STOCHASTIC WASSERSTEIN HAMILTONIAN FLOWS

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Abstract. In this paper, we study the stochastic Hamiltonian flow in Wasserstein manifold, the probability density space equipped with $L^2$-Wasserstein metric tensor, via the Wong–Zakai approximation. We begin our investigation by showing that the stochastic Euler-Lagrange equation, regardless it is deduced from either variational principle or particle dynamics, can be interpreted as the stochastic kinetic Hamiltonian flows in Wasserstein manifold. We further propose a novel variational formulation to derive more general stochastic Wasserstein Hamiltonian flows, and demonstrate that this new formulation is applicable to various systems including the stochastic Schrödinger equation, Schrödinger equation with random dispersion, and Schrödinger bridge problem with common noise.

1. Introduction

The density space equipped with $L^2$-Wasserstein metric forms an infinite dimensional Riemannian manifold, often called Wasserstein manifold or density manifold in literature (see e.g. [31]). It plays an important role in optimal transport theory [41]. Many well-known equations, such as Schrödinger equation, Schrödinger bridge problem and Vlasov equation, can be written as Hamiltonian systems on the density manifold. In this sense, they can be considered as members of the so-called Wasserstein Hamiltonian flows ([41, 3, 22, 13, 11, 12, 16]). The study of Wasserstein Hamiltonian flow can be traced back to Nelson’s mechanics ([35, 36, 37, 38]). Recently, it is shown in [12] that the kinetic Hamiltonian flows in density space are probability transition equations of classical Hamiltonian ordinary differential equations (ODEs). In other words, this reveals that the density of a Hamiltonian flow in sample space is a Hamiltonian flow on density manifold.

In the existing works on Wasserstein Hamiltonian flows, random perturbations to the Lagrangian functional are not considered. Consequently, the theory is neither directly applicable to the Wasserstein Hamiltonian flows subjected to random perturbations, nor to the systems whose parameters are not given deterministically. The main goal of this article is developing a theory to cover these scenarios in which the stochasticity is presented. More precisely, we mainly focus on the stochastic perturbation of the Wasserstein Hamiltonian flow,

\[ d\rho_t = \frac{\delta}{\delta S_t} H_0(\rho_t, S_t) dt, \]

\[ dS_t = -\frac{\delta}{\delta \rho_t} H_0(\rho_t, S_t) dt, \]

with a Hamiltonian $H_0$ on the density manifold and $\frac{\delta}{\delta S_t}, \frac{\delta}{\delta \rho_t}$ being the variational derivatives, which is proposed by only imposing randomness on the initial position of the phase space [12].
This is different from the Hamiltonian flows considered in [3], where the authors consider and construct the solutions of the ODEs in the measure space of even dimensional phase variables equipped with the Wasserstein metric.

To study the stochastic variational principle on density manifold, we may confront several challenges. First and the foremost, the Wasserstein Hamiltonian flow studied in [12] is induced based on the principle that the density of a Hamiltonian flow in sample space is a Wasserstein Hamiltonian flow in density manifold. This principle may no longer hold if the Hamiltonian flow in sample space is perturbed by random noise. Second, the stochastic variational framework must be carefully designed in order to induce stochastic dynamics that possess Hamiltonian structures on Wasserstein manifold. Last by not the least, it is not clear at all that how to introduce the Christoffel symbol, a tool that plays the vital role in the typical kinetic dynamics, in the noise perturbed Wasserstein Hamiltonian flows on the density manifold.

To overcome the difficulties, we begin our study by investigating the classical Lagrangian functional perturbed by the Wong–Zakai approximation (see e.g. [44, 39]) on phase space, and show that the critical point of the new Lagrangian functional is convergent to the known stochastic Hamiltonian flow driven by Wiener process. We further prove that the stochastic Wasserstein Hamiltonian flow is the critical point of a stochastic variational principle (see e.g. [42]). Meanwhile, the macro behaviors of this convergence indicates that the critical point of the macro Lagrangian functional corresponding to Wong–Zakai approximation is convergent to the stochastic Euler–Lagrange equation in density space.

Furthermore, a general variational principle is proposed to derive a large class of stochastic Hamiltonian equations on density manifold via Wong–Zakai approximation, such as stochastic nonlinear Schrödinger equation (see, e.g., [4, 19, 29, 40]), nonlinear Schrödinger equation with white noise dispersion (see, e.g., [1, 2]), and the mean-field game system with common noise (see, e.g., [23, 6, 7]). We would like to mention that although the Wong–Zakai approximation of stochastic differential equations has been studied for many years (see, e.g., [44, 39, 5, 43]), few result is known for the convergence on the density manifold. In this work, we also provide some new convergence results of Wong–Zakai approximation for the continuity equation induced by stochastic Hamiltonian system and the stochastic Schrödinger equation on density space under suitable assumptions.

Another main message that we would like to convey in this paper is that the stochastic Hamiltonian flow on phase space, when viewed through the lens of conditional probability, induces the stochastic Wasserstein Hamiltonian flow on density manifold, and it is hard to observe those stochastic Hamiltonian structures in the density manifold without the help of conditional probability.

The organization of this article is as follows. In section 2, we review the formulation and derivation of Hamiltonian ODE, and use the Wong–Zakai approximation of the Lagrangian functional to connect the classic and stochastic variational principles on phase space. In section 3, we study the macro behaviors of stochastic Hamiltonian ODE and its Wong–Zakai approximation, including the stochastic Euler–Lagrange equation on density space, Vlasov equation, as well as the generalized stochastic Wasserstein Hamiltonian flow. Several examples are demonstrated in section 4. Throughout this paper, we denote $C$ and $c$ as positive constants which may differ from line to line.

2. Stochastic Hamiltonian ODEs

In this section, we briefly review the classical and stochastic Hamiltonian flows on a finite dimensional Riemannian manifold.
The classical Hamiltonian flow on a smooth $d$-dimensional Riemannian manifold $(\mathcal{M}, g)$ with $g$ being the metric tensor of $\mathcal{M}$, is derived by the variational problem
\[
I(x_0, x_T) = \inf_{(x(t))_{t\in[0,T]}} \left\{ \int_0^T L_0(x, \dot{x}) dt : x(0) = x_0, x(T) = x_T \right\}.
\]
Here the Lagrangian $L_0$ is a functional (also called Lagrange action functional) defined on the tangent bundle of $\mathcal{M}$. Its critical point induces the Euler-Lagrange equation
\[
\frac{d}{dt} \frac{\partial}{\partial \dot{x}} L_0(x, \dot{x}) - \frac{\partial}{\partial x} L_0(x, \dot{x}) = 0.
\]
When $L_0(x, \dot{x}) = \frac{1}{2} \dot{x}^T g(x) \dot{x} - f(x)$ with a smooth potential functional $f$ on $\mathcal{M}$, the Euler-Lagrange equation can be rewritten as a Hamiltonian system,
\[
\dot{x} = g(x)^{-1} p, \quad \dot{p} = -\frac{1}{2} p^T d_x g^{-1}(x) p - d_x f(x)
\]
Here $\top$ denotes the transpose, $p = g(x) \dot{x}$ and the Hamiltonian is
\[
H_0(x, p) = \frac{1}{2} p^T g^{-1}(x) p + f(x).
\]
However, the Lagrange action functional $L_0(x, \dot{x})$ may not be homogeneous or it can by impacted by random perturbations in some problems, which is the reason to introduce stochastic Hamiltonian flows.

Let us start with the case that $L(x, \dot{x})$ is composed by the deterministic Lagrange functional $L_0(x, \dot{x})$ and a random perturbation $\eta \sigma(x) \xi_\delta(t)$. Here $\xi_\delta$ can be chosen as a piecewise continuous differentiable function which obeys certain distribution law in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, $\sigma(x)$ is a potential function and $\eta \in \mathbb{R}$ characterizes the noise intensity. In this paper, $\xi_\delta$ is taken as a Wong-Zakai approximation (see e.g. [44]) of the standard Brownian motion such that $\xi_\delta$ is a real function. When $\delta \to 0$, $\xi_\delta(t)$ is convergent to the Brownian motion $B(t)$ in pathwise sense or strong sense. For fixed $\omega \in \Omega$, since $\xi_\delta(t)$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with piecewise continuous trajectory, the value of the action functional
\[
\int_0^T L_0(x, \dot{x}) - \eta \sigma(x) \xi_\delta(t) dt
\]
is finite for any given $x(0) = x_0, x(T) = x_T$.

Newton’s law can be used to derive the Euler–Lagrange equation or the Hamiltonian system in the stochastic case. In order to find out the critical point of $\int_0^T L_0(x, \dot{x}) - \eta \sigma(x) \xi_\delta(t) dt$, we calculate its Gâteaux derivative (see, e.g., [24]). Set $x_\epsilon(t) = x(t) + \epsilon h(t), h(0) = h(T) = 0$, the Newton’s law indicates the critical point satisfies
\[
\frac{d}{dt} \frac{\partial}{\partial \dot{x}} L(x, \dot{x}) - \frac{\partial}{\partial x} L(x, \dot{x}) = \frac{\partial}{\partial x} L_0(x, \dot{x}) - \eta \frac{\partial}{\partial x} \sigma(x) \xi_\delta,
\]
which is equivalent to the integral equation
\[
\frac{\partial}{\partial x} L(x(t), \dot{x}(t)) - \frac{\partial}{\partial x} L(x(0), \dot{x}(0)) = \int_0^t \frac{\partial}{\partial x} L_0(x, \dot{x})ds - \eta \int_0^t \frac{\partial}{\partial x} \sigma(x) d\xi_\delta.
\]
One can also introduce the Legendre transformation $p = g(x) \dot{x}$, and get
\[
\dot{\dot{x}} = g(x)^{-1} p, \quad \dot{\dot{p}} = -\frac{1}{2} p^T d_x g^{-1}(x) p - d_x f(x) - \eta d_x \sigma(x) \xi_\delta.
\]
Since it can be rewritten as
\[
\dot{x} = \frac{\partial}{\partial p} H_0(x, p) + \frac{\partial}{\partial p} H_1(x, p) \xi_\delta, \quad \dot{p} = -\frac{\partial}{\partial x} H_0(x, p) - \frac{\partial}{\partial x} H_1(x, p) \xi_\delta,
\]
where $H_1(x, p) = \sigma(x)$, the equations form a stochastic Hamiltonian system.
Remark 2.1. When $\xi_d$ is a constant, the Hamilton’s principle gives a Hamiltonian system with a homogenous perturbation. Otherwise, for a fixed $\omega$, the Hamilton’s principle leads to a Hamiltonian system with an inhomogenous perturbation.

2.1. Wong–Zakai approximation in $\mathcal{M} = \mathbb{R}^d$. In this part, we show that the limit of the Wong-Zakai approximation (2.1) is a stochastic Hamiltonian system.

**Lemma 2.1.** Let $\mathcal{M} = \mathbb{R}^d$ and $T > 0$, $g$ be the identity matrix $I_{d \times d}$. Assume that $f, \sigma \in C^2_b(\mathcal{M})$, $\xi_d$ is the linear interpolation of $B(t)$ with width $\delta$ and that $x_0, p_0$ is $\mathcal{F}_0$-adapted. Then (2.1) on $[0, T]$ is convergent to

$$
(2.2) \quad dx = p, \ dp = -d_x f(x) - \eta d_x \sigma(x) \circ dB(t), \ a.s.,
$$

where $\circ$ denotes the stochastic integral in the Stratonovich sense.

**Proof.** The condition of $\sigma, f$ ensures the global existence of a unique strong solution for (2.1) and (2.2) by using standard Picard iterations. Then one can follow the classical arguments (see e.g. [39]) to show that the solution of (2.1) is convergent to that of (2.2) and that the right hand side of (2.1) is convergent to that of (2.2). \qed

The following lemma relaxes the classical conditions on the convergence of Wong-Zakai approximation whose proof is presented in Appendix. We call that $g$ is equivalent to $I_{d \times d}$ if $g \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^d)$ is symmetric satisfying $\Lambda I_{d \times d} \succeq g(x) \succeq \lambda I_{d \times d}$ for some constant $0 < \lambda \leq \Lambda$. In the following, we will use the standard notation for the matrix product, that is, $g(x) \cdot (y, z) = y^\top g(x)z$ and $g(x) \cdot y = g(x)y$.

**Lemma 2.2.** Let $\mathcal{M} = \mathbb{R}^d$, $T > 0$, $g$ be equivalent to $I_{d \times d}$. Assume that $f, \sigma \in C^2_b(\mathcal{M})$, $\xi_d$ is the linear interpolation of $B(t)$ with the width $\delta$, that $x_0, p_0$ is $\mathcal{F}_0$-adapted and possess any finite $q$-moment, $q \in \mathbb{N}^+$, and that

$$
H_0(x, p) \geq c_0|p| + c_1|x|, \text{ for large enough } |x|, |p|
$$

$$
(2.3) \quad \eta^2|\nabla_{pp} H_0(x, p) \cdot (\nabla_x \sigma(x), \nabla_x \sigma(x))| + \eta|\nabla_{pp} H_0(x, p) \cdot (p, \nabla_x \sigma(x))| + \eta|\nabla_{pp} H_0(x, p) \cdot (\nabla_x \sigma, \nabla^{-1}_x(x)p)| + \eta|\nabla_{pp} H_0(x, p) \cdot \nabla_x \sigma(x)g^{-1}(x)p| \leq c_1 + c_1 H_0(x, p).
$$

Then the solution of (2.1) on $[0, T]$ is convergent in probability to the solution of

$$
(2.4) \quad dx = g^{-1}(x)p, \ dp = -\frac{1}{2}p^\top d_x g^{-1}(x)p - d_x f(x) - \eta d_x \sigma(x) \circ dB(t).
$$

Denote the solution of (2.1) by $(x^\delta(\cdot, x_0, p_0), p^\delta(\cdot, x_0, p_0))$. According to Lemma 2.2, by studying the equation of $\frac{\partial}{\partial x^0} x^\delta(t, x_0, p_0)$ and $\frac{\partial}{\partial p_0} x^\delta(t, x_0, p_0)$, one could obtain the following convergence result.

**Corollary 2.1.** Under the condition of Lemma 2.2, let $f, \sigma \in C^2_b(\mathcal{M})$. Then for any $\epsilon > 0$, it holds that

$$
\lim_{\delta \to 0} P \left( \sup_{t \in [0, T]} \left| \frac{\partial}{\partial x^0} x^\delta(t, x_0, p_0) - \frac{\partial}{\partial x^0} x(t, x_0, p_0) \right| + \sup_{t \in [0, T]} \left| \frac{\partial}{\partial p_0} x^\delta(t, x_0, p_0) - \frac{\partial}{\partial p_0} x(t, x_0, p_0) \right| \geq \epsilon \right) = 0.
$$
Remark 2.2. One may impose more additional conditions on the coefficients $f, \sigma$ to obtain the strong convergence order $\frac{1}{2}$ of the Wong–Zakai approximation, that is,

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} |x^\epsilon(t) - x(t)|^2 \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} |\dot{x}^\epsilon(t) - \dot{x}(t)|^2 \right] \leq C \delta^\frac{1}{2}.
$$

The convergence in probability yield that there exists a pathwise convergent subsequence. In this sense, the limit equation of (2.1) is (2.4) on $[0, T]$. When the growth condition (2.3) fails, one could also obtain the convergence in probability of $(\dot{x}^\epsilon, p^\epsilon)$ before the stopping time $\tau_R \wedge \tau_{R_1}$ (see Appendix for the definition of $\tau_R$ and $\tau_{R_1}$). One could also choose different type of Wong–Zakai approximation of the Wiener process and obtain similar results (see, e.g., [44]).

