Ensemble of Vortex Loops in the
Abelian-Projected SU(3)-Gluodynamics

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Abstract

Grand canonical ensemble of small vortex loops emerging in the London limit of the effective Abelian-projected theory of the SU(3)-gluodynamics is investigated in the dilute gas approximation. An essential difference of this system from the SU(2)-case is the presence of two interacting gases of vortex loops. Two alternative representations for the partition function of such a grand canonical ensemble are derived, and one of them, which is a representation in terms of the integrals over vortex loops, is employed for the evaluation of the correlators of both kinds of loops in the low-energy limit.

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In a recent paper [1], the grand canonical ensemble of small vortex loops, existing in the Abelian Higgs model, have been investigated. Such loops are nothing else but the 4D analogue of the vortex dipoles, which are present in the usual Ginzburg-Landau theory. In particular, it has been demonstrated that the summation over the grand canonical ensemble of small vortex loops leads to an effective Sine-Gordon type theory of the massive Kalb-Ramond field [2] (cf. 2D- and 3D cases studied in Refs. [3, 4]). This field describes a dual vector boson, which therefore acquires an additional mass due to the Debye screening in the gas of vortex loops. Furthermore, a representation of the partition function of such a gas directly in terms of the integral over vortex loops as well as the related effective potential of those have been discussed. Such a representation then turned out to be useful for the evaluation of the bilocal correlator of vortex loops in the low-energy limit. These calculations demonstrate the importance of treating the topological defects in the Abelian Higgs model and Ginzburg-Landau theory as forming the ensembles, rather than individual ones.

An interest to the study of vortex loops in the Abelian Higgs model is motivated by the fact that the dual Abelian Higgs model is discussed to be relevant to the description of confinement in the \( SU(2) \)-gluodynamics [5]. This agreement is based on the method of Abelian projections [6] (for a recent progress see [7, 8, 9, 10, 11, 12, 13, 14, 15], for a review see [16]) and the so-called Abelian dominance hypothesis [17], according to which off-diagonal (in the sense of the Cartan decomposition) degrees of freedom are inessential for confinement and can be disregarded. In the spirit of this hypothesis, the following partition function describing an effective \([U(1)]^2\) gauge invariant Abelian-projected theory of the realistic \( SU(3) \)-gluodynamics has been proposed [18, 19]

\[
Z = \int \mathcal{D} \tilde{B}_\mu \mathcal{D} \chi_a \mathcal{D} \chi_a^* \delta \left( \sum_{a=1}^3 \theta_a \right) \times \\
\times \exp \left\{ -\int d^4 x \left[ \frac{1}{4} \tilde{F}^2_{\mu \nu} + \sum_{a=1}^3 \left[ \left| \left( \partial - i g_m \tilde{e}_a \tilde{B}_\mu \right) \chi_a \right|^2 + \lambda \left( |\chi_a|^2 - \eta^2 \right)^2 \right] \right] \right\}. \tag{1}
\]

Here, \( \tilde{F}^2_{\mu \nu} = \partial_\mu \tilde{B}_\nu - \partial_\nu \tilde{B}_\mu \) stands for the field strength tensor of the magnetic vector potential \( \tilde{B}_\mu = (B^3_\mu, B^8_\mu) \) dual to the electric one \( \tilde{A}_\mu = (A^3_\mu, A^8_\mu) \). Next in Eq. (1), \( \chi_a = |\chi_a| e^{i \theta_a} \), \( a = 1, 2, 3 \), are effective Higgs fields describing condensed magnetic monopoles, whose magnetic charge \( g_m \) is expressed via the QCD coupling constant \( g_{QCD} \) as \( g_m = 4\pi / g_{QCD} \). Finally in Eq. (1),

\[
\tilde{e}_1 = (1, 0), \quad \tilde{e}_2 = \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \quad \tilde{e}_3 = \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right)
\]

stand for the so-called root vectors, which play the role of the structural constants in the algebra \( [\tilde{H}, E_{\pm a}] = \pm \tilde{e}_a E_{\pm a} \). Here, the operators \( \tilde{H} \equiv (H_1, H_2) = (T_3, T_8) \) generate the Cartan subalgebra, where from now on \( T_i \equiv \frac{T_i}{2}, \) \( i = 1, \ldots, 8 \), are just the \( SU(3) \)-generators. We have also introduced the so-called step operators \( E_{\pm a} \)'s (else called raising operators for positive \( a \)'s and lowering operators otherwise) by redefining the rest (non-diagonal) \( SU(3) \)-generators as follows

