SINGULAR LIMIT OF THE GENERALIZED BURGERS EQUATION WITH ABSORPTION

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ABSTRACT. We prove the convergence of the solutions $u_{mp}$ of the equation $u_t + (u^m)_x = -u^p$ in $\mathbb{R} \times (0, \infty)$, $u(x,0) = u_0(x) \geq 0$ in $\mathbb{R}$, as $m \to \infty$ for any $p > 1$ and $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ or as $p \to \infty$ for any $m > 1$ and $u_0 \in L^\infty(\mathbb{R})$. We also show that in general $\lim_{p \to \infty} \lim_{m \to \infty} u_{mp} \neq \lim_{m \to \infty} \lim_{p \to \infty} u_{mp}$.

1. Introduction

Recently there is a lot of studies on the singular limit of solutions of partial differential equations. Singular limit of solutions of the porous medium equation,

$$\left\{ \begin{array}{ll} u_t = \Delta u^m & \text{in } \mathbb{R}^n \times (0, T) \\ u(x,0) = u_0 \geq 0 & \text{in } \mathbb{R}^n \end{array} \right.$$ (1.1)

as $m \to \infty$ is proved by L.A. Caffarelli and A. Friedman in [CF] when $u_0$ satisfies some appropriate conditions. Later P. Bénilan, L. Boccardo and M. Herrero [BBH] and P.E. Sacks [S] extended this result to more general initial value $0 \leq u_0 \in L^1(\mathbb{R}^n)$. Singular limits of the solutions of the porous medium equation with absorption or drift term were proved by K.M. Hui in [H1], [H2] and [H3].

Singular limit as $p \to \infty$ of the solutions of the one dimensional nonlinear wave equation

$$\phi_{tt} - \phi_{xx} = -|\phi|^{p-1}\phi$$ (1.2)

with initial data $\phi(x,0) = \phi_0(x)$, $\phi_t(x,0) = \phi_1(x)$, was proved by T. Tao in [1]. Singular limit of solutions of the hyperbolic equation

$$\left\{ \begin{array}{ll} u_t + (u^m)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = u_0 \geq 0 & \text{in } \mathbb{R} \end{array} \right.$$ (1.3)

as $m \to \infty$ was proved by X. Xu in [X]. Recently B. Perthame, F. Quiros and J.L. Vazquez [PQV] proved the singular limit of solutions of the following system of equations, which arises in the Hele-Shaw models of tumor growth [P], [PTV],

$$\left\{ \begin{array}{l} \rho_t + \text{div} (\rho \nabla p) = \rho \Phi(p, c) \\ c_t - \Delta c = -\rho \Psi(p, c) \\ c(x,t) \to c_B > 0 \text{ as } |x| \to \infty, \end{array} \right.$$ as $m \to \infty$, where $p = k \rho^{m-1}$ for some constant $k > 0$ and $\Phi, \Psi$, are smooth functions that satisfy some structural conditions.

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In this paper we will study the singular limit of solutions $u_{m,p}$ of the generalized Burgers equation with absorption,

$$
\begin{align*}
  u_t + (u^m)_x &= -u^p & \text{in } \mathbb{R} \times (0, \infty) \\
  u(x,0) &= u_0(x) \geq 0 & \text{in } \mathbb{R}
\end{align*}
$$

(1.4)

when either $m \to \infty$ or $p \to \infty$. We will prove that under some mild conditions on the initial data $u_0$, as $m \to \infty$ or $p \to \infty$, the singular limit of solutions of (1.4) exists.

More precisely we will prove the following three results.

**Theorem 1.1.** Let $0 \leq u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For any $p > 1$, $m > 1$, let $u_{m,p}$ be the solution of (1.4) in $\mathbb{R} \times (0, \infty)$ given by Lemma [L.4]. Then as $m \to \infty$, $u_{m,p}$ converges in $C([t_0, T]; L^1_{\text{loc}}(\mathbb{R}))$ for any $T > t_0 > 0$ to some function $u_{\infty,p}$, $0 \leq u_{\infty,p} \leq 1$, which satisfies

$$
  u_t = -u^p \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, \infty))
$$

(1.5)

with initial value $u_{\infty}^0(x)$, $0 \leq u_{\infty}^0 \leq 1$, that satisfies

$$
  u_{\infty}^0(x) + \psi^x(x) = u_0(x) \quad \text{in } \mathcal{D}'(\mathbb{R})
$$

(1.6)

for some function $0 \leq \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

$$
  \psi(x) = 0 \quad \text{a.e. } x \in \{x \in \mathbb{R} : u_{\infty}^0(x) < 1\}.
$$

(1.7)

**Theorem 1.2.** Let $0 \leq u_0 \in L^\infty(\mathbb{R})$. For any $p > 1$, $m > 1$, let $u_{m,p}$ be the solution of (1.4) in $\mathbb{R} \times (0, \infty)$ given by Lemma [L.4]. Then as $p \to \infty$, $u_{m,p}$ converges in $C([t_0, T]; L^1_{\text{loc}}(\mathbb{R}))$ for any $T > t_0 > 0$ to the solution $u_{m,\infty}$ of the equation,

$$
\begin{align*}
  u_t + (u^m)_x &= 0 & \text{in } \mathbb{R} \times (0, \infty) \\
  u(x,0) &= \min(u_0(x),1) & \text{in } \mathbb{R}.
\end{align*}
$$

(1.8)

**Theorem 1.3.** Let $0 \leq u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For any $p > 1$, $m > 1$, let $u_{m,p}$ be the solution of (1.4) and let $u_{\infty,p}, u_{\infty,\infty}$, be given by Theorem [L.1] and Theorem [L.2] respectively. Then the following holds:

(i) as $m \to \infty$, $u_{m,\infty}$ converges in $L^1_{\text{loc}}(\mathbb{R} \times (0, \infty))$ to some function $v_1$ on $\mathbb{R}$, $0 \leq v_1 \leq 1$, which satisfies

$$
  v_1(x) + \psi_1(x) = \min \{u_0(x), 1\} \quad \text{in } \mathcal{D}'(\mathbb{R})
$$

(1.9)

for some function $0 \leq \psi_1 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\psi_1(x) = 0$ for a.e. $x \in \{x : v_1(x) < 1\}$.

