Persistence versus extinction under a climate change in mixed environments

Hoang-Hung Vo *

December 3, 2014

Abstract

This paper is devoted to the study of the persistence versus extinction of species, which are modeled by the reaction-diffusion equation:

$$u_t - \Delta u = f(t, x_1 - ct, y, u) \quad t > 0, \quad x \in \Omega,$$

where $\Omega$ is of cylindrical type or partially periodic domain, $f$ is of KPP-type and the scalar $c > 0$ is a given forced speed. This type of equation originally comes from a model in population dynamics (see [1, 22, 25]) to study the impact of climate change on the persistence versus extinction of species. From those works, we know that the dynamics is governed by the traveling fronts $u(t, x_1, y) = U(x_1 - ct, y)$, thus characterizing the set of traveling fronts plays a major role. In this paper, we first consider a more general model than the model of [1] in higher dimensional space, where the environment is only assumed to be globally unfavorable with favorable pockets extending to infinity in two frameworks: the reaction term is time-independent or time-periodic dependent. For the later, we study the concentration of the species when the environment outside $\Omega$ becomes extremely unfavorable and further prove the symmetry breaking of the fronts.

Mathematical Subject Classification (2010): 35C07, 35J15, 35B09, 35P20, 92D25.

Key words: KPP equations, traveling wave solutions, eigenvalue problems, unfavorable, compactness argument, concentration, cylindrical domains.

Contents

1 Introduction and main results
   1.1 Introduction and definitions .........................................................
   1.2 Hypotheses and main results .......................................................

*Centre d’Analyse et de Mathématique Sociales, 190-198, Avenue de France 75244 Paris Cedex 13, France. Email: hhvo@ehess.fr.
1 Introduction and main results

1.1 Introduction and definitions

In the pioneering paper [1], Berestycki et al. studied the influence of climate change (global warming) on the population dynamics of biological species, who are strongly sensitive to temperature conditions. The authors proposed a mathematical model in $\mathbb{R}$, which is formulated as a reaction-diffusion equation with a forced speed $c$:

$$ u_t - u_{xx} = f(x - ct, u) \quad x \in \mathbb{R}, \quad (1) $$

where $f$ is such that

$$ f(x, s) = \begin{cases} -sm & x < 0 \text{ and } x > L \\ sm' \left(1 - \frac{s}{K}\right) & 0 \leq x \leq L, \end{cases} \quad (2) $$

for some positive constants $m, m', L, K$.

This nonlinearity refers to environment of the species being unfavorable outside a compact set and favorable inside. The higher dimensional versions with more general type of $f$ were studied later in [9], [10]. Beside that the same type of model was also considered in the context of competing species by Potapov and Lewis [22], where the authors investigated the co-existence of two species under the effect of climate change and moving range boundaries on habitat invasibility.
The main purpose of this paper is to study the criterion for persistence and extinction of species in more general frameworks than the ones considered in previous works \[1,10,22\] and further provide some applications of this theory. Basically, in \[1,10,22\] the authors investigated the environments, which are unfavorable outside a compact set, namely there exist \( R, m > 0 \) such that

\[
f_s(x, 0) \leq -m, \quad \forall |x| \geq R. \tag{3}
\]

Note that \( f_s(x, 0) \) is understood as the initial per capita rate of growth.

The present paper deals with the new cases in which condition (3) is no longer true. We extend the model of (2) in two frameworks. The first is for an infinite cylindrical domain with Neumann boundary condition:

\[
\begin{cases}
    u_t - \Delta u = f(x_1 - ct, y, u) & t > 0, \ x \in \Omega \\
    \partial_\nu u(t, x_1, y) = 0 & t > 0, \ x \in \partial \Omega,
\end{cases}
\tag{4}
\]

where \( \Omega = \mathbb{R} \times \omega, \omega \) is an open bounded and smooth domain in \( \mathbb{R}^{N-1} \), \( \nu \) denotes the exterior unit normal vector field to \( \Omega \). In this framework, the environments are assumed to be independent of time. Moreover, we are especially interested in considering the environments of mixed type, which are only assumed to be globally unfavorable at infinity.

To understand the model, one can think of the environment containing both favorable and unfavorable regions that extend all the way to infinity, namely \( f_s(x_1, y, 0) > 0 \) and \( f_s(x_1, y, 0) < 0 \) respectively as \( x_1 \to \pm \infty \), depending on the location of \( y \). The competitive and mutual influence between these regions play a major role in characterizing the persistence and extinction of the species in the whole domain. Such environments are pointed out in \[22\] that they are important to investigate. For example, if the species are of alpine ecosystem, their habitats are very complex, heterogeneous or fragmentary and the climate change may give different effects on each patch of their environment. Our paper presents a new look on such environments.

Mathematically, a global condition should be looked for to deal with the more general problem. We shall use a global condition in terms of spectral property to describe that the environment is globally unfavorable at infinity. The more detailed explanations of this condition will be given in subsection 1.2.1.

In the second framework, we investigate another type of mixed environment with periodic dependence on \( y \) and \( t \). More precisely, the equation is now of following type

\[
u_t - \Delta u = f(t, x_1 - ct, y, u) \quad t > 0, \ x = (x_1, y) \in \mathbb{R}^N,
\]

where the nonlinearity reaction is assumed to be periodic in \( y \) and \( t \). The time-periodic dependent reaction has been previously investigated in various frameworks, the interested reader are referred to \[14,19,20,23\]. The main difference of the present work with respect to these papers is that here \( f \) is not assumed to be periodic in \( x_1 \)-direction of but be shifted with the forced speed \( c \), which can be seen as an effect of climate change. Moreover, as considered in the first framework, we allow the medium \( f_s(t, x_1, y, 0) \) to be sign-changing depending on the location of \( y \in \mathbb{R}^{N-1} \) at the time \( t \) and we only require it to satisfy a global condition as \( x_1 \to \pm \infty \). The additional difficulties are due to the fact that we do not a priori require the solutions to be periodic in \( y \) nor in \( t \) and also we do
not impose any boundary conditions as \( x_1 \to \pm\infty \). The time-periodic dependence of reaction term can be thought as representation of a seasonal dependence of environment.

We further investigate the concentration of the species in more favorable region. More precisely, our aim is to describe the dynamics of the species in the first framework not only in the cylindrical domain \( \Omega \) but in the whole space \( \mathbb{R}^N \) under the assumption that the environment outside \( \Omega \) becomes more and more unfavorable. From the biological point of view one may wonder whether the species still survive if some parts of the environment becomes extremely unfavorable. This question can be addressed by solving the following mathematical problem. We consider equation (4) in the whole space \( \mathbb{R}^N \) and study the limit of the sequence of traveling fronts with the reaction terms \( F_n(x,s) \) such that their growth rates are negative outside the cylindrical domain \( \Omega \) and tend to \( -\infty \) as \( n \to \infty \). These solutions solve the equations

\[
\Delta U_n + c\partial_1 U_n + F_n(x,U_n) = 0, \quad x \in \mathbb{R}^N,
\]

where \( F_n(x,s) = f(x,s) \) for \( x \in \overline{\Omega} \) and \( \frac{\partial F_n}{\partial s}(x,0) \to -\infty \) as \( n \to \infty \) locally uniformly in \( \mathbb{R}^N \setminus \Omega \). If the species survives, we aim to characterize the limit. This is the object of section 4. We point out that, very recently, Guo and Hamel [16] have studied the similar problem on the periodic and not necessarily connected domains without the effect of climate change, namely \( c = 0 \). From a different point of view, the current investigation will consider the concentration of the species facing a climate change in the infinite cylindrical domain \( \Omega \) when the exterior domain \( \mathbb{R}^N \setminus \Omega \) becomes extremely unfavorable. To this aim, we first need to validate the existence and uniqueness of traveling front for problem (4) with Dirichlet boundary condition on \( \partial \Omega \). The lack of compactness of \( \Omega \) as well as the presence of \( c \neq 0 \) and non-constant unfavorability near infinity of \( \Omega \) are the central difficulties to be overcome thanks to some recent advances of spectral theory in [11].

Finally, the last result is devoted to the study of symmetry breaking of the fronts in \( \Omega \). The main reason leading to the symmetry breaking is the difference of asymptotic behaviors near \( \pm\infty \).

In the remainder of this section, we give notations and definitions that are used in the paper. The set \( \Omega \) denotes an infinite straight cylindrical domain \( \Omega = \mathbb{R} \times \omega \), where \( \omega \) is an open bounded and smooth domain in \( \mathbb{R}^{N-1} \). We sometimes use \( x \in \Omega \) instead of \( (x_1,y) \in \mathbb{R} \times \omega \) and denote :

\[
\Omega^+ = \{x \in \Omega, x_1 \geq 0, y \in \omega\} \quad \Omega^- = \{x \in \Omega, x_1 \leq 0, y \in \omega\};
\]

\[
\Omega_r = \{x \in \Omega, -r < x_1 < r, y \in \omega\}.
\]

Let \( \mathcal{O} \subset \mathbb{R}^N \) and \( L \) be a uniformly elliptic operator with coefficients bounded on \( \mathcal{O} \)

\[
Lu = a_{ij}(x)\partial_{ij}u(x) + b_i(x)u_i(x) + c(x)u.
\]

If \( \mathcal{O} \) is smooth and bounded, it is classical that \( L \) admits a unique eigenvalue \( -\lambda_D \) (respectively \( -\lambda_N \)) and a unique (up to multiplication) eigenfunction with Dirichlet (respectively Neumann) boundary condition i.e :

\[
\left\{ \begin{array}{ll}
L\varphi = -\lambda_D \varphi & x \in \mathcal{O} \\
\varphi = 0 & x \in \partial\mathcal{O}.
\end{array} \right.
\]

\[
\left\{ \begin{array}{ll}
L\varphi = -\lambda_N \varphi & x \in \mathcal{O} \\
\partial_{\nu} \varphi = 0 & x \in \partial\mathcal{O}.
\end{array} \right.
\]
As is known, the principal eigenpair (eigenvalue and eigenfunction) for an associated elliptic operator plays an important role in deriving persistence results and long time dynamics. In 1994, Berestycki, Nirenberg and Varadhan [4] gave a very simple and general definition of the principal eigenvalue of $L$ for general domains whose boundaries are not necessarily smooth and later Berestycki, Hamel and Rossi [2] used this approach to define generalized principal eigenvalues in unbounded domains. More precisely, now allowing $\mathcal{O}$ to be a smooth and possibly unbounded domain, they defined the generalized Neumann principal eigenvalue as follow

$$
\lambda_{N}(-L, \mathcal{O}) := \sup \{ \lambda \in \mathbb{R} : \exists \phi \in W^{2,\text{loc}}_{0}(\mathcal{O}), \phi > 0, (L + \lambda)\phi \leq 0 \text{ a.e in } \mathcal{O}, \partial_{\nu}\phi \geq 0 \text{ on } \partial\mathcal{O} \}. \quad (5)
$$

When $\mathcal{O}$ is bounded, then two notions coincide : $\lambda_{N}(-L, \mathcal{O}) = \lambda_{N}$. We adopt this definition in our paper. Under the assumption $a_{ij}, b_{i}, c \in L^{\infty}(\mathcal{O})$, it is easily seen that $\lambda_{1}(-L, \mathcal{O})$ is well defined. For related definition and more properties of generalized eigenvalue, the reader is referred to [11].

### 1.2 Hypotheses and main results

#### 1.2.1 The cylindrical environment without seasonal dependence

The function $f(x_{1}, y, s) : \mathbb{R} \times \omega \times [0, +\infty) \mapsto \mathbb{R}$ is assumed to be continuous in $x_{1}$, measurable in $y$, and locally Lipschitz continuous in $s$. In addition, the map $s \mapsto f(x, s)$ is of class $C^{1}(0, s_{0})$ for some positive constant $s_{0}$, uniformly in $x$. We assume that $f(x, 0) = 0, \forall x \in \Omega$.

The dynamics is controlled by traveling fronts. Thus, we look for the solutions of Eq. (4) of the type $u(t, x) = U(x_{1} - ct, y)$, which are called the traveling front solutions with forced speed $c$. Such solutions are given by the equation:

$$
\begin{cases}
\Delta U + c\partial_{1}U + f(x, U) = 0 & \text{in } \Omega \\
\partial_{\nu}U = 0 & \text{on } \partial\Omega \\
U > 0 \text{ in } \Omega
\end{cases}
$$

(6)

In the results below, we will require the following hypotheses on $f$:

$$
\exists S > 0 \text{ such that } f(x, s) \leq 0 \text{ for } s \geq S, \forall x \in \Omega, \quad (7)
$$

$$
s \mapsto f(x, s)/s \text{ is nonincreasing a.e in } \Omega \text{ and there exist } D \subset \Omega, |D| > 0 \text{ such that it is strictly decreasing in } D. \quad (8)
$$

Both of these conditions are classical in the context of population dynamics. The first condition means that there is a maximum carrying capacity effect : when the population density is very large, the death rate is higher than the birth rate and the population decreases. The second condition means the intrinsic growth rate decreases when the population density is increasing. This is due to the intraspecific competition for resources.

As has been already mentioned, we are looking for a condition that applies to mixed environments. We assume that there exists a measurable bounded function $\mu : \omega \mapsto \mathbb{R}$ such that

$$
\mu(y) = \limsup_{|x_{1}| \to \infty} f_{s}(x_{1}, y, 0) \quad \text{and} \quad \lambda_{\mu} := \lambda_{N}(-\Delta_{y} - \mu(y), \omega) > 0. \quad (9)
$$

5
Condition (9) means that in the mixed environment is globally unfavorable at $x_1$-infinity. This generalizes the following condition

$$f_s(x_1, y, 0) \leq -m < 0 \quad \text{for } |x_1| \text{ large enough, } y \in \omega,$$

which is used in [10] since if $\mu(y) \leq -m < 0$ one gets immediately $\lambda_1(-\Delta_y - \mu(y), \omega) \geq m > 0$. Our generalization here aims at allowing $f_s(x_1, y, 0)$ to change sign when $|x_1|$ large and obviously (10) satisfies (9) by choosing $\mu(y) \equiv -m$ and 1 is a test-function. An illustration of condition (9) is given in Section 2.1.

We are ready to state the results of this section:

### 1.2.2 The existence and uniqueness of traveling front

The existence and uniqueness results are directly conditioned by the amplitude of the speed of climate change and the sign of the principal eigenvalue $\lambda_0 := \lambda_1(-L_0, \Omega)$, where

$$L_0 \varphi = \Delta \varphi + f_s(x, 0)\varphi.$$

**Theorem 1.1.** Assume that (7)-(9) hold. Then there exists a unique critical speed $c^*$ such that Eq. (6) admits solutions which are traveling fronts of (4) if and only if $0 \leq c < c^*$. Moreover, the front is unique once exists.

This theorem yields an analogous result to the ones in [1], [10], [22], where the assumption of type (10) is primarily applied. However, more complex environments are investigated and thus makes the problem nontrivial. We further point out that to the uniqueness of Eq. (6) is achieved in the class of bounded solutions without necessarily prescribing the boundary condition as $x_1 \to \pm \infty$.

**Definition of the critical speed $c^*$**

By using the Liouville transformation $V(x_1, y) := U(x_1, y)e^{\frac{c^2}{2}x_1}$, problem (6) is equivalent to

$$\begin{cases}
\Delta V + f(x_1, y, V(x_1, y)e^{-\frac{c^2}{2}x_1})e^{\frac{c^2}{2}x_1} - \frac{c^2}{4}V = 0 & x \in \Omega \\
\partial_\nu V = 0 & x \in \partial \Omega \\
V > 0 \text{ in } \Omega \\
V(x_1, y)e^{-\frac{c^2}{2}x_1} \text{ is bounded.}
\end{cases}$$

Linearizing this equation about 0, one gets a self-adjoint operator :

$$\tilde{L}w := \Delta w + (f_s(x, 0) - c^2/4)w.$$

We set $L_0 \varphi = \Delta \varphi + f_s(x, 0)\varphi$ and $\lambda_0 := \lambda_N(-L_0, \Omega)$ is the generalized Neumann principal eigenvalue of $L_0$ in $\Omega$. Since $f_s(x, 0)$ is bounded, $\lambda_0$ is well defined and finite. We are led to

**Definition 1.2.** We define the critical speed by

$$c^* := 2\sqrt{-\lambda_0} \quad \text{if } \lambda_0 < 0$$

(12)
Proposition 1.3. The eigenvalue $\lambda_1(-\Delta - c\partial_1 - f_s(x,0), \Omega) < 0$ iff $0 \leq c < c^*$. 

Proof. Let $\mathcal{L} = \Delta + c\partial_1 + f_s(x,0)$. Since we do not assume the test-function of (5) to be bounded. It immediately follows from the definition (5) that $\lambda_N(-\mathcal{L}, \Omega) = \lambda_N(-\tilde{\mathcal{L}}, \Omega)$. \hfill $\square$

The next two results deal with the long time dynamics of the evolution equation (4) in $L^\infty(\Omega)$ and $L^1(\Omega)$.

1.2.3 Long time dynamics

Theorem 1.4. Let $u(t,x)$ be the solution of (4) with initial condition $u(0,x) \in L^\infty(\Omega)$, which is nonnegative and not identically equal to zero. Assume that (7)−(9) hold.

i) If $c \geq c^*$ then

$$\lim_{t \to \infty} \|u(t,x)\|_{\infty,\Omega} = 0;$$

ii) if $0 \leq c < c^*$ then

$$\lim_{t \to \infty} \|(u(t,x_1+ct,y) - U(x_1,y))\|_{\infty,\Omega} = 0,$$

where $U$ is the unique solution of (7) and $\| \cdot \|_{\infty,\Omega}$ denotes the sup-norm on $\Omega$.

This theorem means that, under the effect of climate change, a species cannot keep up with the climate change if its pace is too large. This theorem generalizes the results in [1], [10]. Note that, in [1], [10], condition (10) was actually used in the proofs, in particular, to derive the exponential behavior at infinity. Therefore, here we have to use different arguments to treat the more general situation when (10) is not necessarily fulfilled. Although our approaches are similar to those in [1], [10], extra complications arise from the non-constant unfavorable characterization at infinity, especially to obtain the comparison principle.

The next result is concerned with the $L^1(\Omega)$ convergence of the traveling fronts. This result describes the long time dynamics of total population.

Theorem 1.5. Let $u(t,x)$ be the solution of (4) with initial condition $u(0,x) \in L^\infty(\Omega) \cap L^1(\Omega)$, which is nonnegative and not identically equal to zero. Assume that (7)−(9) are satisfied then the same conclusions as in Theorem 1.4 hold with $L^1(\Omega)$ norm instead of $L^\infty(\Omega)$.

1.2.4 The partially periodic environment with seasonal dependence

We now consider problem (4) in partially periodic environments with seasonal dependence. Namely, the reaction term $f$ now depends periodically in time variable and Eq. (4) becomes

$$u_t - \Delta u = f(t,x_1-ct,y,u) \quad t \in \mathbb{R}, x = (x_1,y) \in \mathbb{R}^N,$$

where $c > 0$ is given forced speed and $f$ is now assumed to be periodic in $y$. More precisely, we say that the environment is partially periodic in $y$ and depends seasonally on time if:
1) \( \forall i \in \{1, ..., N - 1\} \), there exist the constants \( L_1, ..., L_{N-1} \) such that
\[
f(t, x + L_i e_{i+1}, s) = f(t, x, s) \quad \forall t \in \mathbb{R}, s \in \mathbb{R}, x \in \mathbb{R}^N
\]
where \( \{e_1, ..., e_N\} \) denotes the unit normal orthogonal basis of \( \mathbb{R}^N \).

2) There exists \( T > 0 \), such that
\[
f(t + T, x, s) = f(t, x, s) \quad \forall t \in \mathbb{R}, s \in \mathbb{R}, x \in \mathbb{R}^N.
\]

We assume in addition that \( f(t, x, 0) = 0 \), \( f \) is Lipschitz continuous with respect to \( s \) and Holder-continuous with respect to \( x \) and \( t \), precisely
\[
\forall s > 0, \quad f(\cdot, \cdot, s), f_s(\cdot, \cdot, 0) \in C^\alpha_t,\alpha_x(\mathbb{R} \times \mathbb{R}^N),
\]
where \( C^\alpha_t,\alpha_x(I \times \mathcal{H}), I \subset \mathbb{R}, \mathcal{H} \subset \mathbb{R}^N \) denotes the space of functions \( \phi(t, x) \) such that \( \phi(\cdot, x) \in C^\alpha_t(I) \) and \( \phi(t, \cdot) \in C^\alpha(\mathcal{H}) \) uniformly with respect to \( x \) and \( t \) respectively.

