1. Introduction

The flow of coupling parameters of an action (or equivalently a Hamiltonian) under renormalization group (RG) transformations plays a central role in understanding the low-energy properties of the system described by the action. This also provides us with a way of understanding the concept of universality which refers to the fact that systems described by different microscopic Hamiltonians show identical scale independent low-energy behavior, specially near critical points. The microscopic action describing a quantum system may have many parameters and thus be complicated; however, many of these parameters might turn out to be irrelevant for phenomena involving low energy or low momenta, i.e. they flow to zero under the RG transformations. This leads to a simpler effective action with fewer parameters which describes the low-energy properties of the system. This concept is central to understanding the validity of attempts to explain, for example, the low-temperature experimental data of a quantum system based on simple model actions. The procedure for obtaining such an effective action is well known for equilibrium systems [1]. For weakly interacting systems, where the interaction term in the action can be treated perturbatively, this can be done analytically; the analysis of the resultant RG equations provides useful information of the coupling parameters, and hence the effective action, of the system at an arbitrary length scale [2].

In recent years, there has been a lot of theoretical and experimental interest in studying the intrinsic quantum dynamics of strongly interacting many-body systems [3]. This interest is largely due to the recent experimental realization of such isolated quantum systems in the form of ultracold atoms in optical lattices [4] which act as perfect test beds for such dynamics. The uniqueness of these systems in this regard originates from their near-perfect isolation from their surroundings which leads to long time scale over which quantum dynamics can be observed. However, we note here that more recently pump-probe experiments have also started to probe non equilibrium dynamics in the context of standard materials-based condensed matter systems [5].
The equilibrium properties of ultracold atoms are generically described by using simple model Hamiltonians such as the Bose–Hubbard model [6] or Ising model in transverse and longitudinal fields [7]; indeed, one of the main interests in ultracold atom systems stems from their role as emulators of well-studied models of quantum statistical mechanics. However, the description of a complicated coupled atom-laser system in terms of simple quantum models at low energies invariably relies on the concept of universality. This procedure is conceptually justified by invoking standard RG arguments in equilibrium which leads to an effective action using the following steps. First, one imposes an ultraviolet momentum cutoff \( \Lambda \), which, in a typical condensed matter system, is roughly the inverse of the lattice spacing. Second, this cutoff is lowered from \( \Lambda \) to \( \Lambda - d\Lambda \) and the field modes within the momentum shell \( \Lambda \) and \( \Lambda - d\Lambda \) are integrated out perturbatively (in the simplest case to one-loop order in interaction) to obtain an effective action describing the field modes below the cutoff \( \Lambda - d\Lambda \). Next, the momentum and the frequency in the action are rescaled appropriately so as to offset the change in the cutoff. Finally, one reads out the change in parameters of the action due to the set of transformations described above (rescaling and integrating out the field modes within the momentum shell) and obtains the resultant RG flow equation for the parameters of the action. Such a flow leads to either increase (relevant) or decrease (irrelevant) of an Hamiltonian parameter; the low-energy effective Hamiltonian is thus determined by only the relevant parameters which leads to universality.

However, a well-defined RG procedure for a generic out-of-equilibrium system which can justify universality in the long-time behavior of an out-of-equilibrium system described by an arbitrary \( d \)-dimensional field theory is not yet available in the literature. In fact, one of the central questions in this field concerns the applicability of universality in a driven quantum system for an arbitrary drive protocol. This question has been partially addressed in a recent work studying the role of a periodic potential in the time evolution following a sudden quench of interaction parameter of a one-dimensional Luttinger liquid [8]. Such an interaction is known to be irrelevant for the equilibrium situation; in contrast, [8] found that such a term can play an important role in generation of dissipation and eventual thermalization of such a system and can therefore not be neglected as irrelevant during evolution after a quench. Similar studies have been carried out for other non-equilibrium low-dimensional driven systems using generalizations of Hamiltonian flow methods [9, 10]. In addition there have been several works on RG analysis of out-of-equilibrium systems where the system is in a steady state and the action is time independent [11–17]. More recently, functional renormalization group (FRG) techniques have been used to study the RG behavior of a quantum dot coupled to a bath with a time-dependent coupling and several quantities of physical interest have been computed numerically [18, 19]. Such studies have also been carried out in the context of pump-probe experiments [20]. However, the situation for higher dimensional systems with finite-rate protocols is presently far from clear.

In this work, we consider a driven bosonic system which is described by a \( \phi^4 \) field theory with the action:

\[
S_0 = \frac{1}{(2\pi)^{d+1}} \int d\omega d^d k \phi'^4(k, \omega),
\]

\[
S_1 = -\frac{1}{(2\pi)^{d+1}} \int d\omega d^d k \lambda(t) \phi(k, \omega),
\]

where \( g(\omega) \) depends on the dynamical critical exponent \( \gamma \) of the theory and takes values \( \frac{\omega}{\omega^2} \) for \( \gamma = 2(1) \), \( \omega_0 \) is the velocity and \( r \) is the square of the mass of the bosons and \( \lambda(t) = \lambda f(\omega_0 t) \) is the time dependent interaction parameter, \( f \) is an arbitrary function, and \( \omega_0 \) is the drive frequency. We carry out a perturbative momentum-shell RG analysis of this action. There are numerous concrete examples of such effective field theories in condensed matter physics; several quantum models (such as the Bose–Hubbard and the Ising models) near their critical point are described by such a field theory. In fact, almost all quantum critical systems which are described by a Landau–Ginzburg action of a single component order parameter field admits an analogous description in their ordered phase near the quantum critical point. We expect our RG analysis to hold for such systems. In particular, the dynamics of the Bose–Hubbard model near its superfluid-insulator transition point may provide an experimentally realizable test bed for the present analysis. Our analysis leads to the following results. First, we show that the drive frequency \( \omega_0 \) scales in the same manner as temperature in equilibrium systems [2] and provides a new cutoff scale for the RG flow. Second, we show that the RG procedure naturally generates additional terms in the action which leads to coupling between modes of the free field theory. By analyzing the RG equations for \( r \) and \( \lambda \) and these additional generated terms, we identify two regimes for such driven systems; in the first regime the drive can be treated perturbatively and the concept of universality holds similar to that in the equilibrium situation while in the second, the drive dominates the physics and determines the cutoff scale (similar to temperature in an equilibrium system) for RG flow. We provide a criterion for crossover between these two regimes for arbitrary drive protocol. Third, we show that in the second regime, the presence of the drive may take the system out of the gaussian regime (where the interaction term of the effective low-energy action can be treated perturbatively). At the onset of this non-gaussian regime, the coupling between the different field modes due to the interaction becomes comparable to the mass term in the action. We provide an analytical condition involving \( r \), \( \lambda \) and \( \omega_0 \) for this phenomenon to take place and discuss its relation to the onset of dynamical transition studied in [21]. Fourth, we supplement the above-mentioned results by obtaining their analog from an equation of motion method and discuss the relevance of our analysis for near-critical systems described by time-dependent Landau–Ginzburg theories. Finally, we present a computation of correlators of the effective field theory for a specific drive protocol in the Gaussian regime and discuss their physical significance. We would like to emphasize here that the goal of this work is to compute the form of the effective action in terms of the low-momentum modes; we do not aim to analyze this action at a quantitative level. Such an analysis, which is supposed to yield, for example, the state reached at long times due to the dynamics, is beyond the scope of the present work.
The plan of the rest of the paper is as follows. In section 1, we analyze $S$ (equation (1)) using a Keldysh formalism and obtain the RG equations for its parameters. This is followed by analysis of these equations in section 3 where we obtain analytical condition for the onset of the non-Gaussian regime. Next, we analyze the equation of motion for the bosons in section 4. This is followed by calculation of boson correlation function in section 5. Finally, we discuss our main results and conclude in section 6 and provide some detail of the calculations in the appendix.