(ii) as $p \to \infty$, $u_{\infty,p}$ converges weakly in $L^1(\mathbb{R} \times (0, \infty))$ to $u_{\infty}^0$.

Note that as a consequence of Theorem [L.3] in general we have

$$
\lim_{p \to \infty} \lim_{m \to \infty} u_{m,p} \neq \lim_{m \to \infty} \lim_{p \to \infty} u_{m,p}.
$$

The plan of the paper is as follows. We will prove Theorem [L.1] and Theorem [L.2] in section two and section three respectively. In section four we will prove Theorem [L.3].

We start with some definitions. We will use the definition of solution in [K] for (1.4). For any $\varphi \in C^1([0, \infty))$, we say that a function $0 \leq u \in L^\infty(\mathbb{R} \times (0, \infty))$ is a solution of

$$
\begin{align*}
  u_t + (u^m)_x &= \varphi(u) & \text{in } \mathbb{R} \times (0, \infty) \\
  u(x,0) &= u_0(x) \geq 0 & \text{in } \mathbb{R}
\end{align*}
$$

(1.10)

if it satisfies the following two conditions:
Let $m > 1$, $p > 1$ and $0 \leq u_0 \in L^\infty(\mathbb{R})$. Then there exist unique solutions $u_{m,p}$, $v_m$, of (1.4) and (1.12), respectively which satisfy

\begin{align*}
0 \leq u_{m,p} \leq v_m \leq \|u_0\|_{L^\infty(\mathbb{R})} \quad \text{in} \quad \mathbb{R} \times (0, \infty).
\end{align*}

If $0 \leq u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, then

\begin{align*}
\int_0^\infty u_{m,p}(x,t) \, dx \leq \int_0^\infty v_m(x,t) \, dx = \int_\mathbb{R} u_0(x) \, dx \quad \forall t > 0.
\end{align*}

**Lemma 1.5.** Let $m > 1$, $p > 1$, $0 \leq u_0 \in L^\infty(\mathbb{R})$, and $u_{m,p}$ be the unique solution of (1.4) in $\mathbb{R} \times (0, \infty)$. Then for any $R > 1$ and $T > t_0 > 0$ there exists a monotone increasing function $\omega_R \in C([0, \infty))$, $\omega_R(0) = 0$, depending only on $R$, $\|u_0\|_{L^\infty}$, $m$, $\|u_{m,p}\|_{L^\infty(\mathbb{R} \times (t_0, T])}$ and $p \|u_{m,p}\|_{L^\infty(\mathbb{R} \times (t_0, T])}$ such that

\begin{align*}
\int_{|x| < R} |u_{m,p}(x+t_0, t) - u_{m,p}(x, t)| \, dx \leq \omega_R(|x_0|) \quad \forall |x_0| \leq 1, \ t_0 \leq t \leq T
\end{align*}

and

\begin{align*}
\int_{|x| < R} |u_{m,p}(x, t_1) - u_{m,p}(x, t_2)| \, dx \leq \omega_R(|t_1 - t_2|) \quad \forall t_1, t_2 \in [t_0, T].
\end{align*}

By Theorem 1 of [K] and Lemma 1.4 we have the following result.

**Lemma 1.6.** Let $m > 1$, $p > 1$, and $u_{0,1}, u_{0,2} \in L^\infty(\mathbb{R})$ be non-negative functions on $\mathbb{R}$. Suppose $u_1$, $u_2$, are the solutions of (1.4) in $\mathbb{R} \times (0, \infty)$ with initial value $u_0 = u_{0,1}$, $u_{0,2}$, respectively. Let

\begin{align*}
N = \max \left\{ m \|u_{0,1}\|_{L^\infty(\mathbb{R})}^{m-1}, m \|u_{0,2}\|_{L^\infty(\mathbb{R})}^{m-1} \right\}.
\end{align*}

Then

\begin{align*}
\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^1(B_{R-N})} \leq \|u_{0,1} - u_{0,2}\|_{L^1(B_R)} \quad \forall 0 < t < R/N, R > 0, p > 1.
\end{align*}

We will now assume that $0 \leq u_0 \in L^\infty(\mathbb{R})$ and let $u_{m,p}$, $v_m$, be the solutions of (1.4) and (1.12), respectively for the rest of the paper. For any $x_0 \in \mathbb{R}$ and $R > 0$, we let $B_R(x_0) = \{ x \in \mathbb{R} : |x - x_0| < R \}$ and $B_R = B_R(0)$. 
2. Singular limit as $m \to \infty$

In this section we will prove Theorem 1.1. For fixed $p > 1$, we will write $u_m := u_{m,p}$ for any $m > 1$. We will also assume that $0 \leq u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and let

$$
\psi_m(x,t) = \int_0^t u_m(x,\tau)^m d\tau
$$

in this section. Let $\{m_i\}_{i=1}^\infty \in \mathbb{Z}^+$ be a sequence such that $m_i \to \infty$ as $i \to \infty$. By (1.13) and the result on P. 64 of [X],

$$
0 \leq u_m(x,t)^m \leq u_m(x,t)^m \leq \frac{2\|u_0\|_{L^1(\mathbb{R})}}{(m-1)t}, \quad \text{a.e. } (x,t) \in \mathbb{R} \times (0,\infty).
$$

Then

$$
m(u_m)^{m-1} \leq m\left(\frac{2\|u_0\|_{L^1(\mathbb{R})}}{(m-1)t}\right)^{-m} = \left(m \frac{m}{m-1}\right)(m-1)^\frac{1}{m} \left(\frac{2\|u_0\|_{L^1(\mathbb{R})}}{t}\right)^{m-1}.
$$

Hence for any $t_0 > 0$ there exists a constant $M_{t_0} > 0$ such that

$$
m\|u_m\|_{L^\infty(\mathbb{R} \times (t_0,\infty))} \leq M_{t_0} \quad \forall m \geq 2.
$$

Thus for any $R > 1$ and $T > t_0 > 0$, we can choose the function $\omega_R$ in Lemma 1.5 to be independent of $m \geq 2$. Hence, by (1.13), (1.15) and (1.16), the sequence $\{u_m\}_{i=1}^\infty$ is equi-continuous in $C\left([t_0,T];L^1_{\text{loc}}(\mathbb{R})\right)$ for any $T > t_0 > 0$. Thus by (2.1), the Ascoli theorem and a diagonalization argument the sequence $\{u_m\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges in $C\left([t_0,T];L^1_{\text{loc}}(\mathbb{R})\right)$ for any $T > t_0 > 0$ to some function $u_{\infty,p} \in C\left((0,\infty);L^1_{\text{loc}}(\mathbb{R})\right)$, $0 \leq u_{\infty,p} \leq 1$, as $i \to \infty$. When there is no ambiguity we will drop the subscript $p$ and write $u_\infty$ for $u_{\infty,p}$.