We are interested in looking for the pulsating fronts of (13), namely the solutions of the form
\[
u(t, x) = U(t, x_1 - ct, y) > 0.
\]
They are obtained from the equation
\[
\begin{align*}
U_t &= \Delta U + c\partial_1 U + f(t, x, U) \\
U &= \text{is bounded.}
\end{align*}
\]
\tag{14}

To obtain the results, we will need to study the principal eigenvalue of linearized operator of Eq. (14). Generally, we consider the operators of the form:
\[
Lu = \partial_t u - a_{ij}(t, x)\partial_{ij} u(t, x) - b_i(t, x)u_i(t, x) - c(t, x)u(t, x), \quad x = (x_1, y) \in \mathbb{R}^N,
\]
\tag{15}

where \( a_{ij}, b_i, c_i \) are \( T \)-periodic in \( t \) and partially periodic in \( y \) with the same period. To define the generalized principal eigenvalue of \( L \), we assume that the coefficients satisfy the regularity condition as mentioned above and the matrix \( (a_{ij}(t, x)) \) is uniformly elliptic, namely \( a_{ij}, b_i, c_i \in C^\alpha_t,\alpha_x(\mathbb{R} \times \mathbb{R}^N) \) and there exist some positive constants \( E_1, E_2 \) such that for all \( \xi \in \mathbb{R}^N \) and \( (t, x) \in \mathbb{R} \times \mathbb{R}^N \) such that
\[
E_1\|\xi\|^2 \leq \sum_{1 \leq i \leq j \leq N} a_{ij}(t, x)\xi_i\xi_j \leq E_2\|\xi\|^2,
\]
where \( \|\xi\| = \sqrt{\xi_1^2 + ... + \xi_N^2} \).

**Definition 1.6.** Let \( \mathcal{O} \subset \mathbb{R} \) and \( Q = \{(t, x) = (t, x_1, y) \in \mathbb{R} \times \mathcal{O} \times \mathbb{R}^{N-1}\} \), the generalized principal eigenvalue of \( L \) on \( Q \) is defined by:
\[
\hat{\lambda}_1(L, Q) = \sup \{ \lambda \in \mathbb{R} : \exists \phi > 0, \phi \in C^1_{t,x}(Q), \phi \text{ is } T\text{-periodic in } t \text{ and partially periodic in } y \text{ such that } (L - \lambda)\phi \geq 0 \text{ in } Q \}.
\]
\tag{16}
By assuming in addition that \(a_{ij}, b_i, c_i \in L^\infty(\mathbb{R}^{N+1})\), one can take \(\lambda = -\sup_{\mathbb{R} \times \mathbb{R}^N} f_s(t,x,0)\) and 1 as a test function to see that \(\tilde{\lambda}_1\) is well-defined and \(-\sup_{\mathbb{R} \times \mathbb{R}^N} f_s(t,x,0) \leq \tilde{\lambda}_1\). We point out that this definition does not make sense if we do not assume that the test functions are periodic in \(t\). Indeed, since \((\mathcal{L} - \lambda)(\phi e^{\alpha t}) = (\mathcal{L} + \alpha - \lambda)(\phi e^{\alpha t}), \forall \alpha \in \mathbb{R}\), if we do not force the periodicity in \(t\), it would yield \(\tilde{\lambda}_1 = \lambda_1 + \alpha\) for all \(\alpha\). When \(\mathcal{O} = \mathbb{R}\), the class of admissible test-function may contain the functions, which decay in \(x_1 \in \mathbb{R}\).

We further need the two following conditions that are similar to \([7,8]\), but here \(f\) takes into account the time periodic dependence:

\[
\exists S > 0 \text{ such that } f(t,x,s) \leq 0 \text{ for } s \geq S, \forall t \in \mathbb{R}, x \in \mathbb{R}^N, \quad (17)
\]

\[
s \to \frac{f(t,x,s)}{s} \text{ is nonincreasing and for all } t_0 \in \mathbb{R} \text{ there exist } D \subset (-\infty, t_0) \times \mathbb{R}^N, |D| > 0 \text{ such that it is strictly decreasing in } D. \quad (18)
\]

Suppose that a parabolic operator \(\tilde{\mathcal{L}}\) is defined on \(\mathbb{R} \times \mathbb{R}^{N-1}\) and has the form

\[
\tilde{\mathcal{L}}\phi = \partial_t \phi - a_{ij}(t,y)\partial_{ij}\phi(t,y) - b_i(t,y)\phi_i(t,y) - c(t,y)\phi(t,y), \quad y \in \mathbb{R}^{N-1}.
\]

Under assumptions that \(a_{ij}(t,y), b_i(t,y), c_i(t,y) \in L^\infty(\mathbb{R} \times \mathbb{R}^{N-1})\) and the matrix \(a_{ij}(t,x)\) satisfies the uniform elliptic condition, we define the generalized space-time periodic eigenvalue of \(\tilde{\mathcal{L}}\):

\[
\tilde{\lambda}_1(\tilde{\mathcal{L}}, \mathbb{R} \times \mathbb{R}^{N-1}) = \sup\{\lambda \in \mathbb{R} : \exists \phi > 0, \phi \in C^{1,2}_{t,y}(\mathbb{R} \times \mathbb{R}^{N-1}), \phi \text{ is } T\text{-periodic in } t
\]

and periodic in \(y\) such that \((\tilde{\mathcal{L}} - \lambda)\phi \geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^{N-1}\}.

This kind of eigenvalue seems analogous to the ones introduced by Berestycki and Rossi \([9]\) and Nadin \([19]\). However, the difference is that here we force the test functions to be periodic in \(t\) and \(y\) but not in \(x_1\) while in \([9]\), the test functions are not periodic in any direction of \(x = (x_1, y)\) and in \([19]\), the test functions must be periodic in both \(t\) and \(x = (x_1, y)\). On the other hand, in \([19]\), the author has shown the existence and uniqueness of the space-time periodic eigenpair \((\lambda_p, \varphi_p)\) of the eigenvalue problem

\[
\begin{cases}
\tilde{\mathcal{L}}\varphi_p = \lambda_p\varphi_p \\
\varphi_p > 0 \\
\varphi_p(\ldots + T) = \varphi_p \\
\varphi_p(\ldots + L_ie_i, \ldots) = \varphi_p.
\end{cases}
\]

See further in Theorems 2.7, \([19]\). Obviously, \(\lambda_p \leq \tilde{\lambda}_1(\tilde{\mathcal{L}}, \mathbb{R} \times \mathbb{R}^{N-1})\).

Using this notion, we assume that there exists a function \(\gamma(t,y) \in L^\infty(\mathbb{R} \times \mathbb{R}^{N-1})\), which is periodic in \(y\) and \(T\)-periodic in \(t\) such that

\[
\gamma(t,y) = \lim_{|x_1| \to \infty} \sup f_s(t,x_1,y,0) \quad \text{and} \quad \tilde{\lambda}_1(\partial_t - \Delta_y - \gamma(t,y), \mathbb{R} \times \mathbb{R}^{N-1}) > 0. \quad (19)
\]

Condition \((19)\) yields an important characterization of the environment near infinity. The meaning of this condition has been mentioned in the first section. The difficulties of this problem arise
since we deal with the solution in the unbounded domain (whole space) without assuming a priori that it is periodic in $y$ nor in $t$. Moreover, the monotonicity in time of solutions of parabolic operators starting by a stationary sub (or super) solution no longer holds, but this obstacle can be overcome by the help of time-periodic assumption of the reaction term.

Let us call $P \varphi = \partial_t \varphi - \Delta \varphi - c \partial_1 \varphi - f_s(t, x, 0) \varphi$ the linearized operator associated to Eq. (14) about 0. In the sequel, we will briefly denote by $\lambda_1 = \lambda_1(P, \mathbb{R} \times \mathbb{R}^{N+1})$. We are now able to state the results of this section:

**Theorem 1.7.** Assume that (17) − (19) hold, then there exists a positive pulsating front $U \in C^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ of Eq. (14) if and only if $\lambda_1 < 0$. If it exists, it is unique, $T$-periodic in $t$, partially periodic in $y$ and decays exponentially in $|x_1|$, uniformly in $y$ and $t$.

One of the interesting points of this theorem is the loss of compactness since $f$ is not periodic in $x_1$ and the solution is not a priori assumed to be periodic in $y$ nor in $t$. We will prove that the uniqueness of Eq. (14) holds in the larger class of solution, that is the class of nonnegative bounded solutions but not necessarily periodic. As pointed out in section 1.5 [20], one cannot expect to show a general uniqueness in the class of nonnegative bounded solutions, even when the coefficients of Eq. (14) are periodic in $x = (x_1, y)$ and in $t$ with only condition $\lambda_1 < 0$. Some extra assumptions are needed. Here the uniqueness holds due to assumption (19), which is a key ingredient to derive the exponential behavior of solutions of (14) as $x_1 \to \pm \infty$.

**Theorem 1.8.** Let $u(t, x)$ be the solution of (13) with nonnegative initial datum $u_0(x) \in L^\infty(\mathbb{R}^N)$ and not identically equal to 0. Assume that (17) − (19) hold.

i) If $\lambda_1 \geq 0$ then

$$\lim_{t \to \infty} u(t, x) = 0,$$

uniformly in $x \in \mathbb{R}^N$.

ii) if $\lambda_1 < 0$ then

$$\lim_{t \to \infty} (u(t, x_1 + ct, y) - U(t, x_1, y)) = 0,$$

where $U$ is the unique solution of (14), uniformly in $x_1$, locally uniformly in $y$. If, in addition, $u_0$ is periodic in $y$ or satisfies

$$\forall r > 0, \quad \inf_{|x_1|<r,y\in \mathbb{R}^{N-1}} u_0(x_1, y) > 0,$$

then above convergence is uniform also in $y$.

The fact that the convergence holds uniformly in $x_1$ is a consequence of condition (19), which implies that solutions of (14) decay exponentially as $x_1 \to \pm \infty$. The effect of the climate change condition (via the constant $c > 0$) on the persistence or extinction of a species is implicitly included in the bifurcation of $\lambda_1$.

**Organization of the paper.** We divide the rest of paper into four sections. Section 2 deals with problem (14) on the cylindrical domain, where no seasonal dependence is considered. Section 3 investigates of problem (13) on partially periodic domain with seasonal dependence. In section 4, first we study the concentration of the species in cylindrical domain of Section 2, when the exterior domain becomes extremely unfavorable and further prove the symmetry breaking of the fronts. Finally, some auxiliary results are contained in the Appendix.
2 The cylindrical environment without seasonal dependence

2.1 An illustration

Before proving the main results of this section, let us provide an illustration of how theorems (1.1)-(1.5) apply and why condition (9) is useful to describe the heterogeneity of habitat of the species facing a climate change.

We consider the invasion of favorable zone that may lead to a certain persistence of species although a climate change presents. Let \( \Omega = \{(x_1, y) \in \mathbb{R} \times \omega = \mathbb{R} \times (0, 2), \alpha \in (0, 2), \text{ and } c > 0 \} \) is an arbitrary constant. Consider the equations

\[
\begin{aligned}
\Delta u + c \partial_1 u + f_\alpha(x_1, y, u) &= 0 \quad \text{in } \Omega \\
\partial_\nu u &= 0 \quad \text{on } \partial \Omega, \\
\end{aligned}
\]

(21)

where the nonlinearities \( f_\alpha(x_1, y, s) = a_\alpha(x_1, y)s - s^2 \) such that for some \( \varepsilon > 0 \)

\[
a_\alpha(x_1, y) \geq \mu_\alpha(y) = \begin{cases} 
\frac{c\pi}{2} - \frac{7c\pi\alpha}{6} & y \in [0, \alpha - \varepsilon] \\
\frac{c\pi}{2} & y \in (\alpha - \varepsilon, \alpha + \varepsilon) \\
\frac{c\pi}{2} & y \in [\alpha + \varepsilon, 2]
\end{cases}
\]

with \( \sigma_\alpha \) is a smooth connecting function such that \( \mu_\alpha(y) \in C^2(\omega) \) and \( \frac{c\pi}{2} - \frac{7c\pi\alpha}{6} \leq \sigma_\alpha \leq \frac{c\pi}{2} \). For every \( \alpha \), we see that the environment modeled by \( f_\alpha \) is unfavorable in \( \mathbb{R} \times [0, \alpha - \varepsilon] \) and favorable in \( \mathbb{R} \times [\alpha + \varepsilon, 2] \) and the favorable zone invades the unfavorable zone as \( \alpha \to 0 \). Consider the generalized Neumann principal eigenvalue defined in (5) of the operator

\[
-L_\alpha \phi = -\Delta x_1, y \phi - c \partial_1 \phi - a_\alpha(x_1, y)\phi.
\]

Taking \( \phi(x, y) = \phi_\alpha(x_1, y) = e^{-\pi\alpha|x_1|} \sin(\frac{c\pi}{2}y) \) to be a test function, we see that for \( \alpha \leq 1/9, \phi_\alpha > 0 \) in \( \Omega \) and \( \partial_\nu \phi_\alpha \geq 0 \) on \( \partial \Omega \) and

\[
L_\alpha \phi_\alpha = \left(-c\frac{\pi\alpha}{3} \text{sign}(x_1) + a_\alpha(x_1, y)\right) \phi_\alpha \geq \frac{c\pi}{3} \phi_\alpha.
\]

Obviously, \( \lambda_N(-L_\alpha, \Omega) \leq -\frac{c\pi}{3} \) for \( \alpha \leq 1/9 \) and 0 < \( \varepsilon < \alpha \). There exists \( S_0 > 0 \) to be a supersolution of Eq. (21). By choosing \( \kappa > 0 \) small enough, we have \( \kappa \phi_\alpha \leq S_0 \). The argument of sub-super solution implies that there exists a positive solution of Eq. (21) between \( \kappa \phi_\alpha \) and \( S_0 \). This is fit to the fact to be proved later that if \( \lambda_N(-L_\alpha, \Omega) < 0 \), Eq. (21) admits at least a positive solution.

On the other hand, when \( \alpha > 3/2, \phi_\alpha(x_1, 2) < 0 \), thus \( \phi_\alpha \) can no longer be a test-function. For \( \alpha \) close to 2 and \( \varepsilon \) small enough, \( \lambda_N(-\Delta_y - \mu_\alpha(y), \omega) > 0 \). Our theory applies in this case, where it is difficult to find an explicit subsolution for proving existence. Even the existence of positive solutions is known, the nonexistence and uniqueness are still indeed the delicate questions.

2.2 The existence and uniqueness of the front

To achieve the existence and uniqueness of Eq. (6), the key property is the exponential decay of solution. This estimate is the object of the following section.
2.2.1 Exponential decay

Proposition 2.1. Let $U$ be a nonnegative bounded solution Eq. (4). Assume that (5)-(7) hold, then for all $0 < \alpha < \alpha^*$ with $\alpha^* = \frac{-c + \sqrt{c^2 + 4\lambda\mu}}{2}$, there exists a positive constant $C(\alpha)$ such that

$$U(x_1, y) + |\nabla U(x_1, y)| \leq C(\alpha)e^{-\alpha|x_1|}.$$

We point out that this proposition generalizes the Proposition 3, [10] for which $f_s(x_1, y, 0)$ possibly changes sign as $|x_1| \to \infty$. Moreover, our proof is simpler than the proof of Berestycki and Rossi in the sense that we do not necessarily use the Liouville transformation.

Proof. Consider $\varphi \in W^{2,p}(\omega)$ the eigenfunction associated with $\lambda_\mu$

$$\begin{cases} \Delta \varphi + \mu(y)\varphi + \lambda_\mu \varphi = 0 & \text{in } \omega \\ \partial_\nu \varphi(y) = 0 & \text{on } \partial \omega \end{cases}$$

where $\nu = \nu(y)$ is the outward unit normal on $\omega$. As is well-known by the Hopf lemma, $\inf_{\omega} \varphi > 0$.

For any $\delta \in (0, \lambda_\mu)$, let $Lw = \Delta w + c\partial_1 w + (\mu(y) + \delta)w$ be defined in $\Omega$. Due to (3)-(9), one has

$$\Delta U + c\partial_1 U + f_s(x, 0)\leq 0 \quad \text{in } \Omega,$$

and there exists $R = R(\delta) > 0$ such that

$$f_s(x_1, y, 0) \leq \mu(y) + \delta \quad \text{in } \Omega \setminus \Omega_R,$$

therefore $Lw \geq 0$ in $\Omega \setminus \Omega_R$. For any $p > 0$, set

$$w_p(x_1, y) = e^{(R+p)(\tau-\alpha)}e^{\alpha|x_1|}\varphi(y) + e^{R(\tau-\alpha)}e^{-\alpha|x_1|}\varphi(y) = C_1e^{\alpha|x_1|}\varphi(y) + C_2e^{-\alpha|x_1|}\varphi(y),$$

where $R, \tau, \alpha > 0$ will be chosen. Direct computation shows that

$$\begin{align*}
\frac{Lw_p}{\varphi} &= C_1 \left( \alpha^2 + \frac{\Delta \varphi}{\varphi} + \text{cosh}x_1 + \mu(y) + \delta \right) e^{\alpha|x_1|} + \\
&\quad + C_2 \left( \alpha^2 + \frac{\Delta \varphi}{\varphi} - \text{cosh}x_1 + \mu(y) + \delta \right) e^{-\alpha|x_1|} \\
&\leq \left( \alpha^2 + c\alpha - \lambda_\mu + \delta \right) (C_1e^{\alpha|x_1|} + C_2e^{-\alpha|x_1|})
\end{align*}$$

(22)

To assign $w_p$ to be a supersolution of $L$ in $\Omega \setminus \Omega_R$, it suffices to take

$$\alpha = \alpha(\delta) = \frac{-c + \sqrt{c^2 + 4\lambda\mu - 4\delta}}{2} > 0.$$

Clearly, $\alpha(\delta)$ is decreasing with respect to $\delta \in (0, \lambda_\mu)$. Choosing $\tau = \alpha/2$ and $R$ large enough such that

$$\begin{cases} w_p(x_1, y) \geq e^{R\tau}\varphi(y) \geq e^{R\tau}\inf_{\omega}\varphi \geq U(x_1, y) \quad \text{as } |x_1| = R, y \in \omega \\
w_p(x_1, y) \geq e^{(R+p)\tau}\varphi(y) \geq e^{(R+p)\tau}\inf_{\omega}\varphi \geq U(x_1, y) \quad \text{as } |x_1| = R + p, y \in \omega.
\end{cases}$$

12
Fix $\alpha$, $\tau$ and $R$, we set $z(x_1, y) = \frac{w_p(x_1, y) - U(x_1, y)}{\varphi(y)}$. Routine computation yields

$$\frac{\mathcal{L}[z\varphi]}{\varphi} = \Delta z + 2\nabla_y\varphi \cdot \nabla_y z + c\partial_1 z + \frac{\Delta \varphi + (\mu(y) + \delta)\varphi}{\varphi} z \leq 0$$

Observe that $z \geq 0$ when $|x_1| \in \{R, R + p\}$, $y \in \omega$ and $\partial_\nu z = 0$ on $\partial \omega$. Moreover, the zero-order’s coefficient of the operator satisfied by $z$ is negative, we imply by the maximum principle that

$$U(x_1, y) \leq w_p(x_1, y) = e^{-(R+p)\alpha/2}e^{\alpha|x_1|}\varphi(y) + e^{3R\alpha/2}e^{-\alpha|x_1|}\varphi(y) \quad \text{in } \Omega_{R+p} \setminus \Omega_R.$$ 

