TOPOLOGY OF A SYSTEM OF TWO QUADRATIC INEQUALITIES

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Abstract. We study the topology of the set of the (spherical, projective and affine) solutions of a system of two quadratic inequalities: we give explicit formulas for its Betti numbers and give some sharp estimates for them; we also give a bound linear in \( n \) for the topological complexity of the intersection of two quadrics in \( \mathbb{R}P^n \). We study the geometry of generic pencils of quadrics. We discuss some applications of the previous results.

1. Introduction

The main object of this paper will be pairs \((q,K)\):

\[
q : \mathbb{R}^m \to \mathbb{R}^2 \quad \text{and} \quad K \subset \mathbb{R}^2
\]

where \( q \) is a homogeneous quadratic map and \( K \) is a closed polyhedral cone. Such a pair describes a system of two quadratic inequalities in the following sense: if we let \( K \) be defined by \( \eta_0 \leq 0, \ldots, \eta_l \leq 0 \) for certain covectors \( \eta_0, \ldots, \eta_l \), then the set \( q^{-1}(A) \) coincides with the set of the solutions to the system \( \eta_0 q \leq 0, \ldots, \eta_l q \leq 0 \).

Thus in principle the number of inequalities arising by presenting \( K \) as a polyhedral cone can be greater than two, and our definition is slightly more general, including also the cases of equalities.

The previous setting naturally leads to the study of quadratic maps between vector spaces. Among all, the case of our interest, i.e. when the target space is two-dimensional, has some peculiar properties; in a certain sense the complexity of this case is one order less than the general case and we should expect all the computations to be simpler. To give an idea of this lower complexity consider a semialgebraic set \( X \) in \( \mathbb{R}P^n \) defined by \( k+1 \) quadratic inequalities. If we let \( b(X) \) be the sum of the Betti numbers of \( X \), the standard Oleinik-Petrovskii-Thom-Milnor bound\(^1\) would give \( b(X) \leq O((k+1)^{n+1}) \), but the fact that \( X \) has quadratic equations allows to switch the numbers \( k \) and \( n \) in the previous inequality and to give the following bound\(^2\):

\[
b(X) \leq O(n+1)^{2(k+1)}
\]

This evidence suggests that the number of quadratic inequalities defining our set, which appears at the exponent, gives a naive measure of its complexity and the case of two quadrics is thus the simplest to study (behind that of one single quadric). The effectiveness of this simplicity will become clear in a while. For the moment

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\(^1\)Of course this bound is not sharp; we worsened it to have more symmetry when compared with the next one.

\(^2\)Also this second bound is not sharp and for a sharper estimate we refer the reader to [5]; in a slightly different formulation, this was firstly proved by Barvinok in [3].
we explain the idea which is behind this kind of duality between the number of quadratic equations and the number of variables: let

$$q : V \to W$$

be a quadratic map between two vector spaces and for every covector $\eta$ in $W^*$ consider the quadratic form $\eta q$ over $V$. As we let $\eta$ vary we can reconstruct the map $q$ itself from the various $\eta q$. As long as we are concerned with the topology of $A = q^{-1}(K)$, where $K$ is a convex polyhedral cone in $W$, we can replace this complicated object with a simpler one: for each vector $\eta$ in $W^*$ consider the quadratic form $\eta q$ over $V$. As we let $\eta$ vary we can reconstruct the map $q$ itself from the various $\eta q$. As long as we are concerned with the topology of $A = q^{-1}(K)$, where $K$ is a convex polyhedral cone in $W$, we can replace this complicated object with a simpler one: for each vector $\eta$ in $W^*$ consider only the positive inertia index $i^+(\eta q)$, i.e. the maximal dimension of a subspace of $V$ on which $\eta q$ is positive definite. Thus, instead of dealing with a map with values in the space of quadratic forms, we have a function to the natural numbers. We let $K^\circ$ be the polar cone of $K$ and we define the following subsets of $S^k$:

$$\Omega = K^\circ \cap S^k \quad \text{and} \quad \Omega^j = \{ \omega \in \Omega \mid i^+(\omega q) \geq j \}, \quad j \in \mathbb{N}$$

The spirit of the mentioned duality is in this procedure of replacing the original framework with the above filtration $\Omega^j$. In the general case this filtration is a “continuous” object, in the sense that the sets of points where the function $i^+$ changes its value is an algebraic subset $Z$ of $S^k$; in the case of a quadratic map to the plane, $Z$ reduces to a finite number of points on $S^1$ and our object becomes something “discrete”.

Moving to topology, the crucial point is that there is a relation between the geometry of the previous filtration and that of $A$. To give an example, in the case $X$ is the intersection of quadrics in $\mathbb{R}P^n$, the following formula relates the Euler characteristic of $X$ with that of the previously defined sets (here $K$ is the zero cone and the components of $q$ are given by the quadratic forms defining $X$):

$$(1) \quad (-1)^n \chi(X) = \chi(S^n) + \sum_{j=0}^{n} (-1)^{j+1} \chi(\Omega^j).$$

In the case $X$ is the intersection of two quadrics the topology of the sets $\Omega^j$, $j \in \mathbb{N}$, being semialgebraic subsets of $S^1$, is very easy to compute, since such subsets are finite union of points and arcs.

**Example** (The bouquet of three circles). Consider the map $q : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$x \mapsto (x_1^2 + 2x_0x_2 - x_3^2, x_1x_2)$$

and the zero cone in $\mathbb{R}^2$. The subset $X$ of $\mathbb{R}P^3$ defined by $\{q = 0\}$ consists of two projective line and an ellipse meeting at one point; this set is homeomorphic to a bouquet of three circles. Associating to a quadratic form a symmetric matrix by means of a scalar product, the family $\eta q$ for $\eta \in \Omega = S^1$ is represented by the matrix:

$$\eta S = \begin{pmatrix} 0 & 0 & \eta_0 & 0 \\ 0 & \eta_0 & \eta_1 & 0 \\ \eta_0 & \eta_1 & 0 & 0 \\ 0 & 0 & 0 & -\eta_0 \end{pmatrix} \quad \eta = (\eta_0, \eta_1) \in S^1$$

The determinant of this matrix vanishes only at the points $\omega = (0,1)$ and $-\omega = (0,-1)$; outside of these points the index function must be locally constant. Then it is easy to verify that $i^+$ equals 2 everywhere except at this two points, where it equals 1:

$$\Omega^1 = S^1, \quad \Omega^2 = S^1\setminus\{\omega, -\omega\}, \quad \Omega^3 = \Omega^4 = \emptyset.$$
Applying formula (1) to this example gives:

\[-\chi(X) = \chi(S^3) + \chi(S^1) - \chi(S^1 \setminus \{\omega, -\omega\}) = 2.\]

A formula similar to (1) holds in the case of the intersection \(Y\) of two quadrics on the sphere (since \(Y\) double covers \(X\) we simply have to multiply the right hand side by 2). In this case we can be even more precise; if we set \(Y = q^{-1}(K) \cap S^k\) the following holds for the reduced Betti numbers of \(Y\):

\[(2) \tilde{b}_k(Y) = b_0(\Omega^{n-k}, \Omega^{n-k+1}) + b_1(\Omega^{n-k-1}, \Omega^{n-k}), \quad k < n - 2\]

This formula was essentially proved by Agrachev in [1] under some nondegeneracy hypothesis.

To stress again the difference with the general case, we must say the analogous formula for the Betti numbers of intersection of more than two quadrics only gives bounds for them (see [2]).

If we let \(p : S^n \to \mathbb{R}P^n\) be the covering map, the formula to compute the Betti numbers of \(X = p(Y)\)

is slightly different (notice that since \(q\) is homogeneous of degree two, then the set \(\{x \in \mathbb{R}P^n \mid q(x) \in K\}\) is well defined and coincides with \(X\)). The reason for this lies in the structure of the homology of \(\mathbb{R}P^n\) which is richer than that of the sphere. We can write easily the Betti numbers of \(\mathbb{R}P^n \setminus X\) by the formula:

\[(3) b_k(\mathbb{R}P^n \setminus X) = b_0(\Omega^{k+1}) + b_1(\Omega^{k}), \quad k \in \mathbb{N}\]

Using (3) we can prove classical results in convexity theory, such as the quadratic convexity theorem: the image of the sphere \(S^n\) under a homogeneous quadratic map \(q : \mathbb{R}^{n+1} \to \mathbb{R}^2\) with \(n \geq 2\) is a convex subset of the plane; this result is certainly false in the case the target space is three dimensional, as the map \(x \mapsto (x_0x_1, x_0x_2, x_1x_2)\) shows.

To know the Betti numbers of \(X\) we need to compute the rank of the homomorphism induced on the homology by the inclusion \(i : X \hookrightarrow \mathbb{R}P^n\) (this computation was not necessary for the sphere because of Alexander duality). If we let \(\mu\) be the maximum of \(i^+\) on \(\Omega\) we have:

\[(4) \text{rk}(i_*|_k) = b_0(C\Omega, \Omega^{k+1}), \quad k \neq n - \mu\]

where \(C\Omega\) is the topological space cone of \(\Omega\). The critical case \(k = n - \mu\) is more subtle and we need an extra information. For this purpose we introduce the bundle \(L_\mu \to \Omega^{\mu}\) whose fiber over the point \(\eta \in \Omega^{\mu}\) equals \(\text{span}\{x \in \mathbb{R}^{n+1} \mid \exists \lambda > 0 \text{ s.t. } (\eta Q)x = \lambda x\}\) and whose vector bundle structure is given by its inclusion in \(\Omega^{\mu} \times \mathbb{R}^{n+1}\). The extra information we need is the first Stiefel-Whitney class of \(L_\mu\):

\[w_1(L_\mu) \in H^1(\Omega^{\mu}).\]

Once we have this data we can compute also the rank of \((i_*|_{n-\mu})\):

\[(5) \text{rk}(i_*|_{n-\mu}) = 1 \quad \text{iff} \quad w_1(L_\mu) = 0.\]

\[^3\text{from now on we work with } \mathbb{Z}_2 \text{ coefficients.}\]
Consider now the table $E = (e^j_i)_{i,j \in \mathbb{Z}}$ whose nonzero part is the following:

$$E' = \begin{array}{cccc}
1 & 0 & 0 \\
\vdots & \vdots & \vdots \\
1 & 0 & 0 \\
c & 0 & 0 \\
0 & b_0(\Omega^\mu) - 1 & d \\
\vdots & \vdots & \vdots \\
0 & b_0(\Omega^1) - 1 & b_1(\Omega^1) \\
\end{array}$$

where $c = e^{0,\mu}$ and we have $(c, d) = (1, b_1(\Omega^\mu))$ if $w_{1,\mu} = 0$ and $(c, d) = (0, 0)$ otherwise. In term of the previous table it is easy to write the formula for the Betti numbers of $X$: if $\mu = n + 1$ then $X$ is empty; in the contrary case for every $k \in \mathbb{Z}$ we have:

$$b_k(X) = e^{0,n-k} + e^{1,n-k-1} + e^{2,n-k-2}.$$  

Notice that according to what we already stated we have $\text{rk}(i_*)_k = e^{0,n-k}$.

**Example** (The bouquet of three circles; continuation). In this case the previous table is the following:

$$E = \begin{array}{ccc}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}$$

This is because $\mu = 2$ and $\Omega^2$ is not the whole $S^2$ (thus $w_1(L_2) = 0$); in particular we have:

$$b_0(X) = e^{0,3} + e^{1,2} + e^{2,1} = 1, \quad b_1(X) = e^{0,2} + e^{1,1} + e^{2,0} = 3.$$  

Notice also that $\text{rk}(i_*)_1 = e^{0,2} = 1$, in accordance with the fact that $X$ contains a projective line.

For the reader familiar with spectral sequences we can say that the previous table gives the ranks of the second term of a spectral sequence converging to the homology of $X$: the class $w_1(L_\mu)$ gives the second differential and for the case of two quadrics that’s enough (the spectral sequence degenerates at the third step); otherwise higher differentials have to be calculated (the reader is referred to [2] for a detailed treatment of the general case); this evidence again confirms the difference of complexity with the case of more than two quadrics.

Incidentally we notice that it is possible to “send the number of variables to infinity” and our procedure is stable with respect to this limit: formulas analogous to the previous one hold for the set of the solutions of a system of two quadratic inequalities on the infinite dimensional sphere

If we were interested in the level sets of a quadratic map to the plane, namely in the set

$$A_c = q^{-1}(c) \subseteq \mathbb{R}^{n+1}, \quad c = (c_0, c_1)$$

4In this case some nondegeneracy condition has to be assumed in order to guarantee we are computing the cohomology with respect to the induced topology.
we define for $k \in \mathbb{N}$ the sets $C_k = \{ \omega \in S^1 | i^+(-\omega) \leq k \text{ and } \langle \omega, c \rangle < 0 \}$. If $A_c$ is nonempty and $0 \leq k \leq n + 1$ we have the formula:

$$
\tilde{b}_k(A_c) = b_0(C_{k+1}, C_k) + b_1(C_{k+2}, C_{k+1}).
$$

The ideas we discussed so far for the case of two quadrics leads us to explore the beautiful geometry of the index function on a circle. In fact the space of generic linear systems of two quadrics has a kind of algebraic extra structure; this extra structure allows us to label each pencil with a binary array in such a way that performing some rules (i.e. admitted permutations) on its characters corresponds to make generic homotopies of pencils. The combinatorial nature of these ideas finally leads to a bound on each specific Betti number of the set of solutions of a system of two proper quadratic inequalities (i.e. equalities are not permitted) $X$ in $\mathbb{R}P^n$ and of its double cover $Y$, improving the bounds of [5]:

$$
b_k(X) \leq k + 2 \text{ and } b_k(Y) \leq 2k + 4.
$$

Notice that summing over $k \geq 0$ all the previous bounds would give something of the form $b(X) \leq O(n)^2$, which is essentially the standard bound (see [3]). It is remarkable that, in the case $X$ is the intersection of two quadrics in $\mathbb{R}P^n$, using the same ideas as before, this bound can be made linear, improving the standard one:

$$
b(X) \leq 3n + 2
$$

To the author’s knowledge there are no such bounds in the literature; on the other hand, as pointed out in [4] and [5], there is no hope to get similar improvement for the case of more then two quadrics, since the standard bounds are asymptotically (in the number of quadrics) sharp.

