TESTING ISOMORPHISM OF CHORDAL GRAPHS OF BOUNDED LEAFAGE IS FIXED-PARAMETER TRACTABLE

VIKRAMAN ARVIND, ROMAN NEDELA, ILIA PONOMARENKO, AND PETER ZEMAN

Abstract. The computational complexity of the graph isomorphism problem is considered to be a major open problem in theoretical computer science. It is known that testing isomorphism of chordal graphs is polynomial-time equivalent to the general graph isomorphism problem. Every chordal graph can be represented as the intersection graph of some subtrees of a representing tree, and the leafage of a chordal graph is defined to be the minimum number of leaves in a representing tree for it. We prove that chordal graph isomorphism is fixed parameter tractable with leafage as parameter. In the process we introduce the problem of isomorphism testing for higher-order hypergraphs and show that finding the automorphism group of order-$k$ hypergraphs with vertex color classes of size $b$ is fixed parameter tractable for any constant $k$ and $b$ as fixed parameter.

1. Introduction

The graph isomorphism problem is one of the few natural problems in NP that is neither known to be NP-complete nor is known to be polynomial-time solvable. In a fairly recent breakthrough, Babai [4] proved that the graph isomorphism problem is solvable in quasipolynomial time, i.e., in time $n^\text{poly}(\log n)$, where $n$ is the number of vertices.

A significant line of research concerns the parameterized complexity of the graph isomorphism problem with respect to some natural graph parameter. These include treewidth [19], degree [21, 15], genus [23, 22], excluded minors [24, 16], etc. It is worth mentioning that in several of these cases, Babai’s new techniques have yielded new algorithms with improved running time. For example, Luks’s original algorithm with running time $n^{O(k)}$ for degree-$k$ graphs has a modified $n^{\text{poly}(\log k)}$-time algorithm [21, 15]. However, in some of these cases a fixed-parameter tractable (FPT) algorithm, i.e., an algorithm with running time $f(k)\text{ poly}(n)$, have remained elusive. Such an improvement likely cannot be obtained using known techniques and would require some new techniques and ideas.

In our work, we deal with parameterized complexity of the graph isomorphism problem for the class of chordal graphs. An undirected graph is said to be chordal if it has no chordless cycle of length at least four. Every chordal graph admits a representation as the intersection graph of subtrees of some tree $T$ [14]. We say that a chordal graph $X$ has representing tree $T$ if $X$ can be represented as the intersection graph of subtrees of $T$. The leafage $\ell(X)$ of a chordal graph $X$ is the least positive integer such that $X$ has a representing tree with $\ell(X)$ leaves. The notion of leafage was introduced in [18] and is a natural graph parameter for chordal graphs.
It is interesting to note that the well-studied interval graphs are precisely the intersection graphs of paths. It follows that \( \ell(X) \leq 2 \) if and only if \( X \) is an interval graph (and \( \ell(X) = 1 \) if and only if \( X \) is complete). Thus, the leafage of a chordal graph \( X \) measures how far it is from being an interval graph, which has interesting algorithmic consequences. For instance, efficient solutions to certain NP-hard problems on interval graphs naturally extend to chordal graphs of bounded leafage; e.g., [26].

Graph Isomorphism restricted to chordal graphs is polynomial-time equivalent to Graph Isomorphism for general graphs [20, Theorem 5]. On the other hand, the problem can be solved in polynomial (even linear time) for interval graphs [20]. The main result of the present paper can be considered as a substantial generalization of the latter.

Results of this paper

**Theorem 1.1.** Testing isomorphism of chordal graphs of leafage \( \ell \) is fixed parameter tractable, with \( \ell \) as fixed parameter.

The leafage of chordal graphs is known to be polynomial-time computable [17]. Denote by \( \mathcal{R}_\ell \) the class of all chordal graphs of leafage at most \( \ell \). In particular, the graph class \( \mathcal{R}_\ell \) is polynomial-time recognizable.

In order to test if two connected graphs \( X, Y \in \mathcal{R}_\ell \) are isomorphic, it suffices to check if there is a generator of the automorphism group of their disjoint union \( X \cup Y \), which swaps \( X \) and \( Y \). Since the graph \( X \cup Y \) belongs to the class \( \mathcal{R}_{2\ell} \), the graph isomorphism problem for the graphs in \( \mathcal{R}_\ell \) is reduced to the problem of determining the automorphism group of a given graph in \( \mathcal{R}_{2\ell} \). Thus Theorem 1.1 is an immediate consequence of the following theorem which is proved in the paper.

**Theorem 1.2.** Given an \( n \)-vertex graph \( X \in \mathcal{R}_\ell \), a generating set of the group \( \text{Aut}(X) \) can be found in time \( t(\ell) \text{poly}(n) \), where \( t(\cdot) \) is a function independent of \( n \).

The function \( t \) from Theorem 1.2 is bounded from above by a polynomial in \( (2\ell)! \). The running time bound, especially the function \( t \), does not appear to be final and, most likely, it can be significantly improved.

We emphasize that our algorithm does not require that the input \( X \) is given by an intersection representation. Indeed, the algorithm works correctly on all chordal graphs and the leafage bound \( \ell \) is required only to bound the running time for inputs from the class \( \mathcal{R}_\ell \).

The proof of Theorem 1.2 is given in Section 7. The main steps involved in the algorithm are: (a) to efficiently transform the given graph \( X \) into an order-3 hypergraph \( H = H(X) \) (see below), (b) to give an algorithm for computing a generating set for \( \text{Aut}(H) \), and (c) to recover from it a generating set for \( \text{Aut}(X) \).

This brings us to the notion of higher-order hypergraphs. A usual hypergraph with vertex set \( V \) has hyperedge set contained in the power set \( \mathcal{E}_1 = 2^V \). The hyperedges of an order-3 hypergraph \( H \) will, in general, include order-2 and order-3 hyperedges. These are elements of \( \mathcal{E}_2 = 2^{\mathcal{E}_1} \) and \( \mathcal{E}_3 = 2^{\mathcal{E}_2} \), respectively. The hyperedge set \( E \) of \( H \) is contained in \( \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \) and can be of triple-exponential size in \( |V| \). However, the input size of \( H \) is defined to be \( |V| \cdot |E| \), for \( H \) given as input to an algorithm. The efficient reduction from finding \( \text{Aut}(X) \) to finding
Testing isomorphism of chordal graphs of bounded leafage  

\({\text{Aut}}(H)\) is presented in Sections 4 and 5. The key point of the reduction is a graph-theoretical analysis of the vertex coloring of the chordal graph \(X\) obtained by the 2-dimensional Weisfeiler-Leman algorithm \[27\]. The reduction takes \(X\) as input and computes the colored order-3 hypergraph \(H\) such that each vertex color class of \(H\) has size at most \(b = \ell 2^\ell\), where \(\ell = \ell(X)\).

At this point, we deal with the general problem of determining the automorphism group of a colored order-\(k\) hypergraph \(H\) (\(k \geq 1\)) by an FPT algorithm with respect to the parameter \(b\) which bounds the size of each vertex color class. This problem seems interesting in itself and could find other applications. For ordinary hypergraphs, it was shown to be fixed parameter tractable in \([3]\). A generalization of that result to order-\(k\) hypergraphs is given in Section 6. The running time bound we obtain is not FPT in terms of the parameter \(k\). Whether or not the problem is in FPT with both \(b\) and \(k\) as fixed parameters seems to be an interesting problem.

We complete the introduction with some remarks about \(H\)-graphs introduced in \([5]\). An \(H\)-graph \(X\) is an intersection graph of connected subgraphs of a subdivision of a fixed graph \(H\). Every graph is an \(H\)-graph for a suitable \(H\), which gives a parametrization for all graphs. It is interesting to note that we can get well-known graph classes as \(H\)-graphs for suitable choices of \(H\). For instance, interval graphs are \(K_2\)-graphs, circular-arc graphs are \(K_3\)-graphs, and chordal graphs are the union of all \(T\)-graphs, where \(T\) is a tree.

Basic algorithmic questions on \(H\)-graphs, including their recognition and isomorphism testing, have been studied, e.g., \([4, 8, 13]\). It is shown in \([1]\) that isomorphism testing for \(S_d\)-graphs, where \(S_d\) is a star of degree \(d\), is fixed parameter tractable. Since \(S_d\)-graphs are chordal graphs of leafage at most \(d\), our FPT algorithm applied to chordal graphs with bounded leafage significantly extends that result \([1]\) \(1\)

On the other hand, the isomorphism problem for \(H\)-graphs is as hard as the general graph isomorphism problem if \(H\) is not unicyclic \([9]\). Thus, it remains open whether isomorphism can be solved in polynomial time for the unicyclic case with fixed number of leaves, which would provide a dichotomy for the parametrization by \(H\)-graphs. Our work can be also considered a step towards this dichotomy.

2. Preliminaries

2.1. General notation. Throughout the paper, \(\Omega\) is a finite set. Given a bijection \(f\) from \(\Omega\) to another set and a subset \(\Delta \subseteq \Omega\), we denote by \(f^\Delta\) the bijection from \(\Delta\) to its image \(\Delta_f = \{\delta_f : \delta \in \Delta\}\). For a set \(S\) of bijections from \(\Delta\) to another set, we put \(S^\Delta = \{f^\Delta : f \in S\}\).

The group of all permutations of a set \(\Omega\) is denoted by \(\text{Sym}(\Omega)\). When a group \(G\) acts on \(\Omega\), we set \(G^\Omega = \{g^\Omega : g \in G\}\) to be the permutation group induced by this action. Concerning standard permutation group algorithms we refer the reader to \([25]\).

Let \(\pi\) be a partition of \(\Omega\). The set of all unions of the classes of \(\pi\) is denoted by \(\pi^\cup\). The partition \(\pi\) is a refinement of a partition \(\pi'\) of \(\Omega\) if each class of \(\pi'\) belongs to \(\pi^\cup\); in this case, we write \(\pi \geq \pi'\), and \(\pi > \pi'\) if \(\pi \geq \pi'\) and \(\pi \neq \pi'\). The partition of \(\Delta \subseteq \Omega\) induced by \(\pi\) is denoted by \(\pi^\Delta\).

\(1\)Paper \([2]\) appeared in the arXiv some time after our paper was published there, contains an FPT algorithm testing isomorphism of T-graphs for every fixed tree T. This result gives an alternative FPT algorithm for chordal graphs of leafage \(\ell\).
2.2. Graphs. Let $X$ be an undirected graph. The vertex and edge sets of $X$ are denoted by $\Omega(X)$ and $E(X)$, respectively. The automorphism group of $X$ is denoted by $\text{Aut}(X)$. The set of all isomorphisms from $X$ to a graph $X'$ is denoted by $\text{Iso}(X, X')$.

