Light leaves and Lusztig’s conjecture

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Abstract We give a combinatorial algorithm to find, for any given Weyl
or affine Weyl group, the set of primes for which Soergel’s conjecture hold.
This conjecture for Weyl groups is equivalent to a part of Lusztig’s con-
jecture and for affine Weyl groups implies (and is probably equivalent to)
the full Lusztig conjecture. The forementioned algorithm is based on the light
leaves basis, a combinatorial basis introduced by the author in a previous
paper for the Hom spaces between two Bott-Samelson-Soergel bimodules.
The light leaves basis has already found spectacular applications in the re-
cent proof by B. Elias and G. Williamson of the positivity of the coefficients
of Kazhdan-Lusztig polynomials for any Coxeter system and in their alge-
braic proof of Kazhdan-Lusztig conjecture.

1 Introduction

1.1 Lusztig’s Conjecture

Let $n \geq 2$ be an integer and $p$ a prime number. Consider the following ques-
tion.

Q: What are the characters of the irreducible rational representations of $GL_n(\mathbb{F}_p)$ over $\mathbb{F}_p$?

This natural question gained interest in the year 1963 when Steinberg proved
that all the irreducible representations of the finite groups $GL_n(\mathbb{F}_q)$ (with $q$ a
power of $p$) could be obtained from the irreducible representations of $GL_n(\mathbb{F}_p)$
by restriction (see [St]).

The groups $GL_n(\mathbb{F}_q)$ are examples of finite groups of Lie type, i.e. groups
$G(\mathbb{F}_q)$ of rational points of a reductive linear algebraic group $G$ defined over
$\mathbb{F}_q$. In the mid-seventies it became likely that (minor modifications of) the finite
groups of Lie type, together with the cyclic and alternating groups, would give
all the infinite families of finite simple groups; this gave a new impulse to the
study of Q.
A big breakthrough came in 1979 when Lusztig [Lu1] gave a conjectural answer for $\mathbb{Q}$ when $p > n$, and more generally, he conjectured a formula for the characters of the irreducible rational representations of any reductive algebraic group $G$ over $\mathbb{F}_p$ when $p$ is bigger than $h$, the Coxeter number. This is known as Lusztig’s Conjecture for algebraic groups [1].

### 1.2 Main approaches toward Lusztig’s Conjecture

There have been several approaches to Lusztig’s conjecture throughout the years. Let us recall some of them.

One approach is due to E. Cline, B. Parshall and L. Scott. They have many papers, starting in the late eighties, related to Lusztig’s conjecture and to the algebraic understanding of the geometry involved in Kazhdan-Lusztig theory (see [Sc] for an overview). Let $G$ be a reductive algebraic group and $A$ the quasi-hereditary algebra associated to the category of $G$-modules whose composition factors have regular high weights in a poset $\Gamma$ taken to be the Jantzen region; see [CPS]. They prove [CPS, remark 2.3.5] that if $A$ is Kozul (in a suitably strong sense) then Lusztig’s conjecture is true. To the date it has been impossible to prove the Kozulity of $A$. Parshall and Scott [PS] have recently made a step in this approach by showing that it is sometimes possible to transfer the good Lie theoretic properties to $\text{gr}A$, thereby artificially creating much the same situation that would be obtained from an actual Koszul grading on $A$ itself.

In 1990 Lusztig himself outlined a program ([Lu2], [Lu3]) for proving this conjecture when $p$ is “large enough”. This program used deep algebraic geometry. It was fulfilled in several steps by Kashiwara-Tanisaki ([KT1], [KT2]), Kazhdan-Lusztig ([KL1], [KL2], [KL3], [KL4]) and Andersen-Jantzen-Soergel ([AJS]). They essentially reduced the problem to the one of calculating the local intersection cohomology over the complex numbers of some finite dimensional Schubert varieties in an affine flag variety. However this approach seems not to lead to any reasonable bounds on $p$. Thus, apart from the cases

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1Actually Lusztig made his conjecture only for those irreducible modules with high weights in the “Jantzen region”, a finite collection of weights containing all “restricted” weights for $p \geq 2h - 3$, and thus giving formulas for any high weight by previous work of Steinberg. This conjecture was later generalized by Kato [Ka] to all restricted weights for $p \geq h$. This generalization is what we call “Lusztig’s conjecture”. In the literature is either called “Lusztig’s conjecture”, “Kato’s conjecture” or “Kato’s extension of the Lusztig conjecture”.

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$A_1, A_2, A_3, B_2, G_2$, the characters of the irreducible modules of $G$ remained unknown for a given characteristic.

A more recent trend is R. Bezrukavnikov geometric approach, in collaboration with I. Mirkovic, D. Rumynin and S. Arkhipov. This program is implicit in Bezrukavnikov’s ICM address [Be]. In 2010 Bezrukavnikov and Mircović [BM] prove Lusztig’s conjectures [Lu4] relating canonical basis in the homology of a Springer fiber to modular representations of semi-simple Lie algebras. Their proof is based on the works [BMR1], [BMR2] and [AB]. These conjectures of Lusztig are what at some point was known as “Lusztig’s Hope”. Lusztig’s conjecture (on representations of algebraic groups in positive characteristic) is essentially equivalent to the particular case of restricted representations, when the relevant Springer fiber is the full flag variety, so with this method they give a new proof of Lusztig’s conjecture for $p \gg h$.

In 2010 P. Fiebig made an improvement in the known results. In the paper [Fi4], he finds, for any given root system $R$, an explicit number $N(R)$ such that Lusztig’s conjecture is true for $p > N(R)$. The only problem is that $N(R)$ is enormously big compared with the Coxeter number. To find this number $N(R)$ he introduces a category of sheaves on moment graphs and translates the problem into that language.

1.3 Soergel’s approach

Let us explain Soergel’s approach to the problem dating back to 1990. For any Coxeter group $W$, Soergel constructs a polynomial ring $R$ with coefficients in the real numbers $\mathbb{R}$. Then he constructs a concrete category $\mathcal{B} = \mathcal{B}(W, \mathbb{R})$ of graded bimodules over $R$, called the category of Soergel bimodules.

Soergel proved in [So2] that $\mathcal{B}$ categorifies the Hecke algebra $\mathcal{H}$ of $W$ in the sense that he constructs a ring isomorphism

$$\varepsilon : \mathcal{H} \to \langle \mathcal{B} \rangle,$$

where $\langle \mathcal{B} \rangle$ denotes the split Grothendieck group of $B$. He stated a beautiful conjecture (that we call Soergel’s 0-conjecture, even though it is a theorem for some months now) saying that, if $\{C'_x\}_{x \in W}$ is the Kazhdan-Lusztig basis of the Hecke algebra, then $\varepsilon(C'_x) = \langle B_x \rangle$, where $B_x$ is an indecomposable object of $B$.

The reader might ask why is there a 0 in “Soergel’s 0-conjecture”. The answer is that the characteristic of the field $\mathbb{R}$ is 0 and we can do similar constructions
for fields with positive characteristic, but the situation there is more complicated.

If $W$ is a Weyl group then everything works as well over a field of characteristic $p$ and Soergel’s conjecture in this context will be called Soergel’s $p$-conjecture. If $W$ is an affine Weyl group, then there is a finite subset $W^\circ$ of $W$ (for details see Section 7.1) such that all the theory works if one replaces $W$ by $W^\circ$ in characteristic $p$. This version of Soergel’s conjecture will be called affine Soergel’s $p$-conjecture and it will mean that $\varepsilon(C'_x) = \langle B_x \rangle$ for elements $x \in W^\circ$. The naive generalization of Soergel’s conjecture to this context, i.e. to conjecture that this equation is true for every element $x \in W$, is not true.

We would like to note that Soergel never conjectured any statements for positive characteristic and, as we already observed, the naive positive characteristic analogue of Soergel’s conjecture is false in general. One would like to know, for any Coxeter group $W$, in what subset of $W$ does this analogue of Soergel’s conjecture hold.

Soergel himself proved Soergel’s 0-conjecture for Weyl groups in [So1] and M. Härterich proved it for affine Weyl groups in [Ha]. Soergel’s 0-conjecture for universal Coxeter groups was proved independently by P. Fiebig in [Fi2] and the author in [Li6].

A revolution in this subject was done recently by the beautiful algebraic proof of Soergel’s 0-conjecture given by B. Elias and G. Williamson in [EW1]. By previous results of Soergel, their work gives a proof of the longstanding and fundamental Kazhdan-Lusztig positivity conjecture for every Coxeter system and an algebraic proof of Kazhdan-Lusztig conjecture. With their method they not only solved these conjectures, but they invented a new mathematical subject: “algebraic Hodge theory” by looking at Soergel bimodules as intersection cohomology of some (non-existent) spaces. When $W$ is a Weyl or affine Weyl group, these spaces do exist.

Soergel proved in [So3] that Soergel’s $p$-conjecture is equivalent to a part of Lusztig’s conjecture (for weights around the Steinberg weight). We prove in this paper that affine Soergel’s $p$-conjecture is equivalent to Fiebig’s conjecture about shaves on Bruhat graphs. Fiebig proved that his conjecture implies the full Lusztig conjecture. We believe that the converse is also true: Lusztig’s conjecture implies Fiebig’s conjecture, but we are not yet able to prove it.
1.4  Double leaves basis

Now we explain our contributions to Soergel’s approach. For each reduced expression $s$ of an element $x \in W$ there is an explicit Soergel bimodule $M_s$ called the Bott-Samelson bimodule (see Section 2.3 for details). For each $x \in W$ we fix a reduced expression of $x$ and we call $M_x$ the corresponding Bott-Samelson. In [Li1] we constructed combinatorially, for $x \in W$ a perfect binary tree $T_x$ (i.e. a tree where every node other than the leaves has two children and in which all leaves are at the same depth) where the nodes are colored by Bott-Samelson bimodules and the edges are colored by morphisms from the corresponding parent to child. The root node of $T_x$ is colored by $M_x$ and if a leaf is colored by $M_y$ then the composition of all the morphisms in the path from the root to that leaf gives an element in $\text{Hom}(M_x, M_y)$ that we call a leaf from $x$ to $y$. We identify each leaf in $T_x$ with the corresponding morphism between Bott-Samelson bimodules.

For each leaf $l$ from $x$ to $y$ there exist its adjoint leaf $l^o \in \text{Hom}(M_y, M_x)$ (in fact, every morphism between Bott-Samelson bimodules has an adjoint morphism inverting the sense of the arrow). If $x, y \in W$ and we have leaves $l_u$ from $x$ to $u$ and $l_v$ from $y$ to $v$, we define the product $l_v \cdot l_u$ as $l_v^o \circ l_u \in \text{Hom}(M_x, M_y)$ if $u = v$ and empty set if $u \neq v$. The first theorem of this paper is the following:

**Theorem 1.1** The set $\{l' \cdot l | l \text{ leaf of } T_x \text{ and } l' \text{ leaf of } T_y\}$ is a basis of $\text{Hom}(M_x, M_y)$ as a left $R$-module.

