Dynamical System of Kaluza-Klein Brane Cosmology with Gauss-Bonnet Term in a Bulk

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Abstract. Brane-world cosmological model in higher-dimensional spacetime is studied with additional Gauss-Bonnet term in the bulk. By using Gauss-Codazzi equations, we derive the (4+n)-dimensional gravitational field equations. The (4+n)-dimensional gravitational field equations can be formulated to general Einstein field equation with Gauss-Bonnet term and extra component. In the following, we take FRW metric and choose a relation between the external and internal scale factors of the form $b(t) = a^{\gamma}(t)$ in which the brane world evolves with two scale factors. Finally, a dynamical analysis is performed to determine the stability of this model.

1. Introduction

One of the most popular research in particle physics and gravitational theories is how to describe quantum theory of gravity [1]. The possibel candidate for this theory is superstring theory. In superstring theory extend the new picture of our universe, is called brane world. The Braneworld model was introduced firstly by Arkani-Hamed et al., In 1998 [2], Called the ADD (Arkani-Hamed-Dimopoulos-Dvali) model. This Braneworld model aims to explain the problem of hierarchy in particle physics. In this model is introduced an extra dimension with large size macroscopically by assuming that the electroweak scale is the fundamental scale and the plank mass has the same orde as this scale.

In 1999 Randall and Sundrum [3.4] introduced two braneworld models, Randall-Sundrum I (RS I) and Randall-Sundrum II (RS II). In RS I, The braneworld model consists of two branes which each have a positive and negative brane tension embedded into a 5-dimensional anti-sitter (AdS) bulk. In this model, all matter fields are localized to a brane which is located at fixed points. RS I model succeeds in explaining the problem of hierarchy through a metric solution that contains warped factor. In RS II introduced braneworld model with a brane that has positive brane tension. In this model can explain newton potential 4-dimension with correction term $1/r^3$.

To obtain an effective Einstein equation in the brane there are two approaches performed namely through the formulation of covarian curvature and gradient expansion methods. In the year of 2010 Shiromizu, et al., [5] through covariance curvature formulation obtained effective Einstein equation in the brane. This effective Einstein equation was obtained by projecting fields in a 5-dimensional bulk to a 4-dimensional brane field through Gauss-Codazi equation. The field equations in this brane modify Einstein equation in general relativity with two additional terms, the quadratic of energy-momentum.
tensor and non-local of the 5-dimensional Weyl tensor projection. In 2002 Kanno and Soda [6] proposed a method for obtaining effective Einstein equation in the brane with gradient expansion method. This model implements expansion of perturbatif a metric and define a low energy limit as the limit where the density of matter energy on the brane is smaller than the brane tension. Then a 5-dimensional field equation is solved on some order of on a brane because the Weyl tensor term can be removed in a manner directly.

2. Model and Effective Einstein Equation
The model which is studied in this research is brane $4 + n$ dimension embeded into bulk $5 + n$ dimension that lies at the fixed point $y = 0$. In this model the bulk space-time is added Gauss-Bonnet term. This model can be described through the action equation

$$ S = \int d^{5+n}x \sqrt{-g} \left[ \frac{\mathcal{R}}{2\kappa^2} - \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) + f(\phi) L_{GB} \right] + \int d^{5+n}x \sqrt{-h} \left[ -\sigma(\phi) + L_m \right] \delta(y), $$

where $\mathcal{R}$ is a scalar Ricci $5+n$ dimension, $g_{ab}$ is a metric in bulk, $h_{\alpha\beta}$ is induction metrics in the Brane and $\sigma$ is brane tension. The 4th term of the above action equation is the equation Gauss-Bonnet action [7] with

$$ L_{GB} = \mathcal{R}_{abcd} \mathcal{R}^{abcd} - 4 \mathcal{R}_{ab} \mathcal{R}^{ab} + \mathcal{R}^2. $$

