Genera of non-algebraic leaves of polynomial foliations of $\mathbb{C}^2$

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In this article, we prove two results. First, we construct a dense subset in the space of polynomial foliations of degree $n$ such that each foliation from this subset has a leaf with at least \((n+1)(n+2)/2 - 4\) handles. Next, we prove that for a generic foliation invariant under the map \((x, y) \mapsto (-x, y)\) all leaves have infinitely many handles.

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1 Introduction

Consider a polynomial differential equation in $\mathbb{C}^2$ (with complex time),
\begin{align}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y),
\end{align}
where $\max(\deg P, \deg Q) = n$. The splitting of $\mathbb{C}^2$ into trajectories of this vector field defines a singular analytic foliation of $\mathbb{C}^2$.

Denote by $\mathcal{A}_n$ the space of foliations of $\mathbb{C}^2$ defined by vector fields (1) of degree at most $n$ with coprime $P$ and $Q$. Two vector fields define the same foliation if they are proportional, hence $\mathcal{A}_n$ is a Zariski open subset of the projective space of dimension $\frac{n(n+1)(n+2)}{2} - 1$.

Denote by $\mathcal{B}_n$ the space of foliations of $\mathbb{C}P^2$ defined by a polynomial vector field (1) of degree at most $n$ in each affine chart. It is easy to show that $\mathcal{A}_n \subset \mathcal{B}_{n+1} \subset \mathcal{A}_{n+1}$.

Numerous studies in this field are devoted to the properties of generic foliations from $\mathcal{A}_n$ and $\mathcal{B}_n$. Another classical question concerns degree and genus of an algebraic leaf of a polynomial foliation. We study genera of non-algebraic leaves.

For a generic analytic foliation, the question about the topology of a leaf was studied by T. Firsova and T. Golenishcheva–Kutuzova.

**Theorem** ([2, 1]). Among leaves of a generic analytic foliation, countably many are topological cylinders, and the rest are topological discs.

For a generic polynomial foliation, the analogous result is not known. The fact that almost all leaves are topological discs would follow from Anosov conjecture on identical cycles.

**Definition.** Identical cycle on a leaf $L$ is an element $[\gamma]$ of the free homotopy group of $L$ such that the holonomy along (any) its representative $\gamma$ is identical.

**Conjecture** (D. Anosov). A generic polynomial foliation has no identical cycles.

In [Section 2] [A leaf with many handles], we give a partial answer to the question “How complicated can be a topological structure of a leaf of a foliation from some dense subset in $\mathcal{A}_n$?”. We prove the following theorem.

**Theorem 1.** For each $n \geq 2$, the set of polynomial foliations having a leaf with at least $\frac{(n+1)(n+2)}{2} - 4$ handles is dense in $\mathcal{A}_n$.

This theorem is inspired by the following theorem due to Denis Volk [15].

**Theorem** (Density of foliations with separatrix connection). For each $n \geq 2$, the set of polynomial foliations having a separatrix connection is dense in $\mathcal{A}_n$.

We shall discuss the latter theorem in more details in [Section 2.4] [Volk Theorem] below. In [Section 3] [Leaves of infinite genus], we get the following result:

**Theorem 2.** Let $\mathcal{A}_n^{sym}$ (resp., $\mathcal{B}_n^{sym}$) be the subspace of $\mathcal{A}_n$ (resp., $\mathcal{B}_n$) given by
\[ P(-x, y) = P(x, y), \quad Q(-x, y) = -Q(x, y). \]
(2)

Take $n \geq 2$. For any foliation $\mathcal{F}$ from some open dense subset of $\mathcal{A}_n^{sym}$ (resp., $\mathcal{B}_n^{sym}$), all leaves of $\mathcal{F}$ (except for a finite set of algebraic leaves) have infinite genus.

\(^1\)From now on, “dimension” means “complex dimension”.

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There are some unpublished earlier results in this direction. For generic homogeneous vector fields, almost all leaves have infinite genus; the proof is due to Yu.Ilyashenko, but it was never written down. We write it in Section 3.3 [Proof of Ilyashenko Theorem].

In the unpublished draft version of his thesis, V. Moldavskis [10] proves that for a generic vector field of degree $n \geq 5$ with real coefficients and the symmetry (2) each leaf has infinitely-generated first homology group. However this is only a draft text, so the proof lacks some details and has some gaps.

2 A leaf with many handles

First, we recall some results that we use in the proof of Theorem 1. In some cases we formulate their refined versions or give explicit constructions.

2.1 Extension to infinity

Let us extend a polynomial foliation $\mathcal{F} \in \mathcal{A}_n$ given by (1) to $\mathbb{CP}^2$. For this end, make the coordinate change $u = \frac{1}{x}$, $v = \frac{y}{x}$, and the time change $d\tau = -u^{n-1}dt$. The vector field takes the form

\[
\begin{align*}
\dot{u} &= u\tilde{P}(u,v) \\
\dot{v} &= v\tilde{P}(u,v) - \tilde{Q}(u,v)
\end{align*}
\]

where $\tilde{P}(u,v) = P\left(\frac{1}{u}, \frac{v}{u}\right) u^n$ and $\tilde{Q}(u,v) = Q\left(\frac{1}{u}, \frac{v}{u}\right) u^n$ are two polynomials of degree at most $n$.

Since $\dot{u}(0,v) \equiv 0$, the infinite line $L_\infty = \{u = 0\}$ is invariant under this vector field. Denote by $h(v)$ the polynomial $\dot{v}(0,v) = v\tilde{P}(0,v) - \tilde{Q}(0,v)$. There are two cases.