The set of all leaves and of all connected components of $X$ are denoted by $L(X)$ and $\text{Conn}(X)$, respectively. For a vertex $\alpha$, we denote by $\alpha X$ the set of neighbors of $\alpha$ in $X$. The vertices $\alpha$ and $\beta$ are called twins in $X$ if every vertex other than $\alpha$ and $\beta$ is adjacent either to both $\alpha$ and $\beta$ or neither of them. The graph $X$ is said to be twinless if no two distinct vertices of $X$ are twins.

Let $\Delta, \Gamma \subseteq \Omega(X)$. We denote by $X_{\Delta, \Gamma}$ the graph with vertex set $\Delta \cup \Gamma$ in which two vertices are adjacent if and only if one of them is in $\Delta$, the other one is in $\Gamma$, and they are adjacent in $X$. Thus, $X_\Delta = X_{\Delta, \Delta}$ is the subgraph of $X$ induced by $\Delta$, and $X_{\Delta, \Gamma}$ is bipartite if $\Delta \cap \Gamma = \emptyset$.

Let $\Delta \subseteq \Omega(X)$ and $Y = X_\Delta$. The set of all vertices adjacent to at least one vertex of $\Delta$ and not belonging to $\Delta$ is denoted by $\partial Y$. The subgraph of $X$, induced by $\Delta \cup \partial Y$ is denoted by $\overline{Y}$.

For a tree $T$, let $S(T) = \{\Omega(T') : T'$ is a subtree of $T\}$ be the set of all vertex sets of the subtrees of $T$. A representation of a graph $X = (\Omega, E)$ on the tree $T$ (called tree-representation) is a function $R: \Omega \rightarrow S(T)$ such that for all $u, v \in \Omega$,

$$R(u) \cap R(v) \neq \emptyset \iff \{u, v\} \in E.$$ 

It is known that a graph $X$ is chordal if and only if $X$ has a tree-representation \cite{14}. The leafage $\ell(X)$ of $X$ is defined to be the minimum of $|L(T)|$ over all trees $T$ such that $X$ has a tree-representation on $T$.

2.3. Colorings. A partition $\pi$ of $\Omega$ is said to be a coloring (of $\Omega$) if the classes of $\pi$ are indexed by elements of some set, called colors. In this case, the classes of $\pi$ are called color classes and the color class containing $\alpha \in \Omega$ is denoted by $\pi(\alpha)$. Usually the colors are assumed to be linearly ordered. A bijection $f$ from $\Omega$ to another set equipped with coloring $\pi'$ is said to be color preserving if the colors of $\pi(\alpha)$ and $\pi'(f(\alpha))$ are the same for all points $\alpha \in \Omega$.

A graph equipped with a coloring of the vertex set (respectively, edge set) is said to be vertex colored (respectively, edge colored); a graph that is both vertex and edge colored is said to be colored. The isomorphisms of vertex/edge colored graphs are ordinary isomorphisms that are color preserving. To emphasize this, we sometimes write $\text{Aut}(X, \pi)$ for the automorphism group of a graph $X$ with coloring $\pi$.

Let $X$ be a colored graph with vertex coloring $\pi$. Consider the application of the Weisfeiler-Leman algorithm (2-dim WL) to $X$ \cite{27}. For the purpose of the paper, it suffices to understand that 2-dim WL iteratively colors pairs of vertices of $X$ until the coloring satisfies a specific regularity condition (where the vertex coloring corresponds to the coloring of diagonal pairs $(\alpha, \alpha)$). The resulting coloring of pairs is just what is called a coherent configuration.

The output of 2-dim WL defines a new vertex coloring $\text{WL}(X, \pi) \geq \pi$ of $X$. We say that $\pi$ is stable if $\text{WL}(X, \pi) = \pi$. In the language of coherent configurations, $\pi$ is stable precisely when the classes of $\pi$ are the fibers of a coherent configuration (details can be found in the monograph \cite{10}). In the sequel, we will use some
elementary facts from theory of coherent configurations. The following statement summarizes relevant properties of stable colorings.

**Lemma 2.1.** Let $X$ be a graph and $\pi$ be a stable coloring of $X$. Then

1. for $\Delta, \Gamma \in \pi$, the number $|\delta X \cap \Gamma|$ does not depend on $\delta \in \Delta$,
2. if $\Delta \in \pi^U$ or $X_\Delta \in \text{Conn}(X)$, then the coloring $\pi_\Delta$ is stable.

A coloring $\pi$ of the vertices of a graph $X$ is said to be invariant if every class of $\pi$ is $\text{Aut}(X)$-invariant. In this case, the coloring $\text{WL}(X, \pi)$ is also invariant and stable. Since the coloring of the vertices in one color is invariant and the Weisfeiler-Leman algorithm is polynomial-time, in what follows we deal with invariant stable colorings.

**2.4. Hypergraphs.** Let $V$ be a finite set. The set $\mathcal{E}_k = \mathcal{E}_k(V)$ of the order-$k$ hyperedges on $V$ is defined recursively as follows:

$$
\mathcal{E}_0 = V, \quad \mathcal{E}_k = \mathcal{E}_{k-1} \cup 2^\mathcal{E}_{k-1} \quad \text{for } k > 1.
$$

So, we consider elements of $V$ as order-0 hyperedges and the order-$k$ hyperedges include all order-$(k-1)$ hyperedges and their subsets.

Let $U \subseteq V$ and $e \in \mathcal{E}_k$ ($k \geq 1$). We recursively define the projection of $e$ on $U$ as the multiset

$$
e^U = \begin{cases} 
e \cap U & \text{if } k = 1, \\
\{\{\ne^U : \ne \in e\}\} & \text{if } k > 1.
\end{cases}
$$

We extend this definition to all sets $E \subseteq \mathcal{E}_k$ by putting $E^U = \{e^U : e \in E\}$.

**Definition 2.2** (order-$k$ hypergraph). An order-$k$ hypergraph ($k \geq 1$) on $V$ is a pair $H = (V, E)$, where $E \subseteq 2^{\mathcal{E}_k}$; the elements of $V$ and $E$ are called vertices and hyperedges of $H$, respectively.

Clearly, order-1 hypergraphs are usual hypergraphs. Moreover, higher-order hypergraphs (i.e., order-$k$ hypergraph for some $k$) are combinatorial objects in the sense of [11]. The concepts of isomorphism and coloring extend to higher-order hypergraphs in a natural way.

Let $k \geq 2$. The $(k-1)$-skeleton of an order-$k$ hypergraph $H = (V, E)$ is an order-$(k-1)$ hypergraph $H''(k-1)$ on $V$ with the hyperedge set

$$
E''(k-1) = \{\ne \in \mathcal{E}_{k-1} : \ne \text{ is an element of some } e \in \mathcal{E}_k \cap E\}.
$$

It is easily seen that for every order-$k$ hypergraph $H' = (V', E')$

$$
(1) \quad \text{Iso}(H, H') = \{f \in \text{Iso}(H''(k-1), H''(k-1)) : e \in E''(k) \iff e^f \in E''(k)\},
$$

where for each order-$k$ hyperedge $e = \{e_1, \ldots, e_a\}$ we set $e^f = \{e_1^f, \ldots, e_a^f\}$.

Let $H_1 = (V_1, E_1)$ be an order-$k$ hypergraph for some $k$ and $H_2 = (V_2, E_2)$ be a usual hypergraph such that $V_2 = E_1$. Then each hyperedge $e \in E_2$ is a subset of hyperedges of $H_1$. We define the hypergraph composition of $H_1$ and $H_2$ to be the order-$(k + 1)$ hypergraph

$$
H := H_1 \uparrow H_2 = (V, E_1 \cup E_2).
$$

When the hypergraphs $H_1$ and $H_2$ are colored, the vertex coloring of $H$ is defined in the obvious way. The color $c(e)$ of $e \in E(H)$ is defined as follows: if $e \in E_1 \setminus E_2$ then $c(e)$ is the color $c_1(e)$ of $e$ in $H_1$. If $e \in E_1 \cap E_2$ then $c(e)$ is defined as the
3. Chordal graphs

3.1. Stable colorings in chordal graphs. In this subsection, we prove several auxiliary statements about the structure of subgraphs of a chordal graph, induced by one or two color classes of a stable coloring.

Lemma 3.1. Let \( X \) be a chordal graph and \( \pi \) a stable coloring of \( X \). Then for every \( \Delta, \Gamma \in \pi \), the following statements hold:

1. \( \text{Conn}(X_\Delta) \) consists of cliques of the same size,
2. if \( |\text{Conn}(X_\Delta)| \leq |\text{Conn}(X_\Gamma)| \), then \( \text{Conn}(X_\Delta) = \{ Y : Y \in \text{Conn}(X_{\Delta \cup \Gamma}) \} \),
3. if the graphs \( X_\Delta \) and \( X_\Gamma \) are complete, then \( X_{\Delta \cup \Gamma} \) is either complete bipartite or empty.

Proof. (1) By Lemma 2.1(1) for \( \Delta = \Gamma \), the graph \( X_\Delta \) is regular. It suffices to verify that every graph \( Y \in \text{Conn}(X_\Delta) \) is complete. Since \( Y \) is chordal it contains a simplicial vertex, i.e., a vertex whose neighborhood induces a complete graph. As \( Y \) is regular, all its vertices are simplicial. Thus, \( Y \) is complete.

(2) Let \( X' \) be a bipartite graph with parts \( \Delta' = \text{Conn}(X_\Delta) \) and \( \Gamma' = \text{Conn}(X_\Gamma) \) in which two vertices \( \alpha' \in \Delta' \) and \( \beta' \in \Gamma' \) are adjacent if and only if there are vertices \( \alpha \in \alpha' \) and \( \beta \in \beta' \) adjacent in \( X \). By statement (1), the components of \( X_{\Delta \cup \Gamma} \) are in one-to-one correspondence with the components of \( X' \). Denote by \( Y' \) the component of \( X' \), corresponding to the component \( Y \in \text{Conn}(X_{\Delta \cup \Gamma}) \).

The partition \( \pi' = \{ \Delta', \Gamma' \} \) is a stable coloring of \( X' \). Indeed, it suffices to find a coherent configuration on \( \Omega(X') \), for which \( \Delta' \) and \( \Gamma' \) are fibers. As such a configuration, one can take the quotient of the coherent configuration corresponding to \( \pi \), modulo the equivalence relation on \( \Delta \cup \Gamma \) the classes of which are vertex sets of the graphs belonging to \( \Delta' \) and \( \Gamma' \), see [10, Section 3.1.2].

