QUASI-FUCHSIAN VS NEGATIVE CURVATURE METRICS ON SURFACE GROUPS

BY

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To Benjy Weiss with gratitude and admiration

ABSTRACT
We compare two families of left-invariant metrics on a surface group \( \Gamma = \pi_1(\Sigma) \) in the context of course-geometry. One family comes from Riemannian metrics of negative curvature on the surface \( \Sigma \), and another from quasi-Fuchsian representations of \( \Gamma \). We show that the Teichmüller space \( \mathcal{T}(\Sigma) \) is the only common part of these two families, even when viewed from the coarse-geometric perspective.

1. Introduction and statement of the main result

1.A. INTRODUCTION AND BACKGROUND. Let \( \Sigma \) be a closed surface of genus at least two, and \( \Gamma = \pi_1(\Sigma) \) its fundamental group. The Teichmüller space \( \mathcal{T}(\Sigma) \) has several equivalent descriptions: as the moduli space of (i) complex structures, or (ii) conformal structures, or (iii) Riemannian structures of constant curvature \(-1\) on \( \Sigma \), or as (iv) the space of discrete cocompact representations \( \Gamma \to \text{PSL}_2(\mathbb{R}) \), up to conjugation. The latter two points of view can be extended as follows:

- \( \mathcal{R}(\Sigma) \)—the space of all Riemannian structures of possibly variable negative curvature, up to isotopy and scaling.
- \( \mathcal{QF}(\Sigma) \)—the space of all convex cocompact representations

\[
\Gamma = \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3),
\]

up to conjugation.
Both $\mathcal{R}(\Sigma)$ and $\mathcal{QF}(\Sigma)$ arise from convex cocompact isometric $\Gamma$-actions on CAT(-1) spaces: the $\Gamma$-action by deck transformations on the universal cover $(\tilde{\Sigma}, d_{\tilde{\gamma}})$ in the Riemannian case, and the $\Gamma$-action on $H^3$ in the quasi-Fuchsian case.

We can put these notions into an even broader context by looking at the space $\mathcal{D}_{\Gamma}$ of equivalence classes $[d]$ of left-invariant metrics $d$ on $\Gamma$ obtained from restricting the metric of the underlying Gromov-hyperbolic space to a $\Gamma$-orbit. Here two metrics $d, d'$ on $\Gamma$ are equivalent if they are bounded distance from each other after scaling:

$$d \sim d' \text{ if } \exists k, A : \left| d'(\gamma_1, \gamma_2) - k \cdot d(\gamma_1, \gamma_2) \right| \leq A.$$ 

This perspective, introduced by the second author in [11] (see also more recent treatment in Bader–Furman [1]), allows to observe possible “geometries” of $\Sigma$ from the “outside” by studying the corresponding classes $[d] \in \mathcal{D}_{\Gamma}$ of metrics $d$ on $\Gamma$. The space $\mathcal{D}_{\Gamma}$ can be defined for a general non-elementary Gromov hyperbolic group $\Gamma$, and $\mathcal{D}_{\Gamma}$ contains classes of metrics on $\Gamma$ from various sources, such as word metrics on $\Gamma$, Green metrics associated with symmetric generating random walks on $\Gamma$ (see Blachère–Haïssinsky–Mathieu [3, 4]), Anosov representations of $\Gamma$ in higher rank simple Lie groups (see Dey–Kapovich [10]), etc.

To avoid ambiguity in scaling we can normalize metrics $d$ by the growth

$$h_d = \lim_{R \to \infty} \frac{1}{R} \log \# \{ \gamma \in \Gamma \mid d(\gamma, e) < R \},$$

replacing $d$ by $\hat{d} = h_d \cdot d$, so that $h_{\hat{d}} = 1$. For $\delta \in \mathcal{D}_{\Gamma}$ we can define:

- **Marked Length Spectrum** $\ell_{\delta} : \Gamma \to \mathbb{R}_+$ given by the limit

$$\ell_{\delta}(\gamma) = \lim_{n \to \infty} \frac{\hat{d}(\gamma^n, e)}{n}$$

where $\delta = [d]$ and $\hat{d} = h_d \cdot d$. Note that $\ell_{\delta}$ is constant on conjugacy classes, so we can write it as $\ell_{\delta} : C_{\Gamma} \to \mathbb{R}_+$.

- **Patterson–Sullivan-like $\Gamma$-invariant measure class** $[\nu^PS_{\delta}]$ on $\partial \Gamma$ (see Coorneart [7], and [11, 1]).

- **Bowen–Margulis–Sullivan-like $\Gamma$-invariant Radon measure** $m^BMS_{\delta}$ on the space $\partial^{(2)} \Gamma$ of distinct pairs $(\xi, \eta)$ of points on $\partial \Gamma$ (see [11, 1]).