Variation of the action equation (1) to the $g_{ab}$ metric and by assuming the Einstein equation applies to bulk $5 + n$-dimension, then we get the field equation on bulk

$$ g_{ab} = \kappa^2 T_{ab}^\phi + \kappa^2 T_{ab}^{GB} + \kappa^2 \left( -\sigma g_{\mu\nu} + t_{\mu\nu} \right) \delta(y), $$

where $T_{ab}^\phi$ and $T_{ab}^{GB}$ are the energy momentum tensor of the scalar field and the Gauss-Bonnet term that are defined

$$ T_{ab}^\phi = \nabla_a \phi \nabla_b \phi - g_{ab} \left( \frac{1}{2} \mathcal{R} c \phi \nabla_c \phi - V(\phi) \right) $$

$$ T_{ab}^{GB} = 4 \left[ \nabla_a \nabla_b f(\phi) \mathcal{R} - g_{ab} \left( \nabla^c \nabla_c f(\phi) \right) \mathcal{R} - 2 \left( \nabla^c \nabla_a f(\phi) \right) \mathcal{R}_{bc} - 2 \left( \nabla^c \nabla_b f(\phi) \right) \mathcal{R}_{ac} + 2 \left( \nabla^c \nabla_c f(\phi) \right) \mathcal{R}_{ab} + 2 g_{ab} \left( \nabla^d \nabla_d f(\phi) \right) \mathcal{R}_{cd} - 2 \left( \nabla^d \nabla_d f(\phi) \right) \mathcal{R}_{abcd} \right]. $$

The variation of the action equation (1) to the scalar field is obtained the equation of motion of scalar field

$$ \nabla^a \nabla_a \phi - V'(\phi) - f_{,\phi} \left( \mathcal{R}^2 - 4 \mathcal{R}_{ab} \mathcal{R}^{ab} + \mathcal{R}_{abcd} \mathcal{R}^{abcd} \right) - \sigma'(\phi) \delta(y) = 0. $$

For the field equations in brane can be obtained by projecting field quantity in the bulk. The projection tensor on the brane is defined

$$ h_{ab} = g_{ab} - n_a n_b, $$

where $n$ is the normal unit vector in the brane which is toward the positive y axis. The other four dimension tensor is very important in obtaining gravitational equations on the brane is extrinsic curvature which is defined [8]

$$ K_{ab} = h_a^c h_b^d \nabla_c n_d. $$
where $\nabla_a$ is derivative covariant in $5+n$-dimension for $g_{ab}$ metric. By using Gauss-Codazzi equation [9] we get Einstein field equation components

$$G^\gamma_\nu = -\frac{1}{4} R + \frac{1}{2} K^2 - \frac{1}{2} K^{\alpha \beta} K_{\alpha \beta} = \kappa^2 T^\gamma_\nu + \kappa^2 T^{GB}_\nu$$

(8)

$$G^\mu_\mu = -\nabla_\alpha K^\alpha + \nabla_\mu K = \kappa^2 T^\phi_\mu + \kappa^2 T^{GB}_\mu$$

(9)

$$G^\mu_\nu = G^\mu_\nu + (K^\mu_\nu - \delta^\mu_\nu K)_{\gamma \nu} - K K^\mu_\nu + \frac{1}{2} \delta^\mu_\nu (K^2 + K^{\alpha \beta} K_{\alpha \beta})$$

$$= \kappa^2 T^\phi_\nu + \kappa^2 T^{GB}_\nu + \kappa^2 (-\sigma \delta^\mu_\nu + t^\mu_\nu) \delta (\gamma).$$

(10)

By using equation (8) and (10) as well as the Gauss-Codazzi equation is obtained equation of tensor Einsten brane $4+n$ dimensions.

$$G^\nu_\mu = \frac{2 + n}{3 + n} \kappa^2 \left[ T^\nu_\mu - \frac{1}{4 + n} \delta^\nu_\mu T^\alpha_\alpha + \frac{3 + n}{4 + n} \delta^\nu_\mu T^\nu_\nu - K^{\mu \rho} K_{\nu \rho} + K^\nu_\nu K + \frac{1}{2} \delta^\nu_\nu (K^{\alpha \beta} K_{\alpha \beta} - K^2) E^\mu_\nu \\ + \frac{4(2 + n)}{3 + n} \kappa^2 \left[ \frac{5}{4 + n} (5 + n) \nabla^\mu \nabla_\nu f (\phi) + \frac{2(3 + n)(5 + n)}{(4 + n)^2} + \frac{2 - 2(3 + n)}{(4 + n)^2} \delta^\mu_\nu \nabla^2 f (\phi) \right] \right]$$

$$+ \left( \frac{5 + n}{4 + n} \right) \kappa^2 \left( K^\nu_\alpha K^\alpha_\nu - K_{\alpha \beta} K \right) \delta^\mu_\nu \nabla^a \nabla f (\phi)$$

$$- \frac{4(5 + n)}{(4 + n)^2} \left( K^\nu_\alpha K^\alpha_\nu - K_{\alpha \beta} K \right) \nabla^a \nabla f (\phi) + 2(5 + n) \left( K^\nu_\nu K^{\mu \beta} - K^\mu_\nu K \right) \nabla^c \nabla f (\phi)$$