The paper is organized as follows. In section 2 we introduce some notations. Formulas (2), (3), (4), (5) and (6) are proved in section 3; the formula (1) for the Euler characteristic is a direct consequence of (6). In section 4 we discuss some classical applications. Formula (7) is proved in section 5. In section 6 we discuss some generalizations to the infinite dimensional case. In section 7 we study the combinatorics of nondegenerate pencils. The bounds (8) are proved in section 8 and section 9 is devoted to (9).

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2. Preliminary notations

We denote with the symbol $Q(\mathbb{R}^m)$ the set of all quadratic forms over $\mathbb{R}^m$; it is a vector space of dimension $m(m+1)/2$ and can be identified with the space $\mathbb{R}[x_1, \ldots, x_m]_{(2)}$ of homogeneous polynomials of degree two. If we introduce a scalar product on $\mathbb{R}^m$, then the identity $q(x) = \langle x, Qx \rangle$ defines a map $q \mapsto Q$ giving a linear isomorphism between $Q(\mathbb{R}^m)$ and the space of real symmetric matrices of

$^{5}$Generic with respect to a certain nondegeneracy condition.

$^{6}$Indeed the following bound holds for the set of solutions of any systems of two quadratic inequalities
order m.
Given a quadratic form $q$ we will denote by $\text{ind}^+(q)$ its positive inertia index:

$$\text{ind}^+(q) = \max\{\dim(V) \mid V \text{ is a subspace of } \mathbb{R}^m \text{ and } q|_V > 0\}.$$  

**Definition 1.** We define $\mathcal{Q}(m, 2) \cong \mathcal{Q}(\mathbb{R}^m) \times \mathcal{Q}(\mathbb{R}^m)$ with its product topology. In other words $q \in \mathcal{Q}(m, 2)$ is given by a pair $(q_0, q_1)$ of quadratic forms.

We can interpret $q \in \mathcal{Q}(m, 2)$ also as a quadratic map $q : \mathbb{R}^m \to \mathbb{R}^2$ whose components are $(q_0, q_1)$.

Given $q \in \mathcal{Q}(m, 2)$ it is defined a linear map $\tilde{q} : \mathbb{R}^2 \to \mathcal{Q}(\mathbb{R}^m)$ by the correspondence $\omega = (\omega_0, \omega_1) \mapsto \omega_0q_0 + \omega_1q_1$; to shortening notations we will often write $\omega \mapsto \omega q$ for the previous map. As above the scalar product gives an identification between $\mathcal{Q}(m, 2)$ and the space of pairs of symmetric matrices of order $m$. Again we will use the notations $q = (q_0, q_1) \mapsto Q = (Q_0, Q_1)$ for this identification; in a similar way if $\omega \in \mathbb{R}^2$, then $\tilde{q} : \omega \mapsto \omega Q = \omega_0Q_0 + \omega_1Q_1$.

**Definition 2.** Let $K \subset \mathbb{R}^k$ be a convex cone; we define its polar $K^\circ \subset (\mathbb{R}^k)^*$ by

$$K^\circ = \{\eta \in (\mathbb{R}^k)^* \mid \eta(y) \leq 0 \ \forall y \in K\}.$$  

If $K$ is closed and convex, then $(K^\circ)^\circ = K$; using a scalar product we can also mean $K$ to be $\{x \in \mathbb{R}^k \mid \langle x, y \rangle \leq 0 \ \forall y \in K\}$.

**Definition 3.** Given $q \in \mathcal{Q}(m, 2)$ and $K \subset \mathbb{R}^2$ we define $\Omega = K^\circ \cap S^1$ and the function $i^+ : \Omega \to \mathbb{N}$ by

$$\omega \mapsto \text{ind}^+(\omega q).$$  

Moreover for every $j \in \mathbb{N}$ we define also

$$\Omega^j = \{\omega \in \Omega \mid i^+(\omega) \geq j\}.$$  

Thus given $q \in \mathcal{Q}(m, 2)$ and $K \subset \mathbb{R}^2$ we are considering a linear pencil of quadratic forms $\omega \mapsto \omega q$ and the restriction of the index function to $\Omega = K^\circ \cap S^1$ in the space of the parameters of the pencil.

From now on all homology and cohomology groups are with $\mathbb{Z}_2$ coefficients; if $(X, Y)$ is a pair of spaces, then we will denote by $b_k(X, Y)$ the rank of $H^k(X, Y)$ and by $b_k(X)$ that of $H^k(X)$ (in our cases they will always be defined); $b(X)$ will denote the topological complexity of $X$, i.e. the sum $\sum_k b_k(X)$.

### 3. Systems of two quadratic inequalities

Throughout this section we consider $q = (q_0, q_1) \in \mathcal{Q}(n + 1, 2)$, $K \subset \mathbb{R}^2$ a convex polyhedral cone and the set $Y \subset S^n$ defined by

$$Y = \{x \in S^n \mid q(x) \in K\}.$$  

**Theorem 4.** The following formula holds for $k < n - 2$:

$$\tilde{b}_k(Y) = \tilde{b}_{n-k-1}(S^n \setminus Y) = b_0(\Omega^{n-k}, \Omega^{n-k+1}) + b_1(\Omega^{n-k-1}, \Omega^{n-k}).$$  

**Proof.** The first equality follows from Alexander duality. For the second consider the set

$$B = \{(\omega, x) \in \Omega \times S^n \mid (\omega q)(x) > 0\}.$$  

The projection $p_2 : B \to S^n$ gives a homotopy equivalence $B \sim p_2(B) = S^n \setminus Y$ (the fibers are contractible). On the other side for $\epsilon > 0$ sufficiently small the inclusion

$$B(\epsilon) = \{(\omega, x) \in \Omega \times S^n \mid (\omega q)(x) \geq \epsilon\} \to B$$  

satisfies the hypotheses of Theorem 4.4.3; in consequence $\tilde{b}_k(Y) = \tilde{b}_{n-k-1}(S^n \setminus Y)$. The proof is completed.
is a homotopy equivalence. Consider \( \pi = p_{1\mid B(\epsilon)} : B(\epsilon) \to \Omega \) and the Leray spectral sequence associated to it:
\[
(E_r(\epsilon), d_r) \Rightarrow H^*(B(\epsilon); \mathbb{Z}_2), \quad E_2(\epsilon)^{i,j} = \tilde{H}^i(\Omega, \mathcal{F}^j(\epsilon)),
\]
where \( \mathcal{F}^j(\epsilon) \) is the sheaf associated to the presheaf \( V \mapsto H^j(\pi^{-1}(V)) \). Since \( B(\epsilon) \) and \( \Omega \) are locally compact and \( \pi \) is proper (\( B(\epsilon) \) is compact) then the following isomorphism holds for the stalk of \( \mathcal{F}^j(\epsilon) \) at each point \( \omega \in \Omega \):
\[
\mathcal{F}^j(\epsilon)_\omega \cong H^j(\pi^{-1}(\omega)).
\]
Let \( g \in \mathbb{R}[x_0, \ldots, x_n]_{(2)} \) such that \( S^n = \{ g(x) = 1 \} \), then \( \pi^{-1}(\omega) \simeq \{ x \in S^n \mid (\omega q - \epsilon g)(x) \geq 0 \} \) has the homotopy type of a sphere of dimension \( n - \text{ind}^- (\omega q - \epsilon g) \); thus if we set \( i^- (\epsilon) \) for the function \( \omega \mapsto \text{ind}^- (\omega q - \epsilon g) \), we have that for \( j > 0 \) the sheaf \( \mathcal{F}^j(\epsilon) \) is locally constant with stalk \( \mathbb{Z}_2 \) on \( \Omega_{n-j}(\epsilon) \setminus \Omega_{n-j-1}(\epsilon) \), where \( \Omega_{n-j}(\epsilon) = \{ i^- (\epsilon) \leq n - j \} \), and zero on its complement. Since \( \Omega_{n-j-1}(\epsilon) \) is closed in \( \Omega_{n-j}(\epsilon) \), we have for \( j > 0 \):
\[
\tilde{H}^i(\Omega, \mathcal{F}^j(\epsilon)) = \tilde{H}^i(\Omega_{n-j}(\epsilon), \Omega_{n-j-1}(\epsilon)).
\]
Since the sets \( \{ \Omega_{n-j}(\epsilon) \}_{j \in \mathbb{N}} \) are CW-subcomplex of the one-dimensional complex \( S^1 \) (covers such that triple intersections of their open sets are empty are cofinal), then \( E_2^{ij}(\epsilon) = 0 \) for \( i \geq 2 \) and the Leray spectral sequence of \( \pi \) degenerates at \( E_2(\epsilon) \). By semialgebraicity the topology of \( \Omega_{n-j}(\epsilon) \) is definitely constant in \( \epsilon \) and form small \( \epsilon \) we have
\[
E_2^{ij}(\epsilon) \simeq \lim_{\epsilon} \{ \tilde{H}^i(\Omega_{n-j}(\epsilon), \Omega_{n-j-1}(\epsilon)) \}, \quad j > 0.
\]
The following lemma implies for \( j > 0 \) the isomorphism \( E_2^{ij}(\epsilon) \simeq \tilde{H}^i(\Omega^{j+1}, \Omega^{j+2}) \) and the conclusion follows.

**Remark 1.** The anomalous behaviour for \( j = 0 \) is due to the fact that there is no canonical choice for the generator of \( H^0(S^0) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

**Lemma 5.** For every \( j \in \mathbb{N} \) we have \( \Omega^{j+1} = \bigcup_{\epsilon > 0} \Omega_{n-j}(\epsilon) \); moreover every compact subset of \( \Omega^{j+1} \) is contained in some \( \Omega_{n-j}(\epsilon) \) and in particular
\[
\lim_{\epsilon} \{ H_*(\Omega_{n-j}(\epsilon)) \} = H_*(\Omega^{j+1}).
\]

**Proof.** Let \( \omega \in \bigcup_{\epsilon > 0} \Omega_{n-j}(\epsilon) \); then there exists \( \tau \) such that \( \omega \in \Omega_{n-j}(\epsilon) \) for every \( \epsilon < \tau \). Since for \( \epsilon \) small enough
\[
i^- (\epsilon)(\omega) = i^- (\omega) + \dim(\ker \omega p)
\]
then it follows that
\[
i^+ (\omega) = n + 1 - i^- (\omega) - \dim(\ker \omega p) \geq j + 1.
\]

Viceversa if \( \omega \in \Omega^{j+1} \) the previous inequality proves \( \omega \in \Omega_{n-j}(\epsilon) \) for \( \epsilon \) small enough, i.e. \( \omega \in \bigcup_{\epsilon > 0} \Omega_{n-j}(\epsilon) \).

Moreover if \( \omega \in \Omega_{n-j}(\epsilon) \) then, eventually choosing a smaller \( \epsilon \), we may assume \( \epsilon \) properly separates the spectrum of \( \omega p \) and thus, by algebraicity of the map \( \omega \mapsto \omega p \), there exists \( U \) open neighborhood of \( \omega \) such that \( \epsilon \) properly separates also the spectrum of \( \omega p' \) for every \( \omega' \in U \) (see [13]). Hence \( \omega' \in \Omega_{n-j}(\epsilon) \) for every \( \omega' \in U \).
From this consideration it easily follows that each compact set in $\Omega^{j+1}$ is contained in some $\Omega_{n-j}(\epsilon)$ and thus
\[
\lim_{\epsilon \to 0} \{ H_*(\Omega_{n-j}(\epsilon)) \} = H_*(\Omega^{j+1}).
\]

In Theorem 4 we essentially computed the Betti numbers of $B$ which was seen to be homotopy equivalent to $S^n \setminus Y$. Replacing $B$ with $B = \{ (\omega, [x]) \in \Omega \times \mathbb{R}P^n \mid (\omega \eta)(x) > 0 \}$ the same argument yields the following theorem (here we use the Leray sheaf $\mathcal{F}_j(V) = H^j(\pi^{-1}(V) \cap B(\epsilon))$, which is locally constant with stalk $\mathbb{Z}_2$ on $\Omega_{n-j}(\epsilon)$ and zero on its complement. We set $X = \{ [x] \in \mathbb{R}P^n \mid q(x) \in K \}$.

**Theorem 6.** $b_k(\mathbb{R}P^n \setminus X) = b_0(\Omega^{k+1}) + b_1(\Omega^k)$ for every $k \in \mathbb{N}$.

In this case we cannot apply Alexander duality directly, since we have to know the map induced by the inclusion $c : \mathbb{R}P^n \setminus X \to \mathbb{R}P^n$ on the cohomology.

**Proposition 7.** Set $\mu = \max_{\omega \in \Omega} i^+(\omega)$. Then for $k \leq \mu - 1$
\[
H^k(\mathbb{R}P^n) \xrightarrow{i_*} H^k(\mathbb{R}P^n \setminus X)
\]
is injective and for $k \geq \mu + 1$ is zero.

Notice that the case $k = \mu$ is excluded from this statement: it deserves a special treatment.

