The $\phi^4_4$: quantum field theory, I.
Wave operator, holomorphy and Wick kernel. *

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Abstract

With the help of the complex structure and the wave operator of the nonlinear classical Klein-Gordon equation with the interaction $\phi^4_4$ we define the Wick kernel of the interacting quantum field in four-dimensional space-time and consider its properties. In particular, the diagonal of this Wick kernel is (real) solutions of the classical nonlinear Klein-Gordon equation with the interaction $\phi^4_4$.

1. Introduction.

In the paper [22], see also [17, 36], we have constructed the solution $\phi$ of the quantum wave equation

$$\Box \phi(t, x) + m^2 \phi(t, x) + \lambda : \phi^3(t, x) := 0$$

in four-dimensional space-time. In the integral form this equation is the following

$$\phi(t, x) = \phi_{in}(t, x) - \lambda \int_{-\infty}^{t} \int R(t - \tau, x - y) : \phi^3(\tau, y) : d\tau d^3y,$$  \hspace{1cm} (1.2)

$$\phi_{out}(t, x) = \phi_{in}(t, x) - \lambda \int_{-\infty}^{+\infty} \int R(t - \tau, x - y) : \phi^3(\tau, y) : d\tau d^3y,$$  \hspace{1cm} (1.3)

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1 The constructed solution is nontrivial [22, Theorem 4.3], [29], but here we do not discuss the question of the nontriviality of the $\phi^4_4$: quantum theory [3, 8, 11, 17, 35, 22].
where the in-field $\phi_{in}$ is the in-coming free quantum field and $:$ is the Wick normal ordering with respect to the free field $\phi_{in}$.

In the papers [17, 22] this solution has been correctly defined as a bilinear form on $D_{coh}(\vartheta) \times D_{coh}(\vartheta)$, where $D_{coh}(\vartheta)$ is a subspace in the Fock space of the in-field $\phi_{in}$. The subspace $D_{coh}(\vartheta)$ is generated by linear combinations of coherent vectors which are near to the vacuum.

In the present paper we consider the solution of the quantum nonlinear wave equation (1.1), which was constructed in [17, 22], from the other point of view.

We introduce explicitly the Wick kernel for the interacting field

$$\phi(e_{z_1}, e_{z_2}) = \frac{1}{2} \exp(\langle z_1, z_2 \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}) (RW_{in} R^{-1}(\bar{z}_1 + z_2) + RW_{in} R^{-1}((\bar{z}_1 + z_2))) \quad (1.4)$$

and for the out-going field

$$\phi_{out}(e_{z_1}, e_{z_2}) = \frac{1}{2} \exp(\langle z_1, z_2 \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}) (RSR^{-1}(\bar{z}_1 + z_2) + RSR^{-1}((\bar{z}_1 + z_2))) \quad (1.5)$$

where $R$ is the isomorphism,

$$R : H^1(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3) \to H^1(\mathbb{R}^3, \mathcal{C}),$$

and $W_{in}, W_{out}$ are the in- and out-wave operator, respectively, of the classical nonlinear equation with the interaction of the 4th degree. This Wick symbol is defined for all coherent vectors with finite energy. Due to holomorphity the expression (1.4) is holomorphic in $\bar{z}_1, z_2$ for $z_1, z_2 \in H^1(\mathbb{R}^3, \mathcal{C})$, see [23].

The expression (1.4) gives, in fact, the mathematical realization of a physical idea to write an interacting field with the help of a wave operator. Due to holomorphity (1.4) has the quite correct sense written as

$$\phi = \frac{1}{2} ( : R W_{in} R^{-1}(\phi_{in}) : + : R W_{out} R^{-1}(\phi_{in}) :),$$

i.e.

$$\phi = \sum_{n=1}^{\infty} \phi_n( : \phi_{in} \ldots \phi_{in} : ) = \sum_{n=1}^{\infty} \frac{1}{2} (R_n + \overline{R}_n)( : \phi_{in} \ldots \phi_{in} :), \quad (1.6)$$

where $R_n$ is the unique generalized function from $\mathcal{S}(\mathbb{R}^{3n}, \mathcal{C})$, which corresponds to the $n$-th Frechét derivative of the function $R W_{in} R^{-1}$ at zero in the complex Hilbert space $H^1(\mathbb{R}^3, \mathcal{C})$ and $\overline{R}_n$ is its complex conjugation, i.e. (we denote later $W_{in}$ as $W$ without the subscript “in”)

$$R_n(f_1 \otimes \ldots \otimes f_n) = \frac{1}{n!} d^n R W R^{-1}(0)(f_1, \ldots, f_n),$$

$f_j \in \mathcal{S}(\mathbb{R}^{3j}, \mathcal{C})$. In other words, $R_n$ is equal to the $n$-th derivative of the function $R W R^{-1}$ at zero and the derivative is taken over the positive frequency part of in-solution.
positive frequency part of in-solution may be defined with the help of the isomorphism \( R \) which gives the one-to-one correspondence with the initial in-data.

The expansion (1.6) is defined by the Taylor expansion of the (classical) wave operator at zero and it is convergent on coherent vectors near to the vacuum. The expansion (1.6) defines the bilinear form and the holomorphity allows to prove that the Wick kernel of (1.6) can be extended uniquely on all coherent vectors with finite energy (and is equal to (1.4)).

Thus, we construct the \( \phi_4^4 \): quantum field theory along the following pathway. To construct the \( \phi_4^4 \): quantum field theory we define the Wick kernel for the interacting quantum field as (1.4). As a consequence we obtain that the interacting quantum field satisfies the expansion (1.5) in the sense of bilinear forms on an appropriate subspace of the Fock space. The complex structure and estimates allow us to prove that (1.4) is defined correctly as a Wick kernel in the sense of Paneitz, Pedersen, Segal, and Zhou [34].