In [11] (see also Bader–Furman [1]), it was shown that each $\delta \in \mathcal{D}_{\Gamma}$ is determined by each of these objects. Furthermore, extending a prior work of Bader–Muchnik [2], Garncarek [12] showed that for each $\delta \in \mathcal{D}_{\Gamma}$ the quasi-regular
unitary $\Gamma$-representation

$$\pi_\delta : \Gamma \longrightarrow U(\partial \Gamma, \nu_\delta^{PS})$$

is irreducible, and that the map $\mathcal{D}_\Gamma \longrightarrow \hat{\Gamma}$, $\delta \mapsto \pi_\delta$, is also injective. Thus $\mathcal{D}_\Gamma$ can be embedded into any one of the following spaces:

$$\mathbb{R}_+^{C_\Gamma}, \text{ Prob}(\partial \Gamma), \text{ Meas}(\partial(\mathcal{L})^2 \Gamma), \hat{\Gamma}.$$ 

The space $\mathcal{D}_\Gamma$ is also equipped with a natural metric: given two classes $\delta = [d], \delta' = [d']$ in $\mathcal{D}_\Gamma$ we can define the (log) Lipschitz distance by

$$\rho_{\text{Lip}}(\delta, \delta') : = \log \left( \inf \left\{ \frac{K}{k} | \exists A, k \cdot d - A \leq d' \leq K \cdot d + A \right\} \right).$$

It is clear from the definition that $\rho_{\text{Lip}}(-,-)$ is symmetric and satisfies the triangle inequality. One can see that for any $a, b \in \Gamma \setminus \{e\}$ one has

$$\left| \log \left( \frac{\ell_\delta(a)}{\ell_\delta(b)} : \ell_{\delta'}(a) : \ell_{\delta'}(b) \right) \right| \leq \rho_{\text{Lip}}(\delta, \delta').$$

This shows that $\rho_{\text{Lip}}(\delta, \delta') = 0$ implies $\ell_\delta = \ell_{\delta'}$, which occurs only when $\delta = \delta'$. So $\rho_{\text{Lip}}(.)$ is indeed a metric on $\mathcal{D}_\Gamma$ (see also a recent work of Cantrell–Tanaka [6] for a more detailed picture).

1.B. RIEMANNIAN AND QUASI-FUCHSIAN STRUCTURES ON SURFACES. In this paper we focus on surface group $\Gamma = \pi_1(\Sigma)$ and two specific sources for $\delta \in \mathcal{D}_\Gamma$: namely $\mathcal{R}(\Sigma)$ and $\mathcal{QF}(\Sigma)$.

For the case of negatively curved Riemannian metric $g$ on $\Sigma$, fix $x \in \tilde{\Sigma}$ and consider the metric on $\Gamma$

$$d_{g,x}(\gamma_1, \gamma_2) : = d_{\delta}(\gamma_1 x, \gamma_2 x).$$

Since $|d_{g,x} - d_{g,x'}| \leq d(x, x')$ the class $[d_{g,x}]$ does not depend on the choice of $x \in \tilde{\Sigma}$, and we can denote this class by $\delta_g = [d_{g,x}]$. Note that $h_{d_{g,x}}$ is the topological entropy of the geodesic flow on the unit tangent bundle $T^1 \Sigma$ to $\Sigma$, and we assume that all $g \in \mathcal{R}(\Sigma)$ are normalized so that $h_{d_{g,x}} = 1$. We have a map

$$i : \mathcal{R}(\Sigma) \longrightarrow \mathcal{D}_\Gamma.$$ 

The Marked Length Spectrum Rigidity Conjecture, that for surfaces was proved by Otal [14] and Croke [8], asserts that a Riemannian structure $g$ of variable negative curvature on a surface $\Sigma$ is uniquely determined by the function $\ell_g : \mathcal{C}_\Gamma \rightarrow \mathbb{R}$. As a consequence, we obtain:
Proposition 1.1: The map $R(\Sigma) \to D_\Gamma$, $i: g \mapsto \delta_g$, is injective.

Our second source of examples are quasi-Fuchsian representations. For $q \in QF(\Sigma)$ choose a representation $\pi: \Gamma \to \text{Isom}^+(H^3) \cong \text{PSL}_2(\mathbb{C})$ in this class and a point $y \in H^3$ and consider the metric on $\Gamma$:

$$d_{\pi,y}(\gamma_1, \gamma_2) := d_{H^3}(\pi(\gamma_1).y, \pi(\gamma_2).y).$$

The class $[d_{\pi,y}]$ does not depend on the choice of $y \in H^3$ and remains unchanged if $\pi$ is replaced by a conjugate $\gamma \mapsto g\pi(g)^{-1}$; thus we write $\delta_q$ for $[d_{\pi,y}]$. This gives a well defined map

$$j: QF(\Sigma) \to D_\Gamma.$$

One can deduce from a work of Burger [5] (or Dal’bo–Kim [9]) the following.

