Study of the family of Nonlinear Schrödinger equations by using the Adler-Kostant-Symes framework and the Tu methodology and their Non-holonomic deformation

Partha Guha*
S.N. Bose National Centre for Basic Sciences
JD Block, Sector III, Salt Lake
Kolkata - 700098, India

Indranil Mukherjee †
Department of Natural Science
West Bengal University of Technology
BF 142, Salt Lake, Kolkata-700064, India.

today

Abstract

The objective of this work is to explore the class of equations of the Non-linear Schrödinger type by employing the Adler-Kostant-Symes theorem and the Tu methodology. In the first part of the work, the AKS theory is discussed in detail showing how to obtain the non-linear equations starting from a suitably chosen spectral problem. Equations derived by this method include different members of the NLS family like the NLS, the coupled KdV type NLS, the generalized NLS, the vector NLS, the Derivative NLS, the Chen-Lee-Liu and the Kundu-Eckhaus equations. In the second part of the paper, the steps in the Tu methodology that are used to formulate the hierarchy of non-linear evolution equations starting from a spectral problem, are outlined. The AKNS, Kaup-Newell, and generalized DNLS hierarchies are obtained by using this algorithm. Several reductions of the hierarchies are illustrated. The famous trace identity is then applied to obtain the Hamiltonian structure of these hierarchies and establish their complete integrability. In the last part of the paper, the non-holonomic deformation of the class of integrable systems belonging to the NLS family is studied. Equations examined include the NLS, coupled KdV-type NLS and Derivative NLS (both Kaup-Newell and Chen-Lee-Liu equations). NHD is also applied to the hierarchy of equations in the

*E-mail: partha@bose.res.in
†E-mail: indranil.m11@gmail.com
AKNS system and the KN system obtained through application of the Tu methodology. Finally, we discuss the connection between the two formalisms and indicate the directions of our future endeavour in this area.

Mathematics Subject Classifications (2000): 35Q53, 14G32.

Keywords and Keyphrases. Adler-Kostant-Symes scheme, Nonlinear Schrödinger equation, loop groups, bihamiltonian system, Tu methodology, Trace Identity, Non-holonomic deformation, Differential constraints.

Contents

1 Introduction 3
  1.1 History of NLSE 5
  1.2 Background of the NHD formalism 7
  1.3 Organization 8

2 Adler-Kostant-Symes Scheme 8
  2.1 AKS theory and loop algebra 10
  2.2 Applications to loop group 11
  2.3 AKS equation with cocycle 12
  2.4 Hermitian Symmetric Spaces and Integrable Systems 14

3 Examples of integrable systems 15
  3.1 Non-linear Schrodinger Equation 16
  3.2 Coupled KdV type NLS equations 17
  3.3 Generalized Nonlinear Schrödinger Equation 17
  3.4 Dimensionless Vector Nonlinear Schrödinger Equation (VNLSE), a Manakov system 18
  3.5 Derivative NLS equation (DNLS Eqn) 19
  3.6 Chen-Lee-Liu (CLL) type DNLS equation 19
  3.7 Kundu-Eckhaus Equation 20

4 Tu methodology 21
  4.1 The Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy 21
  4.2 The Derivative Non-linear Schrödinger Equation (Kaup-Newell hierarchy) 23
  4.3 The Derivative Non-linear Schrödinger equations - a general structure 25

5 The trace identity 27
  5.1 The Hamiltonian structure of the AKNS hierarchy 28
  5.2 Hamiltonian structure of the Kaup-Newell hierarchy 30
  5.3 The Hamiltonian formalism for the general structure of the DNLS equations 31
6 The Non-holonomic Deformation of Integrable Systems

6.1 Non-Holonomic Deformation (NHD) of the Non-linear Schrödinger Equation (NLSE) .................................................. 34
6.2 NHD of coupled KdV type NLSE .................................................. 36
6.3 NHD of Derivative NLS equation (DNLS) : Kaup-Newell (KN) system .................................................. 37
6.4 NHD of Chen-Lee-Liu (CLL) system .................................................. 38
6.5 NHD of the hierarchy of equations in the AKNS system .................................................. 39
6.6 NHD of the hierarchy of equations in the DNLS system (Kaup-Newell hierarchy) .................................................. 40

7 Discussion 41

1 Introduction

Completely integrable systems play an important role in many physical applications including water waves, plasma physics, field theory and nonlinear optics. An important feature of many integrable evolution equations is that a large class of their exact solutions, particularly the solitons, can be derived by applying the method of inverse scattering transform (IST) in appropriate variables [1, 2, 10]. One of the most fascinating features of integrable hierarchies is the fact that they possess a local bihamiltonian structure [41]; these, in turn, yield the recursion operator and an infinite set of conserved quantities. The bihamiltonian structure is a consequence of the existence of classical $r$-matrices on the loop algebra. The applications of Gelfand-Zakharevich [17, 18, 19] bi-Hamiltonian structure, which is an extension of a Poisson-Nijenhuis structure on phase space, has been extensively explored by Falqui, Magri and Pedroni [11, 12, 13] in the context of separation of variables. In [24] we unveil the connection between Adler-Kostant-Symes (AKS) formalism applied to loop algebra and the Gelfand-Zakharevich bi-Hamiltonian structure by superposition of the results of Fordy and Kulish [16] in the AKS scheme. Fordy-Kulish decomposition has been demonstrated for the third-order flow in [5]. Athorne and Fordy [6] generalized this to $(2 + 1)$-dimensions and demonstrated how $N$-wave, Davey-Stewartson, and Kadomtsev-Petviashvili (KP) equations are associated with homogeneous and symmetric spaces. We have also shown [21, 22] that the AKS scheme also yields various $(1 + 1)$ dimensionl integrable equations which are various reductions of the SDYM equation.

It is well known that a systematic procedure of obtaining most finite dimensional completely integrable systems is given by the Adler, Kostant and Symes (AKS) theorem [3, 4, 31] applied to some Lie algebra $\mathfrak{g}$ equipped with an ad-invariant non-degenerate bi-linear form. AKS scheme provides a family of integrable systems, each consisting of a homogeneous space with a hierarchy of flows generated by the $ad^*$-invariant functions. We assume $\mathfrak{g}$ be a vector space, presented as the linear sum of two subalgebras $\mathfrak{g} = \mathfrak{k} + \mathfrak{l}$. The bilinear form induces an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$. Hence with the help of the bi-linear form $\langle , \rangle$ we can identify $\mathfrak{k}^* \sim \mathfrak{l}^\perp$ and $\mathfrak{l}^* \sim \mathfrak{k}^\perp$ where

\[ \langle \mathfrak{k}^\perp, \mathfrak{k} \rangle = \langle \mathfrak{l}^\perp, \mathfrak{l} \rangle = 0. \]  (1)
So $\mathfrak{t}^\perp$ acquires a Poisson structure from that of $\mathfrak{l}^*$. The co-adjoint action of $L$ on $\mathfrak{t}^\perp \sim \mathfrak{l}^*$ is given by

$$g \circ p = \pi_{\mathfrak{k}^\perp}(gpg^{-1})$$

for $g \in L$ and $p \in \mathfrak{t}^\perp$. Then the infinitesimal action is

$$\eta(p) = \pi_{\mathfrak{t}^\perp}[\eta, p]$$

for $\eta \in \mathfrak{l}$.

The symplectic manifold here is some co-adjoint $L$-orbit $\mathcal{M} \subset \mathfrak{t}^\perp \simeq \mathfrak{l}^*$. We associate to it a Hamiltonian equation of suitable ad-invariant function $f : \mathfrak{g} \rightarrow \mathbb{R}$ for all $f|_\mathcal{M}$. Note that in this paper our Lie algebra $\mathfrak{g}$ is a loop algebra.

As we mentioned previously, many important equations can be derived from this approach, e.g. Adler and van Moerbeke [3] obtained the Euler-Arnold equation as a geodesic flow on an ellipsoid, Ratiu [46] obtained C. Neumann equation and so on. These are all finite-dimensional systems. In order to use this technique to obtain partial differential equations, it is necessary to work with infinite-dimensional Lie algebras (IDLA). This was demonstrated on loop algebras by Reyman et al. [47, 48] and Flashchka et al. [14]. Hence AKS proves to be a very general systematic procedure of obtaining many completely integrable Hamiltonian system.

It is also a well known fact that starting from a properly chosen spectral problem, one can set up a hierarchy of non-linear evolution equations. Obviously one of the most important challenges in the study of integrable systems is to find new such systems associated with non-linear evolution equations of physical significance. Another important issue in this context is to demonstrate the bi-Hamiltonian structure of the derived non-linear evolution equations which proves its complete integrability. When a set of non-linear evolution equations can be formulated as a Hamiltonian system in two distinct but compatible ways, then by a theorem due to Magri [41], they lead to an infinite sequence of conserved Hamiltonians that are in involution w.r.t either one of these two symplectic structures. One powerful approach for constructing infinite-dimensional Liouville integrable Hamiltonian systems is the one due to Tu [51, 52]. In this method, one uses the Trace Identity to derive the Hamiltonian structure of many integrable systems starting from an appropriate spectral problem. The related hierarchy of non-linear evolution equations can also be derived.

The motivation for the present work is to explore the family of Non-linear Schrödinger equations by using both the Adler-Kostant-Symes technique and the Tu methodology in parallel. The equations derived in the AKS framework are the Non-linear Schrödinger equation (NLSE), the coupled KdV type NLSE, the generalized NLSE, the vector NLSE, the Derivative NLSE, the Chen-Lee-Liu (CLL) type DNLS and the Kundu-Eckhaus equations.

The Tu methodology is used to first establish the AKNS hierarchy and then derive the NLS equations and coupled KdV type NLS equations as special cases. The AKNS
hierarchy was introduced in 1974 in [2]. In that work, the authors generalized the inverse scattering approach of Zakharov and Shabat that was developed in [55] for a solution of the NLS equation (for details, see [1]). The GNLS is interpreted as a combination of the ordinary NLS and the coupled KdV type NLS equations. The Tu formalism is next used to derive the hierarchy of Kaup-Newell (KN) type non-linear evolution equations and in the lowest order the coupled KN system. After this, the spectral problem is expanded and after imposing the appropriate constraint, the coupled Kundu type equation is obtained. Under suitable reduction, this gives rise to the coupled KN, coupled CLL and the GI equations. The multi-Hamiltonian structures of these systems of equations are examined using the trace identity.

An attempt is also made to rigorously examine the connection between the AKS formalism and the Tu methodology and thus to unravel the relationship between these two powerful approaches to the construction and analysis of integrable systems.

At this point, we may mention that Non-holonomic Deformation (NHD) of integrable systems has come to occupy an important place in the literature on Integrable Systems. This is an interesting phenomenon in which an integrable system is perturbed in such a way that under suitable differential constraints on the perturbing function, the system maintains its integrability. In the last part of this paper, NHD of the family of NLS equations is studied including the NLS, coupled KdV type NLS, Derivative NLS (both Kaup-Newell and Chen-Lee-Liu systems). NHD is also applied to the hierarchy of equations in the AKNS system and the Kaup-Newell system obtained through the application of the Tu methodology.

1.1 History of NLSE

The Nonlinear Schrödinger (NLS) equation is a very well-known soliton equation [1, 10]. Various modifications and generalizations of the NLS eqn have been considered. The four most important DNLS eqns are the Kaup-Newell (KN), Chen-Lee-Liu (CLL), the GI and the Kundu eqns. The first generalization of the DNLS equation was considered by Kaup and Newell [28]

\[
i q_t = q_{xx} + i \beta (|q|^2 q)_x,
\]

In our earlier work we showed that this equation and their generalizations are the Euler-Poincaré flow on the space of first order scalar (or matrix) differential operators [23].

The immediate generalization of the Kaup-Newell equation was given by Chen, Lee and Liu [8]

\[
i q_t = q_{xx} + i \alpha |q|^2 q_x.
\]

The equivalence of Kaup-Newell [28] and Chen-Lee-Liu equation [8] was apparently first noticed by Wadati and Sogo [54], although it was believed by some mathematicians that this was implicit in the work of Kaup and Newell.

Gerdjikov and Ivanov [20] and independently Kundu [32, 33] proposed another version of DNLS equation

\[
i q_t = q_{xx} + i \beta q^2 q_x + \frac{1}{2} \beta^2 q^3 q^*^3.
\]
It must be noted that Eckhaus [7, 9] also derived this equation independently. It is known that the KN, CLL and GI equations are described by using a unified generalized derivative Schrödinger equation involving a parameter, and their Hamiltonian structure and Lax pairs are also given by unified and explicit formulae. Using suitable gauge transformations these equations can be transformed into one another.

The method of gauge transformations can be applied to some more generalized derivative nonlinear Schrödinger equation. Using this process Kundu [32, 33] obtained

$$i q_t = q_{xx} + \beta |q|^2 q + i \alpha (|q|^2 q)_x, \quad (5)$$

which is a hybrid of the NLS equation and Kaup-Newell system. Actually this equation was proved to be integrable by Wadati, Konno and Ichikawa \(^1\) and was further transformed by Kakei, Sasa and Satsuma [29] into

$$i Q_T = Q_{XX} + 2i \gamma |Q|^2 Q_X + 2i(\gamma - 1)Q^2 Q_X^* + (\gamma - 1)(\gamma - 2)|Q|^4 Q, \quad (6)$$

by means of change of variables

$$q(x, t) = \sqrt{2 \alpha} Q(X, T) \exp\left(i \frac{\beta}{\alpha} X + i \frac{\beta^2}{\alpha^2} T\right),$$

$$x = X + 2 \beta \frac{\alpha}{\alpha} T, \quad t = T, \quad \gamma = 4 \frac{\delta}{\alpha} + 2.$$ Soliton solutions for \(\gamma = 2\) were already known. Kakei et al. [29] have given the multi-soliton solutions for the general case.

Recently, while attempting to classify certain non-commutative generalizations of classical integrable soliton equations, Olver and Sokolov [44] made a detailed investigation on the DNLS type systems of the form

$$P_t = P_{xx} + f(P, S, P_x, S_x) \quad S_t = -S_{xx} + g(P, S, P_x, S_x), \quad (7)$$

where \(P\) and \(S\) take values in the associative algebra. These two systems can be interpreted as nonabelian analogues of the generalized derivative nonlinear Schrödinger equation.

In a recent paper, Tsuchida and Wadati [53] studied the Lax pair of the matrix generalization of the Chen-Lee-Liu equation

$$i P_t = P_{xx} - i P S P_x, \quad i S_t = -S_{xx} - i S_x P S, \quad (8)$$

which is a member of the list by Olver and Sokolov [44]. In that paper they studied the Lax pair of Chen-Lee-Liu equation.

