The $B_c$ Decays to $P$-wave Charmonium by Improved Bethe-Salpeter Approach

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Abstract

We re-calculate the exclusive semileptonic and nonleptonic decays of $B_c$ meson to a $P$-wave charmonium in terms of the improved Bethe-Salpeter (B-S) approach, which is developed recently. Here the widths for the exclusive semileptonic and nonleptonic decays, the form factors, and the charged lepton spectrums for the semileptonic decays are precisely calculated. To test the concerned approach by comparing with experimental measurements when the experimental data are available, and to have comparisons with the other approaches the results obtained by the approach and those by some approaches else as well as the original B-S approach, which appeared in literature, are comparatively presented and discussed.

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I. INTRODUCTION

The meson $B_c$ is the ground state of the double heavy (both of the components are heavy) quark-antiquark binding system ($\bar{b}c$). In Stand Model (SM) it is an unique meson which carries two different heavy flavors explicitly, thus it decays weakly, that is very different from the ground states of flavor-hidden double heavy mesons, such as $\eta_c$ and $\eta_b$. Namely $B_c$ decays via weak interaction (via virtual $W$ emitting or annihilating) only, while the ground states of flavor-hidden double heavy mesons decay dominantly by annihilating to gluons (strong interaction) or/and photons (electronic interaction). The meson $B_c$ has very rich and experimentally accessible decay channels, so to study the decays of $B_c$ meson is specially interesting. By comparing the experimental and theoretical results of the decays of $B_c$ meson, we can also reach some insight into the binding effects of the heavy quark-antiquark system, which are of QCD nature, besides the knowledge of the weak interaction such as the CKM matrix elements etc.

The meson $B_c$ was first experimentally discovered by the CDF collaboration at Fermilab through the semileptonic decay $B_c \rightarrow J/\psi + l + \nu$ [1], and soon it is confirmed not only by CDF itself via another decay channel $B_c \rightarrow J/\psi + \pi$ [2], but also by the other collaboration D0 at Fermilab [3]. The latest experimental report for its lifetime and mass in PDG [4] is $M_{B_c} = 6.277 \pm 0.006$ GeV and $\tau_{B_c} = (0.453 \pm 0.041) \times 10^{-12}$ s. Because the cross section of $B_c$ production is comparatively small, so to discover it is quite difficult in experiment. Whereas according to the estimates [2-7], that LHC will produce about $5 \times 10^{10} B_c$ events per year, it is expected that more measurements of decays and production of the meson $B_c$ are available soon at LHC (LHCb, CMS, ATLAS), and it must push more studies of the decays of $B_c$ meson forward. So both experimental and theoretical studies on $B_c$ meson now become more interesting.

In fact, the decays of $B_c$ meson can be divided into three categories: i). The anti-bottom quark $\bar{b}$ decays into $\bar{c}$ (or $\bar{u}$) with $c$-quark being as a spectator; ii). The charm quark $c$ decays into $s$ (or $d$) with $\bar{b}$-quark being as a spectator; iii). The two components, $\bar{b}$ and $c$, annihilate weakly. According to the decay products we may realize which one or two even three of the categories play roles in a concerned decay, thus one can measure the CKM elements such as $V_{bc}$, $V_{ub}$, $V_{cs}$, $V_{cd}$ through the decays. In the present paper, we are highlighting the decays of $B_c$ meson to a $P$-wave charmonium, and one may easily to realize that the decays being considered here belong to the category i). Since the lepton spectrum and the weak form factors, which relate to the binding effects (wave functions) precisely, may be measurable in
semi-lepton decays as long as the experimental sample of decay events is great enough, so we will share quite a lot of lights on them.

In fact, one may find a lot of theoretical methods to treat the semi-leptonic and non-leptonic decays of $B_c$ meson, such as the varieties of relativistic constituent quark models [8–15] and QCD sum rules [16, 17] etc in the literature, and moreover one may realize that among the relativistic constituent quark models, the method presented in Ref. [18] and adopted in Refs. [9, 10] is based on the instantaneous version [19] of the Bethe-Salpeter (B-S) equation [20], and the ‘instantaneous treatment’ is also extended to the weak-current matrix elements using the Mandelstam formulation [21], while the adopted approach in Ref. [8] is different from the one presented in Ref. [18], only in the kernel of the B-S equation and the ‘instantaneous treatment’ etc. Recently in [22] an improvement to that of [18] is proposed, and the relativistic effects in the binding systems and decays between the systems may be considered by the new development more properly, especially, considering the fact that, of the new development, the part (factor) for dealing with the binding effects has been applied to study (test) the spectra of positronium (a QED binding system) [23] and double heavy flavor binding systems (QCD binding systems) [24] and quite satisfied results are obtained (see Refs. [23, 24]), so to test the new development [22] when experimental data are available in foreseeable future, in this paper we try to apply the development to the decays of $B_c$ meson to a $P$-wave charmonium and to compare the obtained results with those obtained by old method in Ref. [18] and obtained by other theoretical approaches. Since we suspect that the decays of $B_c$ meson to a $P$-wave charmonium might be more sensitive in testing the effects caused by the improvement than the decays of $B_c$ meson to an $S$-wave charmonium, so here we focus our attention on the decays of $B_c$ meson to a $P$-wave charmonium.

The new development [22] contains two factors: one is about relativistic wave functions which describe bound states with definite quantum numbers, i.e. a relativistic form of wave functions (see Appendix C) which are solutions of the full Salpeter equation (see Appendix B). Note that here we solve the full equations Eqs. (B9, B10, B11), not only the first one Eq. (B9) as other authors did. The other factor of the improvement is about computing the weak-current matrix elements for the decays with the obtained relativistic wave functions as input. It is more 'complete' than that as done in Refs. [9, 10, 18], i.e. the 'complete' formula in Eq. (15).

The paper is organized as follows: the formulations of the exclusive semi-leptonic and non-leptonic decays are outlined in Sec. II. The newly developed formulations, mainly for the
matrix elements of the hadron weak decays, are presented in Sec. III. In Sec. IV, numerical calculations for the exclusive semi-leptonic decays and non-leptonic decays are described, the results and comparisons among the various approaches are presented. Finally the Sec. V is attributed to discussions. In Appendices, the formulations as necessary pieces for the calculations of the decays are given.

II. THE FORMULATIONS FOR EXCLUSIVE SEMI-LEPTONIC DECAYS AND NON-LEPTONIC DECAYS

Let us now derive the formulations for the exclusive semi-leptonic and non-leptonic decays precisely (mainly quoted from Ref. [22]) for numerical calculations later on.

In the following subsections we will focus light on the matrix elements of weak currents, and show how to present the amplitudes of the semileptonic or nonleptonic decays via the matrix elements of weak currents precisely. In fact, one may see that the newly developed method mainly is about the matrix elements of weak currents.

A. The semileptonic decays of $B_c$ meson

The Fig. 1 is a typical Feynman diagram responsible for a semileptonic decay of $B_c$ meson to a charmonium. The corresponding amplitude for the decay can be written as:

$$T = \frac{G_F}{\sqrt{2}} V_{bc} \bar{u}_\nu(p_\nu) \gamma_\mu (1 - \gamma_5) v_l(p_l) \langle \chi_{c}(h_c)(P_f) | J^\mu | B_c(P) \rangle,$$

where $V_{bc}$ is the CKM matrix element, $\langle \chi_{c}(h_c)(P_f) | J^\mu | B_c(P) \rangle$ is the hadronic weak-current matrix element responsible for the decay, and $P, P_f, p_\nu$ and $p_l$ are the momenta of initial

![FIG. 1: The Feynman diagram of a semileptonic decay of $B_c$ meson to a charmonium.](image-url)
state $B_c$, the final $P$-wave state of $(c\bar{c})$ (i.e. $h_c$, $\chi_{c0}$, $\chi_{c1}$, $\chi_{c2}$ and their excited states), the neutrino and the charged lepton respectively.

Generally, the form factors are defined in terms of the matrix elements of weak current responsible for the decays appearing in Eq. (1). Namely for the decay of $B_c$ meson to scalar charmonium $\chi_{c0}$, the form factors $s_+$ and $s_-$ are defined as follows:

\[
\langle \chi_{c0}(P_f)|V^\mu|B_c(P)\rangle = 0, \\
\langle \chi_{c0}(P_f)|A^\mu|B_c(P)\rangle = s_+(P + P_f)^\mu + s_-(P - P_f)^\mu. \tag{2}
\]

For the decay of $B_c$ meson to vector charmonium $\chi_{c1}$, the relevant form factors $f$, $u_1$, $u_2$ and $g$ are defined as follows:

\[
\langle \chi_{c1}(P_f)|V^\mu|B_c(P)\rangle = f(M + M_f)\varepsilon^\mu + [u_1P^\mu + u_2P_f^\mu] \frac{\varepsilon \cdot P}{M}, \\
\langle \chi_{c1}(P_f)|A^\mu|B_c(P)\rangle = \frac{2g}{M + M_f}i\varepsilon^{\mu\rho\sigma\alpha} \epsilon_\rho P_\sigma P_f^\alpha. \tag{3}
\]

For the decay of $B_c$ meson to tensor charmonium $h_c$, the relevant form factors $V_0$, $V_1$, $V_2$ and $V_3$ are defined as follows:

\[
\langle h_c(P_f)|V^\mu|B_c(P)\rangle = V_0(M + M_f)\varepsilon^\mu + [V_1P^\mu + V_2P_f^\mu] \frac{\varepsilon \cdot P}{M}, \\
\langle h_c(P_f)|A^\mu|B_c(P)\rangle = \frac{2V_3}{M + M_f}i\varepsilon^{\mu\rho\sigma\alpha} \epsilon_\rho P_\sigma P_f^\alpha. \tag{4}
\]

For the decay of $B_c$ meson to tensor charmonium $\chi_{c2}$, the relevant form factors $k$, $c_1$, $c_2$ and $h$ are defined as follows:

\[
\langle \chi_{c2}(P_f)|A^\mu|B_c(P)\rangle = k(M + M_f)\varepsilon^{\mu\alpha} \frac{P_\alpha}{M} + \varepsilon_{\alpha\beta} \frac{P_\alpha P_\beta}{M^2}(c_1P^\mu + c_2P_f^\mu), \\
\langle \chi_{c2}(P_f)|V^\mu|B_c(P)\rangle = \frac{2h}{M + M_f}i\varepsilon^{\mu\alpha\beta\sigma} \frac{P_\alpha}{M} \varepsilon_{\alpha\beta\rho\sigma} P_\rho P_f^\sigma. \tag{5}
\]

In the case without considering polarization, we have the squared decay-amplitude with the polarizations in final states being summed:

\[
\Sigma_{s_\nu,s_t} S_{\chi_{c(hc)}} |T|^2 = \frac{G_F^2}{2} |V_{bc}|^2 l_{\mu\nu} h^{\mu\nu}, \tag{6}
\]

where $l_{\mu\nu}$ is the leptonic tensor:

\[
l_{\mu\nu} = \Sigma_{s_\nu,s_t} \bar{v}_l(p_t)\gamma_\mu(1 - \gamma_5)u_{\nu}(p_\nu)\bar{u}_\nu(p_\nu)\gamma_\nu(1 - \gamma_5)u_l(p_t),
\]
and the hadronic tensor relating to the weak-current in Eq. (1) is

\[ h^{\mu\nu} \equiv \Sigma_{\chi_c(h_c)} \langle B_c(P) | J^{\mu+} | \chi_c(h_c)(P_f) \rangle \langle \chi_c(h_c)(P_f) | J^{\nu} | B_c(P) \rangle \]

\[ = -\alpha g^{\mu\nu} + \beta_{++}(P + P_f)^\mu(P + P_f)^\nu + \beta_{+-}(P + P_f)^\mu(P - P_f)^\nu + \beta_{-+}(P - P_f)^\mu(P + P_f)^\nu + \beta_{--}(P - P_f)^\mu(P - P_f)^\nu + i\gamma \epsilon^{\mu\nu\rho\sigma}(P + P_f)_\rho(P - P_f)_\sigma, \]

(7)

where the functions \( \alpha, \beta_{++}, \beta_{+-}, \beta_{-+}, \beta_{--}, \gamma \) are related to the form factors and we put the relations in Appendix A precisely.

