SPECTRAL GAP ESTIMATES IN MEAN FIELD SPIN GLASSES

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Abstract. We show that mixing for local, reversible dynamics of mean field spin glasses is exponentially slow in the low temperature regime. We introduce a notion of free energy barriers for the overlap, and prove that their existence imply that the spectral gap is exponentially small, and thus that mixing is exponentially slow. We then exhibit sufficient conditions on the equilibrium Gibbs measure which guarantee the existence of these barriers, using the notion of replicon eigenvalue and 2D Guerra Talagrand bounds. We show how these sufficient conditions cover large classes of Ising spin models for reversible nearest-neighbor dynamics and spherical models for Langevin dynamics. Finally, in the case of Ising spins, Panchenko’s recent rigorous calculation of the free energy for a system of “two real replica” enables us to prove a quenched LDP for the overlap distribution, which gives us a wider criterion for slow mixing directly related to the Franz-Parisi-Virasoro approach. This condition holds in a wider range of temperatures.

1. Introduction

We prove here that local, reversible dynamics for a general class of mean field spin glasses are exponentially slow in the low temperature, or Replica Symmetry Breaking (RSB), phase for a broad class of Ising and spherical models. More precisely, we give sufficient conditions for the spectral gap of these dynamics to be exponentially small. In the case of Ising spin models, we provide a wider criterion that holds in a broader range of temperatures.

The study of the convergence to equilibrium for dynamics of mean-field spin glasses has a rich history in the physics literature, and it is impossible to give here anything close to an exhaustive description of these results. We refer instead to the general surveys. We concentrate here on a basic aspect, the time to equilibrium of reversible dynamics for these models should scale exponentially in the system size. This is what we aim to prove.

The long-time behavior of spin glass dynamics has a very rich phenomenology. Along with the time to equilibrium, there is the phenomenon of aging, which occurs on timescales that are very long but shorter than the time to equilibrium. Aging for mean-field spin glasses has been extensively studied in the mathematical literature, mostly for a simple class of dynamics, the Random Hopping Time (RHT) dynamics. This was done first for the Random Energy Model (REM), see, following the seminal works in physics. This was later extended to $p$-spin models again for simple RHT dynamics. Understanding aging for Metropolis dynamics for general spin glasses is still an important open question, except in the case of the REM, where this has been achieved in the recent remarkable works. For more on this, see also.

The mathematics literature related to the spectral gap or for the mixing time at low temperature, however, is sparser than the one related to aging. The behavior of the mixing time has been understood in detail in the simple case of the Random Energy Model since the early work. An upper bound for the mixing time (or for the related notion of thermalization time) is given by for Glauber dynamics for hard spin models. Bounds on the correlation time for a single spin were studied in for Glauber dynamics of dilute $p$-spin models at moderate temperatures. Finally, very recently, gives exponential bounds for the spectral gap for the spherical pure $p$-spin model. For the sake of completeness, we mention here that the related question of mixing times or spectral

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gaps for short-range spin glass models goes back at least to [52] and [39]. We finally note here that the spectral approach to dynamics can also be useful for the study of aging. This was observed initially in the physics literature by [69], and detailed in a simple context (the REM-like trap model) by [23].

Let us now explain the core of our approach to proving slow mixing in this work. In order to understand the long-time dynamical behavior, we study the evolution of the overlap of two replica. This point of view is naturally inspired by the well-known fact that the order parameter for the study of equilibrium Gibbs measures of mean-field spin glasses is the distribution of the overlap of two replica. This was the seminal insight of Parisi [82, 70] and has been developed in a monumental work by Talagrand [92, 93, 91], building on work by Guerra [49], and much further expanded recently by Panchenko [75, 79, 76], following the work of Aizenman-Sims-Starr [1] and Aizenman-Arguin [3]. For more recent results see also [6, 58, 35, 10].

To this end, we introduce “replicated” dynamics, i.e., dynamics of two replica evolving independently. Our aim is to bound the spectral gap of this replicated dynamics. Using simple adaptations of classical tools like the Cheeger inequality, we first prove that the existence of a free energy barrier for the overlap (to be defined shortly) implies that this spectral gap is exponentially small. We then use Talagrand’s “2-dimensional Guerra interpolation” estimates [91], to provide broad sufficient conditions on the limiting Parisi measure to ensure that these free energy barriers exist. An important role in the formulation of this sufficient condition is played by what is called the “Replicon eigenvalue” which was first introduced by Parisi [81] and recently studied in [58, 55]. Our results are then shown to cover a broad class of spin glass models at low temperatures, both for Ising spins and spherical models.

In the case of Ising spin models, we introduce a deeper tool which is a (quenched) large deviation principle for the overlap distribution. This large deviation principle is based on the recent deep results of [79] which rigorously obtains the free energy for a system of “m real replica” [43, 60]. We then give a more robust sufficient condition based on the rate function for this large deviation principle and show how this approach is related to the Replicon eigenvalue. This approach applies to a much broader family of models, and even implies exponentially slow mixing well within the high temperature, or Replica Symmetric (RS), phase.

It might be worthwhile to compare our approach with the recent work of Gheissari and one of the authors in [46] for the pure $p$-spin spherical model. In [46], the basic tool is the recent understanding of the complexity of the geometry of the random landscape at zero temperature introduced in [5] and in [4], and deepened in [85] and [87]. This approach has allowed Subag to obtain a very detailed understanding of the Gibbs measure at very low temperatures [81] along the lines of the Thouless-Anderson-Palmer (TAP) approach, and to show that extensive barriers for the Hamiltonian exist in this regime. The recent work [46] builds on this fact to show slow mixing for Langevin dynamics. Our approach, however, builds on the existence of free energy barriers for the overlap of two replica, rather than on energy barriers for one replica. In the language of physics, [46] uses a “complexity based” approach similar to the dynamical TAP approach of [19] whereas this paper uses a “replica” approach. It is thus applicable to cases where a detailed understanding of the energy landscape is lacking whereas the understanding of the overlap behavior is sharper.

1.1. Ising Spin Models. We begin here by describing succinctly our results for the dynamics of Ising mean-field spin glasses. Let $\Sigma_N = \{-1, 1\}^N$ be the discrete hypercube in dimension $N$. The mixed $p$-spin glass on the hypercube is the Gaussian process, $(H_N(\sigma))$, indexed by $\Sigma_N$ with mean and covariance

$$
\begin{align*}
\mathbb{E}H_N(\sigma) &= \frac{h}{N} \sum \sigma_i \\
\text{Cov}(\sigma^1, \sigma^2) &= N\xi \left( \frac{1}{N} \sum \sigma_i^1 \sigma_i^2 \right)
\end{align*}
$$

(1.1)
Here $\xi(t) = \sum_{p \geq 1} \beta_p^2 t^p$ is a power series with positive coefficients, which we call the model, and $h \geq 0$ is called the external field. We assume that $\xi(1 + \epsilon) < \infty$ so that $H_N$ is well defined in all dimensions. An important quantity in the following will be the overlap,

$$R(\sigma^1, \sigma^2) = \frac{1}{N} \sum \sigma^1_i \sigma^2_i,$$

which we also denote by $R_{12}$. We call $H_N$ the Hamiltonian, $\Sigma_N$ the configuration space, and corresponding to $H_N$ we define the Gibbs measure,

$$\pi_N(d\sigma) = \frac{e^{-H_N(\sigma)}}{Z_N} d\sigma,$$

where $d\sigma$ is the uniform measure on $\Sigma_N$. We call such models Ising spin models.

We now turn to the class of dynamics that we will consider in this paper. We study nearest neighbor dynamics that are reversible with respect to the Gibbs measure, $\pi_N$, both in discrete and continuous time. More precisely, let $Q(\sigma^1, \sigma^2)$ be a (random) Markov transition matrix on $\Sigma_N$. We assume that $Q$ satisfies detailed balance with respect to $\pi_N$: $\pi_N(\sigma^1)Q(\sigma^1, \sigma^2) = \pi(\sigma^2)Q(\sigma^2, \sigma^1)$ for all $\sigma^1, \sigma^2 \in \Sigma_N$, and that $Q$ is nearest-neighbor, i.e., $Q(x, y) = 0$ if $x$ and $y$ differ in more than one coordinate.

Since these results hold for rather general dynamics, let us consider an example of dynamics to which these results will apply.

**Example 1.1.** Let $P_{SRW}$ denote the transition kernel for the simple random walk (SRW) on $\Sigma_N$,

$$P_{SRW}(\sigma^1, \sigma^2) = \frac{1}{N} \mathbb{1}_{\exists i \in [N]: \sigma^1(i) \neq \sigma^2(i) \text{ and } \sigma^1(j) = \sigma^2(j) \text{ for all } j \in [N] \setminus \{i\}}.$$

Consider the transition matrix

$$Q_{Met,SRW}^{Met}(\sigma^1, \sigma^2) = \begin{cases} P_{SRW}(\sigma^1, \sigma^2) \left(1 \wedge \frac{\pi_N(\sigma^2)}{\pi_N(\sigma^1)}\right) & \sigma^1 \neq \sigma^2 \\ 1 - \sum_{\sigma \neq \sigma^1} P_{SRW}(\sigma^1, \sigma) \left(1 \wedge \frac{\pi_N(\sigma)}{\pi_N(\sigma^1)}\right) & \sigma^1 = \sigma^2. \end{cases}$$

We then consider the following two processes. The discrete time Metropolis chain with base chain the SRW is the Markov chain, $(\sigma_d(n))_{n \geq 1}$, with transition matrix $Q_{Met,SRW}^{Met}$. The continuous time Metropolis chain with base chain the SRW is the continuous time Markov process, $(\sigma_c(t))$, on $\Sigma_N$ with infinitesimal generator $I - Q_{Met,SRW}^{Met}$.

The spectral gap of $I - Q$, call it $\lambda_1$, is the first non-trivial eigenvalue of $I - Q$. Our goal is to prove that in a certain regime, called the spin glass or Replica Symmetry Breaking (RSB) phase, $\lambda_1$ will be exponentially small, so that the corresponding induced dynamics will mix slowly. (For a brief reminder how spectral gap estimates relate to mixing see Section 6)

To this end, we begin by introducing the (static) notion of free energy barriers for the overlap. In Section 2 we introduce a much broader notion of the difficulty of the landscape of the Gibbs measure and use this to bound the spectral gap. For the sake of exposition in this introduction, however, we restrict it here to the following simpler notion. Heuristically, a number $q_2$ is a free energy barrier for the overlap if the probability that two replica have an overlap close to $q_2$ is exponentially small, whereas two other values $q_1$ and $q_3$ on either side of $q_2$ are probable for the overlap. More precisely, we have the following.

**Definition 1.2.** We say that there exists a free energy barrier of height $C > 0$ for the overlap if there exists a triple $-1 \leq q_1 < q_2 < q_3 \leq 1$, $0 < \epsilon < \frac{1}{4} \min\{q_3 - q_2, q_2 - q_1\}$ such that

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left( \pi_N^{q_2} (R_{12} \in (q_2 - \epsilon, q_2 + \epsilon)) > e^{-CN} \right) < 0$$
and such that, for \( i = 1 \) and \( i = 3 \)
\[
\lim_{N \to \infty} \mathbb{E}_{\pi_N^2} (R_{12} \in (q_i - \epsilon, q_i + \epsilon)) > 0
\]  
(1.5)
If there is a free energy barrier of height \( C \) for some \( C > 0 \), then we say that FEB holds.

Our first result is then that the existence of a free energy barrier for the overlap implies that the spectral gap is exponentially small.

**Theorem 1.3.** If there exists a free energy barrier for the overlap of height \( C > 0 \), then
\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left( \frac{1}{N} \log \lambda_1 > -C \right) < 0.
\]  
(1.6)
We will provide a stronger bound that generalizes the above shortly. Said bound, however, will have a more limited range of applicability for technical reasons. For the sake of exposition, we postpone this to Section 1.2.

The next step is naturally to find good sufficient conditions to insure the existence of free energy barriers for the overlap. In order to do, consider the notion of a *limiting overlap distribution*. Let
\[
\zeta_N(\cdot) = \mathbb{E}_{\pi_N^2} (R_{12} \in \cdot).
\]  
(1.7)
Since \(-1 \leq R_{12} \leq 1\), the sequence \((\zeta_N)\) is tight. A *limiting overlap distribution* is any weak limit point of this sequence
\[
\lim_{N \to \infty} \mathbb{E}_{\pi_N^2} (R_{12} \in \cdot) = \zeta.
\]  
(1.8)
In the following, it is convenient to make the following technical assumption.

**Assumption. A** There is a unique limiting overlap distribution, \( \zeta \).

It is known that this assumption holds in a large class of models. For more on this see Section 6. For the remainder of this paper we will assume A, and we will refer to \( \zeta \) simply as the *limiting overlap distribution*.

One is most interested in the properties of the support of \( \zeta \) particularly its topology. When \( \zeta \) is an atom, \((\xi, h)\) is said to be in the *Replica Symmetric (RS)* phase. When \( \zeta \) is not an atom, \((\xi, h)\) is said to be in the *Replica Symmetry Breaking (RSB)* phase. In the language of statistical physics, one thinks of the RS phase as corresponding to classical high temperature behavior and RSB as spin glass, or low temperature, behavior.

We will work in the following regime throughout this paper.

**Definition 1.4.** We say that RSB holds if A holds and \( \zeta \) is not an atom.

With this terminology, we can now restate our goals. We aim to provide an analytical criterion regarding the support of \( \zeta \) and the pair \((\xi, h)\) when RSB holds that will imply that \( \lambda_1 \) decays exponentially in \( N \). To this end, we introduce the following analytical tools from the study of spin glasses.

For \( \nu \in \text{Pr}([0, 1]) \), the *Parisi functional*, \( P_f(\nu) \), is
\[
P_f(\nu) = \phi_{\nu}(0, h) - \frac{1}{2} \int_0^1 \xi''(s) s \nu[0, s] ds
\]  
(1.9)
where \( \phi_{\nu} \) is the unique weak solution of
\[
\begin{align*}
\partial_t \phi_{\nu} + \frac{\xi''}{2} (\Delta \phi_{\nu} + \nu([0, s])(\partial_x \phi_{\nu})^2) &= 0, \\
\phi_{\nu}(1, x) &= \log \cosh(x)
\end{align*}
\]  
(1.10)
(For the definition of weak solution and basic properties of \( \phi \), see Appendix A or [56].) It is known that \( P_I \) is continuous and strictly convex [7], and in particular has a unique minimizer. The Parisi functional provides a variational formula for what is called the free energy:

\[
F = \lim_{N \to \infty} \frac{1}{N} \log \int e^{-H(s)} \, d\sigma = \min_{\nu \in \text{Pr}([0,1])} P_I(\nu),
\]  

where \( d\sigma \) is the uniform measure on \( \Sigma_N \). This formula, called the Parisi formula, was proved by Talagrand [91] for even \( \xi \) and Panchenko [77] for general \( \xi \). The minimizer of this problem will play an important role in our analysis.

**Definition 1.5.** The Parisi measure is the minimizer of (1.11), which we denote by \( \mu \).

We now turn to defining the main analytical quantity of interest, the replicon eigenvalue. For \( \nu \in \text{Pr}([0,1]) \), consider the solution of the SDE

\[
dX_t = \xi''(s)\nu(s)\partial_x \phi_{\nu}(s, X_s) \, ds + \sqrt{\xi''(s)} \, dW_s
\]  

with initial data \( X_0 = h \), where \( W_s \) is a standard Brownian motion. Following the physics literature [70], we will refer to \( X_t \) as the local field process. We note here that \( \partial_x \phi \) is continuous in time and smooth and bounded in space (see Appendix A below or [56]) so this solution exists in the Itô sense.

For \( \nu \in \text{Pr}([0,1]) \) and \( q \in \text{supp}(\nu) \), the replicon eigenvalue is

\[
\Lambda_R(q, \nu) = 1 - \xi''(q) \mathbb{E}_h (\Delta \phi_{\nu})^2 (q, X_q).
\]  

Here and in the following we denote the support of a probability measure, \( \nu \), by \( \text{supp}(\nu) \). We can now define the main analytical condition for our results.

**Definition 1.6.** A pair \((\xi, h)\) is said to satisfy PREV, if:

- RSB holds.
- There are at least two points in \( E = \text{supp}(\zeta) \cap \text{supp}(\mu) \).
- There is a \( q \in E \) with a positive replicon eigenvalue:

\[
\Lambda_R(q, \mu) > 0.
\]

**Remark 1.7.** The condition PREV is related to a generalization of the de Almeida-Thouless line [38]. More precisely, it can be shown [6, 58] that if the replicon eigenvalue is positive for some \( q \in \text{supp}(\mu) \), then \( q \) is an isolated element of \( \text{supp}(\mu) \). As we will point out in Theorem 1.1, the condition that \( \Lambda_R(q, \mu) > 0 \) implies that \( q \) is an isolated element of the support of \( \zeta \) when PREV holds.

With these notions in hand, we then have the following theorem.

**Theorem 1.8.** If \( \xi \) is convex and \((\xi, h)\) satisfy PREV, then FEB holds. In particular, for some \( C > 0 \),

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\frac{1}{N} \log \lambda_1 > -C) < 0.
\]

Let us now turn to an example of models to which our result applies. One class of models to which our results hold are the following. We say that a model is **generic** if the family of monomials

\[
\mathcal{F} = \{ t^p : \beta_p \neq 0 \} \cup \{1\}
\]

is total in \((C([-1,1]), \text{sup}||\cdot||)\). We say that a model is **even generic** if \( \mathcal{F} \) is total in \( C([0,1], \text{sup}||\cdot||) \) and \( \xi \) is even. (For instance, the latter case holds if \( \sum p; \beta_p \neq 0 \mathcal{P}^{-1} = \infty \) by the Müntz-Szász theorem.)
Theorem 1.9. Suppose that $\xi = \beta^2 \xi_0$ has $\xi''_0(0) = 0$ and is convex and either generic or even generic. Then there is an $h_0(\beta, \xi) > 0$ such that for $h \leq h_0$, if the Parisi measure, $\mu$, is not an atom, then PREV holds. Consequently, 

$$\lim_{N \to \infty} \frac{1}{N} \log P \left( \frac{1}{N} \log \lambda_1 > -c \right) < 0$$

for some $c > 0$. In particular, this holds if $\beta$ is sufficiently large.

As we have shown that the order of decay of $\lambda_1$ is at least exponential, it is natural to ask if this is indeed the correct order of growth. Of course, simply knowing that the dynamics are local is insufficient to determine this question; one needs more assumptions. A natural assumption is coercivity with respect to some base Markov process.

Definition 1.10. Let $P$ be a transition matrix on $\Sigma_N$ that satisfies detailed balance with respect to the uniform measure. A transition matrix $Q$ for some reversible Markov chain on $\Sigma_N$ is said to be $P$-coercive if there is some constant $A$ such that

$$AP(\sigma^1, \sigma^2) \leq Q(\sigma^1, \sigma^2).$$

This is simply asserting a form of coercivity between the corresponding Dirichlet forms (see Section 2.2.2 below). With this definition in hand, we then have the following

Theorem 1.11. If $Q_N$ is $P_{SRW}$-coercive with constant $A_N$ then there is some constant $c' > 0$ such that

$$\lim_{N \to \infty} \frac{1}{N} \log P \left( \frac{1}{N} \log \lambda_1 < -c' + \frac{1}{N} \log A_N \right) < 0$$

almost surely for all $N$.

As an example, note the following

Corollary 1.12. $Q^M,SRW$ is $P_{SRW}$-coercive. In particular, there is a constant $c > 0$, such that

$$\lim_{N \to \infty} \frac{1}{N} \log P \left( \frac{1}{N} \log \lambda_1 < -c \right) < 0$$

for all $N$ almost surely.

