A Geometrical Construction of Rational Boundary States in Linear Sigma Models

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Abstract

Starting from the geometrical construction of special Lagrangian submanifolds of a toric variety, we identify a certain subclass of A-type D-branes in the linear sigma model for a Calabi-Yau manifold and its mirror with the A- and B-type Recknagel-Schomerus boundary states of the Gepner model, by reproducing topological properties such as their labeling, intersection, and the relationships that exist in the homology lattice of the D-branes. In the non-linear sigma model phase these special Lagrangians reproduce an old construction of 3-cycles relevant for computing periods of the Calabi-Yau, and provide insight into other results in the literature on special Lagrangian submanifolds on compact Calabi-Yau manifolds. The geometrical construction of rational boundary states suggests several ways in which new Gepner model boundary states may be constructed.

1 Introduction

The study of string theory on Calabi-Yau manifolds provides a potential point of contact with low energy ("real-world") physics. One can obtain four-dimensional gauge theories with $\mathcal{N} = 1$ supersymmetry (which is usually desirable to maintain a degree of computational control over the theory, as well as for phenomenological reasons) by compactifying heterotic string theory on $\mathbb{R}^{3,1} \times CY_3$ where $CY_3$ is a Calabi-Yau 3-fold: this point of view was studied extensively in the 1980s and 1990s. Since the discovery of D-branes as extended objects carrying gauge fields on their world-volume, one may also obtain four-dimensional gauge theories by considering Type II string theory on $\mathbb{R}^{3,1} \times CY_3$ in the presence of D-branes. In order to obtain a four-dimensional gauge theory with $\mathcal{N} = 1$ supersymmetry, these D-branes must be BPS. BPS D-branes come in two types, labeled A- and B-type according to which type of supersymmetry is preserved.

The linear sigma model (LSM) is a useful tool for studying the phase structure that exists in the Kähler moduli space of string theory compactified on Calabi-Yau manifolds. Although
the LSM is not itself a conformally-invariant theory, it flows to the desired conformal field theories in the infrared via renormalization group flow. Typically the Kähler moduli space phase structure contains a geometrical phase where the infrared CFT is described by string theory on a Calabi-Yau manifold, as well as one or more “non-geometrical” phases where the IR CFT does not have an obvious geometrical description, but is instead described by an abstract CFT such as the IR limit of a Landau-Ginzburg (LG) theory (in a certain limit of Kähler moduli space this is a Gepner model \[1,2\], an exactly solvable CFT), or hybrid phases such as a LG theory fibred over a geometrical base space.

In a geometrical phase, A-type D-branes correspond to flat vector bundles over special Lagrangian 3-cycles, while B-type D-branes correspond to stable holomorphic vector bundles over holomorphic (even-dimensional) cycles \[3,4,5\]. A similar A-/B-type classification of D-branes exists in Gepner models, as studied in \[5,6\]. D-branes were studied in the linear sigma model in \[5,6,10\]. The spectrum of BPS D-branes for a given Calabi-Yau has been studied in \[11,12,13,14\] and elsewhere, and their stability under variations of the moduli \[15,16,17,18,19,20\] is of central importance.

In this paper I consider special Lagrangian submanifolds of noncompact Calabi-Yau toric varieties, and their restriction to a compact Calabi-Yau embedded within it. The linear sigma model construction and results of mirror symmetry are used to study the properties of a certain class of these D-branes in the two phases of the Kähler moduli space of the quintic hypersurface in \(\mathbb{P}^4\), as well as at special points in the complex structure moduli space of the mirror manifold.

One of my aims in this paper is to show how existing results on D-branes in conformal field theories may be obtained from the linear sigma model picture which flows to (and interpolates between) these conformal field theories in the infrared. The general principle is that quantities that are controlled under renormalization group flow can be safely computed in the LSM framework: for example, since the \((n,0)\)-form \(\Omega\) is holomorphic, its functional form is not renormalized under RG flow, and we can compute string theory quantities that depend on \(\Omega\) within the linear sigma model. Therefore A-type D-branes can be constructed in the LSM and descend to A-type D-branes in string theory. On the other hand quantities that depend only on the Kähler structure are renormalized, and in general we do not have direct control over or explicit knowledge of them in the infrared limit of the LSM.

The main new results of this work are as follows:

At the Gepner point of Kähler moduli space, a class of special Lagrangian submanifolds of the linear sigma model target space – those that span the toric base of the target space, referred to as base-filling D-branes – are shown to reproduce the labelling and intersection properties of A-type rational boundary states of the Gepner model (the boundary states first constructed by Recknagel and Schomerus \[3\] and further developed in \[7\]). The corresponding set of special Lagrangian submanifolds of the mirror reproduce the properties of the B-type states at the Gepner point of the quintic, in accordance with mirror symmetry (the analysis was performed for B-type states directly in \[21\], where they were associated to fractional branes of the Landau-Ginzburg orbifold theory).

In this paper I will refer to the set of boundary states constructed in \[3,7\] (which preserve the full tensor product \(\mathcal{N} = 2\) supersymmetry algebra of the Gepner model) as “rational boundary states”, although it should be noted that there may be more general boundary states of the Gepner models (which do not preserve the full algebra, but only a diagonal
\( \mathcal{N} = 2 \) which are also rational.

The Lagrangian D-branes of the Landau-Ginzburg model associated to a single minimal model were obtained in [8] by studying BPS solitons in Landau-Ginzburg theories; the toric geometry construction naturally produces the extension of this result to the full Gepner model.

The base-filling D-branes of the linear sigma model can be thought of as providing a geometrical description of the rational boundary states of the Gepner model, which are defined in abstract CFT and do not have an obvious intrinsic geometrical description. The construction of rational boundary states following [6, 7] relies on preservation of the \( \mathcal{N} = 2 \) supersymmetry in each of the minimal model factors of the Gepner model, and it appears difficult to construct a more general class of boundary states within conformal field theory alone. However, preservation of these extra symmetry algebras translates into a simple geometrical property of the base-filling D-branes, and it should be possible to relax this constraint to obtain a much larger class of boundary states than have previously been constructed in CFT.

The base-filling D-branes have obvious relations between their homology classes, and by representing the D-branes as polynomials these relationships can be quantified in terms of relations between the polynomials. The large-volume homology classes of the B-type rational boundary states of the quintic were computed in [11], and the relations between these classes are reproduced by the polynomial encoding.

The base-filling D-branes of the linear sigma model restrict to A-type D-branes of string theory, and can be followed from the LG phase to the geometrical phase. They are shown to reproduce the construction [22, 23] of 3-cycles relevant for computing periods of the compact Calabi-Yau.

It was proposed in [11] that some of the A-type states of the quintic should be identified in the geometrical phase with certain \( \mathbb{RP}^3 \) submanifolds of the quintic, by comparing their intersection forms. The corresponding base-filling D-branes from the toric geometry construction have the correct intersection properties, but do not coincide with the \( \mathbb{RP}^3 \)s in the geometrical phase and instead produce a distinct special Lagrangian submanifold with the same intersection form. This is compatible with results from deformation theory.

The layout of the paper is as follows: section 2 contains a brief review of some existing results on D-branes in conformal field theories (rational boundary states in Gepner models, and supersymmetric D-branes on Calabi-Yau manifolds). Section 3 provides an introduction to toric geometry; the presentation follows that of [26, 27] and should be more accessible to beginners than the usual mathematical treatments of toric geometry. The linear sigma model [28] is briefly recalled in section 4 as a natural consequence of the geometrical construction of toric varieties. The construction of A-type D-branes in the LSM is discussed in section 4.1. The properties of a certain class of these D-branes are analyzed in the Landau-Ginzburg orbifold phase of the LSM in section 4.2, and in the non-linear sigma model phase in section 4.3. Section 4.4 studies the relationships that exist in the homology lattice of the D-branes. The possibilities for construction of new CFT boundary states based on the geometry of the linear sigma model are discussed in section 5, and I bring together the results obtained in previous sections with a proposal that the boundary states of the Gepner model should be thought of as the “latent geometry” of the special Lagrangian submanifolds of the linear sigma model in a limit which confines the theory to a single point. Finally, section 6 summarizes some unresolved problems and possibilities for further work.
2 D-branes in Conformal Field Theories

There are two important classes of conformally-invariant string compactification (conformal field theory): the non-linear sigma model (NLSM), describing a string propagating on a Calabi-Yau manifold, and the Gepner models, which are exactly solvable conformal field theories built from a tensor product of $\mathcal{N} = 2$ minimal models with the correct central charge to saturate the conformal anomaly of string theory.

The properties of D-branes in these two CFTs have been much studied in recent years, and I will now review some of the relevant results.

2.1 D-branes on Calabi-Yau manifolds

A BPS D-brane in Type II string theory preserves $1/2$ of the spacetime supersymmetry; i.e. it is invariant under an $\mathcal{N} = 1$ subalgebra of the $\mathcal{N} = 2$ spacetime supersymmetry algebra.

The BPS conditions were worked out from the point of view of both the worldsheet CFT and the target space geometry in [3, 4, 5], which I now briefly review.

A Lagrangian submanifold $L$ of a Kähler manifold is one for which the Kähler form pulls back to 0 on the submanifold:

$$\omega|_L = 0$$

(1)

In terms of the worldsheet theory, D-branes that wrap Lagrangian submanifolds preserve half of the worldsheet supersymmetry (i.e. they preserve $\mathcal{N} = 2$ worldsheet supersymmetry algebra), but not necessarily spacetime supersymmetry; the condition for preserving half of the spacetime supersymmetry (i.e. $\mathcal{N} = 1$ in spacetime) is that the submanifold must further be special Lagrangian.

A special Lagrangian submanifold of a Calabi-Yau manifold is a Lagrangian submanifold for which the pullback of the holomorphic $n$-form of the Calabi-Yau $n$-fold has a constant phase on the submanifold:

$$\text{Im } \log \Omega|_L = \epsilon$$

(2)

where $\epsilon$ is called the $U(1)$ grade of the D-brane and is defined modulo $2\pi$: it is associated to the relative phase of the left- and right- spectral flow operators of the worldsheet theory when they are glued together with an A-type automorphism on the boundary of the worldsheet [3]. Two A-type D-branes that have the same $U(1)$ grade will be mutually supersymmetric; D-branes with a different grade will break spacetime supersymmetry and will therefore not be stable (there exists a tachyon in the spectrum of open strings stretching between the branes, which causes the system to decay into a BPS system of D-branes with the same total topological charges, e.g. in the same total homology class).

B-type D-branes correspond to holomorphic submanifolds of the target space, however this paper will not discuss B-type D-branes directly. Under mirror symmetry the A- and B-type D-branes will interchange, so the A-type states on $\mathcal{M}$ are exchanged with the B-type states on the mirror manifold $\mathcal{W}$, and vice versa. This operation also interchanges the role of Kähler moduli and complex structure moduli on the manifold and its mirror. Therefore we can restrict our attention to the A-type D-branes at the expense of considering both $\mathcal{M}$
and deformations of one type (Kähler or complex structure) and \( W \) with deformations of the other type. This is the approach I will take in this paper.

A general construction of special Lagrangian submanifolds of a Calabi-Yau manifold is not known, but one known construction is the fixed-point set of a real involution of the manifold (a “reality condition”: the special Lagrangian submanifolds are “real” objects).

The example that will be used in section 4.3 involves the quintic hypersurface in \( \mathbb{P}^4 \) (a compact Calabi-Yau manifold) defined by

\[
\sum_{i=1}^{5} z_i^5 = 0
\]  

where \( z_i \) are homogeneous coordinates on the \( \mathbb{P}^4 \).

This equation may be deformed by addition of monomials of degree 5; these correspond to complex structure deformations of the Calabi-Yau manifold. The complex structure deformation that is relevant for the mirror quintic is:

\[
\sum_{i=1}^{5} z_i^5 - \psi z_1 \ldots z_5 = 0
\]  

where \( \psi \) is a complex parameter.

If we impose the reality condition

\[
z_i = \overline{z_i} \quad \Leftrightarrow \quad \text{Im} \ z_i = 0
\]  

on each \( z_i \), (i.e. \( z_i \equiv x_i \in \mathbb{R} \)) then we obtain the real equation

\[
\sum_{i=1}^{5} x_i^5 = 0
\]  

which has a unique solution for one of the coordinates in each projective coordinate patch in terms of the remaining three. Therefore, this 3-dimensional real submanifold is diffeomorphic to \( \mathbb{R}\mathbb{P}^3 \).

More generally we can extend the reality condition (5) to:

\[
\text{Im} \ \omega_i z_i = 0
\]  

where \( \omega_i^5 = 1 \), which gives a total of \( 5^{(5-1)} = 625 \) \( \mathbb{R}\mathbb{P}^3 \)s inside the quintic (since a common phase rotation \( z_i \mapsto \omega z_i \) acts trivially in projective space).

Another construction of special Lagrangians when \( \mathcal{M} \) is a toric variety will be presented in section 4.2. There may be other real involutions which can be imposed to construct special Lagrangians, as well as more general constructions.
2.2 D-branes in Gepner Models

The work of Recknagel and Schomerus [6] used the techniques of Boundary Conformal Field Theory (BCFT) developed by Cardy [29, 30], Ishibashi [31, 32] and others, to formulate states corresponding to D-branes in terms of “boundary states” of the world-sheet (open string) theory (in the closed string channel). Their construction was clarified and extended in [7] using the framework of simple current extensions.

If we choose to preserve the total tensor product algebra \( A \otimes \), i.e. the \( \mathcal{N} = 2 \) superconformal algebra in each minimal model factor of the Gepner model, the resulting CFT is rational and its study is tractable. Since all we require physically is an overall \( \mathcal{N} = 2 \) supersymmetry, this construction preserves much more symmetry than we need (it preserves a separate \( \mathcal{N} = 2 \) algebra in each minimal model factor), but preserving less symmetry renders the theory non-rational and therefore difficult to study from the point of view of CFT.

