UNIQUENESS AND SUPERPOSITION OF THE DISTRIBUTION-DEPENDENT ZAKAI EQUATIONS*

MEIQI LIU¹ AND HUIJIE QIAO¹,²

1. Department of Mathematics, Southeast University, Nanjing, Jiangsu 211189, P.R.China
2. Department of Mathematics, University of Illinois at Urbana-Champaign Urbana, IL 61801, USA

Abstract. The work concerns the Zakai equations from nonlinear filtering problems of McKean-Vlasov stochastic differential equations with correlated noises. First, we establish the Kushner-Stratonovich equations, the Zakai equations and the distribution-dependent Zakai equations. And then, the pathwise uniqueness, uniqueness in joint law and uniqueness in law of weak solutions for the distribution-dependent Zakai equations are shown. Finally, we prove a superposition principle between the distribution-dependent Zakai equations and distribution-dependent Fokker-Planck equations. As a by-product, we give some conditions under which distribution-dependent Fokker-Planck equations have unique weak solutions.

1. Introduction

McKean-Vlasov (distribution-dependent or mean-field) stochastic differential equations (SDEs in short) describe the evolution rules of particle systems perturbed by noises. And the difference between McKean-Vlasov SDEs and general SDEs lies in that McKean-Vlasov SDEs depend on the positions and probability distributions of these particles. Therefore, McKean-Vlasov SDEs are widely applied in many fields, such as biology, game theory and control theory. Moreover, more and more results about McKean-Vlasov SDEs appear. Let us mention some results associated with our work. Ding and Qiao [3, 4] investigated the well-posedness and stability of weak solutions for McKean-Vlasov SDEs under non-Lipschitz conditions. Lacker, Shkolnikov and Zhang [5] studied superposition principles for conditional McKean-Vlasov equations. Ren and Wang [14] proved that additive functionals of McKean-Vlasov SDEs have path-independence.

Nonlinear filtering problems are to extract some useful information of unobservable phenomenon from observable ones and then estimate and predict these unobservable phenomenon (c.f. [1, 8, 9, 10, 11, 12, 13, 15, 16]). And nonlinear filtering theory plays an important role in many areas including stochastic control, financial modeling, speech and image processing, and Bayesian networks. Although McKean-Vlasov SDEs have widespread applications, the result about nonlinear filtering problems of McKean-Vlasov

AMS Subject Classification(2010): 60G35; 35K55.

Keywords: McKean-Vlasov SDEs; the distribution-dependent Zakai equations; the pathwise uniqueness; distribution-dependent Fokker-Planck equations; a superposition principle.

*This work was partly supported by NSF of China (No. 11001051, 11371352) and China Scholarship Council under Grant No. 201906095034.

Corresponding author: Huijie Qiao, hjqiaogean@seu.edu.cn.
SDEs is seldom. Sen and Caines [17,18] studied nonlinear filtering problems of McKean-Vlasov SDEs with independent noises and then applied the results to mean field game problems.

In the paper, we focus on nonlinear filtering problems of McKean-Vlasov SDEs with correlated noises. Let us explain them in detail. Fix $T > 0$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a complete filtered probability space and \{W_t, t \geq 0\}, \{V_t, t \geq 0\} be $d$-dimensional and $m$-dimensional standard Brownian motions defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$, respectively. Moreover, $W_v, V$ are mutually independent.

Consider the following McKean-Vlasov signal-observation system $(X_t, Y_t)$ on $\mathbb{R}^n \times \mathbb{R}^m$:

$$
\begin{align*}
\frac{dX_t}{dt} &= b_1(t, X_t, \mathcal{L}_X_t)dt + \sigma_0(t, X_t, \mathcal{L}_X_t)\sigma_0(t, X_t, \mathcal{L}_X_t)dt + \sigma_1(t, X_t, \mathcal{L}_X_t)dV_t, \\
\frac{dY_t}{dt} &= b_2(t, X_t, \mathcal{L}_X_t, Y_t)dt + \sigma_2(t, Y_t)dW_t, \quad 0 \leq t \leq T,
\end{align*}
$$

where $\mathcal{L}_X_t$ denotes the distribution of $X_t$, and these coefficients $b_1 : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \mapsto \mathbb{R}^n$, $\sigma_0 : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \mapsto \mathbb{R}^{n \times d}$, $\sigma_1 : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \mapsto \mathbb{R}^{n \times m}$, $b_2 : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^m \mapsto \mathbb{R}^{m \times m}$ are Borel measurable. The initial value $X_0$ is assumed to be a $p$-order ($p > 2$) integrable random variable independent of $Y_0, W_v, V$. Nonlinear filtering problems of the systems like (1) are called as nonlinear filtering problems of McKean-Vlasov SDEs with correlated noises. And then we deduce the Kushner-Stratonovich equation (4), the Zakai equation (10) and the distribution-dependent Zakai equation (12) about the system (1). Next, we view Eq. (12) as a SDE and define its weak solutions, and the pathwise uniqueness, uniqueness in joint law and uniqueness in law of weak solutions. Moreover, we prove that weak solutions of Eq. (12) have the pathwise uniqueness, uniqueness in joint law and uniqueness in law under some suitable conditions. Finally, we set up a correspondence between weak solutions of Eq. (12) and weak solutions of a distribution-dependent Fokker-Planck equation (24) (c.f. Section 5).

Here is a summary of our results. First, we establish the Kushner-Stratonovich equation (4) (See Theorem 3.2), the Zakai equation (10) (See Theorem 3.3) and the distribution-dependent Zakai equation (12) about the system (1). Theorem 3.2 and 3.3 generalize Theorem 3.1 and 3.2 in [17] and Theorem 2.6 and 2.8 for the case without jumps in [10]. Second, we prove the pathwise uniqueness (See Theorem 4.6), uniqueness in joint law (See Theorem 4.9) and uniqueness in law (See Corollary 4.10) of weak solutions to Eq. (12). Theorem 4.6 and 4.9 are more general than Theorem 3.9 and 3.10 for the case without jumps in [10] and Theorem 3.4 for the case without jumps in [12]. Finally, we show a superposition principle between weak solutions of Eq. (12) and weak solutions of Eq. (24) (See Theorem 5.2). Theorem 5.2 covers Theorem 6.4 in [11].

It is worthwhile to mention that our methods can be applied to nonlinear filtering problems of McKean-Vlasov SDEs with correlated sensor noises. Concretely speaking, consider the following signal-observation system $(\tilde{X}_t, \tilde{Y}_t)$ on $\mathbb{R}^n \times \mathbb{R}^m$:

$$
\begin{align*}
\frac{d\tilde{X}_t}{dt} &= \tilde{b}_1(t, \tilde{X}_t, \mathcal{L}_{\tilde{X}_t})dt + \tilde{\sigma}_1(t, \tilde{X}_t, \mathcal{L}_{\tilde{X}_t})d\tilde{V}_t, \\
\frac{d\tilde{Y}_t}{dt} &= \tilde{b}_2(t, \tilde{X}_t, \mathcal{L}_{\tilde{X}_t}, \tilde{Y}_t)dt + \tilde{\sigma}_2d\tilde{W}_t + \tilde{\sigma}_3d\tilde{W}_t, \quad 0 \leq t \leq T,
\end{align*}
$$

where the initial value $\tilde{X}_0$ is assumed to be a $p$-order ($p > 2$) integrable random variable independent of $Y_0, W_v, V$. The mappings $\tilde{b}_1 : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \mapsto \mathbb{R}^n$, $\tilde{\sigma}_1 : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \mapsto \mathbb{R}^{n \times d}$ and $\tilde{b}_2 : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^m \mapsto \mathbb{R}^{m \times m}$ are all Borel measurable. $\tilde{\sigma}_2, \tilde{\sigma}_3$ are $m \times m$ and $m \times d$ real matrices, respectively. By the same way to that for the system (11), we can also establish the distribution-dependent Zakai equation (31), and study the
pathwise uniqueness, uniqueness in joint law, uniqueness in law and the superposition principle of weak solutions for Eq. (31) (c.f. Section 4).

In Section 2, we introduce notation and L-derivative for functions on $\mathcal{P}_2(\mathbb{R}^n)$ used in the sequel. After this, we introduce nonlinear filtering problems for McKean-Vlasov signal-observation systems with correlated noises, and derive the Kushner-Stratonovich equations, the Zakai equations and the distribution-dependent Zakai equations. In Section 4 the pathwise uniqueness, uniqueness in joint law and uniqueness in law for weak solutions to the Zakai equation (12) are shown. We place the superposition principle between weak solutions of Eq. (12) and weak solutions of Eq. (24) in Section 5. Finally, in Section 6 we summarize our results and apply our methods to the system (2).

The following convention will be used throughout the paper: $C$ with or without indices will denote different positive constants whose values may change from one place to another.

2. Preliminary

In the section, we introduce notation and L-derivative for functions on $\mathcal{P}_2(\mathbb{R}^n)$.

2.1. Notation. In the subsection, we introduce notation used in the sequel.

For convenience, we shall use $| \cdot |$ and $\| \cdot \|$ for norms of vectors and matrices, respectively. Let $A^*$ denote the transpose of the matrix $A$.

Let $\mathcal{B}(\mathbb{R}^n)$ be the Borel $\sigma$-field on $\mathbb{R}^n$. Let $\mathcal{B}_b(\mathbb{R}^n)$ denote the set of all real-valued uniformly bounded $\mathcal{B}(\mathbb{R}^n)$-measurable functions on $\mathbb{R}^n$. $C^2(\mathbb{R}^n)$ stands for the space of continuous functions on $\mathbb{R}^n$ which have continuous partial derivatives of order up to 2, and $C^2_b(\mathbb{R}^n)$ stands for the subspace of $C^2(\mathbb{R}^n)$, consisting of functions whose derivatives up to order 2 are bounded. $C^2_c(\mathbb{R}^n)$ is the collection of all functions in $C^2(\mathbb{R}^n)$ with compact supports and $C^\infty_c(\mathbb{R}^n)$ denotes the collection of all real-valued $C^\infty$ functions of compact supports.

Let $\mathcal{M}(\mathbb{R}^n)$ be the set of all bounded Borel measures defined on $\mathcal{B}(\mathbb{R}^n)$ carrying the usual topology of weak convergence. Let $\mathcal{P}(\mathbb{R}^n)$ be the space of all probability measures defined on $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{P}_2(\mathbb{R}^n)$ be the collection of all the probability measures $\mu$ on $\mathcal{B}(\mathbb{R}^n)$ satisfying

$$\|\mu\|_2 := \int_{\mathbb{R}^n} |x|^2 \mu(dx) < \infty.$$ 

We put on $\mathcal{P}_2(\mathbb{R}^n)$ a topology induced by the following metric:

$$\mathcal{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \pi(dx, dy), \quad \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^n),$$

where $\mathcal{C}(\mu_1, \mu_2)$ denotes the set of all the probability measures whose marginal distributions are $\mu_1, \mu_2$, respectively. Thus, $(\mathcal{P}_2(\mathbb{R}^n), \mathcal{W}_2)$ is a Polish space.

2.2. L-derivative for functions on $\mathcal{P}_2(\mathbb{R}^n)$. In the subsection we recall the definition of L-derivative for functions on $\mathcal{P}_2(\mathbb{R}^n)$. And the definition was first introduced by Lions [1]. Moreover, he used some abstract probability spaces to describe the L-derivatives. Here, for the convenience to understand the definition, we apply a straight way to state it ([14]). Let $I$ be the identity map on $\mathbb{R}^n$. For $\mu \in \mathcal{P}_2(\mathbb{R}^n)$ and $\phi \in L^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu; \mathbb{R}^n)$, $< \mu, \phi > := \int_{\mathbb{R}^n} \phi(x) \mu(dx)$. Moreover, by simple calculation, it holds that $\mu \circ (I + \phi)^{-1} \in \mathcal{P}_2(\mathbb{R}^n)$. 

