Jacobi fields of the Tanaka-Webster connection on Sasakian manifolds

Elisabetta Barletta\textsuperscript{1} \quad Sorin Dragomir

Abstract. We build a variational theory of geodesics of the Tanaka-Webster connection $\nabla$ on a strictly pseudoconvex CR manifold $M$. Given a contact form $\theta$ on $M$ such that $(M, \theta)$ has nonpositive pseudohermitian sectional curvature ($k_\theta(\sigma) \leq 0$) we show that $(M, \theta)$ has no horizontally conjugate points. Moreover, if $(M, \theta)$ is a Sasakian manifold such that $k_\theta(\sigma) \geq k_0 > 0$ then we show that the distance between any two consecutive conjugate points on a lengthy geodesic of $\nabla$ is at most $\pi/(2\sqrt{k_0})$. We obtain the first and second variation formulae for the Riemannian length of a curve in $M$ and show that in general geodesics of $\nabla$ admitting horizontally conjugate points do not realize the Riemannian distance.

1. Introduction

Sasakian manifolds possess a rich geometric structure (cf. [5], p. 73-80) and are perhaps the closest odd dimensional analog of Kählerian manifolds. In particular the concept of holomorphic sectional curvature admits a Sasakian counterpart, the so called $\phi$-sectional curvature $H(X)$ (cf. [5], p. 94) and it is a natural problem (as well as in Kählerian geometry, cf. e.g. [18], p. 171, and p. 368-373) to investigate how restrictions on $H(X)$ influence upon the topology of the manifold. An array of findings in this direction are described in [5], p. 77-80. For instance, by a result of M. Harada, [12], for any compact regular Sasakian manifold $M$ satisfying the inequality $h > k^2$ the fundamental group $\pi_1(M)$ is cyclic. Here $h = \inf\{H(X) : X \in T_x(M), \|X\| = 1, x \in M\}$ and it is also assumed that the least upper bound of the sectional curvature of $M$ is $1/k^2$. Moreover, if additionally $M$ has minimal diameter $\pi$ then $M$ is isometric to the standard sphere $S^{2n+1}$, cf. [13], p. 200.

\textsuperscript{1}Authors’ address: Università degli Studi della Basilicata, Dipartimento di Matematica, Campus Macchia Romana, 85100 Potenza, Italy, e-mail: barletta@unibas.it, dragomir@unibas.it
In the present paper we embrace a different point of view, that of pseudohermitian geometry (cf. [28]). To describe it we need to introduce a few basic objects (cf. [5], p. 19-28). Let $M$ be a $(2n + 1)$-dimensional $C^\infty$ manifold and $(\varphi, \xi, \eta, g)$ a contact metric structure i.e. $\varphi$ is an endomorphism of the tangent bundle, $\xi$ is a tangent vector field, $\eta$ is a differential 1-form, and $g$ is a Riemannian metric on $M$ such that
\[
\varphi^2 = -I + \eta \otimes \xi, \quad \varphi(\xi) = 0, \quad \eta(\xi) = 1,
\]
and $\Omega = d\eta$ (the contact condition) where $\Omega(X, Y) = g(X, \varphi Y)$. Any contact Riemannian manifold $(M, (\varphi, \xi, \eta, g))$ admits a natural almost CR structure $T_{1,0}(M) = \{X - iJX : X \in \text{Ker}(\eta)\}$ $(i = \sqrt{-1})$ i.e. it satisfies (2) below. By a result of S. Ianu, [14], if $(\varphi, \xi, \eta)$ is normal (i.e. $[\varphi, \varphi] + 2(d\eta) \otimes \xi = 0$) then $T_{1,0}(M)$ is integrable, i.e. it obeys to (3) in Section 2. Cf. [5], p. 57-61, for the geometric interpretation of normality, as related to the classical embeddability theorem for real analytic CR structures (cf. [11]). Integrability of $T_{1,0}(M)$ is required in the construction of the Tanaka-Webster connection of $(M, \eta)$, cf. [25], [28] and definitions in Section 2 (although many results in pseudohermitian geometry are known to carry over to arbitrary contact Riemannian manifolds, cf. [27] and more recently [2], [6]). A manifold carrying a contact metric structure $(\varphi, \xi, \eta, g)$ whose underlying contact structure $(\varphi, \xi, \eta)$ is normal is a Sasakian manifold (and $g$ is a Sasakian metric). The main tool in the Riemannian approach to the study of Sasakian geometry is the availability of a variational theory of geodesics of the Levi-Civita connection of $(M, g)$ (cf. e.g. [13], 194-197). In this paper we start the elaboration of a similar theory regarding the geodesics of the Tanaka-Webster connection $\nabla$ of $(M, \eta)$ and give a few applications (cf. Theorems 6-7 and 13 below). Our motivation is twofold. First, we aim to study the topology of Sasakian manifolds under restrictions on the curvature of $\nabla$ and conjecture that Carnot-Carathéodory complete Sasakian manifolds whose pseudohermitian Ricci tensor $\rho$ satisfies $\rho(X, X) \geq (2n - 1)k_0\|X\|^2$ for some $k_0 > 0$ and any $X \in \text{Ker}(\eta)$ must be compact. Second, the relationship between the sub-Riemannian geodesics of the sub-Riemannian manifold $(M, \text{Ker}(\eta), g)$ and the geodesics of $\nabla$ (emphasized by our Corollary [1] together with R.S. Strichartz’s arguments (cf. [24], p. 245 and 261-262) clearly indicates that a variational theory of geodesics of $\nabla$ is the key requirement in bringing results such as those in [25] or [28]...
into the realm of subelliptic theory. In [3] one obtains a pseudohermitian version of the Bochner formula (cf. e.g. [4], p. 131) implying a lower bound on the first nonzero eigenvalue $\lambda_1$ of the sublaplacian $\Delta_b$ of a compact Sasakian manifold

$$-\lambda_1 \geq 2nk/(2n-1)$$

(a CR analog to the Lichnerowicz theorem, [20]). It is likely that a theory of geodesics of $\nabla$ may be employed to show that equality in (1) implies that $M$ is CR isomorphic to a sphere $S^{2n+1}$ (the CR analog to Obata's result, [23]).

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2. Sub-Riemannian geometry on CR manifolds

Let $M$ be an orientable $(2n + 1)$-dimensional $C^\infty$ manifold. A CR structure on $M$ is a complex distribution $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$, of complex rank $n$, such that

$$T_{1,0}(M) \cap T_{0,1}(M) = (0)$$

and

$$Z, W \in T_{1,0}(M) \implies [Z, W] \in T_{1,0}(M)$$

(the formal integrability property). Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (overbars denote complex conjugates). The integer $n$ is the CR dimension. The pair $(M, T_{1,0}(M))$ is a CR manifold (of hypersurface type). Let $H(M) = \text{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ be the Levi distribution. It carries the complex structure $J : H(M) \to H(M)$ given by $J(Z + \overline{Z}) = i(Z - \overline{Z}) (i = \sqrt{-1})$. Let $H(M)^\perp \subset T^*(M)$ the conormal bundle, i.e. $H(M)^\perp_x = \{\omega \in T^*_x(M) : \text{Ker}(\omega) \supseteq H(M)_x\}$, $x \in M$. A pseudohermitian structure on $M$ is a globally defined nowhere zero cross-section $\theta$ in $H(M)^\perp$. Pseudohermitian structures exist as the orientability assumption implies that $H(M)^\perp \approx M \times \mathbb{R}$ (a diffeomorphism) i.e. $H(M)^\perp$ is a trivial line bundle. For a review of the main notions of CR and pseudohermitian geometry one may see [8].

Let $(M, T_{1,0}(M))$ be a CR manifold, of CR dimension $n$. Let $\theta$ be a pseudohermitian structure on $M$. The Levi form is

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$
$M$ is nondegenerate if $L_{\theta}$ is nondegenerate for some $\theta$. Two pseudohermitian structures $\theta$ and $\hat{\theta}$ are related by

$$\hat{\theta} = f \theta$$

for some $C^\infty$ function $f : M \to \mathbb{R} \setminus \{0\}$. Since $L_{\hat{\theta}} = fL_{\theta}$ nondegeneracy of $M$ is a CR invariant notion, i.e. it is invariant under a transformation (4) of the pseudohermitian structure. The whole setting bears an obvious analogy to conformal geometry (a fact already exploited by many authors, cf. e.g. [10], [26]-[28]). If $M$ is nondegenerate then any pseudohermitian structure $\theta$ on $M$ is actually a contact form, i.e. $\theta \wedge (d\theta)^n$ is a volume form on $M$. By a fundamental result of N. Tanaka and S. Webster (cf. op. cit.) on any nondegenerate CR manifold on which a contact form $\theta$ has been fixed there is a canonical linear connection $\nabla$ (the Tanaka-Webster connection of $(M, \theta)$) compatible to the Levi distribution and its complex structure, as well as to the Levi form. Precisely, let $T$ be the globally defined nowhere zero tangent vector field on $M$, transverse to $H(M)$, uniquely determined by $\theta(T) = 1$ and $T \cdot d\theta = 0$ (the characteristic direction of $d\theta$). Let

$$G_{\theta}(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M),$$

(the real Levi form) and consider the semi-Riemannian metric $g_{\theta}$ on $M$ given by

$$g_{\theta}(X, Y) = G_{\theta}(X, Y), \quad g_{\theta}(X, T) = 0, \quad g_{\theta}(T, T) = 1,$$

for any $X, Y \in H(M)$ (the Webster metric of $(M, \theta)$). Let us extend $J$ to an endomorphism of the tangent bundle by setting $JT = 0$. Then there is a unique linear connection $\nabla$ on $M$ such that i) $H(M)$ is parallel with respect to $\nabla$, ii) $\nabla g_{\theta} = 0$, $\nabla J = 0$, and iii) the torsion $T_{\nabla}$ of $\nabla$ is pure, i.e.

$$T_{\nabla}(Z, W) = T_{\nabla}(Z, \overline{W}) = 0, \quad T_{\nabla}(Z, \overline{W}) = 2iL_{\theta}(Z, \overline{W})T,$$

for any $Z, W \in T_{1,0}(M)$, and

$$\tau \circ J + J \circ \tau = 0,$$

where $\tau(X) = T_{\nabla}(T, X)$ for any $X \in T(M)$ (the pseudohermitian torsion of $\nabla$). The Tanaka-Webster connection is a pseudohermitian analog to both the Levi-Civita connection in Riemannian geometry and the Chern connection in Hermitian geometry.

A CR manifold $M$ is strictly pseudoconvex if $L_{\theta}$ is positive definite for some $\theta$. If this is the case then the Webster metric $g_{\theta}$ is a Riemannian metric on $M$ and if we set $\varphi = J$, $\xi = -T$, $\eta = -\theta$ and $g = g_{\theta}$ then $(\varphi, \xi, \eta, g)$ is a contact metric structure on $M$. Also $(\varphi, \xi, \eta, g)$ is
normal if and only if \( \tau = 0 \). If this is the case \( g_\theta \) is a Sasakian metric and \((M, \theta)\) is a Sasakian manifold.

We proceed by recalling a few concepts from sub-Riemannian geometry (cf. e.g. R.S. Strichartz, [24]) on a strictly pseudoconvex CR manifold. Let \((M, T_{1,0}(M))\) be a strictly pseudoconvex CR manifold, of CR dimension \( n \). Let \( \theta \) be a contact form on \( M \) such that the Levi form \( G_\theta \) is positive definite. The Levi distribution \( H(M) \) is bracket generating i.e. the vector fields which are sections of \( H(M) \) together with all brackets span \( T_x(M) \) at each point \( x \in M \), merely as a consequence of the nondegeneracy of the given CR structure. Indeed, let \( \nabla \) be the Tanaka-Webster connection of \((M, \theta)\) and let \( \{ T_\alpha : 1 \leq \alpha \leq n \} \) be a local frame of \( T_{1,0}(M) \), defined on the open set \( U \subseteq M \). By the purity property [4] (where \( \tau \) normal if and only if \( U \) shows that \( \{ \tau \} \) on \( U \) and \( \{ \tau \} \) on \( U \)).

\[
\Gamma^\alpha_{\beta\gamma} T_\gamma - \Gamma^\gamma_{\beta\alpha} T_\gamma - [T_\alpha, T_\beta] = 2ig_{\alpha\beta} T_\gamma,
\]

where \( \Gamma^A_{BC} \) are the coefficients of \( \nabla \) with respect to \( \{ T_\alpha \} \)

\[
\nabla_{T_\beta} T_C = \Gamma^A_{BC} T_A
\]

and \( g_{\alpha\beta} = L_\theta(T_\alpha, T_\beta) \). Our conventions as to the range of indices are \( A, B, C, \cdots \in \{0, 1, \cdots, n, \overline{1}, \cdots, \overline{n}\} \) and \( \alpha, \beta, \gamma, \cdots \in \{1, \cdots, n\} \) (where \( T_0 = T \)). Note that \( \{ T_\alpha, T_{\overline{\alpha}}, T \} \) is a local frame of \( T(M) \otimes \mathbb{C} \) on \( U \). If \( T_\alpha = X_\alpha - iJX_\alpha \) are the real and imaginary parts of \( T_\alpha \) then (7) shows that \( \{ X_\alpha, JX_\alpha \} \) together with their brackets span the whole of \( T_x(M) \), for any \( x \in U \). Actually more has been proved. Given \( x \in M \) and \( v \in H(M)_x \setminus \{0\} \) there is an open neighborhood \( U \subseteq M \) of \( x \) and a local frame \( \{ T_\alpha \} \) of \( T_{1,0}(M) \) on \( U \) such that \( T_1(x) = v - iJ_x v \), hence \( v \) is a 2-step bracket generator so that \( H(M) \) satisfies the strong bracket generating hypothesis (cf. the terminology in [24], p. 224).

Let \( x \in M \) and \( g(x) : T_x(M) \rightarrow H(M)_x \) determined by

\[
G_{g,x}(v, g(x)\xi) = \xi(v), \quad v \in H(M)_x, \quad \xi \in T^*_x(M).
\]

Note that the kernel of \( g \) is precisely the conormal bundle \( H(M)^\perp \). In other words \( G_\theta \) is a sub-Riemannian metric on \( H(M) \) and \( g \) its alternative description (cf. also (2.1) in [24], p. 225). If \( \hat{\theta} = e^{u}\theta \) is another contact form such that \( G_{\hat{\theta}} \) is positive definite (\( u \in C^\infty(M) \)) then \( \hat{g} = e^{-u}g \). Clearly if the Levi form \( L_\theta \) is only nondegenerate then \((M, H(M), G_\theta)\) is a sub-Lorentzian manifold, cf. the terminology in [24], p. 224.

Let \( \gamma : I \rightarrow M \) be a piecewise \( C^1 \) curve (where \( I \subseteq \mathbb{R} \) is an interval). Then \( \gamma \) is a lengthy curve if \( \dot{\gamma}(t) \in H(M)_{\gamma(t)} \) for every \( t \in I \) such that \( \gamma(t) \) is defined. For instance, any geodesic of \( \nabla \) (i.e. any \( C^1 \) curve \( \gamma(t) \) such that \( \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \) of initial data \( (x, v) \), \( v \in H(M)_x \), is
lengthy (as a consequence of $\nabla g_\theta = 0$ and $\nabla T = 0$). A piecewise $C^1$ curve $\xi : I \to T^*(M)$ is a cotangent lift of $\gamma$ if $\xi(t) \in T_{\gamma(t)}^*(M)$ and $g(\gamma(t))\xi(t) = \dot{\gamma}(t)$ for every $t$ (where defined). Clearly cotangent lifts of a given lengthy curve $\gamma$ exist (cf. also Proposition 1 below). Also, cotangent lifts of $\gamma$ are uniquely determined modulo sections of the conormal bundle $H(M)^\perp$ along $\gamma$. That is, if $\eta : I \to T^*(M)$ is another cotangent lift of $\gamma$ then $\eta(t) - \xi(t) \in H(M)^\perp_{\gamma(t)}$ for every $t$. The length of a lengthy curve $\gamma : I \to M$ is given by

$$L(\gamma) = \int_I \{\xi(t) [g(\gamma(t))\xi(t)]\}^{1/2} dt.$$ 

The definition doesn’t depend upon the choice of cotangent lift $\xi$ of $\gamma$. The Carnot-Carathéodory distance $\rho(x, y)$ among $x, y \in M$ is the infimum of the lengths of all lengthy curves joining $x$ and $y$. That $\rho$ is indeed a distance function on $M$ follows from a theorem of W.L. Chow, [7], according to which any two points $x, y \in M$ may be joined by a lengthy curve (provided that $M$ is connected).

Let $g_\theta$ be the Webster metric of $(M, \theta)$. Then $g_\theta$ is a contraction of the sub-Riemannian metric $G_\theta$ ($G_\theta$ is an expansion of $g_\theta$), cf. [24], p. 230. Let $d$ be the distance function corresponding to the Webster metric. The length $L(\gamma)$ of a lengthy curve $\gamma$ is precisely its length with respect to $g_\theta$ hence

$$d(x, y) \leq \rho(x, y), \quad x, y \in M.$$ 

While $\rho$ and $d$ are known to be inequivalent distance functions, they do determine the same topology. For further details on Carnot-Carathéodory metrics see J. Mitchell, [22].

Let $(U, x^1, \ldots, x^{2n+1})$ be a system of local coordinates on $M$ and let us set $G_{ij} = g_\theta(\partial_i, \partial_j)$ (where $\partial_i$ is short for $\partial/\partial x^i$) and $[G^{ij}] = [G_{ij}]^{-1}$. Using

$$G_\theta(X, g \, dx^i) = (dx^i)(X), \quad X \in H(M),$$

for $X = \partial_k - \theta_k T$ (where $\theta_i = \theta(\partial_i)$) leads to

$$g^{ij}(G_{jk} - \theta_j \theta_k) = \delta^i_k - \theta_k T^i$$

where $g \, dx^i = g^{ij} \partial_j$ and $T = T^i \partial_i$. On the other hand $g^{ij} \theta_j = \theta(g \, dx^i) = 0$ so that (9) yields

$$g^{ij} = G^{ij} - T^i T^j.$$ 

As an application we introduce a canonical cotangent lift of a given lengthy curve on $M$. 
Proposition 1. Let \( \gamma : I \to M \) be a lengthy curve and let \( \xi : I \to T^*(M) \) be given by \( \xi(t)T_{\gamma(t)} = 1 \) and \( \xi(t)X = g_\theta(\dot{\gamma}, X) \), for any \( X \in H(M)_{\gamma(t)} \). Then \( \xi \) is a cotangent lift of \( \gamma \).

Proof. Let \( x^i(t) \) be the components of \( \gamma \) with respect to the chosen local coordinate system. By the very definition of \( \xi \)

\[
\xi_j = G_{ij} \frac{dx^i}{dt} + \theta_j.
\]

Hence

\[
g \xi = \xi_j g^{ij} \partial_i = g^{ij}(G_{jk} \frac{dx^k}{dt} + \theta_j) \partial_i = g^{ij} G_{jk} \frac{dx^k}{dt} \partial_i = (G^{ij} - T^i T^j) G_{jk} \frac{dx^k}{dt} \partial_i = (\delta^i_k - T^i \theta_k) \frac{dx^k}{dt} \partial_i = \dot{\gamma}(t) - \theta(\dot{\gamma}(t)) T = \dot{\gamma}(t).
\]

We recall (cf. [24], p. 233) that a sub-Riemannian geodesic is a \( C^2 \) curve \( \gamma(t) \) in \( M \) satisfying the Hamilton-Jacobi equations associated to the Hamiltonian function \( H(x, \xi) = \frac{1}{2} g^{ij}(x) \xi_i \xi_j \) that is

\[
\frac{dx^i}{dt} = g^{ij}(\gamma(t)) \xi_j(t),
\]

\[
\frac{d\xi_k}{dt} = -\frac{1}{2} \frac{\partial g^{ij}}{\partial x^k}(\gamma(t)) \xi_i(t) \xi_j(t),
\]

for some cotangent lift \( \xi(t) \in T^*(M) \) of \( \gamma(t) \). Our purpose is to show that

Theorem 1. Let \( M \) be a strictly pseudoconvex CR manifold and \( \theta \) a contact form on \( M \) such that \( G_\theta \) is positive definite. A \( C^2 \) curve \( \gamma(t) \in M \), \( |t| < \epsilon \), is a sub-Riemannian geodesic of \( (M, H(M), G_\theta) \) if and only if \( \gamma(t) \) is a solution to

\[
\nabla_{\dot{\gamma}} \dot{\gamma} = -2b(t) J \dot{\gamma}, \quad b'(t) = A(\dot{\gamma}, \dot{\gamma}), \quad |t| < \epsilon,
\]

with \( \gamma(0) \in H(M)_{\gamma(0)} \), for some \( C^2 \) function \( b : (-\epsilon, \epsilon) \to \mathbb{R} \). Here \( A(X, Y) = g_\theta(\tau X, Y) \) is the pseudohermitian torsion of \( (M, \theta) \).

