Dynamics in Two Complex Dimensions

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Abstract

We describe results on the dynamics of polynomial diffeomorphisms of \( \mathbb{C}^2 \) and draw connections with the dynamics of polynomial maps of \( \mathbb{C} \) and the dynamics of polynomial diffeomorphisms of \( \mathbb{R}^2 \) such as the Hénon family.

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1. Introduction

The subject of this article is part of the larger subject area of higher dimensional complex dynamics. This larger area includes the dynamical study of holomorphic maps of complex projective space, automorphisms of \( K3 \) surfaces, birational maps, automorphisms of \( \mathbb{C}^n \) and higher dimensional Newton’s method. Our particular topic of research is polynomial automorphisms of \( \mathbb{C}^2 \). This area is particularly interesting because of its connections to some fundamental questions of dynamical systems via two real dimensional dynamics and because of its connection to some powerful techniques via one dimensional complex dynamics. I will begin by describing some of these connections. The reader is encouraged to consult [17] for a more thorough discussion of the historical background summarized here.

Over one hundred years after Poincaré observed chaotic behavior in the dynamics of surface diffeomorphisms the problem of creating a comprehensive theory of the dynamics of diffeomorphisms remains unsolved. Though the objective is to create a theory that would apply to diffeomorphisms in any dimension the focus remains on the two dimensional case. On the one hand the chaotic behavior which makes these problems challenging first appears for diffeomorphisms in dimension

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two, on the other hand there is a sense that if the tools can be developed to solve the problem in dimension two then the higher dimensional problem will be approachable. There are reasons to believe that if the tools can be developed to thoroughly analyze one specific interesting family of diffeomorphisms then one would be in a good position to attack the general problem. If we were to suggest a family to play the role of a “test case” there is one particular family which stands out. This is the family of diffeomorphisms of $\mathbb{R}^2$ introduced by the French astronomer Hénon:

$$f_{a,b}(x, y) = (a - by - x^2, x).$$

The parameter $b$ is the Jacobian determinant of $f_{a,b}$. When $b \neq 0$ these maps are diffeomorphisms. When $b = 0$ then $f_{a,0}$ is a map with a one dimension range and the behavior of $f_{a,0}$ is essentially that of the quadratic unimodal map $f_a(x) = a - x^2$.

In singling out the Hénon family we are following a well established tradition. This family has appeared often both in the physics and mathematics literature. It has been studied theoretically and numerically.

Virtually all interesting dynamical behavior which is known to occur for two dimensional diffeomorphisms is known to occur in this family. Hénon’s original question involved an apparent strange attractor, and this is the first family in which the existence of strange attractors was proved ([11]). For certain parameter values this family exhibits hyperbolic behavior such as the Smale horseshoe ([14]). For other parameters it exhibits persistently nonhyperbolic behavior ([19]).

There is also a great deal that is not understood about the Hénon family. Despite the fact that many different types of dynamic behavior occur it is not known whether the union of these behaviors accounts for a large set of parameter values. There are also open questions about how the complexity of behavior varies with the parameters. When $a \ll 0$ the behavior is non-chaotic. When $a \gg 0$, $f_{a,b}$ exhibits a horseshoe, a model for chaotic behavior. What happens for intermediate values? How is chaos created? (cf [13])

Another reason for looking at the Hénon family is its connection with the one dimensional family of unimodal maps $f_a$. One dimensional diffeomorphisms exhibit only regular behavior but one dimensional maps exhibit a wealth of chaotic behavior. In contrast to the situation for the Hénon family, the most fundamental questions for the unimodal family have been answered. In the language of [17] the family $f_a$ provides a “qualitatively solvable model of chaos” which is to say that there is a good understanding of attractors, strange and otherwise, for large sets of parameters and there is a good understanding of the transition to chaos.

