Commutator of Marcinkiewicz Integral Operators on Herz-Morrey-Hardy Spaces with Variable Exponents

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Abstract

In this paper, our aim is to prove the boundedness of commutators generated by the Marcinkiewicz integrals operator \([b, \mu_\Omega]\) and obtain the result with Lipschitz function and BMO function \(f\) on the Herz-Morrey-Hardy spaces with variable exponents \(H_{\mu_\Omega}(\mathbb{R}^n)\).

Keywords

Marcinkiewicz Integral Operator, Herz-Morrey-Hardy Space, Commutator, Variable Exponent, Lipschitz Space

1. Introduction

Firstly in 1938, Marcinkiewicz [1] introduced the Marcinkiewicz integral. Next, the Marcinkiewicz integral operator has been studied extensively by many mathematicians in various fields. For example, Stein in [2] introduced the Marcinkiewicz integral operator related to the littlewood-Paley \(g\) function on \(\mathbb{R}^n\) and proved that \(\mu_\Omega\) is of type \((p, p)\) for \(1 < p \leq 2\) and of week type \((1, 1)\). In [3], Ding, Fan and Pan improved the above result and obtained the boundedness of the Marcinkiewicz cussed the boundedness for the commutator generated by the Marcinkiewicz integral \(\mu\) under some weak conditions. Torchinsky and Wang in [4] discussed integral \(\mu_\Omega\) and BMO function on Lebesgue spaces \(L^p(\mathbb{R}^n)\).

On the other hand, a class of functional spaces called Herz-Morrey-Hardy spaces with variable exponent has attracted great interest in recent years. We find that in successive studies in this field, in [5] [6] Xu, Yang introduced Herz-
Morrey-Hardy spaces with variable exponents and their some applications. He obtained that certain singular integral operators are bounded from Herz-Morrey-Hardy spaces with variable exponents into Herz-Morrey spaces with variable exponents as an application of the atomic characterization. Also, he established their molecular decomposition, and by using their atomic and molecular decompositions, he gave the boundedness of a convolution type singular integral on Herz-Morrey-Hardy spaces with variable exponents. Omer in [7] proved the boundedness of commutators generated by the Calderón-Zygmund and used properties of variable exponent, BMO(R^n) function and Lipschitz function to prove this boundedness. Also, Yang in [8] established some boundedness for \(T^\gamma - T_a^\gamma\) and \((T^r - T^r_a)D^r\) on the homogeneous Morrey-Herz-type Hardy spaces with variable exponents and studied Boundedness of Calderón-Zygmund operator on these spaces.

Suppose \(S^{n-1}(n \geq 2)\) denotes the unit sphere in \(\mathbb{R}^n\) equipped with the normalized measure \(d\sigma\). Let \(\Omega\) be homogenous function of degree zero and satisfies

\[
\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,
\]

(1.1)

where \(x' = x/|x|\) for any \(x \neq 0\).

Then the Marcinkiewicz integral operator \(\mu_\Omega\) is defined by

\[
\mu_\Omega(f)(x) = \left(\int_0^1 \left[ F_{\Omega,t}(f)(x) \right]^2 \frac{dt}{t}\right)^{1/2},
\]

(1.2)

where

\[
F_{\Omega,t}(f)(x) = \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y)dy.
\]

(1.3)

Let \(b \in Lip_\gamma(\mathbb{R}^n)\) and \(b \in BMO\) be a locally integrable function on \(\mathbb{R}^n\), the commutator generated by the Marcinkiewicz integral \(\mu_\Omega\) and \(b\) is defined by

\[
[b, \mu_\Omega](f)(x) = \left(\int_0^1 \left[ F_{\Omega,t}(f)(x) \right]^2 \frac{dt}{t}\right)^{1/2}.
\]

(1.4)

Motivated by [6] and [7], the aim of this paper is to study the boundedness for the commutator of Marcinkiewicz integral operator \([b, \mu_\Omega]\) on the Herz-Morrey-Hardy space with variable exponent where \(\Omega \in L'(S^{n-1})\) for \(s \geq 1\), with BMO function and Lipschitz function, we will define The definitions of the Morrey-Herz spaces with variable exponents, the Morrey-Herz-Hardy spaces with variable exponents (which will be defined in the next section), and the preliminary lemmas are presented in Section 2. In Section 3, we will prove the boundedness of the commutator of Marcinkiewicz integrals on Herz-Morrey-Hardy spaces with variable exponent with \(b \in Lip_\gamma(\mathbb{R}^n)\). Lastly, in Section 4 we will prove the boundedness of the commutator of Marcinkiewicz integrals on Herz-Morrey-Hardy spaces with variable exponent with function \(b \in BMO(\mathbb{R}^n)\).
A given open set \( \Omega \subset \mathbb{R}^n \) and a measurable function \( p(\cdot) : \Omega \to [1, \infty) \), \( L^{p(\cdot)}(\Omega) \) denotes the set of measurable function \( f \) on \( \Omega \) such that for some \( \lambda > 0 \),
\[
L^{p(\cdot)}(\Omega) = \left\{ f \text{ is measurable} : \int_{\Omega} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx < \infty \text{ for some constant } \eta > 0 \right\}.
\]
(1.5)

The space \( L^{p(\cdot)}_{\text{loc}}(\Omega) \) is defined by
\[
L^{p(\cdot)}_{\text{loc}}(\Omega) = \left\{ f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset \Omega \right\}.
\]
(1.6)

The Lebesgue spaces \( L^p(\Omega) \) is Banach spaces with the norm defined by
\[
\|f\|_{L^p(\Omega)} = \inf \left\{ \eta > 0 : \int_{\Omega} \left( \frac{|f(x)|}{\eta} \right)^p \, dx \leq 1 \right\},
\]
(1.7)

where \( p_+ = \inf \{ p(x) : x \in \Omega \} > 1 \), \( p_- = \esssup \{ p(x) : x \in \Omega \} < \infty \).

Denotes \( p'(x) = p(x) / (p(x) - 1) \). Let \( M \) be the Hardy-Littlewood maximal operator. We denote \( B(\Omega) \) to be the set of all functions \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfying the \( M \) is bounded on \( L^{p(\cdot)}(\Omega) \).

**Definition 1.1.** [6]

Let \( 0 < q \leq \infty \), \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), \( 0 \leq \lambda < \infty \). Let \( \alpha(\cdot) \) be a bounded real-valued measurable function on \( \mathbb{R}^n \). The nonhomogeneous Morrey-Herz space \( MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n) \) and homogeneous Morrey-Herz space with variable exponents \( MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}_\text{h}(\mathbb{R}^n) \) are respectively defined by
\[
MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)} < \infty \right\},
\]
(1.8)

and
\[
MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}_\text{h}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}_\text{h}(\mathbb{R}^n)} < \infty \right\},
\]
(1.9)

where
\[
\|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(\mathbb{R}^n)} := \sup_{L \in \mathcal{L}_h} 2^{-\lambda L} \left( \sum_{k=0}^L 2^{|L|} f \mathcal{X}_L \right)^{1/q},
\]
(1.10)

\[
\|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}_\text{h}(\mathbb{R}^n)} := \sup_{L \in \mathcal{L}_h} 2^{-\lambda L} \left( \sum_{k=-\infty}^L 2^{|L|} f \mathcal{X}_L \right)^{1/q}.
\]
(1.11)

**Definition 1.2.** [9]

For all \( 0 < q \leq 1 \), the Lipschitz space \( \text{Lip}_\gamma(\mathbb{R}^n) \) is defined by
\[
\text{Lip}_\gamma = \left\{ f : \|f\|_{\text{Lip}_\gamma} = \sup_{x,y \in \mathbb{R}^n : x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma} < \infty \right\}.
\]
(1.12)

**Definition 1.3.** [5]

Let \( \alpha(\cdot) \in L^\infty(\mathbb{R}^n) \), \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), \( 0 < q \leq \infty \), \( 0 \leq \lambda < \infty \) and \( N > n + 1 \). The
nonhomogeneous Herz-Morrey-Hardy space with variable exponent $\text{HMK}^{\alpha(\cdot), \lambda(\cdot)}_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ and homogeneous Herz-Morrey-Hardy space with variable exponents $\text{HMK}^{\alpha(\cdot), \lambda(\cdot)}_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ are respectively defined by

$$\text{HMK}^{\alpha(\cdot), \lambda(\cdot)}_{p(\cdot), q(\cdot)}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{\text{HMK}^{\alpha(\cdot), \lambda(\cdot)}_{p(\cdot), q(\cdot)}} := \| G_\lambda f \|_{\text{MK}^{\alpha(\cdot), \lambda(\cdot)}_{p(\cdot), q(\cdot)}} < \infty \right\}.$$  \hfill (1.13)

$$\text{HMK}^{\alpha(\cdot), \lambda(\cdot)}_{p(\cdot), q(\cdot)}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{\text{HMK}^{\alpha(\cdot), \lambda(\cdot)}_{p(\cdot), q(\cdot)}} := \| G_\alpha f \|_{\text{MK}^{\alpha(\cdot), \lambda(\cdot)}_{p(\cdot), q(\cdot)}} < \infty \right\}.$$  \hfill (1.14)

**Definition 1.4.** [10] (Hölder’s inequality) Let $\alpha > 1$ and $1/\alpha + 1/\beta = 1$. Then the discrete and integral forms of Hölder’s inequality are given as

$$\sum_{x \in [a,b]} |f(x)g(x)| \leq \left( \sum_{x \in [a,b]} |f(x)|^\alpha \right)^{1/\alpha} \left( \sum_{x \in [a,b]} |g(x)|^\beta \right)^{1/\beta},$$  \hfill (1.15)

for continuous function $f$ and $g$ on $[a,b]$.

