Pareto optimization of resonances and minimum-time control

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Abstract

The aim of the paper is to reduce one spectral optimization problem, which involves the minimization of the decay rate $|\text{Im}\kappa|$ of a resonance $\kappa$, to an optimal control problem on a 2-D manifold. This reduction allows us to apply methods of extremal synthesis to the structural optimization of layered optical cavities. We start from a dual problem of minimization of the resonator length and give several reformulations of this problem that involve Pareto optimization of the modulus $|\kappa|$ of a resonance, a minimum-time control problem on the Riemann sphere, and associated Hamilton-Jacobi-Bellman equations. Various types of controllability properties are studied in connection with the existence of optimizers and with the relationship between the Pareto optimal frontiers of minimal decay and minimal modulus. We give explicit examples of optimal resonances and describe qualitatively properties of the Pareto frontiers near them. A special representation of bang-bang controlled trajectories is combined with the analysis of extremals to obtain various bounds on optimal widths of layers.

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1 Introduction

1.1 Resonance optimization and motivations for its study

The mathematical study of the problem of optimization of an individual resonance was initiated in [15] with the aims to obtain an optimal bound on the resonance width and to estimate resonances of random Schrödinger Hamiltonians. For the time-independent 3-dimensional (3-D) Schrödinger equation, the resonance width, roughly speaking, can be measured via the negative of the imaginary part of a resonance in the second ‘nonphysical’ sheet of the two-sheeted Riemann surface for $\sqrt{\cdot}$, the first sheet of which is the ‘energy plane’ (see, e.g., [29]).

The problem of minimization of resonance width falls in the class of nonselfadjoint spectral optimization problems, which include also other types of optimization of transmission
properties [24] and of eigenvalues of nonselfadjoint operators or matrices [10, 8]. Such problems are much less studied in comparison with selfadjoint spectral optimization, which go back to Lagrange’s problem on the shape of the strongest column and to the Faber-Krahn solution of Lord Rayleigh’s problem on the lowest tone of a drum. We refer to [12, 13, 22, 32] for reviews and more recent studies of variational problems for eigenvalues of selfadjoint operators and would like to note that some of these studies (e.g., [12, 13, 22]) are directly or indirectly connected with resonance optimization, in particular, because square roots \( \kappa \in i\mathbb{R}^+ := \{ ic : c \in \mathbb{R}^+ \} \) of nonpositive eigenvalues \( \kappa^2 \) are often considered to be resonances for associated selfadjoint operators. Nonselfadjointness brings new difficulties into eigenvalue optimization. The two of these difficulties noticed in the pioneering paper [15] are connected with the existence of optimizers and with appearance of multiple eigenvalues (see the discussions of these points in [17, p. 425] and the introductions to [18, 19]).

During the last two decades variation problems for transmission and resonance effects attracted considerable attention in connection with active studies of photonic crystals [16] and high quality-factor (high-Q) optical cavities [2, 21, 26, 9]. The problem of design of a high-Q optical cavity was partially motivated (see e.g. [33]) by rapid theoretical and experimental advances in the fields of cavity quantum electrodynamics.

1.2 Review of decay rate and Q-factor optimization

For the idealized model involving a layered optical cavity and normally passing electromagnetic (EM) waves, the Maxwell system can be reduced to the wave equation of a nonhomogeneous string

\[
\varepsilon(s)\partial_t^2 v(s, t) = \partial_s^2 v(s, t), \quad s \in \mathbb{R},
\]

where \( \varepsilon(\cdot) \) is the spatially varying dielectric permittivity of layers (assuming that the speed of light in vacuum is normalized to be equal to 1). It is a step function that can take several positive values \( \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_m \), corresponding to the materials available for fabrication. The function \( \varepsilon(\cdot) \) varies in a finite interval \( s \in [s^-, s^+] \), which represents the nonhomogeneous structure of the resonator. Outside of this interval \( \varepsilon(\cdot) \) equals to the constant permittivity \( \varepsilon_\infty = n_\infty^2 \) (where \( n_\infty > 0 \)) of the homogeneous outer medium. Resonances associated with (1.1) can be defined in several equivalent ways. Following [3], one can define the resonances as the poles of the meromorphic extension of the integral kernel of the resolvent \( \left( -\frac{1}{\varepsilon(s)} \partial_s^2 - \kappa^2 \right)^{-1} \), where the extension is done from the upper complex half-plane \( \mathbb{C}^+ := \{ \kappa \in \mathbb{C} : \text{Im} \kappa > 0 \} \) to the whole plane \( \mathbb{C} \), and then to associate them with zeros of a specially constructed analytic function [10, 18, 20]. Another way is to introduce resonances as poles of the cut-off resolvent (see [34] and references therein). These definitions are equivalent to time-harmonic eigenvalue problem \( \partial_s^2 y(s) = -\kappa^2 \varepsilon(s) y(s) \) equipped with outgoing (radiation) boundary conditions, which in the 1-D case can be written as local boundary conditions at \( s^\pm \) depending on the spectral parameter \( \kappa \) (see Section 2.1 for details).

We call nonzero resonances \( \kappa \) quasi-(normal)-eigenvalues. The set of quasi-eigenvalues lies in the lower complex half-plane \( \mathbb{C}_- := \{ \text{Im} \kappa < 0 \} \) and will be denoted by \( \Sigma(\varepsilon) \) because we assume always that the constant values of \( \varepsilon(\cdot) \) in the outer semi-infinite intervals \( (-\infty, s^-) \) and \( (s^+, +\infty) \) are fixed, and so \( \Sigma(\varepsilon) \) depends only on the function \( \varepsilon(\cdot) \) in \( [s^-, s^+] \), to which we refer simply as the resonator. The real part \( \text{Re} \kappa \) and the negative \( (-\text{Im} \kappa) \) of the imaginary part of \( \kappa \) in the context of the wave equation (1.1) correspond to the (real angular) frequency and
the (exponential) decay rate of eigenoscillations $e^{-\lambda t}y(s)$, respectively. The value $(-2\Im\kappa)$ is called the *bandwidth* of a resonance $\kappa$.

Simulations for perspective designs of high-Q optical cavities [2, 26, 9] have lead to the mathematical question [17, 14] of minimization of the decay rate $Dr(\kappa; \varepsilon) := -\Im\kappa$ (or maximization of the $Q$-factor $Q = \frac{\Re\kappa}{-2\Im\kappa}$) of a quasi-eigenvalue $\kappa \in \Sigma(\varepsilon)$ by structural changes of $\varepsilon(\cdot)$. Mathematically, it is convenient to consider this optimization over the relaxed family $\mathbb{F}_{s^-,s^+}$ of feasible resonators that consists of all $L^\infty(s^-,s^+)$-functions $\varepsilon(\cdot)$ (rigorously, of all $L^\infty(s^-,s^+)$-equivalence classes) satisfying the constraints

$$0 < n_1^2 \leq \varepsilon(s) \leq n_2^2 < +\infty \text{ for almost all (a.a.) } s \in (s^-,s^+),$$

where $n_j > 0$, $j = 1, 2$, and $\varepsilon_1 = n_1^2$ ($\varepsilon_2 = n_2^2$) is the minimal (resp., maximal) of the admissible permittivities (see Sections 2.3 and 5). From the point of view of optimization of a single cavity the following relation between the parameters $n_1$, $n_2$, and $n_\infty$ is reasonable

$$n_1 \leq n_\infty \leq n_2 \quad \text{(see [20] for the explanation and examples).} \quad (1.3)$$

Over the relaxed feasible family $\mathbb{F}_{s^-,s^+}$ the problem is well-posed in the Pareto sense of the paper [18] (cf. also various definitions of optimizers in [15, 19]). This means that an additional constraint on the frequency $\Re\kappa = \alpha$ of a quasi-eigenvalue $\kappa$ is imposed, and, provided that this frequency $\alpha$ is achievable under condition (1.2), the function $\varepsilon(\cdot) \in \mathbb{F}_{s^-,s^+}$ that generates $\kappa$ on the line $\alpha + i\mathbb{R}$ with the minimal possible (over $\mathbb{F}_{s^-,s^+}$) rate of decay $Dr(\kappa; \varepsilon)$ is called the **resonator of minimal decay for the frequency $\alpha$** (for details, see [18] and Section 5) note that the above definition does not involve a special partial ordering and so differs from the standard definition of Pareto optimizers for vector-valued cost functions [7].

The steepest descent numerical experiments of [17, 14, 28] suggest that global minimizers of $Dr(\cdot; \cdot)$ (without constraints imposed on the frequency) do not exist in the usual sense since, in the iterative process of improvement of the resonator in the direction of the gradient of $\Im\kappa$, the quasi-eigenvalue $\kappa$ itself slides to $\infty$ together with its real part $\Re\kappa$. Therefore it was conjectured in [28] (and in a more explicit form in [20]) that $\arg\min_{\varepsilon \in \mathbb{F}_{s^-,s^+}} Dr(\kappa; \varepsilon) = \emptyset \quad \kappa \in \Sigma(\varepsilon)$

(here and below we use the standard arg min notation [7] for the set of minimizers).

It was noticed in the numerical experiments of [17, 14] and rigorously proved in [18] that in these settings the relaxed problem is equivalent to the problem with a finite number of admissible permittivities and, moreover, to the problem when only two extreme values $n_1^2$ and $n_2^2$ are admissible for $\varepsilon(s)$. It was shown in [18] that resonator of minimal decay switches between $n_1^2$ and $n_2^2$ according to a nonlinear eigenvalue problem with a special bang-bang term (see equation (5.2) in Section 5), which can be seen as an analogue of the Euler–Lagrange equation (or, more precisely, of the Karush-Kuhn-Tucker condition because it handles the influence of the constraints (1.2) on the first order necessary condition of optimality). In [20] this result was extended on all resonators $\varepsilon(\cdot)$ generating $\kappa$ on the boundary $\text{bd} \Sigma[\mathbb{F}_{s^-,s^+}]$ of the set of all achievable quasi-eigenvalues $\Sigma[\mathbb{F}_{s^-,s^+}] := \bigcup_{\varepsilon \in \mathbb{F}_{s^-,s^+}} \Sigma(\varepsilon)$.

The steepest descent numerical experiments of [17, 14, 28] and the shooting method of [20], which is based on the analogue of Euler-Lagrange equation, suggest that Pareto optimizers $\varepsilon(\cdot)$ are close to structures that consist of periodic repetitions of two layers with extreme admissible permittivities $n_1^2$ and $n_2^2$ with a possible introduction of defects. This is in good agreement with Photonics studies of high-Q cavities [2, 26, 9], which are based on
sophisticated designs of defects in periodic Bragg reflectors, and with Physics intuition \cite{16}, which says that oscillations of EM field are expected to accumulate on the defects if their frequency $\text{Re} \kappa$ is in a stopband of the Bragg reflector.

However presently available analytic and numerical methods do not give clear answers about optimal designs of the defects and the lengths of alternating layers of the background periodic structure. On one side, the size modulated 1-D stack design, which was suggested in \cite{26} on the base of simulations with several modulation parameters and further studied in \cite{9}, introduces defects to 27 silicon layers in the alternating periodic structure. The widths of these defects are given by a quadratic polynomial function of the layer’s number counting from the center of the cavity. The width of each of these defects is small in comparison with the period of the original structure without defects. On the other side, resonances of periodic structures with one defect in the center were considered in \cite{2, 14, 23, 28}. This defect is not necessarily small in comparison with the period. The numerical experiment with the shooting method \cite{20} suggests that Pareto optimal resonators may involve combination of the both types of defects mentioned above.

Other important questions are concerned with the uniqueness and symmetry of optimizers (these questions are obviously connected, see the discussion in \cite{4}). Most of research for 1-D photonic crystals have been done under the assumption \cite{17, 26, 18, 28} that

$$\varepsilon(\cdot) \text{ is symmetric with respect to the resonator center } s^{\text{centr}} = (s^- + s^+)/2. \quad (1.4)$$

For such symmetric $\varepsilon(\cdot)$, the resonant mode $y(\cdot)$ is either an even, or odd function with respect to (w.r.t.) $s^{\text{centr}}$ in the sense that $y(\cdot - s^{\text{centr}})$ is even, or odd, and therefore satisfies

either the condition $y'(s^{\text{centr}}) = 0$, or the condition $y(s^{\text{centr}}) = 0, \quad (1.5)$

which can be treated as boundary conditions and can be used to simplify the problem. We will say that the corresponding $\kappa$ is an even-mode quasi-eigenvalue, or an odd-mode quasi-eigenvalue, respectively.

The above open questions show that new theoretical tools and more accurate numerical approaches are needed to understand the structure of optimal resonators.

To address these goals several other approaches were proposed. One of directions \cite{13, 23} suggests to study resonant properties with the help of certain associated selfadjoint spectral problems avoiding in this way the difficulties of nonselfadjoint spectral optimization. One more direction employs ‘solvable’ models of Schrödinger operators with point interactions \cite{27, 4}.

1.3 Overview of aims, methods, and results of the paper

The aim of the present paper is to propose a different approach. We connect the problem of optimization of resonances with optimal control theory, more precisely, with one special minimum-time problem on a smooth 2-D manifold.

On one hand, this brings a large number of tools of optimal control. On the other hand, the two problems are not completely equivalent, and so, we have to systematically look on these tools from new points of view and to adjust them to the needs of resonance optimization. The application of this approach leads to a variety of conclusions about structures of optimal resonators and, in particular, explains why it is difficult to expect that one particular type of
design covers all reasonable cases (for example, one can compare structures of abnormal and normal controls in Sections 5, 7).

However, the main goal of this paper is rather to trace the main ideas and to shape the main instruments, than to give all consequences (which would make the size of the paper unreasonable). To demonstrate the effectiveness of the approach and the use of various its tools, we choose three particular aims: (i) the comparison of optimal resonators with quarter-wave stacks, i.e., with structures consisting of periodic alternating repetitions of layers with permittivities $\epsilon_1$ and $\epsilon_2$ and widths equal to the quarter of wavelength in each of these media (see [16]), (ii) analytically calculated explicit examples of optimal resonators and resonances, and (iii) qualitative descriptions of Pareto optimal frontiers.

In more details, as a starting point, we introduce in Section 2.1 a dual problem, which consists of minimization of the length $s^+ - s^-$ of a cavity under the assumption that it generates a given resonance $\kappa$. On one hand, this problem is equivalent to a modified Pareto optimization problem, whose scalarized version is to minimize the modulus $|\kappa|$ of a quasi-eigenvalue for a fixed complex argument Arg $\kappa$ (see Section 2.3). On the other hand, we reformulate the problem of length minimization as a minimum-time control problem $x' = i\kappa(-x^2 + \varepsilon)$, where the state $x$ is connected with resonant mode $y$ by $x = y/(i\kappa y)$ and evolves in the Riemann sphere $\hat{\mathbb{C}} = \{\infty\} \cup \mathbb{C}$, which is the state space. This leads to the applicability of all the variety of tools of optimal control theory [1, 30] including Hamilton-Jacobi-Bellman (HJB) equations [5], their theory on manifolds [11], and the extremal synthesis [6]. In particular, we obtain a system of HJB equations on $\hat{\mathbb{C}}$ for the corresponding direct and backward value functions (Section 2.4). The unique solvability of this system in the sense of proximal analysis follows from [11].

The question of the existence of quasi-eigenvalues of minimum modulus $|\kappa|$ becomes equivalent to the existence of minimum time controls and raises the question of controllability. While our study is not restricted to the case (1.3), we pay a special attention to it and to the most interesting case of position of the resonance $\kappa \in \mathbb{C} - i\mathbb{R}$, and show that (1.3) leads to many ‘good’ properties of the control system. In particular, it leads to the global controllability (Theorem 3.3). Note that resonance optimization in the case $\kappa \in i\mathbb{R}$ is simpler because it has some features of selfadjoint spectral optimization [18, 21].

The small time local controllability (STLC) for our control system can be easily characterized with the use of the Kalman test and the result of [31] on singular equilibrium points. However, the property that is needed to return from the dual problem to the original problem of minimization of the decay rate is more delicate. It is a locally uniform in $\kappa$ version of STLC (see Section 3.3 and Theorem 8.1).

To obtain more information about the structure of minimum-time controls, and so, more information about the resonators of minimum modulus $|\kappa|$, we generalize the results of [15, 18, 20] about rotation of resonant eigenmodes $y$ (see Lemma 3.2) and combine them with the Pontryagin Maximum Principle (PMP). This gives in Theorem 4.1 the absence of singular arcs and another derivation of the bang-bang eigenproblem of [18, 20] supplemented by the additional condition $\lambda_0 \geq 0$ of PMP, where $\lambda_0$ is the constant component of the multiplier (for the terminology and the main principles of geometric optimal control, we refer to [11, 15, 6, 30]). In Sections 5 and 6 the inequality $\lambda_0 \geq 0$ plays an important role for the structural analysis of extremals. Note that Theorem 4.1 does not imply and does not follow from the corresponding results of [18, 20] since quasi-eigenvalues of minimal decay are not necessarily of minimum modulus and vice versa (to see this one can compare Theorem 8.4 with the cloud structure of a part of $\Sigma[F_{s^-},s^+]$ in [20]).
To obtain estimates on the widths of layers (Theorems 5.5 [5,8 and Corollaries 7.1 7.3], we introduce the *ilog-phase* \( \vartheta \), which is function in \( s \) that appears in special representations of \( x \) and \( y \) solutions convenient from the point of view of iterative analytic calculation of the positions of switch points (see Sections 3.2 and 5.2). To demonstrate effectiveness of the tools of extremal synthesis, we give an example of a resonator of minimal length in Theorem 7.4 for a special value \( \kappa_0 \in \mathbb{C}_4 := \{ z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z < 0 \} \) and qualitative descriptions of the Pareto frontiers of minimal decay and minimal modulus near \( \kappa_0 \).

In Section 8 we return to the problem of minimization of decay rate and show that it can be completely reduced to the dual problem of minimization of the resonator length in the case \( n_1 < n_\infty < n_2 \) (Theorem 8.1), and partially reduced (Theorems 8.3 and 8.4) in the cases \( n_\infty = n_1 \) and \( n_\infty = n_2 \). An important tool for the proofs of Theorems 8.1-8.4 is the study of the properties of the resonance free region \( \mathbb{C} \setminus \Sigma_{F_{s^-,s^+}} \) (Section 8.2), which requires a combination of the optimal control approach with the multi-parameter perturbation results of [19] 20].

Note that, while the ideas of optimal synthesis [6, 30] are behind many statements of Sections 5-8, we use only a small part of the related terminology. The reason is that our terminology and notation are oriented to the specific of the resonance optimization problems.

**Notation.** We use the convention that \( \inf \emptyset = +\infty \). The sets \( \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \) and \( \mathbb{C}_4 \) are the open quadrants in \( \mathbb{C} \) corresponding to the combinations of signs \((+,+), (-,+), (-,-), \) and \((+,-)\) for \((\operatorname{Re} z, \operatorname{Im} z)\). Other sets used in the paper are: the compactifications \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) perceived as smooth real 1-D and 2-D manifolds, resp., open half-lines \( \mathbb{R}_± = \{ s \in \mathbb{R} : \pm s > 0 \} \) and half-planes \( \mathbb{C}_± = \{ z \in \mathbb{C} : \pm \operatorname{Im} z > 0 \} \), closed line-segments \( [z_1, z_2] := \{ (1-t)z_1 + tz_2 : t \in [0,1] \} \) (which we call \( \mathbb{C} \)-intervals), \( \mathbb{C} \)-intervals \( (z_1, z_2) = [z_1, z_2) \cap \{ z \in \mathbb{C} \} \) with excluded endpoints \( z_{1,2} \in \mathbb{C} \), circles \( T_\delta(\zeta) := \{ z \in \mathbb{C} : |z - \zeta| = \delta \} \) with \( \delta > 0 \), open discs \( D_\delta(\zeta) := \{ z \in \mathbb{C} : |z - \zeta| < \delta \} \) with \( \delta > 0 \) and \( \zeta \in \mathbb{C} \), \( T = T_1(0), D = D_1(0) \), the infinite sector (without the origin \( z = 0 \))

\[
\text{Sec}(\xi_1, \xi_2) := \{ ce^{i\xi} : c > 0 \text{ and } \xi \in (\xi_1, \xi_2) \}, \quad \xi_1 < \xi_2, \quad \xi_{1,2} \in \mathbb{R}. \tag{1.6}
\]

For a normed space \( W \) over \( \mathbb{C} \) with a norm \( \| \cdot \|_W \), \( \mathcal{B}_\delta(w_0; W) := \{ w \in W : \| w - w_0 \|_W < \delta \} \) are nonempty balls with \( w_0 \in W \) and \( \delta > 0 \). For \( E \subset W \) and \( z \in \mathbb{C} \), we write \( zE + w_0 := \{ zb + w_0 : w \in E \} \). The closure (the interior) of a set \( E \) in the norm topology is denoted by \( \overline{E} \) (resp., \( int E \)).

For a function \( f \) defined on a set \( E \), \( f[E] \) is the image of \( E \) and, in the case when \( f \) maps to \((-\infty, +\infty]\), the domain of \( f \) is \( \operatorname{dom} f := \{ s \in E : f(s) < +\infty \} \). A line over a complex number \( z \) or over a \( \widehat{\mathbb{C}} \)-valued function denotes complex conjugation, i.e., \( \overline{\eta} \) means that \( y(s) = \overline{y(s)} \) for every \( s \), where \( f \) is defined; \( \overline{\infty} = \infty \) and \( f^\dagger(s) = (f(s))^\dagger \), while \( f^{(-1)}(\cdot) \) is the inverse function. By \( \partial_s f, \partial_z f, \) etc., we denote (ordinary or partial) derivatives with respect to (w.r.t.) \( s, z, \) etc.; \( \varepsilon(s \pm 0) \) are the one-side limits of a function \( \varepsilon \) at a point \( s \in \mathbb{R} \). For an interval \( I \subset \mathbb{R} \), \( L^p(I) \) and \( W^{k,p}(I) := \{ y \in L^p(I) : \partial_j^\alpha y \in L^p(I), j \leq k \} \) are the complex Lebesgue and Sobolev spaces with standard norms \( \| \cdot \|_p \) and \( \| \cdot \|_{W^{k,p}} \). To denote the corresponding Lebesgue and and Sobolev spaces of real-valued functions, we use subscript \( \mathbb{R} \) \( (L^p_{\mathbb{R}} \), etc.). The space of continuous complex valued functions with the uniform norm is denoted by \( C(s^-,s^+) \). The loc-notation \( y \in W^{k,p}_{\mathbb{C}_{\text{loc}}}(\mathbb{R}) \) means that \( y \in W^{k,p}(s^-,s^+) \) for every finite interval \( (s^-,s^+) \subset \mathbb{R} \) (the same is applied to the space \( C_{\text{loc}}(\mathbb{R}) \)).

By \( z^{1/2}, \text{Arg}_0 z, \text{Ln} z \) we denote the continuous in \( \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) branches of multi-functions \( \sqrt{z}, \) complex argument \( \text{Arg} z, \) and complex natural logarithm \( \text{Ln} z \) fixed by \( 1^{1/2} = 1 \) and
\[ \operatorname{Ln} 1 = i \operatorname{Arg}_0 1 = 0. \] For the multifunction \( \sqrt{\cdot} \) we use also the notation \( [\cdot]^{1/2} \). For \( z \in \mathbb{R}_- \), we put \( \operatorname{Im} \ln z = \operatorname{Arg}_0 z := \pi \) and \( z^{1/2} := i(-z)^{1/2} \). The characteristic function of a set \( E \) is denoted by \( \chi_E(\cdot) \), i.e., \( \chi_E(s) = 1 \) if \( s \in E \) and \( \chi(s) = 0 \) if \( s \notin E \).

The notation \((x_1;x_2)\) stands for row-vectors. So \((x_1;x_2)^\top\) is the column vector \[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

obtained from \((x_1;x_2)\) by the transposition.

## 2 Dual problem and its equivalent reformulations

In this section we consider the problem of minimization of length of a resonator under the assumption that it produces a given quasi-eigenvalue \( \kappa \) and gives several reformulations of this problem. These different points of views will provide analytical tools, which will complement each other through the rest of the paper.

### 2.1 Minimization of length of a cavity for a fixed resonance \( \kappa \)

The family \( \mathcal{F} \) of feasible (permittivity) coefficients consists, by definition, of positive functions \( \varepsilon(\cdot) \in L_\mathbb{R}^\infty(\mathbb{R}) \) such that there exists \( s^\pm \in \mathbb{R} \) satisfying \( s^- \leq s^+ \) and the following conditions:

\[ \varepsilon(s) = n_\infty^2 \quad \text{for a.a.} \ s \in \mathbb{R} \setminus [s^-, s^+], \quad (2.1) \]

\[ n_1^2 \leq \varepsilon(s) \leq n_2^2 \quad \text{for} \ s \in (s^-, s^+), \quad (2.2) \]

where \( n_\infty, n_1, n_2 \) are fixed constants satisfying \( 0 < n_\infty \) and \( 0 < n_1 < n_2 \). In the sequel, we will omit ‘almost all’ (a.a.) and ‘almost everywhere’ (a.e.) where these words are clearly expected from the context.