2.2. Wong–Zakai approximation on a differential manifold $\mathcal{M}$. Assume that $\mathcal{M} \subset \mathbb{R}^k$ is a $d$-dimensional differential manifold of class $\mathcal{C}^\infty, \alpha \in \mathbb{N}^+ \cup \infty$ without boundary. Given a $\mathcal{C}^\infty$-diffeomorphism $\phi: W \to V \subset \mathcal{M}$ from an open subset $W$ of $\mathbb{R}^d$ to an open set $V$ of $\mathcal{M}$, the inverse $\phi^{-1} : V \to W$ is called a chart or coordinate system on $\mathcal{M}$. The coordinate components are denoted by $\Phi_1, \Phi_2, \ldots, \Phi_d, d \in \mathbb{N}^+$. The tangent bundle of $\mathcal{M}$ is denoted by $T\mathcal{M} := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^k | x \in \mathcal{M}, y \in T_x(\mathcal{M})\}$. Moreover, $dimT_x(\mathcal{M}) = d$. The canonical projection is denoted by $\pi : T\mathcal{M} \to \mathcal{M}$.

In the following, we start from the deterministic Hamiltonian system

$$
\dot{x} = p, \\
\dot{p} = -d_x f(x),
$$

where the vector field $(p, -d_x f(x)) \in T_{(x,p)} T\mathcal{M}$ for all $(x, p) \in T\mathcal{M}$. We show how the random force can be added to the system so that $(\dot{x}, \dot{p}) \in \mathbb{R}^k \times \mathbb{R}^k$ is still tangent to $T\mathcal{M}$ at $(x, p)$. As a physical interpretation, this tangent condition represents the constrain of the motion equations and is to ensure that the physical motion is living in $T\mathcal{M}$ by the Kampe property of the maximal solutions (see e.g. [21, Chapter 3]). Consider $\mathcal{M}$ which is regularly defined as the zero level set of a $\mathcal{C}^\infty$ map $F$ from $\mathbb{R}^k$ to $\mathbb{R}^{k-d}$. Then we have that the tangent space to $\mathcal{M}$ at $x$ is $T_x \mathcal{M} := \{ p \in \mathbb{R}^k | F'(x)p = 0 \}$, and $T\mathcal{M} = \{(x, p) \in \mathbb{R}^k \times \mathbb{R}^k | F(x) = 0, F'(x)p = 0 \}$. We can also obtain

$$
TT\mathcal{M} = \{(x, p, \dot{x}, \dot{p}) | F(x) = 0, F'(x)p = 0, F''(x)(\dot{x}, p) + F'(x)\dot{p} = 0 \}.
$$

Therefore, if the added random force satisfies,

$$(2.5) \quad F'(x)\dot{p} = -F''(x)(\dot{x}, p) = \psi(x; p, \dot{x}), \quad \dot{x} \in T_x(\mathcal{M}),$$

we have $(\dot{x}, \dot{p}) \in T_{(x,p)} (T\mathcal{M})$. Following [21], we denote a smooth mapping $\psi$ from the vector bundle $\{(x; u, v) \in \mathbb{R}^k \times (\mathbb{R}^k \times \mathbb{R}^k) | x \in \mathcal{M}, u, v \in T_x(\mathcal{M})\}$ to $\mathbb{R}^{k-d}$. Given any vector $z \in \mathbb{R}^{k-d}$, denote by $A_z \in (\ker F'(x))^\perp = (T_x \mathcal{M})^\perp$ the unique solution of $F'(x)\dot{p} = z$. Hence, the solution of (2.5) satisfies

$$
\dot{p} = \mu(x; p, \dot{x}) + w,
$$

where $\mu(x; p, \dot{x}) = A\psi(x; p, \dot{x}) \in (T_x(\mathcal{M}))^\perp$ and $w \in T_x(\mathcal{M})$. We observe that to ensure $(\dot{x}, \dot{p}) \in T_{(x,p)} (T\mathcal{M})$, it suffices to take $u, w \in T_x(\mathcal{M})$ and define $(\dot{x}, \dot{p}) = (u, \mu(x; p, u) + w)$. In Eq. (2.1) with the driving noise being $-d_x \sigma(x)\xi_\delta$, using the above condition, we can verify that it satisfies that $(\dot{x}, \dot{p}) \in T_{(x,p)} (T\mathcal{M})$. Similarly, a second order differential equation with random force satisfies

$$
\ddot{x} = \mu(x; \dot{x}, \dot{x}) + \mathcal{R}(t, x, \dot{x}),
$$

where $\mathcal{R} : T\mathcal{M} \ni (x, \dot{x}) \mapsto \mathcal{R}(t, x, \dot{x}) \in \mathbb{R}^k$ is a tangent vector field on $\mathcal{M}$. A typical example is that $\mathcal{R} = -\alpha \dot{x} + a(t, x)$ with the frictional force $-\alpha \dot{x}$ and applied random force $a(t, x) = \ldots$
Hamiltonians satisfies that Hamiltonian flows of a standard Brownian motion, and that $x_0, p_0$ are $F_0$-adapted and possess any finite $q$-moment, $q \in \mathbb{N}^+$. Then $(x^\delta, p^\delta)$ is convergent in probability to the solution $(x, p)$ of (2.4).

**Proof.** The existence and uniqueness of $(x, p)$ can be found in [25]. The global existence of $(x^\delta, p^\delta)$ could be also obtained by the fact that $g = \mathbb{I}$, $f$ and $\sigma$ are globally Lipschitz and that the growth condition (2.3) holds. We only need to show the convergence of $(x^\delta, p^\delta)$ in probability to $(x, p)$. Since $T^1\mathcal{M}$ is 2d-dimensional manifold which could be embedding to $\mathbb{R}^{2k}$, we can extend the vector field $V(x, p) := (p, -d_x f(x) - \eta d_x \sigma(x))$ to a vector field $\tilde{V}(\cdot, \cdot)$ on $\mathbb{R}^{2k}$. And thus the equations of $(x, p)$ and $(x^\delta, p^\delta)$ can be viewed as the equations on $\mathbb{R}^{2k}$. The global existence of $(x, p)$ and $(x^\delta, p^\delta)$, together with Lemma 2.2, yield the convergence in probability of $(x^\delta, p^\delta)$. □

**Remark 2.3.** The above result relies on the particular structure of $g = \mathbb{I}$ and the growth condition (2.3). If this condition (2.3) fails, the explosion time $\epsilon(x^\delta, p^\delta)$ of $(x^\delta, p^\delta)$ may depend on $\delta$. And the convergence in probability may only hold before $\epsilon(x, p) \wedge \inf_{\delta > 0} \epsilon(x^\delta, p^\delta)$. When applying different type of Wong–Zakai approximations, the different type of stochastic ODEs could be derived (see e.g. [26]).

To end this section, we give a special example of stochastic Hamiltonian flows which concentrates on a submanifold with conserved quantities.

**Example 2.1.** Let $\mathcal{M} = \mathbb{R}^d$, $g$ and $\tilde{g}$ be metrics equivalent to $\mathbb{I}_{d \times d}$. Define an action functional with random perturbation in dual coordinates,

$$\frac{-1}{2} \int_0^T (\langle p, \dot{x} \rangle - H_0(x, p)) dt + \int_0^T H_1(x, p) d\xi(t),$$

where $H_0(x, p) = \frac{1}{2} p^T g^{-1}(x)p + f(x)$, $H_1(x, p) = \frac{1}{2} p^T \tilde{g}^{-1}(x)p + \eta \sigma(x)$ with smooth potentials $f$ and $\sigma$. Then the critical points under the constrain $x(0) = x_0, x(T) = x_T$ satisfies the stochastic Hamiltonian flows

$$\dot{x} = \frac{\partial H_0}{\partial p}(x, p) + \frac{\partial H_1}{\partial x}(x, p) \xi,$$

$$\dot{p} = -\frac{\partial H_0}{\partial x}(x, p) - \frac{\partial H_1}{\partial p}(x, p) \xi.$$

Its limit $(x, p)$ lie on the manifold $\{H_0(x, p) = H_0(x_0, p_0), H_1(x, p) = H_1(x_0, p_0)\}$ when the Hamiltonians satisfies that $\{H_0, H_1\} = 0$ with $\{\cdot, \cdot\}$ being the Poisson bracket. Similar to Lemma 2.2, it can be shown that $(x^\delta, p^\delta)$ converges globally to $(x, p)$ in probability if $H_0$ or $H_1$ satisfies the growth condition (2.3).

### 3. Stochastic Wasserstein Hamiltonian flow

In this section, we study the behaviors of the inhomogenous Hamiltonian system (2.1) and stochastic Hamiltonian system (2.4) on the density manifold. To illustrate the strategy, let us focus on the case that $(\mathcal{M}, g)$ equals $(\mathbb{R}^d, \mathbb{I})$ or $(\mathbb{R}^d, \mathbb{I})$. Given the filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, we assume that $\xi(t)$ is the piecewisely linear Wong–Zakai approximation of a standard Brownian motion, and that $x_0$ is a random variable with the density $\rho_0$ on another complete probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$. For a fixed $\tilde{\omega} \in \widehat{\Omega}$, we denote $t^\delta := \inf\{t \in (0, T) | x^\delta_t \text{ is not a smooth diffeomorphism on } \mathcal{M}\}$, $p^\delta_t = v(t, x^\delta_t)$ is the vector field depending on
Taking any test function $C^3(\mathbb{R}^d)$ which implies that for the conservation law with random influence ($\rho$), we denote $\sigma(t) = |\nabla \sigma(x(t))| \xi(t)$.

Differentiating $v(t, x(t))$ before $\tau^\delta$ leads to
$$
\partial_t v(t, x(t)) + \nabla v(t, x(t)) \cdot \frac{d}{dt} x(t) = \partial_t v(t, x(t)) + \nabla v(t, x(t)) \cdot v(t, x(t)) = -\nabla f(x(t)) - \eta \nabla \sigma(x(t)) \xi(t).
$$

Taking $x_0 = (x(t))^{-1}(x)$, we obtain the following conservation law with random perturbation,
$$
\partial_t v(t, x) + \nabla v(t, x) \cdot v(t, x) = -\nabla f(x) - \eta \nabla \sigma(x) \xi(t).
$$

Taking any test function $\psi$ in $C^\infty(M)$, it holds that
$$
\frac{d}{dt} \int_M \psi(x) \rho(t, x) dx = \int_M \nabla \psi(x) \cdot v(t, x) \rho_0(x) dx
$$
which implies that for $\omega = \Omega, \rho_t = x(t)^\eta \rho_0$, i.e., $\rho_t$ equals the distribution generated by the map $x_t(\cdot)$ push-forward $\rho_0$, satisfies the continuity equation,
$$
\partial_t \rho(t, x) + \nabla \cdot (\rho(t, x) v(t, x)) = 0.
$$

Introducing the pseudo inverse $(-\Delta)^{-1}$ (see e.g. [12]) of $-\Delta = -\nabla \cdot (\rho \nabla)$ for a positive density $\rho$, we denote $S_t = (-\Delta)^{-1} \partial_t \rho_t$. When there exists a potential $S$ such that $v = \nabla S$, the conservation law with random influence (3.1) and the continuity equation (3.2) induce a Hamiltonian system in density manifold before $\tau^\delta$,
$$
\partial_t \rho_t = \frac{\delta}{\delta S_t} H_0(\rho_t, S_t) = -\nabla \cdot (\rho_t \nabla S_t),
$$
$$
\partial_t S_t = -\frac{\delta}{\delta \rho_t} H_0(\rho_t, S_t) - \frac{\delta}{\delta \rho_t} H_1(\rho_t, S_t) \xi(t) + C(t)
$$
$$
= -\frac{1}{2} |\nabla S_t|^2 - \frac{\delta}{\delta \rho_t} F(\rho_t) - \frac{\delta}{\delta \rho_t} \eta \Sigma(\rho_t) \xi(t) + C(t),
$$
where $C(t)$ is an arbitrary stochastic process on $(\Omega, F, \mathbb{P})$ independent of the spatial position $x$ and $v(0, \cdot) = \nabla S(0, \cdot)$. Here the dominated average energy is
$$
H_0(\rho, S) := K(\rho) + F(\rho) = \int_M \frac{1}{2} |\nabla S(x)|^2 \rho(x) dx + \int_M f(x) \rho(x) dx,
$$
and the perturbed average energy is
$$
H_1(\rho, S, t) = \eta \Sigma(\rho_t) = \eta \int_M \sigma(x) \rho(x) dx.
$$

Taking $\delta \to 0$, the limit system becomes a stochastic Hamiltonian system,
$$
d\rho_t = \frac{\delta}{\delta S_t} H_0(\rho_t, S_t) dt,
$$
$$
dS_t = -\frac{\delta}{\delta \rho_t} H_0(\rho_t, S_t) - \frac{\delta}{\delta \rho_t} H_1(\rho_t, S_t) \ast d\xi + C(t) dt,
$$
where $\xi$ is the limit process of $\xi_\delta$ in path-wise sense. We would like to remark that the solution of (3.4) is not predictable in general. In our particular case, since $\xi_\delta(t)$ is a piecewisely linear Wong-Zakai approximation of $B(t)$, the limit of (3.1), (3.2) is the following system in Stratonovich sense,

\begin{align}
(3.5) \quad dp_t &= -\nabla \cdot (\rho(t,x)v(t,x))dt,

& \quad dv(t,x) + \nabla v(t,x) \cdot v(t,x)dt = -\nabla f(x)dt - \eta \nabla \sigma(x) \circ dB_t.
\end{align}

We would like to emphasize that the above analysis indicates a principle for deriving the stochastic Hamiltonian system on Wasserstein manifold: The conditional probability density of stochastic Hamiltonian flow in phase space is a stochastic Hamiltonian flow in density manifold almost surely. In the following we always assume that the initial distribution $\rho(0, \cdot)$ of $x_0$ and the initial velocity $v(0, \cdot)$ are smooth and bounded.

**Proposition 3.1.** Suppose that $\mathcal{M}$ is a d-dimensional compact smooth differential submanifold and $T > 0$. Let $g = I, \nu(0, \cdot)$ be a smooth vector field, $f, \sigma$ be smooth function on $\mathcal{M}$, $\xi_\delta$ be the linear interpolation of $(\rho, v)$ to $(\rho_0, v_0)$, which converges in probability to the solution $(\rho, v)$ of (3.5) before $\tau$.