The quadratic family is distinguished in the family of unimodal maps because it has a natural extension to the complex numbers. In the family $f_a(x) = a - x^2$ both $x$ and $a$ can be taken to be complex. The use of complex methods stands out as a reason for the success of the analysis of the quadratic family and unimodal maps more generally. While there are important results about unimodal maps that do not use complex techniques, these techniques do play a central role in the monotonicity results and in the analysis of attractors for the quadratic family.
Because the Hénon family is also given by polynomial equations it also has a
natural complex extension. My first introduction to the importance of the complex
Hénon family was through lectures of J. H. Hubbard in the mid 1980’s. Another
contributor who brought new ideas to the subject was N. Sibony. Hubbard and
his co-authors as well as Fornaess and Sibony and many others have continued to
make fundamental contributions to this area and it is not possible to do justice to
all of this work in the space provided. I will focus here on work that was carried
out jointly with E. Bedford and, in some cases, M. Lyubich over the past 15 years.

2. Basic definitions in one and two variables

The fundamental paper of Friedland and Milnor [15] shows that a natural class
of holomorphic diffeomorphisms to consider is the family of polynomial diffeomor-
phisms of $C^2$. This class contains the Hénon family and the tools that we use to
analyze the Hénon family work equally well for all diffeomorphisms in this class.
In studying polynomial maps of $C$ one focuses on those of degree greater than one
because these exhibit chaotic behavior. One way of quantifying chaotic behavior is
through the topological entropy, $h_{\text{top}}(f)$. In one complex dimension the entropy is
the logarithm of the degree so the distinction between degree one and higher degree
is the distinction between entropy zero and positive entropy.

For polynomial diffeomorphisms in dimension two the algebraic degree is not
a conjugacy invariant and hence not a dynamical invariant. One way to create a
conjugacy invariant is to define the following “dynamical degree”:

$$d = \lim_{n \to \infty} \left( \text{algebraic degree } f^n \right)^{\frac{1}{n}}.$$

It is again true that the topological entropy of a complex diffeomorphism is the
logarithm of its dynamical degree, so dynamical degree seems to be the appropriate
two dimensional analog of degree. The Hénon diffeomorphisms have the property
that the algebraic degree of $f^n$ is $2^n$ so the dynamical degree is two. Friedland and
Milnor show that any diffeomorphism with dynamical degree one is conjugate to an
affine or elementary diffeomorphism. They also show that a diffeomorphism with
dynamical degree greater than one is, like the Hénon diffeomorphism, conjugate
to an explicit diffeomorphism whose actual degree is equal to its dynamical degree.
When we refer to the degree of a diffeomorphism we will mean the dynamical degree.
We make the standing assumption that all of our polynomial diffeomorphisms have
degree greater than one.

Let us review some standard definitions for polynomial maps. Let $f : C \to C$
be a polynomial map with degree $d > 1$. The set $K$ is the set of points with
bounded orbits. The Julia set, $J$ is the boundary of $K$. In dimension one all
recurrent behavior is contained in $K$. All chaotic recurrent behavior is contained in
$J$. The ease with which this set can be defined leaves one unprepared for the range
of intricate behavior that it exhibits.
Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial diffeomorphism with dynamical degree $d > 1$. The set $K^+$ is the set of points with bounded forward orbits. Following Hubbard we take the set $K^-$ to be the set of points with bounded backward orbits. The sets $J^\pm$ are defined as the boundaries of $K^\pm$. The set $J$ is defined to be $J^+ \cap J^-$. In dimension two all chaotic recurrent behavior is contained in $J$. Thus $J$ seems to be a good analog of the one dimensional Julia set. (In fact there is an alternative analog of $J$ but we will not deal with that here.)

