On the expansivity gap of integer polynomials

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Abstract

Expansive polynomials (whose roots are greater than 1 in modulus) often arise in dynamical systems and other computational problems. This paper examines the expansivity gap (the gap between 1 and the smallest modulus of the roots) of these polynomials, assuming that the coefficients are integers.

We give lower bounds on the expansivity gap, using the degree and the coefficient size as parameters. We also construct a family of polynomials which indicate the sharpness of these bounds. As a side-result, we present an explicit condition for deciding expansivity of polynomials, which we find superior to the existing recursive methods for our purpose.

1 Introduction

A polynomial is called expansive if all (real and complex) roots have modulus greater than 1. A square matrix is expansive if its characteristic polynomial is expansive. The present paper investigates the question of how tightly a polynomial can fulfil this condition if we require that all coefficients are integers. More precisely, we are interested in the gap between 1 and the smallest absolute value of the roots, which we call the expansivity gap, as it can be considered as a measure of how expansive the polynomial is. We give lower bounds on this gap using the degree and the maximal size of the coefficients. We also show that for fixed degree, our bound is asymptotically sharp in the coefficient size (up to a constant), by constructing families of polynomials having the appropriate order of expansivity gap.

Expansive polynomials or matrices – or their contractive inverse – are mainly used in continuous mathematics, as it is often required in a wide
range of problems involving convergence, where the convergence rate depends on the expansivity gap. Our problem is special, as we restrict our attention to polynomials/matrices with integer coefficients, therefore working in the field of discrete mathematics instead. Our main motivation comes from the complexity analysis of some algorithms in numeration systems. In so-called matrix numeration systems \[3,\ Definition\ 3.2\], one considers a matrix-vector generalization of number systems. Relevant properties of these systems (if all vectors have a representation, the representation is unique, etc.) are related to dynamical properties of a discrete dynamical system obtained using an expansive matrix. The convergence properties of this dynamical system and thus the running time of algorithms are influenced by the expansivity gap of the characteristic polynomial of the matrix.

An \( n \times n \) matrix \( M \) corresponds to a linear map on \( \mathbb{R}^n \), which is repelling if and only if \( M \) is expansive. Given a rounding function on \( \mathbb{R}^n \), the discretized linear map associated to \( M \) is a map on \( \mathbb{Z} \) that applies first \( M \) and then the rounding function. The investigation of these discrete maps comes up in several contexts, e.g. shift radix systems \[11\], periodicity of discretized rotations \[2\] etc. We believe that our results are relevant in these areas for the analysis of dynamical properties and the related algorithms.

We also mention a related, and seemingly very deep problem, Lehmer’s Mahler measure problem (for surveys of results see e.g. \[4\] \[5\] \[6\]). The Mahler measure of a monic polynomial is defined as the product of the absolute values of the roots which are larger than 1 in absolute value (for a detailed discussion, see the book \[7\]). Lehmer investigated how small the Mahler measure can be for specific integer polynomials. The primary motivation for investigating the Mahler measure is also its connection to dynamical systems, namely it gives the topological entropy of a \( \mathbb{Z}^n \)-dynamical system canonically associated to the polynomial.

In this paper, we present two theorems about the order of the expansivity gap of integer polynomials. As an important tool for the proofs, but also as an interesting result on its own, we formulate a necessary and sufficient condition for the expansivity of a polynomial (with not necessarily integer coefficients). We obtain this by expanding the Schur–Cohn algorithm, which is a recursive method to decide expansivity. With careful analysis, by observing how the symbolic expressions obtained from the expansion can be simplified, we give a concise, closed-form condition which gives much smaller coefficients than the naive expansion of the Schur–Cohn algorithm, and which, to our knowledge, has not yet appeared in this form. The construction borrows the general idea from Bareiss’s algorithm \[8\], which is a modification of Gaussian elimination for exact rational arithmetic. Bareiss makes similar symbolic simplifications to avoid exponential coefficient growth, and therefore control the running time of the algorithm.

The paper is built up as follows. Section 2 gives the basic concepts and states the main results of this paper. Section 3 is devoted to the proof of
the expansivity condition and the lower bound on the expansivity gap. In Section 4, we construct a family of polynomials and prove that they have asymptotically optimal expansivity gaps. Finally, in Section 5, we give a summary and further directions.

2 The main results

In this paper, \( f(x) \) denotes a polynomial of degree \( n \):

\[
    f(x) = a_n x^n + \ldots + a_1 x + a_0 \quad (a_n \neq 0),
\]

with mostly \( a_i \in \mathbb{Z} \) and \( n \geq 2 \), except when noted. Denote the coefficient bound by \( A := \max_{i=0}^n |a_i| \). Denote the roots of \( f(x) \) by \( x_1, x_2, \ldots, x_n \) (either real or complex, in any order, counted with multiplicities). Then \( f \) is expansive if and only if \( \forall x_i : |x_i| > 1 \), and the expansivity gap is \( \varepsilon := \min_{i=1}^n |x_i| - 1 \).

We use the convention throughout this paper that any index outside of the allowed range indicates a zero value (e.g. \( a_i = 0 \) for \( i < 0 \) and \( i > n \)).

A well-known tool to decide expansivity of a polynomial is the Schur–Cohn algorithm, which relies on the following lemma:

**Lemma 2.1** (Schur–Cohn test) Let \( f(x) \) be any non-zero real polynomial, and define \( g(x) = \sum_{k=0}^n b_k x^k \) as follows:

\[
    b_k := a_0 a_k - a_n a_{n-k}.
\]

Then \( f(x) \) is expansive if and only if \( |a_0| > |a_n| \) and \( g(x) \) is expansive.

This lemma can be turned into an algorithm by recursively generating this so-called Schur transform and checking the condition \( |a_0| > |a_n| \) for each one in the sequence. Note that since \( b_n = 0, \) \( \deg g(x) < n \), so the recursion terminates with a constant polynomial in at most \( n \) iterations.

This algorithm is however not suitable for exact integer calculations, as the coefficient size doubles each time, which leads to an exponential time. Neither is it sufficient for our purposes, as it would give a much worse bound on the expansivity gap than we give below. Our first result is a replacement for the Schur–Cohn test with only polynomially growing coefficients.

**Lemma 2.2** Let \( a_i \in \mathbb{R} \) and \( n \geq 1 \). Define \( F_{n,k}^\pm \) (with \( 1 \leq k \leq n-1 \), and \( \pm \) is either + or −) to be a determinant of size \( k \times k \) with the following elements:

\[
    d_{ij} = a_{j-i} \pm a_{i+j-n-k-1}, \quad (1 \leq i, j \leq k)
\]

For example:

\[
    F_{7,6}^{+,+} = \begin{vmatrix}
        a_0 & a_1 & a_3 & a_4 & a_6 & a_7 & a_8 \\
        -a_2 & a_0 & a_4 & a_5 & a_6 & a_7 & a_8 \\
        -a_3 & -a_1 & a_0 & a_4 & a_5 & a_6 & a_7 \\
        -a_4 & -a_2 & -a_0 & a_6 & a_1 & a_7 & a_3 \\
        -a_5 & -a_3 & -a_1 & -a_0 & a_1 & a_7 & a_2 \\
        -a_6 & -a_4 & -a_2 & -a_1 & -a_0 & a_1 & a_7 \\
        -a_7 & -a_5 & -a_3 & -a_2 & -a_1 & -a_0 & a_1
    \end{vmatrix}
\]

Assume that \( a_0 > 0 \). Then \( f(x) \) is expansive if and only if:
1. \( f(\pm 1) > 0 \), and
2. \( \forall k \in \{1, 2, \ldots, n - 1\}: F_{n,k}^\pm > 0. \)

Furthermore, the last condition can be weakened so that it is only required for every second \( k \) counting back from \( n - 1 \).