**Dicritical case, $h(v) \equiv 0$** In this case (3) vanishes identically on $L_\infty$. Thus it is natural to consider the time change $d\tau = -u^ndt$ instead of $d\tau = -u^{n-1}dt$, and study the vector field

\[
\begin{align*}
\dot{u} &= \tilde{P}(u,v) \\
\dot{v} &= \frac{v\tilde{P}(u,v) - \tilde{Q}(u,v)}{u}
\end{align*}
\]

whose trajectories are almost everywhere transverse to $L_\infty$. This case corresponds to $\mathcal{B}_n \subset \mathcal{A}_n$.

**Non-dicritical case, $h(v) \not\equiv 0$** In this case (3) has isolated singular points $a_j \in L_\infty$ at the roots of $h$, and $L_\infty \setminus \{a_1, a_2, \ldots\}$ is a leaf of the extension of $\mathcal{F}$ to $\mathbb{CP}^2$.

Denote by $\mathcal{A}_n'$ the set of foliations $\mathcal{F} \in \mathcal{A}_n$ such that $h$ has $n+1$ distinct roots $a_j$, $j = 1, \ldots, n+1$. In particular, all these foliations are non-dicritical.

For each $j$, let $\lambda_j$ be the ratio of the eigenvalues of the linearization of (3) at $a_j$ (the eigenvalue corresponding to $L_\infty$ is in the denominator). One can show that $\sum \lambda_j = 1$, and this is the only relation on $\lambda_j$.

2.2 Monodromy group and rigidity

For $\mathcal{F} \in \mathcal{A}_n'$, fix a non-singular point $O \in L_\infty$ and a cross-section $S$ at $O$ given by $v = \text{const}$. For a loop $\gamma \in \pi_1(L_\infty, O)$, denote by $M_\gamma : (S, O) \rightarrow (S, O)$ the monodromy map along $\gamma$. It is easy to see that the map $\gamma \mapsto M_\gamma$ reverses the order of multiplication,

$$M_{\gamma\gamma'} = M_{\gamma'} \circ M_\gamma.$$
The set of all possible monodromy maps $M_\gamma, \gamma \in \pi_1(L_\infty, O)$, is called the monodromy pseudogroup $G = G(\mathcal{F})$. The word “pseudogroup” means that there is no common domain where all elements of $G$ are defined. However we will follow the tradition and write “monodromy group” instead of “monodromy pseudogroup”.

Remark. This construction does not work for a foliation from $\mathcal{B}_n$, because a generic foliation from $\mathcal{B}_n$ has no algebraic leaves. That’s why $\mathcal{B}_n$ is worse studied than $\mathcal{A}_n$.

Choose $n + 1$ loops $\gamma_j \in \pi_1(L_\infty, O), j = 1, 2, \ldots, n + 1$, passing around points $a_j$, respectively. Then the pseudogroup $G(\mathcal{F})$ is generated by monodromy maps $M_j = M_{\gamma_j}$. It is easy to see that the multipliers $\mu_j = M_j'(0)$ are equal to $\exp 2\pi i \lambda_j$.

A generic foliation of $\mathbb{R}^2$ is structurally stable, i.e. its small perturbation is topologically conjugate to the initial foliation. For a generic polynomial foliation of $\mathbb{C}^2$, we have the opposite property, called rigidity. Informally, a close foliation is topologically conjugate to the initial one only if it is affine conjugate to it.

There are few different theorems of the form “topological conjugacy of polynomial foliations plus some assumptions imply affine conjugacy of these foliations”, see [3, 12, 11]. These theorems are called Rigidity Theorems with various adjectives that depend on the extra assumptions on the foliations and conjugating homeomorphism. We shall need the following theorem.

**Theorem 3.** There exists an open dense subset $\mathcal{A}_n^R \subset \mathcal{A}_n'$ such that for each $\mathcal{F}_0 \in \mathcal{A}_n^R$ the following holds. There exists a neighborhood $U \ni \mathcal{F}_0$ such that for $\mathcal{F} \in U$ the analytic conjugacy of their monodromy groups at infinity $G(\mathcal{F}_0), G(\mathcal{F})$ (as groups with marked generators) implies the affine conjugacy of foliations.

This theorem easily follows from the proof of Theorem 28.32 in [5]. Theorem 28.32 states that the analogue of [Theorem 3] holds for a full-measure subset of $\mathcal{A}_n$ even if we require only a topological conjugacy of monodromy groups. First, the authors prove that for $\mathcal{F}_0$ from some full-measure subset of $\mathcal{A}_n$ the topological conjugacy of monodromy groups implies the analytic conjugacy. This is the only place where they use the assumption that the multipliers $\mu_i$ of $\mathcal{F}_0$ generate a dense multiplicative subgroup in $\mathbb{C}^*$. Other assumptions on $\mathcal{F}_0$ define an open dense set in $\mathcal{A}_n$, thus the rest of the proof of Theorem 28.32 yields [Theorem 3].

### 2.3 Infinite number of limit cycles

The following definition generalizes the notion of a limit cycle of a foliation of $\mathbb{R}^2$.

**Definition.** Limit cycle on a leaf $L$ is an element $[\gamma]$ of the free homotopy group of $L$ such that the holonomy along (any) its representative $\gamma$ is non-identical (cf. with the definition of identical cycle).

Note that each isolated fixed point $z_0$ of some monodromy map $M_\gamma \in G$ gives us a limit cycle, namely we can take the lifting of the loop $\gamma$ that starts from $z_0$.

**Definition.** A set of limit cycles of a foliation is called homologically independent, if for any leaf $L$ all the cycles located on this leaf are linearly independent in $H_1(L)$.

The following result was obtained in [13].

**Theorem 4.** For $n \geq 3$, for an open dense set $\mathcal{A}^{LC}_n \subset \mathcal{A}_n$, each $\mathcal{F} \in \mathcal{A}^{LC}_n$ possesses an infinite number of homologically independent limit cycles.
The next lemma was proved in [3], and was heavily used in the same paper to establish new properties of generic polynomial foliations. In particular, it was used in the proof of a weaker version of Theorem 4.

