APPROXIMATING THE SPECTRUM OF MATRICES AND HYPERMATRICES

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Abstract. We describe a general approach for computing generators for elimination ideals associated with matrix and hypermatrix spectral decomposition constraints. We derive from these generators iterative procedures for approximating the spectral decomposition of matrices and hypermatrices.

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1. Introduction

Many combinatorial optimization problems including instances of the subgraph-isomorphism problem are of great interest and well known to be NP-hard. One therefore seeks to identify special families of graphs for which efficient combinatorial algorithms can be devised. The complexity of the ensuing algorithms is often determined by combinatorial and algebraic properties of graph encodings. A good illustration for this fact is provided by the graph property of being an expander graph. The complexity of many combinatorial algorithms for such a graph can be expressed in terms of its expansion parameter [AC88, Alo86, ASS08, AM85, BL06, Che70, Chu97, Li01]. As is well known, many graph properties, including the property of being an expander graph, are closely tied to the spectral decomposition of matrices deduced from incidence structures present in the associated graph [Alo86, Che70, Chu97, ST11]. The well known graph adjacency matrix is constructed such that the \((i_1, i_2)\) matrix entry equals 1 if the associated graph admits a directed edge connecting vertex \(i_1\) to vertex \(i_2\), and equals 0 otherwise. Following a construction proposed in [FW95] by Friedman and Widgerson, one also associates with

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a graph a \((k - 1)-\text{path adjacency hypermatrix}\). The \((k - 1)\)-path adjacency hypermatrix corresponds to a \(k\)-th order hypermatrix (note that ordinary matrices are second order hypermatrices) whose \((i_1, i_2, \cdots, i_k)\) entry equals 1 if the ordered tuple \((i_1, i_2, \cdots, i_k)\) denotes a directed path of length \(k - 1\) in the associated graph, and equals 0 otherwise. Consequently, graph algebraic invariants derived from the corresponding adjacency matrix can be extended to include polynomial relations between entries of path adjacency hypermatrices. The polynomial relations which are retained as algebraic and combinatorial invariants are usually the ones which are invariant under some group of unitary transformations. Such algebraic relations are to be thought of as generalizations of the classical Cayley-Hamilton matrix polynomial. Just as it is done for matrices, the algebraic varieties defined by the algebraic and combinatorial invariants will be referred to as the spectrum or the spectral decomposition. The invariants associated with path adjacency hypermatrices enable us to distinguish some non-isomorphic graphs with isospectral adjacency matrices.

The spectral analysis of hypermatrices is considerably more difficult to define and to compute [Chu93, Lub14, PRT12, FGL+11, GER11, CD13, QSW13, LSQ14, FW95, Qi12, IGZ94] when compared with the analysis of matrices. Mesner and Bhattacharya in [MB90, MB94] introduced an \(m\) operands product for \(m\)-th order hypermatrices. E. Gnang, V. Retakh and A. Elgammal proposed in [GER11] a generalization to hypermatrices of the notions of Hermicity and unitarity. E. Gnang, V. Retakh and A. Elgammal also proved in [GER11] a conjecture of Bhattacharya by using these new definitions to extend the spectral decomposition to hypermatrices of arbitrary order. In the present work we show that the spectral decomposition introduced in [GER11] for a hypermatrix is mostly determined by the spectral decomposition its minors in a similar spirit to Cauchy’s interlacing theorem. We present new algorithms based on resolutions of the identity for deriving generators of the elimination ideals associated with matrix and hypermatrix spectral decomposition constraints. We deduce from these procedure new approaches for approximating the spectral decomposition of matrices and hypermatrices. We extend to even order hypermatrices the self-adjoint argument for the existence of real solutions to spectral constraints. Finally we deduce from the the spectral decomposition of hypermatrices upper and lower bounds for hypermatrix eigenvalues introduced by L.H. Lim and L.Q. Qi in [Lim05, Qi05].

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2. Notation

We describe here for the convenience of the reader the notation that used throughout the paper. The Hadamard product of the column vectors \(\mathbf{a}, \mathbf{b} \in \mathbb{C}^{n \times 1}\) noted \(\mathbf{a} \star \mathbf{b}\), corresponds to a column vector of the same dimensions whose entries correspond to the product of corresponding entries of \(\mathbf{a}\) and \(\mathbf{b}\); we write

\[k - \text{th entry of } \mathbf{a} \star \mathbf{b} \text{ is } a_k b_k.\]
Let us recall here the notation used for the vector product of \( \mathbf{a}, \mathbf{b} \in \mathbb{C}^{n \times 1} \) with background matrix \( \mathbf{M} \). We write\
\[
\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} := \sum_{0 \leq k_0, k_1 < n} a_{k_0} m_{k_0, k_1} b_{k_1},
\]
in particular it follows that\
\[
\langle \mathbf{a}, \mathbf{b} \rangle := \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{I}} = \sum_{0 \leq k < n} a_k b_k
\]
where the entries of \( \mathbf{I} \) are given by\
\[
[I]_{i,j} = \delta_{i,j} = \begin{cases} 1 & \text{if } 0 \leq i = j < n \\ 0 & \text{otherwise} \end{cases}.
\]
The inner-product of \( \mathbf{a}, \mathbf{b} \in \mathbb{C}^{n \times 1} \) is expressed as \( \langle \mathbf{a}, \mathbf{b} \rangle \). Furthermore we have\
\[
\mathbf{a} \ast \left( \sum_{0 \leq t < m} \mathbf{b}_t \right) = \sum_{0 \leq t < m} \mathbf{a} \ast \mathbf{b}_t, \quad \text{and} \quad \langle \mathbf{1}_{n \times 1}, \mathbf{a} \ast \mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{a} \ast \mathbf{b}, \mathbf{1}_{n \times 1} \rangle.
\]

For a set of \( n \) vectors \( \{\mathbf{v}_j\}_{0 \leq j < n} \subset \mathbb{C}^{n \times 1} \) we define the multiple vector correlation product noted \( \langle \mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{n-1} \rangle \) to be\
\[
\langle \mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{n-1} \rangle := \langle \mathbf{1}_{n \times 1}, \mathbf{v}_0 \ast \cdots \ast \mathbf{v}_{n-1} \rangle.
\]
It shall also be convenient to adopt the notation convention\
\[
\mathbf{a}^* := (a_k^*)_{0 \leq k < n}.
\]
The \( n \) dimensional vector \( \mathbf{w} \) will denote the vector collecting powers of the primitive \( n \)-th roots of unity with entries given by\
\[
\mathbf{w} := \left( w_k = \exp \left\{ \frac{2 \pi i k}{n} \right\} \right)_{0 \leq k < n}.
\]
Finally, for an arbitrary vector \( \mathbf{v} \in \mathbb{C}^{n \times 1} \), we associate to it the \( n \times n \) Vandermonde matrix whose entries are specified as follows\
\[
[V_{\text{Vandermonde}}(\mathbf{v})]_{i,j} = (v_i)^j.
\]

3. Overview of the Bhattacharya-Mesner algebra and its dual

We present here the basic elements of the Bhattacharya-Mesner (BM) algebra proposed as a generalization of matrix algebra.