Proposition 1.2: The map $QF(\Sigma) \to D_\Gamma$, $j: q \mapsto \delta_q$, is injective.

Hence one might view each of $R(\Sigma)$ and $QF(\Sigma)$ as being embedded in $D_\Gamma$.

Remark 1.3: We note in passing that the uniformization theorem allows us to view $R(\Sigma)$ as a bundle over $\mathcal{I}(\Sigma)$ with fibers that can be identified with the positive cone $C^+_\infty(\Sigma)/R_+$; in particular $R(\Sigma)$ is connected. One can show that the map (1.1) is continuous, and so the image $i(R(\Sigma))$ in $D_\Gamma$ is connected.

Ahlfors and Bers showed that $QF(\Sigma)$ can be identified with $\mathcal{I}(\Sigma) \times \mathcal{I}(\Sigma)$, and is in particular connected. The map (1.2) can be shown to be continuous; hence the image $j(QF(\Sigma))$ is a connected subset of $D_\Gamma$.

It is natural to wonder whether the intersection

$$i(R(\Sigma)) \cap j(QF(\Sigma)) \subset D_\Gamma$$

contains anything except for the image of $\mathcal{I}(\Sigma)$. In other words, is it true that given a quasi-Fuchsian representation $\pi: \Gamma \to \text{PSL}_2(\mathbb{C})$ and a negatively curved metric $g$ on the surface $\Sigma$, there exist constants $k, A$ and points $x \in \tilde{\Sigma}$, $y \in H^3$, so that

$$k \cdot d_\gamma(x,x) - A \leq d_{H^3}(\pi(\gamma).y, y) \leq k \cdot d_\gamma(x,x) + A \quad (\gamma \in \Gamma)$$

only if $g$ has constant curvature, $\pi$ is conjugate to a subgroup of $\text{PSL}_2(\mathbb{R})$, and $(\Sigma, g)$ and $\pi$ represent the same point in $\mathcal{I}(\Sigma)$?

Our main result answers this affirmatively.
Theorem A: The images of $\mathcal{R}(\Sigma)$ and $\mathcal{DF}(\Sigma)$ in $\mathcal{D}$ have only $\mathcal{J}(\Sigma)$ in common. Moreover, for any $q \in \mathcal{DF}(\Sigma) \setminus \mathcal{J}(\Sigma)$ there is $\alpha_q > 0$ so that

$$\rho_{\text{Lip}}(\delta_q, \delta_g) \geq \alpha_q > 0$$

for all $g \in \mathcal{R}(\Sigma)$.

The following natural question remains open.

Question 1.4: Is it true that for any $g \in \mathcal{R}(\Sigma) \setminus \mathcal{J}(\Sigma)$ there is $\beta_g > 0$ so that

$$\rho_{\text{Lip}}(\delta_q, \delta_g) \geq \beta_g > 0$$

for all $q \in \mathcal{DF}(\Sigma)$?

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2. Length inequalities for negatively curved surfaces

Consider the topological picture first. Let $\Sigma$ be a closed surface of genus at least two, $\Gamma = \pi_1(\Sigma)$ the corresponding surface group, that acts on the universal cover $\tilde{\Sigma}$ by deck transformations. This action extends to the action of $\Gamma$ on the boundary circle $\partial \tilde{\Sigma}$, which is also the Gromov boundary $\partial \Gamma$ of $\Gamma$. Every $\gamma \neq 1$ in $\Gamma$ has two fixed points on the topological circle $\partial \tilde{\Sigma}$: a repelling point $\gamma^-$ and an attracting point $\gamma^+$. We shall consider a pair $a, b \in \Gamma$ where $a^-, a^+, b^-, b^+$ are four distinct points on the circle.

Let $A = (\alpha_1, \alpha_2)$ and $B = (\beta_1, \beta_2)$ be two ordered pairs on a circle $C$, where all four points are distinct. The action of $\text{Homeo}(C)$ on such pairs has 3 orbits corresponding to 3 possible relative positions of the two pairs $A, B$:

- The pairs are linked, meaning that $\beta_1$ and $\beta_2$ lie in distinct arcs defined by $\{\alpha_1, \alpha_2\}$—connected components of $C \setminus \{\alpha_1, \alpha_2\}$. The relation of being linked is symmetric: $A$ is linked with $B$ iff $B$ is linked with $A$. 
The order within the pairs \( A = (\alpha_1, \alpha_2) \) and \( B = (\beta_1, \beta_2) \) does not change the status of being linked. We say that disjoint pairs \( A \) and \( B \) are **unlinked** if they are not linked.

