Theorem. There exists a smooth oriented fibre bundle $\mathbb{H}^2 \to E^2 \to S^4$ with $\hat{A}(E) \neq 0$.

Remark. The argument we give also shows that this fibre bundle may be assumed to have a section with trivial normal bundle (see Remark 2.2), and provides analogous $\mathbb{H}^n$-bundles over $S^4$ for all even $n \geq 2$ (see Remark 3). It can certainly be extended further.

The standard Riemannian metric $g_{st}$ on $\mathbb{H}^2$ has positive sectional curvature, so pulling back $g_{st}$ along orientation-preserving diffeomorphisms yields a map

$$(-)^* g_{st} : \text{Diff}(\mathbb{H}^2) \to \mathcal{R}^{\sec > 0}(\mathbb{H}^2) \subset \mathcal{R}^{\text{Ric} > 0}(\mathbb{H}^2) \subset \mathcal{R}^{\text{scal} > 0}(\mathbb{H}^2)$$

from the group of diffeomorphisms of $\mathbb{H}^2$ in the smooth topology to the spaces of Riemannian metrics on $\mathbb{H}^2$ having positive sectional, Ricci, or scalar curvature. By an argument of Hitchin [Hit74] (see for instance [BEW20, p. 3999] for an explanation of this), the theorem has the following corollary, which answers a question of Schick [OWL17, p. 30] and provides an example as asked for in [BEW20, Remark 2.2].

Corollary. The induced map

$$\pi_3((-)^* g_{st}) \otimes \mathbb{Q} : \pi_3(\text{Diff}(\mathbb{H}^2); \text{id}) \otimes \mathbb{Q} \to \pi_3(\mathcal{R}^{\text{scal} > 0}(\mathbb{H}^2); g_{st}) \otimes \mathbb{Q}$$

is nontrivial, so in particular $\pi_3(\mathcal{R}^{\text{sec} > 0}(\mathbb{H}^2); g_{st}) \otimes \mathbb{Q} \neq 0$ and $\pi_3(\mathcal{R}^{\text{Ric} > 0}(\mathbb{H}^2); g_{st}) \otimes \mathbb{Q} \neq 0$.

Proof of the Theorem. Smooth $\mathbb{H}^2$-bundles over $S^4$ (together with an identification of the fibre over the basepoint with $\mathbb{H}^2$) are classified by $\pi_4(\text{BDiff}(\mathbb{H}^2))$, so our task is to show that the morphism $\hat{A} : \pi_4(\text{BDiff}(\mathbb{H}^2)) \to \mathbb{Q}$ assigning an $\mathbb{H}^2$-bundle $E \to S^4$ the $\hat{A}$-genus of the total space is nontrivial. This morphism factors over the map $\pi_4(\text{BDiff}(\mathbb{H}^2)) \to \pi_4(\text{BDiff}(\mathbb{H}^2))$ induced by the canonical comparison map $\text{Diff}(\mathbb{H}^2) \to \text{Diff}(\mathbb{H}^2)$ to the block diffeomorphism group of $\mathbb{H}^2$. This follows for instance from [ERW14, Theorem 1], but there is also a more direct argument: via the canonical isomorphism $\pi_4(\text{BDiff}(\mathbb{H}^2)) \cong \pi_4(\text{Diff}(\mathbb{H}^2); \text{id})$, the morphism $\hat{A} : \pi_4(\text{BDiff}(\mathbb{H}^2)) \to \mathbb{Q}$ is given by mapping a diffeomorphism $\phi : D^3 \times \mathbb{H}^2 \to D^3 \times \mathbb{H}^2$ that is the identity on the...
boundary and commutes with the projection to $S^3$ to the $\hat{\Delta}$-genus of the glued manifold $D^4 \times \mathbb{H}P^2 \cup_{\partial, \text{id}} D^4 \times \mathbb{H}P^2$. This description of the morphism makes clear that it factors through the map $\pi_3(\text{Diff}(\mathbb{H}P^2); \text{id}) \to \pi_3(\text{Diff}(D^3 \times \mathbb{H}P^2); \text{id}) \cong \pi_3(\text{Diff}(\mathbb{H}P^2); \text{id})$ that only remembers the underlying isotopy class of $\phi$.

