Lectures on Hopf Algebras, Quantum Groups and Twists*  

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Abstract  

Lead by examples we introduce the notions of Hopf algebra and quantum group. We study their geometry and in particular their Lie algebra (of left invariant vectorfields). The examples of the quantum $sl(2)$ Lie algebra and of the quantum (twisted) Poincaré Lie algebra $iso_\theta(3, 1)$ are presented.

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1 Introduction

Hopf algebras were initially considered more than Half a century ago. New important examples, named quantum groups, where studied in the 80’s [1–3], they arose in the study of the quantum inverse scattering method in integrable systems. Quantum groups can be seen as symmetry groups of noncommutative spaces, this is one reason they have been investigated in physics and mathematical physics (noncommutative spaces arise as quantization of commutative ones). The emergence of gauge theories on noncommutative spaces in open string theory in the presence of a NS 2-form background [4] has further motivated the study of noncommutative spaces, and of their symmetry properties.

We here introduce the basic concepts of quantum group and of its Lie algebra of infinitesimal transformations. We pedagogically stress the connection with the classical (commutative) case and we treat two main examples, the quantum $sl(2)$ Lie algebra and the quantum Poincaré Lie algebra.

Section 2 shows how commutative Hopf algebras emerge from groups. The quantum group $SL_q(2)$ is then presented and its corresponding universal enveloping algebra $U_q(sl(2))$ discussed. The relation between $SL_q(2)$ and $U_q(sl(2))$ is studied in Section 5. The quantum $sl(2)$ Lie algebra, i.e. the algebra of infinitesimal transformations, is then studied in Section 6. Similarly the geometry of Hopf algebras obtained from (abelian) twists is studied via the example of the Poincaré Lie algebra.

In the appendix for reference we review some basic algebra notions and define Hopf algebras diagramatically.

One aim of these lectures is to concisely introduce and relate all three aspects of quantum groups:

- deformed algebra of functions [3],
- deformed universal enveloping algebra [1, 2],
- quantum Lie algebra [5].

Quantum Lie algebras encode the construction of the (bicovariant) differential calculus and geometry, most relevant for physical applications. A helpful review for the first and second aspects is [6], for quantum Lie algebras we refer to [7] and [8]. The (abelian) twist case, that is an interesting subclass, can be found in [9] and in [10].

2 Hopf algebras from groups

Let us begin with two examples motivating the notion of Hopf algebra. Let $G$ be a finite group, and $A = Fun(G)$ be the set of functions from $G$ to complex numbers
\( \mathbb{C} \). \( A = \text{Fun}(G) \) is an algebra over \( \mathbb{C} \) with the usual sum and product \((f + h)(g) = f(g) + h(g),\) \((f \cdot h) = f(g)h(g),\) \((\lambda f)(g) = \lambda f(g)\), for \( f, h \in \text{Fun}(G),\) \( g \in G; \) \( \lambda \in \mathbb{C} \).

The unit of this algebra is \( I, \) defined by \( I(g) = 1, \) \( \forall g \in G. \) Using the group structure of \( G \) (multiplication map, existence of unit element and inverse element), we can introduce on \( \text{Fun}(G) \) three other linear maps, the coproduct \( \Delta, \) the counit \( \varepsilon, \) and the coinverse (or antipode) \( S: \)

\[
\Delta(f)(g, g') \equiv f(gg'), \quad \Delta : \text{Fun}(G) \to \text{Fun}(G) \otimes \text{Fun}(G) \quad (2.1)
\]

\[
\varepsilon(f) \equiv f(1_G), \quad \varepsilon : \text{Fun}(G) \to \mathbb{C} \quad (2.2)
\]

\[
(Sf)(g) \equiv f(g^{-1}), \quad S : \text{Fun}(G) \to \text{Fun}(G) \quad (2.3)
\]

where \( 1_G \) is the unit of \( G. \)

In general a coproduct can be expanded on \( \text{Fun}(G) \otimes \text{Fun}(G) \) as:

\[
\Delta(f) = \sum_i f_i^1 \otimes f_i^2 \equiv f_1 \otimes f_2, \quad (2.4)
\]

where \( f_i^1, f_i^2 \in A = \text{Fun}(G) \) and \( f_1 \otimes f_2 \) is a shorthand notation we will often use in the sequel. Thus we have:

\[
\Delta(f)(g, g') = (f_1 \otimes f_2)(g, g') = f_1(g)f_2(g') = f(gg'). \quad (2.5)
\]

It is not difficult to verify the following properties of the co-structures:

\[
(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta \quad \text{(coassociativity of \( \Delta \))} \quad (2.6)
\]

\[
(id \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes id)\Delta(a) = a \quad (2.7)
\]

\[
m(S \otimes id)\Delta(a) = m(id \otimes S)\Delta(a) = \varepsilon(a)I \quad (2.8)
\]

and

\[
\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(I) = I \otimes I \quad (2.9)
\]

\[
\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(I) = 1 \quad (2.10)
\]

\[
S(ab) = S(b)S(a), \quad S(I) = I \quad (2.11)
\]

where \( a, b \in A = \text{Fun}(G) \) and \( m \) is the multiplication map \( m(a \otimes b) = ab. \) The product in \( \Delta(a)\Delta(b) \) is the product in \( A \otimes A: \)

\[(a \otimes b)(c \otimes d) = ab \otimes cd.\]

For example the coassociativity property \((2.6), (id \otimes \Delta)\Delta(f) = (\Delta \otimes id)\Delta(f) \) reads \( f_1 \otimes (f_2)_1 \otimes (f_2)_2 = (f_1)_1 \otimes (f_1)_2 \otimes f_2, \) for all \( f \in A. \) This equality is easily seen to hold by applying it on the generic element \((g, g', g'')\) of \( G \times G \times G, \) and then by using associativity of the product in \( G. \)

An algebra \( A \) (not necessarily commutative) endowed with the homomorphisms \( \Delta : A \to A \otimes A \) and \( \varepsilon : A \to \mathbb{C}, \) and the linear and antimultiplicative map \( S : A \to A \)
satisfying the properties (2.6)-(2.11) is a Hopf algebra. Thus \( \text{Fun}(G) \) is a Hopf algebra, it encodes the information on the group structure of \( G \).

As a second example consider now the case where \( G \) is a group of matrices, a subgroup of \( GL \) given by matrices \( T^a_b \) that satisfy some algebraic relation (for example orthogonality conditions). We then define \( A = \text{Fun}(G) \) to be the algebra of polynomials in the matrix elements \( T^a_b \) of the defining representation of \( G \) and in \( \det T^{-1} \); i.e. the algebra is generated by the \( T^a_b \) and \( \det T^{-1} \).

