The Mean Field Kinetic Equation for a Pedestrian Flow Model:  
The Global Existence of Weak Solution

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Abstract

In this paper we prove the global existence of the weak solution to the mean field kinetic equation derived from the \( N \)-particle pedestrian Newtonian system. For \( L^1 \cap L^\infty \) initial data, the solvability of the mean field kinetic equation can be obtained by using uniform estimates and compactness arguments while the difficulties arising from the non-local non-linear interaction are tackled appropriately using the Aubin-Lions compact embedding theorem.

Keywords: pedestrian flow, mean field limit, weak solution, compact embedding.

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1 Introduction

Social phenomena and in particular pedestrian flow models have been experienced a great increase in interest over the last few years. Different models have been developed and investigated from a numerical and theoretical point of view, see for example [1, 8, 20] for a general overview. The modeling of pedestrian crowds is inspired by fluid dynamics and the consideration of humans interactions, so-called social fields [18] or social forces [16]. Comparisons with empirical data [16, 31] allow then for the observation of new spatiotemporal patterns (e.g. line formation, freezing by heating) in pedestrian motions. Further applications of such behavioral models include group dynamics [2], minimal travel times [10, 17] or evacuation scenarios [23, 30].

Model hierarchies for pedestrian models have been introduced in [9] or [12]. Therein, macroscopic equations are formally derived from a microscopic pedestrian Newtonian system. Depending on the closure assumption, different non-local continuum models can occur, cf. [7]. However, from an analytical point of view, there are still several open problems that need to be thoroughly investigated as for instance the detailed derivation from the \( N \)-particle pedestrian Newtonian system to its mean field limit or Vlasov equation, see [5]. Instead of the formal derivation with the help of the BBGKY hierarchy [12, 27], the kinetic description has been rigorously derived by a probabilistic method [3, 4, 14, 15, 22, 28].

In this paper, we now aim to prove the global existence of the weak solution to the mean field kinetic equation proposed in [12]. However, the proposed pedestrian model involves a singularity comparable to the Coulomb potential in 2-d, resulting from the total interaction

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force. That means this singularity, or in other words the non-local term, needs extra care in the final limiting process. For more information about the Coulomb potential and the Vlasov-Poisson system we refer to [21, 24, 25].

We now briefly explain our approach. In order to obtain the existence of the weak solution, we consider an approximate problem (kinetic equation with cut-off) as a starting point and show that the approximate problem has a weak solution, where the mean field characteristic flow is of great importance. Unlike the 3-d Vlasov-Poisson equation [11, 19], the non-local operator in the pedestrian flow model cannot be decoupled into an elliptic equation. Hence, the Calderón-Zygmund continuity theorem [13] for second order elliptic equations is not applicable in this case and we have to find an alternative way to fix the desired compactness arguments. The idea is now to use the Aubin-Lions lemma [6, 26] and to argue that due to that compact embedding theorem, we are able to pass the limit especially in the non-local term. We also remark that the result obtained in the present paper plays a crucial role in the proof of the rigorous derivation of the mean field equation in [5].

This article is organized as follows: In Section 2 we start with the background setting of the mean field equation for the pedestrian flow model. Then, in Section 3 some notations and preliminary work will be introduced to show that the characteristic flow associated with the cut-off mean field equation admits a unique solution. We also prove the existence and uniqueness of the weak solution to the cut-off mean field equation. Section 4 is concerned with the compactness arguments that are needed to pass the limit and to obtain the desired weak formulation of the non-cut-off kinetic equation, however the corresponding uniqueness can no longer be kept during the limiting procedure. Finally, we summarize our results.

2 Model Equations

Following the pedestrian flow model originally introduced in [12], we consider a two-dimensional kinetic mean field equation with position $x \in \mathbb{R}^2$ and velocity $v \in \mathbb{R}^2$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f)f] + \nabla_v \cdot (Gf) = 0,$$

(2.1)

where $F(x,v)$ denotes the total interaction force and $G(x,v)$ the desired velocity and direction acceleration. More precisely, $F(x,v)$ is composed of the interaction force $F_{\text{int}}(x)$ and the dissipative force $F_{\text{diss}}(x,v)$, i.e.,

$$F(x,v) = (F_{\text{int}}(x) + F_{\text{diss}}(x,v))\mathcal{H}(x,v)$$

(2.2)

with $F_{\text{int}}(x) = 2Rk_n \frac{x}{|x|} - k_n x, F_{\text{diss}}(x,v) = \frac{\langle v, x \rangle}{|x|^2} (\gamma_t - \gamma_n) x - \gamma_t v$ and $\mathcal{H}(x,v) := \mathcal{H}_{2R(|x|)} \cdot \mathcal{H}_{\tilde{2R}(|v|)}$, where $\mathcal{H}_{2R(|x|)}$ and $\mathcal{H}_{\tilde{2R}(|v|)}$ are smooth functions with compact support such that

$$\mathcal{H}_{2R(|x|)} = \begin{cases} 0, & |x| > 2R, \\ 1, & |x| < R, \end{cases} \quad \text{and} \quad \mathcal{H}_{\tilde{2R}(|v|)} = \begin{cases} 0, & |v| > 2\tilde{R}, \\ 1, & |v| < \tilde{R}. \end{cases}$$

Here, $F_{\text{diss}}^n(x,v)$ and $F_{\text{diss}}^t(x,v)$ are the normal dissipative force and the tangential friction force, respectively. Moreover, $k_n$ is the interaction constant and $\gamma_n, \gamma_t$ are suitable positive friction constants.
Remark 2.1. To cover a realistic behavior of pedestrians, the functions $\mathcal{H}_{2R}(|x|)$ and $\tilde{\mathcal{H}}_{2R}(|v|)$ are used to express that the interaction force and the pedestrian velocity are of finite range. So the total force is considered on a bounded domain.