There is also the higher order nonlinear Schrödinger equation for the propagation of short light pulses in an optical fibre. Theoretical prediction of Hasegawa and Tappert [27] that an optical pulse in a dielectric fibre form an envelope soliton and subsequent

\(^{1}\)This was pointed out to me by late Professor Miki Wadati
Experimental verification by Mollenauer et al. \cite{43} have made a significant impact in ultra high speed telecommunications. This equation is given by

$$\partial_z E = i(\alpha_1 \partial_t + \alpha_2 |E|^2 E) + \alpha_3 \partial_{tt} E + \alpha_4 \partial_t (|E|^2 E) + \alpha_5 \partial_t (|E|^2),$$

\hspace{1cm} (9)

where $E$ is the envelope of the electric field propagating in the $z$ direction at a time $t$. The coefficients $\alpha_3$, $\alpha_4$, $\alpha_5$ respectively represent third order dispersion, self steepening related to Kerr effect and the self frequency shifting via stimulated Raman scattering\cite{55}. It is the last term which plays an important role in the propagation of distortionless optical pulse over long distance.

### 1.2 Background of the NHD formalism

It was shown by Karasu-Kalkani et al\cite{30} that the integrable 6th order KdV equation represented a Non-holonomic deformation (NHD) of the celebrated KdV equation preserving its integrability band giving rise to an integrable hierarchy. The equation is given by

$$(\partial_x^2 + 8u_x \partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0$$

\hspace{1cm} (10)

With the change of variables $v = u_x$, $w = u_t + u_{xxx} + 6u_x^2$, equation (10) can be rewritten as a pair of equations,

$$v_t + v_{xxx} + 12vv_x - w_x = 0$$

$$w_{xxx} + 8vw_x + 4vw_x = 0$$

\hspace{1cm} (11)

The authors of \cite{30} obtained the lax pair as well as an auto-B"acklund transformation for equation (11). They claimed that equation (11) was different from the KdV equation with the self consistent sources and wanted to explore the higher symmetries, higher conserved densities and Hamiltonian formalism for equation (11). In reference \cite{45} Ramani et al bilinearized the KdV6 equation and deduced a new and simpler auto-B"acklund transformation.

The terminology "nonholonomic deformation" was used by Kuperschmidt in \cite{37}. Kuperschmidt rescaled $v$ and $t$ and modified equation (11) to take the following form

$$u_t - 6uu_x - u_{xxx} + w_x = 0$$

$$w_{xxx} + 4uw_x + 2u_x w = 0$$

\hspace{1cm} (12)

The pair of equations given by (12) can be converted into a bi-Hamiltonian system

$$u_t = B_1 (\frac{8H_3}{8u}) - B_1 (w)$$

$$w_x = B_2 (\frac{8H_3}{8u}) - B_1 (w),$$

\hspace{1cm} (13)

where the Hamiltonian operators are given by

$$B_1 \equiv \partial \equiv \partial_x$$

$$B_2 \equiv \partial^3 + 2(u\partial + \partial u)$$

\hspace{1cm} (14)

and $H_n$ denote the conserved densities.
In reference [34], a matrix Lax pair, the N-soliton solution using the Inverse Scattering Transform (IST) technique as well as a two-fold integrable hierarchy were obtained by Kundu for the non-holonomic deformation of the KdV equation. The work was carried forward in reference [36] by Kundu et al to include the non-holonomic deformation of both KdV and mKdV equations as well as their symmetries, hierarchies and integrability. One of the authors of reference [36] extended the study to the NHD of the DNLS and the Lenells-Fokas equation in reference [35]. Non-holonomic deformation of generalized KdV type equations were studied by Guha in reference [25] wherein a geometric insight was provided into the KdV6 equation. In this paper, Kirillov’s theory of co-adjoint representation of the Virasoro algebra was used to generate a large class of KdV-6 type equations equivalent to the original equation. It was further shown that the Adler-Kostant-Symes approach provided a geometric formalism to obtain non-holonomic deformed integrable systems. NHD for the coupled KdV system was thereby generated. In reference [26], Guha extended Kupershmidt’s infinite-dimensional construction to generate nonholonomic deformation of a wide class of coupled KdV systems, all of which follow from the Euler-Poincare-Suslov flows. In this paper, the author also derives a nonholonomic deformation of the N=1 supersymmetric KdV equation, also known as the sKdV6 equation.

1.3 Organization

We give a brief introduction to the Adler-Kostant-Symes (AKS) theory in Section 2. We apply this scheme to current algebra over $S^1$ of a loop with a central extension given by a two cocycle. The AKS scheme yields a hierarchy of commuting Hamiltonians. We construct various types of nonlinear Schrödinger equations in Section 3, they are associated to different symmetric spaces. Tu methodology is explained in Section 4. Note that Tu’s method is mainly confined to the Chinese group. In this paper we present a description of this method and show how hierarchies of different integrable evolution equations can be constructed using this technique. Section 5 is dedicated to the trace identity method. We derive the Hamiltonian structures for NLSE equations using this method. Section 6 deals with the application of the NHD formalism to the class of equations belonging to the NLS family. We complete our work with a modest outlook in Section 7.

2 Adler-Kostant-Symes Scheme

Let $G$ be a connected compact semi-simple Lie group with the Lie algebra $\mathfrak{g}$, endowed with a non-degenerate ad-invariant and symmetric inner product $<\cdot,\cdot>: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, that is,

\[ <X, [Y, Z]> = <[X, Y], Z> \quad \forall X, Y, Z \in \mathfrak{g}. \]

Its dual space $\mathfrak{g}^*$ has a natural Poisson structure

\[ \{g_1, g_2\}(\mu) = < \alpha, \frac{\delta g_1}{\delta \mu}, \frac{\delta g_2}{\delta \mu} >, \]
of two smooth functions \( g_1 \) and \( g_2 \) on \( \mathfrak{g}^* \). The functional derivative of \( g \) (or gradient of \( g \)) at \( \mu \) is the unique element \( \frac{\delta f}{\delta \mu} \) of \( \mathfrak{g} \) defined by

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} [f(\mu + \epsilon \delta \mu) - f(\mu)] = <\delta \mu, \frac{\delta f}{\delta \mu} >.
\]

Here the gradient of \( g_i \) are interpreted as elements of \( \mathfrak{g} \) due to the identification of \( \mathfrak{g} \simeq \mathfrak{g}^{**} \).

Let us introduce an additional structure from which, in addition to the ordinary bracket, a modified bracket can be defined as follows.

Let

\[ R : \mathfrak{g} \to \mathfrak{g} \]

be an \( R \)-matrix, and this defines another Lie bracket on \( \mathfrak{g} \)

\[
[X,Y]_R = \frac{1}{2}([RX,Y] + [X,RY]),
\]

such a pair \((\mathfrak{g}, R)\) is called a double Lie algebra. It is known that \((\mathfrak{g}, R)\) is a double Lie algebra if and only if the following bilinear map

\[
B_R : (\mathfrak{g}, R) \times (\mathfrak{g}, R) \to (\mathfrak{g}, R)
\]

given by

\[
B_R(X,Y) = [RX,RY] - R([X,Y]_R)
\]

is ad-invariant, that is, the equation

\[
[X,B_R(Y,Z)] + [Y,B_R(Z,X)] + [Z,B_R(X,Y)] = 0
\]

holds for all \( X,Y,Z \in \mathfrak{g} \).

It is clear that the trivial solution \( B_R(X,Y) = 0 \) yields the Yang-Baxter equation. The second solution satisfies the so called modified Yang-Baxter equation

\[
B_R(X,Y) = -[X,Y].
\]

The best known class of \( R \)-matrices arises when the Lie algebra \( \mathfrak{g} \) split into a direct sum of two subalgebras \( \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \). Since there is a vector space decomposition of \( \mathfrak{g} \) into a direct sum of two Lie subalgebras, hence, we put

\[
R = P_+ - P_-,
\]

where \( P_\pm \) denotes the corresponding projection onto \( \mathfrak{g}_\pm \). Under this identification the above bracket boils down to

\[
[X,Y]_R = [X_+,Y_+] - [X_-,Y_-],
\]

where \( X_\pm = P_\pm X \).
By $g^*$ and $g^*_R$ we denote the dual of $g$ endowed with the Lie-Poisson structures arising from $[,]_R$ and $[,]_R^*$ respectively. The Poisson bivectors arising from the Lie brackets $[,]_R$ and $[,]_R^*$ are related by $P_R = R^*P + PR$, where $R$ is considered to be a pointwise lift of the map $R$ on $g$ to the vector fields over $g$ and $R^*$ is the transpose of this map.

The $R$-matrix construction on $g$ allows us to define an additional Lie-Poisson bracket of the following form:

$$\{f, g\}(\mu) = \langle \mu, [R(\nabla f), (\nabla g)] + [\nabla f, R(\nabla g)] \rangle \quad f, g \in C^\infty(g^*). \tag{20}$$

We wish to take a look at the special case of an $R$-structure given by a splitting into two subalgebras. With $g = g_+ \oplus g_-$, $R = P_+ - P_-$, $\mu \in g^*$ one computes the Lie-Poisson bracket arising from $[,]_R^*$:

$$\{f, g\}(\mu) = 2 \langle \mu, [(\nabla f)_+, (\nabla g)_+] \rangle - 2 \langle \mu, [(\nabla f)_-, (\nabla g)_-] \rangle. \tag{21}$$

**Definition 2.1** We say a smooth function $H : g^* \to \mathbb{R}$ on any Lie algebra $g$ is $Ad^*$ invariant if $H(Ad^*_g \alpha) = H(\alpha)$ for all $\alpha \in g^*$ and $g \in G$.

**Theorem 2.2** (AKS) Let $g$ be Lie algebra with $R$-matrix $R : g \to g$, then the $ad^*$ invariant functions on $g^*$ are in involution with respect to

$$\{f, g\}(\mu) = \langle \mu, [R(\nabla f), (\nabla g)] + [\nabla f, R(\nabla g)] \rangle.$$

Then the Hamiltonian flow on the coadjoint orbits in $L \subset g^*$ is

$$\frac{d}{dt}L = ad^*_R(\nabla H)L + R^*ad^*_{\nabla H}L, \tag{22}$$

where $R^*$ is the transpose of $R$.

### 2.1 AKS theory and loop algebra

The Adler-Kostant-Symes (AKS) theory produces hierarchies of completely integrable partial (or ordinary) differential equations. This scheme is quite general and is based on the following ingredients.

(a) A Lie algebra $g$, with a non-degenerate bilinear form $< . , >$ which allows us to identify $g$ with its dual $g^*$. The Lie algebra $g$ splits into $g = g^+ \oplus g^-$ i.e. two subalgebras $g^+$ and $g^-$. The bilinear form is used to identify $g^{-*}$ with $g^{+\perp}$.

(b) The phase space is an $ad^*$ invariant finite dimensional submanifold $\Gamma \subset g^{-*} \equiv g^{+\perp}$. The Poisson structure on $\Gamma$ is the Kostant-Kirillov structure associated to $g^{-*}$.

(c) The complete set of commuting constants of motion will be elements of the algebra $A(\Gamma)$ of ad-invariant functions on $g^*$ restricted to $\Gamma$.
2.2 Applications to loop group

Let us apply this scheme to a loop group. Let $\Omega G$ be the space of based loop, then the corresponding Lie algebra, called loop algebra, is the Laurent polynomials in the variable $\lambda$ with coefficients in $g$:  

\[ \Omega g = \{ X(\lambda) = \sum_i x_i \lambda^i; x_i \in g \}, \]

with Lie bracket  

\[ [X(\lambda), Y(\lambda)] := \sum_{i,j} [x_i, y_j] \lambda^{i+j}, \quad \text{where } X(\lambda) = \sum x_i \lambda^i, \ Y(\lambda) = \sum y_j \lambda^j. \]

Here we can define the projection operator in the following way:

\[ P_{\pm} X = \begin{cases} X & \text{if } X = \sum_{n\geq 0} X_n \lambda^n \\ -X & \text{if } X = \sum_{n<0} X_n \lambda^n \end{cases} \]

We define the bilinear form on $\Omega g$ as

\[ \langle X(\lambda), Y(\lambda) \rangle := \text{tr} \left( \sum_{i+j=-1} x_i y_j \right) = \oint \text{tr}(X(\lambda)Y(\lambda)) d\lambda. \]

The two subalgebras of $\Omega g$ are given as

\[ \Omega g_+ := \{ \sum_{i=0}^k g_i \lambda^i : g_i \in g \}, \quad \Omega g_- := \{ \sum_{i=-\infty}^{-1} g_i \lambda^i : g_i \in g \}. \]

With the above choice of inner product, one can verify easily $\Omega g_-^* = \Omega g_+^\perp$, so that $\Gamma$ can be identified with a submanifold of $\Omega g_+^\perp$:

\[ \Gamma := \{ A(\lambda) = \sum_{n=0}^n a_{n-i} \lambda^i, \ n \text{ fixed } \}. \]

The Kostant-Kirillov bracket for $\hat{\Omega} g^*$ is given by

\[ \{ f, g \}(\mu) = \langle \mu, [\nabla f(\mu), \nabla g(\mu)] \rangle, \quad \text{where } \mu \in \Omega g^* \quad (23) \]

The gradient of a function $f : g^* \rightarrow \mathbb{C}$ is the vector field $\nabla f : g^* \rightarrow g$ such that  

\[ \langle \nabla f(\mu), X(\mu) \rangle = df(X(\mu)) \quad \forall \mu \in g^*. \]

But this does not restrict to $\Omega g^*$. In fact, with respect to this bracket, the Hamiltonian vector fields of elements of $A(\Gamma)$ are identically zero, one justifies this by  

\[ \langle ad^*_X \mu, \nabla H \rangle = \langle \mu, [X, \nabla H] \rangle = 0 \]

for all $X \in \Omega g$. 