By Lemma 2.1(1), any two vertices of \( X' \) that are from the same part have the same degree. Moreover, the graph \( X' \) is obviously chordal. Consequently, it is acyclic: otherwise, \( X' \) being bipartite contains an induced cycle of length at least 4, which is impossible for a chordal graph. Hence, \( X' \) has a vertex \( \alpha' \) of degree 1. Since all vertices of the part containing \( \alpha' \) have the same degree, each \( Y' \in \text{Conn}(X'(\alpha')) \) is a star. The center of this star lies in \( \Delta' \), because \( |\Delta'| \leq |\Gamma'| \). Thus, \( Y_{\Delta} \in \text{Conn}(X_\Delta) \), which implies the required statement.

(3) Without loss of generality we may assume that the bipartite graph \( X_{\Delta \cup \Gamma} \) is not empty and \( \Delta \neq \Gamma \). Suppose to the contrary that there are \( \delta_1, \delta_2 \in \Delta \) such that \( \delta_1 X \cap \Gamma \neq \delta_2 X \cap \Gamma \). By Lemma 2.1(1), we have \( |\delta_1 X \cap \Gamma| = |\delta_2 X \cap \Gamma| \). Thus there exist \( \gamma_1 \in \delta_1 X \cap \Gamma \) and \( \gamma_2 \in \delta_2 X \cap \Gamma \) such that

\[ \gamma_1 \notin \delta_2 X \quad \text{and} \quad \gamma_2 \notin \delta_1 X. \]

By assumption, the graphs \( X_\Delta \) and \( X_\Gamma \) are complete. Hence, \( \delta_1 \) and \( \delta_2 \) are adjacent, and also \( \gamma_1 \) and \( \gamma_2 \) are adjacent. Therefore, the vertices \( \delta_1, \gamma_1, \gamma_2, \delta_2 \) form an induced 4-cycle of \( X \), which is a contradiction. Thus any two vertices that are in \( \Delta \) have the same neighborhoods in the bipartite graph \( X_{\Delta \cup \Gamma} \). As \( X_{\Delta \cup \Gamma} \) has no isolated vertices in \( \Gamma \) (by Lemma 2.1), it follows that \( X_{\Delta \cup \Gamma} \) is a complete bipartite graph. \( \square \)
Remark 3.2. Recall that stable colorings are defined via the 2-dimensional Weisfeiler-Leman algorithm. While the 1-dimensional Weisfeiler-Leman algorithm suffices for the first and third parts of Lemma 3.1, it is worth noting that the second part requires the 2-dimensional algorithm.

Lemma 3.3. Let $X$ be a connected chordal graph and let $\pi$ be a stable partition of $\Omega$. There exists $\Delta \in \pi$ such that the graph $X_\Delta$ is complete.

Proof. The statement immediately follows from Lemma 3.1(1) if $|\pi| = 1$. Assume that $|\pi| > 1$. Since $\pi$ is stable, the classes of $\pi$ are the fibers of some coherent configuration on $\Omega$, see Subsection 2.3.

Suppose to the contrary that the graph $X_\Delta$ is not complete for any $\Delta \in \pi$. Let $\Delta$ be a class of $\pi$ that contains a simplicial vertex of $X$. Then all vertices in $\Delta$ are simplicial, see [12, Lemma 8.1]. It follows that the graph $X' := X_{\Omega \setminus \Delta}$ is connected and chordal. Moreover, the partition $\pi' := \pi_{\Omega \setminus \Delta}$ is stable by Lemma 2.1(2). Since $|\pi'| < |\pi|$, we conclude by induction that there is $\Delta' \in \pi'$ such that $X_{\Delta'}$ is complete, which is not possible, because $\Delta' \in \pi$. □

3.2. Estimates depending on the leafage. The two lemmas in this subsection show bounds that are crucial for estimating the complexity of the main algorithm.

Lemma 3.4. Let $X$ be a chordal graph, $\Delta$ a subset of its vertices, $X - \Delta$ is the subgraph of $X$ induced by the complement of $\Delta$, and

$$(2) \quad S = S(X, \Delta) = \{ Y \in \text{Conn}(X - \Delta) : \ Y \text{ is not interval}\}.$$  

Then $|S| \leq \ell(X) - 2$.

Proof. Let $R$ be a tree-representation of $X$ on a tree $T$ such that $|L(T)| = \ell(X)$. Let $n_3$ be the number of all vertices of $T$ of degree at least 3. Clearly,

$$(3) \quad n_3 \leq \ell - 2,$$  

where $\ell = \ell(X)$.

Let $Y \in S$, and let $R(Y)$ be the union of all subtrees $R(\alpha), \alpha \in \Omega(Y)$. Then $R(Y)$ is a subtree of $T$. We claim that $R(Y)$ contains a vertex $t_Y$ of degree at least 3. Indeed, otherwise, $R(Y)$ is a path in $T$. Moreover, if $\alpha \in \partial Y$, then either $R(\alpha)$ is a subpath of $P$, or $R(\alpha)$ contains at least one end of $P$. This implies that the restriction of $R$ to the set $\Omega(Y) \cup \partial Y$ is a tree-representation of $Y$ on $P$. But then $Y$ is interval, a contradiction.

To complete the proof, we note that the sets $R(Y), \ Y \in S$, are pairwise disjoint. Therefore the vertices $t_Y$ are pairwise distinct. By inequality (3), this yields

$$|S| = |\{ t_Y : \ Y \in S \}| \leq n_3 \leq \ell - 2,$$  

as required. □

Let $\pi$ be a vertex coloring of $X$. Given a pair $(\Delta, \Gamma) \in \pi \times \pi$, we define an equivalence relation $e_{\Delta, \Gamma}$ on $\Delta$ by setting

$$(4) \quad (\delta, \delta') \in e_{\Delta, \Gamma} \iff \delta \text{ and } \delta' \text{ are twins in } X_{\Delta, \Gamma}.$$  

Note that the equivalence relation $e_{\Gamma, \Delta}$ is defined on $\Gamma$, and coincides with $e_{\Delta, \Gamma}$ only if $\Gamma = \Delta$. The sets of classes of $e_{\Delta, \Gamma}$ and $e_{\Gamma, \Delta}$ are denoted by $\Delta/e_{\Delta, \Gamma}$ and $\Gamma/e_{\Gamma, \Delta}$, respectively.
Lemma 3.5. Let $X$ be a chordal graph, $\pi$ a stable coloring, and $\Delta, \Gamma \in \pi$. Assume that the graph $X_\Delta$ is complete. Then

$$|\Delta/e_{\Delta, \Gamma}| \leq 2^\ell \quad \text{and} \quad |\Gamma/e_{\Gamma, \Delta}| \leq \ell,$$

where $\ell = \ell(X)$.

Proof. Without loss of generality we may assume that $X = X_{\Delta, e_t}$ (because $\ell(X_{\Delta, e_t}) \leq \ell(X)$), and the graph $X_{\Delta, \Gamma}$ is neither complete bipartite, nor empty (otherwise, $|\Delta/e_{\Delta, \Gamma}| = 1$ and $|\Gamma/e_{\Gamma, \Delta}| = 1$, and both statements are trivial). Thus, $X_\Gamma$ is not complete by Lemma 3.4 (3) and $\Delta$ is a maximal clique of $X$; in particular, $\Delta \neq \Gamma$.

Let $R : \Omega \rightarrow S(T)$ be a tree-representation of the graph $X$ on a tree $T$ with $\ell$ leaves. Without loss of generality, we may assume that the set $\Omega(T)$ is the minimum possible. Since $\Delta$ is a clique of $X$, the intersection of the subtrees $R(\delta)$, $\delta \in \Delta$, contains at least one point $t$.

Let $\gamma \in \Gamma$. Then $t \notin R(\gamma)$ by the maximality of $\Delta$. Denote by $t_{\gamma}$ the point of $R(\gamma)$, lying at the minimum distance from $t$ in $T$. Let $P_{\gamma}$ be the path connecting $t$ and $t_{\gamma}$; note that $P_{\gamma}$ has at least two vertices, because $t \neq t_{\gamma}$.

Let us define a partial order on $T = \{t_\gamma : \gamma \in \Gamma\}$ by setting $t_\gamma \preceq t_{\gamma'}$ if and only if $t_{\gamma}$ lies in $P_{\gamma'}$ (in particular, either $t_{\gamma} = t_{\gamma'}$ or $t_{\gamma}$ is closer to $t$ than $t_{\gamma'}$), or equivalently, $P_{\gamma} \subseteq P_{\gamma'}$.

Claim. If $t_\gamma \preceq t_{\gamma'}$, then $(\gamma, \gamma') \in e_{\Gamma, \Delta}$.

Proof. Let $\delta \in \gamma'X \cap \Delta$. Then the intersection $R(\delta) \cap R(\gamma')$ is not empty. Moreover, it contains $t_{\gamma'}$: for otherwise, because $t \in R(\delta)$, the set $R(\gamma')$ contains a vertex which is closer to $t$ than $t_{\gamma'}$. Consequently, $P_{\gamma} \subseteq R(\delta)$. Since $P_{\gamma} \subseteq P_{\gamma'}$ we have

$$t_{\gamma} \in P_{\gamma} \subseteq P_{\gamma'} \subseteq R(\delta).$$

That is, the intersection $R(\delta) \cap R(\gamma) \ni t_{\gamma}$ is not empty; in particular, $\delta \in \gamma X \cap \Delta$. It follows that $\gamma'X \cap \Delta \subseteq \gamma X \cap \Delta$. Since also $|\gamma'X \cap \Delta| = |\gamma X \cap \Delta|$ by Lemma 2.1 (1), we are done.

Let $T_{\text{min}} \subseteq T$ be the set of all minimal points with respect to the partial order on $T$. By the claim, for every $\gamma' \in T \setminus T_{\text{min}}$ there is $\gamma \in T_{\text{min}}$ such that $(\gamma, \gamma') \in e_{\Gamma, \Delta}$.

Thus,

$$|\Gamma/e_{\Gamma, \Delta}| \leq |T_{\text{min}}|.$$

On the other hand, by the minimality of $T$, every leaf of $T$ belongs to $R(\gamma)$ for some $\gamma \in \Gamma$. Consequently, the path from any leaf of $T$ to $t$ contains at most one point of $T_{\text{min}}$. Thus, $|T_{\text{min}}| \leq \ell$ and so

$$|\Gamma/e_{\Gamma, \Delta}| \leq |T_{\text{min}}| \leq \ell,$$

which proves the second inequality in (5).