\[
E_{\pm 1} = \frac{1}{\sqrt{2}} (T_1 \pm iT_2), \quad E_{\pm 2} = \frac{1}{\sqrt{2}} (T_4 \pm iT_5), \quad E_{\pm 3} = \frac{1}{\sqrt{2}} (T_6 \pm iT_7).
\]

\(^1\) Throughout the present Letter, all the investigations will be performed in the Euclidean space-time.
Clearly, these operators are non-Hermitean in the sense that \((E_a)^\dagger = E_{-a}\). For bookkeeping purposes, it is worth listing the remaining commutation relations, completing the Lie algebra, which read

\[
[E_{\pm a}, E_{\pm b}] = \mp \frac{1}{\sqrt{2}} \varepsilon_{abc} E_{\mp c} \quad \text{and} \quad [E_{a}, E_{-b}] = \delta_{ab} \varepsilon_{a} \vec{H}.
\]

Notice also that due to the fact that the original \(SU(3)\) group is special, the phases of the three magnetic Higgs fields are not independent and should obey the constraint \(\sum_{\alpha=1}^{3} \theta_{\alpha} = 0\). The latter one has been imposed by the introduction of the corresponding \(\delta\)-function into the functional integral on the R.H.S. of Eq. (1).

Before proceeding with the study of the model (1), it is worth mentioning its certain feature owing to which this model is not quite adequate to the description of confinement in the real \(SU(3)\)-gluodynamics. Its essence is that the model (1) describes only the sector of the full yet unknown Abelian-projected theory of the \(SU(3)\)-gluodynamics, where antimonopoles are completely absent. Clearly, the expected full theory should contain the antimonopole sector as well. Possible interference between these two sectors is up to now unclear and should be clarified by further investigations.

In what follows, we shall be interested in the study of the model (1) in the London limit, \(i.e.,\) the limit of infinitely large Higgs coupling constant \(\lambda\). Analogously to the \(SU(2)\)-case, in this limit the model under study allows for an exact reformulation in terms of the integral over closed Abrikosov-Nielsen-Olesen type strings [13]. In this limit, the radial parts of the Higgs fields can be integrated out, and the partition function (1) takes the form

\[
Z = \int \mathcal{D}\vec{B}_\mu \mathcal{D}\theta_a \delta \left( \sum_{\alpha=1}^{3} \theta_{\alpha} \right) \exp \left\{ -\int d^4x \left[ \frac{1}{4} \vec{F}^2_{\mu\nu} + \eta^2 \sum_{a=1}^{3} \left( \partial_\mu \theta_a - g_m \varepsilon_{a} \vec{B}_{\mu} \right)^2 \right] \right\}. \tag{2}
\]

Next, the total phases \(\theta_a\)'s of magnetic Higgs fields should be decomposed into the singular and regular parts, \(\theta_a = \theta_{a}^{\text{sing}} + \theta_{a}^{\text{reg}}\) [12, 13, 14] \((\text{cf. also Refs. [20, 21, 15] for the } SU(2)-\text{case}). Here, \(\theta_{a}^{\text{sing}}\)'s describe a certain configuration of electric strings and are unambiguously related to their world-sheets \(\Sigma_a\)'s according to the equation (see the above cited Refs.)

\[
\varepsilon_{\mu\nu\lambda\rho} \partial_\lambda \partial_\rho \theta_{a}^{\text{sing}}(x) = 2\pi \sum_{\mu\nu} \theta_{a}^{\text{sing}}(x) \equiv 2\pi \int d\sigma_{\mu\nu} \left( x^{(a)}(\xi) \right) \delta \left( x - x^{(a)}(\xi) \right). \tag{3}
\]

This equation is just the covariant formulation of the 4D analogue of the Stokes theorem for the gradient of the field \(\theta_a\), written in the local form. In Eq. (3), \(x^{(a)}(\xi) \equiv x_a^{(a)}(\xi)\) is a vector parametrizing the world-sheet \(\Sigma_a\) with \(\xi = (\xi^1, \xi^2) \in [0, 1] \times [0, 1]\) standing for the two-dimensional coordinate.

