Some Remarks on Multiresolution Analyses
Containing Compactly Supported Functions

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1. **Introduction.** In [7], P. G. Lemarié proved that if a multiresolution analysis contains a compactly supported function, then it contains a minimal (pre)scale function. That is, a function \( \phi(x) \) of compact support such that

1. the integer translates, \( \phi(x - k), \ k \in \mathbb{Z} \), are a Riesz basis for the space \( V_0 \);
2. every function in \( V_0 \) that is compactly supported may be written as a finite linear combination of translates of \( \phi \).

The most basic examples of minimal scale functions of this type are the \( B \)-splines, and the compactly supported scale functions constructed by I. Daubechies [3]. An important property of these minimal-scale functions was first proved by Y. Meyer [10] for Daubechies’ functions, and subsequently stated by Lemarié [7] in the general case: *The translates of \( \phi \), restricted to the unit interval, form a linearly independent set.*

The purpose of this note is to prove the following stronger version of the above for a minimal scale function \( \phi \) that is continuous on the unit interval: the translates of \( \phi \) are linearly independent over any subset of positive measure contained in the unit interval. The stronger version is of interest because it may be used to obtain a local convergence theorem for multiresolution analyses with continuous minimal scale functions. The first version of this type of local convergence theorem was proved by Gundy and Kazarian [2] for a class of wavelet expansions that include the spline wavelets. Their theorem assumed a regularity condition (condition (M-Z) of [2]). It turns out that this regularity condition is, in fact, a property of all multiresolution analyses with continuous minimal scale functions, as a consequence of the above strong linear independence of these functions.

2. **Notation.** We suppose that a multiresolution analysis is given. That is, we have a scale of subspaces of \( L^2(\mathbb{R}) \), \( V_j, \ j \in \mathbb{Z} \), such that \( V_j \subset V_{j+1} \), and \( f(\cdot) \in V_j \) iff \( f(2^{-j}\cdot) \in V_0 \). Furthermore, we are given a function \( \phi \in V_0 \) such that the integer translates \( \phi(\cdot - k) \) form a Riesz basis for \( V_0 \): any function \( f(\cdot) \in V_0 \) has a representation

\[
f(x) = \sum a_k \phi(x - k)
\]

with

\[
\sum a_k^2 \approx \|f\|_2^2.
\]
If \( \phi \) has a nonzero integral, then it follows that the increasing sequence of subspaces exhausts \( L^2(\mathbb{R}) \). (See [5, Chapter 2].) Let \( P_j \) be the orthogonal projection operator from \( L^2 \) onto \( V_j \).

Now we impose another restriction on the multiresolution analysis. We require that the space \( V_0 \) contain a nontrivial continuous function that is compactly supported. (The continuity restriction omits the Haar case, for which the theorems below are obviously true. However, a more tedious formulation of assumptions, designed to include this case, hardly seems worthwhile. Our class of multiresolution analyses does include the spline wavelets, the compactly supported Daubechies wavelets, and those obtained from these classes by integration, as indicated in Lemarié [7].) With this additional assumption, the techniques of Lemarié [7] may be used to show that there exists a minimally supported, real-valued, continuous function \( \phi \in V_0 \) such that every compactly supported function in \( V_0 \) admits a representation as a finite linear combination of integer translates of \( \phi \). If we agree to normalize \( \phi \) by setting its integral equal to one, then \( \phi \) is unique, up to integer translates.

3. Linear Independence of Translates. In this section, we state the theorem on linear independence.

**Theorem 1.** Let \( \phi \) be a continuous, minimal (pre)scale function supported on the interval \([0, N]\). Then the translates \( \phi(\cdot - k), k = 0, \ldots, N - 1 \) are linearly independent over any set of positive measure of the unit interval.

**Remarks.** As we noted above, this line of investigation was initiated by Y. Meyer [10] and pursued by P. G. Lemarié in [7]. These authors treated the case where the "set of positive measure" was the entire unit interval. Lemarié and Margouyres [8] gave another simplified proof that showed the translates were linearly independent on any subinterval of the unit interval. Finally, Lemarié [7] showed that this property characterizes minimal scale functions. Those authors made no continuity assumptions.

**Proof.** We give a proof by contradiction as follows: If the translates are linearly dependent over a set of positive measure, we show that they are dependent over a set of
measure one in the unit interval. Since the function \( \phi \) is continuous, this means that the translates of \( \phi \) are dependent over the unit interval itself, thus contradicting the theorem of Meyer.

