FRAME OF OPERATORS IN QUATERNIONIC HILBERT SPACES

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Abstract. In this paper, we introduce and study frame of operators in quaternionic Hilbert spaces as a generalization of $g$-frames which in turn generalized various notions like Pseudo frames, bounded quasi-projectors and frame of subspaces (fusion frames) in separable quaternionic Hilbert spaces.

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [12]:

“A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is said to be a frame for a Hilbert space $\mathcal{H}$ if there exist positive constants $r_1$ and $r_2$ such that

$$r_1 \|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq r_2 \|x\|^2, \text{ for all } x \in \mathcal{H}.$$ (1)

The positive real constants $r_1$ and $r_2$, respectively, are called lower and upper frame bounds for the frame $\{x_n\}_{n\in\mathbb{N}}$. The inequality (1) is called the frame inequality for the frame $\{x_n\}_{n\in\mathbb{N}}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is a tight frame if it is there exist $r_1, r_2$ satisfying inequality (1) with $r_1 = r_2$ and a Parseval frame if it is a tight frame with $r_1 = r_2 = 1$. For more details related to frames, one may refer to [7, 10].

Some important classes of sequences that are closely related to frames are the Bessel sequences and Riesz bases. Bessel sequence are the sequences which only satisfies upper inequality of (1). These sequences are in general need not be bases but posses stable reconstruction. On the other-hand, Riesz basis being a bounded image of an orthonormal basis in $\mathcal{H}$ is always a frame for $\mathcal{H}$.

After being observed by Daubechies, Grossmann and Meyer [11] in 1986 that, frames provides a stable reconstruction of functions in $L^2(\mathbb{R})$, frame theory became popular among researchers. Being redundant in nature, frame representation has many more benefits over a basis representation in $L^2(\mathbb{R})$, namely signal and image processing [4], filter bank theory [6], wireless communications [15] and sigma - delta quantization [5]. Due to this prospective, one can observe frames as some kind of extension for orthonormal bases in Hilbert spaces.

Keeping more applications in mind, various generalizations of frames have been introduced and studied namely:

- **Fusion frames** by Casazza and Kutyniok [8]

  **Definition 1.** Let $\{W_i\}_{i\in\mathbb{N}}$ be a sequence of closed subspaces in $\mathcal{H}$ and $\{v_i\}_{i\in\mathbb{N}}$ be a family of weights, i.e., $v_i > 0$, for all $i \in \mathbb{N}$. Then $\{(W_i, v_i)\}_{i\in\mathbb{N}}$ is called a frame of subspaces (fusion frame), if there exist constants $0 < C \leq D < \infty$ such that

  $$C \|x\|^2 \leq \sum_{i \in \mathbb{N}} v_i^2 \|\pi_{W_i}(x)\|^2 \leq D \|x\|^2, \text{ for all } x \in \mathcal{H},$$

  where $\pi_{W_i}$ is an orthogonal projection onto the subspace $W_i$. The constants $C$ and $D$, respectively, are called lower and upper frame bounds for the fusion frame.

- **Pseudo-frames** by Li and Ogawa [17]

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Definition 2. Let $\mathcal{X}$ be a closed subspace of a separable Hilbert space $\mathcal{H}$. Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{x_n^*\}_{n \in \mathbb{N}}$ be sequences in $\mathcal{H}$. The sequence $\{x_n\}_{n \in \mathbb{N}}$ is a pseudo frame for the subspace $\mathcal{X}$ with respect to $\{x_n^*\}_{n \in \mathbb{N}}$ if
\[
x = \sum_{n=1}^{\infty} (x, x_n) x_n^*, \quad \text{for all } x \in \mathcal{X}.
\]
The sequence $\{x_n^*\}_{n \in \mathbb{N}}$ is called a dual pseudo frame to $\{x_n\}_{n \in \mathbb{N}}$ for the subspace $\mathcal{X}$.

- Bounded quasi-projectors by Fornasier [13]

Definition 3. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{W}_0$ be a closed subspace of $\mathcal{H}$. Let $(\mathcal{W}_0, \{D_j : \mathcal{H} \to \mathcal{H}\}_{j \in \mathbb{N}})$ be a decomposition of $\mathcal{H}$. Then a system of bounded quasi-projectors or a Bessel resolution of the identity is a set $\mathcal{P} = \{P_j\}_{j \in \mathbb{N}}$ of operators such that
  
  (I) for each $j \in \mathbb{N}$, $P_j : \mathcal{H} \to \mathcal{W}_j (= D_j(\mathcal{W}_0))$,
  (II) $\sum_{j \in \mathbb{N}} P_j = I_H$, in the strong operator topology and
  (III) there exists positive constant $B$ such that
  \[
  \sum_{j \in \mathbb{N}} \|P_j(x)\|^2 \leq B \|x\|^2, \quad \text{for all } x \in \mathcal{H}.
  \]
The system is called self-adjoint and compatible with the canonical projections if
  
  (I) $P_j = P_j^*$, for all $j \in \mathbb{N}$.
  (II) $P_j \circ \pi_{\mathcal{W}_j} = P_j$, for all $j \in \mathbb{N}$.

- $G$-frames by Sun [20]

Definition 4. Let $\mathcal{H}$ be a Hilbert space. A sequence $\{A_j \in L(\mathcal{H}, \mathcal{V}_j) : j \in \mathbb{N}\}$ is said to be a generalized frame, or simply a $g$-frame for $\mathcal{H}$ with respect $\{\mathcal{V}_j : j \in \mathbb{N}\}$ if there exist positive constants $A$ and $B$ such that
\[
A \|x\|^2 \leq \sum_{j \in \mathbb{N}} \|A_j x\|^2 \leq B \|x\|^2, \quad \text{for all } x \in \mathcal{H},
\]
where $\{\mathcal{V}_j : j \in \mathbb{N}\}$ is a sequence of subspaces of $\mathcal{H}$.

Sun also proved that frames of subspaces (fusion frames), pseudo frames, bounded quasi-projectors and oblique frames are special cases of $g$-frames. Fusion frames and $g$-frames are also studied in [1, 3, 16, 20].

