Completeness of
Unbounded Best-First Game Algorithms

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Abstract
In this article, we prove the completeness of the following game search algorithms: unbounded best-first minimax with completion and descent with completion, i.e. we show that, with enough time, they find the best game strategy. We then generalize these two algorithms in the context of perfect information multiplayer games. We show that these generalizations are also complete: they find one of the equilibrium points.

1 Introduction
Unbounded Best-First Minimax [12, 6, 4] is an old game tree search algorithm that is used very little. Unlike classic Minimax, this one performs a search at an unbounded depth, making it possible to explore the game tree non-uniformly and thus to anticipate many turns in advance with regard to the part of the game tree assumed the most interesting. Most recently, it has been successfully applied in the context of reinforcement learning without knowledge [5, 6]. More precisely, for many games, at least combined with a search algorithm dedicated to reinforcement learning, called Descent, the unbounded minimax gives better results than the state of the art of reinforcement learning without knowledge (i.e. the AlphaZero algorithm [17], based on Monte Carlo Tree Search (MCTS) [7, 3], another search algorithm, which is stochastic). The Descent algorithm is a modification of Unbounded Best-first Minimax that builds the search tree more in depth in order to more effectively propagate endgame information during learning. Unbounded Minimax and Descent in their basic version are however not complete in certain contexts like that of reinforcement learning, that is to say they do not allow to calculate an equilibrium point of the complete game tree (i.e. a winning strategy for each player) subject to having sufficient calculation time [5]. For example, the basic algorithms, not using the fact that certain states are resolved, will always choose to play during the exploration a resolved state rather than an unresolved state less well evaluated (whereas the
evaluation of a unresolved state is only an estimate, so the unresolved state may be better). This then blocks the exploration and prevents, when this scenario occurs, calculating the minimax value and therefore determining the best action to play. In [5], a modification of Descent and Unbounded Minimax, which is called completion, has been proposed.

We show in this article on the one hand that the completion makes the two algorithms complete (Section 2). These two algorithms being limited to two-player games, we propose in this article, for each of these two algorithms, two possible generalizations to the multi-player framework, i.e. to the Max\textit{n} framework [13, 11, 18] the equivalent of Minimax in the multi-player framework. The multiplayer algorithms that we propose are therefore unbounded versions and variants of the algorithm Max\textit{n} (which searches at a fixed depth). In addition, we show that these new algorithms are also complete (Section 3).

Note that it has been proven that the standard MCTS is complete [9, 10] and that it has also been generalized to the multi-player framework [20, 17, 11]. Several other search algorithms for multiplayer games have also been proposed [19, 10, 14, 8, 2], but they are all at fixed depth.

2 Completeness in Perfect Two Player Games

We start by defining the two player games that we consider in this article. Then, we recall the Unbounded Best-First Minimax and Descent algorithms using the completion technique. Finally, we show that these two algorithms are indeed complete.

2.1 Perfect Two Player Games

In this section, we are interested in two-player games with perfect information (games without hidden information or chance where players take turns playing), that is to say formally:

**Definition 1.** A perfect two-player game is a tuple \((S, A, j, f_0)\) where

- \((S, A)\) is a finite directed acyclic graph,
- \(j\) is a function from \(S\) to \(\{1, 2\}\),
- \(f_0\) is a function from \(\{s \in S \mid t(s)\} \to \{-1, 0, 1\}\), where :
  - \(t\) is a predicate such that \(t(s)\) is true if and only if \(|A(s)| = 0\);
  - \(A(s)\) is the set defined by \(\{s' \in S \mid (s, s') \in A\}\), for all \(s \in S\).

The set \(S\) is the set of game states. \(A\) encodes the actions of the game : \(A(s)\) is the set of states reachable by an action from \(s\). The function \(j\) indicates, for each state, the number of the player whose turn it is, i.e. the player who must play. The predicate \(t(s)\) indicates if \(s\) is a terminal state (i.e. an end-of-game state). Let \(s \in S\) such that \(t(s)\) is true, the value \(f_0(s)\) is the payout for the
first player in the terminal state $s$ (the gain for the second player is $-f_b(s)$; we are in the context of zero-sum games, non-zero-sum games are included in the multiplayer generalization introduced in the next section). We have $f_b(s) = 1$ if the first player wins, $f_b(s) = 0$ in the event of a draw and $f_b(s) = -1$ if the first player loses.

The terminal evaluation $f_b$ is often not very informative about the quality of a game. A more expressive terminal evaluation function, with values in $\mathbb{R}$, can be used to favor some games over others, which can improve the level of play. We denote such functions by $f_t$. For example, in score games, $f_t(s)$ maybe the score of the endgame $s$ (see [5] for other terminal evaluation functions). In order to guide the search, it is necessary to be able to evaluate non-terminal states. To do this, an evaluation function from $S$ to $\mathbb{R}$, denoted by $f_\theta(s)$, is used. A good evaluation function $f_\theta(s)$ provides for example an approximation of the minimax value of $s$. Such a function $f_\theta(s)$ can be determined by reinforcement learning, using $f_t(s)$ as “reinforcement heuristic” [5].

### 2.2 Algorithms

#### 2.2.1 Unbounded Best-First Minimax with Completion

Unbounded (Best-First) Minimax (noted more succinctly UBFM) is an algorithm which builds a (partial) tree of the game to then decide which is the best action to play given the current knowledge about the game (i.e. given the partial tree of the game). Each state $s$ of the partial game tree is associated with three values. The first value, the completion value $c(s)$, indicates the exact minimax value of $s$ compared to the classic gain of the game (if it is not known $c(s) = 0$). The (exact) minimax value of a state $s$ with respect to a certain terminal evaluation function (such as the classic game gain) is the terminal evaluation of the end-of-game state obtained by starting from $s$ where the players play optimally (i.e. the first player maximizes the end-of-game value and the second player minimizes it; each by having complete knowledge of the full game tree). The second value, the heuristic evaluation $v(s)$, is an estimate of the true minimax value of $s$ with respect to a certain reinforcement heuristic (a terminal evaluation function, at least, more expressive than the classic gain of the game). More precisely, the heuristic evaluation $v(s)$ is the minimax value of $s$ in the partial game tree where the terminal leaves of the tree are evaluated by the reinforcement heuristic and the other leaves by an adaptive evaluation function (learned, for example by reinforcement, to estimate the minimax value corresponding to the reinforcement heuristic). Finally, the third value, the resolution value $r(s)$, indicates whether the state $s$ is (weakly) resolved (case $r(s) = 1$) or not (case $r(s) = 0$). A terminal state is resolved. A non-terminal leaf is not resolved. An internal state is resolved if the best child is a winning resolved state for the current player or if all of its children are resolved ($c(s)$ is then exact). The algorithm iteratively builds the partial tree of the game by extending each time the best sequence of unresolved states (in a state $s$ the first player (resp. second player) chooses the state $s'$ which maximizes (resp. minimizes) the lexicograph-
ically ordered pair \((c(s), v(s))\). In other words, it adds the child states of the principal variation of the partial game tree deprived of resolved states. An iteration of Unbounded Minimax is described in Algorithm 2. This iteration is performed (from a game state) as long as there is some search time left (\(\tau\) : search time) or until this state is resolved (and therefore its minimax value is determined, as we will prove later).

\[
\text{Function } \text{completed\_best\_action}(s, A) \quad \text{if } j(s) = 1 \text{ then} \\
\qquad \qquad \text{return } \arg \max_{s' \in A} (c(s'), v(s'), n(s, s')) \\
\text{else} \\
\qquad \qquad \text{return } \arg \min_{s' \in A} (c(s'), v(s'), -n(s, s')) \\
\]

\[
\text{Function } \text{completed\_best\_action\_dual}(s, A) \quad \text{if } j(s) = 1 \text{ then} \\
\qquad \qquad \text{return } \arg \max_{s' \in A} (c(s'), v(s'), -n(s, s')) \\
\text{else} \\
\qquad \qquad \text{return } \arg \min_{s' \in A} (c(s'), v(s'), n(s, s')) \\
\]

\[
\text{Function } \text{backup\_resolution}(s) \quad \text{if } |c(s)| = 1 \text{ then} \\
\qquad \qquad \text{return } 1 \\
\text{else} \\
\qquad \qquad \text{return } \min_{s' \in A(s)} r(s') \\
\]

Algorithm 1: Definition of the algorithms \(\text{completed\_best\_action}(s, A)\), which computes the \textit{a priori} best child state by using completion from a set of child states \(A\), and \(\text{backup\_resolution}(s)\), which updates the resolution of \(s\) from its child states.