**Proof.** Applying Lemma 2.3, we have that $(x_\delta^t, v(t, x_\delta^t))$ is convergent to $(x_t, v(t, x_t))$ in $[0, T]$, a.s., up to a subsequence. Define the stopping time $\tau = \inf\{t \in [0, T] | x_t \text{ is not smooth diffeomorphism on } \mathcal{M}\}$. For convenience, let us take a subsequence such that for almost $\omega \in \Omega$, $(x_\delta^t, v(t, x_\delta^t))$ converges to $(x_t, v(t, x_t))$ and $\frac{\partial}{\partial x_\delta^t} x_\delta^t(x_0)$ converges to $\frac{\partial}{\partial x_0} x_0(x_0)$. Before $\tau(\omega)$, there exists $\alpha > 0$ such that $\det(\frac{\partial}{\partial x_\delta^t} x_\delta^t(x_0)) > \alpha$. The pathwise convergence of $x_\delta^t$ implies that for any $\epsilon > 0$ there exists $\delta_0 = \delta(\epsilon, \omega) > 0$ such that when $\delta \leq \delta_0$, $\det(\frac{\partial}{\partial x_\delta^t} x_\delta^t(x_0)) > \alpha - \epsilon > 0$. Notice that the density function $\rho^\delta(t, y)$ of $x^\delta_t$ satisfies $\rho^\delta(t, y) = |\det(\nabla x_\delta^t(y))| \rho(0, x_0^\delta(y))$. Since $\rho(0, \cdot)$ is smooth for any fixed $\omega$ and the pathwise convergence of $x^\delta_t$ holds, it follows that $\rho^\delta(t, y)$ converges to the density function of $x_t$, which is $\rho(t, y) = |\det(\nabla x_t(y))| \rho(0, x_t(y))$. Similarly, the pathwise convergence of $v^\delta(t, x^\delta_t(y))$ to $v(t, x_t(y))$, together with invertibility of $x^\delta_t$ and $x_t$, implies the convergence of $v^\delta(t, x)$ to $v(t, x)$. Consequently, the solution of $(\rho^\delta, v^\delta)$ is convergent to $(\rho, v)$ in pathwise sense up to a subsequence. \qed

### 3.1. Vlasov equation

We would like to present the connections and differences between the classic Vlasov equation and stochastic Wasserstein Hamiltonian flow in this part. For simplicity, let us consider the case that $\mathcal{M} = \mathbb{R}^d$. We fix $\tilde{\omega} \in \Omega$, and consider (2.1). Taking differential on $\mathbb{E}_\mathcal{Q}[\phi(x^\delta_t, p^\delta_t)]$ where $\phi$ is a sufficient smooth test function, we get

\[
\frac{d}{dt} \mathbb{E}_\mathcal{Q}[\phi(x^\delta_t, p^\delta_t)] = \mathbb{E}_\mathcal{Q}[\nabla_x \phi(x^\delta_t, p^\delta_t) \frac{d}{dt} x^\delta_t + \nabla_p \phi(x^\delta_t, p^\delta_t) \frac{d}{dt} p^\delta_t]
\]

\[
= \mathbb{E}_\mathcal{Q}[\nabla_x \phi(x^\delta_t, p^\delta_t) p_t + \nabla_p \phi(x^\delta_t, p_t)(-\nabla_x f(x^\delta_t) - \eta \nabla_x \sigma(x^\delta_t) \xi^\delta_t)].
\]

Denoting the initial joint probability density function by $F_0(x, p)$, it holds that

\[
\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x^\delta_t, p^\delta_t) F_0(x, p) dx dp = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \nabla_x \phi(x^\delta_t, p^\delta_t) p_t + \nabla_p \phi(x^\delta_t, p^\delta_t)(-\nabla_x f(x^\delta_t) - \eta \nabla_x \sigma(x^\delta_t) \xi^\delta_t) \right) F_0(x, p) dx dp.
\]
Thus the joint distribution on $\Omega$, $F^\delta_t = (x^\delta, p^\delta)^\# F_0$, satisfies

$$
\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x, p) \frac{d}{dt} F^\delta_t (x, p) dx dp$

$$
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \nabla_x \phi(x, p)p + \nabla_p \phi(x, p)(-\nabla x f(x)) \right) F^\delta_t (x, p) dx dp$

$$
+ E_\Omega [\nabla_p \phi(x^\delta_t, p^\delta_t)(-\eta \nabla x \sigma(x^\delta_t)) \dot{\xi^\delta}_t(t)].$

Notice that the solution process $x^\delta_t$ is $\mathbb{F}_{t_k+1}$-measurable when $t \in (t_k, t_{k+1}]$, $t_k = k\delta t$ and $\mathbb{F}_{t_k}$-measurable when $t = t_k$, and $x_t$ is $\mathbb{F}_t$-measurable. By applying the chain rule, we have that for $t \in (t_k, t_{k+1}]$,

$$
\int_0^t E_\Omega [\nabla_p \phi(x^\delta_s, p^\delta_s)(-\eta \nabla x \sigma(x^\delta_s)) \dot{\xi^\delta}_s(s)] ds$

$$
= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} E_\Omega [\nabla_p \phi(x^\delta_s, p^\delta_s)(-\eta \nabla x \sigma(x^\delta_s)) \dot{\xi^\delta}_s(s)] ds$

$$
+ \int_{t_k}^t E_\Omega [\nabla_p \phi(x^\delta_s, p^\delta_s)(-\eta \nabla x \sigma(x^\delta_s)) \dot{\xi^\delta}_s(s)] ds$

$$
= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} E_\Omega [\nabla_p \phi(x^\delta_s, p^\delta_s)(-\eta \nabla x \sigma(x^\delta_s)) \frac{B_{t_{j+1}} - B_{t_j}}{\delta}] ds$

$$
+ \int_{t_k}^t E_\Omega [\nabla_p \phi(x^\delta_s, p^\delta_s)(-\eta \nabla x \sigma(x^\delta_s)) \frac{B_{t_{k+1}} - B_{t_k}}{\delta}] ds$

Then repeating similar arguments in the proof of Lemma 2.2, we have that

$$
\int_0^t E_\Omega [\nabla_p \phi(x^\delta_s, p^\delta_s)(-\eta \nabla x \sigma(x^\delta_s)) \dot{\xi^\delta}_s(t)] ds$

$$
= \int_0^t E_\Omega [\nabla_p \phi(x^\delta_t, x^\delta_t, p^\delta_t, p^\delta_t)(-\eta \nabla x \sigma(x^\delta_t)) \dot{\xi^\delta}_t(t)] ds$

$$
+ \int_0^t \frac{1}{2} E_\Omega [(\Delta p^\delta \phi(x^\delta_t, p^\delta_t) \nabla x \sigma(x^\delta_t))(-\eta \nabla x \sigma(x^\delta_t)) \dot{\xi^\delta}_t(t)^2] ds$

$$
+ o(\delta^\beta),$

where $\beta \in (0, \frac{1}{2})$. Taking $\delta \to 0$ yield that the second order Vlasov equation

$$
\partial_t F(t, x, p) = -\nabla x \cdot (F(t, x, p) \frac{\partial H_0}{\partial p}) + \nabla_p \cdot (F(t, x, p) \frac{\partial H_0}{\partial x})$

$$
+ \frac{1}{2} \Delta_{pp} F(t, x, p) \cdot \left( \frac{\partial H_1}{\partial x}, \frac{\partial H_1}{\partial x} \right).$

This implies that when we consider the joint distribution on $\Omega$, the density function satisfies the second order Vlasov equation. However, when we consider the conditional probability on $\hat{\Omega}$
instead of $\Omega$, the conditional joint probability of Wong–Zakai approximation satisfies the following first order Vlasov equation,
\[
\partial_t F^\delta(t, x, p) = -\nabla_x \cdot (F^\delta(t, x, p) \frac{\partial H_0}{\partial p}) + \nabla_p \cdot (F^\delta(t, x, p) \frac{\partial H_0}{\partial x}) + \nabla_p \cdot (F^\delta(t, x, p) \frac{\partial H_1}{\partial x} \xi_t).
\]

Its limit equation becomes
\[
dF(t, x, p) = -\nabla_x \cdot (F(t, x, p) \frac{\partial H_0}{\partial p}) dt + \nabla_p \cdot (F(t, x, p) \frac{\partial H_0}{\partial x}) dt + \nabla_p \cdot (F(t, x, p) \frac{\partial H_1}{\partial x}) dB_t.
\]

### 3.2. Stochastic Euler–Lagrange equation in density space.

In this section, we consider the Wasserstein Hamiltonian flow with random perturbation, i.e., the second order stochastic Euler–Lagrange equation from the Lagrange functional on density manifold. The density space $\mathcal{P}(M)$ is defined by
\[
\mathcal{P}(M) = \{ \rho \text{dvol}_M | \rho \in C^\infty(M), \rho \geq 0, \int_M \rho \text{dvol}_M = 1 \}.
\]

Its interior of $\mathcal{P}(M)$ is denoted by $\mathcal{P}_o(M)$. The tangent space at $\rho \in \mathcal{P}_o(M)$ is defined by
\[
T_{\rho} \mathcal{P}_o(M) = \{ \kappa \in C^\infty(M) | \int_M \kappa \text{dvol}_M = 0 \}.
\]

Define the quotient space of smooth functions $\mathcal{F}(M)/\mathbb{R} = \{ [\Phi] | \Phi \in C^\infty(M) \}$, where $[\Phi] = \{ \Phi + c | c \in \mathbb{R} \}$. Then one could identify the element in $\mathcal{F}(M)/\mathbb{R}$ as the tangent vector in $T_{\rho} \mathcal{P}_o(M)$ by using the map $\Theta : \mathcal{F}(M)/\mathbb{R} \to T_{\rho} \mathcal{P}_o(M)$, $\Theta_{[\Phi]} = -\nabla \cdot (\rho \nabla \Phi)$. The boundaryless condition of $M$ and the property of elliptical operator $\Delta_{\rho}(-) = -\nabla \cdot (\rho \nabla (-))$ ensures that $\Psi$ is one to one and linear. This implies that $\mathcal{F}(M)/\mathbb{R} \cong T_{\rho}^* \mathcal{P}_o(M)$, where $T_{\rho}^* \mathcal{P}_o(M)$ is the cotangent space of $\mathcal{P}_o(M)$. $L^2$-Wasserstein metric on density manifold $g_W : T_{\rho} \mathcal{P}(M) \times T_{\rho} \mathcal{P}(M) \to \mathbb{R}$ is define by
\[
g_W(\kappa_1, \kappa_2) = \int_M (\nabla \Phi_1, \nabla \Phi_2) \text{dvol}_M = \int_M \kappa_1(-\Delta_{\rho})^\dagger \kappa_2 \text{dvol}_M,
\]

where $\kappa_1 = \Theta_{[\kappa_1]}, \kappa_2 = \Theta_{[\kappa_2]}$, and $(-\Delta_{\rho})^\dagger$ is the pseudo inverse operator of $-\Delta_{\rho}$. In deterministic case, it is known that the critical point of the Wasserstein metric
\[
\frac{1}{2} W^2(\rho_0, \rho_1)^2 := \inf_{\rho \in \mathcal{P}_o(M)} \left\{ \int_0^1 \int_M \frac{1}{2} g_W(\partial_t \rho_t, \rho_t) \text{dvol}_M dt \right\}
\]
satisfies the geodesic equation in cotangent bundle on density manifold (see e.g. [14]), that is,
\[
\partial_t \rho_t = -\nabla \cdot (\rho_t \nabla \Phi_t),
\]
\[
\partial_t \Phi_t = -\frac{1}{2} \nabla \Phi_t^2 + C_t,
\]

where $\Phi_t = (-\Delta_{\rho_1})^\dagger \partial_t \rho_1$, $C_t$ is independent of $x \in M$. The above geodesic equation in primal coordinates is the Euler–Lagrange equation,
\[
\partial_t \frac{\delta}{\delta \partial_t \rho_t} \mathcal{L}(\rho_t, \partial_t \rho_t) = \frac{\delta}{\delta \rho_t} \mathcal{L}(\rho_t, \partial_t \rho_t) + C(t),
\]

where $\mathcal{L}(\rho_t, \partial_t \rho_t) = \frac{1}{2} g_W(\partial_t \rho_t, \partial_t \rho_t)$. 
Next, we consider the Lagrangian in density manifold with random perturbation,

\[ \mathcal{L}(\rho_t, \partial_t \rho_t) = \frac{1}{2} \sigma_W(\partial_t \rho_t, \partial_t \rho_t) - \mathcal{F}(\rho_t) - \mathcal{G}(\rho_t) \xi_s(t), \]

and its variational problem \( I_{\delta}(\rho^0, \rho^T) = \inf \{ \int_0^T \mathcal{L}(\rho_t, \partial_t \rho_t) dt | \rho_0 = \rho^0, \rho_T = \rho^T \}. \)

**Theorem 3.1.** The Euler Lagrangian equation of the variational problem \( I_{\delta}(\rho^0, \rho^T) \) satisfies

\[ \partial_t \rho_t + \nabla \cdot (\rho_t \nabla \Phi_t) = 0, \]

\[ \partial_t \Phi_t + \frac{1}{2} | \nabla \Phi_t |^2 = -\frac{\delta}{\delta \rho_t} \mathcal{F}(\rho_t) - \frac{\delta}{\delta \rho_t} \mathcal{G}(\rho_t) \xi_s(t), \]

where \( \Phi_t = (-\Delta_{\rho_t})^\dagger \partial_t \rho_t \) up to a spatially constant stochastic process shift.

**Proof.** Consider a smooth perturbation \( ch_t \) satisfying \( \int_M h_t \text{dvol}_M = 0, t \in [0, T] \) and \( h_0 = h_T = 0 \). Applying Taylor expansion with respect \( \epsilon \) and integration by parts, using \( h_0 = h_T = 0 \) and the fact that \( M \) is compact, we get

\[ \int_0^T \mathcal{L}(\rho_t, \partial_t \rho_t) dt = \int_0^T \mathcal{L}(\rho_t, \partial_t \rho_t) dt + \epsilon \int_0^T \int_M \left( \frac{\delta}{\delta \rho_t} \mathcal{L}(\rho_t, \partial_t \rho_t) - \frac{\delta}{\delta \rho_t} \mathcal{L}(\rho_t, \partial_t \rho_t) \right) h_t \text{dvol}_M dt + o(\epsilon). \]

Direct calculations lead to

\[ \frac{\partial}{\partial \rho_t} \frac{\delta}{\delta \rho_t} \mathcal{L}(\rho_t, \rho_t) = \partial_t (-\Delta_{\rho_t})^\dagger \partial_t \rho_t = (-\Delta_{\rho_t})^\dagger \partial_t \rho_t - (-\Delta_{\rho_t})^\dagger (-\Delta_{\rho_t})^\dagger \partial_t \rho_t, \]

\[ \frac{\delta}{\delta \rho_t} \mathcal{L}(\rho_t, \rho_t) = \frac{1}{2} | (-\Delta_{\rho_t})^\dagger \partial_t \rho_t |^2 - \frac{\delta}{\delta \rho_t} \mathcal{F}(\rho_t) - \frac{\delta}{\delta \rho_t} \mathcal{G}(\rho_t) \xi_s(t), \]

which, together with the property \( \int_M h_t \text{dvol}_M = 0 \), yields (3.6) up to a spatially-constant stochastic process shift by multiplying \( \Delta_{\rho_t} \) on both sides. By introducing the Legendre transformation \( \Phi_t = (-\Delta_{\rho_t})^\dagger \partial_t \rho_t \), we obtain Eq. (3.7) from Eq. (3.6).

