ON HAUSDORFF DIMENSION OF THE SET OF NON-ERGODIC DIRECTIONS OF TWO-GENUS DOUBLE COVER OF TORI

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Abstract. Cheung, Hubert and Masur [Invent. Math., 183(2011), no.2, pp. 337-383] proved that the Hausdorff dimension of the set of nonergodic directions of billiards in a kind of rectangle with barrier is either 0 or \( \frac{1}{2} \). As an application of their argument, we prove that there exist the third-kind two-genus double covers of tori in which the set of minimal and non-ergodic directions have Hausdorff dimension \( \frac{1}{2} \).

1. Introduction. A translation surface has a collection of directional flows which preserves the Lebesgue measure induced by its translation structure. It is classical that the subset of ergodic directions is of full measure. See [7].

For quantifying the subset of non-ergodic directions further, it is naturally to consider the Hausdorff dimension. In fact, the Hausdorff dimension provides a functor on the moduli space of translation surfaces, which is invariant under the action of \( G = \text{GL}_+ (2, \mathbb{R}) \) and is constant almost everywhere with respect to its natural measure. See [8].

Non-ergodic directions in two-genus translation surfaces are studied extensively. In [4], Cheung and Masur proved that for a two-genus translation surface either it is a Veech surface, or it has uncountable non-ergodic directions. On the other hand, the Hausdorff dimension of the set of non-ergodic directions is equal to \( \frac{1}{2} \) for almost every translation surface in the stratum \( \mathcal{H}(2) \). See [1].

For any \( \lambda \in (0, 1) \) let \( Q_\lambda \) be a \( \frac{1}{2} \)-by-1 rectangle with a horizontal barrier of length \( \frac{1-\lambda}{2} \) initiated from the middle point of the vertical side. There is a standard procedure to turn billiards in \( Q_\lambda \) into the directional flows on a translation surface. Denote the translation surface by \( X_\lambda \). It is easy to see that \( X_\lambda \) be the resulting surface by gluing two copies of \( T^2 \) together along a horizontal geodesic segment of length \( \lambda \). See [2], [3] and [5]. It seems that by now \( X_\lambda \) is the only translation surface, of which the Hausdorff dimension of the set of non-ergodic directions is calculated concretely, except Veech surfaces.

Let \( \mathcal{E} \) be the space of two-genus translation surfaces, which is a double cover of a flat torus. Then \( \mathcal{E} \subset \mathcal{H}(1,1) \) is an affine invariant sub-manifold of complex dimension 3. Currently, the study of affine invariant sub-manifold is active. See

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It should be interesting to consider the Hausdorff dimension functor restricting on affine invariant sub-manifolds.

A saddle connection $\gamma$ of a two-genus translation surface is said to be \textit{separating} if the complement of $\gamma \cup j(\gamma)$ has two connected components, where $j$ is the hyper-elliptic involution. The corresponding direction is also said to be \textit{separating}. It is known that a two-genus translation surface always has infinitely many separating directions. See [10] and [9]. Recall that a direction in a translation surface is periodic if there is a closed geodesic in this direction.

\textbf{Definition 1.1.} $X \in E$ is of the first kind if there are two separating and periodic directions; the second kind if there is only one separating and periodic direction; the third kind if there is no separating and periodic direction at all.

It is obviously that the classification of surfaces in $E$ is invariant under the $G$-action. Moreover, $X \in E$ is of the first kind if and only if $X$ is a Veech surface, which means that its non-ergodic directions are countable; the second kind if and only if $X$ belongs to the $G$-orbit of $X_\lambda$ for a unique irrational $\lambda$. In 2011 Cheung, Hubert and Masur [3] proved that the set of nonergodic directions on $X_\lambda$ has Hausdorff dimension either $\frac{1}{2}$ or 0, depending on the sequence of $k$-th convergent of $\lambda$ satisfies the Pérez-Marco condition

$$\sum \log \log q_{k+1} < \infty$$

or not. Hence it would be desirable to know whether the set of non-ergodic directions on any double cover $X \in E$ has Hausdorff dimension either $\frac{1}{2}$ or 0. In fact, we get the following result:

\textbf{Theorem 1.2.} \textit{There exist $X \in E$ of the third kind such that the Hausdorff dimension of the set of its minimal and non-ergodic directions is equal to $\frac{1}{2}$}.

The paper is organized as following: In Section 2 one classifies two-genus double covers of tori; Theorem 3.1 is proved in Section 3 under the existence of good trees of slits; Section 4 is devoted to prove the existence of such trees.

2. \textbf{Double cover of tori.} To generalize $X_\lambda$, denote by $X_\phi$, $0 \leq \Re \phi, \Im \phi < 1$, the resulting surface of gluing two copies of the flat torus $T^2$ along an oriented geodesic segment with two different end points, of which the holonomy is equal to $\phi$. See Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure1.png}
\caption{The double cover}
\end{figure}

\textbf{Proposition 1.} \textit{Suppose that $X \in E$. Then $X$ belongs to the $G$-orbit of $X_\phi$ for some $\phi$.}
Proof. The famous Riemann-Hurwitz’s formula implies that \( \mathcal{E} \subset \mathcal{H}(1,1) \). Since the moduli space of one-genus translation surfaces \( \mathcal{H}_1 = G/SL(2,Z) \), there exists \( Y \) in the \( G \)-orbit of \( X \) such that \( Y \) is a double cover of the flat torus \( T^2 \). Without loss of generality, assume that the images of two singular points of \( Y \) are \( [0] \) and \( [\phi] \).