Separately, we prove that the solution of Eq. (1.3), defined as a bilinear form, has the Wick kernel equal to (1.4) on coherent vectors near to the vacuum, i.e. in a zero neighbourhood, [26], see also [22, 17, 34]. In addition, this Wick kernel (1.4) defines the unique operator–valued generalized function from \( S^1(\mathbb{R}^4) \times S^1(\mathbb{R}^4) \) into \( L(\Phi \omega \kappa) \). Here \( S^\alpha(\mathbb{R}^4) \) are the Gelfand spaces of test functions [14, v. 2], [15, v. 4], \( L(\Phi \omega \kappa) \) is the space of linear bounded operators in the space \( \Phi \omega \kappa \) and \( \Phi \omega \kappa \) denotes the Fock space of the free quantum in-field. Moreover, this operator-valued generalized function can be extended on a Gelfand space \( S^\alpha, \alpha < 6/5 \), and these Gelfand spaces with \( \alpha > 1 \) contain dense subspaces of functions with compact supports in coordinate space [28]. This result allows us to construct Wightman functions (as generalized function of Jaffe type) and matrix elements of the quantum scattering operator. It is possible to consider Wightman axioms: the conditions of positivity, spectrality, Poincaré invariance, locality, cluster property, and asymptotic completeness [29].

In this paper we introduce the Wick kernel (1.4) for the interacting quantum field of the \( \phi_4^4 \): quantum field theory and consider its properties.

2. Complex structure.

In this section we formulate and prove the assertions connected with the complex structure of solutions of the classical (real) nonlinear wave equation with the cubic nonlinearity in four–dimensional Minkowski space–time,

\[
\Box u + m^2 u + \lambda u^3 = 0, \quad m > 0, \quad \lambda > 0.
\]

We use these statements for the construction of the \( \phi_4^4 \): quantum field theory. To introduce the Wick kernel of the interacting quantum field of the \( \phi_4^4 \): theory we need the assertion about holomorphity (Theorem 2.1) and about \( T \) (and/or \( CPT \)) symmetry (Theorem 2.2).

\footnote{We use the usual “physical” terminology for this notion. It is defined uniquely for the massive linear Klein–Gordon equation, i.e. for free solutions.}
The solution of Eq. (2.1) is uniquely determined by its Cauchy data
\[ u|_{t=0} = \varphi, \quad \partial_t u|_{t=0} = \pi. \]
Let \( W_{in} (= W) \) be the incoming wave operator (the in–wave operator, or the backward wave operator) of Eq. (2.1). The operator \( W_{in} \) maps in–data at time zero into initial data at time zero. Let \( W_{out} \) be the outgoing wave operator (the out–wave operator, or the forward wave operator). The operator \( W_{out} \) maps out–data into initial data. Let \( S \) be the scattering operator of the classical nonlinear wave equation (2.1).

**Theorem 2.1** [25, Theorem 1.1].

Let \( R(\varphi, \pi) = \varphi + i \mu^{-1} \pi, \mu = (-\Delta + m^2)^{1/2}, \) be the invertible map of \( S_{Re}(\mathbb{R}^3) \oplus S_{Re}(\mathbb{R}^3) \) onto \( S(\mathbb{R}^3, \mathbb{C}) \). This map \( R \) defines also isomorphisms \( (H^{1/2}(\mathbb{R}^3) \oplus H^{-1/2}(\mathbb{R}^3), J)) \) onto \( H^{1/2}(\mathbb{R}^3, \mathbb{C}) \) and of \( (H^1(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3), J) \) onto \( H^1(\mathbb{R}^3, \mathbb{C}) \). Here
\[ J = R^{-1}iR = \begin{pmatrix} 0 & -\mu^{-1} \\ \mu & 0 \end{pmatrix}. \]

The map \( RW_{out}R^{-1} \) is defined correctly as the mapping from \( H^1(\mathbb{R}^3, \mathbb{C}) \) onto \( H^1(\mathbb{R}^3, \mathbb{C}) \) and is the complex analytic mapping of the complex Hilbert space \( H^1(\mathbb{R}^3, \mathbb{C}) \) onto itself. In particular, for \( z_{in}(\alpha) = \sum_{j=1}^N \alpha_j z_{in,j}, \alpha_j \in \mathbb{C}, z_{in,j} \in H^1(\mathbb{R}^3, \mathbb{C}), h \in H^{1/2}(\mathbb{R}^3, \mathbb{C}), \) the functions \( \langle h, RW_{out}R^{-1}(z_{in}(\alpha)) \rangle_{H^1/2(\mathbb{R}^3, \mathbb{C})} \) are entire holomorphic function on \((\alpha_1, ..., \alpha_N) \in \mathbb{C}^N\).

The same assertions are valid for the transformation \( RW_{out}R^{-1} \) and \( RSR^{-1} \).

Theorem 2.1 is proved in the paper [25, Theorem 1.1].

Let \( \mathcal{P} \) be the Poincaré group. Let \( \mathcal{P}_0 \) be the connected component of the Poincaré group. Given \((a, \Lambda) \in \mathcal{P}\) and a finite–energy solution \( u \) of (2.1), define \((a, \Lambda)u\) by
\[ ((a, \Lambda)u)(t, x) = u((a, \Lambda)^{-1}(t, x)). \]
Then \((a, \Lambda)u\) is also a finite–energy solution of (2.1), and the map \((a, \Lambda, u) \rightarrow (a, \Lambda)u\) defines an action of the group \( \mathcal{P}\) on the Hilbert space \( H^1 \oplus L_2 \) of initial data. We denote this map by \( U(a, \Lambda) \). When the coupling constant in (2.1) \( \lambda = 0 \), i.e. for the case of the linear Klein–Gordon equation, the maps \( U_0(a, \Lambda) \) are linear.

The wave operators have the following basic properties.

**Theorem 2.2.**

The wave operator \( W_{in} \) and \( W_{out} \) are correctly defined on the Hilbert space \( H^1 \oplus L_2 \). Moreover, they intertwine the free and interacting action of the Poincaré group, i.e.
\[ U(a, \Lambda) = W_{in}U_0(a, \Lambda)W_{in}^{-1}, \]
\[ U(a, \Lambda) = W_{out}U_0(a, \Lambda)W_{out}^{-1} \]
for all \((a, \Lambda) \in \mathcal{P}_0\).
Proof of Theorem 2.2. See, for instance, [4]. Theorem 2.2 is proved.