The total decay width \( \Gamma \) can be written as:

\[ \Gamma = \frac{1}{2M(2\pi)^9} \int \frac{d^3\vec{P}_f}{2E_f} \frac{d^3\vec{P}_i}{2E_i} \frac{d^3\vec{P}_\nu}{2E_\nu} (2\pi)^4 \delta^4(P - P_f - p_l - p_\nu) \Sigma_{s_\nu, s_l, s_{\chi_c(h_c)}} |T|^2, \]

(8)

where \( E_f, E_i \) and \( E_\nu \) are the energies of the charmonium, the charged lepton and the neutrino respectively. If we define \( x \equiv E_i/M, \ y \equiv (P - P_f)^2/M^2 \), the differential width of the decay can be reduced to:

\[ \frac{d^2\Gamma}{dx dy} = |V_{bc}|^2 \frac{G_F^2 M_5}{64\pi^3} \left( \begin{array}{c} \frac{2\alpha}{M^2} \left( y - \frac{m_i^2}{M^2} \right) \\ + \beta_{++} \left[ 4 \left( 2x(1 - \frac{M_f^2}{M^2} + y) - 4x^2 - y \right) + \frac{m_i^2}{M^2} \left( 8x + 4 \frac{M_f^2}{M^2} - 3y - \frac{m_i^2}{M^2} \right) \right] \\ + (\beta_{+-} + \beta_{-+}) \frac{m_i^2}{M^2} \left( 2 - 4x + y - 2 \frac{M_f^2}{M^2} + \frac{m_i^2}{M^2} \right) + \beta_{--} \frac{m_i^2}{M^2} \left( y - \frac{m_i^2}{M^2} \right) \\ - 2\gamma y \left( 1 - \frac{M_f^2}{M^2} - 4x + y + \frac{M_f^2}{M^2} \right) + 2\gamma \frac{M_f^2}{M^2} \left( 1 - \frac{M_f^2}{M^2} \right) \end{array} \right), \]

(9)

here \( M \) is the mass of the meson \( B_c \), \( M_f \) is the mass of the charmonium in final state, and the total width of the decay is just an integration of the differential width i.e. \( \Gamma = \int dx \int dy \frac{d\Gamma}{dx dy} \).

Thus the key problem for calculating the semileptonic decays is turned to calculating the hadronic weak-current matrix elements.

**B. The nonleptonic decays of \( B_c \) meson**

In this subsection we mainly consider the nonleptonic two-body decays to a \( P \)-wave charmonium, i.e. \( \) decays \( B_c \rightarrow M_1 M_2 \) where \( M_1 \) is a \( P \)-wave charmonium and \( M_2 \) is a common meson. Fig. 2 is the Feynman diagram for the decays via the relevant effective Hamiltonian \( H_{eff} \) [25, 26]:

\[ H_{eff} = \frac{G_F}{\sqrt{2}} \left\{ V_{cb} \left[ c_1(\mu)O_1^{cb} + c_2(\mu)O_2^{gb} \right] - V_{tb} V_{tq}^{*} \left( \sum_{i=3}^{10} C_i(\mu)O_i \right) \right\} + h.c., \]

(10)
FIG. 2: The Feynman diagram of a nonleptonic decay of $B_c$ to two mesons $M_1$ (a charmonium) and $M_2$ (a common meson).

where $G_F$ is the Fermi constant, $q = d, s$, $V_{ij}$ are the CKM matrix elements and $c_i(\mu)$ are the scale-dependent Wilson coefficients. $O_i$ are the operators constructed by four quark fields and have $J^\mu J_\mu$ structure as follows:

$$O_1^{cb} = [V_{ud}(\bar{d}_a u_\alpha)_{V-A} + V_{us}(\bar{s}_a u_\alpha)_{V-A} + V_{cd}(\bar{d}_a c_\alpha)_{V-A} + V_{cs}(\bar{s}_a c_\alpha)_{V-A}] (\bar{c}_\beta b_\beta)_{V-A},$$
$$O_2^{cb} = [V_{ud}(\bar{d}_a u_\beta)_{V-A} + V_{us}(\bar{s}_a u_\beta)_{V-A} + V_{cd}(\bar{d}_a c_\beta)_{V-A} + V_{cs}(\bar{s}_a c_\beta)_{V-A}] (\bar{c}_\beta b_\alpha)_{V-A},$$
$$O_3 = (\bar{q}_a b_\alpha)_{V-A} \sum_{q'} (\bar{q}_\beta' q_\beta')_{V-A}, \quad O_4 = (\bar{q}_\beta b_\alpha)_{V-A} \sum_{q'} (\bar{q}_\alpha q_\beta')_{V-A},$$
$$O_5 = (\bar{q}_a b_\alpha)_{V-A} \sum_{q'} (\bar{q}_\beta' q_\beta')_{V+A}, \quad O_6 = (\bar{q}_\beta b_\alpha)_{V-A} \sum_{q'} (\bar{q}_\alpha q_\beta')_{V+A},$$
$$O_7 = \frac{3}{2} (\bar{q}_a b_\alpha)_{V-A} \sum_{q'} e_{q'} (\bar{q}_\beta' q_\beta')_{V+A}, \quad O_8 = \frac{3}{2} (\bar{q}_\beta b_\alpha)_{V-A} \sum_{q'} e_{q'} (\bar{q}_\alpha q_\beta')_{V+A},$$
$$O_9 = \frac{3}{2} (\bar{q}_a b_\alpha)_{V-A} \sum_{q'} e_{q'} (\bar{q}_\beta' q_\beta')_{V-A}, \quad O_{10} = \frac{3}{2} (\bar{q}_\beta b_\alpha)_{V-A} \sum_{q'} e_{q'} (\bar{q}_\alpha q_\beta')_{V-A},$$

where $(\bar{q}_1 q_2)_{V-A} = \bar{q}_1 \gamma^\mu (1 - \gamma_5) q_2$. The operators $O_1$ and $O_2$ are the current-current (tree) operators, $O_3, ..., O_6$ are the QCD-penguin operators and $O_7, ..., O_{10}$ are the electroweak penguin operators. Since we calculate the decay up to leading order, we just consider the contribution of $O_1$ and $O_2$.

Here we apply the so-called naive factorization to $H_{eff}$ i.e. the operators $O_i$ [27], so the nonleptonic two-body decay amplitude $T$ can be reduced to a product of a transition matrix element of a weak current $(M_1|J^\mu|B_c)$ and an annihilation matrix element of another weak current $(M_2|J_\mu|0)$:

$$T = \langle M_1 M_2 | H_{eff} | B_c \rangle \approx \langle M_1 | J^\mu | B_c \rangle \langle M_2 | J_\mu | 0 \rangle,$$

while the annihilation matrix element is relating to a decay constant directly. The reason why we adopt the naive factorization here is that it works well enough due to the fact that
all the decays concerned in this paper are ‘constrained’ to those in them the quark $c$ as a
‘spectator’ goes from initial $B_c$ meson into the final meson $M_1$ always, thus as pointed by
the authors of [28, 29], in the concerned cases the corrections to the naive factorization are
suppressed.

Since $M_1 = \chi_c(h_c)$, the matrix element $\langle M_1|J^\mu|B_c(\bar{P})\rangle$ is just the hadronic weak-current
matrix element appearing in the previous subsection, but different from it by momentum
transfer being fixed (owing to the decays are of one to two-body). The annihilation matrix
element $\langle M_2|J_\mu|0\rangle$ with $J^\mu = (\bar{q}_1q_2)V_{-A}$ is related to the decay constant of a ‘common meson’
$M_2$ and can be measured via proper processes generally.

Precisely, let us now ‘restrict ourselves’ to analyze the $B_c$ nonleptonic decays to the 
$P$-wave charmonium and the $\pi^+, \rho^+$, etc, which are governed by the weak decay $\bar{b} \rightarrow \bar{c}ud$, or to
the $P$-wave charmonium and $K^+, K^*$, etc, which are governed by the weak decay $\bar{b} \rightarrow \bar{c}us\bar{s}$. As an example, under naive factorization, we have the decay amplitude of $B_c \rightarrow \chi_{c0}\rho^+$ as follows:

$$T(B_c \rightarrow \chi_{c0}\rho^+) = \frac{G_F}{\sqrt{2}}V_{cb}V_{ud}^* a_1(\mu)\langle \chi_{c0}|J^\mu|B_c\rangle \langle \rho^+|J_\mu|0\rangle,$$

(13)

here $a_1 = c_1 + \frac{1}{N_c} c_2$ and $N_c = 3$ is the number of colors.

Since $\langle M_2|J_\mu|0\rangle$ is relating to the decay constant of the meson $M_2$ directly, so to calculate
the widths of the non-leptonic decays is straightforward when the weak-current transition
matrix elements $\langle M_1|J^\mu|B_c(\bar{P})\rangle$ are well calculated. Thus one may see that the problem
to calculate the non-leptonic decays is essentially attributed to calculating the hadronic
weak-current matrix elements $\langle M_1|V^\mu|B_c(\bar{P})\rangle$ and $\langle M_1|A^\mu|B_c(\bar{P})\rangle$ appearing in the above
subsection for semileptonic decays.

III. COMPUTATION OF THE TRANSITION-MATRIX ELEMENTS FOR WEAK-CURRENTS

From the section above, we can see that to calculate the weak currents matrix elements
$\langle M_1|J^\mu|B_c(\bar{P})\rangle$ is the key problem for the concerned semileptonic and nonleptonic decays,
so let us now explain the reason why and show how to apply the newly developed method
[22] to calculate the matrix elements. In fact it is also to prepare necessary formulae for
final numerical calculations.

Here the weak-current matrix elements are for ‘transitions’ from a state of a double heavy
meson to another double heavy meson. Due to the mass difference of the two states, the
relativistic effects for the transitions are great, that a proper formulation to deal with the relativistic effects is desired. It is known that the approach of relativistic B-S equation for the bound states and Mandelstam formulation for the transition matrix elements may be taken into account quite well, and furthermore the B-S equation and Mandelstam formulation even under ‘instantaneous approximation’ still works, because here the involved mesons are double heavy. While the newly developed method [22], which applies the ‘instantaneous approximation’ to the current matrix elements and B-S equation completely, should be better than the original one in Ref. [18], where the ‘instantaneous approximation’ is applied incompletely. The ‘completeness’ here means to apply it to the B-S equation, the solutions (B-S wave functions) and the transition matrix element (under Mandelstam formulation) properly, and let us outline it below.

According to the Mandelstam formulation [21], the corresponding hadronic matrix elements of weak current between the double heavy meson $B_c$ in initial state and the double heavy meson $\chi_c(h_c)$ in final state, appearing in Eq. (1), Eq. (12) and Eq. (13), can be written as:

$$
\langle \chi_c(h_c)(P_f)|J^\mu|B_c(P)\rangle
= i \int \frac{d^4q d^4q'}{(2\pi)^4} Tr \left[ \chi_{\chi_c(h_c)}(P', q')(\not p_1 - m_1)\chi_{\beta_c}(P, q)V_{cb}\gamma^\mu(1 - \gamma_5)\delta(p_1 - p'_1) \right]
$$

$$
= i \int \frac{d^4q}{(2\pi)^4} Tr \left[ \chi_{\chi_c(h_c)}(P', q')(\not \alpha_1 P + \not q - m_1)\chi_{\beta_c}(P, q)V_{cb}\gamma^\mu(1 - \gamma_5) \right],
$$

(14)

where $p_1 = \alpha_1 P + q$ ($\alpha_1 \equiv \frac{m_1}{m_1 + m_2}$), $p_2 = \alpha_2 P - q$ ($\alpha_2 \equiv \frac{m_2}{m_1 + m_2}$) are the momenta of $c$-quark and $\bar{b}$-quark respectively inside $B_c$ meson; $p'_1 = \alpha'_1 P_f + q'$ ($\alpha'_1 \equiv \frac{m'_1}{m'_1 + m_2}$), $p'_2 = \alpha'_2 P_f - q'$ ($\alpha'_2 \equiv \frac{m'_2}{m'_1 + m_2}$) are the momenta of $c$-quark and $\bar{c}$-quark respectively inside the $P$-wave charmonium $\chi_c(h_c)$; moreover, for the final result (the last line of Eq. (14)) we have $P = P_f + p_l + p_v$ and $q' = \alpha_1 P + q - \alpha'_1 P_f$.