**Proof.** Consider the commutative diagram of maps
\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{i} & \Omega \times \mathbb{R}P^n \\
\downarrow{P_2|_B} & & \downarrow{p_2} \\
\mathbb{R}P^n \setminus X & \xrightarrow{c} & \mathbb{R}P^n
\end{array}
\]
Since $p_2|_B$ is a homotopy equivalence, then $c^* = i^* \circ p_2^*$. If $k \leq \mu - 1$, then $\Omega^{k+1} \neq \emptyset$; thus let $\eta \in \Omega^{k+1}$. Then $p_2^{-1}(\eta) \cap B$ deformation retracts to $\{ \eta \} \times P^{d_\eta}$, where $P^{d_\eta}$ is a projective space of dimension $d_\eta = i^+(\eta) - 1 \geq k$; in particular the inclusion $P^{d_\eta} \xrightarrow{i_\eta} \mathbb{R}P^n$ induces isomorphism on the $k$-th cohomology group. The following factorization of $i_\eta$ concludes the proof of the first part (all the maps are the natural ones):
\[
\begin{array}{ccc}
H^k(\mathbb{R}P^n) & \xrightarrow{i_\eta^*} & H^k(P^{d_\eta}) \\
\downarrow & & \downarrow \\
H^k(\Omega \times \mathbb{R}P^n) & \longrightarrow & H^k(\mathcal{B})
\end{array}
\]
For the second statement simply observe that for $k \geq \mu + 1$ we have $\Omega^k = \emptyset$ and thus
\[
H^k(\mathbb{R}P^n \setminus X) \cong H^k(\Omega^{k+1}) \oplus H^1(\Omega^k) = 0.
\]
It remains to study $H^\mu(\mathbb{R}P^n \setminus X) \to H^\mu(\mathbb{R}P^n)$. For this purpose we introduce the bundle $L_\mu \to \Omega^\mu$ whose fiber at the point $\eta \in \Omega^\mu$ equals span\{ $x \in \mathbb{R}^{n+1} \mid \exists \lambda > 0$ s.t. $(\eta Q)x = \lambda x$\} and whose vector bundle structure is given by its inclusion in $\Omega^\mu \times \mathbb{R}^{n+1}$. We let $w_{1,\mu} \in H^1(\Omega^\mu)$ be the first Stiefel-Whitney class of $L_\mu$. We have the following result.

**Proposition 8.** $rk(c^*)_{\mu} = 0 \iff w_{1,\mu} = 0$.

**Proof.** In the case $\Omega^\mu \neq S^1$, then clearly $w_{1,\mu}$ is zero and also $rk(c^*)_{\mu}$ is zero since $H^*(\mathbb{R}P^n \setminus X) = 0$. If $\Omega^\mu = S^1$, then $i^+$ is constant and we consider the projectivization $P(L_\mu)$ of the bundle $L_\mu$. In this case it is easily seen that the inclusion

$$P(L_\mu) \xrightarrow{i} \mathbb{B}$$

is a homotopy equivalence and, since $rk(c^*)_{\mu} = rk(i^+ \circ p_2^*)$ we have $rk(c^*)_{\mu} = rk(\lambda^* \circ i^+ \circ p_2^*)$. Let us call $I$ the map $p_2 \circ i \circ \lambda$; then $I : P(L_\mu) \to \mathbb{R}P^n$ is a map which is linear on the fibres and if $y \in H^1(\mathbb{R}P^n)$ is the generator, we have by Leray-Hirsch

$$H^*(P(L_\mu)) \simeq H^*(S^1) \otimes \{1, l^*y, \ldots, l^*y^{\mu-1}\}.$$ 

By the Whitney formula we get

$$l^*y^\mu = w_{1,\mu}(l^*y)^{\mu-1}$$

which proves $(c^*)_{\mu}$ is zero iff $w_{1,\mu} = 0$. □

Collecting together Theorem 8 and the previous two propositions allows us to split the long exact sequence of the pair $(\mathbb{R}P^n, \mathbb{R}P^n \setminus X)$ and, since $H_\ast(X) \simeq H^{n-\ast}(\mathbb{R}P^n, \mathbb{R}P^n \setminus X)$, to compute the Betti numbers of $X$. We first define the table $E = (e_{i,j})_{i,j \in \mathbb{Z}}$ with $e_{i,j} \in \mathbb{N}$, and whose nonzero part $E' = \{e_{i,j} \mid 0 \leq i \leq 2, 0 \leq j \leq n\}$ is the following table:

$$
\begin{array}{cccc}
1 & 0 & 0 \\
\vdots & \vdots & \vdots \\
1 & 0 & 0 \\
\vdots & \vdots & \vdots \\
c & 0 & 0 \\
0 & b_0(\Omega^\mu) - 1 & d \\
\vdots & \vdots & \vdots \\
0 & b_0(\Omega^\mu) - 1 & b_1(\Omega^\mu)
\end{array}
$$

where $c = e_{0,\mu}$ and we have $(c, d) = (1, b_1(\Omega^\mu))$ if $w_{1,\mu} = 0$ and $(c, d) = (0, 0)$ otherwise.

**Theorem 9.** If $\mu = n + 1$ then $X$ is empty; in the contrary case for every $k \in \mathbb{Z}$ the following formula holds:

$$b_k(X) = e^{0,n-k} + e^{1,n-k-1} + e^{2,n-k-2}.$$ 

Moreover if $i : X \to \mathbb{R}P^n$ is the inclusion map and $i_\ast$ is the map induced on homology, then

$$e^{0,n-k} = rk(i_\ast)_k.$$ 

The last statement follows from the formula

$$b_{n-k}(\mathbb{R}P^n) = rk(c^*)_{n-k} + rk(j_\ast)_k.$$
**Example 1** (The complex squaring). Consider the quadratic forms

\[ q_0(x) = x_0^2 - x_1^2, \quad q_1(x) = 2x_0x_1. \]

Identifying \( \mathbb{R}^2 \) with \( \mathbb{C} \) via \( (x_0, x_1) \mapsto x_0 + ix_1 \), the map \( q = (q_0, q_1) \) is the complex squaring \( z \mapsto z^2 \). We easily see that the common zero locus set of \( q_0 \) and \( q_1 \) in \( \mathbb{R}P^1 \) is empty. The image of the linear map \( \overline{\eta} : \mathbb{R}^2 \to \mathbb{Q} \langle 2 \rangle \) defined by \( \eta \mapsto \eta q \) consists of a plane intersecting the set of degenerate forms \( Z \) only at the origin; we identify \( \mathbb{Q} \langle 2 \rangle \) with the space of \( 2 \times 2 \) real symmetric matrices. Thus \( \overline{\eta}(S^1) \) is a circle looping around \( Z = \{ \det = 0 \} \) and the index function is constant:

\[ i^+ (\omega q) = 1, \quad \omega \in S^1. \]

Thus \( \Omega^1 = S^1 \); and the table \( E \) in this case has the following picture:

\[
E = \begin{pmatrix}
    c & 0 & 0 \\
    0 & d & 0 \\
    0 & 0 & d
\end{pmatrix}
\]

On the other hand the first Stiefel-Whitney class of the bundle \( L_1 \to \Omega^1 \) in this case is nonzero; hence in this case \((c, d) = (0, 0)\) and we have that \( b_k(X) = 0 \) for every \( k \), as confirmed from the fact that \( X = \emptyset \).

Alternatively we could give a direct proof of the previous theorem using Theorems A, B and C of [2]; the reader should recognize in the previous table the structure of some spectral sequence.

The previous theorem raises the question when can happen \( w_{1, \mu} \neq 0 \). Since \( \mu = \max i^+ \), then clearly \( \Omega = S^1 \) and \( i^+ \equiv \mu \). Moreover since \( \mu = i^+ (\eta) = n + 1 - \ker (\eta Q) - i^+ (-\eta) = n + 1 - \ker (\eta Q) - \mu \) it follows \( \mu \leq \frac{n+1}{2} \).

It is interesting to classify pairs of quadratic forms \((q_0, q_1)\) such that \( i^+ \) is constant; this classification follows from a general theorem on the classification up to congruence of pencils of real symmetric matrices (see [13]).

4. **Classical applications**

We discuss here some applications of the previous results; the reader is referred to [3] for a detailed treatment using different techniques. We start with the following.

**Theorem 10** (Calabi). Let \( q_0, q_1 \) be real quadratic forms over \( \mathbb{R}^{n+1} \) with \( n+1 \geq 3 \). If the only \( x \in \mathbb{R}^{n+1} \) satisfying \( q_0(x) = q_1(x) = 0 \) is \( x = 0 \), then there exists a real linear combination \( \omega q_0 + \omega_1 q_1 \) which is positive definite.

**Proof.** The hypothesis is equivalent to \( n+1 \geq 3 \) and \( X = \{ \tau \in \mathbb{R}P^n \mid q_0(x) = 0 = q_1(x) \} = \emptyset \) and the thesis to \( \Omega^{n+1} \neq \emptyset \).

First notice that for every \( k \geq 2 \) we have \( b_1 (\Omega^k) = 0 \) : if it was the contrary, then \( b_0 (\Omega^k) = 1 = b_1 (\Omega^{k-1}) \) and Theorem [3] would give \( b_{k-1} (\mathbb{R}P^n \setminus X) = b_{k-1} (\mathbb{R}P^n) = b_0 (\Omega^k) + b_1 (\Omega^{k-1}) = 2 \), which is absurd. Thus if \( n+1 \geq 2 \) we have

\[
1 = b_0 (\mathbb{R}P^n) = b_n (\mathbb{R}P^n \setminus X) = b_0 (\Omega^{n+1}) + b_1 (\Omega^n) = b_0 (\Omega^{n+1})
\]

which implies \( \Omega^{n+1} \neq \emptyset \). \( \square \)

Thus the previous theorem states that for \( n+1 \geq 3 \)

\[ X = \emptyset \Rightarrow \Omega^{n+1} \neq \emptyset. \]

Also the contrary is true, with no restriction on \( n \) : if \( X \neq \emptyset \) then \( 0 = b_n (\mathbb{R}P^n \setminus X) = b_0 (\Omega^{n+1}) + b_1 (\Omega^n) \) which implies \( \Omega^n \neq S^1 \) and \( \Omega^{n+1} = \emptyset \). Thus we have the following corollary.
Corollary 11. If \( n + 1 \geq 3 \), then \( X = \emptyset \iff \Omega^{n+1} \neq \emptyset \).

Using the previous we can prove the well known quadratic convexity theorem.

**Theorem 12.** If \( n + 1 \geq 3 \) and \( q : \mathbb{R}^{n+1} \to \mathbb{R}^2 \) is defined by \( x \mapsto (q_0(x), q_1(x)) \), where \( q_0, q_1 \) are real quadratic forms, then

\[
q(S^n) \subset \mathbb{R}^2 \text{ is a convex set.}
\]

**Proof.** First observe that if \( S^n = \{ g(x) = 1 \} \) with \( q \) quadratic form, then for a given \( c = (c_0, c_1) \) we have \( S^n \cap q^{-1}(c) \neq \emptyset \) iff \( S^n \cap q_c^{-1}(0) \neq \emptyset \) iff \( X(q_c) = \emptyset \), where \( q_c \) is the quadratic map whose components are \( (q_0 - c_0 g, q_1 - c_1 g) \) and \( X(q_c) = \{ \varpi \in \mathbb{R}^n \mid q_c(x) = 0 \} \). Thus by Corollary 11 we have \( X(q_c) \neq \emptyset \) iff \( \Omega^{n+1}(q_c) = \emptyset \) (here \( n + 1 \geq 3 \)).

Let now \( a = (a_0, a_1) \) and \( b = (b_0, b_1) \) be such that \( X(q_a) \neq \emptyset \neq X(q_b) \) and suppose there exists \( T \in [0, 1] \) such that \( aT + (1 - T)b \notin q(S^n) \). Then by Corollary 11 there exists \( \eta \in \mathbb{R}^2 \) such that

\[
\eta Q - \langle \eta, aT + (1 - T)b \rangle I > 0.
\]

Assume \( \langle \eta, a - b \rangle \geq 0 \), otherwise switch the role of \( a \) and \( b \). We have \( 0 < \eta Q - \langle \eta, aT + (1 - T)b \rangle I = \eta Q + \langle \eta, T(b - a) \rangle I - \langle \eta, b \rangle I \leq \eta Q - \langle \eta, b \rangle I \). Thus we got

\[
\eta Q - \langle \eta, b \rangle I > 0,
\]

which implies \( \Omega^{n+1}(q_b) \neq \emptyset \), but this is impossible by corollary 11 since \( X(q_b) \neq \emptyset \). Hence for every \( t \in [0, 1] \) we have \( at + (1 - t)b \in q(S^n) \).

The conclusion of the previous theorems are false if \( n + 1 = 2 \): pick \( q_0(x, y) = x^2 - y^2 \) and \( q_1(x, y) = 2xy \), then \( q_0(x) = q_1(x) = 0 \) implies \( x = 0 \) but any real linear combination of \( q_0 \) and \( q_1 \) is sign indefinite. Moreover \( q(S^1) = S^1 \) which of course is not a convex subset of \( \mathbb{R}^2 \).

**Corollary 13.** If \( q : \mathbb{R}^{n+1} \to \mathbb{R}^2 \) has homogeneous quadratic components, then \( q(\mathbb{R}^{n+1}) \) is closed and convex.

**Proof.** Since \( q(\mathbb{R}^{n+1}) \) is the positive cone over \( q(S^n) \), then it is closed and convex.

\[\Box\]

The previous proof works only for \( n + 1 \geq 3 \), but the theorem is actually true with no restriction on \( n \). The number of quadratic forms is indeed important, as the following example shows: let \( q : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by \( (x_0, x_1, x_2) \mapsto (x_0x_1, x_0x_2, x_1x_2) \); then the image of \( \mathbb{R}^3 \) under \( q \) consists of the four hortants \( \{ x_0 \geq 0, x_1 \geq 0, x_2 \geq 0 \} \), \( \{ x_0 \leq 0, x_1 \leq 0, x_2 \geq 0 \} \), \( \{ x_0 \leq 0, x_1 \geq 0, x_2 \leq 0 \} \), \( \{ x_0 \geq 0, x_1 \leq 0, x_2 \leq 0 \} \).