Let $p$ be a periodic saddle point of period $n$ in $\mathbb{C}^2$. Let $W_u^p$ denote the unstable manifold of $p$. This is the set of points that converge to $p$ under iteration of $f^{-1}$. Since this definition involves $f^{-1}$ it is less clear what the one variable analog should be. Let us examine the situation more carefully. The set $W_u^p$ is holomorphically equivalent to $\mathbb{C}$. We can find a parameterization $\phi_p : \mathbb{C} \to W_u^p$ which satisfies the functional equation $f^n(\phi_p(z)) = \lambda \cdot z$ where $\lambda$ is the expanding eigenvalue of $Df^n_p$. Now if $p$ is a periodic point in $\mathbb{C}$ then the functional equation still makes sense. A function $\phi_p$ which satisfies this equation is called a linearizing coordinate and this is a good analog of the parameterized unstable manifold in two dimensions.

Hubbard made the key observation that this construction gives a natural way to draw pictures of the sets $W_u^p \cap K^+$ in two variables and a natural way to compare them to the pictures of $K$ in one variable. In both cases we identify a region in $\mathbb{C}$ with the computer screen and choose a color scheme where the color for a pixel corresponding to $z$ is related to the rate of escape of $\phi_p(z)$. The general convention is that points that do not escape (those points in $\phi_p^{-1}(K)$) are colored black. (See [http://www.math.cornell.edu/~dynamics/].)

There is an abstract construction which makes it easier to compare invertible systems such as diffeomorphisms with non-invertible systems. Given a non-invertible system such as $f : \mathbb{C} \to \mathbb{C}$ there is a closely related invertible system called the natural extension. Let us denote this by $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The points in $\hat{\mathbb{C}}$ consist of sequences $(\ldots z_{-1}, z_0, z_1, \ldots)$ such that $f(z_j) = z_{j+1}$. The map $\hat{f}$ acts by shifting such a sequence to the left.

The natural extension gives us a way of justifying the analogy between linearizing coordinates and unstable manifolds. Corresponding to a periodic saddle point $p$ in $\mathbb{C}$ there is a unique periodic point $\hat{p}$ in $\hat{\mathbb{C}}$. Since $\hat{f}$ is invertible we can make sense of the unstable manifold $W_{\hat{p}}^u$ and the linearizing coordinate can be used to parameterize this unstable manifold.

Though $\hat{\mathbb{C}}$ contains “leaves” such as $W_{\hat{p}}^u$ it is a mistake to think of $\hat{\mathbb{C}}$ as a lamination. When $f$ is expanding $\hat{\mathbb{C}}$ is a lamination near the Julia set but the more complicated the dynamics of $f$, the more degenerate this structure becomes. This complexity arises from recurrent behavior of the critical point for $f$. This suggests a certain connection between regularity of unstable manifolds in two dimensions and recurrence of critical points in one dimension that we will return to later.

Since points in $\hat{\mathbb{C}}$ have bounded unstable manifolds, we should think of $\hat{\mathbb{C}}$ as an analog of $J^-$. Let $f_a$ be an expanding one dimensional map and consider a
A standard construction in potential theory associates to nice sets a measure \( \mu \) called the harmonic measure or equilibrium measure. The harmonic measure associated to the Julia set turns out to be a measure of dynamical interest. The potential theory construction starts with the Green function. The Green function of \( K \) has a dynamical description:

\[
G(p) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(p)|.
\]

The Green function is non-negative and equal to zero precisely on the set \( K \). The harmonic measure \( \mu \) is obtained by applying the Laplacian to \( G \). The support of \( \mu \) is the boundary of \( K \) which is the set \( J \). The connection between polynomial maps and potential theory first appears in the work of Brolin ([12]). It reappears in a paper of Manning ([18]) and is nicely summarized in [20].

The harmonic measure has connections to entropy and to the connectivity of \( J \). These connections do not play a major role in the one dimensional theory because entropy and connectivity can be approached more directly. In the two dimensional theory these connections are much more important.

