A COUNTEREXAMPLE TO THE STRONG VERSION OF FREEDMAN'S CONJECTURE

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Abstract. A long-standing conjecture due to Michael Freedman asserts that the 4-dimensional topological surgery conjecture fails for non-abelian free groups, or equivalently that a family of canonical examples of links (the generalized Borromean rings) are not $A - B$ slice. A stronger version of the conjecture, that the Borromean rings are not even weakly $A - B$ slice, where one drops the equivariant aspect of the problem, has been the main focus in search for an obstruction to surgery. We show that the Borromean rings, and more generally all links with trivial linking numbers, are in fact weakly $A - B$ slice. This result shows the lack of a non-abelian extension of Alexander duality in dimension 4, and of an analogue of Milnor’s theory of link homotopy for general decompositions of the 4-ball.

1. Introduction

Surgery and the s-cobordism conjecture, central ingredients of the geometric classification theory of topological 4-manifolds, were established in the simply-connected case and more generally for elementary amenable groups by Freedman [1], [7]. Their validity has been extended to the groups of subexponential growth [8], [13]. A long-standing conjecture of Freedman [2] asserts that surgery fails in general, in particular for free fundamental groups. This is the central open question, since surgery for free groups would imply the general case, cf [7].

There is a reformulation of surgery in terms of the slicing problem for a special collection of links, the untwisted Whitehead doubles of the Borromean rings and of a certain family of their generalizations, see figure 2. (We work in the topological category, and a link in $S^3 = \partial D^4$ is slice if its components bound disjoint embedded locally flat disks in $D^4$.) An “undoubling” construction [3] allows one to work with a more robust link, the Borromean rings, but the slicing condition is replaced in this formulation by a more general $A-B$ slice problem. Freedman’s conjecture pinpoints the failure of surgery in a specific example and states that the Borromean rings are not $A - B$ slice. This approach to surgery has been particularly attractive since it is amenable to the tools of link-homotopy theory and nilpotent invariants of links, and partial obstructions are known in restricted cases, cf [6], [10], [11]. At the same time

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it is an equivalent reformulation of the surgery conjecture, and if surgery holds there must exist specific $A - B$ decompositions solving the problem.

The $A - B$ slice conjecture is a problem at the intersection of 4-manifold topology and Milnor’s theory of link homotopy [14]. It concerns codimension zero decompositions of the 4-ball. Here a decomposition of $D^4$, $D^4 = A \cup B$, is an extension of the standard genus one Heegaard decomposition of $\partial D^4 = S^3$. Each part $A, B$ of a decomposition has an attaching circle (a distinguished curve in the boundary: $\alpha \subset \partial A, \beta \subset \partial B$) which is the core of the solid torus forming the Heegaard decomposition of $\partial D^4$. The two curves $\alpha, \beta$ form the Hopf link in $S^3$.

![Diagram of a 2-dimensional example of a decomposition](image)

**Figure 1.** A 2-dimensional example of a decomposition $(A, \alpha), (B, \beta)$: $D^2 = A \cup B$, $A$ is shaded; $(\alpha, \beta)$ are linked 0-spheres in $\partial D^2$.

Figure 1 is a schematic illustration of a decomposition: an example drawn in two dimensions. While the topology of decompositions in dimension 2 is quite simple, they illustrate important basic properties. In this dimension the attaching regions $\alpha, \beta$ are 0-spheres, and $(\alpha, \beta)$ form a “Hopf link” (two linked 0-spheres) in $\partial D^2$. Alexander duality implies that exactly one of the two possibilities holds: either $\alpha$ vanishes as a rational homology class in $A$, or $\beta$ does in $B$. In dimension 2, this means that either $\alpha$ bounds an arc in $A$, as in the example in figure 1, or $\beta$ bounds an arc in $B$. (See figure 12 in section 5 for additional examples in 2 dimensions.)

Algebraic and geometric properties of the two parts $A, B$ of a decomposition of $D^4$ are tightly correlated. The geometric implication of Alexander duality in dimension 4 is that either (an integer multiple of) $\alpha$ bounds an orientable surface in $A$ or a multiple of $\beta$ bounds a surface in $B$. Alexander duality does not hold for homotopy groups, and this difference between being trivial homologically (bounding a surface) as opposed to homotopically (bounding a disk) is an algebraic reason for the complexity of decompositions of $D^4$.

A geometric refinement of Alexander duality is given by handle structures: under a mild condition on the handle decompositions which can be assumed without loss of
general, there is a one-to-one correspondence between 1— handles of each side and 2— handles of its complement. In general the interplay between the topologies of the two sides is rather subtle. Decompositions of $D^4$ are considered in more detail in sections 2 and 4 of this paper.

