FOURIER DECAY BOUND AND DIFFERENTIAL IMAGES OF SELF-SIMILAR MEASURES

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ABSTRACT. In this note, we investigate $C^2$ differential images of the homogeneous self-similar measure associated with an IFS $\mathcal{I} = \{\rho x + a_j\}_{j=1}^m$ satisfying the strong separation condition and a positive probability vector $\vec{p}$. It is shown that the Fourier transforms of such image measures have power decay for any contractive ratio $\rho \in (0, 1/m)$, any translation vector $\vec{a} = (a_1, \ldots, a_m)$ and probability vector $\vec{p}$, which extends a result of Kaufman on Bernoulli convolutions. Our proof relies on a key combinatorial lemma originated from Erdős, which is important in estimating the oscillatory integrals. An application to the existence of normal numbers in fractals is also given.

1. Introduction

Let $\mu$ be a Borel probability measure on $\mathbb{R}$. The Fourier transform of $\mu$ is defined by

$$\hat{\mu}(\xi) = \int \exp(2\pi i \xi t) d\mu(t), \quad \xi \in \mathbb{R}. \tag{1.1}$$

It is known that the asymptotic behaviors of $\hat{\mu}(\xi)$ at infinity give information to absolute continuity or singularity, the geometric or arithmetic structure, and the size of the support of $\mu$ (see for instance [17] for more details). In this note, we are concerned with a special class of Borel probability measures, that is, the differential images of some homogeneous self-similar measures. The motivation is to see how the asymptotic behaviors of the Fourier transform are affected when a self-similar measure is smoothly perturbated.

Recall that an iterated function system (IFS) is a finite family $\mathcal{I} = \{f_j\}_{j=1}^m$ $(m \geq 2)$ of strictly contracting maps on some complete metric space $X$ (here we always assume that $X = \mathbb{R}$). An IFS $\mathcal{I}$ is called linear (resp. of class $C^\alpha$) if all the maps in $\mathcal{I}$ are affine (resp. of class $C^\alpha$). An IFS $\mathcal{I}$ is called $C^\alpha$-conjugate to another IFS $\mathcal{J}$ if $\mathcal{J} = \{\varphi \circ f_j \circ \varphi^{-1}\}_{j=1}^m$ for some $C^\alpha$-diffeomorphism $\varphi$. It is well known that there exists a unique non-empty compact set $K = K(\mathcal{I}) \subset \mathbb{R}$, called the attractor of $\mathcal{I}$, such that $K = \bigcup_{j=1}^m f_j(K)$. We say that the strong separation condition holds for $\mathcal{I}$ if the pieces $\{f_j(K)\}_{j=1}^m$ are pairwise disjoint. For a measurable map $f : \mathbb{R} \to \mathbb{R}$ and a Borel probability measure $\mu$, we use $f\mu$ to denote the push-forward measure of $\mu$ by $f$, that is, $f\mu(A) = \mu(f^{-1}(A))$ for all Borel subsets $A \subset \mathbb{R}$. $f\mu$ is also called the image measure of $\mu$ under $f$. Given an IFS $\mathcal{I} = \{f_j\}_{j=1}^m$ and a positive probability vector $\vec{p} = (p_1, \ldots, p_m)$, namely, $\sum_{j=1}^m p_j = 1$ and $p_j > 0$.

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for each $j$, there exists a unique Borel probability measure $\mu$ supported on $K$ such that
\[
\mu = \sum_{j=1}^{m} p_j \cdot f_j \mu. \tag{1.2}
\]