**Definition 1.5.** [10] (Minkowski’s inequality) Let $u > 1$. Then the discrete and integral forms of Minkowski’s inequality are given as

$$\left( \sum_{x \in [a,b]} |f(x)+g(x)|^u \right)^{1/u} \leq \left( \sum_{x \in [a,b]} |f(x)|^u \right)^{1/u} + \left( \sum_{x \in [a,b]} |g(x)|^u \right)^{1/u},$$  \hfill (1.16)

for continuous function $f$ and $g$ on $[a,b]$, for more general functions can be obtained naturally. A further generalization is: If $u > 1$, then

$$\left( \int \left( \int |f(x,y)|^u \, dx \right)^{1/u} \, dy \right) \leq \left( \int \left( \int |f(x,y)|^u \, dx \right) \, dy \right)^{1/u}.$$  \hfill (1.17)

2. Preliminaries

In this section, we give some preliminaries which we used to prove theorems.

**Lemma 2.1.** [11] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then for any $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$, we have

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq C_{p} \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| g \|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C_p = 1 + \frac{1}{p(\cdot)} - \frac{1}{p'(\cdot)}$.

This inequality is called the generalized Hölder inequality with respect to the variable $L^{p(\cdot)}$ spaces.

**Lemma 2.2.** [12] Given $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $g \in L^{p_1(\cdot)}(\mathbb{R}^n)$, when $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, we get

$$\| f(x)g(x) \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \| f \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| g \|_{L^{p_2(\cdot)}(\mathbb{R}^n)},$$

where $C_{p_1, p_2} = \left[ 1 + \frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} \right]^{1/p_2(\cdot)}.$

**Proposition 2.3.** [13] If $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies

$$|q(x) - q(y)| \leq \frac{C}{\log |x-y|}, \quad |x-y| \leq 1/2,$$
\[ |q(x) - q(y)| \leq \frac{C}{\log(e + |x|)}, \quad |y| \geq |x|, \]

then \( q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \).

**Lemma 2.4.** [14] Let \( k \) be a positive integer and \( B \) be a ball in \( \mathbb{R}^n \). Then we have that for all \( b \in BMO(\mathbb{R}^n) \) and \( i, j \in \mathbb{Z} \) with \( i < j \), we have

1. \[ C^{-1} \|\mathcal{H}^k_b\|_{L^1(\mathbb{R}^n)} \leq \sup_{i} \frac{1}{|X_b|} \|b - b_b\| \chi_{X_b} \|\mathcal{H}^k_{X_b}\|_{L^1(\mathbb{R}^n)} \leq C \|\mathcal{H}^k_b\|_{L^1(\mathbb{R}^n)}, \]
2. \[ \left( b - b_b \right)^k \chi_{X_b} \|\mathcal{H}^k_{X_b}\|_{L^1(\mathbb{R}^n)} \leq C (j - i)^k \|\mathcal{H}^k_{X_b}\|_{L^1(\mathbb{R}^n)}, \]

where \( B_i = \{ x \in \mathbb{R}^n : |x| \leq 2^i \} \) and \( B_j = \{ x \in \mathbb{R}^n : |x| \leq 2^j \} \).

**Lemma 2.5.** [15] Let \( q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), then there exist positive constants \( \delta_1, \delta_2 > 0 \), such that for all balls \( B \subset \mathbb{R}^n \) and all measurable subset \( R \subset B \),

\[ \frac{|X_b|}{X_b} \leq C \left( \frac{|R|}{|B|} \right)^{\delta_1}, \quad \frac{\|\mathcal{H}^k_{X_b}\|_{L^1(\mathbb{R}^n)}}{\|\mathcal{H}^k_{X_b}\|_{L^1(\mathbb{R}^n)}} \leq C \left( \frac{|R|}{|B|} \right)^{\delta_2}, \]

where \( \delta_1, \delta_2 \) are constants with \( 0 < \delta_1, \delta_2 < 1 \).

**Lemma 2.6.** [16] If \( q(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), then there exists a constant \( C > 0 \) such that for any balls \( B \) in \( \mathbb{R}^n \),

\[ \frac{|X_b|}{X_b} \leq C. \]

**Lemma 2.7.** [6] Let \( 0 < q < \infty, p(\cdot) \in \mathcal{B}(\mathbb{R}^n), 0 < \lambda < \infty \), and \( \alpha(\cdot) \in L^\infty(\mathbb{R}^n) \) be log-Hölder continuous both at the origin and infinity, \( 2\lambda \leq \alpha(\cdot), n\delta_1 \leq \alpha(0), \alpha_\infty = \alpha_\infty = \alpha_\infty = \delta_2 \) as in lemma 2.4. Then \( f \in \text{HMK}^{\alpha(\cdot), p(\cdot)}(\mathbb{R}^n) \) (or \( \text{HMK}^{\alpha(\cdot), p(\cdot)}(\mathbb{R}^n) \)) if and only if \( f = \sum_{k=1}^\infty \lambda_k f_k \) (or \( f = \sum_{k=1}^\infty \lambda_k f_k \)), in the sense of \( f \in S'(\mathbb{R}^n) \), where each \( a_k \) is a central \( \alpha(\cdot), p(\cdot) \) atom with support contained in \( B_k \) and

\[ \sup_{L \leq \lambda, L \in \mathcal{Z}} 2^{-Ld} \sum_{k=L}^\infty |\lambda_k|^\theta < \infty \quad \text{or} \quad \left( \sup_{L \leq \lambda, L \in \mathcal{Z}} 2^{-Ld} \sum_{k=L}^\infty |\lambda_k|^\theta \right) \]

moreover

\[ |f|_{\text{HMK}^{\alpha(\cdot), p(\cdot)}(\mathbb{R}^n)} \approx \inf \sup_{L \leq \lambda, L \in \mathcal{Z}} 2^{-Ld} \left( \sum_{k=L}^\infty |\lambda_k|^\theta \right)^\frac{1}{\theta}, \]

or

\[ |f|_{\text{HMK}^{\alpha(\cdot), p(\cdot)}(\mathbb{R}^n)} \approx \inf \sup_{L \leq \lambda, L \in \mathcal{Z}} 2^{-Ld} \left( \sum_{k=L}^\infty |\lambda_k|^\theta \right)^\frac{1}{\theta}, \]

where infimum is taken over all above decomposition of \( f \).

**Lemma 2.8.** [17] Let \( q(\cdot) \in \mathcal{P}(\mathbb{R}^n), q \in (0, \infty] \) and \( \lambda \in [0, \infty) \). If \( \alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{B}(\mathbb{R}^n) \cap \mathcal{B}(\mathbb{R}^n) \), then

\[ |f|_{\text{HMK}^{\alpha(\cdot), p(\cdot)}(\mathbb{R}^n)} = \max \left\{ \sup_{L \leq \lambda, L \in \mathcal{Z}} 2^{-Ld} \sum_{k=1}^L 2^{kq(0)} \|f X_k\|_{L^p}, \quad \sup_{L \leq \lambda, L \in \mathcal{Z}} 2^{-Ld} \left( \sum_{k=1}^L 2^{kq(0)} \|f X_k\|_{L^p} + \sum_{k=0}^L 2^{kq(0)} \|f X_k\|_{L^p} \right) \right\}. \]
Lemma 2.9. [18] Let $\Omega$ satisfies $L'$-Dini condition with $r \in [1, \infty)$. If there exist constants $C > 0$ and $R > 0$ such that $|y| < R/2$, then for every $x \in \mathbb{R}^n$, we have

$$\left( \int_{|x-y| < R/2} \left| \frac{\Omega(x-y) - \Omega(x)}{|x-y|} \right|^r \, dx \right)^{1/r} \leq CR^{1-\frac{n}{r}} \left( |y| R \right) + \int_{|x-y| \leq |y|} \frac{\partial_y (\delta)}{\delta} \, d\delta \right).$$

Lemma 2.10. [15] Given $E$, let $q(\cdot) \in \mathcal{P}(E) \times E \to \mathbb{R}^n$ be a measurable function (with respect to product measure) such that for almost every $y \in E, f(., y) \in L^1(E)$. Then

$$\left( \int_{|x-y| = |y|} |y| \left| \frac{\Omega(x-y) - \Omega(x)}{|x-y|} \right|^d \, dy \right)^{1/d} \leq C \left( \int_{|x-y| = |y|} |\Omega(x-y)| d\delta \right).$$

Lemma 2.11. [19] If $a > 0, 1 \leq s \leq \infty, 0 \leq d \leq s$ and $-n + (n-1)d/s < v < \infty$, then

$$\left( \int_{|x-y| = |y|} |y| \left| \frac{\Omega(x-y) - \Omega(x)}{|x-y|} \right|^d \, dy \right)^{1/d} \leq C \left( \int_{|x-y| = |y|} |\Omega(x-y)| d\delta \right).$$

Lemma 2.12. [19] Let $q(\cdot) \in \mathcal{P}$ satisfies Proposition 2.3. Then

$$\|X_\Omega \|_{\mathcal{L}^1(\mathbb{R}^n)} \approx \begin{cases} \frac{1}{|\Omega|^{1/n}} & \text{if } |\Omega| \leq 2^n \text{ and } x \in Q \\ \frac{1}{|\Omega|^{1/n}} & \text{if } |\Omega| \geq 1 \\
\end{cases}$$

for every cube (or ball) $Q \in \mathbb{R}^n$, where $p(\infty) = \lim_{x \to \infty} p(x)$.

