CHARACTERISTIC CLASSES OF GAUGE SYSTEMS

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Abstract. We define and study invariants which can be uniformly constructed for any gauge system. By a
gauge system we understand an (anti-)Poisson supermanifold endowed with an odd Hamiltonian self-commuting
vector field called a homological vector field. This definition encompasses all the cases usually included into
the notion of a gauge theory in physics as well as some other similar (but different) structures like Lie or Courant
algebroids. For Lagrangian gauge theories or Hamiltonian first class constrained systems, the homological vector
field is identified with the classical BRST transformation operator. We define characteristic classes of a gauge
system as universal cohomology classes of the homological vector field, which are uniformly constructed in
terms of this vector field itself. Not striving to exhaustively classify all the characteristic classes in this work,
we compute those invariants which are built up in terms of the first derivatives of the homological vector field.
We also consider the cohomological operations in the space of all the characteristic classes. In particular, we
show that the (anti-)Poisson bracket becomes trivial when applied to the space of all the characteristic classes,
instead the latter space can be endowed with another Lie bracket operation. Making use of this Lie bracket
one can generate new characteristic classes involving higher derivatives of the homological vector field. The
simplest characteristic classes are illustrated by the examples relating them to anomalies in the traditional BV
or BFV-BRST theory and to characteristic classes of (singular) foliations.

1. Introduction

Most of the fundamental physical models, especially those from the field theory, are the gauge systems in the
traditional sense, i.e. the models whose action functionals are invariant under the local (gauge) transformations
of the configuration space. At the level of local geometry, the conventional gauge theories were well studied
in seventies and eighties in the framework of the BRST-BV-BFV approach [1, 2, 3, 4, 5, 6] which had been initially
designed as a universal method for quantizing these theories. With time this method has evolved
into the theory having much broader range of physical and mathematical applications including computation
of anomalies, constructing consistent interactions of gauge fields, quantizing wide classes of Poisson manifolds
related with Lie algebroids, etc.

The BRST method works with a gauge system by constructing its special super-extension in such a way as
to convert the gauge invariance into a special global symmetry generated by an odd vector field \( Q \) (a generator
of the BRST transformation). Being one-dimensional and odd, the corresponding Lie algebra is necessarily
Abelian,

\[
[Q, Q] = 0 .
\]

The physical observables of a conventional gauge theory are then identified with certain cohomology classes
of the BRST differential \( Q \), signifying the fundamental role this operator plays in physics. In mathematics,
a growing trend is observed to study the Lie algebroids and the Courant algebroids through their embedding
into an extended target superspace equipped with the nilpotent odd vector field, called a homological vector
field \[\mathfrak{g}\] \[\mathfrak{n}\] \[\mathfrak{h}\]. Though the homological description of the algebroids is somewhat different from the BRST
theory of traditional gauge systems, both the types of gauge theories can be put into the uniform homological
scheme. In this paper we hold a wider understanding of the gauge system (the accurate definition is given in
the next section) encompassing both traditional models of field theory and a range of other problems. All the
cases covered have in common that the original manifold of the theory can be embedded into a supermanifold
endowed with a nilpotent odd vector field \( Q \) which “absorbs” the algebraic structure behind the original system.
In doing so, the distinctions between different types of original gauge systems are encoded in the ghost number
grading prescription for the extended supermanifold.

The global properties of the gauge systems remain yet much less studied even though the BRST methods offer
promise as a processing tool for that. One of the obstacles to studying the global geometry of gauge systems
by the BRST tools is that the BRST embedding procedure, as it was originally proposed, is not explicitly
covariant with respect to natural automorphisms of the gauge algebra: the structure functions entering the
BRST generator are neither tensors nor connections, that makes the covariance of the BRST embedding not so evident. This difficulty is not crucial, however, for the BRST theory and, as we show below, it can be easily overcome by introducing an appropriate connection on the bundle of the ghost variables.

Once the covariant formulation is established, various questions can be addressed concerning the global geometry of the gauge systems, e.g.: Given the gauge system, to which simplest form it can be brought to by a suitable automorphism? Given two gauge systems, defined on the same supermanifold, are they equivalent to each other? In the local setting, the exhaustive answers to these questions are provided by the Abelization theorem [10,11] which can be regarded as a generalization of the Darboux theorem to the case of the (anti-)symplectic supermanifolds endowed with a regular Poisson nilpotent superfunction (BRST charge or master action, depending on the parity of the Poisson bracket). In general, we cannot expect to find a globally defined Abelian basis for the gauge algebra generators as the existence of this basis imposes very strong restrictions on the global geometry. In the finite dimensional case, for instance, the gauge orbits should cover a torus to make the system globally abelianizable. In the field theory, dealing with infinite-dimensional manifolds, the very definition of a global equivalence represents a problem, not to mention the in-depth study of the topology of the infinite-dimensional gauge orbits. This suggests to look for other topological invariants which could be both easily computable and sufficiently informative to perceive locally invisible differences between gauge systems.

In this paper, we begin studying the characteristic classes which reflect the global properties of the gauge systems. The characteristic classes are defined by the cocycles of the homological vector field which are explicitly built up in terms of this operator itself. In the particular case of Lie algebroids some of our characteristic classes seem to coincide with the characteristic classes introduced by Fernandes [12], and, in the more special situation of an abstract Lie algebra, they are reduced to the primitive generators of the Lie algebra cohomology. For other types of gauge systems falling into this uniform scheme, no general expressions were known before for the characteristic classes, as far as we know.

The paper is organized as follows. In Sec. 2 we formulate the general notion of the gauge system and consider some important examples, demonstrating how different types of physical and geometrical models can be brought into the uniform framework of the gauge system theory. Some of the considered examples contain new results, e.g., we propose a new Hamiltonian gauge embedding for the Lie algebroids, for the BV and BFV formalisms we give slightly modified, explicitly covariant formulations introducing a connection in the corresponding ghost bundles. In Sec. 3 we discuss the cohomology of gauge systems with tensor coefficients. It is argued that the tensor cohomology can be reduced, in a sense, to the scalar one by an appropriate extension of the original gauge system. The opposite operation, i.e. the reduction on an invariant submanifold, gives rise to additional cohomological invariants which can be related with a gauge system. The notions of the universal cocycles and characteristic classes are introduced in Sec. 4. In this section we also explicitly construct a simplest series of characteristic classes. Sec. 5 is devoted to natural cohomological operations which can be introduced on the space of all characteristic classes. We show that the characteristic classes are Poisson commuting, so the Poisson bracket induces a trivial cohomological operation among the topological invariants; instead, the linear space of all characteristic classes carries a special Lie bracket operation, which can be used, for example, to generate new characteristic classes from the already known ones. Sec. 6 is devoted to the interpretation of simplest characteristic classes as one-loop quantum anomalies in the BV and BRST-BFV formalism as well as to application of these classes to the theory of (singular) foliations. In concluding section we briefly summarize the results and discuss some prospects of further studies. Appendix collects basic formulas and conventions on the differential geometry of supermanifolds used throughout the text.

2. Gauge systems: definition and examples.

In this paper we work in the category of the supermanifolds endowed with an even or odd Poisson structure\(^1\). Recall that the Poisson structure on a supermanifold \(M\) is a bilinear map \(\{\cdot,\cdot\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)\), called bracket, satisfying the following axioms:

\[
\begin{align*}
\epsilon(\{f, g\}) &= \epsilon(f) + \epsilon(g) + \epsilon \\
\{f, g\} &= -\{g, f\} \quad (mod \; 2), \\
\{f, gh\} &= \{f, g\} h + \{f, h\} g - \{g, h\} f - \{f, g\} \quad (symmetry), \\
\{f, gh\} &= \{f, g\} h + \{f, h\} g - \{g, h\} f - \{f, g\} \quad (Leibnitz rule), \\
\{f, \{g, h\}\} &= (-1)^{(\epsilon(f)+\epsilon)(\epsilon(h)+\epsilon)} + \text{cycle}(f, g, h) = 0 \quad (Jacobi identity),
\end{align*}
\]

where \(\epsilon = \epsilon(\{\cdot, \cdot\})\) is the Grassman parity of the bracket (\(\epsilon = 0\) for the even Poisson structure, and \(\epsilon = 1\) for the odd one). The pair \((M, \{\cdot, \cdot\})\) is called a Poisson manifold.

\(^1\)In the physical literature the odd Poisson bracket is usually called anti-bracket.
Hereinafter we omit the prefix “super” whenever possible, e.g. the terms like manifold, vector bundle, matrix, function, etc. will actually mean the corresponding notions of super-mathematics.

Now, let us specify what we understand by the gauge system.

**Definition 2.1.** A gauge system is a triple \((M, \{ \cdot, \cdot \}, Q)\) constituted by a smooth manifold \(M\), endowed with a Poisson bracket \(\{ \cdot, \cdot \}\) and a function \(Q \in C^\infty(M)\), such that
\[
\epsilon(Q) = \epsilon(\{ \cdot, \cdot \}) + 1, \quad \{ Q, Q \} = 0.
\]
As the parity of \(Q\) is opposite to the parity of the bracket, the last relation does not hold identically, it is a nontrivial condition imposed on the function \(Q\). From \((2.2)\) also follows that the Hamiltonian vector field \(Q = \{ Q, \cdot \}\) is always odd and integrable
\[
\epsilon(Q) = 1, \quad [Q, Q] = 2Q^2 = 0.
\]

In this paper we adopt the following terminology. Any odd and nilpotent vector field is called a homological vector field. In the case of a Hamiltonian homological vector field \(Q = \{ Q, \cdot \}\), the Hamiltonian \(Q\) is referred to as a homological potential. The condition \(Q\) on the homological potential is called a master equation (odd or even, depending on the parity of the Poisson bracket involved).

The morphisms can be obviously defined in the category of gauge systems: Given two gauge systems \(G = (M, \{ \cdot, \cdot \}, Q)\) and \(G' = (M', \{ \cdot, \cdot \}', Q')\), a smooth map \(\varphi : M \rightarrow M'\) defines a morphism \(G \rightarrow G'\) if
\[
\varphi^*\{ (f, g) \} = \{ \varphi^*(f), \varphi^*(g) \}, \quad \varphi^*(Q') = Q', \quad \forall f, g \in C^\infty(M').
\]
In other words, the morphisms are just Poisson maps relating homological potentials. When \(\varphi\) is a diffeomorphism, the gauge systems \(G\) and \(G'\) are said to be equivalent.

One can also pick out several suitable subcategories in the category of all gauge systems. Given a homological vector field \(Q = \{ Q, \cdot \}\) and an affine connection \(\nabla\), introduce the \((1, 1)\)-tensor field
\[
\Lambda = (\nabla_i Q^j)\cdot
\]
If \(p \in M\) is a stationary point of \(Q\), i.e. \(Q^p(p) = 0\) then, treating \(\Lambda(p) = \Lambda_p\) as the matrix of an odd linear operator \(\Lambda_p : T_p^o M \rightarrow T_p^o M\), one can see that
\[
\Lambda_p^2 = 0.
\]
As \(\Lambda_p\) is a nilpotent operator, one can define the corresponding cohomology group
\[
H_p = \ker \Lambda_p / \text{im} \Lambda_p.
\]
The set of all stationary points of the homological vector field \(Q\) is called a stationary shell and denoted \(\Sigma = \{ p \in M \mid Q^p(p) = 0 \}\).

**Definition 2.2.** A gauge system \((M, \{ \cdot, \cdot \}, Q)\) is said to be regular if the cohomology group \(H_p\) has the same dimension at any stationary point:
\[
\dim H_p = \text{const.}, \quad \forall p \in \Sigma.
\]

**Remark 2.3.** The regularity condition ensures \(\Sigma \subset M\) to be a smooth submanifold. The usual notion of a corank of the Jacobi matrix associated to the system of equations \(Q^p(p) = 0\), is not directly applicable to the odd matrix \(\Lambda_p\) and the dimension of the cohomology group \(H_p\) substitutes this notion, in a sense.

The local structure of a regular gauge system was established by different methods in Refs. 3 10 11 13 14 15. (In fact, these works addressed either BV theory or the BRST-BFV one, although their results hold true for other types of the gauge systems as well). Given a regular gauge system \((M, \{ \cdot, \cdot \}, Q)\), then a coordinate system \((x^i, \xi^\alpha, \eta^a)\) exists in a neighbourhood of every point \(p \in \Sigma\), such that
\[
\epsilon(\xi^\alpha) = \epsilon(x^i) + 1, \quad \alpha = 1, ..., \dim H_p,
\]
and the homological vector field has the form
\[
Q = \xi^\alpha \frac{\partial}{\partial x^\alpha}.
\]
In these adapted coordinates the stationary shell \(\Sigma\) is determined by the equations \(\xi^\alpha = 0\). The geometry of the stationary shell has been further studied in 16 17 using both the BRST-BFV and the BV formalism.

