ON THE ADDITION OF SQUARES OF UNITS MODULO N

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Abstract. Let \( \mathbb{Z}_n \) be the ring of residue classes modulo \( n \), and let \( \mathbb{Z}_n^* \) be the group of its units. 90 years ago, Brauer obtained a formula for the number of representations of \( c \in \mathbb{Z}_n \) as the sum of \( k \) units. Recently, Yang and Tang in [Q. Yang, M. Tang, On the addition of squares of units and nonunits modulo \( n \), J. Number Theory., 155 (2015) 1–12] gave a formula for the number of solutions of the equation \( x_1^2 + x_2^2 = c \) with \( x_1, x_2 \in \mathbb{Z}_n^* \). In this paper, we generalize this result. We find an explicit formula for the number of solutions of the equation \( x_1^2 + \cdots + x_k^2 = c \) with \( x_1, \ldots, x_k \in \mathbb{Z}_n^* \).

1. Introduction

Let \( \mathbb{Z}_n \) be the ring of residue classes modulo \( n \), and let \( \mathbb{Z}_n^* \) be the group of its units. Let \( c \in \mathbb{Z}_n \), and let \( k \) be a positive integer. Brauer in [1] gave a formula for the number of solutions of the equation \( x_1 + \cdots + x_k = c \) with \( x_1, \ldots, x_k \in \mathbb{Z}_n^* \). In [4] Sander found the number of representations of a fixed residue class mod \( n \) as the sum of two units in \( \mathbb{Z}_n \), the sum of two non-units, and the sum of mixed pairs, respectively. In [3] the results of Sander were generalized into an arbitrary finite commutative ring, as sum of \( k \) units and sum of \( k \) non-units, with a combinatorial approach.

The problem of finding explicit formulas for the number of representations of a natural number \( n \) as the sum of \( k \) squares is one of the most interesting problems in number theory. For example, if \( k = 4 \), then Jacobi’s four-square theorem states that this number is \( 8 \sum_{m \mid c} m \) if \( c \) is odd and

\begin{equation}
8 \sum_{m \mid c} m = \begin{cases} 
4c & \text{if } c \text{ is even}, \\
2c & \text{if } c \text{ is odd}.
\end{cases}
\end{equation}
24 times the sum of the odd divisors of $c$ if $c$ is even. See [5] and the references given there for historical remarks.

Recently, Tóth [5] obtained formulas for the number of solutions of the equation

$$a_1x_1^2 + \cdots + a_kx_k^2 = c,$$

where $c \in \mathbb{Z}_n$, and $x_i$ and $a_i$ all belong to $\mathbb{Z}_n$.

Now, consider the equation

$$x_1^2 + \cdots + x_k^2 = c,$$  \hspace{1cm} (1)

where $c \in \mathbb{Z}_n$, and $x_i$ are all units in the ring $\mathbb{Z}_n$. We denote the number of solutions of this equation by $S_{sq}(\mathbb{Z}_n, c, k)$. In [6] Yang and Tang obtained a formula for $S_{sq}(\mathbb{Z}_n, c, 2)$. In this paper we provide an explicit formula for $S_{sq}(\mathbb{Z}_n, c, k)$, for an arbitrary $k$. Our approach is combinatorial with the help of spectral graph theory.

2. Preliminaries

In this section we present some graph theoretical notions and properties used in the paper. See, e.g., the book [2]. Let $G$ be an additive group with identity $0$. For $S \subseteq G$, the Cayley graph $X = Cay(G, S)$ is the directed graph having vertex set $V(X) = G$ and edge set $E(X) = \{(a, b); b - a \in S\}$. Clearly, if $0 \notin S$, then there is no loop in $X$, and if $0 \in S$, then there is exactly one loop at each vertex. If $-S = \{-s; s \in S\} = S$, then there is an edge from $a$ to $b$ if and only if there is an edge from $b$ to $a$.

Let $\mathbb{Z}_n^* = \{x^2; x \in \mathbb{Z}_n^*\}$. The quadratic unitary Cayley graph of $\mathbb{Z}_n$, $G_{\mathbb{Z}_n}^2 = Cay(\mathbb{Z}_n; \mathbb{Z}_n^*)$, is defined as the directed Cayley graph on the additive group of $\mathbb{Z}_n$ with respect to $\mathbb{Z}_n^*$; that is, $G_{\mathbb{Z}_n}^2$ has vertex set $\mathbb{Z}_n$ such that there is an edge from $x$ to $y$ if and only if $y - x \in \mathbb{Z}_n^2$. Then the out-degree of each vertex is $|\mathbb{Z}_n^2|$.

