Bilinear equations and $q$-discrete Painlevé equations satisfied by variables and coefficients in cluster algebras

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Abstract
We construct cluster algebras the variables and coefficients of which satisfy the discrete mKdV equation, the discrete Toda equation and other integrable bilinear equations, several of which lead to $q$-discrete Painlevé equations. These cluster algebras are obtained from quivers with a finite number of vertices or with the mutation-period property. We will also show that a suitable transformation of quivers corresponds to a reduction of the difference equation.

Keywords: discrete integral system, cluster algebra, discrete Painlevé equation

1. Introduction
In this article, we deal with cluster algebras, which were introduced by Fomin and Zelevinsky [1, 2]. A cluster algebra is a commutative ring described by cluster variables and coefficients. A generating set of the cluster algebra is defined by mutation, which is a transformation of a seed consisting of a set of cluster variables, coefficients, and a quiver. Cluster variables and coefficients obtained from a mutation of an initial seed satisfy some difference equations. It is known that cluster variables can satisfy the discrete KdV equation [4] and the Hirota–Miwa equation [5], when the initial seed includes suitable quivers with infinite vertices [6]. These quivers have the property that an infinite number of mutations gives a permutation of its vertices. This property is called ‘mutation-period’ and a quiver with this property is called a mutation-periodic quiver. Several results concerning mutation-periodic quivers have been reported in [3]. All the mutation-periodic quivers in which a permutation of its vertices is achieved by a single mutation have already obtained. The quiver which gives the discrete KdV equation is obtained from a transformation of the quiver of the Hirota–Miwa equation. This transformation corresponds to a reduction from the Hirota–Miwa equation to the discrete
KdV equation. In this paper, we construct the cluster algebras whose variables and coefficients satisfy the discrete mKdV equation, the discrete Toda equation [7], and some $q$-discrete Painlevé equations [8]. We introduce the quiver which generalizes the one that corresponds to the discrete KdV equation and the Hirota–Miwa equation. Quivers of $q$-Painlevé I, II equations and their higher order analogues have been obtained in [6, 9]. We shall introduce the quivers for the $q$-Painlevé III, VI equations, which are mutation-periodic and are obtained from transformations of quivers for the discrete KdV equation and the discrete mKdV equation.

2. Cluster algebras

In this section, we briefly explain the notion of cluster algebra which we use in the following sections. Let $x = (x_1, x_2, \ldots, x_N), y = (y_1, y_2, \ldots, y_N)$ be $N$-tuple variables. Let $Q$ be a quiver with $N$ vertices. Consider the quiver whose vertices correspond to the cluster variables. We assume that the quiver does not have a loop ($i \to i$) or a two-cycle ($i \to j \to i$). Each $x_i$ is called a cluster variable and each $y_i$ is called a coefficient. The triple $(Q, x, y)$ is called a seed.

Let $\lambda_{ij}$ be the number of arrows from $i$ to $j$ of the quiver $Q$. We define $\lambda_{ij}$ for all $i, j \leq N$ as $\lambda_{ij} = -\lambda_{ji}$. As a quiver $Q$ does not have a loop, $\lambda_{ii} = 0$. A mutation is a particular transformation of seeds.

**Definition 2.1.** Let $\mu_k: (Q, x, y) \mapsto (Q', x', y')$ ($k = 1, 2, \ldots, N$) be the mutation at the vertex $k$ of the quiver $Q$, defined as follows.

- $Q'$ is a new quiver, obtained by three operations on the quiver $Q$.
  - * For all $(i, j)$ such that $\lambda_{i,k} > 0, \lambda_{k,i} > 0$, we add $\lambda_{i,k} \lambda_{k,i}$ arrows from $i$ to $j$.
  - * If two-cycles appear by the operation 1, we remove all of them.
  - * We reverse the direction of all directed arrows which have edges at the vertex $k$.
- New coefficients $y' = (y'_1, y'_2, \ldots, y'_N)$ are defined from $Q$ and $y$ as:
  $$
  y'_i = \begin{cases}
  y^{-1}_k, & (k = i) \\
  y_i (y_k^{-1} + 1)^{-\lambda_{i,k}} & (\lambda_{i,k} > 0), \\
  y_i (y_k + 1)^{\lambda_{i,k}} & (\lambda_{i,k} > 0), \\
  y_i & (\lambda_{i,k} = 0).
  \end{cases}
  \tag{2.1}
  $$
- New cluster variables $x' = (x'_1, x'_2, \ldots, x'_N)$ are defined from $Q$, $y$ and $x$ as:
  $$
  x'_i = \frac{1}{(y_k + 1)x_k} \left( \prod_{\lambda_{j,k} > 0} x_j^{\lambda_{j,k}} + y_k \prod_{\lambda_{j,k} > 0} x_j^{\lambda_{j,k}} \right),
  $$
  $$
  x'_i = x_i \quad (i \neq k).
  \tag{2.2}
  $$

A mutation $\mu_k$ denotes the mutation at $k$ or $x_k$. For any seed $(Q, x, y)$, it holds that $\mu_k^2(Q, x, y) = (Q, x, y)$. For any seed $(Q, x, y)$ and $(i, j)$ such that $\lambda_{i,j} = 0$, it holds that $\mu_i \mu_j(Q, x, y) = \mu_j \mu_i(Q, x, y)$. 