$$- \frac{2}{(4 + n)^2} \left( K^\nu_\alpha K^\alpha_\nu - K_{\alpha \beta} K^\nu_\nu \right) + \frac{4(2 + n)}{3 + n} \kappa^2 \left[ \frac{5 + n}{4 + n} R \left( 5 + n \right) \nabla^a \nabla f (\phi) \right]$$

$$+ \left( \frac{5 + n}{4 + n} + \frac{2(5 + n)(3 + n) - 1}{4 + n} \right) \delta^\mu_\nu \nabla^a \nabla f (\phi) + \left( \frac{5 + n}{4 + n} + 2 \right) \frac{R_{\alpha \beta}}{(4 + n)^2} \delta^\mu_\nu \nabla^a \nabla f (\phi)$$

$$+ \frac{2(5 + n)}{(4 + n)^3} R^\mu_\alpha \nabla^a \nabla f (\phi) + \frac{4(5 + n)}{(4 + n)^2} R^\mu_\alpha \nabla^a \nabla f (\phi) + \frac{2(5 + n)}{(4 + n)^3} R^\mu_\alpha \nabla^a \nabla f (\phi)$$

$$- \frac{2(5 + n)}{(4 + n)^3} R^\mu_\alpha \nabla^a \nabla f (\phi) + \frac{2(5 + n)(5 + n)(3 + n) - 1}{(4 + n)^3} R^\mu_\alpha \nabla^a \nabla f (\phi)$$

(11)

where $E^\mu_\nu$ is Is a projection of Weyl tensor $5+n$ dimensions to $4+n$ dimension. The Einstein tensor equation on a brane $4+n$ dimension above has been expressed in quantity $4+n$-dimension. The extrinsic curvature terms of the Einstein tensor equation above can be eliminated by using Israel Junction condition [10] and By assuming $Z2$ symmetry is obtained

$$[K^\nu_\mu - \delta^\nu_\mu K] = \frac{\kappa^2}{2} \left( \sigma \delta^\nu_\mu + t^\nu_\mu \right)$$

$$\partial_\gamma \phi = \frac{1}{2} \sigma (\phi).$$

(12)

(13)
By using equation (12) and (13) in equation (11) we find Einstein tensor equation

\[
G^\mu_\nu = \frac{2 + n}{4 + n} \kappa^2 \left( \nabla^\mu \nabla_\nu \phi - \frac{5 + n}{2(4 + n)} \delta^\mu_\nu \nabla^a \phi \nabla_a \phi + \frac{3 + n}{8(4 + n)} \delta^\mu_\nu \sigma^a \nabla^a \phi + \frac{3 + n}{4 + n} \delta^\mu_\nu \nabla^1 \phi \right)
- \frac{2 + n}{8(3 + n)} \kappa^1 \delta^\mu_\nu + \frac{2 + n}{3 + n} \frac{\kappa^1 \sigma^\mu_\nu}{4} + \frac{1}{3 + n} \frac{\kappa^2}{4} t^\mu_\nu - \frac{1}{4(3 + n)} \kappa^1 \delta^\mu_\nu - \frac{1}{4} \kappa^1 \delta^\mu_\nu - \frac{1}{4} \kappa^1 \delta^\mu_\nu - E^\mu_\nu
+ \frac{\kappa^1}{8} \delta^\mu_\nu \alpha^a \nabla^a \phi - \frac{4(2 + n)}{3 + n} \kappa^1 (\xi^\mu_\nu + \xi^\mu_\nu + \xi^\mu_\nu + \xi^\mu_\nu) + \frac{4(2 + n)}{3 + n} \kappa^1 \left( \frac{5 + n}{3 + n} \nabla^a \nabla^b \phi + \frac{5 + n}{4 + n} \nabla^a \nabla^b \phi \right)
+ \left( \frac{5 + n}{3 + n} \right) \frac{1}{3 + n} \delta^\mu_\nu \nabla^a \nabla^b \phi + \frac{2(5 + n)}{(4 + n)^3} \delta^\mu_\nu \nabla^a \nabla^b \phi
\]

\[
+ \frac{2}{(4 + n)^2} \delta^\mu_\nu \nabla^a \nabla^a \phi + \frac{2(5 + n)}{(4 + n)^3} \delta^\mu_\nu \nabla^a \nabla^a \phi
\]

\[
= \frac{2}{(4 + n)^3} R^\mu_\nu \nabla^a \nabla^b \phi + \frac{2(5 + n)(5 + n)(3 + n)^{-1}}{(4 + n)^3} R^\mu_\nu \nabla^a \nabla^b \phi.
\]