11
Let us consider Hamiltonian equation with respect to \{\ldots\} where \( H \) can be expressed in terms of linear coordinates \( \mu_r = \langle \mu, X_r \rangle \), where \( X_r \) form the basis in \( \Omega g_- \). Thus the Hamiltonian equation becomes

\[
\langle \dot{\mu}, X_r \rangle = \{ H, \mu_r \} = \langle \mu, [\nabla H(\mu)_-, (\nabla \mu_r(\mu))_-] \rangle \\
\implies \langle \dot{\mu}, X_r \rangle = \langle \mu, [(\nabla H(\mu))_-, X_r] \rangle \\
\implies \langle \dot{\mu}, X_r \rangle = \langle [\nabla H(\mu)_+, \mu], X_r \rangle ,
\]

hence we obtain

\[
\dot{\mu} = [(\nabla H(\mu))_+, \mu]. \tag{24}
\]

**Hierarchy equation** Let us consider the Hamiltonians

\[
H_i(\mu) = \frac{1}{2} tr(\lambda^{-p-i})\mu^2, \quad 0 \geq i \geq p. \tag{25}
\]

The gradient of \( H \) is given by

\[
\langle \nabla H_i, X \rangle = dH_i(X) = tr (\lambda^{-p-i})\mu X),
\]

hence \( \nabla H_i = \lambda^{-p-i}\mu \).

Therefore, the Hamiltonian equations motion for \( H_i \) are given as

\[
\frac{d\mu}{dt_i} = [(\lambda^{-p-i})\mu]_+ , \mu]. \tag{26}
\]

### 2.3 AKS equation with cocycle

Sometimes it is necessary to define orbit starting from the higher powers of \( \lambda \). In order to meet such demand we alter the Adler-Kostant-Symes (AKS) scheme slightly. Instead of using the bilinear form in the previous section we use

\[
tr_n(X(\lambda)Y(\lambda)) := tr_0(\lambda^nX(\lambda)Y(\lambda)) \tag{27}
\]

such that

\[
\Omega g_+ = \{ X(\lambda) = \sum_{i\geq-n} x_i\lambda^i \} \quad \Omega g_- = \{ Y(\lambda) = \sum_{i\leq-n-1} y_i\lambda^i \}.
\]

Hence the previous computation becomes

\[
\langle \dot{\mu}, X_r \rangle = \langle [\nabla H(\mu)_+, \mu], X_r \rangle \quad \text{for all } X_r \in \Omega g_-.
\]

So that again we have \( \dot{\mu} = [(\nabla H(\mu))_+, \mu] \).

Our first aim is to extend the loop algebra \( \Omega g \). Let us introduce a non-trivial two cocycle on \( \Omega g \), known as Maurer-Cartan cocycle. Then corresponding to the centrally
extended Lie group $\widehat{\Omega G} = \Omega G \times \mathbb{R}$. The Lie algebra is $\widehat{\Omega g} = \Omega g \oplus \mathbb{R}$. This is a centrally extended loop algebra associated with 2-cocycle $\omega(X, Y) = (X, \frac{dY}{dx})$. Loop algebra $\widehat{\Omega g}$ satisfies the following commutation relation

$$[[X, a], [Y, b]] = ([X, Y], \int_{S^1} \text{tr}(XY'))$$

where $(X, a), (Y, b) \in \widehat{\Omega g}$. We also define the bilinear form on $\widehat{\Omega g}$ by

$$< (X, a), (Y, b) > = ab + \int tr(XY').$$

In this case the ad-invariant function satisfies:

**Lemma 2.3** Suppose $H$ is Ad* invariant function on $\Omega g^*$ then

$$\text{ad}^*(\nabla H(\alpha), a)(\mu, 1) = ((\text{ad}^*(\nabla H(\mu))(\mu) + (\nabla H)', 0)$$

The co-adjoint representation leaves invariant the hyperplanes $e = \text{constant}$. Note that from the above proposition and lemma we can conclude two things

1. The centre of the $\widehat{\Omega g}$ acts trivially on $\widehat{\Omega g}^*$, the space of $\widehat{\Omega g}^*$ is a natural $G$-module.
2. $\widehat{\Omega G}$ acts on $\widehat{\Omega g}^*$ by a gauge transformation.

The Poisson bracket is

$$\{ f, g \}(\mu + cI) = < \mu, [\hat{R}(\nabla f), \nabla g] + [\nabla f, \hat{R}(\nabla g)] >$$

where $\hat{R}$ is the R-matrix on $\hat{g}$, it satisfies

$$\hat{R} : \widetilde{\Omega g} \rightarrow \widetilde{\Omega g} \quad \hat{R}(k + aI) := R(k).$$

**Proposition 2.4** The Poisson bracket in the space of $\widetilde{\Omega g}^*$ for the two smooth functions has the form

$$\{ f_1, f_2 \}(Y) = < [\hat{R}(\nabla f_1), \nabla f_2], Y > + [\nabla f_1, \nabla f_2]$$

$$+ \int_{S^1} R\nabla f_1 \frac{d\nabla f_2}{dx} + \int_{S^1} \nabla f_1 R \frac{d\nabla f_2}{dx},$$

where $Y \in \Omega g$.

If we repeat the previous steps we arrive at

**Theorem 2.5** The Hamiltonian equations of motion on the hyperplane of $\widetilde{\Omega g}^*$ generated by the gradient of the Hamiltonian $H(L)$, the ad-invariant function, have the form

$$\frac{d\mu}{dt} = d(\nabla H(\mu)) - \frac{d(\nabla H(\mu))}{dx} + [\nabla H]_-, \mu_m \quad \mu \in \widetilde{\Omega g}^*$$

so it denotes that the connection $\mu dx + \nabla H dt$ on a cylinder $S^1 \times \mathbb{R}$ is flat.
2.4 Hermitian Symmetric Spaces and Integrable Systems

Let \( G \) be a semi-simple Lie group and \( \mathfrak{g} \) be the corresponding Lie algebra. Let \( M \) be a homogeneous space of \( G \), so, \( M \) is a differentiable manifold on which \( G \) acts transitively. There is a homeomorphism of the coset space \( G/K \) onto \( M \) for some isotropy subgroup \( K \) of \( G \) at a point of \( M \). Let \( k \) be the Lie algebra of \( K \) and \( \mathfrak{g} \) satisfies

\[
\mathfrak{g} = k \oplus m \quad \text{and} \quad [k, k] \subset k
\]

where \( m \) is the vector space complement of \( k \). The Lie algebra \( \mathfrak{g} \) splits in such a way that \( M \) is equipped with two kinds of extra structure, these are:

1. left translation of \( m \) around \( G \) gives rise to a canonical connection on the principle \( K \)-bundle: \( G \rightarrow G/K \).
2. When \( x \in M \), the map \( \mathfrak{g} \rightarrow T_x M \) given by

\[
\eta \mapsto \frac{d}{dt}\bigg|_{t=0}\exp(t\eta.x),
\]

restricts to give an isomorphism \([m]_x \rightarrow T_x M\).

The inverse map

\[
\omega_x : T_x M \rightarrow [m]_x
\]

defines a \( \mathfrak{g} \)-valued one form on \( M \), known as Maurer Cartan form.

If \( k \) and \( m \) satisfy

\[
[k, m] \subset m
\]

then \( G/K \) is called reductive homogeneous space [42]. We can associate to these spaces a canonical connection with curvature and torsion. Curvature and torsion at a fixed point \( p \) are given purely in terms of the Lie bracket operation,

\[
(R(X,Y)Z)_p = -[[X,Y]_k, Z], \quad X,Y,Z \in m
\]

\[
T(X,Y)_p = -[X,Y]_m, \quad X,Y \in m.
\]

If \( k \) and \( m \) satisfy above two conditions and also satisfy

\[
[m, m] \subset k,
\]

then \( G/K \) is a symmetric space. Here the curvature satisfies

\[
(R(X,Y)Z)_p = -[[X,Y], Z], \quad X,Y,Z \in m.
\]

Here \([X,Y] \in k\) happens automatically due to \([m,m] \in k\).

Let \( h \) be the Cartan subalgebra of \( \mathfrak{g} \) which is the maximal abelian subalgebra of diagonalizable elements of \( \mathfrak{g} \). In terms of the Weyl basis

\[
[H_i, H_j] = 0, \quad [H_i, X_\alpha] = \alpha(H_i)X_\alpha
\]

\[
[X_\alpha, X_\beta] = N_{\alpha, \beta}X_{\alpha+\beta} \quad (\alpha + \beta \in \Delta)
\]
\[ \sum_{i=1}^{mh} C_{\alpha,i} H_i \quad (\alpha + \beta = 0) \]
\[ = 0 \quad (\alpha + \beta \notin \Delta, \alpha + \beta \neq 0) \]

For any \( H_i \in h \) and \( X_\alpha \in \mathfrak{g} h \) and \( N_{\alpha,\beta} \) and \( C_{\alpha,i} \) are structure constants and \( \Delta \) is a set all roots.

The components \( R_{ijkl}^i \) and \( T_{jk}^i \) of the curvature and torsion with respect to a basis \( X_i \) of \( T_p M \) are defined by
\[
R(X_k, X_l)X_j = R_{ijkl}^i X_i, \quad T(X_j, X_k) = T_{jk}^i X_i
\]
and the component of the metric \( g(X, Y) = \text{tr} \ (ad(X)ad(Y)) \) is \( g_{ij} = g(X_i, X_j) \).

Let \( \varrho \) be an element of \( h \). We select the isotropy group \( K \) such that its Lie algebra is \( k \). This is given by the centralizer
\[
C_g(\varrho) = \{ X \in g \mid [X, \varrho] = 0 \}
\]
If \( \varrho \) is regular i.e. the eigenvalues \( \alpha(\varrho) \) of \( \text{ad} \varrho \) are mutually distinct then \( C_g(\varrho) = h \) and here \([h, m] \subset m\).

In this case since \( k = h \) hence the corresponding coset space \( G/K \) decomposition is essentially a Cartan decomposition.

When \( k = C_g(\varrho) \supset h \), then the eigenvalues \( \alpha(\varrho) \) coalesce, and thus \( C_g(\varrho) \) becomes larger than \( h \). Hence in this case the homogeneous space \( G/K \) becomes smaller.

In the case of Hermitian symmetric spaces \( \alpha(\varrho) \) have eigenvalues \( \{0, \pm \alpha\} \). Thus \( g \) splits up into
\[
g = k \oplus m^+ \oplus m^-.
\]
If we set \( X^0 \in k, X^\pm \in m^\pm \) for any \( X \in g \).
\[
[\varrho, X^0] = 0, \ [\varrho, X^\pm] = \pm \alpha X^\pm
\]
here eigenvalues \( \alpha(\varrho) \) take the same eigenvalue for all \( X^\pm \in m^\pm \). From the second commutation relation we can assert that hermitian symmetric space has almost complex structure.

### 3 Examples of integrable systems

In this section we derive several equations of the Nonlinear Schrödinger family by using the Adler-Kostant-Symes (AKS) framework.
3.1 Non-linear Schrodinger Equation

In this case we consider the orbit

\[ L = \lambda A_1 + A_2 \]

\[ A_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \]

\[ A_2 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \] (29)

which is basically the space part of the Lax pair.

To obtain the temporal part of the Lax pair we set

\[ \nabla H = \sum_{j=0}^{\infty} \lambda^{2+j} h_{2+j} \] (30)

The \( h_i \) are obtained from the equation,

\[ [L, \nabla H] - (\nabla H)_x = 0 \] (31)

Equating different powers of \( \lambda \) and solving recursively, we obtain

\[ h_2 = A_1, \quad h_1 = A_2, \quad h_0 = \begin{pmatrix} h_{03} & \frac{i}{2} q_x \\ -\frac{i}{2} r_x & -h_{03} \end{pmatrix} \] (32)

where \( h_{03} \) is as yet undeterminate.

Now the AKS flow is given by,

\[ L_t = (\Pi_n(\nabla H))_x - [L, \Pi_n(\nabla H)] \] (33)

where, \( \Pi_n(\nabla H) \) represents the projection of \( \nabla H \) on the subalgebra containing non-negative powers of \( \lambda \) i.e.

\[ \Pi_n(\nabla H) = \lambda^2 h_2 + \lambda h_1 + h_0 \] (34)

Equating the terms containing \( \lambda \), we get \( h_{03} = -\frac{i}{2} qr \) while the \( \lambda \)-free terms lead to the following dynamical equations,

\[ q_t = \frac{i}{2} q_{xx} - iq^2 r \]

\[ r_t = -\frac{i}{2} r_{xx} + iqr^2 \] (35)

which is the coupled NLS equations and reduce to the conventional NLS upon putting \( r = q^* \).
3.2 Coupled KdV type NLS equations

We start with the same orbit viz that given by (29) but take

\[ \nabla H = \sum_{j=0}^{-\infty} \lambda^{3+j} h_{3+j} \]  

(36)

and use it in equation (31).

Equating various powers of \( \lambda \) and solving recursively we get,

\[ h_3 = A_1, h_2 = A_2, h_1 = \left( -\frac{i}{2} q r - \frac{i}{2} q_x, \frac{1}{2} q r, \right), h_0 = \left( \frac{h_{03}}{2}, -\frac{1}{4} q_{xx} - h_{03} \right) \]  

(37)

Next using the AKS flow equation (33), using \( \Pi_n(\nabla H) \) and equating different powers of \( \lambda \), we obtain

\[ h_{03} = \frac{1}{4} \left( r q - q r x \right) \]  

which completes the determination of \( h_0 \).