**Lemma 5.** Let \( g : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) be an expanding analytic germ, \(|g'(0)| > 1\). Let \( \mu, |\mu| < 1 \), be a number such that the multiplicative semigroup generated by \( \mu \) and \( g'(0) \) is dense in \( \mathbb{C}^* \). Then for each \( \nu \in \mathbb{C}^* \), the linear map \( z \mapsto \nu z \) can be approximated by a map of the form \( z \mapsto \mu^{-s}g'(\mu^{r+s}z) \) with any prescribed accuracy. Moreover, if \( g \) depends analytically on some parameter \( \varepsilon \in (\mathbb{C}^n, 0) \), then this approximation is uniform in \( \varepsilon \).

**Idea of the proof.** Due to the condition on \( \mu \) and \( g'(0) \), one can approximate \( \nu \) by a number of the form \((g'(0))^t \mu^r\). Thus the multiplier of the map \( z \mapsto g'(\mu^r z) \) is close to \( \nu \). Then we conjugate this map by a strongly contracting linear map \( z \mapsto \mu^s z \). The obtained map has the form \( z \mapsto \mu^{-s}g'(\mu^{r+s}z) \) and is close to its linear part \( z \mapsto (g'(0))^t \mu^r z \), hence to the map \( z \mapsto \nu z \).

For details including dependence on \( \varepsilon \) see, e.g., [15].

We shall need another lemma that was used in [3] to prove a weaker version of Theorem 4 (for full Lebesgue measure in \( \mathbb{A}_n \) instead of an open dense set). In some assumptions on the monodromy group, it gives an explicit construction of a monodromy map \( M_\nu \in \mathcal{G} \) having an isolated fixed point arbitrarily close to zero.

**Lemma 6.** Suppose that two monodromy maps \( M_1 \) and \( M_2 \) do not commute, and their multipliers satisfy

- \(|\mu_1| < 1, |\mu_2| > 1\);
- the multiplicative semigroup generated by \( \mu_1 \) and \( \mu_2 \) is dense in \( \mathbb{C}^* \).

Then the set of hyperbolic fixed points of compositions of the form \( M_1^{-s}M_2M_1^{r+s}M_2 \) is dense in a small neighborhood of the origin.

**Proof.** We will work in a linearizing chart for \( M_1 \). Let \( z_0 \) be close enough to zero; since \( M_1 \) and \( M_2 \) do not commute, we can assume, after a small perturbation of \( z_0 \), that \(|M_2'(z_0)| \neq \left| \frac{M_2'(z_0)}{z_0} \right|\).

Note that the map \( z \mapsto \frac{z_0}{M_2'(z_0)} M_2(z) \) has an isolated hyperbolic fixed point at \( z_0 \). Due to Lemma 5, we can approximate the linear map \( z \mapsto \frac{z_0}{M_2'(z_0)} z \) by a map of the form \( M_1^{-s}M_2M_1^{r+s} \). If this map is close enough to \( z \mapsto \frac{z_0}{M_2'(z_0)} z \), then the map \( M_1^{-s}M_2M_1^{r+s}M_2 \) also has a hyperbolic fixed point close to \( z_0 \).

We shall also need the following theorem, see [3].

**Theorem 7.** There exists a subset \( \mathcal{A}_n^F \subset \mathcal{A}_n^F \) of full Lebesgue measure such that for each \( F \in \mathcal{A}_n^F \), the monodromy group at infinity is the free group with \( n \) generators \( M_j, j = 0, \ldots, n-1 \).

This theorem implies that there are no identical cycles on \( L_\infty \), thus proves Anosov conjecture for cycles on \( L_\infty \).

### 2.4 Volk Theorem

The following statement is a generalized version of the main theorem from [15].

**Theorem.** Let \( \mathcal{F} \) be a polynomial foliation of degree \( n \geq 2 \). Let \( a, b \) be holomorphic maps of a neighborhood of \( \mathcal{F} \) in \( \mathcal{A}_n \) to \( \mathbb{C}^2 \). Then there exists \( \mathcal{F} \) arbitrarily close to \( \mathcal{F} \) such that the points \( a(\mathcal{F}) \) and \( b(\mathcal{F}) \) belong to the same leaf of \( \mathcal{F} \).
D. Volk proved this theorem for the case when $a(F)$ and $b(F)$ belong to two different separatrices of $F$ (see Section 1 Introduction). Actually, his arguments work for the general case as well, but we shall need a slightly more precise statement.

**Theorem 8.** Let $M$ be a germ of an analytic submanifold of $A^R_n$ at $\tilde{F}$, $\dim M > \dim \text{Aff}(C^2) = 6$, such that

- $\mu_1 = \text{const}$ and $\mu_2 = \text{const}$ on $M$;
- $|\mu_1| < 1$ and $|\mu_2| < 1$;
- the multiplicative semigroup generated by $\mu_1$ and $\mu_2^{-1}$ is dense in $C^*$;

Let $S$ be a cross-section at infinity that is included by the Schröder chart for $M_1$. Let $a, b : M \to S$ be two holomorphic functions. Then there exist two loops $\gamma$ and $\gamma'$ such that the condition $M_{\gamma'}(a(F)) = b(F)$ defines a non-empty submanifold $M' \subset M$ of codimension one.

Moreover, the loops can be constructed in the following way. There exists an index $i$ such that for each sufficiently large $p$ we can choose either $\gamma = \gamma_1^i \gamma_i$ or $\gamma = \gamma_1'$. After $i$ and $p$ are fixed, there exists a triple of arbitrarily large numbers $(r, s, t)$ such that we can take $\gamma' = \gamma_1^{r+s} \gamma_2^{-t} \gamma_1^{-s}$.

