The Singular Behaviour of QCD Amplitudes at Two-loop Order

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Abstract

We discuss the structure of infrared singularities in on-shell QCD amplitudes at two-loop order. We present a general factorization formula that controls all the $\epsilon$-poles of the dimensionally regularized amplitudes. The dependence on the regularization scheme is considered and the coefficients of the $1/\epsilon^4$, $1/\epsilon^3$ and $1/\epsilon^2$ poles are explicitly given in the most general case. The remaining single-pole contributions are also explicitly evaluated in the case of amplitudes with a $q\bar{q}$ pair.
1 Introduction

Perturbative QCD predictions are important to test QCD and to measure its fundamental parameters (e.g. the strong coupling $\alpha_s$), to estimate Standard Model backgrounds for new-physics signals, and to understand the interplay between perturbative and non-perturbative phenomena.

Calculations at the leading order (LO) in perturbation theory are straightforward: the matrix elements at tree level can be easily computed by using helicity techniques and colour-subamplitude decompositions and can then be numerically or analytically integrated.

Nowadays, also next-to-leading order (NLO) calculations are feasible in a direct manner, as witnessed by the accelerated production rate of new NLO computations. This achievement is the result of much theoretical progress in the last few years. This progress regards both efficient techniques for the evaluation of one-loop matrix elements and completely general algorithms to handle and cancel infrared singularities when combining tree-level and one-loop contributions in the evaluation of physical quantities.

In contrast, the calculation of jet observables at next-to-next-to-leading order (NNLO), although feasible in principle, remains a challenge in practice. Indeed, no computations have appeared that involve more than two kinematic variables. The general obstacles that have to be overcome to perform NNLO calculations are the evaluation of two-loop amplitudes and the cancellation of infrared singularities between real and virtual contributions. There is only one QCD amplitude that is known at two-loop order, namely the electromagnetic form factor of the quark. The extension to higher loops of cutting techniques has recently led to the computation of the two-loop four-gluon amplitude in the simplified case of $N=4$ supersymmetric QCD. Other approaches for efficiently calculating higher-loop amplitudes taken by various authors include string theory, first quantized and recursive approaches. Regarding the cancellation of infrared singularities, much work still has to be done to extend the NLO algorithms to higher orders.

In the case of calculations at NLO, an important ingredient of the theoretical progress mentioned above has been the complete understanding of the factorization properties of tree-level and one-loop amplitudes in the soft and collinear limits. An analogous understanding at NNLO is important as well. In the case of tree-level amplitudes, the factorization structure in the double-soft and double-collinear limits is known. The collinear limit of one-loop amplitudes is also known. The singular behaviour of two-loop amplitudes has not been investigated so far.

In this paper we discuss the structure of infrared singularities in on-shell QCD amplitudes at two-loop order and, in particular, we present a universal factorization formula that gives the coefficients of the $\epsilon$-poles of the dimensionally regularized divergences. Our results have been obtained by improving and extending the coherent-state approach of Ref. to higher infrared accuracy. More details on the derivation of the results presented here and on the singular behaviour of all-loop amplitudes will be given elsewhere.

The outline of the paper is as follows. In Sect. we introduce our notation and, in
particular, we point out the subtleties related to the regularization scheme of the infrared divergences in the general context of dimensional regularization. Section 3 is devoted to a review of the known results on the singular behaviour of one-loop amplitudes. Our general factorization formula for two-loop amplitudes is presented and discussed in Sect. 4 and applications to some particular cases are considered in Sects. 5 and 6. At present, we are unable to explicitly compute the coefficient of the single poles in general terms, but we can do that in the particular case of amplitudes with two quarks, as shown in Sect. 5.2. Our results are summarized in Sect. 7.

2 Notation

2.1 Dimensional regularization

In the evaluation of loop amplitudes one encounters ultraviolet and infrared singularities that have to be properly regularized. The most efficient method to simultaneously regularize both singularities in gauge theories is to use dimensional regularization [22, 24].

The key ingredient of dimensional regularization is the analytic continuation of loop momenta to \( d = 4 - 2 \varepsilon \) space-time dimensions. Having done this, one is left with some freedom regarding the dimensionality of the momenta of the external particles as well as the number of polarizations of both external and internal particles. This leads to different regularization schemes (RS) within the dimensional-regularization prescription.

The RS known as conventional dimensional regularization (CDR) is, in a sense, the most natural scheme because no distinction is made between particles in the loops and external particles. All particle momenta are \( d \)-dimensional and one considers \( d - 2 \) helicity states for gluons and 2 helicity states for massless quarks. Nonetheless, other RS are known to be useful for field theoretical reasons (e.g. to explicitly preserve supersymmetric Ward identities [25]) and/or because of practical simplifications in actual calculations [4].