Another important subcategory of gauge systems is related to the notion of \(Z\)-grading, traditionally called in physics the ghost-number grading. In mathematics, \(Z\)-graded supermanifolds are characterized by introducing an additional integer grading in the structure sheaf 3 12. Simply stated, a \(Z\)-graded manifold \(M\) is a supermanifold possessing an atlas of affine charts in which each local coordinate is assigned an integer number called weight and the admissible coordinate transformations respect the total weight. It is appropriate mention that the supplementary \(Z\)-grading is in no way related to the underlying \(Z_2\)-grading (the Grassman parity) of
a supermanifold. In what follows we prefer to use the physical terminology adopted in the BRST theory and refer to \( \mathbb{Z} \)-grading as the ghost grading; accordingly, the weights of local coordinates \( x^i \), denoted by \( \text{gh}(x^i) \), will be called the ghost numbers.

Notice that on the graded manifold \( M \) all geometric objects (tensor fields, connections, etc.) carry ghost numbers, which can be conveniently described by means of the Euler vector field

\[
\hat{G} = \sum_i \text{gh}(x^i) x^i \frac{\partial}{\partial x^i}, \quad \text{gh}(x^i) \in \mathbb{Z}.
\]

The ghost numbers are just the eigenvalues of the operator of the Lie derivative along \( \hat{G} \): the tensor field \( S \) on \( M \) is said to have a ghost number \( n \in \mathbb{Z} \) if \( \mathcal{L}_{\hat{G}} S = nS \). For coordinate functions on a local chart this gives the definition of \( \text{gh}(x^i) \).

The stationary points of the Euler vector field \( \hat{G} \) form a smooth submanifold \( N \subset M \), which we will call the body of \( M \). According to this definition the local coordinates on \( N \) carry zero ghost number. In physics, it is the body \( N \) which is usually considered as “an original” space which is then “BRST-embedded” into the super-extension \( M \).

In the graded category, the homological vector field \( Q \) is usually required to carry the ghost number \( +1 \), i.e.

\[
[\hat{G}, Q] = Q,
\]

and this results in the following correlation between ghost numbers of the homological potential and the bracket:

\[
\text{gh}(Q) = 1 - \text{gh}(\{\cdot, \cdot\}).
\]

It is the Lie superalgebra \( \mathfrak{B} \), sp\( \mathfrak{G} \)\( \mathfrak{R} \), spanned by two generators \( Q \) and \( G \), which is known as the BRST-algebra.

The cohomology group \( \mathcal{H} \) associated to the stationary shell \( \Sigma \subset M \) inherits the ghost grading from \( M \): If the spectrum of ghost numbers ranges form \( -n \) to \( m \), then

\[
H^p = \bigoplus_{k=-n}^{m} H^k_p.
\]

Besides the notion of regularity, an important notion of the properness of a gauge system can be formulated in this case.

**Definition 2.4.** A regular gauge system is said to be a proper if

\[
H^k_p = 0, \quad \forall k < 0.
\]

For instance, the properness condition \( 2.13 \) is automatically satisfied if there are no coordinates with negative ghost numbers on \( M \). In the conventional BRST theory it is the properness condition which provides the unique existence of the solution to the master equation with a given boundary condition.

**2.1. Examples.** Here we exemplify the general definitions by reviewing some known constructions which have been intensively studied both in physics and mathematics, and which all fit into the concept of the gauge system explained above. The basic characteristics of a gauge system are the spectrum of ghost numbers, the parity and ghost number of the homological potential \( Q \) (that automatically defines the ghost number and parity of the bracket). Except of the case of Hamiltonian first-class systems, a \( \mathbb{Z} \)-graded manifold \( M \) underlying a gauge system involves the cotangent bundle \( T^*N \) to the body of \( M \), the fibers of which carry a non-zero ghost number\(^2\). To get a general impression how different types of gauge systems are embedded in the uniform homological framework, it is convenient to arrange the corresponding data in the table:

\(^2\)The more precise description of a gauge system includes also the specification of the ghost number and parity distribution among all the coordinates on \( M \).
1. Irreducible Lagrangian gauge system

\[ -2, -1, 0, 1 \]
\[ \epsilon(Q) = 0 \]
\[ \epsilon(T_p^* N) = -1 \]

2. \( n \)-times reducible Lagrangian gauge system

\[ -2 - n, \ldots, n + 1 \]
\[ \epsilon(Q) = 0 \]
\[ \epsilon(T_p^* N) = -1 \]

3. Hamiltonian first class constrained system

\[ -1, 0, 1 \]
\[ \epsilon = 1 \]
\[ \epsilon(T_p^* N) = -1 \]

4. \( n \)-times reducible first class system

\[ -n - 1, \ldots, n + 1 \]
\[ \epsilon = 1 \]
\[ \epsilon(T_p^* N) = -1 \]

5. Lie algebroid

\[ 0, 1, 2, 3 \]
\[ \epsilon = 4 \]
\[ \epsilon(T_p^* N) = 3 \]

6. \( n \)-times reducible Lie algebroid

\[ 0, \ldots, 2n + 1 \]
\[ \epsilon = 2n + 2 \]
\[ \epsilon(T_p^* N) = 2n + 1 \]

7. Courant algebroid

\[ 0, 1, 2 \]
\[ \epsilon = 3 \]
\[ \epsilon(T_p^* N) = 2 \]

8. Weak Poisson bracket

\[ -1, 0, 1, 2 \]
\[ \epsilon = 2 \]
\[ \epsilon(T_p^* N) = 1 \]

Examples 1, 3 are well known in physics and we present them below in a conventional way. A slightly novel point here is that we use explicitly covariant (even or odd) Poisson brackets involving the connection to describe the gauge models both in the Hamiltonian and Lagrangian formalisms. Examples 2, 4 are the extensions of the previous two ones to the case of reducible gauge algebra generators. Although the Lie algebroid may appear, for instance, as a particular solution to the BV-master equation of the Lagrangian gauge theory, the latter solution would be neither proper nor invariant w.r.t. automorphisms of the BRST. The genuine BRST-like imbedding for the Lie algebroid requires another spectrum of ghost numbers, indicated in fifth line of the table, and we elaborate on this example below. The case of \( n \)-times reducible Lie algebroid (Example 6), extends the previous one in the same sense as 2 and 4 extend 1 and 3 respectively. This will be considered elsewhere, here we just mention the grading of the appropriate manifold to embed the reducible algebroid into the master equation. Example 7 is due to Roytenberg, who has first translated the Courant algebroid into the language of the master equation. The last example concerns the embedding of a Poisson manifold into an almost-Poisson manifold so that the imbedding map to relate the corresponding bivectors.

2.1.1. BV anti-field formalism. The Lagrangian description of a field-theoretical model with an irreducible gauge symmetry is implemented in terms of a \( \mathbb{Z} \)-graded manifold \( M \), equipped with an odd symplectic structure. Topologically, \( M \) is given by a direct sum of vector bundles \( \Pi T^*[1] N \oplus \Pi E[1] \oplus E^*[2] \), where \( N \) is a configuration space (more accurately, the space of trajectories) of the original gauge invariant model and \( E \to N \) is a vector bundle whose sections are the parameters of the gauge transformations. Local coordinates on the original configuration space (fields, in physics) are denoted \( \phi_A \). Linear coordinates on the fibres of \( \Pi T^*[1] N \), \( \Pi E[1] \) and \( E^*[2] \) are, respectively, \( \phi_A^* \) (anti-fields), \( C^\alpha \) (ghosts) and \( C_\beta^* \) (anti-ghosts). By definition,

\[
\epsilon(\phi_A) = \epsilon_A + 1, \\
\epsilon(C^\alpha) = \epsilon_\alpha + 1, \\
\epsilon(C_\beta^*) = \epsilon_\beta, \\
\text{gh}(\phi_A) = 0, \\
\text{gh}(\phi_A^*) = -1, \\
\text{gh}(C^\alpha) = 1, \\
\text{gh}(C_\beta^*) = -2, \\
(\text{mod } 2),
\]

where \( \epsilon_A = \epsilon(\phi_A) \), \( \epsilon_\alpha = \epsilon(C^\alpha) \), and \( \epsilon_\beta \) are the parameters of the gauge transformations. Below, for the sake of simplicity the original configuration space \( N \) is assumed to be an ordinary (even) manifold.

\[ ^3 \text{Similar brackets for the ghosts have been found earlier by L.A.Batalin, M.A.Grigoriev and one of us (SLL) for another type of the gauge system as a part of a different work which still remains unfinished. We also truly appreciate Maxim Grigoriev’s comments on the relevance of the connection in the ghost bundle for this construction.} \]

\[ ^4 \text{Hereinafter, the numbers in square brackets point on the ghost number of the fiber coordinates.} \]
Upon choosing a linear connection \( \nabla \) on \( E \to N \) one can endow \( M \) with the structure of an odd Poisson manifold. The nonvanishing Poisson brackets among the local coordinates are

\[
\{ \phi_B^*, \phi_A^* \} = \delta_B^A, \quad \{ C_\beta^, C_\alpha^ \} = \delta_\beta^\alpha,
\]

(2.15)

\[
\{ \phi_A^, C_\alpha^ \} = \Gamma^A_{\beta\alpha}(\phi)C_\beta^, \quad \{ \phi_A^, C_\beta^ \} = -\Gamma^A_{\beta\alpha}(\phi)C_\alpha^,
\]

\[
\{ \phi_A^, \phi_B^* \} = R_{AB}^\beta(\phi)C_\beta^C_\alpha^,
\]

where \( \Gamma^\alpha_{\beta\gamma}(\phi) \) and \( R_{AB}^\gamma(\phi) \) are components of the connection \( \nabla \) and its curvature tensor, respectively. The Poisson bracket (2.16) increases the ghost number by 1,

\[
\text{gh}(\{ f, g \}) = \text{gh}(f) + \text{gh}(g) + 1, \quad \forall f, g \in C^\infty(M),
\]

and comes from the following (exact) odd symplectic structure\(^5\)

\[
\omega = d(\phi_A^*d\phi^A + C_\alpha^\nabla C^\alpha) = d\phi_A^*d\phi^A + \nabla C_\alpha^\nabla C^\alpha + \frac{1}{2}d\phi^A d\phi^B R_{AB}^\beta C_\beta^C_\alpha^,
\]

\[
\nabla C_\alpha^ = dC_\alpha^ + d\phi^A \Gamma^\alpha_{\beta\gamma} C_\gamma^B, \quad \nabla C_\beta^ = dC_\beta^ - d\phi^A \Gamma^\beta_{\alpha\gamma} C_\gamma^B.
\]

Considering \( M \) as an abstract \( \mathbb{Z} \)-graded manifold it is natural to extend the group of coordinate transformations (which are originally linear in the fiber coordinates \( \phi^*, C \) and \( C^* \)) to an arbitrary smooth coordinate changes respecting the Grassmann parity and the ghost-number grading. In this case, different choices for the connection \( \nabla \) will lead to equivalent symplectic structures on \( M \). Namely, any two symplectic structures \( \omega \) and \( \omega' \) of the form (2.16) are related to each other by the coordinate transform \( \phi_A^* \to \phi_A^* + \Delta \Gamma^A_{\beta\alpha}(\phi)C_\beta^C_\alpha^, \) where \( \Delta \Gamma = \nabla - \nabla' \) is the difference between corresponding connections. Moreover, one can prove an odd counterpart of Rothstein’s theorem [21], stating that any odd symplectic structure of the ghost number 1 can be brought into the form (2.16) by a suitable diffeomorphism of \( M \).

A solution to the master equation (2.2), denoted usually by \( S \), is called a master action. As the odd bracket carries the ghost number 1, \( \text{gh}(S) = 0 \). The corresponding homological vector field \( Q = \{ S, \cdot \} \) is known as the generator of the BRST transformation. The properness condition can be written as

\[
\text{rank}(d^2S)|_{dS=0} = \dim E = \frac{1}{2} \dim M,
\]

where \( d^2S = (\partial_\alpha \partial_\beta S) \) is the Hesse matrix and the equation \( dS(\phi) = 0 \) is supposed to have a solution.

With the account of ghost numbers and parities of the local coordinates we can write

\[
S(x) = S(\phi) + C^\alpha R_A^\alpha(\phi)\phi_A^* + \frac{1}{2}C^\alpha C^\beta T_{\beta\gamma}(\phi)C_\gamma^ + \frac{1}{4}C^\alpha C^\beta E^AB(\phi)\phi_A^\phi_B^ + O(C^3).
\]

Substituting this expansion into the odd master equation

\[
\{ S, S \} = 0,
\]

one can find all the higher order ghost and anti-field terms provided (2.19) is satisfied up to the second order in \( C \)'s. In this lower order in ghosts the equation (2.19) is reduced to the relations

\[
R_A^\alpha \partial_\alpha S = 0, \quad [R_\alpha, R_\beta] = U_{\alpha\beta}^\gamma R_\gamma + \partial_\alpha S E^A_B \partial_B ,
\]

where

\[
U_{\alpha\beta}^\gamma = T_{\alpha\beta}^\gamma - R_\alpha^A \Gamma_{\alpha\beta}^A + R_\beta^A \Gamma_{\alpha\beta}^A .
\]

The first relation means that the function \( S(\phi) \), identified with the action function of a gauge model, is invariant under the action of the (local) vector fields \( R_\alpha = R_\alpha^A \partial_A \), while the second relation implies these vector fields to form an integrable distribution on the surface \( \Sigma' \subset N : dS = 0 \). Notice that \( \Sigma' \) is the projection on \( N \) of the stationary shell \( \Sigma : dS = 0 \). In the physical literature the relations (2.20) are called a gauge algebra. Accounting the rank condition (2.17), one can see that (i) the local vector fields \( R_\alpha \) are linearly independent on \( \Sigma' \), and (ii) the corank of the Hesse matrix \( d^2S \) on \( \Sigma' \) is equal to the rank \( E \), so that \( \Sigma' \subset N \) is a smooth surface indeed.

