SIMPSON TYPE INEQUALITIES FOR FUNCTIONS WHOSE THIRD DERIVATIVES IN THE ABSOLUTE VALUE ARE $s$–CONVEX AND $s$–CONCAVE FUNCTIONS

M.E. ÖZDEMİR⋆, MERVE AVCI⋆♦, AND HAVVA KAVURMACI⋆

Abstract. In this paper, we established some new inequalities via $s$–convex and $s$–concave functions.

1. INTRODUCTION

The following inequality is well known in the literature as Simpson’s inequality:

$\left| \int_a^b f(x)dx - \frac{b-a}{3} \left[ \frac{f(a)+f(b)}{2} + 2f \left( \frac{a+b}{2} \right) \right]\right| \leq \frac{1}{2880} \left\| f^{(4)} \right\|_\infty (b-a)^5,$

where the mapping $f : [a, b] \rightarrow \mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on $(a, b)$, that is,

$\left\| f^{(4)} \right\|_\infty = \sup_{t \in (a,b)} \left| f^{(4)}(t) \right| < \infty.$

For some results which generalize, improve and extend the inequality (1.1), see the papers [9]–[13].

In [3], Hudzik and Maligranda considered among others the class of functions which are $s$–convex in the second sense. This class is defined in the following way: A function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be $s$–convex in the second sense if

$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of $s$–convex functions in the second sense is usually denoted by $K_s^2$.

It can be easily seen that for $s = 1$, $s$–convexity reduces to ordinary convexity of functions defined on $[0, 1)$.

Some interesting and important inequalities for $s$–convex functions can be found in [3]–[8].

In [4], Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for $s$–convex functions in the second sense.

Key words and phrases. Simpson inequality, $s$–convex function, $s$–concave function, Hölder inequality, Power-mean inequality.

♦corresponding author.
Theorem 1. Suppose that \( f : [0, \infty) \to [0, \infty) \) is an \( s \)-convex function in the second sense, where \( s \in (0, 1] \) and let \( a, b \in [0, \infty) \), \( a < b \). If \( f' \in L^1[a, b] \), then the following inequalities hold:

\[
2^{s-1} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}.
\]

The constant \( k = \frac{1}{s+1} \) is the best possible in the second inequality in (1.2). The above inequalities are sharp.

In [2], Barani et. al obtained the following results.

Theorem 2. Let \( f : I \to \mathbb{R} \) be a function such that \( f''' \) be absolutely continuous on \( I^0 \). Assume that \( a, b \in I^0 \), with \( a < b \) and \( f''' \in L[a, b] \). If \( |f'''| \) is a \( P \)-convex function on \( [a, b] \) then, the following inequality holds:

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \leq \frac{(b-a)^4}{1152} \left\{ |f'''(a)| + |f'''\left( \frac{a+b}{2} \right)| + |f'''(b)| \right\}.
\]

Corollary 1. Let \( f \) as in Theorem 2. If \( f''' \left( \frac{a+b}{2} \right) = 0 \), then we have

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \leq \frac{(b-a)^4}{1152} \left\{ |f'''(a)| + |f'''(b)| \right\}.
\]

The main purpose of this paper is to establish some new inequalities for functions whose third derivatives in the absolute value are \( s \)-convex and \( s \)-concave.

2. INEQUALITIES FOR \( s \)-CONVEX FUNCTIONS IN THE SECOND SENSE

To prove our new results we need the following lemma (see [1]).

Lemma 1. Let \( f : I \to \mathbb{R} \) be a function such that \( f''' \) be absolutely continuous on \( I^0 \), the interior of \( I \). Assume that \( a, b \in I^0 \), with \( a < b \) and \( f''' \in L[a, b] \). Then, the following equality holds:

\[
\int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] = (b-a)^4 \int_0^1 p(t)f'''(ta + (1-t)b)dt,
\]

where

\[
p(t) = \begin{cases} 
\frac{1}{8}t^2 \left( t - \frac{1}{2} \right), & t \in [0, \frac{1}{2}] \\
\frac{1}{8}(t-1)^2 \left( t - \frac{1}{2} \right), & t \in (\frac{1}{2}, 1].
\end{cases}
\]