We consider also the family of even (or symmetric) feasible coefficients

\[ \mathcal{F}^{\text{sym}} = \{ \varepsilon \in \mathcal{F} : \varepsilon(s) = \varepsilon(-s) \} \]

**Definition 2.1.** For any given \( \varepsilon(\cdot) \) that is not equal to the constant function \( n_\infty^2 \), we denote by \([s^-_\varepsilon, s^+_\varepsilon]\) the shortest interval \([s^-, s^+]\) satisfying \((2.1)\), and by \( \ell(\varepsilon) := s^+_\varepsilon - s^-_\varepsilon \) the effective length of the resonator defined by the coefficient \( \varepsilon(\cdot) \). If \( \varepsilon(\cdot) = n_\infty^2 \) (in \( L_\mathbb{R}^\infty(\mathbb{R}) \)-sense), we put \( s^-_\varepsilon = s^+_\varepsilon = 0 \) and \( \ell(\varepsilon) = 0 \).

A resonance of a coefficient \( \varepsilon(\cdot) \) is a number \( \kappa \in \mathbb{C} \) such that the (generalized) eigenproblem

\[ \begin{align*}
y''(s) &= -\kappa^2 \varepsilon(s)y(s) \quad \text{for} \ s \in \mathbb{R}, \quad (2.3) \\
y'(s) &= \pm i \kappa n_\infty \ y(s) \quad \text{for} \ s = s^\pm_\varepsilon \quad (2.4)
\end{align*} \]

has a nontrivial solution \( y \in W^{2,\infty}_{\text{loc}}(\mathbb{R}) \) (nontrivial means that \( y \) is not identically 0 in the \( L_\mathbb{R}^\infty \)-sense). Such a solution \( y \) is called a (resonant) mode associated with \( \kappa \) and \( \varepsilon(\cdot) \).

Note that for a nontrivial solution \( y(\cdot) \) to \((2.3)\), equality \((2.4)\) is satisfied for \( s = s^\pm_\varepsilon \) if and only if it is satisfied for certain \( s \) such that \( \pm s > \pm s^\pm_\varepsilon \). Indeed, in the both cases, \( y(s) = C^\pm_\varepsilon \exp(\pm i \kappa s) \) for all \( \pm s \geq \pm s^\pm_\varepsilon \) with certain constants \( C^\pm_\varepsilon \).

One can see that \( \kappa = 0 \) is a resonance for every \( \varepsilon(\cdot) \in \mathcal{F} \). The zero resonance corresponds to the background level of the EM field and is often excluded form Physics considerations by a change of coordinates (see, e.g., \cite{10}).
Therefore, we concentrate our attention on nonzero resonances and call them quasi-eigenvalues. The set of quasi-eigenvalues of $\varepsilon(\cdot)$ is denoted by $\Sigma(\varepsilon)$ and is a subset of $\mathbb{C}_-$ (for basic properties of resonances and quasi-eigenvalues, see [3, 10, 20, 34] and references therein).

The main objects of this paper are the two following length minimization problems

$$\begin{align*}
\arg\min_{\varepsilon \in \mathbb{F}} \ell(\varepsilon), \\
\arg\min_{\varepsilon \in \mathbb{F}} \ell(\varepsilon),
\end{align*}$$

where the resonance $\kappa \in \mathbb{C}_-$ and the material parameters $n_\infty$, $n_1$, $n_2$ are fixed.

**Remark 2.1.** The symmetric optimization problem (2.6) can be split into two, the odd-mode and even-mode problems. Indeed, if $\varepsilon(\cdot) \in \mathbb{F}_{\text{sym}}$ and $y$ is an associated with $\kappa \in \Sigma(\varepsilon)$ mode, then $y$ is either odd and satisfy $y(0) = 0$, or even and satisfy $y'(0) = 0$. In the first case, we write $\kappa \in \Sigma_{\text{odd}}(\varepsilon)$, in the second $\kappa \in \Sigma_{\text{even}}(\varepsilon)$. Note that $\Sigma_{\text{odd}}(\varepsilon) \cap \Sigma_{\text{even}}(\varepsilon) = \emptyset$ (otherwise, the odd- and even-modes form a fundamental system of solutions and at least one of them does not satisfy (2.4)). Hence, $\varepsilon(\cdot)$ is a minimizer for (2.6) if and only if it is a minimizer for exactly one of the two problems:

$$\begin{align*}
\arg\min_{\varepsilon \in \mathbb{F}_{\text{sym}}} \ell(\varepsilon), \\
\arg\min_{\varepsilon \in \mathbb{F}_{\text{sym}}} \ell(\varepsilon).
\end{align*}$$

Without a priori knowledge if any minimizers $\varepsilon(\cdot)$ for the problems (2.5), (2.7) do exist, one can define the corresponding minimum lengths by

$$\ell_{\text{min}}(\kappa) := \inf_{\varepsilon \in \mathbb{F}} \ell(\varepsilon), \quad \ell_{\text{min}}^{\text{odd(even)}}(\kappa) := \inf_{\varepsilon \in \mathbb{F}_{\text{sym}}} \ell(\varepsilon).$$

### 2.2 Minimum-time control problem and Riccati equations

In this subsection, the variable $s$ will be interpreted as time, $\kappa \neq 0$ is a fixed parameter. Functions $\varepsilon(\cdot) \in L^\infty_{\mathbb{R},\text{loc}}(s^-; +\infty)$ will be called *controls*.

The family $\mathbb{F}_{s^-}$ of feasible controls is defined by

$$\mathbb{F}_{s^-} := \{\varepsilon(\cdot) \in L^\infty_{\mathbb{R}}(s^-; +\infty) : n_1^2 \leq \varepsilon(s) \leq n_2^2 \text{ for } s > s^-\}.$$

To modify the differential equation (2.3) into a control system in a state-space $\mathbb{C}^2$, we denote $Y_0(s) = y(s)$, $Y_1(s) = \frac{y'(s)}{\imath \kappa}$, form a column vector $Y(s) = (Y_0; Y_1)^\top \in \mathbb{C}^2$, and write (2.3) as the control system $Y'(s) = \imath \kappa \begin{pmatrix} 0 & 1 \\ \varepsilon(s) & 0 \end{pmatrix} Y(s)$.

Since this system is linear one can consider the associated dynamics on the complex projective line, which we identify with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. From the point of view of elementary ODEs, this is the standard reduction to the associated Riccati differential equation. Namely, for a nontrivial solution $y(\cdot)$ to (2.3), the dynamics of the function $x(\cdot)$ defined by

$$x(s) = \frac{y'(s)}{\imath \kappa y(s)} \text{ if } y(s) \neq 0, \quad x(s) = \infty \text{ if } y(s) = 0,$$

(2.8)
is described by the control system
\[ x'(s) = f(x(s), \varepsilon(s)), \text{ with } f(x, \varepsilon) := i\varepsilon(-x^2 + \varepsilon) \] (2.9)
and the control function \( \varepsilon(\cdot) \). The solution \( x(\cdot) \) of (2.9) blows-up in the time-points \( s \) such that \( y(s) = 0 \). The simplest way to describe the evolution of \( x \) near \( \infty \) is to see that, when \( x \) evolves in the neighborhood \( \mathcal{C} \setminus \{0\} \) of \( \infty \), the dynamics of
\[ \tilde{x}(s) = -1/x(s) \] satisfies \( \tilde{x}' = \tilde{f}(\tilde{x}, \varepsilon) \), where \( \tilde{f}(\tilde{x}, \varepsilon) := i\varepsilon(-1 + \varepsilon\tilde{x}^2) \). (2.10)

Recall that \( [5] \) (see also \( [11] [30] \)) a state \( x(s^-) \) of the system (2.9)-(2.10) is said to be in the time-\( t \)-controllable set \( \mathcal{C}_{t}(\eta_+;\kappa) \) to a state \( \eta_+ \) with \( t \geq 0 \) if there exists a feasible control \( \varepsilon \in \mathcal{F}_- \) such that \( x(s^- + t) = \eta_+ \). For \( t \in [0, +\infty) \), we put \( \mathcal{C}_{[0,t]}(\eta_+;\kappa) := \bigcup_{0 \leq t \leq t} \mathcal{C}_{t}(\eta_+;\kappa) \), and write \( \eta_+ \in \mathcal{R}_{[0,t]}(\eta_-;\kappa) \) if \( \eta_- \in \mathcal{C}_{[0,t]}(t;\eta_+) \). A state \( \eta_+ (\eta_-) \) is said to be reachable from \( \eta_- \) (resp., controllable to \( \eta_+ \)) if it belongs to the set \( \mathcal{R}_{[0,\infty]}(\eta_-) \) (resp., \( \mathcal{C}_{[0,\infty]}(\eta_+) \)).

A feasible control \( \varepsilon \in \mathcal{F}_- \) is said to be a minimum-time control from \( x(s^-) \) to \( \eta_+ \) if \( x(s^- + t) = \eta_+ \) in the minimum possible time \( t = T_{\kappa}^{\min}(x(s^-), \eta_+) \), which can be defined by
\[ T_{\kappa}^{\min}(x(s^-), \eta_+) := \inf \{ t \geq 0 : x(s^-) \in \mathcal{C}_{t}(\eta_+, \kappa) \}. \] (2.11)

If \( \eta_- \) is not controllable to \( \eta_+ \), we put by definition \( T_{\kappa}^{\min}(\eta_-; \eta_+) := +\infty \).

For \( (\eta_-; \eta_+) \in \mathbb{C}^2 \), we say that \( \kappa \in \mathbb{C} \setminus \{0\} \) is an \( (\eta_-; \eta_+) \)-eigenvalue of \( \varepsilon(\cdot) \) on an interval \( (s^-; s^+) \) with \( s^- < s^+ \) if equation (2.3) has a nontrivial solution \( y \) satisfying the two boundary conditions
\[ \frac{y'(s^\pm)}{iky(s^\pm)} = \eta_\pm \quad \text{(which are understood as } y(s^\pm) = 0 \text{ when } \eta_\pm = \infty). \] (2.12)

We denote the set of \( (\eta_-; \eta_+) \)-eigenvalues by \( \Sigma_{\eta_-, \eta_+}(\varepsilon) \).

One sees that the following statements are equivalent for \( t > 0 \):
(C1) \( \kappa \in \Sigma_{\eta_-, \eta_+}(\varepsilon) \), where \( s^+ = s^- + t \);
(C2) \( \kappa \neq 0 \) and the control \( \varepsilon(\cdot) \) steers the system (2.9)-(2.10) from the initial state \( \eta_- \) to the target state \( \eta_+ \) in time \( t \).

**Proposition 2.1.** For a fixed \( \kappa \neq 0 \), the following minimization problems are equivalent.

(i) Problem (2.7) is equivalent to the problem of minimum-time control of the system (2.9)-(2.10) from \( -n_\infty \) to \( n_\infty \) in the sense that \( t_{\min}(\kappa) = T_{\kappa}^{\min}(-n_\infty, n_\infty) \) and the two families of \( L^\infty(s^-_\varepsilon, s^+_\varepsilon) \) functions produced by the restrictions of minimizers \( \varepsilon(\cdot) \) for these two problems to the interval \( (s^-_\varepsilon, s^+_\varepsilon) \) coincide after a suitable shift of \( s^- \) to \( s^-_\varepsilon \).

(ii) The odd-mode (even-mode) problem in (2.7) is equivalent to the minimum-time problem for (2.9)-(2.10) from \( -n_\infty \) to \( \infty \) (resp., to \( 0 \)) in the sense that \( t_{\min}(\kappa) = 2T_{\kappa}^{\min}(-n_\infty, 0) \) (resp., \( t_{\min}(\kappa) = 2T_{\kappa}^{\min}(-n_\infty, 0) \)) and, for their minimizers \( \varepsilon(\cdot) \), the intervals \( (s^-_\varepsilon, 0) \) and \( (s^-_\varepsilon, s^+_\varepsilon) \) and the corresponding restrictions of \( \varepsilon(\cdot) \) can be identified by the shift of \( s^- \) to \( s^-_\varepsilon \).

The proposition follows from the obvious identities:
\[ \Sigma(\varepsilon) = \Sigma_{-n_\infty, n_\infty}(\varepsilon) \quad \text{if } \varepsilon \in \mathcal{F}; \] (2.13)
\[ \Sigma^{\text{odd}}(\varepsilon) = \Sigma_{-n_\infty, 0}(\varepsilon) = \Sigma_{0, n_\infty}(\varepsilon) \quad \text{and } \Sigma^{\text{even}}(\varepsilon) = \Sigma_{-n_\infty, 0}(\varepsilon) = \Sigma_{0, n_\infty}(\varepsilon) \quad \text{if } \varepsilon \in \mathcal{F}_{\text{sym}}. \] (2.14)
Assuming that there exists a minimum-time control $\varepsilon$ from $x(s^-) = \eta_-$ to $\eta_+$, one sees that the trajectory of the associated optimal solution $x$ of (2.9), (2.10) has no self-intersections in the sense that

$$x(s) \neq x(s)$$

if $s \neq \bar{s}$ and $s, \bar{s} \in (s^-, s^+ + T^\text{min}_\kappa(\eta_-, \eta_+)).$ \hfill (2.15)

### 2.3 Pareto frontier of resonances of minimal $|\kappa|$  

Recall that $\text{Arg}_0(\cdot)$ is a continuous in $\mathbb{C} \setminus \mathbb{R}^-$ branch of multi-valued complex argument $\text{Arg}(\cdot)$ fixed by $\text{Arg}_0 1 = 0$.

In this subsection, we reformulate the problems of length minimization (2.5)-(2.7) and, more generally, of the minimum-time control of (2.9)-(2.10) as the problem of minimization of modulus $|\kappa|$ of an $(\eta_-; \eta_+)$-eigenvalue $\kappa$ for a given complex argument $\gamma = \text{Arg}_0 k$ over the family

$$F_{s^-, s^+} := \{\varepsilon(\cdot) \in L^\infty_\mathbb{R}(s^-, s^+) : n_1^2 \leq \varepsilon(s) \leq n_2^2 \text{ for } s \in (s^-, s^+)\},$$

where the finite interval $(s^-, s^+)$ with $s^- < s^+$ and the tuple $(\eta_-; \eta_+)$ are fixed.

The main tool for this reformulation is the natural scaling of eigenproblem (2.3), (2.12):

$$\text{if } \kappa \in \Sigma_{s^-, s^+}^+(\varepsilon) \text{ and } \tilde{\varepsilon}(s) = \varepsilon(\tau s) \text{ for } \tau \in \mathbb{R}_+, \text{ then } \tau \kappa \in \Sigma_{s^-, s^+}^{\tau^{-1}s^-, \tau^{-1}s^+}(\tilde{\varepsilon}).$$

Let us introduce the set

$$\Sigma_{s^-, s^+}^\circ[F_{s^-}] := \bigcup_{\varepsilon \in F_{s^-, s^+}} \Sigma_{s^-, s^+}^\circ[\varepsilon]$$

of achievable $(\eta_-; \eta_+)$-eigenvalues (over $F_{s^-, s^+}$). Let us define the set of achievable $(\eta_-; \eta_+)$-arguments by

$$\text{Arg}_0 \Sigma_{s^-, s^+}^\circ[F_{s^-}] := \{\text{Arg}_0 \kappa : \kappa \in \Sigma_{s^-, s^+}^\circ[\varepsilon] \text{ for certain } \varepsilon \in F_{s^-, s^+}\},$$

and the minimal modulus $\rho_{\text{min}}(\gamma) = \rho_{\text{min}}(\gamma, \eta_-, \eta_+) \text{ by}$

$$\rho_{\text{min}}(\gamma) := \inf\{\kappa : \kappa \in \Sigma_{s^-, s^+}^\circ[F_{s^-}] \text{ and } \text{Arg}_0 \kappa = \gamma\}.$$ \hfill (2.19)

The function $\rho_{\text{min}}$ takes values in $[0, +\infty]$ (actually in $(0, \infty]$ because of the assumption $\eta_- \neq \eta_+$, see Theorem (2.2) and depends on $\gamma$, $\eta_\pm$, and $s^\pm$. We will omit $s^\pm$ and sometimes $\eta_\pm$ from the list of variables of $\rho_{\text{min}}$ when they are fixed.

If $k_{\gamma}^\text{min} := e^{i\gamma} \rho_{\text{min}}(\gamma)$ belongs to $\Sigma_{s^-, s^+}^\circ[\bar{\varepsilon}]$ for a certain $\bar{\varepsilon}_\gamma^\text{min}(\cdot) \in F_{s^-, s^+}$, i.e., if minimum is achieved in (2.19), then we say that

$$\varepsilon_\gamma^\text{min}(\cdot) \text{ is a resonator of minimal modulus } |\kappa| \text{ for (the complex argument) } \gamma.$$ \hfill (2.20)

The set $\{e^{i\gamma} \rho_{\text{min}}(\gamma) : \gamma \in \text{Arg}_0 \Sigma_{s^-, s^+}^\circ[F_{s^-, s^+}]\}$ forms the Paredo optimal frontier for the problem of minimization of the modulus $|\kappa|$ of an $(\eta_-; \eta_+)$-eigenvalue $\kappa$ over $F_{s^-, s^+}$ (see [10], [4] for the discussion on the notion of Paredo optimizer and Sections 11 and 8 for another Pareto frontier).

The minimum-time control problem for the system (2.9)-(2.10) and the problem of finding of resonators of minimal modulus for given $\gamma$ over $F_{s^-, s^+}$ are equivalent in the sense of the following theorem, which includes also a result on the existence of optimizers.
Theorem 2.2. Let $\eta_\neq \neq \eta_+, \kappa \neq 0$, and $\gamma = \arg \kappa$. Then the following statements are equivalent:

(i) $\eta_\neq \in \mathcal{E}_{[0, +\infty]}(\eta_+, \kappa)$, i.e., (2.9)-(2.10) is controllable from $\eta_\neq$ to $\eta_+$;

(ii) there exists a minimum-time control $\varepsilon(\cdot) \in F_{\eta_\neq}$ for (2.9)-(2.10) that steers $\eta_\neq$ to $\eta_+$ in the minimal time $T_{\kappa}^{\min}(\eta_-, \eta_+)$. 

(iii) $\gamma \in \arg \sum_{\eta_\neq} \mathbb{R}_+ \mathbb{F}_{s_\neq, s_+}$, i.e., the complex argument $\gamma$ is achievable over $\mathbb{F}_{s_-, s_+}$;

(iv) there exist at least one resonator $\varepsilon^{\min}_\gamma(\cdot)$ of minimal modulus for $\gamma$ over $\mathbb{F}_{s_-, s_+}$.

If statements (i)-(iv) hold true, then

$$T_{\kappa}^{\min}(\eta_-, \eta_+) = \frac{(s^+ - s^-) \rho_{\min}(\gamma, \eta_-, \eta_+)}{|\kappa|}. \quad (2.21)$$

If, additionally, $s^\pm$ are chosen so that $s^+ - s^- = T_{\kappa}^{\min}(-\eta_-, \eta_+)$, then the families of minimum-time controls $\varepsilon(\cdot)$ and of resonators of minimal modulus $\varepsilon^{\min}_\gamma(\cdot)$ coincide.

Remark 2.2. In the last statement of the theorem, we take into account only the parts of minimizers $\varepsilon(\cdot)$ on the interval $[s_-, s_+]$, on which (2.9)-(2.10) is controlled from $\eta_-$ to $\eta_+$ in the minimal time.

Proof of Theorem 2.2. The statement (i) $\Leftrightarrow$ (ii) on the existence of optimal control is a standard application of Filippov’s theorem (see, e.g., [1]). Equivalences (i) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iv) formula (2.21), and the last statement of the theorem follow from the scaling (2.17) and equivalence (C1) $\Leftrightarrow$ (C2) of Section 2.2.

2.4 Proximal solution to Hamilton–Jacobi–Bellman equations

Let $\eta_\pm \in \tilde{\mathbb{C}}$ and $\kappa \in \mathbb{C} \setminus \{0\}$ be fixed. Let us define for all states $x \in \tilde{\mathbb{C}}$ the backward value function $V_{\eta_+}^-(x) := T_{\kappa}^{\min}(x, \eta_+)$, which, according to (2.11), takes values in $[0, +\infty]$ and, in the standard terminology [5, 11] is called simply the value function. Recall that the domain of $V_{\eta_+}^-(\cdot)$ is the set

$$\text{dom} V_{\eta_+}^-(\cdot) := \{x \in \tilde{\mathbb{C}} : V_{\eta_+}^-(x) < +\infty\} = \mathcal{E}_{[0, +\infty]}(\eta_+, \kappa).$$

Similarly, we define the forward value function $V_{\eta_+}^+(x) := T_{\kappa}^{\min}(\eta_-, x), x \in \tilde{\mathbb{C}}$.

Bellman’s Dynamic Programming Principle associates with an optimal control problem a special Hamilton-Jacobi-Bellman (HJB) equation such that properly posed boundary value problems for this equation has value functions as its solutions in various generalized senses of non-smooth analysis. The related theory addresses also the uniqueness of these solutions (see [5, 11, 30] and references therein). The HJB equation formally associated with the backward value function for the minimum time problem of Section 2.2 can be written in the settings of [30] as

$$0 = 1 - \max \{-\nabla V(x) \cdot (f_1(x, \epsilon); f_2(x, \epsilon))^\top : \epsilon = \eta_j^2, j = 1, 2\}, \quad x \in \mathbb{C} = \mathbb{R}^2. \quad (2.22)$$

Here and below $V : \tilde{\mathbb{C}} \to (-\infty, +\infty]$ and we use the notation $x = x_1 + ix_2 = (x_1; x_2)^\top \in \mathbb{C}$,

$$f(x, \epsilon) = f_1(x, \epsilon) + if_2(x, \epsilon) = (f_1(x, \epsilon); f_2(x, \epsilon))^\top = (\text{Re} f(x, \epsilon); \text{Im} f(x, \epsilon))^\top \in \mathbb{R}^2,$$
The similar notation is used for the coordinates of $\tilde{x} = -1/x$ and $\tilde{f}(\tilde{x})$.

The difficulties connected with (2.22) are that it has to be equipped with boundary conditions at $\infty$ and on the boundary $\partial \text{dom } V$ of the domain of $V$, and that $V$ is not necessarily continuous in $\text{dom } V$, while $\text{dom } V$ itself can be a ‘thin’ set for some values of $\kappa$. The boundary condition on $\partial \text{dom } V$ can be handled with the use of proximal analysis, but the condition at $x = \infty$ is encoded in the HJB equation for (2.10) in the $\infty$-state space.

That is why we employ the manifold state-space approach of [11] to consider (2.9), (2.10) and the corresponding HJB equation on the Riemann sphere $\hat{\mathbb{C}}$, which is understood as a smooth 2-D real manifold.

For our needs, it is enough to consider only the minimal atlas for $\hat{\mathbb{C}}$ that consists of the cover $\{ \mathbb{C}, \hat{\mathbb{C}} \setminus \{0\} \}$ and the associated charts $\varphi : x \mapsto x = (x_1; x_2)^\top$, $\hat{\varphi} : x \mapsto \tilde{x} = (\tilde{x}_1; \tilde{x}_2)^\top$ onto $\mathbb{C} = \mathbb{R}^2$. To write (2.9), (2.10) as the differential inclusion $x' \in F_N(x)$ on $\hat{\mathbb{C}}$ in the sense of [11] (see also [1]), we define the multifunction $F_N$, which maps each $x \in \hat{\mathbb{C}}$ to a subset $F_N(x)$ of the tangent space $T_x \hat{\mathbb{C}}$, via its representation in the local coordinates

$$
\varphi_* F_N(x) = \{ f_1(x, \epsilon) \partial_{x_1} + f_2(x, \epsilon) \partial_{x_2} : \epsilon \in [n_1^2, n_2^2] \}, \quad x \in \mathbb{C},
$$

$$
\hat{\varphi}_* F_N(\tilde{x}) = \{ \hat{f}_1(\tilde{x}, \epsilon) \partial_{\tilde{x}_1} + \hat{f}_2(\tilde{x}, \epsilon) \partial_{\tilde{x}_2} : \epsilon \in [n_1^2, n_2^2] \}, \quad \tilde{x} = -x^{-1} \in \hat{\mathbb{C}} \setminus \{0\}.
$$

Note that $f$ and $\tilde{f}$ depend also on the spectral parameter $\kappa$ and that this dependence for $F_N$ is shown explicitly as the lower index, which is important for Corollary 2.3.