The newly developed method [22] essentially is to apply the ‘instantaneous approximation’ to the current matrix elements and the B-S equation completely, to outline it and for ‘applying the instantaneous approximation’ in a covariant way, we need to decompose the relative momentum $q$ into two components: the time-like one $q^\mu_\parallel$ and the space-like one $q^\mu_\perp$ as follows:

$$
q^\mu = q^\mu_\parallel + q^\mu_\perp, \quad q^\mu_\parallel \equiv \frac{P \cdot q}{M^2} P^\mu, \quad q^\mu_\perp \equiv q^\mu - q^\mu_\parallel,
$$

$$
P'^\mu = P'^\mu_\parallel + P'^\mu_\perp, \quad P'^\mu_\parallel \equiv \frac{(P \cdot P')}{M^2} P^\mu, \quad P'^\mu_\perp \equiv P'^\mu - P'^\mu_\parallel;
$$
and
\[ q^\mu = q_\|^{\mu} + q_\|^\mu, \quad q_\|^{\mu} \equiv (P \cdot q^*/M^2)P^\mu, \quad q_\|^\mu \equiv q^\mu - q_\|^{\mu}, \]
where \( M \) is the mass of the meson \( B_c \), and we may further have two Lorentz invariant variables \( q_P \equiv \frac{P^\mu}{M} \) and \( q_T \equiv \sqrt{-q_\|^2} \).

The ‘instantaneous approximation’ applying to the matrix element is just to carry out the integration of \( dq_\| \) by a contour one on Eq. (14) precisely and to obtain the result below:

\[
\langle \chi_c(h_c)(P_f) | J^\mu | B_c(P) \rangle = i \int \frac{d^3q_\perp}{(2\pi)^3} Tr \left[ \chi_c(h_c)(P', q') (\alpha_1 P + \not{q} - m_1) \chi_{B_c}(P, q) V_{cb} \gamma^\mu (1 - \gamma_5) \right]
\]
\[
= \int \frac{d^3q_\perp}{(2\pi)^3} Tr \left\{ \left[ \varphi^{\mu+}(q_\perp) \frac{P}{M} \varphi^{\mu+}(q_\perp) + \varphi^{\mu+}(q_\perp') \frac{P}{M} \varphi^{\mu-}(q_\perp) \right.ight.
\]
\[
- \bar{\varphi}^{\mu-}(q_\perp) \frac{P}{M} \varphi^{\mu+}(q_\perp) - \bar{\varphi}^{\mu+}(q_\perp') \frac{P}{M} \varphi^{\mu-}(q_\perp) \right. \}
\[
\left. + \varphi^{\mu-}(q_\perp') \frac{P}{M} \varphi^{\mu-}(q_\perp) - \bar{\varphi}^{\mu-}(q_\perp') \frac{P}{M} \varphi^{\mu-}(q_\perp) \right\} \gamma^\mu (1 - \gamma_5) \right\}, \tag{15}
\]
where:
\[
\varphi^{\mu+}(q_\perp) = \frac{\Lambda^{+}_1(q_\perp) \eta(q_\perp) \Lambda^{+}_2(q_\perp)}{M - \omega_1 - \omega_2}, \quad \varphi^{\mu+}(q_\perp') = \frac{\Lambda^{+}_2(q_\perp') \bar{\eta}(q_\perp') \Lambda^{+}_1(q_\perp')}{E_f - \omega'_1 - \omega'_2},
\]
\[
\varphi^{\mu-}(q_\perp) = \frac{\Lambda^{-}_1(q_\perp) \eta(q_\perp) \Lambda^{-}_2(q_\perp)}{M + \omega_1 + \omega_2}, \quad \varphi^{\mu-}(q_\perp') = \frac{\Lambda^{-}_2(q_\perp') \bar{\eta}(q_\perp') \Lambda^{-}_1(q_\perp')}{E_f + \omega'_1 + \omega'_2},
\]
\[
\psi^{\mu+}(q_\perp) = \frac{\Lambda^{+}_1(q_\perp) \eta(q_\perp) \Lambda^{+}_2(q_\perp)}{M - \omega_2 - \omega'_2 - E_f}, \quad \psi^{\mu+}(q_\perp') = \frac{\Lambda^{+}_2(q_\perp') \bar{\eta}(q_\perp') \Lambda^{+}_1(q_\perp')}{M - \omega_2 - \omega'_2 - E_f},
\]
\[
\psi^{\mu-}(q_\perp) = \frac{\Lambda^{-}_1(q_\perp) \eta(q_\perp) \Lambda^{-}_2(q_\perp)}{M + \omega_2 + \omega'_2 - E_f}, \quad \psi^{\mu-}(q_\perp') = \frac{\Lambda^{-}_2(q_\perp') \bar{\eta}(q_\perp') \Lambda^{-}_1(q_\perp')}{M + \omega_2 + \omega'_2 - E_f}, \tag{16}
\]

\( \varphi^{\mu+}(q_\perp), \psi^{\mu+}(q_\perp), \varphi^{\mu-}(q_\perp), \psi^{\mu-}(q_\perp) \) are B-S wave functions as the B-S equation solutions under ‘complete instantaneous approximation’ \cite{23, 24} and with ‘energy projection’ \( \Lambda^{\pm} \) of the mesons in initial and final states properly. The precise definitions of the ‘energy projection’ and the B-S ‘vertex’ \( \eta_P, \bar{\eta}_P (\eta'_P, \bar{\eta}'_P) \) are presented in Appendix B. One may also see that the four equations, Eqs. \cite{39, 40, 41, 42}, are B-S equations under the complete instantaneous approximation, instead of the incomplete instantaneous approximation which only considering the Eq. \cite{39}.

Namely the ‘improvements’ from the ‘newly development method’ are attributed to: 1) with the complete instantaneous approximation to current matrix element, as a result, there are six terms in the squared bracket of Eq. \cite{15} instead of the first term

\[
\langle \chi_c(h_c)(P_f) | J^\mu | B_c(P) \rangle = \int \frac{d^3q_\perp}{(2\pi)^3} Tr \left\{ \varphi^{\mu+}(q_\perp) \frac{P}{M} \varphi^{\mu+}(q_\perp) \gamma^\mu (1 - \gamma_5) \right\}, \tag{17}
\]
is only kept; ii). the B-S wave functions hidden in \( \varphi^{ij}(q_{\perp}) \), \( \psi^{ij}(q_{\perp}) \) and \( \bar{\varphi}^{ij}(q'_{P_{\perp}}) \), \( \bar{\psi}^{ij}(q'_{P_{\perp}}) \) are solved under complete instantaneous approximation to the B-S equation. For the point i), since the considered double heavy meson, \( B_c \) or \( \chi_c(h_c) \), is weak binding system i.e. the binding energy \( \varepsilon \equiv M - \omega_1 - \omega_2 \) (or \( \varepsilon \equiv E_f - \omega'_1 - \omega'_2 \)) is small \( (\frac{\varepsilon}{M} \ll O(1)) \), thus from Eq. (16) we are sure that \( \varphi^{++}(q_{\perp}) \) and \( \bar{\varphi}^{++}(q'_{P_{\perp}}) \) are much greater than the others \( \varphi^{ij}(q_{\perp}) \), \( \psi^{ij}(q_{\perp}) \) and \( \bar{\varphi}^{ij}(q'_{P_{\perp}}) \), \( \bar{\psi}^{ij}(q'_{P_{\perp}}) \), so that using the Eq. (17) instead of Eq. (15) is a very good approximation, which we have precisely examined by considering the decay \( B_c \rightarrow \chi_c l \bar{\nu}_l \) as an example: in fact, the contributions of the second term and third term of Eq. (15) to the form factor are less than the one of first term of Eq. (15) roughly by a factor \( 10^{-2} \sim 10^{-3} \) times. If the first three terms are considered, the decay width is \( 1.85 \times 10^{-15} \) GeV, while if only the first term is considered, the decay width is \( 1.87 \times 10^{-15} \) GeV, i.e. the two results are very similar. So the approximation is very good and we may use Eq. (17) instead of Eq. (15) to compute the weak-current matrix elements safely.

IV. NUMERICAL CALCULATIONS AND RESULTS WITH PROPER COMPARISONS

In this section, based on the formulations obtained in the paper, we evaluate the decay widths for semileptonic and nonleptonic decays and some interesting quantities else for semileptonic decays, such as form factors and charged lepton spectrum etc and then discuss them briefly.

First of all, we need to fix the parameters appearing in the framework. We adjusted the parameters \( a = e = 2.7183 \), \( \lambda = 0.21 \) GeV\(^2\), \( \Lambda_{QCD} = 0.27 \) GeV, \( m_b = 4.96 \) GeV, \( m_c = 1.62 \) GeV and \( V_0 \) for the B-S kernel as those in Refs. \[24, 30, 31\], which as the best input for spectroscopy, then the spectra of the mesons and the masses \( M_{B_c} = 6.276 \) GeV, \( M_{\chi_{c0}} = 3.414 \) GeV, \( M_{\chi_{c1}} = 3.510 \) GeV, \( M_{\chi_{c2}} = 3.555 \) GeV, \( M_{h_c} = 3.526 \) GeV etc \[24\], which are used in this paper, are obtained, moreover the decay constants, average energies as well as annihilations of quarkonia are fitted \[30, 32\].

With the obtained B-S wave functions (under the formulation defined in Appendix B) and as a next step, we substitute the functions into \( \varphi^{++}(q_{\perp}) \) and \( \bar{\varphi}^{++}(q'_{P_{\perp}}) \), so that they are related to the components of the B-S wave functions precisely as depicted in Appendix C. With the formula Eq. (17), finally we represent the hadronic transition weak-current matrix elements as proper integrations of the components of the B-S wave functions. As final re-
TABLE I: The semileptonic decay widths (in the unit $10^{-15}$GeV)

| Mode              | This work | [12] | [13] | [15] | [10] | [16] | [17] |
|-------------------|-----------|------|------|------|------|------|------|
| $B_c^+ \rightarrow \chi_c0\nu$ | 1.87 ± 0.46 | 1.27 | 2.52 | 1.55 | 1.69 | 2.60 ± 0.73 |
| $B_c^+ \rightarrow \chi_c0\tau
$ | 0.23 ± 0.12 | 0.11 | 0.26 | 0.19 | 0.25 | 0.7 ± 0.23 |
| $B_c^+ \rightarrow \chi_c1\nu$ | 1.52 ± 0.45 | 1.18 | 1.40 | 0.94 | 2.21 | 2.09 ± 0.60 |
| $B_c^+ \rightarrow \chi_c1\tau
$ | 0.14 ± 0.10 | 0.13 | 0.17 | 0.10 | 0.35 | 0.21 ± 0.06 |
| $B_c^+ \rightarrow \chi_c2\nu$ | 1.50 ± 0.39 | 2.27 | 2.92 | 1.89 | 2.73 |
| $B_c^+ \rightarrow \chi_c2\tau
$ | 0.12 ± 0.07 | 0.13 | 0.20 | 0.13 | 0.42 |
| $B_c^+ \rightarrow h_c\nu$ | 3.98 ± 1.10 | 1.38 | 4.42 | 2.4 | 2.51 | 2.03 ± 0.57 |
| $B_c^+ \rightarrow h_c\tau
$ | 0.28 ± 0.20 | 0.11 | 0.38 | 0.21 | 0.36 | 0.20 ± 0.05 |

Suits of this paper, the decay widths for the semileptonic and nonleptonic decays and some interesting quantities else for the semileptonic decays, such as form factors and charged lepton spectrum etc, are straightforwardly calculated numerically. In the following subsections we present the results for the semileptonic decays and nonleptonic decays separately.

A. The semi-leptonic decays

When the weak current transition matrix element for a definite semi-leptonic decay is calculated precisely and the values of the CKM matrix elements $|V_{ud}| = 0.974$, $|V_{us}| = 0.225$, $|V_{bc}| = 0.0406$ [4] are given, not only the decay width can be calculated straightforwardly, but also the form factors may be extracted out. Moreover as ‘semifinished product’, the spectrum of the charged lepton which may be measurable experimentally can be also acquired too. Namely the functions $\alpha$, $\beta_{++}$, $\beta_{+-}$, $\beta_{-+}$, $\beta_{--}$, $\gamma$ appearing in the spectrum of the charged lepton (see Eq. (9)) are related to the form factors directly as shown in Appendix A precisely. Therefore when we calculate and present the results for semi-leptonic decays, not only those of the decay widths but also the spectrums of the charged lepton in the decays are considered. Since $\tau$ lepton is quite massive and $m_\tau \approx m_e$ is quite a good approximation for the $B_c$ meson decays, so when we calculate and present the widths and the spectrums of the charged lepton for the decays, only the cases that the lepton being electron or $\tau$ are considered.