Throughout the proof, we will use matrices \( P_0 \) and \( P_1 \). To define these matrices, let us write the dilation equation for \( \phi \) as

\[
\phi(x/2) = \sum_{k=0}^{N} p_k \phi(x - k).
\]

First, let us define the \((N - 1) \times (N - 1)\) matrix \( P \) whose first row consists of the vector of odd numbered coefficients, \( p_{2k+1} \), followed by the appropriate number of zeros to give the vector \( N - 1 \) components. The second row of \( P \) is defined in the same way, using the even number coefficients, \( p_{2k} \), followed by the appropriate number of zeros. Third and fourth rows are obtained from the first two rows by a cyclic permutation of the indices: each entry is shifted to the right, with the final entry, a zero, moving to first position. This procedure is continued until \( N - 1 \) rows are obtained. (Thus if \( N = 2k \), the second row will contain the \( k + 1 \) entries \( p_0, p_2, \ldots, p_{2k} \) followed by \( k - 2 \) zeros. The last row will contain \( k - 1 \) zeros followed by the \( k \) coefficients \( p_1, p_3, \ldots, p_{2k-1} \). If \( N = 2k + 1 \), then the last row of the matrix consists of \( k - 1 \) zeros, followed by the \( k + 1 \) entries \( p_0, p_2, \ldots, p_{2k} \).) Now define the two \( N \times N \) matrices

\[
P_0 = \begin{pmatrix} p_0 & p_t \\ 0 & P \end{pmatrix} \quad \text{and} \quad P_1 = \begin{pmatrix} P & 0 \\ p_b & p_N \end{pmatrix}
\]

where \( p_t \) is the \( N - 1 \) vector consisting of the even numbered \( p_k \), starting with \( p_2 \), followed by the appropriate number of zeros; \( p_b \) consists of zeros followed by the coefficients \( p_k \) where \( k \) has the same parity as \( N \), where the final entry of the vector \( p_b \) is the coefficient \( p_{N-2} \).

The roles of \( P_0 \) and \( P_1 \) are as follows: consider a general linear combination of translates \( \sum c_k \phi(x + k) \). If we take account of the fact that \( \phi \) is supported on \([0, N]\) and restrict attention to \( x \in [0, 1] \), this sum is, in fact, finite and may be expressed as \( \sum_{k=0}^{N-1} c_k \phi(x + k) \). If we apply the dilation equation to express each \( \phi(\cdot + k) \) in terms of a sum of translates of \( \phi(2x) \), the resulting double sum is a certain linear combination
of translates of $\phi(2x)$ and $\phi(2x - 1)$, depending on whether $x$ is in $[0, \frac{1}{2}]$ or in $[\frac{1}{2}, 1]$. The coefficients of this linear combination are given by the matrices $P_0$ or $P_1$, acting on the vector $c = (c_0, \ldots, c_{N-1})$. (These matrices are implicit in the reconstruction-decomposition schemes in the wavelet literature, and appear explicitly, in the $3 \times 3$ case in Daubechies [4, section 7.2]. We summarize the above in the following proposition. Let $\Phi(x) = (\phi(x), \phi(x+1), \ldots, \phi(x+N-1))^t$ for $x \in [0,1]$, $\epsilon_k(x)$, $k = 1, 2, \ldots$ be the digits in the binary expansion of $x$. That is, $x = \sum \epsilon_i/2^k$, with $\epsilon_k = 0$ or 1. Let $T$ be the plus-one shift on the $\epsilon$-sequence: $T : (\epsilon_1, \epsilon_2, \ldots) \to (\epsilon_2, \epsilon_3, \ldots)$. We write $Tx = \sum_{k=1}^\infty \epsilon_{k+1}/2^k$.