Using the elements \( T^a_b \) we can write an explicit formula for the expansion (2.4) or (2.5): indeed (2.1) becomes

\[
\Delta(T^a_b)(g, g') = T^a_c(g) T^c_b(g'), \quad (2.12)
\]

since \( T \) is a matrix representation of \( G \). Therefore:

\[
\Delta(T^a_b) = T^a_c \otimes T^c_b. \quad (2.13)
\]

Moreover, using (2.2) and (2.3), one finds:

\[
\begin{align*}
\varepsilon(T^a_b) &= \delta^a_b, \quad (2.14) \\
S(T^a_b) &= (T^{-1})^a_b. \quad (2.15)
\end{align*}
\]

Thus the algebra \( A = \text{Fun}(G) \) of polynomials in the elements \( T^a_b \) and \( \det T^{-1} \) is a Hopf algebra with co-structures defined by (2.13)-(2.15) and (2.9)-(2.11).

The two example presented concern commutative Hopf algebras. In the first example the information on the group \( G \) is equivalent to that on the Hopf algebra \( A = \text{Fun}(G) \). We constructed \( A \) from \( G \). In order to recover \( G \) from \( A \) notice that every element \( g \in G \) can be seen as a map from \( A \) to \( \mathbb{C} \) defined by \( f \rightarrow f(g) \). This map is multiplicative because \( fh(g) = f(g)h(g) \). The set \( G \) can be obtained from \( A \) as the set of all nonzero multiplicative linear maps from \( A \) to \( \mathbb{C} \) (the set of characters of \( A \)).

Concerning the group structure of \( G \), the product is recovered from the coproduct in \( A \) via (2.5), i.e. \( gg' \) is the new character that associates to any \( f \in A \) the complex number \( f_1(g)f_2(g') \). The unit of \( G \) is the character \( \varepsilon \); the inverse \( g^{-1} \) is defined via the antipode of \( A \).

In the second example, in order to recover the topology of \( G \), we would need a \( C^* \)-algebra completion of the algebra \( A = \text{Fun}(G) \) of polynomial functions. Up to these topological (\( C^* \)-algebra) aspects, we can say that the information concerning a matrix group \( G \) can be encoded in its commutative Hopf algebra \( A = \text{Fun}(G) \).

In the spirit of noncommutative geometry we now consider noncommutative deformations \( \text{Fun}_q(G) \) of the algebra \( \text{Fun}(G) \). The space of points \( G \) does not exist anymore, by
noncommutative or quantum space $G_q$ is meant the noncommutative algebra $\text{Fun}_q(G)$. Since $G$ is a group then $\text{Fun}(G)$ is a Hopf algebra; the noncommutative Hopf algebra obtained by deformation of $\text{Fun}(G)$ is then usually called Quantum group. The term quantum stems for the fact that the deformation is obtained by quantizing a Poisson (symplectic) structure of the algebra $\text{Fun}(G)$ [1].

3 Quantum groups. The example of $SL_q(2)$

Following [3] we consider quantum groups defined as the associative algebras $A$ freely generated by non-commuting matrix entries $T^a_b$ satisfying the relation

$$R^{ab}_{\;\;ef} T^e_c T^f_d = T^b_f T^a_e R^{ef}_{\;\;cd}$$

and some other conditions depending on which classical group we are deforming (see later). The matrix $R$ controls the non-commutativity of the $T^a_b$, and its elements depend continuously on a (in general complex) parameter $q$, or even a set of parameters. For $q \to 1$, the so-called “classical limit”, we have

$$R^{ab}_{\;\;cd} \xrightarrow{q \to 1} \delta^a_c \delta^b_d,$$

i.e. the matrix entries $T^a_b$ commute for $q = 1$, and one recovers the ordinary $\text{Fun}(G)$. The $R$-matrices for the quantum group deformation of the simple Lie groups of the $A, B, C, D$ series were given in [3].

The associativity of $A$ leads to a consistency condition on the $R$ matrix, the quantum Yang–Baxter equation:

$$R^{a_1 b_1}_{\;\;a_2 b_2} R^{a_2 c_1}_{\;\;a_3 c_2} R^{b_2 c_2}_{\;\;b_3 c_3} = R^{b_1 c_1}_{\;\;b_2 c_2} R^{a_1 c_2}_{\;\;a_2 c_3} R^{a_2 b_2}_{\;\;a_3 b_3}.$$  

(3.3)

For simplicity we rewrite the “RTT” equation [3.1] and the quantum Yang–Baxter equation as

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}$$  

(3.4)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},$$  

(3.5)

where the subscripts 1, 2 and 3 refer to different couples of indices. Thus $T_1$ indicates the matrix $T^a_b$, $T_1 T_1$ indicates $T^a_c T^c_b$, $R_{12} T_2$ indicates $R^{ab}_{\;\;cd} T^d_e$ and so on, repeated subscripts meaning matrix multiplication. The quantum Yang–Baxter equation (3.5) is a condition sufficient for the consistency of the RTT equation (3.4). Indeed the product of three distinct elements $T^a_b$, $T^c_d$ and $T^e_f$, indicated by $T_1 T_2 T_3$, can be reordered as $T_3 T_2 T_1$ via two different paths:

$$T_1 T_2 T_3 \xrightarrow{T_1 T_3 T_2 \to T_3 T_1 T_2} T_3 T_2 T_1 \xrightarrow{T_2 T_1 T_3 \to T_2 T_3 T_1} T_2 T_3 T_1.$$

(3.6)
by repeated use of the RTT equation. The relation (3.3) ensures that the two paths lead to the same result.

The algebra $A$ ("the quantum group") is a noncommutative Hopf algebra whose co-structures are the same of those defined for the commutative Hopf algebra $Fun(G)$ of the previous section, eqs. (2.13)-(2.15), (2.9)-(2.11).

**Note** Define $\hat{R}_{ab}^{cd} = R_{ba}^{cd}$. Then the quantum Yang-Baxter equation becomes the braid relation

$$
\hat{R}_{23} \hat{R}_{12} \hat{R}_{23} = \hat{R}_{12} \hat{R}_{23} \hat{R}_{12}.
$$

(3.7)

If $\hat{R}$ satisfies $\hat{R}^2 = \text{id}$ we have that $\hat{R}$ is a representation of the permutation group. In the more general case $\hat{R}$ is a representation of the braid group. The $\hat{R}$-matrix can be used to construct invariants of knots.