As shown in Figure 2, it turns out that either \( Y = X_\phi \), \( Y = X_{1-\phi} \), \( Y = X_{i-\phi} \), or \( Y = X_{1+i-\phi} \).

Corollary 1. Let \( X \in \mathcal{E} \) be a double cover. Then \( X \) is of the first kind if and only if it is Veech surface; \( X \) is of the second kind if and only its \( G \)-orbit contain \( X_\lambda \) with \( \lambda \not\in \mathbb{Q} \), and such \( \lambda \) is unique.

A slit of \( X_\phi \) is defined to be the holonomy of an oriented separating saddle connection in \( X_\phi \). Denote by \( S(\phi) \) the set of all slits of \( X_\phi \).

Proposition 2. If \( \phi \not\in \mathbb{Q}[i] \), then \( S(\phi) = \{ \pm \phi + 2v : v \in \mathbb{Z}[i] \} \).

Proof. Cut the flat torus \( T^2 \) with two marked points \( [0] \) and \( [\phi] \) along three saddle connections with holonomies \( \phi, 1 + i - \phi \) and \( 1 - \phi \). one gets a hexagon. Because \( X_\phi \) is a double cover of \( T^2 \) branched at \( [0] \) and \( [\phi] \). \( X_\phi \) can be realized as two copies of the hexagons gluing together along the side.

Take any \( \phi + m + in \), where \( m + in \in \mathbb{Z}[i] \). Cutting the two hexagons along the geodesic line determined by \( \phi + m + in \) results a finite number of pieces of polygon. To get \( X_{\phi+m+in} \), at first glue these polygons along the sides determined by \( \phi + m + in \) to get two new hexagons; next glue these new hexagon along sides determined by \( \phi, 1 + i - \phi \) and \( 1 - \phi \). See Figure 3. Then some tedious manipulation yields that \( X_{\phi+m+in} = X_\phi \) if and only \( m + in \in 2\mathbb{Z}[i] \).

Notation. For any \( w \in S(\phi) \) denoted by \( \Re(w) \) the real part of \( w \) and \( \Im(w) \) its imaginary part; denote by \( |w| \), named the height of \( w \), the absolute value of \( \Im(w) \); and denote by \( \alpha(w) \), named the inverse slope of \( w \), the quotient \( \frac{\Re(w)}{\Im(w)} \).
3. **Proof of Theorem 1.2.** Let $NE(\phi)$ be the set of inverse slopes of minimal and non-ergodic directions in $X_\phi$ and let $H\dim(\phi)$ be the Hausdorff dimension of $NE(\phi)$. Recall that an irrational number $\alpha$ is Diophantine if there exist constants $c > 0$ and $e > 0$ such that
\[ |\alpha - \frac{p}{q}| \geq \frac{c}{q^e} \]
for any integers $p$ and $q > 0$.

**Theorem 3.1.** Suppose that $\Im(\phi)$ is a reduced rational number $\frac{\xi}{\eta}$. If $\Re(\eta\phi)$ is Diophantine, then $H\dim(\phi) = \frac{1}{2}$.

**Remark 1.** If $\Im\phi = 0$, i.e. $\xi = 0$ and $\eta = 1$, Theorem 3.1 is just Theorem 2 in [2]; if otherwise, $X_\phi$ is of the third kind. Hence Theorem 3.1 implies the main theorem.

**Proposition 3.** Suppose that $w_j \in S(\phi)$ with $|w_j| > |w_j+1|$ for any $j \geq 0$. If
\[ \sum |w_j \times w_{j+1}| < \infty, \quad (1) \]
then $\alpha_{w_j}$ converges to some $\alpha \in NE(\phi)$.

**Proof.** Since
\[
|\alpha_{w_j} - \alpha_{w_{j+1}}| \leq |\alpha_{w_j} - \alpha_{w_{j+1}}| + \ldots + |\alpha_{w_{j+l}} - \alpha_{w_{j+l+1}}|
\]
\[
= \frac{|w_j \times w_{j+1}|}{|w_j||w_{j+1}|} + \ldots + \frac{|w_{j+l} \times w_{j+l+1}|}{|w_{j+l-1}||w_{j+l}|}
\]
\[
\leq \frac{1}{|w_j|^2} (|w_j \times w_{j+1}| + \ldots + |w_{j+l-1} \times w_{j+l}|),
\]
it follows from (1) that $\{\alpha_{w_j}\}$ is a Cauchy sequence.

Without loss of generality, assume that $\lim_{j \to +\infty} \alpha_{w_j} = 0$. Using Masur and Smillie’s Theorem (c.f. [3]), the vertical direction is a non-ergodic direction provided that $\lim_{j \to +\infty} |\Re w_j| = 0$. Since $\lim_{j \to +\infty} \alpha_{w_j} = 0$ and $|w_j| \leq |w_{j+1}|$,
\[
|\alpha_{w_j}| \leq \sum_{k \geq j} |\alpha_{w_k} - \alpha_{w_{k+1}}|
\]
\[
= \sum_{k \geq j} \frac{|w_k \times w_{k+1}|}{|w_k||w_{k+1}|}
\]
\[
\leq \sum_{k \geq j} \frac{|w_k \times w_{k+1}|}{|w_j|}.
\]
Hence
\[
|\Re w_j| = |w_j||\alpha_{w_j}|
\]
\[
\leq \sum_{k \geq j} |w_k \times w_{k+1}|.
\]
Using (1) again, $\lim_{j \to +\infty} |\Re w_j| = 0$. \(\square\)

**Definition 3.2.** Let $w$ and $w'$ be two slits in $S(\phi)$. For $\delta > 0$, $w'$ is a $\delta$-child of $w$ if
\[
|q\alpha_w - p| \leq \frac{1}{|w| \log |w|} \quad (2)
\]
and

\[ |w|^{1+\delta} \leq q \leq 2|w|^{1+\delta}, \quad (3) \]

where \( p \) and \( q \) are co-prime integers such that \( w' = w + 2(p + iq) \) and \( \Im(qw) > 0 \).