Throughout this paper, we will denote $\Omega$ to be a non-commutative field of quaternions, $\mathbb{N}$ be the set of natural numbers, $\mathbb{H}^R(\Omega)$ be a separable right quaternionic Hilbert space, $\{\mathbb{H}^R(\Omega)\}_{j \in \mathbb{N}}$ be the sequence of separable right quaternionic Hilbert spaces, by the term “right linear operator”, we mean a “right $\Omega$-linear operator”, $\mathfrak{B}(\mathbb{H}^R(\Omega))$ denotes the set of all bounded (right $\Omega$-linear) operators of $\mathbb{H}^R(\Omega)$ and $\mathfrak{B}(\overline{\mathbb{H}^R(\Omega)}, \mathbb{H}^R(\Omega))$ be the set of all bounded (right $\Omega$-linear) operators from $\mathbb{H}^R(\Omega)$ to $\overline{\mathbb{H}^R(\Omega)}$.

The non-commutative field of quaternions $\Omega$ is a four dimensional real algebra with unity. In $\Omega$, 0 denotes the null element and 1 denotes the identity with respect to multiplication. It also includes three so-called imaginary units, denoted by $i, j, k$. i.e.,
\[
\Omega = \{r_0 + r_1i + r_2j + r_3k : r_0, r_1, r_2, r_3 \in \mathbb{R}\}
\]
where $i^2 = j^2 = k^2 = -1; \; ij = -ji = k; \; jk = -kj = i$ and $ki = -ik = j$. For each quaternion $q = r_0 + r_1 i + r_2 j + r_3 k \in \Omega$, define conjugate of $q$ denoted by $\overline{q}$ as $\overline{q} = r_0 - r_1 i - r_2 j - r_3 k \in \Omega$. If $q = r_0 + r_1 i + r_2 j + r_3 k$ is a quaternion, then $r_0$ is called the real part of $q$ and $r_1 i + r_2 j + r_3 k$ is called the imaginary part of $q$. The modulus of a quaternion $q = r_0 + r_1 i + r_2 j + r_3 k$ is defined as
\[
|q| = (\overline{q}q)^{1/2} = (q\overline{q})^{1/2} = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}.
\]
For more literature related to quaternionic Hilbert spaces one may refer to [2, 14].
Frames in separable right quaternionic Hilbert spaces $\mathbb{H}^R(\Omega)$ are defined as:

**Definition 5 ([13])**. Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space and $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $\mathbb{H}^R(\Omega)$. Then $\{u_i\}_{i \in \mathbb{N}}$ is said to be a frame for $V_R(\Omega)$, if there exist two finite real constants with $0 < r_1 \leq r_2$ such that

$$r_1\|u\|^2 \leq \sum_{i \in \mathbb{N}} |\langle u_i | u \rangle|^2 \leq r_2\|u\|^2, \text{ for all } u \in V_R(\Omega). \tag{2}$$

The positive constants $r_1$ and $r_2$, respectively, are called lower frame and upper frame bounds for the frame $\{u_i\}_{i \in \mathbb{N}}$. The inequality (2) is called frame inequality for the frame $\{u_i\}_{i \in \mathbb{N}}$. A sequence $\{u_i\}_{i \in \mathbb{N}}$ is called a Bessel sequence for the right quaternionic Hilbert space $\mathbb{H}^R(\Omega)$ with bound $r_2$, if $\{u_i\}_{i \in \mathbb{N}}$ satisfies the right hand side of the inequality (2). A sequence $\{u_i\}_{i \in \mathbb{N}}$ is a tight frame for right quaternionic Hilbert space $V_R(\Omega)$ if there exist positive $r_1$, $r_2$ satisfying inequality (2) with $r_1 = r_2$, Parseval frame if it is tight with $r_1 = r_2 = 1$ and exact if it ceases to be a frame in case any one of its element is removed.

If $\{u_i\}_{i \in \mathbb{N}}$ is a frame for $V_R(\Omega)$. Then, the right linear operator $T : \ell_2(\Omega) \to V_R(\Omega)$ defined by

$$T(\{q_i\}_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} u_i q_i, \quad \{q_i\} \in \ell_2(\Omega)$$

is called the (right) synthesis operator and the adjoint operator $T^*$ is called the (right) analysis operator given by

$$T^*(u) = \{\langle u_i | u \rangle\}_{i \in \mathbb{N}}, \quad u \in V_R(\Omega).$$

Also, the (right) frame operator $S : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)$ for the frame $\{u_i\}_{i \in I}$ is a right linear operator given by

$$S(u) = TT^*(u) = T(\{\langle u_i | u \rangle\}_{i \in I}) = \sum_{i \in I} u_i \langle u_i | u \rangle, \quad u \in \mathbb{H}^R(\Omega).$$

**Various Generalizations of Frames in Quaternionic Hilbert Spaces**

In this section, we define various generalizations of frames in a right quaternionic Hilbert space. We begin with the notion of frame of subspaces in a right quaternionic Hilbert space:

**Definition 6.** Let $\{W_i^R\}_{i \in \mathbb{N}}$ be a sequence of right closed subspaces of a right separable quaternionic Hilbert space $\mathbb{H}^R(\Omega)$ and $\{v_i\}_{i \in \mathbb{N}}$ be a family of weights, i.e., $v_i > 0$, for all $i \in \mathbb{N}$. Then $\{(W_i^R, v_i)\}_{i \in \mathbb{N}}$ is called a frame of subspaces (fusion frame), if there exist constants $0 < C \leq D < \infty$ such that

$$C\|x\|^2 \leq \sum_{i \in \mathbb{N}} v_i^2 \|\pi_{W_i^R}(x)\|^2 \leq D\|x\|^2, \text{ for all } x \in \mathbb{H}^R(\Omega),$$

where each $\pi_{W_i}$ is an orthogonal projection onto the subspace $W_i$. The constants $C$ and $D$, respectively, are called lower and upper frame bounds for the fusion frame.