Letting $p \to \infty$, we obtain

$$U(x_1, y) \leq e^{3\alpha/2}e^{-\alpha|x_1|} \quad \text{in } \Omega.$$ 

Set $\alpha^* = -\frac{c + \sqrt{c^2 + 4\lambda}}{2}$, one sees that when $\delta$ goes to $0$, $\alpha$ is arbitrarily close to $\alpha^*$. Since $\varphi$ is bounded, obviously for all $0 < \alpha < \alpha^*$ we can choose $C(\alpha) = e^{3R\alpha/2}\sup_\omega \varphi$ such that

$$U(x_1, y) \leq C(\alpha)e^{-\alpha|x_1|} \quad \text{in } \Omega.$$ 

This implies that $U(x_1, y)$ decays exponentially as $|x_1| \to \infty$, uniformly in $y$. On the other hand, since $U$ is a solution of Eq. (6), the estimate on $|\nabla U|$ follows from the standard argument. Indeed, by Sobolev embedding for $p > N$, $L^p$ estimates and Harnack inequality, one has

$$\|\nabla U\|_{L^\infty(B_1(x))} \leq C_1\|U\|_{W^{2,p}(B_1(x))} \leq C_2\|U\|_{L^\infty(B_2(x))} \leq C_3U(x).$$

These inequalities end the proof. $\square$

### 2.2.2 Proof of Theorem 1.1

Let us consider the first case $c < c^*$. By Proposition [13], we know that

$$\tilde{\lambda}_1 := \lambda_N(-\Delta - c\partial_1 - f_s(x_1, y, 0), \Omega) < 0.$$ 

Thanks to Proposition 1, [10], we have the limit $\lim_{R \to \infty} \lambda_R = \tilde{\lambda}_1 < 0$, where $\lambda_R$ is the unique eigenvalue of problem:

$$\begin{cases}
-\Delta \varphi - c\partial_1 \varphi - f_s(x, 0)\varphi = \lambda_R\varphi & x \in \Omega_R \\
\varphi_R(x) > 0 & x \in \Omega_R \\
\partial_\nu \varphi_R(x_1, y) = 0 & |x_1| < R, y \in \partial \omega \\
\varphi_R(\pm R, y) = 0 & y \in \omega.
\end{cases}$$

Moreover there exists an eigenfunction $\varphi_\infty \in W^{2,N}(\Omega)$ associated with $\tilde{\lambda}_1$. Fix $R > 0$ large enough such that $\lambda_R < 0$, we define $\phi(x)$ as following:

$$\phi(x) = \begin{cases}
\varphi_R(x) & x \in \Omega_R \\
0 & \text{otherwise}.
\end{cases}$$

13
Since \( f(x, s) \) is of \( C^1[0, s_0] \) with respect to \( s \), for \( \varepsilon > 0 \) small enough, we see that
\[
\Delta(\varepsilon \phi) + c\partial_1(\varepsilon \phi) + f(x_1, y, \varepsilon \phi) = \varepsilon \phi \left[ -\lambda_R + \frac{f(x_1, y, \varepsilon \phi)}{\varepsilon \phi} - f_s(x_1, y, 0) \right] > 0.
\]
Hence, \( \varepsilon \phi \) is a subsolution of Eq. (23). Since \( \phi \) is compactly supported, we can choose \( \varepsilon \) small such that \( \varepsilon \sup \phi \leq S \), where \( S \) is a super solution of Eq. (6) given by (7). Therefore, by the classical iteration method, there exists a nonnegative solution \( U \) satisfying \( \varepsilon \phi \leq U \leq S \). Furthermore, thanks to the strong maximum principle, \( U \) is strictly positive.

The nonexistence and uniqueness are direct consequences of the following comparison principle. Let \( U \) and \( V \) be respectively super and subsolutions of (6). We will show now that \( V(x) \leq U(x) \) in \( \Omega \). Indeed, by condition (7), there exists an eigenfunction \( \varphi \) associated with \( \lambda_\mu \), namely
\[
\begin{cases}
-\Delta \varphi - \mu(y)\varphi = \lambda_\mu \varphi & \text{in } \omega \\
\partial_\nu \varphi = 0 & \text{on } \partial \omega.
\end{cases}
\] (23)
The Hopf lemma yields \( \inf_{\partial \omega} \varphi > 0 \). On the other hand, Proposition (2.1) implies that \( V \) decays exponentially as \( |x_1| \to \infty \), uniformly in \( y \), therefore, for any \( \varepsilon > 0 \), there exist \( R(\varepsilon) > 0 \) such that \( V(x_1, y) \leq \varepsilon \varphi(y) \) in \( \Omega \setminus \Omega_{R(\varepsilon)} \). Then, the set
\[
K_\varepsilon := \{ k > 0 : kU \geq V - \varepsilon \varphi \text{ in } \bar{\Omega} \},
\]
is nonempty. Let us call \( k(\varepsilon) := \inf K_\varepsilon \). Obviously, the function \( k(\varepsilon) : \mathbb{R}^+ \to \mathbb{R} \) is nonincreasing. Assume by a contradiction
\[
k^* = \lim_{\varepsilon \to 0^+} k(\varepsilon) > 1.
\]
Take \( 0 < \varepsilon < \sup_\omega V/\sup_\omega \varphi \), we have \( k(\varepsilon) > 0 \), \( k(\varepsilon) U - V + \varepsilon \varphi \geq 0 \). By the definition of \( k(\varepsilon) \), there exists a sequence \((x_{1,n}^\varepsilon, y_n^\varepsilon)\) in \( \bar{\Omega} \) such that
\[
\left( k(\varepsilon) - \frac{1}{n} \right) U(x_{1,n}^\varepsilon, y_n^\varepsilon) < V(x_{1,n}^\varepsilon, y_n^\varepsilon) - \varepsilon \varphi(y_n^\varepsilon).
\]
Fix \( \varepsilon > 0 \), we have \((x_{1,n}^\varepsilon, y_n^\varepsilon) \in \Omega_{R(\varepsilon)} \) for \( n \) large enough, therefore \((x_{1,n}^\varepsilon, y_n^\varepsilon)\) converges up to subsequence to some \((x_1(\varepsilon), y(\varepsilon))\) in \( \Omega_{R(\varepsilon)} \). This limiting point must satisfy:
\[
(k(\varepsilon) U - V + \varepsilon \varphi)(x_1(\varepsilon), y(\varepsilon)) = 0.
\] (24)
Without loss of generality, we assume \( \lim_{\varepsilon \to 0^+} y(\varepsilon) = y_0 \in \bar{\omega} \). The case that there exists \( x_0 \) such that \( |x_0| = \liminf_{\varepsilon \to 0^+} |x_1(\varepsilon)| < \infty \) is ruled out. Indeed, from (24), \( k^* < \infty \), the function \( W = k^* U - V \) is nonnegative and vanishes at \((x_0, y_0)\). Since \( f \) is Lipschitz continuous with respect to second variable and \( k^* > 1 \), we have
\[
-\Delta W - c\partial_1 W \geq k^* f(x, U) - f(x, V) \geq f(x, k^* U) - f(x, V) \geq z(x) W,
\]
where \( z(x) \in L^\infty_{\text{loc}}(\Omega) \). Thanks to condition \( \text{(3)} \), this inequality holds strictly in \( D \subset \Omega \), with \( |D| > 0 \). The strong maximum principle implies that \( W \) cannot achieve the minimum value in the interior of \( \Omega \). This means \( y_0 \in \partial \omega \), but the Hopf lemma yields another contradiction \( \partial_\nu W(x_0, y_0) < 0 \).

It remains to consider the case \( \lim_{\varepsilon \to 0^+} |x_1(\varepsilon)| = \infty \). Set \( W^\varepsilon = k(\varepsilon)U - V + \varepsilon \varphi \), we have \( W^\varepsilon \geq 0 \) and vanishes at \( (x_1(\varepsilon), y(\varepsilon)) \). There thus exists a neighborhood \( \mathcal{O} \) of \( (x_1(\varepsilon), y(\varepsilon)) \) such that \( k(\varepsilon)U < V \) in \( \mathcal{O} \). For \( \varepsilon \) small enough, \( k(\varepsilon) > 1 \), we derive from \( \text{(5)} \) for \( x \in \mathcal{O} \):

\[
(\Delta + c\partial_t)W^\varepsilon \leq f(x, V) - k(\varepsilon)f(x, U) - (\mu(y) + \lambda_\mu)\varepsilon \varphi \leq f(x, V) - f(x, k(\varepsilon)U) - (\mu(y) + \lambda_\mu)\varepsilon \varphi \leq -\frac{f(x, k(\varepsilon)U)}{k(\varepsilon)U}(k(\varepsilon)U - V + \varepsilon \varphi - \frac{\lambda_\mu}{2} \varepsilon \varphi - \left( \frac{\lambda_\mu}{2} + \mu(y) - \frac{f(x_1, y, k(\varepsilon)U)}{k(\varepsilon)U} \right) \varepsilon \varphi).
\]

Take \( 0 < \varepsilon \ll 1 \), then \( |x_1(\varepsilon)| \gg 1 \), we have

\[
\frac{f(x_1, y, k(\varepsilon)U)}{k(\varepsilon)U} < \mu(y) + \frac{\lambda_\mu}{2}, \quad \forall (x_1, y) \in \mathcal{O},
\]

shrinking \( \mathcal{O} \) to be smaller if necessary. Since \( \lambda > 0 \), we get from \( \text{(25)} \)

\[
-\Delta W^\varepsilon - c\partial_t W^\varepsilon - \varrho(x)W^\varepsilon > \frac{\lambda_\mu}{2} \varepsilon \varphi > 0 \quad \text{in} \ \mathcal{O},
\]

where \( \varrho(x) = \frac{f(x, k(\varepsilon)U)}{k(\varepsilon)U} \) is bounded. The strong maximum principle asserts that \( (x_1(\varepsilon), y(\varepsilon)) \) cannot be an interior point of \( \Omega \). Hence \( (x_1(\varepsilon), y(\varepsilon)) \in \partial \Omega \), but then the Hopf lemma yields another contradiction \( \partial_\nu W^\varepsilon(x_1(\varepsilon), y(\varepsilon)) < 0 \).

We have proved that \( k^* = \lim_{\varepsilon \to 0^+} k(\varepsilon) \leq 1 \). Letting \( \varepsilon \to 0^+ \), we derive

\[
V \leq \lim_{\varepsilon \to 0^+} (k(\varepsilon)U + \varepsilon \varphi) \leq U \quad \text{in} \ \Omega.
\]

The uniqueness of Eq. \( \text{(6)} \) is obviously achieved by exchanging the roles of \( U \) and \( V \). We end the proof by showing the nonexistence when \( c \geq c^* \). Assume by contradiction that \( \text{(6)} \) possesses a positive solution \( U \). One has \( \lambda_1 = \lambda_1(-\Delta - c\partial_t - f_s(x_1, y, 0), \Omega) \geq 0 \). Let \( \varphi_\infty \) be a generalized principal eigenfunction with Neumann boundary condition associated with \( \lambda_1 \). Without loss of generality, we may assume that \( 0 < \varphi_\infty(0) < U(0) \). We derive, from \( \text{(3)} \), that

\[
-\Delta \varphi_\infty - c\partial_t \varphi_\infty = (f_s(x_1, y, 0) + \tilde{\lambda}_1)\varphi_\infty \geq f(x_1, y, \varphi_\infty) \quad \text{in} \ \Omega.
\]

By Proposition \( \text{(24)} \), \( U \) decays exponentially as \( |x_1| \to \infty \) uniformly in \( y \). Above comparison principle implies that \( U(x) \leq \varphi_\infty(x) \) for all \( x \in \Omega \). This contradiction ends the proof of this theorem.

**Remark 1.** The unfavorability of environment near infinity plays the key role to derive the uniqueness of Eq. \( \text{(6)} \). Indeed, if \( f \) is homogeneous in the traveling direction, that is independent of \( x_1 \), then for all \( a \in \mathbb{R} \), \( U(x_1 + a, y) \) are the solutions of \( \text{(6)} \).
2.3 Long time dynamics

In order to study the long time dynamics of Eq. (4), we first prove the following Liouville type theorem for entire solutions (solutions for all \( t \in \mathbb{R} \)). Consider the evolution problem in the cylindrical domain with Neumann boundary condition

\[
\begin{aligned}
\partial_t u^* &= \Delta u^* + c \partial_1 u^* + f(x, u^*) & t \in \mathbb{R}, \; x \in \Omega \\
\partial_\nu u^* &= 0 & t \in \mathbb{R}, \; x \in \partial \Omega,
\end{aligned}
\]  

(26)

we have the following auxiliary result

**Theorem 2.2.** Assume that conditions (7) – (9) are satisfied, then Eq. (26) admits a positive bounded entire solution only if \( c < c^* \), where \( c^* \) is defined in Proposition (1.3). Conversely, if \( c < c^* \) and there exist a sequence \( (t_n) \in \mathbb{R} \) as \( n \to \infty \) and a point \( x_0 \in \Omega \) such that

\[
\lim_{n \to \infty} t_n = +\infty, \quad \liminf_{n \to \infty} u^*(-t_n, x_0) > 0,
\]

(27)

then \( u^*(t, x) \equiv U(x) \), where \( U(x) \) is the unique solution of (21), given by Theorem 1.1.

**Proof.** Set \( S^* = \max\{S, \|u^*\|_{L^\infty(\Omega)}\} \), where \( S \) is the positive constant given in (7), obviously \( S^* \) is a super solution of stationary equation of Eq. (26). Let \( v(t, x) \) be the solution of (26) starting by \( v(0, x) = S^* \), the parabolic maximum principle and standard estimates imply that \( v \) is nonincreasing in \( t \) and converges locally uniformly in \( \Omega \) to a stationary solution \( V(x) \) of (26). That \( V(x) \) solves Eq. (26). For any \( h \in \mathbb{R} \), we define \( v_h(t, x) = v(t - h, x) \). This function is a solution of (26) in \((h, +\infty) \times \Omega\) and satisfies \( v_h(h, x) = S^* \geq u^*(h, x) \). The parabolic comparison principle thus yields

\[
0 \leq u^*(t, x) \leq \lim_{h \to +\infty} v_h(t, x) = V(x) \quad \forall t \in \mathbb{R}, \; x \in \Omega.
\]

(28)

We consider separately two different cases:

Case 1. \( c \geq c^* \).

Theorem 1.1 asserts that the stationary equation of Eq. (26) only has zero-solution. Namely, \( V(x) \equiv 0 \) in \( \Omega \). Therefore, the necessary condition for existence of nontrivial entire solution of (26) is \( c < c^* \).

Case 2. \( c < c^* \) and (27) holds.

Theorem 1.1 again asserts that the stationary equation of Eq. (26) admits unique positive solution \( U \). We will prove that \( u^*(t, x) \equiv U(x) \). Assume by contradiction that there exists \( x_0 \in \Omega \) such that \( u^*(t, x_0) \neq U(x_0) \). We will reach a contradiction by proving the following claim.

**Claim.** There exist \( \varepsilon \in (0, 1] \) and \( n_0 \in \mathbb{N} \) such that for \( n \geq n_0 \), one has \( \varepsilon U(x) \leq u^*(-t_n, x) \).

Assume for a moment that this claim holds true, the concluding argumentation goes as follows. Thanks to (8), for any \( \varepsilon \in (0, 1] \), \( \varepsilon U \) is a subsolution of stationary equation of Eq. (26). Let \( w(t, x) \) be a solution of (26) with initial condition \( w(0, x) = \varepsilon U(x) \) and \( w_n(t, x) = w(t + t_n, x) \). We know, by the standard parabolic estimates, that as \( t \to \infty \), \( w(t, x) \) is nondecreasing, bounded
from above by $S^*$ and converges locally uniformly in $\Omega$ to the unique stationary solution $W(x)$ of Eq. (20). The strict positivity of $W$ is derived from the condition $c < c^*$. By the way of setting, one has $w_n(-t_n, x) = \varepsilon U(x) \leq u^*(t_n, x)$. The parabolic comparison principle implies that $w_n(t, x) \leq u^*(t, x)$ in $(-t_n, +\infty) \times \Omega$. Therefore, by letting $n \to \infty$, one has

$$u^*(t, x) \geq \lim_{n \to \infty} w_n(t, x) = W(x) \quad \text{locally in } \mathbb{R} \times \Omega.$$ 

Combining this inequality with (23), we obtain $W(x) \leq u^*(t, x) \leq V(x), \forall t \in \mathbb{R}, \forall x \in \Omega$. The uniqueness result of Theorem 1.1 yields $u^* \equiv W \equiv V$.

It remains to prove the claim. Assume by contradiction that for all $\varepsilon \in (0, 1]$ and for all $n_0 \in \mathbb{N}$ there exist $n(\varepsilon) > n_0$ and $x(n_\varepsilon) \in \Omega$ so that $\varepsilon U(x(n_\varepsilon)) \geq u^*(-t_n(x), x(n_\varepsilon))$. Since $U$ is bounded, choosing a sequence $\varepsilon_k \to 0$ as $k \to \infty$, by a diagonal extraction, one finds sequences $(t_k) \in \mathbb{R}^+$ and $(x_k) \in \Omega$ such that $t_k \to +\infty$ and $u^*(t_k, x_k) \to 0$ as $k \to \infty$. We set

$$\tilde{u}_k(t, x) = u^*(t + t_k, x + x_k).$$

Obviously, $\tilde{u}_k(t, x)$ is bounded from above by $S^*$ and satisfies the equation

$$\begin{cases}
\partial_t \tilde{u}_k = \Delta \tilde{u}_k + c\partial_1 \tilde{u}_k + f(x + x_k, \tilde{u}_k) & t \in \mathbb{R}, x \in \Omega \\
\partial_n \tilde{u}_k = 0 & t \in \mathbb{R}, x \in \partial \Omega,
\end{cases}$$

By standard parabolic estimates, we get $\tilde{u}_k \to \tilde{u}_\infty$ (up to subsequences) as $k \to \infty$. Thanks to the Lipschitz continuity of $f(x, s)$ with respect to $s$, there exists a negative constant $-M$ so that $\tilde{u}_\infty$ satisfies the equation

$$\begin{cases}
\partial_t \tilde{u}_\infty \geq \Delta \tilde{u}_\infty + c\partial_1 \tilde{u}_\infty - M\tilde{u}_\infty & t \in \mathbb{R}, x \in \Omega \\
\partial_n \tilde{u}_\infty = 0 & t \in \mathbb{R}, x \in \partial \Omega.
\end{cases}$$

Moreover, $\tilde{u}_\infty(0, 0, 0) = 0$. The strong maximum principle implies that $\tilde{u}_\infty(t, x) = 0, \forall t \leq 0, x \in \Omega$. Choosing $t = -2t_k$, we get $\lim_{k \to \infty} u^*(-t_k, x) = 0, \forall x \in \Omega$. This contradicts to assumption (27). We complete the proof.

**Remark 2.** In this proof, we use different arguments as of Berestycki and Rossi, Lemma 3.4 [10]. More precisely, we choose a solution of Eq. (20), $w(t, x)$ starting by a subsolution of stationary equation $\varepsilon U(x)$, which is not necessarily compactly supported but bounded. On the other hand, we reach the contradiction by showing that $\lim_{k \to \infty} u^*(-t_k, x) = 0, \forall x \in \Omega$, which differs from the way to show that for all $r > 0$, $\liminf_{n \to \infty} u^*(t_n, x) > 0$ in [10].

We are now ready to prove Theorem 1.4 to derive long time behavior of solution of (4) in $L^\infty(\Omega)$.