This brief overview of dimensional regularization is important to discuss the singular behaviour of loop amplitudes, because only the leading singularities are RS-independent. In general, the coefficients of all the other singular terms as well as the finite parts do depend on the RS. The RS dependence regards both ultraviolet and infrared singularities, but it affects them in different ways.

In principle the regularized ultraviolet divergences have to be removed from off-shell Green functions via renormalization of wave functions and coupling constant. This leads to the introduction of the running coupling \( \alpha_S(\mu^2) \), whose definition in terms of the bare coupling depends on the regularization (and renormalization) scheme. The ensuing scheme dependence in renormalized on-shell amplitudes is nonetheless trivial. It simply amounts to (and can be controlled by) an overall perturbative shift in \( \alpha_S \) (cf. Sect. 2.2). This perturbative shift has no effect on tree-amplitudes and is universal in loop amplitudes.

†The RS that are mostly used in one-loop computations are the ’t Hooft and Veltman scheme [22], the dimensional-reduction scheme [25] and the four-dimensional helicity scheme [26].
that is, independent of the number and type of particles.

Infrared singularities behave differently with respect to dimensional-regularization prescriptions. Although one can keep track of their RS dependence in general terms [27], to relate loop amplitudes in different RS one has to introduce proper transition factors that depend on the number and type of external partons. In the context of one-loop amplitudes, this feature was pointed out in Ref. [28], where the (one-loop) transition factors between the main schemes in current usage were explicitly computed. Beyond one-loop order, care has to be taken in the operational definition of some RS. We do not explicitly consider this problem, but in Sects. 4–6 we discuss the RS dependence of two-loop amplitudes within the general class of schemes specified at the end of Sect. 2.2.

Note also that, in order to fulfil unitarity in calculations of physical quantities, the RS dependence of loop amplitudes has to be consistently matched to that of tree amplitudes. This issue is discussed on quite a general basis in Ref. [27] and is no longer considered in this paper.

2.2 Renormalized amplitudes

We consider amplitudes \( \mathcal{M}_m \) that involve \( m \) external QCD partons (gluons and massless quarks) with momenta \( p_1, \ldots, p_m \) and an arbitrary number and type of particles with no colour (photons, leptons, vector bosons, ...). Note that, by definition, we always consider incoming and outgoing parton momenta in the physical region, i.e. any \( p_i \) is massless and with positive-definite energy (in particular, \( p_i \cdot p_j > 0 \)). The amplitudes are denoted by \( \mathcal{M}_m(p_1, \ldots, p_m) \) (or, shortly, \( \mathcal{M}_m(\{p\}) \)) and the dependence on the momenta and quantum numbers of non-QCD particles is always understood. Besides that, the unrenormalized amplitude \( \mathcal{M}_m \) depends on powers of \( \alpha_u^2 \mu_0^2 \epsilon \), where \( \alpha_u^2 \) is the bare coupling constant and \( \mu_0 \) is the dimensional-regularization scale.

We find it convenient to use renormalized amplitudes. These are obtained from the unrenormalized ones by just expressing the bare coupling \( \alpha_u^2 \) in terms of the running coupling \( \alpha_S(\mu^2) \) evaluated at the arbitrary renormalization scale \( \mu^2 \). In order to avoid renormalization-scheme ambiguities, we always consider the running coupling \( \alpha_S(\mu^2) \) as that defined in the standard \( \overline{\text{MS}} \) scheme. Thus, we use the following expression

\[
\alpha_S^2 \mu_0^{2\epsilon} S_\epsilon = \alpha_S(\mu^2) \mu^{2\epsilon} \left[ 1 - \alpha_S(\mu^2) \frac{\beta_0}{\epsilon} + \alpha_S^2(\mu^2) \left( \frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) + \mathcal{O}(\alpha_S^3(\mu^2)) \right],
\]

where \( \beta_0, \beta_1 \) are the first two coefficients of the QCD beta function

\[
\beta_0 = \frac{11C_A - 4T_R N_f}{12\pi}, \quad \beta_1 = \frac{17C_A^2 - 10C_AT_R N_f - 6C_F T_R N_f}{24\pi^2},
\]

and \( S_\epsilon \) is the typical phase-space volume factor in \( d = 4 - 2\epsilon \) dimensions (\( \gamma_E = -\psi(1) = 0.5772 \ldots \) is the Euler number)

\[
S_\epsilon = \exp \left[ \epsilon (\ln 4\pi + \psi(1)) \right].
\]

Note that, as recalled in Sect. 2.1, unrenormalized amplitudes contain some RS dependence of ultraviolet origin. Using the \( \overline{\text{MS}} \) renormalization scheme, we have eliminated this
harmless dependence from renormalized amplitudes. In practical terms, our renormalization procedure works as follows. If the bare amplitude is evaluated in CDR, one simply replaces \( \alpha_s \) according to Eq. (1). If a different RS is used, the minimally-subtracted substitution in Eq. (1) still applies in terms of an RS-dependent coupling \( \alpha_{S}^{RS}(\mu^2) \). To obtain the MS amplitudes, one then has to perform the additional substitution:

\[
\alpha_{S}^{RS} = \alpha_S \left[ 1 + d_1 \alpha_S + d_2 \alpha_S^2 + \mathcal{O}(\alpha_S^3) \right],
\]

where the coefficients \( d_1, d_2 \) depend on the RS and are finite for \( \epsilon \to 0 \).

