Entropy Gain in $p$-Adic Quantum Channels

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Abstract—A configuration of a $p$-adic quantum linear bosonic Gaussian channel is proposed. The entropy gain of such a channel is calculated. An adelic formula for the entropy gain is derived.

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1. INTRODUCTION

Starting from [1, 2], non-Archimedean analysis has been actively used to build physical models. Thereby, the branch in mathematical physics ($p$-adic mathematical physics) has arisen. One can find a bibliography on this subject in the book [3] and reviews [4, 5]. The article is organized as follows. Firstly, we give the necessary facts of the $p$-adic analysis. Secondly, we give a definition of the $p$-adic Gaussian state and the $p$-adic Gaussian linear bosonic channel and also give their properties. Basically, this information follows [6]; the results are presented without proof. New results are presented below. Thirdly, we prove a formula for the magnitude of the entropy gain in the $p$-adic Gaussian channel. Fourthly, we give the adelic formula for the entropy gain and its possible applications.

2. $p$-ADIC NUMBERS

AND SYMPLECTIC GEOMETRY

In this section, a number of known facts concerning the geometry of lattices in a two-dimensional symplectic space over the field $\mathbb{Q}_p$ of $p$-adic numbers are presented without proof. The necessary information from the $p$-adic analysis is contained, for example, in [7]. Most statements regarding the geometry of lattices can be found in [8].

Let $F$ be a two-dimensional vector space over the field $\mathbb{Q}_p$. A nondegenerate symplectic form $\Delta : F \times F \to \mathbb{Q}_p$ is given in the space $F$. A free module of rank two over the ring $\mathbb{Z}_p$ of integer $p$-adic numbers considered as a subset of $F$ will be called a lattice in $F$. A lattice is a compact set in the natural topology in the space $F$.

We introduce the duality relation on the set of lattices. Let $L$ be a lattice; then the notion of the dual lattice $L^*$ means a subset of the space $F$ in the form $u \in L^*$ if and only if condition $\Delta(u, v) \in \mathbb{Z}_p$ is satisfied for all $v \in L$.

If $L$ coincides with $L^*$, then the lattice $L$ will be called a self-dual lattice.

For any self-dual lattice $L$, there exists a symplectic basis $\{e_1, e_2\}$ in the space $F$ such that the lattice $L$ has the form

$$L = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2,$$

that is, $L$ is the unit ball in this basis.

By $Sp(F, \Delta)$ denote the symplectic group (the group of nondegenerate linear transformations of the space $F$ that preserve the form $\Delta$). The group $Sp(F, \Delta)$ is isomorphic to the group $SL_2(\mathbb{Q}_p)$.

In the space $F$, there exists a unique translation-invariant measure (Haar measure) up to normalization. We will normalize the measure in such a way that the measure of the unit ball is equal to unity. The action of the symplectic group preserves the measure; therefore, the measure of any self-dual lattice is equal to unity. The measure of the lattice $L$ will be denoted by $|L|$. If $L$ is a self-dual lattice, then, as noted above, $|L| = 1$; the converse is also true, if $|L| = 1$, then the lattice $L$ is self-dual. It is easy to verify the validity of the relation $|L||L^*| = 1$.

Let $S \in Sp(F, \Delta)$, $L \subset F$ be an arbitrary lattice. As already noted, the action of the symplectic group preserves measure, that is, $|SL| = |L|$. The converse is also true, if $L_1, L_2$ are arbitrary lattices in $F$ with the same measure, $|L_1| = |L_2|$, then there exists a symplectic transformation $S \in Sp(F, \Delta)$ such that $SL_1 = L_2$. 
3. $p$-adic Linear Bosonic Channels

Let $\mathcal{H}$ be a separable Hilbert space over the field $\mathbb{C}$ of complex numbers. The scalar product in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle$, and we will consider it antilinear in the first argument.

The state of the system is described by the density matrix $\rho$ in the space $\mathcal{H}$. Denote by $\mathcal{M}$ the set of all states.