Confining and topological properties of the model (2) have been studied in Refs. [12, 13, 14]. This has been done by making use of the so-called path-integral duality transformation, elaborated in Refs. [4, 20] for the usual Abelian Higgs model, which casts the partition function (2) into the following form,

\[
Z = \int \mathcal{D}x_{\mu}^{(a)}(\xi) \delta \left( \sum_{\alpha=1}^{3} \Sigma_{\alpha} \right) \mathcal{D}A_{\mu}^a \mathcal{D}h_{\mu\nu}^a \exp \left\{ -\int d^4x \left[ \frac{1}{12\eta^2} \left( H_{\mu\nu\lambda}^a \right)^2 + \right. \right. \]

Notice that according to the lattice data [8, 16], it is this limit of the Abelian-projected theories, in which they reveal properties similar to the real QCD.
with \( A_\mu^a \equiv \varepsilon_a \bar{A}_\mu \). Here, \( H_{\mu\nu}^a = \partial_\mu h_{\nu}^a + \partial_\nu h_{\mu}^a + \partial_\lambda h_{\mu\nu}^a \) stands for the field strength tensor of the antisymmetric tensor field \( h_{\mu\nu}^a \) (the so-called Kalb-Ramond field \( \bar{h}_{\mu\nu} \)). The integration over this field came about via some constraints resulting from the integration over \( \theta_a^\text{reg} \)'s, whereas the integration over \( \theta_a^\text{sing} \)'s has transformed into the integration over \( x_\mu^a (\xi) \)'s by virtue of Eq. (1) (Notice that since in what follows we shall be interested in effective actions rather than the integration measures, the Jacobian appearing during the change of the integration variables, \( \theta_a^\text{sing} \rightarrow x_\mu^a (\xi) \), which has been evaluated in Ref. [21], will not be discussed below and is assumed to be included into the measure \( D x_\mu^a (\xi) \) ). Also, due to this one-to-one correspondence between \( \theta_a^\text{sing} \)'s and \( \Sigma_a \)'s, the constraint imposed by the \( \delta \)-function on the R.H.S. of Eqs. (1) and (2) has gone over into the constraint imposed by the \( \delta \)-function on the R.H.S. of Eq. (3), which relates the world-sheets of three types to each other, making only two of them really independent.

The aim of the present Letter is to treat Abrikosov-Nielsen-Olesen type strings in the model (2) in the sense of the grand canonical ensemble of small vortex loops, rather than as individual (i.e., noninteracting) ones. To understand why one might expect in this case the appearance of some nontrivialities \( w.r.t. \) the Abelian-projected \( SU(2) \)-gluodynamics, let us begin with considering noninteracting vortex loops. This can be done by gauging the field \( A_\mu^a \) away from Eq. (4) by performing the hypergauge transformation \( h_{\mu\nu}^a \rightarrow h_{\mu\nu}^a - \frac{2}{g_m \sqrt{3}} \left( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) \) and subsequent integration over the Kalb-Ramond fields (see the first paper from Ref. [23] for details of this integration). The result has the form

\[
Z = \int D x_\mu^a (\xi) \delta \left( \sum_{a=1}^3 \Sigma_{\mu\nu}^a \right) \exp \left\{ -g_m \frac{\sqrt{3}}{2} \eta^3 \int d^4 x d^4 y \Sigma_{\mu\nu}^a(x) \frac{K_1(m|x-y|)}{|x-y|} \Sigma_{\mu\nu}^a(y) \right\},
\]

where \( m = \sqrt{3} g_m \eta \) is the mass of the fields \( B_\mu^3 \) and \( B_\mu^8 \), which they acquire due to the Higgs mechanism, and \( K_1 \) stands for the modified Bessel function. Finally, one of the three world-sheets, for concreteness \( x_\mu^3 (\xi) \), can be integrated out, which yields

\[
Z = \int D x_\mu^{(1)} (\xi) D x_\mu^{(2)} (\xi) \times
\]

\[
\times \exp \left\{ -g_m \eta^3 \sqrt{3} \int d^4 x d^4 y \left[ \Sigma_{\mu\nu}^1(x) \Sigma_{\mu\nu}^1(y) + \Sigma_{\mu\nu}^1(x) \Sigma_{\mu\nu}^2(y) + \Sigma_{\mu\nu}^2(x) \Sigma_{\mu\nu}^2(y) \right] \frac{K_1(m|x-y|)}{|x-y|} \right\}. \tag{5}
\]