It is known that the latter limit exists almost surely and is given by a constant [50]. For a variational representation of this constant in this setting see [8].

This bound is rather coarse and makes no mention of either the temperature or the overlap distribution. As an application of the deep study by Mathieu [67], it can be shown that if we study $\xi$ of the form $\xi = \beta^2 \xi_0$, then the thermalization time (which is related to, but slight different from the mixing time) can be bounded in terms of the free energy $F(\beta)$. In particular, it is bounded by $F'(\beta)$, which is known [73] to satisfy

$$F'(\beta) = \beta \int \xi_0(1) - \xi_0(t) d\mu.$$
Observe that $\zeta_N = \mathbb{E} \mathcal{E}_N$.) With this in hand, we have the following theorem.

**Theorem 1.13.** Let $\xi$ be convex. The sequence $\{\mathcal{E}_N\}$ satisfies an large deviation principle with a rate function $I$ and rate $N$ almost surely.

This large deviation principle is a consequence of a recent deep result of Panchenko [79] which proves the sharpness of the Guerra-Talagrand bounds. Indeed, the rate function here is given by

$$I(q) = -\inf_{\lambda \in \mathbb{R}, Q \in \mathcal{Q}_q, \nu \in \Pr([0,1])} P(\nu, Q, \lambda) + 2 \min_{\nu} P_I(\nu)$$

where $P_I$ is from (1.9) and the set $\mathcal{Q}_q$ and the functional $P$ are defined in Section 4.1. For a related result, see [25]. In the physics literature, this rate function is referred to as the free energy for a system of “two real replica” [43, 60].

With this in hand, we may easily improve upon the spectral bound provided in Theorem 1.3 through FEB. To this end, define the following quantity

$$H = \sup_{q_1 < q_2 < q_3} -I(q_1) + I(q_2) - I(q_3).$$

(1.15)

Heuristically, $H$ encodes the “length” of a passage between $q_1$ and $q_3$ though the point $q_2$. We then have the following generalization of a free energy barrier.

**Definition 1.14.** We say that Generalized FEB holds if $H > 0$.

Generalized FEB is of course natural related to FEB. Indeed, the following holds.

**Proposition 1.15.** FEB implies Generalized FEB

We introduce the notion of Generalized FEB as it may hold in broader generality. Furthermore, it implies the following stronger result.

**Theorem 1.16.** We have that

$$\lim \frac{1}{N} \log \lambda_1 \leq -H.$$

(1.16)

almost surely. In particular, if $\xi$ is convex and Generalized FEB holds, this is strictly negative.

The question of slow mixing then reduces to showing that Generalized FEB holds. To this end, we note the following equivalent statement. Recall that by [79] Theorem 1, there is a $q$ such that $I(q) = 0$. (In fact, this holds for every $q$ in the support of any limit point of $\zeta$, see Lemma 7.2.) Thus we see that Generalized FEB is equivalent to the following property of the rate function.

**Proposition 1.17.** Generalized FEB holds if and only if there is some $q_0$ with $I(q_0) = 0$ such that $I$ is not both:

- non-increasing on $[-1, q_0]$, and
- non-decreasing on $[q_0, 1]$.

It is now a good time to relate this approach the the one of the previous section. To show that $I$ is not monotone, we find two points, $q_1, q_3$ that satisfy (1.5). By an elementary argument, see Lemma 7.2 this implies that $I(q_i) = 0$. The main observation is that if one of these points, say $q_1$, has a positive replicon eigenvalue, then $I > 0$ in a neighborhood of that point.

**Theorem 1.18.** Suppose that $q_* \in \text{supp}(\mu)$ has a positive replicon eigenvalue, $\Lambda_R(q_*, \mu) > 0$. Then there is a punctured neighborhood of $q_*$, $E = (q_* - \epsilon_0, q_* + \epsilon_0) \cap (0,1) \setminus \{q_*\}$, such that for $q \in E$, $I(q) > 0$

Consequently, $I$ will not be monotone.

In this setting, there is a natural analogue of Theorem 1.8. To this end, we introduce the following definitions which are modifications of the previous conditions.
Definition 1.19. We say that GRSB holds if the Parisi measure, \( \mu \), has at least two points in its support.

Observe that this is different from RSB as the former requires assumptions regarding the overlap distribution. Secondly, we have a modification of PREV.

Definition 1.20. We say that GPREV holds if for some \( q \in \text{supp}(\mu) \),

\[ \Lambda_R(q, \mu) > 0. \]

We then have the following theorem.

Theorem 1.21. Suppose that \( \xi \) is convex. If GRSB and GPREV hold, then GFEB holds.

As an example, of models to which this applies we note that this of course subsumes Theorem 1.9, without the requirement that the model be generic.

Theorem 1.22. Suppose that \( \xi = \beta^2 \xi_0 \) has \( \xi_0''(0) = 0 \) and is convex. Then there is an \( h_0 \) such that for \( h \leq h_0 \), if the Parisi measure is not an atom, then GPREV and GRSB hold. Consequently,

\[ \lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left( \frac{1}{N} \log \lambda_1 > -c \right) < 0 \]

for some \( c > 0 \). In particular, this holds if \( \beta \) is sufficiently large.

It is natural to ask if the condition GFEB holds in models even when GRSB does not hold, i.e., even in the replica symmetric phase. This is discussed presently.

**Dynamical Phase transitions in Ising spin models.** We end our discussion of Ising spin models by observing the following. An important consequence of these results is that they resolve a natural question raised in the physics literature namely if the static glass transition is always below the dynamical glass transition. To make this precise, let \( \nu_\beta \) denote the minimizer of (1.11) for \( \xi \) of the form \( \xi = \beta^2 \xi_0 \). Define

\[ \beta_s = \max \{ \beta > 0 : \nu_\beta = \delta_q \text{ for some } q \in [0,1] \} \]

\[ \beta_d = \min \{ \beta > 0 : \lim \mathbb{P}(-\frac{1}{N} \log \lambda_1(\beta) \geq C) = 1 \text{ for some } C > 0 \}. \]

It is predicted that \( \beta_s \geq \beta_d \) \[27, 70\]. This is a consequence of Theorem 1.9. In fact, we may go further. If we let

\[ \beta_{GFEB} = \min \{ \beta > 0 : \text{GFEB holds} \} \]

then as a consequence of the above, we have

**Corollary 1.23.** If \( h = 0 \), \( \xi_0''(0) = 0 \) and \( \xi_0 \) is convex, then

\[ \beta_d \leq \beta_{GFEB} < \beta_s \]

We end this section with the following natural question.

**Question 1.** Is it true that \( \beta_d = \beta_{GFEB} \)?

1.3. **Spherical Models.** Let us now consider spherical mixed \( p \)-spin glasses, which we refer to as spherical models for short. For these models the configuration space will be \( S_N = S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N \), which we equip with the usual, induced metric, \( g \), and the normalized volume measure, \( d\sigma \). Let \( \xi \) and \( h \) be as before. The Hamiltonian for this model will be the Gaussian process on \( S_N \) with mean and covariance given by (1.1), and the Gibbs measure, \( \pi_N \), will be as in (1.2) with respect to the normalized volume measure as opposed to the uniform measure.
Our goal in this section is to understand the relaxation time of the Langevin dynamics of this model. The Langevin dynamics is the heat flow induced by
\[ \mathcal{L}_H = -\Delta + g(\nabla H, \nabla \cdot) \]  
(1.19)
It is known that \( H \) is (a.s.) smooth and Morse, so that \( \mathcal{L}_H \) is essentially self-adjoint with pure point spectrum. In particular, this dynamics is uniquely defined. We wish to analyze the asymptotics of \( \lambda_1 \), the first nontrivial eigenvalue of \( \mathcal{L}_H \).

Again our starting point is by relating the spectral gap to free energy barriers for the overlap. Define free energy barriers as in Definition 1.2. We then have the following.

**Theorem 1.24.** If there exists a free energy barrier for the overlap of height \( C > 0 \), then
\[ \lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left( \frac{1}{N} \log \lambda_1 > -C \right) < 0. \]  
(1.20)

As in the Ising spin setting, we wish to show that FEB holds under an PREV-type condition. To this end, let
\[ \zeta_N(\cdot) = \mathbb{E} \pi_N^{\otimes 2}(R_{12} \in \cdot), \]
as in (1.7) except here \( \pi_N \) is the Gibbs measure for the spherical model. We may then define \( \zeta \), \( A \), and RSB from above analogously. It remains to define the analogue of PREV, specifically the replicon eigenvalue.

To this end, consider the Crisanti-Sommers functional. For \( \nu \in \Pr([0,1]) \), the Crisanti-Sommers functional, \( C(\nu) \), is given by
\[ C(\nu) = \frac{1}{2} \left( \int \xi''(s)\varphi_{\nu}(s) + \int \frac{1}{\varphi_{\nu}(s)} - \frac{1}{1 - s}ds + h^2\varphi_{\nu}(0) \right), \]  
(1.21)
where
\[ \varphi_{\nu}(s) = \int_s^1 \nu([0,s])ds. \]  
(1.22)
Observe that \( C \) is lower semicontinuous and strictly convex, so that the existence and uniqueness of this minimizer are guaranteed. The Crisanti-Sommers functional provides a variational formula for the free energy:
\[ F = \lim_{N \to \infty} \frac{1}{N} \log \int e^{-H} d\sigma = \min_{\nu} C(\nu), \]  
(1.23)
where \( d\sigma \) is the normalized volume measure on the sphere. This formula, called the Crisanti-Sommers formula, was proved by Talagrand [89] in our setting and Chen [33] for more general \( \xi \).

For every \( \nu \in \Pr([0,1]) \) and \( q \in \text{supp}(\nu) \), the replicon eigenvalue for spherical models is
\[ \Lambda_R(s, \nu) = \frac{1}{\varphi_{\nu}^2(s)} - \xi''(s), \]
and is related to the case of optimality in a certain obstacle problem [55, 57]. With this definition, we may then define the PREV condition analogously to Definition 1.6.

Our main result in this setting is the following.

**Theorem 1.25.** If \( \xi \) is convex and \( (\xi,h) \) satisfy PREV, then FEB holds. In particular, for some \( C > 0 \),
\[ \lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left( \frac{1}{N} \log \lambda_1 > -C \right) < 0. \]

Let us now turn to an example of models for which these results apply.

**Theorem 1.26.** Suppose that either:
(1) $\xi = \beta^2 \xi_0$ is convex and has $\xi_0''(0) = 0$ and is generic or even generic, or
(2) $\xi(t) = \beta^2 t^p$ for even $p \geq 4$.

Then there is an $h_0 > 0$ such that for $h \leq h_0$, if the Parisi measure, $\mu$, is not an atom, then PREV holds. Consequently,

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left( \frac{1}{N} \log \lambda_1 > -c \right) < 0$$

for some $c > 0$. In particular, this holds if $\beta$ is sufficiently large.

Remark 1.27. We note here that the main result does not use the form of $L_H$ in an essential way. For example, if $L$ is the infinitesimal generator for any other reversible dynamics for $\pi_N$, then the result still holds provided the corresponding Carré du champ operator, $\Gamma_1(f)(x)$, satisfies the gradient estimate $\Gamma_1(f)(x) \leq C g(Df, Df)$ for some $C = C(N)$ that grows at most polynomially in $N$. See Remark 2.8

In our setting, the matching exponential lower bound has been proved in [46]. The proof provided there works for all $\xi$, though it is stated only for Pure $p$-spin models.

**Theorem 1.28.** There is a constant $c(\xi, h)$ such that

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left( \frac{1}{N} \log \lambda_1 < -c \right) < 0,$$

(1.24)

**Dynamical phase transitions for spherical models.** An important consequence of these results is that they resolve a natural question raised in the physics literature namely if the static glass transition is always below the dynamical glass transition. To make this precise, let $\nu_\beta$ denote the minimizer of (1.21), for $\xi$ of the form $\xi = \beta^2 \xi_0$. Define

$$\beta_s = \max \{ \beta > 0 : \nu_\beta = \delta_q \text{ for some } q \in [0, 1] \}$$

$$\beta_d = \min \{ \beta > 0 : \lim \mathbb{P} \left( -\frac{1}{N} \log \lambda_1(\beta) \geq C \right) = 1 \text{ for some } C > 0 \}.$$

It is predicted that $\beta_s \geq \beta_d$ [27, 70]. This is a consequence of Theorem 1.26.

**Corollary 1.29.** If $h = 0$, $\xi_0(0) = 0$, and $\xi_0$ is convex and generic or even generic, or $\xi_0 = t^p$ for some even $p \geq 4$, then $\beta_s \geq \beta_d$.

1.4. **Outline of Paper.** The paper is organized as follows. We begin by introducing, in Section 2, the needed bounds on spectral gap, along the classical lines of Cheeger inequalities. We begin in Section 2.1 by introducing the notion of the landscape difficulty of a function, in a very general context for reversible dynamics on metric measure spaces, which cover the cases needed here of weighted graphs or Riemannian manifolds. This notion quantifies how long it takes for a given function to be “equilibrated” by the dynamics, in an exponential time scale. We then define the landscape difficulty of a metric measure space as the maximal landscape difficulty, among Lipschitz functions. In Section 2.2, we show how this notion of landscape difficulty can be applied to the case of finite graphs, and get upper bounds on the spectral gap in Section 2.2.1. We also give a short treatment to lower bounds on the spectral gap through Poincaré inequalities in Section 2.2.2. In Section 2.3, we give the analogous bounds for the spectral gap for compact Riemannian manifolds.

Section 3 is devoted to proving that free energy barriers for mean-field spin glasses imply an exponential bound for the spectral gap for their dynamics using the abstract tools introduced in Section 2. The main idea is here is that the overlap is a difficult function to equilibrate for “replicated” dynamics. We do this first in Section 3.1 for Ising Spin glasses. We show there that the existence of free energy barriers for the overlap imply an exponential upper bound for the spectral gap of Ising Spin glasses. We complete the analogous task for spherical spin glasses in Section 3.1.
In the brief Section 3.3, we prove lower bounds on the spectral gap for both Ising and spherical spin glasses.

Section 4 is devoted to the proof of existence of free energy barriers for the overlap at low temperature, for Ising spin glasses. This section only deals with equilibrium quantities. We show how the behavior of the overlap distribution, for large $N$, encodes the existence of free energy barriers, and thus show that the overlap is difficult to equilibrate. This section uses recent deep tools about the static behavior of spin glasses, like the 2D Guerra-Talagrand bounds, which we recall in Section 4.1. In Section 4.2 we show how a positive replicon eigenvalue can help bound the 2D Guerra-Talagrand functional. In Section 4.3 and 4.4 we show how this information allows us to prove quickly inequalities like (1.4) as well as Theorem 1.8.

Section 5 extends the results of Section 4 to the case of spherical spin glasses, using the Crisanti-Sommers formula and the spherical version of the 2D Guerra-Talagrand bounds.

In Section 6, we show that wide classes of models satisfy our sufficient conditions for slow mixing.

In Section 7, we improve upon our spectral approach by introducing a generalized FEB. In the process, we will prove an LDP for the overlap distribution which follows from a recent deep result of Panchenko [79] on the matching lower bound to the 2D Guerra-Talagrand bound.

We have pushed some of the most technical properties of the Parisi PDE to Appendix A (resp. Appendix B) needed in Section 4 (resp. Section 5).

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2. The landscape difficulty and spectral gap bounds for metric measure spaces

We introduce here the notion of the landscape difficulty of a function and the maximal landscape difficulty of a measure in our setting. The results of this section do not depend of the rest of the of the paper and will be for deterministic in a fixed class of metric measure spaces.

Our goal is to produce upper bounds on the spectral gap of the infinitesimal generator of a Markov process on some metric measure space. For us this will be either a finite graph with a metric and measure or a compact (weighted) Riemannian manifold. We wish for these estimates to be intrinsic to the metric measure space. A classical approach to prove such bounds is through isoperimetric methods: through Cheeger’s and Buser’s inequalities [31, 26] in the manifold setting and what is called alternately the Cheeger constant, Bottleneck ratio, or conductance bound in the setting of graphs [61, 59, 66, 2].

Instead of working with sets, their complements, and surface areas, we work with the volumes of a specific family of sets. We work with the level sets of Lipschitz functions. Unlike, for example, the bottleneck ratio, we want to be able to take these level sets to correspond to disjoint energy windows, i.e. $\{f \in (E_i - \epsilon, E_i + \epsilon)\}$ for $\epsilon$ small enough and $E_i - E_j$ large enough. To this end, we will use a modification of [46, Proposition 21]. Though in principle, this estimate is highly suboptimal asymptotically as $\epsilon \to 0$ (see Remark 2.10), provided the Lipschitz constant of the function is itself smaller this is not a major issue.

Let us now turn to the main results of this section. In the following we say that $f \lesssim g$ if there is a universal constant $C$ such that $f \leq Cg$ and that $f \lesssim_a g$ if $C$ depends at most on $a$. If $A$ is a Borel subset of a metric space $(X, d)$ then we define the $\epsilon$-dilation of this set by $A_{\epsilon} = \{x \in X : d(x, A) \leq \epsilon\}$. Finally, $Lip_K$ denotes the space of $K$-Lipschitz functions.
2.1. The notion of landscape difficulty for a metric measure space. Let \((X, d, \nu)\) be a metric measure space. Let \(f\) be a \(K\)-Lipschitz function on \((X, d, \nu)\). For \(E \in \mathbb{R}\) and \(\epsilon > 0\), let
\[
S(E, \epsilon; f) = \log \nu(f \in (E - \epsilon, E + \epsilon)).
\] For every \(C > 0\), let
\[
\mathcal{R}_C = \left\{ (E_1, E_2, E_3) \in \mathbb{R}^3 : E_1 < E_2 < E_3, \ C \leq \frac{1}{4} \min\{E_2 - E_1, E_3 - E_2\} \right\}.
\] Define the function \(\Phi : \mathcal{R}_\epsilon \to \mathbb{R}\) by
\[
\Phi(E_1, E_2, E_3; \epsilon; f) = S(E_1, \epsilon; f) + S(E_3, \epsilon; f) - S(E_2, \epsilon; f).
\] Define the \(\epsilon\)-landscape difficulty of \(f\), or simply the \(\epsilon\)-difficulty of \(f\), by
\[
\mathcal{D}_\epsilon(f) = \sup_{a, b, c \in \mathcal{R}_\epsilon} \Phi(a, b, c; \epsilon; f).
\] Finally, define the \((K, \epsilon)\)-maximal landscape difficulty of \(\nu\), or simply the \((K, \epsilon)\)-maximal difficulty of \(\nu\), by
\[
\mathcal{D}_\epsilon(\nu, K) = \sup_{f \in \text{Lip}_K} \mathcal{D}_\epsilon(f).
\] Here \(\text{Lip}_K\) is the space of \(K\)-Lipschitz functions. Note that \(\mathcal{D}_\epsilon(f)\) depends on \(\nu\) as well, however, to distinguish this notation from \(\mathcal{D}_\epsilon(\nu, K)\), we omit the dependence. We then have the following definition.