3
Definition 2.1. (i) A function \( f : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \) is called \( L \)-differentiable at \( \mu \in \mathcal{P}_2(\mathbb{R}^n) \), if the functional
\[
L^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu; \mathbb{R}^n) \ni \phi \mapsto f(\mu \circ (I + \phi)^{-1})
\]
is Fréchet differentiable at \( 0 \in L^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu; \mathbb{R}^n) \); that is, there exists a unique \( \gamma \in L^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu; \mathbb{R}^n) \) such that
\[
\lim_{\|\mu,|\phi|^2\| \rightarrow 0} \frac{f(\mu \circ (I + \phi)^{-1}) - f(\mu) - \mu(\gamma \cdot \phi)}{\sqrt{\|\mu,|\phi|^2\|}} = 0.
\]
In the case, we denote \( \partial_\mu f(\mu) = \gamma \) and call it the \( L \)-derivative of \( f \) at \( \mu \).

(ii) A function \( f : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \) is said to be \( L \)-differentiable on \( \mathcal{P}_2(\mathbb{R}^n) \) if \( L \)-derivative \( \partial_\mu f(\mu) \) exists for all \( \mu \in \mathcal{P}_2(\mathbb{R}^n) \).

(iii) By the same way, \( \partial_\mu^2 f(\mu)(y, y') \) for \( y, y' \in \mathbb{R}^n \) can be defined.

Next, we introduce some related spaces.

Definition 2.2. The function \( f \) is said to be in \( C^2(\mathcal{P}_2(\mathbb{R}^n)) \), if \( \partial_\mu f \) is continuous, for any \( \mu \in \mathcal{P}_2(\mathbb{R}^n) \), \( \partial_\mu f(\mu)(\cdot) \) is differentiable, and its derivative \( \partial_\phi \partial_\mu f : \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n \) is continuous, and for any \( y \in \mathbb{R}^n \), \( \partial_\mu f(y) \) is differentiable, and its derivative \( \partial_\mu^2 f : \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n \) is continuous.

Definition 2.3. (i) The function \( F : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \) is said to be in \( C^{2,2}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)) \), if \( F(x, \mu) \) is \( C^2 \) in \( x \in \mathbb{R}^n \) and \( \mu \in \mathcal{P}_2(\mathbb{R}^n) \) respectively, and its derivatives
\[
\partial_x F(x, \mu), \partial_x^2 F(x, \mu), \partial_\mu F(x, \mu)(y), \partial_\mu^2 F(x, \mu)(y, y')
\]
are jointly continuous in the corresponding variable family \( (x, \mu) \), \( (x, \mu, y) \) or \( (x, \mu, y, y') \).

(ii) The function \( F : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R} \) is said to be in \( \mathcal{S}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)) \), if \( F \in C^{2,2}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)) \) and for any compact set \( K \subset \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \),
\[
\sup_{(x,\mu) \in K} \int_{\mathbb{R}^n} (|\partial_y \partial_\mu F(x, \mu)(y)|^2 + |\partial_\mu F(x, \mu)(y)|^2) \mu(dy) < \infty.
\]

3. Nonlinear filtering problems for McKean-Vlasov signal-observation systems with correlated noises

In this section, we introduce nonlinear filtering problems for McKean-Vlasov signal-observation systems with correlated noises, and derive the Kushner-Stratonovich equations, the Zakai equations and the distribution-dependent Zakai equations.

3.1. The framework. In the subsection, we introduce McKean-Vlasov signal-observation systems.

Assume:

\((H_{1,\sigma_0,\kappa_1})\) For \( t \in [0, T] \) and \( x_1, x_2 \in \mathbb{R}^n \), \( \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^n) \)

\[
\begin{align*}
|b_1(t, x_1, \mu_1) - b_1(t, x_2, \mu_2)| & \leq L_1(t) \left( |x_1 - x_2| \kappa_1(|x_1 - x_2|) + \mathbb{W}_2(\mu_1, \mu_2) \right), \\
||\sigma_0(t, x_1, \mu_1) - \sigma_0(t, x_2, \mu_2)||^2 & \leq L_1(t) \left( |x_1 - x_2|^2 \kappa_2(|x_1 - x_2|) + \mathbb{W}_2^2(\mu_1, \mu_2) \right), \\
||\sigma_1(t, x_1, \mu_1) - \sigma_1(t, x_2, \mu_2)||^2 & \leq L_1(t) \left( |x_1 - x_2|^2 \kappa_3(|x_1 - x_2|) + \mathbb{W}_2^2(\mu_1, \mu_2) \right),
\end{align*}
\]
where \( L_1(t) > 0 \) is an increasing function and \( \kappa_i \) is a positive continuous function, bounded on \([1, \infty)\) and satisfies
\[
\lim_{x \downarrow 0} \frac{\kappa_i(x)}{\log x^{-1}} < \infty, \quad i = 1, 2, 3.
\]

\((H_{b_1,\sigma_0,\sigma_1})\) For \( t \in [0, T] \) and \( x \in \mathbb{R}^n, \mu \in \mathcal{P}_2(\mathbb{R}^n)\)
\[
|b_1(t, x, \mu)|^2 + \|\sigma_0(t, x, \mu)\|^2 + \|\sigma_1(t, x, \mu)\|^2 \leq K_1(t)(1 + |x| + \|\mu\|)^2,
\]
where \( K_1(t) > 0 \) is an increasing function.

\((H_{\sigma_2})\) For \( t \in [0, T] \) and \( y_1, y_2 \in \mathbb{R}^m, \)
\[
\|\sigma_2(t, y_1) - \sigma_2(t, y_2)\|^2 \leq L_2(t)|y_1 - y_2|^2,
\]
where \( L_2(t) > 0 \) is an increasing function.

\((H_{b_2,\sigma_2})\) For \( t \in [0, T], y \in \mathbb{R}^m, \sigma_2(t, y) \) is invertible, and
\[
|b_2(t, x, \mu, y)|\|\sigma_2(t, 0)\|\|\sigma_2^{-1}(t, y)\| \leq K_2, \text{ for all } t \in [0, T], x \in \mathbb{R}^n, \mu \in \mathcal{P}_2(\mathbb{R}^n), y \in \mathbb{R}^m,
\]
where \( K_2 > 0 \) is a constant.

Under the assumptions \((H_{b_1,\sigma_0,\sigma_1})\) \((H_{b_2,\sigma_0,\sigma_1})\) \((H_{\sigma_2})\) \((H_{b_2,\sigma_2})\), by Theorem 3.1 in [3], it holds that the system \((\mathbf{I})\) has a pathwise unique strong solution denoted as \((X_t, Y_t)\). Set
\[
h(t, X_t, \mathcal{L}_X, Y_t) := \sigma_2^{-1}(t, Y_t)b_2(t, X_t, \mathcal{L}_X, Y_t),
\]
\[
\Gamma_t^{-1} := \exp \left\{ -\int_0^t h^i(s, X_s, \mathcal{L}_X, Y_s) dV^i_s - \frac{1}{2} \int_0^t |h(s, X_s, \mathcal{L}_X, Y_s)|^2 ds \right\}.
\]
Here and hereafter, we use the convention that repeated indices imply summation. By \((H_{b_2,\sigma_2})\), we know that
\[
\mathbb{E} \left( \int_0^T |h(s, X_s, \mathcal{L}_X, Y_s)|^2 ds \right) < \infty,
\]
and then \( \Gamma_t^{-1} \) is an exponential martingale. Define a measure \( \tilde{\mathbb{P}} \) via
\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \Gamma_T^{-1},
\]
and under the measure \( \tilde{\mathbb{P}} \),
\[
\tilde{V}_t := V_t + \int_0^t h(s, X_s, \mathcal{L}_X, Y_s) ds
\]
(3)
is an \((\mathcal{F}_t)\)-adapted Brownian motion. Moreover, the \( \sigma \)-algebra \( \mathcal{F}_t^{Y_0} \) generated by \( \{Y_s, 0 \leq s \leq t\} \), can be characterized as
\[
\mathcal{F}_t^{Y_0} = \mathcal{F}_t^{\tilde{V}} \vee \mathcal{F}_0^{Y_0},
\]
where \( \mathcal{F}_t^{\tilde{V}} \) denotes the \( \sigma \)-algebra generated by \( \{\tilde{V}_s, 0 \leq s \leq t\} \). And then \( \mathcal{F}_t^{Y} \) denotes the usual augmentation of \( \mathcal{F}_t^{Y_0} \).
3.2. The Kushner-Stratonovich equation. Set

\[ < \Lambda_t, F > := \mathbb{E}[F(X_t, \mathcal{L}_{X_t}) | \mathcal{F}_t^Y], \quad F \in \mathcal{B}_0(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)), \]

where \( \mathcal{B}_0(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)) \) denotes the set of all bounded measurable functions on \( \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \), and then \( \Lambda_t \) is called as the nonlinear filtering of \( (X_t, \mathcal{L}_{X_t}) \) with respect to \( \mathcal{F}_t^Y \).

Moreover, the equation satisfied by \( \Lambda_t \) is called the Kushner-Stratonovich equation. In order to derive the Kushner-Stratonovich equation, we need the following result (c.f. Lemma 2.2 in [10]).

**Lemma 3.1.** Under the measure \( \mathbb{P} \), \( \tilde{V}_t := \tilde{V}_0 - \int_0^t < \Lambda_s, h(s, \cdot, \cdot, Y_s) > ds \) is an \( (\mathcal{F}_t^Y) \)-adapted Brownian motion.

Now, it is the position to establish the Kushner-Stratonovich equation.

**Theorem 3.2.** (The Kushner-Stratonovich equation)

For \( F \in \mathcal{S}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)) \), the Kushner-Stratonovich equation of the system (7) is given by

\[
< \Lambda_t, F > = < \Lambda_0, F > + \int_0^t < \Lambda_s, \mathbb{L}_s F > ds + \int_0^t < \Lambda_s, \partial_x F \sigma_y^{ij}(s, \cdot, \cdot) > d\tilde{V}_s^j \\
+ \int_0^t (< \Lambda_s, F h^j(s, \cdot, \cdot, Y_s) > - < \Lambda_s, F > < \Lambda_s, h^j(s, \cdot, \cdot, Y_s) >) d\tilde{V}_s^j,
\]

where the operator \( \mathbb{L}_s \) is defined as

\[
(\mathbb{L}_s F)(x, \mu) = \partial_x F(x, \mu)b^i_1(s, x, \mu) + \frac{1}{2} \partial^2_{x,x_j} F(x, \mu) (\sigma_0 \sigma_0^*)^{ij}(s, x, \mu) \\
+ \frac{1}{2} \partial^2_{x,x_j} F(x, \mu) (\sigma_1 \sigma_1^*)^{ij}(s, x, \mu) + \int_{\mathbb{R}^n} (\partial_{\mu} F)(x, \mu)(y)b^i(s, y, \mu) \mu(dy) \\
+ \frac{1}{2} \int_{\mathbb{R}^n} \partial_{\mu}(\partial_{\mu} F)(x, \mu)(y)\sigma_0 \sigma_0^*(s, y, \mu) \mu(dy) \\
+ \frac{1}{2} \int_{\mathbb{R}^n} \partial_{\mu}(\partial_{\mu} F)(x, \mu)(y)\sigma_1 \sigma_1^*(s, y, \mu) \mu(dy).
\]

**Proof.** By the extended Itô’s formula in [4, Proposition 2.8], we know that for \( F \in \mathcal{S}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)) \)

\[
F(X_t, \mathcal{L}_{X_t}) = F(X_0, \mathcal{L}_{X_0}) + \int_0^t (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s}) ds \\
+ \int_0^t \partial_x F(X_s, \mathcal{L}_{X_s}) \sigma_0^{ij}(s, X_s, \mathcal{L}_{X_s}) dW_s^j \\
+ \int_0^t \partial_x F(X_s, \mathcal{L}_{X_s}) \sigma_1^{ik}(s, X_s, \mathcal{L}_{X_s}) dV_s^k \\
=: F(X_0, \mathcal{L}_{X_0}) + \int_0^t (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s}) ds + \Pi_t,
\]