The \( \lambda \) independendent terms lead to the following equations:

\[ q_t = -\frac{1}{4} q_{xxx} + \frac{3}{2} q q_x r \\
q_t = -\frac{1}{4} r_{xxx} + \frac{3}{2} r r_x q \]  

(38)

Putting \( r = q^* \), we obtain the coupled KdV type NLS equations viz

\[ q_t = -\frac{1}{4} q_{xxx} + \frac{3}{2} q q_x^* \]

\[ q_t = -\frac{1}{4} r_{xxx} + \frac{3}{2} r r_x^* \]  

(39)

3.3 Generalized Nonlinear Schrödinger Equation

Choose the orbit \( L \) as before but now take

\[ \nabla H(\lambda) = \nabla H_1(\lambda) + \xi \nabla H_2(\lambda) \]

where

\[ \nabla H_1(\lambda) = \sum_{j=0}^{-\infty} \lambda^{2+j} h_{2+j} \quad \text{and} \quad \nabla H_2(\lambda) = \sum_{j=0}^{-\infty} \lambda^{3+j} g_{3+j} \]  

(40)

and use it in equation (31). Now equating various powers of \( \lambda \) we obtain the following:

\( \lambda^4 \) : \( g_3 = A_1 \)

\( \lambda^3 \) : \( [A_1, h_2] + \xi [A_1, g_2] + \xi [A_2, A_1] + \xi A_{1x} = 0 \)

which is satisfied on choosing,

\( g_2 = A_2, \quad h_2 = A_1 \) By equating the other powers of \( \lambda \) , we obtain after some algebra,

\[ h_1 = A_2, \quad g_1 = \left( -\frac{i}{2} q r - \frac{i}{2} q_x, \frac{1}{2} q r, \right), \quad h_0 = \left( \frac{h_{03}}{2}, -\frac{1}{4} q_{xx} - h_{03} \right) \]

\[ g_0 = \left( \frac{1}{4} (q_x r - q r x), \frac{1}{4} q^2 r - \frac{1}{4} q_{xx} \right) \]

(41)
Taking projection on the subalgebra containing non-negative powers of $\lambda$, we obtain the coupled system of Generalized NLS equations as,

$$
q_t = -\frac{i}{2} q_{xx} + i q^2 r + \xi (-\frac{1}{4} q_{xxx} + \frac{3}{2} q q_x r)
$$

$$
r_t = \frac{i}{2} r_{xx} - i q r^2 + \xi (-\frac{1}{4} r_{xxx} + \frac{3}{2} q r_x r_x)
$$

(41)

Putting $r = q^*$ leads us to

$$
q_t = -\frac{i}{2} q_{xx} + i|q|^2 q + \xi (-\frac{1}{4} q_{xxx} + \frac{3}{2} |q|^2 q_x)
$$

(42)

### 3.4 Dimensionless Vector Nonlinear Schrödinger Equation (VNLSE), a Manakov system

In this case we choose a different orbit

$$
L = \lambda A_1 + A_2
$$

where,

$$
A_1 = \begin{pmatrix}
i & 0 & 0 \\
0 & -i & 0 \\
0 & 0 & -i
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & -u^{(1)*} & -u^{(2)*} \\
u^{(1)} & 0 & 0 \\
u^{(2)} & 0 & 0
\end{pmatrix},
$$

(43)

that is we move to a matrix representation in higher dimension. Take $\nabla H(\lambda) = \sum_{j=0}^{-\infty} \lambda^{2+j} h_{2+j}$ and use it in (31) leading to

$$
h_2 = A_1, \quad h_1 = A_2, \quad h_0 = \begin{pmatrix}
h_0^{11} & \frac{i}{2} u_x^{(1)*} & \frac{i}{2} u_x^{(2)*} \\
\frac{i}{2} u_x^{(1)} & h_0^{22} & h_0^{23} \\
\frac{i}{2} u_x^{(2)} & h_0^{32} & h_0^{33}
\end{pmatrix}
$$

(44)

where entries in $h_0$ remain undetermined.

Now using the AKS flow equation (33), after some lengthy calculations, we obtain

$$
\begin{align*}
h_0^{11} &= -\frac{i}{2} (u^{(1)*} u^{(1)} + u^{(2)*} u^{(2)}) \\
h_0^{22} &= \frac{i}{2} (u^{(1)} u^{(1)*}) \\
h_0^{23} &= \frac{i}{2} (u^{(1)} u^{(2)*}) \\
\end{align*}
$$

(45)

and the dynamical systems

$$
\begin{align*}
i u_t^{(1)} + (|u^{(1)}|^2 + |u^{(2)}|^2) u^{(1)} + \frac{1}{2} u_{xx}^{(1)} &= 0 \\
i u_t^{(2)} + (|u^{(1)}|^2 + |u^{(2)}|^2) u^{(2)} + \frac{1}{2} u_{xx}^{(2)} &= 0
\end{align*}
$$

(46)

which are the desired equations.
3.5 Derivative NLS equation (DNLS Eqn)

We now begin to explore the different types of Derivative NLS equations using the AKS formalism.

To obtain the DNLS eqn using the AKS technique we start with the orbit

\[ L = \lambda^2 A_1 + \lambda A_2 \]  \hspace{1cm} (47)

where \( A_1 \) and \( A_2 \) are defined previously.

We choose

\[ \nabla H(\lambda) = \sum_{j=0}^{-\infty} \lambda^{4+j} h_{4+j} \]  \hspace{1cm} (48)

Following the usual procedure, we obtain

\[ h_4 = A_1, \quad h_3 = A_2, \quad h_2 = \frac{1}{2} qr A_1, \quad h_1 = \begin{pmatrix} 0 \\ \frac{1}{2} q^2 r + \frac{i}{2} q_x \\ \frac{1}{2} q^2 r - \frac{i}{2} r_x \end{pmatrix} \]  \hspace{1cm} (49)

Next we impose the condition that \( \Pi_n(\nabla H) \) should contain positive powers of \( \lambda \) only, so that the \( h_0 \) term gets dropped.

Now using eqn (33), we arrive at

\[ q_t = \frac{i}{2} q_{xx} + \frac{1}{2} (q^2 r)_x \]
\[ r_t = -\frac{i}{2} r_{xx} + \frac{1}{2} (qr^2)_x \]  \hspace{1cm} (50)

On putting \( r = q^* \), the above equations reduce to,

\[ q_t = \frac{i}{2} q_{xx} + \frac{1}{2} (|q|^2 q)_x \]  \hspace{1cm} (51)

This is the Kaup-Newell type DNLS eqn.

3.6 Chen-Lee-Liu (CLL) type DNLS equation

Here we take the orbit to be

\[ L = \lambda^2 A_1 + \lambda A_2 + A_0 \]

where,

\[ A_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{i}{2} q r \end{pmatrix} \]
and

\[ \nabla H = \sum_{k=0}^{-\infty} \lambda^{4+k} h_{4+k} \]  \tag{52} \]

The standard procedure outlined above leads to the following values of \( h_i \),

\[ h_4 = 2A_1, \quad h_3 = 2A_2, \quad h_2 = qr A_1 \quad \text{(by choice)} \]

\( h_1 \) is chosen to be off-diagonal, and

\[ h_1 = \begin{pmatrix} 0 & iq_x + \frac{1}{2} q^2 r \\ -ir_x + \frac{1}{2} qr^2 & 0 \end{pmatrix} \]

The off-diagonal elements of \( h_1 \) are determined by equating the powers of \( \lambda^3 \).

Equating powers of \( \lambda^2 \), leads to \([A_1, h_0] = 0\)

whence, \( h_0 \) is determined to be

\[ h_0 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}(rq_x - r_x q) + \frac{i}{4} r^2 q^2 \end{pmatrix} \]

and thereby we obtain the CLL type DNLS eqn

\[ \begin{align*}
q_t &= iq_{xx} + qq_x r + \frac{1}{2} q^2 r_x + \frac{i}{4} q^2 r^2 \\
r_t &= -ir_{xx} + rr_x q + \frac{1}{2} r^2 q_x - \frac{1}{4} q^2 r^3
\end{align*} \tag{53} \]

### 3.7 Kundu-Eckhaus Equation

Kundu [32, 33] and Eckhaus [7, 9] independently derived what can now be called the Kundu-Eckhaus equation as a linearizable form of the nonlinear Schrödinger equation.

The orbit \( L \) and \( \nabla H \) are taken to be the same as in (52) but while taking the projection \( \Pi_n(\nabla H) \) , only the terms containing the positive powers of \( \lambda \) are considered i.e. we take

\[ \Pi_n(\nabla H) = \lambda^4 h_4 + \lambda^3 h_3 + \lambda^2 h_2 + \lambda h_1 \]  \tag{54} \]

Using (31) and equating powers of \( \lambda \), we obtain

\[ A_{2t} = [h_1, A_0] + h_{1x} \]

which leads to the Kundu DNLS equations:

\[ \begin{align*}
q_t &= iq_{xx} + \frac{1}{2} qq_x r + \frac{1}{7} q^2 r_x + \frac{i}{4} q^2 r^2 \\
r_t &= -ir_{xx} + \frac{1}{2} rr_x q + \frac{1}{2} r^2 q_x - \frac{1}{4} q^2 r^3
\end{align*} \tag{55} \]
4 Tu methodology

The Tu method allows one to derive a hierarchy of non-linear evolution equations and also to obtain the Hamiltonian structure of these equations by using the trace identity. Let us first focus on the method of obtaining the hierarchy of equations. To this end consider an isospectral problem of the form

$$\psi_x = U(\lambda)\psi$$ (56)

with $\lambda$ being the spectral parameter for which $\lambda_t = 0$.

Suppose that $U$ can be expressed in the form

$$U = R + u_1e_1 + u_2e_2 + \ldots + u_Pe_P$$ (57)

where the variables $u_1, u_2, \ldots, u_P \in S$ (S: Schwartz space) and $R, e_1, e_2, \ldots, e_P \in \Omega g$ ($\Omega g$: the loop algebra corresponding to the finite dimensional Lie algebra $g$) and satisfy the conditions,

(i) $R, e_1, e_2, \ldots, e_P$ are linearly independent

(ii) $R$ is pseudoregular.

We first solve the stationary eqn,

$$V_x = [U, V]$$ (58)

for $V$,

and then search for a $\triangle_n \in \Omega g$ such that for

$$V^{(n)} = (\lambda^n V)_+ + \triangle_n$$ (59)

it is found that

$$-V_x^{(n)} + [U, V^{(n)}] \in e_1 + e_2 + \ldots + e_P$$

Once $\triangle_n$ is determined, the hierarchy of non-linear evolution eqns is determined from the zero-curvature representation,

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0$$ (60)

4.1 The Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy

One starts with $U = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}$ and take

$$V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$ (61)
Using these in (58), we get the following relations:

\[
\begin{align*}
    a_x &= qc - rb \\
    b_x &= -2i\lambda b - 2qa \\
    c_x &= 2i\lambda c + 2ra
\end{align*}
\]  \quad (62)

Now expand \(a, b, c\) as

\[
\begin{align*}
    a &= \sum_{m \geq 0} a_m \lambda^{-m} \\
    b &= \sum_{m \geq 0} b_m \lambda^{-m} \\
    c &= \sum_{m \geq 0} c_m \lambda^{-m}
\end{align*}
\]
and put these in (62), there by obtaining,

\[
\begin{align*}
    a_{mx} &= qc_m - rb_m \\
    b_{mx} &= -2ib_{m+1} - 2qa_m \\
    c_{mx} &= 2ic_{m+1} + 2ra_m
\end{align*}
\]  \quad (63)

Choosing \(b_0 = c_0 = 0\) and solving recursively we obtain the following values for \(a_i, b_i\) and \(c_i\).

\[
\begin{align*}
    a_0 &= \alpha \text{ (constant)} \\
    b_1 &= i\alpha q \\
    c_1 &= i\alpha r \\
    a_1 &= 0 \\
    b_2 &= -\frac{\alpha}{2} q_x, \quad c_2 = \frac{\alpha}{2} r_x, \quad a_2 = \frac{\alpha}{2} qr
\end{align*}
\]  \quad (64)

\[
\begin{align*}
    b_3 &= i\alpha \left( -\frac{\alpha}{4} q x x + q^2 r \right) \\
    c_3 &= i\alpha \left( -\frac{\alpha}{4} r x x + qr^2 \right) \\
    a_3 &= \frac{i\alpha}{4} (rq_x - qr_x) \\
    b_4 &= \frac{\alpha}{8} (q_{xxx} - 6q q x r) \\
    c_4 &= \frac{\alpha}{8} (-r_{xxx} + 6q r r_x)
\end{align*}
\]  \quad (66)

and so on.

Next we calculate the following expression

\[
(\lambda^n V)_+ - [U, (\lambda^n V)_+]
\]

Here \((\lambda^n V)_+\) denotes the terms in \((\lambda^n V)\) carrying non-negative powers of \(\lambda\) only.

\[
(\lambda^n V)_+ = \sum_{m=0}^{n} \lambda^{n-m} \begin{pmatrix} a_m & b_m \\ c_m & -a_m \end{pmatrix}
\]  \quad (68)

Using (61) and (68) we obtain,

\[
(\lambda^n V)_+ - [U, (\lambda^n V)_+] = \sum_{m=0}^{n} \lambda^{n-m} \begin{pmatrix} a_{mx} - q c_m + rb_m & b_{mx} + 2i\lambda b_m + 2qa_m \\ c_{mx} - 2i\lambda c_m - 2ra_m & -(a_{mx} - q c_m + rb_m) \end{pmatrix}
\]

On using the recurrence relations and simplifying the above matrix reduces to,

\[
\begin{pmatrix}
    0 & -2ib_{n+1} \\
    2ic_{n+1} & 0
\end{pmatrix}
\]  \quad (69)
It is obvious that $\triangle_n = 0$ and therefore we obtain the AKNS hierarchy,

\[
q_t = -2ib_{n+1} \\
r_t = 2ic_{n+1}
\]  

(70)

Successive equations can be generated by putting $n = 1, 2, 3$ etc

Putting $n = 2$, we obtain

\[
q_t = -2ib_3 \\
r_t = 2ic_3
\]  

(71)

Using the values of $b_3$ and $c_3$ from (66) we get

\[
q_t = \alpha(-\frac{1}{2}q_{xx} + q^2r) \\
r_t = \alpha(\frac{1}{2}r_{xx} - qr^2)
\]  

(72)

which constitute a system of NLS eqns.

Further, setting $n = 3$ leads to

\[
q_t = i\alpha(-\frac{1}{4}q_{xxx} + \frac{3}{4}qq_xr) \\
r_t = i\alpha(-\frac{1}{4}r_{xxx} + \frac{3}{4}qrr_x)
\]  

(73)

which are a pair of coupled KdV type NLS equation.

The Generalised Non-linear Schrödinger is essentially a combination of the ordinary NLS and the coupled KdV type NLS equation and can be obtained similarly.