Let $z = z_F : S \to (C, 0)$ be the Schröder chart for $M_1$ such that the change of coordinates with respect to some fixed chart is parabolic. Then $z_F$ depends analytically on $F$. In the rest of this section, $M_i, a(F), b(F)$ etc. are written in the corresponding chart $z = z_F$. In particular, $M_1(z) = \mu_1 z$.

The objects corresponding to $\tilde{F}$ will be denoted by the tilde above, e.g., $\tilde{a}, \tilde{b}, \tilde{M}_i$.

**Theorem 8** is an immediate consequence of the following two lemmas.

**Lemma 9.** In the assumptions of **Theorem 8**, suppose that the ratio $\frac{a}{b} = \frac{z_F(a(F))}{z_F(b(F))}$ is a non-constant function of $F$. Then we can choose a triple of arbitrarily large numbers $(r, s, t)$ such that for $\gamma' = \gamma_1^{r+s} \gamma_2^{-t} \gamma_1^{-s}$ the condition $M_{\gamma'}(a(F)) = b(F)$ defines a non-empty submanifold $M' \subset M$ of codimension one.

**Lemma 10.** In the assumptions of **Theorem 8**, we can find an index $i$ such that for each sufficiently large $p$ either for $\gamma = \gamma_1^i \gamma_i$ or for $\gamma = \gamma_1^p$, the equality $\frac{M_i(a)}{b} = \frac{\tilde{M}_i(a)}{\tilde{b}}$ does not hold.

Indeed, it is sufficient to take $\gamma$ from **Lemma 10** and substitute $M_{\gamma}(a)$ for $a$ in **Lemma 9**.

**Remark.** **Lemma 10** is a refined version of the union of Lemmas 6 and 7 in [15]. The proof of Lemma 6 in [15] deals separately with $n \geq 3$ and $n = 2$; unfortunately, the proof for the case $n = 2$ has a gap. We give another proof which works for all $n \geq 2$.

Now let us prove the lemmas.

**Proof of Lemma 9.** Since the equality $\frac{a}{b} = \frac{\tilde{a}}{\tilde{b}}$ is not trivial, it defines a codimension-one submanifold in $(M, \tilde{F})$. This submanifold is non-empty, because it contains $\tilde{F}$.

Let us approximate the linear map $z \mapsto \frac{\tilde{b}}{\tilde{a}} z$ in the chart $z$ by a map of the form $M_{\gamma'} = M_1^{r} M_2^{-t} M_1^{-s}$. This approximation is uniform with respect to $F$ (see **Lemma 5**). If $M_{\gamma'}$ is sufficiently close to this linear map, then the holomorphic function $M_{\gamma'}(a) - b$ on $(M, \tilde{F})$ is close to the function $\frac{\tilde{b}}{\tilde{a}} a - b$, thus the condition $M_{\gamma'}(a) - b = 0$ also defines a codimension-one nonempty submanifold $M' \subset M$. \qed
Remark. In the chart $z$, $M_\gamma$ approximates the linear map $z \mapsto \frac{b}{M_\gamma(a)}z$ in $C^0$ in a larger domain, hence it approximates this linear map in $C^\infty$ in a smaller neighborhood of $M_\gamma(a)$. In particular, the derivative $\frac{d}{dz}M_\gamma|_{M_\gamma(a)}$ can be made arbitrarily close to $\frac{b}{M_\gamma(a)}$ uniformly in $\mathcal{F} \in \mathcal{M}'$. Further, $M_\gamma'(\mu_p a) \approx \mu_i$ and $\frac{M_i(\mu_p a)}{\mu_p a} \approx \mu_i$ for large $p$, hence $M_\gamma''(a)$ can be made arbitrarily close to $\frac{b}{a}$.

Proof of Lemma 10. Since $\dim \mathcal{M} > \dim \text{Aff}(\mathbb{C}^2)$, there exists $\mathcal{F} \in \mathcal{M}$ close to $\tilde{\mathcal{F}}$ which is not affine conjugated to $\tilde{\mathcal{F}}$. Since $\tilde{\mathcal{F}} \in \mathcal{A}_R$, Theorem 3 implies that the monodromy groups at infinity of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are not analytically conjugated as groups with marked generators.

Hence there exists $i$ such that $M_i$ is not conjugate to $\tilde{M}_i$ by the map $z \mapsto \frac{\tilde{a}}{a}z$. Fix a punctured neighborhood $U$ of the origin such that
\[
\frac{\tilde{a}}{a}M_i(z) \neq \tilde{M}_i\left(\frac{\tilde{a}}{a}z\right) \quad \forall z \in U.
\] 
Let $p$ be a large integer number such that $\mu_1^p a \in U$. Note that $M_i^p(a) = \mu_1^p a$, $\tilde{M}_i^p(\tilde{a}) = \mu_1^p \tilde{a}$. If the assertion of the lemma fails for $p$, then we have both $\frac{\mu_1^p a}{b} = \frac{\tilde{a}}{a} \mu_1^p a$ and $\frac{M_i(\mu_1^p a)}{b} = \frac{\tilde{M}_i(\mu_1^p \tilde{a})}{b}$, thus $\frac{\tilde{a}}{a}M_i(\mu_1^p a) = \tilde{M}_i\left(\frac{\tilde{a}}{a} \cdot \mu_1^p a\right)$, and this contradicts the inequality above. \hfill \Box

2.5 Proof of Theorem 1

Consider an open subset $U \subset \mathcal{A}_n$. Shrinking $U$ if necessary, we may and will assume that $U \subset \mathcal{A}_R^\infty$.

