Abstract

The purpose of this note is to rephrase Speyer’s elegant topological proof for Kasteleyn’s Theorem in a simple graph theoretical manner.
Speyer’s elegant topological proof for Kasteleyn’s Theorem

Markus Fulmek

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Speyer [2] recently published a very short and elegant proof for Kasteleyn’s Theorem [1]. This proof involves some higher topology arguments; and the purpose of this note is to replace them by elementary graph–theoretical arguments. We shall, however, present only a sketch of a proof, since the technical details of a fully rigorous proof would likely conceal the beauty and simplicity of the basic idea.

1 Basic definitions and notation

For reader’s convenience and to fix our notation, we shall briefly recall some basic definitions.

Graph

A finite graph $G$ consists

- of a finite set of vertices $V(G)$
- and a finite (multi–)set of edges $E(G)$,

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where each $e \in E(G)$ joins two different vertices $v_1, v_2 \in V(G)$ (we also say that $e$ is incident with $v_1$ and $v_2$), which we denote by $e = \{v_1, v_2\}$. We shall not consider loops, which are edges "joining a single vertex" ($e = \{v\}$): If such loop arises during the constructions described below, we shall remove it immediately.

**Graph embedding**

A finite graph $G$ can always be drawn in the plane $\mathbb{R}^2$, such that

- the vertices correspond bijectively to points in the plane (we shall call them vertex–points),
- and the edges correspond bijectively to smooth curves $[0, 1] \to \mathbb{R}^2$ in the plane (we shall call them edge–curves) which connect the respective vertex–point, i.e., some edge $e = \{v_1, v_2\}$ corresponds to an edge–curve
  - which starts in the vertex–point corresponding to $v_1$ and ends in the vertex–point corresponding to $v_2$ (or vice versa: the orientation of the curve is irrelevant in our context),
  - but does not "intersect itself" (i.e., does not pass twice through the same point; in particular, edge–curves do not “return” to their starting point or end point),
  - and does not pass through any other vertex–point.

Such drawing is called an embedding of the graph $G$ in the plane.

By definition, the edge–curves of edges incident to the same vertex $v$ have the vertex–point corresponding to $v$ in common, but there might be other such intersections in nonvertex–points.

For our purpose, we additionally assume that:

1. Any two curves intersecting in some nonvertex–point $p$ do actually cross each other (i.e., they are not tangent in $p$), whence we call such point $p$ a crossing of the two curves.
2. There is only a finite number of such crossings.
Figure 1: Example of a proper embedding of some graph $G$ in the plane. Let $V(G) = \{v_1, v_2, \ldots, v_6\}$ and $E(G) = \{\{v_1, v_2\}, \{v_2, v_3\}^2, \{v_3, v_4\}, \{v_5, v_6\}\}$, where the notation $\{v_2, v_3\}^2$ indicates that there are two edges joining vertices $v_2$ and $v_3$. The picture shows a proper embedding of $G$ with a single crossing $p$: The points corresponding to the vertices are indicated by black dots, the curves corresponding to edges are indicated by gray lines, and the crossing is indicated by a small white circle.

We call an embedding which fulfills these additional assumptions a *proper embedding*.

See Figure 1 for an illustration of these concepts: In all graphical representations of proper embeddings in this note, we shall indicate

- vertex–points by *black dots*,
- curves corresponding to edges by *gray lines*,
- and crossings by *small white circles*.

**Simplified notation**

If we are given some graph $G$ and an embedding $\eta$ of $G$, let us denote

- by $\eta(v)$ the vertex–point representing vertex $v \in V(G)$,
- by $\eta(e)$ the edge–curve representing edge $e \in E(G)$.

But we shall abuse this notation in the following by making no distinction
• between \( v \in V(G) \) and \( \eta(v) \), i.e., instead of saying “the vertex–point in the plane representing \( v \)” we simply shall say “\( v \)”,

• and between \( e \in E(G) \) and \( \eta(e) \), i.e., instead of saying “the edge–curve in the plane representing \( e \)” we simply shall say “\( e \)”.

This slight imprecision should cause no confusion in our context.