The desired velocity and direction acceleration is given by

$$G(x, v) := G(x, v, \rho) = \frac{1}{T} \left( -U(\rho) \frac{\nabla \Phi(x)}{\nabla \Phi(x)} - v \right),$$

(2.3)

$$\rho = \rho(x) = \frac{1}{N_{\text{max}}} \sum_{j, |x - x_j| < R} 1.$$  

Here $N_{\text{max}}$ depends on the time $t$ via the coupling to the positions $x_j$. For a fixed time $t$, $N_{\text{max}}$ describes the maximal number of particles in a ball of radius $R$ and is used here as a normalization parameter. This means, we only scale the number of particles in this region in the sense how compressed they are. Further, $\Phi$ is given by the solution of the eikonal equation

$$U(\rho(\Phi(x)) |\nabla \Phi|) - 1 = 0,$$

(2.4)

where $U : [0, 1] \to [0, U_{\text{max}}]$ is a density-dependent velocity function. The reaction time $T$ might also depend on the density $\rho$. But for simplicity, we take $T = 1$. Here, we would also like to point out that the desired velocity and direction acceleration $G(x, v)$ can be further written as

$$G(x, v) = g(x) - v,$$

(2.5)

where $g(x) = -U(\rho) \frac{\nabla \Phi(x)}{\nabla \Phi(x)}$. For any solution $\Phi(x)$ of the eikonal equation (2.4), it is apparent that $\|g\|_{L^\infty} \leq U_{\text{max}}$.

Definition 2.1. Let $f_0(x, v) \in L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. A function $f = f(t, x, v)$ is said to be a weak solution to the kinetic mean field equation (2.1) with initial data $f_0$, if there holds

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) \varphi(x, v) \, dx \, dv = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x, v) \varphi(x, v) \, dx \, dv$$

$$+ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} v f(s, x, v) \cdot \nabla_x \varphi(x, v) \, dx \, dv \, ds$$

$$+ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} (F(x, v) * f(s, x, v)) f(s, x, v) \cdot \nabla_v \varphi(x, v) \, dx \, dv \, ds$$

$$+ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} G(x, v) f(s, x, v) \cdot \nabla_v \varphi(x, v) \, dx \, dv \, ds$$

(2.6)

for all $\varphi(x, v) \in C^\infty_0(\mathbb{R}^2 \times \mathbb{R}^2)$ and $t \in \mathbb{R}_+$.

Now, we present the main theorem of this paper. In the following, $G(x, v)$ is given by (2.5) while $F(x, v)$ is defined by (2.2).

Theorem 2.1. Let $f_0(x, v)$ be a nonnegative function in $L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2} |v|^2 f_0(x, v) \, dx \, dv =: \mathcal{E}_0 < \infty.$$
Then, there exists a weak solution \( f \in L^\infty(\mathbb{R}_+;L^1(\mathbb{R}^2 \times \mathbb{R}^2)) \) to the mean field equation \( (2.1) \) with initial data \( f_0 \). Moreover this solution satisfies

\[
0 \leq f(t,x,v) \leq \|f_0\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} \quad \text{for a.e.} \ (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2, \ t \geq 0 \tag{2.7}
\]

together with the mass conservation

\[
\iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(t,x,v) \, dx \, dv = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x,v) \, dx \, dv =: \mathcal{M}_0 \tag{2.8}
\]

and the kinetic energy bound

\[
\mathcal{E}(t) := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2}|v|^2 f(t,x,v) \, dx \, dv \leq C, \ \forall \ t \geq 0, \tag{2.9}
\]

where the constant \( C \) is independent of \( t \).

One of the main difficulties in this context is that the interaction force \( F(x,v) \) not only depends on the position \( x \) but also on the velocity \( v \). This leads to a totally different structure compared to the Vlasov-Poisson equation, where the \( W^{2,p} \) theory for Poisson equations is generally used. The proof of Theorem \( 2.1 \) is therefore not as straightforward and intuitive as one might expect and needs to be dedicatedly handled step by step within the next sections.

On the other hand, the self-generating force (or desired velocity and direction acceleration) \( G(x,v) \) is not Lipschitz continuous, which requires an additional work of mollification.

### 3 Mean Field Equation with Cut-off

We briefly recall essential assumptions and properties, cf. [5], that are necessary for the existence proof.

#### 3.1 Notations and Preliminary Work

We consider the pedestrian flow model with cut-off of order \( N^{-\theta} \) with \( 0 < \theta < \frac{1}{4} \), i.e.,

\[
F^N(x,v) = \begin{cases} 
2Rk_n \frac{x}{|x|} - k_n x + \frac{(v,x)}{|x|^2} (\gamma_t - \gamma_n) x - \gamma_t v \mathcal{H}(x,v), & |x| \geq N^{-\theta}, \\
(2Rk_n N^\theta - k_n) x + N^{2\theta} (v,x) (\gamma_t - \gamma_n) x - \gamma_t v \mathcal{H}(x,v), & |x| < N^{-\theta}.
\end{cases} \tag{3.1}
\]

Then, the mean field cut-off equation becomes

\[
\partial_t f^N + v \cdot \nabla_x f^N + \nabla_v \cdot [(F^N \ast f^N) f^N] + \nabla_v \cdot (G^N f^N) = 0, \tag{3.2}
\]

where we also take the cut-off of \( G(x,v) \) into consideration, i.e.,

\[
G^N(x,v) = j_{\frac{1}{N}} \ast g(x) - v
\]

with \( j_{\frac{1}{N}}(x) \) being the standard mollifier.

We also point out several properties for the interaction force \( F^N(x,v) \) and the acceleration \( G(x,v) \), namely
(a) $F^N(x, v)$ is bounded, i.e., $|F^N(x, v)| \leq C$.