**Proposition.** For $c = (c_b, c_1, \ldots, c_{N-1})$ we have

$$c \circ \Phi(x) = (P_{\epsilon_1}c^t) \circ \Phi(Tx).$$

More generally, for any $m$,

$$c \circ \Phi(x) = (P_{\epsilon_1} \cdots P_{\epsilon_m}c^t) \circ \Phi(T^m x).$$

**Proof.** Recall that the support of $\phi(\cdot + m)$ is the interval $[-m, -m + N]$ in the following computation. For $x \in [0, 1]$,

$$\sum_{k=0}^{N-1} c_k \phi(x_k) = \sum_k c_k \left( \sum_j p_j \phi(2(x + k) - j) \right)$$

$$= \sum_m \left( \sum_k c_k p_{2k-m} \right) \phi(2x + m).$$

The inner sum is taken over all $k$ with the provision that $p_{2k-m} = 0$ if $2k - m$ is not one of the integers $0, 1, \ldots, N - 1$. The outer sum with index $m$ changes according to whether $0 < 2x \leq 1$ or $1 < 2x \leq 2$, due to the support condition mentioned above. In the first case, when $\epsilon_1 = 0$, we have $0 \leq m \leq N - 1$; in the second case, when $\epsilon_1 = 1$, $-1 \leq m \leq N - 2$. Thus, the transformation takes two forms, with matrices $P_0$ and $P_1$. This proves the Proposition.
Now fix $c = (c_0, \ldots, c_{N-1})$ and let $K_c = \{x : c \circ \Phi(x) = 0\}$. The continuity of $\Phi(x)$ implies that $K_c$ is closed. We assume that $c \neq 0$, and that $K_c$ has positive measure in $[0, 1]$; we seek to contradict Meyer-Lemarié theorem. There are two cases to consider.

Case 1. There exists a finite sequence $P_{\epsilon_k}$, $k = 1, \ldots, m$ such that $P_{\epsilon_1} \cdots P_{\epsilon_m} c = 0$. If this is the case, we have our contradiction since $c \circ \Phi \equiv 0$ on the dyadic interval

$$\{x : \epsilon_1(x) = \epsilon_1, \ldots, \epsilon_m(x) = \epsilon_m\}.$$

Case 2. The vector $c$ is such that $P_{\epsilon_1} \cdots P_{\epsilon_m} c \neq 0$ for every finite sequence $\epsilon_1, \ldots, \epsilon_m$. In this case, we say that $c$ is a “never zero” vector.

**Lemma 1.** Let $c$ be a never zero vector, and suppose $m(K_c) > 0$. Then, for every $\eta$, $0 < \eta < 1$, there exists a never zero vector $b$ such that $m(K_b) > 1 - \eta$.

**Proof.** Since $K_c$ has positive measure, we can find dyadic interval $I_j = \{x : \epsilon_1(x) = \epsilon_1, \ldots, \epsilon_j(x) = \epsilon_j\}$ such that $m(K_c \cap I_j)/2^{-j} > 1 - \eta$. This is a consequence of the maximal martingale inequality, or alternatively, a point of density argument. Now apply the Proposition to points $x \in I_j$ to obtain a nonzero vector $P_{\epsilon_1} \cdots P_{\epsilon_j} c = b$. The set $K_b$ has measure greater than $1 - \eta$ since $m(\{x : x = T^j y \text{ for some } y \in I_j\}) = 1$. This proves the Lemma.

Now set

$$A_c = \left\{ a \in \mathbb{R}^N : a = \frac{b}{\|b\|_2} \text{ for some } b = P_{\epsilon_1} \cdots P_{\epsilon_j} c, \; j \in \mathbb{Z} \right\}.$$

By the Lemma, we have

$$\sup_{a \in A_c} m(K_c) = 1.$$

Now we claim that the supremum is achieved: there is an $a \in \mathbb{R}^N$ with $\|a\|_2 = 1$ such that $m(K_a) = 1$. To this end, we topologize the class $\mathcal{K}$ of compact sets $K \subset [0, 1]$ with a metric $\rho$ such that $(\mathcal{K}, \rho)$ is a compact (Hausdorff) metric space. Let

$$\rho(A, B) = \sup_{x \in [0, 1]} |d(x, A) - d(x, B)|$$

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where \( d(x, D) = \inf \{|x - y| : y \in D\} \). This metric is equivalent to

\[
\sigma(A, B) = \inf \{ \epsilon > 0 : A \subset V_\epsilon(B) \text{ and } B \subset V_\epsilon(A) \}
\]

where \( V_\epsilon(D) = \{ z \in [0, 1] : d(z, D) < \epsilon \} \). Proofs of this equivalence and the fact that \( \mathcal{K} \) is a compact space may be found in Kornum [6, section 6.2].

Let \( K_{a_n} \) be a sequence of sets such that \( m(K_{a_n}) \) tends to one. From this sequence, we may extract a convergent subsequence \( K_{a_{n_k}} \). Since \( \|a_n\|_2 \equiv 1 \), we may extract a convergent subsequence \( a_{m_k} \), so that, finally, we obtain a sequence \( K_{a_n} \rightarrow K \) and \( a_n \rightarrow a \).