Let $G$ be a graph, and let $V(G) = \{v_1, \ldots, v_n\}$. The adjacency matrix $A_G$ of $G$ is defined in a natural way. Thus, the rows and the columns of $A_G$ are labeled by $V(G)$. For $i, j$, if there is an edge from $v_i$ to $v_j$ then $a_{v_i v_j} = 1$; otherwise $a_{v_i v_j} = 0$. We will write it simply $A$ when no confusion can arise. For the graph $G_{\mathbb{Z}_n}^2$ the matrix $A$ is symmetric, provided that -1 is a square mod $n$.

We write $J_m$ for the $m \times m$ all 1-matrix. The identity $m \times m$ matrix will be denoted by $I_m$. 
The complete graph on \( m \) vertices with loop at each vertex is denoted by \( K^l_m \). Thus, the adjacency matrix of \( K^l_m \) is \( J_m \).

A \textit{walk} in a graph \( G \) is a sequence \( v_0, e_1, v_1, e_2, \ldots, e_n, v_n \) so that \( v_i \in V(G) \) for every \( 0 \leq i \leq n \), and \( e_i \) is an edge from \( v_{i-1} \) to \( v_i \), for every \( 1 \leq i \leq n \). We denote by \( w_k(G, i, j) \) the number of walks of length \( k \) from \( i \) to \( j \) in the graph \( G \).

One application of the adjacency matrix is to calculate the number of walks between two vertices.

**Lemma 2.1.** [\cite{2}, Lemma 8.1.2] Let \( G \) be a directed graph, and let \( k \) be a positive integer. Then the number of walks from vertex \( i \) to vertex \( j \) of length \( k \) is the entry on row \( i \) and column \( j \) of the matrix \( A^k \), where \( A \) is the adjacency matrix.

The next theorem provides the connection between \( S_{sq}(\mathbb{Z}_{p^\alpha}, c, k) \) and \( w_k(G_{\mathbb{Z}_{p^\alpha}}, 0, c) \).

**Theorem 2.2.** Let \( p \) be an odd prime number and \( \alpha \) be a positive integer. Then

\[
S_{sq}(\mathbb{Z}_{p^\alpha}, c, k) = 2^k w_k(G_{\mathbb{Z}_{p^\alpha}}, 0, c).
\]

**Proof.** Consider the graph \( G_{\mathbb{Z}_{p^\alpha}} \). Let \((x_1, \ldots, x_k) \in (\mathbb{Z}_{p^\alpha})^k \) such that \( x_1^2 + x_2^2 + \cdots + x_k^2 = c \). Then \( 0, x_1^2, x_1^2 + x_2^2, \ldots, x_1^2 + x_2^2 + \cdots + x_k^2 = c \) is a walk of length \( k \) from 0 to \( c \).

Now, let \( 0 = a_0, a_1, \ldots, a_k = c \) be a walk of length \( k \). Then \( a_i - a_{i-1} = y_i \), where \( y_i \in \mathbb{Z}_{p^\alpha} \) for \( i = 1, \ldots, k \). Hence \( y_1^2 + y_2^2 + \cdots + y_k^2 = c \). Then the set \( \{(\epsilon_k y_1, \ldots, \epsilon_k y_k) ; \epsilon_i \in \{1, -1\}\} \) is a set of solutions of size \( 2^k \), which proves the theorem. \( \square \)

The \textit{tensor product} \( G_1 \otimes G_2 \) of two graphs \( G_1 \) and \( G_2 \) is the graph with vertex set \( V(G_1) \times V(G_2) := V(G_1) \times V(G_2) \), with edges specified by putting \((u, v)\) adjacent to \((u', v')\) if and only if \( u \) is adjacent to \( u' \) in \( G_1 \) and \( v \) is adjacent to \( v' \) in \( G_2 \). It can be easily verified that the number of edges in \( G_1 \otimes G_2 \) is equal to the product of the number of edges in the graphs \( G \) and \( H \).

**Lemma 2.3.** The adjacency matrix of \( G \otimes H \) is the tensor product of the adjacency matrices of \( G \) and \( H \).

