PARTIAL REGULARITY OF SOLUTIONS TO THE 3D CHEMOTAXIS-NAVIER-STOKES EQUATIONS AT THE FIRST BLOW-UP TIME

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Abstract. As Dombrowski et al. showed in [11] (see also [37]), suspensions of aerobic bacteria often develop flows from the interplay of chemotaxis and buoyancy, which is described as the chemotaxis-Navier-Stokes flow, and they observed self-concentration occurs as a turbulence by exhibiting transient, reconstituting, high-speed jets. Moreover, local concentration leads to a jet descending faster than its surroundings, which entrains nearby fluid to produce paired, oppositely signed vortices. In this note, we investigate the Hausdorff dimension of these vortices (singular points) by considering partial regularity of weak solutions of the three dimensional chemotaxis-Navier-Stokes equations, and obtain the $\frac{5}{3}$-dimensional Hausdorff measure of the possible singular set is vanishing at the first blow-up time, which generalizes the Caffarelli-Kohn-Nirenberg’s partial regularity theory to the chemotaxis-fluid model. The new ingredients are to establish certain type of local energy inequality and deal with the non-scaling invariant quantity of $n \ln n$, where $n$ represents the cell concentration, which seems to be the first description for the singular set of weak solutions of the model.

Keywords: chemotaxis-Navier-Stokes, partial regularity, local energy inequality

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1. The question

Consider a PDE model on $Q_T = \mathbb{R}^3 \times (0, T)$ describing the dynamics of oxygen, swimming bacteria, and viscous incompressible fluids, which was proposed by Tuval et al. [37] as follows:

$$
\begin{align*}
\partial_t n + u \cdot \nabla n - \Delta n &= -\nabla \cdot (\chi(c)n \nabla c), \\
\partial_t c + u \cdot \nabla c - \Delta c &= -\kappa(c)n, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -n \nabla \phi, \quad \nabla \cdot u = 0
\end{align*}
$$

(1.1)

where $c(x, t) : Q_T \to \mathbb{R}^+$, $n(x, t) : Q_T \to \mathbb{R}^+$, $u(x, t) : Q_T \to \mathbb{R}^3$ and $p(x, t) : Q_T \to \mathbb{R}$ denote the oxygen concentration, cell concentration, the fluid velocity and the associated pressure, respectively. Moreover, the gravitational potential $\phi$, the chemotactic sensitivity $\chi(c) \geq 0$ and the per-capita oxygen consumption rate $\kappa(c) \geq 0$ are sufficiently smooth given functions (see also [11]).
Due to the significance of the biological background (see [11], [37]), the model could be used to predict the large-scale bioconvection affecting clearly the overall oxygen consumption in the above experiments (see, for example, [28]). Many mathematicians have studied this model and made much progress, such as the existence of weak solutions, the chemotaxis-Navier-Stokes system with a nonlinear diffusion, blow-up criteria, stability and so on. Here we just mention some related works for the result in this paper.

Firstly, for the existence of weak solutions, in [14], global classical solutions near constant steady states are constructed for the full chemotaxis-Navier-Stokes system by Duan-Lorz-Markowich. In [28], for the case of bounded domain in $\mathbb{R}^n$ with $n = 2, 3$, the local existence of weak solutions for problem (1.1) is obtained by Lorz. Later, Winkler proved the existence of global weak solution in [42] by assuming that

$$\left(\frac{\kappa}{\chi}\right)' > 0, \quad \left(\frac{\kappa}{\chi}\right)'' \leq 0, \quad (\kappa \chi)' \geq 0.$$

By assuming $\chi', \kappa' \geq 0$ and $\kappa(0) = 0$. In [6] and [7], local well-posed results and blow-up criteria were established by Chae-Kang-Lee. Recently, Winkler proved the global existence of weak solutions of the system (1.1) in bounded domain with large initial data, and obtained much better a priori estimates such as $|\nabla c|^4 \in L^1$ in [43]. For the two-dimensional system of (1.1), the system is better understood. Liu and Lorz [27] proved the global existence of weak solutions to the two-dimensional system of (1.1) for arbitrarily large initial data, under the assumptions on $\chi$ and $f$ made in [14]. See [13, 18, 25, 39–41] and the references therein for more results. For more references about the existence of solutions, we refer to [2, 5, 9, 10, 21, 44] and the references therein.

As for the case of the chemotaxis-Navier-Stokes system with a nonlinear diffusion, that means $\Delta n$ is replaced by $\Delta n^m$, there are also many results. In [15], Lorz-Francesco-Markowich showed the global existence of a bounded solution to porous medium equation, on a bounded domain in $\mathbb{R}^2$, with the boundary conditions $\partial_n n^m = \partial_n c = u = 0$ and the condition $m \in (\frac{3}{2}, 2]$. In [33], Tao and Winkler extended the result to the case $m > 1$ on a bounded domain in $\mathbb{R}^2$. In [27], Lorz and Liu proved the global existence of a weak solution to porous medium equation in $\mathbb{R}^3$ when $m = \frac{4}{3}$. For more references, one can refer to [8, 34, 46] and so on.

For the consideration of boundary conditions, inhomogeneous Dirichlet boundary conditions for the signal may affect global regularity in the three-dimensional full Navier-Stokes version, which is founded by Black-Winkler in [3], we referred the recent result.

As Winkler said in [43], “For the full three-dimensional chemotaxis-Navier-Stokes system, even at the very basic level of global existence in generalized solution frameworks, a satisfactory solution theory is entirely lacking.” In this paper our aim is to explore partial regularity properties of weak solutions. For simplicity, we consider the
case \( \kappa(c) = c \) and \( \chi(c) = 1 \). Then the three dimensional chemotaxis-Navier-Stokes system (1.1) is reduced to

\[
\begin{align*}
\partial_t n + u \cdot \nabla n - \Delta n &= -\nabla \cdot (n \nabla c), \\
\partial_t c + u \cdot \nabla c - \Delta c &= -cn, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -n \nabla \phi, \quad \nabla \cdot u = 0,
\end{align*}
\]

(1.2)

Recently, global weak solution for this system was obtained in 2D and 3D (see, for example, [17, 45]), respectively, where they established a priori estimate

\[
U(t) + \int_0^t V(t) \, d\tau \leq Ce^{Ct},
\]

where

\[
U = \|n\|_{L^1 \cap L_{\log} L} + \|\nabla \sqrt{c}\|_2^2 + \|u\|_2^2,
\]

and

\[
V = \|\nabla \sqrt{n + 1}\|_2^2 + \|\Delta \sqrt{c}\|_2^2 + \|\nabla u\|_2^2 + \int_{\mathbb{R}^d} (\sqrt{c})^{-2} |\nabla \sqrt{c}|^4 \, dx + \int_{\mathbb{R}^d} n |\nabla \sqrt{c}|^2 \, dx,
\]

where \( d = 2, 3 \), the definition of \( L_{\log} L \) is given by Definition 1.2, and we write the norm of \( \|f\|_{L^q(\mathbb{R}^d)} \) as \( \|f\|_q \) for simplicity. We also refer to the recent existence result of global weak solution in \( \mathbb{R}^3 \) in [19] by assuming \( (n_0 + 1) \ln(n_0 + 1) \in L^1 \). They established a different priori estimate as following:

\[
\frac{d}{dt} F_\varepsilon(t) + \frac{1}{2} D_\varepsilon(t) \leq C.
\]

Here,

\[
F_\varepsilon(t) = \int_{B_1^\varepsilon} (n_\varepsilon + 1) \ln(n_\varepsilon + 1) + \frac{1}{2} \int_{B_1^\varepsilon} \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{b}{2} \int_{B_1^\varepsilon} |u_\varepsilon|^2,
\]

and

\[
D_\varepsilon(t) = \frac{1}{4} \int_{B_1^\varepsilon} \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon + 1} + K_1 \int_{B_1^\varepsilon} \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + K_1 \frac{1}{4} \int_{B_1^\varepsilon} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}
\]

\[
+ \frac{1}{2} \int_{B_1^\varepsilon} \frac{F_\varepsilon(n_\varepsilon)}{c_\varepsilon} |\nabla c_\varepsilon|^2 + \frac{b}{2} \int_{B_1^\varepsilon} |\nabla u_\varepsilon|^2,
\]

with \( F_\varepsilon(s) = \varepsilon^{-1} \ln(1 + \varepsilon s) \). However, up to now more information about these weak solutions is still not known, especially the interior singular vortices as described in [11] and the self-organized generation of a persistent hydrodynamic vortex that traps cells near the contact line(see [37]). Motivated by the recent progress on the non-uniqueness of suitable Leray-Hopf solutions to the Navier-Stokes equations with identical body force by Albritton-Brüe-Colombo in [1], it’s interesting to consider suitable weak solutions of (1.2) as Caffarelli-Kohn-Nirenberg in [4], and one may ask naturally:
Q1: Whether does there exist a suitable weak solution for the system of (1.1) or (1.2)?
Q2: How to characterize the singular points of weak solutions?

In this note we aim to answer the second question. Recall that these so-called partial regularity or $\varepsilon$-regularity theory, it can be traced back to the well-known work by Caffarelli-Kohn-Nirenberg [4] for the analysis of suitable weak solutions of the three dimensional time-dependent Navier-Stokes equations, where they showed that the set $S$ of possible interior singular points of a suitable weak solution is one-dimensional parabolic Hausdorff measure zero by improving Scheffer’s results in [29–31]. The suitable weak solution is better than Leray-Hopf weak solution introduced by Leray in [24] and if the local strong solution blows up, then the solution may be continued as a suitable weak solution (see Proposition 30.1 in [23]).

Besides, it is worth mentioning the interesting approach of Katz and Pavlović (20) for studying the dimension of the singular set, where they considered the Navier-Stokes equation with dissipation $(-\Delta)^{\alpha}$ with the condition of $1 < \alpha < \frac{5}{4}$ and proved the Hausdorff dimension of the singular set at time of first blow up is at most $5 - 4\alpha$. More references on simplified proofs and improvements, we refer to Lin [26], Ladyzhenskaya-Seregin [22], Tian-Xin [35], Seregin [32], Gustafson-Kang-Tsai [16], Vasseur [38] and the references therein. Here we consider the partial regularity of the system (1.2) at the first blow-up time as Dong-Du in [12], since the first question of (Q1) is still unknown, which is an open question.

Recall the well-posed results in [6] or [17].

**Theorem 1.1.** Assume that $n_0 \geq 0, c_0 \geq 0$ and $\nabla^k \phi \in L^\infty$ with $1 \leq k \leq m$. There exists a constant $T^*$, the maximal existence time, which depends on the norm of initial data, such that for any $t < T^*$, if the initial data $(n_0, c_0, u_0) \in H^{m-1}(\mathbb{R}^3) \times H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3)$ with $m \geq 3$ satisfy (1.2) in $\mathbb{R}^3 \times (0, T^*)$, then there exists a unique regular solution $(n, c, u)$ of (1.2) satisfying $n \geq 0, c \geq 0$ and

$$(n, c, u) \in L^\infty(0, t); H^{m-1}(\mathbb{R}^3) \times H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3),$$

$$(\nabla n, \nabla c, \nabla u) \in L^2(0, t); H^{m-1}(\mathbb{R}^3) \times H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3),$$

$$(\partial_t n, \partial_t c, \partial_t u) \in L^\infty(0, t); H^{m-1}(\mathbb{R}^3) \times H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3).$$

Firstly, we give the definition of $L \log L$ norm, which will be used in the following time.

**Definition 1.2.** The Zygmund classes with $A(t) = t \log^{+} t$, is defined as the set all functions $f$ such that

$$\int_{\mathbb{R}^3} A(|f(x)|) dx < \infty.$$ 

the corresponding Zygmund space $L \log L(\mathbb{R}^3)$ is defined as the linear hull of the Zygmund class, which is equipped with the Luxemburg norm

$$||f||_{L \log L} = \inf \left\{ k \left| \int_{\mathbb{R}^3} A(\frac{f}{k}) dx \leq 1 \right\}. $$
and
\[ \log^+ t = \begin{cases} \log t, & t \geq 1, \\ 0, & \text{otherwise.} \end{cases} \] (1.3)

For the given initial data, there exists a global weak solution (see [43] or [17]), which is defined as follows:

**Definition 1.3.** \((n, c, u)\) is called a weak solution to the Cauchy problem (1.2) with the initial data \((n_0, c_0, u_0)\) satisfying

\[ n_0 \in L^1 \cap L \log L, n_0 > 0, c_0 \in L^1 \cap L^\infty, \sqrt{c_0} \in L^2, u_0 \in L^2, \text{div} u_0 = 0 \]

and \(\nabla \phi \in L^\infty\), if the following conditions hold:

(i) \(n(t, x) > 0, c(t, x) > 0, \) for \(t > 0\) and \(x \in \mathbb{R}^3\),

(ii) \((n, c, u)\) satisfies the system (1.2) in the sense of distribution;

(iii) \((n, c, u)\) satisfies: for any \(t > 0\), the following inequality is true

\[ U(t) + \int_0^t V(\tau) d\tau \leq C e^{Ct}; \]

where

\[ U(t) = \|n\|_{L^1 \cap L \log L} + \|\nabla \sqrt{c}\|_2^2 + \|u\|_2^2; \]
\[ V(t) = \|\nabla \sqrt{n + 1}\|_2^2 + \|\nabla^2 \sqrt{c}\|_2^2 + \|\nabla u\|_2^2 + \int_{\mathbb{R}^3} (\sqrt{c})^{-2} |\nabla \sqrt{c}| dx + \int_{\mathbb{R}^3} n |\nabla \sqrt{c}|^2 dx \]

Motivated by [4], we consider the partial regularity property of weak solutions, which is so-called \(\varepsilon\)-regularity criteria. First, we say a point \((x_0, t_0)\) is a regular point if \((n, \nabla c, u) \in L^\infty(Q_0(x_0, t_0))\) for some \(r_0 > 0\), where \((Q_0(x_0, t_0)) = B_{r_0}(x_0) \times (t_0 - r_0^2, t_0)\) and \(B_{r_0}(x_0) = \{x, |x - x_0| < r_0\}\). It is worth noting that the definition here is consistent with the global regularity criterion proved in [7], where the regularity was ensured by

\[ u \in L^2(0, T; L^\infty(\mathbb{R}^3)), \ n \in L^1(0, T; L^\infty(\mathbb{R}^3)), \]

(See Theorem 1 in [7]). When \((x_0, t_0) = (0, 0)\), we write \((0, 0) = 0, Q_{t_0}(0) = Q_{t_0}^0\) and \(B_{r_0}(0) = B_{r_0}\) for simplicity.

Our first theorem is as follows:

**Theorem 1.4.** Assume that \((n, c, u)\) is a regular solution of (1.2) in \(\mathbb{R}^3 \times (-1, 0)\) with the initial data \((n(x, -1), c(x, -1), u(x, -1)) \in H^2(\mathbb{R}^3) \times H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)\) as in Theorem 1.1, which is also a weak solution as in Definition 1.3. Then \(z_0 = (x_0, 0)\) is a regular point, if there exists an absolute constant \(\varepsilon_1\) such that

\[ \sup_{-1 < t < 0} \int_{B_1(x_0)} n + |n \ln n| + |\nabla \sqrt{c + 1}|^2 + |u|^2 dx \]
\begin{align}
+ \int_{Q_1(\varepsilon_0)} |\nabla \sqrt{n}|^2 + |\nabla u|^2 + |\nabla^2 \sqrt{c+1}|^2 + |p|^2 \, dx \, dt \leq \frac{\varepsilon_1}{(\Lambda_0 \Lambda_1)^{4+4\alpha_0}}, \quad (1.4)
\end{align}

where \( \alpha_0 > 0 \) is an absolute constant, \( \Lambda_0 = 108\|c(\cdot,-1) + 1\|_{L^\infty(\mathbb{R}^3)} \) and \( \Lambda_1 = (\|\nabla \phi\|_{\infty} + 1) \).