The \( \overline{\text{MS}} \)-renormalized amplitude has the following perturbative expansion:

\[
\mathcal{M}_m(\alpha_S(\mu^2), \mu^2; \{p\}) = \left( \frac{\alpha_S(\mu^2)}{2\pi} \right)^q \left[ \mathcal{M}^{(0)}_m(\mu^2; \{p\}) + \frac{\alpha_S(\mu^2)}{2\pi} \mathcal{M}^{(1)}_m(\mu^2; \{p\}) \right. \\
+ \left. \left( \frac{\alpha_S(\mu^2)}{2\pi} \right)^2 \mathcal{M}^{(2)}_m(\mu^2; \{p\}) + \mathcal{O}(\alpha_S^3(\mu^2)) \right],
\]

where the overall power \( q \) is half-integer (\( q = 0, 1/2, 1, 3/2, \ldots \)), in general. This equation fixes the normalization of the tree-level (\( \mathcal{M}^{(0)}_m(\mu^2) \)), one-loop (\( \mathcal{M}^{(1)}_m(\mu^2) \)) and two-loop (\( \mathcal{M}^{(2)}_m(\mu^2) \)) coefficient amplitudes that we use in the rest of the paper. These coefficient amplitudes are infrared-singular when \( \epsilon \to 0 \) and behave as

\[
\mathcal{M}^{(n)}_m \sim \left( \frac{1}{\epsilon} \right)^{2n} + \ldots,
\]

where the dots stand for \( \epsilon \)-poles of lower order. The coefficient of the poles are still RS-dependent. The results presented in the following sections are valid in CDR and in any other RS that is consistently defined as ultraviolet regulator. By that, we mean in any other scheme whose minimally-subtracted coupling is related to the \( \overline{\text{MS}} \) coupling via Eq. (4) in a universal way (i.e. with coefficients \( d_1, d_2 \) that do not depend on the computed amplitude).

### 2.3 Colour space

The colour structure of QCD amplitudes can be handled in two different and equivalent ways. One method consists in projecting out the amplitude onto a particular set of non-orthogonal colour vectors, thus extracting colour subamplitudes. The other method works directly in colour space. We prefer to use the colour-space formalism and, in particular, the same notation as in Ref. [8].

The colour indices of the \( m \) partons in the amplitude \( \mathcal{M}_m \) are generically denoted by \( c_1, \ldots, c_m \): \( c_i = \{ a \} = 1, \ldots, N_c^2 - 1 \) for gluons and \( c_i = \{ \alpha \} = 1, \ldots, N_c \) for quarks and antiquarks. We formally introduce an orthogonal basis of unit vectors \( \{ c_1, \ldots, c_m \} \) in the \( m \)-parton colour space, in such a way that the colour amplitude can be written as follows:

\[
\mathcal{M}^{c_1 \ldots c_m}_{m}(p_1, \ldots, p_m) \equiv \langle c_1, \ldots, c_n | \mathcal{M}_m(p_1, \ldots, p_m) \rangle.
\]

\(^4\) Precisely speaking, \( \mathcal{M}^{(0)}_m \) is not necessarily a tree amplitude, but rather the lowest-order amplitude for that given process; \( \mathcal{M}^{(1)}_m, \mathcal{M}^{(2)}_m \) are the corresponding one-loop and two-loop corrections. For instance, in the case of \( gg \to \gamma \gamma \), \( \mathcal{M}^{(0)} \) involves a quark loop.

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Thus $|\mathcal{M}_m(p_1, \ldots, p_m)|$ is an abstract vector in colour space and, in particular, the square amplitude summed over colours is:

$$|\mathcal{M}_m({p})|^2 = \langle \mathcal{M}_m({p}) | \mathcal{M}_m({p}) \rangle .$$

(8)

Colour interactions at the QCD vertices are represented by associating a colour charge $T_i$ with the emission of a gluon from each parton $i$. The colour charge $T_i = \{T_i^a\}$ is a vector with respect to the colour indices $a$ of the emitted gluon and an $SU(N_c)$ matrix with respect to the colour indices of the parton $i$. More precisely, its action onto the colour space is defined by

$$\langle c_1, \ldots, c_i, \ldots, c_m | T_i^a | b_1, \ldots, b_i, \ldots, b_m \rangle = \delta_{c_i b_i} \cdots T_{c_i b_i}^a \cdots \delta_{c_m b_m} ,$$

(9)

where $T_{cb}^a \equiv i f_{cab}$ (colour-charge matrix in the adjoint representation) if the emitting particle $i$ is a gluon and $T_{a\beta}^a \equiv t_{a\beta}^a$ (colour-charge matrix in the fundamental representation) if the emitting particle $i$ is a final-state quark (in the case of a final-state antiquark $T_{a\beta}^a \equiv t_{a\beta}^a = -t_{\beta a}^a$). Note that the colour-charge operator of an initial-state parton is defined by crossing symmetry, that is by $(T_i)^a_{a\beta} = t_{a\beta}^2 = -t_{\beta a}^2$ if $i$ is an initial-state quark and $(T_i)^a_{a\beta} = t_{\beta a}^2$ if $i$ is an initial-state antiquark.