2. Computation of RG equations

In this section, we analyze $S$ (equation (1)) using the Keldysh technique which is ideally suited for handling out-of-equilibrium quantum systems [22]. To this end, we follow standard procedure to introduce the fields $\phi_n(k, \omega)$ and $\phi_n(k, \omega)$ living on the forward and backward time contours. In terms of these fields, the zero temperature partition function for a system of interacting bosons can be written as

$$Z = \int D\phi_0 D\phi_r e^{i\langle S'\{k\} - S\{k\} \rangle},$$

(2)

where $S[\phi]$ is given by equation (1). Next, for computational convenience, we define classical and quantum components of the bosonic fields $\phi$ as

$$\phi_{c}(\omega) = (\phi_{r} + (-\phi_{l}))/2$$

(3)

and write the partition function as

$$Z = \int D\phi_0 D\phi_r e^{i\langle S_c\{k\}, \phi_{c}\rangle},$$

(4)

where the action $S' = S_0 + S_1$ is given by

$$S_0 = \frac{2}{(2\pi)^d} \int d^d k d\omega_0 \psi^*(k, \omega)(g(\omega) - v_f^2 k^2 - r) \sigma_0 \phi(k, \omega)$$

$$S_1 = -\frac{4}{(2\pi)^d} \int d^d x d\tau \psi \left[ \phi_{c}^{*}(x, \tau) \phi_{c}(x, \tau) \phi(x, \tau) \phi_{c}(x, \tau) + \phi_{c}^{*}(x, \tau) \phi_{c}(x, \tau) \right] + h.c.$$

(5)

Here $\phi^* = (\phi_{c}^*, \phi_{r}^*)$ is the two component bosonic field and $\sigma_0$ is the Pauli matrix acting in $c-q$ space.

To analyze this action using perturbative RG, we first rewrite $S_1$ in momentum-frequency space. To this end, we define a dimensionless kernel

$$K(\alpha) = \int_{-\infty}^{\infty} dy \exp(\alpha iy)$$

(6)

and rewrite $\int_{-\infty}^{\infty} df(\omega_0) \exp(i\omega_0 t) = K(\omega_0/\omega_0)$. In terms of this dimensionless kernel, one can write

$$S_1' = -\frac{4}{(2\pi)^d} \sum_{i=1}^{3} \int d^d k d\omega_0 \lambda K(\omega_0/\omega_0)$$

$$\left[ \phi_{c}(k_1, \omega_1) \phi_{c}(k_2, \omega_2) \left\{ \phi_{c}(k_3, \omega_3) \phi_{c}(k_1 + k_2 - k_3, \omega_3) \right\} + \phi_{r}^{*}(k_1, \omega_3) \phi_{r}(k_1 + k_2 - k_3, \omega_3) \right] + h.c.$$  

(7)

where $\omega = \omega_1 + \omega_2 - \omega_3 - \omega_4$. We note that the physical significance of $K$ is that it encodes the manner in which different modes are coupled by the interaction. For example, for a periodic drive with $f(\omega_0) = a + b \cos(\omega_0)$, one can show that $K(\omega_0/\omega_0) = \sum_{n} \alpha_n \delta(\omega_0 - \omega_n)$ with $\alpha_n = a$, $\alpha_1 = \omega_0 - \omega_n$ with amplitude $\alpha_n$. In contrast, for a gaussian drive profile with $f(\omega_0) \approx \exp(-\omega_0^2/2)$, one finds $K(\omega_0) \approx \exp(-\omega^2/(2\omega_0^2))$; here, as we shall derive subsequently, any two field modes with frequencies $\omega$ and $\omega'$ are coupled to each other with a strength $\exp[-(\omega - \omega')^2/(2\omega_0^2)]$. We would like to stress that values of $K(\omega_0)$ (or $\alpha_n$ for periodic drive) depends on the drive protocol. In what follows we shall keep $f(\omega_0)$ arbitrary except for the requirement that $\int df(y) \exp(i\alpha y)$ is well defined. We also note that we envisage a situation in which the drive decays to zero with a characteristic time scale $T_0$. The presence of $T_0^{-1}$ changes the expressions for $K(\omega)$ but, as we shall see, do not influence the RG flow otherwise provided $T_0^{-1} \ll \omega_0$. For example, for periodic drives with a gaussian decaying profile $f(\omega_0) = (a + b \cos(\omega_0)) \exp[-t^2/2\omega_0^2]$, one has $K(\omega_0; T_0) \equiv K(\omega_0) \approx a \exp(-\omega_0^2 T_0^2/2) + b/2 \left( \exp[-(\omega - \omega_0)^2 T_0^2/2] + \exp[-(\omega + \omega_0)^2 T_0^2/2] \right)$ which reduces to earlier derived results for large $T_0$. In general, as can be seen from the example above, all information about $T_0$ is encoded in the function $K(\omega_0)$. We note that having a finite $T_0$ is important in the present case since in the absence of a reservoir, for $T_0 \to \infty$, the system will heat up indefinitely. A similar situation would occur if $\omega_0 \to \infty$ for a fixed $T_0$. In these cases, the system reaches the infinite temperature fixed point where the low-energy effective action loses its meaning. In this work, we shall concentrate on the case where both $T_0$ and $\omega_0$ are finite; further we shall keep $T_0$ fixed and vary $\omega_0$ in the rest of this work.

Next, we present our rationale for feasibility of a RG analysis of the driven system. We consider the system to be in the ground state of $S'$ at the start of the drive labeled by a momentum $k_0$. The central assumption of the RG analysis that follows is that for any generic action, there will be a finite set of states in the Hilbert space around $k_0$, as schematically shown in figure 1, which will actively participate in the dynamics. The number of such states depends on the drive frequencies and amplitude. The other states in the Hilbert space do not participate in the dynamics and may thus be systematically integrated out to obtain an effective action for the system in terms of the active modes. In what follows, we are going to implement this procedure. In doing so, we follow the convention of imposing a finite momentum cutoff leaving the frequency cutoff to infinity [1]. The first step of the RG transformation is scaling which constitutes lowering of the momentum cutoff $\Lambda$ to $\Lambda e^{-t}$ leading to the slow and the fast modes given by

$$\phi(k) = \begin{cases} \phi^c & 0 < k < \Lambda e^{-t} \\ \phi^r & \Lambda e^{-t} < k < \Lambda \end{cases}$$

(8)
In perturbative RG, the fast modes are eliminated by integrating out perturbatively keeping only one-loop terms in the interaction $\lambda$, followed by a standard rescaling of the resulting effective action. Such an elimination of the fast modes leads to

$$S'(\phi^\upsilon) = S_0(\phi^\upsilon) + \langle S_1(\phi^\upsilon, \phi^\upsilon) \rangle_{\theta_0} + \frac{1}{2} \left[ (S_1)_{\theta_0}^{2} - (S_1)_{\theta_0}^{2} \right] + \ldots$$

$$= S_0(\phi^\upsilon) + S_1(\phi^\upsilon) + S_2(\phi^\upsilon),$$

where $S_2$ results from one-loop corrections from the interaction terms and is derived in appendix A.

We first consider scaling of $S_0$ and $S_1$. To this end, we follow the standard procedure of rescaling, namely, $k \rightarrow k \exp(\epsilon t)$, $\phi^\upsilon \rightarrow \phi^\upsilon \exp(-\alpha t)$, and $\omega \rightarrow \omega \exp(\zeta t)$.

The invariance of $S_0$ under this rescaling demands $r \rightarrow r' = r \exp(2t)$ and $\alpha = (d + z + 2)t/2$ which fixed the scaling of the fields. The invariance of $S_1$ is slightly more tricky; for this we note that $K$ is a dimensionless function which does not scale under RG. Thus the invariance of $S_1$ requires $\lambda \theta_0 \rightarrow (\lambda \theta_0) \exp[\epsilon t]$, where $\epsilon = 4 - d - z$. We choose the simplest possible protocol independent solution (demanding that $\omega_0 \theta_0$, being dimensionless, will remain invariant under scaling) of this equation which is given by $\alpha \rightarrow \lambda \exp(\epsilon t)$ and $\omega_0 \exp(\zeta t)$. This leads to the tree level RG equations:

$$\frac{d\lambda}{dt} = 2r(\lambda), \quad \frac{d\omega_0(t)}{dt} = \omega_0(t), \quad \frac{d\lambda(t)}{dt} = c\lambda(t)$$

within the initial condition $r(0) = r$, $\lambda(0) = \lambda$ and $\omega_0(0) = \omega_0$. From equation (9), we note that the drive frequency $\omega_0(t)$ scales as $\omega_0(t) = \omega_0 \exp(\zeta t)$ showing that it is relevant under RG. The scaling of $\omega_0$ is reminiscent of the scaling of physical temperature in equilibrium systems [2] which is known to scale as $T(t) = T \exp(\zeta t)$.

