MIXED DETERMINANTS AND THE KADISON-SINGER PROBLEM

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Abstract. We adapt the arguments of Marcus, Spielman and Srivastava in their proof of the Kadison-Singer problem to prove improved paving estimates. Working with Anderson’s paving formulation of Kadison-Singer instead of Weaver’s vector balancing version, we show that the machinery of interlacing polynomials due to Marcus, Spielman and Srivastava works in this setting as well. The relevant expected characteristic polynomials turn out to be related to the so called “mixed determinants” that have been carefully studied by Borcea and Branden.

This technique allows us to show that any projection with diagonal entries strictly less than $\frac{1}{4}$ can be two paved, matching recent results of Bownik, Casazza, Marcus and Speegle, though our estimates are asymptotically weaker. We also show that any projection with diagonal entries at most $\frac{1}{2}$ can be four paved, yielding improvements over currently known estimates.

We also relate the problem of finding optimal paving estimates to bounding the root intervals of a natural one parameter deformation of the characteristic polynomial of a matrix that turns out to have some remarkable combinatorial properties.

1. Introduction

The Kadison-Singer problem, posed in 1959 [15], asked if extensions of pure states on the diagonal subalgebra, $\ell^\infty(\mathbb{N})$ to $B((\ell^2(\mathbb{N})))$ are unique. This esoteric looking problem was shown to be equivalent to a fundamental combinatorial problem concerning finite matrices by Joel Anderson in 1979 [2].

**Question 1.1** (Anderson’s Paving formulation). Are there universal constants $\epsilon < 1$ and $r \in \mathbb{N}$ so that for any zero diagonal hermitian matrix $A \in M_n(\mathbb{C})$, there are diagonal projections $Q_1, \cdots, Q_r$ with $Q_1 + \cdots + Q_r = I$ such that

$$||Q_iAQ_i|| < \epsilon ||A||, \quad 1 \leq i \leq r?$$

Charles Akemann and Joel Anderson gave an alternate formulation of this problem in terms of paving projections in 1991 [1] and showed that a positive solution to it implies a positive solution to the Kadison-Singer problem.

**Question 1.2** (Akemann-Anderson’s projection formulation). Are there universal constants $\delta$ and $\epsilon < \frac{1}{2}$ so that whenever $P$ is a projection in $M_n(\mathbb{C})$ with diagonal entries at most $\alpha$, there is a diagonal projection $Q$ such that

$$||QPQ + (I-Q)P(I-Q)|| < \frac{1}{2} + \epsilon?$$

Nik Weaver gave a interpretation of this in terms of partitioning sets of isotropic vectors in 2004 [23], a conjecture was solved by Adam Marcus, Dan Spielman and Nikhil Srivastava in 2013 [17], who showed that one may take $\epsilon = \sqrt{2\alpha + \alpha}$. This result of Marcus, Spielman and Srivastava was improved by Bownik, Casazza, Marcus and Speegle [6], who showed that one may take $\epsilon = \sqrt{(2\alpha)(1 - 2\alpha)}$ whenever $\alpha \leq \frac{1}{4}$. Consequently, any projection matrix with diagonal entries all less than $\frac{1}{4}$ can be 2 paved.

Marcus, Spielman and Srivastava applied their result [17] to Anderson’s paving problem by means of a theorem of Casazza, Edidin, Kalra and Paulsen [9] to show that any zero diagonal hermitian matrix has a two sided paving of size 144. This estimate is far from optimal and finding optimal...
estimates is a problem of some theoretical interest. In the opposite direction, there is a result of Casazza et al. from the same paper [9] that projections with constant diagonal $\frac{1}{2}$ cannot be paved and it is expected that this is the best that one can do, that projections with diagonal less than $\delta < \frac{1}{2}$ can be $2$ paved. While we do not prove this conjecture in this paper, we prove some weaker paving estimates. We show the following,

**Theorem 1.3.** For any $\alpha \leq \frac{1}{2}$ so that whenever $P$ is a projection in $M_n(\mathbb{C})$ with diagonal entries at most $\alpha$, there is a diagonal projection $Q$ such that

$$||QPQ + (I - Q)P(I - Q)|| \leq \frac{1}{4} \left( \sqrt{\alpha} + \sqrt{3(1 - \alpha)} \right)^2.$$ 

This estimate is suboptimal, being inferior to the estimates proved by Bownik et al [6]. For one, it is easy to see that the constant should go to $\frac{1}{2}$ as $\alpha$ goes to 0, rather than $\frac{3}{4}$ as above. Another result along these lines that we prove is showing that positive contractions with diagonal at most $\frac{1}{2}$ can be $4$ paved.

**Theorem 1.4.** Let $A \in M_n(\mathbb{C})_1^+$ be a positive contraction with diagonal entries all at most $\delta \leq \frac{1}{2}$. Then, there are diagonal projections $Q_1, Q_2, Q_3, Q_4$ such that $Q_1 + Q_2 + Q_3 + Q_4 = I$ and,

$$||Q_1AQ_1 + Q_2AQ_2 + Q_3AQ_3 + Q_4AQ_4|| \leq \frac{(3 + \sqrt{7})^2}{32} \approx 0.996.$$ 

Together with a well known result from [9], this implies that any zero diagonal hermitian can be $16$ paved with paving constant $\approx 0.984$. This is also far from optimal, but hopefully our technique can be finetuned to get optimal results.

As mentioned above, we approach the Kadison-Singer problem through Anderson’s paving formulation in this paper, rather than Akemann-Anderson’s projection formulation or Weaver’s influential vector balancing version. It turns out that both the major innovations in the work of MSS [17], the method of interlacing polynomials and the multivariate barrier method can be directly applied to pavings of matrices. Estimates on the required size of pavings follow from estimates on the locations of roots of certain extremely natural multivariate polynomials, that are closely related to mixed determinants (these are distinct from the more familiar mixed discriminants). Even more interestingly, one can also relate these to natural univariate polynomials, that are related to expressions that have appeared in several works independently, called alpha permanents [21, 22, 8] by some and fermionants [18, 10] by others.

Let us now briefly outline our approach. Let $A$ be a matrix in $M_n(\mathbb{C})$. For any partition $\mathcal{X} = \{X_1, \ldots, X_r\}$ where $X_1 \cup X_2 \cdot \cdot \cdot \cup X_r = [n]$, we use the notation $A_{\mathcal{X}}$ to denote the corresponding $r$ paving of $A$,

$$A_{\mathcal{X}} = P_{X_1}AP_{X_1} + P_{X_2}AP_{X_2} + \cdots + P_{X_r}AP_{X_r}.$$ 

There are $r^n$ possible pavings of a $n \times n$ matrix and we use the expression $\mathcal{P}_r(A)$ or just $\mathcal{P}_r$ in short, to denote the set of all $r$ pavings. We will show that when $A$ is hermitian, the characteristic polynomials $r$ pavings form an interlacing family in the sense of Marcus, Spielman and Srivastava [17]. As a consequence of their method, one can prove

**Theorem 1.5.** Let $A \in M_n(\mathbb{C})$ be hermitian and let $r \in \mathbb{N}$. Then, the sum of the characteristic polynomials of all the $r$ pavings is real rooted and further, there is a paving $\mathcal{X} \in \mathcal{P}_r$ such that

$$\max_{\chi} \chi[A_{\mathcal{X}}] \leq \max_{\chi} \sum_{\mathcal{X} \in \mathcal{P}_r} \chi[A_{\mathcal{X}}].$$ 

The expression on the right has several delightful combinatorial expressions. The first expression is especially pretty and will lead to some unexpected combinatorial consequences, though we will use a different expression, a multivariate specialization, in proving paving estimates.
Definition 1.6. Given a matrix $A \in M_n(\mathbb{C})$, define
\[
\det_r(A) := \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)} (-1)^{\text{sgn}(\sigma)} r^{c(\sigma)},
\]
where $c(\sigma)$ denotes the number of cycles in $\sigma$.