4.2 The Derivative Non-linear Schrödinger Equation (Kaup-Newell hierarchy)

Here we assume $U$ to have the form

\[
U = \begin{pmatrix}
-i\lambda^2 & \lambda q \\
\lambda r & i\lambda^2
\end{pmatrix}
\]  

(74)

and use it in (58) to solve for $V$ with $V$ chosen as

\[
\begin{pmatrix}
a & b \\
c & -a
\end{pmatrix}
\]

Simple algebra yields

\[
a_x = \alpha(qc - rb) \\
b_x = -2i\alpha^2b - 2\lambda qa \\
c_x = 2i\lambda^2c + 2\lambda ra
\]  

(75)

Expanding the elements $a, b, c$ of the matrix $V$ as

\[
a = \sum_{m \geq 0} a_m \lambda^{-m} , \quad b = \sum_{m \geq 0} b_m \lambda^{-m} , \quad c = \sum_{m \geq 0} c_m \lambda^{-m}
\]

we arrive at the following recurrence relationships:

\[
a_{mx} = qc_{m+1} - rb_{m+1} \\
b_{mx} = -2ib_{m+2} - 2qa_{m+1} \\
c_{mx} = 2ic_{m+2} + 2ra_{m+1}
\]  

(76)
Choosing \( b_0 = 0, c_0 = 0, a_1 = 0 \) we obtain
\[
\begin{align*}
a_0 &= \alpha, b_1 = 2q, c_1 = 2r \\
b_2 &= 0, c_2 = 0, a_2 = -iqr \\
b_3 &= iq_x + q^2 r, c_3 = -i r_x + qr^2, a_3 = 0 \\
b_4 &= 0, c_4 = 0, a_4 = \frac{1}{2}(r q_x - q r_x) - \frac{3}{2} i r x q
\end{align*}
\] (77)

In general it is found that
\[
\begin{align*}
a_{2j+1} &= 0 \\
b_{2j} &= 0 \\
c_{2j} &= 0
\end{align*}
\] (78)

for \( j = 0, 1, 2, \ldots \), hence let us write
\[
\begin{align*}
a &= \sum_{j \geq 0} a_{2j} \lambda^{-2j} \\
b &= \sum_{j \geq 0} b_{2j+1} \lambda^{-(2j+1)} \\
c &= \sum_{j \geq 0} c_{2j+1} \lambda^{-(2j+1)}
\end{align*}
\] (79)

so that
\[
V = \sum_{j \geq 0} \left( \begin{array}{cc} a_{2j} \lambda^{-2j} & b_{2j+1} \lambda^{-(2j+1)} \\
-2c_{2j} \lambda^{-(2j+1)} & a_{2j} \lambda^{-(2j+1)} \end{array} \right)
\] (80)

We now construct \( \tilde{V}^{(n)} \) such that it contains positive powers of \( \lambda \) only by defining
\[
\tilde{V}^{(n)} = (\lambda^{2n+2} V)_+ = \sum_{j=0}^{n} \left( \begin{array}{cc} a_{2j} \lambda^{2(n-j)+2} & b_{2j+1} \lambda^{2(n-j)+1} \\
-2c_{2j+1} \lambda^{2(n-j)+1} & a_{2j} \lambda^{2(n-j)+2} \end{array} \right)
\] (81)

We now calculate
\[
\tilde{V}^{(n)} - [U, \tilde{V}^{(n)}]
\]
and obtain,
\[
\begin{align*}
c_{11} &= \sum_{j=0}^{n} \lambda^{2(n-j)+2}[a_{2j} - (qc_{2j+1} - rb_{2j+1})] \\
c_{12} &= \sum_{j=0}^{n} b(2j+1) x \lambda^{2(n-j)+1} + 2 \sum_{j=0}^{n} (ib_{2j+1} \lambda^{2(n-j)+3} + qa_{2j}\lambda^{2(n-j)+3}) \\
c_{21} &= \sum_{j=0}^{n} c(2j+1) x \lambda^{2(n-j)+1} - 2 \sum_{j=0}^{n} (ic_{2j+1} + ra_{2j}) \lambda^{2(n-j)+3}
\end{align*}
\] (82)

On using the recurrence relations, the diagonal element vanishes while the off-diagonal elements yield \( \lambda b_{(2n+1)x} \) and \( \lambda c_{(2n+1)x} \) respectively. Thus
\[
\tilde{V}^{(n)} - [U, \tilde{V}^{(n)}] = \lambda \left( \begin{array}{cc} 0 & b_{(2n+1)x} \\
c_{(2n+1)x} & 0 \end{array} \right)
\] (83)

from the above it is clear that \( \Delta_n = 0 \) whence, \( \tilde{V}^{(n)} \) becomes equal to \( V^{(n)} = (\lambda^{2n+2} V)_+ \). The AKS equation now yields
\[
\begin{align*}
q_t &= b_{(2n+1)x} \\
r_t &= c_{(2n+1)x}
\end{align*}
\] (84)
which is the hierarchy of non-linear evolution equations for Kaup-Newell system. Putting \( n = 1 \), we obtain

\[
\begin{align*}
q_t &= b_{3x} = iq_{xx} + q^2r_x + 2qq_x r \\
r_t &= c_{3x} = -ir_{xx} + r^2q_x + 2rr_x q
\end{align*}
\]  

(85)

Setting \( n = 2 \) gives a higher order eqn of the system,

\[
\begin{align*}
q_t &= b_{5x} = -\frac{1}{2}q_{xxx} + \frac{3}{4}(q^2r^2)_x + \frac{3i}{2}(qqr)_x \\
r_t &= c_{5x} = -\frac{1}{2}r_{xxx} + \frac{3}{4}(q^2r^3)_x - \frac{3i}{2}(rrq)_x
\end{align*}
\]  

(86)

4.3 The Derivative Non-linear Schrödinger equations - a general structure

Let us expand the orbit by including another field variable \( s \) in the diagonal term, whence \( U \) takes the form

\[
\left( \begin{array}{ccc}
- i\lambda^2 - is & \lambda q \\
\lambda r & i\lambda^2 + is
\end{array} \right)
\]  

(87)

as before, taking \( V = \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right) \) and expanding the elements \( a \), \( b \) and \( c \) we obtain the recurrence relations

\[
\begin{align*}
a_{mx} &= qc_{m+1} - rb_{m+1} \\
b_{mx} &= -2ib_{m+2} - 2isb_m - 2qa_{m+1} \\
c_{mx} &= 2ic_{m+2} + 2isc_m + 2ra_{m+1}
\end{align*}
\]  

(88)

from which it can be shown that

\[
a_{(m+1)x} = (rsb_m - qsc_m) - \frac{i}{2}(qc_{mx} + rb_{mx})
\]  

(89)

Solving the system (88) we obtain successively,

\[
\begin{align*}
a_0 &= -2i, b_0 = 0, c_0 = 0 \\
a_1 &= 0, b_1 = 2q, c_1 = 2r \\
a_2 &= -iqr, b_2 = 0, c_2 = 0 \\
a_3 &= 0, b_3 = iq_x - 2qs + q^2r, c_3 = -ir_x - 2rs + qr^2 \\
a_4 &= \frac{1}{2}(rq_x - qr_x) - \frac{3i}{4}q^2r^2 + 2isqr, b_4 = 0, c_4 = 0
\end{align*}
\]  

(90)

and so on

In general then it is found that

\[
a_{2j+1} = 0, b_{2j} = 0, c_{2j} = 0
\]  

(91)

for \( j = 0, 1, 2, 3, \ldots \)

\[
V = \sum_{j \geq 0} \left( \begin{array}{ccc}
a_{2j} \lambda^{-2j} & b_{2j+1} \lambda^{-(2j+1)} \\
c_{2j+1} \lambda^{-(2j+1)} & -a_{2j} \lambda^{-2j}
\end{array} \right)
\]  

(92)
As before taking $\tilde{V}^{(n)} = (\lambda^{2n+2}V)_+$ and evaluating the expression $\tilde{V}_x^{(n)} - [U, \tilde{V}^{(n)}]$ we obtain

$$\tilde{V}_x^{(n)} - [U, \tilde{V}^{(n)}] = \lambda(b(2n+1)x + 2isb_{2n+1})e_{12} + \lambda(c(2n+1)x + 2isc_{2n+1})e_{21}$$

(93)

Since there is no diagonal element on the RHS of the above, we define

$$V^{(n)} = \tilde{V}^{(n)} + \Delta_n$$

(94)

where $\Delta_n$ is taken to be of the form

$$\Delta_n = \begin{pmatrix} \delta_n & 0 \\ 0 & -\delta_n \end{pmatrix}$$

(95)

Using equations (92) – (94) in the zero curvature equation (60) , we are led to the following dynamical equations.

$$-is_t = \delta_{nx}$$

$$q_t = b_{(2n+1)x} + 2isb_{2n+1} + 2\delta_n$$

$$r_t = c_{(2n+1)x} - 2isc_{2n+1} - 2r\delta_n$$

(96)

But $\delta_n$ is yet undetermined. To determine $\delta_n$, let us impose the condition

$$s = \beta qr$$

(97)

where $\beta$ is a constant. Using (97) in the system of equations (96) we obtain after simplification,

$$\delta_n = 2\beta\partial^{-1}[(rsb_{2n+1} - qsc_{2n+1}) - \frac{i}{2}(qc_{(2n+1)x} + rb_{(2n+1)x})]$$

$$= 2\beta\partial^{-1}a_{2(2n+1)}$$

$$= 2\beta a_{2(2n+1)}$$

(98)

where we have used (89) to simplify the expression in the square bracket. Putting back (98) in (96) we obtain the following dynamical systems

$$q_t = b_{(2n+1)x} + 2i\beta qr b_{2n+1} + 4\beta qa_{2(n+1)}$$

$$r_t = c_{(2n+1)x} - 2i\beta qrc_{2n+1} - 4\beta ra_{2(n+1)}$$

(99)

These represent a coupled system of hierarchy of equations. Putting $n = 1$, we obtain,

$$q_t = b_{3x} + 2i\beta qr b_{3} + 4\beta qa_4$$

$$r_t = c_{3x} - 2i\beta qrc_{3} - 4\beta ra_4$$

(100)

After using $s = \beta qr$ , we obtain the following expressions for $b_3, c_3$ and $a_4$ viz.

$$b_3 = iq_x - (2\beta - 1)q^2r$$

$$c_3 = -ir_x - (2\beta - 1)qr^2$$

$$a_4 = \frac{1}{4}(rq_x - qr_x) + (2\beta - \frac{3}{4})iq^2r^2$$

(101)
Hence (100) yields,
\[
q_t = iq_{xx} - (4\beta - 1)q^2 r_x - 2(2\beta - 1)qq_x r + i\beta(4\beta - 1)q^3 r^2
\]
\[
r_t = -ir_{xx} - (4\beta - 1)r^2 q_x - 2(2\beta - 1)rr_x q - i\beta(4\beta - 1)q^2 r^3
\]

Eqns (102) represent coupled Kundu type systems.
Several reductions of (102) are possible.
Putting \( \beta = 0 \), we get,
\[
q_t = iq_{xx} + (q^2 r)_x
r_t = -ir_{xx} + (qr^2)_x
\]
which form a coupled Kaup-Newell (KN) system.
\( \beta = \frac{1}{4} \) leads to,
\[
q_t = iq_{xx} + qq_x r
r_t = -ir_{xx} + rr_x q
\]
which is the coupled Chen-Lee-Liu (CLL) system.
Finally, \( \beta = \frac{1}{2} \) yields,
\[
q_t = iq_{xx} - q^2 r_x + \frac{i\beta}{2} q^3 r^2
r_t = -ir_{xx} - r^2 q_x - \frac{i\beta}{2} q^2 r^3
\]
a coupled GI system.
Putting \( r = q^* \) in the above system of equations leads to further reductions.
\[ \text{eg. setting } r = q^* \text{ in (97) leads to} \]
\[
q_t = iq_{xx} - (4\beta - 1)q^2 q^*_x - 2(2\beta - 1)|q|^2 q_x + i\beta(4\beta - 1)|q|^4 q
\]
which represents Kundu type equation.

5 The trace identity

The trace identity is a powerful tool for constructing infinite-dimensional Liouville integrable Hamiltonian systems. Starting from a properly chosen spectral problem, many integrable hierarchies and their Hamiltonian structures can be obtained by using trace identity method.

As noted before, let \( g \) be a finite dimensional semi-simple Lie algebra and \( \Omega g \) the corresponding loop algebra.
The Killing-Cartan form \( \langle x, y \rangle \) is taken to be equal to \( tr(xy) \) where \( x, y \in g \) i.e. \( \langle x, y \rangle = tr(xy) \)
Let \( U = U(\lambda, u) \) be an element of \( \Omega g \) that depends on \( \lambda \) and \( u = (u_i) \), where \( u_i \) are the field variables.
For any solution \( V \) of the stationary eqn (58), which is of homogeneous rank, there exists a constant \( \gamma \) such that for \( \bar{V} = \lambda^\gamma V \) which is again a solution of (58) it is true that
\[
\frac{\delta}{\delta u_i} < V, \frac{\partial U}{\partial \lambda} > = \frac{\partial}{\partial \lambda} < V, \frac{\partial U}{\partial u_i} >
\]
This is the famous trace identity which on putting \( \bar{V} = \lambda^\gamma V \) reduces to
\[
\frac{\delta}{\delta u_i} < V, \frac{\partial U}{\partial \lambda} > = (\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma) < V, \frac{\partial U}{\partial u_i} >
\]
Here \( \frac{\delta}{\delta u} \) stands for the variational derivative given by

\[
\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \left( \frac{\partial F}{\partial u_x} \right)_x + \left( \frac{\partial F}{\partial u_{xx}} \right)_{xx} - \ldots
\]  

(109)

Further let \( J \) and \( L \) be two linear operators mapping \( S^M \) into itself. Here \( S \) denotes the Schwartz space over \( \mathbb{R} \) and \( S^M = S \otimes \ldots \otimes S \) (\( M \) times) where \( \otimes \) denotes the direct/outer/tensor product.

Suppose that

(i) both \( J \) and \( JL \) are skew symmetric i.e.

\[ J^* = -J \quad \text{and} \quad JL = L^*J \]  

(110)

(ii) there exists a series of scalar functions \( \{H_n\} \), for which it is true that

\[ L^n f(u) = \frac{\delta H_n}{\delta u} \]  

(111)

for some \( f(u) \in S^M \).

Then \( \{H_n\} \) is a common series of conserved derivatives for the whole hierarchy of equations.

\[ u_t = JL^n f(u) \]  

(112)

and further

\[ \{H_n, H_m\} = 0 \]  

(113)

The above conditions are used to investigate the Hamiltonian structures of several integrable systems in the following sections.

### 5.1 The Hamiltonian structure of the AKNS hierarchy

In this case, \( U \) and \( V \) are given as in (61) and the hierarchy is given by (70).