The full RG procedure which involves integrating out the fast modes is worked out in detail in the appendix. The resultant RG equations are given by

$$\frac{d\lambda(t)}{dt} = 2\lambda(t) + cK(0)\lambda(t), \quad \omega_0(t) = \omega_0 \exp(\epsilon t),$$

$$\frac{d\lambda(t)}{dt} = \epsilon\lambda(t) - c_2\lambda^2(t),$$

$$\frac{d\omega_0(t)}{dt} = c_1\omega_0(t), \quad n \neq 0,$$

$$\frac{d\lambda(t)}{dt} = -c_2\lambda\omega_0^2(t), \quad m \neq 0.$$  (12)

Here $c_1$ and $c_2$ are constants whose expressions are given in the appendix, $e = 4 - d - z$, and $r(\omega; t) \equiv r(\omega)$ and $\lambda(\omega, \omega'; t) \equiv \lambda(\omega, \omega')$ are coefficients of terms generated in the action due to integrating out the field modes within the shell $\Lambda$ and $\Lambda \exp(-t)$. These terms are derived in the appendix and are given by

$$\delta S_1 = -2\int \frac{d^d k \omega_0 d\omega'}{(2\pi)^d} \phi^\nu(\omega_1) \phi^\nu(\omega_2) \phi^\nu(\omega_3) \phi^\nu(\omega_4) \times [1 - \delta(\omega' / \omega_0)]$$

$$\delta S_2 = \frac{1}{4d!} \left[ \phi^\nu(k_1, \omega_1) \phi^\nu(k_2, \omega_2) \phi^\nu(k_3, \omega_3) \phi^\nu(k_4, \omega_4) \right]$$

$$\times [1 - \delta(\omega / \omega_0)] \times [1 - \delta(\omega' / \omega_0)].$$

They are not present in the original action but are spontaneously generated due to the RG flow. They represent quadratic and quartic couplings between field modes with different frequencies due to the time dependent drive. These terms keep track of the transfer of energy between the field modes due to the drive and have no analog in equilibrium RG. We also note that for periodic drive, $K(0)$ and $r(\omega)$ should be carefully defined since $K(\omega \omega_0)$ has supports on discrete points where $\omega / \omega_0 = n$. As shown in the appendix, in this case one obtains

$$\frac{d\lambda(t)}{dt} = 2\lambda(t) + c\alpha_0 \lambda(t), \quad \omega_0(t) = \omega_0 \exp(\epsilon t),$$

$$\frac{d\lambda(t)}{dt} = \epsilon\lambda(t) - c_2\lambda^2(t),$$

$$\frac{d\omega_0(t)}{dt} = c_1\omega_0(t), \quad n \neq 0,$$

$$\frac{d\lambda(t)}{dt} = -c_2\lambda\omega_0^2(t), \quad m \neq 0.$$  (12)
$$\delta S^1 = -2 \sum_{n \neq 0} \int d^d k d\omega_n \phi^\dagger(k, \omega_n) r_n \varphi \phi(k, \omega_n - n \omega_0)$$
$$\delta S^2 = 4 \sum_{i=1}^{3} \sum_{j=1}^{4} \sum_{m} \int d^d k d\omega \lambda \lambda_{mn} \left[ \phi^\dagger(k_1, \omega_1) \phi(k_2, \omega_2) \times \{ \phi^\dagger(k_3, \omega_3 - m \omega_0) \phi(k_4, \omega_4) + \phi^\dagger(k_3, \omega_3 - m \omega_0) \phi(k_2, \omega_2 - \omega_3 + m \omega_0) \} + \text{h.c.} \right] \tag{13}$$

The RG equations derived here show that the drive frequency provides a natural cutoff scale for the RG flow. We note that when $\omega_0(t) \sim \Lambda(t)$, all the states in the Hilbert space below the momentum cutoff participate in the dynamics and hence one can not integrate out states any further. This cutoff scale $\ell_2$ satisfies $\omega_0(\ell_2) = 0$, and is given by
$$\ell_2 = \frac{1}{z + 1} \ln (\Lambda_0 / \omega_0). \tag{14}$$

Note that there are other cutoff scales in the problem stem from the mass term $r$ since RG stops when the momentum cutoff reaches the inverse of the correlation length or when the interaction term grows (for $\epsilon > 0$) such that the perturbative RG analysis ceases to hold. We shall provide an explicit expression for these scales in section 3; here we simply note that for large $\omega_0$, the RG flow stops at $\ell_2$. Beyond $t = \ell_2$, the property of the system is determined essentially by the drive term and this regime has no analog in equilibrium RG. We shall derive this explicitly in the next section.

Before moving on to the analysis of the RG equations, we note that the one-loop correction terms to $r(\ell)$ and $\lambda(\ell)$ depend crucially on the driving protocol through $K(0)$ or $\omega_0$. This feature in turns leads to a protocol-dependent fixed-point structure for the RG equations. For example, drive protocols with $K(0) = 0$, the equations for $r(\ell)$ and $\lambda(\ell)$ do not have a Wilson-Fisher fixed point for relevant interactions ($\epsilon > 0$); only the Gaussian fixed point exists in this case.

### 3. Analysis of RG equations

The solutions of the RG equations (equation (10)) depend crucially on the relevance/irrelevance of the interaction. We begin with the case when the interaction term is irrelevant, i.e. $d + z > 4$. In this case since $\epsilon < 0$, it is possible to ignore the second term in the right side of the RG equation for $\lambda(\ell)$. Denoting $r$ and $\lambda$ to be the bare values of $r(\ell)$ and $\lambda(\ell)$ and scaling all frequencies (momenta) in units in $\Lambda_0$ ($\lambda$), we get

$$r(\ell) = r_{\text{eff}} e^{2\ell} - c_1 K(0) \lambda e^{4\ell} / (d + z - 2)$$
$$\lambda(\ell) = \lambda e^{\ell}$$
$$r(\omega; \ell) = -c_1 K(0) \lambda e^{4\ell} / |k|$$
$$\lambda(\omega, \omega'; \ell) = c_2 K(0) \lambda e^{4\ell} / |k|$$

where the effective mass $r_{\text{eff}} = r + c_1 K(0) \lambda / (d + z - 2)$. For periodic drive, the solution of the RG equations can be easily read off from equation (15) by replacing $K(0) \rightarrow \omega_0$, $r(\omega; \ell) \rightarrow r_{\text{eff}}(\ell)$, $K(0) \lambda \rightarrow \lambda_{\text{eff}}$, $\lambda(\omega, \omega'; \ell) \rightarrow \lambda_{\text{eff}}$, and $K(0) \lambda \rightarrow \lambda_{\text{eff}}$.

To analyze equation (15), we note that the RG flow stops when the momentum cutoff reaches the cutoff scale set by the drive frequency $\omega_0$ or when it reaches the inverse of the correlation length. The former occurs at $t = \ell_{\text{eff}} = \ln(1/\omega_0)(z + 1)$ while the latter happens at a RG time $t_{\ell} = 1$. This leads to $t_{\ell} = \ln(1/\omega_0)/2$. With these two scales, there are two distinct regimes. In the first regime, $t_{\ell} \ll \ell_2$, so that $r_{\text{eff}} \geq 0^2(z + 1)$, and the RG stops when the momentum cutoff reaches the inverse correlation length. In this regime $r_{\text{eff}} \approx 1$, and the drive frequency remain small compared to $r_{\text{eff}}$ if $\omega_0(t_{\ell}) \ll 1$ which leads to the condition
$$\omega_0 \ll r_{\text{eff}}^2 \omega_0^2. \tag{17}$$

We note that if the condition given by equation (17) is satisfied, then the drive can be treated perturbatively; this condition becomes analogous to the condition for the existence of a perturbative quantum regime in equilibrium systems where the role of $\omega_0$ is played by the temperature $T$. In this perturbative regime, when the RG flow stops, $\lambda(t_{\ell}) = \lambda_{\text{eff}}^0$ and is thus small provided $\lambda_{\text{eff}} \ll 1$. All also higher powers of interaction remain small and can therefore be ignored; thus we conclude that the universality of the driven system remains qualitatively similar to that in the equilibrium situation in this regime.