The rest of paper is organized as follows. In section 3 we reduce the case \( S_{sq}(\mathbb{Z}_n, c, k) \) to the cases \( S_{sq}(\mathbb{Z}_p, c, k) \) and \( S_{sq}(\mathbb{Z}_{2^\alpha}, c, k) \). We show that if \( p \) is an odd prime number, then \( G_{\mathbb{Z}_{p^\alpha}}^2 \cong G_{\mathbb{Z}_{p^\alpha}}^2 \otimes K_{p^\alpha - 1}^l \).

Section 4 is devoted to the study of \( S_{sq}(\mathbb{Z}_p, c, k) \), where \( p \equiv 1 \mod 4 \). In this section, we write \( A^k \)
as a linear combination of matrices $A, J_p$ and $I_p$, and then we obtain a formula for $S_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$. Similarly, we find a formula for $S_{sq}(\mathbb{Z}_{p^\alpha}, c, k)$, where $p \equiv 3 \mod 4$, in section 5. Last section, provides an explicit formula for $S_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$ by direct counting.

3. General results

In this section, we reduce the case $S_{sq}(\mathbb{Z}_n, c, k)$ to the cases $S_{sq}(\mathbb{Z}_p, c, k)$ and $S_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$.

The next lemma shows that the function $n \rightarrow S_{sq}(\mathbb{Z}_n, c, k)$ is multiplicative.

**Lemma 3.1.** Let $m, n$ be coprime numbers. Then $S_{sq}(\mathbb{Z}_{mn}, c, k) = S_{sq}(\mathbb{Z}_m, c, k) \cdot S_{sq}(\mathbb{Z}_n, c, k)$.

**Proof.** The proof follows using the Chinese remainder theorem. □

**Lemma 3.2.** Let $p$ be an odd prime number, and let $m$ be the ideal generated by $p$ in the ring $\mathbb{Z}_{p^\alpha}$. Let $u \in \mathbb{Z}_{p^\alpha}^2$ and $r \in m$. Then $u + r \in \mathbb{Z}_{p^\alpha}^2$.

**Proof.** For this to happen, it is enough to show that $1 + r$ belongs to $\mathbb{Z}_{p^\alpha}^2$. We know that $r$ is a nilpotent element of $\mathbb{Z}_{p^\alpha}$. Let $\lambda$ be a sufficiently large integer. Then $(1 + r)^{p^\lambda} = 1$. Hence, $(1 + r)^{p^{\lambda+1}} = 1 + r$. □

**Theorem 3.3.** Let $p$ be an odd prime number, and let $\alpha$ be a positive integer. Then $G_{\mathbb{Z}_{p^\alpha}}^2 \cong G_{\mathbb{Z}_p}^2 \otimes K_{p^{\alpha-1}}^l$.

**Proof.** Let $m$ be the ideal generated by $p$, and $\mathbb{Z}_{p^\alpha} = \bigcup_{i=1}^m (m + r_i)$, where $m + r_i$ is a coset of the maximal ideal $m$ in $\mathbb{Z}_{p^\alpha}$. The ring $\mathbb{Z}_{p^\alpha}/m$ is isomorphic to the field $\mathbb{Z}_p$. Then for each $r \in \mathbb{Z}_{p^\alpha}$ there is a unique $i$ and $n_r \in m$ such that $r = r_i + n_r$. Let $\psi : G_{\mathbb{Z}_{p^\alpha}}^2 \rightarrow G_{\mathbb{Z}_p}^2 \otimes K_{p^{\alpha-1}}^l$ be defined by $\psi(r) := (r_i + m, n_r)$. Obviously, this map is a bijection. Now, let $(r, r')$ be a directed edge in $G_{\mathbb{Z}_{p^\alpha}}^2$. We show that $(\psi(r), \psi(r'))$ is also a directed edge in $G_{\mathbb{Z}_p}^2 \otimes K_{p^{\alpha-1}}^l$. By definition, $\psi(r) = (r_i + m, n_r)$ and $\psi(r') = (r_j + m, n_{r'})$. We have $r' - r \in \mathbb{Z}_{p^\alpha}^2$. Thus, $r_j - r_i + n_{r'} - n_r \in \mathbb{Z}_{p^\alpha}^2$. Hence by Lemma 3.2, $r_j - r_i \in \mathbb{Z}_{p^\alpha}^2$. Then $r_j - r_i + m \in (\mathbb{Z}_{p^\alpha}/m)^2$. Since the number of edges of $G_{\mathbb{Z}_{p^\alpha}}^2$ and $G_{\mathbb{Z}_p}^2 \otimes K_{p^{\alpha-1}}^l$ are the same, the proof is complete. □

By the aforementioned theorem, we see

$$A_{G_{\mathbb{Z}_{p^\alpha}}^2}^k = A_{G_{\mathbb{Z}_p}^2}^k \otimes A_{K_{p^{\alpha-1}}^l}^k$$

$$= A_{G_{\mathbb{Z}_p}^2}^k \otimes J_{p^{\alpha-1}}^k.$$
4. $S_{sq}(Z_p^*, c, k)$ where $p \equiv 1 \mod 4$

In this section, we find $S_{sq}(Z_p^*, c, k)$, where $p$ is a prime number with $p \equiv 1 \mod 4$.