We now turn to the main subject of the paper, the $A - B$ slice reformulation of the surgery conjecture. An $n$— component link $L$ in $S^3$ is $A - B$ slice if there exist $n$ decompositions $(A_i, B_i)$ of $D^4$ and disjoint embeddings of all $2n$ manifolds $A_1, B_1, \ldots, A_n, B_n$ into $D^4$ so that the attaching curves $\alpha_1, \ldots, \alpha_n$ form the link $L$ and the curves $\beta_1, \ldots, \beta_n$ form an untwisted parallel copy of $L$. Moreover, the re-embeddings of $A_i, B_i$ are required to be standard—topologically equivalent to the ones coming from the original decompositions of $D^4$. The connection of the $A - B$ slice problem for the Borromean rings to the surgery conjecture is provided by considering the universal cover of a hypothetical solution to a canonical surgery problem [3], [4]. The action of the free group by covering transformations is precisely encoded by the fact that the re-embeddings of $A_i, B_i$ are standard. A formal definition and a more detailed discussion of the $A - B$ slice problem are given in section 2. The following is the statement of Freedman’s conjecture [2], [4] concerning the failure of surgery.

**Figure 2.** The Borromean rings and their untwisted Whitehead double.

**Conjecture 1.** The untwisted Whitehead double of the Borromean rings (figure 2) is not a freely slice link. Equivalently, the Borromean rings are not $A - B$ slice.

Here a link is freely slice if it is slice, and in addition the fundamental group of the slice complement in the 4—ball is freely generated by meridians to the components of the link. An affirmative solution to this conjecture would exhibit the failure of surgery, since surgery predicts the existence of the free-slice complement of the link above.

A stronger version of Freedman’s conjecture, that the Borromean rings are not even weakly $A - B$ slice, has been the main focus in search for an obstruction to surgery. Here a link $L$ is weakly $A - B$ slice if the re-embeddings of $A_i, B_i$ are required
to be disjoint but not necessarily standard in the definition above. To understand
the context of this conjecture, consider the simplest example of a decomposition
\( D^4 = A \cup B \) where \((A, \alpha)\) is the 2–handle \((D^2 \times D^2, \partial D^2 \times \{0\})\) and \(B\) is just
the collar on its attaching curve \(\beta\). This decomposition is trivial in the sense that
all topology is contained in one side, \(A\). It is easy to see that a link \(L\) is weakly
\(A-B\) slice with this particular choice of a decomposition if an only if \(L\) is slice.
The Borromean rings is not a slice link (cf [14]), so it is not weakly \(A-B\) slice with
the trivial decomposition. However to find an obstruction to surgery, one needs to
find an obstruction for the Borromean rings to be weakly \(A-B\) slice for all possible
decompositions.

Freedman’s program in the \(A-B\) slice approach to surgery could be roughly sum-
marized as follows. First consider model decompositions, defined using Alexander
duality and introduced in [6] (see also section 4). The main step is then to show that
any decomposition is algebraically approximated, in some sense, by the models – in
this case a suitable algebraic analogue of the partial obstruction for model decom-
positions should give rise to an obstruction to surgery. The first step, formulating
an obstruction for model decompositions, was carried out in [11], [12]. We now state
the main result of this paper which shows that the second step is substantially more
subtle than previously thought, involving not just the submanifolds but also their
embedding information.

**Theorem 1.** Let \(L\) be the Borromean rings or more generally any link is \(S^3\) with
trivial linking numbers. Then \(L\) is weakly \(A-B\) slice.

The linking numbers provide an obstruction to being weakly \(A-B\) slice (see section
3), so in fact Theorem 1 asserts that a link is weakly \(A-B\) slice if and only if it has
trivial linking numbers.

To formulate the main ingredient in the proof of this result in the geometric context
of link homotopy, it is convenient to introduce the notion of a robust \(4-\)manifold.
Recall that a link \(L\) in \(S^3\) is homotopically trivial if its components bound disjoint
maps of disks in \(D^4\). \(L\) is called homotopically essential otherwise. (The Borromean
rings is a homotopically essential link [14] with trivial linking numbers.) Let \((M, \gamma)\)
be a pair \((4-\)manifold, attaching curve in \(\partial M)\). The pair \((M, \gamma)\) is robust if whenever
several copies \((M_i, \gamma_i)\) are properly disjointly embedded in \((D^4, S^3)\), the link formed
by the curves \(\{\gamma_i\}\) in \(S^3\) is homotopically trivial. The following question relates this
notion to the \(A-B\) slice problem: Given a decomposition \((A, \alpha), (B, \beta)\) of \(D^4\), is one
of the two pairs \((A, \alpha), (B, \beta)\) necessarily robust? The answer has been affirmative
for all previously known examples, including the model decompositions [11], [12]. In
contrast, we prove

**Lemma 2.** There exist decompositions \(D^4 = A \cup B\) where neither of the two sides
\(A, B\) is robust.
This result suggests an intriguing possibility that there are 4−manifolds which are not robust, but which admit \textit{robust embeddings} into $D^4$. (The definition of a robust embedding $e: (M, \gamma) \hookrightarrow (D^4, S^3)$ is analogous to the definition of a robust pair above, with the additional requirement that each of the embeddings $(M_i, \gamma_i) \subset (D^4, S^3)$ is equivalent to $e$.) Then the question relevant for the surgery conjecture is: \textit{given a decomposition $D^4 = A \cup B$, is one of the given embeddings $A \hookrightarrow D^4$, $B \hookrightarrow D^4$ necessarily robust?}

Theorem 1 has a consequence in the context of \textit{topological arbiters}, introduced in [5]. Roughly speaking, it points out a substantial difference in the structure of the invariants of submanifolds of $D^4$, depending on whether they are endowed with a specific embedding or not. We refer the reader to that paper for the details on this application.