Let us enumerate the singular points at $L_\infty$ so that $(|\mu_1| - 1)(|\mu_2| - 1) < 0$ and $(|\mu_1| - 1)(|\mu_3| - 1) > 0$. Without loss of generality, we may assume that $|\mu_1| < 1$, $|\mu_2| > 1$ and $|\mu_3| < 1$. Indeed, if $|\mu_1| > 1$, then we can pass to the conjugated coordinates $(\bar{x}, \bar{y})$ in $\mathbb{C}^2$. Let us choose $\tilde{\mathcal{F}} \in \mathcal{U}$ such that

- the multiplicative semigroup generated by $\mu_1$ and $\mu_2$ is dense in $\mathbb{C}^*$;
- the multiplicative semigroup generated by $\mu_1^{-1}$ and $\mu_3$ is dense in $\mathbb{C}^*$;
- $M_1 \circ M_2 \neq M_2 \circ M_1$ and $M_1 \circ M_3 \neq M_3 \circ M_1$.

This is possible since each of three conditions defines a subset of $\mathcal{A}_n$ of full Lebesgue measure. For the first two conditions it is clear, and for the third one it follows from Theorem 7. Shrinking $U$ if necessary, we may and will assume that the third condition holds for the whole $U$.

Let $\mathcal{M}_0 \subset U$ be the submanifold given by $\mu_1 = \text{const}$, $\mu_2 = \text{const}$ and $\mu_3 = \text{const}$ passing through $\tilde{\mathcal{F}}$. For $n = 2$, $\text{codim} \mathcal{M}_0 = 2$, otherwise $\text{codim} \mathcal{M}_0 = 3$.

The following lemma is a key step in the proof of Theorem 1.

Lemma 11. Let $\mathcal{M}_0$ and $\mu_i$, $i = 1, 2, 3$ be as above. Let $\mathcal{M} \subset \mathcal{M}_0$ be an analytic submanifold of dimension at least 7. Then for any $\varepsilon > 0$ there exists a submanifold $\mathcal{M}' \subset \mathcal{M}$ of codimension one such that each $\mathcal{F} \in \mathcal{M}'$ has a leaf with a handle $\varepsilon$-close to $L_\infty$. More precisely, for each $\mathcal{F} \in \mathcal{M}'$ there exists a leaf $L \in \mathcal{F}$ and two hyperbolic cycles $c_1, c_2 \subset L \cap U_\varepsilon(L_\infty)$ analytically depending on $\mathcal{F}$ with intersection index $c_1 \cap c_2 = 1$.

We shall postpone the proof of this lemma till the end of this section. Now let us deduce Theorem 1 from this lemma. First, we obtain many handles on different leaves.
Corollary 12. For each $0 \leq g \leq \dim \mathcal{M}_0 - 6$, there exists an analytic submanifold $\mathcal{M}_g \subset \mathcal{M}_0$ of codimension at most $g$ such that each $\mathcal{F} \in \mathcal{M}_g$ possesses $g$ handles with hyperbolic generating cycles $(c_1, c_2), (c_3, c_4), \ldots, (c_{2g-1}, c_{2g})$, and the generators of different handles do not intersect.

Proof. Let us prove the assertion by induction. For $g = 0$, we just take $\mathcal{M}_0$. Suppose that we already have $\mathcal{M}_g, g < \dim \mathcal{M}_0 - 7$. Then $\dim \mathcal{M}_g \geq 7$. Using Lemma 11 we get a submanifold $\mathcal{M}_{g+1} \subset \mathcal{M}_g$ of codimension 1 such that each $\mathcal{F} \in \mathcal{M}_{g+1}$ possesses a handle which is closer to $L_\infty$ than all the handles guaranteed by $\mathcal{M}_g$. Hence, $\mathcal{M}_{g+1}$ satisfies the assertion of this corollary. This completes the proof. \hfill $\square$

Proof of Theorem 1. Let us apply the previous corollary to $g = \frac{(n+1)(n+2)}{2} - 4$. This is possible since for $n \geq 2$ we have

$$\frac{(n+1)(n+2)}{2} - 4 \leq (n+1)(n+2) - 10 \leq \dim \mathcal{M}_0 - 6.$$ 

Let $\widetilde{\mathcal{M}}_0 \subset \mathcal{U}, \widetilde{\mathcal{M}}_0 \supset \mathcal{M}_0$ be the submanifold defined by $\mu_1 = \text{const}, \mu_2 = \text{const}$. It is easy to see that $\dim \mathcal{M}_0 = (n+1)(n+2) - 3$ and $\mathcal{M}_g$ extends to a submanifold $\mathcal{M}_g \subset \mathcal{M}_0$ of the same codimension $g$. Indeed, all the cycles $c_1, \ldots, c_{2g}$ provided by the corollary are hyperbolic, hence they survive under a small perturbation in $\widetilde{\mathcal{M}}_0$. The submanifold $\widetilde{\mathcal{M}}_g$ is defined by $g$ equations of the form “$c_{2i-1}$ intersects $S$ at the same point as $c_{2i}$”. Since $\mathcal{M}_g$ is not empty, it is a submanifold of codimension $g$,

$$\dim \widetilde{\mathcal{M}}_g = \dim \widetilde{\mathcal{M}}_0 - g = \frac{(n+1)(n+2)}{2} + 1 = (g - 1) + 6$$

Applying Theorem 8 $(g - 1)$ times, we obtain a 6-dimensional submanifold $\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{M}}_0$ such that for each $\mathcal{F} \in \mathcal{M}$ all the handles guaranteed by the corollary are located on the same leaf. \hfill $\square$

Now let us prove Lemma 11.

Proof of Lemma 11. Due to Lemma 6 there exist $k, l$ and $m$ such that $M_1^{-k}M_2^{-m}M_1^{k+l} \circ M_2$ has a hyperbolic fixed point $p_0$ near the origin. We require some additional conditions on $p_0$. More precisely,

- first we choose a domain for $p_0$ sufficiently close to the origin, so that $|M_2(p_0)| > |p_0|$, hence $M_2(p_0) \neq p_0$;
- then we shrink this domain so that in the linearizing chart for $M_3$ we have $|M_1'(p_0)| \neq \left| \frac{M_1(p_0)}{p_0} \right|$;
- finally, we choose $k, l, m$ so large that $|M_1^{k+l}(M_2(p_0))| < |p_0|$, hence $M_1^{k+l}(M_2(p_0)) \neq p_0$.