An strongly regular graph with parameters $(n, k, \lambda, \mu)$ is a simple graph with $n$ vertices that is regular of valency $k$ and has the following properties:
- For any two adjacent vertices $x, y$, there are exactly $\lambda$ vertices adjacent to both $x$ and $y$.
- For any two non-adjacent vertices $x, y$, there are exactly $\mu$ vertices adjacent to both $x$ and $y$.

Let $p$ be a fixed prime number with $p \equiv 1 \mod 4$. The Paley graph $P_p$ is defined by taking the field $Z_p$ as vertex set, with two vertices $x$ and $y$ joined by an edge if and only if $x - y$ is a nonzero square in $Z_p$.

As in well known (see e.g., [2, Page 221]), the Paley graph is strongly regular with parameters $(p, \frac{p^2 - 1}{2}, \frac{p^2 - 1}{4}, \frac{p - 1}{4})$. The fact that Paley graph is strongly regular shows that $A^2$ can be written as a linear combination of matrices $A$, $J_p$ and $I_p$.

**Lemma 4.1.** [2, Page 219] Let $p$ be a prime number such that $p \equiv 1 \mod 4$. Then the adjacency matrix of the Paley graph $P_p$ satisfies

$$A^2_{P_p} = -A_{P_p} + \left(\frac{p - 1}{4}\right)J_p + \left(\frac{p - 1}{4}\right)I_p. \quad (2)$$

Although the graph $G_{Z_p^*}^2$ is a directed graph and $P_p$ is a simple graph, they share the same adjacency matrix. Then $A_{G_{Z_p^*}^2}^n$ can be written as a linear combination of $A_{G_{Z_p^*}^2}$, $I_p$ and $J_p$.

Let

$$A^{n+1} = a_{n+1, p}A + b_{n+1, p}J_p + c_{n+1, p}I_p. \quad (3)$$

Then

$$A^{n+2} = a_{n+2, p}A^2 + \frac{p - 1}{2}b_{n+2, p}J_p + c_{n+2, p}A.$$

Now, by equation (2), we have

$$A^{n+2} = (a_{n+1, p}a_{1, p} + c_{n+1, p})A + \left(\frac{p - 1}{2}\right)b_{n+1, p} + a_{n+1, p}b_{1, p})J_p + (a_{n+1, p}c_{1, p})I_p.$$