Let us give the example of the quantum group $SL_q(2) \equiv Fun_q(SL(2))$, the algebra freely generated by the elements $\alpha, \beta, \gamma$ and $\delta$ of the $2 \times 2$ matrix

$$
T^a_b = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
$$

(3.8)

satisfying the commutations

\[
\alpha\beta = q\beta\alpha,
\alpha\gamma = q\gamma\alpha,
\beta\delta = q\delta\beta,
\gamma\delta = q\delta\gamma
\]

\[
\beta\gamma = \gamma\beta,
\alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma,
q \in \mathbb{C}
\]

(3.9)

and

$$
\det_q T \equiv \alpha\delta - q\beta\gamma = I.
$$

(3.10)

The commutations (3.9) can be obtained from (3.1) via the $R$ matrix

$$
R^{ab}_{cd} = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & q^{-1} & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}
$$

(3.11)

where the rows and columns are numbered in the order 11, 12, 21, 22.

It is easy to verify that the "quantum determinant" defined in (3.10) commutes with $\alpha, \beta, \gamma$ and $\delta$, so that the requirement $\det_q T = I$ is consistent. The matrix inverse of $T^a_b$ is

$$
(T^{-1})^a_b = (\det_q T)^{-1} \begin{pmatrix}
\delta & -q^{-1}\beta \\
-q\gamma & \alpha
\end{pmatrix}
$$

(3.12)
The coproduct, counit and coinverse of \( \alpha, \beta, \gamma \) and \( \delta \) are determined via formulas (2.13)-(2.15) to be:

\[
\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta \\
\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta \\

(3.13)
\]

\[
\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0 \\
S(\alpha) = \delta, \quad S(\beta) = -q^{-1} \beta, \quad S(\gamma) = -q \gamma, \quad S(\delta) = \alpha. \\

(3.14)
\]

\[3.15\]

Note  The commutations (3.9) are compatible with the coproduct \( \Delta \), in the sense that \( \Delta(\alpha \beta) = q \Delta(\beta \alpha) \) and so on. In general we must have

\[
\Delta(R_{12} T_{1} T_{2}) = \Delta(T_{2} T_{1} R_{12}), \\

(3.16)
\]

which is easily verified using \( \Delta(R_{12} T_{1} T_{2}) = R_{12} \Delta(T_{1}) \Delta(T_{2}) \) and \( \Delta(T_{1}) = T_{1} \otimes T_{1} \). This is equivalent to proving that the matrix elements of the matrix product \( T_{1} T_{1}' \), where \( T' \) is a matrix [satisfying (3.1)] whose elements commute with those of \( T_{a b} \), still obey the commutations (3.14).

Note  \( \Delta(\det_{q} T) = \det_{q} T \otimes \det_{q} T \) so that the coproduct property \( \Delta(I) = I \otimes I \) is compatible with \( \det_{q} T = I \).

Note  The condition (3.10) can be relaxed. Then we have to include the central element \( \zeta = (\det_{q} T)^{-1} \) in \( A \), so as to be able to define the inverse of the \( q \)-matrix \( T_{a b} \) as in (3.12), and the coinverse of the element \( T_{a b} \) as in (2.15). The \( q \)-group is then \( GL_{q}(2) \). The reader can deduce the co-structures on \( \zeta \): \( \Delta(\zeta) = \zeta \otimes \zeta, \ \varepsilon(\zeta) = 1, \ S(\zeta) = \det_{q} T \).

4 Universal enveloping algebras and \( U_{q}(sl(2)) \)

Another example of Hopf algebra is given by any ordinary Lie algebra \( g \), or more precisely by the universal enveloping algebra \( U(g) \) of a Lie algebra \( g \), i.e. the algebra, with unit \( I \), of polynomials in the generators \( \chi_{i} \) modulo the commutation relations

\[
[\chi_{i}, \chi_{j}] = C_{ij}^{\ k} \chi_{k}. \\

(4.1)
\]

Here we define the co-structures on the generators as:

\[
\Delta(\chi_{i}) = \chi_{i} \otimes I + I \otimes \chi_{i} \quad \Delta(I) = I \otimes I \\
\varepsilon(\chi_{i}) = 0 \quad \varepsilon(I) = 1 \\
S(\chi_{i}) = -\chi_{i} \quad S(I) = I \\

(4.2) \quad (4.3) \quad (4.4)
\]

and extend them to all \( U(g) \) by requiring \( \Delta \) and \( \varepsilon \) to be linear and multiplicative, \( \Delta(\chi \chi') = \Delta(\chi) \Delta(\chi'), \ \varepsilon(\chi \chi') = \varepsilon(\chi) \varepsilon(\chi') \), while \( S \) is linear and antimultiplicative. In
order to show that the construction of the Hopf algebra \( U(g) \) is well given, we have to check that the maps \( \Delta, \varepsilon, S \) are well defined. We give the proof for the coproduct. Since \([\chi, \chi']\) is linear in the generators we have
\[
\Delta[\chi, \chi'] = [\chi, \chi'] \otimes I + I \otimes [\chi, \chi'] ,
\] (4.5)
on the other hand, using that \( \Delta \) is multiplicative we have
\[
\Delta[\chi, \chi'] = \Delta(\chi)\Delta(\chi') - \Delta(\chi')\Delta(\chi)
\] (4.6)
it is easy to see that these two expressions coincide.

The Hopf algebra \( U(g) \) is noncommutative but it is cocommutative, i.e. for all \( \zeta \in U(g) \), \( \zeta_1 \otimes \zeta_2 = \zeta_2 \otimes \zeta_1 \), where we used the notation \( \Delta(\zeta) = \zeta_1 \otimes \zeta_2 \). We have discussed deformations of commutative Hopf algebras, of the kind \( A = \text{Fun}(G) \), and we will see that these are related to deformations of cocommutative Hopf algebras of the kind \( U(g) \) where \( g \) is the Lie algebra of \( G \).