(5)
where \((\Pi_t)\) is an \((\mathcal{F}_t^Y)\)-adapted local martingale. And then, by taking the conditional expectation with respect to \(\mathcal{F}_t^Y\) on two hand sides of the above equality, it holds that

\[
\mathbb{E}[F(X_t, \mathcal{L}_X)|\mathcal{F}_t^Y] = \mathbb{E}[F(X_0, \mathcal{L}_{X_0})|\mathcal{F}_t^Y] + \mathbb{E} \left[ \int_0^t (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s}) ds | \mathcal{F}_t^Y \right] + \mathbb{E}[\Pi_t|\mathcal{F}_t^Y].
\]

We rewrite the above equality to furthermore obtain that

\[
\mathbb{E}[F(X_t, \mathcal{L}_X)|\mathcal{F}_t^Y] - \mathbb{E}[F(X_0, \mathcal{L}_{X_0})|\mathcal{F}_t^Y] - \int_0^t \mathbb{E} \left[ (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s}) | \mathcal{F}_s^Y \right] ds = \int_0^t \Phi_s^j d\tilde{V}_s,
\]

and then

\[
\mathbb{E}[F(X_t, \mathcal{L}_X)|\mathcal{F}_t^Y] = \mathbb{E}[F(X_0, \mathcal{L}_{X_0})|\mathcal{F}_0^Y] + \int_0^t \mathbb{E} \left[ (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s}) | \mathcal{F}_s^Y \right] ds + \int_0^t \Phi_s^j d\tilde{V}_s.
\]

Since \(X_0\) is independent of \((\mathcal{F}_t^Y)\), it holds that

\[
\mathbb{E}[F(X_0, \mathcal{L}_{X_0})|\mathcal{F}_0^Y] = \mathbb{E}[F(X_0, \mathcal{L}_{X_0})] = \mathbb{E}[F(X_0, \mathcal{L}_{X_0})|\mathcal{F}_0^Y].
\]

From this, it follows that

\[
\mathbb{E}[F(X_t, \mathcal{L}_X)|\mathcal{F}_t^Y] = \mathbb{E}[F(X_0, \mathcal{L}_{X_0})|\mathcal{F}_0^Y] + \int_0^t \mathbb{E} \left[ (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s}) | \mathcal{F}_s^Y \right] ds + \int_0^t \Phi_s^j d\tilde{V}_s. \tag{6}
\]

In the following, we determine the process \((\Phi_t)\). On one side, one can apply the Itô formula to \(\mathbb{E}[F(X_t, \mathcal{L}_X)|\mathcal{F}_t^Y]\tilde{V}_t^j\) and obtain that

\[
\mathbb{E}[F(X_t, \mathcal{L}_X)|\mathcal{F}_t^Y]\tilde{V}_t^j = \int_0^t \mathbb{E}[F(X_s, \mathcal{L}_{X_s})|\mathcal{F}_s^Y] d\tilde{V}_s^j + \int_0^t \tilde{V}_s^j d\mathbb{E}[F(X_s, \mathcal{L}_{X_s})|\mathcal{F}_s^Y] + \int_0^t \Phi_s^j ds
\]

\[
= \int_0^t \mathbb{E}[F(X_s, \mathcal{L}_{X_s})|\mathcal{F}_s^Y] \mathbb{E}[h^j(s, X_s, \mathcal{L}_{X_s}, Y_s)|\mathcal{F}_s^Y] ds
\]

\[
+ \int_0^t \tilde{V}_s^j \mathbb{E} \left[ (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s}) | \mathcal{F}_s^Y \right] ds + \int_0^t \Phi_s^j ds + I_1^j, \tag{7}
\]

where \((I_1^j)\) is an \((\mathcal{F}_t^Y)\)-adapted local martingale.

On the other side, by (3) (5) and the Itô formula for \(F(X_t, \mathcal{L}_X)\tilde{V}_t^j\), we get that for \(j = 1, 2, \ldots, m,\)

\[
F(X_t, \mathcal{L}_X)\tilde{V}_t^j = \int_0^t F(X_s, \mathcal{L}_{X_s}) d\tilde{V}_s^j + \int_0^t \tilde{V}_s^j dF(X_s, \mathcal{L}_{X_s}) + \int_0^t \partial_x F(X_s, \mathcal{L}_{X_s}) \sigma_{1}^j(s, X_s, \mathcal{L}_{X_s}) ds
\]

\[
= \int_0^t F(X_s, \mathcal{L}_{X_s}) h^j(s, X_s, \mathcal{L}_{X_s}, Y_s) ds + \int_0^t \tilde{V}_s^j (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s}) ds.
\]
where ˜Λ is measurable with respect to 𝓡^Y_t. Thus, by taking the conditional expectation with respect to 𝓡^Y_t on two sides of the above equality, it holds that

\[ \mathbb{E}[F(X_t, \mathcal{L}_{X_t})|\mathcal{F}_t^Y] \hat{V}_t^j = \int_0^t \mathbb{E} \left[ F(X_s, \mathcal{L}_{X_s}) h^j(s, X_s, \mathcal{L}_{X_s}, Y_s)|\mathcal{F}_s^Y \right] ds \\
+ \int_0^t \hat{V}_s^j \mathbb{E} \left[ (\mathbb{L}_s F)(X_s, \mathcal{L}_{X_s})|\mathcal{F}_s^Y \right] ds \\
+ \int_0^t \mathbb{E} \left[ \partial_{x_i} F(X_s, \mathcal{L}_{X_s}) \sigma^ij(s, X_s, \mathcal{L}_{X_s})|\mathcal{F}_s^Y \right] ds + I_t^j, \quad (8) \]

where \((I_t^j)\) denotes an \((\mathcal{F}_t^Y)\)-adapted local martingale.

Since the left side of (7) is the same to that of (8), bounded variation parts of their right sides should be the same. Therefore,

\[ \Phi_t^j = \mathbb{E} \left[ F(X_s, \mathcal{L}_{X_s}) h^j(s, X_s, \mathcal{L}_{X_s}, Y_s)|\mathcal{F}_s^Y \right] - \mathbb{E} \left[ F(X_s, \mathcal{L}_{X_s})|\mathcal{F}_s^Y \right] \mathbb{E} \left[ h^j(s, X_s, \mathcal{L}_{X_s}, Y_s)|\mathcal{F}_s^Y \right] \\
+ \mathbb{E} \left[ \partial_{x_i} F(X_s, \mathcal{L}_{X_s}) \sigma^ij(s, X_s, \mathcal{L}_{X_s})|\mathcal{F}_s^Y \right], \quad a.s.\mathbb{P}. \quad (9) \]

Inserting (9) in (6) and noting \(\Lambda_t(F) = \mathbb{E}[F(X_t, \mathcal{L}_{X_t})|\mathcal{F}_t^Y]\), we can get (4). Thus, the proof is complete. \(\square\)

3.3. The Zakai equation. Set

\[ < \tilde{\Lambda}_t, F > := \mathbb{E}^\tilde{p}[F(X_t, \mathcal{L}_{X_t}) \Gamma_t|\mathcal{F}_t^Y], \quad F \in \mathcal{B}_b(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)), \]

where \(\mathbb{E}^\tilde{p}\) denotes expectation under the probability measure \(\tilde{p}\). The equation satisfied by \(\tilde{\Lambda}_t\) is called the Zakai equation. Moreover, by the Kallianpur-Striebel formula and the Kushner-Stratonovich equation (4), we can obtain the following Zakai equation.

**Theorem 3.3. (The Zakai equation)**

The Zakai equation of the system (7) is given by

\[ < \tilde{\Lambda}_t, F > = < \tilde{\Lambda}_0, F > + \int_0^t < \tilde{\Lambda}_s, \mathbb{L}_s F > ds + \int_0^t < \tilde{\Lambda}_s, F h^j(s, \cdot, \cdot, Y_s) > d\tilde{V}_s^j \\
+ \int_0^t < \tilde{\Lambda}_s, \partial_{x_i} F \sigma^ij(s, \cdot, \cdot) > d\tilde{V}_s^j, \quad F \in \mathcal{S}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)), t \in [0, T]\]

**Proof.** Although the deduction of the Zakai equation (10) is the same to that in (10) Theorem 2.8], we give the proof to the readers’ convenience.

By the Kallianpur-Striebel formula, it holds that

\[ < \Lambda_t, F >= \mathbb{E}[F(X_t, \mathcal{L}_{X_t})|\mathcal{F}_t^Y] = \frac{\mathbb{E}^\tilde{p}[F(X_t, \mathcal{L}_{X_t}) \Gamma_t|\mathcal{F}_t^Y]}{\mathbb{E}^\tilde{p}[\Gamma_t|\mathcal{F}_t^Y]} = < \tilde{\Lambda}_t, F >, \quad (10) \]

and \(< \tilde{\Lambda}_t, F >=< \Lambda_t, F >= < \tilde{\Lambda}_t, 1 >= < \Lambda_t, 1 >\). So, to establish the Zakai equation (10), we investigate \(< \tilde{\Lambda}_t, 1 >\).

First of all, the Itô formula admits us to get that

\[ \Gamma_t = 1 + \int_0^t \Gamma_s h^j(s, X_s, \mathcal{L}_{X_s}, Y_s) d\tilde{V}_s^j. \]
Taking the conditional expectation with respect to $\mathcal{F}_t^Y$ under the probability measure $\tilde{P}$, one can have that

$$\mathbb{E}^\tilde{P}[\Gamma_t | \mathcal{F}_t^Y] = 1 + \int_0^t \mathbb{E}^\tilde{P}[\Gamma_s h^i(s, X_s, L_{X_s}, Y_s) | \mathcal{F}_s^Y] d\tilde{V}_s^i,$$

namely,

$$<\tilde{\Lambda}_t, 1> = 1 + \int_0^t <\tilde{\Lambda}_s, 1 > <\Lambda_s, h^i(s, \cdot, \cdot, Y_s) > d\tilde{V}_s^i. \quad (11)$$

Next, combining (4) and (11) and applying the Itô formula to $<\Lambda_t, F><\tilde{\Lambda}_t, 1>$, we obtain that

$$<\Lambda_t, F><\tilde{\Lambda}_t, 1> = <\Lambda_0, F><\tilde{\Lambda}_0, 1> + \int_0^t <\Lambda_s, F > d<\tilde{\Lambda}_s, 1>$$

$$+ \int_0^t <\tilde{\Lambda}_s, 1 > d<\Lambda_s, F>$$

$$+ \int_0^t <\tilde{\Lambda}_s, 1 > <\Lambda_s, h^i(s, \cdot, \cdot, Y_s) > \Phi^i_s ds$$

$$= <\Lambda_0, F><\tilde{\Lambda}_0, 1> + \int_0^t <\Lambda_s, F > <\tilde{\Lambda}_s, 1 > <\Lambda_s, h^i(s, \cdot, \cdot, Y_s) > d\tilde{V}_s^i$$

$$+ \int_0^t <\Lambda_s, F > <\Lambda_s, L_s F > ds + \int_0^t <\tilde{\Lambda}_s, 1 > \Phi^j_s d\tilde{V}_s^j$$

$$+ \int_0^t <\tilde{\Lambda}_s, 1 > <\Lambda_s, h^i(s, \cdot, \cdot, Y_s) > \Phi^i_s ds$$

$$= <\Lambda_0, F><\tilde{\Lambda}_0, 1> + \int_0^t <\Lambda_s, L_s F > ds$$

$$+ \int_0^t <\Lambda_s, F > <\tilde{\Lambda}_s, 1 > <\Lambda_s, \partial_x^i F \sigma^{ij}_1 (s, \cdot, \cdot) > d\tilde{V}_s^j$$

$$+ \int_0^t <\tilde{\Lambda}_s, 1 > <\Lambda_s, F h^j(s, \cdot, \cdot, Y_s) > d\tilde{V}_s^j,$$