Let $\Theta^T$ be the time reflection operator on the space of initial data, $\Theta^T(\varphi, \pi) = (\varphi, -\pi)$.
Let $\Theta^P$ be the space reflection operator on the space of initial data, $\Theta^P(\varphi, \pi) = (\varphi^P, \pi^P)$, where $\varphi^P(x) = \varphi(-x)$, $\pi^P(x) = \pi^P(-x)$.

Theorem 2.3 ($T$-symmetry).
The equalities
$$W_{\text{out}} = \Theta^T W_{\text{in}} \Theta^T, \quad S^{-1} = \Theta^T S \Theta^T$$
are fulfilled.

Corollary 2.4 ($PT$- and $CPT$-symmetry).
The equalities
$$W_{\text{out}} = \Theta^T \Theta^P W_{\text{in}} \Theta^T \Theta^P, \quad S^{-1} = \Theta^T \Theta^P S \Theta^T \Theta^P$$
are fulfilled.

Remarks.
1. The symmetry $\Theta$ for the classical system means the following. $\Theta$ is a mapping of initial data, $(\varphi, \pi) \rightarrow \Theta(\varphi, \pi)$, if $u$ is a solution of Eq. (2.1) with initial data $(\varphi, \pi)$, then $\Theta(\varphi, \pi)$ are initial data of some solution $u^\Theta$. In particular, if $u(t,x)$ is a solution of Eq. (2.1), then $u^{\Theta^T}(t,x) = u(-t, x)$ is the corresponding solution of Eq. (2.1).
2. Since $P$-symmetry (the space reflection symmetry) is obvious, so $T$-symmetry implies $PT$-symmetry and $CPT$-symmetry for the nonlinear equation (2.1). Of course, $CPT$-symmetry is more fundamental.
3. On the other hand Theorem 2.2 is connected with the correspondence between the time reflection and the complex conjugation. Theorem 2.2 appears in the fact that the interacting quantum field is a Hermitian bilinear form. For the bilinear form–solution this was pointed out in [17], [22].
4. The indexation of free solutions with the help of positive frequency part corresponds to the fact that the real diagonal of Wick symbol of the interacting quantum field satisfies the classical nonlinear wave equation.

Proof of Theorem 2.3.
Let $u(t,x)$ be a solution of classical nonlinear equation (2.1) with finite energy. Then $u(-t,x)$ (and $u(-t,-x)$) is a solution of the same equation also. The solution $u$ defines the unique initial, $\text{in}$-, and $\text{out}$-data at time zero,

$$(\varphi(x), \pi(x)) = (u(0,x), \dot{u}(0,x)), \quad (\varphi_{\text{in}}(x), \pi_{\text{in}}(x)) = (u_{\text{in}}(0,x), \dot{u}_{\text{in}}(0,x)), \quad (\varphi_{\text{out}}(x), \pi_{\text{out}}(x)) = (u_{\text{out}}(0,x), \dot{u}_{\text{out}}(0,x)),$$

where $u_{\text{in}}$ $u_{\text{out}}$ are the unique free solutions such that

$$\| (u(t,\cdot), \dot{u}(t,\cdot)) - (u_{\text{in}}(t,\cdot), \dot{u}_{\text{in}}(t,\cdot)) \|_{H^1 \oplus L^2} \quad \text{for} \quad t \rightarrow -\infty$$
and
\[ \| (u(t,\cdot),\dot{u}(t,\cdot)) - (u_{out}(t,\cdot),\dot{u}_{out}(t,\cdot)) \|_{H^1 \oplus L^2} \quad \text{for} \quad t \to +\infty. \]

This implies immediately that the solution \( u^\Theta_T = u(-t,x) \) has the initial data equal to \((\varphi, -\pi)\) at time \( t = 0 \), the in–data equal to \((\varphi_{out}, -\pi_{out})\), and the out–data equal to \((\varphi_{in}, -\pi_{in})\).

Therefore, since by definition the wave operators satisfy the following equalities
\[ W_{in}(\varphi_{in}, \pi_{in}) = (\varphi, \pi), \quad W_{out}(\varphi_{out}, \pi_{out}) = (\varphi, \pi), \]
\[ S(\varphi_{in}, \pi_{in}) = (\varphi_{out}, \pi_{out}), \quad S = W_{out}^{-1} W_{in}, \]
the previous assertion implies the equality
\[ \Theta^T W_{in}(\varphi_{in}, \pi_{in}) = W_{out} \Theta^T(\varphi_{in}, \pi_{in}), \]
i.e.
\[ W_{out} = \Theta^T W_{in} \Theta^T. \]
The last equality implies that
\[ S^{-1} = \Theta^T S \Theta^T. \]
Theorem 2.3 is proved.

**Remark.**
It is well known that \( S \) is not the identity operator \([20]\). This implies that \( W_{out} \neq W_{in} \).

3. Wick kernel of the interacting quantum field.

The existence of classical wave operator and holomorphity allow us to introduce the Wick kernel of the interacting quantum field and to construct the bilinear form given by this kernel.

For this purpose we introduce the Hilbert Fock space for the free quantum field \( \phi_{in} \). We introduce the Fock space as it described in \([36]\), but instead of the basis used by Rączka \([30]\) we take a basis in the complex Hilbert space \( H^{1/2}(IR^3, \mathbb{C}) \) with explicit introduction of its pure real and pure imaginary parts (for the integral by Paneitz, Pedersen, Segal, Zhou \([34]\), see also \([5]\), we may use any basis in the complex Hilbert space).

Let us take a (canonical and standard) basis in the space \( H^{1/2}(IR^3, \mathbb{C}) \) analogous to the basis introduced in \([4]\, ch. 5.1, p. 141\) (the space \( H^{1/2}(IR^3, \mathbb{C}) \) corresponds to the dual space with dualization \((.,.)_{L^2}\) of the one-particle subspace of the Fock space, this one-particle subspace is denoted by Bogoliubov, Logunov, Todorov \([4]\, ch. 5.1, p. 141\) as \( \mathcal{H}_1 \)).