**Lemma 2.1.** $u_\infty$ satisfies (1.5).

**Proof.** By (2.1), $(u_m)^m \to 0$ uniformly on $\mathbb{R} \times [T_0,\infty)$ for any fixed $T_0 > 0$ as $m \to \infty$. Putting $u = u_m$, $\varphi(u) = -(u_m)^p$, $m = m_i$, in (1.11) and letting $i \to \infty$, we get

$$
\int_0^\infty \int_{-\infty}^\infty u_\infty \eta dx dt = \int_0^\infty \int_{-\infty}^\infty (u_\infty)^p \eta dx dt \quad \forall 0 \leq \eta \in C^0_0(\mathbb{R} \times (0,\infty)).
$$

and (1.5) follows. \qed

**Lemma 2.2.** For any $T > 0$ the sequence of functions $\{\psi_m(x,t)\}_{m>0}$ is equi-continuous in $C([0,T];L^1(\mathbb{R}))$.

**Proof.** We will use a modification of the technique of [X] to prove the lemma. We first extend $u_m$ to a function on $\mathbb{R}^2$ by letting $u_m(x, t) = 0$ for all $t < 0$, $x \in \mathbb{R}$. Since $u_m$ satisfies (1.11) with $\varphi(u) = -(u_m)^p$, by (1.11) and an approximation argument,

$$
\int_{-\infty}^\infty \int_{-\infty}^\infty u_m \eta dx dt + \int_{-\infty}^\infty \int_{-\infty}^\infty (u_m)^m \eta dx dt + \int_{-\infty}^\infty u_0 \eta dx = \int_{-\infty}^\infty \int_{-\infty}^\infty (u_m)^p \eta dx dt \quad \forall 0 \leq \eta \in C^0_0(\mathbb{R}^2).
$$

We choose $\Phi \in C^0_0(\mathbb{R}^2)$, $0 \leq \Phi \leq 1$, such that $\int_{\mathbb{R}} \int_{\mathbb{R}} \Phi d\xi d\tau = 1$ and let $J_\varepsilon(x,t) = \frac{\Phi(\xi - x, \tau - t)}{\varepsilon}$ for any $\varepsilon > 0$. Putting $\eta(x,t) = J_\varepsilon(x,t)$ in (2.2),

$$
-(A_m(\xi, \tau) - B_m(\xi, \tau))_\tau + \int_{-\infty}^\infty J_\varepsilon(\xi - x, \tau)u_0(x) dx = C_m(\xi, \tau)
$$

(2.3)
where
\[ A_{m,\epsilon}(\xi, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_m(x, t) f_\epsilon(\xi - x, \tau - t) \, dx \, dt, \]
\[ B_{m,\epsilon}(\xi, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u_m)^m(x, t) f_\epsilon(\xi - x, \tau - t) \, dx \, dt, \]
and
\[ C_{m,\epsilon}(\xi, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u_m)^p(x, t) f_\epsilon(\xi - x, \tau - t) \, dx \, dt. \]

Integrating (2.3) first with respect to \( \xi \) over \((x, x + h), h > 0\), and then with respect to \( \tau \) over \((\sigma, t), t > \sigma > 0\),
\[ \int_{\sigma}^{t} B_{m,\epsilon}(x + h, \tau) \, d\tau - \int_{\sigma}^{t} B_{m,\epsilon}(x, \tau) \, d\tau + \int_{\sigma}^{t} \int_{x}^{x+h} C_{m,\epsilon}(\xi, \tau) \, d\xi \, d\tau \]
\[ = - \int_{x}^{x+h} (A_{m,\epsilon}(\xi, t) - A_{m,\epsilon}(\xi, \sigma)) \, d\xi + \int_{-\infty}^{\infty} \int_{x}^{x+h} \int_{\sigma}^{t} f_\epsilon(\xi - z, \tau) u_0(z) \, d\tau \, d\xi \, dz. \] (2.4)

Similar to the proof on P.63–64 of [X], letting \( \epsilon \to 0 \) in (2.4),
\[ \int_{\sigma}^{t} [u_m(x + h, \tau)^m - u_m(x, \tau)^m] \, d\tau \]
\[ = - \int_{x}^{x+h} (u_m(\xi, t) - u_m(\xi, \sigma)) \, d\xi - \int_{\sigma}^{t} \int_{x}^{x+h} u_m(\xi, \tau)^p \, d\xi \, d\tau \quad \text{a.e. } (x, t) \in \mathbb{R} \times (0, \infty). \] (2.5)

Letting \( \sigma \to 0 \),
\[ \psi_m(x + h, t) - \psi_m(x, t) = \int_{x}^{x+h} u_0(\xi) \, d\xi - \int_{x}^{x+h} u_m(\xi, t) \, d\xi - \int_{\sigma}^{t} \int_{x}^{x+h} u_m(\xi, \tau)^p \, d\xi \, d\tau \]
for a.e. \((x, t) \in \mathbb{R} \times (0, \infty)\). By (1.14) and (2.7),
\[ \int_{\mathbb{R}} \int_{\sigma}^{t} \int_{x}^{x+h} u_m^p(\xi, \tau) \, d\xi \, d\tau \, dx \leq h \int_{\sigma}^{t} \int_{\mathbb{R}} u_m^p(\xi, \tau) \, d\xi \, d\tau \]
\[ \leq \left( \frac{2 \|u_0\|_{L^1(\mathbb{R})}}{m-1} \right)^{\frac{p-1}{m}} h \int_{\sigma}^{t} \left( \int_{\mathbb{R}} u_m \, d\xi \right) \, d\tau \]
\[ \leq h \left( \frac{1}{1 - \frac{p-1}{m}} \right) \left( \frac{2 \|u_0\|_{L^1(\mathbb{R})}}{m-1} \right)^{\frac{p-1}{m}} \|u_0\|_{L^1(\mathbb{R})} t^{1 - \frac{p-1}{m}} \quad \forall m > p - 1. \] (2.6)

