A PARTIAL PROOF OF A CONJECTURE OF DRIS

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Abstract. Euler showed that if an odd perfect number $N$ exists, it must consist of two parts $N = q^kn^2$, with $q$ prime, $q \equiv k \equiv 1 \pmod{4}$, and $\gcd(q, n) = 1$. Dris conjectured in [5] that $qk < n$. We first show that $q < n$ for all odd perfect numbers. Afterwards, we show $qk < n$ holds in many cases.

1. Introduction

We define $\sigma(N)$ to be the sum of the positive divisors of $N$ and note the following properties of $\sigma$, which we will use freely:

1. $\sigma(p^b) = 1 + p + p^2 + \ldots + p^b$ for powers of primes.
2. $\sigma(ab) = \sigma(a)\sigma(b)$, whenever $\gcd(a, b) = 1$.
3. $(\frac{p-1}{p})\sigma(p^{2b}) < p^{2b}$. (Note that $\left(\frac{2}{3}\right)\sigma(p^{2b}) < p^{2b}$ works for any odd prime $p$.)

We say $N$ is perfect when $\sigma(N) = 2N$. Euler showed that if an odd perfect number $N$ exists, then its factorization consists of two parts. A special prime $q$ appearing an odd number, say $k$ times, such that $q \equiv k \equiv 1 \pmod{4}$. The rest of the primes in the factorization appear an even number of times, which we represent as $n^2$. It is understood that $\gcd(q, n) = 1$. When written as $N = q^kn^2$ we say $N$ is written in Eulerian form. The condition $\gcd(q, n) = 1$, implies $n \neq q$. Thus, it is interesting to determine conditions requiring and consequences of $n < q$ and $q < n$.

A well known result of Nielsen [8] states that $N$ must consist of at least 9 different odd primes, i.e. $n$ must have at least 8 unique factors. At first glance, it would seem reasonable to guess that $q < n$. However, a quick consideration of Descartes famous “spoof” odd perfect number:

$$N = 3^27^211^213^222021$$

where if one pretends for a moment that 22021 is prime, and that $\sigma(22021) = 22022$, then $\sigma(N) = 2N$. For this example, $q = 22021$ and $q > n$. So it seems a plausible question to ask that if an odd perfect number exists, is it necessary that the special prime dominate the rest of the factors?

Our initial intuitions turn out to be correct. Dris proved in [6] that $k > 1 \implies q < n$. Acquaah and Konyagin [1] later showed $k = 1 \implies q < (3N)^{1/3}$ from which it is immediate that $q < \sqrt{3n}$. (Their proof having been modified from Luca and Pomerance [7].) In [3], Dagal and Dris, utilize Acquaah and Konyagin’s results to show $q < n$ so long as $3 \nmid N$. In section 2, we utilize Neilsen’s result to make a simple adjustment to Acquaah and Konyagin’s argument to conclude $k = 1 \implies q < n$, which allows us to conclude $q < n$ (and mildly stronger results) for all odd perfect numbers.

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Dris conjectured in [5] that $q^k < n$. In section 3, we endeavor to prove this conjecture adjusting Acquaah and Konyagin’s argument even further. We start with a proof of the simplest case and show the argument can be massaged to conclude $k > 1 \implies q^k < n$. However, limitations in the method prevent a complete proof without additional assumptions in the second and third case.

2. Proof of $q < n$

**Theorem 1.** Let $N = q^k n^2$ be an odd perfect number written in Eulerian form, then $q < n$.

**Proof.** As mentioned above, the case $k > 1$ has been established, so we assume $k=1$. Rewrite $N$ in full as

$$N = q p^{2b} r_1^{2\beta_1} r_2^{2\beta_2} \cdots r_j^{2\beta_j}.$$ 

Where $p$ is the unique prime whereby $q \mid \sigma(p^{2b})$, $r_i$ for $1 \leq i \leq j$, represent the rest of the primes dividing $N$. When convenient, we will truncate $N$ as