According to the terminology in [24], p. 237, the canonical cotangent lift \( \xi(t) \) of a given lengthy curve \( \gamma(t) \) is the one determined by the orthogonality requirement

\[
V_j(\xi) \Gamma^j(\xi, v) = 0,
\]

for any \( v \in H(M)_{\gamma(t)}^\perp \) and any \( |t| < \epsilon \), where

\[
V_k(\xi) = \frac{d\xi_k}{dt} + \frac{1}{2} \frac{\partial g^{ij}}{\partial x^k} \xi_i \xi_j,
\]
\[
\Gamma^i(\xi, v) = \Gamma^{ijk}\xi_jv_k, \quad \Gamma^{ijk} = \frac{1}{2}(g^{ij}_k \partial g^{jk}_i + g^{ik}_j \partial g^{ij}_k - g^{kj}_i \partial g^{jk}_i).
\]

Let \(\gamma(t)\) be a lengthy curve and \(\xi_0(t)\) the cotangent lift of \(\gamma(t)\) furnished by Proposition 1. Then any other cotangent lift \(\xi(t)\) is given by

(16) \[
\xi(t) = \xi_0(t) + a(t) \theta_{\gamma(t)}, \quad |t| < \epsilon,
\]

for some \(a : (-\epsilon, \epsilon) \to \mathbb{R}\). We shall need the following result (a replica of Lemma 4.4. in [24], p. 237)

**Lemma 1.** The unique cotangent lift \(\xi(t)\) of \(\gamma(t)\) satisfying the orthogonality condition (15) is given by (16) where

(18) \[
a(t) = -\frac{1}{2} |\dot{\gamma}(t)|^{-2} g_\theta(\nabla_\gamma \dot{\gamma}, J \dot{\gamma}) - 1, \quad |t| < \epsilon.
\]

**Proof.** By (11) and (16)

\[
V_k(\xi) = V_k(\xi_0) + a'(t) \theta_k + a(t) \frac{\partial \theta_k}{\partial x^\ell} \frac{dx^\ell}{dt} +
\]

(17) \[
+ \frac{1}{2} \frac{\partial g^{ij}}{\partial x^k} [a(t)(\xi_0^j \theta_i + \xi_0^i \theta_j) + a(t)^2 \theta_i \theta_j]
\]

we obtain

(18) \[
V_i(\xi_0) \Gamma^i(\xi_0, v) + 2a(t)(d\theta)(\dot{\gamma}, \Gamma(\xi_0, v)) = 0,
\]

where \(\Gamma(\xi, v) = \Gamma^i(\xi, v) \partial_i\). On the other hand, a calculation based on (10)-(11) shows that

(19) \[
V_i(\xi_0) = G_{ij}(D_\gamma \dot{\gamma})^j + 2(d\theta)(\dot{\gamma}, \partial_i),
\]

hence
where $D$ is the Levi-Civita connection of $(M, g_\theta)$. Then (18)-(19) yield
\[
g_\theta(D\dot{\gamma}, \Gamma(\xi_0, v)) + 2(a(t) + 1)(d\theta)(\dot{\gamma}, \Gamma(\xi_0, v)) = 0,
\]
for any $v \in H(M)_{\xi_0}^\perp$. Yet $H(M)_{\xi_0}^\perp$ is the span of $\theta$ hence
\[
g_\theta(\Gamma(\xi_0, \theta), D\dot{\gamma} + 2(a(t) + 1)J\dot{\gamma}) = 0
\]
and
\[
\Gamma^i(\xi_0, \theta) = -G^{ij}(d\theta)(\dot{\gamma}, \partial_j),
\]
(because of $T[d\theta = 0)$ yields
\[
(20) \quad 2(a(t) + 1)|\dot{\gamma}(t)|^2 + g_\theta(D\dot{\gamma}, J\dot{\gamma}) = 0.
\]
Lemma 1 is proved. At this point we may prove Theorem 1. Let $\gamma(t) \in M$ be a sub-Riemannian geodesic of $(M, H(M), G_\theta)$. Then there is a cotangent lift $\xi(t) \in T^*(M)$ of $\gamma(t)$ (given by (16) for some $a : (-\epsilon, \epsilon) \to \mathbb{R}$) such that $V(\xi) = 0$ (where $V(\xi) = V^i(\xi)\partial_i$). In particular the orthogonality condition (15) is identically satisfied, hence $a(t)$ is determined according to Lemma 1. Using (17) and (19) the sub-Riemannian geodesics equations are
\[
G_{ij}(D\dot{\gamma})^j + a'(t)\theta_i + 2(a(t) + 1)(d\theta)(\dot{\gamma}, \partial_i) = 0
\]
or
\[
(21) \quad D\dot{\gamma} + a'(t)T + 2(a(t) + 1)J\dot{\gamma} = 0.
\]
We recall (cf. e.g. [10]) that $D = \nabla - (d\theta + A) \otimes T$ on $H(M) \otimes H(M)$ hence (by the uniqueness of the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$) the equations (21) become
\[
\nabla\dot{\gamma} + 2(a(t) + 1)J\dot{\gamma} = 0, \quad a'(t) = A(\dot{\gamma}, \dot{\gamma}),
\]
(and we set $b = a + 1$). Theorem 1 is proved.

**Corollary 1.** Let $M$ be a strictly pseudoconvex CR manifold and $\theta$ a contact form on $M$ with vanishing pseudohermitian torsion ($\tau = 0$). Then any lengthy geodesic of the Tanaka-Webster connection $\nabla$ of $(M, \theta)$ is a sub-Riemannian geodesic of $(M, H(M), G_\theta)$. Viceversa, if every lengthy geodesic $\gamma(t)$ of $\nabla$ is a sub-Riemannian geodesic then $\tau = 0$.

Indeed, if $\nabla\dot{\gamma} = 0$ then the equations (14) (with $b = 0$) are identically satisfied.

**Proposition 2.** Let $\gamma(t) \in M$ be a sub-Riemannian geodesic and $s = \phi(t)$ a $C^2$ diffeomorphism. If $\gamma(t) = \tau(\phi(t))$ then $\tau(s)$ is a sub-Riemannian geodesic if and only if $\phi$ is affine, i.e. $\phi(t) = \alpha t + \beta$, for
some $\alpha, \beta \in \mathbb{R}$. In particular, every sub-Riemannian geodesic may be reparametrized by arc length $\phi(t) = \int_0^t |\dot{\gamma}(u)| \, du$.

**Proof.** Set $k = |\dot{\gamma}(0)|^2 > 0$. By taking the inner product of the first equation in (14) by $\dot{\gamma}(t)$ it follows that $d|\gamma(t)|^2/dt = 0$, hence $|\gamma(t)|^2 = k$, $|t| < \epsilon$. Throughout the proof an overbar indicates the similar quantities associated to $\overline{\gamma}(s)$. In particular $\overline{k} = \phi(0)^{-2}k$. Locally

$$d^2x^i/dt^2 + \Gamma_{jk}^i dx^j/dt dx^k/dt = -2(a + 1)J^j \frac{dx^j}{dt}.$$  

On the other hand, using (20) and

$$d^2x^i/dt^2 + \Gamma_{jk}^i dx^j/dt dx^k/dt = \phi''(t) \frac{d^2\pi}{ds^2} + \phi'(t)^2 (\frac{d^2\pi}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds})$$

we obtain

$$k(a + 1) = \overline{k}(a + 1)\phi'(t)^3.$$  

Then (22) may be written

$$k\phi''(t) \frac{d\pi}{ds} + 2(\pi + 1)\phi'(t)^2[\overline{k}\phi'(t)^2 - k]J \frac{d\pi}{ds} = 0$$

hence $\phi''(t) = 0$. Proposition 2 is proved.

Let $S^1 \to C(M) \overset{\pi}{\to} M$ be the canonical circle bundle over $M$ (cf. e.g. [8], p. 104). Let $\Sigma$ be the tangent to the $S^1$-action. Next, let us consider the 1-form $\sigma$ on $C(M)$ given by

$$\sigma = \frac{1}{n+2} \{dr + \pi^* (i\omega^\alpha - \frac{i}{2} g^{\alpha\beta} d\alpha d\beta - \frac{R}{4(n+1)} \theta)\},$$

where $r$ is a local fibre coordinate on $C(M)$ (so that locally $\Sigma = \partial/\partial r$) and $R = g^{\alpha\beta} R_{\alpha\beta}$ is the pseudohermitian scalar curvature of $(M, \theta)$. Then $\sigma$ is a connection 1-form in $S^1 \to C(M) \to M$. Given a tangent vector $v \in T_x(M)$ and a point $z \in \pi^{-1}(x)$ we denote by $v^\dagger$ its horizontal lift with respect to $\sigma$, i.e. the unique tangent vector $v^\dagger \in \text{Ker}(\sigma_z)$ such that $(d_z\pi)v^\dagger = v$. The **Fefferman metric** of $(M, \theta)$ is the Lorentz metric on $C(M)$ given by

$$F_\theta = \pi^* \tilde{G}_\theta + 2(\pi^* \theta) \odot \sigma,$$

where $\tilde{G}_\theta = G_\theta$ on $H(M) \otimes H(M)$ and $\tilde{G}_\theta(X, T) = 0$, for any $X \in T(M)$. Also $\odot$ is the symmetric tensor product. We close this section by demonstrating the following geometric interpretation of sub-Riemannian geodesics (of a strictly pseudoconvex CR manifold).

**Theorem 2.** Let $M$ be a strictly pseudoconvex CR manifold, $\theta$ a contact form on $M$ such that $G_\theta$ is positive definite, and $F_\theta$ the Fefferman metric of $(M, \theta)$. For any geodesic $z : (-\epsilon, \epsilon) \to C(M)$ of $F_\theta$ if the
projection $\gamma(t) = \pi(z(t))$ is lengthy then $\gamma : (-\epsilon, \epsilon) \to M$ is a sub-Riemannian geodesic of $(M, H(M), G_{\theta})$. Viceversa, let $\gamma(t) \in M$ be a sub-Riemannian geodesic. Then any solution $z(t) \in C(M)$ to the ODE

\[
\dot{z}(t) = \dot{\gamma}(t)^\uparrow + ((n + 2)/2)b(t)\Sigma_{z(t)},
\]

where $b(t) = a(t) + 1$ is given by (20), is a geodesic of $F_{\theta}$.

Here $\dot{\gamma}(t)^\uparrow \in \text{Ker}(\sigma_{z(t)})$ and $(d_{\dot{z}(t)\pi})\dot{\gamma}(t)^\uparrow = \dot{\gamma}(t)$. To prove Theorem 2 we shall need the following

**Lemma 2.** For any $X, Y \in H(M)$

\[
\nabla_{X^\uparrow}^{C(M)}Y^\uparrow = (\nabla_{X}Y)^\uparrow - (d\theta)(X, Y)T^\uparrow - (A(X, Y) + (d\sigma)(X^\uparrow, Y^\uparrow))\hat{\Sigma},
\]

\[
\nabla_{T^\uparrow}^{C(M)}T^\uparrow = (\tau X + \phi X)^\uparrow,
\]

\[
\nabla_{X^\uparrow}^{C(M)}X^\uparrow = (\nabla_{T}X + \phi X)^\uparrow + 2(d\sigma)(X^\uparrow, T^\uparrow)\hat{\Sigma},
\]

\[
\nabla_{T^\uparrow}^{C(M)}X^\uparrow = (JX)^\uparrow,
\]

\[
\nabla_{\Sigma}C(M)\hat{\Sigma} = 0,
\]

\[
\nabla_{\Sigma}C(M)\hat{\Sigma} = 0,
\]

where $\phi : H(M) \to H(M)$ is given by $G_{\theta}(\phi X, Y) = (d\sigma)(X^\uparrow, Y^\uparrow)$, and $V \in H(M)$ is given by $G_{\theta}(V, Y) = 2(d\sigma)(T^\uparrow, Y^\uparrow)$. Also $\hat{\Sigma} = ((n + 2)/2)\Sigma$.

This relates the Levi-Civita connection $\nabla^{C(M)}$ of $F_{\theta}$ to the Tanaka-Webster connection of $(M, \theta)$. Cf. [9] for a proof of Lemma 2.

**Proof of Theorem 2** Let $z(t) \in C(M)$ be a geodesic of $\nabla^{C(M)}$ and $\gamma(t) = \pi(z(t))$. Assume that $\dot{\gamma}(t) \in H(M)_{\gamma(t)}$. Note that $\dot{z}(t) - \dot{\gamma}(t)^\uparrow \in \text{Ker}(d_{\dot{z}(t)\pi})$ hence $\dot{z}(t)$ is given by (23), for some $b : (-\epsilon, \epsilon) \to \mathbb{R}$. Then (by Lemma 2)

\[
0 = \nabla_{\dot{z}}^{C(M)}\dot{z} = \nabla_{\dot{\gamma}}^{C(M)}\dot{\gamma} + b'(t)\hat{\Sigma} + 2b(t)(J\dot{\gamma})^\uparrow =
\]

\[
= (\nabla_{\dot{\gamma}}\dot{\gamma})^\uparrow + [b'(t) - A(\dot{\gamma}, \dot{\gamma})]\hat{\Sigma} + 2b(t)(J\dot{\gamma})^\uparrow
\]

hence (by $T(C(M)) = \text{Ker}(\sigma) \oplus \mathbb{R}\Sigma$) $\gamma(t)$ satisfies the equations (14), i.e. $\gamma(t)$ is a sub-Riemannian geodesic. The converse is obvious.
3. Jacobi fields on CR manifolds

Let $M$ be a strictly pseudoconvex CR manifold endowed with a contact form $\theta$ such that $G_\theta$ is positive definite. Let $\nabla$ be the Tanaka-Webster connection of $(M, \theta)$. Let $\gamma(t) \in M$ be a geodesic of $\nabla$, parametrized by arc length. A Jacobi field along $\gamma$ is vector field $X$ on $M$ satisfying the second order ODE

\[
\nabla^2_\gamma X + \nabla_\gamma T_\nabla(X, \gamma) + R(X, \gamma)\gamma = 0.
\]

Let $J_\gamma$ be the real linear space of all Jacobi fields of $(M, \nabla)$. Then $J_\gamma$ is $(4n + 2)$-dimensional (cf. Prop. 1.1 in [18], Vol. II, p. 63). We denote by $\hat{\gamma}$ the vector field along $\gamma$ defined by $\hat{\gamma}(t) = t\dot{\gamma}(t)$ for every value of the parameter $t$. Note that $\dot{\gamma}, \hat{\gamma} \in J_\gamma$. We establish

**Theorem 3.** Every Jacobi field $X$ along a lengthy geodesic $\gamma$ of $\nabla$ can be uniquely decomposed in the following form

\[
X = a\dot{\gamma} + b\hat{\gamma} + Y
\]

where $a, b \in \mathbb{R}$ and $Y$ is a Jacobi field along $\gamma$ such that

\[
g_\theta(Y, \hat{\gamma}(t)) = -\int_0^t \theta(X)\gamma(s) A(\dot{\gamma}, \dot{\gamma})\gamma(s) ds.
\]

In particular, if i) $X_\gamma(t) \in H(M)_{\gamma(t)}$ for every $t$, or ii) $(M, \theta)$ is a Sasakian manifold (i.e. $\tau = 0$), then $Y$ is perpendicular to $\gamma$.

We need the following

**Lemma 3.** For any Jacobi field $X \in J_\gamma$

\[
\frac{d}{dt}\{g_\theta(X, \dot{\gamma})\} + \theta(X)\gamma(t) A(\dot{\gamma}, \dot{\gamma})\gamma(t) = \text{const}.
\]

**Proof.** Let us take the inner product of the Jacobi equation \([24]\) by $\dot{\gamma}$ and use the skew symmetry of $g_\theta(R(X,Y)Z,W)$ in the arguments $(Z,W)$ (a consequence of $\nabla g_\theta = 0$) so that to get

\[
\frac{d^2}{dt^2}\{g_\theta(X, \dot{\gamma})\} + \frac{d}{dt}\{g_\theta(T_\nabla(X, \dot{\gamma}), \dot{\gamma})\} = 0.
\]

On the other hand, let us set $X_H = X - \theta(X)T$ (so that $X_H \in H(M)$). Then

\[
g_\theta(T_\nabla(X, \dot{\gamma}), \dot{\gamma}) = -2\Omega(X_H, \dot{\gamma})g_\theta(T, \dot{\gamma}) + \theta(X)g_\theta(\tau(\dot{\gamma}), \dot{\gamma})
\]

or (as $\gamma$ is lengthy)

\[
g_\theta(T_\nabla(X, \dot{\gamma}), \dot{\gamma}) = \theta(X)A(\dot{\gamma}, \dot{\gamma}).
\]
Lemma 3 is proved. Throughout the section we adopt the notation $X' = \nabla_{\dot{\gamma}} X$ and $X'' = \nabla^2 \dot{\gamma} X$.

**Proof of Theorem 3** We set by definition

$$a = g_\theta(X, \dot{\gamma})_{\gamma(0)}, \quad b = g_\theta(X', \dot{\gamma})_{\gamma(0)} + \theta(X)_{\gamma(0)} A(\dot{\gamma}, \dot{\gamma})_{\gamma(0)},$$

and $Y = X - a\dot{\gamma} - b\dot{\gamma}$. Clearly $Y \in J_\gamma$. Then, by Lemma 3

$$\frac{d}{dt}(g_\theta(Y, \dot{\gamma})) + \theta(Y) A(\dot{\gamma}, \dot{\gamma}) = \alpha,$$

for some $\alpha \in \mathbb{R}$. Next we integrate from 0 to $t$

$$g_\theta(Y, \dot{\gamma})_{\gamma(t)} - g_\theta(Y, \dot{\gamma})_{\gamma(0)} + \int_0^t \theta(Y)_{\gamma(s)} A(\dot{\gamma}, \dot{\gamma})_{\gamma(s)} ds = \alpha t$$

and substitute $Y$ from (25) (and use $\dot{\gamma}, \dot{\gamma} \in H(M)$). Differentiating the resulting relation with respect to $t$ at $t = 0$ gives $\alpha = 0$. Hence

$$g_\theta(Y, \dot{\gamma}) + \int_0^t \theta(X)_{\gamma(s)} A(\dot{\gamma}, \dot{\gamma})_{\gamma(s)} ds = 0.$$

The existence statement in Theorem 3 is proved. We need the following terminology. Given $X \in J_\gamma$ a Jacobi field $Y \in J_\gamma$ satisfying (26) is said to be slant at $\gamma(t)$ relative to $X$. Also $Y$ is slant if it slant at any point of $\gamma$. To check the uniqueness statement let $X = a'\dot{\gamma} + b'\dot{\gamma} + Z$ be another decomposition of $X$, where $a', b' \in \mathbb{R}$ and $Z \in J_\gamma$ is slant (relative to $X$). Then

$$(a + bt)\dot{\gamma(t)} + Y_{\gamma(t)} = (a' + b't)\dot{\gamma(t)} + Z_{\gamma(t)}$$

and taking the inner product with $\dot{\gamma}(t)$ yields $a + bt = a' + b't$, i.e. $a = a', b = b'$ and $Y_{\gamma(t)} = Z_{\gamma(t)}$. Q.e.d.

**Corollary 2.** Suppose a Jacobi field $X \in J_\gamma$ is slant at $\gamma(r)$ and at $\gamma(s)$ relative to itself, for some $r \neq s$. Then $X$ is slant. In particular, if i) $X_{\gamma(t)} \in H(M)_{\gamma(t)}$ for every $t$, or ii) $(M, \theta)$ is a Sasakian manifold, and $X$ is perpendicular to $\gamma$ at two points, it is perpendicular to $\gamma$ at every point of $\gamma$.

**Proof.** By Theorem 3 we may decompose $X = a\dot{\gamma} + b\dot{\gamma} + Y$, where $Y \in J_\gamma$ is slant (relative to $X$). Taking the inner product of $X_{\gamma(r)} = (a + br)\dot{\gamma(r)} + Y_{\gamma(r)}$ with $\dot{\gamma}(r)$ gives $a + br = 0$. Similarly $a + bs = 0$ hence (as $r \neq s$) $a = b = 0$, so that $X = Y$. Q.e.d.

4. CR manifolds without conjugate points

Two points $x$ and $y$ on a lengthy geodesic $\gamma(t)$ are horizontally conjugate if there is a Jacobi field $X \in J_\gamma$ such that $X_{\gamma(t)} \in H(M)_{\gamma(t)}$ for
every \( t \) and \( X_x = X_y = 0 \). As \( T_\gamma \) is pure, the Jacobi equation (24) may also be written

\[
(27) \quad X'' - 2\Omega(X', \dot{\gamma})T + \theta(X')\tau(\dot{\gamma}) + \theta(X)(\nabla_\gamma \tau)\dot{\gamma} + R(X, \dot{\gamma})\dot{\gamma} = 0.
\]

Given \( X \in J_\gamma \) one has (by (27))

\[
\frac{d}{dt}\{g_{\theta}(X', X)\} = g_{\theta}(X''', X) + g_{\theta}(X', X') = |X'|^2 + 2\theta(X)\Omega(X', \dot{\gamma}) - \theta(X')A(\dot{\gamma}, X) - \\
-\theta(X)g_{\theta}(\nabla_\gamma \tau)\dot{\gamma}, X) - g_{\theta}(R(X, \dot{\gamma})\dot{\gamma}, X).
\]

On the other hand (again by \( \nabla g_{\theta} = 0 \))

\[
\theta(X')A(\dot{\gamma}, X) + \theta(X)g_{\theta}(\nabla_\gamma \tau)\dot{\gamma}, X) = \\
= \theta(X')A(\dot{\gamma}, X) + \theta(X)\frac{d}{dt}\{A(\dot{\gamma}, X)\} - \theta(X)A(\dot{\gamma}, X') = \\
= \frac{d}{dt}\{\theta(X)A(\dot{\gamma}, X)\} - \theta(X)A(\dot{\gamma}, X')
\]

hence

\[
(28) \quad \frac{d}{dt}\{g_{\theta}(X', X) + \theta(X)A(\dot{\gamma}, X)\} = \\
= |X'|^2 - g_{\theta}(R(X, \dot{\gamma})\dot{\gamma}, X) + \theta(X)[A(\dot{\gamma}, X') + 2\Omega(X', \dot{\gamma})].
\]

S. Webster (cf. [28]) has introduced a notion of pseudohermitian sectional curvature by setting

\[
(29) \quad k_{\theta}(\sigma) = \frac{1}{4} G_{\theta}(X, X)^{-2} g_{\theta,x}(R_x(X, J_x X)J_x X, X),
\]

for any holomorphic 2-plane \( \sigma \) (i.e. a 2-plane \( \sigma \subset H(M)_x \) such that \( J_x(\sigma) = \sigma \)), where \( \{X, J_x X\} \) is a basis of \( \sigma \). The coefficient \( 1/4 \) makes the sphere \( \iota : S^{2n+1} \subset \mathbb{C}^{n+1} \) (endowed with the contact form \( \theta_0 = \iota^\ast[\frac{1}{2}(\bar{\partial} - \partial)]|z|^2] \) have constant curvature +1. Clearly, this is a pseudohermitian analog to the notion of holomorphic sectional curvature in Hermitian geometry. On the other hand, for any 2-plane \( \sigma \subset T_x(M) \) one may set

\[
k_{\theta}(\sigma) = \frac{1}{4} g_{\theta,x}(R_x(X, Y)Y, X)
\]

where \( \{X, Y\} \) is a \( g_{\theta,x} \)-orthonormal basis of \( \sigma \). Cf. [18], Vol. I, p. 200, the definition of \( k_{\theta}(\sigma) \) doesn’t depend upon the choice of orthonormal basis in \( \sigma \) because the curvature \( R(X, Y, Z, W) = g_{\theta}(R(Z, W)Y, X) \) of the Tanaka-Webster connection is skew symmetric in both pairs \( (X, Y) \) and \( (Z, W) \). We refer to \( k_{\theta} \) as the pseudohermitian sectional curvature of \( (M, \theta) \). A posteriori the restriction (29) of \( k_{\theta} \) to holomorphic 2-planes
is referred to as the holomorphic pseudohermitian sectional curvature of \((M, \theta)\). As an application of (28) we may establish

**Theorem 4.** If \((M, \theta)\) has nonpositive pseudohermitian sectional curvature then \((M, \theta)\) has no horizontally conjugate points.