(14)

with

\[
\xi^\nu_\nu = \sigma(\frac{1}{4} \left( \frac{6(5 + n)^2}{(3 + n)(4 + n)^2} - \frac{7}{(3 + n)(4 + n)^2} \right) \delta^\nu_\nu \nabla^a \nabla^a \phi + \frac{5(n + 7 + n)}{2(5 + n)(3 + n)^2} \delta^\nu_\nu \nabla^a \nabla^a \phi)
+ \frac{2(5 + n)(5 + n)(3 + n)}{(3 + n)^2} \delta^\nu_\nu \nabla^a \nabla^a \phi
+ \frac{4(5 + n)(5 + n)(3 + n)}{(3 + n)^2} \delta^\nu_\nu \nabla^a \nabla^a \phi
\]

(15)

\[
e^\nu_\nu = \sigma(\frac{4}{4} \left( \frac{5 + n}{(3 + n)(4 + n)} \right) \delta^\nu_\nu \nabla^a \nabla^a \phi + \frac{5 + n}{3 + n} \left( \frac{5 + n}{(3 + n)(4 + n)^2} + \frac{3 + n}{(3 + n)(4 + n)^2} \right) \delta^\nu_\nu \nabla^a \nabla^a \phi)
\]

(16)

\[
\xi^\nu_\nu = \frac{5 + n}{3 + n} \left( \frac{5 + n}{3 + n} \right) \delta^\nu_\nu \nabla^a \nabla^a \phi + \frac{5 + n}{3 + n} \left( \frac{5 + n}{(3 + n)(4 + n)^2} + \frac{3 + n}{(3 + n)(4 + n)^2} \right) \delta^\nu_\nu \nabla^a \nabla^a \phi
\]

(17)
Equation (14) is the Einstein tensor equation in the brane already stated in quantity 4+n-dimension, then the above equation can be rewritten to

\[ G^\mu_\nu = \frac{2 + n}{4 + n} \kappa^4 \left[ \frac{1}{4} \sigma t^\mu_\nu - 4 \kappa^2 \xi^\mu_\nu \right] + \frac{2 + n}{3 + n} \kappa^2 \left[ \nabla^\mu \phi \nabla^\nu \phi - \frac{5 + n}{2(4 + n)} \delta^\mu_\nu \nabla^a \phi \nabla_a \phi \right] - \Lambda_\nu \delta^\mu_\nu + \kappa^4 \Pi^\mu_\nu - E^\mu_\nu \]

where

\[ \Lambda_\nu = \frac{2 + n}{4 + n} \kappa^2 \left( V(\phi) + \frac{4 + n}{8(3 + n)} \kappa^2 \sigma^2 - \frac{1}{8} \sigma^2 \right) + \frac{4(4 + n)}{(3 + n)} \kappa^4 \epsilon^\mu_\nu \]  

\[ \Pi^\mu_\nu = -\frac{1}{4} \delta^\mu_\nu + \frac{1}{4(3 + n)} t^\mu_\nu \frac{1}{8} \delta^\alpha_\beta (t^\mu_\alpha t^\nu_\beta - \frac{1}{3 + n} t^2) - \frac{4(2 + n)}{3 + n} \kappa^2 \xi^\mu_\nu. \]

The brane Einstein field equation above differs from Einstein field equations in General relativity theory. Where the dependence of Einstein's tensor on brane matter on the equation (18) is quadratic and contain non-local term of Weyl tensor bulk projection.

The equation of motion of scalar field (5) can be written to

\[ -\nabla^\alpha \nabla_\alpha \phi + V_\phi + f_\phi (R^2 - 4 R_{\alpha \beta} R^{\alpha \beta} + (4 + n) R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta}) = J_n, \]

where \( J_n \) is possible energy leak out from the brane to the bulk, which is defined

\[ J_n = \partial_\gamma \phi - f_\phi \left[ -4 (\mathcal{R}_{\gamma \lambda} \mathcal{R}^{\lambda \beta} + \mathcal{R}_{\lambda \gamma} \mathcal{R}^{\lambda \beta}) + \mathcal{R}_{\alpha \beta \gamma \delta} \mathcal{R}^{\alpha \beta \gamma \delta} + (5 + n) (\mathcal{R}^{\alpha \beta} \mathcal{R}^{\alpha \beta} + 2 \mathcal{R}^{\alpha \beta \gamma} \mathcal{R}^{\alpha \beta \gamma}) \right]. \]