Applying the standard technique of the cohomological perturbation theory [10] it is possible to prove the inverse: given the classical action function \( S \in C^\infty(N) \) with the Hesse matrix \( d^2S \) having a constant rank on \( \Sigma' \), there exists a unique (up to symplectomorphisms of \( M \)) proper solution to the Batalin-Vilkovisky master equation (2.19) starting with \( S \).

We see that any gauge algebra (2.20) defines a Lie algebroid \( \mathcal{E}|_{\Sigma'} \) over the stationary surface \( \Sigma' \) with the anchor \( R|_{\Sigma'} : \mathcal{E}|_{\Sigma'} \to T\Sigma' \). The action of the algebroid foliates \( \Sigma' \) onto the gauge orbits. Mention that the gauge algebra does not necessarily define a Lie algebroid structure beyond \( \Sigma' \) (and hence it defines no foliation structure in a tubular neighbourhood of \( \Sigma' \subset N \)) because of possible contributions of the trivial gauge generators

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\(^5\)We omit the sign of the wedge product treating the differentials \( dx^i \) as basis sections of \( \Pi T^* M \) (see Appendix A).
(second term in the r.h.s. of (2.20)). In physical literature, this more general structure, with an admixture of the trivial gauge generators (vanishing on the equations of motion \(d\mathcal{S} = 0\)) in the r.h.s., is called an open algebra. The BV theory encodes all the structure relations of this more complicated algebra by means of a single master equation (2.20) supplemented with the properness condition (2.17).

2.1.2. BRST-BFV Hamiltonian formalism. Consider BFV description of the Hamiltonian systems subject to irreducible first class constraints \([1], [3], [9]\). The description involves a graded manifold \(M\) associated to the total space of the direct sum \(\Pi E[1] \oplus \Pi E^*[−1]\), where \(E \to N\) is a vector bundle over a symplectic manifold \((N, ω^0)\). Let \(x^i\) denote local coordinates on the base \(N\), and let \(C^α\) and \(P_α\) denote linear coordinates on fibres of \(\Pi E\) and \(\Pi E^*\), respectively. In physical literature \(C^α\)'s and \(P^α\)'s are called the ghost coordinates and momenta (or just ghosts). According to the definition of \(M\),

\[
\text{gh}(x^i) = 0, \quad \text{gh}(C^α) = 1, \quad \text{gh}(P_α) = −1.
\]

Again, to simplify the exposition, we assume \(N\) to be an even manifold, so that \(ε(x^i) = 0, ε(C^α) = ε(P_α) = 1\). Using a linear connection \(\nabla\) on \(E \to N\) one can extend the even symplectic structure \(ω^0\) from \(N\) to \(M\) as follows:

\[
(2.21) \quad \omega = dx^i dx^j ω^0_{ij} + d(P_α \nabla C^α) = dx^i dx^j ω^0_{ij} + \nabla P_α \nabla C^α + \frac{1}{2} dx^i dx^j R_{ij}^α C^α P_β,
\]

\[
\nabla C^α = dC^α + dx^i Γ^α_β C^β, \quad \nabla P_α = dP_α - dx^i Γ^α_β P_β,
\]

where \(Γ^α_β(x), R_{ij}^α(x)\) are the components of the connection \(\nabla\) and its curvature tensor. The corresponding Poisson bracket on \(M\) reads

\[
\{x^i, x^j\} = Π^{ij}, \quad \{C^α, P_β\} = δ^α_β - Π^{ij} Γ^α_β C^μ P_ν,
\]

\[
\{x^i, C^α\} = Π^{ij} Γ^α_β C^β, \quad \{P_α, P_β\} = Π^{ij} Γ^μ_α Γ^ν_β C^μ P_ν,
\]

\[
\{C^α, C^β\} = Π^{ij} Γ^μ_α Γ^ν_β C^μ C^ν, \quad \{P_α, P_β\} = Π^{ij} Γ^μ_α Γ^ν_β C^μ C^ν.
\]

where

\[
Π^{ij} = π m=0 \infty (R^m)^j_k, \quad R = (R^j_k) = \left(\frac{1}{2} π m k R_{kj}^β C^α P_β\right),
\]

and \(π\) is the Poisson bivector dual to the symplectic 2-form \(ω^0\) on \(N\). An adaptation of Rothstein’s theorem [23] to the setting of \(Z\)-graded manifolds shows that any Poisson structure on \(M\) is equivalent to that given by (2.22).

Mention that, unlike the previous example, the bracket does not change the ghost number, i.e. \(\text{gh}\{f, g\} = \text{gh}(f) + \text{gh}(g)\), and the ghost number operator \(G\) can be realized in the adjoint manner

\[
(2.25) \quad \hat{G} f = \{G, f\} = \text{gh}(f) f, \quad G = C^α P_α,
\]

for any homogeneous function \(f \in C^∞(M)\). Since the bracket is even, an odd homological potential \(Ω\) is to be introduced. In the BRST-BFV theory \(Ω\) carries the ghost number 1, and is called the BRST charge. So, one has

\[
(2.26) \quad \{Ω, Ω\} = 0, \quad \{G, Ω\} = Ω.
\]

The last relations are known as the BRST algebra. Since \(M\) is a symplectic manifold the properness condition for the generator of the BRST transformations \(Q = \{Ω, \cdot\}\) can be written as

\[
(2.27) \quad \text{rank}(d^2 Ω)|_{dΩ=0} = 2 \text{rank} E,
\]

where \(d^2 Ω = (∂_A ∂_B Ω)\) is an even matrix if one changes the parity of the first index. Accounting the ghost number distribution, the general expression for the BRST charge reads

\[
(2.28) \quad Ω = C^α T_α(x) + \frac{1}{2} C^α C^β T^γ_{βα}(x) P_γ + O(P^2).
\]

The coefficients \(T_α(x)\) at the first order in ghosts are identified with the Hamiltonian constraints on the original phase space \(N\). The properness condition (2.27) allows to conclude that equations \(T_α(x) = 0\) single out a smooth submanifold (the constraint surface) \(S' \subset N\), being the projection on \(N\) of the stationary shell \(Σ = ΠE^*[−1]|_{S'}\).

In the lowest order in \(C^α\), the master equation (2.20) gives the involution relations for the first class constraints

\[
(2.29) \quad \{T_α, T_β\}_0 = U^γ_{αβ} T_γ,
\]
where the Poisson bracket is defined by the bivector $\pi$, and
\begin{equation}
U_{\alpha\beta}^\gamma = T_{\alpha\beta}^\gamma - \pi^{ij} \partial_i T_{\alpha \beta} \Gamma_j^\gamma + \pi^{ij} \partial_i \Gamma_{\beta \gamma}^j .
\end{equation}
In other words, $\Sigma' \subset N$ is a coisotropic submanifold [23].

Again, using a sort of homological perturbation theory [9], [19] one can see that (i) all the higher orders in (2.28) can be iteratively restored provided $\Sigma \subset N$ is a smooth coisotropic submanifold, and (ii) any two solutions for $\Omega$, associated with the same $\Sigma$, are related to each other by a simplectomorphism of $M$ [6], [19].

2.1.3. Lie algebroids. Let $E \to N$ be a Lie algebroid over a smooth manifold $N$ with the anchor $A : E \to TN$. Define the graded supermanifold $M = T^* [3] N \oplus \Pi E^*[1] \oplus \Pi E^*[2]$. By the definition,
\begin{equation}
\epsilon(x^i) = 0, \quad \epsilon(p_i) = 0, \quad \epsilon(C^\alpha) = 1, \quad \epsilon(P_\alpha) = 1 ,
\end{equation}
\begin{equation}
\text{gh}(x^i) = 0, \quad \text{gh}(p_i) = 3, \quad \text{gh}(C^\alpha) = 1, \quad \text{gh}(P_\alpha) = 2 .
\end{equation}
Here $x^i$ are local coordinates on $N$, while $p_i$, $C^\alpha$ and $P_\alpha$ are linear coordinates on the fibers of $T^* [3] N$, $\Pi E^*[1]$ and $\Pi E^*[2]$, respectively.

Choosing a linear connection $\nabla$ on $E \to N$ the manifold $M$ can be endowed with an even symplectic structure of the ghost number 3. The corresponding Poisson brackets of local coordinates read
\begin{equation}
\{p_i, x^j\} = \delta_i^j , \quad \{P_\alpha, C^\beta\} = \delta_\alpha^\beta ,
\end{equation}
\begin{equation}
\{p_i, C^\beta\} = \Gamma^{\beta}_{\alpha\gamma} C^\gamma , \quad \{p_i, P_\beta\} = -\Gamma^{\alpha}_{i\beta} P_\alpha ,
\end{equation}
and the other brackets vanish. Here $R_{ij\alpha}^\beta$ is the curvature tensor of the connection $\nabla = \partial + \Gamma$. Since the even bracket is introduced with the ghost number -3, the homological potential $Q$ should be odd and carrying the ghost number 4. Taking into account the ghost number and parity of the homological potential on this manifold, the most general admissible expression for $Q$ reads
\begin{equation}
Q = C^\alpha A^\alpha_a(x) p_i + \frac{1}{2} C^{\alpha\beta} T_{\alpha\beta}^\gamma (x) P_\gamma .
\end{equation}
The usual Lie algebroid relations for the anchor $A^\alpha_a \in \Gamma (E^* \otimes TN)$ and the torsion $T_{\alpha\beta}^\gamma \in \Gamma (E \otimes E \wedge E)$ immediately follow from the master equation $\{Q, Q\} = 0$ with the bracket (2.32). Notice that our BRST-like imbedding for the Lie algebroid differs from that considered in [25], where the ghost numbers are assigned in a different way and more than one ghost grading is imposed.

2.1.4. Courant algebroids. Consider the manifold $M$ given by the direct sum of vector bundles $T^* [2] N \oplus \Pi E^*[1]$ over an even manifold $N$. If $x^i$ are local coordinates on $N$, and $p_i$ and $\xi^a$ are local coordinates on fibers of $T^* [2] N$ and $\Pi E$, then we have
\begin{equation}
\epsilon(x^i) = 0, \quad \epsilon(p_i) = 0, \quad \epsilon(\xi^a) = 1 ,
\end{equation}
\begin{equation}
\text{gh}(x^i) = 0, \quad \text{gh}(p_i) = 2, \quad \text{gh}(\xi^a) = 1 .
\end{equation}
Any symplectic structure on $M$, having the ghost number 2 is equivalent to the following one:
\begin{equation}
\omega = d(p_i dx^i + \xi^a g_{ab} \nabla \xi^b) = dp_i dx^i + g_{ab}(x) \nabla \xi^a \nabla \xi^b + \frac{1}{2} dx^i dx^j R_{ijab}(x) \xi^b \xi^a ,
\end{equation}
where $\nabla$ is a linear connection on $E$ respecting a pseudo-Euclidean metric $g_{ab}$ and having the curvature $R_{ijab} = g_{ca} R_{ijcb}$. The corresponding Poisson brackets of local coordinates read
\begin{equation}
\{x^i, x^j\} = 0 , \quad \{\xi^a, x^i\} = 0 , \quad \{p_i, \xi^a\} = \Gamma^a_{\alpha i}(x) \xi^b ,
\end{equation}
\begin{equation}
\{p_i, x^j\} = \delta_i^j , \quad \{\xi^a, \xi^b\} = \frac{1}{2} g^{ab}(x) , \quad \{p_i, p_j\} = R_{ijab}(x) \xi^a \xi^b .
\end{equation}
The most general expression for the homological potential, compatible with the ghost number grading and parity, is given by
\begin{equation}
Q = \xi^a A^a_b(q) p_i - \frac{1}{6} \phi_{abc}(q) \xi^a \xi^b \xi^c , \quad \text{gh}(Q) = 3 ,
\end{equation}
where $A \in \Gamma (E^* \otimes TN)$ and $\phi \in \Gamma (\Lambda^3 E^*)$. Given a connection $\nabla$ on $N$, the master equation $\{Q, Q\} = 0$ imposes certain relations on the structure functions $q, A$ and $\phi$, which prove to be equivalent to the structure relations of the Courant algebroid [27], [8], (see also [28]) upon identification of $A$ as the respective anchor.
3. Cohomology of a gauge system

Any homological vector field $Q$ gives rise to the nilpotent operator $\delta : \mathcal{T}(M) \to \mathcal{T}(M)$ on the space of smooth tensor fields:

\begin{equation}
\delta S = \mathcal{L}_Q S, \quad \forall S \in \mathcal{T}(M),
\end{equation}

$\mathcal{L}_Q$ being the Lie derivative along $Q$. Let $Z_\delta(\mathcal{T}(M)) = \ker \delta$ to denote the group of $\delta$-cocycles and let $B_\delta(\mathcal{T}(M)) = \text{im} \delta$ be the group of $\delta$-coboundaries. The $\delta$-cohomology group is the quotient

\begin{equation}
H_\delta(\mathcal{T}(M)) = Z_\delta(\mathcal{T}(M))/B_\delta(\mathcal{T}(M)).
\end{equation}

Clearly, the subgroup $Z_\delta(\text{Vect}(M))$ has the distinguished element $Q$. When $Q$ is a Hamiltonian vector field, one has an additional pair of cocycles: the homological potential $Q \in Z_\delta(C^\infty(M))$ and the Poisson bivector $\pi \in Z_\delta(\wedge^2 \text{Vect}(M))$.