We thus have to show nontriviality of the composition

$$
\pi_3(\text{BDiff}(\mathbb{H}P^2)) \to \pi_3(\text{BDiff}(\mathbb{H}P^2)) \to \hat{\Delta} \to Q.
$$

It suffices to check this after rationalisation, which makes the first map surjective:

**Lemma.** The map $\pi_3(\text{BDiff}(\mathbb{H}P^2)) \otimes \mathbb{Q} \to \pi_3(\text{BDiff}(\mathbb{H}P^2)) \otimes \mathbb{Q}$ surjective.

**Proof.** Choosing an embedded disc $D^8 \subset \mathbb{H}P^2$, we consider the commutative square

$$
\begin{array}{ccc}
\text{BDiff}_\partial(D^8) & \longrightarrow & \text{BDiff}(\mathbb{H}P^2) \\
\downarrow & & \downarrow \\
\text{BDiff}_\partial(D^8) & \longrightarrow & \text{BDiff}(\mathbb{H}P^2)
\end{array}
$$

whose horizontal maps are induced by extending (block) diffeomorphisms of $D^8$ that are the identity on the boundary to $\mathbb{H}P^2$ by the identity. The claim follows by showing that the third rational homotopy group of the right vertical homotopy fibre vanishes for which we note that, since $\mathbb{H}P^2$ is 3-connected, the square is 4-cartesian by Morlet’s lemma of disjunction [BLR75, Corollary 3.2, p. 29], so it suffices to show that the third rational homotopy group of the left vertical map is trivial. Since $\pi_3(\text{BDiff}_\partial(D^8)) \cong \pi_3(\text{Diff}(D^2)) \cong \Theta_{2n-i}$ vanishes rationally as the group $\Theta_{2n+i}$ of homotopy $(2n + i)$-spheres is finite, the claim follows from $\pi_3(\text{BDiff}_\partial(D^8)) \cong \mathbb{Q} = 0$ which holds by [RW17, Theorem 4.1].

Given the lemma, we are left to show that the map $\hat{\Delta} : \pi_3(\text{BDiff}(\mathbb{H}P^2)) \to Q$ is nontrivial, which we shall do after precomposition with the map

$$
\pi_3(\text{hAut}(\mathbb{H}P^2)/\text{Diff}(\mathbb{H}P^2); \text{id}) \longrightarrow \pi_3(\text{BDiff}(\mathbb{H}P^2))
$$

induced by the inclusion of the homotopy fibre of the comparison map $\text{BDiff}(\mathbb{H}P^2) \to \text{hAut}(\mathbb{H}P^2)$, where $\text{hAut}(\mathbb{H}P^2)$ is the topological monoid of homotopy automorphisms of $\mathbb{H}P^2$. Considering this homotopy fibre is advantageous since the $h$-cobordism theorem provides an isomorphism

$$
\pi_3(\text{hAut}(\mathbb{H}P^2)/\text{Diff}(\mathbb{H}P^2)) \cong \mathbb{S}_\partial(D^4 \times \mathbb{H}P^2)
$$

to the *structure group* of $D^4 \times \mathbb{H}P^2$ relative to $\partial D^4 \times \mathbb{H}P^2$ in the sense of surgery theory (see [Wal99] for background on surgery theory, especially Chapter 10), which in turn fits into the surgery exact sequence of abelian groups

$$
0 = L_{13}(Z) \xrightarrow{\partial} \mathbb{S}_\partial(D^4 \times \mathbb{H}P^2) \xrightarrow{\eta} N_\partial(D^4 \times \mathbb{H}P^2) \xrightarrow{\sigma} L_{12}(Z) \cong Z
$$

featuring the surgery obstruction map $\sigma$ from the normal invariants $N_\partial(D^4 \times \mathbb{H}P^2)$ to the $L$-group $L_{12}(Z) \cong Z$. The standard smooth structure on $D^4 \times \mathbb{H}P^2$ provides an isomorphism

$$
N_\partial(D^4 \times \mathbb{H}P^2) \cong [S^4 \wedge \mathbb{H}P^2_+, \mathbb{G}/\mathbb{O}],
$$

where $[\cdot, \cdot]$ stands for based homotopy classes and $\mathbb{G}/\mathbb{O}$ is the homotopy fibre of the map $BO \to BG$ classifying the underlying stable spherical fibration of a stable vector bundle.