**Proposition 3.2.** The Euler–Lagrange equation of the variational problem \( I(\rho^0, \rho^T) \),

\[ I(\rho_0, \rho_T) = \int_0^T \left( \frac{1}{2} \sigma_W(\partial_t \rho_t, \partial_t \rho_t) - \mathcal{F}(\rho_t) \right) dt - \int_0^T \mathcal{G}(\rho_t) \circ dB(t) \]

satisfies

\[ \partial_t \rho_t + \nabla \cdot (\rho_t \nabla \Phi_t) = 0, \]

\[ \partial_t \Phi_t + \frac{1}{2} | \nabla \Phi_t |^2 = -\frac{\delta}{\delta \rho_t} \mathcal{F}(\rho_t) - \frac{\delta}{\delta \rho_t} \mathcal{G}(\rho_t) \circ dB_t, \]

where \( \Phi_t = (-\Delta_{\rho_t})^\dagger \partial_t \rho_t \) up to a spatially constant stochastic process shift.
Proof. Consider a smooth perturbation $ch_t$ satisfying $\int_M h_t d\text{vol}_M = 0$, $t \in [0, T]$ and $h_0 = h_T = 0$. Notice that there exists $\Phi_t = (-\Delta_{\rho_t})^\dagger \partial_t \rho_t$. Using the equivalence of stochastic integral between Itô sense and Stratonovich sense (see e.g. [28]), we have that

$$
\int_0^T \frac{1}{2} \partial_t W(\partial_t \rho_t + ch_t, \partial_t \rho_t + ch_t) - F(\rho_t + ch_t) dt - \int_0^T \Sigma(\rho_t + ch_t) dB_t \\
= \int_0^T \mathcal{L}_0(\rho_t, \partial_t \rho_t) dt + \int_0^T \Sigma(\rho_t) dB_t \\
+ \epsilon \int_0^T \int_M \left( \frac{\delta}{\delta \rho_t} \mathcal{L}_0(\rho_t, \partial_t \rho_t) - \partial_t \frac{\delta}{\delta \rho_t} \mathcal{L}_0(\rho_t, \partial_t \rho_t) \right) \cdot h_t d\text{vol}_M dt \\
+ \epsilon \int_0^T \int_M \frac{\delta}{\delta \rho_t} \Sigma(\rho_t) \cdot h_t d\text{vol}_M dB_t + o(\epsilon).
$$

Similar to the proof of Theorem 3.1, we obtain (3.8) and its equivalent Hamiltonian system (3.9).

3.3. Generalized stochastic Wasserstein–Hamiltonian flow. In the last section, we show that the density of a Hamiltonian ODE with random perturbation satisfies the stochastic Wasserstein Hamiltonian flow. In this section, We derived the stochastic Wasserstein Hamiltonian flow via the random perturbation in the dual coordinates in density space. It provides a more general framework that can derive a large class of stochastic Wasserstein Hamiltonian flows which cannot be obtained from the classic dynamics with perturbations.

We introduce the following variational problem

(3.10) \[ I_\delta(\rho^0, \rho^T) = \inf \{ S(\rho_t, \Phi_t) | (-\Delta_{\rho_t})^\dagger \Phi_t \in T_{\rho_t} \mathcal{P}_\alpha(M), \rho(0) = \rho^0, \rho(T) = \rho^T \} \]

whose action functional is given by the dual coordinates,

\[ S(\rho_t, \Phi_t) = - \int_0^T \langle \Phi(t), \partial_t \rho_t \rangle - \mathcal{H}_0(\rho_t, \Phi_t) dt + \int_0^T \mathcal{H}_1(\rho_t, \Phi_t) d\xi_\delta(t). \]

Here $\mathcal{H}_0(\rho_t, \Phi_t) = \int_M \frac{1}{2} |\nabla \Phi_t|^2 \rho_t d\text{vol}_M + \mathcal{F}(\rho_t)$, $\mathcal{H}_1(\rho_t, \Phi_t) = \eta \int_M \frac{1}{2} |\nabla \Phi_t|^2 \rho_t d\text{vol}_M + \eta \Sigma(\rho_t)$, $\mathcal{F}$ and $\Sigma$ are smooth potential functions.

Theorem 3.2. The critical point of the variational problem $I_\delta(\rho^0, \rho^T)$ satisfies the following Hamiltonian system

(3.11) \[ \partial_t \rho_t + \nabla \cdot (\rho_t \nabla \Phi_t) + \eta \nabla \cdot (\rho_t \nabla \Phi_t) \dot{\xi}_\delta = 0, \]

\[ \partial_t \Phi_t + \frac{1}{2} |\nabla \Phi_t|^2 + \frac{1}{2} \eta |\nabla \Phi_t|^2 \dot{\xi}_\delta = - \frac{\delta}{\delta \rho_t} \mathcal{F}(\rho_t) - \eta \frac{\delta}{\delta \rho_t} \Sigma(\rho_t) \dot{\xi}_\delta, \]

where $(1 + \dot{\xi}_\delta(t)) \Phi_t = (-\Delta_{\rho_t})^\dagger \partial_t \rho_t$ up to a spatially constant stochastic process shift.
Proof. Consider the perturbations on $\rho$ and $\Phi$. Following the arguments in the proof of Proposition 3.2, the critical point satisfies
\[
S(\rho_t + \epsilon \delta \rho, \Phi_t + \epsilon \delta \Phi) = S(\rho_t, \Phi_t) - \epsilon \int_0^T \langle \Phi(t), \partial_t \delta \rho(t) \rangle dt
+ \epsilon \int_0^T \left( \frac{\delta}{\delta \rho_t} \mathcal{H}_0(\rho_t, \Phi_t) \right) \delta \rho_t
+ \epsilon \int_0^T \left( \frac{\delta}{\delta \Phi_t} \mathcal{H}_0(\rho_t, \Phi_t) \right) \delta \Phi_t dt
+ \epsilon \int_0^T \left( \frac{\delta}{\delta \rho_t} \mathcal{H}_1(\rho_t, \Phi_t) \right) \delta \rho_t
+ \epsilon \int_0^T \left( \frac{\delta}{\delta \Phi_t} \mathcal{H}_1(\rho_t, \Phi_t) \right) \delta \Phi_t d\xi(t) + o(\epsilon).
\]
Taking $\epsilon \to 0$, we obtain that
\[
\partial_t \rho_t = -\frac{\delta}{\delta \rho_t} \mathcal{H}_0(\rho_t, \Phi_t) - \frac{\delta}{\delta \rho_t} \mathcal{H}_0(\rho_t, \Phi_t) \xi(t)
\]
\[
\partial_t \Phi_t = \frac{\delta}{\delta \Phi_t} \mathcal{H}_0(\rho_t, \Phi_t) - \frac{\delta}{\delta \rho_t} \mathcal{H}_0(\rho_t, \Phi_t) \xi(t),
\]
which leads to (3.11).

Similarly, consider the action functional
\[
S_B(\rho_t, \Phi_t) = \langle \rho(0), \Phi(0) \rangle - \langle \rho(T), \Phi(T) \rangle + \int_0^T \langle \rho_t, \partial_t \Phi(t) \rangle dt + \mathcal{H}_0(\rho_t, \Phi_t) dt
+ \int_0^T \mathcal{H}_1(\rho_t, \Phi_t) \circ dB_t
\]
over the $\mathbb{F}_t$-adapted feasible set, we obtain the following stochastic system.

Theorem 3.3. The critical point of the variational problem $I(\rho^0, \rho^T)$ defined by
\[
I(\rho^0, \rho^T) = \inf \{ S_B(\rho_t, \Phi_t) | \rho(0) = \rho^0, \rho(T) = \rho^T \}
\]
satisfies the following Hamiltonian system
\[
\partial_t \rho_t + \nabla \cdot (\rho_t \nabla \Phi_t) + \eta \nabla \cdot (\rho_t \nabla \Phi_t) \circ dB_t = 0,
\]
\[
\partial_t \Phi_t + \frac{1}{2} | \nabla \Phi_t |^2 + \frac{1}{2} \nabla | \nabla \Phi_t |^2 \circ dB_t = -\frac{\delta}{\delta \rho_t} \mathcal{F}(\rho_t) - \eta \frac{\delta}{\delta \rho_t} \Sigma(\rho_t) \circ dB_t
\]
up to a spatially constant stochastic process shift on $\Phi_t$.

Next, we show that the continuity equation and the velocity equation generated by $\Phi$,
\[
\partial_t \rho_t + \nabla \cdot (\rho_t v_t) + \eta \nabla \cdot (\rho_t v_t) \xi(t) = 0,
\]
\[
\partial_t v_t + \nabla v_t \cdot v_t + \eta \nabla v_t \cdot v_t \xi(t) = -\nabla \frac{\delta}{\delta \rho_t} \mathcal{F}(\rho_t) - \eta \frac{\delta}{\delta \rho_t} \nabla \Sigma(\rho_t) \xi(t)
\]
is convergent to the corresponding system driven by the Brownian motion.
Proposition 3.3. Assume that \( v(0, \cdot), \rho(0, \cdot) \) is \( \mathbb{F}_0 \)-measurable and smooth, \( \mathcal{F}(\rho_t) = \int_M f \rho_t d\nu(M) \)
and \( \Sigma(\rho_t) = \int_M \sigma \rho_t d\nu(M) \), with \( f, \sigma \in C_0^1(\mathcal{M}) \). Let \( \rho^\delta, v^\delta \) be the solution of (3.13), and \( \rho, v \) be the solution of

\[
\begin{align*}
\partial_t \rho_t + \nabla \cdot (\rho_t v_t) + \eta \nabla \cdot (\rho_t v_t) & = d B_t, \\
\partial_t v_t + \nabla v_t \cdot v_t + \eta \nabla v_t \cdot v_t & = -\nabla f(\rho_t) - \eta \frac{\delta}{\delta \rho} \Sigma(\rho_t) \circ d B_t,
\end{align*}
\]

Then there exists a stopping time \( \tau > 0 \) such that

\[
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, \tau)} \left[ |\rho^\delta_t - \rho_t|_{L^\infty(\mathcal{M})} + |v^\delta_t - v_t|_{L^\infty(\mathcal{M}^d)} \right] > \epsilon \right) = 0.
\]

Proof. Since \( \mathcal{M} \) is compact, \( f, \sigma \in C_0^1(\mathcal{M}) \), similar to the proofs of Lemma 2.2 and Lemma 2.3, we can obtain the global well-posedness of the particle ODE systems

\[
\begin{align*}
dX_t &= v(t, X_t)dt + \eta v(t, X_t) \circ dB_t, \\
dv(t, X_t) &= -\nabla f(X_t)dt - \eta \nabla \sigma(X_t) \circ dB_t,
\end{align*}
\]

and

\[
\begin{align*}
dX^\delta_t &= v^\delta(t, X^\delta_t)dt + \eta v^\delta(t, X^\delta_t) d\xi^\delta, \\
dv^\delta(t, X^\delta_t) &= -\nabla f(X^\delta_t)dt - \eta \nabla \sigma(X^\delta_t) d\xi^\delta.
\end{align*}
\]

Following the arguments in the proof Proposition 3.1, we can obtain that there exists a stopping time \( \tau > 0 \) such that \( X_t \) is a smooth diffeomorphism before \( \tau \). Notice that the density function \( \rho^\delta(t, y) \) of \( X^\delta_t \) satisfies \( \rho^\delta(t, y) = |\det(\nabla X^\delta_t(y))| \rho(0, X^\delta_t(y)) \). Since \( \rho(0, \cdot) \) is smooth for any fixed \( \omega \) and the pathwise convergence of \( X^\delta \) holds, it follows that \( \rho^\delta(t, y) \) converges to the density function of \( X_t \) before \( \tau \), which is \( \rho(t, y) = |\det(\nabla X_t(y))| \rho(0, X_t(y)) \). Similarly, the pathwise convergence of \( v^\delta(t, X^\delta_t(y)) \) to \( v(t, X_t(y)) \), together with invertibility of \( X^\delta_t \) and \( X_t \), implies the convergence of \( v^\delta(t, x) \) to \( v(t, x) \) before \( \tau \).

Remark 3.1. If one obtains the convergence of the Wong–Zakai approximations of the mean-field SODEs,

\[
\begin{align*}
dX_t &= v(t, X_t)dt + \eta v(t, X_t) \circ dB_t, \\
dv(t, X_t) &= -\nabla \frac{\delta}{\delta \rho(t, X_t)} \mathcal{F}(\rho(t, X_t))dt - \eta \nabla \frac{\delta}{\delta \rho(t, X_t)} \Sigma(t, X_t) \circ dB_t,
\end{align*}
\]

then the convergence of (3.13) to (3.14) can be shown similarly before the stopping time \( \tau \), that is, the first time \( X_t \) is not a smooth diffeomorphism on \( \mathcal{M} \) or \( X_t \) escapes \( \mathcal{M} \).