### 2.2.2 Descent with Completion

Descent is a variant of Unbounded Minimax. Its difference is that it will extend the best sequence of unresolved game states until it reaches a terminal state or a resolved state. So unlike Unbounded Minimax, it can add the children of several states at each iteration. It explores the game tree in a different order, much more in depth first (this exploration priority is interesting from a learning point of view \[5\]). An iteration of Descent is described in Algorithm 4.

### 2.2.3 Proof of Completeness

We now show that the two algorithms are complete. We start by formalizing precisely what we call being complete in the context of these two algorithms
and then we establish the completeness result.

**Definition 2.** The minimax value of a state \( s \in S \) (compared to the terminal evaluation \( f_b \) in the complete game tree \( S \)) is the value \( M(s) \) recursively defined by

\[
M(s) = \begin{cases} 
\max_{s' \in \mathcal{A}(s)} M(s') & \text{if } t(s) = f_b(s) \\
\min_{s' \in \mathcal{A}(s)} M(s') & \text{if } t(s) = 1
\end{cases}
\]

**Lemma 3.** If \( r(s) = 1 \) before an iteration of descent (resp. UBFM) then after the iteration, \( r(s) \) and \( c(s) \) have not changed.

**Lemma 4.** Let \((S_p, A)\) be a game tree built by the algorithm UBFM or by the algorithm descent from a certain state. Let \( s \in S_p \). We have the following properties:

- If \( r(s) = 1 \), then either \(|c(s)| = 1 \) or for all \( s' \in A(s) \), \( r(s') = 1 \).

**Algorithm 2:** UBFM tree search algorithm with completion (see Section 2 for the definitions of symbols; see Algorithm 1 for the definitions of completed_best_action(s) and backup_resolution(s)). Note: \( T = (v,c,r) \).
• If $|c(s)| = 1$, then $r(s) = 1$;
• If for all $s' \in A(s)$, $r(s') = 1$, then $r(s) = 1$.

Proof. By definition of the algorithm (in particular by the definition and by the use of the method $\text{backup\_resolution}(s)$ and because as soon as $r(s) = 1$, $r(s)$ and $c(s)$ do not change anymore (Lemma 3)).

Proposition 5. Let $(S, A)$ be a game tree built by UBFM or descent from a certain state and let $s \in S$.
If $r(s) = 1$, then $c(s) = M(s)$.

Proof. Let $(S_p, A)$ be a game tree built by UBFM (resp. descent) from a certain state (i.e. the algorithm has been applied $k$ times on that state). We show by induction this property. Let $s \in S_p$ such that $r(s) = 1$. We first show that this property is true for terminal states.

Function `descent\_iteration(s, S_p, T, f_\theta, f_t)`

```plaintext
if t(s) then
    $S_p \leftarrow S_p \cup \{s\}$
    $r(s), c(s), v(s) \leftarrow 1, f_b(s), f_t(s)$
else
    if $s \notin S_p$ then
        $S_p \leftarrow S_p \cup \{s\}$
        foreach $s' \in A(s)$ do
            if $t(s')$ then
                $S_p \leftarrow S_p \cup \{s'\}$
                $r(s'), c(s'), v(s') \leftarrow 1, f_b(s'), f_t(s')$
            else
                if $s' \notin S_p$ then
                    $r(s'), c(s'), v(s') \leftarrow 0, 0, f_\theta(s')$
                    $s' \leftarrow \text{completed\_best\_action}(s, A(s))$
                    $c(s), v(s) \leftarrow c(s'), v(s')$
                    $r(s) \leftarrow \text{backup\_resolution}(s)$
            if $r(s) = 0$ then
                $A \leftarrow \{s' \in A(s) \mid r(s') = 0\}$
                $s' \leftarrow \text{completed\_best\_action\_dual}(s, A)$
                $n(s, s') \leftarrow n(s, s') + 1$
                $\text{descent\_iteration}(s', S_p, T, f_\theta, f_t)$
                $s' \leftarrow \text{completed\_best\_action}(s, A(s))$
                $c(s), v(s) \leftarrow c(s'), v(s')$
                $r(s) \leftarrow \text{backup\_resolution}(s)$
```

Algorithm 3: Descent tree search algorithm with completion (see Section 2 for the definitions of symbols and Algorithm 1 for the definitions of $\text{completed\_best\_action}(s)$ and $\text{backup\_resolution}(s)$). Note: $T = (v, c, r)$. 

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Suppose in addition that \( t(s) \) is true. Thus, \( c(s) = f_b(s) \). Consequently, \( c(s) = f_b(s) = M(s) \).

We now show this property for non-terminal states: therefore suppose instead that \( t(s) \) is false.

- Suppose on the one hand that \( j(s) = 1 \).

If \( r(s) = 1 \), then either \( |c(s)| = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \), by Lemma 4.
If \( c(s) = -1 \), then, at the iteration that calculated \( c(s) = -1 \), we had: for all \( s' \in \mathcal{A}(s) \), \( c(s') = -1 \) (as at this moment \( c(s) = \max_{s' \in \mathcal{A}(s)} c(s') \)). Therefore, since this iteration, for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \) (by Lemma 4 and Lemma 3).
Thus, we have two cases: either \( c(s) = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \).

- Now suppose on the other hand, instead that \( j(s) = 2 \).

If \( r(s) = 1 \) then either \( |c(s)| = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \), by Lemma 4.
If \( c(s) = 1 \), then at the iteration that calculated \( c(s) = 1 \), we had: for all \( s' \in \mathcal{A}(s) \), \( c(s') = 1 \) (as at this moment \( c(s) = \min_{s' \in \mathcal{A}(s)} c(s') \)). Therefore, since this iteration, for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \) (by Lemma 4 and Lemma 3).
Thus, we have two cases: either \( c(s) = -1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \).

- Now suppose on the other hand, instead that \( j(s) = 2 \).

If \( r(s) = 1 \) then either \( |c(s)| = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \), by Lemma 4.
If \( c(s) = 1 \), then at the iteration that calculated \( c(s) = 1 \), we had: for all \( s' \in \mathcal{A}(s) \), \( c(s') = 1 \) (as at this moment \( c(s) = \max_{s' \in \mathcal{A}(s)} c(s') \)). Therefore, since this iteration, for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \) (by Lemma 4 and Lemma 3).
Thus, we have two cases: either \( c(s) = -1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \).

- Now suppose on the other hand, instead that \( j(s) = 2 \).

If \( r(s) = 1 \) then either \( |c(s)| = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \), by Lemma 4.
If \( c(s) = 1 \), then at the iteration that calculated \( c(s) = 1 \), we had: for all \( s' \in \mathcal{A}(s) \), \( c(s') = 1 \) (as at this moment \( c(s) = \min_{s' \in \mathcal{A}(s)} c(s') \)). Therefore, since this iteration, for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \) (by Lemma 4 and Lemma 3).
Thus, we have two cases: either \( c(s) = -1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \).

- Now suppose on the other hand, instead that \( j(s) = 2 \).