3. Lipschitz Boundedness for the Commutator of Marcikiewicz Integrals Operator

In this section, we prove the boundedness of the commutator of Marcikiewicz integrals on Herz-Morrey-Hardy spaces with variable exponent so whenever $b \in Lip_\gamma(\mathbb{R}^n)$ under some conditions.

Theorem 3.1.

Suppose that $b \in Lip_\gamma(\mathbb{R}^n)$ with $0 < \gamma \leq 1$. If $q_i(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies proposition 2.3 with $q_i^* < n/\gamma, 1/q_i(x) - 1/q_i(x) = \gamma/n$, $\Omega \in L^s(S^{n-1})(s > q_i^*)$ with $1 \leq s^* < q_i^*$ and satisfies

$$\int_0 \Omega (\delta) \, d\delta < \infty,$$

let $0 < p_i \leq q_i^* < \infty$ and $n\delta_i \leq \alpha < n\delta_i + \gamma$ or

$(0 < \max(n\delta_i, \alpha_i) \leq \alpha_i < n\delta_i + \gamma)$. Then the commutator $[b, \mu_\Omega]$ is bounded from $HMK^{\alpha_i}\mathcal{H}^s(\mathbb{R}^n)$ (or $HMK^{\alpha_i}\mathcal{H}^s(\mathbb{R}^n)$) to $MK^{\alpha_i}\mathcal{H}^s(\mathbb{R}^n)$ (or $MK^{\alpha_i}\mathcal{H}^s(\mathbb{R}^n)$).

To the proof the above theorem, we will recall the following lemma.

Lemma 3.1. [15]

Suppose that $b \in Lip_\gamma(\mathbb{R}^n)$ with $0 < \gamma \leq 1$. If $q_i(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies Proposition 2.3 with $q_i^* < n/\gamma, 1/q_i(x) - 1/q_i(x) = \gamma/n$ with $\Omega \in L^s(S^{n-1})(s > q_i^*)$. Then the commutator $[b, \mu_\Omega]$ is bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_1}(\mathbb{R}^n)$.
Next, we will give the Lipschitz estimate about the commutator \([b, \mu_\Omega]\) on Herz-Morrey-Hardy spaces with variable exponent.

**Proof Theorem 3.1:**

To prove this theorem, we only prove the homogeneous case. Let \(f \in \text{HM}_{p(\cdot),q_{(\cdot)}}(\mathbb{R}^n)\). By lemma 2.6 we have \(f = \sum_{j=-\infty}^{\infty} \lambda_j f_j\) converged in \(\mathcal{S}'(\mathbb{R}^n)\), where each \(b_j\) is a central \((\alpha(\cdot), p(\cdot))\) atom with support contained in \(B_j\) and

\[
\|f\|_{\text{HM}_{p(\cdot),q_{(\cdot)}}(\mathbb{R}^n)} \approx \inf_{L_0, L \in \mathbb{Z}} \sup_{L_0 \leq L \leq Z} 2^{Lq_0} \left( \sum_{j=-\infty}^{\infty} |\lambda_j|^{q_0} \right)^{1/q}.
\]

Here we denote \(\Delta = \sup_{L_0 \leq L \leq Z} 2^{Lq_0} \sum_{k=-\infty}^{L} |\lambda_k|^{q_0}\). By lemma 2.8 we have

\[
\|b(\mu_\Omega) - b(\mu_\Omega)\|_{L_0, L_0, L, L} = \sup_{L_0 \leq L \leq Z} 2^{Lq_0} \left( \sum_{k=-\infty}^{L} \left| 2^{kq_0(0)} \left[ b(\mu_\Omega) \right] \chi_k \right| \right),
\]

\[
= \sum_{k=-\infty}^{L} 2^{kq_0(0)} \left[ b(\mu_\Omega) \right] \chi_k \right|_{L, L}.
\]

In beginning, we examine a function which we will use in proving

\[
\left\| \left[ b(\mu_\Omega) \right] (x) \right\| \leq \left\{ \int \left[ \int_{|\cdot| < 1} \frac{\Omega(x-y)}{|x-y|^{n-1}} |b(x) - b(y)| |b_j(y)| \right] \right\}^{1/2}.
\]

When \(x \in A_k\) and \(|x - y| \leq t\) with \(t \leq |x|\), it follows from \(j \leq k-2\) that \(|x - y| \sim |x|\). We have

\[
\left\| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right\| \leq \frac{|y|}{|x|^3}.
\]

Then by (3.1), the Minkowski’s inequality, the generalized Hölder’s inequality and the vanishing of the moment of \(b_j\) we have

\[
Y_j \leq C \int_{|y| \leq |x|^2} \frac{\Omega(x-y)}{|x-y|^2} |b(x) - b(y)| |b_j(y)| dy
\]

\[
\leq C \int_{|y| \leq |x|^2} \frac{\Omega(x-y)}{|x-y|^2} |b(x) - b(y)| |b_j(y)| \left( \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right) dy
\]
Similarly, we consider $\gamma_3$. Noting that $|x-y| \sim |x|$. By the Minkowski’s inequality, the generalized Hölder’s inequality and the vanishing moments of $b_j$ we have

$$\gamma_3 \leq C \int_{\mathbb{R}^n} \frac{|\Omega\omega(x)|}{|x|} \left| \frac{|\Omega\omega(x)|}{|x|} \right| \left| \frac{|b(x)-b(y)|}{|x|} \right| \left| \frac{|b(x)-b(y)|}{|x|} \right| dy.$$

So we have

$$\left\| b, \mu_3 \right\|(b_j)(x) \leq C \int_{\mathbb{R}^n} \left\| \frac{|\Omega\omega(x)|}{|x|} \right\| \left\| \frac{|\Omega\omega(x)|}{|x|} \right\| \left| \frac{|b(x)-b(y)|}{|x|} \right| \left| \frac{|b(x)-b(y)|}{|x|} \right| dy.$$

From lemma 2.10 and the Minkowski’s inequality we have

$$\left\| b, \mu_3 \right\|(b_j)(x) Z_1 \left( \mathbb{R}^n \right) \leq C \int_{\mathbb{R}^n} \left\| \frac{|\Omega\omega(x)|}{|x|} \right\| \left\| \frac{|\Omega\omega(x)|}{|x|} \right\| \left| \frac{|b(x)-b(y)|}{|x|} \right| \left| \frac{|b(x)-b(y)|}{|x|} \right| \left. \right|_{L^1(\mathbb{R}^n)} dy \leq \gamma_1 + \gamma_2.$$

For $\gamma_1$, noting $s > p'$, we denote $\bar{p}'(\cdot) > 1$ and $\frac{1}{p'(x)} = \frac{1}{p'(x)} = \frac{1}{s}$. By lemma 2.2 we have

$$\left\| \frac{|\Omega\omega(x)|}{|x|} \right\| \left\| \frac{|\Omega\omega(x)|}{|x|} \right\| \left| \frac{|b(x)-b(y)|}{|x|} \right| \left| \frac{|b(x)-b(y)|}{|x|} \right| \left. \right|_{L^1(\mathbb{R}^n)} dy \leq \gamma_3 + \gamma_4.$$
\[
\|X_{B_k}\|_{L^p(|\mathbb{R}^n|)} \approx |B_k|^{\frac{1}{p(n)}} \approx X_{B_k} \|\mathcal{X}^{(\gamma)}(\mathbb{R}^n)\| |B_k|^{\frac{1}{p} - \frac{n}{n}}.
\]

When \(|B_k| \geq 1\) we have
\[
\|X_{B_k}\|_{L^p(|\mathbb{R}^n|)} \approx |B_k|^{\frac{1}{p(n)}} \approx X_{B_k} \|\mathcal{X}^{(\gamma)}(\mathbb{R}^n)\| |B_k|^{\frac{1}{p} - \frac{n}{n}}.
\]

So we obtain
\[
\|X_{B_k}\|_{L^p(|\mathbb{R}^n|)} \approx |B_k|^{\frac{1}{p} - \frac{n}{n}}.
\]