This is the same as the determinant save for the $r^{c(\sigma)}$ term and in particular, when $r = 1$, we have the determinant. This expression has appeared several times in the mathematical literature and has also shown up in recent work of theoretical computer scientists and physicists \[18, 10\]. A different scaling of this expression, which has been studied in several papers goes under the name of the $\alpha$ permanent \[21, 22\]. We will write down a polynomial, analogous to the way we define the characteristic polynomial of a matrix. This natural operation has, as far as I know, not been studied so far.

Definition 1.7. Given a matrix $A \in M_n(\mathbb{C})$, define
\[
\chi_r[A] := \det_r[xI - A].
\]

When $r = 1$, this is the characteristic polynomial. One remarkable feature of this polynomial is that for any positive integer $r$, the polynomial $\chi_r[A]$ is real rooted for hermitian $A$. This can be deduced from the interlacing polynomials machinery of MSS and one can write down several pleasing expressions for this polynomial. For non-integer values of $r$, this polynomial is not real rooted but even in this case, there is an interesting alternate combinatorial expression for $\chi_r[A]$, an expression that is a consequence of McMahon’s master theorem, see \[12, 21\] or \[8\]. We will also show that a direct analogue of the Cauchy interlacing theorem holds for any value of $r \in \mathbb{N}$. And fundamentally, the expected characteristic polynomial over all pavings turns out to be given by the $r$ characteristic polynomial.

Theorem 1.8. Let $A \in \mathbb{C}^n$ be hermitian. Then, for any $r \in \mathbb{N}$, we have that,
\[
\sum_{\lambda \in \mathcal{P}_r} \chi[\lambda A] = \chi_r[A].
\]

The required paving estimates will therefore follow from bounds on the roots of the $r$ characteristic polynomial. To derive bounds for the roots of the $r$ characteristic polynomial, we use an alternate combinatorial interpretation in terms of a specialization of a multivariate polynomial.

Theorem 1.9. Let $A \in M_n(\mathbb{C})$. Let $Z$ be the diagonal matrix with diagonal entries $(z_1, \cdots, z_n)$ where the $z_k$ are variables. Then, for any integer value or $r$,
\[
\det_r[A] = \frac{\partial^{(r-1)n}}{\partial z_1^r \cdots \partial z_n^r} \det[A + Z] \bigg|_{z_1=\cdots=z_n=0}.
\]

Consequently,
\[
\chi_r[A](x) = \frac{\partial^{(r-1)n}}{\partial z_1^r \cdots \partial z_n^r} \det[Z - A] \bigg|_{z_1=\cdots=z_n=x}.
\]

The multivariate barrier function method is a general technique to study the evolution of roots of real stable polynomials that was introduced by MSS \[17\]. A polynomial $p(z_1, \cdots, z_n)$ is said to be real stable if its coefficients are real and it has no zeroes in $\mathbb{H}^n$ where $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$. Stability is defined algebraically, but MSS, see also Branden \[7\] have shown how stable polynomials also enjoy several convexity properties.

Given a real stable polynomial $p \in \mathbb{C}[z_1, \cdots, z_n]$, a point $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$ is said to be above the roots of $p$ , denoted $z \in \text{Ab}_p$ if $p$ is non-zero in the positive orthant based at $z$, that is,
\[
p(z + t) \neq 0, \quad \forall t \in \mathbb{R}^n_+.
\]
The Gauss-Lucas theorem implies that the positive orthant based at $z$ is zero free for any partial derivative $\frac{\partial p}{\partial z_i}$, (which we denote $\partial_i p$ in short) as well, unless this partial derivative is zero. In particular, if $z \in \text{Ab}_p$, then $z \in \text{Ab}_{\partial_i p}$. However, more is true. Taking the partial derivatives of a real stable polynomial with respect to $z_i$ shifts zero free orthants to the left along the direction $e_i$. In other words, one can show that there is a $\delta > 0$ such that $z - \delta e_i \in \text{Ab}_{\partial_i p}$ as well. The multivariate barrier method is a powerful method of getting concrete estimates for how large $\delta$ can be.

MSS used the multivariate barrier method to get estimates for how zero free orthants evolve under applying operators of the form $1 - \partial_i$ in their proof of the Kadison-Singer problem. We apply this method to derivative operators instead. For optimal estimates, we exploit the special structure of the polynomials relevant to Kadison-Singer, in particular their degree restrictions, something that was also done by Bownik et. al. in [6].

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2. Characteristic polynomials of pavings

Let $A$ be a matrix in $M_n(\mathbb{C})$; Given a subset $S \subset [n]$, we let $A_S$ be the principal submatrix of $A$ with rows and columns corresponding to elements in $S$ removed. Also, let $Z$ be the diagonal matrix $Z = \text{diag}(z_1, \cdots, z_n)$, where the $z_i$ are variables. Let us consider the polynomial,

$$\det[Z + A]$$

This is a multi affine polynomial in the $z_i$ and it is easy to see that the coefficient of $z^S$ for any subset $S \subset [n]$ equals the determinant of $A_S$. Consequently, we have,

**Lemma 2.1.** Given $A \in M_n(\mathbb{C})$ and a subset $S \subset [n]$, we have that,

$$\chi[A_S](x) = \frac{\partial^S}{\partial z^S} \det[Z - A] \mid_{Z=xI}.$$ 

The celebrated Cauchy-Poincare interlacing theorem says that any defect 1 principal submatrix of a hermitian matrix has the property that its eigenvalues interlace those of the parent matrix. This implies in particular that if $S_1$ and $S_2$ are two equal sized subsets of $[n]$ that differ in at most one element, then $\chi[A_{S_1}]$ and $\chi[A_{S_2}]$ have a common interlacer. The property of having a common interlacer can be read from the polynomials at hand, without having to compute a common interlacer, thanks to Obreschkoff’s theorem, see [11].

**Theorem 2.2** (Obreshkoff). Two real rooted univariate polynomials with positive leading coefficient have a common interlacer iff every convex combination of the two is real rooted.

Also, if two polynomials $p, q$ have a common interlacer, then it is a folklore result that

$$\min\{\max\text{root } p, \max\text{root } q\} \leq \max\text{root}(p + q).$$

Given two subsets $S \subset [n]$ and $T \subset [n]$, lemma 2.1 shows that we have, letting $Y = \text{diag}(y_1, \cdots, y_n)$,

$$\chi[A_S \oplus A_T] = \frac{\partial^S}{\partial z^S} \frac{\partial^T}{\partial y^T} \det[(Z - A)(Y - A)] \mid_{Z=Y=xI}.$$
We will use the notation $p(Z, Y)$ to denote,

$$p(Z, Y) = \det \left( (Z - A)(Y - A) \right).$$

Suppose now that $S, T \subset [n]$ are two subsets neither of which contain the element $i$. We then have that for any $\alpha \in [0, 1],

$$q(x) := \alpha \chi[A_{S \cup \{i\}} \oplus A_T] + (1 - \alpha) \chi[A_S \oplus A_{T \cup \{i\}}]$$

$$= \left[ \alpha \frac{\partial}{\partial z_i} \frac{\partial^T}{\partial y_T} + (1 - \alpha) \frac{\partial}{\partial y_i} \frac{\partial^T}{\partial z_T} \right] p(Z, Y) \mid_{Z=Y=xI}$$

$$= \left( \alpha \frac{\partial}{\partial z_i} + (1 - \alpha) \frac{\partial}{\partial y_i} \right) \frac{\partial^T}{\partial z_T} \frac{\partial^T}{\partial y_T} p(Z, Y) \mid_{Z=Y=xI}.$$

Now, note that $p$ is real stable and partial derivatives as well as positive linear combinations of partial derivatives preserve real stability and further, specializing variables to real scalars preserves real stability, the polynomial $q$ is real rooted.

We now construct an binary tree as follows:

- **Levels:** This tree will have $n + 1$ levels denoted 0 to $n$.
- **Nodes:** The nodes at the bottom or the $n$'th level will correspond to (ordered) partitions of $[n]$ into two subsets, that is, $S \sqcup T = [n]$. There are $2^n$ such partitions. There will be 2$^k$ nodes at level $k$ and they will be indexed by (ordered) two partitions of $[k]$. The top node (the single node at level 0) will be denoted $\{\phi\}$. Let us denote the nodes by tuples $(S, T)$, where the level can be read off by finding $k$ such that $S \sqcup T = [k]$.
- **Edges:** Each node save for those at level $n$ will have two children: Given $S, T$ such that $S \sqcup T = [k]$, the node $(S, T)$ at level $k$ will have as children $(S \cup \{k + 1\}, T)$ and $(S, T \cup \{k + 1\})$.
- **Markings:** To each node, we will attach a polynomial, which we will denote by $q(S, T)$. Given a node at the bottom level, the polynomial will be $q(S, T) = \chi[A_S \oplus A_T]$, where $S \sqcup T = [n]$.