Hence, by (1.14), (2.5) and (2.6),
\[ \int_{\mathbb{R}} \left| \psi_m(x + h, t) - \psi_m(x, t) \right| \, dx \]
\[ \leq h \left[ 2 + \left( \frac{1}{1 - \frac{p-1}{m}} \right) \left( \frac{2 \|u_0\|_{L^1(\mathbb{R})}}{m-1} \right)^{\frac{p-1}{m}} t^{1 - \frac{p-1}{m}} \right] \|u_0\|_{L^1(\mathbb{R})} \quad \text{a.e. } t > 0, \forall m > p - 1, h > 0. \] (2.7)
By (1.14) and (2.1),
\[
\int_R \left| \psi_m(x, t + h) - \psi_m(x, t) \right| dx = \int_t^{t+h} \int_R u_m(x, \tau)^m d\tau \leq \frac{m}{m-1} (m-1)^\frac{1}{m} 2^{1-\frac{1}{m}} \|u_0\|_{L^1(R)}^2 (t+h)^\frac{1}{m} - t^\frac{1}{m}
\]
(2.8)
for a.e \( t > 0, h > 0, \) and any \( m > p - 1 \). By (1.13) and (2.1),
\[
0 \leq \psi_m(x, t) = \int_0^t u_m(x, \tau)^m d\tau \leq \frac{m}{m-1} (m-1)^\frac{1}{m} 2^{1-\frac{1}{m}} \|u_0\|_{L^1(R)}^2 \|u_0\|_{L^\infty(R)} t^\frac{1}{m}
\]
a.e. \( x \in \mathbb{R}, t > 0 \). (2.9)
By (2.7), (2.8) and (2.9), the lemma follows. □

By (2.9) and Lemma 2.2 the sequence \( \{\psi_{m_i}\}_{i=1}^\infty \) has a subsequence which we may assume without loss of generality to the sequence itself such that \( \psi_{m_i} \) converges in \( C([0, T]; L^1(\mathbb{R})) \) to some function \( \psi \in C([0, \infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (0, T)) \) for any \( T > 0 \) as \( i \to \infty \).

By an argument similar to the proof of Theorem 1.2 of [X], the following result holds.

**Proposition 2.3.** The function \( \psi \) is independent of \( t \).

By Proposition 2.3, \( 0 \leq \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). We are now ready for the proof of Theorem 1.1

**Proof of Theorem 1.1.** By the previous arguments it remains to prove the uniqueness of \( u_\infty \). Let \( \eta \in C_0^\infty(\mathbb{R}^2) \). We first claim that
\[
\lim_{i \to \infty} \int_0^\infty \int_R \left( u_{m_i} \right)^{m_i} \eta_x \ dx \ dx dt = \int_R \psi(x) \eta(x, 0) \ dx. \tag{2.10}
\]
To prove the claim we choose \( R_0 > 0, T_0 > 0 \), such that \( \text{supp} \ \eta \subset [-R_0, R_0] \times [-T_0, T_0] \). Then
\[
\int_0^\infty \int_R (u_{m_i})^{m_i} \eta_x \ dx \ dx dt = \int_0^{T_0} \int_R (u_{m_i})^{m_i} \eta_x \ dx \ dx dt + \int_0^\infty \int_R (u_{m_i})^{m_i} (\eta_x(x, t) - \eta_x(x, 0)) \ dx \ dx dt
\]
\[
+ \int_R \eta_x(x, 0) \int_0^\infty (u_{m_i}(x, t)^{m_i} \ dx \ dx dt
\]
\[
= I_1 + I_2 + I_3 \quad \forall 0 < \delta < T_0. \tag{2.11}
\]
By (2.1),
\[
I_1 \to 0 \quad \text{as } i \to \infty. \tag{2.12}
\]
By the mean value theorem, for any \( x \in \mathbb{R}, t > 0 \), there exists a constant \( t_x \in (0, t) \) such that
\[
\eta_x(x, t) - \eta_x(x, 0) = \eta_{tx}(x, t_x).
\]
Then by (2.1),
\[
|I_2| = \left| \int_0^\delta \int_R (u_{m_i})^{m_i} \eta_{tx}(x, t_x) \ dx \ dx dt \right| \leq \int_0^\delta \int_{-R_0}^{R_0} \frac{2\|u_0\|_{L^1(R)} \|\eta_{tx}\|_{L^\infty(R)}}{m_i - 1} \ dx \ dx dt
\]
\[
\leq \frac{4\delta R_0\|u_0\|_{L^1(R)} \|\eta_{tx}\|_{L^\infty(R)}}{m_i - 1} \to 0 \quad \text{as } i \to \infty. \tag{2.13}
\]
By Proposition 2.3,
\[
\int_0^\delta u_{m_i}(x, t)^{m_i} \ dx \ dx dt \to \psi(x) \quad \text{in } L^1(\mathbb{R}) \quad \text{as } i \to \infty. \tag{2.14}
\]
Letting \( i \to \infty \) in (2.11), by (2.12), (2.13) and (2.14), the claim (2.10) follows.
Since \( u_\infty \) satisfies (1.5), \( u_\infty(x, t) \) is monotone decreasing in \( t > 0 \). Hence
\[
u'_\infty(x) := u_\infty(x, 0) = \lim_{t \to 0} u_\infty(x, t) \quad \text{exists.}
\]

Putting \( m = m_i \) in (2.2) and letting \( i \to \infty \),
\[
\int_0^\infty \int_\mathbb{R} u_\infty \eta_t \, dx \, dt + \int_\mathbb{R} u_0(x) \eta(x, 0) \, dx + \int_\mathbb{R} \psi(x) \eta_t(x, 0) \, dx = \int_0^\infty \int_\mathbb{R} (u_\infty)^p \eta \, dx \, dt \quad \forall \eta \in C_0^\infty(\mathbb{R}^2).
\]