In view of Definition 4 we have the following example:

**Example 7.** Let $\mathbb{H}^R(\Omega)$ be a right separable quaternionic Hilbert space and $\{z_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathbb{H}^R(\Omega)$. Define $W_1^R = [z_1]$, $W_i^R = [z_{i-1}]$, $i \geq 2$, and $v_i = 1$, $i \in \mathbb{N}$. Then $\{(W_i^R, v_i)\}_{i \in \mathbb{N}}$ is a frame for $\mathbb{H}^R(\Omega)$ with lower and upper fusion frame bounds $r_1 = 1$ and $r_2 = 2$, respectively.

Next, we define the notion of pseudo-frames using two Bessel sequences in a right quaternionic Hilbert space.
**Definition 8.** Let $X^R$ be a right closed subspace of a right separable quaternionic Hilbert space $\mathbb{H}^R(\Omega)$. Let $\{x_i\}_{i \in \mathbb{N}}$ and $\{x^*_i\}_{i \in \mathbb{N}}$ be sequences in $\mathbb{H}^R(\Omega)$. We say that $\{x_i\}_{i \in \mathbb{N}}$ is a *pseudo frame* for the subspace $X^R$ with respect to $\{x^*_i\}_{i \in \mathbb{N}}$ if
\[
x = \sum_{n=1}^{\infty} x^*_n \langle x_n | x \rangle,
\]
for all $x \in X^R$.

The sequence $\{x^*_n\}_{n \in \mathbb{N}}$ is called a *dual pseudo frame* to $\{x_n\}_{n \in \mathbb{N}}$ for the subspace $X^R$.

In view of Definition (8) we have a following example:

**Example 9.** Let $\mathbb{H}^R(\Omega)$ be a right separable quaternionic Hilbert space and $\{z_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathbb{H}^R(\Omega)$. For each $i \in \mathbb{N}$, define $x_i = z_i$, $x^*_i = z_{2i-1}$, and $X^R \subseteq \{x^*_i\}_{i \in \mathbb{N}}$. Then $\{x_i\}_{i \in \mathbb{N}}$ is a *pseudo frame* for the subspace $X^R$ with respect to $\{x^*_i\}_{i \in \mathbb{N}}$.

The following definition is an extension of bounded quasi-projectors in right quaternionic Hilbert spaces.

**Definition 10.** Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $W_0^R$ be a right closed subspace of $\mathbb{H}^R(\Omega)$. Let $(W_0^R, \{D_j : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\})_{j \in \mathbb{N}}$ be a right decomposition of $\mathbb{H}^R(\Omega)$. Then a system of bounded quasi-projectors or a Bessel resolution of the identity is a set $\mathcal{P} = \{P_j\}_{j \in \mathbb{N}}$ of operators satisfying

(I) for each $j \in \mathbb{N}$, $P_j : \mathbb{H}^R(\Omega) \to W_j (= D_j(W_0^R))$,

(II) $\sum_{j \in \mathbb{N}} P_j = I_{\mathbb{H}^R(\Omega)}$, in the strong operator topology and

(III) there exists a positive constant $r_2$ such that
\[
\sum_j \|P_j(x)\|^2 \leq r_2 \|x\|^2,
\]
for all $x \in \mathbb{H}^R(\Omega)$.

The system $\{P_j\}_{j \in \mathbb{N}}$ is called *self-adjoint* and *compatible* with the canonical projections if

(I) $P_j = P^*_j$, for all $j \in \mathbb{N}$.

(II) $P_j \circ \pi_{W_j} = P_j$, for all $j \in \mathbb{N}$.

In view of Definition (10) we have a following example:

**Example 11.** Let $\mathbb{H}^R(\Omega)$ be a right separable quaternionic Hilbert space and $\{z_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathbb{H}^R(\Omega)$. Define $W_0^R = \mathbb{H}^R(\Omega)$, $D_j : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)$ as
\[
D_j \left( x = \sum_{i \in \mathbb{N}} z_i q_i \right) = z_j q_j, x \in \mathbb{H}^R(\Omega), j \in \mathbb{N}.
\]
Then the system of operators $\mathcal{P} = \{P_j\}_{j \in \mathbb{N}}$ is a *bounded quasi-projectors* with $r_2 = 1$. Further, the system of operators $\mathcal{P} = \{P_j\}_{j \in \mathbb{N}}$ is self-adjoint and compatible with the canonical projections.

Next, we define frame of operators in a right quaternionic Hilbert space as follows:

**Definition 12.** Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\{\mathbb{H}^R_i(\Omega)\}_{i \in \mathbb{N}}$ be a sequence of right quaternionic Hilbert spaces, and $\{\Sigma_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R_i(\Omega)\}_{i \in \mathbb{N}}$ be a sequence of bounded right linear operators. Then $\{\Sigma_i\}_{i \in \mathbb{N}}$ is called a *frame of operators* for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}^R_i(\Omega)\}_{i \in \mathbb{N}}$ if there exist real constants $r_1$ and $r_2$ with $0 < r_1 \leq r_2 < \infty$ such that
\[
r_1 \|u\|^2 \leq \sum_{i=1}^{\infty} \|\Sigma_i(u)\|^2 \leq r_2 \|u\|^2,
\]
for all $u \in \mathbb{H}^R(\Omega)$.

The positive constants $r_1$ and $r_2$, respectively, are called *lower* and *upper* bounds for the frame of operators $\{\Sigma_i\}_{i \in \mathbb{N}}$. A sequence $\{\Sigma_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ is said to be a *Bessel sequence of operators* for $\{\mathbb{H}^R_i(\Omega)\}_{i \in \mathbb{N}}$ with respect to $\mathbb{H}^R(\Omega)$ if righthand side of inequality (3) is satisfied.

A frame of operators $\{\Sigma_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R_i(\Omega)\}_{i \in \mathbb{N}}$ for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}^R_i(\Omega)\}_{i \in \mathbb{N}}$ is said to be
• **tight** if it is possible to choose $r_1$, $r_2$ satisfying inequality (3) with $r_1 = r_2$.
• **Parseval** if it is possible to choose $r_1$, $r_2$ satisfying inequality (4) with $r_1 = r_2 = 1$.