As $BG$ has trivial rational homotopy groups, the map

$$
[S^4 \wedge \mathbb{H}P^2_+, \mathbb{G}/\mathbb{O}] \longrightarrow [S^4 \wedge \mathbb{H}P^2_+, \text{BO}] = \mathbb{KO}^0(S^4 \wedge \mathbb{H}P^2)
$$

is rationally an isomorphism. Furthermore the Pontrjagin character gives an isomorphism $\text{ph}(-) = \text{ch}(- \otimes C) : \mathbb{KO}^0(S^4 \wedge \mathbb{H}P^2) \otimes \mathbb{Q} \longrightarrow \bigoplus_{i \geq 0} \hat{H}^i(S^4 \wedge \mathbb{H}P^2; \mathbb{Q}) = u \cdot \mathbb{Q}[z]/(z^2)$,
where \( u \in \tilde{H}^4(S^4; \mathbb{Q}) \) denotes the cohomological fundamental class, and \( z \in H^4(HP^2; \mathbb{Q}) \) is the usual generator. Therefore for any triple \((A, B, C) \in \mathbb{Q}^3\) there exists a nonzero \( \lambda \in \mathbb{Z} \) and a normal invariant \( n \in N_0(D^4 \times HP^2) \) whose underlying stable vector bundle \( \xi \) has \( ph(\xi) = \lambda \cdot u \cdot (A + Bz + Cz^2) \). Since \( S^4 \land HP^2 \) has no nontrivial cup-products among elements of positive degree, we have \( ph_1(\xi) = (1)^{y+1} / (2i - 1)! \cdot p_1(\xi) \) and hence

\[
(1) \quad p_1(\xi) = \lambda A \cdot u \quad p_2(\xi) = -6A \cdot B \cdot z \quad p_3(\xi) = 120 \lambda C \cdot u \cdot z^2.
\]

To evaluate the surgery obstruction map \( \sigma \), recall that a normal invariant \( n \) with underlying stable vector bundle \( \xi \) is represented by a degree 1 normal map

\[
v_M \xrightarrow{\hat{f}} v_{D^4 \times HP^2} \oplus \xi
\]

\[
M^{12} \xrightarrow{\hat{f}} D^4 \times HP^2,
\]

where \( \partial M = \partial D^4 \times HP^2 \) and \( \hat{f} \) and \( \hat{f} \) restrict to the identity maps on the boundary. Here \( v_{(\_)} \) denotes the stable normal bundle of a manifold. The surgery obstruction is unchanged by gluing into \( M \) and \( D^4 \times HP^2 \) a copy of \( D^4 \times HP^2 \) along the identification of their boundaries with \( \partial D^4 \times HP^2 \), and extending \( \hat{f} \) and \( \hat{f} \) trivially, giving rise to a degree 1 normal map to \( f' : M' \rightarrow S^4 \times HP^2 \). The surgery obstruction may then be expressed in terms of the signatures of these manifolds, as

\[
\sigma(n) = \frac{1}{2} \left( \text{sign}(M') - \text{sign}(S^4 \times HP^2) \right).
\]

The signature of \( S^4 \times HP^2 \) is trivial, and that of \( M' \) may be computed in terms of the Hirzebruch signature theorem as the evaluation \( \int_M L(TM') \) of the \( L \)-class. As \( f' \) has degree 1 and pulls back \( \nu_{S^4 \times HP^2} \oplus \xi \) to the stable inverse of \( TM' \), we have

\[
(3) \quad \text{sign}(M') = \int_{M'} L(TM') = \int_{S^4 \times HP^2} L(TS^4) \cdot L(THP^2) \cdot L(-\xi).
\]