We call this the double leaves basis (LL) of $\text{Hom}(M_x, M_y)$. We can “see” this basis as gluing the tree $T_x$ with the tree $T_y$ inverted. This construction is heavily based on that of the light leaves basis in [Li1].

One spectacular application of this theorem is the previously cited proof of Soergel’s 0-conjecture.

1.5  How to find the good primes

Let $W$ denote a Weyl group or an affine Weyl group. If $W$ is a Weyl group we define $W^o = W$ and if $W$ is an affine Weyl group $W^o$ is the finite subset

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2In a personal communication G. Williamson told the author "the original proof of Soergel’s conjecture was in the light leaves language, which allowed us to do many calculations, and appears more natural at several steps in the proof. It will appear explicitly in [EW4]. We could not publish it right away because the results relied on the long papers [EW2], [EW3] which are still in preparation. Hence we decided to make the proof purely algebraic so that it only relied on citable results".
considered in Section 1.3.

For every \( x \in W^\circ \), there is an indecomposable Soergel bimodule \( B_x \in \mathcal{B}(W, \mathbb{Q}) \) appearing only once in the direct sum decomposition of \( M_x \). We find an algorithm to express the projector \( p_x \) corresponding to \( B_x \) as a linear combination with coefficients in \( \mathbb{Q} \) of the elements in the LL of \( \text{End}(M_x) \). Let \( d(x) \) be the set of primes dividing the denominators of the above-mentioned coefficients and

\[
D = \bigcup_{x \in W^\circ} d(x).
\]

In the category \( \mathcal{B}(W^\circ, \mathbb{F}_p) \) (defined as the full subcategory of \( \mathcal{B}(W, \mathbb{F}_p) \) with objects direct sums of shifts of objects of the type \( B_x \) with \( x \in W^\circ \)) the LL are also well defined, so if \( p \notin D \), you can reduce mod \( p \) the coefficients in the expansion of \( p_x \) in terms of the LL so as to produce projectors \( p'_x \in \mathcal{B}(W^\circ, \mathbb{F}_p) \) for every \( x \in W^\circ \). The corresponding bimodules \( \text{Im}(p'_x) \in \mathcal{B}(W^\circ, \mathbb{F}_p) \) have the same decategorification in \( \mathcal{H} \) as \( \text{Im}(p_x) \in \mathcal{B}(W^\circ, \mathbb{Q}) \). As Soergel’s 0-conjecture is true, with this method one can prove that it is true in \( \mathcal{B}(W^\circ, \mathbb{F}_p) \) if and only if \( p \notin D \).

Let us roughly explain the algorithm mentioned above. Consider \( x, y \in W^\circ \) and \( l, l' \) two degree zero leaves from \( x \) to \( y \). The restriction of the degree zero endomorphism \( l \circ l'^a \) to the one dimensional degree zero part of \( M_y \) is a scalar multiple of the identity that we denote \( \lambda(l, l') \in \mathbb{Q} \). For \( x, y \in W^\circ \) there exist a subset \( L_{x,y} = \{l_1, \ldots, l_n\} \) of the set of degree zero leaves from \( x \) to \( y \) such that if \( d(x, y) \) is the determinant of the matrix with \( (i, j) \)-entry \( \lambda(l_i, l_j) \), then we have the following theorem.

**Theorem 1.2** Let \( R \) be a root system with affine Weyl group \( W \) (resp. Weyl group \( W \)). Lusztig’s conjecture (resp. Lusztig’s conjecture around the Steinberg weight) for algebraic groups with root system \( R \) is true if (resp. if and only if) the characteristic is not contained in the set

\[
D = \{d(x, y) \mid x \in W^\circ, y \leq x\}
\]

where \( \leq \) stands for the Bruhat order.

We remark that (as we said before) we believe that Fiebig’s conjecture is equivalent to Lusztig’s conjecture and this would imply that in this theorem, when \( W \) is an affine Weyl group we could replace “if” by “if and only if”. The missing part (the only if) could be relevant only if Lusztig’s conjecture is false and we could find counterexamples in weights not around the Steinberg weight.
Theorem 1.2 (proved in Section 4.3) is not exactly the algorithm we mentioned in the abstract, because the set $L_{x,y}$ is not explicitly constructed. The explicit way to calculate the ”good primes” is very similar, but a little bit more involved. For details look in Section 4.2. We remark that a better understanding of the numbers $\lambda(l_i, l_j)$ could give a good bound (at least exponentially better than Fiebig’s bound) for Lusztig’s conjecture. And of course, if we could control the determinants $d(x, y)$ in some way, for instance by finding relations between $\lambda(l_i, l_j)$, $\lambda(l_j, l_k)$ and $\lambda(l_i, l_k)$ or similar ideas, this might eventually lead to a proof of Lusztig’s conjecture.

An algorithm to see when Soergel’s $p$-conjecture or affine Soergel’s $p$-conjecture is implied by Soergel’s 0-conjecture was found independently by G. Williamson using parity sheaves. He announced his results in [Wi1] and [Wi2].

### 1.6 Structure of the paper

The structure of the paper is as follows. In Section 2 we give the definitions and first properties of Hecke algebras and Soergel bimodules. In Section 3 we construct the double leaves basis and in Section 5 we prove that it is indeed a basis for arbitrary Coxeter systems. In Section 4.2 we find the above-mentioned algorithm to express $p_x$ as a linear combination of the elements of the DLB and in Section 4.3 we define $L_{x,y}$ and prove Theorem 1.2. In section 4.4 we explain the relation between idempotents in Soergel’s theory over $\mathbb{Q}_p$ and idempotents in Soergel’s theory over $\mathbb{F}_p$. In section 6 we find, as a corollary of the proof in Section 5 a basis (that we call leaves basis) of the spaces $\text{Hom}(M_x, R_y)$, with $x, y \in W$ (see 2.3 for the definition of $R_y$). These Hom spaces appear in the formulas to decategorify Soergel bimodules. Finally, in Section 7 we prove that affine Soergel’s $p$-conjecture is equivalent to Fiebig’s conjecture in Bruhat graph theory.

### 1.7 Acknowledgements

We would like to thank Geordie Williamson, Francois Digne and Wolfgang Soergel for helpful comments.

The results of this paper have been exposed from 2011 to 2013 (Freiburg, Erlangen, Buenos Aires, Córdoba, Valparaíso, Olmué, Pucón, Tucumán).

This work was partially supported by the Von Humboldt foundation during the author’s postdoctoral stay in Freiburg and by Fondecyt project number 11121118.
2 Preliminaries

In this section we consider \((W, S)\) an arbitrary Coxeter system unless explicitly stated. We define the corresponding Hecke algebra and Soergel bimodules.

2.1 Hecke algebras

Let \(A = \mathbb{Z}[v, v^{-1}]\) be the ring of Laurent polynomials with integer coefficients. The Hecke algebra \(H = H(W, S)\) is the \(A\)-algebra with generators \(\{T_s\}_{s \in S}\), and relations

\[
T_s^2 = v^{-2} + (v^{-2} - 1)T_s \quad \text{for all} \ s \in S \quad \text{and} \quad \frac{T_s T_r T_s \ldots}{m(s, r) \text{ terms}} = \frac{T_r T_s T_r \ldots}{m(s, r) \text{ terms}} \quad \text{if} \ s, r \in S \quad \text{and} \ sr \text{ is of order } m(s, r).
\]

If \(x = s_1 s_2 \cdots s_n\) is a reduced expression of \(x\), we define \(T_x = T_{s_1} T_{s_2} \cdots T_{s_n}\) (\(T_x\) does not depend on the choice of the reduced expression). The set \(\{T_x\}_{x \in W}\) is a basis of the \(A\)-module \(H\). We put \(q = v^{-2}\) and \(\tilde{T}_x = v^{l(x)} T_x\).

There exists a unique ring involution \(d: H \to H\) with \(d(v) = v^{-1}\) and \(d(T_x) = (T_{x^{-1}})^{-1}\). Kazhdan and Lusztig [KL1] proved that for \(x \in W\) there exist a unique \(C'_x \in H\) with \(d(C'_x) = C'_x\) and

\[
C'_x \in \tilde{T}_x + \sum_y v\mathbb{Z}[v] \tilde{T}_y.
\]

The set \(\{C'_x\}_{x \in W}\) is the so-called Kazhdan-Lusztig basis of the Hecke algebra. It is a basis of \(H\) as an \(A\)-module.

2.2 Reflection representations

Recall that \((W, S)\) is an arbitrary Coxeter system. A reflection faithful representation of \(W\) over \(k\) as defined by Soergel is a finite dimensional \(k\)-representation that is faithful, and such that, for \(w \in W\) the fixed point set \(V^w\) has codimension one in \(V\) if and only if \(w\) is a reflection, i.e. a conjugate of a simple reflection (we call this set \(T \subset W\)). For \(k = \mathbb{R}\) Soergel constructed [So4] such a representation for any Coxeter system.

For the purposes of this paper we need to define a class of representations in which all of Soergel’s theory works but that are local in nature. For \(x \in W\) consider the reversed graph

\[
\text{Gr}(x) = \{(xv, v) \mid v \in V\} \subseteq V \times V.
\]
**Definition 1** Let $W'$ be a subset of $W$ closed under $\leq$ (where $\leq$ is the Bruhat order). This means that for every $w' \in W'$, we have
\[
\{ w \in W \mid w \leq w' \} \subseteq W'.
\]
An $n$–dimensional representation $V$ of $W$ is called $W'$–reflection if the two following conditions hold.

- For $x, y \in W'$, we have $\dim(\text{Gr}(x) \cap \text{Gr}(y)) = n - 1$ if and only if $x^{-1}y \in T$.
- There is no $x, y, z \in W'$ different elements such that
  \[
  \text{Gr}(x) \cap \text{Gr}(y) = \text{Gr}(z) \cap \text{Gr}(y),
  \]
  both sets having dimension $n - 1$.

We remark that a $W$–reflection representation $V$ of $W$ is reflection faithful, and that all of Soergel’s theory works for a $W$-reflection representation and also for a $W'$-reflection representation in the sense of Proposition 2.1(5).

We now make a brief terminology detour.

**Terminology 1** The term ”geometric representation” defined in [Bo] and used generally in the literature seems flawed to us. This representation is not more geometric than the contravariant one. We propose (with W. Soergel) to call it rootic representation as you can see the lines generated by the roots as pairwise disjoint $(-1)$-eigenspaces of reflections. We also propose to call the contravariant representation alcovic representation, since the alcoves are visible.

We proved in [Li2] that all of Soergel’s theory over $k = \mathbb{R}$ works as well if you choose the rootic representation (that it is not a reflection faithful representation).