Theorem 3. Let \( f : I \subset [0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^0 \) such that \( f''' \in L[a, b] \), where \( a, b \in I^0 \) with \( a < b \). If \( |f'''| \) is \( s \)-convex in the second sense
on \([a, b]\) and for some fixed \(s \in (0, 1]\), then

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq \frac{(b-a)^4}{6} \left[ \frac{2^{-4-s}((1+s)(2+s) + 34 + 2^{4+s}(-2+s) + 11s + s^2)}{(1+s)(2+s)(3+s)(4+s)} \right] \\
\times [f'''(a)] + [f'''(b)].
\]

**Proof.** From Lemma \(\text{(1)}\) and \(s\)-convexity of \(|f'''|\), we have

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq \frac{(b-a)^4}{6} \left\{ \int_0^{\frac{1}{2}} \frac{1}{6} t^2 \left( t - \frac{1}{2} \right) \left| f'''(ta + (1-t)b) \right| dt \\
+ \int_{\frac{1}{2}}^1 \frac{1}{6} (t-1)^2 \left( t - \frac{1}{2} \right) \left| f'''(ta + (1-t)b) \right| dt \right\}
\]

\[
= \frac{(b-a)^4}{6} \left[ \frac{2^{-4-s}((1+s)(2+s) + 34 + 2^{4+s}(-2+s) + 11s + s^2)}{(1+s)(2+s)(3+s)(4+s)} \right] \\
\times [f'''(a)] + [f'''(b)],
\]

where we use the fact that

\[
\int_0^{\frac{1}{2}} t^{2+s} \left( \frac{1}{2} - t \right) dt = \int_{\frac{1}{2}}^1 (1-t)^{s+2} \left( t - \frac{1}{2} \right) = \frac{2^{-4-s}}{(3+s)(4+s)}
\]

and

\[
\int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) (1-t)^s dt = \int_{\frac{1}{2}}^1 (t-1)^2 \left( t - \frac{1}{2} \right) t^s dt = \frac{2^{-4-s} (34 + 2^{4+s}(-2+s) + 11s + s^2)}{(1+s)(2+s)(3+s)(4+s^2)}.
\]

\(\square\)

**Remark 1.** With all the assumptions of Theorem \(\text{(3)}\) if we choose \(s = 1\), we have the inequality in \(\text{(1.3)}\).

**Theorem 4.** Let \(f : I \subset [0, \infty) \to \mathbb{R}\) be a differentiable function on \(I^o\) such that \(f''' \in L[a, b]\), where \(a, b \in I^o\) with \(a < b\). If \(|f'''|^q\) is \(s\)-convex in the second sense on \([a, b]\) and for some fixed \(s \in (0, 1]\) and \(q > 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\), then the following
inequality holds:
\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \leq \frac{(b-a)^4}{48} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right) \\
\times \left\{ \left[ \frac{1}{2s+1(s+1)} |f'''(a)|^q + \frac{2^{s+1}-1}{2s+1(s+1)} |f'''(b)|^q \right]^{\frac{1}{q}} \\
+ \left[ \frac{2^{s+1}-1}{2s+1(s+1)} |f'''(a)|^q + \frac{1}{2s+1(s+1)} |f'''(b)|^q \right]^{\frac{1}{q}} \right\}.
\]

**Proof.** From Lemma [1] using the \( s \)-convexity of \( |f'''|^q \) and the well-known Hölder’s inequality we have

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \\
\leq \frac{(b-a)^4}{6} \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}\Gamma(3p+2)} \right)^{\frac{1}{p}} \\
\times \left\{ \left[ \int_0^{\frac{1}{2}} \left( t^2 \left( \frac{1}{2} - t \right) \right)^p dt \right]^{\frac{1}{p}} \right\} \\
\leq \frac{(b-a)^4}{48} \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right)^{\frac{1}{p}} \\
\times \left\{ \left[ \frac{1}{2s+1(s+1)} |f'''(a)|^q + \frac{2^{s+1}-1}{2s+1(s+1)} |f'''(b)|^q \right]^{\frac{1}{q}} \\
+ \left[ \frac{2^{s+1}-1}{2s+1(s+1)} |f'''(a)|^q + \frac{1}{2s+1(s+1)} |f'''(b)|^q \right]^{\frac{1}{q}} \right\}.
\]

where

\[
\int_0^{\frac{1}{2}} \left( t^2 \left( \frac{1}{2} - t \right) \right)^p dt = \int_{\frac{1}{2}}^{1} \left( t-1 \right)^2 \left( t - \frac{1}{2} \right)^p dt = \frac{8^{-p}\Gamma(2p+1)\Gamma(p+1)}{2\Gamma(3p+2)}
\]

and \( \Gamma \) is the Gamma function.
Corollary 2. If we choose \( s = 1 \) in Theorem 4, we have