where $\Phi$ is defined in (9). Thus, the above equality together with $<\tilde{\Lambda}_t, F >=<\Lambda_t, F><\tilde{\Lambda}_t, 1>$ yields that

$$<\Lambda_t, F> = <\Lambda_0, F> + \int_0^t <\Lambda_s, L_s F > ds + \int_0^t <\tilde{\Lambda}_s, \partial_x^i F \sigma^{ij}_1 (s, \cdot, \cdot) > d\tilde{V}_s^j$$

$$+ \int_0^t <\tilde{\Lambda}_s, F h^j(s, \cdot, \cdot, Y_s) > d\tilde{V}_s^j,$$

which is just the Zakai equation (11). The proof is over. \hfill \Box

In the following, set

$$<\tilde{P}_t, \varphi> := \mathbb{E}^\tilde{P}[\varphi(X_t) \Gamma_t | \mathcal{F}_t^Y], \quad \varphi \in \mathcal{B}_b(\mathbb{R}^n),$$
and then by the above theorem, it holds that $< \hat{P}_t, \varphi >$ satisfies the distribution-dependent Zakai equation

$$< \hat{P}_t, \varphi > = < \hat{P}_0, \varphi > + \int_0^t < \hat{P}_s, (L_s \varphi)(\cdot, \mathcal{L}_X) > ds + \int_0^t < \hat{P}_s, \varphi h^j(s, \cdot, \mathcal{L}_X, Y_s) > d\hat{V}^j_s$$

$$+ \int_0^t < \hat{P}_s, \partial_x \varphi \sigma^j_i(s, \cdot, \mathcal{L}_X) > d\hat{V}^i_s, \quad \varphi \in C^\infty_c(\mathbb{R}^n), \quad t \in [0, T],$$

where the operator $L_s$ is defined as

$$(L_s \varphi)(x, \mu) = \partial_x \varphi(x) b_1(s, x, \mu) + \frac{1}{2} \partial^2_{x_i x_j} \varphi(x) (\sigma_0 \sigma^*_0)^{ij}(s, x, \mu) + \frac{1}{2} \partial^2_{x_i x_j} \varphi(x) (\sigma_1 \sigma^*_1)^{ij}(s, x, \mu).$$

Remark 3.4. If $\sigma_1 = 0$, Eq. (12) becomes Eq. (3.14) in [17] Theorem 3.1]. And if $b_1, \sigma_0, \sigma_1, b_2$ are independent of the distribution of $X_t$, Eq. (13) is the same to Eq. (15) without jumps in [10]. Therefore, our result is more general.

4. THE PATHWISE UNIQUE AND UNIQUENESS IN JOINT LAW OF WEAK SOLUTIONS FOR THE DISTRIBUTION-DEPENDENT ZAKAI EQUATION (12)

In the section we require that $b_2(t, x, \mu, y), \sigma_2(t, y)$ are independent of $y$. First of all, we define weak solutions, the pathwise uniqueness and uniqueness in joint law of weak solutions for the distribution-dependent Zakai equation (12). After this, a family of operators is introduced and applied to show the pathwise uniqueness, uniqueness in joint law and uniqueness in law for weak solutions to the distribution-dependent Zakai equation (12).

Definition 4.1. $\{(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0, T]}, \hat{P}), (\hat{\mu}_t, \hat{V}_t)\}$ is called a weak solution of the distribution-dependent Zakai equation (12), if the following holds:

(i) $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0, T]}, \hat{P})$ is a complete filtered probability space;

(ii) $\hat{\mu}_t$ is a $\mathcal{M}(\mathbb{R}^n)$-valued $(\hat{\mathcal{F}}_t)$-adapted continuous process and $\hat{\mu}_0 \in \mathcal{P}(\mathbb{R}^n)$;

(iii) $\hat{V}_t$ is a m-dimensional $(\hat{\mathcal{F}}_t)$-adapted Brownian motion;

(iv) For any $t \in [0, T]$,

$$\hat{P} \left( \int_0^t \int_{\mathbb{R}^n} \left( |b_1(r, z, \mathcal{L}_X) | + |h(r, z, \mathcal{L}_X)|^2 + \|\sigma_1(r, z, \mathcal{L}_X)\|^2 \right) \hat{\mu}_r(\cdot) dr < \infty \right) = 1;$$

(v) $(\hat{\mu}_t, \hat{V}_t)$ satisfies the following equation

$$< \hat{\mu}_t, \varphi > = < \hat{\mu}_0, \varphi > + \int_0^t < \hat{\mu}_s, (L_s \varphi)(\cdot, \mathcal{L}_X) > ds + \int_0^t < \hat{\mu}_s, \varphi h^j(s, \cdot, \mathcal{L}_X) > d\hat{V}^j_s$$

$$+ \int_0^t < \hat{\mu}_s, \partial_x \varphi \sigma^j_i(s, \cdot, \mathcal{L}_X) > d\hat{V}^i_s, \quad t \in [0, T], \quad \varphi \in C^\infty_c(\mathbb{R}^n).$$

Remark 4.2. By the deduction in Section 3, it is obvious that $\{(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \in [0, T]}, \bar{P}), (\bar{\mu}_t, \bar{V}_t)\}$ is a weak solution of the distribution-dependent Zakai equation (12).

Definition 4.3. The pathwise uniqueness of weak solutions for the distribution-dependent Zakai equation (12) means that if there exist two weak solutions $\{(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0, T]}, \hat{P}), (\hat{\mu}_t^1, \hat{V}_t^1)\}$
and \(\{(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P}}), (\hat{\mu}^1_t, \hat{V}^1_t)\}\) with \(\hat{\mathbb{P}}\{\hat{\mu}_0^1 = \hat{\mu}_0^2\} = 1\), then
\[
\hat{\mu}_1^t = \hat{\mu}_2^t, \quad t \in [0,T], \text{ a.s. } \hat{\mathbb{P}}.
\]

**Definition 4.4.** The uniqueness in joint law of weak solutions for the distribution-dependent Zakai equation \((12)\) means that if there exist two weak solutions \(\{(\hat{\Omega}_t, \hat{\mathcal{F}}_t, \{\hat{\mathcal{F}}^1_t\}_{t \in [0,T]}, \hat{\mathbb{P}}^1), (\hat{\mu}^1_t, \hat{V}^1_t)\}\) and \(\{(\Omega^2_t, \mathcal{F}^2_t, \{\mathcal{F}^2_t\}_{t \in [0,T]}, \mathbb{P}^2), (\hat{\mu}^2_t, \hat{V}^2_t)\}\) with \(\hat{\mathbb{P}}^1 \circ (\hat{\mu}^1_0)^{-1} = \hat{\mathbb{P}}^2 \circ (\hat{\mu}^2_0)^{-1}\), then \(\{(\hat{\mu}_1^t, \hat{V}^1_t), t \in [0,T]\}\) and \(\{(\hat{\mu}_2^t, \hat{V}^2_t), t \in [0,T]\}\) have the same finite-dimensional distributions.

**Definition 4.5.** The uniqueness in law of weak solutions for the distribution-dependent Zakai equation \((12)\) means that if there exist two weak solutions \(\{(\hat{\Omega}_t, \hat{\mathcal{F}}_t, \{\hat{\mathcal{F}}^1_t\}_{t \in [0,T]}, \hat{\mathbb{P}}^1), (\hat{\mu}^1_t, \hat{V}^1_t)\}\) and \(\{(\Omega^2_t, \mathcal{F}^2_t, \{\mathcal{F}^2_t\}_{t \in [0,T]}, \mathbb{P}^2), (\hat{\mu}^2_t, \hat{V}^2_t)\}\) with \(\hat{\mathbb{P}}^1 \circ (\hat{\mu}^1_0)^{-1} = \hat{\mathbb{P}}^2 \circ (\hat{\mu}^2_0)^{-1}\), then \(\hat{\mathbb{P}}^1 \circ (\hat{\mu}_1^t)^{-1} = \hat{\mathbb{P}}^2 \circ (\hat{\mu}_2^t)^{-1}\) for any \(t \in [0,T]\).

Next, in order to obtain the uniqueness for weak solutions to the distribution-dependent Zakai equation \((12)\), we also assume:

\((\mathcal{H}^{1}_{b_1,\sigma_0,\sigma_1})\) There exists an increasing function \(L_1(t) > 0\) such that for \(t \in [0,T]\) and \(x_1, x_2 \in \mathbb{R}^n, \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^n)\)
\[
|b_1(t, x_1, \mu_1) - b_2(t, x_2, \mu_2)| \leq L_1(t) \left( |x_1 - x_2| + \mathbb{W}_2(\mu_1, \mu_2) \right).
\]
\[
||\sigma_0(t, x_1, \mu_1) - \sigma_0(t, x_2, \mu_2)|| \leq L_1(t) \left( |x_1 - x_2| + \mathbb{W}_2(\mu_1, \mu_2) \right)
\]
\[
||\sigma_1(t, x_1, \mu_1) - \sigma_1(t, x_2, \mu_2)|| \leq L_1(t) \left( |x_1 - x_2| + \mathbb{W}_2(\mu_1, \mu_2) \right)
\]

\((\mathcal{H}^{2}_{b_1,\sigma_0,\sigma_1})\) There exists an increasing function \(K_1(t) > 0\) such that for \(t \in [0,T]\) and \(x \in \mathbb{R}^n, \mu \in \mathcal{P}_2(\mathbb{R}^n)\)
\[
|b_1(t, x, \mu)| + ||\sigma_0(t, x, \mu)|| + ||\sigma_1(t, x, \mu)|| \leq K_1(t).
\]

\((\mathcal{H}^{3}_{b_2})\) There exists an increasing function \(L_3(t) > 0\) such that for \(t \in [0,T]\) and \(x_1, x_2 \in \mathbb{R}^n, \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^n)\)
\[
|b_2(t, x_1, \mu_1) - b_2(t, x_2, \mu_2)| \leq L_3(t) \left( |x_1 - x_2| + \mathbb{W}_2(\mu_1, \mu_2) \right)
\]

Now, it is the position for us to state and prove the main result in the section.

**Theorem 4.6.** (The pathwise uniqueness)

Suppose that \((\mathcal{H}^{1}_{b_1,\sigma_0,\sigma_1}) \ (\mathcal{H}^{2}_{b_1,\sigma_0,\sigma_1}) \ (\mathcal{H}^{3}_{b_2,\sigma_2}) \ (\mathcal{H}^{3}_{b_3})\) hold. If \(\{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P}), (\hat{\mu}_t, \hat{V}_t)\}\) with \(\hat{\mu}_0 = \hat{\mathbb{P}}_0\) is another weak solution for the distribution-dependent Zakai equation \((12)\), then \(\hat{\mu}_t = \hat{\mathbb{P}}_t\) for any \(t \in [0,T]\) a.s. \(\hat{\mathbb{P}}\).