We need

**Lemma 4.** For every Jacobi field \(X \in J_{\gamma}\)

\[
\frac{d}{dt}\{\theta(X)\} - 2\Omega(X, \dot{\gamma})\gamma(t) = c = \text{const}.
\]

To prove Lemma 4 one merely takes the inner product of (27) by \(T\).

**Proof of Theorem 4** The proof is by contradiction. If there is a lengthy geodesic \(\gamma(t) \in M\) (parametrized by arc length) and a Jacobi field \(X \in J_{\gamma}\) such that \(X_{\gamma(a)} = X_{\gamma(b)} = 0\) for two values \(a\) and \(b\) of the parameter then we may integrate in (28) so that to obtain

\[
\int_{a}^{b}\{|X'|^2 - g_\theta(R(X, \dot{\gamma})\gamma, X) + \theta(X)[A(\dot{\gamma}, X') + 2\Omega(X', \dot{\gamma})]\}dt = 0.
\]

On the other hand

\[
\theta(X)\Omega(X', \dot{\gamma}) = \theta(X)\frac{d}{dt}\{\Omega(X, \dot{\gamma})\} =
\]

\[
= \frac{d}{dt}\{(\theta(X)\Omega(X, \dot{\gamma})) - \Omega(X, \dot{\gamma})\theta(X')\}.
\]

Then (by Lemma 4)

\[
2\int_{a}^{b}\theta(X)\Omega(X', \dot{\gamma})dt = -2\int_{a}^{b}\Omega(X, \dot{\gamma})\frac{d}{dt}\{\theta(X)\}dt =
\]

\[
= c\int_{a}^{b}\frac{d}{dt}\{\theta(X)\}dt - \int_{a}^{b}\theta(X')^2dt = - \int_{a}^{b}\theta(X')^2dt
\]

hence (30) becomes

\[
\int_{a}^{b}\{|X'|^2 - g_\theta(R(X, \dot{\gamma})\gamma, X) + \theta(X)A(\dot{\gamma}, X') - \theta(X')^2\}dt = 0.
\]

Finally, if \(X \in H(M)\) then \(X' \in H(M)\) and then (under the assumptions of Theorem 4) \(X' = 0\), a contradiction.
5. Jacobi fields on CR manifolds of constant pseudohermitian sectional curvature

As well known (cf. Example 2.1 in [18], Vol. II, p. 71) one may determine a basis of $J_\gamma$ for any elliptic space form (a Riemannian manifold of positive constant sectional curvature). Similarly, we shall prove

**Proposition 3.** Let $M$ be a strictly pseudoconvex CR manifold of CR dimension $n$, $\theta$ a contact form with $G_\theta$ positive definite and constant pseudohermitian sectional curvature. Let $\gamma(t) \in M$ be a lengthy geodesic of the Tanaka-Webster connection $\nabla$ of $(M, \theta)$, parametrized by arc length. For each $v \in T_{\gamma(0)}(M)$ we let $E(v)$ be the space of all vector fields $X$ along $\gamma$ defined by $X_{\gamma(t)} = (at + b)Y_{\gamma(t)}$, where $a, b \in \mathbb{R}$, $\nabla_{\dot{\gamma}}Y = 0$, $Y_{\gamma(0)} = v$. Assume that $(M, \theta)$ has parallel pseudohermitian torsion, i.e. $\nabla \tau = 0$. Then $T \in J_\gamma$. Let $\{v_1, \ldots, v_{2n-2}\} \subset H(M)_{\gamma(0)}$ such that $\{\dot{\gamma}(0), J_{\gamma(0)}\dot{\gamma}(0), v_1, \ldots, v_{2n-2}\}$ is a $G_{\theta, \gamma(0)}$-orthonormal basis of $H(M)_{\gamma(0)}$. Then

$E(\dot{\gamma}(0)) \oplus E(v_1) \oplus \cdots \oplus E(v_{2n-2}) \subseteq H_{\gamma} := J_\gamma \cap \Gamma^\infty(\gamma^{-1}H(M))$

if and only if

$A_{\gamma(0)}(\dot{\gamma}(0), \dot{\gamma}(0)) = 0, \quad A_{\gamma(0)}(v_i, \dot{\gamma}(0)) = 0, \quad 1 \leq i \leq 2n-2,$

where $\gamma^{-1}H(M)$ is the pullback of $H(M)$ by $\gamma$. If additionally $(M, \theta)$ has vanishing pseudohermitian torsion (i.e. $(M, \theta)$ is Tanaka-Webster flat) then $E(T_{\gamma(0)}) \subset J_\gamma$.

The proof of Proposition 3 requires the explicit form of the curvature tensor of the Tanaka-Webster connection of $(M, \theta)$ when $k_\theta = \text{const}$. This is provided by

**Theorem 5.** Let $M$ be a strictly pseudoconvex CR manifold and $\theta$ a contact form on $M$ such that $G_\theta$ is positive definite and $k_\theta(\sigma) = c$, for some $c \in \mathbb{R}$ and any 2-plane $\sigma \subset T_x(M), x \in M$. Then $c = 0$ and the curvature of the Tanaka-Webster connection of $(M, \theta)$ is given by

$$R(X,Y)Z = \Omega(Z,Y)\tau(X) - \Omega(Z,X)\tau(Y) + A(Z,Y)JX - A(Z,X)JY,$$

for any $X, Y, Z \in T(M)$. In particular, if $(M, \theta)$ has constant pseudohermitian sectional curvature and CR dimension $n \geq 2$ then the Tanaka-Webster connection of $(M, \theta)$ is flat if and only if $(M, \theta)$ has vanishing pseudohermitian torsion ($\tau = 0$).

The proof of Theorem 5 is given in Appendix A. By Theorem 5 there are no “pseudohermitian space forms” except for those of zero pseudohermitian sectional curvature and these aren’t in general flat. Cf.
the term pseudohermitian space form is reserved for manifolds of constant holomorphic pseudohermitian sectional curvature (and then examples with arbitrary $c \in \mathbb{R}$ abound, cf. [10], Chapter 1).

Proof of Proposition By \ref{eq:pseudohermitian_form}

\[ R(X, \gamma)\dot{\gamma} = \Omega(X, \dot{\gamma})\tau(\dot{\gamma}) + A(\dot{\gamma}, \dot{\gamma})JX - A(X, \gamma)J\dot{\dot{\gamma}} \]

hence the Jacobi equation \ref{eq:jacobi_equation} becomes

\[ X'' - 2\Omega(X', \dot{\gamma})T + \theta(X')\tau(\dot{\gamma}) + \nabla f(X, \dot{\gamma}) = 0 \]

or (by taking the inner product with $Y$) \[ f''(t)Y + f(t)[A(\dot{\gamma}, \dot{\gamma})JY - A(Y, \dot{\gamma})J\dot{\gamma}] = 0 \]

We look for solutions to \ref{eq:pseudohermitian_form} of the form $X_{\gamma(t)} = f(t)T_{\gamma(t)}$. The relevant equation is

\[ f''(t)\dot{T} + f'(t)\tau(\dot{\gamma}) + f(t)(\nabla f(\dot{\gamma}) = 0 \]

(by $\nabla T = 0$) or $f''(t) = 0$ and $f'(t)\tau(\dot{\gamma}) + f(t)(\nabla f(\dot{\gamma}) = 0$. Therefore, if $\nabla \tau = 0$ then $T \in J_{\gamma}$ while if $\tau = 0$ then $T, \dot{T} \in J_{\gamma}$, where $T_{\gamma(t)} = tT_{\gamma(t)}$.

Next, we look for solutions to \ref{eq:pseudohermitian_form} of the form $X_{\gamma(t)} = f(t)Y_{\gamma(t)}$ where $Y$ is a vector field along $\gamma$ such that $\nabla Y = 0$, $Y_{\gamma(0)} = v \in H(M)_{\gamma(0)}$, $|v| = 1$, and $g_{\gamma(0)}(v, \dot{J}\gamma(0)) = 0$. Substitution into \ref{eq:pseudohermitian_form} gives

\[ f''(t)Y + f(t)[A(\dot{\gamma}, \dot{\gamma})JY - A(Y, \dot{\gamma})J\dot{\gamma}] = 0 \]

or (by taking the inner product with $Y$) \[ f''(t)Y = 0 \]

\[ f(t) = \text{a constant} \]

\[ a, b \in \mathbb{R} \]

Therefore (with the notations in Proposition \ref{eq:pseudohermitian_form} $E(v_i) \subset J_{\gamma} \cap \Gamma^1(\gamma^{-1}H(M))$ if and only if $A_{\gamma(0)}(\dot{\gamma}(0), \dot{\gamma}(0)) = 0$ and $A_{\gamma(0)}(v_1, \dot{\gamma}(0)) = 0$. Also, to start with, $E(\dot{\gamma}(0))$ (the space spanned by $\dot{\gamma}$ and $\dot{\gamma}$) consists of Jacobi fields lying in $H(M)$. As $\{\dot{\gamma}(t), J_{\gamma(t)}\dot{\gamma}(t), Y_{1,\gamma(t)}, \ldots, Y_{2n-2,\gamma(t)}\}$ is an orthonormal basis of $H(M)_{\gamma(t)}$ (where $Y_i$ is the unique solution to $\nabla Y_{\gamma(t)} = 0$, $Y_{\gamma(0)} = v_i$) it follows that the sum $E(\dot{\gamma}(0)) + E(v_1) + \cdots + E(v_{2n-2})$ is direct. Q.e.d.

Let $(M,(\varphi, \xi, \eta, g))$ be a contact Riemannian manifold. Let $X \in T_{\varphi}(M)$ be a unit tangent vector orthogonal to $\xi$ and $\sigma \subset T_{\varphi}(M)$ the 2-plane spanned by $\{X, \varphi X\}$ (a $\varphi$-holomorphic plane). We recall (cf. e.g. [5], p. 94) that the $\varphi$-sectional curvature is the restriction of the sectional curvature $k$ of $(M, g)$ to the $\varphi$-holomorphic planes. Let us set $H(X) = k(\sigma)$. A Sasakian manifold of constant $\varphi$-sectional curvature $H(X) = c$, $c \in \mathbb{R}$, is a Sasakian space form. Compact Sasakian space forms have been classified in [17]. By a result in [5], p. 97, the Riemannian curvature $R^D$ of a Sasakian space form $M$ (of $\varphi$-sectional curvature $c$) is given by

\[ R^D(X, Y)Z = \frac{c + 3}{4} \{g(Y, Z)X - g(X, \varphi X)Y\} + \]
A calculation based on (34) leads to
\[ D = \nabla + (\Omega - A) \otimes T + \tau \otimes \theta + 2\theta \otimes J. \]
for any \( X, Y, Z \in T(M) \). Given a strictly pseudoconvex CR manifold \( M \) and a contact form \( \theta \) we recall (cf. e.g. (1.59) in [11]) that (34) shows that
\[ R(X, Y)Z = R(X, Y)Z + (LX \wedge LY)Z - 2\Omega(X, Y)JZ - g_\theta(S(X, Y), Z)T + \theta(Z)S(X, Y) - 2g_\theta((\theta \wedge O)(X, Y), Z)T + 2\theta(Z)(\theta \wedge O)(X, Y) \]
for any \( X, Y, Z \in T(M) \), relating the Riemannian curvature \( R^D \) of \( (M, g_\theta) \) to the curvature \( R \) of the Tanaka-Webster connection. Here
\[ L = \tau + J, \quad O = \tau^2 + 2J\tau - I, \]
and \( (X \wedge Y)Z = g_\theta(X, Z)Y - g_\theta(Y, Z)X. \) Also \( S(X, Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X. \) Let us assume that \( (M, \theta) \) is a Sasakian manifold \( (\tau = 0) \) whose Tanaka-Webster connection is flat \( (R = 0) \). Then \( S = 0, L = J \) and \( O = -I \) hence
\[ R^D(X, Y)Z = (JX \wedge JY)Z - 2\Omega(X, Y)JZ + 2g_\theta((\theta \wedge I)(X, Y), Z)T - 2\theta(Z)(\theta \wedge I)(X, Y) \]
and a comparison to (33) shows that

**Proposition 4.** Let \( (M, \theta) \) be a Sasakian manifold. Then its Tanaka-Webster connection is flat if and only if \( (M, (J, -T, -\theta, g_\theta)) \) is a Sasakian space form of \( \varphi \)-sectional curvature \( c = -3 \).

By Lemma 8 below the dimension of \( \mathcal{H}_\gamma \) is at most \( 4n \). On a Sasakian space form we may determine \( 4n - 1 \) independent vectors in \( \mathcal{H}_\gamma \). Indeed, by combining Propositions 3 and 4 we obtain

**Corollary 3.** Let \( (M, \theta) \) be a Sasakian space form of \( \varphi \)-sectional curvature \( c = -3 \) and \( \gamma(t) \in M \) a lengthy geodesic of the Tanaka-Webster connection \( \nabla \), parametrized by arc length. Let \( \{v_1, \ldots, v_{2n-2}\} \subset H(M)_{\gamma(0)} \) such that \( \{\dot{\gamma}(0), J_{\gamma(0)} \dot{\gamma}(0), v_1, \ldots, v_{2n-2}\} \) is a \( G_{\theta, \gamma(0)} \)-orthonormal basis of \( H(M)_{\gamma(0)} \). Let \( X_i \) be the vector field along \( \gamma \) determined by
\[ \nabla_{\dot{\gamma}(0)} X_i = 0, \quad X_i(\gamma(0)) = v_i, \]
for \( 1 \leq i \leq 2n - 2 \). Then \( S = \{\dot{\gamma}, \ddot{\gamma}, J_{\gamma} \dot{\gamma}, X_i, \ddot{X}_i : 1 \leq i \leq 2n - 2\} \) is a free system in \( \mathcal{H}_\gamma \) while \( S \cup \{T, \ddot{T}\} \) is free in \( J_\gamma \). Here if \( Y \) is a vector field along \( \gamma(t) \) we set \( \dot{X}_\gamma(t) = tY_{\gamma(t)} \) for every \( t \).
6. Conjugate points on Sasakian manifolds

Let $(M, \theta)$ be a Sasakian manifold and $\gamma : [a, b] \to M$ a geodesic of the Tanaka-Webster connection $\nabla$, parametrized by arc length. Given a piecewise differentiable vector field $X$ along $\gamma$ we set

$$I_a^b(X) = \int_a^b \{g_\theta(\nabla_\gamma X, \nabla_\gamma X) - g_\theta(R(X, \gamma)\dot{\gamma}, X)\}_{\gamma(t)}dt$$

where $R$ is the curvature of $\nabla$. We shall prove the following

**Proposition 5.** Let $(M, \theta)$ be a Sasakian manifold and $\gamma(t) \in M$, $a \leq t \leq b$, a lengthy geodesic of $\nabla$, parametrized by arc length, such that $\gamma(a)$ has no conjugate point along $\gamma$. Let $Y \in H_\gamma$ be a horizontal Jacobi field along $\gamma$ such that $Y_{\dot{\gamma}(a)} = 0$ and $Y$ is perpendicular to $\gamma$. Let $X$ be a piecewise differentiable vector field along $\gamma$ such that $X_{\dot{\gamma}(a)} = 0$ and $X$ is perpendicular to $\gamma$. If $X_{\gamma(b)} = Y_{\gamma(b)}$ then

$$I_a^b(X) \geq I_a^b(Y)$$

and the equality holds if and only if $X = Y$.

**Proof.** Let $J_{\gamma,a}$ be the space of all Jacobi fields $Z \in J_\gamma$ such that $Z_{\gamma(a)} = 0$. By Prop. 1.1 in [14], Vol. II, p. 63, $J_{\gamma,a}$ has dimension $2n + 1$. Moreover, let $J_{\gamma,a,\perp}$ be the space of all $Z \in J_{\gamma,a}$ such that $g_\theta(Z, \dot{\gamma})_{\gamma(t)} = 0$, for every $t$. Then by Theorem 3 it follows that $J_{\gamma,a,\perp}$ has dimension $2n$. We shall need the following

**Lemma 5.** For every Sasakian manifold $(M, \theta)$ the characteristic direction $T$ of $(M, \theta)$ is a Jacobi field along any geodesic $\gamma : [a, b] \to M$ of $\nabla$. Also, if $T_a$ is the vector field along $\gamma$ given by $T_{a,\gamma(t)} = (t - a)T_{\gamma(t)}$, $a \leq t \leq b$, and $\gamma$ is lengthy then $T_a \in J_{\gamma,a,\perp}$ and $T_{a,\gamma(t)} \neq 0$, $a < t \leq b$.

**Proof.** Let $\mathcal{J}_\gamma$ be the Jacobi operator. Then

$$\mathcal{J}_\gamma T = T'' - 2\Omega(T', \dot{\gamma})T + R(T, \dot{\gamma})\dot{\gamma} = R(T, \dot{\gamma})\dot{\gamma}$$

as $T' = \nabla_\gamma T = 0$. On the other hand, on any nondegenerate CR manifold with $S = 0$ (i.e. $S(X, Y) \equiv (\nabla_X \tau)Y - (\nabla_Y \tau)X = 0$, for any $X, Y \in T(M)$) the curvature of the Tanaka-Webster connection satisfies

$$R(T, X)X = 0, \quad X \in T(M),$$

hence $\mathcal{J}_\gamma T = 0$. As $R(T, T) = 0$ and $R(X, Y)T = 0$ it suffices to check (30) for $X \in H(M)$, i.e. locally $X = Z^\alpha T_\alpha + Z^\tau T_\tau$. Then

$$R(T, X)X = \{R^B_\beta Z^\alpha Z^\beta + R^B_\beta Z^\tau Z^\beta\}T_\gamma +$$

$$+ R^\tau_{\alpha 0} Z^\alpha Z^\beta + R^\tau_{0 \alpha} Z^\tau Z^\beta \}T_\tau$$
The proof is by contradiction. Assume that \( a < t \) for some \( \alpha \).

To complete the proof of Lemma 5 let \( u(t) = t - a \). Then (by \( T \mid \Omega = 0 \) and (36))

\[
\mathcal{J}_\gamma T_a = u''T - 2u'\Omega(T, \dot{\gamma})T + uR(T, \dot{\gamma})\gamma = 0.
\]

**Lemma 6.** Let \((M, \theta)\) be a Sasakian manifold and \( \gamma(t) \in M \) a geodesic of \( \nabla \). If \( X \in J_\gamma \) then \( X_H \equiv X - \theta(X)T \) satisfies the second order ODE

\[
\nabla^2_{\dot{\gamma}}X_H + R(X_H, \dot{\gamma})\dot{\gamma} = 0.
\]

**Proof.**

\[
0 = \mathcal{J}_\gamma X = \nabla^2_{\dot{\gamma}}X_H + \theta(X'')T - 2\Omega(\nabla_{\dot{\gamma}}X_H, \dot{\gamma})T + R(X_H, \dot{\gamma})\dot{\gamma}
\]

hence (by the uniqueness of the direct sum decomposition \( T(M) = H(M) \oplus \mathbb{R}T \)) \( X_H \) satisfies (37).