Similar to \([4]\, ch.5.1, p. 141\) we introduce our basis and note that all basic vectors in our basis are pure real–valued functions in coordinate space. Thus, we define for the (relativistic) coordinate operator (in momentum space)
\[ q = \sqrt{\mu(p)}i \nabla_p - \frac{1}{\sqrt{\mu(p)}} = i \nabla_p - \frac{i}{2} \frac{p}{m^2 + p^2}, \]
in this case the momentum operator is the operator of multiplication on $p$ (in momentum space).

The operators $q^j$ and $p^j$, $j = 1, 2, 3$, satisfy the canonical commutation relations

$$[q^j, p^{j'}] = i\delta_{jj'}, \quad [q^j, q^{j'}] = [p^j, p^{j'}] = 0,$$

are hermitian (and essentially self–adjoint on $S(\mathbb{R}^3, C)$) in the Hilbert space $\mathcal{H}_1$ with the scalar product

$$(\psi_1, \psi_2) = \int \frac{d^3p}{\mu(p)} \psi_1(p) \overline{\psi_2(p)},$$

the Hilbert space $\mathcal{H}_1$ coincides with the Hilbert space $H^{-1/2}(\mathbb{R}^3, C)$ in coordinate space.

We define the operators

$$b_j = \frac{1}{\sqrt{2}}(q^j + ip^j), \quad b^*_j = \frac{1}{\sqrt{2}}(q^j - ip^j), \quad j = 1, 2, 3.$$  

These operators are mutually adjoint and correspond to the analogous operators in [4, ch. 5.1, p. 141],

$$b_j = ib_j^{BLT} \quad ([4, ch. 5.1, p. 141]), \quad b^*_j = -ib_j^{BLT} \quad ([4, ch. 5.1, p. 141]).$$

They satisfy the commutation relations

$$[b_j, b^*_{j'}] = \delta_{jj'}, \quad [b_j, b_{j'}] = [b^*_j, b^*_{j'}] = 0.$$  

Let $e_0$ be the normalized vector, satisfying the equalities

$$b_j e_0 = 0, \quad j = 1, 2, 3.$$  

In the considered momentum space

$$e_0(p) = \pi^{-3/4} (m^2 + p^2)^{1/4} e^{-p^2/2}.$$  

The orthonormalized basic vectors in $\mathcal{H}_1$ we define as the Hermite functions (the Hermite polynomials) defined on the mass hyperboloid,

$$e_k \equiv e_{k_1 k_2 k_3} = \frac{(b^*_k)^{k_1} (b^*_k)^{k_2} (b^*_k)^{k_3}}{\sqrt{k_1! k_2! k_3!}} e_0, \quad k_1, k_2, k_3 = 0, 1, ... .$$

In $x$–space the vector $e_0(x)$ is a real–valued function and the vectors $e_k(x)$ are real–valued functions also. This follows from the fact, that in the $p$–space

$$b^*_j = \frac{1}{\sqrt{2}} (\mu^{1/2} i \nabla_p \mu^{1/2} - ip^j) = \frac{1}{\sqrt{2}} (\mu^{1/2} x^j \mu^{-1/2} - \partial / \partial x^j)$$

and from the reality of operators $\mu$ as convolution operators. Here

$$q = \mu^{1/2} x \mu^{-1/2} = \mu^{1/2} i \nabla_p \mu^{-1/2}$$
is the operator of relativistic coordinate in $p$–space (= in momentum space), see [7, ch. 5.1, (2.5.6), p. 140], $\mu^{-1/2} = (-\Delta + m^2)^{-1/2}$ and gives an isomorphism $L^2(\mathbb{R}^3) \to H^{1/2}(\mathbb{R}^3)$.

In $x$-space the coordinate operator has the form

$$q = \mu^{-1/2} x \mu^{1/2}$$

because the function $F(x)$ from the one-particle space of the Fock space corresponds to the function

$$\mu(\mathcal{F}F)(p)$$

in the $p$–space, where $\mathcal{F}$ is the Fourier transform. If, correspondingly, $F(p)$ is a function that is square integrable over $d^3p/\mu(p)$, then in $x$–space this function corresponds to the function $\mu^{-1}(\mathcal{F}F)(x)$ and the operator of relativistic coordinate is $\mu^{-1/2} x^j \mu^{1/2}$ (here all identifications appear as the consideration of the same solution in different indexing spaces).

In this case, all functions that define the basis are orthonormal and real–valued in the $x$–space. In the $p$–space these functions satisfy the following relation

$$F(p) = \overline{F(-p)}.$$

The functions

$$\varphi_k(x) = c \int \exp(-ipx)e_k(p)d\rho(p),$$

with

$$c = [2(2\pi)^3]^{-1/2}, \quad d\rho(p) = d^3p/\mu(p), \quad \mu(p) = (p^2 + m^2)^{1/2},$$

are orthonormal real in the complex Hilbert space $H^{1/2}(\mathbb{R}^3, \mathbb{C})$ and generate a basis in this Hilbert space.

We use the following representation of the free (scalar, hermitian) quantum field in the Fock space, see [5], [7, ch. 5.1], [36]. We denote the Fock space for the free field as $\Phi_0\kappa$ and chose it as

$$\Phi_0\kappa = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_1 = H^{-1/2}(\mathbb{R}^3, \mathbb{C})$, $\mathcal{H}_n = \text{sym} \otimes_n H^{-1/2}(\mathbb{R}^3, \mathbb{C})$. The Fock space $\Phi_0\kappa$ is associated with the free quantum field $\phi_{in}$. Let $\phi^-_{in}(0, x)$, $\phi^+_{in}(0, x)$ be the annihilation and creation part of the free quantum field, i.e. the negative and positive frequency part of the field $\phi_{in}$ at time zero. Using the chosen basis $\{e_k(p)\}_{k=0}^{\infty}$ we define the smeared annihilation and creation operators $a_k$ and $a^*_k$,

$$a_k = \int \phi^-_{in}(0, x)e_k(x)d^3x, \quad a^*_k = \int \phi^+_{in}(0, x)e_k(x)d^3x,$$

where $e_k(x)$ is a basis vector in coordinate space (i.e. the Fourier transform of the vector $e_k(p)$).