For other nodes, the polynomial will the sum of the polynomials attached to all the leaves under that node. Note that given $S \sqcup T = [k]$, we have that,

$$q(S, T) = \sum_{U \sqcup V = [k+1, \ldots, n]} \chi[A_{S \cup U} \oplus A_{T \cup V}]$$

$$= \frac{\partial^T}{\partial z_T} \frac{\partial^T}{\partial y_T} \left( \prod_{m=k+1}^{n} (\partial z_m + \partial y_m) \right) p(Z, Y) \mid_{Z=Y=xI}$$

We now show that this family is an interlacing family in the sense of MSS. One needs to show that given a product distribution $\mu$ on $\mathcal{P}([n])$, that is, there are scalars $p_1, \cdots, p_n$ such that,

$$\mu(S) = \left( \prod_{i \in S} p_i \right) \left( \prod_{i \notin S} 1 - p_i \right),$$

then, the polynomial

$$q := \mathbb{E}_\mu \chi[A_S \oplus A_T],$$
is real rooted. We write this out as

\[ q = \sum_{S \cup T = [n]} \left( \prod_{i \in S} p_i \right) \left( \prod_{i \in T} (1 - p_i) \right) \chi[A_S \oplus A_T] \]

\[ = \sum_{S \cup T = [n]} \left( \prod_{i \in S} p_i \partial_{z_i} \right) \left( \prod_{i \in T} (1 - p_i) \partial_{y_i} \right) p(Z,Y) \]

\[ = \left( \prod_{m=1}^{n} (p_m \partial_{z_m} + (1 - p_m) \partial_{y_m}) \right) p(Z,Y) \mid_{Z=Y=xI} \]

This last polynomial is clearly real rooted and the interlacing property follows. We conclude,

**Theorem 2.3.** Let \( A \in M_n(\mathbb{C}) \) be hermitian. Then, the sum of the characteristic polynomials of all the 2 pavings of \( A \) is real rooted and satisfies,

\[ \sum_{S \cup T = [n]} \chi[A_S \oplus A_T] = \left[ \prod_{m=1}^{n} (\partial_{z_m} + \partial_{y_m}) \right] p(Z,Y) \mid_{Z=Y=xI}. \]

Further, there is a paving \((S,T) \in \mathcal{P}_2\) such that

\[ \lambda_{\max} \chi[A_S \oplus A_T] \leq \lambda_{\max} \sum_{S \cup T = [n]} \chi[A_S \oplus A_T]. \]

This analysis can be carried out for \( r \) pavings as well for any \( r \in \mathbb{N} \). The proof is similar and we omit it.

**Theorem 2.4.** Let \( A \in M_n(\mathbb{C}) \) be hermitian. Then, the sum of the characteristic polynomials of all the \( r \) pavings of \( A \) is real rooted and satisfies,

\[ \sum_{X \in \mathcal{P}_r([n])} \chi[A_X] = \left( \frac{1}{(r-1)!} \right) \prod_{m=1}^{n} \left( \sum_{i=1}^{r} \partial_{z_i}^{(m)} \right)^{r-1} p(Z_1, \cdots, Z_r) \mid_{Z_1=\cdots=Z_r=xI}. \]

Further, there is a paving \( X \in \mathcal{P}_r([n]) \) such that

\[ \lambda_{\max} \chi[A_X] \leq \lambda_{\max} \sum_{X \in \mathcal{P}_r([n])} \chi[A_X]. \]

In the next section, we will derive other useful expressions for this expected characteristic polynomial.

We now show how this fact, that one can use expected characteristic polynomials to get estimates about one paving can be understood in a more general framework. The concept of a *Strongly Rayleigh* measure was introduced by Borcea, Branden and Liggett \[5\] in order to develop a systematic theory of negative dependance in probability. The main MSS theorem was extended to the setting of Strongly Rayleigh measures by Anari and Oveis Gharan in \[13\] and we would like to point out how a version of the algebraic component of their results holds in our setting.

Recall that a probability distribution \( \mu \) on \( \mathcal{P}([n]) \) is said to be *Strongly Rayleigh* if the generating polynomial,

\[ P_\mu = \sum_{S \subset [n]} \mu(S) z^S, \]

is real stable. A simple adaptation of the proof of theorem \[23\] shows the following,
Theorem 2.5. Let \( \mu \) be a Strongly Rayleigh distribution on \( \P([n]) \) and let \( A \in M_n(\mathbb{C})^{sa} \) be hermitian. Then,

\[
\mathbb{E} \chi[A_S] = \sum_{S \subset [n]} \mu(S) \chi[A_S],
\]

is real rooted and further,

\[
\mathbb{P} \left[ \lambda_{\max} \chi[A_S] \leq \lambda_{\max} \mathbb{E} \chi[A_S] \right] > 0.
\]

Further, we have the following elegant formula for the expected characteristic polynomial,

\[
\mathbb{E} \chi[A_S] = P_\mu(\partial_1, \cdots, \partial_n) \det[Z - A]_{z=xI}.
\]

This theorem can be specialized to both the restricted invertibility problem as well as Kadison-Singer. For restricted invertibility, the measure we work with is the uniform measure \( \mu \) over all \( n-k \) element subsets, whose generating polynomial is,

\[
P_\mu = \binom{n}{k}^{-1} \sum_{|S|=n-k} z^S = \binom{n}{k}^{-1} (\partial_1 + \cdots + \partial_n)^k z_1 \cdots z_n.
\]

For Kadison-Singer, let’s look at two paving first. Let \( A \in M_n(\mathbb{C})^{sa} \) be a hermitian matrix. Consider the set \([2n]\), which we write as \([n] \times [n]\). We now choose the measure \( \mu_2 \) that is uniform on subsets of the form \( S \times S^c \), where \( S \subset [n] \). More generally, when it comes to \( r \) paving, we look at \([n] \llbracket \Pi \cdots \Pi [n] \rrbracket \sim [rn] \) and consider the uniform measure, which we denote \( \mu_r \) on \( S_1 \times \cdots \times S_r \) where \( S_1^c \cdots \Pi S_r^c = [n] \).

Using variables \( (z_1^{(i)}, \cdots, z_n^{(i)}) \) to represent the atoms in the \( i^{th} \) copy of \([n] \), we have that the generating polynomial is

\[
P_{\mu_r} = r^{-n} \prod_{i=1}^r \left( \partial_{z_1^{(i)}} + \cdots + \partial_{z_n^{(i)}} \right) \prod_{i=1}^r z_1^{(i)} \cdots z_n^{(i)}.
\]

We now apply the theorem (2.5) to the \( r n \times r n \) matrix \( A \oplus \cdots \oplus A \). It is easy to see that it gives us that there is a \( r \) paving \( X = X_1 \cup \cdots \cup X_r \) such that

\[
\lambda_{\max} A_X \leq \mathbb{E} \chi[A_S].
\]

In the next section, we will show that the above expected characteristic polynomials, the ones that are relevant for Kadison-Singer have other, even more pleasant expressions.

3. The \( r \) Characteristic Polynomial

The expression for the sum of the characteristic polynomials over all \( r \) pavings in theorem (2.4) will allow us to prove strong estimates on its roots, but might seem unwieldy. In this section, we give three other expressions for this polynomial, which will hopefully make a case for its being a natural and interesting object.