Let $k$ be a field of characteristic different from 2 and let $G \supset B \supset T$ be a semisimple split simply connected algebraic group over $k$ with a Borel and a maximal torus. Let $(W, S)$ be the finite Weyl group of $G \supset B$. The representation of $W$ on the Lie algebra $\text{Lie}(T)$ is reflection faithful.

Let $\widehat{W}$ be the affine Weyl group associated to $G \supset B$. Let $V$ be a realization of the affine Cartan matrix. This representation of $\widehat{W}$ is not reflection faithful but $\widehat{W}^\triangledown$–reflection as defined before, where we define $\widehat{W}^\triangledown$ as the finite subset of $\widehat{W}$ consisting of the elements that are lesser or equal than the longest element in the set of antidominant restricted elements.
So, summing up, if $W$ is a Weyl group of $G \supset B \supset T$ we take $V = \text{Lie}(T)$ as representation of $W$, if $\widehat{W}$ is an affine Weyl group we take $V$ a realization of the affine Cartan matrix and if $W$ is a general Coxeter group that is not a Weyl nor an affine Weyl group and $k = \mathbb{R}$ we take $V$ as the rootic representation.

2.3 Soergel bimodules

For any $\mathbb{Z}$-graded object $M = \bigoplus_i M_i$, and every $n \in \mathbb{Z}$, we denote by $M(n)$ the shifted object defined by the formula

$$(M(n))_i = M_{i+n}.$$

Let $R = R(V)$ be the algebra of regular functions on $V$ with the following grading: $R = \bigoplus_{i \in \mathbb{Z}} R_i$ with $R_2 = V^*$ and $R_i = 0$ if $i$ is odd. The action of $W$ on $V$ induces an action on $R$. For $s \in S$ consider the graded $(R, R)$–bimodule

$$B_s = R \otimes_{R^s} R(1),$$

where $R^s$ is the subspace of $R$ fixed by $s$.

The category of Soergel bimodules $\mathcal{B} = \mathcal{B}(W, k)$ is the category of $\mathbb{Z}$–graded $(R, R)$–bimodules with objects the finite direct sums of direct summands of objects of the type

$$B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_n}(d)$$

for $(s_1, \ldots, s_n) \in S^n$, and $d \in \mathbb{Z}$.

Given $M, N \in \mathcal{B}$ we denote their tensor product simply by juxtaposition: $M \otimes N$.

If $\underline{s} = (s_1, \ldots, s_n) \in S^n$, we will denote by $B_{\underline{s}}$ the $(R, R)$–bimodule

$$B_{s_1} \otimes_R B_{s_2} \cdots B_{s_n} \cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R(n).$$

We use the convention $B_{[1]} = R$. Bimodules of the type $B_{\underline{s}}$ will be called Bott-Samelson bimodules.

Given a Laurent polynomial with positive coefficients $P = \sum a_i v^i \in \mathbb{N}[v, v^{-1}]$ and a graded bimodule $M$ we set

$$P \cdot M := \bigoplus M(-i)^{\oplus a_i}.$$

For every essentially small additive category $\mathcal{A}$, we call $\langle \mathcal{A} \rangle$ the split Grothendieck group. It is the free abelian group generated by the objects of $\mathcal{A}$ modulo the relations $M = M' + M''$ whenever we have $M \cong M' \oplus M''$. Given an object $A \in \mathcal{A}$, let $\langle A \rangle$ denote its class in $\langle \mathcal{A} \rangle$.

In [So2] Soergel proves that there exists a unique ring isomorphism $\varepsilon : \mathcal{H} \to \langle \mathcal{B} \rangle$ such that $\varepsilon(v) = \langle R(1) \rangle$ and $\varepsilon(T_s + 1) = \langle R \otimes_{R^s} R \rangle$ for all $s \in S$. 

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2.4 Support

For any finite subset $A$ of $W$ consider the union of the corresponding graphs

$$\text{Gr}(A) = \bigcup_{x \in A} \text{Gr}(x) \subseteq V \times V.$$ 

If $A$ is finite, we view $\text{Gr}(A)$ as a subvariety of $V \times V$. If we identify $R \otimes_k R$ with the regular functions on $V \times V$ then $R_A$, the regular functions on $\text{Gr}(A)$, are naturally $\mathbb{Z}$-graded $R$-bimodules. We will also write $R_{\{x\}} = R_x$. One may check that given $x \in W$ the bimodule $R_x$ has the following simple description: $R_x \cong R$ as a left module, and the right action is twisted by $x$: $m \cdot r = mx(r)$ for $m \in R_x$ and $r \in R$.

For any $R$-bimodule $M \in R\text{-Mod-}R$ we can view $M$ as an $R \otimes_k R$-module (because $R$ is commutative) and hence as a quasi coherent sheaf on $V \times V$. Given any finite subset $A \subseteq W$ we define

$$M_A := \{m \in M \mid \text{supp } m \subseteq \text{Gr}(A)\}$$

to be the subbimodule consisting of elements whose support is contained in $\text{Gr}(A)$ (we remark that in [So4] the bimodule $M_A$ is denoted $\Gamma_A M$). For $x \in W$ we denote $M_{\{x\}} = M_x$.

Given any Soergel bimodule $M$ we define $M^x$ the restriction of $M$ to $\text{Gr}(x)$, and $M^{x \cap y}$ its restriction to $\text{Gr}(x) \cap \text{Gr}(y)$. For $x, y \in W$ we call $\rho_{x,y} : M^x \to M^{x \cap y}$ the restriction map.

In the following we will abuse notation and write \( \leq x \) for the set \( \{ y \in W \mid y \leq x \} \).

2.5 Indecomposable Soergel bimodules

We start by recalling the most important features of the indecomposable Soergel bimodules, central in Soergel’s theory. Consider $W$ a Weyl group and $B = B(W, k)$ with $k = \mathbb{Q}$ or $k = \mathbb{F}_p$. The following can be found in [So2, Theorem 2] and [So4, Satz 6.16].

**Proposition 2.1**  
(1) For all $w \in W$ there is, up to isomorphism, a unique indecomposable $B_w \in B$ with support in $\text{Gr}(\leq w)$ and such that $B_w \cong R_w(l(w))$.

(2) The map $(w, i) \mapsto B_w(i)$ defines a bijection from the set $W \times \mathbb{Z}$ to the set of indecomposable objects of $B$ up to isomorphism.
(3) For all \( x, y \in W \), we have the following formula
\[
\text{Hom}(B_x, B_y) \cong \begin{cases} 
  k & \text{if } x = y, \\
  0 & \text{otherwise},
\end{cases}
\]
where \( \text{Hom}(B_x, B_y) \) denote the set of degree zero elements in \( \text{Hom}(B_x, B_y) \).

(4) If \( s = (s_1, \ldots, s_n) \) is a reduced expression of \( w \in W \) then there are polynomials \( p_y \in \mathbb{N}[v, v^{-1}] \) such that
\[
B_s \cong B_w \bigoplus_{y < x} p_y B_y.
\]

(5) If we replace everywhere in this proposition the Weyl group \( W \) by the finite subset \( \hat{W}^0 \) of the affine Weyl group \( \hat{W} \) then point (1) remains true, so if we replace \( B(W, k) \) by the full subcategory \( B^0 \) of \( B(\hat{W}, k) \) with objects finite direct sums of elements in the set
\[
\{ B_w(d) \mid w \in \hat{W}^0, d \in \mathbb{Z} \},
\]
then all points (1) through (4) remain true.

We can now state \([\text{So}2, \text{Conjecture 1.13}]\), the central conjecture of Soergel.

**Conjecture 2.2** (Soergel’s 0-conjecture) If \( k = \mathbb{R} \) and \( W \) is any Coxeter system, for every \( w \in W \), there exists an indecomposable bimodule \( B_w \in \mathcal{B} \) such that \( \varepsilon(C'_w) = \langle B_w \rangle \).

Its generalization to positive characteristic (not explicitly stated by Soergel but considered in \([\text{So}3]\)).

**Conjecture 2.3** (Soergel’s \( p \)-conjecture) If \( k = \mathbb{F}_p \) and \( W \) is a Weyl group, for every \( w \in W \), there exists an indecomposable bimodule \( B_w \in \mathcal{B} \) such that \( \varepsilon(C'_w) = \langle B_w \rangle \).

And now its affine generalization in positive characteristic. This conjecture was never considered by Soergel, but it seems reasonable for us to call it

**Conjecture 2.4** (affine Soergel’s \( p \)-conjecture) If \( k = \mathbb{F}_p \) and \( W \) is an affine Weyl group, for every \( w \in W^0 \), there exists an indecomposable bimodule \( B_w \in \mathcal{B} \) such that \( \varepsilon(C'_w) = \langle B_w \rangle \).

It is this last conjecture that implies (and should be equivalent to) Lusztig’s conjecture.
3  Double leaves basis

In this section $(W, S)$ is again an arbitrary Coxeter system. Let us fix for the rest of this section the sequences $\underline{s} = (s_1, \ldots, s_n) \in S^n$ and $\underline{r} = (r_1, \ldots, r_p) \in S^p$. In this section we will define the double leaves basis (DLB), a basis of the space $\text{Hom}(B_{\underline{s}}, B_{\underline{r}})$ closely related with the light leaves basis (LLB) constructed in [Li1]. The DLB will be more useful for the purposes of this paper than the LLB because of its symmetry properties. We start by recalling the construction of the tree $T_{\underline{s}}$.

3.1  The tree $T_{\underline{s}}$

3.1.1  Three basic morphisms

In this subsection we will introduce three important morphisms between Soergel bimodules. Let $x_s \in V^*$ be an equation of the hyperplane fixed by $s \in S$. We have a decomposition $R \simeq R^s \oplus x_s R^s$, corresponding to

$$R \ni p = \frac{p + s \cdot p}{2} + \frac{p - s \cdot p}{2}.$$  

We define the Demazure operator, a morphism of graded $R^s$-modules

$$\partial_s : R(2) \to R, \quad p_1 + x_s p_2 \mapsto p_2.$$  

We will now define three morphisms in $B$ that are the basic ingredients in the construction of the double leaves basis. The first one is the multiplication morphism

$$m_s : B_s \to R \quad R \otimes_{R^s} R(1) \ni p \otimes q \mapsto pq.$$  

The second one is the only (up to non-zero scalar) degree $-1$ morphism from $B_s B_s$ to $B_s$:

$$j_s : B_s B_s \to B_s \quad R \otimes_{R^s} R \otimes_{R^s} R(2) \ni p \otimes q \otimes r \mapsto p \partial_s(q) \otimes r.$$  

Consider the bimodule $X_{sr} = B_s B_r B_s \cdots$ the product having $m(s, r)$ terms (recall that $m(s, r)$ is the order of $sr$). We define $f_{sr}$ as the only degree $0$ morphism from $X_{sr}$ to $X_{rs}$ sending $1 \otimes 1 \otimes \cdots \otimes 1$ to $1 \otimes 1 \otimes \cdots \otimes 1$. In [Li2], [Li4] and [EK] there are different explicit formulas for $f_{sr}$.
3.1.2 Some choices

For \( x \in W \) consider the set \( \mathcal{R}(x) \) of all reduced expressions of \( x \). The graph of reduced expressions of \( x \) (or \( \text{Gre}_x \)) is the graph with nodes the elements of \( \mathcal{R}(x) \) and edges between nodes that are connected by a braid move. It is a known fact that this graph is connected. Moreover, for every couple \((s, t)\) with \( s \in S \) and \( t \) a reduced expression of some \( x \in W \) satisfying that \( l(xs) < l(x) \), there exist a path in \( \text{Gre}_x \) starting at \( t \) and ending in an element \( u \) satisfying that the last element in the sequence \( u \) is \( s \). In fact there exist many paths satisfying this property, but we choose arbitrarily one of them and call it \( P(s, t) \).