The entropy of the measure \( \mu \), \( h(\mu) \), happens to be \( \log d \) which is equal to the topological entropy of the map. The topological entropy dominates the measure theoretic entropy of any invariant measure. A measure for which equality holds is called a measure of maximal entropy. For polynomial maps of \( \mathbb{C} \) the measure \( \mu \) can be characterized as the unique measure of maximal entropy.

The dimension of a measure \( \nu \), \( \dim_H(\nu) \), is the minimum of the Hausdorff dimensions of subsets of full \( \nu \) measure. The dimension of the harmonic measure of a planar set is always less than or equal to one. If the set is connected then the dimension is one. For Julia sets the converse is true: the dimension of the harmonic measure is one if and only if \( J \) is connected.

The Lyapunov exponent, \( \lambda(\mu) \), of \( f \) with respect to an ergodic measure measures the rate of growth of tangent vectors under iteration (for a set of full \( \mu \) measure). The Lyapunov exponent is related to Hausdorff dimension of the measure by the formula:

\[
\dim_H(\mu) = h(\mu)/\lambda(\mu).
\]

Since \( h(\mu) = \log d \), \( \lambda(\mu) = \log d \) if and only if \( J \) is connected. We will return to this in the next section.

In dimension two we have two rate of escape functions:

\[
G^\pm(p) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^{\pm n}(p)|.
\]
Potential theory in one variable centers on the behavior of the Laplacian. The Laplacian is not holomorphically invariant in two variables but it has a close relative which is. This is the operator $dd^c$ which takes real valued functions to real two forms. The $d$ that appears here is just the exterior derivative and the $d^c$ is a version of the exterior derivative twisted by using the complex structure. In one variable we have:

$$dd^c g = (\Delta g)dx \wedge dy.$$ 

Not only is $dd^c$ holomorphically natural but it is well defined on complex manifolds of any dimension. Of course, in the two variable context as in the one variable, the functions to which these operators are applied are not smooth and the result has to be interpreted appropriately. The theory connected with the operator $dd^c$ is referred to as pluripotential theory. It was an observation of Sibony that the methods of pluripotential can be profitably applied to the complex Hénon diffeomorphisms.

Define $\mu^\pm = \frac{1}{2\pi} dd^c G^\pm$. These are dynamically significant currents supported on $J^\pm$. Define $\mu = \mu^+ \wedge \mu^-$. This measure $\mu$ is the analog of the harmonic measure in one dimension. The following result suggest that “$\mu$” defined above is the dynamical analog as well as the pluripotential theoretic analog.

**Theorem 3.1** ([4]) *The measure $\mu$ is the unique measure of maximal entropy.*

### 4. Connectivity and critical points

We want to consider the way in which the dynamical behavior of a polynomial diffeomorphism such as $f_{a,b}$ depends on the parameter. Looking at pictures of $W^u_p \cap K^+$ shows that there are indeed many things that do change. If we want to focus on one fundamental property we might start by looking at connectivity. In one variable the connectivity of the Julia set of $f_a$ defines the Mandelbrot set which is the fundamental object of study for quadratic maps. In two variables there are several notions of connectivity that we could consider. The following has proved useful. We say that $f$ is *stably/unstably connected* if $W^{s/u}_p \cap K^-/+$ has no compact components. We can ask about the relation between stable connectivity, unstable connectivity and the connectivity of $J$. A priori the property of being stably/unstably connected depends on the saddle point $p$. In fact we show that these properties are independent of $p$.

Let us look at the situation in one variable. The basic result about connectivity is the following.

**Theorem 4.1** (Fatou) *Let $f$ be a polynomial map of $\mathbb{C}$. Then $J$ is connected if and only if every critical point of $f$ has a bounded orbit.*

The following formula makes a connection between the Lyapunov exponent and critical points ([20]):

$$\lambda(\mu) = \log d + \sum_{\{c_j: f'(c_j)=0\}} G(c_j).$$

The function $G$ is non-negative and zero precisely on the set $K$. In light of the theorem above we see that $J$ is connected if and only if the Lyapunov exponent is
log \( d \). This proves an assertion made in Section 1 about the relation between the Lyapunov exponents of \( \mu \) and the connectivity of \( J \).