**Planar graph**

A proper embedding without any intersection of edge–curves in non–vertex points is called a *planar embedding*. A graph which admits a planar embedding is called a *planar graph*.

Not all graphs are planar, but we may view every proper embedding of some graph \( G \) as planar embedding of a planar graph \( G' \) by reinterpreting all crossings as vertex–points: \( G = G' \) (in the sense of graph isomorphisms) if the embedding has no crossings at all. (In Figure 1, this reinterpretation would amount to replacing the single crossing \( p \) by a new vertex–point \( v_7 \).)

**Perfect matching**

A *perfect matching* \( M \subseteq E(G) \) of \( G \) is a subset of edges such that every vertex of \( G \) is incident with precisely one edge in \( M \).

**Edge weight**

We assume that we are given some (nowhere–zero) *weight function* (called *edge weight*) \( \omega \)

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\omega : E(G) \to R \setminus \{0\}
\]
on the multiset of edges of \( G \), where \( R \) is some (nontrivial) commutative ring (in most cases, \( R \) is \( \mathbb{Z} \) or some polynomial ring; the constant edge weight \( \omega \equiv 1 \) is used for enumeration purposes).
Generating function of perfect matchings

The weight of some perfect matching $M \subseteq E(G)$ is the product of the weights of the edges in $M$:

$$\omega(M) := \prod_{e \in M} \omega(e).$$

The generating function of perfect matchings of some graph $G$ with edge weight $\omega$ is defined as

$$m(G, \omega) := \sum_{M} \omega(M),$$

where $M$ ranges over all perfect matchings of $G$.

Sign of perfect matchings

Assuming a fixed proper embedding $\eta$ of graph $G$, the sign of some perfect matching $M$ of $G$ in the embedding $\eta$ is defined as

$$\text{sgn}(M, \eta) := (-1)^{C(M, \eta)},$$

where $C(M, \eta)$ is the number of all crossings of all edges in $M$, i.e., denoting by $\text{img}(e)$ the image of the (edge–curve corresponding to) edge $e$:

$$C(M, \eta) := \sum_{\{e_1, e_2\} \subseteq M} |\text{img}(e_1) \cap \text{img}(e_2)|. \quad (1)$$

It is possible to modify some proper embedding $\eta$ to another embedding $\eta'$ so that the signs of all perfect matchings are the same for $\eta$ and $\eta'$: See Figure 2 for a simple example of such sign-preserving modification.

Signed generating function of perfect matchings

The signed generating function of perfect matchings $s(G, \omega, \eta)$ (which depends not only on the graph $G$, but also on the proper embedding $\eta$) is defined as

$$s(G, \omega, \eta) := \sum_{M} \text{sgn}(M, \eta) \cdot \omega(M).$$
Figure 2: Untangling of a multiple crossing. The left picture shows four edge–curves passing through the same nonvertex–point: Observe that this situation can be “modified locally” (in the sense that the embedding is left unchanged outside some small neighbourhood of the crossing) to a situation like the one shown in the right picture, and that such “local modification” does not change the sign of any perfect matching. For instance, if we assume that the edges shown here belong to some perfect matching $M$, both “local situations” contribute a factor $(-1)$ to the sign of $M$ for every two–element subset of the four curves (i.e., $(-1)^{\binom{4}{2}} = (-1)^6 = 1$).

where $M$ ranges over all perfect matchings of $G$.

Clearly, $m(G, \omega) = s(G, \omega, \eta) \equiv 0$ if $|V(G)|$ is odd.

**Stembridge’s embedding and the Pfaffian**

For every graph $G$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ there is a specific proper embedding which we shall call Stembridge’s embedding (see [3]): Every edge $e = \{v_i, v_j\}$, $i \neq j$, is represented by the half–circle in the upper half–plane with center $\left(\frac{i+j}{2}, 0\right)$ and radius $\frac{|i-j|}{2}$, and every vertex $v_i$ is represented by the point $(i, 0)$. (See the left picture in Figure 3 for an example of Stembridge’s embedding.)