4. Examples

In this section, we show that both the stochastic nonlinear Schrödinger (NLS) equation, which models the propagation of nonlinear dispersive waves in random or inhomogenous media in quantum physics (see e.g. [4, 19, 29, 40]), and nonlinear Schrödinger equation with random dispersion, which describes the propagation of a signal in an optical fibre with dispersion management (see e.g. [1, 2]), are stochastic Wasserstein-Hamiltonian flows. We also discuss that the mean-field game system with common noise (see e.g. [44, 39, 43]) is a stochastic Wasserstein-Hamiltonian flow under suitable transformations.
4.1. **Stochastic NLS equation.** The dimensionless stochastic NLS equation is given by

\begin{equation}
\tag{4.1} du = i\Delta u dt + i\lambda f(|u|^2)u dt + iu \circ dW_t,
\end{equation}

where $W_t$ is a Wiener process on the Hilbert space $L^2(\mathcal{M}; \mathbb{R})$ and $f$ is a real-valued continuous function. Since $Q$-Wiener process $W$ has the Karhunen–Loève expansion $W(t, x) = \sum_{i \in \mathbb{N}^*} Q^2\hat{e}_i(x)\beta_i(t)$ (see e.g. [17]), where \{\hat{e}_i\}_{i \in \mathbb{N}} is an orthonormal basis of $L^2(\mathcal{M}; \mathbb{R})$, and \{\beta_i\}_{i \in \mathbb{N}} is a sequence of linearly independent Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. We denote $W_\delta(t, x) = \sum_{i \in \mathbb{N}^*} Q^2\hat{e}_i(x)\beta_i^\delta(t)$ as the piecewise linear Wong–Zakai approximation (or other type Wong–Zakai approximation) of $W$ and consider the approximated NLS equation of (4.1)

\begin{equation}
\tag{4.2} \partial_t u(t, x) = i\Delta u(t, x) + i\lambda f(|u(t, x)|^2)u(t, x) + iu(t, x)\hat{W}_\delta(t, x).
\end{equation}

We aim to prove that (4.2) is a stochastic Wasserstein Hamiltonian flow for any $\delta > 0$, and thus its limit (4.1) is also a stochastic Wasserstein Hamiltonian flow. In the following, we assume that $f$ is a real-value function, $W$ is smooth with respect to the space variable, and (4.2) possesses a mild solution or a strong solution on $[0, T]$. Denote the $L^2$-inner product by $\langle u, v \rangle = \mathbb{R} \int_{\mathcal{M}} u \overline{v} d\text{vol}_M$, where $\mathbb{R}$ is the real part of a complex number. The variational problem on density manifold of (4.2) is

\begin{equation}
\tag{4.3} I_\delta(\rho^0, \rho^T) = \inf \{S(\rho_t, \Phi_t)|(-\Delta_{\rho_t})^\frac{1}{2}\Phi_t \in T_{\rho_t}\mathcal{P}_\rho(\mathcal{M}), \rho(0) = \rho^0, \rho(T) = \rho^T\}
\end{equation}

whose action functional is given by the dual coordinates,

\[ S(\rho_t, \Phi_t) = -\int_0^T \langle \Phi(t), \partial_t \rho_t \rangle dt + \int_0^T H_0(\rho_t, \Phi_t) dt + \sum_{i \in \mathbb{N}^*} \int_0^T H_i(\rho_t, \Phi_t) \beta_i^\delta(t). \]

Here $H_0(\rho_t, \Phi_t) = \int_{\mathcal{M}} |\nabla \Phi_t|^2 \rho_t d\text{vol}_M + \frac{1}{4} I(\rho) + \mathcal{F}(\rho_t), H_i(\rho_t, \Phi_t) = -\Sigma_i(\rho_t) - \int_{\mathcal{M}} Q^2\hat{e}_i \rho_t d\text{vol}_M, \mathcal{F}(\rho) = -\frac{1}{2} \int_{\mathcal{M}} \int_0^\rho \frac{f(s)}{s} ds d\text{vol}_M$ with a smooth function $f$, and $I(\rho) = \int_{\mathcal{M}} |\nabla \log(\rho)|^2 \rho d\text{vol}_M$.

In the following, we show the relationship between the the variational problem (4.3) and nonlinear Schrödinger equation with Wong–Zakai approximation (4.2) by using the Madelung transform [34].

**Proposition 4.1.** The critical point of the variational problem (4.3) satisfies the Madelung system of (4.2) on the support of $\rho_t$. Conversely, the Madelung transform of (4.2) satisfies the critical point of (4.3) on the support of $|u_t|$.

**Proof.** By studying the perturbation on the dual coordinates, the arguments in the proof of Theorem 3.2 yield that the critical point of (4.3) satisfies

\[ \partial_t \rho_t + 2\nabla \cdot (\rho_t \nabla \Phi_t) = 0, \]

\[ \partial_t \Phi_t + |\nabla \Phi_t|^2 = -1/4 \frac{\delta}{\delta \rho_t} I(\rho_t) - \frac{\delta}{\delta \rho_t} \mathcal{F}(\rho_t) - \hat{W}_\delta. \]

Define a complex valued function by $\tilde{u}(t, x) = \sqrt{\rho(t, x)}e^{i\Phi(t, x)}$. One obtains the equation of $\tilde{u}(t, x)$ satisfying (4.2) on the support of $\rho_t$ by direct calculations.
Conversely, using the Madelung transform of the solution \( \sqrt{\rho(t,x)} e^{iS(t,x)} = u(t,x) \) where \( \rho = |u|^2 \) for (4.2). Then direct calculation leads to
\[
e^{\frac{1}{2} \log(\rho)+iS} \left( \frac{1}{2} \frac{\partial \rho}{\rho} + i \partial_t S \right) \\
= i e^{\frac{1}{2} \log(\rho)+iS} \left( \frac{1}{2} \nabla \rho + i \nabla S \right)^2 + i e^{\frac{1}{2} \log(\rho)+iS} \left( \frac{1}{2} \Delta \rho + i \Delta S - \frac{1}{2} |\nabla \rho|^2 \right) \\
+ i e^{\frac{1}{2} \log(\rho)+iS} (\lambda f(\rho) + \dot{W}_\delta)
\]
This implies that on the support or \(|u|\), it holds that
\[
\partial_t \rho = -2 \nabla \cdot (\rho \nabla S), \\
\partial_t S = -|\nabla S|^2 - \frac{1}{4} \frac{\delta}{\rho} I(\rho) + \lambda f(\rho) + \dot{W}_\delta.
\]

Based on the above result, taking spatial gradient on the potential \( S \), we get the following system with the conservation law
\[
\partial_t \rho = -\nabla \cdot (\rho \nabla v), \\
\partial_t v = -\nabla_x v \cdot \nabla_x \frac{1}{2} \frac{\delta}{\rho} I(\rho) + 2 \lambda \nabla_x f(\rho) + 2 \nabla_x \dot{W}_\delta,
\]
where \( v(t, x) = 2\nabla S(t, x) \).

The following theorem indicates that the stochastic NLS equation is a stochastic Wasserstein Hamiltonian flow due to the convergence of the Wong–Zakai approximation. For convenience, let us assume that \( \mathcal{M} = \mathbb{T}^d \) or \( \mathbb{R}^d \) and consider the case that \( W \) consists of a finite combinations of independent Brownian motions, i.e., \( W(t, x) = \sum_{k=1}^N q_k(x) / \delta(t) \), with \( q_k(x) \in \mathbb{H}^m(\mathcal{M}) \cap W^{k,\infty}(\mathcal{M}) \) for some \( m \in \mathbb{N} \) and \( k \in \mathbb{N}^+ \). Here \( \mathbb{H}^m(\mathcal{M}), W^{k,\infty}(\mathcal{M}) \) are the standard Sobolev space.

**Theorem 4.1.** Let \( m \in \mathbb{N} \) and \( k \in \mathbb{N}^+ \). Suppose that the initial value of (4.2) and (4.1) \( u_0 \in \mathbb{H}^m \) is \( F_0 \)-measurable and has any finite \( p \)-moment, \( p \in \mathbb{N}^+ \), and that \( f \) is a real-valued continuous function satisfies
\[
\| f(|u|^2) u - f(|v|^2) v \| \leq L_f(R) \| u - v \|, \quad \| u \|, \| v \| \leq R, \\
\| f(|u|^2) u \|_{H^1} \leq L_f(R)(1 + \| u \|_{H^1}), \quad \| u \|_{H^1} \leq R,
\]
where \( \lim_{R \to \infty} L_f(R) = \infty \). The Wong–Zakai approximation (4.2) is convergent almost surely to the stochastic NLS equation (4.1) up to a subsequence.

**Proof.** Since the driving noise is real-valued, the skew-symmetry of the NLS equation leads to the mass conservation laws for both (4.2) and (4.1). By the local Lipschitz property of \( f(\cdot, \cdot) \), one can obtain the existence of the unique mild solutions for both (4.2) and (4.1) in \( \mathcal{C}([0, T], L^2) \) by a standard argument in [17]. In order to study the converge in \( L^2 \), let us define an approximation sequence \( u_{0,R}^i \in \mathbb{H}^1, R_1 \to \infty \) of the initial value \( u_0 \), which can be taken by using truncated Fourier series or spectral Galerkin method (see e.g. [15]). The growth condition of \( f \) in \( \mathbb{H}^1 \) and
the uniform boundedness assumption of $q_k$ lead to
\[
\mathbb{E}\left[\sup_{t \in [0,T]} \|u_t^{R_1}\|_{L^2}^{2p}\right] \leq C(T, R_1, p) < \infty, \quad \mathbb{E}\left[\sup_{t \in [0,T]} \|u_t^{R_1, \delta}\|_{L^2}^{2p}\right] \leq C(T, R_1, \delta, p) < \infty,
\]
where $p \geq 1$, $\lim_{R_1 \to \infty} C(T, R_1, p) = \infty$, $\lim_{R_1 \to \infty} C(T, R_1, \delta, p) = 0$. Meanwhile, $u_t^{R_1}, u_t^{\delta, R_1}$ are convergent to $u_t, u_t^{\delta}$, a.s. in $\mathcal{C}([0,T]; L^2)$ as $R_1 \to \infty$, respectively up to a subsequence. The continuity estimate of $u_t^{R_1}, u_t^{R_1, \delta}$,
\[
\mathbb{E}\left[\|u_t^{R_1}(t) - u_t^{R_1}(s)\|^{2p}\right] \leq C(T, R_1, p)|t - s|^p,
\]
\[
\mathbb{E}\left[\|u_t^{R_1, \delta}(t) - u_t^{R_1, \delta}(s)\|^{2p}\right] \leq C(T, R_1, \delta, p)(|t - s|^p + |\delta|^p),
\]
can be obtained due to the mass conservation law and the continuity of $e^{4\Delta t}$. However, to get the convergence of (4.2), we need a priori estimate of $u_t^{R_1, \delta}$, which is independent of $\delta$. To this end, we use the energy of the Wong-Zakai approximation, $H(u) = \int_M \frac{1}{2} |\nabla u|^2 d\text{vol}_M - \frac{1}{2} \int_M \int_0^t |u|^2 f(s) d\text{vol}_M$, and obtain
\[
H(u^\delta(t)) = H(u^\delta(0)) + \int_0^t \langle \nabla u^\delta(s), iu^\delta(s) \rangle dW^\delta(s).
\]
By taking expectation, we get that
\[
\mathbb{E}\left[\sup_{t \in [0,T]} H(u^\delta(t))\right] \\ \leq \mathbb{E}\left[H(u^\delta(0))\right] + \mathbb{E}\left[\sup_{t \in [0,T]} \int_0^{|t|} \langle \nabla u^\delta([s]_\delta), iu^\delta([s]_\delta) \rangle dW^\delta(s)\right] \\ + \mathbb{E}\left[\sup_{t \in [0,T]} \int_0^{|t|} \langle \nabla u^\delta([s]_\delta), iu^\delta([s]_\delta) \rangle dW^\delta(s)\right] \\ + \mathbb{E}\left[\sup_{t \in [0,T]} \int_0^{|t|} \langle \nabla u^\delta([s]_\delta), iu^\delta([s]_\delta) \rangle dW^\delta(s)\right] \\ + \mathbb{E}\left[\sup_{t \in [0,T]} \int_0^{|t|} \langle \nabla u^\delta([s]_\delta), iu^\delta([s]_\delta) \rangle dW^\delta(s)\right] \\ + \mathbb{E}\left[\sup_{t \in [0,T]} \int_0^{|t|} \langle \nabla u^\delta([s]_\delta), iu^\delta([s]_\delta) \rangle dW^\delta(s)\right] \\ = \mathbb{E}\left[H(u^\delta(0))\right] + V_1 + V_2 + V_3 + V_4 + V_5 + V_6.
\]
Below we show the estimates of $V_i$ ($i = 1, \ldots, 6$). The Burkholder’s inequality and mass conservation law lead to
\[
V_i \leq \mathbb{E}\left[\int_0^T C(H(u^\delta([s]_\delta)) + C(\|u_{0\|})) ds\right].
\]
Applying the Burkholder and Minkowski inequalities, and the mass conservation law, we achieve that for $T = K\delta$,

$$
V_2 \leq 1 + \mathbb{E}\left[\sup_{t \in [0,T]} \left| \int_{[t_s]}^t \langle \nabla u^\delta([s]_\delta), iu^\delta([s]_\delta) \nabla dW^\delta(s) \rangle \right|^2 \right]
$$

$$
\leq 1 + \mathbb{E} \sum_{k=0}^{K-1} \sup_{t_k \in [t_k, t_{k+1}]} \left| \int_{t_k}^t \langle \nabla u^\delta(t_k), iu^\delta(t_k) \nabla dW(s) \rangle \right|^2
$$

$$
\leq 1 + C \sum_{k=0}^{K-1} \mathbb{E} \sum_{i=1}^N \int_{t_k}^{t_{k+1}} \langle \nabla u^\delta(t_k), iu^\delta(t_k) \nabla q_i(x) \rangle^2 dt
$$

$$
\leq 1 + C \mathbb{E} \sum_{i=1}^N \left\| \nabla u^\delta([t]_\delta) \right\|^2 \left\| u^\delta([t]_\delta) \right\|^2 \left\| q_i \right\|^2_{L^2} \int_0^T \mathbb{E} \left[ \left\| \nabla u^\delta([t]_\delta) \right\|^2 \right] dt
$$

The definition of $H$ leads to that there exists a constant $C(\|u_0\|)$ depending on $\|u_0\|$ such that

$$
\mathbb{E} \left[\sup_{t \in [0,T]} \left| \int_{[t_s]}^t \langle \nabla u^\delta([s]_\delta), iu^\delta([s]_\delta) \nabla dW^\delta(s) \rangle \right|^2 \right]
$$

$$
\leq 2C\|u_0\|^2 \sum_{i=1}^N \left\| q_i \right\|^2_{L^2} \int_0^T \mathbb{E} [H(u^\delta([t]_\delta))] dt + C(\|u_0\|).
$$

The mild form of $u^\delta(s) - u^\delta([s]_\delta)$,

$$
u^\delta(s) - u^\delta([s]_\delta) = (e^{i\Delta(s-[s]_\delta)} - I)u^\delta([s]_\delta) + \int_{[s]_\delta}^s e^{i\Delta(s-r)} i\lambda f(|u^\delta(r)|^2)u^\delta(r) dr
$$

$$
+ \int_{[s]_\delta}^s i\lambda e^{i\Delta(s-r)} u^\delta(r) dW^\delta(r),
$$

together with the mass conservation law and $\|e^{i\Delta t} - I\|_{L(H^2, L^2)} \leq C t^{\frac{1}{2}}$ (see, e.g., [17]), yields that

$$(4.6)\quad \|u^\delta(s) - u^\delta([s]_\delta)\| \leq C \|u^\delta([s]_\delta)\|_{H^2}^\delta + L_f(\|u_0\|)(1 + \|u_0\|) \delta + C\|W([s]_\delta + \delta) - W([s]_\delta)\|_{L^2} \|u_0\|.
$$

By making use of (4.6) and the Burkholder's inequality, we obtain

$$
V_4 \leq C(1 + \mathbb{E} \int_0^T \|\nabla u^\delta([s]_\delta)\|^2 ds)
$$

$$
+ C(\|u_0\|) \mathbb{E} \int_0^T \|\nabla u^\delta([s]_\delta)\|^2 (1 + \|u_0\|) \left( \frac{\|W([s]_\delta + \delta) - W([s]_\delta)\|^2_{L^2}}{\delta} + \|W([s]_\delta + \delta) - W([s]_\delta)\|_{L^2} \right) ds
$$

$$
\leq C(\|u_0\|)(1 + \mathbb{E} \int_0^T H(u^\delta([s]_\delta)) ds).
$$
Similar arguments yield that

\[
V_4 \leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \int_t^T \| \nabla u^\delta([s],\delta) \| ^2 \| W([s],\delta + \delta) - W([s],\delta) \| \delta^{-\frac{1}{2}} ds \right]
+ C(\| u_0 \|) \mathbb{E} \left[ \sup_{t \in [0,T]} \int_t^T \| \nabla u^\delta([s],\delta) \| (1 + \| u_0 \|) \left( \frac{\| W([s],\delta + \delta) - W([s],\delta) \| _L^\infty}{\delta} \right) + \| W([s],\delta + \delta) - W([s],\delta) \| _L^\infty ds \right]
\leq C \delta \mathbb{E} \left[ \sup_{s \in [0,T]} H(u^\delta([s],\delta)) \right] + C(\| u_0 \|) \delta.
\]