In the second regime, RG flow stops at $\ell = \ell_2$ where $\omega_0(\ell_2) = v_0(\ell_2)$, and in this regime one finds

$$r(\ell_2) = r_{\text{eff}}(0) e^{-2\epsilon(\ell + 1)} - c_1 K(0) \lambda e^{4\ell} / (d + z - 2)$$
$$\lambda(\ell_2) = \lambda_0 e^{\epsilon(\ell + 1) / 2}.$$

Thus the condition for non-gaussian behavior $\lambda(\ell_2) \geq r(\ell_2)$ occurs when

$$\lambda' \geq r_{\text{eff}}^2(2 - z - d) e^{\epsilon(\ell + 1) / 2}.$$

where $\lambda' = \lambda[1 + c_1 K(0)(z + d - 2)]$. In this regime $\ell_2 \ll \ell$, and so we have $r_{\text{eff}} \ll \omega_0^2 e^{\epsilon(\ell + 1) / 2}$; thus a sufficient (but not necessary) condition for violation of the Gaussian regime is given by

$$\lambda' \geq \omega_0^2 e^{\epsilon(\ell + 1) / 2}. \tag{20}$$

Equations (19) and (20) constitute the central result of this work. These equations show that the presence of a drive frequency may stop the RG flow at a scale $\ell = \ell_2$. At this scale, the system will exhibit non-gaussian behavior for a range of $\lambda$ and $r_{\text{eff}}$ for which equation (19) is satisfied. As shown in figures 2 and 3, the condition for such a non-gaussian regime, for $d = 3$ is given by $\lambda' \omega_0 \geq r_{\text{eff}}$ for any $z$. The sufficient condition for $d = 3$ and $z = 2$ is given by $\lambda' \geq \omega_0^2 e^{\epsilon(\ell + 1) / 2}$. In contrast for $d = 2$, both the necessary and the sufficient conditions depend on $z$; for $d = 2$ and $z = 3$, they are given by $\lambda' \omega_0^3 / 4 \geq r_{\text{eff}}$ and $\lambda' \geq \omega_0^{-1 / 4}$. Further, in this non-gaussian regime, one has

$$r(\omega; \ell_2) = -c_1 K(0) \lambda e^{4\ell} / |k|$$
$$\lambda(\omega, \omega'; \ell_2) = c_2 K(0) \lambda e^{4\ell} / |k|$$

(21)
This indicates that in the frequency range where \( K(\omega/\omega_0) \approx 1 \), \( r(\omega, \ell_2) \) may become comparable to the mass \( r(\ell_2) \). Thus the onset of the non-gaussian regime indicates that the drive may effectively transfer energy between different modes. This is reminiscent of a dynamical energy delocalization transition [21] and we shall discuss this point further in section 6. We also note that since \( K(0) \) (or equivalently \( \alpha_0 \) for periodic drive) depends on the protocol, the condition of the onset of the non-gaussian regime may vary drastically depending on the drive protocol. For larger values of \( K(0) \), one may have a regime where the non-gaussian behavior does not show up for any finite \( \lambda/\ell \) below a critical drive frequency. This is reflected in figure 2 where we sketch the condition on \( r/\lambda \) as a function of \( \omega_0 \) for several representative values of \( c_1K(0) \) or \( c_1\alpha_0 \). The corresponding sufficiency condition for the onset of the non-gaussian regime (equation (20)) is plotted in figure 3.

Next, we discuss the RG equation for the case of marginally irrelevant interaction with \( \epsilon = 0 \). For this, after some straightforward algebra, one obtains the solution of the RG equations to be

\[
\begin{align*}
r(t) &= r'_{\text{eff}} e^{2t} - c_1K(0)\lambda I(\ell; \lambda) \\
\lambda(t) &= \lambda(1 + c_2K(0)\lambda t)^{-1} \\
r(\omega; t) &= \frac{K(\omega/\omega_0)c_1}{K(0)c_2} \ln(1 + c_2K(0)\lambda t) \\
\lambda(\omega, \omega; t) &= \frac{K(\omega/\omega_0)K(\omega'/\omega_0)\lambda}{K(0)[1 + c_2K(0)\lambda t]}, \quad K(0) \neq 0, \\
&= c_2K(\omega/\omega_0)K(\omega'/\omega_0)\lambda^2 t, \quad K(0) = 0,
\end{align*}
\]

where \( r'_{\text{eff}} \) and \( I(\ell; \lambda) \) are given by

\[
\begin{align*}
r'_{\text{eff}} &= r + c_1K(0)\lambda I(0; \lambda), \\
I(\ell; \lambda) &= \int_0^\ell dt' e^{-2t'}/(1 + c_2K(0)\lambda t').
\end{align*}
\]

The analysis of the RG equations proceeds along the same line as the one carried out for the earlier case. The RG flow stops at
\[ r = \ell_1 \text{ if } r'_{\text{eff}} \geq a_0^{2/(1+\epsilon)} \]. In this regime the drive can be treated perturbatively provided \( a_0/r'_{\text{eff}}^{2/2} \ll 1 \). In the other regime, where \( r'_{\text{eff}} \leq a_0^{2/(1+\epsilon)} \), the flow stops at \( \ell = \ell_2 \). In this regime, one finds

\[
\begin{align*}
\lambda(\ell_2) &= \lambda \left[ 1 - c_2 K(0) \ln(a_0)/(z + 1) \right]^{-1} \\
r(\omega; \ell_2) &= \frac{K(0)c_1}{K(0)c_2} \ln \left[ 1 - c_2 \lambda K(0) \frac{\ln(a_0)}{z + 1} \right] \\
\lambda(\omega, \omega'; \ell_2) &= \frac{K(0)c_1}{1 - c_2 K(0) \ln(a_0)/(z + 1)} \left( c_2 K(0) \frac{\ln(a_0)}{z + 1} \right) K(0) = 0, \tag{24}
\end{align*}
\]

The necessary and sufficient conditions for the violation of the Gaussian regime \( \lambda(\ell_2) \geq r'_{\text{eff}}(\ell_2) \), is then given by

\[
\lambda'_{\text{eff}} = a_0^{2/(1+\epsilon)} \quad \text{and} \quad \lambda'_{\text{m}} \geq 1 \tag{25}
\]

respectively, where \( \lambda'_{\text{eff}} = \lambda \left[ 1 - c_2 K(0) \ln(a_0)/(z + 1) + c_1 K(0) I \left( -\ln(a_0)/(z + 1); \ell \right) \right] \). Using equation (25), a plot of limiting values of \( r\ell \hat{a} \) which separates the gaussian and the non-gaussian regimes versus the drive frequency \( a_0 \) is shown in figure 4. These relations become particularly simple for drive protocols for which \( K(0) = 0 \) (or equivalently \( a_0 = 0 \)). For these protocols, the necessary condition for violation of the gaussian regime is given by \( r = a_0^{-2/1+\epsilon} \). It is easy to see from equation (24), that in this limit \( r(\omega; \ell_2) \sim \lambda \) becomes comparable to \( r(\ell_2) \) leading to onset of transfer of energies between different field modes. The corresponding sufficiency condition for the onset of the non-gaussian regime is plotted in figure 5.

Finally, we consider the case for \( \epsilon > 0 \). For theories with \( \epsilon > 0 \), the interaction grows with RG time. Consequently, in equilibrium, the flow equations run towards the well-known Wilson-Fisher fixed point at which \( \lambda' = \epsilon/\ell_2 \) and \( r' = -c_1\ell c_1/2c_2) \). For the driven system, the position of the fixed point depends on \( K(0) \); indeed, for \( K(0) = 0 \), the fixed point does not exist. The solution of the RG equations in this case is straightforward and is given by

\[ r(\ell) = r_0 e^{2\ell}, \quad \lambda(\ell) = \lambda e^{\ell} \]

\[ r(\omega; \ell) = -c_1 K(0)/\epsilon \]

\[ \lambda(\omega, \omega'; \ell) = -c_2 K(0)/\epsilon \lambda^2(\ell)/2c_e \] \( \tag{26} \)

Clearly, equation (26) is valid till \( \lambda(\ell) \approx 1 \) after which the system flows towards the strong-coupling fixed point and the perturbative RG does not hold any more. This happens at
\[ \ell_1 = \ln(1/\ell') \ell. \text{ Thus we analyze the regime where } \ell_2 \leq \ell_1, \ell_3 \text{ which requires the frequency to satisfy} \]

\[ \omega_0 \geq \frac{1}{\ell^{(z+1)/2}} \ell^{(z+1)/6}. \]

If the drive frequency satisfies equation (27), RG stops at \( \ell = \ell_2 \) and in this regime the condition of non-Gaussian behavior is given by \[ \lambda l r \geq \omega_0^{2d-2}(z+1)/2. \]

Equation (28) shows that the onset of the non-gaussian regime occurs in a qualitatively different manner for \( d + z < 2 \) since here \( \ell (t) \) grows faster than \( r (t) \). For smaller \( \omega_0 \), where the RG flow stops at larger RG time \( \ell_2 \), \( \ell (t) r (t) \) may become large even for a smaller initial value \( \lambda l r \) and lead to the onset of the non-gaussian regime. Thus for any given \( \lambda l r \) there exists an upper critical frequency \( \omega_0^c \approx (\lambda l r )^{(z+1)/(2d-2)} \) below which the system sees the onset of the non-gaussian regime. In contrast to \( d + z > 2 \), where the interaction either grows slower than the mass term or decays, one needs a finite drive frequency greater than a lower critical frequency \( \omega_0^c \approx (r/\ell) \ell^{(z+1)/(2d-2)} \) to achieve the non-gaussian regime. For \( z + d = 2 \), the onset of the non-gaussian regime requires \( \lambda > r \) since both \( r (t) \) and \( \ell (t) \) scale in the same way. We note however, that at higher loops in RG this relation is expected to be modified due to the presence of a non-zero anomalous exponent \( \eta \); this point is discussed in detail in section 6.