**Proof of Theorem 1.4.** Let $S' := \max\{S, \|u_0\|_{L^\infty(\Omega)}\}$, where $S$ is the positive constant in (7). Then 0 and $S'$ are respectively sub and super solution of (4). It follows from [18], by the standard theory of semilinear parabolic equations, that there exists a unique (weak) solution to (4) satisfying $0 \leq u \leq S'$ with initial condition $u_0(x)$. We deduce, from the parabolic strong maximum principle,
that $u(t, x) > 0 \forall t > 0, x \in \overline{\Omega}$ (by extending $u(t, x)$ to larger cylinder to make the "corner" smooth). The locally long time behavior of $u$ follows by applying directly Theorem 2.2 and the standard parabolic estimates. Actually, one sets $\tilde{u}(t, x, y) = u(t, x_1 + ct, y)$. The solution of this type satisfies $\tilde{u}(0, x) = u_0(x)$ and reads the equation
\[
\left\{ \begin{array}{ll}
\partial_t \tilde{u} = \Delta \tilde{u} + c \partial_1 \tilde{u} + f(x, \tilde{u}) & t \in \mathbb{R}, x \in \Omega \\
\partial_\nu \tilde{u} = 0 & t \in \mathbb{R}, x \in \partial \Omega.
\end{array} \right.
\tag{29}
\]
To apply Theorem 2.2, we only need to verify condition (27) when $c < c^*$. Indeed, the first case $c \geq c^*$ is easily seen. Let $(t_n)$ be a sequence such that $t_n \to +\infty$ as $n \to \infty$, we imply, by the parabolic estimates and embedding theorems, that the sequence $\tilde{u}(t_n, x)$ converges (up to subsequences) to some nonnegative bounded solution $u^*(t, x)$ of Eq. (29) as $n \to \infty$ locally in $\Omega$. By Theorem 2.2, this limit is identically equal to 0 when $c \geq c^*$. Consider the case $c < c^*$, we necessarily verify condition (27). Let $U$ be the unique solution of stationary solution of (29) and $(t_n)$ be such that $t_n \to -\infty$ as $n \to \infty$. Fix $R > 0$, the Hopf lemma implies that $\inf_{t_n, x} \tilde{u}(1, x) > 0$. For $\varepsilon > 0$, the function $\varepsilon U$ is a subsolution to stationary equation of (29) when $\varepsilon \leq 1$. Take $\varepsilon$ small enough such that $\varepsilon U \leq \tilde{u}(1, x)$ and in $\overline{\Omega_R}$. Hence $(t, x) \mapsto \varepsilon U(x)$ is a subsolution to (29) in $\mathbb{R} \times \Omega_R$. The parabolic comparison yields $\varepsilon U(x) \leq \tilde{u}(t + 1, x)$ for $t > 0$ and $x \in \Omega_R$. As a consequence
\[
\inf_{t \in \mathbb{R}} u^*(t, 0, y_0) \geq U(0, y_0) > 0, \quad \text{for some } y_0 \in \omega.
\]
It remains to show that the convergences hold uniformly in $\Omega$. Assume by contradiction that
\[
\lim_{t \to \infty} \tilde{u}(t, x) = U(x)
\]
is not uniform in $x \in \Omega$. This means that there exist $\varepsilon > 0$, $(t_n) \in \mathbb{R}^+$ and $(x_{1,n}, y_n) \in \Omega$ such that
\[
\lim_{t \to \infty} t_n = \infty, \quad |\tilde{u}(t_n, x_{1,n}, y_n) - U(x_{1,n}, y_n)| \geq \varepsilon \quad \forall n \in \mathbb{N}.
\]
Since $y_n \in \omega$, which is bounded, one may assume that $y_n$ converges (up to subsequences) to $\zeta \in \overline{\omega}$. The locally uniform convergences yields $\lim_{n \to \infty} |x_{1,n}| = \infty$, therefore $\lim_{n \to \infty} U(x_{1,n}, y_n) = 0$ in both cases $c \geq c^*$ and $c < c^*$. Then we get
\[
\lim_{n \to \infty} \inf_{t \in \mathbb{R}} \tilde{u}(t_n, x_{1,n}, y_n) \geq \varepsilon.
\]
The standard parabolic estimates and compact injections again imply that $\tilde{u}(t + t_n, x_1 + x_{1,n}, y_n)$ converges (up to subsequences) to $\tilde{u}_{\infty}(t, x_1, \zeta)$ uniformly in $(-\rho, \rho) \times \Omega_\rho$, for any $\rho > 0$. In particular, $\tilde{u}_{\infty}$ satisfies $\tilde{u}_{\infty}(0, 0, \zeta) \geq \varepsilon$ and reads the following equation
\[
\left\{ \begin{array}{ll}
\partial_t \tilde{u}_{\infty} \leq \Delta \tilde{u}_{\infty} + c \partial_1 \tilde{u}_{\infty} + \mu(y) \tilde{u}_{\infty} & t \in \mathbb{R}, x \in \Omega \\
\partial_\nu \tilde{u}_{\infty} = 0 & t \in \mathbb{R}, x \in \partial \Omega.
\end{array} \right.
\tag{30}
\]
By condition (19), there exists an eigenpair $(\lambda_\mu, \varphi)$ of Eq. (23) satisfying $\lambda_\mu > 0$. Setting $\omega(t, y) = S^\mu e^{-\lambda_\mu(t+h)} \varphi(y)$, we have
\[
\partial_t \omega - \Delta \omega - c \partial_1 \omega - \mu(y) \omega \geq 0.
\]

We know, by the Hopf lemma, that \( \inf_{\omega} \varphi(y) > 0 \). Hence, the function \( W(t, x) = \tilde{u}_\infty(t, x) - \omega(t, y) \) satisfies \( W(-h, x) \leq 0 \) for \( S'' \) large enough. Let us call \( \mathcal{L}_1 = \partial_t - \Delta - c\partial_t - \mu(y) \), then

\[
\begin{cases}
\mathcal{L}_1 W \leq 0 & t \in \mathbb{R}, x \in \Omega \\
\partial_{y} W \leq 0 & t \in \mathbb{R}, x \in \partial\Omega.
\end{cases}
\]

Set \( W(t, x) = z(t, x)\varphi(y) \) we have \( \partial_{y}z \leq 0 \) on \( \partial\Omega \), \( z(-h, x) \leq 0 \ \forall x \in \Omega \) and \( z \) satisfies

\[
0 \geq \frac{\mathcal{L}_1 W}{\varphi} = z_t - \Delta z - \frac{2}{\varphi} \nabla_y \varphi \cdot \nabla_y z - c\partial_t z + \lambda_{\mu}z.
\]

Since \( \lambda_{\mu} > 0 \), the parabolic maximum principle implies that \( z \leq 0 \) in \( (-\infty, -h) \times \bar{\Omega} \). As a consequence

\[
0 < \varepsilon \leq \tilde{u}_\infty(0, 0, \zeta) \leq \lim_{h \to -\infty} \omega(0, \zeta) = 0.
\]

This is a contradiction. We conclude the proof. \( \square \)

The next result involves in the long time behavior of the solution of Eq. (1) in \( L^1(\Omega) \). The central difficulty to deal with is sign-changing condition of \( f_s(x, 0) \). To overcome, we decompose the solution of (1) into the summation of two integrable functions. The following lemma plays the key role.

**Lemma 2.3.** Let \( w(t, x) \) be a nonnegative bounded solution of

\[
\begin{cases}
\partial_t w = \Delta w + c\partial_t w + \zeta(t, x)w, & t > 0, x \in \Omega \\
\partial_{y} w(t, x) = 0, & t > 0, x \in \partial\Omega
\end{cases}
\]

with initial function \( w(0, \cdot) = w_0 \in L^1(\Omega) \cap L^\infty(\Omega) \). We assume, in addition that, \( \lim_{t \to \infty} w(t, x) = 0 \), pointwise in \( x \in \Omega \), \( \zeta(x) \in L^\infty(\Omega) \) and there exist \( \mu \in L^\infty(\omega) \) such that

\[
\mu(y) = \lim_{R \to \infty} \sup_{|x| \geq R} \zeta(t, x, y), \quad \text{and} \quad \lambda_1(-\Delta_y - \mu(y), \omega) > 0.
\]

There holds

\[
\lim_{t \to \infty} \|w(t, x)\|_{L^1(\Omega)} = 0.
\]

**Proof of Lemma 2.3.** From Eq. (31), for any \( \delta > 0 \), we have

\[
\partial_t w - \Delta w - c\partial_t w - (\mu(y) + \delta)w = (\zeta(t, x) - \mu(y) - \delta)w.
\]

Let us call

\[
P := \partial_t - \Delta - c\partial_t - \mu(y) - \delta, \quad g(t, x) := (\zeta(t, x) - \mu(y) - \delta)w(t, x).
\]

Then, we imply, from the superposition principle, that \( w = w_1 + w_2 \), where \( (w_1, w_2) \) is the solution of the system

\[
\begin{align*}
\begin{cases}
Pw_1 = 0 & t > 0, x \in \Omega \\
\partial_{y} w_1 = 0 & t > 0, x \in \partial\Omega,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
Pw_2 = g(t, x) & t > 0, x \in \Omega \\
\partial_{y} w_2 = 0 & t > 0, x \in \partial\Omega,
\end{cases}
\end{align*}
\]

with the initial condition \( (w_1, w_2)(0, x) = (w_0(x), 0) \). From condition (32), for any \( \delta > 0 \), there exist \( R > 0 \), such that

19
\( \zeta(t, x_1, y) \leq \mu(y) + \delta, \quad \forall t > 0, \forall (x_1, y) \in \Omega \setminus \Omega_R. \)

and there exists an eigenpair \((\lambda_\mu, \varphi)\) of Eq. (23) satisfying \(\lambda_\mu > 0\).

Letting \(\delta < \lambda_\mu\), we set \(v_1(t, x) = e^{(\lambda_\mu-\delta)t}w_1(t, x)/\varphi(y)\). Then \(v_1\) reads the equation

\[
\begin{aligned}
&\partial_t v_1 - \Delta_x v_1 - 2 \nabla_y \varphi \cdot \nabla_y v_1 - c\partial_\nu v_1 \leq 0, \quad x \in \Omega. \\
&\partial_\nu v_1 \geq 0 \quad x \in \partial \Omega.
\end{aligned}
\]  

We know, by the Hopf lemma, that \(\inf_{\partial \Omega} \varphi > 0\), then \(\|v_1(0, \cdot)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} / \inf_{\partial \Omega} \varphi\). Then, the parabolic maximum principle yields \(\|v_1\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} / \inf_{\partial \Omega} \varphi\). It follows immediately that \(\|w_1(t, \cdot)\|_{L^\infty(\Omega)} \to 0\) as \(t \to \infty\). On the other hand, set \(v^*_1(t, \rho, y) = \int_{\rho}^{\infty} v_1(t, x_1 + \rho, y) dx_1\), we obtain

\[
\partial_t v^*_1 - \Delta v^*_1 - 2 \nabla_y \varphi \cdot \nabla_y v^*_1(t, \rho, y) - c\partial_\nu v^*_1 \leq 0 \quad (\rho, y) \in \Omega.
\]

Since \(w_0(0, \cdot) \in L^1(\Omega) \cap L^\infty(\Omega), v^*_1(0, \rho, y)\) is well-defined a.e on \(\Omega, \forall \rho > 0\). Moreover, there exists a constant \(M\) such that \(v^*_1(0, \rho, y) \leq M, \text{a.e in } \Omega, \forall \rho > 0\), \(M\) is a supersolution of Eq. (34). Then, we imply, by the parabolic comparison principle [18], that

\[
v^*_1(t, \cdot) \leq M, \quad \text{a.e in } \Omega, \forall t > 0, \forall \rho > 0.
\]

Therefore, \(\|v_1(t, \cdot)\|_{L^1(\Omega)} = \lim_{t \to \infty} \int_{\omega} \int_r^{\infty} v_1(t, x_1 + \rho, y) dx_1 dy \leq M|\omega|, \forall t > 0\). As a consequence, we get

\[
\lim_{t \to \infty} \|w_1(t, \cdot)\|_{L^\infty(\Omega)} = \lim_{t \to \infty} \|v_1(t, \cdot)\|_{L^1(\Omega)} = 0.
\]

On the other hand, by assumption, \(w\) is bounded, then \(w_2\) is bounded and so it is integrable on any compact set. The same argumentation as of the Proposition (2.1) enables us to find supersolution of the problem satisfied by \(w_2\) of the form \(\xi(x_1, y) = Ce^{-r|x_1|}\varphi(y)\) such that \(P(\xi) \geq 0 \geq g(t, x)\) in \(\Omega \setminus \Omega_R\). We have \(w_2(0, x) = 0 < \xi(x), \partial_\nu w_2 \leq \partial_\nu \xi\) for \(y \in \partial \Omega\). Moreover, the parabolic maximum principle implies that \(w_2(t, x) \leq \xi(x), \forall x \in \Omega \setminus \Omega_R\). Since \(w_2\) is bounded, one can choose \(C\) large enough so that \(\partial_\nu w_2(t, x) \leq \xi(x), \forall x \in \Omega\). Moreover,

\[
\forall x \in \Omega \quad \lim_{t \to \infty} w^+_2(t, x) = \lim_{t \to \infty} (w - w_1)^+(t, x) = 0.
\]

Hence \(0 \leq w = w_1 + w_2 \leq w_1 + w^+_2\), which is integrable on \(\Omega\). It follows from Lebesgue’s dominated convergence theorem \(\lim_{t \to \infty} \|w(t, x)\|_{L^1(\Omega)} = 0\) because \(\lim_{t \to \infty} w(t, x) = 0\) for \(x \in \Omega\). We conclude the proof. \(\square\)

We are in the position to prove Theorem 1.5.

**Proof of Theorem 1.5** The proof is a direct consequence of Lemma (2.3). Let \(u\) be the solution of (4) with \(u(0, x) = u_0(x) \in L^\infty(\Omega) \cap L^1(\Omega)\). The function \(\tilde{u}(t, x_1, y) := u(t, x_1 + ct, y)\) satisfies Eq. (34) with the same initial condition \(u_0\). Let \(W\) be defined as following:

\[
W(x) = \begin{cases}
0 & \text{if } c \geq c^* \\
U(x) & \text{if } c < c^*,
\end{cases}
\]

(35)
where $U(x)$ is the unique positive solution of Eq. (6) when $c < c^*$. 

Let $\bar{u}, \underline{u}$ be respectively the solutions of (31) with initial conditions $\bar{u}(0, x) = \max\{u_0(x), W(x)\}$ and $\underline{u}(0, x) = \min\{u_0(x), W(x)\}$. We know, from Theorem 1.4 that the functions $\bar{u}(t, x_1 - ct, y)$ and $\underline{u}(t, x_1 - ct, y)$ converge to $W(x)$ as $t \to \infty$, uniformly with respect to $x \in \Omega$. Moreover, the parabolic maximum principle yields

$$\forall t > 0, x \in \Omega \quad \bar{u}(t, x) \geq \max\{\hat{u}(t, x), W(x)\} \quad \underline{u}(t, x) \leq \min\{\hat{u}(t, x), W(x)\}.$$ 

Therefore, the functions $\bar{w}(t, x) := \bar{u}(t, x) - W(x)$ and $\underline{w}(t, x) := W(x) - \underline{u}(t, x)$ is nonnegative bounded solution of Eq. (31) with

$$\zeta(t, x) = f(x, \bar{u}) - f(x, W), \quad \hat{\zeta}(t, x) = f(x, W) - f(x, \underline{u}).$$

Thanks to condition (8), one easily sees that $\hat{\zeta}$ and $\hat{\zeta}$ are less than $f_s(x, 0)$. Thanks to condition (9), $\zeta$ and $\hat{\zeta}$ satisfy (32). The initial conditions $\bar{w}(0, x), \underline{w}(0, x) \in L^1(\Omega)$ allow one to apply Lemma (2.3) to derive

$$\lim_{t \to \infty} \|\bar{u} - W\|_{L^1(\Omega)} = 0; \quad \lim_{t \to \infty} \|W - \underline{u}\|_{L^1(\Omega)} = 0.$$

This completes the proof because $\underline{u} \leq \hat{u} \leq \bar{u}$. 

The next section is of independent interest. We are concerned with the existence, uniqueness, long time behavior of pulsating fronts, which are $T$-periodic in $t$ and partially periodic in $y$.

3 The partially periodic environment with seasonal dependence

Before proving the main results, let us introduce some new definition and and preliminary results, that are needed in this section.

Proposition 3.1. Let $\mathcal{O}_r = \mathbb{R} \times (-r, r) \times \mathbb{R}^{N-1}$, then for any $r > 0$, there exists a unique real number $\lambda_p$ such that the eigenvalue problem

$$\begin{cases} L\chi_r = \lambda_p(r)\chi_r & a.e \ in \ \mathcal{O}_r \\ \chi_r = 0 & on \ \partial \mathcal{O}_r \\ \chi_r \ is \ periodic \ both \ in \ y \ and \ t \end{cases}$$

admits a positive solution $\chi_r(t, x) \in C^{1,2}_{r_1}((\mathbb{R} \times (-r, r) \times \mathbb{R}^{N-1})$, where $L$ is the parabolic operator of the form defined in (13). The function $\chi_r$ (unique up to a multiplication) is called the principal eigenfunction associated with eigenvalue $\lambda_p(r)$ of $L$ in $\mathcal{O}_r$.

Proof. The existence and uniqueness of principal eigenvalue for time periodic operator with Dirichlet boundary condition have been derived in [17]. For the framework of space-time periodic operator, one can refer to the work of Nadin [19] and the proof of this Theorem is essentially similar to the proof of Theorem 2.7, [19]. We omit the proof here.
Proposition 3.2. There holds $\lambda_p(r) \rightarrow \lambda_p$ as $r \rightarrow \infty$, strictly decreasing in $r$, where $\lambda_p = \tilde{\lambda}_1(\mathcal{L}, \mathbb{R} \times \mathbb{R}^N)$ defined in (15). Moreover, there exists an eigenfunction $\chi \in C^{1,2}_{t,x}(\mathbb{R} \times \mathbb{R}^N)$ associated with $\tilde{\lambda}_1$ such that $\mathcal{L}\chi = \tilde{\lambda}_1 \chi$ a.e in $\mathbb{R}^{N+1}$.

Proof. Assume by contradiction that there exist $0 < r_1 < r_2$ but $\lambda_p(r_1) \geq \lambda_p(r_2)$. Thanks to the periodicity in $y$ and $t$ and the boundedness of domain in $x$, there exist $\alpha > 0$ and $x_0 \in \mathcal{O}_{r_1}$ such that $\alpha \chi_{r_1} \geq \chi_{r_2}$ and $\alpha \chi_{r_1}(x_0) = \chi_{r_2}(x_0)$. It follows immediately that

$$\mathcal{L}(\alpha \chi_{r_1} - \chi_{r_2}) \geq \lambda_p(r_1)(\alpha \chi_{r_1} - \chi_{r_2}) \quad \text{a.e } x \in \mathcal{O}_{r_1}.$$  

The parabolic strong maximum principle implies $\alpha \chi_{r_1} \equiv \chi_{r_2}$, contradiction. Moreover, it is easily seen, by strong maximum principle, that $\lambda_p(r) > -\sup_{\mathcal{O}_r} f_s(t,x,0) \geq -\sup_{\mathbb{R} \times \mathbb{R}^N} f_s(t,x,0), \forall r > 0$. This implies that the limit $\chi'_p = \lim_{r \rightarrow \infty} \lambda_p(r)$ does exist and bounded from below by $\lambda_p$.

Let us argue that there exists an eigenfunction $\chi \in C^{1,2}_{t,x}(\mathbb{R} \times \mathbb{R}^N)$ which is periodic in $y$ and $t$ such that $\mathcal{L}\chi = \chi'_p \chi$, thanks to the periodicity in $y$ and $t$, one can apply the Harnack inequality for the family $(\chi(r))_r > 0$ with normalization $\chi_r(0,0) = 1$ to derive that it is uniformly bounded on any the compact set of $\mathbb{R} \times \mathbb{R}^N$. Then, the standard parabolic estimates imply that $\chi_r \rightarrow \chi$, as $r \rightarrow \infty$ locally uniformly in $\mathbb{R} \times \mathbb{R}^N$ and $\chi$ satisfies $\mathcal{L}\chi = \chi'_p \chi$. Moreover, $\chi$ is strictly positive, periodic in $y$ and $t$ and satisfies $\chi(0,0) = 1$ by the strong maximum principle. Lastly, taking $\chi$ as a test super solution for $\lambda_p$, one finds $\chi'_p = \lambda_p$.

We are now able to prove Theorem 1.7.