The first terms of the total \( L \)-class are given as

\[
L = 1 + \frac{p_1}{7} + \frac{7p_2 - p_1^2}{40} + \frac{63p_3 - 13p_2p_1 + 2p_1^3}{945} + \cdots
\]

which we combine with \( p(THP^2) = 1 + 2z + 7z^2 \) from [Hir53, Satz 1] to compute

\[
L(TS^4) = 1 \quad L(T(THP^2)) = 1 + \frac{1}{12} z + z^2 \quad L(-\xi) = 1 + \lambda(-\frac{1}{12} A \cdot u + \frac{15}{576} B \cdot (u \cdot z) - \frac{49}{63} C \cdot (u \cdot z^2))
\]

and thus

\[
8\sigma(n) = \text{sign}(M') = \lambda(-\frac{1}{12} A + \frac{28}{45} B - \frac{49}{63} C).
\]

It follows that for each triple \((A, B, C) \in \mathbb{Q}^3\) satisfying \( \frac{1}{12} A - \frac{28}{45} B + \frac{49}{63} C = 0 \) there exists a non-zero \( \lambda \in \mathbb{Z} \) and a degree 1 normal map as in \( (2) \) with \( f \) a homotopy equivalence and with \( \xi \) having Pontrjagin classes as in \( (1) \). This gives a smooth block \( HP^2 \)-bundle structure on the composition

\[
M' \xrightarrow{f} S^4 \times HP^2 \xrightarrow{\nu} S^4
\]

giving rise to a class in \( \pi_4(\text{BDiff}(HP^2)) \), so it remains to evaluate \( \hat{A}(M') \). As in \( (3) \), we get

\[
\hat{A}(M') = \int_{M'} \hat{A}(TM') = \int_{S^4 \times HP^2} \hat{A}(TS^4) \cdot \hat{A}(THP^2) \cdot \hat{A}(-\xi),
\]

which we combine with the formula for the first terms of the total \( \hat{A} \)-class

\[
\hat{A} = 1 - \frac{p_1}{24} + \frac{-4p_2 + 7p_1^2}{5760} + \frac{-16p_3 + 44p_2p_1 - 31p_1^3}{967680} + \cdots
\]

to compute

\[
\hat{A}(TS^4) = 1 \quad \hat{A}(THP^2) = 1 - \frac{1}{12} z \quad \hat{A}(-\xi) = 1 + \lambda(-\frac{1}{24} A \cdot u + \frac{11}{288} B \cdot (u \cdot z) + \frac{1}{576} C \cdot (u \cdot z^2))
\]

from which we conclude

\[
\hat{A}(M') = \lambda(\frac{1}{288} B + \frac{1}{576} C).
\]
As there are clearly triples $(A, B, C) \in \mathbb{Q}^3$ satisfying
\[-\frac{1}{4}A + \frac{\pi}{4\pi}B - \frac{\pi}{4\pi}C = 0 \quad \text{and} \quad \frac{1}{2\pi\pi}B + \frac{1}{5\pi\pi}C \neq 0,
\]
this finishes the argument.

**Remark 1.** A fibre bundle $\pi: E \to S^4$ constructed in this way is fibre homotopy equivalent to the trivial bundle $\pi_1: S^4 \times \mathbb{H}P^2 \to S^4$, and under this fibre homotopy trivialisation we have $p_1(TE) = 2 \cdot (1 \otimes z) - \lambda A \cdot (u \otimes 1)$. Thus $p_1(TE)^3 = -12\lambda A \cdot (u \otimes z^2)$, and so
\[\int_E p_1(TE)^3 = -12\lambda A \quad \text{and} \quad \int_E \hat{A}(TE) = \lambda\left(\frac{1}{2\pi\pi}B + \frac{1}{5\pi\pi}C\right).
\]
This argument therefore guarantees the existence of a 2-dimensional subspace of the group $\pi_3(B\text{Diff}(\mathbb{H}P^2)) \otimes \mathbb{Q}$, detected by the characteristic numbers $\int_E p_1(TE)^3$ and $\int_E \hat{A}(TE)$.

**Remark 2.** The Theorem can be slightly strengthened: we may in addition assume that the smooth $\mathbb{H}P^2$-bundle $\pi: E \to S^4$ admits a smooth section with trivial normal bundle, which may be helpful for fibrewise surgery constructions.