5. Level sets of quadratic maps

In this section we discuss the topology of the level sets of a homogeneous quadratic map. We start with the following observation: in the case we are given a semialgebraic subset \( A \) in \( \mathbb{R}^n \) defined by inequalities involving polynomials of degree two (the presence of degree one polynomials reduce to this case by restricting to affine subspaces), then we can find a semialgebraic subset \( A' \) in \( \mathbb{R}^n \) such that the inclusion of \( A \) in \( A' \) is a homotopy equivalence and \( A' \) is defined by quadratic inequalities in \( \mathbb{R}^n \). Consider first the projective closure \( \overline{A} \) of \( A \) in \( \mathbb{R}^n \), which amounts to consider the system of quadratic inequalities in \( \mathbb{R}^n \) defined by the homogenization of the polynomials defining \( A \). Then \( \overline{A} \) is obtained from \( A \) by adding the set of the solutions of a system of quadratic inequalities at infinity, namely on
the hyperplane \( \{x_0 = 0\} \), where \( x_0 \) is the new variable we added by homogenization (the restriction of the homogenization of the system to this hyperplane is clearly homogeneous). Consider now the inequality

\[
l_c(x_0, \ldots, x_n) = \epsilon (x_1^2 + \cdots + x_n^2) - x_0^2 \leq 0.
\]

We want to show that for \( \epsilon > 0 \) small enough \( A \) and \( \overline{A} \cap \{l_c \leq 0\} \) are homotopy equivalent. Notice first that \( l_c = 0 \) has no solutions with \( x_0 = 0 \): in fact in this case it must be \( x_1^2 + \cdots + x_n^2 \leq 0 \) which implies \( x_1 = \cdots = x_n = 0 \), but this is impossible on \( \mathbb{RP}^n \). Thus on the projective space the inequality \( l_c \leq 0 \) is equivalent to the one \( x_0^{-2}(x_1^2 + \cdots + x_n^2) \leq R \) where \( R = \epsilon^{-1} \). In non-homogeneous coordinates on \( \mathbb{R}^n \) we can rewrite the last inequality as \( y_1^2 + \cdots + y_n^2 \leq R \), hence in particular:

\[
\overline{A} \cap \{l_c \leq 0\} = A \cap \{||y||^2 \leq R\}.
\]

Consider now the semialgebraic map \( \psi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\} \) defined by sending \( x \) to \( x||x||^{-2} \), and the semialgebraic set \( S = \psi(A) \cup \{0\} \). Then \( \psi \) maps \( A \cap \{||y||^2 \leq R\} \) to \( S \cap \{||y||^2 \leq R^{-1}\} \) and the conclusion follows from the local conic structure of semialgebraic sets (see [8] or [7] Proposition 5.49 for a direct argument).

In summary this argument shows that \( A \) is homotopy equivalent to the set \( A(\epsilon) \) defined in the projective space by homogenization of the inequalities defining \( A \) and by adding the inequality \( l_c \leq 0 \) for \( \epsilon > 0 \) small enough.

Thus we reduce the problem of studying the topology of \( A \) to that of studying the set of projective solutions of a system of \textit{three} homogeneous quadratic inequalities. For this purpose we recall the following theorem from [2].

**Theorem 14.** Let \( q : \mathbb{R}^{n+1} \to \mathbb{R}^k \) be a homogeneous quadratic map, \( K \subset \mathbb{R}^{k+1} \) be a closed convex polyhedral cone and define

\[
X = \{[x] \in \mathbb{RP}^n \mid q(x) \in K\}.
\]

Moreover let \( K^\circ \subset \mathbb{R}^{k+1} \) be the polar of \( K \) and \( \Omega = K^\circ \cap S^k \); let also \( i^+ : \Omega \to \mathbb{N} \) be the function defined by \( \omega \mapsto \text{ind}^+(\omega q) \), and

\[
\Omega^k = \{\omega \in \Omega \mid i^+(\omega) \geq k\}, \quad k \in \mathbb{N}.
\]

Then there exists a spectral sequence \( (E_r, d_r) \Rightarrow H_{n-*}(X) \) such that

\[
E_{2, j}^n = H^j(C\Omega, \Omega^{j+1})
\]

where \( C\Omega \) is the topological space cone of \( \Omega \).

Moreover \( E_r \) degenerates at the \((k+2)-\text{th}\) step and if \( i_* : H_*(X) \to H_*(\mathbb{RP}^n) \) is the map induced by the inclusion, then

\[
\text{rk}(i_*) = \dim(E_{0, n-*}^\infty).
\]

In view of the previous discussion we are led to consider \( \tilde{q}_c : \mathbb{R}^{n+1} \to \mathbb{R}^3 \) whose components are \((h^0 q_0, h^1 q_1, l_c)\) and the cone \( \tilde{K} = K \times (-\infty, 0] \subset \mathbb{R}^3 \). Moreover if we let \( \tilde{\Omega} = \tilde{K}^\circ \cap S^2 \subset \mathbb{R}^2 \times \mathbb{R} \) then theorem [14] gives, for small \( \epsilon \), a spectral sequence \( (\tilde{E}_r(\epsilon), \tilde{d}_r(\epsilon)) \) such that

\[
(\tilde{E}_r(\epsilon), \tilde{d}_r(\epsilon)) \Rightarrow H_{n-*}(A(\epsilon)) \simeq H_{n-*}(A) \quad \text{and} \quad \tilde{E}_{2, j}^n(\epsilon) = H^j(C\tilde{\Omega}, \tilde{\Omega}^{j+1}(\epsilon)),
\]

where \( \tilde{\Omega}^{j+1}(\epsilon) = \{\omega, t) \in \tilde{\Omega} | i^+(\omega^p + tl_c) \geq j + 1\} \).

In the case we were interested in the level sets of homogeneous quadratic maps, namely

\[
A = q^{-1}(c)
\]
where \( q \in \mathcal{Q}(n, 2) \) and \( c = (c_0, c_1) \in \mathbb{R}^2 \), we have the following theorem.

**Theorem 15.** Let \( q : \mathbb{R}^n \to \mathbb{R}^2 \) be defined by \((q_0, q_1)\) with \( q_0, q_1 \) homogeneous of degree two, \( c = (c_0, c_1) \in \mathbb{R}^2 \) and \( A_c = q^{-1}(c) \subset \mathbb{R}^n \).

Moreover define for every \( k \in \mathbb{N} \) the set

\[
C_k = \{\omega = (\omega_0, \omega_1) \in S^1 | \langle \omega, c \rangle < 0 \text{ and } i^+(-\omega_0 q_0 - \omega_1 q_1) \leq k\}
\]

and the number

\[
\nu_c = \min \{i^+(-\omega_0 q_0 - \omega_1 q_1) | (\omega, c) \in S^1, \langle \omega, c \rangle < 0\}.
\]

Then we have

(i) \( A_c = \emptyset \iff \nu_c = 0 \);

(ii) if \( \nu_c \neq 0 \), then \( b_k(A_c) = b_0(C_{k+1}, C_k) + b_1(C_{k+2}, C_{k+1}) \) for \( 0 \leq k \leq n \).

**Proof.** First notice that the condition \( \nu_c = 0 \) is equivalent to

\[
\exists \eta \in q(\mathbb{R}^n)^o \quad \text{s.t.} \quad \langle \eta, c \rangle > 0.
\]

Suppose \( A_c = \emptyset \); then if \( \nu_c \neq 0 \) we would have \( \forall \eta \in q(\mathbb{R}^n)^o \) the inequality \( \langle \eta, c \rangle \leq 0 \), namely \( c \in q(\mathbb{R}^n)^{co} = q(\mathbb{R}^n) \) which is absurd - remember that \( q(\mathbb{R}^n) \) is a closed convex (polyhedral) cone by corollary [13]. On the contrary if \( A_c \neq \emptyset \), then \( c \in q(\mathbb{R}^n) \) and hence \( \{te \}_{t \geq 0} \supset q(\mathbb{R}^n)^o \); thus \( \langle \eta, c \rangle \leq 0 \) for every \( \eta \in q(\mathbb{R}^n)^o \). This proves part (i).

For part (ii) we are substantially going to prove that

\[
\hat{E}_2^j(\epsilon) = H^i(C\hat{\Omega}, \hat{\Omega}^{j+1}(\epsilon)) \simeq H^i(C_{n-j+1}, C_{n-j})
\]

for small \( \epsilon \) and that if \( A_c \neq \emptyset \), then \( \hat{E}_2(\epsilon) \simeq \hat{E}_{\infty}(\epsilon) \).

Notice also that for \( i \geq 1 \) we have

\[
H^i(C\hat{\Omega}, \hat{\Omega}^{j+1}(\epsilon)) \simeq H^{i-1}(\hat{\Omega}^{j+1}(\epsilon)).
\]

Set \( \hat{\Omega}(\epsilon) = \{(\omega, t) \in \hat{\Omega} | \langle (c, \epsilon), (\omega, t) \rangle \geq 0 \} \) and \( \hat{\Omega}(\epsilon) = \{(\omega, t) \in \hat{\Omega} | \langle (c, \epsilon), (\omega, t) \rangle \leq 0 \} \). Notice that if \( (\omega, t) \in \hat{\Omega}(\epsilon) \cap \hat{\Omega}(\epsilon) \) for \( k \leq n \) then for every \( t' \geq t \) we have \( (\omega, t') \in \hat{\Omega}(\epsilon) \cap \hat{\Omega}(\epsilon) \). Define \( \Omega(\epsilon) = \partial\hat{\Omega}(\epsilon) \sim \Omega \) and \( \Omega(\epsilon) = \{(\omega, t) \in \Omega(\epsilon) | i^+(\omega, t) \geq k \} \). Then since \( i^+(0, 1) = n \), for \( k \leq n \) we have

\[
\hat{\Omega}(\epsilon) \sim (\hat{\Omega}(\epsilon) \cup \Omega(\epsilon)) \cup C\Omega(\epsilon).
\]

Thus we derive the following chain of isomorphisms:

\[
H^i(\hat{\Omega}(\epsilon)) \simeq H^i((\hat{\Omega}(\epsilon) \cup \Omega(\epsilon)) \cup C\Omega(\epsilon)) \simeq H^i(\hat{\Omega}(\epsilon) \cup \Omega(\epsilon)), \Omega(\epsilon)).
\]

We define now the set \( \Omega_{\geq}(\epsilon) = \Omega(\epsilon) \cap \hat{\Omega}(\epsilon) \subset \Omega(\epsilon) \) and notice that its closure is contained in the interior of \( \Omega(\epsilon) \); thus we can apply the excision theorem and get:

\[
\hat{H}^i(\Omega(\epsilon)) \simeq H^i((\hat{\Omega}(\epsilon) \cup \Omega(\epsilon)) \cup \hat{\Omega}(\epsilon), \Omega(\epsilon), \Omega(\epsilon) \cup \hat{\Omega}(\epsilon))
\]

If we denote by \( \hat{\Omega}(\epsilon) \) the set \( \hat{\Omega}(\epsilon) \setminus \{(\omega, t) \geq 0 \} \) we finally have the isomorphism:

\[
H^i(\hat{\Omega}(\epsilon)) \simeq H^i(\hat{\Omega}(\epsilon), \hat{\Omega}(\epsilon) \cup \partial\hat{\Omega}(\epsilon)).
\]

Now we consider the set \( C = \{(\omega, c) < 0 \} \) and the function \( \theta : C \to \mathbb{N} \) defined by

\[
\omega \mapsto i^+(\omega Q - \langle c, \omega \rangle I).
\]

We call \( C(\epsilon) \) the set \( \{\theta(\epsilon) \geq k \} \) and notice that for \( \epsilon \) small we have isomorphisms:

\[
H^i(\hat{\Omega}(\epsilon), \hat{\Omega}(\epsilon) \cup \partial\hat{\Omega}(\epsilon)) \simeq H^i(C(\epsilon), C(\epsilon)).
\]
Since for $\epsilon_1 \leq \epsilon_2$ we have $C^{k-1}(\epsilon_1) \subset C^{k-1}(\epsilon_2)$, then for small $\epsilon > 0$
$$\hat{H}^k(C^{k-1}(\epsilon), C^k(\epsilon)) \simeq \lim_{\epsilon \to 0} \hat{H}^k(C^{k-1}(\epsilon), C^k(\epsilon)).$$
Moreover $\bigcap_{\epsilon} C^k(\epsilon) = \{ \omega \in C | i^- (\omega) \leq n - k \}$ (notice that $i^- (\omega) = i^+ (-\omega)$) and thus setting $C_{i} = \{ \omega \in C | i^- (\omega) \leq i \}$ we finally end up with
$$\hat{E}_{k,i}^{1,j}(\epsilon) \simeq H^{i-1}(\mathbb{R}^n, C_{i-j+1}, C_{i-j}) \quad i \geq 1, \epsilon > 0 \text{ small}.$$ We have $\max_1 i^+ \geq n$ and thus $\hat{E}_{k,i}^0(j) = 0$ for $j \leq n - 1$ and small $\epsilon$; on the other side if $A_c \neq \emptyset$ then by theorem 13 we must have $\hat{E}_{k,i}^0(\epsilon) = \mathbb{Z}_2$ for small $\epsilon$ and the only possibly nonzero differential is $d_{2}(\epsilon)^{0,n} : \mathbb{Z}_2 \to \hat{E}_{k,i}^{-1}(\epsilon)$.

As an easy corollary we get the following.

**Corollary 16.** Let $q$ be in $Q(n,2)$. Then $q(\mathbb{R}^n) = \{ tv | v \in -\Omega_0^* \}$.

**Proof.** By property (i) of theorem 13 we have
$$q(\mathbb{R}^n) = \{ c \in \mathbb{R}^n | \nu_c \neq 0 \} = \{ c \in \mathbb{R}^n | c \in -\{ i^- \leq 0 \} \},$$
where clearly $\{ i^- \leq 0 \}$ is a convex cone. \hfill $\square$

**Remark 2.** The statement of the previous theorem still holds for systems of inequalities: if $A = (q_0 \leq c_0, q_1 \leq c_1)$ then $A = q_{c,1}(K)$ for $q_c = (q_0 - c_0, q_1 - c_1)$ and $K$ a certain cone and the result is the same by setting $C_k = \{ \omega \in \Omega | K = S^1 | \langle \omega, c \rangle < 0, i^+ (-\omega) \leq k \}$.

## 6. Infinite Dimensional Case

We consider here the case $H$ is a Hilbert space and $q_0, q_1$ are continuous quadratic forms on on $H$:
$$q_i(x) = \langle x, Q_i x \rangle \quad Q_i \text{ is linear, continuous and selfadjoint.}$$

In this case we easily prove the following generalization of theorem 13.

**Theorem 17.** Let $q_0, q_1$ be two quadratic forms on $H$ and $q : H \to \mathbb{R}^2$ the map $x \mapsto (q_0(x), q_1(x))$. Then $q(H)$ is a convex subset of $\mathbb{R}^2$, but not necessarily closed.