In order to proceed from the individual strings to the grand canonical ensemble of interacting vortex loops, one should replace \( \Sigma_{\mu\nu}^a(x) \), where from now on on \( a = 1, 2 \), in Eq. (4) by

\[
\Sigma_{\mu\nu}^a (x) = \sum_{k=1}^N \eta_k^{(a)} \int d \sigma_{\mu\nu} \left( x_k^{(a)}(\xi) \right) \delta \left( x - x_k^{(a)}(\xi) \right).
\]

Here, \( \eta_k^{(a)} \)'s stand for winding numbers, which we shall set to be equal 1 (cf. Ref. [21]). Performing such a replacement, one can see the crucial difference of the grand canonical ensemble of small

\[^3\text{This is just the essence of the dipole approximation.}\]
vortex loops in the model under study from that in the Abelian-projected $SU(2)$-gluodynamics \cite{1}. Namely, the system has now the form of two interacting gases consisting of the vortex loops of two kinds, while in the $SU(2)$-case the gas was built out of vortex loops of the only one kind.

Analogously to that case, we shall treat such a grand canonical ensemble of vortex loops in the dilute gas approximation. According to it, characteristic sizes of loops are much smaller than characteristic distances between them, which in particular means that the vortex loops are short living objects. Then the summation over this grand canonical ensemble can be most easily performed by inserting the unity into the R.H.S. of Eq. (5) (with $\Sigma^a_{\mu\nu}$ replaced by $\Sigma^a_{\mu\nu}^{gas}$) and representing the $\delta$-functions as the integrals over Lagrange multipliers. Then, the contribution of $N$ vortex loops of each kind to the full grand canonical ensemble takes the following form

$$1 = \int D S^a_{\mu\nu} \delta \left( S^a_{\mu\nu} - \Sigma^a_{\mu\nu}^{gas} \right)$$

(6)

into the R.H.S. of Eq. (3) and representing the $\delta$-functions as the integrals over Lagrange multipliers. Then, the contribution of $N$ vortex loops of each kind to the full grand canonical ensemble takes the following form

$$Z \left[ \Sigma^a_{\mu\nu}^{gas} \right] = \int D S^a_{\mu\nu} D \lambda^a_{\mu\nu} \times$$

$$\times \exp \left\{ - \int d^4 x d^4 y \left[ S^1_{\mu\nu}(x) S^1_{\mu\nu}(y) + S^1_{\mu\nu}(x) S^2_{\mu\nu}(y) + S^2_{\mu\nu}(x) S^2_{\mu\nu}(y) \right] - \frac{K_1(m |x - y|)}{|x - y|} +$$

$$+ i \int d^4 x d^4 y \left( S^1_{\mu\nu} - \Sigma^1_{\mu\nu}^{gas} \right) \right\}. \quad (7)$$

After that, the desired summation is straightforward, since it technically parallels the one of Abelian-projected $SU(2)$-gluodynamics described in Ref. \cite{1}. We have

$$\left\{ 1 + \sum_{N=1}^{\infty} \zeta^N N! \prod_{i=1}^{N} \int d^4 y_i^{(1)} \sum_{N_1} \int D z_{i}^{(1)}(\mu) \left[ z_{i}^{(1)} \right] \right\} \times$$

$$\times \sum_{n_{k}^{(1)}=\pm 1} \exp \left\{ i \sum_{k=1}^{N} n_{k}^{(1)} \int d \sigma_{\mu\nu} \left( z_{k}^{(1)}(\xi) \right) \lambda_{\mu\nu}^{1} \left( x_{k}^{(1)}(\xi) \right) \right\} \times$$

$$\left\{ \text{the same term with the replacements } (1) \rightarrow (2) \text{ and } \lambda_{\mu\nu}^{1} \rightarrow \lambda_{\mu\nu}^{2} \right\} =$$

$$\exp \left\{ 2 \zeta \int d^4 y \left[ \cos \left( \frac{\lambda_{\mu\nu}^{1}(y)}{\Lambda^2} \right) + \cos \left( \frac{\lambda_{\mu\nu}^{2}(y)}{\Lambda^2} \right) \right] \right\}. \quad (8)$$