(b) $F^N(x, v)$ satisfies

$$|F^N(x, v) - F^N(y, v)| \leq q^N(x, v)|x - y|,$$

where $q^N$ has compact support in $B_{2R} \times B_{2\tilde{R}}$ with

$$q^N(x, v) := \begin{cases} C \cdot \frac{1}{|x|} + C, & |x| \geq N^{-\theta}, \\ C \cdot N^\theta, & |x| < N^{-\theta}. \end{cases}$$

(c) $\nabla_v F^N(x, v)$ is uniformly bounded in $N$.

(d) $|G^N(x, v) - G^N(y, v)| \leq C \cdot N \cdot |x - y|$.

Here, we use $C$ as a universal constant that might depend on all the given constants $k_n, R, \tilde{R}, \gamma_n, \gamma_t$.

Furthermore, if there is a singularity in the velocity $v$ in the interaction potential similar to property (b), it can be treated by using the same method as above and the results also apply.

### 3.2 Mean Field Characteristic Flow with Cut-off

Before we start to prove the existence of the unique weak solution to the equation (3.2), we need first the following definition.

**Definition 3.1.** Let $(X_1, \Sigma_1)$ and $(X_2, \Sigma_2)$ be measurable spaces (meaning that $\Sigma_1$ and $\Sigma_2$ are $\sigma$-algebras of the subsets of $X_1$ and $X_2$, respectively). Let $T : X_1 \to X_2$ be a $(\Sigma_1, \Sigma_2)$-measurable map and $\mu$ be a positive measure on $(X_1, \Sigma_1)$. Then, the formula

$$\nu(B) := \mu(T^{-1}(B)), \quad \forall B \in \Sigma_2$$

defines a positive measure on $(X_2, \Sigma_2)$, denoted by

$$\nu =: T\#\mu,$$

and is referred to as the push-forward of the measure $\mu$ under the map $T$.

Due to the property of the transport equation, we know that solving the equation (3.2) is equivalent to investigating the corresponding characteristic system, i.e.,

$$\begin{cases} \frac{d}{dt}Z(t, z_0, \mu_0) = \int_{\mathbb{R}^4} K \left(Z(t, z_0), z'\right) \mu(t, dz'), \\ Z(0, z_0, \mu_0) = z_0, \end{cases} \tag{3.3}$$

where

$$K^N(z, z') = K^N(x, v, x', v') := \left(v, F^N(x - x', v - v') + G^N(x, v)\right)$$

and $\mu(t, \cdot)$ is the push-forward of the measure $\mu_0$. Here, for the sake of convenience, we use $z = (x, v)$ and $Z$ as the four-dimensional vector.
The solvability of the cut-off problem can be then obtained via the standard argument using Banach Fixed-Point Theorem. For completeness, we present the proof in the following proposition and theorem.

We denote \( P(\mathbb{R}^4) \) as the set of Borel probability measures on \( \mathbb{R}^4 \) and \( P_1(\mathbb{R}^4) \) is defined by

\[
P_1(\mathbb{R}^4) := \left\{ \mu \in P(\mathbb{R}^4) \left| \int_{\mathbb{R}^4} |v| \mu(dx, dv) < \infty \right. \right\}.
\]

**Proposition 3.1.** Assume that the interaction kernel \( K(z, z') \in C(\mathbb{R}^4 \times \mathbb{R}^4; \mathbb{R}^4) \) is Lipschitz continuous in \( z \), uniformly in \( z' \) (and conversely), i.e., there exists a constant \( L > 0 \) such that

\[
\sup_{z' \in \mathbb{R}^4} |K(z_1, z') - K(z_2, z')| \leq L|z_1 - z_2|,
\]

\[
\sup_{z \in \mathbb{R}^4} |K(z, z_1) - K(z, z_2)| \leq L|z_1 - z_2|.
\]

For any given \( z_0 = (x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \) and Borel probability measure \( \mu_0 \in P_1(\mathbb{R}^4) \), there exists a unique \( C^1 \)-solution, denoted by

\[
\mathbb{R}_+ \ni t \mapsto Z(t, z_0, \mu_0) \in \mathbb{R}^4,
\]

to the problem

\[
\begin{align*}
\frac{d}{dt}Z(t, z_0, \mu_0) &= \int_{\mathbb{R}^4} K \left( Z(t, z_0), z' \right) \mu(t, dz'), \\
Z(0, z_0, \mu_0) &= z_0,
\end{align*}
\]

where \( \mu(t, \cdot) \) is the push-forward of the measure \( \mu_0 \), i.e., \( \mu(t, \cdot) = Z(t, \cdot, \mu_0)\#\mu_0 \).

**Proof.** Let \( \mu_0 \in P_1(\mathbb{R}^4) \) and denote

\[
\kappa := \int_{\mathbb{R}^4} |v| \mu_0(dx, dv).
\]

For \( t^* := \frac{1}{2L(2 + \kappa)} \), let

\[
\mathcal{X} := \left\{ Z(t, z) \in C([0, t^*]; C(\mathbb{R}^4; \mathbb{R}^4)) \left| \sup_{0 \leq t \leq t^*} \sup_{z=(x,v)\in\mathbb{R}^4} \frac{|Z(t, z)|}{1 + |v|} < \infty \right. \right\}
\]

be a Banach space equipped with the norm

\[
\|Z\|_{\mathcal{X}} := \sup_{0 \leq t \leq t^*} \sup_{z=(x,v)\in\mathbb{R}^4} \frac{|Z(t, z)|}{1 + |v|}.
\]