Rational boundary states of minimal models are in 1-1 correspondence with the chiral primary fields of the bulk minimal model. The Gepner model boundary states constructed by Recknagel and Schomerus are therefore labeled (before GSO projection) by

\[
|\psi\rangle_\Omega = |L_1, \ldots, L_r; M_1, \ldots, M_r; S_1, \ldots, S_r\rangle_\Omega
\]

where \( \Omega = A, B \) labels which type of supersymmetry is to be preserved by the worldsheet boundary, \( r \) is the number of minimal model factors of the Gepner model, and

\[
L_i \in \{0, \ldots, k_i\} \\
M_i \in \{-k_i - 1, \ldots, k_i + 2\} \\
S_i \in \{-1, 0, 1, 2\} \\
L_i + M_i + S_i = 0 \mod 2
\]

where \( k_i \) is the choice of model in the \( \mathcal{N} = 2 \) minimal series, for the \( i^{th} \) tensor product factor. The \( S_i \) distinguish between the NS (\( S_i = 0, 2 \)) and R sectors (\( S_i = -1, 1 \)) of the minimal model, and the two values in each sector correspond to a brane and its anti-brane.

The labels (8) overcount the physically distinct boundary states in several ways. First, for a generic model there is a “field identification”

\[
(L_i, M_i, S_i) \sim (k_i - L_i, M_i + k_i + 2, S_i + 2)
\]

which reduces the number of distinct boundary states by half in each minimal model factor. This has a simple geometrical interpretation in terms of the geometrical D-branes derived later.

The choice of NS/R sectors are constrained by the GSO projection (which ensures that the boundary states preserve \( \mathcal{N} = 1 \) spacetime supersymmetry) to be the same in each factor, and hence the distinct states are labelled only by a single \( S \) label [11, 7].

There may be other over-counting depending on the symmetries of the particular Gepner model chosen. In general these will correspond to geometrical symmetries of the linear sigma

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1When one or more of the minimal model levels \( k_i \) is even there are subtleties to do with fixed points under the field identification. They have been studied in [33, 7], but I will not address these issues here since I mainly focus on the quintic model \((k = 3)\).
model that are inherited by the conformal field theory at the infrared fixed point of each phase (i.e. which are unbroken by the vacuum submanifold of the linear sigma model).

For example, the \((k = 3)^5\) model corresponds to a point in the extended Kähler moduli space of the quintic hypersurface in \(\mathbb{P}^4\). One finds that the A-type boundary states are physically indistinguishable under a simultaneous shift \(M_i \mapsto M_i + 2\) of all of the \(5\) \(M_i\) labels, and therefore this can be used to fix one of the \(M_i\) leaving four free. This corresponds in the LSM to a simultaneous \(\mathbb{Z}_5\) rotation of each of the coordinates of the D-brane, which acts trivially in the LSM target space as we will see later. In the LG orbifold phase of the quintic the target space is \(\mathbb{C}^5/\mathbb{Z}_5\) where the \(\mathbb{Z}_5\) is precisely this common rotation, and in the NLSM phase the target space is a line bundle over \(\mathbb{P}^4\), and the homogeneous coordinates on \(\mathbb{P}^4\) are invariant under the same shift.

The B-type boundary states on the quintic are invariant under a \(\mathbb{Z}_4\) action on the \(M_i\), which implies that the equivalence classes are labeled by a single \(M\) value. Geometrically this is best understood by looking at the symmetries of the mirror manifold, which by the Greene-Plesser construction is given by an \((\mathbb{Z}_5)^3\) orbifold of the quintic. Therefore, the symmetry of the mirror models is \((\mathbb{Z}_5)^4\) including the \(\mathbb{Z}_5\) that acts “projectively”.

The redundancies in the boundary state labeling will be explicit in the geometrical representatives that will be constructed later.

The \(U(1)\) grade of the boundary state \(\theta\) is given by:

\[
\frac{\theta}{\pi} \equiv \left( \sum_{i=1}^{r} \left( -\frac{M_i}{k_i + 2} + \frac{S_i}{2} \right) \right) \mod 2 \quad (11)
\]

The intersection form of the D-brane boundary states can be computed; in conformal field theory this is computed in the open string channel by the Witten index

\[
Tr_R(-1)^F
\]

for a worldsheet with two given boundary conditions (boundary states), where \(F\) is the worldsheet fermion number operator.

This quantity was computed for the rational boundary states in \([\Pi]\). For two such states

\[
|L_1, \ldots, L_r; M_1, \ldots, M_r; S\rangle_\Omega
\]

and

\[
|\overline{L}_1, \ldots, \overline{L}_r; \overline{M}_1, \ldots, \overline{M}_r; \overline{S}\rangle_\Omega
\]

the intersection form was found to be:

\[
I_A = \frac{1}{C} \left(-1\right)^{\frac{s - \pi}{2}} \sum_{\nu_0=0}^{K-1} \prod_{j=1}^{r} N_{L_j, L_j}^{2\nu_0 + M_j - \overline{M}_j} \quad (13)
\]

for the A-type states, and

\[
I_B = \frac{1}{C} \left(-1\right)^{\frac{s - \pi}{2}} \sum_{m_j} \delta^{(K')}_{\frac{L_j - M_j + \sum \frac{k_j^m}{\nu'}}{2}} \prod_{j=1}^{r} N_{L_j, L_j}^{m_j - 1} \quad (14)
\]

for the B-type states.
for the B-type states, where $C$ is a normalization constant, and

$$K' = \text{lcm}\{k_j + 2\}$$

$$M = \sum_j \frac{K'M_j}{k_j + 2}$$

$$\delta^{(K')}_n \equiv \delta((n \mod K'))$$

(15)

$N^l_{L_j,T_j}$ are the extended $SU(2)_k$ fusion coefficients [11]:

$$N^l_{L,T} = \begin{cases} 1 & \text{if } |L - T| \leq l \leq \min\{L + T, 2k - L - T\} \\ 0 & \text{otherwise} \end{cases}$$

$$N^{-l-2}_{L,T} = -N^l_{L,T}$$

(16)

We will see later that these formulae can be derived easily from geometrical considerations in the LSM.

### 3 Toric Geometry

In this section I will give an intuitive description of the framework of toric geometry using the “symplectic quotient” construction, following [26] and [27]. This description differs from the usual mathematical presentation of toric geometry, but allows a more direct understanding of the geometrical constructions of toric varieties and their (special) Lagrangian submanifolds, as well as directly carrying over to the linear sigma model construction: for any given toric variety expressed in this way, the corresponding linear sigma model can be written down immediately. See [26] for more details on the connections between this framework and the more traditional approach to toric geometry.

Essentially, a toric variety is a $T^n$-fibration (hence the name toric) over some (not necessarily compact) linear base space with boundary, where the $T^n$ fibres are allowed to degenerate over the boundary of the base. In the case where the base space is compact, the resulting toric variety will also be compact.

A simple example of a toric variety is $\mathbb{C}^n$, which can be parametrized by $z_i = |z_i|e^{i\theta_i}, i = 1, \ldots n$. This is a Kähler manifold with Kähler form given by

$$\omega = i \sum_i dz_i \wedge d\bar{z}_i$$

$$= \sum_i d(|z_i|^2) \wedge d\theta_i$$

(17)

(18)

A Lagrangian submanifold $L$ of a Kähler manifold $\mathcal{M}$ is defined by

$$\omega|_L = 0$$

(19)

i.e. the Kähler form vanishes identically upon restriction to the submanifold. Since we will be interested in Lagrangian (and special Lagrangian) submanifolds in later applications to D-branes, it is convenient to write $\mathbb{C}^n$ in a way that makes the Lagrangian structure clear.
Using the natural $U(1)$ action
\[ z_i \mapsto z_i e^{i\theta_i} \]
we can express $\mathbb{C}^n$ as a $T^n$ fibration over a Lagrangian submanifold $L$ of $\mathbb{C}^n$ defined by taking $\theta_i$ constant for all $i$; $L$ is then isomorphic to the positive segment of $\mathbb{R}^n$ parametrized by $|z_i|^2 \geq 0, i = 1, \ldots, n$ (see figure 1). We parametrize the base by $|z_i|^2$ instead of $|z_i|$ in order to make the base space linear. The $T^n \simeq U(1)^n$ action acts on the $|z_i|^2$ to recover the manifold $\mathbb{C}^n$. The boundary of $L$ is given by the union of the hyperplanes $|z_i|^2 = 0$, and the $U(1)$ action has fixed points along each of these boundary segments, corresponding to the degeneration of the $T^n$ fibre to a $T^{n-d}$ at points on the boundary where $d$ hyperplanes intersect.

Note that in choosing this parametrization of $\mathbb{C}^n$ and fixing each of the angular coordinates $\theta_i$, the Kähler form $\Omega$ in fact vanishes term by term, since $d\theta_i \equiv 0$ for all $i$.

In order to obtain a more general toric variety of complex dimension $n$ we use the method of symplectic quotient $\mathbb{C}^{n+r}/G$, where $G \simeq U(1)^r$ for some $r$. Concretely, we proceed in two steps: first restricting to a certain linear subspace in the $|z_i|^2$ and then dividing by the group action $G$.

The choice of subspace and group action $G$ is defined by a set of $r$ vectors of integral weights or “charges” $Q^a$
\[ Q^a = (Q_1^a, \ldots, Q_{n+r}^a), \quad a = 1, \ldots, r \]
For each $a$ we define the hyperplane
\[ \sum_{i=1}^{n+r} Q_i^a |z_i|^2 = r^a \]

\footnote{The projection to the base space $z_i \mapsto |z_i|^2$ is also called the moment map.}

\footnote{The term comes from the related linear sigma model construction to be described in section \ref{sec:lsm}, where the $Q^a$ define the charges of the LSM chiral superfields under the $U(1)^r$ gauge group.}
The intersection of these $r$ hyperplanes is (assuming the $Q^a$ are linearly independent vectors) a real space $D$ (with boundary) of dimension $n+r-r = n$. Each parameter $r^a$ is a deformation modulus for the toric variety (it is just the normal translation modulus for the hyperplane).

In order to obtain a Kähler manifold we quotient the $T^{n+r}$ bundle over $D$ by the $U(1)^r$ action generated by a simultaneous phase rotation of the coordinates:

$$z_i \mapsto e^{iQ^a_i} z_i$$

for each $a = 1, \ldots, r$, where $e^a$ are the generators of the $U(1)$ factors. This fixes $r$ of the phases and gives a $T^n$ bundle over $D$. This construction preserves the Kähler form on the toric variety induced from (18), and the quotient space is therefore a Kähler manifold of complex dimension $n$. The translation moduli $r^a$ become the Kähler moduli of the Kähler manifold, so the manifold has $\dim \mathbb{C}H^{1,1}(M) = r$. When $r = 1$ we obtain a weighted projective space with weights $Q_i$ (if one or more of the $Q_i$ are negative then this is a non-compact generalization of the usual compact weighted projective spaces); the cases $r > 1$ are more general toric varieties.

If in addition the charges $Q^a$ satisfy

$$\sum_{i=1}^{n+r} Q^a_i = 0$$

for all $a$, then the toric variety is moreover a Calabi-Yau manifold, since the holomorphic $(n+r,0)$-form of $\mathbb{C}^{n+r}$ is $\mathbb{C}^*$-invariant and pulls back to a non-vanishing $(n,0)$-form on the toric variety [28]. Note that (24) implies that for each $a$ at least one of the $Q^a_i$ must be negative: this implies that the Calabi-Yau manifold will therefore be non-compact since the hyperplane (22) is unbounded above in this coordinate direction along $D$.

In later sections we will make use of the non-compact Calabi-Yau $n$-fold

$$\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$$

the holomorphic line bundle of degree $-n$ over $\mathbb{P}^{n-1}$ (equivalently, the normal bundle to $\mathbb{P}^{n-1}$). Starting with $\mathbb{C}^{n+1}$ as a $T^{n+1}$ fibration over $L = (\mathbb{R}^+)^{n+1}$ we choose a real codimension-1 subspace of the base space $L$ defined by the vector $Q = (1, 1, \ldots, 1, -n)$, i.e. the hyperplane

$$\sum_{i=1}^{n+1} |z_i|^2 - n|z_{n+1}|^2 = r$$

where $r$ is an arbitrary real parameter.

At $|z_{n+1}|^2 = 0$ the equation becomes

$$\sum_{i=1}^{n} |z_i|^2 = r$$

which defines an $(n-1)$-simplex of size $r$ (see Figure 3). The size of the simplex increases along the $|z_{n+1}|^2$ direction. Therefore, this subspace describes a family of $(n-1)$-simplices parametrized by $|z_{n+1}|^2$:
Figure 2: The simplex at $|z_4|^2 = 0$ for the case $n = 3$. For nonzero $|z_4|^2$ the simplex has size $r + |z_4|^2$

$$\sum_{i=1}^{n} |z_i|^2 = r + n|z_{n+1}|^2$$  \hspace{1cm} (28)

as shown in Figure 3. The base space therefore has one non-compact direction and $n - 1$ compact directions.

After dividing by the $U(1)$ symmetry, we are left with a $T^n$-fibration over a real $n$-dimensional base space, which is a toric Calabi-Yau manifold. As before, the $T^n$ fibration will degenerate along the boundary of the base. The total space over the $(n-1)$-simplex at $z_{n+1} = 0$ is isomorphic to $\mathbb{P}^{n-1}$ by construction: along this face we have $z_{n+1} = 0$, and the symplectic quotient construction fixes the radius $|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 = r$ of an $S^{2n-1}$ inside $\mathbb{C}^n \cong \mathbb{R}^{2n}$, as well as a $U(1)$ acting by common phase rotation. Altogether we have taken the quotient of $\mathbb{C}^n$ by a $\mathbb{C}^*$ action; this is the usual construction of $\mathbb{P}^{n-1}$.

The total space of the non-compact direction transverse to the simplex is given by an $S^1$ fibre over $\mathbb{R}^+$. Therefore, corresponding to every point in $\mathbb{P}^{n-1}$ we have a copy of $\mathbb{C} \cong \mathbb{R}^+ \times S^1$. The toric variety is therefore a (holomorphic) line bundle, and the $U(1)$ action on the coordinate $z_{n+1}$ defined by the charge $Q_{n+1} = -n$ means it is the promised line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$. Alternatively, the space can be viewed as the weighted projective space $\mathbb{W}\mathbb{P}^n(1, 1, \ldots, 1, -n)$ (which contains $\mathbb{P}^{n-1}$ as a compact submanifold).