**Definition 2.1.** Let \((X, d, \nu)\) be a metric measure space. It is said that there is a free energy barrier corresponding to \(f\) if for some \(\epsilon > 0\),
\[
\mathcal{D}_\epsilon(f) > \log 4.
\] It is said that \(\nu\) is \((K, \epsilon)\)-difficult if the maximal \((K, \epsilon)\)-difficulty of \(\nu\) satisfies
\[
\mathcal{D}_\epsilon(\nu, K) > \log 4.
\]

2.2. Spectral Gap bounds for Finite Graphs. Let \(G = (V, E)\) be a finite graph with a metric \(d\) and measure \(\nu\) on the vertex set \(V\). We call the triple \((G, d, \nu)\) a metric measure graph. For two points \(x, y \in V\), we say that \(x \sim y\) if \(x\) and \(y\) are connected by an edge. Let \(\Omega(x) = \{y : x \sim y\}\) denote the set of nearest neighbors of \(x\). Let \(D = \max_{x \sim y} d(x, y)\). For a function \(f : V \to \mathbb{R}\) on the vertex set of \(G\), we define the discrete gradient by
\[
\nabla f = (f(x) - f(y))_{y \in \Omega(x)}.
\] For a set of vertices, \(S\), let \(\partial S\) be those vertices in \(S\) that have at least one edge leaving \(S\).

Let \(Q\) be a transition matrix on \(V\) with invariant measure \(\nu\). We say that \(Q\) is nearest neighbor if \(Q(x, y) > 0\) only if \(x \sim y\) or \(x = y\). Let the Dirichlet form be given by
\[
\mathcal{E}(f, g) = \langle (I - Q)f, g \rangle_\nu.
\] Recall that since \(\nu\) is reversible,
\[
\mathcal{E}(f, f) = \frac{1}{2} \sum_x \sum_y (f(x) - f(y))^2 Q(x, y) \nu(x).
\] Recall from Rayleigh’s min-max principle [66] that, the spectral gap for \(Q\), call it \(\gamma_Q\), satisfies
\[
\gamma_Q = \min_{\text{Var}_\nu(f) \neq 0} \frac{\mathcal{E}(f, f)}{\text{Var}_\nu(f)}.
\]
2.2.1. Upper bounds through the landscape difficulty: graphs. In order to prove spectral gap upper bounds, we will wish to take the following modification of the Conductance bound with respect to sub-level sets of regular functions. Define the difficulty and maximal difficulty for the metric measure space \((V,d,\nu)\) as in Definition 2.1. We then have the following theorem which allows us to bound the spectrum of \(I - Q\) using quantities that relate only to \((\nu,d)\).

**Theorem 2.2.** Let \((G,d,\nu)\) be a metric measure graph. Let \(Q\) be a transition matrix for that satisfies detailed balance with respect to \(\nu\) that is nearest-neighbor and has spectral gap \(\gamma_Q\). Let \(K > 0\) and \(\epsilon > 2 \cdot K \cdot D\). If \(\nu\) is \((K,\epsilon)\)-difficult, then the spectral gap and the maximum \((K,\epsilon)\)-difficulty satisfy the relation

\[
\gamma_Q \leq 2 \left( \frac{K \cdot D}{\epsilon} \right)^2 \frac{e^{-D_A(\nu,K)}}{1 - 4e^{-D_A(\nu,K)}}.
\]

(2.7)

**Remark 2.3.** This result uses the nearest-neighbor property rather weakly. In particular, in the language of Bakry-Émery theory, if we let \(\Gamma_1(f)(x)\) denote the Carré du champ

\[
\Gamma_1(f)(s) = \frac{1}{2} \sum_{y}(f(x) - f(y))^2 Q(x,y)
\]

then we use here simply that

\[
\Gamma_1(f)(x) \leq \frac{1}{2} \|\nabla f\|^2_{\infty}(x),
\]

This argument should extend to local dynamics, and even non-local dynamics provided one assumes that \(Q(x,y)\) decays sufficiently fast as \(d(x,y)\) increases, though we do not pursue this direction here.

In order to prove this theorem, we start with the following estimate. This is a modification and discretization of [46, Proposition 21].

**Lemma 2.4.** Let \(Q\) be the transition matrix on a metric measure graph \((G,d,\nu)\) that satisfies detailed balance with respect to \(\nu\), is nearest-neighbor, and has spectral gap, \(\gamma_Q\). Let \(A \subset V\) and let \(B = \partial A \cup \partial A^c \cup A^c \setminus A\). Suppose that

\[
\nu(A) \nu(A^c) - 4 \nu(B)^2 > 0.
\]

Then \(\gamma_Q\) satisfies

\[
\gamma_Q \leq \frac{D^2}{2\epsilon^2} \frac{\nu(B)}{\nu(A^c)\nu(A) - 4 \nu(B)^2}.
\]

**Proof.** We begin by the following simple observation. Since \(Q\) is a transition matrix, \(\|Q(x,\cdot)\|_1 = 1\) for every \(x\). Thus by Hölder’s inequality and the fact that \(Q\) is nearest-neighbor, we have that for any function \(f\) on \(V\),

\[
\mathcal{E}(f,f) = \frac{1}{2} \sum_x \sum_y (f(x) - f(y))^2 Q(x,y)\nu(x) \leq \frac{1}{2} \sum_x \|\nabla f\|^2_{\infty}(x)\nu(x).
\]

(2.8)

Let

\[
\psi(x) = \begin{cases} 
\nu(A) & x \in A^c \\
-\nu(A^c) & x \in A \\
-\nu(A^c) + \min\{ \frac{d(x,A)}{\epsilon}, 1 \} & x \in A^c \setminus A
\end{cases}.
\]

We now bound the gradient of \(\psi\). From the form of \(\psi\), we obtain the gradient estimate

\[
\|\nabla \psi\|^2_{\infty}(x) \leq \max_{y \in \Omega(x)} \frac{d(x,y)^2}{\epsilon^2} \mathbb{1}_{x \in B}.
\]
Applying this to the estimate, \[(2.8)\], on the Dirichlet form gives the bound

\[\mathcal{E}(\psi, \psi) \leq \frac{D^2}{2\epsilon^2} \nu(B).\]  

(2.9)

On the other hand,

\[\left| \int \psi d\nu \right| \leq \left| \int_{(A_e \setminus A)^c} \psi d\nu \right| + \left| \int_{A_e \setminus A} \psi d\nu \right| \leq 2\nu(A_e \setminus A),\]

and

\[\int \psi^2 d\nu \geq \int_{(A_e \setminus A)^c} \psi^2 d\nu = \nu(A)^2 (\nu(A^c) - \nu(A_e \setminus A)) + \nu(A^c)^2 \nu(A) = \nu(A)\nu(A^c) - \nu(A)^2 \nu(A_e \setminus A)\]

Thus

\[\text{Var}_\nu(\psi) \geq \nu(A)\nu(A^c) - \nu(A)^2 \nu(A_e \setminus A) - 4\nu(A_e \setminus A)^2 - \nu(A^c)(\nu(A_e \setminus A) - \nu(A)^2 \nu(A_e \setminus A) - 4\nu(A_e \setminus A)^2\]

Since \(A_e \setminus A \subset B\), it follows that

\[\text{Var}_\nu(\psi) \geq \nu(A)\nu(A^c) - 4\nu(B)^2.\]

Thus the Rayleigh quotient satisfies

\[\frac{\mathcal{E}(\psi, \psi)}{\text{Var}_\nu(\psi)} \leq \frac{D^2}{2\epsilon^2} \frac{\nu(B)}{\nu(A)\nu(A^c) - 4\nu(B)^2}.\]

The result then follows by Rayleigh’s min-max principle, \[(2.6)\].

□

In the following, we will use a specific form of this estimate.

**Corollary 2.5.** Let \((G, d, \nu), Q\), and \(\gamma_Q\) be as in Lemma \[2.4\]. Let \(L \in \mathbb{R}\), let \(f\) be a \(K\)-Lipschitz function on \(V\). Suppose that

\[\nu(f \geq L)\nu(f \leq L - 2K\delta \vee D) - 4\nu(f \in (L - 2K\delta \vee D, L + 2K\delta \vee D))^2 > 0.\]

(2.10)

Then

\[\gamma_Q \leq \frac{D^2}{2\delta^2} \frac{\nu(f \in (L - 2K\delta \vee D, L + 2K\delta \vee D))}{\nu(f \geq L)\nu(f \leq L - 2K\delta \vee D) - 4\nu(f \in (L - 2K\delta \vee D, L + 2K\delta \vee D))^2}.\]

**Proof.** Let \(A = \{f \geq L\}\). Observe that

\[A_\delta^c \subset \{f \geq L - K\delta\}, \quad \partial A \subset \{f \in [L, L + K\delta]\}.\]

Suppose that \(x \in \partial A_\delta^c\). Then there is a \(y \in A_\delta\) such that

\[d(y, x) \leq D.\]

Then

\[f(x) \geq f(y) - Kd(x, y) \geq L - K(\delta + D).\]

Thus

\[\partial A_\delta^c \subset \{f \geq L - 2K\delta \vee D\}.\]

Thus if we let

\[\tilde{B} = \{f \in (L - 2K\delta \vee D, L + 2K\delta \vee D)\},\]
it follows that $B$ from Lemma 2.4 satisfies $B \subset \tilde{B}$. Applying Lemma 2.4 then yields

$$\gamma_Q \leq \frac{D^2}{2\delta^2} \frac{\nu(\tilde{B})}{\nu(A)\nu(A_\delta) - 4\nu(B)^2}.$$  

The result then follows by set containment. □

We can now prove Theorem 2.2.

**Proof of Theorem 2.2.** Suppose that $\nu$ is $(K, \epsilon)$-difficult. Then there is a $K$-Lipschitz $f$, an $\epsilon > 2K \cdot D$, and a triple $(E_1, E_2, E_3) \in \mathcal{R}_\epsilon$ such that

$$\Phi(E_1, E_2, E_3; \epsilon) > \log 4,$$

and such that $S(E_1, \epsilon; f), S(E_2, \epsilon; f) > -\infty$. Set $\delta = \frac{\epsilon}{2K}$, then $\delta > D$. Thus, if we let $L = E_2$, then

$$\nu(f \geq L)\nu(f \leq L - 2K\delta \vee D) - 4\nu(f \in (L - 2K\delta \vee D, L + 2K\delta \vee D))^2 = \nu(f \geq E_2)\nu(f \leq E_2 - \epsilon) - 4\nu(f \in (E_2 - \epsilon, E_2 + \epsilon))^2 \geq \nu(f \in (E_3 - \epsilon, E_3 + \epsilon))\nu(f \in (E_1 - \epsilon, E_1 + \epsilon))(1 - 4e^{-\Phi}).$$

By assumption on $\Phi$ and $S$, and since $\delta > D$, we see that (2.10) is positive. We may then apply Corollary 2.5 to obtain

$$\gamma_Q \leq \frac{(2 \cdot K \cdot D)^2}{2\epsilon^2} \frac{e^{-\Phi}}{1 - 4e^{-\Phi}}.$$

Minimizing the right hand side in $E_1, E_2, E_2, \epsilon$, and $f$ and using the fact that the function

$$x \mapsto \frac{e^{-x}}{1 - 4e^{-x}}$$

is decreasing for $x \geq \log 4$, yields the result. □

2.2. Lower bound by stability of Poincaré inequalities: graphs . This inequality allows us to prove spectral gap upper bounds. Correspondingly it will be useful to obtain lower bounds. To this end, we remind the reader of the following classical stability property of Poincaré inequalities due to Holley and Stroock [54]. See also [53, 83].

**Proposition 2.6.** Let $(X, d, \mu)$ be a finite metric measure space and let $d\nu = \frac{e^{-U(x)}}{Z}d\mu$, as before. Suppose that $Q(x, y)$ is a transition matrix that satisfies detailed balance with respect to $\nu$, and that $P(x, y)$ is a transition matrix for a Markov chain that satisfies detailed balanced with respect to $\mu$. Suppose furthermore that $AP(x, y) \leq Q(x, y)$. Then if $P$ has spectral gap $\gamma_P$, the spectral gap of $Q$ satisfies

$$Ae^{-2(\max U - \min U)}\gamma_P \leq \gamma_Q.$$

2.3. Spectral Gap bounds for Compact Riemannian Manifolds. Let $(M, g)$ be a smooth compact boundary-less Riemannian manifold equipped with some measure

$$\nu = \frac{e^{-U}d\text{vol}}{Z},$$

where $U \in C^\infty(M)$. For a function $f \in C^\infty(M)$ we let $Df$ denote the usual gradient and we let $\Delta f$ denote the Laplace-Beltrami operator. Let $L = \Delta - g(DU, D\cdot)$ be the corresponding Langevin operator, and define the corresponding Dirichlet form

$$\mathcal{E}(f, h) = \int g(Df, Dh)d\nu.$$

Note that $L$ is uniformly elliptic with smooth bounded coefficients so its eigenfunctions are smooth by standard elliptic regularity [17, 111]. Furthermore, $L$ is symmetric on $C^\infty(M)$ with respect to $\nu$
so that in fact by this regularity result, one can show that it is essentially self-adjoint there and has pure point spectrum \(0 = \lambda_0 \leq \lambda_1 \leq \ldots\) As a result, the corresponding heat flow \(P_t = e^{tL}\) is well defined.

By the Courant-Fischer min-max principle [62, 30], recall that the first non-trivial eigenvalue of \(L\) satisfies

\[
\lambda_1 = \min_{f \in C^\infty(M), \Var_{\nu}(f) \neq 0} \frac{\mathcal{E}(f, f)}{\Var_{\nu}(f)}.
\]

That this is a minimum and not an infimum can be seen by elliptic regularity [41, 30]. It will be useful to note that one can relax this minimization problem to being over the space \(H^1(\nu) \cap \{\|Df\|_{L^2(\nu)} \neq 0\}\).

2.3.1. Upper bound using the landscape difficulty: Riemannian manifolds. As before, we seek to bound \(\lambda_1\) using quantities that are intrinsic to the metric measure space \((M,dg,\nu)\). Define the difficulty and maximal difficulty as in Definition 2.1 for \((M,dg,\nu)\). We then have the following theorem.

**Theorem 2.7.** Let \((M,g)\) be a smooth compact, boundary-less Riemannian manifold, and let \(\nu = e^{-U}dvol\) for some smooth \(U\). Let \(L = (\Delta - g(DU,D\cdot))\) with first nontrivial eigenvalue \(\lambda_1\). Let \(K,\epsilon > 0\). If \(\nu\) is \((K,\epsilon)\)-difficult, then the spectral gap and the maximum difficulty satisfy the relation

\[
\lambda_1 \leq K^2 \frac{e^{-\mathcal{D}_\nu(\nu,K)}}{\epsilon^2} \cdot \frac{1}{1 - 4e^{-\mathcal{D}_\nu(\nu,K)}}.
\]

**Remark 2.8.** Again, this result uses the form of \(L\) rather weakly. In particular, if we study a general reversible dynamics with infinitesimal generator \(L\), and let \(\Gamma_1(f)(x)\) denote the corresponding Carré du champ, then the above result holds, for example, if

\[
\Gamma_1(f)(x) \leq Cg(Df,Df)^2(x),
\]

where the above inequality will have an additional factor of \(C\).

The proof of this is similar to the discrete setting. We begin, as before, with the following which is a small modification of [46, Proposition 21].

**Lemma 2.9.** Let \((M,g)\) be smooth compact, boundary-less Riemannian manifold, and let \(A \subset M\) be Borel. Let \(\nu = e^{-U}dvol\) for some smooth \(U\), let \(L = (\Delta - g(DU,D\cdot))\), and let \(\mathcal{E}\) be its corresponding Dirichlet energy. Let \(\lambda_1\) be first eigenvalue for \(L\) restricted to the the orthogonal complement of the constant functions. Let \(B = A_c \setminus A\). Then provided \(\nu(A)\nu(A_c^c) - 4\nu(B)^2 > 0\) we have that

\[
\lambda_1 \leq \frac{\nu(B)}{\epsilon^2 \nu(A) \nu(A_c^c) - 4\nu(B)^2}.
\]

**Remark 2.10.** As observed in [46, Proposition 21], this estimate is highly suboptimal in \(\epsilon\). Indeed as \(\epsilon \to 0\), the numerator scales like \(\epsilon\) so that the expression scales like \(\epsilon^{-1}\). See for example [11, 63]. In our applications, however, this will be irrelevant.

**Proof.** Consider the test function

\[
\psi(x) = \begin{cases} 
\nu(A) & \text{on } (A_c^c) \\
-\nu(A_c^c) & \text{on } A \\
-\nu(A_c) + \min \left\{ \frac{d(x,A)}{\epsilon}, 1 \right\} & \text{on } B
\end{cases}
\]
Observe that since $d(x, A)$ is Lipschitz, $\psi \in H^1(d\text{vol})$ and thus $H^1(d\nu)$. Observe furthermore that since $d(x, A)$ is Lipschitz, we have that
\[
\|\nabla \psi\|_\infty \leq \frac{1}{\epsilon}.
\]
Thus if we evaluate this on the Dirichlet form, we get
\[
\mathcal{E}(\psi, \psi) = \int_M g(D\psi, D\psi) d\nu = \int_B g(D\psi, D\psi) d\nu \leq \frac{1}{\epsilon^2} \nu(B).
\]
The variance lower bound is identical to that in Lemma 2.4. Thus by the Courant-Fischer min-max principle [62],
\[
\lambda_1 \leq \frac{\mathcal{E}_R(\psi, \psi)}{\text{Var}_\nu(\psi)} \leq \frac{\nu(B)}{\epsilon^2 \nu(A) \nu(A^c) - 4 \nu(B)^2},
\]
as desired. □

We apply this for Lipschitz statistics.

**Corollary 2.11.** Let $(M, g)$ and $\lambda_1$ be as in Lemma 2.9. Let $L \in \mathbb{R}$, let $f$ be a $K$-Lipschitz function on $M$. Suppose that
\[
\nu(f \geq L) \nu(f \leq L - K\delta) - 4 \nu(f \in [L - K\delta, L])^2 > 0.
\]
Then
\[
\lambda_1 \leq \frac{\nu(f \in [L - K\delta, L])}{\delta^2 \nu(f \geq L) \nu(f \leq L - K\delta) - 4 \nu(f \in [L - K\delta, L])^2}
\]
provided the denominator is positive.

**Proof.** Let
\[
A = \{f \geq L\}
\]
Then
\[
A_\epsilon \subset \{f \geq L - K\delta\} \quad A \setminus A_\epsilon \subset \{f \in [L - K\delta, L]\} \quad A^c_\epsilon \supset \{f \leq L - K\delta\}.
\]
The result then follows by Lemma 2.9. □

We may then prove Theorem 2.7.

**Proof of Theorem 2.7** Suppose that $\nu$ is $(K, \epsilon)$-difficulty. Then there is a $K$-Lipschitz $f$, and $\epsilon > 0$, and a triple $(E_1, E_2, E_3) \in \mathcal{R}_\epsilon$ such that
\[
\Phi(E_1, E_2, E_3, \epsilon; f) > \log 4,
\]
and such that $S(E_1, \epsilon; f), S(E_3, \epsilon; f) > 0$. Set $\delta = \frac{\epsilon}{K}$. If we let $L = E_2$, then
\[
\nu(f \geq L) \nu(f \leq L - K\delta) - 4 \nu(f \in [L - K\delta, L])^2 \geq \nu(f \in (E_3 - \epsilon, E_3 + \epsilon)) \nu(f \in (E_1 - \epsilon, E_1 + \epsilon)) (1 - 4e^{-\Phi}).
\]
By the assumption on $\Phi$ and $S$, we see that $(2.14)$ is positive. We may then apply Corollary 2.11 to obtain
\[
\lambda_1 \leq \frac{K^2 e^{-\Phi}}{\epsilon^2 (1 - 4e^{-\Phi})}.
\]
Then by Corollary 2.11 if we let $\delta = \frac{\epsilon}{K}$,
\[
\lambda_1 \leq \frac{K^2}{\epsilon^2} \nu(f \geq E_2) \nu(f \leq E_2 - \epsilon) - 4 \nu(f \in [E_2 - \epsilon, E_2])^2.
\]
Minimizing the right hand side in $(E_1, E_2, E_3, \epsilon, f)$ and using the fact that the function $(2.11)$ is decreasing for $x \geq \log 4$ yields the result. □
2.3.2. Lower bound by stability of Poincaré inequalities: manifolds. We end this section again by noting the following classical result regarding the stability of Poincaré inequalities due to Holley and Stroock [54]. See also [53].