The $\mathbb{R}$-valued Hamiltonian $H_N(x, \lambda) = \sup_{v \in F_N(x)} \{ v, \lambda \}$ is defined for all $x \in \hat{\mathbb{C}}$ and $\lambda \in T_x^* \hat{\mathbb{C}}$. Let $\eta \in \hat{\mathbb{C}}$. A lower semicontinuous function $V : \hat{\mathbb{C}} \to (-\infty, +\infty]$ is called [11] a proximal solution to the HJB boundary value problem

$$
0 = 1 - H_N(x, -dV(x)), \quad x \in \hat{\mathbb{C}} \setminus \{ \eta \},
$$

$$
0 = V(\eta),
$$

if the following three conditions hold: (i) $0 = 1 - H(x, -\lambda)$ for all $\lambda \in \partial^P V(x)$ and all $x \neq \eta$, (ii) $0 \leq 1 - H(x, -\lambda)$ for all $\lambda \in \partial^P V(\eta)$ and all $x$, and (iii) $0 = V(\eta)$, where $\partial^P V(x) \subset T_x \hat{\mathbb{C}}$ is the set of all proximal subgradients of $V$ at $x$. Here a cotangent vector $\lambda \in T_x^* \hat{\mathbb{C}}$ is called a proximal subgradient of $V$ at $x$ if $x \in \text{dom } V$ and there exists a $C^2$-function $m$ defined in a certain neighborhood $\Omega \subset \hat{\mathbb{C}}$ of $x$ that satisfies $\lambda = dm(x)$ and $V(z) - V(x) \geq m(z) - m(x)$ for all $z \in \Omega$.

**Corollary 2.3.** Let $\kappa \in \mathbb{C} \setminus \{0\}$. Then $T_{\kappa}^{\min}(\cdot, \cdot)$ coincides on whole $\hat{\mathbb{C}}^2$ with the unique solution $W : \hat{\mathbb{C}}^2 \to (-\infty, +\infty]$ to the system

$$
W(\eta_-, \eta_+) = V_{\eta_+}^-(\eta_-) = V_{\eta_-}^+(\eta_+), \quad \eta_- \in \hat{\mathbb{C}}, \quad \eta_+ \in \hat{\mathbb{C}},
$$

$$
0 = W(x, x), \quad x \in \hat{\mathbb{C}},
$$

$$
0 = 1 - H_N(x, -dV_{\eta_+}^-(x)), \quad x \in \hat{\mathbb{C}} \setminus \{ \eta_+ \},
$$

$$
0 = 1 - H_{-\kappa}(x, -dV_{\eta_-}^+(x)), \quad x \in \hat{\mathbb{C}} \setminus \{ \eta_- \},
$$

where the combination of (2.27)-(2.28) with (2.30) (with (2.29)) is understood in the sense that, for each $\eta_- \in \hat{\mathbb{C}}$ (resp., $\eta_+ \in \hat{\mathbb{C}}$), the function $V_{\eta_+}^+(\cdot)$ (resp., $V_{\eta_-}^-(\cdot)$) is a proximal solution to the boundary value problem $0 = V_{\eta_-}^-(\eta_-)$ for the equation (2.30) (resp., to the boundary value problem (2.27)-(2.28)).
Proof. By the definition, the system \((2.27)-(2.30)\) consists of the two families of the HJB boundary value problems, namely, \((2.25)-(2.26)\) for \(V_{\eta^+}(\cdot)\) and \((2.30)\), \(0 = V_{\eta^-}(\eta_-)\) for \(V_{\eta^-}(\cdot)\), parametrized by \(\eta_{\pm} \in \hat{C}\) and satisfying the compatibility condition \(V_{\eta^+}(\eta_+) = V_{\eta^+}(\eta_-)\). The compatibility condition makes it possible to define the notion of the solution \(W(\eta_-^-, \eta_+^+)\), \(\eta_{\pm} \in \hat{C}\), to \((2.27)-(2.30)\) assuming that, for each value of the parameter \(\eta_{\pm}\), the corresponding representative in the family of the HJB boundary value problems has a proximal solution \((V_{\eta^-}(\cdot) \text{ or } V_{\eta^+}(\cdot))\).

For each \(\eta_{\pm}\) such proximal solutions exist and are unique \([11]\). To prove this existence and uniqueness result for \((2.25)-(2.26)\) it is enough to show that the conditions (H1)-(H3) of \([11]\) are satisfied. Note that (H1) and (H2) are standard compactness, convexity, local boundedness, and local Lipschitz continuity assumptions, which are obviously fulfilled for our control system. The assumption (H3) is a geometric version of the linear growth assumption. Its validity follows from the fact that \(\hat{C}\) is compact (indeed, it is enough to choose the function \(s : \hat{C} \to \mathbb{R}\) in (H3) to be identically equal to zero). Finally, note that \(V_{\eta^+}(\cdot)\) is the classical (i.e., backward) value function corresponding to the time-reversal version of our control system (cf. the symmetry \((S0) \iff (S2)\) in Section 3.1), and so the existence and uniqueness result of \([11]\) is applicable to \(V_{\eta^+}(\cdot)\).

It also follows from \([11]\) that the unique proximal solutions for each of the HJB equations considered above are the backward or forward value functions in the sense that \(T_{\kappa}^{\text{min}}(\eta_-, \eta_+) = V_{\eta^+}(\eta_-) = V_{\eta^-}(\eta_+)^{\text{for}} \eta_{\pm} \in \hat{C}\). This completes the proof.

\section{Controllability and existence of minimizers}

The plan of this section is following. In the preparatory Subsection 3.1 we consider symmetries of the system \((2.9)-(2.10)\) and explain why, in the subsequent sections, we restrict our attention to the case \(\kappa \in C_4\). We also give technical Lemma 3.2 needed for Section 4 and use it to show that \((2.9)-(2.10)\) is not completely controllable for any \(\kappa \in C_\kappa\), i.e., for each \(\kappa \in C_\kappa\) there exists \((\eta_-, \eta_+) \in \hat{C}^2\) so that \(\eta_+ \notin \mathcal{R}_{[0, +\infty)}(\eta_-, \kappa)\) (in the degenerate case \(\kappa = 0\), this is obvious since \(f \equiv f \equiv 0\)).

The proof of the global controllability result of Theorem 3.3 in Subsection 3.2 is based on the study of trajectories generated by the constant controls. Subsection 3.3 deals with local controllability and its uniform in \(\kappa\) version.

One of the applications of the results of this section is the following existence statement, which immediately follows from Theorem 2.2 and the controllability result of Theorem 3.3.

**Corollary 3.1** (existence of minimizers). Let \(\kappa \in C_\kappa \setminus i\mathbb{R}\) and \(n_1 \leq n_\infty \leq n_2\). Then each of the four sets

\[
\text{arg min}_{\ell(\varepsilon)} \in \mathbb{F} \quad \text{arg min}_{\ell(\varepsilon)} \in \mathbb{F}_{\text{sym}} \quad \text{arg min}_{\ell(\varepsilon)} \in \mathbb{F}_{\text{odd(even)}} \quad \text{arg min}_{\ell(\varepsilon)} \in \mathbb{F}_{\text{odd(even)}}
\]

is nonempty.

\subsection{Lines of no return, monotonicities, and symmetries}

Let us consider first the symmetries of \((2.9)-(2.10)\), which appear because the right hand side in \((2.9)\) is an even function in \(x\) and \(\mathbb{F}_{s_-s_+}\) is symmetric w.r.t. \(s_{\text{centr}}\).
Considering dynamics of \((-x(-s)), x(-s),\) and \(\bar{x}(s)\), one sees that the statements that \(\varepsilon(s) = \varepsilon_0(s)\) for the following choices of \((\varepsilon; \varepsilon_0) \in \mathbb{C}^2, \kappa, \) and \(\varepsilon(s) = \varepsilon_0(s)\) are equivalent:

(S0) \((\eta_+; \eta_-) = (\eta_1; \eta_2), \kappa = \kappa_0, \) and \(\varepsilon(s) = \varepsilon_0(s)\);

(S1) \((\eta_+; \eta_-) = (-\eta_2; -\eta_1), \kappa = \kappa_0, \) and \(\varepsilon(s) = \varepsilon_0(s)\);

(S2) \((\eta_+; \eta_-) = (\eta_2; \eta_1), \kappa = -\kappa_0, \) and \(\varepsilon(s) = \varepsilon_0(s)\);

(S3) \((\eta_+; \eta_-) = (\eta_1; \eta_2), \kappa = -\kappa_0, \) and \(\varepsilon(s) = \varepsilon_0(s)\).

Equivalence \((S0) \Leftrightarrow (S1)\) implies that
\[
\eta_+ \in \mathfrak{R}_{[0, t]}(\eta_-, \kappa) \text{ if and only if } (-\eta_-) \in \mathfrak{R}_{[0, t]}(-\eta_+, \kappa).
\] (3.2)

Equivalence \((S0) \Leftrightarrow (S3)\) implies that \(\eta_+; \eta_-) \in \mathbb{R}^2\) and \(\varepsilon(s) \in \mathbb{F}_{s \to s^+}\), then
\[
\Sigma_{s, s^+}^s \varepsilon(s) \text{ and } \Sigma_{s, s^+}^s \varepsilon(s) \text{ are symmetric w.r.t. the imaginary axis } i\mathbb{R}.
\] (3.3)

**Lemma 3.2.** Let \(\varepsilon(s) \in \mathbb{F}_{s \to s^+}, \kappa \in \text{Sec}(-\pi/2, 0),\) and \(y\) be a nontrivial solution to \((2.3)\) in \([-s, s^+].\) Then, for each \(\tau \in [0, -2\text{Arg}_0 \kappa],\) the function
\[
G_\tau(s) = \text{Re}(\varepsilon^{i(\tau-\pi/2)} e^{i(\tau-\pi/2)} y(s))
\]
is strictly increasing in \(s\). In particular, there exists at most one point \(p_\tau \in [s^-, s^+]\) such that \(G_\tau(p_\tau) = 0\). If \(G_\tau(p_\tau) = 0,\) then \(p_\tau\) is the only point where the trajectory \(x([s^-, s^+]) := \{x(s) : s \in [s^-, s^+]\} \) \(\text{Re}(\varepsilon^{i(\tau-\pi/2)} e^{i(\tau-\pi/2)} y(s)) \geq 0.\) (3.4)

It is easy to see that \(G_\tau\) cannot vanish on an interval for a nontrivial \(y\). The equality \(G_\tau(s) = |y|^2|\kappa| \text{Re}(\varepsilon^{i(\tau+\text{Arg}_0 \kappa)} x(s))\) valid for \(y(s) \neq 0 \) completes the proof.

**Remark 3.1.** In the settings of Lemma 3.2. \(\mathcal{L}_\tau\) is an *line of no return* in the following sense: if \(x(s_0)\) is in the \(\mathbb{C}\)-closure \(\mathbb{C}_s \) of the half-plane \(\mathbb{C}_s \) of \(\text{Arg}_0 \kappa\) \(\subset \mathcal{L}_\tau \cup i\text{Arg}_0 \kappa \mathbb{C}_s\), then \(x(s) \in \mathbb{C}_s \) for \(s > s_0\). This shows that \((2.9)-(2.10)\) is not controllable from \(\eta_- \in \mathbb{H}_s \) to \(\eta_+ \in \mathbb{H}_s \setminus \langle \eta_- \rangle,\) where \(\mathbb{H}_s := -\mathbb{H}_s^+.

**Remark 3.2.** (i) Let \(\kappa \in \mathbb{I}_-\). Then Lemma 3.2 and Remark 3.1 are valid for \(\tau \in (0, \pi).

(ii) Let \(\kappa \in \mathbb{R}\) and \(\varepsilon(s) \in \mathbb{F}_{s \to s^+}\). Then (3.4) applied to \(\tau = 0\) implies that \(G_0(s)\) is a constant function and that \((2.9)-(2.10)\) is not controllable from \(\eta_- \in \mathbb{I}_- \) to \(\eta_+ \in (-\mathbb{H}_s^+).

**Remark 3.3.** Let \(\kappa \in \mathbb{I}_-\). Let \(\eta_- \in \mathbb{R}\) like in (2.13) or in (2.14). Then the control problem and the resonance optimization problems are essentially simpler since the trajectory of \(x\) lies in \(\mathbb{R}\). On one hand, the control problems can be solved by the application of the simple no-selfintersection principle (2.15). On the other hand, the solution of the problems (3.1) for \(n_\infty \in \mathbb{R}_+\) and \(\kappa \in \mathbb{I}_-\) can be obtained from the results of [21].

**Remark 3.4.** Note that the constancy of \(G_0(\cdot)\) for \(\kappa \in \mathbb{R},\) the existence of no-return lines \(\mathcal{L}_\tau\) for \(\kappa \in \mathbb{C}_4 \cup \mathbb{I}_-,\) and the symmetries (S0)–(S3) imply the well-known fact that \(\Sigma(s) \subset \mathbb{C}_s\) for \(s \in \mathbb{F}.\) This, the symmetries (S0)–(S3), and Remark 3.3 on the case \(\kappa \in \mathbb{I}_-\) show that we can restrict our attention to the case \(\kappa \in \mathbb{C}_4.\)
3.2 Global controllability to stable equilibria

Let us denote by $\mathcal{J} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the Kutta-Zhukovskii transform $\mathcal{J}(z) = (z + z^{-1})/2$ and by $\mathcal{I} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the linear fractional transform of the form $\mathcal{I}(z) := (1 - z)/(1 + z)$. Note that

$$\mathcal{I}(-z) = (\mathcal{I}(z))^{-1}, \quad \mathcal{I}(z^{-1}) = -\mathcal{I}(z),$$

and that the inverse $\mathcal{J}^{-1}(\cdot)$ of $\mathcal{J}$ given by the expression

$$\mathcal{J}^{-1}(\zeta) := \zeta + \sqrt{\zeta^2 - 1}$$

is a 2-valued analytic function with the branching points $\pm 1$. For $\zeta \in \mathbb{C} \setminus [-1, 1]$, two branches $\mathcal{J}^{-1}_\pm(\zeta)$ of $\mathcal{J}^{-1}$ can be defined by (3.6) and the condition $\pm(|\mathcal{J}^{-1}_\pm(\zeta)| - 1) > 0$.

Let us consider the forward and backward evolution of (2.9)-(2.10) under the constant control $\epsilon(\cdot) \equiv \epsilon \in \mathbb{R}_+$ from an initial state $x(s^-) = \eta_\epsilon \in \hat{\mathbb{C}}$. Let us define the continuous $\hat{\mathbb{C}}$-valued function

$$\vartheta_\epsilon(s) = \mathcal{I}(\epsilon^{-1/2}x(s)).$$

The evolution of $x(\cdot)$ and of the associated solution $y$ to (2.3) have the form

$$y(s) = c_+ e^{i\kappa s} + c_- e^{-i\kappa s} \quad \text{and} \quad x(s) = \epsilon^{1/2} \frac{c_+ - c_- e^{-2i\kappa s}}{c_+ + c_- e^{-2i\kappa s}}$$

with $\kappa := \epsilon^{1/2} \kappa$.

and certain $c_\pm \in \mathbb{C}$ satisfying $|c_-| + |c_+| \neq 0$. The identity $\mathcal{I}(\mathcal{I}(z)) \equiv z$ implies

$$\vartheta_\epsilon(s) = \frac{c_-}{c_+} e^{-2i\kappa(s-s^-)} = \vartheta_\epsilon(s^-) e^{-2i\kappa(s-s^-)}, \quad s \in \mathbb{R},$$

which have to be understood as $\vartheta_\epsilon(s) \equiv \infty$ if $c_+ = 0$. The latter case corresponds the unique unstable equilibrium solution $x(\cdot) \equiv -\epsilon^{1/2}$. The equation $x'(s) = i\kappa(-x^2 + \epsilon)$ has also one stable equilibrium solution $x(\cdot) \equiv \epsilon^{1/2}$.

Assume now that $\eta_- \neq \pm \epsilon^{1/2}$. Then

$$y(s) = A \cos \left(\frac{i}{2} \ln \vartheta_\epsilon(s)\right) = A \cos \left(\kappa(s-s^-) + \frac{i}{2} \ln \vartheta_\epsilon(s^-)\right),$$

$$x(s) = i\epsilon^{1/2} \tan \left(\frac{i}{2} \ln \vartheta_\epsilon(s)\right) = i\epsilon^{1/2} \tan \left(\kappa(s-s^-) + \frac{i}{2} \ln \vartheta_\epsilon(s^-)\right),$$

where $\ln \vartheta_\epsilon(\cdot)$ is an arbitrary continuous on $\mathbb{R}$ branch of the multifunction $\ln \vartheta_\epsilon(\cdot)$ and $A \in \mathbb{C} \setminus \{0\}$ is a constant depending on $c_\pm$ and the choice of the branch $\ln \vartheta_\epsilon(\cdot)$.

Because of the right sides of (3.10), (3.11) we will say that $\vartheta_\epsilon(\cdot)$ is the *ilog-phase* for the constant control $\epsilon$. A curve $x(s)$ produced by the constant control $\epsilon$ has the form $C_1 \tan \left(\epsilon^{1/2} \kappa(s-s^-) + C_2\right)$. In the case $\kappa \in \mathbb{C} \setminus i \mathbb{R}$, $x[\mathbb{R}]$ is the image of the logarithmic spiral (3.9) under the linear fractional map $z \mapsto \epsilon^{1/2} \mathcal{I}(z)$. This gives the rigorous sense to the statement that, as $s \to \pm \infty$, the trajectory of $x(s)$ asymptotically approaches a logarithmic spiral with the pole at $\pm 1/\epsilon$.

Using (3.9) it is easy to see that, if $\eta_- \neq \eta_+$ and each of $\vartheta_\epsilon(s^\pm) = \mathcal{I}(\epsilon^{-1/2} \eta_\pm)$ is not in the set $\{0, \infty\}$, then the $(\eta_-, \eta_+)$-spectrum produced by the constant resonator $\epsilon(\cdot) = \epsilon$ is

$$\Sigma_{\eta_-, \eta_+}(\epsilon) = \{k_n(\epsilon)\}_{n \in \mathbb{Z}}, \quad k_n(\epsilon) := -\frac{i}{2(s^+ - s^-)\epsilon^{1/2}} \ln \frac{\vartheta_\epsilon(s^-)}{\vartheta_\epsilon(s^+)} + \frac{\pi n}{(s^+ - s^-)\epsilon^{1/2}}.$$
Theorem 3.3. Let $\kappa \in \mathbb{C}_3 \cup \mathbb{C}_4$ and $\eta_+ \in [n_1, n_2]$. Then (2.9), (2.10) is globally controllable to $\eta_+$ in the sense $\mathcal{E}_{[0, +\infty)}(\eta_+, \kappa) = \hat{\mathbb{C}}$.

Proof. It is enough to consider the case $\kappa \in \mathbb{C}_4$. Geometrically, $\mathcal{E}_{[0, +\infty)}(\eta, \kappa) = \hat{\mathbb{C}}$ can be proved using the fact that $x(\cdot) \equiv \eta_+$ is a stable equilibrium solution to (2.9) with the constant control $\epsilon(\cdot) \equiv \eta_+^2$. The corresponding trajectory $x(s)$ with $x(s^-) = \eta_+ \neq \eta_+$ asymptotically approaches a logarithmic spiral with the pole at $\eta_+$ as $s \to +\infty$. One can concatenate this asymptotic spiral with a backward trajectory through $\eta_+$ under another feasible constant control $\epsilon(\cdot) \equiv \epsilon_0 \neq \eta_+^2$. This argument can be made rigorous with the use of the map from (3.7) and the dynamics of the ilog-phases for $\epsilon = \eta_+^2$ and $\epsilon = \epsilon_0$.

Let us give another proof that can be used in less geometrically transparent situations and is based on the Pareto optimization equivalence of Section 2.3. Let us fix an interval $(s^-, s^+)$ and $\eta_- \neq \eta_+$. We want to show that

$$\operatorname{Arg}_0 \sum_{\eta_-, \eta_+} \left[ F_{s^-, s^+} \right] \supset (-\pi/2, 0)$$

and use Theorem 2.2. Consider for $n \in \mathbb{N}$ the behavior of $k_n(\epsilon)$ from (3.12) as the constant $\epsilon$ runs through $[n_1^2, n_2^2] \setminus \{\eta_+^2\}$. On one hand, such $k_n(\epsilon)$ lie in the strip $\Re z \in \frac{-\pi}{s^+ - s} \left[ \frac{n-1/2}{n_2}, \frac{n+1/2}{n_1} \right]$. On the other, $\Im k_n(\epsilon) \to -\infty$ as $\epsilon \to \eta_+^2$ since $x(s^+) = \eta_+$ and $\eta_+ \neq \eta_+$. We can choose $\epsilon_0$ such that

$$\{ \operatorname{Arg}_0 k_n(\epsilon) : \epsilon \in [n_1^2, n_2^2], \epsilon \neq \eta_+^2 \} \supset (-\pi/2, \operatorname{Arg}_0 k_n(\epsilon_0)]$$

where $\epsilon_0$ is any fixed number in $[n_1^2, n_2^2] \setminus \{\eta_+^2\}$. Since $\operatorname{Arg}_0 k_n(\epsilon_0) \to 0$ as $n \to +\infty$, we see from (3.12) that (3.13) holds true. 

3.3 Uniform in $\kappa$ small-time local controllability

Let us recall the definition of regular equilibria [31], which determines the choice of $\eta$ in this section. If there exists a feasible control $\epsilon \in \mathbb{F}_{s^+}$ such that the constant function $x(s) = \eta_+$, $s > s^+$, is a solution to (2.9), (2.10), then $\eta$ is an equilibrium point for (2.9), (2.10). The set of equilibrium points of (2.9), (2.10) equals $[-n_2, -n_1] \cup [n_1, n_2]$. The points of the set $(-n_2, -n_1) \cup (n_1, n_2)$ are called regular equilibrium points, the points $\pm n_1$ and $\pm n_2$ are called singular equilibrium points.

Recall that the system (2.9), (2.10) is small-time local controllable (STLC) from $\eta_-$ (STLC to $\eta_+$) if $\eta_- \in \bigcap_{t > 0} \mathbb{R} \mathcal{E}_{[0, t]}(\eta_-, \kappa)$ (resp., $\eta_+ \in \bigcap_{t > 0} \mathcal{E}_{[0, t]}(\eta_+, \kappa)$), where $\mathbb{R} \mathcal{E}$ is the interior of a set $E$. Since we need a version of this definition that is uniform over a set $\Omega$ of spectral parameters $\kappa$, we quantify STLC with the use of the function

$$r_{\Omega, \kappa}^{\max}(t; \eta_+, \kappa) := \sup \{ r > 0 : \mathbb{D}_r(\eta_+) \subset \mathcal{E}_{[0, t]}(\eta_+, \kappa) \}$$

where $\eta_+ \in \mathbb{C}$ and $\mathbb{D}_r(z_0) := \{ |z - z_0| < r \}$. As a function of $t$, $r_{\Omega, \kappa}^{\max}(\cdot; \eta_+, \kappa)$ is nondecreasing, takes finite values for small enough $t$, and satisfies $r_{\Omega, \kappa}^{\max}(0; \eta_+, \kappa) = 0$.

Definition 3.1. Let $\Omega$ be a subset of $\mathbb{C}$. We say that (2.9), (2.10) is STL to $\eta \in \mathbb{C}$ uniformly over $\Omega$ if there exists $\delta > 0$ and a strictly increasing continuous function $r_{\Omega, \kappa}^{\max} : [0, \delta) \to [0, +\infty)$ such that $r_{\Omega, \kappa}^{\max}(t) \leq r_{\Omega, \kappa}^{\max}(t, \eta, \kappa)$ for all $\kappa \in \Omega$ and $t \in [0, \delta)$. The uniform over $\Omega$ version of STL-controllability from $\eta$ can be defined similarly.
Let us denote by \( x_{\eta,-}(s, \varepsilon, \kappa) \) the solution to (2.9), (2.10) satisfying \( x(s^-) = \eta_- \). Let \( \kappa \in \mathbb{C}, \eta_- \in \mathbb{C} \), and \( s^+ = s^- + t_0 \). Then for small enough \( \delta_1 > 0 \) and small enough \( t_0 \in (0, T_{\kappa}^{\min}(\eta_-, \infty)) \), the map \( (\varepsilon(\cdot); \kappa) \mapsto x_{\eta,-}(s, \varepsilon, \kappa) \) is analytic from a certain neighborhood of \( \mathbb{F}_{s^-, s^+} \times \mathbb{D}_{\delta_1}(\kappa_0) \) in the Banach space \( L_{C}^\infty(s^-, s^+) \times \mathbb{C} \) to \( W_{C}^1, \infty[s^-, s^+] \) (here \( \max\{\|\varepsilon\|_\infty, |\kappa|\} \) can be taken as the norm in \( L_{C}^\infty(s^-, s^+) \times \mathbb{C} \)). For \( \varepsilon_0 \in \mathbb{F}_{s^-, s^+} \) and \( \varepsilon_1 \in L_{C}^\infty(s^-, s^+) \), the directional derivative \( \frac{\partial x_{\eta,-}(s, \varepsilon_0, \kappa)}{\partial \varepsilon}(\varepsilon_1) = \partial_s x_{\eta,-}(s, \varepsilon_0 + \varepsilon_1, \kappa) \) is equal to

\[
\frac{\partial x_{\eta,-}(s, \varepsilon_0, \kappa)}{\partial \varepsilon}(\varepsilon_1) = i \kappa \int_{s^-}^s \varepsilon_1(\tilde{s}) \exp \left( -2i\kappa \int_{\tilde{s}}^s x(\sigma, \varepsilon_0(\sigma), \kappa)d\sigma \right) d\tilde{s}. \tag{3.15}
\]

According to [31], if (2.9), (2.10) is STLC to (from) \( \eta \in \bar{\mathbb{C}} \), then \( \eta \) is a regular equilibrium point, i.e., \( \eta \in \{-n_2, -n_1\} \cup (n_1, n_2) \). In the case \( \kappa \not\in \mathbb{R} \), the next theorem gives a strengthened uniform version of the inverse implication.