If \( r(s) = 1 \) then either \( |c(s)| = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \), by Lemma 4.
If \( c(s) = 1 \), then at the iteration that calculated \( c(s) = 1 \), we had: for all \( s' \in \mathcal{A}(s) \), \( c(s') = 1 \) (as at this moment \( c(s) = \max_{s' \in \mathcal{A}(s)} c(s') \)). Therefore, since this iteration, for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \) (by Lemma 4 and Lemma 3).
Thus, we have two cases: either \( c(s) = -1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \).

- Now suppose on the other hand, instead that \( j(s) = 2 \).

If \( r(s) = 1 \) then either \( |c(s)| = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \), by Lemma 4.
If \( c(s) = 1 \), then at the iteration that calculated \( c(s) = 1 \), we had: for all \( s' \in \mathcal{A}(s) \), \( c(s') = 1 \) (as at this moment \( c(s) = \min_{s' \in \mathcal{A}(s)} c(s') \)). Therefore, since this iteration, for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \) (by Lemma 4 and Lemma 3).
Thus, we have two cases: either \( c(s) = -1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \).

- Now suppose on the other hand, instead that \( j(s) = 2 \).

If \( r(s) = 1 \) then either \( |c(s)| = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \), by Lemma 4.
If \( c(s) = 1 \), then at the iteration that calculated \( c(s) = 1 \), we had: for all \( s' \in \mathcal{A}(s) \), \( c(s') = 1 \) (as at this moment \( c(s) = \max_{s' \in \mathcal{A}(s)} c(s') \)). Therefore, since this iteration, for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \) (by Lemma 4 and Lemma 3).
Thus, we have two cases: either \( c(s) = -1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \).

Proposition 6. Let \( S \) be the set of states of a two-player game. It exists \( N \in \mathbb{N} \) such that after applying \( N \) times the algorithm descent (resp. UBFM) on a state \( s \in S \), we have \( r(s) = 1 \).

Proof. We show that with at most \( N = 2 |S| \) iterations of descent (resp. UBFM) applied on a same state \( s \in S \), we have \( r(s) = 1 \). Note first that if \( s \) is terminal or satisfies \( r(s) = 1 \), then after having applied the algorithm, we have \( r(s) = 1 \). Now suppose that \( s \) is not terminal and satisfies \( r(s) = 0 \). To show this property
we show that each iteration adds in $S_p$ at least one state of $S$ not being in $S_p$ or marks as “solved” an additional state, that is, a state $s' \in S$ satisfying $r(s') = 0$, satisfies $r(s') = 1$ after the iteration. This is sufficient to show the property, because either after one of the iterations, we have $r(s) = 1$, or the iterative application of the algorithm ends up adding in $S_p$ all descendants of $s$ and/or by marking as “solved” all states of $S_p$. Indeed, if all the descendants of $s$ are added then necessarily $r(s) = 1$ (since by induction all descendants verify $r(s) = 1$; by definition and use of backup_resolution($s$)). Since $S$ is finite, with at most $2|S|$ iterations, $r(s) = 1$.

We therefore show, under the assumptions $r(s) = 0$ and $\neg t(s)$, that each iteration adds at least one new state of $S$ in $S_p$ or change the value $r(s')$ from 0 to 1 for a some state $s' \in S$. Let $\bar{s}$ be the current state analyzed by the algorithm (at the beginning $\bar{s} = s$). If $\bar{s}$ is not in $S_p$, then $\bar{s}$ is added in $S_p$. Otherwise for UBFM and then for descent, the algorithm recursively chooses the best child of the current state satisfying $r(\bar{s}') = 0$, which we denote $\bar{s}'$. For UBFM, this recursion is performed until $\bar{s}$ is not in $S_p$ (and adds it ) or that $\bar{s}$ is terminal or that there is no child state $\bar{s}'$ satisfying $r(\bar{s}') = 0$. Given that $\bar{s}$ necessarily satisfies $r(\bar{s}) = 0$ at the start of each recursion, $\bar{s}$ is not terminal. Therefore, this recursion is performed until $\bar{s}$ is not in $S_p$ or that there is no child state $\bar{s}'$ satisfying $r(\bar{s}') = 0$. In the latter case, all the children $\bar{s}'$ of the state $\bar{s}$ satisfy $r(\bar{s}') = 1$, and thus, at the end of the iteration we have $r(\bar{s}) = 1$ while at the beginning we have $r(\bar{s}) = 0$. Therefore, with UBFM, each iteration effectively adds a new state in $S_p$ or mark as solved a new state. With descent this recursion is performed until the state $\bar{s}$ is terminal or satisfies $r(\bar{s}) = 1$ after the block of the test “$\bar{s} \in S_p$”. Note that with descent, if $r(\bar{s}) = 0$ after the block of the test “$\bar{s} \in S_p$”, then there still exists a child state $\bar{s}'$ satisfying $r(\bar{s}') = 0$ (otherwise the block changes the value of $\bar{s}$ to $r(\bar{s}) = 1$). Since $r(\bar{s}) = 0$ at the start of each descent recursion step (and that $s$ is not terminal), the recursion is performed until the state $\bar{s}$ satisfies $r(\bar{s}) = 1$ after the test $\bar{s} \in S_p$. Thus, when the iteration ends, before the test, we have $r(\bar{s}) = 0$ and after the test, we have $r(\bar{s}) = 1$. Therefore, necessarily before the test, $\bar{s}$ was not in $S_p$ and thus $\bar{s}$ has been added. Consequently, for the two algorithms, an iteration adds at least one new state of $S$ in $S_p$, or marks as resolved a new state (under the assumption $s$ is neither terminal nor solved).

**Theorem 7.** The algorithm descent and the algorithm UBFM are complete, i.e. applying descent (resp. UBFM) on a state $s \in S$, with a search time $\tau$ large enough, gives $r(s) = 1$ and $c(s) = M(s)$.

**Proof.** By Proposition 6 then by Proposition 5.

## 3 Perfect Multiplayer Games

We now generalize to the multiplayer framework the two algorithms, unbounded minimax and descent with completion, to obtain respectively Unbounded Max$^n$ (that we note more succinctly UBFM$^n$) and descent$^n$ (both with completion).
We propose two possible generalizations for each of the two algorithms and we show that the 4 algorithms are all complete. Note that there is a difference between the two generalizations only if the game have draws.

In the multi-player framework (or two players with non-zero sum), the gain of the first player is no longer sufficient to characterize the end of the game. Each player must therefore have their own end-of-game gain. A terminal state is therefore evaluated by a n-uplet of values where the i-th component is the gain of i-th player and n is the total number of players. The goal of each player is then obviously to maximize its final gain. Unbounded Max^n and descent^n consist, in the context of each of the two variants, in iteratively extending the best sequence of unresolved states. For this, each player maximizes its own gain, that is to say it plays the state s' from the state s which lexicographically maximizes the pair \((c(s')_{j(s)}, v(s')_{j(s)})\) until reaching a leaf of the tree for Unbounded Max^n and and until reaching a terminal state or a resolved state for descent^n. As in the two-player framework, during each iteration, states that are not in the partial game tree are added.

However, generalizing to the multiplayer framework is not so easy, as a fundamental property is lost. Unlike the two-player framework, if the best child completion value is non-zero, then the current state is not necessarily resolved. Formally, the property \(\arg \max_{s' \in A(s)} c(s'_{j(s)}) \neq (0, \ldots, 0) \Rightarrow r(s) = 1\) is lost (or \(r(s) = 1\) no longer means to be solved). It is therefore necessary to modify the way in which either the completion value or the resolution value is calculated. Each of the two possibilities leads to a different algorithm, which builds a different game tree. The first possibility is to propagate the best completion value to the current state, only if the current state is resolved. With this approach, we get a weaker property \(c(s) \neq (0, \ldots, 0) \Rightarrow r(s) = 1\), but we also lose the weaker property \(c(s) = \arg \max_{s' \in A(s)} c(s'_{j(s)})\). The other possibility (introduced later) is to always propagate the best completion value, which requires that a state is marked as solved only if its best child is marked as solved. With this variant, we do not have the property \(c(s) \neq (0, \ldots, 0) \Rightarrow r(s) = 1\), but we recover the property \(c(s) = \arg \max_{s' \in A(s)} c(s'_{j(s)})\). It is not clear whether one of the two algorithms is better than the other. With the second approach, one can use exact information about what is likely to happen (instead of being limited to an estimate of what will happen). This information is the value \(c(s)\), which is, with this variant, the Max^n value in the partial game tree based on the classic terminal evaluation, i.e. the completion value of the last state of the best states sequence (which is terminal if \(c(s) \neq (0, \ldots, 0)\); in this case \(c(s)\) is the n-uplet of end-of-game gains). This additional information, although being the exact value of a terminal state, is not necessarily the exact value of that state. This happens if one of the players prefers to have the guarantee of a draw to a possibility of losing the game.