To complete the proof, we observe that if $\delta \in \Delta$, then the set $\delta X \cap \Gamma$ is a union of some classes of $\Gamma/e_{\Gamma, \Delta}$. Denote this union by $\Gamma_\delta$. Note that if $\delta, \delta' \in \Delta$, then $\Gamma_\delta = \Gamma_{\delta'}$ if and only if $(\delta, \delta') \in e_{\Delta, \Gamma}$. Therefore, the number $|\Delta/e_{\Delta, \Gamma}|$ is at most

$$|\Delta/e_{\Delta, \Gamma}| \leq |2^\Gamma/e_{\Gamma, \Delta}| = 2|\Gamma/e_{\Gamma, \Delta}| \leq 2^\ell,$$

which proves the first inequality in (5). \qed
4. Critical set of a chordal graph

Let $X$ be a chordal graph and $\pi$ a stable coloring. Denote by $\Omega^* = \Omega^*(X, \pi)$ the union of all $\Delta \in \pi$ such that

\begin{equation}
|\text{Conn}(X_\Delta)| \leq \ell(X).
\end{equation}

By Lemma 3.1(1), the graph $X_\Delta$ is a disjoint union of cliques; thus the above condition means that the number of them is at most $\ell(X)$. By Lemma 3.3, the set $\Omega^*$ is not empty if the graph $X$ is connected.

**Theorem 4.1.** Let $X$ be a chordal graph and $\Omega^* = \Omega^*(X, \pi)$. Then one of the following statements holds:

(i) for every $Y \in \text{Conn}(X - \Omega^*)$, the graph $Y$ is interval,

(ii) there is an invariant stable coloring $\pi' > \pi$.

Moreover, in case (ii), the coloring $\pi'$ can be found in polynomial time in $|\Omega|$.

**Proof.** Assume that (i) does not hold. Then the set the set $S = S(X, \Delta)$ defined by formula (2) for $\Delta = \Omega^*$ is not empty. By Lemma 3.4, we have

\begin{equation}
|S| \leq \ell - 2,
\end{equation}

where $\ell = \ell(X)$. Take an arbitrary $Y \in S$. By Lemma 3.1(2), the coloring $\pi_Y := \pi_{\Omega(Y)}$ is stable. By Lemma 3.3 there is $\Gamma' \in \pi_Y$ such that the graph $Y_{\Gamma'}$ is complete. Let $\Gamma$ be the class of $\pi$, containing $\Gamma'$. Then

\begin{equation}
\Gamma \cap \Omega^* = \emptyset,
\end{equation}

because $\Gamma$ intersects $\Omega \setminus \Omega^* \in \pi^\cup$. Moreover, every automorphism of $X$ preserves the sets $S$ and $\Gamma$ and hence preserves the set

\begin{equation}
S' = \{Z \in S : Z_{\Gamma \cap \Omega(Z)} \text{ is complete}\}.
\end{equation}

Thus the union $\Gamma_0$ of all sets $\Gamma \cap \Omega(Z)$, $Z \in S'$, is a nonempty $\text{Aut}(X)$-invariant set contained in $\Gamma$. Now if $\Gamma_0 \neq \Gamma$, then we come to case (ii) with

$$
\pi' = (\pi \setminus \{\Gamma\}) \cup \{\Gamma \setminus \Gamma_0, \Gamma \setminus \Gamma_0\}.
$$

To complete the proof, assume that $\Gamma_0 = \Gamma$. Then by inequality (7), the graph $X_{\Gamma}$ is the union of at most $|S'| \leq |S| \leq \ell - 2$ cliques. By the definition of $\Omega^*$, this yields $\Gamma \subseteq \Omega^*$, which contradicts relation (8). \qed

We say that $\Omega^*$ is a *critical set* of $X$ (with respect to $\pi$) if statement (i) of Theorem 4.1 holds. In the rest of the section we define a hypergraph $H^*$ associated with the critical set $\Omega^*$ and show that the groups $\text{Aut}(H^*)^{\Omega^*}$ and $\text{Aut}(X)^{\Omega^*}$ are closely related.

The vertices of $H^*$ are set to be the elements of the disjoint union

$$
V = \bigcup_{\Delta \in \pi_{\Omega^*}} \bigcup_{\Gamma \in \pi} \Delta/e_{\Delta, \Gamma},
$$

where $e_{\Delta, \Gamma}$ is the equivalence relation on $\Delta$, defined by formula (4). Thus any vertex of $H^*$ is a class of some $e_{\Delta, \Gamma}$. Taking the disjoint union means, in particular, that if $\Lambda$ is a class of $e_{\Delta, \Gamma}$ and $e_{\Delta, \Gamma'}$, then $V$ contains two vertices corresponding to $\Lambda$. The partition

$$
\pi = \{\Delta/e_{\Delta, \Gamma} : \Delta \in \pi_{\Omega^*}, \Gamma \in \pi\}
$$

of the set $V$ is treated as a coloring of $V$. 


Let us define the hyperedges of $H^*$. First, let $\alpha \in \Omega^*$. Denote by $\Delta$ the class of $\pi$, containing $\alpha$. Then $\Delta \in \pi_{\Omega^*}$. Moreover, for every $\Gamma \in \pi$, there is a unique class $\Lambda_\alpha(\Delta, \Gamma)$ of the equivalence relation $e_{\Delta, \Gamma}$, containing $\alpha$. Put

$$\bar{\pi} = \{\Lambda_\alpha(\Delta, \Gamma) : \Gamma \in \pi\},$$

in particular, $\bar{\pi} \subseteq V$. It is easily seen that $\bar{\pi} = \bar{\beta}$ if and only if the vertices $\alpha$ and $\beta$ are twins in $X$, lying in the same class of $\pi$. Next, let $\beta \in \Omega^*$ be adjacent to $\alpha$ in $X$, and $\Gamma$ the class of $\pi$, containing $\beta$. Then every vertex in $\Lambda_\alpha(\Delta, \Gamma)$ is adjacent to every vertex of $\Lambda_\beta(\Gamma, \Delta)$. Put

$$\{\alpha, \beta\} = \{\Lambda_\alpha(\Delta, \Gamma), \Lambda_\beta(\Gamma, \Delta)\},$$

again $\{\alpha, \beta\} \subseteq V$. With this notation, the hyperedge set of $H^*$ is defined as the union:

$$E^* = \{\bar{\pi} : \alpha \in \Omega^*\} \cup \{\{\alpha, \beta\} : \alpha, \beta \in \Omega^*, \beta \in \alpha X\}.$$

As we are interested in only automorphisms of $E^*$ that stabilize the two parts $\{\bar{\pi} : \alpha \in \Omega^*\}$ and $\{\{\alpha, \beta\} : \alpha, \beta \in \Omega^*, \beta \in \alpha X\}$, we can color the hyperedges in $E^*$ using two distinct colors to ensure this. Clearly, the hypergraph $H^* = (V, E^*)$ and the coloring $\bar{\pi}$ can be constructed in polynomial time in $|\Omega|$.

**Theorem 4.2.** Let $X$ be a chordal graph, $\pi$ an invariant stable vertex coloring of $X$, $\Omega^* = \Omega^*(X, \pi)$ the critical set, and $H^* = (V, E^*)$ is the above hypergraph with vertex coloring $\bar{\pi}$. Then

(i) $\max|\Delta| : \Delta \in \pi| \leq \ell 2^\ell$, where $\ell = \ell(X)$,
(ii) if $X$ is twinless, then the mapping $f : \Omega^* \to E^*$, $\alpha \mapsto \bar{\pi}$, is an injection,
(iii) if $X$ is twinless and $G = G(H^*)$ is the group induced by the natural action of $\text{Aut}(H^*)$ on $\text{Im}(f) = \{\bar{\pi} : \alpha \in \Omega^*\} \subseteq E^*$, then

$$\text{Aut}(X)^{\Omega^*} \leq G^f \leq \text{Aut}(X_{\Omega^*}),$$

where $G^f^{-1} = fGf^{-1}$.

**Proof.** (i) The color classes of $\bar{\pi}$ are the sets $\Delta/e_{\Delta, \Gamma}$, where $\Delta \in \pi_{\Omega^*}$ and $\Gamma \in \pi$. By the definition of $\Omega^*$, we have $|\text{Conn}(X_\Delta)| \leq \ell$, and Lemma 3.1(2) yields

$$\text{Conn}(X_{\Delta \cup \Gamma}) \leq \min\{|\text{Conn}(X_\Delta)|, |\text{Conn}(X_\Gamma)|\} \leq \ell. \tag{10}$$

Further, let $Y \in \text{Conn}(X_{\Delta \cup \Gamma})$. Then by Lemma 2.1(2), the coloring $\pi_Y$ is stable. It has two classes, one inside $\Delta$ and the other one inside $\Gamma$; denote them by $\Delta_Y$ and $\Gamma_Y$, respectively. Note that by Lemma 2.1(2), at least one of the graphs $X_{\Delta_Y}$, $X_{\Gamma_Y}$ is complete. From Lemma 3.3 we obtain

$$|\Delta/e_{\Delta, \Gamma}| \leq \max\{\ell, 2^\ell\} \leq 2^\ell. \tag{11}$$

Since the equivalence relation $e_{\Delta, \Gamma}$ is the union of the equivalence relations $e_{\Delta_Y, \Gamma_Y}$, $Y \in \text{Conn}(X_{\Delta \cup \Gamma})$, inequalities (10) and (11) imply

$$|\Delta/e_{\Delta, \Gamma}| \leq \sum_{Y \in \text{Conn}(X_{\Delta \cup \Gamma})} |e_{\Delta_Y, \Gamma_Y}| \leq \ell 2^\ell,$$

as required.

---

\footnote{Note that the composition $fGf^{-1}$ is defined from left to right.}
(ii) Assume that $X$ is twinless. Let $\alpha \in \Omega^*$ and let $\Delta \in \pi$ contain $\alpha$. Denote by $\Lambda_\alpha$, the intersection of all $\Lambda_\alpha(\Delta, \Gamma)$, $\Gamma \in \pi$. Note that every $\beta \in \Lambda_\alpha$ belongs to $\Delta$. Moreover,

$$\alpha X \cap \Gamma = \beta X \cap \Gamma$$

for all $\Gamma \neq \Delta$, and

$$(\alpha X \cap \Delta) \setminus \{\beta\} = (\beta X \cap \Delta) \setminus \{\alpha\}.$$  

It follows that $\alpha$ and $\beta$ are twins in $X$. Since $X$ is twinless, we conclude that $\alpha = \beta$. Thus,

$$\Lambda_\alpha = \{\alpha\} \quad \text{for all } \alpha \in \Omega^*.$$  

Now assume that $f(\alpha) = f(\beta)$ for some $\alpha, \beta \in \Omega^*$. Then $\Lambda_\alpha = \Lambda_\beta$ and the above formula implies $\{\alpha\} = \Lambda_\alpha = \Lambda_\beta = \{\beta\}$. Thus, $\alpha = \beta$ and $f$ is injective.