Let us go back to the proof of Proposition 5. Let us complete \( T_a \) to a linear basis \( \{T_a, Y_2, \cdots, Y_{2n}\} \) of \( J_{\gamma,a} \) and set \( Y_1 = T_a \) for simplicity. Then \( Y = a^iY_i \) for some \( a^i \in \mathbb{R}, \ 1 \leq i \leq 2n \). Let us observe that for each \( a < t \leq b \) the tangent vectors

\[
\{T_{a,\gamma(t)}, Y^H_{2,\gamma(t)}, \cdots, Y^H_{2n,\gamma(t)}\} \subset [\mathbb{R}\dot{\gamma}(t)]^\perp \subset T_{\gamma(t)}(M)
\]

are linearly independent, where \( Y^H_j := Y_j - \theta(Y_j)T, \ 2 \leq j \leq 2n \). Indeed

\[
0 = aT_{a,\gamma(t)} + \sum_{j=2}^{2n} \alpha^j Y^H_{j,\gamma(t)} = \{\alpha - \sum_{j=2}^{2n} \frac{\alpha^j}{t-a} \theta(Y_j)\gamma(t)\}T_{a,\gamma(t)} + \sum_{j=2}^{2n} \alpha^j Y^H_{j,\gamma(t)}
\]

implies \( \alpha^j = 0 \), and then \( \alpha = 0 \), because \( \{Y_{i,\gamma(t)} : 1 \leq i \leq 2n\} \) are linearly independent, for any \( a < t \leq b \). At their turn, the vectors \( Y_{i,\gamma(t)} \) are independent because \( \gamma(a) \) has no conjugate point along \( \gamma \). The proof is by contradiction. Assume that

\[
(38) \quad \lambda^i Y_{i,\gamma(t_0)} = 0,
\]

for some \( a < t_0 \leq b \) and some \( \lambda = (\lambda^1, \cdots, \lambda^{2n}) \in \mathbb{R}^{2n} \setminus \{0\} \). Let us set \( Z_0 = \lambda^i Y_i \in J_\gamma \). Then

\[
\lambda \neq 0 \implies Z_0 \neq 0,
\]

\[
Z_0 \in J_{\gamma,a} \implies Z_{0,\gamma(a)} = 0, \quad (38) \implies Z_{0,\gamma(t_0)} = 0,
\]
hence \( \gamma(a) \) and \( \gamma(t_0) \) are conjugate along \( \gamma \), a contradiction. Yet \([\mathbb{R}\gamma(t)]^\perp\) has dimension \( 2n \) hence

\[
X_{\gamma(t)} = f(t)T_{a,\gamma(t)} + \sum_{j=2}^{2n} f^j(t)Y^H_{j,\gamma(t)},
\]

for some piecewise differentiable functions \( f(t), f^j(t) \). We set \( f^1 = f \), \( Z_1 = T_a \) and \( Z_j = Y_{j}^H \), \( 2 \leq j \leq 2n \), for simplicity. Then

\[
\left| X' \right|^2 = \left| \frac{df}{dt} Z_i \right|^2 + \left| f^i Z'_i \right|^2 + 2g_\theta \left( \frac{df}{dt} Z_i, f^j Z'_j \right).
\]

Also (by (36) and Lemma 6)

\[
-g_\theta(R(X, \dot{\gamma}) \dot{\gamma}, X) = -f^i g_\theta(R(Z_i, \dot{\gamma}) \dot{\gamma}, X) = -\sum_{j=2}^{2n} f^j g_\theta(R(Z_j, \dot{\gamma}) \dot{\gamma}, X) = \sum_{j=2}^{2n} f^j g_\theta(Z''_j, X)
\]
or (as \( T'_a = 0 \))

\[
-g_\theta(R(X, \dot{\gamma}) \dot{\gamma}, X) = g_\theta(f^i Z''_i, f^j Z_j).
\]

Finally, note that

\[
g_\theta \left( \frac{df}{dt} Z_i, f^j Z'_j \right) + \left| f^i Z'_i \right|^2 + g_\theta \left( f^i Z_i, f^j Z''_j \right) =
\]
\[
= \frac{d}{dt} g_\theta(f^i Z_i, f^j Z'_j) - g_\theta(f^i Z_i, \frac{df}{dt} Z'_j).
\]

Summing up (by (39)-(41))

\[
\left| X' \right|^2 - g_\theta(R(X, \dot{\gamma}) \dot{\gamma}, X) =
\]
\[
= \frac{d}{dt} g_\theta(f^i Z_i, f^j Z'_j) - g_\theta(f^i Z_i, \frac{df}{dt} Z'_j) + g_\theta \left( \frac{df}{dt} Z_i, f^j Z'_j \right) + \left| \frac{df}{dt} Z_i \right|^2.
\]

**Lemma 7.** Let \((M, \theta)\) be a Sasakian manifold and \( \gamma(t) \in M \) a geodesic of \( \nabla \). If \( X \) and \( Y \) are solutions to \( \nabla^2 X + g_\theta(X, \dot{\gamma}) \dot{\gamma} = 0 \) then

\[
\frac{d}{dt} \{ g_\theta(X, Y') - g_\theta(X', Y) \} = 0.
\]

In particular, if \( X_{\gamma(a)} = 0 \) and \( Y_{\gamma(a)} = 0 \) at some point \( \gamma(a) \) of \( \gamma \) then

\[
g_\theta(X, Y') - g_\theta(X', Y) = 0.
\]
Proof. As \( \tau = 0 \) the 4-tensor \( R(X, Y, Z, W) \) possesses the symmetry property \( R(X, Y, Z, W) = R(Z, W, X, Y) \) (cf. (12) in Appendix A) one may subtract the identities

\[
\frac{d}{dt} g_{\theta}(X, Y') = g_{\theta}(X', Y') - g_{\theta}(X, R(Y, \dot{\gamma})\dot{\gamma}),
\]
\[
\frac{d}{dt} g_{\theta}(X', Y) = g_{\theta}(X', Y') - g_{\theta}(R(X, \dot{\gamma})\dot{\gamma}, Y)
\]

so that (44) becomes

\[
\frac{d}{dt} \frac{d}{dt} g_{\theta}(X, Y') = \frac{d}{dt} g_{\theta}(X', Y') - \frac{d}{dt} g_{\theta}(X, R(Y, \dot{\gamma})\dot{\gamma}, Y)
\]

so that to obtain (13). Q.e.d.

By Lemma (3) the fields \( Z_j, 2 \leq j \leq 2n \) satisfy \( \nabla^2 \gamma Z_j + R(Z_j, \dot{\gamma})\dot{\gamma} = 0 \). Then we may apply Lemma 7 to conclude that

\[
\frac{d}{dt} \left( g_{\theta}(Z_j, Z'_j) - g_{\theta}(Z'_j, Z_i) \right) = 0
\]

so that (12) becomes

\[
|X'|^2 - g_{\theta}(R(X, \dot{\gamma})\dot{\gamma}, X) = \frac{d}{dt} g_{\theta}(f^i Z_i, f^j Z'_j) + \frac{d}{dt}|Z_i|^2
\]

and integration gives

\[
I^b_a(X) = g_{\theta}(f^i Z_i, f^j Z'_j)_{\gamma(b)} + \int_a^b \frac{d}{dt}|Z_i|^2 dt.
\]

We wish to apply (14) to the vector field \( X = Y \). If this is the case the functions \( f^j \) are \( f^1(t) = a^1 + (1/(t-a)) \sum_{i=2}^{2n} a^2 \theta(Y_j)_{\gamma(t)} = 0 \) (because of \( Y_{\gamma(t)} \in H(M)_{\gamma(t)} \)) and \( f^j = a^j \) (so that \( df^j/dt = 0 \)) for \( 2 \leq j \leq 2n \). Then (by (14))

\[
I^b_a(Y) = g_{\theta}(\sum_{i=2}^{2n} a^i Z_i, \sum_{j=2}^{2n} a^j Z'_j)_{\gamma(b)}.
\]

As \( X_{\gamma(t)} = Y_{\gamma(t)} \) it follows that \( f^1(b) = 0 \) and \( f^j(b) = a^j, 2 \leq j \leq 2n \), so that by subtracting (14) and (15) we get

\[
I^b_a(X) - I^b_a(Y) = \int_a^b \frac{d}{dt}|Z_i|^2 dt \geq 0
\]

and (35) is proved. The equality \( I^b_a(X) = I^b_a(Y) \) yields \( df^i/dt = 0 \), i.e. \( f^1(t) = f^1(b) = 0 \) and \( f^j(t) = f^j(b) = a^j, 2 \leq j \leq 2n \), hence

\[
X_{\gamma(t)} = \sum_{j=2}^{2n} a^j Y^H_{j,\gamma(t)} = \sum_{j=2}^{2n} a^j \{ Y_{j,\gamma(t)} - \theta(Y_j)_{\gamma(t)} T_{\gamma(t)} \} =
\]
\[= \sum_{j=2}^{2n} a_j^i Y_{i,j}(t) + (t-a) a^1 T_{i,j}(t) = a^i Y_{i,j}(t) = Y_{i,j}(t).\]

Q.e.d.

Setting \( Y = 0 \) in Proposition 3 leads to

**Corollary 4.** Let \((M, \theta)\) be a Sasakian manifold and \( \gamma : [a, b] \to M \) a lengthy geodesic of the Tanaka-Webster connection, parametrized by arc length and such that \( \gamma(a) \) has no conjugate point along \( \gamma \). If \( X \) is a piecewise differentiable vector field along \( \gamma \) such that \( X_{\gamma(a)} = X_{\gamma(b)} = 0 \) and \( X \) is perpendicular to \( \gamma \) then \( I_{a}^{b}(X) \geq 0 \) and equality holds if and only if \( X = 0 \).

Corollary 4 admits the following application

**Theorem 6.** Let \((M, \theta)\) be a Sasakian manifold and \( \nabla \) its Tanaka-Webster connection. Assume that the pseudohermitian sectional curvature satisfies \( k_\theta(\sigma) \geq k_0 > 0 \), for any \( 2 \)-plane \( \sigma \subset T_x(M), \ x \in M \). Then for any lengthy geodesic \( \gamma(t) \in M \) of \( \nabla \), parametrized by arc length, the distance between two consecutive conjugate points of \( \gamma \) is less equal than \( \pi/(2\sqrt{k_0}) \).

**Proof.** Let \( \gamma : [a, c] \to M \) be a geodesic of \( \nabla \), parametrized by arc length, such that \( \gamma(c) \) is the first conjugate point of \( \gamma(a) \) along \( \gamma \). Let \( b \in (a, c) \) and let \( Y \) be a unit vector field along \( \gamma \) such that \( (\nabla_i Y)_{\gamma(t)} = 0 \) and \( Y \) is perpendicular to \( \gamma \). Let \( f(t) \) be a nonzero smooth function such that \( f(a) = f(b) = 0 \). Then we may apply Corollary 4 to the vector field \( X = f Y \) so that

\[
0 \leq I_{a}^{b}(X) = \int_{a}^{b} \{ f'(t)^2 |Y|^2 - f(t)^2 g_\theta(R(Y, \dot{\gamma})\dot{\gamma}, Y) \} dt =
\]

\[
= \int_{a}^{b} \{ f'(t)^2 - 4 f(t)^2 k_\theta(\sigma) \} dt \leq \int_{a}^{b} \{ f'(t)^2 - 4k_0 f(t)^2 \} dt
\]

where \( \sigma \subset T_{\gamma(t)}(M) \) is the 2-plane spanned by \( \{Y_{\gamma(t)}, \dot{\gamma}(t)\} \). Finally, we may choose \( f(t) = \sin[\pi(t-a)/(b-a)] \) and use \( \int_{0}^{\pi} \cos^2 x \ dx = \int_{0}^{\pi} \sin^2 x \ dx = \pi/2 \). We get \( b - a \leq \pi/\sqrt{4k_0} \) and let \( b \to c \). Q.e.d.

We may establish the following more general version of Theorem 6

**Theorem 7.** Let \((M, \theta)\) be a Sasakian manifold of CR dimension \( n \) such that the Ricci tensor \( \rho \) of the Tanaka-Webster connection \( \nabla \) satisfies

\[ \rho(X, X) \geq (2n - 1)k_0 g_\theta(X, X), \quad X \in H(M), \]

for some constant $k_0 > 0$. Then for any geodesic $\gamma$ of $\nabla$, parametrized by arc length, the distance between any two consecutive conjugate points of $\gamma$ is less than $\pi/\sqrt{k_0}$.

**Remark.** The assumption on $\rho$ in Theorem 7 involves but the pseudohermitian Ricci curvature. Indeed (cf. (1.98) in [10], section 1.4)

\[
\text{Ric}(T_\alpha, T_\beta) = g_{\alpha\beta} - \frac{1}{2} R_{\alpha\beta},
\]

\[
R_{\alpha\beta} = i(n-1)A_{\alpha\beta}, \quad R_{0\beta} = S^\alpha_{\alpha\beta}, \quad R_{\alpha0} = R_{00} = 0,
\]

hence (by $\tau = 0$) $\rho(X, X) = 2R_{\alpha\beta}Z^\alpha Z^\beta$, for any $X = Z^\alpha T_\alpha + Z^\beta T_\beta \in H(M)$. Here Ric is the Ricci tensor of the Riemannian manifold $(M, g_\theta)$ (whose symmetry yields $R_{\alpha\beta} = R_{\beta\alpha}$). Note that $S = 0$ alone implies $T\mid \rho = 0$. Also, if $(M, g_\theta)$ is Ricci flat then $(M, \theta)$ is pseudo-Einstein (of pseudohermitian scalar curvature $R = 2$), in the sense of [10].

**Proof of Theorem 7.** Let $\gamma(t) \in M$ as in the proof of Theorem 6. Let \{\(Y_1, \ldots, Y_{2n-1}\)\} be parallel (i.e. $(\nabla_\gamma Y_i)_{\gamma(t)} = 0$) vector fields such that $Y_i \in H(M)$ and \{\(\dot{\gamma}(t), Y_{1,\gamma(t)}, \ldots, Y_{2n-1,\gamma(t)}\)\} is an orthonormal basis of $H(M)_{\gamma(t)}$ for every $t$. Let $f(t)$ be a nonzero smooth function such that $f(a) = f(b) = 0$ and let us set $X_i = fY_i$. Then (by Corollary 4)

\[
0 \leq \sum_{i=1}^{2n-1} I^b_a(X_i) = \sum_{i=1}^{2n-1} \int_a^b \{f'(t)^2|Y_i|^2 - f(t)^2 g_\theta(R(Y_i, \dot{\gamma})\dot{\gamma}, Y_i)\} dt =
\]

\[
\int_a^b \{(2n-1)f'(t)^2 - f(t)^2 \rho(\dot{\gamma}, \dot{\gamma})\} dt \leq (2n-1) \int_a^b \{f'(t)^2 - k_0 f(t)^2\} dt
\]

and the proof may be completed as that of Theorem 6.

**Remark.** The assumption in Theorem 7 is weaker than that in Theorem 6. Indeed, let $X \in H(M), X \neq 0$, and $V = |V|^{-1}V$. Let \{\(X_j : 1 \leq j \leq 2n\)\} be a local orthonormal frame of $H(M)$ and $\sigma_j \subset T_z(M)$ the 2-plane spanned by \{\(Y_{j,x}, X_x\)\}, where $Y_j := X_j - g_\theta(V, X_j)V$. Then $k_\theta(\sigma_j) = \frac{1}{4}g_\theta(R(V_j, V)V, V_j)_x$ where $V_j = |Y_j|^{-1}Y_j$ and $k_\theta(\sigma_j) \geq k_0/4$ yields

\[
\rho(X, X)_x = 4|X|_x^2 \sum_{j=1}^{2n} k_\theta(\sigma_j)|Y_j|_x^2 \geq (2n-1)k_0|X|_x^2.
\]

As another application of Proposition 5 we establish
Theorem 8. Let \((M, \theta)\) be a Sasakian manifold, of CR dimension \(n\). Let \(\gamma : [a, b] \rightarrow M\) be a lengthy geodesic of the Tanaka-Webster connection \(\nabla\), parametrized by arch length. Assume that i) there is \(c \in (a, b)\) such that the points \(\gamma(a)\) and \(\gamma(c)\) are horizontally conjugate along \(\gamma\) and ii) for any \(\delta > 0\) such that \([c - \delta, c + \delta] \subset (a, b)\) one has \(\dim \mathcal{H}_{\gamma_{\delta}} = 4n\), where \(\gamma_{\delta}\) is the restriction of \(\gamma\) to \([c - \delta, c + \delta]\). Then there is a piecewise differentiable horizontal vector field \(X\) along \(\gamma\) such that

1) \(X\) is perpendicular to \(\dot{\gamma}\) and \(J\dot{\gamma}\),
2) \(X_{\gamma(a)} = X_{\gamma(b)} = 0\), and
3) \(I^b_a(X) < 0\).

In general, we have

Lemma 8. Let \((M, \theta)\) be a Sasakian manifold of CR dimension \(n\) and \(\gamma(t) \in M\) a lengthy geodesic of \(\nabla\), parametrized by arch length. Then

\[
2n + 1 \leq \dim \mathcal{H}_\gamma \leq 4n.
\]

Hence the hypothesis in Theorem 8 is that \(\mathcal{H}_{\gamma_{\delta}}\) has maximal dimension. We shall prove Lemma 8 later on. As to the converse of Theorem 8, Corollary 4 guarantees only that the existence of a piecewise differentiable vector field \(X\) as above implies that there is some point \(\gamma(c)\) conjugate to \(\gamma(a)\) along \(\gamma\).

Proof of Theorem 8. Let \(a < c < b\) such that \(\gamma(a)\) and \(\gamma(c)\) are horizontally conjugate and let \(Y \in \mathcal{H}_\gamma\) such that \(Y_{\gamma(a)} = Y_{\gamma(c)} = 0\). By Corollary 2 (as \((M, \theta)\) is Sasakian) \(Y\) is perpendicular to \(\gamma\). Let \((U, x^i)\) be a normal (with respect to \(\nabla\)) coordinate neighborhood with origin at \(\gamma(c)\). By Theorem 8.7 in [18], Vol. I, p. 149, there is \(R > 0\) such that for any \(0 < r < R\) the open set

\[
U(\gamma(c); r) \equiv \{y \in U : \sum_{i=1}^{2n+1} x^i(y)^2 < r^2\}
\]

is convex\(^1\) and each point of \(U(\gamma(c); r)\) has a normal coordinate neighborhood containing \(U(\gamma(c); r)\). By continuity there is \(\delta > 0\) such that \(\gamma(t) \in U(\gamma(c); r)\) for any \(c - \delta \leq t \leq c + \delta\). Let \(\gamma_{\delta}\) denote the restriction of \(\gamma\) to the interval \([c - \delta, c + \delta]\). We need the following

Lemma 9. The points \(\gamma(c \pm \delta)\) are not conjugate along \(\gamma_{\delta}\).

The proof is by contradiction. If \(\gamma(c + \delta)\) is conjugate to \(\gamma(c - \delta)\) along \(\gamma_{\delta}\) then (by Theorem 1.4 in [18], Vol. II, p. 67) there is \(v \in T_{\gamma(c-\delta)}(M)\)

\(^1\)That is any two points of \(U(\gamma(c); r)\) may be joined by a geodesic of \(\nabla\) lying in \(U(\gamma(c); r)\).
such that $\exp_{\gamma(c-\delta)} v = \gamma(c + \delta)$ and the linear map
\[
d_v \exp_{\gamma(c-\delta)} : T_v(T_{\gamma(c-\delta)}(M)) \to T_{\gamma(c+\delta)}(M)
\]
is singular, i.e. $\ker(d_v \exp_{\gamma(c-\delta)}) \neq 0$. Yet $\gamma(c - \delta) \in U(\gamma(c); r)$ hence there is a normal (relative to $\nabla$) coordinate neighborhood $V$ with origin at $\gamma(c - \delta)$ such that $V \supseteq U(\gamma(c); r)$. In particular $\exp_{\gamma(c-\delta)} : V \to M$ is a diffeomorphism on its image, so that $d_v \exp_{\gamma(c-\delta)}$ is a linear isomorphism, a contradiction. Lemma 9 is proved.

Let us go back to the proof of Theorem 8. The linear map
\[
\Phi : \mathcal{H}_\gamma \to T_{\gamma(c-\delta)}(M) \oplus T_{\gamma(c+\delta)}(M), \quad Z \mapsto (Z_{\gamma(c-\delta)}, Z_{\gamma(c+\delta)}),
\]
is a monomorphism. Indeed $\ker(\Phi) = 0$, otherwise $\gamma(c \pm \delta)$ would be conjugate (in contradiction with Lemma 9). Both spaces are $(4n + 2)$-dimensional so that $\Phi$ is an epimorphism, as well. By hypothesis $\mathcal{H}_{\gamma_\delta}$ is $4n$-dimensional hence $\Phi$ descends to an isomorphism
\[
\mathcal{H}_{\gamma_\delta} \approx H(M)_{\gamma(c-\delta)} \oplus H(M)_{\gamma(c+\delta)}.
\]
Let then $Z \in \mathcal{H}_{\gamma_\delta}$ be a horizontal Jacobi field such that
\[
Z_{\gamma(c-\delta)} = Y_{\gamma(c-\delta)}, \quad Z_{\gamma(c+\delta)} = 0.
\]
We set
\[
X = \begin{cases} 
  Y & \text{on } \gamma|_{[a,c-\delta]}, \\
  Z & \text{on } \gamma|_{[c,\delta]}, \\
  0 & \text{on } \gamma|_{[c+\delta,b]}.
\end{cases}
\]
By the very definition $X$ is horizontal, i.e. $X_{\gamma(t)} \in H(M)_{\gamma(t)}$ for every $t$. Moreover (by $\mathcal{J}_\gamma Y = 0$ and $\theta(Y) = 0$)
\[
I^c_a(Y) = \int_a^c \{ |\nabla_{\gamma} Y|^2 - g_\theta(R(Y, \dot{\gamma})\dot{\gamma}, Y) \} dt =
= \int_a^c \{ |\nabla_{\gamma} Y|^2 + g_\theta(\nabla^2_{\gamma} Y, Y) \} dt =
= g_\theta(\nabla_{\gamma} Y, Y)_{\gamma(c)} - g_\theta(\nabla_{\gamma} Y, Y)_{\gamma(a)} = 0
\]
i.e. $I^c_{\gamma-\delta}(Y) = -I^c_{\gamma+\delta}(Y)$. Hence
\[
I^b_a(X) = I^c_{\gamma-\delta}(Y) + I^{c+\delta}_{\gamma-\delta}(Z) = -I^c_{\gamma-\delta}(Y) + I^{c+\delta}_{\gamma-\delta}(Z).
\]
Finally, let us consider the vector field along $\gamma_\delta$
\[
W = \begin{cases} 
  Y & \text{on } \gamma|_{[e-\delta,c]}, \\
  0 & \text{on } \gamma|_{[c,c+\delta]}.
\end{cases}
\]
Note that $W_{\gamma(c+\delta)} = 0$, $W_{\gamma(c-\delta)} = Z_{\gamma(c-\delta)}$ and $W$ is perpendicular to $\gamma$. Thus we may apply Proposition 5 to $W$ and to $Z \in \mathcal{H}_{\gamma_\delta}$ to conclude
that \( I^c_{-\delta}(Y) = I^c_{-\delta}(W) \geq I^c_{-\delta}(Z) \). Consequently \( I^b_a(X) < 0 \). Let us show that \( X \) is orthogonal to \( J\gamma \). By Lemma \( \text{[Lemma 4]} \) (as \( \dot{Y} \in J\gamma \))

\[
\theta(Y'(t) - 2\Omega(Y', \dot{\gamma})_{\gamma(t)} = \text{const.} = \theta(Y'(a) - 2\Omega(Y, \dot{\gamma})_{\gamma(a)}
\]

hence (as \( Y_{\gamma(a)} = 0 \) and \( Y_{\gamma(t)} \in H(M)_{\gamma(t)} \implies Y'_{\gamma(t)} \in H(M)_{\gamma(t)} \))

\[
2\Omega(Y, \dot{\gamma})_{\gamma(t)} = \theta(Y'(t) - \theta(Y'(a) = 0
\]

for any \( a \leq t \leq c - \delta \). Similarly (as \( Z_{\gamma(c+\delta)} = 0 \) and \( Z \) is horizontal) \( \Omega(Z, \dot{\gamma})_{\gamma(t)} = 0 \) for any \( c - \delta \leq t \leq c + \delta \). Therefore \( \Omega(X, \dot{\gamma})_{\gamma(t)} = 0 \) for every \( t \). Theorem \( \text{[Theorem 8]} \) is proved.