The assumption for the Lipschitz continuity of the kernel \( K(z, z') \) actually implies that \( K \) grows at most linearly at infinity, i.e.,

\[
|K(z, z')| \leq L(|z| + |z'|), \quad z, z' \in \mathbb{R}^4.
\]
The map $T : \mathcal{X} \to \mathcal{X}$, defined by

$$TZ(t, z) := z + \int_0^t \int_{\mathbb{R}^4} K(Z(s, z), Z(s, \zeta)) \mu_0(d\zeta) ds,$$

constitutes a contraction which can be seen from the following estimates. For each $Z, \hat{Z} \in \mathcal{X}$, we have for $0 \leq s \leq t^*$

$$\left| \int_{\mathbb{R}^4} K(Z(s, z), Z(s, \zeta)) \mu_0(d\zeta) - \int_{\mathbb{R}^4} K(\hat{Z}(s, z), \hat{Z}(s, \zeta)) \mu_0(d\zeta) \right|$$

$$\leq L \int_{\mathbb{R}^4} \left( |Z(s, z) - \hat{Z}(s, z)| + |Z(s, \zeta) - \hat{Z}(s, \zeta)| \right) \mu_0(d\zeta)$$

$$\leq L(1 + |v| + 1 + \kappa) \sup_{z = (x, v) \in \mathbb{R}^4} \frac{|Z(s, z) - \hat{Z}(s, z)|}{1 + |v|}.$$

Consequently, we get

$$\|TZ(t, \cdot) - T\hat{Z}(t, \cdot)\|_{\mathcal{X}}$$

$$= \left\| \int_0^t \int_{\mathbb{R}^4} K(Z(s, z), Z(s, \zeta)) \mu_0(d\zeta) ds - \int_0^t \int_{\mathbb{R}^4} K(\hat{Z}(s, z), \hat{Z}(s, \zeta)) \mu_0(d\zeta) ds \right\|_{\mathcal{X}}$$

$$\leq L \|Z(t, \cdot) - \hat{Z}(t, \cdot)\|_{\mathcal{X}}(2 + \kappa)t^*$$

$$\leq \frac{1}{2} \|Z(t, \cdot) - \hat{Z}(t, \cdot)\|_{\mathcal{X}}.$$

Then, by Banach Fixed-Point Theorem, there exists a unique $Z^* \in \mathcal{X}$ such that $TZ^* = Z^*$, i.e.,

$$Z^*(t, z) = z + \int_0^t \int_{\mathbb{R}^4} K(Z^*(s, z), Z^*(s, \zeta)) \mu_0(d\zeta) ds.$$  (3.5)

The interaction kernel is globally Lipschitz continuous, i.e., $t^*$ is a fixed constant, which implies that the solution can be easily extended to all time $t$. Since $Z^* \in C(\mathbb{R}_+: C(\mathbb{R}^4; \mathbb{R}^4))$, $K \in C(\mathbb{R}^4 \times \mathbb{R}^4; \mathbb{R}^4)$ and $\mu_0 \in \mathcal{P}_1(\mathbb{R}^4)$, the function

$$s \mapsto \int_{\mathbb{R}^4} K(Z^*(s, z_0), Z^*(s, \zeta)) \mu_0(d\zeta)$$

is continuous on $\mathbb{R}_+$ for all $z_0 \in \mathbb{R}^4$. Exploiting the integral equation $\text{(3.5)}$ shows that the function $t \mapsto Z(t, z_0)$ is $C^1$ in $t$ and satisfies

$$\begin{cases}
\frac{d}{dt}Z^*(t, z_0) = \int_{\mathbb{R}^4} K(Z^*(t, z), Z^*(t, \zeta)) \mu_0(d\zeta), \\
Z(0, z_0) = z_0.
\end{cases}$$

Substituting $z' = Z^*(t, \zeta)$ in the integral above leads to

$$\int_{\mathbb{R}^4} K(Z^*(t, z_0), Z^*(t, \zeta)) \mu_0(d\zeta) = \int_{\mathbb{R}^4} K(Z^*(t, z_0), z') Z^*(t, \cdot) \# \mu_0(dz'),$$

which means that the function $Z^*$ is the solution to the problem $\text{(3.4)}$. \hfill \blacksquare
Thanks to Proposition 3.1, we are now able to prove that there exists a unique weak solution to the Vlasov equation with cut-off (3.2).

**Theorem 3.1.** Let \( f_0^N \) be a nonnegative compactly supported function in \( L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) satisfying

\[
\|f_0^N\|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2)} = M_0 \quad \text{and} \quad f_0^N(x, v) \leq \|f_0\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}
\]

and

\[
\int_{\mathbb{R}^2} \frac{1}{2} |v|^2 f_0^N(x, v) \, dx \, dv \leq \mathcal{E}_0 < \infty.
\]

Then, there exists a unique weak solution \( f^N \in C^1(\mathbb{R}^+; L^1(\mathbb{R}^2 \times \mathbb{R}^2)) \) to the mean field cut-off equation (3.2) with initial data \( f_0^N \), i.e., \( f^N(t, x, v) \) satisfies

\[
\int_{\mathbb{R}^2} \partial_t f^N(t, x, v) \varphi(x, v) \, dx \, dv = \int_{\mathbb{R}^2} v f^N(t, x, v) \cdot \nabla_x \varphi(x, v) \, dx \, dv
\]

\[
+ \int_{\mathbb{R}^2} \left( F^N(x, v) \ast f^N(t, x, v) \right) f^N(s, x, v) \cdot \nabla_v \varphi(x, v) \, dx \, dv
\]

\[
+ \int_{\mathbb{R}^2} G^N(x, v) f^N(t, x, v) \cdot \nabla_v \varphi(x, v) \, dx \, dv \quad (3.6)
\]

for all \( \varphi(x, v) \in C^\infty_0(\mathbb{R}^2 \times \mathbb{R}^2) \). Moreover this solution satisfies

\[
\lim_{t \to 0} f^N(t, x, v) = f_0^N(x, v), \quad \text{for a.e.} \ (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2,
\]

\[
0 \leq f^N(t, x, v) \leq \|f_0\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}, \quad \text{for a.e.} \ (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2, \ t \geq 0 \quad (3.7)
\]

together with the mass conservation

\[
\int_{\mathbb{R}^2} f^N(t, x, v) \, dx \, dv = \int_{\mathbb{R}^2} f_0^N(x, v) \, dx \, dv =: M_0 \quad (3.8)
\]

and the kinetic energy bound

\[
\int_{\mathbb{R}^2} \frac{1}{2} |v|^2 f^N(t, x, v) \, dx \, dv \leq C, \quad \forall \ t \geq 0, \quad (3.9)
\]

where the constant \( C \) is independent of \( N \) and \( t \).