Here, the world-sheet coordinate of the $k$-th vortex loop of the $a$-th type \cite{1} $x_{k}^{(a)}(\xi)$ has been decomposed as $x_{k}^{(a)}(\xi) = y_{k}^{(a)} + z_{k}^{(a)}(\xi)$, where the vector $y_{k}^{(a)} \equiv \int d^2 \xi x_{k}^{(a)}(\xi)$ describes the position of the vortex loop, whereas the vector $z_{k}^{(a)}(\xi)$ describes its shape. Next, on the L.H.S. of Eq. (5), $\mu \left[ z_{i}^{(a)} \right]$ stands for a certain rotation- and translation invariant measure of integration over the shapes of the world-sheets of the vortex loops, and $\zeta \propto e^{-S_0}$ is the so-called fugacity (Boltzmann factor).

\footnote{For brevity, we omit the Lorentz index.}
factor of a single vortex loop of dimension (mass)$^4$ with $S_0$ denoting the action of a single loop. In Eq. (8), we have also introduced the UV momentum cutoff $\Lambda \equiv \sqrt{L/a}$ ($\gg a^{-1}$), where $a$ is a typical size of the vortex loop, and $L$ is a typical distance between loops, so that in the dilute gas approximation under study $a \ll L$. Finally in Eq. (8), we have denoted $|\lambda_{\mu\nu}| \equiv \sqrt{(\lambda_{\mu\nu})^2}$. The reader is referred to Ref. [1] for details of a derivation of Eq. (8).

Note that the value of $S_0$ is approximately equal to $\sigma a^2$, where we have estimated the area of a vortex loop as $a^2$, and $\sigma$ stands for an analogue of the string tension for the loop, i.e., its energy per unit area. This energy can be evaluated from Eq. (5) by virtue of the results of Ref. [22] and has the form

$$\sigma = 2\eta^2 \int d^2t \frac{K_1(|t|)}{|t|} \approx 2\pi \eta^2 \ln \left(\frac{\lambda}{\eta m^2}\right).$$

Here, we have in the standard way [19] set for a characteristic small dimensionless quantity in the model under study the value $\frac{\Lambda}{\sqrt{\lambda}}$, which is of the order of the ratio of $m$ to the masses of magnetic Higgs fields. Moreover, it has been assumed that not only $\frac{\Lambda}{\eta m} \gg 1$, but also $\ln \frac{\Lambda}{\eta m} \gg 1$, i.e., the last equality on the R.H.S. of Eq. (9) is valid with the logarithmic accuracy.

Next, it is possible to integrate out the Lagrange multipliers by solving the saddle-point equations following from Eqs. (7) and (8),

$$\frac{\lambda_{\mu\nu}}{|\lambda_{\mu\nu}|} \sin \left(\frac{|\lambda_{\mu\nu}|}{\Lambda^2}\right) = - \frac{i\Lambda^2}{2\zeta} S_{\mu\nu}^a.$$

After that, we arrive at the following representation for the partition function of the grand canonical ensemble

$$Z_{\text{grand}} = \int D S_{\mu\nu}^a \exp \left\{-\frac{g_m \eta^3 \sqrt{3}}{2} \int d^4x d^4y \left[S_{\mu\nu}^1(x)S_{\mu\nu}^1(y) + S_{\mu\nu}^1(x)S_{\mu\nu}^2(y) + S_{\mu\nu}^2(x)S_{\mu\nu}^2(y)\right] \times \right.$$}

$$\times \frac{K_1(m|x-y|)}{|x-y|} + V \left[S_{\mu\nu}^1\right] + V \left[S_{\mu\nu}^2\right]\right\},$$