For every couple of nodes \((u, t)\) in \( \text{Gre}_x \) that are connected by an edge (i.e. where \( u \) and \( t \) differ by a braid move) there is an associated morphism in \( \text{Hom}(B_u, B_t) \), of the type \( \text{id} \otimes f_{sr} \otimes \text{id} \) (we look here the morphism \( f_{sr} \) as the categorification of the braid relation). In this way, every directed path in \( \text{Gre}_x \) gives a morphism from the starting node to the ending node. We define \( F_i(t) \) the morphism between Bott-Samelson bimodules associated to \( P(s, t) \).

Let us fix, for every \( x \in W \) a reduced expression \( x \in \mathcal{R}(x) \) such that the sequence \( x \) is the same as the sequence \( x^{-1} \) read from right to left. Also, for any reduced expression \( u \in \mathcal{R}(x) \) we fix a directed path in \( \text{Gre}_x \) starting in \( u \) and ending in \( x \). We denote by \( F(u, x) \) the corresponding morphism from \( B_u \) to \( B_x \).

3.1.3 Construction of \( \mathbb{T}_\Lambda \)

We construct a perfect binary tree with nodes colored by Bott-Samelson bimodules and arrows colored by morphisms from parent to child nodes. We construct it by induction on the depth of the nodes. In depth one we have the following tree:

![Diagram](image)

Let \( k < n \) and \( t = (t_1, \cdots, t_i) \in S^i \) be such that a node \( N \) of depth \( k - 1 \) is colored by the bimodule \((B_{t_1} \cdots B_{t_i})(B_{s_k} \cdots B_{s_n})\), then we have two cases.
(1) If we have the inequality $l(t_1 \cdots t_is_k) > l(t_1 \cdots t_i)$, then the child nodes and child edges of $N$ are colored in the following way:

\[
(B_{t_1} \cdots B_{t_i})(B_{sk} \cdots B_{sn})
\]

\[
\xymatrix{(B_{t_1} \cdots B_{t_i})(B_{sk} \cdots B_{sn}) & (B_{t_1} \cdots B_{t_i})(B_{sk+1} \cdots B_{sn}) \ar[l]^{(B_{t_1} \cdots B_{t_i})} \ar[r]_{(B_{t_1} \cdots B_{t_i})} & (B_{t_1} \cdots B_{t_i})(B_{sk} \cdots B_{sn})}
\]

(2) If we have the opposite inequality $l(t_1 \cdots t_is_k) < l(t_1 \cdots t_i)$, then the child nodes and child edges of $N$ are colored in the following way (arrows are the composition of the corresponding pointed arrows):

\[
(B_{t_1} \cdots B_{t_i})(B_{sk} \cdots B_{sn})
\]

\[
\xymatrix{(B_{t_1} \cdots B_{t_i})(B_{sk} \cdots B_{sn}) & (B_{t_1} \cdots B_{t_i})(B_{sk+1} \cdots B_{sn}) \ar[l]^{(B_{t_1} \cdots B_{t_i})} \ar[r]_{(B_{t_1} \cdots B_{t_i})} & (B_{t_1} \cdots B_{t_i})(B_{sk} \cdots B_{sn})}
\]

In the last step of the construction, i.e. $k = n$, we do exactly the same arrows as in cases (1) and (2) but we compose each one of the lower Bott-Samelsons with a morphism of the type $F(u, x)$ (see Section 3.1.2) with $u$ being a reduced expression of $x \in W$. So we have that each leaf of the tree is colored by a bimodule of the form $B_x$ for some $x \in W$. This finishes the construction of $T_x$.

By composition of the corresponding arrows we can see every leaf of the tree $T_x$ colored by $B_x$ as a morphism in the space $\text{Hom}(B_x, B_x)$. Consider the set $L_x(id)$, the leaves of $T_x$ that are colored by the bimodule $R$. In [Li1] the set $L_x(id)$ is called light leaves basis and the following theorem is proved.

**Theorem 3.1** The set $L_x(id)$ is a basis of $\text{Hom}(B_x, R)$ as a left $R$-module.
3.2 Construction of the double leaves basis

In [Li1] the problem of finding a basis of the space $H = \text{Hom}(B_2, B_2)$ was solved using Theorem 3.1 and using repeatedly the adjunction isomorphism

$$\text{Hom}(MB_s, N) \cong \text{Hom}(M, NB_s),$$

which is an isomorphism of graded left $R$-modules. This gives one basis that we called light leaves basis in [Li1] of the space $H$. In this section we will introduce a new basis, that we consider to be the natural generalization of the construction made in Section 5.3 and that we will call the double leaves basis of $H$. This basis can be thought of as follows. Take $T_s$ and ”paste” it with the tree $T_r$ inverted. This is the image we have to keep in mind.

Let us be more precise. We need to introduce the adjoint morphisms (in the sense of the adjunction (1)) of $m_s, j_s, f_{sr}$. The adjoint of $m_s$ is the morphism

$$\epsilon_s : R \rightarrow B_s$$

$$1 \mapsto x_s \otimes 1 + 1 \otimes x_s.$$  

The adjoint of $j_s$ is the morphism

$$p_s : B_s \rightarrow B_s B_s$$

$$a \otimes b \mapsto a \otimes 1 \otimes b.$$  

And finally the adjoint morphism of $f_{sr}$ is $f_{rs}$.

For any leaf $f : B_2 \rightarrow B_2$ in $T_s$ we can find its adjoint leaf $f^a : B_2 \rightarrow B_2$ by replacing each morphism in the set $\{m_s, j_s, f_{sr}\}$ by its adjoint. So we obtain a tree $T^a$ where the arrows go from children to parents.

Let $x, y \in W$. If $f \in \text{Hom}(B_2, B_2)$ and $g \in \text{Hom}(B_2, B_2)$, we define

$$g \cdot f = \begin{cases} 
    g \circ f & \text{if } x = y \\
    \emptyset & \text{if } x \neq y
\end{cases}$$

Let $L_2$ be the set of leaves of $T_2$, this is

$$L_2 \subset \prod_{x \in W} \text{Hom}(B_2, B_2).$$

We call the set $L_2^a \cdot L_2$ the double leaves basis (or DLB) of $\text{Hom}(B_2, B_2)$.

**Theorem 3.2** The DLB is a basis as left $R$-module of the space $\text{Hom}(B_2, B_2)$.

The proof of Theorem 3.2 is very similar to the proof of [Li1, Théorème 5.1]. We will write down all the details in Section 5.
4 Soergel’s conjecture for affine Weyl groups

In sections 4.1 through 4.3 we consider $W$ an arbitrary Coxeter system and $k$ a field of characteristic zero. We will find a recursive algorithm to find the set of “good primes” for Soergel’s conjecture.

4.1 The favorite projector in any Bott-Samelson

Let $x \in W$ and $s \in S$ be such that $l(xs) > l(x)$. Then we have the following formula in the Hecke algebra

$$C'_x C'_s = C'_{xs} + \sum_{y < xs} m_y C'_y,$$

with $m_y \in \mathbb{N}$.

As Soergel’s 0-conjecture is now a theorem, this formula implies

$$B_x B_s \cong B_{xs} \oplus \bigoplus_{y < xs} B_{ym_y}^y,$$

with $m_y \in \mathbb{N}$. (2)

By Proposition 2.1 (3) we see that there is only one projector $p_{xs}$ in $\text{End}(B_x B_s)$ whose image is $B_{xs}$. Moreover, if $z \in Z := \{y \in W | m_y \neq 0\}$, there is only one projector $p_{zs}$ in $\text{End}(B_x B_s)$ whose image is $B_{zs}^z$.

By the previous paragraph, for $t$ a reduced expression of $y \in W$, there is an obvious way to define by induction on the length of $t$ a projector $p_{t'}$ in $\text{End}(B_t)$ whose image is isomorphic to $B_y$. Let us give the details. If $t = (t_1)$ then $p(t_1) = \text{id} \in \text{End}(B_{t_1})$.

Let $t = (t_1, \ldots, t_k) \in S^k$ be a reduced expression of $y \in W$ and suppose that we have defined $p_{t_1'}$ in $B_{t_1} \cdots B_{t_{k-1}}$, with $t_1' = (t_1, \ldots, t_{k-1})$ and $x = t_1 \cdots t_{k-1}$. As the image of $p_{t_1'} \otimes \text{id} \in \text{End}(B_y)$ is isomorphic to $B_x B_{t_k}$, we define

$$p_t = p_{x,t_k} \circ (p_{t'} \otimes \text{id}).$$

4.2 Explicit construction of the favorite projector

We will construct the favorite projector by induction in the length of the Bott-Samelson. We want to find the favorite projector of the expression $s = (s_1, \ldots, s_n)$ and we suppose that we have explicitly constructed the favorite projector of every Bott-Samelson having less than $n$ terms. Let $s' = (s_1, \ldots, s_{n-1})$, $x = s_1 \cdots s_{n-1} \in W$ and $s_n = s$. 

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If \( q \) and \( t \) are reduced expressions of some elements in \( W \) and \( f \in \text{Hom}(B_q, B_t) \) then we denote 
\[
pf := p_q \circ f \in \text{Hom}(B_q, B_t),
\]
and 
\[
fp := f \circ p_t \in \text{Hom}(B_q, B_t).
\]
We will see in Section 5 (Corollary 5.2) that the set \( \mathcal{L}_q \cdot p \mathcal{L}_s \) is also a basis of \( \text{Hom}(B_s, B_t) \). We call it \( p \)-double leaves basis.

Let \( P = p_s' \otimes \text{id} \in \text{End}(B_s) \). By definition of \( p_s' \) we have \( \text{Im}(P) \cong B_x B_s \).