As we take $r \to 0$, the simplex forming the base of the $\mathbb{P}^{n-1}$ shrinks to zero size, and for $r < 0$ the geometry is isomorphic, up to a translation along the $|z_{n+1}|^2$ axis (see Figure 3). Geometrically, taking $r \to 0$ corresponds to “blowing down” the $\mathbb{P}^{n-1}$ at the base space of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$, and the geometry becomes isomorphic to $\mathbb{C}^n/\mathbb{Z}_n$. This transition changes the topology of the space, and is an example of a birational equivalence [28]. The induced metric from $\mathbb{C}^{n+1}$ on the line bundle is not Ricci-flat; however in this case the Ricci-flat metric is known explicitly and is given by the Calabi metric [33]

$$ds^2 = \frac{1}{\rho} \left\{ dy_i dy_i^j - \frac{1}{\rho} y_i dy_i^j y_j dy_j^i \right\} + \left\{ \rho^n d\bar{w} dw + n\rho^{n-1} \left( \bar{w} dw y_i dy_i^j + w d\bar{w} y_i dy_i^j \right) \right\}$$  \hspace{1cm} (29)
Figure 3: a) The geometry of the base of $\mathcal{O}_{\mathbb{P}^2}(-3)$ as a subset of $\mathbb{R}^3$. b) The same geometry projected onto the plane, showing the 2-dimensional boundary of the base space. The plane is divided into various regions labeled by the coordinates that vanish in each, describing the embedding of the plane into the various boundary hyperplanes of $(\mathbb{R}^+)^4$.

Figure 4: Cross-section of the toric base of the $Q = (1,1,1,\ldots,1,-n)$ toric variety showing how the target space geometry changes for $r \leq 0$. At $r = 0$ the simplex at the “tip” of the toric base (shown in figures 2 and 3) shrinks to zero size, and the topology of the space changes from $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ to $\mathbb{C}^n/\mathbb{Z}_n$. Throughout the phase $r \leq 0$ the target space geometry stays the same up to a shift along the $z_{n+1}$ axis, because of the requirement that $|z_i|^2 \geq 0$. 
where \( \rho = 1 + y_i y^i \). The coordinate \( w \) parametrizes the non-compact direction of the line bundle, and the metric (29) reduces at \( w = 0 \) to the Fubini-Study metric on \( \mathbb{P}^{n-1} \):

\[
ds^2 = \frac{1}{\rho} \left\{ dy_i dy^i - \frac{1}{\rho} y_i y^i y_j y^j \right\}
\]

(30)

where \( y_i, i = 1, \ldots, n - 1 \) are inhomogeneous coordinates on the \( \mathbb{P}^{n-1} \), in a coordinate patch \( U_n \).

The linear sigma model for the line bundle, which I discuss in the next section, will have the induced non-Ricci-flat target space metric at high energies, but it is expected to flow to the Calabi metric under worldsheet RG flow to the infrared, i.e. the linear sigma model becomes conformal in this limit (and correctly describes perturbative string theory on the line bundle).

For a given non-compact Calabi-Yau manifold described by a line bundle over a \( \mathbb{P}^{n-1} \) (or \( \mathbb{WP}^{n-1} \)) we can obtain a family of compact Calabi-Yau manifolds of complex dimension \( n - 2 \) by restricting to the zero locus of a quasi-homogeneous degree-\( n \) polynomial inside the \( \mathbb{P}^{n-1} \) (more generally, we can consider “complete intersections” of multiple polynomials). From the point of view of the compact Calabi-Yau, the presence of the line bundle is irrelevant, but it is needed for the LSM construction. The simplest case for \( \mathbb{P}^{n-1} \) is a single degree-\( n \) polynomial of Fermat type:

\[
n \sum_{i=1}^{n} z_i^n = 0
\]

(31)

which is a Calabi-Yau hypersurface of complex dimension \( n - 2 \) within the \( \mathbb{P}^{n-1} \).

Note that in general these hypersurfaces do not respect the Lagrangian or toric descriptions of the ambient space in which they are embedded. However a toric description may be recovered in a limit when the hypersurface degenerates, such as \( \psi \to \infty \) for the mirror quintic (4) [26]. In this limit the defining equation of the quintic (4) becomes (after scaling out by \( \psi \))

\[
z_1 \ldots z_5 = 0
\]

(32)

which is solved by taking one or more of the \( z_i = 0 \). In terms of the toric description of \( \mathbb{P}^4 \) the quintic is restricted to the boundary of the 4-simplex, and the \( T^3 \) fibration over this boundary produces (before dividing out by the extra orbifold symmetry) 5 intersecting \( \mathbb{P}^3 \)s (faces of the 3-skeleton of the 4-simplex), which intersect in 10 \( \mathbb{P}^2 \)s (faces of the 2-skeleton), which in turn meet in 10 \( \mathbb{P}^1 \)s (faces of the 1-skeleton), which meet in 5 points (faces of the 0-skeleton) [22].

Just as in the non-compact case, the metric on a hypersurface inherited from the ambient \( \mathbb{P}^{n-1} \) is not Ricci-flat, even if we use the Ricci-flat metric on the \( \mathcal{O}_{\mathbb{P}^{n-1}}(-n) \) in which \( \mathbb{P}^{n-1} \) is embedded. Unfortunately, in contrast to the non-compact case the Ricci-flat metric is not known explicitly for any compact Calabi-Yau manifolds of dimension 3 or higher (although its existence is guaranteed by Yau’s theorem). However, the holomorphic \( n \)-form of the non-compact ambient space (which is obtained by pullback from the original \( \mathbb{C}^{n+1} \)) pulls back to the correct holomorphic \( (n-2) \)-form on the hypersurface.

In order to later check whether a submanifold is special Lagrangian one needs the explicit form of the holomorphic \( n \)-form on the Calabi-Yau. This is induced from the holomorphic
\((n + r)\)-form on \(\mathbb{C}^{n+r}\)

\[ \Omega^{(n+r)} = dz_1 \wedge \ldots \wedge dz_{n+r} \]  

(33)

For a general weighted projective \(n\)-space \(\mathbb{WP}^n(k_1, \ldots, k_n)\) obtained by projection from \(\mathbb{C}^{n+1}\), an \(n\)-form is obtained \([34]\) by contraction of \(\Omega^{(n+1)}\) with the vector field that generates the \(\mathbb{C}^*\) action

\[ (z_1, \ldots, z_{n+1}) \sim (\lambda^{k_1}z_1, \ldots, \lambda^{k_{n+1}}z_{n+1}) \]  

(34)

giving

\[ \Omega^{(n)} = \frac{1}{(N+1)!} \epsilon^{i_1 \ldots i_{n+1}} k_{i_{n+1}} z_{i_{n+1}} dz_{i_1} \ldots dz_{i_n} \]  

(35)

For a general projective space \(\Omega^{(n)}\) is not well-defined, since under the \(\mathbb{C}^*\) action \(\text{(34)}\) it transforms like

\[ \Omega^n \rightarrow \lambda^{\sum_{i=1}^{n+1} k_i} \Omega^n \]  

(36)

In the special case where \(\sum_{i=1}^{n+1} k_i = 0\) it is globally well-defined; these are precisely the line bundles described above. \(\Omega^{(n)}\) can further be shown to be non-vanishing; therefore these spaces are non-compact Calabi-Yau manifolds.

In order to produce a well-defined \(n\)-form for a compact \(\mathbb{WP}^n\) we can choose a collection of polynomials \(P_1, \ldots, P_\alpha\) and we take instead

\[ \Omega^{(n-\alpha)'} = \int_\Gamma \frac{\Omega^{(n)}}{P_1 \ldots P_\alpha} \]  

(37)

where \(\Gamma\) is a real \(\alpha\)-dimensional contour that is the product of small circles around each of the surfaces \(P_i = 0\), so the integral picks up the residue coming from the poles at \(P_i = 0\). This form will be scale-invariant, and hence globally defined, if the \(P_i\) are chosen to have appropriate degree to compensate for the transformation of \(\Omega^{(n)}\). It can be shown that the algebraic variety defined by the intersection of the polynomial vanishing loci \(\{P_i = 0\}\) is a compact Calabi-Yau manifold.

4 Linear Sigma Models

The linear sigma model was introduced in [28], and will not be described in detail here. The reader will observe that the construction in [28] precisely follows the symplectic quotient construction of non-compact toric Calabi-Yau manifolds from the previous section: a toric Calabi-Yau variety is obtained as the vacuum manifold of the theory (parametrized by the scalar fields), with the linear, i.e. “wrong” metric on the Calabi-Yau in the UV; localization to a compact Calabi-Yau hypersurface within the non-compact Calabi-Yau is implemented by an appropriate superpotential term in the LSM. Therefore using the construction of the previous section we can use the associated linear sigma model to describe string propagation on non-compact toric Calabi-Yau \(n\)-folds\(^4\), as well as compact Calabi-Yau manifolds of lower

\(^4\)Studying non-compact manifolds is useful for providing local descriptions of string compactifications where we neglect the rest of the compact manifold “at infinity”, e.g. for studying the neighbourhood of a singularity.
dimension that can be embedded in a non-compact Calabi-Yau as a hypersurface or complete intersection of hypersurfaces.

Essentially, under RG flow to the infrared the target space metric of the \( d = 2 \) linear sigma model flows from the linear metric to the Ricci-flat metric (and the coupling constant tends to infinity, localizing onto classical vacua of the theory). We can avoid the complexities of working with the Ricci-flat metric directly by using the linear sigma model, providing we only work with quantities that are protected or controlled under RG flow, so that we can follow them to the CFT limit. For example, since the \((n,0)\)-form \( \Omega \) is holomorphic, its functional form is not renormalized under RG flow, and we can therefore hope to identify special Lagrangian submanifolds of the linear sigma model target space with A-type boundary states of the conformal field theory upon restriction to the vacuum submanifold of the LSM.

The real Kähler moduli \( r \) of a toric variety are complexified by the \( \theta \)-angles of the LSM (which becomes the B-field in string theory) through the combination \( \theta + ir \), and for the 1-parameter models the complexified Kähler moduli space has two phases. When \( r > 0 \) the infrared fixed point of the linear sigma model is a non-linear sigma model with the “correct” Ricci-flat metric on the target space (e.g. \( \mathcal{O}_{\mathbb{P}^4}(-5) \), or the quintic hypersurface inside the \( \mathbb{P}^4 \)) and this is called a geometrical phase. The phase \( r < 0 \) corresponds formally to an analytic continuation to negative Kähler class. For \( \mathcal{O}_{\mathbb{P}^{n-1}}(-n) \) this means “negative size” of the \( \mathbb{P}^{n-1} \) in which the Calabi-Yau hypersurface is embedded, i.e. we pass to the blown-down phase where the \( \mathbb{P}^{n-1} \) has been collapsed to a point, and the target space is \( \mathbb{C}^n/\mathbb{Z}_n \) (the Kähler modulus for moving around in this phase is hidden within the kinetic term of the LG model, which is only defined implicitly through RG flow). If we also have a superpotential in the theory then this phase is a Landau-Ginzburg orbifold theory. The singularity at \( r = 0 \) can be avoided by turning on a non-zero \( \theta \)-angle.

The Gepner model exists at the infrared fixed point of the LSM in the limit \( r \to -\infty \), the “deep interior point” of the LG phase of Kähler moduli space, and is an exactly solvable CFT. In the opposite limit \( r \to \infty \) of the geometrical phase (the “large volume limit”), closed string instanton corrections are suppressed since the volume of all 2-cycles is large, and the infrared fixed point is a nonlinear sigma model described by classical geometry.

A-type D-branes may decay under variation of complex structure [20], but are stable under Kähler deformations [21, 22]. Therefore when we consider A-type D-branes in the various phases of Kähler moduli space, they will remain stable throughout since the special Lagrangian condition that defines A-type branes depends on the complex structure, which is kept fixed.

Mirror symmetry of Calabi-Yau manifolds exchanges A- and B-type D-branes, as well as the role of Kähler and complex structure moduli on the manifold and its mirror. Therefore, in order to understand the behaviour of B-type D-branes under variation of Kähler structure we can equivalently consider A-type D-branes of the mirror theory under variation of complex structure. There are issues of stability of these A-type D-branes to consider, but this is a purely geometrical problem (in contrast to the B-type D-branes, which are destabilized by instanton effects). It is possible to obtain concrete results in certain limits, for example the mirror to the deep interior points of the Kähler moduli space of the quintic.

Mirror symmetry can be studied in the LSM framework, and essentially has the interpretation of T-duality along the \( T^n \) fibres of the non-compact toric variety [36]. Starting with the LSM for a given Calabi-Yau manifold (including compact Calabi-Yau manifolds
embedded in a non-compact Calabi-Yau via a superpotential), the authors of that paper were able to derive the corresponding LSM for the proposed mirror Calabi-Yau (as well as the mirror for more general situations). I will not make use of their derivation explicitly, but will instead use known results on mirror symmetry for the quintic (see [37], [22]).

### 4.1 A-type D-branes in the LSM

Recall from section 2.1 that A-type D-branes are associated to special Lagrangian \( n \)-cycles of a Calabi-Yau \( n \)-fold. We would like to realise a class of A-type D-branes in the linear sigma model; using the symplectic quotient construction this is equivalent to describing special Lagrangian submanifolds of the toric variety. The starting point of the symplectic quotient construction was the description of \((\mathbb{R}^+)^n\) as a Lagrangian submanifold of \(\mathbb{C}^n\): we can immediately describe other Lagrangian and special Lagrangian D-branes as submanifolds of this space (there may be other possibilities that cannot be obtained by this method).

Lagrangian submanifolds are obtained by taking additional hyperplane constraints in the toric base [27]: the intersection of \(p\) linearly independent hyperplanes will give an \((n-p)\)-dimensional subspace of the base, and taking the orthogonal subspace of the fibres gives a \(T^p\) fibration over this base space, producing a real \(n\)-dimensional submanifold of the complex \(n\)-dimensional toric variety.