Note that the assumption \( a_0 > 0 \) makes no real restriction, since if \( a_0 < 0 \), the polynomial can be multiplied by \(-1\) without changing its roots, and if \( a_0 = 0 \), the polynomial is not expansive.

This lemma is proved in Section 3 using the Schur–Cohn test.

The main results of this paper regarding the expansivity gap are the following two theorems. Note that we use the notations from the beginning of this section.

**Theorem 2.3 (Lower bound theorem)**

If \( f(x) \) is expansive, then the expansivity gap is greater than \( \varepsilon \) (i.e. all \( |x_i| > 1 + \varepsilon \)) with:

\[
\varepsilon := \frac{1}{n^2 \cdot n! \cdot A^{n-1}}.
\]

**Theorem 2.4 (Existence theorem)**

For each \( n \geq 2 \) and for each sufficiently large \( A \), there exists an \( f(x) \) with these parameters whose expansivity gap is:

\[
\varepsilon = \frac{1}{2A^{n-1}} + O\left(\frac{1}{A^n}\right).
\]

The proof of these theorems use the expansivity condition described above (Lemma 2.2), and are detailed in Sections 3 and 4, respectively.

### 3 Lower bound theorem

#### 3.1 Proof of Lemma 2.2

For \( n = 1 \), the Schur–Cohn algorithm terminates after one step, so the only condition is \( a_0 > |a_1| \), which is equivalent to \( f(\pm 1) > 0 \).

For \( n = 2 \), the Schur–Cohn algorithm performs two steps, and the two conditions are expanded to \( a_0 > |a_2| \) and \( |a_0^2 - a_2^2| > |a_0a_1 - a_2a_1| \). The first is equivalent to \( F_{2,1}^\pm = a_0 \pm a_2 > 0 \). After dividing the latter with (the positive) \( a_0 - a_2 \), it is \( a_0 + a_2 > |a_1| \), which is equivalent to \( f(\pm 1) > 0 \).

Generally, the lemma is proved by induction on \( n \). It is sufficient to show that assuming \( a_0 > |a_n| \), our conditions for \( f(x) \) are equivalent to the conditions for its Schur transform, \( g(x) \).
First we show that, assuming $a_0 > |a_n|$, the sign of $f(\pm1)$ and $g(\pm1)$ are the same:

$$g(\pm1) = \sum_{j=0}^{n} (\pm1)^j b_j = \sum_{j=0}^{n} (\pm1)^j (a_0 a_j - a_n a_{n-j}) =$$

$$(a_0 \pm a_n) \sum_{j=0}^{n} (\pm1)^j a_j = (a_0 \pm a_n) f(\pm1)$$

where $\pm$ is a distinct $\pm$ from $(\pm1)$, and depends on the parity of $n$ and the sign of $(\pm1)$. Since, by the assumption, both $a_0 + a_n$ and $a_0 - a_n$ are positive, the sign of $g(\pm1)$ and $f(\pm1)$ are the same.

Now we show that, still assuming $a_0 > |a_n|$, $F_{n,k}^\pm$ also has the same sign as $G_{n-1,k-1}^\pm$ (defined similarly, but for $g(x)$). For example, the case for $n = 5$, $k = 4$ and sign $(-)$ is (using formally the zero $b_5$):

$$G_{4,3}^\pm = \begin{vmatrix} b_0 - b_2 & b_1 - b_4 & b_2 - b_4 \\ -b_3 & b_0 - b_4 & b_1 - b_5 \\ -b_4 & -b_5 & b_0 \end{vmatrix} = \begin{vmatrix} a_0 - a_2 & a_5 - a_3 & a_1 - a_3 & a_4 - a_2 & a_2 - a_4 & a_3 - a_1 \\ a_5 & a_0 & a_5 & a_0 & a_5 & a_0 \\ -a_3 & -a_2 & a_0 - a_4 & a_5 - a_3 & a_1 - a_5 & a_4 - a_0 \\ a_5 & a_0 & a_5 & a_0 & a_5 & a_0 \\ -a_4 & -a_1 & -a_5 & -a_0 & a_0 & a_5 \\ a_5 & a_0 & a_5 & a_0 & a_5 & a_0 \end{vmatrix}.$$

It can be easily shown that this kind of hyperdeterminant, with only one different rows in the inner determinants, can be written in one big determinant, using the repeated rows only in the block-diagonal. We continue the example case, and also describe the general case formally.

$$G_{4,3}^- = \begin{vmatrix} a_0 - a_2 & a_5 - a_3 & a_1 - a_3 & a_4 - a_2 & a_2 - a_4 & a_3 - a_1 \\ a_5 & a_0 & a_5 & a_0 & a_5 & a_0 \\ -a_3 & -a_2 & a_0 - a_4 & a_5 - a_3 & a_1 - a_5 & a_4 - a_0 \\ a_5 & a_0 & a_5 & a_0 & a_5 & a_0 \\ -a_4 & -a_1 & -a_5 & -a_0 & a_0 & a_5 \\ a_5 & a_0 & a_5 & a_0 & a_5 & a_0 \end{vmatrix}$$

$$G_{n-1,k-1}^\pm : \begin{cases} e^{2i-1,2j-1} = a_{j-i} \pm a_{n+i+j-k-1} \\ e^{2i-1,2j} = a_{n-j+i} \pm a_{-i-j+k+1} \\ e^{2i,2j-1} = a_n \delta_{ij} \\ e^{2i,2j} = a_0 \delta_{ij} \end{cases}$$

Now we arrange the odd rows and columns to the top and left halves, and multiply the other rows and columns by $-1$ (neither change the sign of the determinant since the same number of actions are performed for the rows and the columns):

$$G_{4,3}^- = \begin{vmatrix} a_0 - a_2 & a_1 - a_3 & a_2 - a_4 & a_3 - a_5 & a_2 - a_4 & a_1 - a_3 \\ -a_3 & a_0 - a_4 & a_1 - a_5 & a_2 & a_1 - a_5 & a_0 - a_4 \\ -a_4 & -a_5 & a_0 & a_1 & a_0 & -a_5 \\ -a_5 & -a_5 & a_0 & a_0 & a_0 \\ a_5 & -a_5 & a_0 & a_0 & a_0 \end{vmatrix}.$$
It can be easily seen that in this matrix, the \( j \)th column is the same as the \((2k - j)\)th in the first \( k \) rows. Now subtract the \( 2 \leq j \leq k - 1 \) columns from the appropriate similar columns:

\[
G_{n-1,k-1}^{\pm}: e_{ij} = \begin{cases} 
  a_{j-i} \pm a_{n+i+j-k-1}, & 1 \leq i, j \leq k - 1 \\
  a_{i-j+2k} \pm a_{n+i-j+k-1}, & 1 \leq i \leq k - 1, \ k \leq j \leq 2k - 2 \\
  \pm a_n \delta_{i-k+1,j}, & k \leq i \leq 2k - 2, \ 1 \leq j \leq k - 1 \\
  a_0 \delta_{i,j}, & k \leq i, j \leq 2k - 2 
\end{cases}
\]