Note that since the input B-S wave functions by solving the B-S equation for the double
FIG. 3: The form factors of the $B_c$ decays to a $P$-wave charmonium defined as in Eq. (A1), Eq. (A2), Eq. (A3) and Eq. (A4) and $t = q^2 = (P - P_f)^2 = M_c^2 + M_f^2 - 2ME_f$ ($t_m$ is the maximum of $t$).

heavy mesons which are involved in the transition matrix elements of weak current have uncertainties, due to the parameters fitting to fix the B-S kernel and quark masses, the way to solve the B-S equation numerically, and the approximation from Eq. (15) to Eq. (17) for the transition matrix elements of the weak currents is taken etc, so in the numerical results obtained finally there are certain errors. To consider the uncertainties caused by the input parameters, we changed all the input parameters simultaneously within 5% of the center values, then we get the uncertainties of numerical results for the semi-leptonic decays and the non-leptonic decays shown in Table. I. We find that the uncertainties of the decays $B_c \rightarrow h_c(\chi_c) + e + \nu_e$ vary up to 30% of center values, while the uncertainties of
FIG. 4: The energy spectrums of the charged lepton in the \( B_c \) semileptonic decays to \( P \)-wave charmoniums. The left figure is for \( B_c \to \chi_{c0,1,2}(h_c)e\nu \) and the right figure is for \( B_c \to \chi_{c0,1,2}(h_c)\tau\nu \).

Where the solid lines are the results for \( \chi_{c0} \), the dash lines are for \( \chi_{c1} \), the dot lines are for \( \chi_{c2} \) and the dot-dash lines are for \( h_c \).

\( B_c \to h_c(\chi_c) + \tau + \nu_\tau \) are up to 60\% in Table. I, the reason is that the phase spaces for \( B_c \to h_c(\chi_c) + \tau + \nu_\tau \) are smaller than the ones for \( B_c \to h_c(\chi_c) + e + \nu_e \) because of the heavy \( \tau \) lepton, and the the uncertainties for the former are more sensitive to the changes of the phase space than the latter.

To compare with the results obtained by the other approaches, we present the decay widths calculated out this work with error bar and the results obtained by the other approaches by putting them together in a table i.e. Table I.

In addition we also present the obtained form factors and the spectrums of the charged lepton in the decays in Fig. 3 and Fig. 4 respectively. To compare with the results of the previous work Ref. [10], we draw the curves of the spectrums of charged lepton obtained by this work and the work Ref. [10] in Fig. 5. Whereas in order to see the tendency of the form factors and the lepton spectrum clearly and we suspect that at present stage it is enough, so in the figures we draw the curves with the center values but not involve the errors precisely.
FIG. 5: The energy spectrums of the charged lepton in the $B_c$ semileptonic decays to $P$-wave charmoniums respectively. The solid lines are the results of this work, the dash lines are the results of Ref. [10].

B. The non-leptonic decays

The exclusive non-leptonic decays are of two-body in final states, thus the hadronic transition matrix elements of weak-currents appearing in Eq. (13) have a fixed momentum transfer $t = m_2^2$ (the mass squared of the other meson $M_2$ in the decay $B_c \to M_1 M_2$ and $M_1 = \chi_c$ or $h_c$). In fact the transition matrix elements have been already calculated in the above subsection of semi-leptonic decays. To calculate the decay widths, from Eq. (13), now we need to calculate the annihilation matrix element of the weak current such as $\langle M_2 | J_{\mu} | 0 \rangle$ additionally. It is known that the annihilation matrix element is related to the 'decay
constant’ $f_{M_2}$ directly, and the decay constant $f_P$, $f_V$ or $f_A$ of a pseudoscalar meson, a vector meson or an axial vector meson may be extracted from experimental data for the pure leptonic decays of the relevant mesons, but they may also be calculated by models, such as the one in Ref. [30] although there are some debates. In this work we adopt the values of the decay constants: $f_\pi = 0.130$ GeV, $f_\rho = 0.205$ GeV, $f_K = 0.156$ GeV, $f_{K^*} = 0.217$ GeV etc for numerical calculations. Then the relevant decay widths for the concerned non-leptonic decays are calculated. As the final results, we present the decay widths by our method and the others’ methods else in Table II. Note that the uncertainties in Table II are estimated as done in the previous subsection for semileptonic decays.

For comparison precisely with the other approaches and experimental measurements in future, we take the values $a_1 = 1.14$ for non-leptonic decays as done in most references, and

| Mode                  | This work          | [12]    | [13]    | [15]    | [10]    |
|-----------------------|--------------------|---------|---------|---------|---------|
| $B_c^+ \rightarrow \chi_0\pi^+$ | $(0.34 \pm 0.04)a_1^2$ | $0.23a_1^2$ | $0.622a_1^2$ | $0.28a_1^2$ | $0.317a_1^2$ |
| $B_c^+ \rightarrow \chi_1\pi^+$ | $(0.023 \pm 0.002)a_1^2$ | $0.22a_1^2$ | $0.076a_1^2$ | $0.0015a_1^2$ | $0.0815a_1^2$ |
| $B_c^+ \rightarrow \chi_2\pi^+$ | $(0.24 \pm 0.05)a_1^2$ | $0.41a_1^2$ | $0.518a_1^2$ | $0.24a_1^2$ | $0.277a_1^2$ |
| $B_c^+ \rightarrow h_\pi^+$ | $(1.10 \pm 0.16)a_1^2$ | $0.51a_1^2$ | $1.24a_1^2$ | $0.58a_1^2$ | $0.569a_1^2$ |
| $B_c^+ \rightarrow \chi_0\rho^+$ | $(0.85 \pm 0.10)a_1^2$ | $0.64a_1^2$ | $1.47a_1^2$ | $0.73a_1^2$ | $0.806a_1^2$ |
| $B_c^+ \rightarrow \chi_1\rho^+$ | $(0.25 \pm 0.02)a_1^2$ | $0.16a_1^2$ | $0.326a_1^2$ | $0.11a_1^2$ | $0.331a_1^2$ |
| $B_c^+ \rightarrow \chi_2\rho^+$ | $(0.62 \pm 0.19)a_1^2$ | $1.16a_1^2$ | $3.26a_1^2$ | $0.71a_1^2$ | $0.579a_1^2$ |
| $B_c^+ \rightarrow h_\rho^+$ | $(2.50 \pm 0.50)a_1^2$ | $1.11a_1^2$ | $2.78a_1^2$ | $1.41a_1^2$ | $1.40a_1^2$ |
| $B_c^+ \rightarrow \chi_0K^+$ | $(0.026 \pm 0.003)a_1^2$ | $0.018a_1^2$ | $0.0472a_1^2$ | $0.022a_1^2$ | $0.0035a_1^2$ |
| $B_c^+ \rightarrow \chi_1K^+$ | $(0.0018 \pm 0.0002)a_1^2$ | $0.016a_1^2$ | $0.0057a_1^2$ | $0.00012a_1^2$ | $0.0058a_1^2$ |
| $B_c^+ \rightarrow \chi_2K^+$ | $(0.018 \pm 0.0003)a_1^2$ | $0.031a_1^2$ | $0.0384a_1^2$ | $0.018a_1^2$ | $0.00199a_1^2$ |
| $B_c^+ \rightarrow h_\omega K^+$ | $(0.082 \pm 0.012)a_1^2$ | $0.039a_1^2$ | $0.0939a_1^2$ | $0.045a_1^2$ | $0.0043a_1^2$ |
| $B_c^+ \rightarrow \chi_0K^{*+}$ | $(0.050 \pm 0.006)a_1^2$ | $0.045a_1^2$ | $0.0787a_1^2$ | $0.041a_1^2$ | $0.00443a_1^2$ |
| $B_c^+ \rightarrow \chi_1K^{*+}$ | $(0.018 \pm 0.001)a_1^2$ | $0.01a_1^2$ | $0.0201a_1^2$ | $0.008a_1^2$ | $0.00205a_1^2$ |
| $B_c^+ \rightarrow \chi_2K^{*+}$ | $(0.037 \pm 0.007)a_1^2$ | $0.082a_1^2$ | $0.0732a_1^2$ | $0.041a_1^2$ | $0.00348a_1^2$ |
| $B_c^+ \rightarrow h_\omega K^{*+}$ | $(0.14 \pm 0.02)a_1^2$ | $0.077a_1^2$ | $0.146a_1^2$ | $0.078a_1^2$ | $0.0076a_1^2$ |
TABLE III: Branching ratios (in %) of $B_c$ decays calculated for the $B_c$ lifetime $\tau_{B_c} = 0.453$ ps and $a_1 = 1.14$.

| Decay | Br         | Decay | Br         |
|-------|------------|-------|------------|
| $B_c^+ \rightarrow \chi_{c0} e\nu$ | $0.13 \pm 0.03$ | $B_c^+ \rightarrow \chi_{c0} \tau\nu$ | $0.016 \pm 0.008$ |
| $B_c^+ \rightarrow \chi_{c1} e\nu$ | $0.11 \pm 0.03$ | $B_c^+ \rightarrow \chi_{c1} \tau\nu$ | $0.0097 \pm 0.0065$ |
| $B_c^+ \rightarrow \chi_{c2} e\nu$ | $0.10 \pm 0.03$ | $B_c^+ \rightarrow \chi_{c2} \tau\nu$ | $0.0082 \pm 0.0048$ |
| $B_c^+ \rightarrow h_c e\nu$ | $0.28 \pm 0.08$ | $B_c^+ \rightarrow h_c \tau\nu$ | $0.019 \pm 0.013$ |
| $B_c^+ \rightarrow \chi_{c0} \pi^+$ | $0.031 \pm 0.004$ | $B_c^+ \rightarrow \chi_{c0} \rho^+$ | $0.076 \pm 0.009$ |
| $B_c^+ \rightarrow \chi_{c1} \pi^+$ | $0.0021 \pm 0.0002$ | $B_c^+ \rightarrow \chi_{c1} \rho^+$ | $0.023 \pm 0.002$ |
| $B_c^+ \rightarrow \chi_{c2} \pi^+$ | $0.021 \pm 0.005$ | $B_c^+ \rightarrow \chi_{c2} \rho^+$ | $0.056 \pm 0.011$ |
| $B_c^+ \rightarrow h_c \pi^+$ | $0.098 \pm 0.015$ | $B_c^+ \rightarrow h_c \rho^+$ | $0.22 \pm 0.04$ |
| $B_c^+ \rightarrow \chi_{c0} K^+$ | $0.0023 \pm 0.0003$ | $B_c^+ \rightarrow \chi_{c0} K^{*+}$ | $0.0045 \pm 0.0006$ |
| $B_c^+ \rightarrow \chi_{c1} K^+$ | $0.00016 \pm 0.000002$ | $B_c^+ \rightarrow \chi_{c1} K^{*+}$ | $0.0017 \pm 0.0001$ |
| $B_c^+ \rightarrow \chi_{c2} K^+$ | $0.0016 \pm 0.0003$ | $B_c^+ \rightarrow \chi_{c2} K^{*+}$ | $0.0033 \pm 0.0006$ |
| $B_c^+ \rightarrow h_c K^+$ | $0.0074 \pm 0.0011$ | $B_c^+ \rightarrow h_c K^{*+}$ | $0.013 \pm 0.002$ |

the experimental value of $B_c$ lifetime $\tau_{B_c} = 0.453$ ps as well, we calculate branching ratios of the decays and put them in Table III.

V. DISCUSSIONS AND CONCLUSIONS

In Sec. IV, the form factors (Fig. 3), energy spectrums of the charge leptons (Fig. 4 and Fig. 5), decay widths (Table I) for the semileptonic decays, and the decay widths for non-leptonic decays (Table II) are presented. Specially in tables some comparisons with other approaches elsewhere are also given. Thus one may read off a lot of interesting matters already.