Our first observation, which is a trivial application of the chain rule, is that we may write this polynomial out in a simpler way, that is reminiscent of the mixed characteristic polynomial of MSS,

**Lemma 3.1.** Let \( A \in M_n(\mathbb{C}) \). Then, for any positive integer \( r \), we have that,

\[
\sum_{X \in \P_r([n])} \chi[A_X] = \left( \frac{1}{(r-1)!} \right)^n \partial_z^{(r-1)n} \det[Z - A]^r \bigg|_{z_1 = \cdots = z_n = x}.
\]
Let us derive another expression for this polynomial. Consider the expression,

\[
\left(\frac{1}{(r-1)!}\right)^n \frac{\partial^{(r-1)n}}{\partial z_1^{r-1} \cdots \partial z_n^{r-1}} \det[Z + A]^r |_{z_1 = \cdots = z_n = 0}
\]

This is the coefficient of \((z_1 \cdots z_r)^{r-1}\) in the polynomial \(\det[Z + A]^r\). We may expand out \(\det[Z + A]\) as

\[
\det[Z + A] = \sum_{S \in [n]} z^S \det[A_S].
\]

Expanding out \(\det[A_S]\), we have,

\[
\det[Z + A] = \sum_{S \in [n]} \sum_{\sigma \in \text{Aut}(S^c)} z^S(-1)^{\text{sgn}(\sigma)} \prod_{i \in [S^c]} a_{\sigma(i)}
\]

Let us use the term \(PS_n\) to denote the set of all partial permutations of \([n]\), that is,

\[
PS_n = \bigcup_{S \subseteq [n]} \text{Aut}(S).
\]

We also use the notation \(PS_n^r\) to refer to the products of \(r\) partial permutations of \([n]\). Finally, let us use the expression \((\sigma)\) to denote the subset that \(\sigma\) is a permutation of. With this notation, we may write

\[
\det[Z + A] = \sum_{\sigma \in PS_n} z^{(\sigma)}(-1)^{\text{sgn}(\sigma)} \prod_{i \in (\sigma)} a_{\sigma(i)}
\]

We now have,

\[
\det[Z - A]^r = \sum_{\sigma \in PS_n^r} z^{(\sigma)}(-1)^{\text{sgn}(\sigma)} \prod_{i \in (\sigma)} a_{\sigma(i)}
\]

If an element \(\sigma \in PS_n^r\) is such that \(z^{(\sigma)} = (z_1 \cdots z_n)^{r-1}\), then we must have that \(\sigma\) is actually a permutation of \([n]\). The number of ways a permutation \(\sigma\) can be written as the product of \(r\) partial permutations is clearly equal to \(r\cdot\text{sgn}(\sigma)\). Recalling the definition of \(\chi_r\) from definition \ref{def:chi_r}, we conclude,

**Lemma 3.2.** Let \(A \in M_n(\mathbb{C})\). Then, we have that,

\[
\mathbb{E}_{X \in \mathcal{P}_r([n])} \chi[A X] = \chi_r[A].
\]

The \(r\) characteristic polynomial, for positive integer values of \(r\) shares some of the features of the regular characteristic polynomial (the case when \(r = 1\)). We have already seen that the roots of \(\chi_r[A]\) for any hermitian matrix \(A\) are real (just combine theorem \ref{thm:hermitian} and lemma \ref{lem:interlacing}). We now show that these enjoy the same interlacing properties that the characteristic polynomial does.

**Proposition 3.3** (Cauchy Interlacing). Let \(A \in M_n(\mathbb{C})_{sa}\) be hermitian and let \(r \in \mathbb{N}\). For any \(i \in [n]\) we have that the roots of \(\chi[A]\) and \(\chi[A_i]\) interlace. Here, \(\chi[A_i]\) is the principal submatrix of \(A\) with the \(i\)th row and column removed.

**Proof.** The classical Hermite-Biehler theorem says that two real rooted polynomials with real coefficients \(f\) and \(g\) interlace each other iff \(f + ig\) has all its roots either in the upper half plane or in the lower half plane.

Without loss of generality, we may take \(i = 1\). We have that,

\[
\chi_r[A] = \left(\frac{1}{(r-1)!}\right)^n \frac{\partial^{(r-1)n}}{\partial z_1^{r-1} \cdots \partial z_n^{r-1}} \det[Z - A]^r |_{z_1 = \cdots = z_n = x},
\]
Since,
\[ \det[Z_1 - A_1]^r = \frac{\partial^r}{\partial z_1^r} \det[Z - A]^r, \]
we have that,
\[
\chi_r[A_i] = \left( \frac{1}{(r-1)!} \right)^{n-1} \frac{\partial^{(r-1)(n-1)}}{\partial z_2^{r-1} \cdots \partial z_n^{r-1}} \det[Z_1 - A_1]^r \bigg|_{z_2 = \cdots = z_n = x}
\]
\[ = \frac{\partial}{\partial z_1} \left( \frac{1}{(r-1)!} \right)^{n-1} \frac{\partial^{(r-1)(n-1)}}{\partial z_2^{r-1} \cdots \partial z_n^{r-1}} \det[Z_1 - A_1]^r \bigg|_{z_2 = \cdots = z_n = x}
\]
\[ = \frac{\partial}{\partial z_1} \left( \frac{1}{(r-1)!} \right)^{n-1} \frac{\partial^{(r-1)n}}{\partial z_1^{r-1} \cdots \partial z_n^{r-1}} \det[Z - A]^r \bigg|_{z_1 = z_2 = \cdots = z_n = x}
\]
Therefore,
\[
\chi_r[A_1] + i \chi_r[A] = \left( (r-1)! \frac{\partial}{\partial z_1} + i \right) \left( \frac{1}{(r-1)!} \right)^n \frac{\partial^{(r-1)n}}{\partial z_1^{r-1} \cdots \partial z_n^{r-1}} \det[Z - A]^r \bigg|_{z_1 = \cdots = z_n = x}.
\]
The polynomial,
\[ p(Z) = \left( \frac{1}{(r-1)!} \right)^n \frac{\partial^{(r-1)n}}{\partial z_1^{r-1} \cdots \partial z_n^{r-1}} \det[Z - A]^r, \]
is stable and since the symbol of \((r-1)! \frac{\partial}{\partial z_1} + i\), namely \((r-1)!z + i\) has all its roots (its single root) in the lower half plane, it preserves stability(add reference). Specializing this polynomial to real numbers preserves stability as well and we conclude that \(\chi_r[A_1] + i \chi_r[A]\) is stable, which is what was required. \(\square\)

Another feature the \(r\) characteristic polynomial shares with the regular characteristic polynomial is an analogue of Thompson’s formula, see \([20]\), that the sum of characteristic polynomials of defect 1 submatrices equals the derivative of the characteristic polynomial of the original matrix.

**Proposition 3.4 (Thompson type formula).** Let \(A \in M_n(\mathbb{C})\) be a not necessarily hermitian matrix and let \(r \in \mathbb{N}\). Then,
\[ r \sum_{i \in [n]} \chi_r[A_i] = \chi'_r[A]. \]

**Proof.** The simplest proof of this uses the univariate representation for \(\chi_r[A]\). We have,
\[
\chi_r[-A] = \det_r[z I + A] = \det_r[A] + r x \sum_{i \in [n]} \det_r[A_i] + (r x)^2 \sum_{i \neq j \in [n]} \det_r[A_{ij}] + \cdots + (r x)^n
\]
\[ = \sum_{k=0}^{n} (rx)^k \sum_{S \subset [n], |S| = k} \det_r[A_S]. \]

We have therefore,
\[
\sum_{m \in [n]} r \chi_r[-A_m] = r \sum_{m \in [n]} \sum_{k=0}^{n} (rx)^{k-1} \sum_{S \subset [n]\setminus\{m\}, |S| = k} \det_r[A_S] \]
\[ = \sum_{k=0}^{n} r (rx)^{k-1} \sum_{S \subset [n], |S| = k} \det_r[A_S] \]
\[ = \chi'_r[-A]. \]
A simple induction argument, then shows that sums of $r$ characteristic polynomials of defect $k$ principal sub matrices can be related to the $k$’th derivative of $\chi_r[A]$.

**Corollary 3.5.** Let $A \in M_n(\mathbb{C})$ be a not necessarily hermitian matrix and let $r \in \mathbb{N}$. Then, for any $k \in [n]$,

$$r^kk! \sum_{S \subset [n], |S| = k} \chi_r[A_S] = \chi_r^{(k)}[A].$$

We finally record a simple combinatorial fact.