In two variables there are two Lyapunov exponents, \( \lambda^\pm(\mu) \), of \( f \) with respect to harmonic measure. The following result establishes the connection between stable/unstable connectivity and these exponents.

**Theorem 4.2 ([7])** We have \( \lambda^+(\mu) \geq \log d \); and \( \lambda^+(\mu) = \log d \) if and only if \( f \) is unstably connected. Similarly \( \lambda^-(\mu) \leq -\log d \); and \( \lambda^-(\mu) = -\log d \) if and only if \( f \) is stably connected.

It is clear from this result that neither exponent is zero. Pesin theory shows that stable and unstable manifolds exist for \( \mu \) almost every point. Let \( \mathcal{C}^u \) be the set of critical points of the restriction of \( G^+ \) to these unstable manifolds. We define \( \mathcal{C}^s \) in the corresponding way.

**Theorem 4.3 ([7])** The diffeomorphism \( f \) is unstably connected if and only if \( \mathcal{C}^u = \emptyset \). The diffeomorphism \( f \) is stably connected if and only if \( \mathcal{C}^s = \emptyset \).

In [6] we prove an analog of the critical point formula where the role of the critical point is played by critical points is played by \( \mathcal{C}^u \). This formula leads to proofs of the two theorems above.

The following result makes the connection between stable and unstable connectivity and the connectivity of \( J \). Note that in this two variable situation the Jacobian of \( f \) enters the picture.

**Theorem 4.4 ([7])** If \( |\det Df| < 1 \) then \( f \) is never stably connected. In this case \( J \) is connected if and only if \( f \) is unstably connected. If \( |\det Df| = 1 \) then \( f \) is stably connected iff \( f \) is unstably connected iff \( J \) is connected.

(The case \( |\det Df| > 1 \) is analogous to the case \( |\det Df| < 1 \).) The Jacobian enters the proof through the relation: \( \lambda^+(\mu) + \lambda^-(\mu) = \log |\det Df| \). We see for example that \( |\det Df| < 1 \) implies that \( \lambda^-(\mu) < -\log d \) which, by Theorem 4.2 implies that \( f \) is unstably disconnected.

Using this result J. H. Hubbard and K. Papadantonakis have developed a computer program that uses the set \( \mathcal{C}^u \) to draw pictures of the connectivity locus in parameter space. (See [http://www.math.cornell.edu/~dynamics/].)

5. The boundary of the horseshoe locus

Hyperbolic behavior, as exhibited by the horseshoe, is structurally stable. This implies that the set of \((a, b)\) for which \( f_{a,b} \) exhibits a horseshoe is open. Let us call this set the horseshoe locus. Standard techniques from dynamical systems can be used to analyze the dynamical behavior inside the horseshoe locus. These techniques break down on the boundary of the horseshoe locus however. By contrast complex techniques from [4], [9] and [10] can be applied on the closure of the horseshoe locus. Thus the analysis of this boundary provides a setting for demonstrating that these techniques derived from complex analysis are not without interest in the real setting.

Let us look at the one dimensional case \( f_a \). We say that \( f_a \) exhibits a horseshoe
if \( f_a | J_a \) is expanding and topologically conjugate to the one sided two shift. The horseshoe locus here is the set \( a > 2 \). The boundary of the horseshoe locus is \( a = 2 \). The map \( f_2 \) is the well known example of Ulam and von Neumann. The failure of expansion is demonstrated by the fact that the critical point 0 is in the Julia set, \([-2, 2]\). In fact the critical point maps to the fixed point \(-2\) after two iterates.

The following result describes the failure of hyperbolicity on the boundary of the horseshoe locus for Hénon diffeomorphisms. Note that the property of eventually mapping to the fixed point \( p \) in dimension one corresponds to belonging to \( W^s_p \) in dimension two.