$$N = q p^{2b} r_1^{2\beta_1} w^2$$

Let $c_i \geq 0$ be the integer whereby $p^{c_i} \mid \sigma(r_i^{2\beta_i})$ for $1 \leq i \leq j$. Where we give “$\mid$” its standard meaning that $p^{c_i} \mid \sigma(r_i^{2\beta_i})$, but $p^{c_i+1} \nmid \sigma(r_i^{2\beta_i})$. It is possible that $p^{c_i} = \sigma(r_i^{2\beta_i})$ for any particular $i$, but since we know $n$ has at least eight components, at least one of the $\sigma(r_i^{2\beta_i})$ has to have factors other than $p$. Thus, we may rewrite subscripts and assume:

$$p^{c_1} r_2 \mid \sigma(r_1^{2\beta_1})$$

**Case 1:** $p \nmid \sigma(q)$

$$2N = \sigma(N) = \sigma(q) \sigma(p^{2b}) \sigma(r_1^{2\beta_1}) \sigma(w^2)$$

Observe that $p \nmid \sigma(q)$ implies $p^{2b-c_1} \mid \sigma(w^2)$. Thus,

$$2N > (q + 1) q (p^{c_1} r_2) (p^{2b-c_1})$$

We now utilize the fact that $p^{2b} > \frac{2}{3} \sigma(p^{2b})$ and $r_2$ being an odd prime means $r_2 \geq 3$.

$$2N > q^2 \left(3 \frac{2}{3} \sigma(p^{2b})\right)$$

$$2N > q^2 \left(3 \frac{2}{3} q\right)$$

$$N > q^3$$

from which $q < n$ easily follows.

**Case 2:** $p \mid \sigma(q)$
Let $p^c \mid \sigma(q)$. Let $u = \sigma(p^{2b})/q$. Since

$$\sigma(p^{2b}) \equiv 1 \pmod{p}, \quad q \equiv -1 \pmod{p}$$

(2.6)

we know $u \equiv -1 \pmod{p}$. Since $u$ is odd, we know $u \neq p - 1$, and thus $u \geq 2p - 1$. By construction, we have $p^{2b-c_1} \mid \sigma(w^2)$, which implies

$$\sigma(w^2) \geq p^{2b-c_1}$$

(2.7)

Observe now,

$$p^{2b+1} - 1 = (p - 1)\sigma(p^{2b}) = (p - 1)uq = (p - 1)u\sigma(q) - (p - 1)u$$

(2.8)

Therefore, $(p - 1)u \equiv 1 \pmod{p^{c_1}}$. Which implies $(p - 1)u > p^{c_1}$.

Combining the last two inequalities yields,

$$\sigma(w^2)(p - 1)u > p^{2b-c_1} \iff \sigma(w^2)u > \frac{p^{2b-c_1}}{p - 1}$$

(2.9)

This should be all we need:

$$2N = \sigma(N) = \sigma(q)\sigma(p^{2b})\sigma(r_1^{2\beta_1})\sigma(w^2)$$

(2.10)

$$2N > (q + 1)uq(p^{c_1}r_2)\sigma(w^2)$$

(2.11)

$$2N > q^2\frac{p^{2b-c_1}}{p - 1}p^{c_1}r_2$$

(2.12)

$$2N > q^2r_2\frac{p^{2b}}{p - 1}$$

(2.13)

Again, we utilize $p^{2b} > \frac{2}{3}\sigma(p^{2b})$.

$$2N > q^2r_2\frac{2\sigma(p^{2b})}{3(p - 1)}$$

(2.14)

$$2N > q^2r_2\frac{2uq}{3(p - 1)}$$

(2.15)

Recall, $u \geq 2p - 1$ and again $r_2$ being an odd prime means $r_2 \geq 3$.

$$2N > q^33\left(\frac{2}{3}\right)\frac{2p - 1}{(p - 1)}$$

(2.16)

$$N > 2q^3$$

(2.17)

And again, we get $q < n$.

\[\square\]