The top left \( k \times k \) is exactly \( F_{n,k}^{\pm} \), and the bottom right is the following:

\[
G_{n-1,k-1}^{\pm}: e_{ij} = \begin{cases} 
  a_{j-i} \pm a_{n+i+j-k-1}, & 1 \leq i, j \leq k \\
  0, & 1 \leq i \leq k, \ k + 1 \leq j \leq 2k - 2 \\
  \pm a_n \delta_{i-k+1,j}, & k + 1 \leq i \leq 2k - 2, \ 1 \leq j \leq k \\
  a_0 \delta_{i,j} + a_n \delta_{i-k+1,2k-j}, & k + 1 \leq i, j \leq 2k - 2 
\end{cases}
\]

Since \( a_0 > |a_n| \) is assumed, this subdeterminant is positive, which proves that the whole determinant (which is \( G_{n-1,k-1}^{\pm} \)) and \( F_{n,k}^{\pm} \) has the same sign.

Note that only one condition for \( f(x) \) was not covered, namely \( F_{n,1}^{\pm} > 0 \), but that is \( a_0 \pm a_n > 0 \), i.e. \( a_0 > |a_n| \).

The statement that only every second \( F_{n,k}^{\pm} \) is required is inherited by induction from \( G_{n-1,k-1}^{\pm} \). The only problem is that the induction step, as described above, requires \( a_0 > |a_n| \), i.e. the condition for \( k = 1 \), but that is only explicitly included when \( n \) is even. If \( n \) is odd, then the first included condition is \( F_{n,2}^{\pm} > 0 \), i.e. the \( k = 2 \) case. We show that this implies \( F_{n,1}^{\pm} > 0 \). Indeed, \( F_{n,2}^{\pm} = a_0(a_0 \pm a_n) - a_n(a_n \pm a_1) > 0 \), which is equivalent to \( a_0^2 - a_n^2 > |a_0a_n - a_na_1| \), and it implies that \( a_0^2 - a_n^2 > 0 \), i.e. \( a_0 > |a_n| \), which is \( F_{n,1}^{\pm} > 0 \).}

### 3.2 Finishing the proof of the Lower bound theorem

Theorem 2 assumes that all \(|x_i| > 1\), i.e. the polynomial is expansive. The statement \(|x_i| > 1 + \epsilon\) is equivalent to that \( f^\epsilon(x) := f((1 + \epsilon)x) \) is also expansive. In both cases, we use Lemma 2.2. In the first case, the coefficients of \( f(x) \) are integers, so any \( Q > 0 \) condition of the lemma implies \( Q \geq 1 \) (since \( Q \) is a multivariate polynomial of the coefficients). Therefore, to prove
the appropriate \( Q(\varepsilon) > 0 \), it is sufficient to show that \(|Q(\varepsilon) - Q| < 1\) for each \( Q \).

Consider \( |F_{n,k}^\varepsilon - F_{n,k}^\pm| \) first. Since \( F_{n,k}^\pm \) is a \( k \times k \) determinant whose elements are of the form \( a_i^\varepsilon \pm a_j \), we can expand it:

\[
F_{n,k}^\pm = \sum_{i=1}^{N} \left( \pm \prod_{j=1}^{k} a_{j'} \right),
\]

where \( j' \) is an index dependent on \( i \) and \( j \). Similarly:

\[
|F_{n,k}^\varepsilon - F_{n,k}^\pm| \leq \sum_{i=1}^{N} \left| \prod_{j=1}^{k} a_{j'} - \prod_{j=1}^{k} a_j \right| \leq \sum_{i=1}^{N} \prod_{j=1}^{k} |a_{j'} - a_j| \leq E \sum_{i=1}^{N} \prod_{j=1}^{k} |a_{j'}| \leq E \sum_{i=1}^{N} \prod_{j=1}^{k} A^{l-1}(A + E)^{k-l},
\]

where \( E := \max_{j=0}^{n} |a_{j'} - a_j| \). We can obtain the following bound for \( E \):

\[
E = \max_{j=0}^{n} |(1 + \varepsilon)^i a_j - a_{j'}| \leq A ((1 + \varepsilon)^n - 1) = A \sum_{i=1}^{n} \left( \frac{n}{i} \right) \varepsilon^i < A \sum_{i=1}^{\infty} \left( \frac{n}{i} \right) \varepsilon^i = \frac{An}{1 - n\varepsilon} = \frac{A}{n\varepsilon - 1} < \frac{A}{nn!A^{n-1} - 1}
\]

Now bounding \( (A + E)^{k-l} \):

\[
(A + E)^{k-l} = \sum_{i=0}^{k-l} \binom{k-l}{i} E^i A^{k-l-i} < \sum_{i=0}^{k-l} (kE)^i A^{k-l-i} < A^{k-l} \sum_{i=0}^{\infty} \left( \frac{kE}{A} \right)^i = A^{k-l+1} \frac{A}{A - kE}
\]

Continuing the original calculation:

\[
|F_{n,k}^\varepsilon - F_{n,k}^\pm| < E \sum_{i=1}^{N} \sum_{j=1}^{k} \frac{A^k}{A - kE} = Nk \frac{AE}{A - kE}.
\]

Now we give an upper bound on \( N \), i.e. the number of terms in the expansion of \( F_{n,k}^\pm \). Note that not only the determinant itself is expanded, but also any \( a_i \pm a_j \) expression in that determinant, so that any term is in the form \( \pm \prod_{j=1}^{k} a_{j'} \). Expand the determinant starting from the last row, going back to the first. The last row gives a choice of 2 entries. The next one contains 4 entries, but one of them is excluded by the previous choice, so there are only 3 possibilities. On each next row, there are 2 more entries, but one more position is excluded, so there are at most 1 more possibilities (the excluded entry can be a double-entry, which reduces the number further, but it cannot be a zero entry). This gives a bound for the number of possibilities as \( 2 \cdot 3 \cdot 4 \cdots \cdot (k+1) \), i.e. \( N \leq (k+1)! \).
Continuing the calculation using that \( k \leq n - 1 \) and \( N \leq n! \):

\[
|F_{n,k}^\pm - F_{n,k}^\pm| < N k \frac{A_k}{E - k} < n! (n - 1) \frac{A^{n-1}}{nn! A^{n-1} - 1 - (n - 1)}
\]

\[
= \frac{(n - 1)n! A^{n-1}}{nn! A^{n-1} - n} \leq 1,
\]

Now prove the same for \(|f^\varepsilon(\pm 1) - f(\pm 1)|\):

\[
|f^\varepsilon(\pm 1) - f(\pm 1)| \leq \sum_{j=1}^{n} |a_j^\varepsilon - a_j| \leq nE < \frac{nA}{nn! A^{n-1} - 1} < 1.
\]

This finishes the proof of the theorem. ■

4 Existence theorem

In this section, we will prove Theorem 2.4 and explicitly construct the stated polynomials. For this, we define some recursively defined table of values and prove some simple equalities involving them.