Since $\delta$ is essentially the generator of an infinitesimal diffeomorphism, applying various tensor operations to $\delta$-cocycles one can induce the corresponding operations in the $\delta$-cohomology. Again, in the case of the gauge systems one has distinguished cohomological operations e.g. taking the Poisson bracket of two scalar cocycles or multiplying a tensor cocycle by the Poisson bivector.

In what follows we will mostly deal with the scalar cocycles and their cohomology groups $H_\delta(C^\infty(M)) = H_\delta(M)$. The reason is that any tensor cocycle can be always realized as a scalar cocycle on an appropriately extended manifold $M'$. Moreover, it is always possible to convert the corresponding homological vector field $Q'$ into a Hamiltonian one by further extending $M'$. Indeed, one can always identify $T^{(n,m)}(M)$ with the space of the polyniolar functions on the total space $M'$ of $T^*M \otimes \otimes T^M$. The homological vector field $Q$ is then lifted to $M'$ by usual formulas of tensor calculus (i.e. as the Lie derivative). At the next step one can lift $Q'$ to the Hamiltonian vector field $Q''$ on $T^*M'$ (or $\Pi T^*M'$) w.r.t. the canonical symplectic structure:

\begin{equation}
Q''(x,p) = Q^n(x)p_i, \quad Q''(x,x^*) = Q'^i(x)x^i_*,
\end{equation}

where $x^i$ are the local coordinates on $M'$, while $p_i$ and $x^i_*$ are the linear coordinates on the fibers of $T^*M'$ and $\Pi T^*M'$, respectively. Thus, studying the general structure of the homological algebra associated with nilpotent $Q$, we can always restrict consideration to the scalar cocycles.

One may regard the above procedure as an extension of the homological vector field $Q$ from $M$ to $M'$ (of course, there can be many other procedures achieving the same goal). Let us describe an inverse procedure, namely, the reduction of a homological vector field to a submanifold. Given a set of smooth functions $\{f_\alpha\}$ on the manifold $M$ equipped with a homological vector field $Q$, consider the ideal $I \subset C^\infty(M)$ generated by the extended set of functions $\{f_\alpha, \delta f_\beta\} = (\phi_A)$. The equations $\phi_A = 0$ extract a smooth submanifold $N \subset M$ provided the standard regularity condition is satisfied:

\begin{equation}
\text{rank } \left( \frac{\partial \phi_A}{\partial x^i} \right)_N = \text{const}.
\end{equation}

By construction, $N$ is the $Q$-invariant submanifold endowed with the homological vector field $Q' = Q|_N$. Thus, to any $\delta$-closed ideal of functions $I$ satisfying the regularity condition, we can associate a cohomology group $H_\delta(T(N))$ together with the natural homomorphism $H_\delta(T^{(0,*)}(M)) \to H_\delta(T^{(0,*)}(N))$ induced by the restriction of covariant tensor fields on the submanifold $N$. In algebraic terms, the commutative algebra of functions $C^\infty(N)$ can be equivalently described as the quotient $C^\infty(M)/I$. Respectively, the group $H_\delta(N)$ is identified with cohomology classes of relative cocycles: $c \subset C^\infty(M)$ defines an element $[c] \in H_\delta(N)$ iff $\delta c \in I$.

One can generalize this construction by replacing $f_\alpha$ with a set of tensor fields $S_{\alpha}$ and their $\delta$-differentials. Again, under the regularity condition, the equations $S_{\alpha} = \delta S_{\alpha} = 0$ define a smooth submanifold equipped with a homological vector field. An important example of an invariant submanifold is the stationary shell $\Sigma \subset M$ of a homological vector field $Q$.

Existence of the invariant submanifolds provides in that way an additional source for cohomological invariants one can relate with the gauge system.

In the category of $\mathbb{Z}$-graded manifolds the group $H_\delta(M)$ is naturally graded with respect to the ghost number. In field theory one usually deals with the scalar cohomology of the ghost numbers 0, 1 and 2. The group $H^0_\delta(M)$ is identified with the space of physical observables of a gauge system, while $H^1_\delta(M)$ is considered as a source for possible anomalies in the Lagrangian quantization. In the Hamiltonian description the same quantum anomalies are nested in $H^2_\delta(M)$. We refer to [20], [21] for general results on the local BRST cohomology as well as its

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6) Like tensor multiplication, permutation and contraction of indices, Schouten brackets of polyvector fields, concomitant of symmetric contravariant tensors, exterior derivative of differential forms, etc.
application to the Yang-Mills type theories. In Sec. 6 we will put the problem of anomalies in the general context of characteristic classes to be developed in the next section.

4. Characteristic classes

In this section we consider scalar cocycles which can be universally defined for any gauge system \((M, \{\cdot, \cdot\}, Q)\). For this reason we call them universal cocycles. The formal definition is as follows.

**Definition 4.1.** Given a Poisson manifold \((M, \{\cdot, \cdot\})\), a local map \(U : C^\infty(M) \to C^\infty(M)\) is said to be a universal cocycle if for any \(\Theta \in C^\infty(M)\) with \(\epsilon(\Theta) = \epsilon(\{\cdot, \cdot\}) + 1\) there exists a linear differential operator \(\hat{U} : C^\infty(M) \to C^\infty(M)\) (depending perhaps on \(\Theta\)) such that

\[
(4.1) \quad \{\Theta, U(\Theta)\} = \hat{U}(\Theta, \Theta) .
\]

The mentioned property of locality means that

\[
(4.2) \quad \text{supp } U(\Theta) \subset \text{supp } \Theta , \quad \forall \Theta \in C^\infty(M) .
\]

When \(M\) is compact, this amounts to saying that \(U(\Theta)\) is specified by some symmetric polydifferential operator (i.e. symmetric w.r.t. permutations of its arguments, which all are then substituted by one and the same function \(\Theta\)).

The equation \((4.1)\) merely says that once a homological potential \(Q\) is substituted into \(U\) instead of an arbitrary \(\Theta\), the function \(U(Q)\) becomes a \(\delta\)-cocycle. Note that Definition 4.1 involves the Poisson structure underlying a gauge system but not the homological potential. Taking another homological potential on the same Poisson manifold, one can find its \(\delta\)-cocycles making use of the same map \(U\). That is why we name \(U(Q)\) a universal cocycle.

By characteristic classes of a gauge system we mean the \(\delta\)-cohomology classes of its universal cocycles. These form a subgroup \(UH_\delta(M) \subset H_\delta(M)\) in the full group of \(\delta\)-cohomology. The natural grading in the space of polydifferential operators induces the grading in \(UH_\delta(M)\):

\[
(4.3) \quad UH^j_\delta(M) \ni [U] \iff U(tQ) = t^n U(Q) , \quad t \in \mathbb{R} .
\]

For example, \(U(Q) = Q^n\) is the universal cocycle of degree \(n\). Less trivial examples of universal cocycles can be obtained as follows. Let \(\omega \in \Omega^n(M)\) be a closed \(n\)-form on \(M\), put

\[
(4.4) \quad U(Q) = i_Q^n \omega ,
\]

where \(i_Q\) denotes the contraction of a differential form with the homological vector field \(Q\). It is easy to check that \(i_Q\) is a \(\delta\)-cocycle. Indeed, accounting that \(i_Q\) is an even operator we can write

\[
\begin{align*}
\delta U(Q) &= \mathcal{L}_Q i_Q^n \omega = -i_Q d i_Q^n \omega = -(i_Q \mathcal{L}_Q i_Q^{n-1} + i_Q^2 d i_Q^{n-1}) \omega = -(2 \mathcal{L}_Q i_Q^n + i_Q^3 d i_Q^{n-2}) \omega = \cdots \\
&= -n \mathcal{L}_Q i_Q^n \omega - i_Q^{n+1} d \omega = -n \delta U(Q) - i_Q^{n+1} d \omega .
\end{align*}
\]

Therefore,

\[
(4.6) \quad \delta U(Q) = \frac{1}{n+1} i_Q^{n+1} d \omega = 0 .
\]

On the other hand, for an exact \(\omega = d \theta\) we have

\[
(4.7) \quad U(Q) = i_Q^n d \theta = i_Q^{n-1} (d i_Q + \mathcal{L}_Q) \theta = \cdots = n \mathcal{L}_Q i_Q^{n-1} \theta = \delta(n i_Q^{n-1} \theta) .
\]

Together, Rel. \((4.6)\) and \((4.7)\) imply the natural homomorphism \(h : H^j_\delta(M) \to UH^j_\delta(M)\) from \(n\)’th group of the De Rham cohomology to \(n\)’th group of \(\delta\)-cohomology. This motivates to consider the factor group

\[
(4.8) \quad \tilde{U}H_\delta(M) = UH_\delta(M) / h(H_\delta(M)) ,
\]

i.e. equivalence classes of scalar \(\delta\)-cocycles modulo those coming from De Rham’s cohomology classes. In the next subsection we construct a large number of universal cocycles irreducible to the De Rham ones.

**Principal series of the characteristic classes.** Let \(M\) be a smooth manifold endowed with a homological vector field \(Q\) (not necessarily Hamiltonian) and a symmetric connection \(\nabla\). Using these data, we define the following pair of \((1,1)\)-tensor fields:

\[
(4.9) \quad \Lambda = (\nabla_i Q^j) , \quad R = (R^j_i) = \left( -\frac{1}{2} (-1)^{i+j} Q^j Q^k R^l_{ijk} \right) ,
\]
exists a $C(4.17)$ is a $\delta$

Theorem 4.3. For each $f(4.16)$ $R$ in matrices $\Lambda$

Corollary 4.4. If $n = 2m - 1$ for $m = 1, 2, \ldots,$ or $n = 2m$ and 2m \textquoteleft{}th Pontrjagin class is trivial, then there exists a $(2n - 1)$-form $\phi_n$ such that

\begin{equation}
C_n = f_n + i^{2n-1}_Q \phi_n
\end{equation}

\textit{7}Informally speaking, Pontrjagin\textapos;s classes of the tangent bundle $TM$ represent obstructions for a half of the functions $\text{4.11}$ to be extendible to $\delta$-cocycles on a non-parallelizable $Q$-manifold $M$.}
Proof. Under the assumptions made
\[ \text{str}(R^n) = d \omega_n, \]
for some \((2n - 1)\)-form \(\omega_n\). On the other hand,
\[ (2n)! \text{str}(R^n) = (2n)!(-2)^n \text{str}(R^n), \]
and Corollary 4.4 readily follows from the relations (4.4), (4.6) if one put
\[ \phi_n = -\left( \frac{1}{2} \right)^n \frac{c_n \omega_n}{(2n - 1)!}. \]

We will prove the above theorem using an auxiliary algebraic construction which can also be understood in terms of spectral sequences. Consider a bigraded associative algebra
\[ A = \bigoplus_{n=0}^{\infty} A_{n,m}, \]
freely generated by elements \(a_n, n = 0, 1, \ldots\). The general element of \(A\) is given by a linear combination of homogeneous monomials
\[ A = a_{n_1}a_{n_2} \cdots a_{n_k}. \]
By definition we set
\[ \deg_1 A = n_1 + n_2 + \cdots + n_k, \quad \deg_2 A = k. \]
It is also convenient to define the total degree \(\deg = \deg_1 + 2\deg_2\) and the corresponding decomposition
\[ A = \bigoplus_{m=2}^{\infty} A_m, \quad A_m = \bigoplus_{n+2k=m} A_{n,k}. \]
The algebra \(A\) can be endowed with a pair of nilpotent differentials \(\delta_1 : A_{n,m} \to A_{n+1,m}\) and \(\delta_2 : A_{n,m} \to A_{n-1,m+1}\) defined by
\[ \delta_1(AB) = \delta_1 A \cdot B + (-1)^{\deg_1 A} A \delta_1 B, \quad \delta_2 A = k, \]
\[ \delta_1 a_n = \begin{cases} a_{n+1}, & \text{for odd } n, \\ 0, & \text{for even } n, \end{cases} \quad \delta_2 a_n = \begin{cases} \sum_{s=0}^{n-1} a_s a_{n-s-1}(-1)^s, & \text{for } n > 0, \\ 0, & \text{for } n = 0, \end{cases} \]
for any homogeneous elements \(A, B \in A\). It is easy to see that
\[ (\delta_1)^2 = (\delta_2)^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0, \]
and hence \(\delta = \delta_1 + \delta_2 : A_n \to A_{n+1}\) is a nilpotent differential increasing the total degree by 1.