By lemma 2.9 we have
\[
\left\| \Omega(-y) - \Omega(\cdot) \right\|_{L^1(|\mathbb{R}^n|)} \leq 2 \left( k^{-1} \right) \left\{ \frac{1}{2^n} \int_{\mathbb{R}^n} \frac{\partial \Omega(\delta)}{\partial \delta} \, d\delta \right\}
\]
\[
\leq 2 \left( k^{-1} \right) \left\{ 2^{j-k+1} + 2^{(j-k+1)^-} \int_0^\infty \frac{\alpha_0(\delta)}{\delta} \, d\delta \right\}
\]
\[
\leq 2 \left( k^{-1} \right) 2^{(j-k)^-}.
\]

Now, by using the generalized Hölder’s inequality we get:
\[
\int_{\mathbb{R}^n} \left| b_j(\cdot) - b_j(\cdot) \right| \Omega(\cdot) \mathcal{X}_k(\cdot) \, dy \leq C \|X_{\mathcal{B}_k}\|_{L^p(|\mathbb{R}^n|)} \left| B_k \right|^{\frac{1}{p} - \frac{n}{n}} \int_{\mathbb{R}^n} \left| b_j(\cdot) \right| \, dy \quad (3.2)
\]

For \(Y_1\) similar to the method of \(Y_1\) we have
\[
\left\| \Omega(-y) - \Omega(\cdot) \right\|_{L^1(|\mathbb{R}^n|)} \leq \left\| \Omega(-y) - \Omega(\cdot) \right\|_{L^1(|\mathbb{R}^n|)} \mathcal{X}_k(\cdot) \left\| X_{\mathcal{B}_k}\|_{L^p(|\mathbb{R}^n|)} \right\|
\]
\[
\leq \left\{ k^{-1} \right\} \left\{ 2^{j-k+1} + 2^{(j-k+1)^-} \int_0^\infty \frac{\alpha_0(\delta)}{\delta} \, d\delta \right\}
\]
\[
\leq 2 \left( k^{-1} \right) 2^{(j-k)^-} \|X_{\mathcal{B}_k}\|_{L^p(|\mathbb{R}^n|)}
\]

Now, by using the generalized Hölder’s inequality we get:
\[ Y_2^* \leq \int_{\mathbb{R}} \left| \frac{\Omega(-y)}{|-y|^r} \right| \left| \frac{\Omega(\cdot)}{|\cdot|^r} \right| X_1(\cdot) \left| p(0) - b(y) \right| \left| p_j(y) \right| dy \]

\[ \leq C \left\| \mathcal{L}_{b,y} \right\|_{L^{p_1}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_2}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_3}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_4}([\mathbb{R}^n])} \]

\[ \leq C \left\| \mathcal{L}_{b,y} \right\|_{L^{p_1}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_2}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_3}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_4}([\mathbb{R}^n])} \]

Now by (3.3), (3.4), and lemmas 2.5 and 2.6, we have

\[ \left\| b, \mu_{\Omega} \right\|_{L^{p_1}([\mathbb{R}^n])} \left\| b \right\|_{L^{p_2}([\mathbb{R}^n])} \]

\[ \leq C \left\| \mathcal{L}_{b,y} \right\|_{L^{p_1}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_2}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_3}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_4}([\mathbb{R}^n])} \]

\[ \leq C \left\| \mathcal{L}_{b,y} \right\|_{L^{p_1}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_2}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_3}([\mathbb{R}^n])} \left\| \mathcal{L}_{b,y} \right\|_{L^{p_4}([\mathbb{R}^n])} \]

Firstly we estimate \( I \). We need to show that there exists a positive constant \( C \), such that \( I \leq CA \), we consider

\[ I = \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left| \lambda_j \left\| b, \mu_{\Omega} \right\|_{L^{p_1}([\mathbb{R}^n])} \left\| \frac{f}{|x|^r} \right\|_{X_1([\mathbb{R}^n])} \right|^{\theta} \]

\[ \leq \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \left\| b, \mu_{\Omega} \right\|_{L^{p_1}([\mathbb{R}^n])} \right|^{\theta} \right) \]

\[ + \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \left\| b, \mu_{\Omega} \right\|_{L^{p_1}([\mathbb{R}^n])} \right|^{\theta} \right) \]

By the \( L^{p_1}([\mathbb{R}^n]), L^{p_2}([\mathbb{R}^n]) \), boundedness of the commutator \( [b, \mu_{\Omega}] \) on \( L^{p_1} \) (see [15]), we have the following. Therefore, when \( 0 < q \leq 1 \)

\[ I_1 = \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \left\| b, \mu_{\Omega} \right\|_{L^{p_1}([\mathbb{R}^n])} \right|^{\theta} \right) \]

\[ \leq \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \left\| b, \mu_{\Omega} \right\|_{L^{p_1}([\mathbb{R}^n])} \right|^{\theta} \right) \]

\[ \leq \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \right|^{\theta} \right) \]

By [15], we have the following. Therefore, when \( 0 < q \leq 1 \)

\[ I_2 = \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \right|^{\theta} \right) \]

\[ \leq \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \right|^{\theta} \right) \]

By [15], we have the following. Therefore, when \( 0 < q \leq 1 \)

\[ I_3 = \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \right|^{\theta} \right) \]

By [15], we have the following. Therefore, when \( 0 < q \leq 1 \)

\[ I_4 = \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \right|^{\theta} \right) \]

By [15], we have the following. Therefore, when \( 0 < q \leq 1 \)

\[ I_5 = \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \right|^{\theta} \right) \]

By [15], we have the following. Therefore, when \( 0 < q \leq 1 \)

\[ I_6 = \sup_{L \geq 0} 2^{-Lq} \sum_{k=-\infty}^{\infty} 2^{kq} \left( \sum_{j \in \mathbb{Z}} \left| \lambda_j \right|^{\theta} \right) \]
\[
0 < q < \infty, \text{ let } 1/q + 1/q' = 1 \text{ we have }
\]
\[
I_1 = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k = -\infty}^{L} 2^{b_{q}(0)/2} \left( \sum_{j = 0}^{K} |\mathcal{L}_j| \|b_j, \mathcal{H}_0, \mathcal{J}_0\|_p \mathcal{O}(|x|^p) \right) ^q
\]
\[
\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k = -\infty}^{L} 2^{b_{q}(0)} \left( \sum_{j = 0}^{K} |\mathcal{L}_j| 2^{-ja_0} \right) ^q
\]
\[
\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k = -\infty}^{L} \left( \sum_{j = 0}^{K} |\mathcal{L}_j| \right) \left( \sum_{j = 0}^{K} 2^{a(0)(j - j')/2} \right)^{q'/q}\]
\[
\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k = -\infty}^{L} \sum_{j = 0}^{K} |\mathcal{L}_j|^{q'} 2^{(j - j')q'/2}
\]
\[
\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k = -\infty}^{L} \sum_{j = 0}^{K} |\mathcal{L}_j|^{q'} 2^{(j - j')q'/2}
\]
\[
\leq \Delta.
\]

When \(0 < q < \infty\), let \(1/q + 1/q' = 1\) we have

\[
I_2 = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k = -\infty}^{L} 2^{b_{q}(0)(k - j)q} \left( \sum_{j = 0}^{K} |\mathcal{L}_j| \|b_j, \mathcal{H}_0, \mathcal{J}_0\|_p \mathcal{O}(|x|^p) \right) ^q
\]
\[
\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k = -\infty}^{L} 2^{b_{q}(0)(k - j)q} \left( \sum_{j = 0}^{K} |\mathcal{L}_j| 2^{a(0)(k - j)q} \right) ^q + \Delta
\]
\[
\leq \Delta + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{j = 0}^{K} |\mathcal{L}_j| 2^{a(0)(j - j')q} + \Delta
\]
\[
\leq \Delta + \Delta
\]
\[
\leq \Delta.
\]
When $0 < q < \infty$, let $1/q + 1/q' = 1$. Since $n\delta_2 \leq \alpha(0) < \gamma + n\delta_2$, by Hölder's inequality, we have

$$I_2 = \sup_{L \geq 0, L \in \mathbb{Z}} 2^{-Ld_q} \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left( \sum_{j=0}^{k-1} |\beta_j| \left\| b_j, \mu_\alpha \right\| \mathcal{X}_k \right)^q$$

$$\leq C \| \mathcal{L}_p \|_{L_p} \sup_{L \geq 0, L \in \mathbb{Z}} 2^{-Ld_q} \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left( \sum_{j=0}^{k-1} 2^{-ja} 2^{(j-k)(\gamma + n\delta_2)} |\beta_j| \right)^q$$

$$\leq C \| \mathcal{L}_p \|_{L_p} \sup_{L \geq 0, L \in \mathbb{Z}} 2^{-Ld_q} \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left( \sum_{j=0}^{k-1} 2^{-ja} 2^{(j-k)(\gamma + n\delta_2)} |\beta_j| \right)^q 2^{-\delta(0) q/2}$$

$$\leq C \| \mathcal{L}_p \|_{L_p} \sup_{L \geq 0, L \in \mathbb{Z}} 2^{-Ld_q} \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left( \sum_{j=0}^{k-1} 2^{-ja} 2^{(j-k)(\gamma + n\delta_2)} |\beta_j| \right)^q 2^{-\delta(0) q/2}$$

$$\leq C \| \mathcal{L}_p \|_{L_p} \sup_{L \geq 0, L \in \mathbb{Z}} 2^{-Ld_q} \sum_{j=0}^{Lq} |\beta_j|^q \sum_{k=j+1}^{L} 2^{-\delta(0) q/2} \leq \Delta. \quad (3.7)$$