The hierarchy (70) can be rewritten as

\[
\begin{align*}
 u_t &= \begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -2ib_{n+1} \\ 2ic_{n+1} \end{pmatrix} = J \begin{pmatrix} ic_{n+1} \\ ib_{n+1} \end{pmatrix} \\
 \text{where} \\
 J &= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}
\end{align*}
\]  

(114)

Further the operator

\[
L = \begin{pmatrix} -\frac{1}{2}\partial + ir\partial^{-1}q & -ir\partial^{-1}r \\ iq^{-1}\partial & \frac{1}{2}\partial - iq\partial^{-1}r \end{pmatrix}
\]  

(116)

is such that

\[
L^n \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix}
\]  

(117)
To check this we note that

\[ L \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} \partial + ir^{-1}q & -ir^{-1}r \\ iq^{-1}q & \frac{i}{2} \partial - iq^{-1}r \end{pmatrix} \begin{pmatrix} i\alpha r \\ iq \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} r_x \\ -\frac{\alpha}{2} q_x \end{pmatrix} = \begin{pmatrix} c_2 \\ b_2 \end{pmatrix} \]

and

\[ L^2 \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} \partial + ir^{-1}q & -ir^{-1}r \\ iq^{-1}q & \frac{i}{2} \partial - iq^{-1}r \end{pmatrix} \begin{pmatrix} \frac{\alpha}{2} r_x \\ -\frac{\alpha}{2} q_x \end{pmatrix} = \begin{pmatrix} -\frac{i\alpha}{4} r_{xx} + \frac{i\alpha}{2} q r^2 \\ -\frac{i\alpha}{4} q_{xx} + \frac{i\alpha}{2} q r^2 \end{pmatrix} = \begin{pmatrix} c_3 \\ b_3 \end{pmatrix} \]

which proves assertion (117) in these special cases. With

\[ J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad \text{we have} \quad J^* = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \]

Thus,

\[ J^* = -J \]

Further,

\[ L^* = \begin{pmatrix} \frac{i}{2} \partial - iq^{-1}r & -iq^{-1}q \\ ir^{-1}r & -\frac{i}{2} \partial + ir^{-1}q \end{pmatrix} \]

We also note that

\[ JL = L^*J = \begin{pmatrix} -2iq^{-1}r & -2(\frac{i}{2} \partial - iq^{-1}r) \\ 2(-\frac{i}{2} \partial + ir^{-1}q) & -2ir^{-1}r \end{pmatrix} \]

For using the trace identity we compute the following,

\[ \langle V, \frac{\partial U}{\partial \lambda} \rangle = -2ia, \]
\[ \langle V, \frac{\partial U}{\partial q} \rangle = c, \]
\[ \langle V, \frac{\partial U}{\partial r} \rangle = b \]

The trace identity now yields

\[ \frac{\delta}{\delta q}(-2ia) = \lambda^{-\gamma} \frac{\delta}{\delta x}(\lambda^\gamma c) \]

or, \[ -2i \frac{\delta}{\delta q} \left( \sum_{m \geq 0} a_m \lambda^{-m} \right) = \lambda^{-\gamma} \frac{\delta}{\delta x} (\lambda^\gamma \sum_{m \geq 0} c_m \lambda^{-m}) \]

or, \[ -2i \frac{\delta}{\delta q} \left( \sum_{m \geq 0} a_m \lambda^{-m} \right) = \sum_{m \geq 0} c_m (\gamma - m) \lambda^{-m-1} \]

Similarly, we have

\[ -2i \frac{\delta}{\delta r} \left( \sum_{m \geq 0} a_m \lambda^{-m} \right) = \sum_{m \geq 0} b_m (\gamma - m) \lambda^{-m-1} \]

Equating the coefficients of \( \lambda^{-n-2} \) on both sides of (121) and (122), we obtain

\[ -2i \left( \frac{\partial}{\partial q}, \frac{\partial}{\partial r} \right) (a_{n+2}) = (\gamma - n - 1)(c_{n+1}, b_{n+1}) \]

To determine the unknown constants \( \gamma \), we put \( n = 0 \) in (123) and obtain

\[ -2i \frac{\delta}{\delta q} (a_2) = (\gamma - 1)c_1 \]
Using the values of $a_2$ and $c_1$ and the definition of the variational derivative in the above equation we obtain

$$-2i \frac{\delta}{\delta q} (\frac{\alpha}{2} qr) = (\gamma - 1)i\alpha r$$

or, $$-2(\frac{\alpha}{2}r) = (\gamma - 1)i\alpha r$$

which leads to $\gamma = 0$

Thus we are left with

$$\left( \frac{\partial}{\partial q}, \frac{\partial}{\partial r} \right) (2i \frac{a_{n+2}}{n+1}) = (c_{n+1}, b_{n+1})$$

(124)

where,

$$H_n = 2i \left( \frac{a_{n+2}}{n+1} \right)$$

(125)

The AKNS hierarchy can now be cast in the Hamiltonian form

$$u_t = \left( \begin{array}{c} q \\ r \end{array} \right)_t = JL^n \left( \begin{array}{c} c_1 \\ b_1 \end{array} \right) = J \frac{\delta H_n}{\delta u}$$

(126)

where $J$ and $L$ are defined previously and $H_n$ are hierarchy of commuting conserved functionals.

### 5.2 Hamiltonian structure of the Kaup-Newell hierarchy

This hierarchy is defined by

$$U = \begin{pmatrix} -i\lambda^2 & \lambda q \\ \lambda r & i\lambda^2 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

The hierarchy can be expressed as

$$u_t = \left( \begin{array}{c} q \\ r \end{array} \right)_t = \left( \begin{array}{c} b_{(2n+1)x} \\ c_{(2n+1)x} \end{array} \right) = J \left( \begin{array}{c} c_{2n+1} \\ b_{2n+1} \end{array} \right)$$

(127)

with $J$ given by

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}$$

(128)

The operators $L_1$ and $L_2$ defined by

$$L_1 = \frac{1}{2} \begin{pmatrix} r\partial^{-1}r & -i + r\partial^{-1}q \\ i + q\partial^{-1}r & q\partial^{-1}q \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}$$

(129)

are such that

$$L_1 L_2 \left( \begin{array}{c} c_{2n-1} \\ b_{2n-1} \end{array} \right) = \left( \begin{array}{c} c_{2n+1} \\ b_{2n+1} \end{array} \right)$$

(130)

As a check one notes that

$$\frac{1}{2} \begin{pmatrix} r\partial^{-1}r & -i + r\partial^{-1}q \\ i + q\partial^{-1}r & q\partial^{-1}q \end{pmatrix} \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} -irx + r^2q \\ iqx + q^2r \end{pmatrix} = \begin{pmatrix} c_3 \\ b_3 \end{pmatrix}$$
In general then

\[(L_1 L_2)^n \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} c_{2n+1} \\ b_{2n+1} \end{pmatrix}\]  

(131)

To use the trace identity, we compute the following

\[\langle V, \frac{\partial U}{\partial \lambda} \rangle = -4i\lambda a + rb + qc,\]
\[\langle V, \frac{\partial U}{\partial q} \rangle = \lambda c,\]
\[\langle V, \frac{\partial U}{\partial r} \rangle = \lambda b\]

(132)

The trace identity gives

\[\frac{\delta}{\delta q} (-4i\lambda a + rb + qc) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} (\lambda^\gamma \lambda c)\]
and
\[\frac{\delta}{\delta r} (-4i\lambda a + rb + qc) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} (\lambda^\gamma \lambda b)\]

(133)

Expanding \(a, b\) and \(c\) in the first of the above expressions and equating the coefficients of \(\lambda^{-(2m+1)}\) on both sides, we obtain,

\[\frac{\delta}{\delta q} [-4ia_{2m+2} + rb_{2m+1}qc_{2m+1}] = (\gamma - 2m)c_{2m+1}\]

Putting \(m = 0\) in the above equation we obtain \(\gamma = 0\)

Using the trace identity we have

\[\left( \frac{\delta}{\delta q}, \frac{\delta}{\delta r} \right) H_m = (c_{2m+1}, b_{2m+1})\]

(134)

with

\[H_m = \frac{4ia_{2m+2} - rb_{2m+1} - qc_{2m+1}}{2m}\]

(135)

and the hierarchy given by

\[u_t = J \frac{\delta H_m}{\delta u}\]

5.3 The Hamiltonian formalism for the general structure of the DNLS equations

In the general case, one starts with the following spectral problem,

\[U = \begin{pmatrix} -i\lambda^2 - i\beta qr & \lambda q \\ \lambda r & i\lambda^2 + i\beta qr \end{pmatrix}\]

(136)

as already discussed in some detail in section 4.3

Since the steps to be followed in order to obtain the Hamiltonian structure have already been explained in the preceding two subsections, for this example we just provide the outline of the procedure.
The coupled system of the hierarchy of equations are given by (99) and can be cast in the form
\[
\begin{pmatrix} q \\ r \end{pmatrix}_t = L_3 L_2 \begin{pmatrix} c_{2n+1} \\ b_{2n+1} \end{pmatrix}
\] (137)
where \( L_3 \) and \( L_2 \) are given by
\[
L_3 = \begin{pmatrix} 1 - 2i\beta q \partial^{-1}r & -2i\beta q \partial^{-1}q \\ 2i\beta r \partial^{-1}r & 1 + 2i\beta r \partial^{-1}q \end{pmatrix}, L_2 = \begin{pmatrix} 0 & \partial + 2i\beta qr \\ \partial - 2i\beta qr & 0 \end{pmatrix}
\] (138)
The coefficients in the expansion of \( b \) and \( c \) are related by
\[
L_1 L_2 \begin{pmatrix} c_{2j-1} \\ b_{2j-1} \end{pmatrix} = \begin{pmatrix} c_{2j+1} \\ b_{2j+1} \end{pmatrix}
\] (139)
where
\[
L_1 = \frac{1}{2} \begin{pmatrix} r \partial^{-1}r & -i + r \partial^{-1}q \\ i + q \partial^{-1}r & q \partial^{-1}q \end{pmatrix},
\] (140)
and \( L_2 \) is already defined above. Here \( L_1 \) and \( L_2 \) are skew-symmetric operators i.e.
\[
L_1^* = -L_1
\]
and
\[
L_2^* = -L_2
\]
Extending (139) we can write
\[
(L_1 L_2)^n \begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} c_{2n+1} \\ b_{2n+1} \end{pmatrix}
\] (141)
where
\[
c_1 = 2r \quad \text{and} \quad b_1 = 2q
\]. Let us now introduce the function
\[
P_{2j+1} = \begin{pmatrix} c_{2j+1} - 2i\beta ra_{2j} \\ b_{2j+1} - 2i\beta qa_{2j} \end{pmatrix}
\] (142)
It can be shown that,
\[
L_3^* P_{2j+1} = \begin{pmatrix} c_{2j+1} \\ b_{2j+1} \end{pmatrix}, \quad j \geq 0
\] (143)
where we need to use the relation
\[
a_{2jx} = q c_{2j+1} - rb_{2j+1}
\]
given by equation (88) and \( L_3^* \) denotes the conjugate of \( L_3 \).
The hierarchy of equations can now be written as
\[
u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = L_3 L_2 \begin{pmatrix} c_{2n+1} \\ b_{2n+1} \end{pmatrix} = L_3 L_2 L_3^* P_{2n+1}
\]
or,
\[
u_t = JP_{2n+1} \quad \text{where}
\]
\[
J = L_3 L_2 L_3^*
\] (144)
It may further be shown that the operators $JL^k (k = 0, 1, 2, \ldots m)$ are skew-symmetric. For the trace identity, we compute

$$\langle V, \frac{\partial U}{\partial \lambda} \rangle = -4i\lambda a + rb + qc$$

$$\langle V, \frac{\partial U}{\partial q} \rangle = c\lambda - 2i\beta ra$$

$$\langle V, \frac{\partial U}{\partial r} \rangle = b\lambda - 2i\beta qa$$

(145)

The trace identity gives,

$$\frac{\delta}{\delta u} (-4i\lambda + rb + qc) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} [\lambda^{\gamma} (\lambda c - 2i\beta ra, \lambda b - 2i\beta qa)]$$

(146)

Expanding a, b and c in negative powers of $\lambda$ and equating coefficients of $\lambda^{-(2n+1)}$, we arrive at the relation

$$\frac{\delta}{\delta u} (-4ia_{2n+2} + rb_{2n+1} + qc_{2n+1}) = (\gamma - 2n)(c_{2n+1} - 2i\beta ra_{2n}, b_{2n+1} - 2i\beta qa_{2n})$$

(147)

Putting $n = 0$, we find $\gamma = 0$.

Therefore,

$$\frac{\delta}{\delta u} \left[ \frac{4ia_{2n+2} - rb_{2n+1} + qc_{2n+1}}{2n} \right]$$

$$= (c_{2n+1} - 2i\beta ra_{2n}, b_{2n+1} - 2i\beta qa_{2n})$$

(148)

or,$$P_{2n+1} = \frac{\delta H_n}{\delta u}$$

where

$$H_0 = 2qr, \quad H_n = \frac{4ia_{2n+2} - rb_{2n+1} + qc_{2n+1}}{2n}, \quad n \geq 1$$

(149)

This means that the hierarchy can be expressed as

$$u_t = J \frac{\delta H_n}{\delta u}$$

(150)

In view of the theory outlined previously, the complete integrability of the system is established.

It is worthwhile to note that the hierarchy of the evolution equations in all the cases discussed above can be put in the form

$$u_t = J \frac{\delta H_n}{\delta u} = JL \frac{\delta H_{n-1}}{\delta u} = \ldots = JL^n \frac{\delta H_0}{\delta u}, \quad n = 1, 2, 3, \ldots$$

(150)

in the light of the foregoing discussion where $J$ and $L$ are already defined along with their properties.

Equation (150) emphasizes the multi-Hamiltonian formulation of the hierarchy of nonlinear evolution equations in all the different cases discussed above.

6 The Non-holonomic Deformation of Integrable Systems

It would be pertinent to explain what exactly is meant by the Non-holonomic deformation (NHD) of integrable systems. Perturbation generally disturbs the integrability
of a system. However, when we consider NHD of an integrable system, the system gets perturbed with a deforming function in such a way that under suitable differential constraints on the perturbing function, the system maintains its integrability. The constraints are furnished in the form of differential relations and they turn out to be equivalent to a non-holonomic constraint.

To construct these non-holonomic deformations, one starts with a lax pair, keeping the space part $U(\lambda)$ unchanged but modifying the temporal component $V(\lambda)$. This implies that the scattering problem remain unchanged, but the time evolution of the spectral data becomes different in the perturbed models. Corresponding to these deformed systems, it is possible to generate some kind of two-fold integrable hierarchy. One method is to keep the perturbed equations the same but increase the order of the differential constraints in a recursive manner, thus generating a new integrable hierarchy for the deformed system. Alternatively, the constraint may be kept fixed at its lowest level, but the order of the original equation may be increased in the usual way, thereby leading to new hierarchies of integrable systems.