The analysis of the RG equations for \( e > 0 \) and \( K (0) \neq 0 \) turns out to be more complicated and we do not attempt it here.

4. Analysis of equations of motion

In this section, we shall derive the RG equations from an equation of motion approach. Although the end results are the same, we carry out this analysis to establish a connection between these two approaches; the latter being widely used in the statistical mechanics community for studying classical non-equilibrium phenomena. Our approach here will be along the same lines as [23].

The saddle point equations of the \( \phi^4 \) action are obtained by \( \delta S / \delta \phi^a \phi^b(k, k) = 0 = \delta S / \delta \phi_0^a \phi_0^b(k, k) \) which yields the equation of motion for the fields

\[
G_0^{\phi^4}(k, k_0) \phi(k, k_0) = \sum_{\alpha=0}^{3} \frac{1}{\alpha_0^{2}} \int \frac{d^d k_{0} d\omega}{(2\pi)^{d+1}} K \left( \frac{\omega}{\alpha_0} \right) \left[ \phi_0^\alpha(k_0, k) \phi(k + k_0 - k, \omega_0) + \phi_0^\alpha(k_0, k) \phi(k + k_0 - k, \omega_0) + 2 \phi_0^\alpha(k_0, k) \phi(k + k_0 - k, \omega_0) \right].
\]

Next, we carry lower the momentum cutoff from \( A \) to \( A e^{-\omega} \). To this end, we separate the field into slow and fast modes \( \phi (k, k_0) = \phi^\alpha(k, k_0) + \phi^\alpha(k, k_0) \). Using equation (29), we write down the equations for \( \phi^\alpha(k, k_0) \) and find the propagator for the fast modes. In doing so, we make the following approximations. First, we retain only part of the interaction term for which \( \Theta (k | - \lambda) \) is satisfied. Second, we ignore the terms in the right side of equation (29) which have more than one \( \phi^\alpha(k, k_0) \). This approximation is equivalent to replacing the full \( G^\alpha \) in the action formalism by the free propagator \( G^\alpha_0 \) which is the standard approximation in perturbative RG procedure. This leaves out terms with one \( \phi^\alpha \) and two \( \phi^\alpha \) which constitutes the self energy of \( \phi^\alpha \) fields due to the interaction between the fast \( \phi^\alpha \) and the slow \( \phi^\alpha \) field modes. This yields

\[
\int \frac{d^d k_{0} d\omega}{(2\pi)^{d+1}} K \left( \frac{\omega}{\alpha_0} \right) \left[ \phi_0^\alpha(k_0, k) \phi(k + k_0 - k, \omega_0) + \phi_0^\alpha(k_0, k) \phi(k + k_0 - k, \omega_0) + 2 \phi_0^\alpha(k_0, k) \phi(k + k_0 - k, \omega_0) \right] = 0,
\]

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\[
\left[ G_0^{-1}(k) \delta_{a'b}(k - k') - \Sigma_{a'b}(k, k') \right] \phi^\alpha(k') = 0,
\]

where \( k \equiv (k, k_0), k' \equiv (k, k_0) \), the indices \( \alpha_a/b \) take values \( 1/2 \) for \( c(q) \), all repeated indices are summed or integrated over, and \( \phi^\alpha \equiv (\phi^\alpha, \phi^\alpha) \). The self-energy \( \Sigma_{a'b} \) satisfies \( \Sigma_{11} = \Sigma_{22} \equiv \Sigma_1 \) and \( \Sigma_{12} = \Sigma_{21} \equiv \Sigma_2 \) which are given by

\[
\Sigma(k, k_2) = \frac{2\lambda}{\omega_0} \int \frac{d^d k_{0} d\omega}{(2\pi)^{d+1}} K \left( \frac{\omega}{\alpha_0} \right) \left[ \phi_0(k_0, k) \phi_0(k + k_0 - k, \omega_0) + \phi_0(k_0, k) \phi_0(k + k_0 - k, \omega_0) + 2 \phi_0(k_0, k) \phi_0(k + k_0 - k, \omega_0) \right] = 0
\]

The Keldysh Green function for the \( \phi^\alpha \) fields can thus be obtained as

\[
G^\alpha(k, k_2) = \left( \begin{array}{cc} G_0(k, k_2) & G_0(k, k_2) \\ G_0(k, k_2) & 0 \end{array} \right)
\]

\[
G_0(k, k_2) = G_0(k, k_2) \delta(k - k_2) + G_0(k, k_2) \Sigma_1(k_1, k_2) G_0(k_2) + G_0(k, k_2) \Sigma_2(k_1, k_2) G_0(k_2) + G_0(k, k_2) \Sigma(k_1, k_2) G_0(k_2) + G_0(k, k_2) \Sigma(k_1, k_2) G_0(k_2)
\]

\[
G_0(k, k_2) = G_0(k, k_2) \delta(k - k_2) + G_0(k, k_2) \Sigma_1(k_1, k_2) G_0(k_2) + G_0(k, k_2) \Sigma_2(k_1, k_2) G_0(k_2) + G_0(k, k_2) \Sigma(k_1, k_2) G_0(k_2) + G_0(k, k_2) \Sigma(k_1, k_2) G_0(k_2)
\]

(32)
\[ G^{-1}_0(k) \phi^c(k) = \frac{2i}{\omega_0} \int \frac{d^4 k d \omega_1 d \omega_2}{(2\pi)^3} \frac{\omega}{\omega_0} K^{\omega, \omega_1, \omega_2} \left\{ \phi^c(k, \omega_3 - \omega) \times \phi^c(k, \omega_3 + \omega) \right\} \]

\[ \times \int \frac{d^4 k d \omega_1 d \omega_2}{(2\pi)^3} G_G(k_2, k_2 + k + k_3) + \phi^c(k, \omega_3) \]

\[ \times \int \frac{d^4 k d \omega_1 d \omega_2}{(2\pi)^3} G_G(k_2, k_2 + k + k_3) \}

\[ + \frac{\lambda}{\omega_0} \left( \prod_{i=2}^4 \int \frac{d^4 k d \omega_1 d \omega_2}{(2\pi)^3} K^{\omega, \omega_1, \omega_2} \left\{ \phi^c(k, \omega_3) \phi^c(k, \omega_3) \times \phi^c(k, k_2 + k_3, \omega_1) + \phi^c(k, k_2 + k_3, \omega_3) \right\} \]

\[ \times \phi^c(k, k_2 + k_3, \omega_3) + 2 \phi^c(k, k_2 + k_3, \omega_3) \phi^c(k, k_3, \omega_3) \]

\[ \times \phi^c(k, k_2 + k_3, \omega_3) \right\} \].

The equation for \( \phi^c(k) \) is obtained by \( \phi^c(k) \rightarrow \phi^c(k) \) in the above equation.

The additional term arising from replacing the \( \phi^c \) fields with their averages is the first of the two terms in the right side of equation (33). Substituting \( G \) from equation (32), we find that to \( O(\lambda^2) \), the additional term in the equation of motion for the \( \phi^c \) fields is given by

\[ G^-_0(k) \phi^c(k) = \frac{2i}{\omega_0} \int \frac{d^4 k d \omega_1 d \omega_2}{(2\pi)^3} K^{\omega, \omega_1, \omega_2} \sigma, \phi^c(k, \omega_3 - \omega) Tr[G_K] + \ldots , \]

(34)

where \( Tr[G_K] = \int \frac{d^4 k d \omega_1}{(2\pi)^3} G_G(k, k_0) \) and the ellipses denote all other terms in the right side of equation (33) and its counterpart for \( \phi^c(k) \) which is obtained by substituting \( \phi^c(k) \rightarrow \phi^c(k) \) in equation (33). Thus we find that in exact accordance with the results obtained by implementing RG on the action, the \( \omega = 0 \) part of this term provides the one loop correction to the \( r \) while the \( \omega \neq 0 \) part generates new terms in the equation of motion which are the same as those obtained from equation (13).