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq \frac{(b-a)^4}{96} \left( \frac{1}{4} \right)^{\frac{1}{p}} \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right)^{\frac{1}{p}}
\]

\[
\times \left\{ \left( |f'''(a)|^q + 3|f'''(b)|^q \right)^{\frac{1}{q}} + \left( 3|f'''(a)|^q + |f'''(b)|^q \right)^{\frac{1}{q}} \right\}.
\]

Theorem 5. Suppose that all the assumptions of Theorem 4 are satisfied. Then

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq \frac{(b-a)^4}{6} \left( \frac{1}{192} \right)^{1-\frac{1}{q}}
\]

\[
\times \left\{ \left( \frac{2^{-4-s}}{(3+s)(4+s)} |f'''(a)|^q + \frac{2^{-4-s} (34 + 2^{4+s}(-2 + s) + 11s + s^2)}{(1 + s)(2 + s)(3 + s)(4 + s)} |f'''(b)|^q \right)^{\frac{1}{q}}
\]

\[
+ \left( \frac{2^{-4-s} (34 + 2^{4+s}(-2 + s) + 11s + s^2)}{(1 + s)(2 + s)(3 + s)(4 + s)} |f'''(a)|^q + \frac{2^{-4-s}}{(3+s)(4+s)} |f'''(b)|^q \right)^{\frac{1}{q}} \right\}.
\]

Proof. From Lemma 1 and using the well-known power-mean inequality we have

\[
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq \frac{(b-a)^4}{6} \left\{ \left( \int_0^\frac{1}{2} t^2 \left( \frac{1}{2} - t \right) dt \right)^{1-\frac{1}{q}} \left( \int_0^\frac{1}{2} t^2 \left( \frac{1}{2} - t \right) |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_\frac{1}{2}^1 (t-1)^2 \left( t - \frac{1}{2} \right) dt \right)^{1-\frac{1}{q}} \left( \int_\frac{1}{2}^1 (t-1)^2 \left( t - \frac{1}{2} \right) |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}.
\]

Since \( |f'''| \) is \( s \)-convex, we have

\[
\int_0^\frac{1}{2} t^2 \left( \frac{1}{2} - t \right) |f'''(ta + (1-t)b)|^q dt \leq \int_0^\frac{1}{2} t^2 \left( \frac{1}{2} - t \right) \left( t^s |f'''(a)| + (1-t)^s |f'''(b)| \right) dt
\]

\[
= \frac{2^{-4-s}}{(3+s)(4+s)} |f'''(a)|^q + \frac{2^{-4-s} (34 + 2^{4+s}(-2 + s) + 11s + s^2)}{(1 + s)(2 + s)(3 + s)(4 + s)} |f'''(b)|^q.
\]
Theorem 6. Therefore we have

\[
\begin{align*}
\int_1^b (t - 1)^2 \left( t - \frac{1}{2} \right) |f'''(ta + (1 - t)b)|^q \, dt \\
\leq \int_1^b (t - 1)^2 \left( t - \frac{1}{2} \right) (t^s |f'''(a)| + (1 - t)^s |f'''(b)|) \, dt \\
= \frac{2^{-4-s} (34 + 2^{4+s}(-2 + s) + 11s + s^2)}{(1 + s)(2 + s)(3 + s)(4 + s)} |f'''(a)|^q + \frac{2^{-4-s}}{(3 + s)(4 + s)} |f'''(b)|^q.
\end{align*}
\]