Proof of Theorem 1.7. Recall that $\tilde{\lambda}_1 = \tilde{\lambda}_1(\mathcal{P}, \mathbb{R} \times \mathbb{R}^N)$. We first consider the case that $\tilde{\lambda}_1 < 0$. It then follows from Proposition 3.2 that for $r > 0$ large enough, $\lambda_p(r) < 0$, where $\lambda_p(r)$ is the space-time periodic principal eigenvalue on $\mathcal{O}_r$, with Dirichlet boundary condition, which is defined in Proposition 3.1. Let $\chi_r$ be an eigenfunction associated to $\lambda_p(r)$ in $\mathcal{O}_r$, we define the function:

$$\phi(t,x) = \begin{cases} \eta \chi_r(t,x) & x \in \mathcal{O}_r \\ 0 & \text{otherwise.} \end{cases}$$

For $\eta \in \mathbb{R}$ small enough, one obtains immediately that

$$\partial_t(\phi) - \Delta(\phi) - c\partial_t(\phi) - f(t,x,\phi) = (f_s(t,x,0) + \lambda_p(r))\phi - f(t,x,\phi) < 0.$$  

That is, $\phi$ is a subsolution of Eq. (14) while the constant $S$ given in (17) is a super solution of Eq. (14). Let us consider the solution $u$ of Eq. (14) with the initial condition $u(0,x) = \phi(0,x)$. The standard parabolic theory and maximum principle imply that there exist such solution for any $t > 0$ and satisfies $\phi(t,x) \leq u(t,x) \leq S, \forall (t,x) \in \mathbb{R} \times \mathbb{R}^{N+1}$. In particular, $\phi(T,x) \leq u(T,x)$, where $T$ is the period of $\phi$ and $f$ with respect to $t$. Consider the function $u(t+T,x)$, it is also a solution of Eq. (14) with initial condition $u(T,x) \geq u(0,x)$, then $u(t+T,x) \geq u(t,x)$. By induction, one sees that the sequence $u_n(t,x) = u(t+nT,x)$ is nondecreasing in $n$ and uniformly bounded by $S$. Therefore, $u_n(t,x)$ converges pointwise to a bounded function $U(t,x)$ such that $U$ is $T$-periodic in $t$, $\phi \leq U \leq S$, $U$ solves Eq. (14). The partial periodicity in $y$ of solution follows from the construction.
Let us postpone for a moment the proof of necessary condition to prove the uniqueness of solutions. We emphasize that the uniqueness of Eq. (14) is proved to hold the class of nonnegative bounded solutions.

Assume that \( \overline{U} \) and \( \underline{U} \) are the positive bounded solution of Eq. (14). Theorem A.2, Appendix, yields
\[
\lim_{|x_1| \to \infty} \overline{U}(t, x_1, y) = \lim_{|x_1| \to \infty} \underline{U}(t, x_1, y) = 0,
\]
uniformly in \( y \) and \( t \). By condition (13), there exists a pair \((\lambda, \phi)\) such that \( \partial_t \phi - \Delta \phi - \gamma(t, y) \phi \geq \lambda \phi \), \( \phi \) is periodic in \( y \) and \( t \), and \( \lambda > 0 \). Thanks to the periodicity, we have \( \inf_{\mathbb{R}^N} \varphi(t, y) > 0 \). For any \( \varepsilon > 0 \), there exists \( R(\varepsilon) > 0 \) such that
\[
\overline{U}(t, x_1, y) \leq \varepsilon \varphi(t, y), \quad \forall |x_1| \geq R(\varepsilon), y \in \mathbb{R}^{N-1}, t \in \mathbb{R},
\]
and therefore the set
\[
K_{\varepsilon} := \{ k > 0 : k \overline{U} \geq \underline{U} - \varepsilon \varphi \text{ in } \mathbb{R} \times \mathbb{R}^N \}
\]
is nonempty. Set \( k(\varepsilon) := \inf K_{\varepsilon} \). Obviously, the function \( k(\varepsilon) : \mathbb{R}^+ \to \mathbb{R} \) is nonincreasing. Assume by way of contradiction
\[
k^* = \lim_{\varepsilon \to 0^+} k(\varepsilon) > 1.
\]
Take \( 0 < \varepsilon < \sup_{\mathbb{R}^{N+1}} \overline{U}/\varphi \), we see that \( k(\varepsilon) > 0 \), \( k(\varepsilon) \overline{U} - \underline{U} + \varepsilon \varphi \geq 0 \). The definition of \( k(\varepsilon) \) yields that there exists a sequence \((t_{n}^\varepsilon, x_{1,n}^\varepsilon, y_n^\varepsilon)\) in \( \Omega \) such that
\[
\left( k(\varepsilon) - \frac{1}{n} \right) \overline{U}(t_{n}^\varepsilon, x_{1,n}^\varepsilon, y_n^\varepsilon) - \underline{U}(t_{n}^\varepsilon, x_{1,n}^\varepsilon, y_n^\varepsilon) - \varepsilon \varphi(t_{n}^\varepsilon, y_n^\varepsilon).
\]
From (35), we have \((t_{n}^\varepsilon, x_{1,n}^\varepsilon, y_n^\varepsilon) \in \mathcal{O}(R(\varepsilon)) \) for \( n \) large enough. Taking the sequences \((\tau_n^\varepsilon)\) and \((z_n^\varepsilon)\) such that \( t_{n}^\varepsilon - \tau_n^\varepsilon \in [0, T) \) and \( y_n^\varepsilon - z_n^\varepsilon \in [0, L_1) \times \ldots \times [0, L_{N-1}) \). For any \( \varepsilon > 0 \), one sees that
\[
\overline{U}_{n}(t, x_1, y) = \overline{U}(t + \tau_n^\varepsilon, x_1, y + z_n^\varepsilon) \quad \text{and} \quad \underline{U}_{n}(t, x_1, y) = \underline{U}(t + \tau_n^\varepsilon, x_1, y + z_n^\varepsilon)
\]
are the solutions of following equation
\[
\partial_t U_{n}^\varepsilon - \Delta U_{n}^\varepsilon - c \partial_t U_{n}^\varepsilon = f(t + \tau_n^\varepsilon, x_1, y + z_n^\varepsilon, U_{n}^\varepsilon).
\]
Using the priori estimates of solutions (Theorem A.2, Appendix), we deduce that as \( n \to \infty \), up to extractions, \( \overline{U}_{n}^\varepsilon \to \overline{U}_{\infty}^\varepsilon \) and \( \underline{U}_{n}^\varepsilon \to \underline{U}_{\infty}^\varepsilon \) locally uniformly in \( \mathbb{R}^{N+1} \). Moreover, since \( f \) is partially periodic in \( y \) and \( T \)-periodic in \( t \), the standard parabolic estimates yield that there exist \( \tau_{\infty}^\varepsilon, z_{\infty}^\varepsilon \) such that \( \overline{U}_{\infty}^\varepsilon \) and \( \underline{U}_{\infty}^\varepsilon \) are the solutions of:
\[
\partial_t U_{\infty}^\varepsilon - \Delta U_{\infty}^\varepsilon - c \partial_t U_{\infty}^\varepsilon = f(t + \tau_{\infty}^\varepsilon, x_1, y + z_{\infty}^\varepsilon, U_{\infty}^\varepsilon).
\]
By passing to the limit, \( W_{\infty}^\varepsilon = k(\varepsilon) \overline{U}_{\infty}^\varepsilon - \underline{U}_{\infty}^\varepsilon + \varepsilon \varphi_{\infty}^\varepsilon \geq 0, \) where \( \varphi_{\infty}^\varepsilon(y) = \lim_{n \to \infty} \varphi(y + z_n^\varepsilon) \) satisfying
\[
\partial_t \phi_{\infty}^\varepsilon - \Delta \phi_{\infty}^\varepsilon - \gamma(t, y + z_{\infty}^\varepsilon) \phi_{\infty}^\varepsilon \geq \lambda \phi_{\infty}^\varepsilon, \phi_{\infty}^\varepsilon \text{ is periodic in } y \text{ and } t, \text{ and } \lambda > 0.
\]
Moreover, there exist \((t(\varepsilon), x_1(\varepsilon), y(\varepsilon))\) such that
\[
(k(\varepsilon) \overline{U}_{\infty}^\varepsilon - \underline{U}_{\infty}^\varepsilon + \varepsilon \varphi_{\infty}^\varepsilon)(t(\varepsilon), x_1(\varepsilon), y(\varepsilon)) = 0.
\]
Note that $t(\varepsilon) \in [0, T)$ and $y(\varepsilon) \in [0, L_1) \times \ldots \times [0, L_{N-1})$ are bounded with respect to $\varepsilon$. The case that $\liminf_{\varepsilon \to 0^+} |x_1(\varepsilon)| < \infty$ is ruled out. Indeed, if $\liminf_{\varepsilon \to 0^+} |x_1(\varepsilon)| < \infty$, there exists a sequence $(\varepsilon_n) \to 0$ as $n \to \infty$ such that $(t(\varepsilon_n), x_1(\varepsilon_n), y(\varepsilon_n)) \to (t_0, x_0, y_0)$ as $n \to \infty$, up to subsequences. The prior estimates of solutions again yield that $U_{\infty}^0, U_{\infty}^0, U_0 \in \mathbb{R}^{N+1}$. Moreover, by the partial periodicity in $y$ and the periodicity in $t$ of $f$, the standard parabolic estimates yield that $\overline{U}_{\infty}^0, \underline{U}_{\infty}^0$ satisfy the following equation

$$
\partial_t U_{\infty}^0 - \Delta U_{\infty}^0 - c\partial_t U_{\infty}^0 = f(t + t_0^0, x_1, y + z_0^0, U_{\infty}^0).
$$

for some $t_0^0 \in [0, T)$ and $z_0^0 \in [0, L_1) \times \ldots \times [0, L_{N-1})$. From (37), $k^* < \infty$, then the function $W = k^*U_{\infty}^0 - U_{\infty}^0$ is nonnegative and vanishes at $(t_0, x_0, y_0)$. Since $k^* > 1$, The Lipschitz continuity of $f$ with respect to $s$ and condition (18) yield

$$
\partial_t W - \Delta W - c\partial_t W \geq k^* f(t + t_0^0, x_1, y + z_0^0, U_{\infty}^0) - f(t + t_0^0, x_1, y + z_0^0, U_{\infty}^0)
\geq f(t + t_0^0, x_1, y + z_0^0, k^*U_{\infty}^0) - f(t + t_0^0, x_1, y + z_0^0, U_{\infty}^0)
\geq z(t + t_0^0, x_1, y + z_0^0)W, \quad (38)
$$

where $z(t, x) \in L_{loc}^{\infty}(\mathbb{R} \times \mathbb{R}^N)$. Hence, the parabolic strong maximum principle implies $W = 0$ in $(-\infty, t_0) \times \mathbb{R}^N$. This is a contradiction because from condition (18), the inequality (38) hold strictly in some $D \subset (-\infty, t_0) \times \mathbb{R}^N$ with $|D| > 0$. Otherwise, we consider the case $\lim_{\varepsilon \to 0^+} |x_1(\varepsilon)| = \infty$.

We have shown that $W_\infty(\varepsilon) \geq 0$ and $W_\infty(\varepsilon)$ vanishes at $(t(\varepsilon), x_1(\varepsilon), y(\varepsilon))$, then $k(\varepsilon)U < \underline{U}$ in $\mathcal{O}$, shrinking $\mathcal{O}$ if necessary. Since $k^* > 1$, for $\varepsilon$ small enough, $k^*(\varepsilon) > 1$, we derive from (18) for $x \in \mathcal{O}$

$$
\partial_t W_\infty(\varepsilon) - \Delta W_\infty(\varepsilon) - c\partial_t W_\infty(\varepsilon)
\geq k(\varepsilon)f(t + \tau_{\infty}^\varepsilon, x_1, y + z_\infty^\varepsilon, U_\infty) - f(t + \tau_{\infty}^\varepsilon, x_1, y + z_\infty^\varepsilon, U_\infty^\varepsilon) + (\gamma(t + \tau_{\infty}^\varepsilon, y + z_\infty^\varepsilon) + \lambda)\varepsilon\varphi_\varepsilon
\geq f(t + \tau_{\infty}^\varepsilon, x_1, y + z_\infty^\varepsilon, k(\varepsilon)U_\infty) - f(t + \tau_{\infty}^\varepsilon, x_1, y + z_\infty^\varepsilon, U_\infty^\varepsilon) + (\gamma(t + \tau_{\infty}^\varepsilon, y + z_\infty^\varepsilon) + \lambda)\varepsilon\varphi_\varepsilon
\geq f(t + \tau_{\infty}^\varepsilon, x_1, y + z_\infty^\varepsilon, k(\varepsilon)U_\infty)\left(\frac{(k(\varepsilon)U_\infty - U_\infty^\varepsilon + \varepsilon\varphi_\varepsilon)}{k(\varepsilon)U_\infty}ight) + \left(\frac{\lambda}{2} + \gamma(t + \tau_{\infty}^\varepsilon, y + z_\infty^\varepsilon) - f(t + \tau_{\infty}^\varepsilon, x_1, y + z_\infty^\varepsilon, k(\varepsilon)U_\infty)\right)\varepsilon\varphi_\varepsilon. \quad (39)
$$

Condition (19) implies, for $\varepsilon$ small enough, that

$$
\frac{f(t + \tau_{\infty}^\varepsilon, x_1, y + z_\infty^\varepsilon, k(\varepsilon)U_\infty)}{k(\varepsilon)U_\infty} < \gamma(t + \tau_{\infty}^\varepsilon, y + z_\infty^\varepsilon) + \frac{\lambda}{2}, \quad \forall(x_1, y) \in \mathcal{O}.
$$

Then, it follows from (39) that

$$
\partial_t W_\infty - \Delta W_\infty - c\partial_t W_\infty - \varrho(t + \tau_{\infty}^\varepsilon, x_1, y + z_\infty^\varepsilon)W_\infty > \frac{\lambda}{2}\varepsilon\varphi_\varepsilon > 0 \quad \text{in} \quad \mathcal{O},
$$

24
where \( g(t, x) = \frac{f(t, x, k(\varepsilon)\tilde{U})}{k(\varepsilon)} \) is bounded. This is a contradiction because the strong maximum principle implies that \( W(t, x_1, y) = 0 \) in \( \mathcal{O} \). As a consequence, we have proved that \( k^* = \lim_{\varepsilon \to 0^+} k(\varepsilon) \leq 1 \). Therefore

\[
\frac{U}{\varepsilon} \leq \lim_{\varepsilon \to 0^+} (k(\varepsilon)U + \varepsilon \varphi) \leq U \quad \text{in} \quad \mathbb{R}^{N+1}.
\]

We derive the uniqueness due to the equivalence of \( \mathcal{U} \) and \( \mathcal{U} \). Note that we do not use the partial periodicity in \( y \) and the periodicity in \( t \) of solution in the proof of uniqueness. This therefore infers that any nonnegative bounded solutions of Eq. (14) must be partially periodic in \( y \) and \( T \)-periodic in \( t \).

To conclude the proof of Theorem 1.7, it only remains to prove the necessary condition. Assume by contradiction that \( \lambda_1 \geq 0 \) and Eq. (14) admits a solution \( U \), which is \( T \)-periodic in \( t \) but not necessarily partially periodic in \( y \). Let \( \chi \) be a principal eigenfunction associated with \( \lambda_1 \) (Proposition 3.2) with normalization \( \chi(0, 0) < U(0, 0) \). Then

\[
\varpartial \chi - \Delta \chi - c\varpartial \chi - f(t, x, \chi) = \lambda_1 \chi + f_\varepsilon(t, x, 0)\chi - f(t, x, \chi) \geq 0.
\]

The same argument for the uniqueness is applied to achieve the contradiction : \( U \leq \chi \) in \( \mathbb{R}^{N+1} \). \( \square \)

Before considering the large time behaviour, we point out that the monotonicity in time of solutions starting by a stationary sub (or super) solution of parabolic operator with time-dependent coefficients no longer holds. In addition, the boundedness of initial datum does not suffice to guarantee that the solutions of Eq. (13) converge uniformly to the unique solution of Eq. (14) as \( t \to \infty \). However, by taking advantage of the periodicity in \( t \) of solutions, we obtain the locally uniform convergence and with some extra restrictions (part (ii), Theorem 1.8) we can actually derive the uniform convergence as \( t \to \infty \).

**Proof of Theorem 1.8** Set \( S' := \max\{S, \|u_0\|_{L^\infty(\Omega)}\} \), \( S \) is the positive constant given in (17). Then, the function \( \tilde{u}(t, x) = u(t, x + ct, y) \) satisfies \( 0 < \tilde{u} \leq S' \) in \( \mathbb{T}^+ \times \mathbb{R}^N \) and solves

\[
\varpartial_t \tilde{u} = \Delta \tilde{u} + c\varpartial_x \tilde{u} + f(t, x, \tilde{u}) \quad t > 0, x \in \mathbb{R}^N,
\]

with initial condition \( \tilde{u}(0, x) = u_0(x) \). Let \( w \) be the solution to (40) with initial condition \( w_0(x) = S' \). Clearly, the constant \( S' \) is \( T \)-periodic in \( t \) and partially periodic in \( y \). Arguing as the proof of Theorem 1.7, we deduce that the sequence \( w_n(t, x) = w(t + nT, x) \) is nonincreasing and converges locally uniformly to \( W(t, x) \), which is a solution of

\[
\varpartial_t W - \Delta W - c\varpartial_x W - f(t, x, W) = 0 \quad \forall t > 0, x \in \mathbb{R}^N.
\]

Moreover, \( W(t, x) \) is \( T \) periodic in \( t \) and partially periodic in \( y \). Then

\[
\forall r > 0, \lim_{t \to \infty} \sup_{x \in \tilde{\mathcal{O}}} (\tilde{u}(t, x) - W(t, x)) \leq \lim_{t \to \infty} \sup_{x \in \tilde{\mathcal{O}}} (w(t, x) - W(t, x)) = 0.
\]

If \( \lambda \geq 0 \), then \( W \equiv 0 \) in \( \mathbb{R} \times \mathbb{R}^N \). Therefore, \( \tilde{u}(t, x) \to 0 \) as \( t \to \infty \) locally uniformly with respect to \( x \in \mathbb{R}^N \). This convergence is uniform in \( x \in \mathbb{R}^N \) due to the following claim.
Claim.

\[
\lim_{\min(t,|x_1|) \to \infty} \tilde{u}(t, x, y) = 0 \quad \text{uniformly in } y \in \mathbb{R}^{N-1}.
\] (42)

Let us postpone for a moment the proof of claim to consider the case \( \lambda_1 < 0 \). From propositions (3.1) and (3.2), there exists \( \rho > \lambda \) such that \( \lambda(\rho) < 0 \). Let \( \chi(\rho, t, x) \) be an associated principal eigenfunction of \( \lambda(\rho) \), for \( \gamma > 0 \) small enough, one sees that the function

\[ V(t, x) = \begin{cases} \gamma \chi(t, x) & x \in O_{\rho} \\ 0 & \text{otherwise} \end{cases} \]

is a subsolution of Eq. (11). Then if (20) holds, there exist \( \gamma \) small enough such that \( V(0, x) \leq u_0(x) \). Alternatively, \( u_0(x) \) is partially periodic in \( y \), then \( \tilde{u}(t, x) \) is strictly positive, partially periodic in \( y \) and \( T \)-periodic in \( t \), then the parabolic strong maximum principle yields \( \tilde{V}(T, x) \leq \tilde{u}(T, x) \). In both cases, we always can define \( \tilde{v}(t, x) \) is such that \( \tilde{v}(0, x) = V(0, x) \) or \( \tilde{v}(T, x) = V(T, x) \). It follows immediately by parabolic strong maximum principle that \( \tilde{v}(t, x) \leq u(t, x), \forall t > T, x \in \mathbb{R}^N \). Arguing again as the proof of Theorem (1.7) we deduce that the sequence \( v_n(t, x) = \tilde{v}(t + nT, x) \) is nondecreasing and converges locally uniformly to \( P(t, x) \), which is a solution of

\[
\partial_t P - \Delta P - c \partial_1 P - f(t, x, P) = 0 \quad \forall t > 0, x \in \mathbb{R}^N.
\]

Moreover, \( P \) is strictly positive, partially periodic in \( y \) and \( T \)-periodic in \( t \). The uniqueness of Theorem (1.7) implies that \( W = P = U \) in \( \mathbb{R}^{N+1} \), which is a solution of Eq. (14). Assume by contradiction that this convergence is not uniform in \( x \), this means that there exist \( \varepsilon > 0 \) and a sequence \( (t_n, x_{1n}, y_n) \in \mathbb{R}^+ \times \mathbb{R}^N \) such that

\[
\lim_{n \to \infty} t_n = \infty, \quad \forall n \in \mathbb{N}, \quad |\tilde{u}(t_n, x_{1n}, y_n) - U(x_{1n}, y_n)| \geq \varepsilon. \quad (43)
\]

Due to the locally uniform convergence of \( \tilde{u} \), necessarily, the sequence \( (x_{1n}) \) is divergent. We get, by the a priori estimates of \( U \), that \( U(x_{1n}, y_n) \to 0 \) as \( n \to \infty \), uniformly in \( y \). But from (43), this inference contradicts with the claim (42). Therefore, in order to conclude the proof, it remains to prove the claim (42).