We here introduce the basic example of deformed universal enveloping algebra: \( \mathbb{U}_q(sl(2)) \) [1, 2], which is a deformation of the usual enveloping algebra of \( sl(2) \),
\[
[X^+, X^-] = H , \quad [H, X^\pm] = 2X^\pm .
\] (4.7)
The Hopf algebra \( U_q(sl(2)) \) is generated by the elements \( K_+, K_-, X_+ \) and \( X_- \) and the unit element \( I \), that satisfy the relations
\[
[X_+, X_-] = \frac{K_+^2 - K_-^{-2}}{q - q^{-1}} ,
\] (4.8)
\[
K_+X_+K_- = q^{\pm 1}X_- ,
\] (4.9)
\[
K_+K_- = K_-K_+ = I .
\] (4.10)
The parameter \( q \) that appears in the right hand side of the first two equations is a complex number. It can be checked that the algebra \( U_q(sl(2)) \) becomes a Hopf algebra by defining the following costructures
\[
\Delta(X_\pm) = X_\pm \otimes K_+ + K_- \otimes X_\pm , \quad \Delta(K_\pm) = K_\pm \otimes K_\pm
\] (4.11)
\[
\varepsilon(X_\pm) = 0 \quad \varepsilon(K_\pm) = 1
\] (4.12)
\[
S(X_\pm) = -q^{\pm 1}X_\mp \quad S(K_\pm) = K_\mp
\] (4.13)
If we define \( K_+ = 1 + \frac{q^2}{2}H \) then we see that in the limit \( q \to 1 \) we recover the undeformed \( U(sl(2)) \) Hopf algebra.

The Hopf algebra \( U_q(sl(2)) \) is not cocommutative, however the noncocommutativity is under control, as we now show. We set \( q = e^h \), consider \( h \) a formal parameter and
allow for power series in $h$. We are considering a topological completion of $U_q(sl(2))$, this is equivalently generated by the three generators $X_\pm$ and $H$, where we have $K_\pm = e^{\pm h H/2}$. In this case there exists an element $\mathcal{R}$ of $U_q(sl(2)) \hat{\otimes} U_q(sl(2))$ (also the usual tensor product $\otimes$ has to be extended to allow for power series), called universal $R$-matrix that governs the noncocommutativity of the coproduct $\Delta$,

$$\sigma \Delta(\zeta) = \mathcal{R}\Delta(\zeta)\mathcal{R}^{-1},$$

where $\sigma$ is the flip operation, $\sigma(\zeta \otimes \xi) = \xi \otimes \zeta$. The element $\mathcal{R}$ explicitly reads

$$\mathcal{R} = q^{H \otimes H} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]!} (q^{H/2} X_+ \otimes q^{-H/2} X_-)^n q^{n(n-1)/2}$$

(4.15)

where $[n] \equiv \frac{aq^n - q^{-n}}{q - q^{-1}}$, and $[n]! = [n][n-1] \ldots 1$.

The universal $\mathcal{R}$ matrix has further properties, that structure $U_q(sl(2))$ to be a quasitriangular Hopf algebra. Among these properties we mention that $\mathcal{R}$ is invertible and that it satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

(4.16)

where we used the notation $\mathcal{R}_{12} = \mathcal{R} \otimes I$, $\mathcal{R}_{23} = I \otimes \mathcal{R}$ and $\mathcal{R}_{13} = \mathcal{R}^a \otimes I \otimes \mathcal{R}_\alpha$, where $\mathcal{R} = \mathcal{R}^a \otimes \mathcal{R}_\alpha$ (sum over $\alpha$ understood).

\section{Duality}

Consider a finite dimensional Hopf algebra $A$, the vector space $A'$ dual to $A$ is also a Hopf algebra with the following product, unit and costructures [we use the notation $\psi(a) = \langle \psi, a \rangle$ in order to stress the duality between $A'$ and $A$]: $\forall \psi, \phi \in A'$, $\forall a, b \in A$

$$\langle \psi \phi, a \rangle = \langle \psi \otimes \phi, \Delta a \rangle, \quad (I, a) = \varepsilon(a)$$

(5.1)

$$\langle \Delta(\psi), a \otimes b \rangle = \langle \psi, ab \rangle, \quad \varepsilon(\psi) = \langle \psi, I \rangle$$

(5.2)

$$\langle S(\psi), a \rangle = \langle \psi, S(a) \rangle$$

(5.3)

where $\langle \psi \otimes \phi , a \otimes b \rangle \equiv \langle \psi, a \rangle \langle \phi, b \rangle$. Obviously $(A')' = A$ and $A$ and $A'$ are dual Hopf algebras.

In the infinite dimensional case the definition of duality between Hopf algebras is more delicate because the coproduct on $A'$ might not take values in the subspace $A' \otimes A'$ of $(A \otimes A)'$ and therefore is ill defined. We therefore use the notion of pairing: two Hopf
algebras $A$ and $U$ are paired if there exists a bilinear map $\langle \ , \ \rangle : U \otimes A \to \mathbb{C}$ satisfying (5.1) and (5.2), (then (5.3) can be shown to follow as well).

The Hopf algebras $\text{Fun}(G)$ and $U(g)$ described in Section 3 and 4 are paired if $g$ is the Lie algebra of $G$. Indeed we realize $g$ as left invariant vectorfields on the group manifold. Then the pairing is defined by

$$\forall t \in g, \forall f \in \text{Fun}(G), \quad \langle t, f \rangle = t(f)|_{1_G},$$

where $1_G$ is the unit of $G$, and more in general is well defined by\(^1\)

$$\forall t t'...t'' \in U(g), \forall f \in \text{Fun}(G), \quad \langle t t'...t'', f \rangle = t(t'...(t''(f)))|_{1_G}.$$

The duality between the Hopf algebras $\text{Fun}(\text{Sl}(2))$ and $U(\text{sl}(2))$ holds also in the deformed case, so that the quantum group $\text{Sl}_q(2)$ is dual to $U_q(\text{sl}(2))$. In order to show this duality we introduce a subalgebra (with generators $L^\pm$) of the algebra of linear maps from $\text{Fun}_q(\text{Sl}(2))$ to $\mathbb{C}$. We then see that this subalgebra has a natural Hopf algebra structure dual to $\text{SL}_q(2) = \text{Fun}_q(\text{Sl}(2))$. Finally we see in formula (5.22) that this subalgebra is just $U_q(\text{sl}(2))$. This duality is important because it allows to consider the elements of $U_q(\text{sl}(2))$ as (left invariant) differential operators on $\text{Sl}_2(2)$. This is the first step for the construction of a differential calculus on the quantum group $\text{Sl}_q(2)$.