First of all, we claim that \( m(K) = 1 \). If not, there a \( \delta > 0 \) such that \( m(K) \leq m(V_\delta(K)) = d < 1 \). In this case, there is an \( n_0 \) such that for all \( n \geq n_0 \), \( K_{a_n} \subset V_\delta(K) \), and so \( m(K_{a_n}) \leq m(V_\delta(K)) = d < 1 \). Since \( m(K_{a_n}) \rightarrow 1 \), this is a contradiction. Second, we claim that \( K \subset K_a \). Since \( K_{a_n} \rightarrow K \), we have

\[
\sup_{y \in [0,1]} |d(y, K) - d(y, K_{a_n})| \rightarrow 0.
\]

Therefore, if \( x \in K \), \( d(x, K_{a_n}) \rightarrow 0 \). Choose \( x_n \in K_{a_n} \) so that \( |x - x_n| \rightarrow 0 \). Then, by the Cauchy-Schwarz and triangle inequalities,

\[
|a \circ \Phi(x)| \leq |(a - a_n) \circ \Phi(x)| + |a_n \circ (\Phi(x) - \Phi(x_n))| + |a_n \circ \Phi(x_n)|
\leq \|a - a_n\|_2 \cdot \|\Phi(x)\|_2 + \|\Phi(x) - \Phi(x_n)\|_2.
\]

Since both terms on the right tend to zero, we have the inclusion \( K \subset K_a \).

4. Local Convergence of Wavelet Expansions. In [1], the following local convergence theorem is proved for Haar series, using martingale methods.

**Theorem A.** Let \( f(x) = (f_0(x), f_1(x), \ldots) \) be a sequence of functions such that

(a) \( f_j \in V_j \) where \( \{V_k\} \) is the Haar multiresolution analysis;

(b) \( P_j(f_{j+1})(x) = f_j(x) \) for \( j \geq 0 \).

Let \( S^2(f)(x) = \sum (f_{j+1}(x) - f_j(x))^2 + f_0^2(x) \) and \( f^*(x) = \sup_j |f_j(x)| \). Then, the following sets are equivalent almost everywhere:
(a) \( \{ x : \lim_{j \to \infty} f_j(x) \text{ exists and is finite} \} \);
(b) \( \{ x : S(f)(x) < +\infty \} \);
(c) \( \{ x : \sup_j |f_j(x)| < \infty \} \).

Gundy and Kazarian [2] extended this local convergence theorem to the class of multiresolution analyses arising from the basic splines. In fact, the proof did not appear to use properties specific to the spline family. The basic regularity condition essential to the proof is a two-norm condition, reminiscent of a condition first proposed by Marcinkiewicz and Zygmund [9] in their study of series of independent random variables. This condition, called condition (M-Z) is as follows:

Let \( \phi \) be a compactly supported scale function, supported on \([0,N]\). We suppose that, for every \( \delta, 0 < \delta < 1 \), there exist constants \( B_\delta \) and \( C_\delta \) such that
\[
(M-Z) \text{ For every measurable subset } E \subset [0,1] \text{ of measure greater than } \delta, \text{ and any sequence } a_k; k = 0, 1, \ldots, N-1, \text{ we have }
\]
\[
C_\delta \sum_{k=0}^{N-1} |a_k| \leq \sup_{x \in E} \left| \sum_{k=0}^{N-1} a_k \phi(x + k) \right| \leq B_\delta \sum_{k=0}^{N-1} |a_k|. 
\]

The constants \( B_{\delta}, C_\delta \) depend only on \( \phi \) and the measure of the set \( E \). The condition holds for the class of \( B \)-spline scale functions, as pointed out in [2]. However, the scope of the condition was not known, and left as an open problem in [2]. The following theorem answers this question.

**Lemma 2.** Let \( \{V_j\} \) be a multiresolution analysis such that \( V_0 \) contains continuous functions of compact support. Then the minimal scale function \( \phi \) satisfies condition (M-Z).

Before proving the lemma, we state the following theorem, in which we use the definitions in Theorem A, for multiresolution analyses more general than the Haar system.