Notation 4.1 If \( l, l' \in \mathcal{L}_s \) we denote by \((l; l')\) the element \( P \cdot l \cdot pl' \vert_{\text{Im}(P)} \in \text{End}(\text{Im}(P)) \).

Notation 4.2 Let \( X \) be a set of homogeneous elements of graded vector spaces. We define \( X_0 \) as the subset of degree zero elements of \( X \).

It is clear that the set \( (\mathcal{L}_x; \mathcal{L}_x)_0 \) generates \( \text{End}(B_x B_s) \subseteq \text{End}(B_s) \).

Let \( l' \cdot pl \in (\mathcal{L}_x^a \cdot p \mathcal{L}_x)_0 \). If \( \deg(l) < 0 \) then by Proposition 2.1 (3), we have \( pl\vert_{\text{Im}(P)} = 0 \). On the other hand, if \( \deg(l') < 0 \) then by Proposition 2.1 (3), we have \( Plp = 0 \). So we deduce the equality 
\[
(\mathcal{L}_x; \mathcal{L}_x)_0 = P (\mathcal{L}_x^a \cdot p \mathcal{L}_x)_0 \vert_{\text{Im}(P)}
\]

Definition 2 For \( x \in W \) we define \( \mathcal{L}_x(x) \) as the subset of \( \mathcal{L}_x \) consisting of the elements belonging to \( \text{Hom}(B_x, B_x) \). For \( z \in Z \) define \( \mathcal{L}_z \) as the set of leaves \( l \in (\mathcal{L}_z(z))_0 \) satisfying that \( pl\vert_{\text{Im}(P)} \neq 0 \).

Lemma 4.3 The set \( (\mathcal{L}_x; \mathcal{L}_x) \) generates \( \text{End}(B^\oplus m_z) \subseteq \text{End}(B_x) \), where we see the bimodule \( B^\oplus m_z \) included in \( B_x \) via the formula 
\[
B^\oplus m_z = \text{Im}(p^z_{x,s} \circ (p_{z'} \otimes \text{id})).
\]

Proof If \( l \in (\mathcal{L}_x(z))_0 \), by Proposition 2.1 (3) we know that both the images of \( pl\vert_{\text{Im}(P)} \) and of \( Pl^a p_\z \) are either isomorphic to \( B_z \) or to zero. On the other hand, by adjunction arguments \( l \in \mathcal{L}_s \) if and only if \( l \in (\mathcal{L}_s(z))_0 \) and \( Pl^a p_s \neq 0 \), so we conclude that \( l_1, l_2 \in (\mathcal{L}_x(z))_0 \) satisfy that \((l; l') \neq 0 \), if and only if \( l, l' \in \mathcal{L}_z \).

\[\square\]
A consequence of Lemma 4.3 is that
\[
\sum_{l \in L} \text{Im}(P l^a p_z) = B_z^{\oplus m_z} \subseteq B_z.
\]

Let \( L_z^{\oplus} = \{l_1, \ldots, l_b\} \subseteq L_z \) be such that
\[
\bigoplus_{l \in L_z^{\oplus}} \text{Im}(P l^a p_z) = B_z^{\oplus m_z} \subseteq B_z.
\]

This set can be explicitly constructed. Let us be more precise. We have that \( \text{End}(B_z^{\oplus m_z}) \) can be identified with the full ring of \( m_z \) by \( m_z \) matrices with coefficients in \( k \). If we have that \( l_1, \ldots, l_p \) belong to \( L_z^{\oplus} \), we need to know if a new element \( l \) of \( L_z \) will belong or not to \( L_z^{\oplus} \). It is easy to see that \( \text{Im}(P l^a p_z) \) is included or intersects only at zero the set \( \bigoplus_{l=1}^p \text{Im}(P l_i^a p_z) \). To distinguish between these cases we just need to evaluate at one element, for example the minimum degree element in \( B_z \) (this element goes to itself under \( p_z \)).

For \( l \in L_z \) we have seen that \( \text{Im}(P l_i^a p_z) \cong B_z \), so for \( 1 \leq i \leq b \) we define
\[
\text{Im}(P l_i^a p_z) = B_i^z \subseteq B_z.
\]

We will find the projector \( p_i^j \) in \( \text{End}(B_z) \) onto \( B_i^z \) as a linear combination of elements in the set \( (l_i; L_z^{\oplus}) \). Define
\[
p_i^j = \sum_{j=1}^b \eta_i^j (l_i; l_j).
\]

We have the equation
\[(l_{i'}; l_j) \circ (l_i; l_j') = \lambda_{i'j}^j (l_{i'}; l_j'),\]
where \( \lambda_{i'j}^j \in \mathbb{Z} \) are uniquely defined by the equations
\[
\lambda_{i'j}^j p_z = p_{i'j} P l_i^a p_z.
\]

The fact that the sum \( \bigoplus_i B_i^z \) is direct translates into the set of equations
\[
p_i^j p_j^i = \delta_{ij} p_i^j,
\]
and this translates into the matrix equation
\[
\lambda_z \eta_z = \text{Id}, \tag{3}
\]
where \( \lambda_z \) is the matrix with \((i, j)\) entry \( \lambda_{ij}^j \) and \( \eta_z \) is the matrix with \((i, j)\) entry \( \eta_{ij}^j \). As \( p_z = \sum_{i=1}^b p_i^j \) is the projector onto \( B_z^{\oplus m_z} \), we finally obtain the projector we were searching for
\[
p_z = P - \sum_{z \in \mathbb{Z}} p_z.
\]
4.3 Non-explicit construction of the favorite projector

In this section we give a non explicit construction of the favorite projector. We think that this construction if better suited for some calculations, for example, for finding bounds on the bad primes, and could eventually be made explicit by using perverse filtrations in Soergel bimodules.

We use the same notations and hypothesis as in Section 4.2. For \( y \in W \) let \( t_y \in \mathbb{N}[v, v^{-1}] \) be such that we have an isomorphism

\[
B_{s_1} \cdots B_{s_n} \cong \bigoplus_{y \in W} t_y B_y.
\]

Of course, this isomorphism is not unique but we choose one. Let \( P' \) be the projector onto the bimodule

\[
\bigoplus_{z \in Z} t_z(0) B_z = B_{xs} \oplus \bigoplus_{z \in Z} B_z^{\oplus m'_z},
\]

with \( m'_z \geq m_z \) (the set \( Z \) was defined in Section 4.2). Now repeat all the reasoning of section 4.2 but replacing \( P \) by \( P' \) and \( m_z \) by \( m'_z \). We have the corresponding objects \( L'_z, (L_z^{\oplus})' \) and \( \lambda'_z \). We obtain the formula

\[
p'_z = P' - \sum_{z \in Z} p_z.
\]

The projector \( P' \) is not constructed inductively, so the reader might ask why to do a similar, but non-explicit construction? The important point is how to calculate the coefficients of \( \lambda'_z \). If \( (L_z^{\oplus})' = \{ l'_1, \ldots, l'_c \} \subseteq L'_z \), for \( 1 \leq i, j \leq c \) we have that \( (\lambda'_z)^{ij} \) is, as before, defined by the equation

\[
(\lambda'_z)^{ij} p_z = pl'_j P' (l'_i)^a p_z.
\]

As \( \ker(P') \) has no direct summand isomorphic to \( B_{xs} \) nor to \( B_z \) for any \( z \in Z \), by Proposition 2.1(2) we know that

\[
pl'_j \ker(P) l'_i p_z = 0,
\]

so we obtain the simpler formula

\[
(\lambda'_z)^{ij} p_z = pl'_j (l'_i)^a p_z.
\]

4.4 From characteristic zero to positive characteristic

In this section we consider \( W \) to be a Weyl group (or an affine Weyl group, in which case you have to replace everywhere \( W \) by \( W^\circ \)). We have defined a
representation $V = \oplus_{s \in S} \mathbb{Z} e_s$ of $W$ over $\mathbb{Z}$. We define the symmetric algebra $R$ of $V^* = \text{Hom}(V, \mathbb{Z})$.

We can define the category $\mathcal{B}_Z$ of Soergel bimodules over $\mathbb{Z}$ just as before, with objects direct sums of direct summands of shifts of elements of the form $R \otimes_R R \otimes_R \cdots \otimes_R R$ for $sr \cdots t$ a reduced expression in $W$. Let $\mathbb{F}_p$ be the finite field with $p$ elements, $\mathbb{Z}_p$ the $p$-adic integers and $\mathbb{Q}_p$ the $p$-adic numbers. The categories $\mathcal{B}_{\mathbb{F}_p}$ (resp. $\mathcal{B}_{\mathbb{Z}_p}, \mathcal{B}_{\mathbb{Q}_p}$) have objects $b_{\mathbb{Z}} \otimes_{\mathbb{F}_p} b_{\mathbb{Z}}$ (resp. $b_{\mathbb{Z}} \otimes_{\mathbb{Z}_p} b_{\mathbb{Z}} \otimes_{\mathbb{Q}_p}$) with $b_{\mathbb{Z}} \in \mathcal{B}_Z$.

As the double leaves is an $\mathbb{F}_p$ and $\mathbb{Q}_p$ basis respectively of the Hom spaces between the Bott-Samelsons in $\mathcal{B}_{\mathbb{F}_p}$ and $\mathcal{B}_{\mathbb{Q}_p}$, by Nakayama’s lemma it is a $\mathbb{Z}_p$-basis of the Hom’s in $\mathcal{B}_{\mathbb{Z}_p}$.

Consider the following functors

- $\otimes_{\mathbb{Z}_p} \mathbb{F}_p : \mathcal{B}_{\mathbb{Z}_p} \to \mathcal{B}_{\mathbb{F}_p}$
- $\otimes_{\mathbb{Z}_p} \mathbb{Q}_p : \mathcal{B}_{\mathbb{Z}_p} \to \mathcal{B}_{\mathbb{Q}_p}$

Let $B_\mathbf{s}$ be a Bott-Samelson in $\mathcal{B}_Z$, with $\mathbf{s}$ a reduced expression of $x \in W$. Let $p_\mathbf{s} \in \text{End}_{\mathcal{B}_{\mathbb{Q}_p}}(B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ be the favorite projector onto $B_\mathbf{s}$ as defined in Section 4.1. There are two cases.

1. Let us suppose that in the expansion of $p_\mathbf{s}$ in the LL the coefficients have no denominators that are divisible by $p$. Then we can lift $p_\mathbf{s}$ to another primitive idempotent in $\text{End}_{\mathcal{B}_{\mathbb{F}_p}}(B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ and then apply the functor $\otimes_{\mathbb{Z}_p} \mathbb{F}_p$ to obtain a primitive idempotent $p'_\mathbf{s}$ in $\text{End}_{\mathcal{B}_{\mathbb{F}_p}}(B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$.