**Example 8.** The first player has the choice between two states \(s'\) and \(s''\) in a certain state \(s\). The state \(s'\), which is terminal, is evaluated by \(c(s') = (0, 0, 0, -1)\) and \(v(s') = (0, 0, 0, -1)\). The state \(s''\) is evaluated by \(c(s'') = (0, 0, 0, 0)\) and \(v(s'') = (-1, 1, 1, 1)\). With the current state of knowledge about
the game, a reasonable choice for the first player is to choose the state $s'$ (choose the state $s''$ is also a reasonable choice but it is a risky choice which seems less interesting although it could actually be more interesting). With the first variant, since the state $s$ is not resolved, $s$ is evaluated by $c(s) = (0, \ldots, 0)$. With the second variant, $c(s) = (0, 1, 0, -1)$, although $s$ is still not resolved.

With the first approach, this information, the completion value $c(s)$, which may be misleading or advantageous, is not used to distinguish unresolved states. Note that with the second approach, it is necessary to adapt the calculation of the best child to favor the resolved states over the unresolved states with the same completion value.

In the multi-player framework, there is another property that is lost, it is the uniqueness of the completion value of a state. This property can however be recovered under a certain assumption. In particular, it is necessary to consider the equilibrium point with respect to the pairs $(c(s), v(s))$, i.e. set the value $\text{Max}^n$ of a state $s$ as the pair $(f_b(s'), f_t(s'))$ of the last state $s'$ of the best states sequence starting from the state $s$ in the complete game tree. In addition, the terminal evaluation $f_t$ must verify a particular property, which guarantees to be able to decide between two terminal states of different gains. Otherwise, without these two conditions, two states can have the same gain for a same player but different gains for the other players. Thus, without these two conditions, there would be multiple optimal strategies with an uncertain outcome (i.e. a "king-making" effect).

### 3.1 Definition

We are now interested in multi-player perfect information games, that is to say formally:

**Definition 9.** A perfect multiplayer game with $n$ players ($n > 1$), is a tuple $(S, A, j, f_b)$ where

- $(S, A)$ is a finite acyclic directed graph,
- $j$ is a function from $S$ to $\{1, \ldots, n\}$,
- $f_b$ is a function from $\{s \in S | t(s)\}$ to $\{-1, 0, 1\}^n$, with :
  - $t$ a predicate such as $t(s)$ is true if and only if $|A(s)| = 0$ and
  - $A(s)$ the set defined by $\{s' \in S | (s, s') \in A\}$, for all $s \in S$.

The set $S$ is the set of game states. $A$ encodes the actions of the game : $A(s)$ is the set of states reachable by an action from $s$. The function $j$ indicates, for each state, the number of the player whose turn it is, that is to say the player who must play. The predicate $t(s)$ indicates if $s$ is a terminal state (i.e. an end-of-game state). Let $s \in S$ such that $t(s)$ is true. The value $f_b(s)_j$ is the gain for the $j$-th player in the terminal state $s$. We have $f_b(s)_j = 1$ if the
j-th player is winning, \( f_b(s)_j = 0 \) in the event of a draw for the j-th player and \( f_b(s)_j = -1 \) if the j-th player is losing.

The terminal evaluation \( f_s \) is often not very informative about the quality of a game. A more expressive terminal evaluation function, valuable in \( \mathbb{R}^n \), can be used to favor some games to others, which can improve the level of play. We denote such functions by \( f_t \). For example, in score games, \( f_t(s) \) can be the endgame scores of each player in the terminal state \( s \). In order to guide the search of the best action, it is necessary to be able to evaluate non-terminal states. To do this, an evaluation function from \( S \) to \( \mathbb{R}^n \), denoted by \( f_\theta(s) \), is used. A good evaluation function \( f_\theta(s) \) provides for example an approximation of the value \( \text{Max}^{n} \) of \( s \). Such a function \( f_\theta(s) \) can be determined by reinforcement learning, by using \( f_t(s) \) as a “reinforcement heuristic” and using the descent framework [5] with the algorithm descent” instead of descent.

### 3.2 First Multi-player Generalization

We start by introducing the first generalization, which consists in modifying \( c(s) \) only if \( s \) is resolved and which introduces an additional evaluation \( c'(s) \). The evaluation \( c'(s) \) is calculated from the values \( c(s') \) of the children \( s' \) of \( s \) in the same way as \( c(s) \) in the classic case. If \( s \) is resolved, then \( c(s) = c'(s) \) and \( c(s) \) is the value \( \text{Max}^{n} \) in the partial game tree whose leaves are labeled by the classic game gain \( f_b \). Otherwise, \( c(s) = (0, \ldots, 0) \). This ensures that if \( c(s) \) (resp. \( c'(s) \)) is not zero, then this value corresponds to a resolved child. To calculate an equilibrium point (an optimal strategy) in the multi-player case, it is also necessary that a state be considered resolved if it has a winning child state for the current player and the value \( v(s') \) of this child state is maximum (or that all the children are resolved). An iteration of Unbounded \( \text{Max}^{n} \) is described in Algorithm [6]. Once the partial game tree has been built, as in the two-player framework [5], there are two strategies for deciding which action to play: the one leading to the best value state (we choose the child \( s' \) of \( s \) which lexicographically maximizes \( (c(s')_j(s), v(s')_j(s)) \)) (see Algorithm [7]) or the safest action (we choose the action leading to a winning state of higher value if it exists, otherwise we choose the child \( s' \) of \( s \) which maximizes the number of times it has been selected from \( s \) and since the start of the game (see Algorithm [8]). An iteration of descent” is described in Algorithm [9] and the complete code of descent” is described in Algorithm [10].

Remark 10. Some variants of UBFM” and descent” are possible, which perhaps have a practical interest, at the cost of completeness. First, we can consider that a state is resolved as soon as \( c'(s)_j(s) = 1 \) (instead of imposing in addition \( v(s)_j(s) = \max_{s' \in S} f_t(s)_j(s') \)). Another variant is to stop an iteration of descent when the best action leads to a draw, i.e. choosing \( s' \leftarrow \text{best_action_n}(s, A(s)) \) instead of \( s' \leftarrow \text{best_action_n}(s, A) \). This could be interesting in the context of games with a lot of draws.

Regarding the choice of the action to play after having carried out a possibly
partial search, an alternative to the best action can be
\[
\operatorname{arg\,max}_{s' \in \mathcal{A}(s)} \left( \sum_{j} c(s'_j), c'(s'_j), v(s'_j), n(s, s') \right)
\]
and an alternative to the safest action may be
\[
\operatorname{arg\,max}_{s' \in \mathcal{A}(s)} \left( \sum_{j} c(s'_j), c'(s'_j), n(s, s'), v(s'_j) \right).
\]
Finally, note that the criterion
\[
v(s_j) = \max_{s' \in \{ s' \in S \mid t(s') \}} f_t(s'_j)
\]
can be replaced, while keeping the completeness, by a criterion of local maximum compared to the terminal states which are descendants of \( s \), if the local maximum is known, i.e.
\[
v(s_j) = \max_{s' \in D_t(s)} f_t(s'_j)
\]
where
\[
D_t(s) = \{ s' \in S \mid t(s') \land \exists s_1, \ldots, s_k \in \mathcal{S} \ s' \in \mathcal{A}(s_k) \land s_k \in \mathcal{A}(s_{k-1}) \land \cdots \land s_2 \in \mathcal{A}(s_1) \land s_1 \in \mathcal{A}(s) \}
\]

### 3.2.1 Proof of Completeness

We now show that the two algorithms are complete. We start by formalizing precisely what we call being complete in the context of these two algorithms and then we establish the completeness result.