Each hyperplane (indexed by \(\alpha\)) is defined by a normal vector \(\vec{q}^\alpha\) and a translation modulus \(c^\alpha\) fixing the location of the hyperplane:

\[
\sum_{i=1}^{n+r} q_i^\alpha |z_i|^2 = c^\alpha, \quad \alpha = 1, \ldots, p
\]  

(38)

To obtain a rational (Hausdorff) subspace of the \(T^n\) the entries \(q_i^\alpha\) of the normal vector are constrained to be integers. These Lagrangian submanifolds are therefore characterized by an integer \(p\) which specifies the number of “D-term-like” constraints defining the base of the submanifold as an intersection with the base of the toric variety.

The orthogonality conditions on the angular coordinates which define the \(T^p\) fibre of the Lagrangian submanifold are [27]

\[
\vec{v}^\beta \cdot \vec{\theta} = 0 \mod 2\pi, \quad \beta = 1, \ldots, n-p
\]  

(39)

where \(\vec{\theta} = (\theta_1, \ldots, \theta_{n+r})\) are the angular coordinates on \(T^{n+r}\), and \(\vec{v}^\beta\) are integral vectors that span the intersection of the hyperplanes, i.e. which satisfy

\[
\vec{v}^\beta \cdot \vec{q}^\alpha = 0
\]  

(40)

In order to be well-defined after dividing by the \(U(1)^r\) gauge-symmetry of the toric variety we also require the \(\vec{v}^\beta\) to satisfy

\[
\vec{v}^\beta \cdot \vec{Q}^a = 0
\]  

(41)

The condition [24] for a toric variety to be Calabi-Yau is

\[
\sum_{i=1}^{n+r} Q_i^a = 0, \quad a = 1, \ldots, r
\]  

(42)
which has a similar form to the constraint that a Lagrangian submanifold be special Lagrangian:

\[ \sum_{i=1}^{n+r} q_i^\alpha = 0, \quad \alpha = 1, \ldots, p \]  

(43)

In this paper I will focus on the class \( p = 0 \): in the next section these submanifolds will be related to the set of rational boundary states of the Gepner model. They are submanifolds with no additional hyperplane constraints in the base and which will therefore span the toric base of the toric variety: I will sometimes refer to them as “base-filling D-branes” to emphasize this property. If the toric variety is Calabi-Yau then these submanifolds are furthermore special Lagrangian submanifolds. To obtain the \( p = 0 \) submanifolds we can choose \( n \) vectors \( \vec{v}^\beta \) that span the hyperplane defining the Calabi-Yau, i.e. which satisfy (11).

Recall that \( \mathcal{O}_{n-1}(-n) \) is described by a single set of charges

\[ Q = (1, 1, \ldots, 1, -n) , \]  

(44)

which gives one D-term constraint (which fixes the base of the \( T^{n+1} \) fibration to lie within a hyperplane in \((\mathbb{R}^+)^{n+1}\), and one \( U(1) \) gauge invariance (which reduces the fibre from \( T^{n+1} \) to \( T^n \)). If we are interested in studying D-branes on a compact Calabi-Yau such as the quintic hypersurface in \( \mathbb{P}^4 \), we need to consider special Lagrangian submanifolds of \( \mathcal{O}_{p=1}(-n) \) that intersect the \( \mathbb{P}^{n-1} \), as well as the hypersurface within it. Submanifolds that do not intersect the hypersurface will not be visible to the string theory at the infrared fixed point, which is constrained to lie within the hypersurface. In the LG phase the constraint is that the submanifolds must intersect the orbifold point.

For \( \mathcal{O}_{p=1}(-n) \) we can take the \( \vec{v}^\beta \) to be

\[ \begin{align*}
\vec{v}^1 & = (n, 0, \ldots, 0, 1) \\
\vec{v}^2 & = (0, n, \ldots, 0, 1) \\
& \vdots \\
\vec{v}^n & = (0, 0, \ldots, n, 1)
\end{align*} \]  

(45)

These span the hyperplane (although they do not form an orthonormal basis). The fibre constraints (39) reduce to

\[ n\theta_i + \theta_{n+1} = 2\pi a_i, \quad a_i \in \mathbb{Z}, \quad i = 1, \ldots, n \]  

(46)

or

\[ \theta_i = \frac{2\pi a_i - \theta_{n+1}}{n} \]  

(47)

This formal similarity seems to be at the foundation of recent studies of open string mirror symmetry (mirror symmetry for Calabi-Yau manifolds including D-branes) \[38, 39\], in which a non-compact Calabi-Yau 3-fold together with a certain type of special Lagrangian D-brane (\( p = 2 \) in my notation) is promoted into a Calabi-Yau 4-fold without D-branes, to which closed string mirror symmetry can be applied to compute exact disc instanton sums of the original theory. This is called “open/closed string duality” in \[38\].
Using the $U(1)$ symmetry we can set $\theta_{n+1} = 0$; the $\theta_i$s become:

$$\theta_i = \frac{2\pi a_i}{n}$$

i.e. they are $n^{th}$ roots of unity. These are the same constraints obtained in [8] for D-branes of the Landau-Ginzburg theory associated to a single $\mathcal{N} = 2$ minimal model (i.e. single $\theta$), which were obtained by studying the BPS solitons of the LG theory. Since the Gepner model is constructed from the tensor product of $\mathcal{N} = 2$ minimal models (with certain identifications and projections) it is natural to expect that the rational boundary states should have a similar form in the tensor product theory, however note that they are derived here from pure geometry.

Along the submanifold the values of $\theta_i$ are constant. Therefore, these submanifolds are isomorphic to the toric base of the Calabi-Yau, and in particular they have a boundary isomorphic to the boundary of the base. For comparison to existing results in the literature to which I will later relate them, I will refer to these $n$-dimensional submanifolds with boundary (i.e. submanifolds which are described by an $n^{th}$ root of unity in each angular coordinate) as “spokes”. The presence of the boundary means that these submanifolds are 3-chains, and one needs to take appropriate differences of two such branes in order to obtain a special Lagrangian 3-cycle, and I will therefore also refer to them as “half-branes” as context suggests.

Since this class of special Lagrangian submanifold spans the simplex that is the toric base of the $\mathbb{P}^{n-1}$, appropriately chosen special Lagrangians will intersect the hypersurface embedded within it. However in order to obtain a good D-brane, we need to take the difference of two such submanifolds with a common boundary in the vacuum submanifold of the LSM, so that a string in the infrared will see a D-brane with no boundary. For this D-brane to be BPS the two half-branes must have the same $U(1)$ grade; i.e., for A-type D-branes the pair of submanifolds with common boundary must also be a special Lagrangian submanifold.

As will be shown in the next section, this class of A-type D-branes in the LSM is in 1-1 correspondence with the set of A-type boundary states of the Gepner model constructed in [6, 7] and reproduces their symmetries and intersection form. When we blow up the origin of $\mathbb{C}^n/\mathbb{Z}_n$ (i.e. pass to the NLSM phase) and restrict to the Calabi-Yau hypersurface in the blown up $\mathbb{P}^{n-1}$ it is possible to relate a particular subset of the blown-up LSM D-branes to known special Lagrangian cycles in the compact Calabi-Yau hypersurface.

By construction, a pair of half-branes will have a constant phase of the holomorphic $n$-form for each half-brane, but in general it need not be the same phase for both. In other words, the pair of submanifolds are only piecewise special Lagrangian, not necessarily globally special Lagrangian. From the point of view of the worldsheet theory, since the piecewise special Lagrangian submanifolds are only Lagrangian submanifolds they only preserve $\mathcal{N} = 2$ world-sheet supersymmetry, and do not preserve space-time supersymmetry.

Since they are not BPS objects, we expect there to be a tachyon in the string excitation spectrum of a single brane which will drive a flow to a state in the same homology class that is BPS and therefore stable (tachyon-free). In other words, there should be a special Lagrangian submanifold in the same homology class as the non-special Lagrangian submanifold we started with. A priori this may not be a special Lagrangian submanifold of the
type considered here (i.e. a pair of spokes aligned along \(n^{\text{th}}\) roots of unity), but in fact for each homology class obtained by the spoke construction for the quintic there exists a special Lagrangian submanifold in the same homology class that is also a pair of spokes. Therefore, even the pairs of spokes that are not themselves special Lagrangian are representatives of other spoke pairs in the same homology class that are special Lagrangian.

The A-type D-brane construction described here is only strictly valid at the “deep interior point” of each LSM phase (the large volume limit for the geometrical phase, and the Gepner point for the LG phase): away from the large volume limit the superpotential of the D-brane world-volume theory receives instanton corrections coming from open string worldsheets ending on the D-brane (which wind a nontrivial \(S^1\) of the D-brane and wrap a nontrivial Riemann surface in the bulk), and control is also lost away from the Gepner point of the LG phase. The disc instanton corrections have been analyzed for non-compact Calabi-Yau manifolds in [40, 41, 27, 58] using open string mirror symmetry, for the special Lagrangians labeled by \(p = 2\) in the notation used above, but this analysis has not yet been extended to the other \(p\) classes. Contributions to the instanton-generated superpotential for the quintic were analyzed in the B-model in [23].

4.2 D-branes in the LG phase

In this section I specialize to the quintic for definiteness, although the results should generalize in a straightforward manner to the other Calabi-Yau manifolds with \(h^{1,1} = 1\), except as noted. In the Landau-Ginzburg orbifold phase of the quintic, we will consider pairs of half-branes of the type constructed in the previous section, and I will show how they are related to the A-type rational boundary states at the Gepner point. The corresponding construction of B-type boundary states will be related to A-type states in the mirror Landau-Ginzburg theory [37], in accordance with mirror symmetry.

In the LG phase the target space is \(\mathbb{C}^5/\mathbb{Z}_5\), and the special Lagrangian submanifolds (half-branes) constructed in the previous section are particularly simple. In each coordinate \(z_i\) of the orbifold space \(\mathbb{C}^5/\mathbb{Z}_5\) they are parametrized along rays that are aligned along fifth roots of unity through the origin (modulo the \(\mathbb{Z}_5\) orbifold symmetry that acts by a common phase rotation of the \(z_i\)). The rays are oriented, inducing an orientation for the submanifold. Together, the independent coordinates \(r_i\) along the 5 rays parametrize an oriented 5-dimensional real submanifold of the target space, which is isomorphic to the base of \(\mathbb{C}^5/\mathbb{Z}_5\) in the toric construction of section 3, and has a boundary over the boundary of the toric base space (when one or more of the coordinates \(r_i = 0\)). The submanifolds are referred to as “spokes” because their image in each of the coordinates \(z_i\) is a 1-dimensional ray, but note that the submanifolds as a whole are 5-dimensional objects.

The \(\theta_i\) are angular coordinates on \(\mathbb{C}^5\), i.e. they are not single-valued on \(\mathbb{C}^5/\mathbb{Z}_5\). However, when we fixed \(\theta_6 = 0\) there was a residual \(\mathbb{Z}_5\) symmetry left unfixed, which acts on the \(\theta_i\) by a common \(\mathbb{Z}_5\) phase rotation. Acting with this symmetry to bring \(\theta_1\) into the range \([0, \frac{2\pi}{5})\) gives the single-valued angular coordinates on \(\mathbb{C}^5/\mathbb{Z}_5\); since the \(\theta_i\) are fifth roots of unity this fixes the coordinate \(\hat{\theta}_1 = 0\) and leaves the other 4 coordinates as arbitrary fifth roots of unity: I will refer to the \(\hat{\theta}_i\) as the “reduced” angular coordinates on the orbifold.

We can represent the D-branes in \(\mathbb{C}^5\) (graphically, as 5 copies of \(\mathbb{C} \simeq \mathbb{R}^2\), each of which contains a ray from the origin along one of the fifth roots of unity) providing we remember
the orbifold condition that the target space is actually $\mathbb{C}^5/\mathbb{Z}_5$, in which one of the rays is aligned with the positive real axis.

In the mirror LG picture the orbifold group contains an additional $(\mathbb{Z}_5)^3$: the combined $(\mathbb{Z}_5)^4$ symmetry can be taken to act in the following way:

\[
\begin{align*}
\mathbb{Z}_5^{(1)} & : (4, 0, 0, 0, 1) \\
\mathbb{Z}_5^{(2)} & : (0, 4, 0, 0, 1) \\
\mathbb{Z}_5^{(3)} & : (0, 0, 4, 0, 1) \\
\mathbb{Z}_5^{(4)} & : (0, 0, 0, 4, 1)
\end{align*}
\] (49)

where the notation $(\alpha_1, \ldots, \alpha_n)$ means that the symmetry acts on the coordinates by

\[
(z_1, \ldots, z_n) \mapsto (\lambda^{\alpha_1} z_1, \ldots, \lambda^{\alpha_n} z_n), \quad \lambda = e^{\frac{2\pi}{5}}.
\] (50)

We can use these symmetries to align four of the rays with the positive real axis, with a compensating rotation of the fifth ray leaving it along some other fifth root of unity. See Figure 5 for an example of a pair of spokes in $\mathbb{C}^5$, and its reduced image in the $\mathbb{C}^5/\mathbb{Z}_5$ orbifold and in the $\mathbb{C}^5/(\mathbb{Z}_5)^4$ orbifold of the mirror LG model.

As noted above, a single half-brane contains a boundary, and we therefore need to study the possibilities for cancelling this boundary to form a cycle.

Consider two half-branes specified by $\hat{\theta}_1, \ldots, \hat{\theta}_5$ and $\tilde{\theta}_1, \ldots, \tilde{\theta}_5$. Along one of the boundary segments $|z_i|^2 = 0$ of the half-branes, the $S^1_{(i)}$ fibre degenerates, and so different values of $\hat{\theta}_i$ in fact represent the same point. If the other four $\hat{\theta}_j$ coordinates are distinct, the two half-branes do not meet and therefore do not have a common boundary segment there.