To prove the above theorem, we introduce a family of operators on \(\mathbb{H} := \mathcal{L}^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dx)\). For \(\varepsilon > 0\), set
\[
(S_{\varepsilon} \mu)(x) := \int_{\mathbb{R}^n} (2\pi \varepsilon)^{-\frac{n}{2}} \exp \left\{-\frac{|x - y|^2}{2\varepsilon}\right\} \mu(dy), \quad \mu \in \mathcal{M}(\mathbb{R}^n),
\]
\[
(S_{\varepsilon} \varphi)(x) := \int_{\mathbb{R}^n} (2\pi \varepsilon)^{-\frac{n}{2}} \exp \left\{-\frac{|x - y|^2}{2\varepsilon}\right\} \varphi(y)dy, \quad \varphi \in \mathbb{H},
\]
and then one can justify \(S_{\varepsilon} \mu, S_{\varepsilon} \varphi \in \mathbb{H}\). In the following, we deduce some results by means of \(S_{\varepsilon}\).
Lemma 4.7. Assume that \( \{\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathcal{P}}, (\hat{\mu}_t, \hat{V}_t) \} \) is a weak solution for the distribution-dependent Zakai equation (12). Set \( Z^\varepsilon_t := S^\varepsilon \hat{\mu}_t \), and then it holds that

\[
\mathbb{E}^\hat{\mathcal{P}} \| Z^\varepsilon_t \|_{\mathbb{H}}^2 = \| Z^\varepsilon_0 \|_{\mathbb{H}}^2 - 2 \int_0^t \mathbb{E}^\hat{\mathcal{P}} < Z^\varepsilon_t, \partial_{x_i} \left( S^\varepsilon(b^j_i(s, \cdot, \mathcal{L}_s \hat{\mu}_s)) \right) >_{\mathbb{H}} \, ds \\
+ \int_0^t \mathbb{E}^\hat{\mathcal{P}} < Z^\varepsilon_t, \partial^2_{x_i x_j} \left( S^\varepsilon \left( (\sigma_0^* \sigma_0^*)^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right) \right) >_{\mathbb{H}} \, ds \\
+ \int_0^t \mathbb{E}^\hat{\mathcal{P}} < Z^\varepsilon_t, \partial^2_{x_i x_j} \left( S^\varepsilon \left( (\sigma_1^* \sigma_1^*)^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right) \right) >_{\mathbb{H}} \, ds \\
+ \sum_{j=1}^m \int_0^t \mathbb{E}^\hat{\mathcal{P}} \| S^\varepsilon \left( h^j(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right) \|_{\mathbb{H}}^2 \, ds \\
+ \sum_{j=1}^m \int_0^t \mathbb{E}^\hat{\mathcal{P}} \| \partial_{x_i} \left( S^\varepsilon \left( \sigma_1^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right) \right) \|_{\mathbb{H}}^2 \, ds \\
- \int_0^t \mathbb{E}^\hat{\mathcal{P}} < S^\varepsilon \left( h^j(s, \cdot, \mathcal{L}_s \hat{\mu}_s), \partial_{x_i} \left( S^\varepsilon \left( \sigma_1^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right) \right) \right) >_{\mathbb{H}} \, ds. \tag{14}
\]

**Proof. Step 1.** We take \( \varphi \in C^\infty_c(\mathbb{R}^n) \) and establish the following equation

\[
< Z^\varepsilon_t, \varphi >_{\mathbb{H}} = < Z^\varepsilon_0, \varphi >_{\mathbb{H}} - \int_0^t < \partial_{x_i} \left( S^\varepsilon(b^j_i(s, \cdot, \mathcal{L}_s \hat{\mu}_s)) \right), \varphi >_{\mathbb{H}} \, ds \\
+ \frac{1}{2} \int_0^t < \partial^2_{x_i x_j} \left( S^\varepsilon \left( (\sigma_0 \sigma_0^*)^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right) \right), \varphi >_{\mathbb{H}} \, ds \\
+ \frac{1}{2} \int_0^t < \partial^2_{x_i x_j} \left( S^\varepsilon \left( (\sigma_1 \sigma_1^*)^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right) \right), \varphi >_{\mathbb{H}} \, ds \\
+ \int_0^t < S^\varepsilon \left( h^j(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right), \varphi >_{\mathbb{H}} \, d\hat{V}^j_s \\
- \int_0^t < \partial_{x_i} \left( S^\varepsilon \left( \sigma_1^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) \right) \right), \varphi >_{\mathbb{H}} \, d\hat{V}^j_s, \quad t \in [0,T]. \tag{15}
\]

By Definition 4.11 it holds that for \( \varphi \in C^\infty_c(\mathbb{R}^n) \)

\[
< \hat{\mu}_t, \varphi > = < \hat{\mu}_0, \varphi > + \int_0^t < \hat{\mu}_s, (\mathcal{L}_s \varphi)(\cdot, \mathcal{L}_s \hat{\mu}_s) > \, ds + \int_0^t < \hat{\mu}_s, \varphi h^j(s, \cdot, \mathcal{L}_s \hat{\mu}_s) > \, d\hat{V}^j_s \\
+ \int_0^t < \hat{\mu}_s, \partial_{x_i} \varphi \sigma_1^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) > \, d\hat{V}^j_s, \quad t \in [0,T].
\]

Replacing \( \varphi \) by \( S^\varepsilon \varphi \) and using Lemma 3.1, we obtain that

\[
< S^\varepsilon \hat{\mu}_t, \varphi >_{\mathbb{H}} = < S^\varepsilon \hat{\mu}_0, \varphi >_{\mathbb{H}} + \int_0^t < \hat{\mu}_s, (\mathcal{L}_s S^\varepsilon \varphi)(\cdot, \mathcal{L}_s \hat{\mu}_s) > \, ds \\
+ \int_0^t < \hat{\mu}_s, (S^\varepsilon \varphi) h^j(s, \cdot, \mathcal{L}_s \hat{\mu}_s) > \, d\hat{V}^j_s \\
+ \int_0^t < \hat{\mu}_s, \partial_{x_i} (S^\varepsilon \varphi) \sigma_1^{ij}(s, \cdot, \mathcal{L}_s \hat{\mu}_s) > \, d\hat{V}^j_s. \tag{16}
\]
Set

\[ I_1 := \langle \mu_s, (L_x S_\varepsilon \varphi)(\cdot, \mathcal{L}_{X_s}) \rangle, \quad I_2 := \langle \mu_s, (S_\varepsilon \varphi) h^j(s, \cdot, \mathcal{L}_{X_s}) \rangle, \quad I_3 := \langle \mu_s, \partial_x (S_\varepsilon \varphi) \sigma_1^{ij}(s, \cdot, \mathcal{L}_{X_s}) \rangle, \]

and then we investigate them one by one.

For \( I_1 \), from the definition of \( L_s \) and [10] Lemma 3.1, it follows that

\[ I_1 = \langle \mu_s, \partial_x (S_\varepsilon \varphi) b_i^j(s, \cdot, \mathcal{L}_{X_s}) \rangle + \frac{1}{2} \langle \mu_s, \partial_{x_ix_j}^2 (S_\varepsilon \varphi) (\sigma_0 \sigma_0^*)^{ij}(s, \cdot, \mathcal{L}_{X_s}) \rangle + \frac{1}{2} \langle \mu_s, \partial_{x_ix_j}^2 (S_\varepsilon \varphi) (\sigma_1 \sigma_1^*)^{ij}(s, \cdot, \mathcal{L}_{X_s}) \rangle \]

Combining (17)-(19) with (16) and noting \( <\sigma_0 \sigma_0^*|^2 \leq 2 \langle \sigma_0 \sigma_0^* \rangle \), we have

\[ <\sigma_1 \sigma_1^*|^2 \leq 2 \langle \sigma_1 \sigma_1^* \rangle \]

where in the last equality the formula for integration by parts is used.

For \( I_2, I_3 \), we apply [10] Lemma 3.1 to have that

\[ I_2 = \langle h^j(s, \cdot, \mathcal{L}_{X_s}) \mu_s, S_\varepsilon \varphi \rangle = \langle S_\varepsilon \left( h^j(s, \cdot, \mathcal{L}_{X_s}) \mu_s \right), \varphi \rangle \]

and

\[ I_3 = \langle \sigma_1^{ij}(s, \cdot, \mathcal{L}_{X_s}) \mu_s, \partial_x (S_\varepsilon \varphi) \rangle \]

Combining (17)-(19) with (16) and noting \( Z_t^\varepsilon = S_\varepsilon \mu_t \), one can get (15).

**Step 2** We prove (14).

Applying the Itô formula to \( | < Z_t^\varepsilon, \varphi >_H |^2 \), we obtain that

\[ | < Z_t^\varepsilon, \varphi >_H |^2 = | < Z_0^\varepsilon, \varphi >_H |^2 - 2 \int_0^t < \varphi >_H \partial_x \left( S_\varepsilon \left( b_i^j(s, \cdot, \mathcal{L}_{X_s}) \mu_s \right) \right), \varphi >_H ds \]

\[ + \int_0^t < Z_s^\varepsilon, \varphi >_H \partial_{x_ix_j}^2 \left( S_\varepsilon \left( \sigma_0 \sigma_0^* \right)^{ij}(s, \cdot, \mathcal{L}_{X_s}) \mu_s \right), \varphi >_H ds \]

\[ + \int_0^t < Z_s^\varepsilon, \varphi >_H \partial_{x_ix_j}^2 \left( S_\varepsilon \left( \sigma_1 \sigma_1^* \right)^{ij}(s, \cdot, \mathcal{L}_{X_s}) \mu_s \right), \varphi >_H ds \]
Thus, (14) is obtained by taking the expectation under the measure $\hat{\mu}^j$.

\begin{align*}
+2 \int_0^t < Z^\varepsilon_s, \varphi >_H &< S_\varepsilon \left( h^j(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right), \varphi >_H d\hat{V}^j_s \\
-2 \int_0^t < Z^\varepsilon_s, \varphi >_H &< \partial_{x_i} \left( S_\varepsilon \left( \sigma^{ij}_1(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right) \right), \varphi >_H d\hat{V}^j_s \\
+ \sum_{j=1}^m \int_0^t | < S_\varepsilon \left( h^j(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right), \varphi >_H |^2 ds \\
+ \sum_{j=1}^m \int_0^t | < \partial_{x_i} \left( S_\varepsilon \left( \sigma^{ij}_1(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right) \right), \varphi >_H |^2 ds \\
- \int_0^t < S_\varepsilon \left( h^j(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right), \varphi >_H < \partial_{x_i} \left( S_\varepsilon \left( \sigma^{ij}_1(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right) \right), \varphi >_H ds \\

t &\in [0, T], \quad \varphi \in C_c^\infty (\mathbb{R}^n).
\end{align*}

And then we take a complete orthogonal basis $\{\phi_j, j = 1, 2, ...\}$ in $\mathbb{H}$ and fix it in the sequel. Letting $\varphi = \phi_j, j = 1, 2, ...$ in the above equality and using the equality $\|Z^\varepsilon_t\|_H^2 = \sum_{j=1}^\infty | < Z^\varepsilon_i, \phi_j >_H |^2$, one can have that

\begin{align*}
\|Z^\varepsilon_t\|_H^2 &= \|Z^\varepsilon_0\|_H^2 - 2 \int_0^t < Z^\varepsilon_s, \partial_{x_i} \left( S_\varepsilon \left( b^1_i(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right) \right), \varphi >_H ds \\
+ \int_0^t < Z^\varepsilon_s, \partial^2_{x_i,x_j} \left( S_\varepsilon \left( \sigma^{ij}_0 s^0_0 s^0_0 \right) \left( s, \cdot, \mathcal{L}_{X_s} \right) \hat{\mu}_s \right), \varphi >_H ds \\
+ \int_0^t < Z^\varepsilon_s, \partial^2_{x_i,x_j} \left( S_\varepsilon \left( \sigma^{ij}_1 s^1 s^1 \right) \left( s, \cdot, \mathcal{L}_{X_s} \hat{\mu}_s \right) \right), \varphi >_H ds \\
+ 2 \int_0^t < Z^\varepsilon_s, S_\varepsilon \left( h^j(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right), \varphi >_H d\hat{V}^j_s \\
- 2 \int_0^t < Z^\varepsilon_s, \partial_{x_i} \left( S_\varepsilon \left( \sigma^{ij}_1(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right) \right), \varphi >_H d\hat{V}^j_s \\
+ \sum_{j=1}^m \int_0^t \|S_\varepsilon \left( h^j(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right)\|_H^2 ds + \sum_{j=1}^m \int_0^t \|\partial_{x_i} \left( S_\varepsilon \left( \sigma^{ij}_1(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right) \right)\|_H^2 ds \\
- \int_0^t < S_\varepsilon \left( h^j(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right), \partial_{x_i} \left( S_\varepsilon \left( \sigma^{ij}_1(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right) \right), \varphi >_H ds.
\end{align*}

Thus, (14) is obtained by taking the expectation under the measure $\hat{\mu}$ on two hand sides of the above equality. The proof is complete. \hfill \Box

For $\mu \in \mathcal{M}(\mathbb{R}^n)$, $\mu \in \mathbb{H}$ means that $\|\mu\|_H^2 := \sum_{j=1}^\infty < \mu, \phi_j >^2 < \infty$.