In this note we mainly consider linear IFSs, that is, $f_j(x) = \rho_j x + a_j$ with $0 < \rho_j < 1$ and $a_j \in \mathbb{R}$ for all $j = 1, \ldots, m$. Then each $f_j$ is a similitude in the sense that $|f_j(x) - f_j(y)| = \rho_j |x - y|$ for any $x, y \in \mathbb{R}$. The attractor $K$ and the measure $\mu$ satisfying (1.2) are termed as a self-similar set and self-similar measure respectively. We are particularly interested in IFS of the form $\mathcal{I} = \{\rho x + a_j\}_{j=1}^{m}$, where $\rho \in (0, 1)$ is the common contractive ratio and $\vec{a} = (a_1, \ldots, a_m)$ denotes the translation vector. We call the corresponding self-similar measure $\mu_{\rho, \vec{a}}$ a homogeneous self-similar measure and write it as $\mu_{\rho}$ for simplicity. When $m = 2$ and $p = (1/2, 1/2)$, $\mu_{\rho}$ is the so-called (infinite) Bernoulli convolution, which was studied extensively but is still not completely understood (see [14] for a good survey). It is well known that $\mu_{\rho}$ can be viewed as the weak limit of the ($N + 1$)-fold convolution product $\ast_{k=0}^{N}(p_1 \rho^k a_1 + \cdots + p_m \rho^k a_m)$ [17]. Moreover, the Fourier transform $\hat{\mu}_{\rho}$ has the form of infinite product:
\[
\hat{\mu}_{\rho}(\xi) = \int \exp(2\pi i \xi t) d\mu_{\rho}(t) = \prod_{k=0}^{\infty} \sum_{j=1}^{m} p_j \exp(2\pi i a_j \rho^k \xi), \quad \xi \in \mathbb{R}. \tag{1.3}
\]

Asymptotic behaviors of $\hat{\mu}_{\rho}$ were studied by many authors. For the Bernoulli convolutions, the Erdős-Salem theorem [5, 21] states that $\hat{\mu}_{\rho}$ has no decay at infinity if and only if $\rho^{-1}$ is a Pisot number and $\rho \neq 1/2$ (recall that an algebraic integer $\beta > 1$ is called a Pisot number if all its conjugate roots have modulus strictly less than one). For general homogeneous self-similar measures with $m > 2$, Hu [10] obtained similar criterions in some special cases, where the decay properties of $\hat{\mu}_{\rho}$ is also affected by the translation vector $\vec{a}$ and the probability vector $\vec{p}$. Explicit examples of $\mu_{\rho}$ whose Fourier transforms have certain decay rate were given in [2, 7, 15] (Bernoulli convolutions) and [3, 8, 23] (general homogeneous self-similar measures). Note that in all cases the arithmetic properties of $\rho$ play an important role. Furthermore, Shmerkin [22] showed that there exists a set $E \subset (0, 1)$ of Hausdorff dimension 0 such that for all $\rho \in (0, 1) \setminus E$, all translation vectors $\vec{a}$ with distinct coordinates and all positive probability vector $\vec{p}$, the Fourier transform $|\hat{\mu}_{\rho}(\xi)|$ tends to 0 polynomially.

In this note, we study the Fourier decay properties of the image measure of $\mu_{\rho}$ under a differential function. The strong separation condition is assumed to hold for the corresponding IFS, thus $\rho \in (0, 1/m)$ and $\mu_{\rho}$ is supported on a set of Cantor type.

**Theorem 1.1.** Let $\mu_{\rho}$ be the homogeneous self-similar measure associated with an IFS $\mathcal{I} = \{\rho x + a_j\}_{j=1}^{m}$ that satisfies the strong separation condition and a positive probability vector $\vec{p}$. Then for any $\varphi \in \mathcal{C}^{2}(\mathbb{R})$ with $\varphi'' > 0$ and any $g \in \mathcal{C}^{1}(\mathbb{R})$, we have
\[
\left| \int \exp(2\pi i \varphi(t)) g(t) d\mu_{\rho}(t) \right| = O(|\xi|^{-\gamma}), \quad |\xi| \to \infty \tag{1.4}
\]
for some constant $\gamma > 0$. In particular, the Fourier transform $\hat{\varphi \mu}_{\rho}$ of the image measure $\varphi \mu_{\rho}$ has a power decay.

**Remark 1.2.** (1) The above theorem indicates that the Fourier decay bound of $\hat{\varphi \mu}_{\rho}$ is polynomial for any parameters $\rho, \vec{p}, \vec{a}$, even though the Fourier transform $\hat{\mu}_{\rho}$ of the original measure $\mu_{\rho}$ admits no decay at infinity. This means that smooth permutations could change the Fourier decay behaviors of measures, where the non-linearity of the phase function is...
crucial for obtaining the power decay (otherwise one can easily find a counterexample, for instance, \( \varphi(x) = x \) and \( \mu_\rho \) is the Cantor-Lebesgue measure on the middle-third Cantor set). We remark that the theorem also holds in the case of \( \varphi'' < 0 \), where the proof needs only a slight modification. It seems that the condition could be weakened as long as the graph of \( \varphi \) is curved enough.