Since the form factors for the semi-leptonic decays, which are directly related to overlapping integrations of the components of the B-S wave functions of the initial and final states as shown in Appendix C, are comparatively difficult to be measured, so in Fig. 3 we show the behaviors of the form factors briefly (without errors). Whereas the energy spectrums of the charged lepton in the decays may be measured not so difficult, as long as the event example is great enough and the abilities of the detector are strong enough, and to see the differences
between the spectrums of electron and τ lepton clearly in Fig. 4 we plot the curves with center values without theoretical uncertainties. Moreover to see the differences between this work and the ones [10], in Fig. 5 we plot the spectrums of electron obtained by this work vs the ones [10] obtained by previous approach and for both of them only center value without theoretical uncertainties are taken. Since the spectrums of muon (μ) is very similar to that of electron in exclusive semi-leptonic decays, thus we do not present the spectrums of muon at all. From Fig. 4 we can see the difference in the energy spectrums among the $B_c$ decays to different $P$-wave charmonia clearly, although the results of electron is greater than the one of τ lepton. From Fig. 5 we can see that the difference in the energy spectrums of electron due to different approaches: the difference caused by newly improved approach and by the previous approach can be quite sizable and can be tested experimentally in future. For the widths of the decays, from Table I and Table II, both the semi-leptonic decays and the non-leptonic decays, one may see that in general the results of this work fall into the region of the predictions by various models, but the distribution of the predictions is quite wide, so future experimental data will be critical and may conclude which one of the predictions is more reliable.

Considering the fact that the substantial tests of the $B_c$-meson decays have not been started yet, although the meson $B_c$ has been observed at Tevatron for years and LHC is running now, according to the estimates of the production at LHC, one may believe reasonably that the tests of the predictions on the $B_c$ decays will be started with LHC more measurements available. From theoretical point of view, we think that the newly improved approach works better than the previous one, this trust need to be tested by experiments. We would also like to note here that according to the estimates [33–36] of the production at an $e^- e^+$ collider running at CM energy $\sqrt{S} \simeq m_Z$ ($m_Z$ is Z-boson mass) with very high luminosity ($L = 10^{34-36}$cm$^{-2}$s$^{-1}$) i.e. a “Super-Z-Factory” and considering the advantages, may be more suitable to test the approaches by measuring the decays precisely than that to do them at hadronic collider such as Tevatron or LHC, because at such a Super-Z-Factory numerous $B_c$ mesons may be produced and the energy-momentum of the produced $B_c$ meson, as the $e^- e^+$ one of the collider, is precisely known in an $e^+-e^-$ collider environment.
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Appendix A: The functions $\alpha, \beta_{++}, \beta_{+-}, \beta_{-+}, \beta_{--}, \gamma$

Here according to the $P$-wave charmonium appearing in the final state we present the useful functions $\alpha, \beta_{++}, \beta_{+-}, \beta_{-+}, \beta_{--}, \gamma$ how precisely to relate to the form factors in turn.

a). When $B_c$ decays to $\chi_c0$:

Since the matrix elements of weak currents are described in terms of two form factors ($s_+, s_-)$:

$$\langle \chi_c0(P_f)|V^\mu|B_c(P)\rangle = 0,$$
$$\langle \chi_c0(P_f)|A^\mu|B_c(P)\rangle = s_+(P + P_f)^\mu + s_-(P - P_f)^\mu,$$

then the functions are read as

$$\beta_{++} = s_+^2, \quad \beta_{+-} = \beta_{-+} = s_+s_-, \quad \beta_{--} = s_-^2. \quad (A1)$$

b). When $B_c$ decays to $\chi_c1$:

Since the matrix elements of weak currents can be described in terms of four form factors ($f, u_1, u_2, g$):

$$\langle \chi_c1(P_f)|V^\mu|B_c(P)\rangle = f(M + M_f)\varepsilon^\mu + [u_1P^\mu + u_2P_f^\mu] \varepsilon \cdot P,$$
$$\langle \chi_c1(P_f)|A^\mu|B_c(P)\rangle = \frac{2g}{M + M_f}i\epsilon^{\mu\nu\rho\sigma}\varepsilon_\nu P_\mu P_f_\rho P_f_\sigma,$$

then the functions are read as

$$\alpha = f_1^2 + 4M^2g_1^2p_f^2,$$
$$\beta_{++} = \frac{f_1^2}{4M_f^2} - M^2g_1^2y + \frac{1}{2}\left[\frac{M^2}{M_f^2}(1 - y) - 1\right]f_1u_+ + \frac{M^2p_f^2}{M_f^2}u_+^2,$$
$$\beta_{+-} = \beta_{-+} = g_1^2(M^2 - M_f^2) - \frac{f_1^2}{4M_f^2} - \frac{1}{2}f_1(u_+ + u_-) - \frac{1}{2}ME_f f_1(u_+ - u_-) + u_+u_-\frac{M^2p_f^2}{M_f^2},$$
$$\beta_{--} = -g_1^2(M^2 + 2ME_f + M_f^2) + \frac{f_1^2}{4M_f^2} - \left(\frac{ME_f}{M_f^2} + 1\right)f_1u_- + u_-\frac{M^2p_f^2}{M_f^2},$$
$$\gamma = -2f_1g_1 \quad (A2)$$
when setting \( f_1 = f(M + M_f), u_+ = \frac{(u_1 + u_2)}{2M}, u_- = \frac{(u_1 - u_2)}{2M}, g_1 = \frac{g}{M+M_f}. \)

**c). When \( B_c \) decays to \( h_c \):**

Since the matrix elements of weak currents can be described in terms of four invariant form factors \((V_0, V_1, V_2, V_3)\):

\[
\langle h_c(P_f) | V^\mu | B_c(P) \rangle = V_0(M + M_f) \varepsilon^\mu + [V_1 P^\mu + V_2 P_f^\mu] \varepsilon \cdot P, \]

\[
\langle h_c(P_f) | A^\mu | B_c(P) \rangle = \frac{2V_3}{M + M_f} i\varepsilon^{\mu \nu \rho \sigma} \varepsilon_\nu P_\rho P_{f\sigma},
\]

then the functions are read as

\[
\alpha = f_1^2 + 4M^2 g_1^2 P_f^2,
\]

\[
\beta_{++} = \frac{f_1^2}{4M_f^2} - M^2 g_1^2 y + \frac{1}{2} \left[ \frac{M^2}{M_f^2} (1 - y) - 1 \right] f_1 a_+ + \frac{M^2 P_f^2}{M_f^2} a_+^2,
\]

\[
\beta_{+-} = \beta_{-+} = g_1^2 (M^2 - M_f^2) - \frac{f_1^2}{4M_f^2} - \frac{1}{2} f_1 (a_+ + a_-) - \frac{1}{2} \frac{M E_f}{M_f^2} f_1 (a_+ - a_-) + a_+ a_- \frac{M^2 P_f^2}{M_f^2},
\]

\[
\beta_{--} = -g_1^2 (M^2 + 2M E_f + M_f^2) + \frac{f_1^2}{4M_f^2} - \left( \frac{M E_f}{M_f^2} + 1 \right) f_1 a_+ + a_-^2 \frac{M^2 P_f^2}{M_f^2},
\]

\[
\gamma = -2f_1 g_1,
\]

when setting \( f_1 = V_0(M + M_f), a_+ = \frac{(V_1 + V_2)}{2M}, a_- = \frac{(V_1 - V_2)}{2M}, g_1 = \frac{V_3}{M + M_f}. \)

**d). When \( B_c \) decays to \( \chi_{c2} \):**

Since the matrix elements of weak currents can be described in terms of four form factors \((k, c_1, c_2, h)\):

\[
\langle \chi_{c2}(P_f) | A^\mu | B_c(P) \rangle = k(M + M_f) \varepsilon^{\alpha \mu} P_\alpha M + \varepsilon_{\alpha \beta} \frac{P^\alpha P^\beta}{M^2} (c_1 P^\mu + c_2 P_f^\mu),
\]

\[
\langle \chi_{c2}(P_f) | V^\mu | B_c(P) \rangle = \frac{2h}{M + M_f} i\varepsilon^{\mu \rho \sigma} \frac{P^\rho}{M} \varepsilon^{\beta \rho \sigma} P_\rho P_{f\sigma},
\]

where \( \varepsilon_{\alpha \beta}(\varepsilon^{\alpha \mu}) \) is the polarization tensor of tensor meson, then the functions are read as

\[
\alpha = \frac{c}{2} \left( k_1^2 + 4M h_1^2 P_f^2 \right),
\]

\[
\beta_{++} = \frac{c k_1^2}{8M_f^2} - \frac{c h_1^2}{2} M^2 y + \frac{2}{3} c^2 c_+^2 + \frac{4}{3} c k_1 c_+ \left( \frac{M^2(1 - y) + M_f^2}{4M_f^2} - \frac{1}{2} \right) + \frac{k_1^2}{6} \left( \frac{M^2(1 - y) + M_f^2}{4M_f^2} - \frac{1}{2} \right)^2,
\]

\[
\beta_{+-} = \beta_{-+} = -\frac{c k_1^2}{8M_f^2} + \frac{c h_1^2}{2} (M^2 - M_f^2) + \frac{k_1^2}{6} \left[ \left( \frac{M^2(1 - y) + M_f^2}{4M_f^2} \right)^2 - \frac{1}{4} \right] + \frac{2}{3} c^2 c_+ c_-.
\]
\[-2\frac{ck_1}{3}c_+ \left( \frac{M^2(1 - y) + M_j^2}{4M_f^2} + \frac{1}{2} \right) + \frac{2}{3}ck_1c_- \left( \frac{M^2(1 - y) + M_j^2}{4M_f^2} - \frac{1}{2} \right), \]

\[\beta_- = \frac{ck_1^2}{8M_f^2} - \frac{ch_1^2}{2}(2(M^2 + M_f^2) - M^2y) + \frac{2}{3}c^2c_-, \]

\[+ \frac{4}{3}ck_1c_- \left( \frac{M^2(1 - y) + M_j^2}{4M_f^2} - \frac{1}{2} \right) + \frac{k_1^2}{6} \left( \frac{M^2(1 - y) + M_f^2}{4M_f^2} + \frac{1}{2} \right)^2,\]

\[\gamma = -ch_1k_1, \quad (A4)\]

when setting \(c = \frac{M_j^2\beta_1^2}{M_f^2}, k_1 = k(1 + \frac{M_f}{M}), c_+ = \frac{c_1 + c_2}{2M_f^2}, c_- = \frac{c_1 - c_2}{2M_f^2}, h_1 = \frac{h}{M(M + M_f)}.\)

### Appendix B: The B-S equation under ‘complete instantaneous approximation’

In this appendix we outline the ‘complete instantaneous approximation’ onto the Bethe-Salpeter equation when it has an instantaneous kernel, which describes a double heavy meson quite well.

