Hall Conductivity in a Spin-Triplet Superconductor

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We calculate the Hall conductivity for a spin-triplet superconductor, using a generalized pairing symmetry dependent on an arbitrary phase, $\varphi$. A promising candidate for such an order parameter is Sr$_2$RuO$_4$, whose superconducting order parameter symmetry is still subject to investigation. The value of this phase can be determined through Kerr rotation and DC Hall conductivity measurements. Our calculations impose significant constraints on $\varphi$.

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Superconducting strontium ruthenate (Sr$_2$RuO$_4$) [1] is remarkable for a variety of reasons: it is a layered compound without copper, and, while its transition temperature is relatively low, the symmetry of the superconducting order parameter is most certainly non-conventional [2]. It is believed [3] that Sr$_2$RuO$_4$ is a spin-triplet superconductor [4]; thus, the orbital part of the Cooper pair should have the odd parity [5]. Moreover, muon spin-relaxation measurements indicate that the superconducting state of Sr$_2$RuO$_4$ breaks time-reversal symmetry [6]. The diffraction patterns of Josephson junctions made from Sr$_2$RuO$_4$ also illustrate this phenomenon [7]. In the interpretation of these experimental observations, a spin-triplet superconductor with a $(k_x \pm i k_y)$-wave gap symmetry has been used [2]. On the other hand, characteristics of the penetration depth [8] in Sr$_2$RuO$_4$ are not consistent with a pure nodeless $p$-wave gap. Furthermore, ultrasound attenuation measurements [9] in the superconducting state of Sr$_2$RuO$_4$ exclude the possibility of a nodeless $p$-wave gap, while they seem to imply a possible fourfold gap modulation. In this sense, an $f$-wave gap has been also proposed [10].

Recently, Xia et al. [11] have observed a Kerr rotation developing in Sr$_2$RuO$_4$ as the temperature is lowered below $T_c = 1.5K$. They tried to understand this observation based on a theoretical analysis [12] using a nodeless $p$-wave gap. However, their theoretical estimate gives a Kerr angle of the order of $10^{-3}$ nanorad, while the measured value is as big as 65 nanorad. Since the Kerr angle is proportional to the imaginary part of the Hall conductivity [11], Yakovenko [13] derived a Chern-Simons like term in the action associated with the Hall conductivity, and estimated a Kerr angle of about 230 nanorad. Depending on $\beta_2$ and $\beta_3$, we see what values of $\varphi$ would be possible. For example, if $\beta_2 > 0$ and $4\beta_2 > \beta_3$, $\varphi = \pm \pi/2$. When, however, $\beta_2 = 0$ and $\beta_3 < 0$, one can consider an arbitrary value of the relative phase $\varphi$.

In this letter we propose a generalized $p$-wave (we also consider $f$-wave) gap with a relative phase ($\varphi$) between the momenta along the $x$ and $y$ directions; namely $\Delta_x = \Delta_0(\hat{c}_x + e^{i\varphi}\hat{c}_y)$ for the $p$-wave gap and $\Delta_k = \Delta_0(\hat{c}_x + e^{i\varphi}\hat{c}_y)(\hat{c}_{k_x}^2 - \hat{c}_{k_y}^2)$ for the $f$-wave gap. We derive an expression for the Hall conductivity and show that the Kerr angle is indeed proportional to $\Delta_0$ as experimentally observed. As in the phenomenological model used by Xia et al., our derivation reveals that impurity scattering plays an important role in the problem. The actual value of $\varphi$ we use can be identified by comparison with experimental results for the Kerr angle. The DC Hall conductivity at zero temperature is also computed because it is less sensitive to impurity scattering but demonstrates a strong $\varphi$ dependence. Consequently, the DC Hall conductivity would be another ideal experiment to determine $\varphi$. We also discuss the chirality [14] and the density of states (DOS) for these gaps. The DOS of the $p$-wave gap provides a mapping of $\varphi$ onto a tiny gap [10] associated with the shape of the Fermi surface in Sr$_2$RuO$_4$.