We call $\{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to a right quaternionic Hilbert space $\mathbb{U}^R(\Omega)$ if $\mathbb{H}^R(\Omega) = \mathbb{U}^R(\Omega)$, $i \in \mathbb{N}$ and simply, frame of operators for a right quaternionic Hilbert space $\mathbb{H}^R(\Omega)$ if $\mathbb{H}^R_i(\Omega) = \mathbb{H}^R(\Omega)$, $i \in \mathbb{N}$.

Regarding the existence of frame of operators in a right quaternionic Hilbert space, we give the following examples:

**Example 13.** Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space and $N = \{z_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathbb{H}^R(\Omega)$. Define $\{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \Omega\}_{i \in \mathbb{N}}$ by

\[
\mathcal{T}_i(u) = \langle z_i | u \rangle, \quad \text{for all } u \in \mathbb{H}^R(\Omega), \quad i \in \mathbb{N}.
\]

Then $\{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \Omega\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}^R(\Omega)$ with respect $\Omega$. □

**Example 14.** Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space and $N = \{z_i\}_{i \in \mathbb{N}}$ be a Hilbert basis for $\mathbb{H}^R(\Omega)$. Define a sequence $\{u_i\}_{i \in \mathbb{N}}$ in $\mathbb{H}^R(\Omega)$ by

\[
\begin{align*}
    u_1 &= z_1, \\
    u_i &= z_{i-1}, \quad i \geq 2,
\end{align*}
\]
and $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ by

\[
\begin{align*}
    \mathbb{H}_1^R(\Omega) &= \{z_1\}, \\
    \mathbb{H}_i^R(\Omega) &= \{z_{i-1}\}, \quad i \geq 2.
\end{align*}
\]

Now, for each $i \in \mathbb{N}$, define $\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)$ by $\mathcal{T}_i(u) = u_i(u_i | u)$, for all $u \in \mathbb{H}^R(\Omega)$. Then $\{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ with bounds $r_1 = 1$ and $r_2 = 2$. □

**Remark 15.** Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space and $\{u_i\}_{i \in \mathbb{N}}$ be a frame for $\mathbb{H}^R(\Omega)$ such that $0 < \inf_{i \in \mathbb{N}} \|u_i\| \leq \sup_{i \in \mathbb{N}} \|u_i\| < \infty$. Then $\mathbb{H}^R(\Omega)$ has a frame of operators.

Above discussion, give rise to the following observations:

(I) Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\mathcal{X}^R$ be a right closed subspace of $\mathbb{H}^R(\Omega)$ and $\{x_n\}_{n \in \mathbb{N}}$ be a pseudo frame for the subspace $\mathcal{X}^R$ with respect to $\{x_n^*\}_{n \in \mathbb{N}}$. Then, there exist constants $0 < r_1 \leq r_2 < \infty$ such that

\[
    r_1 \|x\|^2 \leq \sum_{i=1}^{\infty} |\langle x_i | x \rangle|^2 \leq r_2 \|x\|^2, \quad \text{for all } x \in \mathbb{H}^R(\Omega).
\]

Define $\{\mathcal{T}_i : \mathcal{X} \to \Omega\}_{i \in \mathbb{N}}$ by $\mathcal{T}_i(x) = \langle x_i | x \rangle$, for all $x \in \mathcal{X}^R$ and $i \in \mathbb{N}$. Then, $\{\mathcal{T}_i : \mathcal{X} \to \Omega\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathcal{X}^R$ with respect $\Omega$.

(II) Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\mathcal{W}_0^R$ be a right closed subspace of $\mathbb{H}^R(\Omega)$. Let $\{\mathcal{W}_j^R, \{D_j : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{j \in \mathbb{N}}\}$ be a right decomposition of $\mathbb{H}^R(\Omega)$ and $\mathcal{P} = \{\mathcal{P}_j : \mathbb{H}^R(\Omega) \to \mathcal{W}_j^R(= D_j(\mathcal{W}_0^R))\}_{j \in \mathbb{N}}$ be a system of bounded quasi-projectors such that $\mathcal{P}_j = \mathcal{P}_j^\circ \mathcal{P}_j$ and $\mathcal{P}_j \circ \pi_{\mathcal{W}_j^R} = \mathcal{P}_j$ for all $j \in \mathbb{N}$. Then there exist constants $0 < r_1 \leq r_2 < \infty$ such that

\[
    r_1 \|x\|^2 \leq \sum_{j \in \mathbb{N}} \|\mathcal{P}_j(x)\|^2 \leq r_2 \|x\|^2, \quad x \in \mathbb{H}^R(\Omega).
\]

Define $\{\mathcal{T}_j : \mathbb{H}^R(\Omega) \to \Omega\}_{j \in \mathbb{N}}$ by $\mathcal{T}_j(x) = \mathcal{P}_j(x)$, for all $x \in \mathbb{H}^R(\Omega)$ and $j \in \mathbb{N}$. Then $\{\mathcal{T}_j : \mathbb{H}^R(\Omega) \to \Omega\}_{j \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}^R(\Omega)$ with respect $\Omega$.

(III) Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\{\mathcal{W}_i^R\}_{i \in \mathbb{N}}$ be a sequence of right closed subspaces, $\{v_i\}_{i \in \mathbb{N}}$ be a sequence of weights, and $\{\mathcal{W}_i^R\}_{i \in \mathbb{N}}$ be a frame of subspaces for $\mathbb{H}^R(\Omega)$
with respect to \( \{u_i\}_{i \in \mathbb{N}} \). Then there exist constants \( 0 < r_1 \leq r_2 < \infty \) such that
\[
    r_1 \|x\|^2 \leq \sum_{i=1}^{\infty} u_i^2 \|\pi_{\mathcal{W}_R}(x)\|^2 \leq r_2 \|x\|^2, \quad x \in R(H)
\]
Define \( \{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathcal{D}\}_{i \in \mathbb{N}} \) by \( \mathcal{T}_i(x) = u_i \pi_{\mathcal{W}_R}(x) \), for all \( x \in \mathbb{H}^R(\Omega) \) and \( i \in \mathbb{N} \). Then \( \{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathcal{D}\}_{i \in \mathbb{N}} \) is a frame of operators for \( \mathbb{H}^R(\Omega) \) with respect \( \mathbb{C} \).