By $\mathcal{B}(F)$ we denote the $\sigma$-algebra of Borel subsets of $F$ space; by $\mathfrak{B}(\mathcal{H})$ we denote the algebra of bounded operators on $\mathcal{H}$ space. A quantum observable $M : \mathcal{B}(F) \to \mathfrak{B}(\mathcal{H})$ is a projection-valued measure in $\mathcal{B}(F)$. Denote by $\mathfrak{M}$ the set of quantum observables.

The probability distribution of the observable $M$ in state $\rho$ is given by the Born–von Neumann formula

$$\mu^M_\rho = \text{Tr} (\rho M(B)), \quad B \in \mathcal{B}(F).$$

Actually, we do not go beyond the framework of the standard statistical model of quantum mechanics ([9]), since the set of real numbers and the set of $p$-adic numbers are Borel isomorphic. As noted above, the state of a quantum system is described by a density matrix $\rho$ in a Hilbert space $\mathcal{H}$. Let an irreducible representation of the CCR described by a density matrix $\rho$ in a Hilbert space $\mathcal{H}$.

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As noted above, the state of a quantum system is described by a density matrix $\rho$ in a Hilbert space $\mathcal{H}$. Let an irreducible representation of the CCR $\{W(z), z \in F\}$ be given in this space. Each density operator can be associated with a function $\pi_\rho$ in the space $F$ by the following formula

$$\pi_\rho(z) = \text{Tr} (\rho W(z)).$$

The function $\pi_\rho$ is called the characteristic function of the quantum state $\rho$ and defines this state uniquely. To reconstruct state $\rho$, the characteristic function is used according to the relation

$$\rho = \int_F \pi_\rho(z) W(-z) dz.$$

The characteristic function of a quantum state has the property of $\Delta$-positive definiteness: for any finite sets $z_1, z_2, \ldots, z_n$ of points in the space $F$ and $c_1, c_2, \ldots, c_n$ of complex numbers, the inequality

$$\sum_{i,j=1}^n c_i \overline{c_j} \pi_\rho(z_i - z_j) \chi \left( -\frac{1}{2} \Delta(z_i, z_j) \right) \geq 0,$$

is satisfied.

As in the case of the representation of CCR over a real symplectic space, a noncommutative analog of the Bochner–Khinchin theorem holds for the $p$-adic case; this theorem establishes a one-to-one correspondence between states and $\Delta$-positive definite functions ([10]).

We give the following definition.

**Definition 1.** A state $\rho$ will be called a $p$-adic Gaussian state if its characteristic function $\pi_\rho$ is the characteristic function of some lattice, that is,

$$\pi_\rho(z) = \text{Tr} (\rho W(z)) = h_L(z) = \begin{cases} 1, & \text{if } z \in L; \\ 0, & \text{if } z \notin L. \end{cases}$$

This definition is natural in the following context. Let $\mathfrak{F}$ be the Fourier transform in $\mathcal{L}(F)$ defined by the formula

$$\mathfrak{F} [f](z) = \int_F \chi (\Delta(z, s)) f(s) ds,$$

$L$ is a lattice in $F$. Then, the formula

$$|L|^{-1/2} \mathfrak{F} [h_L] = |L|^ {1/2} h_L,$$

is satisfied.

In other words, the Fourier transform turns the characteristic function of the lattice into the characteristic function of the dual lattice up to a factor. In particular, the characteristic function of the self-dual lattice is invariant under the action of the Fourier transform. In this context, the characteristic function of the lattice is an analog of the Gaussian function in real analysis.

$p$-adic Gaussian states are very simple in structure. Namely, the following statements are true.

**Proposition 1.** In order for the characteristic function $h_L$ of the lattice $L$ to determine the quantum state, it is necessary and sufficient that the condition $|L| \leq 1$ be satisfied.

A Gaussian state $\rho$ having a characteristic function $\pi_\rho = h_L$ is $|L| P_L$, where $P_L$ is an orthogonal projector of rank $1/|L|$.

Some obvious properties of Gaussian states are given below.

**Proposition 2.** The following statements are true.
- A Gaussian state is pure if and only if the corresponding lattice is self-dual.
- The entropy of the Gaussian state defined by the lattice $L$ is $-\log |L|$. 


• Gaussian states $\rho_1$ and $\rho_2$ are unitarily equivalent if and only if the corresponding lattices $L_1$ and $L_2$ have the same measure.