Since this is true over each boundary segment $|z_i|^2 = 0$, the only way to produce two half-branes with common boundary is to take $\hat{\theta}_i = \tilde{\theta}_i$ for all $i$. These are the reduced coordinates on the $\mathbb{C}^5/\mathbb{Z}_5$ orbifold; in terms of the unreduced coordinates $\theta_i$ on $\mathbb{C}^5$ we can take a pair of rays that differ by a common $\mathbb{Z}_5$ phase rotation. The two sets of rays will reduce to the same image in the orbifold, but since they are only identified up to a $\mathbb{Z}_5$ rotation in $\mathbb{C}^5$ they are in a twisted sector of the orbifold, and therefore the pair of half-branes carries a $\mathbb{Z}_5$ topological charge which encodes the twist value of the D-brane. Heuristically speaking the pair of half-branes produce a D-brane that comes in from infinity, twists around the orbifold fixed point and heads off to infinity again along the same path in the orbifold space (but a different path in the covering space).

For other values of $\hat{\theta}_i$, the two half-branes will only meet at the origin, so they are topologically two copies of $(\mathbb{R}^+)^5$ touching at the origin inside $\mathbb{C}^5/\mathbb{Z}_5$. The generic pair of half-branes therefore still has a boundary. We can complete this submanifold to one without boundary by taking the “doubled images” of each half-brane under $\hat{\theta}_i \mapsto \hat{\theta}_i + \pi$, together with a reversal of orientation to cancel the common boundary segment. This is the same thing as taking $z_i \mapsto -z_i$, i.e. adjoining another $(\mathbb{R}^+)^n$ along one of the boundary hyperplanes; if we take $2^n$ such doublings we complete $(\mathbb{R}^+)^n$ to $\mathbb{R}^n$ and obtain a submanifold without boundary, which looks like two copies of $\mathbb{R}^n$ intersecting at the origin inside $\mathbb{C}^n/\mathbb{Z}_n$. Note that the doubling preserves the phase of the holomorphic 5-form, since both the shift in $\hat{\theta}_i$ and the orientation reversal shift the phase by $\pi$, leaving it invariant: the doubled half-brane is therefore still special Lagrangian.
Figure 5: (a) A typical pair of “spokes” in the five coordinate planes \( \mathbb{C} \) of \( \mathbb{C}^5 \): the incoming and outgoing rays in each coordinate plane are aligned with fifth roots of unity and parametrize the two half-branes in \( \mathbb{C}^5 \). (b) The image of the same spoke in \( \mathbb{C}^5/\mathbb{Z}_5 \): using the \( \mathbb{Z}_5 \) orbifold symmetry, which acts by a common phase rotation of each of the coordinates, we can “collapse” the pair in \( z_1 \), i.e. align both the incoming and outgoing ray in the \( z_1 \) coordinate patch with the real axis, with a corresponding rotation of the other coordinates. (c) The image of the same spoke in the \( \mathbb{C}^5/(\mathbb{Z}_5)^4 \) mirror model; the symmetry action \( (51) \) can be used to align the first four pairs of rays with the real axis at the expense of a compensating rotation of the fifth coordinate. The first four outgoing rays need to be rotated clockwise by a combined total of \( 1 + 1 + 1 + 3 = 6 \) units to align them all; the orbifold symmetry causes the fifth to rotate counterclockwise by 6 units to the position shown. Similarly, the incoming rays are rotated clockwise by \( 0 + 2 + 3 + 0 = 5 \) units so the incoming fifth ray is rotated counterclockwise by 5 (and comes back to the same position).
The fact that the generic pair of half-branes in the LG phase only meet at the origin is not a problem for string theory, since we are only interested in the infrared fixed point of the LSM. Under RG flow to the infrared the LSM coupling constant $g \to \infty$, so the conformally-invariant string is confined to the classical vacuum of the LSM (the orbifold fixed point). The configuration space of the infrared string is therefore the single point at the intersection of the two half-branes, as expected for the LG models.

In the mirror model, because of the extra orbifold symmetry arising from the Greene-Plesser construction there is no problem with boundary cancellation of pairs of spokes, and in fact any two pairs of half-branes will have a common boundary, without need to double the geometry by taking $\theta_i \mapsto \theta_i + \pi$: using the orbifold symmetry, we can collapse any four of the five pairs of rays to align them with the real axis, and the fifth pair of rays will have different values of $\theta_i$. Since different values of $\theta_i$ over any boundary $z_i = 0$ still represent the same point, and all of the other angular coordinates are equal by the orbifold reduction process, the two boundaries are identified automatically.

However, even when a pair of half-branes have a common boundary, we still have to double the geometry in order to obtain submanifolds that are special Lagrangian. This will be apparent in two places below: in the following paragraphs when we consider the construction of globally special Lagrangian submanifolds from a pair of half-branes, and in the next section when we consider the image of these submanifolds in the NLSM phase, and their intersection with the quintic.

Given this construction of (special) Lagrangian submanifolds of the LG orbifold target space, I will now derive their relationship to the A-type rational boundary states of the Gepner model.

Consider a pair of rays in a single coordinate $z_i$. They are labeled by $(k + 2)^{th}$ roots of unity where $n$ labels the incoming ray and $\overline{n}$ labels the outgoing. The correspondence between these labels and the labeling $(L_i, M_i, S_i)$ of the A-type minimal model boundary states is as follows [8]:

\begin{align*}
L_i &= |\overline{n} - n| - 1 \\
M_i &= n + \overline{n} + \eta \\
S_i &= \text{sign}(\overline{n} - n) + \eta
\end{align*}

(51)

where $\eta = 0, 1$ for the R/R, NS/NS sectors respectively. Since we are interested in constructing BPS D-branes, which are objects with R/R charge, I henceforth restrict to boundary states with $\eta = 0$. Geometrically, $L_i$ is related to the opening angle of the rays, $M_i$ is related to the rigid rotation angle of the pair, and $S_i$ gives the orientation (see figure 6).

This correspondence reproduces the field identification (11) as follows. There are two possible ways of interpreting the opening angle of the pair of rays: we can either start from $n$ and proceed counterclockwise around the circle to $\overline{n}$ (giving $L_i = |\overline{n} - n| - 1$), or we can start at $\overline{n}$ and proceed counterclockwise around the circle to $n$, and then reverse the orientation.
Figure 6: The Lagrangian submanifolds corresponding to A-type rational boundary states are composed of a pair of rays aligned along roots of unity in each coordinate $z_i$ of the linear sigma model target space (fifth roots of unity for the quintic), with compatible orientation. The submanifold represented here corresponds to the $L = 1, M = 2, S = 1$ boundary state of the $(k = 3)$ $\mathcal{N} = 2$ minimal model; the A-type rational boundary states of the Gepner model are obtained by taking this class of boundary condition on each of the coordinates $z_i$.

to obtain the same submanifold. Using the fact that the $n$ are defined mod $k + 2$, this gives

$$
L'_i = (n + k + 2) - \overline{n} - 1 = k - (\overline{n} - n - 1) = k - L_i
$$

$$
S'_i = S_i + 2
$$

$$
M'_i = (n + k + 2) + \overline{n} = M + k + 2
$$

as desired.

Furthermore, the geometrical intersection of the pair of rays gives the extended $SU(2)_k$ fusion coefficients [8]. Two D-branes $(n, \overline{n})$, $(m, \overline{m})$ which are each pairs of rays will intersect with positive orientation if they “overlap”, i.e. if the rays alternate or are paired between one D-brane and the other as one proceeds counterclockwise around the circle (see figure 6).

$$
n \leq m < \overline{m} < \overline{n} < n + k + 2
$$

Some simple algebra [8] brings these inequalities into the form (14). To obtain a negative intersection number we can just reverse the orientation of one of the D-branes of a pair that intersects positively; this can be expressed as a prefactor $(-1)^{S - \overline{S}}$. In other words, the intersection form of two pairs of rays in a single coordinate is compactly summarized by the extended $SU(2)_k$ fusion coefficients:

$$
I_{m.m.}(L, \overline{L}, M, \overline{M}) = (-1)^{S - \overline{S}} N_{L, \overline{L}}^{M - \overline{M}}
$$

(54)
This can be extended to the intersection form of the A-type rational boundary states of the Gepner model, which are built up from A-type boundary states in each of the minimal model factors: the intersection number of two sets of rays in \( n \) coordinates is just the product of the intersection number in each coordinate, and since the target space is a \( \mathbb{Z}_5 \) orbifold we must sum over the intersections in the twisted sectors, which all project to the same image in the orbifold space. Since the \( \mathbb{Z}_5 \) acts by a common \( M_j \mapsto M_j + 2 \) this is the same as summing over all shifts of one of the sets of \( M_j \) values by an even integer, i.e. relative \( \mathbb{Z}_5 \) rotations of the two pairs of spokes.

Therefore,

\[
I_A = \frac{1}{C}(-1)^{\frac{K-2}{2}} \sum_{\nu_0=0}^{r} \prod_{j=1}^{K-1} I_{\text{m.m.}}(L_j, \overline{L}_j, M_j + 2\nu_0, \overline{M}_j)
\]

\[
= \frac{1}{C}(-1)^{\frac{K-2}{2}} \sum_{\nu_0=0}^{r} \prod_{j=1}^{K-1} N_{L_j, \overline{L}_j}^{2\nu_0 + M - \overline{M}}
\]

where \( C \) is an overall normalization constant, and we have identified the \( S_i \) values in the individual minimal models, as discussed in section 2.2. This is the desired intersection form (13). Upon dividing by the additional \((\mathbb{Z}_5)^3\) orbifold symmetry to get to the A-type intersection form on the mirror model, one obtains the intersection form (14), which is also the intersection form \( I_B \) of the B-type states on the original manifold (as expected by mirror symmetry). Note that the presence of the \( SU(2)_k \) fusion rules in both A- and B-type intersection forms is related to a geometrical description in terms of spokes.

Following [11] the intersection forms \( I_A, I_B \) can be expressed as a polynomial in \( \mathbb{Z}_5 \) shift generators

\[
g_i : M_i \mapsto M_i + 2
\]

The operator \( g_i^{1/2} \) corresponds to a shift \( M_i \mapsto M_i + 1 \), which is needed when \( L_j + \overline{L}_j = 1 \mod 2 \) to satisfy the relation (9). The intersection form can be expressed as a matrix where the entries are labelled by the \((M_j, \overline{M}_j)\) values.
Algorithmically, the intersection matrix is built up by considering the intersection number of two spoke pairs according to the rules (53), summing over the $\mathbb{Z}_5$ shifts generated by $g_i$ (i.e. the coefficient of each $g_i^a$ in the polynomial is given by the intersection number of the first spoke pair and the second spoke pair rotated by $g_i^a$). For the A-type states the $g_i$ are subject to the relation $\prod_{i=1}^5 g_i = 1$ which implements the triviality of a common $\mathbb{Z}_5$ rotation, and for B-type states the $g_i$ are all identified with a single $g$ by the $(\mathbb{Z}_5)^4$ symmetry. Thus, the polynomial encodes the intersections of physically distinct combinations of $M$ labels for a given set of $L_i$ labels.

For example, consider the intersection of two spokes with $L = 1$ in the $(k = 3)$ minimal model, where the roots of unity of the two spoke pairs are given by $(n, n + 2 \mod 5)$ and $(m, m + 2 \mod 5)$. The intersections are summarized in Table 1.

### Table 1: The intersection number of two pairs of spokes, aligned along roots of unity given by $(n, n + 2 \mod 5)$ and the same spoke pair rotated by $g^a$, i.e. $(n + a, n + a + 2 \mod 5)$, for the 5 values of $a$. The intersection numbers are calculated according to the rules (53).

| Rel. shift | State 1 | State 2 | Intersection |
|------------|---------|---------|--------------|
| 1          | (0, 2)  | (0, 2)  | 1            |
| $g$        | (0, 2)  | (1, 3)  | 0            |
| $g^2$      | (0, 2)  | (2, 4)  | -1           |
| $g^3$      | (0, 2)  | (3, 0)  | -1           |
| $g^4$      | (0, 2)  | (4, 1)  | -1           |

Since $0 \leq 0 < 2 \leq 2$, $0 \leq 1 < 2 \leq 3$, $0 \leq 2 < 2 \leq 4$, $0 \leq 0 < 2 \leq 3$ (Note orientation reversal), $0 \leq 1 < 2 \leq 4$

Therefore

$$I_A(L = 1) = (1 + g - g^3 - g^4)$$ (57)

for the single minimal model. Therefore for the A-type rational boundary state of the full Gepner model the intersection matrix is:

$$I_A(L = \{11111\}) = \prod_{i=1}^5 (1 + g_i - g_i^3 - g_i^4)$$ (58)

subject to $\prod_{i=1}^5 g_i = 1$. I will make use of this intersection form in the next section.

The phase of the holomorphic 5-form (33) (also known as the “$U(1)$ grade”) of a half-brane $L$ is valued in the $\mathbb{Z}_5$ subgroup of $U(1)$ and is given by the sum of the angles in each coordinate plane.

$$G = \text{Im} \log \Omega^{(5)}|_L = \sum_{i=1}^5 \theta_i = \frac{2\pi}{5} \sum_{i=1}^5 n_i$$ (59)

where $n_i$ label the roots of unity of the half-brane. Note that it is invariant under the $\mathbb{Z}_5$ orbifold symmetry. When we take a second half-brane $\overline{L}$ we must reverse its orientation in
order to have a compatible boundary orientation at the origin; this shifts the assignment of
the grade by \(\pi\):

\[
\overline{G} = \text{Im} \log \Omega^{(5)}|_L = \sum_{i=1}^{5}(\theta_i + \pi) = (\sum_{i=1}^{5}\theta_i) + \pi
\]  

(60)

It is easy to verify that with this assignment of grades a D-brane made up of two half-branes
with opening angle \(\pi\) in each coordinate (i.e. which is isomorphic to flat \(\mathbb{R}^n\)) will have a
constant grade everywhere; and conversely a half-brane and the same half-brane taken with
opposite orientation (i.e. the antibrane to the first) have a relative grade of \(\pi\).