The $L^\pm$ functionals
The linear functionals $L^\pm_{a\ b}$ are defined by their value on the elements $T^a_{\ b}$:

$$L^\pm_{a\ b}(T^c_{\ d}) = \langle L^\pm_{a\ b}, T^c_{\ d} \rangle = R^\pm_{\ ac}b_d, \quad (5.4)$$

where

$$(R^\pm)_{ac}^{\ bd} \equiv q^{-1/2}R^{ac}_{\ db}, \quad (5.5)$$

$$R_{ac}^{\ bd} \equiv q^{1/2}(R^{-1})_{ac}^{\ bd}, \quad (5.6)$$

\(^1\)In order to see that relations (5.1), (5.2) hold, we recall that $t$ is left invariant if $TL_g(t)|_{1_G} = t|_g$, where $TL_g$ is the tangent map induced by the left multiplication of the group on itself: $L_g g' = gg'$. We then have

$$t(f)|_g = (TL_g(t)|_{1_G})(f) = t[f(g\tilde{g})]|_{\tilde{g}} = t[f_1(g)f_2(\tilde{g})]|_{\tilde{g}} = f_1(g)\ t(f_2)|_{1_G}$$

and therefore

$$\langle \tilde{t}, f \rangle = \tilde{t}(t(f))|_{1_G} = \tilde{t}f_1|_{1_G}f_2|_{1_G} = \langle \tilde{t} \otimes t, \Delta f \rangle,$$

and

$$\langle t, fh \rangle = t(f)|_{1_G}h|_{1_G} = (\Delta(t), f \otimes h).$$


The inverse matrix $R^{-1}$ is defined by

$$R^{-1 \alpha \beta}_{\gamma \delta} R^{\gamma \delta}_{\epsilon \eta} \equiv \delta^\alpha_\gamma \delta^\beta_\delta \equiv R^{\alpha \beta}_{\gamma \delta} R^{-1 \gamma \delta}_{\epsilon \eta}. \quad (5.7)$$

To extend the definition (5.4) to the whole algebra $A$ we set:

$$L^\pm_{\alpha \beta}(ab) = L^\pm_{\alpha \gamma}(a)L^\pm_{\gamma \beta}(b), \quad \forall a, b \in A \quad (5.8)$$

so that, for example,

$$L^\pm_{\alpha \beta}(T^c_d T^e_f) = R^\pm_{\alpha \gamma} R^\pm_{\gamma \beta}. \quad (5.9)$$

In general, using the compact notation introduced in Section 2,

$$L^\pm_{\alpha \beta}(T_2 T_3 ... T_n) = R^\pm_{12} R^\pm_{13} ... R^\pm_{1n}. \quad (5.10)$$

As it is easily seen from (5.9), the quantum Yang-Baxter equation (3.5) is a necessary and sufficient condition for the compatibility of (5.4) and (5.8) with the RTT relations:

$$L^\pm_{12}(R_{23} T_3 - T_3 R_{23}) = 0. \quad (5.11)$$

Finally, the value of $L^\pm$ on the unit $I$ is defined by

$$L^\pm_{\alpha \beta}(I) = \delta^\alpha_\beta. \quad (5.12)$$

It is not difficult to find the commutations between $L^\pm_{\alpha \beta}$ and $L^\pm_{\gamma \delta}$:

$$R_{12} L^\pm_{21} = L^\pm_{21} R_{12} \quad (5.13)$$

where the product $L^\pm_{21}$ is by definition obtained by duality from the coproduct in $A$ for all $a \in A$.

For example consider

$$R_{12}(L^+_{21})(T_3) = R_{12}(L^+_{21} \otimes L^+_{11}) \Delta(T_3) = R_{12}(L^+_{21} \otimes L^+_{11})(T_3 \otimes T_3) = q R_{12} R_{32} R_{31}$$

and

$$L^+_{11}(T_3) R_{12} = q R_{31} R_{32} R_{12}$$

so that the equation (5.12) is proven for $L^+$ by virtue of the quantum Yang–Baxter equation (3.5), where the indices have been renamed $2 \rightarrow 1, 3 \rightarrow 2, 1 \rightarrow 3$. Similarly, one proves the remaining “RLL” relations.

**Note** As mentioned in [3], $L^+$ is upper triangular, $L^-$ is lower triangular (this is due to the upper and lower triangularity of $R^+$ and $R^-$, respectively). From (5.12) and (5.13) we have

$$L^\pm_{\alpha \beta} L^\pm_{\gamma \delta} = L^\pm_{\beta \gamma} L^\pm_{\alpha \delta}; \quad L^\pm_{\alpha \beta} L^\pm_{\gamma \delta} = L^\pm_{\beta \gamma} L^\pm_{\alpha \delta} = \epsilon. \quad (5.14)$$
we also have

\[ L^\pm_1 L^\pm_2 = \varepsilon . \]  

(5.15)

The algebra of polynomials in the \( L^\pm \) functionals becomes a Hopf algebra paired to \( Sl_q(2) \) by defining the costructures via the duality (5.4):

\[ \Delta(L^a_b)(T^d_c \otimes T^e_f) \equiv L^a_g(T^d_c) L^g_b(T^e_f) \]

(5.16)

\[ \varepsilon(L^a_b) \equiv L^a_b(I) \]

(5.17)

\[ S(L^a_b)(T^d_c) \equiv L^a_b(S(T^d_c)) \]

(5.18)

cf. [5.2], [5.3], so that

\[ \Delta(L^a_b) = L^a_g \otimes L^g_b \]

(5.19)

\[ \varepsilon(L^a_b) = \delta^a_b \]

(5.20)

\[ S(L^a_b) = L^a_b \circ S . \]

(5.21)

This Hopf algebra is \( U_q(sl(2)) \) because it can be checked that relations (5.12), (5.13), (5.14), (5.15) fully characterize the \( L^\pm \) functionals, so that the algebra of polynomials in the symbols \( L^a_b \) that satisfy the relations (5.12), (5.13), (5.14), (5.15) is isomorphic to the algebra generated by the \( L^\pm \) functionals on \( U_q(sl(2)) \). An explicit relation between the \( L^\pm \) matrices and the generators \( X^\pm \) and \( K^\pm \) of \( U_q(sl(2)) \) introduced in the previous section is obtained by comparing the “RLL” commutation relations with the \( U_q(sl(2)) \) Lie algebra relations, we obtain

\[ L^+ = \begin{pmatrix} K_+ & q^{-1/2}(q - q^{-1})X_+ \\ 0 & K_+ \end{pmatrix}, \quad L^- = \begin{pmatrix} 0 & K_+ \\ q^{1/2}(q^{-1} - q)X_- & K_- \end{pmatrix}. \]

(5.22)

6 Quantum Lie algebra

We now turn our attention to the issue of determining the Lie algebra of the quantum group \( Sl_q(2) \), or equivalently the quantum Lie algebra of the universal enveloping algebra \( U_q(sl(2)) \).