Denote this specific embedding by $\eta$: The Pfaffian of a graph $G$ is defined as the signed generating function for the embedding $\eta$, i.e.:

$$\text{Pf}(G, \omega) := s(G, \omega, \eta).$$
Figure 3: Stembridge’s embedding and the Pfaffian.
The left picture shows Stembridge’s embedding $\eta$ of a graph $G$.
The right picture shows a specific perfect matching $M$ of $G$ for which the number $C(M,\eta)$ of crossings according to (1) equals 3, whence the sign $\text{sgn}(M,\eta)$ of this matching is $(-1)^3 = (-1)$.

Sign–modifications of edge weights and Kasteleyn’s Theorem

Let $G$ be some simple graph with edge weight $\omega$: Another edge weight $\omega'$ is called a sign–modification of $\omega$ if for all $e \in E(G)$ there holds

$$\omega'(e) = \omega(e) \text{ or } \omega'(e) = -\omega(e).$$

Then we may formulate Kasteleyn’s Theorem [1] as follows:

**Theorem 1** (Kasteleyn’s Theorem). Let $G$ be a planar finite simple graph with edge weight $\omega$. Let $\eta$ be an arbitrary proper embedding of $G$. Then there exists a sign–modification $\omega'$ of $\omega$ such that

$$m(G, \omega) = \text{Pf}(G, \omega').$$

Note that for any planar embedding $\eta$ of $G$ we have $m(G, \omega) = s(G, \omega, \eta)$.

We will rephrase Speyer’s elegant argument [2] to give a sketch of proof for the following slight generalization:

**Theorem 2.** Let $G$ be a finite simple graph with edge weight $\omega$. Let $\eta$ be a proper embedding of $G$. Then for every proper embedding $(G, \eta')$ there is a
sign–modification $\omega'$ of $\omega$ such that

$$s(G, \omega, \eta) \equiv s(G, \omega', \eta').$$

2 Speyer’s argument, in simple pictures

The simple idea for the proof of Theorem 2 is to successively transform the embedding $\eta$

- by local modifications of the embedding
- and corresponding modifications of the edge weight (if necessary)

such that the signed generating function of perfect matchings stays unchanged, until the embedding $\eta'$ is obtained.

Note that smooth bijections $\mathbb{R}^2 \to \mathbb{R}^2$ will transform the embedding without changing the (numbers of) crossings, thus leaving the signed generating function of perfect matchings unchanged. More informally: If we think of the embedding as a “web of infinitely thin and ductile strings” (corresponding to the edges) glued together at their endpoints (corresponding to the vertices), then we can imagine that we may “deform and drag around” these strings, and such transformation will not change the signed generating function of perfect matchings if we do not remove crossings or introduce new ones.

Moreover, there are modifications which leave the sign of every perfect matching unchanged, but cannot be achieved by a continuous mapping $\mathbb{R}^2 \to \mathbb{R}^2$. One example is the “untangling of a multiple crossing” illustrated in Figure 2.

In the following, we shall consider “local modifications of proper embeddings”. For simplicity, we assume that every crossing is the intersection of precisely two edge–curves (which we can achieve by the “untangling of a multiple crossing” illustrated in Figure 2) without changing the signed generating function we are interested in).
Crossing edges incident to the same vertex

If two edges $e_1, e_2$ have a vertex $v$ in common, then they can never both belong to the same perfect matching, hence crossings of $e_1, e_2$ are irrelevant for the sign of any perfect matching. If the curve–segments between $v$ and some crossing $p$ do not contain any other crossing, then $p$ can simply be removed (or introduced) by the modification illustrated in the following picture:

This modification has no effect on the sign of any perfect matching. If we modify the situation “from left to right”, we shall call this straightening out a single crossing of $e_1$ with $e_2$.

Self–crossing edges

We ruled out self–intersections of edge–curves for proper embeddings, but they might arise by one of the modifications described below. The following picture makes clear that we can “change the situation locally” and remove (or introduce) such self–crossing.

This modification has no effect on the sign of any perfect matching, since only crossings of different edges contribute to the sign. If we modify the situation “from left to right”, we shall call this straightening out a single self–crossing of $e$. 