**Theorem 5.1 ([10]).** For \( f_{a,b} \) on the boundary of the horseshoe locus there are fixed points \( p \) and \( q \) so that \( W^s_p \) and \( W^u_q \) have a quadratic tangency. When \( b > 0 \) we have \( p = q \). When \( b < 0 \), \( p \neq q \).

The next result gives additional information about the precise nature of the dynamics of maps on the boundary of the horseshoe locus:

**Theorem 5.2 (Bedford-Smillie)** For any \((a, b)\) in the boundary of the horseshoe locus the restriction of \( f_{a,b} \) to its non-wandering set is conjugate to the full two-shift with precisely two orbits identified. Given \((a, b)\) and \((a', b')\) in the boundary of the horseshoe locus, the restrictions of \( f_{a,b} \) and \( f_{a', b'} \) to their non-wandering sets are conjugate if and only if \( b \) and \( b' \) have the same sign.

There are many techniques which work only for \( b \) small. Note that the result above applies for all values of \( b \) including the volume preserving case \( b = \pm 1 \).

We can ask how the dynamics of \( f_{a,b} \) for \((a, b)\) on the boundary of the horseshoe regions \( b > 0 \) and \( b < 0 \) compares with the dynamics of \( f_2 \) which corresponds to the boundary of the horseshoe region when \( b = 0 \). The sets \( J_{a,b} \) for \( b \neq 0 \) are totally disconnected while the set \( J_2 \) is connected. In particular the inverse limit system \( \hat{J}_2 \) is not conjugate to either system with \( b \neq 0 \). This is an example where the insights gained from looking at the inverse limit system need to be interpreted cautiously.

I will touch on the techniques used in the proofs of these theorems. Our fundamental approach to proving these results was to exploit the relationship between the real mapping \( f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 \) and its complex extension \( f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2 \). In passing from \( \mathbb{C}^2 \) to \( \mathbb{R}^2 \) something may be lost. The first question to ask is how much chaotic behavior do we lose? One way to measure this is through the topological entropy function. If we denote \( f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( f_{\mathbb{R}} \) and \( f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2 \) by \( f_{\mathbb{C}} \) then we have

\[
h_{\text{top}}(f_{\mathbb{R}}) \leq h_{\text{top}}(f_{\mathbb{C}}) = \log 2.
\]

If we want to study the real Hénon diffeomorphisms most closely connected to their complex extensions we should focus our attention on those \( f \) with \( h_{\text{top}}(f_{\mathbb{R}}) = \log 2 \). We say that these examples have *maximal entropy*. This is an interesting set to look at. The horseshoe locus is contained in the maximal entropy locus but the maximal entropy locus is larger than the horseshoe locus. The horseshoe locus is open, and the maximal entropy locus is closed. In particular the maximal entropy locus contains the boundary of the horseshoe locus.
For maximal entropy diffeomorphisms the relation between the real and complex dynamics is as close as one could want:

**Theorem 5.3** ([4]) \( f_R \) has maximal entropy if and only if \( J \) is contained in \( \mathbb{R}^2 \).

This theorem is a consequence of the fact that \( \mu \) is the unique measure of maximal entropy and the fact that the support of \( \mu \) is contained in \( J \). The fact that the real and complex dynamics are closely related for this class of maps means that it is a good starting point for applying complex techniques to the real case. It also provides us with useful techniques from harmonic analysis. For example the Green functions of real sets satisfy certain growth conditions, and these translate into conditions insuring expansion and regularity of unstable manifolds. This allows us to show that maximal entropy diffeomorphisms are quasi-expanding ([9]). Quasi-expansion is the two dimensional analog of \( f \) having non-recurrent critical points. The exploitation of the properties of quasi-expanding diffeomorphisms leads to the proofs of the Theorems 5.1 and 5.2.