**Lemma 4.8.** Suppose that $(\mathcal{H}^1_{b_1,s_0,s_1})$ $(\mathcal{H}^\varepsilon_{b_1,s_0,s_1})$ $(\mathcal{H}^{b_2,s_2})$ $(\mathcal{H}^{b_2})$ hold. Then for a weak solution $\{(\Omega, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\})_{t \in [0, T]}, \hat{\mu}, (\hat{V}_t)\}$ of the distribution-dependent Zakai equation (12) with $\hat{\mu}_0 \in \mathbb{H}$, it holds that for $Z^\varepsilon_t := S_\varepsilon \hat{\mu}_t$,

$$
\mathbb{E}^\hat{\mu} \|Z^\varepsilon_t\|_H^2 \leq \|Z^\varepsilon_0\|_H^2 + C \int_0^t \mathbb{E}^\hat{\mu} \|Z^\varepsilon_s\|_H^2 ds,
$$

(20)

and $\hat{\mu}_t \in \mathbb{H}$, $\hat{\mu}$ a.s. for $t \in [0, T]$. 

14
Proof. By Lemma \ref{lem1}, it holds that
\[
\mathbb{E}^\hat{\mathbb{P}} \|Z_t^\varepsilon\|_H^2 = \|Z_0^\varepsilon\|_H^2 - 2 \int_0^t \mathbb{E}^\hat{\mathbb{P}} < Z_s^\varepsilon, \partial_{x_i} \left( S_s^\varepsilon \left( b_s^1(s, \cdot, \mathcal{L}_{X_s}) \hat{\mu}_s \right) \right) >_H ds
+ \int_0^t \mathbb{E}^\hat{\mathbb{P}} < Z_s^\varepsilon, \partial_{x_i}^2 \left( S_s^\varepsilon \left( \left( \sigma_0^s \sigma_0^s \right)^{ij} \left( s, \cdot, \mathcal{L}_{X_s} \right) \hat{\mu}_s \right) \right) >_H ds
+ \int_0^t \mathbb{E}^\hat{\mathbb{P}} < Z_s^\varepsilon, \partial_{x_i}^2 \left( S_s^\varepsilon \left( \left( \sigma_1^s \sigma_1^s \right)^{ij} \left( s, \cdot, \mathcal{L}_{X_s} \right) \hat{\mu}_s \right) \right) >_H ds
+ \sum_{j=1}^m \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|S_s^\varepsilon \left( h^j \left( s, \cdot, \mathcal{L}_{X_s} \right) \hat{\mu}_s \right) \|_H^2 ds
+ \sum_{j=1}^m \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|\partial_{x_i} \left( S_s^\varepsilon \left( \left( \sigma_1^s \sigma_1^s \right)^{ij} \left( s, \cdot, \mathcal{L}_{X_s} \right) \hat{\mu}_s \right) \right) \|_H^2 ds
- \int_0^t \mathbb{E}^\hat{\mathbb{P}} < S_s^\varepsilon \left( h^j \left( s, \cdot, \mathcal{L}_{X_s} \right) \hat{\mu}_s \right), \partial_{x_i} \left( S_s^\varepsilon \left( \left( \sigma_1^s \sigma_1^s \right)^{ij} \left( s, \cdot, \mathcal{L}_{X_s} \right) \hat{\mu}_s \right) \right) >_H ds
=: \|Z_0^\varepsilon\|_H^2 + J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \tag{21}
\]
By \cite{10} Lemma 3.2, we know that
\[
J_1 + J_4 + J_6 \leq C \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|Z_s^\varepsilon\|_H^2 ds. \tag{22}
\]
And then \cite{10} Lemma 3.3 admits us to obtain that
\[
J_2 + J_3 + J_5 \leq C \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|Z_s^\varepsilon\|_H^2 ds. \tag{23}
\]
By combining (22)(23) with (21), it holds that
\[
\mathbb{E}^\hat{\mathbb{P}} \|Z_t^\varepsilon\|_H^2 \leq \|Z_0^\varepsilon\|_H^2 + C \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|Z_s^\varepsilon\|_H^2 ds.
\]
This is just right (20).

Next, (20) and the Gronwall inequality admit us to have that
\[
\mathbb{E}^\hat{\mathbb{P}} \|\hat{\mu}_t\|_H^2 \leq \liminf_{\varepsilon \to 0} \|Z_0^\varepsilon\|_H^2 e^{Ct} = \|\hat{\mu}_0\|_H^2 e^{Ct} < \infty.
\]
That is, $\hat{\mu}_t \in H, \hat{\mathbb{P}} a.s.$ for $t \in [0, T]$.

\[\Box\]

**The proof of Theorem \ref{thm1}**
Set $\Sigma_t := \hat{\mu}_t - \tilde{\mu}_t$, and then $\Sigma_t$ satisfies Eq.\ref{eq13}. Thus, it follows from Lemma \ref{lem8} that
\[
\mathbb{E}^\hat{\mathbb{P}} \|S_s^\varepsilon \Sigma_t\|_H^2 \leq C \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|S_s^\varepsilon (|\Sigma_s|)\|_H^2 ds \leq C \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|\Sigma_s\|_H^2 ds = C \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|\Sigma_s\|_H^2 ds,
\]
and furthermore
\[
\mathbb{E}^\hat{\mathbb{P}} \|\Sigma_t\|_H^2 \leq C \int_0^t \mathbb{E}^\hat{\mathbb{P}} \|\Sigma_s\|_H^2 ds,
\]
where we use the Fatou lemma. By the Gronwall inequality and the continuity of \( \bar{\mu}_t, \bar{P}_t \) in \( t \), it holds that
\[
\bar{\mu}_t = \bar{P}_t, \quad \forall t \in [0, T], \text{ a.s. } \bar{P}.
\]
The proof is over.

**Theorem 4.9.** (*The uniqueness in joint law*)
Under \( (H_{b_1,\sigma_1}) (H_{b_2,\sigma_1}^2) (H_{b_2,\sigma_2}^2) (H_{b_2}^3) \), it holds that weak solutions of the distribution-dependent Zakai equation (12) have the uniqueness in joint law.

Since the proof of the above theorem is similar to that of Theorem 4 (ii) in [12], we omit it. Besides, by Theorem 4.9, we can obtain the following corollary.

**Corollary 4.10.** Under \( (H_{b_1,\sigma_0,\sigma_1}) (H_{b_1,\sigma_0,\sigma_1}^2) (H_{b_2,\sigma_2}^2) (H_{b_2}^3) \), it holds that weak solutions of the distribution-dependent Zakai equation (12) have the uniqueness in law.

5. **Superposition between the distribution-dependent Zakai equations (12) and distribution-dependent Fokker-Planck equations**

In the section we also require that \( b_2(t, x, \mu, y), \sigma_2(t, y) \) are independent of \( y \). We introduce distribution-dependent Fokker-Planck equations and then prove a superposition principle between the distribution-dependent Zakai equations (12) and them.

5.1. **The Fokker-Planck equations associated with the distribution-dependent Zakai equations (12).** In the subsection, we introduce distribution-dependent Fokker-Planck equations associated with the distribution-dependent Zakai equations (12) and define their weak solutions.

First of all, set
\[
\mathcal{G} := \big\{ \nu \in \mathcal{M}(\mathbb{R}^n) \mapsto G(\nu) = g(\nu, \varphi_1, \cdots, \nu, \varphi_k) : k \in \mathbb{N}, g \in C^2_b(\mathbb{R}^k), \\
\varphi_1, \cdots, \varphi_k \in C^\infty_c(\mathbb{R}^n) \big\}.
\]
And then for any \( G(\nu) = g(\nu, \varphi_1, \cdots, \nu, \varphi_k) =: g(\nu, \varphi) \in \mathcal{G} \), we define an operator \( \mathbf{L}_t \) on \( \mathcal{G} \) as follows:
\[
\mathbf{L}_t G(\nu) = \frac{1}{2} \partial_{y_i y_i} g(\nu, \varphi) < \nu, \varphi h^l(t, \cdot, \mathcal{L}_{X_i}) + \partial_{x_i} \varphi_1^u(t, \cdot, \mathcal{L}_{X_i}) > \\
\times < \nu, \varphi h^l(t, \cdot, \mathcal{L}_{X_i}) + \partial_{x_i} \varphi_1^u(t, \cdot, \mathcal{L}_{X_i}) > < \nu, \varphi > + \partial_{y_i} g(\nu, \varphi) < \nu, \varphi h^l(t, \cdot, \mathcal{L}_{X_i}) + \partial_{x_i} \varphi_1^u(t, \cdot, \mathcal{L}_{X_i}) > \\
\times < \nu, \varphi h^l(t, \cdot, \mathcal{L}_{X_i}) + \partial_{x_i} \varphi_1^u(t, \cdot, \mathcal{L}_{X_i}) > < \nu, \varphi > + \partial_{y_i} g(\nu, \varphi) < \nu, \varphi b_i^j(t, \cdot, \mathcal{L}_{X_i}) > \\
\times < \nu, \varphi b_i^j(t, \cdot, \mathcal{L}_{X_i}) > < \nu, \varphi > + \partial_{y_i} g(\nu, \varphi) < \nu, \varphi (\sigma_0 \sigma_1^u)^{ij} (t, \cdot, \mathcal{L}_{X_i}) > \\
\times < \nu, \varphi (\sigma_0 \sigma_1^u)^{ij} (t, \cdot, \mathcal{L}_{X_i}) > < \nu, \varphi > + \partial_{y_i} g(\nu, \varphi) < \nu, \varphi (\sigma_0 \sigma_1^u)^{ij} (t, \cdot, \mathcal{L}_{X_i}) > \\
\times < \nu, \varphi (\sigma_0 \sigma_1^u)^{ij} (t, \cdot, \mathcal{L}_{X_i}) > < \nu, \varphi >.
\]

Consider the following Fokker-Planck equation (FPE in short):
\[
\partial_t \Xi_t = \mathbf{L}_t^* \Xi_t, \quad (24)
\]
called a weak solution of the FPE (24) if

$$\int_0^T \int_{\mathcal{M}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left( |b_1(r,z,L_X)| + |h(r,z,L_X)|^2 + \|\sigma_1(r,z,L_X)\|^2 + \|\sigma_0^*\sigma_0^*(r,z,L_X)\| \right) \nu(dz)\Xi_t(d\nu)dr < \infty,$$

and for any $G \in \mathcal{U}$ and $0 \leq t \leq T$,

$$\int_{\mathcal{M}(\mathbb{R}^n)} G(\nu)\Xi_t(d\nu) = \int_{\mathcal{M}(\mathbb{R}^n)} G(\nu)\Xi_0(d\nu) + \int_0^t \int_{\mathcal{M}(\mathbb{R}^n)} \mathbb{L}_t G(\nu)\Xi_t(d\nu)dr. \quad (26)$$

The uniqueness of the weak solutions to Eq. (24) means that, if $(\Xi_t)_{t \in [0,T]}$ and $(\hat{\Xi}_t)_{t \in [0,T]}$ are two weak solutions to Eq. (24) with $\Xi_0 = \hat{\Xi}_0$, then $\Xi_t = \hat{\Xi}_t$ for any $t \in [0,T]$.

It is easy to see that under the condition (25), the integral in the right side of Eq. (26) is well defined.

5.2. A superposition principle between Eq. (12) and Eq. (24). In the subsection, we prove the following superposition principle between Eq. (12) and Eq. (24).