**Proof.** Without loss of generality, we assume that \( M_0 = 1 \). If we choose the interaction kernel \( K \) as

\[
K^N(z, z') = K^N(x, v, x', v') := \left( v, F^N(x - x', v - v') + G^N(x, v) \right),
\]

the mean field cut-off equation (3.2) can be put into the form

\[
\partial_t f^N(t, z) + \text{div}_z \left( f^N(t, z) \int_{\mathbb{R}^2} K^N(z, z') f^N(t, z') \, dz' \right) = 0.
\]

Notice that the non-linear non-local dynamical system that appears in Proposition 3.1 is exactly the equation of characteristics for the mean field kinetic equation with cut-off (3.2), which we refer to as the mean field characteristic flow (with cut-off). The existence and
uniqueness of the solution to (3.2) are therefore achieved as a direct result of the construction of the mean field characteristic flow. By Proposition 3.1, there exists a unique map
\[ \mathbb{R}_+ \times \mathbb{R}^4 \times \mathcal{P}_1(\mathbb{R}^4) \ni (t, z_0, \mu_0) \mapsto Z^N(t, z_0, \mu_0) \in \mathbb{R}^d \]
such that \( t \mapsto Z^N(t, z_0, \mu_0) \) is the integral curve of the vector field
\[ z \mapsto \int_{\mathbb{R}^2 \times \mathbb{R}^2} K^N(z, \zeta') \mu^N(t, d\zeta') \]
passing through \( z_0 \) at time \( t = 0 \), where \( \mu^N(t) := Z^N(t, \cdot, \mu_0) \# \mu_0 \). For the given initial data \( f_0^N \), letting \( d\mu_0 = f_0^N dz \) results in
\[ f^N(t, z) := f_0^N \left( Z^N(t, \cdot)^{-1}(z) \right), \quad \forall t \geq 0. \]
Applying this formulation, properties (3.7) and (3.8) are straightforward. Property (3.9) is left to be proven. For the kinetic energy estimate, we will use the property of the acceleration \( G^N(x, v) \), i.e., \( G^N(x, v) = j_{\eta/\mu} \ast g(x) - v \), where \( j_{\eta/\mu} \ast g(x) \) is a \( L^\infty \)-function. We remark that \( v \) in \( G^N(x, v) \) is critical in the estimate because it serves as a damping term. We now choose \( \{ \varphi_\eta(x) \phi_\eta(v) \} \) to be a smooth function which satisfies
\[ \varphi_\eta(x) = \begin{cases} 0, & |x| > \frac{1}{\eta}, \\ 1, & |x| < \frac{1}{2\eta}, \end{cases} \quad \text{and} \quad \phi_\eta(v) = \begin{cases} 0, & |v| > \frac{1}{\eta}, \\ 1, & |v| < \frac{1}{2\eta}. \end{cases} \]
and
\[ \left| \nabla_z \left( \varphi_\eta(x) \phi_\eta(v) \right) \right| \leq \eta \left| \varphi_\eta(x) \phi_\eta(v) \right|. \]
Since \( \varphi_\eta(x) \phi_\eta(v) \) is monotone and converges to one for almost all \( x \) and \( v \) as \( \eta \) goes to zero, we have
\[ \int_{\mathbb{R}^2 \times \mathbb{R}^2} v^2 f^N(t, x, v) \varphi_\eta(x) \phi_\eta(v) \, dxdv \to \int_{\mathbb{R}^2 \times \mathbb{R}^2} v^2 f^N(t, x, v) \, dxdv, \quad \text{as} \ \eta \to 0. \]
The compact support of \( f_0^N \) implies that \( f^N(t, x, v) \) has compact support in \((x, v)\) for any fixed time \( t \). By the definition of weak solution for test functions \( v^2 \varphi_\eta(x) \phi_\eta(v) \), we have
\[ \frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2} v^2 f^N(t, x, v) \varphi_\eta(x) \phi_\eta(v) \, dxdv \]
\[ = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} v f^N(t, x, v) \cdot \nabla_x \left( v^2 \varphi_\eta(x) \phi_\eta(v) \right) \, dxdv \]
\[ + \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( F^N(x, v) - f^N(t, x, v) \right) f^N(s, x, v) \cdot \nabla_v \left( v^2 \varphi_\eta(x) \phi_\eta(v) \right) \, dxdv \]
\[ + \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} G^N(x, v) f^N(t, x, v) \cdot \nabla_v \left( v^2 \varphi_\eta(x) \phi_\eta(v) \right) \, dxdv \]
\[
\begin{align*}
&= \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} v^2 f_N(t, x, v) \phi(v) v \cdot \nabla_x (\varphi(v(x))) \, dxdv \\
&\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2} v \left( F^N(x, v) * f^N(t, x, v) \right) f^N(t, x, v) \varphi(v(x)) \phi(v) \, dxdv \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} v^2 \left( F^N(x, v) * f^N(t, x, v) \right) f^N(s, x, v) \cdot \nabla_v \left( \varphi(v(x)) \phi(v) \right) \, dxdv \\
&\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2} v \cdot G^N(x, v) f^N(t, x, v) \varphi(v(x)) \phi(v) \, dxdv \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} v^2 G^N(x, v) f^N(t, x, v) \cdot \nabla_v \left( \varphi(v(x)) \phi(v) \right) \, dxdv \\
&=: \sum_{j=1}^{5} I_j.
\end{align*}
\]