The Einstein field equation (18) can be converted to standard Einstein field equations in general relativity with additional energy-momentum tensor of Gauss-Bonnet term and extra component

\[ G^\mu_\nu = \kappa^2 \left( \nabla^\mu \phi \nabla_\nu \phi - \delta^\mu_\nu \left( \frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V_{\text{eff}} \right) \right) + 4 \kappa^2 \left[ (\nabla^\mu \nabla_\nu f) R - (\nabla^\nu \nabla_\nu f) R - 2 (\nabla^\mu \nabla_\nu f) R^\nu_\nu - 2 (\nabla^\nu \nabla_\nu f (\phi) R^\nu_\nu + 2 \delta^\mu_\nu (\nabla_\sigma \nabla^\alpha f (\phi) R^\alpha_\sigma ) - 2 (\nabla^\nu \nabla_\nu f (\phi) R^\nu_\nu \right] + X^\mu_\nu, \]

where

\[ X^\mu_\nu = Y^\mu_\nu - Z^\mu_\nu \]

\[ V_{\text{eff}} = \frac{\Lambda_\nu}{\kappa^2} \]

\[ Z^\mu_\nu = E^\mu_\nu + \frac{1}{3 + n} (\nabla^\mu \phi \nabla_\nu \phi - \frac{1}{4 + n} \delta^\mu_\nu (\nabla^\alpha \phi \nabla_\alpha \phi) \]

The quantity \( Z^\mu_\nu \) is the Bonnet term and extra component of Weyl tensor bulk projection.
The Einstein field component becomes of external and internal dimension factor.

From the above metric equations, the (00) Einstein tensor component are obtained as follows:

\[ \mathcal{G}^{0}_{0} = \frac{2 + n}{4 + n} \kappa^4 \left[ \frac{1}{4} \sigma_{\nu}^{\mu} - 4 \kappa^2 \chi_{\nu}^{\mu} + \frac{3 + n}{4 + n} \Pi_{\nu}^{\mu} \right] \frac{4(2 + n)}{3 + n} \kappa^2 \frac{2(5 + n)}{(4 + n)^3} R_{\nu}^{\mu} \partial_{\phi}^2 f(\phi) + 2(5 + n) \frac{((5 + n)(3 + n) - 1)}{(4 + n)^3} R\delta_{\nu}^{\mu} \partial_{\phi}^2 f(\phi) + 8 \kappa^2 R_{\nu}^{\mu} (\nabla_{\nu} \nabla_{\mu} f(\phi)) \\
+ \left( \frac{4(2 + n)(5 + 4)^2}{(3 + n)(4 + n)^2} - 4 \right) \kappa^2 R(\nabla_{\nu} \nabla_{\mu} f(\phi) + \frac{8(2 + n)(5 + n)}{(3 + n)(4 + n)^3} - 8 \right) \kappa^2 R_{\nu}^{\mu} (\nabla_{\nu} \nabla_{\mu} f(\phi)) \\
+ \left( \frac{4(2 + n)(5 + n)}{(3 + n)(4 + n)^2} - \frac{8((5 + n)(3 + n) - 1)}{4 + n} \right) + 4 \delta_{\nu}^{\mu} \nabla_{\nu} \nabla_{\mu} f(\phi) \\
+ \left[ 8 - \frac{16(5 + n)(2 + n)}{(3 + n)^2} \right] \kappa^2 R_{\alpha}^{\mu} \nabla_{\nu} \nabla_{\mu} f(\phi) - \left( \frac{8(2 + n)}{(3 + n)(4 + n)^2} - 8 \right) \kappa^2 R_{\alpha \beta}^{\mu} \nabla_{\mu} \nabla_{\beta} f(\phi) \\
- \frac{8((2 + n)(5 + n)(3 + n) + 2)}{(3 + n)(4 + n)^2} - 8 \right) \kappa^2 R_{\alpha \beta}^{\mu} \nabla_{\mu} \nabla_{\beta} f(\phi). \tag{27} \]

The Einstein field equation in the brane (23) still contains Weyl tensor projected in bulk term $E_{\nu}^{\mu}$. Tensor Weyl brings information about the geometry of the bulk, so the geometry of the bulk may affect the dynamics of the brane. This causes the equation of Einstein field on the brane is not closed. To solve the Einstein field equations in the brane, we must know firstly the bulk geometry completely.