**Definition 1.** The Bhattacharya-Mesner [MB90, MB94] algebra generalizes the matrix product\
\[
\mathbf{B} = \mathbf{A}^{(1)} \cdot \mathbf{A}^{(2)}
\]
where \( \mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{B} \) are matrices of sizes \( n_1 \times k, k \times n_2, n_1 \times n_2 \), respectively, and\
\[
b_{i_1 i_2} = \sum_{1 \leq j \leq k} a_{i_1 j}^{(1)} a_{j i_2}^{(2)},
\]
to the hypermatrix product\
\[
\mathbf{B} = \text{Prod} \left( \mathbf{A}^{(1)}, \cdots, \mathbf{A}^{(m)} \right)
\]
where $B$ is an $n_1 \times \cdots \times n_m$ hypermatrix, $A^{(i)}$ for $i = 1, \ldots, m - 1$ is a hypermatrix of size obtained by replacing $n_{i+1}$ by $k$ in the size of $B$, and $A^{(m)}$ is a hypermatrix of the size $k \times n_2 \times \cdots \times n_m$. Here

$$b_{i_1, \ldots, i_m} = \sum_{1 \leq j \leq k} a^{(1)}_{i_1, j, i_3, \ldots, i_m} \cdots a^{(t)}_{i_1, i_t, i_{t+2}, \ldots, i_m} \cdots a^{(m)}_{j, i_2, \ldots, i_m}.$$

The Mesner-Bhattacharya algebra generalizes the matrix product to an $m$-operand product for order $m$ hypermatrices. We recall, that the $(i_1, i_2)$ entry of the $k$-th power of the adjacency matrix enumerates the number of paths of length $k$ from vertex $i_1$ to vertex $i_2$ in the associated graph. This combinatorial interpretation extends to the Mesner-Bhattacharya algebra where for cubic hypermatrices (the analog of square matrices) the composition of products which we call powers enumerates the ways of gluing simplices by identifying some of their vertices [Gna14]. As a result of the non-associativity of the Mesner-Bhattacharya algebra, the number of ways of computing hypermatrix products which involves some fixed number of hypermatrices is given by the Fuss-Catalan numbers [Lin11]. In particular, a third order hypermatrix $A$ admits 3 different ways of computing fifth degree hypermatrix powers

$$\text{Prod}(A, A, \text{Prod}(A, A, A)),$$

$$\text{Prod}(A, \text{Prod}(A, A, A), A),$$

$$\text{Prod}(\text{Prod}(A, A, A), A, A).$$

In [GER11] a slight variation of the BM product was introduced and is expressed here as

$$C = \text{Prod}_B \left( A^{(1)}, \ldots, A^{(m)} \right).$$

The resulting hypermatrix $C$ is an $n_1 \times \cdots \times n_m$ hypermatrix. Similarly to the BM product, each one of the input hypermatrices $A^{(i)}$ for $i = 1, \ldots, m - 1$ is a hypermatrix of size obtained by replacing $n_{i+1}$ by $k$ in the size of $C$, and $A^{(m)}$ is a hypermatrix of size $k \times n_2 \times \cdots \times n_m$. The difference with the BM product is seen in the fact that the product involves a background cubic hypermatrix $B$ of $m$-th order with sides of length $k$ (meaning that the hypermatrix $B$ is of size $k \times k \times \cdots \times k$), we have

$$c_{i_1, \ldots, i_m} = \sum_{1 \leq j_1, \ldots, j_m \leq k} a^{(1)}_{i_1, j_1, i_3, \ldots, i_m} \cdots a^{(t)}_{i_1, i_t, i_{t+2}, \ldots, i_m} \cdots a^{(m)}_{j_1, i_2, \ldots, i_m} b_{j_1, j_2, \ldots, j_m}.$$

In the particular case of matrices we write

$$C = A^{(1)} \cdot_B A^{(2)}. \tag{3.1}$$

Note that the original BM product is recovered by setting the background hypermatrix $B$ to the Kronecker hypermatrix (namely the hypermatrix whose non zero entries all equal one and correspond to entries whose indices all have the same values in particular the Kronecker matrix is the identity matrix).

In particular the product

$$D = C * \text{Prod}_B \left( A^{(1)}, \ldots, A^{(m)} \right),$$

eythwise given by

$$c_{i_1, \ldots, i_m} = \sum_{1 \leq j_1, \ldots, j_m \leq k} a^{(1)}_{i_1, j_1, i_3, \ldots, i_m} \cdots a^{(t)}_{i_1, i_t, i_{t+2}, \ldots, i_m} \cdots a^{(m)}_{j_1, i_2, \ldots, i_m} b_{j_1, j_2, \ldots, j_m}.$$
is dual the product entrywise specified by
\[ b_{j_1,j_2,\cdots,j_m} = \sum_{1 \leq i_1, \ldots, i_m \leq k} a_{i_1,j_2,\cdots,j_m}^{(1)} \cdots a_{i_1,\ldots,i_t,j_{t+1},\ldots,j_m}^{(t)} \cdots a_{i_1,\ldots,i_{m}}^{(m)} c_{i_1,\ldots,i_m}. \] (3.2)

The duality arises from interchanging the summands and the hypermatrix entry indices. Note that in the matrix case, the product \( D \ast (A \cdot_C B) \) is dual to the product \( C \ast (A^T \cdot_D B^T) \). As a result matrix multiplication is self dual. More generally, the algebra dual to the BM algebra for higher order hypermatrices yields a different product. This new hypermatrix product was independently introduced by R. Kerner in [Ker08]. Their proposed ternary product definition was motivated by Theoretical physics considerations and can be shown to share many properties with the BM algebra via the duality argument.

4. Constrained inverse pair.

We discuss here an algorithm for solving the constrained matrix inverse pair problem. The importance of the constrained inverse pair problem stems from the fact that it’s solution yields a general orthogonalization procedure which extends to hypermatrices of all order. A solution to the constrained inverse pair problem for input non-zero \( n \times n \) matrix pair \((U, V)\) is obtained by identifying a pair of non-zero \( n \times n \) matrices \((X, Y)\) for which we have
\[ (X \cdot Y)^T \ast (X \cdot Y^T) = (X \cdot Y^T) \cdot (X \cdot Y^T) \]
subject to the inner-product constraints
\[ \left\{ \begin{array}{l}
\forall \ 0 \leq k < n \\
\ 0 \leq i \neq j < n \\
\langle u_i \ast v_j, w^{(-k)} \rangle = \langle x_i \ast y_j, w^{(-k)} \rangle
\end{array} \right. \].

more explicitly we express the constraints above as
\[ \left\{ \begin{array}{l}
\forall \ 0 \leq i \neq j < n, \ x_i \ast y_j = u_i \ast v_j - \left( \frac{u_i^T \cdot v_j}{n} \right) 1_{n \times 1}
\end{array} \right. \]. (4.1)

These constraints can be reformulated as a set of \( n^2(n-1) \) sparse linear constraints in \( 2n^2 \) variables. The sparse linear constraints are obtained by applying the logarithm function to both sides of each equations
\[ \left\{ \begin{array}{l}
\forall \ 0 \leq k < n \\
\ 0 \leq i \neq j < n \\
\ln (x_{ik}) + \ln (y_{kj}) = \ln \left( u_{ik} v_{kj} - \frac{u_i^T \cdot v_j}{n} \right)
\end{array} \right. \]. (4.2)

**Proposition 2.** A solution to the constrained matrix inverse pair problem exist if and only if the right hand side \( n^2(n-1) \times 1 \) vector lies in the column space of the sparse binary entered \( n^2(n-1) \times 2n^2 \) matrix associated with the sparse linear constraints.