Next, we estimate the expressions \( I_j, j = 1, \ldots, 5 \) individually. It is easy to see

\[
|I_1| \leq \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left| v^2 f^N(t, x, v) \phi(v) v \cdot \nabla_x (\varphi(v(x))) \right| \, dxdv
\leq \frac{1}{2} \eta \int_{\mathbb{R}^2 \times \mathbb{R}^2} |v|^3 f^N(t, x, v) |\phi(v)\varphi(v)| \, dxdv.
\]

Due to the fact that \( f^N_0 \) is compactly supported, i.e., \( f^N \) has also compact support for any finite time \( t \), \( I_1 \) converges to zero as \( \eta \to 0 \) for fixed \( N \). The same argument holds for \( I_3 \) and \( I_5 \), i.e., \( I_3 \) and \( I_5 \) converge to zero as \( \eta \to 0 \):

\[
|I_3| \leq \frac{1}{2} \cdot C\eta \| F^N * f^N \|_{L^\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} v^2 f^N(t, x, v) \varphi(v(x)) \phi(v) \, dxdv
\]

\[
I_5 \leq \frac{1}{2} \cdot \eta \| \frac{1}{\pi} * g \|_{L^\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} v^2 f^N(t, x, v) \varphi(v(x)) \phi(v) \, dxdv
\]

\[
- \frac{1}{2} \eta \int_{\mathbb{R}^2 \times \mathbb{R}^2} |v|^3 f^N(t, x, v) |\phi(v)\varphi(v)| \, dxdv.
\]

However, for the other integral estimates, we need some extra calculations. Using the properties of the desired velocity and direction acceleration \( G^N(x, v) \), we arrive at

\[
I_2 \leq \| F^N * f^N \|_{L^\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \frac{1}{4\varepsilon} + \varepsilon v^2 \right) f^N(t, x, v) \varphi(v(x)) \phi(v) \, dxdv
\]

\[
I_4 \leq \| \frac{1}{\pi} * g \|_{L^\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \frac{1}{4\varepsilon} + \varepsilon v^2 \right) f^N(t, x, v) \varphi(v(x)) \phi(v) \, dxdv
\]

\[
- \int_{\mathbb{R}^2 \times \mathbb{R}^2} v^2 f^N(t, x, v) \varphi(v(x)) \phi(v) \, dxdv
\]

Combining all the five terms, taking \( \eta \) to zero in the inequality above and setting \( \varepsilon \) small enough such that

\[
\varepsilon \leq \frac{1}{2(\| F^N * f^N \|_{L^\infty} + \| g \|_{L^\infty})},
\]

where the fact that \( \| \frac{1}{\pi} * g \|_{L^\infty} \leq \| g \|_{L^\infty} \) has been used, we end up with

\[
\frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2} v^2 f^N(t, x, v) \, dxdv \leq C - \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2} v^2 f^N(t, x, v) \, dxdv,
\]
where $C$ does not depend on $N$. A direct computation shows that the kinetic energy is bounded uniformly in $t$ and $N$.

\section{Compactness Arguments}

In this section, we aim to achieve all the compactness arguments that are needed to pass the limit and to obtain the desired weak formulation of the non-cut-off kinetic equation, namely to prove the main result Theorem 2.1.

For given initial data $f_0$, let $f^N_0$ be a sequence of functions with compact support which are w.l.o.g. assumed to be in $B_N$, i.e., a ball of radius $N$ centered at origin. Furthermore $f^N_0$ satisfies

$$
\|f^N_0 - f_0\|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} \to 0, \text{ as } \ N \to \infty.
$$

Let $f^N(t, x, v)$ be the solution obtained from Theorem 3.1 with initial data $f^N_0(x, v)$. Then, we know

$$
0 \leq f^N(t, x, v) \leq \|f_0\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}, \text{ for a.e. } (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2, \ t \geq 0,
$$

and there exists a subsequence of $f^N$, still denoted by $f^N$ for simplicity, such that

$$
f^N \rightharpoonup f \text{ in } L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)).
$$

Moreover, we notice that the total mass is preserved, i.e.,

$$
\iint_{\mathbb{R}^2 \times \mathbb{R}^2} f^N(t, x, v) \, dx \, dv = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f^N_0(x, v) \, dx \, dv =: M_0.
$$

By the definition of weak* convergence for characteristic functions $\chi_{|x|+|v| \leq r} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, we have for each $a < b \in \mathbb{R}_+$

$$
\int_a^b \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_{|x|+|v| \leq r} f(t, x, v) \, dx \, dv \, dt = \lim_{N \to \infty} \int_a^b \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_{|x|+|v| \leq r} f^N(t, x, v) \, dx \, dv \, dt \leq \lim_{N \to \infty} \int_a^b \int_{\mathbb{R}^2 \times \mathbb{R}^2} f^N(t, x, v) \, dx \, dv \, dt = M_0(b - a).
$$

Letting $r \to \infty$ and applying Fatou’s lemma yields

$$
\int_a^b \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) \, dx \, dv \, dt \leq \lim_{r \to \infty} \int_a^b \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_{|x|+|v| \leq r} f(t, x, v) \, dx \, dv \, dt \leq \lim_{r \to \infty} \int_a^b \int_{\mathbb{R}^2 \times \mathbb{R}^2} f^N(t, x, v) \, dx \, dv \, dt = M_0(b - a).
$$

By a similar argument for test functions of type $\chi_{|x|+|v| \leq r} |v|^2$, we can show that

$$
\int_a^b \int_{\mathbb{R}^2 \times \mathbb{R}^2} |v|^2 f(t, x, v) \, dx \, dv \, dt \leq C(b - a)
$$

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by using
\[ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2} |v|^2 f^N(t, x, v) \, dx \, dv \leq C(b - a), \quad \forall t \geq 0. \]

Since the above two inequalities hold for all \( a < b \in \mathbb{R}_+ \), they also hold for a.e. \( t \in \mathbb{R}_+ \).