**Proof.** Let $a = q(\alpha)$ and $b = q(\beta)$ be in the image of $q$. Consider $V = \text{span}(\alpha, \beta)$; then $q|_V (V)$ is convex by theorem 13 and thus for every $t \in [0,1]$ there exists $v_t \in V \subset H$ such that $ta + (1-t)b = q|_V (v_t) = q(v_t)$. \hfill $\square$

If $c = (c_0, c_1) \in \mathbb{R}^2$ we are interested in the set
$$A_c = \{ x \in H | q_0(x) = c_0, q_1(x) = c_1 \}$$
with its induced topology from $A_c \subset H$. Without any regularity assumption the set $A_c$ can be very wild, but we can however attach to it some algebraic invariant, namely
$$\mathcal{H}_+(A_c) \doteq \lim_{V \to H} \{ \mathcal{H}_+(A_c \cap V) \},$$
where \( \mathcal{F} = \{ V \subset H \mid V \text{ finite dimensional subspace of } H \} \), and then give conditions for which \( \mathcal{H}_s(A_c) \) coincides with \( \tilde{H}_s(A_c) \). We recall the definition of positive inertia index for a quadratic form \( q \) on \( H \):

\[
\text{ind}^+(q) = \max \{ \dim(V) \mid V \subset H, q|_V > 0 \}
\]

and we define also, using the notation of the previous section,

\[
\mathcal{C} = \{ \omega \in C \mid i^+(-\omega) < \infty \} \quad \text{and} \quad \mathcal{C}_k = C_k \cap \mathcal{C}.
\]

The set \( \mathcal{C} \) happens to be a convex subset of \( C \), but the subsets \( \mathcal{C}_k \) are not in general euclidean neighborhood retracts and thus their Cech cohomology may not coincide with their singular cohomology.

**Lemma 18.** If \( A_c = \emptyset \) then \( \mathcal{H}_s(A_c) = 0 \). If \( A_c \neq \emptyset \) then

\[
\mathcal{H}_k(A_c) = \tilde{H}^0(C_{k+1}, C_k) \oplus \tilde{H}^1(C_{k+2}, C_{k+1})
\]

**Proof.** If \( A_c = \emptyset \) then clearly for every \( V \subset H \) we have \( A_c \cap V = \emptyset \) and \( \mathcal{H}_s(A_c \cap V) = 0 \) which implies \( \mathcal{H}_s(A_c) = 0 \).

On the contrary if \( A_c \neq \emptyset \), then setting \( C_k(W) = \{ \omega \in C \mid \text{ind}^-(\omega q|_W) \leq k \} \) for \( V \subset W \) subspaces we have \( C_k(W) \xrightarrow{\iota^W} C_k(V) \). We refer to [1] for the proof that \( \mathcal{H}_s(A_c \cap V) \to \mathcal{H}_s(A_c \cap W) \) induces on the graded complex associated to spectral sequence of theorem [1] the maps

\[
H^*(C_{k+1}(V), C_k(V)) \xrightarrow{\iota_{\iota}^W} H^*(C_{k+1}(W), C_k(W)).
\]

It follows from the properties of Cech cohomology that

\[
\lim_{\longrightarrow} \{ H^*(C_{k+1}(V), C_k(V)) \} = \tilde{H}^*(\bigcap_{V \in \mathcal{F}} C_{k+1}(V), \bigcap_{V \in \mathcal{F}} C_k(V))
\]

and since \( \bigcap_{V \in \mathcal{F}} C_k(V) = C_k \) then the conclusion follows. \( \square \)

Notice that the proof of part (i) of theorem [1] here does not apply, because in general \( q(H) \) is not closed and hence \( q(H)^{\circ\circ} \) can be different from \( q(H) \). The following proposition gives a sufficient condition for \( \mathcal{H}_s(A_c) \simeq \tilde{H}_s(A_c) \).

**Proposition 19.** Suppose \( c = (c_0, c_1) \in \mathbb{R}^2 \) is a regular value for the homogeneous quadratic map \( q : H \to \mathbb{R}^2 \). Then

\[
\mathcal{H}_s(A_c) \simeq \tilde{H}_s(A_c).
\]

**Proof.** We give only a sketch; for details the reader is advised to see [1]. If \( c \) is a regular value, then \( A_c \) is a Hilbert submanifold of \( H \) and has a tubular neighborhood \( U_c \). Thus \( \mathcal{H}_s(U_c) \simeq \tilde{H}_s(A_c) \) and any singular chain in \( A_c \) can be turned in a chain lying in a finite dimensional subspace of \( H \) without leaving \( U_c \). The conclusion follows. \( \square \)

In the case \( c = 0 \), then \( A_0 \) is contractible and is possible to study the topology of \( A_0 \cap \{ x \in H \mid \|x\| = 1 \} \) in a similar way; for a precise treatment in the nondegenerate case the reader is referred again to [1].
7. Nondegenerate pencils

We introduce here for a given \( q \in Q(m, 2) \) a nondegeneracy condition with respect to a cone \( K \subset \mathbb{R}^2 \). The set of nondegenerate maps has a very rich structure, which we discuss here; starting from this we will derive some estimates also for the general case.

The following definition was given in [1].

**Definition 20.** Let \( K \subset \mathbb{R}^2 \) be a closed polyhedral cone. We say that \( q \in Q(m, 2) \) is degenerate with respect to \( K \) if there exists \( \omega \in K^2 \setminus \{0\} \) and \( x \in \mathbb{R}^m \setminus \{0\} \) such that \( q(x) \in K \) and \( x \in \ker(\omega q) \).

We denote with the symbol \( Q(m, 2; K) \) the set of \( q \in Q(m, 2) \) that are nondegenerate with respect to \( K \); it is an open subset of \( Q(m, 2) \) whose complement (the set of degenerate maps) is closed and semialgebraic. If \( q \in Q(m, 2; K) \), then \( Y(q) = \{ x \in S^{m-1} \mid q(x) \in K \} \) and \( X(q) = \{ [x] \in \mathbb{RP}^{m-1} \mid q(x) \in K \} \) are topological submanifolds (eventually with boundary) respectively of \( S^{m-1} \) and \( \mathbb{RP}^{m-1} \); moreover we have the following result.

**Lemma 21.** Let \( q_t, t \in [0, 1] \) be a homotopy of maps such that \( q_t \in Q(m, 2; K) \) for every \( t \in [0, 1] \); then \( Y(q_0) \) and \( Y(q_1) \) are homotopy equivalent. The same holds true for \( X(q_0) \) and \( X(q_1) \).

For the proof of the previous statements the reader is referred to [1] or [14]. Following Lemma 21 if \( q_0 \) and \( q_1 \) are in the same connected component of \( Q(m, 2; K) \) then we will say they are \( K \)-homotopic (there is a curve joining them).

From now on for this section we assume

\[
K = \{ x_0 \leq 0, x_1 \leq 0 \}.
\]

Thus, according to the previous notation, if \( q = (q_0, q_1) \), we have

\[
Y(q) = \begin{cases} 
q_0(x) \leq 0 \\
q_1(x) \leq 0
\end{cases}
\]

We begin by studying \( K \)-homotopy classes in \( Q(2, 2; K) \).

**Lemma 22.** If \( q \in Q(2, 2; K) \) then it is \( K \)-homotopic to \( p : \mathbb{R}^2 \to \mathbb{R}^2 \simeq \mathbb{C} \)

\[
(x_0, x_1) \mapsto x_0^2 e^{i\theta_0} + x_1^2 e^{i\theta_1}.
\]

such that \( \theta_0 \neq \pm \theta_1 \) and \( \theta_1, \theta_0 \neq k\pi/2, k \in \mathbb{Z} \).

**Proof.** Consider the following equation for \( [\omega] \in \mathbb{RP}^1 \):

\[
det(\omega p) = 0
\]

and let \( \Delta : Q(2, 2) \to \mathbb{R} \) its discriminant. Then \( \Delta \) is a polynomial function not identically zero and \( \{ \Delta(p) = 0 \} \) is a proper algebraic subset of \( Q(2, 2) \); since \( Q(2, 2; K) \) is open, we may assume \( \Delta(q) \neq 0 \).

If \( \Delta(q) > 0 \) then there are two noncollinear roots \([\omega_0] \) and \([\omega_1] \) in \( \mathbb{RP}^1 \).

This means that the image of \( \pi : \omega \mapsto \omega q \) intersects the set \( Z \) of degenerate quadratic forms in two distinct lines.

Since \( \det(\omega_j Q) = 0 \), for \( j = 0, 1 \), then there exist \( x_0 \) and \( x_1 \) in \( \mathbb{R}^2 \) different from zero and such that

\[
x_0^T (\omega_0 Q) = 0, \quad x_1^T (\omega_1 Q) = 0.
\]
Moreover, since $\omega_0$ and $\omega_1$ are linearly independent, then
$$\omega_0(x_0^TQx_1) = \omega_1(x_0^TQx_1) = 0$$
It follows that
$$x_0^TQx_1 = 0.$$ 
Moreover if $x_0$ and $x_1$ were collinear, then writing $\eta \in K^2 \setminus \{0\}$ as a linear combination of $\omega_0$ and $\omega_1$,
$$\eta = c_0\omega_0 + c_1\omega_1,$$
we would have
$$x_0^T(\eta Q) = x_0^T(c_0\omega_0 + c_1\omega_1)Q = x_0^T(c_0\omega_0Q) + x_0^T(c_1\omega_1Q) = 0$$
against the nondegeneracy hypothesis on $q$.
The condition $x_0^T(\omega_j Q) = 0$ tells that the quadratic form $\omega_j q$ restricted to $\{\lambda x_j\}$ is zero; nevertheless $\omega_j q$ is not identically zero: if for example it was $(\omega_1 q)(x) = 0$ for every $x \in \mathbb{R}^2$, then in coordinates $(\omega_0, \omega_1)$ we would have
$$q(x) = q(x_0, x_1) = (ax_0^2, 0), \quad Jq(x_0, x_1) = \begin{pmatrix} 2ax_0 & 0 \\ 0 & 0 \end{pmatrix}$$
and for every $\lambda \neq 0$ it would be $q(0, \lambda) = 0 \in K$ and $Jq(0, \lambda) \equiv 0$, against the nondegeneracy assumption (here $Jq(x_0, x_1)$ denotes the Jacobian matrix of $q$ at $(x_0, x_1)$).
Thus $q : \mathbb{R}^2 \to \mathbb{R}^2 \simeq \mathbb{C}$ is of the form
$$q(x) = \langle a_0, x \rangle^2 e^{i\theta_0} + \langle a_1, x \rangle^2 e^{i\theta_1}$$
with $\theta_0 \neq \pm \theta_1$ (since $\Delta(q) \neq 0$) and $a_0, a_1 \in \mathbb{R}^2$ such that
$$\langle a_0, x_1 \rangle = \langle a_1, x_0 \rangle = 0 \quad \text{and} \quad q(\lambda x_j) = \lambda^2 \langle a_j, x_j \rangle^2 e^{i\theta_j} \quad \text{for } j = 0, 1.$$ 
The nondegeneracy condition implies none of $e^{i\theta_0}$ and $e^{i\theta_1}$ is a generator of $K$ and thus slightly perturbing (which does not affect nondegeneracy) we obtain $e^{i\theta_j} \neq k\pi/2, k \in \mathbb{Z}$. We can clearly change $a_0$ and $a_1$ through $K$-homotopies as to arrive to $p$.
If $\Delta(q) < 0$ there are no real roots of the previous equation: $[\omega_0], [\omega_1] \in \mathbb{CP}^1$; moreover since the coefficients of the equations are real, then $[\omega_0] = [\overline{\omega_1}]$. In this case the nonexistence of real roots guarantees automatically nondegeneracy. We exhibit now a $K$-homotopy between $q$ and a map with positive discriminant. First notice that we have $\det(\omega q) \neq 0$ for every $\omega \neq 0$ and thus $d\eta_{\mathbb{R}^2 \setminus \{0\}}$ is surjective; moreover for every $\eta \neq 0$ we have $i^+(\eta q) = 1$. Thus let $\eta \in \text{int}(K)$ and $e^\perp$ orthogonal to $e$. In coordinates $(e, e^\perp)$ we have
$$q(x) = (\langle e, q(x) \rangle, \langle e^\perp, q(x) \rangle).$$
Diagonalizing the first component we find a basis $(y_0, y_1)$ of $\mathbb{R}^2$ such that in coordinates we have
$$q(x) = (x_0^2 - x_1^2, ax_0^2 + bx_1^2 + c x_0 x_1).$$
We define the homotopy $q_t$ through the equation:
$$q_t(x) = (x_0^2 - x_1^2, t(ax_0^2 + bx_1^2 + c x_0 x_1)).$$
Naturally we have
$$Jq_t(x_0, x_1) = \begin{pmatrix} 2x_0 & 2tax_0 + tcx_1 \\ -2x_1 & 2tx_1 + tcx_0 \end{pmatrix}$$
\[ \det(J q_t(x_0, x_1)) = t \det(J q_1(x_0, x_1)) \]

and thus \( q_t \) is nondegenerate for every \( t \): for \( t \neq 0 \) the differential of \( q_t : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^2 \) is surjective; for \( t = 0 \) we have \( \Delta(q_0) = 0 \) but the choice of \( e \) guarantees nondegeneracy. Thus after this homotopy \( q \) will be of the form

\[ q_0(x) = (a_0, x)^2 e^{i\theta} + (b_0, x)^2 e^{-i\theta}, \]

with \( e^{i\theta} = \lambda^2 e \) and \( a_0 \) and \( b_0 \) nonzero; a small rotation of one of the two vectors \( e^{i\theta} \) or \( e^{-i\theta} \) gives the \( K \)-homotopy between \( q_0 \) and a map with positive discriminant, to which the previous part applies. \( \Box \)

Using the previous lemma we can attach to each \( q \in Q(2, 2; K) \) a word \( s(q) \) of three characters from the sets \( \{\omega, \hat{\omega}, z\} \) in the following way. First notice that by assumption on \( K \) we have

\[ K^0 = \text{cone}\{z = (0^1), w = (1_0)\}. \]

Let \( p \) be given by the previous lemma, and consider the solutions \( \omega_0, \ldots, \omega_3 \) on the unit circle of the equation \( \det(\omega p) = 0 \) (these solutions occurs in opposite pairs). These points on the circle are the points where the index function \( i^+ \) for \( p \) can changes its value; outside of them \( i^+ \) must be locally constant. If we move clockwise from \( w \) to \( -w \) on the circle we meet exactly two of these points: this is because the previous lemma implies no one of them belongs to \( \{z, w, -w\} \). Each time we meet one of them we label it \( \hat{\omega} \) or \( \omega \) according to the fact that the index function \( i^+ \) increases or decreases after passing it going clockwise. We define \( s(q) \) to be the word obtained writing the letters of the points we meet going from \( -w \) to \( w \) clockwise. Thus in a certain sense \( s(q) \) is the “derivative” of the index function: it takes into account the signed jumps of \( i^+ \).