It remains that we prove Lemma \( \text{[Lemma 8]} \). Let \( \gamma(t) \in M, |t| < \epsilon \), be a lengthy geodesic of \( \nabla \). Let \( X \in \mathcal{H}_\gamma \) and \( \{Y_j : 1 \leq j \leq 4n + 2\} \) a linear basis in \( J\gamma \). Then \( X = c^j Y_j = c^j Y_j^H + c^j \theta(Y_j) T \) (where \( Y_j^H = Y_j - \theta(Y_j) T \)) for some \( c^j \in \mathbb{R} \). As \( X_{\gamma(t)} \in H(M)_{\gamma(t)} \) one has i) \( c^j \theta(Y_j)_{\gamma(t)} = 0 \) on one hand, and ii) \( c^j f^j_a(\gamma(t)) = f^a(\gamma(t)), 1 \leq a \leq 2n \), on the other, where \( X = f^a X_a, Y_j^H = f^j_a X_a \) and \( \{X_a : 1 \leq a \leq 2n\} \) is a local frame of \( H(M) \). One may think of (i)-(ii) as a linear system in the unknowns \( c^j \).

Let \( \gamma(t) \) be its rank. Then \( \dim_{\mathbb{R}} \mathcal{H}_\gamma = 4n + 2 - r(t) \geq 2n + 1 \). To prove the remaining inequality in Lemma \( \text{[Lemma 8]} \) it suffices to observe that \( \mathcal{H}_\gamma \) is contained in the space of all solutions to \( X'' + R(X, \dot{\gamma})_{\gamma} = 0 \) obeying \( X_{\gamma(0)} \in H(M)_{\gamma(0)} \) and \( X'_{\gamma(0)} \in H(M)_{\gamma(0)} \), which is \( 4n \)-dimensional.

7. The First Variation of the Length Integral

Let \( M \) be a strictly pseudoconvex CR manifold and \( y, z \in M \). Let \( \Gamma \) be the set of all piecewise differentiable curves \( \gamma : [a, b] \rightarrow M \) parametrized proportionally to arc length, such that \( \gamma(a) = y \) and \( \gamma(b) = z \). As usual, for each \( \gamma \in \Gamma \) we let \( T\gamma(\Gamma) \) be the space of all piecewise differentiable vector fields along \( \gamma \) such that \( X_y = X_z = 0 \).

Given \( X \in T\gamma(\Gamma) \) let \( \gamma^s : [a, b] \rightarrow M, |s| < \epsilon \), be a family of curves such that i) \( \gamma^s \in \Gamma, |s| < \epsilon \), ii) \( \gamma^0 = \gamma \), iii) there is a partition \( a = t_0 < t_1 < \cdots < t_k = b \) such that the map \( (t, s) \mapsto \gamma^s(t) \) is differentiable on each rectangle \( [t_j, t_{j+1}] \times (-\epsilon, \epsilon), 0 \leq j \leq k - 1 \), and iv) for each fixed \( t \in [a, b] \) the tangent vector to

\[
\sigma_t : (-\epsilon, \epsilon) \rightarrow M, \sigma_t(s) = \gamma^s(t), |s| < \epsilon,
\]

at the point \( \gamma(t) \) is \( X_{\gamma(t)} \). We set as usual

\[
(d_{\gamma^s} L)X = \frac{d}{ds} \{L(\gamma^s)\}_{s=0}.
\]

Here \( L(\gamma^s) \) is the Riemannian length of \( \gamma^s \) with respect to the Webster metric \( g_\theta \) (so that \( \gamma^s \) need not be lengthy to start with). One scope of this section is to establish the following
Theorem 9. Let $\gamma^s : [a, b] \to M$, $|s| < \epsilon$, be a 1-parameter family of curves such that $(t, s) \mapsto \gamma^s(t)$ is differentiable on $[a, b] \times (-\epsilon, \epsilon)$ and each $\gamma^s$ is parametrized proportionally to arc length. Let us set $\gamma = \gamma^0$. Then

$$
\frac{d}{ds}\{L(\gamma^s)\}_{s=0} = \frac{1}{r}\{g_\theta(X, \dot{\gamma})\gamma(t) - g_\theta(X, \dot{\gamma})\gamma(a) - \int_a^b [g_\theta(X, \nabla_{\dot{\gamma}}) - g_\theta(T_{\nabla}(X, \dot{\gamma}), \dot{\gamma})]_{\gamma(t)} dt}\}
$$

where $X_{\gamma(t)} = \dot{\gamma}(0)$, $a \leq t \leq b$, and $r = |\dot{\gamma}(t)|$ is the common length of all tangent vectors along $\gamma$.

This will be shortly seen to imply

Theorem 10. Let $\gamma \in \Gamma$ and $X \in T_\gamma(\Gamma)$. Let $a = c_0 < c_1 < \cdots < c_h < c_{h+1} = b$ be a partition such that $\gamma$ is differentiable on each $[c_j, c_{j+1}]$, $0 \leq j \leq h$. Then

$$
(d_{\gamma}L)X = \frac{1}{r}\{\sum_{j=1}^{b} g_\theta,\gamma(c_j)(X_{\gamma(c_j)}, \dot{\gamma}(c_j^+) - \dot{\gamma}(c_j^-)) - \int_a^b [g_\theta(X, \nabla_{\dot{\gamma}}) - g_\theta(T_{\nabla}(X, \dot{\gamma}), \dot{\gamma})]_{\gamma(t)} dt\}
$$

where $\dot{\gamma}(c_j^+) = \lim_{t \to c_j^+} \dot{\gamma}(t)$.

Consequently, we shall prove

Corollary 5. A lengthy curve $\gamma \in \Gamma$ is a geodesic of the Tanaka-Webster connection if and only if

$$
(d_{\gamma}L)X = \frac{1}{r} \int_a^b \theta(X)_{\gamma(t)}A(\dot{\gamma}, \dot{\gamma})_{\gamma(t)} dt
$$

for all $X \in T_\gamma(\Gamma)$. In particular, if $(M, \theta)$ is a Sasakian manifold then lengthy geodesics belonging to $\Gamma$ are the critical points of $L$ on $\Gamma$.

The remainder of this section is devoted to the proofs of the results above. We adopt the principal bundle approach in [II], Vol. II, p. 80-83. The proof is a verbatim transcription of the arguments there, except for the presence of torsion terms.

Let $\pi : O(M, g_\theta) \to M$ be the $O(2n + 1)$-bundle of $g_\theta$-orthonormal frames tangent to $M$. Let $Q = [a, b] \times (-\epsilon, \epsilon)$. Let $f : Q \to O(M, g_\theta)$ be a parametrized surface in $O(M, g_\theta)$ such that i) $\pi(f(t, s)) = \gamma^s(t)$, $(t, s) \in Q$, and ii) $f^0 : [a, b] \to O(M, g_\theta)$, $f^0(t) = f(t, 0)$, $a \leq t \leq b$, is a horizontal curve. Precisely, the Tanaka-Webster connection $\nabla$ of $(M, \theta)$ induces an infinitesimal connection in the principal bundle $GL(2n +$
$1, \mathbb{R}) \to L(M) \to M$ (of all linear frames tangent to $M$) descending (because of $\nabla g_\theta = 0$) to a connection $H$ in $O(2n+1) \to O(M, g_\theta) \to M$.

The requirement is that $(df^0/dt)(t) \in H_{f^0(t)}, a \leq t \leq b$.

Let $S, T \in \mathcal{X}(Q)$ be given by $S = \partial/\partial s$ and $T = \partial/\partial t$. Let

\[ \mu \in \Gamma^\infty(T^*(O(M, g_\theta)) \otimes \mathbb{R}^{2n+1}), \quad \Theta = D\mu, \]
\[ \omega \in \Gamma^\infty(T^*(O(M, g_\theta)) \otimes \mathfrak{o}(2n + 1)), \quad \Omega = D\omega, \]

be respectively the canonical 1-form, the torsion 2-form, the connection 1-form, and the curvature 2-form of $H$ on $O(M, g_\theta)$. We denote by

\[ \mu^* = f^*\mu, \quad \Theta^* = f^*\Theta, \quad \omega^* = f^*\omega, \quad \Omega^* = f^*\Omega, \]

the pullback of these forms to the rectangle $Q$. We claim that

\[ [S, T] = 0, \]

(49)

(50) \[ \omega^*(T)_{(t,0)} = 0, \quad a \leq t \leq b. \]

Indeed (49) is obvious. To check (50) one needs to be a bit pedantic and introduce the injections

\[ \alpha^s : [a, b] \to Q, \quad \beta_t : (-\epsilon, \epsilon) \to Q, \]
\[ \alpha^s(t) = \beta_t(s) = (t, s), \quad a \leq t \leq b, \quad |s| < \epsilon, \]

so that $f^0 = f \circ \alpha^0$. Then

\[ H_{f^0(t)} \supset \frac{df^0}{dt}(t) = (d_{(t,0)}f)(d_t\alpha^0)\frac{d}{dt}t = (d_{(t,0)}f)T_{(t,0)}, \]
\[ \omega^*(T)_{(t,0)} = \omega_{f(t,0)}((d_{(t,0)}f)T_{(t,0)}) = 0. \]

Next, we claim that

(51) \[ S(\mu^*(T)) = T(\mu^*(S)) + \omega^*(T) \cdot \mu^*(S) - \omega^*(S) \cdot \mu^*(T) + 2\Theta^*(S, T), \]

(52) \[ S(\omega^*(T)) = T(\omega^*(S)) + \omega^*(T)\omega^*(S) - \omega^*(S)\omega^*(T) + 2\Omega^*(S, T). \]

The identities (51)-(52) follow from Prop. 3.11 in [18], Vol. I, p. 36, our identity (49), and the first and second structure equations for a linear connection (cf. e.g. Theor. 2.4 in [18], Vol. I, p. 120). Let us consider the $C^\infty$ function $F : Q \to [0, +\infty)$ given by

\[ F(t, s) = \langle \mu^*(T)_{(t,s)}, \mu^*(T)_{(t,s)} \rangle^{1/2}, \quad (t, s) \in Q. \]

Here $\langle \xi, \eta \rangle$ is the Euclidean scalar product of $\xi, \eta \in \mathbb{R}^{2n+1}$. Note that

\[ \mu^*(T)_{(t,s)} = \mu_f(t,s)((d_{(t,s)}f)T_{(t,s)}) =
\]
\[ = f(t, s)^{-1}(d_{(t,s)}\pi)(d_{(t,s)}f)T_{(t,s)} = f(t, s)^{-1}d_t(\pi \circ f \circ \alpha^s)\frac{d}{dt}|_t. \]
On the other hand

Yet

onto \((T, \langle , \rangle)\)

\(f(t, s) \in O(M, g_0)\), i.e. \(f(t, s)\) is a linear isometry of \(\mathbb{R}^{2n+1}\) onto \((T, \langle , \rangle, g_\gamma(t))\), so that

\[ F(t, s) = g_{\gamma(t)}(\gamma^s(t), \gamma^s(t))^{1/2} \]

and then

\[ L(\gamma^s) = \int_a^b F(t, s) \, dt. \]

As \(\gamma^s\) is parametrized proportionally to arc length \(F(t, s)\) doesn’t depend on \(t\). In particular

\[ F(t, 0) = r. \]

We claim that

\[ \mathcal{S}(F) = \frac{1}{r} \left\{ \langle T(\mu^*(S)), \mu^*(T) \rangle + 2 \langle \Theta^*(S, T), \mu^*(T) \rangle \right\} \]

at all points \((t, 0) \in Q\). Indeed, by (51)

\[ 2F \mathcal{S}(F) = \mathcal{S}(F^2) = \mathcal{S}(\langle \mu^*(T), \mu^*(T) \rangle) = 2 \langle \mathcal{S}(\mu^*(T)), \mu^*(T) \rangle = \]

\[ = 2\langle T(\mu^*(S)), \mu^*(T) \rangle + 2\langle \omega^*(T) \cdot \mu^*(S), \mu^*(T) \rangle - 2\langle \omega^*(S) \cdot \mu^*(T), \mu^*(T) \rangle + 4\langle \Theta^*(S, T), \mu^*(T) \rangle. \]

On the other hand \(\omega\) is \(\mathfrak{o}(2n + 1)\)-valued (where \(\mathfrak{o}(2n + 1)\) is the Lie algebra of \(O(2n + 1)\), i.e. \(\omega^*(S)(t, s) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}\) is skew symmetric, hence the last-but-one term vanishes. Therefore (53) follows from (50) and (54). We may compute now the first variation of the length integral

\[ \frac{d}{dt}(L(\gamma^s))_{s=0} = \int_a^b \mathcal{S}(F)(t, 0) \, dt = \text{ (by (55))} \]

\[ = \frac{1}{r} \int_a^b \left\{ \langle T(\mu^*(S)), \mu^*(T) \rangle_{(t, 0)} + 2\langle \Theta^*(S, T), \mu^*(T) \rangle_{(t, 0)} \right\} dt. \]

On the other hand

\[ \mu^*(S)(t, 0) = \mu^\omega(t)((d_{(t, 0)} f) S_{(t, 0)}) = \]

\[ = f(t, 0)^{-1} d_0(\pi \circ f \circ \beta_t) \frac{d}{ds} \bigg|_0 = f(t, 0)^{-1} d\sigma_t \bigg|_0 \]

i.e.

\[ \mu^*(S)(t, 0) = f^0(t)^{-1} X_\gamma(t). \]

Note that given \(u \in C^\infty(Q)\) one has \(T(u)(t, 0) = (u \circ \alpha^0)'(t)\). Then

\[ T(\mu^*(T))(t, 0) = \lim_{h \to 0} \frac{1}{h} (\mu^*(T)(t+h, 0) - \mu^*(T)(t, 0)) \quad = \text{ (by (53))} \]
\[ = \lim_{h \to 0} \frac{1}{h} \{ f^0(t + h)^{-1} \dot{\gamma}(t + h) - f^0(t)^{-1} \dot{\gamma}(t) \}. \]

Yet, as \( f^0 \) is an horizontal curve

\[ f^0(t + h)^{-1} \dot{\gamma}(t + h) = f^0(t)^{-1} \tau_t^{t+h} \dot{\gamma}(t + h), \]

where \( \tau_t^{t+h} : T_{\gamma(t+h)}(M) \to T_{\gamma(t)}(M) \) is the parallel displacement operator along \( \gamma \) from \( \gamma(t + h) \) to \( \gamma(t) \). Hence

\[ T(\mu^*(\nabla))(t,0) = f^0(t)^{-1} \left( \lim_{h \to 0} \frac{1}{h} \{ \tau_t^{t+h} \dot{\gamma}(t + h) - \dot{\gamma}(t) \} \right) \]
i.e.

\[ T(\mu^*(\nabla))(t,0) = f^0(t)^{-1}(\nabla_{\dot{\gamma}}\gamma)(t). \]

To compute the torsion term we recall (cf. [18], Vol. I, p. 132)

\[ T_{\nabla,X}(X,Y) = 2v(\Theta_v(X^*,Y^*)), \]

for any \( X, Y \in T_x(M) \), where \( v \) is a linear frame at \( x \) and \( X^*, Y^* \in T_v(L(M)) \) project respectively on \( X, Y \). Note that \( (d(t,0)f)S(t,0) \) and \( (d(t,0)f)T(t,0) \) project on \( X_{\gamma(t)} \) and \( \dot{\gamma}(t) \), respectively. Then

\[ 2\Theta^*(S, T)(t,0) = f^0(t)^{-1}T_{\nabla}(X, \dot{\gamma})(t). \]

Finally (by [13] and (56) - (58))

\[ \frac{d}{dt}\{L(\gamma^s)\}_{s=0} = \frac{1}{r} \int_a^b \{ T(\mu^*(S), \mu^*(\nabla)) - \langle \mu^*(S), T(\mu^*(\nabla)) \rangle + 2\langle \Theta^*(S,T), \mu^*(\nabla) \rangle \} dt = \]

\[ = \frac{1}{r} \{ \langle \mu^*(S), \mu^*(\nabla) \rangle_{(b,0)} - \langle \mu^*(S), \mu^*(\nabla) \rangle_{(a,0)} \} - \]

\[ -\frac{1}{r} \int_a^b \{ g_\theta(X, \nabla_{\dot{\gamma}}) - g_\theta(T_{\nabla}(X, \dot{\gamma}), \dot{\gamma}) \} \gamma(t) dt \]

and (46) is proved.

**Proof of Theorem 10** Let \( c_j = t_0^{(j)} < t_1^{(j)} < \cdots < t_k^{(j)} = c_{j+1} \) be a partition of \([c_j, c_{j+1}]\) such that \( X \) is differentiable along the restriction of \( \gamma \) at each \([t_i^{(j)}, t_{i+1}^{(j)}], 0 \leq i \leq k_j - 1 \). Moreover, let \( \{ \gamma^s \}_{|s|<\epsilon} \) be a family of curves \( \gamma^s \in \Gamma \) such that \( \gamma^0 = \gamma \), the map \((t,s) \mapsto \gamma^s(t)\) is differentiable on \([c_j, c_{j+1}] \times (-\epsilon, \epsilon)\) for every \( 0 \leq j \leq h \), and \( X_{\gamma^s(t)} = (d\sigma_t/ds)(0) \) for every \( t \) (with \( \sigma_t(s) = \gamma^s(t) \)). Let \( \gamma^s_j \) (respectively \( \gamma^s_{j+1} \)) be the restriction of \( \gamma^s \) (respectively of \( \gamma^s_{j+1} \)) to \([c_j, c_{j+1}]\) (respectively to \([t_i^{(j)}, t_{i+1}^{(j)}]\)). We
may apply Theorem 9 (to the interval \([t_{i+1}^{(j)}, t_i^{(j)}]\) rather than \([a, b]\)) so that to get

\[
\frac{d}{ds}\{L(\gamma_{ji}^s)\}_{s=0} = \frac{1}{r}\left\{g_\theta(X, \hat{\gamma})_{\gamma(t_i^{(j)}+1)} - g_\theta(X, \hat{\gamma})_{\gamma(t_i^{(j)})} - \int_{t_i^{(j)}}^{t_{i+1}^{(j)}} F(X, \hat{\gamma})dt\right\}
\]

where \(F(X, \hat{\gamma})\) is short for \(g_\theta(X, \nabla \hat{\gamma})_{\gamma(t)} - g_\theta(T_{\nabla}(X, \hat{\gamma}), \hat{\gamma})_{\gamma(t)}\). Let us take the sum over \(0 \leq i \leq k_j - 1\). The lengths \(L(\gamma_{ji}^s)\) ad up to \(L(\gamma^s_j)\). Taking into account that at the points \(\gamma(c_j)\) only the lateral limits of \(\dot{\gamma}\) are actually defined, we obtain

\[
\frac{d}{ds}\{L(\gamma_{ji}^s)\}_{s=0} = \frac{1}{r}\left\{g_\theta,\gamma(c_{j+1}) (X_{\gamma(c_{j+1})}, \dot{\gamma}(c_{j+1}^-)) - g_\theta,\gamma(c_j) (X_{\gamma(c_j)}, \dot{\gamma}(c_j^+)) - \int_{c_j}^{c_{j+1}} F(X, \dot{\gamma})dt\right\}
\]

and taking the sum over \(0 \leq j \leq h\) leads to (47) (as \(X_{\gamma(0)} = 0\) and \(X_{\gamma(h+1)} = 0\)). Q.e.d.

**Proof of Corollary 3.** Let \(\gamma(t) \in M\) be a lengthy curve such that \(\gamma \in \Gamma\). If \(\gamma\) is a geodesic of \(\nabla\) then \(\nabla \hat{\gamma} = 0\) implies (by Theorem 9)

\[
(d,\gamma)LX = \frac{1}{r}\int_a^b g_\theta(T_{\nabla}(X, \hat{\gamma}), \hat{\gamma})_{\gamma(t)}dt
\]

for any \(X \in T_\gamma(\Gamma)\) and then

\[
T_{\nabla}(X, \hat{\gamma}) = -2\Omega(X_{H\gamma}, \hat{\gamma}) T + \theta(X)\tau(\hat{\gamma}), \quad g_\theta(T, \hat{\gamma}) = 0,
\]

yield (18). Viceversa, let \(\gamma \in \Gamma\) be a lengthy curve such that (18) holds. There is a partition \(a = c_0 < c_1 < \cdots < c_{h+1} = b\) such that \(\gamma\) is differentiable in \([c_j, c_{j+1}]\), \(0 \leq j \leq h\). Let \(f\) be a continuous function defined along \(\gamma\) such that \(f(\gamma(c_j)) = 0\) for \(1 \leq j \leq h\) and \(f(\gamma(t)) > 0\) elsewhere. We may apply (17) in Theorem 10 to the vector field \(X = f \nabla \hat{\gamma}\) so that to get

(59) \( (d,\gamma)LX = -\frac{1}{r}\int_a^b f \{|\nabla \hat{\gamma}|^2 - g_\theta(T_{\nabla}(\nabla \hat{\gamma}, \hat{\gamma}), \hat{\gamma})\}dt \). 