We start with the current operator $j$

$$j = \frac{1}{2e} \sum_k v_k \hat{\psi}_k \hat{\psi}_k$$  (1)

where $\hat{\psi}_k = \left( C_{k\downarrow} \ C_{k\uparrow} \ C_{-k\downarrow} \ C_{-k\uparrow} \right)$. Following the standard formalism we obtain the current-current correlation
\[ \Pi(i\Omega) \text{ in the Matsubara representation as follows:} \]
\[ \Pi(i\Omega) = \frac{1}{4\pi^2} \sum_{k} v_k v_k^* T \sum_{\omega} \text{Tr} \left[ G_k(i\omega + i\Omega) G_k^*(i\omega) \right] \quad (2) \]
For a spin-triplet superconductor, it is necessary to introduce the \((4 \times 4)\) Green function \(G(k, i\omega)\)
\[ G = \begin{pmatrix} \hat{G} & -\hat{F} \\ -\hat{F}^* & -\hat{G}^* \end{pmatrix} \quad (3) \]
with
\[ \hat{G}(k, i\omega) = -\frac{i\omega + \xi_k}{\omega^2 + E_k^2} \hat{I} \]
\[ \hat{F}(k, i\omega) = \frac{\hat{\Delta}_k}{\omega^2 + E_k^2} \]
where \(\xi_k = k^2/2m - \epsilon_f, \hat{\Delta}_k = i(d(k) \cdot \hat{\sigma})\hat{\sigma}_y, \hat{I} \) is the \((2 \times 2)\) unit matrix, and \(E_k = \sqrt{\xi_k^2 + \text{tr}[\hat{\Delta}_k \hat{\Delta}^*_k]}/2\). Since \(\hat{\Delta}_k \hat{\Delta}^*_k = |d(k)|^2 \hat{1} + i(d(k) \times d^*(k)) \cdot \hat{\sigma}\), depending on \(d(k) \times d^*(k)\), the pairing state is called unitary if \(d(k) \times d^*(k) = 0\); otherwise it is non-unitary. It is commonly assumed that the unitary state is relevant to \(\text{Sr}_2\text{RuO}_4\) and \(d(k) = \Delta_k \hat{z}^*\): \[
\hat{\Delta}_k = \begin{pmatrix} 0 \\ \Delta_k \\ 0 \end{pmatrix} \quad (4)
\]
For this state, the net spin average of a Cooper pair \(\text{tr}[\hat{\Delta}_k \hat{\Delta}^*_k] = 0\), and \(E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}\). It is also possible to represent the \(d\)-wave gap in the \((4 \times 4)\) matrix formalism as follows: \(\hat{\Delta}_k = i\Delta_k \hat{\sigma}_y\) with \(\hat{\Delta}_k = \Delta_0(k_x^2 - k_y^2)\).

Defining \(G(k, i\omega) = \hat{G}_{11}\) and \(F(k, i\omega) = \hat{F}_{12}\), the \(xy\) component of the current-current correlation becomes, at the bare bubble level,
\[ \Pi_{xy}(i\Omega) = \epsilon^2 \sum_k v_x v_y T \sum_{\omega} \left[ G(k, i\omega + i\Omega) G(k, i\omega) + F(k, i\omega + i\Omega) F^*(k, i\omega) \right]. \quad (5) \]