Let $c_1$ be the corresponding hyperbolic cycle, i.e., the lifting of $\gamma_1^{k+l} \gamma_2^{-m} \gamma_1^{-k}$ starting at $p_0$. Figure “First cycle” shows its projection to $L_\infty$. Clearly, $c_1$ survives under a small perturbation.
Figure 1: First cycle for $k = 2$, $l = 3$, $m = 6$

Figure 2: Second cycle for $p = 2$, $q = 1$, $r = 1$, $s = 2$, $t = 5$

Now let us apply Theorem 8 to the points $a(\mathcal{F}) = M_1(p_0(\mathcal{F}))$ and $b(\mathcal{F}) = p_0(\mathcal{F})$ and the maps $M_3$ and $M_1$. Due to this theorem, there exist $i \in \{1, \ldots, n + 1\}$, $p \in \mathbb{N}$, $q \in \{0, 1\}$, $r \in \mathbb{N}$, $s \in \mathbb{N}$, $t \in \mathbb{N}$ such that the equality $M(p_0(\mathcal{F})) = p_0(\mathcal{F})$ defines a codimension one submanifold $\mathcal{M}' \subset \mathcal{M}$, where $M = M_3^{-s} \circ M_1^{-t} \circ M_i^{r+s} \circ M_1^q \circ M_3^p \circ M_1$. Let $c_2$ be the corresponding cycle (see Figure "Second cycle" for its projection to $L_\infty$).
Let us prove that one can choose the numbers in \textbf{Theorem 8} so that \(c_2\) is a hyperbolic cycle and it intersects \(c_1\) transversally at exactly one point \(p_0(\mathcal{F})\).

Since \(s\) and \(p\) can be made arbitrarily large, we may and will assume that \(M_3(p_0(\mathcal{F}))\) and \(M_3^p(M_1(p_0(\mathcal{F})))\) are much closer to the origin than all points of \(c_1\). Then the parts of \(c_2\) that go along \(\gamma_1\) and \(\gamma_2\) are closer to the origin than \(c_1\), in particular these parts of \(c_2\) do not intersect \(c_1\). Therefore, \(c_2\) may intersect \(c_1\) only above points \(A, B,\) and \(C\). Due to the choice of \(c_1, p_0\) is not a fixed point for \(M_2\) and \(M_4^{k+1} \circ M_2\), hence the loops do not intersect over \(C\) and \(A\). Also, recall that \(p_0\) belongs to the domain of the linearizing chart of \(M_1\), thus \(M_3^k(p_0) \neq p_0\), so \(c_1\) does not intersect \(c_2\) over \(B\).

Therefore, \(p_0(\mathcal{F})\) is the only intersection point of \(c_1\) and \(c_2\). Note that this intersection is transverse, because the projections of \(c_1\) and \(c_2\) to \(L_\infty\) intersect transversally, and (holomorphic) projection to \(L_\infty\) preserves angles. Hence \(c_1 \cap c_2 = 1\).

Now let us prove that for sufficiently large numbers in \textbf{Theorem 8}, \(c_2\) is a hyperbolic cycle. Indeed, due to the \textbf{remark to Volk Theorem}, the derivative \(M'(p_0(\mathcal{F}))\) can be made arbitrarily close to \(\frac{M'_2(p_0(\mathcal{F})))}{M_3(p_0(\mathcal{F}))}\) (the fraction is evaluated in the Schröder chart for \(M_3\)). Due to the choice of \(p_0\), this ratio does not belong to the unit circle, hence one can choose \(\gamma\) and \(\gamma'\) in \textbf{Theorem 8} so that \(c_2\) is a hyperbolic cycle.

This completes the proof of the lemma, hence, the proof of \textbf{Theorem 1}.

\[\square\]

3 Leaves of infinite genus

3.1 Preliminaries

We shall need to prove that a generic leaf of a generic foliation from \(A_n^{sym}\) or \(B_n^{sym}\) intersects the line \(x = 0\) in infinitely many points. The proof will be based on the following two statements. First, we use theorem due to Jouanolou to estimate the number of algebraic leaves.

\textbf{Theorem 13 ([\(]]\). If a polynomial foliation \(\mathcal{F} \in A_n\) has at least \(\frac{1}{2} n(n+1)+2\) algebraic irreducible invariant curves, then it has a rational first integral.\]

Then we prove that a non-algebraic leaf intersects a generic line in infinitely many points.

\textbf{Lemma 14. Consider a polynomial foliation \(\mathcal{F}\) of \(\mathbb{C}P^2\), its non-algebraic leaf \(L\) and a line \(S \subset \mathbb{C}P^2\) such that}

\begin{itemize}
  \item the singular points of \(\mathcal{F}\) which are located at \(S\) (including those at infinity) are all non-degenerate, with the ratio of eigenvalues \(\lambda \notin \mathbb{R}_+;\)
  \item \(S\) is not a leaf of \(\mathcal{F}\).
\end{itemize}

Then \(#(L \cap S) = \infty\).

The proof is completely analogous to the proof of Lemma 28.10 in [5]. This lemma states that a non-algebraic leaf of a foliation \(\mathcal{F} \in A_n\) cannot approach the infinite line only along the separatrices of singular points. However, we repeat the proof here for completeness.

\textbf{Proof.} Suppose the contrary, i.e., \(L\) is not algebraic and \(#(L \cap S) < \infty\). Let us replace \(S\) by a close line \(S'\) that contains no singular points of \(\mathcal{F}\). Since \(S\) is not a leaf of \(\mathcal{F}\), each point of \(L \cap S'\) is located either on a germ of a leaf passing through a point of \(L \cap S'\), or on a germ of a separatrix of a singular point located at \(S\). Due to the first assumption on \(S\), the latter part of \(L \cap S'\) is finite, hence \(#(L \cap S') < \infty\).
Make a projective coordinate change such that \( S' \) is mapped to the infinite line \( L_\infty = \{ u = 0 \} \), and the point \( v = \infty \) of \( L_\infty \) does not belong to \( L \).