Let us call \( (t_n, x_{1n}, y_n) \in \mathbb{R}^+ \times \mathbb{R}^N \) be a sequence such that the claim (42) is not true:

\[
\lim_{n \to \infty} t_n = \lim_{n \to \infty} x_{1n} = \infty, \quad \forall n \in \mathbb{N}, \quad \liminf_{n \to \infty} \tilde{u}(t_n, x_{1n}, y_n) \geq \varepsilon \quad \text{for some } \varepsilon > 0.
\]

For any \( n \in \mathbb{N} \), we define the functions \( \tilde{u}_n(t, x) = \tilde{u}(t + t_n, x_1 + x_{1n}, y + y_n) \). It holds that \( 0 \leq \tilde{u}_n \leq S' \) and

\[
\partial_t \tilde{u}_n = \Delta \tilde{u}_n + c \partial_1 \tilde{u}_n + f(t + t_n, x_1 + x_{1n}, y + y_n, \tilde{u}_n) \quad t > -t_n, x \in \mathbb{R}^N.
\]

Since \( f \) is partially periodic in \( y \) and \( T \)-periodic in \( t \), we can assume without loss of generality that \( t_n \to t_0 \) and \( y_n \to y_0 \) (up to a subsequence) as \( n \to \infty \). Thanks to (18) and (19), we imply, by the parabolic estimates and embedding theorems that \( \tilde{u}_n \) converges (up to a subsequence) to \( \tilde{u}_\infty \) locally uniformly in \( \mathbb{R} \times \Omega \) satisfying

\[
\partial_t \tilde{u}_\infty \leq \Delta \tilde{u}_\infty + c \partial_1 \tilde{u}_\infty + \gamma(t + t_0, y + y_0) \tilde{u}_\infty \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^N.
\]
and \( \tilde{u}_\infty(0, 0) \geq \varepsilon \). Moreover, condition (19) implies that there exists a pair \((\lambda, \phi)\) such that \( \partial_t \phi - \Delta \phi - \gamma(t + t_0, y + y_0)\phi \geq \lambda \phi \), \( \phi \) is periodic in \( y \) and \( t \), and \( \lambda > 0 \). Let us define the function 
\[
\omega(t, x) = S'\phi(t, y) e^{-\lambda(t-h)}
\]
we have
\[
\partial_t \omega \geq \Delta \omega + c\partial_1 \omega + \gamma(t + t_0, y + y_0)\omega.
\]
Then, set 
\[
\tilde{L} = \partial_t - \Delta - c\partial_1 - \gamma(t + t_0, y + y_0),
\]
\( W(t, x) = \omega - \tilde{u}_\infty \) and 
\( W(t, x) = z(t, x) \phi(t, y) \), we obtain
\[
0 \leq \frac{\tilde{L}W}{\phi} = z_t - \Delta z - \nabla_y z \cdot \nabla_y \phi + \lambda z.
\]
Since the periodicity yields \( \inf_{\mathbb{R}^N} \phi(t, y) > 0 \) and \( \tilde{u} \leq S \), we can enlarge \( S' \) so that \( z(h, x) \geq 0 \) in \( \mathbb{R}^N \). The parabolic comparison principle yields \( z(t, x) \geq 0 \), \( \forall t \leq h, x \in \mathbb{R}^N \). As a consequence
\[
\varepsilon \leq \tilde{u}(0, 0) = S'\phi(0, 0)e^{\lambda h}.
\]
Letting \( h \) to \( -\infty \), we get a contradiction. This concludes the proof.

\[\Box\]

4 Further results and applications

4.1 Similarity of the problem with Dirichlet boundary condition

In this subsection, we aim at proving an analogous result of Theorem (1.1), where Dirichlet instead of Neumann condition is imposed on the boundary of \( \Omega \). This is to prepare for the main goal in the next subsection. To this end, let us define the generalized Dirichlet principal eigenvalue
\[
\lambda_D(-L, \Omega) := \sup \{ \lambda \in \mathbb{R} : \exists \phi \in W^{2, N}_{loc}(\Omega), \phi > 0, (L + \lambda)\phi \leq 0 \text{ a.e in } \Omega \text{ and } \phi = 0 \text{ on } \partial\Omega \}. \tag{44}
\]
Note that, if \( \Omega \) is bounded, \( \lambda_D \) coincides with the classical Dirichlet eigenvalue (111). Similar to assumption (9), we assume that there exists a measurable bounded function \( \mu : \omega \to \mathbb{R} \) such that
\[
\mu(y) = \limsup_{|x_1| \to \infty} f_s(x_1, y, 0) \quad \text{and} \quad \lambda_D(-\Delta_y - \mu(y), \omega) > 0. \tag{45}
\]
Then, we obtain an analogue of Theorem (1.1) as following

**Theorem 4.1.** Assume that conditions (7)-(8) and (45) are satisfied. Then there exists a unique critical speed \( c^* \) such that the equation
\[
\begin{aligned}
\Delta U + c\partial_1 U + f(x, U) &= 0 \quad \text{in } \Omega \\
U &= 0 \quad \text{on } \partial\Omega \\
U &> 0 \quad \text{in } \Omega \\
U &\text{ is bounded.}
\end{aligned} \tag{46}
\]
admits a solution if and only if \( 0 \leq c < c^* \). Moreover, the solution is unique once exists.
Proof. The proof of this theorem is essentially similar to the proof of Theorem (1.1). However, there are significant differences to be outlined here:

i) Existence.

Since the problem is set up with the Dirichlet boundary condition, the Proposition 1, in [10], cannot be applied. However, the Dirichlet boundary condition allows us to use Theorem 1.9, [11] to prove the existence of Eq. (46). Indeed, let us call \( \lambda_D = \lambda_D(-\Delta - c \partial_1 - f_s(x,0), \Omega) \) and let \( c^* \) be defined as Definition 1.2, then the Proposition 1.3 yields that \( \lambda_D < 0 \) iff \( 0 \leq c < c^* \). Let \( (\lambda_R, \varphi_R) \) be the Dirichlet principal eigenvalue and eigenfunction of the problem

\[
\begin{cases}
-\Delta \varphi_R - c \partial_1 \varphi_R - f_s(x,0) \varphi_R = \lambda_R \varphi_R & x \in \Omega_R \\
\varphi_R(x_1, y) = 0 & |x_1| < R, y \in \partial \omega \\
\varphi_R(\pm R, y) = 0 & y \in \omega,
\end{cases}
\]

we deduce, by Theorem 1.9 in [11], that \( \lambda_D = \lim_{R \to \infty} \lambda_R \). Note that Theorem 1.9 in [11] also deals with the case of nonsmooth domain as the set \( \Omega_R \) of ours. Then the existence of Eq. (46) can be obtained in the same way with Theorem (1.1).

ii) Nonexistence and Uniqueness.

Let \( \tilde{U}(x) = U(x)e^{\tilde{x}x_1} \), then \( U \) solves Eq. (46) if and only if \( \tilde{U} \) solves the equation

\[
\begin{align*}
\Delta \tilde{U} + f(x_1, y, \tilde{U}(x_1, y)e^{-\tilde{x}x_1})e^{\tilde{x}x_1} - \frac{c^2}{4} \tilde{U} &= 0 & x \in \Omega \\
\tilde{U} &= 0 & x \in \partial \Omega \\
\tilde{U} &> 0 & \text{in } \Omega \\
\tilde{U}(x_1, y)e^{-\tilde{x}x_1} &\text{ is bounded.}
\end{align*}
\]

The argument of Theorem (1.1) for the nonexistence result can be applied if one can prove that \( \tilde{U} \) decays exponentially as \(|x_1| \to \infty\). Moreover, if \( \tilde{U} \) decays exponentially, the uniqueness can be obtained by using variational argument as Theorem 2.3 in [9]. Note that the sliding argument as of Theorem (1.1) does not work in this case due to the lack of the Hopf lemma under Dirichlet condition. We end the proof by showing that \( \tilde{U} \) really decays exponentially.

From assumption (15), we know that for any \( \delta > 0 \) there exists \( R = R(\delta) > 0 \) such that \( f_s(x_1, y, 0) \leq \mu(y) + \delta \) when \(|x_1| \geq R\). Since \( \omega \) is bounded, \( \lambda_D(-\Delta_y - \mu(y), \omega) \) coincides with the classical Dirichlet principal eigenvalue in \( \omega \), says \( \lambda \). There exists an eigenfunction \( \phi \in L^\infty(\omega) \), associated with \( \lambda \), positive in \( \omega \), such that

\[
\Delta_y \phi + \mu(y) \phi + \lambda \phi = 0 \quad \text{in } \omega ; \quad \phi = 0 \quad \text{on } \partial \omega.
\]

One can actually find a positive function \( \tilde{\phi} > 0 \) in \( \overline{\omega} \) such that \( \Delta_y \tilde{\phi} + (\mu(y) + \lambda) \tilde{\phi} = 0 \in \omega \), with \( \tilde{\mu}(y) \) sufficiently close to \( \mu(y) + \lambda \) in such the way \( \sup_{\overline{\omega}} |\tilde{\mu} - \mu| < \lambda \). Set \( L + \delta = \Delta + \mu(y) - c^2/4 + \delta \), conditions (8) and (15) yield \( (L+\delta)\tilde{U} \geq 0 \) in \( \Omega \setminus \Omega_R \). On the other hand, we set \( w(x) = C\theta_a(x)\tilde{\phi}(y) \), where \( \theta_a \) is the solution of equation

\[
\begin{cases}
\theta''_a = (\kappa + \delta)\theta_a & \text{in } (R, R + a) \\
\theta_a(R) = Ce^{\sqrt{\kappa}R} \\
\theta_a(R + a) = Ce^{\sqrt{\kappa}(R+a)},
\end{cases}
\]

28
with \( \kappa = \lambda + c^2/4 - 2\delta \) and \( C = \sup_{\Omega} \tilde{U}(x)e^{-\frac{\tilde{\omega}}{\kappa}/2} \). Then, choosing \( \delta < (\lambda - \sup_{\Omega}\tilde{\mu} - \mu)/2 \), direct computation leads to \( (\mathcal{L} + \delta)w \leq 0 \) in \( \Omega \setminus \Omega_R \). The argumentation of Proposition (4.1) enables one to conclude that \( U(x_1, y) \leq Ce^{-(\sqrt{\kappa c}/2)|x_1|}\tilde{\phi}(y) \) in \( \Omega \). This concludes the proof of theorem. \( \square \)

By this preliminary, we are ready to present the main theorem of this section. Some ideas in the proof are inspired from [16].

### 4.2 Concentration of species in the more favorable region

Based on the characterizations of the persistence and extinction of the species in the cylindrical domain \( \Omega \) under Dirichlet boundary condition, we study the behavior of the species when the environment outside \( \Omega \) changes to be very bad. More precisely, we consider Eq. (46) in the domain \( \Omega \) under Dirichlet boundary condition, we study the behavior of the species when the death rate in \( \Omega \) becomes extremely high.

Our goal is to characterize the limit of the sequence \( U_n(x) \), which are solutions of the equations

\[
\begin{aligned}
\Delta U_n + c\partial_1 U_n + F_n(x, U_n) &= 0 \quad x \in \mathbb{R}^N, \\
U_n &> 0 \text{ and bounded in } \mathbb{R}^N.
\end{aligned}
\]

(48)

For any \( n \), the nonlinearities \( F_n(x, s) \) are assumed to be continuous with respect to \( x \) and of class \( C^1 \) with respect to \( s \), \( F_n(x, 0) = 0, \forall x \in \mathbb{R}^N \). Moreover, \( F_n(x, s) \) are assumed to satisfy

\[
\exists S > 0 \text{ such that } F_n(x, s) \leq 0 \text{ for } s \geq S, \forall x \in \mathbb{R}^N,
\]

(49)

\[
s \to F_n(x, s)/s \text{ is nonincreasing a.e in } \mathbb{R}^N \text{ and there exist } D \subset \mathbb{R}^N, |D| > 0 \text{ such that it is strictly decreasing in } D.
\]

(50)

Let \( f(x, s) : \Omega \times [0, +\infty) \rightarrow \mathbb{R} \) satisfy [17]-[8] and [15], we assume further that:

\[
\begin{aligned}
F_n(x, s) &= f(x, s) \quad x \in \Omega, s \in \mathbb{R}^+ \text{ for all } n \in \mathbb{N} \\
F_n(x, s) &\text{ is nonincreasing in } n \quad \forall (x, s) \in \overline{\Omega} \times \mathbb{R} \\
\rho_n(x) &= \frac{\partial F_n}{\partial s}(x, 0) \to -\infty \text{ as } n \to \infty \text{ locally uniformly in } \Omega^c. \\
\limsup_{x \in \Omega^c, |x| \to \infty} \rho_n(x) &< \min\{0, \inf_{\Omega^c} \mu(y)\} \quad \forall n \in \mathbb{N}.
\end{aligned}
\]

(51)

Before stating the result, let us explain the meaning of this condition. This condition means that the living environment of the species outside \( \Omega \) is unfavorable and it becomes extremely unfavorable as \( n \to \infty \). We will prove that no species can be persistent outside \( \Omega \) under such condition as \( n \to \infty \). As is proved in Subsection 4.1 that the species is persistent in \( \Omega \) if and only if \( 0 \leq c < c^* \). In the following result, we will see that as \( n \to \infty \) the species can only persist in \( \Omega \) and it is immediately mortal outside \( \Omega \). Moreover, we shall prove that the limit as \( n \to \infty \) coincides with the unique solution of Eq. (46) in \( \Omega \) and zero in \( \Omega^c \).

We derive the following result :
**Theorem 4.2.** Let $U_n(x)$ be the sequence of traveling front solution of Eq. (48) with $F_n(x, s)$ is given as above. If $c < c^*$ with $c^*$ is given in Theorem (4.1), then the following limit holds

$$U_n(x) \to U_\infty(x) \quad \text{as } n \to \infty,$$

uniformly for $x \in \mathbb{R}^N$, where $U_\infty \in W^{2,N}(\mathbb{R}^N)$ is nonnegative, vanishing in $\Omega^c$ and coincides with the unique positive solution of the following equation

$$\begin{cases}
\Delta U + c\partial_1 U + f(x, U) = 0 & x \in \Omega \\
U(x) = 0 & x \in \partial\Omega.
\end{cases} \quad (52)$$

**Proof.** Let us call $L_n = \Delta + c\partial_1 + \rho_n(x)$, defined in $\mathbb{R}^N$, and the generalized principal eigenvalue of $L_n$ as follows

$$\lambda_n = \sup \{ \lambda \in \mathbb{R} : \exists \phi \in W^{2,N}_{loc}(\mathbb{R}^N), \phi > 0, (L_n + \lambda)\phi \leq 0 \text{ a.e in } \mathbb{R}^N \}.$$ 

We also denote by $\lambda_D$ the generalized Dirichlet principal eigenvalue of the operator $\Delta + c\partial_1 + f_s(x, 0)$ in $\Omega$. We have the following lemma:

**Lemma 4.3.** There holds that $\lambda_n$ converges increasingly to $\lambda_D$.

Let us postpone the proof of this lemma for a moment to continue the proof of theorem. As is known, $c < c^*$ if and only if $\lambda_D < 0$. By Lemma (4.3) and assumption, we have $\lambda_n < \lambda_D < 0$. Then Theorem 1.1 [9] yields that the equation

$$\begin{cases}
\Delta U_n + c\partial_1 U_n + F_n(x, U_n) = 0 & x \in \mathbb{R}^N \\
0 \leq U_n \leq S & x \in \mathbb{R}^N.
\end{cases} \quad (53)$$

admits a strictly positive solution $U_n$. Let $V_n(x_1, y) = U_n(x_1, y)e^{\bar{\xi}x_1}$, then $U_n$ is a solution of Eq. (53) if and only if $V_n$ is a solution of

$$\begin{cases}
\Delta V_n + F_n(x_1, y, V_n(x_1, y))e^{\bar{\xi}x_1} - \frac{c^2}{4}V_n = 0 & x \in \mathbb{R}^N \\
V_n(x_1, y)e^{\bar{\xi}x_1} \text{ is bounded}.
\end{cases} \quad (54)$$

For any $n \in \mathbb{N}$, one has $\lim \sup_{|x| \in \Omega^c, |x| \to \infty} \rho_n(x) < 0$, The Proposition 4, [9] yields that $V_n(x)$ decays exponentially for $x \in \mathbb{R}^N \setminus \Omega$ and since $F_n(x, s) = f(x, s)$ satisfies condition (15) in $\Omega$, Theorem (4.1) yields that $V_n(x)$ also decays exponentially for $x \in \Omega$. Thanks to condition (50), we derive, by Theorem 1.1 of [9], that $U_n(x)$ is unique. Moreover, since $F_n$ is nonincreasing, for $m, k \in \mathbb{N}, k \geq m$, one has

$$\Delta U_m + c\partial_1 U_m + F_k(x, U_m) = -F_m(x, U_m) + F_k(x, U_m) \leq 0, \quad x \in \mathbb{R}^N.$$ 

Thus $U_m$ is a supersolution of equation satisfied by $U_k$. One can apply the comparison principle, Theorem 2.3 [9], to imply that $U_k \leq U_m$ in $\mathbb{R}^N$ for $k \geq m$. Then $U_n$ is nonincreasing with respect
to \( n \) and converges pointwise to a nonnegative function \( U_\infty \leq S \). We shall show now that \( U_\infty = 0 \) in \( \Omega^c \).

From above arguments, \( V_n(x) \) decays exponentially as \( |x| \to \infty \). Multiplying \( V_n(x) \) to Eq. (54), we derive, by applying the Stokes formula, that

\[
\int_{\mathbb{R}^N} \nabla V_n \cdot \nabla V_n = \int_{\mathbb{R}^N} \nabla F_n(x_1, y, V_n e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1} V_n - \frac{c^2}{4} V_n^2 \leq \int_{\mathbb{R}^N} F_0(x_1, y, V_n e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1} V_n \leq \max_{x \in \mathbb{R}^N} \partial_s F_0(x, 0) \int_{\mathbb{R}^N} V_n^2(x) \leq M < \infty.
\]

This implies, by Lebesgue monotone convergence theorem, that the sequence \( V_n \) converges monotonically to some \( V_\infty \in H^1(\mathbb{R}^N) \) as \( n \to \infty \), weakly in \( H^1(\mathbb{R}^N) \) and strongly in \( L^2(\mathbb{R}^N) \). Moreover, taking an arbitrary compact set \( K \subset \Omega^c \), one gets

\[
-(\max_K \rho_n) \int_K V_n^2 \leq -\int_K \rho_n V_n^2 \leq -\int_K F_n(x, V_n e^{-\frac{c}{2}x_1}) e^{\frac{c}{2}x_1} V_n = -\int_{\mathbb{R}^N} |\nabla V_n|^2 - \int_{\mathbb{R}^N} \frac{c^2}{4} V_n + \int_{\mathbb{R}^N \setminus K} F_n(x, V_n e^{-\frac{c}{2}x_1}) e^{\frac{c}{2}x_1} V_n \leq \int_{\mathbb{R}^N \setminus K} F_n(x, V_n e^{-\frac{c}{2}x_1}) e^{\frac{c}{2}x_1} V_n \leq M
\]

Then, from assumption (51), we have \( \max \rho_n \to -\infty, \forall K \subset \Omega^c \), whence \( V_\infty = 0 \) for all compact set in \( \Omega^c \). This implies \( V_\infty = 0 \) a.e in \( \Omega^c \) or in the other words \( U_\infty = 0 \) a.e in \( \Omega^c \). As a consequence, the restriction of \( U_\infty \) in \( \Omega \) belongs to \( H^1_0(\Omega) \). Moreover, since \( F_n(x, s) = f(x, s) \) in \( \Omega \), we have

\[
\Delta U_n + c\partial_1 U_n + f(x, U_n) = 0 \quad x \in \Omega.
\]

The standard elliptic estimates imply that \( U_n \to U_\infty \) as \( n \to \infty \) locally uniformly in \( \Omega \) and moreover \( U_\infty \) is a solution of the same equation in \( \Omega \) in the weak \( H^1_0(\Omega) \) sense. Thanks to Theorem 4.1, we know that \( U_\infty \) is unique and \( U_\infty(x) \to 0 \) as \( |x| \to \infty \), uniformly in \( y \in \omega \). Assume by contradiction that the convergence \( U_n \to U_\infty \) as \( n \to \infty \) is not uniform, one finds a positive constant \( \varepsilon > 0 \) and a sequence \( (x_k) \in \mathbb{R}^N \) such that for some \( n_0 \in \mathbb{N} \),

\[
|U_n(x_k) - U_\infty(x_k)| \geq \varepsilon, \quad \forall n \geq n_0.
\]

Since the convergence is already locally uniform in \( \mathbb{R}^N \), we deduce that \( |x_n| \to \infty \) as \( n \to \infty \). Since \( U_n \) is nonincreasing with respect to \( n \), we imply that for some \( 0 < \varepsilon_1 < \varepsilon \), one has

\[
U_{n_0}(x_k) \geq \varepsilon_1 + U_\infty(x_k).
\]

This is a contradiction since we know that \( U_\infty(x), U_{n_0}(x) \to 0 \) as \( |x| \to \infty \).