As \(\gamma\) is lengthy and \(H(M)\) is parallel with respect to \(\nabla\) one has \(\nabla \hat{\gamma} \in H(M)\) hence (by (18)) \( (d,\gamma)L(f \nabla \hat{\gamma}) = 0 \) and

\[
g_\theta(T_{\nabla}(\nabla \hat{\gamma}, \hat{\gamma}), \hat{\gamma}) = -2\Omega(\nabla \hat{\gamma}, \hat{\gamma}) g_\theta(T, \hat{\gamma}) = 0
\]

so that (by (59)) it must be \(\nabla \hat{\gamma} = 0\) whenever \(\nabla \hat{\gamma}\) makes sense, i.e. \(\gamma\) is a broken geodesic of \(\nabla\). It remains that we prove differentiability of \(\gamma\) at the points \(c_j, 1 \leq j \leq h\). Let \(j \in \{1, \cdots, h\}\) be a fixed index and let us consider a vector field \(X_j \in T_\gamma(\Gamma)\) such that \(X_j,\gamma(c_j) = \hat{\gamma}(c_j^-) - \hat{\gamma}(c_j^+)\)
and \( X_{j, \gamma(c_k)} = 0 \) for any \( k \in \{1, \ldots, h\} \setminus \{j\} \). Then (by (47)-(48)) one has \( |\dot{\gamma}(c_j^+) - \dot{\gamma}(c_j^-)|^2 = 0 \). Q.e.d.

**Remark.** The following alternative proof of Theorem 9 is also available. Since \((M, g_{\theta})\) is a Riemannian manifold and \(L(\gamma^s)\) is the Riemannian length of \(\gamma^s\) we have (cf. Theorem 5.1 in [18], Vol. II, p. 80)

\[
\frac{d}{ds} \{L(\gamma^s)\}_{s=0} = \frac{1}{r} \{ g_{\theta}(X, \dot{\gamma})_{\gamma(b)} - g_{\theta}(X, \dot{\gamma})_{\gamma(a)} \} - \frac{1}{r} \int_a^b g_{\theta}(X, D\dot{\gamma})_{\gamma(t)} \, dt
\]

where \( D \) is the Levi-Civita connection of \((M, g_{\theta})\). On the other hand (cf. e.g. [10], section 1.3) \( D \) is related to the Tanaka-Webster connection of \((M, \theta)\) by

\[
D = \nabla + (\Omega - A) \otimes T + \tau \otimes \theta + 2\theta \otimes J
\]

hence

\[
g_{\theta}(X, D\dot{\gamma}) = g_{\theta}(X, \nabla \dot{\gamma}) - \theta(X) A(\dot{\gamma}, \dot{\gamma}) + \theta(\dot{\gamma}) A(X, \dot{\gamma}) + 2\theta(\dot{\gamma}) \Omega(X, \dot{\gamma}) = g_{\theta}(X, \nabla \dot{\gamma}) - g_{\theta}(T_{\nabla}(X, \dot{\gamma}), \dot{\gamma})
\]

so that (60) yields (46). Q.e.d.

**8. The Second Variation of the Length Integral**

We introduce the Hessian \( I \) of \( L \) at a geodesic \( \gamma \in \Gamma \) as follows. Given \( X \in T_{\gamma}(\Gamma) \) let us consider a 1-parameter family of curves \( \{\gamma^s\}_{|s| < \epsilon} \) as in the definition of \((d_{\gamma} L) X\). Let \( I(X, X) \) be given by

\[
I(X, X) = \frac{d^2}{ds^2} \{L(\gamma^s)\}_{s=0}
\]

and define \( I(X, Y) \) by polarization. By analogy to Riemannian geometry (cf. e.g. [18], Vol. II, p. 81) \( I(X, Y) \) is referred to as the index form. The scope of this section is to establish

**Theorem 11.** Let \((M, \theta)\) be a Sasakian manifold. If \( \gamma \in \Gamma \) is a lengthy geodesic of the Tanaka-Webster connection \( \nabla \) of \((M, \theta)\) and \( X, Y \in T_{\gamma}(\Gamma) \) then

\[
I(X, Y) = \frac{1}{r} \int_a^b \left\{ g_{\theta}(\nabla_{\dot{\gamma}} X^\perp, \nabla_{\dot{\gamma}} Y^\perp) - g_{\theta}(R(X^\perp, \dot{\gamma}) \dot{\gamma}, Y^\perp) - 2\Omega(X^\perp, \dot{\gamma}) \nabla_{\dot{\gamma}} Y^\perp - 2[\theta(\nabla_{\dot{\gamma}} X^\perp) - 2\Omega(X^\perp, \dot{\gamma})] \Omega(Y^\perp, \dot{\gamma}) \right\} dt
\]

where \( X^\perp = X - (1/r^2) g_{\theta}(X, \dot{\gamma}) \dot{\gamma} \).

We shall need the following reformulation of Theorem 11
Theorem 12. Let \((M, \theta), \gamma\) and \(X, Y\) be as in Theorem \[\text{11}\] Then

\[
I(X, Y) = -\frac{1}{r} \int_{a}^{b} \left\{ g_{\theta}(\mathcal{J}_{\gamma}X^-, Y^-) + 2[\theta(\nabla_{\gamma}X^-) - 2\Omega(X^-, \dot{\gamma})]\Omega(Y^-, \dot{\gamma}) \right\} dt + \\
+\frac{1}{r} \sum_{j=1}^{b} g_{\theta, \gamma(t_j)}((\nabla_{\gamma}X^-)_{\gamma(t_j)}^- - (\nabla_{\gamma}X^+)_{\gamma(t_j)}^+, Y^+)_{\gamma(t_j)}
\]

where \(\mathcal{J}_{\gamma}X \equiv \nabla_{\gamma}^2 X - 2\Omega(X^+, \dot{\gamma}) T + R(X, \dot{\gamma}) \dot{\gamma}\) is the Jacobi operator and \(a = t_0 < t_1 < \cdots < t_h < t_{h+1} = b\) is a partition of \([a, b]\) such that \(X\) is differentiable in each interval \([t_j, t_{j+1}], 0 \leq j \leq h\), and \((\nabla_{\gamma}X^+)_{\gamma(t_j)}^+ = \lim_{t \to t_j^+}(\nabla_{\gamma}X^+)_{\gamma(t)}\).

This will be seen to imply

Corollary 6. Let \((M, \theta)\) be a Sasakian manifold, \(\gamma \in \Gamma\) a lengthy geodesic of the Tanaka-Webster connection of \((M, \theta)\), and \(X \in T_{\gamma}(\Gamma)\). Then \(X^\perp\) is a Jacobi field if and only if there is \(\alpha(X) \in \mathbb{R}\) such that

\[
\frac{d}{dt}\{\theta(X^\perp) \circ \gamma\}(t) - 2\Omega(X^\perp, \dot{\gamma})_{\gamma(t)} = \alpha(X)
\]

for any \(a \leq t \leq b\), and

\[
I(X, Y) = -(2/r)\alpha(X) \int_{a}^{b} \Omega(Y^\perp, \dot{\gamma})_{\gamma(t)} \, dt,
\]

for any \(Y \in T_{\gamma}(\Gamma)\).

Proof of Theorem \[\text{11}\] We adopt the notations and conventions in the proof of Theorem \[\text{9}\] As a byproduct of the proof of \([\text{55}]\) we have the identity

\[
\frac{1}{2} S(F^2) = \langle T(\mu^*(S)), \mu^*(T) \rangle + \\
+\langle \omega^*(T) \cdot \mu^*(S), \mu^*(T) \rangle + 2\langle \Theta^*(S, T), \mu^*(T) \rangle.
\]

Applying \(S\) we get

\[
\frac{1}{2} S^2(F^2) = \langle ST(\mu^*(S)), \mu^*(T) \rangle + \langle T(\mu^*(S)), S(\mu^*(T)) \rangle + \\
+\langle \omega^*(T) \cdot \mu^*(S), \mu^*(T) \rangle + \langle \omega^*(S) \cdot S(\mu^*(S)), \mu^*(T) \rangle + \\
+\langle \omega^*(T) \cdot \mu^*(S), S(\mu^*(T)) \rangle + \\
+2\langle \Theta^*(S, T), \mu^*(T) \rangle + 2\langle \Theta^*(S, T), S(\mu^*(T)) \rangle.
\]
When calculated at points of the form \((t, 0) \in Q\) the 4th and 5th terms vanish (by (50)). We proceed by calculating the remaining terms (at \((t, 0)\)). By (49)

\[
1^{st} \text{ term } = \langle S T (\mu^* (S)), \mu^* (T) \rangle = \langle T S (\mu^* (S)), \mu^* (T) \rangle = \\
= T (\langle S (\mu^* (S)), \mu^* (T) \rangle) - \langle S (\mu^* (S)), T (\mu^* (T)) \rangle.
\]

Yet \(\gamma \in \Gamma\) is a geodesic hence (by (57)) \(T (\mu^* (T))_{(t, 0)} = 0\). Hence

\[
1^{st} \text{ term } = T (\langle S (\mu^* (S)), \mu^* (T) \rangle)_{(t, 0)}.
\]

Next (by (51))

\[
2^{nd} \text{ term } = \langle T (\mu^* (S)), S (\mu^* (T)) \rangle = \langle T (\mu^* (S)), T (\mu^* (S)) \rangle + \\
+ \langle T (\mu^* (S)), \omega^* (T) \cdot \mu^* (S) \rangle - \langle T (\mu^* (S)), \omega^* (S) \cdot \mu^* (T) \rangle + \\
+ 2 \langle T (\mu^* (S)), \Theta^* (S, T) \rangle.
\]

Again terms are evaluated at \((t, 0)\) hence \(\omega^* (T) = 0\) (by (50)). On the other hand \(\omega^* (S)\) is \(o(2n + 1)\)-valued hence

\[
2^{nd} \text{ term } = \langle T (\mu^* (S)), T (\mu^* (S)) \rangle + \langle \omega^* (S) \cdot T (\mu^* (S)), \mu^* (T) \rangle + \\
+ 2 \langle T (\mu^* (S)), \Theta^* (S, T) \rangle
\]

at each \((t, 0) \in Q\). Next (by (52))

\[
3^{rd} \text{ term } = \langle S (\omega^* (T)), \mu^* (S), \mu^* (T) \rangle = \langle T (\omega^* (S)), \mu^* (S), \mu^* (T) \rangle + \\
+ \langle \omega^* (T), \omega^* (S) \cdot \mu^* (S), \mu^* (T) \rangle - \langle \omega^* (T), \omega^* (S) \cdot \mu^* (S), \mu^* (T) \rangle + \\
+ 2 \langle \Omega^* (S, T), \mu^* (S), \mu^* (T) \rangle
\]

or (by (50))

\[
3^{rd} \text{ term } = \langle T (\omega^* (S)), \mu^* (S), \mu^* (T) \rangle + 2 \langle \Omega^* (S, T), \mu^* (S), \mu^* (T) \rangle
\]

at each \((t, 0) \in Q\). Finally (by (51))

\[
7^{th} \text{ term } = 2 \langle \Theta^* (S, T), S (\mu^* (T)) \rangle = 2 \langle \Theta^* (S, T), T (\mu^* (S)) \rangle + \\
+ 2 \langle \Theta^* (S, T), \omega^* (T), \mu^* (S) \rangle - 2 \langle \Theta^* (S, T), \omega^* (S), \mu^* (T) \rangle + 4 |\Theta^* (S, T)|^2
\]

or (by (50) and the fact that \(\omega^* (S)\) is skew)

\[
7^{th} \text{ term } = 2 \langle \Theta^* (S, T), T (\mu^* (S)) \rangle + \\
+ 2 \langle \omega^* (S), \Theta^* (S, T), \mu^* (T) \rangle + 4 |\Theta^* (S, T)|^2
\]

at each \((t, 0) \in Q\). Summing up the various expressions and noting that (again by (57))

\[
T (\langle S (\mu^* (S)), \mu^* (T) \rangle) + \langle \omega^* (S) \cdot T (\mu^* (S)), \mu^* (T) \rangle + \\
+ \langle T (\omega^* (S)), \mu^* (S), \mu^* (T) \rangle = \\
= T (\langle S (\mu^* (S)) + \omega^* (S), \mu^* (S), \mu^* (T) \rangle)
\]
we obtain

\begin{equation}
\frac{1}{2} S^2(F^2) = |\mathcal{T}(\mu^*(S))|^2 + 2 \langle \Omega^*(S, T) \cdot \mu^*(S), \mu^*(T) \rangle +
\end{equation}

\begin{align*}
&+ \mathcal{T} \left( \langle S(\mu^*(S)) + \omega^*(S) \cdot \mu^*(T) \rangle \right) \mu^*(S) + \\
&+ 4 \langle \Theta^*(S, T), \mathcal{T}(\mu^*(S)) \rangle + 2 \langle S(\Theta^*(S, T), \mu^*(T)) \rangle + \\
&+ 2 \langle \omega^*(S) \cdot \Theta^*(S, T), \mu^*(T) \rangle + 4 |\Theta^*(S, T)|^2
\end{align*}

at each \((t, 0) \in Q\). Since \(FS^2(F) = \frac{1}{2}S^2(F^2) - S(F)^2\) we get (by \(65\) and \(66\))

\begin{equation}
rS^2(F) = |\mathcal{T}(\mu^*(S))|^2 + 2 \langle \Omega^*(S, T) \cdot \mu^*(S), \mu^*(T) \rangle +
\end{equation}

\begin{align*}
&+ \mathcal{T} \left( \langle S(\mu^*(S)) + \omega^*(S) \cdot \mu^*(T) \rangle \right) \mu^*(S) + \\
&+ 4 \langle \Theta^*(S, T), \mathcal{T}(\mu^*(S)) \rangle + 2 \langle S(\Theta^*(S, T), \mu^*(T)) \rangle + \\
&+ 2 \langle \omega^*(S) \cdot \Theta^*(S, T), \mu^*(T) \rangle + 4 |\Theta^*(S, T)|^2 - \\
&- \frac{1}{r^2} \{ \mathcal{T}(\mu^*(S)), \mu^*(T) \}^2 + 4 \langle \Theta^*(S, T), \mu^*(T) \rangle + \\
&+ 4 \langle \mathcal{T}(\mu^*(S)), \mu^*(T) \rangle \langle \Theta^*(S, T), \mu^*(T) \rangle
\end{align*}

at any \((t, 0) \in Q\). Moreover (by \(65\))

\[ \mathcal{T}(\mu^*(S))(t, 0) = \frac{d}{dt} \left\{ \mu^*(S) \circ \alpha^0 \right\}(t) = \]

\[ = \lim_{h \to 0} \frac{1}{h} \left\{ \mu^*(S)(t+h, 0) - \mu^*(S)(t, 0) \right\} = \]

\[ = \lim_{h \to 0} \frac{1}{h} \left\{ f^0(t+h)^{-1} X_{\gamma(t+h)} - f^0(t)^{-1} X_{\gamma(t)} \right\} = \]

(as \(f^0 : [a, b] \to O(M, g_\theta)\) is a horizontal curve)

\[ = \lim_{h \to 0} \frac{1}{h} \left\{ f^0(t)^{-1} \tau_t^{t+h} X_{\gamma(t+h)} - f^0(t)^{-1} X_{\gamma(t)} \right\} = \]

\[ = f^0(t)^{-1} \left( \lim_{h \to 0} \frac{1}{h} \left\{ \tau_t^{t+h} X_{\gamma(t+h)} - X_{\gamma(t)} \right\} \right) \]

that is

\begin{equation}
\mathcal{T}(\mu^*(S))(t, 0) = f^0(t)^{-1}(\nabla_{\dot{\gamma}} X)_{\gamma(t)} .
\end{equation}

Consequently (by \(65\) and \(68\))

\begin{equation}
|\mathcal{T}(\mu^*(S))|^2 + 4 \langle \Theta^*(S, T), \mathcal{T}(\mu^*(S)) \rangle -
\end{equation}

\begin{align*}
&- \frac{1}{r^2} \{ \mathcal{T}(\mu^*(S)), \mu^*(T) \}^2 + 4 \langle \Theta^*(S, T), \mu^*(T) \rangle^2 + \\
&+ 4 \langle \mathcal{T}(\mu^*(S)), \mu^*(T) \rangle \langle \Theta^*(S, T), \mu^*(T) \rangle = \\
&= |\nabla_{\dot{\gamma}} X|^2 + 2 g_\theta(T_{\nabla}(X, \dot{\gamma}), \nabla_{\dot{\gamma}} X) - \frac{1}{r^2} \left\{ g_\theta(\nabla_{\dot{\gamma}} X, \dot{\gamma})^2 + \\
&+ g_\theta(T_{\nabla}(X, \dot{\gamma}), \dot{\gamma})^2 + 2 g_\theta(\nabla_{\dot{\gamma}} X, \dot{\gamma}) g_\theta(T_{\nabla}(X, \dot{\gamma}), \dot{\gamma}) \right\}
\end{align*}
\[ = |\nabla_\gamma X^\perp|^2 + 2g_\theta(T_{\nabla}(X^\perp, \dot{\gamma}), \nabla_\gamma X) - \frac{1}{r^2}\{g_\theta(T_{\nabla}(X^\perp, \dot{\gamma}), \dot{\gamma})^2 + 2g_\theta(\nabla_\gamma X, \dot{\gamma})g_\theta(T_{\nabla}(X^\perp, \dot{\gamma}), \dot{\gamma})\} \]

and (by (53) and (56))

\[ (70) \quad \text{the curvature term } = 2 \langle \Omega^*(S, \mathbb{T}) \cdot \mu^*(S), \mu^*(\mathbb{T}) \rangle = g_\theta(R(X, \dot{\gamma})X, \dot{\gamma})_{\gamma(t)} = -g_\theta(R(X^\perp, \dot{\gamma})\dot{\gamma}, X^\perp). \]

On the other hand \( \pi(f(a, s)) = y \) and \( \pi(f(b, s)) = z \) imply that \((d_{(a, s)}f)S_{(a, s)}\) and \((d_{(b, s)}f)S_{(b, s)}\) are vertical hence

\[ (71) \quad \mu^*(S)_{(a, s)} = 0, \quad \mu^*(S)_{(b, s)} = 0. \]

Next, we wish to compute \( S(\mu^*(S))_{(t, 0)} \). To do so we need to further specialize the choice of \( f(t, s) \). Precisely, let \( v \in \pi^{-1}(\gamma(a)) \) be a fixed orthonormal frame and let

\[ f(t, s) = \sigma_t^+(s), \quad a \leq t \leq b, \quad |s| < \epsilon, \]

where \( \sigma_t^+ : (-\epsilon, \epsilon) \to O(M, g_\theta) \) is the unique horizontal lift of \( \sigma_t : (-\epsilon, \epsilon) \to M \) issuing at \( \sigma_t(0) = \gamma^+(t) \). Also \( \gamma^+ : [a, b] \to O(M, g_\theta) \) is the horizontal lift of \( \gamma : [a, b] \to M \) determined by \( \gamma^+(a) = v \). Therefore \( f^0 = \gamma^+ \) is a horizontal curve, as required by the previous part of the proof. In addition \( \gamma^+ \) possesses the property that for each \( t \) the curve \( s \mapsto f(t, s) \) is horizontal, as well. Then

\[ S(\mu^*(S))_{(t, 0)} = \frac{d}{ds}\{\mu^*(S) \circ \beta_t\}(0) = \]

\[ \frac{1}{s \to 0} \frac{1}{s} \{f(t, s)^{-1}\dot{\sigma}_t(s) - f(t, 0)^{-1}\dot{\sigma}_t(0)\} = \]

(as \( f_t : (-\epsilon, \epsilon) \to O(M, g_\theta), f_t(s) = f(t, s), |s| < \epsilon, \) is horizontal)

\[ \frac{1}{s \to 0} \frac{1}{s} \{f(t, 0)^{-1}\tau^s\dot{\sigma}_t(s) - f(t, 0)^{-1}\dot{\sigma}_t(0)\} = \]

\[ = f(t, 0)^{-1}\left(\lim_{s \to 0} \frac{1}{s} \{\tau^s\dot{\sigma}_t(s) - \dot{\sigma}_t(0)\}\right) \]

where \( \tau^s : T_{\sigma_t(s)}(M) \to T_{\sigma_t(0)}(M) \) is the parallel displacement along \( \sigma_t \) from \( \sigma_t(s) \) to \( \sigma_t(0) \), i.e.

\[ (73) \quad S(\mu^*(S))_{(t, 0)} = f(t, 0)^{-1}(\nabla_{\dot{\sigma}_t}\dot{\sigma}_t)_{\gamma(t)}. \]