To see this, note that a fibre bundle $π: E \to S^4$ as constructed above is fibre homotopy equivalent to the trivial bundle $π_1: S^4 \times \mathbb{H}P^2 \to S^4$, so it admits a smooth section $s: S^4 \to E$ corresponding to a trivial section of the trivial bundle. By the description of $p_1(TE)$ in the previous remark we have $s^* p_1(TE) = -\lambda A \cdot u$. If we choose $(A = 0, B = \frac{4\pi}{\pi}, C = \frac{2\pi}{\pi})$, which is a triple whose surgery obstruction vanishes, then the corresponding bundle has $s^* p_1(TE) = 0$ and $\hat{A}(E) \neq 0$. As the tangent bundle of $S^4$ is stably trivial, it follows that the normal bundle of $s(S^4) \subset E$ has trivial first Pontrjagin class. As $p_1: \pi_1(BSO(n)) \to Z$ is injective, this implies that this normal bundle is trivial.

**Remark 3.** The argument can be generalised to prove that for any even $n \geq 2$ there is a smooth $\mathbb{H}P^n$-bundle $E^{4n+4} \to S^4$ with $\hat{A}(E) \neq 0$, so the Corollary holds for $\mathbb{H}P^n$ as well.

Indeed, the application of Morlet’s lemma only required the fibre to be at least 8-dimensional and 3-connected. Similar to the above, one argues that for any pair $(A, C) \in \mathbb{Q}^2$ there exists a nonzero $λ \in Z$ and a normal invariant $n \in N_0(D^4 \times \mathbb{H}P^n)$ with underlying stable vector bundle $ξ$ such that
\[p_1(ξ) = λA \cdot u \quad p_i(ξ) = 0 \quad \text{for} \quad 1 < i < n, \quad p_{n+1}(ξ) = λ(2n+1)!(-1)^n C \cdot u \cdot z^n.
\]
Using that the coefficient of $z^n$ in $L(\mathbb{H}P^n)$ is 1 by the Hirzebruch’s signature theorem, we see that the surgery obstruction satisfies $8\sigma(n) = λ(2n+1)!(-1)^nC$ where $h_n$ is the coefficient of $p_n$ in the total $L$-class. In particular, we can always find pairs $(A, C)$ with $C \neq 0$ and $8\sigma(n) = 0$. As the coefficient of $z^n$ in $\hat{A}(\mathbb{H}P^n)$ vanishes, because $\mathbb{H}P^n$ admits a metric of positive scalar curvature, it holds $\hat{A}(E) = λa_n(2n+1)!(-1)^n C \neq 0$ with $a_n$ the coefficient of $p_n$ in the total $\hat{A}$-class, which is easily proved to be non-zero using [Hir95, Ch. 1. §1 (10)].

**Remark 4.** If one is willing to instead consider $\mathbb{H}P^n$-bundles over $S^{4m}$ for $n$ sufficiently large compared with $m$, then one may replace the appeal to the Lemma by the more classical [BL82, Corollary D], which implies that the map
\[\pi_{4m}(h\text{Aut}(\mathbb{H}P^n)/\text{Diff}(\mathbb{H}P^n); id) \otimes Z[\frac{1}{2}] \to \pi_{4m}(h\text{Aut}(\mathbb{H}P^n)/\text{Diff}(\mathbb{H}P^n); id) \otimes Z[\frac{1}{2}]
\]
is (split) surjective as long as $4m - 1$ lies in the pseudoisotopy stable range for $\mathbb{H}P^n$ (so $3m < n$ suffices, by [Igu88]). See also [Bur79, Theorem 1]. One must still produce an appropriate element of $S_3(D^{4m} \times \mathbb{H}P^n)$, which may be approached as in Remark 3.

**Acknowledgements.** We thank Thomas Schick for encouraging us to write this note, and Johannes Ebert for a comment which reminded us of [Bur79] and [BL82]. The first and third author were supported by a Philip Leverhulme Prize from the Leverhulme Trust. The third author was also supported by the ERC under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 756444).
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