The estimates of \( V_5 \) and \( V_6 \) are omitted here since they are very similar to those of \( V_3 \) and \( V_4 \). We conclude that

\[
V_1 + V_2 + V_3 + V_4 + V_5 + V_6 \leq C \delta \mathbb{E} \left[ \sup_{t \in [0,T]} H(u^\delta(t)) \right] + C \mathbb{E} \left[ \int_0^T (H(u^\delta([t],\delta))) dt \right] + C(\| u_0 \|).
\]

Thus, we obtain \( \mathbb{E} \left[ \sup_{t \in [0,T]} H(u^\delta(t)) \right] \leq C(T, R_1, \| u_0 \|) \) by using Gronwall’s inequality and taking \( \delta \) small enough. Similarly, it holds that for any \( p \geq 1 \),

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} H^p(u^\delta(t)) \right] \leq C(T, R_1, \| u_0 \|, p),
\]

\[
\mathbb{E} \left[ \| u^{R_1,\delta}(t) - u^{R_1,\delta}(s) \| ^{2p} \right] \leq C(T, R_1, p)(|t - s|^p + |\delta|^p).
\]

Next, it suffices to prove the convergence of the Wong–Zakai approximation. To this end, we consider a stopping time \( \tau = \inf \{ t \in [0,T] \| \| u^{R_1}(t) \| \geq R \) or \( \| u^{R,\delta}(t) \| \geq R \} \). In the following,
we omit the supindex $R_1$. Applying the chain rule, we obtain that for $t \leq \tau$,

$$
\|u(t) - u^\delta(t)\|^2 = \|u(0) - u^\delta(0)\|^2 + 2 \int_0^t \langle fi(|u(s)|^2)u(s) - fi(|u^\delta(s)|^2)u^\delta(s), u(s) - u^\delta(s)\rangle ds \\
+ 2 \int_0^t \langle u(s) - u^\delta(s), 1u(s)dW(s) - iu^\delta(s)dW_\delta(s) \rangle \\
+ \int_0^t \sum_{k=1}^N \int_{\mathcal{S}_M} |u(s)|^2 |q_k|^2 dvol_M ds \\
\leq \int_0^t 2L_f(\|u(0)\|) \|u(s) - u^\delta(s)\|^2 ds + \int_0^t \sum_{k=1}^N |q_k|^2 u(s) ds \\
- 2 \int_0^t \langle u(s), iu^\delta([s]\delta)dW^\delta(s) \rangle - 2 \int_0^t \langle u(s), iu^\delta(s) - u^\delta([s]\delta) dW^\delta(s) \rangle \\
- 2 \int_0^t \langle u^\delta([s]\delta), iu(s)dW(s) \rangle - 2 \int_0^t \langle u^\delta(s) - u^\delta([s]\delta), iu(s)dW(s) \rangle \\
=: \int_0^t 2L_f(\|u(0)\|) \|u(s) - u^\delta(s)\|^2 ds + III_1 + III_2 + III_3 + III_4 + III_5.
$$

For the term $III_2$, the property of Wiener process, the mass conservation law, Hölder’s and Young’s inequality, as well as the property of the martingale, yield that

$$
\mathbb{E}[III_2] \leq -2 \int_0^{[t]s} \mathbb{E}\left[ (u(s) - u([s]\delta), iu^\delta([s]\delta)dW^\delta(s)) \right] \\
- 2 \int_0^{[t]s} \mathbb{E}\left[ (u([s]\delta), iu^\delta([s]\delta)dW^\delta(s)) \right] + C\delta^{\frac{\gamma}{2}} \\
\leq C(1 + C_R)\delta^{\frac{\gamma}{2}} - 2 \int_0^{[t]s} \mathbb{E}\left[ \left( \int_{[s]\delta}^s \mathbb{E}\left[ (\int_{[r]\delta}^s iu([r]\delta)dW(r), iu^\delta([s]\delta)dW^\delta(s)) \right] \\
- 2 \int_0^{[t]s} \mathbb{E}\left[ \left( \int_{[s]\delta}^s iu([r]\delta)dW(r), iu^\delta([s]\delta)dW^\delta(s)) \right] + C(1 + C_R)\delta^{\frac{\gamma}{2}}.
$$

Similar to $III_2$, we have that $\mathbb{E}[III_4] \leq C(1 + C_R)\delta^{\frac{\gamma}{2}}.$
For the terms $III_3$ and $III_5$, by taking expectation and using the property $\|e^{\Delta t} - I\|_{\mathcal{L}(\mathcal{W},L^2)} \leq C t^{\frac{1}{2}}$, the continuity estimate of $u$ and the property of martingale, we arrive at

$$E[III_3] \leq - \int_0^{[t]} 2E\left[\langle u(s) - u([s]_\delta), i(u^\delta(s) - u^\delta([s]_\delta))dW^\delta(s)\right]$$

$$= - \int_0^{[t]} 2E\left[\langle u([s]_\delta), i(u^\delta(s) - u^\delta([s]_\delta))dW^\delta(s)\right] + C(1 + C_R)\delta^{\frac{1}{2}}.$$
Remark which implies that
Similarly, following the above arguments, we further obtain (4.7)

\[ (\text{tic ODEs, one may expect the particle version of the stochastic non linear Schr"{o}dinger equation} \]

\[ B \]

Brownian motion \( T > \) and such that for any \( \epsilon \) it is expected that the limiting model when \( \epsilon \to 0 \) is the following stochastic NLS equation with random dispersion.

But we have not found a rigorous way to prove it. This will be studied in the future.

It follows that
\[ \mathbb{P}(\|u(t) - u^\delta(t)\| > \epsilon) \]
\[ \leq \mathbb{P}(\|u^{R_1}(t) - u(t)\| > \frac{\epsilon}{3}) + \mathbb{P}(\|u^{R_1,\delta}(t) - u^\delta(t)\| > \frac{\epsilon}{3}) \]
\[ + \mathbb{P}(\|u^{R_1}(t) - u^{R_1,\delta}(t)\| > \frac{\epsilon}{3}, t \leq \tau) + \mathbb{P}(\|u^{R_1}(t) - u^{R_1,\delta}(t)\| > \frac{\epsilon}{3}, t > \tau). \]

Taking limit on \( \delta \to 0, R, R_1 \to \infty \), using the strong convergence estimate and Chebyshev’s inequality, we obtain
\[ \lim_{\delta \to 0} \mathbb{P}(\|u(t) - u^\delta(t)\| > \epsilon) \]
\[ \leq \lim_{\delta \to 0} \frac{9}{\epsilon^2} C(1 + C_R) \exp(2L_f(\|u_0\|)T)\delta^\frac{1}{2} \]
\[ + \lim_{R \to \infty} \mathbb{P}(\sup_{s \in [0,t]} \|u(s)\| \geq R) + \lim_{R \to \infty} \mathbb{P}(\sup_{s \in [0,t]} \|u^\delta([s])\| \geq R) = 0. \]

Similarly, following the above arguments, we further obtain
\[ \lim_{\delta \to 0} \mathbb{E}[\sup_{t \in [0,T]} \|u(t) - u^\delta(t)\|^2] = 0, \]
which implies that
\[ \lim_{\delta \to 0} \mathbb{P}(\sup_{t \in [0,T]} \|u(t) - u^\delta(t)\| > \epsilon) = 0. \]

\[ \square \]

\textbf{Remark 4.1.} Similar to the stochastic Wasserstein Hamiltonian flow induced by classical Stochastic ODEs, one may expect the particle version of the stochastic nonlinear Schrödinger equation (4.1), that is,

\[ dX_t = v(t, X_t), \]
\[ dv(t, X_t) = -\nabla X_t \frac{1}{2} \delta \rho f(\rho(t, X_t)) + 2\lambda \nabla X_t f(\rho(t, X_t)) + 2\nabla X_t \circ dW(t). \]

But we have not found a rigorous way to prove it. This will be studied in the future.

\textbf{4.2. NLS equation with random dispersion.} The dimensionless NLS equation with random dispersion is given by

\[ du = i\Delta u - \frac{1}{\epsilon} m(\frac{t}{\epsilon})dt + i\lambda f(|u|^2)udt, \]

where \( m \) is a real-valued centered stationary random process. Under ergodic assumptions on \( m \), it is expected that the limiting model when \( \epsilon \to 0 \) is the following stochastic NLS equation with white noise dispersion

\[ du = \sigma_0 i\Delta u \circ dB_t + i\lambda f(|u|^2)udt, \]

where \( \sigma_0^2 = 2 \int_0^\infty \mathbb{E}[m(0)m(t)]dt \) (see e.g. [18]). For simplicity, we set \( \sigma_0 = 1 \) in (4.9) throughout this section.

To see (4.9) as a stochastic Wasserstein Hamiltonian flow, let us use (4.8) instead of Wong–Zakai approximations. Assume that the real valued centered stationary process \( m(t) \) is continuous and such that for any \( T > 0, t \mapsto \epsilon \int_0^t \epsilon^2 m(s)ds \) converges in distribution to a standard real-valued Brownian motion \( B \) in \( \mathcal{C}([0, T]) \) (see e.g. [18]).
First, using Madelung transform \( u(t, x) = \sqrt{\rho(t, x)} e^{iS(t, x)} \) gives

\[
e^{\frac{i}{\epsilon} \log(\rho) + iS} \left( \frac{1}{2} \frac{\partial \rho}{\rho} + i \partial_t S \right)
= ie^{\frac{i}{\epsilon} \log(\rho) + iS} \left( \frac{1}{2} \frac{\nabla \rho}{\rho} + i \nabla S \right)^2 + \left( \frac{1}{2} \frac{\Delta \rho}{\rho} + i \Delta S - \frac{1}{2} \frac{\nabla \rho \cdot \nabla S}{\rho} \right) \frac{1}{\epsilon} m(\frac{t}{\epsilon^2})
+ ie^{\frac{i}{\epsilon} \log(\rho) + iS} \lambda f(\rho)
= ie^{\frac{i}{\epsilon} \log(\rho) + iS} \left( \frac{1}{4} \left( \frac{\nabla \rho}{\rho} \right)^2 - (\nabla S)^2 + \frac{1}{2} \frac{\nabla \rho}{\rho} \cdot \nabla S \right) + \left( \frac{1}{2} \frac{\Delta \rho}{\rho} + i \Delta S - \frac{1}{2} \frac{\nabla \rho \cdot \nabla S}{\rho} \right) \frac{1}{\epsilon} m(\frac{t}{\epsilon^2})
+ ie^{\frac{i}{\epsilon} \log(\rho) + iS} \lambda f(\rho).
\]

We obtain that

\[
\partial_t \rho = -2 \nabla \cdot (\rho \nabla S) \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}),
\]

\[
\partial_t S = -|\nabla S|^2 - \frac{1}{4} \frac{\delta}{\delta \rho} I(\rho) \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}) + \lambda f(\rho),
\]

which can be rewritten as

\[
\partial_t \rho = -\nabla \cdot (\rho v) \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}),
\]

\[
\partial_t v = -(\nabla_x v \cdot v - \nabla_x \frac{1}{2} \frac{\delta}{\delta \rho} I(\rho) \frac{1}{\epsilon} m(\frac{t}{\epsilon^2})) + 2\lambda \nabla_x f(\rho).
\]

Based on the above calculations, following the similar steps in the proof of Proposition 4.1, we conclude the following result.

**Proposition 4.2.** The critical point of the variational problem

\[
I_c(\rho^0, \rho^T) = \inf \{ S(\rho_t, \Phi_t) | (\Delta_{\rho_t})^t \Phi_t \in T_{\rho_t} \mathcal{P}_o(M), \rho(0) = \rho^0, \rho(T) = \rho^T \}
\]

whose action functional is given by the dual coordinates,

\[
S(\rho_t, \Phi_t) = -\int_0^T (\Phi(t), \partial_t \rho_t) dt + \int_0^T H_0(\rho_t, \Phi_t) dt + \int_0^T H_1(\rho_t, \Phi_t) \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}) dt,
\]

satisfies (4.10). Here \( H_0(\rho_t, \Phi_t) = -\lambda \int_M \rho_0 f(s) ds dvol_M \) with a smooth function \( f \), \( H_1(\rho_t, \Phi_t) = \int_M |\nabla \Phi_t|^2 \rho_t dvol_M + \frac{1}{2} I(\rho), I(\rho) = \int_M |\nabla \log(\rho)|^2 \rho dvol_M \).

It has been shown in [18] that the limit of (4.10) is the NLS equation with white noise dispersion. Therefore, (4.10) is also a stochastic Wasserstein Hamiltonian flow on density manifold.

**Remark 4.2.** The above system is also expected to have a particle version. By applying the push-forward map in section 3 on \( \hat{\Omega} \), the particle version of (4.9) is expected to be

\[
dX_t = v(t, X_t) \circ dB_t
\]

\[
dv(t, X_t) = -\nabla X_t \frac{1}{2} \frac{\delta}{\delta \rho} I(\rho(t, X_t)) \circ dB_t + 2\lambda \nabla X_t f(\rho(t, X_t)).
\]

And this will be studied in the future.
4.3. Schrödinger Bridge Problem (SBP) with common noise. In this part, we indicates that the critical point of the Schrödinger bridge problem (SBP) with common noise may also be a stochastic Wasserstein Hamiltonian flow. The SBP with common noise is inspired by [6, 45] for the Schrödinger Bridge type problem in stochastic case, where the common noise is added into the classical Schrödinger Bridge type problem [33, 9]. This problem can be formulated as a stochastic control problem on Wasserstein manifold:

\[
\min_{\{\nu_t\}_{t \in [0,T]}} \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \nu_t(x)^2 \rho_t(x, \omega) \, dx \, dt
\]

Subject to:

\[
\frac{\partial \rho_t(x, \omega)}{\partial t} + \nabla \cdot (\rho_t(x, \omega)(\nu_t + A(x, t)\dot{W}_t(\omega))) = \Delta \rho_t,
\]

and \(\rho_0(\cdot, \omega) = \rho_a, \rho_T(\cdot, \omega) = \rho_b\).

The continuity equation (4.13) can be viewed as an SDE on the Wasserstein manifold \(\mathcal{P}_2(\mathbb{R}^d)\), which reads

\[
dX_t = v(t, X_t) dt + \sqrt{2} dB(t) + A(t, X_t) dW(t).
\]

Here \(B\) is the Brownian motion which corresponding to the diffusion effect in (4.13), and \(W\) is another Brownian motion which is independent of \(B\) and is called the common noise.