From equation (23) and by using bianchi identity we get

\[ \nabla_{\mu} X_{\nu}^{\mu} = -\kappa^2 f_n \nabla_{\nu} \phi. \tag{28} \]

Equation (28) shows that the energy-momentum tensor of the extra component is not conservative. In this circumstance we interpret the extra component as a energy-momentum tensor of dark radiation [11].

3. Cosmology Applications

For cosmological applications we will review space-time homogeneous and isotropic in the brane. On this case we will take the Friedmann-Robertson-Walker metric on the brane [12] which written down

\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j + b^2(t) \delta_{\alpha \beta} dz^\alpha dz^\beta, \tag{29} \]

where $\delta_{ij}$ represents a 3-dimensional space metric with spatial coordinates $x^i (i = 1, 2, 3)$ and $\delta_{\alpha \beta}$ represents a compact $n$-dimensional space metric with spatial coordinates $z^\alpha (\alpha = 4, \ldots, 3 + n)$. Scale factor $a$ is scale factor for internal dimension and scale factor $b$ is scale factor for external dimension.

From the above metric equations, the (00) Einstein tensor component are obtained as follows

\[ G_0^0 = -3 \left( H_a^2 + \frac{k_a}{a^2} - n \left( \frac{n - 1}{2} \right) H_b^2 + 3H_aH_b + \frac{n - 1}{2} \right). \tag{30} \]

Assuming the scalar field $\phi$ depends on $t$ only, extra dimension is flat ($k_b = 0$) and taken the relation of external and internal dimension factor $b(t) = a(t)^{\gamma}$ [13], then $H_b = \gamma H_a = \gamma H$ and the (00) Einstein field component becomes

\[ G_0^0 = -\kappa^2 \left( \frac{1}{2} \phi^2 + V_{eff} \right) + X_{\phi} + 24 \kappa^2 (\gamma y + 1)f(\phi)H \frac{k_a}{a^2} - 4H^3 f(\phi)(-6 - 9n^2 y^2 \\
+ 9ny^2 - 18ny + 3n^2 y^3 - n^3 y^3 - 2ny^3) \tag{31} \]
and Einstein tensor equation

\[ G^0_0 = -H^2 \left( 3 + \frac{n(n-1)y^2}{2} + 3ny \right) - \frac{3k_a}{a^2}. \]  \quad (32)

From equation (31) and equation (32), the Friedmann equation is obtained

\[ 3(1 + \alpha_0)H^2 = \kappa^2 \left( \frac{1}{2} \dot{\phi}^2 + V_{\text{eff}} \right) + 3(1 + \rho_0) \frac{k_a}{a^2} + 24(1 + \beta_0)H^3 \dot{f}(\phi) + X^0_0, \]  \quad (33)

where

\[ \alpha_0 = \frac{n(n-1)}{6} + ny \] \quad (34)

\[ \rho_0 = -24(ny + 1)\dot{f}(\phi)H \] \quad (35)

\[ \beta_0 = \frac{1}{6}ny(9ny - 9\gamma + 18 - 3ny^2 + n^2y^2 + 2y^2). \] \quad (36)

By assuming \( k_a = 0 \) [14], the Friedmann equation becomes

\[ 3(1 + \alpha_0)H^2 = \kappa^2 \left( \frac{1}{2} \dot{\phi}^2 + V_{\text{eff}} \right) + 24(1 + \beta_0)H^3 \dot{f}(\phi) + X^0_0 \] \quad (37)

and the equation of motion of scalar field

\[ \ddot{\phi} + 3(1 + \alpha_3)H\dot{\phi} + V_{\text{eff}}\phi + 24(1 + \alpha_1)f,\phi H^4 + 24(1 + \alpha_2)f,\phi HH^2 = f_n, \] \quad (38)

where

\[ \alpha_1 = 3ny - ny^2 + 2(ny)^2 - \frac{1}{2}(ny)^2 - \frac{1}{24}(ny)^2 - \frac{1}{12}n^2y^4 + \frac{1}{24}(ny)^4 + \frac{1}{12}ny^4 \]

\[ \alpha_2 = 3ny - \frac{7}{6}ny^2 + \frac{3}{2}(ny)^2 - \frac{1}{2}(ny)^2 + \frac{1}{6}(ny)^3 \]

\[ \alpha_3 = \frac{ny}{3}. \]

4. Dynamical Analysis

Furthermore, dynamic analyst is conducted to determine the stability of this model. This analysis is taken for the case of the absence of extra component (\( X^0_0 = 0 \)).