Using the symmetry of \(\Pi_{xy}(i\Omega)\), one can see \(\Pi_{xy}(i\Omega) = 0\) for a pure nodeless \(p\)-wave gap. A similar analysis has been done for order parameters with various symmetries. The Hall conductivity follows readily from this expression: \(\sigma_{xy}(\Omega) = \frac{\pi^2}{2} \Pi_{xy,ret}(\Omega) \equiv \sigma_{xy}'(\Omega) + i\sigma_{xy}''(\Omega)\). Introducing the spectral functions \(\mathcal{A}(k, \omega) = -2\text{Im}[G_{ret}(k, i\omega)]\) and \(\mathcal{B}(k, \omega) = -2\text{Im}[F_{ret}(k, i\omega)]\), one obtains
\[ \sigma_{xy}''(\Omega) = \frac{\epsilon^2}{\Omega^3} \sum_k v_x v_y \int \frac{d\omega' d\omega''}{(2\pi)^2} \frac{f(\omega'') - f(\omega')}{\omega'' - \omega' + \Omega} \times \left[ \mathcal{A}(k, \omega'), \mathcal{A}(k, \omega'') + \mathcal{B}(k, \omega') \mathcal{B}^*(k, \omega'') \right] \quad (6) \]
In the clean limit, the spectral functions are
\[ \mathcal{A}(k, \omega) = 2\pi |u_k|^2 \delta(\omega - E_k) + 2\pi |v_k|^2 \delta(\omega + E_k) \]
\[ \mathcal{B}(k, \omega) = 2\pi u_k v_k \left[ \delta(\omega + E_k) - \delta(\omega - E_k) \right], \quad (7) \]
where \(u_k = \sqrt{(1 + \xi_k/E_k)/2}\) and \(v_k = \sqrt{(1 - \xi_k/E_k)/2}\). In this instance, \(\sigma_{xy}''(\Omega)\) vanishes regardless of the gap symmetry because the \(AA\) term (quasiparticle contribution) is exactly canceled by the \(BB\) term (Cooper pair contribution). Nonetheless, impurity scattering prevents a complete cancellation. With impurity scattering rate \(\gamma\), the corresponding spectral functions can be approximated by \[ \mathcal{A}(k, \omega) = |u_k|^2 D(\omega - E_k) + |v_k|^2 D(\omega + E_k) \]
\[ \mathcal{B}(k, \omega) = u_k v_k [D(\omega + E_k) - D(\omega - E_k)], \quad (8) \]

\[ D(\omega \pm E_k) = \frac{2\gamma}{(\omega \pm E_k)^2 + \gamma^2}. \quad (9) \]

When the self-energy \(\Sigma\) due to impurity scattering is considered, \(\omega \rightarrow \omega = \omega - \Sigma\). Using the Born approximation for simplicity, one can readily evaluate the frequency integrals in Eq. (6). Taking the low temperature and high frequency limits, \(T \rightarrow 0\) (or \(T \ll \Delta_0\)), and \(\Omega \gg \Delta_0 \gg \gamma\), as in Ref. [11], we obtain
\[ \sigma_{xy}''(\Omega) \approx \frac{\epsilon^2}{2\pi^3} \sum_k v_x v_y \ln \left[ 1 + \frac{\Omega^4 - 2\Omega^2(E_k^2 - \gamma^2)}{(E_k^2 + \gamma^2)^2} \right]. \quad (10) \]
Changing the summation to an integration over \(k\), we arrive at the high frequency result,
\[ \sigma_{xy}''(\Omega) \approx \frac{\epsilon^2}{2\pi^2} v_f^2 N(0) \frac{\gamma_{\Delta_0}}{\Omega^3} I(\varphi), \quad (11) \]
where \(v_f\) is the Fermi velocity, \(N(0)\) the DOS of the normal state, and \(I(\varphi)\) is for the \(p\)-wave gap,
\[ I(\varphi) = \frac{4\cos(\varphi)}{\sin(\varphi)} \sqrt{1 - \frac{2}{\sin(\varphi)^2}} |E(\nu) - |\sin(\varphi)||K(\nu)|] \]
where \(\nu = (1 - |\sin(\varphi)|)/(1 + |\sin(\varphi)|)\), \(K\) and \(E\) are the complete elliptic integrals of the first and the second kind. The corresponding result for the \(f\)-wave gap is
\[ I(\varphi) = \frac{8\sqrt{2}}{15} \frac{1 + |\sin(\varphi)|}{1 + |\sin(\varphi)|^2} \left[ |\sin(\varphi/2)| - |\cos(\varphi/2)| \right]. \]
In Fig. 1, we plot $I(\varphi)$ for the $p$-wave and $f$-wave gap. Using a numerical integration of Eq. (10) with actual values of $\Omega$ and $\gamma$ taken from experiment [2, 11] gives results indistinguishable from those in Fig. 1. Note that $I(\varphi)$ changes its sign depending on $\varphi$ and vanishes at $\varphi = \pi/2$ and $3\pi/2$. From this plot, we estimate $\varphi \approx \pi/\sqrt{2} \approx 5\pi/6$, for which $I(\varphi) \approx 1$. These estimates will depend on the precision as clearly the $p$-wave gap gives greater values than the $f$-wave gap. Since $\psi^2 N(0) \sim \Omega$ from parameters in Ref.[2], we obtain