In the undeformed case the Lie algebra of a universal enveloping algebra \( U \) (for example \( U(sl(2)) \)) is the unique linear subspace \( g \) of \( U \) of primitive elements, i.e. of elements \( \chi \) that have coproduct:

\[ \Delta(\chi) = \chi \otimes 1 + 1 \otimes \chi . \]

(6.1)
Of course $g$ generates $U$ and $g$ is closed under the usual commutator bracket $[ , ]$, 
$$[u, v] = uu - vu \in g \quad \text{for all } u, v \in g .$$
(6.2)

The geometric meaning of the bracket $[u, v]$ is that it is the adjoint action of $g$ on $g$, 
$$[u, v] = ad_u v$$
(6.3)
$$ad_u v := u_1 v S(u_2)$$
(6.4)
where we have used the notation $\Delta(\xi) = \sum_\alpha u_1^{\alpha} \otimes u_2^{\alpha} = u_1 \otimes u_2$, so that a sum over $\alpha$ is understood. Recalling that $\Delta(u) = u \otimes 1 + 1 \otimes u$ and that $S(u) = -u$, from (6.4) we immediately obtain (6.3). In other words, the commutator $[u, v]$ is the Lie derivative of the left invariant vectorfield $u$ on the left invariant vectorfield $v$. More in general the adjoint action of $U$ on $U$ is given by
$$ad_\xi \zeta = \xi_1 \zeta S(\xi_2) ,$$
(6.5)
where we used the notation (sum understood) $\Delta(\xi) = \xi_1 \otimes \xi_2$.

In the deformed case the coproduct is no more cocommutative and we cannot identify the Lie algebra of a deformed universal enveloping algebra $U_q$ with the primitive elements of $U_q$, they are too few to generate $U_q$. We then have to relax this requirement. There are three natural conditions that according to [5] the $q$-Lie algebra of a $q$-deformed universal enveloping algebra $U_q$ has to satisfy, see [8, 11], and [12] p. 41. It has to be a linear subspace $g_q$ of $U_q$ such that
$$i) \quad g_q \text{ generates } U_q ,$$
(6.6)
$$ii) \quad \Delta(g_q) \subset g_q \otimes 1 + U_q(\text{sl}(2))_q \otimes g_q ,$$
(6.7)
$$iii) \quad [g_q, g_q] \subset g_q .$$
(6.8)

Here now $\Delta$ is the coproduct of $U_q$ and $[ , ]$ denotes the $q$-bracket
$$[u, v] = ad_u v = u_1 v S(u_2) .$$
(6.9)
where we have used the coproduct notation $\Delta(u) = u_1 \otimes u_2$. Property $iii)$ is the closure of $g_q$ under the adjoint action. Property $ii)$ implies a minimal deformation of the Leibnitz rule.

From these conditions, that do not in general single out a unique subspace $g_q$, it follows that the bracket $[u, v]$ is quadratic in $u$ and $v$, that it has a deformed antisymmetry property and that it satisfies a deformed Jacoby identity.

In the example $U_q = U_q(\text{sl}(2))$ we have that a quantum $\text{sl}(2)$ Lie algebra is spanned by the four linearly independent elements
$$\chi^{c_1 c_2} = \frac{1}{q - q^{-1}} [L^{+ c_1 \alpha} b S(L^{- b \alpha} c_2) - \delta^{c_1 c_2} \epsilon] .$$
(6.10)
In the commutative limit $q \to 1$, we have $\chi_2^2 = -\chi_1^1 = H/2$, $\chi_1^2 = X_+$, $\chi_2^1 = X_-$, and we recover the usual $sl(2)$ Lie algebra.

The $q$-Lie algebra commutation relations explicitly are

\[
\begin{align*}
\chi_1 \chi_+ - \chi_+ \chi_1 + (q^4 - q^2) \chi_+ \chi_2 &= q^3 \chi_+ \\
\chi_1 \chi_- - \chi_- \chi_1 - (q^2 - 1) \chi_- \chi_2 &= -q \chi_- \\
\chi_1 \chi_2 - \chi_2 \chi_1 &= 0 \\
\chi_+ \chi_- - \chi_- \chi_+ - (1 - q^2) \chi_1 \chi_2 + (1 - q^2) \chi_2 \chi_2 &= q(\chi_1^1 - \chi_2^1) \\
\chi_2 \chi_+ - q^2 \chi_+ \chi_2 &= -q \chi_+ \\
\chi_2 \chi_- - q^{-2} \chi_- \chi_2 &= q^{-1} \chi_-
\end{align*}
\]

where we used the composite index notation

\[
a_1 a_2 \to i, \quad b_1 b_2 \to j, \quad \text{and} \quad i, j = 1, +, -, 2.
\]

These $q$-lie algebra relations can be compactly written \[^2\]

\[
[[\chi_i, \chi_j], \chi_r] = [[\chi_i, \chi_j], \chi_r] + \Lambda^{kl} \chi_k [\chi_l, [\chi_i, \chi_r]] .
\]

(6.12)

7 Deformation by twist and quantum Poincaré Lie algebra

In this last section, led by the example the Poincaré Lie algebra, we review a quite general method to deform the Hopf algebra $U(g)$, the universal enveloping algebra of a given Lie algebra $g$. It is based on a twist procedure. A twist element $\mathcal{F}$ is an invertible element in $U(g) \otimes U(g)$. A main property $\mathcal{F}$ has to satisfy is the cocycle condition

\[
(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} .
\]

Consider for example the usual Poincaré Lie algebra $iso(3,1)$:

\[
\begin{align*}
[P_{\mu}, P_{\nu}] &= 0 , \\
[P_{\rho}, M_{\mu\nu}] &= i(\eta_{\rho\mu} P_{\nu} - \eta_{\rho\nu} P_{\mu}) , \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) .
\end{align*}
\]

Relation to the conventions of [5, 8] (here underlined): $\chi_i = -S^{-1} \chi_i$, $f^i_j \rightarrow S^{-1} f^i_j$. \[13\]
A twist element is given by
\[ F = e^\frac{i}{2} \theta^{\mu\nu} P_\mu \otimes P_\nu, \]
where \( \theta^{\mu\nu} \) (despite the indices \( \mu\nu \) notation) is a real antisymmetric matrix of dimensionful constants (the previous deformation parameter \( q \) was a constant too). We consider \( \theta^{\mu\nu} \) fundamental physical constants, like the velocity of light \( c \), or like \( \hbar \). In this setting symmetries will leave \( \theta^{\mu\nu}, c \) and \( \hbar \) invariant. The inverse of \( F \) is
\[ F = e^{-\frac{i}{2} \theta^{\mu\nu} P_\mu \otimes P_\nu}. \]
This twist satisfies the cocycle condition (7.1) because the Lie algebra of momenta is abelian.