For what follows the reader should read the notations in Section 5.1. Let $M, N \in \mathcal{B}_Z$. As 

$$\dim \text{Hom}_{\mathcal{B}_{\mathbb{F}_p}}(M, N) = \text{rk} \text{Hom}_{\mathcal{B}_{\mathbb{Z}_p}}(M, N) = \dim \text{Hom}_{\mathcal{B}_{\mathbb{Q}_p}}(M, N),$$

(where we consider $M$ and $N$ inside the brackets as tensored with the corresponding field), we have that

$$\eta_{\mathbb{Q}_p}(\text{Im}(p_\mathbf{s})) = \eta_{\mathbb{F}_p}(\text{Im}(p'_\mathbf{s})) \in \mathcal{H}.$$ 

(2) Let us suppose that in the expansion of $p_\mathbf{s}$ in the LL there is at least one coefficient that has a denominator that is divisible by $p$. Then, as every idempotent in $\text{End}_{\mathcal{B}_{\mathbb{F}_p}}(B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ can be lifted to an idempotent in $\text{End}_{\mathcal{B}_{\mathbb{Q}_p}}(B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ (passing through $\mathbb{Z}_p$), then Soergel’s conjecture over $\mathbb{F}_p$ can not be true for all $y \leq x$ because if this was true then $B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ would have the same number of indecomposable summands but this can not be true because $\text{End}_{\mathcal{B}_{\mathbb{Q}_p}}(B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ has all the lifted idempotents of $\text{End}_{\mathcal{B}_{\mathbb{F}_p}}(B_\mathbf{s} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ and also $p_\mathbf{s}$. The fact that $p_\mathbf{s}$ is
different from the other idempotents mentioned is due to the fact that
the other idempotents as are lifted can not have a coefficient divisible by
\( p \) in the expansion of them in the LL.

So we conclude that Soergel’s \( p \)-- conjecture (resp. affine Soergel’s \( p \)-- conjecture)
is true if and only if \( p \) does not belong to the set of primes that divide at least
one of the denominators of the coefficients appearing in the expansion of all
the \( p_\alpha \) for \( \alpha \) a reduced expression of an element of \( W \) (resp. \( W^0 \)).

5 The double leaves basis is a basis

In this section we prove Theorem 3.2. We fix for the rest of this section the
Coxeter system \((W, S)\) and the sequences \( s = (s_1, \ldots, s_n) \in S^n \) and \( r = (r_1, \ldots, r_p) \in S^p \). We first prove that the graded degrees are the correct ones.

5.1 Some notation

The notations introduced in this section will be useful to understand what do
we mean when we say that the graded degrees are the correct ones.

Given a \( \mathbb{Z} \)-graded vector space \( V = \bigoplus_i V_i \), with \( \dim(V) < \infty \), we define its
graded dimension by the formula
\[
\dim V = \sum_i (\dim V_i) v^{-i} \in \mathbb{Z}[v,v^{-1}].
\]

We define the graded rank of a finitely generated \( \mathbb{Z} \)-graded \( R \)-module \( M \) as follows
\[
\text{rk} M = \dim(M/MR_+) \in \mathbb{Z}[v,v^{-1}],
\]
where \( R_+ \) is the ideal of \( R \) generated by the homogeneous elements of non
zero degree. We have \( \dim(V(1)) = v(\dim V) \) and \( \text{rk}(M(1)) = v(\text{rk} M) \). We define \( \text{rk} M \) as the image of \( \text{rk} M \) under \( v \mapsto v^{-1} \).

Now that we have introduced these notations we can explain Soergel’s inverse
theorem. Soergel [So4, Theorem 5.3] proves that the categorification \( \varepsilon : \mathcal{H} \to \langle B \rangle \) admits an inverse \( \eta : \langle B \rangle \to \mathcal{H} \) given by
\[
\langle B \rangle \to \sum_{x \in W} \text{rk} \text{Hom}(B, R_x) T_x. \quad (4)
\]
Recall that $R_x$ was defined in section 2.4. The following notations will be necessary to understand the graded rank formula for the Hom spaces given by Soergel.

- Let $X$ be a set of homogeneous elements of graded vector spaces. We define the **degree of $X$**

$$d(X) = \sum_{x \in X} q^{\deg(x)/2}.$$ 

- For every $x \in W$, the trace $\tau_x : \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ is defined by

$$\tau_x \left( \sum_{y \in W} p_y \beta_y \right) = p_x \quad (p_y \in \mathbb{Z}[v, v^{-1}] \ \forall y \in W).$$

We denote $\tau_{id}$ simply by $\tau$.

- Finally we define the element $C_{s_1^t} = C_{s_1^t} \cdots C_{s_n^t} \in \mathcal{H}$.

### 5.2 Graded degrees

Let $t$ be an arbitrary sequence of elements in $S$. We recall [Li1, Lemma 5.6]

$$d(\mathbb{L}_t(x)) = \tau_x(C_{s_1^t}). \quad (5)$$

We remark that the construction of $T_t$ is motivated by (and can be seen as a categorification of) this formula. As $\eta$ is the inverse of $\varepsilon$ we obtain

$$\tau_x(C_{s_1^t}) = \tau_x \circ \eta(B_{s_1^t}) = \overline{\text{rkHom}}(B_{s_1^t}, R_x). \quad (6)$$

So we have

$$d(\mathbb{L}_t(x)) = \overline{\text{rkHom}}(B_{s_1^t}, R_x), \quad (7)$$

and in particular

$$d(\mathbb{L}_t(id)) = \overline{\text{rkHom}}(B, R) \quad (8)$$

as Theorem 3.1 asserts. This formula says that the double leaves basis has the correct degrees. We will prove that the double leaves basis for $\text{Hom}(B_{s_1^t}, B_{s_2^t})$ as defined in 3.2 also has the correct degrees, i.e. we will prove the formula

$$d(\mathbb{L}_{s_1^t} \cdot \mathbb{L}_{s_2^t}) = \overline{\text{rkHom}}(B_{s_1^t}, B_{s_2^t}). \quad (9)$$
Let $\mathfrak{T}^{\text{op}} = (r_p, \ldots, r_2, r_1)$. Using repeatedly the adjunction isomorphism (1) we obtain
\[
\text{Hom}(B_s, B_r) \cong \text{Hom}(B_s B_{\mathfrak{T}^{\text{op}}}, R).
\] (10)

Using equations (5), (8) and (10) we obtain
\[
\overline{\tau} \text{Hom}(B_s, B_r) = \tau(C'^t_s C'^r_{\mathfrak{T}^{\text{op}}}).
\]

For any sequence $\underline{u}$ of elements of $S$ and every $y \in W$ we define the polynomials $p_{\underline{u}}^y$ by the formulas
\[
C'^t_{\underline{u}} = \sum_y p_{\underline{u}}^y \Delta y
\]
It is easy to check that $p_{\underline{u}}^{y, \text{op}} = p_{\underline{u}}^y - 1$. Using the following equation (see [GP, proposition 8.1.1])
\[
\tau(\overline{T}_x \overline{T}_{y^{-1}}) = \delta_{x, y}
\] (11)
we conclude that
\[
\tau(C'^t_s C'^r_{\mathfrak{T}^{\text{op}}}) = \sum_{x \in W} p_x^s p_x^r.
\]

It is easy to see that the degrees of the generating morphisms $f_{sr}, j_s, m_s$ are equal to the degrees of their adjoints, so we conclude that the degree of any morphism between Bott-Samelson bimodules is equal to the degree of its adjoint, so
\[
d(\mathbb{L}_{\mathfrak{T}}(x)^a) = p_x^r.
\]
We thus obtain
\[
d(\mathbb{L}_{\mathfrak{T}}(x)^a \cdot \mathbb{L}_{\mathfrak{A}}(x)) = p_x^s p_x^r
\]
and finally
\[
d(\mathbb{L}_{\mathfrak{T}}^a \cdot \mathbb{L}_{\mathfrak{A}}) = \sum_{x \in W} d(\mathbb{L}_{\mathfrak{T}}(x)^a \cdot \mathbb{L}_{\mathfrak{A}}(x)) = \sum_{x \in W} p_x^s p_x^r.
\]
Thus proving formula (9).
5.3 An important Lemma

Let us suppose that the elements of $L_a \cdot L_s$ are linearly independent for the left action of $R$. Let $M$ be the sub $R$-module of $\text{Hom}(B_s, B_r)$ generated by the elements of $L_a \cdot L_s$. In every degree the modules $M$ and $\text{Hom}(B_s, B_r)$ are finite dimensional $k$-vector spaces, and they have the same dimension as we saw in Section 5.2, so they are equal. This being true in every degree we deduce that $M = \text{Hom}(B_s, B_r)$, and this would complete the proof of Theorem 3.2.

So we only need to prove that the elements of $L_a \cdot L_s$ are linearly independent for the left action of $R$. Before we can do this we have to introduce an order in $L_a \cdot L_s$. The linear independence will follow from a triangularity condition with respect to this order.

Recall that $n$ is the length of $s$. To every element $l \in L_s$, we associate two elements $i = (i_1, \ldots, i_n)$ and $j = (j_1, \ldots, j_n)$ of $\{0, 1\}^n$ in the following way.

By construction $l$ is a composition of $n$ morphisms, say $l = l_n \circ \cdots \circ l_1$. For $1 \leq k \leq n$ we put $i_k = 1$ if in $l_k$ (that is a composition of morphisms of the type $m_s, j_s$ and/or $f_{sr}$) appears a morphism of the type $m_s$ and we put $i_k = 0$ otherwise. We put $j_k = 1$ if in $l_k$ appears a morphism of the type $j_s$ and we put $j_k = 0$ otherwise.

On the other hand, the leaf $l$ is completely determined by $s$, $i$ and $j$, (moreover, it is completely determined by $s$ and $i$), so we denote $l = f^i_j$. The following lemma is key for defining the mentioned order.

**Lemma 5.1** Let $(i, i') \in \{0, 1\}^n \times \{0, 1\}^n$ with $i \neq i'$. If $f^i_j \in L_s(x)$ and $f^{i'}_j \in L_s(x')$, then $x \neq x'$.