Section 2 reviews the background material on surgery and the $A - B$ slice problem. The $A - B$ slice problem for two-component links is considered in section 3; it is shown that Alexander duality provides an obstruction for links with non-trivial linking numbers. The proof of theorem 1 starts in section 4 with a construction of the relevant decompositions of $D^4$. The final section completes the proof of the theorem.

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\section{4−dimensional surgery and the the $A - B$ slice problem}

The surgery conjecture asserts that given a 4−dimensional Poincaré pair $(X, N)$, the sequence

\[ S^h_{\text{TOP}}(X, N) \longrightarrow N_{\text{TOP}}(X, N) \longrightarrow L^h_4(\pi_1 X) \]

is exact (cf [FQ], Chapter 11). This result, as well as the 5−dimensional topological s-cobordism theorem, is known to hold for a class of \textit{good} fundamental groups. The simply-connected case followed from Freedman’s disk embedding theorem \cite{1} allowing one to represent hyperbolic pairs in $\pi_2(M^4)$ by embedded spheres. Currently the class of good groups is known to include the groups of subexponential growth \cite{8}, \cite{13} and it is closed under extensions and direct limits. There is a specific conjecture for the failure of surgery for free groups \cite{2}:

\textbf{Conjecture 2.1.} There does not exist a topological 4−manifold $M$, homotopy equivalent to $\vee^3 S^1$ and with $\partial M$ homeomorphic to $S^0(Wh(Bor))$, the zero-framed surgery on the Whitehead double of the Borromean rings.
This statement is seen to be equivalent to Conjecture III in the introduction by considering the complement in $D^4$ of the slices for $Wh(Bor)$. This is one of a collection of canonical surgery problems with free fundamental groups, and solving them is equivalent to the surgery theorem without restrictions on the fundamental group. The $A − B$ slice problem, introduced in [3], is a reformulation of the surgery conjecture, and it may be roughly summarized as follows. Assuming on the contrary that the manifold $M$ in the conjecture above exists, consider its universal cover $\tilde{M}$. It is shown in [3] that the end point compactification of $\tilde{M}$ is homeomorphic to the 4−ball. The group of covering transformations (the free group on three generators) acts on $D^4$ with a prescribed action on the boundary, and roughly speaking the $A − B$ slice problem is a program for finding an obstruction to the existence of such actions. To state a precise definition, consider decompositions of the 4−ball:

**Definition 2.2.** A decomposition of $D^4$ is a pair of compact codimension zero submanifolds with boundary $A, B \subset D^4$, satisfying conditions (1)−(3) below. Denote $\partial^+ A = \partial A \cap \partial D^4$, $\partial^+ B = \partial B \cap \partial D^4$, $\partial A = \partial^+ A \cup \partial^− A$, $\partial B = \partial^+ B \cup \partial^− B$.

1. $A \cup B = D^4$,
2. $A \cap B = \partial^− A = \partial^− B$,
3. $S^3 = \partial^+ A \cup \partial^+ B$ is the standard genus 1 Heegaard decomposition of $S^3$.

Recall the definition of an $A − B$ slice link [4], [6]:

**Definition 2.3.** Given an $n$−component link $L = (l_1, \ldots, l_n) \subset S^3$, let $D(L) = (l_1, l'_1, \ldots, l_n, l'_n)$ denote the $2n$−component link obtained by adding an untwisted parallel copy $l'$ to $L$. The link $L$ is an $A − B$ slice if there exist decompositions $(A_i, B_i), i = 1, \ldots, n$ of $D^4$ and self-homeomorphisms $\phi_i, \psi_i$ of $D^4, i = 1, \ldots, n$ such that all sets in the collection $\phi_1 A_1, \ldots, \phi_n A_n, \psi_1 B_1, \ldots, \psi_n B_n$ are disjoint and satisfy the boundary data: $\phi_i(\partial^+ A_i)$ is a tubular neighborhood of $l_i$ and $\psi_i(\partial^+ B_i)$ is a tubular neighborhood of $l'_i$, for each $i$.

The surgery conjecture is equivalent to the statement that the Borromean rings (and a family of their generalizations) are $A − B$ slice. The idea of the proof of one implication is sketched above; the converse is also true: if the generalized Borromean rings were $A − B$ slice, consider the complement of the entire collection $\phi_i(A_i), \psi_i(B_i)$. Gluing the boundary according to the homeomorphisms, one gets solutions to the canonical surgery problems (see the proof of theorem 2 in [3].)