(iii) Assume that $X$ is twinless. By (ii), the mapping $f$ is an injection. Let $g \in \text{Aut}(X)$ and $\alpha \in \Omega^*$. Then $\alpha$ lies in some $\Delta \in \pi_{\Omega^*}$. Since $\pi$ is invariant and stable, we have $e_{\Delta^*, \Gamma^*} = e_{\Delta, \Gamma}$ and so

$$\Lambda_\alpha(\Delta, \Gamma) = \Lambda_\alpha(\Delta, \Gamma)$$

for every $\Gamma \in \pi$, where $\bar{g} \in \text{Sym}(V)$ is the permutation induced by $g$. Note that $\bar{g}$ preserves the coloring $\pi$. Moreover,

$$(\pi)^{\bar{g}} = \{\Lambda_\alpha(\Delta, \Gamma)^{\bar{g}} : \Gamma \in \pi\} = \{\Lambda_{\alpha^g}(\Delta, \Gamma) : \Gamma \in \pi\} = \overline{\pi^g}$$

and

$$\{\alpha, \beta\}^{\bar{g}} = \{\Lambda_\alpha(\Delta, \Gamma)^{\bar{g}}, \Lambda_\beta(\Delta, \Gamma)^{\bar{g}}\} = \{\Lambda_{\alpha^g}(\Delta, \Gamma), \Lambda_{\beta^g}(\Delta, \Gamma)\} = \overline{\{\alpha^g, \beta^g\}}.$$  

Consequently, $\bar{g} \in \text{Aut}(\mathcal{H}^*)$. Since $\overline{\pi^{f^{-1}gf}} = (\alpha^g)^f = \overline{\alpha^g} = \overline{\pi^g}$, it follows that $f^{-1}gf \in G$, which proves the left-hand side inclusion in (9).

Let $\alpha, \beta \in \Omega^*$. Denote by $\Delta$ and $\Gamma$ the classes of $\pi$, containing $\alpha$ and $\beta$, respectively. Then $\alpha$ and $\beta$ are adjacent in $X$ if and only if every vertex in $\Lambda_\alpha(\Delta, \Gamma)$ is adjacent to every vertex of $\Lambda_\beta(\Gamma, \Delta)$, or equivalently, $\{\alpha, \beta\} \in E^*$. Thus, the right-hand side inclusion in (9) follows from (12).  

\section{The Hypergraph Associated with Complement of the Critical Set}

The goal of this section is to provide some tools related to the critical set that will help design the algorithm for computing the automorphism group of a chordal graph in $\mathcal{K}_e$.

Suppose $X$ is a chordal graph on $\Omega$ and $\pi$ an invariant stable coloring of $X$. Further, let $\Omega^*$ denote the critical set of $X$ with respect to $\pi$. Let $G^* = G^*(X)$ denote the kernel of the restriction homomorphism $\text{Aut}(X) \to \text{Aut}(X)^{\Omega^*}$. We claim that a generating set for $G^*$ can be efficiently computed.

\begin{theorem}
A generating set for the kernel $G^* \leq \text{Sym}(\Omega)$ of the restriction homomorphism from $\text{Aut}(X)$ to $\text{Aut}(X)^{\Omega^*}$ can be found in polynomial time in $|\Omega|$.
\end{theorem}

\begin{proof}
Without loss of generality, we may assume that the set $\Omega^* = \Omega \setminus \Omega^*$ is not empty. Let us define a vertex coloring $\pi^*$ of the graph $X^* = X|_{\Omega^*}$, such that $\pi^*(\alpha) = \pi^*(\beta)$ if and only if $\pi(\alpha) = \pi(\beta)$ and $\alpha X \cap \Omega^* = \beta X \cap \Omega^*$. It is not hard to see that

$$(G^*)^{\Omega^*} = \text{Aut}(X^*, \pi^*).$$

Since also the graph $X^*$ is interval (see the definition of the critical set), a generating set of $(G^*)^{\Omega^*}$ can be found by the algorithm in [11, Theorem 3.4], which constructs...
a generating set of the automorphism group of a vertex colored interval graph efficiently. Since \((G^\ast)\Omega^\ast = \{id_{\Omega^\ast}\}\), the theorem is proved. \( \square \)

In what follows, \(X\) is a chordal graph, \(\pi\) a stable coloring of \(X\), \(\Omega^*\) the critical set of \(X\) with respect to \(\pi\), and \(\partial \Omega^* = \Omega^* \setminus \Omega^\ast\). Recall that by the definition of critical set, every graph \( \overrightarrow{Y} \subseteq \text{Conn}(X_{\Omega^*}) \), is interval and

\[
\partial Y = \Omega(\overrightarrow{Y}) \cap \Omega^*.
\]

**Lemma 5.2.** For every \(Y \in \text{Conn}(X_{\Omega^*})\), there is a colored hypergraph \(H = H_Y\) whose vertex set is \(\partial Y\) colored by \(\pi_{\partial Y}\), and such that

\[
\text{Iso}(H_Y, H_{Y'}) = \text{Iso}(\overrightarrow{Y}, \overrightarrow{Y'})^{\partial Y}, \quad Y' \in \text{Conn}(X_{\Omega^*}).
\]

Moreover, in time polynomial in \(|Y|\) one can

(a) construct the hypergraph \(H_Y\),

(b) given \(\overline{g} \in \text{Iso}(H_Y, H_{Y'})\), find \(g \in \text{Iso}(\overrightarrow{Y}, \overrightarrow{Y'})\) such that \(g^{\partial Y} = \overline{g}\).

**Proof.** We make use of the results of [12]. Namely, let \(Z\) be an interval graph and \(\pi_Z\) a stable coloring of \(Z\). From [12] Theorem 6.10 and Proposition 6.4, it follows that there exists a canonical rooted tree \(T = T(Z)\) and a stable coloring \(\pi_T\) of \(T\) such that

\[
L(T) = \Omega(Z) \quad \text{and} \quad \pi_Z = (\pi_T)_{L(T)}.
\]

The term “canonical” means that for every interval vertex colored graph \(Z'\), the isomorphisms between \(Z\) and \(Z'\) are related with the isomorphisms between \(T\) and \(T' = T(Z')\) as follows:

\[
\text{Iso}(T, T')^{L(T)} = \text{Iso}(Z, Z').
\]

Moreover, the proof of [12] Proposition 6.4 shows that the sizes of \(T\) and \(\pi_T\) are polynomials in \(|\Omega(Z)|\), and \(T\) and \(\pi_T\) can be constructed in polynomial time.

Now let \(Y \in \text{Conn}(X_{\Omega^*})\). Since the graph \(\overrightarrow{Y}\) is interval, one can define the rooted tree \(T = T(\overrightarrow{Y})\) as above. Next, for each vertex \(x\) of \(T\), we introduce the following notation:

- \(L(x)\) is the set of all descendants of \(x\) in \(T\), lying in \(L(T)\),
- \(T_x\) is the subtree of \(T\) rooted at \(x\) and such that \(L(T_x) = L(x) \setminus \partial Y\),
- \(F(T_x)\) is a string encoding the isomorphism type of the rooted tree \(T_x\).

Now for each \(x\) with \(L(x) \cap \partial Y \neq \emptyset\) and \(L(x) \setminus \partial Y \neq \emptyset\), we delete from \(T\) all the vertices of \(T_x\), except for \(x\), and define the new color of \(x\) to be equal to \((\pi_T(x), F(T_x))\). Denote the resulting tree and its vertex coloring by \(T_1 = T_1(Y)\) and \(\pi_1 = \pi_1(Y)\), respectively. Then

\[
L(T_1) = \partial Y.
\]

It is not hard to see that \(T_1\) and \(\pi_1\) can efficiently be constructed, and \(T\) and \(\pi_T\) are uniquely recovered from \(T_1\) and \(\pi_1\). In particular,

\[
\text{Iso}(T_1, T_1')^{\partial Y} = \text{Iso}(\overrightarrow{Y}, \overrightarrow{Y'}),
\]

where \(Y' \in \text{Conn}(X - \Omega^*)\) and \(T_1' = T_1(Y')\), respectively, cf., (14) and (15).

At this point we can define the required hypergraph \(H_Y = (\partial Y, E_Y)\), where

\[
E_Y = \{L(x) : x \in \Omega(T_1(Y))\}.
\]
Note that $L(x) = L(y)$ if and only if $x = y$ or one of $x, y$ is the descendants of the other in $T_1$, and if, say $y$ is the descendant of $x$, then for each vertex $z \neq y$ of the path $P_{yx}$ connecting $y$ with $x$, we have $L(x) = L(z)$; moreover, in the latter case, $z$ has a unique child in $T_1$. Thus for each $e \in E_Y$ there exist uniquely determined vertex $x_e$ and its descendant $y_e$ such that

$$L(z) = e \iff z \in \Omega(P_{y_0x_e}) \text{ and } z \neq y_e \text{ if } x_e \neq y_e.$$ In particular, if $x_e = y_e := x$, then $L(x) = L(z)$ if and only if $x = z$.

To define the color of the hyperedge $e \in E_Y$, let $\Omega(P_{y_0x_e}) = \{y_0, y_1, \ldots, y_k\}$, where $k \geq 0$ is the length of $P_{y_0x_e}$, $y_0 = y_e$, $y_k = x_e$, and $y_i$ is the child of $y_{i+1}$, $i = 0, \ldots, k - 1$. Then the color of $e$ is set to be the tuple

$$(\pi_1(y_1), \ldots, \pi_1(y_k)).$$

Again, it is clear that the hypergraph $H_Y$ and its coloring can be constructed efficiently and that they determine the colored tree $T_1$ in a unique way. Thus the statement of the lemma is a consequence of formulas (11) and (17).