Secondly, we estimate $II$. We need to show that there exists a positive constant $C$, such that $II \leq C\Delta$, we consider

$$II = \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left\| b_j, \mu_\alpha \right\| (f) \mathcal{X}_k \right\|_{L_p}^q$$

$$\leq \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left( \sum_{j=0}^{k-1} |\beta_j| \left\| b_j, \mu_\alpha \right\| (f_j) \mathcal{X}_k \right\|_{L_p}^q$$

$$+ \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left( \sum_{j=0}^{k-1} |\beta_j| \left\| b_j, \mu_\alpha \right\| (f_j) \mathcal{X}_k \right\|_{L_p}^q$$

$$:= II_1 + II_2.$$

When $0 < q \leq 1$, we get

$$II_1 = \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left( \sum_{j=0}^{k-1} |\beta_j| \left\| b_j, \mu_\alpha \right\| (f_j) \mathcal{X}_k \right\|_{L_p}^q$$

$$\leq \sum_{k=0}^{Lq} 2^{k|\alpha(0)|} \left( \sum_{j=0}^{k-1} 2^{-ja} |\beta_j| \right)^q$$
\[ \leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \Delta + \Delta \sum_{j=0}^{\alpha} 2^{(\alpha - \alpha_j) q} \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \\
\leq \Delta. \]

When \( 0 < q < \infty \), let \( 1/q + 1/q' = 1 \) we have

\[ H_1 = \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \Delta + \Delta \sum_{j=0}^{\alpha} 2^{(\alpha - \alpha_j) q} \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \\
\leq \Delta. \]

For \( H_2 \), when \( 0 < q \leq 1 \), by \( n\alpha_2 \leq \alpha(0) < \gamma + n\alpha_2 \) we get

\[ H_2 = \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \left( \sum_{j=0}^{\alpha} |b_j|^{\gamma} \right)^{\gamma} \\
\leq \Delta + \Delta \sum_{j=0}^{\alpha} 2^{(\alpha - \alpha_j) q} \sum_{k=1}^{\alpha} 2^{\alpha q(0)} \\
\leq \Delta. \]
When $1 < q < \infty$, let $1/q + 1/q' = 1$. Since $n\delta_2 \leq \alpha(0) < \gamma + n\delta_2$, by Hölder’s inequality, we have

$$H_2 = \sum_{k=1}^{\infty} 2^{kq_0(q)} \left[ \sum_{j=1}^{\infty} |\lambda_j| \|b_j \mu_\alpha\|_{L^q} \right]^{q}$$

$$\leq \sum_{k=1}^{\infty} 2^{kq_0(q)} \left( C \|b_k \mu_\alpha\|_{L^q} \sum_{j=1}^{\infty} |\lambda_j| \right)^{q}$$

$$\leq C \|b_k \mu_\alpha\|_{L^q} \sum_{k=1}^{\infty} 2^{kq_0(q)} \left[ \sum_{j=1}^{\infty} |\lambda_j| \left( 2^{-ja_q + j(\gamma + n\delta_2)} \right)^{q/2} \right]$$

$$\leq C \|b_k \mu_\alpha\|_{L^q} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |\lambda_j| \right)^{q} \left( 2^{-ja_q + j(\gamma + n\delta_2)} \right)^{q/2}$$

$$\leq C \|b_k \mu_\alpha\|_{L^q} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |\lambda_j| \right)^{q} \left( 2^{-ja_q + j(\gamma + n\delta_2 - \sigma(0))} \right)^{q/2}$$

$$\leq C \|b_k \mu_\alpha\|_{L^q}$$

(3.11)

Thirdly, we estimate $III$, we need to show that there exists a positive constant $C$, such that $III \leq CA$

$$III = \sup_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} 2^{kq_0(q)} \left[ \sum_{j=1}^{\infty} |\lambda_j| \|b_j \mu_\alpha\|_{L^q} \right]^{q}$$

$$\leq \sup_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} 2^{kq_0(q)} \left( \sum_{j=1}^{\infty} |\lambda_j| \left( 2^{-ja_q + j(\gamma + n\delta_2)} \right)^{q/2} \right)$$

$$\leq \sup_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} \left( \sum_{j=1}^{\infty} |\lambda_j| \right)^{q} \left( 2^{-ja_q + j(\gamma + n\delta_2 - \sigma(0))} \right)^{q/2}$$

$$\leq \sup_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} \left( \sum_{j=1}^{\infty} |\lambda_j| \right)^{q} \left( 2^{-ja_q + j(\gamma + n\delta_2 - \sigma(0))} \right)^{q/2}$$

$$\leq \Delta + \Delta \sum_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} 2^{-ja_q + j(\gamma + n\delta_2 - \sigma(0))}$$

When $0 < q \leq 1$, by the boundedness of $[b, \mu_\alpha]$ in $L^p(\Omega)$ ([20]), we have

$$III = \sup_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} 2^{kq_0(q)} \left[ \sum_{j=1}^{\infty} |\lambda_j| \|b_j \mu_\alpha\|_{L^q} \right]^{q}$$

$$\leq \sup_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} \left( \sum_{j=1}^{\infty} |\lambda_j| \right)^{q}$$

$$\leq \sup_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} \left( \sum_{j=1}^{\infty} |\lambda_j| \right)^{q}$$

$$\leq \sup_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} \left( \sum_{j=1}^{\infty} |\lambda_j| \right)^{q}$$

$$\leq \Delta + \Delta \sum_{L \leq 0, \Omega \in Z} 2^{-Lq(\gamma)} \sum_{k=0}^{\infty} \left( \sum_{j=1}^{\infty} |\lambda_j| \right)^{q}$$

(3.12)
When $0 < q \leq \infty$, by $n\delta_2 \leq \alpha(0), \alpha_\infty < \gamma + n\delta_2$ and the boundedness of $[b, \mu_\Omega]$ in $L^{\gamma}(\Omega)$ ([20]) and Hölder’s inequality, we get

\[
III_1 = \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| \lambda_j \| \left\| [b, \mu_\Omega] (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\times \left( \sum_{j=0}^{\infty} \| [b, \mu_\Omega] (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)}^{q/2} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \Delta + \Delta \sup_{L \leq L, L \in \mathbb{Z}} \sum_{L} 2^{(j-L)q(a-a_\infty/2)} 2^L \sum_{j=L}^{(j-L)q(a-a_\infty/2)}
\]

\[\leq \Delta \tag{3.13}\]

When $0 < q \leq 1$, by $n\delta_2 \leq \alpha(0), \alpha_\infty < \gamma + n\delta_2$ we get

\[
III_2 = \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| \lambda_j \| \left\| [b, \mu_\Omega] (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \sup_{L \leq L, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{kq(\kappa)} \left( \sum_{j=0}^{\infty} \| B_j \| \left\| (b_j) \left( \chi_k \right) \right\|_{L^{\gamma}(\Omega)} \right)^q
\]

\[
\leq \Delta \tag{3.14}\]
When \( 1 < q < \infty \), let \( 1/q + 1/q' = 1 \). Since \( n\delta_2 \leq \alpha(0) \), \( \alpha < r + n\delta_2 \), and by Hölder’s inequality, we have

\[
III_2 = \sup_{L>0, \mathbb{L} \subset \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{\infty} 2^{kp_0(q)} \left( \sum_{j=0}^{k-1} |\mathcal{A}_j| \|b_j \mu_{\alpha}\| \|b_j\| \Omega \right)_{r(1)}^q \\
\leq \sup_{L>0, \mathbb{L} \subset \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{\infty} 2^{kp_0(q)} \left( \sum_{j=0}^{k-1} \|\mathcal{A}_j\| \|\mathcal{B}_j\| \Omega \right)_{r(1)}^q \\
\leq C \|\mathcal{B}\|_{L^\infty} \sup_{L>0, \mathbb{L} \subset \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{\infty} 2^{kp_0(q)} \left( \sum_{j=0}^{k-1} \|\mathcal{A}_j\| \|\mathcal{B}_j\| \Omega \right)_{r(1)}^q \\
\times \left( \sum_{j=0}^{k-1} \|\mathcal{A}_j\| \|\mathcal{B}_j\| \Omega \right)_{r(1)}^{q/q'}. \\
\leq C \|\mathcal{B}\|_{L^\infty} \sup_{L>0, \mathbb{L} \subset \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{\infty} 2^{kp_0(q)} \left( \sum_{j=0}^{k-1} \|\mathcal{A}_j\| \|\mathcal{B}_j\| \Omega \right)_{r(1)}^{q/q'}. \\
\leq C \|\mathcal{B}\|_{L^\infty} \sup_{L>0, \mathbb{L} \subset \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{\infty} 2^{kp_0(q)} \left( \sum_{j=0}^{k-1} \|\mathcal{A}_j\| \|\mathcal{B}_j\| \Omega \right)_{r(1)}^{q/q'}. \\
\leq C \|\mathcal{B}\|_{L^\infty} \sup_{L>0, \mathbb{L} \subset \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{\infty} 2^{kp_0(q)} \left( \sum_{j=0}^{k-1} \|\mathcal{A}_j\| \|\mathcal{B}_j\| \Omega \right)_{r(1)}^{q/q'}. \\
\leq C \|\mathcal{B}\|_{L^\infty} \Delta. \\
\end{align}