**Proposition 3.6 (Multilinearization).** Let $A \in M_n(\mathbb{C})$ be a not necessarily hermitian matrix and let $r \in \mathbb{N}$. Then, letting $Z$ as usual be a diagonal matrix of variables, $Z = \text{diag}(z_1, \ldots, z_n)$, we have,

$$\det[r][Z + A] = \left(\frac{1}{(r-1)!}\right)^n \frac{\partial^{(r-1)n}}{\partial z_1^{r-1} \cdots \partial z_n^{r-1}} \det[Z + A]^r.$$

**Proof.** Both polynomials are multilinear in the variables $z_i$. Let $S \subset [n]$; We compare the coefficients of $z^S$ in both the polynomials above. It is easy see that the coefficient of $z^S$ in $\det[r][Z + A]$ is $r^{|S|} \det[r][A_S]$. As for the polynomial on the right hand side, recalling that

$$\det[Z + A] = \sum_{S \subset [n]} z^S \det[A_S],$$

we see that the coefficient of $z^{[n]^{r-1} \setminus S}$ in $\det[Z + A]^r$ equals,

$$\sum_{S_1 \cup \cdots \cup S_r = S} \prod_{i \in [r]} \det[A_{S_i}] r^{|S_i|}.$$

After taking derivatives and scaling, we then have that the coefficient of $z^S$ in the right hand polynomial is,

$$\sum_{S_1 \cup \cdots \cup S_r = S} \prod_{i \in [r]} \det[A_{S_i}] r^{|S_i|}.$$

Specializing the equality between the $r$ characteristic polynomial and the expected characteristic polynomial over all pavings to $x = 0$, we see that this equals $r^{|S|} \det[r][A_S]$. □

We summarize this in a separate corollary,

**Corollary 3.7.** Let $A \in M_n(\mathbb{C})$ be a not necessarily hermitian matrix and let $r \in \mathbb{N}$. Then,

$$\det[r][Z + A] = \sum_{S \subset [n]} z^S r^{|S|} \det[r][A_S]$$

We now use the observation that $\det[r](Z + A)$ is a stable to polynomial to conclude that the measure $\mu$ defined on $P([n])$ by

$$\mu(S) = r^{|S|} \det[r][A_S],$$

is a *Strongly Rayleigh* measure., see [5] where these were introduced for the definition and a discussion. This immediately implies an analogue of the Hadamard-Fischer-Koteljanskii inequalities.

**Proposition 3.8.** Let $A \in M_n(\mathbb{C})^+$ be PSD and let $r \in \mathbb{N}$. Then, for any two subsets $S, T \subset [n]$, we have that,

$$\det[r][A_S] \det[r][A_T] \geq \det[r][A_{S \cap T}] \det[r][A_{S \cup T}].$$
The $r$ characteristic polynomial is closely related to the mixed determinant (not to be confused with the mixed discriminant), that is defined for tuples of matrices \([3]\). Given $n \times n$ matrices $A_1, \ldots, A_k$, the mixed determinant of the tuple is defined as,

$$D(A_1, \ldots, A_k) = \sum_{S_1 \cup \cdots \cup S_k = [n]} \det[A(S_1)] \cdots \det[A(S_k)].$$

It is immediate that,

$$\chi_r[A](x) = D(xI - A, \ldots, xI - A).$$

Borcea and Branden \([3]\) proved a variety of interlacing results for polynomials of the form $D(xA, B)$, which generalize the regular characteristic polynomial, because of the identity, $\chi[A] = D(xI, -A)$.

There is another expression for the $r$ determinant, a consequence of MacMahon’s Master theorem, see \([12, 21]\) or \([8]\) which works for non-integral values of $r$ as well,

**Theorem 3.9.** Let $A \in M_n(\mathbb{C})$. Let $Z$ be the diagonal matrix with diagonal entries $(z_1, \ldots, z_n)$ where the $z_k$ are variables. Then, for any $r \in \mathbb{R}$, we have that,

$$\det_r[A] = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det[I - ZA]^r \bigg|_{z_1=\cdots=z_n=0}.$$

Consequently,

$$\chi_r[A](x) = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det[I - xZ + ZA]^r \bigg|_{z_1=\cdots=z_n=0}.$$

**Remark 3.10.** For non-integer values of $r$, this polynomial is not real rooted. For instance, let $J_4$ be the $4 \times 4$ matrix with $J_{ij} = 1$, $i, j \in [4]$. Then, it is possible to check that $\chi_r[J_4]$ is not real rooted for $r \in (1, 2)$. It is also possible to show using matrices of the form $J_k$ that the only values of $r$ such that $\chi_r[A]$ is real rooted for every hermitian matrix are positive integers.

In the next section, we prove estimates on the maximum roots of $\chi_2$, $\chi_3$ and $\chi_4$ and discuss plausible estimates for general $r$. We feel the following is true,

**Conjecture 3.11.** Let $r \in \mathbb{N} \setminus \{1\}$ and let $A \in M_n(\mathbb{C})^+$ be a positive contraction and let the diagonal entries of $A$ all be at most $\delta$. Then,

$$\max_{z_i \in [r]} \chi_r[A] \leq \frac{1}{r} \left( \sqrt{1 - \delta} + \sqrt{(r - 1)\delta} \right)^2, \quad i \in [r].$$

While we are unable to prove this, we do prove partial results.

### 4. The Multivariate Barrier Method

In this section, we prove bounds for the largest root of the two characteristic polynomial. Let $A \in M_n(\mathbb{C})^{sa}$ be a hermitian matrix and recall that the two characteristic polynomial is equal to,

$$\chi_2[A](x) = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det[Z - A]^2 \bigg|_{Z=xI}.$$

Let $p$ be the polynomial,

$$p(z) = \det[Z - A]^2.$$

For any subset $S \subset [n]$, we define,

$$p_S(z) = \frac{\partial^S p}{\partial z^S}.$$

Each of these polynomials $p_S$ is of degree at most $2$ in $z_i$ for $i \in S^c$ and of degree at most one in the $z_i$ for $i \in S$. All of these polynomials are further, real stable.
For a real stable polynomial $q(z)$, define the barrier function in the direction $i$ at a point $z$ that is above the roots of $q$ by,

$$\Phi^i_q(z) := \frac{\partial_i q}{q}.$$ 

As pointed out by MSS [17], at any point $z \in \text{Ab}_q$, we have,

$$\Phi^i_q(z) \geq 0, \quad \partial_j \Phi^i_q(z) \leq 0, \quad \partial^2_j \Phi^i_q(z) \geq 0, \quad i, j \in [n].$$

We further have that,

$$(1) \quad \Phi^i_{\partial_i q} = \Phi^i_q + \frac{\partial_i \Phi^i_q}{\Phi^i_q}.$$ 

This shows that $\Phi^i_{\partial_i q} \leq \Phi^i_q$ and the essence of the barrier function method is to get estimates on the largest $\delta$ such that

$$(2) \quad \Phi^i_{\partial_i q}(z - \delta e_i) \leq \Phi^i_q(z).$$

Let $q(z)$ now be a real stable polynomial of degree at most 2 in the variable $z_i$ and let $z \in \text{Ab}_q$. We then have that, for any $z_j$, exploiting the fact that the degree of $q$ in $z_i$ is at most 2 that,

$$\Phi^i_{\partial_i q}(z - \delta e_i) = \frac{\partial_i \partial_j q(z - \delta e_i)}{\partial_i q(z - \delta e_i)} = \frac{\partial_i \partial_j q - \delta \partial_i \partial^2_j q}{\partial_i q - \delta \partial^2_i q}(z)$$

We make a simple calculation: We have that,

$$\frac{\partial_i \partial_j q - \delta \partial_i \partial^2_j q}{\partial_i q - \delta \partial^2_i q}(z) \leq \frac{\partial_i q}{q}(z),$$

if we have,

$$(3) \quad [q(\partial_j \partial_i q) - (\partial_i q)(\partial_j q)](z) \leq \delta[q(\partial_i \partial^2_j q) - (\partial^2_i q)(\partial_j q)](z).$$