**Definition 11.** Let \((S, \mathcal{A}, t, j, f_b)\) be a perfect multi-player game. Let \( f_t \) be a terminal evaluation function for this game.

**Algorithm 4:** Best action computation for \( n \) players (see Section 3.1 for the definitions of symbols).

```
Function best_action_n(s, T)
    return \operatorname{arg\,max}_{s' \in \mathcal{A}(s)} \left( \sum_{j} c(s'_j), v(s'_j), n(s, s') \right)

Function best_action_n_dual(s, T)
    return \operatorname{arg\,max}_{s' \in \mathcal{A}(s)} \left( \sum_{j} c(s'_j), v(s'_j), -n(s, s') \right)
```

**Algorithm 5:** Definition of backup_resolution_n, which updates the resolution value of the state \( s \) from its child states.

```
Function backup_resolution_n(s)
    if \( c'(s_j) = 1 \land v(s_j) = \max_{s' \in \{ s' \in S \mid t(s') \}} f_t(s'_j) \) then
        return 1
    else
        return \min_{s' \in \mathcal{A}(s)} r(s')
```

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The evaluation $f_t$ is said to be tie-breaking for the game if for all $s, s' \in S$ such that $t(s)$ and $t(s')$, we have either for all $j \in \{1, \ldots, n\}$, $f_b(s)_j = f_b(s')_j \implies f_t(s)_j \neq f_t(s')_j$ or $f_b(s) = f_b(s') \land f_t(s) = f_t(s')$.

**Definition 12.** The value $\text{Max}^n$ of a state $s \in S$ with respect to the terminal evaluation $f_t$ (in the complete game tree $S$) is the value $M(s)$ recursively defined by

$$M(s) = \begin{cases} M\left(\arg\max_{s' \in A(s)} \left(M(s')_{0,j(s')} , M(s')_{1,j(s')}\right)\right) & \text{si } \neg t(s) \\ (f_b(s) , f_t(s)) & \text{si } t(s) \end{cases}$$

**Function UBFM$^n$ _iteration$(s, S_p, T, f_\theta, f_t)$**

if $t(s)$ then
  $S_p \leftarrow S_p \cup \{s\}$
  $r(s), c(s), v(s) \leftarrow 1, f_b(s), f_t(s)$
else
  if $s \notin S_p$ then
    $S_p \leftarrow S_p \cup \{s\}$
    foreach $s' \in A(s)$ do
      if $t(s')$ then
        $S_p \leftarrow S_p \cup \{s'\}$
        $r(s'), c(s'), v(s') \leftarrow 1, f_b(s'), f_t(s')$
      else
        if $s' \notin S_p$ then
          $r(s'), c(s'), v(s') \leftarrow 0, (0, \ldots, 0), f_\theta(s')$
        else
          $A \leftarrow \{s' \in A(s) \mid r(s') = 0\}$
      if $|A| > 0$ then
        $s' \leftarrow \text{best_action_n_dual}(s, A)$
        $n(s, s') \leftarrow n(s, s') + 1$
        UBFM$^n$ _iteration$(s', S_p, T, f_\theta, f_t)$
    $s' \leftarrow \text{best_action_n}(s, A(s))$
    $c'(s), v(s) \leftarrow c(s'), v(s')$
    $r(s) \leftarrow \text{backup_resolution_n}(s)$
    if $r(s)$ then
      $c(s) \leftarrow c'(s)$

**Algorithm 6:** Iteration algorithm of UBFM$^n$ with completion (see Section 3.1 for the definitions of symbols, Algorithm 4 for the definitions of completed_best_action_n$(s)$ and Algorithm 5 for the definitions of backup_resolution_n$(s)$). Note: $T = (v, c, r)$, each $c(s)$ is initialized to $(0, \ldots, 0)$ and each number of selection of $s'$ from $s$ $n(s, s')$ is initialized to 0.
Lemma 13. Let \((S_p, A)\) be a game tree built by the algorithm UBFM\(^n\) or by the algorithm descent\(^n\) from a certain state and a tie-breaking terminal evaluation function for the game. Let \(s \in S_p\).

If \(r(s) = 1\) before an iteration of descent (resp. UBFM) then after the iteration, \(r(s)\), \(c(s)\) and \(v(s)\) have not changed.

Proposition 14. Let \((S_p, A)\) be a game tree built by the algorithm UBFM\(^n\) or by the algorithm descent\(^n\) from a certain state and a tie-breaking terminal evaluation function for the game. Let \(s \in S_p\). We have the following property:

- if \(r(s) = 1\) then either \(c'(s)_{j(s)} = 1\) and \(v(s)_{j(s)} = \max_{s'\in S | t(s')} f_t(s')_{j(s)}\) or for all \(s' \in A(s)\), \(r(s') = 1\);
- if \(c'(s)_{j(s)} = 1\) and \(v(s)_{j(s)} = \max_{s'\in S | t(s')} f_t(s')_{j(s)}\) then \(r(s) = 1\);
- if for all \(s' \in A(s)\), \(r(s') = 1\) then \(r(s) = 1\).

Proof. By definition of the algorithm (in particular by the definition and by the use of the method backup_resolution\(_n)(s)\) and because as soon as we have \(r(s) = 1\), \(r(s)\), \(v(s)\), \(c'(s)\) and \(c(s)\) do not change anymore (Lemma 13). \(\square\)

Function UBFM\(^n\)(\(s, S_p, T, f_\theta, f_t, \tau\))

\[
\begin{align*}
  t &= \text{time}() \\
  \text{while} \text{time}() - t < \tau \land r(s) = 0 \text{ do UBFM\(_n\)_iteration}(s, S_p, T, f_\theta, f_t) \\
  \text{return best\_action\(_n\)(s, T)}
\end{align*}
\]

Algorithm 7: The algorithm UBFM\(^n\) (see Section 3.1 for the definitions of symbols ; see Algorithm 6 for the code of UBFM\(_n\)_iteration\(s, S_p, T, f_\theta, f_t\); time() returns the current time in seconds ; \(\tau\): search time per action).

Function safest\_action\(_n\)(\(s, T\))

\[
\begin{align*}
  \text{return arg max}_{s' \in A(s)} \left( c(s')_{j(s)}, n(s, s'), v(s')_{j(s)} \right)
\end{align*}
\]

Function UBFM\(_s\)(\(s, S_p, T, f_\theta, f_t, \tau\))

\[
\begin{align*}
  t &= \text{time}() \\
  \text{while} \text{time}() - t < \tau \land r(s) = 0 \text{ do UBFM\(_n\)_iteration}(s, S_p, T, f_\theta, f_t) \\
  \text{return safest\_action\(_n\)(s, T)}
\end{align*}
\]

Algorithm 8: The algorithm UBFM\(_s\) (see Section 3.1 for the definitions of symbols ; see Algorithm 6 for the code of UBFM\(_n\)_iteration\(s, S_p, T, f_\theta, f_t\); time() returns the current time in seconds ; \(\tau\): search time per action).
Lemma 15. Let \((S_p,A)\) be a game tree built by the algorithm UBFM\(^n\) or by
the algorithm descent\(^n\) from a certain state. Let \(s \in S_p\).