Let \( \{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}} \) be a sequence of right quaternionic Hilbert spaces. Define the space
\[
    \mathcal{H} = \bigoplus_{i \in \mathbb{N}} \mathbb{H}_i^R(\Omega) = \left\{ u_i \in \mathbb{H}_i^R(\Omega) : \sum_{i=1}^{\infty} \|u_i\|^2_{\mathbb{H}_i^R(\Omega)} < \infty \right\}.
\]
Then \( \mathcal{H} \) is a right quaternionic Hilbert space with the norm given by
\[
    \|u\|_{\mathcal{H}} = \sum_{i=1}^{\infty} \|u_i\|^2_{\mathbb{H}_i^R(\Omega)}, \quad \text{for all } u \in \mathcal{H}.
\]

Next, we give a characterization for a Bessel sequence of operators.

**Theorem 16.** Let \( \mathbb{H}^R(\Omega) \) be a right quaternionic Hilbert space, \( \{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}} \) a sequence of right quaternionic Hilbert spaces, and \( \{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}} \) a sequence of bounded linear operators. Then \( \{\mathcal{T}_i\}_{i \in \mathbb{N}} \) is a Bessel sequence of operators with Bessel bound \( r_2 \) for \( \mathbb{H}^R(\Omega) \) with respect to \( \{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}} \) if and only if the operator \( \mathcal{G} : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega) \) defined by
\[
    \mathcal{G}(x) = \sum_{i=1}^{\infty} \mathcal{T}_i^* \mathcal{T}_i(x), \quad \text{for all } x \in \mathbb{H}^R(\Omega)
\]
is a well-defined bounded linear operator on \( \mathbb{H}^R(\Omega) \) with \( \|\mathcal{G}\| \leq B \).

**Proof.** Let \( x \in \mathbb{H}^R(\Omega) \). Then, for \( p, q \in \mathbb{N} \) with \( p > q \), we have
\[
    \left\| \sum_{i=1}^{p} \mathcal{T}_i^* \mathcal{T}_i(x) - \sum_{i=1}^{q} \mathcal{T}_i^* \mathcal{T}_i(x) \right\| = \sup_{\|y\|=1} \left| \left\langle y, \sum_{i=1}^{p} \mathcal{T}_i^* \mathcal{T}_i(x) - \sum_{i=1}^{q} \mathcal{T}_i^* \mathcal{T}_i(x) \right\rangle \right|
    \leq \sup_{\|y\|=1} \left[ \left( \sum_{i=1}^{p} \|\mathcal{T}_i(x)\|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{p} \|\mathcal{T}_i(y)\|^2 \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{q} \|\mathcal{T}_i(x)\|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{q} \|\mathcal{T}_i(y)\|^2 \right)^{\frac{1}{2}} \right]
    \leq \sqrt{B} \left[ \left( \sum_{i=1}^{p} \|\mathcal{T}_i(x)\|^2 \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{q} \|\mathcal{T}_i(x)\|^2 \right)^{\frac{1}{2}} \right].
\]
Thus \( \sum_{i=1}^{\infty} \mathcal{T}_i^* \mathcal{T}_i(x) \) exists in \( \mathbb{H}^R(\Omega) \). Therefore \( \mathcal{G} : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega) \) is a well-defined operator on \( \mathbb{H}^R(\Omega) \). Further, we compute
\[
    \|\mathcal{G}\| = \sup_{\|x\|=1} \left| \langle x, \mathcal{G}(x) \rangle \right| = \sup_{\|x\|=1} \sum_{i=1}^{\infty} \|\mathcal{T}_i(x)\|^2 \leq r_2.
\]

Conversely, for each \( x \in \mathbb{H}^R(\Omega) \), we have
\[
    \sum_{i=1}^{\infty} \|\mathcal{T}_i(x)\|^2 = |\langle x, \mathcal{G}(x) \rangle| \leq \|\mathcal{G}\| \|x\|^2 \leq r_2 \|x\|^2.
\]

Hence, \( \{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}} \) is a Bessel sequence of operators with Bessel bound \( r_2 \) for \( \mathbb{H}^R(\Omega) \) with respect to \( \{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}} \). \( \square \)
In view of Theorem 16 if $\mathbb{H}_R(\Omega)$ is a right quaternionic Hilbert space, $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of right quaternionic Hilbert spaces, and $\{T_i : \mathbb{H}_i^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}_R(\Omega)$ with respect to $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$. Then there exist an operator $\mathcal{S} : \mathbb{H}_R(\Omega) \to \mathbb{H}_R(\Omega)$ defined by

$$\mathcal{S}(x) = \sum_{i=1}^{\infty} T_i^* T_i(x), \text{ for all } x \in \mathbb{H}_R(\Omega).$$

We call $\mathcal{S}$ to be the frame operator for the frame of operators $\{T_i : \mathbb{H}_i^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$.

Also, if $\mathbb{H}_R(\Omega)$ is a right quaternionic Hilbert space, $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of right quaternionic Hilbert spaces, $\{T_i : \mathbb{H}_i^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of bounded linear operators, and $\{e_k^i\}_{k \in \mathbb{N}}$ is a Hilbert basis in $\mathbb{H}_i^R(\Omega)$. Then, for each $i \in \mathbb{N}$, $h_i^R : \mathbb{H}_R(\Omega) \to \Omega$ given by $h_i^R(x) = \langle e_k^i | T_i(x) \rangle$, $x \in \mathbb{H}_R(\Omega)$ is a right bounded linear functional on $\mathbb{H}_R(\Omega)$. Therefore, by Riesz representation Theorem for quaternions, there exists $x_k^i \in \mathbb{H}_R(\Omega)$ such that

$$h_i^R(x) = \langle x_k^i | x \rangle, \text{ for all } x \in \mathbb{H}_R(\Omega).$$