(10) which owing to Eq. (3) is natural to be referred to as the representation in terms of the vortex loops. In Eq. (10), the effective potential of vortex loops reads

$$V \left[S_{\mu\nu}^a\right] = \sum_{n=-\infty}^{\infty} \int d^4x \left\{ \Lambda^2 \left|S_{\mu\nu}^a\right| \ln \left[\frac{\Lambda^2}{2\zeta} \left|S_{\mu\nu}^a\right| + \sqrt{1 + \left(\frac{\Lambda^2}{2\zeta} \left|S_{\mu\nu}^a\right|\right)^2} + 2\pi i\right] - \right.$$}

$$-2\zeta \sqrt{1 + \left(\frac{\Lambda^2}{2\zeta} \left|S_{\mu\nu}^a\right|\right)^2}. \right\}.$$ (11)

It is further instructive to illustrate the difference of such a partition function of two interacting gases of vortex loops from the case of Abelian-projected $SU(2)$-gluodynamics by studying a related

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$^5$It is natural to assume that the vortex loops of different kinds have the same fugacity, since different $\theta_a^{\text{sing}}$'s enter the initial partition function (3) in the same way.
representation in terms of a certain effective Sine-Gordon theory. This can be done by introducing the new integration variables \( S^1_{\mu \nu} = \frac{\sqrt{3}}{2} \left( S^1_{\mu \nu} + S^2_{\mu \nu} \right) \) and \( S^2_{\mu \nu} = \frac{1}{2} \left( S^1_{\mu \nu} - S^2_{\mu \nu} \right) \), which diagonalize the quadratic form in square brackets on the R.H.S. of Eq. (7). Then Eqs. (4) and (5) yield

\[
Z_{\text{grand}} = \int \mathcal{D}S^a_{\mu \nu} \mathcal{D} \lambda^a_{\mu \nu} \exp \left\{ -g m \eta^3 \sqrt{3} \int d^4x d^4y S^a_{\mu \nu}(x) \frac{K_1(m|x-y|)}{|x-y|} S^a_{\mu \nu}(y) + 2 \zeta \int d^4x \left[ \cos \left( \frac{|\lambda^a_{\mu \nu}(x)|}{\Lambda^2} \right) + \cos \left( \frac{|\lambda^2_{\mu \nu}(x)|}{\Lambda^2} \right) \right] - i \int d^4x h^a_{\mu \nu} S^a_{\mu \nu} \right\},
\]

where we have denoted \( h^1_{\mu \nu} = \frac{1}{\sqrt{3}} \left( \lambda^1_{\mu \nu} + \lambda^2_{\mu \nu} \right) \) and \( h^2_{\mu \nu} = \lambda^1_{\mu \nu} - \lambda^2_{\mu \nu} \). The partition function of the desired Sine-Gordon theory can be obtained from Eq. (12) by making use of the following equality:

\[
\int \mathcal{D}S^a_{\mu \nu} \exp \left\{ - \left[ g m \eta^3 \sqrt{3} \int d^4x d^4y S^a_{\mu \nu}(x) \frac{K_1(m|x-y|)}{|x-y|} S^a_{\mu \nu}(y) + i \int d^4x h^a_{\mu \nu} S^a_{\mu \nu} \right] \right\} = \\
\exp \left\{ - \frac{1}{4\pi^2} \int d^4x \left[ \frac{1}{12\eta^2} \left( H^a_{\mu \nu,\lambda} \right)^2 + \frac{3g_m^2}{4} \left( h^a_{\mu \nu} \right)^2 \right] \right\}
\]

(cf. the R.H.S. with the quadratic part of the action of the Kalb-Ramond field on the R.H.S. of Eq. (11) with the field \( A_{\mu}^a \) gauged away). Substituting this equality into Eq. (12) and performing the rescaling \( h^a_{\mu \nu} \rightarrow h^a_{\mu \nu} \), we arrive at the following representation for the partition function of the grand canonical ensemble of vortex loops in terms of the local Sine-Gordon theory, equivalent to the nonlocal theory (11):

\[
Z_{\text{grand}} = \int \mathcal{D}h^a_{\mu \nu} \exp \left\{ - \int d^4x \left[ \frac{1}{12\eta^2} \left( H^a_{\mu \nu,\lambda} \right)^2 + \frac{3g_m^2}{4} \left( h^a_{\mu \nu} \right)^2 - 2 \zeta \left[ \cos \left( \frac{\pi}{\Lambda^2} \sqrt{3}h^1_{\mu \nu} + h^2_{\mu \nu} \right) + \cos \left( \frac{\pi}{\Lambda^2} \sqrt{3}h^1_{\mu \nu} - h^2_{\mu \nu} \right) \right] \right\}.
\]