The operators $a_k, a^*_k$, satisfy the canonical commutation relations $[a_k, a^*_l] = \delta_{kl}$ and form an infinite-dimensional nilpotent Lie algebra, which is irreducible in the Fock space $\Phi_0\kappa$. 

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The free quantum field \( \phi_{in}(t, x) \) has the following form in terms of annihilation \( a_k \) and creation \( a_k^* \) operators:

\[
\phi_{in}(t, x) = \sum_k (\varphi_k(t, x)a_k^* + \overline{\varphi_k(t, x)}a_k),
\]

where

\[
\varphi_k(t, x) = c \int \exp(i\mu(p)t - ipx)e_k(p)d\rho(p),
\]

\[
d\rho(p) = d^3p/\mu(p), \quad \mu(p) = (p^2 + m^2)^{1/2}, \quad c = \left[2(2\pi)^3\right]^{-1/2},
\]

\[
\varphi_k(0, x) \equiv \varphi_k(x) = c \int \exp(-ipx)e_k(p)d\rho(p),
\]

\[
\varphi_k(x) = \overline{\varphi_k(x)},
\]

with our choice of the real basis, \( e_k(p) = e_k(-p) \). In this case

\[
\varphi_k(t, x) = \varphi_k(-t, x).
\]

The vacuum averages of a sum of products of the field \( \phi_{in} \) coincide with the corresponding value of the free Wightman functional.

Now we construct convenient dense subspaces contained in the Fock space. Let

\[
e_z \equiv e(z) = \exp(za^*)\Omega,
\]

\[
|z\rangle = \exp(-\frac{1}{2}\|z\|^2)\exp(za^*)\Omega,
\]

where

\[
\|z\|^2 = \sum_{k=0}^{\infty} |z_k|^2, \quad za^* = \sum_{k=0}^{\infty} z_k a_k^*,
\]

\( z \) is an element of the complex Hilbert space \( H^{1/2}(IR^3, C) \), \( z \equiv \langle \varphi_k, z \rangle_{H^{1/2}(IR^3, C)} \). The vectors \( e_z, z \in H^{1/2}(IR^3, C) \), are called the coherent vectors and the vectors \( |z\rangle \) are called the coherent state vectors. It follows from Eq. (3.1) that

\[
(\exp(z'a^*)\Omega, \exp(za^*)\Omega) = \exp(\langle z', z \rangle_{H^{1/2}(IR^3, C)}) = \exp(\sum_{k=0}^{\infty} \overline{z'_k}z_k)
\]

and

\[
\langle z'|z \rangle = \exp\left\{ -\sum_{k=0}^{\infty} \left[ \frac{1}{2}|z'_k - z_k|^2 - i \text{Im} \left( \overline{z'_k}z_k \right) \right] \right\}
\]

\[
= \exp\left( \sum_{k=0}^{\infty} \left( \overline{z'_k}z_k - \frac{1}{2}|z'_k|^2 - \frac{1}{2}|z_k|^2 \right) \right)
\]

in the chosen earlier basis

\[
z = \sum_k z_k \varphi_k, \quad z_k = \langle \varphi_k, z \rangle_{H^{1/2}(IR^3, C)}.
\]
Let $A$ be a subset in $H^{1/2}(\mathbb{R}^3, \mathcal{C})$. Let $D_{coh}(A)$ be the subspace in the Fock space of the in-field generated by finite linear combinations of coherent vectors from $A$, i.e. if $\chi \in D_{coh}(A)$, then

$$D_{coh}(A) = \{ \chi \in D_{coh}(A) \mid \chi = \sum \alpha_j \epsilon(z_j), \text{the sum is finite}, \alpha_j \in \mathcal{C}, z_j \in A \}.$$ 

We define the Wick kernel for the interacting quantum field as the function $(z_1, z_2) \to \phi(e_{z_1}, e_{z_2})$ from $H^1(\mathbb{R}^3, \mathcal{C}) \times H^1(\mathbb{R}^3, \mathcal{C})$ into $H^1(\mathbb{R}^3, \mathcal{C})$,

$$\phi(e_{z_1}, e_{z_2}) = \exp(\langle z_1, z_2 \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}) \frac{1}{2}(RW R^{-1}(\bar{z}_1 + z_2) + \overline{RW R^{-1}(\bar{z}_1 + z_2)}),$$

where $RW R^{-1}$ maps $H^1(\mathbb{R}^3, \mathcal{C})$ into $H^1(\mathbb{R}^3, \mathcal{C})$. Thus, the Wick kernel smoothed with some test function over space is equal to

$$\int (\phi(e_{z_1}, e_{z_2}))(x)h(x)d^3x = \exp(\langle z_1, z_2 \rangle)$$

$$\frac{1}{2}(\int (\text{Re } RW R^{-1}(\bar{z}_1 + z_2))(x)h(x)d^3x + i \int (\text{Im } RW R^{-1}(\bar{z}_1 + z_2))(x)h(x)d^3x$$

$$+ \int (\text{Re } RW R^{-1}(\bar{z}_1 + z_2))(x)h(x)d^3x - i \int (\text{Im } RW R^{-1}(\bar{z}_1 + z_2))(x)h(x)d^3x)$$

$$= \exp(\langle z_1, z_2 \rangle)\frac{1}{2}(\langle RW R^{-1}(\bar{z}_1 + z_2), \mu^{-1}h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} + \langle RW R^{-1}(\bar{z}_1 + z_2), \mu^{-1}h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}).$$

The smoothed Wick kernel corresponds to the smoothed field at the time zero, i.e. to the bilinear form corresponding to $\int \phi(0, x)h(x)d^3x$.

Analogously, we define the Wick kernel for the out-going quantum field as

$$\phi_{out}(e_{z_1}, e_{z_2}) = \exp(\langle z_1, z_2 \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}) \frac{1}{2}(RS R^{-1}(\bar{z}_1 + z_2) + \overline{RS R^{-1}(\bar{z}_1 + z_2)}).$$

In [24] we show that the interacting and out-going Wick kernel can be reconstructed uniquely with the help of the bilinear form solution, that has been constructed in [22]. In this paper we consider the introduced Wick kernels, their properties and the bilinear form defined by these Wick kernels.