In the following, we consider the Wong–Zakai approximation of (4.12), i.e,

\[
\min_{\{\nu_t\}_{t \in [0,T]}} \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \nu_t(x)^2 \rho_t(x, \omega) \, dx \, dt
\]

Subject to:

\[
\frac{\partial \rho_t(x, \omega)}{\partial t} + \nabla \cdot (\rho_t(x, \omega)(\nu_t + A(x, t)\dot{\xi}_t(t))) = \Delta \rho_t,
\]

and \(\rho_0(\cdot, \omega) = \rho_a, \rho_T(\cdot, \omega) = \rho_b\), and show that its critical point is a Wasserstein Hamiltonian flow.

**Proposition 4.3.** Assume that \(W\) is \(d\)-dimensional Brownian motion, \(\xi\) is the piecewisely linear Wong–Zakai approximation of \(W\). Let \(A(\cdot, t) \in C_b^1(\mathbb{R}^d), \rho_a, \rho_b \in \mathcal{P}_a(\mathbb{R}^d)\) be smooth. Then the critical point of (4.15) satisfies

\[
\partial_t \rho_t = \frac{\delta}{\delta \Phi} \mathcal{H}_0(\rho_t, \Phi_t) + \sum_{i=1}^d \frac{\delta}{\delta \Phi_i} \mathcal{H}_i(\rho_t, \Phi_t)(\dot{\xi}_t)_i(t),
\]

\[
\partial_t \Phi_t = -\frac{\delta}{\delta \rho} \mathcal{H}_0(\rho_t, \Phi_t) - \sum_{i=1}^d \frac{\delta}{\delta \rho} \mathcal{H}_i(\rho_t, \Phi_t)(\dot{\xi}_t)_i(t),
\]

where \(\mathcal{H}_0(\rho, \Phi) = \frac{1}{2} \int_M |\nabla \Phi|^2 d\text{vol}_M - \frac{1}{2} I(\rho), \mathcal{H}_i(\rho, \Phi) = \int_M \rho A_i \partial_x \Phi d\text{vol}_M\). Here \(A_i\) denotes the \(i\)-th column of the matrix \(A_t\).

**Proof.** By using the Lagrangian multiplier method, the critical point satisfies

\[
\partial_t \rho_t + \nabla \cdot (\rho(\nabla S_t + A_t \dot{\xi}_t(t))) = \frac{1}{2} \Delta \rho_t,
\]

\[
\partial_t S_t + \frac{1}{2} |\nabla S_t|^2 + \nabla S_t \cdot A_t \dot{\xi}_t(t) = -\frac{1}{2} \Delta S_t.
\]

Applying the “Hopf-Cole” transform (see e.g. [32]) \(\Phi_t = S_t - \frac{1}{2} \log(\rho_t)\), we obtain

\[
\partial_t \rho_t + \nabla \cdot (\rho_t \nabla \Phi_t) + \nabla \cdot (\rho_t A_t \dot{\xi}_t(t)) = 0,
\]

\[
\partial_t \Phi_t + \frac{1}{2} |\nabla \Phi_t|^2 + \nabla \Phi \cdot A_t \dot{\xi}_t(t) = \frac{1}{8} \frac{\delta}{\delta \rho} I(\rho).
\]
which implies (4.16).

The above result also coincides with the generalized variational principle (3.10) with the action functional

\[ S(\rho_t, \Phi_t) = -\int_0^T \langle \Phi(t), \partial_t \rho_t \rangle dt + \int_0^T \mathcal{H}_0(\rho_t, \Phi_t) dt + \sum_{i=1}^d \int_0^T \mathcal{H}_i(\rho_t, \Phi_t) d\xi_i(t), \]

whose critical point is the stochastic Hamiltonian system (4.16). From the proof of Proposition 4.3, (4.16) is equivalent to the forward and backward system which contains the backward stochastic Hamilton-Jacobi equation (4.18) and a forward stochastic Kolmogorov equation (4.17), and plays the role of characteristics for the master equation [6]. The derivation of (4.16) may be extended to the mean-field game systems with common noise in [6, 8] up to an Itô-Wentzell correction term [30]. If the Wong–Zakai approximation (4.15) is convergent to (4.12), then the critical point of (4.12) is expected to be a stochastic Wasserstein Hamiltonian flow. This will be our future research.

5. Conclusions

In this paper, we study the stochastic Wasserstein Hamiltonian flows, including the stochastic Euler–Lagrange equations and its Hamiltonian flows on density manifold. First, we show that the classical Hamiltonian motions with random perturbations and random initial data induce the stochastic Wasserstein Hamiltonian flows via Wong–Zakai approximation with Lagrangian formalism. Then we propose a generalized variational principle to derive and investigate the generalized stochastic Wasserstein Hamiltonian flows, including the stochastic nonlinear Schrödinger equation, Schrödinger equation with random dispersion and stochastic Schrödinger bridge problem. The study provides rigorous mathematical justification for the principle that the conditional probability density of stochastic Hamiltonian flow in sample space is stochastic Hamiltonian flow on density manifold.

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Proof of Lemma 2.2

Proof. The local existence of (2.4) and (2.1) is ensured thanks to the local Lipschitz condition of $f$ and $\sigma$. To obtain a global solution, a priori bound on $H_0(x, p)$ is needed. Denote the solutions of (2.1) and (2.4) with same initial condition $(x_0, p_0)$ by $(x^\delta_t, p^\delta_t)$, $\delta > 0$ and $x^0_t, p^0_t$, respectively. Applying the chain rule to $H_0(x^\delta_t, p^\delta_t)$ for (2.4) and (2.1), we get that

$$H_0(x^\delta_t, p^\delta_t) = H_0(x_0, p_0) + \int_0^t \eta \nabla p H_0(x^\delta_s, p^\delta_s) \cdot \nabla x \sigma(x_s) \xi_\delta(s) ds$$

$$H_0(x_t, p_t) = H_0(x_0, p_0) + \int_0^t \eta \nabla p H_0(x_s, p_s) \cdot \nabla x \sigma(x_s) dB_s$$

$$+ \frac{1}{2} \int_0^t \eta^2 \nabla p H_0(x_s, p_s) \cdot (\nabla x \sigma(x_s), \nabla \sigma(x_s)) ds.$$

By applying growth condition (2.3) and taking expectation on the second equation, we derive that

$$H_0(x^\delta_t, p^\delta_t) \leq (H_0(x_0, p_0) + \eta C_1 T) \exp(\int_0^t c_1 \eta \xi_\delta(s) ds),$$

$$E[H_0(x_t, p_t)] \leq (E[H_0(x_0, p_0)] + \frac{\eta^2}{2} C_1 T) \exp(\int_0^t c_1 \frac{\eta^2}{2} ds).$$

The first inequality leads to $H_0(x^\delta_t, p^\delta_t) < \infty$ since $\xi_\delta(s) = \frac{B_{s+1} - B_s}{\delta}$, if $s \in [t_k, t_{k+1}]$. Furthermore, taking expectation on the first inequality, applying Fernique’s theorem (see, e.g. [20]) for Gaussian variable and independent increments of $B_t$, we get that

$$E[H_0(x^\delta_t, p^\delta_t)] \leq C(T, \eta, c_1)(2^{\frac{\eta^2}{4}}(E[H_0(x_0, p_0)] + 1),$$

where $[w]$ is the integer part of the real number $w$. The second inequality yield that $H_0(x_t, p_t) < \infty, a.s.$, and the global existence of the strong solution of (2.4). Similarly, for $p \geq 2$, we have that

$$E[H^p_0(x^\delta_t, p^\delta_t)] \leq C(T, \eta, c_1, C_1, p)(E[H^p_0(x_0, p_0)] + 1),$$

$$E[H^p_0(x_t, p_t)] \leq C(T, \eta, c_1, p)(E[H^p_0(x_0, p_0)] + 1).$$

Furthermore, applying the above bounded moment estimate, we obtain that for $s \leq t$,

$$E[|x(t) - x(s)|^{2p} + |p(t) - p(s)|^{2p}] \leq C(T, \eta, c_1, C_1, p, x_0, p_0)|t - s|^p$$

$$E[|x^\delta(t) - x^\delta(s)|^{2p} + |p(t) - p(s)|^{2p}] \leq C(T, \eta, c_1, C_1, p, x_0, p_0)2\frac{\eta^2}{4}|t - s|^p.$$
Assume that \( t \in [t_k, t_{k+1}] \), \( t_k = k\delta \). Then by using the expansion of (2.1), we have that

\[
H_0(x_t^\delta, p_t^\delta) = H_0(x_0^0, p_0^0) - \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \eta \nabla_p H_0(x_t^\delta, p_t^\delta) \cdot \nabla_x \sigma(x_t^\delta) d\xi(s) \\
- \int_{t_k}^t \eta \nabla_p H_0(x_t^\delta, p_t^\delta) \cdot \nabla_x \sigma(x_t^\delta) d\xi(s) \\
= H_0(x_0^0, p_0^0) - \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \eta \nabla_p H_0(x_t^\delta, p_t^\delta) \cdot \nabla_x \sigma(x_t^\delta) d\xi(s) \\
- \int_{t_k}^t \eta \nabla_p H_0(x_t^\delta, p_t^\delta) \cdot \nabla_x \sigma(x_t^\delta) d\xi(s) \\
- \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \eta \left( \int_{t_j}^s \nabla_{p_p} H_0(x_t^\delta, p_t^\delta) \cdot (\nabla_x \sigma(x_t^\delta), -\eta \nabla_x \sigma(x_t^\delta) \dot{\xi}(r)) dr \right) ds \\
+ \int_{t_j}^s \nabla_{p_p} H_0(x_t^\delta, p_t^\delta) \cdot (\nabla_x \sigma(x_t^\delta), -\frac{1}{2}(p_t^\delta)^T d_z g^{-1}(x)p_t^\delta - \nabla_x f(x^\delta)) dr ds \\
+ \int_{t_j}^s \nabla_{p_x} H_0(x_t^\delta, p_t^\delta) \cdot (\nabla_x \sigma(x_t^\delta) \dot{\xi}(s), g^{-1}(x^\delta)p_t^\delta dr ds \\
- \int_{t_k}^t \eta \left( \int_{t_k}^s \nabla_{p_p} H_0(x_t^\delta, p_t^\delta) \cdot (\nabla_x \sigma(x_t^\delta), -\eta \nabla_x \sigma(x_t^\delta) \dot{\xi}(r)) dr \right) ds \\
+ \int_{t_k}^s \nabla_{p_p} H_0(x_t^\delta, p_t^\delta) \cdot (\nabla_x \sigma(x_t^\delta), -\frac{1}{2}(p_t^\delta)^T d_z g^{-1}(x)p_t^\delta - \nabla_x f(x^\delta)) dr ds \\
+ \int_{t_k}^s \nabla_{p_x} H_0(x_t^\delta, p_t^\delta) \cdot (\nabla_x \sigma(x_t^\delta) \dot{\xi}(s), g^{-1}(x^\delta)p_t^\delta dr ds \\
=: H_0(x_0^0, p_0^0) + \sum_{j=0}^{k-1} I_j + I_k(t) \\
+ \sum_{j=0}^{k-1} (I_j^{21} + I_j^{22} + I_j^{23} + I_j^{24}) + I_k^{21}(t) + I_k^{22}(t) + I_k^{23}(t) + I_k^{24}(t).
Making use of the growth condition \((2.3)\), we have that

\[
\sum_{j=0}^{k-1} (I_j^2 + I_j^2 + I_j^2 + I_j^2 + I_j^2 + I_j^2) + I_j^2(t) + I_j^2(t) + I_j^2(t)
\]

\[
\leq \sum_{j=0}^{k-1} \int_t^{t_{j+1}} (C_1 + c_1 H_0(x_{j+1}^p, p^A_{j+1})) |\xi_j(s)|^2 \, ds + \sum_{j=0}^{k-1} \int_t^{t_{j+1}} (C_1 + c_1 H_0(x_{j+1}^p, p^A_{j+1})) |\xi_j(s)| \, ds
\]

\[
+ \int_{t_k}^{t} (C_1 + c_1 H_0(x_{j+1}^p, p^A_{j+1})) |\xi_j(s)|^2 \, ds + \int_{t_k}^{t} (C_1 + c_1 H_0(x_{j+1}^p, p^A_{j+1})) |\xi_j(s)| \, ds
\]

\[
= \int_0^t (C_1 + c_1 H_0(x_{j+1}^p, p^A_{j+1})) |\xi_j(s)|^2 \, ds + \int_0^t (C_1 + c_1 H_0(x_{j+1}^p, p^A_{j+1})) |\xi_j(s)| \, ds.
\]

By using the Gronwall–Bellman inequality, we obtain that

\[
H_0(x^p, p^A) \leq \exp(\int_0^t c_1(|\xi_j(s)|^2 + |\xi_j(s)|) \, ds) (H_0(x_0, p_0) + CT + \sum_{j=0}^{k-1} I_j^2 + I_k^2(t)).
\]

For simplicity, assume that \(T = K\delta\). Denote \([t] = t_k = k\delta\) if \(t \in [t_k, t_{k+1})\). The definition of \(\xi_j(s)\) yields that \(s \in [t_j, t_{j+1}]\)

\[
|\xi_j(s)| \delta = \left| \frac{B(t_{j+1}) - B(t_j)}{\delta} \right|^2 \delta + |B(t_{j+1}) - B(t_j)|.
\]

Define a stopping time \(\tau_R = \inf\{t \in [0, T] \mid \int_0^t |\xi(s)|^2 \, ds \geq R\}\). The stopping time is well-defined since the quadratic variation process of Brownian motion is bounded in \([0, T]\). Then taking \(t \leq \tau_R\) and using Hölder’s inequality, then it holds that

\[
(6.1) \quad H_0(x^p, p^A) \leq \exp(\int_0^t c_1(|\xi_j(s)|^2 + |\xi_j|) \, ds) \exp(C(R + T))(H_0(x_0, p_0) + CT + \sum_{j=0}^{k-1} I_j^2 + I_k^2(t))
\]

\[
\leq \exp(\int_0^t c_1(3 |\xi_j(s)|^2) \, ds) \exp(C(R + T)) H_0(x_0, p_0)
\]

\[
+ \exp(C(R + T)) \exp(\int_0^t c_1(3 |\xi_j(s)|^2) \, ds) \left| \int_0^t -\eta \nabla_p H_0(x^p_{[j+1]}, p^A_{[j+1]}) \cdot \nabla_x \sigma(x_{[j+1]}) dB(s) \right|
\]

\[
+ \exp(C(R + T)) \exp(\int_0^t c_1(3 |\xi_j(s)|^2) \, ds) \left| \int_0^t -\eta \nabla_p H_0(x^p_{[j+1]}, p^A_{[j+1]}) \cdot \nabla_x \sigma(x_{[j+1]}) \xi_j(s) ds \right|.
\]