$$\sigma''_{xy}(\Omega) \approx \frac{e^2 \gamma \Delta_0}{\pi \Omega^2} \tag{12}$$

Note that our result is reduced from that in Ref.[13] by a factor of $\gamma/\Delta_0$. This revises the theoretical estimate for the Kerr angle to 10 ~ 80 nanorad for $\gamma/\Delta_0 \approx 0.05 ~ 0.4$. Its value congruous to the measured Kerr angle would be 0.15 ~ 0.35. Consequently, Eq. (12) correctly illustrates the observed linear dependence of the Kerr angle on the $0$-wave and $15$-wave case, respectively. It is understandable that the DC Hall conductivity as in Ref.[13]:

$$N(\omega) = \frac{\pi}{2} \text{Re} \left[ \frac{\omega \gamma}{\sqrt{\omega^2 - 2\Delta_\varphi^2}} K \left( \frac{2\Delta_\varphi^2 \cos(\varphi)}{\omega^2 - 2\Delta_\varphi^2} \right) \right] \tag{15}$$

where $\Delta_\varphi = \Delta_0 \sin(\varphi/2)$. Fig. 3 shows the DOS of the case with a $p$-wave gap. It is interesting that the peak does not occur at $\omega = \Delta_0$ except for $\varphi = \pi/2$, for which the DOS is $s$-wave-like. In fact, its location is $\omega/\Delta_0 = \sqrt{1 + \cos(\varphi)}$. As mentioned early, the DOS illustrates a tiny gap obtained in a different context[12]. When $\varphi \approx 5\pi/6$, the tiny (minimum) gap is about $0.3\Delta_0$ while the maximum gap (peak) about $1.3\Delta_0$. Note that this value of $\varphi$ also explains the Kerr angle measurement. It is not possible to express the DOS of the case with $f$-wave gap for general $\varphi$ in an analytic form. We plot the DOS in Fig. 4 for values of $\varphi = 3\pi/4, 4\pi/3, and \pi$. The location of the peak is not $\omega = \Delta_0$ either; for example, the peak is at $\omega/\Delta_0 = (4/3)\sqrt{2/3}$ for $\varphi = \pi$. As one can see, the DOS is more or less like the DOS of the $d$-wave gap. When $\varphi = \pi/2$, the DOS is exactly $d$-wave-like. However, for $\varphi = \pi$ the DOS is definitely not a linear function of $\omega$ at low frequency ($\omega \ll \Delta_0$). This is due to the quadratic behavior of the corresponding $f$-wave gap as a function of wave vector near the nodes. In addition, because of this, the nodal approximation breaks down for $\varphi \approx \pi$.