Joint the estimates for I, II and III, we obtain

\[
\|\mathcal{B}_L \mu_{\alpha}\|_{L^2} \leq C \|\mathcal{B}\|_{L^\infty} \|\mathcal{B}\|_{L^\infty} |q|^{1/2} \\
\|\mathcal{B}_L \mu_{\alpha}\|_{L^2} \leq C \|\mathcal{B}\|_{L^\infty} \|\mathcal{B}\|_{L^\infty} |q|^{1/2} \\
\|\mathcal{B}_L \mu_{\alpha}\|_{L^2} \leq C \|\mathcal{B}\|_{L^\infty} \|\mathcal{B}\|_{L^\infty} |q|^{1/2} \\
\|\mathcal{B}_L \mu_{\alpha}\|_{L^2} \leq C \|\mathcal{B}\|_{L^\infty} \|\mathcal{B}\|_{L^\infty} |q|^{1/2}. \\
\end{align}

Then we complete the proof of Theorem 3.1.

4. BMO Boundedness for the Commutator of Marcikiewicz Integrals Operator

In this section, we prove the boundedness of the commutator of Marcikiewicz...
integrals on Herz-Morrey-Hrdy spaces with variable exponent with function \( b \in BMO(\mathbb{R}^n) \).

**Theorem 4.1.**

Suppose that \( b \in BMO(\mathbb{R}^n) \) with \( 0 < \gamma \leq 1 \). If \( \mu(\cdot) \in P(\mathbb{R}^n) \) satisfies proposition 2.3 and \( \Omega \in L^r \left( S^{n-1} \right) (s > q^{-}) \). Let \( 0 < p_1 \leq p_2 < \infty \) and \( 0 < \lambda < n \delta_2 - \gamma - \frac{n}{s} \) (or \( 0 < \lambda < \alpha \leq \alpha_1 < n \delta_2 - \gamma - \frac{n}{s} \)). Then \([b, \mu_\alpha]\) is bounded from \( HMK_{p(\cdot),\gamma}^\mu(\mathbb{R}^n) \) (or \( HMK_{p(\cdot),\gamma}^{\mu_{\alpha}}(\mathbb{R}^n) \)) to \( MK_{p(\cdot),\lambda}^{\mu_{\alpha}}(\mathbb{R}^n) \) (or \( MK_{p(\cdot),\lambda}^{\mu_{\alpha}}(\mathbb{R}^n) \)).

**proof:**

In a way similar to theorem (3.2) we only prove the homogeneous case. Let \( b \in BMO(\mathbb{R}^n) \) and \( f \in HMK_{p(\cdot),\gamma}^{\mu_{\alpha}}(\mathbb{R}^n) \). Let us write

\[
f(x) = \sum_{j=0}^{\infty} f(x) \chi_j(x) = \sum_{j=0}^{\infty} f_j(x).
\]

Then we have

\[
\| [b, \mu_\alpha](f) \|_{MK_{p(\cdot),\lambda}^{\mu_{\alpha}}(\mathbb{R}^n)} = \max \left\{ \sup_{l \leq 0, l \in \mathbb{Z}} 2^{-Lq_l} \sum_{k=-\infty}^{\infty} 2^{kq_0(0)} \| [b, \mu_\alpha](f) \chi_k \|_{L^p(\mathbb{R}^n)} \right\},
\]

\[
= \sup_{l \leq 0, l \in \mathbb{Z}} 2^{-Lq_l} \left( \sum_{k=-\infty}^{\infty} 2^{kq_0(0)} \left\| [b, \mu_\alpha](f) \chi_k \right\|_{L^p(\mathbb{R}^n)} \right) + \sum_{k=0}^{\infty} 2^{kq_0(0)} \left\| [b, \mu_\alpha](f) \chi_k \right\|_{L^p(\mathbb{R}^n)} \right) = \max \left\{ H, HH, HHHH \right\}.
\]

\[
H = \sup_{l \leq 0, l \in \mathbb{Z}} 2^{-Lq_l} \sum_{k=-\infty}^{\infty} 2^{kq_0(0)} \left\| [b, \mu_\alpha](f) \chi_k \right\|_{L^p(\mathbb{R}^n)}^{q},
\]

\[
HH = \sum_{k=-\infty}^{\infty} 2^{kq_0(0)} \left\| [b, \mu_\alpha](f) \chi_k \right\|_{L^p(\mathbb{R}^n)}^{q},
\]

\[
HHHH = \sup_{l \leq 0, l \in \mathbb{Z}} 2^{-Lq_l} \sum_{k=0}^{\infty} 2^{kq_0(0)} \left\| [b, \mu_\alpha](f) \chi_k \right\|_{L^p(\mathbb{R}^n)}^{q}.
\]

From the Hölder’s inequality, we have

\[
\left\| [b, \mu_\alpha](b_j) X_k \right\|_{L^p(\mathbb{R}^n)} \leq C \int_{B_j} \left| \Omega(x-y) \right| \left| b(x) - b(y) \right| \left| f_j(y) \right| dy,
\]

\[
\leq C_2 \left\| \Omega(x-y) \right| \left| b(x) - b(y) \right| \left| f_j(y) \right| dy,
\]

\[
\leq C_2 \left\| \Omega(x-y) \right| \left| b(x) - b(y) \right| \left| f_j(y) \right| dy + \int_{B_j} \left| \Omega(x-y) \right| \left| b_j - b(y) \right| \left| f_j(y) \right| dy,
\]

\[
\leq C_2 \left\| \Omega(x-y) \right| \left| b(x) - b(y) \right| \left| f_j(y) \right| dy + \int_{B_j} \left| \Omega(x-y) \right| \left| b_j - b(y) \right| \left| f_j(y) \right| dy,
\]

\[
\leq C_2 \left\| \Omega(x-y) \right| \left| b(x) - b(y) \right| \left| f_j(y) \right| dy + \int_{B_j} \left| \Omega(x-y) \right| \left| b_j - b(y) \right| \left| f_j(y) \right| dy.
\]

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Noting \( s > q^- \), we denote \( q'(\cdot) > 1 \) and \( \frac{1}{q'(x)} = \frac{1}{q'(x)} + \frac{1}{s} \). By lemmas 3.2, 3.10 we have

\[
\left\| \Omega(x - \cdot) X_j(\cdot) \right\|_{L^q([\mathbb{R}^n])} \leq \left\| \Omega(x - \cdot) X_j(\cdot) \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \\
\leq \left\| \Omega(x - \cdot) X_j(\cdot) \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \\
\leq 2^{-j\beta} \left( \int_{B_j} y^{\nu} \right) \left| \Omega(x - y) \right| dy \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \\
\leq C 2^{-j\beta} \left| 2^{j\frac{\nu}{s}} \right| \left\| \Omega \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])}.
\]

By lemma (2.12), when \( |B_j| \leq 2^s, x_j \in B_j \) and when \( |B_i| \geq 1 \) respectively we have

\[
\left\| X_j \right\|_{L^q([\mathbb{R}^n])} \approx \left| B_j \right|^{\frac{1}{q}} \approx \left\| X_j \right\|_{L^q([\mathbb{R}^n])} \left| B_j \right|^{\frac{1}{q}},
\]

and

\[
\left\| X_j \right\|_{L^q([\mathbb{R}^n])} \approx \left| B_j \right|^{\frac{1}{q}} \approx \left\| X_j \right\|_{L^q([\mathbb{R}^n])} \left| B_j \right|^{\frac{1}{q}},
\]

we obtain

\[
\left\| X_j \right\|_{L^q([\mathbb{R}^n])} \approx \left\| X_j \right\|_{L^q([\mathbb{R}^n])} \left| B_j \right|^{\frac{1}{q}}.
\]

So we have

\[
\left\| \Omega(x - \cdot) X_j(\cdot) \right\|_{L^q([\mathbb{R}^n])} \leq C 2^{-j\left(\frac{k}{s} - \frac{s}{s} \right)} \left\| \Omega \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])}.
\] (4.1)

Similarly by lemma 2.4 we have

\[
\left\| \Omega(x - \cdot)(b_j - b(\cdot)) X_j(\cdot) \right\|_{L^q([\mathbb{R}^n])} \leq C \left\| \Omega(x - \cdot) X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \left\| b_j - b(\cdot) \right\|_{L^q([\mathbb{R}^n])} \\
\leq C \left\| b_j \right\| \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \left\| \Omega(x - \cdot) X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \\
\leq C \left\| b_j \right\| 2^{-j\left(\frac{k}{s} - \frac{s}{s} \right)} \left\| \Omega \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])}.
\] (4.2)