If there exists \(j\) such that \(c(s)_j = 1\) then \(r(s) = 1\).

Proof. By definition of the algorithm, we have \(r(s) = 0 = \Rightarrow c(s) = (0, \ldots, 0)\).

Proposition 16. Let \(f_t\) be a tie-breaking terminal evaluation function. Let
\((S_p,A)\) be a game tree built by UBFM\(^n\) or descent\(^n\) from a certain state using

\[
\begin{align*}
\text{Function } \text{descent}^n_{\text{iteration}}(s, S_p, T, f_\theta, f_t) \\
\text{if } t(s) \text{ then} \\
&\quad S_p \leftarrow S_p \cup \{s\} \\
&\quad r(s), c(s), v(s) \leftarrow 1, f_\theta(s), f_t(s) \\
\text{else} \\
&\quad \text{if } s \notin S_p \text{ then} \\
&\quad \quad S_p \leftarrow S_p \cup \{s\} \\
&\quad \quad \text{foreach } s' \in A(s) \text{ do} \\
&\quad \quad \quad \text{if } t(s') \text{ then} \\
&\quad \quad \quad \quad S_p \leftarrow S_p \cup \{s'\} \\
&\quad \quad \quad \quad r(s'), c(s'), v(s') \leftarrow 1, f_\theta(s'), f_t(s') \\
&\quad \quad \quad \text{else} \\
&\quad \quad \quad \quad \text{if } s' \notin S_p \text{ then} \\
&\quad \quad \quad \quad \quad r(s'), c(s'), v(s') \leftarrow 0, (0, \ldots, 0), f_\theta(s') \\
&\quad \quad \quad s' \leftarrow \text{best_action}_n(s, A(s)) \\
&\quad \quad \quad c'(s), v(s) \leftarrow c(s'), v(s') \\
&\quad \quad \text{else} \\
&\quad \quad \quad r(s) \leftarrow \text{backup_resolution}_n(s) \\
&\quad \quad \quad \text{if } r(s) \text{ then} \\
&\quad \quad \quad \quad c(s) \leftarrow c'(s) \\
&\quad \text{if } r(s) = 0 \text{ then} \\
&\quad \quad A \leftarrow \{s' \in A(s) \mid r(s') = 0\} \\
&\quad \quad s' \leftarrow \text{best_action}_n_{\text{dual}}(s, A) \\
&\quad \quad n(s, s') \leftarrow n(s, s') + 1 \\
&\quad \quad \text{descent}^n_{\text{iteration}}(s', S_p, T, f_\theta, f_t) \\
&\quad \quad s' \leftarrow \text{best_action}_n(s, A(s)) \\
&\quad \quad c'(s), v(s) \leftarrow c(s'), v(s') \\
&\quad \quad \text{else} \\
&\quad \quad r(s) \leftarrow \text{backup_resolution}_n(s) \\
&\quad \quad \text{if } r(s) \text{ then} \\
&\quad \quad \quad c(s) \leftarrow c'(s)
\end{align*}
\]

Algorithm 9: Iteration algorithm of descent\(^n\) with completion (see Section 3.1 for
the definitions of symbols, Algorithm 4 for the definitions of completed_{best_action}_n(s) and
Algorithm 5 for the definitions of backup_resolution_n(s)). Note: \(T = (v, c, r)\) and each \(c(s)\) is initialized
to \((0, \ldots, 0)\).

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Let $s \in S_p$.

If $r(s) = 1$ then there exists a unique value $\text{Max}^n$ of $s$ with respect to $f_t$, denoted by $M(s)$, and we have $(c(s), v(s)) = M(s)$.

**Proof.** Let $(S_p, A)$ be a game tree built by UBFM$^n$ (resp. descent$^n$) from a certain state (i.e. the algorithm has been applied $k$ times on that state). We show this property by induction. Let $s \in S_p$ such that $r(s) = 1$. We first show that this property holds for terminal states.

Suppose in addition that $t(s)$ is true. Thus, $(c(s), v(s)) = (f_b(s), f_t(s))$ and therefore $(c(s), v(s)) = (f_b(s), f_t(s)) = M(s)$.

We now show this property for non-terminal states: we suppose instead that $t(s)$ is false.

Since $r(s) = 1$, we have either for all $s' \in A(s)$, $r(s') = 1$ or $c'(s)_{j(s)} = 1$ and $v(s)_{j(s)} = \max_{s' \in S | t(s')} f_t(s'_{j(s)})$, by Lemma 14. We also have $c(s) = c'(s)$.

If for all $s' \in A(s)$, $r(s') = 1$, then by induction, we have for all $s' \in A(s)$, $(c(s'), v(s')) = M(s')$. But $c(s) = c \left( \arg \max_{s' \in A(s)} \left( c(s'_{j(s)}), v(s'_{j(s)}) \right) \right)$ and $v(s) = v \left( \arg \max_{s' \in A(s)} \left( c(s'_{j(s)}), v(s'_{j(s)}) \right) \right)$, since there is a unique pair $(c(s'), v(s'))$ maximizing $\left( c(s'_{j(s)}), v(s'_{j(s)}) \right)$ (as $f_t$ is tie-breaking and by Lemma 13). Therefore $(c(s), v(s)) = M \left( \arg \max_{s' \in A(s)} \left( M(s'_{0,j(s)}), M(s'_{1,j(s)}) \right) \right)$, hence $(c(s), v(s)) = M(s)$.

If $c'(s)_{j(s)} = 1$ and $v(s)_{j(s)} = \max_{s' \in S | t(s')} f_t(s'_{j(s)})$, there exists $\tilde{s} \in A(s)$ such that we had $c'(s) = c(\tilde{s})$ and $v(s) = v(\tilde{s})$ at the iteration that marked $s$ as resolved. Thus, at this iteration $c(\tilde{s})_{j(s)} = 1$ and therefore $r(\tilde{s}) = 1$.

**Algorithm 10:** The descent$^n$ algorithm (see Section 5.1 for the definitions of symbols; see Algorithm 9 for the code of descent$^n$_iteration$(s, S, T, f_b, f_t)$; bestexplorationaction_n$(s, T)$ is an exploration method, for example an action selection distribution (see [5]).

```
Function completed_best_exploration_action_n(s, T)
    if r(s) then
        return arg max_{s' \in A(s)} \left( c(s')_{j(s)}), v(s')_{j(s)} \right)
    else
        return best_exploration_action_n(s, T)

Function descent^n(s, S_p, T, f_b, f_t, \tau)
    t = time()
    while time() - t < \tau \land r(s) = 0 do iteration_descent^n(s, S_p, T, f_b, f_t)
    return completed_best_exploration_action_n(s, T)
```
by Lemma 15. Thus, we still have \( c'(s) = c(\bar{s}), v(s) = v(\bar{s}) \) and \( r(\bar{s}) = 1 \) (Lemma 13). By induction, \( (c(\bar{s}), v(\bar{s})) = M(\bar{s}) \) and therefore \( (c(s), v(s)) = (c'(s), v(s)) = M(\bar{s}) \). But, since \( M(\bar{s}) \) is maximum for the player \( j(s) \), for all \( s' \in A(s) \) either \( \left[M(\bar{s})_{0,j(s)} , M(\bar{s})_{1,j(s)} \right] > \left(M(s')_{0,j(s)} , M(s')_{1,j(s)} \right) \) or \( M(\bar{s}) = M(s') \) (as \( f_t \) is tie-breaking). Thus,

\[
(c(s), v(s)) = M \left( \arg \max_{s' \in A(s)} \left( M(s')_{0,j(s)} , M(s')_{1,j(s)} \right) \right).
\]

Hence \( (c(s), v(s)) = M(s) \).

\[ \square \]

**Proposition 17.** Let \( S \) be the set of states of a perfect multi-player game. There exists \( N \in \mathbb{N} \) such that after applying \( N \) times the algorithm descent\(^n \) (resp. UB\(FM^n\)) on any state \( s \in S \), we have \( r(s) = 1 \).