We now choose \( \phi \in C^\infty(\mathbb{R}) \), \( 0 \leq \phi \leq 1 \), such that \( \phi(r) = 0 \) for all \( r \leq -1 \) and \( \phi(r) = 1 \) for all \( r \geq 0 \) and let \( \phi_\varepsilon(r) = \phi(r/\varepsilon) \) for any \( r \in \mathbb{R} \) and \( \varepsilon > 0 \). For any \( \eta \in C_0^\infty(\mathbb{R}) \) and \( t_0 > 0 \), by replacing \( \eta \) by \( \phi_\varepsilon(t) \phi_t(t_0 - t) \eta(x) \) in (2.15) and letting \( \varepsilon \to 0 \), we have
\[
- \int_\mathbb{R} u_\infty(x, t_0) \eta(x) \, dx + \int_\mathbb{R} u_0(x) \eta(x) \, dx + \int_\mathbb{R} \psi(x) \eta_t(x) \, dx = \int_0^{t_0} \int_\mathbb{R} (u_\infty)^p \eta \, dx \, dt.
\]

Letting \( t_0 \to 0 \) in (2.16), by the monotone convergence theorem,
\[
- \int_\mathbb{R} u_\infty^0(x) \eta(x) \, dx + \int_\mathbb{R} u_0(x) \eta(x) \, dx + \int_\mathbb{R} \psi(x) \eta_t(x) \, dx = 0 \quad \forall \eta \in C_0^\infty(\mathbb{R})
\]
and (1.6) holds. We are now going to prove (1.7). For any \( k > 1 \) let \( \eta_k(x) = \phi(x + k) \phi(k - x) \). Then \( 0 \leq \phi k \leq 1 \), \( \eta_k(x) = 1 \) for any \( |x| \leq k \) and \( \eta_k(x) = 0 \) for any \( |x| \geq k + 1 \). By (1.6) there exists a constant \( C > 0 \) such that
\[
\left| \int_\mathbb{R} u_\infty^0 \eta_k \, dx - \int_\mathbb{R} u_0 \eta_k \, dx \right| \leq C \int_{k \leq |x| \leq k + 1} \psi \, dx \quad \forall k > 1.
\]

Since \( \psi \in L^1(\mathbb{R}) \), letting \( k \to \infty \) in (2.17),
\[
\int_\mathbb{R} u_\infty^0 \, dx = \int_\mathbb{R} u_0 \, dx.
\]

We now recall that by the result of [X],
\[
v_m(x, t) \to v_\infty(x) \quad \text{and} \quad \int_0^\infty v_m(x, t)^m \, dt \to \bar{\psi}(x) \quad \text{as} \ m \to \infty \quad \text{in} \ L_1^1(\mathbb{R} \times (0, \infty))
\]
for some functions \( v_\infty(x), \bar{\psi}(x) \), which satisfy
\[
0 \leq v_\infty(x) \leq 1, \quad \int_\mathbb{R} v_\infty \, dx = \int_\mathbb{R} u_0 \, dx, \quad 0 \leq \bar{\psi}(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),
\]
and
\[
v_\infty(x) + \bar{\psi}_{x}(x) = u_0(x) \quad \text{in} \ \mathcal{D}'(\mathbb{R}),
\]
with
\[
\bar{\psi}(x) = 0 \quad \text{a.e.} \ x \in \{x \in \mathbb{R} : v_\infty(x) < 1\}.
\]

Since \( u_m(x, t) \leq v_m(x, t) \), we have
\[
0 \leq \psi \leq \bar{\psi}
\]
and
\[
u_\infty(x, t) \leq v_\infty(x) \quad \Rightarrow \quad u'_\infty(x) \leq v_\infty(x).
\]

By (2.18) and (2.19),
\[
\int_\mathbb{R} u_\infty^0 \, dx = \int_\mathbb{R} v_\infty \, dx.
\]

By (2.23) and (2.24),
\[
u_\infty^0(x) = v_\infty(x) \quad \text{a.e.} \ x \in \mathbb{R}.
\]
By (2.21), (2.22) and (2.25), we get (1.7). By the discussion on P.70 of [X], \( u_0 \) is uniquely determined by (1.6) and (1.7). Since \( u_\infty \) satisfies (1.5) with initial value \( u_0 \), the function \( u_\infty \) is unique. Since the sequence \( \{m_i\}_{i=1}^{\infty} \) is arbitrary, \( u_m \) converges to \( u_\infty \) in \( C([t_0, T]; L^1_{\text{loc}}(\mathbb{R})) \) for any \( T > t_0 > 0 \) as \( m \to \infty \) and Theorem 1.1 follows.

3. Singular limit as \( p \to \infty \)

In this section we will prove Theorem 1.2. We will fix \( m > 1 \) and write \( w_p := u_{m,p} \) for any \( p > 1 \). We will assume that \( 0 \leq u_0 \in L^\infty(\mathbb{R}) \) in this section.

Lemma 3.1. \( w_p \) satisfies

\[
w_p(x, t) \leq \frac{1}{(p-1)t + \|u_0\|_{L^\infty}^{1-p}} \quad \text{a.e. } (x, t) \in \mathbb{R} \times (0, \infty) \quad \forall p > 1.
\]

Proof. By direct computation, the function

\[
h(x, t) = \frac{1}{(p-1)t + \|u_0\|_{L^\infty}^{1-p}}
\]

satisfies

\[
\begin{cases}
    u_t + (u^m)_x = \varepsilon u_{xx} - u^p & \text{in } \mathbb{R} \times (0, \infty) \\
    u(x, 0) = \|u_0\|_{L^\infty} & \text{in } \mathbb{R}
\end{cases}
\]

for any \( \varepsilon > 0 \). Let \( w^\varepsilon_{m,p}(x, t) \) be the solution of the problem

\[
\begin{cases}
    u_t + (u^m)_x = \varepsilon u_{xx} - u^p & \text{in } \mathbb{R} \times (0, \infty) \\
    u(x, 0) = \|u_0\|_{L^\infty} & \text{in } \mathbb{R}
\end{cases}
\]