Now, by (4.1), (4.2), lemmas 2.4, 2.5 and 2.3, we have

\[
\left\| \left[ b_j, \mu_{\alpha_j} \right] (f_j) X_j \right\|_{L^q([\mathbb{R}^n])} \leq C 2^{-k\alpha} \left\| b_j \right\| \left\| \Omega \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \left\| b_j(\cdot) - b(\cdot) \right\|_{L^q([\mathbb{R}^n])} \\
+ \left\| b_j \right\| 2^{-j\left(\frac{k}{s} - \frac{s}{s} \right)} \left\| \Omega \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \left\| b_j(\cdot) - b(\cdot) \right\|_{L^q([\mathbb{R}^n])} dy \right)
\leq C 2^{-k\alpha} \left\| b_j \right\| 2^{-j\left(\frac{k}{s} - \frac{s}{s} \right)} \left\| \Omega \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \left\| b_j(\cdot) - b(\cdot) \right\|_{L^q([\mathbb{R}^n])} \left\| X_j(\cdot) \right\|_{L^p([\mathbb{R}^n])} \left\| b_j(\cdot) - b(\cdot) \right\|_{L^q([\mathbb{R}^n])}.
\]
\[\|f_j\|_{L^p(\mathbb{R}^n)} \leq C(k-j) \|\mathcal{M}_{\gamma} f\|_{L^p(\mathbb{R}^n)} \leq C(k-j)^2 \|\mathcal{M}_{\gamma} f\|_{L^p(\mathbb{R}^n)}\]

By the boundedness of \(\mu_\Omega\) in \(L^p(\mathbb{R}^n)\) see [7], we have

\[\|b, \mu_\Omega\|_{L^q(\mathbb{R}^n)} \leq \|b, \mu_\Omega\|_{L^q(\mathbb{R}^n)} \leq 2^{-j\alpha p} = 2^{-j\alpha p}.
\]

So we have

\[\|b, \mu_\Omega\|_{L^q(\mathbb{R}^n)} \leq \|b, (k-j) \|\mathcal{M}_{\gamma} f\|_{L^p(\mathbb{R}^n)} \leq \|b, (k-j)^2 \|\mathcal{M}_{\gamma} f\|_{L^p(\mathbb{R}^n)}\]
\[ \leq \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \sum_{k=0}^{L} \left[ \sum_{j=0}^{L} |b_j| \right]^{\gamma} \left[ \sum_{j=0}^{L} |a_{j-k}^{(0)}| \right]^{\gamma} \]

\[ + \Delta \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \sum_{k=0}^{L} \left[ \sum_{j=0}^{L} |a_{j-k}^{(0)}| \right]^{\gamma} \]

\[ \leq \Delta + \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} |a_{j-k}^{(0)}| \right]^{\gamma} \]

\[ \leq \Delta + \sup_{L \leq L_0, L \leq Z} \sum_{j=0}^{L} 2^{-Lq} \left[ \sum_{j=0}^{L} |a_{j-k}^{(0)}| \right]^{\gamma} \]

\[ \leq \Delta. \]

When \( 1 < q < \infty \), and \( 1/q + 1/q' = 1 \), and let \( \gamma + n\delta_2 - \alpha > 0 \), we have

\[ H_1 = \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} |b_j| \right]^{\gamma} \left[ \sum_{j=0}^{L} |a_{j-k}^{(0)}| \right]^{\gamma} \]

\[ \leq \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} |b_j| \right]^{\gamma} \left[ \sum_{j=0}^{L} \left[ 2^{a(0)(j-k)} \right]^{\gamma} \right] \]

\[ + \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} \left[ 2^{a(0)(j-k)} \right]^{\gamma} \right] \]

\[ \leq \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} \left[ 2^{a(0)(j-k)} \right]^{\gamma} \right] \]

\[ + \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} \left[ 2^{a(0)(j-k)} \right]^{\gamma} \right] \]

\[ \leq \sup_{L \leq L_0, L \leq Z} 2^{-Lq} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} \left[ 2^{a(0)(j-k)} \right]^{\gamma} \right] \]

\[ + \sup_{L \leq L_0, L \leq Z} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} \left[ 2^{a(0)(j-k)} \right]^{\gamma} \right] \]

\[ \leq \sup_{L \leq L_0, L \leq Z} \sum_{j=0}^{L} \left[ \sum_{j=0}^{L} \left[ 2^{a(0)(j-k)} \right]^{\gamma} \right] \]

\[ \leq \Delta. \]
\[
H_2 = \sup_{L \in B, L \in Z} 2^{-Lq} \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} |\lambda_j| \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \right)^q
\]
\[
\leq C \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \sup_{L \in B, L \in Z} 2^{-Lq} \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} |\lambda_j| \left( k-j \right)^q 2^{-jx(s-j-k)2^{\gamma}} \right)^q
\]
\[
\leq C \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \sup_{L \in B, L \in Z} 2^{-Lq} \sum_{k=-\infty}^{k=1} |\lambda_j| \sum_{j=-\infty}^{j=1} (k-j)^q 2^{-jx(s-j-k)2^{\gamma}} \delta_{2^{-2}}(0)
\]
\[
\leq C \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \Delta.
\]

(4.6)

when \( 1 < q < \infty \), let \( 1/q + 1/q' = 1 \). Since \( n\delta_2 < \alpha(0) \leq \gamma + n\delta_2 \), by Hölder’s inequality we have

\[
H_2 = \sup_{L \in B, L \in Z} 2^{-Lq} \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} |\lambda_j| \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \right)^q
\]
\[
\leq C \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \sup_{L \in B, L \in Z} 2^{-Lq} \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} |\lambda_j| \left( k-j \right)^q 2^{-jx(s-j-k)2^{\gamma}} \right)^q
\]
\[
\leq C \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \sup_{L \in B, L \in Z} 2^{-Lq} \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} |\lambda_j| \left( k-j \right)^q 2^{-jx(s-j-k)2^{\gamma}} \right)^q / 2
\]
\[
\times \left( \sum_{j=-\infty}^{j=1} 2^{-jx(s-j-k)2^{\gamma}} \right)^{q/q'}
\]
\[
\leq C \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \sup_{L \in B, L \in Z} 2^{-Lq} \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} |\lambda_j| \left( k-j \right)^q 2^{-jx(s-j-k)2^{\gamma}} \right)^q / 2
\]
\[
\leq C \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \sup_{L \in B, L \in Z} 2^{-Lq} \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} |\lambda_j| \left( k-j \right)^q 2^{-jx(s-j-k)2^{\gamma}} \right)^q / 2
\]
\[
\leq C \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \Delta.
\]

(4.7)

Secondly, we estimate \( HH \). We need to show that there exists a positive constant \( C \), such that \( HH \leq C \Delta \). Consider

\[
HH = \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \]
\[
\leq \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \right)^q
\]
\[
+ \sum_{k=-\infty}^{k=1} 2^{kqo(0)} \left( \sum_{j=-\infty}^{j=1} \left\| b, \mu_\Delta \right\|_{L^p(\mathbb{R})} \right)^q
\]
\[
= HH_1 + HH_2.
\]

When \( 0 < q \leq 1 \), we get
\[ HH_i = \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|b, \mu_1|f_j\|_{L^q}(f_j))^{q} \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ + \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ + C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ + C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ + C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ + C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} \sum_{j=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} \sum_{j=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} \sum_{j=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

Now when \( 1 < q < \infty \), let \( 1/q + 1/q' = 1 \) we have

\[ HH_i = \sum_{k=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|b, \mu_1|f_j\|_{L^q}(f_j))^{q} \]

\[ + C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} \sum_{j=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} \sum_{j=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]

\[ \leq C \|b, \mu_1\|_{(s^1)} \sum_{k=0}^{n} \sum_{j=0}^{n} 2^{k\rho\alpha\beta}(\sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} \sum_{j=0}^{n} \|\lambda_j\|_{(k-j)}^{q} ) \]
\[
\times \left( \sum_{j=1}^{k} (k-j)^{q'} \left( \sum_{j=0}^{k-1} \left( k-j \right)^{q} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right)^{q'}
\]

+ \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right)^{q'}

\leq \left( \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \left( k-j \right)^{q'} \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right)^{q'}

\leq \left( \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \left( k-j \right)^{q'} \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right)^{q'}

\leq \left( \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \left( k-j \right)^{q'} \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right)^{q'}

\leq C \left( \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \left( k-j \right)^{q'} \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right)^{q'}

For \( HH_{2} \), when \( 0 < q \leq 1 \), by \( n_{\delta_{2}} \leq \alpha(0) < s + \delta + n_{\delta_{2}} \) we get

\[
HH_{2} = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \left[ b, \mu_{\alpha_{1}}(f_{j}) \right] X_{k} \int \mu_{\alpha_{1}(j)} \left( \sum_{j=0}^{k-1} \left( k-j \right)^{q'} \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right) \]

\leq C \left( \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \left( k-j \right)^{q'} \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right)^{q'}