The Poincaré Hopf algebra \( U^F(\text{iso}(3,1)) \) is a deformation of \( U(\text{iso}(3,1)) \). As algebras \( U^F(\text{iso}(3,1)) = U(\text{iso}(3,1)) \); but \( U^F(\text{iso}(3,1)) \) has the new coproduct
\[ \Delta^F(\xi) = F \Delta(\xi) F^{-1}, \]
for all \( \xi \in U(\text{iso}(3,1)) \). In order to write the explicit expression for \( \Delta^F(P_\mu) \) and \( \Delta^F(M_{\mu\nu}) \), we use the Hadamard formula
\[
\text{Ad}_e^X Y = e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} [X,\ldots[X,Y]] = \sum_{n=0}^{\infty} \frac{(adX)^n}{n!} Y
\]
and the relation \([P \otimes P', M \otimes 1] = [P,M] \otimes P'\), and thus obtain [15], [14]
\[
\Delta^F(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu, \\
\Delta^F(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} + \frac{1}{2} \theta^{\alpha\beta} \left( (\eta_{\alpha\mu} P_\nu - \eta_{\alpha\nu} P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu} P_\nu - \eta_{\beta\nu} P_\mu) \right).
\]
We have constructed the Hopf algebra \( U^F(\text{iso}(3,1)) \): it is the algebra generated by \( M_{\mu\nu} \) and \( P_\mu \) modulo the relations (7.2), and with coproduct (7.6) and counit and antipode that are as in the undeformed case:
\[
\varepsilon(P_\mu) = \varepsilon(M_{\mu\nu}) = 0, \quad S(P_\mu) = -P_\mu, \quad S(M_{\mu\nu}) = -M_{\mu\nu}.
\]
This algebra is a symmetry algebra of the noncommutative spacetime \( x^\mu x'^\nu - x'^\mu x^\nu = i\theta^{\mu\nu} \).

In general given a Lie algebra \( g \), and a twist \( F \in U(g) \otimes U(g) \), formula (7.5) defines a new coproduct that is not cocommutative. We call \( U(g)^F \) the new Hopf algebra with coproduct \( \Delta^F \), counit \( \varepsilon^F = \varepsilon \) and antipode \( S^F \) that is a deformation of \( S \) [13]. By definition as algebra \( U(g)^F \) equals \( U(g) \), only the costructures are deformed.

---

3Explicitly, if we write \( F = f^a \otimes f_a \) and define the element \( \chi = f^a S(f_a) \) (that can be proven to be invertible) then for all elements \( \xi \in U(g) \), \( S^F(\xi) = \chi S(\xi) \chi^{-1} \).
We now construct the quantum Poincaré Lie algebra $iso^F(3,1)$. Following the previous section, the Poincaré Lie algebra $iso^F(3,1)$ must be a linear subspace of $U^F(iso(3,1))$ such that if $\{t_i\}_{i=1,...,n}$ is a basis of $iso^F(3,1)$, we have (sum understood on repeated indices)

1. $\{t_i\}$ generates $U^F(iso(3,1))$
2. $\Delta^F(t_i) = t_i \otimes 1 + f^j_i \otimes t_j$
3. $[t_i, t_j]_F = C^k_{ij} t_k$

where $C_{ij}^k$ are structure constants and $f^j_i \in U^F(iso(3,1))$ ($i,j = 1,...,n$). In the last line the bracket $[,]_F$ is the adjoint action:

$$[t_i, t'_j]_F := ad^F_{t_j} t'_i = t_i t'_j S(t_j), \quad (7.8)$$

where we used the coproduct notation $\Delta^F(t) = t \otimes t + 1 \otimes t$. The statement that the Lie algebra of $U^F(iso(3,1))$ is the undeformed Poincaré Lie algebra $(7.2)$ is not correct because conditions ii) and iii) are not met by the generators $P_\mu$ and $M_{\mu\nu}$. There is a canonical procedure in order to obtain the Lie algebra $iso^F(3,1)$ of $U^F(iso(3,1))$ [9,10].

Consider the elements

$$P^F_\mu := \Gamma^\nu(P_\mu) \Gamma_\alpha = P_\mu, \quad (7.9)$$

$$M^F_{\mu\nu} := \Gamma^\sigma(M_{\mu\nu}) \Gamma_\alpha = M_{\mu\nu} - \frac{i}{2} \theta^{\alpha\sigma}[P_\rho, M_{\mu\nu}] P_\sigma$$

$$= M_{\mu\nu} + \frac{1}{2} \theta^{\rho\sigma}(\eta_{\mu\rho} P_\nu - \eta_{\nu\mu} P_\rho) P_\sigma \quad (7.10)$$

Their coproduct is

$$\Delta^F(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu, \quad \Delta^F(M^F_{\mu\nu}) = M^F_{\mu\nu} \otimes 1 + 1 \otimes M^F_{\nu\mu} + i\theta^{\alpha\beta} P_\alpha \otimes [P_\beta, M_{\mu\nu}] \quad (7.11)$$

The counit and antipode are

$$\varepsilon(P_\mu) = \varepsilon(M^F_{\mu\nu}) = 0, \quad S(P_\mu) = -P_\mu, \quad S(M^F_{\mu\nu}) = -M^F_{\mu\nu} - i\theta^{\mu\sigma}[P_\rho, M_{\mu\nu}] P_\sigma \quad (7.12)$$

The elements $P^F_\mu$ and $M^F_{\mu\nu}$ are generators because they satisfy condition i) (indeed $M_{\mu\nu} = M^F_{\mu\nu} + \frac{i}{2} \theta^{\mu\nu}[P_\rho, M^F_{\mu\nu}] P_\sigma$). They are deformed infinitesimal generators because they satisfy the Leibniz rule ii) and because they close under the Lie bracket iii).

Explicitly

$$[P_\mu, P_\nu]_F = 0,$$

$$[P_\rho, M^F_{\mu\nu}]_F = i(\eta_{\rho\nu} P_\mu - \eta_{\rho\mu} P_\nu),$$

$$[M^F_{\mu\nu}, M^F_{\rho\sigma}]_F = -i(\eta_{\rho\nu} M^F_{\mu\sigma} - \eta_{\mu\rho} M^F_{\nu\sigma} - \eta_{\nu\rho} M^F_{\mu\sigma} + \eta_{\mu\sigma} M^F_{\nu\rho}) \quad (7.13)$$
We notice that the structure constants are the same as in the undeformed case, however the adjoint action \([M^F_{\mu\nu}, M^F_{\rho\sigma}]_x\) is not the commutator anymore, it is a deformed commutator quadratic in the generators and antisymmetric:

\[
[M^F_{\mu\nu}, M^F_{\rho\sigma}]_x = [M^F_{\mu\nu}, M^F_{\rho\sigma}] - i\theta^{\alpha\beta}[P_\alpha, M^F_{\rho\sigma}][P_\beta, M^F_{\mu\nu}].
\] (7.14)

From (7.13) we immediately obtain the Jacobi identities:

\[
[t, [t', t'']]_x + [t', [t'', t]]_x + [t'', [t, t']]_x = 0,
\] (7.15)

for all \(t, t', t'' \in \text{iso}^F(3, 1)\).