• The entropy of the Gaussian state determines this state uniquely up to unitary equivalence.

• The Gaussian state has the maximum entropy among all states with a fixed rank $p^m$, $m \in \mathbb{N}$.

We will use the notation $\gamma(L)$ for the density operator of the Gaussian state defined by the lattice $L$. We considered only centered Gaussian states. We can similarly consider general Gaussian states $\gamma(L, h_0)$, which are defined by a characteristic function in the form

$$\pi_{\gamma(L,\alpha)} = \chi(\Delta(\alpha, z)) h_L(z).$$

It is easy to see that

$$\gamma(L, \alpha) = W(\alpha)\gamma(L)W(-\alpha).$$

Let $W$ be the irreducible representation of the CCR in the Hilbert space $\mathcal{H}$; $\mathcal{Z}(\mathcal{H})$ is the set of states.

By analogy with the real case ([11]), a linear bosonic channel (in the Schrodinger representation) is a linear completely positive trace-preserving mapping $\Phi : \mathcal{Z}(\mathcal{H}) \to \mathcal{Z}(\mathcal{H})$ such that the characteristic function $\pi_\rho$ of any state $\rho \in \mathcal{Z}(\mathcal{H})$ is transformed by the formula

$$\pi_{\Phi(\rho)}(z) = \pi_\rho(Kz)k(z)$$
for some linear transformation $K$ of the space $F$ and some complex-valued function $k$ in $F$.

Generally speaking, expression (1) does not always determine the channel; to do this, additional conditions on the transformation $K$ and the function $k$ are necessary.

A $p$-adic Gaussian channel is a linear bosonic channel for which the function $k$ is the characteristic function of some lattice $L \subset F$, that is, $k(z) = h_L(z), z \in F$.

The following theorem holds.

**Proposition 3.** Let $K$ be a nondegenerate linear transformation of the space $F$, $L$ be a lattice in the space $F$, $k(z) = h_L(z)$. In this case, expression (1) defines a channel if and only if the inequality

$$|L - \det K| \leq 1$$

is satisfied.

Note that in the case of $\det K = 1$, the transformation $K$ is symplectic and, therefore, unitarily representable. Next, we consider the case of $\det K \neq 1$.

4. ENTROPY GAIN

The entropy $H(\rho)$ of state $\rho$ is defined by the following expression

$$H(\rho) = -\text{Tr} \rho \log \rho.$$
5. ADELIC CHANNELS AND ENTROPY GAIN

Now let $F$ be a two-dimensional vector space over the field $\mathbb{Q}$ of rational numbers, $\Delta$ be a nondegenerate symplectic form in $F$ taking values in the field $\mathbb{Q}$, and $K$ be a nondegenerate linear transformation of the space $F$.

For each prime $p$, we construct the corresponding linear bosonic Gaussian channel $\Phi_p$ and calculate the entropy gain $G(\Phi_p)$ of such a channel. We can also construct a real linear bosonic Gaussian channel $\Phi_\infty$ and calculate the entropy gain $G(\Phi_\infty)$. The following statement is true.

**Theorem 2.**

$$\sum_{\{p\text{-prime}\} \cup \{p=\infty\}} G(\Phi_p) = 0.$$  

Note that $\det K \in \mathbb{Q}$. The further follows from a simple adelic formula, which is valid for an arbitrary nonzero rational number (for example, see [7])

$$|\det K| \prod_{p\text{-prime}} |\det K|_p = 1.$$  

Theorem 2 can be interpreted as follows. Consider the adelic linear bosonic Gaussian channel generated by the linear transformation $K$ of a two-dimensional vector space over the field of rational numbers, that is, the tensor product of the channels $\Phi_p$ over all primes $p$ and the real channel $\Phi_\infty$. A nontrivial entropy gain $G(\Phi_p)$ is possible in each component of this adelic channel. However, the total entropy gain in the adelic channel is zero. There is a nontrivial exchange of information between the components of the adelic channel when the total entropy is conserved.

If we accept the hypothesis that “at the fundamental level, our world ... is adelic” ([13]), then Theorem 2 can be used, for example, to interpret the information paradox of a black hole.

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