In this notation, each vector $|\mathcal{M}_m(p_1, \ldots, p_m)|$ is a colour singlet, so colour conservation is simply

$$\sum_{i=1}^m T_i |\mathcal{M}_m\rangle = 0 .$$

(10)

The colour-charge algebra for the product $(T_i)^a (T_j)^a \equiv T_i \cdot T_j$ is:

$$T_i \cdot T_j = T_j \cdot T_i \quad \text{if} \quad i \neq j; \quad T_i^2 = C_i ,$$

(11)

where $C_i$ is the Casimir operator, and we have $C_i = C_A = N_c$ if $i$ is a gluon and $C_i = C_F = T_R(N_c^2 - 1)/N_c = (N_c^2 - 1)/2N_c$ if $i$ is a quark or an antiquark (we are using the customary normalization $T_R = 1/2$).

### 3 Singular behaviour at one-loop order

In this section we recall the known results [17, 8, 8] on the infrared-singular behaviour of QCD amplitudes at one-loop order. The one-loop coefficient subamplitude $\mathcal{M}_m^{(1)}(\mu^2; \{p\})$ has double and single poles in $1/\epsilon$. The coefficients of these poles are universal and can be given by the following formula

$$|\mathcal{M}_m^{(1)}(\mu^2; \{p\})|_{RS} = I^{(1)}(\epsilon, \mu^2; \{p\}) |\mathcal{M}_m^{(0)}(\mu^2; \{p\})|_{RS} + |\mathcal{M}_m^{(1)\text{fin}}(\mu^2; \{p\})|_{RS} .$$

(12)

The contribution $\mathcal{M}_m^{(1)\text{fin}}$ on the right-hand side is finite for $\epsilon \to 0$ and, hence, in Eq. (12) all one-loop singularities are factorized in colour space with respect to the tree-level amplitude $\mathcal{M}_m^{(0)}$. The singular dependence is embodied in the factor $I^{(1)}$ that acts as
a colour-charge operator onto the colour vector $|\mathcal{M}_{m}^{(0)}\rangle$. Its explicit expression in terms of the colour charges of the $m$ partons is

$$I^{(1)}(\epsilon, \mu^2; \{p\}) = \frac{1}{2 \Gamma(1-\epsilon)} \sum_i \frac{1}{T_i^2} \gamma^{\text{sing}}_i(\epsilon) \sum_{j \neq i} T_i \cdot T_j \left( \frac{\mu^2 e^{-i\lambda_{ij}\pi}}{2 p_i \cdot p_j} \right)^\epsilon, \quad (13)$$

where $e^{-i\lambda_{ij}\pi}$ is the unitarity phase ($\lambda_{ij} = +1$ if $i$ and $j$ are both incoming or outgoing partons and $\lambda_{ij} = 0$ otherwise) and the singular (for $\epsilon \to 0$) function $\gamma^{\text{sing}}_i(\epsilon)$ depends only on the parton flavour and is given by

$$\gamma^{\text{sing}}_i(\epsilon) = T_i^2 \frac{1}{\epsilon^2} + \gamma_i \frac{1}{\epsilon}. \quad (14)$$

The flavour coefficients $T^2_i$ and $\gamma_i$ are

$$T^2_q = T^2_{\bar{q}} = C_F, \quad T^2_g = C_A, \quad \gamma_q = \gamma_{\bar{q}} = \frac{3}{2} C_F, \quad \gamma_g = \frac{11}{6} C_A - \frac{2}{3} T_R N_f. \quad (15)$$

Note that in Eq. (12) the double poles $1/\epsilon^2$ are factorized completely and not only in colour space. The simplest way to see that is to expand Eq. (13) in powers of $\epsilon$ and then use the colour conservation relation (10), i.e. $\sum_{j \neq i} T_j = -T_i$. One obtains the result

$$I^{(1)}(\epsilon, \mu^2; \{p\}) = \frac{1}{2} \sum_i \frac{1}{\epsilon^2} \sum_{j \neq i} T_i \cdot T_j + \mathcal{O}(1/\epsilon) = -\frac{1}{2 \epsilon^2} \sum_i T_i^2 + \mathcal{O}(1/\epsilon), \quad (16)$$

that explicitly shows the absence of colour correlations at $\mathcal{O}(1/\epsilon^2)$. Nonetheless, single poles $1/\epsilon$ are still colour-correlated.