- The pairs \( A \) and \( B \) are **unlinked and aligned**, if in the arc \( \overline{\alpha_1, \alpha_2} \) determined by \( \{\alpha_1, \alpha_2\} \) on \( C \) containing \( \beta_1 \) and \( \beta_2 \) one has linear order \( \alpha_1 < \beta_1 < \beta_2 < \alpha_2 \). We note that \( A \) is unlinked and aligned with \( B \) iff \( B \) is unlinked and aligned with \( A \). In this case flipping the order in both pairs \( A \) and \( B \) simultaneously does not change the status of being aligned.

- The pairs \( A \) and \( B \) are **unlinked and misaligned**, if in the arc \( \overline{\alpha_1, \alpha_2} \) determined by \( \{\alpha_1, \alpha_2\} \) on \( C \) containing \( \beta_1 \) and \( \beta_2 \) one has linear order \( \alpha_1 < \beta_2 < \beta_1 < \alpha_2 \). We note that \( A \) is unlinked and misaligned with \( B \) iff \( B \) is unlinked and misaligned with \( A \). In this case flipping the order in both of \( A \) and \( B \) simultaneously does not change the status of being misaligned. Yet flipping the order in either \( A \) or \( B \) makes the pair unlinked and aligned.

Let us now choose a negatively curved Riemannian metric \( g \) on \( \Sigma \), and let \( \tilde{g} \) be its lift to \( \tilde{\Sigma} \). Denote by \( d_{\tilde{g}} \) the corresponding distance on \( \tilde{\Sigma} \), and by \( \ell_g : \Gamma \to [0, \infty) \) the associated **stable length**

\[
\ell_g(\gamma) := \lim_{n \to \infty} \frac{1}{n} d_{\tilde{g}}(\gamma^n.p,p)
\]

where \( p \in \tilde{\Sigma} \) is arbitrary.
Theorem 2.1: Let \( a, b \in \Gamma \) be non-trivial elements with distinct fixed points \( a^-, a^+, b^-, b^+ \) on the boundary circle \( \partial \Gamma \). Then:

1. If \((a^-, a^+), (b^-, b^+)\) are linked, then
\[
\ell_g(ab) < \ell_g(a) + \ell_g(b).
\]
2. If \((a^-, a^+), (b^-, b^+)\) are unlinked and aligned, then
\[
\ell_g(ab) > \ell_g(a) + \ell_g(b).
\]
3. If \((a^-, a^+), (b^-, b^+)\) are unlinked and misaligned, then
\[
\ell_g(a^{-1}b) > \ell_g(a) + \ell_g(b).
\]

Proof. First recall that in the case of negatively curved manifolds, such as \((\Sigma, g)\), the stable length \( \ell_g(\gamma) \) can also be defined as the \textbf{minimal translation length}
\[
\ell_g(\gamma) = \inf_{p \in \tilde{\Sigma}} d_{\tilde{g}}(\gamma.p, p).
\]

Moreover, when \( \ell_g(\gamma) > 0 \), which is the case of any non-trivial \( \gamma \neq 1 \), the inf is attained and the set
\[
\text{Ax}_\gamma := \{p \in \tilde{\Sigma} | d_{\tilde{g}}(\gamma.p, p) = \ell_g(\gamma)\}
\]
is the geodesic line \((\gamma^-, \gamma^+)\) in \( \tilde{\Sigma} \). It is called the \textbf{axis} of \( \gamma \).

Elementary topology of the disc \( \tilde{\Sigma} \) implies that when \((a^-, a^+)\) and \((b^-, b^+)\) are linked, the axes \(\text{Ax}_a\) and \(\text{Ax}_b\) must intersect in \( \tilde{\Sigma} \). Due to negative curvature the intersection is a singleton: \( \text{Ax}_a \cap \text{Ax}_b = \{p\} \). Since \( p \in \text{Ax}_b \), we have \( x = b^{-1}.p \in \text{Ax}_b \). Similarly, we have \( p \) and \( y = a.p \) are in \( \text{Ax}_a \) as well. To prove part (1) we use the triangle inequality to obtain for \( x = b^{-1}.p \):
\[
\ell_g(ab) \leq d_{\tilde{g}}(x, ab.x) < d_{\tilde{g}}(x, b.x) + d_{\tilde{g}}(b.x, ab.x) = d_{\tilde{g}}(b^{-1}.p, p) + d_{\tilde{g}}(p, a.p) = \ell_g(b) + \ell_g(a).
\]

We observe that the second inequality is strict and will sharpen it in the proof of Theorem A below.