By the construction of solution in [K], \( w^\varepsilon_{m,p} \) converges almost everywhere in \( \mathbb{R} \times (0, \infty) \) to \( w_p \) as \( \varepsilon \to 0^+ \). By the maximum principle for parabolic equation,

\[
w^\varepsilon_{m,p}(x, t) \leq h(x, t) \quad \forall (x, t) \in \mathbb{R} \times (0, \infty)
\]

\[
\Rightarrow w_p(x, t) \leq h(x, t) \quad \text{a.e. } (x, t) \in \mathbb{R} \times (0, \infty) \quad \text{as } \varepsilon \to 0^+
\]

and the lemma follows. \( \square \)

Let \( \{p_i\}_{i=1}^\infty \subset \mathbb{R}^+ \), \( p_i > 2 \), \( \forall i, \cdots \), be such that \( p_i \to \infty \) as \( i \to \infty \). By (3.1) for any \( t_0 > 0 \),

\[
p\|w_{p_i}\|_{L^\infty([t_0, \infty))}^{p_i-1} \leq \frac{p}{(p-1)t_0} \quad \forall p > 1.
\]

By (3.3) for any \( R > 0 \), \( T > t_0 > 0 \), we can choose the function \( \omega_R \) in Lemma 1.5 to be independent of \( p \geq 2 \). Hence by Lemma 1.5 the sequence \( \{w_{p_i}\}_{i=1}^\infty \) is equi-continuous in \( C([t_0, T]; L^1_{\text{loc}}(\mathbb{R})) \) for any \( T > t_0 > 0 \). Hence by [13], the Ascoli theorem and a diagonalization argument the sequence \( \{w_{p_i}\}_{i=1}^\infty \) has a subsequence which we may assume without loss of generality to be the sequence itself that converges in \( C([t_0, T]; L^1_{\text{loc}}(\mathbb{R})) \) for any \( T > t_0 > 0 \) to some non-negative function \( w_\infty \in C((0, \infty); L^1_{\text{loc}}(\mathbb{R})) \) as \( i \to \infty \). Putting \( p = p_i \) in (3.1) and letting \( i \to \infty \),

\[
w_\infty \leq 1 \quad \text{a.e. in } \mathbb{R} \times (0, \infty).
\]
Lemma 3.2. \( w_{\infty} \) satisfies
\[
\int_0^T \int_\mathbb{R} \left( |w_{\infty}(x,t) - k| \eta_t + \left| w_{\infty}(x,t) \right|^m - k^m |\eta_x| \right) \, dx \, dt \geq 0 \quad \forall k \in \mathbb{R}, \ 0 \leq \eta \in C_0^\infty(\mathbb{R} \times (0, \infty)). \tag{3.5}
\]

Proof. Let \( 0 \leq \eta \in C_0^\infty(\mathbb{R} \times (0, \infty)) \). Since \( w_p \) is the solution of (1.4),
\[
\int_0^T \int_\mathbb{R} \left( |w_p(x,t) - k| \eta_t + \left| w_p(x,t) \right|^m - k^m |\eta_x| - \text{sign}(w_p(x,t) - k)w_p(x,t)^p \eta \right) \, dx \, dt \geq 0 \quad \forall k \in \mathbb{R}. \tag{3.6}
\]
We now choose \( T > t_0 > 0 \) and \( R_1 > 0 \) such that
\[
\text{supp} \ \eta \subset B_{R_1} \times (t_0, T).
\]
By (3.1),
\[
\left| w_p(x,t)^p \right| \leq \frac{1}{((p-1)t_0)^p} \quad \text{a.e.} \ (x,t) \in \mathbb{R} \times [t_0, \infty), \ \forall p > 1. \tag{3.7}
\]

Since the right hand side of (3.7) converges to 0 as \( p \to \infty \), letting \( p = p_i \) and \( i \to \infty \) in (3.6), by (1.13), (3.7) and the Lebesgue Dominated Convergence Theorem, (3.5) follows. \( \square \)

Lemma 3.3. Let \( 0 \leq u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}) \). Suppose there exists \( x_0 \in \mathbb{R} \) and \( \delta > 0 \) such that
\[
u(x) < 1, \quad \forall x \in B_{2\delta}(x_0).
\]
Then,
\[
\lim_{i \to 0} \| w_{\infty}(\cdot,t) - u_0(\cdot) \|_{L^1(B_\delta(x_0))} = 0. \tag{3.8}
\]

Proof. We divide the proof into two cases.

Case 1. \( \| u_0 \|_{L^\infty(\mathbb{R})} \leq 1. \)

By (1.13),
\[
\left| \int_\mathbb{R} w_p(x,t) \eta(x) \, dx - \int_\mathbb{R} u_0(x) \eta(x) \, dx \right| \leq \int_0^T \int_\mathbb{R} \left[ (w_p)^m |\eta_x| + (w_p)^p |\eta| \right] \, dx \, dt \leq C_\eta t, \quad \forall p > 1, \ \eta \in C_0^\infty(\mathbb{R})
\]
\[
\Rightarrow \left| \int_\mathbb{R} w_{\infty}(x,t) \eta(x) \, dx - \int_\mathbb{R} u_0(x) \eta(x) \, dx \right| \leq C_\eta t, \quad \forall \eta \in C_0^\infty(\mathbb{R}) \quad \text{as} \ p = p_i \to \infty.
\]

Then
\[
w_{\infty} \to u_0 \quad \text{weakly in} \ L^1(\mathbb{R}) \ \text{as} \ t \to 0. \tag{3.9}
\]

Let \( \{t_i\}_{i=1}^\infty \subset \mathbb{R}^+ \) be such that \( t_i \to 0 \) as \( i \to \infty \). Then by (3.9), there exists the sequence \( \{t_i\}_{i=1}^\infty \) has a subsequence which we may assume without loss of generality to be the sequence itself such that
\[
w_{\infty}(x,t_i) \to u_0(x) \quad \text{a.e.} \ x \in \mathbb{R} \ \text{as} \ i \to \infty. \tag{3.10}
\]

By (3.10) and Lebesgue Dominated Convergence Theorem,
\[
\| w_{\infty}(\cdot,t_i) - u_0(\cdot) \|_{L^1(B_R)} \to 0 \quad \forall R > 0 \quad \text{as} \ i \to \infty.
\]
Since the sequence \( \{t_i\}_{i=1}^\infty \) is arbitrary, (3.8) follows.