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### A Algebras, Coalgebras and Hopf algebras

In the introduction we have motivated the notion of Hopf algebra. We here review some basic definitions in linear algebra and show how Hopf algebras merge algebras and coalgebras structures in a symmetric (specular) way ([16]).

We recall that a module by definition is an abelian group. The group operation is denoted + (additive notation). A vector space \(A\) over \(\mathbb{C}\) (or \(\mathbb{R}\)) is a \(\mathbb{C}\)-module, i.e. there is an action \((\lambda, a) \rightarrow \lambda a\) of the group \((\mathbb{C} - \{0\}, \cdot)\) on the module \(A\),

\[
(\lambda \lambda') a = \lambda (\lambda' a),
\] (A.1)

and this action is compatible with the addition in \(A\) and in \(\mathbb{C}\), i.e. it is compatible with the module structure of \(A\) and of \(\mathbb{C}\):

\[
\lambda(a + a') = \lambda a + \lambda a', \quad (\lambda + \lambda') a = \lambda a + \lambda' a.
\] (A.2)

An algebra \(A\) over \(\mathbb{C}\) with unit \(I\), is a vector space over \(\mathbb{C}\) with a multiplication map, that we denote \(\cdot\) or \(\mu\),

\[
\mu : A \times A \rightarrow A
\] (A.3)
that is \( \mathbb{C}\)-bilinear: \((\lambda a) \cdot (\lambda' b) = \lambda \lambda'(a \cdot b)\), that is associative and that for all \(a\) satisfies \(a \cdot I = I \cdot a = a\).

These three properties can be stated diagrammatically. \(\mathbb{C}\)-bilinearity of the product \(\mu : A \times A \to A\), is equivalently expressed as linearity of the map \(\mu : A \otimes A \to A\). Associativity reads,

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\
\downarrow \text{id} \otimes \mu & & \downarrow \mu \\
A \otimes A & \xrightarrow{\mu} & A \\
\end{array}
\]

Finally the existence of the unit \(I\) such that for all \(a\) we have \(a \cdot I = I \cdot a = a\) is equivalent to the existence of a linear map

\[
i : \mathbb{C} \to A
\]  

(A.4)

such that

\[
\begin{array}{ccc}
\mathbb{C} \otimes A & \xrightarrow{i \otimes \text{id}} & A \otimes A \\
\downarrow \cong & & \downarrow \mu \\
A & \xrightarrow{\text{id}} & A \\
\end{array}
\]

and

\[
\begin{array}{ccc}
A \otimes \mathbb{C} & \xrightarrow{\text{id} \otimes i} & A \otimes A \\
\downarrow \cong & & \downarrow \mu \\
A & \xrightarrow{\text{id}} & A \\
\end{array}
\]

where \(\cong\) denotes the canonical isomorphism between \(A \otimes \mathbb{C}\) (or \(\mathbb{C} \otimes A\)) and \(A\). The unit \(I\) is then recovered as \(i(1) = I\).

A **coalgebra** \(A\) over \(\mathbb{C}\) is a vector space with a linear map \(\Delta : A \to A \otimes A\) that is coassociative, \((\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta\), and a linear map \(\varepsilon : A \to \mathbb{C}\), called counit that satisfies \((\text{id} \otimes \varepsilon)D(a) = (\varepsilon \otimes \text{id})\Delta(a) = a\). These properties can be expressed diagrammatically by reverting the arrows of the previous diagrams:

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xleftarrow{\Delta \otimes \text{id}} & A \otimes A \\
\uparrow \text{id} \otimes \Delta & & \uparrow \Delta \\
A \otimes A & \xleftarrow{\Delta} & A \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{C} \otimes A & \xleftarrow{\varepsilon \otimes \text{id}} & A \otimes A \\
\uparrow \cong & & \uparrow \Delta \\
A & \xleftarrow{\text{id}} & A \\
\end{array}
\]
We finally arrive at the

**Definition** A bialgebra $A$ over $C$ is a vectorspace $A$ with an algebra and a coalgebra structure that are compatible, i.e.

1) the coproduct $\Delta$ is an algebra map between the algebra $A$ and the algebra $A \otimes A$, where the product in $A \otimes A$ is $(a \otimes b)(c \otimes d) = ac \otimes bd$,

$$\Delta(ab) = \Delta(a)\Delta(b) \quad \Delta(I) = I \otimes I \quad (A.5)$$

2) The counit $\varepsilon : A \rightarrow C$ is an algebra map

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad \varepsilon(I) = 1 \quad (A.6)$$

**Definition** A Hopf algebra is a bialgebra with a linear map $S : A \rightarrow A$, called antipode (or coinverse), such that

$$\mu(S \otimes \text{id})\Delta(a) = \mu(\text{id} \otimes S)\Delta(a) = \varepsilon(a)I \quad (A.7)$$

It can be proven that the antipode $S$ is unique and antimultiplicative

$$S(ab) = S(b)S(a) .$$

From the definition of bialgebra it follows that $\mu : A \otimes A \rightarrow A$ and $i : C \rightarrow A$ are coalgebra maps, i.e., $\Delta \circ \mu = \mu \otimes \mu \circ \Delta$, $\varepsilon \otimes \mu = \underline{\varepsilon}$ and $\Delta \circ i = i \otimes i \circ \Delta_C$, $\varepsilon \circ i = \varepsilon_C$, where the coproduct and counit in $A \otimes A$ are given by $\Delta(a \otimes b) = a_1 \otimes b_1 \otimes a_2 \otimes b_2$ and $\underline{\varepsilon} = \varepsilon \otimes \varepsilon$, while the coproduct in $C$ is the map $\Delta_C$ that identifies $C$ with $C \otimes C$ and the counit is $\varepsilon_C = id$. Viceversa if $A$ is an algebra and a coalgebra and $\mu$ and $i$ are coalgebra maps then it follows that $\Delta$ and $\varepsilon$ are algebra maps.

One can write diagrammatically equations $(A.5), (A.6), (A.7)$, and see that the Hopf algebra definition is invariant under inversion of arrows and exchange of structures with costructures, with the antipode going into itself. In this respect the algebra and the coalgebra structures in a Hopf algebra are specular.
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