The restrictions $\phi_i|_{A_i}, \psi_i|_{B_i}$ in the definition above provide disjoint embeddings into $D^4$ of the entire collection of $2n$ manifolds $\{A_i, B_i\}$. Moreover, these re-embeddings are standard: they are restrictions of self-homeomorphisms of $D^4$, so in particular the complement $D^4 \smallsetminus \phi_i(A_i)$ is homeomorphic to $B_i$, and $D^4 \smallsetminus \psi_i(B_i) \cong A_i$. This requirement that the re-embeddings are standard is removed in the following definition:
Definition 2.4. A link \( L = (l_1, \ldots, l_n) \) in \( S^3 \) is weakly \( A-B \) slice if there exist decompositions \(((A_1, \alpha_1), (B_1, \beta_1)), \ldots, ((A_n, \alpha_n), (B_n, \beta_n))\) of \( D^4 \) and disjoint embeddings of all manifolds \( A_i, B_i \) into \( D^4 \) so that the attaching curves \( \alpha_1, \ldots, \alpha_n \) form the link \( L \) and the curves \( \beta_1, \ldots, \beta_n \) form an untwisted parallel copy of \( L \).

3. Abelian versus non-abelian Alexander duality

This section uses Alexander duality to show that the vanishing of the linking numbers is a necessary condition in theorem \( \Box \). Specifically, we prove

Proposition 3.1. Let \( L \) be a link with a non-trivial linking number. Then \( L \) is not weakly \( A-B \) slice.

Proof. It suffices to consider 2-component links, since any sub-link of a weakly \( A-B \) slice link is also weakly \( A-B \) slice. Let \( L = (l_1, l_2) \) with \( \text{lk}(l_1, l_2) \neq 0 \), and consider any two decompositions \( D^4 = A_1 \cup B_1 = A_2 \cup B_2 \).

Consider the long exact sequences of the pairs \((A_i, \partial^+ A_i), (B_i, \partial^+ B_i)\), where the homology groups are taken with rational coefficients:

\[
0 \rightarrow H_2 A_i \rightarrow H_2 (A_i, \partial^+ A_i) \rightarrow H_1 \partial^+ A_i \rightarrow H_1 A_i \rightarrow H_1 (A_i, \partial^+ A_i) \rightarrow 0
\]

\[
0 \rightarrow H_2 B_i \rightarrow H_2 (B_i, \partial^+ B_i) \rightarrow H_1 \partial^+ B_i \rightarrow H_1 B_i \rightarrow H_1 (B_i, \partial^+ B_i) \rightarrow 0
\]

Recall that \( \partial^+ A_i, \partial^+ B_i \) are solid tori (regular neighborhoods of the attaching curves \( \alpha_i, \beta_i \)). The claim is that for each \( i \), the attaching curve on exactly one side vanishes in its first rational homology group. Both of them can’t vanish simultaneously, since the linking number is 1. Suppose neither of them vanishes. Then the boundary map in each sequence above is trivial, and \( \text{rk} H_2 (A_i) = \text{rk} H_2 (A_i, \partial^+ A_i) \). On the other hand, by Alexander duality \( \text{rk} H_2 (A_i) = \text{rk} H_1 (B_i, \partial^+ B_i), \text{rk} H_2 (A_i, \partial^+ A_i) = \text{rk} H_1 (B_i) \). This is a contradiction, since \( H_1 \partial^+ B_i \cong \mathbb{Q} \) is in the kernel of \( H_1 B_i \rightarrow H_1 (B_i, \partial^+ B_i) \).

Now to show that the link \( L = (l_1, l_2) \) is not weakly \( A-B \) slice, set \((C_i, \gamma_i) = (A_i, \alpha_i)\) if \( \alpha_i = 0 \in H_1 (A_i; \mathbb{Q}) \) or \((C_i, \gamma_i) = (B_i, \beta_i)\) otherwise. If \( L \) were weakly \( A-B \) slice, there would exist disjoint embeddings \((C_1, \gamma_1) \subset (D^4, S^3), (C_2, \gamma_2) \subset (D^4, S^3)\) so that \( \gamma_1 \) is either \( l_1 \) or its parallel copy, and \( \gamma_2 \) is \( l_2 \) or its parallel copy. Then \( \text{lk}(\gamma_1, \gamma_2) \neq 0 \), a contradiction. \( \Box \)

Proposition 3.1 should be contrasted with theorem \( \Box \). Milnor’s link-homotopy invariant of the Borromean rings, \( \overline{\pi}_{123} (\text{Bor}) \), equals 1 \( \Box \). \( \overline{\pi}_{123} \), defined using the quotient \( \pi_1 / (\pi_1)^3 \) of the fundamental group by the third term of the lower central series, is a non-abelian analogue of the linking number of a link. Our result, theorem \( \Box \) shows the lack of a non-abelian extension of Alexander duality in dimension 4.
This section starts the proof of theorem 1 by constructing the relevant decompositions of $D^4$. The simplest decomposition $D^4 = A \cup B$ where $A$ is the $2$–handle $D^2 \times D^2$ and $B$ is just the collar on its attaching curve, was discussed in the introduction. Now consider the genus one surface $S$ with a single boundary component $\alpha$, and set $A_0 = S \times D^2$. Moreover, one has to specify its embedding into $D^4$ to determine the complementary side, $B$. Consider the standard embedding (take an embedding of the surface in $S^3$, push it into the $4$–ball and take a regular neighborhood.) Note that given any decomposition, by Alexander duality the attaching curve of exactly one of the two sides vanishes in it homologically, at least rationally. Therefore the decomposition under consideration now may be viewed as the first level of an “algebraic approximation” to an arbitrary decomposition.