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Dragggin a segment of an edge–curve over another segment

Assume that segments of edges $e$, $e'$ can be “dragged over one another” such that precisely two crossings arise (or vanish), as the following picture illustrates:

This modification has no effect on the sign of any perfect matching. If we modify the situation “from left to right”, we shall call this straightening out a double crossing of $e$ with $e'$.

Transition of a vertex through an edge

Assume that a segment of some edge $e$ can be “dragged over some vertex $v$” such that precisely one crossing arises (or vanishes) for every edge incident with $v$, as the following picture illustrates:

Since every perfect matching $M$ must contain precisely one edge incident to $v$, the sign of $M$ is reversed by this modification if and only if $e \in M$: But we can easily offset this by replacing $\omega(e)$ by $-\omega(e)$.
Figure 4: Three different embeddings of the same graph $G$.

The graph $G$ we are considering here is the complete bipartite graph $K_{3,3}$: The fact that $K_{3,3}$ is not planar corresponds to the fact that we are unable to present an embedding without crossings.

3 Sketch of proof by example

A rigorous proof of Theorem 2 would involve tedious technical details: We shall avoid them by only illustrating the idea of proof, i.e, the “successive modification of embeddings”, in an example.

The example we shall consider is the complete bipartite graph $K_{3,3}$: Three different proper embeddings of this graph are shown in Figure 4.

Observe that the embedding shown in the middle of Figure 4 can easily be transformed to the embedding shown in the right by the “untangling of a multiple crossing” illustrated in Figure 2.

Figure 5 illustrates the process of successively modifying the embedding $\eta$ shown in the left picture of Figure 4 until the embedding $\eta'$ shown in the middle picture of Figure 4 is obtained. We shall explain the single steps in the following, where we denote the edge connecting vertices $v_i$ and $v_j$ by $e_{i,j}$.

The left upper picture in Figure 5 shows the result of a smooth deformation of the plane, where vertices $v_2$, $v_4$ and $v_6$ already have arrived at their desired positions (according to the hexagonal configuration in $\eta'$). Now we want to drag $v_5$ to its desired position along the path indicated by the dashed arrow. This involves a transition of $v_5$ through edge $e_{3,6}$, whose weight must therefore change its sign.

The upper picture in the middle shows the result of this operation: Note
Figure 5: Example: Successive modification of embeddings.
Consider the embeddings of the graph $G$ depicted in Figure 4. We want to transform the embedding $\eta$ shown in the left picture to the embedding $\eta'$ shown in the middle picture of Figure 4.
that we may now straighten out the double crossing of $e_{4,5}$ with $e_{3,6}$ and the single crossing of $e_{5,6}$ with $e_{3,6}$.

Now we want to drag $v_1$ to its desired position along the path indicated by the dashed arrow. This involves a transition of $v_1$ through edge $e_{2,3}$, whose weight must therefore change its sign. Note that after performing this operation, we may straighten out the double crossing of $e_{1,4}$ with $e_{2,3}$.

The right upper picture shows the result of these operations: Observe that we may now straighten out the double crossing of $e_{2,3}$ with $e_{1,6}$ and the single crossing of $e_{1,2}$ with $e_{2,3}$.

Now we want to drag $v_3$ to its desired position along the path indicated by the dashed arrow: This can be done without introducing or removing any crossing.

The left lower picture shows the result of these operations. By dragging $e_{1,6}$ over vertex $v_4$ and straightening out the single crossing with $e_{1,4}$ thus introduced, we arrive at the lower picture in the middle: This involves a transition of $v_4$ through $e_{1,6}$, whose weight must therefore change its sign.

Now dragging $e_{3,6}$ over vertices $v_4$ and $v_5$ (and changing the sign of $\omega(e_{3,6})$ twice, accordingly), and straightening out

- the two single crossings of $e_{3,6}$ with $e_{5,6}$ and $e_{3,4}$
- and the double crossing of $e_{3,6}$ with $e_{4,5}$

introduced by this modification gives the right lower picture.

Repeating the last step by dragging $e_{2,5}$ over vertices $v_3$ and $v_4$, we arrive at the right proper drawing in Figure 4.

References

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