By taking the case of absence of extra component and to excite the writing is taken \( k = 1 \), then the Friedmann equation (37) can be written to be

\[ 1 = \frac{\dot{\phi}^2}{6(1 + \alpha_0)H^2} + \frac{V_{\text{eff}}}{3(1 + \alpha_0)H^2} + \frac{8(1 + \beta_0)}{(1 + \alpha_0)f,\phi H} \] \quad (39)

and the equation of motion of scalar field becomes

\[ \ddot{\phi} + 3(1 + \alpha_3)H\dot{\phi} + V_{\text{eff}}\phi + 24(1 + \alpha_1)f,\phi H^4 + 24(1 + \alpha_2)f,\phi HH^2 = 0. \] \quad (40)

Then we take efective potensial \( V_{\text{eff}} \) and parameter \( f(\phi) \) in a form
\[ V_{eff} = V_0 e^{-\lambda \phi}, \quad f(\phi) = \left(\frac{\phi}{\mu}\right) e^{\mu \phi} \quad \text{with} \quad \lambda > 0. \]

Next is defined the new variable
\[ x_1 = \frac{\phi}{\sqrt{6(1 + a_0)} H}, \quad x_2 = \frac{\sqrt{V_{eff}}}{\sqrt{3(1 + a_0)} H}, \quad x_3 = f_\phi H^2. \]

By taking the above variable, then equation (39) becomes
\[ 1 = x_1^2 + x_2^2 + \frac{8((1 + \beta_0)\sqrt{\phi})}{\sqrt{1 + a_0}} x_1 x_3. \]

Density parameter is specified
\[ \Omega_\phi = x_1^2 + x_3^2; \quad \Omega_{GB} = \frac{8((1 + \beta_0)\sqrt{\phi})}{\sqrt{1 + a_0}} x_1 x_3. \]

Eliminating $\lambda$ of equation (40) by using equation (39) is obtained the equation
\[ (1 + \frac{96((1 + a_2)(1 + \beta_0))}{(1 + a_0)} x_1^2 - \frac{4\sqrt{6(1 + a_0)}}{1 + a_0} (3(1 + \beta_0) - (1 + a_2)) \frac{H}{H^2} = -3(1 + a_3) x_1^2 - \frac{4\sqrt{6(1 + a_0)}}{1 + a_0} (1 + a_1 + 1 + a_3)(1 + \beta_0) x_1 x_3 + 24(1 + \beta_0) \mu x_1^2 x_3 + 12 x_2^2 x_3 - \frac{96((1 + \beta_0)(1 + a_1))}{(1 + a_0)} x_3^2. \]

Furthermore, autonomous equations are given by
\[ \frac{dx_1}{dN} = -3(1 + a_1) x_1 - \frac{4\sqrt{6(1 + a_2)}}{\sqrt{1 + a_0}} x_3 + \frac{\sqrt{6(1 + a_0)}}{2} x_2^2 - (x_1 + \frac{4\sqrt{6(1 + a_2)}}{\sqrt{1 + a_0}} x_3) \frac{H}{H^2} \]
\[ \frac{dx_2}{dN} = -x_2 \frac{\sqrt{6(1 + a_0)}}{2} \mu x_1 + \frac{H}{H^2} \]
\[ \frac{dx_3}{dN} = 2 x_3 \frac{\sqrt{6(1 + a_0)}}{2} \mu x_1 + \frac{H}{H^2}. \]

From the above autonomous equation looks for each critical point. For $n = 0$, $\lambda = 0.1$ and $\mu = 2$

Is obtained critical point
\[ a1) (x_1, x_2, x_3) = \left(\frac{\sqrt{60}}{60}, \frac{\sqrt{59}}{60}, 0\right) \quad \text{b3) } (x_1, x_2, x_3) = (4.13, 0, -0.14) \]
\[ a2) (x_1, x_2, x_3) = \left(\frac{\sqrt{60}}{60}, -\frac{\sqrt{59}}{60}, 0\right) \quad \text{c1) } (x_1, x_2, x_3) = (0, 1, 0.1) \]
\[ b1) (x_1, x_2, x_3) = (0.19, 0, -0.19) \quad \text{c2) } (x_1, x_2, x_3) = \left(0, -1, \frac{0.1}{8}\right) \]
\[ b2) (x_1, x_2, x_3) = (0.41, 0, 22.37) \]