6.1 Non-Holonomic Deformation (NHD) of the Non-linear Schrödinger Equation (NLSE)

The spatial and temporal components of the Lax pair of the NLS equation are given by,

$$U = -i\lambda \sigma_3 + q\sigma_+ + r\sigma_- \quad (151)$$

$$V_{\text{original}} = -i\lambda^2 \sigma_3 + \lambda(q\sigma_+ + r\sigma_-) - \left(\frac{i}{2}\right) qr\sigma_3 + \left(\frac{i}{2}\right) qx \sigma_+ - \left(\frac{i}{2}\right) rx \sigma_- \quad (152)$$

To obtain the deformation of the NLS equation, let us introduce

$$V_{\text{deformed}} = \frac{i}{2}\lambda^{-1} G^{(1)} \quad (153)$$

where

$$G^{(1)} = a\sigma_3 + g_1 \sigma_+ + g_2 \sigma_- \quad (154)$$

So that the time part of the Lax pair takes the form

$$\tilde{V} = V_{\text{original}} + V_{\text{deformed}} \quad (155)$$

Let us now impose the zero curvature or the flatness condition

$$U_t - \tilde{V}_x + [U, \tilde{V}] = 0$$

with $U$ and $V$ as in (151) and (155) respectively.

We observe from the zero curvature condition that while the positive powers of \lambda are trivially satisfied, the zeroth power (or the \lambda free term) leads to the perturbed dynamical systems (equations), while the negative powers of \lambda give rise to the differential
constraints.

For example, the deformed pair of the NLS equations are given by,

\[ q_t - \frac{i}{2}q_{xx} + iq^2r = -g_1 \]  \hspace{1cm} (156)

\[ r_t + \frac{i}{2}r_{xx} - iq^2 = g_2 \]  \hspace{1cm} (157)

Considering the \( \lambda^{-1} \) terms and equating the coefficients of the generators \( \sigma_3, \sigma_+, \sigma_- \) successively, we obtain the following individual constraint conditions on the functions \( a, g_1 \) and \( g_2 \)

\[ a_x = qg_2 - rg_1 \]  \hspace{1cm} (158)

\[ g_{1x} + 2aq = 0 \]  \hspace{1cm} (159)

\[ g_{2x} - 2ar = 0 \]  \hspace{1cm} (160)

The foregoing equations can be shown to give rise to the differential constraint

\[ \hat{L}(g_1, g_2) = r g_{1xx} + q_x g_{2x} + 2qr(qg_2 - r g_1) = 0 \]  \hspace{1cm} (161)

Eliminating the deforming functions \( g_1 \) and \( g_2 \), we can derive a new higher order equation as

\[
\begin{align*}
& -r(q_t - \frac{i}{2}q_{xx} + iq^2r)_{xx} + q_x(r_t + \frac{i}{2}r_{xx} - iq^2)_{x} \\
& + 2qr[q(r_t + \frac{i}{2}r_{xx} - iq^2) + r(q_t - \frac{i}{2}q_{xx} + iq^2r)] = 0
\end{align*}
\]  \hspace{1cm} (162)

We can now consider a double deformation of the NLS equation by taking

\[ V_{\text{deformed}}(\lambda) = i\frac{1}{2}(\lambda^{-1}G^{(1)} + \lambda^{-2}G^{(2)}) \]  \hspace{1cm} (163)

where the function \( G^{(2)} \) is given by

\[ G^{(2)} = b\sigma_3 + f_1\sigma_+ + f_2\sigma_- \]  \hspace{1cm} (164)

and \( G^{(1)} \) is already defined in equation (154).

The zero-curvature condition is now applied with \( U \) as before but \( V_{\text{deformed}} \) as defined in (163). The following results arise:

(i) No change occurs in the deformed NLS equations.

(ii) Picking up the terms in \( \lambda^{-1} \) and equating the coefficients of the generators \( \sigma_3, \sigma_+, \sigma_- \) successively, we are led to the following individual constraints

\[ a_x = qg_2 - rg_1 \]  \hspace{1cm} (165)

\[ g_{1x} + 2if_1 + 2aq = 0 \]  \hspace{1cm} (166)

\[ g_{2x} - 2if_2 - 2ar = 0 \]  \hspace{1cm} (167)

The preceding set of equations finally lead to the following differential constraint
with

\[ \hat{L}(g_1, g_2) = rg_{1xx} + g_2xq_x + 2qr(gg_2 - rg_1) \]  

(iii) The terms in \( \lambda^{-2} \) give rise to a second constraint

\[ L(f_1, f_2) = 0 \]  

where the functional form of the above expression is already given by (169) while \( f_1, f_2 \) make up the argument in (170).

Thus, this is an example where the perturbed equations are kept the same, but the order of the differential constraint is increased recursively, thereby creating a new integrable hierarchy for the NLS equation.

### 6.2 NHD of coupled KdV type NLSE

For the coupled KdV type NLSE, the space and time components of the Lax pair are given by,

\[
U = -i\lambda\sigma_3 + q\sigma_+ + r\sigma_- \quad \text{and} \\
V_{\text{original}} = -i\lambda^3\sigma_3 + \lambda^2(q\sigma_+ + r\sigma_-) + \lambda[(-\frac{i}{2}qr)\sigma_3 + \frac{1}{2}g_+\sigma_+ - \frac{1}{2}g_-\sigma_-] + \frac{1}{4}(rq_x - qr_x)\sigma_3 + (\frac{1}{2}q^2r - \frac{1}{2}g_{xx})\sigma_+ + (\frac{1}{2}qr^2 - \frac{1}{2}r_{xx})\sigma_- 
\]

Note that \( V_{\text{original}} \) now includes a term in \( \lambda^3 \) as compared to \( \lambda^2 \) in the previous example of the NLS equation. This would lead to a higher order dispersion term.

Take \( V_{\text{deformed}} = \frac{1}{2}\lambda^{-1}G(1) \)

depth therefore, \( \hat{V} = V_{\text{original}} + V_{\text{deformed}}. \)

Using the zero-curvature condition, we arrive at the following deformed equations:

\[ q_t + \frac{1}{4}q_{xxx} - \frac{3}{2}g_{xx}r = -g_1 \]  

and

\[ r_t + \frac{1}{4}r_{xxx} - \frac{3}{2}rr_xq = g_2 \]  

along with the differential constraint

\[ \hat{L}(g_1, g_2) = 0 \]

In this example, the constraint is held fixed at its lowest level, but the order of the NLS equation is increased (terms enter with higher order dispersion) and thus a new integrable hierarchy can be formed.

The generalized NLSE is actually a combination of the ordinary NLSE and the coupled KdV type NLSE. NHD of such a system can be carried out in the manner already outlined previously.
6.3 NHD of Derivative NLS equation (DNLS) : Kaup-Newell (KN) system

In this case, the Lax pair are given by

\[
U = -i\lambda^2 \sigma_3 + \lambda (q \sigma_+ + r \sigma_-)
\]  
(175)

\[
V_{\text{original}} = -i\lambda^4 \sigma_3 + \lambda^3 (q \sigma_+ + r \sigma_-) - \lambda^2 qr (\sigma_+ + \sigma_-)
\]  
(176)

The modified temporal component of the Lax pair is given as

\[
\tilde{V} = V_{\text{original}} + V_{\text{deformed}}
\]

where

\[
V_{\text{deformed}} = i(G(0) + \lambda^{-1}G(1) + \lambda^{-2}G(2))
\]  
(177)

and

\[
G(0) = w \sigma_3 + m_1 \sigma_+ + m_2 \sigma_-
\]  
(178)

\[
G(1) = a \sigma_3 + g_1 \sigma_+ + g_2 \sigma_-
\]  
(179)

\[
G(2) = b \sigma_3 + f_1 \sigma_+ + f_2 \sigma_-
\]  
(180)

Using the zero-curvature relation, we obtain the following deformed DNLS equations:

\[
q_t - \frac{i}{2} q_{xx} - \frac{1}{2} (q^2 r)_x + 2g_1 - 2iqw = 0
\]  
(181)

\[
r_t + \frac{i}{2} r_{xx} - \frac{1}{2} (qr^2)_x - 2g_2 + 2irw = 0
\]  
(182)

Further, we obtain the following conditions on the different components of the deforming functions \( G(i) \): \( m_1 = 0, m_2 = 0, a = 0, f_1 = 0, f_2 = 0 \) and \( b_x = 0 \) which implies that \( b = b(t) \) only.

We are, therefore, left with the following deforming functions:

\[
G(0) = w(x,t) \sigma_3
\]

\[
G(1) = g_1(x,t) \sigma_+ + g_2(x,t) \sigma_-
\]

\[
G(2) = b(t) \sigma_3
\]  
(183)

Moreover, the following constraints are obtained:

\[
g_{1x} = -2q(x,t)b(t)
\]

\[
g_{2x} = 2r(x,t)b(t)
\]

\[
w_x = qg_2 - rg_1
\]  
(184)

It is possible to obtain new non-linear integrable equations by resolving the constraint relations and expressing all the perturbing functions through the basic field variables. To this end, we put

\[
q = u_x
\]

\[
r = v_x
\]  
(185)
where $u = u(x,t)$ and $v = v(x,t)$

Equation (185) used in equation (184) allows us to express $g_1$, $g_2$ and $w$ in terms of $b(t)$, $u$ and $v$ only as follows:

$$g_1 = -2b(t)u$$
$$g_2 = 2b(t)v$$
$$w = 2b(t)uv + K(t)$$

(186)

where $K$ is again a function of $t$ only.

Eliminating $g_1$, $g_2$ and $w$ from equations (181) and (182), we can rewrite the coupled perturbed (deformed) DNLS equations in the following form:

$$u_{xt} - rac{i}{2}u_{xxx} - rac{1}{2}(u^2_v)_x - 4ub(t) - 2iu_x(2b(t)uv + K(t)) = 0$$

(187)

$$v_{xt} + rac{i}{2}v_{xxx} - rac{1}{2}(u_xv^2)_x - 4v_b(t) + 2iv_x(2b(t)uv + K(t)) = 0$$

(188)

These are coupled evolution equations which are non-autonomous with arbitrary time-dependent coefficients $b(t)$ and $K(t)$. Clearly, no more constraints are left at this stage. Equations (187) and (188) generalize the coupled system of Lenells-Fokas equations [38], [39] by including a non-linear derivative term as well as a higher order dispersion term.

### 6.4 NHD of Chen-Lee-Liu (CLL) system

The Lax pair for the CLL equations is given by,

$$U = \lambda^2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ iqr & 0 \end{pmatrix}$$

(189)

$$V = 2\lambda^4 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + 2\lambda^3 \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \lambda^2 qr \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ -ir_x + \frac{1}{2}qr^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2}(rq_x - r_xq) + \frac{1}{4}r^2q^2 \end{pmatrix}$$

(190)

We take

$$V_{\text{deformed}} = i(G^{(0)} + \lambda^{-1}G^{(1)} + \lambda^{-2}G^{(2)})$$

(191)

where $G^{(0)} = w\sigma_3$, $G^{(1)} = g_1\sigma_+ + g_2\sigma_-$, $G^{(2)} = b\sigma_3$

Here we have taken the cue from the discussion in the previous section in choosing the form of the matrices $G^{(0)}$, $G^{(1)}$ and $G^{(2)}$.  

Taking $\tilde{V} = V + V_{\text{deformed}}$, and imposing the zero-curvature condition, we are led to the following deformed CLL equations:

$$q_t = iq_{xx} + qq_xr - 2g_1 + 2iqw$$

(192)

$$r_t = -i - r_{xx} + rr_xq + 2g_2 - 2irw$$

(193)
The following differential constraints are also obtained:

\[ iw_x = qg_2 - rg_1 \]  \hspace{1cm} (194)
\[ g_{1x} + 2qb + \frac{1}{2} qrg_1 = 0 \]  \hspace{1cm} (195)
\[ g_{2x} - 2rb - \frac{1}{2} qrg_2 = 0 \]  \hspace{1cm} (196)

We also get \( b_x = 0 \) which implies that \( b \) is a function of \( t \) only.

However, it may be noted that it is not possible in this case to resolve the constraints and express the perturbing functions through the basic field variables by re-defining these variables. This is due to the presence of a non-linear term in (195) and (196).

It may be mentioned in passing that the NHD of the Kundu-Eckhaus equation can be worked out in an exactly similar manner. However, we are not reproducing the details of that calculation here.

### 6.5 NHD of the hierarchy of equations in the AKNS system

The hierarchy of dynamical equations in the AKNS system is given by

\[ q_t = -2ib_{n+1} \]
\[ r_t = 2ic_{n+1} \]  \hspace{1cm} (197)

Successive equations can be generated by putting \( n = 1, 2, 3 \) etc.

Since the time part of the Lax pair is given by

\[ V^{(n)} = (\lambda^n V)_+ + \Delta_n = (\lambda^n V)_+ + \sum_{m=0}^{n} \lambda^{n-m}(a_m \sigma_3 + b_m \sigma_+ + c_m \sigma_-) \]  \hspace{1cm} (198)

we introduce the non-holonomic deformation by taking

\[ V_{deformed} = \frac{i}{2}(\lambda^{-1}G^{(1)} + \lambda^{-2}G^{(2)} + \lambda^{-3}G^{(3)} + .......) \]  \hspace{1cm} (199)

where

\[ G^{(1)} = a_1 \sigma_3 + g_1 \sigma_+ + g_2 \sigma_- \]

with similar expressions for \( G^{(2)}, G^{(3)} \) etc.