**Theorem 2.** (Theorem B of [2]) Let \( \{V_j\} \) be a multiresolution analysis that contains continuous functions of compact support. Then the following sets are equivalent almost everywhere:
(a) \( \{ x : \lim_{j \to \infty} f_j(x) \text{ exists and is finite} \} \);
(b) \( \{ x : S(f)(x) < +\infty \} \);
(c) \( \{ x : \sup_j |f_j(x)| < \infty \} \).

**Proof of Lemma 2.** Since \( \phi \) is continuous on \([0, 1]\), the issue is to show the existence of \( C_\delta \) that is uniform over all sets \( E \subset [0, 1] \) of measure greater than \( \delta \). First, observe that, since the translates of \( \phi \) are linearly independent over \( E \) (Theorem 1), there is a constant \( C(E) \) such that

\[
C(E) \sum_{k=0}^{N-1} |a_k| \leq \sup_{x \in E} \left| \sum_{k=0}^{N-1} a_k \phi(x+k) \right|.
\]

This follows from the fact that the right-hand side defines a norm on \( \mathbb{R}^N \): the linear independence of the translates of \( \phi \) guarantees that this norm is strictly positive on \( \mathbb{R}^N - \{0\} \). Since the left-hand side is also a norm, the existence of a constant is assured by the open mapping theorem. Now we must show that

\[
\inf\{ C(E) : m(E) \geq \delta \} > 0.
\]

It is enough to show this for closed sets. To this end, we show that

\[
C(E) = \sup_{x \in E} \frac{|a \circ \Phi(x)|}{\|a\|_2}
\]

is a continuous function of \( E \) for the topology \((\mathcal{K}, \rho)\) introduced above. Let \( \epsilon > 0 \) be given, and let \( \{A_n\} \) be a sequence of sets converging to \( A \) in \( \mathcal{K} \). The vector \( \Phi(x) \) is uniformly continuous on \([0, 1]\), so that

\[
\|\Phi(x) - \Phi(y)\|_2 \leq \frac{\epsilon\|a\|_1}{\|a\|_2}
\]

whenever \( |x - y| \leq \delta(\epsilon) \). Let \( n_0 \) be an integer such that

\[
A_n \subset V_\delta(A) \quad \text{and} \quad A_n \subset V_\delta(A_n)
\]

for all \( n \geq n_0 \). Since \( \Phi \) is continuous on \( \mathbb{R}^N \) and \( \overline{V_\delta(A_n)} \) is compact, it follows that for \( n \geq n_0 \), there exists an \( x_n \in \overline{V_\delta(A_n)} \) for which

\[
C(A) \leq C(\overline{V_\delta(A_n)}) = \frac{|a \circ \Phi(x_n)|}{\|a\|_1}.
\]
By the uniform continuity of $\Phi(\cdot)$, $|C(A_n) - C(V_\delta(A_n))| \leq \epsilon$, so that

$$C(A) \leq C(A_n) + \epsilon.$$ 

If we reverse the roles of $A_n$ and $A$ in the above argument, we see that $C(A_n) \leq C(A) + \epsilon$. Thus, $C(E)$ is continuous on $\mathcal{K}$. The collection $\{E \in \mathcal{K} : m(E) \geq \delta\}$ is closed in $\mathcal{K}$ by a similar argument, so there is a closed $E_0$, $m(E_0) \geq \delta$ such that $0 < C(E_0) \leq C(E)$ for any $E$, $m(E) \geq \delta$.

5. Concluding Remarks. The quadratic variation functional $S(f)$ of Theorem 3 is invariant under changes of scale functions $\phi$ for $V_0$ and $\psi$ for $V_1$: $S(f)$ is defined from the sequence of projections $\{P_j\}$ without specific reference to the choice of scale function. However, $S(f)$ is an “incomplete” square function in the sense that if the prewavelet family $\{\psi(2^jx - k)\}$ is orthogonalized in $k$, to obtain a family $\{\tilde{\psi}(2^jx - k)\}$ that is orthonormal in both variables $j, k \in \mathbb{Z}$, then one could consider a quadratic variation functional $S^2(f)(x) = \sum_{j,k} (a_{j,k}\tilde{\psi}(2^jx - k))^2$. If we have a multiresolution analysis that admits a compactly supported, continuous orthonormal family $\{\tilde{\psi}(2^jx - k)\}$, then one can show that $S(f)(x)$ and $S(f)(x)$ are finite on the same set, up to a set of measure zero. The proof of this fact follows the same lines as the proof in [2]. Since the details are given there, we will not repeat them here.
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