Suppose that the leaf \( L \) is given by \( y = \varphi_j(x), j = 1, 2, \ldots, k \), in neighborhoods of \( k \) intersection points. Note that \( \varphi_j \) have a linear growth at infinity. Consider the product \( \prod_{j=1}^k (y - \varphi_j(x)) \); this is a polynomial in \( y \), with symmetric functions \( \sigma_1 = \sum_{j=1}^k \varphi_j, \sigma_2 = \sum_{j=1}^k \sum_{l=1}^k \varphi_j \varphi_l \), \ldots, \( \sigma_k = \prod_{j=1}^k \varphi_j \) as coefficients.

Let \( p_i \) be projections of finite singularities of \( \mathcal{F} \) to \( x \)-plane. It is possible to extend \( \varphi_j \) holomorphically to \( \mathbb{C} \setminus \{ p_1, \ldots, p_l \} \) by the symmetric combinations of intersections \( L \cap \{ x = c \} \), with multiplicities. Indeed, the number of these intersections stays locally the same, thus equals \( k \) for any \( c \). The intersections depend holomorphically on \( c \) and stay bounded, otherwise the leaf \( L \) would approach the infinite line along \( x = \text{const} \).

Since \( \varphi_j \) are bounded in any compact, \( p_i \) are removable singularities of \( \varphi_j \).

So, the symmetric combinations of \( \varphi_j \) extend holomorphically to \( \mathbb{C} \) and have a polynomial growth at infinity. Thus they are polynomials in \( x \), and the function \( F = \prod_{j=1}^k (y - \varphi_j(x)) \) is a polynomial in \( x, y \). Thus \( F = 0 \) is a polynomial equation defining the leaf \( L \), and \( L \) is algebraic.

### 3.2 Proof of Theorem 2

The following lemma explicitly describes the open and dense subset of \( \mathcal{A}_n^{sym} \) (or \( \mathcal{B}_{n+1}^{sym} \)) that satisfies the assertion of Theorem 2.

**Lemma 15.** Consider a foliation \( \mathcal{F} \in \mathcal{A}_n^{sym} \) such that

- \( x = 0 \) satisfies assumptions of Lemma 14.
- \( \mathcal{F} \) has no rational first integral.

Then \( \mathcal{F} \) has finitely many (probably, zero) algebraic leaves, and all other leaves have infinite genus.

**Remark.** We can also take the saturation of the set constructed above by the orbits of affine group. This adds 3 to the dimension, but this saturation will be more complicated object than an open dense subset in a linear subspace.

**Proof.** Consider the map \( F_2 : \mathbb{C}^2 \to \mathbb{C}^2 \) given by \( (x, y) \mapsto (x^2, y) \). Since \( \mathcal{F} \in \mathcal{A}_n^{sym} \), the image of \( \mathcal{F} \) is a well-defined foliation given by

\[
\dot{u} = \hat{p}(u, v); \\
\dot{v} = \hat{q}(u, v),
\]

where \( P(x, y) = \hat{p}(x^2, y) \) and \( Q(x, y) = x \hat{q}(x^2, y) \).

Since \( \mathcal{F} \) has no rational first integral, Theorem 13 implies that all but a finite number of leaves are non-algebraic. Let \( L \) be a non-algebraic leaf of \( \mathcal{F} \). Due to Lemma 14, \( L \cap \{ x = 0 \} \) is infinite. Note that there is at most finite number of non-transverse intersections, hence there is an infinite number of transverse intersections of \( L \) and \( \{ x = 0 \} \).

The restriction \( F_2|_L : L \to F_2(L) \) is a ramified double covering. It is easy to see that the points of transverse intersection \( L \cap \{ x = 0 \} \) are ramification points of \( F_2|_L \). Hence the covering \( F_2|_L \) has countably many ramification points. It is easy to see that for a ramified covering of a surface, the number of handles of the covering space is at least half the number of ramification points minus one. Therefore, \( L \) has an infinite genus. 

\[\square\]
Since $B_n \subset A_n$, this lemma is applicable to foliations $F \in B^{sym}_n$ as well.

Now let us deduce Theorem 2 from the above lemma.

**Proof of Theorem 2.** It is sufficient to prove that for $n \geq 2$ the subset of $A^{sym}_n$ (resp., $B^{sym}_{n+1}$) defined by the additional assumptions from Lemma 15 is open and dense in the ambient projective space. For the first assumption of Lemma 14 this is trivial. For the second assumption of the same lemma, note that these singularities correspond to the roots $y_1, y_2, \ldots, y_n$ of $P(0, y)$, and the linearization matrix of (1) at $(0, y_j)$ equals

\[
\begin{pmatrix}
P'_x & P'_y \\
Q'_x & Q'_y
\end{pmatrix}
(0, y_j)
= \begin{pmatrix}
0 & \tilde{q}'_j(0, y_j) \\
\tilde{q}'(0, y_j) & 0
\end{pmatrix}
\]

If the vector field satisfies $\tilde{q}(0, y_j) \neq 0$, $\tilde{q}'(0, y_j) \neq 0$ for all roots of $P(0, y)$, the ratio of the eigenvalues of each of these singularities equals $-1 \notin \mathbb{R}_+$. This defines open and dense set in $A^{sym}_n$ (resp., $B^{sym}_{n+1}$).