4.1 Some coefficient tables

Let \( b_{i,j} \) and \( c_{i,j} \) for all \( i, j \in \mathbb{N} \) be defined as follows. First some initial values are given (also on negative indices for convenience), and a recurrence relation for each set:

\[
b_{-1,-1} = 1, \quad b_{i,-1} = 0, \quad b_{-1,j} = 0 \quad (i, j \geq 0)
\]

(4.1)

\[
b_{i,j} = b_{i-1,j-1} + b_{i-1,j} + b_{i-1,j+1} \quad (i, j \geq 0)
\]

\[
c_{i,-2} = 0, \quad c_{i,-1} = 0, \quad c_{0,0} = 1, \quad c_{0,j} = 0 \quad (i \geq 0, j \geq 1)
\]

(4.2)

\[
c_{i,j} = c_{i+1,j-2} + c_{i+1,j-1} + c_{i+1,j} \quad (i, j \geq 0)
\]

The following table shows the first few values (for nonnegative indices):

| \( b_{i,j} \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| \( i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 0 | 1 | 1 | 2 | 3 | 4 | 3 | 1 |
| 1 | 1 | 1 | 2 | 1 | 1 | -3 | 3 | 1 |
| 2 | 1 | 1 | 2 | 4 | 3 | 2 | 9 | 3 |
| 3 | 2 | 4 | 5 | 4 | 3 | 2 | 9 | 3 |
| 4 | 9 | 12 | 9 | 9 | 4 | 1 | 6 | 3 |
| 5 | 30 | 25 | 14 | 14 | 5 | 1 | 18 | -5 |
| 6 | 196 | 189 | 133 | 70 | 27 | 7 | 1 | 18 |

| \( c_{i,j} \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| \( j \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | -1 | 0 | 1 | -1 | 0 | 1 | -1 | 0 |
| 2 | 1 | -2 | 1 | 2 | -4 | 2 | 3 | -6 | 2 |
| 3 | 1 | -3 | 3 | 2 | -9 | 9 | 3 | -18 | 3 |
| 4 | 4 | -5 | 6 | 0 | 15 | 24 | -6 | -36 | 4 |
| 5 | 5 | -6 | 10 | -5 | -20 | 49 | -35 | -50 | 5 |
| 6 | 6 | -7 | 15 | -14 | -21 | 84 | -98 | -36 | 6 |
| 7 | 7 | -8 | 21 | -28 | -14 | 126 | -210 | 48 | 7 |

These coefficients have the following properties.
Lemma 4.1

\[ (4.3) \quad \sum_{j=0}^{n-k} b_{n,k+j}c_{l,j} = b_{n-l,k-l} \quad (-1 \leq l - 1 \leq k \leq n) \]

\[ (4.4) \quad \sum_{j=0}^{k} c_{n,j}c_{m,k-j} = c_{n+m,k} \quad (n, m, k \geq 0) \]

\[ (4.5) \quad \sum_{j=0}^{\infty} b_{n,j}b_{m,j} = b_{n+m,0} \quad (n, m \geq 0) \]

\[ (4.6) \quad \sum_{i=j}^{k} b_{i,j}c_{i+1,k-i} = \delta_{j,k} \quad (j, k \geq -1) \]

\[ (4.7) \quad \sum_{j=0}^{m-l} b_{n+l,m-l-j}c_{l,j} + \sum_{i=0}^{l-1} b_{n+i,0}c_{i+1,m-i} = b_{n,m} \quad (n, l \geq 0, m \geq l - 1) \]

Proof

(4.3) is proved by induction on \( l \). The case \( l = 0 \) is trivial, and the induction step from \( l \) to \( l + 1 \) is:

\[ \sum_{j=0}^{n-k} b_{n,k+j}c_{l+1,j} = \sum_{j=0}^{n-k} b_{n-1,k+j+d}c_{l+1,j} = \sum_{d=-1}^{n-k+d} b_{n-1,k+j}c_{l+1,j} = \sum_{j=0}^{n-k-1} b_{n-1,k+j}c_{l+1,j} + b_{n-1,k-1} \]

(4.4) is proved by double-induction on \( m \) and \( k \). The cases \( m = 0, k = 0 \) and \( k = -1 \) are trivial. Assume the statement for \((m, k + 1), (m + 1, k)\) and \((m + 1, k - 1)\), and prove it for \((m + 1, k + 1)\):

\[ \sum_{j=0}^{k+1} c_{n,j}c_{m+1,k-j+1} = \sum_{j=0}^{k+1} c_{n,j}(c_{m,k-j+1} - c_{m+1,k-j} - c_{m+1,k-j-1}) = \]

\[ = \sum_{j=0}^{k+1} c_{n,j}c_{m,k-j+1} - \sum_{j=0}^{k} c_{n,j}c_{m+1,k-j} - \sum_{j=0}^{k-1} c_{n,j}c_{m+1,k-j-1} \]

(4.5) is proved by double-induction on \( j \) and \( k \). The cases \( j = -1, k = -1 \) and \( k = 0 \) are trivial. Assume the statement for \((j, k), (j + 1, k), (j + 2, k)\)
and \((j + 1, k - 1)\), and prove it for \((j + 1, k + 1)\):
\[
\sum_{i=j+1}^{k+1} b_{i,j+1} c_{i+1,k-i+1} = \sum_{i=j+1}^{k+1} b_{i,j+1} (c_{i,k-i+1} - c_{i+1,k-i} - c_{i+1,k-i-1}) = \\
= \sum_{i=j+1}^{k+1} b_{i,j+1} c_{i,k-i+1} - \sum_{i=j+1}^{k} b_{i,j+1} c_{i+1,k-i} - \sum_{i=j+1}^{k-1} b_{i,j+1} c_{i+1,k-i-1} \quad \text{(4.7), ind.}
\]
\[
= \sum_{i=j+1}^{k+1} \sum_{d=0}^{2} b_{i-1,j+d} c_{i,k-i+1} - \delta_{j+1,k} - \delta_{j+1,k-1} = \sum_{d=0}^{2} \sum_{i=j}^{k} b_{i,j+d} c_{i+1,k-i} - \\
- \delta_{j+1,k} - \delta_{j+1,k-1} \quad \text{ind.}
\]

\text{(4.7)} is proved by induction on \(l\). The \(l = 0\) case is trivial, and the induction step from \(l\) to \(l + 1\) is:
\[
\sum_{j=0}^{m-l-1} b_{n+l+1,m-l-j-1} c_{l+1,j} = \sum_{j=0}^{m-l-1} b_{n+l,m-l-j-d} c_{l+1,j} = \\
= \sum_{d=0}^{2} \sum_{j=0}^{m-l-1+d} b_{n+l,m-l-j} c_{l+1,j-d} = \sum_{j=0}^{m-l} \sum_{d=0}^{2} b_{n+l,m-l-j} c_{l+1,j-d} - \\
- b_{n+l,0} c_{l+1,m-l} \quad \text{(4.6)}
\]
\[
= (b_{n,m} - \sum_{i=0}^{l-1} b_{n+i,0} c_{i+1,m-i}) - b_{n+l,0} c_{l+1,m-l} = b_{n,m} - \sum_{i=0}^{l} b_{n+i,0} c_{i+1,m-i}.
\]

\text{(4.6)} is proved by induction on \(m\). Note that the sums are finite, since \(b_{i,j} = 0\) for \(j > i\). The case \(m = 0\) is trivial. For the induction step, instead of using the defining recursion of \(b_{i,j}\)'s (4.1), we use the relation (4.6) with \(k := m + 1\), which can be written as follows:
\[
b_{m+1,j} = \delta_{j,m+1} - \sum_{i=j}^{m} b_{i,j} c_{i+1,m-i+1}
\]