![Diagram](image3.png)

**Figure 3.**

**Proposition 4.1.** Let $A_0 = S \times D^2$, where $S$ is the genus one surface with a single boundary component $\alpha$. Consider the standard embedding $(A_0, \alpha \times \{0\}) \subset (D^4, S^3)$. Then the complement $B_0$ is obtained from the collar on its attaching curve, $S^1 \times D^2 \times I$, by attaching a pair of zero-framed $2$–handles to the Bing double of the core of the solid torus $S^1 \times D^2 \times \{1\}$, figure 3.

The proof is a standard exercise in Kirby calculus, see for example [6]. A precise description of these $4$–manifolds is given in terms of Kirby diagrams in figure 4. Rather than considering handle diagrams in the $3$–sphere, it is convenient to draw them in the solid torus, so the $4$–manifolds are obtained from $S^1 \times D^2 \times I$ by attaching the $1$– and $2$–handles as shown in the diagrams. To make sense of the “zero framing” of curves which are not null-homologous in the solid torus, recall that the solid torus is embedded into $S^3 = \partial D^4$ as the attaching region of a $4$–manifold, and the $2$–handle framings are defined using this embedding.

This example illustrates the general principle that (in all examples considered in this paper) the $1$–handles of each side are in one-to-one correspondence with the
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2−handles of the complement. This is true since the embeddings in $D^4$ considered here are all standard, and in particular each 2−handle is unknotted in $D^4$. The statement follows from the fact that 1−handles may be viewed as standard 2−handles removed from a collar, a standard technique in Kirby calculus (see Chapter 1 in [9].) Moreover, in each of our examples the attaching curve $\alpha$ on the $A$−side bounds a surface in $A$, so it has a zero framed 2−handle attached to the core of the solid torus. On the 3−manifold level, the zero surgery on this core transforms the solid torus corresponding to $A$ into the solid torus corresponding to $B$. The Kirby diagram for $B$ is obtained by taking the diagram for $A$, performing the surgery as above, and replacing all zeroes with dots, and conversely all dots with zeroes. (Note that the 2−handles in all our examples are zero-framed.)

Note that a distinguished pair of curves $\alpha_1, \alpha_2$, forming a symplectic basis in the surface $S$, is determined as the meridians (linking circles) to the cores of the 2−handles $H_1, H_2$ of $B_0$ in $D^4$. In other words, $\alpha_1, \alpha_2$ are fibers of the circle normal bundles over the cores of $H_1, H_2$ in $D^4$.

An important observation [6] is that this construction may be iterated: consider the 2−handle $H_1$ in place of the original 4−ball. The pair of curves ($\alpha_1$, the attaching
circle $\beta_1$ of $H_1$) form the Hopf link in the boundary of $H_1$. In $H_1$ consider the standard genus one surfaces bounded by $\beta_1$. As discussed above, its complement is given by two zero-framed $2$–handles attached to the Bing double of $\alpha_1$. Assembling this data, consider the new decomposition $D^4 = A_1 \cup B_1$, figures 5, 6. As above, the diagrams are drawn in solid tori (complements in $S^3$ of unknotted circles drawn dashed in the figures.) The handlebodies $A_1, B_1$ are examples of model decompositions [6] obtained by iterated applications of the construction above. It is shown in [11], [12] that such model handlebodies are robust, or in other words the Borromean rings are not weakly $A - B$ slice when restricted to the class of model decompositions.

We are now in a position to define the decomposition $D^4 = A \cup B$ used in the proof of theorem [1]

**Definition 4.2.** Consider $B = (B_1 \cup$ zero-framed $2$–handle) attached as shown in the Kirby diagram in figure 7. The effect of this $2$–handle on the complement $A = D^4 \setminus B$ is shown in figure 8: it adds a $1$–handle to the diagram of $A_1$. Figure 9 shows a handle diagram of $A$ after a handle slide. Note that a $(1-, 2-)$ handle pair
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may be canceled, the result is given on the left in figure 12. This fact will be used in
the proof of theorem 1.

Figure 8.

Figure 9. A handle diagram for $A$ after a handle slide.

Imprecisely (up to homotopy, on the level of spines) $B$ may be viewed as $B_1 \cup 2$–cell
attached along (the attaching circle $\beta$ of $B_1$, followed by a curve representing a
generator of $H_1$ of the second stage surface of $B_1$). This 2–cell is schematically
shown in the spine picture of $B$ in the first part of figure 7 as a cylinder connecting
the two curves. The shading indicates that the new generator of $\pi_1$ created by
adding the cylinder is filled in with a disk. Similarly, one checks that the effect of
this operation on the $A$–side is that one of the 2–handles at the second stage is
connected-summed with the first stage surface, figure 8. (This is seen in the handle
diagram by canceling a 1–, 2–handle pair, as shown in figure 12.) Again, the shading
indicates that no new generators of $\pi_1$ are created. The figures showing the spines
are provided only as a motivation for the construction; a precise description of $A, B$
is of course given by their handle diagrams. While the proof of theorem 1 below is
given in terms of Kirby diagrams, it can easily be followed at the level of spines.
5. Proof of theorem I: a relative slice problem

We start this section by recalling the technique which will be useful in completing the proof of theorem I, the relative slice problem, introduced in [6]. The setup in our context is as follows: suppose two codimension zero submanifolds $M, N$ of $D^4$ are given; each one has an attaching circle $\gamma \subset \partial M, \delta \subset \partial N$. The submanifolds are proper in the sense that one has embeddings of pairs $(M, \gamma) \subset (D^4, S^3), (N, \delta) \subset (D^4, S^3)$, where each circle $\gamma, \delta$ is unknotted in the $3-$sphere.