Then we see that

\[
\begin{align*}
    a_{n+1, p} &= a_{n, p}a_{1, p} + c_{n, p}, & a_{1, p} &= -1, a_{2, p} = \frac{p + 3}{4}, \\
    b_{n+1, p} &= \frac{p - 1}{2}b_{n, p} + a_{n, p}b_{1, p}, & b_{1, p} &= \frac{p - 1}{4}, b_{2, p} = \left(\frac{p - 1}{4}\right)\left(\frac{p - 3}{2}\right), \\
    c_{n+1, p} &= a_{n, p}c_{1, p}, & c_{1, p} &= \frac{p - 1}{4}, c_{2, p} = -\frac{p - 1}{4}.
\end{align*}
\]
From the first and last equations, we have the following homogeneous linear recurrence relation
\[ a_{n,p} = \frac{p-1}{4} a_{n-2,p} - a_{n-1,p}. \]
Since \( a_1 = -1 \) and \( a_2 = \frac{p+3}{4} \), we deduce
\[ a_{n,p} = \left( \frac{\sqrt{p}-1}{2\sqrt{p}} \right)(-1+\sqrt{p})^n + \left( \frac{\sqrt{p}+1}{2\sqrt{p}} \right)(-1-\sqrt{p})^n. \]
Then
\[ a_{n,p} = \left( \frac{1}{2n+1} \right) (-1 + \sqrt{p})^{n+1} + (-1)^n (1 + \sqrt{p})^{n+1}. \]
Now, we have
\[ c_{n,p} = \left( \frac{p-1}{2n+1} \right) (-1 + \sqrt{p})^n + (-1)^{n-1} (1 + \sqrt{p})^n. \]
Thus, for \( b_{n,p} \) we have the following non-homogeneous linear recurrence relation
\[ b_{n,p} = \frac{p-1}{2} b_{n-1,p} + \left( \frac{p-1}{2n+1} \right) (-1 + \sqrt{p})^{n-1} + (-1)^{n-2} (1 + \sqrt{p})^{n-1}. \]
Then
\[ b_{n,p} = \beta \left( \frac{p-1}{2} \right)^n + \left( \frac{p-1}{2n+1} \right) \left( \frac{\sqrt{p}+1}{\sqrt{p}-1} \right) (-1 + \sqrt{p})^{n-1} + (-1)^{n-2} \frac{\sqrt{p}-1}{\sqrt{p}+1} (1 + \sqrt{p})^{n-1}. \]
Since \( b_1 = \frac{p-1}{4} \), it follows that
\[ b_{n,p} = \frac{(p-5)(p-1)}{2(p-1)} \left( \frac{p-1}{2} \right)^n + \left( \frac{p-1}{2n+1} \right) \left( \frac{\sqrt{p}+1}{\sqrt{p}-1} \right) (-1 + \sqrt{p})^{n-1} + (-1)^{n-2} \frac{\sqrt{p}-1}{\sqrt{p}+1} (1 + \sqrt{p})^{n-1}. \]
We can now find \( S_{sq}(\mathbb{Z}_p, c, k) \).
\[ S_{sq}(\mathbb{Z}_p, c, k) = \begin{cases} 2^k(b_{k-1,p} + c_{k-1,p}), & \text{if } c = 0; \\ 2^k(a_{k-1,p} + b_{k-1,p}), & \text{if } c = x^2, \text{ for some } x \in \mathbb{Z}_p^*; \\ 2^kb_{k-1,p}, & \text{otherwise}. \end{cases} \]
\[ (4) \]
The last theorem of this section provides a formula for \( S_{sq}(\mathbb{Z}_{p^*}, c, k) \).

**Theorem 4.2.** Let \( p \) be a prime number such that \( p \equiv 1 \pmod{4} \). Let \( \alpha \) be a positive integer. Then
\[ S_{sq}(\mathbb{Z}_{p^*}, c, k) = \begin{cases} p^{(\alpha-1)(k-1)/2} (b_{k-1,p} + c_{k-1,p}), & \text{if } c \equiv 0 \pmod{p}; \\ p^{(\alpha-1)(k-1)/2} (a_{k-1,p} + b_{k-1,p}), & \text{if } c = x^2, \text{ for some } x \in \mathbb{Z}_{p^*}^*; \\ p^{(\alpha-1)(k-1)/2} b_{k-1,p}, & \text{otherwise}, \end{cases} \]
where \(a_{k-1,p}, c_{k-1,p}\) and \(b_{k-1,p}\) are defined by equations (i), (ii) and (iii), respectively, (putting \(n = k - 1\)).

**Proof.** By Theorem 3.3 and Lemma 2.3, \(A_{G^2_{p\alpha}}^k = A_{G^2_{p\alpha}} \otimes A_{K^{p\alpha-1}}\). Then

\[
A_{G^2_{p\alpha}}^k = A_{G^2_{p\alpha}} \otimes J_{p\alpha-1}^k
\]

\[
= A_{G^2_{p\alpha}}^k \otimes p^{(a-1)(k-1)}J_{p\alpha-1}.
\]

Then equation (4) and Lemma 2.1, complete the proof. \(\square\)

5. \(S_{sq}(Z_{p\alpha}, c, k)\) where \(p \equiv 3 \mod 4\)

In this section, we find \(S_{sq}(Z_{p\alpha}, c, k)\), where \(p\) is a prime number with \(p \equiv 3 \mod 4\). The main idea is similar to that used in the previous section. We try to write \(A_{G^2_{p\alpha}}^2\) as a linear combination of matrices \(A_{G^2_{p\alpha}}, I_p\) and \(J_p\).

The field \(Z_p\), has no square root of -1. Then for each pair of \((x, y)\) of distinct elements of \(Z_p\), either \(x - y\) or \(y - x\), but not both, is a square of a nonzero element. Hence in the graph \(G^2_{Z_p}\), each pair of distinct vertices is linked by an arc in one and only one direction. Therefore, \(A_{G^2_{Z_p}} + A_{G^2_{Z_p}}^T = J_p - I_p\). The entry on row \(a\) and column \(b\) of the matrix \(A_{G^2_{Z_p}}^2\) equals to the size of the set \((a + Z_{p}^2) \cap (b - Z_{p}^2)\). The goal of following lemmas is to find \(|(a + Z_{p}^2) \cap (b - Z_{p}^2)|\).