There is nothing special about this method of proof and the result of $q < n$ compared to Acquaah and Kanyugin’s estimate $q < \sqrt{3}n$. If one is prepared to do the bookkeeping to account for extra unaccounted for factors $r_3$, $r_4$, $r_5$, etc., one can estimate them as $r_3 \geq 5$, $r_4 \geq 11$, $r_5 \geq 13$, etc. to get $q < \frac{n}{(\sqrt{5}+11+13+\cdots)}$. 
3. Partial Proof of Dris’s Conjecture

Assume \( N = q^k n^2 \) is an odd perfect number written in Eulerian form. The case \( k = 1 \) is proven in Theorem 1, so we assume \( k \geq 5 \). Our goal is to prove \( q^k < n \) with as few assumptions as possible. Rewrite \( N \) in full as

\[
N = q^k p_1^{2b_1} \cdots p_s^{2b_s} w^2
\]

(3.1)

Where the \( p_i \) are the primes whereby \( q^i \mid \sigma(p_i^{2b_i}) \), for integers \( t_i \geq 0 \) and \( 1 \leq i \leq s \). It is convenient to name another prime \( r \) separate from the \( p_i \)’s and allow \( w^2 \) to represent the rest of the primes dividing \( N \). We do not assume a priori that \( r \) and \( w \) necessarily exist and in such cases we simply take one or both to be one.

Let \( c_i \geq 0 \) be the integer whereby \( p_i^{c_i} \mid \sigma(p_i^{2b_i}) \) for \( 1 \leq i \leq s \).

**Case 1:** \( p_i \mid \sigma(q^k) \) for each \( i \)

Because \( k \) is odd, \( \sigma(q^k) = (1 + q)(1 + q^2 + q^4 \ldots + q^{k-1}) \). It is straightforward to show \( (1 + q^2 + q^4 \ldots + q^{k-1}) \) is coprime to its formal derivative, which makes the polynomial separable, and as such has no repeated roots modulo any prime. Thus any prime dividing \( (1 + q^2 + q^4 \ldots + q^{k-1}) \), divides at most once. Let \( r \) be a prime dividing \( \sigma(q^k) \) such that \( r \mid \sigma(q^k) \). By assumption \( r \) is not any of the \( p_i \)’s. Also note that we may assume \( r \geq 7 \). If \( r = 1 + q^2 + q^4 \ldots + q^{k-1} \), then clearly \( r > 7 \). Otherwise, we may assume \( r \equiv 1 \) (mod \( \frac{k+1}{2} \)), by virtue of the fact that \( r \) divides a cyclotomic polynomial. For the smallest exponent, \( k = 5 \), \( \frac{k+1}{2} = 3 \); and the smallest 1 (mod 3) prime is 7.

\[
2N = \sigma(N) = \{ \sigma(q^k) \} \{ \sigma(p_i^{2b_i}) \} \{ \sigma(r^{2\beta} w^2) \}
\]

(3.2)

\[
2N > \{ q^k \} \{ q^k p_1^{c_1} \cdots p_s^{c_s} \} \{ p_1^{2b_1 - c_1} \cdots p_s^{2b_s - c_s} \} \{ r \}
\]

(3.3)

Note that the quantity in each brace on the right hand side is less than or equal to the quantity in the respective brace in the previous line, with the exception of \( \{ r \} \). Since \( r \) divides \( N \) an even number of times, but can only divide \( \sigma(q^k) \) once, we know \( r \) must divide either \( \sigma(p_1^{2b_1} \cdots p_s^{2b_s}) \) or \( \sigma(r^{2\beta} w^2) \), in addition to our previous assumptions.

\[
2N > q^{2k}(r)p_1^{2b_1} \cdots p_s^{2b_s}
\]

(3.4)

We utilize the fact that \( (\frac{p-1}{p})\sigma(p^{2b}) < p^{2b} \) and that \( r \geq 7 \).

\[
2N > q^{2k}(7) \prod_{i=1}^{s} \left( \frac{p_i - 1}{p_i} \right) \sigma(p_i^{2b_i})
\]

(3.5)

\[
2N > q^{2k}(7) \prod_{i=1}^{s} \left( \frac{p_i - 1}{p_i} \right) q^k p_i^{c_1} \cdots p_s^{c_s}
\]