Now the induction step assumes the statement for all \(0 \leq i \leq m\), and proves it for \(m + 1\):
\[
\sum_{j=0}^{\infty} b_{n,j} b_{m+1,j} = \sum_{j=0}^{\infty} b_{n,j} (\delta_{j,m+1} - \sum_{i=j}^{m} b_{i,j} c_{i+1,m-i+1}) = \\
= b_{n,m+1} - \sum_{i=0}^{m} c_{i+1,m-i+1} b_{n,j} b_{i,j} \quad \text{ind.} = b_{n,m+1} - \sum_{i=0}^{m} c_{i+1,m-i+1} b_{n,i+1} \quad \text{(4.7)}
\]
\[
= \sum_{i=0}^{m+1} b_{n+i,0} c_{i+1,m-i+1} - \sum_{i=0}^{m} b_{n+i,0} c_{i+1,m-i+1} = b_{n+m+1,0}.
\]

where (4.7) was used with the substitution \(m := m + 1\) and \(l := m + 2\). ■
4.2 The construction

Now we define the family of polynomials for which Theorem 2.4 will be proved. For each \( n \geq 2 \) and \( A \geq 1 \), define the coefficients of \( f(x) \) as follows:

\[
\begin{align*}
a_0 &= A \\
a_1 &= A - (b_{n-3,0} + 1) \\
a_2 &= A - b_{n-2,0} \\
a_i &= -b_{n-2,i-2} & (3 \leq i \leq n)
\end{align*}
\]

(4.8)

The first few examples of these polynomials are:

\[
\begin{align*}
n = 2 : & \quad (A - 1)x^2 + (A - 1)x + A \\
n = 3 : & \quad -x^3 + (A - 2)x^2 + (A - 2)x + A \\
n = 4 : & \quad -x^4 - 2x^3 + (A - 4)x^2 + (A - 3)x + A \\
n = 5 : & \quad -x^5 - 3x^4 - 5x^3 + (A - 4)x^2 + (A - 3)x + A \\
n = 6 : & \quad -x^6 - 4x^5 - 9x^4 - 12x^3 + (A - 9)x^2 + (A - 5)x + A \\
n = 7 : & \quad -x^7 - 5x^6 - 14x^5 - 25x^4 - 30x^3 + (A - 21)x^2 + (A - 10)x + A
\end{align*}
\]

An interesting side-note is that the constant 1 in the definition of \( a_1 \) can sometimes be replaced by \(-1\), more precisely by an \( \omega \) such that \( \omega^{n-2} = 1 \): \( a_1 := A - (b_{n-3,0} + \omega) \). For odd \( n \), it remains only \( \omega = +1 \), but for even \( n \), it can be either \(+1\) or \(-1\), moreover if \( n = 2 \), it can be any \( \omega \in \mathbb{Z} \).

The crux in proving the Existence theorem is the following properties of these polynomials, especially the third one.

**Lemma 4.2** Define \( F_{n,k}^\pm \) as in Lemma 2.2. Then:

1. \( f(\pm 1) > 0 \) for sufficiently large \( A \) parameters.
2. For any \( 1 \leq k \leq n - 1 \) and any sign in \( \{+, -\} \), \( F_{n,k}^\pm > 0 \) for sufficiently large \( A \).
3. \( F_{n,n-1}^- = 1 \) for all \( A \in \mathbb{N}^+ \).

4.3 Proof of the properties

The first statement of the lemma is trivial.

Denote the \( k \times k \) components of \( F_{n,k}^\pm \) by \( d_{i,j} \), which is, by (2.2) and (4.8):

\[
d_{i,j} = a_{j-i} \pm a_{i+j+n-k-1} =
\begin{cases}
0, & j < i \\
A \pm A, & j = i = 1 \land k = n - 1 \\
A, & j = i, \text{ otherwise} \\
A - b_{n-3,0} - \omega, & j = i + 1 \\
A - b_{n-2,0}, & j = i + 2 \\
-b_{n-2,i-2}, & j \geq i + 3
\end{cases}
\]

(4.9)
For example:

\[
F_{7,6}^- = 
\begin{vmatrix}
21 & A + 20 & A + 4 & -16 & -20 & -13 \\
30 & A + 25 & A + 4 & A - 16 & -29 & -25 \\
25 & 14 & A + 5 & A - 9 & A - 21 & -30 \\
14 & 5 & 1 & A & A - 10 & A - 21 \\
5 & 1 & A & A - 10 & \\
1 & A & \\
\end{vmatrix}
\]

These determinants are polynomials in \( A \) of degree at most \( k \). \( A \) occurs only in the main diagonal, the superdiagonal (1-diagonal) and the 2-diagonal. The coefficient of all occurrences is 1 except on the top left corner, where it is 1 if \( k < n - 1 \), 2 if \( k = n - 1 \) with the + sign, and 0 if \( k = n - 1 \) with the − sign. In the former two cases, the main term of the polynomial is therefore \( A^k \) or \( 2A^k \), which proves that the determinant is positive for sufficiently large \( A \).

The remaining of this proof will show that in the third case, the determinant is the constant polynomial 1.

From now on, the determinant \( D := F_{n,n-1}^- \) (for \( n \geq 2 \)) will be examined. The method of calculating it is to reduce it to a trivial case by a sequence of determinant-preserving transformations.

### 4.3.1 First step

The first step in simplifying the determinant \( D \) is to eliminate the \( A \)’s from the off-diagonals by row transformations. If we denote the new determinant by \( D^{(0)} \) and its elements by \( d_{i,j}^{(0)} \), then the following operations are performed starting from the last row \((i = n - 1)\) upwards to the first one \((i = 1)\):

\[
d_{i,j}^{(0)} := d_{i,j} - d_{i+1,j}^{(0)} - d_{i+2,j}^{(0)}. 
\]

For example, \( F_{7,6}^- \) above becomes this determinant:

\[
\begin{vmatrix}
0 & -1 & 1 & 0 & -1 & 1 \\
9 & A + 12 & -1 & -7 & -8 & -5 \\
12 & 9 & A + 4 & -9 & -11 & -9 \\
9 & 4 & 1 & A & -10 & -11 \\
4 & 1 & A & -10 & \\
1 & A & \\
\end{vmatrix}
\]

Now we prove that the recursive definition (4.10) above is equivalent to the following explicit formula:

\[
d_{i,j}^{(0)} = \sum_{k=1}^{n-1} d_{k,j}c_{1,k-i},
\]

where the \( c_{i,j} \) values were defined in (4.2). It is trivial for \( i = n - 1 \) and for
the invalid \( i = n \). Proof by induction:
\[
d_{i,j}^{(0)} \overset{\text{(4.10)}}{=} d_{i,j} - d_{i+1,j}^{(0)} - d_{i+2,j}^{(0)} \overset{\text{ind.}}{=} d_{i,j} - \sum_{k=i+1}^{n-1} d_{k,j}c_{1,k-i-1} - \sum_{k=i+2}^{n-1} d_{k,j}c_{1,k-i-2} =
\]
\[
= \sum_{k=i}^{n-1} d_{k,j}(c_{0,k-i} - c_{1,k-i-1} - c_{1,k-i-2}) \overset{\text{(4.12)}}{=} \sum_{k=i}^{n-1} d_{k,j}c_{1,k-i}.
\]