**Proof.** The proof follows from the reduction of the constrained inverse pair problem to solving a systems of linear equation. \( \square \)

The orthogonalization procedure devised for orthogonal matrices from the constrained inverse pair problem is given by the constraints in the \( n^2 \) variables \( \{\ln q_{ij}\}_{0 \leq i, j < n} \)
\[ \left\{ \begin{array}{l}
\ln q_{ut} + \ln q_{vt} = \ln \left( a_{ut} a_{vt} - n^{-1} \sum_{0 \leq j < n} a_{uj} a_{vj} \right)
\end{array} \right. \] \( 0 \leq u < v < n \),
\[ \ 0 \leq t < n \).
The orthogonalization procedure devised for unitary matrices from the constrained inverse pair problem are slightly different and specified by the following linear constraints in the $2n^2$ variables $\{\ln r_{ij}, \theta_{ij}\}_{0 \leq i,j < n}$

$$\begin{cases} \ln r_{ut} + \ln r_{vt} = \Re \left\{ \ln \left( a_{ut}a_{vt}e^{i(a_{ut} - a_{vt})} - \sum_{0 \leq j < n} \frac{a_{uj}a_{vj}e^{i(a_{uj} - a_{vj})}}{n} \right) \right\} \\ \theta_{ut} - \theta_{vt} = 2k\pi + \Im \left\{ \ln \left( a_{ut}a_{vt}e^{i(a_{ut} - a_{vt})} - \sum_{0 \leq j < n} \frac{a_{uj}a_{vj}e^{i(a_{uj} - a_{vj})}}{n} \right) \right\} \end{cases} \text{ for } u < v \quad t < n$$

for $k \in \mathbb{Z}$. The technique devised for solving the constrained matrix inverse pair problem extends to hypermatrices of arbitrary order. For $k$-th order hypermatrices, the corresponding problem amount to identify a $k$-th tuple of hypermatrices. In particular for a given triplet of third order hypermatrices $(A, B, C)$ the corresponding problem will amount to identify a new triplet of third order hypermatrices $(X, Y, Z)$ subject to the constraints

$$\left[ \text{Prod} \left( X, Y^{T^2}, Z^{T} \right) \right]_{u_0, u_1, u_2} = \begin{cases} \lambda_k \neq 0 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

and for all integers $0 < t < n$ and indices $u_0, u_1, u_2$ which are not all equal we require that

$$\left\{ \left( a_{u_0,u_2} \star b_{u_1,u_0} \star c_{u_2,u_1} \star w^{(t)} \right) \right\} = \left( x_{u_0,u_2} \star y_{u_1,u_0} \star z_{u_2,u_1} \star w^{(t)} \right).$$

In the expressions of the constraints above, the indexing of the vectors follows from the viewpoint of third order hypermatrix as matrices of vectors.

5. Spectral decomposition from the spectrum of minors.

We discuss here an argument for reducing the spectral decomposition of an arbitrary cubic $k$-th order hypermatrices to the decomposition of cubic minors of same order having sides of length $k$. The hypermatrices considered here are associated with directed and weighted $k$-uniform hypergraph having no degenerate edges. In other words, the collection of $k$ vertices associated with any hyperedge of the hypergraph must all be distinct. Such hypermatrices arise as $(k - 1)$-path adjacency hypermatrices of rooted trees whith edges are directed towards the leaf nodes and away from the root. They also arise as $(k - 1)$-path adjacency hypermatrices of directed acyclic graphs.

**Theorem 3.** Let $H$ denote a directed weighted $k$-uniform hypergraph having no degenerate hyperedges. Then the spectral decomposition of its $k$-vertex sub-hypergraphs determine the spectral decomposition of a larger $k$-uniform hypergraph on $n(n^k)$ vertices which admit $H$ as a sub-hypergraph.

We will first present the detail proof in the case of graphs and subsequently proceed to extend the argument to hypergraphs.

**Proof.** In the case of graphs, Theorem 3 asserts that for some arbitrary directed and weighted graph

$$G_1 := (V_1 = \{0, 1, \cdots, n - 1\}, E_1 \subset V_1 \times V_1)$$
having no loop edges, the spectral decomposition of the adjacency matrices of its 2 vertex subgraphs determine the spectral decomposition of the adjacency matrix of a larger graph

\[ G_2 := \left( V_2 = \left\{ 0, 1, \cdots, n \left( \frac{n}{2} \right) - 1 \right\}, E_2 \subset V_2 \times V_2 \right) \]

such that

\[ G_1 \subset G_2. \]

Let \( A \) denote the \( n \times n \) adjacency matrix of the graph \( G_2 \). We seek to determine the spectral decomposition of the adjacency matrix of \( G_2 \) expressed as

\[
\begin{pmatrix}
A & B_{01} \\
B_{10} & B_{11}
\end{pmatrix} = (U \text{ diag } \{ \mu \}) \left( V \text{ diag } \{ \nu \} \right)^{\dagger}, \quad U V^{\dagger} = I_{n \choose 2}
\]

where the sub matrices \( B_{01}, B_{10} \) and \( B_{11} \) are matrices of size respectively given by \( n \times (n-1) \binom{n}{2}, (n-1) \binom{n}{2} \times n \), and \( (n-1) \binom{n}{2} \times (n-1) \binom{n}{2} \). Incidentally the matrices \( U \) and \( V \) denotes \( n \binom{n}{2} \times n \binom{n}{2} \) matrix which are respectively associated with a basis for the left and right eigenspaces respectively. Finally the vectors \( \mu, \nu \) are \( n \binom{n}{2} \) dimensional vectors such that the entries of their Hadamard product \( \mu \otimes \nu \) yield the eigenvalues of the adjacency matrix of \( G_2 \). The \( k \)-th column vector of the matrices \( U \text{ diag } \{ \mu \} \) and \( V \text{ diag } \{ \nu \} \) denote the scaled left eigenvectors and right eigenvectors respectively.

For \( 0 \leq j_1 < j_2 < n \) let \( A^{[j_0, j_1]} \) denote \( n \times n \) matrices defined by

\[
A^{[j_1, j_2]} = A \sum_{\sigma \in S_2} e_{\sigma(j_1)} \otimes e_{\sigma(j_2)}^{\dagger}
\]

where \( \{ e_j \}_{0 \leq j < n} \) denotes column vectors of the identity matrix \( I_n \). By construction we have

\[
A = \sum_{0 \leq j_1 < j_2 < n} A^{[j_1, j_2]}
\]

Furthermore let the spectral decomposition of the matrix \( A^{[j_1, j_2]} \) be expressed as

\[
A^{[j_1, j_2]} = \left( U^{[j_1, j_2]} \text{ diag } \{ \mu^{[j_1, j_2]} \} \right) \left( V^{[j_1, j_2]} \text{ diag } \{ \nu^{[j_1, j_2]} \} \right)^{\dagger}
\]

we have

\[
\forall 0 \leq i_1, i_2 < n, \quad \delta_{i_1 i_2} = \sum_{0 \leq k < n} \left( \frac{u^{[j_1, j_2]}_{i_1 k}}{\sqrt{n-1}} \right) \left( \frac{u^{[j_1, j_2]}_{i_2 k}}{\sqrt{n-1}} \right)
\]

and

\[
\forall 0 \leq i_1, i_2 < n, \quad \delta_{i_1 i_2} = \sum_{0 \leq k < n} \left( \frac{u^{[j_1, j_2]}_{i_1 k}}{\sqrt{n-1}} \right) \left( \frac{v^{[j_1, j_2]}_{i_2 k}}{\sqrt{n-1}} \right)
\]