Let us now extend the cochain complex \((A, \delta)\) to the following one:
\[ A' = A \oplus B \oplus C, \]
where \(B = \text{span}(b_1, b_2, \ldots), \quad C = \text{span}(c_1, c_2, \ldots), \quad \text{and}
\[ \delta c_n = a_n^n, \quad \delta b_n = \begin{cases} a_{n-1}, & \text{for odd } n, \\ 0, & \text{for even } n. \end{cases} \]
By definition we set \(\deg b_n = n\) and \(\deg c_n = 2n - 1\). Let \(H^n(A')\), \(n = 1, 2, \ldots\), denote the corresponding cohomology groups.

Lemma 4.5. \(H^n(A') \simeq B_n = \text{span}(b_n)\).

Proof. Define a partial homotopy operator for \(\delta_1\) by the rule
\[ \delta_1^* A = \frac{1}{\deg_2 A} \sum_{l=1}^{k} a_{n_1} \cdots a_{n_{l-1}} (\delta_1^* a_{n_l}) a_{n_{l+1}} \cdots a_{n_k} (-1)^{n_1 + \cdots + n_{l-1}}, \]
(4.27)
\[ \delta_1^* a_n = \begin{cases} a_{n-1}, & \text{for even } n > 0, \\ 0, & \text{for odd } n \text{ or } n = 0, \end{cases} \]
where \(A\) is given by (4.21). One can easily see that \((\delta_1^*)^2 = 0\) and
\[ \pi \delta_1^* + \delta_1 \delta_1^* + \pi = 1, \]
\(\pi\) being the canonical projection onto the subspace of nontrivial \(\delta_1\)-cocycles \(A_0 = \text{span}(a_0, a_0^2, \ldots)\).
Now we introduce the (twisting) operator \( G = 1 + \delta_1^* \delta_2 : \mathcal{A} \to \mathcal{A} \). Since \( \delta_1^* \delta_2 \) decreases the first degree by 2, the operator \( G \) is invertible. One can check that
\[
\delta|_{\mathcal{A}} = G^{-1} \delta_1 G + \pi \delta_2 .
\]

Let us define \( \delta^* = G^{-1} \delta_1 G : \mathcal{A} \to \mathcal{A} \). From \eqref{eq:delta} it then follows that
\[
(\delta \delta^* + \delta^* \delta)a = a - \pi(G + \delta_2 \delta^*)a, \quad \forall a \in \mathcal{A} .
\]

Consider a general \( \delta \)-cocycle \( z = a + b + c \), where \( a \in \mathcal{A}, \ b \in \mathcal{C}, \ c \in \mathcal{C} \). Applying \( \delta^* \) to the equation \( \delta z = 0 \) and using \eqref{eq:delta}, we obtain
\[
\delta^* (a + b + c) = a - \delta \delta^* a - \pi(G + \delta_2 \delta^*)a + \delta^* \delta b = 0 ,
\]
and hence \( a \) is cohomologous to \(-\delta^* \delta b\). Substituting \( a = -\delta^* \delta b \) to the cocycle condition \( \delta z = 0 \) and using \eqref{eq:delta}, we find
\[
\pi(G + \delta_2 \delta^*) \delta b + \delta c = 0 .
\]
Since the restriction \( \delta|_{\mathcal{C}} \) is bijective, we can write \( c = -\delta^{-1} \pi(G + \delta_2 \delta^*) \delta b \). Gathering all together, we finally get \( z = Ab \), where
\[
A = 1 - \delta^* \delta - \delta^{-1} \pi(G + \delta_2 \delta^*) \delta .
\]
It remains to note that the operator \( A \) is invertible and preserves the total degree. \( \square \)

Now the proof of Theorem \ref{thm} immediately follows from the obvious identifications:
\[
b_n = \frac{1}{n} \Lambda^n, \quad a_n = R \Lambda^n, \quad c_n = i_Q^{2n-1} \phi_n, \quad \delta = \nabla_Q ,
\]
and from the cyclicity property of the supertrace. Making use of the operator \eqref{eq:delta} gives us explicit expressions for the cocycles \( \mathcal{C}_n \), provided the corresponding Pontrjagin’s class (if any) is trivial. For a few first cocycles we find
\[
\mathcal{C}_1 = \text{str}(\Lambda) + \psi_1 ,
\]
\[
\mathcal{C}_2 = \text{str}(\Lambda^2 + 3R \Lambda) + \psi_3 ,
\]
\[
\mathcal{C}_3 = \text{str}(\Lambda^3 + 5R \Lambda^3 + 10R^2 \Lambda) + \psi_5 ,
\]
\[
\mathcal{C}_4 = \text{str}(\Lambda^4 + 7R \Lambda^5 + 14R^2 \Lambda^3 + 7R^3 \Lambda^2 + 35R^3 \Lambda) + \psi_7 ,
\]
where \( \psi_n = i_Q^{2n-1} \phi_n \). In what follows we will refer to \( \chi_n = [\mathcal{C}_n] \), defined by \eqref{eq:chi}, as the principal series of characteristic classes.

**Remark 4.6.** The form \( \phi_n \), entering the definition of \( \mathcal{C}_n \), is determined by Relns. \eqref{eq:phi} up to the adding any closed \((2n - 1)\)-form. This ambiguity, however, does not effect on the characteristic class \( \chi_n = [\mathcal{C}_n] \) being considered as an element of \( \overline{UH}_s(M) \).

**Remark 4.7.** As is seen, the class \( \mathcal{C}_1 \) is nothing but the divergence of the homological vector field and thereby it can be equivalently defined in terms of a nowhere vanishing density \( \rho \) instead of the connection:
\[
\mathcal{C}_1 = \text{div}_\rho Q = \rho^{-1} \partial_i (\rho Q^i) .
\]

If \( \rho' \) is another density on \( M \), then \( \mathcal{C}'_1 = \text{div}_{\rho'} Q \) is cohomologous to \( \mathcal{C}_1 \):
\[
\mathcal{C}'_1 - \mathcal{C}_1 = \delta f , \quad f = \ln(\rho'/\rho) .
\]
In particular, this class appears to be trivial for any gauge system \((M, \{\cdot, \cdot\}, Q)\) associated with an even symplectic structure: the divergence of the Hamiltonian vector field \( Q \) vanishes identically when it is evaluated w.r.t. the Liouville volume form.

Meanwhile, for an odd Poisson bracket this class can be nontrivial. For instance, with every Poisson manifold \((N, \pi)\), \( \pi \in \wedge^2 T^* N \) being a Poisson bivector, one can associate a gauge system \((\Pi T^* N, \{\cdot, \cdot\}, Q)\) w.r.t. the canonical Poisson bracket on \( \Pi T^* N \) and
\[
Q = \frac{1}{2} \pi^{ij}(x)x^i_\ast x^j_\ast .
\]

As is known \[23\], every density \( \sigma \) on \( N \) corresponds to the density \( \rho = \sigma^2 \) on \( \Pi T^* N \). Using this special density one can see that \( \chi_1 \) is proportional to the modular cocycle of a Poisson manifold \[30\]:
\[
\chi_1 = 2[\sigma^{-1} \partial_i (\sigma \pi^{ij})^i] .
\]
Vanishing of this class is known to be the necessary and sufficient condition for the Poisson manifold \((N, \pi)\) to admit a continuous trace density.

Having in mind this example, \(C_1\) is referred to as a modular class of a gauge system \((M, \{ \cdot, \cdot \}, Q)\).

**Theorem 4.8.** The characteristic classes \(\chi_n\) do not depend on the choice of the symmetric connection \(\nabla\).

**Proof.** Given two symmetric connections \(\nabla\) and \(\nabla'\), define the one-parametric family of connections \(\nabla^{(t)} = (1 - t)\nabla + t\nabla'\), \(t \in \mathbb{R}\), such that \(\nabla^{(0)} = \nabla\) and \(\nabla^{(1)} = \nabla'\). Consider the manifold \(M' = M \times \mathbb{R}_{1,1}\), where \(\mathbb{R}_{1,1}\) is a linear superspace with one even coordinate \(t \in \mathbb{R}\) and one odd coordinate \(\theta\). The homological vector field \(\hat{Q}\) and the connection \(\nabla^{(t)}\) are extended from \(M\) to \(M'\) as follows:

\[
\hat{Q} = Q + \partial \theta \partial_t, \quad \nabla_X^{(t)} = \nabla_X \nabla_t = \partial_t, \quad \nabla_{\partial_t} = \partial_t \nabla_t = \partial_t \theta,
\]

where \(\partial_t = \partial / \partial t\), \(\partial_\theta = \partial / \partial \theta\), and \(X \in \text{Vect}(M)\). The matrices \((4.40)\) take the form

\[
\hat{\Lambda} = \begin{pmatrix}
\Lambda^{(t)} & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad \hat{R} = \begin{pmatrix}
R^{(t)} + \theta \Psi & 0 \\
-\theta \Psi & 0 \\
0 & 0
\end{pmatrix},
\]

where \(\Lambda^{(t)}\) and \(R^{(t)}\) are constructed from \(\nabla^{(t)}\) and \(Q\) by the same formula \((4.41)\), and

\[
\Psi = (Q^k \Delta \Gamma_{kj}), \quad \Delta \Gamma = \nabla' - \nabla.
\]

The principal cocycles associated to \(\hat{M}, \hat{Q}\) and \(\nabla\) are given by

\[
(4.43) \quad \hat{C}_n = f_n(\Lambda^{(t)}, R^{(t)} + \theta \Psi) + \psi^{(t)}_{2n-1} = C^{(t)} + \theta \text{str}(\Psi F_n),
\]

where

\[
(4.44) \quad C^{(t)} = f_n(\Lambda^{(t)}, R^{(t)}) + \psi^{(t)}_{2n-1}, \quad F_n(\Lambda^{(t)}, R^{(t)}) = (F_n^j) = (\partial f_n / \partial R^{(t)}). \quad \text{By construction,}
\]

\[
(4.45) \quad \mathcal{L}_Q \hat{C}_n = \mathcal{L}_Q C^{(t)} + \theta[\partial_t C^{(t)} - \mathcal{L}_Q \text{str}(\Psi F_n)] = 0,
\]

and hence

\[
(4.46) \quad \mathcal{L}_Q C_n^{(t)} = 0, \quad \partial_t C^{(t)} = \mathcal{L}_Q \text{str}(\Psi F_n).
\]

Integrating the last equation with respect to \(t\) from 0 to 1, we finally get

\[
(4.47) \quad C_n^{(t)} - C_n^{(0)} = \delta \int_0^1 \text{str}(\Psi F_n) dt,
\]

i.e., the cocycles associated to different connections belong to the same class of \(\delta\)-cohomology. \(\square\)

5. COHOMOLOGICAL OPERATIONS

As we have mentioned in Sec.\(5\) every tensor operation in \(\mathcal{T}(M)\) induces an operation in the \(\delta\)-cohomology group \(H_{\delta}(\mathcal{T}(M))\). In particular, scalar cocycles of any gauge system \((M, \{ \cdot, \cdot \}, Q)\) form a Poisson algebra with respect to the point-wise multiplication and taking the Poisson bracket. It is shown below that the latter operation turns out to be trivial when applied to characteristic classes; instead, the space of all universal cocycles carry a natural Lie algebra structure with respect to another Lie bracket:

\[
(5.1) \quad [U, V] = U \circ V - (-1)^{(\epsilon(U) + \epsilon(V))} (\epsilon(V) + \epsilon(Q)) V \circ U,
\]

where

\[
(5.2) \quad U \circ V = \frac{d}{d\mu} V(Q + \mu U(Q)) \bigg|_{\mu=0}, \quad \epsilon(\mu) = \epsilon(Q) + \epsilon(U).
\]

Notice the parity shift in the definition of the commutator of two universal cocycles. In particular,

\[
(5.3) \quad \epsilon([U, V]) = \epsilon(U) + \epsilon(V) + \epsilon(Q).
\]

The Jacobi identity for the commutator \((5.1)\) readily follows from the definition above. As is seen the commutator of two universal cocycles is a local map \([U, V] : C^\infty(M) \to C^\infty(M)\). So, it remains to check that \(\delta([U, V]) = 0\). The last fact can be established as follows.