Let us define a colored order-2 hypergraph $H^\circ$ with vertex set $\Omega^*$ and hyperedge set $E_1 \cup E_2$, where

$$E_1 = \bigcup_{Y \in \text{Conn}(X_{\Omega^*})} E(H_Y) \quad \text{and} \quad E_2 = \{E(H_Y) : Y \in \text{Conn}(X_{\Omega^*})\}.$$ 

The vertex coloring of $H^\circ$ is set to be $\pi^\circ$. Note that the union in the definition of $E_1$ is not disjoint; the color $\pi^\circ(e)$ of a hyperedge $e \in E_1$ is defined to be the multiset of the colors of $e$ in $H_Y$, where $Y$ runs over all graphs $Y \in \text{Conn}(X_{\Omega^*})$ such that $e \in E(H_Y)$.

To define a coloring of $E_2$, denote by $\sim$ the equivalence relation on $\text{Conn}(X_{\Omega^*})$ by setting

$$Y \sim Y' \iff H_Y = H_{Y'}.$$ 

Condition (13) implies that $Y \sim Y'$ if and only if there exists an isomorphism $g \in \text{Iso}(Y, Y')$ such that the bijection $g^{\partial Y}$ is identical. The color $\pi^\circ(e)$ of the hyperedge $e \in E_2$ is defined to be so that if $e = \{E(H_Y)\}$ and $e' = \{E(H_{Y'})\}$, then

$$(\pi^\circ(e) = \pi^\circ(e')) \iff \text{Iso}(Y, Y') \neq \emptyset \quad \text{and} \quad n_Y = n_{Y'},$$

where $n_Y$ and $n_{Y'}$ are the cardinalities of the classes of the equivalence relation $\sim$, containing $Y$ and $Y'$, respectively.

**Remark 5.3.** Let $e \in E_2$ and $Y \in \text{Conn}(X - \Omega^*)$ be such that $e = E(H_Y)$. In general, the coloring $\pi_e$ of the hyperedges of $E_1$, contained in $e$, is different from the coloring $\pi_Y$ of the corresponding hyperedges of $H_Y$. However, $\pi_e \geq \pi_Y$ and $\pi_Y$ is uniquely determined by $\pi_e$.

**Lemma 5.4.** Let $X'$ be a colored graph obtained from $X$ by deleting all edges of the induced subgraph $X_{\Omega^*}$. Then

$$\text{Aut}(H^\circ) = \text{Aut}(X')^{\Omega^*}.$$ 

Moreover, given $\overline{g} \in \text{Aut}(H^\circ)$ one can construct $g \in \text{Aut}(X')$ such that $g^{\Omega^*} = \overline{g}$ in polynomial time in $|\Omega|$.  

Proof. Let $g \in \text{Aut}(X')$. Since the set $\Omega^*$ is $\text{Aut}(X')$-invariant, the permutation $\overline{g} = g^{\Omega^*}$ preserves the coloring $\pi_{1\Omega}$. Moreover, $g$ induces a permutation
\begin{equation}
Y \mapsto Y', \quad Y \in \text{Conn}(X - \Omega^*),
\end{equation}
such that $(\partial Y)^g = \partial Y'$ for all $Y$, and the isomorphisms
\[ g_Y \in \text{Iso}(\overline{Y}, \overline{Y'}), \quad Y \in \text{Conn}(X - \Omega^*). \]
By formula (13), we have $(g_Y)^{\partial Y} = g^{\partial Y} \in \text{Iso}(H_Y, H_{Y'})$. Now, let $e \in E_1$. Then $e \in E(H_Y)$ for some $Y \in \text{Conn}(X - \Omega^*)$. It follows that
\[ e^g = e^{g_Y} \in E(H_{Y'}) \quad \text{for all } e \in E(H_Y). \]
Consequently, the permutation $\overline{g}$ preserves the hyperedges of $E_1$. Because the isomorphism $g_Y$ is color preserving, $\overline{g}$ preserves also the colors of them. Finally, the automorphism $g \in \text{Aut}(X')$ preserve the relations on the right-hand side of formula (18) and hence the permutation (19) leaves the equivalence relation $\sim$ fixed. Since $g$ induces the same permutation, we conclude that $\overline{g}$ preserves the colors of the hyperedges of $E_2$. Thus, $\overline{g} \in \text{Aut}(\mathcal{H}^\circ)$.

Conversely, let $\overline{g} \in \text{Aut}(\mathcal{H}^\circ)$. Formula (18) implies that $\overline{g}$ induces a cardinality preserving permutation of the classes of the equivalence relation $\sim$. Consequently, there is a permutation (19) such that $\text{Iso}(H_Y, H_{Y'}) \neq \emptyset$; although such a permutation is not necessarily unique, one can efficiently find at least one such permutation.

Recall that $\mathcal{E}_1 = \mathcal{E}_1$. Moreover, the hyperedges from $E(H_Y) \in E_2$ go to the edges from $E(H_{Y'}) \in E_2$. Therefore (see Remark 5.3),
\begin{equation}
\partial Y := \partial^{g_Y} \in \text{Iso}(H_Y, H_{Y'}). 
\end{equation}
By formula (13), there exists a bijection $g_Y \in \text{Iso}(\overline{Y}, \overline{Y'})$ such that
\begin{equation}
(g_Y)^{\partial Y} = \overline{g}_Y, 
\end{equation}
and this bijection can efficiently be found (Lemma 5.2(b)). Now we define a permutation $g \in \text{Sym}(\Omega)$ by setting $\alpha^g = \alpha^{g_Y}$, where $Y$ is an arbitrary element of $\text{Conn}(X - \Omega^*)$, for which $\alpha \in \Omega(\overline{Y})$. The permutation $g$ is well defined, because by (20) and (21),
\[ \alpha^{g_Y} = \alpha^{\overline{g}_Y} = \alpha^{\overline{g}} = \alpha^{g_Z} = \alpha^g \]
for all $Z \in \text{Conn}(X - \Omega^*)$ and all $\alpha \in \partial Y \cap \partial Z$. It remains to note that $g \in \text{Aut}(X')$, because $g$ moves edges of each $\overline{Y}$ to $\overline{Y'}$, and $E(X')$ is the union of the sets $E(\overline{Y})$.

The following theorem is the main result of the section, which together with Theorem 5.1 essentially provides a polynomial-time reduction of finding the group $\text{Aut}(X)$ to finding the groups $\text{Aut}(\mathcal{H}^\circ)$ and $\text{Aut}(\mathcal{H})$.

**Theorem 5.5.** In the conditions and notation of Theorem 4.2 set $G^* = G(\mathcal{H}^\circ)^{-1}$. Then
\[ \text{Aut}(X)^{\Omega^*} = \text{Aut}(\mathcal{H}^\circ) \cap G^*. \]
Moreover, every permutation $\overline{g} \in \text{Aut}(\mathcal{H}^\circ) \cap G^*$ can be lifted in polynomial time to an automorphism $g \in \text{Aut}(X)$ such that $g^{\Omega^*} = g$.

**Proof.**
By Theorem 4.2, we have $\text{Aut}(X)^{\Omega^*} \leq G^*$. Furthermore, $\text{Aut}(X) \leq \text{Aut}(X')$, where $X'$ is the graph from Lemma 5.4. By that lemma, this implies that $\text{Aut}(X)^{\Omega^*} \leq \text{Aut}(X')^{\Omega^*} = \text{Aut}(H^*)$. Thus,

$$\text{Aut}(X)^{\Omega^*} \leq \text{Aut}(H^*) \cap G^*.$$

Conversely, let $\overline{g} \in \text{Aut}(H^*) \cap G^*$. By Lemma 5.4 one can efficiently find $g \in \text{Aut}(X')$ such that $g^{\Omega^*} = \overline{g}$. Now, by Theorem 4.2 the permutation $g$ preserves the edges of $X$ contained in $E(X_{\Omega^*})$. The other edges of $X$ are exactly those in $E(X')$ and $g$ preserves them by Lemma 5.4. Thus,

$$E(X)^g = (E(X_{\Omega^*}) \cup E(X'))^g = E(X_{\Omega^*})^g \cup E(X')^g = E(X_{\Omega^*}) \cup E(X') = E(X),$$

i.e., $g \in \text{Aut}(X)$, as required. \hfill $\Box$

6. Order-$k$ Hypergraph Isomorphism: Bounded Color Classes

The goal of this section is to design an FPT algorithm for testing isomorphism of colored $k$-hypergraphs in which the sizes of vertex color classes are bounded by a fixed parameter; no assumption is made on the hyperedge color classes. The algorithm we present is a generalization of the one for usual hypergraphs [3].

**Theorem 6.1.** Let $k \geq 1$. Given two colored order-$k$ hypergraphs $H$ and $H'$, the isomorphism coset $\text{Iso}(H, H')$ can be computed in time $(b!s)^{O(k)}$, where $b$ is the maximal size of a vertex color class of $H$ and $s$ is the size of $H$. In particular, the group $\text{Aut}(H)$ can be found within the same time.

The proof of Theorem 6.1 is given at the end of the section. We start with some notation and definitions; most of them go back to those in [3]. In what follows, we fix a finite set $V$ and the decomposition of $V$ into the disjoint union of its color classes,

$$V = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m,$$

where $m \geq 1$ and $|C_i| \leq b$ for each $i$. For every higher order hyperedge $e$, we consider its projections to unions of the color classes,

$$e^{(i)} = e^{C_1 \sqcup C_2 \sqcup \cdots \sqcup C_i}, \quad 0 \leq i \leq m,$$

see Subsection 2.4. Obviously, $e^{(0)} = \emptyset$ and $e^{(m)} = e$.

**$i$-equivalence.** Let $i \in \{0, \ldots, m\}$. Two order-$k$ hyperedges $e$ and $e'$ are said to be $i$-equivalent if the multisets $e^{(i)}$ and $e'^{(i)}$ are equal. The following statement is straightforward.

**Proposition 6.2.**

1. any two high order hyperedges are 0-equivalent,
2. for $i \geq 1$, any two $i$-equivalent high order hyperedges are $(i-1)$-equivalent,
3. two high order hyperedges are $m$-equivalent if and only if they are equal.

**$i$-blocks.** Let $H = (V, E)$ be an order-$k$ hypergraph. For every $i \in \{0, \ldots, m\}$, the $i$-equivalence partitions the set $E$ into equivalence classes called $i$-blocks; the set of all of them is denoted by $\overline{E}_i$. From Proposition 6.2 it follows that

$$\overline{E}_0 = \{E\} \quad \text{and} \quad \overline{E}_m = E.$$
Hypergraphs $A[i]$ associated with $i$-blocks. Each $i$-block $A \in \hat{E}_i$ defines an order-$k$ hypergraph $(V, A)$, which is just $H$ if $i = 0$, and is essentially the order-$(k - 1)$ hypergraph $(V, e)$ if $i = m$ and $A = \{e\}$ for some $e \in E$. Denote by $A[i]$ the order-$k$ hypergraph on the set

$$V_i = C_i \sqcup C_{i+1} \sqcup \cdots \sqcup C_m,$$

obtained from the projection $A^{V_i}$ of $A$ to $V_i$ by replacing each multiset $e^{V_i}$, $e \in A$ with the corresponding set (without repetitions). Then $A[0] = H$.