In the factorization formula (12) we have introduced the subscripts $\text{RS}$ to explicitly recall the RS dependence of the various quantities. Note that the insertion operator $I^{(1)}$ is RS-independent. The RS dependence of the one-loop amplitude $\mathcal{M}^{(1)}$ is completely taken into account by that of the tree amplitude $\mathcal{M}^{(0)}_m$ and by the finite remainder $\mathcal{M}^{(1)\text{fin}}_m$. In particular, the interference between RS-dependent terms of $\mathcal{O}(\epsilon)$ in $\mathcal{M}^{(0)}_m$ and double poles $1/\epsilon^2$ in $I^{(1)}$ produces, in general, an RS dependence of $\mathcal{M}^{(1)}_m$ that begins at $\mathcal{O}(1/\epsilon)$.

The RS-dependent terms of $\mathcal{M}^{(1)\text{fin}}_m$ are not colour-correlated. Since they are completely factorized, we can avoid the colour-space notation and write

$$\mathcal{M}^{(1)\text{fin}}_{m, \text{RS}}(\mu^2; \{p\}) = \frac{1}{2} \left( \sum_i \vec{\gamma}_{i}^{\text{RS}} \right) \mathcal{M}^{(0)}_{m, \text{RS}}(\mu^2; \{p\}) + F^{(1)}_m(\mu^2; \{p\}) + \mathcal{O}(\epsilon), \quad (17)$$

where the finite coefficients $\vec{\gamma}_{i}^{\text{RS}}$ depend only on the flavour of the external partons in $\mathcal{M}_m$. The transition coefficients $\vec{\gamma}_{i}^{\text{RS}}$ that relate the RS mostly used in one-loop computations were first calculated in Ref. [28]. Using the normalization in Eq. (17) and assuming CDR as reference scheme (i.e. setting $\vec{\gamma}_{i}^{\text{CDR}} = 0$, by definition), the explicit expressions of $\vec{\gamma}_{i}^{\text{RS}}$ can be found in Ref. [27].

The residual RS dependence on the right-hand side of Eq. (17) affects only the terms of $\mathcal{O}(\epsilon)$. In particular, the $\epsilon$-independent function $F^{(1)}_m(\mu^2; \{p\})$ is RS-independent [28, 27].

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Note that the $\epsilon$-dependence of any amplitude $\mathcal{M}_m$ is always understood.
4 Factorization formula at two-loop order

The two-loop coefficient subamplitude $\mathcal{M}_m^{(2)}$ has $1/\epsilon^4, 1/\epsilon^3, 1/\epsilon^2$ and $1/\epsilon$ poles. Because of the increased degree of singularities, it is not a priori guaranteed that all of them can be controlled by a universal factorization formula as in the one-loop case. The main result presented in this paper is that such a factorization formula does exist and can be written in the following form

$$|\mathcal{M}_m^{(2)}(\mu^2; \{p\})\rangle_{\text{RS}} = I^{(1)}(\epsilon, \mu^2; \{p\}) |\mathcal{M}_m^{(1)}(\mu^2; \{p\})\rangle_{\text{RS}} + I^{(2)}_{\text{RS}}(\epsilon, \mu^2; \{p\}) |\mathcal{M}_m^{(0)}(\mu^2; \{p\})\rangle_{\text{RS}} + |\mathcal{M}_m^{(2)\text{fin}}(\mu^2; \{p\})\rangle_{\text{RS}}. \quad (18)$$

The factorization structure of Eq. (18) is not trivial and is somehow analogous to that of the collinear limit of one-loop amplitudes considered in Ref. [20].

The main features of Eq. (18) are the following:

- In the first term on the right-hand side of Eq. (18), the one-loop insertion operator $I^{(1)}$ of Eq. (13) acts onto the one-loop matrix element. The corresponding contribution thus contains poles of the type $1/\epsilon^n$ with $n = 1, \ldots, 4$, coming from the single and double poles in $I^{(1)}$ and $\mathcal{M}_m^{(1)}$.