Therefore we have

\[
\left| \int_a^b f(x) \, dx - \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] \right|
\leq \frac{(b-a)^4}{6} \left( \frac{1}{192} \right)^{1-s/4} \times \left\{ \left( \frac{2^{-4-s}}{(3 + s)(4 + s)} |f'''(a)|^q + \frac{2^{-4-s} (34 + 2^{4+s}(-2 + s) + 11s + s^2)}{(1 + s)(2 + s)(3 + s)(4 + s)} |f'''(b)|^q \right)^{\frac{s}{4}} + \left( \frac{2^{-4-s} (34 + 2^{4+s}(-2 + s) + 11s + s^2)}{(1 + s)(2 + s)(3 + s)(4 + s)} |f'''(a)|^q + \frac{2^{-4-s}}{(3 + s)(4 + s)} |f'''(b)|^q \right)^{\frac{s}{4}} \right\},
\]

which is the required result. \[\Box\]

Corollary 3. If we choose \( s = 1 \) in Theorem 3 we have

\[
\left| \int_a^b f(x) \, dx - \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] \right|
\leq \frac{(b-a)^4}{1152} \left\{ \left( \frac{3 |f'''(a)|^q + 7 |f'''(b)|^q}{10} \right)^{\frac{s}{4}} + \left( \frac{7 |f'''(a)|^q + 3 |f'''(b)|^q}{10} \right)^{\frac{s}{4}} \right\},
\]

The following result holds for \( s \)-concave functions.

Theorem 6. Let \( f : I \subset [0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^o \) such that \( f''' \in L[a,b] \), where \( a, b \in I^o \) with \( a < b \). If \( |f'''|^{\frac{q}{3}} \) is \( s \)-concave on \([a,b]\) for \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| \int_a^b f(x) \, dx - \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] \right|
\leq \frac{(b-a)^4}{48} \left( \frac{1}{2} \right)^{\frac{s}{4}} \left( 2^{\frac{s+2}{2}} \right)^{\frac{s}{4}} \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \left\{ |f''\left( \frac{a+3b}{4} \right)| + |f''\left( \frac{3a+b}{4} \right)| \right\}.\]
Proposition 1. Let \( a, b \in \mathbb{R} \), \( a < b \) and \( 0 < s < 1 \). Then, for all \( q > 1 \), we have

\[
\left| L_s^*(a, b) - \frac{1}{3} A^*(a, b^s) - \frac{2}{3} A^*(a, b) \right|
\leq \frac{(b-a)^3}{96} \left( \frac{1}{2} \right)^{1-q} s (1-s) (2-s) \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right)^{\frac{1}{2}}
\times \left\{ \left| a \right|^{(s-3)q} + 3 \left| b \right|^{(s-3)q} \right\}^{\frac{1}{2}} + \left( 3 \left| a \right|^{(s-3)q} + \left| b \right|^{(s-3)q} \right)^{\frac{1}{2}}.
\]

Proof. From Lemma 1 and using the Hölder’s inequality, we have

\[
\left( \int_a^b f(x) \, dx \right) \leq \left( \frac{b-a}{6} \right)^{\frac{1}{2}} \left( \int_a^b \left( f(x) + \frac{f(a) + f(b)}{2} \right) \right) \left( \int_a^b |f''(x)|^q \, dx \right)^{\frac{1}{q}} \left( \int_a^b f'''(x) \, dx \right)^{\frac{1}{q}}.
\]

Since \( f''''(x) \) is \( s \)-concave, using the inequality (1.2), we have

\[
\int_0^1 \left| f''''(x) \right| \, dx \leq 2^{s-2} \left| f'''' \left( \frac{a+3b}{4} \right) \right|^q.
\]

and

\[
\int_0^1 \left| f''''(x) \right| \, dx \leq 2^{s-2} \left| f'''' \left( \frac{3a+b}{4} \right) \right|^q.
\]

From (2.1), (2.2), and (2.3), we get

\[
\left| \int_a^b f(x) \, dx - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right|
\leq \frac{(b-a)^4}{48} \left( \frac{1}{2} \right)^{\frac{3}{2}} \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right)^{\frac{1}{2}} 2^{-\frac{3s}{2}} \left\{ \left| f'''' \left( \frac{a+3b}{4} \right) \right| + \left| f'''' \left( \frac{3a+b}{4} \right) \right| \right\}
\]

which completes the proof. \( \square \)

3. Applications to Special Means

We consider the means for nonnegative real numbers \( a < b \) as follows:

1. Arithmetic mean:

\[
A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}.
\]

2. Generalized log-mean:

\[
L_s(\alpha, \beta) = \left( \frac{\beta^{s+1} - \alpha^{s+1}}{(s+1)(\beta - \alpha)} \right)^{\frac{1}{s}}, \quad \alpha, \beta \in \mathbb{R} \setminus \{-1, 0\}, \quad s \in \mathbb{R} \setminus \{-1, 0\}.
\]

We give some applications to special means of real numbers by using the results of Section 2.