Coloring of $A[i]$. Assume that the hypergraph $H$ is colored. The vertex coloring of the hypergraph $A[i]$ is defined in a natural way, whereas the color of the hyperedge corresponding to $e^{V_i}$ is defined as a multiset

$$\{\{\pi(\tilde{e}) : \tilde{e}^{V_i} = e^{V_i}, \tilde{e} \in A\},$$

where $\pi$ is the coloring of $E(H)$.

Proof of Theorem 6.1. Let $H = (V, E)$ and $H' = (V', E')$ be colored order-$k$ hypergraphs. Without loss of generality we may assume that there is a decomposition of $V'$ similar to (22) with the same $m$ and $b$. Our aim is to design an algorithm of running time $x(k, s, b) = (bls)^{O(k)}$ for computing the coset $\text{Iso}(H, H')$.

Inductively, assume that $k \geq 2$ and we have such an algorithm for order-$(k - 1)$ hypergraphs of running time $x(k - 1, s, b)$. As the base case for the induction, by [3] Corollary 9, we already have

$$x(1, s, b) = 2^{O(b)} \text{poly}(s).$$

The algorithm for order-$k$ hypergraphs will invoke as subroutine the algorithm for order-$(k - 1)$ hypergraphs. Put

$$C(k, i; H, H') = \{\text{Iso}(A[i], A'[i]) : A \in \hat{E}_i, A' \in \hat{E}_i'\}, \quad 0 \leq i \leq m.$$

The algorithm below computes the collections $C(k, i; H, H')$ for decreasing values of $i$ from $m$ down to $0$. Specifically, for each $i$, it first computes the set $C(k, i + 1; H, H')$ and uses it for computing the set $C(k, i; H, H')$. Since $A[0] = H$ and $A'[0] = H'$, notice that we will finally have computed $\text{Iso}(H, H') = C(k, 0; H, H')$ as required.

Algorithm for computing $C(k, 0; H, H')$

Input: colored order-$k$ hypergraphs $H = (V, E)$ and $H' = (V', E')$, $k > 1$.

Output: the table of all $C(k, i; H, H')$, $0 \leq i \leq m$.

For $i := m$ down to 1 do

for all $A \in \hat{E}_i$ and $A' \in \hat{E}_i'$ add to $C(k, i; H, H')$ the coset $\text{Iso}(A[i], A'[i])$ computed below.

Step 0. If $i = m$

then $A[i]$ and $A'[i]$ are order-$k$ hypergraphs on the sets $C_m$ and $C'_m$ of cardinality at most $b$. In this case, $\text{Iso}(A[i], A'[i])$ can be computed in time $O(b^{s})$ by inspecting all bijections from $V_i$ to $V'_i$.

else

Step 1. Construct the $(k - 1)$-skeleton hypergraphs $Y = A[i]^{(k-1)}$ and $Y' = A'[i]^{(k-1)}$ (see Section 2.4).
Step 2. Compute $K\tau := \text{Iso}(Y, Y') = C(k - 1, 0; Y, Y')$ by using the algorithm for order-$(k - 1)$ hypergraphs as subroutine.

Step 3. Computation of $\text{Iso}(A[i], A'[i])$:

Step 3.1. Let $A_1, A_2, \ldots, A_i$ and $A'_1, A'_2, \ldots, A'_i$ be the $(i+1)$-blocks contained in $A$ and $A'$, respectively; if $\ell \neq \ell'$, then set $\text{Iso}(A[i], A'[i]) = \emptyset$.

Step 3.2. Find the set $P \leq \text{Sym}(\ell)$ of all permutations induced by $K\tau$ as the bijections from $C_{i+1}$ to $C'_{i+1}$ which maps the set $\{A_1, A_2, \ldots, A_i\}$ to $\{A'_1, A'_2, \ldots, A'_i\}$; note that $|P| \leq b!$.

Step 3.3. Using the algorithm in [3, Theorem 5], compute the coset

$$\text{Iso}(A[i], A'[i]) = \bigcup_{P \in P} \bigcup_{j=1}^{\ell} \text{Iso}(A_j[i+1], A'_j[i+1]),$$

where the cosets on the right-hand side are available from the set $C(k, i+1; H, H')$ found earlier.

end-for

Correctness and Analysis. By induction, it suffices to see how Step 3 computes $\text{Iso}(A[i], A'[i])$. Notice that the union on the right-hand side of (25) with $P$ replaced by the set of all bijections from $C_{i+1}$ to $C'_{i+1}$ gives the coset $H\nu$ of all isomorphisms from $A[i]$ to $A'[i]$ projected to $V_{i+1}$ and $V'_{i+1}$. Since $A$ and $A'$ are $i$-blocks, they are single order-$k$ hyperedges on color class $C_i$ and $C'_i$, respectively. In view of formula (1), the coset $K\tau = \text{Iso}(Y, Y')$ restricted to $C_i$ and $C'_i$ precisely includes all the isomorphisms from $A[i]$ to $A'[i]$ restricted to $C_i$ and $C'_i$. Hence, $K\tau \cap H\nu$ is precisely $\text{Iso}(A[i], A'[i])$ which is computed at Steps 3.3.

We analyze the running time $x(k, s, b)$ for the computation of the set $C(k, 0; H, H')$. The outer for-loop executes $m$ times and the inner for-loop executes at most $|E|^2$ times (for each pair $A, A'$ of $i$-blocks).

We now bound the time required for computing each $C(k, i; H, H')$. By induction, each iteration of Steps 0-2 require time

$$O(|E|^2 \cdot b! + x(k - 1, s, b) + \text{poly}(s)).$$

The number $\ell$ in Step 3.1 is at most $|E|$. Therefore the cost of Steps 3.1-3.2 is at most $|E| |P| \text{poly}(s) \leq b! \text{poly}(s)$. Finally, in Step 3.3, we compute at most $b!$ intersections of $\ell$ cosets available in the already computed set $C(k, i+1, H, H')$. Since the intersection of two such cosets by the algorithm from [3, Theorem 5] requires $2^{O(b)} \cdot \text{poly}(s)$ time, the overall cost of Step 3 is at most $O(b!) \text{poly}(s)$. Putting it together, the time spent in computing $C(k, i; H, H')$, given the pre-computed table entries for $C(k, i+1, H, H')$, is bounded by $x(k - 1, s, b) \cdot O(b!) \text{poly}(s)$. It follows that the overall time for computing $C(k; H, H')$ is bounded by $m \cdot |E|^2 \cdot x(k - 1, s, b) \cdot O(b!) \text{poly}(s)$. Thus, we have

$$x(k, s, b) \leq m \cdot |E|^2 \cdot x(k - 1, s, b) \cdot O(b!) \text{poly}(s) \leq x(k - 1, s, b) \cdot (b! \cdot s)^c,$$

for a suitable constant $c > 0$. By induction hypothesis $x(k - 1, s, b) \leq (b! \cdot s)^{c(k-1)}$. Hence, we obtain an overall upper bound of $(b! \cdot s)^{c(k-1)}$ for the running time of the algorithm for order-$k$ hypergraphs. 

□
7. **Main algorithm and the proof of Theorem 7.2**

Based on the results obtained in the previous sections, we present an algorithm that constructs the automorphism group of a chordal twinless graph.

**Main Algorithm**

- **Input:** a chordal twinless graph $X$ and vertex coloring $\pi$ of $X$.
- **Output:** the group $\text{Aut}(X, \pi)$.

**Step 1.** Construct $\pi = \text{WL}(X, \pi)$ and $\Omega^* = \Omega^*(X, \pi)$.

**Step 2.** While the set $\Omega^*$ is not critical with respect to $\pi$, find $\pi := \text{WL}(X, \pi')$ and set $\Omega^* := \Omega^*(X, \pi)$, where $\pi'$ is the coloring from Theorem 4.1(ii).

**Step 3.** If $\Omega^* = \emptyset$, then $X$ is interval and we output the group $\text{Aut}(X, \pi)$ found by the algorithm from [20, Theorem 5].

**Step 4.** Construct the mapping $f$ and colored hypergraph $\mathcal{H}^*$ on $(\Omega^*)^f$, defined in Section 4, and the colored hypergraph $H^\circ$ on $\Omega^*$, defined in Section 5.

**Step 5.** Using the algorithm from Theorem 6.1, find a generating set $\mathcal{S}$ of the automorphism group of the colored order-3 hypergraph $\mathcal{H}^* \uparrow (H^\circ)^f$.

**Step 6.** For each $g \in \mathcal{S}$ find a lifting $g \subset \text{Aut}(X, \pi)$ of $[\overline{\mathcal{S}}]_f \subset \text{Sym}(\Omega^*)$ by the algorithm from Theorem 6.3 let $S$ be the set of all these automorphisms $g$'s.

**Step 7.** Output the group $\text{Aut}(X, \pi) = \langle G^\circ, S \rangle$, where $G^\circ$ is the group defined in Theorem 5.1.

**Theorem 7.1.** The Main Algorithm correctly finds the group $\text{Aut}(X, \pi)$ in time $t(\ell) \cdot n^{O(1)}$, where $n = |\Omega(X)|$, $t$ is a function independent of $n$, and $\ell = \ell(X)$.

**Proof.** Note that the number of iterations of the loop at Step 2 is at most $n$, because $|\pi| \leq n$ and $|\pi'| > |\pi|$. Next, the running time at each other step, except for Step 5, is bounded by a polynomial in $n$, see the time bounds in the used statements. On the other hand, at Step 5, the cardinality of each vertex color class of the order-3 hypergraph $\mathcal{H}^* \uparrow (H^\circ)^f$ is at most $\ell 2^\ell$ (Theorem 4.1(ii)). By Theorem 6.1 for $b = \ell 2^\ell$ and $k = 3$, the running time of the Main Algorithm is at most $t(\ell) \cdot n^{O(1)}$ with $t(\ell) = (\ell 2^\ell)!^{O(1)}$.