Lastly, to conclude the proof, it remains to prove Lemma 4.3, \( \lambda_n \geq \lambda_D \). To this end, we first show that \( \lambda_n < \lambda_D \), \( \forall n \in \mathbb{N} \). Since \( \rho_n(x) \) is nonincreasing in \( n \), one sees that \( \lambda_n \) is nondecreasing in \( n \). Assume by contradiction that \( \lambda_n \geq \lambda_D \) for some \( n \). Let us denote by \( \varphi_n \) and \( \varphi \) respectively the principal eigenfunctions associated with \( \lambda_n \) and \( \lambda_D \), it holds that:

\[
\Delta \varphi_n + c\partial_1 \varphi_n + f(x, 0) \varphi_n = -\lambda_n \varphi_n \leq -\lambda_D \varphi_n, \quad x \in \Omega,
\]
and \( \varphi_n > 0 \) in \( \Omega \). Note that the existence of a positive eigenfunction associated with the generalized principal eigenvalue \( \lambda_n \) in unbounded domain is given in [11]. Because \( \varphi_n \) is a supersolution of equation satisfied by \( \varphi \), if there exists \( 0 < \kappa < +\infty \) such that \( \kappa \varphi_n \leq \varphi \) in \( \Omega \), one can enlarge \( \kappa \) until \( \kappa \varphi \) touches \( \varphi_n \) from below at some point. The strong maximum principle implies \( \kappa \varphi \equiv \varphi_n \) in \( \Omega \), which is impossible because \( \varphi = 0 \) on \( \partial \Omega \). In other words, \( \sup \{ \kappa \in (0, +\infty), \kappa \varphi \leq \varphi_n \) in \( \Omega \} = +\infty \). This yields another contradiction since \( \varphi > 0 \) in \( \Omega \). As a result, \( \lambda_n < \lambda_D, \forall n \in \mathbb{N} \). Next, we aim to show the limit \( \lim_{n \to \infty} \lambda_n = \lambda_D \). Since \( \lambda_n \) is nondecreasing and bounded from above, there exists \( \lambda_\infty = \lim_{n \to \infty} \lambda_n \leq \lambda_D \). We shall prove that \( \lambda_\infty = \lambda_D \).

Observe that, by the transformation \( \tilde{\varphi}_n = \varphi_n e^{\frac{x^2}{4}} \), we see that \( \tilde{\varphi}_n \) satisfies the equation

\[
\Delta \tilde{\varphi}_n + \rho_n(x) \tilde{\varphi}_n - \frac{c^2}{4} \tilde{\varphi}_n + \lambda_n \tilde{\varphi}_n = 0, \quad x \in \mathbb{R}^N.
\]  

(55)

Let us show that \( \tilde{\varphi}_n \) decays exponentially. Indeed, as in the proof of Theorem [11], let \( \tilde{\phi} \) be the function such that \( \inf_{\Omega} \tilde{\phi} > 0 \) and \( \Delta_{\Omega} \tilde{\phi} + \tilde{\mu}(y) \tilde{\phi} = 0 \) in \( \omega \), with \( \tilde{\mu}(y) \) sufficiently close to \( \mu(y) - \lambda \), where \( \lambda = \lambda_D(-\Delta_y - \mu(y), \omega) > 0 \). Then, one sees that, \( (\lambda_n, \tilde{\varphi}_n) \) is the principal eigenpair of the operator \( \tilde{L}_n = \Delta + \rho_n(x) - c^2/4 \) if and only if \( (\lambda_n, \phi_n) \), with \( \phi_n = \tilde{\varphi}_n/\tilde{\phi} \), is the principal eigenpair of the following operator

\[
\Delta + \frac{2\nabla \tilde{\phi} \cdot \nabla \tilde{\varphi}_n}{\tilde{\phi}} + \rho_n(x) \phi_n - \mu(y) \phi_n - \lambda \phi_n.
\]

Beside that, by assumptions (45) and (51), one has

\[
\limsup_{|x| \to \infty} \rho_n(x) \phi_n - \mu(y) \phi_n - \lambda \phi_n < 0 \quad < -\lambda_n.
\]

(56)

Hence, applying the Proposition 1.11 [11], we know that \( \lambda_n \) is simple, moreover \( \phi_n \) is unique (up to multiplications) and decays exponentially as \( |x| \to \infty \). It follows immediately that \( \tilde{\varphi}_n \) also decays exponentially.

On the other hand, since \( \lambda_n < \lambda_D < 0 \) and \( \rho_n(x) \) is nonincreasing in \( n \), one has

\[
\Delta \tilde{\varphi}_n + \rho_0(x) \tilde{\varphi}_n \geq 0.
\]

From above, \( \tilde{\varphi}_n \) is bounded and \( \tilde{\varphi}_n \) solves linear equation (55), we can normalize \( \varphi_n \) in such the way \( \sup_{\mathbb{R}^N} \varphi_n = 1 \). Thanks to conditions (45) and \( \limsup_{x \in \Omega, |x| \to \infty} \rho_0(x) < 0 \), the same argumentation of Proposition 8.6 in [11] may be applied to derive that there exists an exponential decay function \( \varphi \) depending only on \( \rho_0(x) \) such that \( \varphi_n \leq \varphi \) in \( \mathbb{R}^N \). From the equation (55), one has

\[
\int_{\mathbb{R}^N} |\nabla \varphi_n|^2 \leq \int_{\mathbb{R}^N} \rho_n(x) \varphi_n^2(x) + \lambda_n \int_{\mathbb{R}^N} \varphi_n^2(x) \leq \int_{\mathbb{R}^N} \rho_0(x) \varphi^2(x) < \infty
\]

(57)

This implies that there exists \( \varphi_\infty \in H^1(\mathbb{R}^N) \) such that \( \varphi_n \) converges up to subsequence to \( \varphi_\infty \) weakly in \( H^1(\mathbb{R}^N) \) and strongly in \( L^2(K) \) for all compact set \( K \subset \mathbb{R}^N \). For any compact set \( K \subset \Omega^c \), we derive from (57)

\[
-\max_K \rho_n \int_K \varphi_n^2 dx \leq -\int_K -\varphi_n^2 dx \leq -\int_{\mathbb{R}^N} |\nabla \varphi_n|^2 + \int_{\mathbb{R}^N \setminus K} \rho_n \varphi_n^2 dx \leq \sup_{\mathbb{R}^N} \rho_0 \int_{\mathbb{R}^N} \varphi_\infty^2 dx.
\]

32
Since, from (51) for all \( K \subset \Omega^c \), \(-\max_K \rho_n \to \infty\) as \( n \to \infty \), we have \( \varphi_\infty = 0 \) a.e in \( K \) and then a.e in \( \Omega^c \). Lastly, again from (55), one has

\[
\int_{\mathbb{R}^N} |\nabla \tilde{\varphi}_n|^2 \leq \int_{\mathbb{R}^N} \rho_n(x) \varphi_n^2(x) + \lambda_n - \frac{c^2}{4} \leq \int_{\mathbb{R}^N} \rho_0(x) \varphi_n^2(x) dx + \lambda_\infty - \frac{c^2}{4}.
\]

Since \( \tilde{\varphi}_n \leq \tilde{\varphi} \), we derive, by Lebesgue dominated convergence theorem that

\[
\int_{\mathbb{R}^N} \rho_0 \varphi_n^2(x) dx \to \int_{\mathbb{R}^N} \rho_0 \varphi_\infty^2(x) dx = \int_{\Omega} f_s(x, 0) \varphi_\infty^2(x) dx.
\]

Whence, the lower semicontinuity property yields

\[
\int_\Omega |\nabla \varphi_\infty|^2 = \int_{\mathbb{R}^N} |\nabla \varphi_\infty|^2 \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} |\nabla \tilde{\varphi}_n|^2 \leq \int_{\Omega} f_s(x, 0) \varphi_\infty^2(x) dx + \lambda_\infty - \frac{c^2}{4} \tag{58}
\]

By the Liouville transformation, \( \lambda_D \) is the principal eigenvalue of a self-adjoint operator. We know from [11] that it has a variational structure.

\[
\lambda_D = \inf_{w \in C^1_c(\Omega) \mid \|w\|_{L^2(\Omega)} = 1} \int_\Omega |\nabla w|^2 - f_s(x, 0) w^2 dx + \frac{c^2}{4}.
\]

Since \( C^1_c(\Omega) \) is dense in \( H^1_0(\Omega) \) and \( \varphi_\infty \in H^1_0(\Omega) \), there exists a sequence \( w_n \in C^1_c(\Omega) \) of \( L^2(\Omega) \)-norm equal to 1 converges to \( \varphi_\infty \) in \( H^1(\Omega) \). Combining with (58), we derive

\[
\lambda_D \leq \int_\Omega |\nabla w_n|^2 - f_s(x, 0) w_n^2 dx + \frac{c^2}{4} \to \int_\Omega |\nabla \varphi_\infty|^2 - f_s(x, 0) \varphi_\infty^2 dx + \frac{c^2}{4} \leq \lambda_\infty. \tag{59}
\]

Eventually, we obtain \( \lambda_D = \lambda_\infty \). This completes the proof. \( \square \)

**Remark 3.** In the proof, we have proved a result, which is stronger than what we need. In fact, to obtain the conclusion of Theorem (4.2), one only needs to prove \( \lambda_n < \lambda_D \). However, by proving Lemma (4.3), we obtain a more interesting result on the convergence of the eigenvalues and eigenfunctions. This indeed makes Theorem (4.2) more transparent and more interesting.

**Remark 4.** Inequality (59) in fact implies that

\[
\lambda_D = \inf_{w \in H^1_0(\Omega) \mid \|w\|_{L^2(\Omega)} = 1} \int_\Omega |\nabla w|^2 - f_s(x, 0) w^2 dx + \frac{c^2}{4}.
\]

By assumption (45) and the same arguments as (53)-(59), we can actually prove that \( \lambda_D \) is simple. This proof is referred to Proposition 1.11 [11]. In this case, due to the density of \( C^1_c(\Omega) \) in \( H^1_0(\Omega) \), the infima of variational characterization of \( \lambda_D \) taken over \( H^1_0(\Omega) \) and \( C^1_c(\Omega) \) are equivalent and equal to the simple \( \lambda_D \). This fact is not true in general, especially when (45) does not hold. Here, thanks to linear structure of Eq. (55), we can normalize \( \tilde{\varphi}_n \) of the sup-norm equal 1. Therefore, we can find a uniform exponential decay, \( \varphi \geq \tilde{\varphi}_n \), depending only on \( \rho_0(x) \), which helps us to compensate the lack of compactness of \( \Omega \) and \( \mathbb{R}^N \).

33
The last result is devoted to further investigating qualitative properties of the fronts of Eq. (6), namely the symmetry breaking in $x_1$ axis. To this aim, the monotonicity and exact asymptotic behavior of the fronts play the crucial role. From Proposition (2.1), we know that the fronts $U(x_1, y)$ decay exponentially as $x \to \pm \infty$. Therefore, natural questions may arise, which are the right conditions such that the fronts are monotone when $|x_1|$ large enough and whether they are symmetric in $x_1$ axis. These questions are addressed in the following by studying the asymptotic behavior of solutions as $x_1 \to \pm \infty$.

### 4.3 Symmetry breaking of the fronts

**Theorem 4.4.** Let $U$ be a traveling front solution of Eq. (4) with $f$ is such that

\[\begin{align*}
|f_s(x_1, y, 0) - \alpha(y)| &= O(e^{p|x|}) \quad \text{as } x_1 \to -\infty, \quad \text{and} \quad \lambda_\alpha = \lambda_1(-\Delta_y - \alpha(y), \omega) > 0 \quad \text{as } x_1 \to +\infty, \\
|f_s(x_1, y, 0) - \beta(y)| &= O(e^{-q|x|})
\end{align*}\]

uniformly in $y \in \omega$, for some $\alpha, \beta \in L^\infty(\omega)$, $p, q > 0$. We assume further that $s \to f(x, s) \in C^{1,r}(0, \delta)$ for some $r, \delta > 0$. Then, $U$ is asymmetric if

\[
\lambda_\beta \neq \lambda_\alpha + c^2 - 2c\sqrt{\lambda_\alpha + \frac{c^2}{4}},
\]

where $c$ is the given forced speed of traveling front.

**Proof.** We investigate at first the precise asymptotic behavior of solution of Eq. (6) on the branch $\Omega^-$. By analogy, we derive also the asymptotic behavior on the branch $\Omega^+$. According to Proposition (2.1) and Theorem A.1, Appendix, for any $\delta > 0$, we have shown that

\[C_{2,\delta} e^{\kappa_\delta x_1} \leq U(x_1, y) \leq C_{1,\delta} e^{\tau_\delta x_1} \quad \forall (x_1, y) \in \Omega^-,\]

where $\kappa_\delta = \sqrt{\lambda_\alpha + \delta + \frac{c^2}{4} - \frac{c}{2}}$ and $\tau_\delta = \sqrt{\lambda_\alpha - \delta + \frac{c^2}{4} - \frac{c}{2}}$. Since $\omega$ is bounded, we refer to [4], that there exists a unique (up to a multiplication) eigenfunction $\varphi$ with Neumann boundary condition associated to $\lambda_\alpha$:

\[
\begin{align*}
-(\Delta_y + \alpha(y))\varphi &= \lambda_\alpha \varphi \quad \text{in } \omega, \\
\partial_\nu \varphi &= 0 \quad \text{on } \partial \omega.
\end{align*}
\]

We rewrite Eq. (6) as in the following form

\[
\begin{align*}
MU &= -\Delta U - c\partial_1 U - \alpha(y)U = H(x_1, y) \\
H_f(x_1, y) &= f(x_1, y, U) - \alpha(y)U
\end{align*}
\]

By the regularity condition $s \to f(x, s) \in C^{1,r}(0, \delta)$ and condition (60), we have

\[
\begin{align*}
|H_f(x_1, y)| &\leq |f(x_1, y, U) - f_s(x_1, y, 0)\alpha(y)| + |f_s(x_1, y, 0)U - \alpha(y)U| \\
&\leq C_1 e^{(r+1)\tau_\delta x_1} + C_2 e^{(p+\tau_\delta) x_1} \leq C_3 e^{\sigma(x_1)} \quad \text{as } x \to -\infty,
\end{align*}
\]

34
where \( m(\tau_3) = \min\{(r + 1)\tau_3, p + \tau_3\} \). By Theorem 4.3 of [3], we can write \( U \) as follow

\[
U = u^0(x_1, y) + u^*(x_1, y),
\]

where \((u^0, u^*)\) is a solution of system

\[
\begin{align*}
\mathcal{M}u^0 &= 0 \quad x \in \Omega^- \\
\mathcal{M}u^* &= H_f(x_1, y) \quad x \in \Omega^- \\
\partial_n u^0 &= 0 \quad \partial \Omega^- \\
\partial_n u^* &= 0 \quad \partial \Omega^-.
\end{align*}
\]

(63)

Moreover, \( u^0 \) has a precisely exponential asymptotic behavior as \( x_1 \to -\infty \), namely there exist \( \lambda > 0 \) and \( \psi(x_1, y) = (-x_1)^k \psi_k(y) + \ldots + \psi_0(y) \neq 0 \) such that

\[
\begin{align*}
u^0(x_1, y) &= e^{\lambda x_1} \psi(x_1, y) + O(e^{\lambda x_1}) \\
\nabla u^0(x_1, y) &= \nabla (e^{\lambda x_1} \psi(x_1, y)) + O(e^{\lambda x_1}),
\end{align*}
\]

(64)

and for any \( \varepsilon > 0 \), \( u^* \) satisfies the inequality

\[
|u^*(x_1, y)| + |\nabla u^*(x_1, y)| \leq C_{c, \delta} e^{(m(\tau_3) - \varepsilon)x_1} \quad \text{for some } C_c > 0.
\]

(65)

Let us define

\[
\tau_0 = \sup\{\tau : \exists C_\tau \text{ such that } u(x_1, y) \leq C_\tau e^{\tau x_1} \text{ in } \Omega^-\}.
\]

The inequalities \((61)\) yields \( \kappa_\delta \leq \tau_0 \leq \tau_3 \), for any \( \delta > 0 \), thus \( \tau_0 \) is indeed a real number. We want to prove that \( \tau_0 = \sqrt{\lambda_\alpha + \frac{c^2}{4} - \frac{c}{2}} \). Taking \( \tau < \tau_0 \), then \( 0 \leq u(x_1, y) \leq C_\tau e^{\tau x_1} \) and moreover

\[
|\nabla u(x_1, y)| \leq C'_\tau e^{\tau x_1}
\]

by the Harnack inequality. Doing as \((62)\), we get : \( |H_f(x_1, y)| \leq C_4 e^{m(\tau)x_1} \), where \( m(\tau) = \min\{(r + 1)\tau, p + \tau\} > \tau \). As a result of \((65)\), we have

\[
|u^*(x_1, y)| + |\nabla u^*(x_1, y)| \leq D_\tau e^{(r + m(\tau))x_1}.
\]

One sees that as \( \tau \nearrow \tau_0 \), \( \frac{\tau + m(\tau)}{2} \nearrow \frac{\tau_0 + m(\tau_0)}{2} > \tau_0 \). Therefore, there exist \( \varepsilon > 0 \) and \( C_c > 0 \) such that

\[
|u^*(x_1, y)| + |\nabla u^*(x_1, y)| \leq C_c e^{(\tau_0 + \varepsilon)x_1} \quad \text{in } \Omega^-.
\]

(66)

It follows immediately that \( \forall \tau < \tau_0 \)

\[
|u^0(x_1, y)| \leq |u(x_1, y)| + |u^*(x_1, y)| \leq C_\tau e^{\tau x_1} + C_c e^{(\tau_0 + \varepsilon)x_1} \leq (C_\tau + C_c) e^{\tau x_1} \quad \text{in } \Omega^-.
\]

On the other hand, for \( \delta \) small enough

\[
u_0(x_1, y) = u(x_1, y) - u^*(x_1, y) \geq C_{2, \delta} e^{(\sqrt{\lambda_\alpha + \delta + \frac{c^2}{4} - \frac{c}{2}} + \frac{c}{2})x_1} - C_c e^{(\tau_0 + \varepsilon)x_1} \geq C_{3, \delta} e^{(\sqrt{\lambda_\alpha + \delta + \frac{c^2}{4} - \frac{c}{2}} + \frac{c}{2})x_1}.
\]

Applying Theorem 4.2 of [3] to \( u^0 \), we imply that there is exactly one positive constant \( \lambda \) such that \( \tau \leq \lambda \leq \sqrt{\lambda_\alpha + \delta + \frac{c^2}{4} - \frac{c}{2}} \), \( \forall \tau < \tau_0 \) and \((64)\) holds for a suitable exponential solution \( w(x_1, y) = e^{\lambda x_1} \psi(y) \). From \((66)\), \( \lambda \) cannot be strictly bigger than \( \tau_0 \), therefore we must have
\[ \lambda = \tau_0 > 0. \] Since \( u > 0 \), we deduce \( \psi_k > 0 \) and thus Theorem 2.4 of [5] yields that \( \psi(y) \) is a solution of
\[
\begin{cases}
- (\Delta_y + \alpha(y))\psi = (\lambda^2 + c\lambda)\psi & \text{in } \omega \\
\psi = 0 & \text{on } \partial \omega.
\end{cases}
\] (67)
Since \( \lambda_\alpha > 0 \), Theorem 2.1 of [5] implies that (67) possesses exactly one positive principal eigenvalue, that is \( \lambda = \frac{-c + \sqrt{c^2 + 4\lambda_\alpha}}{2} = \tau_0 \). We obtain the precisely asymptotic behavior of \( U(x_1, y) \) as \( x_1 \to -\infty \).