**Remark 1.5.** (i) The term of \( \nabla \sqrt{n} \in L^2 \) is reasonable, see the a priori estimates of weak solutions by Winkler in \([43]\), or it can also be derived from Lemma 2.14 and the definition of weak solutions. The pressure \( p \) is well-defined due to the Calderón-Zygmund estimates and the equations of (1.2). Especially, there holds

\[
\phi(t) \leq C \int_{-1}^{t} |\sqrt{\ln n}|^{1+\frac{4}{9}} \, dx \leq C \int_{-1}^{t} |\sqrt{\ln n}|^{\frac{14}{9}} \, dx \leq C \Lambda_1 \|\nabla \phi\|_{L^1_t L^3_x} \|\nabla n\|_{L^1_t L^2_x}^{\frac{1}{3}} \|\nabla n\|_{L^2_t L^q_x}^{\frac{2}{3}}
\]

since \( u, \sqrt{n} \in L^\infty_t L^2_x \) and \( \nabla u, \sqrt{n} \in L^2_t L^2_x \) imply that

\[
u, \sqrt{n} \in L^s_t L^2_x, \quad \frac{2}{s} + \frac{3}{q} = \frac{3}{2}, \quad 2 \leq q \leq 6.
\]

(ii) The new observation of this theorem is the local energy inequality of (2.2) (See Lemma [2.14]), which is indeed a local a priori estimate for weak solutions, which is of independent interest.

(iii) The difficulty mainly lies in dealing with the term including \( \ln n \), which is not scaling invariant under the embedding inequality. We establish the local a priori estimates of \( \int_{B_1}(n \ln n)(\cdot,t) \) firstly by estimating the local energy inequality, then use the equation of \( n \) by estimating the term of \( \int_{B_1}(n \psi)(\cdot,t) \), which combined with the embedding inequality in a fixed sphere imply the estimate of \( \int_{B_1} n \mid \ln n \mid \).

(iv) In \([4, 7]\), the estimate of the term \( \int n \mid \ln n \mid \) in whole space is that

\[
\int n \mid \ln n \mid \leq \int n \ln n + 2 \int n(\ln n)_- \leq \int n \ln n + C \int n(x). \quad (1.5)
\]

Here \( (\ln n)_- \) is a negative part of \( \ln n \) and \( \langle x \rangle^2 = 1 + |x|^2 \). We use a different method to deal with this term without the weight \( \langle x \rangle \), since the following estimate holds locally:

\[
\int_{B_k} n \mid \ln n \mid \leq \int_{B_k} n \ln n + 2 \int_{B_k} n(\ln n)_- \leq \int_{B_k} n \ln n + C \int_{B_k} n^{1-\alpha}. \quad (1.6)
\]

**Remark 1.6.** To the authors’ best knowledge, whether the weak solution considered as in \([17]\) verifies the local energy estimate is still unknown, and it seems that the existence of such weak solutions is still an open problem. The main obstacle lies in the right hand term of \( -\nabla \cdot (n \nabla c) \), since the nonlinear term could not be controlled by the energy norm of \( n \) and \( \nabla \sqrt{c+1} \) under the Sobolev imbedding theorem in the
energy inequality. Here we prove a local energy inequality of weak type with the term of \( n \ln n \), which may have an uncertain sign.

Similar as Lin’s regularity version in \([26]\), we have the following interior regularity criteria.

**Theorem 1.7.** Taking the same assumptions as Theorem 1.4, there exists a constant \( \varepsilon_2(\Lambda_0, \Lambda_1) = \frac{\varepsilon^2}{C(\Lambda_0, \Lambda_1)^{15+12500}} \) and \( \varepsilon'_2(\Lambda_0, \Lambda_1) = \frac{\varepsilon'^2}{C(\Lambda_0, \Lambda_1)^{15+12500}} \) with an absolute constant \( C \) such that \( z_0 = (x_0, 0) \) is a regular point of \((n, c, u)\), if one of the following conditions holds

\[
(i) \left( \int_{Q_1(z_0)} n^{\frac{3}{2}}(|\ln n| + 1)^{\frac{3}{2}} + |\nabla \sqrt{n + 1}|^2 + |u|^2 + |p|^2 \right) \leq \varepsilon_2, \tag{1.7}
\]

\[
(ii) \left( \int_{Q_1(z_0)} n^{\frac{8}{3}} + |\nabla \sqrt{n + 1}|^{\frac{10}{3}} + |u|^{\frac{10}{3}} + |p|^{\frac{10}{3}} \right) \leq \varepsilon'_2. \tag{1.8}
\]

Recall the definition of Hausdorff measure and the parabolic version:

**Definition 1.8** (see \([36]\) Chapter 6). For a set \( E \subset \mathbb{R}^{n+1} \) and \( \alpha \geq 0 \), \( Q_r(z_0) = B_r(x_0) \times (t_0 - r^2, t_0) \) for \( z_0 = (x_0, t_0) \). Denote by \( \mathcal{P}^\alpha(E) \) its \( \alpha \)-dimensional parabolic Hausdorff measure, namely,

\[
\mathcal{P}^\alpha(E) = \lim \inf_{\delta \to 0^+} \left\{ \sum_{j=1}^{\infty} r_j^\alpha : E \subset \bigcup_j Q(z_j, r_j), r_j \leq \delta \right\}.
\]

Immediately, it follows from the above theorem that

**Corollary 1.9.** Taking the same assumptions as Theorem 1.4, there holds \( \mathcal{P}^\frac{5}{3}(S) = 0 \), where \( S \) is the singular set at time 0.

**Remark 1.10.** At a fixed time, the possible singular set is described via the parabolic Hausdorff measure, since the known a priori estimates for weak solutions are parabolic norms. The above conclusion indicates that the concentration of cells may appear in a linear form, but not in a two-dimensional region, which is closely related to the interior singular vortices as described in \([11]\). The estimate on the Hausdorff dimension of the singular set here is weaker than the Navier-Stokes case (see \([4]\)). In fact, it is still unknown that whether the condition of

\[
\limsup_{r \to 0} \frac{1}{r} \int_{Q_r} |\nabla \sqrt{n}|^2 + |\nabla u|^2 + |\nabla^2 \sqrt{c}|^2 \leq \varepsilon \tag{1.9}
\]

implies the regularity, which is similar as \([3]\). The main obstacle comes from the terms including \( n \ln n \) on the right hand side of the local energy inequality (2.2), since “\( n \ln n \)” seems to be not cancelled by the term of “\( |\nabla \sqrt{n}| \)” with the help of the embedding inequality in the scaling sense.
In fact, the condition of the pressure can be removed, which is stated as follows.

**Theorem 1.11.** Taking the same assumptions as Theorem 1.4, there exists an absolute constant $\varepsilon_3$ such that $z_0 = (x_0, 0)$ is a regular point if,

$$
\limsup_{r \to 0} r^{-1} \left( \sup_{-r^2 < t < 0} \int_{B_r(x_0)} n + |n \ln n| + |\nabla \sqrt{c + 1}|^2 + |u|^2 \right)
\leq \frac{\varepsilon_3}{(\Lambda_0 \Lambda_1)^{1/(4 + \alpha_0)}}(1.10)
$$

The paper is organized as follows, in Section 2, we introduce some definitions and technical lemmas, especially including the new local energy inequality. In Section 3, we prove Theorem 1.4 which is divided into four steps. Theorem 1.7, Corollary 1.9 and Theorem 1.11 are proved in Section 4 and Section 5, respectively.

Throughout this article, $C$ denotes an absolute constant independent of $(n, c, u)$ and may be different from line to line.

## 2. Preliminaries and some technical lemmas

Let $(n, c, u, p)$ be a solution to the chemotaxis-Navier-Stokes equations (1.2). Without loss of generality, let $z_0 = (0, 0)$. Set the following scaling:

$$
n_\lambda(x, t) = \lambda^2 n(\lambda x, \lambda^2 t); \quad c_\lambda(x, t) = c(\lambda x, \lambda^2 t);
\quad u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t); \quad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)
$$

then $(n_\lambda, c_\lambda, u_\lambda, p_\lambda)$ is also a solution of (1.2).

Now define some quantities which are invariant under the scaling (2.1):

$$
A_u(r) = r^{-1} \|u\|^2_{L_1^\infty L_2^1(Q_r)}; \quad E_u(r) = r^{-1} \|\nabla u\|^2_{L_1^2 L_2^2(Q_r)};
A_c(r) = r^{-1} \|\nabla c\|^2_{L_1^2 L_2^2(Q_r)}; \quad E_c(r) = r^{-1} \|\nabla^2 c\|^2_{L_1^2 L_2^2(Q_r)};
A_n(r) = r^{-1} \|\sqrt{n}\|^2_{L_1^\infty L_2^2(Q_r)}; \quad E_n(r) = r^{-1} \|\nabla (\sqrt{n})\|^2_{L_1^2 L_2^2(Q_r)};
C_u(r) = r^{-2} \|u\|^3_{L_1^3 L_2^2(Q_r)}; \quad \tilde{C}_u(r) = r^{-2} \|u - (u)_r\|^3_{L_1^3 L_2^2(Q_r)};
D(r) = r^{-2} \|p\|_{L_1^2 L_2^2(Q_r)}^\frac{3}{2}.
$$

Recall a property of harmonic function.

**Lemma 2.12 (See [26]).** Let $f$ be a harmonic function in $B_1 \subset \mathbb{R}^3$, for $1 \leq p, q \leq \infty$, $0 < r < \rho < 1$ and $k \geq 1$, there holds

$$
\|\nabla^k f\|_{L^q(B_\rho)} \leq C \frac{r^\frac{3}{2}}{(\rho - r)^{\frac{3}{4} + k}} \|f\|_{L^p(B_r)}.
$$
Lemma 2.13 (A priori estimates). Under the assumptions of Theorem 1.4 there holds
\[ \nabla \sqrt{c+1} \in L^\infty L^2 \cap L^2 \dot{H}^1, \]
and
\[ \int_{\mathbb{R}^3 \times (-1,0)} (\sqrt{c+1})^{-2} |\nabla \sqrt{c+1}|^4 < \infty. \]

Proof. Direct calculations imply
\[ |\nabla \sqrt{c+1}| = \left| \frac{1}{2} (c+1)^{-\frac{1}{2}} \nabla c \right| = \left| \frac{\sqrt{c}}{\sqrt{c+1}} \nabla \sqrt{c} \right| \leq |\nabla \sqrt{c}|, \]
and
\[ |\nabla^2 \sqrt{c+1}| \leq \frac{|\nabla \sqrt{c}|^2}{\sqrt{c+1}} + |\nabla^2 \sqrt{c}| + \frac{1}{\sqrt{c+1}} |\nabla \sqrt{c+1}||\nabla \sqrt{c}|. \]

Hence we arrive at
\[ \int_{\mathbb{R}^3 \times (-1,0)} |\nabla^2 \sqrt{c+1}|^2 \leq \int_{\mathbb{R}^3 \times (-1,0)} |\nabla^2 \sqrt{c}|^2 + \int_{\mathbb{R}^3 \times (-1,0)} \left| \frac{\nabla \sqrt{c}}{\sqrt{c+1}} \right|^2. \]

Under the assumptions of Theorem 1.4 there hold
\[ \nabla \sqrt{c} \in L^\infty L^2 \cap L^2 \dot{H}^1, \]
and
\[ \int_{\mathbb{R}^3 \times (-1,0)} (\sqrt{c})^{-2} |\nabla \sqrt{c}|^4 < +\infty. \]

Noting that \((\sqrt{c+1})^{-2} \leq (\sqrt{c})^{-2}\), we have
\[ \nabla \sqrt{c+1} \in L^\infty L^2 \cap L^2 \dot{H}^1. \]

The second inequality is obviously due to the relation \(|\nabla \sqrt{c+1}| \leq |\nabla \sqrt{c}|\).

Next we establish a new local energy inequality including \(\ln n\). Moreover, consider the equation of \(c+1\) instead of \(c\), and we obtain some new estimates of \(c+1\), which is slightly different with those in [45] and [17].

Lemma 2.14 (Local energy inequality). Let \(\psi\) be a cut-off function, which vanishes on the parabolic boundary of \(Q_1^1\). Then for any \(t \in (-1,0)\), the following inequality holds under the assumptions of Theorem 1.4:
\[
\int_{B_1} (n \ln n\psi)(\cdot, t)dx + 2 \int_{Q_1^t} |\nabla \sqrt{n}|^2 \psi dx dt \\
+ 2 \int_{B_1} (|\nabla \sqrt{c}|^2 \psi)(\cdot, t)dx + \frac{4}{t} \int_{Q_1^t} |\nabla^2 \sqrt{c}|^2 \psi dx dt
\]
\[ +2 \int_{Q_1^t} |\nabla \sqrt{c}|^2 n \psi dxdt + \frac{1}{4} \sum_{i,j} \int_{Q_1^t} (\sqrt{c})^{-2} (\partial_i \sqrt{c})^2 (\partial_j \sqrt{c})^2 \psi dxdt \]
\[ + 112 \| \bar{c}(\cdot, -1) \|_\infty \int_{B_1} (|u|^2)(\cdot, t) \psi dxdt + 112 \| \bar{c}(\cdot, -1) \|_\infty \int_{Q_1^t} |\nabla u|^2 \psi dxdt \quad (2.2) \]
\[ \leq \int_{Q_1^t} n \ln n (\partial_t \psi + \Delta \psi) dxdt + \int_{Q_1^t} n \ln n \nabla \psi dxdt \]
\[ + \int_{Q_1^t} n \ln n c \cdot \nabla \psi dxdt + \int_{Q_1^t} n \nabla c \cdot \nabla \psi dxdt \]
\[ + 2 \int_{Q_1^t} |\nabla \sqrt{c}|^2 (\partial_t \psi + \Delta \psi) dxdt + 2 \int_{Q_1^t} |\nabla \sqrt{c}|^2 u \cdot \nabla \psi dxdt \]
\[ - \frac{4}{7} \int_{Q_1^t} \bar{c}(\cdot, -1) |\nabla \sqrt{c}|^2 |\nabla \sqrt{c}| \cdot \nabla \psi dxdt + 112 \| \bar{c}(\cdot, -1) \|_\infty \int_{Q_1^t} |u|^2 (\partial_t \psi + \Delta \psi) dxdt \]
\[ + 112 \| \bar{c}(\cdot, -1) \|_\infty \int_{Q_1^t} |u|^2 u \cdot \nabla \psi dxdt + 112 \| \bar{c}(\cdot, -1) \|_\infty \int_{Q_1^t} (p - \bar{p}) u \cdot \nabla \psi dxdt \]
\[ - 224 \| \bar{c}(\cdot, -1) \|_\infty \int_{Q_1^t} n \nabla \phi \cdot u \psi dxdt, \]

where \( Q_1^t = (-1, t) \times B_1 \) and \( \bar{c} = c + 1 \).