To each universal cocycle \(U(Q)\) we assign a flow \(Q(\mu) = \varphi^{(t)}(\mu)\): the evolution with respect to the “time” \(\mu\) is described by the nonlinear PDE

\[
(5.4) \quad \frac{\partial Q}{\partial \mu} = U(Q) - \frac{1}{2} \mu [U, U](Q), \quad \epsilon(\mu) = \epsilon(Q) + \epsilon(U).
\]
Here we are interested in solutions given by formal power series in $\mu$ with coefficients in $C^\infty(M)$. Clearly, given $U(Q)$, there is a unique $Q(\mu) \in C^\infty(M)[[\mu]]$ satisfying \[Q(\mu) = Q + \mu U(Q),\] and the boundary condition $Q(0) = Q$. In particular, for the odd $\mu$ the general solution reads
\[Q(\mu) = Q + \mu U(Q),\]
and we see that $\{Q(\mu), Q(\mu)\} = 0$. For the even $\mu$ the commutator $[U, U]$, entering the r.h.s. of \[Q(\mu), Q(\mu)\], vanishes identically and we can write
\[\{\varphi^U_\mu(Q), \varphi^U_\mu(Q)\} = \varphi^U_\mu(\{Q, Q\}) = \{Q, Q\} + \int_0^\mu \varphi^U_\nu(\{U, Q\})d\nu = 0.\]

In either case, the flow $\varphi^U_\mu$ respects the master equation. In geometric terms, one can think of $U(Q)$ as a vector field on the space $C^\infty(M)$ (perhaps nonintegrable when $[U, U] \neq 0$), which is tangent to the “surface” of all solutions to the master equation $\{Q, Q\} = 0$.

Consider now the composition of two flows. Using the definition of $\circ$-product \[\circ\], we can write
\[Q(\mu, \nu) = \varphi^U_\nu(\varphi^U_\mu(Q)) = Q + \mu U + \nu V + \mu \nu V \circ U + O(\mu^2, \nu^2).\]
Since
\[0 = \{Q(\mu, \nu), Q(\mu, \nu)\} = 2\mu \nu \left((-1)^{\epsilon(\mu)}\{V, U\} + (-1)^{\epsilon(\mu) + \epsilon(\nu)}\{Q, V \circ U\} + O(\mu^2, \nu^2)\right),\]
we have
\[\{V, U\} = \delta((-1)^{\epsilon(\nu)+1}V \circ U).\]
Taking into account the symmetry property of the Poisson bracket \[\circ\], we finally get
\[\delta([V, U]) = 0.\]
Thus, the commutator of two universal cocycles is an universal cocycle again. As a byproduct of this observation we have proved the following statement

**Theorem 5.1.** The characteristic classes of any gauge system Poisson commute.

It is instructive to compute the commutator of two universal cocycles when one of them is a coboundary. We have
\[\{U, \delta V\} = U \circ \{Q, V\} - (-1)^{(\epsilon(U)+\epsilon(Q))(\epsilon(V)+\epsilon(Q)+1)}\{Q, V\} \circ U =
\{U, V\} - (-1)^{(\epsilon(U)+\epsilon(Q))(\epsilon(V)+\epsilon(Q)+1)}\{Q, V\} \circ U + (-1)^{\epsilon(U)+\epsilon(Q)}\{Q, U \circ V\} =
= (-1)^{(\epsilon(V)+\epsilon(Q)+1)(\epsilon(U)+\epsilon(Q)+1)}(\mathcal{L}_V U)(Q) + (-1)^{\epsilon(U)+\epsilon(Q)}\delta(U \circ V),\]
where $V = \{V, \cdot\}$ is the Hamiltonian vector field corresponding to $V \in C^\infty(M)$, and $\mathcal{L}_V U(Q)$ is the Lie derivative of the polydifferential operator $U$ acting on $Q$. Although the last expression need not be a $\delta$-coboundary in general, it is not easy to come with a counterexample. Indeed, every cocycle of the principal series is determined by a polydifferential operator $C_n(\pi, \nabla)(Q)$ constructed from the Poisson bivector $\pi$ and the connection $\nabla$ by means of tensor operations. Then
\[\mathcal{L}_V C_n(Q) = C_n(\mathcal{L}_V \pi, \nabla)(Q) + C_n(\pi, \mathcal{L}_V \nabla)(Q).\]
The first term vanishes because $V$ is a Hamiltonian vector field, while the second term is a $\delta$-coboundary in consequence of Theorem 1.8.

If now $c = i_\omega^\delta$ is the $\delta$-cocycle, associated to a closed $n$-form $\omega \in \Omega^n(M)$, then one can check that
\[\mathcal{L}_V c(Q) = i_\omega^\delta d\mathcal{L}_V \omega = \delta(n2^{n-1}i_\omega \omega),\]
and we come with a coboundary again.

\[\omega\]From Rel. \[\circ\] also follows that the commutator of two coboundaries is always a coboundary. For if $U = \delta W$ we have
\[\mathcal{L}_V \delta W(Q) = \delta((-1)^{\epsilon(V)+\epsilon(Q)+1}(\mathcal{L}_V W)(Q)).\]
Let $B \in \mathbb{L}$ denote the subalgebra of $\delta$-coboundaries in the Lie algebra $\mathbb{L}$ of universal cocycles, let $F \subset C^\infty(M)$ denote the supercommutative subalgebra of functions generated by $\{C_n\}$ and let $L \in \mathbb{L}$ denote the Lie subalgebra generated by all the elements of $F$. In view of the Jacobi identity for the Lie bracket \[\circ\] and the Leibnitz rule for the Lie derivative \[\circ\] we arrive at the following

\[^8\]Notice that the adjoint action of the Lie algebra \[\mathbb{L}\] does not differentiate the point-wise product of universal cocycles.
Theorem 5.2. The subalgebra $L \subset \mathbb{L}$ belongs to the normalizer of $B$ in $L$ and therefore the Lie algebra structure on $L$ is descended to its $\delta$-cohomology space.

6. Applications and interpretations of the characteristic classes

As the global geometric properties of a gauge system are embodied in the corresponding characteristic classes, the latter can find applications every time when the global geometry becomes important for the study of the system. Here we address two types of such problems: the anomalies, and the theory of foliations.

It is a common knowledge that the anomalies appearing in the quantum field theory have a topological nature (see e.g. [31]). In a wide sense, the term anomaly means breaking of a classical gauge symmetry upon quantization. In practice, the anomalies manifest themselves as nontrivial BRST-cocycles in the ghost number 1 or 2 (depending on which formalism, Lagrangian BV or Hamiltonian BFV-BRST, is used) and present cohomological obstructions to solvability of the quantum counterparts to the classical master equations. In this context, the very existence of the nontrivial cocycles in the ghost number 1 or 2 provides an indirect evidence for possible obstructions to solvability of the quantum counterparts to the classical master equations. In this Section we show that the modular class of the Lagrangian gauge system appears to be the first obstruction to the existence of a quantum master action. Examining the same problem for the quantum BRST charge we arrive at a new universal cocycle having the ghost number 2 and depending on a Kähler metric of the phase space to be quantized. This analysis exhausts all the possible anomalies at the level of the first quantum correction. Here we also discuss the applications of our characteristic classes to theory of (singular) foliations and give an explicit example of a regular foliation with a nontrivial modular class.

6.1. Anomalies in the BV-formalism. The quantum master action $S$, governing the quantum dynamics of a gauge model, is defined on the same odd Poisson manifold $(M, \{\cdot, \cdot\})$ as the classical one (see Example 2.1.1), but obeys the quantum master equation

$$\{S, S\} = 2i\hbar \Delta S,$$

$h$ being the Planck constant. Here $\Delta : C^\infty(M) \to C^\infty(M)$ is a second order differential operator, called the odd Laplacian, defined by

$$\Delta f = \frac{1}{2} \text{div}_\rho X_f,$$

where $X_f = \{f, \cdot\}$ is the Hamiltonian vector field associated to $f \in C^\infty(M)$ and $\rho$ is a nowhere vanishing density on $M$. It is possible to choose $\rho$ in such a way that one can make $\rho = 1$ in a neighbourhood of each point on $M$ by an appropriate choice of Darboux coordinates for the odd symplectic structure (this unimodularity condition is always assumed satisfied in the BV-theory). Such a density is called normal, and for each normal density $\rho$ one has

$$\Delta^2 = 0.$$

For example, any density $\rho = \sigma^2$, obtained by squaring a density $\sigma$ on the Lagrange submanifold $\Pi \Sigma$, is automatically normal.

In the Darboux coordinates the odd Laplacian was first introduced in the works [1], [2] for purposes of the Lagrangian quantization. The covariant definition [6], involving a density function, was given in [13], [14], [22]. Various properties of odd Laplacians have been further studied and systematized in [23].

The fundamental property of the odd Laplacian is that it differentiates the odd Poisson bracket:

$$\Delta \{f, g\} = \{\Delta f, g\} + (-1)^{\ell(f)+1}\{f, \Delta g\}.$$  

This relation can be easily derived from another important formula expressing the odd bracket via the odd Laplacian:

$$\Delta (f \cdot g) = \Delta f \cdot g + (-1)^{\ell(f)}\{f, g\} + (-1)^{\ell(f)}f \cdot \Delta g.$$

One can regard the quantum master equation (6.1) as a one-parametric deformation of the classical one (2.19) with the Plank constant being the deformation parameter. This suggests to look for the solution in the form of a formal power series in $\hbar$:

$$C^\infty(M)[[\hbar]] \ni S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$$

In view of (6.3) and (6.4) the quantum master equation is the Maurer-Cartan type equation and thereby it is algebraically consistent. The local solvability of Eq. (6.1) was proven in Ref. [3]. The existence of the globally
defined solution is a more difficult question due to possible cohomological obstructions. Indeed, substituting the expansion (6.6) into (6.1) one gets the following chain of equations:

\[
\begin{align*}
\{S_0, S_0\} &= 0, \\
\{S_0, S_1\} &= i\Delta S_0, \\
\{S_0, S_n\} &= i\Delta S_{n-1} + \sum_{k=1}^{n-1} \{S_k, S_{n-k}\}, \quad n \geq 2.
\end{align*}
\]

(6.7)

The first equation identifies \(S_0\) as a classical master action. Then the rest equations take the cohomological form \(\delta S_n = B_n(S_0, ..., S_{n-1})\). By induction in \(n\), one can see that \(B_n\) is \(\delta\)-closed, provided \(S_0, ..., S_{n-1}\) obey the first \((n-1)\)'th equations of (6.7). Thus the existence problem for a solution to the quantum master equation appears to be equivalent to vanishing of the sequence of the \(\delta\)-cohomology classes \([B_n]\); in so doing, \(n\)'th cohomology class is defined, provided all the previous classes vanish. In particular, the second equation in (6.7) expresses the triviality of the modular class \(\chi_1 = [\Delta S_0]\) associated to the classical gauge system \((M, \{\cdot, \cdot\}, S_0)\). For the general gauge theory specified by the classical master action (2.18) we have the explicit formula

\[
\chi_1 = \left[ C^a (\sigma^{-1} \nabla_A (\sigma R^n_A) + T^\beta_{\alpha\beta}) + \frac{1}{2} \sigma^{-1} C^a C^b (\sigma E^A_{\alpha\beta}) \phi^*_A + \cdots \right],
\]

(6.8)

where \(\nabla\) is a connection on \(\mathcal{E} \to N\) and \(\sigma\) is a density on \(N\).

It is appropriate mention that in the field-theoretical context the modular class as well as the other characteristic classes are not well-defined objects. The matter is that actual computations of these classes for local functionals, like the master action, will result in singular expressions proportional to the “value” of the Dirac \(\delta\)-function at zero, \(\delta(0)\). Nonetheless, dividing formally by these infinite overall constants one can obtain well-defined \(\delta\)-cocycles. A more rigourous way to handle these infinities is to apply a suitable regularization scheme (see, for instance, [32]).

6.2. Anomalies in the Hamiltonian BRST-BFV formalism. In this case, the quantum master equation reads

\[
\frac{1}{2} [\Omega, \Omega] = \Omega \ast \Omega = 0,
\]

(6.9)

where \(\ast\) stands for a associative product in the algebra \(C^\infty(M)[[\hbar]]\) of symbols of operators associated to a Poisson manifold \((M, \{\cdot, \cdot\})\). As for any element of this algebra, the quantum BRST charge is supposed to be given by a formal power series in \(\hbar\):

\[
C^\infty(M)[[\hbar]] \ni \Omega = \Omega_0 + \hbar \Omega_1 + \hbar^2 \Omega_2 + \cdots.
\]

(6.10)

Once the second group of De Rham’s cohomology \(H^2(M)\) is nontrivial there are infinitely many inequivalent quantizations (\(\ast\)-products). We refer to [33, 34] for the classification and the explicit description of inequivalent \(\ast\)-products on symplectic manifolds.

It is generally accepted that the consistent quantization of field-theoretical models should rely on the Wick symbols of quantum observables (the representation of creation-annihilation operators, in the simplest case). This implies the phase space of the model \(M\) to be endowed with an integrable complex structure \(J = \delta x^i J^i_j \partial_j\) compatible with the given Poisson structure in the sense that

\[
J^k_i \pi^k_l = \pi^{ij}, \quad \nabla_k \pi^{ij} = 0.
\]

(6.11)

Here \(\pi^{ij}\) is the nondegenerate Poisson bivector and \(\nabla\) is the symmetric connection constructed by the Kähler metric \(g^{ij} = J^k_i \pi^k_j\).