- The second term on the right-hand side of Eq. (18) contains a new colour-charge operator $I^{(2)}$ that acts onto the tree-level subamplitude. The two-loop insertion operator $I^{(2)}$ is given as follows

$$I^{(2)}_{\text{RS}}(\epsilon, \mu^2; \{p\}) = -\frac{1}{2} I^{(1)}(\epsilon, \mu^2; \{p\}) \left( I^{(1)}(\epsilon, \mu^2; \{p\}) + 4\pi\beta_0 \frac{1}{\epsilon} \right) + \frac{e^{+\epsilon\gamma_E}}{\Gamma(1-\epsilon)} (2\pi\beta_0 \frac{1}{\epsilon} + K) I^{(1)}(2\epsilon, \mu^2; \{p\}) \quad (19)$$

where the coefficient $K$ is:

$$K = \left( \frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R N_f. \quad (20)$$

The first and second line\footnote{Note that on the second line the argument of the operator $I^{(1)}$ is $2\epsilon$ rather than $\epsilon$ as on the first line.} on the right-hand side of Eq. (19) lead to $\epsilon$ poles that are at most of 4th- and 3rd-order, respectively. Moreover, these two lines control all the poles up to $1/\epsilon^2$. In fact, the remaining term $H^{(2)}_{\text{RS}}(\epsilon, \mu^2; \{p\})$ contains only single poles:

$$H^{(2)}_{\text{RS}}(\epsilon, \mu^2; \{p\}) = \mathcal{O}(1/\epsilon). \quad (21)$$

- The last term, $\mathcal{M}_m^{(2)\text{fin}}$, on the right-hand side of Eq. (18) is a non-singular remainder in the limit $\epsilon \to 0$. 


Note that the two-loop operator $I_{RS}^{(2)}$ in Eq. (19) depends on the RS only through the single-pole contributions in $H_{RS}^{(2)}$. At present, we cannot give an explicit expression for $H_{RS}^{(2)}$ that is valid for any QCD amplitude. The particular case of a $q\bar{q}$ pair is considered in Sect. 5.2.

Using the factorization formula (18) and Eq. (19), all the coefficients of the poles $1/\epsilon^4, 1/\epsilon^3, 1/\epsilon^2$ can be explicitly evaluated in terms of the one-loop operator $I^{(1)}$, the first coefficient $\beta_0$ of the beta function and the constant $K$ in Eq. (20).

In particular, the coefficient $K$ allows one to control the double poles $1/\epsilon^2$ in a universal way, that is, independently of the given QCD amplitude. The same coefficient typically appears in the resummation program of higher-order logarithmic corrections of the Sudakov type [29]. This universality [30] follows from the fact that $K$ measures the renormalization (in the MS renormalization scheme) of the intensity of the lowest-order soft-gluon emission. The origin of $K$ is thus essentially ultraviolet, which explains its independence of the RS of infrared singularities.

The non-singular remainder $\mathcal{M}_{m}^{(2)\text{fin}}$ is analogous to $\mathcal{M}_{m}^{(1)\text{fin}}$ in Eq. (12). However, its RS-dependent part is in general not factorized in colour space and cannot be expressed in factorized form as in Eq. (17).

5 Two-loop amplitudes with two partons

5.1 Colour structure

Amplitudes with only two QCD partons have a trivial colour structure to any loop order. The colour-space formulae of Sects. 3 and 4 thus become ‘true’ factorization formulae because all the insertion operators are proportional to the identity matrix in colour space.

In the case of the one-loop insertion operator $I^{(1)}$, one can use colour conservation as in Eq. (10) and explicitly perform the colour algebra, that is $T_1 \cdot T_2 = -T_2 \cdot T_1 = -T_2$. Then one obtains

$$I_{ij}^{(1)}(\epsilon, \mu^2; p_1, p_2) = -\gamma_{ij}\frac{e^{-\epsilon \psi(1)}}{\Gamma(1-\epsilon)} \left( \frac{\mu^2 e^{-i\lambda_{12}\pi}}{2p_1 \cdot p_2} \right)^{\epsilon},$$

where the subscript $ij$ denotes the flavour of the two partons: $ij = gg$ or $ij = q\bar{q}, qq, \bar{q}\bar{q}$ for the various crossed channels with two fermions. Inserting Eq. (22) into Eq. (19), one can thus rewrite Eq. (18) in an explicitly factorized form.

5.2 Amplitudes with a $q\bar{q}$ pair: complete structure of singular terms in CDR

The only QCD amplitude that has been computed so far at two-loop order is the electromagnetic form factor of the quark [10, 11]. The result of Ref. [11] and Eq. (22) can be used to check our Eqs. (18) and (19) in this particular case. As a by-product of this check,
we can moreover derive the explicit expression of the $O(1/\epsilon)$ operator $H_{q\bar{q}}^{(1)}$ for the QCD amplitudes with a $q\bar{q}$ pair or, in general, for those related to them by crossing symmetry. We obtain

$$H_{q\bar{q},\text{CDR}}^{(2)}(\epsilon, \mu^2; p_1, p_2) = \frac{1}{4\epsilon} \frac{e^{-\epsilon\psi(1)}}{\Gamma(1 - \epsilon)} \left( \frac{\mu^2 e^{-i\lambda_{12} \pi}}{2p_1 \cdot p_2} \right)^{2\epsilon} \cdot \left[ \frac{1}{4} \gamma(1) + 3C_F K + 5\zeta_2 \pi \beta_0 C_F - \frac{56}{9} \pi \beta_0 C_F - \left( \frac{16}{9} - 7\zeta_3 \right) C_F C_A \right],$$

where

$$\gamma(1) = (-3 + 24\zeta_2 - 48\zeta_3) C_F^2 + \left( -\frac{17}{3} - \frac{88}{3} \zeta_2 + 24\zeta_3 \right) C_F C_A + \left( \frac{4}{3} + \frac{32}{3} \zeta_2 \right) C_F T_R N_f.$$  

Note that the $C_F^2$-part of $H_{q\bar{q}}^{(2)}$ is entirely included in the coefficient $\gamma(1)$. This is exactly the same coefficient as controls the virtual contribution (i.e. the term proportional to $\delta(1 - z)$) to the NLO Altarelli–Parisi splitting function in the flavour non-singlet sector [31].