This assignment of grades forces us to take at least one of the coordinates of the half-
brane to its doubled image in order to obtain a pair of half-branes with the same grade: if all
5 \(\theta_i\) angles are valued in fifth roots of unity for both incoming and outgoing half-branes, then
there will be an extra shift of \(5\pi \simeq \pi\) between the two grades, and for models with \(k = \text{odd}\)
there is no way to make the two grades equal (since \(\pi\) is not a \((k+2)^{th}\) root of unity). This is
remedied by taking an odd number of the \(\theta_i\) to their doubled image \(\theta_i \mapsto \theta_i + \pi\), which
causes the two grades to be valued in the same \(\mathbb{Z}_5\) subgroup of \(U(1)\) so they can potentially be equal.

Thus, in each minimal model (\(z_i\) coordinate) the boundary conditions are isomorphic to those
constructed in [8], but there are additional constraints on how the boundary conditions on
each of the \(z_i\) can be glued together to form supersymmetric boundary conditions for the
full Gepner model (\(\mathbb{C}^3/\mathbb{Z}_5\) orbifold).

The grade of the submanifold is the same in both phases of the LSM, because the \(\mathbb{Z}_5\)
action on the roots of unity becomes part of the projective action on the coordinates of
\(\mathbb{P}^4\). Therefore D-branes that are mutually special Lagrangian in one phase will still be
in the other. This is to be expected because the special Lagrangian submanifolds do not
decay under variation of Kähler structure [15]. Their stability depends only on the complex
structure of the Calabi-Yau manifold and not the Kähler structure (except through the
Lagrangian condition), and we have fixed the 101 complex structure moduli of the quintic
to 0 throughout.

The condition for two half-branes \(\{\theta_i\}, \{\overline{\theta}_i\}\) to preserve the same A-type supersymmetry
is therefore:

\[
\sum_{i=1}^{r}\overline{\theta}_i = \sum_{i=1}^{r}\theta_i
\]

\[
\Leftrightarrow \sum_{i=1}^{r}(\overline{n}_i - n_i) = 0
\]

\[
\Leftrightarrow \sum_{i=1}^{r}L_i = r
\]

\[
\equiv 0
\]  

(61)

using the labeling identification \(L_i = \overline{n}_i - n_i - 1\), and where the last equivalence is true for
the quintic since \(k + 2 = r = 5 \equiv 0 \text{ mod } 5\), but may not be true in a more general model.

D-branes with \(\sum_i L_i \neq r\) are not special Lagrangian since the two half-branes have
different grades, and we expect them to flow to another state in the same homology class
that is special Lagrangian. For the quintic (and the \((k = 1)^3\) torus model) every pair of
spokes has a special Lagrangian in the same homology class that is also another pair of
spokes (see Appendix A for the proof of this result). For more general Gepner models,
construction of the special Lagrangian appears to be more subtle, because some of the spoke
pairs may fail to have a special Lagrangian representative that is also a spoke pair. Since the
spoke pairs reproduce the topological properties of the A-type boundary states, they are still
good topological representatives of this unknown special Lagrangian, but I do not presently
know how to explicitly construct it for those that fail.

Comparison of (51), (59) and (61) shows that for special Lagrangian spokes of the quintic,
the grade (11) of the Gepner model boundary states from CFT reduces to
\[
\sum_{i=1}^{r} \frac{M_i}{k_i} + 2 = \sum_{i=1}^{5} n_i + \frac{n_5}{5} = \frac{2}{5} \sum_{i=1}^{5} n_i = \frac{G}{\pi}
\]  
(62)
i.e. it is in agreement with the geometrical grade of the special Lagrangian (the phase of the
holomorphic 5-form).

4.3 D-branes in the large volume limit

Having characterized the properties of the spokes in the LG phase, where they are related
to the rational boundary states of the Gepner model, I now turn to their properties in the
geometrical phase \( r > 0 \).

Recall that the transition from the LG phase to the NLSM phase of the quintic blows up
the orbifold fixed point at the origin of \( \mathbb{C}^5/Z_5 \) into a \( \mathbb{P}^4 \), which becomes the zero section of
the line bundle \( \mathcal{O}_{\mathbb{P}^4}(-5) \). The vacuum submanifold of the linear sigma model in this phase is
the quintic hypersurface in \( \mathbb{P}^4 \), and we will see that the 5-dimensional special Lagrangian
submanifolds of \( \mathcal{O}_{\mathbb{P}^4}(-5) \) (pairs of spokes) intersect this hypersurface to form 3-dimensional
special Lagrangian submanifolds of the quintic.

Also recall from section 3 that the Ricci-flat metric on \( \mathcal{O}_{\mathbb{P}^{n-1}}(-n) \) is the Calabi metric,
which has a copy of the Fubini-Study metric (30) on the \( \mathbb{P}^{n-1} \) at \( w = 0 \):
\[
ds^2 = \frac{1}{\rho} \left( dy_i dy^i - \frac{1}{\rho} y_i dy^i y_j dy^j \right)
\]  
(63)
where \( \rho = 1 + y_i y^i \), and \( y_i \) are inhomogeneous coordinates on the \( \mathbb{P}^{n-1} \) in a coordinate patch
\( \mathcal{U}_n \). \( \mathbb{P}^{n-1} \) may be described topologically by a copy of \( \mathbb{C}^{n-1} \), plus a \( \mathbb{P}^{n-2} \) at infinity that
compactifies the space. In a local coordinate patch we do not see the \( \mathbb{P}^{n-2} \) at infinity and
the patch is diffeomorphic to \( \mathbb{C}^{n-1} \).

We need to find the change of coordinates from the coordinates induced from \( \mathbb{C}^{n+1} \) on
the line bundle, to the inhomogeneous coordinates on \( \mathbb{P}^{n-1} \). Consider a vertex of the \((n-1)\)-
simplex that is the toric base of the \( \mathbb{P}^{n-1} \) in the induced metric from \( \mathbb{C}^{n+1} \). There are \( n-1 \)
lines meeting at the vertex, and the face opposite to the vertex is an \((n-2)\)-simplex. The
Fubini-Study metric on \( \mathbb{P}^{n-1} \) effectively stretches out the \( n-1 \) lines meeting the vertex
to infinite coordinate distance; the opposite \((n-2)\)-simplex is pushed off to infinity. The
\((n-2)\)-simplex is the base of a \( \mathbb{P}^{n-2} \) when we include the (degenerate) \( T^{n-1} \) fibres, and if
we delete it we are left with a non-compact space which is diffeomorphic to \( \mathbb{C}^{n-1} \) since each
of the coordinates along the lines meeting our vertex also comes with an \( S^1 \) fibre (recall
the toric construction of \( \mathbb{C}^n \) in section 3). We can repeat this construction for any of the \( n \) vertices of the \((n-1)\)-simplex; these are the \( n \) coordinate charts of the \( \mathbb{P}^{n-1} \).

We can parametrize the simplex at \( |z_{n+1}|^2 = 0 \) by the \( n-1 \) coordinates \((|z_1|^2, \ldots, |z_{n-1}|^2)\). The remaining coordinate \( |z_n|^2 \) depends on the first \( n-1 \) according to

\[
|z_n|^2 = r - \sum_{i=1}^{n-1} |z_i|^2 \tag{64}
\]

in order to satisfy the D-term constraint \((28)\). We can obtain the other coordinate patches by choosing a different set of \( n-1 \) independent coordinates to parametrize the simplex. Therefore, the \( \mathbb{C}^n \) coordinates \( z_i \) at \( |z_{n+1}|^2 = 0 \) are similar to the usual homogeneous coordinates on \( \mathbb{P}^{n-1} \), except that the \( |z_i|^2 \) range from 0 to \( r \).

The change of variables into the Fubini-Study coordinates is given by:

\[
y_i = \frac{z_i}{z_n} \quad \Leftrightarrow \quad |y_i|^2 = \frac{|z_i|^2}{r - \sum_{i=1}^{n-1} |z_i|^2} \tag{65}
\]

\[
y_i = |y_i|^2 e^{i(\theta_i - \theta_n)} \tag{66}
\]

which are just the usual projective coordinates on \( \mathbb{P}^{n-1} \). Note that since the angles are shifted by roots of unity the projected spokes again look like spokes in the coordinate patch, except that they are only spokes in \( n-1 \) coordinates; the remaining coordinate is pushed off to infinity in the coordinate patch on \( \mathbb{P}^{n-1} \), but it is visible by considering the image of the submanifold in two different coordinate patches.

In projective coordinates the quintic polynomial becomes

\[
1 + \sum_{i=1}^{4} |y_i|^5 = 0 \tag{67}
\]

If the submanifolds are parametrized along \( n^{th} \) roots of unity, then this equation has no solution on the submanifold, because \( y_i^5 \in \mathbb{R}^+ \). Therefore, in order for the D-branes to intersect the quintic hypersurface in \( \mathbb{P}^{n-1} \) we must take one or more of the \( \theta_i \mapsto \theta_i + \pi \) to introduce a relative minus sign into one of the terms in \((67)\). This is the same prescription that was required to cancel the boundary of two piecewise special Lagrangian submanifolds and to obtain globally special Lagrangian submanifolds from the pair.

This construction reproduces an old construction of 3-cycles on the mirror quintic from \cite{22}, which are used to compute the periods of the holomorphic 3-form. The construction was further analyzed in \cite{23}, where the 3-cycles were termed “spokes”, and in \cite{43}. Those

\footnote{There is a slight over-generalization in the discussion of these 3-cycles in the appendix to that paper, which implies that an arbitrary pair of spokes on a general Calabi-Yau manifold will have a common boundary. As shown in section 4.2 this is in fact only true for the Greene-Plesser mirror where there is additional orbifold symmetry. This correction does not affect the results of that paper, since they only use the construction to study periods of the mirror manifold.}
papers were concerned with the intrinsic geometry of the compact Calabi-Yau manifold itself, however the analysis naturally fits into the LSM framework discussed in this paper since the holomorphic 3-form on the compact Calabi-Yau manifold is obtained by pullback from $\mathbb{C}^{n+r}$ via the construction outlined in section $[3]$, so A-type D-branes of the LSM descend to A-type D-branes of the compact Calabi-Yau manifold in the infrared. The results of $[22]$ were also used in $[11]$ to study the B-type rational boundary states of the quintic at the Gepner point and at large volume, which are mirror to A-type states of the mirror quintic. Those results will be analyzed in more detail in the following section.

Topological properties of the spokes such as intersection numbers should not change on transition from the LG phase to the NLSM phase, since from the point of view of the LSM we are considering submanifolds defined by constant $\theta_i$, and the transition between the two phases is implemented by translating the defining hyperplane (26) away from the origin in $\mathbb{C}^6$ to truncate the “tip” of the toric base space into a simplex (see section $[3]$). In other words the transition does not modify the fibre coordinates and the passage from LG to NLSM phase simply “chops off the tip” of the 5-dimensional half-branes.

This accords with the observation in $[11]$ that the intersection form of the $L = (11111)$ A-type states in the $(k = 3)^5$ Gepner model is the same as the intersection form of certain special Lagrangian 3-cycles on the quintic (the 625 $\mathbb{RP}^3$'s constructed in section $[2.1]$), however there is an important difference which I will now discuss.

Recall from section $[2.1]$ that the $\mathbb{RP}^3$ special Lagrangians were constructed as the fixed-point set of a real involution, and their image in each of the coordinates $z_i$ of $\mathbb{C}^5$ is a straight line aligned along the $\omega_i^{th}$ fifth root of unity (i.e. two rays through the origin with opening angle of $\pi$). However, the construction of the $L = (11111)$ spokes gives a submanifold of the quintic that is “bent” and has an opening angle of $\frac{4\pi}{5}$ in each of the $z_i$ planes. It is also a special Lagrangian submanifold since the two half-branes have the same grade, as discussed above.

As discussed in section $[2.2]$ the spoke construction correctly reproduces the intersection form $I_A$ of the A-type rational boundary states and is equal to

$$I_A(11111) = \prod_{i=1}^5 (1 + g_i^2 - g_i^3 - g_i^4)$$

The intersection form of the $\mathbb{RP}^3$'s was calculated in $[11]$ to be

$$I_{\mathbb{RP}^3} = \prod_{i=1}^5 (g_i^2 - g_i^3 - g_i^4)$$

which is equal to $I_A$ up to the relation $\prod_{i=1}^5 g_i = 1$ and an overall minus sign (which presumably comes from a relative change of orientation between the conventions used to define the $\mathbb{RP}^3$'s and the $L = (11111)$ spokes). Therefore, we have two distinct special Lagrangians with the same intersection numbers.

In terms of the gauge theory living on the world-volume of the D-brane, stability of the D-brane (i.e. existence of a stable vacuum) is governed by the D-terms of the gauge theory, whereas the moduli space of deformations of the D-brane is determined by the F-flatness
conditions on the world-volume superpotential:
\[
\frac{\partial W}{\partial \psi} = 0
\]  
(70)

where \( \psi \) are the massless chiral fields of the world-volume theory. In other words, flat directions of the superpotential correspond to exactly marginal deformations.

For A-type D-branes, the world-volume D-terms depend only on the complex structure moduli of the Calabi-Yau, and similarly the superpotential (and therefore the moduli space of deformations of a given stable D-brane) depends only on the Kähler moduli (as usual, the role of Kähler and complex structure moduli are interchanged by mirror symmetry, i.e. for the mirror B-model). Away from the large volume limit the world-volume superpotential receives corrections from open string instantons, since the area of the open string world-sheet is measured by the Kähler form.

McLean [44] showed that the moduli space of deformations of a compact special Lagrangian submanifold \( L \) is (locally) a smooth manifold of dimension \( b_1(L) \) (the first Betti number of \( L \)). In particular, special Lagrangian submanifolds with vanishing first Betti number are rigid. This is the case for the special Lagrangian submanifolds of the quintic under discussion, since they are diffeomorphic to \( \mathbb{R}P^3 \) which has \( H_1(\mathbb{R}P^3, \mathbb{Z}) \cong \mathbb{Z}_2 \). Since we have two rigid special Lagrangians in the same homology class, the 0-dimensional moduli space has (at least) two components. In fact, all of the special Lagrangian submanifolds associated to rational boundary states of the quintic are topologically \( \mathbb{R}P^3 \), by a piecewise version of the argument given in section \[27\]. Therefore all the A-type D-branes associated to A-type rational boundary states of the quintic are rigid in the large volume limit.