**Theorem 3.4 (uniform STLC).** Let \( \kappa_0 \in \mathbb{C} \setminus \mathbb{R} \). Then (2.9), (2.10) is STLC to (from) \( \eta \in \bar{\mathbb{C}} \) uniformly over \( \mathbb{D}_{\delta}(\kappa_0) \) with a certain \( \delta > 0 \) if and only if \( \eta \in \{-n_2, -n_1\} \cup (n_1, n_2) \).

**Proof.** The part ‘only if’ of the theorem follows from [31]. Let us put \( \eta = \{-n_2, -n_1\} \cup (n_1, n_2) \) and prove the part ‘if’.

For \( \delta_0 > 0 \), we define the nondecreasing function \( r_{\delta_0}^{\max} : [0, +\infty) \to [0, +\infty) \) by

\[
r_{\delta_0}^{\max}(t) := \inf_{|\kappa - \kappa_0| < \delta_0} r_{\kappa_0}(t, \eta, \kappa).
\]

Note that \( \text{dom} r_{\delta_0}^{\max} \supset [0, T_{\min}(\infty, \eta)] \), and that \( T_{\min}(\infty, \eta) = T_{\min}(\kappa_0, \kappa_0) \) is the escape time from \( (-\eta) \) to \( \infty \) and is positive. Assume that for a certain \( \delta_0 > 0 \) and \( \delta \) satisfying \( 0 < \delta < \delta_1 := \min\{1, T_{\min}(\infty, \eta)\} \),

\[
r_{\delta_0}^{\max}(t) > 0 \quad \text{for all} \quad t \in (0, \delta).
\]

Then the function \( \rho_{\Omega, \eta_0}^{\max} \) defined on \( [0, \delta) \) by \( \rho_{\Omega, \eta_0}^{\max}(t) := \int_0^t r_{\delta_0}^{\max}(s)ds \) satisfies Definition 3.1 with \( \Omega = \mathbb{D}_{\delta_0}(\kappa_0) \). Hence, to prove (ii) it is enough to prove (3.16) for certain \( \delta_0, \delta > 0 \).

Put \( \eta_- = -\eta \). Using the symmetry \( (S0) \Leftrightarrow (S1) \), it is easy to see that if, for each \( t \in (0, \delta) \),

\[\mathfrak{R}_{[0, t]}(\eta_-) \supset \mathbb{D}_r(\eta_-) \quad \text{with a certain} \quad r > 0 \quad \text{(depending on} \quad t) \quad \text{for all} \quad \kappa \in \mathbb{D}_{\delta_0}(\kappa_0), \tag{3.17}\]

then (3.16) is fulfilled, and so (2.9), (2.10) is STLC to \( \eta \) and from \( \eta_- \) uniformly over \( \mathbb{D}_{\delta_0}(\kappa_0) \).

Let us prove (3.17) for \( \kappa_0 \not\in \mathbb{R} \). Put \( d_0 = \min\{|\eta|^2 - n_1^2|, |\eta|^2 - n_2^2|\} \), and \( \varepsilon_0(s) = \eta_0^2 \). So \( x_{\eta,-}(s, \varepsilon_0, \kappa_0) \equiv \eta_- \) and (3.15) takes the form

\[
\frac{\partial x_{\eta,-}(s, \varepsilon_0, \kappa_0)}{\partial \varepsilon}(\varepsilon_1) = i \kappa \int_{s^-}^s \varepsilon_1(\tilde{s})e^{2i\kappa_0(\tilde{s}-s)}d\tilde{s}. \tag{3.18}
\]

Let us take \( \delta_0, \delta > 0 \) such that \( 2\delta_0 < |\text{Re} \kappa_0|, |\delta| \text{Re} \kappa_0| < \pi/6 \), and let \( s^+ \in (s^-, s^- + \delta) \).

Then

\[
\frac{\pi}{2} < \inf_{|\kappa - \kappa_0| \leq \delta_0} \text{Arg}(e^{2i\kappa_0s}) < \sup_{|\kappa - \kappa_0| \leq \delta_0} \text{Arg}(e^{2i\kappa_0s}) < \frac{\pi}{2}
\]

and it follows from (3.18) that taking as \( \varepsilon_j(\cdot), j = 1, 2 \), characteristic (indicator) functions of disjoint small enough subintervals of \( [s^-, s^+] \) we can ensure that the Fréchet derivatives \( \frac{\partial x_{\kappa_0}(0)}{\partial \kappa} \) at \( \zeta = (\zeta_1; \zeta_2) = 0 \) of the functions \( h_{\kappa} : (d_0, d_0^2) \to \mathbb{R}^2 \),

\[
h_{\kappa}(\zeta_1, \zeta_2) := x_{\eta,-}(s^+, \varepsilon_0 + \zeta_1 \varepsilon_1 + \zeta_2 \varepsilon_2, \kappa)
\]
satisfy sup_{s \in [\varepsilon - \kappa_0, \varepsilon]} \| (\frac{\partial h_\kappa(x)}{\partial x})^{-1} \|_{\mathbb{R}^2 \to \mathbb{R}^2} < +\infty. Since the images of \((-d_0, d_0)\) under \(h_\kappa\) are subsets of the images of \(F_{s^-, s^+}\) under the mapping \(\varepsilon \mapsto x_\eta(s^+, \varepsilon, \kappa)\), one can see (e.g., from the local analyticity of the map \((\zeta_1; \zeta_2; \kappa) \mapsto h_\kappa(\zeta_1, \zeta_2)\) and the Graves theorem) that \(\Omega_{[0, s^+]}(\eta_-, \kappa) \supset \mathbb{D}_{\delta_2}(\eta_-)\) with certain \(\delta_2 > 0\) for all \(\kappa \in \mathbb{D}_{\delta_0}(\kappa_0)\). This completes the proof of (3.16) and of the theorem.

Remark 3.5. It can be seen from Theorem 3.3 and its proof that for \(\eta \in [-n_2, -n_1] \cup [n_1, n_2]\) and \(\kappa \in \mathbb{C}_4\), (2.9), (2.10) is locally controllable to \(\eta\) in the sense that \(\eta \in \text{int} \mathbb{C}_{[0, +\infty]}(\eta, \kappa)\). Thus, for \(\eta = \pm n_j, j = 1, 2\), the system (2.9), (2.10) is locally controllable to \(\eta\), but is not STLC to \(\eta\).

Corollary 3.5. Let \(k \in \mathbb{C}_4\). Then the value function \(V_{\eta_+^-}\) (see Section 2.4) has the following properties:

(i) \(\text{dom} V_{\eta_+^-}(\cdot) = \widehat{\mathbb{C}}\) whenever \(\eta_+ \in [n_1, n_2]\);

(ii) If \(\eta_+ \in (-n_2, -n_1) \cup (n_1, n_2)\), then \(\text{dom} V_{\eta_+^-}(\cdot)\) is an open subset of \(\widehat{\mathbb{C}}\) and \(V_{\eta_+^-}(\cdot)\) is continuous on \(\text{dom} V_{\eta_+^-}\).

Proof. (i) is a reformulation of Theorem 3.3. Statement (ii) follows from Theorem 3.4 and [5] Chapter 4.

4 Maximum principle and rotation of modes

In the rest of the paper it is assumed that \(\kappa \in \mathbb{C}_4\) if it is not explicitly stated otherwise (see Remark 3.4 for explanations). For the terminology of the optimal control theory used in this section, we refer to [30].

In this section, we combine the monotonicity result of Lemma 3.2 and the properties of the complex argument \(\text{Arg} y\) of an eigenfunction \(y\) of (2.3), (2.12) with the Pontryagin Maximum Principle (PMP) in the form of [30] Section 2.8 with The rotational properties of \(y\) are important to show that singular arcs and the chattering effect are absent, and so minimum-time controls are of bang-bang type in the sense described below.

Let \([s^-, s^+]\) be a bounded interval in \(\mathbb{R}\). Recall that a function \(\varepsilon(\cdot)\) is called piecewise constant in \([s^-, s^+]\) if there exist a finite partition \(s^- = b_0 < b_1 < b_2 < \cdots < b_n < b_{n+1} = s^+\), \(n \in \{0\} \cup \mathbb{N}\), such that, after a possible correction of \(\varepsilon(\cdot)\) on a set of zero (Lebesgue) measure,

\[\varepsilon(\cdot)\text{ is constant on each interval } (b_j, b_{j+1}), 0 \leq j \leq n.\] (4.1)

For such functions, we always assume that the abovementioned correction that ensures (4.1) has been done. A point \(b_j, 1 \leq j \leq n\), of the partition is called a switch point of \(\varepsilon(\cdot)\) if \(\varepsilon(b_j - 0) \neq \varepsilon(b_j + 0)\). (Throughout the paper, \(\varepsilon(s \pm 0)\) are the one-side limits of a function \(\varepsilon\) at a point \(s \in \mathbb{R}\).)

Suppose \(\varepsilon(\cdot) \in \mathbb{F}_{s^-, s^+}\). Then \(\varepsilon(\cdot)\) is said to be a bang-bang control on \([s^-, s^+]\) if it is a piecewise constant function that takes only the values \(n_1^2\) and \(n_2^2\) (after a possible correction at switch points). Note that this definition excludes the cases with the chattering effects.

When \(\varepsilon(\cdot)\) represents a layered structure of an optical resonator, it is natural to say that the maximal intervals of constancy \((b_j, b_{j+1})\) of a piecewise constant control \(\varepsilon(\cdot) \in \mathbb{F}_{s^-, s^+}\) are layers of width \(b_{j+1} - b_j\) with the constant permittivity equal to \(\varepsilon(s), s \in (b_j, b_{j+1})\).
Assume that \( \varepsilon(\cdot) \in F_{s^- s^+} \) and that \( y \) is a nontrivial solution to (2.3) on \([s^-, s^+]\). If the point \( p_0 \) introduced in Lemma 3.2 exists in \([s^-, s^+]\) it is called the turning point of \( y \) [20]. Note that

\[
\text{if } y(s_0) y'(s_0) = 0 \text{ for a certain } s_0 \in [s^-, s^+], \text{ then } s_0 = p_0 = p_t
\]

for every \( \tau \in [0, -2 \arg_0 \kappa] \) (for definition of \( p_t \), see Lemma 5.2). In particular, the set \( \{ s : y(s) = 0 \} \) consists of at most one point. If the point \( p_0 \) does not exist in the interval \([s^-, s^+]\), we will assign to \( p_0 \) a special value outside of \([s^-, s^+]\), which will be specified later.

The special role of \( p_0 \) is that the trajectory of \( y \) rotates clockwise for \( s < p_0 \), and counterclockwise for \( s > p_0 \). More rigorously, the multifunction \( \arg y(\cdot) \) has a branch \( \arg_\* y(\cdot) \in W^{1, \infty}[s^-, s^+] \) that is defined and continuously differentiable on \([s^-, s^+] \setminus \{ s : y(s) = 0 \} \) and possesses the following properties:

(A1) if an interval \( I \subset [s^-, s^+] \) does not contain \( p_0 \), then the derivatives \( \partial_s \arg_\* y(s) \) are nonzero and of the same sign for all \( s \in I \);

(A2) if \( p_0 \in [s^-, s^+] \), then

\[
\partial_s \arg_\* y(s) < 0 \text{ if } s < p_0, \quad \partial_s \arg_\* y(s) > 0 \text{ if } s > p_0.
\]

(A3) if \( p_0 \in (s^-, s^+) \) and \( y(p_0) = 0 \), there exist finite limits \( \arg_\* y(p_0 \pm 0) \) satisfying

\[
\arg_\* y(p_0 + 0) = \pi + \arg_\* y(p_0 - 0). \tag{4.4}
\]

The existence of such \( \arg_\* y(\cdot) \) with properties (A1), (A2) is essentially proved in [20]. Formally, [20] works with the case where \( \eta_{-} \in \mathbb{R}_{\pm} \) in (2.12). However the proof can be extended without changes. It is based on the facts that \( \text{Im}(i \kappa x(s)) = \partial_s \text{Im} \ln y(s) = \partial_s \arg_\* y(s) \) for a suitable branch of \( \ln y(s) \) differentiable on \( \mathbb{R} \setminus \{ s : y(s) = 0 \} \), and that the function \( |y|^2 \text{Im}(i \kappa x) = \text{Im}(\overline{y} \partial_s y) = G_0 \) is strictly increasing (see Lemma 3.2 for \( \tau = 0 \)). Concerning the property (A3), one sees that \( y'(p_0) \neq 0 \) in the case \( y(p_0) = 0 \) and so \( \arg_\* y(p_0 \pm 0) = \pm \arg y'(p_0) \pmod{2\pi} \). This ensures that \( \arg_\* y \) can be chosen such that (A3) holds.

From now on, we assume that, for each \( y \), the function \( \arg_\* y \) is a certain fixed branch of \( \arg y \) satisfying the above properties.

**Theorem 4.1.** Let \( \kappa \in \mathbb{C}_4, \eta_{\pm} \in \mathbb{C}, \eta_- \neq \eta_+ \). Let \( \varepsilon(\cdot) \) be a minimum-time control from the initial state \( x(s^-) = \eta_- \) of the system (2.3), (2.10) to the target state \( x(s^+) = \eta_+ \). Then \( \varepsilon(\cdot) \) is a bang-bang control on \([s, s^+]\), and there exist a constant \( \lambda_0 \in [0, +\infty) \) and an eigenfunction \( y \) of the problem (2.3), (2.12) such that

\[
\varepsilon(s) = \begin{cases} n_2^2, & \text{if } \text{Im}(y^2(s)) < 0; \\ n_3^2, & \text{if } \text{Im}(y^2(s)) > 0; \end{cases} \quad (4.5)
\]

\[
\text{Im}(\varepsilon(s)y^2(s) + \kappa^{-2}(y'(s))^2) = \lambda_0 \text{ for all } s \in [s^-, s^+]. \tag{4.6}
\]

**Remark 4.1.** If \( \varepsilon(\cdot) \in F_{s^-, s^+} \) (which is not assumed here to be minimum-time) and an eigenfunction \( y(\cdot) \) of (2.3), (2.12) satisfy (4.5), then \( \varepsilon(\cdot) \) is a bang-bang control. To see this one needs the following simple consequence from (4.3) and the inclusion \( \partial_s \arg_\* y \in L^\infty(s^-, s^+) \):

for each eigenfunction \( y \) of (2.3), (2.12) the set \( \{ s \in (s^-, s^+) : \text{Im} y^2(s) = 0 \} \) is finite.
Note that the switch points of $\varepsilon(\cdot)$ in this case are exactly the points $\{s \in (s^-, s^+) \setminus \{p_0\}: y^2(s) \in \mathbb{R}\}$ (cf. [18, 20]). Indeed, from the fact that the direction of rotation of $y$ changes at the turning point $p_0$, one sees that

$p_0$ is not a switch point.

**Proof of Theorem 4.4.** By Lemma 3.2, the trajectory $x([s^-, s^+])$ intersects $i\hat{\mathbb{R}}$ at most once. So we can apply PMP in the form of [30] Section 2.2 to the following two cases with restricted state spaces:

$$x([s^-, s^+]) \subset \mathbb{C} \text{ and } \mathbb{C} \text{ is the state space for dynamics of } x(\cdot);$$

(4.7)

$$x([s^-, s^+]) \subset \hat{\mathbb{C}} \setminus \{0\} \text{ and } \mathbb{C} \text{ is the state space for dynamics of } \hat{x}(\cdot)$$

(4.8)

(recall that $\hat{x} = -1/x$). In each of the cases, we perceive the $\mathbb{C}$-valued adjoint variable $\lambda(\cdot)$ and the state variable $x(\cdot)$ as $\mathbb{R}^2$-valued functions defined on $[s^-, s^+]$ with real-valued components $\lambda_1$, $\lambda_2$ and $x_1, x_2$, i.e., $\lambda = \lambda_1 + i\lambda_2$, etc. To perceive $\mathbb{C}$ as $\mathbb{R}^2$, we equip it with the $\mathbb{R}$-valued scalar product $\langle \lambda, x \rangle = \text{Re}(\lambda x) = \text{Re}(\lambda\overline{x})$.

Let $\varepsilon(\cdot)$ be a minimum time control that steers optimal $x(\cdot)$ from $\eta_-$ to $\eta_+$.\footnote{Note that the switch points of $\varepsilon(\cdot)$ in this case are exactly the points $\{s \in (s^-, s^+) \setminus \{p_0\}: y^2(s) \in \mathbb{R}\}$ (cf. [18, 20]). Indeed, from the fact that the direction of rotation of $y$ changes at the turning point $p_0$, one sees that $p_0$ is not a switch point.}

**Case (4.7).** We assume that optimal trajectory $x([s^-, s^+])$ lies in $\mathbb{C} = \mathbb{R}^2$. The time-independent Hamiltonian function $H$ equals $H(\lambda_0, \lambda, x, \varepsilon) = \lambda_0 + \langle \lambda, f(x, \varepsilon) \rangle$, where $\lambda_0 \geq 0$.

The adjoint equation is $\partial_s \lambda = -\partial_x \langle \lambda, f(x, \varepsilon) \rangle$, where

$$\partial_x \langle \lambda, f \rangle_C = \langle \lambda, \partial_x f \rangle_C + i\langle \lambda, \partial_x f \rangle_C = \langle \lambda, (\partial_x f) \rangle_C + i\langle \lambda, (i\partial_x f) \rangle_C$$

$$= \text{Re}(\lambda[2i\kappa x]) + i\text{Im}(\lambda[-2i\kappa x]) = -2i\kappa x \lambda$$

(the analyticity of $f$ in $x$ is used here, namely, $\partial_x f = i\partial_x f$). From (4.7), we see that $y_* \neq 0$ everywhere for any eigenfunction $y_*$ of (2.3), (2.12), and $\lambda' = 2(ikx)\lambda = 2\lambda y_*/|y_*|$. The adjoint variable $\lambda(\cdot)$ takes the form

$$\lambda(s) = \overline{y_*^2} \quad \text{for a certain eigenfunction } y_* \text{, and so}$$

$$H = \lambda_0 + \text{Re}(i\kappa y_*^2[-x^2 + \varepsilon]) = \lambda_0 + \text{Im}(\kappa y_*^2[x^2 - \varepsilon]) = \lambda_0 + \text{Im}(y^2[x^2 - \varepsilon]),$$

(4.9)

(4.10)

where $y := \kappa^{1/2}y_*$ is another eigenfunction. Since the Hamiltonian is time-independent, the PMP for minimum-time problems [30] Section 2.8.1] implies $0 = H = \lambda_0 - \text{Im}(\kappa^{-2}(y)^2 + y^2\varepsilon)$ for all $s$. This gives (4.6). The minimum condition takes the form $\text{Im}(-y^2\varepsilon(s)) = \min_{\varepsilon \in [\varepsilon_1, \varepsilon_2]} \text{Im}(-y^2\varepsilon)$ and gives (4.5).

**Case (4.8).** The scheme is the same with the change in of the state variable to $\tilde{x} = -1/x$. This leads to a different adjoint variable $\lambda = \overline{y_*^2}$, where $y_*$ is a certain eigenfunction, but to the same formulas at the end.

Remark 4.1 completes the proof in the both cases.

## 5 Extremals and quarter-wave stacks

### 5.1 Definitions and the notation for the analysis of extremals

Let an eigenfunction $y$ of (2.3)- (2.12), a constant $\lambda_0 \in [0, +\infty)$, and functions $\varepsilon \in \mathbb{F}_{s^-, s^+}$ and $x$ satisfy (4.5), (4.6), and (2.9)- (2.10). Then $y(\cdot)$, $x(\cdot)$, and $\varepsilon(\cdot)$ are called a $y$-extremal,
an $x$-extremal, and an extremal control ($\varepsilon$-extremal), resp., associated with the extremal tuple $(x, \varepsilon, \lambda_0, y)$ (our terminology is adapted to the needs of resonance optimization and differs slightly from the standard terminology of extremal lifts \[6, 30\]) If, additionally, $\lambda_0 > 0$ ($\lambda_0 = 0$), these extremals are called normal (resp., abnormal).

We say that a nondegenerate interval $[\tilde{s}^-, \tilde{s}^+]$ is stationary for $x$ if $x(s) = c$ for all $s \in [\tilde{s}^-, \tilde{s}^+]$ and a certain constant $c$. If this is the case,

$$c \in \{n_1, n_2\} \text{ and } \varepsilon(s) = c^2 \text{ for all } s \in [\tilde{s}^-, \tilde{s}^+].$$

(5.1)

Remark 5.1. Stationary intervals cannot be discarded as pathologies irrelevant to the minimization problems. Their role for calculation of resonances of minimal decay is elucidated in Theorems \[8.3\] and \[8.4\].

Let $(x, \varepsilon, \lambda_0, y)$ be an extremal tuple on $[s^-, s^+]$. Then, by Remark \[4.1\] and \[4.5\], $\varepsilon$ is a bang-bang control on $[s^-, s^+]$ and $y$ is a nontrivial solution to the autonomous equation

$$-y''(s) = \kappa^2 y(s) E(y)(s), \quad \text{where } E(y)(\cdot) := n_1^2 + (n_2^2 - n_1^2)\chi_{\mathbb{C}_+}(y^2(\cdot))$$

(5.2)

and $\chi_{\mathbb{C}_+}(\cdot)$ is the characteristic function of $\mathbb{C}_+$, i.e., $\chi_{\mathbb{C}_+}(z) = 1$ if $z \in \mathbb{C}_+$, and $\chi_{\mathbb{C}_+}(z) = 0$ if $z \in \{\text{Im } z \leq 0\}$. Besides, $\varepsilon(\cdot) = E(y)(\cdot)$ on $[s^-, s^+]$.

Any nontrivial solution $y$ to equation (5.2) on $[s^-, s^+]$ can be extended from this interval to whole $\mathbb{R}$ in a unique way due to the existence and uniqueness theorem \[20\] for initial value problems for (5.2). Then it is easy to see from (5.2) that $\text{Im}(\varepsilon(s)y^2(s) + \kappa^{-2}(y'(s))^2)$ is a constant independent of $s$. Let us denote this constant by $\tilde{\lambda}_0$ and define $x(\cdot)$ by (2.8). If $\tilde{\lambda}_0 \geq 0$, the solution $y$ (the tuple $(x(\cdot), E(y)(\cdot), \tilde{\lambda}_0, y)$) defined on the whole $\mathbb{R}$ will be called an extended $y$-extremal (resp., extended extremal tuple). The reason for this is that its restriction to any non-degenerate interval $[s^-, s^+]$ is an extremal tuple on this interval (associated with the values $\eta_{\pm} = x(s^\pm)$ in the boundary conditions (2.12)).

By Theorem \[4.1\] and \[2.15\], every minimum time control is a restriction of an extended extremal control on a finite interval $[s^-, s^+]$ containing no stationary subintervals. This statement explains the logic of definitions and studies of this section.

In the rest of the section we assume (if it is not explicitly stated otherwise) that $y$ is an extended $y$-extremal, $(x, \varepsilon, \lambda_0, y^2)$ is the associated extended extremal tuple, and $\{b_j\}$ is the set of switch points of $\varepsilon(\cdot)$ indexed by the following lemma.

Lemma 5.1. Let $\varepsilon(\cdot)$ be an extended extremal control. Then the set of switch points of $\varepsilon$ has a form of strictly increasing sequence $\{b_j\}_{j=-\infty}^{+\infty}$ satisfying $\lim_{j \to \pm\infty} b_j = \pm\infty$.

Proof. Let us show that for arbitrary $s_0 \in \mathbb{R}$, there exists a switch point $a$ of $\varepsilon$ in $(s_0, +\infty)$.

Indeed, assume converse. Then there are no switch points in $(s_0, +\infty)$ and $\varepsilon(s) = \varepsilon$ for $s > s_0$ with a constant $\varepsilon$, which is equal either to $n_1^2$, or $n_2^2$. Then, for $s > s_0$, (3.8) holds with $|c_-| + |c_+| \neq 0$. Since $i\varepsilon \in \mathbb{C}_1$, one sees that either $x(s) = -\varepsilon^{-1/2}$ for all $s > s_0$ in the case $c_+ = 0$ (an unstable equilibrium solution), or $x(s) \to \varepsilon^{1/2}$ (which is a stable equilibrium solution) as $s \to +\infty$ in the case $c_+ \neq 0$. In the both cases, there exists $\delta > 0$ and $s_1 > s_0$ such that, on the whole interval $[s_1, +\infty)$, $|\partial_s \text{Arg}_* y| \geq \delta$ and $\partial_{s_1} \text{Arg}_* y$ is of the same sign. Thus, there exists at least one switch point in the interval $[s_1, s_1 + \frac{\pi}{2\varepsilon}]$.