**Theorem 3.1.**

The Wick kernel

$$\phi(e_{z_1}, e_{z_2}) = \exp(\langle z_1, z_2 \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}) \frac{1}{2}(RW R^{-1}(\bar{z}_1 + z_2) + \overline{RW R^{-1}(\bar{z}_1 + z_2)})$$

is correctly defined on $H^1(\mathbb{R}^3, \mathcal{C}) \subset H^{1/2}(\mathbb{R}^3, \mathcal{C})$ as the map from

$$H^1(\mathbb{R}^3, \mathcal{C}) \times H^1(\mathbb{R}^3, \mathcal{C})$$

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into $H^1(\mathbb{R}^3, \mathbb{C})$ and is complex antiholomorphic on $z_1$ and complex holomorphic on $z_2$. In particular, for $z_1, z_2$ belonging to finite-dimensional subspaces in $H^1(\mathbb{R}^3, \mathbb{C})$,

$$z_1(\alpha_1) = \sum_{j=1}^{n} \alpha_{1,j} z_{1,j}, \quad z_2(\alpha_2) = \sum_{j=1}^{n} \alpha_{2,j} z_{2,j}, \quad z_{1,j}, z_{2,j} \in H^1(\mathbb{R}^3, \mathbb{C}), \quad \alpha_{1,j}, \alpha_{2,j} \in \mathbb{C},$$

$h \in H^1(\mathbb{R}^3, \mathbb{C})$, the function

$$\langle h, \phi(e(z_1(\alpha_1)), e(z_2(\alpha_2))) \rangle_{H^{1/2}(\mathbb{R}^3, \mathbb{C})}$$

is an entire complex antiholomorphic function on $\alpha_1 \in \mathbb{C}^m$ and an entire complex holomorphic function on $\alpha_2 \in \mathbb{C}^m$ in the usual sense. Furthermore,

$$\overline{\langle h, \phi(e_{z_1}, e_{z_2}) \rangle}_{H^{1/2}(\mathbb{R}^3, \mathbb{C})} = \langle \overline{h}, \phi(e_{z_1}, e_{z_2}) \rangle_{H^{1/2}(\mathbb{R}^3, \mathbb{C})},$$

i.e.

$$\overline{\phi(e_{z_1}, e_{z_2})} = \phi(e_{z_2}, e_{z_1}),$$

where a complex conjugation is defined as the complex conjugation of a function with values in $H^1(\mathbb{R}^3, \mathbb{C})$. It is valid the following estimate

$$|\langle h, \phi(e_{z_1}, e_{z_2}) \rangle_{H^{1/2}(\mathbb{R}^3, \mathbb{C})}| \leq c \|h\|_{L^2(\mathbb{R}^3, \mathbb{C})} \exp(\text{Re}(\langle \langle z_1, z_2 \rangle_{H^{1/2}(\mathbb{R}^3, \mathbb{C})} \rangle)) \|\bar{z}_1 + z_2\|_{H^1(\mathbb{R}^3, \mathbb{C})}.$$
is a bilinear form with values in $H^1(\mathbb{R}^3, \mathcal{C})$. In addition, the expression
\[
\langle h, \phi(\chi_1, \chi_2) \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} = \sum \overline{\alpha_{1,j}} \overline{\alpha_{2,j}} \langle h, \phi(e_{z_{1,j}}, e_{z_{2,j}}) \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}
\]
defines the bilinear form also. The bilinear forms $\phi(\chi_1, \chi_2)$ and $\langle h, \phi(\chi_1, \chi_2) \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}$ are Hermitian symmetric
\[
\overline{\phi(\chi_1, \chi_2)} = \phi(\chi_1, \chi_2),
\]

\[
\langle h, \phi(\chi_1, \chi_2) \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} = \langle h, \phi(\chi_1, \chi_2) \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}.
\]

The analogous assertion is also valid for the out-going quantum field $\phi_{\text{out}}$.

Let $H_{\text{in}}$ be the Hamiltonian of the (free quantum) in-coming field $\phi_{\text{in}}$. Due to the energy conservation it is equal to the total Hamiltonian.

**Theorem 3.3 (Energy positivity and holomorphy).**

Let $\chi = \sum \alpha_{1,j} e_{z_{1,j}}, \chi_2 = \sum \alpha_{2,j} e_{z_{2,j}}, \chi_1, \chi_2 \in D_{\text{coh}}(H^1(\mathbb{R}^3, \mathcal{C})$. The expression
\[
\phi(\exp(it_1 H_{\text{in}} - s_1 H_{\text{in}}) \chi_1, \exp(it_2 H_{\text{in}} - s_2 H_{\text{in}}) \chi_2)
\]

\[
= \sum \overline{\alpha_{1,j}} \overline{\alpha_{2,j}} \phi(e(\exp(\imath t_1 - \mu s_1)z_{1,j}), e(\exp(\imath t_2 - \mu s_2)z_{2,j}))
\]

(3.3)
is correctly defined for $\Im s_1 \geq 0$, $\Im s_2 \geq 0$ as a bilinear form on
\[
D_{\text{coh}}(H^1(\mathbb{R}^3, \mathcal{C})) \times D_{\text{coh}}(H^1(\mathbb{R}^3, \mathcal{C}))
\]

with values in $H^1(\mathbb{R}^3, \mathcal{C})$. Expression (3.3) depends antiholomorphically on $t_1 + \imath s_1$ and holomorphically on $t_2 + \imath s_2$.

The analogous assertion is also valid for the out-going quantum field $\phi_{\text{out}}$.

**Proof of Theorem 3.2 and 3.3.** It is evident that Theorem 3.2 is the consequence of Theorem 3.3 with $t_1 = t_2 = s_1 = s_2 = 0$. Therefore, it is sufficient to prove Theorem 3.3.