So, we get

$$T_i(x) = \sum_{k=1}^{\infty} e_k^i \langle e_k^i | T_i(x) \rangle = \sum_{k=1}^{\infty} e_k^i \langle x_k^i | x \rangle, \ x \in \mathbb{H}_R(\Omega).$$

Thus, for each $x, y \in \mathbb{H}_R(\Omega)$, we obtain

$$\langle T_i^* (y) | x \rangle = \left\langle y \left| \sum_{k=1}^{\infty} e_k^i \langle x_k^i | x \rangle \right. \right\rangle = \left\langle \sum_{k=1}^{\infty} x_k^i \langle e_k^i | y \rangle \left| x \right. \right\rangle.$$

Hence

$$T_i^* (y) = \sum_{k=1}^{\infty} x_k^i \langle e_k^i | y \rangle, \text{ for all } y \in \mathbb{H}_R(\Omega).$$

We call the sequence $\{x_k^i\}_{i \in \mathbb{N}}$ as the sequence induced by the sequence of right bounded linear operators $\{T_i : \mathbb{H}_R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$.

In the next result, we show that if the sequence induced by $\{T_i\}_{i \in \mathbb{N}}$ is a frame for $\mathbb{H}_R(\Omega)$, then $\{T_i\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}_R(\Omega)$ and vice-versa.

**Theorem 17.** Let $\mathbb{H}_R(\Omega)$ be a right quaternionic Hilbert space, $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of right quaternionic Hilbert spaces, and $\{T_i : \mathbb{H}_i^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of bounded linear operators. Then $\{T_i : \mathbb{H}_i^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}_R(\Omega)$ with respect to $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ with bounds $r_1$ and $r_2$ if and only if the sequence induced by $\{T_i\}_{i \in \mathbb{N}}$ is a frame for $\mathbb{H}_R(\Omega)$ with bounds $r_1$ and $r_2$.

Further, the operator $\mathcal{S}$ of frame of operators $\{T_i : \mathbb{H}_i^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ coincides with the frame operator of the frame induced by $\{T_i\}_{i \in \mathbb{N}}$.

**Proof.** For each $x \in \mathbb{H}_R(\Omega)$, we have

$$\sum_{i=1}^{\infty} \|T_i(x)\|^2 = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |\langle x_k^i | x \rangle|^2.$$
Further, for each $x \in \mathbb{H}^R(\Omega)$, we compute
\[
\mathcal{S}(x) = \sum_{i=1}^{\infty} \mathcal{T}_i^* \mathcal{T}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} x_k^i \langle e_k^i | \sum_{j=1}^{\infty} e_j^i(x_j^i|x) \rangle = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} x_k^i \langle e_k^i | x \rangle.
\]
Hence, the operator $\mathcal{S}$ of the frame of operators $\{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ coincides with the frame operator of the frame induced by $\{\mathcal{T}_i\}_{i \in \mathbb{N}}$. \qed

Next, we give some properties of the frame operator for frame of operators.

**Theorem 18.** Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of right quaternionic Hilbert spaces, and $\{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ be a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$. Then the frame operator $\mathcal{S}$ for the frame of operators $\{\mathcal{T}_i\}_{i \in \mathbb{N}}$ is bounded, invertible, self-adjoint and positive.

**Proof.** In view of Theorem 16, $\mathcal{S}$ is bounded. For each $x, y \in \mathbb{H}^R(\Omega)$, we have
\[
\langle y | \mathcal{S}(x) \rangle = \sum_{i=1}^{\infty} \langle y | \mathcal{T}_i^* \mathcal{T}_i(x) \rangle = \sum_{i=1}^{\infty} \langle \mathcal{T}_i^* \mathcal{T}_i(y) | x \rangle = \langle \mathcal{S}(y) | x \rangle
\]
Thus, the operator $\mathcal{S}$ is self-adjoint. Also
\[
\langle x | \mathcal{S}(x) \rangle = \sum_{i=1}^{\infty} ||\mathcal{T}_i(x)||^2, \text{ for all } x \in \mathbb{H}^R(\Omega).
\]
So, by inequality (3), we have
\[
r_1 \langle x | \mathcal{I}(x) \rangle \leq \langle x | \mathcal{S}(x) \rangle \leq r_2 \langle x | \mathcal{I}(x) \rangle, \text{ for all } x \in \mathbb{H}^R(\Omega),
\]
where, $\mathcal{I}$ is the identity operator. This gives $r_1 \mathcal{I} \preceq \mathcal{S} \preceq r_2 \mathcal{I}$. Thus $\mathcal{S}$ is a positive operator. Further, since
\[
0 \leq \mathcal{I} - r_2^{-1} \mathcal{S} \leq \frac{r_2 - r_1}{r_2} \mathcal{I},
\]
$||\mathcal{I} - r_2^{-1} \mathcal{S}|| < 1$. Hence $\mathcal{S}$ is invertible. \qed

Next, we show that if we have a frame of operators $\{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ with the frame operator $\mathcal{S}$ and bounds $r_1$ and $r_2$, then one can construct another frame of operators with frame operator $\mathcal{S}^{-1}$ and with bounds $r_1^{-1}$ and $r_2^{-1}$.

**Theorem 19.** Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of right quaternionic Hilbert spaces and $\{\mathcal{T}_i : \mathbb{H}^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ with bounds $r_1$ and $r_2$. Then $\{\mathcal{U}_i = \mathcal{T}_i \mathcal{S}^{-1} : \mathbb{H}^R(\Omega) \to \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ with bounds $r_1^{-1}$ and $r_2^{-1}$ and with the frame operator $\mathcal{S}^{-1}$.