**Proof** Let us prove this lemma by contradiction. We will suppose that $x = x'$.

Let $1 \leq p \leq n$. If the node of depth $p-1$ corresponding to the leaf $f^i_j$ is colored by the bimodule $(B_{t_1} \cdots B_{t_a})(B_{s_{p'}} \cdots B_{s_{n'}})$, then we define

$$y_{p-1} = t_1 t_2 \cdots t_a \in W,$$

and $y_n = x$. In a similar way we define, for $1 \leq p \leq n$, the element $y'_{p-1} \in W$ associated to the leaf $f^{i'}_j$ and $y'_n = x'$. By construction we have

$$l(y_{p-1} s_p) = l(y_{p-1}) - 1 \iff l(y'_{p-1} s_p) = l(y'_{p-1}) - 1.$$

We have supposed that $y_n = y'_n$. We will prove by descendent induction that $y_p = y'_p$ for all $1 \leq p \leq n$ and this will imply the equality $i = i'$, yielding the contradiction we are looking for.
Let \( 1 \leq p \leq n \). Let us suppose that \( y_p = y'_p \), we will prove by contradiction that \( y_{p-1} = y'_{p-1} \), so we will suppose \( y_{p-1} \neq y'_{p-1} \). By construction of \( T_s \), we know that \( y_p = y_{p-1} \) or \( y_p = y_{p-1}s_p \), then we can suppose without loss of generality that \( y_p = y_{p-1}s_p \) and \( y'_p = y'_{p-1} \). Let us consider the two possible cases:

- **Case 1:** \( l(y_{p-1}s_p) = l(y_{p-1}) - 1 \). As we have seen, this implies \( l(y'_{p-1}s_p) = l(y'_{p-1}) - 1 \). But this yields to a contradiction:

\[
\begin{align*}
l(y_{p-1}) &= l(y_{p-1}s_p) \\
&= l(y_{p-1}) + 1 \\
&= l(y_{p-1}s_p) + 1 \\
&= l(y_{p-1}) + 1.
\end{align*}
\]

- **Case 2:** \( l(y_{p-1}s_p) = l(y_{p-1}) + 1 \). This implies \( l(y'_{p-1}s_p) = l(y'_{p-1}) + 1 \). This also yields to a contradiction:

\[
\begin{align*}
l(y_{p-1}) &= l(y_{p-1}s_p) \\
&= l(y_{p-1}) + 1 \\
&= l(y_{p-1}s_p) + 1 \\
&= l(y_{p-1}) + 2.
\end{align*}
\]

This finishes the proof of Lemma 5.1.

### 5.4 The order

Consider the element \((f^j_l)^a \cdot f^j_k \in L_l^a(x)^a \cdot L_k^a(x)\). Recall that the sequences \( r \) and \( s \) have already been fixed. As the couple \((i, r)\) determine \( l \) and \( x \) and the triplet \((j, s, x)\) determine \( k \) by Lemma 5.1, we conclude that the morphism \((f^j_l)^a \cdot f^j_k\) is completely determined by \((i, j)\). We will call it \( f_{i,j} \).

Now we can introduce a total order \( \preceq \) in \( \{0, 1\}^n \times \{0, 1\}^n \). The symbol \( \leq \) will be used for the lexicographical order in \( \{0, 1\}^n \). Let \( \underline{i} = (i_1, \ldots, i_n), \underline{j} = (j_1, \ldots, j_n) \) and \( \underline{i} = (i'_1, \ldots, i'_n), \underline{j} = (j'_1, \ldots, j'_n) \). The order is defined as follows.

- if \( \underline{i} < \underline{j} \) then \((\underline{i}, \underline{j}) \preceq (\underline{i}', \underline{j}')\)
- if \( \underline{i} = \underline{j} \) then
  - if \( \sum_{p=1}^n i_p < \sum_{p=1}^n j'_p \) then \((\underline{i}, \underline{j}) \preceq (\underline{i}', \underline{j}')\)
- if $\sum_{p=1}^{n} i_p = \sum_{p=1}^{n} i'_p$ and $i < i'$ then $(i, j) \prec (i', j')$

**Remark 1** In the notation of [Li1], if $j = j'$, then $(i, j) \triangleright (i', j')$ if and only if $i \triangleright i'$.

This order in $\{0, 1\}^n \times \{0, 1\}^n$ induces an order in $\mathbb{L}_a \cdot \mathbb{L}_b$:

$$f_{i,j}f_{i',j'} \leq f_{i,j}'f_{i',j}' \iff (i,j) \leq (i',j').$$

In the next paragraphs we will see why this order particularly interesting for the double leaves basis.

### 5.5 Linear independence

We recall that if $t = (t_1, \ldots, t_a) \in S^a$, then the opposite sequence is $t^{\text{op}} = (t_a, \ldots, t_1) \in S^a$. Let us define $\alpha_s = p_s \circ e_s$. The category $B$ of Soergel bimodules is generated as tensor category by the morphisms in the set

$$\{m_s, j_s, \alpha_s, f_{sr}\}_{s \neq r \in S}$$

because of Theorem [3.1] and adjunction [1]. So to any morphism between Bott-Samelson bimodules $f \in \text{Hom}(B_t, B_q)$ there is an associated morphism that we denote $f^{\text{op}} \in \text{Hom}(B_{t^{\text{op}}}, B_{q^{\text{op}}})$. Moreover, this gives a bijection between the spaces $\text{Hom}(B_t, B_q)$ and $\text{Hom}(B_{t^{\text{op}}}, B_{q^{\text{op}}})$.

Recall that in the construction of the tree $\mathbb{T}_a$ we fixed for every element $x \in W$ a reduced expression $x = (x_1, \ldots, x_a)$ of $x$. We define the following morphism in $\text{Hom}(B_x B_{x^{\text{op}}}, R)$:

$$\rho_x = (m_{x_1} \circ j_{x_1}) \circ (\text{id}^1 \otimes (m_{x_2} \circ j_{x_2}) \otimes \text{id}^1) \circ \cdots \circ (\text{id}^{a-1} \otimes (m_{x_a} \circ j_{x_a}) \otimes \text{id}^{a-1}).$$

In the isomorphism [10] the subset $\mathbb{L}_a \cdot \mathbb{L}_b \subseteq \text{Hom}(B_a B_b, R)$ corresponds to

$$\mathbb{L}_a \bullet \mathbb{L}_b^{\text{op}} \subseteq \text{Hom}(B_a B_{b^{\text{op}}}, R),$$

where $\bullet$ is defined as follows. If $f \in \mathbb{L}_a(x)$ and $g \in \mathbb{L}_b(y)$, then

$$f \bullet g^{\text{op}} = \begin{cases} 
0 & \text{if } x \neq y \\
\rho_x \circ (f \otimes g^{\text{op}}) & \text{if } x = y
\end{cases}$$

So it is enough to prove that $\mathbb{L}_a \bullet \mathbb{L}_b^{\text{op}}$ is a linearly independent set. For any sequence $(u_1, \ldots, u_p) \in S^p$, let us denote

$$1^\otimes = 1 \otimes 1 \otimes \cdots \otimes 1 \in B_{u_1} \cdots B_{u_p}$$
and
\[ x^\otimes = 1 \otimes x_{u_1} \otimes \cdots \otimes x_{u_p} \in B_{u_1} \cdots B_{u_p}. \]

Let us recall that \( R_+ \) is the ideal of \( R \) generated by the homogeneous elements of non zero degree. An element in \( B_{u_1} \cdots B_{u_p} \) is called superior if it belongs to the set \( x^\otimes + R_+ B_{u_1} \cdots B_{u_p} \) and it is called normalsup if it belongs to the set \( x^\otimes + R_+ B_{u_1} \cdots B_{u_p}. \) For \( s \in S \), let us denote \( x_s^0 = 1 \) and \( x_s^1 = x_s. \) If \( \underline{j} = (j_1, \ldots, j_n) \in \{0, 1\}^n, \) we put
\[ x_{\underline{j}} = x_{s_1}^{j_1} \otimes \cdots \otimes x_{s_n}^{j_n} \otimes 1 \in B_{s_1} \cdots B_{s_n}. \]

By construction of \( f_{\underline{k}} \in \mathbb{L}_{\underline{s}} \), we have that \( f_{\underline{k}}(x_{\underline{j}}) = 1^\otimes \) and \( f_{\underline{k}}(x_{\underline{j}'}) = 0 \) if \( \underline{j}' < \underline{j}. \) We remark that this is independent of \( k. \) On the other hand, if
\[ x_{\underline{i}} = 1 \otimes x_{r_1}^{i_1} \otimes \cdots \otimes x_{r_p}^{i_p} \in B_{r_1} \cdots B_{r_p}, \]
then \( f_{\underline{k}}(x_{\underline{i}}) \) is normalsup [\text{Lem. 5.10}] and \( f_{\underline{k}}(x_{\underline{i}'}) \) is a superior element if \( \underline{i}' \prec \underline{i}. \) Again, this is independent of \( k. \)

Let us define
\[ x_{\underline{j}}^{op} = x_{r_p}^{j_p} \otimes \cdots \otimes x_{r_1}^{j_1} \otimes 1 \in B_{r_p} \cdots B_{r_1}. \]

If \( f_{\underline{k}} \in \mathbb{L}_{\underline{s}} \cdot \mathbb{L}_{\underline{s}} \), by explicit calculation (using the fact that \( f_{sr}(1^\otimes) = 1^\otimes \) and some simple degree arguments) we obtain the formula
\[ f_{\underline{k}} \cdot f_{\underline{k}}^{op} (x_{\underline{j}} \otimes x_{\underline{j}}^{op}) = \begin{cases} 1 & \text{if } (\underline{j}', \underline{j}) = (\underline{i}, \underline{j}) \\ 0 & \text{if } (\underline{j}', \underline{j}) < (\underline{i}, \underline{j}) \end{cases} \quad (12) \]

This unitriangularity formula proves that \( \mathbb{L}_{\underline{s}} \cdot \mathbb{L}_{\underline{s}}^{op} \) is a linearly independent set and thus we finish the proof of Theorem 3.2. \( \square \)

**Corollary 5.2** The set \( \mathbb{L}_{\underline{s}}^{op} \cdot p \mathbb{L}_{\underline{s}} \) is a basis of \( \text{Hom}(B_{\underline{s}}, B_{\underline{r}}). \)

**Proof** The graded degrees of \( \mathbb{L}_{\underline{s}}^{op} \cdot p \mathbb{L}_{\underline{s}} \) are the same as the graded degrees of \( \mathbb{L}_{\underline{s}} \cdot \mathbb{L}_{\underline{s}} \) because \( p_{\underline{s}} \) is a degree zero morphism for every \( \underline{s} \) reduced expression of an element of \( \bar{W}. \)

By degree reasons, using Proposition 2.1 (3) we can see that the element \( 1^\otimes \in B_{\underline{s}} \) is in the image of \( p_{\underline{s}}. \) So we have a unitriangularity formula analogous to (12) that gives us the linear independence. \( \square \)
6 Leaves basis

In the rest of this section we prove a result that is not needed for the rest of the paper (the reader only interested in Lusztig’s Conjecture can skip this section), but it is a simple corollary of Lemma 5.1. We expect that it will help in the understanding of Kazhdan-Lusztig polynomials because it gives an important step in the calculation of the character of any Soergel bimodule that is given as the image of a projector of a Bott-Samelson in the spirit of [Li4] or [El] (see the formula for \( \eta \) in section 5.1).