Theorem 5.2. (The superposition principle on $\mathcal{M}(\mathbb{R}^n)$)

(i) If $\{\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P}}, (\hat{\mu}_t, \hat{V}_t)\}$ is a weak solution for Eq. (12), then $(L_{\hat{\mu}_t})$ is a weak solution for Eq. (24).

(ii) If $(\Xi_t)$ is a weak solution for Eq. (24), then there exists a weak solution $\{\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P}}, (\hat{\mu}_t, \hat{V}_t)\}$ for Eq. (12) such that $\Xi_t = L_{\hat{\mu}_t}$ for any $t \in [0,T]$.

(iii) If weak solutions for Eq. (12) are unique in law, then Eq. (24) has at most a weak solution.

(iv) If weak solutions for Eq. (24) are unique, then weak solutions for Eq. (12) have the uniqueness in law.

The proof of Theorem 5.2 (i).

We verify that $L_{\hat{\mu}_t}$ satisfies the condition (25). Indeed, by (iv) in Definition 4.1, it holds that

$$\int_0^T \int_{\mathcal{M}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left( |b_1(r,z,L_X)| + |h(r,z,L_X)|^2 + \|\sigma_1(r,z,L_X)\|^2 
+ \|\sigma_0^*\sigma_0^*(r,z,L_X)\| \right) \nu(dz)L_{\hat{\mu}_t}(d\nu)dr$$

$$= \int_0^T \mathbb{E}_{\hat{\mathbb{P}}} \int_{\mathbb{R}^n} \left( |b_1(r,z,L_X)| + |h(r,z,L_X)|^2 + \|\sigma_1(r,z,L_X)\|^2 
+ \|\sigma_0^*\sigma_0^*(r,z,L_X)\| \right) \hat{\mu}_t(dz)dr$$

$$= \mathbb{E}_{\hat{\mathbb{P}}} \int_0^T \int_{\mathbb{R}^n} \left( |b_1(r,z,L_X)| + |h(r,z,L_X)|^2 + \|\sigma_1(r,z,L_X)\|^2 
+ \|\sigma_0^*\sigma_0^*(r,z,L_X)\| \right) \hat{\mu}_t(dz)dr$$

$$< \infty.$$

17
In the following, we justify that \( \hat{\mathcal{L}}_{\hat{\mu}_t} \) satisfies Eq. (26). Eq. (13) in Definition 4.1 yields that \( \hat{\mu}_t \) satisfies the following equation

\[
< \hat{\mu}_t, \varphi_u > = < \hat{\mu}_0, \varphi_u > + \int_0^t < \hat{\mu}_s, (\mathcal{L}_s \varphi_u)(\cdot, \mathcal{L}_X) > ds + \int_0^t < \hat{\mu}_s, \partial_x \varphi_u \sigma_1^d(s, \cdot, \mathcal{L}_X) > d\hat{V}_s^t
\]

and furthermore

\[
\mathcal{L}_t \hat{\mu}_t = G(\hat{\mu}_t) = G(\hat{\mu}_0) + \int_0^t \partial_y g(< \hat{\mu}_s, \varphi >) < \hat{\mu}_s, (\mathcal{L}_s \varphi_u)(\cdot, \mathcal{L}_X) > ds
\]

And then we arbitrarily choose \( G(\nu) = g(< \nu, \varphi_1 >, \cdots, < \nu, \varphi_k >) = g(< \nu, \varphi >) \in \mathcal{V} \), and apply the Itô formula to the process \( \hat{\mu}_t \) to have that

\[
\mathbb{E}^\hat{\nu} G(\hat{\mu}_t) = \mathbb{E}^\hat{\nu} G(\hat{\mu}_0) + \int_0^t \mathbb{E}^\hat{\nu} \partial_y g(< \hat{\mu}_s, \varphi >) < \hat{\mu}_s, (\mathcal{L}_s \varphi_u)(\cdot, \mathcal{L}_X) > ds
\]

and furthermore

\[
\int_{\mathcal{M}(\mathbb{R}^n)} G(\nu) \mathcal{L}_{\hat{\mu}_t}(d\nu) = \int_{\mathcal{M}(\mathbb{R}^n)} G(\nu) \mathcal{L}_{\hat{\mu}_0}(d\nu) + \int_0^t \int_{\mathcal{M}(\mathbb{R}^n)} L_\nu G(\nu) \mathcal{L}_{\hat{\mu}_t}(d\nu) dr.
\]

The proof is complete.

Combining Remark 4.2 and Theorem 5.2 (i), we have the following conclusion.

**Corollary 5.3.** Under the assumptions \( (H^1_{\nu_1, \varphi_1, \sigma_1})(H^2_{\nu_1, \varphi_1, \sigma_1})(H^2_{\nu_2, \varphi_2}) \), Eq. (27) has a weak solution.
and \( \Psi \circ \mathcal{T} \in \mathbb{Y} \). By the assumption, we know that for \( \Psi \circ \mathcal{T} \in \mathbb{Y} \)
\[
\int_{\mathcal{M}(\mathbb{R}^n)} (\Psi \circ \mathcal{T})(\nu) \Xi_t(d\mu) = \int_{\mathcal{M}(\mathbb{R}^n)} (\Psi \circ \mathcal{T})(\nu) \Xi_0(d\nu) + \int_0^t \int_{\mathcal{M}(\mathbb{R}^n)} \mathbf{L}_r(\Psi \circ \mathcal{T})(\nu) \Xi_r(d\nu)dr,
\]

namely,
\[
\int_{\mathbb{R}^n} \Psi(x)Q_t(dx) = \int_{\mathbb{R}^n} \Psi(x)Q_0(dx) + \int_0^t \int_{\mathbb{R}^n} \sum_{u=1}^\infty \partial_{x_u} \Psi(x) < T^{-1}(x), (\mathcal{L}_r \varphi_u)(\cdot, \mathcal{L}_X) > Q_r(dx)dr
\]
\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} \sum_{u=1}^\infty \sum_{v=1}^\infty \partial_{x_u x_v} \Psi(x) < T^{-1}(x), \varphi_u h^l(r, \cdot, \mathcal{L}_X) + \partial_{x_l} \varphi_u \sigma^u_1(r, \cdot, \mathcal{L}_X) > \times < T^{-1}(x), \varphi_v h^l(r, \cdot, \mathcal{L}_X) + \partial_{x_l} \varphi_v \sigma^u_1(r, \cdot, \mathcal{L}_X) > Q_r(dx)dr,
\]

where \( Q_t := \Xi_t \circ \mathcal{T}^{-1} \).

Next, set for \( u, v \in \mathbb{N} \)
\[
\beta^u(r, x) := < T^{-1}(x), (\mathcal{L}_r \varphi_u)(\cdot, \mathcal{L}_X) >, \\
\alpha^{uv}(r, x) := < T^{-1}(x), \varphi_u h^l(r, \cdot, \mathcal{L}_X) + \partial_{x_l} \varphi_u \sigma^u_1(r, \cdot, \mathcal{L}_X) > \times < T^{-1}(x), \varphi_v h^l(r, \cdot, \mathcal{L}_X) + \partial_{x_l} \varphi_v \sigma^u_1(r, \cdot, \mathcal{L}_X) >,
\]

and then \( \beta : [0, T] \times \mathbb{R}^\infty \mapsto \mathbb{R}^\infty \) and \( \alpha : [0, T] \times \mathbb{R}^\infty \mapsto \mathbb{R}^\infty \times \mathbb{R}^\infty \) are Borel measurable and
\[
\int_0^T \int_{\mathbb{R}^\infty} |\beta^u(r, x)|Q_r(dx)dr < \infty, \quad \int_0^T \int_{\mathbb{R}^\infty} |\alpha^{uv}(r, x)|Q_r(dx)dr < \infty,
\]

where the condition (25) is used. Thus we rewrite Eq. (27) as
\[
\int_{\mathbb{R}^n} \Psi(x)Q_t(dx) = \int_{\mathbb{R}^n} \Psi(x)Q_0(dx) + \int_0^t \int_{\mathbb{R}^n} \sum_{u=1}^\infty \partial_{x_u} \Psi(x) \beta^u(r, x)dr
\]
\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} \sum_{u=1}^\infty \sum_{v=1}^\infty \partial_{x_u x_v} \Psi(x) \alpha^{uv}(r, x)dr
\]
\[
=: \int_{\mathbb{R}^n} \Psi(x)Q_0(dx) + \int_0^t \mathcal{L}(\alpha, \beta) \Psi(x)dr.
\]

The above deduction and [11] Theorem 3.3 admit us to obtain that there exists a solution \( \zeta \) to the martingale problem associated with \( \mathcal{L}(\alpha, \beta) \) with the initial law \( Q_0 \) at time 0 such that \( \zeta \circ e_t^{-1} = Q_t \) for any \( t \in [0, T] \), where
\[
e_t : C([0, T], \mathbb{R}^\infty) \mapsto \mathbb{R}^\infty, \quad e_t(w) = w_t, \quad w \in C([0, T], \mathbb{R}^\infty).
\]

By the similar deduction to that in [7] Proposition 4.6], it holds that there is an \( m \)-dimensional Brownian motion \( \tilde{W} \) defined on an extension \( (\bar{\Omega}, \bar{\mathcal{F}}, \{ \bar{\mathcal{F}}_t \}_{t \in [0, T]}, \bar{\mathbb{P}}) \) of \( (C([0, T], \mathbb{R}^\infty), \mathcal{B}, \{ \mathcal{B}_t \}_{t \in [0, T]}, \mathcal{L}, \zeta) \), where \( \mathcal{B}_t := \sigma\{ w_s : s \in [0, t] \}, \mathcal{B}_t := \cap_{s \geq t} \mathcal{B}_s, \) and \( \mathcal{B} := \mathcal{B}_T, \) such that
\[
\{ (\bar{\Omega}, \bar{\mathcal{F}}, \{ \bar{\mathcal{F}}_t \}_{t \in [0, T]}, \bar{\mathbb{P}}, (\xi_t = w_t, \mathcal{V}_t) \}
\]

is a weak solution of the following stochastic differential equation on \( \mathbb{R}^\infty \) for \( u \in \mathbb{N} \)
\[
d\xi^u_t = < T^{-1}(\xi_t), (\mathcal{L}_t \varphi_u)(\cdot, \mathcal{L}_X) > dt + < T^{-1}(\xi_t), \varphi_u h^l(t, \cdot, \mathcal{L}_X) > d\tilde{W}^l_t
\]
\[< \hat{\mu}_t, \varphi_u > = < \hat{\mu}_0, \varphi_u > + \int_0^t < \hat{\mu}_s, (\mathcal{L}_s \varphi_u)(\cdot, \mathcal{L}_X) > ds \]

\[+ \int_0^t < \hat{\mu}_s, \varphi_u h^l(s, \cdot, \mathcal{L}_X) + \partial_{x_i} \varphi_u \sigma^i(t, \cdot, \mathcal{L}_X) > d\hat{V}_s. \quad (29)\]

Moreover, \( \mathcal{L}_{\hat{\mu}_t} = \Xi_t \). Thus, the remainder is to prove that \((\hat{\mu}_t)\) satisfies Eq. (13).