The problem that has to be analyzed is: can $(M, \gamma), (N, \delta)$ be embedded disjointly into $(D^4, S^3)$ so that the curves $\gamma, \delta$ form the Hopf link in the $3-$sphere? Assume that $M, N$ have handle decompositions, relative to the attaching regions $S^1 \times D^2$, with only 1- and 2- handles. Let $\gamma, \delta$ form the Hopf link in $\partial D^4$, and consider the 4-ball $D' = D^4 \setminus (\text{collar on } \partial D^4)$. To be precise, denote the 1-handles of $M, N$ by $H_1, H_1'$, and their 2-handles by $H_2, H_2'$. As usual, we view the 1-handles of $M, N$ as standard slices removed from their collars. Denote these slices by $H_1^s, H_2^s$. Then $M, N$ embed disjointly into $D^4$ if and only if there are disjoint embeddings of the 2-handles $H_2 \cup H_2'$, attached to the collars, in the handlebody $D' \cup H_1 \cup H_1'$.

![Figure 10.](image)

An example of $M, N$ drawn in two dimensions is given in figure 10, and a solution to this relative-slice problem – disjoint embeddings of $M, N$ in $D^4$ with their attaching circles $\gamma, \delta$ forming a Hopf link in $\partial D^4$ – is shown in figure 11. Note that the handle $H_2$ of $M$ in the solution goes over the “helping” handle $H_1^s$ attached to $D'$.

Consider the decomposition $D^4 = A \cup B$ constructed in definition 4.2. The proof of theorem I follows from lemmas 5.1 and 5.2 below.

**Lemma 5.1.** Let $S$ denote the genus one surface with one boundary component, $\gamma = \partial S$. Denote by $S_0$ its untwisted 4-dimensional thickening, $S_0 = S \times D^2$, and set $\gamma_0 = \gamma \times \{0\}$. Then there exists a proper embedding $(A, \alpha) \subset (S_0, \gamma_0)$.
Proof. Kirby diagrams of $A$ are given in figures 8, 9. Observe that a $(1-, 2-$ handle) pair in the diagram in figure 9 may be canceled, the result is shown on the left in figure 12.

In light of proposition 4.1, to prove that $A$ embeds in $S_0$ it suffices to show that $(A, \alpha)$ embeds in the complement of a standard embedding of two zero-framed $2-$ handles attached to the Bing double of a meridian to $\alpha$ in $S^3$. This is an instance of the relative-slice problem discussed above, where $(M, \gamma) = (A, \alpha)$ and $N$ is obtained from a collar on $\delta$ by attaching $2-$ handles to the Bing double of the core. (Note $(N, \delta)$ equals $(B_0, \beta)$ considered in sections 4, see figures 3, 4.) This relative-slice problem is shown on the right in figure 12. The link is considered in the $3-$ sphere boundary of the $4-$ ball $D'$, and the link $l_1, \ldots, l_4$ has to be sliced in the handlebody $D' \cup (2-$ handles $H'_1)$ where the handles are attached with zero framings along $r_1, r_2$. Here $l_1, l_2$ are the attaching curves for the $2-$ handles of $N$ and $l_3, l_4$ are the attaching curves for the $2-$ handles of $M$. Note that the slices for $l_1, l_2$ constructed in the
proof are required to be standard in $D^4$, to make sure that their complement is the thickened surface $S_0$.

A solution to this relative-slice problem is given in figures 13, 14. The slices are described in terms of the Morse function given by the radial coordinate in the 4-ball $D'$. Denote the $3-$sphere at the radius $R$ from the origin by $S^3_R$, $0 < R \leq 1$. The link on the right in figure 12 lies in $\partial D' = S^3_1$. The link components move by an isotopy for $1 > R > 3/4$, and at $R = 3/4$ the component $l_4$ is connected-summed with a parallel copy of $r_2$. The result is denoted by $l'_4$, figure 13. Note that $l'_4$ bounds a disk in $S^3_{3/4}$ in the complement of all other curves. To make the slice non-degenerate in terms of the Morse function, let $l'_4$ bound a disk as $R$ decreases from $3/4$ to $1/2$, while all other curves move by an isotopy. The link in $S^3_{1/2}$ is shown on the right in figure 13.