**Lemma 5.1.** Let \(a\) and \(b\) be elements of \(Z_p\). Then \(|(a + Z_{p}^2) \cap (b - Z_{p}^2)| = |(a - b + Z_{p}^2) \cap -Z_{p}^2|\).

**Proof.** Let \(\psi : (a + Z_{p}^2) \cap (b - Z_{p}^2) \longrightarrow (a - b + Z_{p}^2) \cap -Z_{p}^2\) be defined by \(\psi(r) = r - b\). Obviously, \(\psi\) is well-defined and injective. Now, let \(c \in (a - b + Z_{p}^2) \cap -Z_{p}^2\), so there exists \(s \in Z_{p}^2\) such that \(c = a - b + s\). Then \(\psi(c + b) = c\), which completes the proof. \(\square\)

**Lemma 5.2.** Let \(a\) be a non-zero element of \(Z_p\). Then \(|(a^2 + Z_{p}^2) \cap -Z_{p}^2| = |(1 + Z_{p}^2) \cap -Z_{p}^2|\) and \(|(-a^2 + Z_{p}^2) \cap -Z_{p}^2| = |(-1 + Z_{p}^2) \cap -Z_{p}^2|\).

**Proof.** Let \(\psi : (a^2 + Z_{p}^2) \cap -Z_{p}^2 \longrightarrow (1 + Z_{p}^2) \cap -Z_{p}^2\) be defined by \(\psi(r) = ra^{-2}\). Obviously, \(\psi\) is well-defined and injective. Now, let \(c \in (1 + Z_{p}^2) \cap -Z_{p}^2\). Thus, there exists \(s \in Z_{p}^2\) such that \(c = 1 + s^2\). Then \(\psi(cs^2) = c\), which completes the proof. \(\square\)

The proof for the second part is similar.
Then by Lemmas 5.1 and 5.2, one can easily see that \( A^2 \) is a linear combination of matrices \( A \), \( J_p \) and \( I_p \).

**Lemma 5.3.** \(|(1 + Z_p^{s_2}) \cap (-Z_p^{s_2})| = \frac{p+1}{4} \).

**Proof.** We know that \( (1+Z_p^{s_2}) \cap (-Z_p^{s_2}) = 1+Z_p^{s_2} \), and \( (1+Z_p^{s_2}) \cap Z_p^{s_2} = 0 \). Then \(|(1+Z_p^{s_2}) \cap (-Z_p^{s_2})| = \frac{p-1}{2} - |(1+Z_p^{s_2}) \cap Z_p^{s_2}| \). Now, \( a \in (1+Z_p^{s_2}) \cap Z_p^{s_2} \) if and only there exist \( b, c \in Z_p^* \) such that \( a = 1+b^2 = c^2 \). Thus, \((c-b)(c+b) = 1 \). Hence \( c = \frac{u+v^{-1}}{2} \) and \( b = \frac{u-v^{-1}}{2} \), for \( u \in Z_p^* - \{1, -1\} \). Then \((1+Z_p^{s_2}) \cap (Z_p^{s_2}) = \{(\frac{u+v^{-1}}{2})^2; u \in Z_p^* \} - \{1\} \).

If \((\frac{u+v^{-1}}{2})^2 = (\frac{u+v^{-1}}{2})^2\), then we have two cases:

(i) \( \frac{u+v^{-1}}{2} = \frac{u+v^{-1}}{2} \). A trivial verification shows that \( u = v \) or \( u = v^{-1} \).

(ii) \( \frac{u+v^{-1}}{2} = -\frac{u+v^{-1}}{2} \). Then \( u = -v \) or \( u = -v^{-1} \).

Then \(|(1+Z_p^{s_2}) \cap (Z_p^{s_2})| = \frac{p-1}{4} \), and the lemma follows. \( \square \)

The following lemma may be proved in much the same way as Lemma 5.3.