A similar result for the one-loop corrections to \( O(\lambda^2) \) terms can be obtained by gathering the terms originating from \( G \) to \( O(\lambda^2) \). For example, the terms which contribute to the correction of the term \( \phi^c(k) \phi^c(k) \) are given by

\[ \left( \frac{2i}{\omega_0} \right)^2 \int \frac{d^4 k d^2 k d \omega_1 d \omega_2}{(2\pi)^6} K^{\omega, \omega_1, \omega_2} \left\{ \phi^c(k, \omega_3 - \omega) \phi^c(k, \omega_3 + \omega) \times \phi^c(k, k_2 + k + k_3, \omega_3) + G_G(k_2 + k + k_3) + G_G(k_2 + k + k_3) + G_G(k_2 + k + k_3) \right\} \]

\[ + G_G(k_2) G_G(k_2 + k + k_3) + G_G(k_2) G_G(k_2 + k + k_3) + G_G(k_2) G_G(k_2 + k + k_3) \}

The last three terms cancel in the limit of zero external frequency and momenta. The rest of the terms provide the same loop corrections to \( \lambda \) and generate the new terms in the equation of motion as those obtained from equation (13).

Finally, we complete the RG procedure by scaling the momenta and frequency by \( k \rightarrow k e^{\lambda t} \) and \( \omega \rightarrow \omega e^{\lambda t} \). The scaling of the fields are the same as those obtained in section 2 and leads to \( \omega_0 \rightarrow \omega_0 e^{\lambda t} \) for the drive frequency. Gathering the RG terms generated from the scaling and the one-loop corrections described above, we finally obtain equations (10) and (11).

5. Correlation function of the bosons

In this section, we shall compute the boson correlation functions using the effective action derived using the RG analysis. To do this we note that the effective action contains the off-diagonal terms in frequency space at the quadratic level (equation (13)); consequently inverting propagator of such an action to obtain the Green function and hence the boson correlation function amounts to, even at the Gaussian level, inverting a large matrix (whose size depends on the drive protocol through \( K(\omega) \) which needs to be done numerically in general. Further in the non-Gaussian regime, the quartic interaction term \( \lambda(\ell) \) becomes comparable to the mass term \( \sim r(\ell) \) and any perturbative calculation of boson correlations fails. In this work, we shall not address these issues; instead we shall compute the boson correlation functions for a specific drive \( f(\omega, \omega) = \cos(\omega \omega) \) and in the Gaussian regime.

For the specified periodic drive protocol, the quadratic action is of the form

\[ S_2 = \frac{2}{\omega_0^2} \sum_n \phi^*(k, \omega) \phi(k, \omega + n \omega_0), \]

(35)

where \( r_n \equiv r(\omega) \) and \( r_1 \equiv r(\ell) \) are to be evaluated at the RG time scale \( \ell \) where the RG flow stops.

The Green functions can be obtained from \( S_2 \) in a straightforward manner. Below we provide the expressions for the Keldysh Green functions. The advanced and the retarded Green functions can be easily read off from these expressions by replacing \( \delta(x - y) \rightarrow 1/(x - y + i\eta) \) in each of the terms below, where \( \eta \) is an infinitesimal positive quantity and \((+,-)\) corresponds to advanced(retarded) components of the Green function. The expressions for the Green functions are

\[ \Gamma \phi(k, \omega) \equiv \langle \phi(k, \omega - \omega_0) \phi(k, \omega + \omega_0) \rangle = [1 + 2 \eta(\omega)][g \delta(\omega - E_k) \]

\[ + g^2 \delta(\omega - E_k_1) + g^2 \delta(\omega - E_{k-1})], \]

\[ \Gamma \phi(k, \omega) \equiv \langle \phi(k, \omega + \omega_0) \phi(k, \omega + \omega_0) \rangle = [1 + 2 \eta(\omega)][g \delta(\omega - E_k) \]

\[ + g^2 \delta(\omega - E_k_1) + g^2 \delta(\omega - E_{k-1})]. \]

\[ \phi(k, \omega) \equiv \langle \phi(k, \omega - \omega_0) \phi(k, \omega + \omega_0) \rangle = [1 + 2 \eta(\omega)][f \delta(\omega - E_k) \]

\[ + f^2 \delta(\omega - E_k_1) + f^2 \delta(\omega - E_{k-1})]. \]

\[ \phi(k, \omega) \equiv \langle \phi(k, \omega - \omega_0) \phi(k, \omega + \omega_0) \rangle = [1 + 2 \eta(\omega)][-f \delta(\omega - E_k) \]

\[ - f^2 \delta(\omega - E_k_1) + f^2 \delta(\omega - E_{k-1})]. \]

where \( E_k = \sqrt{\omega_0^2 + r} \) denotes free boson dispersion and \( c_{1,2k} = E_k \pm \sqrt{\omega_0^2 + 2r_k^2} \). The spectral weights of the different
poles of these Green functions are given by \( g_i = r_i^2/(\omega_i^2 + 2r_i^2) \), \( s_{2,3} = [\omega_i^2 + r_i^2 \pm \omega_0 \sqrt{\omega_i^2 + r_i^2}]/(2(\omega_i^2 + 2r_i^2)) \), \( f_1 = -\omega_0 g_1 r_1 \), and \( f_{2,3} = r_1 (\omega_0 \pm \omega_0)^2 + 2r_i^2)/(2(\omega_0^2 + 2r_i^2)) \). Note that the presence of the drive terms leads to off-diagonal components in the boson Green functions in frequency space since such terms lead to a coupling of the different field modes.

Using the Green functions, we can obtain the boson occupation and the off-diagonal boson correlators at zero temperature. These are given by

\[
\begin{align*}
\bar{n}_k = \langle \phi^*(k, \omega) \phi(k, \omega) \rangle &= \int \frac{d\omega}{2\pi} \text{Tr}(G^R(k, \omega - 1)/2) \\
&= \Theta(\sqrt{\omega_i^2 + 2r_i^2} - E_k) \\
\bar{n}_k' = \langle \phi^*(k, \omega) \phi(k, \omega - \omega_0) \rangle &= r_1 (\omega_0 - \omega_0) / (2(\omega_0^2 + 2r_i^2)) \\
&\times \Theta(\sqrt{\omega_i^2 + 2r_i^2} - E_k) \\
\bar{n}_k'' = \langle \phi^*(k, \omega) \phi(k, \omega + \omega_0) \rangle &= -r_1 (\omega_0 + \omega_0) / (2(\omega_0^2 + 2r_i^2)) \\
&\times \Theta(\sqrt{\omega_i^2 + 2r_i^2} - E_k)
\end{align*}
\]  

(36)

Note that the expression for \( n_k \) reproduces the well-known limit for non-interacting theory for \( r_1 \sim \lambda = 0 \), where production of excitation requires \( \omega_0 \gg \sqrt{r} \). The presence of \( r_1 \) facilitates excitation production which can now occur at lower drive frequencies. In addition, it also leads to off-diagonal boson correlators \( \bar{n}_k' \) and \( \bar{n}_k'' \).