The next step to show is that \( D^{(0)} \), as the example suggests, has a similar, but simpler structure than \( D \):
\[
d_{i,j}^{(0)} = b_{n-3,i+j-3} + \begin{cases} 0, & j < i \\ 0, & j = i = 1 \\ A, & j = i \geq 2 \\ -b_{n-3,j-i-1} - \omega c_{1,j-i-1}, & j > i \end{cases}
\]

Proving it for \( j < i \) or \( j = i = 1 \):
\[
d_{i,j}^{(0)} \overset{\text{(4.11)}}{=} \sum_{k=i}^{n-1} d_{k,j}c_{1,k-i} \overset{\text{(4.9)}}{=} \sum_{k=i}^{n-1} b_{n-2,k+j-2}c_{1,k-i} = \sum_{k=0}^{n-i-j} b_{n-2,k+i+j-2}c_{1,k} \overset{\text{(4.3)}}{=}
\]
\[
= b_{n-3,i+j-3}.
\]

The case \( j \geq 2 \) is similar, but an additional \( A \) comes from the first term in the sum due to (4.9).

When \( j = i + 1 \), more terms appear additionally:
\[
d_{i,j}^{(0)} = \ldots = b_{n-3,i+j-3} + (A - b_{n-3,0} - \omega) - A = b_{n-3,i+j-3} - b_{n-3,0} - \omega c_{1,0}.
\]

Now consider the case \( j \geq i + 2 \):
\[
d_{i,j}^{(0)} \overset{\text{(4.11)}}{=} \sum_{k=0}^{n-i-1} d_{k+i,j}c_{1,k} \overset{\text{(4.9)}}{=} \sum_{k=0}^{n-i-1} b_{n-2,k+i+j-2}c_{1,k} - \sum_{k=0}^{j-i-2} b_{n-2,j-i-k-2}c_{1,k} +
\]
\[
+ A(c_{1,j-i+2} + c_{1,j-i} + c_{1,j-i}) - b_{n-3,0}c_{1,j-i-1} - \omega c_{1,j-i-1} \overset{\text{(4.3) (4.12)}}{=}
\]
\[
= b_{n-3,i+j-3} - \left( \sum_{k=0}^{j-i-2} b_{n-2,j-i-k-2}c_{1,k} + b_{n-3,0}c_{1,j-i-1} \right) - \omega c_{1,j-i-1} \overset{\text{(4.7) (4.12)}}{=}
\]
\[
= b_{n-3,i+j-3} - b_{n-3,j-i-1} - \omega c_{1,j-i-1},
\]

where the last equality used (4.7) with the substitutions \( l := 1 \), \( j := k \), \( m := j - i - 1 \) and \( n := n - 3 \). This finishes the proof of (4.12).

### 4.3.2 Reduction step

The next set of transformations creates successively smaller and smaller determinants starting from \( D^{(0)} \). Denote these by \( D^{(1)}, D^{(2)}, \ldots, D^{(n-2)} \), where each \( D^{(m)} \) is of size \((n-m-1) \times (n-m-1)\).

Each step can be described informally as performing the following three transformations:

1. adding multiples of the second column to the columns afterwards to make their first entry 0,
2. adding multiples of rows to the second row to remove any off-diagonal
   \( A \)'s created by the previous substep,
3. and removing the first row and second column, reducing the size by one.

Continuing the example:

\[
\begin{bmatrix}
0 & -1 & 1 & 0 & -1 & 1 \\
9 & A + 12 & -1 & -7 & -8 & -5 \\
12 & 9 & A + 4 & -9 & -11 & -9 \\
9 & 4 & 1 & A & -10 & -11 \\
4 & 1 & A & -10 & & \\
1 & & & & & \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
0 & A + 4 & -1 & 2 & -1 & -2 \\
12 & 9 & A + 13 & -9 & -20 & 0 \\
9 & 4 & 5 & A & -14 & -7 \\
4 & 1 & 1 & A & -1 & -9 \\
1 & & & & & \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & -1 & 2 & -1 & -2 \\
12 & A + 13 & -19 & -20 & 0 \\
9 & 5 & A & -14 & -7 \\
4 & 1 & A & -1 & -9 \\
1 & & & & & \\
\end{bmatrix}
\]

Repeating these steps in the example:

\[
\begin{bmatrix}
0 & -1 & 1 & 0 & -1 & 1 \\
9 & A + 12 & -1 & -7 & -8 & -5 \\
12 & 9 & A + 4 & -9 & -11 & -9 \\
9 & 4 & 1 & A & -10 & -11 \\
4 & 1 & A & -10 & & \\
1 & & & & & \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & -1 & 3 & -3 \\
9 & A + 10 & -19 & -17 \\
4 & 2 & A & -11 & -17 \\
& & & & & \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & -1 & 4 \\
4 & A + 4 & -17 \\
& & & & & \\
1 & & & & & \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & -1 \\
1 & A \\
& & & & & \\
\end{bmatrix}
= |1|
\]

The remaining of the proof will show that the informal description above
is equivalent to the formal definition below, these transformations preserve
the determinant, and they eventually lead to the 1 \( \times \) 1 determinant 1.

\[
(4.13) \quad d_{i,j}^{(m+1)} := \sum_{k=2}^{n-m-1} d_{i,j}^{(m)} c_{m+1,k-2}, \quad (i = 1, j = 1)
\]

\[
(4.14) \quad d_{i,j}^{(m+1)} := \sum_{k=2}^{n-m-1} (d_{i,j}^{(m)} - d_{i+1,j}^{(m)}) c_{m+1,k-2}, \quad (i = 1, j \geq 2)
\]

\[
(4.15) \quad d_{i,j}^{(m+1)} := d_{i+1,j}, \quad (i \geq 2, j = 1)
\]

\[
(4.16) \quad d_{i,j}^{(m+1)} := d_{i+1,j+1} - d_{i+1,j}^{(m)} c_{m+1,j-1}, \quad (i \geq 2, j \geq 2)
\]

\[0 \leq m \leq n - 3, \quad 1 \leq i, j \leq n - m - 2\]
Furthermore, the following explicit formulas will be shown:

(4.17) \( d^{(m)}_{1,1} = 0 \)

(4.18) \( d^{(m)}_{1,j} = -\omega c_{m+1,j-2} \)

(4.19) \( d^{(m)}_{i,1} = b_{n-3,i+m-2} \ (i \geq 2) \)

(4.20) \( d^{(m)}_{i,2} = \sum_{k=0}^{m} b_{n-3,i+m+k-1} b_{m,k} + A\delta_{i,2} \ (i \geq 2) \)

(4.21) \( d^{(m)}_{i,j} = b_{n-3,i+j+2m-3} - \sum_{l=k}^{m-1} \sum_{l=k}^{m-1} b_{l,k} c_{l+1,j+m-l-2} b_{n-3,i+m+k-1} + \)

\[ + \begin{cases} 0, & j < i \\ A, & j = i \\ -b_{n-3,j-i-1} - \omega c_{1,j-i-1}, & j > i \end{cases} \]

\((0 \leq m \leq n-3, \ 1 \leq i, j \leq n - m - 1)\)

These formulas are proved simultaneously by induction on \( m \). For \( m = 0 \), they coincide with (4.12).