**Proposition 4.** Suppose that \( w' \) and \( w'' \) are two different \( \delta \)-children of \( w \in \mathcal{S}(\phi) \). Then \(|\alpha_{w'} - \alpha_{w''}| \geq \frac{1}{8|w|^{2+2\delta}}\) provided that \(|w|\) is large enough.

**Proof.** Assume that \( w' = w + 2(p + iq) \) and \( w'' = w + 2(p' + iq') \). Since \( w' \neq w'' \),

\[ \left| \frac{p}{q} - \frac{p'}{q'} \right| \geq \frac{1}{|qq'|} \geq \frac{1}{4|w|^{2+2\delta}}. \]

On the other hand,

\[ |\alpha_{w'} - \frac{p}{q}| = \frac{|w' \times (p + iq)|}{|w'| |q|} = \frac{|w \times (p + iq)|}{(|w| + 2|q|) |q|} \leq \frac{|w||q\alpha_w - p|}{2|q|^2} \leq \frac{1}{2|w|^{2+2\delta} \log |w|} \]

and similarly,

\[ |\alpha_{w''} - \frac{p'}{q'}| \leq \frac{1}{2|w|^{2+2\delta} \log |w|}. \]

As a result,

\[ |\alpha_{w'} - \alpha_{w''}| \geq |\alpha_{w'} - \frac{p}{q} + \frac{p}{q} - \frac{p'}{q'} + \frac{p'}{q'} - \alpha_{w''}| \geq \frac{1}{4|w|^{2+2\delta}} - \frac{2}{2|w|^{2+2\delta} \log |w|} \]

\[ = \frac{1}{|w|^{2+2\delta}} (1/4 - 1/ \log |w|) \geq \frac{1}{8|w|^{2+2\delta}} \]

for sufficiently large \(|w|\).

**Definition 3.3.** For any \( \delta > 0 \), a \( \delta \)-tree \( V \) of slits is a disjoint union of subsets \( V_j, j \geq 0 \), of \( \mathcal{S}(\phi) \) such that there is a unique 0-level slit and each \((j + 1)\)-level slit is a \( \delta \)-child of some \( j \)-level slit. And \( V \) is good if there exists a constant \( C \) such that each slit \( w \in V \) has at least \( C|w|^{\delta}/|w| \) \( \delta \)-children in \( V \).

For any \( w \in \mathcal{S}(\phi) \) denote by \( I(w) \) the closed interval centered at \( \alpha_w \) with diameter \( \frac{4}{|w|^{2+\delta}} \). For any \( \delta \)-tree \( V = \bigcup V_j \) of slits denote \( K(V) = \bigcap_j K(V_j) \), where \( K(V_j) = \bigcup_{w \in V_j} I(w) \).
Proposition 5. Let $V$ be a $\delta$-tree and $|w_0|$ its 0-level slit. Then

(i) $K(V) \subset \text{NE}(\phi)$; furthermore,

(ii) if $V$ is good and $|w_0|$ is sufficiently large, $H.\text{dim}K(V) \geq \frac{1}{2+3\delta+\delta^2}$.

Proof. As mentioned above, for any $\alpha \in K(V)$ there exist a sequence of $j$-level slits $w_j$ such that $w_{j+1}$ is a $\delta$-child of $w_j$ and $\alpha \in I(w_j)$. Suppose that $w_j+1 = w_j + 2(p_j + iq_j)$. Since $I(w_j) = 4|p_j| (1+\delta)^2$,

$$|\alpha_{w_j} - \alpha| \leq \frac{2}{|w_j|^{2+\delta}}.$$  

Hence $\lim_{j \to +\infty} \alpha_{w_j} = \alpha$. On the other hand,

$$|w_j \times w_{j+1}| = 2|w_j \times (p_j + iq_j)| = 2|w_j||q_j \alpha_{w_j} - p_j| \leq \frac{2}{\log |w_j|} \leq \frac{2}{\log |w_{j-1}|^{1+\delta}} \leq \frac{2}{(1+\delta)^2 \log |w_0|}.$$  

Therefore, $\sum |w_j \times w_{j+1}| < +\infty$ which implies that $\alpha \in \text{NE}(\phi)$ by Proposition 3.

If $w' = w + 2(p + iq)$ is a $\delta$-child of $w$, then it follows that $|w'| = |w| + 2q \geq 2|w|^{1+\delta}$ and

$$|\alpha_{w'} - \alpha_w| = \frac{|w \times w'|}{|w||w'|} = \frac{2|w \times (p + iq)|}{|w||w'|} = \frac{2|q \alpha_{w} - p|}{|w'|} \leq \frac{1}{|w|^{2+\delta}}.$$  

Hence $I(w') \subset I(w)$. Assume that $w'' = w + 2(p' + iq')$ is another child of $w$. Let $d(I(w'), I(w''))$ be the distance between $I(w')$ and $I(w'')$. Then Proposition 4 implies that

$$d(I(w'), I(w'')) = |\alpha_{w'} - \alpha_{w''}| - \frac{2}{|w'|^{2+\delta}} - \frac{2}{|w''|^{2+\delta}} \geq \frac{4}{8|w|^{2+2\delta}} - \frac{2}{|w'|^{2+\delta}} - \frac{2}{|w''|^{2+\delta}} \geq \frac{4}{8|w|^{2+2\delta}} - \frac{2}{|w'|^{2+\delta}} - \frac{2}{|w''|^{2+\delta}} \geq \frac{4}{8|w|^{2+2\delta}} - \frac{4}{(2|w|^{1+\delta})^{2+\delta}} \geq 16|w|^{2+2\delta},$$  

which completes the proof.
provided that $|w|^{\delta(1+\delta)}$ is larger than 16.