![Figure 13.](image)

The curves $r_i \subset \partial D'$ bound disjoint embedded disks $\Delta_i$: the cores of the zero-framed 2-handles $H^*_1$ attached to $D'$. As the Morse function $R$ changes from 1 to 0, it is important to note that the curves $r_i$ move by an isotopy and no other curves intersect them. Therefore, $r_1, r_2$ at each radius $R_0$ bound disjoint disks: the disks $\Delta_i$ as above, union with the annuli corresponding to the isotopy of $r_i$ for $1 > R > R_0$. Moreover, since the handles $H^*_1$ attached to $D'$ are zero-framed, untwisted parallel copies of $r_i$ also bound disjoint embedded disks.

Morse-theoretically the connected sum at $R = 3/4$ in figure 13 corresponds to a saddle point of the slice for $l_4$. This slice is of the form shown in figure 15 (disregard the labels in that figure, which are used for a later argument.)

To finish the proof of the relative-slice problem, let the link in $S^3_{1/2}$ move by an isotopy for $1/2 > R > 1/4$, and at $R = 1/4$ the components $l_1, l_2$ are connected-summed with $r_1, r_2$ as shown on the left in figure 14. Denote the resulting curves by $l'_1, l'_2$. The components $l'_1, l'_2, l_3$ form the unlink. This is seen by performing an isotopy (at $1/4 > R > 1/8$) to the link on the right in figure 14. Now let all curves bound disks
at $1/8 > R > 0$. The slices for $l_1, l_2$ again have the form shown in figure 15; the slice for $l_3$ has just a single critical point.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure14}
\caption{Figure 14.}
\end{figure}

This concludes the proof of the relative-slice problem. It remains to show that the slices $S_1, S_2$ for $l_1, l_2$ constructed above are standard. We start by recording the data involved in their construction. $(\gamma, \delta)$ is a Hopf link in $\partial D^4$, $D' = D \setminus (\text{collar on } \partial D^4)$. Since the slices were described in terms of the radial Morse function on $D'$, to be specific consider $D'$ as the 4-ball of radius 1 in $D^4$ of radius 2. The curves $l_1, l_2$ are in the boundary of $D'$; extending them by the product $l_i \times [1, 2]$ we will consider them as curves in $\partial D^4$.

For the rest of this argument, we only need to consider the curves $l_1, l_2$ and their slices; the slices for the other components are disregarded. $l_1, l_2$ form the unlink and therefore bound disjoint embedded disks $D_1, D_2$ in $\partial D^4$. We will show that the slices $S_1, S_2$ for $l_1, l_2$ are standard by constructing disjoint embedded 3-balls $B_1, B_2$ in $D^4$, with $\partial B_i = D_i \cup S_i$ for each $i$. The existence of these 3-balls provides an isotopy in $D^4$ from $S_1, S_2$ to $D_1, D_2$ and shows that the slices are standard.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure15}
\caption{Figure 15. The 3-ball $B_i$.}
\end{figure}

The construction of $S_i$ is illustrated in figure 15. The vertical axis in this figure corresponds to the radial component in $D^4$. There is a single maximum point given
by the core $\Delta_i$ of the 2-handle attached to $D'$ along $r_i$. Recall that $\Delta_1$, $\Delta_2$ are embedded in $D^4$ in a standard way, and so they are isotopic to disjoint embedded disks $D'_1$, $D'_2$ bounded by $r_1$, $r_2$ in the 3-sphere slice $\partial D' = S^3_1$. The curves $l_i$, $r_i$, and the disks bounded by them: $D_i$, $D'_i$ move by an isotopy as $R$ decreases from 1 to $1/4$ until the index 1 critical points of the slices at $R = 1/4$ (shown in figure 14).

The analysis of the disks at the level $R = 1/4$ is presented in figure 16. At the level of these critical points, the disks $D_i$ and $D'_i$ are band-summed, and the result: the disks $D''_1$, $D''_2$ are disjoint. The component $l'_1$ on the left in figure 16 bounds $D''_1$, the component $l'_2$ on the right bounds $D''_2$. (Figure 16 has two copies of the link $(l'_1, l'_2)$ just for convenience of visualization of the disks $D''_1$, $D''_2$.) Finally, at $R < 1/4$ the disks $D''_i$ move by an isotopy and shrink to points.

We summarize the construction of the disjoint 3-balls $B_i$, $i = 1, 2$: in the 3-sphere $S^3_R$, each component of $S^3_R \cap S_i$ bounds a disk: $l_i = \partial D_i$, $r_i = \partial D'_i$, $l'_i = \partial D''_i$. These disks are the levels of the radial Morse function restricted to $B_i$. This concludes the proof that $(A, \alpha)$ embeds into $(S_0, \gamma_0)$. \hfill $\square$

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**Figure 16.** Disjoint disks $D''_1, D''_2$: on the left $D''_1$ is a band sum of $D_1, D'_1$, on the right $D''_2$ is a band sum of $D_2, D'_2$.

**Figure 17.**
Lemma 5.2. \( B \) embeds in a collar on its attaching curve. More precisely, there exists a proper embedding \( (B, \beta) \subset (S^1 \times D^2 \times [0,1], S^1 \times \{0\} \times \{0\}) \).

Figure 18.