**Proposition 2.12.** Suppose that $-\Delta$ has first non-trivial eigenvalue $\lambda_1(M)$. Then $\lambda_1$, satisfies

$$e^{-2(\max U - \min U)}\lambda_1(M) \leq \lambda_1$$

3. Spectral Gap bounds in the presence of a free energy barrier

In this section, we show how a free energy barrier will imply spectral gap upper bounds for dynamics of mean field spin glasses. In particular, we aim to prove Theorem 1.3 and Theorem 1.24.

Before turning to these proofs, we begin by observing the following elementary consequence of concentration of measure. Recall that by Gaussian concentration [64] for both spherical and Ising spin models, if

$$Z_N(A) = \int \int_{R^{12}} e^{-H(\sigma_1) - H(\sigma_2)} d\sigma_1 d\sigma_2,$$

then there is a $K = K(\xi,h)$ such that

$$\mathbb{P}\left(\left| \frac{1}{N} \log Z_N(A) - \mathbb{E}\left(\frac{1}{N} \log Z_N(A)\right) \right| > \epsilon \right) \leq K e^{-N\epsilon/K}, \quad (3.1)$$

for all $\epsilon > 0$. As a consequence, we have the following.

**Lemma 3.1.** Fix $\xi,h$. There is a constant $K = K(\xi,h) > 0$ such that the following holds for both Ising spin and spherical models. Suppose that there is a relatively open subset $A \subset [-1,1]$ such that

$$\lim \mathbb{E}_{\pi_N^{\otimes 2}}(R_{12} \in A) > 0.$$

Then

$$\mathbb{P}\left(\frac{1}{N} \log \int \int_{R_{12} \in A} e^{-H(\sigma_1) - H(\sigma_2)} d\sigma_1^{\otimes 2} - 2F < -\epsilon \right) \leq K e^{-N\epsilon/K}.$$ 

**Proof.** We prove this by contradiction. Let

$$\Delta_N = \frac{1}{N} \log \int \int_{R_{12} \in A} e^{-H(\sigma_1) - H(\sigma_2)} d\sigma_1^{\otimes 2} - 2F.$$ 

Suppose, for contradiction, that

$$\mathbb{P}(\Delta_N < -\epsilon) \geq 2K e^{-N\epsilon/K}, \quad (3.2)$$

where $K$ is from (3.1). Then

$$\mathbb{P}(\mathbb{E}\Delta_N < -\epsilon/2) \geq \mathbb{P}( |\Delta_N - \mathbb{E}\Delta_N| < \epsilon/2, \Delta_N < -\epsilon) \geq K e^{-N\epsilon/K}$$

by the inclusion-exclusion principle combined with (3.1) and (3.2). Thus

$$\mathbb{E}\Delta_N \leq -\epsilon/2.$$

Applying (3.1) again, this implies that

$$\mathbb{P}(\Delta_N \geq -\epsilon/4) \leq \mathbb{P}( |\Delta_N - \mathbb{E}\Delta_N| \geq \epsilon/4) \leq K e^{-N\epsilon/4K}.$$

Thus

$$\mathbb{E}_{\pi_N^{\otimes 2}}(R_{12} \in A) = \mathbb{E}e^{-N\Delta_N} \to 0$$

which is a contradiction. \qed
3.1. Free energy barriers and the landscape difficulty of the overlap for Ising spin models. To prove Theorem 1.3 let us first relate FEB to the difficulty. In the following, we let $d_H$ denote the unnormalized Hamming distance on $\Sigma^n$.

**Theorem 3.2.** For every $n \geq 1$ the following holds. Suppose that for some $(K, \epsilon)$ with $\epsilon > 2K$, $\pi_{\otimes n}^N$ is $(K, \epsilon)$-difficult. Then

$$\frac{\lambda_1}{n} \leq 2 \left( \frac{K}{\epsilon} \right)^2 \frac{e^{-D_\epsilon(\pi_{\otimes n}^N, K)}}{1 - 4e^{-D_\epsilon(\pi_{\otimes n}^N, K)}}.$$ 

**Proof.** Let us start with $n = 1$. This follows immediately from Theorem 2.2. Indeed $Q$ is by assumption reversible with respect to $\pi_N$ and nearest neighbor. Furthermore we can think of $(\Sigma_N, d_H, \pi_N)$ as a metric measure graph in the obvious way.

Let us now take $n = 2$. The case $n \geq 3$ is identical. Recall the elementary observation that if we consider the replicated transition matrix, which is the transition matrix

$$Q_r = \frac{1}{2} (Q \otimes Id + Id \otimes Q),$$

then $Q_r$ satisfies detailed balance with respect to $\pi_{\otimes 2}^N$ and the spectral gap of $I - Q_r$,

$$\Lambda_1 = \{ \min_{\mathcal{V},\mathcal{E},(f) \neq 0} \frac{(I - Q_r)\sigma,\sigma}{\text{Var}_{\pi_{\otimes 2}}(f)} \}$$

satisfies

$$\Lambda_1 = \frac{1}{2} \lambda_1.$$ (3.4)

This follows from the fact that the eigenbasis for $Q_r$ consists of tensor products of the eigenbasis for $Q$. In the study of Markov chains, $Q_r$ is often referred to as the transition matrix for a product chain. (See [66] for this terminology.)

Observe that by (3.4), it suffices to prove that

$$\Lambda_1 \leq 2 \left( \frac{K}{\epsilon} \right)^2 \frac{e^{-D_\epsilon(\pi_{\otimes 2}^N, K)}}{1 - 4e^{-D_\epsilon(\pi_{\otimes 2}^N, K)}}.$$ (3.5)

This follows immediately from Theorem 2.2. To see that we are in this setting, observe that we may view $(\Sigma_N \times \Sigma_N, d_{H}, \pi_{\otimes 2}^N)$ as a metric measure graph as follows. Let $G = (V, E)$ have vertex set $V = \Sigma_N \times \Sigma_N$ and edge set

$$E = \{ (\sigma, \sigma') \in V \times V : d_H(\sigma, \sigma') = 1 \}.$$ 

Thus $(G, d, \pi_{\otimes 2}^N)$ is a metric measure graph. Observe that, $Q_r$ from (3.3) is a transition matrix that satisfies detailed balance with respect to $\pi_{\otimes 2}^N$ and is nearest neighbor since $Q$ satisfies both of these properties. We are thus in the setting of Theorem 2.2 for any $\epsilon > 2K$, from which (3.5) follows. \(\square\)

With this in hand, we may then prove Theorem 1.3.

**Proof of Theorem 1.3.** View the overlap map, $(\sigma^1, \sigma^2) \mapsto R_{12}$, as a map on the metric measure graph $(\Sigma_N^2, d_H, \pi_{\otimes 2}^N)$ (we view this as a metric measure graph as in Theorem 3.2). Observe that $R_{12}$ is $N^{-1}$-Lipschitz. Then the $\epsilon$-difficulty of $R_{12}$ satisfies

$$D_\epsilon(\pi_{\otimes 2}^N, N^{-1}) \geq D_\epsilon(R_{12})$$

Suppose now that there is a free energy barrier of height $C > 0$ corresponding to some $q_1, q_2, q_3$ and $\epsilon > 0$. By (1.4), we have that for $N$ sufficiently large,

$$S(q_2, \epsilon; R_{12}) = \frac{1}{N} \log \pi_{\otimes 2}^N(R_{12} \in (q_2 - \epsilon, q_2 + \epsilon)) < -C.$$
with probability $1 - K_1 e^{-N/K_1}$ for some $K_1 > 0$. Similarly, by (1.3) and Lemma 3.4 it follows that
\[
\frac{1}{N} \left( S(q_1, \epsilon) + S(q_3, \epsilon) \right) \geq -C/2
\]
with probability $1 - K_2 e^{-N/(2K_2)}$ for some $K_2 > 0$. Thus on the intersection of these events,
\[
D_\epsilon(R_{12}) \geq \Phi(q_1, q_2, q_3, \epsilon; R_{12}) \geq N \frac{C}{2},
\]
In particular,
\[
P \left( D_\epsilon < \frac{C}{2} N \right) \leq K_3 e^{-N/K_3},
\]
in this case for some $K_3 > 0$. On the complement of this event,
\[
\frac{1}{N} \log(\lambda_1) < -C/2
\]
by Theorem 3.2. The result then follows.

3.2. Free energy barriers and the landscape difficulty of the overlap for Spherical models. To prove Theorem 1.24, let us first relate FEB to the difficulty.

We then have the following.

Theorem 3.3. For every $n \geq 1$ the following holds. Suppose that for some $(K, \epsilon)$ with $K, \epsilon > 0$, $\pi_N^{\otimes 2}$ is $(K, \epsilon)$-difficult. Then
\[
\lambda_1 \leq \left( \frac{K}{\epsilon} \right)^2 \frac{e^{-D_\epsilon(\pi_N^{\otimes 2}, K)}}{1 - 4e^{-D_\epsilon(\pi_N^{\otimes 2}, K)}}.
\]

Proof. In the case $n = 1$ this immediately follows from Theorem 2.7.

Let us now take $n = 2$. The case $n \geq 3$ is identical. As in the Ising spin setting, it will be helpful to introduce the replicated dynamics. The replicated dynamics for spherical models is the heat flow on the product space $S_N \times S_N$ induced by the generator
\[
\mathcal{L}_R = \mathcal{L}_H \otimes \text{Id} + \text{Id} \otimes \mathcal{L}_H,
\]
on $S_N^2$. Since $\mathcal{L}_H$ is uniformly elliptic and essentially self-adjoint, the same is true for $\mathcal{L}_R$. In particular, its spectrum is non-positive and pure point. Thus this heat flow is uniquely defined. Heuristically, this corresponds to two particles, $(X_t, Y_t)$, independently flowing with respect to the flow for $\mathcal{L}_H$.

Recall that $\lambda_1$ is the first nontrivial eigenvalue of $\mathcal{L}_H$ and, correspondingly, let $\Lambda_1$ denote the first non-trivial eigenvalue of $\mathcal{L}_R$. The starting point for our analysis is the simple observation that
\[
\lambda_1 = \Lambda_1. \tag{3.6}
\]
To see this observe that the eigenfunctions of $\mathcal{L}_H$ are a complete basis for $L^2(S_N, d\nu)$, so their products are a complete basis of $L^2(S_N \times S_N, d\nu^{\otimes 2})$ by density of tensor products. The result then follows by (2.12).

Again, by (3.6), observe that it suffices to prove that
\[
\Lambda_1 \leq \left( \frac{K}{\epsilon} \right)^2 \frac{e^{-D_\epsilon(\pi_N^{\otimes 2}, K)}}{1 - 4e^{-D_\epsilon(\pi_N^{\otimes 2}, K)}}.
\]
This is a consequence of Theorem 2.7. To see that we are in this setting. Observe that $M = S_N \times S_N$ with the natural product metric is a compact boundary-less Riemannian manifold and that
\[
\nu = \pi_N^{\otimes 2} = \frac{e^{-U}}{Z} d\nu_M
\]
where \( U(\sigma^1, \sigma^2) = H(\sigma^1) + H(\sigma^2) \). Finally observe that
\[
\mathcal{L}_R = -\Delta + g(DU, D\cdot).
\]
Thus we are in the setting of Theorem 2.7 for any \( K, \epsilon > 0 \).

Finally we note the following.

**Proof of Theorem 1.24** This result follows from Theorem 3.3 after observing that the overlap map is \( N^{-1/2} \)-Lipschitz. The proof is identical to Theorem 1.3 so it is omitted.

### 3.3. Spectral gap lower bounds

We end this section by briefly mentioning the spectral gap lower bounds for Ising spin and spherical models.

We first briefly turn to the proof of Theorem 1.11. Recall from [4], that by an application of Borell’s and Slepian’s inequalities,
\[
\mathbb{P}(-CN \leq \min_{\sigma \in \mathcal{S}_N} H(\sigma) \leq \max_{\sigma \in \mathcal{S}_N} H(\sigma) \leq CN) \geq 1 - \frac{1}{C} e^{-cN}
\]
for some \( C = C(\xi, h) > 0 \) and \( c = c(\xi, h) > 0 \). The same bound then holds for the Ising spin setting since \( \Sigma_N \subset \mathcal{S}_N \).

In order to obtain an exponential lower bound in the Ising spin setting, recall that we needed coercivity. Recall that the spectral gap of the simple random walk [40] is
\[
\lambda_{SRW} = \frac{2}{N}.
\]

**Proof of Theorem 1.11** Observe that by Proposition 2.6, if \( Q \) is \( P_{SRW} \)-coercive with constant \( A_N \), then
\[
\lambda_1(Q) \geq A_N e^{-2(\max H - \min H)\lambda_{SRW}}.
\]
Taking logs and using (3.7), we see that
\[
\lambda_1(Q) \geq \frac{1}{N} \log A_N - 2C
\]
for \( N \) sufficiently large.

**Proof of Corollary 1.12** It suffices to show that
\[
\frac{1}{N} \log(A_N) = -(\max H_N - \min H_N)
\]
To see this, simply observe that when \( \sigma^1 \neq \sigma^2 \),
\[
Q(\sigma^1, \sigma^2) = P_{SRW}(\sigma^1, \sigma^2)(1 \land e^{H(\sigma^2) - H(\sigma^1)}),
\]
and \( P_{SRW}(\sigma^1, \sigma^1) = 0 \).

In the setting of spherical models we have a similar result. Recall that the first non-trivial eigenvalue for the Laplacian on \( \mathcal{S}_N \) satisfies [30]
\[
\lambda_1(\mathcal{S}_N) = 1 - \frac{1}{N}.
\]

**Proof of Theorem 1.28** To obtain the spectral gap lower bound from (1.24) in the spherical spin setting, observe that (3.7) still applies. The result then follows by Proposition 2.12.
In this section, we aim to prove Theorem 1.8. The main difficulty is showing that certain regions of overlap values are exponentially rare as in (1.4). This is the content of the following theorem, which is the goal of this section.

**Theorem 4.1.** Suppose that for some \( q^* \) in the support of \( \mu \), \( \Lambda(q^*, \mu) > 0 \). Then there is an \( \epsilon_0 \) such that for every \( q \) in the punctured neighborhood \( (q^* - \epsilon_0, q^* + \epsilon_0) \cap (0,1) \setminus \{q^*\} \), there is an \( \epsilon(q) \) and a \( c(q) > 0 \) such that

\[
\lim_{N \to \infty} \frac{1}{N} \log P \left( \frac{1}{N} \log \pi_2^N (R_{12} \in (q-\epsilon,q+\epsilon)) > -c \right) < 0
\]

If, furthermore, \( q^* \) is in the support of \( \zeta \) from (1.8), then it must be isolated.

To prove this estimate, we control constrained free energies:

\[
F_{2,N}(A) = \frac{1}{N} \log \int \int_{R_{12} \in A} e^{-H(\sigma^1) - H(\sigma^2)} d\sigma \otimes^2
\]

where \( A \) is some Borel set. More precisely, taking \( A = (q-\epsilon,q+\epsilon) \), we will show that

\[
F_{2,N}((q-\epsilon,q+\epsilon)) = \frac{1}{N} \log \int \int_{|R_{12} - q| < \epsilon} e^{-H(\sigma^1) - H(\sigma^2)} d\sigma \otimes^2
\]

\[
F_N = \frac{1}{N} \log \int e^{-H(\sigma)} d\sigma
\]

satisfy

\[
F_{2,N}((q-\epsilon,q+\epsilon)) - 2F_N < -c
\]

with high probability. This will follow by application of the 2D Guerra-Talagrand bounds. The key ideas in this proof can already be seen in [91] and [93]. For completeness, we present here an alternative, stochastic analysis and PDE based approach following Bovier-Klimovsky [25] and Chen [32].

**Notation:** Here and in the following, for a probability measure \( \nu \) we make the abuse of notation \( \nu(t) = \nu([0,t]) \). All matrix norms will be Frobenius/Hilbert-Schmidt norms.

### 4.1. 2D Guerra-Talagrand Bounds

Let \( \mathcal{P}_d \) be the space of \( d \times d \) positive semidefinite matrices. Fix \( q \in [-1,1] \). Let \( Q_t : [0,1] \to \mathcal{P}_2 \), be a continuous, weakly differentiable, non-decreasing path in \( \mathcal{P}_2 \) with boundary conditions

\[
Q_0 = 0
\]

\[
Q_1 = \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}.
\]

(Here, by weakly differentiable we mean in the sense that its derivative in \( t \) is \( W^{1,1}(\mathbb{R}; (\mathcal{P}_2, ||\cdot||)) \).)

Let the space of such paths be denoted by \( \mathcal{Q}_q \). Let the space of such paths with arbitrary final data \( Q_1 \) be denoted by \( \mathcal{Q} \).

Let \( \nu \in \Pr([0,1]) \). Finally, let

\[
A = \frac{d}{dt} (\xi'(Q_t)) = \xi''(Q_t) \odot \dot{Q}_t
\]

where \( \odot \) denotes the Hadamard product and function evaluations are to be understood component wise. Since \( Q \) was assumed to be non-decreasing, \( \dot{Q} \) is positive semidefinite. We observe here the following lemma.

**Lemma 4.2.** For any \( Q \in \mathcal{Q} \), \( \xi''(Q_t) \) and \( A_t \) are positive semidefinite for each \( t \).
This follows by Schur’s product theorem after observing that $\xi''(Q_t)$ can be viewed as a power series in $Q$ in the Hadamard product sense and $\dot{Q}$ is positive semidefinite.

Let us begin by supposing that $A_t$ is strictly positive definite for all $t$. We may then consider the weak solution, $u$, of

$$
\begin{align*}
\partial_t u + \frac{1}{2} \left( (A, D^2 u) + \nu(t)(Du, ADu) \right) &= 0 \\
u(1, x) &= f(x)
\end{align*}
$$

(4.4)

where

$$
f(x) = \log \left( \frac{1}{4} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \exp(\epsilon_1 x_1 + \epsilon_2 x_2 + \lambda \epsilon_1 \epsilon_2) \right).
$$

For the existence, uniqueness, and basic regularity of $u$, see Appendix A. For any $Q \in Q$ and $\nu \in \operatorname{Pr}([0,1])$ and $\lambda \in \mathbb{R}$, define the quantities

$$
L(\nu, Q) = \frac{1}{2} \sum_{ij} \nu(t) \xi''(q_{ij}(t)) q_{ij}(t) \dot{q}_{ij}(t) dt,
$$

(4.5)

and

$$
P(\nu, Q, \lambda) = u(0, h) - \lambda q - \frac{1}{2} \sum_{ij} \nu(t) \xi''(q_{ij}(t)) q_{ij}(t) \dot{q}_{ij}(t) dt.
$$

(4.6)

Finally, let $\mathcal{R}_N$ denote the set of allowed overlaps,

$$
\mathcal{R}_N = \{ q \in [-1,1] : \exists \sigma^1, \sigma^2 \in \Sigma_N : R_{12} = q \}.
$$

then have Talagrand’s 2D Guerra-Talagrand bound [91, 93].