We recall that the operator in Eq. (23) is not only related to the electromagnetic form factor of the quark. The same operator applies to the two-loop factorization formula of any QCD amplitude with two quarks such as, for instance, the amplitude $q\bar{q} \rightarrow \gamma\gamma$, which is relevant for the hadroproduction of diphotons. Note also that expression (23) is valid only in CDR because it is obtained by a calculation [11] that uses this RS.

6 Two-loop amplitudes with three partons

6.1 Colour structure

In the case of QCD amplitudes with three partons, the one-loop insertion operator $I_{(1)}^{(1)}$ is again factorizable in colour space. Using colour conservation (i.e. Eq. (14)), we have $2T_1 \cdot T_2 = T_3^2 - T_1^2 - T_2^2$ and likewise for the other permutations. Then, in the case of amplitudes with three gluons, we obtain:

$$I_{ggg}^{(1)}(\epsilon, \mu^2; p_1, p_2, p_3) = -\frac{1}{2} \Gamma_g^\text{sing}(\epsilon) \frac{e^{-\epsilon\psi(1)}}{\Gamma(1 - \epsilon)} \left( \frac{\mu^2 e^{-i\lambda_{12} \pi}}{2p_1 \cdot p_2} \right)^{\epsilon} + \left( \frac{\mu^2 e^{-i\lambda_{23} \pi}}{2p_2 \cdot p_3} \right)^{\epsilon} + \left( \frac{\mu^2 e^{-i\lambda_{31} \pi}}{2p_3 \cdot p_1} \right)^{\epsilon},$$

and, for amplitudes with two fermions of momenta $p_1, p_2$ and a gluon of momentum $p_3$, we get:

$$I_{qgq}^{(1)}(\epsilon, \mu^2; p_1, p_2, p_3) = -\frac{1}{2} \frac{e^{-\epsilon\psi(1)}}{\Gamma(1 - \epsilon)} \left\{ \Gamma_g^\text{sing}(\epsilon) \frac{1}{C_F} (2C_F - C_A) \left( \frac{\mu^2 e^{-i\lambda_{12} \pi}}{2p_1 \cdot p_2} \right)^{\epsilon} \right\} + \left( \frac{\mu^2 e^{-i\lambda_{23} \pi}}{2p_2 \cdot p_3} \right)^{\epsilon} + \left( \frac{\mu^2 e^{-i\lambda_{31} \pi}}{2p_3 \cdot p_1} \right)^{\epsilon} \left\{ \right\}.$$  

The fact that the insertion operator $I_{(1)}^{(1)}$ is proportional to the identity matrix in colour space does not imply, however, that formula (18) is exactly factorizable in the three-parton.
In the case. In fact, colour correlations can still be present in $I^{(2)}$ through the insertion operator $H^{(2)}$. When acting onto the tree-level matrix element $M^{(0)}_m$, the operator $H^{(2)}$ can lead to colour transitions between its possible colour states.

In this respect, $qar{q}g$ and $ggg$ amplitudes behave in a different way. In the $qar{q}g$ case, there is only one possible colour state, namely $M^{qar{q}g} \propto \ell_{ab}$, and thus $H^{(2)}$ exactly factorizes. In the $ggg$ case, the gluons can either be in the symmetric ($M^{ggg} \propto f_{abc}$) colour configuration, and the colour structure of $H^{(2)}$ has to be properly taken into account.

6.2 $q\bar{q}g$ amplitudes and $e^+e^- \rightarrow 3$ jets at NNLO

Perturbative QCD predictions at NNLO for three-jet observables in $e^+e^-$ annihilation are strongly demanded by the high experimental accuracy of the LEP and SLC data [3]. These predictions require the calculation of the corresponding scattering amplitude at two-loop order. Since the knowledge of its singularity structure can help to set up and to check the calculation, we apply in this section the factorization formula (18) to this particular case.