We rewrite this as,

$$\partial_j \left( \frac{\partial_i q}{q} \right)(z) \leq \delta \partial_j \left( \frac{\partial^2_i q}{q} \right)(z) \iff \partial_j \Phi^i_q(z) \leq \delta \partial_j \left( \frac{\partial^2_i q}{q} \right)(z).$$

We summarize this,

**Proposition 4.1.** Let $q$ be a real stable polynomial of degree at most 2 in the variable $z_i$ and let $z$ be above the roots of $q$. Then if,

$$\partial_j \left( \frac{\partial_i q}{q} \right)(z) \leq \delta \partial_j \left( \frac{\partial^2_i q}{q} \right)(z),$$

we have that,

$$\Phi^i_{\partial_i q}(z - \delta e_i) \leq \Phi^i_q(z).$$

The expression on the right in the hypothesis can be written as,

$$\partial_j \left( \frac{\partial^2_i q}{q} \right)(z) = \partial_j [\Phi^i_q \Phi^i_{\partial_i q}](z) = \partial_j [(\Phi^i_q)^2 + \partial_i \Phi^i_q](z) = 2\Phi^i_q(z) \partial_j \Phi^i_q(z) + \partial_j \Phi^i_q(z).$$

For the second implication, we used (1). Since $\partial_i \Phi^i_q = \partial_j \Phi^i_q$, we need to find a $\delta$ such that,

$$\partial_j \Phi^i_q(z) \leq 2\delta \Phi^i_q(z) \partial_j \Phi^i_q(z) + \delta \partial^2_i \Phi^i_q(z).$$

Since $\partial^2_i \Phi^i_q(z) \geq 0$, we see that taking $\delta = \frac{1}{2\Phi^i_q(z)}$ works. We summarize this,
Proposition 4.2. Let $q$ be a real stable polynomial of degree at most 2 in the variable $z_i$ and let $z$ be above the roots of $q$. Then, letting $\delta = \frac{1}{2\Phi_i(z)}$, we have that $z - \delta e_i$ is above the roots of $\partial z_i p$ and further,

$$
\Phi_i^j(z - \delta e_i) \leq \Phi_i^j(z).
$$

Upon iteration, this implies the following.

Theorem 4.3. Let $A \in M_n(\mathbb{C})^{sa}$ and let $p$ be the polynomial,

$$
p(z) = \det[Z - A]^2.
$$

Let $z$ be above the roots of $p$ and let

$$
\delta = \min_{i \in [n]} \left\{ \frac{1}{2\Phi_i(z)} \right\}.
$$

Then, we have that $z - \delta 1$ is above the roots of $\frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det[Z - A]^2$.

The potentials of the function $\det[Z - A]^2$ are easy to calculate.

Lemma 4.4. Let $A \in M_n(\mathbb{C})^{sa}$ and let $p = \det[Z - A]^2$. Then,

$$
\Phi_i^j(z) = \frac{\partial p}{p}(z) = 2 \epsilon_i^*(Z - A)^{-1} e_i.
$$

We conclude,

Theorem 4.5. Let $A \in M_n(\mathbb{C})^{sa}$ be hermitian. We then have that,

$$
\max_{\text{root}} \chi_2[A] \leq \inf_{b \geq \lambda_{max}(A)} \left( b - \min_{i \in [n]} \frac{1}{4 \epsilon_i^*(bI - A)^{-1} e_i} \right).
$$

The quantity on the right can be controlled by the diagonal entries of the matrix $A$. We give here a first order estimate.

Lemma 4.6. Let $A \in M_n(\mathbb{C})^+$ be a positive contraction and let the diagonal entries all be at most $\delta$. Then, for any $b \geq 1$,

$$
\epsilon_i^*(bI - A)^{-1} e_i \leq \frac{\delta}{b - 1} + \frac{1 - \delta}{b}, \quad i \in [n].
$$

Proof. Let $D$ be the diagonal matrix of eigenvalues of $A$, $D = \text{diag}(\lambda_1, \cdots, \lambda_n)$ and let $U$ be a unitary matrix such that $A = UDU^*$. We see that

$$
\epsilon_i^*(bI - A)^{-1} e_i = \sum_j \frac{|U_{ij}|^2}{b - \lambda_j}
$$

The condition that the diagonal entries are all at most $\delta$ translates to,

$$
\sum_j |U_{ij}|^2 \lambda_j \leq \delta, \quad i \in [n].
$$

Since $U$ is unitary, we also have that,

$$
\sum_j |U_{ij}|^2 = 1, \quad i \in [n].
$$

Let us maximize the expression,

$$
\sum_j \frac{\alpha_j}{b - \lambda_j},
$$

where
where the $\alpha_j$ are constants such that
\[ \sum_j \alpha_j = 1, \quad \alpha_j \geq 0, \]
and the $\lambda_j$ are variables, subject to,
\[ \sum_j \alpha_j \lambda_j \leq \delta, \quad \sum_j \lambda_j = \tau, \quad 0 \leq \lambda_j \leq 1, \]
It is easy to see that the expression is maximized when all the $\lambda_j$ are either 0 or 1 and when equality holds in the second inequality above. We then have that $\tau = n \delta$ and it is then easy to see that the expression is exactly, \( \frac{\delta}{b-1} + \frac{1-\delta}{b} \).

We now perform some simple optimization. Given a positive contraction $A \in M_n(\mathbb{C})^+_1$, we have that,
\[ \lambda_{\text{max}} \chi_2[A] \leq \inf_{b \geq 1} b - \frac{1}{4\varphi}, \quad \varphi = \frac{\delta}{b-1} + \frac{1-\delta}{b}. \]
We now see that,
\[ b - \frac{1}{4\varphi} = \left[ \frac{3(1-\delta)}{4} + \frac{\delta}{4} \right] + \frac{3(b-1+\delta)}{4} + \frac{\delta(1-\delta)}{4(b-1+\delta)} \]
This is minimized when $b = 1 - \delta + \sqrt{\delta(1-\delta)}/3$ and this minimizer is greater than 1 whenever $\delta \leq \frac{1}{4}$.
If $\delta \geq \frac{1}{4}$, the minimizer for (4) is 1 yielding the trivial estimate, $\lambda_{\text{max}} \chi_2[A] \leq 1$. For $\delta \leq \frac{1}{4}$, we get the estimate,
\[ \lambda_{\text{max}} \chi_2[A] \leq \frac{1}{4} \left( \sqrt{3(1-\delta)} + \sqrt{\delta} \right)^2, \]
which is strictly less than 1 if $\delta < \frac{1}{4}$. We conclude,

**Theorem 4.7.** Let $A \in M_n(\mathbb{C})^+_1$ be a positive contraction with diagonal entries all at most $\delta \leq \frac{1}{4}$. Then, there is a partition $X_1 \sqcup X_2 = [n]$ such that,
\[ \lambda_{\text{max}} P_{X_i} AP_{X_i} \leq \frac{1}{4} \left( \sqrt{3(1-\delta)} + \sqrt{\delta} \right)^2, \quad i = 1, 2. \]

5. The multivariate barrier method: Continued

We apply the barrier method to polynomials of degree more than 2 in its variables. What follows is quite ad hoc, but we were unable to find a more systematic method to get the required estimates. A highlight of the argument in this section is theorem (5.5) that shows that projections with diagonal at most $\frac{1}{2}$ can be 4 paved. An argument identical to the proof of proposition (4.1) yields,

**Proposition 5.1.** Let $q$ be a real stable polynomial of degree at most $k$ in the variable $z_i$ and let $z$ be above the roots of $q$. Then, for any $j$,
\[ \Phi_{\partial_i^{k-1} q}(z - \delta e_i) \leq \Phi_{\partial_i^k q}(z) \iff \partial_j \left( \frac{\partial_i^{k-1} q}{q} \right)(z) \leq \delta \left( \frac{\partial_i k q}{q} \right)(z). \]

With this in hand, we prove estimates for polynomials of degree 3 and 4.
Proposition 5.2. Let $q$ be a real stable polynomial that has degree at most 3 in the variable $z_i$. Then, for each $j$, we have that,

$$\partial_j \left( \frac{\partial^2 q}{q} \right)(z) \leq \delta \left( \frac{\partial^3 q}{q} \right)(z), \text{ if } \delta = \frac{4}{3\Phi_q(z)}.$$