**Theorem 4.3.** We have the following:

1. If $\xi$ is convex on $[-1,1]$, then for every $q \in \mathcal{R}_N, Q_t \in Q_q$ positive definite, $\nu \in \operatorname{Pr}([0,1])$ and $\lambda \in \mathbb{R}$,

$$
\mathbb{E} F_{2,N}(\{q\}) \leq P(\nu, Q, \lambda)
$$

(4.7)

2. In particular, if $\xi$ is convex on $[-1,1]$, then for every $q \in [-1,1]$,

$$
\lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{E} F_{2,N}(\{(q - \epsilon, q + \epsilon) \cap [0,1]\}) \leq P(\nu, Q, \lambda).
$$

(4.8)

Let us now turn to the setting in which we will apply this class of estimates. In our applications, we will be interested in cases where $Q_t$ is allowed to be positive semi-definite. We will focus on a specific form. In particular, take $q \geq 0$ and define $Q_t(q) \in Q_q$ by

$$
Q_t(q) = \begin{cases} 
\begin{pmatrix} t & t \\ t & t \end{pmatrix} & t \leq q \\
\begin{pmatrix} t & q \\ q & t \end{pmatrix} & t \geq q
\end{cases}.
$$

(4.9)

In this case

$$
A(t) = \begin{cases} 
\xi''(t) \mathbb{I} & t \leq q \\
\xi''(t) \operatorname{Id} & t \geq q
\end{cases},
$$

(4.10)

where $\mathbb{I}$ is the matrix of all 1's. Define

$$
P(\nu, Q(q), \lambda) = u(0, h) - \lambda q - L(\nu, Q),
$$

(4.11)
where \( v(t, x) = u(t, x, x) \) for \( t \geq q \) and \( v(t, x) \) is the unique weak solution of

\[
\begin{aligned}
\partial_t v + \frac{\xi''(t)}{2} (\Delta v + v(t)(\partial_x v)^2) &= \xi''(t) \left\{ \partial_{x_1} \partial_{x_2} u(t, x, x) + v(t) \partial_{x_1} u(t, x, x) \partial_{x_2} u(t, x, x) \right\} 1_{t \geq q} \\
(4.12)
\end{aligned}
\]

\( (t, x) \in [0, 1] \times \mathbb{R}^2 \).

For the notion of weak solution in this setting and the existence and uniqueness see Appendix [A].

We then have the following, which is proved by a standard extension argument. See, e.g., [32].

**Corollary 4.4.** For every \( q \in [0, 1] \), \( Q \) as in \((4.9)\), \( \nu \in \Pr([0, 1]) \), and \( \lambda \in \mathbb{R} \), we have

\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{E} F_{2, N}(\{(q - \epsilon, q + \epsilon) \cap [0, 1]\}) \leq P(\nu, Q, \lambda). \tag{4.13}
\]

### 4.2. Bounding the 2D Guerra-Talagrand functional under the assumption of a positive replicon eigenvalue.

Consider the probability measure \( \nu \), defined by the cumulative distribution function

\[
\nu(t) = \begin{cases} 
\frac{\mu(t)}{2} & t \leq q \\
\mu(t) & t \geq q,
\end{cases} \tag{4.14}
\]

where \( q \in [0, 1] \) and \( \mu \) is the Parisi measure. Let \( Q_t(q) \) be as in \((4.9)\), and let \( A_t \) be as in \((4.10)\). Corollary 4.4 applies in this setting. Observe that in this setting the functional \((4.11)\), is a function of \( q \) and \( \lambda \) alone, so we denote it by

\[
\mathbb{P}(\lambda, q) = P(\nu, Q, \lambda). \tag{4.15}
\]

We aim to prove the following theorem. Recall that \( \mu \) is the Parisi measure from Definition 1.5.

**Theorem 4.5.** Let \( q_\ast \in \text{supp}(\mu) \) be such that \( \Lambda_R(q_\ast, \mu) > 0 \). Then there is an \( \epsilon_0 \) such that for all \( q \in (q_\ast - \epsilon_0, q_\ast + \epsilon_0) \cap (0, 1) \) with \( q \neq q_\ast \), there is a \( \lambda_\ast(q) \) satisfying

\[
\mathbb{P}(\lambda_\ast, q) < 2P_I(\mu). \tag{4.16}
\]

We begin the proof of Theorem 4.5 with the following elementary observations. Observe that by \((4.9)\) and \((4.14)\), \( L \) from \((4.5)\) is constant in \( (\lambda, q) \) and satisfies

\[
L = \int \xi''(t) t \mu(t) dt. \tag{4.17}
\]

Observe furthermore, that at \( \lambda = 0 \), \( u_\nu \) from \((4.9)\) with parameters \((4.14)\) and \((4.10)\), factorizes for \( t \geq q \) as

\[
u(t, x, y) = \phi_\mu(t, x) + \phi_\mu(t, y), \tag{4.18}
\]

where \( \phi_\mu \) is the solution of the Parisi initial value problem, \((1.10)\), corresponding to \( \mu \). By a scaling argument applied to the Parisi PDE, since \( \phi_\mu \) satisfies \((1.10)\) and \( \nu \) satisfies \((4.14)\), \( 2\phi_\mu \) is the solution, \( v \), of \((4.12)\),

\[
v = 2\phi_\mu \tag{4.19}
\]

for all \( (t, x) \in [0, 1] \times \mathbb{R} \). Thus

\[
P(0, q) = 2 \left( \phi_\mu(0, h) - \frac{1}{2} L \right) = 2P_I(\mu), \tag{4.20}
\]

for all \( q \).

Let us now explain, formally, the argument behind Theorem 4.5. By \((4.18)\), \( P \) is constant on the \( \lambda = 0 \) axis. As we will soon see, the point \( (\lambda, q) = (0, q') \) is a critical point for \( P \) for any \( q' \) in \( \text{supp}(\mu) \). Evidently, \( \partial_{q'}^2 P(0, q') = 0 \) for such \( q' \). Thus, formally, the Hessian of \( P \) is of the form

\[
\text{Hess}(P) = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix} \tag{4.21}
\]
for some $a, b \in \mathbb{R}$. Note that this has a negative eigenpair

$$
\begin{align*}
\lambda &= \frac{1}{2} \left( a - \sqrt{a^2 + 4b^2} \right) \\
v &= \left( \frac{-a - \sqrt{a^2 + 4b^2}}{b}, 1 \right)
\end{align*}
$$

(4.20)

provided $b \neq 0$. What we will find is that,

$$
b = \partial_q \partial_{x} P = -\Lambda_R(q) > 0,
$$

which will yield the result.

Rigorously, it is cumbersome to check that $\mathcal{P}$ is jointly $C^2$. To avoid this issue we recall the following basic result of calculus which is a minor modification of the second derivative test.

**Lemma 4.6.** Let $f(x, y)$ be a continuous function of two variables such that:

1. it has a partial derivative in $x$ at $(x_0, y_0)$ that vanishes at that point,
2. it has a locally bounded, continuous second partial derivative in $x$ for all $(x, y)$,
3. it has a nonzero mixed partial derivative at $(x_0, y_0)$, $\partial_y \partial_x f$,
4. $f(x_0, y_0)$ is constant in $y$.

Then there is an $r$ such that for all $y \in B_r(y_0) \setminus \{y_0\}$, there is an $x_*(y)$ with

$$
f(x_*, y) < f(x_0, y_0).
$$

Furthermore, the same holds if the mixed partial in $y$ is only a right (left) derivative except with $0 < y - y_0 < r$ (resp. $0 > y - y_0 > -r$).

With this in mind, let us begin the proof.

4.2.1. **Derivatives of the Multidimensional Parisi PDE in the Lagrange multiplier.** We start with the following result, regarding the differentiability of $v$ in $\lambda$. Such results are standard in the spin glass literature. See, e.g., [34, 93]. Let $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ be the solution of

$$
d\tilde{X}_t = \nu ADu_{\nu}(t, \tilde{X}_t)dt + \sqrt{A}dW_t,
$$

(4.21)

for $t \geq q$, where $W_t$ is standard Brownian motion in $\mathbb{R}^2$ and let $\tilde{X}_t$ be the solution of

$$
d\tilde{X}_t = \xi''(t)\partial_x v(t, \tilde{X}_t)dt + \frac{\xi''(t)}{2}dW_t,
$$

(4.22)

where $W_t$ is standard Brownian motion for $t \leq q$. Note that since $Du$ and $\partial_x v$ are bounded measurable in time and uniformly Lipschitz in space (in fact they are smooth and bounded in space) by Lemma A.1 and Lemma A.3 these solutions exist in the Itô sense. (The regularity of $u, v$, and the Parisi PDE are discussed in Appendix A)

**Lemma 4.7.** We have the following.

1. The solution $u$ of (4.11) with parameters given by (4.10) and (4.12) is twice differentiable in $\lambda$ for $(t, x, y) \in [q, 1] \times \mathbb{R}^2$ for each $q$. Furthermore, $\partial_\lambda u$ satisfies

$$
\partial_\lambda u(t, x, y) = \mathbb{E} \left( \partial_\lambda f(1, \tilde{X}_1) | \tilde{X}_t = (x, y) \right).
$$

(4.23)

2. The solution $v$ of (4.12) with parameter given by (4.13) is twice differentiable in $\lambda$ for each $(t, x)$ with $t < q$ and each $q$. Furthermore, $\partial_\lambda v$ satisfies

$$
\partial_\lambda v(0, h) = \mathbb{E} \left( \partial_\lambda u(q, \tilde{X}_1, \tilde{X}_q) | \tilde{X}_0 = h \right).
$$

Finally, the first and second derivatives are continuous in $(\lambda, q)$ and uniformly bounded in $(t, x)$ and $(\lambda, q)$. 

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The proof of this result is a standard differentiable dependence argument. Since it is technical, we defer it to Section A.3.

Let us now compute the derivatives in which we are interested. In the following, we let $X_t$ denote the solution to the local fields process (1.12).

**Lemma 4.8.** For every $q \geq 0$, at $\lambda = 0$,  
$$
\partial_\lambda u(q, x, x) = (\partial_x \phi_\mu)^2(q, x) \quad (4.24)
$$  
$$
\partial_\lambda v(0, x) = \mathbb{E}_x (\partial_x \phi_\mu)^2(q, X_q) \quad (4.25)
$$
where $X_t$ is the local field process (1.12) with initial data $X_0 = x$. Furthermore, at $\lambda = 0$, $\partial_\lambda v(0, h)$ has a partial derivative in $q$ and the derivative satisfies  
$$
\partial_q \partial_\lambda v(0, h) = \xi''(q) \mathbb{E} (\Delta \phi_\mu)^2(q, X_q),
$$
where if $q = 0$, this is a right-partial derivative, and if $q = 1$ this is a left-partial derivative.

**Proof.** Observe that at $\lambda = 0$,  
$$
\partial_\lambda f_\lambda(x, y) = \tanh(x) \cdot \tanh(x).
$$
Recall from (1.10), that $\partial_x \phi_\mu(1, x) = \tanh(x)$. Thus  
$$
\partial_\lambda u(q, x, x) = \mathbb{E} \left( \partial_x \phi_\mu(1, \tilde{X}_t^1) \cdot \partial_x \phi_\mu(1, \tilde{X}_t^2) \right) \big| (\tilde{X}_t^1, \tilde{X}_t^2) = (x, x),
$$
by Lemma 4.7. Since $t \geq q$, $u$ satisfies (4.16) and $A(t) = \xi''(t)Id$. Thus $\tilde{X}$ from (4.24) is two independent copies of the local field process, (1.12), corresponding to $\mu$, which we denote by $(X_1^t, X_2^t)$. Thus  
$$
\partial_\lambda u(q, x, x) = \mathbb{E} \left( \partial_x \phi_\mu(1, X_1^t) \cdot \partial_x \phi_\mu(1, X_2^t) \right) \big| (X_1^t, X_2^t) = (x, x).
$$
Observe that $\partial_x \phi_\mu$ weakly solves  
$$
\left( \partial_t + \frac{1}{2} \mathcal{L} \right) \partial_x \phi_\mu = 0,
$$
where $\mathcal{L}$ is the infinitesimal generator of $X_t$,  
$$
\mathcal{L} = \frac{\xi''(s)}{2} (\Delta + 2\mu(s)\partial_x \phi_\mu(s, x) \partial_x).
$$
Thus, $\partial_x \phi_\mu(s, X_s)$ is a martingale. By the martingale property and independence, we then obtain  
$$
\partial_\lambda u(q, x, x) = \partial_x \phi_\mu(q, x)^2.
$$
This is the first equality.

We now turn to the second. With (4.24), we see that (4.23) satisfies  
$$
\partial_\lambda v(0, h) = \mathbb{E} (\partial_x \phi_\mu)^2(q, \tilde{X}_q),
$$
where $\tilde{X}_t$ solves (4.22). Differentiating (4.17) in space and applying (4.14), we see that $\tilde{X}_t = X_t$. This yields the second result.

By an application of Itô’s lemma, we have that  
$$
\frac{d}{dt} \mathbb{E}(\partial_x \phi_\mu)^2(t, X_t) = \xi''(t) \mathbb{E} (\Delta \phi_\mu)^2(t, X_t),
$$
for every $t \geq 0$, where if $t = 0$ this is a right derivative and if $t = 1$ this is a left derivative. Since $\partial_\lambda v(0, h)$ satisfies (4.25) at $\lambda = 0$ for every $q \geq 0$, we see that (4.28) implies that the partial derivative in $q$ at $\lambda = 0$ satisfies (4.26). $\square$
4.2.2. **Proof of Theorem 4.5.** With these results in hand we may then prove Theorem 4.5.

**Proof of Theorem 4.5.** We note the following. Firstly, by Lemma 4.8 and (4.18), $P(\lambda, q)$ has a partial derivative in $\lambda$ at 0 that satisfies $$\frac{\partial}{\partial \lambda} P(0, q) = \mathbb{E} (\partial_x \phi_{\mu})^2 (q, X_q) - q.$$ Since $\mu$ is the Parisi measure by assumption, for every $q'$ in the support of $\mu$, we have the fixed point relation $$\mathbb{E} (\partial_x \phi_{\mu})^2 (q', X_{q'}) = q',$$ by (6.2). Thus for $q'$ in the support of $\mu$,$$\frac{\partial}{\partial \lambda} P(0, q') = 0.$$

Fix $\lambda = 0$, and now differentiate again in $q$. We then obtain$$\frac{\partial}{\partial q} \frac{\partial}{\partial \lambda} P(0, q) = \xi''(q) \mathbb{E} (\Delta \phi_{\mu})^2 (q, X_q) - 1 = -\Lambda_R(q)$$by Lemma 4.8 where if $q = 0$, this is understood to be a right derivative and if $q = 1$ this is understood to be a right derivative. This is negative at $q = q_*$ by assumption.

By Lemma 4.7, $\partial^2 P$ is uniformly bounded and continuous in a neighborhood of $(0, q')$ for any $q'$ in the support of $\mu$. Recall that $P(0, q)$ is constant in $q$. The result then follows by Lemma 4.6 applied to $P$ at the point $(0, q_*)$.

4.3. **Proof of Theorem 4.1.** As a consequence of the previous section, we may also prove Theorem 4.1.

**Proof of Theorem 4.1.** Let $\epsilon_0$ and

$$E = (q_* - \epsilon_0, q_* + \epsilon_0) \cap (0, 1) \setminus \{q_*\}$$
be as in Theorem 4.5. Fix $q \in E$ and let $\lambda_*(q)$ be as in Theorem 4.5. By Theorem 4.5 (1.11), and Corollary 4.4, it follows that

$$\mathbb{E} F_{2, N}((q - \epsilon, q + \epsilon)) - 2\mathbb{E} F_N \leq P(\lambda_*, q) - 2P_1(\mu) + o(1) < -c$$
for some $c > 0$ and $\epsilon$ sufficiently small.

By Gaussian concentration (3.1), this implies that with probability at least $1 - K e^{-N/K}$

$$F_{2, N}((q - \epsilon, q + \epsilon)) - 2F_N < -c.$$ 

Taking $N$ sufficiently large and then $\epsilon$ sufficiently small, shows that

$$\mathbb{P} \left( \frac{1}{N} \log \pi_N^{\otimes 2} (R_{12} \in (q - \epsilon, q + \epsilon)) < -c' \right) \geq 1 - K e^{-N\epsilon/K}$$

for some $c'$, where we again apply (3.1). The result then follows by taking complements and limits. \(\square\)
4.4. Proof of Theorem 1.8. We finally turn to the proof of Theorem 1.8. In the following proof, $K > 0$ will denote a constant that depends at most on $\xi, h$ and possibly varies from line to line.

Proof of Theorem 1.8. Suppose that PREV holds. Then there are two points in the support of the overlap distribution, $\zeta$, that are also in the support of the Parisi measure, $\mu$. Call these two points $q_1$ and $q_3$. Without loss of generality, $0 < q_1 < q_3 < 1$. Furthermore at least one of these points satisfy $\Lambda_R(q, \mu) > 0$.

We begin by showing (1.4). Suppose first that $\Lambda_R(q_3, \mu) > 0$. Then by Theorem 4.1 there is a $q_2$ with $q_1 < q_2 < q_3$ and $c, \epsilon$ such that

$$\frac{1}{N} \log \pi^{\otimes N}_N(R_{12} \in (q_2 - \epsilon, q_2 + \epsilon)) < -c,$$

with probability $1 - K e^{-N/K}$. Furthermore, we may take $\epsilon$ sufficiently small that

$$\epsilon < \frac{1}{4} \min \{q_2 - q_1, q_3 - q_2\}.$$

This yields (1.4). The case $\Lambda_R(q_1, \mu) > 0$ is the same by symmetry.

We now show (1.5). Since $q_1, q_3 \in \text{supp}(\zeta)$, we have that

$$\zeta(q_1 - \epsilon, q_1 + \epsilon), \zeta(q_3 - \epsilon, q_3 + \epsilon) > 0.$$

Since $\zeta$ is by assumption the unique limit point of $\mathbb{E} \pi^{\otimes N}_N(R_{12} \in \cdot)$ and the sets $(q_i - \epsilon, q_i + \epsilon)$ are relatively open,

$$\lim_{N \to \infty} \mathbb{E} \pi^{\otimes N}_N(q_i - \epsilon, q_i + \epsilon) > 0$$

by the portmanteau lemma for $i = 1, 3$, as desired. \hfill \square

5. Free energy barriers in Spherical Models

In this section, we prove Theorem 1.25. As in the Ising spin setting, the main obstruction in proving the this result will be to show that that certain regions of overlap values are exponentially rare as in (1.4). The arguments are analogous but technically simpler in this spherical setting.

The main result of this section is the following theorem. Recall that $\mu$ denotes the minimizer of (1.21).

Theorem 5.1. Suppose that for some $q_*$ in the support of $\mu$, $\Lambda(q_*, \mu) > 0$. Then there is an $\epsilon_0$ such that for every $q$ in the punctured neighborhood $(q_* - \epsilon, q_* + \epsilon) \cap (0, 1) \setminus \{q_*\}$, there is an $\epsilon(q)$ and a $c(q) > 0$ such that

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left( \frac{1}{N} \log \pi^{\otimes N}_N(R_{12} \in (q - \epsilon, q + \epsilon)) > -c \right) < 0.$$

If, furthermore, $q_*$ is in the support of $\zeta$ from (1.3), then it must be isolated.