(2) The fact that the Fourier decay properties of push-forward measures are closely related to the smoothness of the phase functions date back to van der Corput’s lemma in harmonic analysis (see [14]). Kaufman [14], by rediscovering a method of Erdös [6], proved that the Fourier transform of \( C^2 \) differential images of a class of Bernoulli convolutions has power decay. Applying the same method, we extend Kaufman’s result to general homogeneous self-similar measures. Moreover, our proof shows that the constant \( \gamma \) in (1.4) could be taken explicitly. In fact, one can choose appropriate \( \beta \in (1/2, 1) \) and \( \epsilon \in (0, \delta) \) with \((2 - \alpha)\beta < 1\) such that 

\[
\min\{2\beta - 1, (1 - \beta)\epsilon \log \delta \log \rho \}
\]

attains the maximum, and then let \( \gamma \) be this maximum.

The main ingredients of the proof are the convolution nature of the homogeneous self-similar measure \( \mu_\rho \) and a combinatorial lemma originated from Erdös [6]. The former allows us to approximate \( \mu_\rho \) by a sequence of discrete measures; the latter can be used to control the “bad” set for the purpose of estimating the oscillatory integral. Recently, by using an estimate on decay of exponential sums due to Bourgain, Bourgain-Dyatlov [1] proved similar results for the Patterson-Sullivan measure on the limit set \( \Lambda \Gamma \leq SL(2, \mathbb{Z}) \), where the nonlinearity of transformations in \( \Gamma \) is very important. Measures whose Fourier transforms admit power decay were also studied for stationary measures [16] and for Gibbs measure with respect to the Gauss map [11, 13, 20].

This note is part of the second author’s PhD thesis (in Chinese). When we are going to submit this note, we hear that a version of Theorems 1.1 is simultaneously and independently proved by Mosquera and Shmerkin (see Theorem 3.1 of [18]) among some other results. However, we remark that our method allows us to obtain a better decay bound (see Remark 1.1.(2)), besides the combinatorial lemma established in this note is different from theirs.

The asymptotic behavior of \( \hat{\mu}_\rho \) is closely related to the arithmetic structure of \( \mu_\rho \). We give an application of Theorem 1.1 in this spirit. Recall that a sequence \( \{x_n\}_{n \geq 1} \) in \( \mathbb{R} \) is called equidistributed modulo 1 if for any subinterval \( I \subset [0, 1] \), we have

\[
\lim_{N \to \infty} \frac{1}{N} \# \{n : 1 \leq n \leq N, \{x_n\} \in I \} = |I|,
\]

where \( \{x_n\} \) stands for the fractional part of \( x_n \) and \( |I| \) denotes the length of \( I \). For a fixed integer \( b > 1 \), if the sequence \( \{b^n x\}_{n \geq 1} \) is equidistributed modulo 1, then we say that \( x \) is normal in base \( b \), or equivalently \( b \)-normal. Weyl’s equidistribution criterion provides a useful tool to deal with the problem of normality. Motivated by this, Davenport-Erdős-Leveque [4] established a criterion to check whether a random number with respect to a Borel probability measure \( \mu \) is normal or not. It was reformulated by Queffélec-Ramaré [20] as follows:

**Theorem 1.3. (Davenport-Erdős-Leveque [4] [20])** Let \( \mu \) be a Borel probability measure on \( \mathbb{R} \) and \( \{s_n\}_{n \geq 1} \) a sequence of positive integers. If for any integer \( h \neq 0 \),

\[
\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m,n=1}^{N} \hat{\mu}(h(s_n - s_m)) < \infty,
\]

(1.5)

then the sequence \( \{s_n x\}_{n \geq 1} \) is equidistributed modulo 1 for \( \mu \) almost every \( x \).
In particular, when $|\hat{\mu}(\xi)| = O(|\xi|^{-\gamma})$ for some $\gamma > 0$ as $|\xi| \to \infty$, then (1.5) holds for every strictly increasing sequence $\{s_n\}_{n \geq 1}$, which implies that $\mu$ almost every $x$ is normal in any bases.