We believe that the connections made so far do not represent the end of the story but only the beginning. We trust that the picture of two dimensional complex dynamics will become clearer with time and, as it does, there will be valuable interactions with the theory of real dynamics.

**References**

[1] E. Bedford & J. Smillie, *Polynomial diffeomorphisms of \( \mathbb{C}^2 \): currents, equilibrium measures and hyperbolicity*, Inventiones Math. **103** (1991), 69–99.

[2] E. Bedford & J. Smillie, *Polynomial diffeomorphisms of \( \mathbb{C}^2 \) II: stable manifolds and recurrence*, 4 No. 4 Journal of the A.M.S. (1991), 657–679.

[3] E. Bedford & J. Smillie, *Polynomial diffeomorphisms of \( \mathbb{C}^2 \) III: ergodicity, exponents and entropy of the equilibrium measure*, Math. Annalen **294** (1992), 395–420.

[4] E. Bedford, M. Lyubich, and J. Smillie, Polynomial diffeomorphisms of \( \mathbb{C}^2 \) IV: the measure of maximal entropy and laminar currents, *Inventiones Math.* **112** (1993), 77–125.

[5] E. Bedford, M. Lyubich, and J. Smillie, Distribution of periodic points of polynomial diffeomorphisms of \( \mathbb{C}^2 \), Inventiones Math. **114** (1993), 277–288.

[6] E. Bedford & J. Smillie, *Polynomial diffeomorphisms of \( \mathbb{C}^2 \) V: Critical points and Lyapunov exponents*, J. Geom. Anal. 8 no. 3, (1998), 349–383.

[7] E. Bedford & J. Smillie, Polynomial diffeomorphisms of \( \mathbb{C}^2 \) VI: Connectivity of \( J \), *Annals of Mathematics*, 148 (1998), 695–735.

[8] E. Bedford & J. Smillie, Polynomial diffeomorphisms of \( \mathbb{C}^2 \). VII: Hyperbolicity and external rays, *Ann. Sci. Ecole Norm. Sup.* 4 série 32 (1999), 455–497.

[9] E. Bedford & J. Smillie, Polynomial diffeomorphisms of \( \mathbb{C}^2 \). VIII: Quasi-expansion, *American Journal of Mathematics* **124** (2002), 221–271.

[10] E. Bedford & J. Smillie, Real polynomial diffeomorphisms with maximal en-
tropy: tangencies, (available at http://arXiv.org).

[11] M. Benedicks & L. Carleson, The dynamics of the Hénon map, *Annals of Mathematics*, 133, (1991), 73–179.

[12] H. Brolin, Invariant sets under iteration of rational functions, *Ark. Mat.*, 6, (1965), 103–144.

[13] A. de Carvalho & T. Hall, How to prune a horseshoe, *Nonlinearity*, 15 no. 3, (2002), R19–R68.

[14] R. Devaney & Z. Nitecki, Shift automorphisms in the Hénon mapping. Comm. Math. Phys. 67 (1979), no. 2, 137–146.

[15] S. Friedland & J. Milnor, Dynamical properties of plane polynomial automorphisms, *Ergodic Theory Dyn. Syst.* 9, (1989), 67–99.

[16] J. Hubbard & R. Oberste-Vorth, Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials, in *Real and complex dynamical systems* Kluwer, 1995.

[17] M. Lyubich, The quadratic family as a qualitatively solvable model of chaos. *Notices Amer. Math. Soc.* 47 (2000), no. 9, 1042–1052.

[18] A. Manning, The dimension of the maximal measure for a polynomial map, *Ann. Math.* 119 (1984), 425–430.

[19] S. Newhouse, The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms. *Inst. Hautes tudes Sci. Publ. Math.* No. 50 (1979), 101–151.

[20] F. Przytycki, Hausdorff dimension of the harmonic measure on the boundary of an attractive basin for a holomorphic map, *Invent. math.* 80 (1985), 161–179.