We remind the reader here of the Parisi-type formulation of the Crisanti-Sommers formula. For $\nu \in \Pr([0, 1])$ and $b \geq 1$, define the spherical Parisi functional,

$$P_S(\nu, b) = \begin{cases} \frac{h^2}{b - \psi_0(0)} + \int_0^1 \frac{\xi''(t)d\nu}{b - \psi_0(t)}dt + b - 1 - \log b - \int_0^1 t \xi''(t)\nu(t)dt, & b - \psi_0(0) \geq 0, \\ \infty, & \text{otherwise} \end{cases}$$

where $\psi_0(t) = \int_t^1 \xi''(s)\nu(s)ds$. Let $\mathcal{A} = \{(\nu, b) \in \Pr([0, 1]) \times [1, \infty) : b \geq \psi_0(0)\}$. With these in hand, we also have the spherical Parisi formula is given by

$$2F = \min_{\nu, b} P_S(\nu, b). \quad (5.1)$$

That these are equivalent was proved by Talagrand in [89]. We remind the reader of the following basic facts regarding the optimization of this functional. Recall $\varphi_0$ from (1.22).
**Lemma 5.2.** $P_S(\nu, b')$ is strictly convex and lower semicontinuous on $\Pr([0, 1]) \times [1, \infty)$ equipped with the product topology, where $\Pr([0, 1])$ is equipped with the weak* topology. In particular, there is a unique minimizing pair $(\mu, b)$. This pair satisfies:

$$b > \psi_\mu(0) \text{ and } b > 1 \quad (5.2)$$

$$q = \int_0^q \frac{\xi''(t)}{(b - \psi_\mu(t))^2} \, dt + \frac{h^2}{(b - \psi_\mu(0))^2} \quad \forall q \in \text{supp}(\mu) \quad (5.3)$$

$$b = \xi'(1) - \xi'(q_{EA}) + \frac{1}{1 - q_{EA}} \quad (5.4)$$

$$\varphi_\mu(q) = \frac{1}{b - \psi_\mu(q)} \quad \forall q \in \text{supp}(\mu) \quad (5.5)$$

$$P(\mu, b) = C(\mu) = \min_{\nu \in \text{Pr}([0, 1])} C(\nu) \quad (5.6)$$

In particular, $\mu$ is the minimizer of $\frac{1}{2}\lambda_2^2$.

**Remark 5.3.** This was proved under the assumption that $\mu$ is $k$-atomic in [89, Section 4]. One can perform a first variation argument directly to $P_S$ and $C$ to obtain these equalities for general $\mu$. For the reader’s convenience we sketch this in Appendix B.

We now remind the reader here of the Guerra-Talagrand bound for spherical models, with the choice of parameters (4.9), (4.10), and (4.14). Let $(\mu, b)$ be the optimizers of (5.1), and define

$$P(\lambda, q) = \log \left( \frac{b^2}{b^2 - \lambda^2} \right) + \int_0^q \frac{\xi''(t)}{b - \lambda - \psi_\mu(t)} \, dt$$

$$+ \frac{1}{2} \int_0^1 \xi''(t) \left( \frac{1}{b - \lambda - \psi_\mu(t)} + \frac{1}{b + \lambda - \psi_\mu(t)} \right) \, dt$$

$$- \lambda q + b - 1 - \log b - \int_0^1 t \xi''(t) \mu(t) \, dt - \frac{h^2}{b - \lambda - \psi_\mu(0)}$$

where $\psi = \psi_\mu$ and where $b$ is taken to solve (5.1). Observe that

$$P(0, q, b) = 2P_S(\mu, b).$$

We then have the following analogue of Talagrand’s 2D Guerra bound, Corollary [4.4] for the spherical setting, which is from [89]. See also [80, 36].

**Theorem 5.4.** For $\lambda$ such that $b - \psi_\mu(0) + |\lambda| > 0$ and every $q \geq 0$, we have that

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} E F_{2,N}((q - \epsilon, q + \epsilon) \cap [0, 1]) \leq P(\lambda, q). \quad (5.7)$$

With this in hand we then have the following theorem which is an analogue of Theorem 4.5

**Theorem 5.5.** Let $q_* \in \text{supp}(\mu)$ be such that $\Lambda_R(q_*, \mu) > 0$. Then there is an $\epsilon_0$ such that for all $q \in (q_* - \epsilon_0, q_* + \epsilon_0) \cap (0, 1)$ with $q \neq q_*$, there is a $\lambda_*(q)$ satisfying

$$P(\lambda, q) < P(0, q_*).$$

**Proof.** Observe that by Lemma 5.2

$$b - \psi_\mu(0) = b - \psi_\mu(q_0) = \frac{1}{\varphi_\mu(q_0)} > 0$$

where $q_0 = \min \text{supp}(\mu)$. Thus for $\lambda$ in a neighborhood of 0 we may apply Theorem 5.4. Differentiating first in $\lambda$ and setting $\lambda = 0$, we see that

$$\frac{\partial}{\partial \lambda} P|_{\lambda=0} = \int_0^q \frac{\xi''(t)}{(b - \psi_\mu(t))^2} \, dt + \frac{h^2}{(b - \psi_\mu(0))^2} - q.$$
Taking \( q \in \text{supp}(\mu) \), we see that this is zero by (5.3). Differentiating this expression in \( q \), we see that for \( q \in \text{supp}(\mu) \),

\[
\frac{\partial}{\partial q} \frac{\partial}{\partial \lambda} P = \frac{\xi''(q)}{(b - \psi_{\mu}(q))^2} - 1 = \xi''(q) \varphi_{\mu}^2 - 1 = -\varphi_{\mu}^2 \Lambda_{R}(q, \mu),
\]

where the second equality follows by Lemma 5.2. Taking \( q = q_* \) implies that this is strictly negative. Observe finally that

\[
|\partial^2_{\lambda} P| < \infty,
\]
in a neighborhood of \( \lambda = 0 \) for all \( q \). The result then follows by Lemma 4.6.

Alternatively, note here that \( P \) is \( C^2 \), and that \( \partial_q P(0, q_*) = \partial^2_q P(0, q_*) = 0 \), so that the Hessian is of the form (4.19). Thus it has a negative eigenpair as in (4.20), with eigen-direction in the first quadrant. Thus, \( P \) decreases for \( (\lambda, q) \) in this direction. \( \square \)

Finally we note the following.

**Proof of Theorem 5.1** The proof of this theorem is identical to that of Theorem 4.1. In particular, it is an immediate consequence of Theorem 5.5 and (3.1). \( \square \)

**Proof of Theorem 1.25** The proof that this result follows from Theorem 5.1 is identical to the proof that Theorem 4.1 implies Theorem 1.8 so it is omitted. \( \square \)

### 6. Examples and Applications

The main motivation for the above results is to understand the dynamics of spin glass models. In this section, we will remind the reader of the connection between spectral gaps and dynamics. We then discuss regimes under which the assumptions A, RSB, GRSB, PREV, and GPREV are known to hold. In particular, we prove Theorem 1.9 and Theorem 1.26.

#### 6.1. Spectral Gaps and Mixing

Our interest in \( \lambda_1 \) is that it is a classical measure of the time to equilibrium. For example, in the Ising spin setting, it measures the rate of \( L^2 \)-mixing of the continuous time Markov chain induced by \( Q \). That is, consider the semigroup \( P_t \) with infinitesimal generator \( L = I - Q \). Since \( \lambda_1 \) is the first nontrivial eigenvalue of \( L \), we have the inequality

\[
\text{Var}_{\pi_N}(P_t f) \leq e^{-\lambda_1 t} \text{Var}_{\pi_N}(f) \quad \forall f \in L^2(\pi_N)
\]  

(6.1)

If the spectrum of \( Q \) is non-negative, then for the discrete time Markov chain \((\sigma(t))\) in \( \Sigma_N \) induced by \( Q \), one has the \( \lambda_1 \) also measures both the \( L^2 \)-mixing and the Total variation mixing.

**Lemma 6.1.** If the spectrum of \( Q \) is non-negative then,

\[
\text{Var}_\pi(Q^n f) \leq (1 - \lambda_1)^{2n} \text{Var}_\pi(f).
\]

Furthermore, if we denote the total variation mixing time by \( t_{\text{mix}} \), then there exists constants \( c, C > 0 \) such that

\[
\mathbb{P} \left( \frac{1}{N} \left| \log t_{\text{mix}} - \log \left( \frac{1}{\lambda_1} \right) \right| > \epsilon \right) \leq Ce^{-cN\epsilon^2}.
\]

**Proof.** The first inequality as well as the inequality

\[
\frac{1}{N} \left| \log t_{\text{mix}} - \log \frac{1}{\lambda_1} \right| \leq \frac{1}{N} \log \log \pi_N \leq \frac{1}{N} \log (|\max H_N| + |\min H_N|)
\]

are classical [66]. That the second term is less than \( \epsilon \) with probability \( 1 - Ce^{-cN\epsilon^2} \) follows by (5.7) combined with Gaussian concentration. \( \square \)
The assumption that $Q$ has non-negative spectrum is not particularly stringent. It is common in the literature to circumvent this issue by working with the “Lazy” version of the chain, i.e., $\tilde{Q} = \frac{1}{2}(I + Q)$ which only makes an $O(1)$ to the mixing. Alternatively note that, as we are mainly interested in lower bounds on the mixing time then if $\lambda_1 < 1$ which will be true in our applications, $(\lambda_1)^{-1}$ is still a good lower bound on the mixing time.

In the spherical setting, $\lambda_1$ is of interest as it measures the $L^2$ mixing through the inequality (6.1) as well, where here $P_t$ is the heat semigroup

$$P_t = e^{-t\mathcal{L}_H}$$

induced by $\mathcal{L}_H$ from (1.19).

6.2. Verifying A, RSB, GRSB, PREV, and GPREV for Ising spin models. In this section, we will discuss a family of models to which these results apply. In particular, we aim to prove Theorem 1.9. To understand how this result holds, we explain here how each of the assumptions can be shown to hold. The proof of this theorem is at the end of the subsection.

**Assumption A:** In general it is expected that $\zeta$ exists and is in fact characterized by the following additional assumption

**Definition 6.2.** We say that $(\xi, h)$ satisfies $P$ if the push forward of the overlap distribution through the map $f(x) = |x|$ is the minimizer of (1.11).

If one could show $P$ for any limiting overlap distribution, then Assumption A would be an immediate consequence of the strict convexity of the Parisi functional for even models. A class of models which are known to satisfy $P$ are the even generic models.

**RSB:** Most results regarding when RSB hold currently focus on the challenging analytical question of the phase diagram for the Parisi measure. See [6, 58, 90] for results in this direction.

**PREV:** It is known [58, 32] that for the optimizer, $\mu$, of (1.11), we have the following: for every $q \in \text{supp}(\mu)$,

$$\begin{align*}
\mathbb{E}_h \left( \partial_x \phi_\mu \right)^2(q, X_q) &= q \\
\Lambda_R(q, \mu) &\geq 0
\end{align*}$$

(6.2)

In our application, we are most interested in the case where $\mu$ is not an atom and the above inequality is strict.

The following result is a collection of several results already appearing in the literature.

**Lemma 6.3.** Suppose that $\xi = \beta^2 \xi_0$ is convex and that $\xi_0''(0) = 0$. Then there is an $h_0$ such that for $h \leq h_0$.

- $q_0 = \min \text{supp}(\mu)$ satisfies $\Lambda_R(q_0, \mu) > 0$.
- For $\beta$ sufficiently large, $\mu$ is not an atom.
- If $\mu$ is not an atom, then GRSB, and GPREV hold.

Suppose furthermore that $\xi$ is either generic or even generic. Then $P$ and Assumption A hold, and in particular, if $\mu$ is not an atom then RSB and PREV hold.

**Proof.** That $P$ holds for $\xi_0$ generic and even generic is shown in [29, 74, 76]. That $P$ implies Assumption A for even generic models at zero external field is clear by symmetry. For generic models and for even generic models with non-zero external field this follows by the positivity of the overlap distribution which is well-known: for generic models this follows by Talagrand’s positivity principle [88, 72], and for even generic models with nonzero external field this follows by [32, Theorem 7].
It remains to prove that there is such an $h_0$. Suppose first that $h = 0$. In this case, it was shown in [6] that $q_0 = 0$. It is also known for all $\xi_0$ [6, 9] that if $\beta$ is sufficiently large, $\mu$ is not a single atom. GRSB then immediately follows. To see that $\Lambda_R(0, \mu) > 0$, recall from [56] Lemma 16 that for all $\nu$, $0 < \Delta \phi < 1$. Thus

$$\Lambda_R(0) > 1 - \xi''(0) > 0.$$  

Thus GPREV holds. For $h > 0$ we now argue by continuity. Observe that $P_I(\nu; h)$ is jointly continuous in the pair $(\nu, h)$. (Here we have made the dependence of $P_I$ on $h$ explicit.) To see this, metrize the weak-* topology on $\text{Pr}([0,1])$ with $d(\mu, \nu) = \int |\mu([0,t]) - \nu([0,t])| \, dt$. It is well-known [48, 56] that $\|\phi_\mu - \phi_\nu\| \leq d(\mu, \nu)$, and that $|\partial_x \phi_\mu| \leq 1$.

Thus $P_I$ is Lipschitz in the usual product metric. By a standard argument (e.g., the fundamental theorem of $\Gamma$-convergence) $\mu_h$, the optimizer of $P_I(\cdot; h)$, is continuous in $h$. (Recall again that the minimizer is unique.)

Observe furthermore that the map $q_0 : \text{Pr}([0,1]) \to \mathbb{R}_+$ defined by

$$q_0(\mu) = \min \text{supp}(\mu)$$

is upper semicontinuous in the weak-* topology. Thus

$$\Lambda_R(q_0, \mu_h) \geq 1 - \xi''(q_0(h)) > 0$$

for $h$ sufficiently small. We used here that $0 < \Delta \phi < 1$ for all $\nu$ and $\xi$ [56]. Thus the first point holds. The second is then immediate. The final point holds by continuity. The results regarding PREV and RSB for generic and even generic models are then immediate.

Proof of Theorem 1.9. This follows by applying Theorem 1.8 and Lemma 6.3.

6.3. Verifying A, RSB, and PREV for Spherical models. In this section, we will discuss a family of spherical models to which these results apply. In particular, we aim to prove Theorem 1.26. To understand how this result holds, we explain here how each of the assumptions can be shown to hold. The proof of this theorem is at the end of the subsection. Many of these results are common with the Ising spin setting.

Assumption A: Define $P$ as in Definition 6.2 except for $\xi$ corresponding to a spherical model. As with Ising spin models, $P$ holds for generic models. It was also shown in [80], that $P$ holds for the so called Pure $p$-spin models i.e., models of the form $\xi(t) = \beta_2 t^p p \geq 4$ and even.

RSB: As in the Ising spin setting, most of the analysis regarding RSB concerns the phase diagram for the Parisi measure, $\mu$. In the spherical setting, far more is known about the Parisi measure $\mu$ [6, 55, 89]. It is also known that Pure $p$-spin models are 1RSB [89]. An explicit, finite dimensional characterization of the space in which $\mu$ lives for general $\xi$ is described in [55]. In particular, one can numerical check the class of ansatzes provided there.

PREV: For $\mu$, we have the following two relations [55, 89] for every $q \in \text{supp}(\mu)$:

$$\begin{cases} -h^2 + \int_0^q \frac{d}{ds} \Lambda_R(q, \mu) \, ds = \xi'(q) \\ \Lambda_R(q, \mu) \geq 0 \end{cases}.$$  

We are interested in understanding when $\Lambda_R(q, \mu)$ is strictly positive.

Consider the following result which collects results already appearing in the literature.
Lemma 6.4. Suppose that either:

1. $\xi = \beta^2 \xi_0$ with $\xi_0''(0) = 0$ is convex and generic or even generic, or
2. $\xi(t) = \beta^2 t^p$ for some even $p \geq 4$.

Then $P$ and Assumption $A$ hold. Furthermore, there is an $h_0$ such that for $h \leq h_0$,

- $q_0 = \min \text{supp}(\mu)$ satisfies $\Lambda_R(q_0, \mu) > 0$.
- If $\mu$ is not an atom, then RSB and PREV hold.
- For $\beta$ sufficiently large, $\mu$ is not an atom.

Proof. As in Lemma 6.3 that $P$ holds and implies Assumption $A$ in our setting is well-known by the same argument for generic and even generic models. The only points to note are that: the differentiation argument provided there holds using using the differentiability of the Crisanti-Sommers formula, which follows by an application of an envelope-type theorem as in [89, Theorem 1.2] and holds even if $h = 0$. In the case of even generic models when $h \neq 0$, use [89, Theorem 7.2] to enforce positivity. That it holds for Pure $p$-spin models was proved by [80, Theorem 4]. There the authors prove that the support of the Parisi measure and the absolute overlap distribution coincide to check that they are the same, note that by the same differentiation argument, the $p$-th moment of these two measures coincide.

It remains to prove the existence of $h_0$. Suppose first that $h = 0$. It is known [55, Corollary 1.3] that $0 \in \text{supp}(\mu)$ since $\xi(t) \neq \beta^2 t^2$ for some $\beta \leq 1$ by assumption. Recall that by [89, Proposition 2.3], $\mu$ is a single atom if and only if for every $s \in (0, 1)$,

$$\beta^2 \xi_0(s) + \log(1 - s) + s < 0,$$

which is evidently violated for

$$\beta > \sqrt{\frac{\log(2) - \frac{1}{2}}{\xi_0(1/2)}},$$

by taking $s = 1/2$. Thus for all $\beta$ sufficiently large $\mu$ is not a single atom. Thus RSB immediately follows. Finally,

$$\Lambda_R(0, \mu) > 0,$$

so that PREV holds. The result for $h > 0$ then holds by continuity as before after noting that $P_S$ is jointly lower semicontinuous in $(\nu, b, h)$ and continuous in $h$ for $(\nu, b)$ fixed.

Proof of Theorem 1.26. This follows by applying Theorem 1.25 and Lemma 6.4.

7. Proof of results from Section 1.2

In this section we collect the proofs of the results from Section 1.2. We provide the proofs in the order that the corresponding theorems are stated in the introduction.

Proof of Theorem 1.13. By the Guerra-Talagrand upper bound, [79, Lemma 2], we have that for every $q \in [-1, 1]$

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \mathcal{Q}_N(B_\epsilon(q)) \leq -I(q).$$

(7.1)

By Panchenko’s lower bound [79, Theorem 2], we have that for every $q \in [-1, 1]$

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \mathcal{Q}_N(B_\epsilon(q)) \geq -I(q).$$

(7.2)
By Gaussian concentration, (3.1), we then obtain,
\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \mathcal{D}_N(B_\epsilon(q)) \leq -I(q)
\]
\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \mathcal{D}_N(B_\epsilon(q)) \geq -I(q),
\]
almost surely. The proof then follows by the following very elementary result from large deviations theory, whose proof is left to the reader.

**Lemma 7.1.** Let \( \mathcal{X} \) be a Polish metric space. Let \( \{P_N\} \) be a sequence of Borel probability measures on \( \mathcal{X} \). Let \( J : \mathcal{X} \to [0, \infty] \) be a measurable function such that for every \( x \in \mathcal{X} \),
\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log P_N(B_\epsilon(x)) \leq -J(x)
\]
\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log P_N(B_\epsilon(x)) \geq -J(x).
\]
Then \( J \) is rate function and \( P_N \) satisfies a weak large deviation principle (LDP) with rate \( N \) and rate function \( J \).

Indeed, by Lemma 7.1, \( I \) is a rate function, and \( Q_N \) almost surely has a weak LDP with rate function \( I \) and rate \( N \). Since \( \mathcal{X} = [-1, 1] \) is compact, this is in fact an LDP and \( I \) is good. \( \square \)

Let us now prove Proposition 1.15.