Proof. Without loss of generality, we may set $i = 1$ and $j = 2$. The Helton-Vinnikov theorem [14, 16] says that any bivariate real stable polynomial $p$ with leading term positive has a determinantal representation,

$$p(z_1, z_2) = \det[A + Bz_1 + Cz_2],$$

where $B, C$ are PSD and $A$ is hermitian. Fix a point $z = (z_1, z_2)$ that is above the roots of $p$ and let $M = A + Bz_1 + Cz_2$. Since $(z_1, z_2)$ is above the roots of $p$, the matrix $M$ is PSD. Further, let,

$$X = M^{-1/2}BM^{-1/2}, \quad Y = M^{-1/2}CM^{-1/2}.$$

Note that both of these are PSD. It is easy to check that,

$$\frac{\partial p}{p} = \text{Tr}[X], \quad \frac{\partial^2 p}{p} = \text{Tr}[X^2] - \text{Tr}[X^2],$$

as well as,

$$\frac{\partial^3 p}{p} = \text{Tr}[X]^3 - 3 \text{Tr}[X] \text{Tr}[X^2] + 2 \text{Tr}[X^3].$$

We also have,

$$\frac{\partial^2}{p} \left( \frac{\partial p}{p} \right) = -2 \text{Tr}[X] \text{Tr}[XY] + 2 \text{Tr}[X^2Y],$$

as well as,

$$\frac{\partial^2}{p} \left( \frac{\partial^2 q}{p} \right) = -3 \text{Tr}[X]^2 \text{Tr}[XY] + 3 \text{Tr}[X^2] \text{Tr}[XY] + 6 \text{Tr}[X] \text{Tr}[X^2Y] - 6 \text{Tr}[X^3Y].$$

Writing out our claim and rearranging the terms, we need to show that,

$$4 \text{Tr}[X^2] \text{Tr}[XY] + 6 \text{Tr}[X] \text{Tr}[X^2Y] \geq 2 \text{Tr}[X]^2 \text{Tr}[XY] + 8 \text{Tr}[X^3Y].$$

The condition that $p$ is of degree at most 3 in the variable $z_1$ shows that the matrix $X$ is of rank 3. Choose a basis so that $X$ is diagonal and of the form $X = \text{diag}(\lambda_1, \lambda_2, \lambda_3, 0, \cdots)$ and let $Y_{ii} = x_i$ for $i \in [3]$. We need to show that,

$$4(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) + 6(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 x_1 + \lambda_2^2 x_2 + \lambda_3^2 x_3) \geq 2(\lambda_1 + \lambda_2 + \lambda_3)^2(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) + 8(\lambda_1^2 x_1 + \lambda_2^2 x_2 + \lambda_3^2 x_3)$$

Simplifying, we need to show that,

$$2\lambda_1 x_1[(\lambda_2 - \lambda_3)^2 + \lambda_1 (\lambda_2 + \lambda_3)] + 2\lambda_2 x_2[\cdot] + 2\lambda_3 x_3[\cdot] \geq 0,$$

which is manifestly true. □

We now prove a result concerning 3 paving.

Theorem 5.3. Let $A \in M_n(\mathbb{C})_+^1$ be a positive contraction with diagonal entries all at most $\delta \leq \frac{4}{9}$. Then, there is a partition $X_1 \amalg X_2 \amalg X_3 = [n]$ such that,

$$\maxroot_{\chi_i[A]} \leq \left( \sqrt{\delta(1 - \delta)} + 2\sqrt{\delta} \right)^2, \quad i = 1, 2, 3.$$
Proof. Using proposition (5.2), arguing analogously with the argument leading upto (4.5), we see that,

\[
\maxroot \chi_3[A] \leq \inf_{b \geq \lambda_{\max}(A)} b - \frac{4}{9} \left( \frac{\delta}{b - 1} + \frac{1 - \delta}{b} \right)^{-1}
\]

Boilerplate optimization shows that this quantity is 1 for \( \delta \geq \frac{4}{9} \) and for other values, we get,

\[
\maxroot \chi_3[A] \leq \frac{1}{9} \left( \sqrt{5(1 - \delta)} + 2\sqrt{\delta} \right)^2, \quad \delta \leq \frac{4}{9}.
\]

We continue this analysis for four pavings.

**Proposition 5.4.** Let \( q \) be a real stable polynomial that has degree at most 4 in the variable \( z_i \). Then, for each \( j \), we have that,

\[
\partial_j \left( \frac{\partial^3 q}{q} \right) \leq \delta \left( \frac{\partial^4 q}{q} \right), \quad \text{if} \quad \delta = \frac{9}{4\Phi_q^4(z)}.
\]

**Proof.** We note that,

(8) \[
\frac{\partial^4 p}{p} = \text{Tr}[X]^4 - 6 \text{Tr}[X]^2 \text{Tr}[X^2] + 8 \text{Tr}[X] \text{Tr}[X^3] + 3 \text{Tr}[X^2]^2 - 6 \text{Tr}[X^4].
\]

Also,

(9) \[
\partial_2 \left( \frac{\partial^4 p}{p} \right) = -4 \text{Tr}[X]^3 \text{Tr}[XY] + 12 \text{Tr}[X] \text{Tr}[X^2] \text{Tr}[XY] - 8 \text{Tr}[X^3] \text{Tr}[XY] + 12 \text{Tr}[X]^2 \text{Tr}[X^2Y] - 24 \text{Tr}[X] \text{Tr}[X^3Y] + 24 \text{Tr}[X^2Y].
\]

We need to show that,

\[
(24 \text{Tr}[X]^2 \text{Tr}[X^2] - 6 \text{Tr}[X]^3 - 18 \text{Tr}[X^3]) \text{Tr}[XY] + (21 \text{Tr}[X]^2 - 27 \text{Tr}[X^2]) \text{Tr}[X^2Y] - 48 \text{Tr}[X] \text{Tr}[X^3Y] + 54 \text{Tr}[X^4Y] \geq 0.
\]

The condition that \( p \) is of degree at most 4 in the variable \( z_1 \) variables shows that the matrix \( X \) is of rank 4. Choose a basis so that \( X \) is diagonal and of the form \( X = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, 0, \cdots) \) and let \( Y_{ii} = x_i \) for \( i \in [4] \). A straightforward calculation shows that this expression reduces to,

\[
6\lambda_1 x_1 \left[ \lambda_2(\lambda_3 - \lambda_4)^2 + \lambda_3(\lambda_2 - \lambda_4)^2 + \lambda_4(\lambda_2 - \lambda_3)^2 + 6\lambda_1 \sum_{i \neq j \in [2,4]} \lambda_i \lambda_j \right] + 6\lambda_2 x_2 [\cdot] + \cdots,
\]

which is indeed positive.

Exactly in the argument in the proof of theorem (5.3), we have the following result for four paving,

**Theorem 5.5.** Let \( A \in M_n(\mathbb{C})^+ \) be a positive contraction with diagonal entries all at most \( \delta \leq \frac{9}{16} \). Then, there is a partition \( X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4 = [n] \) such that,

\[
\maxroot \chi_i[A] \leq \frac{1}{16} \left( \frac{\sqrt{7(1 - \delta)} + 3\sqrt{\delta}}{\delta} \right)^2, \quad i = 1, 2, 3, 4.
\]
In particular, if \( P \) is a projection with diagonal entries at most \( \frac{1}{2} \), we have that,

\[
\max_{\text{root}} \chi_4[P] \leq \left( \frac{3 + \sqrt{7}}{32} \right)^2 \approx 0.996, \quad i = 1, 2, 3, 4.
\]

It is quite plausible that one can show the following, along the same lines,

**Conjecture 5.6.** Let \( A \in M_n(\mathbb{C})_1^+ \) be a positive contraction with diagonal entries all at most \( \delta \leq \left( \frac{r - 1}{r} \right)^2 \). Then, there is a partition \( X_1 \sqcup \cdots \sqcup X_r = [n] \) such that,

\[
\max_{\text{root}} \chi_r[A] \leq \frac{1}{r^2} \left( \sqrt{\left(2r - 1\right)(1 - \delta) + (r - 1)\sqrt{\delta}} \right)^2, \quad i \in [r].
\]

However, this would very likely be suboptimal as well and rather than trying to prove this, we discuss the problem of trying to get optimal estimates in the next section.