**Proof.** Multiplying \((1 + \ln n)\psi\) in (1.2), integration by parts yields that
\[
\int_{Q_1^t} \partial_t n (1 + \ln n) \psi dxdt + \int_{Q_1^t} u \cdot \nabla n (1 + \ln n) \psi dxdt - \int_{Q_1^t} \Delta n (1 + \ln n) \psi dxdt
\]
\[ + \int_{Q_1^t} \nabla \cdot (n \nabla c)(1 + \ln n) \psi dxdt = T_1 + \cdots + T_4 = 0, \]
where
\[
T_1 = \int_{Q_1^t} \partial_t n (1 + \ln n) \psi dxdt = \int_{B_1} (n \ln n \psi)(\cdot, t) dx - \int_{Q_1^t} n \ln n \partial_t \psi dxdt; \]
\[
T_2 = - \int_{Q_1^t} u \cdot \nabla n \psi dxdt - \int_{Q_1^t} n (1 + \ln n) u \cdot \nabla \psi dxdt = - \int_{Q_1^t} n \ln n u \cdot \nabla \psi dxdt; \]
\[
T_3 = \int_{Q_1^t} \frac{1}{n} \nabla n \cdot \nabla n \psi dxdt + \int_{Q_1^t} \nabla n \cdot \nabla \psi dxdt + \int_{Q_1^t} \ln n \nabla n \cdot \nabla \psi dxdt
\]
\[ = \int_{Q_1^t} \frac{1}{n} \nabla n \cdot \nabla n \psi dxdt + \int_{Q_1^t} \nabla n \cdot \nabla \psi dxdt - \int_{Q_1^t} \frac{1}{n} \nabla n \cdot \nabla \psi dxdt - \int_{Q_1^t} \n \ln n \Delta \psi dxdt
\]
\[ = 4 \int_{Q_1^t} |\nabla \sqrt{n}|^2 \psi dxdt - \int_{Q_1^t} n \ln n \Delta \psi dxdt; \]
and
\[ T_4 = - \int_{Q_t^1} \nabla c \cdot \nabla n \psi dx dt - \int_{Q_t^1} n \nabla c \cdot \nabla \psi dx dt - \int_{Q_t^1} n \ln n \nabla c \cdot \nabla \psi dx dt. \]

Then we have
\[
\int_{B_t^1} (n \ln n \psi) (\cdot, t) dx + 4 \int_{Q_t^1} |\nabla \sqrt{n}|^2 \psi dx dt \\
= \int_{Q_t^1} n \ln n (\partial_t \psi + \Delta \psi) dx dt + \int_{Q_t^1} n \ln n \cdot \nabla \psi dx dt \\
+ \int_{Q_t^1} n \ln n \nabla c \cdot \nabla \psi dx dt + \int_{Q_t^1} \nabla n \cdot \nabla c \psi dx dt + \int_{Q_t^1} n \nabla c \cdot \nabla \psi dx dt \quad (2.3)
\]

Due to \( \tilde{c} = c + 1 \), it follows from the equation (1.2) that
\[
\partial_t \tilde{c} + u \cdot \nabla \tilde{c} - \Delta \tilde{c} = -\tilde{c} n + n.
\]

Note that
\[
\Delta \tilde{c} = 2|\nabla \sqrt{\tilde{c}}|^2 + 2 \sqrt{\tilde{c}} \Delta \sqrt{\tilde{c}},
\]
and dividing \( 2\sqrt{\tilde{c}} \) on both sides, we get
\[
\partial_t \sqrt{\tilde{c}} + u \cdot \nabla \sqrt{\tilde{c}} = \frac{|\nabla \sqrt{\tilde{c}}|^2}{\sqrt{\tilde{c}}} - \Delta \sqrt{\tilde{c}} = -\frac{1}{2} \sqrt{\tilde{c}} n + \frac{1}{2} \frac{n}{\sqrt{\tilde{c}}}. \quad (2.4)
\]

Multiplying the above equation (2.4) by \(-\partial_t (\partial_t \sqrt{\tilde{c}})\) and integration by parts, there holds
\[
- \int_{Q_t^1} \partial_t \sqrt{\tilde{c}} \partial_t (\partial_t \sqrt{\tilde{c}}) dx dt - \int_{Q_t^1} u \cdot \nabla \sqrt{\tilde{c}} \partial_t (\partial_t \sqrt{\tilde{c}}) dx dt + \int_{Q_t^1} \frac{|\nabla \sqrt{\tilde{c}}|^2}{\sqrt{\tilde{c}}} \partial_t (\partial_t \sqrt{\tilde{c}}) dx dt \\
+ \int_{Q_t^1} \Delta \sqrt{\tilde{c}} \partial_t (\partial_t \sqrt{\tilde{c}}) dx dt - \int_{Q_t^1} \frac{1}{2} \sqrt{\tilde{c}} n \partial_t (\partial_t \sqrt{\tilde{c}}) dx dt + \int_{Q_t^1} \frac{1}{2} \frac{n}{\sqrt{\tilde{c}}} \partial_t (\partial_t \sqrt{\tilde{c}}) dx dt \\
\doteq J_1 + J_2 + \cdots + J_6 = 0,
\]
where
\[
J_1 = \frac{1}{2} \int_{Q_t^1} \partial_t (\partial_t \sqrt{\tilde{c}})^2 \psi dx dt = \frac{1}{2} \int_{B_t^1} (|\nabla \sqrt{\tilde{c}}|^2 \psi) (\cdot, t) dx - \frac{1}{2} \int_{Q_t^1} |\nabla \sqrt{\tilde{c}}|^2 \partial_t \psi dx dt;
\]
\[
J_2 = \int_{Q_t^1} \partial_t u_j \partial_j \sqrt{\tilde{c}} (\partial_t \sqrt{\tilde{c}}) dx dt + \int_{Q_t^1} u_j \partial_j \sqrt{\tilde{c}} (\partial_t \sqrt{\tilde{c}}) dx dt \\
= \int_{Q_t^1} \nabla u : (\nabla \sqrt{\tilde{c}} \otimes \nabla \sqrt{\tilde{c}}) \psi dx dt - \frac{1}{2} \int_{Q_t^1} u \cdot \nabla \psi \nabla \sqrt{\tilde{c}} \cdot \nabla \sqrt{\tilde{c}} dx dt;
\]
\[
J_3 = - \int_{Q_t^1} \partial_t ((\sqrt{\tilde{c}})^{-1} |\nabla \sqrt{\tilde{c}}|^2) (\partial_t \sqrt{\tilde{c}} \psi) dx dt \\
= \int_{Q_t^1} ((\sqrt{\tilde{c}})^{-1} |\nabla \sqrt{\tilde{c}}|^2) \Delta \sqrt{\tilde{c}} \psi dx dt + \int_{Q_t^1} ((\sqrt{\tilde{c}})^{-1} |\nabla \sqrt{\tilde{c}}|^2) \nabla \sqrt{\tilde{c}} \cdot \nabla \psi dx dt;
\]
\[ J_4 = - \int_{Q_1} \partial_{ij} \sqrt{c} \partial_i \sqrt{c} \psi \, dx \, dt + \int_{Q_1} \sqrt{\nabla^2 \sqrt{c}^2} \psi \, dx \, dt + \int_{Q_1} \partial_{ij} \sqrt{c} \partial_i \sqrt{c} \partial_j \psi \, dx \, dt \]
\[ = \int_{Q_1} \sqrt{\nabla^2 \sqrt{c}^2} \psi \, dx \, dt - \frac{1}{2} \int_{Q_1} \sqrt{\nabla^2} \Delta \psi \, dx \, dt; \]
\[ J_5 = \frac{1}{2} \int_{Q_1} \sqrt{c} n \left( - \partial_i (\partial_i \sqrt{c}) \right) \, dx \, dt \]
\[ = \frac{1}{2} \int_{Q_1} \sqrt{\nabla^2 \sqrt{c}^2} n \psi \, dx \, dt + \frac{1}{2} \int_{Q_1} \sqrt{c} n \cdot \nabla \sqrt{c} \psi \, dx \, dt \]
\[ = \frac{1}{2} \int_{Q_1} \sqrt{\nabla^2 \sqrt{c}^2} n \psi \, dx \, dt + \frac{1}{4} \int_{Q_1} \nabla c \cdot n \psi \, dx \, dt; \]

and
\[ J_6 = - \frac{1}{2} \int_{Q_1} \nabla \left( \frac{n}{\sqrt{c}} \right) \cdot \nabla \sqrt{c} \psi \, dx \, dt \]

Combining all of them, we have
\[
\frac{1}{2} \int_{B_1} (|\nabla \sqrt{c}|^2 \psi)(\cdot, t) \, dx + \int_{Q_1} \sqrt{\nabla^2 \sqrt{c}^2} \psi \, dx \, dt \\
= \frac{1}{2} \int_{Q_1} \sqrt{\nabla^2 \sqrt{c}^2} (\partial_i \psi + \Delta \psi) \, dx \, dt + \frac{1}{2} \int_{Q_1} |\nabla \sqrt{c}|^2 u \cdot \nabla \psi \, dx \, dt \\
- \int_{Q_1} \nabla \sqrt{c} \cdot \nabla u \cdot \nabla \sqrt{c} \psi \, dx \, dt - \int_{Q_1} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 \Delta \sqrt{c} \psi \, dx \, dt \\
- \int_{Q_1} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 \nabla \sqrt{c} \cdot \nabla \psi \, dx \, dt - \frac{1}{2} \int_{Q_1} |\nabla \sqrt{c}|^2 n \psi \, dx \, dt \\
- \frac{1}{4} \int_{Q_1} \nabla n \cdot \nabla c \psi \, dx \, dt + \frac{1}{2} \int_{Q_1} \nabla \left( \frac{n}{\sqrt{c}} \right) \cdot \nabla \sqrt{c} \psi \, dx \, dt.
\]

(2.5)

We remark here the bad terms are those integrals without \( \nabla \psi \). One bad term of all the above terms is \( I = - \int_{Q_1} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 \Delta \sqrt{c} \psi \), and next we estimate it in details. Firstly, integration by parts yields that
\[
I = - \int_{Q_1} (\sqrt{c})^{-1} (\partial_j \sqrt{c})^2 \partial_i \sqrt{c} \psi \, dx \, dt \\
= - \int_{Q_1} (\sqrt{c})^{-2} (\partial_j \sqrt{c})^2 (\partial_i \sqrt{c})^2 \psi \, dx \, dt + 2 \int_{Q_1} (\sqrt{c})^{-1} \partial_j \sqrt{c} \partial_i \sqrt{c} \partial_j \sqrt{c} \psi \, dx \, dt \\
+ \int_{Q_1} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 \nabla \sqrt{c} \cdot \nabla \psi \, dx \, dt
\]
\[
\begin{align*}
= & \ - \sum_{i,j} \int_{Q_{1,t}} (\sqrt{c})^{-2}(\partial_j \sqrt{c})^2(\partial_i \sqrt{c})^2 \psi \ dx \ dt + 2 \sum_{i,j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_j \sqrt{c} \partial_i \sqrt{c} \partial_j \sqrt{c} \psi \ dx \ dt \\
+ & \ 2 \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_{ij} \sqrt{c} \partial_i \sqrt{c} \partial_j \sqrt{c} \psi \ dx \ dt + \int_{Q_{1,t}} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 \nabla \sqrt{c} \cdot \nabla \psi \ dx \ dt.
\end{align*}
\]

Noting that
\[
\sum_{i,j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_{ij} \sqrt{c} \partial_i \sqrt{c} \partial_j \sqrt{c} \psi \ dx \ dt = - I - \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_{ii} \sqrt{c} \partial_j \sqrt{c} \partial_j \sqrt{c} \psi \ dx \ dt,
\]
we have
\[
I = \ - \frac{1}{3} \sum_{i,j} \int_{Q_{1,t}} (\sqrt{c})^{-2}(\partial_j \sqrt{c})^2(\partial_i \sqrt{c})^2 \psi \ dx \ dt - \frac{2}{3} \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_{ii} \sqrt{c} \partial_j \sqrt{c} \partial_j \sqrt{c} \psi \ dx \ dt \\
+ & \ 2 \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_{ij} \sqrt{c} \partial_i \sqrt{c} \partial_j \sqrt{c} \psi \ dx \ dt + \frac{1}{3} \int_{Q_{1,t}} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 \nabla \sqrt{c} \cdot \nabla \psi \ dx \ dt.
\]

Secondly, using Young inequality of \(ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}\), it follows that
\[
\frac{2}{3} \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_{ij} \sqrt{c} \partial_i \sqrt{c} \partial_j \sqrt{c} \psi \ dx \ dt \leq \ \frac{1}{3} \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-2}|\partial_i \sqrt{c}|^2 |\partial_j \sqrt{c}|^2 \psi \ dx \ dt \\
+ & \ \frac{1}{3} \sum_{i \neq j} \int_{Q_{1,t}} |\partial_{ij} \sqrt{c}|^2 \psi \ dx \ dt,
\]
and
\[
\begin{align*}
& \ - \frac{2}{3} \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_{ii} \sqrt{c} \partial_j \sqrt{c} \partial_j \sqrt{c} \\
= & \ - \frac{2}{3} \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-2}(\partial_j \sqrt{c})^2(\partial_i \sqrt{c})^2 \psi + \frac{4}{3} \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-1}\partial_{ij} \sqrt{c} \partial_i \sqrt{c} \partial_j \sqrt{c} \psi \\
+ & \ 2 \sum_{i \neq j} \int_{Q_{1,t}} (\sqrt{c})^{-1} |\nabla j \sqrt{c}|^2 \nabla i \sqrt{c} \nabla \psi \\
\leq & \ \frac{2}{3} \sum_{i \neq j} \int_{Q_{1,t}} |\partial_{ij} \sqrt{c}|^2 \psi + \frac{2}{3} \int_{Q_{1,t}} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 \nabla \sqrt{c} \cdot \nabla \psi
\end{align*}
\]
Then we have
\[
I \leq \ - \frac{1}{3} \sum_{i,j} \int_{Q_{1,t}} (\sqrt{c})^{-2}(\partial_j \sqrt{c})^2(\partial_i \sqrt{c})^2 \psi + \sum_{i \neq j} \int_{Q_{1,t}} |\partial_{ij} \sqrt{c}|^2 \psi \\
+ \int_{Q_{1,t}} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 \nabla \sqrt{c} \cdot \nabla \psi \quad (2.6)
\]
Submitting it to (2.5), we get
\[ \frac{1}{2} \int_{B_1} (|\nabla \sqrt{\tilde{c}}|^2 \psi)(\cdot, t) + \frac{1}{3} \int_{Q_{1,t}} |\Delta \sqrt{\tilde{c}}|^2 \psi \]
\[ + \frac{1}{2} \int_{Q_{1,t}} |\nabla \sqrt{\tilde{c}}|^2 n \psi + \frac{1}{3} \sum_{i=j} \int_{Q_{1,t}} (\sqrt{\tilde{c}})^{-2} (\partial_j \sqrt{\tilde{c}})^2 (\partial_i \sqrt{\tilde{c}})^2 \psi \]
\[ \leq \frac{1}{2} \int_{Q_{1,t}} |\nabla \sqrt{\tilde{c}}|^2 (\partial_t \psi + \Delta \psi) + \frac{1}{2} \int_{Q_{1,t}} |\nabla \sqrt{\tilde{c}}|^2 u \cdot \nabla \psi \]
\[ - \int_{Q_{1,t}} \nabla \sqrt{\tilde{c}} \cdot \nabla u \cdot \nabla \sqrt{\tilde{c}} \psi \]
\[ - \frac{1}{4} \int_{Q_{1,t}} \nabla n \cdot \nabla c \psi + \frac{1}{2} \int_{Q_{1,t}} \nabla \left( \frac{n}{\sqrt{\tilde{c}}} \right) \cdot \nabla \sqrt{\tilde{c}} \psi. \tag{2.7} \]

Note that
\[ \int_{Q_{1,t}} (\sqrt{\tilde{c}})^{-2} |\nabla \sqrt{\tilde{c}}|^4 \psi = \sum_{i,j} \int_{Q_{1,t}} (\sqrt{\tilde{c}})^{-2} (\partial_j \sqrt{\tilde{c}})^2 (\partial_i \sqrt{\tilde{c}})^2 \psi \]
\[ \leq 3 \sum_{i,j} \int_{Q_{1,t}} (\sqrt{\tilde{c}})^{-2} (\partial_j \sqrt{\tilde{c}})^2 (\partial_i \sqrt{\tilde{c}})^2 \psi \]

and we have
\[ I = - \sum_{i,j} \int_{Q_{1,t}} (\sqrt{\tilde{c}})^{-1} (\partial_j \sqrt{\tilde{c}})^2 \partial_i \sqrt{\tilde{c}} \psi \]
\[ \leq \frac{3}{2} \int_{Q_{1,t}} |\Delta \sqrt{\tilde{c}}|^2 \psi + \frac{1}{2} \sum_{i,j} \int_{Q_{1,t}} (\sqrt{\tilde{c}})^{-2} (\partial_j \sqrt{\tilde{c}})^2 (\partial_i \sqrt{\tilde{c}})^2 \psi. \tag{2.8} \]