**Lemma 5.4.** \(|(-1 + Z_p^{s_2}) \cap (-Z_p^{s_2})| = \frac{p-3}{4} \).

**Lemma 5.5.** Let \( p \) be a prime number with \( p \equiv 3 \mod 4 \). Let \( A \) be the adjacency matrix of the graph \( G_{Z_p^2}^2 \). Then

\[
A^2 = -A + \left(\frac{p+1}{4}\right)J_p - \left(\frac{p+1}{4}\right)I_p.
\]

**Proof.** Let \( a, b \in Z_p \). By Lemma 5.1,

\[
(A)_{ab} = |(a + Z_p^{s_2}) \cap (b - Z_p^{s_2})| = |(a - b + Z_p^{s_2}) \cap (-Z_p^{s_2})|.
\]

If there is an edge from \( a \) to \( b \), then by Lemmas 5.2 and 5.4,

\[
(A)_{ab} = |(-1 + Z_p^{s_2}) \cap (-Z_p^{s_2})| = \frac{p-3}{4}.
\]

If \( a \neq b \) and there is no edge from \( a \) to \( b \), then by a similar argument, we have \((A)_{ab} = \frac{p+1}{4} \). If \( a = b \), then by Lemma 5.1,

\[
(A)_{ab} = |(a + Z_p^{s_2}) \cap (b - Z_p^{s_2})| = |(Z_p^{s_2}) \cap (Z_p^{s_2})| = 0,
\]

which establishes equation (5). \( \square \)
Let
\[ A^{n+1} = a_{n,p}A + b_{n,p}J_p + c_{n,p}I_p. \]

Hence
\[ A^{n+1} = a_{n,p}A^2 + b_{n,p}\frac{p-1}{2}J_p + c_{n,p}A. \]

Then
\[ A^{n+1} = (c_{n+1,p} - a_{n,p})A + (a_{n,p}\frac{p+1}{4} + b_{n+1,p}\frac{p-1}{2})J_p + (-a_{n,p}\frac{p+1}{4})I_p. \]

Thus, we have
\[
\begin{align*}
  a_{n+1,p} &= c_{n,p} - a_{n,p}, \quad a_{1,p} = -1, a_{2,p} = \frac{3-p}{4}; \\
  b_{n+1,p} &= \frac{p-1}{2}b_{n,p} + a_{n,p}\frac{p+1}{4}, \quad b_{1,p} = \frac{p+1}{4}, b_{2,p} = \frac{p+1}{4}(\frac{p-1}{2} - 1); \\
  c_{n+1,p} &= -a_{n,p}\frac{p+1}{4}, \quad c_{1,p} = -\frac{p+1}{4}, c_{2,p} = \frac{p+1}{4}.
\end{align*}
\]

From the first and last equations, we have the following homogeneous linear recurrence relation
\[ a_{n+1,p} + a_{n,p} + \frac{p+1}{4}a_{n-1,p} = 0. \]

Since \(a_{1,p} = -1\) and \(a_{2,p} = \frac{3-p}{4}\), we deduce
\[ a_{n,p} = (\frac{\sqrt{p} + i}{2\sqrt{p}})(\frac{-1 + i\sqrt{p}}{2})^n + (\frac{\sqrt{p} - i}{2\sqrt{p}})(\frac{-1 - i\sqrt{p}}{2})^n, \]
where \(i = \sqrt{-1}\). Then
\[ c_{n,p} = -\frac{p+1}{4}((\frac{\sqrt{p} + i}{2\sqrt{p}})(\frac{-1 + i\sqrt{p}}{2})^{n-1} + (\frac{\sqrt{p} - i}{2\sqrt{p}})(\frac{-1 - i\sqrt{p}}{2})^{n-1}). \]

Thus, for \(b_{n,p}\) we have the following non-homogeneous linear recurrence relation
\[ b_{n,p} = \frac{p-1}{2}b_{n-1,p} + \frac{p+1}{4}((\frac{\sqrt{p} + i}{2\sqrt{p}})(\frac{-1 + i\sqrt{p}}{2})^{n-1} + (\frac{\sqrt{p} - i}{2\sqrt{p}})(\frac{-1 - i\sqrt{p}}{2})^{n-1}). \]

Then
\[ b_{n,p} = \alpha(p-1)^n + \frac{p+1}{8\sqrt{p}}((\frac{\sqrt{p} + i}{i\sqrt{p} - p})(\frac{-1 + i\sqrt{p}}{2})^{n-1} + (\frac{\sqrt{p} - i}{i\sqrt{p} + p})(\frac{-1 - i\sqrt{p}}{2})^{n-1}). \]