Assume that $w \in V_j$ is a slit of level $j$ and $w'$ and $w''$ are two $\delta$-children in $V_{j+1}$ of $w$. Since there is a unique 0-level slit $w_0$ in $V$ and each $(j+1)$-level slit is a $\delta$-child of some $j$-level slit, it turns out that $K(V_j)$ is a disjoint union of finitely many closed intervals such that $K(V_{j+1}) \subset K(V_j)$ and

$$|w_0|^{(1+\delta)^j} \leq |w| \leq (5^{\frac{1}{\delta}}|w_0|)^{(1+\delta)^j} \leq |w_0|^{(1+\delta)^{j+1}},$$

provided that $5^{\frac{1}{\delta}} \leq |w_0|$. Then the distance $dist(I(w'), I(w''))$ between $I(w')$ and $I(w'')$ satisfies

$$dist(I(w'), I(w'')) \geq \frac{1}{16|w|^{2(1+\delta)}} \geq \frac{1}{16|w_0|^{2(1+\delta)^{j+2}}} := \epsilon_j.$$ 

If $V$ is good, then there is a constant $C$ such that $I(w)$ contains at least

$$\frac{C|w|^\delta}{\log |w|} \geq \frac{C}{\log |w_0|} \frac{|w_0|^{\delta(1+\delta)^j}}{(1+\delta)^{j+1}} := m_j,$$

closed intervals in $K(V_{j+1})$, provided that $|w_0|$ is large enough.

Using the classical estimation of Hausdorff dimension (c.f. [6]),

$$\text{H.dim}(K(V)) \geq \liminf_{j \to +\infty} \frac{\log(m_0 \ldots m_{j-1})}{-\log(m_j \epsilon_j)}.$$ 

Calculating directly, it follows that

$$\liminf_{j \to +\infty} \frac{\log(m_0 \ldots m_{j-1})}{-\log(m_j \epsilon_j)} = \liminf_{j \to +\infty} \frac{\delta \log |w_0| \sum_{k=0}^{j-1} (1+\delta)^k - \sum_{k=0}^{j-1} (k+1) \log(1+\delta) + j \log(C \log |w_0|)}{(1+\delta)^j \log |w_0| (2 + 3\delta + 2\delta^2) + (j+1) \log(1+\delta) + \log 16 - \log C}$$

$$= \liminf_{j \to +\infty} \frac{(1+\delta)^j \log |w_0| - j(j+1) \log(1+\delta) + j \log C - \log |w_0|}{(1+\delta)^j (2 + 3\delta + 2\delta^2) + (j+1) \log(1+\delta) + \log 16 - \log C}$$

$$= \frac{1}{2 + 3\delta + \delta^2}.$$

4. Existence of good tree. The section is devoted to the existence of good $\delta$-tree, of which the unique 0-level slit has arbitrarily large height.

**Theorem 4.1.** If $\Re(\eta \phi)$ is Diophantine, then for any $\delta > 0$ and $M > 0$ there exists a good $\delta$-tree in $S(\phi)$ such that the height of its 0-level slit is larger than $M$.

For any irrational number $\alpha$ denote by $\text{Spec}(\alpha)$ the sequence formed by heights of the convergents of $\alpha$. Let $\frac{p_k}{q_k}$ be the $k$-th convergent of $\alpha$. It is well-known that

$$\frac{1}{q_k(q_k + q_{k+1})} \leq |\alpha - \frac{p_k}{q_k}| \leq \frac{1}{q_k q_{k+1}} \quad (4)$$
and
\[ p_{k+1} + iq_{k+1} = a_{k+1}(p_k + iq_k) + p_{k-1} + iq_{k-1}, \] (5)
where \( a_k \) is the \( k \)-th partial quotient of \( \alpha \). Conversely, if \( p \) and \( q \) are integers satisfying
\[ |\alpha - \frac{p}{q}| \leq \frac{1}{2q^2}, \] (6)
then \( \frac{p}{q} \) is a convergent of \( \alpha \), where \( \frac{p}{q} \) is unnecessarily reduced.

The hypothesis that \( \mathcal{R}(\eta\phi) \) is Diophantine implies that there are constants \( c_0 > 0 \) and \( e_0 > 0 \) such that
\[ |w \times (p + iq) - n| \geq \frac{c_0}{|q|^{e_0}} \] (7)
for any \( w \in \mathcal{S}(\phi) \), \( p + iq \in \mathbb{Z}[i] \) with \( |q| > 0 \) and \( n \in \mathbb{Z} \). Fix a real number \( N \) such that \( e_0 < N\delta \).

**Definition 4.2.** A slit \( w \in \mathcal{S}(\phi) \) is normal if for every real number \( n, 1 \leq n \leq 1 + N, \ Spec(\alpha_w) \cap [e^{n\delta}|w|\log|w|, |w|^{1+n\delta}] \neq \emptyset \); a slit \( w \in \mathcal{S}(\phi) \) is \( n \)-good, \( n > 0 \), if \( \ Spec(\alpha_w) \cap [e^{n\delta}|w|\log|w|, |w|^{1+n\delta}] \neq \emptyset \).