**Proof of Proposition 1.15** Suppose that FEB holds for some \( q_1 < q_2 < q_3 \) and \( \epsilon > 0 \). By Lemma 3.1, (1.5) implies that
\[
\lim_{N \to \infty} \frac{1}{N} \log Q_N(R_{12} \in (q_i - \epsilon, q_i + \epsilon)) = 0
\]
for \( i = 1, 3 \). Since \( I \) is the rate function of the quenched LDP for \( R_{12} \) by Theorem 1.13, we have that \( I(\tilde{q}_i) = I(\tilde{q}_3) = 0 \) for some \( \tilde{q}_i \) in these respective neighborhoods, by the LDP upper bound. Furthermore, by (1.4), and since \( I \) is the rate function,
\[
-I(q_2) < \lim_{N \to \infty} \frac{1}{N} \log Q_N(R_{12} \in (q_2 - \epsilon, q_2 + \epsilon)) < -C
\]
by the LDP lowerbound. Thus
\[
\mathcal{H} \geq I(q_2) > C
\]
so that Generalized FEB holds. \( \square \)

**Proof of Theorem 1.16** Observe that by definition of the difficulty,
\[
\lim_{N \to \infty} \frac{1}{N} D_\epsilon(p_N^{\otimes 2}, \epsilon / 4) \geq \lim_{N \to \infty} \frac{1}{N} D_\epsilon(R_{12}) \geq \lim_{N \to \infty} \frac{1}{N} \Phi(q_1, q_2, q_3; \epsilon)
\]
for every \( q_1 < q_2 < q_3 \in \mathcal{R}_\epsilon \) almost surely. By Theorem 1.13
\[
-\inf_{x \in (q \epsilon, q + \epsilon)} I(x) \leq \lim_{N \to \infty} \frac{1}{N} S(q; \epsilon) \leq \lim_{N \to \infty} \frac{1}{N} S(q; \epsilon) \leq -\inf_{x \in (q \epsilon, q + \epsilon)} I(x),
\]
amost surely. Combining this with the above, using that \( I \) is lower semicontinuous, and taking suprema, we obtain
\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} D_\epsilon(p_N^{\otimes 2}, \epsilon / 4) \geq \mathcal{H},
\]
almost surely. By Theorems 3.2 and (7.3), we see that
\[
\lim_{N \to \infty} \frac{1}{N} \log \lambda_1 \leq -\mathcal{H}
\]
almost surely, yielding the desired inequality. The result is then immediate by definition. □

Proof of Proposition 1.17 Suppose that Generalized FEB holds. Then there is some $q_1 < q_2 < q_3$ such that

$$I(q_2) > I(q_1) + I(q_3).$$

Consequently, either there is some $\tilde{q}_1 \in [-1, q_2]$ with $I(\tilde{q}_1) = 0$ or there is some $\tilde{q}_3 \in [q_2, 1]$ such that $I(\tilde{q}_3) = 0$. In the first case, $I$ is not monotone to the right of $\tilde{q}_1$. In the second case, $I$ is not monotone to the left of $\tilde{q}_2$. Thus Generalized FEB is a sufficient condition for this monotonicity. That it is necessary is immediate by definition. □

Proof of Theorem 1.18 This is simply a restatement of Theorem 4.5. □

Finally we have the following lemma which is an immediate consequence of Lemma 3.1 and the large deviation principle, Theorem 1.13.

Lemma 7.2. Fix $q \in [-1, 1]$. Suppose that for every $\epsilon > 0$,

$$\lim_{\epsilon \to 0} \mathbb{E} \mathcal{D}_N((q - \epsilon, q + \epsilon)) > 0.$$

Then $I(q) = 0$.

7.1. Regularity of the rate function. Before turning to the proof or Theorem 1.21, we note here the following regularity results regarding the rate function $I$.

Lemma 7.3. For every $\xi$ convex, $I$ is continuous. In particular, it is $(1/2)$–Hölder.

Proof. Without loss of generality we may take $h = 0$. The case $h > 0$ is identical. We begin by showing that if $\eta < \epsilon$, then there is a $C = C(\xi)$ such that

$$|\mathbb{E} F_{2,N}(u - \epsilon, u + \epsilon) - \mathbb{E} F_{2,N}(u - \eta, u + \eta)| \leq J(\epsilon - \eta) + C \cdot (\epsilon - \eta)$$

where

$$J(x) = -x \log x - (1 - x) \log(1 - x).$$

To this end, for each $\sigma \in \Sigma_N$, let

$$B_{u,\epsilon}(\sigma) = \{\sigma' : R(\sigma, \sigma') \in (u - \epsilon, u + \epsilon)\}.$$

Recall that,

$$\frac{1}{N} \log |B_{1,\epsilon}(\sigma)| \leq J(\epsilon/2).$$

Fix $\epsilon > \eta > 0$ and let $\pi' : B_{u,\epsilon}(\sigma) \to B_{u,\eta}(\sigma)$ be the map that takes $\sigma'$ to $\pi(\sigma') \in B_{u,\eta}(\sigma)$ such that the Euclidean distance, $d(\pi(\sigma'), \sigma')$, is minimal. As $\Sigma_N$ is finite, this map is well-defined. Furthermore, we can choose $\pi(\sigma')$ so that $d(\pi(\sigma'), \sigma') \leq 2\sqrt{N}(\epsilon - \eta)$.

Let $A_{\sqrt{N}}$ denote the ball in $\mathbb{R}^N$ of radius $\sqrt{N}$. By Dudley’s entropy bound [63], for any $\delta > 0$, then

$$\mathbb{E} \sup_{d(\sigma^1, \sigma^2) \leq \delta \sqrt{N}, \sigma^1, \sigma^2 \in A_N^2} |H(\sigma^1) - H_N(\sigma^2)| \lesssim_N N \delta.$$
Combining these estimates yields
\[
\mathbb{E} F_N(u - \epsilon, u + \epsilon) = \frac{1}{N} \mathbb{E} \log \int_{\sum} \int_{\sigma^2 \in B_{u,\epsilon}(\sigma^1)} e^{H(\sigma^1) + H(\sigma^2)} d\sigma^2 d\sigma^1 \\
\leq \frac{1}{N} \mathbb{E} \log \int_{\sum} \int_{\sigma^2 \in B_{u,\epsilon}(\sigma^1)} e^{H(\sigma^1) + H(\sigma^2)} d\sigma^2 d\sigma^1 + C \cdot (\epsilon - \eta) \\
\leq \frac{1}{N} \mathbb{E} \log \int_{\sum} \int_{\sigma^2 \in B_{u,\eta}(\sigma^1)} e^{H(\sigma^1) + H(\sigma^2)} |B_{1,2}(\epsilon - \eta)(\sigma^2)| d\sigma^2 d\sigma^1 + C \cdot (\epsilon - \eta) \\
= \mathbb{E} F_N(u - \eta, u + \eta) + C \cdot (\epsilon - \eta) + J(\epsilon - \eta).
\]
Since
\[J(\epsilon - \eta) \leq K \sqrt{\epsilon - \eta}\]
for \(\epsilon\) sufficiently small and \(K > 0\), we obtain the desired inequality.

If \(u, v\) are such that \(|u - v| = \delta\), then for any \(\eta < \delta < 1/2\), the above yields,
\[
\mathbb{E} F_N(u - \eta, u + \eta) - F_N(v - \eta, v + \eta) \leq \mathbb{E} F_N(v - 2\delta, v + 2\delta) - F_N(v - \eta, v + \eta) \lesssim \sqrt{2\delta - \eta}.
\]
Thus by symmetry
\[
|\mathbb{E} F_N(u - \eta, u + \eta) - \mathbb{E} F_N(v - \eta, v + \eta)| \lesssim \sqrt{2\delta - \eta}.
\]
Combining the above bounds with Guerra–Talagrand and Panchenko’s bounds (7.1)-(7.2), if we send \(N \to \infty\) and then \(\eta \to 0\), we obtain
\[
|I(u) - I(v)| \lesssim \sqrt{\delta}
\]
as desired. \(\square\)

**Lemma 7.4.** Suppose that \(\xi\) is convex. The map \((\beta, h, \xi) \mapsto I_{\beta,h,\xi}\) is strongly continuous from \(\mathbb{R}^2_+ \times C([-1, 1])\) to \(C([-1, 1])\).

**Proof.** Again by (7.1)-(7.2), it suffices to show that for \((\beta_1, h_1, \xi_1), (\beta_2, h_2, \xi_2)\), and for every \(u \in [-1, 1]\) and \(\epsilon > 0\) sufficiently small,
\[
|\mathbb{E} F_{2,N}^{\beta_1,h_1,\xi_1}(u - \epsilon, u + \epsilon) - \mathbb{E} F_{2,N}^{\beta_2,h_2,\xi_2}(u - \epsilon, u + \epsilon)| \leq \|\beta_1^2 \xi_1 - \beta_2^2 \xi_2\|_{\infty} + |h_1 - h_2|,
\]
for some universal \(c\). Furthermore, by Jensen’s inequality, it suffices to take the case \(h_1 = h_2\). This case follows by a standard interpolation estimate. Indeed, fix such a \(u\) and \(\epsilon\) and let \(H_1\) denote the Hamiltonian corresponding to \(\beta_1^2 \xi_1\) and \(H_2\) that corresponding to \(\beta_2^2 \xi_2\). Then, if we define the interpolating Hamiltonian
\[H_t(\sigma) = \sqrt{t} H_1(\sigma) + \sqrt{1-t} H_2(\sigma),\]
and let
\[\phi(t) = \mathbb{E} F_{2,N}^t(u - \epsilon, u + \epsilon).
\]
Gaussian integration by parts, see, e.g., [76, Lemma 1], implies that
\[\phi'(t) = \mathbb{E} \int C(\sigma^1, \sigma^1) - C(\sigma^1, \sigma^2) d\pi_t^{\otimes 2}\]
where \(C(\sigma^1, \sigma^2) = \beta_1^2 \xi_1(R_{12}) - \beta_2^2 \xi_2(R_{12})\) and \(\pi_t\) is the Gibbs measure corresponding to \(H_t\). The inequality (7.5) is the immediate. \(\square\)

**Theorem 7.5.** Suppose that \(\xi\) is convex. We have that
\[
\text{supp}(\mu) \subset \{I = 0\}.
\]
Proof. Suppose first that $\xi$ is even generic. Since $P$ and $A$ hold by Lemma 6.3
\[ \text{supp}(\mu) \subset \text{supp}(\zeta), \]
from which the result follows by Lemma 7.2. Suppose now that $\xi$ is only even. Let $\xi_\epsilon = \xi + \epsilon \eta$ where $\epsilon = \sum \frac{1}{2^t} t^p$. Then $\xi_\epsilon$ is even generic. Thus
\[ \text{supp}(\mu_\epsilon) \subset \{ I_\epsilon = 0 \}. \]
It is well-known [56] that $\mu_\epsilon \rightarrow \mu$ weakly. Recall the following basic fact.

Lemma 7.6. If $\nu_\epsilon, \nu \in \text{Pr}([0,1])$ and $\nu_\epsilon \rightarrow \nu$ weakly as measures. Then for every $q \in \text{supp}(\nu)$ there is a sequence $q_\epsilon \rightarrow q$ with $q_\epsilon \in \text{supp}(\nu_\epsilon)$.

Thus for $q$ in $\text{supp}(\mu)$ if we take $(q_\epsilon)$ as above, Lemma 7.3 and Lemma 7.4 yield
\[ 0 \leq I(q) \leq \lim_{\epsilon \to 0} I(q_\epsilon) \leq \lim_{\epsilon \to 0} (I(q_\epsilon) - I_\epsilon(q_\epsilon) + I_\epsilon(q_\epsilon) - I(q_\epsilon)) \leq \lim_{\epsilon \to 0} \epsilon \| \eta \|_{\infty} = 0 \]
as desired. The case $\xi$ is convex is dealt with analogously, by adding a nonzero external field and sending $h \to 0$. The only difference is to note that, since the perturbed model $\xi + \epsilon \eta$ is such that the collection $\{ \nu_\epsilon : \beta_\epsilon \neq 0 \} \cup \{ 1 \}$ is total in $C([0,1])$, conditions $P$ and $A$ still hold when $h > 0$ by the same argument from Lemma 6.3. \qed

With this in hand, we may now prove Theorem 1.24

Proof of Theorem 1.21 By GRSB, there are at least two points in the support of $\mu$ call them $q_1 < q_3$. By Theorem 7.5
\[ I(q_1) = I(q_3) = 0. \]
By GPREV, at least one of these points has positive replicon eigenvalue. Thus $I$ is positive in a punctured neighborhood of this point by Theorem 1.18. In particular, there is a point $q_2$ between $q_1$ and $q_3$ for which $I(q_2) > 0$. Thus $H > 0$, that is, GFEB holds. \qed

7.2. Applications of GFEB. Let us now turn to the proof of our main examples.

First we have the following.

Proof of Theorem 1.22 By Lemma 6.3 GPREV holds for these models. The first result then follows by Theorem 1.21. The remaining follows by the second point of Lemma 6.3. \qed

Let us now turn to the proof of Corollary 1.23. Recall the following theorem of Auffinger–Chen [6] Theorem 4].

Theorem 7.7. We have that $0 \in \text{supp}(\mu)$ for all $\beta$. Furthermore, if $q_{AC}$ denotes the solution of $\xi(q_{AC}) = 1$, then $\mu([0, q_{AC} \wedge 1)) = \mu(\{0\})$.

Corollary 1.23 then follows by a straightforward continuity argument.

Proof of Corollary 1.23 By Theorem 7.7 and Theorem 7.5 we know that $I(0) = 0$ for all $\beta > 0$. For every $\beta > \beta_s$, we know that there is some $q_3 > 0$ such that $I(q_3) = 0$. Let
\[ q_s = \lim_{\beta \downarrow \beta_s} q_3(\beta). \]
By Lemma 7.3 and Lemma 7.4 $I(q_s) = 0$. Furthermore, by Theorem 7.7 $q_s \geq q_{AC} > 0$. Thus there are $q_1, q_3$ such that $I(q_1) = 0$ for $\beta = \beta_s$.

By Theorem 1.18 for every $\beta > 0$, there is an $\epsilon_0(\beta) > 0$ such that for all $q \in (-\epsilon_0, \epsilon_0) \setminus \{0\}$,
\[ I(q) > 0. \]
Thus there is a $q_2 \in (q_1, q_3)$ such that $I(q_2) > 0 = I(q_1) + I(q_3)$ for $\beta = \beta_s$. The result the follows by Lemma 7.4 and the intermediate value theorem. \qed
Appendix A. Analytical properties of the Parisi PDE

In this section, we collect basic facts about the Parisi PDE and its multidimensional analogues. Basic results regarding this functional are treated in many different fashions and are scattered throughout the literature [30, 25, 1, 56, 32]. For a systematic review of the 1-dimensional setting see [56]. For the sake of completeness and as we imagine it will be useful for future research, we state these results in a general setting. Most of these results follow from arguments already appearing in the literature, so our presentation will be brief. In the following, we say that a function $f$ on $\mathbb{R}^d$ has at most linear growth at infinity if there are constants $a, b$ such that $|f(x)| \leq a\|x\| + b$.

Consider the following Cauchy problem. Let $T > t_0 \geq 0$. Suppose that $A(t) : [0, T] \to \mathcal{P}_d$ is a $d \times d$ positive semidefinite matrix that is strictly positive definite on $(t_0, T]$. Suppose furthermore that there is a non-decreasing function $\alpha(s) : (t_0, T] \to \mathbb{R}_+$ and a constant $\kappa$ such that
\[
\kappa \text{Id} \geq A(t) \geq \alpha(t) \text{Id}
\] (A.1)

Finally, let $\nu(t) \in L^\infty([0, T])$ and $g \in C^\infty$ with uniformly bounded derivatives. Consider the Cauchy problem
\[
\begin{aligned}
\partial_t u + \frac{1}{2} (A(D^2 u) + \nu(t) (Du, ADu)) &= 0 \\
u(T, x) &= g(x).
\end{aligned}
\] (A.2)

We say that $u$ is a weak solution to (A.2) if $u$ is continuous in space and time with essentially bounded weak spatial derivative $Du$, and solves
\[
\int_{t_0}^T \int_{\mathbb{R}^d} -u\partial_t \varphi + \frac{1}{2} (u (A(D^2 \varphi) + \nu(t) (Du, ADu) \varphi)) \ dx \ dt + \int_{\mathbb{R}^d} \varphi(T, x) g(x) \ dx
\] (A.3)
for any test function $\varphi \in C^\infty([t_0, T] \times \mathbb{R}^d)$.

A.1. Existence, uniqueness, and regularity of weak solutions. The following is a minor modification of [56] Theorem 2. The arguments provided there extend to the higher dimensional setting and also extends, with minor modifications to the setting where the initial data is only bounded and not also square-integrable. In particular the heat kernel estimates from [56] Eq. (3)] still hold so the arguments there still hold in any dimension.

**Lemma A.1.** Suppose that $\nu(t) \in L^\infty$ and $A(t)$ are as above. Let $g_\lambda$ be a one parameter family of functions that are smooth and have uniformly bounded derivatives. Then there is a unique weak solution, $u_\lambda$, to (A.2) for each $\lambda$. Furthermore, we have the following:

1. $u_\lambda$ is continuous in space, smooth in time, with uniformly bounded spatial derivatives.
2. $u_\lambda$ and its derivatives are once weakly differentiable in time with $\partial_t \partial_\lambda u \in L^\infty_{t,x}$.
3. There are constants $K_n$ that depend at most on $A$ and $\lambda$ such that
\[
\|D^n u\|_{L^\infty_{t,x}} \leq K_n(A, \lambda) \quad \forall n \geq 1.
\]

If the derivatives of $g$ in $\lambda$ are uniformly bounded in $\lambda$, then $K_n$ depends on $A$ alone.

**Remark A.2.** When $d = 1$, $A = \xi''(t)$, and $g = \log \cosh(x)$ we are studying (1.10). When $d = 2$, $A$ is strictly positive definite on $[0, T]$ and $g = f(x)$ we are in the setting of (4.4). When $d = 2$, $t_0 = q$, $A$ is as in (4.10), and $g = f_x$ we are in the setting used in Lemma 4.7. In this case, we note that we have the estimate on $A$ with $\alpha(s) = (\xi''(s))^{-1}$.

Observe that this result applies to $u$ from (4.11). Furthermore, we note here that $v$ from (4.12), satisfies the same bounds by the same argument.

**Lemma A.3.** We have that $v$ from (4.12) exists and is unique. Furthermore, $v \in C_t C^\infty_x$, $\partial_t \partial_\lambda v \in L^\infty_{t,x}$, and there are constants $K_n$ that depend at most on $\xi$ such that
\[
\|\partial^n v\|_{L^\infty_{t,x}} \leq K_n \quad \forall n \geq 1
\]
Remark A.4. We note here again that $K_n$ does not depend on $\lambda$.

A.2. Continuous and Differentiable dependence of the solution of the Parisi PDE. The proof of Theorem A.3 requires differentiable dependence of the solution of (4.14) on $\lambda$ and $q$.

A.2.1. Differentiable dependence in initial data. We aim to differentiate the solution of (4.14) in $\lambda$. This follows by classical differentiable dependence arguments. This type of derivative has already been used many times in the literature [32] [34] [35] [78].

More generally, we have the following result regarding the differentiable dependence of the Parisi PDE on its initial data.