Then with the help of (2.8), by (2.5) + 6 \times (2.7) we get
\[ \frac{7}{2} \int_{B_1} (|\nabla \sqrt{\tilde{c}}|^2 \psi)(\cdot, t)dx + \int_{Q_{1,t}} |\nabla^2 \sqrt{\tilde{c}}|^2 \psi dxdt \]
\[ + \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{\tilde{c}}|^2 n \psi dxdt + \frac{1}{2} \int_{Q_{1,t}} (\sqrt{\tilde{c}})^{-2} |\nabla \sqrt{\tilde{c}}|^4 \psi dxdt \]
\[ \leq \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{\tilde{c}}|^2 (\partial_t \psi + \Delta \psi) dxdt + \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{\tilde{c}}|^2 u \cdot \nabla \psi dxdt \]
\[ - 7 \int_{Q_{1,t}} \nabla \sqrt{\tilde{c}} \cdot \nabla u \cdot \nabla \sqrt{\tilde{c}} \psi dxdt - \int_{Q_{1,t}} (\sqrt{\tilde{c}})^{-1} |\nabla \sqrt{\tilde{c}}|^2 \nabla \sqrt{\tilde{c}} \cdot \nabla \psi dxdt \]
\[ - \frac{7}{4} \int_{Q_{1,t}} \nabla n \cdot \nabla c \psi dxdt + \frac{7}{2} \int_{Q_{1,t}} \nabla \left( \frac{n}{\sqrt{\tilde{c}}} \right) \cdot \nabla \sqrt{\tilde{c}} \psi dxdt, \tag{2.9} \]

where we neglected the term of \( \frac{1}{2} \int_{Q_{1,t}} |\Delta \sqrt{\tilde{c}}|^2 \psi dxdt. \)
Moreover, using Young’s inequality

\[
\left| \int_{Q_{1,t}} \nabla \sqrt{\bar{c}} \cdot \nabla u \cdot \nabla \sqrt{\bar{c}} \psi \right| \leq \frac{1}{112} \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-2} |\nabla \sqrt{\bar{c}}|^4 \psi + 28 \int_{Q_{1,t}} (\sqrt{\bar{c}})^2 |\nabla u|^2 \psi \\
\leq \frac{1}{112} \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-2} |\nabla \sqrt{\bar{c}}|^4 \psi + 28 \|ar{c}\|_\infty \int_{Q_{1,t}} |\nabla u|^2 \psi
\]

(2.10)

and

\[
\frac{7}{2} \int_{Q_{1,t}} \nabla \left( \frac{n}{\sqrt{\bar{c}}} \right) \cdot \nabla \sqrt{\bar{c}} \psi \\
= -\frac{7}{2} \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-2} n |\nabla \sqrt{\bar{c}}|^2 \psi + 7 \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-1} \sqrt{n} \nabla \sqrt{n} \cdot \nabla \sqrt{\bar{c}} \psi \\
\leq -\frac{7}{2} \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-2} n |\nabla \sqrt{\bar{c}}|^2 \psi + \frac{7}{2} \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-2} n |\nabla \sqrt{\bar{c}}|^2 \psi + \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{n}|^2 \psi \\
\leq \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{n}|^2 \psi,
\]

(2.11)

Substitute these estimates (2.10)-(2.11) to the inequality of (2.9), and we get

\[
\frac{7}{2} \int_{B_1} (|\nabla \sqrt{\bar{c}}|^2 \psi)(\cdot, t) dx + \int_{Q_{1,t}} |\nabla^2 \sqrt{\bar{c}}|^2 \psi dx dt \\
+ \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{\bar{c}}|^2 n \psi + \frac{7}{16} \sum_{i,j} \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-2} (\partial_i \sqrt{\bar{c}})^2 (\partial_j \sqrt{\bar{c}})^2 \psi \\
\leq \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{\bar{c}}|^2 (\partial_i \psi + \Delta \psi) dx dt + \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{\bar{c}}|^2 u \cdot \nabla \psi dx dt \\
- \frac{7}{4} \int_{Q_{1,t}} \nabla n \cdot \nabla \psi dx dt - \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-1} |\nabla \sqrt{\bar{c}}|^2 \nabla \sqrt{\bar{c}} \cdot \nabla \psi dx dt \\
+ 196 \|ar{c}\|_\infty \int_{Q_{1,t}} |\nabla u|^2 \psi + \frac{7}{2} \int_{Q_{1,t}} |\nabla \sqrt{n}|^2 \psi dx dt.
\]

(2.12)

Recall the local estimate of \( n \) in (2.3), by taking (2.3) + \( \frac{4}{7} \times (2.12) \), we arrive at

\[
\int_{B_1} (n \ln n \psi)(\cdot, t) dx + 2 \int_{Q_{1,t}} |\nabla \sqrt{n}|^2 \psi dx dt \\
+ 2 \int_{B_1} (|\nabla \sqrt{\bar{c}}|^2 \psi)(\cdot, t) dx + \frac{4}{7} \int_{Q_{1,t}} |\nabla^2 \sqrt{\bar{c}}|^2 \psi dx dt \\
+ 2 \int_{Q_{1,t}} |\nabla \sqrt{\bar{c}}|^2 n \psi + \frac{1}{4} \sum_{i,j} \int_{Q_{1,t}} (\sqrt{\bar{c}})^{-2} (\partial_i \sqrt{\bar{c}})^2 (\partial_j \sqrt{\bar{c}})^2 \psi
\]
\[
\begin{align*}
&\leq \int_{Q_1^t} n \ln n (\partial_t \psi + \Delta \psi) dx dt + \int_{Q_1^t} n \ln n u \cdot \nabla \psi dx dt \\
&+ \int_{Q_1^t} n \ln n \nabla c \cdot \nabla \psi dx dt + \int_{Q_1^t} n \nabla c \cdot \nabla \psi dx dt \\
&+ 2 \int_{Q_1^t} |\nabla \sqrt{\tilde{c}}|^2 (\partial_t \psi + \Delta \psi) dx dt + 2 \int_{Q_1^t} |\nabla \sqrt{\tilde{c}}|^2 u \cdot \nabla \psi dx dt \\
&- \frac{4}{7} \int_{Q_1^t} (\sqrt{\tilde{c}})^{-1} |\nabla \sqrt{\tilde{c}}|^2 \nabla \sqrt{\tilde{c}} \cdot \nabla \psi dx dt + 112 \|\tilde{c}\|_\infty \int_{Q_1^t} |\nabla u|^2 \psi.
\end{align*}
\]

Multiplying the third equation of (1.2) by \(2u\psi\) and integration by parts, we have
\[
\int_{B_1} (|u|^2)(\cdot, t) \psi + 2 \int_{Q_1^t} |\nabla u|^2 \psi \leq \int_{Q_1^t} |u|^2 (\partial_t \psi + \Delta \psi) + \int_{Q_1^t} |u|^2 u \cdot \nabla \psi \\
+ \int_{Q_1^t} (p - \bar{p}) u \cdot \nabla \psi - 2 \int_{Q_1^t} n \nabla \phi \cdot u \psi
\] (2.14)
where \(\bar{p}\) is independent of the space variable \(x\). Taking (2.13) + 112\|\tilde{c}(\cdot, -1)\|_\infty \times (2.14),
which yields (2.2) via maximum principle of \(c\)'s equation. The proof is complete.

3. Proof of Theorem 1.4

Without loss of generality, let \(z_0 = (0, 0)\). Before we begin the proof, for \(r_k = 2^{-k}\) with \(k \in \mathbb{N}\), we introduce the following backward heat kernel \(\Psi_n\) as in [4]:
\[
\Psi_n(x, t) = \frac{1}{(r_n^2 - t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4(r_n^2 - t)}\right),
\]
where \((x, t) \in \mathbb{R}^3 \times (-\infty, r_n^2)\). Take a suitable cut-off function \(\xi(x, t)\) in \(Q_{r_3}\), which satisfies
\[
\xi(x, t) = \begin{cases} 
1, & \text{in } Q_{r_4} \\
0, & \text{in } Q_{r_3}^-.
\end{cases}
\] (3.1)
It is easy to check the following properties of \(\phi_n = \Psi_n \xi\).

Proposition 3.15. There exist two absolute constants \(C_1\) and \(C_2\) such that
(i). \(C_1 r_n^{-3} \leq \phi_n(x, t) \leq C_2 r_n^{-3}\) on \(Q_{r_n}\) for \(n \geq 2\);
(ii). \(\phi_n(x, t) \leq C_2 r_k^{-3}\) for \((x, t) \in Q_{r_k} \setminus Q_{r_{k+1}}\), \(1 < k \leq n\);
(iii) \(|\nabla \phi_n(x, t)| \leq C_2 r_n^{-1}\) in \(Q_{r_n}\), \(n \geq 2\);
(iv). \(|\nabla \phi_n(x, t)| \leq C_2 r_k^{-1}\) on \(Q_{r_k} \setminus Q_{r_{k+1}}\), \(1 < k \leq n\);
(v). \(\partial_t \phi + \Delta \phi \leq C\) on \(Q_{r_3}\);
(vi). \(\partial_t \phi + \Delta \phi = 0\) on \(Q_{r_4}\).

Proof of Theorem 1.4.
In the following proof, by mathematical induction we are aimed to prove the following inequality

$$
\begin{align*}
    r_k^{-3} \sup_{-r_k^2 < t < 0} \int_{B_{r_k}} (n \ln n) + |\nabla \sqrt{\tilde{c}}|^2 + |u|^2 \\
    + r_k^{-3} \int_{Q_{r_k}} |\nabla \sqrt{n}|^2 + |\nabla^2 \sqrt{\tilde{c}}|^2 + |\nabla u|^2 \leq C_0 \varepsilon_0^\frac{1}{3}, \quad (3.2)
\end{align*}
$$

for any $k \geq 1$, where $C_0 > 1$ is an absolute constant and $\varepsilon_0 = \frac{\varepsilon}{(\alpha_0 \Lambda_1)^{1+\alpha_0}}$. Obviously, (1.4) implies that (3.2) holds for $k = 1$. Assume that (3.2) holds for the case of $k = 1, 2, \ldots, N$. Next we prove (3.2) the case of $k = N + 1$.

**Step 1: Estimates from the local energy inequality.**

Taking $\psi = \phi_{N+1}$ as a test function in the local energy inequality (2.2), we have

$$
\begin{align*}
    \int_{B_{r_{N+1}}^3} (n \ln n \psi)(\cdot, t) + r_{N+1}^{-3} \int_{B_{r_{N+1}}^3} |\nabla \sqrt{\tilde{c}}|^2(\cdot, t) + \Lambda_0 r_{N+1}^{-3} \int_{B_{r_{N+1}}^3} |u|^2(\cdot, t) \\
    + r_{N+1}^{-3} \int_{Q_{r_{N+1}}^2} |\nabla \sqrt{n}|^2 + \Lambda_0 r_{N+1}^{-3} \int_{Q_{r_{N+1}}^2} |\nabla u|^2 \\
    + r_{N+1}^{-3} \int_{Q_{r_{N+1}}^2} |\nabla^2 \sqrt{\tilde{c}}|^2 + r_{N+1}^{-3} \int_{Q_{r_{N+1}}^2} (\sqrt{\tilde{c}})^{-2} |\nabla \sqrt{\tilde{c}}|^4 + r_{N+1}^{-3} \int_{Q_{r_{N+1}}^2} |\nabla \sqrt{\tilde{c}}|^2 n
\end{align*}
$$

\begin{align*}
\leq C \int_{Q_{r_3}^2} |n \ln n (\partial_t \psi + \Delta \psi)| + C \int_{Q_{r_3}^2} |n \ln n u \cdot \nabla \psi| + C \int_{Q_{r_3}^2} |n \ln n \nabla c \cdot \nabla \psi| \\
+ C \int_{Q_{r_3}^2} |n \nabla c \cdot \nabla \psi| + C \int_{Q_{r_3}^2} |\nabla \sqrt{\tilde{c}}|^2 (\partial_t \psi + \Delta \psi) + C \int_{Q_{r_3}^2} |\nabla \sqrt{\tilde{c}}|^2 u \cdot \nabla \psi| \\
+ C \int_{Q_{r_3}^2} (\sqrt{\tilde{c}})^{-1} |\nabla \sqrt{\tilde{c}}|^2 |\nabla \sqrt{\tilde{c}} \cdot \nabla \psi| + C \Lambda_0 \int_{Q_{r_3}^2} |u|^2 |(\partial_t \psi + \Delta \psi)| \\
+ C \Lambda_0 \int_{Q_{r_3}^2} |u|^2 |u \cdot \nabla \psi| + C \Lambda_0 \int_{Q_{r_3}^2} |n \nabla \phi \cdot u \psi| \\
:= I_1 + I_2 + \cdots + I_{11}.
\end{align*}

For the term $I_1, I_2$ and $I_3$, we need to deal with the part $n \ln n$. For $1 \leq k \leq N$, by the embedding inequality

$$
||n^{\frac{1}{2}}||_{L^2_t L^\infty_x(Q_{r_k})} \leq C ||n^{\frac{1}{2}}||_{L^\infty_t L^2_x(Q_{r_k})} ||\nabla n^{\frac{1}{2}}||_{L^2_t L^2_x(Q_{r_k})} + C ||n^{\frac{1}{2}}||_{L^\infty_t L^2_x(Q_{r_k})}
$$

and by (3.2) we have

$$
r_k^{-3} ||n||_{L^2_x(Q_{r_k})} \leq C_0 \varepsilon_0^\frac{1}{3}.
$$

(3.3)
Decomposing $Q_{r_k}$ into $Q_{r_k} \cap \{ n \leq A \}$ and $Q_{r_k} \cap \{ n > A \}$ with a constant $A$, we arrive at
\[
\int_{Q_{r_k}} |n \ln n|^{\frac{5}{3} - \delta} = \int_{Q_{r_k} \cap \{ n(x) \leq A \}} |n \ln n|^{\frac{5}{3} - \delta} + \int_{Q_{r_k} \cap \{ n(x) > A \}} |n \ln n|^{\frac{5}{3} - \delta}.
\]
Since
\[
\lim_{n \to 0} n^{\frac{\delta}{3}} \ln n = 0,
\]we know that for $0 < \delta < 1/3$, $n^\delta \ln n|^{\frac{5}{3} - \delta} \leq C(\delta, A)$ in the domain $\{ x : n(x) \leq A \}$.
On the other hand,
\[
\lim_{n \to \infty} n^{-\delta} |\ln n|^{\frac{5}{3} - \delta} = 0.
\]
Thus $n^{-\delta} |\ln n|^{\frac{5}{3} - \delta} \leq C(\delta, A)$ in the domain $\{ x : n(x) > A \}$ for a large constant $A$.

**Fixed $A$ and $\delta$, for example one can choose $A = 100$ and $\delta = \frac{1}{6}$.** From (3.4) and (3.5), we have
\[
\int_{Q_{r_k}} |n \ln n|^{\frac{5}{3}} \leq C \int_{Q_{r_k} \cap \{ n(x) \leq 100 \}} |n|^{\frac{5}{3}} + C \int_{Q_{r_k} \cap \{ n(x) > 100 \}} |n|^{\frac{5}{3}},
\]
which is controlled by
\[
\int_{Q_{r_k}} |n \ln n|^{\frac{5}{3}} \leq C \int_{Q_{r_k}} |n|^{\frac{5}{3}} + r_k^5 C_0 r_k^5 \varepsilon_0^{\frac{5}{3}} \leq C r_k^5 C_0 \varepsilon_0^{\frac{5}{3}} \leq C r_k^5 C_0 \varepsilon_0^{\frac{5}{3}},
\]
due to (3.2) and Hölder inequality.