**Proof.** Since
\[
x = \mathcal{S} \mathcal{S}^{-1}(x) = \sum_{i=1}^{\infty} \mathcal{S}^{-1} \mathcal{T}_i \mathcal{T}_i(x), \text{ } x \in \mathbb{H}^R(\Omega).
\]
For each $x \in \mathbb{H}^R(\Omega)$, we obtain

$$x = \sum_{i=1}^{\infty} T_i^* T_i(x) = \sum_{i=1}^{\infty} U_i^* U_i(x).$$

Also, we compute

$$\sum_{i=1}^{\infty} \|U_i(x)\|^2 = \sum_{i=1}^{\infty} \left\langle T_i, \mathcal{S}^{-1}(x) \right| T_i, \mathcal{S}^{-1}(x) \right\rangle$$

$$= \sum_{i=1}^{\infty} \left\langle \mathcal{S}^{-1}(x) \right| T_i^* T_i \mathcal{S}^{-1}(x) \right\rangle$$

$$\leq \frac{1}{r_1} \|x\|^2, \text{ for all } x \in \mathbb{H}^R(\Omega).$$

Further, for each $x \in \mathbb{H}^R(\Omega)$

$$\|x\|^2 = \sum_{i=1}^{\infty} \left| \langle U_i^* T_i(x) \rangle \right|^2$$

$$\leq \left( \sum_{i=1}^{\infty} \|T_i(x)\|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} \|U_i(x)\|^2 \right)^{\frac{1}{2}}$$

$$\leq \sqrt{r_2} \|x\| \left( \sum_{i=1}^{\infty} \|U_i(x)\|^2 \right)^{\frac{1}{2}}.$$

This gives

$$\left( \sum_{i=1}^{\infty} \|U_i(x)\|^2 \right)^{\frac{1}{2}} \geq \frac{1}{r_2} \|x\|^2, \text{ for all } x \in \mathbb{H}^R(\Omega).$$

Hence $\{U_i : \mathbb{H}^R(\Omega) \rightarrow \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ is a frame of operators with bounds $1/r_1$ and $1/r_2$. Furthermore, let $\mathcal{U}$ denotes the operator for the frame of operators $\{U_i\}_{i \in \mathbb{N}}$. Then, for each $x \in \mathbb{H}^R(\Omega)$, we have

$$\mathcal{S} \mathcal{U}(x) = \sum_{i=1}^{\infty} \mathcal{S} U_i^* U_i(x) = \sum_{i=1}^{\infty} T_i^* T_i \mathcal{S}^{-1}(x) = x.$$

Hence $\mathcal{U} = \mathcal{S}^{-1}$. \qed

In view of Theorem 19 every frame of operators $\{T_i : \mathbb{H}^R(\Omega) \rightarrow \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ produces a frame of operators, namely $\{U_i = T_i \mathcal{S}^{-1} : \mathbb{H}^R(\Omega) \rightarrow \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$, where $\mathcal{S}$ is the frame operator for the frame of operators $\{T_i : \mathbb{H}^R(\Omega) \rightarrow \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$. We call $\{U_i\}_{i \in \mathbb{N}}$ as dual frame of operators for $\{T_i\}_{i \in \mathbb{N}}$. In such a case $\{T_i\}_{i \in \mathbb{N}}$ and $\{U_i\}_{i \in \mathbb{N}}$ are refer to as a pair of dual frame of operators.

In the next result, we give a relation between a pair of dual frame of operators and their induced sequences.

**Theorem 20.** Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space and $\{\mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ be a sequence of right quaternionic Hilbert spaces. If $\{T_i : \mathbb{H}^R(\Omega) \rightarrow \mathbb{H}_i^R(\Omega)\}_{i \in \mathbb{N}}$ and $\{U_i : \mathbb{H}^R(\Omega) \rightarrow \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ forms a pair of dual frame of operators, then the sequences induced by them is a pair of dual frames and vice-versa.

**Proof.** Follows from Theorem 17 and Theorem 19. \qed

Next by using frame of operators, we construct a Parseval frame of operators.
Theorem 21. Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\{\mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of right quaternionic Hilbert spaces, and $\{T_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$. Then $\{R_i = T_i\mathcal{S}^{-1/2} : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ is a Parseval frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$.

Proof. Straight forward. □

Finally, in this section we prove that in order to construct a frame of operator for a right quaternionic Hilbert space, it is enough to construct it for a dense subset.

Theorem 22. Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\{\mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of right quaternionic Hilbert spaces and $\{T_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of bounded linear operators such that there exist positive constants $r_1$ and $r_2$ satisfying

$$r_1 \|x\|^2 \leq \sum_{i=1}^{\infty} \|T_i(x)\|^2 \leq r_2 \|x\|^2,$$

for all $x$ in a dense subset $\mathcal{V}$ of $\mathbb{H}^R(\Omega)$.

Then $\{T_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ with bounds $r_1$ and $r_2$.

Proof. Suppose that there exist $x \in \mathbb{H}^R(\Omega)$ such that $\sum_{i=1}^{\infty} \|T_i(x)\|^2 > r_2 \|x\|^2$. Then, there exists some $n \in \mathbb{N}$ such that $\sum_{i=1}^{n} \|T_i(x)\|^2 > r_2 \|x\|^2$. Since $\mathcal{V}$ is dense in $\mathbb{H}^R(\Omega)$, there exist $y \in \mathcal{V}$ such that $\sum_{i=1}^{n} \|T_i(y)\|^2 > r_2 \|y\|^2$. This a contradiction. Also, note that

$$r_1 \|x\|^2 \leq \|x\|\mathcal{S}(x)\|, \quad \text{for all } x \in \mathcal{V}. \quad (4)$$

Since $\mathcal{S}$ is bounded and $\mathcal{V}$ is dense in $\mathbb{H}^R(\Omega)$, we conclude that $\mathcal{H}$ holds for all $x \in \mathbb{H}^R(\Omega)$. □

Stability of frame of Operators

In this section, we prove two results related to the stability of frame of operators.

Theorem 23. Let $\mathbb{H}^R(\Omega)$ be a right quaternionic Hilbert space, $\{\mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ a sequence of right quaternionic Hilbert spaces, and $\{T_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ with bounds $r_1$ and $r_2$. Let $\{R_i : \mathbb{H}^R(\Omega) \to \mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ be a sequence of operators such that there exist constants $\lambda_1$, $\lambda_2$, $\mu \geq 0$ satisfying

$$\left(\sum_{i=1}^{\infty} \|T_i - R_i\|/\|x\|\right)^{1/2} \leq \lambda_1 \left(\sum_{i=1}^{\infty} \|T_i(x)\|^2\right)^{1/2} + \lambda_2 \left(\sum_{i=1}^{\infty} \|R_i(x)\|^2\right)^{1/2} + \|x\|,$$

for all $x \in \mathbb{H}^R(\Omega)$.