To prove the correctness of the algorithm, we exploit the natural restriction homomorphism

$$
\varphi : \text{Aut}(X) \rightarrow \text{Sym}(\Omega^*), \quad g \mapsto g^{\Omega^*}.
$$

Given a generating set $S'$ of the group $\text{Im}(\varphi)$, we have $\text{Aut}(X) = \langle \ker(\varphi), S \rangle$, where $S \subseteq \text{Aut}(X)$ is a set of cardinality $|S|$ such that $S' = \{\varphi(g) : g \in S\}$.

According to Step 7, $\ker(\varphi) = G^\circ$. Thus, it suffices to verify that as the set $S'$ one can take the set $\{f[\overline{\mathcal{S}}]_f^{-1} : \mathcal{S} \in \overline{\mathcal{S}}\}$, where $f$ is the bijection found at Step 4 and $\mathcal{S}$ is the generating set of the group $\text{Aut}(\mathcal{H}^* \uparrow (H^\circ)^f)$, found at Step 5. By Theorem 5.5 we need to check that

$$
\text{Aut}(\mathcal{H}^* \uparrow (H^\circ)^f)^{-1} = G^* \cap \text{Aut}(H^\circ).
$$

Notice that

$$
h \in \text{Aut}(\mathcal{H}^* \uparrow (H^\circ)^f) \iff h \in \text{Aut}(\mathcal{H}^*) \quad \text{and} \quad (E(H^\circ)^f)^h = E(H^\circ)^f
$$

$$
\iff fhf^{-1} \in G^* \quad \text{and} \quad fhf^{-1} \in \text{Aut}(H^\circ)
$$

$$
\iff fhf^{-1} \in G^* \cap \text{Aut}(H^\circ),
$$

where $E(A)$ is the edge set of the graph $A$. This completes the proof.

\[\square\]
which proves equality \[26\].

Proof of Theorem 1.2. Denote by \(e_X\) the equivalence relation on \(\Omega = \Omega(X)\) such that \((\alpha, \beta) \in e_X\) if and only if the vertices \(\alpha\) and \(\beta\) are twins in \(X\). Since \(e_X\) is \(\text{Aut}(X)\)-invariant, there is a natural homomorphism

\[\varphi : \text{Aut}(X) \to \text{Sym}(\Omega/e_X)\]

To find the group \(\text{Aut}(X)\), it suffices to construct generating sets of the groups \(\ker(\varphi)\) and \(\text{Im}(\varphi)\), and then to lift every generator of the latter to an automorphism of \(X\).

First, we note that every class of the equivalence relation \(e_X\) consists of twins of \(X\). Consequently,

\[\ker(\varphi) = \prod_{\Delta \in \Omega/e_X} \text{Sym}(\Delta),\]

and this group can efficiently be found.

Now let \(X'\) be the graph with vertex set \(\Omega/e\), in which the classes \(\Delta\) and \(\Gamma\) are adjacent if and only if some (and hence each) vertex in \(\Delta\) is adjacent to some (and hence each) vertex of \(\Gamma\). Note that \(X'\) is isomorphic to an induced subgraph of \(X\), and hence belongs to the class \(K_\ell\). Let \(\pi'\) be the vertex coloring of \(X'\) such that \(\pi'(\Delta) = \pi'(\Gamma)\) if and only if \(X_\Delta\) and \(X_\Gamma\) are isomorphic, which is easy to check because each of \(X_\Delta\) and \(X_\Gamma\) is either empty or complete. Then

\[\text{Im}(\varphi) = \text{Aut}(X', \pi'),\]

and this group can efficiently be found in time \(t(\ell) \cdot n^{O(1)}\) by Theorem 7.1.

To complete the proof, we need to show that given \(g' \in \text{Aut}(X', \pi')\), one can efficiently find \(g \in \text{Aut}(X)\) such that \(\varphi(g) = g'\). To this end, choose an arbitrary bijection \(g_\Delta : \Delta \to \Delta^{\pi'}\); recall that \(\pi'(\Delta) = \pi'(\Delta^{\pi'})\) and so \(|\Delta| = |\Delta^{\pi'}|\). Then the mapping \(g\) taking a vertex \(\alpha \in \Omega\) to the vertex \(\alpha^{g_\Delta}\), where \(\Delta\) is the class of \(e_X\), containing \(\alpha\) is a permutation of \(\Omega\). Moreover, from the definition of \(e_X\), it follows that \(g \in \text{Aut}(X)\). It remains to note that \(g\) can efficiently be constructed. \(\Box\)

8. Concluding Remarks

In this paper we have presented an isomorphism testing algorithm for \(n\)-vertex chordal graphs of leafage \(\ell\) which has running time \(t(\ell) \cdot n^{O(1)}\), where \(t(\ell)\) is a double exponential function not depending on \(n\). A natural question is to improve the running time dependence on the leafage.

The other problem of interest is isomorphism testing of order-\(k\) hypergraphs for \(b\)-bounded color classes. Can we obtain an \(FPT\) algorithm with both \(k\) and \(b\) as parameters, or with \(k\) as parameter for fixed \(b\)?

9. Acknowledgements

We thank the anonymous referees of an earlier version for their valuable comments and corrections. Roman Nedela was supported by GAČR 20-15576S. Peter Zeman was supported by GAČR 20-15576S, GAUK 1224120, and by the Charles University project PRIMUS/21/SCI/014.
References

1. D. Agaoglu and P. Hlinéný, Isomorphism problem for $\omega d$-graphs, in: 45th International Symposium on Mathematical Foundations of Computer Science, MFCS (2020), pp. 4:1–4:14.
2. D. Agaoglu and P. Hlinéný, Isomorphism testing for $T$-graphs is FPT, https://arxiv.org/abs/2111.10910.
3. V. Arvind, B. Das, J. Köbler, and S. Toda, Colored hypergraph isomorphism is fixed parameter tractable, Algorithmica, 71(1):120–138 (2015).
4. L. Babai, Groups, Graphs, Algorithms: The Graph Isomorphism Problem, Proc. ICM 2018, Rio de Janeiro, Vol. 3, 3303–3320.
5. M. Biro, M. Hujter, and Z. Tuza, Precoloring extension. I. Interval graphs, Discrete Math., 100, no. 1-3, 267–279 (1992).
6. N. Brand, Isomorphisms of cyclic combinatorial objects, Discrete Math., 78, 73–81 (1989).
7. S. Chaplick, M. Toepfer, J. Voborník, and P. Zeman, On $H$-topological intersection graphs, in: Graph-theoretic concepts in computer science, Lecture Notes in Comput. Sci., 10520, Springer, Cham (2017), pp. 167–179.
8. S. Chaplick and P. Zeman, Combinatorial problems on $H$-graphs, Electron. Notes Discrete Math., 61, 223–229 (2017).
9. S. Chaplick and P. Zeman, Isomorphism-completeness for $H$-graphs, https://kam.mff.cuni.cz/~pizet/gic.pdf (2021).
10. G. Chen and I. Ponomarenko, Coherent Configurations, Central China Normal University Press, Wuhan (2019); a draft is available at http://www.pdmi.ras.ru/~inp/ccNOTES.pdf.
11. C. J. Colbourn and K. S. Booth, Linear time automorphism algorithms for trees, interval graphs, and planar graphs, SIAM J. Comput., 10(1):203–225 (1981).
12. S. Evdokimov, I. Ponomarenko, and G. Tinhofer, Forestal algebras and algebraic forests (on a new class of weakly compact graphs), Discrete Math., 225(1-3):149–172 (2000).
13. F. V. Fomin, P. A. Golovach, and J.-F. Raymond, On the tractability of optimization problems on $H$-graphs, Algorithmica, 82, no. 9, 2432–2473 (2020).
14. F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, J. Combin. Theory, B16, no. 1, 47–56 (1974).
15. M. Grohe, D. Neuen, P. Schweitzer, and D. Wiebking, An improved isomorphism test for bounded-tree-width graphs, ACM Trans. Algorithms, 16, no. 3, 34:1–34:31 (2020).
16. M. Grohe, D. Neuen, and D. Wiebking, Isomorphism testing for graphs excluding small minors, in 61st IEEE FOCS, Durham, NC, USA (2020), pp. 625–636.
17. M. Habib and J. Stacho, Polynomial-Time Algorithm for the Leafage of Chordal Graphs, Proceedings European Symposium on Algorithms (ESA 2009), pp. 290–300.
18. I.-J. Lin, T. A. McKee, and D. B. West, The leafage of a chordal graph, Discuss. Math. Graph Theory, 18, no. 1, 23–48 (1998).
19. D. Lokshtanov, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, Fixed-parameter tractable canonization and isomorphism test for graphs of bounded treewidth, SIAM J. Comput., 46, no. 1, 161–189 (2017).
20. G. S. Lueker and K. S. Booth, A linear time algorithm for deciding interval graph isomorphism, J. ACM, 26(2):183–195 (1979).
21. E. M. Luks, Isomorphism of graphs of bounded valence can be tested in polynomial time, J. Comput. Syst. Sci., 25, no. 1, 42–65 (1982).
22. D. Neuen, Hypergraph isomorphism for groups with restricted composition factors, in: 47th International Colloquium on Automata, Languages, and Programming, ICALP, 168, Schloss Dagstuhl-Leibniz-Zentrum für Informatik (2020), pp. 88:1–88:19.
23. I. Ponomarenko, Polynomial isomorphism algorithm for graphs which do not pinch to $K_{3,3}$, J. Soviet Math., 34(4):1819–1831 (1986).
24. I. Ponomarenko, The isomorphism problem for classes of graphs closed under contraction, J. Soviet Math., 55(2):1621–1643 (1991).
25. A. Seress, Permutation Group Algorithms, Cambridge Tracts in Mathematics, 152, Cambridge University Press (2003).
26. J. Stacho, On 2-subcolourings of chordal graphs, Lecture Notes in Computer Science, 4957, 544–554 (2008).
27. B. Weisfeiler and A. Leman, *Reduction of a graph to a canonical form and an algebra which appears in the process*, NTI, Ser. 2 (1968), no. 9, 12–16 (Russian); English translation is available at [https://www.iti.zcu.cz/wl2018/pdf/wl_paper_translation.pdf](https://www.iti.zcu.cz/wl2018/pdf/wl_paper_translation.pdf)

The Institute of Mathematical Sciences (IHNI), Chennai, India  
*Email address: arvind@imsc.res.in*

Faculty of Applied Sciences, University of West Bohemia, Technická 8, Pilsen, Czech Republic  
*Email address: nedela@savbb.sk*

V. A. Steklov Institute of Mathematics, Russian Academy of Sciences, St. Petersburg, Russia  
*Email address: inp@pdmi.ras.ru*

Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic  
*Email address: zeman@kam.mff.cuni.cz*