By Remark \[4.1\] there exists at most finite number of switch points in each finite interval. The symmetry $(S0) \iff (S1)$ completes the proof.
Remark 5.2. The special role of the no-return line $L_0$ and the extended half-planes $H_{p_0}^\pm$ of Remark 3.1 is that

$$x(s) \in L_0 \text{ if and only if } s \text{ is the turning point } p_0 \text{ of } y.$$  

(5.3)

If the trajectory $x[\mathbb{R}]$ does not intersect $L_0$, then either $x[\mathbb{R}] \subset \text{int } H_{p_0}^-$ and we put $p_0 = +\infty$, or $x[\mathbb{R}] \subset \text{int } H_{p_0}^+$ and we put $p_0 = -\infty$.

Let $p_0 \in \mathbb{R}$. Then we can introduce an additional enumeration for switch points denoting by $b_j$, $j \in \mathbb{N}$ ($j \in \mathbb{N}$), all switch points greater than $p_0$ (resp., all switch points less than $p_0$) in increasing order. We put $b_0 := p_0$, but recall that $p_0$ is not a switch point due to (4.3) and (4.5).

5.2 Iterative calculation of switch points

For all $s \in \mathbb{R}$, let us define the $\hat{C}$-valued function

$$\vartheta(s) = \mathcal{I}(\varepsilon^{-1/2}(s)x(s)) \quad \text{(recall that } \mathcal{I}(z) := (1 - z)/(1 + z)),$$

(5.4)

which we call the *ilog-phase* for the control $\varepsilon(\cdot)$. Let us take an arbitrary layer $I = (b_j, b_{j+1})$ of $\varepsilon(\cdot)$, then, on $I$, the function $\vartheta(\cdot)$ is continuous and equals to $\vartheta_c(\cdot)$ of (3.7) with $\epsilon := \varepsilon(b_j + 0)$. On $I$, the representations (3.8) and (3.9) with $\zeta := \varepsilon^{1/2}\kappa$ give

$$\vartheta(s) = \vartheta(s) e^{-2i\kappa(s-s)}, \quad s, \bar{s} \in I,$$

(5.5)

which have to be understood as $\vartheta(s) \equiv \infty$ on $I$ if $\vartheta(s) = \infty$. Note that

$$I \text{ is an interval of constancy for } x \text{ exactly when } c_+ = 0, \text{ or } c_- = 0 \text{ in (3.8).}$$

(5.6)

Hence, in the case when $I$ is not an interval of constancy, (3.10) and (3.11) holds in $I$.

Lemma 5.2. Assume that $I = (b_j, b_{j+1})$ is not a stationary interval for $\varepsilon(\cdot)$. Let $\epsilon = \varepsilon(b_j + 0)$ and $\kappa = \varepsilon^{1/2}\kappa$. Let a function $\Phi(\cdot) \in C[b_j, +\infty)$ satisfy

$$\Phi(s) = \vartheta^{1/2}(b_j + 0)e^{-i\kappa(s-b_j)}.$$  

(5.7)

Denote by $t^+(\zeta)$ the minimal $s \in (b_j, +\infty)$ so that

$$\Phi(s) \in \mathcal{J}^{-1}([c]_{\mathbb{R}_+}), \text{ where } \mathcal{J}^{-1}([c]_{\mathbb{R}_+}) := \{z \in \mathbb{C} : \mathcal{J}(z) = \zeta c \text{ for a certain } c \in \mathbb{R}_+\}$$

if such $s$ exists, otherwise put $t^+(\zeta) = +\infty$.

(i) Let $p_0 \notin I$ and $\mp \partial_\ast \text{ Arg}_\ast y(s) > 0$ on $I$. Then $b_{j+1} = t^+(\mp i\mathcal{J}(\Phi(b_j)))$.

(ii) Let $p_0 \in \mathcal{T} = [b_j, b_{j+1}]$ and $x(p_0) \neq \infty$. Then $b_{j+1} = t^+(\mathcal{J}(\Phi(b_j)))$.

(iii) Let $p_0 \in \overline{\mathcal{T}}$ and $x(p_0) = \infty$. Then $b_{j+1} = t^+(\mathcal{J}(\Phi(b_j)))$.

Proof. By the definition of $\Phi(\cdot)$ and (3.9),

$$\vartheta(s) = \Phi^2(s) \text{ for } s \in I.$$

(5.8)
It follows from (3.10) that, for a certain constant $\tilde{A} \neq 0$,
\[ y(s) = \tilde{A} \mathcal{J}(\Phi(s)), \quad s \in I. \] (5.9)
Since $b_j$ is a switch point, $y^2(b_j) \in \mathbb{R} \setminus \{0\}$. The switch point $b_{j+1}$ is the smallest time point after the time point $b_j$ when the trajectory of $y^2$ again reaches $\mathbb{R} \setminus \{0\}$.

In the case of statement (i), the rotation of $y$ around 0 goes in one direction. So Arg, $y$ changes on $\pi/2$ on $I$. This and (5.9) give the desired statement.

In the case (ii), $y(s)$ rotates in one direction till $s = p_0$, then rotate back (see (4.3)) and Arg, $y$ reaches at $s = b_{j+1}$ the value of Arg, $y(b_j)$. So (5.9) gives statement (ii).

In the case (iii), $y(s)$ passes through 0 = $y(p_0)$ changing the direction of rotation and, due to (1.4), Arg, $y(s)$ reaches at $s = b_{j+1}$ the value $\pi/2 + \text{Arg, } y(s)$. Due to (5.9) this translates to (iii) if $\text{Arg, } y(s) > 0$ if and only if $\pm |\partial_s y(s = 0) - Z_0| \mp R_0 > 0$;
\[ x \in iC_{\pm} \text{ if and only if } \pm |\partial_s y(s = 0)| \mp 1 > 0. \] (5.11)
(5.12)

5.3 Abnormal extremals and quarter wave layers

**Proposition 5.3.** Let $(x, \varepsilon, \lambda_0, y)$ be an extended extremal tuple.

(i) Let $\lambda_0 = 0$ (i.e., the extremal tuple is abnormal). Then $s$ is a switch point of $\varepsilon(\cdot)$ if and only if $x^2(s) \in \mathbb{R} \setminus \{0\}$ and $s$ is not an interior point of a stationary interval for $x(\cdot)$.

(ii) Let $x^2(b_j) \in \mathbb{R}$ for at least one switch point $b_j$, then $\lambda_0 = 0$.

**Proof.** (i) Since $H \equiv 0$ in (4.10) and $\lambda_0 = 0$, we get for a $y$-extremal $y$ the equality
\[ \text{Im}(y^2[x^2 - \varepsilon]) = 0. \] (5.13)
Consider a switch point $b_j$. By Remark 4.1, $b_j$ is not the turning point $p_0$ of $y$, $y^2(b_j) \in \mathbb{R} \setminus \{0\}$, and $x(b_j) \notin \{0, \infty\}$. From this and (5.13), we see that $x^2(b_j) \in \mathbb{R} \setminus \{0\}$.

It follows from (5.13) that, in the case $\varepsilon(s = 0) = x^2(s) = \varepsilon(s \mp 0)$, $s$ is an internal point of a stationary interval and is not a switch point. In the case $\varepsilon(s = 0) = x^2(s) \neq \varepsilon(s \mp 0)$, $s$ is a switch point and is an end point of a certain maximal stationary interval. Thus, a point $s$ of a maximal stationary interval $I$ for $x$ is a switch point for $\varepsilon$ exactly when $s$ is an end point of $I$. This completes the proof of the ‘only if’ part.

Assume now that $x^2(s) \in \mathbb{R} \setminus \{0\}$, but $x^2(s) \notin \{\varepsilon(s - 0), \varepsilon(s + 0)\}$. Then $\text{Im} y^2(s) = 0$, and $s$ is a switch point due to Remark 4.1 and the fact that $s \neq p_0$ (the latter follows from Remark 5.2).

(ii) It follows from $H = 0$ and (4.10), that $\lambda_0 = - \text{Im}(y^2(b_j)[x^2(b_j) - \varepsilon(b_j \pm 0)])$. Since $y^2(b_j)$ and $x^2(b_j)$ are real, one gets $\lambda_0 = 0$. \[ \square \]
Proposition 5.4. Let \((x, \varepsilon, 0, y)\) be an extended abnormal extremal tuple. Then:

(i) The intersection of the trajectory \(x[\mathbb{R}]\) with the interval \([n_1, n_2]\) (with \([-n_2, -n_1]\)) consists of at most one point \(x_+\) (resp., \(x_-\)).

(ii) Assume that the point \(x_+\) defined in (i) does exist. Then one of the two cases takes place:

(ii.a) In the case \(x_+ \in (n_1, n_2)\), there exists a unique \(s \in \mathbb{R}\) such that \(x(s) = x_+\). This \(s\) equals to \(b_j\) for a certain \(j \in \mathbb{N}\), moreover, \(\text{Im} x'(b_j - 0)\) and \(\text{Im} x'(b_j + 0)\) are nonzero and of the same sign. If \(\pm \text{Im} x'(b_j - 0) > 0\) and \(\pm \text{Im} x'(b_j + 0) > 0\), then \(\varepsilon(b_j - 0) = n_1^2\) and \(\varepsilon(b_j + 0) = n_2^2\).

(ii.b) In the case \(x_+ = n_1\) (\(x_+ = n_2\)), the set of \(s \in \mathbb{R}\) such that \(x(s) = x_+\) consists of one stationary interval of the form \([b_j, b_{j+1}]\), and \(\varepsilon(b_j - 0) = \varepsilon(b_{j+1} + 0) \neq x_+^2\).

(iii) If \(x(b_m) \in (n_1, +\infty)\) and \(\varepsilon(b_m + 0) = n_1^2\) (for a certain \(m \in \mathbb{Z}\)), then \(x(b_{m+1}) \in (0, n_1)\) (and \(\varepsilon(b_{m+1} + 0) = n_2^2\)).

(iv) If \(x(b_m) \in (0, n_2)\) and \(\varepsilon(b_m + 0) = n_2^2\), then \(x(b_{m+1}) \in (n_2, +\infty)\) (and \(\varepsilon(b_{m+1} + 0) = n_1^2\)).

Remark 5.3. One can formulate an analogue of statement (ii) for the point \(x_-\) using, e.g., the time reversal symmetry \((S_0) \Leftrightarrow (S_1)\).

Proof of proposition 5.4. Statements (iii)-(iv) follows from Remark 3.1 about the lines \(\Sigma_\tau\) of no-return and the sign of the value of the bracket \((-x^2 + \varepsilon)\) in (2.9), for \(x \in \mathbb{R}\). Statements (i)-(ii) are simple consequences of (iii)-(iv) as it is explained in the following remark.

Remark 5.4. It follows from statements (iii)-(iv) of Proposition 5.3 that in the cases (1) \(x(b_m) \in (n_2, +\infty), \varepsilon(b_m + 0) = n_1^2\) and (2) \(x(b_m) \in (0, n_1), \varepsilon(b_m + 0) = n_2^2\), we get for \(b_{m+1}\) the cases (2) and (1), respectively. So the alternation of the cases (1) and (2) is stable in the sense that, provided it has been started once, it continues infinitely. Now note that if for a certain \(j\) we get \(x(b_j) \in (n_1, n_2)\), then for \(b_{j+1}\) we are in the case (2) or in the case (1) depending on value of \(\varepsilon(b_j + 0)\). This implies (ii.a) of Proposition 5.3. The modification of this argument for the case when the trajectory of \(x\) gets into one of stationary points \(n_1, n_2\) is straightforward.

When the values \(x(b_j)\) and \(x(b_{j+1})\) of an abnormal \(x\)-extremal in two consecutive switch points are real, the relation between them and the corresponding length \(b_{j+1} - b_j\) of the layer is given by the next theorem.

With a matrix \(M = (a_{jm})_{j,m=1}^2\) belonging to \(\text{GL}_2(\mathbb{C}) = \{M \in \mathbb{C}^{2 \times 2} : \det M \neq 0\}\), one can associate the Möbius transformation \(f_M : z \mapsto \frac{a_{11} z + a_{12}}{a_{21} z + a_{22}}\). The map \(M \mapsto f_M\) is a group homomorphism. Since \(f_{cM} = f_M\) for \(c \in \mathbb{C} \setminus \{0\}\), every Möbius transformation can be represented as \(f_M\), with \(M_1 \in \text{SL}_2(\mathbb{C}) := \{M \in \mathbb{C}^{2 \times 2} : \det M = 1\}\).

Theorem 5.5. Let \((x, \varepsilon, 0, y)\) be an extended abnormal extremal tuple. Let \(\varepsilon = \varepsilon(b_j + 0)\) be the value of \(\varepsilon(\cdot)\) in the layer \(I = (b_j, b_{j+1})\). Let

\[
q := -\frac{\text{Im} \kappa}{\text{Re} \kappa} \quad \text{and} \quad q_1 := \mathcal{I}(-e^{-q\pi}) \quad (\text{note that } q_1 = \frac{1 + e^{-q\pi}}{1 - e^{-q\pi}} > 1).
\]

(i) If the part of trajectory \(x[\mathbb{R}] := \{x(s) : s \in [b_j, b_{j+1}]\}\) does not intersect \(\mathbb{R}\), then

\[
b_{j+1} - b_j = \frac{\pi}{2e^{1/2} \text{Re} \kappa} \quad (5.14)
\]
(i.e., the layer width is 1/4 of the wavelength in the material of the layer) and

\[ x(b_{j+1}) = f_{M_{(\varepsilon)}}(x(b_j)), \quad \text{where} \quad M(\varepsilon) := \begin{pmatrix} 1 & \varepsilon^{1/2} q_1 \\ \varepsilon^{-1/2} q_1 & 1 \end{pmatrix}. \quad \text{(5.15)} \]

(ii) Assume that \( x(s_0) \in \{0, \infty\} \) for a certain \( s_0 \in [b_j, b_{j+1}] \). Then \( s_0 = (b_j + b_{j+1})/2 \) and

\[ b_{j+1} - b_j = \frac{\pi}{\varepsilon^{1/2} \Re \kappa} \]

(i.e., the layer width is 1/2 of the corresponding wavelength). Besides,

\[ -x(b_j) = x(b_{j+1}) \in (0, \varepsilon^{1/2}) \text{ in the case } x(s_0) = \infty; \quad \text{(5.17)} \]
\[ -x(b_j) = x(b_{j+1}) \in (\varepsilon^{1/2}, +\infty) \text{ in the case } x(s_0) = 0. \quad \text{(5.18)} \]

Proof. (i) First, note that the combination of (5.14) with (3.9) gives

\[ \mathcal{I}(\varepsilon^{-1/2} x(b_{j+1})) = -e^{-q_2 \mathcal{I}(\varepsilon^{-1/2} x(b_j))}, \]

and, in turn, (5.15). Let us prove (5.14).

Assume that \( I \) is not a stationary interval for \( x \). Then \( x(b_j) \notin \{\pm \varepsilon^{1/2}\} \). Since \( J[I] \cap i\mathbb{R} = \emptyset \), it is easy to see from Remark 5.1 and Proposition 5.3 that \( p_0 \notin I \). So Lemma 5.2 (i) is applicable. The case \( \partial_s \text{Arg}_s y(s) > 0 \) on \( I \) is equivalent to \( \{x(b_j), x(b_{j+1})\} \subset \mathbb{R}_+ \), and so, due to (5.12), is equivalent to \( \{(\vartheta(b_j + 0))^{\pm 1}, (\vartheta(b_{j+1} - 0))^{\pm 1}\} \in (-1, 1) \setminus \{0\}. \)

To be specific, consider the case when \( \partial_s \text{Arg}_s y(b_{j+1}) < 0 \) and \( \vartheta(b_j + 0) \in (-\infty, -1) \). Then we know that \( \vartheta(b_{j+1} - 0) \in \mathbb{R} \setminus [-1, 1] \), \( \Phi(b_j) \in i(1, +\infty) \), and \( \Phi(b_{j+1}) \in \mathbb{R} \cup i\mathbb{R} \setminus \mathbb{R}_+ \), where \( \Phi(s) := \vartheta^{1/2}(b_j + 0)e^{-i\kappa(s-b_j)} \) is from Lemma 5.2. Since \( \mathcal{J}(\Phi(b_j)) \in i\mathbb{R}_+ \), applying Lemma 5.2 (i), we see from the well-known properties of the Kutta-Zhukovskii transform that

\[ \mathcal{J}^{-1}[\mathcal{I}(\Phi(b_j))]_{\mathbb{R}_+} = \mathcal{J}^{-1}[\mathcal{I}^{-1}]_{\mathbb{R}_+} = \mathbb{R}_+ \cup (\mathbb{T} \cap i\mathbb{C}_-). \]

Since \( |\Phi(b_{j+1})| > 1 \) and \( b_{j+1} \) is the first point where the trajectory of \( \Phi \) crosses \( \mathbb{R}_+ \cup (\mathbb{T} \cap i\mathbb{C}_-) \), we see that \( \Phi(b_{j+1}) \in (1, +\infty) \). Thus, the definition of \( \Phi(\cdot) \) implies (5.14). The case \( \partial_s \text{Arg}_s y(b_j) < 0 \), \( \vartheta(b_j + 0) \in (1, +\infty) \), and the cases when \( \partial_s \text{Arg}_s y(b_j) > 0 \) can be considered similarly.

Assume now that \( I \) is a stationary interval for \( x \). Then (3.8) holds on \( I \) with \( c_- = 0 \), or \( c_- > 0 \). Thus, (5.14) follows from (4.5).

(ii) It follows from \( x(s_0) \in \{0, \infty\} \) that \( \varepsilon(\cdot) \) is symmetric w.r.t. \( s_0 \), \( s_0 = p_0 = (b_j + b_{j+1})/2 \), and \( x(b_j) = -x(b_{j+1}) \in \mathbb{R}_- \). In particular, either the case (ii), or the case (iii) of Lemma 5.2 is applicable.

Consider the case \( x(p_0) = \infty \) and \( x(b_j) \in (-\varepsilon^{1/2}, 0) \). Then \( \vartheta(p_0) = -1 \) and

\[ \vartheta(b_j + 0) = (\vartheta(b_{j+1} - 0))^{-1} \in (1, +\infty). \quad \text{(5.20)} \]

So \( \Phi(b_j) \) and \( \mathcal{J}(\Phi(b_j)) \) are in \( (1, +\infty) \). From Lemma 5.2 (iii), we see that \( b_{j+1} \) is the smallest point in \( (b_j, +\infty) \), where the trajectory of \( \Phi(\cdot) \) reaches the set

\[ \mathcal{J}^{-1}[\mathcal{I}(\Phi(b_j))]_{\mathbb{R}_+} = \mathcal{J}^{-1}[\mathcal{I}^{-1}]_{\mathbb{R}_-} = \mathbb{R}_- \cup (\mathbb{T} \cap i\mathbb{C}_+). \]

By (5.8) and (5.20), \( \vartheta(b_{j+1} - 0) = \Phi^2(b_{j+1}) \in (0, 1) \). So \( \Phi(b_{j+1}) \in (-1, 0) \). Thus, the definition of \( \Phi \) implies (5.16).
Let us show that in the case \( x(p_0) = \infty \), the inclusion \( x(b_j) \in (-\infty, -\epsilon^{1/2}] \) is impossible. If \( x(b_j) = -\epsilon^{1/2} \), this is obvious since \( I \) is stationary interval and \( p_0 \notin I \). Suppose \( x(b_j) \in (-\infty, -\epsilon^{1/2}) \). Then (2.10) implies \( \tilde{x}'(b_j) \in \mathbb{C}_+ \), \( \tilde{x}'(p_0) = 0 \), and \( \tilde{x}'(p_0) \in \mathbb{C}_+ \). So there exists \( s \in (b_j, p_0) \) so that \( x(s) \in \mathbb{R}_- \). Since there are no switch points between \( b_j \) and \( b_{j+1} \), we get a contradiction with Proposition 5.5.3.

The case \( x(s_0) = 0 \) can be treated similarly. \( \square \)

Denote
\[ \eta_j := \varepsilon^{1/2}(b_j + 0), \quad \tau_{j+1} := \eta_j/\eta_{j+1}, \quad \text{and} \quad r = n_2/n_1. \]

**Lemma 5.6.** The transform \( f_M(\eta_{j+1}^2)M(\eta_{j}^2) \) has exactly two fixed points equal to \( x_{-}^{\pm} \), where
\[ x_r^{\pm} = \frac{(n_1n_2)^{1/2}}{2} \left[ q_1(r^{-1/2} - r^{1/2}) \pm \left( q_1^2(r^{-1/2} - r^{1/2})^2 + 4 \right)^{1/2} \right] \in \mathbb{R}_\pm. \]

Besides, \( f_M(\eta_{j+1}^2)M(\eta_{j}^2) = f_M(\eta_{j+1}) \), where
\[ M_{j+1} = S_{j+1}^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} S_{j+1}, \]
\[ \lambda = J_1^{-1} \left( \frac{q_1^2 J(r) + 1}{q_1^2 - 1} \right) > 1, \quad \text{and} \quad S_r := \begin{pmatrix} 1 & -x_r^- \\ 0 & 1 \end{pmatrix}. \]

(In particular, the transform \( f_M(\eta_{j+1}^2)M(\eta_{j}^2) \) is of hyperbolic type with the repulsive fixed point \( x_{-}^{\pm} \) and the attractive fixed point \( x_{+}^{\pm} \), for the classification see [25] and also figure 31 therein.)

**Proof.** The determinant of the matrix
\[ M(\eta_{j+1}^2)M(\eta_{j}^2) = \begin{pmatrix} 1 + q_1^2\tau_{j+1}^{-1} & q_1(\eta_j + \eta_{j+1}) \\ q_1(\eta_j^{-1} + \eta_{j+1}^{-1}) & 1 + q_1^2\tau_{j+1} \end{pmatrix}, \]
equals \( q_1^2 - 1 \). Then \( f_M(\eta_{j+1}^2)M(\eta_{j}^2) = f_M \), where \( M := \frac{1}{q_1 - 1} M(\eta_{j+1}^2)M(\eta_{j}^2) \) and \( \det M = 1 \). So the matrix \( M \) has two eigenvalues \( \lambda^{\pm} \) with \( \lambda \neq 0 \) and its trace is equal to
\[ 2J(\lambda) = 2 + q_1^2(\eta_{j+1}/\eta_j + \eta_j/\eta_{j+1}) = 2 + q_1^2(r + r^{-1}) > 2. \]

So we can assume that \( \lambda \in (1, +\infty) \) and
\[ \lambda = J_1^{-1} \left( \frac{q_1^2 J(r) + 1}{q_1^2 - 1} \right) = \frac{q_1^2 J(r) + 1}{q_1^2 - 1} + \sqrt{\left( \frac{q_1^2 J(r) + 1}{q_1^2 - 1} \right)^2 - 1}. \]

The decomposition \( 5.21 \) can be obtained by standard calculations, see [25]. \( \square \)

**Corollary 5.7.** Let \((x, \varepsilon, 0, y)\) be an extended abnormal extremal tuple.

(i) If \( x(b_j) \in \mathbb{R}_\pm \), then \( f_{S_{j+1}}(x(b_j \pm 2m)) = \lambda^{\pm 2m} f_{S_{j+1}}(x(b_j)) \) for \( m \in \mathbb{N} \).

(ii) If \( \tilde{b}_0 = p_0 \in \mathbb{R} \) and \( x(\tilde{b}_0) \in \{0, \infty\} \), then \( f_{S_t}(x(\tilde{b}_{2m})) = \lambda^{2m} f_{S_t}(x(\tilde{b}_0)) \) for \( m \in \mathbb{N} \).

(iii) \( \lim_{m \to \pm\infty} x(b_j \pm 2m) = x_{j,j+1}^{\pm} \).

(Here the notation of Lemma 5.6 and the notation introduced in the end of Section 5.1 are used.)

**Proof.** The corollary follows from Lemma 5.6 and Theorem 5.5. \( \square \)
5.4 Normal extremals

If \((x, \varepsilon, \lambda_0, y)\) is a normal extremal tuple, then without loss of generality we assume that \(\lambda_0 = 1\). Indeed, since \(\lambda_0 > 0\), one can divide \([0,1]\) on \(\lambda_0\) and replace \(y\) by \(y/\lambda_0^{1/2}\).

The no-return line \(L_0\) (which contains all possible states \(x(p_0)\) at turning points \(p_0\)), and the two axes \(\mathbb{R}\) and \(i\mathbb{R}\), split \(\mathbb{C}\) into sectors \(\text{Sec}_j, j = 1, \ldots, 6\), which we number in the following way:

\[
\text{Sec}_1 = \mathbb{C}_1, \quad \text{Sec}_2 = \text{Sec}(\pi/2, \pi/2 - \text{Arg}_0 \kappa), \quad \text{Sec}_3 := (\pi/2 - \text{Arg}_0 \kappa, \pi), \\
\text{Sec}_4 = \mathbb{C}_3, \quad \text{Sec}_5 = \text{Sec}(3\pi/2, 3\pi/2 - \text{Arg}_0 \kappa), \quad \text{Sec}_6 := (3\pi/2 - \text{Arg}_0 \kappa, 2\pi)
\]  

(5.23)

(this splitting is connected with extremal synthesis [6] and related notions of ordinary and non ordinary points.)