(4.19) is trivially inherited by induction due to (4.15).

(4.17) is an application of (4.3):

(4.20) follows from (4.21) with \( j = 2 \):

\[ d^{(m)}_{i,2} = b_{n-3,i+2m-1} - \sum_{k=0}^{m-1} \left( \sum_{l=k}^{m-1} b_{l,k} c_{l+1,m-l} \right) b_{n-3,i+m+k-1} + A\delta_{i,2} = \]

\[ = b_{n-3,i+2m-1} - \sum_{k=0}^{m-1} (\delta_{k,m} - b_{m,k} c_{m+1,0}) b_{n-3,i+m+k-1} + A\delta_{i,2} = \]

\[ = \sum_{k=0}^{m} b_{m,k} b_{n-3,i+m+k-1} + A\delta_{i,2}. \]

(4.21) is proved by inductively using (4.20) and (4.21):

\[ d^{(m)}_{i,j} = d^{(m-1)}_{i+1,j+1} - d^{(m-1)}_{i+1,j} c_{m,j-1} d^{(m-1)}_{i,j} - c_{m,j-1} \sum_{k=0}^{m-1} b_{n-3,i+m+k-1} b_{m-1,k} + \]

\[ + b_{n-3,i+j+2m-3} - \sum_{k=0}^{m-2} \sum_{l=k}^{m-2} b_{l,k} c_{l+1,j+m-l-2} b_{n-3,i+m+k-1} + \{ \ldots \} = \]

\[ = b_{n-3,i+j+2m-3} - \sum_{k=0}^{m-1} \sum_{l=k}^{m-1} b_{l,k} c_{l+1,j+m-l-2} b_{n-3,i+m+k-1} + \{ \ldots \} , \]

where \( \ldots \) indicates the cases in (4.21).

Now we can prove the most important part, (4.18). For \( j = 1 \), it is (4.17).
Let $j \geq 2$:

\[
d_{1,j}^{(m)} \overset{\text{(4.14)}}{=} \sum_{i=0}^{n-m-2} d_{i+2,j+1}^{(m-1)} c_{m,i} - \sum_{i=0}^{n-m-2} d_{i+2,2}^{(m-1)} c_{m,j-1} c_{m,i}
\]

Denote the two sums by $X$ and $Y$, respectively. Using (4.21) and (4.20), split them into several expressions, $X = X_1 + X_2 + X_3$ and $Y = Y_1 + Y_2$, respectively, and calculate them separately:

\[
X_1 = \sum_{i=0}^{n-m-2} b_{n-3,i+j+2m-2} c_{m,i} \overset{\text{(4.3)}}{=} b_{n-m-3,m+j-2}
\]

\[
X_2 = -\sum_{k=0}^{m-2} \sum_{l=k}^{m-2} b_{l,k} c_{l+1,j+m-l-2} \sum_{i=0}^{n-m-2} b_{n-3,i+m+k} c_{m,i} \overset{\text{(4.3)}}{=}
\]

\[
= -\sum_{k=0}^{m-2} \sum_{l=k}^{m-2} b_{l,k} c_{l+1,j+m-l-2} b_{n-m-3,k} =
\]

\[
= -\sum_{i=0}^{n-m-2} c_{i+1,m+j-i-2} \sum_{i=0}^{j-2} b_{i,k} b_{n-m-3,k} \overset{\text{(4.5)}}{=}
\]

\[
= -\sum_{i=0}^{j-2} b_{n-3,j-i-2} c_{m,j-i-2} - \omega c_{1,j-i-2}) c_{m,i} \overset{\text{(4.4)}}{=}
\]

\[
X_3 = \sum_{i=0}^{n-m-2} (A - \sum_{i=0}^{j-2} b_{n-3,j-i-2} c_{m,i} - \omega c_{m+1,j-2}) c_{m,i} = c_{m,j-1} A
\]

\[
Y_1 = \sum_{k=0}^{m-1} b_{m-1,k} c_{m,j-1} \sum_{i=0}^{n-m-2} b_{n-3,i+m+k} c_{m,i} \overset{\text{(4.3)}}{=}
\]

\[
= \sum_{k=0}^{m-1} b_{m-1,k} c_{m,j-1} b_{n-m-3,k} \overset{\text{(4.5)}}{=}
\]

\[
Y_2 = \sum_{i=0}^{n-m-2} A c_{m,j-1} c_{m,i} = c_{m,j-1} A
\]

Combining all together:

\[
d_{1,j}^{(m)} = X_1 + X_2 + X_3 - Y_1 - Y_2 =
\]

\[
= b_{n-m-3,m+j-2} - \sum_{i=0}^{m-2} b_{n-3,i+1,m+j-i-2} - \sum_{i=0}^{j-2} b_{n-3,j-i-2} c_{m,i} -
\]

\[
- \omega c_{m+1,j-2} - c_{m,j-1} b_{n-4,0} = -\omega c_{m+1,j-2} -
\]

\[
- \left( \sum_{i=0}^{j-2} b_{n-3,j-i-2} c_{m,j} + \sum_{i=0}^{m-1} b_{n-m+i-3,0} c_{i+1,m+j-i-2} - b_{n-m-3,m+j-2} \right) \overset{\text{(4.7)}}{=}
\]

\[
= -\omega c_{m+1,j-2},
\]

where the last equality used (4.7) with the substitutions $j := i$, $m := m+j-2$, respectively.
Now using these results, compare the recursive formulas (4.13)-(4.16) to the informal descriptions 1-3. above them. It is clear that they both describe transformations of the same structure, only the exact coefficients that give the expected result are in question, and that removing the appropriate row and column preserves the determinant.

1. (4.18) tells that the first row contains the values \( c_{m+1,j-2} \) multiplied by a constant, so the column transformations with the coefficients \( c_{m+1,j-2} \), as in the formulas, indeed make the first entry zero in these columns.

2. It can be seen from the explicit formulas (4.17)-(4.21) that after the column and row transformations with the specified coefficients, there are indeed no off-diagonal values.

3. Since now the entries in the first row are all zeros except the second, which is \( -\omega \), the first row and the second column can be removed without changing the determinant if \( \omega = 1 \), otherwise the determinant multiples by \( \omega \). In the latter case, since there are \( n - 2 \) steps and \( \omega^{n-2} = 1 \) by definition, the first and the last determinants are still the same.

This proves that the original determinant \( D = D^{(0)} = D^{(n-2)} \), and the latter has size 1 \( \times 1 \), and its only element is:

\[
\begin{bmatrix}
0 \\
c_n - 2, 0 \\
b_{n-3,n-3} c_{n-2,0}
\end{bmatrix} = 1.
\]

4.4 Finishing the proof of the Existence theorem

Theorem 2.4 will be proved for the family of polynomials \( f(x) \) in (4.8). It is already clear that they are expansive for sufficiently large \( A \), since all quantities of \( f(x) \) required to be positive by Lemma 2.2 are positive due to Lemma 4.2. To find the smallest root, we need to examine the same quantities for \( f(1 + \varepsilon)x \), and the smallest \( \varepsilon \) for which one of these quantities becomes non-positive will give the size of the smallest root as \( 1 + \varepsilon \).