**Proposition 6.** A \( N \)-good slit \( w \in \mathcal{S}(\phi) \) is normal if \( |w| \) is sufficiently large.

**Proof.** By definition a \( (N+1) \)-good slit is always normal. So it suffices to prove the proposition in the case that \( w \) is not \( (N+1) \)-good.

Let \( q_k \) be the largest in \( \ Spec(\alpha_w) \cap [1, |w|^{1+\delta}] \). Then \( q_k = e^{n_1\delta}|w|\log|w| \) with \( N \leq n_1 < N+1 \). According to the right hand of (4) and the Diophantine condition (7), it follows that \( q_{k+1} \leq |q_k \alpha - p_k|^{-1} \leq (1/c_0)|w|^c_0 \leq (1/c_0)|w|(e^{N+1}|w|\log|w|)^{e_0} \). On the other hand, \( \lim_{|w| \to +\infty} |w|^{1-N\delta} \log|w| = 0 \). Therefore, \( q_{k+1} \leq |w|^{1+N\delta} \) provided that \( |w| \) is large enough.

Since \( N \leq n_1 \), it is clear that \( q_{k+1} \in [e^{n\delta}|w|\log|w|, |w|^{1+n\delta}] \) for any \( n_1 < n \leq (N+1) \). It follows that \( w \) is normal.

**Proposition 7.** For any \( M > 0 \) there exist normal slits \( w \in \mathcal{S}(\phi) \) with \( |w| \geq M \).

**Proof.** By Proposition 6, it suffices to prove the existence of \( N \)-good slits with arbitrarily large height.

At first, assert that there exists a slit \( w_1 \) with \( |w_1| > M \) such that \( \ Spec(\alpha_{w_1}) \cap [1, |w_1|^{1+\delta}] \neq \emptyset \). Take a slit \( w_2 \) with \( |w_2| > M \). Suppose that \( \ Spec(\alpha_{w_2}) \cap [1, |w_2|^{1+\delta}] = \emptyset \). Set \( w_1 = w_2 + 2m(p + iq) \), where \( m \) is a positive integer and \( (p, q) \) is a convergent of \( \alpha_{w_2} \). Then \( |q| \leq |w_1|^{1+\delta} \) for \( m \) large enough. On the other hand, the RHS of (4) implies that
\[ |\alpha_{w_1} - \frac{p}{q}| = \frac{|w_1 \times (p + iq)|}{|w_1||q|} = \frac{|w_2 \times (p + iq)|}{|w_1||q|} \leq \frac{1}{|q|(|w| + 2m|q|)|w|^{\delta}} \leq \frac{1}{2|q|^2}. \]

Since \( \gcd(p, q) = 1 \), it follows from (6) that \( (p, q) \) is a convergent of \( \alpha_{w_1} \). The assertion follows.
Now it remains to get a $N$-good slit from $w_1$. Assume that $w_1$ is not $N$-good. Let $q_k$ be the largest height in $\text{Spec}(\alpha w_1) \cap [1, |w_1|^{1+\delta}]$. Since $q_{k+1} > |w_1|^{1+\delta}$, the RHS of (4) implies $\Delta := (|w_1||q_k \alpha w_1 - p_k|^{-1}) > |w_1|^\delta$. Then there exists a positive integer $m$ such that $w = w_1 + 2m(p_k + iq_k)$ satisfies that $e^{N\delta} \log |w| + 1/2 \leq \Delta \leq |w|^\delta$.

Indeed, if $m$ is smallest for the right hand of the above sequence, then

$$\Delta > (|w_1| + 2(m - 1)|q_k|)^\delta$$

$$\geq (|w_1| + 2(m - 1))^{\delta}$$

$$\geq (2(m - 1))^{\delta}.$$

It follows from (7) that

$$\Delta = \frac{1}{q_k \times (p_k + iq_k)}$$

$$\leq \frac{c_0}{c_0}$$

$$\leq \frac{|w_1|^{c_0(1+\delta)}}{c_0}.$$

Then $m \leq \frac{1}{2c_0^\delta} |w_1|^{(1+\delta)c_0} + 1$, which means that

$$e^{N\delta} \log |w| + 1/2 = e^{N\delta} \log(|w_1| + 2m|q_k|) + 1/2$$

$$\leq e^{N\delta} \log \{|w_1| + 2[\frac{1}{2c_0^\delta} |w_1|^{(1+\delta)c_0} + 1]|w_1|^{1+\delta}\} + 1/2.$$

Since

$$\lim_{|w_1| \to \infty} |w_1|^{-\delta} \log \{|w_1| + 2[\frac{1}{2c_0^\delta} |w_1|^{(1+\delta)c_0} + 1]|w_1|^{1+\delta}\} = 0,$$

it follows that

$$e^{N\delta} \log |w| + 1/2 \leq |w_1|^\delta$$

$$\leq \Delta,$$

provided $|w_1|$ is large enough.