The instanton corrections to the superpotential may be thought of as “stringy” modifications to the classical deformation theory of the special Lagrangian submanifolds. In other words, McLean’s theorem is only true perturbatively in \( \alpha' \) and may be violated non-perturbatively in \( \alpha' \) by instanton corrections [40, 41].

In the large volume limit, marginal operators of the world-volume gauge theory correspond to geometrical deformations of the special Lagrangian submanifold and the flat \( U(1) \) gauge bundle living on it, which are both classified by \( H_1(L, \mathbb{Z}) \) [45] and pair to form complex moduli fields [21] (when this group is finite there are \( |H_1(L, \mathbb{Z})| \) distinct choices of flat \( U(1) \) bundle [20]). Away from the large volume limit, open string instanton contributions to the superpotential are enumerated by a choice of 1-cycle of the special Lagrangian upon which the open string world-sheet ends (as well as a 2-cycle of the bulk Calabi-Yau around which the interior of the world-sheet wraps) [27]. In particular, for the rigid special Lagrangians under discussion, \( H_1(L, \mathbb{Z}) \cong \mathbb{Z}_2 \) and the instanton contribution dramatically simplifies (an open string worldsheet can only wind once around the single 1-cycle to give a nontrivial contribution).

Since there are no deformations of these special Lagrangians in the large volume limit (i.e. massless fields), the only way they can arise at the Gepner point is due to instanton effects. The counting of massless boundary fields at the Gepner point was described in [11]. One finds that the only A-type rational boundary states of the quintic which possess massless fields are the \( L = (11111) \) states (which have one), and all other boundary states have no massless fields in their spectrum.

\[ \text{I am grateful to I. Brunner for a discussion on this point.} \]
This massless field may have an instanton-generated superpotential. A superpotential was postulated in \[11\] of the form
\[
W = \psi^3 + \psi \phi \tag{71}
\]
where \(\psi\) is the massless boundary field and \(\phi\) is the (bulk) Kähler modulus. The cubic term in this superpotential was calculated explicitly in \[24\] and was indeed found to be non-vanishing. The term linear in \(\psi\) was not calculated explicitly in that paper, but it is not forbidden by selection rules and is therefore likely to also be non-vanishing. Thus away from the Gepner point \((\phi = 0)\) the superpotential has two distinct vacua corresponding to the distinct gauge bundles on a special Lagrangian in the \(L = (11111)\) homology class, and these vacua combine and become degenerate at the Gepner point. Therefore all of the A-type rational boundary states of the quintic are rigid at the Gepner point as well as in the large volume limit.

The story for the A-type states of the mirror quintic is more complicated, since these special Lagrangians may have \(b_1 > 0\) due to identifications under the Greene-Plesser orbifold action. Indeed, the boundary spectrum of these D-branes at the Gepner point generally contains large numbers of massless fields \([11]\), and explicit computations in the B-model \([25]\) shows that these fields often remain exactly marginal at the Gepner point (i.e. the flat classical superpotential \(W = 0\) is not completely lifted by the instanton contributions).

4.4 Relationships within the charge lattice

I now turn to the question of relationships between the rational boundary states, from the point of view of the D-branes of the linear sigma model constructed in section 4.2. I will show that this construction correctly reproduces the numerous relationships which exist between the large-volume homology classes of the rational boundary states; by exploiting the “spoke” structure of the D-branes this reduces to a simple problem involving the addition of polynomials. This serves as a nontrivial check of the construction, and the discussion may be a useful starting point for describing the dynamics of tachyon condensation of an unstable pair of boundary states into a bound state.

The A- and B- type rational boundary states of the Gepner model associated to the quintic were studied in \([11]\) and were related to geometrical objects in the NLSM phase using results from \([22]\) (the main results from \([11]\) on A-type states were reproduced in the previous two sections). By essentially ignoring the problem of stability of the B-type states as they are transported through Kähler moduli space, a set of large-volume topological invariants (Chern classes, or equivalently, the D-brane Ramond-Ramond charges) can be associated to these B-type D-branes using an algorithmic procedure.

The analysis of \([11]\) mostly focused on B-type states since these are both fewer in number for the quintic and easy to associate to known geometrical objects (such as bundles and sheaves on \(\mathbb{P}^4\)); however since the B-type rational boundary states map under mirror symmetry to the A-type rational boundary states of the mirror model we can hope to reproduce these results using the LSM construction of A-type D-branes of the mirror model.

Computing the spectrum of stable B-type D-branes at a generic point in Kähler moduli space is a difficult task because the B-type D-branes can decay under Kähler deformations
away from the deep interior points where the spectrum is known \cite{15}. The instability of B-type D-branes under variation of Kähler structure is due to instanton effects and is difficult to study directly (it is formulated in terms of “\(\Pi\)-stability” of vector bundles \cite{15}, and the formalism of the derived category of coherent sheaves \cite{16, 17, 18}); however under mirror symmetry this is mapped into the geometrical problem of stability of special Lagrangian submanifolds under variation of complex structure. It is known \cite{20} that there are “walls” (hypersurfaces of real codimension one) in the complex structure moduli space of Calabi-Yau 3-folds; as one deforms towards such a wall, a family of special Lagrangian submanifolds becomes singular and it can cease to exist on the other side of the wall.

Thus, we also expect there to be stability problems in transporting A-type D-branes from the mirror to the Gepner point \(\psi = 0\) to the large complex structure limit \(\psi \to \infty\), but we can study the topology of such objects in the same sense as in \cite{11}.

The spoke pairs constructed in the previous section descend to special Lagrangian submanifolds of the compact Calabi-Yau when the complex structure moduli are fixed to zero: away from this point the singular, V-shaped spokes are deformed and become smoothed out. This process was studied briefly in \cite{22, 8}, and it may be possible to obtain a more complete understanding of the stability of BPS D-branes by studying complex structure deformations of A-type D-branes in more detail.

The spectrum of D-brane charges of the B-type Reckangel Schomerus states was computed in \cite{11}. They form an overcomplete basis for the D-brane charge lattice of the quintic, but they are not an integral basis because the D0 charge of the B-type states only occurs in multiples of 5. This is a generic feature of the B-type rational boundary states and is true even for the simplest Gepner models, which are associated to 2-tori.

The complete spectrum of A-type D-branes on a torus \(T^2\) is given by circles \(S^1\) with all possible integer winding numbers; however (roughly speaking) the rational boundary states only correspond to the D-branes with unit winding numbers, and not the higher winding cycles. Under mirror symmetry (T-duality) they become even-dimensional cycles with D0 charge given by multiples of some integer (multiples of 3 for the \((k = 1)^3\) torus model). The D0 charge comes from the complex structure parameter \(\tau\) of the torus: a D1-brane wrapping along the lattice vector \(\tau\) dualizes into a D-brane system with flux coming from the angle \cite{46}.

In other words, the A-type rational boundary states are an integral basis for the middle-dimensional homology, but under mirror symmetry the B-type states at the Gepner point are not an integral basis for \(H_0\) but form a sublattice of finite index within it. This can be explained by the additional orbifold symmetry that exists in the LG phase: for the quintic we cannot consider just a single D0-brane in the LG orbifold model, but must consider its images under the \(\mathbb{Z}_5\) symmetry as well. In the NLSM phase there is no orbifold symmetry, and a single D0-brane can potentially be BPS (as expected, since in the limit of large volume the Calabi-Yau becomes approximately flat, and a single D0-brane is BPS in flat space).

The main result of this paper (see also \cite{21}) is that the rational boundary states descend from certain submanifolds of the linear sigma model in both phases. Therefore it is no great surprise that they produce objects with multiple D0 charge even in the NLSM phase: for example a generic curve in \(\mathbb{P}^4\) (i.e. a D-brane inherited from the non-compact linear sigma model target space) will intersect the quintic in 5 points, giving a D-brane with D0 charge of 5.
Table 2: Large-volume D-brane (Ramond-Ramond) charges of the B-type rational boundary states, computed using the procedure in [11]. Permutations of the set of \( L_i \) values have the same charges, and performing a “field identification” on \( n \) of the \( L_i \) to interchange \( L_i = 0 \leftrightarrow L_i = 3 \) or \( L_i = 1 \leftrightarrow L_i = 2 \) (which also shifts the \( M \) labels) introduces an overall sign of \((-1)^n\). Therefore the remaining combinations of \( \{L_i\} \) not listed here also have these charges up to an overall minus sign.

The rational boundary states are only *generators* for the homology lattice, and it seems hard in general to construct a larger class of boundary states purely within the Gepner models\(^8\), but since we have associated these boundary states to a certain subclass of the D-branes in the LSM it is clear that a more general construction should be possible: presumably these more general D-branes will give rise to states of higher D-brane charge. I will discuss the possibilities for constructing more general Gepner model boundary states in section 5.

I present the complete list of D-brane (Ramond-Ramond) charges of the B-type rational boundary states on the quintic in Table 2. These were obtained using the algorithm in [11]. In order to look for relationships between the A-type D-branes (and therefore the B-type D-branes on the mirror) it is convenient to encode the submanifolds as certain polynomials. Recall that for the models under consideration the D-branes in the LSM description are classified by a pair of rays aligned along \( n^{th} \) roots of unity (with compatible orientation) in each coordinate \( z_i, \ i = 0, \ldots, n-1 \), where not all of the choices in each of the \( z_i \) coordinates are distinct due to the orbifold symmetry of the target space which also acts by some combination of \( \mathbb{Z}_n \) rotations.

For each independent coordinate, we introduce a \( \mathbb{Z}_n \) rotation generator \( R_i \) (i.e. satisfying

---

8See [17] for a construction of the higher-winding boundary states in the \( T^2 \) models; unfortunately this construction does not seem to generalize to the higher-dimensional Gepner models.

9The partial list of D-brane topological invariants in that paper is given in terms of the rank and Chern numbers \( \text{Ch}_n \) of the vector bundles instead of the D-brane charges.
$R_i^n = 1$) which acts on the positive real axis in $\mathbb{C}$ to produce an outgoing ray aligned in the $e^{2\pi i/n}$ direction (recall that the special Lagrangian submanifolds we are considering look like copies of $(\mathbb{R}^+)^n$, i.e. rays $\mathbb{R}^+$ in each coordinate. Reversing the orientation of the ray (to obtain the incoming ray) is represented by taking $-R_i$ instead of $R_i$.

In this formalism, the A-type D-branes constructed in section 4.2 are represented by completely factorizable polynomials which can be written as the product of $n$ terms:

$$P(R_1, \ldots, R_n) = \prod_{i=1}^{n} (R_i^{k_i} - R_i^{k_i'})$$  \hspace{1cm} (72)

which corresponds to rays along the $(k_i, k_i')$ roots of unity in the $i$th coordinate. The labelling identification (51) can be used to translate back and forth between the polynomials, the $(L_i, M_i, S_i)$ labels of the boundary states, and the geometrical image of the D-brane. Note that these polynomials are different from the polynomials discussed in section 4.2; those polynomials encoded the intersection of two boundary states; the polynomials currently under discussion describe the geometrical image of a single D-brane.

The $\mathbb{Z}_5$ symmetries of the quintic and its mirror are manifested as relations between the generators $R_i$: for the quintic the generators satisfy

$$\prod_{i=1}^{5} R_i = 1$$  \hspace{1cm} (73)

while for the mirror the $(\mathbb{Z}_5)^4$ symmetry implies

$$R_i^4 R_5 = 1, \ i = 1, \ldots, 4$$
$$\Leftrightarrow R_i^5 R_5 = R_i$$
$$\Leftrightarrow R_5 = R_i$$  \hspace{1cm} (74)

i.e. the generators are all identified and we are left with a polynomial in a single generator.

In this language, it is clear how one can look for relationships between the D-branes corresponding to rational boundary states: find two factorizable polynomials (rational boundary states) that, when added together, produce a third polynomial that is also factorizable (another rational boundary state), up to the relations (73) or (74).

For example, consider the two A-type D-branes on the mirror quintic represented by

$$P_1 = (R - 1)^5$$
$$P_2 = (R - 1)^4(R^2 - R)$$

which are equivalent by mirror symmetry to B-type rational boundary states on the quintic, in this case two states in the $L = (00000)$ orbit with $M = (1 \cdot 5) + (0 \cdot 5) = 5$ and $M = (1 \cdot 4 + 2) + (0 \cdot 4 + 1) = 7$ respectively. Under addition:

$$P_{(1+2)} = P_1 + P_2 = (R - 1)^4 \left[(R - 1) + (R^2 - R)\right] = (R - 1)^4(R^2 - 1)$$  \hspace{1cm} (76)
which is the polynomial corresponding to the state \( \{ L = (00001), M = 6 \} \). Referring to Table 2 it is seen that addition of the D-brane charges is indeed satisfied:

\[
\{ L = (00000), M = 5 \} + \{ L = (00000), M = 7 \} = \{ L = (00001), M = 6 \}
\]

\[
(1, 0, 0, 0) + (-4, -1, -8, 5) = (-3, -1, -8, 5)
\]

A more complicated example is:

\[
P_1 = (R - 1)^3(R^2 - 1)^2
\]

\[
P_2 = (R - 1)^2(R^2 - 1)^2(R^4 - R^2)
\]

\[
P_{(1+2)} = (R - 1)^2(R^2 - 1)^2 [(R - 1) + (R^4 - R^2)]
\]

\[
= (R - 1)^2 [(R - 1)^2(R + 1)^2 [(R - 1) + (R^2 - 1)R^2]
\]

\[
= (R - 1)^4 [(R + 1)^2(R - 1) + (R + 1)^3(R - 1)R^2]
\]

\[
= (R - 1)^4(R^5 - R^3)
\]

which corresponds to the charge relation

\[
\{ L = (00011), M = 7 \} + \{ L = (00111), M = 2 \} = \{ L = (00001), M = 2 \}
\]

\[
(-5, -2, -11, 10) + (6, 3, 14, -15) = (1, 1, 3, -5)
\]

The other relationships within the charge lattice may be obtained similarly.