Taking

\[ V_{final} = V^{(n)} + V_{deformed} \]

and using the zero-curvature condition with \( V_{final} \) as the (new) time part of the Lax pair, we get the following deformed equations:

\[ q_t = -2ib_{n+1} - g_1 \]
\[ r_t = 2ic_{n+1} + g_2 \]  \hspace{1cm} (200)

along with the constraint conditions given as a hierarchy of recursive relations as follows:

\[ iG_x^{(1)} = [\sigma_3, G^{(2)}] + i[q\sigma_+ + r\sigma_- , G^{(1)}] \]  \hspace{1cm} (201)
\[ iG_x^{(2)} = [\sigma_3, G^{(3)}] + i[q\sigma_+ + r\sigma_- , G^{(2)}] \]  \hspace{1cm} (202)

and so on.
6.6 NHD of the hierarchy of equations in the DNLS system (Kaup-Newell hierarchy)

In this section, we show how non-holonomic deformation may be applied to the equations of the DNLS hierarchy (KN system) obtained by using the Tu methodology. The space and time components of the Lax pair are given as follows:

\[ U = -i\lambda^2 \sigma_3 + \lambda q \sigma_+ + \lambda r \sigma_- \] (203)

\[ V^{(n)} = \sum_{j=0}^{n}(a_{2j} \lambda^{2(n-j)+2} \sigma_3 + b_{2j+1} \lambda^{2(n-j)+1} \sigma_+ + c_{2j+1} \lambda^{2(n-j)+1} \sigma_-) \] (204)

On using the zero curvature equation, we get the hierarchy of equations for the Kaup-Newell system:

\[ q_t = b(2n+1)x \]
\[ r_t = c(2n+1)x \] (205)

To carry out the non-holonomic deformation, we take

\[ V_{deformed} = \frac{i}{2}(G^{(0)} + \lambda^{-1}G^{(1)} + \lambda^{-2}G^{(2)} + \lambda^{-3}G^{(3)} + ......) \] (206)

so that the time part of the Lax pair becomes

\[ V_{final} = V^{(n)} + V_{deformed} \] (207)

Now applying the zero curvature condition again with \( U \) and \( V_{final} \) as the Lax pair, we get the deformed equations of the Kaup-Newell hierarchy as:

\[ q_t = b(2n+1)x - g_1 + iqa_0 \]
\[ r_t = c(2n+1)x + g_2 - ira_0 \] (208)

where it has been deduced that

\[ G^{(0)} = a_0 \sigma_3 \] (209)

and \( G^{(1)} \) is taken to be

\[ G^{(1)} = a_1 \sigma_3 + g_1 \sigma_+ + g_2 \sigma_- \] (210)

The differential constraints are given recursively by a series of equations as follows:

\[ G_x^{(0)} = -i[\sigma_3, G^{(2)}] + [q \sigma_+ + r \sigma_-, G^{(1)}] \] (211)

\[ G_x^{(1)} = -i[\sigma_3, G^{(3)}] + [q \sigma_+ + r \sigma_-, G^{(2)}] \] (212)

etc.
7 Discussion

The family of Non-linear Schrödinger equations have been studied exhaustively by using two different techniques viz. the AKS framework and the Tu methodology. In this section we try to explore the connection between these two formalisms. The construction of an integrable dynamical system is accomplished by using the zero-curvature equation. This means we need to obtain both the spatial and temporal components of the Lax pair. Applying the zero-curvature condition on the Lax pair would lead us to the integrable system in one space and one temporal dimension. Both the methods used in this work start by identifying a properly chosen spectral problem through the space part of the Lax pair. In the AKS method, this object, i.e. the orbit can be constructed by suitable co-adjoint action of the Lie group acting on an element of the Lie algebra. Thus the underlying geometry of the integrable model gets emphasized in the AKS method. In order to obtain the temporal component of the Lax pair in an Infinite Dimensional Lie Algebra, in the AKS method, one expands this temporal component in powers of the spectral parameter and take a suitable projection on a particular subalgebra of the IDLA. Once the temporal component is determined by this technique, application of zero-curvature leads to the desired equation for the dynamical system. The commuting flows of the AKS hierarchy can also be obtained in this framework. In the Tu methodology also, we expand the time component $V$ in negative powers of the spectral parameter and use it in the stationary zero-curvature equation to obtain the different elements of $V$ in a recursive manner. After this, the expansion in negative powers of $\lambda$ is multiplied by a suitable positive degree of $\lambda$ and the projection taken on non-negative powers of $\lambda$. Zero curvature is subsequently applied, with a suitable constraint to obtain the hierarchy of non-linear evolution equations. Thus, in the Tu method we need to choose the spectral problem judiciously and the hierarchy results naturally when the sequence of steps outlined above is applied. While both the AKS and Tu methods rely on an expansion of the temporal component of the Lax pair in powers of $\lambda$, the AKS method definitely stresses on the geometry underlying the construction of the integrable system, whereas the Tu method is more algebraic in spirit. Use of the trace identity to determine the Hamiltonian and then application of the operators $J$ and $L$ to set up the complete Hamiltonian structure is another remarkable feature of the Tu method. The AKS scheme also endows an Integrable System with a Hamiltonian structure. But the trace identity method is definitely a more convenient tool to set up the Hamiltonian or rather the multi-Hamiltonian structure of the Integrable System. Associated results then guarantee the complete integrability of the system. One feels that using the AKS theorem and the Tu methodology in tandem will help us unearth rich results in the domain of non-linear Integrable Systems. In particular, applying both techniques to the same class or family of problems will help us in understanding the problem in finer detail. We have also carried out a detailed analysis covering Non-holonomic deformation of different equations of the NLS family. In particular, NHD has been applied to the hierarchy of equations (AKNS and DNLS-Kaup-Newell systems) obtained by using the Tu methodology. It may be mentioned that the structure of the DNLS system may be made more general and NHD may be applied on the resulting hierarchy. The topics covered above and related problems will be the subject of our future investigation.
Acknowledgements

The work owes a lot to our past discussions, correspondences and collaboration with Walter Oevel, Franco Magri, Marco Pedroni, Darryl Holm, Tudor Ratiu, Alfred Ramani, Victor Kac, Peter Olver, Sarbarish Chakravarty, Wen Xiu Ma, Asesh Roy Chowdhury and Sudipto Roy Choudhury. Finally we are also grateful to Allan Fordy for various references.

References

[1] M.J. Ablowitz M J and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, (Cambridge: Cambridge University Press, 1991).
[2] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, The inverse scattering transformFourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249-315.
[3] M. Adler, On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg de Vries type equations. Invent. Math. 50, 219-248, 1979.
[4] Adler, M., van Moerbeke, P. Completely integrable systems, Euclidean Lie algebras and curves. Adv. Math. 38, 1980 , 267.
[5] C. Athorne and A. Fordy, Generalised KdV and MKdV equations associated with symmetric spaces, J. Phys. A 20 (1987), no. 6, 1377-1386.
[6] C. Athorne and A. Fordy, Integrable equations in (2 + 1) dimensions associated with symmetric and homogeneous spaces, J. Math. Phys. 28 (1987), no. 9, 2018-2024.
[7] F. Calogero and W. Eckhaus,Nonlinear evolution equations, rescalings, model PDES and their integrability: I Inverse Problems 3 (1987) 229-62.
[8] H. H. Chen, Y.C. Lee, and C.S. Liu, Integrability of nonlinear Hamiltonian systems by inverse scattering method. Special issue on solitons in physics, Phys. Scripta 20 (1979) 490-492.
[9] W. Eckhaus, The long-time behaviour for perturbed wave-equations and related problems, Preprint no. 404 Department of Mathematics, University of Utrecht (Published in part in Lecture Notes in Physics vol 246 (Berlin: Springer) 1986).
[10] L. Faddeev and L. Takhtajan, Hamiltonian methods in the theory of solitons. Springer-Verlag, Berlin, 1987.
[11] G. Falqui, Separation of variables for Lax systems: a bihamiltonian point of view. Fourth Italian-Latin American Conference on Applied and Industrial Mathematics (Havana, 2001), 393–403, Inst. Cybern. Math. Phys., Havana, 2001.
[12] G. Falqui, F. Magri and M. Pedroni, Soliton equations, bi-Hamiltonian manifolds and integrability. 21º Coloquio Brasileiro de Matematica. [21st Brazilian Mathematics Colloquium] Instituto de Matemica Pura e Aplicada (IMPA), Rio de Janeiro, 1997.
[13] G. Falqui, F. Magri and M. Pedroni, *Soliton equations, bi-Hamiltonian manifolds and integrability*. 21st Colloquio Brasileiro de Matematica. [21st Brazilian Mathematics Colloquium] Instituto de Matemtica Pura e Aplicada (IMPA), Rio de Janeiro, 1997.

[14] H. Flaschka, A.C. Newell and T. Ratiu, *Kac-Moody Lie algebras and soliton equations. II. Lax equations associated with $A^{(1)}_1$*, Phys. D 9 (1983), no. 3, 300-323.

[15] Fordy, Allan P.  *Derivative Nonlinear Schrodinger equations and Hermitian symmetric spaces*, J. Phys. A 17 (1984), no. 6, 1235-1245.

[16] Fordy, A.P., Kulish, P.P.  *Nonlinear Schrödinger equations and simple Lie algebras*. Comm. Math. Phys. 89 (1983), no. 3, 427-443.

[17] I.M. Gelfand and I. Zakharevich.  *webs, Veronese curves, and bihamiltonian systems*. J. of Func. Anal. 99 (1991), 150–178.

[18] I.M. Gelfand and I. Zakharevich.  *On the local geometry of bihamiltonian structures*. The Gelfand Mathematical Seminar, 1990–1992 (Boston). Birkhäuser, 1993, pp. 51–112.

[19] I.M. Gelfand and I. Zakharevich.  *webs, Lenard schemes, and the local geometry of bi-Hamiltonian Toda and Lax structures*. Selecta Math. (N.S.) 6 (2000), no. 2, 131–183.

[20] V.S. Gerdjikov and M.I. Ivanov, *The quadratic bundle of general form and the nonlinear evolution equations: II. Hierarchies of Hamiltonian structures* (in Russian) Bulg. J. Phys. 10 (1983) 130.

[21] P. Guha,  *On Commuting flows of AKS hierarchy and twistor correspondence*. Journal of Geom. Phys. 20 (1996) 207-217.

[22] P. Guha,  *Adler-Kostant-Symes construction, bi-Hamiltonian manifolds, and KdV equations*. J. Math. Phys. 38 (1997) 5167-5182.

[23] P. Guha,  *Geometry of the Kaup-Newell equation*. Rep. Math. Phys. 50 (2002), no. 1, 112.

[24] P. Guha,  *AKS hierarchy and bi-Hamiltonian geometry of Gelfand-Zakharevich type*. , J. Math. Phys. 45 (2004), no. 7, 2864–2884.

[25] P. Guha,  *Nonholonomic deformation of generalized KdV type equations*. J.Phys.A: Math. Theor 42 (2009) 345201.

[26] P.Guha,  *Nonholonomic deformation of coupled and supersymmetric KdV equation and Euler-Poincare-Suslov method*, IHES Preprint. IHES/M/13/15, June 2013.

[27] A Hasegawa and F Tappert  *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion* Appl. Phys. Lett. 23, 142 (1973)

[28] D. J. Kaup and A.C. Newell,  *An exact solution for a derivative nonlinear Schrödinger equation*, J. Math. Phys. 19 (1978) 798.

[29] S. Kakei, N. Sasa and J. Satsuma,  *Bilinearization of a generalized derivative nonlinear Schrödinger equation*, Phys. Soc. Japan 64 (1995) 1519-1523.
[30] A. Karasu-Kalkani, A.Karasu, A. Sakovich, S.Sakovich and R.Turhan A new integrable generalization of the Korteweg-de Vries equation. J. Math. Phys. 49 (2008) 073516 arXiv: 0708.3247[nlin].

[31] B. Kostant, Quantization and representation theory, in Representation theory of Lie groups, Lond. Math. Soc. Lect. Note 34, edited by M.F. Atiyah.

[32] A. Kundu, Landau-Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger type equations, J. Math. Phys. 25 (1984) 3433.

[33] A. Kundu, Exact solutions to higher-order nonlinear equations through gauge transformation, Physica D 25 (1987) 399-406.

[34] A. Kundu, Exact accelerating solitons in non-holonomic deformation of the KdV equation with two-fold integrable hierarchy. J. Phys. A: Math. Theor. 41 (2008) 495201

[35] A. Kundu, Two-fold integrable hierarchy of non-holonomic deformation of the DNLS and the Lenells-Fokas equation arXiv:0910:0383v1 [nlin. SI] 2009

[36] A. Kundu, R.Sahadevan and L. Nalinidevi, Nonholonomic deformation of KdV and mKdV equations and their symmetries, hierarchies and integrability J. Phys. A. 42 (2009) 115213

[37] B.A.Kupershmidt KdV6: an integrable system, Phys. Lett. A 372 (2008) 2634-9

[38] J. Lenells Exactly solvable model for nonlinear pulse propagation in optical fibers. Stud. Appl.Math. 123 (2009) 215-232

[39] J. Lenells and A.S.Fokas On a novel integrable generalization of the nonlinear Schrödinger equation. Nonlinearity 22 (2009) 11-27

[40] Wen Xiu Ma, Integrable couplings of vector AKNS soliton equations, J. Math. Phys. 46 (2005), no. 3, 033507, 19 pp.

[41] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978), no. 5, 1156-1162

[42] I. Marshall, Some integrable systems related to affine Lie algebras and homogeneous spaces. Phys. Lett. 127A , 19, 1988.

[43] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon Experimental Observation of Picosecond Pulse Narrowing and Solitons in Optical Fibers Phys. Rev. Lett. 45, 1095 (1980).

[44] P. J. Olver and V. V. Sokolov, Integrable evolution equations on associative algebras, Commun. Math. Phys. 193 (1998) 245.

[45] A.Ramani, B. Grammaticos and R. Willox Bilinearization and solutions of the KdV6 equation. Anal. Appl. 6 (2008) 401-12

[46] T. Ratiu, The C. Neumann problem as a complete integrable system on an adjoint orbit. Trans. Amer. Math. Soc. 264, 321-9, 1981.

[47] Reiman, A.G., Semenov-Tian-Sanskii, M.A. Reduction of Hamiltonian systems, affine Lie algebras and Lax equations I. Invent. Math. 54, 81-100, 1979.
[48] Reiman A.G., M.A. Semenov-Tian-Sanskii, Soviet Math. Dokl. 21, *Current algebras and nonlinear partial differential equations*. 630-634, 1980.

[49] W. Symes, *Systems of Toda type, inverse spectral problems and representation theory*. Invent. Math. 59, 13-51, 1980.

[50] W. Symes, Hamiltonian group actions and integrable systems. Physica D1, 339-374, 1980.

[51] Gui-zhang Tu *The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems*. J. Math. Phys. 30 (1989) 330338.

[52] Gui-zhang Tu, *A trace identity and its applications to the theory of discrete integrable systems*, J. Phys. A: Math. Gen. 23 (1990) 39033922.

[53] T Tsuchida and M Wadati *New integrable systems of derivative nonlinear Schrödinger equations with multiple components*. Phys. Lett. A Vol 287, Issue 1-2, 53-64

[54] M. Wadati and K. J. Sogo, *Gauge transformations in soliton theory*, J. Phys. Soc. Japan 52 (1983) 394-398.

[55] V.E. Zakharov V E and A.B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Zh. Eksp. Teor. Fiz. 61 (1971) 11834. V.E. Zakharov and A.B. Shabat, Sov. Phys.JETP 34 (1972) 629. (Engl. Transl)