The right-hand side of the expressions in 5.1 should be viewed as expressing inner-products of \( n \binom{n}{2} \)-dimensional vectors. The first \( n \) rows of the matrices \( U \) and \( V \) are therefore
determined by the expansion above. The remaining \((n-1)\binom{n}{2}\) rows of the matrix \(U\) and \(V\) are determined by either the Gram-Schmidt process or by solving instances of the constrained inverse matrix pair problem. The proposed construction used for adjacency matrices of graphs is easily extended to higher order hypermatrices via the BM algebra. For notational convenience we discuss here only the third order hypermatrix case. We recall from [GER11] that by analogy to the matrix case the spectral decomposition a third order hypermatrix \(A\) is expressed in terms of scaled eigenmatrices as follows

\[
A = \text{Prod} \left( \text{Prod} \left( Q, D_1, D_1^T \right), \left[ \text{Prod} \left( U, D_2, D_2^T \right) \right]^T, \left[ \text{Prod} \left( V, D_3, D_3^T \right) \right]^T \right).
\]

The collection of row-depth matrix slices of the hypermatrices \(Q, U\) and \(V\) yield bases for the eigenmatrices of \(A\) which are also subject to the constraints

\[
\left[ \text{Prod} \left( Q, U^{T_2}, V^T \right) \right]_{i_1,i_2,i_3} = \begin{cases} 1 & \text{if } i_1 = i_2 = i_3 \\ 0 & \text{otherwise} \end{cases}.
\]

The scaling hypermatrices \(\{D_i\}_{1 \leq i \leq 3}\) correspond to the hypermatrix analog of diagonal matrices. Consequently, the row-depth slices of the hypermatrices \(\text{Prod} \left( Q, D_1, D_1^T \right),\ \text{Prod} \left( U, D_2, D_2^T \right),\ \text{and} \ \text{Prod} \left( V, D_3, D_3^T \right)\), correspond to scaled eigenmatrices of \(A\). By analogy to the matrix case, for all triplets \((j_1, j_2, j_3)\) of vertex label satisfying the inequality \(0 \leq j_1 < j_2 < j_3 < n\) we define the matrix \(A_{[j_1,j_2,j_3]}\) to be the third order hypermatrix expressed by the equality

\[
A_{[j_1,j_2,j_3]} = A \star \sum_{\sigma \in S_3} e_{\sigma(j_1)} \otimes e_{\sigma(j_2)}^T \otimes e_{\sigma(j_3)}^{T_2},
\]

where the set \(\{e_j\}_{0 \leq j < n}\) denotes column vectors of the identity matrix \(I_n\) and the transpose operation here refers to the hypermatrix transpose introduced in [GER11]. Similarly we have

\[
A = \sum_{0 \leq j_1 < j_2 < j_3 < n} A_{[j_1,j_2,j_3]}
\]

by appropriately concatenating the spectral decomposition of the hypermatrix minors \(A_{[j_0,j_1,j_2]}\) and using the generalization to hypermatrices of the constrained inverse matrix pair problem, we deduce the spectral decomposition of a larger \(n\binom{n}{3} \times n\binom{n}{3} \times n\binom{n}{3}\) which admits the hypermatrix \(A\) as a sub-hypermatrix.

More generally a similar construction is easily devised for higher order hypermatrices. The argument therefore provides a way to reduce the spectral decomposition of \(k\)-th order cubic hypermatrices to the spectral decomposition cubic sub-hypermatrices with sides of length \(k\).

\[
\square
\]

6. Computing spectral elimination ideals

For the purpose of introducing new iterative algorithms to approximate the spectral decomposition of matrices and hypermatrices, we will need to isolate variables associated with entries of the scaling hypermatrices. To achieve this we describe elimination procedures for computing generators of elimination ideals associated with the spectral decomposition constraints. Recall that the well known matrix spectral decomposition for
an \( n \times n \) matrix \( A \) is obtained by solving for the matrices \( U, V, \) and \( \{ D_i \}_{0 \leq i < 2} \) subject to the constraints

\[
\begin{align*}
A &= (U D_1) (VD_2)^T, \\
U V^T &= I_n, \\
D_i D_i^T &= D_i^T D_i, \quad \forall 0 \leq i < 2.
\end{align*}
\]  

(6.1)

For notational convenience we reformulate the diagonality constraints in the spectral constraints in terms of \( n \)-dimensional vectors \( \mu \) and \( \nu \) as follows

\[
\begin{align*}
A &= (U \text{ diag } \{ \mu \}) (V \text{ diag } \{ \nu \})^T, \\
U V^T &= I_n.
\end{align*}
\]

The two elimination ideals of interest associated with the spectral decomposition constraints are:

\[
I_{\mu, \nu} := \mathbb{C} [\mu * \nu] \cap \text{Ideal generated by } \left\{ (U \text{ diag } \{ \mu \}) (V \text{ diag } \{ \nu \})^T - A, U V^T - I \right\}
\]

and

\[
I_{U, \nu} := \mathbb{C} [U, \nu] \cap \text{Ideal generated by } \left\{ (U \text{ diag } \{ \mu \}) (V \text{ diag } \{ \nu \})^T - A, U V^T - I \right\}.
\]

Let us start by describing the derivation of generators for \( I_{\mu, \nu} \). Consider the identity

\[
\forall 0 \leq k \leq n, \quad A^k = \left( U \text{ diag } \{ \mu^k \} \right) \left( V \text{ diag } \{ \nu^k \} \right)^T.
\]

Let \( \{ u_i \}_{0 \leq i < n} \) and \( \{ \nu_j \}_{0 \leq j < n} \) respectively denote row vectors of the matrices \( U \) and \( V \), we have

\[
\begin{pmatrix}
[u_i * \nu_j]_0 \\
\vdots \\
[u_i * \nu_j]_{n-1}
\end{pmatrix}_{0 \leq i,j < n}
= (I_{n^2} \otimes \text{Vandermonde (} \mu * \nu \))^{-1} \cdot \begin{pmatrix} [A^0]_{i,j} \\
\vdots \\
[A^{n-1}]_{i,j}
\end{pmatrix}_{0 \leq i,j < n}.
\]

It is implicitly assumed in the identity above that \( A \) has distinct eigenvalues, otherwise the identity above is expressed using the Penrose matrix inverse instead as follows

\[
\begin{pmatrix}
[u_i * \nu_j]_0 \\
\vdots \\
[u_i * \nu_j]_{n-1}
\end{pmatrix}_{0 \leq i,j < n}
= (I_{n^2} \otimes \text{Vandermonde (} \mu * \nu \))^+ \cdot \begin{pmatrix} [A^0]_{i,j} \\
\vdots \\
[A^{n-1}]_{i,j}
\end{pmatrix}_{0 \leq i,j < n}.
\]

For simplicity, let us assume that \( A \) has distinct eigenvalues, in which case generators for \( I_{\mu, \nu} \) are prescribed by

\[
I_{\mu, \nu} = \text{Ideal generated by } \left\{ (u_i * \nu_j) * (u_j * \nu_j) - (u_i * \nu_j) * (u_j * \nu_j) \right\}_{0 \leq i,j < n}.
\]