As we now see, an essential property of the obtained Sine-Gordon theory, which distinguishes it from an analogous theory describing the grand canonical ensemble of vortex loops in the Abelian-projected \( SU(2) \)-gluodynamics [1], is the presence of two interacting Kalb-Ramond fields, while in the \( SU(2) \)-case there was only one self-interacting field. Notice that upon the expansion of the cosines on the R.H.S. of Eq. (13), it is straightforward to see that only the interaction terms of the type \( (h^1_{\mu \nu})^{2n} (h^2_{\mu \nu})^{2k} \) survive. In another words, despite of the mixing of the Kalb-Ramond fields in the arguments of the cosines, no terms linear in any of these fields appear in the action. In particular, the masses of both Kalb-Ramond fields, \( M_1 \) and \( M_2 \), can be read off from Eq. (13) by expanding the cosines up to the quadratic terms. The result reads \( M_1^2 = m^2 + m_a^2 \equiv Q_1^2 \eta^2 \), where \( m_1 = \frac{2\eta}{\Lambda^2} \sqrt{6\zeta} \), \( m_2 = \frac{2\eta}{\Lambda^2} \sqrt{2\zeta} \) are the Debye masses, and we have introduced the corresponding magnetic charges \( Q_1 = \sqrt{3g_m^2 + \frac{24\pi^2}{\Lambda^2}} \), \( Q_2 = \sqrt{3g_m^2 + \frac{8\pi^2}{\Lambda^2}} \).

Eqs. (11) and (11) can now be used for the evaluation of correlators of vortex loops, which due to Eq. (6), are nothing else but the correlators of \( S^a_{\mu \nu} \)'s. Those are calculable in the low-energy

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\(^6\)This equality can straightforwardly be proved by mentioning that \( \partial_{\mu} h^a_{\mu \nu} = 0 \), which follows from the equation of motion corresponding to its L.H.S. and Eq. (6), according to which \( \partial_{\mu} S^a_{\mu \nu} = 0 \).
The correlators of vortex loops following from this expression have the form

where \( J_{\mu\nu} \) is a source of \( S_{\mu\nu}^a \), and \( J_{\mu\nu}^\pm \equiv J_{\mu\nu}^1 \pm J_{\mu\nu}^2 \). Such two Gaussian integrals can be calculated by virtue of the following equality

\[
\int \mathcal{D} S_{\mu\nu} \exp \left\{ - \left[ g_m \eta^3 \sqrt{3} \int d^4 x d^4 y S_{\mu\nu}(x) \frac{K_1(m|x-y|)}{|x-y|} S_{\mu\nu}(y) + \frac{\Lambda^4}{2\zeta} \int d^4 x \left[ \left( S_{\mu\nu}^1 \right)^2 + \left( S_{\mu\nu}^2 \right)^2 \right] + \frac{\Lambda^4}{2\zeta} \int d^4 x \left[ S_{\mu\nu}^1 \frac{J_{\mu\nu}^+}{\sqrt{3}} + S_{\mu\nu}^2 J_{\mu\nu}^- \right] \right]\} =
\exp \left\{ - \frac{M_2 \zeta}{8\pi^2 \Lambda^4} \int d^4 x d^4 y J_{\mu\nu}(x) J_{\mu\nu}(y) \left( \partial_x^2 - m^2 \right) \frac{K_1(M_2|x-y|)}{|x-y|} \right\},
\]

and the result reads

\[
\mathcal{Z} \left[ J_{\mu\nu}^a \right] = \mathcal{Z}_{\text{grand}} \exp \left\{ - \frac{\zeta}{8\pi^2 \Lambda^4} \int d^4 x d^4 y \left[ M_1 J_{\mu\nu}^+(x) J_{\mu\nu}^+(y) \left( \partial_x^2 - m^2 \right) \frac{K_1(M_1|x-y|)}{|x-y|} + M_2 J_{\mu\nu}^-(x) J_{\mu\nu}^-(y) \left( \partial_x^2 - m^2 \right) \frac{K_1(M_2|x-y|)}{|x-y|} \right] \right\}.
\]