(3.6)

\[
2N > q^{3k}(7) \prod_{i=1}^{s} \left( 1 - \frac{1}{p_i} \right)
\]

(3.7)

Next we use the well known result, if \( 0 < \theta_i < 1 \) for \( i = 1, \ldots, s \), then

\[
\prod_{i=1}^{s} (1 - \theta_i) \geq 1 - \sum_{i=1}^{s} \theta_i.
\]
While not the first to prove $\sum_{p|N} \frac{1}{p} < \ln(2)$, Cohen gives a simple proof of this fact in ([2]).

$$2N > q^{3k} \left(1 - \sum_{i=1}^{s} \frac{1}{p_i}\right) \quad (3.8)$$

Since $7(1 - \ln(2)) > 2.14$, we have

$$q^{3k} < N = q^k n^2 \implies q^k < n \quad (3.9)$$

as required.

**Case 2: $s = 1$ and $p_1|\sigma(q^k)$**

In Section 2, case 2, the result relied on being able to find two inequalities

$u \geq 2p - 1$ and $(p - 1)u \geq p^q$. The former depending on $p$ being unique and the latter depending on $k = 1$, which made $\sigma(q^k) = q + 1$. To give a full proof of Dris’s conjecture using this methodology, these two obstacles will have to be overcome.

In this case, with $s = 1$, we get $p_1$ is unique.

We proceed as before, let $c_{1q} \geq 0$ be the integer for which $p_1^{-c_{1q}} \equiv \sigma(p_1^{2b_1}) \mod p_1$. We may again conclude $u \equiv -1 \pmod{p_1}$ and $u \geq 2p_1 - 1$, however,

$$p_1^{2b_1} - 1 = (p_1 - 1)\sigma(p_1^{2b_1}) = (p_1 - 1)u \sigma(q^k) - (p_1 - 1)u \sigma(q^{k-1}) \quad (3.11)$$

allows us to, at best, conclude $(p_1 - 1)u \sigma(q^{k-1}) \equiv 1 \pmod{p_1^{c_{1q}}}$; which seems to be unhelpful.

We push on, let $v = \frac{\sigma(w^2)}{p_1^{c_{1q}}}$

$$2N = \sigma(N) = \sigma(q^k)\sigma(p_1^{2b_1})\sigma(w^2) \quad (3.12)$$

$$2N > q^k u q^k v p_1^{2b_1 - c_{1q}} \quad (3.13)$$

$$2N > q^{2k} u v p_1^{2b_1 - c_{1q}} \quad (3.14)$$

$$2N > q^{2k} u v p_1^{-1} \sigma(p_1^{2b_1}) \quad (3.15)$$

$$2N > q^{2k} u v p_1^{-1} u q^k p_1^{-c_{1q}} \quad (3.16)$$

$$2N > q^{2k} u v p_1^{-1} p_1^{-c_{1q}} \quad (3.17)$$

$$2N > q^{3k} (2p_1 - 1)^2 v p_1^{-1} p_1^{-c_{1q}} \quad (3.18)$$

$$N > q^{3k} (2p_1^2 - 4p_1 + \frac{5}{2} - \frac{1}{2p_1}) v p_1^{-c_{1q}} \quad (3.19)$$

We see Dris’s conjecture follows immediately whenever $c_{1q} \leq 2$ or, with more knowledge about $N$, when $vp_1^2 > p_1^{c_{1q}}$. By Neilsen’s result, $w$ must have at least 7 components, which makes the latter inequality seem quite likely. Since these are
amongst the first theorems relating components of an odd perfect number, more research is clearly needed.