By analogy, we obtain the precisely exponential behavior of \( U(x_1, y) \) as \( x_1 \to +\infty \). It is precisely exponentially asymptotic as \( x_1 \to +\infty \) with the exponent \( \lambda' = \frac{-c - \sqrt{c^2 + 4\lambda_\beta}}{2} \). As a consequence, we have proved that
\[ U(x_1, y) \sim C_1 e^{-\left(\frac{c + \sqrt{c^2 + 4\lambda_\beta}}{2}\right)x_1} \text{ as } x \to +\infty; \quad U(x_1, y) \sim C_2 e^{\left(\frac{-c + \sqrt{c^2 + 4\lambda_\alpha}}{2}\right)x_1} \text{ as } x \to -\infty, \]
uniformly in \( y \). This result, in particular, implies that \( U(x_1, y) \) is increasing in \((-\infty, -R) \times \omega \) and decreasing in \((R_1, \infty) \times \omega \) for \( R, R_1 \) large enough. To achieve the symmetry in \( x_1 \), necessarily, we have
\[ \frac{c + \sqrt{c^2 + 4\lambda_\beta}}{2} = \frac{-c + \sqrt{c^2 + 4\lambda_\alpha}}{2} \iff \lambda_\beta = \lambda_\alpha + c^2 - 2c\sqrt{\lambda_\alpha + \frac{c^2}{4}}. \]
In other words \( U \) is asymmetric if \( \lambda_\beta \neq \lambda_\alpha + c^2 - 2c\sqrt{\lambda_\alpha + \frac{c^2}{4}} \).

**Remark 5.** We see that if \( c \neq 0 \), the asymmetry holds even when \( \lambda_\beta = \lambda_\alpha \). The drift term is therefore the main cause making the fronts asymmetric. However, we do not know that whether the front is symmetric when
\[ \lambda_\beta = \lambda_\alpha + c^2 - 2c\sqrt{\lambda_\alpha + \frac{c^2}{4}}. \]
The answer of this question requires more involved analysis. We state this as an open question. Our result applies, in particular, to show that the asymmetry holds when condition (60) becomes as (2), namely \( \lambda_\beta = \lambda_\alpha = m > 0 \) and
\[ |f_s(x_1, y, 0) + m| = O(e^{-p|x_1|}) \text{ as } |x_1| \to \infty, \text{ for some } m, p > 0. \]

**Remark 6.** We point out that, in assumption (60), the exponential rate of \( f_s(x_1, y, 0) \) converges to \( \alpha(y) \) and \( \beta(y) \) is important. Indeed, the precise exponential behavior of \( u_0 \) satisfying Eq. (63) in general may not be obtained like (64). For instance, in one dimensional space, \( \beta(y) \equiv -1/2 \), if \( f_s(x, 0) \) converges slowly to \(-1/2\), we can take \( w(x) = xe^{-x} \), which is a solution of
\[ w'' + \frac{1}{2}w' + g(x)w = 0 \quad \text{in } \mathbb{R} \setminus (2, -\infty), \quad g(x) = -\frac{1}{2} + \frac{1}{2x} \to \frac{1}{2} \text{ as } x \to +\infty. \]

5 Appendix

Theorem 5.1. Let $U$ be a traveling front solution of (64). Assume that (64) holds and $f$ is such that

$$\liminf_{a \to \pm \infty} f_s(x_1, y, 0) \geq a_\pm(y) \quad \text{and} \quad \lambda_{a_\pm} = \lambda_1(-\Delta_y - a_\pm(y), \omega) > 0,$$

for some functions $a_\pm \in L^\infty(\omega)$. Then, for any $\delta > 0$, there exist $A_\pm > 0$ and $\tau_{a_\pm} \geq \sqrt{\lambda_{a_\pm} + \delta + \frac{c^2}{4}}$ such that

$$U(x_1, y) \geq A_- e^{(\tau_{a_-} - \frac{c}{2})x_1} \quad \forall (x_1, y) \in \Omega^- \quad \text{and} \quad U(x_1, y) \geq A_+ e^{-(\tau_{a_+} + \frac{c}{2})x_1} \quad \forall (x_1, y) \in \Omega^+.$$

Proof. Since (64) holds, from Proposition (21), we know that $U(x_1, y)$ decays exponentially as $|x_1| \to \infty$. Let us denote $I = \{a_-, a_+\}$. We know from (64) that there exist the principal eigenfunctions $\varphi_i$ associated with $\lambda_i$ such that for $i \in I$

$$\begin{cases} -\Delta \varphi_i - i(y) \varphi_i = \lambda_i \varphi_i & \text{in } \omega \\ \partial_\nu \varphi_i = 0 & \text{on } \partial \omega. \end{cases}$$

For $i \in I, \delta > 0$, we set $\tilde{L}_i = \Delta_x + c\partial_1 + i(y) - \delta$. By assumption (68), there exists $R = R(\delta) > 0$ such that $\tilde{L}_a U \leq 0$ in $\Omega^- \setminus \Omega_R$ and $\tilde{L}_a U \leq 0$ in $\Omega^+ \setminus \Omega_R$. Define the functions

$$\omega_{a_-}(x) = e^{-\tau_{a_-} x_1} \varphi_{a_-}(y) \quad \text{and} \quad \omega_{a_+}(x) = e^{-\tau_{a_+} x_1} \varphi_{a_+}(y),$$

direct computation yields

$$\tilde{L}_{a_-} \omega_{a_-} = (\tau_{a_-}^2 - c\tau_{a_-} - \lambda_{a_-} - \delta) \omega_{a_-} \geq 0 \quad \text{in } \Omega^- \setminus \Omega_R \quad \text{if } \tau_{a_-} \geq \frac{\sqrt{c^2 + 4(\lambda_{a_-} + \delta)} - c}{2};$$

$$\tilde{L}_{a_+} \omega_{a_+} = (\tau_{a_+}^2 - c\tau_{a_+} - \lambda_{a_+} - \delta) \omega_{a_+} \geq 0 \quad \text{in } \Omega^+ \setminus \Omega_R \quad \text{if } \tau_{a_+} \geq \frac{\sqrt{c^2 + 4(\lambda_{a_+} + \delta)} + c}{2}.$$
Since \( z_{a_+}(x_1, y) \to 0 \) as \( x_1 \to -\infty \), \( z_{a_-}(x_1, y) \to 0 \) as \( x_1 \to +\infty \), and zero-order coefficients of elliptic operators with respect to \( z_i \) are negative, the weak maximum principle is applied to derive \( z_i \geq 0 \) in \( \Omega^+ \setminus \Omega_R \). As a consequence, there exist \( \tau_i \geq \sqrt{\lambda_i + \delta + \frac{c^2}{4}} \) such that

\[
\begin{cases}
C_{\delta, \alpha_i} e^{-(\tau_{a_+} + \frac{c}{2}) x_1} \varphi_{a_+}(y) \leq U(x_1, y) & \text{in } \Omega^+ \setminus \Omega_R, \\
C_{\delta, \alpha_i} e^{-(\tau_{a_-} - \frac{c}{2}) x_1} \varphi_{a_-}(y) \leq U(x_1, y) & \text{in } \Omega^- \setminus \Omega_R.
\end{cases}
\]

(69)

The Harnack inequality is applied to derive \( \inf_{|x_1| \leq R} U(x_1, y) > 0 \), we imply that \( C_{\delta, i} \) indeed can be chosen such that the inequality (69) holds respectively in \( \Omega^\pm \). The proof is complete.

**Theorem 5.2.** Let \( U \in W^{1,2}_{N+1, \text{loc}}(\mathbb{R} \times \mathbb{R}^N) \) be a solution of (14), where \( f \) is such that condition (12) holds true. Then there exist two positive constants \( k \) and \( \varepsilon \) such that

\[
\forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \quad U(t, x, y) \leq ke^{-\varepsilon |x_1|}.
\]

Proof. We only need to prove that the statement holds for \( x_1 \geq 0 \) and by analogy we also derive the result for \( x_1 \leq 0 \). Using the transformation \( V(t, x, y) = U(t, x_1, y)e^{\frac{c}{2} x_1} \), we see that \( V(t, x) \) is partially periodic in \( y \), \( T \)-periodic in \( t \) and satisfies the following equation

\[
\begin{cases}
V_t = \Delta V + f(t, x, Ve^{\frac{c}{2} x_1})e^{\frac{c}{2} x_1} - \frac{c^2}{4} V & t \in \mathbb{R}, x \in \mathbb{R}^N \\
Ve^{\frac{c}{2} x_1} \text{ is bounded.}
\end{cases}
\]

For any \( R > 0 \), \( \delta > 0 \), we denote \( Q_R = \mathbb{R} \times [0, R] \times \mathbb{R}^{N-1} \) and set \( \tilde{L} = \partial_t - \Delta - \gamma(y) - \delta \). From condition (19), there exist \( R = R(\delta) > 0 \) such that \( \tilde{L} V \leq 0 \), \( \forall (t, x) \in Q_R \). Moreover, since \( \lambda_\gamma = \lambda_1(\partial_t - \Delta - \gamma(y), \mathbb{R} \times \mathbb{R}^{N-1}) > 0 \), we refer to Theorems 2.7, [19] that there exists a unique space-time periodic eigenpair \( (\lambda_\gamma, \varphi) \) satisfying

\[
\begin{cases}
\tilde{L} \varphi = \lambda_\gamma \varphi \\
\varphi > 0 \\
\varphi(\cdot, \cdot + T) = \varphi \\
\varphi(\cdot + L_1 \varepsilon_1, \cdot) = \varphi 
\end{cases}
\]

for \( i \in \{1, N-1\} \).

Fix \( \tau \in \mathbb{R} \) and define the function \( \omega(t, x) = \theta_a(x_1) \varphi(t, y)e^{(\tau - \delta) \theta_a} \), where \( \theta_a : [R, R + a] \rightarrow \mathbb{R} \) is the solution of

\[
\begin{cases}
\theta''_a = (\kappa + \delta) \theta_a & \text{in } (R, R + a) \\
\theta_0(R) = C e^{\sqrt{\kappa} R} \\
\theta_0(R + a) = C e^{\sqrt{\kappa} (R + a)},
\end{cases}
\]

where \( C = \sup_{R \times \mathbb{R}^N} V(t, x)e^{-\frac{c}{2} x_1} / \inf_{R \times \mathbb{R}^{N-1}} \varphi(t, y) \), \( \kappa > 0 \) would be chosen later. Note that \( \inf_{R \times \mathbb{R}^{N-1}} \varphi(t, y) > 0 \) thanks to the periodicity of \( \varphi \) in \( y \) and \( t \). Direct calculation yields

\[
\theta_a(\rho) = C(e^{(\sqrt{\kappa} + \sqrt{\kappa + \delta}) R}) \left( 1 - \frac{e^{\sqrt{\kappa + \delta} a} - e^{-\sqrt{\kappa + \delta} a}}{e^{\sqrt{\kappa + \delta} a} - e^{-\sqrt{\kappa + \delta} a}} \right) e^{-\sqrt{\kappa + \delta} \rho} + \\
+ C(e^{(\sqrt{\kappa} - \sqrt{\kappa + \delta}) R}) \frac{e^{\sqrt{\kappa + \delta} a} + e^{\sqrt{\kappa} a}}{e^{\sqrt{\kappa + \delta} a} - e^{-\sqrt{\kappa + \delta} a}} e^{\sqrt{\kappa + \delta} \rho}.
\]

38
On the other hand, we have \( \tilde{L}\omega = (-\kappa - 3\delta + \lambda_\gamma + c^2/4) \omega \geq 0 \) if and only if \( \kappa \leq \lambda_\gamma + c^2/4 - 3\delta \).
Moreover, it is easily seen that \( \omega(t, x_1, y) \geq V(t, x_1, y) \) for \( t \leq \tau, x_1 \in \{R, R+a\} \). Taking \( 0 < \delta < \lambda_\gamma/3 \) we can choose \( \kappa \in (c^2/4, \lambda_\gamma + c^2/4 - 3\delta) \). Let \( W(t, x) = \omega(t, x) - V(t, x) = z(t, x)\varphi(t, y) \), one sees that \( z(t, x) \geq 0 \) for \( t \leq \tau, \; \varphi(t, y) > 0 \), and \( z(t, x) \neq 0 \) for \( t \leq \tau, \; \varphi(t, y) = 0 \). Since \( \inf_{t \leq \tau, x \in (R,R+a) \times R^{N-1}} \omega(t, x) > 0 \), there exists \( t_0(a) \ll \tau \), which may depend on \( a \) and sufficiently close to \( -\infty \) such that \( z(t_0(a), x) \geq 0 \) for \( x \in (R, R+a) \times R^{N-1} \). In addition that, we have

\[
0 \leq \frac{\tilde{L}W}{\varphi} = \partial_t z - \Delta z - 2\nabla z \cdot \nabla \varphi + \left( \lambda_\gamma - \delta + \frac{c^2}{4} \right) z.
\]

Since the zero-order’s coefficient of parabolic operator with respect to \( z \) is positive, we imply from the parabolic weak maximum principle that \( z(t, x) \geq 0 \) in \( (t_0(a), \tau) \times Q_{R+a} \setminus Q_R \) for every \( a > 0 \). Finally, the classical parabolic regularity implies that \( V(t, x) \leq \omega(t, x) \) for \( x \in Q_{R+a} \setminus Q_R \). Therefore,

\[
U(\tau, x) \leq \lim_{a \to +\infty} \theta_a(x_1)\varphi(\tau, y)e^{-\frac{a}{2}x_1} = C(e^{(\sqrt{R+\sqrt{\kappa+\delta}})R}e^{-(\sqrt{\kappa+\delta}+\frac{a}{2})x_1}.
\]

Since \( \tau \) can be chosen arbitrarily, we complete the proof. \( \square \)

**Remark 7.** In the proof of this theorem, we need not assume that the solution \( U \) is partially periodic in \( y \) and \( T \)-periodic in \( t \), but the local regularity of solutions plays an important role. On the other hand, as seen from above, it is possible to choose \( \kappa = \lambda_\gamma + c^2/4 - 3\delta \) to obtain

\[
U(\tau, x) \leq C_1e^{-\left(\sqrt{\lambda_\gamma + \frac{c^2}{4}} - 2\delta \right)x_1} \quad \text{for } \tau \in \mathbb{R}, x_1 \geq R, \; y \in \mathbb{R}^{N-1}
\]

Using the same arguments, we derive that there exist \( C_2 > 0 \):

\[
U(\tau, x) \leq C_2e^{\left(\sqrt{\lambda_\gamma + \frac{c^2}{4}} - 2\delta \right)x_1} \quad \text{for } \tau \in \mathbb{R}, x_1 \leq -R, \; y \in \mathbb{R}^{N-1}.
\]

Since \( U \) is bounded, one can choose \( C_1, C_2 \) large enough such that these inequalities hold in \( \mathbb{R}^{N+1} \).

**Acknowledgements.** The research presented in this paper is a part of the PhD work. The author thanks professor Henri Berestycki for suggesting the problem and useful advices. He is also thankful to Luca Rossi for many interesting discussions and proof reading. This work is supported by FIRST program of Marie Curie 7th framework of European Commission, grant agreement 238702. He also thanks Technion-Israel Institute of Technology and Technische Universiteit Eindhoven for their encouragements, friendly and stimulating atmosphere during his visits.

**References**

[1] H. Berestycki, O. Diekman, K. Nagelkerke, P. Zegeling. Can a Species Keep Pace with a Shifting Climate ? In *Bull. Mathematical Biology* 71, 2008, p. 399.
[2] H. Berestycki, F. Hamel, L. Rossi. Liouville type results for semilinear elliptic equations in unbounded domains. *Annali Mat. Pura Appl.*, (4), 186 (2007), pp 469-507.

[3] H. Berestycki, F. Hamel, and L. Roques. Analysis of the periodically fragmented environment model. I. Species persistence. *J. Math. Biol.*, 51(1):75–113, 2005.

[4] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.*, 47(1):47–92, 1994.

[5] H. Berestycki, L. Nirenberg. traveling fronts in cylinders. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9 (1992), no. 5, 497-572.

[6] H. Berestycki, L. Nirenberg. Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains, *Analysis, Et Cetera ed. P. Rabinowitz, et al.*, Academic Press, Boston, 1990, pp. 115164.

[7] H. Berestycki, P.-L. Lions. Nonlinear scalar field equations, I - Existence of a ground state. *Arch. Rational Mech. Anal.* 82, (1983), p.313345.

[8] H. Berestycki and P.-L. Lions. Some applications of the method of super and subsolutions. *Bifurcation and nonlinear eigenvalue problems. (Proc., Session, Univ. Paris XIII, Villetaneuse, 1978), Lecture Notes in Math.*, 782, Springer, Berlin, 1980, 1641.

[9] H. Berestycki, L. Rossi. Reaction-diffusion equations for population dynamics with forced speed, I - The case of the whole space. *Discrete and Continuous Dynamical System Series B*, 21, 2008, p. 41-67.

[10] H. Berestycki, L. Rossi. Reaction-diffusion equations for population dynamics with forced speed, II - Cylindrical type domains. *Discrete and Continuous Dynamical Systems Series A*, 25, 2009, p. 19-61.

[11] H. Berestycki, L. Rossi. Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains. *Comm. Pure Appl. Math.*, to appear.

[12] H. Brezis. Functional analysis, Sobolev spaces and partial differential equations. *Universitext. Springer*, New York, 2011. xiv+599 pp. ISBN: 978–0–387–70913–0.

[13] G. Chapuisat Existence and nonexistence of curved front solution of a biological equation. J. Differential Equations 236 (2007), no. 1, 237–279.

[14] Y. Du, R. Peng. The periodic logistic equation with spatial and temporal degeneracies. *Trans. Amer. Math. Soc.* 364 (2012), no. 11, 6039–6070.

[15] D. Gilbarg and N. S. Trudinger. "Elliptic partial differential equations of second order 2nd edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Vol. 224, Springer-Verlag, Berlin, 1983. 48(3):497–521, 1981.
[16] JS. Guo and F. Hamel. Propagation and blocking in periodically hostile environments. *Arch. Ration. Mech. Anal.* 204 (2012), no. 3, 945–975.

[17] P. Hess. Periodic-parabolic boundary value problems and positivity vol. 247. *Longman Scientific and Technical*, London (1991).

[18] G. M. Lieberman, Second Order Parabolic Differential Equations, *World Scientific Publishing Co. Inc.*, River Edge, NJ, 1996.

[19] G. Nadin The principal eigenvalue of a space-time periodic parabolic operator, *Ann. Mat. Pura Appl*. 188(4) (2009).

[20] G. Nadin Existence and uniqueness of the solutions of a space-time periodic reaction-diffusion equation, *J. Differential Equations*. 249(6) (2010), pp 1288-1304.

[21] R.Peng, D.Wei The periodic-parabolic logistic equation on $\mathbb{R}^N$. *Discrete Contin. Dyn. Syst*. 32 (2012), no. 2, 619–641.

[22] A. B. Potapov, M. A. Lewis, Climate and competition: the effect of moving range boundaries on habitat invasibility. *Bull. Math. Biol*, 66 (2004), no. 5, 9751008.

[23] M. H. Protter and H. F. Weinberger. Maximum principles in differential equations. *Book*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.

[24] D.H. Sattinger. Topics in stability and bifurcation theory *Book*, Springer, 1973.

[25] N. Shigesada and K. Kawasaki Biological Invasions; Theory and Practice. Oxford University Press, 1997.