The constants \( r_1 \), \( \omega_0 \), and \( \omega_0 \) in the expressions of the correlators are the values at the scale where we stop the RG. As we have seen in the section 3, the RG flow stops at two scales \( \ell_1 \) and \( \ell_2 \). In the Gaussian regime where the RG flow stops at \( \ell = \ell_1 = \ln(1/r)/2 \), the occupation numbers at these scales are given by

\[
\bar{n}_k(\ell_1) = \Theta(\sqrt{\omega_i^2 r_i^2 + (2c_i^2 \alpha_i^2 \lambda_i^2)/(\epsilon_i^2 r_i^2)} - \sqrt{v_0^2 |k|^2 + 1})
\]  

(37)

In this perturbative quantum regime (equation (17)), \( \omega_0/r_i \ll 1 \) and \( \lambda(\ell_1) \ll r(\ell_1) \), so that the system does not produce excitations. By contrast, in the drive dominated regime, where the RG flow stops at \( \ell = \ell_2 \), one finds

\[
\bar{n}_k(\ell_2) = \Theta(\sqrt{\omega_0^2 \lambda_i^2 (\epsilon_i^2 r_i^2) + (2c_i^2 \alpha_i^2 \lambda_i^2)/(\epsilon_i^2 r_i^2)} - \sqrt{v_0^2 |k|^2 + r_0^2 \lambda_i^2 (\epsilon_i^2 r_i^2) + 1}).
\]  

(38)

where we have scaled all energies in units of \( \Lambda v_0 \). Equation (38) clearly demonstrates the facilitation of excitation production as \( \lambda \) and \( \omega_0 \) increases. In the limit where \( r(\ell_2) \rightarrow \lambda(\ell_2) \), the present analysis predicts that the threshold frequency for excitation production approaches zero. This indicates a rise in the energy absorption at the onset of the gaussian regime. We note that the present analysis does not hold when the non-gaussian regime sets in since the interaction can not be treated perturbatively any more.

Finally we evaluate the off-diagonal correlators at \( \ell = \ell_1 \) and \( \ell = \ell_2 \). These are given by

\[
\bar{n}_k^{(\ell_1)}(\ell_1) = \frac{c_i \alpha_i \lambda_i^2}{2 \epsilon_i} - \Omega(\lambda_i^2 - \epsilon_i^2 r_i^2) - \frac{\alpha_i^2}{\epsilon_i^2 r_i^2} + \frac{\epsilon_i^2 r_i^2}{\epsilon_i^2 r_i^2}
\]

(39)

6. Discussion

In this work, we have aimed at providing a perturbative RG approach to understanding the properties of a driven quantum system. We have illustrated the main points of our work by deriving and analyzing the RG equations for a system described by a scalar bosonic \( \lambda \phi^4 \) theory. The key results that emerge from our analysis are the following. First we show that the drive frequency scales like the physical temperature in equilibrium systems and sets the cutoff scale for RG given by \( \ell_2 = \ln(\Lambda v_0/\omega_0)/(\epsilon_i + 1) \). Second, when the drive frequency \( \omega_0 \) and the effective mass term \( r_0 \) satisfies \( \omega_0/2r_0 < 1 \), the drive can be treated perturbatively and one expect the universality of such a system to be analogous to its equilibrium counterpart. In this regime, the RG flow stops when the cutoff reaches the system correlation length at \( \ell = \ell_1 \leq \ell_2 \) and the drive does not qualitatively alter the behavior of the flow. Third, we show that in the other regime where \( \ell_2 \leq \ell_1 \) which occurs when \( r_0 \approx \omega_0/2r_0 < 1 \), the drive dominates the physics and may lead to setting in of a non-gaussian regime. For drive protocols with \( K(0) = 0 \) (or \( \alpha_0 = 0 \) for periodic protocols), the condition for setting in such a regime is \( |d| > \omega_0^2 \approx \omega_0^2 (\epsilon_i^2 r_i^2) \) for any \( d + z \). This relation clearly distinguishes between the behaviors of systems with \( d + z < 2 \) and \( d + z > 2 \). For the former, there exists an upper critical drive frequency below which the non-gaussian regime sets in while for the latter such a setting occurs above a lower critical drive frequency. For drive protocols with \( K(0) \), \( \alpha_0 = 0 \), we have shown that the analogous condition for irrelevant or marginal interaction is \( d + z \leq 4 \) is \( \lambda(r) > \omega_0/2r_0 < 1 \). Finally, we note that the present scheme can be easily generalized to drive protocols with multiple frequencies; for those drives, the highest characteristic frequency scale assumes the role of \( \omega_0 \).
We also note that the setting in of the non-gaussian regime occurs concomitantly with \( r(\omega) \) (or \( r_n \) for periodic drive) becoming comparable with \( r \). This indicates that the effective Hamiltonian which describes the dynamics in this regime will have non-perturbative mode coupling terms. Consequently, one expects that the energy pumped in the system due to the drive will be efficiently distributed between the different modes. Thus the system can effectively absorb large amounts of energy at long time. In contrast, in the gaussian regime, the mode coupling terms can be treated perturbatively and the presence of a large mass gap prevents the system from having large amounts of excess energy. The crossover between the two regimes has been argued in [23], using a Magnus expansion approach for a one-dimensional spin chain, to be the signature of an energy localization-delocalization transition. Our RG analysis shows similar behavior and provides a criterion for such a crossover to occur; however, deciphering the precise relation of the present general analysis with the specific quantitative study of [23] would require further study. We also note that for the present system such a crossover is not expected to occur if the drive term involved a time-dependent \( r \); in this case the mode coupling terms are present in the starting Hamiltonian and grow under RG. Consequently, the system is expected to continue to absorb energy indefinitely and hence be always delocalized in energy. This expectation receives support from our analysis of the boson correlators in section 5, where we show that the threshold frequency for excitation production approaches zero with the onset of the non-Gaussian regime where \( \lambda(t) \sim r(t) \).

Another generalization of our work would involve working out the RG equations to two loops. One expects such a calculation to unravel the dependence of the condition of setting in of the non-Gaussian regime on the anomalous dimension \( \eta \). The lowest-order non-zero contribution to \( \eta \) comes from two-loop RG diagrams and hence such a dependence can not be studied within the one-loop RG analysis carried out here. The simplest guess to the nature of such a correction is as follows. The scaling dimension of the fields \( \phi(k, \omega) \) in the presence of finite \( \eta \), is given by \( d' = (2 + d + z + \eta)/2 \). Using this, a straightforward power counting shows that \( \epsilon \to \epsilon' = 4 - d - z + 2\eta \). This means that \( \lambda \sim \epsilon' \) (for all protocols with \( K(0) \) or \( a_0 = 0 \) and \( r \sim \epsilon^2 \)). Thus at a scale \( \ell = \ell_2 \), the condition for the non-Gaussian behavior would be modified to \( \lambda/r \geq \omega_0 \langle \omega_0 \rangle \). Note that this is extremely important for systems with \( d + z = 2 \) where \( \eta \) is expected to provide the entire frequency dependence. However, this guess needs to be substantiated with full two-loop RG calculations which is left as a topic for future study.

Finally, we note that the present RG technique allows for several other extensions. First, it will be interesting to study the consequence of driving an open quantum system in the presence of a bath at a finite temperature which would allow for noise and dissipation using this scheme. Such a study has recently been carried out using functional RG in [11]; however, their work did not involve a time-dependent drive protocol which is the main focus of the present study. Second, the RG procedure could be easily generalized to actions describing bosonic fields with \( N > 1 \) components. Finally, it would be interesting to carry out a similar RG for fermions where the presence of a Fermi surface is expected to provide new features for the RG flow. We plan to undertake these studies in future.

### Appendix A. RG calculation

In this section, we provide a detailed derivation of the RG equations. Our analysis will be primarily carried out for theories with \( z = 2 \), but similar results can be obtained for \( z = 1 \). From equation (1), we note that \( S_0 \) represents the quadratic part of the action and leads to the Green functions given by

\[
G_k(k, \omega) = \langle \phi^*_k(k, \omega) \phi_k(k, \omega) \rangle = \left[ 1 + 2n_B(\omega) \right] \delta(\omega - E_k) \\
G_{\omega}(k, \omega) = \langle \phi^*_\omega(k, \omega) \phi_\omega(k, \omega) \rangle = \frac{1}{\omega - E_k + i\eta} = G_{\omega}^0(k, \omega). \tag{A.1}
\]

The interaction term \( S_1 \), in frequency and momentum space is given by equation (7).