**Case 3: s > 1 and \( p_i | \sigma(q^k) \) for at least one \( i \)**

Let \( c_{iq} \geq 0 \) be the integer for which \( p_i^{c_{iq}} \| \sigma(q^k) \) for \( 1 \leq i \leq s \). We begin as before,

\[
2N = \sigma(N) = \sigma(q^k)\sigma(p_1^{2b_1} \ldots p_s^{2b_s})\sigma(w^2)
\]

(3.20)

\[
2N > q^k q^k p_1^{c_{1q}} \ldots p_s^{c_{sq}} p_1^{2b_1-c_{1q}} \ldots p_s^{2b_s-c_{sq}}
\]

(3.21)

\[
2N > q^{2k} p_1^{2b_1} \ldots p_s^{2b_s} p_1^{-c_{1q}} \ldots p_s^{-c_{sq}}
\]

(3.22)

\[
2N > q^{2k} \prod_{i=1}^{s} \left( \frac{p_i-1}{p_i} \right) \sigma(p_1^{2b_1} \ldots p_s^{2b_s}) p_1^{-c_{1q}} \ldots p_s^{-c_{sq}}
\]

(3.23)

\[
2N > \prod_{i=1}^{s} \left( \frac{p_i-1}{p_i} \right) q^k p_1^{c_{1q}} \ldots p_s^{c_{sq}} p_1^{-c_{1q}} \ldots p_s^{-c_{sq}}
\]

(3.24)

\[
2N > q^{3k} \prod_{i=1}^{s} \left( \frac{p_i-1}{p_i} \right) q p_1^{c_{1q}} \ldots p_s^{c_{sq}}
\]

(3.25)

Using \( \prod_{i=1}^{s} \left( \frac{p_i-1}{p_i} \right) > 1 - \ln(2) \), we see now the result \( q^k < n \) follows whenever

\[
p_1^{c_{1q}} \ldots p_s^{c_{sq}} > \frac{2}{1 - \ln(2)} p_1^{c_{1q}} \ldots p_s^{c_{sq}}
\]

(3.26)

We recap our results thus far in the following.

**Theorem 2.** Let \( N = q^k n^2 \) be an odd perfect number written in Eulerian form, then \( q < n \).

Write \( N = q^k p_1^{2b_1} \ldots p_s^{2b_s} w^2 \), where \( q | \sigma(p_i) \) for \( 1 \leq i \leq s \). Let \( c_i, c_{iq} \geq 0 \) be integers where \( p_i^{c_{iq}} \| \sigma(p_1^{2b_1} \ldots p_s^{2b_s}) \) and \( p_i^{c_{iq}} \| \sigma(q^k) \) for \( 1 \leq i \leq s \).

If \( k > 1 \) and one of the following holds:

1. \( p_i \nmid \sigma(q^k) \) for \( 1 \leq i \leq s \);
2. \( s = 1, p_1 | \sigma(q^k) \), and \( c_{1q} \leq 2 \);
3. \( s > 1, \) and \( p_i | \sigma(q^k) \) for at least one \( p_i \) and

\[
p_1^{c_{1q}} \ldots p_s^{c_{sq}} > 7p_1^{c_{1q}} \ldots p_s^{c_{sq}}
\]

then \( q^k < n \).

4. FURTHER CONSIDERATIONS

The condition \( p_1^{c_{1q}} \ldots p_s^{c_{sq}} \geq 7p_1^{c_{1q}} \ldots p_s^{c_{sq}} \) in Theorem 2 seems to suggest Dris’s conjecture holds if

\[
\prod_{i \neq j} \gcd(\sigma(p_i^{2b_i} p_j^{2b_j}), p_i^{2b_i} p_j^{2b_j}) > \prod_{i=1}^{s} \gcd(\sigma(q^k), p_i^{2b_i}).
\]

Again, at first glance, it seems nothing can be said about this situation, but Dandapat, Hunsucker, and Pomerance in [4], Theorem 2 implies for each fixed \( i \), there is a \( j \) where

\[
\gcd(\sigma(p_i^{2b_i} p_j^{2b_j}), p_i^{2b_i} p_j^{2b_j}) > 1 \text{ for } i \neq j
\]
holds for most non-special components of \( n \), not just for the restricted \( p_i \)’s as we have defined them.

The next obvious question may be for \( N = q^kp^{2b}w^2 \), an odd perfect number where \( q \) is the special prime, \( p \) is any other prime dividing \( N \), and \( w^2 \) encompasses the rest of the components of \( N \), in the same way we showed \( q < n \), can we show \( p < w \), \( p^b < w \), or even \( p^{2b} < w \)?

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