Note that the characteristic polynomial is easily derived from the generators of \( I_{\mu, \nu} \) by observing that the Hadamard products \( u_i * \nu_j \) is given by

\[
u_i * \nu_j = \begin{pmatrix}
\prod_{t \neq 0} \left( \frac{A - \mu^n \nu_t}{\mu^n - \mu t} \right)_{ij} \\
\vdots \\
\prod_{t \neq n-1} \left( \frac{A - \mu^n \nu_t}{\mu_n - \mu_{t+1}} \right)_{ij}
\end{pmatrix},
\]
\[ I_n = [\langle u_i, v_j \rangle]_{ij} = [\langle u_i \ast v_j, 1_{n \times 1} \rangle]_{ij}, \]
\[ I_n = \sum_{0 \leq k < n} \prod_{0 \leq t \neq k < n} \left( \frac{A - \mu_t v_I}{\mu_k v_k - \mu_t v_I} \right). \]

It follows from well known properties of the Lagrange interpolation construction that
\[ O_{n \times n} = \prod_{0 \leq t < n} (A - \mu_t v_I). \]

which expresses the classical Cayley-Hamilton theorem. Generators for \( I_{\mu} v \) as well as the classical characteristic polynomial have been historically devised using the determinant polynomial. In order to extend the spectral decomposition to hypermatrices it is useful to establish methods for deriving generators for spectral elimination ideals without assuming a priori knowledge of the determinant polynomial. Note that resultant computations or alternatively Groebner basis computation can also be used to compute generators for \( I_{\mu} v \) starting from the constraints in 6.1 as pointed out in [GER11]. However these approaches are less direct and considerably less efficient because one must identify for spectral constraints optimal monomial orderings.

The derivation of generators for the elimination ideal \( I_{U, V} \) is achieved by eliminating the variables associated with the eigenvalues determined by the entries of the vectors \( \mu \) and \( \nu \). The elimination of variables (associated with the eigenvalues) from spectral constraints is achieved via resolutions of the identity. We recall that the spectral constraints in 6.1 can alternatively be expressed as a collection of \( n^2 \) inner product constraints of the form
\[ \{ a_{i,j} = \langle u_i \ast \mu, v_j \rangle \}_{0 \leq i,j < n} \quad (6.2) \]

The use of resolution of the identity amounts to rewriting each of the \( n^2 \) constraints as follows
\[ \left\{ \sum_{0 \leq k < n} \langle u_i \ast \mu, v_j \rangle \langle v_k \rangle^{-1} u_k = a_{i,j} = \sum_{0 \leq s,t < n} \mu_s v_t f_{n,s+t,n,i+j} (U, V) \right\}_{0 \leq i,j < n} \quad (6.3) \]

where \( \{ f_{i,j} (U, V) \}_{0 \leq i,j < n^2} \subset C[U, V] \). The constraints result in a system of \( n^2 \) equations in \( n^2 \) unknowns \( \{ \mu_t v_I \}_{0 \leq i,j < n} \). We may therefore express the polynomial constraints in 6.3 in matrix form as follows
\[ \begin{pmatrix} f_{0,0} & \cdots & f_{0,j} & \cdots & f_{0,(n^2-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{i,0} & \cdots & f_{ij} & \cdots & f_{i,(n^2-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{(n^2-1),0} & \cdots & f_{(n^2-1),j} & \cdots & f_{(n^2-1),(n^2-1)} \end{pmatrix} \begin{pmatrix} \mu_0 v_0 \\ \vdots \\ \mu_i v_j \\ \vdots \\ \mu_{n-1} v_{n-1} \end{pmatrix} = \begin{pmatrix} a_{0,0} \\ \vdots \\ a_{i,j} \\ \vdots \\ a_{(n-1),(n-1)} \end{pmatrix} \quad (6.4) \]

For any integer \( 0 \leq k < n^2 \), let \( F_k \) denote the \( n^2 \times n^2 \) matrix constructed by substituting the \( k \)-th column of the left hand side matrix \( F \) above by the righthand side \( n^2 \times 1 \) vector of entries of \( A \) as prescribed by Cramer's rule. The generators for \( I_{U, V} \) are determined by rational function identities derived from the tautologies
\[ \{ (\mu_t v_I) (\mu_j v_j) = (\mu_t v_I) (\mu_j v_j) \}_{0 \leq i,j < n} \quad (6.5) \]
It follows from Cramers’ rule these that rational identities can be expressed as
\[
\left\{ \frac{\det(F_{n+i}F_{n-j} + F_{n-j}F_{n+i}) - \det(F_{n+i}F_{n-j} - F_{n-j}F_{n+i})}{\det(F^2)} = 0 \right\}_{0 \leq i < j < n}.
\] (6.6)

The sought after generators are thus given by
\[
I_{U,V} = \text{Ideal generated by } \{ \det(F_{n+i}F_{n-j} + F_{n-j}F_{n+i}) - \det(F_{n+i}F_{n-j} - F_{n-j}F_{n+i}) = 0 \} \quad (6.7)
\]

The elimination ideal \( I_{U,V} \) is seldom used in the literature however it’s analog will be of crucial importance for higher order hypermatrices. Finally we remark that \( I_{U,V} \) and \( I_{\mu,\nu} \) conveniently separates polynomial constraints in the entries of the eigenvector from the polynomial constraints in the entries of the eigenvalues. It is relatively straight forward to use the generators for \( I_{U,V} \) and \( I_{\mu,\nu} \) to set up an optimization problem which allows for the approximation of the spectral decomposition provided of course that some contractive map can be deduced from the generators of \( I_{U,V} \) and \( I_{\mu,\nu} \) respectively.

7. Approximating the spectral decompositions of matrices and hypermatrices.

We describe here a recursive algorithm for approximating the spectral decomposition of matrices and hypermatrices whose main diagonal entries are all equal. Such matrices arise as adjacency matrices of weighted directed graph, from normalized covariance matrices, normalized skewness hypermatrices, normalized kurtosis hypermatrices as well as path adjacency hypermatrices associated with directed acyclic graphs. Without loss of generality we can assume that the diagonal entries are all zero. We start by discussing the matrix case. Let \( A \in \mathbb{C}^{n \times n} \) be such a matrix, furthermore let \( A_k \) denote the \( n-1 \) minor matrix which result from setting to zero all the entries of the \( k \)-th row and all the entries of the \( k \)-th column \( A \). By construction we have
\[
A = (n-2)^{-1} \sum_{0 \leq k < n} A_k.
\]

Let us assume the spectral decomposition of the matrices \( A_k \) to be known and expressed by
\[
(n-2)^{-1} A_k = \left( U^{[k]} \text{diag} \{ \mu^{[k]} \} \right) \left( V^{[k]} \text{diag} \{ \nu^{[k]} \} \right)^{\dagger}, \quad I_n - e_k \cdot e_k^T = U^{[k]} \left( V^{[k]} \right)^{\dagger}.
\]

We deduce from the spectral decomposition of the matrices \( \{ A_k \}_{0 \leq k < n} \) that for all \( 0 \leq i_0, i_1 < n \)
\[
\begin{align*}
a_{i_0 i_1} &= \sum_{0 \leq t < n} \left( \sqrt{n-1} \mu_t^{[k]} \frac{u_{i_0 t}^{[k]}}{\sqrt{n-1}} \right) \left( \sqrt{n-1} \nu_t^{[k]} \frac{v_{i_1 t}^{[k]}}{\sqrt{n-1}} \right) \\
\delta_{i_0 i_1} &= \sum_{0 \leq t < n} \left( \frac{u_{i_0 t}^{[k]}}{\sqrt{n-1}} \right) \left( \frac{v_{i_1 t}^{[k]}}{\sqrt{n-1}} \right)
\end{align*}
\]
As was done in the proof of theorem 3 we construct the spectral decomposition of a larger \( n^2 \times n^2 \) matrix by concatenating the spectral decomposition of the matrices \( \{A_k\}_{0 \leq k < n} \) which admits the matrix decomposition for third order hypermatrices is expressed by

\[
\begin{pmatrix}
A & B_{01} \\
B_{10} & B_{11}
\end{pmatrix} = (U \text{ diag } \{\mu\}) (V \text{ diag } \{\nu\})^\dagger, \quad I_{n^2} = UV^\dagger.
\]