Twelve possibilities can happen and we partition them into four disjoint subsets (the reason for this partition will become clear in a while):

1. \( [\omega \omega z] = \{\omega \omega z\} \)
2. \( [\omega \hat{\omega} z] = \{\omega \hat{\omega} z, \omega \hat{\omega} \omega, \hat{\omega} \omega z, \hat{\omega} \omega \omega\} \)
3. \( [\hat{\omega} \omega z] = \{\hat{\omega} \omega z, \hat{\omega} \omega \omega, \hat{\omega} \omega \hat{\omega}, \hat{\omega} \omega \hat{\omega} \hat{\omega} \hat{\omega}\} \)
4. \( [\hat{\omega} \hat{\omega} \hat{\omega}] = \{\hat{\omega} \hat{\omega} \hat{\omega}\} \).

For a given \( q \in Q(2, 2; K) \) we define

\[ \sigma(q) = [s(q)] \]

and prove the following result, which classify \( K \)-homotopy classes (i.e. connected components) of \( Q(2, 2; K) \).

**Theorem 23.** Two maps \( q_0, q_1 \in Q(2, 2; K) \) are \( K \)-homotopic if and only if

\[ \sigma(q_0) = \sigma(q_1). \]

In particular \( P(2, 2; K) \) has four connected components.

**Proof.** Notice first that the four cases we described correspond to the following situations:

1. \( e^{i\theta_0} \) and \( e^{i\theta_1} \) belong to \( \text{int}(K) \);
2. one among \( e^{i\theta_0} \) and \( e^{i\theta_1} \) belongs to \( \text{int}(K) \) and the other does not;
3. \( e^{i\theta_0} \) and \( e^{i\theta_1} \) do not belong to \( K \) and \( p(\mathbb{R}^2) \cap K = \{0\} \);
4. \( e^{i\theta_0} \) and \( e^{i\theta_1} \) do not belong to \( K \) and \( p(\mathbb{R}^2) \supset K \).
Clearly if $\sigma(q_0) = \sigma(q_1)$ then $q_0$ and $q_1$ are $K$-homotopic: first make a homotopy from $q_0$ to the map $p_0$ given by lemma 22. Then rotating the vectors $e^{i\theta_0}$ and $e^{i\theta_1}$ gives a homotopy between $p_0$ and $p_1$, where $p_1$ comes from the lemma applied to $q_1$; this homotopy is a $K$-homotopy because $\sigma(q_0) = \sigma(q_1)$ (the reader can check it simply drawing a picture). Finally perform the homotopy from $p_1$ to $q_1$.

On the contrary if $q_0$ and $q_1$ are $K$-homotopic, then also $p_0$ and $p_1$ are $K$-homotopic. If $\sigma(q_0) \neq \sigma(q_1)$, then the homotopy joining $p_0$ and $p_1$ must have zero discriminant at a certain point $p_s$, $s \in [0, 1]$. Let $\overline{p}_s : \mathbb{R}^2 \to Q(\mathbb{R}^2) \simeq \mathbb{R}^3$ be the map $\omega \mapsto \omega p_s$; then $\overline{p}_s$ is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$. Since the set of linear maps $L : \mathbb{R}^2 \to \mathbb{R}^3$ with rank less than or equal to one has codimension two, then we may assume $\text{rk}(\overline{p}_s) = 2$; if there is a $K$-homotopy $p_t$ between $p_0$ and $p_1$ then there also is a $K$-homotopy avoiding the codimension two set of maps with not maximal rank.

The nondegeneracy condition of $p_s$ traduces in the following condition for the linear map $\overline{p}_s$ (see [1]):

$$\forall \eta \in K^\circ \backslash \{0\}, \forall y \in \ker(qp_s) \backslash \{0\} \ \exists v \in T_y K^\circ \ \text{s.t.} \ (d_\eta \overline{p}_s)(v) > 0.$$ 

Thus if we set $Z = \{ q \in Q(\mathbb{R}^2) | \det(q) = 0 \}$, then we have $\overline{p}_s(\mathbb{R}^2)$ intersects $Z$ in a line $l$. Now in principle three possibilities can happen: (i) $\overline{p}_s(K^\circ) \cap l \subset \text{int}(K^\circ)$, in which case $p_s$ would be degenerate with respect to $K$; (ii) $\overline{p}_s(K^\circ) \cap l \subset \partial K^\circ$, a case which has codimension at least two and thus that can be avoided; (iii) $\overline{p}_s(K^\circ) \cap l = \{0\}$, in which case $p_s$ is nondegenerate with respect to $K$.

Thus if the discriminant of $p_s$ vanishes performing a $K$-homotopy between maps $p_0, p_1$ with positive discriminant, then it can happen only in the way described in lemma 22 and thus, recalling the proof of this lemma, we have $\sigma(p_0) = \sigma(p_1)$ which concludes the proof.

We move now to the classification of $K$-homotopy classes in $Q(m, 2; K)$. We adopt the following convention: if $q_1, \ldots, q_k$ are quadratic maps with $q_j \in Q(n_j, 2)$ for $j = 1, \ldots, k$, we define the quadratic map $q_1 \oplus \cdots \oplus q_k = q \in Q(n_1 + \cdots + n_k, 2)$ by the formula

$$q(x) = q(x_1, \ldots, x_k) = \sum_{j=1}^k q_j(x_j).$$

The following is a classical result.

**Lemma 24.** Let $q \in Q(m, 2)$ such that $\Delta(q) \neq 0$. Then there exist $q_j \in Q(2, 2)$ for $j = 1, \ldots, l$ and $p_k \in Q(1, 2)$ for $k = 1, \ldots, b$ such that $2l + b = m$ and

$$q = (\bigoplus_{j=1}^l q_j) \oplus (\bigoplus_{k=1}^b p_k).$$

**Proof.** See [1]

In particular lemma 24 implies that if $q \in Q(m, 2; K)$ then each $q_j$ must belong to $Q(2, 2; K)$ and each $p_k$ to $Q(1, 2; K)$.

For our purpose we need the following lemma.

**Lemma 25.** Let $q_0 \in Q(m, 2; K)$ such that $q_0 = s_0 \oplus r$ with $r \in Q(m - 2, 2; K)$, $s \in Q(2, 2; K)$ and $\Delta(s_0) < 0$.

Then $q_0$ is $K$-homotopic to a map $q_1 = s_1 \oplus r$ such that $\sigma(s_1) = [\omega \omega_2]$ and $\Delta(s_1) > 0$. 

\[ \square \]
Proof. Consider the $K$-homotopy $s_t$ we built when we proved that $\Delta(s_0) < 0$ then $\sigma(s_0) = [\omega \bar{\omega} z]$ and stop this homotopy once we reach a map $\tilde{s} = s_T$ with zero discriminant. Thus suppose we have a $K$-homotopy $s_t$ between $s_0$ and $\tilde{s}$. We define $q_t = s_t \oplus r$; then we have $q_t(x) = (x^T Q_1(t)x, x^T Q_2(t)x)$ with

$$Q_j(t) = \begin{pmatrix} S_j(t) & 0 \\ 0 & R_j \end{pmatrix} \quad j = 1, 2$$

If $w = (w_1, w_2)$, then

$$w Q(t) = w_1 Q_1(t) + w_2 Q_2(t) = \begin{pmatrix} w_1 S_1(t) + w_2 S_2(t) & 0 \\ 0 & w_1 R_1 + w_2 R_2 \end{pmatrix}$$

Suppose there exists $\tau \in (0, T)$ such that $q_T$ is degenerate with respect to $K$; then there would exist a nonzero vector $x = (x_s, x_r) \in \mathbb{R}^2 \times \mathbb{R}^{m-2}$ and a covector $w \in K^\circ \setminus \{0\}$ such that $q_T(x) = K$ and $w(dq_T)x = 0$. Since $(dq_T)x = x^T (\omega Q(\tau))$ then $x_s = 0$, because for $x_s \neq 0$ the linear map $w(dx_s) = x_s^T (w_1 S_1(\tau) + w_2 S_2(\tau))$ is nonzero; thus $r(x_r) = q(x) \in K$ and $x_r^T (w_1 R_1 + w_2 R_2) = 0$ against the fact that $r$ is nondegenerate with respect to $K$. Thus we showed that for $t \neq T$ the map $q_t$ is nondegenerate with respect to $K$.

On the other side for $t = T$ we have $(dq_T)(x_s, x_r) = (ds_T)x_s, P_s + dx_r, P_r$, where $P_s$ and $P_r$ are the projections on the subspace respectively of the first 2 coordinates and the remaining $m - 2$.

Thus suppose $(x_s, x_r) \neq (0, 0)$ and $q_T(x_s, x_r) \in K$. Then two cases can happen: $x_s \neq 0$ and $x_s = 0$. If $x_s \neq 0$ then no supporting hyperplane for $K$ contains the image of the differential $(dq_T)(x_s, x_r)$ because no supporting hyperplane for $K$ contains the image of the differential $(ds_T)x_s$; if $x_s = 0$, since $r$ is nondegenerate with respect to $K$, then no supporting hyperplane of $K$ contains the image of the differential $(dq_T)(0, x_r) = dx_r$. Thus in both cases $q_T$ is nondegenerate with respect to $K$.

Let now $\{ s_n \}_{n>1} \subset Q(2, 2; K)$ be a sequence of maps such that for every $n$ we have $\sigma(s_n) = [\omega \bar{\omega} z], \Delta(s_n) > 0$ and $s_n \to s_T$.

If we define $q_n = s_n \oplus r$, then clearly $q_n \to q_T$. Since $Q(m, 2; K)$ is open in $Q(n, 2)$ and $q_T$ is nondegenerate with respect to $K$, then there exists $\pi$ such that $q_{\pi}$ is nondegenerate with respect to $K$ and $q_{\pi}$ is $K$-homotopic to $q_T$.

Let finally $s_1 = s_{\pi}$, $q_1 = s_1 \oplus r = q_{\pi}$ and $q_t$ be the composition of the two $K$-homotopies from $q_0$ to $q_T$ and from $q_T$ to $q_{\pi}$. Then $\sigma(s_1) = [\omega \bar{\omega} z], \Delta(s_1) > 0$ and $q_T$ is the required $K$-homotopy. $\square$

We describe now a procedure to associate to each $q \in Q(m, 2; K)$ a word of $m+1$ letters on the set of characters $\{ \omega, \bar{\omega}, z \}$.

Again let $\Delta : Q(m, 2) \to \mathbb{R}$ the discriminant of the equation $\det(\omega p) = 0$ : it is a polynomial function and $\{ \Delta(p) = 0 \}$ is a proper algebraic set; hence $q$ is $K$-homotopic to $q'$ with $\Delta(q') \neq 0$. Applying lemma 24 we get that $q$ is $K$-homotopic to a map of the form $(\oplus q_j) \oplus (\oplus p_k)$ with each $q_j$ and each $p_k$ nondegenerate with respect to $K$. Lemma 24 allows now to change each $q_j$ with $\Delta(q_j) < 0$ in a $q'_j$ with $\Delta(q'_j) > 0$ without losing nondegeneracy w.r.t. $K$. Thus there exist $e^{i \theta_1}, \ldots, e^{i \theta_m}$ such that $q$, up to $K$-homotopies, is of the form:

$$q(x) = q(x_1, \ldots, x_m) = \sum_{j=1}^m e^{i \theta_j} x_j^2$$
Slightly perturbing the $e^{i\theta_j}$'s (which does not affect nondegeneracy w.r.t. $K$) we may assume $\theta_i \neq \pm \theta_j$ for $i \neq j$ and $\theta_j \neq k\pi/2$ for $k \in \mathbb{Z}$ and $j = 1, \ldots, m$.

If we consider the index function $i^+$ for such a $q$, then we can mark the points where it jumps by $\hat{\omega}$ if the jump is positive and by $\omega$ is the jump is negative; exactly as we did for the case $q \in Q(2, 2; K)$ we associate a word $s(q)$ to $q$ by writing the characters we meet by moving clockwise on the circle from $w$ to $-w$.

Again the word we obtained can be thought as the “derivative” of $i^+$ (as it keeps track of the jumps of this function) and we can obtain the value of $i^+$ by “integrating” this word; for the moment this concept is purely intuitive, but we will discuss this later.

We notice that a lot of possibilities can happen and we introduce the following rules to change one word into another:

**Rules.**

(A) $s_1\hat{\omega}s_2 = s_1\hat{\omega}s_2$: we can commute $\hat{\omega}$ and $z$;
(B) $s\omega = \hat{\omega}s$ for every word $s$ with characters in $\{z, \hat{\omega}, \omega\}$: if $\hat{\omega}$ is the last character of one word, we can cancel it and place $\hat{\omega}$ at the beginning of the word as the first character;
(C) $s_1\hat{\omega}s_2\omega s_3\hat{\omega}s_4 = s_1\hat{\omega}s_2\omega s_3\hat{\omega}s_4$ for every choice of words $s_1, s_2, s_3, s_4$ with characters in $\{\omega, \hat{\omega}\}$: we can commute $\hat{\omega}$ and $\omega$ to the left of $z$.

We will see that each rule correspond to a precise $K$-homotopy between two quadratic maps and that the previous are exactly the $K$-homotopies we can perform.

In view of this idea we give the following definition.