Proposition 1. Let \( a, b \in \mathbb{R} \), \( a < b \) and \( 0 < s < 1 \). Then, for all \( q > 1 \), we have

\[
\left| L_s^*(a, b) - \frac{1}{3} A^*(a, b^s) - \frac{2}{3} A^*(a, b) \right|
\leq \frac{(b-a)^3}{96} \left( \frac{1}{2} \right)^{1-q} s (1-s) (2-s) \left( \frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right)^{\frac{1}{2}}
\times \left\{ \left| a \right|^{(s-3)q} + 3 \left| b \right|^{(s-3)q} \right\}^{\frac{1}{2}} + \left( 3 \left| a \right|^{(s-3)q} + \left| b \right|^{(s-3)q} \right)^{\frac{1}{2}}.
\]
Proof. The assertion follows from Corollary 2 applied to the $s-$convex mapping $f : [0, 1] \to [0, 1]$, $f(x) = x^s$. □

Proposition 2. Let $a, b \in I^o$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have

\[
\left| L_s^2(a, b) - \frac{1}{3} A(a^s, b^s) - \frac{2}{3} A^s(a, b) \right| \leq \frac{(b-a)^3}{1152} s(1-s)(2-s) \left\{ \frac{3 |a|^{(s-3)q} + 7 |b|^{(s-3)q}}{10} \right\} + \left( \frac{7 |a|^{(s-3)q} + 3 |b|^{(s-3)q}}{10} \right) .
\]

Proof. The assertion follows from Corollary 3 applied to the $s-$convex mapping $f : [0, 1] \to [0, 1]$, $f(x) = x^s$. □

References

[1] M. Alomari and S. Hussain, Two inequalities of Simpson type for quasi-convex functions and applications, Appl. Math. E-notes, 11 (2001) 110-117.

[2] A. Barani, S. Barani and S.S. Dragomir, Simpson’s type inequalities for functions whose third derivatives in the absolute values are $P-$convex, RGMIA Res. Rep. Coll., 14(2011), Preprints, Article 95.

[3] H. Hudzik and L. Maligranda, Some remarks on $s-$convex functions, Aequationes Math., 48 (1994) 100–111.

[4] S.S. Dragomir and S. Fitzpatrick, The Hadamard’s inequality for $s-$convex functions in the second sense, Demonstratio Math., 32(4) (1999) 687–696.

[5] U.S. Kirmaci et al, Hadamard-type inequalities for $s-$convex functions, Appl. Math. Comp., 193 (2007) 26–35.

[6] M. Alomari, M. Darus and S.S. Dragomir, New inequalities of Simpson’s type for $s$-convex functions with applications, RGMIA Res. Rep. Coll., 12 (4) (2009) Article 9.

[7] M.Z. Sarikaya, E. Set and M.E. Özdemir, On new inequalities of Simpson’s type for $s$-convex functions, Comput. Math. Appl., 60 (2010) 2191-2199.

[8] M. Avci, H. Kavurmaci and M.E. Özdemir, New inequalities of Hermite–Hadamard type via $s$--convex functions in the second sense with applications, Appl. Math. and Comput., 217(2011) 5171-5176.

[9] S.S. Dragomir, R.P. Agarwal and P. Cerone, On Simpson’s inequality and applications, J. Inequal. Appl., 5(2000), no.6, 533–579.

[10] K.L. Tseng, G.S. Yang and S.S. Dragomir, OnWeighted Simpson Type Inequalities and Applications, RGMIA Res. Rep. Coll., 7(1) (2004), Article 17.

[11] M.Z. Sarikaya, E. Set and M.E. Özdemir, On new inequalities of Simpson’s type for convex functions, RGMIA Res. Rep. Coll., 13 (2) (2010) Article 2.

[12] B.Z. Liu, An inequality of Simpson type, Proc. R. Soc. A., 461 (2005) 2155–2158.

[13] N. Ujević, New bounds for Simpson’s inequality, Tamkang J. of Math., 33 (2002), no.2, 129–138.