In the case where the pairs \((a^-, a^+)\) and \((b^-, b^+)\) are unlinked and aligned, we remind ourselves of the definition, that \( a^-, a^+ \) define an arc \( a^-a^+ \) on the boundary circle containing both \( b^- \) and \( b^+ \), which can be equipped with a linear order (anti-clockwise in the figure) so that
\[
a^- < b^- < b^+ < a^+.
\]
The action of $b$ on the arc/interval from $b^+$ to $a^+$ is decreasing towards the fixed point $b^+$, while the action of $a$ is increasing towards $a^+$. Thus $ab$ maps this interval into itself, and therefore the attracting point $(ab)^+$ satisfies $b^+ < (ab)^+ < a^+$. Moreover, we have

$$b^+ = b.b^+ < b.(ab)^+ < (ab)^+.$$ 

Since the repelling fixed point of an element is the attracting fixed point of its inverse, the same argument gives $a^- < (ab)^- < b^-$. We claim that $a^- < b.(ab)^- < (ab)^-$. Indeed, in the linear order on the arc $b^+b^-$ that contains $a^\pm$ so that $b^+ < a^+$, $a^- < b^-$ the map $b$ is decreasing, and thus $\xi = b.(ab)^- < (ab)^-$. Since $a.\xi = (ab). (ab)^- = (ab)^- > \xi$ we deduce that $a^- < \xi < (ab)^-$. Hence

$$a^- < b.(ab)^- < (ab)^-.$$ 

We conclude that the pair $((ab)^-, (ab)^+)$ is linked with its image under $b$. Denote by $p$ the intersection of $Ax_{ab}$ and $b. Ax_{ab}$ in $\tilde{\Sigma}$, and let $x = b^{-1}.p$. Since $p \in b. Ax_{ab}$ we have $x \in Ax_{ab}$ and $ab.x \in Ax_{ab}$ as well. Thus the points $x$, $p = b.x$, $ab.x = a.p$ lie on the geodesic line $Ax_{ab}$, and in fact in this linear order. This can be seen by inspecting the projections of these points to $Ax_a$ and $Ax_b$, making use of the assumption that the pairs are aligned. Hence

$$\ell_g(ab) = d_g(x, ab.x) = d_g(x, b.x) + d_g(p, a.p) > \ell_g(b) + \ell_g(a).$$
The strict inequality here occurs because \( p \not\in Ax_a \) and \( x \not\in Ax_b \). This proves statement (2).

Statement (3) follows from (2) by replacing \( a \) by \( a^{-1} \). This completes the proof of Theorem 2.1. \( \blacksquare \)

3. Spiraling of the boundary of a quasi-Fuchsian embedding

Let \( \Gamma = \pi_1(\Sigma) \) be a surface group, and \( q \in \mathcal{QF}(\Sigma) \) be defined by a representation \( \pi : \Gamma \to \text{PSL}_2(\mathbb{C}) \). For \( \gamma \in \Gamma \) the element \( g = \pi(\gamma) \in \text{PSL}_2(\mathbb{C}) \) has two preimages \( \pm \hat{g} \) in \( \text{SL}_2(\mathbb{C}) \). Since the traces \( \pm \text{tr}(\hat{g}) \) are invariant under conjugation, we can denote them by \( \pm \text{tr}_q(\gamma) \). The following is a particular case of a lemma of Vinberg [15] (see [13, Corollary 3.2.5]).

**Lemma 3.1:** Let \( \Gamma = \pi_1(\Sigma) \) be a surface group, and \( q \in \mathcal{QF}(\Sigma) \setminus \mathcal{T}(\Sigma) \). Then there exists \( \gamma \in \Gamma \) with \( \pm \text{tr}_q(\gamma) \in \mathbb{C} \setminus \mathbb{R} \).

Let \( \pi : \Gamma \to \text{PSL}_2(\mathbb{C}) \) be a quasi-Fuchsian representation. There exists a \( \Gamma \)-equivariant continuous map
\[
\phi : \partial \Gamma \longrightarrow \mathbb{P}_1^1(\mathbb{C}), \quad \phi \circ \gamma = \pi(\gamma) \circ \phi
\]
that is a homeomorphism between the topological circle \( \partial \Gamma \) and the Jordan curve on the sphere \( \mathbb{P}_1^1(\mathbb{C}) \) formed by the limit set \( L_{\pi}(\Gamma) \) of \( \pi(\Gamma) \).