### 6. Towards optimal estimates: Issues with the barrier method

Let \( q \) be a univariate real stable polynomial. It is a simple fact, see [19] that for every \( z \) above the roots of \( q \), we have that,

\[
\Phi_{q'}(z - \delta) \leq \Phi_q(z), \quad \delta = \frac{1}{\Phi_q(z)}.
\]

Let \( q(z_1, \ldots, z_n) \) be a real stable polynomial and let \( z \) be above the roots of \( q \). If we had this same inequality, namely, that for every \( i, j \),

\[
\Phi_{\delta_i q}(z - \delta e_i) \leq \Phi_{q}(z), \quad \delta = \frac{1}{\Phi_{q}(z)}
\]

then, if \( q \) were of degree \( r \) in \( z_i \), iterating this \( r - 1 \) times, we would have that,

\[
\Phi_{\delta_i - 1 q}(z - \delta e_i) \leq \Phi_{q}(z), \quad \delta = \frac{r - 1}{\Phi_{q}}.
\]

This estimate would in turn yield optimal paving estimates. The following can be proved in the same way that theorem (5.3) follows from proposition (5.2).

**Theorem 6.1.** Let \( r \in \mathbb{N} \) and let \( p = \det[Z - A]^r \) and suppose for every \( S \subset [n] \) and \( i, j \in S^c \), we have, letting \( q = \frac{\partial^S p}{\partial z^S} \), for every point \( z \) above the roots of \( q \), that,

\[
\Phi_{\delta_i - 1 q}(z - \delta e_i) \leq \Phi_{q}(z),
\]

for

\[
\delta = \frac{r - 1}{\Phi_{q}}.
\]

Then, we have that,

\[
\max_{\text{root}} \chi_r[A] \leq \frac{1}{r} \left( \sqrt{1 - \delta} + (r - 1)\sqrt{\delta} \right)^2, \quad i \in [r].
\]

This last estimate is the best possible; Note that when \( r = 2 \), this would give that,

\[
\max_{\text{root}} \chi_2[A] \leq \frac{1}{2} + \sqrt{\delta(1 - \delta)}.
\]

This would in turn, by theorem (6.1) yield the desired optimal paving estimates by taking \( q = \det[Z - A]^r \) to start off with and iterating this. However, [10] is not true in general. It is false.
even when the degree of \( q \) is constrained, say, by saying that it is of degree at most 2 in the variable \( z_i \) or even in all the \( z_i \).

For instance, the polynomial,

\[
p(x, y) = \det \left( \begin{bmatrix} 7 & 0 & 8 \\ 0 & 0 & 4 \\ 1 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} y \right) = (7+8x+y)(8+4x+4y),
\]

is real stable and it is easy to see that (1, 1) is above the roots of \( p \). We have that,

\[
\Phi^x_p(1, 1) = \frac{3}{4}, \quad \Phi^y_p(1, 1) = \frac{5}{16}, \quad \Phi_{\partial p}^y(1 - \frac{1}{\Phi^x_p(1, 1)}, 1) = \Phi_{\partial p}^y \left( -\frac{4}{3}, 1 \right) = \frac{27}{73} > \frac{5}{16},
\]

showing that one cannot expect to shift the barrier to the left by \( \frac{1}{\varphi} \) and still have that the potential for the derivative is less than the original potential.

Even in the special of the polynomials at hand, of the form \( p(Z) = \det(\begin{bmatrix} Z - A \end{bmatrix})^r \) and their derivatives, this fact does not hold. Let us restrict our attention to the 2 characteristic polynomial, so let us consider \( p(Z) = \det(Z - A)^2 \). Recall that \( A(S) \) for a matrix \( A \in M_n(\mathbb{C}) \) and a subset \( S \subset [n] \) is the principal submatrix composed of rows and columns from the subset \( S \). We use the same expression with the same definition when \( S \) is a multiset rather than just a set. We have the following pleasant combinatorial fact,

**Proposition 6.2.** Let \( A \in M_n(\mathbb{C})^{sa} \) be hermitian. Then, for any multiset \( S \) composed of elements in \([n]\), we have,

\[
\frac{\partial^S}{\partial z^S} p(z) = p(z) \det_2[(Z - A)^{-1}(S)].
\]

**Proof.** It is a standard fact from linear algebra that given a matrix \( A \in M_n(\mathbb{C}) \) and a subset \( S \subset [n] \), we have,

\[
\det(A_S) = \det(A) \det(A^{-1}(S)).
\]

We also have,

\[
\frac{\partial^S}{\partial z^S} \det(Z - A)^2 = \sum_{S_1 \sqcup S_2 = S} \frac{\partial^{S_1}}{\partial z^{S_1}} \det(Z - A) \frac{\partial^{S_2}}{\partial z^{S_2}} \det(Z - A) \\
= \sum_{S_1 \sqcup S_2 = S} \det([Z - A]_{S_1}) \det([Z - A]_{S_2}) \\
= \det([Z - A]^2) \sum_{S_1 \sqcup S_2 = S} \det([Z - A]^{-1}(S_1)) \det([Z - A]^{-1}(S_2)) \\
= \det([Z - A]^2) \det_2([Z - A]^{-1}(S)).
\]

\[\square\]

Let us remark here that it is a fact due to Vere-Jones \cite{21} that for a multiset \( S \) composed of elements from \([n]\), the expression \( \det_2(A(S)) \) is zero whenever there is an element from \([n]\) that appears more than twice in \( S \), something that also follows from the above proposition.

Let \( q = \frac{\partial^S q}{\partial z^S} \) of a subset \( S \subset [n] \) and let \( z \) be above the roots of \( q \). Recall the expression \( \Phi \) that is equivalent to our desired inequality, \( \Phi_{\partial q}^y(z - \delta e_i) \leq \Phi_{\partial q}^y(z) \), for \( \delta = \frac{1}{\Phi_{\partial q}^y(z)} \),

\[
\partial_i q [\partial_j \partial_i q - (\partial_i q)(\partial_j q)](z) \leq q \left( \partial_j \partial_i^2 q - (\partial_i^2 q)(\partial_j q) \right)(z)
\]
\[ \frac{\det_2[B(S \cup i)] \det_2[B(S \cup i \cup j)]}{\det_2[B(S)]} + \frac{\det_2[B(S \cup i)] \det_2[B(S \cup j)]}{\det_2[B(S)]} \leq \frac{\det_2[B(S \cup i)] \det_2[B(S \cup j)]}{\det_2[B(S)]} \]

This might seem like a diabolically complicated expression, but consider the case when in place of \( \det_2 \), we have the regular determinant. The expressions corresponding to multisets that are not sets vanish and we would need to show that,

\[ \det[B(S)] \det[B(S \cup i \cup j)] \leq \det[B(S \cup i)] \det[B(S \cup j)] \]

If \( B \) were a PSD matrix, this is just the classical Koteljanskii inequality. It is a remarkable fact, due to Borcea and Branden \[4, \text{Corollary 4.2}\], that this inequality holds for arbitrary hermitian matrices as well.

The following purely linear algebraic conjecture would thus yield optimal estimates in Kadison-Singer, see theorem (6.1).

**Statement 6.3.** Let \( B \in M_n(\mathbb{C})^{sa} \) be hermitian and let \( S \subset [3, n] \) be a subset. Then, we have that,

\[ \frac{\det_2[B(S \cup 1)] \det_2[B(S \cup 2)]}{\det_2[B(S)]} \leq \frac{\det_2[B(S \cup 1 \cup 2)]}{\det_2[B(S)]} \]

Alas, numerical simulations show that this is false in general. When the set \( S \) is empty, this does hold, something that can be checked directly. However, even when \( |S| = 1 \), this statement as written down is false. We feel strongly that there is a purely qualitative linear algebraic statement that would imply the necessary barrier estimates, but are currently unable to come up with the right candidate.

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