**Estimate of $I_1$.** Noting that Proposition 3.15 and (1.4), we have
\[
I_1 \leq C \int_{Q_{r_3}} |n \ln n| \leq C \varepsilon_0.
\]

**Estimate of $I_2$.** Using Proposition 3.15 (3.2) and (3.6), we have
\[
I_2 \leq C \sum_{k=1}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+1}}} |n \ln \nu \cdot \nabla \psi| + C \int_{Q_{r_{N+1}}} |n \ln \nu \cdot \nabla \psi|
\]
\[
\leq \sum_{k=1}^{N} C r_k^{-4} ||n \ln n||_{L^\infty(Q_{r_k})} ||u||_{L^{10}(Q_{r_k})} + C r_N^{-1} ||n \ln n||_{L^\infty(Q_{r_N})} ||u||_{L^{10}(Q_{r_N})} r_N^{-1}
\]
\[
\leq C C_0^{29} \sum_{k=1}^{N} r_k^{-4} r_k^{10} r_k^{4} \varepsilon_0^{\frac{4}{3}} r_k^{4} e_0^{\frac{4}{3}} + C C_0^{29} r_N^{-1} r_N^{10} r_N^{4} e_0^{\frac{4}{3}} e_0^{\frac{4}{3}}
\]
\[
\leq C C_0^{29} \varepsilon_0^{\frac{4}{3}} e_0^{\frac{4}{3}}.
\]
Estimate of $I_3$. Since $u$ is similar as $\nabla \sqrt{c}$, using Proposition 3.15 (3.2) and (3.6) again, we have

$$I_3 \leq CC_0^{\frac{22}{18}} \varepsilon_0^{\frac{26}{9}} \Lambda_0.$$ 

Estimate of $I_4$. Using Proposition 3.15 (1.4) and (3.3), we have

$$I_4 \leq C \sum_{k=1}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+1}}} |n \nabla c \cdot \nabla \psi| + C \int_{Q_{r_{N+1}}} |n \nabla c \cdot \nabla \psi|$$

$$\leq C \sqrt{\Lambda_0} \sum_{k=1}^{N} r_k^{-4} \|n\|_{L^3(Q_{r_k})} \|\nabla \sqrt{c}\|_{L^{\frac{10}{3}}(Q_{r_k})} r_k^{\frac{1}{2}} + Cr_{N+1}^{-4} \|n\|_{L^3(Q_{r_{N+1}})} \|\nabla \sqrt{c}\|_{L^{\frac{10}{3}}(Q_{r_{N+1}})} r_{N+1}^{\frac{1}{2}}$$

$$\leq C \sqrt{\Lambda_0} C_0^{\frac{3}{2}} \sum_{k=1}^{N} r_k^{-4} r_k^{-\frac{5}{2}} r_k^{-\frac{1}{2}} r_k^{-\frac{1}{2}} + CC_0^{\frac{3}{2}} r_{N+1}^{-4} r_N^{-\frac{3}{2}} r_N^{-\frac{1}{2}} r_N^{-\frac{1}{2}}$$

$$\leq C \sqrt{\Lambda_0} C_0^{\frac{3}{2}}.$$ 

Estimate of $I_5$. Noting that Proposition 3.15 and (1.4), we have

$$I_5 \leq C \int_{Q_{r_3}} |\nabla \sqrt{c}|^2 \leq C \varepsilon_0.$$ 

Estimate of $I_6$. Similar as the estimate of $I_4$, by Proposition 3.15 (3.2) and embedding inequality, we have

$$I_6 \leq C \sum_{k=1}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+1}}} |\nabla \sqrt{c}|^2 |u \cdot \nabla \psi| + C \int_{Q_{r_{N+1}}} |\nabla \sqrt{c}|^2 |u \cdot \nabla \psi|$$

$$\leq C \sum_{k=1}^{N} r_k^{-4} \|\nabla \sqrt{c}\|_{L^{\frac{10}{3}}(Q_{r_k})}^2 \|u\|_{L^{\frac{10}{3}}(Q_{r_k})} r_k^{\frac{1}{2}} + Cr_{N+1}^{-4} \|\nabla \sqrt{c}\|_{L^{\frac{10}{3}}(Q_{r_{N+1}})}^2 \|u\|_{L^{\frac{10}{3}}(Q_{r_{N+1}})} r_{N+1}^{\frac{1}{2}}$$

$$\leq CC_0^{\frac{3}{2}} \sum_{k=1}^{N} r_k^{-4} r_k^{-\frac{5}{2}} r_k^{-\frac{1}{2}} r_k^{-\frac{1}{2}} + CC_0^{\frac{3}{2}} r_{N+1}^{-4} r_N^{-\frac{3}{2}} r_N^{-\frac{1}{2}} r_N^{-\frac{1}{2}}$$

$$\leq CC_0^{\frac{3}{2}}.$$ 

Estimate of $I_7$. Noting that $\sqrt{c} \geq 1$, we have

$$I_7 \leq C \sum_{k=1}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+1}}} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 |\nabla \sqrt{c} \cdot \nabla \psi| + C \int_{Q_{r_{N+1}}} (\sqrt{c})^{-1} |\nabla \sqrt{c}|^2 |\nabla \sqrt{c} \cdot \nabla \psi|$$

$$\leq C \sum_{k=1}^{N} r_k^{-4} \|\nabla \sqrt{c}\|_{L^{\frac{10}{3}}(Q_{r_k})}^3 + Cr_{N+1}^{-4} \|\nabla \sqrt{c}\|_{L^{\frac{10}{3}}(Q_{r_{N+1}})}^3,$$
which is similar as $I_4$ and $I_6$. Then

$$I_7 \leq C \varepsilon_0^3.$$

**Estimate of $I_8$.** For this term, noting that Proposition 3.15 and (1.4), we have

$$I_8 \leq C \Lambda_0 \int_{Q_{r_3}} |u|^2 \leq C \Lambda_0 \varepsilon_0.$$

**Estimate of $I_9$.** Using Proposition 3.15 and (3.2), we have

$$I_9 \leq \Lambda_0 \sum_{k=1}^N \int_{Q_{rk}\backslash Q_{rk+1}} |u|^2 |u \cdot \nabla \psi| + \Lambda_0 \int_{Q_{r_{N+1}}} |u|^2 |u \cdot \nabla \psi|$$

$$\leq C \Lambda_0 \sum_{k=1}^N r_k^{-4} \|u\|_{L^4(Q_{rk})}^3 r_k^{\frac{1}{2}} + C \Lambda_0 r_{N+1}^{-4} \|u\|_{L^4(Q_{r_{N+1}})}^3 r_{N+1}^{\frac{1}{2}}$$

$$\leq C \Lambda_0 \sum_{k=1}^N r_k^{-4} \frac{2}{3} C_0^\frac{3}{2} r_k^{\frac{1}{2}} + C \Lambda_0 r_{N+1}^{-4} \frac{2}{3} C_0^\frac{3}{2} r_{N+1}^{\frac{1}{2}}$$

$$\leq C \Lambda_0 C_0^\frac{3}{2} \varepsilon_0^3.$$

**Estimate of $I_{10}$.** First of all, we estimate the decomposition of pressure $p$. For $0 < 2r < \rho \leq 1$, let $\eta \geq 0$ be supported in $B_\rho$ with $\eta \equiv 1$ in $B_{\frac{3}{2}}$. The divergence of (1.2) gives $-\Delta p = \partial_i \partial_j (u_i u_j) + \nabla \cdot (n \nabla \phi)$ in the sense of distribution. Let

$$p_1 = \int_{\mathbb{R}^3} \frac{1}{4\pi |x-y|} [\partial_i \partial_j (u_i - (u_i)_\rho)(u_j - (u_j)_\rho)\eta] + \nabla \cdot (n \nabla \phi \eta)] (y, t) dy \quad (3.7)$$

where $(u)_\rho$ denotes the mean value of $u$ in $B_\rho$. Then $p_2 = p - p_1$ and $\Delta p_2 = 0$ in $B_{\frac{3}{2}}$. Choose $\rho = 1$ from now on.

By Lemma 2.12 and Hölder inequality,

$$\int_{B_r} |p_2 - (p_2)_{B_r}|^\frac{2}{3} dx \leq C r^\frac{2}{3} \int_{B_r} |\nabla p_2|^\frac{2}{3} dx \quad (3.8)$$

$$\leq C \left(\frac{r}{\rho}\right)^\frac{2}{3} \int_{B_\rho} |p|^\frac{2}{3} dx + C \left(\frac{r}{\rho}\right)^\frac{2}{3} \int_{B_\rho} |p_1|^\frac{2}{3} dx$$

By the Calderon-Zygmund estimate and Riesz potential estimate, we have

$$\int_{B_\rho} |p_1|^\frac{2}{3} dx \leq C \int_{B_\rho} |u - (u)_{B_\rho}|^3 + C \rho^\frac{2}{3} \left(\int_{B_\rho} |n \nabla \phi|^\frac{2}{3} dx\right)^\frac{5}{4}. \quad (3.9)$$
Second, we choose $\chi_k$ to be a cut-off function which vanishes outside of $Q_{r_k}$, equals 1 in $Q_{2r_k}$ and satisfies $|\nabla \chi_k| \leq Cr_k^{-1}$. Noting that $\chi_1 \phi_n = \phi_n$.

\[
I_{10} = C \Lambda_0 \left| \int_{-1}^{t} \int_{B_{r_3}} p u \cdot \nabla \psi \right|
\leq C \Lambda_0 \left| \sum_{k=1}^{N} \int_{Q_{r_k}^t} (p - (p)_{B_{r_k}}) u \cdot \nabla((\chi_k - \chi_{k+1}) \psi) \right|
+ C \Lambda_0 \left| \int_{Q_{r_{N+1}^+}} (p - (p)_{B_{r_{N+1}^+}}) u \cdot \nabla(\chi_{N+1} \psi) \right|
\leq C \Lambda_0 (I'_{10} + I''_{10})
\]

where

\[
I'_{10} = \left| \sum_{k=1}^{N} \int_{Q_{r_k}^t} (p_1 - (p_1)_{B_{r_k}}) u \cdot \nabla((\chi_k - \chi_{k+1}) \psi) \right| + \left| \int_{Q_{r_{N+1}^+}} (p_1 - (p_1)_{B_{r_{N+1}^+}}) u \cdot \nabla(\chi_{N+1} \psi) \right|
\]

and

\[
I''_{10} = \left| \sum_{k=1}^{N} \int_{Q_{r_k}^t \setminus Q_{r_k+2}^t} (p_2 - (p_2)_{B_{r_k}}) u \cdot \nabla((\chi_k - \chi_{k+1}) \psi) \right| + \left| \int_{Q_{r_{N+1}^+}} (p_2 - (p_2)_{B_{r_{N+1}^+}}) u \cdot \nabla(\chi_{N+1} \psi) \right|
\]

For $I'_{10}$, there holds

\[
I'_{10} \leq \left| \sum_{k=1}^{N} \int_{Q_{r_k}^t \setminus Q_{r_k+2}^t} (p_1 - (p_1)_{B_{r_k}}) u \cdot \nabla((\chi_k - \chi_{k+1}) \psi) \right| + \left| \int_{Q_{r_{N+1}^+}} (p_1 - (p_1)_{B_{r_{N+1}^+}}) u \cdot \nabla(\chi_{N+1} \psi) \right|
:= T'_1 + T'_2.
\]

In order to estimate $T'_1$, we introduce a new cut-off function $\xi(x) = \eta(\frac{x}{r_k})$ and $\xi_0 = 1$.

Then by (3.7) and $1 = \sum_{\ell=0}^{k}(\xi_\ell - \xi_{\ell+1}) + \xi_{k+1}$, in $B_1$ for $1 \leq k \leq N$, we have

\[
T'_1 = \sum_{k=1}^{N} \int_{Q_{r_k}^t \setminus Q_{r_k+2}^t} \left| \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \left( \partial_i \partial_j (u_i u_j \eta) + \nabla \cdot (n \nabla \phi \eta)(y,t) \right) dy \right| u \cdot \nabla((\chi_k - \chi_{k+1}) \psi)
\]

\[
= \sum_{k=4}^{N} \int_{Q_{r_k}^t \setminus Q_{r_k+2}^t} \left| \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \partial_i \partial_j \left( u_i u_j \left[ \sum_{\ell=0}^{k} (\xi_\ell - \xi_{\ell+1}) + \xi_{k+1} \right] \eta \right)(y,t) dy \right| u \cdot \nabla((\chi_k - \chi_{k+1}) \psi)
\]

\[
+ \sum_{k=4}^{N} \int_{Q_{r_k}^t \setminus Q_{r_k+2}^t} \left| \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \nabla \cdot \left( n \nabla \phi \left[ \sum_{\ell=0}^{k} (\xi_\ell - \xi_{\ell+1}) + \xi_{k+1} \right] \eta \right)(y,t) dy \right| u \cdot \nabla((\chi_k - \chi_{k+1}) \psi)
\]

\[
+ \sum_{k=4}^{3} \int_{Q_{r_k}^t \setminus Q_{r_k+2}^t} \left| \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \left( \partial_i \partial_j (u_i u_j \eta) + \nabla \cdot (n \nabla \phi \eta)(y,t) \right) dy \right| u \cdot \nabla((\chi_k - \chi_{k+1}) \psi)
\]
\[ := M_1 + M_2 + M_3, \tag{3.10} \]

Noting that the support set of \( \xi_\ell \), we have

\[
M_1 = \sum_{k=4}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+2}}} \left| \sum_{\ell=0}^{k-3} \int_{B_{r_\ell} \setminus B_{r_{\ell+2}}} \frac{1}{4\pi|x-y|} \left( \partial_i \partial_j (u_i u_j (\xi_\ell - \xi_{\ell+1}) \eta) \right) (y, t) dy \cdot \nabla \left( (\chi_k - \chi_{k+1}) \psi \right) \right| \\
+ \sum_{k=4}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+2}}} \left| \int_{B_{r_{k-2}}} \frac{1}{4\pi|x-y|} \left( \partial_i \partial_j (u_i u_j \xi_{k-2} \eta) \right) (y, t) dy \cdot \nabla \left( (\chi_k - \chi_{k+1}) \psi \right) \right| \\
:= M_{11} + M_{12}.
\]

Since \(|x - y| \geq r_{\ell+3}\) for the term \(M_{11}\) and \(|\nabla ((\chi_k - \chi_{k+1}) \psi)| \leq C r_k^{-4}\), by using integration by parts, Hölder’s inequality and (3.2), there holds

\[
M_{11} \leq C \sum_{k=4}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+2}}} \left| \sum_{\ell=0}^{k-3} \int_{B_{r_\ell}} r_{\ell+3}^{-3} (u_i u_j (\xi_\ell - \xi_{\ell+1}) \eta) (y, t) dy \cdot \nabla \left( (\chi_k - \chi_{k+1}) \psi \right) \right| \\
\leq C \sum_{k=4}^{N} r_k^{-4} \|u\|_{L^3(Q_{r_k})} \left( \int_{Q_{r_k}} \left| \sum_{\ell=0}^{k-3} \int_{B_{r_\ell}} r_{\ell+3}^{-3} |u|^2 dy \right| \right)^{\frac{2}{3}} \\
\leq C \sum_{k=4}^{N} r_k^{-4} \|u\|_{L^3(Q_{r_k})} \left( \int_{Q_{r_k}} \left| \sum_{\ell=0}^{k-3} \int_{B_{r_\ell}} \frac{r_{\ell+3}^{-3} |u|^2 dy}{r_k} \right| \right)^{\frac{2}{3}} \\
\leq C \sum_{k=4}^{N} r_k^{-3} \|u\|_{L^3(Q_{r_k})} \left( \int_{Q_{r_k}} \left| \sum_{\ell=0}^{k-3} \int_{B_{r_\ell}} \frac{r_{\ell+3}^{-3} r_{k}^{-1} |C_0 \chi_{l}^0|}{r_k} \right| \right)^{\frac{2}{3}} \\
\leq C \sum_{k=4}^{N} k r_k \|C_0 \chi_{l}^0\|_{L^2} \leq C C r_k^{-\frac{3}{2}}. \tag{3.11} \]