**Proposition 3.2:** Let \( q \in \mathcal{QF}(\Sigma) \setminus \mathcal{T}(\Sigma) \) be given by a quasi-Fuchsian representation \( \pi : \Gamma \longrightarrow \text{PSL}_2(\mathbb{C}) \). Then there exists an isometrically embedded hyperbolic plane \( \mathbb{H}^2 \subset \mathbb{H}^3 \) and a sequence \( \xi_1, \xi_2, \ldots \rightarrow \xi_* \in \partial \Gamma \) whose cyclic order with respect to the circle \( \partial \Gamma \) is
\[
\xi_1, \xi_2, \xi_3, \xi_4, \ldots, \xi_*
\]
and whose images \( \phi(\xi_n) \in \mathbb{P}_1^1(\mathbb{C}) \) lie on the boundary circle \( \partial \mathbb{H}^2 \) in the following cyclic order:
\[
\phi(\xi_1), \phi(\xi_3), \phi(\xi_5), \ldots, \phi(\xi_*), \ldots, \phi(\xi_6), \phi(\xi_4), \phi(\xi_2).
\]
In particular, we have:
- \((\xi_1, \xi_4)\) and \((\xi_2, \xi_3)\) are unlinked and aligned in \( \partial \Gamma \), while \((\phi(\xi_1), \phi(\xi_4))\) and \((\phi(\xi_2), \phi(\xi_3))\) are linked in \( \partial \mathbb{H}^2 \).
- \((\xi_1, \xi_3)\) and \((\xi_2, \xi_4)\) are linked in \( \partial \Gamma \), while \((\phi(\xi_1), \phi(\xi_3))\) and \((\phi(\xi_2), \phi(\xi_4))\) are unlinked and aligned in \( \partial \mathbb{H}^2 \).
Proof. Fix an element $\gamma \in \Gamma$ with $\pm \text{tr}_q(\gamma) \in \mathbb{C} \setminus \mathbb{R}$ as in Lemma 3.1. Note that $\gamma$ must be hyperbolic, and denote by $\xi_*$ the attracting point $\gamma^+ \in \partial \Gamma$. At the same time $\pi(\gamma) \in \text{PSL}_2(\mathbb{C})$ is loxodromic with an attracting point $\phi(\gamma^+)$. Identifying $\mathbb{P}^1_\mathbb{C}$ with $\mathbb{C} \cup \{\infty\}$ and replacing $\pi: \Gamma \rightarrow \text{PSL}_2(\mathbb{C})$ by an appropriate conjugate we may assume $\phi(\gamma^+)=\infty$ and $\phi(\gamma^-)=0$. Then the action of $\pi(\gamma)$ on $\mathbb{C}$ is given by the linear map $z \mapsto \lambda e^{2\pi i \theta} \cdot z$ with $\lambda > 1$, $\theta \in \mathbb{R} \setminus \mathbb{Z}$.

Identify $\partial \Gamma \setminus \{\gamma^+\}$ with $\mathbb{R}$ so that $\gamma^-$ corresponds to $0 \in \mathbb{R}$. With a slight abuse of notation we write $\gamma$ and $\phi$ for the corresponding homeomorphism of $\mathbb{R}$, and an equivariant injective continuous map $\mathbb{R} \rightarrow \mathbb{C}$. Note that $\gamma(0)=0$, $\gamma$ is strictly increasing on $[0, \infty)$ (and strictly decreasing on $(-\infty, 0]$), while $\phi$ satisfies

$$
\phi(\gamma(t)) = \lambda e^{2\pi i \theta} \cdot \phi(t) \tag{3.1}
$$

and

$$
|\phi(t)| \to \infty \quad \text{as } |t| \to \infty.
$$

Since $\phi(t) \neq 0$ for all $t \in (0, \infty)$ there exist continuous functions $r:(0, \infty) \rightarrow (0, \infty)$ and $s:(0, \infty) \rightarrow \mathbb{R}$ so that

$$
\phi(t) = r(t) \cdot e^{2\pi i \cdot s(t)} \quad (t > 0).
$$

Thus (3.1) implies that

$$
\begin{align*}
r(\gamma^n(t)) &= \lambda^n \cdot r(t), \\
\gamma^n(t) &= s(t) + n\Theta
\end{align*}
$$

where $\Theta \in \theta + \mathbb{Z}$. Note that the assumption that $q \in \mathcal{D}(\Sigma) \setminus \mathcal{F}(\Sigma)$ gives $\theta \notin \mathbb{Z}$ (Lemma 3.1), implying $\Theta \neq 0$.

Fix $t_0 > 0$ and use points $t_n = \gamma^n(t_0)$, $n \in \mathbb{Z}$, to partition the ray $(0, \infty)$. Let

$$
R_0 = \max\{r(t) \mid 0 \leq t \leq t_0\}, \quad r_0 = \min\{r(t) \mid t_0 \leq t < \infty\}.
$$