(7.1)

For \( \{u_i\}_{0 \leq i < n^2} \) and \( \{v_j\}_{0 \leq j < n^2} \) denoting respectively the \( n^2 \)-dimensional row vectors of \( U \) and \( V \) we have

\[
\begin{pmatrix}
A & B_{01} \\
B_{01} & B_{11}
\end{pmatrix} = \sum_{0 \leq k < n^2} (\mu_k u_k) (\nu_k v_k)^\dagger
\]

the eigenvalue are labeled such that

\[
|\mu_0 \nu_0| \geq |\mu_1 \nu_1| \geq \cdots \geq |\mu_{n^2-1} \nu_{n^2-1}|.
\]

Initial conditions for gradient descent or other iterative procedure are to be set to the truncated matrix expression

\[
\begin{pmatrix}
\tilde{A} & 0_{n \times (n^2-n)} \\
0_{(n^2-n) \times n} & 0_{(n^2-n) \times (n^2-n)}
\end{pmatrix} = \sum_{0 \leq k < n} (\mu_k u_k^* \left( \begin{array}{c} 1_{n 	imes 1} \\ 0_{(n^2-n) \times 1} \end{array} \right)) \cdot (\nu_k v_k^* \left( \begin{array}{c} 1_{n \times 1} \\ 0_{(n^2-n) \times 1} \end{array} \right))^\dagger.
\]

The total error incurred by the truncation from the original larger matrix is

\[
\left\| \begin{pmatrix}
A & B_{01} \\
B_{01} & B_{11}
\end{pmatrix} - \begin{pmatrix}
\tilde{A} & 0_{n \times (n^2-n)} \\
0_{(n^2-n) \times n} & 0_{(n^2-n) \times (n^2-n)}
\end{pmatrix} \right\|^2 = \sum_{0 \leq k < n^2} (\mu_k u_k) \cdot (\nu_k v_k)^\dagger + \left\| \sum_{0 \leq k < n^2} (\mu_k u_k^* (1_{n \times 1})) \cdot (\nu_k v_k^* (1_{(n^2-n) \times 1})) \right\|^2
\]

The approximation error may be further reduced by the use of iterative methods such as gradient descent. The recursive approximation scheme presented here allows us to build up an approximation for the spectral decomposition of an \( n \times n \) matrix starting from the spectral decomposition of it’s \( \binom{n}{2} \) matrix minors of size \( 2 \times 2 \) all the way up to the \( n \) matrix minors of size \( (n-1) \times (n-1) \) from which we deduce the spectral decomposition as described above. The proposed algorithm is also of interest because it extends to hypermatrices of all order. For notational convenience we will restrict the discussion to third order hypermatrices. As we did for matrices, we build up the spectral decomposition of an arbitrary third order hypermatrix from the spectral decomposition of its \( 2 \times 2 \times 2 \) hypermatrix minors. It will therefore suffice to discuss in detail the derivation of the corresponding spectral elimination ideals associated with the spectral constraints for \( 2 \times 2 \times 2 \) hypermatrices which will be required by iterative procedures. The spectral decomposition for third order hypermatrices is expressed by

\[
A = \text{Prod} \left( \text{Prod} (Q, D_1, D_1^T), \left[ \text{Prod} (U, D_2, D_2^T) \right]^T, \left[ \text{Prod} (V, D_3, D_3^T) \right]^T \right).
\]
The collection of matrix slices of the hypermatrices \( Q, U \) and \( V \) collects the eigen-matrices of \( A \) which are subject to the constraints

\[
\left[ \prod \left( Q, U^T, V^T \right) \right]_{i_1, i_2, i_3} = \begin{cases} 
1 & \text{if } i_1 = i_2 = i_3 \\
0 & \text{otherwise}
\end{cases}
\]

The scaling hypermatrices \( \{D_i\}_{1 \leq i \leq 3} \) are hypermatrix analog of diagonal matrices. Third order hypermatrix diagonality constraint are similar to matrix diagonality constraint and expressed by

\[
\forall 1 \leq i \leq 3, \quad D_i \ast D_j \ast D_k = \prod \left( D_i^T, D_j^T, D_k^T \right),
\]

similarly to the matrix diagonality constraints. The the slices of \( \prod (Q, D_1, D_1^T) \), \( \prod (U, D_1, D_1^T) \), and \( \prod (V, D_3, D_3^T) \), correspond to scaled eigenmatrices of \( A \). Moreover the spectral decomposition constraints can rewritten in terms of inner product constraints of the form

\[
\forall 0 \leq i, j, k < 2, \quad a_{ijk} = \langle (\alpha_i \ast q_{ik} \ast \alpha_k), (\beta_j \ast u_{ji} \ast \beta_i), (\gamma_k \ast v_{kj} \ast \gamma_j) \rangle
\]

\[
\langle q_{ik}, u_{ji}, v_{kj} \rangle = \begin{cases} 
1 & \text{if } i = j = k \\
0 & \text{otherwise}
\end{cases} \quad \forall 0 \leq i, j, k < 2
\]

Just as we did in the matrix case we isolate the Hadamard product of vectors as follows

\[
q_{00} \ast u_{00} \ast v_{00} = \left[ \text{Vandermonde} \left( \begin{array}{c}
\alpha^2_{00} \beta^2_{00} \\
\alpha^2_{01} \beta^2_{01}
\end{array} \right) \right]^{-1} \left( \begin{array}{c}
1 \\
a_{000}
\end{array} \right)
\]

\[
q_{01} \ast u_{00} \ast v_{10} = \left[ \text{Vandermonde} \left( \begin{array}{c}
\alpha_{00} \alpha_{01} \beta_{00} \beta_{01} \\
\alpha_{01} \alpha_{11} \beta_{01} \beta_{11}
\end{array} \right) \right]^{-1} \left( \begin{array}{c}
0 \\
a_{001}
\end{array} \right)
\]

\[
q_{00} \ast u_{10} \ast v_{01} = \left[ \text{Vandermonde} \left( \begin{array}{c}
\alpha_{00} \alpha_{01} \beta_{00} \beta_{01} \\
\alpha_{01} \alpha_{11} \beta_{01} \beta_{11}
\end{array} \right) \right]^{-1} \left( \begin{array}{c}
0 \\
a_{010}
\end{array} \right)
\]

\[
q_{01} \ast u_{10} \ast v_{11} = \left[ \text{Vandermonde} \left( \begin{array}{c}
\alpha_{00} \alpha_{01} \beta_{00} \beta_{01} \\
\alpha_{01} \alpha_{11} \beta_{01} \beta_{11}
\end{array} \right) \right]^{-1} \left( \begin{array}{c}
0 \\
a_{011}
\end{array} \right)
\]

\[
q_{10} \ast u_{01} \ast v_{00} = \left[ \text{Vandermonde} \left( \begin{array}{c}
\alpha_{00} \alpha_{01} \beta_{00} \beta_{01} \\
\alpha_{01} \alpha_{11} \beta_{01} \beta_{11}
\end{array} \right) \right]^{-1} \left( \begin{array}{c}
0 \\
a_{100}
\end{array} \right)
\]