The Bethe-Salpeter equation [20] is read as

\[(p_1 - m_1)\chi_p(q)(p_2 + m_2) = i \int \frac{d^4k}{(2\pi)^4} V(P, k, q)\chi_p(k), \quad (B1)\]

where \(\chi_p(q)\) is B-S wave function of the relevant bound state, \(P\) is the four momentum of the meson state and \(p_1, p_2, m_1, m_2\) are the momenta and constituent masses of the quark and anti-quark respectively. From the definition, they relate to the total momentum \(P\) and relative momentum \(q\) as follows:

\[p_1 = \alpha_1P + q, \quad \alpha_1 \equiv \frac{m_1}{m_1 + m_2}, \]

\[p_2 = \alpha_2P - q, \quad \alpha_2 \equiv \frac{m_2}{m_1 + m_2}.\]

The interaction kernel \(V(P, k, q)\) for a double heavy system, being instantaneous approximately, can be treated as a potential after doing instantaneous approximation, i.e. the kernel take the simple form (in the rest frame) [19]

\[V(P, k, q) \Rightarrow V(\|k - q\|).\]

For various usages, we divide the relative momentum \(q\) into two parts,

\[q^\mu = q_{\|}^\mu + q_{\perp}^\mu, \quad q_{\parallel}^\mu \equiv \frac{P \cdot q}{M^2}P^\mu, \quad q_{\perp}^\mu \equiv q^\mu - q_{\parallel}^\mu, \]

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where \( M \) is the mass of the meson, and we may have two Lorentz invariant variables:

\[
q_P \equiv \frac{P \cdot q}{M}, \quad q_T \equiv \sqrt{-q_T^2}.
\]

For the convenience below, let us introduce the definitions:

\[
\varphi_p(q_\perp^\mu) \equiv i \int \frac{dq_P}{2\pi} \chi_p(q_\parallel, q_\perp^\mu), \quad \eta(q_\perp^\mu) \equiv \int \frac{dk_\perp^3}{(2\pi)^3} V(k_\perp, q_\perp) \varphi(p(k_\perp)),
\]

then the B-S equation can be rewritten as

\[
\chi(q_\parallel, q_\perp) = S_1(p_1) \eta(q_\perp) S_2(p_2).
\]

Owing to Eqs. (B2, B3), it is reasonable and for convenience we may call \( \eta(q_\perp) \) as ‘instantaneous B-S vertex’. The propagator of quark or anti-quark may be decomposed:

\[
S_i(p_i) = \frac{\Lambda_1^+(q_\perp)}{J(i) q_P + \alpha_i M - w_i + i\epsilon} + \frac{\Lambda_1^-(q_\perp)}{J(i) q_P + \alpha_i M - w_i - i\epsilon},
\]

where \( i=1, 2 \) for quark and anti-quark respectively, and \( J(i) = (-1)^{i+1}, \omega_1 = \sqrt{m_1^2 + q_T^2}, \quad \omega_2 = \sqrt{m_2^2 + q_T^2} \), and \( \Lambda_1^+, \Lambda_2^\pm \) are the generalized energy projection operators,

\[
\Lambda_1^+(q_\perp) \equiv \frac{1}{2\omega_1} \left[ \frac{P}{M} \omega_1 \pm (m_1 + q_\perp) \right], \quad \Lambda_2^\pm(q_\perp) \equiv \frac{1}{2\omega_2} \left[ \frac{P}{M} \omega_2 \mp (m_2 + q_\perp) \right],
\]

and have the properties:

\[
\Lambda_1^+(q_\perp) + \Lambda_2^- (q_\perp) = \frac{P}{M}, \quad \Lambda_2^+(q_\perp) \frac{P}{M} \Lambda_1^-(q_\perp) = 0,
\]

\[
\Lambda_1^+(q_\perp) \frac{P}{M} \Lambda_1^-(q_\perp) = \Lambda_2^+(q_\perp),
\]

The instantaneous approximation to the B-S equation is to do contour integration over \( q_P \) on both sides of Eq. (B3), and obtains:

\[
\varphi_p(q_\perp) = \frac{\Lambda_1^+(q_\perp) \eta(q_\perp) \Lambda_2^+(q_\perp)}{M - \omega_1 - \omega_2} - \frac{\Lambda_1^-(q_\perp) \eta(q_\perp) \Lambda_2^-(q_\perp)}{M + \omega_1 + \omega_2},
\]

If we introduce the notations:

\[
\varphi_p^{\pm \pm}(q_\perp) \equiv \Lambda_1^\pm(q_\perp) \frac{P}{M} \varphi_p(q_\perp) \frac{P}{M} \Lambda_2^\pm(q_\perp),
\]

we have

\[
\varphi_p(q_\perp) = \varphi_p^{++}(q_\perp) + \varphi_p^{+-}(q_\perp) + \varphi_p^{-+}(q_\perp) + \varphi_p^{--}(q_\perp),
\]
With the properties Eq. (B5) and notations Eq. (B7), the full Salpeter equation Eq. (B6) can be written as

\[(M - \omega_1 - \omega_2)\varphi^+_p(q_\perp) = \Lambda_1^+(q_\perp)\eta(q_\perp)\Lambda_2^+(q_\perp), \]

(B9)

\[(M + \omega_1 + \omega_2)\varphi^-_p(q_\perp) = -\Lambda_1^-(q_\perp)\eta(q_\perp)\Lambda_2^-(q_\perp), \]

(B10)

\[\varphi^+_p(q_\perp) = \varphi^-_p(q_\perp) = 0. \]

(B11)

The normalization condition for the B-S equations now is read as:

\[
\int \frac{q^2 dq_T}{2\pi^2} Tr[\frac{P}{M}\varphi^+_p - \frac{P}{M}\varphi^-_p] = 2P_0. \]

(B12)

The couple equations Eq. (B9), Eq. (B10) and Eq. (B11) with the normalization condition Eq. (B12) are the final B-S (Salpeter) equation under ‘complete instantaneous approximation’ vs the previous one i.e. Salpeter equation [19] where only Eq. (B9) is considered.

In addition, note that in the model used here for the double heavy quark-antiquark systems, the QCD-inspired interaction kernel \(V\), being instantaneous approximately and dictating the Cornell potential which is composed by a linear scalar interaction plus a vector interaction, is read as:

\[
V(\vec{q}) = V_s(\vec{q}) + V_v(\vec{q})\gamma^0 \otimes \gamma^0, \\
V_s(\vec{q}) = -\left(\frac{\lambda}{\alpha} + V_0\right)\delta^3(\vec{q}) + \frac{\lambda}{\pi^2} \frac{1}{(q^2 + \alpha^2)^2}, \\
V_v(\vec{q}) = -\frac{2}{3\pi^2} \frac{\alpha_s(\vec{q})}{(q^2 + \alpha^2)}, \]

(B13)

where the QCD running coupling constant \(\alpha_s(\vec{q}) = \frac{12\pi}{33-2N_f \log(a+q^2/\Lambda_{QCD}^2)}\); the constants \(\lambda, \alpha, a, V_0\) and \(\Lambda_{QCD}\) are the parameters characterizing the potential.

Appendix C: The reduced wave functions \(\varphi^{++}(\vec{q})\) and the form factors

In the appendix we present the reduced wave functions \(\varphi^{++}(\vec{q})\) (and \(\psi^{++}(\vec{q})\)) which directly relate to the solutions by newly solving the obtained coupled equations Eq. (B9), Eq. (B10) and Eq. (B11) under a new approach. The key point of the new approach is to solve the B-S equation according to the quantum numbers of the concerned bound states respectively [24, 30, 31], i.e. to solve the equation under the new approach we need to give
the most general formulation for the wave function first. Therefore for the present usage, in this appendix, we precisely quote the solutions for the low-laying bound states $B_c$ meson with quantum numbers $J^P = 0^-$, $\chi_{c0}$ with quantum numbers $J^{PC} = 0^{++}$, $\chi_{c1}$ with quantum numbers $J^{PC} = 1^{++}$, $h_c$ with quantum numbers $J^{PC} = 1^{+-}$ from [24, 30, 31], and then we write down the reduced wave functions $\varphi^{++}(\vec{q})$ and the form factors accordingly.

When the weak-current matrix elements are computed precisely, as an intermediate step, the form factors can be represented as overlapping integrations of the components appearing in the B-S solutions, thus in this appendix we also give the formulas of the form factors in terms of the ‘overlapping integrations’.

a). For $B_c$ meson with quantum numbers $J^P = 0^-$

The B-S wave function (solution of Eq. (B9), Eq. (B10) and Eq. (B11) of $B_c$ meson with $J^P = 0^-$ is read as:

$$
\varphi_{B_c}(\vec{q}) = M \left[ \frac{P}{M} f_1(\vec{q}) \left\{ 1 - \frac{q_\perp (w_1 + w_2)}{m_2 w_1 + m_1 w_2} \right\} + f_2(\vec{q}) \left\{ 1 + \frac{q_\perp (w_2 - w_1)}{m_1 w_2 + m_2 w_1} \right\} \right] \gamma_5, \quad (C1)
$$

where $M$, $P$ are the mass and the total momentum of the meson $B_c$, $q_\perp = (0, \vec{q})$, $\vec{q}$ is the relative momentum of quark and anti-quark in the meson, so $q^2_\perp = -\vec{q}^2$.

Then we can rewrite the reduced wave function:

$$
\varphi^{++}_{B_c}(\vec{q}) = b_1 \left[ b_2 + \frac{P}{M} + b_3 q_\perp + b_4 \frac{q_\perp P}{M} \right] \gamma_5, \quad (C2)
$$

where

$$
b_1 = \frac{M}{2} \left( f_1(\vec{q}) + f_2(\vec{q}) \frac{m_1 + m_2}{w_1 + w_2} \right), \quad b_2 = \frac{w_1 + w_2}{m_1 + m_2},
$$

$$
b_3 = -\frac{(m_1 - m_2)}{m_1 w_2 + m_2 w_1}, \quad b_4 = \frac{(w_1 + w_2)}{(m_1 w_2 + m_2 w_1)}.
$$

In Appendix. B in Eq. (B2), we have

$$
\eta(q_\perp) = \int d^3 k V(\vec{k}) M \left[ \frac{P}{M} f_1(\vec{k}) \left\{ 1 - \frac{k_\perp (w_{11} + w_{21})}{m_2 w_{11} + m_1 w_{21}} \right\} + f_2(\vec{k}) \left\{ 1 + \frac{k_\perp (w_{21} - w_{11})}{m_1 w_{21} + m_2 w_{11}} \right\} \right] \gamma_5, \quad (C3)
$$

where $w_{11} = \sqrt{m_1^2 - k_1^2}$, $w_{21} = \sqrt{m_2^2 - k_1^2}$, $V(\vec{k}) = V_s(\vec{k}) + V_v(\vec{k}) \gamma^0 \otimes \gamma^0$.

According to Eq. (C1),

$$
\eta(q_\perp) = \int d^3 k (V_s(\vec{k}) + V_v(\vec{k}) \gamma^0 \otimes \gamma^0)
$$
So we can also write down the wave function of \( \psi^{+\pm}(q_{\perp}) \),

\[
\psi^{+\pm}(q_{\perp}) = \frac{\Lambda_{1}^{+}(q_{p_{\perp}}) \eta(q_{\perp}) \Lambda_{2}^{-}(q_{p_{\perp}})}{M + \omega_{2} + \omega'_{2} - E_{f}} = \left[ n_{1} \frac{P}{M} + n_{2} + n_{3} q_{\perp} + n_{4} q_{\perp} \frac{P}{M} \right] \gamma_{5}.
\]

Set \( tt = \frac{1}{4w_{1}w_{2}(M + \omega_{2} + \omega'_{2} - E_{f})} \), where the symbol \( ' \) denotes the final state, and

\[
n_{1} = tt[g_{1}M(-q^{2} + m_{1}m_{2} - w_{1}w_{2}) + g_{2}M(m_{2}w_{1} - m_{1}w_{2}) + g_{3}(w_{1} + w_{2})q^{2} + g_{4}(m_{1} + m_{2})q^{2}],
\]

\[
n_{2} = tt[g_{1}M(m_{2}w_{1} - m_{1}w_{2}) + g_{2}M(q^{2} + m_{1}m_{2} - w_{1}w_{2}) + g_{3}(m_{1} - m_{2})q^{2} + g_{4}(w_{1} - w_{2})q^{2}],
\]

\[
n_{3} = tt[-g_{1}M(w_{1} + w_{2}) - g_{2}M(m_{1} - m_{2}) + g_{3}(q^{2} + m_{1}m_{2} + w_{1}w_{2}) + g_{4}(m_{2}w_{1} + m_{1}w_{2})],
\]

\[
n_{4} = tt[g_{1}M(m_{1} + m_{2}) + g_{2}M(w_{1} - w_{2}) - g_{3}(m_{2}w_{1} + m_{1}w_{2}) - g_{4}(-q^{2} + m_{1}m_{2} + w_{1}w_{2})].
\]