Translating back to the geometrical language, this again has a simple interpretation; two rational D-branes that combine in such a way to produce another rational D-brane contain a pair of rays which align with opposite orientation, with the other rays in those coordinates being distinct, and in the other coordinate planes both pairs of rays coincide with the same orientation (see Figure 8). The pair of rays with opposite orientation define a homologically trivial submanifold and can be collapsed to give a representative element of the same homology class (which is the sum of the homology classes of the original two D-branes). This is the D-brane corresponding to the sum of the two polynomials.

In physical language, the process of erasing anti-aligned D-brane segments is very reminiscent of tachyon condensation of a coincident D-brane and anti-D-brane pair \[ \text{[48, 49, 50, 51]} \]. We consider two D-branes that are individually BPS and therefore stable (i.e. each of them preserve a particular linear combination of the \( \mathcal{N} = 2 \) supersymmetry generators of the target space), but taken together they do not preserve supersymmetry (i.e. they do not preserve the \textit{same} linear combination of supersymmetry generators, so there is no combination of supersymmetries under which the combined system is invariant). One finds that there is a tachyon in the spectrum of string excitations between the two branes, and this tachyon causes the system to decay into a stable configuration in which supersymmetry is restored.

In our system, the two A-type D-branes we start with preserve a different phase in the A-type linear combination of supercharges and hence break supersymmetry. Since they are special Lagrangian branes, they are minimal volume elements in their homology class with respect to the Kähler structure of the LSM target space, but their sum is not minimal volume in its homology class because of the homologically trivial piece. There will be a maximally tachyonic mode in the spectrum of strings stretched between the anti-aligned segments of the D-brane which will drive a process of tachyon condensation causing these segments to annihilate to the vacuum. After annihilation of the common anti-aligned line segment(s)
Figure 8: Tachyon condensation of the $L = \{00000\}, M = 5$ state with the $L = \{00000\}, M = 7$ state to give the $L = \{00001\}, M = 6$ state, as described in (76) in terms of polynomials. The spoke pairs in the first 4 coordinates of $\mathbb{C}^5$ are the same, but in the fifth coordinate the outgoing ray of the first state is anti-aligned with the incoming ray of the second state. Addition of the two states is achieved by superposition of the pairs of spokes, and the anti-aligned rays are erased (shown as dashed lines in the above figure) to give the endpoint of the condensation process.

we are again left with a D-brane that is minimal volume in its homology class (i.e. another special Lagrangian brane associated to a rational boundary state). One can presumably make this tachyon condensation argument more rigorous in these simple examples.

In terms of the tachyon condensation picture, when we add together two D-branes that have both rays anti-aligned in a coordinate plane (with the rays in the other coordinates all equal), they will annihilate completely to give a D-brane that is trivial in this coordinate. Topologically, two such D-branes have opposite orientation, and they are indeed found to have opposite D-brane charges which therefore cancel to give the vacuum.

5 New boundary states from geometry

In section 4.2 I constructed the set of special Lagrangian submanifolds of $\mathbb{C}^5/\mathbb{Z}_5$ that span the toric base of the orbifold; these are the “base-filling D-branes” or “spokes” discussed in that section. As I have shown, this class of D-brane reproduces the properties of the rational
boundary states of the Gepner model, to which the linear sigma model flows in the infrared at the point \( r \to -\infty \) in Kähler moduli space.

Recall that these submanifolds are characterized by a constant value of the angular coordinates \( \theta_i \) along the submanifold. Therefore, the Kähler form \((18)\) will vanish term by term upon restriction to the submanifold; i.e. these submanifolds are separately Lagrangian with respect to each coordinate \( z_i \) of the target space.

This is directly analogous to the Recknagel-Schomerus construction of boundary states in Gepner models and its further analysis by Fuchs et. al.: recall that this construction preserves an \( \mathcal{N} = 2 \) supersymmetry algebra in each minimal model factor of the Gepner model. Since Lagrangian submanifolds correspond to boundary states of CFT that preserve \( \mathcal{N} = 2 \) worldsheet supersymmetry, the preserved supersymmetry algebras of the CFT and geometrical constructions are in direct agreement.

It is then clear how one might proceed to study a more general class of “non-rational” boundary states of the Gepner model, by relaxing this symmetry condition in the linear sigma model and studying special Lagrangian submanifolds that do not preserve the Lagrangian condition in each coordinate separately, but only for the expression \((18)\) as a whole (and which intersect the vacuum submanifold of the LG phase, i.e. the origin in \( \mathbb{C}^5/\mathbb{Z}_5 \)). This symmetry relaxation seems difficult to study directly in conformal field theory since in general it renders the theory non-rational, but the mapping of this problem into geometry shows how it may be approached geometrically.

One large class of more general special Lagrangian submanifolds are those that respect the toric structure of the linear sigma model target space, i.e. the special Lagrangian submanifolds with \( p \neq 0 \) in the notation of section \( 4.2 \). Recall that they are defined by taking additional hyperplane constraints in the toric base, and corresponding orthogonal subspaces of the torus fibre. Since the special Lagrangians with \( p \neq 0 \) do not have fixed values of the angular coordinates \( \theta_i \) they meet the requirement discussed in the previous paragraph. Note that there are a countably infinite number of these special Lagrangian submanifolds since they are defined by vectors of integer charges \((38)\) that specify the normal vectors to the hyperplanes. In general the translation moduli of these hyperplanes are fixed in the linear sigma model by the requirement that they intersect the orbifold fixed point (the vacuum submanifold of the LG phase, where the CFT lives in the infrared). At large volume these special Lagrangians possess moduli since they have \( b_1 \neq 0 \); these moduli may survive at the Gepner point if they are not lifted by an instanton-generated superpotential.

Taking hyperplanes defined by normal vectors with non-integral entries induce subtori of the \( T^n \) fibres that are non-rational, i.e. they foliate the \( T^n \). This is a familiar example in non-commutative geometry and it may be possible to understand these “non-commutative” special Lagrangians (and corresponding “non-commutative boundary states”) in that context.

There may be other more general classes of special Lagrangian submanifolds that can be studied in the linear sigma model framework and which could be used to define boundary states in CFT: for example special Lagrangian submanifolds that do not respect the toric description of the target space, or that are constructed using more general involutions of the target space (one such possibility currently under study in CFT is \( 52 \)).

There are no obviously defined notions of geometry within the Gepner model itself. However, the structure of the bulk Gepner model as a conformal field theory (on a worldsheet
without boundary) can be thought of as a remnant of the geometry of the bulk linear sigma model (i.e. without D-branes) in this limit. For example, the superpotential of the Landau Ginzburg orbifold theory (which generates the chiral ring) survives as the spectrum of chiral primary operators of the Gepner model [53, 54], and the tensor product structure of the Gepner model descends from the \( \mathbb{C}^5/\mathbb{Z}_5 \) geometry of the linear sigma model target space as explained above.

In the same way that the bulk Gepner model retains a “memory” of the bulk linear sigma model, I propose that the boundary states of the Gepner model should be thought of as remnants of the special Lagrangian submanifolds of the linear sigma model. The construction of Recknagel and Schomerus and Fuchs et. al. amounts to reconstructing this “latent geometry” for the class of \( p = 0 \) special Lagrangians, which are simple enough to construct from first principles in CFT because of their high degree of symmetry.

For more general special Lagrangian submanifolds of the linear sigma model target space, a corresponding boundary state of the CFT would also be defined by RG flow, but certain properties could be studied within the linear sigma model directly, as was done in previous sections for the class of special Lagrangian that descend to the rational boundary states in the infrared.

6 Conclusions and future directions

I have studied the simplest class of toric special Lagrangian submanifolds of the target space of a linear sigma model, which descend to A-type D-branes of string theory on the compact Calabi-Yau in the infrared and which reproduce the topological properties of the rational boundary states of the Gepner model. Some of these submanifolds are only piecewise special Lagrangian, but since they are in the same homology class as a true special Lagrangian we can still use them for topological purposes. Furthermore, for the quintic a true special Lagrangian in this homology class can always be found explicitly.

For the LG model associated to an \( \mathcal{N} = 2 \) minimal model, the behaviour of Lagrangian D-branes under variation of complex structure was considered in [5]. The decay of special Lagrangian submanifolds is a classical geometrical problem which has been studied in [19, 21], and it is not corrected by instantons in string theory. It should be possible to analyze this problem in the linear sigma model, and it may be easier to study than the equivalent B-type problem, where destabilization may be caused by instanton effects and the mathematical description of the decay process is more complicated [15, 16, 17, 18].

The discussion in section 4.4 about relationships within the lattice of rational D-brane states has much of the flavour of a tachyon condensation argument, in which two rational D-brane states condense to another state (possibly another rational boundary state) that is of minimal volume in the same total homology class (hence an A-type D-brane). It would be interesting to analyze the dynamics of this process concretely from the point of view of boundary RG flows in the boundary linear sigma model [53, 57], and to use it to study the fate of unstable intersecting rational boundary states that do not share a common boundary (i.e. for which the tachyon is non-maximal), which are expected to decay to a (presumably non-rational) bound state.

Perhaps the most interesting possibility to emerge from the linear sigma model construc-
tion of A-type D-branes is the construction of new boundary states of the Gepner model. It would be interesting to investigate these possibilities in more detail.

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A Existence of special Lagrangians in the set of spokes

Theorem: For any pair of half-branes described by \((k + 2)\)th roots of unity \(\{n_i\}, \{\overline{n}_i\}\) in \(\mathbb{C}^{k+2}/\mathbb{Z}_{k+2}\), where \((k + 2)\) is a prime, there is a special Lagrangian submanifold in the same homology class that is also described by pairs of \((k + 2)\) roots of unity.

Proof: Performing a field identification \((10)\):

\[
(L_i, M_i, S_i) \sim (k_i - L_i, M_i + k_i + 2, S_i + 2)
\]

on some subset of the \(i\) labels does not change the homology class of a D-brane (up to an overall minus sign for an odd number of field identifications, corresponding to an anti-brane), although it does change the value of \(L_i\) and therefore the grade of the brane (one can verify this by computing the charges of the D-branes according to \([11]\)).

Geometrically, the field redefinition corresponds to changing the orientation of the D-brane in one of the coordinates \(z_i\), which interchanges \(n_i\) from the set of angles labeling the “incoming” ray with \(\overline{n}_i\) labeling the “outgoing” ray; the new pair of half-branes is a different choice of cycle in the same homology class, which has a different grade on each of the half-branes since we have reassigned the \(n_i\).

Suppose the initial grades \(G = \sum_{i=1}^{k+2} \theta_i\), \(\overline{G} = \sum_{i=1}^{k+2} \overline{\theta}_i\). The \(\theta_i\) are valued in \(U(1)\), but if we are looking for a solution amongst the \((k + 2)\)th roots of unity then we restrict to the \(\mathbb{Z}_{k+2}\) subgroup of \(U(1)\) generated by the roots of unity, labeled by the \(n_i\). If \(G = \overline{G}\) then the submanifold is already special Lagrangian and we are done. For \(G \neq \overline{G}\) we want to find a subset \(I \subset \{1, \ldots, k+2\}\) such that interchanging \(n_i \leftrightarrow \overline{n}_i\) for each \(i \in I\) gives

\[
G' = \sum_{i=1}^{k+2} n'_i = \overline{G'} = \sum_{i=1}^{k+2} \overline{n}'_i
\]

The interchange operation on \(I\) shifts \(G\) and \(\overline{G}\) by

\[
G' = G - S
\]

\[
\overline{G}' = \overline{G} + S
\]

\[
S = \sum_{i \in I} (n_i - \overline{n}_i) = \sum_{i \in I} \Delta n_i
\]

Therefore

\[
G - \overline{G} = 2S
\]

The LHS is equal to \(\sum_{i=1}^{k+2} L_i\) and is given. If \(k + 2\) is even, then there exist elements \(G - \overline{G}\) of \(\mathbb{Z}_{k+2}\) for which \([81]\) has no solution for \(S\) (namely the odd elements of \(\mathbb{Z}_{k+2}\)), because
the operation “division by two” is not well-defined on $\mathbb{Z}_{2m}$. Therefore for $k$ even there is a subset of the spoke pairs that cannot be brought into special Lagrangian form by interchange operations.

Restricting to $k$ odd, the element $S$ exists for all values of $\sum_i L_i = G - \overline{G}$ and the problem may be reformulated as follows: find a subset $I \in \{1, \ldots, k+2\}$ such that

$$\sum_{i \in I} \Delta n_i = S$$

$$\sum_{i=1}^{k+2} \Delta n_i = 2S = \sum_{i=1}^{k+2} L_i$$

(82)

If $(k+2)$ is prime, $\mathbb{Z}_{k+2}$ has no proper subgroup, and any $r \geq k+2$ elements of $\mathbb{Z}_{k+2}$ will generate the entire group by taking all possible partial sums. Therefore, for $(k+2)$ prime and $r \geq k+2$ every choice of $\{L_i\}$ has a set of interchange operations to bring it into special Lagrangian form and the theorem is proven. If $r < k+2$ then there may exist elements $2S \in \mathbb{Z}_{k+2}$ for which the corresponding $S$ cannot be obtained by partial sum, because the $\{\Delta n_i\}$ do not span the entire group under partial summation.

If $\mathbb{Z}_{k+2}$ has a proper subgroup, i.e. $k+2$ is composite, then the elements $\{\Delta n_i\}$ may again fail to span the entire group under partial summation, because they can span the subgroup $H$ and possibly some of its cosets $g + H$ without spanning the entire group.

Therefore, for cases other than $\mathbb{Z}_{k+2}$ prime, $r \geq k+2$ there exist spoke pairs in the model that are in the same homology class as a special Lagrangian submanifold (the existence of the A-type rational boundary states ensures the existence of such a special Lagrangian), but for which that special Lagrangian does not exist in the set of spoke pairs. It is generally only a subset of spoke pairs that fail to be globally special Lagrangian, and there are also true special Lagrangian submanifolds in the set of spoke pairs.

\[\square\]

The conditions of this theorem are true for the $(k = 1)^3$ and $(k = 3)^5$ Gepner models, which correspond respectively to a torus $T^2$ with periodicity given by the $su(3)$ root lattice, and the quintic hypersurface in $\mathbb{P}^4$. 

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