Let us prove that a generic foliation $F \in A^{sym}_n$ or $F \in B^{sym}_{n+1}$ has no rational first integral. Note that a complex hyperbolic singular point is not locally integrable, hence a foliation with complex hyperbolic singular point cannot have a rational first integral. Since a complex hyperbolic singular point survives under small perturbations, it is sufficient to prove that the set of foliations from $A^{sym}_n$ (resp., $B^{sym}_{n+1}$) having a complex hyperbolic singular point is dense in the ambient space.

Consider a foliation $F_0$ from $A^{sym}_n$ or $B^{sym}_{n+1}$, $n \geq 2$. Let $(x_0, y_0)$ be one of its singular points with $x_0 \neq 0$. Let $A_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be its linearization matrix at $(x_0, y_0)$. Consider the two-parametric perturbation $F_{\varepsilon, \delta}$ of $F_0$ given by

\[
\begin{align*}
\dot{x} &= P_0(x, y) + (x^2 - x_0^2)\varepsilon; \\
\dot{y} &= Q_0(x, y) + x(y - y_0)\delta.
\end{align*}
\]

It is easy to see that the perturbed foliation belongs to the same class ($A^{sym}_n$ or $B^{sym}_{n+1}$) and has a singularity at the same point $(x_0, y_0)$. The linearization matrix of $F_{\varepsilon, \delta}$ at $(x_0, y_0)$ is $A_{\varepsilon, \delta} = \begin{pmatrix} a + 2x_0\varepsilon & b \\ c & d + x_0\delta \end{pmatrix}$. Clearly,

\[
\begin{align*}
\text{tr} A_{\varepsilon, \delta} - \text{tr} A_0 &= x_0(2\varepsilon + \delta); \\
\det A_{\varepsilon, \delta} - \det A_0 &= x_0(2\varepsilon d + \delta a + 2x_0\varepsilon \delta).
\end{align*}
\]

It is easy to see that we can achieve any small perturbation of the trace and determinant of the linearization matrix. Therefore, we can achieve any small perturbation of the eigenvalues. In particular, after some perturbation the singular point at $(x_0, y_0)$ becomes complex hyperbolic.

**Remark.** The arguments above do not work for $A^{sym}_1$ and $B^{sym}_2$ because generic foliations from these spaces have rational first integrals. Indeed, a generic foliation from the former space is affine equivalent to a foliation of the form

\[
\begin{align*}
\dot{x} &= y; \\
\dot{y} &= ax,
\end{align*}
\]

which has the first integral $y^2 - ax^2$. A generic foliation from $B^{sym}_2$ is affine equivalent to a foliation of the form

\[
\begin{align*}
\dot{x} &= x^2 + y + a; \\
\dot{y} &= xy,
\end{align*}
\]
which has the first integral \( \frac{(y+a)^2-ax^2}{y} \).

### 3.3 Proof of Ilyashenko Theorem

In this Section we shall prove the following theorem.

**Theorem.** Let \( \mathcal{A}_n^h \subset \mathcal{A}_n \) be the space of foliations given by homogeneous polynomials \( P \) and \( Q \). For a foliation \( \mathcal{F} \) from some open dense subset of \( \mathcal{A}_n^h \), all its leaves except for a finite set have infinite genus.

It seems that this theorem was proved by Ilyashenko many years ago, but he has never written the proof, though told it to various people orally.

**Proof.** Take a homogeneous foliation \( \mathcal{F} \). Note that the polynomials \( \tilde{P} \) and \( \tilde{Q} \) in (3) do not depend on \( u \), hence in the chart \((u, v) = (\frac{1}{x}, \frac{y}{x})\) our foliation \( \mathcal{F} \) is given by

\[
\begin{align*}
\dot{u} &= up(v) \\
\dot{v} &= h(v)
\end{align*}
\]

Clearly, the monodromy group at infinity is generated by linear maps \( M_j : u \mapsto \mu_j u \).

Fix a cross-section \( S \) given by \( v = \text{const} \) and a point \( p \in S \setminus L_\infty \). Let us find a handle passing through \( p \).

The monodromy maps along loops \([\gamma_2, \gamma_1] = \gamma_2 \gamma_1 \gamma_2^{-1} \gamma_1^{-1}\) and \([\gamma_3, \gamma_2^{-1}] = \gamma_3 \gamma_2^{-1} \gamma_3^{-1} \gamma_2\) are identity maps, hence the lifts \( c_1 \) and \( c_2 \) of these loops starting at \( p \) are closed loops.

![Figure 3: Cycles \( c_1 \) (dashed) and \( c_2 \) (dotted)](image)

Note that these loops intersect only at \( p \). Indeed, if \( c_1 \) and \( c_2 \) intersect above one of 7 other intersection points of \([\gamma_2, \gamma_1]\) and \([\gamma_3, \gamma_2^{-1}]\), then \( p \) is a fixed point of one of the maps \( M_3, M_1^{-1} \circ M_3, M_2^{-1} \circ M_1^{-1} \circ M_3, M_2^{-1} \circ M_3, M_2 \circ M_1 \circ M_2, (M_1^{-1} M_2^{-1}?) \) \( M_1 \) respectively. In a generic case, all these maps are linear non-identical, thus they have no fixed points except zero.

Thus \( c_1 \) and \( c_2 \) are two cycles with \( c_1 \cap c_2 = 1 \), and we have found a handle passing through \( p \).

Consider a leaf \( L \) of \( \mathcal{F} \) which is not a separatrix \( v = a_j \) of a singular point of \( \mathcal{F} \) at \( L_\infty \). In a generic case (say, if \( |\mu_j| \neq 1 \) for some \( j \)), \( L \) intersects \( S \) arbitrarily close to \( L_\infty \). Thus \( L \) includes countably many cycles \( c_1, c_2, \ldots \), such that \( c_{2j-1} \cap c_{2j} = 1 \), and other pairs of cycles do not intersect. Hence any leaf except separatrices at \( a_j \) has infinitely many handles. \( \square \)
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