The assertion about holomorphy on $t_1 + \imath s_1, t_2 + \imath s_2$ is implied by Theorem 3.1 (or by Theorem 2.1) and by holomorphy of the operator $\exp(\imath \mu t - \mu s) = \exp \imath \mu(t + \imath s)$ on $t + \imath s$ for $s > 0$.

Now we prove that the expression $\phi$ is a correctly defined bilinear form. We consider here two variants of the proof for the interacting field $\phi$. The field $\phi_{\text{out}}$ can be considered analogously.

The first variant of the proof uses Theorem 1 [34, Theorem 1]. The Wick kernel is given by nonlinear operators $R W R^{-1}$ and $R \Theta W \Theta^{-1} R^{-1}$. The complex holomorphy of the considered Wick kernel was proved in [25]. In order to use Theorem 1 [34, Theorem 1] we take as an operator $B$ the operator
\[
B = \mu + \mu^{-1/2}(1 + x^2)\mu^{1/2}.
\]
It is evident, that this operator is positive in the coordinate space $H^{1/2}(\mathbb{R}^3, \mathcal{C})$, self-adjoint, and $\exp(-sB)$ is a trace class operator for $s > 0$. 


Remark.

It is possible to use more simple choice of (positive, self-adjoint) operator $B$, for instance,

$$B = \mu^\alpha + \mu^{-1/2}(1 + x^2)^\beta \mu^{1/2},$$

or

$$B = \mu^{\alpha - 1/2}(1 + x^2)^\beta \mu^{1/2},$$

for sufficiently large $\alpha, \beta$, such that the operator $B^{-1}$ is nuclear (as a product of two Hilbert-Schmidt operators).

We now consider the Wick kernel $\langle h, \phi(e^{-s_1 B} z_1), e^{-s_2 B} z_2)\rangle$. It is easily to see that this kernel satisfies the estimate (2) of Theorem 1 [34, Theorem 1]. This is implied by the explicit form of the Wick kernel, conservation of energy (which gives an estimate for the nonlinear terms), and by a simple estimate $B \leq c(s) \exp(sB)$, $s > 0$. Therefore, the conditions of Theorem 1 [34, Theorem 1] are fulfilled and $\phi$ is a correctly defined bilinear form. It is possible to extend this bilinear form by continuity on the subspace generated by finite linear combinations of coherent vectors with finite energy.

The first proof of Theorem 3.3 and Theorem 3.2 is complete.

The second proof of Theorem 3.3. Let $\chi_1(z)$, $\chi_2(z)$ be the holomorphic functions on the space $H^1(\mathbb{R}^3, \mathbb{C})$ ($\subset H^{1/2}(\mathbb{R}^3, \mathbb{C})$) corresponding to the vectors $\chi_1$, $\chi_2$ in the complex wave representation of the Fock space (see [3], ch. 1, Theorem 1.13, p. 67-68]. Let $\theta$ be strictly positive, $\theta > 0$, and such that for $\beta_1, \beta_2 \in \mathbb{C}$, $|\beta_1| + |\beta_2| < \theta$, the Taylor series in $\beta_1, \beta_2$ for the functions $\phi(e(\beta_1 z_{1,j_1}), e(\beta_2 z_{2,j_2}))$ converge in $H^1(\mathbb{R}^3, \mathbb{C})$ for all $j_1, j_2$ ($\chi_1, \chi_2$ are finite linear combinations of coherent vectors). Let

$$\chi_1(\beta_1) = \chi_1(\beta_1 z), \quad \chi_2(\beta_2) = \chi_2(\beta_2 z).$$

For our sufficiently small $\beta_1, \beta_2$, $|\beta_1| + |\beta_2| < \theta$, we write the following equality

$$\sum_{j_1, j_2} \pi_{1,j_1} \alpha_{2,j_2} \phi(e(\beta_1 z_{1,j_1}), e(\beta_2 z_{2,j_2}))$$

$$= \sum_{j_1, j_2} \pi_{1,j_1} \alpha_{2,j_2} \exp(\langle \beta_1 z_{1,j_1}, \beta_2 z_{2,j_2}\rangle) \sum_{n=1}^{\infty} \phi_n(\langle \beta_1 z_{1,j_1}, \beta_2 z_{2,j_2}\rangle)$$

$$= \lim_{N \to \infty} \lim_{V, \sigma} \sum_{n=1}^{N} \sum_{j_1, j_2} \pi_{1,j_1} \alpha_{2,j_2} \exp(\langle \beta_1 z_{1,j_1}, \beta_2 z_{2,j_2}\rangle)$$

$$\phi_n(\langle \beta_1 z_{1,j_1, V, \sigma} + \beta_2 z_{2,j_2, V, \sigma}\rangle) \cdots \langle \beta_1 z_{1,j_1, V, \sigma} + \beta_2 z_{2,j_2, V, \sigma}\rangle). \quad (3.4)$$

Here $\sigma$ is an ultraviolet cut-off and $V$ is a space cut-off (i.e. $\sigma$ is the smoothing with some test function and $V$ is the multiplication on some test function, $\sigma$ tends to the $\delta$-function and $V$ tends to 1). Further,

$$\phi_n = \frac{1}{2}(R_n + \overline{R}_n)$$

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is a tempered distribution (i.e. it is a generalized function from the Schwartz space), $R_n$ is the tempered distribution defined uniquely by the $n$-linear continuous form
\[ \frac{1}{n!} d^n W R^{-1}(0) \]
on $H^1(\mathbb{R}^3, \mathcal{C}) \otimes ... \otimes H^1(\mathbb{R}^3, \mathcal{C})$ and by the Schwartz nuclear theorem. We denote this $n$-linear form and the generalized function by the same notation $R_n$. The generalized function $R_n$ is such that
\[ R_n(f_1 \otimes ... \otimes f_n) = \frac{1}{n!} d^n W R^{-1}(0)(f_1, ..., f_n). \]

A Wick polynomial is a correctly defined bilinear form, for instance, on
\[ D_{\text{coh}}(S(\mathbb{R}^3, \mathcal{C})) \times D_{\text{coh}}(S(\mathbb{R}^3, \mathcal{C})). \]
This is implied easily by the explicit form of Wick monomial
\[ : \phi_{in}(f_1) ... \phi_{in}(f_n) : (\chi_1, \chi_2) = 2^{-n/2} \sum_{K \subseteq \{1, ..., n\}} \left( \prod_{k \in K} a(f_k) \chi_1, \prod_{k \in \{1, ..., n\} \setminus K} a(f_k) \chi_2, \right) \]
where $a$ is an annihilation operator. This explicit form allows to extend these expressions by continuity and linearity on tempered generalized functions, see, for instance, Reed, Simon [38], v. 2, [3], Theorem 3 with appropriate operator $A$.