**Definition 26.** We define $S(m, 2; K)$ to be the set of equivalence classes of words of maps $q \in Q(m, 2; K)$ under the relation that two words are equivalent if and only if we can change one into the other with the previous rules. We let $\sigma(q)$ be the class of $s(q)$ in $S(m, 2; K)$.

Before proving the main theorem of this section, we first prove one useful lemma. If $q \in Q(m, 2)$ is given by

$$q(x) = q(x_1, \ldots, x_n) = \sum_{j=1}^{n} e^{i\theta_j} x_j^2$$

then for every pair of distinct indices $(a, b)$ we define $q_{ab} \in Q(2, 2)$ by

$$q_{ab}(x_a, x_b) = e^{i\theta_a} x_a^2 + e^{i\theta_b} x_b^2.$$

**Lemma 27.** Let $q \in Q(m, 2)$ be defined by

$$q(x) = q(x_1, \ldots, x_m) = \sum_{j=1}^{m} e^{i\theta_j} x_j^2.$$

Then $q$ is nondegenerate with respect to $K$ if and only if $q_{ab}$ is nondegenerate w.r.t. $K$ for every pair of distinct indices $(a, b)$.

**Proof.** Clearly if $q$ is nondegenerate w.r.t. $K$ then for every pair $(a, b)$ of distinct indices $q_{ab}$ is nondegenerate w.r.t. $K$.

Viceversa suppose $q$ is degenerate w.r.t. $K$ and let us prove that there exists a pair of distinct indices $(a, b)$ such that $q_{ab}$ is degenerate w.r.t. $K$.

Degeneracy of $q$ implies that there exists a nonzero vector $x = (x_1, \ldots, x_m)$ and a covector $\omega \in K^m \setminus \{0\}$ such that $q(x) \in K$ and $\omega dq_x \equiv 0$. 


If all the components of $x$ but $x_j$ were zero, then for every $l \neq j$ we have $q_{lj}$ degenerate w.r.t. $K$. If $x$ has $k > 1$ nonzero components, the first $k$ for example, then since

$$dq_x = \sum_{j=1}^{k} 2x_j e^{i\theta_j} dx_j$$

all the vectors $e^{i\theta_1}, \ldots, e^{i\theta_k}$ must be collinear, otherwise the rank of $dq_x$ would be 2 (against the fact that there exists $\omega \in K^2 \setminus \{0\}$ such that $\omega dq_x \equiv 0$).

If $e^{i\theta_1} = e^{i\theta_2} = \cdots = e^{i\theta_k}$ then it must be $e^{i\theta_1} \in \partial K$ and thus $q_{12}$ is degenerate w.r.t. $K$; if among $e^{i\theta_1}, \ldots, e^{i\theta_k}$ there are two vectors with different signs, for example $e^{i\theta_1}$ and $e^{i\theta_2}$, then nondegeneracy of $q$ implies no one among $e^{i\theta_1}, \ldots, e^{i\theta_k}$ belongs to $\mathrm{int}(K)$; thus either one among them coincides with one generator of the cone $K$ or $q(x) = 0$ and thus $q_{12}(x_1, x_2) = 0 \in K$; in both cases $q_{12}$ is degenerate w.r.t. $K$.

Everything is ready now for the proof of the following theorem, which classifies $K$-homotopy classes of $Q(m, 2; K)$.

**Theorem 28.** The set $S(m, 2; K)$ classifies $K$-homotopy classes of $Q(m, 2; K)$: two maps $q_0, q_1 \in Q(m, 2; K)$ are $K$-homotopic if and only if

$$[s(q_0)] = [s(q_1)].$$

Moreover the sequence of rules we have to apply to change $s(q_0)$ to $s(q_1)$ describes one possible $K$-homotopy.

**Proof.** Thanks to lemma 27 if $q \in Q(m, 2; K)$ and we perform a rotation of the $e^{i\theta_j}$’s such that for every pair of distinct indices $(a, b)$ the map $q_{ab}$ is nondegenerate, then the result is a $K$-homotopy. Thus every rule corresponds to a precise $K$-homotopy and $\sigma(q_0) = \sigma(q_1)$ implies $q_0$ and $q_1$ are $K$-homotopic. Moreover from the proof of lemma 26 it follows that if $q = r + s$ with $s \in Q(2, 2)$ and $\Delta(s) < 0$ then $q$ is nondegenerate w.r.t. $K$ if and only if $r$ is; thus iterating the reasoning, if $q = v_1 \oplus \cdots \oplus v_k \oplus s_1 \oplus \cdots \oplus s_l$ with the $v_j$’s representing maps in $Q(1, 2; K)$ and the $s_j$’s maps in $Q(2, 2; K)$ with negative discriminant, then $q$ is nondegenerate w.r.t. $K$ if and only if $v_1 \oplus \cdots \oplus v_k$ is nondegenerate w.r.t. $K$. Moreover if

$$s(v_1 \oplus \cdots \oplus v_k) = u_1zu_2$$

with $u_1$ and $u_2$ words in $\{\omega, \hat{\omega}\}$, then we have

$$\sigma(q) = [s(v_1 \oplus \cdots \oplus v_k \oplus s_1 \oplus \cdots \oplus s_l)] = [(\omega \hat{\omega})^l u_1zu_2]$$

where by $(\omega \hat{\omega})^l$ we mean the word $\omega \hat{\omega}$ repeated $l$ times.

We prove now that if $q_0$ and $q_1$ are $K$-homotopic, then $\sigma(q_0) = \sigma(q_1)$.

First notice that we may assume $q_0$ and $q_1$ are in the form given by lemma 24. As before we may suppose the $K$-homotopy is generic (i.e. we can avoid sets of codimension grater or equal to two). To a given $q \in Q(m, 2)$ we can associate a linear map $\hat{q} : \mathbb{R}^2 \to Q(\mathbb{R}^m) \simeq \mathbb{R}^{m+1}$ by the correspondence $\omega \mapsto \omega q$. The set of linear maps $L : \mathbb{R}^2 \to Q(\mathbb{R}^m)$ with rank less or equal to one is an algebraic subset of codimension greater than one, hence it can be avoided (i.e. if there is a $K$-homotopy $q_t$ then there is also one with $\mathrm{rk}(q'_t) = 2$ for every $t$). The set of linear maps $L : \mathbb{R}^2 \to Q(\mathbb{R}^m)$ such that the image of $L$ is tangent to $Z = \{q \in Q(\mathbb{R}^m) \mid \det(q) = 0\}$ in at least two distinct lines has codimension greater than one,
hence can be avoided: generically a $K$-homotopy will meet \{\Delta(q) = 0\} only a finite number of time and in these cases only two roots of $\text{det}(\omega q) = 0$ will coincide. Let $A$ be the set of maps in $\mathcal{Q}(m, 2; K)$ with exactly two equal roots of the equation $\text{det}(\omega q) = 0$. Thus let $q_t$ be a generic $K$-homotopy (in particular $\Delta(q_1) \neq 0$ and $\Delta(q_2) \neq 0$).

It is sufficient to show that each time we meet $A$ the class of the word does not change. Suppose $q_{t_1} = v_1 \oplus \cdots \oplus v_k \oplus s_1 \oplus \cdots \oplus s_l$ for $t_1 < T$, $q_T \in A$ and $q_{t_2} = v_1' \oplus \cdots \oplus v'a' \oplus s_1' \oplus \cdots \oplus s'_b$ for $t_2 > T$, where the $v_j$'s and the $v'_j$'s represent maps in $\mathcal{Q}(1, 2; K)$ and the remaining $s'_j$'s and $s_j$'s are in $\mathcal{Q}(2, 2; K)$ and have negative discriminant. We adopt the convention that if there are no maps of a certain type, then the corresponding number in $\{k, l, a, b\}$ is zero. Assume between $t_1$ and $t_2$ the discriminant of $q_t$ vanishes only at $T$.

When $q_t$ meet $A$ two roots happen to coincide. These could be real before $T$ and real after, or real before $T$ and complex after or vice versa.

In the first case $q_{t_2} = v_1' \oplus \cdots \oplus v'_k \oplus s_1' \oplus \cdots \oplus s'_l$ and $\sigma(v_1 \oplus \cdots \oplus v_k) = \sigma(v'_1 \oplus \cdots \oplus v'_k)$ (we simply performed a rule); thus recalling what we stressed at the beginning of the proof, we have $\sigma(q_{t_1}) = \sigma(q_{t_2})$.

In the second case two real roots became switching $(t_1$ and $t_2$ we get the other case): then it must be $\sigma(q_{t_1}) = [(\omega \hat{\omega})^l u_1 z u_2]$ with $l > 1$ and $[u_1 z u_2] = \sigma(v_1 \oplus \cdots \oplus v_k)$. In this case $q_{t_2} = v_1' \oplus \cdots \oplus v'_{k-2} \oplus s_1' \oplus \cdots \oplus s'_{l+1}$ and thus $\sigma(q_{t_2}) = [(\omega \hat{\omega})^{l+1} u_1 z u_2']$ with $\sigma(v'_1 \oplus \cdots \oplus v'_{k-2}) = [u_1' z u_2']$. On the other side, assuming the last two roots became complex, then because of nondegeneracy they could have done it only in the way described in lemma \[22\].

Moreover from lemma \[27\] it follows that the $K$-homotopy between $q_{t_1} \in q_{t_2}$ induces a $K$-homotopy between $v_1 \oplus \cdots \oplus v_{k-2}$ and $v'_1 \oplus \cdots \oplus v'_{k-2}$. Since during this last homotopy the discriminant never vanishes, then $\sigma(v_1 \oplus \cdots \oplus v_{k-2}) = \sigma(v'_1 \oplus \cdots \oplus v'_{k-2})$ and thus $\sigma(q_{t_1}) = \sigma(q_{t_2})$.

This concludes the proof. \qed

We can choose a canonical representative for $[s(q)] \in \mathcal{S}(n, 2; K)$ and adopt the convention that $x^r$, with $r \in \mathbb{N}$, means that the character $x$ is repeated $r$ times. In this way we have that each $q \in \mathcal{Q}(m, 2; K)$ is $K$-homotopic to a map $q'$ of the form:

$$s(q') = \omega^a z \omega^b \omega^{c_1} \omega^{d_1} \cdots \omega^{c_r} \omega^{d_r}$$

with $a + b + \sum c_j + \sum d_j = m$.

Here is the “integration” method we mentioned to reconstruct the index function: if $\eta \in K^o \cap S^1$ and $\eta \neq \omega, \neq \hat{\omega}$:

$$\omega^a z \omega^b \omega^{c_1} \omega^{d_1} \cdots (\eta) \cdots \omega^{c_r} \omega^{d_r}$$

then we have

$$i^+(\eta) = \lambda(\eta) + \rho(\eta),$$

where $\lambda(\eta)$ is the number of $\hat{\omega}$ in $s(q)$ on the left of $\eta$ and $\rho(\eta)$ is the number of $\omega$ in $s(q)$ to the right of $\eta$.

Thus we can reduce the study of possibilities for the topology of $Y$ to the combinatorics of words as above; in particular we have the following estimate for $X = p(Y) \subset \mathbb{R}^n$, where $p : S^n \to \mathbb{R}^n$ is the covering map.

**Proposition 29.** Let $q : \mathbb{R}^{n+1} \to \mathbb{R}^2$ be non degenerate with respect to $K = \{x_0 \leq 0, x_1 \leq 0\}$ and $X = \{x \in \mathbb{R}^n | q_0(x) \leq 0, q_1(x) \leq 0\}$. Then for every $k \in \mathbb{N}$ we have

$$b_k(X) \leq k + 2.$$
Moreover, in the case $b_k(X) = k + 2$, then $\text{rk}(j_* : H_k(X) \to H_k(\mathbb{RP}^n)) = 1$.

**Proof.** We start by proving the inequality

$$b_0(\Omega^{n-k}) \leq k + 2$$

for the canonical representative $q'$ of a map $q \in \mathcal{Q}(n + 1, 2; K)$.

Assuming $b_0(\Omega^{n-k}) \geq 2$, we have that there exist $\eta_1, \eta_2 \in \Omega$ such that $i^+(\eta_1) = i^+(\eta_2) = n - k$ and the index function decreases and increases at least once between them; thus the word $s(q')$ must contain the string of characters $(\hat{\omega}\omega)^r$ for a certain $r > 0$ between $\eta_1$ and $\eta_2$. Since we are searching for the maximum of $b_0(\Omega^{n-k})$ this implies that the word for $q'$ must be the following:

$$s(q') = \omega^a z^{b}(\omega\hat{\omega})^r(\eta_2)$$

for certain $a, b, r \geq 0$, where the $\eta_2$ in parenthesis indicates the position of $\eta_2$ on $\Omega$. In particular we may assume $a = 0$ and since $i^+(\eta_2) = n - k$ we have $b + r = n - k$.

On the other hand $b + 2r = n + 1$; combined together we get $r = k + 1$ and $b = n - 2k + 1$. For such a choice we see that $b_0(\Omega^{n-k}) = k + 2$ and since it is maximal the inequality follows. Now, using theorem 9 we have that

$$b_k(X) = e^{0,n-k} + b_0(\Omega^{n-k}) - 1 \leq b_0(\Omega^{n-k}) \leq k + 2$$

where $e^{0,n-k} = \text{rk}(j_*)_k$; finally notice that in the first inequality we have equality if and only if $\text{rk}(j_*)_k = 1$.

\[\square\]

As a corollary, using the transfer exact sequence (which certainly exists since $X$ and $Y$ are semialgebraic compact) with $\mathbb{Z}_2$ coefficients for the double covering $Y \to p(Y)$ (see [12]) we have the following. In fact this sequence yields for every $k \geq 0$ the inequality $b_k(Y) \leq 2b_k(X)$; notice that this inequality is not sharp, as the case $S^n \to \mathbb{RP}^n$ shows.

**Proposition 30.** Let $q : \mathbb{R}^{n+1} \to \mathbb{R}^2$ be non degenerate with respect to $K = \{x_0 \leq 0, x_1 \leq 0\}$ and $Y = q^{-1}(K) \cap S^n$. Then for every $k \in \mathbb{N}$ we have

$$b_k(Y) \leq 2k + 4.$$

We deal now with the topological complexity of $X$. For $i^+ : \Omega \to \mathbb{N}$ we define the number

$$c(\Omega) = \sum_{j \geq 1} b(\Omega^j)$$

and we prove the following.