**Theorem 5.8.** Let \((x, \varepsilon, 1, y)\) be an extended normal extremal tuple. Let \(\epsilon = \epsilon(b_j + 0)\) be the value of \(\epsilon(\cdot)\) in the layer \(L = (b_j, b_{j+1})\). Assume \(x[I] \subset \text{Sec}_0 \cup \text{Sec}_1 \cup \text{Sec}_3 \cup \text{Sec}_4\). Then:

(i) If \(x(b_j) \in \mathbb{C}_+,\) then \(x(b_{j+1}) \in \mathbb{C}_-\) and \(b_{j+1} - b_j < \frac{\pi}{2\epsilon^{1/2} \text{Re} \kappa}\).

(ii) If \(x(b_j) \in \mathbb{C}_-\), then \(x(b_{j+1}) \in \mathbb{C}_+\) and \(b_{j+1} - b_j > \frac{\pi}{2\epsilon^{1/2} \text{Re} \kappa}\).

**Proof.** Let us consider the case when \(x[I] \subset \text{ie}^{-\text{Arg}_0 \kappa} \mathbb{C}_+ \cap i\mathbb{C}_+\) and \(x(b_j) \in \mathbb{C}_+\).

Since \(\mathcal{I}[\mathbb{C}_+] = \mathbb{C}_-\), \(x(b_j) \in \mathbb{C}_+\) yields \(\vartheta(b_j + 0) \in \mathbb{C}_-\) and \(\Phi(b_j) \in \mathbb{C}_+\). Further, on one hand, \((5.12)\) yields \(\Phi(b_j) \in \mathbb{C}_+ \cap \mathbb{D}\) and \(\Phi(b_j + 1) \notin \mathbb{D}\). On the other hand, \((5.10)\) and \((5.11)\) imply that the case \(\partial \text{Arg} y(s) < 0\) of Lemma 5.2 (i) is applicable to \(L\). So \(b_{j+1} = t^+(-i\mathcal{J}(\Phi(b_j)))\) is the minimal \(s > b\) at which \(\Phi(s)\) intersects \(\mathcal{J}_1^{-1}[-i\mathcal{J}(\Phi(b_j))\mathbb{R}_+]\)

(we use here \(\Phi(b_j + 1) \notin \mathbb{D}\)). Thus, on one hand

\[
\text{Arg}_0 \mathcal{J}(\Phi(b_j)) - \text{Arg}_0 \mathcal{J}(\Phi(b_{j+1})) = \pi/2,
\]

and on the other hand, the squeezing properties of \(\mathcal{J}(\cdot)\) imply that

\[
\text{Arg}_0 \Phi(b_j) < \text{Arg}_0 \mathcal{J}(\Phi(b_j)) < 0 \quad \text{and} \quad \text{Arg}_0 \mathcal{J}(\Phi(b_{j+1})) < \text{Arg}_0 \Phi(b_{j+1}) < -\pi/2.
\]

Combining these inequalities we get \(\text{Arg}_0 \Phi(b_j + \pi/2) < \text{Arg}_0 \Phi(b_j) - \text{Arg}_0 \Phi(b_{j+1}) < \pi/2\)

and, applying the properties of \(\Phi\) from Lemma 5.2

\[
\pi/2 + \frac{1}{2} \text{Arg}_0 \vartheta(b_j + 0) < (b_{j+1} - b_j)\epsilon^{1/2} \text{Re} \kappa < \pi/2.
\]

This gives \(b_{j+1} - b_j < \frac{\pi}{2\epsilon^{1/2} \text{Re} \kappa}\). As a by-product, we get \(\vartheta(b_{j+1} - 0) \in \mathbb{C}_-\) from \(\Phi(b_j) \in \mathbb{C}_+\) and the connection \((5.7)\), \((5.9)\) between \(\Phi\) and \(\vartheta\). Thus, \(x(b_{j+1}) \in \mathbb{C}_-\).

The case when \(x[I] \subset \text{ie}^{-\text{Arg}_0 \kappa} \mathbb{C}_+ \cap i\mathbb{C}_+\) and \(x(b_j) \in \mathbb{C}_-\) can be considered similarly.

The cases when \(x[I] \subset \text{ie}^{-\text{Arg}_0 \kappa} \mathbb{C}_- \cap i\mathbb{C}_-\) can be obtained, e.g., by the time reversal symmetry (S0) \(\Leftrightarrow\) (S1). \(\Box\)

**Theorem 5.9.** Let \((x, \varepsilon, 1, y)\) be an extended normal extremal tuple. Then:

(i) If \(x(b_j) \in \text{Sec}_1 \cup \text{Sec}_3 \cup \text{Sec}_5\), then \(\varepsilon(b_j - 0) = n_2^2\) and \(\varepsilon(b_j + 0) = n_2^2\).

(ii) If \(x(b_j) \in \text{Sec}_2 \cup \text{Sec}_4 \cup \text{Sec}_6\), then \(\varepsilon(b_j - 0) = n_1^2\) and \(\varepsilon(b_j + 0) = n_2^2\).

**Proof.** By \((4.5)\) and \((4.3)\), \(y^2(b_j) \in \mathbb{R} \setminus \{0\}\). By \((4.10)\), \(-1 = \text{Im}(y^2(b_j)(x^2(b_j) - \varepsilon(b_j \pm 0))) = y^2(b_j) \text{Im}(x^2(b_j))\). Thus, \(\text{Im}(x^2(b_j)) = -1/y^2(b_j)\). Combining this equality with the direction \((4.3)\) of rotation of \(y\) one gets the desired statements. \(\Box\)
6 Elements of extremal synthesis

Let $\kappa \in \mathbb{C}_x$. Let us classify all possible $x$-extremals on an interval $[s^-, s^+]$ that satisfy the following properties

$$ x(s^-) \in [-n_2, -n_1], \quad x(s^+) \in [n_1, n_2], \quad \text{and} \quad \text{there are no stationary subintervals in } [s^-, s^+]. \quad (6.1) $$

Let us consider a corresponding extended extremal tuple $(x, \varepsilon, \lambda_0, y)$ on $\mathbb{R}$ (note that such extension is not necessarily unique and that sometimes abnormal and normal extensions are possible simultaneously). We will say that this extended extremal tuple is symmetric w.r.t. $p_0$ if $x(p_0) \in \{0, \infty\}$. The reason is that $x(s - p_0) = -x(s + p_0)$ and $\varepsilon(s - p_0) = \varepsilon(-s + p_0)$. Note that (6.1) implies $p_0 \in (s^-, s^+)$, but in this section $s^{\text{entr}} := (s^- + s^+)/2$ does not necessarily coincide with $p_0$ because the case $x(s^-) \neq x(s^+)$ is allowed.

We consider the partition of $[s^-, s^+]$ by the points belonging to the two classes: switch points and the points $s$ where the trajectory $x|[s^-, s^+]$ crosses the set $\mathbb{R} \cup i\mathbb{R} \cup \mathcal{L}_0$. Each interval of this partition is encoded by signs ‘$-$’ and ‘$+$’ depending on the constant value of the control $\varepsilon(\cdot)$ in this interval; the sign ‘$-$’ corresponds to $\varepsilon_1 = n_1^2$, ‘$+$’ to $\varepsilon_2 = n_2^2$. Above these signs we write the index $j$ of one of the sectors $\text{Sec}_j$, $j = 1, \ldots, 6$ (see (5.39)), where the trajectory of $x(s)$ evolves for $s$ in the interval. The points where the trajectory of $x$ crosses $\mathbb{R} \cup i\mathbb{R} \cup \mathcal{L}_0$ are of special importance. Therefore, for every such intersection we write as the lower index the subset of the corresponding extended line $(\mathbb{R}, i\mathbb{R}, \text{or } \mathcal{L}_0)$ where the intersection takes place. If some pattern in the description repeats $m \in \{0\} \cup \mathbb{N}$ times in row, we put it in the square brackets with the upper index $m$, i.e., $\cdots \cdot \cdot \cdot [m] \cdots$.

**Example 6.1.** The notation

$$ (-n_2, -n_1) \quad + \quad \begin{array}{c} 3 \quad 6 \end{array} \quad \begin{array}{c} (\infty) \quad \{n_1, n_2\} \end{array} $$

(6.2)

means that the extremal trajectory of $x$ starts at a point $x(s^-) \in (-n_2, -n_1]$, the trajectory goes from $x(s^-)$ to the sector $\text{Sec}_3$, the first and the only layer of $\varepsilon(\cdot)$ is of permittivity $\varepsilon_2$, the trajectory passes through $\infty$ at $s = p_0 = s^{\text{entr}} = (s^- + s^+)/2$ in the middle of this layer, goes to the sector $\text{Sec}_6$, and ends at $x(s^+) \in (n_1, n_2)$. In this particular case, it follows from $p_0 = s^{\text{entr}}$ and the symmetry w.r.t. $p_0$ that $x(s^-) = -x(s^+)$. Let $(x, \varepsilon, \lambda_0, y)$ be an extended extremal tuple such that (6.1) holds. Then the structure of this extremal tuple in the interval $[s^-, s^+]$ belongs to one of the classes described by the following statements:

(E1) If $(x, \varepsilon, \lambda_0, y)$ is abnormal and symmetric w.r.t. $p_0$, then it has on $[s^+, s^-]$ either the structure (6.2),

or one of the following structures with a certain number $m \in \mathbb{N}$ of repetitions:

$$ (-n_2, -n_1) \quad [3 \quad 4 \quad 1 \quad 4 \quad 1] \quad \begin{array}{c} 4 \quad 1 \quad 0 \quad (n_1, n_2); \end{array} $$

(6.3)

$$ (-n_2, -n_1) \quad \begin{array}{c} 3 \quad 4 \quad 3 \quad m \quad 6 \quad m \quad 6 \quad 6 \quad \{n_1, n_2\}; \end{array} $$

(6.4)

$$ (-n_2, -n_1) \quad \begin{array}{c} 3 \quad 4 \quad 3 \quad m \quad 6 \quad 1 \quad (n_2, +\infty) \quad (0, n_1) \quad (n_1, n_2); \end{array} $$

(6.5)
(E2) If \((x, \varepsilon, \lambda_0, y^2)\) is normal and symmetric w.r.t. \(p_0\), then it has on \([s^+, s^-]\) one of the following structures with certain numbers \(m_1, m_2 \in \{0\} \cup \mathbb{N}\) of repetitions:

\[
(-n_2, -n_1) + \left[ \frac{3}{(-\infty, -n_1)} + \frac{4}{(-n_2, 0)} + \frac{3}{\{\infty\}} \right]^{m_1} 3 \left[ \frac{6}{(0, n_2)} + \frac{1}{(n_1, +\infty)} - \frac{6}{m_2} \frac{1}{6} \right]^{m_2} 6 \frac{[n_1, n_2]}{; (6.6)}
\]

\[
(-n_2, -n_1) + \left[ \frac{3}{(-\infty, -n_1)} + \frac{4}{(-n_2, 0)} + \frac{3}{\{\infty\}} \right]^{m_1} 4 \left[ \frac{1}{(0, n_2)} + \frac{1}{(n_1, +\infty)} - \frac{6}{m_2} \frac{1}{6} \right]^{m_2} 6 \frac{[n_1, n_2]}{; (6.7)}
\]

\[
[-n_2, -n_1) + \left[ \frac{4}{(-n_2, 0)} + \frac{3}{(-\infty, -n_1)} + \frac{3}{\{\infty\}} \right]^{m_1} 4 \left[ \frac{6}{(0, n_2)} + \frac{1}{(n_1, +\infty)} - \frac{6}{m_2} \frac{1}{6} \right]^{m_2} 6 \frac{[n_1, n_2]}{; (6.8)}
\]

\[
[-n_2, -n_1) + \left[ \frac{4}{(-n_2, 0)} + \frac{3}{(-\infty, -n_1)} + \frac{4}{\{\infty\}} \right]^{m_1} 4 \left[ \frac{1}{(0, n_2)} + \frac{1}{(n_1, +\infty)} - \frac{6}{m_2} \frac{1}{6} \right]^{m_2} 6 \frac{[n_1, n_2]}{; (6.9)}
\]

\[
(-n_2, -n_1) + \left[ \frac{3}{(-\infty, -n_1)} + \frac{4}{(-n_2, 0)} + \frac{3}{\{\infty\}} \right]^{m_1} 6 \left[ \frac{1}{(0, n_2)} + \frac{1}{(n_1, +\infty)} - \frac{6}{m_2} \frac{1}{6} \right]^{m_2} 6 \frac{[n_1, n_2]}{; (6.10)}
\]

\[
[-n_2, -n_1) + \left[ \frac{4}{(-n_2, 0)} + \frac{3}{(-\infty, -n_1)} + \frac{3}{\{\infty\}} \right]^{m_1} 6 \left[ \frac{6}{(0, n_2)} + \frac{1}{(n_1, +\infty)} - \frac{6}{m_2} \frac{1}{6} \right]^{m_2} 6 \frac{[n_1, n_2]}{; (6.11)}
\]

(E3) If \((x, \varepsilon, \lambda_0, y^2)\) is abnormal and is not symmetric w.r.t. \(p_0\), then, on \([s^+, s^-]\), it has in
the case \(x(p_0) \in \mathbb{C}_+ \cap \mathcal{L}_0\) one of the following structures

\[
\begin{aligned}
(-n_2,-n_1) & \left[ \begin{array}{c}
3 \\
4 \\
n_2
\end{array} \right] + (-\infty,-n_2) \left[ \begin{array}{c}
3 \\
4 \\
n_2
\end{array} \right] + (-n_1,0) \\
\mathcal{L}_0 + i\mathbb{R}_+ & \left[ \begin{array}{c}
1 \\
6 \\
0 \\
n_1
\end{array} \right] + (n_2,+\infty) + (0,n_1) \\
m_1 3 & m_2 1 \\
m_1 3 & m_2 1
\end{aligned}
\]  \quad (6.12)

\[
\begin{aligned}
(-n_2,-n_1) & \left[ \begin{array}{c}
3 \\
4 \\
n_2
\end{array} \right] + (-\infty,-n_2) \left[ \begin{array}{c}
3 \\
4 \\
n_2
\end{array} \right] + (-n_1,0) \\
\mathcal{L}_0 + i\mathbb{R}_+ & \left[ \begin{array}{c}
1 \\
6 \\
0 \\
n_1
\end{array} \right] + (n_2,+\infty) + (0,n_1) \\
m_1 3 & m_2 1 \\
m_1 3 & m_2 1
\end{aligned}
\]  \quad (6.13)

\[
\begin{aligned}
(-n_2,-n_1) & \left[ \begin{array}{c}
4 \\
3 \\
n_2
\end{array} \right] + (-\infty,-n_2) \\
\mathcal{L}_0 + i\mathbb{R}_+ & \left[ \begin{array}{c}
1 \\
6 \\
0 \\
n_1
\end{array} \right] + (n_2,+\infty) + (0,n_1) \\
m_1 4 & m_2 1 \\
m_1 4 & m_2 1
\end{aligned}
\]  \quad (6.14)

\[
\begin{aligned}
(-n_2,-n_1) & \left[ \begin{array}{c}
4 \\
3 \\
n_2
\end{array} \right] + (-\infty,-n_2) \\
\mathcal{L}_0 + i\mathbb{R}_+ & \left[ \begin{array}{c}
1 \\
6 \\
0 \\
n_1
\end{array} \right] + (n_2,+\infty) + (0,n_1) \\
m_1 4 & m_2 1 \\
m_1 4 & m_2 1
\end{aligned}
\]  \quad (6.15)

and in the case when \(x(p_0) \in \mathbb{C}_- \cap \mathcal{L}_0\) one of the following structures

\[
\begin{aligned}
(-n_2,-n_1) & \left[ \begin{array}{c}
3 \\
4 \\
n_2
\end{array} \right] + (-\infty,-n_2) \\
\mathcal{L}_0 + i\mathbb{R}_+ & \left[ \begin{array}{c}
6 \\
1 \\
0 \\
n_1
\end{array} \right] + (n_2,+\infty) + (0,n_1) \\
m_1 3 & m_2 6 \\
m_1 3 & m_2 6
\end{aligned}
\]  \quad (6.16)

\[
\begin{aligned}
(-n_2,-n_1) & \left[ \begin{array}{c}
3 \\
4 \\
n_2
\end{array} \right] + (-\infty,-n_2) \\
\mathcal{L}_0 + i\mathbb{R}_+ & \left[ \begin{array}{c}
6 \\
1 \\
0 \\
n_1
\end{array} \right] + (n_2,+\infty) + (0,n_1) \\
m_1 3 & m_2 6 \\
m_1 3 & m_2 6
\end{aligned}
\]  \quad (6.17)

\[
\begin{aligned}
(-n_2,-n_1) & \left[ \begin{array}{c}
4 \\
3 \\
n_2
\end{array} \right] + (-\infty,-n_2) \\
\mathcal{L}_0 + i\mathbb{R}_+ & \left[ \begin{array}{c}
6 \\
1 \\
0 \\
n_1
\end{array} \right] + (n_2,+\infty) + (0,n_1) \\
m_1 4 & m_2 6 \\
m_1 4 & m_2 6
\end{aligned}
\]  \quad (6.18)

\[
\begin{aligned}
(-n_2,-n_1) & \left[ \begin{array}{c}
4 \\
3 \\
n_2
\end{array} \right] + (-\infty,-n_2) \\
\mathcal{L}_0 + i\mathbb{R}_+ & \left[ \begin{array}{c}
6 \\
1 \\
0 \\
n_1
\end{array} \right] + (n_2,+\infty) + (0,n_1) \\
m_1 4 & m_2 6 \\
m_1 4 & m_2 6
\end{aligned}
\]  \quad (6.19)

with certain numbers \(m_1, m_2 \in \{0\} \cup \mathbb{N}\) of repetitions.

(E4) If the extremal tuple \((x, \varepsilon, \lambda_0, y)\) is normal, is not symmetric w.r.t. \(p_0\), and \(x(p_0) \in \mathbb{C}_+ \cap \mathcal{L}_0\), then it passes the no-return lines \(\mathcal{L}_0\) and \(i\mathbb{R}\) in one of the two following ways:

\[
\begin{aligned}
\ldots & + \mathcal{L}_0 + \mathcal{L}_1 + \ldots \\
\ldots & + \mathcal{L}_0 + \mathcal{L}_1 + \ldots
\end{aligned}
\]  \quad (6.20)

If the extremal tuple \((x, \varepsilon, \lambda_0, y)\) is normal, is not symmetric w.r.t. \(p_0\), and \(x(p_0) \in \mathbb{C}_- \cap \mathcal{L}_0\), then it passes the no-return lines \(\mathcal{L}_0\) and \(i\mathbb{R}\) in one the two following ways:

\[
\begin{aligned}
\ldots & + \mathcal{L}_0 + \mathcal{L}_1 + \ldots \\
\ldots & + \mathcal{L}_0 + \mathcal{L}_1 + \ldots
\end{aligned}
\]  \quad (6.21)

All the possible structures of the extremal tuple \((x, \varepsilon, \lambda_0, y)\) on \([s^-, s^+]\) in the case when it is normal and is not symmetric w.r.t. \(p_0\) can be obtained by concatenation of the 4 sequences in
In this section, we summarize some of the consequences of the above analysis for optimizers of symmetric problems \((2.6), (2.7)\).

The proof of the statements \((E1)-(E4)\) can be obtained from \((5.17), (5.18), \text{Lemma 5.2, Theorems 5.8, 5.9, and Propositions 5.3, 5.4}\) taking into account the values of the vector fields \(f(x, n_1^2)\) and \(f(x, n_2^2)\) at the points \(x \in \mathbb{R} \cup \mathbb{R}\), and the equality \(f(0, n_1^2) = f(0, n_2^2) = -\kappa\).

**Remark 6.1.** A special clarification is needed for the sequence

\[
\cdots (-\infty, -n_1) - + (-n_2, 0) + \infty + (0, n_2) + - (n_1, +\infty) \cdots
\]

in the structures \((6.10)\) and \((6.11)\). When a normal \(x\)-extremal passes through \(\infty = p_0 = \tilde{b}_0\) the values of \(x(b_{-1})\) and \(x(b_1)\) at the preceding and the succeeding switch points do not necessarily belong to Sec\(_5\) and Sec\(_6\), resp.; these values may also be in Sec\(_4\) and Sec\(_1\), respectively. The reason is that \(y(p_0) = 0\), and so \(\text{Arg}_x y(p_0)\) does not exist. So the bounds on the positions of switch points \(b_{\pm 1}\) are determined not by \(\text{Arg}_x y(p_0)\), as in the case \(x(p_0) = 0\), but by the limits \(\text{Arg}_x y(p_0 \pm 0)\) (see \((6.1)\) and statement (iii) of \(\text{Lemma 5.2}\)).

### 7 Symmetric case: corollaries and an example

In this section, we summarize some of the consequences of the above analysis for optimizers of symmetric problems \((2.6), (2.7)\).

Let

\[
\varepsilon(\cdot) \in \mathbb{P}^{\text{sym}}, s^+ = -s^- = \pm s_\varepsilon^\pm > 0, \text{ and } (x, \varepsilon, \lambda, y) \text{ be an extremal tuple on } [s^-, s^+] \text{ so that } x(s_\varepsilon^\pm) = \pm n_\infty \quad (7.1)
\]

with a certain \(n_\infty > 0\). Then \(p_0 = s_{\text{centr}}\) is not a switch point of \(\varepsilon\). Moreover, either \(x(p_0) = 0\), or \(x(p_0) = \infty\). Let \(\{\tilde{b}_j\}_{j \in \mathbb{Z}} = \{b_j\}_{j \in \mathbb{Z}}\) be the set of switch point of the extended extremal tuple associated with \((x, \varepsilon, \lambda, y)\) (we assume that the enumeration of Section 5.1 is used for \(\{\tilde{b}_j\}_{j \in \mathbb{Z}}\)).

The control \(\varepsilon(\cdot)\) on the interval \((s^-, s^+)\) defines a layered resonator, which is symmetric w.r.t. 0. Three special layers can be naturally considered in this resonator \((28): \text{the central layer and two edge layers (for a formalization of the notion of a layer, see Section 4). They can be defined in the following way.}

By definition, the *central layer* \(I_{\text{centr}}\) of the resonator \(\varepsilon(\cdot)\) (on the interval \((s^-, s^+)\)) equals \((\tilde{b}_{-1}, \tilde{b}_1)\) in the case \(\tilde{b}_{-1} \in [s^-, s^+]\), and equals \((s^-, s^+)\) in the case \(\tilde{b}_1 \geq s^+\). Note that \(p_0 = 0 \in I_{\text{centr}}\). The right (left) *edge layer* \(I^+\) (resp., \(I^-\)) of the symmetric resonator \(\varepsilon(\cdot)\) is a largest subinterval \(I\) of \((s^-, s^+)\) that satisfies the following properties: \(I \neq I_{\text{centr}}\), \(I\) has the form \((s, s^+)\) (resp., the form \((s^-, s)\)), and \(I\) does not contain switch points (i.e., does not contain the points of the set \(\{\tilde{b}_j\}_{j \in \mathbb{Z}}\)).

By \(|I|\) we denote the length of an interval \(I\). The constant value of \(\varepsilon(\cdot)\) in \(I_{\text{centr}}\) will be denoted by \(\varepsilon_{\text{centr}}\), the constant value of \(\varepsilon(\cdot)\) in the layers \(I^-\) and \(I^+\) coincide and will be denoted by \(\varepsilon_{\text{edge}}\).

Note that the edge layers \(I^\pm\) do not exist exactly when \(I_{\text{centr}} = (s^-, s^+)\).

**Corollary 7.1.** Assume \(\kappa \in \mathbb{C}_4\), \((7.1)\), and \(I_{\text{centr}} = (s^-, s^+)\). Then:
(i) There exists a complex constant \( c \neq 0 \) such that \((x, \varepsilon, 0, cy)\) is an abnormal extremal tuple satisfying (7.1).

(ii) \(|I_{\text{centr}}|/2 = s^+ = -s^- = \frac{\pi}{2\varepsilon_{\text{edge}} \Re \kappa}\)

(iii) If \( x(0) = \infty \) (\( x(0) = 0 \)), then \( \varepsilon_{\text{centr}} = n_2^2 \neq n_1^2 \) (resp., \( \varepsilon_{\text{centr}} = n_1^2 \neq n_2^2 \)).

Proof. Since there are no switch points in \((s^-, s^+)\), one can easily replace the \((-n_\infty; n_\infty)-\)eigenfunction \( y \) by another eigenfunction \( y_1 = cy \) such that \( y_1(s^+) \in \mathbb{R} \) and \( y_1(s) \notin \mathbb{R} \) for all \( s \in (s^-, s^+) \). Then \((x, \varepsilon, 0, \Re^2)\) is an abnormal extremal tuple on \([s^-, s^+]\) and (ii) follows from Theorem 5.5. Statement (iii) follows from statement (E1) of Section 6. Note that \( \varepsilon_{\text{centr}} \neq n_\infty^2 \) because \( \pm n_\infty \) are equilibrium solutions to the equation \( x' = f(x, n_\infty^2) \).

\[ \boxed{} \]

Remark 7.1. The case \( x(0) = \infty \) of Corollary 7.1 takes place for the example of a minimal time extremal that is constructed below in Theorem 7.4.