\leq C \left( \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \left( k-j \right)^{q'} \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right)^{q'}

\leq C \left( \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \left( k-j \right)^{q'} \left( \sum_{j=1}^{k} \sum_{j=0}^{k-1} \left( \frac{n_{\delta_{2}} - n_{\alpha}}{q} - a(0) \right) \right) \right)^{q'}

(4.9)

Now \( 1 < q < \infty \), let \( 1/q + 1/q' = 1 \). Since \( n_{\delta_{2}} \leq \alpha(0) < s + \delta + n_{\delta_{2}} \), by Hölder’s inequality we have
\[ HHH_2 = \sum_{k=0}^{1} 2^{Lq(0)} \left( \sum_{j=0}^{k-1} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \right)^q \]

\[ \leq C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \sum_{k=0}^{1} 2^{Lq(0)} \left( \sum_{j=0}^{k-1} |b_j|^q (k-j)^y 2^{\frac{-ja(0) + j-k}{2}} \right)^q \]

\[ \leq C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \sum_{k=0}^{1} 2^{Lq(0)} \left( \sum_{j=0}^{k-1} |b_j|^q (k-j)^y 2^{\frac{-ja(0) + j-k}{2}} \right)^q \]

\[ \times \left( \sum_{j=0}^{k-1} (k-j)^2 \right)^{y/q} \]

\[ \leq C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \sum_{k=0}^{1} 2^{Lq(0)} \left( \sum_{j=0}^{k-1} |b_j|^q (k-j)^y 2^{\frac{-ja(0) + j-k}{2}} \right)^q \]

\[ = C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \sum_{k=0}^{1} |b_j|^q \sum_{k=1}^{j} (k-j)^q \frac{1}{2} \left(\frac{ja(0) + j-k}{2}\right)^{y/2} \]

\[ \leq C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \Delta. \]

Thirdly, we estimate \( HHH \), we need to show that there exists a positive constant \( C \), such that \( HHH \leq C \Delta \)

\[ HHH = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \left( \sum_{j=0}^{k} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \right)^q \]

\[ \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \left( \sum_{j=0}^{k} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \right)^q \]

\[ + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \left( \sum_{j=0}^{k} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \right)^q \]

\[ = HHH_1 + HHH_2. \]

When \( 0 < q \leq 1 \) by boundedness of \( [b, \mu_\alpha] \) in \( L_q^\alpha \)

\[ HHH_1 = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \left( \sum_{j=0}^{k} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \right)^q \]

\[ \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \]

\[ \leq C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \]

\[ \leq C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \]

\[ \leq C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \]

\[ \leq C \| \mathcal{L} \| (\mathcal{L}^\alpha)^{s+1} \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-Lq} \sum_{k=0}^{L} 2^{Lq(0)} \| b, \mu_\alpha \| (f_j) \mathcal{X}_k \]

When \( 0 < q \leq 1 \) by boundedness of \( [b, \mu_\alpha] \) in \( L_q^\alpha \)
\[ + C \| b \|_L^{s-1} \sup_{L \in L(u, L)} \sum_{k=0}^{L} 2^{jL} \sum_{j=0}^{L} \left( \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \right) 2^{q(j-L)\left( \alpha_{0} - \alpha_{2} \right)} \times 2^{q(j-L)\left( \alpha_{0} - \alpha_{2} \right)} \]
\[ \leq C \| b \|_L^{s-1} \Delta + C \| b \|_L^{s-1} \Delta \sum_{k=0}^{L} \left( k - j \right)^{q} \times 2^{q(j-L)\left( \alpha_{0} - \alpha_{2} \right)} \]

Now when \( 0 < q < \infty \), by boundedness of \( [b, \mu_{2}] \) in \( L^p() \), see ([20]) by Hölder’s inequality we have
\[ HHH = \sup_{L \in L(u, L)} 2^{jL} \sum_{k=0}^{L} \sum_{j=0}^{L} \left( \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \right) \times 2^{q(j-L)\left( \alpha_{0} - \alpha_{2} \right)} \times 2^{q(j-L)\left( \alpha_{0} - \alpha_{2} \right)} \]
\[ \leq \sup_{L \in L(u, L)} 2^{jL} \sum_{k=0}^{L} \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \times 2^{q(j-L)\left( \alpha_{0} - \alpha_{2} \right)} \]
\[ \leq \sup_{L \in L(u, L)} 2^{jL} \sum_{k=0}^{L} \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \times 2^{q(j-L)\left( \alpha_{0} - \alpha_{2} \right)} \]
\[ \leq \Delta + \Delta \sum_{k=0}^{L} \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \]
\[ \leq \Delta. \]

We have \( 0 < q \leq 1 \), by \( n\delta_{2} \leq \alpha(0), \alpha_{2} < s + \delta + n\delta_{2} \) we get
\[ HHH = \sup_{L \in L(u, L)} 2^{jL} \sum_{k=0}^{L} \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \]
\[ \leq C \| b \|_L^{s-1} \sum_{k=0}^{L} \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \]
\[ \leq C \| b \|_L^{s-1} \sum_{k=0}^{L} \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \]
\[ \leq C \| b \|_L^{s-1} \sum_{k=0}^{L} \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \]
\[ \leq C \| b \|_L^{s-1} \sum_{k=0}^{L} \sum_{j=0}^{L} \sum_{j=0}^{L} 2^{(j-L)q} 2^{-(j-L)q} \sum_{k=0}^{L} \left( k - j \right)^{q} \]
\[ \leq C \| b \|_L^{s-1} \Delta. \]

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Now when $1 < q < \infty$, let $1/q + 1/q' = 1$. Since $\alpha_2 \leq \alpha(0), \alpha(\infty) < s + \delta + \alpha_2$, by Hölder’s inequality, we have

$$H_{HHH} = \sup_{L \geq 0, k \geq 2} 2^{L \log k} \sum_{k=0}^{L} \left( \left\| b, \mu_{\Omega} \right\| \left( f_{j} \right) \right)_{L^2}^{q}$$

$$\leq C \left\| b \right\|_{L^q} \left\| \mu_{\Omega} \right\|_{L^q}^{q} \sum_{k=0}^{L} \left( \sum_{j=0}^{k} \left( \frac{a_{n_{k}}}{{n_{k}}^{\frac{n}{q}}} \right) \right)^{q}$$

$$\leq C \left\| b \right\|_{L^q} \left\| \mu_{\Omega} \right\|_{L^q}^{q} \sum_{k=0}^{L} \left( \sum_{j=0}^{k} \left( \frac{a_{n_{k}}}{{n_{k}}^{\frac{n}{q}}} \right) \right)^{q}$$

$$\leq C \left\| b \right\|_{L^q} \left\| \mu_{\Omega} \right\|_{L^q}^{q} \sum_{k=0}^{L} \left( \sum_{j=0}^{k} \left( \frac{a_{n_{k}}}{{n_{k}}^{\frac{n}{q}}} \right) \right)^{q}$$

$$\leq C \left\| b \right\|_{L^q} \left\| \mu_{\Omega} \right\|_{L^q}^{q} \sum_{k=0}^{L} \left( \sum_{j=0}^{k} \left( \frac{a_{n_{k}}}{{n_{k}}^{\frac{n}{q}}} \right) \right)^{q}$$

$$\leq C \left\| b \right\|_{L^q} \left\| \mu_{\Omega} \right\|_{L^q}^{q} \sum_{k=0}^{L} \left( \sum_{j=0}^{k} \left( \frac{a_{n_{k}}}{{n_{k}}^{\frac{n}{q}}} \right) \right)^{q}$$

$$\leq C \left\| b \right\|_{L^q} \left\| \mu_{\Omega} \right\|_{L^q}^{q} \sum_{k=0}^{L} \left( \sum_{j=0}^{k} \left( \frac{a_{n_{k}}}{{n_{k}}^{\frac{n}{q}}} \right) \right)^{q}$$

$$\leq C \left\| b \right\|_{L^q} \left\| \mu_{\Omega} \right\|_{L^q}^{q} \sum_{k=0}^{L} \left( \sum_{j=0}^{k} \left( \frac{a_{n_{k}}}{{n_{k}}^{\frac{n}{q}}} \right) \right)^{q}$$

Joint the estimates for $H$, $HH$ and $HHH$, we obtain

$$\left\| b, \mu_{\Omega} \right\|_{L^q} \left( f_{j} \right)_{L^2}^{q} \leq C \left\| b \right\|_{L^q} \left\| \mu_{\Omega} \right\|_{L^q}^{q} \left( f_{j} \right)_{L^2}^{q}.$$
Then we complete the proof of Theorem 4.1.

5. Conclusion

The study concluded that we can proof of boundedness for commutator of Marcinkiewicz integrals on Herz-Morrey-Hrdy spaces with variable exponent, which we use The main tools are properties of variable exponent in theorem 3.1 when $b \in Lip_{\gamma}(\mathbb{R}^n)$, in theorem 4.1 when $b \in BMO(\mathbb{R}^n)$. We can obtain a solution for proof that commutator of Marcinkiewicz integrals are boundedness.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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