Here we consider the perturbative corrections that originate from integrating out the field modes. To linear order in \( \lambda \) such a term is given by the diagram shown in figure A1 which leads to the one-loop correction to \( r \). One such term is given by

\[
\delta S_2 = -4\lambda \int \frac{d^d k_1 d^d k d a_0 d a_1}{(2\pi)^{d+2}} \phi^*_\omega(k_1, \omega_1) \phi^*_k(k_2, \omega_2) \\
\times \left( \frac{a_1 - a_2 + a_3 - a_4}{a_0} \right) \\
\times \phi^*_\omega(k_3 - k_2 + k_4, a_4) S_0. \tag{A.2}
\]

Other terms can be obtained in a similar fashion and finally we obtain

\[
\delta S_2 = -4\lambda \int \frac{d^d k d a_0 d a_2}{(2\pi)^{d+2}} K \left( \frac{a_1 - a_2}{a_0} \right) \\
\times \phi^*_\omega(k, \omega_1) \phi^*_\omega(k, \omega_2) \text{Tr}[G_k]. \tag{A.3}
\]

To make further progress we divide the terms into a piece for which \( a_1 = a_2 \) contributing to the renormalization of \( r \) and other terms for which \( a_1 \neq a_2 \). This is formally done by writing

\[
K(a_1/a_0) = K(a_1/a_0) \left[ \delta(a_1/a_0) + (1 - \delta(a_1/a_0)) \right]. \tag{A.4}
\]

Figure A1. Tadpole diagram for one-loop correction to mass term.
The first terms yields the one loop correction to \( r \) in the action

\[
G_R(k, k, 0) - G_R(k_1, k_1, 0) + G_R(k_2, k_2, 0)\]

leading to the RG equation for \( r \)

\[
\frac{dr(t)}{dt} = 2r + c_1(\Lambda) K(0) \dot{\lambda}(t),
\]

where \( c_1 = 4 \Gamma_d \Lambda^{d-1} [1 + 2n_B(E_\lambda)] \), and \( \Gamma_d \) is the angular integral in \( d \) dimensions. The second term generates new coupling terms in the action which is of the form

\[
\delta S_2 = - \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \phi^*(k_1, \omega_1) r(\omega_1) \phi^*(k_2, \omega_2 - \omega_1) \times \delta(\omega_1/\omega_0) \times \phi^*(k_3, \omega_3 - \omega_1),
\]

with the RG equation for \( r(\omega) \equiv r(\omega, t) \) given by

\[
\frac{dr(\omega, t)}{dt} = c(\Lambda) K(\omega/\omega_0) \dot{\lambda}(t).
\]

Next, we consider the one-loop correction to the quartic coupling \( \lambda \). The diagrams which contribute to the 1-loop correction to the term \( \delta S_2 \) are shown in the figure A2.

\[
\phi^*(k_1, \omega_1) \phi^*(k_2, \omega_2) \phi^*(k_3, \omega_3) \times \delta(\omega_1/\omega_0) \times \phi^*(k_4, \omega_4) \times \delta(\omega_4/\omega_0)
\]

Figure A2. 1-loop correction to the interaction term in the effective action.
where $\text{Tr}[G_K G_K + G_A G_K]$ is

\[
\int_{-\Lambda}^{\Lambda} \frac{d^d k k_0}{(2\pi)^d+1} \left[ G_K(k, k_0) G_K(k_1 + k, \omega_3 - \omega_1 + k_0) + G_A(k, k_0) G_A(k_2 - k + k_0, \omega_3 - \omega_1 + k_0) \right]
\]

\[
= \Gamma d^{d-1} d\Lambda \left[ 1 + 2\omega_0(E_{i\omega_1}) ] - [ 1 + 2\omega_0(E_{i\omega_1+k_0}) \right] \frac{\omega_3 - \omega_1 + E_1 - E_{i\omega_1+k_0}}{\omega_3 - \omega_1 + E_1 - E_{i\omega_1+k_0}}
\]  

(A.10)

Taking the limits of zero external frequencies and momenta, one gets

\[
\text{Tr}[G_K G_K + G_A G_K]_{k_0=0} = 2\Gamma d^{d-1} d\Lambda \frac{\partial \rho}{\partial E}
\]  

(A.11)

The other terms can be evaluated in a similar manner. Once again, we find that one can split $K(\omega)/\omega_0$ in to $\omega = 0$ and $\omega \neq 0$ parts using equation (A.4). The former provides correction to the $\lambda$ whereas the latter generates new terms in the action. The correction to the $\lambda$ coupling given by

\[
\delta S_0 = 4\lambda^2 K(0) \frac{d^d k d^d k_0 d^d k_1 d^d k_2 d^d \omega d^d \omega'}{(2\pi)^d+1} \omega_0 \omega_1 \omega_2 \omega_3 \omega_4
\]

\[
\times K(\omega_0) \text{Tr}[G_K G_K + G_A G_K]_{k_0=0}
\]

\[
\times \phi_4^{\lambda}(k_1, \omega) \phi_4^{\lambda}(k_2, \omega_2) \phi_4^{\lambda}(k_3, \omega_4)
\]

\[
(\omega - \omega_0) \times \phi_4^{\lambda}(k_3 - k_2 + k_0, \omega_0 + \omega_2 - \omega_3 - \omega_5 - 12)
\]  

(A.12)

leading to RG equation for $\lambda$

\[
\frac{d\lambda(t)}{dt} = c_2(\lambda) K(0) (\lambda(t) - c_2(\lambda) K(0))
\]  

(A.13)

where $c_2(\lambda) K(0) = -4 \text{Tr}[G_K G_K + G_A G_K]_{k_0=0}$. The expression for $c_2(\lambda)$ can thus be directly read off from equation (A.11).

The new terms in the action are of the form

\[
\delta S_{\lambda} = \int d^d k d^d k_0 d^d k_1 d^d \omega d^d \omega' \frac{d^d k_2 d^d \omega_1 d^d \omega_2 d^d \omega_3 d^d \omega_4 d^d \omega_5 d^d \omega_6}{(2\pi)^d+1} \omega_0 \omega_1 \omega_2 \omega_3 \omega_4 \omega_5 \omega_6
\]

\[
\times K(\omega_0) \text{Tr}[G_K G_K + G_A G_K]_{k_0=0}
\]

\[
\times \phi_4^{\lambda}(k_1, \omega_0) \phi_4^{\lambda}(k_2, \omega_2) \phi_4^{\lambda}(k_3, \omega_4)
\]

\[
\times \phi_4^{\lambda}(k_3 - k_2 + k_0, \omega_0 + \omega_1 - \omega_3 - \omega_5 - 12)
\]  

(A.14)

with the RG equations for $\lambda(\omega, \omega', \ell)$ given by

\[
\frac{d\lambda(\omega, \omega', \ell)}{d\ell} = -c_2(\lambda) K(\omega/\omega_0) K(\omega/\omega_0) \lambda(\omega, \omega', \ell)
\]  

(A.15)

Finally, we note that when the drive is periodic, the function $K(\omega/\omega_0)$ has support only on a set of discrete points. In this case, it is easier to write $K(\omega/\omega_0) = \sum_{n} a_n \delta(\pi + \omega/\omega_0 - n^2)$. The analysis for this form of $K$ is then easily carried out and obtains equation (12) with the identification $K(\omega/\omega_0) \rightarrow a_0$. This completes the derivation of the RG equations used in section 2.

Before ending this section, we would like to note that since $r_n$ and $\lambda_{mn}$ are spontaneously generated by the RG flow, one expects to include this term in the effective action as customary in the usual RG procedure. We have checked that at least for a simple drive protocol such as $f(t) = \exp(\omega_0 t)$, this leads to additional contribution to the loop diagrams shown in figures A1 and A2 which are $O(r_n^2/\Lambda^4)$ and $O(a_0^2/\Lambda^4)$ and can thus be ignored. We expect this feature to hold for other protocols as well.

References

[1] Shankar R 1994 Rev. Mod. Phys. 66 129
[2] Herziger J 1976 Phys. Rev. B 14 1165
[3] Millis A 1993 Phys. Rev. B 48 7183
[4] Polkovnikov A, Sengupta K, Silva A and Vengalattore M 2011 Rev. Mod. Phys. 83 853
[5] Bloch I, Dalibard J and Zwerger W 2008 Rev. Mod. Phys. 80 885
[6] Muller M et al 2009 Nature Mater. 5 56
[7] Mitra A, Giamarchi T 2011 Phys. Rev. Lett. 105 150602
[8] Forst M et al 2011 Phys. Rev. B 84 241104
[9] Singh R, Simonig C, Forst M, Prabhakaran D, Cavalleri A L and Cavalleri A 2013 Phys. Rev. B 88 075107
[10] Greiner M, Mandel O, Esslinger T, Hens J W, Bloch I 2002 Nature 415 39
[11] Orzel C, Tuchman A K, Fenselau L M, Yasuda M and Kasevich M A 2001 Science 291 2386
[12] Simon J, Bakr W, Ma R, Tai M E, Preiss P M and Greiner M 2011 Nature 472 307
[13] Mitra A and Giamarchi T 2012 Phys. Rev. Lett. 108 150203
[14] Metzner W, Salmhofer M, Honerkamp C, Meden V, Mathey L and Polkovnikov A 2010 Phys. Rev. A 81 033605