One needs to show that \( (B, \beta) \) embeds in the complement of a standard disk bounded by the meridian to \( \beta \). The proof is again a relative-slice problem, shown in figure 17. Here \( l_1 \) is the meridian which is required to bound a standard disk; \( l_2, l_3, l_4 \) are the attaching curves of the 2−handles of \( B \), and \( r_1, r_2 \) are the attaching curves for the 2−handles attached to \( D' \). Therefore the link \( l_1, \ldots, l_4 \) has to be sliced in \( D' \cup r_1, r_2 \) (zero-framed 2−handles), so that the slice for \( l_1 \) is standard in \( D^4 \).

Taking a connected sum of \( l_1 \) and \( r_1 \) as shown in figure 17, one gets the link on the left in figure 18. Now taking a connected sum of \( l_2 \) and \( r_2 \) results in the trivial link, and the components are capped off with disjoint disks in \( D' \). The proof that the slice for \( l_1 \) is standard is directly analogous to the corresponding proof in lemma 5.1. \( \square \)

Figure 19.

Proof of theorem 1 in the central case \( L = \text{Bor} \), the Borromean rings, follows from lemmas 5.1, 5.2. The components \( l_i \) of \( \text{Bor} \) bound in \( D^4 \) disjoint embedded surfaces \( S_i \): \( S_1 \) is a genus one surface, and \( S_2, S_3 \) are disks. Thinking of the radial coordinate
of $D^4$ as time where $\partial D^4$ corresponds to time 0, $l_1$ bounds a surface $S_1$ (shaded in figure 19) at time $1/2$ and the other two components bound disjoint disks at $t > 1/2$. Consider three decompositions of $D^4$: $(A_1, B_1)$ is the decomposition constructed in section 4. Define $(A_2, B_2)$ and $(A_3, B_3)$ to be the trivial decomposition: $A_2 = A_3 = D^2 \times D^2$, $B_2 = B_3$ are collars on their attaching curves. Lemmas 5.1, 5.2 imply that the Borromean rings are weakly $A - B$ slice with these decompositions.

To prove theorem 1 for all links with trivial linking numbers, a variation of lemmas 5.1, 5.2 is needed, for higher genus surfaces. That is, given any $g$ there is a decomposition $D^4 = A_g \cup B_g$ such that $A_g$ embeds in (surface of genus $g$) $\times D^2$ and $B_g$ embeds in a collar. These are variations of the decompositions $A, B$ in definition 4.2; the case $g = 2$ is shown in figure 20. The proof is analogous to the proof of lemmas 5.1, 5.2. To complete the proof of theorem 1 note that the components of any link with trivial linking numbers bound disjoint embedded surfaces in $D^4$. □

Figure 20.

Remark. If the embeddings $(A, \alpha) \subset (S_0, \gamma_0) \subset (D^4, S^3)$ and $(B, \beta) \subset (M, \gamma) \subset (D^4, S^3)$, constructed in lemmas 5.1, 5.2 were standard, then taking the complement of the six submanifolds (three copies of each of $A$ and $B$) bounding the Borromean rings and their parallel copy in $D^4$ and gluing up the boundary one would get a solution to a canonical surgery problem. Considering the generalized Borromean rings, one would get solutions to all canonical problems, and therefore a proof of the topological 4−dimensional surgery conjecture for all fundamental groups. However the embeddings constructed in the proof are not standard. This raises the question mentioned in the introduction: Given a decomposition $D^4 = A \cup B$, is one of the embeddings $A \hookrightarrow D^4$, $B \hookrightarrow D^4$ necessarily robust?
6. A decomposition of the 4—sphere

The two parts $A, B$ of any decomposition of $D^4$ have inherently different properties. For example, due to Alexander duality the attaching circle $\alpha, \beta$ on exactly one side vanishes in its rational first homology group. However it turns out that when completed to a decomposition of the 4—sphere, the construction in the proof of theorem 1 is quite symmetric: the decomposition $D^4 = A \cup B$ extends to $S^4 = B \cup B$. We record this observation here since it seems likely that this symmetry plays a role in the properties of the decomposition used above.

Consider the decomposition $D^4 = A \cup B$ constructed in definition 4.2, see figures 7, 8. The attaching curves $\alpha, \beta$ form the Hopf link in $\partial D^4$. Consider $S^4 = D^4 \cup (\text{another copy } D' \text{ of the } 4-\text{ball})$, and let $\alpha$ bound the standard disk in $D'$. In terms of the handlebodies, we attach a zero-framed $2-$handle (a thickening of this disk) to $A$ along $\alpha \times D^2$. Denote the result by $A_1$. Considering the handle diagram for $A$ in figures 8, 12, one sees that the Kirby diagram for $A_1$ is given in the first part of figure 21.

Note that the complement of $A_1$ in $S^4$ is homeomorphic to $B$, since the complement to the disk bounded by $\alpha$ in $D'$ is just a collar on $\beta$. The handle diagram for $B$ is
presented in the second part of figure 21. (The dashed circle in figure 7 has to be replaced by a circle with the dot, since the diagrams in figure 21 are drawn in $S^3$, not in the solid torus. Then a $1-, 2-$handle pair canceled, and one gets the diagram in figure 21.) The proof is concluded by the observation that this is a symmetric link: both diagrams in figure 21 are isotopic to the one in figure 22. □

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