For the term \(M_{12}\), noting that \(|x - y| = 0\) for some \(y \in B_{r_k}\) and \((x, t) \in Q_{r_{k+1}} \setminus Q_{r_{k+2}}\), the method of the estimate of \(M_{11}\) is fail. Noting that the operator \(T\) operates on any function \(F\), and \(TF = \int_{\mathbb{R}^3} |x - y|^{-1} \partial_i \partial_j F_{ij}\). \(T\) satisfies the conditions of singular integral theorem, and by Hölder’s inequality, there holds

\[
M_{12} \leq C \sum_{k=4}^{N} r_k^{-4} \|u\|_{L^3(Q_{r_k})} \left( \int_{Q_{r_k}} \left| \int_{B_{r_{k-2}}} \frac{1}{4\pi|x-y|} \left( \partial_i \partial_j (u_i u_j \xi_{k-2} \eta) \right) (y, t) dy \right| \right)^{\frac{2}{3}} \\
\leq C \sum_{k=4}^{N} r_k^{-4} \|u\|_{L^3(Q_{r_k})} \left( \int_{-r_k^2}^{0} \|u\|_{L^2(\mathbb{R}^3)} \left( \int_{-r_k^2}^{0} |\xi_{k-2} \eta|^{\frac{2}{3}} \right)^{\frac{2}{3}} dt \right)^{\frac{2}{3}}.
\]
\[
\leq C \sum_{k=4}^{N} r_k^{-4} \|u\|_{L^3(Q_{r_k})} \|\nabla u\|_{L^3(Q_{r_k})}^2 \leq C C_0^{\frac{3}{2}} \varepsilon_0^{\frac{3}{2}}. \tag{3.12}
\]

Collecting \((3.11), (3.12)\), and we have

\[
M_1 \leq C C_0^{\frac{3}{2}} \varepsilon_0^{\frac{3}{2}}. \tag{3.13}
\]

The estimate of the term \(M_2\) is same as \(M_1\),

\[
M_2 = \sum_{k=4}^{N} \int_{Q_{r_k} \setminus Q_{r_k+2}} \left| \sum_{\ell=0}^{k-3} \int_{B_{r_{\ell+2}} \setminus B_{r_{\ell+3}}} \frac{1}{4\pi |x-y|} \nabla \cdot (n \nabla \phi (\xi_{\ell} - \xi_{\ell+1}) \eta)(y,t) dyu \cdot \nabla ((\chi_k - \chi_{k+1}) \psi) \right| \\
+ \sum_{k=4}^{N} \int_{Q_{r_k} \setminus Q_{r_k+2}} \left| \int_{B_{r_{\ell+3}} \setminus B_{r_{\ell+2}}} \frac{1}{4\pi |x-y|} \nabla \cdot (n \nabla \phi \xi_{\ell-2} \eta)(y,t) dyu \cdot \nabla ((\chi_k - \chi_{k+1}) \psi) \right| \\
:= M_{21} + M_{22}.
\]

For \(M_{21}\), since \(|x-y| \geq r_{\ell+3}\), using \((3.2)\) and Hölder’s inequality, there holds

\[
M_{21} \leq \sum_{k=4}^{N} \int_{Q_{r_k} \setminus Q_{r_k+2}} \left| \sum_{\ell=0}^{k-3} \int_{B_{r_{\ell+2}} \setminus B_{r_{\ell+3}}} r_{\ell+3}^{-2} |\nabla \phi||n| |dyu| \nabla ((\chi_k - \chi_{k+1}) \psi) \right| \\
\leq C \sum_{k=4}^{N} r_k^{-4} \int_{Q_{r_k}} \left| \sum_{\ell=0}^{k-3} \int_{B_{r_{\ell}}} r_{\ell+3}^{-2} |\nabla \phi||n| |dy| |u| \right| \\
\leq C \sum_{k=4}^{N} r_k^{-4} \int_{Q_{r_k}} \sum_{\ell=0}^{k-3} r_{\ell+3}^{-2} C_0 \varepsilon_0^{\frac{3}{2}} |u| \\
\leq C \sum_{k=4}^{N} r_k^{-4} r_k^{\frac{10}{3}} \|u\|_{L^3(Q_{r_k})} C_0 \varepsilon_0^{\frac{1}{2}} \leq C C_0^{\frac{3}{2}} \varepsilon_0^{\frac{3}{2}}. \tag{3.14}
\]

For the term \(M_{22}\), by Hölder’s inequality, Riesz potential estimate and the condition \((3.2)\), there holds

\[
M_{22} \leq \sum_{k=4}^{N} \int_{Q_{r_k} \setminus Q_{r_k+2}} \left| \int_{B_{r_{k-2}}} \frac{1}{4\pi |x-y|} \nabla \cdot (n \nabla \phi \xi_{k-2} \eta)(y,t) dyu \cdot \nabla ((\chi_k - \chi_{k+1}) \psi) \right| \\
\leq C \sum_{k=4}^{N} \int_{Q_{r_k} \setminus Q_{r_k+2}} \left| \int_{B_{r_{k-2}}} |x-y|^{-2} |n \nabla \phi \xi_{k-2} \eta|(y,t) dyu \cdot \nabla ((\chi_k - \chi_{k+1}) \psi) \right| \\
\leq C \sum_{k=4}^{N} r_k^{-4} \|u\|_{L^3(Q_{r_k})} \left( \int_{-r_k^2}^{r_k} \left( \int_{B_{r_{k-2}}} \left| \int_{B_{r_{k-2}}} |x-y|^{-2} |n \nabla \phi \xi_{k-2} \eta|(y,t) dy \right| \, dx \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \, dt \right) r_k^{\frac{4}{5}}
\]
\[ \begin{align*}
\leq C \sum_{k=4}^{N} r_k^{-\frac{8}{3}} \|u\|_{L^3(Q_{r_k})} \left( \int_{-r_k^2}^{0} \|n\nabla \phi \xi_k - 2\eta\|_{L^{\frac{5}{3}}(\mathbb{R}^3)} \right)^{\frac{3}{\frac{5}{3}}}
\leq C \sum_{k=4}^{N} r_k^{-\frac{8}{3}} \|u\|_{L^3(Q_{r_k})} \|n\|_{L^{\frac{5}{3}}(Q_{r_k})}
\leq C \sum_{k=4}^{N} r_k^{-\frac{8}{3}} r_k^{\frac{5}{6}} \varepsilon_0^{\frac{1}{6}} r_k^{\frac{1}{6}} C_0 \varepsilon_0 \leq C C_0^{\frac{3}{5}} \varepsilon_0^{\frac{4}{5}}.
\end{align*} \] (3.15)

Collecting (3.14), (3.15), we have
\[ M_2 \leq C C_0^{\frac{3}{5}} \varepsilon_0^{\frac{4}{5}}. \] (3.16)

As for the term \( M_3 \), by Hölder inequality there holds
\[ \begin{align*}
M_3 &= \sum_{k=1}^{3} \int_{Q_{r_k} \setminus Q_{r_{k+2}}} p_1 u \cdot \nabla ((\chi_k - \chi_{k+1})\psi)
\leq C \sum_{k=1}^{3} r_k^{-4} \|u\|_{L^3(Q_{r_k})} \|p_1\|_{L^\frac{2}{3}(Q_{r_k})} \leq C \|u\|_{L^3(Q_{r_1})} \|p_1\|_{L^\frac{2}{3}(Q_{r_1})}.
\end{align*} \] (3.17)

By (1.4), (3.9) and (3.17), we arrive
\[ \begin{align*}
M_3 &\leq C \|u\|_{L^3(Q_{r_1})} \left( C \int_{I_{r_1}} \left[ \int_{B_{r_1}} |u - (u)_{B_{r_1}}|^3 + \left( \int_{B_{r_1}} |n\nabla \phi|_{L^\frac{5}{3}}^\frac{5}{3} \right) dt \right] \right)^{\frac{3}{4}}
\leq C \|u\|_{L^3(Q_{r_1})}^3 + C \|\nabla \phi\|_{L^\infty} \|u\|_{L^3(Q_{r_1})} \|n\|_{L^{\frac{5}{3}}(Q_{r_1})}^{\frac{3}{5}}
\leq (\|\nabla \phi\|_{L^\infty} + 1) C (\varepsilon_0^{\frac{3}{5}} + \varepsilon_0^{\frac{5}{6}}).
\end{align*} \] (3.18)

To sum up, (3.10), (3.13), (3.16) and (3.18) implies that
\[ T_1' \leq C C_0^{\frac{3}{5}} \varepsilon_0^{\frac{4}{5}} + C \varepsilon_0^{\frac{5}{6}} \leq C C_0^{\frac{3}{5}} \varepsilon_0^{\frac{4}{5}}. \]

The term of \( T_2' \) is same as \( T_1' \), we omit the estimate of \( T_2' \) and arrive
\[ I_{10}' \leq C C_0^{\frac{3}{5}} \varepsilon_0^{\frac{4}{5}}. \]

For the another term \( I_{10}' \), by (2.13), we have
\[ \begin{align*}
I_{10}' &\leq \sum_{k=1}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+2}}} |(p_2 - (p_2)_{B_{r_k}}) u \cdot \nabla ((\chi_k - \chi_{k+1})\psi)|
+ \int_{Q_{r_{N+1}}} |(p_2 - (p_2)_{B_{r_N}}) u \cdot \nabla (\chi_{N+1} \psi)|
\end{align*} \]
\[ \begin{align*}
& \leq C \sum_{k=1}^{N} r^{-4}_k \|p_2 - (p_2)_{B_{r_k}}\|_{L^2 Q_{r_k}} \|u\|_{L^3 Q_{r_k}} \\
& \quad + C r^{-4}_N \|p_2 - (p_2)_{B_N}\|_{L^2 Q_{r_N}} \|u\|_{L^3 Q_{r_N}}.
\end{align*} \]

Using (3.29), (3.30) and (1.4), for any \( k = 1, 2, \ldots \), we have
\[
\|p_2 - (p_2)_{B_{r_k}}\|_{L^2 Q_{r_k}} \leq Cr^2_k \|p_2\|_{L^2 Q_{r_1}}
\]
\[
\leq Cr^2_k \|p_1\|_{L^2 Q_{r_1}} + Cr^2_k \|p\|_{L^2 Q_{r_1}}
\]
\[
\leq Cr^2_k \|u\|_{L^3 Q_{r_1}} + Cr^2_k \int_{I_{r_1}} \left( \int_{B_{r_1}} |n \nabla \phi|^\frac{6}{5} \, dx \right)^{\frac{5}{3}} + C r^2_k \|p\|_{L^2 Q_{r_1}}
\]
\[
\leq Cr^2_k \|\nabla \phi\|_{\infty} + 1 \|p\|_{L^2 Q_{r_1}}.
\]

Then
\[
I''_{10} \leq C A_1 \sum_{k=1}^{N} r^{-4}_k r^3 \|r^3 C_0 \|_{L^2 Q_{r_k}} \|u\|_{L^3 Q_{r_k}} \|\nabla \phi\|_{\infty} + C A_1 r^{-4}_N r^3 \|\nabla \phi\|_{\infty} \leq C C_0 A_1 \Xi_0^{\frac{11}{12}}.
\]

Collecting \( I'_{10} \) and \( I''_{10} \), we have
\[
I_{10} \leq C A_0 A_1 C^2_0 \Xi_0^{\frac{3}{4}} + C A_0 A_1 C^2_0 \Xi_0^{\frac{11}{12}} \leq C A_0 A_1 C^2_0 \Xi_0^{\frac{3}{4}}.
\]

Estimate of \( I_{11} \). Noting that \( \nabla \phi \in L^\infty Q_{r_1} \) and using Proposition 3.15, (3.3) and (3.2), we have
\[
I_{11} \leq C A_0 \sum_{k=1}^{N} \int_{Q_{r_k} \setminus Q_{r_{k+1}}} |n \nabla \phi \cdot u\psi| + C A_0 \int_{Q_{r_{N+1}}} |n \nabla \phi \cdot u\psi|
\]
\[
\leq C A_0 A_1 \sum_{k=1}^{N} r^{-3}_k \|n\|_{L^2 Q_{r_k}} \|u\|_{L^3 Q_{r_k}} r^\frac{2}{5} + C A_0 A_1 r^{-3}_N \|n\|_{L^2 Q_{r_N}} \|u\|_{L^3 Q_{r_N}} r^\frac{2}{5}
\]
\[
\leq C A_0 A_1 \sum_{k=1}^{N} r^{-3}_k C_0 r^\frac{1}{3} C_0 r^\frac{1}{3} + C A_0 A_1 r^{-3}_N C_0 r^\frac{1}{3} C_0 r^\frac{1}{3}
\]
\[
\leq C A_0 A_1 C^2_0 \Xi_0^{\frac{3}{4}}.
\]

Collecting \( I_1, I_2, \ldots, I_{11} \), for any \( t \in (-r^2_{N+1}, 0) \), we arrive at
\[
\int_{B_{r_{N+1}}} (n \ln n\psi)(\cdot, t) + r^{-3}_{N+1} \int_{Q_{r_{N+1}}^t} |\nabla \sqrt{n}|^2 + r^{-3}_{N+1} \int_{B_{r_{N+1}}} |\nabla \sqrt{c}|^2
\]
Recall $\psi$

Multiplying it with $0$ where $\Lambda_0$ using Proposition $3.15$, we have

Collecting the estimates of $K$, for $K$, we have

Using (1.4) and Proposition $3.15$, for $K$, we arrive at

Recall $\psi = \phi_{N+1}$ and using Proposition $3.15$ we arrive at

Using (1.4) and Proposition $3.15$ for $K_1$, we have

For $K_2$, using Proposition $3.15$, direct calculations imply that

The term $K_3$ is similar as the term $K_2$. Noting that $|\nabla c| \leq 2|\sqrt{c}\nabla \sqrt{c}| \leq 2\sqrt{\Lambda_0} \nabla \sqrt{c}|$, using Proposition $3.15$, we have

Collecting the estimates of $K_1 - K_3$, for any $t \in (-r_{N+1}^2, 0)$, we have

(3.20)
Step III. Estimate of \( r_{N+1}^{-3} \int_{B_{r_{N+1}}} n | \ln n | \).

From step II, we know that in order to prove (3.2) for \( k = N + 1 \), it's sufficient to estimate the term of \( r_{N+1}^{-3} \int_{B_{r_{N+1}}} n + n | \ln n | \). Let's prove it. First, combining (3.19) and (3.20), for any \( t \in (-r_{N+1}^2, 0) \), we have

\[
\int_{B_{r_{N+1}}} (n \psi)(\cdot, t) + \int_{B_{r_{N+1}}} (n \ln n \psi)(\cdot, t) + r_{N+1}^{-3} \int_{Q_{r_{N+1}}} \nabla | \sqrt{n} |^2
\]

\[
+ r_{N+1}^{-3} \int_{B_{r_{N+1}}} (| \nabla \sqrt{c} |^2)(\cdot, t) + r_{N+1}^{-3} \int_{Q_{r_{N+1}}} \nabla^2 | \sqrt{\tilde{c}} |^2
\]

\[
+ \Lambda_0 r_{N+1}^{-3} \int_{B_{r_{N+1}}} (| u |^2)(\cdot, t) + \Lambda_0 r_{N+1}^{-3} \int_{Q_{r_{N+1}}} \nabla u |^2
\]

\[
\leq CC_0^9 \varepsilon_0^{-\frac{11}{20}} + CA_0 \Lambda_1 C_3^3 \varepsilon_0^3 + CC_0^2 \varepsilon_0^{-\frac{11}{20}} \leq CC_0^9 \varepsilon_0^{-\frac{11}{20}} + CA_0 \Lambda_1 C_3^3 \varepsilon_0^3 \quad (3.21)
\]

Using \( | \ln n | n^\alpha \leq \alpha^{-1} e^{-1} \) for \( 0 < n < 1 \) and \( 0 < \alpha < \frac{1}{20} \), by (3.20) and the above inequality, we have

\[
Cr_{N+1}^{-3} \int_{B_{r_{N+1}}} (n | \ln n |)(\cdot, t) dx
\]

\[
\leq \int_{B_{r_{N+1}}} (n \ln n \psi)(\cdot, t) dx - 2 \int_{B_{r_{N+1}}} (n \ln n \psi)(\cdot, t) dx
\]

\[
\leq \int_{B_{r_{N+1}}} (n \ln n \psi)(\cdot, t) dx + 2 \alpha^{-1} e^{-1} \int_{B_{r_{N+1}}} (n^{1-\alpha} \psi)(\cdot, t) dx
\]

\[
\leq CC_0^9 \varepsilon_0^{-\frac{11}{20}} + CA_0 \Lambda_1 C_3^3 \varepsilon_0^3 + 2 \alpha^{-1} e^{-1} \left( CC_0^9 \varepsilon_0^{-\frac{11}{20}} + C \sqrt{\Lambda_0} C_3^3 \varepsilon_0^3 \right)^{1-\alpha}
\]

\[
\leq 2 \alpha^{-1} e^{-1} \left( CC_0^9 \varepsilon_0^{-\frac{11}{20}} + CA_0 \Lambda_1 C_3^3 \varepsilon_0^3 \right)^{1-\alpha} \quad (3.22)
\]

where we used the integral of heat kernel is bounded

\[
\int_{B_{r_{N+1}}} \psi dx \leq C,
\]

and we will choose the smallness of the right term \( CC_0^9 \varepsilon_0^{-\frac{11}{20}} + CA_0 \Lambda_1 C_3^3 \varepsilon_0^3 \).