Indeed, in [20], Theorem 1.3 was stated for the Borel probability measures on $[0,1]$. However, the same result holds for any Borel probability measures on $\mathbb{R}$ with a slight modification of their proof. The next corollary is an immediate consequence of the combination of Theorem 1.1 and 1.3, which reproduces some results of Hochman-Shmerkin on the existence of normal numbers in fractals (see Section 1.3.1 in [9]).

**Corollary 1.4.** Let $\mu_\rho$ and $\varphi$ be as in Theorem 1.1, then for $\mu_\rho$ almost every $x$, $\{s_n \varphi(x)\}_{n \geq 1}$ is equidistributed modulo 1 for any strictly increasing sequence $\{s_n\}_{n \geq 1}$. In particular, for $\mu_\rho$ almost every $x$, $\varphi(x)$ is $b$-normal for every base $b$.

In fact, Hochman-Shmerkin presented a general criterion for the $b$-normality of a measure $\mu$, which adapts to many general nonlinear iterated function systems (see Theorem 1.4-1.7 in [9]). However, if $\varphi$ is a homomorphism, then $\varphi_{\mu_\rho}$ is a self-conformal measure associated with the IFS $\{\varphi^{-1} \circ f_j \circ \varphi\}_{j=1}^m$, which is $C^2$-conjugate to the linear IFS $\{f_j\}_{j=1}^m$. Thus, our result extends Theorem 1.6 in [9] in this conjugate-to-linear case by weakening the $C^\infty$ condition.

Hochman-Shmerkin’s approach is ergodic, i.e., to study the ergodic properties of the so-called scenery flow invariant distributions generated by the Gibbs measure; while our method is analytic, which allows us to obtain more quantitative information on the the decay of the Fourier transform of a certain measure. Moreover, the result of Hochman-Shmerkin does not apply to give more general statements on the equidistribution modulo 1 of $\{s_n \varphi(x)\}_{n \geq 1}$ for any strictly increasing sequence $\{s_n\}_{n \geq 1}$, so our result extends theirs in this respect.

2. A Combinatorial Lemma

In this section, we give a combinatorial lemma originated from Erdős [6] (see also [12, 14]), which is important in the estimation of the oscillatory integral in (1.4). Here and below, $\|x\|$ denotes the distance from a real number $x$ to its nearest integer. Let $c_0$ be a fixed real number and $\theta > 1$. For any $\epsilon > 0$ and fixed closed interval $[H_1, H_2] \subset \mathbb{R}$, let

$$\Gamma(\epsilon) = \left\{ x \in [H_1, H_2] : \# \{1 \leq k \leq N : \|c_0 \theta^k x\| < \frac{1}{2(1+\theta)} \} > (1-\epsilon)N \right\}.$$ 

The combinatorial lemma is stated as follows:

**Lemma 2.1.** For any $\epsilon \in (0,1/2)$, $\Gamma(\epsilon)$ could be covered by $O(\exp(\omega(\epsilon)N))$ intervals of length $c_0^{-1} \theta^{-N}$ for $N$ large enough, where $\omega(\epsilon) = -\epsilon \log \epsilon - (1-\epsilon) \log(1-\epsilon) + 2\epsilon \log(\theta+2)$.

The lemma says that the set of parameters $x$ such that $c_0 \theta^k x$ is close to an integer for “most” indices $k \in \{1, \ldots, N\}$ is very small. It turns out that the “bad” set for the sake of estimating the oscillatory integral in (1.4) can be controlled by sets of this kind. For each $x \in [H_1, H_2]$, write

$$c_0 \theta^k x = r_k(x) + \varepsilon_k(x) := r_k + \varepsilon_k,$$

where $r_k \in \mathbb{Z}$ and $-\frac{1}{2} < \varepsilon_k \leq \frac{1}{2}$. The proof of Lemma 2.1 relies on the next lemma.

**Lemma 2.2.** For each value of $r_k$, there are at most $\theta + 2$ possible choices of $r_{k+1}$. Moreover, if $\max \{||\varepsilon_k||, |\varepsilon_{k+1}|\} < \frac{1}{2(1+\theta)}$, then $r_{k+1}$ is uniquely determined by $r_k$.