Then $\{R_i\}_{i \in \mathbb{N}}$ is a frame of operators for $\mathbb{H}^R(\Omega)$ with respect to $\{\mathbb{H}^R(\Omega)\}_{i \in \mathbb{N}}$ with bounds $r_1 \left(1 - \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{T_1}}{1 + \lambda_2}\right)^2$ and $r_2 \left(1 + \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{T_1}}{1 - \lambda_2}\right)^2$.

Proof. For each $x \in \mathbb{H}^R(\Omega)$, we have

$$\left(\sum_{i=1}^{\infty} \|T_i - R_i\|/\|x\|\right)^{1/2} \leq \left(\frac{\lambda_1}{\sqrt{T_1}}\right) \left(\sum_{i=1}^{\infty} \|T_i(x)\|^2\right)^{1/2} + \lambda_2 \left(\sum_{i=1}^{\infty} \|R_i(x)\|^2\right)^{1/2},$$

and

$$\left(\sum_{i=1}^{\infty} \|T_i - R_i\|/\|x\|\right)^{1/2} \geq \left(\frac{\lambda_1}{\sqrt{T_1}}\right) \left(\sum_{i=1}^{\infty} \|T_i(x)\|^2\right)^{1/2} - \left(\sum_{i=1}^{\infty} \|R_i(x)\|^2\right)^{1/2}.$$
Hence the result holds in view of Theorem 17 and Theorem 22.

\[ (1 + \lambda_2) \left( \sum_{i=1}^{\infty} \| \mathcal{R}_i(x) \|^2 \right)^{1/2} \geq \left( 1 - \lambda_1 - \frac{\mu}{\sqrt{r_1}} \right) \left( \sum_{i=1}^{\infty} \| \mathcal{S}_i(x) \|^2 \right)^{1/2} \]

\[ \geq \left( 1 - \lambda_1 - \frac{\mu}{\sqrt{r_1}} \right) \sqrt{r_1} \| x \|, \quad x \in \mathbb{H}^R(\Omega). \]

This gives

\[ \left( \sum_{i=1}^{\infty} \| \mathcal{R}_i(x) \|^2 \right)^{1/2} \geq r_1 \left( 1 - \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{r_1}}{(1 + \lambda_2)} \right)^2 \| x \|^2. \]

On the similar lines, it is easy to verify that

\[ \left( \sum_{i=1}^{\infty} \| \mathcal{R}_i(x) \|^2 \right)^{1/2} \leq r_2 \left( 1 + \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{r_1}}{(1 - \lambda_2)} \right)^2 \| x \|^2, \quad x \in \mathbb{H}^R(\Omega). \]

Finally, we discuss stability of frame of operators in terms of their conjugate operators.

**Theorem 24.** Let \( \mathbb{H}^R(\Omega) \) be a right quaternionic Hilbert space, \( \{ \mathbb{H}^R_i(\Omega) \}_{i \in \mathbb{N}} \) a sequence of right quaternionic Hilbert spaces, and \( \{ \mathcal{S}_i : \mathbb{H}^R(\Omega) \rightarrow \mathbb{H}^R_i(\Omega) \}_{i \in \mathbb{N}} \) a frame of operators for \( \mathbb{H}^R(\Omega) \) with bounds \( r_1 \) and \( r_2 \). Assume that \( \{ \mathcal{R}_i : \mathbb{H}^R(\Omega) \rightarrow \mathbb{H}^R_i(\Omega) \}_{i \in \mathbb{N}} \) is a sequence of bounded linear operators such that there exist constants \( \lambda, \mu \geq 0 \) satisfying \( \left( 1 + \frac{\mu}{\sqrt{r_1}} \right) < 1 \) and

\[ \left\| \sum_{i=1}^{\infty} (\mathcal{S}_i - \mathcal{R}_i^*)(x_i) \right\| \leq \lambda \left\| \sum_{i=1}^{\infty} \mathcal{S}_i(x_i) \right\| + \mu \sum_{i=1}^{\infty} \| x_i \|^2, \quad x_i \in \mathbb{H}^R_i(\Omega), \quad i \in \mathbb{N}. \]

Then \( \{ \mathcal{R}_i : \mathbb{H}^R_i(\Omega) \rightarrow \mathbb{H}^R(\Omega) \}_{i \in \mathbb{N}} \) is a frame of operators for \( \mathbb{H}^R(\Omega) \) with respect to \( \{ \mathbb{H}^R_i(\Omega) \}_{i \in \mathbb{N}} \) with bounds \( r_1 \left( 1 - \frac{\lambda + \mu}{\sqrt{r_1}} \right)^2 \) and \( r_2 \left( 1 + \frac{\lambda + \mu}{\sqrt{r_2}} \right)^2 \).

**Proof.** Note that

\[ x_i = \sum_{k=1}^{\infty} e_k^i q^i_k, \quad \{ q^i_k \} \in \ell^2(\Omega) \quad \text{and} \quad i \in \mathbb{N} \]

where \( \{ e_k^i \}_{k \in \mathbb{N}} \) is a Hilbert basis in \( \mathbb{H}^R_i(\Omega) \). For each \( k \in \mathbb{N} \), let \( \tau_k^i = \mathcal{S}_i(e_k^i) \) and \( \rho_k^i = \mathcal{R}_i^*(e_k^i) \). Then, (5) is equivalent to

\[ \left\| \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (\tau_k^i - \rho_k^i) q_k^i \right\| \leq \lambda \left\| \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \tau_k^i q_k^i \right\| + \mu \sum_{i=1}^{\infty} \| q_k^i \|^2. \]

Hence the result holds in view of Theorem 17 and Theorem 22.

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