**Proof.** We show that with at most \( N = 2|S| \) iterations of descent\(^n \) (resp. UB\(FM^n\)) applied to a certain state \( s \in S \), we have \( r(s) = 1 \). Note first that if \( s \) is terminal or satisfies \( r(s) = 1 \), then after applying the algorithm, we have \( r(s) = 1 \). Now suppose that \( s \) is not terminal and satisfies \( r(s) = 0 \). To show the proposition, we show that each iteration adds in \( S_p \) at least one state of \( S \) which is not in \( S_p \) or marks as solved an additional state, i.e. a state \( s' \in S \) satisfying \( r(s') = 0 \), satisfies \( r(s') = 1 \) after the iteration. This is sufficient to show the property, because either after one of the iterations, we have \( r(s) = 1 \) or the iterative application of the algorithm ends up adding in \( S_p \) all descendants of \( s \) and/or by marking all states of \( S_p \) as resolved. Indeed, if all the descendants of \( s \) are added then necessarily \( r(s) = 1 \) (since by induction all descendants satisfy \( r(s) = 1 \); by definition and use of backup\(_\text{resolution}_n(s) \)). Since \( S \) is finite, with at most \( 2|S| \) iterations, \( r(s) = 1 \).

We therefore show, under the assumption \( r(s) = 0 \) et \(-t(s) \), that each iteration adds at least one new state of \( S \) in \( S_p \) or change the value \( r(s') \) from 0 to 1 for a certain state \( s' \in S \). Let \( \bar{s} \) be the current state analyzed by the algorithm (at the beginning \( \bar{s} = s \)). If \( \bar{s} \) is not in \( S_p \), then \( \bar{s} \) is added in \( S_p \). Otherwise for UBFM and then for descent, the algorithm recursively chooses the best child of the current state satisfying \( r(\bar{s'}) = 0 \), which we denote \( \bar{s'} \). For UB\(FM^n\), this recursion is performed until \( \bar{s} \) is not in \( S_p \) (and adds it) or that \( \bar{s} \) is terminal or that there is no child \( \bar{s}' \) satisfying \( r(\bar{s'}) = 0 \). Given that \( \bar{s} \) necessarily satisfies \( r(\bar{s}) = 0 \) at the beginning of each recursion, \( \bar{s} \) is not terminal. Therefore, this recursion is performed until \( \bar{s} \) is not in \( S_p \) or that there is no child \( \bar{s}' \) satisfying \( r(\bar{s'}) = 0 \). In the latter case, all the children \( \bar{s}' \) of the state \( \bar{s} \) satisfies \( r(\bar{s'}) = 1 \), and therefore at the end of the iteration, we have \( r(\bar{s}) = 1 \) while at the beginning we have \( r(\bar{s}) = 0 \). Thus, with UB\(FM^n\), each iteration adds a new state in \( S_p \) or marks as solved a new state. With descent\(^n \), this recursion is performed until the state \( \bar{s} \) is terminal or satisfies \( r(\bar{s}) = 1 \) after the block of the test "\( \bar{s} \in S_p \)". Note that with descent, if \( r(\bar{s}) = 0 \) after the block of the test "\( \bar{s} \in S_p \)", then there is always a child \( \bar{s}' \) satisfying \( r(\bar{s'}) = 0 \) (otherwise the block would have changed the value of \( \bar{s} \) to \( r(\bar{s}) = 1 \).
\( r(\tilde{s}) = 0 \) at the start of each descent recursion step (and that \( s \) is not terminal), this recursion is performed until the state \( \tilde{s} \) satisfies \( r(\tilde{s}) = 1 \) after the block of the test \( \tilde{s} \in S_p \). Thus, when this iteration ends before the test, we have \( r(\tilde{s}) = 0 \) and after the test, we have \( r(\tilde{s}) = 1 \). Therefore, necessarily before the test, \( \tilde{s} \) is not in \( S_p \), and therefore \( \tilde{s} \) is added. Thus, for the two algorithms, an iteration adds at least one new state of \( S \) in \( S_p \) marks as solved a new state (under the assumption that \( s \) is neither terminal nor solved).

**Theorem 18.** The algorithm \( \text{descent}^n \) and the algorithm \( \text{UBFM}^n \) are “complete”, i.e. applying \( \text{descent}^n \) (resp. \( \text{UBFM}^n \)) on any state \( s \in S \) by using a tie-breaking terminal evaluation \( f_t \), with a search time \( \tau \) large enough, gives \( r(s) = 1 \) and \((c(s), v(s)) = M(s), the unique value Max^n of s with respect to f_t.\)

**Proof.** By Proposition[17] then by Proposition[16]

### 3.3 Second Multi-player Generalization

We now introduce the second generalization, which allows to keep the property \( c(s) = \arg \max_{s' \in \mathcal{A}(s)} c(s') \), and to use the additional information \( c(s) \) about unresolved states to build the partial game tree and decide on the best action to play.

#### 3.3.1 Algorithms

With the second generalization, the two algorithms \( \text{descent}^n \) and Unbounded Max^n are analogous to the algorithms of the first generalization but with several differences described below. At any time \( c(s) = c'(s) \) (so there is no need to \( c' \)). In addition, as with this variant an unresolved state can have a non-zero completion value, it is necessary to be able to separate a winning resolved state from an unresolved “winning” state. Thus, the calculation of the best action consists in choosing the child state \( s' \) of \( s \) maximizing \((r(s') \cdot c(s'), c(s'), v(s')) \) (Algorithm[11]). In addition, choosing the safest action then amounts to maximizing \((r(s') \cdot c(s'), c(s'), n(s, s'), v(s')) \) (Algorithm[12]). Finally, a state is resolved if all its children are resolved or if \( c(s)_{j(s)} = 1 \), if \( v(s)_{j(s)} \) is maximum, and if there is a solved child \( s' \) such that \((c(s), v(s)) = (c(s'), v(s')) \) (Algorithm[12]). The code of an iteration of Unbounded Max^n in the context of this variant is given in Algorithm[14]. The code of an iteration of \( \text{descent}^n \) in the context of this variant is given in Algorithm[15].

#### 3.3.2 Proof of Completeness

We now show that the two algorithms of the second variant are complete.

**Lemma 19.** Let \((S_p, \mathcal{A})\) be a game tree built by the algorithm \( \text{UBFM}^n \) or by the algorithm \( \text{descent}^n \) from a certain state. Let \( s \in S_p \). We have the following property:
• if \( r(s) = 1 \) then either \( c(s)_{j(s)} = 1 \) and \( v(s)_{j(s)} = \max_{s' \in \{s' \in S \mid t(s')\}} f_t(s')_{j(s)} \) and there exists \( \tilde{s} \in \mathcal{A}(s) \) such that \( c(s) = c(\tilde{s}) \), \( v(s) = v(\tilde{s}) \) and \( r(\tilde{s}) = 1 \) or for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \);
• if \( c(s)_{j(s)} = 1 \) and \( v(s)_{j(s)} = \max_{s' \in \{s' \in S \mid t(s')\}} f_t(s')_{j(s)} \) and there exists \( \tilde{s} \in \mathcal{A}(s) \) such that \( c(s) = c(\tilde{s}) \), \( v(s) = v(\tilde{s}) \) and \( r(\tilde{s}) = 1 \) then \( r(s) = 1 \);
• if for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \) then \( r(s) = 1 \).

Proof. By definition of the algorithm (in particular by the definition and by the use of the method backup_resolution_n(s) and because as soon as we have \( r(s) = 1 \), \( r(s) \), \( v(s) \) and \( c(s) \) do not change anymore)).

\[ \Box \]

Proposition 20. Let \( f_t \) be a tie-breaking terminal evaluation function. Let \((\mathcal{S}_p, \mathcal{A})\) be a game tree built by UBFM\(^n\) or descent\(^n\) from a certain state using \( f_t \). Let \( s \in \mathcal{S}_p \).