Using all the estimates presented in Theorem 3.1, we are now ready to pass the limit in (3.2) to the desired weak formulation of the non-cut-off kinetic equation
\[ \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left( [(F * f) f] + \nabla_v \cdot (G f) \right) = 0. \]

However, we need to take special care on the non-linear term, i.e., the consideration of the function \( F^N * f^N \). We remark again that the total force \( F^N \) is considered on a bounded domain, i.e., the function has compact support within a ball of radius \( \tilde{R} := \max\{2R, 2\tilde{R}\} \), denoted by \( B_{\tilde{R}} \), where \( R \) and \( \tilde{R} \) are defined as
\[
\mathcal{H}_2R(|x|) = \begin{cases} 
0, & |x| > 2R, \\
1, & |x| < R,
\end{cases}
\quad \text{and} \quad
\tilde{\mathcal{H}}_{2\tilde{R}}(|v|) = \begin{cases} 
0, & |v| > 2\tilde{R}, \\
1, & |v| < \tilde{R}.
\end{cases}
\]

In the following, we use the notation \( L^p(L^q) \) to denote \( L^p(\mathbb{R}_+; L^q(\mathbb{R}^2 \times \mathbb{R}^2)) \), \( 1 \leq p, q \leq \infty \). It is obvious to see that
\[
\left\| F^N * f^N \right\|_{L^\infty(L^1)} = \left\| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} F^N(x - y, v - w) f^N(t, y, w) \, dy \, dw \right\|_{L^\infty(\mathbb{R}_+)}
\leq C \left( \left\| F \right\|_{L^1, \mathcal{M}_0}, \tilde{R} \right).
\]

Since \( \nabla_v F^N \) is bounded uniformly in \( N \), we get
\[
\left\| \nabla_v \left( F^N * f^N \right) \right\|_{L^\infty(L^1)} = \left\| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla_v F^N(x - y, v - w) f^N(t, y, w) \, dy \, dw \right\|_{L^\infty(\mathbb{R}_+)}
\leq C \left( \left\| \nabla_v F \right\|_{L^1, \mathcal{M}_0}, \tilde{R} \right).
\]

and
\[
\left\| \nabla_v \left( F^N * f^N \right) \right\|_{L^\infty(L^\infty)} = \left\| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla_v F^N(x - y, v - w) f^N(t, y, w) \, dy \, dw \right\|_{L^\infty(L^\infty)}
\leq C \left( \left\| \nabla_v F \right\|_{L^\infty, \mathcal{M}_0} \right).
So far, we can conclude by interpolation that $F^N * f^N$ and $\nabla_v F^N * f^N$ are in $L^\infty(L^2)$. Furthermore, it holds
\[
\| \nabla_x (F^N * f^N) \|_{L^\infty(L^2)} \leq C \left\| \left( \chi_R \cdot \frac{1}{|x|} \right) * f^N \right\|_{L^\infty(L^2)} \leq \| f^N \|_{L^\infty(L^p)}, \quad \forall \ p > 1,
\]
where $\chi_R \cdot \frac{1}{|x|} \in L^\infty(\mathbb{R})$, $\forall \ 1 < r < 2$, and Young’s inequality have been used. Hence, we conclude that $F^N * f^N$ then belongs to $L^\infty(\mathbb{R}^+; W^{1,2}(\mathbb{R}^2 \times \mathbb{R}^2))$. Since
\[
\int_\mathbb{R}^2 \left( v f^N(t, x, v) \right)^2 \, dv \leq \| f^N \|_{L^\infty} \| v^2 f^N \|_{L^\infty(L^1)} \leq C,
\]
we can get for every $\varphi \in C^\infty_0(\mathbb{R}^2 \times \mathbb{R}^2)$ that
\[
\left\| \int_\mathbb{R}^2 v f^N(t, x, v)\nabla_v \varphi(x, v) \, dv \right\|_{L^\infty(\mathbb{R}^+)} \leq \left\| f^N \right\|_{L^\infty(\mathbb{R}^+)} \| v^2 f^N \|_{L^\infty(L^1)} \| \nabla_v \varphi \|_{L^2} \leq C \left\| \nabla_v \varphi \right\|_{L^2}.
\]
(4.1)
Moreover, we have
\[
\left\| \int_\mathbb{R}^2 G^N(x, v) f^N(t, x, v) \nabla_v \varphi(x, v) \, dv \right\|_{L^\infty(\mathbb{R}^+)} \leq \left\| j \chi_R \cdot \frac{g}{L^\infty} \left( f^N \right) \right\|_{L^\infty(\mathbb{R}^+)} \left\| f^N \right\|_{L^\infty(L^1)} \| \nabla_v \varphi \|_{L^2} + \left\| f^N \right\|_{L^\infty(\mathbb{R}^+)} \| v^2 f^N \|_{L^\infty(L^1)} \| \nabla_v \varphi \|_{L^2} \leq C \| \nabla_v \varphi \|_{L^2}.
\]
(4.2)
On the other hand, we know
\[
\left\| \int_\mathbb{R}^2 \partial_t f^N(t, x, v) \varphi(x, v) \, dv \right\|_{L^\infty(\mathbb{R}^+)} \leq \left\| f^N \right\|_{L^\infty(L^\infty)} \| \partial_t \|_{L^\infty(\mathbb{R}^+)} \| \varphi \|_{L^2} \leq C \| \varphi \|_{L^2}.
\]
(4.3)
Combining (4.1)-(4.3), it holds for every $\varphi \in C^\infty_0(\mathbb{R}^2 \times \mathbb{R}^2)$ that
\[
\left\| \int_\mathbb{R}^2 \partial_t f^N(t, x, v) \varphi(x, v) \, dv \right\|_{L^\infty(\mathbb{R}^+)} \leq C \| \varphi \|_{W^{1,2}}.
\]
which implies