The standard \(\ast\)-product of the Wick type [35, 36] has the following structure:

\[
A \ast B = \sum_{n=0}^{\infty} \hbar^n C_n(A, B), \quad A, B \in C^\infty(M)[[\hbar]],
\]

(6.12)

where \(C_n(A, B)\) is a sequence of bilinear differential operators the first three of which are

\[
\begin{align*}
C_0(A, B) &= A \cdot B, \\
C_1(A, B) &= \frac{(-1)^{\epsilon_1}}{2} \lambda^{ij} \nabla_i A \nabla_j B, \\
C_2(A, B) &= \frac{(-1)^{\epsilon_2}}{4} \lambda^{ij} \lambda^{kl} \nabla_i \nabla_k A \nabla_j \nabla_l B.
\end{align*}
\]

\[
\begin{align*}
\epsilon_1 &= \epsilon_j (\epsilon(A) + \epsilon_i), \\
\epsilon_2 &= (\epsilon_j + \epsilon_i) (\epsilon(A) + \epsilon_i) + \epsilon_j \epsilon_i + \epsilon_k \epsilon_i + \epsilon_k \epsilon_i.
\end{align*}
\]

\(^9\)In algebraic terms, the obstruction to extendibility of an \(n\)'th order solution to \((n+1)\)'th order one is represented by \(n\)'th Massey bracket \([S_0, ..., S_0]\) constructed using the Poisson bracket and the Laplacian.
The Hermitian form $\lambda^{ij} = g^{ij} + i\pi^{ij}$ on the complexified cotangent bundle of $M$ is called the Wick tensor. Substituting the expansion (6.10) into (6.9) we get the chain of equations

$$
\{\Omega_0, \Omega_n\} = - \sum_{k+l+m=n+1} C_k(\Omega_l, \Omega_m), \quad i, m < n.
$$

The equation $\{\Omega_0, \Omega_0\} = 0$, appearing at the first order in $\hbar$, is recognized as the master equation for the classical BRST charge $\Omega_0$ (cf. (2.20)). As in the previous example, the system of equations (6.14) has a cohomological form: in order for $n$th-order solution to exist, the r.h.s. should be a $\delta$-coboundary, while a priori one can only ensure its $\delta$-closedness (The latter fact follows from the Jacobi identity for the $\ast$-commutator). In particular, for the first quantum correction $\Omega_1$ to the classical BRST charge we get

$$
\delta \Omega_1 = C_2(\Omega_0, \Omega_0) = \frac{1}{2} \text{str}(J\Lambda^2),
$$

where as before $\delta = \{\Omega_0, \cdot\} = Q^i \partial_i$ and $\Lambda = (\nabla_i Q^i)$. The cocycle $C_2(\Omega_0, \Omega_0)$ has the ghost number 2 and, as we will see, may happen to be non-trivial. Indeed, using the relations (4.10) and the compatibility condition $\nabla J = 0$, one can easily find that

$$
\delta \text{str} (J\Lambda) = \text{str}(JR) - \text{str}(J\Lambda^2).
$$

This shows that $C_2(\Omega_0, \Omega_0)$ is cohomologous to $(1/2)i_{\Omega^0}^2 \rho$, where

$$
\rho = \frac{1}{2} dx^i dx^j R_{ijk}^l J_l^k (-1)^{ck}, \quad d\rho = 0,
$$

is the Ricci form of the Kähler manifold $M$. The De Rham class of a Ricci 2-form is known to be proportional to the first Chern class of a complex manifold. When the latter does not vanish the $\delta$-cohomology class of $i_{\Omega^0}^2 \rho$ can be non-vanishing as well giving the first obstruction for the quantum BRST charge $\Omega$ to exist. However, this conclusion refers only to the chosen $\ast$-product. To overcome the obstruction one can fine tune the $\ast$-product by deforming the original Poisson structure in the “direction” of the Ricci form,

$$
\pi \rightarrow \pi' = (\pi^{-1} + \hbar \rho)^{-1} = \pi - \hbar \pi \rho \pi + O(\hbar^2),
$$

and define then the new $\ast$-product w.r.t. the modified Wick tensor $\lambda' = J\pi' + i\pi'$. In such a way we pass from the standard $\ast$-product of the Wick type, invariantly specified by the characteristic class $\text{cl}(\ast) = \pi^{-1}$, to an inequivalent star-product $\ast'$ with $\text{cl}(\ast') = \pi^{-1} + \hbar [\rho]$. For more details on the equivalence problem of the Wick type star-products see Refs. [33], [36], [37]. It is straightforward to check that the shift (6.18) completes the r.h.s. of (6.15) to the $\delta$-coboundary (6.10). As a result we can write the first-order correction to the classical BRST charge as

$$
\Omega_1 = \frac{1}{2} \text{str} (J\Lambda) + \delta \psi = \frac{1}{2} \Delta \Omega_0 + \delta \psi, \quad \forall \psi \in C^\infty(M),
$$

$\Delta = g^{ij} \nabla_i \nabla_j$ being the Laplace operator associated to the Kähler metric.

We are lead to conclude that, choosing an appropriate Wick $\ast$-product, one can always find a formal solution to the quantum master equation at least in the first order in $\hbar$. This holds true whenever one deals with finite dimensional manifolds. In the field theory, however, the Laplace operator is known to be an ill-defined object: acting on a local functional it gives singular expressions in perfect analogy to the previous case of the odd Laplacian. Of course, one can try to remove these singularities by means of an appropriate regularization but this can (and very so often does) destroy the exactness of the cocycle (6.10). Thus, in the infinite dimensional setting this cocycle may happen to be non-trivial even though one can write it as the formal coboundary (6.10). The conformal anomaly in the string theory [37], [38] is a typical example of this kind.

### 6.3. Characteristic classes of foliations.

Given a regular foliation $F$ on an ordinary manifold $N$, denote by $E \subset TN$ the subbundle given by the union of tangent spaces to leaves of the foliation $F$ (so-called, tangent bundle of a foliation). Since $E$ is integrable, the inclusion map $A : E \rightarrow TM$ defines the Lie algebroid $E$ over $N$. The regularity of $F$ implies the anchor $A$ to be injective. Thus, there is the one-to-one correspondence between the categories of regular foliations and injective Lie algebroids. To any Lie algebroid, in turn, we can associate the gauge system (see Example 2.1.3) and define the corresponding characteristic classes. When the Lie algebroid comes from a regular foliation, these characteristic classes can be attributed to the foliation itself.

In particular, the modular class of the foliation $F$ is determined by the cocycle

$$
C_1 = (\sigma^{-1} \nabla_i (\sigma A^i_\alpha) + T^\beta_\alpha) \bar{C}^\alpha, 
$$

where $\sigma$ is a density on $N$ and $\nabla$ is a connection on $E \rightarrow N$ (cf. (6.8)).

---

10 A good analogy is the integrability problem for a closed 1-form $\theta$ on a multi-connected manifold $M$: there always exist a potential $f$, such that $df = \omega$, if one admits multi-valued (= ill-defined) functions on $M$. 
The nontriviality of the modular class can be illustrated by the following example. Choose a discrete subgroup \( \Gamma \subset SL(2, \mathbb{R}) \) in such a way that the right quotient \( N = SL(2, \mathbb{R})/\Gamma \) to be a compact manifold. (It is well known that such subgroups do exist.) Let \( \{e_i\} \) be Weyl’s basis in the space of right invariant vector fields on \( SL(2, \mathbb{R}) \):

\[
\begin{align*}
[e_{-1}, e_1] & = 2e_0, & [e_0, e_1] & = e_1, & [e_0, e_{-1}] & = -e_{-1}.
\end{align*}
\]

The canonical projection \( \varphi : SL(2, \mathbb{R}) \to SL(2, \mathbb{R})/\Gamma = N \) allows one to descend the vector fields \( e_i \) to \( N \), so that the resulting vector fields \( e'_i \) obey the same commutation relations. Explicitly,

\[
d\varphi e_i(g) = e'_i(\varphi(g)), \quad \forall g \in SL(2, \mathbb{R}).
\]

(The vector fields \( e_i \) and \( e'_i \) are said to be \( \varphi \)-related.) The pair \( (e'_0, e'_1) \) generates the action of the Borel subgroup \( B \subset SL(2, \mathbb{R}) \) on \( N \) and hence defines a two dimensional foliation \( \mathcal{F} \). The corresponding Lie algebroid structure is assigned to the trivial vector bundle \( B \times N \to N \).

Notice that the dual to the 3-vector \( e'_{-1} \wedge e'_0 \wedge e'_1 \) is the \( SL(2, \mathbb{R}) \)-invariant volume form \( \sigma \) on \( N \). Therefore, \( \text{div}_\sigma e'_i = 0 \). Choosing a flat connection on \( B \times N \to N \) and substituting the structure constants into \( e'_0f = 1 \). But every function on a compact manifold has at least two critical points at which \( e'_0f = 0 \). Thus the modular class of the foliation at hands is nontrivial.

7. Conclusion

Let discuss some of the paper results and related issues.

In this paper we use a broad concept of the gauge systems going beyond the traditional physical applications and covering a diversity of other geometric structures like Lie and Courant algebroids, etc. Making use of this universal framework, we study the problem of constructing topological invariants which can be uniformly attributed to every gauge system, characterizing the global geometric properties of the system. As any gauge system is described in this framework by a nilpotent operator, called a homological vector field, the most simple and natural invariants to look for are those belonging to the respective cohomology and constructed from the derivatives of the homological vector field itself. We refer to these invariants as the characteristic classes having in mind the analogous constructions for the vector bundles and foliations and, as we have shown, this is more then just an analogy (see Sec. 5.3). A sequence of characteristic classes, called the principal series, has been explicitly constructed involving the first derivatives of the homological vector field.

The universal concept of the gauge system offers several advantages over the conventional approaches which are specific for every particular type of the gauge models. Even in the well studied case of Lie algebroids this concept seems useful, providing concise and geometrically transparent depiction for the respective structures, their morphisms and cohomological invariants. Besides, it provides a simple description for certain cohomological operations we have found in this paper. Namely, the space of characteristic classes is endowed with a Lie bracket operation which is not induced by the original Poisson bracket (the latter is proved to be trivial when applied to characteristic classes). By making use of this Lie bracket one can derive new characteristic classes depending on higher derivatives of the homological vector field. A general method still remains to be worked out for constructing higher derivative invariants which are not deducible from the principal characteristic classes by means of these cohomological operations. The first derivative invariants appear to be sufficient, however, for many important physical applications like the computation of one-loop anomalies in the quantum field theory.

As is seen, many known types of the gauge systems admit this uniform homological description, though the examples we listed in the paper by no means exhaust all the diversity of the geometric structures which can be treated in this framework. In this homological language, the distinctions between different types of gauge systems are encoded in the ghost number and parity grading in the target space. By scrutinizing all the admissible combinations of ghost number and parity, one can perhaps reveal some new meaningful geometrical constructions emerging from the master equation.

Finally, let us note that the classical gauge systems can be in themselves described in terms of the homological vector field alone, with no reference to the homological potential and the Poisson bracket; and the same holds true for the principal series of characteristic classes. Where the Poisson structure becomes relevant it is in the problem of deforming the classical gauge systems. In the case of the even Poisson structure, the powerful tool for that is provided by the deformation quantization technique which can be applied for this purpose just by replacing the Poisson square of the homological potential with a respective \( * \)-square. In the case of the odd Poisson structure, one can use the BV quantum master equation which is also recognized as an efficient tool for deforming algebraic structures. Although the both methods first appeared as quantization schemes for gauge systems, they can help to construct the deformations which are not necessarily quantum by their nature.
APPENDIX A. TENSOR CALCULUS ON SUPERMANIFOLDS

We list here the notation and conventions we use in the paper, which mostly coincide with those adopted in [40]. Given a smooth supermanifold $M$, denote by $\mathcal{C}^\infty(M)$ the supercommutative algebra of smooth functions on $M$. We fix the basic field to be $\mathbb{R}$, though all the formulae are still valid for an arbitrary algebraic field of characteristic 0.