Now claim that $w = w_1 + 2m(p_k, q_k)$ is $N$-good. In fact,

$$|\alpha_w - p_k/q_k| = \left| \frac{w \times (p_k + iq_k)}{|w||q_k|} \right|$$

$$= \frac{|w_1 \times (p_k + iq_k)|}{|w||q_k|}$$

$$= \frac{1}{|w||q_k|\Delta}$$

$$\leq \frac{1}{|w||q_k||w_1|^{1+\delta}}$$

$$\leq 1/(2q_k^2),$$

which by (6) implies $(p_k, q_k)$ is a convergent of $\alpha_w$. Suppose that $(p_k, q_k)$ is the $k'$-th convergent of $\alpha_w$. Then

$$q_{k+1} \leq |q_k \alpha_w - p_k|^{-1}$$

$$= |w|\Delta$$

$$\leq |w|^{1+\delta}$$
and

\[ q_{k+1} \geq \frac{|q_k \alpha - p_k|^{-1} - q_k}{w|\Delta - q_k} \geq |w|(e^{N\delta \log |w| + 1/2}) - q_k \geq e^{N\delta |w| \log |w|}. \]

So \( w \) is \( N \)-good. \( \square \)

**Figure 4. The parallelogram domain**

As indicated in Figure 4, let \( \Sigma(\alpha, Q, R) \) denote the parallelogram domain

\[ \Sigma(\alpha, Q, R) = \{ x + iy \in \mathbb{C} : |y\alpha - x| < \frac{1}{Q}, R \leq y \leq 2R \}. \]

The density of coprime integers in \( \Sigma(\alpha, Q, R) \) is defined to be

\[ \text{dens}(\Sigma(\alpha, Q, R)) = \frac{\#(\{ a + ic \in \mathbb{Z}[i] \cap \Sigma(\alpha, Q, R) : \gcd(a, c) = 1 \})}{\text{Area}(\Sigma(\alpha, Q, R))} \]

**Lemma 4.3.** [2] There exist constants \( A_0 > 0 \) and \( \rho_0 > 0 \) such that whenever \( \text{area}(\Sigma) \geq A_0 \)

\[ \text{Spec}(\alpha) \cap [Q, R] \neq \emptyset \Rightarrow \text{dens}(\Sigma) \geq \rho_0 \]

In the remainder of this section, assume that \( w \in S(\phi) \) is normal. Let \( q_k \) be the largest in \( \text{Spec}(\alpha_w) \cap [1, |w|^{1+\delta}] \). Define \( n_1 \geq 1 \) by \( q_k = e^{n_1 \delta |w| \log |w|} \) and define \( n_3 > 1 \) uniquely by \( q_{k+1} = |w|^{1+n_3 \delta} \). Let \( n := \min(n_1, N + 1) \) and let \( Q = e^{n \delta |w| \log |w|}, Q' = 8|w|^{1+\delta} \) and \( R = |w|^{1+\delta} \).

**Definition 4.4.** A \( \delta \)-child \( w' \) of \( w \) is nice if \( w' = w + 2(p + iq) \) with \( p + iq \in \Sigma(\alpha, Q, R) \) and \( |q\alpha - p| \geq \frac{1}{Q'} \).

Let \( c_1 = 2\rho_0/e^{(N+1)\delta} \).

**Proposition 8.** A normal slit \( w \) has at least \( (c_1|w|^\delta / \log |w| - 1) \) nice \( \delta \)-children.

**Proof.** Using Lemma 4.3, \( w \) has at least \( c_1|w|^\delta / \log |w| \) \( \delta \)-children \( w' = w + 2(p + iq) \) with \( p + iq \in \Sigma(\alpha, Q, R) \).

It remains to show that there are at most one \( \delta \)-child \( w' = w + 2(p + iq) \) such that \( |q\alpha_w - p| < \frac{1}{8|w|^{1+\delta}} \). Suppose \( w'' = w + 2(p + iq) \) and \( w''' = w + 2(p' + iq') \) are
two different children satisfying the above inequality. Then
\[ \frac{|p - p'|}{q'} \leq |\alpha_w - \frac{p}{q}| + |\alpha_w - \frac{p}{q'}| < \frac{1}{4|w|^{2+2\delta}}. \]

On the other hand,
\[ \frac{|p - p'|}{q'} \geq \frac{1}{qq'} \geq \frac{1}{4|w|^{2+2\delta}}. \]

This leads to a contradiction. \( \Box \)

**Proposition 9.** Suppose that \( w' = w + 2(p + iq) \) is a nice \( \delta \)-child of \( w \). Then \((p, q)\) is the \( k'\)-th convergent of \( \alpha_w \) such that \( q_{k'+1} \in [|w'| \log |w'|, |w'|^{1+\delta}] \).

**Proof.** It is easy to see that
\[ |\alpha_w - \frac{p}{q}| = \frac{|w||q\alpha_w - p|}{|w'|q} \leq \frac{1}{(e^{n\delta} \log |w|)|w'|q} \leq \frac{1}{2q^2}. \]

Since \( \gcd(p, q) = 1 \), it follows from (6) that \((p, q)\) is also a convergent of \( w' \).

Suppose that \((p, q)\) is the \( k'\)-th convergent \((p_{k'}, q_{k'})\) of \( \alpha_{w'} \). Using the LHS of (4), it follows that
\[ q_{k'+1} \geq \frac{1}{|q_{k'}\alpha_{w'} - p_{k'}|} - q_{k'} \]
\[ = \frac{|w'|}{|w||q\alpha_w - p|} - q \]
\[ \geq \frac{e^{n\delta}|w'|}{|w'|(|w'| \log |w| - 1/2)} \]
\[ \geq e^{(n-1)\delta}|w'||w'| \left( \frac{e^{\delta} \log |w|}{\log(|w| + 4|w|^{1+\delta})} - \frac{1}{2 \log |w|} \right). \]

Moreover, due to \( \lim_{|w| \to \infty} \frac{e^{\delta} \log |w|}{\log(|w| + 4|w|^{1+\delta})} = \frac{e^{\delta}}{1+\delta} > 1 \), it follows that
\[ q_{k'+1} \geq e^{(n-1)\delta}|w'||w'| \log |w'| \]
provided that \(|w|\) is large enough.