We are interested in the process $e^+e^- \rightarrow q(p_1) + \bar{q}(p_2) + g(p_3)$, where $Q = p_1 + p_2 + p_3$ denotes the total four-momentum and $y_{ij} = 2p_i \cdot p_j/Q^2$ are the relevant dimensionless invariants. To compute the corresponding cross section at NNLO, one has to evaluate the interference $|M_{qar{q}g}|^2$ between the tree-level and two-loop matrix elements:

$$|M_{qar{q}g}|^2 = (M_{qar{q}g}^{(0)} M_{qar{q}g}^{(2)}) + \text{complex conjugate} .$$  

(27)

The analogous interference $|M_{qar{q}g}|^2$ at the one-loop level,

$$|M_{qar{q}g}|^2 = (M_{qar{q}g}^{(0)} M_{qar{q}g}^{(1)}) + \text{complex conjugate} ,$$  

(28)

was first computed in Refs. [32, 33].

Using the factorization formulae (12) and (18), we can write

$$|M_{qar{q}g}|^2 = 2 |M_{qar{q}g}^{(0)}|^2 \left[ \Re I_{qar{q}g}^{(2)} - (\Im I_{qar{q}g}^{(1)})^2 \right] + |M_{qar{q}g}|^2_{(1\text{-loop})} \Re I_{qar{q}g}^{(1)} + \mathcal{O}(1/\epsilon) .$$  

(29)

Then, we can use the explicit expression for $|M_{qar{q}g}|^2$ and Eqs. (23) and (19) and obtain the final result:

$$|M_{qar{q}g}|^2 = |M_{qar{q}g}|^2_{\text{NRS}} \left\{ \frac{1}{4\epsilon} f_2^2(\epsilon) + \frac{1}{2\epsilon^2} \left[ f_2(\epsilon) f_1(\epsilon) + 6\pi\beta_0 f_2(\epsilon) - \pi\beta_0 f_2(2\epsilon) \right] \\
+ \frac{1}{4\epsilon^2} \left[ f_1^2(\epsilon) - \pi^2 f_2^2(\epsilon) + 8\pi\beta_0 f_1(\epsilon) + 12\pi^2\beta_0^2 + 2L_{\text{RS}} f_2(\epsilon) - K f_2(2\epsilon) \right] \right\} \\
- \frac{1}{2\epsilon^2} f_2(\epsilon) F(y_{12}, y_{23}, y_{13}) + \mathcal{O}(1/\epsilon) ,$$  

(30)

where the function $F(y_{12}, y_{23}, y_{13})$ is given in Eq. (2.21) of Ref. [32] and the functions $f_2, f_1$ and the coefficient $L_{\text{RS}}$ are defined as follows

$$f_2(\epsilon) = \frac{e^{-\epsilon}}{\Gamma(1 - \epsilon)} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \left[ (2C_F - C_A)y_{12} + C_A(y_{13} + y_{23}) \right] ,$$  

(31)
\[ f_1(\epsilon) = \frac{e^{-\epsilon \psi(1)}}{\Gamma(1 - \epsilon)} \left( \frac{\mu^2}{Q^2} \right)^\epsilon 3C_F, \] 

(32)

\[ L_{\text{RS}} = -\frac{\pi^2}{2} (2C_F + C_A) + 8C_F - 2\tilde{\gamma}_{q}^{\text{RS}} - \tilde{\gamma}_{g}^{\text{RS}}. \] 

(33)

The coefficients \( \tilde{\gamma}_{i}^{\text{RS}} \) in Eq. (33) parametrize the RS-dependence of the amplitude (see Eq. (17)). The complete evaluation of the \( \mathcal{O}(1/\epsilon) \)-terms on the right-hand side of Eq. (30) requires the identification of the still unknown operator \( H^{(2)}_{qgq} \).

7 Summary

In this paper we have presented a first discussion of the singular behaviour of on-shell QCD amplitudes at two-loop order. The complete structure of the infrared singularities is described by the colour-space factorization formula given in Sect. 4. The factorization formula is universal, i.e. valid for any amplitude, and the singular factors only depend on the flavour and momentum of the coloured external legs. At present we can explicitly give only the coefficients of the \( 1/\epsilon^4, 1/\epsilon^3 \) and \( 1/\epsilon^2 \) poles of the dimensionally-regularized singular factors. These coefficients are nonetheless known in a form that is manifestly independent of the RS of the infrared singularities. The remaining single-pole contributions, namely the operator \( H^{(2)} \) in Eq. (19), still have to be explicitly evaluated. Owing to their universality, they can be extracted from the calculation of few basic two-loop amplitudes, as discussed in Sect. 5.2 for the case of amplitudes with a \( q\bar{q} \) pair.

Our factorization formula can be useful both to check explicit evaluations of two-loop amplitudes and to organize their calculations in terms of divergent, but analytically computable, parts and finite remainders that can be integrated numerically. In the more general context of NNLO calculations of jet observables, our two-loop results can be used to set up the integration of tree-level and one-loop amplitudes in such a way as to construct process-independent techniques for infrared cancellations.

Acknowledgements. We would like to thank David Kosower, Lorenzo Magnea and Willy van Neerven for discussions.

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