In the case when the extremal tuple \((7.1)\) is abnormal (i.e., \( \lambda_0 = 0 \)) and \( I_{\text{centr}} \neq (s^-, s^+) \), the lengths of all layers are described by Theorem 5.5. That is, \(|I_{\text{centr}}| = \frac{\pi}{2\varepsilon_{\text{centr}} \Re \kappa} \), \(|I^\pm| = \frac{\pi}{2\varepsilon_{\text{edge}} \Re \kappa} \), and for the each layer \( I = (b_j, b_{j+1}) \neq I_{\text{centr}} \) we have \(|I| = \frac{\pi}{2\varepsilon_{\text{edge}} \Re \kappa} (\text{these layers form two quarter-wave stacks, one on the left side of } I_{\text{centr}}, \text{and the other on the right side). Under the additional condition } n_\infty \in [n_1, n_2], \text{ statement (E1) of Section 6 implies that the following interplay between } x(0) \text{ and the values } \varepsilon_{\text{centr}}, \varepsilon_{\text{edge}} \text{ take place: in the case } x(0) = \infty \text{ (in the case } x(0) = 0), \text{ the equality } \varepsilon_{\text{centr}} = \varepsilon_{\text{edge}} = n_2^2 \) (resp., \( \varepsilon_{\text{centr}} = \varepsilon_{\text{edge}} = n_1^2 \)) holds.

Corollary 7.2. Assume \((7.1)\) and \( \kappa \in \mathbb{C}_4 \). Then the following statements hold:

(i) If \( x(0) = \infty \) and \( n_\infty = n_2 \) (if \( x(0) = 0 \) and \( n_\infty = n_1 \)), then \( \lambda_0 > 0 \).

(ii) If \( \lambda_0 > 0 \) and \( x(0) = 0 \), then \( \varepsilon_{\text{centr}} = n_2^2 \).

Proof. (i) follows from statement (E1) of Section 6. (ii) follows from Theorem 4.1.

Consider now the length of the layers in the normal case \( \lambda_0 > 0 \).

Corollary 7.3. Assume \( \kappa \in \mathbb{C}_4 \), \((7.1)\), \( \lambda_0 > 0 \), and \( I_{\text{centr}} \neq (s^-, s^+) \). Then the following statements hold:

(i) Let \( I \subset (s^-, s^+) \) be a layer that is not an edge layer and is not the central layer. Then \( I = (b_j, b_{j+1}) \) for a certain \( j \in \mathbb{Z} \). Moreover, \(|I| < \frac{\pi}{2n_1 \Re \kappa} \) (\(|I| > \frac{\pi}{2n_2 \Re \kappa} \) whenever \( \varepsilon(b_j + 0) = n_1^2 \)) (resp., whenever \( \varepsilon(b_j + 0) = n_2^2 \)).

(ii) If \( \varepsilon_{\text{edge}} = n_2^2 \), then \(|I^\pm| < \frac{\pi}{2n_1 \Re \kappa} \). In particular, this is the case when \( n_\infty = n_2 \).

(iii) If \( x(0) = 0 \), then \(|I_{\text{centr}}| < \frac{\pi}{\varepsilon_{\text{centr}} \Re \kappa} \).

(iv) Let \( x(0) = \infty \) and \( n_\infty \in [n_1, n_2] \). Then, \(|I_{\text{centr}}| > \frac{\pi}{n_2 \Re \kappa} \) in the case \( \varepsilon_{\text{centr}} = n_2^2 \), and \(|I_{\text{centr}}| < \frac{\pi}{n_1 \Re \kappa} \) in the case \( \varepsilon_{\text{centr}} = n_1^2 \).

Proof. (i) follows from the definition of a layer, the definitions of central and edge layers, and from Theorems 5.8 and 5.9.

(ii) follows from Theorems 5.8 and 5.9. Another proof can be obtained from (i). Indeed, one can continue the normal extremal tuple \((x, \varepsilon, \lambda_0, y)\) to a wider symmetric interval.
Consider the extensions of the original edge layers, which become non-edge layers for the extended extremal tuple.

(iii) follows from the assumption that the extremal tuple is normal and the process of computation of the position of the switch point \( b_{-1} \) from the position of the preceding switch point \( \tilde{b}_{-1} \) (see Lemma 5.2).

Consider the case \( \epsilon_{\text{entr}} = n_1^2 \) of statement (iv). It follows from (E2) that \( \epsilon(\cdot) \) corresponds either to the sequence \((6.6)\), or the sequence \((6.8)\). This implies that \( x(s) \) in Sec2 + C for all \( s \) in the left half \( (b_{-1}, 0) \) of the layer \( I_{\text{entr}} \). Considering the corresponding part of the trajectory of \( \vartheta(s) \) (see (5.4) for the definition) in the way similar to the proof of Lemma 5.2, one obtains \( |b_{-1}| < \frac{\pi}{2n_1 \Re \kappa} \). This implies \( |I_{\text{entr}}| < \frac{\pi}{2n_2 \Re \kappa} \).

Let us consider the case \( \epsilon_{\text{entr}} = n_2^2 \) of statement (iv). It follows from (E2) that \( x(\tilde{b}_{-1}) \) in Sec4 + C. Thus, the trajectory of \( x \) passes through \( \mathbb{R}_- \) to Sec3 + C and going through Sec3 reaches \( \infty \) at \( s = 0 \). Considering the rotation of \( \vartheta(s) \) for \( s \in (\tilde{b}_{-1}, 0) \), we see that \( |b_{-1}| > \frac{\pi}{2n_2 \Re \kappa} \), and so \( |I_{\text{entr}}| > \frac{\pi}{2n_2 \Re \kappa} \).

Finally, let us give an explicit example of minimum-time control from \( \infty \) to \( n_{\infty} \in [n_1^2, n_2^2] \).

**Theorem 7.4.** Let \( n_{\infty} \in [n_1, n_2] \), \( s^+ = -s^- > 0 \), and

\[
\kappa_0 = \frac{1}{(s^+ - s^-)n_2} \left( \ln \frac{n_2 + n_{\infty}}{n_2 - n_{\infty}} + \frac{\pi}{(s^+ - s^-)n_2} \right).
\]

Then \( \epsilon_0(s) := \begin{cases} n_1^2, & s \in [s^-, s^+] \\ n_2^2, & s \in \mathbb{R} \setminus [s^-, s^+] \end{cases} \) is the unique minimizer of the problem

\[
\arg\min_{\kappa_0 \in \Sigma_{\text{odd}}(\epsilon)} \ell(\epsilon) \quad \text{and the corresponding minimal value of } \ell(\epsilon) \text{ equals } \epsilon_{\min}^{\text{odd}}(\epsilon_0) = 2s^+.
\]

### 7.1 Proof of Theorem 7.4

It follows from (3.12) that \( \kappa_0 \in \Sigma_{-s^-, s^+}(\epsilon_0) \) and that the control \( \epsilon_0 \) steers \( x(-s^+) = n_{\infty} \) to \( x(0) = \infty \). By (3.11) and (3.7), \( \vartheta(\cdot) \) defined by (5.4) for the control \( \epsilon = \epsilon_0 \) equals \( \vartheta_{n_2}(s) = \frac{1}{(n_2 + n_{\infty}) / (n_2 - n_{\infty})} e^{\pi s^+ / s^-} e^{-\pi s / s^-} \) for \( |s| \leq s^+ \). This implies that \( \vartheta_{n_2}(s) \in \mathbb{C}_- \) and \( x(s) \in \mathbb{C}_+ \) for \( s \in [s^-, 0] \). By Lemma 5.2 (iii), there exists an abnormal extremal tuple \( (x_0, \epsilon_0, 0, y_0) \) on \([s^-, s^+]\) so that \( x_0(s^+) = \pm n_{\infty} \). It is of the type \((6.2)\).

Assume that an extended extremal control \( \epsilon_1 \) is such that \( \epsilon_1(\cdot) \neq n_2^2 \) on a certain interval \([s^-, s^- + t]\) and that \( \epsilon_1 \) steers \( x(s^-) = n_{\infty} \) to \( x(s^- + t) = \infty \). Then \( x(s^- + 2t) = n_{\infty} \), the corresponding extended extremal tuple \( (x_1, \epsilon_1, \lambda_1, y_1) \) is symmetric w.r.t. \( s^- + t \), and is of one of the types \((6.3)-(6.11)\) on \([s^-, s^- + 2t]\). Since \( x_1(\cdot) \) passes through \( \infty \) only the structures \((6.4), (6.6), (6.8), (6.10), \) and \((6.11)\) are possible. The symmetry w.r.t. \( s^- + t \) implies that \( m_1 = m_2 \) for \((6.6)-(6.11)\).

We assume that \( \epsilon_1 \) is a minimum time control that steers \( x_1(s^-) = (-n_{\infty}) \) to \( \infty \) in the minimum possible time \( t \) (in particular, \( t \in (0, s^+] \) and \( s^- + t \leq 0 \)), and, in several steps, show that this leads to a contradiction considering on each step one of the cases \((6.4), (6.8), \) \((6.10), \) \((6.11), \) and \((6.6)\). The case \((6.6)\) is special and will require more technically involved arguments.

**The case \((6.4)\)** assumes \( m \geq 1 \) repetitions and is abnormal. Due to Theorem 5.5, its part \((-n_2, -n_1) \) requires the time \( \frac{\pi}{2n_2 \Re \kappa_0} = s^+ \) (the last equality follows from (7.2)).
Since we have assumed that \( \varepsilon_1 \) is minimum-time from \((-n_\infty, 0)\) to \(\mathbb{C}\), we have \( \varepsilon_1 = 0 \) on \((s^-, 0)\), and so \( x_1(0) = x_1(s^- + s^+) = 0 \). This contradicts to the fact that, according to (6.4), \( x_1(s^- + s^+) \in (-\infty, -n_2) \).

The case (6.8) cannot correspond to the minimal time control from \((-n_\infty)\) to \(\mathbb{C}\) since Corollary 7.3 (i) implies that the part \( +(-n_2, 0) + \) requires time greater than \( s^+ \) (note that this part corresponds to one ordinary layer), and so \( t > s^+ \).

The same argument is applicable to the case (6.11), and almost the same argument (with the use of Corollary 7.3 (iv)) to the sub-case \( m_1 = m_2 \geq 1 \) of (6.10).

Consider the sub-case \( m_1 = m_2 = 0 \) of (6.11). We have \( \varepsilon_0 = \varepsilon_1 \equiv n_2^2 \) and \( x_0(\cdot) = x_1(\cdot) \) on \([s^-, s^+]\). Thus, this case leads to the same minimizer (however, the extremal tuple is normal).

Let the sub-case \( m_1 \geq 1 \) (or \( m_2 \geq 1 \)) of (6.6) take place. Then Corollary 7.3 (i) implies that the sequence \( +(-n_2, 0) + \) requires the time greater than \( s^+ \). Thus, the control \( \varepsilon_1 \) is not minimum-time (the arguments in the case \( m_2 \geq 1 \) are similar).

In the rest of this subsection we assume that the case (6.6) take place and that \( m_1 = m_2 = 0 \). Then there exists maximal \( s_0 \in (s^-, s^- + t) \) such that, for a.a. \( s \in (s^-, s_0) \), we have \( \varepsilon_0(s) = \varepsilon_1(s) \equiv n_2^2 \). So \( x_0(s_0) = x_1(s_0) \in \text{Sec}_3, \varepsilon_1(s) = n_2^2 \) for \( s \in (s_0, s^- + t) \), \( x_0(s) \neq x_1(s) \) for a.a. \( s \in (s_0, 0) \).

Consider an extended abnormal extremal tuple \((x_2, \varepsilon_2, 0, y_2)\) such that for \( s \in (s_0, s^- + t) \), we have \( \varepsilon_2(s) = \varepsilon_1(s) = n_2^2 \) and \( x_2(s) = x_1(s) \). Then \( \varepsilon_2(s) = n_1^2 \) for \( s \in (t_2^-, s^- + t) \), where \( t_2^- = s^- + t - \frac{\pi}{2 \varepsilon_1 \Re \varepsilon} \). On the interval \((t_2^-, t_2^+)\), where \( t_2^+ = s^- + t + (s^- + t - t_2^-) \), the sequence corresponding to \((x_2, \varepsilon_2, 0, y_2)\) is \((-n_1, 0) \rightarrow \{\infty\} \rightarrow (0, n_1)\). In particular, \( x_2(t_2^+) \in (-n_1, 0)\).

Consider a piece-wise smooth Jordan curve \( J_3 \) in \( \mathbb{C} \) consisting of two pieces: \( J_1 : [s^-, 0] \rightarrow \{-n_\infty\} \cup \text{Sec}_3 \cup \{\infty\}, J_1(s) = x_0(s) \), and \( J_2 : [0, n_\infty^-] \rightarrow \{\infty\} \cup (-\infty, -n_\infty], J_2(s) = -1/s \). Its complement in \( \mathbb{C} \) consists of two components \( C_0 \) and \( C_{\mathbb{C}_-} \), where \( C_0 \) is singled out by \( 0 \in C_0 \).

**Lemma 7.5.** The trajectory \( x_2([t_2^-, s^- + t]) \) intersects \( J_1 \) passing through the point \( X_0 := x_2(s_0) = x_1(s_0) \in \text{Sec}_3 \) from \( C_0 \) to \( C_{\mathbb{C}_-} \).

**Proof.** The tangent vector \( f(X_0, n_2^2) \) to \( J_1 \) at \( X_0 \) (strictly speaking, the tangent vector \( f_1(X_0, n_2^2) \partial x_1 + f_2(X_0, n_2^2) \partial x_2 \) in the chart \( \{C, \varphi(x)\} \)) differs from the tangent vector \( f(X_0, n_2^2) \) to the trajectory of \( x_2 \) at \( X_0 \) in the following way \( f(X_0, n_2^2) - f(X_0, n_2^2) = -i\kappa(n_2^2 - n_2^2) \). The angle between this difference and the the tangent vector to \( J_1 \) at \( X_0 \) is

\[
\text{Arg} \frac{-i\kappa(n_2^2 - n_2^2)}{f(X_0, n_2^2)} = \text{Arg} \frac{-i\kappa}{-i\kappa} = -\text{Arg}(-n_2^2 + X_2^2) \in (0, \pi)
\]

(the last inclusion holds since \( X_0 \in \text{Sec}_3 \subset \mathbb{C}_2 \)). This implies the statement of the lemma. \( \square \)

Let us introduce one more abnormal extended extremal tuple \((x_3, \varepsilon_3, 0, y_3)\), which can be considered as a perturbation of \((x_2, \varepsilon_2, 0, y_2)\) because we put \( x_3(t_2^-) = x_2(t_2^-) + \delta_0 \) with small enough \( \delta_0 > 0 \) and assume that \( \varepsilon_3(t_2^+ + 0) = n_2^2 = \varepsilon_2(s) \) for \( s \in (t_2^-, s^- + t) \).

We impose the following conditions on \( \delta_0 \):

(i) \( x_2(t_2^-) + \delta_0 \in (x_2(t_2^-), 0) \),

(ii) \( x_3([t_2^-, s^- + t]) \) intersects \( J_1 \) at a certain point \( X_1 \in \text{Sec}_3 \) passing from \( C_0 \) to \( C_{\mathbb{C}_-} \).
Note that condition (ii) is fulfilled for small enough $\delta_0$ satisfying (i) because of Lemma 7.5.

Consider the point $s_3$ such that $x_3(s_3) \in i\mathbb{R}$. Since $(x_3, \varepsilon_3, 0, y_3)$ is abnormal, we have $\varepsilon_3(s) = n_1^2$ for $s \in (t_2^-, s_3)$. This implies that $x_2[[t_2^-, s^-] + t] \cap x_3[[t_2^-, s_3]] = \emptyset$ (otherwise these trajectories coincide, but this contradicts $\delta_0 > 0$). Hence, $x_3(s_3) \in i\mathbb{R}_+$. This and the condition (ii) on $\delta_0$ implies that

there exist $X_2 \in \mathcal{J}_1 \cap \text{Sec}_3$ such that $x_3[[t_2^-, s^-] + t]$ intersects $\mathcal{J}_1$ at $X_2$

passing from $C_{C_-}$ to $C_0$. \hfill (7.3)

Considering at $X_2$ the values of the tangent vectors to $\mathcal{J}_1$ and to $x_3[[t_2^-, s_3]]$ (which are equal to $f(X_2, n_2^3)$ and $f(X_2, n_1^3)$, resp.), we see that (7.3) is impossible. This contradiction completes the proof of Theorem 7.4

8 Back to the problem of decay rate minimization

The main goal of this section is to prove that the original problem \cite{18, 20} of Pareto optimization of the decay rate of a resonator can be completely reduced to the dual problem of minimization for the length of Section 2 in the case

\begin{equation}
\eta \in \mathbb{R}_- \cup \{0, \infty\} \text{ and } \eta_+ \in \mathbb{R}_+.
\end{equation}

8.1 Main results on the decay rate minimization

Let $s^\pm$ and $\eta^\pm$ be fixed such that $-\infty < s^- < s^+ < +\infty$ and $\eta_- \neq \eta_+$. Recall that the set $\Sigma^s_\eta^-_s^+ \mathbb{F}_s^-$ of achievable $(\eta_-, \eta_+)$-eigenvalues and the associated minimal modulus function $\rho_{\text{min}}(\gamma)$ were introduced in Section 2 and that $\rho_{\text{min}}(\gamma) = +\infty$ if the complex argument $\gamma$ is not achievable (i.e., $\text{dom} \rho_{\text{min}} = \text{Arg} \Sigma^s_\eta^-_s^+ \mathbb{F}_s^-$).

Assume for a time being that

\begin{equation}
\eta_- \in \mathbb{R}_- \cup \{0, \infty\} \text{ and } \eta_+ \in \mathbb{R}_+.
\end{equation}

Then an $(\eta_-, \eta_+)$-eigenvalue $\kappa$ of $\varepsilon(\cdot) \in \mathbb{F}_{s-, s+}$ can be interpreted \cite{20} as a resonance of a self-adjoint in $L^2(\mathbb{R}, \varepsilon(s)ds)$ operator $\frac{i}{\varepsilon} \partial_x^2$, where $\varepsilon(\cdot)$ is extended to the whole $\mathbb{R}$ in a suitable way according to the values of $\eta_{\pm}$ (the set of corresponding resonances may also contain points that are not $(\eta_-, \eta_+)$-eigenvalues, see Section 2 and \cite{20} for details). In the case under consideration, $\text{Re} \kappa$ and $(-\text{Im} \kappa)$ have the 'physical meaning' of frequency and decay rate, respectively. Following \cite{18}, we define the minimal decay rate $\beta_{\text{min}}(\alpha)$ for a frequency $\alpha \in \mathbb{R}$ by

\begin{equation}
\beta_{\text{min}}(\alpha) := \inf \{ \beta \in \mathbb{R} : \alpha - i\beta \in \Sigma^s_\eta^-_s^+ \mathbb{F}_s^- \}.
\end{equation}

Since \cite{8.1} implies $\Sigma^s_\eta^-_s^+ \mathbb{F}_s^- \subset \mathbb{C}_-$, we see that $\beta_{\text{min}}(\alpha) : \mathbb{R} \to [0, +\infty)$ (actually, $\beta_{\text{min}}(\alpha) > 0$ for all $\alpha$ because $\Sigma^s_\eta^-_s^+ \mathbb{F}_s^-$ is closed, see \cite{20} and Subsection 8.2 below). A frequency

\begin{equation}
\alpha \in \mathbb{R} \text{ is called achievable if } \alpha \in \text{dom} \beta_{\text{min}} \left( = \text{Re} \Sigma^s_\eta^-_s^+ \mathbb{F}_s^- \right).
\end{equation}
If $\kappa = \alpha - i\beta_{\min}(\alpha)$ is an $(\eta_-, \eta_+)$-eigenvalue for a certain $\varepsilon(\cdot) \in \mathbb{F}_{s^-, s^+}$ (i.e., the minimum is achieved in (8.2)), we say that $\kappa$ and $\varepsilon(\cdot)$ are of minimal decay for the frequency $\alpha$. The set $\{\alpha - i\beta_{\min}(\alpha) : \alpha \in \text{dom } \beta_{\min}\}$ forms the Pareto optimal frontier for the problem of minimization of the decay rate $(-\Im \kappa)$ of an $(\eta_-, \eta_+)$-eigenvalue $\kappa$ over $\mathbb{F}_{s^-, s^+}$.

**Theorem 8.1.** Assume \((8.1)\) and additionally $\eta_+ \in (n_1, n_2)$. Let $\gamma_0 = \text{Arg}_0 \kappa \in (-\pi/2, 0)$. Then:

(i) The function $\rho_{\min}(\cdot)$ is continuous and $\mathbb{R}$-valued on $(-\pi/2, 0)$.

(ii) Assume additionally that $\kappa$ is the $(\eta_-, \eta_+)$-eigenvalue of minimal decay for the frequency $\Re \kappa$. Then

$$\kappa = \rho_{\min}(\gamma_0)e^{i\gamma_0}, \quad T_{\kappa}^{\min}(\eta_-, \eta_+) = s^+ - s^-,$$

\((i.e., \kappa\) is an $(\eta_-, \eta_+)$-eigenvalue of minimal modulus for $\gamma_0)\), and

the family of minimum time controls for \((2.9), (2.10)\) steering $x(s^-) = \eta_-$ to $\eta_+$ coincides with the family of resonators of minimal decay for $\Re \kappa$. \((8.5)\)

The proof is postponed to Section 8.3.

The next three theorems explain the rigorous meaning of the statement that, in the case $n_\infty \in (n_1, n_2)$, the decay rate minimization can be ‘partially reduced’ to the problem of Pareto optimization of $|\kappa|$, and so to the length minimization.

**Theorem 8.2.** Assume \((8.1)\) and additionally $\eta_+ \in [n_1, n_2]$. Then:

(i) $(-\pi, 0) \setminus \{-\pi/2, 0\} \subset \text{dom } \rho_{\min} \subset (-\pi, 0)$ and

$$\Sigma_{\eta_-, \eta_+}^{s^-, s^+}[\mathbb{F}_{s^-}] = \{ce^{i\gamma} \rho_{\min}(\gamma) : c \in [1, +\infty) \text{ and } \gamma \text{ is achievable} \} \quad (8.6)$$

(ii) The Pareto frontier $\{\alpha - i\beta_{\min}(\alpha) : \alpha \in \text{dom } \beta_{\min}\}$ of minimal decay can be found from the Pareto frontier $\{\rho_{\min}(\gamma)e^{i\gamma} : \gamma \in \text{dom } \rho_{\min}\}$ of minimal modulus via the formulae \((8.0)\) and \((8.2)\).

The proof is postponed to Section 8.3.

**Theorem 8.3.** Suppose \((8.1)\). Assume that $\eta_+ = n_1$ or $\eta_+ = n_2$. Assume that $\eta_- \in \{\infty, 0\} \cup \mathbb{R}_-$, $\gamma_0 = \text{Arg}_0 \kappa_0 \in (-\pi/2, 0)$, $\rho_0 := \rho_{\min}(\gamma_0)$, $\rho_1 := \inf_{\gamma \in (\gamma_0, 0)} \frac{\rho_{\min}(\gamma)}{\cos \gamma_0}$, and that $\kappa_0$ is the $(\eta_-, \eta_+)$-eigenvalue of minimal decay for the frequency $\Re \kappa_0$. Then $\rho_0 \leq \rho_1$ and $\kappa_0 \in \e^{i\gamma_0}[\rho_0, \rho_1]$ (in particular, in the case $\rho_0 = \rho_1$, we have $\kappa_0 = \e^{i\gamma_0}\rho_0$).

Moreover, each of the numbers $\kappa = \rho e^{i\gamma_0}$ with $\rho \in [\rho_0, |\kappa_0|] \cup (\rho_0, \rho_1)$ is the $(\eta_-, \eta_+)$-eigenvalue of minimal decay for the frequency $\rho \cos \gamma_0$ and one of the two following cases take place for such $\kappa$:

(i) In the case $|\kappa| = \rho_0$, statements \((8.4)\) and \((8.5)\) hold.

(ii) In the case $|\kappa| \in (\rho_0, |\kappa_0|] \cup (\rho_0, \rho_1)$, we have $T_{\kappa}^{\min}(\eta_-, \eta_+) < s^+ - s^-$ and, for each minimum time control $\varepsilon_{\gamma_0}^{\min}$ that steer $x(s^-) = \eta_-$ to $x(s^- + T_{\kappa}^{\min}(\eta_-, \eta_+)) = \eta_+$, the function $\varepsilon \in \mathbb{F}_{s^-, s^+}$ defined by

$$\varepsilon(s) := \begin{cases} \varepsilon_{\gamma_0}^{\min}(s), & \text{if } s \in [s^-, s^- + T_{\kappa}^{\min}(\eta_-, \eta_+)], \\ \varepsilon_{\gamma_0}^{\min}(\eta_+), & \text{if } s \in (s^- + T_{\kappa}^{\min}(\eta_-, \eta_+), s^+], \end{cases} \quad (8.7)$$

The proof is postponed to Section 8.3.
is a resonator of minimal decay for frequency $\text{Re} \kappa$ and, simultaneously, is an abnormal $\varepsilon$-extremal on $[s^-, s^+]$ such that the corresponding abnormal $x$-extremal (with the initial point $x(s^-) = \eta_-$) has $[s^- + T^\kappa_{\eta_-}(s^-), s^+]$ as its stationary interval.