\[
q_{11} \ast u_{01} \ast v_{10} = \left[ \text{Vandermonde} \left( \begin{array}{c}
\alpha_{00} \alpha_{01} \beta_{00} \beta_{01} \\
\alpha_{01} \alpha_{11} \beta_{01} \beta_{11}
\end{array} \right) \right]^{-1} \left( \begin{array}{c}
0 \\
a_{101}
\end{array} \right)
\]

\[
q_{10} \ast u_{11} \ast v_{01} = \left[ \text{Vandermonde} \left( \begin{array}{c}
\alpha_{00} \alpha_{01} \beta_{00} \beta_{01} \\
\alpha_{01} \alpha_{11} \beta_{01} \beta_{11}
\end{array} \right) \right]^{-1} \left( \begin{array}{c}
0 \\
a_{110}
\end{array} \right)
\]

\[
q_{11} \ast u_{11} \ast v_{11} = \left[ \text{Vandermonde} \left( \begin{array}{c}
\alpha_{00} \alpha_{01} \beta_{00} \beta_{01} \\
\alpha_{01} \alpha_{11} \beta_{01} \beta_{11}
\end{array} \right) \right]^{-1} \left( \begin{array}{c}
1 \\
a_{111}
\end{array} \right)
\]

and the elimination ideal associated with the scaling factors is determined by the identity

\[
\Delta_2 = \{ (q_{ik} \ast u_{ji} \ast v_{kj}, 1_{2 \times 1}) \}_{0 \leq i, j, k < 2}
\]

where \( \Delta_2 \) denotes the \( 2 \times 2 \times 2 \) Kronecker delta. Finally by analogy to the matrix case the variables which scale the eigenmatrices are eliminated via the resolution of identity induced by the eigenmatrices. The elimination is achieved considering the hypermatrix sequence defined by

\[
G_0 = \Delta_2, \quad G_{k+1} = \prod G_k (Q, U^T, V^T)
\]
in conjunction with the constraints
\[
\left\{ A = \text{Prod}_{G_k} \left( \text{Prod} \left( Q, D_0, D_0^T \right), \left[ \text{Prod} \left( U, D_2, D_2^T \right) \right]^T, \left[ \text{Prod} \left( V, D_1, D_1^T \right) \right]^T \right) \right\}_{0 \leq k < 2}.
\]

For notational convenience we describe the constraints which determine the elimination ideal in the case when \( A \) is cyclically symmetric and admits a spectral decomposition where
\[
D_0 = D_1 = D_2 \text{ and } Q = U = V.
\]

In which case the spectral constraints would be expressed by the following inner product constraints.
\[
\forall 0 \leq i, j, k < 2, \quad a_{ijk} = \langle (\lambda_i * q_{ik} * \lambda_k), (\lambda_j * q_{ji} * \lambda_i), (\lambda_k * q_{kj} * \lambda_j) \rangle
\]
\[
(q_{ik}, q_{ji}, q_{kj}) = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j, k < 2.
\]

The elimination ideal is determined by rational function identities derived from the following tautologies:
\[
\left\{ \begin{array}{l}
\left( \lambda_{km}^2 \lambda_{kp}^2 \right)^3 = \left[ \left( \lambda_{km}^2 \right)^{\frac{3}{2}} \left( \lambda_{kp}^2 \right)^{\frac{3}{2}} \right]^3 \\
\left( \lambda_{km}^2 \lambda_{kn}^2 \lambda_{kp}^2 \right)^3 = \left( \lambda_{km}^2 \right)^3 \left( \lambda_{kn}^2 \right)^3 \left( \lambda_{kp}^2 \right)^3
\end{array} \right.
\]

8. Unitary and Hermitian Hypermatrices.

We describe the spectral decomposition of even order Hermitian hypermatrices. We also introduce here unitary hypermatrices and show how they can be used to extend to hypermatrices the self-adjoint adjoint argument for establishing the existence of real solutions to spectral constraints. The discussion here will also allow us to relate the spectral decomposition to the tensor eigenvalue first defined by Lim [Lim05] and Qi [Qi05]. An even order hypermatrix \( A \) is said to be Hermitian if
\[
\overline{A^T} = A.
\]

The general spectral decomposition of fourth order hypermatrices is expressed by
\[
A = \text{Prod} \left( \text{Prod} \left( Q, \Lambda_1, \Lambda_2, \Lambda_3 \right), \text{Prod} \left( U, \Gamma_1, \Gamma_2, \Gamma_3 \right)^T \right),
\]
\[
\text{Prod} \left( V, \Theta_1, \Theta_2, \Theta_3 \right)^T, \text{Prod} \left( W, \Xi_1, \Xi_2, \Xi_3 \right)^T \right) \left[ \begin{array}{l}
\text{Prod} \left( Q, \overline{U}^T, \overline{V}^T, \overline{W}^T \right) \end{array} \right]_{i_0, i_1, i_2, i_3} = \begin{cases} 1 & \text{if } i_0 = i_1 = i_2 = i_3 \\ 0 & \text{otherwise} \end{cases}
\]

where the entries of the scaling hypermatrices are given by:
\[
[\Lambda_1]_{i_0,i_1,i_2,i_3} := \delta_{i_1,i_2} \lambda_{i_3,i_3}, [\Lambda_2]_{i_0,i_1,i_2,i_3} := \delta_{i_2,i_3} \lambda_{i_1,i_0}, [\Lambda_3]_{i_0,i_1,i_2,i_3} := \delta_{i_1,i_0} \lambda_{i_2,i_2}
\]
\[
[\Gamma_1]_{i_0,i_1,i_2,i_3} := \delta_{i_1,i_2} \gamma_{i_3,i_3}, [\Gamma_2]_{i_0,i_1,i_2,i_3} := \delta_{i_2,i_3} \gamma_{i_1,i_0}, [\Gamma_3]_{i_0,i_1,i_2,i_3} := \delta_{i_1,i_0} \gamma_{i_2,i_2}
\]
\[
[\Theta_1]_{i_0,i_1,i_2,i_3} := \delta_{i_1,i_2} \theta_{i_3,i_3}, [\Theta_2]_{i_0,i_1,i_2,i_3} := \delta_{i_2,i_3} \theta_{i_1,i_0}, [\Theta_3]_{i_0,i_1,i_2,i_3} := \delta_{i_1,i_0} \theta_{i_2,i_2}
\]
\[
[\Xi_1]_{i_0,i_1,i_2,i_3} := \delta_{i_1,i_2} \xi_{i_3,i_3}, [\Xi_2]_{i_0,i_1,i_2,i_3} := \delta_{i_2,i_3} \xi_{i_1,i_0}, [\Xi_3]_{i_0,i_1,i_2,i_3} := \delta_{i_1,i_0} \xi_{i_2,i_2}
\]

where
\[
\delta_{i,j} = \begin{cases} 1 & \text{if } 0 \leq i = j < n \\ 0 & \text{otherwise} \end{cases}
\]
The corresponding entrywise constraints are expressed as 
\[ a_{i_0,i_1,i_2,i_3} = \left( \lambda_{i_0} \ast \lambda_{i_2} \ast \lambda_{i_3} \right) \ast q_{i_0,i_2,i_3} \ast \left( \gamma_{i_1} \ast \gamma_{i_0} \ast \gamma_{i_3} \right) \ast u_{i_1,i_3,i_0} \]
\[ \left( \theta_{i_2} \ast \theta_{i_0} \ast \theta_{i_1} \right) \ast v_{i_2,i_0,i_1} \ast \left( \xi_{i_3} \ast \xi_{i_1} \ast \xi_{i_2} \right) \ast w_{i_3,i_1,i_2} \]
\[ \delta_{i_0,i_1,i_2,i_3} = \langle q_{i_0,i_2,i_3}, u_{i_1,i_3,i_0}, v_{i_2,i_0,i_1}, w_{i_3,i_1,i_2} \rangle. \]