Point a) is the point of dominance of the scalar field, this is indicated by the parameter density $\Omega_\phi = 1$. Point b) is the solution of dominance of scalar kinetic and Gauss-Bonnet, while point c) is a point of scalar field dominance, but only in dominance by the scalar potential ($x_1 = 0$).
To see a stable point, then conduct perturbation around the critical point. Here, conducted perturbation by setting $x_3 = 0$. For $x_3 = 0$ obtained equation of Autonomous

$$\frac{dx_1}{dN} = -3(1 + \alpha_3)x_1 + \frac{\sqrt{6(1 + \alpha_0)}}{2}Ax_2^2 + 3(1 + \alpha_3)x_1^3$$

(46)

$$\frac{dx_2}{dN} = x_2(3(1 + \alpha_3)x_1^2 - \frac{\sqrt{6(1 + \alpha_0)}}{2}Ax_1),$$

(47)

with critical points $(x_1, x_2) = \left( \pm \sqrt{\frac{6(1 + \alpha_0)\lambda}{6(1 + \alpha_3)}}, \pm \sqrt{1 - \frac{(1 + \alpha_0)\lambda^2}{6(1 + \alpha_3)^2}} \right)$.  

To see the stability of the two critical points is executed perturbation

$$\frac{dx_1}{dN} = \frac{d\delta x_1}{dN} + \frac{d\delta x_1}{dN}, \quad \frac{dx_2}{dN} = \frac{d\delta x_2}{dN} + \frac{d\delta x_2}{dN}.$$  

Then the linearization is done

$$\frac{d}{dN} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix} = M \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} \frac{\delta f}{\delta x_1} & \frac{\delta f}{\delta x_2} \\ \frac{\delta g}{\delta x_1} & \frac{\delta g}{\delta x_2} \end{pmatrix}.$$  

(48)

The eigenvalues of the above variable is obtained for $n \neq 0$ and the parameter value $\gamma = -1$. Its eigenvalue

$$M_{1,2} = \frac{1}{2} \left[ -3(1 - \frac{n}{3}) + \frac{3(n(n-7))}{2(1-\frac{n}{3})} \lambda^2 \pm \right.$$

$$\left. \sqrt{(-3(1 - \frac{n}{3}) + \frac{3(n(n-7))}{2(1-\frac{n}{3})} \lambda^2)^2 + 12(1 + \frac{n(n-7)}{6}) \left(1 - \frac{1}{6(1+\frac{n(n-7)}{6})^2} \lambda^2 \right) \lambda^2} \right].$$  

(49)

A stable solution occurs if the eigenvalues of $M_1 < 0$ and $M_2 < 0$. This state occurs only for extra dimensions $n > 6$.

For the extra dimensions $n = 7$ parameter values $\lambda$ that satisfy that is $\lambda > 0$. Figure (1) is trajectory of system in the phase space $(x_1, x_2)$ for the extra dimension $n = 7$ and parameter $\lambda = 1$. In the trajectory it can be seen that both critical points are stable.

Figure 1. Trajectory of system in phase space $(x_1, x_2)$ for $n = 7$ and $\lambda = 1$.
5. Concluding Remarks
In this research has carried out the decrease of gravity equations on braneworld gravity by stating the quantities of $4+n$-dimensional field in expression of $5+n$-dimensional field quantities. Brane field equation is constructed from induced metrics through the projection of the Gauss-Codazzi equation.

Einstein field equation in the brane modifies the field equations of standard gravity in general relativity with some additional terms of the Gauss-Bonnet and extra component. The extra component carries information quadratic of energy-momentum tensor of matter on the brane, so that brane Einstein field equation dependence on quadratic energy-momentum tensor of matter. In addition, extra component also contains bulk Weyl tensor projected. This Weyl tensor brings information about the geometry of bulk $5+n$-dimensions that can affect the dynamics of the $4+n$-dimensional brane. Instead, the presence of matter in the brane can also affect the structure of geometry in bulk. This makes the Einstein brane field equations which is obtained to be uncovered. Tensor Weyl bulk can not be solved from the contribution of local matter on the brane. However, only can be solved by Einstein bulk $5+n$-dimensional equation field solution or we should know the bulk geometry completely first. In this study also conducted dynamic analysis to know the stability of this model. Stability analysis performed is case of absence of extra component ($X^\mu_\nu = 0$). In this case by setting $x_3 = 0$ (scalar field dominance) the only stable solution is given to the dimensions $n > 6$.

Acknowledgements
I.R. gratefully acknowledge support for LPDP scholarship and also thanks to all members of theoretical physics laboratory, Institut Teknologi Bandung for the hospitality and valuable support. F.P.Z. would like to thanks kemenristek DIKTI for financial support.

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