Case 2. \( u_0 \in L^\infty(\mathbb{R}) \).

Let \( \theta = \max_{|x-x_0| < \delta} u_0(x) \). Then, \( \theta < 1 \). We now choose a smooth non-negative function \( v_0 \) on \( \mathbb{R} \) such that
\[
v_0(x) = u_0(x), \quad \forall x \in B_{2\delta}(x_0) \quad \text{and} \quad v_0(x) \leq \frac{\theta + 1}{2}, \quad \forall x \in \mathbb{R}.
\]

Let \( N = m \| u_0 \|_{L^\infty(\mathbb{R})}^{m-1} \) and \( v_p \) be the solution of (1.4) with initial value \( v_0 \). By the same argument as before, the sequence \( \{v_{p_i}\}_{i=1}^\infty \) has a subsequence which we may assume without loss of generality
Hence, by (3.16) and (3.17) and the Lebesgue Dominated Convergence Theorem,

\[ \|w_p(\cdot, t) - v_p(\cdot, t)\|_{L^1(B_\delta(x_0))} = 0 \quad \forall 0 < t < \frac{\delta}{N}, \quad p > 1 \]

\[ \Rightarrow \quad \|w_\infty(\cdot, t) - v_\infty(\cdot, t)\|_{L^1(B_\delta(x_0))} = 0 \quad \forall 0 < t < \frac{\delta}{N}, \quad |x - x_0| \leq \delta. \tag{3.11} \]

Therefore, by (3.11) and Case 1,

\[ \|w_\infty(\cdot, t) - u_0\|_{L^1(B_\delta(x_0))} \to 0 \quad \text{as} \quad t \to 0 \]

and (3.8) follows. \qed

**Lemma 3.4.** Let

\[ w_\infty^0(x) = \min(u_0(x), 1) \quad \forall x \in \mathbb{R}. \]

Then,

\[ \lim_{t \to 0} \int_{B_R} |w_\infty(x, t) - w_\infty^0(x)| \, dx = 0 \quad \forall R > 0. \tag{3.12} \]

**Proof.** We divide the proof into 2 cases.

**Case 1.** \( u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}). \)

Since \( \{x : u_0(x) < 1\} \) is open, by the Lindelöf theorem \([\mathbb{R}], \{x : u_0(x) < 1\} = \bigcup_{j=1}^{\infty} B_{\delta_j}(x_j) \) for some \( x_j \in \{x : u_0(x) < 1\} \) and \( \delta_j > 0, j = 1, 2, \ldots \). By Lemma 3.3, for any \( j \in \mathbb{Z}^+ \) (3.8) holds for \( \delta = \delta_j. \)

Let \( \epsilon > 0 \) and \( u_{0,\epsilon}(x) = \min(u_0(x), 1 - \epsilon). \) For any \( m > 1, \) \( p > 1, \) let \( u_{m,p,\epsilon} \) be the solutions of (1.4) in \( \mathbb{R} \times (0, \infty) \) with initial value \( u_{0,\epsilon}. \) By the same argument as before \( u_{m,p,\epsilon} \) satisfies (3.3). Moreover the sequence \( \{u_{m,p,\epsilon}\}_{m=1}^{\infty} \) is equi-continuous in \( C([0, T]; L^1_{loc}(\mathbb{R})) \) for any \( T > t_0 > 0 \) and has a subsequence which we may assume without loss of generality to be the sequence itself that converges in \( C([0, T]; L^1_{loc}(\mathbb{R})) \) for any \( T > t_0 > 0 \) to some function \( w_{\infty,\epsilon} \in C([0, \infty); L^1_{loc}(\mathbb{R})) \) as \( i \to \infty \) which satisfies

\[ 0 \leq w_{\infty,\epsilon}(x, t) \leq 1 \quad \text{in} \quad \mathbb{R} \times [0, \infty) \tag{3.13} \]

Since \( u_{0,\epsilon} < 1 \) in \( \mathbb{R}, \) by the proof of Lemma 3.3

\[ w_{\infty,\epsilon}(x, t) \to u_{0,\epsilon}(x) \quad \text{in} \quad L^1_{loc}(\mathbb{R}) \quad \text{as} \quad t \to 0. \tag{3.14} \]

Since \( u_{0,\epsilon} \leq u_0, \) by the construction of solutions of (1.4) in \([K],\)

\[ u_{m,p,\epsilon} \leq u_p \quad \text{in} \quad \mathbb{R} \times (0, \infty) \quad \Rightarrow \quad w_{\infty,\epsilon} \leq w_\infty \quad \text{in} \quad \mathbb{R} \times (0, \infty) \quad \text{as} \quad p = p_i, i \to \infty. \tag{3.15} \]

By (3.4), (3.14) and (3.15),

\[ 1 \geq \limsup_{t \to 0} \frac{w_\infty(x, t)}{w_{\infty,\epsilon}(x, t)} \geq \liminf_{t \to 0} \frac{w_{\infty,\epsilon}(x, t)}{w_{\infty,\epsilon}(x, t)} = 1 - \epsilon \quad \text{a.e.} \quad x \in \{x : u_0(x) \geq 1\} \]

\[ \Rightarrow \quad \lim_{t \to 0} w_\infty(x, t) = 1 = w_0^0(x) \quad \text{a.e.} \quad x \in \{x : u_0(x) \geq 1\} \quad \text{as} \quad \epsilon \to 0. \tag{3.16} \]

Since (3.8) holds for \( \delta = \delta_j, \) \( j \in \mathbb{Z}^+, \) any sequence \( \{t_i\}_{i=1}^{\infty}, t_i \to 0 \) as \( i \to \infty, \) will have a subsequence which we may assume without loss of generality to be the sequence itself such that

\[ w_\infty(x, t_i) \to u_0(x) \quad \text{a.e.} \quad x \in \{x : u_0(x) < 1\}. \tag{3.17} \]

Hence, by (3.16), (3.17) and the Lebesgue Dominated Convergence Theorem,

\[ \lim_{i \to \infty} \int_{|x| < R} |w_\infty(x, t_i) - w_0^0(x)| \, dx = 0 \quad \forall R > 0. \]
Since the sequence \( \{t_i\}_{i=1}^{\infty} \) is arbitrary, (3.12) follows.