By (73), \( \dot{\sigma}_a(s) = 0 \) and \( \dot{\sigma}_b(s) = 0 \) (as \( \sigma_a(s) = \gamma^a(s) = y = \text{const.} \) and \( \sigma_b(s) = \gamma^b(s) = z = \text{const.} \)) it follows that

\[ (74) \quad S(\mu^*(S))_{(a, 0)} = 0, \quad S(\mu^*(S))_{(b, 0)} = 0. \]
Using (71) and (74) we may conclude that

\[
\int_a^b T \left( \langle S(\mu^*(S)) + \omega^*(S) \cdot \mu^*(S), \mu^*(T) \rangle \right)_{(t,0)} dt =
\]

\[
= \langle S(\mu^*(S)) + \omega^*(S) \cdot \mu^*(S), \mu^*(T) \rangle_{(b,0)} - \langle S(\mu^*(S)) + \omega^*(S) \cdot \mu^*(S), \mu^*(T) \rangle_{(a,0)} = 0.
\]

Similarly

\[
2 S(\Theta^*(S, T))_{(t,0)} = 2 \lim_{s \to 0} \frac{1}{s} \{ \Theta^*(S, T)_{(t,s)} - \Theta^*(S, T)_{(t,0)} \} =
\]

\[
= \lim_{s \to 0} \frac{1}{s} \{ f(t, s)^{-1} T_{\nabla}(\hat{\sigma}(s), \hat{\gamma}(t)) - f(t, 0)^{-1} T_{\nabla}(\hat{\sigma}(0), \hat{\gamma}(0)) \} =
\]

\[
f(t, 0)^{-1} \left( \lim_{s \to 0} \frac{1}{s} \{ \tau^* V_{\sigma(s)} - V_{\sigma(0)} \} \right)
\]

where \( V \) is the vector field defined at each \( a(t, s) = \sigma_t(s) \) by

\[
V_{a(t,s)} = T_{\nabla,a(t,s)}(\hat{\sigma}(s), \hat{\gamma}(s)), \quad \text{as } a \leq t \leq b, \quad |s| < \epsilon.
\]

Let us assume from now on that \( \tau = 0 \), i.e. \((M, \theta)\) is Sasakian. Then

\[
V_{a(t,s)} = -2\Omega_{a(t,s)}(\hat{\sigma}(s), \hat{\gamma}(s)) T_{a(t,s)}
\]

and \( \nabla T = 0 \) yields

\[
\tau^* V_{a(t,s)} = -2\Omega_{a(t,s)}(\hat{\sigma}(s), \hat{\gamma}(s)) T_{\gamma(t)}.
\]

Finally

\[
2 \langle S(\Theta^*(S, T)), \mu^*(T) \rangle_{(t,0)} =
\]

\[
= \lim_{s \to 0} \frac{1}{s} \{ g_{\theta,a(t,s)}(\tau^* V_{a(t,s)}, \hat{\gamma}(t)) - g_{\theta,a(t,0)}(V_{a(t,0)}, \hat{\gamma}(t)) \} = 0,
\]

as \( \hat{\gamma}(t) \in H(M)_{\gamma(t)} \). It remains that we compute the term \( 2 \langle \omega^*(S) \cdot \Theta^*(S, T), \mu^*(T) \rangle \). As \( f(t, s) \) is a linear frame at \( a(t, s) \)

\[
f(t, s) = (a(t, s), \{ X_{i,a(t,s)} : 1 \leq i \leq 2n + 1 \}),
\]

where \( X_i \in T_{a(t,s)}(M) \). Let \((U, x^i)\) be a local coordinate system on \( M \) and let us set \( X_i = x_i^2 \partial/\partial x^j \). Let \( (\Pi^{-1}(U), \tilde{x}, g^i_j) \) be the naturally induced local coordinates on \( L(M) \), where \( \Pi : L(M) \to M \) is the projection. Then \( g^i_j(f(t, s)) = X_i^j(a(t, s)) \). As \( \omega \) is the connection 1-form of a linear connection

\[
\omega = \omega_i^j \otimes E_j^i.
\]
where $\omega^i_j$ are scalar 1-forms on $L(M)$ and \{${E^i_j : 1 \leq i, j \leq 2n + 1}$\} is the basis of the Lie algebra $\mathfrak{gl}(2n + 1)$ given by $E^i_j = [\delta^i_k \delta^j_\ell]_{1 \leq k, \ell \leq 2n + 1}$. Let \{${e_1, \cdots, e_{2n+1}}$\} be the canonical linear basis of $\mathbb{R}^{2n+1}$. Then

\[
\mu^*(\mathbb{T})_{(t,0)} = f(t,0)^{-1}\dot{\gamma}(t) = \frac{dx_i}{dt}f(t,0)^{-1}\frac{\partial}{\partial x^i} \bigg|_{\gamma(t)} = \frac{dx_i}{dt}Y^j_i e_j
\]

where $[Y^j_i] = [X^j_i]^{-1}$. Therefore

\[
\omega^*(\mathfrak{S})_{(t,0)} : \mu^*(\mathbb{T})_{(t,0)} = \frac{dx^k}{dt} Y^i_k (f^*\omega^i_j)(\mathfrak{S})_{(t,0)} e_j
\]

(because of $E^i_j e_k = \delta^i_k e_j$). On the other hand (by Prop. 1.1 in [18], Vol. I, p. 64) $\omega^*(\mathfrak{S})_{(t,0)} = A$ where the left invariant vector field $A \in \mathfrak{gl}(2n+1)$ is given by

\[
A^i_j(t,0) = (d_{(t,0)}f)\mathfrak{S}_{(t,0)} - \ell_{f(t,0)}\dot{\gamma}(0)
\]

and $\ell_u : T_{H_u}(M) \rightarrow H_u$ is the inverse of $d_u \Pi : H_u \rightarrow T_{H_u}(M)$, $u \in L(M)$ (the horizontal lift operator with respect to $H$). Here $A^*$ is the fundamental vector field associated to $A$, i.e.

\[
A^i_j(t,0) = (d_{e}L_{f(t,0)})A^i_j
\]

where $L_u : GL(2n + 1) \rightarrow L(M), u \in L(M)$, is given by $L_u(g) = ug$ for any $g \in GL(2n + 1)$, and $e \in GL(2n + 1)$ is the unit matrix. If $A = A^i_jE^i_j$ then $A^i_j = (f^*\omega^i_j)(\mathfrak{S})_{(t,0)}$. Let $(g^i_j)$ be the natural coordinates on $GL(2n + 1)$ so that $L_{f(t,0)}$ is locally given by

\[
L^i(g) = \ddot{x}^i, \quad L^i_j(g) = X^i_k g^k_j,
\]

and then $(d_{e}L_{f(t,0)})(\partial/\partial g^i_j) = X^i_k(\partial/\partial g^j_k)_{f(t,0)}$. Next (cf. [18], Vol. I, p. 143)

\[
\ell \frac{\partial}{\partial x^i} = \partial_j - (\Gamma^i_jk \circ \Pi)g^k_\ell \frac{\partial}{\partial g^i_\ell}
\]

(where $\partial_i = \partial/\partial \ddot{x}^i$) and [17] lead to

\[
A^k_\ell X^i_k \frac{\partial}{\partial g^i_\ell}_{f(t,0)} = (d_{(t,0)}f)\mathfrak{S}_{(t,0)} - X^j(\gamma(t))\{\partial_j - (\Gamma^i_jk \circ \Pi)g^k_\ell \frac{\partial}{\partial g^i_\ell}\}_{f(t,0)}
\]

or (by applying this identity to the coordinate functions $g^i_\ell$)

\[
A^k_\ell X^i_k = \mathfrak{S}_{(t,0)}(g^i_\ell \circ f) + X^j(\gamma(t))\Gamma^i_jk(\gamma(t))X^k_\ell.
\]

If $f^i_j = g^i_j \circ f$ then

\[
\mathfrak{S}_{(t,0)}(f^i_j) = \frac{d}{ds}\{f^i_j \circ \beta_t\}(0) = \frac{\partial f^i_j}{\partial s}(t,0)
\]
Therefore (by (78))

\begin{equation}
A_k^i = Y_k^i \left\{ \frac{\partial f^i}{\partial s}(t, 0) + X^j(\gamma(t))\Gamma^i_{jm}(\gamma(t))X^m_t \right\}.
\end{equation}

So far we got (by (79))

\begin{equation}
\omega^*(S)_{(t, 0)} \cdot \mu^*(T)_{(t, 0)} = Y_k^i \frac{dx^k}{dt} \left\{ \frac{\partial f^i}{\partial s}(t, 0) + X^j(\gamma(t))\Gamma^i_{jm}(\gamma(t))X^m_t \right\} f(t, 0)^{-1} \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)}.
\end{equation}

Let us observe that

\[ \frac{\partial f^i}{\partial s}(t, 0) = \frac{\partial X^j}{\partial x^k}(\gamma(t)) \frac{\partial a^k}{\partial s}(t, 0) = \frac{\partial X^i}{\partial x^k}(\gamma(t))X^k(\gamma(t)) \]

hence

\[ \frac{\partial f^i}{\partial s}(t, 0) + X^k(\gamma(t))\Gamma^i_{kt}(\gamma(t))X^t_k = (\nabla X X_j)_{\gamma(t)}^i \]

and we may conclude that

\begin{equation}
\omega^*(S)_{(t, 0)} \cdot \mu^*(T)_{(t, 0)} = Y_k^j \frac{dx^k}{dt} f(t, 0)^{-1}(\nabla X X_j)_{\gamma(t)} = 0.
\end{equation}

Indeed

\[ (\nabla X X_i)_{\gamma(t)} = (\nabla_{\sigma_i} X_i)_{\sigma_i(0)} = \lim_{s \to 0} \frac{1}{s} \left\{ \tau^s X_i_{\sigma_i(s)} - X_{i, \sigma_i(0)} \right\} = \]

\[ = \lim_{s \to 0} \frac{1}{s} \left\{ \tau^s f(t, s)e_i - f(t, 0)e_i \right\} = 0 \]

because \( f_t \) is horizontal (yielding \( \tau^s f(t, s) = f(t, 0) \)). By [69], [70], [75], [76], and [77] the identity (67) may be written

\[ \frac{d^2}{ds^2} \left\{ L(\gamma') \right\}_{s=0} = \frac{1}{r} \int_a^b \left\{ |\nabla_\gamma X^+|^2 - g_\theta(R(X^+, \gamma)\gamma, X^+) + 2g_\theta(TV(X^+, \gamma), \nabla_\gamma X) + |TV(X^+, \gamma)|^2 - \frac{1}{r^2} [g_\theta(TV(X^+, \gamma), \gamma)^2 + 2g_\theta(\nabla_\gamma X, \gamma)g_\theta(TV(X^+, \gamma), \gamma)] \right\} dt \]

or (by \( TV(X^+, \gamma) = -2\Omega(X^+, \gamma)T \) and \( \theta(\gamma) = 0 \))

\begin{equation}
I(X, X) = \frac{1}{r} \int_a^b \left\{ |\nabla_\gamma X^+|^2 - g_\theta(R(X^+, \gamma)\gamma, X^+) + 4\Omega(X^+, \gamma)^2 - 4\Omega(X^+, \gamma)\theta(\nabla_\gamma X) \right\} dt.
\end{equation}

Finally, by polarization \( I(X, Y) = \frac{1}{2} \left\{ I(X + Y, X + Y) - I(X, X) - I(Y, Y) \right\} \) the identity (81) leads to (81).
Proof of Theorem 14: As $\nabla g_\theta = 0$

$$\int_{t_j}^{t_{j+1}} \{g_\theta(\nabla_\gamma X^\perp, \nabla_\gamma Y^\perp) - 2\Omega(X^\perp, \dot{\gamma})\theta(\nabla_\gamma Y^\perp)\} dt =$$

$$= \int_{t_j}^{t_{j+1}} \left\{ \frac{d}{dt}[g_\theta(\nabla_\gamma X^\perp, Y^\perp) - 2\Omega(X^\perp, \dot{\gamma})\theta(Y^\perp)] - g_\theta(\nabla^2_\gamma X - 2\Omega(\nabla_\gamma X^\perp, \dot{\gamma})T, Y^\perp) \right\} dt =$$

$$= g_{\theta, \gamma(t_{j+1})}(\nabla_\gamma X^\perp)_{\gamma(t_{j+1})} Y^\perp_{\gamma(t_{j+1})} - g_{\theta, \gamma(t_j)}(\nabla_\gamma X^\perp)_{\gamma(t_j)} Y^\perp_{\gamma(t_j)} -$$

$$-2\Omega(X^\perp, \dot{\gamma})_{\gamma(t_{j+1})}\theta(Y^\perp)_{\gamma(t_{j+1})} + 2\Omega(X^\perp, \dot{\gamma})_{\gamma(t_j)}\theta(Y^\perp)_{\gamma(t_j)} -$$

$$- \int_{t_j}^{t_{j+1}} \{g_\theta(\nabla^2_\gamma X^\perp - 2\Omega(\nabla_\gamma X^\perp, \dot{\gamma})T, Y^\perp)\} dt$$

and (61) implies (62). Q.e.d.

Proof of Corollary 5: If $X^\perp \in J_\gamma$ then $X^\perp$ is differentiable in $[a, b]$ hence the last term in (62) vanishes. Also $\mathcal{J}_\gamma X^\perp = 0$ and (62) yield

$$I(X, Y) = -\frac{2r}{r} \int_a^b \{\theta(\nabla_\gamma X^\perp) - 2\Omega(X^\perp, \dot{\gamma})\} \Omega(Y^\perp, \dot{\gamma}) dt$$

which implies (by Lemma 4) both (63) and (64). Viceversa, let us assume that (63) holds for some $\alpha(X) \in \mathbb{R}$. Let $f$ be a smooth function on $M$ such that $f(\gamma(t_j)) = 0$ for any $0 \leq j \leq h$ and $f(\gamma(t)) > 0$ for any $t \in [a, b] \setminus \{t_0, t_1, \ldots, t_h\}$ and let us consider the vector field $Y = f \mathcal{J}_\gamma X^\perp$. As $\mathcal{J}_\gamma X^\perp$ is orthogonal to $\dot{\gamma}$ the identity (63) implies

$$\int_a^b f(\gamma(t))|\mathcal{J}_\gamma X^\perp|^2 dt = 0$$

hence $\mathcal{J}_\gamma X^\perp = 0$ in each interval $[t_j, t_{j+1}]$. To prove that $X^\perp \in J_\gamma$ it suffices (by Prop. 1.1 in [18], Vol. II, p. 63) to check that $X^\perp$ is of class $C^1$ at each $t_j$. To this end, for each fixed $j$ we consider a vector field $Y$ along $\gamma$ such that

$$Y_{\gamma(t)} = \begin{cases} \frac{1}{2}(\nabla_\gamma X^\perp)^{-}_\gamma(t) - (\nabla_\gamma X^\perp)^{+}_\gamma(t), & \text{for } t = t_j \\ 0, & \text{for } t = t_k, \; k \neq j. \end{cases}$$

Then (by (64)) $|\nabla_\gamma X^\perp|_{\gamma(t_j)} - (\nabla_\gamma X^\perp)_{\gamma(t_j)}^\perp|^2 = 0$. Q.e.d.

Remark: Let $\gamma(t) \in M$ be a lengthy $C^1$ curve. Then $D_\gamma \dot{\gamma} = \nabla_\gamma \dot{\gamma} - A(\gamma, \dot{\gamma})T$ hence on a Sasakian manifold $\gamma$ is a geodesic of $\nabla$ if and only if $\gamma$ is a geodesic of the Riemannian manifold $(M, g_\theta)$. This observation leads to the following alternative proof of Theorem 11. Let
\( \gamma \in \Gamma \) and \( X, Y \in T_\gamma (\Gamma) \) be as in Theorem 11. By Theorem 5.4 in [18], Vol. II, p. 81, we have

\[ (82) \quad I(X, Y) = \frac{1}{r} \int_a^b \{ g_\theta (D_\gamma X_\perp , D_\gamma Y_\perp ) - g_\theta (R^D (X_\perp , \dot{\gamma}) \dot{\gamma}, Y_\perp ) \} dt. \]

Now on one hand

\[ (83) \quad D_\gamma X_\perp = \nabla_\gamma X_\perp + \Omega (\dot{\gamma}, X_\perp ) T + \theta (X_\perp ) J \dot{\gamma} \]

and on the other the identity

\[ R^D (X, Y) Z = R(X, Y) Z + (JX \wedge JY) Z - 2\Omega (X, Y) J Z + 2g_\theta ((\theta \wedge I)(X, Y), Z) T - 2\theta (Z) (\theta \wedge I)(X, Y), \quad X, Y, Z \in \mathcal{X}(M), \]

yields

\[ (84) \quad R^D (X_\perp , \dot{\gamma}) \dot{\gamma} = R(X_\perp , \dot{\gamma}) \dot{\gamma} - 3\Omega (X_\perp , \dot{\gamma}) J \dot{\gamma} + r^2 \theta (X_\perp ) T. \]

Let us substitute from (83)-(84) into (82) and use the identity

\[ \theta(X_\perp ) \Omega (\nabla_\gamma Y_\perp , \dot{\gamma}) + \theta(Y_\perp ) \Omega (\nabla_\gamma X_\perp , \dot{\gamma}) + \Omega (\dot{\gamma}, X_\perp ) \theta (\nabla_\gamma Y_\perp ) + \Omega (\dot{\gamma}, Y_\perp ) \theta (\nabla_\gamma X_\perp ) = \]

\[ = \frac{d}{dt} \{ \theta(X_\perp ) \Omega (Y_\perp , \dot{\gamma}) + \theta(Y_\perp ) \Omega (X_\perp , \dot{\gamma}) \} - 2\{ \Omega (X_\perp , \dot{\gamma}) \theta (\nabla_\gamma Y_\perp ) + \Omega (Y_\perp , \dot{\gamma}) \theta (\nabla_\gamma X_\perp ) \} \]

(together with \( X_\gamma (a) = X_\gamma (b) = 0 \)) so that to derive (61). Q.e.d.

As an application of Theorems 8 and 11 we shall establish Theorem 13.

**Theorem 13.** Let \((M, \theta)\) be a Sasakian manifold of CR dimension \(n\) and \(\nabla\) its Tanaka-Webster connection. Let \(\gamma: [a, b] \rightarrow M\) be a lengthy geodesic of \(\nabla\), parametrized by arc length. If there is \(c \in (a, b)\) such that the point \(\gamma(c)\) is horizontally conjugate to \(\gamma(a)\) and for any \(\delta > 0\) with \([c - \delta, c + \delta] \subset (a, b)\) the space \(\mathcal{H}_{\gamma_\delta}\) has maximal dimension \(4n\) (where \(\gamma_\delta\) is the geodesic \(\gamma: [c - \delta, c + \delta] \rightarrow M\)) then \(\gamma\) is not a minimizing geodesic joining \(\gamma(a)\) and \(\gamma(b)\), that is the length of \(\gamma\) is greater than the Riemannian distance (associated to \((M, g_\theta)\)) between \(\gamma(a)\) and \(\gamma(b)\).

**Proof.** Let \(\gamma: [a, b] \rightarrow M\) be a geodesic of the Tanaka-Webster connection of the Sasakian manifold \((M, \theta)\), obeying to the assumptions in Theorem 13. Then (by Theorem 8) there is a piecewise differentiable vector field \(X\) along \(\gamma\) such that 1) \(X\) is orthogonal to \(\dot{\gamma}\) and
\[ J\dot{\gamma}, \ 2) \ X_{\gamma(a)} = X_{\gamma(b)} = 0, \ \text{and} \ 3) \ I^b_a(X) < 0. \] Let \( \{\gamma^s\}_{|s| < \epsilon} \) be a 1-parameter family of curves as in the definition of \((L, L)X\) and \(I(X, X)\). By Corollary \(5\) (as \( \gamma \) is a geodesic of \( \nabla \)) one has

\[
\frac{d}{ds} \{L(\gamma^s)\}_{s=0} = 0.
\]

On the other hand (by Theorem \(11\) and \(X^\perp = X\))

\[
I(X, X) = I^b_a(X) + 4 \int_a^b \Omega(X, \dot{\gamma})\{\Omega(X, \dot{\gamma}) - \theta(X')\}dt
\]

hence (as \( X \) is orthogonal to \( J\dot{\gamma} \))

\[
\frac{d^2}{ds^2} \{L(\gamma^s)\}_{s=0} = I^b_a(X) < 0
\]

so that there is \( 0 < \delta < \epsilon \) such that \( L(\gamma^s) < L(\gamma) \) for any \( |s| < \delta \).

**Remark.** If there is a 1-parameter variation of \( \gamma \) (inducing \( X \)) by lengthy curves then \( L(\gamma) \) is greater than the Carnot-Carathéodory distance between \( \gamma(a) \) and \( \gamma(b) \).

9. **Final comments and open problems**

Manifest in R. Strichartz’s paper (cf. \[24\]) is the absence of covariant derivatives and curvature. Motivated by our Theorem \(11\) we started developing a theory of geodesics of the Tanaka-Webster connection \( \nabla \) on a Sasakian manifold \( M \), with the hope that although lengthy geodesics of \( \nabla \) form (according to Corollary \(11\)) a smaller family than that of sub-Riemannian geodesics, the former may suffice for establishing an analog to Theorem 7.1 in \[24\], under the assumption that \( \nabla \) is complete (as a linear connection on \( M \)). The advantage of working within the theory of linear connections is already quite obvious (e.g. any \( C^1 \) geodesic of \( \nabla \) is automatically of class \( C^\infty \), as an integral curve of some \( C^\infty \) basic vector field, while sub-Riemannian geodesics are assumed to be of class \( C^2 \), cf. \[24\], p. 233, and no further regularity is to be expected \( a \ priori \) and doesn’t contradict R. Strichartz’s observation that sub-Riemannian manifolds, and in particular strictly pseudoconvex CR manifolds endowed with a contact form \( \theta \), exhibit no approximate Euclidean behavior (cf. \[24\], p. 223). Indeed, while Riemannian curvature measures the higher order deviation of the given Riemannian manifold from the Euclidean model, the curvature of the Tanaka-Webster connection describes the pseudoconvexity properties of the given CR manifold, as understood in several complex variables analysis. The role as a possible model space played by the tangent cone
of the metric space \((M, \rho)\) at a point \(x \in M\) (such as produced by J. Mitchell’s Theorem 1 in [22], p. 36) is unclear.