Since \(b_{1,p} = \frac{p+1}{4}\), it follows that
\[ b_{n,p} = \frac{p-1}{2p}(p-1)^n + \frac{p+1}{8\sqrt{p}}((\frac{\sqrt{p} + i}{i\sqrt{p} - p})(\frac{-1 + i\sqrt{p}}{2})^{n-1} + (\frac{\sqrt{p} - i}{i\sqrt{p} + p})(\frac{-1 - i\sqrt{p}}{2})^{n-1}). \]
ON THE ADDITION OF SQUARES OF UNITS MODULO N

Then the number of solutions of the equation (1) is

$$S_{sq}(\mathbb{Z}_p, c, k) = \begin{cases} 
2^k(b_{k-1,p} + c_{k-1,p}), & \text{if } c = 0; \\
2^k(a_{k-1,p} + b_{k-1,p}), & \text{if } c = x^2, \text{ for some } x \in \mathbb{Z}_p^*; \\
2^k b_{k-1,p}, & \text{otherwise.}
\end{cases}$$

**Theorem 5.6.** Let $p$ be a prime number such that $p \equiv 3 \mod 4$. Let $\alpha$ be a positive integer. Then

$$S_{sq}(\mathbb{Z}_{p^\alpha}, c, k) = \begin{cases} 
p^{(\alpha-1)(k-1)}2^k(b_{k-1,p} + c_{k-1,p}), & \text{if } c \equiv 0 \mod p; \\
p^{(\alpha-1)(k-1)}2^k(a_{k-1,p} + b_{k-1,p}), & \text{if } c = x^2, \text{ for some } x \in \mathbb{Z}_{p^\alpha}^*; \\
p^{(\alpha-1)(k-1)}2^k b_{k-1,p}, & \text{otherwise,}
\end{cases}$$

where $a_{k-1,p}$, $c_{k-1,p}$ and $b_{k-1,p}$ are defined by equations (i'), (ii') and (iii'), respectively, (putting $n = k - 1$).

**Proof.** The proof is similar to that of Theorem 4.2. \qed

6. $S_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$

In this section we find $S_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$. For $\alpha = 1$ and $\alpha = 2$, this number is easy to find.

**Lemma 6.1.** Let $n = 2^\alpha$ such that $\alpha > 2$. Then $\mathbb{Z}_{n^2}^* = \{8k + 1; k \in \{0, \ldots, \frac{n}{8} - 1\}\}$.

**Proof.** Obviously, $\{8k + 1; k \in \{0, \ldots, \frac{n}{8} - 1\}\} \supseteq \mathbb{Z}_{n^2}^*$. It suffices to show that the set $\mathbb{Z}_{n^2}^*$ has exactly $n/8$ elements. Define the equivalence relation between odd elements of $\mathbb{Z}_n$ as follows. We say $a \sim b$ if and only if $a^2 \equiv b^2 \mod 2^\alpha$. It is easy to check that each equivalence class has exactly 4 elements. Hence the number of equivalence classes is $n/8$, which equals to the size of $\mathbb{Z}_{n^2}^*$. \qed

Now, we are able to find $S_{sq}(\mathbb{Z}_{2^\alpha}, c, k)$.

**Theorem 6.2.** Let $n = 2^\alpha$. Then

$$S_{sq}(\mathbb{Z}_{2^\alpha}, c, k) = \begin{cases} 
1, & \text{if } \alpha = 1 \text{ and } c \equiv k \mod 2; \\
2^k, & \text{if } \alpha = 2 \text{ and } c \equiv k \mod 4; \\
2^{2k + (\alpha-3)(k-1)}, & \text{if } \alpha > 2 \text{ and } c \equiv k \mod 8; \\
0 & \text{otherwise.}
\end{cases}$$
Proof. Let $\alpha > 2$. Let $A = \{(y_1, \ldots, y_k); 8 \sum_{i=1}^{k} y_i = c - k\}$ and $B = \{(x_1, \ldots, x_k); \sum_{i=1}^{k} x_i^2 = c\}$. Then by Lemma 6.1, there exists a $4^k$ to 1 and onto map from $B$ to $A$. It is easy to see that if $c \equiv k \mod 8$, then $|A| = (2^{\alpha-3})^{k-1}$, which establishes the formula.

Remark. Let $n = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$. Then by Lemma 3.1, we conclude that

$$S_{sq}(\mathbb{Z}_n,c,k) = \prod_{i=1}^{t} S_{sq}(\mathbb{Z}_{p_i^{\alpha_i}},c,k),$$

which can be computed easily by Theorems 4.2, 5.6 and 6.2.

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