\[ \left\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \partial_t \left( (F^N \ast f^N)(t,x,v) \right) \varphi(x,v) \, dx \, dv \right\|_{L^\infty(\mathbb{R}_+)} = \left\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \partial_t f^N(t,x,v)(F^N \ast \varphi)(x,v) \, dx \, dv \right\|_{L^\infty(\mathbb{R}_+)} \leq C \|F^N \ast \varphi\|_{W^{1,2}} = C \left\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} F^N(y,w) \varphi(x-y,v-w) \, dy \, dw \right\|_{W^{1,2}} \leq C \|F\|_{L^\infty} \|\varphi\|_{W^{1,2}} \]

or, in other words,

\[ \|\partial_t (F^N \ast f^N)\|_{L^\infty(W^{-1,2})} = \|F^N \ast \partial_t f^N\|_{L^\infty(W^{-1,2})} \leq C. \]

Now for any \( T > 0 \), we then get \( \forall \varphi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \)

\[ F^N \ast f^N \in L^\infty([0,T];W^{1,2}(\Omega)), \quad \partial_t(F^N \ast f^N) \in L^\infty([0,T];W^{-1,2}(\Omega)), \]

where \( \Omega = \text{supp} \varphi \). According to Aubin-Lions compact embedding theorem, e.g. [26], [6], there exists a subsequence and \( h \in L^\infty([0,T];L^2(\Omega)) \) such that

\[ F^N \ast f^N \to h \quad \text{in} \quad L^\infty([0,T];L^2(\Omega)). \]

It is not difficult to check that \( h = F \ast f \). Therefore we obtain the following estimates:

\[
\begin{align*}
&\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \left( (F^N \ast f^N) f^N \right)(s,x,v) \nabla_v \varphi(x,v) - \left( (F \ast f) f \right)(s,x,v) \nabla_v \varphi(x,v) \right) \, dx \, dv \, ds \right| \\
= &\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \left( (F^N \ast f^N) f^N \right)(s,x,v) \nabla_v \varphi(x,v) - \left( (F \ast f) f \right)(s,x,v) \nabla_v \varphi(x,v) \right) + \left( (F \ast f) f \right)(s,x,v) \nabla_v \varphi(x,v) \, dx \, dv \, ds \right| \\
\leq &\left| \int_0^t \int_{\Omega} \left( \left( (F^N \ast f^N) f^N \right)(s,x,v) \nabla_v \varphi(x,v) - \left( (F \ast f) f \right)(s,x,v) \nabla_v \varphi(x,v) \right) \, dx \, dv \, ds \right| \\
&+ \left| \int_0^t \int_{\Omega} \left( (F \ast f) f \right)(s,x,v) \nabla_v \varphi(x,v) \, dx \, dv \, ds \right| \\
= & J_1 + J_2.
\end{align*}
\]

For the first term \( J_1 \), we have

\[ \lim_{N \to \infty} J_1 \leq \lim_{N \to \infty} \left\| F^N \ast f^N - F \ast f \right\|_{L^\infty(L^2(\Omega))} \left\| f^N \right\|_{L^\infty(L^\infty)} \left\| \nabla_v \varphi \right\|_{L^2} = 0 \]

while for the second term \( J_2 \) we use the fact that \( f^N \rightharpoonup f \) in \( L^\infty(\mathbb{R}_+;L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)) \) for \( F \ast f \cdot \nabla_v \varphi \in L^1(\mathbb{R}) \), namely

\[ \lim_{N \to \infty} J_2 = 0. \]

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Finally, we have to examine the initial data. Since $f^N$ is the weak solution to the cut-off mean field equation (3.2), it obviously satisfies

$$
\int_{\mathbb{R}^2 \times \mathbb{R}^2} f^N(t,x,v) \varphi(x,v) \, dx \, dv = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0^N(x,v) \varphi(x,v) \, dx \, dv \\
+ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} v f^N(s,x,v) \cdot \nabla_x \varphi(x,v) \, dx \, dv \, ds \\
+ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( F^N(x,v) * f^N(s,x,v) \right) f^N(s,x,v) \cdot \nabla_x \varphi(x,v) \, dx \, dv \, ds \\
+ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} G^N(x,v) f^N(s,x,v) \cdot \nabla_v \varphi(x,v) \, dx \, dv \, ds
$$

for any test function $\varphi(x,v) \in C^\infty_c(\mathbb{R}^2 \times \mathbb{R}^2)$. We recall

$$
\| f_0^N - f_0 \|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} \to 0, \text{ as } N \to \infty,
$$

and that terms on the right (second till last) hand side are uniformly continuous in time $t$. Then, taking limit $t \to 0^+$ on both sides of the above equation verifies the initial data.

## 5 Summary

This paper deals with the core problem, which is to show existence of the $L^\infty((0, \infty); L^\infty(\mathbb{R}^2 \times \mathbb{R}^2))$-solution to the Vlasov equation for a pedestrian flow model. Our main results, Theorem 2.1 and Theorem 3.1, state that there exists a weak solution to the mean field equation (or approximate equation with cut-off) to the pedestrian flow model. The solution is proven to satisfy the mass conservation and energy bounds, respectively. In particular, this paper addresses technical difficulties caused by the singular interaction force and non-Lipschitz continuous self-generating force.

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