**Proof.** Since $c_0 \theta^k x = r_k + \varepsilon_k$, $c_0 \theta^{k+1} x = r_{k+1} + \varepsilon_{k+1}$, we have

$$|r_{k+1} - \theta r_k| = |\varepsilon_{k+1} - \theta \varepsilon_k|.$$
Thus for a given value of $r_k$, $r_{k+1}$ falls into the interval $[\theta r_k - \frac{\theta-1}{2}, \theta r_k - \frac{\theta+1}{2}]$, which means that $r_{k+1}$ has at most $\theta+2$ choices.

Further, if $\max \{|\varepsilon_k|, |\varepsilon_{k+1}|\} < \frac{1}{2(1+\theta)}$, then

$$|r_{k+1} - \theta r_k| = |\varepsilon_{k+1} - \theta \varepsilon_k| \leq (1+\theta) \max \{|\varepsilon_k|, |\varepsilon_{k+1}|\} < 1/2.$$  

It follows that $r_{k+1}$ is uniquely determined by $r_k$.  

\[ \square \]

**Proof of Lemma 2.1** Observe that $x = \frac{r_N(x)}{c_0\theta^x} + \frac{r_N(x)}{c_0\theta^\ast} \in (\frac{r_N(x)}{c_0\theta^x} - \frac{1}{2c_0\theta^x}, \frac{r_N(x)}{c_0\theta^x} + \frac{1}{2c_0\theta^x})$. Fix $\epsilon$, we first estimate the numbers of integers $r_N$ such that $r_N = r_N(x)$ for some $x \in \Gamma(\epsilon)$.

To begin with, for a fixed $x \in \Gamma(\epsilon)$, we have $\# \{ 1 \leq k \leq N : \|c_0\theta^k x\| \geq \frac{1}{2(1+\delta)} \} < \epsilon N$. Let $1 \leq i_1 < \ldots < i_s \leq N$ be all of the indices $k$ such that $\|c_0\theta^k x\| \geq \frac{1}{2(1+\delta)}$, or equivalently, $|\varepsilon_k| \geq \frac{1}{2(1+\theta)}$. We know from the definition that $s < \epsilon N$, thus the number of choices of such $\{i_1, \ldots, i_s\}$ can not exceed

$$\sum_{j=0}^{[\epsilon N] + 1} \binom{N}{j} \ll \exp (h(\epsilon) N + o(N)),$$  

where the last inequality is by Stirling’s formula and $h(t) = -t \log t - (1-t) \log(1-t)$.

On the other hand, take a $\{i_1, \ldots, i_s\}$ as above and fix it, by Lemma 2.2 if $k$ is of the form $i_l$ or $i_l + 1$ for some $l = 1, \ldots, s$, then there are at most $\theta + 2$ possible choices of $r_k$ after $r_{k-1}$ has been given; conversely, if $k$ is not of the form $i_l$ or $i_l + 1$ for any $l = 1, \ldots, s$, then $r_k$ is uniquely determined by $r_{k-1}$. Note that the number of choices of $r_1$ is less than $c_0\theta(H_2 - H_1) + 1$, thus for each specific $\{i_1, \ldots, i_s\}$, there exists at most $(\theta + 2)^{2s} (c_0\theta(H_2 - H_1) + 1)$ values of $r_N$ such that $r_N = r_N(x)$ for some $x$ with $\# \{ 1 \leq k \leq N : \|c_0\theta^k x\| \geq \frac{1}{2(1+\delta)} \} = s < \epsilon N$.