Using the RHS of (4) again,
\[ q_{k'+1} \leq \frac{|q_{k'}\alpha_{w'} - p_{k'}|^{-1}}{|w'|} \leq \frac{|w||q\alpha - p|}{|w'|} \leq \frac{|w'|}{|w'|/8|w|^{1+\delta}} = 8|w|^{\delta}|w'|. \]
Since \( \lim_{|w| \to \infty} \frac{1}{1+2|w|} = 0 \), it follows that
\[
8|w|^{\delta}|w'| \leq 8\frac{|w|^\delta}{|w'||^{1+\delta}}|w'|^{1+\delta}
= 8\frac{|w|^\delta}{|w||^{1+\delta}}|w'|^{1+\delta}
= 8\left(\frac{|w|}{|w|+2q}\right)^\delta|w'|^{1+\delta}
\leq 8\left(\frac{1}{1+2|w|}\right)^\delta|w'|^{1+\delta}
\leq |w'|^{1+\delta}
\]
provided that \( |w| \) is large enough. Therefore, \( w' \) is \((n-1)\)-good. \( \square \)

Let \( \tilde{V} \) be the collection of nice children of \( w \), which are not normal. It is obviously that \( \tilde{V} = \emptyset \) when \( n_1 \geq N+1 \). It remains to consider the case that \( n_1 < N+1 \).

**Proposition 10.** Suppose that \( n_1 < N + 1 \). Let \( w' \in \tilde{V} \) and \( q_{v'} \) the largest in \( \text{Spec}(\alpha_{w'}) \cap [1, |w'|^{1+\delta}] \). Then
(i) \( q_{v'} \in \left[ |w'| \log |w'|, e^{N_0} |w'| \log |w'| \right] \) and
(ii) there are at most \( e^{N_0} \) positive integers \( a \) such that
\[
|w \times (p_{v'} + iq_{v'})| \leq \frac{1}{|w||n_2+\delta|^2},
\]
where \( n_2 := \max\{1, n_1-1\} \).

**Proof:** Since \( n_1 < N + 1 \), it follows that \( n = n_1 \). The definition of \( q_{v'} \) implies that \( q_{v'-1} = |w'|^{1+n'\delta} \) for some real number \( n' > 1 \). Since \( w' \) is \((n-1)\)-good, it follows that \( q_{v'} = e^{n'\delta} |w'| \log |w'| \) for some real number \( n'' \geq n_1 - 1 \geq 0 \). Using Proposition 6, \( w' \) is not \( N \)-good so that \( n'' < N \); this proves (i).

The fact that \( w' \) is not normal implies that \( n' \geq n'' \). Otherwise, \( q_{v'+1} = |w'|^{1+n''\delta} \in [e^{n\delta} |w'| \log |w'|, |w'|^{1+n\delta}] \) for any \( n \in (n'', N+1) \), and
\[
q_{v'} = e^{n''\delta} |w'| \log |w'| \in [e^{n\delta} |w'| \log |w'|, |w'|^{1+n\delta}]
\]
for any \( n \in [1, n'']. \) Thus \( w' \) is normal, contradicting to the assumption.

Consider \( q_{v'} \in \text{Spec}(\alpha_{w'}) \) in Proposition 9. Since \( q_{k'+1} \in [\log |w'|, |w'|^{1+\delta}] \), the definition of \( q_{v'} \) implies that \( q_{v'} \geq q_{k'+1} \). The recurrence relations satisfied by convergents imply that \( (p_{v'}, q_{v'}) = a(q_{k'+1}, q_{k'+1}) + b(p_{v'}, q_{v'}) \) for some integers \( a > 0 \) and \( b \geq 0 \). By (i),
\[
e^{N_0} |w'| \log |w'| \geq q_{v'} \geq aq_{k'+1} \geq a |w'| \log |w'|.
\]
Hence \( a \leq e^{N_0} \).

Since the cross product of consecutive convergents is \( \pm 1 \),
\[
|w'||q_{v'} |\alpha_{w'} - p_{v'}| = |w' \times (p_{v'} + iq_{v'})|
\]
\[ \begin{align*}
&= \left| [w + 2(p_{k'} + iq_{k'})] \times [a(p_{k'+1} + iq_{k'+1}) + b(p_{k'} + iq_{k'})] \right| \\
&= \left| [w \times (p_{k'} + iq_{k'}) + 2a[(p_{k'} + iq_{k'}) \times (p_{k'+1} + iq_{k'+1})]] \right| \\
&= \left| [w \times (p_{k'} + iq_{k'})] \pm 2a \right|.
\end{align*} \]

Since \( n' > 1, n' \geq n'' \) and \( n'' \geq n' \), then \( n' \geq n_2 \). Because of the RHS of (4) and \(|w'| = |w| + 2q_{k'} \geq |w|^{1+\delta} \),
\[ \begin{align*}
|w'| |q_{w'} \alpha_w - p_{w'}| &\leq \frac{|w'|}{q_{w'} + 1} \\
&\leq \frac{|w'|}{|w'|^{1+\delta}} \\
&\leq \frac{1}{(|w|^{1+\delta})^{1+\delta}} \\
&\leq \frac{1}{|w|^{n_2 \delta + n_2 \delta^2}}.
\end{align*} \]

and (ii) follows. \( \square \)

Thus \( \tilde{V} \) is decomposed as a finite union of subsets \( \tilde{V}_{\pm a} \). Let \( Q_{\pm a} \) denote the corresponding set of heights \( q_{w'} \), associated to the slits in \( \tilde{V}_{\pm a} \) in Proposition 10.