If \( A \) is Hermitian, we say that \( A \) admits a unitary decomposition if \( A \) can be expressed as
\[ A = \text{Prod} \left( \text{Prod} (Q, A_1, A_2, A_3), \text{Prod} (Q, \Gamma_1, \Gamma_2, \Gamma_3)^T \right), \]
\[ \text{Prod} (Q, \Theta_1, \Theta_2, \Theta_3)^T, \text{Prod} (Q, \Theta_1, \Theta_2, \Theta_3)^T \]
where \( Q \) is a unitary hypermatrix subject to the constraints
\[ \left[ \text{Prod} (Q, U^T, V^T, W^T) \right]_{i_0,i_1,i_2,i_3} = \begin{cases} 1 & \text{if } i_0 = i_1 = i_2 = i_3 \\ 0 & \text{otherwise} \end{cases}. \]

The unitary decomposition is entrywise expressed as
\[ a_{i_0,i_1,i_2,i_3} = \left( \lambda_{i_0} \ast \lambda_{i_2} \ast \lambda_{i_3} \right) \ast q_{i_0,i_2,i_3} \ast \left( \gamma_{i_1} \ast \gamma_{i_0} \ast \gamma_{i_3} \right) \ast q_{i_1,i_3,i_0} \]
\[ \left( \theta_{i_2} \ast \theta_{i_0} \ast \theta_{i_1} \right) \ast q_{i_2,i_0,i_1} \ast \left( \xi_{i_3} \ast \xi_{i_1} \ast \xi_{i_2} \right) \ast q_{i_3,i_1,i_2} \]
and the unitarity constraints are entrywise expressed by
\[ \langle q_{i_0,i_2,i_3}, q_{i_1,i_3,i_0}, q_{i_2,i_0,i_1}, q_{i_3,i_1,i_2} \rangle. \]

The vectors \( \{ \lambda_i \ast \lambda_j \ast \lambda_k, \gamma_i \ast \gamma_j \ast \gamma_k, \theta_i \ast \theta_j \ast \theta_k, \xi_i \ast \xi_j \ast \xi_k \}_{0 \leq k < n} \) denote the scaling vectors of the unitary decomposition of \( A \). Note that the unitary decomposition described for even order hypermatrices is analogous to spectral decomposition of Hermitian matrices expressed by
\[ A = (Q \text{ diag } \{ \mu \}) (Q \text{ diag } \{ \nu \})^T, \quad QQ^T = I_n \]
entrywise specified as
\[ a_{i_0,i_1} = \langle \mu_i \ast q_{i_0}, \nu_i \ast q_{i_1} \rangle, \quad \delta_{i_0,i_1} = \langle q_{i_0}, q_{i_1} \rangle \]
where the vectors \( \mu, \nu \) correspond to the scaling vectors. In particular, \( A \) is said to admit slice invariant unitary decomposition if
\[ \forall 0 \leq i < j < n, \quad \lambda_i = \lambda_j; \quad \gamma_i = \gamma_j; \quad \theta_i = \theta_j; \quad \xi_i = \xi_j. \]

In the case of matrices the spectral decomposition of a Hermitian matrix is always slice invariant because the scaling vectors do not change as the index of the the column vectors changes.

**Theorem 4.** Let \( A \) denotes an Hermitian hypermatrix which admits a slice invariant unitary decomposition then it follows that the Hadamard product of the scaling vectors must be real.

The general argument of the proof is well illustrated for hypermatrices of order 2 and 4. It will immediately be apparent how to extend the argument to arbitrary even order hypermatrices.
Proof. In the case of matrices we consider the bilinear form $\langle x, y \rangle_A$ associated with the matrix $A$. Let the spectral decomposition of the matrix $A$ be given by

$$ A = (Q \text{diag} \{\mu\}) (Q \text{diag} \{\nu\})^T, \quad Q Q^T = I_n $$

then the corresponding bilinear form can be expressed as follows

$$ \langle x, y \rangle_A = \sum_{0 \leq k < n} (\mu^* x, x^* y) u_k u_k^* $$

where $u_k$ denotes the $k$-th column of the unitary matrix $Q$. The bilinear form associated with the matrix $A^T$ is therefore given by

$$ \langle x, y \rangle_{A^T} = \sum_{0 \leq k < n} (\mu^* x, \nu^* y) u_k u_k^*. $$

By Hermicity of $A$ we have

$$ \forall x, y \in \mathbb{C}^n, \quad \langle x, y \rangle_A = \langle x, y \rangle_{A^T} \Rightarrow \mu^* \nu^* = \nu^* \mu, $$

thus deriving that the eigenvalues of $A$ must all be real. Similarly we consider the multilinear form associated with Hermitian hypermatrix $A$ which admits a scale invariant unitary decomposition. The corresponding multilinear form is expressed by

$$ \langle x, y, z \rangle_A = \sum_{0 \leq k < n} \langle (\lambda^* x), (\gamma^* y), (\theta^* z), (\xi^* t) \rangle_{\text{Prod}(u_k, u_k^*, u_k^*, u_k^*)} $$

The multilinear form associated with the hypermatrix $A^T$ is therefore given by

$$ \langle x, y, z, t \rangle_{A^T} = \sum_{0 \leq k < n} \langle (\lambda^* x), (\gamma^* y), (\theta^* z), (\xi^* t) \rangle_{\text{Prod}(u_k, u_k^*, u_k^*, u_k^*)}. $$

By Hermicity we have

$$ \forall x, y, z, t \in \mathbb{C}^n, \quad \langle x, y, z, t \rangle_A = \langle x, y, z, t \rangle_{A^T} \Rightarrow \lambda^* \gamma^* \theta^* \xi^* = \lambda^* \gamma^* \theta^* \xi^*. $$

Theorem 4 extends to hypermatrices the self-adjointness argument for establishing the existence of real solutions to spectral constraints. We may also write that

$$ \langle x, x, x, x \rangle_A = \sum_{0 \leq k < n} \langle (\lambda^* x), (\gamma^* x), (\theta^* x), (\xi^* x) \rangle_{\text{Prod}(u_k, u_k^*, u_k^*, u_k^*)}. $$

and therefore if we further make the simplifying assumption that for some positive $\mu$

$$ \forall 0 \leq i, k < n, \quad \mu \leq \min \{\langle e_i, \lambda \rangle, \langle e_i, \gamma \rangle, \langle e_i, \theta \rangle, \langle e_i, \xi \rangle\} $$

and

$$ \max \{\langle e_i, \lambda \rangle, \langle e_i, \gamma \rangle, \langle e_i, \theta \rangle, \langle e_i, \xi \rangle\} \leq \nu $$

the entries of scaling hypermatrix therefore yield upper and lower bounds for the eigenvalues for the symmetrized hypermatrix associated with the multilinear forms $\langle x, x, x, x \rangle_A$ introduced by [Lim05, Qi05]. The corresponding bounds are expressed by the inequality

$$ \|\mu x\|_{\ell_4}^4 \leq \langle x, x, x, x \rangle_A \leq \|\nu x\|_{\ell_4}^4. \quad (8.2) $$
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