**Proposition 31.** Let $q$ be in $\mathcal{Q}(m, 2; K)$ such that

$$s(q) = \omega^{a_0} z^{b_0} \omega^{a_1} \cdots \omega^{a_r} \hat{\omega}^{b_r}.$$

Then the following formula holds:

$$c(\Omega) + a_0 = \sum_{i \geq 0} (a_i + b_i)$$

**Proof.** Let $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1$ and $\Omega_2$ two closed arcs with intersection $\Omega_1 \cap \Omega_2 = \{\eta\}$. The (semialgebraic) Mayer-Vietoris sequence gives for every $j$ the equality

$$b_0(\Omega^j) = b_0(\Omega_1^j) + b_0(\Omega_2^j) - b_0(\{\eta\}^j)$$
where we denote by $C^j$ the set $\Omega^j \cap C$; summing over $j \geq 1$ the previous equation gives:

$$c(\Omega) = c(\Omega_1) + c(\Omega_2) - i^+(\eta).$$

Let now $\Omega$ be decomposed in the two arcs $[z_1 \omega^b \eta]$ and $[\eta \omega^{a_1} \cdots \hat{\omega}^{b_j}]$; then, recalling the formula $i^+(\eta) = \bar{\lambda}(\eta) + \rho(\eta)$, the above computation gives

$$c(\Omega) = c([z_1 \omega^b \eta] + c([\eta \omega^{a_1} \cdots \hat{\omega}^{b_j}]) - i^+(\eta) = c([\eta \omega^{a_1} \cdots \hat{\omega}^{b_j}]).$$

On the other hand writing $[\eta \omega^{a_1} \cdots \hat{\omega}^{b_j}]$ as the union of the two arcs $[\eta \omega^{a_1} \eta']$ and $[\eta' \hat{\omega}^{b_1} \cdots \hat{\omega}^{b_j}]$ the same computation now gives:

$$c([\eta \omega^{a_1} \cdots \hat{\omega}^{b_j}]) = a_1 + c([\eta' \hat{\omega}^{b_1} \cdots \hat{\omega}^{b_j}]).$$

Repeating this procedure until we pass all the $\omega$’s we end up with:

$$c(\Omega) = \sum_{i \geq 1} a_i + i^+(\gamma) = \sum_{i \geq 1} a_i + \sum_{i \geq 0} b_i$$

where $\gamma \in \Omega$ is clockwise after $\hat{\omega}^{b_j}$ and again we used the formula $i^+(\gamma) = \bar{\lambda}(\gamma) + \rho(\gamma)$ for its index. This concludes the proof. \hfill \Box

As a corollary we get the following estimates.

**Corollary 32.** Let $q$ be in $Q(n + 1, 2; K)$; then the following sharp estimates hold

$$b(\mathbb{RP}^n \setminus X) \leq n + 1 \quad \text{and} \quad b(X) \leq n + 1.$$  

**Proof.** The first formula immediately follows from theorem [3]. For the second bound notice that $\mu = \sum_{i \geq 0} b_i$ and $\mu \geq \sum_{i \geq 1} a_i$; since theorem [9] implies

$$b(X) = c(\Omega) + n + 1 - 2\mu$$

the conclusion follows. The quadratic map $q$ such that $s(q) = z(\omega \hat{\omega})^l$ for $n + 1 = 2l$ and $s(q) = \omega z(\omega \hat{\omega})^l$ for $n + 1 = 2l + 1$ attains the maximum, as it is easily verified. \hfill \Box

In the next section we will see that these estimates hold also removing the assumption of nondegeneracy with respect to $K$.

**8. General bounds for systems of two quadratic inequalities**

We start by proving the following lemma.

**Lemma 33.** Consider $q \in Q(n + 1, 2)$ and $K = \{x_0 \leq 0, x_1 \leq 0\} \subset \mathbb{R}^2$; then there exists $q' \in Q(n + 1, 2; K)$ such that $Y(q) = q^{-1}(K) \cap S^n$ has the same homotopy type of $Y(q') = q'^{-1}(K) \cap S^n$. The same result holds true for $X(q) = p(Y(q))$ and $X(q') = p(Y(q'))$ defined as in the previous section.

**Proof.** If $q = (q_0, q_1)$ then $Y(q) = q^{-1}(K) \cap S^n$ coincides with the set of solutions of the following system:

$$\begin{cases} q_0(x) \leq 0 \\ q_1(x) \leq 0 \\ x_1^2 + \cdots + x_{n+1}^2 = 1 \end{cases}$$
By semialgebraicity the set of solutions of the previous system is a deformation retract, for small $\epsilon_0 > 0$ and $\epsilon_1 > 0$, of the set $Y_\epsilon(q)$ of the solutions of the following one:

$$
\begin{cases}
q_0(x) \leq \epsilon_0 \\
q_1(x) \leq \epsilon_1 \\
x_1^2 + \cdots + x_{n+1}^2 = 1
\end{cases}
$$

In other words $Y(q)$ has the same homotopy type of $Y_\epsilon(q)$.

To conclude the proof it is sufficient to show that there exists $q_\epsilon = q' \in \mathbb{Q}(n+1, 2; K)$ such that $Y(q_\epsilon) = Y_\epsilon(q)$.

Thanks to Sard’s Lemma we choose two real numbers $\epsilon_0$ and $\epsilon_1$ such that $(\epsilon_0, \epsilon_1)$ is a regular value of $q$, such that $\epsilon_i$ is not an eigenvalue of $Q_i$, $i = 0, 1$ and such that $Y(q_\epsilon)$ and $Y(q)$ are homotopically equivalent (this last condition is satisfiable since the set of $(\epsilon_0, \epsilon_1)$ satisfying the first two conditions is the complement of a one-dimensional semialgebraic set). In this way the quadratic map $q_\epsilon$, defined by

$$
q_\epsilon(x) = (q_0(x) - \epsilon_0\|x\|^2, q_1(x) - \epsilon_1\|x\|^2)
$$

is nondegenerate with respect to $K$.

The condition $(\epsilon_0, \epsilon_1)$ is a regular value of $q$ guarantees nondegeneracy at $\{0\}$, while the condition that $\epsilon_i$ is not an eigenvalue of $Q_i$, for $i = 0, 1$, guarantees nondegeneracy at $\partial K$.

The set $Y(q_\epsilon)$ coincides with the set of the solutions of

$$
\begin{cases}
q_0(x) - \epsilon_0\|x\|^2 \leq 0 \\
q_1(x) - \epsilon_1\|x\|^2 \leq 0 \\
x_1^2 + \cdots + x_{n+1}^2 = 1
\end{cases}
$$

and thus with the set $Y_\epsilon(q)$.

The proof works the same in the projective case.

In particular the previous lemma tells that for a general $q \in \mathbb{Q}(n + 1, 2)$ and $K = \{x_0 \leq 0, x_1 \leq 0\}$ we still have the estimates of the previous section.

Corollary 34. If $q \in \mathbb{Q}(n + 1, 2)$, $K = \{x_0 \leq 0, x_1 \leq 0\}$ and $Y = q^{-1}(K \cap S^n)$, $p(Y) \subset \mathbb{RP}^n$, then we have

$$b_k(Y) \leq 2k + 4 \quad \text{and} \quad b_k(p(Y)) \leq k + 2.
$$

Similarly using corollary 32 we also have the following linear bound for the topological complexity of $X = p(Y) \subset \mathbb{RP}^n$.

Corollary 35. Let $X$ be the set of the solutions of a system of two quadratic inequalities in $\mathbb{RP}^n$. Then:

$$b(X) \leq n + 1.$$

Remark 3. Estimates on the Betti Numbers of system of quadratic inequalities are given in the general case in [5] and [6]; in the case of intersection of quadrics in projective space estimates on the number of connected components are given in [10]. In particular, following the notations of [10], we can denote by $B_k^r(n)$ the maximum value that the $k$-th Betti number of an intersection (not necessarily complete) of $r + 1$ quadrics in $\mathbb{RP}^n$ can have. There it is proved that for complete intersections

$$B_2^0(n) \leq \frac{3}{2}l(l - 1) + 2, \quad l = \lfloor n/2 \rfloor + 1.$$
9. Bound for the intersection of two quadrics

In this section we mean to give a bound, linear in $n$, for the sum of the Betti numbers of the intersection $X$ of two quadrics in $\mathbb{RP}^n$. This bound will improve the standard Barvinok’s one, which is of the form $O(n)^2$ (see [3]).

**Theorem 36.** Let $X$ be the intersection of two quadrics in $\mathbb{RP}^n$. Then:

$$b(X) \leq 3n + 2$$

**Proof.** Consider the map $\overline{q}$ above defined; it maps the circle $S^1$ to the space $\mathbb{Q}(n + 1)$; this map is real algebraic and by a standard result (see section II.6.2 of [13]) there exist analytic functions $\alpha_1, \ldots, \alpha_{n+1}$ defined on the circle representing the eigenvalues of the previous family; such result is certainly false for an algebraic family of symmetric matrices depending on more than one parameter. Notice that these functions are not ordered, as it was the case for $\lambda_1, \ldots, \lambda_{n+1}$. Let us fix an orientation of the circle and consider a general point $\eta$ on it. Let also $\epsilon > 0$ be small enough; then by analyticity of the functions $\alpha_1, \ldots, \alpha_{n+1}$, the following numbers are well defined: $a(\eta)$ and $b(\eta)$ are the number of $\alpha_i$ that are positive and strictly less than $\epsilon$ respectively in a small left and in a small right neighborhood of $\eta$. We also set $c(\eta) = b(-\eta)$ and $d(\eta) = a(-\eta)$. By $\nu(\eta)$ we denote the multiplicity of $\eta$ as a solution of $\det(\omega q) = 0$. With this setting the following inequalities hold:

$$a(\eta) + c(\eta), b(\eta) + d(\eta) \leq \nu(\eta).$$

Moreover at the point $-\eta$ the same numbers are defined and $(a, b, c, d)(-\eta) = (d, c, b, a)(\eta)$.

Consider now a proper closed convex cone $K$ in $\mathbb{R}^2$ and

$$C = [\eta_0, \eta_1] = K \cap S^1$$

(the notation $[\eta_0, \eta_1]$ indicates that $C$ is the arc on the circle joining $\eta_0$ to $\eta_1$ clockwise). Let now $\eta$ be in $[\eta_0, \eta_1]$ such that there are no points of discontinuity for $i^+$ on $[\eta_0, \eta]$ (thus $\eta$ can be either a point of discontinuity or of continuity for $i^+$). Let also $\eta'$ be a point in $(\eta, \eta_1]$ where $i^+$ is continuous and such that $\eta$ is the only possible point of discontinuity for $i^+$ in $[\eta_0, \eta']$. If we denote, as above, by $c(C)$ the sum $\Sigma_{k \geq 0} b_0(C^{j+1})$, then from the semialgebraic Mayer-Vietoris sequence for the pair $([\eta_0, \eta'], [\eta', \eta_1])$ we get the formula:

$$c([\eta_0, \eta_1]) = c([\eta_0, \eta']) + c([\eta', \eta_1]) + c(\eta').$$

On the other hand $c([\eta_0, \eta'])$ equals $i^+(\eta) + a(\eta) + b(\eta)$ and $c(\eta')$ equals $i^+(\eta') = i^+(\eta) + b$. Thus we have:

$$c([\eta_0, \eta_1]) = a(\eta) + c([\eta', \eta_1]).$$

In a similar way choosing $[\omega', \eta]$ containing only one possible point $\omega$ of discontinuity for $i^+$ we get:

$$c([\eta_0, \eta_1]) = c([\eta_0, \omega']) + b(\omega).$$

If we let now $\eta_1, \ldots, \eta_l$ be the points of discontinuity for $i^+$ on $[\eta_0, \eta_1]$ we get:

$$c([\eta_0, \eta_1]) = i^+(\eta_1) + \sum a(\eta_i) = i^+(\eta_0) + \sum b(\eta_i).$$

Let now $S^1$ be divided into two symmetric arcs $C_1$ and $C_2 = -C_1$ such that $C_1 \cap C_2$ consist of two points $\beta, -\beta$ of continuity for $i^+$. Moreover consider the set
\{\eta_1, \ldots, \eta_s\} of all the points \(\eta \in C_1\) such that \(\eta\) or \(-\eta\) is a discontinuity point for \(i^+\). Then we have

\[
2c(C_1) \leq i^+(\beta) + i^+(-\beta) + \sum_i a(\eta_i) + \sum_i b(\eta_i)
\]

and similarly, recalling that \(a(-\eta) = d(\eta)\) and \(b(-\eta) = c(\eta)\):

\[
2c(C_2) \leq i^+(\beta) + i^+(-\beta) + \sum_i d(\eta_i) + \sum_i c(\eta_i).
\]

In particular, since \(a(\eta_i) + c(\eta_i) \leq \nu(\eta_i)\) and \(b(\eta_i) + d(\eta_i) \leq \nu(\eta_i)\), summing the two previous inequalities gives:

\[
2(c(C_1) + c(C_2)) \leq 2(i^+(\beta) + i^+(-\beta) + \sum_i \nu(\eta_i)).
\]

Since by definition we have that both the quantities \(i^+(\beta) + i^+(-\beta)\) and \(\sum_i \nu(\eta_i)\) are less than \(n + 1\) we have:

\[
c(C_1) + c(C_2) \leq 2n + 2.
\]

Consider now the semialgebraic Mayer-Vietoris sequence for the pair \((C_1, C_2)\); it gives:

\[
c(\Omega) \leq 2n + 2
\]

and since by theorem [9] we get \(b(X) \leq c(\Omega) + n - 2\mu\) (\(\mu\) is the maximum of \(i^+\) and the case of constant index function is excluded since this gives \(b(X) \leq n + 1\)) this finally gives:

\[
b(X) \leq 3n + 2.
\]

The previous bound is clearly not optimal, but it suffices to show that the exponent of the standard bound \(b(X) \leq O(n)^2\) can be lowered; moreover the same estimates clearly holds for any \(X\) the set of the solutions of a system of two quadratic inequalities.

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