Therefore, the number of distinct integers $r_N$ such that $r_N = r_N(x)$ for some $x \in \Gamma(\epsilon)$ is less than

$$\exp (N h(\epsilon) + o(N)) (\theta + 2)^{2s} (c_0\theta(H_2 - H_1) + 1) \ll \exp (\omega(\epsilon) N + o(N)).$$  

Since each such $r_N$ corresponds to a unique interval of length $c_0^{-1} \theta^{-N}$, we conclude the proof.  

\[ \square \]

3. **Proof of Theorem 1.1**

The proof is divided into three steps:

**Step 1.** Decomposition of the self-similar measure $\mu_{\rho}$.

In this step, we decompose $\mu_{\rho}$ by virtue of its convolution nature, which allows us to do subtle analysis. Assume $p_l = \max_{1 \leq j \leq m} p_j$ and $p_s = \min_{1 \leq j \leq m} p_j$, let $\alpha = \frac{\log p_l}{\log \rho}$. Without loss of generality, assume $a_l > a_s$. Notice that $p_l \in [1/m, 1)$ and $\rho \in (0, 1/m)$, we have $0 < \alpha < 1$. Take $\beta \in (1/2, 1)$ such that $(2 - \alpha) \beta < 1$. Let $|\xi|$ be fixed and large enough. Let $N_1, N_2$ be two positive integers satisfying

$$\rho^{-N_1} \leq |\xi|^\beta < \rho^{-N_1 - 1}, \quad \rho^{-N_2} \leq |\xi| < \rho^{-N_2 - 1}.$$  

We decompose $\mu_{\rho}$ into $\mu_{\rho} = \mu_{N_1} * \eta_{N_1}$, where $*$ stands for the convolution operator between measures, $\mu_{N_1} = \sum_{k=0}^{N_1} (p_1 \delta_{\rho^k a_1} + \cdots + p_m \delta_{\rho^k a_m})$ and $\eta_{N_1} = \sum_{k=N_1+1}^{\infty} (p_1 \delta_{\rho^k a_1} + \cdots + p_m \delta_{\rho^k a_m})$, Then
the oscillatory integral in (1.4) could be written as
\[
\int \exp(2\pi i \varphi(t))g(t)d\mu(t)
\]
\[
= \int \exp(2\pi i \varphi(t))g(t)d\mu_N(t) + \int \exp(2\pi i \varphi(t))g(t)d\mu_N(t)d\eta_N(t)
\]
(3.1)

Step 2. Linearizing the phase \(\varphi\).

Let \(\tilde{K} \subset \mathbb{R}\) be the minimal closed interval containing the support of \(\mu\). Put \(H_0 = \sup_{t \in \tilde{K}} |\varphi''(t)|\) and \(M = \sup_{t \in 2\tilde{K}} \{ |g(t)| + |g'(t)| \}\). Split the double integral in (3.1) into three parts:
\[
\int \int \exp(2\pi i \varphi(x+y))g(x+y)d\mu_N(x)d\eta_N(y) = \Upsilon_1 + \Upsilon_2 + \Upsilon_3,
\]
(3.2)

where
\[
\Upsilon_1 = \int \int \left( \exp(2\pi i \varphi(x+y)) - \exp(2\pi i \varphi(x) + \varphi'(x)y) \right)g(x+y)d\mu_N(x)d\eta_N(y),
\]
\[
\Upsilon_2 = \int \int \exp(2\pi i \varphi(x) + \varphi'(x)y) \left( g(x+y) - g(x) \right)d\mu_N(x)d\eta_N(y),
\]
and
\[
\Upsilon_3 = \int \int \exp(2\pi i \varphi(x) + \varphi'(x)y)g(x)d\mu_N(x)d\eta_N(y).
\]

By Taylor’s formula,
\[
|\varphi(x+y) - \varphi(x) - \varphi'(x)y| < H_0y^2.
\]

Combining the inequality \(|\exp(2\pi ix) - 1| \leq 2\pi |x|\), we have
\[
|\Upsilon_1| \leq \int \int M|\exp(2\pi i \varphi(x+y) - \varphi(x) - \varphi'(x)y)| - 1d\mu_N(x)d\eta_N(y)
\]
\[
\leq \int 2\pi M|\xi| \cdot |\varphi(x+y) - \varphi(x) - \varphi'(x)y|d\mu_N(x)d\eta_N(y)
\]
\[
\leq \int 2\pi MH_0|\xi|y^2d\mu_N(x)d\eta_N(y).
\]

Since \(\eta_N\) is supported on the set of points \(\{ \sum_{k=N_1+1}^{\infty} \rho^k X_k \} \) with \(X_k \in \{ a_1, \ldots, a_m \} \) for each \(k\). Thus the support of \(\eta_N\) is of length \(O(\rho^{N_1+1}) = O(|\xi|^{-2\beta})\), it follows that
\[
|\Upsilon_1| = O(|\xi| \cdot |\xi|^{-2\beta}) = O(|\xi|^{1-2\beta}).
\]
(3.3)