Then $|\phi(t)| = r(t) \leq \lambda^n \cdot R_0$ for all $t \in [0, t_n]$, and $|\phi(t)| = r(t) \geq \lambda^n \cdot r_0$ for all $t \geq t_n$. We can now choose integers $n(1) < m(1) < n(2) < m(2) < \cdots$ so that

$$
(m(k) - n(k)) \cdot |\Theta| > 1, \quad \lambda^{n(k+1) - m(k)} > R_0/r_0
$$

for all $k \in \mathbb{N}$. The first condition guarantees that $s(t_{m(k)}) > s(t_{n(k)}) + 1$ and therefore there exist

$$
\xi_k \in [t_{n(k)}, t_{m(k)}] \quad \text{with } e^{2\pi i \cdot s(\xi_k)} = (-1)^k.
$$
Thus $\phi(\xi_k) = (-1)^k r(\xi_k)$ lie on the real line $\mathbb{R} \subset \mathbb{C}$ on both sides of $0 \in \mathbb{R}$ in alternating order. Since $\xi_k \leq t_{m(k)} < t_{n(k+1)} \leq \xi_{k+1}$ we also have

$$|\phi(\xi_k)| = r(\xi_k) \leq \lambda^{m(k)} \cdot R_0 < \lambda^{n(k+1)} \cdot r_0 \leq r(\xi_{k+1}) = |\phi(\xi_{k+1})|.$$ 

Thus the sequence $\{|\phi(\xi_k)|\}$ is monotonically increasing. In particular, we have

$$\cdots < \phi(\xi_5) < \phi(\xi_3) < \phi(\xi_1) < 0 < \phi(\xi_2) < \phi(\xi_4) < \phi(\xi_6) < \cdots$$

on $\mathbb{R} \subset \mathbb{C}$. Recalling that $\phi(\xi_*) = \infty$ we get the required cyclic order. 

4. Proof of Theorem A

Let us first recall two general well-known facts, one related to CAT(-1) spaces $(X, d_X)$, and another to Gromov hyperbolic groups $\Gamma$ acting on their boundary $\partial \Gamma$. We will apply them to $X = \mathbb{H}^3$ and to the surface group $\Gamma = \pi_1(\Sigma)$.

Recall that given a point $p \in \mathbb{H}^3$ and a pair of distinct boundary points $\xi \neq \eta \in \partial \mathbb{H}^3$ the following limit exists:

$$B_p(\xi, \eta) = \lim_{x \to \xi, \; y \to \eta} (d_{\mathbb{H}^3}(p, x) + d_{\mathbb{H}^3}(p, y) - d_{\mathbb{H}^3}(x, y)).$$

Triangle inequality implies that $B_p(\xi, \eta) \geq 0$. Crucial for our purposes is the fact that the strict inequality occurs unless $p$ lies on the geodesic line $(\xi, \eta)$:

$$B_p(\xi, \eta) > 0 \iff p \notin (\xi, \eta).$$

The second fact is a consequence of the topological transitivity of the geodesic flow on the unit tangent bundle to the surface. It can be used to show that for any $\xi \neq \eta$ in $\partial \Gamma$ there exists an infinite sequence $\{\gamma_n\}$ in $\Gamma$ so that

$$\xi = \lim_{n \to \infty} \gamma_n^+ \quad \text{and} \quad \eta = \lim_{n \to \infty} \gamma_n^-$$

where $\gamma_n^-, \gamma_n^+ \in \partial \Gamma$ denote the repelling and the attracting points of $\gamma_n \in \Gamma$.

With these observations we can proceed to the proof of Theorem A. Using Proposition 3.2, let us pick $(\xi_1, \xi_4)$ and $(\xi_2, \xi_3)$ that are unlinked and aligned in $\partial \Gamma$ while $(\phi(\xi_1), \phi(\xi_4))$ and $(\phi(\xi_2), \phi(\xi_3))$ are linked in a copy of a hyperbolic plane $\partial \mathbb{H}^2$ contained in the hyperbolic space $\mathbb{H}^3$. Let $p \in \mathbb{H}^3$ denote the intersection of the linked geodesic lines $(\phi(\xi_1), \phi(\xi_4))$ and $(\phi(\xi_2), \phi(\xi_3))$. Since these two geodesic lines are distinct, $p \notin (\phi(\xi_2), \phi(\xi_4))$, and therefore, using the first fact, we obtain