A.1. Tensor fields on supermanifolds. Recall that an endomorphism $D : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is said to be a differentiation of the Grassman parity $\epsilon(D)$ if

$$D(fg) = D(f)g + (-1)^{\epsilon(D)\epsilon(f)}fD(g).$$  

(A.1)

Similar to the case of ordinary manifolds, the vector fields on $M$ can be identified with elements of the Lie superalgebra $\text{Der}(M)$ of all the differentiations of $\mathcal{C}^\infty(M)$. With this geometrical interpretation the $\mathcal{C}^\infty(M)$-module $\text{Der}(M)$ will be denoted $\text{Vect}(M)$. In each coordinate chart $(U, x^i)$, the restriction $\text{Vect}(U) = \text{Vect}|_U(M)$ is a free $\mathcal{C}^\infty(M)$-module with the basis $\partial_i$, so that

$$X(f) = f^i\partial_i f,$$

$$(A.2)$$

$f, X^i \in \mathcal{C}^\infty(U)$. Hereafter $\partial_i = \partial/\partial x^i$ stands for the left derivative on $\mathcal{C}^\infty(U)$. On each nonempty intersection $U \cap U'$ of two coordinate charts the basis vector fields are related to each other by the rule

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial x'^j}{\partial x^i}\right) \frac{\partial}{\partial x'^j}.$$  

(A.3)

The superspace $\text{Vect}(M) = \text{Vect}_0(M) \oplus \text{Vect}_1(M)$ is decomposed in the direct sum of two subspaces constituted, respectively, by even and odd vector fields:

$$\text{Vect}_0(M) \ni X \iff \epsilon(X) = \epsilon(x^i) + \epsilon_i = 0,$$

$$(A.4)$$

$$\text{Vect}_1(M) \ni Y \iff \epsilon(Y) = \epsilon(x^i) + \epsilon_i = 1,$$

$$\forall i = 1, \ldots, \dim M.$$

Hereafter we put $\epsilon_i = \epsilon(x^i)$. The supercommutator of two vector fields is given by

$$[X, Y] = (x^j\partial_j Y^i - (-1)^{\epsilon(x)\epsilon(y)}Y^j\partial_j X^i)\partial_i.$$  

(A.5)

The space of covector fields $\text{Covect}(M)$ is defined as the $\mathcal{C}^\infty(M)$-module $\text{Hom}(\text{Vect}(M), \mathcal{C}^\infty(M))$. Introduce an even $\mathbb{R}$-linear operator $\delta : \mathcal{C}^\infty(M) \to \text{Covect}(M)$, called differential, as

$$\delta f(X) = X(f), \quad \forall f \in \mathcal{C}^\infty(M), \quad \forall X \in \text{Vect}(M).$$  

(A.6)

By definition, the operator $\delta$ is a differentiation from $\mathcal{C}^\infty(M)$ to $\text{Covect}(M)$, i.e. $\delta(fg) = \delta(f)g + f\delta(g)$. In each coordinate chart $(U, x^i)$ the covector fields $\delta x^i$ form a basis in the $\mathcal{C}^\infty(U)$-module $\text{Covect}(U)$, dual from the right to the basis $\partial_i$. The contraction (or pairing) of a covector field $A = \delta x^iA_i(x)$ with a vector field $X = X^i(x)\partial_i$ is the superfunction

$$\langle X, A \rangle = A(X) = X^iA_i.$$  

(A.7)

Tensoring $\mathcal{C}^\infty(M)$-modules $\text{Vect}(M)$ and $\text{Covect}(M)$, we obtain the tensor algebra

$$\mathcal{T}(M) = \bigoplus_{\epsilon, m, n} \mathcal{T}^{(n,m)}(M), \quad \epsilon = 0, 1, \quad m, n = 0, 1, 2, \ldots,$$

where $\mathcal{T}^{(0,0)}(M) = \mathcal{C}^\infty(M)$, and the general element of $\mathcal{T}^{(n,m)}(M)$ reads

$$S = \delta x^{j_1} \otimes \cdots \otimes \delta x^{j_m} S^{i_1 \cdots i_n}_{j_1 \cdots j_m}(x) \partial_{i_1} \otimes \cdots \otimes \partial_{i_n},$$  

$$\epsilon = \epsilon(S) = \epsilon(S^{i_1 \cdots i_n}_{j_1 \cdots j_m}) + \sum_{k=1}^{n} \epsilon_i \epsilon_{i_k} + \sum_{l=1}^{m} \epsilon_{j_l},$$  

(A.8)

The tensor product $S \otimes T$ is defined using the following conventions:

$$\delta x^i f = (-1)^{\epsilon(f)\epsilon_i} f \delta x^i, \quad \delta x^i \otimes \partial_j = (-1)^{\epsilon_i\epsilon_j} \partial_j \otimes \delta x^i, \quad f \partial_i = (-1)^{\epsilon(f)\epsilon_i} \partial_i \cdot f, \quad \forall f \in \mathcal{C}^\infty(M).$$  

(A.9)

According to the definition, $\epsilon(S \otimes T) = \epsilon(S) + \epsilon(T)$.

Identifying the space $\mathcal{T}^{(1,1)}(M)$ with the space of endomorphisms of $\mathcal{C}^\infty(M)$-module $(\text{Covect}(M))$, we turn the former to a $\mathbb{Z}_2$-graded unital algebra over $\mathcal{C}^\infty(M)$. The product of two elements $S, T \in \mathcal{T}^{(1,1)}(M)$ is the $(1,1)$-tensor $ST$ whose components are

$$(ST)_j^i = S^j_k T^i_k.$$  

(A.10)
Clearly, $\epsilon(ST) = \epsilon(S) + \epsilon(T)$. The contraction of vector and covector fields gives rise to a linear homomorphism $\text{str} : \mathcal{T}^{(1,1)}(M) \to C^\infty(M)$:

$$\text{str} \, (S) = \sum (-1)^{\epsilon(S) + 1} \epsilon_i S^i.$$  

This homomorphism, called supertrace, is characterized by the property of vanishing on supercommutators, i.e.,

$$\text{str} \, ([S, T]) = \text{str} \, (ST) - (-1)^{\epsilon(S)\epsilon(T)} \text{str} \, (TS) = 0, \quad \forall S, T \in \mathcal{T}^{(1,1)}(M).$$

### A.2. The Lie derivative

The space $\mathcal{T}(M)$ can be endowed with a module structure over the Lie superalgebra $\text{Vect}(M)$. The corresponding homomorphism $\mathcal{L} : \text{Vect}(M) \to \text{End}(\mathcal{T}(M))$ is known as the Lie derivative. For any $f \in C^\infty(M)$ and $X, Y \in \text{Vect}(M)$ we set

$$\mathcal{L}_X f = X(f) = (\delta f)(X), \quad \mathcal{L}_X Y = [X, Y],$$

where $\mathcal{L}_X = \mathcal{L}(X)$. Requiring the operator $\mathcal{L}_X$ to be compatible with the pairing, i.e.,

$$\mathcal{L}_X \langle Y, A \rangle = \langle \mathcal{L}_X Y, A \rangle + (-1)^{\epsilon(X)\epsilon(Y)} \langle Y, \mathcal{L}_X A \rangle,$$

and to satisfy the Leibnitz rule

$$\mathcal{L}_X (S \otimes T) = \mathcal{L}_X S \otimes T + (-1)^{\epsilon(X)\epsilon(S)} S \otimes \mathcal{L}_X T,$$

one extends the action of $\mathcal{L}_X$ to the whole tensor algebra $\mathcal{T}(M)$. For example, the Lie derivative of a $(1,1)$-tensor $S = (S^i_j)$ is the $(1,1)$-tensor $\mathcal{L}_X S$ with the components

$$\langle \mathcal{L}_X S \rangle^i_j = (-1)^{\epsilon(X)\epsilon_j} X^k \partial_k S^i_j + \partial_j X^k S^i_k - (-1)^{\epsilon(X)\epsilon(S)} S^k_j \partial_k X^i.$$

It follows from the definition that

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X \mathcal{L}_Y - (-1)^{\epsilon(X)\epsilon(Y)} \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X, Y]}.$$

### A.3. The affine connection

This can be defined as an even map $\nabla : \text{Vect}(M) \otimes \text{Vect}(M) \to \text{Vect}(M)$, which is $C^\infty(M)$-linear in the first argument, additive in the second argument, and such that

$$\nabla_X (fY) = X(f)Y + (-1)^{\epsilon(X)\epsilon(Y)} f \nabla_X Y,$$

for any $f \in C^\infty(M)$, $X, Y \in \text{Vect}(M)$. Here $\nabla_X (Y) = \nabla(X, Y)$. The operator $\nabla_X$ is called the covariant derivative along the vector field $X$. Setting $\nabla_X f = X(f)$, postulating the compatibility of $\nabla_X$ with the pairing,

$$\nabla_X \langle Y, A \rangle = \langle \nabla_X Y, A \rangle + (-1)^{\epsilon(X)\epsilon(Y)} \langle Y, \nabla_X A \rangle$$

as well as the Leibnitz rule

$$\nabla_X (S \otimes T) = \nabla_X S \otimes T + (-1)^{\epsilon(X)\epsilon(S)} S \otimes \nabla_X T,$$

one extends the action of $\nabla_X$ from $\text{Vect}(M)$ to the whole tensor algebra $\mathcal{T}(M)$. In particular, for a $(1,1)$-tensor $S$ we have

$$\langle \nabla_X S \rangle^i_j = (-1)^{\epsilon(X)\epsilon_j} X^k \partial_k S^i_j - (-1)^{\epsilon(X)\epsilon_j} X^k \Gamma^i_{kj} S^j_k + (-1)^{\epsilon(X)\epsilon(S)} S^i_k \partial_k X^i,$$

where $\Gamma^k_{ij}(x) = \delta x^k(\nabla(\partial_i, \partial_j))$ are Christoffel’s symbols. An affine connection $\nabla$ is said to be symmetric if

$$\nabla_X Y - (-1)^{\epsilon(X)\epsilon(Y)} \nabla_Y X = [X, Y].$$

In terms of local coordinates this means that

$$\Gamma^k_{ij}(x) = (-1)^{\epsilon_i \epsilon_j} \Gamma^k_{ij}(x).$$

The curvature of an affine connection is given by the supercommutator of the corresponding covariant derivatives. For a symmetric connection $\nabla$ we have

$$\langle \nabla_i \nabla_j \rangle(\partial_k) = \nabla_i \nabla_j \partial_k - (-1)^{\epsilon_i \epsilon_j} \nabla_j \nabla_i \partial_k = R^l_{ijk}(x) \partial_l,$$

where $\nabla_i = \nabla_{\partial_i}$ and the components of the curvature tensor read

$$R^l_{ijk} = \partial_l \Gamma^j_{ik} - (-1)^{\epsilon_i \epsilon_j} \partial_j \Gamma^l_{ik} + (-1)^{\epsilon_i + \epsilon_k + \epsilon_m} \epsilon_i \epsilon_j \epsilon_m \Gamma^m_{jk} \Gamma^l_{im} - (-1)^{\epsilon_i + \epsilon_m} \epsilon_i \epsilon_m \Gamma^m_{ik} \Gamma^l_{jm}.$$
In terms of local coordinates it is given by
\[ n \Omega \] (A.28) \[ \]

The following identities hold true:

\[ L \]

Clearly, \( \epsilon \) generated by elements of \( \Omega^1 \)

\[ (A.27) \]

According to the definition above (cf. (A.9))

\[ (A.26) \]

The parity reversion operation. Let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}_2 \)-graded linear space (superspace) such that \( \epsilon(v) = 0 \) and \( \epsilon(u) = 1 \) for any \( v \in V_0 \) and \( u \in V_1 \). Reversing parities of all the homogeneous elements of \( V \) we get the new superspace \( \Pi(V) \) with \( \epsilon(v) = 1 \) and \( \epsilon(u) = 0 \). Clearly, \( \Pi(\Pi(V)) = V \). The homomorphism \( \Pi : V \rightarrow \Pi(V) \) is called the parity reversion. Let \( T(V) = \bigoplus V^\otimes n \) denote the tensor algebra of the superspace \( V \). The symmetric algebra \( S(V) \) of \( V \) is defined as the supercommutative algebra generated by elements of \( \Omega^1 \) and the subspace \( \Omega^n \) of \( n \)-forms, \( n \geq 1 \), is multiplicatively generated by elements of \( \Omega^1 \). For any \( f \in C^\infty(M) \) denote by \( df \) the differential 1-form \( \Pi(df) \in \Omega^1 \). Clearly, \( \epsilon(df) = \epsilon(f) + 1 \). In each coordinate chart \( (U, x^i) \) the forms \( dx^i \) constitute a basis in the \( C^\infty(U) \)-module \( \Omega^1(U) \), so that any \( n \)-form \( \omega \in \Omega^n(U) \) can be written as

\[ \omega = dx^{i_1} \cdots dx^{i_n} \omega_{i_1 \cdots i_n}(x), \quad \omega_{i_1 \cdots i_n} \in C^\infty(U). \]

According to the definition above (cf. (A.9))

\[ \]

The exterior derivative is the unique differentiation \( d : \Omega^n(M) \rightarrow \Omega^{n+1}(M) \) possessing the properties:

1. \( \epsilon(d) = 1 \);
2. if \( f \in C^\infty(M) \), then \( df = \Pi(df) \);
3. \( d^2 = 0 \).

In terms of local coordinates it is given by \( d = dx^i \partial_i \).

The interior multiplication of a form by a vector field \( X \in \text{Vect}(M) \) is the unique differentiation \( i_X : \Omega^n(M) \rightarrow \Omega^{n-1}(M) \) obeying conditions:

1. \( \epsilon(i_X) = \epsilon(X) + 1 \);
2. \( i_X f = 0 \), for \( f \in C^\infty(M) \);
3. \( i_X(df) = (-1)^\epsilon(X)f \).

The following identities hold true:

\[ [i_X, i_Y] = 0, \quad [d, i_X] = \mathcal{L}_X, \quad [\mathcal{L}_X, i_Y] = (-1)^\epsilon(X)\epsilon(Y)i_{[X,Y]}, \]

where \( \mathcal{L}_X \) is the Lie derivative along the vector field \( X \in \text{Vect}(M) \).
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