For \(\Upsilon_2\), similarly as \(\Upsilon_1\), we have
\[
|\Upsilon_2| \leq \int \int |g(x+y) - g(x)|d\mu_N(x)d\eta_N(y)
\]
\[
\leq \int \int M y d\mu_N(x)d\eta_N(y) = O(|\xi|^{-\beta}).
\]
(3.4)
For $\Upsilon_3$, by Fubini’s theorem,

$$
|\Upsilon_3| = \left| \int \exp \left( 2\pi i \xi \varphi(x) \right) g(x) \left( \int \exp \left( 2\pi i \xi \varphi'(x)y \right) d\eta_{N_i}(y) \right) d\mu_{N_i}(x) \right|
$$

$$
= \left| \int \exp \left( 2\pi i \xi \varphi(x) \right) g(x) \bar{\eta}_{N_i}(\xi \varphi'(x)) d\mu_{N_i}(x) \right|
$$

$$
\leq \int M |\bar{\eta}_{N_i}(\xi \varphi'(x))| d\mu_{N_i}(x).
$$

The convolution nature of $\eta_{N_i}$ implies

$$
|\bar{\eta}_{N_i}(\xi \varphi'(x))| = \prod_{k=N_i+1}^{\infty} |\Phi(\xi \varphi(x)\rho^k)|,
$$

where $\Phi(t) = \sum_{j=1}^{m} p_j \exp(2\pi i a_j t)$. Therefore,

$$
|\Upsilon_3| \leq M \int \left( \prod_{k=N_i+1}^{\infty} |\Phi(\xi \varphi(x)\rho^k)| \right) d\mu_{N_i}(x).
$$

Recall that $\rho^{-N_2} \leq |\xi| < \rho^{-N_2-1}$, set $N = N_2 - N_1 - 1 > 0$ and $\theta = \rho^{-1} > 1$. Write $\xi = \xi_0 \theta^N$, then $1 \leq |\xi_0| < \theta$. Since $|\Phi(t)| \leq 1$, it is easy to see that

$$
\int \left( \prod_{k=N_i+1}^{\infty} |\Phi(\xi \varphi(x)\rho^k)| \right) d\mu_{N_i}(x)
$$

$$
= \int \left( \prod_{k=N_i+1}^{\infty} |\Phi(\xi_0 \theta^N \varphi(x)\rho^k)| \right) d\mu_{N_i}(x)
$$

$$
\leq \int \left( \prod_{k=1}^{N} |\Phi(\xi_0 \theta^k \varphi(x))| \right) d\mu_{N_i}(x).
$$

Write $\Xi(\xi) = \int \left( \prod_{k=1}^{N} |\Phi(\xi_0 \theta^k \varphi(x))| \right) d\mu_{N_i}(x)$, now we arrive at the conclusion that

$$
|\Upsilon_3| \leq M \Xi(\xi).
$$

**Step 3.** Estimation of the integral $\Xi(\xi)$.

Put $H_1 = \min_{t \in \tilde{K}} (a_l - a_s) \varphi'(t)$ and $H_2 = \max_{t \in \tilde{K}} (a_l - a_s) \varphi'(t)$. Fix a sufficiently small $\epsilon > 0$ which will be specified later, denote

$$
\Lambda(\epsilon) = \left\{ x \in \tilde{K} : (a_l - a_s) \varphi(x) \in \Gamma(\epsilon) \right\},
$$

where $\Gamma(\epsilon)$ is defined as in Section 2 with $c_0 = \xi_0$. Then $\Xi(\xi)$ can be divided into two parts:

$$
\Xi(\xi) = \left( \int_{\Lambda(\epsilon)} + \int_{\Lambda(\epsilon)^c} \right) \left( \prod_{k=1}^{N} |\Phi(\xi_0 \theta^k \varphi'(x))| \right) d\mu_{N_i}(x) := \Xi_1(\xi) + \Xi_2(\xi).
$$
For each \(1 \leq k \leq N\), if \(\|\xi_0 \theta^k (a_1 - a_s) \varphi'(x)\| \geq \frac{1}{2(1+\theta)}\), then
\[
|\Phi(\xi_0 \theta^k \varphi'(x))| = \left| \sum_{j=1}^{m} p_j \exp(2\pi i a_j \xi_0 \theta^k \varphi'(x)) \right| \leq |p_1 \exp (2\pi i (a_1 - a_s) \xi_0 \theta^k \varphi'(x))| + \cdots + |p_m \exp (2\pi i (a_m - a_s) \xi_0 \theta^k \varphi'(x))| + 1 - p_1 - p_s \leq 1 - \frac{2p_1}{1+\theta} := \delta < 1.
\]
Notice that \(N = N_2 - N_1 - 1 = O(\log |\xi|)\), thus for each \(x \in [0, 1] \setminus \Lambda(\epsilon)\),
\[
\prod_{k=1}^{N} |\Phi(\xi_0 \theta^k \varphi'(x))| < \delta^{-N} = |\xi|^{-\frac{c(1-\beta)\log \rho}{\log \rho}}.
\]
As a result, we have \(\Xi_2(\xi) = O(\|\xi|^{-\frac{c(1-\beta)\log \rho}{\log \rho}})\).

On the other hand, denote the intervals of length \(c_0^{-1}\theta^{-N}\) which cover \(\Lambda(\epsilon)\) by \(I_1, \ldots, I_Q\). It is known by Lemma 2.1 that \(Q = O(\exp(\omega(\epsilon)N))\) with \(\omega(\epsilon) = -\epsilon \log \epsilon - (1-\epsilon) \log(1-\epsilon) + 2\epsilon \log(\theta + 2)\). For each \(1 \leq q \leq Q\), let \(J_q = (\varphi)^{-1}((a_1 - a_s)^{-1} I_q)\), since \(\varphi'\) is strictly increasing, we have
\[
\Lambda(\epsilon) \subseteq \bigcup_{q=1}^{Q} J_q.
\]
It follows that
\[
\Xi_1(\xi) \leq \sum_{q=1}^{Q} \int_{J_q} \left( \prod_{k=1}^{N} |\Phi(\xi_0 \theta^k \varphi'(x))| \right) d\mu_{N_1}(x) \ll \exp(\omega(\epsilon)N) \max_{1 \leq q \leq Q} \mu_{N_1}(J_q).
\]
We know from the definition of \(\mu_{N_1}\) and \(J_q\) that, the length of each \(J_q\) satisfies \(|J_q| = O(\rho^{N}) = O(|\xi|^{-\beta-1})\), while the gaps between different atoms of the discrete measure \(\mu_{N_1}\) are at least \(O(\rho^{N}) = O(|\xi|^{-\beta})\). Therefore, the number of atoms of \(\mu_{N_1}\) contained in each \(J_q\) is about \(O(|\xi|^{-\beta})\), which implies that
\[
\mu_{N_1}(J_q) = O(|\xi|^{-\beta-1} \rho N_1) = O(|\xi|^{-\beta-1} \rho \alpha N_1) = O(|\xi|^{-\beta-1}).
\]
So we have
\[
\Xi_1(\xi) = O(\exp(\omega(\epsilon)N)) = O(|\xi|^{-\beta-1}) = O(\exp(|\xi|^{-\frac{c(1-\beta)\log \rho}{\log \rho}} + 1 - (2-\alpha)\beta)).
\]
Taking \(\epsilon \in (0, \delta)\), since \(\omega(\epsilon) + 1 - (2-\alpha)\beta > \frac{1-\beta}{\log \rho}\), it follows that
\[
\Xi(\xi) = \Xi_1(\xi) + \Xi_2(\xi) = O(|\xi|^{-\frac{c(1-\beta)\log \rho}{\log \rho}}).
\]
From (3.1)-(3.6) and the fact that \(\beta \in (1/2, 1)\), we obtain (1.4), where the constant \(\gamma\) is given by \(\min\{2\beta - 1, \frac{1-\beta}{\log \rho}\}\) once we take appropriate \(\beta\) and \(\epsilon\). The second assertion is by taking \(g \equiv 1\).

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