Step IV. The proof of (3.2). Combining (3.20), (3.22), for any \( t \in (-r_{N+1}^2, 0) \), we arrive at

\[
r_{N+1}^{-3} \int_{B_{r_{N+1}}} (n + | n \ln n | + | \nabla \sqrt{n} |^2 + | u |^2)(\cdot, t)
\]

\[
+ r_{N+1}^{-3} \int_{Q_{r_{N+1}}} ^t \nabla | \sqrt{n} |^2 + | \nabla u |^2 + | \nabla ^2 \sqrt{c} |^2
\]
\[
\leq \alpha^{-1} \left(C C_0^{\frac{11}{20}} \varepsilon_0^{\frac{11}{20}} + C \Lambda_0 \Lambda_1 C_0^3 \varepsilon_0^{\frac{3}{20}}\right)^{1-\alpha}
\]
\[
\leq \left(C \alpha^{-1} C_0^2 \varepsilon_0^{\frac{11}{20}} + C \alpha^{-1} C_0^2 (\Lambda_0 \Lambda_1)^{1-\alpha} \varepsilon_0^{\frac{11}{20} - \frac{3}{20} \alpha}\right) C_0 \varepsilon_0^{\frac{1}{2}}
\]

Due to (1.4) and \(\varepsilon_0 = \frac{\varepsilon_1}{\Lambda_0 \Lambda_1} \varepsilon_0^{4+4 \alpha_0}\), we choose \(\alpha = \min\left\{1, \frac{\alpha_0}{4+\alpha_0}\right\}\), and \(\varepsilon_1\) such that
\[
C \alpha^{-1} C_0^2 \varepsilon_0^{\frac{11}{20} - \frac{3}{20} \alpha} \leq C \alpha^{-1} C_0^2 \varepsilon_1^{\frac{11}{20} - \frac{3}{20} \alpha} \leq \frac{1}{2}
\]
and
\[
C \alpha^{-1} C_0^2 (\Lambda_0 \Lambda_1)^{1-\alpha - (\frac{1}{4} - \frac{3}{4} \alpha)(4+\alpha_0)} \varepsilon_1^{\frac{1}{4} - \frac{3}{4} \alpha} \leq \frac{1}{2}
\]
since
\[
1 - \alpha - (\frac{1}{4} - \frac{3}{4} \alpha)(4 + 4 \alpha_0) < 0 \iff 0 < \alpha < \frac{\alpha_0}{2 + 3 \alpha_0}.
\]

Then we have
\[
r_n^{-3} N + 1 \int_{B_{r_n+1}} (n + |n \ln n| + |\nabla \sqrt{c}|^2 + |u|^2)(\cdot, t)\]
\[
+ r_n^{-3} \int_{Q_{r_n+1}} |\nabla \sqrt{n}|^2 + |\nabla u|^2 + |\nabla^2 \sqrt{c}|^2 \leq C_0 \varepsilon_0^{\frac{1}{2}}.
\]

Then, for uniform \(t \in (-r_n^2, 0)\), we have
\[
r_n^{-3} \sup_{-r_n^2 < t < 0} \int_{B_{r_n+1}} (n + |n \ln n| + |\nabla \sqrt{c}|^2 + |u|^2)\]
\[
+ r_n^{-3} \int_{Q_{r_n+1}} |\nabla \sqrt{n}|^2 + |\nabla u|^2 + |\nabla^2 \sqrt{c}|^2 \leq C_0 \varepsilon_0^{\frac{1}{2}}.
\]

To sum up, for any \(k = 1, 2, \ldots, N + 1, N + 2, \ldots\), (3.2) is true.

**Step V. The proof of boundedness of \((n, \nabla c, u)\).**

In the end, by interpolation inequality, we achieve at
\[
\left(\int_{Q_{rk}} |\sqrt{n}|^{\frac{10}{7}} dx dt\right)^{\frac{7}{10}} \leq C \|\sqrt{n}\|_{L^{\frac{2}{7}}(Q_{rk})}^{\frac{2}{7}} \|\nabla \sqrt{n}\|_{L^{\frac{2}{7}}(Q_{rk})}^{\frac{3}{7}} + C \|\sqrt{n}\|_{L^{\infty}(Q_{rk})}^{\frac{2}{7}}.
\]
and
\[
\left(r_k^{-5} \int_{Q_{rk}} |\sqrt{n}|^{\frac{10}{7}} dx dt\right)^{\frac{7}{10}} \leq C_0 \frac{1}{7} \frac{1}{7} r_k^{-5} \varepsilon_0^{\frac{10}{7}} \leq C_0 \varepsilon_0^{\frac{1}{2}}.
\]
Applying Lebesgue differential theorem, there holds
\[
n(0, 0) \leq C_3,
\]
where \( C_3 = C_0 \varepsilon_0^{\frac{1}{2}} \). Similarly, we have
\[
\left( r_k^{-5} \int_{Q_{r_k}} |\nabla \sqrt{\tilde{c}}|^4 \, dx \, dt \right)^{\frac{1}{10}} \leq C_3^{\frac{1}{2}},
\]
and
\[
\left( r_k^{-5} \int_{Q_{r_k}} |u|^4 \, dx \, dt \right)^{\frac{1}{4}} \leq C_3^{\frac{1}{2}},
\]
which means
\[
|\nabla \sqrt{\tilde{c}}(0,0)| \leq C_1 \varepsilon_1, \quad |u(0,0)| \leq C_3^{\frac{1}{2}}.
\] (3.23)

Using (3.23), we have
\[
|\nabla c(0,0)| = |\nabla \tilde{c}(0,0)| = |2\sqrt{\tilde{c}}\nabla \sqrt{\tilde{c}}| \leq CC_3^{\frac{1}{2}}.
\]

Therefore, the proof is complete.

4. The singular set’s estimate

Proof of Theorem 1.7. It suffices to prove the inequality (1.7), since one can use the embedding inequality
\[
\int_{Q_t} |n \ln n|^\frac{3}{2} \leq C \int_{Q_t \cap \{n(x) \leq 100\}} |n|^\frac{3}{2} + C \int_{Q_t \cap \{n(x) > 100\}} |n|^\frac{3}{2}.
\]

Step I. It follows from the local energy inequality that
\[
\int_{B_1} (n \ln n \zeta)(\cdot, t) + 2 \int_{Q_{t^*}} |\nabla \sqrt{n}|^2 \zeta + \frac{1}{2} \int_{B_1} (|\nabla \sqrt{n}|^2 \zeta)(\cdot, t) + \int_{Q_{t^*}} |\nabla^2 \sqrt{n}|^2 \zeta
\]
\[
+ \frac{1}{36} \int_{Q_{t^*}} (\sqrt{\tilde{c}})^{-2}(\partial_i \sqrt{\tilde{c}})^2(\partial_i \sqrt{\tilde{c}})^2 \zeta + \frac{1}{2} \int_{Q_{t^*}} |\nabla \sqrt{\tilde{c}}|^2 n \zeta
\]
\[
+ \Lambda_0 \int_{B_1} (|u|^2 \zeta)(\cdot, t) + \Lambda_0 \int_{Q_{t^*}} |\nabla u|^2 \zeta
\]
\[
\leq \int_{Q_{t^*}} n \ln n (\partial_i \zeta + \Delta \zeta) + \int_{Q_{t^*}} n \ln n u \cdot \nabla \zeta + \int_{Q_{t^*}} n \ln n \nabla c \cdot \nabla \zeta + \int_{Q_{t^*}} n \nabla c \cdot \nabla \zeta
\]
\[
+ 2 \int_{Q_{t^*}} |\nabla \sqrt{n}|^2 (\partial_i \zeta + \Delta \zeta) + 2 \int_{Q_{t^*}} |\nabla \sqrt{n}|^2 u \cdot \nabla \zeta
\]
\[
+ \frac{20}{3} \int_{Q_{t^*}} (\sqrt{\tilde{c}})^{-1} |\nabla \sqrt{\tilde{c}}|^2 |\nabla \sqrt{\tilde{c}}| \cdot \nabla \zeta + \Lambda_0 \int_{Q_{t^*}} |u|^2 (\partial_i \zeta + \Delta \zeta) + \Lambda_0 \int_{Q_{t^*}} |u|^2 u \cdot \nabla \zeta
\]
\[
+ \Lambda_0 \int_{Q_{t^*}} (p - \bar{p}) u \cdot \nabla \zeta - 2\Lambda_0 \int_{Q_{t^*}} n \nabla \phi \cdot u \zeta
\] (4.1)
where \( t \in (-1, 0) \) and \( \zeta \) is a cut-off function on domain \( Q_1 \), which means that \( \zeta = 1 \) on \( Q_1 \) and \( \zeta = 0 \) outside \( Q_1 \). Moreover,

\[
|\nabla \zeta| + |\nabla^2 \zeta| + |\partial_t \zeta| \leq C.
\]

We just compute the first two terms, and other terms are similar. Using Hölder inequality,

\[
A_1 \leq C \int_{Q_1} |n \ln n| \, dx \, dt \leq C \int_{Q_1} |n \ln n| \leq C \left( \int_{Q_1} |n \ln n|^2 \right)^{\frac{1}{2}} \leq C \varepsilon_2^{\frac{3}{2}}.
\]

and

\[
A_2 \leq C \int_{Q_1} |n \ln n| |u| \, dx \, dt \leq C ||n \ln n||_{L^2(Q_1)} ||u||_{L^3(Q_1)} \leq C \varepsilon_2
\]

Hence, we have

\[
\int_{B_1} (n \ln n \zeta)(\cdot, t) = \int_{Q_1} n(\partial_t \zeta + \Delta \zeta) + \int_{Q_1} nu \cdot \nabla \zeta + \int_{Q_1} n \nabla c \cdot \nabla \zeta
\]

Thus we obtain that

\[
\int_{B_1} (n \zeta)(\cdot, t) \leq C \int_{Q_1} |n + |u||^3 + n^2 + \Lambda_0 |\nabla \sqrt{c + 1}| \, dx \, dt \leq C \Lambda_0 \varepsilon_2^{\frac{3}{2}} \quad (4.3)
\]

**Step II.** Similar as (3.20), for any \( t \in (-1, 0) \), we get

\[
\int_{B_1} (n \zeta)(\cdot, t) = \int_{Q_1} n(\partial_t \zeta + \Delta \zeta) + \int_{Q_1} nu \cdot \nabla \zeta + \int_{Q_1} n \nabla c \cdot \nabla \zeta
\]

Thus we obtain that

\[
\int_{B_1} (n \zeta)(\cdot, t) \leq C \int_{Q_1} |n + |u||^3 + n^2 + \Lambda_0 |\nabla \sqrt{c + 1}| \, dx \, dt \leq C \Lambda_0 \varepsilon_2^{\frac{3}{2}} \quad (4.3)
\]

**Step III.** Recalling the estimate of (3.22), for any \( t \in (-1, 0) \), we have

\[
\int_{B_1} (n \log n \zeta)(\cdot, t) \, dx = \int_{B_1} (n \log n \zeta)(\cdot, t) \, dx - 2 \int_{B_1 \cap \{t < 0 < n < 1\}} (n \log n \zeta)(\cdot, t) \, dx
\]

\[
\leq \int_{B_1} (n \log n \zeta)(\cdot, t) \, dx + C \int_{B_1} (\sqrt{n \zeta})(\cdot, t) \, dx.
\]

Combining this with the estimates of (4.2), (4.3) and (4.4), we have

\[
\sup_t \int_{B_1} n(| \ln n| + 2) \zeta + 2 \int_{Q_1} |\nabla \sqrt{n}| \zeta + 2 \sup_t \int_{B_1} |\nabla \sqrt{c}|^2 \zeta
\]

\[
+ \frac{8}{9} \int_{Q_1} |\nabla^2 \sqrt{c}|^2 \zeta + \sup_t \Lambda_0 \int_{B_1} |u|^2 \zeta + \Lambda_0 \int_{Q_1} |\nabla u|^2 \zeta
\]

\[
\leq C \Lambda_0 \Lambda_1 \varepsilon_2^{\frac{1}{2}}.
\]
which and the assumption of $p$ imply the regularity due to Theorem 1.4. Specially, we choose

$$C\Lambda_0\Lambda_1\varepsilon_2^\frac{1}{3} \leq \frac{\varepsilon_1}{(\Lambda_0\Lambda_1)^{1+4\alpha_0}},$$

which means

$$\varepsilon_2 \leq \frac{\varepsilon_1^3}{C(\Lambda_0\Lambda_1)^{15+12\alpha_0}}.$$

For the second assumption (1.8), if

$$(C')^\frac{5}{2}\varepsilon' \leq \frac{\varepsilon_1^{\frac{15}{19}}}{C(\Lambda_0\Lambda_1)^{\frac{23}{19}+15\alpha_0}},$$

we have

$$\int_{Q_1} |n\ln n|^{\frac{5}{2}} \leq C'\varepsilon_2' + C'(\varepsilon_2')^\frac{4}{3} \leq \frac{\varepsilon_1^3}{C(\Lambda_0\Lambda_1)^{15+12\alpha_0}}.$$

The proof is complete.

**Proof of Corollary 1.9.** We will use a parabolic version of the Vitali covering lemma: Let $\{J = Q_{z, r}\}_\alpha$ be any collection of parabolic cylinders contained in a bounded subset of $\mathbb{R}^4$, and noting $J = J_x \times J_t$, there exist disjoint $Q_{z_j, r_j} \in J, j \in N$, such that any cylinder in $J$ is contained in $Q_{z_j, 5r_j}$ for some $j$.

Letting

$$Q^*((x, t), r) = B(x, r) \times (t - \frac{7}{8}r^2, t + \frac{1}{8}r^2),$$

it is a translation in time of $Q((x, t), r)$. Besides, $Q(z, \frac{r}{2}) \in Q^*(z, r)$. Let $S_R = S \cap R$ for any compact set $R \subset Q_{\frac{1}{2}}$. Fix any $\delta > 0$. Assume that for any $z_j = (x_j, t_j) \in S_R$, by Theorem 1.7 there exists $0 < r_{z_j} = r_j < \frac{4}{10}$ such that

$$\left(\int_{Q_{r_{z_j}(z_0)}} n^{\frac{5}{3}} + |\nabla \sqrt{c + 1}|^{\frac{10}{3}} + |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}}\right)^\frac{1}{2} r_{z_j}^\frac{5}{2} \varepsilon_2' = \frac{1}{2} r_{z_j}^\frac{5}{2} \varepsilon_2 \leq \frac{\varepsilon_1^{\frac{15}{19}}}{C(\Lambda_0\Lambda_1)^{\frac{23}{19}+15\alpha_0}}$$

due to scaling. Here, $\Lambda_0^j = 108(\|c\|_{L^\infty((-r_j^2,0) \times \mathbb{R}^3)} + 1) \leq \Lambda_0$, and $\Lambda_1^j = (r_j \|\nabla \phi\|_{L^\infty((-r_j^2,0) \times \mathbb{R}^3)} + 1) \leq \Lambda_1$. Thus,

$$\left(\int_{Q_{r_{z_j}(z_0)}} n^{\frac{5}{3}} + |\nabla \sqrt{c + 1}|^{\frac{10}{3}} + |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}}\right)^\frac{1}{2} r_{z_j}^\frac{5}{2} \varepsilon_4,$$

where $\varepsilon_4 = \frac{\varepsilon_1^{\frac{15}{19}}}{C(\Lambda_0\Lambda_1)^{\frac{23}{19}+15\alpha_0}}$ is independent of $j$.