In conclusion, we have proposed a novel type of superconducting order parameter symmetry, with a relative phase $\varphi$ in the definition of the order parameter symmetry. We then showed that the spontaneous Hall conductivity can be reduced by rotational symmetry breaking as well as time reversal symmetry breaking. We have used the Kerr angle $\theta_K$ expression in terms of the Hall conductivity as in Ref.[13]: $\theta_K = (4\pi/\Omega) \text{Im} \sigma_{xy}/(n^3 - n)$, where $n$ is the complex index of refraction. It is apparent that the actual magnitude and phase of $n$ are important to determine the angle. Another difficulty for a proper theoretical understanding is the issue of the detailed experimental setup discussed recently in Ref.[12]. In fact, we think that the validity of the above expression for $\theta_K$ is still an open question for Sr$_2$RuO$_4$. Our simple analysis is an initial attempt to understand the recent Kerr angle experimental results[11]. Because of these complications, a measurement of the low-frequency Hall conductivity is most desirable — it will provide a definitive guideline for the theoretical modeling of Sr$_2$RuO$_4$. Our $p$-wave model is compatible with the small gap due to the salient shape of the Fermi surface.
FIG. 1: (Color online) $I(\phi)$ for the $p$-wave gap (solid curve) and the $f$-wave gap (dashed curve).

FIG. 2: (Color online) Normalized DC Hall conductivity as a function of $\phi$ at zero temperature for $p$-wave and $f$-wave superconductor with $\gamma/\Delta_0 = 0.05$.

FIG. 3: The density of states of the $p$-wave gap as a function of energy $\omega$.

FIG. 4: The density of states of the $f$-wave gap as a function of energy $\omega$.

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[1] Y. Maeno, H. Hashimoto, K. Yoshida, S. Nishizaki, T. Fujita, J. G. Bednorz, and F. Lichtenberg, Nature 372, 532 (1994).
[2] A. Mackenzie and Y. Maeno, Rev. Mod. Phys. 75, 657 (2003).
[3] K. Ishida, H. Mukuda, Y. Kitaoka, K. Asayama, Z. Q. Mao, Y. Mori, and Y. Maeno, Nature 396, 658 (1998).
[4] M. Sigrist and K. Ueda, Rev. Mod. Phys. 63, 239 (1991).
[5] K. D. Nelson, Z. Q. Mao, Y. Maeno, and Y. Liu, Science 306, 1151 (2004).
[6] G. M. Luke, Y. Fudamoto, K. M. Kojima, M. I. Larkin, J. Merrin, B. Nachumi, Y. J. Uemura, Y. Maeno, and Z. Q. Mao, Nature 394, 558 (1998).
[7] F. Kidwingira, J. D. Strand, D. J. van Harlingen, and Y. Maeno, Science 314, 1267 (2006).
[8] I. Bonalde, Brian D. Yanoff, M. B. Salamon, D. J. Van
Harlingen, E. M. E. Chia, Z. Q. Mao and Y. Maeno, Phys. Rev. Lett. 85, 4775 (2000).

[9] C. Lupien, W. A. MacFarlane, C. Proust, L. Taillefer, Z. Q. Mao, and Y. Maeno, Phys. Rev. Lett. 86, 5986 (2001).

[10] M. J. Graf and A. V. Balatsky, Phys. Rev. B 62, 9697 (2000).

[11] J. Xia, Y. Maeno, P. T. Beyersdorf, M. M. Fejer, and A. Kapitulnik, Phys. Rev. Lett. 97, 167002 (2006).

[12] S. K. Yip and J. A. Sauls, J. Low Temp. Phys. 86, 257 (1992).

[13] V. Yakovenko, Phys. Rev. Lett. 98, 087003 (2007).

[14] G. E. Volovik, JETP Lett. 66, 522 (1997).

[15] A. Furusaki, M. Matsumoto, and M. Sigrist, Phys. Rev. B 64, 054514 (2001).

[16] K. Miyake and O. Narikiyo, Phys. Rev. Lett. 83, 1423 (1999).

[17] Q. P. Li and R. Joynt, Phys. Rev. B 44, 4720 (1991).

[18] J.R. Schrieffer, In: Theory of Superconductivity (Benjamin/Cummings, Don Mills, 1964), p. 122. The more correct form of the Green function in the superconducting state with impurity scattering can be found in F. Mar-siglio and J.P. Carbotte, Aust. J. Phys. 50, 1011 (1997); see also cond-mat/9709242.

[19] R.M. Lutchyn, P. Nagornykh, and V.M. Yakovenko, cond-mat/08014175.