To generalize Theorem 3.1 we need to introduce a new morphism. Consider the \((R, R)\)–bimodule morphism \( \beta : B_s \to R_s \) defined by \( \beta(p \otimes q) = ps(q) \) for \( p, q \in R \). If \( x, y \in W \) we have \( R_x R_y \cong R_{xy} \). If \( \underline{x} = (x_1, \ldots, x_a) \), we can define an \((R, R)\)–bimodule morphism that we also denote \( \beta : B_{\underline{x}} \to R_x \).

For a sequence \( s \) of elements in \( S \) we define the set

\[
\mathbb{L}_s^\beta = \{ \beta \circ l \mid l \in \mathbb{L}_s \} \subseteq \prod_{x \in W} \text{Hom}(B_{\underline{x}}, R_x).
\]

We define \( \mathbb{L}_s^\beta(x) \) as the subset of \( \mathbb{L}_s^\beta \) consisting of the elements belonging to \( \text{Hom}(B_{\underline{x}}, R_x) \). We call the set \( \mathbb{L}_s^\beta \) the leaves basis, and the following generalization of Theorem 3.1 explains this name.

**Proposition 6.1** The set \( \mathbb{L}_s^\beta(x) \) is a basis of \( \text{Hom}(B_{\underline{x}}, R_x) \) as a right \( R \)-module.

**Proof** Because of the equation (7) and using a similar reasoning as in Section 5.3 we only need to prove that the elements of \( \mathbb{L}_s^\beta(x) \) are linearly independent with respect to the right \( R \)-action.

From section 5.3 we know that \( \beta \circ f^j_{x}(x \underline{l}) = 1 \) and that \( \beta \circ f^j_{x}(x \underline{l'}) = 0 \), if \( j \) is bigger than \( j' \) in the lexicographical order. Then Lemma 5.1 allows us to conclude the proof by a triangularity argument.

\[ \Box \]

7 Affine Soergel’s \( p \)-conjecture is equivalent to Fiebig’s conjecture

In this section \( W \) is a Weyl group and \( \widehat{W} \) its associated affine Weyl group. Soergel proves [So3] that Soergel’s conjecture for Weyl groups is equivalent
to a part of Lusztig’s conjecture (weights near the Steinberg weight). We will prove in this section that a version of Soergel’s conjecture (that we called affine Soergel’s $p$-conjecture) implies the full Lusztig conjecture.

More precisely, it is in some sense implicit in the work of Fiebig that affine Soergel’s $p$-conjecture is equivalent to Fiebig’s conjecture (we will explain in detail affine Soergel’s $p$-conjecture, Fiebig’s conjecture and their equivalence in the Section 7.2). Fiebig proves [Fi3] that Fiebig’s conjecture implies Lusztig’s conjecture.

7.1 The category $\mathcal{P}^o$

Let us suppose that $p$ is bigger than the Coxeter number of $W$. We recall that $\hat{W}^o$ is the finite subset of $\hat{W}$ consisting of the elements that are lesser or equal (in the Bruhat order) than the longest element in the set of antidominant restricted elements.

Let $\mathcal{T}$ be the set of reflections in $\hat{W}$, i.e. the orbit of $S$ under conjugacy. For $t \in \mathcal{T}$ define $x_t$ as the equation of the subspace of $V$ fixed by $t$ (as before this is well defined up to a non zero scalar). Define the algebra $Z^o := \{ (z_x) \in \bigoplus_{x \in \hat{W}^o}R \mid z_x \equiv z_{tx} \mod x_t \text{ for all } x \in \hat{W}^o, t \in \mathcal{T} \text{ with } tx \in \hat{W}^o \}$. There is a natural injection of $R$ into $Z^o$ that allows us to see every $Z^o$-module as an $R$-module by restriction of scalars. Let us define $\mathcal{T}^o \subset \mathcal{T}$ as the subset of elements $t \in \mathcal{T}$ satisfying that there exist an element $x \in \hat{W}^o$ such that $tx \in \hat{W}^o$. Denote by $Q$ the localization of $R$ at the multiplicatively closed subset generated by the set $\{\alpha_t \mid t \in \mathcal{T}^o\}$. Given any right $R$-module $M$ we denote by $M_Q$ the right $R$-module $M \otimes_R Q$.

The natural inclusion $Z^o \subset \bigoplus_{x \in \hat{W}^o}R$ induces an inclusion $Z^o_Q \subset \bigoplus_{x \in \hat{W}^o} Q$ that is also [Fi4, Lemma 3.2] a surjection $Z^o_Q = \bigoplus_{x \in \hat{W}^o} Q$. If $\mathcal{M}$ is a $Z^o$-module, then there is [Fi4, Lemma 3.2] a canonical decomposition $\mathcal{M}_Q = \bigoplus_{x \in \hat{W}^o} \mathcal{M}_Q^x$ such that the element $(z_x) \in Z^o_Q$ acts on $\mathcal{M}_Q^x$ as multiplication by $z_x$.

If $\mathcal{M}$ is a $Z^o$-module that is torsion free as an $R$-module, we have a canonical inclusion $\mathcal{M} \subset \mathcal{M}_Q$. For a subset $I \subset \hat{W}^o$ we have $\bigoplus_{x \in I} \mathcal{M}_Q^x \subset \bigoplus_{x \in \hat{W}^o} \mathcal{M}_Q^x$, so we define $\mathcal{M}_I := \mathcal{M} \cap \bigoplus_{x \in I} \mathcal{M}_Q^x \subset \mathcal{M}$. 30
and

\[ \mathcal{M}^I := \mathcal{M}/\mathcal{M}_{W^o \setminus I} \]

It is clear that if \( I' \subseteq I \), then we have a natural surjection \( \mathcal{M}^I \rightarrow \mathcal{M}^{I'} \) and a natural injection \( \mathcal{M}^{I'} \rightarrow \mathcal{M}^I \). We denote \( \mathcal{M}_x := \mathcal{M}_{\{x\}} \) and \( \mathcal{M}_x := \mathcal{M}_{\{x\}} \).

Let us define \( \mathcal{B}_p^\circ \) as the full subcategory of \( \mathcal{B}(\hat{W}, \mathbb{F}_p) \) consisting of objects that are direct sums of objects in the set \( \{ B_w \}_{w \in \hat{W}} \).

We define (following [Fi3, Proof of Theorem 4.3]) a functor \( G : \mathcal{B}_p^\circ \rightarrow Z^\circ \text{-mod} \) by the rule

\[
\mathcal{B}_p^\circ \ni M \mapsto G(M) := \left\{ (m_x) \in \bigoplus_{x \in \hat{W}^o} M_x \mid \rho_{x,tx}(m_x) = \rho_{tx,x}(m_{tx}) \text{ for all } t \in T^o, x, tx \in \hat{W}^o \right\}.
\]

The bimodule \( M^x \) and the morphisms \( \rho_{x,y} \) were defined in section 2.4. \( G(B) \) is a \( Z^\circ \)-module by pointwise multiplication. We define \( \mathcal{P}^o \) as the essential image of \( G \). For \( M \in \mathcal{P}^o \) define \( \text{supp} M := \{ w \in \hat{W} \mid M^w \neq 0 \} \).

### 7.2 Fiebig’s conjecture

The following is Theorem 6.1 of [Fi2], that will be needed to state Fiebig’s conjecture.

**Definition/Theorem 7.1** For all \( w \in \hat{W}^o \) there exist an object \( B(w) \in \mathcal{P}^o \), unique up to isomorphism, with the following properties:

1. \( B(w) \) is indecomposable in \( \mathcal{P}^o \).
2. \( \text{supp} B(w) \subseteq \leq w \) and \( B(w)w \cong R(l(w)) \)

We note that we call \( B(w) \) what is called \( B(w)(l(w)) \) in [Fi4]. Let us recall Fiebig’s conjecture.

**Conjecture 7.2** (Fiebig) \( C'_w = \sum x \text{rk} B(w)xT_x \) for all \( w \in \hat{W}^o \).

We note that because of [Fi4, Thm 2.11] (using the equivalence [Fi4, Prop. 3.4]) we have that Fiebig’s conjecture implies Lusztig’s conjecture for the dual group \( G^\vee_k \).

By [Fi4, Corollary 3.10 (1) (b)] we have that \( \text{rk} B(w)x = \text{rk} B(w)x \). On the one hand this means that part (2) of Definition/Theorem 7.1 is equivalent to
(2') $\text{supp} \mathcal{B}(w) \subseteq \leq w$ and $\mathcal{B}(w)_w \cong R(-l(w))$

and on the other hand we can restate Fiebig’s conjecture:

$$C'_w = \sum_x \text{rk} \mathcal{B}(w)_x T_x \text{ for all } w \in \hat{W}^\circ \quad (13)$$

### 7.3 Equivalence

Let $\eta : \langle \mathcal{B} \rangle \to \mathcal{H}$ be the inverse of the isomorphism of rings $\varepsilon : \mathcal{H} \to \langle \mathcal{B} \rangle$ defined in Section 2.3. Soergel [So4, Prop 5.9] proves that $\eta = d \circ h_\nabla$, (with $d$ as defined in Section 2.1 and $h_\nabla$ defined by the following formula (see [So4, Korollar 5.16])

$$h_\nabla(B) = \sum_{x \in \hat{W}} \text{rk} \text{Hom}(R_x, B)T_x.$$ 

An equivalence between additive categories sends indecomposable objects to indecomposable objects, so looking at Proposition 2.1 (1) and Definition/Theorem 7.1 we can see that the functor $G$ (that restricts to an equivalence between $\hat{B}_p^\circ$ and $P^\circ$) sends the bimodule $B_w$ to the $Z^\circ$-module $B(w)$.

It is easy to verify (see Section 2.4) that for all $M \in \mathcal{B}$ we have

$$\text{Hom}(R_x, M) \cong M_x \quad (14)$$

(the evaluation at $1_x \in R_x$ gives the isomorphism). Moreover $M_x$ is the biggest submodule of $M$ in which $r \in R \otimes R$ acts as multiplication by $r_x \in R_x$. By [Fi4, Lemma 3.3] we have that for $M \in Z^\circ-\text{mod}$, $M_x$ is the biggest submodule of $M$ on which $(z_y) \in Z^\circ$ acts as multiplication by $z_x$, so we have an isomorphism

$$(B_w)_x \cong B(w)_x \quad (15)$$

as graded right $R$-modules.

An equivalent formulation of $p$- affine Soergel’s conjecture is that for all $w \in \hat{W}^\circ$, we have $\eta(\langle B_w \rangle) = C'_w$. As $C'_w$ is self-dual, using equation (14) we see that affine Soergel’s $p$-conjecture is equivalent to

$$C'_w = \sum_x \text{rk} (B_w)_x T_x \text{ for all } w \in \hat{W}^\circ \quad (16)$$

So the isomorphism (15) imply that Fiebig’s conjecture (13) is equivalent to affine Soergel’s $p-$conjecture (16). $\square$
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