The correlators of vortex loops following from this expression have the form

\[
\langle S_{\mu\nu}^1(x) S_{\lambda\rho}^1(0) \rangle = \langle S_{\mu\nu}^2(x) S_{\lambda\rho}^2(0) \rangle = (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) \frac{\zeta}{2\Lambda^4} \left[ 2\delta(x) - \sum_{a=1}^4 m_a^2 M_a \frac{K_1(M_a|x|)}{|x|} \right]
\]

and

\[
\langle S_{\mu\nu}^1(x) S_{\lambda\rho}^2(0) \rangle = (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) \frac{\zeta m_2^2}{2\Lambda^4} \left[ \frac{M_2 K_1(M_2|x|)}{4\pi^2 |x|} - 3 \frac{M_1 K_1(M_1|x|)}{4\pi^2 |x|} \right].
\]

At this point, it is worth recalling that the original theory [1] is an effective theory at large distances [2], where the asymptotic behaviours of the obtained correlators read

\[
\langle S_{\mu\nu}^1(x) S_{\lambda\rho}^1(0) \rangle = \langle S_{\mu\nu}^2(x) S_{\lambda\rho}^2(0) \rangle \rightarrow - \langle S_{\mu\nu}^1(x) S_{\lambda\rho}^2(0) \rangle \rightarrow
\]

\[
- (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) \sqrt{\frac{\pi}{\Lambda^4}} \left( \frac{\eta \zeta}{\Lambda^4} \right)^2 \frac{\sqrt{M_2}}{|x|^2} e^{-M_2|x|}.
\]
This result illustrates how the vortex loops in the grand canonical ensemble under study are correlated to each other. Namely, their correlators decrease according to the Yukawa type law with the screening provided by the lightest of the two full masses, $M_2$.

In conclusion, we have demonstrated that the grand canonical ensemble of vortex loops in the effective Abelian-projected theory of the $SU(3)$-gluodynamics (being treated in the dilute gas approximation) exhibits an essential property distinguishing it from the one of the Abelian-projected $SU(2)$-gluodynamics. Namely, it consists of two interacting subsystems, corresponding to two independent types of strings, which emerge after the Abelian projection. An average over the shapes of the vortex loops with the most general rotation- and translation invariant integration measure leads to two alternative field-theoretical representations of such a grand canonical ensemble. First of them is a representation in terms of an effective Sine-Gordon theory of two interacting Kalb-Ramond fields $[\text{13}]$. It yields the (positive) contribution to the masses $M_a$'s of the Kalb-Ramond fields coming about from the Debye screening as well as to the magnetic charges $Q_a$'s of these fields. The other representation given by Eqs. $[\text{10} \text{ and } \text{11}]$, which is the one in terms of the integral over the vortex loops, is useful for the evaluation of their correlators. While such a calculation is difficult to perform exactly due to the complicated form of the effective potential of the vortex loops $[\text{11}]$, it turns out to be possible to perform it in the low-energy limit within an additional approximation when only the real branch of the potential is taken into account. As a result, the correlators of vortex loops have a Yukawa type asymptotic behaviours at large distances with the screening governed by the lightest of the two full masses of the Kalb-Ramond fields.

It now looks reasonable to find field-theoretical representations for the grand canonical ensembles of vortex loops emerging after the Abelian projection in the general $SU(N)$, $N > 2$, case. In that case, there appear $\frac{N(N-1)}{2} - 1$ independent strings. Indeed, this is just the number of possibilities for the eigenvalues of a certain (adjointly transformed) operator, to be diagonalized during the Abelian projection, to coincide, minus one constraint imposed by the $\delta$-function on the R.H.S. of Eq. $[\text{11}]$ (with the sum over $a$ going from 1 to $N$). It will therefore emerge just this amount of interacting gases of vortex loops, leading to different Debye masses. The study of this system as well as its large-$N$ limit will be the topic of a separate publication.

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