Similarly, one could obtain a analogous estimate of \((6.1)\) with the integral over \([t_k-1, t_k]\), where \(t_k, k \leq K, \ t_k \leq \tau_R\). By the Cauchy inequality and taking expectation on both sides of \((6.1)\), applying the Burkholder–Davis–Gundy inequality (see e.g., \[27\]) and using the independent
Then the Grownall's inequality yield that
\[ E[H^2_0(x_t^\delta, p_t^\delta)] \]
\[ \leq 3E\left[\exp\left(\int_{t_{k-1}}^{t_k} (3c_1|\xi_{s}(s)|^2)ds\right)\exp(2C(R + T))E\left[H^2_0(x_0, p_0)\right]\right] \]
\[ + 3\exp(2C(R + T))E[\exp(\int_{t_{k-1}}^{t_k} 3c_1|\xi_{s}(s)|^2)ds]E\left[\left|\int_0^{t_k} -\eta \nabla_x H_0(x_{s|s\rangle}, p_{s|s\rangle}) \cdot \nabla_x \sigma(x_{s|s\rangle})dB(s)\right|^2\right]\]
\[ + 3\exp(2C(R + T))E\left[\exp(\int_{t_{k-1}}^{t_k} 3c_1|\xi_{s}(s)|^2)ds\right]E\left[H^2_0(x_0, p_0)\right] \]
\[ \times \left|\eta \nabla_x H_0(x_{t_{k-1} - 1}, p_{t_{k-1} - 1}) \cdot \nabla_x \sigma(x_{t_{k-1} - 1})\right|^2 \]
\[ \leq 3E\left[\exp\left(\int_{t_{k-1}}^{t_k} (3c_1|\xi_{s}(s)|^2)ds\right)\exp(2C(R + T))E\left[H^2_0(x_0, p_0)\right]\right] \]
\[ + 3\exp(2C(R + T))E[\exp(\int_{t_{k-1}}^{t_k} 3c_1|\xi_{s}(s)|^2)ds]E\left[\left|\int_0^{t_k} (C_1 + c_1 H_0(x_{s|s\rangle}, p_{s|s\rangle}))^2 ds\right|\right]\]
\[ + 3\exp(2C(R + T))E\left[\exp(\int_{t_{k-1}}^{t_k} 3c_1|\xi_{s}(s)|^2)ds\right]E\left[H^2_0(x_0, p_0)\right] \]
\[ \times E\left[\left|\int_0^{t_k} (C_1 + c_1 H_0(x_{s|s\rangle}, p_{s|s\rangle}))^2 ds\right|\right] . \]

Applying the Fernique theorem and choosing sufficient small \( \delta \) such that \( 12c_1 \delta < 1 \), then we have that
\[ E[\exp(\int_{t_{k-1}}^{t_k} 3c_1|\xi_{s}(s)|^2)ds] \leq C, \]
\[ E\left[\exp(\int_{t_{k-1}}^{t_k} 3c_1|\xi_{s}(s)|^2)ds\right]E\left[\left|\int_0^{t_k} (C_1 + c_1 H_0(x_{s|s\rangle}, p_{s|s\rangle}))^2 ds\right|\right] \]
\[ \leq \sqrt{E[\exp(\int_{t_{k-1}}^{t_k} 6c_1|\xi_{s}(s)|^2)ds]\sqrt{E\left[\left|\int_0^{t_k} (C_1 + c_1 H_0(x_{s|s\rangle}, p_{s|s\rangle}))^2 ds\right|^4\right]} \leq C\delta. \]

The above estimation gives
\[ E[H^2_0(x_{t_k}^\delta, p_{t_k}^\delta)] \leq 3\exp(2C(R + T))CE[H^2_0(x_0, p_0)] \]
\[ + 6\exp(2C(R + T))C \int_0^{t_k} E\left[(C_1^2 + c_1^2 H^2_0(x_{s|s\rangle}, p_{s|s\rangle}))\right]ds \]
\[ + 6\exp(2C(R + T))C\delta E\left[C_1^2 + c_1^2 H^2_0(x_{t_{k-1} - 1}^\delta, p_{t_{k-1} - 1}^\delta)\right] . \]

Then the Grownall's inequality yield that
\[ E[H^2_0(x_{t_k}^\delta, p_{t_k}^\delta)] \leq \exp(6TC C^2 \exp(2C(R + T))) \]
\[ \times \left(3\exp(2C(R + T))CE[H^2_0(x_0, p_0)] + 6C^2 TC \exp(2C(R + T))\right) \]
Combining the above estimates with (6.1) and the Burkholder–Davis–Gundy inequality, we conclude that

\[
\sup_{t \in [0, \tau_{\eta}]} \mathbb{E}[H_0^p(x_t^\delta, p_t^\delta)] \leq \langle \exp(6TCc_2^2 \exp(2C(R + T)) + C) \rangle \times \left(3 \exp(2C(R + T))\mathbb{E}[H_0^2(x_0, p_0)] + 6C^2TC \exp(2C(R + T))\right)
=: C_R.
\]

Similarly, by choosing sufficient small \(\delta\), we have that for \(t \in [0, \tau^R]\),

\[
\mathbb{E}[H_0^p(x_t^\delta, p_t^\delta)] \leq C_{R,p} < \infty.
\]

As a consequence, by again using (6.1), we obtain that

\[
\mathbb{E}\left[\sup_{t \in [0, \tau_{\eta}]} H_0^p(x_t^\delta, p_t^\delta)\right] \leq C_{R,p} < \infty.
\]

Next we show the convergence in probability of the solution of (2.1) to that of (2.4). Introduce another stopping time \(\tau_{R_1} := \inf\{t \in [0, T]| |x_t| + |p_t| \geq R_1, |x_t^\delta| + |p_t^\delta| \geq R_1\}\). Let \(t \in [0, \tau_{R_1} \wedge \tau_{R_2})\). By using the polynomial growth condition of \(f, \sigma\) and the fact that \(\sigma\) is independent of \(p\), we obtain that

\[
|x^\delta(t) - x(t)|^2
\]
\[
= |x^\delta(0) - x(0)|^2 + \int_0^t 2\langle x^\delta(s) - x(s), g^{-1}(x^\delta(s))p^\delta(s) - g^{-1}(x(s))p(s)\rangle ds
\]
\[
\leq |x^\delta(0) - x(0)|^2 + \int_0^t C_\delta(1 + |p(s)|)(|x^\delta(s) - x(s)|^2 + |p^\delta(s) - p(s)|^2)ds,
\]
\[
|p^\delta(t) - p(t)|^2
\]
\[
= \int_0^t \langle -(p^\delta(s))^\top d_s g^{-1}(x^\delta(s))p^\delta(s) + p(s)^\top d_s g^{-1}(x(s))p(s), p^\delta(s) - p(s)\rangle ds
\]
\[
+ \int_0^t 2\langle -\nabla_x f(x^\delta(s)) + \nabla_x f(x(s)), p^\delta(s) - p(s)\rangle ds
\]
\[
- \int_0^t 2\eta(p^\delta(s) - p(s), \nabla_x \sigma(x^\delta(s))d\xi_s(s) - \nabla_x \sigma(x(s))dB_s\rangle
\]
\[
\leq C_\eta \int_0^t (1 + |x^\delta(s)|)(|p^\delta(s)|^2 + |p(s)|^2)(|p^\delta(s) - p(s)|^2 + |x^\delta(s) - x(s)|^2)ds
\]
\[
+ C_\eta \int_0^t (1 + |x(s)|^2 + |x^\delta(s)|^2)(|p^\delta(s) - p(s)|^2 + |x^\delta(s) - x(s)|^2)ds
\]
\[
- \int_0^t 2\eta(p^\delta(s) - p(s), \nabla_x \sigma(x^\delta(s))d\xi_s(s) - \nabla_x \sigma(x(s))dB_s),
\]
where $C_g$ and $C_f$ are constants depending on $f$ and $g$. To deal with the last term, we split it as follows,

\[
\int_0^t 2\eta (p^\delta(s) - p(s), \nabla_x \sigma(x^\delta(s))d\xi_s(s) - \nabla_x \sigma(x(s))dB_s)
= 2\eta \int_0^t (p^\delta([s]_s) - p([s]_s), \nabla_x \sigma(x^\delta(s))d\xi_s(s) - \nabla_x \sigma(x(s))dB_s)
+ 2\eta \int_0^t (p^\delta(s) - p(s) - p^\delta([s]_s) + p([s]_s), \nabla_x \sigma(x^\delta(s))d\xi_s(s) - \nabla_x \sigma(x(s))dB_s)
= 2\eta \int_0^t (p^\delta([s]_s) - p([s]_s), \nabla_x \sigma(x^\delta([s]_s))d\xi_s([s]_s) - \nabla_x \sigma(x([s]_s))dB_s)
+ 2\eta \int_0^t (p^\delta(s) - p(s) - p^\delta([s]_s) + p([s]_s), \nabla_x \sigma(x^\delta([s]_s))d\xi_s(s) - \nabla_x \sigma(x([s]_s))dB_s)
+ 2\eta \int_0^t (p^\delta(s) - p(s) - p^\delta([s]_s) + p([s]_s), (\nabla_x \sigma(x^\delta(s)) - \nabla_x \sigma(x^\delta([s]_s)))d\xi_s(s)
- (\nabla_x \sigma(x(s)) - \nabla_x \sigma(x([s]_s)))dB_s)
\]

Taking expectation on $II^1$, using the property of the discrete martingale, the a prior estimates for $H_0(x_1, p_1)$ and $H_0(x_1^\delta, p_1^\delta)$ and Hölder's inequality, we have that

\[
\mathbb{E}[II^1] = 0,
\mathbb{E}[II^2] \leq 2\eta \int_0^t \mathbb{E} \left[ (p^\delta([s]_s) - p([s]_s), \int_{[s]_s}^s (\nabla_{xx} \sigma(x^\delta(r)) \cdot (g^{-1}(x^\delta(r))p^\delta(r))dr) d\xi_s(s)) \right]
- 2\eta \int_0^t \mathbb{E} \left[ (p^\delta([s]_s) - p([s]_s), \int_{[s]_s}^s (\nabla_{xx} \sigma(x(r)) \cdot (g^{-1}(x^\delta(r))p^\delta(r))dr) dB_s) \right]
\leq C(R_1)\delta^{1/2}.
\]
Similar arguments lead to $E[II'] \leq C(R_1) \delta^\frac{5}{2}$. For the term $II^3$, applying the continuity estimate of $x_t$ and $x_t^\delta$, as well as independent increments of the Brownian motion, we get

$$E[II^3] \leq C(R_1) \delta^\frac{5}{2} + 2\eta^2 E \left[ \int_0^t \left( \int_{[s, \delta]} \nabla_x \sigma(x(s)) d\xi_{s}(s) - \int_{[s+\delta]} \nabla_x \sigma(x(s)) dB_r \right) \right]$$

Then the Gronwall’s inequality implies that

$$\int_0^t \left( \int_{[s, \delta]} |\nabla_x \sigma(x(s))|^2 \delta^\frac{1}{2} (B([s, \delta]) - B([s+\delta]))^2 ds \right)$$

Combining the above estimates, we achieve that

$$E[II^3] \leq C(R_1) \delta^\frac{5}{2} + 2\eta^2 \int_0^t \left[ \int_{[s, \delta]} |\nabla_x \sigma(x(s))|^2 ds \right]$$

where $C(R_1) > 0$ is monotone with $R_1$. Combining the above estimates, we achieve that

$$E[|x^\delta(t) - x(t)|^2] \leq C(R_1) \delta^\frac{5}{2} + 2\eta^2 \int_0^t \left[ \int_{[s, \delta]} |\nabla_x \sigma(x(s))|^2 ds \right]$$

Then the Gronwall’s inequality implies that

$$E[|x^\delta(t) - x(t)|^2] \leq \exp(2(C_g + C_f)(1 + C_{R_1} T)C(R_1) \delta^\frac{5}{2}).$$

By making use of (6.2) and Chebyshev’s inequality, we conclude that

$$\mathbb{P}( |x^\delta(t) - x(t)| + |p^\delta(t) - x(t)| \geq \epsilon)$$

$$\leq \mathbb{P}( |x^\delta(t) - x(t)| + |p^\delta(t) - x(t)| \geq \epsilon) \cap \{ t < \tau_k \} \cap \{ t < \tau_{R_1} \}$$

$$+ \mathbb{P}( |x^\delta(t) - x(t)| + |p^\delta(t) - x(t)| \geq \epsilon) \cap \{ t \geq \tau_k \}$$

$$+ \mathbb{P}( |x^\delta(t) - x(t)| + |p^\delta(t) - x(t)| \geq \epsilon) \cap \{ t \geq \tau_{R_1} \}$$

$$\leq 2 \frac{E \left[ \int_0^t |\xi_{s}(s)|^2 \delta ds \right]}{\epsilon^2} + \frac{E \left[ |x(t)| + |p(t)| + |x^\delta(t)| + |p^\delta(t)| \right]}{R_1}$$

$$\leq 2 \frac{\epsilon^2 \exp(2(C_g + C_f)(1 + C_{R_1} T)C(R_1) \delta^\frac{5}{2})}{R + C + C_{R_1}}.$$
Here, $E[|x(t)| + |p(t)| + |x^\delta(t)| + |p^\delta(t)|] < C(1 + CR)$ is ensured by $E[\sup_{t\in[0,T]} H^\delta_0(x^t, p^t)] \leq CR$.

Taking limit on $\delta \to 0$, $R_1 \to \infty$, and $R \to \infty$ leads to

$$\lim_{\delta \to 0} \mathbb{P}(|x^\delta(t) - x(t)| + |p^\delta(t) - p(t)| > \epsilon) = 0.$$ 

Similarly, one could utilize the properties of martingale and obtain the following estimate, for large enough $q > 0$,

$$E[|x^\delta(t) - x(t)|^q] + E[|p^\delta(t) - p(t)|^q] \leq C_q \exp(C_q(C_g + C_f)(1 + CR_1)T)C(R_1)\delta^{-1}.$$ 

This implies that for large enough $q > 4$,

$$E[\sup_{k \leq K} \sup_{t \in [t_{k-1}, t_k]} |x^\delta(t) - x(t)|^q] + E[\sup_{k \leq K} \sup_{t \in [t_{k-1}, t_k]} |p^\delta(t) - p(t)|^q]$$

$$\leq \sum_{k=0}^{K-1} E[\sup_{t \in [t_{k-1}, t_k]} |x^\delta(t) - x(t)|^q] + E[\sup_{t \in [t_{k-1}, t_k]} |p^\delta(t) - p(t)|^q]$$

$$\leq C_q K \exp(C_q(C_g + C_f)(1 + CR_1)T)C(R_1)\delta^{-1}$$

$$\leq C_q \exp(C_q(C_g + C_f)(1 + CR_1)T)C(R_1)\delta^{-2}.$$ 

Combining the above estimate and applying the Chebyshev’s inequality, we further obtain

$$\lim_{\delta \to 0} \mathbb{P}(\sup_{t \in [0,T]} |x^\delta(t) - x(t)| + \sup_{t \in [0,T]} |p^\delta(t) - p(t)| > \epsilon) = 0.$$ 

□