Then,

$$S_R \subset \bigcup_{j \in N} Q^*(z_j, 2r_{z_j}).$$
Let \( r_j = r_{z_j} \) and \( \{Q_j(z_j)\}_{j \in \mathbb{N}} \) be the countable disjoint subcover guaranteed by the Vitali covering lemma, then
\[
S_R \subset \bigcup_{j \in \mathbb{N}} Q_j(z_j, 10r_j) \quad 10r_j < \delta.
\]
Note that \( Q_j(z_j, \frac{r_j}{2}) \in Q_j(z_j, r_j) \) are disjoint, then
\[
\sum_{j} 10r_j^{\frac{5}{3}} \leq \sum_{j} \frac{20}{\varepsilon_4} \left( \int_{Q_j(z_0)} n^{\frac{4}{3}} + |\nabla \sqrt{c + 1} |^{\frac{10}{3}} + |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}} \right).
\]
Since the bounded-ness of the right part, we have
\[
\sum_{j} 10r_j^{\frac{5}{3}} < +\infty,
\]
which means that \( S_R \) has Lebesgue measure 0. Since the finite covering theorem, we know that for any open neighborhood \( J = J_x \times J_t \subset Q_1 \) of \( S_R \) satisfies \( Q_j(z) \subset J \),
\[
\sum 5r_j^{\frac{5}{3}} \leq \sum \frac{C}{\varepsilon_4} \left( \int_{Q_j(z_0)} n^{\frac{4}{3}} + |\nabla \sqrt{c + 1} |^{\frac{10}{3}} + |u|^{\frac{10}{3}} + |p|^{\frac{5}{3}} \right).
\]
By the arbitrarily of \( J \), we can choose \( J \) with arbitrarily small Lebesgue measure, therefore, the right side is arbitrarily small. Since \( \delta > 0 \) is arbitrary, we have
\[
\mathcal{P}^\frac{5}{3}(S_R) = 0.
\]
By the arbitrarily of \( R \), we have
\[
\mathcal{P}^\frac{5}{3}(S) = 0.
\]
5. Proof of Theorem 1.11

Next we prove Theorem 1.11 with the help of Theorem 1.4.

Proof: It’s sufficient to prove the smallness of \( p \) at a fixed ball, since other terms are the same as those in Theorem 1.4

Step I. The pressure estimate. Recall the equation of \( u \) in (1.2), and taking the divergence yields that
\[
-\Delta p = \partial_i \partial_j (u_i u_j) + \nabla \cdot (n \nabla \phi).
\]
Let \( \eta(x) \geq 0 \) be supported in \( B_{\rho} \) with \( \eta = 1 \) in \( B_{\frac{\rho}{2}} \), and
\[
p_1(x, t) = \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \left( \partial_i \partial_j ((u_i - (u_i)_{\rho})(u_j - (u_j)_{\rho})\eta) + \nabla (n \nabla \phi \eta) \right)(y, t) dy.
\]
Moreover, let
\[
p_2(x, t) = p(x, t) - p_1(x, t)
\]
which implies that
\[
\Delta p_2 = 0 \text{ in } B_{\frac{\rho}{2}}.
\]
Let $0 < 2r < \rho \leq 1$, by the mean value property of harmonic functions and Lemma 2.12, we have

$$\int_{B_r} |p_2|^{\frac{4}{3}} dx \leq C \left(\frac{r}{\rho}\right)^3 \int_{\mathbb{R}^3} |p_2|^{\frac{4}{3}} dx$$

$$\leq C \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |p_2|^{\frac{4}{3}} dx + C \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |p_1|^{\frac{4}{3}} dx. \quad (5.1)$$

and Calderon-Zygmund estimates yields that

$$\int_{B_\rho} |p_1|^{\frac{4}{3}} dx \leq C \int_{B_\rho} |u - (u)_\rho|^3 + C \rho^\frac{3}{4} \left(\int_{B_\rho} |n \nabla \phi|^\frac{6}{5} dx\right)^\frac{5}{6}. \quad (5.2)$$

Combining (5.2) and (5.1) and noting that $r < \rho$, we have

$$r^{-2} \int_{Q_r} |p|^{\frac{4}{3}} \leq r^{-2} \int_{Q_r} |p_1|^{\frac{4}{3}} + r^{-2} \int_{Q_r} |p_2|^{\frac{4}{3}}$$

$$\leq \left(1 + \left(\frac{r}{\rho}\right)^3\right) r^{-2} \int_{Q_\rho} |p_1|^{\frac{3}{2}} + C r^{-2} \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |p_2|^{\frac{4}{3}} dx dt$$

$$\leq C r^{-2} \int_{Q_\rho} |u - (u)_\rho|^3 + C r^{-2} \int_{Q_\rho} \rho^{\frac{3}{2}} \left(\int_{B_\rho} |n \nabla \phi|^\frac{6}{5} dx\right)^\frac{5}{6} dt$$

$$+ C r^{-2} \left(\frac{r}{\rho}\right)^3 \int_{Q_\rho} |p|^{\frac{4}{3}} dx$$

Using Hölder inequality, we have

$$r^{-2} \int_{Q_r} |p|^{\frac{4}{3}} \leq C \left(\frac{\rho}{r}\right)^2 \rho^{-2} \int_{Q_\rho} |u - (u)_\rho|^3 + C \left(\frac{\rho}{r}\right)^2 \rho^{\frac{3}{2}} \left(\rho^{-\frac{3}{2}} \int_{Q_\rho} |n |^{\frac{6}{5}} dx dt\right)^{\frac{10}{9}} \| \nabla \phi \|^\frac{3}{L_{1,\infty}}$$

$$+ C \left(\frac{r}{\rho}\right)^2 \int_{Q_\rho} |p|^{\frac{4}{3}} dx. \quad (5.3)$$

By Gagliardo-Nirenberg and Young inequality, noting that

$$r^{-1} \| n \|_{L^{\frac{10}{3}}(Q_r)} \leq C r^{-1} \| \sqrt{n} \|^2_{L^{\infty}(Q_r)} + C r^{-1} \| \nabla \sqrt{n} \|^2_{L^2(Q_r)} \leq C (A_n(r) + E_n(r)),$$

It follows from (5.3) that

$$D(\rho) \leq C \left(\frac{\rho}{r}\right)^2 \tilde{C}_u(\rho) + C \left(\frac{\rho}{r}\right)^2 (A_n(\rho) + E_n(\rho))^{\frac{3}{2}} + C \left(\frac{r}{\rho}\right) D(\rho), \quad (5.4)$$

where $\rho \leq 1$ is used. Note that

$$\int_{B_\rho} |u - (u)_\rho|^3 dx dt \leq \| u - (u)_\rho \|^\frac{3}{2}_{L^2(B_\rho)} \| u - (u)_\rho \|^\frac{3}{2}_{L^6(B_\rho)} \leq C \| u - (u)_\rho \|^\frac{3}{2}_{L^2(B_\rho)} \| \nabla u \|^\frac{3}{2}_{L^2(B_\rho)}.$$
and integrating in time we get
\[
\rho^{-2} \int_{Q_\rho} |u - (u)_\rho|^3 dx dt \leq C(A_u(\rho) + E_u(\rho))^{\frac{3}{2}}.
\]
Then by (5.4) we have
\[
D(r) \leq C \left( \frac{r}{\rho} \right) D(\rho) + C \left( \frac{\rho}{r} \right)^2 (A_n(\rho) + E_n(\rho))^{\frac{3}{2}} + C \left( \frac{\rho}{r} \right)^2 (A_u(\rho) + E_u(\rho))^{\frac{3}{2}}.
\]
Let \( G(\rho) = A_u(\rho) + E_u(\rho) + A_n(\rho) + E_n(\rho) \), we have
\[
D(r) \leq C \left( \frac{r}{\rho} \right) D(\rho) + C \left( \frac{\rho}{r} \right)^2 G(\rho)^{\frac{3}{2}}.
\]
Moreover, let \( r = \theta_0 \rho \) with \( \theta_0 \in (0, \frac{1}{4}) \) satisfying \( C\theta_0 \leq \frac{1}{2} \) and we arrive at
\[
D(\theta_0 \rho) \leq \frac{1}{2} D(\rho) + C\theta_0^{-2} G(\rho)^{\frac{3}{2}}.
\]
Write \( \varepsilon' = \frac{\varepsilon}{(\Lambda_0 \Lambda_1)^{\frac{3}{4} + 4\alpha_0}} \). Due to (1.10), there exists \( \rho_0 > 0 \) such that \( G(\rho_0) \leq 2\varepsilon' \). Then
\[
D(\theta_0 \rho_0) \leq \frac{1}{2} D(\rho_0) + 4C\theta_0^{-2} \varepsilon'_{\frac{3}{2}},
\]
and
\[
D(\theta_0^2 \rho_0) \leq \frac{1}{2} D(\theta_0 \rho_0) + C\theta_0^{-2} G(\theta_0 \rho_0)^{\frac{3}{2}} \leq \frac{1}{2} D(\theta_0 \rho_0) + 4C\theta_0^{-2} \varepsilon'_{\frac{3}{2}}
\]
\[
D(\theta_0^3 \rho_0) \leq \frac{1}{2} D(\theta_0^2 \rho_0) + 4C\theta_0^{-2} \varepsilon'_{\frac{3}{2}}
\]
\[
\ldots
\]
Consequently, repeating the above progress, one can get
\[
D(\theta_0^k \rho_0) \leq \frac{1}{2^k} D(\rho_0) + \sum_{k \geq 1} \frac{1}{2^{k-1}} (4C\theta_0^{-2} \varepsilon'_{\frac{3}{2}}).
\]
which implies
\[
D(\theta_0^k \rho_0) \leq \frac{1}{2^k} D(\rho_0) + 8C\theta_0^{-2} \varepsilon'_{\frac{3}{2}}.
\]
Recall that \( \varepsilon_0 = \frac{\varepsilon}{(\Lambda_0 \Lambda_1)^{\frac{3}{4} + 4\alpha_0}} \) in Theorem 1.4. Nothing that \( D(\rho_0) < \infty \), there exist a small constant \( \varepsilon' \) satisfies \( \varepsilon' < \frac{1}{128} \varepsilon_0 \) and \( 8C\theta_0^{-2} \varepsilon'_{\frac{3}{2}} \leq \frac{\theta_0^2}{32} \varepsilon_0 \) and the constant \( k = K_0(\theta_0) \) which is large enough and satisfies
\[
2^{-K_0} D(\rho_0) \leq \frac{\theta_0^2}{32} \varepsilon_0,
\]
such that
\[
D(\theta_0^k \rho_0) \leq \frac{\theta_0^2}{16} \varepsilon_0,
\]
holds for all \( k \geq K_0 \). Then for all \( 0 < r \leq \theta_0^{K_0} \rho_0 \), there exist constant \( K_1 > K_0 \) such that \( \theta_0^{K_1+1} \rho_0 < r < \theta_0^{K_1} \rho_0 \). Then

\[
D(r) = r^{-2} \int_{Q_r} |p|^\frac{4}{3} \leq \theta_0^{-2K_1-2} \rho_0^{-2} \int_{Q_0} |p|^\frac{4}{3} \leq \theta_0^{-2} \frac{\theta_0^2}{16} \varepsilon_0 \leq \frac{1}{16} \varepsilon_0.
\]

**Step II. Arguments due to scaling.** Recalling the assumption of (1.4), we arrive at

\[
\sup_t r^{-1} \int_{B_r} n + |n \ln n| + |\nabla \sqrt{c_r}|^2 + |u|^2 \\
+ r^{-1} \int_{Q_r} |\nabla n|^2 + |\nabla u|^2 + |\nabla^2 \sqrt{c_r}|^2 + r^{-2} \int_{Q_r} |p|^\frac{4}{3} \leq \frac{1}{8} \varepsilon_0,
\]

for any \( 0 < r \leq r_0 \). By (2.1), let

\[
n_r(x, t) = r^2 n(rx, r^2 t); \ c_r(x, t) = c(rx, r^2 t); \\
u_r(x, t) = r u(rx, r^2 t); \ p_r(x, t) = r^2 p(rx, r^2 t)
\]

then \((n_r, c_r, u_r, p_r)\) is a solution of (1.2) in \( Q_1^t \). Moreover,

\[
\sup_t \int_{B_1} n_r + |\nabla \sqrt{c_r}|^2 + |u_r|^2 \\
+ \int_{Q_1} |\nabla n_r|^2 + |\nabla u_r|^2 + |\nabla^2 \sqrt{c_r}|^2 + |p_r|^\frac{4}{3} \leq \frac{1}{8} \varepsilon_0.
\]

The remaining part is to estimate the term of \( \sup_t \int_{B_1} |n_r \ln n_r| \), and

\[
\sup_{-1 < t < 0} \int_{B_1} |n_r \ln n_r| \\
\leq \sup_{-2 < t < 0} r^{-1} \int_{B_r} |n \ln(r^2 n)| dx \\
\leq \sup_{-2 < t < 0} r^{-1} \int_{B_r \cap \{n < r^{-\frac{3}{2}}\}} |n \ln(r^2 n)| dx + \sup_{-2 < t < 0} r^{-1} \int_{B_r \cap \{r^{-\frac{3}{2}} \leq n \leq r^{-2}\}} |n \ln(r^2 n)| dx \\
+ \sup_{-2 < t < 0} r^{-1} \int_{B_r \cap \{n \geq r^{-2}\}} |n \ln(r^2 n)| dx \doteq M_1 + \cdots + M_3
\]

Then by (5.5)

\[
M_1 \leq \sup_{-2 < t < 0} r^{-1} \int_{B_r \cap \{n < r^{-\frac{3}{2}}\}} |n \ln n| + 2n |\ln r| dx \leq \frac{1}{8} \varepsilon_0 + 2r^\frac{3}{2} |B_1| |\ln r| \leq \frac{1}{4} \varepsilon_0
\]

where we choose \( r < r_1 \) for a small \( r_1 < r_0 \). Besides,

\[
M_2 \leq \sup_{-2 < t < 0} r^{-1} \int_{B_r \cap \{r^{-\frac{3}{2}} \leq n \leq r^{-2}\}} |n \ln(r^2 n)| dx
\]
\[
\leq \sup_{-r^2 < t < 0} r^{-1} \int_{B_r \cap \{ r^{-\frac{3}{2}} \leq n \leq r^{-2} \}} |n \ln(r^{-\frac{1}{2}})| dx \\
\leq \sup_{-r^2 < t < 0} r^{-1} \int_{B_r} |n \ln n| dx dt \leq \frac{1}{8} \varepsilon_0
\]

and
\[
M_3 \leq \sup_{-r^2 < t < 0} r^{-1} \int_{B_r \cap \{ n \geq r^{-2} \}} |n \ln(r^2n)| dx \\
\leq \sup_{-r^2 < t < 0} r^{-1} \int_{B_r \cap \{ n \geq r^{-2} \}} n \ln n dx \leq \frac{1}{8} \varepsilon_0
\]

where we used \( r < 1 < 1 \). Hence there holds \( \sup_t \int_{B_1} |n_r \ln n_r| \leq \frac{1}{2} \varepsilon_0 \). Combining this with (5.5) we get
\[
\sup_t \int_{B_1} n_r + |n_r \ln n_r| + |\nabla \sqrt{\tilde{c}_r}|^2 + |u_r|^2 \\
+ \int_{Q_1} |\nabla n_r^\frac{1}{2}|^2 + |\nabla u_r|^2 + |\nabla^2 \sqrt{\tilde{c}_r}|^2 + |p_r|^\frac{3}{2} \leq \varepsilon_0.
\] (5.7)

Using Theorem (1.4), it follows that \((n_r, u_r, \nabla c_r)\) is regular at the point 0. The proof is complete.

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