**b). For the charmonium \( \chi_{c0} \) \((J^{PC} = 0^{++})\) and the form factors \( s_{+} \) and \( s_{-} \)**

The B-S wave function (solution of Eq. (B9), Eq. (B10) and Eq. (B11) under new method to solve the coupled equations) of \( \chi_{c0} \) is read as:

\[
\varphi_{\chi_{c0}}(q') = f_{1}'(q') q_{\perp} + f_{2}'(q') \frac{P_{f}}{M_{f}} q_{\perp} + f_{3}'(q') M_{f} + f_{4}'(q') P_{f},
\]

with constraints on the components of wave function, for the charmonium, \( m_{1}' = m_{2}', w_{1}' = w_{2}' \), we get:

\[
f_{3}'(q') = \frac{f_{1}'(q') q_{\perp}^{2}}{M_{f} m_{1}'}, \quad f_{4}'(q') = 0,
\]

where \( M_{f}, P_{f} \) are the mass and the total momentum of final meson \( \chi_{c0}, q_{\perp} = (0, q'), \) \( q' \) is the relative momentum of quark and anti-quark in the meson, so \( q_{\perp}^{2} = -q'^{2} \). Then the reduced wave function \( \varphi_{\chi_{c0}}^{++}(q') \) as:

\[
\varphi_{\chi_{c0}}^{++}(q') = a_{1} \left[ q_{\perp} + a_{2} \frac{P_{f}}{M_{f}} q_{\perp} + a_{3} + a_{4} \frac{P_{f}}{M_{f}} \right],
\]

\[
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\]
with

$$a_1 = \frac{1}{2} \left( f'_1(q^2) + f'_2(q^2) \frac{m_1'}{m_1} \right), \quad a_2 = \frac{w_1'}{m_1}, \quad a_3 = \frac{q_f^2}{m_f}, \quad a_4 = 0.$$  

The wave function of $\bar{\psi}^{J+}(q'_{p_1} \parallel)$ is

$$\bar{\psi}^{J+}(q'_{p_1} \parallel) = N_2' \left( \frac{q'_{p_1} \, \eta'_{p_1} \, N_1'_{+}(q'_{p_1} \parallel)}{M - \omega_2 - \omega'_2 - E_f} \right) = n'_1 \, \eta'_1 + n'_2 \, \frac{\eta'_1 \, P_f}{M_f} + n'_3 + n'_4 \, \frac{P_f}{M_f}. \quad \text{(C8)}$$

Set $tt' = \frac{1}{4w_1' (M - \omega_2 - \omega'_2 - E_f)}$, where

$$n'_1 = tt'[-2g_1 q'^2 + 2g_3 M_f m_1'], \quad n'_2 = 0,$$

$$n'_3 = tt'[-2g_1 m'_1 q^2 + 2g_3 M_f m_1'], \quad n'_4 = tt'[-2g_1 w'_1 q^2 + 2g_3 M_f m_2 w'_1],$$

and

$$g'_1 = \int d^3 k'[V_s - V_v] \frac{\vec{k}' \cdot \vec{q}}{|q'|^2} f'_1(\vec{k}'), \quad g'_2 = \int d^3 k'[V_s - V_v] \frac{\vec{k}' \cdot \vec{q}}{|q'|^2} f'_2(\vec{k}'),$$

$$g'_3 = \int d^3 k'[V_s + V_v] \frac{f'_1(\vec{k}) \, k_f^2}{M_f m_1'}, \quad g'_4 = 0.$$

With Eq. (B7), the form factors may be presented by overlapping integrations:

$$s_+ = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \, 4a_1 b_1 M M_f \left[ a_3 b_2 M_f + \alpha_{11} E_f (a_2 b_2 E_f + M_f + a_2 b_4 \vec{q} \cdot \vec{P}_f) 
+ b_3 (M_f q^2 + \alpha_{11} M_f \vec{q} \cdot \vec{P}_f) + M (a_2 b_4 q^2 - \alpha_{11} a_2 b_2 E_f - \alpha_{11} M_f) 
+ M \frac{q \cos \theta}{|P_f|} (1 - \frac{E_f}{M_f}) (a_2 b_2 E_f - a_2 b_4 M_f + M_f + a_2 b_4 \vec{q} \cdot \vec{P}_f) \right], \quad \text{(C9)}$$

$$s_- = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \, 4a_1 b_1 M M_f \left[ a_3 b_2 M_f + \alpha_{11} E_f (a_2 b_2 E_f + M_f + a_2 b_4 \vec{q} \cdot \vec{P}_f) 
+ b_3 (M_f q^2 + \alpha_{11} M_f \vec{q} \cdot \vec{P}_f) - M (a_2 b_4 q^2 - \alpha_{11} a_2 b_2 E_f - \alpha_{11} M_f) 
- M \frac{q \cos \theta}{|P_f|} (1 + \frac{E_f}{M_f}) (a_2 b_2 E_f - a_2 b_4 M_f + M_f + a_2 b_4 \vec{q} \cdot \vec{P}_f) \right], \quad \text{(C10)}$$

where $\alpha_{11} = \alpha'_1 = \frac{m_1'}{m_1' + m_2'}$.

**c). For the charmonium $\chi_{c1}$ ($J^{PC} = 1^{++}$) and form factors the $f$, $u_1$, $u_2$, $g$**

The B-S wave function (solution of Eq. (B9), Eq. (B10) and Eq. (B11) under new method to solve the coupled equations) of $\chi_{c1}$ is read as:

$$\varphi_{\chi_{c1}}(q') = i \epsilon_{\mu \nu \alpha \beta} P_f^\mu q_{\perp}^{\nu} \varepsilon^{\alpha \beta} [f'_1(q') M_f \gamma^\mu + f'_2(q') P_f \gamma^\mu + f'_3(q') \eta'_1 \gamma^\mu + i f'_{4}(\vec{q}') e^{\mu \nu \rho \delta} P_f \eta'_1 \gamma^\rho \gamma^\delta / M_f] / M_f^2,$$

$$\text{(C11)}$$
where \( \varepsilon \) is the polarization vector of axial vector meson and with the constraint on the components:

\[
f'_3(q^2) = 0, \quad f'_4(q^2) = \frac{f'_2(q^2) M_f}{m'_1},
\]

Then the reduced wave function \( \varphi'_{P_1}(q^2) \) as:

\[
\varphi'_{\chi_{c1}}(q^2) = \frac{\partial}{\partial t} \left[ \frac{1}{2} \left( f'_2(q^2) w'_1 \right) \right]
\]

With Eq. (17), the form factors may be presented by overlapping integrations:

\[
f = \int \frac{d^3q}{(2\pi)^3} \frac{4a_1b_1}{M_f^2(M + M_f)} \left[ \left( a_1M_f^2q^2 - a_2b_1M_f^2q^2 - a_2b_2E_f\vec{q} \cdot \vec{P}_f \right) + \left( a_4 - a_2b_4 \right)(\vec{q} \cdot \vec{P}_f)^2 \right. \\
\left. + \left( a_1a_4E_f^2\vec{P}_f^2 + a_2b_1a_1(\vec{q}^2 - \vec{q} \cdot \vec{q}_f^2) + a_2b_2E_f(\vec{q} \cdot \vec{P}_f) + a_3E_f M_f(q^2 - \vec{q} \cdot \vec{q}_f^2) \right) + \frac{q^2}{2}(\cos^2 \theta - 1) \left( M_f^2(a_4 - a_2b_4) + b_3E_f(M_f + a_4\vec{q} \cdot \vec{P}_f - a_1a_4\vec{P}_f^2) \right) \right]
\]

\[
u_1 = \int \frac{d^3q}{(2\pi)^3} \frac{4a_1b_1M_f}{M_f^2} \left[ \frac{\alpha_1M_f^2}{M^2} \left( a_2b_2M_f^2 + M_f b_3\vec{q} \cdot \vec{P}_f \right) + a_4(\alpha_1E_fM_f^2 + b_3(M_f^2q^2 + (\vec{q} \cdot \vec{P}_f)^2 - \vec{q} \cdot \vec{q}_f^2 - \vec{P}_f E_f^2)) \right) \right] \\
- \frac{E_f q^2}{M^2} \left( a_2b_2M_f^2 + M_f b_3\vec{q} \cdot \vec{P}_f \right) + a_4(\alpha_1E_fM_f^2 + b_3(M_f^2q^2 + (\vec{q} \cdot \vec{P}_f)^2 - \vec{q} \cdot \vec{q}_f^2 - \vec{P}_f E_f^2)) \right) \\
+ \frac{q^2}{2M^2(\vec{P}_f)^2}(-M_f^2 + (2E_f^2 + M_f^2) \cos^2 \theta) \left( M_f^2(a_4 - a_2b_4) + b_3E_f(M_f + a_4\vec{q} \cdot \vec{P}_f - a_1a_4\vec{P}_f^2) \right]
\]

\[
u_2 = \int \frac{d^3q}{(2\pi)^3} \frac{4a_1b_1M_f}{M_f^2} \left[ -\frac{1}{M} \left( b_3M_fq^2 + (a_2b_2 + a_1a_4E_f)(\alpha_1E_f^2 - \vec{q} \cdot \vec{P}_f) \right) \right. \\
+ \frac{E_f q^2}{M|\vec{P}_f|} \left( \alpha_1M_f b_3E_f + (a_2b_2 - a_4)(\vec{q} \cdot \vec{P}_f - a_1E_f^2) \right) \\
+ \frac{1}{M} \frac{q^2}{|\vec{P}_f|} \left( a_2b_2M_f^2 + M_f b_3\vec{q} \cdot \vec{P}_f \right]
\]
Then we have the reduced wave function

\[ \varphi_{h_c}(\vec{q}) = \varphi_1 \cdot e^{i \epsilon} \left[ f''_1(\vec{q}) + f''_2(\vec{q}) \frac{P_f}{M_f} + f''_3(\vec{q}) \varphi_1 + f''_4(\vec{q}) \frac{P_f}{M_f} \varphi_1 \right] \gamma_5, \tag{17} \]

with the constraint on the components of the wave function,

\[ f''_3(\vec{q}) = 0, \quad f''_4(\vec{q}) = -\frac{f''_2(\vec{q}) M_f}{m'_{1}}, \]

Then we have the reduced wave function \( \varphi_{h_c}^{++}(\vec{q}) \):

\[ \varphi_{h_c}^{++}(\vec{q}) = \varphi_1 \cdot e^{i \epsilon} \left[ 1 + a_2 \frac{P_f}{M_f} + a_3 \varphi_1 + a_4 \frac{P_f}{M_f} \varphi_1 \right] \gamma_5, \tag{18} \]

\[ a_1 = \frac{1}{2} \left( f''_1 + f''_2(\vec{q}) \frac{w_{1}'}{m_{1}'} \right), \quad a_2 = \frac{m_{1}'}{w_{1}'} \quad \text{and} \quad a_3 = 0, \quad a_4 = \frac{1}{w_{1}}. \]

With Eq. (17), the form factors may be presented by overlapping integrations:

\[ V_0 = \int \frac{d^3 q}{(2\pi)^3} \frac{4a_1 b_1}{M_f (M + M_f)} q^2 (\cos^2 \theta - 1) \]

\[ \left[ a_4 b_2 E_f + a_2 b_3 E_f - b_4 M_f + a_4 b_3 \vec{q} \cdot \vec{P}_f \right], \tag{19} \]

\[ V_1 = \int \frac{d^3 q}{(2\pi)^3} \frac{4a_1 b_1}{M_f} \left[ E_f \left( \alpha_{11} - \frac{q \cos \theta}{|P_f|} \right) \left( \alpha_{11} a_2 b_2 E_f^2 + \alpha_{11} a_4 b_4 E_f \vec{q} \cdot \vec{P}_f + b_2 M_f + a_2 b_3 \vec{q} \cdot \vec{P}_f \right) \right. \]

\[ \left. \left( \frac{q^2}{2M|P_f|^2} (q^2 (2E_f^2 + |P_f|^2) \cos^2 \theta - \alpha_{11} \frac{E_f^2 q \cos \theta}{M|P_f|}) \right) \right], \tag{20} \]

\[ V_2 = \int \frac{d^3 q}{(2\pi)^3} \frac{4M a_1 b_1}{M_f} \left[ E_f \left( \alpha_{11} - \frac{q \cos \theta}{|P_f|} \right) \left( -a_4 b_4 q^2 + a_2 - a_4 b_2 E_f \alpha_{11} \right) \right. \]

\[ \left. + \left( \frac{E_f q \cos \theta}{M|P_f|} + \frac{E_f q^2}{2M|P_f|^2} (3 \cos^2 \theta - 1) \right) \left( a_4 b_2 E_f + a_2 b_3 E_f - b_4 M_f + a_4 b_3 \vec{q} \cdot \vec{P}_f \right) \right]. \tag{21} \]
\[ V_3 = -\int \frac{d^3q}{(2\pi)^3} \frac{4a_1b_1(M + M_f)q^2}{MM_f} (\cos^2 \theta - 1) \left[ a_4(b_2 + b_4E_f \alpha_{11}) + b_3a_2 \right]. \] (C22)

e). For the charmonium \( \chi_{c2} (J^{PC} = 2^{++}) \) and form factors the \( k, c_1, c_2, h \)