**Case 2.** \( u_0 \in L^\infty(\mathbb{R}) \).

We choose a sequence of functions \( \{u_{0,j}\}_{j=1}^{\infty} \subset C^\infty(\mathbb{R}) \) such that

\[
\begin{align*}
  \|u_{0,j} - u_0\|_{L^1(\mathbb{R})} &\to 0 \quad \text{as } j \to \infty, \quad \forall R > 0 \\
  u_{0,j}(x) &\to u_0(x) \quad \text{a.e. } x \in \mathbb{R} \quad \text{as } j \to \infty \\
  \|u_{0,j}\|_{L^\infty(\mathbb{R})} &\leq \|u_0\|_{L^\infty(\mathbb{R})} + \frac{1}{j} \quad \forall j \in \mathbb{Z}^+.
\end{align*}
\]

(3.18)

For any \( m > 1, p > 1 \), let \( u_{m,p,j} \) be the solutions of (1.4) with initial value \( u_{0,j} \). By the same argument as before for any \( j \in \mathbb{Z}^+ \) the sequence \( \{u_{m,p,j}\}_{i=1}^{\infty} \) has a subsequence which we may assume without loss of generality to be the sequence itself that converges in \( C([0,T]; L^1_{\text{loc}}(\mathbb{R})) \) to some function \( w_{\infty,j} \in C((0,\infty); L^1_{\text{loc}}(\mathbb{R})) \), \( 0 \leq w_{\infty,j} \leq 1 \), for any \( T > t_0 > 0 \) as \( i \to \infty \). Let

\[
  w^0_{\infty,j}(x) = \min\{u_{0,j}(x), 1\}, \quad \forall j \in \mathbb{Z}^+.
\]

By case 1,

\[
  \lim_{t \to 0} \int_{B_R} |w_{\infty,j}(x,t) - w^0_{\infty,j}(x)| \, dx = 0 \quad \forall R > 0, j \in \mathbb{Z}^+. \tag{3.19}
\]

By (3.18) and Lemma 1.6 there exists a constant \( N > 0 \) such that

\[
  \int_{B_{R-N}} |u_{m,p,j}(x,t) - u_{m,p}(x,t)| \, dx \leq \int_{B_R} |u_{0,j}(x) - u_0(x)| \, dx \quad \forall 0 < t < R/N, R > 0, j \in \mathbb{Z}^+, p > 1. \tag{3.20}
\]

Putting \( p = p_i \) in (3.20) and letting \( i \to \infty \),

\[
  \int_{B_{R-N}} |w_{\infty,j}(x,t) - w_\infty(x,t)| \, dx \leq \int_{B_R} |u_{0,j}(x) - u_0(x)| \, dx \quad \forall 0 < t < R/N, R > 0, j \in \mathbb{Z}^+. \tag{3.21}
\]

By (3.21),

\[
\begin{align*}
  \int_{B_{R-N}} |w_\infty(x,t) - w^0_\infty(x)| \, dx \\
  \leq \int_{B_{R-N}} |w_\infty(x,t) - w_{\infty,j}(x,t)| \, dx + \int_{B_{R-N}} |w_{\infty,j}(x,t) - w^0_{\infty,j}(x)| \, dx + \int_{B_{R-N}} |w^0_{\infty,j}(x) - w^0_\infty(x)| \, dx \\
  \leq \int_{B_R} |u_{0,j}(x) - u_0(x)| \, dx + \int_{B_{R-N}} |w_{\infty,j}(x,t) - w^0_{\infty,j}(x)| \, dx + \int_{B_R} |w^0_{\infty,j}(x) - w^0_\infty(x)| \, dx
\end{align*}
\]

(3.22)

for any \( 0 < t < R/N, R > 0 \) and \( j \in \mathbb{Z}^+ \). Letting first \( t \to 0 \) and then \( j \to \infty \) in (3.22), by (3.18) and (3.19), (3.12) follows.

We will now complete the proof of Theorem 1.2.

**Proof of Theorem 1.2** By Lemma 3.2 and Lemma 3.4, \( w_\infty \) is the unique solution of (1.8). Since the sequence \( \{p_i\}_{i=0}^{\infty} \) is arbitrary, \( w_p \) converges to \( w_\infty \) in \( C([t_0,T]; L^1_{\text{loc}}(\mathbb{R})) \) for any \( T > t_0 > 0 \) as \( p \to \infty \) and Theorem 1.2 follows. □
4. Interchange of limits

This section will be devoted to proving Theorem 1.3.

Proof of Theorem 1.3 Note that (i) follows directly by Theorem 1.2 and the result of [X]. Hence we only need to prove (ii). By Theorem 1.1, \( u_{\infty,p} \) satisfies (1.5) with initial value \( u_0^\infty \) that satisfies (1.6) for some function \( 0 \leq \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) which satisfies (1.7) and \( 0 \leq u_{\infty,p} \leq 1 \) on \( \mathbb{R} \times (0,\infty) \). Let \( \{p_i\}_{i=1}^\infty \subset \mathbb{Z}^+ \) be such that \( p_i \to \infty \) as \( i \to \infty \). Since \( 0 \leq u_{\infty,p} \leq 1 \), the sequence \( \{u_{\infty,p_i}\}_{i=1}^\infty \) has a subsequence which we may assume without loss of generality to be the sequence itself such that \( u_{\infty,p_i} \) converges weakly in \( L^1(\mathbb{R} \times (0,\infty)) \) to some function \( v_2 \) as \( i \to \infty \).

On the other hand since \( u_{\infty,p} \) satisfies (1.5),

\[
  u_{\infty,p}(x,t) = \frac{u_0^\infty(x)}{(p-1)t u_0^\infty(x)^{p-1} + 1}^{1/(p-1)} \quad \text{a.e. } (x,t) \in \mathbb{R} \times (0,\infty) \quad \forall p > 1
\]

\[
  \Rightarrow \quad v_2(x,t) = \lim_{p \to \infty} u_{\infty,p}(x,t) = u_0^\infty(x) \quad \text{a.e. } (x,t) \in \mathbb{R} \times (0,\infty)
\]

and Theorem 1.3 follows. \( \square \)

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