Another advantage of our approach stems from the fact that the exponential map on \(M\) thought of as a sub-Riemannian manifold is never a diffeomorphism at the origin (because all sub-Riemannian geodesics issuing at \(x \in M\) must have tangent vectors in \(H(M)_x\)) in contrast with the ordinary exponential map associated to the Tanaka-Webster connection \(\nabla\). In particular cut points (as introduced in [24], p. 260) do not possess the properties enjoyed by conjugate points in Riemannian geometry because (by Theorem 11.3 in [24], p. 260) given \(x \in M\) cut points occur arbitrary close to \(x\). On the contrary (by Theorem 1.4 in [18], Vol. II, p. 67) given \(x \in M\) one may speak about the first point conjugate to \(x\) along a geodesic of \(\nabla\) emanating from \(x\), therefore the concept of conjugate locus \(C(x)\) may be defined in the usual way (cf. e.g. [21], p. 117). The systematic study of the properties of \(C(x)\) on a strictly pseudoconvex CR manifold is an open problem.

Yet another concept of exponential map was introduced by D. Jerison & J. M. Lee, [16] (associated to parabolic geodesics i.e. solutions \(\gamma(t)\) to \((\nabla \dot{\gamma})_{\gamma(t)} = 2cT_{\gamma(t)}\) for some \(c \in \mathbb{R}\)). A comparison between the three exponential formalisms (in [24], [16], and the present paper) hasn’t been done as yet. We conjecture that given a 2-plane \(\sigma \subset T_x(M)\) its pseudohermitian sectional curvature \(k_\theta(\sigma)\) measures the difference between the length of a circle in \(\sigma\) (with respect to \(g_{\theta,x}\)) and the length of its image by \(\exp_x\) (the exponential mapping at \(x\) associated to \(\nabla\)). Also a useful relationship among \(\exp_x\) and the exponential mapping associated to the Fefferman metric \(F_\theta\) on \(C(M)\) should exist (and then an understanding of the singular points of the latter, cf. e.g. M. A. Javaloyes & P. Piccione, [15], should shed light on the properties of singular points of the former).

Finally, the analogy between Theorem 7.3 in [24], p. 245 (producing “approximations to unity” on Carnot-Carathéodory complete sub-Riemannian manifolds) and Lemma 2.2 in [25], p. 50 (itself a corrected version of a result by S.-T. Yau, [29]) indicates that Theorem 7.3 is the proper ingredient for proving that the sublaplacian \(\Delta_b\) is essentially self-adjoint on \(C^\infty_0(M)\) and the corresponding heat operator is given by a positive \(C^\infty\) kernel. These matters are relegated to a further paper.
APPENDIX A. CONTACT FORMS OF CONSTANT PSEUDOHERMITIAN SECTIONAL CURVATURE

The scope of this section is to give a proof of Theorem 5. Let \((M, \theta)\) be a nondegenerate CR manifold and \(\theta\) a contact form on \(M\). Let \(\nabla\) be the Tanaka-Webster connection of \((M, \theta)\). We recall the first Bianchi identity

\[
\sum_{XYZ} R(X,Y)Z = \sum_{XYZ} \{T_{\nabla}(T_{\nabla}(X,Y), Z) + (\nabla_X T_{\nabla})(Y,Z)\}
\]

for any \(X,Y,Z \in T(M)\), where \(\sum_{XYZ}\) denotes the cyclic sum over \(X,Y,Z\). Let \(X,Y,Z \in H(M)\) and note that

\[
T_{\nabla}(T_{\nabla}(X,Y), Z) = -2\Omega(X,Y)\tau(Z),
\]

\[
(\nabla_X T_{\nabla})(Y,Z) = -2(\nabla_X \Omega)(Y,Z)T = 0.
\]

Indeed \(\nabla g_\theta = 0\) and \(\nabla J = 0\) yield \(\nabla \Omega = 0\). Thus (85) leads to

\[
\sum_{XYZ} R(X,Y)Z = -2 \sum_{XYZ} \Omega(X,Y)\tau(Z),
\]

for any \(X,Y,Z \in H(M)\). Let us define a \((1,2)\)-tensor field \(S\) by setting

\[
S(X,Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X.
\]

Next, we set \(X,Y \in H(M)\) and \(Z = T\) in (85) and observe that

\[
T_{\nabla}(T_{\nabla}(X,Y), T) + T_{\nabla}(T_{\nabla}(Y,T), X) + T_{\nabla}(T_{\nabla}(T, X), Y) =
\]

\[
= -T_{\nabla}(\tau(Y), X) + T_{\nabla}(\tau(X), Y) = \quad \text{(as \(\tau\) is \(H(M)\) - valued)}
\]

\[
= 2\{\Omega(\tau(Y), X) - \Omega(\tau(X), Y)\}T = 2g_\theta((\tau J + J \tau)X, Y)T = 0,
\]

(by the purity axiom) and

\[
(\nabla_X T_{\nabla})(Y,T) + (\nabla_Y T_{\nabla})(T,X) + (\nabla_T T_{\nabla})(X,Y) =
\]

\[
= -(\nabla_X \tau)Y + (\nabla_Y \tau)X - 2(\nabla_T \Omega)(X,Y)T = -S(X,Y).
\]

Finally (85) becomes

\[
R(X,T)Y + R(T,Y)X = S(X,Y),
\]

for any \(X,Y \in H(M)\). The 4-tensor \(R\) enjoys the properties

\[
R(X,Y,Z,W) = -R(Y,X,Z,W),
\]

\[
R(X,Y,Z,W) = -R(X,Y,W,Z),
\]

for any \(X,Y,Z,W \in T(M)\). Indeed (88) follows from \(\nabla g_\theta = 0\) while (89) is obvious. We may use the reformulation (86)-(87) of the first Bianchi identity to compute \(\sum_{XYZW} R(X,Y,Z,W)\) for arbitrary vector
fields. For any \( X \in T(M) \) we set \( X_H = X - \theta(X)T \) (so that \( X_H \in H(M) \)). Then

\[
\sum_{YZW} R(X, Y, Z, W) = \sum_{YZW} g_\theta(R(Z, W)Y_H, X) = \\
= \sum_{YZW} g_\theta(R(Z_H, W_H)Y_H + \theta(Y)[R(W_H, T)Z_H + R(T, Z_H)W_H], X)
\]
hence

\[
\sum_{YZW} R(X, Y, Z, W) = \\
= - \sum_{YZW} \{2\Omega(Y, Z)A(W, X) + \theta(Y)g_\theta(X, S(Z_H, W_H))\}
\]
for any \( X, Y, Z, W \in T(M) \). Next, we set

\[
K(X, Y, Z, W) = \sum_{YZW} R(X, Y, Z, W)
\]
and compute (by \(88\) - \(89\))

\[
K(X, Y, Z, W) - K(Y, Z, W, X) - K(Z, W, X, Y) + K(W, X, Y, Z) = \\
= 2R(X, Y, Z, W) - 2R(Z, W, X, Y)
\]
hence (by \(90\))

\[
2R(X, Y, Z, W) - 2R(Z, W, X, Y) = \\
= - \sum_{YZW} \{2\Omega(Y, Z)A(X, W) + \theta(Y)g_\theta(X, S(Z_H, W_H))\} + \\
+ \sum_{ZWX} \{2\Omega(Z, W)A(Y, X) + \theta(Z)g_\theta(Y, S(W_H, X_H))\} + \\
+ \sum_{WXY} \{2\Omega(W, X)A(Y, Z) + \theta(W)g_\theta(Z, S(X_H, Y_H))\} - \\
- \sum_{XYZ} \{2\Omega(X, Y)A(Z, W) + \theta(X)g_\theta(W, S(Y_H, Z_H))\}
\]
or

\[
(91) \\
2R(X, Y, Z, W) - 2R(Z, W, X, Y) = \\
= -4\Omega(Y, Z)A(X, W) + 4\Omega(Y, W)A(X, Z) - \\
-4\Omega(X, W)A(Y, Z) + 4\Omega(X, Z)A(Y, W) + \\
+ \theta(X)[g_\theta(Y, S(Z_H, W_H)) + g_\theta(Z, S(Y_H, W_H)) - g_\theta(W, S(Y_H, Z_H))] + \\
+ \theta(Y)[g_\theta(Z, S(W_H, X_H)) - g_\theta(W, S(Z_H, X_H)) - g_\theta(X, S(Z_H, W_H))] + \\
+ \theta(Z)[g_\theta(Y, S(W_H, X_H)) - g_\theta(X, S(W_H, Y_H)) - g_\theta(W, S(X_H, Y_H))] + \\
+ \theta(W)[g_\theta(Y, S(X_H, Z_H)) - g_\theta(X, S(Y_H, Z_H)) + g_\theta(Z, S(X_H, Y_H))].
\]
As $\nabla_X \tau$ is symmetric one has
\[ g_\theta(Y, S(X, Z)) - g_\theta(X, S(Y, Z)) = g_\theta(S(X, Y), Z) \]
for any $X, Y, Z \in H(M)$, so that (91) may be written
\[ R(X, Y, Z, W) = R(Z, W, X, Y) - \]
\[ -2\Omega(Y, Z)A(X, W) + 2\Omega(Y, W)A(X, Z) - \]
\[ -2\Omega(X, W)A(Y, Z) + 2\Omega(X, Z)A(Y, W) + \]
\[ +\theta(X)g_\theta(S(Z_H, W_H), Y) + \theta(Y)g_\theta(S(W_H, Z_H), X) + \]
\[ +\theta(Z)g_\theta(S(Y_H, X_H), W) + \theta(W)g_\theta(S(X_H, Y_H), Z), \]
for any $X, Y, Z, W \in T(M)$.

The properties (88)-(90) and (92) may be used to compute the full
curvature of a manifold of constant pseudohermitian sectional curva-
ture (the arguments are similar to those in the proof of Prop. 1.2 in [18],
Vol. I, p. 198). Assume from now on that $M$ is strictly pseudoconvex
and $G_\theta$ positive definite. Let us set
\[ R_1(X, Y, Z, W) = g_\theta(X, Z)g_\theta(Y, W) - g_\theta(Y, Z)g_\theta(W, X) \]
so that
\[ (93) \quad R_1(X, Y, Z, W) = -R_1(Y, X, Z, W), \]
\[ (94) \quad R_1(X, Y, Z, W) = -R_1(X, Y, W, Z), \]
\[ (95) \quad \sum_{YZW} R_1(X, Y, Z, W) = 0. \]
Assume from now on that $k_\theta = c = \text{const}$. Let us set $L = R - 4cR_1$
and observe that
\[ (96) \quad L(X, Y, X, Y) = 0 \]
for any $X, Y \in T(M)$. Indeed, if $X, Y$ are linearly dependent then (96)
follows from the skew symmetry of $L$ in the pairs $(X, Y)$ and
$(Z, W)$, respectively. If $X, Y$ are independent then let $\sigma \subset T_x(M)$ be
the 2-plane spanned by $\{X_x, Y_x\}$, $x \in M$. Then
\[ L(X, Y, X, Y)_x = R(X, Y, X, Y)_x - 4cR_1(X, Y, X, Y)_x = \]
\[ = 4k_\theta(\sigma)[|X|^2|Y|^2 - g_\theta(X, Y)^2]_x - 4cR_1(X, Y, X, Y)_x = 0. \]
Next (by (96))
\[ 0 = L(X, Y + W, X, Y + W) = L(X, Y, W) + L(X, W, Y) \]
hence (by (92))

Therefore

\[ L(X, Y, Z, W) = R_1(X, Y, Z, W) = R_1(Z, W, X, Y). \]

Therefore \( L(X, Y, Z, W) - L(Z, W, X, Y) = R(X, Y, Z, W) - R(Z, W, X, Y) \) hence (by (99))

\[ L(X, Y, Z, W) = L(Z, W, X, Y) + \]

\[ + 2\Omega(Y, W)A(X, Z) - 2\Omega(Y, Z)A(X, W) + \]

\[ + 2\Omega(X, Z)A(Y, W) - 2\Omega(X, W)A(Y, Z) + \]

\[ + \theta(X)g_\theta(S(Z_H, W_H), Y) + \theta(Y)g_\theta(S(W_H, Z_H), X) + \]

\[ + \theta(Z)g_\theta(S(Y_H, X_H), W) + \theta(W)g_\theta(S(X_H, Y_H), Z). \]

Applying (99) (to interchange the pairs \((X, W)\) and \((X, Y)\)) we get

\[ L(X, W, X, Y) = L(X, Y, X, W) + \]

\[ + 2\Omega(W, Y)A(X, X) - 2\Omega(W, X)A(X, Y) - 2\Omega(X, Y)A(W, X) + \]

\[ + \theta(X)g_\theta(S(W_H, Y_H), X) + \theta(Y)g_\theta(S(X_H, W_H), X) + \]

\[ + \theta(W)g_\theta(S(Y_H, X_H), X) \]

hence (97) may be written

\[ L(X, Y, X, W) = \Omega(W, X)A(X, Y) + \]

\[ + \Omega(X, Y)A(W, X) - \Omega(W, Y)A(X, X) - \]

\[ - \frac{1}{2}\{\theta(X)g_\theta(S(W_H, Y_H), X) + \theta(Y)g_\theta(S(X_H, W_H), X) + \]

\[ + \theta(W)g_\theta(S(Y_H, X_H), X)\}. \]

Consequently

\[ L(X + Z, Y, X + Z, W) = \Omega(W, X + Z)A(X + Z, Y) + \]

\[ + \Omega(X + Z, Y)A(W, X + Z) - \Omega(W, Y)A(X + Z, X + Z) - \]

\[ - \frac{1}{2}g_\theta(X + Z, \theta(X + Z)S(W_H, Y_H) + \]

\[ + \theta(Y)S(X_H + Z_H, W_H) + \theta(W)S(Y_H, X_H + Z_H) \]

or (using (100) to calculate \( L(X, Y, X, W) \) and \( L(Z, Y, Z, W) \))

\[ L(X, Y, Z, W) + L(Z, Y, X, W) = \Omega(X, Y)A(W, Z) + \]

\[ + \Omega(W, X)A(Z, Y) + \Omega(W, Z)A(X, Y) + \]

\[ + \Omega(Z, Y)A(W, X) - 2\Omega(W, Y)A(X, Z) - \]
\[-\frac{1}{2} g_{0}(X, \theta(Z)S(W_{H}, Y_{H}) + \theta(Y)S(Z_{H}, W_{H}) + \theta(W)S(Y_{H}, Z_{H})) -
\]
\[-\frac{1}{2} g_{0}(Z, \theta(X)S(W_{H}, Y_{H}) + \theta(Y)S(X_{H}, W_{H}) + \theta(W)S(Y_{H}, X_{H})).\]

On the other hand, by the skew symmetry of $L$ in the first pair of arguments and by (99) (used to interchange the pairs $(Y, Z)$ and $(X, W)$)

\[L(Z, Y, X, W) = -L(Y, Z, X, W) = -L(X, W, Y, Z) +
\]
\[+2\Omega(Z, X)A(Y, W) - 2\Omega(Z, W)A(Y, X) +
\]
\[+2\Omega(Y, W)A(Z, X) - 2\Omega(Y, X)A(Z, W) -
\]
\[-\theta(Y)g_{0}(S(X_{H}, W_{H}), Z) - \theta(Z)g_{0}(S(W_{H}, X_{H}), Y) -
\]
\[-\theta(X)g_{0}(S(Z_{H}, Y_{H}), W) - \theta(W)g_{0}(S(Y_{H}, Z_{H}), X)\]

so that (101) becomes

\[
(102) \quad L(X, Y, Z, W) = L(X, W, Y, Z) + 2\Omega(X, Z)A(Y, W) -
\]
\[-\Omega(W, Z)A(X, Y) - \Omega(X, Y)A(Z, W) +
\]
\[+\Omega(W, X)A(Z, Y) + \Omega(Z, Y)A(W, X) +
\]
\[+\frac{1}{2} \theta(X)\{g_{0}(S(Z_{H}, Y_{H}), W) + g_{0}(S(Z_{H}, W_{H}), Y)\} -
\]
\[-\frac{1}{2} \theta(Y)\{g_{0}(S(Z_{H}, W_{H}), X) + g_{0}(S(W_{H}, X_{H}), Z)\} +
\]
\[+\frac{1}{2} \theta(Z)\{g_{0}(S(W_{H}, X_{H}), Y) + g_{0}(S(Y_{H}, X_{H}), W)\} -
\]
\[-\frac{1}{2} \theta(W)\{g_{0}(S(Y_{H}, X_{H}), Z) + g_{0}(S(Z_{H}, Y_{H}), X)\}.\]

By cyclic permutation of the variables $Y, Z, W$ in (102) we obtain another identity of the sort

\[L(X, Y, Z, W) = L(X, Z, W, Y) - 2\Omega(X, W)A(Z, Y) +
\]
\[+\Omega(Y, W)A(X, Z) + \Omega(X, Z)A(W, Y) -
\]
\[-\Omega(Y, X)A(W, Z) - \Omega(W, Z)A(Y, X) -
\]
\[-\frac{1}{2} \theta(X)\{g_{0}(S(W_{H}, Z_{H}), Y) + g_{0}(S(W_{H}, Y_{H}), Z)\} +
\]
\[+\frac{1}{2} \theta(Y)\{g_{0}(S(Z_{H}, X_{H}), W) + g_{0}(S(W_{H}, Z_{H}), X)\} -
\]
\[+\frac{1}{2} \theta(Z)\{g_{0}(S(W_{H}, Y_{H}), X) + g_{0}(S(Y_{H}, X_{H}), W)\} +
\]
\[-\frac{1}{2} \theta(W)\{g_{0}(S(Y_{H}, X_{H}), Z) + g_{0}(S(Z_{H}, X_{H}), Y)\}\]
which together with (102) leads to

\[
3L(X, Y, Z, W) = \sum_{YZW} L(X, Y, Z, W) - 2\Omega(W, Z)A(X, Y) + \\
+3\Omega(X, Z)A(Y, W) - 3\Omega(X, W)A(Y, Z) + \\
+\Omega(Z, Y)A(W, X) + \Omega(Y, W)A(X, Z) + \\
+\frac{3}{2} \theta(X)g_\theta(S(Z_H, W_H), Y) - \frac{1}{2} \theta(Y)g_\theta(S(Z_H, W_H), X) + \\
+\frac{1}{2} \theta(Z)\{2g_\theta(S(Y_H, X_H), W) + \\
+g_\theta(S(W_H, X_H), Y) + g_\theta(S(W_H, Y_H), X)\} - \\
-\frac{1}{2} \theta(W)\{2g_\theta(S(Y_H, X_H), Z) + \\
+g_\theta(S(Z_H, Y_H), X) + g_\theta(S(Z_H, X_H), Y)\}
\]

or

\[
L(X, Y, Z, W) = \Omega(Y, W)A(X, Z) - \Omega(Y, Z)A(X, W) + \\
+\Omega(X, Z)A(Y, W) - \Omega(X, W)A(Y, Z) + \\
+\frac{1}{2} \{\theta(X)g_\theta(S(Z_H, W_H), Y) - \theta(Y)g_\theta(S(Z_H, W_H), X) + \\
+\theta(Z)g_\theta(S(Y_H, X_H), W) - \theta(W)g_\theta(S(Y_H, X_H), Z)\}
\]

or

(103) \[R(X, Y, Z, W) = 4c\{g_\theta(X, Z)g_\theta(Y, W) - g_\theta(Y, Z)g_\theta(X, W)\} + \\
+\Omega(Y, W)A(X, Z) - \Omega(Y, Z)A(X, W) + \\
+\Omega(X, Z)A(Y, W) - \Omega(X, W)A(Y, Z) + \\
+g_\theta(S(Z_H, W_H), (\theta \land I)(X, Y)) - g_\theta(S(X_H, Y_H), (\theta \land I)(Z, W))
\]

for any \(X, Y, Z, W \in T(M)\), where \(I\) is the identical transformation and \((\theta \land I)(X, Y) = \frac{1}{2}\{\theta(X)Y - \theta(Y)X\}\). Using (103) one may prove Theorem 5 as follows. Let \(Y = T\) in (103). As \(R(Z, W)T = 0\) and \(S\) is \(H(M)\)-valued we get

(104) \[0 = 4c\{g_\theta(X, Z)\theta(W) - g_\theta(X, W)\theta(Z)\} - \frac{1}{2} g_\theta(S(Z_H, W_H), X),
\]

for any \(X, Z, W \in T(M)\). In particular for \(Z, W \in H(M)\)

\[S(Z, W) = 0.
\]

Hence \(S(Z_H, W_H) = 0\) and (104) becomes

\[c\{g_\theta(X, Z)\theta(W) - g_\theta(X, W)\theta(Z)\} = 0,
\]

for any \(X, Z, W \in T(M)\). In particular for \(Z = X \in H(M)\) and \(W = T\) one has \(c|X|^2 = 0\) hence \(c = 0\) and (103) leads to (31). Then \(\tau = 0\) yields \(R = 0\). To prove the last statement in Theorem 5 let us
assume that $M$ has CR dimension $n \geq 2$ (so that the Levi distribution has rank > 3). Assume that $R = 0$ i.e.
\[
\Omega(X, Z)\tau(Y) - \Omega(Y, Z)\tau(X) = A(X, Z)JY - A(Y, Z)JX
\]
(by (11)). In particular for $Z = Y$

\[
\Omega(X, Y)\tau(Y) = A(X, Y)JY - A(Y, Y)JX. \tag{105}
\]
Let $X \in H(M)$ such that $|X| = 1$, $g_\theta(X, Y) = 0$ and $g_\theta(X, JY) = 0$. Taking the inner product of (105) with $JX$ gives $A(Y, Y) = 0$, hence $A = 0$ (as $A$ is symmetric). Q.e.d.

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