Let \( Q_0(A) \) be any of these quantities in Lemma 2.2 (i.e. \( F_{n,k}^+ \) or \( f(\pm 1) \)), and let \( Q(A, \varepsilon) \) be the same for \( f(1 + \varepsilon)x \), i.e. the coefficients \( a_0, a_1, \ldots, a_n \) are replaced by \( a_0, a_1(1 + \varepsilon), \ldots, a_n(1 + \varepsilon)^n \), respectively. It is a bivariate polynomial in \( A \) and \( \varepsilon \), and let \( d := \deg_A Q \) and \( N := \deg_\varepsilon Q \). Write its expansion by \( \varepsilon \):

\[
Q(A, \varepsilon) = Q_0(A) + Q_1(A)\varepsilon + Q_2(A)\varepsilon^2 + \ldots + Q_N(A)\varepsilon^N,
\]

where all \( \deg Q_i \leq d \).

We need to find the smallest \( \varepsilon \) for each sufficiently large \( A \) such that \( Q(A, \varepsilon) = 0 \) for any of the examined \( Q \) polynomials. First we prove that such \( \varepsilon \) exists and \( \varepsilon = O(1/A) \).

For any \( \varepsilon = O(1/A) \), the expansion of \( Q(A, \varepsilon) \) can be written as follows, using that \( Q_k(A) = O(A^d) \):

\[
Q(A, \varepsilon) = Q_0(A) + \varepsilon \left( Q_1(A) + O(A^{d-1}) \right).
\]

For those \( Q \) polynomials where \( \deg Q_0 = d \), i.e. \( Q_0(A) = cA^d + O(A^{d-1}) \) with
c > 0 (not negative because of Lemma 4.2), the expansion (4.22) becomes:
\[ Q(A, \varepsilon) = cA^d + O(A^{d-1}) \]
which is positive for sufficiently large A. Using the results from the beginning of the proof of Lemma 4.2, this holds for:
1. \( Q := f(\pm 1), \) where \( d = 1 \) and \( c = 2 \pm 1; \)
2. for all \( Q := F_{n,k}^\pm \) with \( k < n - 1, \) where \( d = k \) and \( c = 1; \)
3. and for \( Q := F_{n,n-1}^\pm, \) where \( d = n - 1 \) and \( c = 2. \)

The only remaining case is \( Q := F_{n,n-1}^-, \) where \( Q_0(A) = 1. \) We prove that here the next polynomial has full degree, i.e. \( \deg Q_1 = n - 1. \) Examine the coefficient of \( A^{n-1} \) in the expansion of \( Q(A, \varepsilon) \) by \( A. \) The structure of \( Q(A, \varepsilon) \) is similar to (4.9), but the \( a_k \) coefficients are replaced by \( a_k(1 + \varepsilon)^k. \) It is still true that it is an \( (n - 1) \times (n - 1) \) determinant where \( A \) appears only in the upper triangular elements, and only as a linear term, so the term \( A^{n-1} \) can come only from the product of the diagonal entries. They are \( q_{i,i} = a_0 - a_{2i}(1 + \varepsilon)^{2i}, \) which is \( A + b_{n-2,2i-2}(1 + \varepsilon)^{2i} \) for \( i \geq 2 \) and \( q_{1,1} = -A\varepsilon(\varepsilon + 2) + b_{n-2,0}(1 + \varepsilon)^2, \) so the coefficient of \( A^{n-1} \) is \( -\varepsilon(\varepsilon + 2). \) Therefore, the leading term of \( Q_1 \) is \(-2A^{n-1}.

Substituting this and \( Q_0(A) = 1 \) to (4.22) gives:
\[ (4.23) \quad Q(A, \varepsilon) = 1 + \varepsilon \left(-2A^{n-1} + O(A^{n-2})\right). \]
Now if \( \varepsilon = 1/A \) exactly, then:
\[ Q(A, \varepsilon) = 1 - 2A^{n-2} + O(A^{n-3}), \]
which is negative for sufficiently large \( A. \) Since \( Q(A, 0) > 0, \) there must be a zero for some \( \varepsilon = O(1/A). \) We can find its order by making (4.23) equal to zero and rearranging:
\[ \varepsilon = \frac{1}{2A^{n-1} + O(A^{n-2})} = \frac{1}{2A^{n-1}} + O \left( \frac{1}{A^n} \right). \]
Since any other \( Q \) polynomials are positive for \( \varepsilon = O(1/A), \) this is the first \( \varepsilon \) for which the conditions of Lemma 2.2 fail. \( \blacksquare \)

## 5 Summary

In this paper we examined expansive polynomials with integer coefficients, and measured their expansivity gap.

First, in Lemma 2.2 we presented an explicit condition for deciding expansivity, which we constructed by expanding the recursive Schur–Cohn algorithm and making smart simplifications. Our condition uses special form of determinants containing the coefficients of the polynomial, with size at most \( (n - 1) \times (n - 1). \) It is an improvement over the naïve expansion of the Schur–Cohn algorithm, i.e. without these simplifications, which would result in determinants of size \( 2^{n-1} \times 2^{n-1}. \)

This condition for expansivity straightforwardly led to Theorem 2.3, i.e. a lower bound on the expansivity gap, namely \( 1/(n^2n!A^{n-1}), \) where the
exponent of $A$ corresponds to the size of the determinants (which would be $2^{n-1}$ by the naive method). Theorem 2.4 proved that this exponent cannot be improved further, because we constructed such polynomials which have the expansivity gap of order $1/(2A^{n-1})$.

Note that exact orders are still different. It is a topic of future work to get the coefficients of $A^{n-1}$ closer to each other, or even make them equal, which would give the exact answer to the lowest possible order of expansivity gaps. Empirical investigation suggests that the answer is closer to the known lower bound ($n^2 n!$) than to the example order, so finding better examples seems more promising than improving the bound.

References

[1] P. Kirschenhofer, J. Thuswaldner: Shift radix systems – a survey. In Numeration and substitution 2012, Research Institute for Mathematical Sciences (RIMS), Kyoto, 2014, pp. 1–59.

[2] S. Akiyama, H. Brunotte, A. Pethő, W. Steiner: Periodicity of certain piecewise affine planar maps. Tsukuba Journal of Mathematics 32 (2008), no. 1, pp. 197–251

[3] G. Barat, V. Berthé, P. Liardet, J. Thuswaldner: Dynamical directions in numeration. Annales de l’Institut Fourier 56 (2006), no. 7, pp. 1987–2092

[4] M. Waldschmidt: Sur le produit des conjugués extérieurs au cercle unité d’un entier algébrique. l’Enseignement Mathématique 26 (1980), no. 3-4, pp. 201–209

[5] D. W. Boyd: Speculations concerning the range of Mahler’s measure. Canadian Mathematical Bulletin 24 (1981), no. 4, pp. 453–469

[6] C. L. Stewart: On a theorem of Kronecker and a related question of Lehmer. Séminaire de Théorie des Nombres 7 (1977–1978), pp. 1-12

[7] G. Everest, T. Ward: Heights of polynomials and entropy in algebraic dynamics. Universitext. Springer-Verlag London, 1999

[8] E. H. Bareiss: Sylvester’s identity and multistep integer-preserving Gaussian elimination. Mathematics of Computation 22 (1968), pp. 565-578

[9] P. Henrici: Applied and Computational Complex Analysis, Volume 1: Power Series – Integration – Conformal Mapping – Location of Zeros. New York: Wiley, 1988.