The proof is given in Section 8.3.

The case (ii) of Theorem 8.3 and statements (iii)-(v) of Corollary 8.8 in the next subsection consider the situation when the minimal-modulus function $\rho_{\min}(\cdot)$ is discontinuous at a certain $\gamma_0 \in (-\pi/2, 0)$. The next result shows that this situation takes place for $\gamma_0 = \text{Arg}_0 \kappa_0$ with $\kappa_0$ from Theorem 7.4.

**Theorem 8.4.** Let $\eta_- = \infty$, $\eta_+ = n_1$, $s^- = 0$, $s^+ > 0$, $\kappa_0 = -\frac{i}{2s^+ n_2} \ln \frac{2s^+ n_2}{n_2 - n_1} + \frac{\pi}{2s^+ n_2}$, $\gamma_0 = \text{Arg}_0 \kappa_0$, $\alpha = \frac{\pi}{2s^+ n_2}$, $\rho_0 = \rho(\gamma_0)$, and $\rho_1 := \inf_{\gamma \in (\gamma_0, 0)} \frac{\rho_{\min}(\gamma) \cos \gamma}{\cos \gamma_0}$. Then

$$\lim_{\gamma \to \gamma_0^+} \rho_{\min}(\gamma) = \rho_0 = |\kappa_0| < \rho_1, \quad \rho_0 < \lim_{\gamma \to \gamma_0^+} \rho_{\min}(\gamma),$$

(8.8)

and the following statements hold:

(i) The number $\kappa_0$ is simultaneously $(\eta_-, \eta_+)$-eigenvalue of minimal decay for the frequency $\alpha$ and $(\eta_-, \eta_+)$-eigenvalue of minimal modulus for the complex argument $\gamma_0$. Moreover, $\varepsilon_{\gamma_0}^{\min}(\cdot)$ defined by $\varepsilon_{\gamma_0}^{\min}(s) = n_2^2$ for all $s \in (0, s^+]$ is the unique function in $F_{0, s^+}$ that generate an $(\eta_-, \eta_+)$-eigenvalue at $\kappa_0$.

(ii) Each of the numbers $\kappa = \rho_0 e^{i\gamma_0}$ with $\rho \in [\rho_0, \rho_1]$ is the $(\eta_-, \eta_+)$-eigenvalue of minimal decay for the frequency $\rho \cos \gamma_0$. One of associated resonators $\varepsilon(\cdot) \in F_{0, s^+}$ of minimal decay can be constructed by the rule (8.7).

The proof is given in Section 8.3.

### 8.2 Maximal star-like quasi-eigenvalue free regions

Recall that a set $\Omega \subset \mathbb{C}$ containing 0 is called star-shaped w.r.t. 0 if $z \in \Omega$ implies $[z, 0] \subset \Omega$.

The set $\mathbb{C} \setminus \left(\{0\} \cup \Sigma_{-n_\infty, n_\infty}[F_{s^-}]\right)$ can be perceived as the resonance free region over $\{\varepsilon \in \mathbb{F} : s^-_\varepsilon = s^-, s^+_\varepsilon = s^+\}$. (We would like to notice that the estimation of resonance free strips for Schrödinger equations was originally [15] one of the main motivation for resonance optimization.) Then

$$\{0\} \cup \{\kappa \in \mathbb{C} : \kappa \neq 0 \text{ and } |\kappa| < \rho_{\min}(\text{Arg}_0(\kappa), -n_\infty, n_\infty)\}$$

(8.9)

is the maximal star-shaped (w.r.t. 0) part of the quasi-eigenvalue free region. If, additionally, $n_1 \leq n_\infty \leq n_2$, the star-shaped set (8.9) exactly equals the quasi-eigenvalue free region as it is shown by the following statement.

**Proposition 8.5.** Let $\eta_- \neq \eta_+$ and $\eta_+ \in [n_1, n_2]$ (or $\eta_- \in [-n_2, -n_1]$). Then $(-\pi, 0) \cup (0, \pi) = \text{Arg}_0 \Sigma_{-\eta_-, \eta_+}[F_{s^-}]$ and formula (8.6) holds true.

**Proof.** Let $\gamma$ be achievable. Then (2.20) and $n_1 \leq n_+ \leq n_2$ imply that an extension $\varepsilon(\cdot)$ of $\varepsilon_{\gamma}^{\min}(\cdot)$ to $(s^+, +\infty)$ by the constant value $\eta^2_\gamma$ belongs to $F_{s^+, s^\gamma}$ for any $s_\gamma \in [s^+, +\infty)$. Moreover, for every such $\gamma$, we have $x(s) = x(s^+) = \eta_+$ for all $s > s^+$. This and the scaling (2.17) imply that $ce^{i\gamma}\rho_{\min}(\gamma)$ is an achievable $(\eta_-, \eta_+)$-eigenvalue for all $c \geq 1$. The global controllability (Theorem 8.3) implies the first statement and completes the proof. □
Let the assumptions of statement (iv) hold and, additionally, assume that

\[
\eta
\]

(iii) Let \(\gamma\) be a point of discontinuity of \(\rho\). Then \(\lim_{\gamma\to\gamma_0}\rho(\gamma) = \rho(\gamma_0)\) if and only if \(\rho(\gamma) = \rho(\gamma_0)\) for all \(\gamma \in (\gamma_0, 0)\). Proposition 8.5 completes the proof. \(\square\)

So \(\Sigma_{s-s+}[F]\) contains its boundary \(\text{bd} \Sigma_{s-s+}[F]\), contains the set \(\{e^{i\gamma}\rho_{\min}(\gamma) : \gamma \in \text{dom} \rho_{\min}\}\), and, in the case (8.1), contains also \(\{\alpha - i\beta(\alpha) : \alpha \in \text{dom} \beta\}.\)

**Proposition 8.7.** Assume that (8.7) holds and \(\eta_+ \in [n_1, n_2]\) (or \(\eta_- \in [-n_2, -n_1]\)). Let \(\text{Arg} \ k_0 = \gamma_0 \in (-\pi/2, 0)\) and \(k_0 \in \text{bd} \Sigma_{\eta_-, \eta_+}[F_{s-}]\). Then \(k_0\) is an \((\eta_-, \eta_+)-eigenvalue\) of minimal decay if and only if \(\text{Re} k_0 < \rho_{\min}(\gamma) \cos(\gamma)\) for all \(\gamma \in (\gamma_0, 0)\). Let \(\text{Arg} \ k_0 = \gamma_0 \in (-\pi/2, 0)\) and \(k_0 \in \text{bd} \Sigma_{\eta_-, \eta_+}[F_{s-}]\). Then \(\gamma_0 \in (\gamma_0, 0)\) does not intersect \(\Sigma_{\eta_-, \eta_+}[F_{s-}]\). Proposition 8.5 completes the proof.

**Corollary 8.8.** Assume (8.7) and \(\gamma_0 \in (-\pi, \pi)\). Then:

(i) \(\lim_{\gamma \to \gamma_0^-} \rho_{\min}(\gamma) = +\infty\) and \(\rho_{\min}(\gamma) = +\infty\) for all \(\gamma \in [0, \pi]\).

(ii) If \(\lim_{\gamma \to \gamma_0^-} \rho_{\min}(\gamma) \neq \rho_{\min}(\gamma_0) \neq \lim_{\gamma \to \gamma_0^+} \rho_{\min}(\gamma)\), then \(\gamma_0 = -\pi/2\) and \(\rho_{\min}(\gamma) < +\infty\).

(iii) Let \(\gamma_0 \in (-\pi/2, 0)\) be a point of discontinuity of \(\rho_{\min}\). Then either \(\lim_{\gamma \to \gamma_0^-} \rho_{\min}(\gamma) = \rho_{\min}(\gamma_0)\), or \(\lim_{\gamma \to \gamma_0^+} \rho_{\min}(\gamma) = \rho_{\min}(\gamma_0)\).

(iv) Assume that \(\eta_+ \in [n_1, n_2]\) (or \(\eta_- \in [-n_2, -n_1]\)), \(\text{Arg} \ k_0 = \gamma_0 \in (-\pi/2, 0)\), \(k_0 \in \text{bd} \Sigma_{\eta_-, \eta_+}[F_{s-}]\), and \(\eta_0 \neq \rho_{\min}(\gamma_0)e^{i\gamma_0}\). Then

\[
\text{either} \quad \lim_{\gamma \to \gamma_0^-} \rho_{\min}(\gamma) = \rho_{\min}(\gamma_0) < \lim_{\gamma \to \gamma_0^+} \rho_{\min}(\gamma),
\]

\[
\text{or} \quad \lim_{\gamma \to \gamma_0^+} \rho_{\min}(\gamma) = \rho_{\min}(\gamma_0) < \lim_{\gamma \to \gamma_0^-} \rho_{\min}(\gamma).
\]

Moreover, \(e^{i\gamma_0}[\rho_{\min}(\gamma_0), |\kappa_0|] \subset \text{bd} \Sigma_{s-s+}[F_{s-}]\).

(v) Let the assumptions of statement (iv) hold and, additionally, \(k_0\) is the \((\eta_-, \eta_+)-eigenvalue\) of minimal decay for the frequency \(\text{Re} k_0\). Then (8.10) takes place and \(\text{Re} k_0 < \rho(\gamma) \cos(\gamma)\) for all \(\gamma \in (\gamma_0, 0)\). Moreover, each \(\kappa \in e^{i\gamma_0}[\rho_{\min}(\gamma_0), |\kappa_0|]\) is the \((\eta_-, \eta_+)-eigenvalue\) of minimal decay for the frequency \(\text{Re} k_0\).

Proof. (i) If \(\gamma \in [0, \pi]\), the equality \(\rho_{\min}(\gamma) = +\infty\) follows from \(\Sigma_{\eta_-, \eta_+}[F_{s-}] \subset C_-\) (see, e.g., Section 3.1 and [20]). Then \(\lim_{\gamma \to \gamma_0^-} \rho_{\min}(\gamma) = +\infty\) follows from the lower semicontinuity of \(\rho_{\min}\) (see Proposition 8.6).

(ii) The lower semicontinuity of \(\rho_{\min}\) also proves that \(\rho_{\min}\) is continuous at each \(\gamma_0 \in \text{dom} \rho_{\min}\) (as a map to the topological space \([0, +\infty]\)). Now, let \(\gamma_0 \in (-\pi, -\pi/2) \cup (-\pi/2, 0)\) be a point of discontinuity of \(\rho_{\min}\). In this case, (ii) follows from (iii).

Let us prove (iii). Since \(\gamma_0\) is a point of discontinuity of \(\rho_{\min}\), there exists the resonance \(\kappa_0 = \rho_{\min}(\gamma_0)e^{i\gamma_0} \in C_4\) of minimum modulus for the argument \(\gamma_0\). It follows from
the perturbation arguments of [20 Appendix A] that there exists a non-degenerate triangle $\mathbf{Tr} \subset \Sigma_{s^{-},s^{+}}^{\eta_{-},\eta_{+}}[F_{s^{-}}]$ with a vertex at $\kappa_{0}$. This implies that at least one of inequalities $\limsup_{\gamma \to \gamma_{0}} \rho_{\min}(\gamma) \leq \rho_{\min}(\gamma_{0})$. The lower semicontinuity completes the proof of (iii), and, in turn, of (ii).

(iv)-(v) By Proposition 8.5 the quasi-eigenvalue free region $C \backslash \Sigma_{s^{-},s^{+}}^{\eta_{-},\eta_{+}}[F_{s^{-}}]$ is star-shaped w.r.t. 0 and open. So $\kappa_{0} \in \operatorname{bd} \Sigma_{s^{-},s^{+}}^{\eta_{-},\eta_{+}}[F_{s^{-}}]$ implies $e^{i\gamma_{0}}[\rho_{\min}(\gamma_{0}),|\kappa_{0}|] \subset \operatorname{bd} \Sigma_{s^{-},s^{+}}^{\eta_{-},\eta_{+}}[F_{s^{-}}]$ (indeed, assuming converse we easily get a contradiction to the star property). The combination of these arguments with (iii) implies that exactly one of the formulæ (8.10), (8.11) holds. Finally, statement (v) easily follows from (iv) and Proposition 8.7.

8.3 Proofs of Theorems 8.1-8.4

Proof of Theorem 8.1 (i) It follows from Proposition 8.5 that $\rho_{\min} \supset (-\pi/2,0)$, and so $\rho_{\min}$ is $\mathbb{R}$-valued in $(-\pi/2,0)$.

Due to Proposition 8.6, to prove $\rho_{\min}(\cdot) \in C_{\text{loc}}(-\pi/2,0)$, it remains to show that $\rho_{\min}$ is upper semicontinuous at each $\gamma_{0} \in (-\pi/2,0)$. This fact follows from the uniform STLC established in Theorem 3.4 (and so uses essentially the assumption $\eta_{\pm} \in (n_{1},n_{2})$). Indeed, for $\gamma \in [\gamma_{0} - \delta, \gamma_{0} + \delta] \subset (-\pi/2,0)$ (with small enough $\delta$) we take $\kappa \in C_{4}$ with $\arg\kappa_{0} = \gamma_{0}$ and put $\overline{\kappa}(\gamma) := e^{i\gamma}|\kappa|$. Then (2.21) takes the form $T_{\overline{\kappa}(\gamma)}^{\rho_{\min}}(\eta_{-},\eta_{+}) = (s^{+} - s^{-})\rho_{\min}(\gamma)/|\kappa|$ and it is enough to prove that

$$
\limsup_{\gamma \to \gamma_{0}} T_{\overline{\kappa}(\gamma)}^{\rho_{\min}}(\eta_{-},\eta_{+}) \leq t_{0}, \text{ where } t_{0} = T_{\kappa}^{\rho_{\min}}(\eta_{-},\eta_{+}). 
$$

Let $\varepsilon_{0}(\cdot)$ be a control that steers the system (2.9), (2.10) from $x(s_{-}) = \eta_{-}$ to $\eta_{+}$ in the minimal time $t_{0}$ for the spectral parameter $\kappa$. Since $f$ and $f$ in (2.9), (2.10) are analytic in $\kappa$, one sees that $x_{\eta_{-}}(s^{-} + t_{0},\overline{\kappa}(\gamma),\varepsilon_{0}) \to \eta_{+}$ as $\gamma \to \gamma_{0}$. For small enough $|\gamma_{0}|$, let us define $t_{1}(\gamma)$ by $|\eta_{+} - x_{\eta_{-}}(s^{-} + t_{0},\overline{\kappa}(\gamma),\varepsilon_{0})| = \frac{1}{2}T_{\kappa_{0}}^{\rho_{\min}}(t_{1}(\gamma))$ assuming that $\Omega$ is a small enough neighborhood of $\kappa_{0}$, where the uniform STLC holds (see Theorem 3.4). Then $T_{\overline{\kappa}(\gamma)}^{\rho_{\min}}(\eta_{-},\eta_{+}) < t_{0} + t_{1}(\gamma)$. From $T_{\kappa_{0}}^{\rho_{\min}}(0) = 0$ and the continuity of $T_{\kappa_{0}}^{\rho_{\min}}$, one gets $\lim_{\gamma \to \gamma_{0}} t_{1}(\gamma) = 0$, and, in turn, (8.12) and the upper semi-continuity of $\rho_{\min}$. This completes the proof of (i).

(ii) Let $\kappa$ be the resonance of minimal decay for $\text{Re} \kappa > 0$. Combining statement (i) and Corollary 8.8 (iv), we see that $\kappa = \rho_{\min}(\gamma_{0})e^{i\gamma_{0}}$. This implies (8.3), (8.5), and completes the proof.

Proof of Theorem 8.3 The description of $\text{dom} \rho_{\min}$ is given in Corollary 8.8. Statement (i) follows from the proposition 8.5. Statement (ii) from statement (i).

Proof of Theorem 8.3 Proposition 8.7 implies $\rho_{0} \leq |\kappa_{0}| \leq \rho_{1}$, and so $\kappa_{0} \in e^{i\gamma_{0}}[\rho_{0},\rho_{1}]$. If $\kappa = \rho e^{i\gamma_{0}}$ with $\rho \in [\rho_{0},|\kappa_{0}|]$, Corollary 8.8 implies that $\kappa$ is of minimal decay. To obtain the same statement for $\rho \in ([\rho_{0},\rho_{1})$ (in the case where this interval is nonempty), one can combine Propositions 8.5, 8.7.

In the case (i), (8.4) and (8.5) are obvious because $\kappa$ is of minimal modulus.

The case (ii) means that $\kappa$ is of minimal decay, but is not of minimal modulus. Let $t_{0} := T_{\kappa}^{\rho_{\min}}(\eta_{-},\eta_{+})$ and $s_{0} := s^{-} + t_{0}$. Consider any control $\varepsilon_{70}^{\min}(\cdot) \in F_{s^{-},s_{0}}$ that steers the system (2.9), (2.10) from $x(s_{-}) = \eta_{-}$ to $\eta_{+}$ in the minimum possible time $t_{0}$. Since $\kappa$ is not of minimal modulus, (2.21) implies $t_{0} < s^{+} - s^{-}$. The control $\overline{\varepsilon} \in F_{s^{-},s^{+}}$ defined by (8.7) is a continuation of $\varepsilon_{70}^{\min}(\cdot)$ to the interval $(s_{0},s^{+})$ by the constant value $\eta_{+}^{2}$. Since $\eta_{+}$ is an
equilibrium solution of \( x' = \text{i}\kappa(-x^2 + \eta_n^2) \), we see that the trajectory of \( \bar{\varepsilon}(s) \) corresponding to the control \( \bar{\varepsilon} \) stays at \( \eta_+ \) for \( s \in (s_0, s_+^-) \). Thus, \( \kappa \in \Sigma_{\eta_-, \eta_+}(\bar{\varepsilon}) \) and so \( \bar{\varepsilon} \) is the resonator of minimal decay for the frequency \( \text{Re} \kappa \).

It remains to show that there exists a solution \( y \) to the equation (5.2) on \([s^-, s^+]\) such that \((\bar{x}, \bar{\varepsilon}, 0, y)\) is an extremal tuple on \([s^-, s^+]\). Indeed, since \( \bar{\varepsilon} \) is the resonator of minimal decay generating \( \kappa \), it follows from (20) that there exists a nontrivial solution \( y \) to (5.2) such that \( \bar{\varepsilon}(\cdot) = \varepsilon(y)(\cdot) \) on \((s^-, s^+)\) and the two boundary conditions (2.1) are satisfied. It follows from Section 2.2 that \( \bar{x} = \frac{\dot{v}}{iky} \) is a solution to (2.9), (2.10) (with \( \varepsilon(\cdot) = \bar{\varepsilon}(\cdot) \)) on \([s^-, s^+]\). It is easy to see from (5.2) that \( \text{Im}(\bar{\varepsilon}(s)y^2(s) + \kappa^{-2}(\dot{y}'(s))^2) = \text{Im}(y^2(s)[\bar{\varepsilon}(s) - x^2(s)]) \) is a constant independent of \( s \). To see that this constant equals 0, it is enough to take any \( s \in (s_0, s_+^-) \). Thus, \((\bar{x}, \bar{\varepsilon}, 0, y)\) satisfies the definition of abnormal extremal tuple (see Section 3.1). \( \square \)

Let us pass to the settings of Theorem 8.4 and give its proof.

To prove (8.8) it is enough to show that \( \rho_0 < \rho_1 \). Indeed, the example considered in this theorem can be seen as ‘the right hand’ of the symmetric example of Theorem 7.4 with \( n_\infty = n_1 \). This implies \( \rho_0 = |\kappa_0| \). On the other side \( \rho_0 < \rho_1 \), implies \( \rho_0 < \lim_\gamma \to \gamma_0 + \rho_{\min}(\gamma) \), and so \( \lim_\gamma \to \gamma_0 - \rho_{\min}(\gamma) = \rho_0 \) follows from Corollary 8.8.

Let us show that \( \rho_0 < \rho_1 \) holds using the method of the proof of Theorem 7.4 and its symmetric settings.

For this we put \( \tilde{s}^+ = -\tilde{s}^- = s^+, \tilde{n}_- = -n_1 \), and \( \tilde{\kappa}(\gamma) = |\kappa_0|e^{i\pi} \) for \( \gamma \in (\gamma_0, 0) \). From \( \rho_0 = |\kappa_0| = |\tilde{\kappa}(\gamma)| \), the equality \( T_{\kappa_0}^{\min}(-n_1, \infty) = s^+ \), and the equalities

\[
T_{\tilde{\kappa}(\gamma)}^{\min}(-n_1, \infty) = T_{\tilde{\kappa}(\gamma)}(\infty, n_1) = \frac{s^+ \rho_0(\gamma, \infty, n_1)}{|\tilde{\kappa}(\gamma)|} = \frac{s^+ \rho_0(\gamma, \infty, n_1)}{\rho_0},
\]

it is easy to see that \( \rho_0 < \rho_1 \) is equivalent to

\[
T_{\kappa_0}^{\min}(-n_1, \infty) \cos \gamma_0 < \inf_{\gamma \in (\gamma_0, 0)} \left( T_{\tilde{\kappa}(\gamma)}^{\min}(-n_1, \infty) \cos \gamma \right). \tag{8.13}
\]

This inequality follows from \( T_{\kappa_0}^{\min}(-n_1, \infty) = \frac{\pi}{2n_2 \rho_0 \cos \gamma_0} \) and the following lemma.

**Lemma 8.9.** There exists \( \delta_1 > 0 \) such that

\[
t_1 > \frac{\pi}{2n_2 \rho_0 \cos \gamma} + \delta_1 \tag{8.14}
\]

for every \( \kappa = \tilde{\kappa}(\gamma) \) with \( \gamma \in (\gamma_0, 0) \) and for every extremal tuple \( (x, \varepsilon, \lambda_0, y) \) on \([s^-, s^+ + 2t_1]\) satisfying \( x(s^-) = -n_1 \) and \( x(s^- + t_1) = \infty \).

**Proof.** Let \( \kappa = \tilde{\kappa}(\gamma) \) with \( \gamma \in (\gamma_0, 0) \) and let \( (x, \varepsilon, \lambda_0, y) \) be an extremal tuple on \([s^-, s^+ + 2t_1]\) satisfying \( x(s^-) = -n_1 \) and \( x(s^- + t_1) = \infty \).

**Case 1.** Assume that \((x, \varepsilon, \lambda_0, y)\) is abnormal. Then it corresponds to the sequence (6.4) with \( m \in \mathbb{N} \) repetitions (due to \( \gamma \in (\gamma_0, 0) \), the case (6.2) can easily excluded by simple calculations, e.g., using the ilog-phase \( \theta_{n_2}^\circ \) (see Section 3.2). Then Theorem 5.5 implies that \( t_1 > \frac{\pi(m+1)}{2n_2 \rho_0 \cos \gamma} \), and so, implies (8.14).

**Case 2.** Assume that \((x, \varepsilon, \lambda_0, y)\) is normal. Since \( x(s^-) = -n_1 \) and \( x(s^- + t_1) = \infty \), we see that only the types (6.6) and (6.10) of extremals are possible. Arguments similar to that of the proof of Theorem 7.4 imply that \( m_1 \geq 1 \). Using Corollary 7.3 it is easy to show that
the part \( +(-n_2,0) \) requires the time \( t_2 > \frac{\pi}{2n_2 \Re \kappa} \) (statement (i) of Corollary 7.3 is needed for the case (6.6), statement (iv) of Corollary 7.3 for (6.10)). The starting part \( (-n_2,-n_1) \) requires certain time \( t_3 > 0 \). This time \( t_3 \) can be bounded from below by a positive number \( \delta > 0 \) that does not depend on \( \gamma \in (\gamma_0,0) \) and on the choice of the extremal tuple. To prove this fact, one can use the arguments of [31] about the absence of STLC to a singular equilibrium point, and make them uniform over \( \gamma \in (\gamma_0,0) \) using the additional information about the structure of normal extremals provided by Theorem 5.8. This completes the proof of the lemma and of the fact that \( \rho_0 < \rho_1 \).

Let us prove statement (i) of Theorem 8.4. The facts that \( \kappa_0 \) is \((\eta_-,\eta_+)-\)eigenvalue of minimal modulus and that the constant function \( \varepsilon_{\gamma_0}^{\min}(\cdot) \) is the unique control in \( F_{0,s} \) such that \( \kappa_0 \in \Sigma_{0,s}^{\eta_-,\eta_+} \) follow from Theorem 7.4 and the symmetry w.r.t. 0. The fact that \( \kappa_0 \) is \((\eta_-,\eta_+)-\)eigenvalue of minimal decay follows from \( \rho_0 < \rho_1 \) and Proposition 8.7.

Statement (ii) of Theorem 8.4 now follows from Theorem 8.3.

This completes the proof of Theorem 8.4.

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