Therefore, the expressions
\[ \phi_n(\cdot, \phi_{in} \cdot), \]
$\phi_n \in S'(\mathbb{R}^{3n}, H^1(\mathbb{R}^3, \mathcal{C}))$, is correctly defined as a bilinear form on
\[ D_{\text{coh}}(S(\mathbb{R}^3, \mathcal{C})) \times D_{\text{coh}}(S(\mathbb{R}^3, \mathcal{C})) \]
with values in $H^1(\mathbb{R}^3, \mathcal{C})$. On vectors $\chi_1, \chi_2 \in D_{\text{coh}}(S(\mathbb{R}^3, \mathcal{C}))$ this bilinear form is equal to
\[ \phi_n(\cdot, \phi_{in} \cdot)(\chi_1, \chi_2) = \sum_{j_1, j_2} \overline{\alpha}_{j_1, j_2} \alpha_{j_2, j_2} \exp((z_{1, j_1}, z_{2, j_2})_{H^1/2(\mathbb{R}^3, \mathcal{C})}) \phi_n((\overline{z}_{1, j_1} + z_{2, j_2}) \otimes ... \otimes (\overline{z}_{1, j_1} + z_{2, j_2})). \]

Now if $\chi_1, \chi_1', \chi_2, \chi_2', \in D_{\text{coh}}(S(\mathbb{R}^3, \mathcal{C}))$,
\[ \chi_1 = \chi_1', \quad \chi_2 = \chi_2', \]
we define $\chi_1(\beta_1), \chi_1'(\beta_1), \chi_2(\beta_2), \chi_2'(\beta_2)$. It is clear that
\[ \chi_1(\beta_1) = \chi_1'(\beta_1), \quad \chi_2(\beta_2) = \chi_2'(\beta_2). \]
Thus, for sufficiently small $\theta$, $|\beta_1| + |\beta_2| < \theta$, Eq. (3.4) implies that
\[ \phi(\chi_1(\beta_1), \chi_2(\beta_2)) = \lim_{N} \lim_{V, \sigma} \sum_{n=1}^{N} \phi_n(\cdot, \phi_{in} \cdot)(\chi_1(\beta_1), \chi_2(\beta_2)) \]
\[ \lim_{N \to \infty} \sum_{n=1}^{N} \phi_n((\phi_{n_1} \cdot \phi_{n_2} \cdots) (\chi'_1(\beta), \chi'_2(\beta))) = \phi(\chi'_1(\beta), \chi'_2(\beta)) \]

i.e.

\[ \phi(\chi_1(\beta_1), \chi_2(\beta_2)) = \phi(\chi'_1(\beta_1), \chi'_2(\beta_2)) \quad (3.5) \]

for small \( \beta_1, \beta_2 \). Since \( \phi(\chi_1(\beta_1), \chi_2(\beta_2)) \) and \( \phi(\chi'_1(\beta_1), \chi'_2(\beta_2)) \) are entire holomorphic functions on \( \beta_1, \beta_2 \), Eq. (3.5) for small \( \beta_1, \beta_2 \) implies that it is fulfilled for all \( \beta_1, \beta_2 \in \mathbb{C} \), in particular, for \( \beta_1 = \beta_2 = 1 \), that is,

\[ \phi(\chi_1, \chi_2) = \phi(\chi_1(1), \chi_2(1)) = \phi(\chi'_1(1), \chi'_2(1)) = \phi(\chi'_1, \chi'_2). \]

The proof of linearity in \( \chi_1, \chi_2 \) is analogous. Therefore, (3.2) and (3.3) are correctly defined bilinear forms. The second proof of Theorem 3.2 and Theorem 3.3 is complete.

**Theorem 3.4 (Poincaré invariance).**

The Wick kernel satisfies the following equalities

\[ \phi(e^{z_1}, e^{z_2})((a, \Lambda)^{-1}(t, x)) = \phi(e(RU_0(a, \Lambda)R^{-1}z_1), e(RU_0(a, \Lambda)R^{-1}z_2))(t, x), \]

\[ \phi_{out}(e^{z_1}, e^{z_2})((a, \Lambda)^{-1}(t, x)) = \phi_{out}(e(RU_0(a, \Lambda)R^{-1}z_1), e(RU_0(a, \Lambda)R^{-1}z_2))(t, x). \]

Here \( (a, \Lambda) \in \mathcal{P}_0, \)

\[ \phi(e^{z_1}, e^{z_2})(t, x)) = \phi(e(RU_0(-t)R^{-1}z_1), e(RU_0(-t)R^{-1}z_2))(0, x), \]

\[ \phi_{out}(e^{z_1}, e^{z_2})(t, x)) = \phi_{out}(e(RU_0(-t)R^{-1}z_1), e(RU_0(-t)R^{-1}z_2))(0, x), \]

\( U_0(t) = U_0(a, \Lambda)|_{(a, \Lambda) = ((t, 0), 1)}. \)

**Proof of Theorem 3.4.** Theorem 3.4 is the direct consequence of Theorem 2.2. Theorem 3.4 is proved.

**4. Conclusion.**

In this paper with the help of the complex structure of the classical nonlinear wave equation we introduce the Wick kernel of the interacting quantum field. In the next paper [28] we prove that the constructed Wick kernel defines the operator-valued generalized function from a Gelfand space \( S^\alpha(\mathbb{R}^4), \alpha > 1 \) [4]. The proof uses mainly the technique of works mentioned in the references.
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