Now, we specialize the sequence \( \{ \varphi_u, u \in \mathbb{N} \} \) as the dense subset of \( C^\infty_c(\mathbb{R}^n) \) (See [5, Lemma 6.1]). And then for any \( \varphi \in C^\infty_c(\mathbb{R}^n) \), there exists a subsequence \( \{ \varphi_{u_k}, k \in \mathbb{N} \} \) such that \( \| \varphi_{u_k} - \varphi \|_{C^2_c(\mathbb{R}^n)} \to 0 \), where

\[\| \varphi \|_{C^2_c(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |\varphi(x)| + \sup_{x \in \mathbb{R}^n} |\partial_{x_i} \varphi(x)| + \sup_{i, j} |\partial_{x_i x_j} \varphi(x)|.\]

Accordingly, \( < \hat{\mu}_t, \varphi_{u_k} > \to < \hat{\mu}_t, \varphi > \) as \( k \to \infty \). Also note that

\[\mathbb{E}^\hat{\mu} \left| \int_0^t < \hat{\mu}_s, (\mathcal{L}_s \varphi_{u_k})(\cdot, \mathcal{L}_X) > ds - \int_0^t < \hat{\mu}_s, (\mathcal{L}_s \varphi)(\cdot, \mathcal{L}_X) > ds \right| \]

\[\leq \mathbb{E}^\hat{\mu} \int_0^t |< \hat{\mu}_s, (\mathcal{L}_s (\varphi_{u_k} - \varphi))(\cdot, \mathcal{L}_X) > | ds \leq \mathbb{E}^\hat{\mu} \int_0^t \int_{\mathbb{R}^n} |(\mathcal{L}_s (\varphi_{u_k} - \varphi))(x, \mathcal{L}_X)\| \hat{\mu}_s(dx) ds \]

\[\leq C \| \varphi_{u_k} - \varphi \|_{C^2_c(\mathbb{R}^n)} \mathbb{E}^\hat{\mu} \int_0^T \int_{\mathbb{R}^n} \left( |b_1(s, x, \mathcal{L}_X)| + |\sigma_0 \sigma_0^* (s, x, \mathcal{L}_X)| \right) \hat{\mu}_s(dx) ds \]

\[+ \| \sigma_1 (s, x, \mathcal{L}_X) \|^2 \hat{\mu}_s(dx) ds \to 0, \quad k \to \infty, \]

and

\[\mathbb{E}^\hat{\mu} \left| \int_0^t < \hat{\mu}_s, \varphi_{u_k} h^l(s, \cdot, \mathcal{L}_X) + \partial_{x_i} \varphi_{u_k} \sigma^i(t, \cdot, \mathcal{L}_X) > d\hat{V}_s \right|^2 \]

\[= \sum_{l=1}^m \mathbb{E}^\hat{\mu} \int_0^t \int_{\mathbb{R}^n} \left( |(\varphi_{u_k} - \varphi) h^l(s, x, \mathcal{L}_X)|^2 + |\partial_{x_i} (\varphi_{u_k} - \varphi) \sigma^i(t, s, x, \mathcal{L}_X)|^2 \right) \hat{\mu}_s(dx) ds \]

\[\leq C \| \varphi_{u_k} - \varphi \|^2_{C^2_c(\mathbb{R}^n)} \mathbb{E}^\hat{\mu} \int_0^T \int_{\mathbb{R}^n} \left( |h(s, x, \mathcal{L}_X)|^2 + |\sigma_1 (s, x, \mathcal{L}_X)|^2 \right) \hat{\mu}_s(dx) ds \]

\[\to 0, \quad k \to \infty.\]
Thus, replacing \( \varphi_u \) in (29) by \( \varphi_{uk} \) and then taking the limit on two sides of (29), we get (13). The proof is complete.

**The proof of Theorem 5.2 (iii) (iv).**

By means of Theorem 5.2 (i) (ii) and Definition 4.5 and 5.1, the proofs of Theorem 5.2 (iii) (iv) are straight. Therefore, we omit them.

By Corollary 4.10 and Theorem 5.2 (iii), we draw the following conclusion.

**Corollary 5.4.** Under \( (H_{b_1,\sigma_0,\sigma_1}) (H_{b_2,\sigma_0,\sigma_1}) (H_{b_2}) (H_{b_2}) \), Eq. (24) has a unique weak solution.

6. Conclusion

In the paper, we consider the Zakai equations from nonlinear filtering problems of McKean-Vlasov stochastic differential equations with correlated noises. First, we establish the Kushner-Stratonovich equations, the Zakai equations and the distribution-dependent Zakai equations. And then, the pathwise uniqueness, uniqueness in joint law and uniqueness in law of weak solutions for the distribution-dependent Zakai equations are shown. Finally, we prove a superposition principle between the distribution-dependent Zakai equations and distribution-dependent Fokker-Planck equations. As a by-product, we give some conditions under which distribution-dependent Fokker-Planck equations have unique weak solutions.

Our methods also can be used to solve nonlinear filtering problems of McKean-Vlasov stochastic differential equations with correlated sensor noises. Concretely speaking, we make the following hypotheses:

(i) \( \tilde{b}_1, \tilde{\sigma}_1 \) satisfy \( (H_{b_1,\sigma_0,\sigma_1}) (H_{b_1,\sigma_0,\sigma_1}) \), where \( \tilde{b}_1, \tilde{\sigma}_1 \) replace \( b_1, \sigma_1 \);
(ii) \( \tilde{b}_2(t, x, \mu, y) \) is bounded for all \( t \in [0, T], x \in \mathbb{R}^n, \mu \in P_2(\mathbb{R}^n), y \in \mathbb{R}^m \);
(iii) \( \tilde{\sigma}_2 \tilde{\sigma}_2^* + \tilde{\sigma}_3 \tilde{\sigma}_3^* = I_m \), where \( \tilde{\sigma}_2 \) stands for the transpose of the matrix \( \tilde{\sigma}_2 \) and \( I_m \) is the \( m \)-order unit matrix.

Under the above assumptions, it holds that the system (2) has a unique strong solution denoted as \( (\tilde{X}_t, \tilde{Y}_t) \). And then set

\[
U_t := \tilde{\sigma}_2 V_t + \tilde{\sigma}_3 W_t,
\]

\[
\tilde{\Gamma}_t^{-1} := \exp \left\{ -\int_0^t \tilde{b}_2(s, \tilde{X}_s, \mathcal{L}_{\tilde{X}_s} \tilde{Y}_s) dU_s^i - \frac{1}{2} \int_0^t |\tilde{b}_2(s, \tilde{X}_s, \mathcal{L}_{\tilde{X}_s} \tilde{Y}_s)|^2 ds \right\},
\]

and then \( \tilde{\Gamma}_t^{-1} \) is an exponential martingale. Moreover, define the probability measure

\[
\frac{d\hat{P}}{dP} := \tilde{\Gamma}_T^{-1},
\]

and set

\[
< \tilde{\Lambda}_t, F > := \mathbb{E}^{\hat{P}}[F(\tilde{X}_t, \mathcal{L}_{\tilde{X}_t})|\mathcal{F}_t],
\]

where \( \mathbb{E}^{\hat{P}} \) stands for the expectation under the probability measure \( \hat{P} \). By the same deduction to that in Theorem 3.3, we obtain the following Zakai equation.

**Corollary 6.1.** *(The Zakai equation)*

The Zakai equation of the system (2) is given by

\[
< \tilde{\Lambda}_t, F > = < \tilde{\Lambda}_0, F > + \int_0^t < \tilde{\Lambda}_s, \mathcal{L}_s F > ds + \int_0^t < \tilde{\Lambda}_s, F \tilde{b}_2(s, \cdot, \cdot, \tilde{Y}_s) > d\tilde{U}_s^j
\]
and by the above corollary, we can derive the following distribution-dependent Zakai equation:

\[
\begin{align*}
\dot{\tilde{P}}_t &= \mathbb{E}^\tilde{P}_t[\varphi(X_t)\tilde{\Gamma}_t], \quad \varphi \in \mathcal{B}_b(\mathbb{R}^n), \\
\end{align*}
\]

and by the above corollary, we can derive the following distribution-dependent Zakai equation:

\[
\begin{align*}
< \tilde{P}_t, \varphi > &= < \tilde{P}_0, \varphi > + \int_0^t < \tilde{P}_s, (\tilde{L}_s\varphi)(\cdot, \mathcal{L}_{X_s}) > ds + \int_0^t < \tilde{P}_s, \varphi \tilde{b}_2(s, \cdot, \mathcal{L}_{X_s}, \tilde{Y}_s) > d\tilde{U}_s^j \\
&\quad + \int_0^t < \tilde{P}_s, \partial_x \varphi \tilde{\sigma}_2^{ijk}(s, \cdot, \mathcal{L}_{X_s}) > d\tilde{U}_s^j, \quad \varphi \in C^\infty_c(\mathbb{R}^n), \ t \in [0, T],
\end{align*}
\]

where

\[
(\tilde{L}_s\varphi)(x, \mu) := \partial_x \varphi(x) \tilde{b}_1^i(s, x, \mu) + \frac{1}{2} \partial^2_{xx} \varphi(x) (\tilde{\sigma}_1 \tilde{\sigma}_1^*)^{ij}(s, x, \mu).
\]

Of course, we can study the pathwise uniqueness, uniqueness in joint law, uniqueness in law and the superposition principle of weak solutions for Eq. (31) by the same means to that in Theorem 4.6, 4.9, Corollary 4.10 and Theorem 5.2.

Acknowledgements:

The second named author would like to thank Professor Xicheng Zhang for valuable discussions and also thanks Professor Renming Song for providing her an excellent environment to work in the University of Illinois at Urbana-Champaign.

References

[1] A. Bain and D. Crisan, Fundamentals of Stochastic Filtering. Springer, New York, 2009.
[2] P. Cardaliaguet: Notes on mean field games (from P.L. Lion’s lectures at Collège de France). https://www.ceremade.dauphine.fr/cardali/MFG100629.pdf, 2013.
[3] X. Ding, H. Qiao: Euler-Maruyama approximations for stochastic McKean-Vlasov equations with non-Lipschitz coefficients, https://arxiv.org/abs/1903.11754.
[4] X. Ding, H. Qiao: Stability for stochastic McKean-Vlasov equations with non-Lipschitz coefficients, https://arxiv.org/abs/1905.07883.
[5] D. Lacker, M. Shkolnikov and J. Zhang: Superposition and mimicking theorems for conditional McKean-Vlasov equations, http://arxiv.org/abs/2004.00099.
[6] J. Jacod and A.N. Shiryaev: Limit Theorems for Stochastic Processes. Springer-Verlag, Berlin, 1987.
[7] I. Karatzas and S. E. Shreve: Brownian motion and stochastic calculus. 2nd ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 2005: 284-295.
[8] T. G. Kurtz and D. L. Ocone: Unique characterization of conditional distributions in nonlinear filtering, *Ann. Prob.*, 16(1988)80-107.

[9] T. G. Kurtz and J. Xiong: Particle representations for a class of nonlinear SPDEs, *Stochastic Process. Appl.* 83(1999)103-126.

[10] H. Qiao: Nonlinear filtering of stochastic differential equations driven by correlated Lévy noises, [https://arxiv.org/abs/1907.06779v3](https://arxiv.org/abs/1907.06779v3).

[11] H. Qiao: Superposition principles for the Zakai equations and the Fokker-Planck equations on measure spaces, [http://arxiv.org/abs/2007.02079](http://arxiv.org/abs/2007.02079).

[12] H. Qiao: Uniqueness for measure-valued equations of nonlinear filtering for stochastic dynamical systems with Lévy noises, *Advances in Applied Probability*, 50(2018)396-413.

[13] H. Qiao and J. Duan: Nonlinear filtering of stochastic dynamical systems with Lévy noises, *Advances in Applied Probability*, 47(2015)902-918.

[14] P. Ren and F.-Y. Wang: Space-Distribution PDEs for path independent additive functionals of McKean-Vlasov SDEs, to appear in *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*

[15] L. C. G. Rogers and D. Williams: *Diffusions, Markov processes and Martingales, vol 2: Itô calculus*, 2nd Ed. Cambridge University Press, 2000.

[16] B. L. Rozovskii: *Stochastic Evolution Systems, Linear Theory and Applications to Nonlinear Filtering*, Math. Appl. (Sov. Ser.), vol. 35, Kluwer Academic Publishers, 1990.

[17] N. Sen and P. E. Caines: Nonlinear filtering theory for McKean-Vlasov type stochastic differential equations, *SIAM J. CONTROL OPTIM.*, 54(2016)153-174.

[18] N. Sen and P. E. Caines: Nonlinear filtering for McKean-Vlasov type SDEs with application to mean field games, in *Proceedings of the 21st International symposium on Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, 2014, pp. 1344-1351.