The B-S wave function (solution of Eq. (B9), Eq. (B10) and Eq. (B11) under new method
to solve the coupled equations) of \( \chi_{c2} \) is read as:

\[
\varphi_{\chi_{c2}}(\vec{q}) = \varepsilon_{\mu\nu} q_\mu'' \left\{ f_1''(\vec{q}) + \frac{P_f}{M_f} f_2(\vec{q}) \right\} + \frac{q_1'}{M_f} f_3'(\vec{q}) + \left( \frac{P_f}{M_f} \right) f_4'(\vec{q})
+ \gamma^\mu [M_f f_5'(\vec{q}) + P_f f_6'(\vec{q}) + q_1'/f_7'(\vec{q})] + \frac{i}{M_f} f_8'(\vec{q}) \epsilon^{\mu\alpha\beta\gamma} P_f a_1 q_1' b_2 f_2 \gamma_3 \gamma_5 \},
\] (C23)

with the constraint on the components of the wave function:

\[
f_1' = \frac{[q_1'^2 f_3'(\vec{q}) + M_f^2 f_4'(\vec{q})]}{M_f m_1'}, f_2' = 0, f_7' = 0, f_8' = \frac{f_6'(\vec{q})M_f}{m_1'},
\]

where \( \varepsilon_{\mu\nu} \) is a tensor for \( J = 2 \). Then we have the reduced wave function \( \varphi_{\chi_{c2}}(\vec{q}) \) as:

\[
\varphi_{\chi_{c2}}^{++}(\vec{q}) = \varepsilon_{\mu\nu} q_\mu'' \left\{ f_1''(\vec{q})a_1 + a_2 P_f + a_3 \frac{q_1'}{M_f} \right\}
+ a_4 \left( \frac{q_1'}{M_f} P_f \right) + \gamma^\mu [a_5 + a_6 \frac{P_f}{M_f} + a_7 \frac{q_1'}{M_f} + a_8 \frac{P_f q_1'}{M_f^2}]
\] (C24)

with

\[
a_1 = \frac{q_1'^2}{2M_f m_1'} n_1 + \frac{(f_3'(\vec{q})w_2' - f_5'(\vec{q})m_2')M_f}{2m_1'w_2'}, \quad a_2 = \frac{(f_6'(\vec{q})w_1' - f_5'(\vec{q})m_1')}{2m_1'w_1'},
\]

\[
a_3 = \frac{1}{2} n_1 + \frac{f_5'(\vec{q})M_f^2}{2m_1'w_2'}, \quad a_4 = \frac{1}{2} (-\frac{w_1'}{m_1'}) n_1 + \frac{f_6'(\vec{q})M_f^2}{2m_1'w_1'},
\]

\[
a_5 = \frac{M_f}{2} n_2, a_6 = \frac{M_f M_1'}{2w_1'n_2}, \quad a_7 = 0, \quad a_8 = \frac{M_f^2}{2w_1' n_2},
\]

\[
n_1 = \frac{1}{2} (f_3'(\vec{q}) + f_4'(\vec{q}) m_1' w_1'), \quad n_2 = \frac{1}{2} (f_5'(\vec{q}) - f_6'(\vec{q}) w_1' m_1').
\]

With Eq. (17), the form factors may be presented by overlapping integrations:

\[
k = \int \frac{d^3q}{(2\pi)^3} \frac{4b_1}{M_f^2(M + M_f)} \left[ \frac{q^2}{2} (\cos^2 \theta - 1) \right. \\
(a_{11}E_f(M_f(-a_3 - a_2b_3E_f + a_1b_4M_f) + a_4(b_2E_f + b_4q \cdot \vec{P}_f))
- (a_{11}a_8b_3E_f^2 - a_5b_3M_f^2 + a_8b_3q \cdot \vec{P}_f) - a_{11}E_f(M_f(-a_3 - a_2b_3E_f + a_1b_4M_f)
+ a_4(b_2E_f + b_4q \cdot \vec{P}_f) + 2a_8b_3E_f))
- E_f(a_{11} - \frac{q \cos \theta}{|\vec{P}_f|})(-a_5M_f^2 + a_8b_3E_f^2 + a_6b_2E_fM_f + a_5b_4M_f q \cdot \vec{P}_f - a_{11}a_8b_3E_f q \cdot \vec{P}_f)
\]

\[
\left. - \frac{E_f q^3 \cos \theta}{|\vec{P}_f|}(1 - \cos^2 \theta)((M_f(-a_3 - a_2b_3E_f + a_1b_4M_f)
+ a_4(b_2E_f + b_4q \cdot \vec{P}_f) + 2a_8b_3E_f)) \right],
\] (C25)
\[ c_1 = \int \frac{d^3q}{(2\pi)^3} \frac{4b_1 M}{M_f^2} \left[ \alpha_1 \frac{E_f}{M} \left( \alpha_11 \frac{E_f}{M} \left(-\alpha_1 a_4 b_2 E_f^2 - \alpha_11 a_4 b_1 E_f q \cdot \vec{P}_f \right) + a_1 b_2 M_f^2 + a_2 b_3 M_f q \cdot \vec{P}_f + a_3 M_f (b_2 q^2 + \alpha_11 E_f - \alpha_1 b_3 q \cdot \vec{P}_f)) -2\alpha_1 a_8 b_3 E_f q \cdot \vec{P}_f/M \right) \right] - \frac{E_f q \cos \theta}{M|\vec{P}_f|} \left( -M(\alpha_11 \frac{E_f}{M})^2 M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) + a_4 (b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f) \right) \\
- \frac{E_f q \cos \theta}{M|\vec{P}_f|} \left( -M(\alpha_11 \frac{E_f}{M})^2 M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) + a_4 (b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f) \right) \\
+ 2\alpha_11 \frac{E_f}{M} (-\alpha_1 a_4 b_2 E_f^2 - \alpha_11 a_4 b_1 E_f q \cdot \vec{P}_f + a_1 b_2 M_f^2 + a_2 b_3 M_f q \cdot \vec{P}_f \\
+ a_3 M_f (b_3 q^2 + \alpha_11 E_f - \alpha_1 b_3 q \cdot \vec{P}_f)) - \alpha_11 a_8 b_3 \vec{q} \cdot \vec{P}_f/M \\
- \alpha_1 \frac{E_f}{M} (\alpha_11 a_8 b_3 E_f^2 - a_5 b_3 M_f^2 + a_8 b_3 \vec{q} \cdot \vec{P}_f) \\
+ a_3 M_f (b_3 q^2 + \alpha_11 E_f - \alpha_1 b_3 q \cdot \vec{P}_f)) - \alpha_11 a_8 b_3 \vec{q} \cdot \vec{P}_f/M \\
+ \frac{q^2}{2M^2|\vec{P}_f|^2} \left( -M_f^2 + (2E_f^2 + M_f^2) \cos^2 \theta \right) \\
\left( M(a_4 b_2 q^2 + a_2 M_f + \alpha_1 a_4 b_2 E_f - \alpha_1 a_3 M_f - a_8 - a_6 b_4 M_f + \alpha_1 a_8 b_3 E_f) \\
- \alpha_11 E_f(M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) + a_4 (b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f)) \\
- \alpha_11 E_f(M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) + a_4 (b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f)) \\
+ 2a_8 b_3 E_f)) \right) - \frac{q^3 E_f \cos \theta}{2M^2|\vec{P}_f|^3} \left( 3M_f^2 - (2E_f^2 + 3M_f^2) \cos^2 \theta \right) \\
\left( (M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) + a_4 (b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f) + 2a_8 b_3 E_f) \right), \quad (C26) \]

\[ c_2 = \int \frac{d^3q}{(2\pi)^3} \frac{4b_1 M}{M_f^2} \left[ \alpha_11 \frac{E_f}{M} \left( \alpha_11 \frac{E_f}{M} \left((a_4 b_2 q^2 + a_2 M_f + \alpha_1 a_4 b_2 E_f - \alpha_1 a_3 M_f) \\
- (a_8 b_3 q^2 - a_6 b_2 M_f + \alpha_1 a_8 E_f)) \right) + \frac{E_f q \cos \theta}{M|\vec{P}_f|} \left( (-\alpha_11 E_f)(M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) + a_4 (b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f)) \\
- \alpha_11 (a_8 b_3 E_f^2 - a_5 b_3 M_f^2 a_8 b_3 \vec{q} \cdot \vec{P}_f)) - \alpha_11 E_f(a_4 b_3 q^2 + a_2 M_f + \alpha_1 a_4 b_2 E_f - \alpha_1 a_3 M_f) \\
+ (a_8 b_3 q^2 - a_6 b_2 M_f + \alpha_1 a_8 E_f) \\
- \alpha_11 E_f(a_4 b_3 q^2 + a_2 M_f + \alpha_1 a_4 b_2 E_f - \alpha_1 a_3 M_f - a_8 - a_6 b_4 M_f + \alpha_1 a_8 b_3 E_f) \right) \right) + \frac{q^2}{2M|\vec{P}_f|^2} \left( -M_f^2 + (2E_f^2 + M_f^2) \cos^2 \theta \right) (a_4 b_3 q^2 + a_2 M_f + \alpha_1 a_4 b_2 E_f \\
- \alpha_1 a_3 M_f - a_8 - a_6 b_4 M_f + \alpha_1 a_8 b_3 E_f - \frac{q^2 E_f}{2M|\vec{P}_f|^2} (3 \cos^2 \theta - 1) \\
\left( -\alpha_1 E_f(M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) + a_4 (b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f)) \\
- \alpha_11 a_8 b_3 E_f^2 - a_5 b_3 M_f^2 + a_8 b_3 \vec{q} \cdot \vec{P}_f) - \alpha_11 E_f(M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) \\
-(\alpha_1 a_8 b_3 E_f^2 - a_5 b_3 M_f^2 + a_8 b_3 \vec{q} \cdot \vec{P}_f) - \alpha_11 E_f(M_f (-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) \\
30 \right) \]
\[\begin{align*}
    &+ a_4(b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f + 2a_8 b_3 E_f)) - \frac{q^2 \cos \theta}{2M|\vec{P}_f|^3}((4E_f^2 + M_f^2) \cos^2 \theta - (2E_f^2 + M_f^2)) \\
    &\left( M_f(-a_3 - a_2 b_3 E_f + a_1 b_4 M_f) + a_4(b_2 E_f + b_4 \vec{q} \cdot \vec{P}_f + 2a_8 b_3 E_f)) \right], \quad (C27)
\end{align*}\]

\[h = \int \frac{d^3q}{(2\pi)^3} \frac{4b_1(M + M_f)}{M_f^2} \left[ \left( \frac{q \cos \theta}{|\vec{P}_f|} - \alpha_{11} \right) \frac{E_f}{M}(a_5 b_3 q^2 + a_6 b_2 M_f + \alpha_{11} a_8 E_f) \\
    + \alpha_{11} \frac{E_f^2 q \cos \theta}{M|\vec{P}_f|}(a_8 - a_6 b_4 M_f + \alpha_{11} a_8 b_3 E_f) - \alpha_{11} \frac{E_f q \cos \theta}{M|\vec{P}_f|} b_3(\alpha_{11} a_8 E_f^2 + a_5 M_f^2 - a_8 \vec{q} \cdot \vec{P}_f) \\
    - \frac{q^2}{2M|\vec{P}_f|^2} (\cos^2 \theta - 1)(b_3(\alpha_{11} a_8 E_f^2 + a_5 M_f^2 - a_8 \vec{q} \cdot \vec{P}_f)) \\
    + \frac{q^2}{2M|\vec{P}_f|^2} (\cos^2 \theta - 1)(2(b_3 M_f(a_3 \alpha_{11} - a_2) + a_4(b_2 + \alpha_{11} b_4 E_f)) + 4a_8 b_3) \\
    + \frac{E_f q^3 \cos \theta}{2M|\vec{P}_f|^2}(1 - \cos^2 \theta)(2(b_3 M_f(a_3 \alpha_{11} - a_2) + a_4(b_2 + \alpha_{11} b_4 E_f)) + 4a_8 b_3) \right]. \quad (C28)\]

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