**Algorithm 11:** Best action function for \( n \) players (see Section 5.1 for the definitions of symbols).

```
Function best_action_n(s, T)
    return arg max_{s' \in \mathcal{A}(s)} \left( r(s') \cdot c(s')_{j(s)} , c(s')_{j(s)} , v(s')_{j(s)} , n(s, s') \right)

Function best_action_n_dual(s, T)
    return arg max_{s' \in \mathcal{A}(s)} \left( r(s') \cdot c(s')_{j(s)} , c(s')_{j(s)} , v(s')_{j(s)} , -n(s, s') \right)
```

Algorithm 12: Definition of backup_resolution_n(s), which updates the resolution value of the state \( s \) from its child states.

```
Function backup_resolution_n(s)
    if \( c(s)_{j(s)} = 1 \) \land \( v(s)_{j(s)} = \max_{s' \in \{s' \in S \mid t(s')\}} f_t(s')_{j(s)} \) \land \( \exists \tilde{s} \in \mathcal{A}(s) \ s.t. \ c(s) = c(\tilde{s}) \) \land \( v(s) = v(\tilde{s}) \) \land \( r(\tilde{s}) = 1 \) then
        return 1
    else
        return min_{s' \in \mathcal{A}(s)} r(s')
```

Algorithm 13: Safest action computation (see Section 5.1 for the definitions of symbols).
If \( r(s) = 1 \) then there exists a unique value \( \text{Max}^s \) of \( s \) with respect to \( f_t \), denoted by \( M(s) \), and we have \( (c(s), v(s)) = M(s) \).

Proof. Let \((S_p, A)\) be a game tree built by UBFM\(^n\) (resp. descent\(^n\)) from a certain state (i.e. the algorithm has been applied \( k \) times on that state). We show this property by induction. Let \( s \in S_p \) such that \( r(s) = 1 \). We first show that this property holds for terminal states.

Suppose in addition that \( t(s) \) is true. Thus, \( (c(s), v(s)) = (f_b(s), f_t(s)) \) and therefore \( (c(s), v(s)) = (f_b(s), f_t(s)) = M(s) \).

We now show this property for non-terminal states: we suppose instead that \( t(s) \) is false.

Since \( r(s) = 1 \), we have either for all \( s' \in A(s) \), \( r(s') = 1 \) or \( c(s') = 1 \), \( v(s') = \max_{s' \in \{s' \mid t(s')\}} f_t(s') \), and there exists \( \tilde{s} \in A(s) \) such that \( c(s) = c(\tilde{s}) \), \( v(s) = v(\tilde{s}) \) and \( r(\tilde{s}) = 1 \), by Lemma \([19]\).

**Function UBFM\(^n\)_iteration\((s, S_p, T, f_\theta, f_t)\)**

```pseudocode
if t(s) then
    \( S_p \leftarrow S_p \cup \{s\} \)
    \( r(s), c(s), v(s) \leftarrow 1, f_b(s), f_t(s) \)
else
if s \( \notin S_p \) then
    \( S_p \leftarrow S_p \cup \{s\} \)
    foreach \( s' \in A(s) \) do
        if t(s') then
            \( S_p \leftarrow S_p \cup \{s'\} \)
            \( r(s'), c(s'), v(s') \leftarrow 1, f_b(s'), f_t(s') \)
        else
            if s' \( \notin S_p \) then
                \( r(s'), c(s'), v(s') \leftarrow 0, (0, \ldots, 0), f_\theta(s') \)
else
    \( A \leftarrow \{s' \in A(s) \mid r(s') = 0\} \)
    if \( |A| > 0 \) then
        \( s' \leftarrow \text{best_action_n_dual}(s, A) \)
        \( n(s, s') \leftarrow n(s, s') + 1 \)
        UBFM\(^n\)_iteration\((s', S_p, T, f_\theta, f_t)\)
    \( s' \leftarrow \text{best_action_n}(s, A(s)) \)
    \( c(s), v(s) \leftarrow c(s'), v(s') \)
    \( r(s) \leftarrow \text{backup_resolution_n}(s) \)
```

**Algorithm 14:** Iteration algorithm of UBFM\(^n\) with completion (see Section 5.1 for the definitions of symbols, Algorithm 11 for the definitions of completed_best_action_n(s) and Algorithm 12 for the definitions of backup_resolution_n(s)). Note: \( T = (v, c, r) \).
If for all \( s' \in \mathcal{A}(s) \), \( r(s') = 1 \), then by induction, we have for all \( s' \in \mathcal{A}(s) \), \( (c(s'), v(s')) = M(s') \). But \( c(s) = c \left( \arg \max_{s' \in \mathcal{A}(s)} \left( c(s')_j(s), v(s')_j(s) \right) \right) \) and \( v(s) = v \left( \arg \max_{s' \in \mathcal{A}(s)} \left( c(s')_j(s), v(s')_j(s) \right) \right) \), since there is a unique pair \( (c(s'), v(s')) \) maximizing \( \left( c(s')_j(s), v(s')_j(s) \right) \) (as \( f_\theta \) is tie-breaking and that the values of a state no longer change as soon as it is marked as solved).

Therefore \( (c(s), v(s)) = M \left( \arg \max_{s' \in \mathcal{A}(s)} \left( M(s')_0(j(s)), M(s')_1(j(s)) \right) \right) \), hence \( (c(s), v(s)) = M(s) \).

Suppose \( c(s)_{j(s)} = 1 \), \( v(s)_{j(s)} = \max_{s' \in \mathcal{S}|t(s')} f_t(s'_{j(s)}) \), and there exists \( \tilde{s} \in \mathcal{A}(s) \) such that \( c(s) = c(\tilde{s}) \), \( v(s) = v(\tilde{s}) \) and \( r(\tilde{s}) = 1 \). By induction, \( (c(\tilde{s}), v(\tilde{s})) = M(\tilde{s}) \) and therefore \( (c(s), v(s)) = M(\tilde{s}) \). But, since \( M(\tilde{s}) \) is

\[
\text{Algorithm 15: Iteration algorithm of descent}^n \text{ with completion (Section 5.1 for the definitions of symbols, Algorithm 11 for the definitions of completed_best_action_n(s), and Algorithm 12 for the definitions of backup_resolution_n(s)). Note: } T = (v, c, r).
\]
maximum for the player \( j(s) \), for all \( s' \in A(s) \) either 
\[
\left( M(s')_{0,j(s)} , M(s')_{1,j(s)} \right) > \\
\left( M(s)_{0,j(s)} , M(s)_{1,j(s)} \right)
\]
or \( M(s) = M(s') \) (as \( f_t \) is tie-breaking). Thus, 
\[
(c(s), v(s)) = M \left( \arg \max_{s' \in A(s)} \left( M(s')_{0,j(s)} , M(s')_{1,j(s)} \right) \right),
\]
hence \( (c(s), v(s)) = M(s) \).

**Proposition 21.** Let \( S \) be the set of states of a perfect multi-player game. There exists \( N \in \mathbb{N} \) such that after applying \( N \) times the algorithm descent\(^n\) (resp. UBFM\(^n\)) on any state \( s \in S \), we have \( r(s) = 1 \).

*Proof.* The proof is analogous to that of Proposition 17.

**Theorem 22.** The algorithm descent\(^n\) and the algorithm UBFM\(^n\) are “complete”, i.e. applying descent\(^n\) (resp. UBFM\(^n\)) on any state \( s \in S \) by using a tie-breaking terminal evaluation \( f_t \), with a search time \( \tau \) large enough, gives \( r(s) = 1 \) and \( (c(s), v(s)) = M(s) \), the unique value \( \text{Max}^n \) of \( s \) with respect to \( f_t \).

*Proof.* By Proposition 21 then by Proposition 21.

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