$$\delta = B_p(\phi(\xi_2), \phi(\xi_4)) > 0.$$
We can now use the second fact, and find sequences \( \{a_n\} \) and \( \{b_n\} \) in \( \Gamma \), so that
\[
a_n^+ \to \xi_1, \quad a_n^- \to \xi_4, \quad b_n^- \to \xi_2, \quad b_n^+ \to \xi_3.
\]
Denote \( A_n = \pi(a_n) \) and \( B_n = \pi(b_n) \) the corresponding elements in \( \text{PSL}_2(\mathbb{C}) \). Note that \( \phi(a_n^\pm) = A_n^\pm \) and \( \phi(b_n^\pm) = B_n^\pm \) are the repelling/attracting points in \( \partial \mathbb{H}^3 \). Upon replacing \( a_n, b_n \) by their powers, we may assume that
\[
\ell_{\mathbb{H}^3}(A_n) \to \infty, \quad \ell_{\mathbb{H}^3}(B_n) \to \infty.
\]
Let \( p_n^A \) denote the projection of point \( p \) to the geodesic line \( (\phi(a_n^-), \phi(a_n^+)) \) which is the axis \( \text{Ax}_{A_n} \) in \( \mathbb{H}^3 \). Since \( \phi : \partial \Gamma \to \partial \mathbb{H}^3 \) is continuous,
\[
A_n^- = \phi(a_n^-) \to \phi(\xi_1) \quad \text{and} \quad A_n^+ = \phi(a_n^+) \to \phi(\xi_4).
\]
This implies
\[
d_{\mathbb{H}^3}(p_n^A, p) \to 0.
\]
Similarly, denoting by \( p_n^B \in \mathbb{H}^3 \) the projection of \( p \) to the geodesic line \( (\phi(\xi_2), \phi(\xi_3)) \) which is the axis \( \text{Ax}_{B_n} \) in \( \mathbb{H}^3 \), we get \( d_{\mathbb{H}^3}(p_n^B, p) \to 0 \).
Now consider the points \( x_n = B_n^{-1}p_n^B \) and \( y_n = A_n.p_n^A \). Since \( p_n^b = B_n.x_n \) and \( x_n \) are on the axis \( Ax_{B_n} \) of \( B_n \) we have \( d_{H^3}(p_n^b, x_n) = \ell_{H^3}(B_n) \) and

\[
|d_{H^3}(p, x_n) - \ell_{H^3}(B_n)| \leq d_{H^3}(p, p_n^B) \to 0. \tag{4.1}
\]

Similarly,

\[
|d_{H^3}(p, y_n) - \ell_{H^3}(A_n)| \leq d_{H^3}(p, p_n^A) \to 0. \tag{4.2}
\]

Hence

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} A^+_{n} = \phi(\xi_2), \quad \lim_{n \to \infty} y_n = \lim_{n \to \infty} B^-_{n} = \phi(\xi_4).
\]

Therefore

\[
\lim_{n \to \infty} (d_{H^3}(x_n, p) + d_{H^3}(p, y_n) - d_{H^3}(x_n, y_n)) = B_p(\phi(\xi_2), \phi(\xi_4)) = \delta > 0. \tag{4.3}
\]

We also have

\[
d_{H^3}((A_n B_n).x_n, y_n) = d_{H^3}(A_n.p_n^B, y_n) = d_{H^3}(A_n.p_n^B, A_n.p_n^A) = d_{H^3}(p_n^B, p_n^A) \leq d_{H^3}(p_n^B, p) + d_{H^3}(p, p_n^A) \to 0.
\]

Using (4.1), (4.2), (4.3) we deduce

\[
\lim_{n \to \infty} (\ell_{H^3}(A_n) + \ell_{H^3}(B_n) - d_{H^3}(A_n B_n.x_n, x_n)) = \delta.
\]

Since \( \ell_{H^3}(A_n B_n) \leq d_{H^3}(A_n B_n.x_n, x_n) \), it follows that

\[
\liminf_{n \to \infty}(\ell_{H^3}(A_n) + \ell_{H^3}(B_n) - \ell_{H^3}(A_n B_n)) \geq \delta.
\]

The latter fact can be rewritten as

\[
\liminf_{n \to \infty}(\ell_q(a_n) + \ell_q(b_n) - \ell_q(a_n b_n)) \geq \delta.
\]

Recall that \( (\xi_1, \xi_4) \) and \( (\xi_2, \xi_3) \) are unlinked and aligned in \( \partial \Gamma \), and are approximated by \( (a_n^-, a_n^+) \) and \( (b_n^-, b_n^+) \) respectively. Thus, we can find \( k \in \mathbb{N} \) large enough, so that the pair of elements \( a = a_k, b = b_k \) satisfy

\[
\ell_q(a) + \ell_q(b) - \ell_q(ab) > \frac{1}{2} \delta
\]

while \( (a^-, a^+) \) and \( (b^-, b^+) \) are unlinked and aligned. By Theorem 2.1 the latter condition implies that for every \( g \in \mathcal{R}(\Sigma) \) we have

\[
\ell_g(a) + \ell_g(b) - \ell_g(ab) < 0.
\]
Thus
\[
\frac{\ell_q(a) + \ell_q(b)}{\ell_q(ab)} ; \frac{\ell_g(a) + \ell_g(b)}{\ell_g(ab)} > \frac{\ell_q(a) + \ell_q(b)}{\ell_q(ab)} > 1 + \frac{\delta}{2\ell_q(ab)}.
\]
We deduce that for every \( g \in \mathcal{R}(\Sigma) \) we get
\[
\rho_{\text{Lip}}(\delta_q, \delta_g) > \log(1 + \frac{\delta}{2\ell_q(ab)}) > 0.
\]
This completes the proof of Theorem A.

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