Lemma A.5. Let $g_\lambda$ be a one parameter family of functions in $C^\infty(\mathbb{R}^d)$ with uniformly bounded derivatives. Suppose that the family of maps $\lambda \mapsto D^k g_\lambda(x)$ is $K$-Lipschitz uniformly in $x \in \mathbb{R}^d$ for $k \in [n]$. Let $u(\lambda)$ be the corresponding solutions to (A.2), with at most linear growth at infinity. The map $\lambda \mapsto (u(\lambda),Du(\lambda),D^2u(\lambda),\ldots,D^n u(\lambda))$ satisfies the Lipschitz property

$$
\|u(\lambda) - u(\lambda')\|_{C([t_0,T];C^0(\mathbb{R}^d))} \lesssim_{\alpha,k,K} |\lambda - \lambda'|.
$$

Furthermore, the map $\lambda \mapsto (\partial_\lambda u, D\partial_\lambda u, D^2\partial_\lambda u)$ is continuous as a map $\mathbb{R} \mapsto C([t_0,T];C^2(\mathbb{R}^2))$.

With this result, we then also have the following result which is an immediate Corollary. Let $v$ be as in (1.12).

Lemma A.6. We have that $\partial_\lambda v$ exists for $t \in [0,T]$ and for $t \leq q$, it is a mild solution to

$$
\begin{aligned}
\partial_t \partial_\lambda v + \frac{\xi''}{2} (\partial_\lambda^2 v + 2 \nu \partial_x \partial_\lambda v) &= 0 \quad (t,x) \in [0,q] \times \mathbb{R} \\
\partial_\lambda v(\tau,x) &= \partial_\lambda u(\tau,x) \quad \tau \geq q
\end{aligned}
$$

Furthermore $\partial_\lambda^2 v$ exists for $t \in [0,T]$ and is a mild solution to

$$
\begin{aligned}
\partial_t \partial_\lambda^2 v + \frac{\xi''}{2} (\partial_\lambda^2 v + 2 \nu \partial_x \partial_\lambda^2 v) &= -\xi''(t)\nu (\partial_x v)^2 \quad (t,x) \in [0,q] \times \mathbb{R} \\
\partial_\lambda v(\tau,x) &= \partial_\lambda u(\tau,x) \quad \tau \geq q
\end{aligned}
$$

That $\partial_\lambda v$ is well-defined for $\tau \geq q$ follows immediately from the existence for $\partial_\lambda u$. In particular, we may also write this as an inhomogeneous heat equation on $[0,T] \times \mathbb{R}^d$.

A.2.2. Lipschitz dependence on ellipticity. To prove Theorem 1.8, it is useful to know that $\mathcal{P}(\lambda,q)$ from (1.15) depends continuously on $q$. To this end, take $q_1, q_2 \in [0,1]$, and let $v_1$ and $v_2$ be the corresponding solutions to (1.12). Observe that $u_\nu$ in that definition may be taken to be the weak solution of (1.4), with $A = \xi''(t)Id$ and $\gamma = \mu$ for all $t \in [0,1]$ since we only evaluate $u_\gamma$ for $t \geq q_i$. Our goal is then to prove the following lemma. Let $w = v_1 - v_2$.

Lemma A.7. We have that

$$
|D^n w(0,x)| \lesssim_{\xi,n} |q_1 - q_2|.
$$

Furthermore, $\partial_\lambda v(0,x)$ and $\partial_\lambda^2 v(0,x)$ have Lipschitz dependence in $q$ uniformly in $\lambda$ as well.
A.3. Proof of Lemma 4.7. We are now in the position to prove Lemma 4.7.

Proof of Lemma 4.7. The existence, uniqueness, and regularity of \(u(\lambda)\) and \(v(\lambda)\) follows from Lemma A.1-A.3. The map \(\lambda \mapsto f_\lambda(x)\) is Lipschitz from \(\mathbb{R} \rightarrow f_0 + C([0,T];C^n(\mathbb{R}^2))\). Indeed, \(f\) is smooth in the pair \((\lambda, x)\), and \(\partial_\lambda^k \partial_x^l f_\lambda(x)\) is bounded by a constant that depends on \(k\) and \(l\) alone. The differentiability of these in \(\lambda\) follows from Lemma A.5 and Lemma A.6. To see that the derivatives satisfy the representation formula, note that \(\partial_\lambda u\) and \(\partial_\lambda v\) are bounded, smooth in space, and weakly differentiable in time with bounded weak derivative by virtue of being the unique solutions of the differentiated equations (A.4) and (A.6) respectively, which are (time inhomogeneous) heat equations with coefficients that are smooth, bounded, Lipschitz in space, and bounded measurable in time. Thus we may apply Itô’s lemma (see, e.g., [84]) after observing that the infinitesimal generators of (4.21) and (4.22) are given by

\[
L_1 = \frac{1}{2} \left( \left( A, D^2 \right) + 2\nu (AUDu, D\cdot) \right) \\
L_2 = \frac{\xi''}{2} \left( \partial_x^2 + 2\nu v_x \partial_x \right).
\]

The continuous dependence in \(\lambda\) of the second derivative follows by Lemma A.5. The continuous dependence in \(q\) is by Lemma A.7.

We note that for these two lemmas, the continuity is uniform in \(\lambda\) since they depend on \(\lambda\) only through the derivative bounds on \(u, \partial_\lambda u, \partial_\lambda^2 u, v\), and \(\partial_\lambda v\), which are themselves uniform in \(\lambda\) by Lemma A.1. □

Appendix B. Optimality Conditions for the spherical Parisi functional

In this section, we briefly sketch the proof of Lemma 5.2.

Sketch of Proof of Lemma 5.2. That \(P_S\) is jointly strictly convex follows from strict convexity of each term. The lower semicontinuity follows from the fact that if \(\mu_n \rightarrow \mu\) weak-* then \(\psi_\mu \rightarrow \psi_\mu\) point-wise almost everywhere, combined with Fatou’s lemma.

Since \(\psi_\mu \leq \xi'(1) - \xi'(t)\), we see that there is a \(C(\beta, h), b_0(\beta, h)\) such that for \(b \geq b_0\),

\[
P(\mu, b) \leq C \cdot b \quad \forall \mu.
\]

Thus we may restrict the optimization to the compact set \(\text{Pr}([0,1]) \times [1, C \cdot b_0]\). Thus the optimal pair exists and is unique.

We now characterize this optimal pair, call it \((\mu, b)\). Suppose \(\gamma_\theta = (\mu_\theta, b_\theta)\) is a path \(\gamma : [0,1] \rightarrow A\) such that \(\gamma_0 = 0\), and such that the right derivative

\[
\frac{d}{dt}|_{t=0} P(\gamma_t)
\]

exists. Then if \((\mu, b)\) is optimal, then

\[
\frac{d}{d\theta}|_{\theta=0} P(\gamma_\theta) \geq 0. \tag{B.1}
\]

We refer to this as the first order optimality condition.

We begin with the first point. Applying (B.1) to the path \((1 + \theta, \mu)\), we see that

\[
\frac{d}{d\theta}|_{\theta=0} P(\gamma_\theta) < 0
\]
so that $b > 1$. Suppose now that $b = \psi_\mu(0)$. Then
\[
\frac{\xi''(t)}{b - \psi_\mu(t)} \geq \frac{1}{t}
\]
which is not integrable. Thus $b > \psi_\mu(0)$.

Now for the second point. Take any $\nu$ and consider the variation that sends $\theta \mapsto (b, \nu_\theta)$, where $\nu_\theta = \theta \nu + (1 - \theta)\mu$ mixes the two measures. For $\theta$ sufficiently small, this path is admissible since $b > \psi_\mu$. Since $\gamma \mapsto P(\gamma, b)$ is strictly convex in on the set $E(b) = \{\nu \in \Pr([0,1]) : b - \psi_\nu(0) \geq 0\}$, we see that
\[
\frac{d}{d\theta} P(\nu_\theta, b) \geq 0.
\]
Computing this derivative and re-arranging, we obtain that
\[
\langle G(t), \nu - \mu \rangle \geq 0
\]
where
\[
G(t) = \int_1^t \xi''(s) \left( \frac{h^2}{(b - \psi_\mu)^2(0)} + \int_0^s \left[ \frac{\xi''(\tau)}{(b - \psi_\mu)^2(\tau)} - 1 \right] d\tau \right) ds.
\]
Thus $\mu$ is optimal only if
\[
\mu(G_0(t) = \min_{t \in [0,1]} G_0(t)) = 1.
\]
In particular $G'(q) = 0$ for all $q \in \text{supp}(\mu)$. This yields (5.3).

For the third point, since $b > \psi_\mu$, we see that we may take a full variation in $b$ so that
\[
\frac{d}{db} P(b, \mu) = 0.
\]
This combined with (6.3) for $q = q_{EA}$ yields (5.4).

For the fourth point, we see that by integrating (5.3),
\[
\varphi_\mu(q') - \varphi(q) = \frac{1}{b - \psi_\mu(q')} - \frac{1}{b - \psi_\mu(q)}
\]
for all $q, q' \in \text{supp}(\mu)$. Taking $q' = q_{EA}$ yields (5.5).

For the finally point, an explicit computation yields
\[
P(\mu, b) = C(\mu).
\]
The result then follows by (5.1).

\section*{References}

[1] Michael Aizenman, Robert Sims, and Shannon L Starr. Extended variational principle for the Sherrington-Kirkpatrick spin-glass model. Physical Review B, 68(21):214403, 2003.

[2] Noga Alon and Vitali D Milman. $\lambda_1$, isoperimetric inequalities for graphs, and superconcentrators. Journal of Combinatorial Theory, Series B, 38(1):73–88, 1985.

[3] Louis-Pierre Arguin and Michael Aizenman. On the structure of quasi-stationary competing particle systems. The Annals of Probability, pages 1080–1113, 2009.

[4] Antonio Auffinger and Gérard Ben Arous. Complexity of random smooth functions on the high-dimensional sphere. Ann. Probab., 41(6):4214–4247, 2013.

[5] Antonio Auffinger, Gérard Ben Arous, and Jiří Černý. Random matrices and complexity of spin glasses. Comm. Pure Appl. Math., 66(2):165–201, 2013.

[6] Antonio Auffinger and Wei-Kuo Chen. On properties of Parisi measures. Probab. Theory Related Fields, 161(3-4):817–850, 2015.

[7] Antonio Auffinger and Wei-Kuo Chen. The Parisi formula has a unique minimizer. Comm. Math. Phys., 335(3):1429–1444, 2015.
[8] Antonio Auffinger and Wei-Kuo Chen. Parisi formula for the ground state energy in the mixed p-spin model. *Ann. Probab.*, (to appear), 2016.
[9] Antonio Auffinger, Wei-Kuo Chen, and Qiang Zeng. The SK model is full-step replica symmetry breaking at zero temperature. *arXiv preprint arXiv:1703.06872*, 2017.
[10] Antonio Auffinger and Aukosh Jagannath. Thouless-Anderson-Palmer equations for generic p-spin glass models. *arXiv preprint arXiv:1612.06359*, 2016.
[11] Dominique Bakry and Michel Ledoux. Lévy-Gromov’s isoperimetric inequality for an infinite-dimensional diffusion generator. *Invent. Math.*, 123(2):259–281, 1996.
[12] Gérard Ben Arous. Aging and spin-glass dynamics. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 3–14. Higher Ed. Press, Beijing, 2002.
[13] Gérard Ben Arous, Anton Bovier, and Jiří Černý. Universality of the REM for dynamics of mean-field spin glasses. *Comm. Math. Phys.*, 282(3):663–695, 2008.
[14] Gérard Ben Arous, Anton Bovier, and Véronique Gayrard. Aging in the random energy model. *Physical review letters*, 88(8):087201, 2002.
[15] Gérard Ben Arous, Anton Bovier, and Véronique Gayrard. Glauber dynamics of the Random Energy Model. *Communications in mathematical physics*, 236(1):1–54, 2003.
[16] Gérard Ben Arous and Onur Gün. Universality and extremal aging for dynamics of spin glasses on subexponential time scales. *Comm. Pure Appl. Math.*, 65(1):77–127, 2012.
[17] Gérard Ben Arous and David S Dean. Aging on parisian’s tree. *Journal de Physique I*, 5(3):265–286, 1995.
[18] Jean-Philippe Bouchaud. Weak ergodicity breaking and aging in disordered systems. *Journal de Physique I*, 2(9):1705–1713, 1992.
[19] Jean-Philippe Bouchaud, Leticia F Cugliandolo, Jorge Kurchan, and Marc Mézard. Out of equilibrium dynamics in spin-glasses and other glassy systems. *Spin glasses and random fields*, pages 161–223, 1998.
[20] Jean-Philippe Bouchaud and David S Dean. Aging on parisian’s tree. *Journal de Physique I*, 5(3):265–286, 1995.
[21] Anton Bovier and Alessandra Faggionato. Spectral characterization of aging: the REM-like trap model. *Ann. Appl. Probab.*, 15(3):1997–2037, 2005.
[22] Anton Bovier and Véronique Gayrard. Glauber dynamics of the Random Energy Model. *Communications in mathematical physics*, 236(1):1–54, 2003.
[23] Anton Bovier and Véronique Gayrard. Convergence of clock processes in random environments and aging in the p-spin SK model. *Ann. Probab.*, 41(2):817–847, 2013.
[24] Anton Bovier and Anton Klimovsky. The Aizenman-Sims-Starr and Guerra’s schemes for the SK model with multidimensional spins. *Electronic Journal of Probability*, 14(8):161–241, 2009.
[25] Peter Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)*, 15(2):213–230, 1982.
[26] Tommaso Castellani and Andrea Cavagna. Spin-glass theory for pedestrians. *Journal of Statistical Mechanics: Theory and Experiment*, 2005(05):P05012, 2005.
[27] Sourav Chatterjee. The Ghirlanda-Guerra identities without averaging. *arXiv preprint arXiv:0911.4520*, 2009.
[28] Isaac Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.
[29] Jeff Cheeger. A lower bound for the smallest eigenvalue of the laplacian. In *Proceedings of the Princeton conference in honor of Professor S. Bochner*, 1969.
[30] Wei Kuo Chen. Variational representations for the Parisi functional and the two-dimensional Guerra-Talagrand bound. *Ann. Probab.*, to appear, [http://arxiv.org/abs/1501.06635](http://arxiv.org/abs/1501.06635).
[31] Wei-Kuo Chen. The Aizenman-Sims-Starr scheme and Parisi formula for mixed p-spin spherical models. *Electron. J. Probab.*, 18:no. 94, 14, 2013.
[32] Wei-Kuo Chen, Partha Dey, and Dmitry Panchenko. Fluctuations of the free energy in the mixed p-spin models with external field. *Probability Theory and Related Fields*, pages 1–13, 2015.
[33] Wei-Kuo Chen, Madeline Handschy, and Gilad Lerman. On the energy landscape of the mixed even p-spin model. *arXiv preprint arXiv:1609.04368*, 2016.
[34] Wei-Kuo Chen, Hsi-Wei Hsieh, Chii-Ruey Hwang, and Yuan-Chung Sheu. Disorder chaos in the spherical mean-field model. *J. Stat. Phys.*, 160(2):417–429, 2015.
[38] J. R. L. de Almeida and David J. Thouless. Stability of the Sherrington-Kirkpatrick solution of a spin glass model. *Journal of Physics A: Mathematical and General*, 11(5):983, 1978.

[39] Emilio De Santis. Glauber dynamics of spin glasses at low and high temperature. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(5):681–710, 2002.

[40] Persi Diaconis, Laurent Saloff-Coste, et al. Logarithmic sobolev inequalities for finite markov chains. *The Annals of Applied Probability*, 6(3):695–750, 1996.

[41] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[42] Luiz Renato Fontes, Marco Isopi, Yoshiharu Kohayakawa, and Pierre Picco. The spectral gap of the REM under Metropolis dynamics. *Ann. Appl. Probab.*, 8(3):917–943, 1998.

[43] Silvio Franz, Giorgio Parisi, and Miguel Angel Virasoro. The replica method on and off equilibrium. *Journal de Physique I*, 2(10):1891–1890, 1992.

[44] Véronique Gayrard. Aging in metropolis dynamics of the REM: a proof. *arXiv preprint arXiv:1602.06081*, 2016.

[45] Véronique Gayrard. Convergence of clock processes and aging in Metropolis dynamics of a truncated REM. *Ann. Henri Poincaré*, 17(3):537–614, 2016.

[46] Reza Gheissari and Aukosh Jagannath. On the spectral gap of spherical spin glass dynamics. *arXiv preprint arXiv:1608.06609*, 2016.

[47] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[48] Francesco Guerra. Sum rules for the free energy in the mean field spin glass model. In *Mathematical physics in mathematics and physics (Siena, 2000)*, volume 30 of *Fields Inst. Commun.*, pages 161–170. Amer. Math. Soc., Providence, RI, 2001.

[49] Francesco Guerra. Broken replica symmetry bounds in the mean field spin glass model. *Comm. Math. Phys.*, 230(1):1–12, 2003.

[50] Francesco Guerra and Fabio Lucio Toninelli. The thermodynamic limit in mean field spin glass models. *Comm. Math. Phys.*, 230(1):71–79, 2002.

[51] Alice Guionnet. Dynamics for spherical models of spin-glass and aging. In *Spin glasses*, pages 117–144. Springer, 2007.

[52] Alice Guionnet and Boguslaw Zegarlisnki. Decay to equilibrium in random spin systems on a lattice. *Comm. Math. Phys.*, 181(3):703–732, 1996.

[53] Alice Guionnet and Boguslaw Zegarlsinski. Lectures on logarithmic Sobolev inequalities. In *Séminaire de Probabilités, XXXVI*, volume 1801 of *Lecture Notes in Math.*, pages 1–134. Springer, Berlin, 2003.

[54] Richard Holley and Daniel Stroock. Logarithmic Sobolev inequalities and stochastic Ising models. *J. Statist. Phys.*, 46(5-6):1159–1194, 1987.

[55] Aukosh Jagannath and Ian Tobasco. Bounding the complexity of replica symmetry breaking for spherical spin glasses. *Proc. Amer. Math. Soc.*, (to appear), 2016.

[56] Aukosh Jagannath and Ian Tobasco. A dynamic programming approach to the Parisi functional. *Proc. Amer. Math. Soc.*, 144(7):3135–3150, 2016.

[57] Aukosh Jagannath and Ian Tobasco. Low temperature asymptotics of spherical mean field spin glasses. *Comm. Math. Phys.*, 352(3):979–1017, 2017.

[58] Aukosh Jagannath and Ian Tobasco. Some properties of the phase diagram for mixed p-spin glasses. *Probab. Theory Related Fields*, 167(3-4):615–672, 2017.

[59] Mark Jerrum and Alistair Sinclair. Approximating the permanent. *SIAM journal on computing*, 18(6):1149–1178, 1989.

[60] Jorge Kurchan, Giorgio Parisi, and Miguel Angel Virasoro. Barriers and metastable states as saddle points in the replica approach. *Journal of Physics I*, 3(8):1819–1838, 1993.

[61] Gregory F Lawler and Alan D Sokal. Bounds on the $L^2$ spectrum for markov chains and markov processes: a generalization of cheeger’s inequality. *Transactions of the American mathematical society*, 309(2):557–580, 1988.

[62] Peter D. Lax. *Functional analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2002.

[63] Michel Ledoux. A simple analytic proof of an inequality by P. Buser. *Proc. Amer. Math. Soc.*, 121(3):951–959, 1994.

[64] Michel Ledoux. *The concentration of measure phenomenon*, volume 89 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.

[65] Michel Ledoux and Michel Talagrand. *Probability in Banach spaces*. Classics in Mathematics. Springer-Verlag, Berlin, 2011. Isoperimetry and processes, Reprint of the 1991 edition.

[66] David Ashler Levin, Yuval Peres, and Elizabeth Lee Wilmer. *Markov chains and mixing times*. American Mathematical Soc., 2009.