**Proposition 11.** \( \hat{V}_{\pm a} \) and \( Q_{\pm a} \) have the same number of elements.

**Proof.** It suffices to show that the map \( \hat{V}_{\pm a} \to Q_{\pm a} \) is injective. Let \( w'' \) be different from \( w' \) with the corresponding image \( q_{w''} \). By the RHS of (4),
\[ \begin{align*}
|\alpha_{w'} - p_{w'}| &\leq \frac{1}{q_{w'} q_{w'+1}} \\
&\leq \frac{1}{q_{w'} |w'|^{1+\delta}} \\
&\leq \frac{1}{(|w| \log |w'|) |w'|^{1+\delta}} \\
&\leq \frac{1}{2^{2+\delta} |w|^{2+3\delta+\delta^2}}.
\end{align*} \]

(The second step is implied by the definition of \( q_{w'} \).) It follows form Proposition 4 that \(|\alpha_{w'} - \alpha_{w''}| \geq \frac{1}{8 |w|^{2+\delta}} \). As a result, the rational numbers \( \frac{p_{w}}{q_{w}} \) and \( \frac{p_{w''}}{q_{w''}} \) are distinct.

It is easy to see that
\[ \begin{align*}
|\frac{p_{w}}{q_{w}} - \frac{p_{w''}}{q_{w''}}| &\leq |\alpha_{w'} - \frac{p_{w'}}{q_{w'}}| + |\alpha_{w''} - \frac{p_{w''}}{q_{w''}}| + |\alpha_{w'} - \alpha_{w''}| \\
&\leq \frac{1}{2^{1+\delta} |w|^{2+3\delta+\delta^2}} + \frac{4}{|w|^{2+\delta}}.
\end{align*} \]

By Proposition 10,
\[ \begin{align*}
\frac{1}{q_{w}} &\geq e^{N \delta |w| \log |w'|} \\
&\geq e^{N \delta (|w| + 4|w|^{1+\delta}) \log (|w| + 4|w|^{1+\delta})} \\
&\geq \frac{1}{16(2 + \delta) e^{N \delta |w|^{1+\delta} \log |w|}}.
\end{align*} \]
Then and representatives of two different clusters are separated by a distance of at least $d$ for some positive integer $p/q$. For any two different $q$ and $q'$, having at most $k$ characters, such that $q$ and $q'$ fall into the same cluster.

Suppose that $q < \bar{q}$, the number $C$ of clusters satisfies that $(C - 1)|w|^{1+\delta} < e^{N\delta}|w'| \log |w'|$. Then

$$
C \leq \frac{e^{N\delta}|w'| \log |w'|}{|w|^{1+\delta}} + 1
$$

provided that $|w|$ is large enough. Thus the proposition follows from the claim.

It remains to prove the claim. Assume that $q < \bar{q}$, and $q_k = \bar{q} = dq_k$ where $d = \gcd(p/q)$. Using Proposition 10 again,

$$
| \bar{q} \alpha_w - \bar{p} | = | q v \alpha_w - p v | \leq | (q v \alpha_w - p v) | \leq | q v \alpha_w - p v | + | q v \alpha_w - p v | = 2 | q v \alpha_w - p v | \leq \frac{2}{|w|^{1+\delta}}.
$$

Since $q < \bar{q} < |w|^{1+\delta}$ and $n_2 \geq 1$, it follows that

$$
| \alpha_w - \frac{p}{\bar{q}} | = | \alpha_w - \frac{p}{\bar{q}} | \leq \frac{2}{|w|^{1+\delta}} + \frac{2}{|w|^{1+\delta}} \leq 1/2q^2
$$

provided that $|w|$ is large enough. Since $\gcd(p, q) = 1$, it follows from (6) that $(p, q)$ is a convergent of $\alpha_w$.

Suppose that $q = q_k \in \text{Spec}(\alpha_w)$ for some index $k'$. By the hypothesis that $q < \bar{q}$, it follows that $q_k \leq q_k$, i.e. $k' \leq k$. In fact, $k' = k$. If otherwise, then

$$
| q_k \alpha_w - \bar{p} | = \frac{1}{q_k + q_k - 1}
$$

and

$$
| q_k \alpha_w - \bar{p} | = \frac{1}{q_k + q_k - 1}
$$
which contradicts the previous inequality.

The LHS of (4) implies that
\[
d = \frac{\bar{q}_k \alpha - \bar{p}}{q_k \alpha - p_k} \\
\leq \frac{2(q_{k+1} + q_k)}{|w|^{1+n_3}\delta+n_2\delta^2} \\
\leq \frac{2(|w|^{1+n_3} + |w|^{1+\delta})}{|w|^{1+n_2\delta+n_2\delta^2}} \\
\leq 4|w|^{(n_3-n_2)\delta-n_2\delta^2} \\
\leq 4|w|^{\delta-\delta^2},
\]

where $1 < n_3$ is defined by $q_{k+1} = |w|^{1+n_3\delta}$, provided that $|w|$ is large enough. Since $w$ is normal, $n_3 \leq n_1$. This together with the definition of $n_2$ in Proposition 10 implies $n_3 - n_2 \leq n_1 - n_2 \leq 1$ and $n_2 \geq 1$. Thus the claim is proved.

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