Definition 2.2. Let us fix a seed \((Q, x, y)\). This seed is called an initial seed. Let \(A(Q, x, y)\) be a cluster algebra (with coefficients) defined as

\[
A(Q, x, y) = \mathbb{Z}(y)[x | x \in X] \subset \mathbb{Q}(y)(x),
\]

where \(X\) is the set of all the cluster variables obtained from iterative mutations to the initial seed.

Now we define the coefficient-free cluster algebra as the pair of a quiver and cluster variables \((Q, x)\), which is also called a seed, and by using its mutation.

Definition 2.3. Let \(\mu_k: (Q, x) \mapsto (Q', x')\) \((k = 1, 2, ..., N)\) be the mutation defined as follows.

- The definition of a new quiver \(Q'\) is the same as in definition 2.1.
- Let \(x' = (x'_1, x'_2, ..., x'_N)\) be the new cluster variables defined by \(Q\) and \(x\) as:

\[
x'_k = \frac{1}{x_k} \left( \prod_{\lambda_i > 0} x_{k^i}^i + \prod_{\lambda_j > 0} x_{j^i}^i \right),
\]

\[
x'_i = x_i \quad (i \neq k).
\]

Definition 2.4. Let us fix an initial seed \((Q, x)\). We define a coefficient-free cluster algebra \(A(Q, x)\) as

\[
A(Q, x) = \mathbb{Z}[x | x \in X] \subset \mathbb{Q}(x),
\]

where \(X\) is the set of cluster variables obtained from a iteration of all mutations to the initial seed.

3. Bilinear equations satisfied by cluster variables and their reductions

In this section, we show that cluster variables satisfy bilinear equations related to discrete integrable systems, if the initial seed includes suitable quivers with infinite vertices. These quivers are obtained from a transformation of quivers which corresponds to a reduction of a certain difference equations.

3.1. The discrete KdV equation and the Hirota–Miwa equation

We construct cluster algebras whose cluster variables satisfy the discrete KdV equation and the Hirota–Miwa equation. The quivers of the initial seeds are generalization of the quivers corresponding to the discrete KdV equation and the Hirota–Miwa equation obtained in [6].

First we consider coefficient-free cluster algebras. For any \(N, M \geq 1\), let \(Q_{KdV}^{N,M}\) be a quiver as shown in figure 1 (the dKdV quiver), where the numbers attached to the arrows of the quivers denote the numbers of arrows pointing in the same direction. Note that each vertex \(x_{in}^m\) corresponds to a cluster variable. We denote by \(I_{KdV}\) the vertex set of \(Q_{KdV}^{N,M}\):

\[
I_{KdV} = \left\{ x_{-iM+n}, x_{iM-n} | i \in [0, N + 1], j \in [0, 1], k \in [1, M - 1], l \in \mathbb{Z} \right\}.
\]

\[\text{(3.1)}\]
Figure 1. The dKdV quiver. (The vertices correspond to cluster variables.)

Figure 2. The dKdV quiver. (The numbers at the vertices denote the order of mutations.)
where \( [i, j] = \{i, i + 1, \ldots, j \} \) \((i \leq j)\). Let \( x \) be the set of these cluster variables. We take \((Q_{KdV}^{N,M}, x)\) as an initial seed. We define \( \mu_i' \) as the iteration of all mutations at the vertices

\[
\{ x_{j+N}^{k-M} \mid j \in [0, N + 1], k \in [0, M - 1], l \in \mathbb{Z}, j + k = i \} \subset I_{KdV} \quad (N \geq M)
\]

(3.2)

or

\[
\{ x_{j+N}^{k-M} \mid j \in [0, N - 1], k \in [0, M], l \in \mathbb{Z}, j + k = i \} \subset I_{KdV} \quad (N < M).
\]

(3.3)

Note that mutations at these vertices are commutative. In fact, \( \mu_i' \) is the iteration of all mutations at the vertices \( i \) in Figure 2. Figure 2 shows the case where \( N \geq M \). We apply the mutation to the initial seed in the order \( \mu'_0, \mu'_1, \ldots, \mu'_{\max[N+1,M+1]}, \mu'_0, \mu'_1, \ldots, \mu'_{\max[N+1,M+1]} \)...

The new cluster variable obtained by mutation at \( x_{0}^{m} \) is denoted by \( x_{1}^{m+1} \). Figure 3 is a part of the dKdV quiver after the mutation \( \mu'_0 \) and the replacement of the vertex \( x_{0}^{0} \) with \( x_{1}^{2} \). Figure 4 is a part of the dKdV quiver after the mutations \( \mu'_0, \mu'_1 \ldots, \mu'_i \) for general \( i \) and replacement of the vertices. The dKdV quiver after the mutations \( \mu'_0, \mu'_1 \ldots, \mu'_{\max[N+1,M+1]} \) is the quiver obtained by replacing the variables \( x_{m}^{n} \) with \( x_{n+1}^{m+1} \) in Figure 1. We then obtain the following proposition from the definition of mutation (2.4).

**Proposition 3.1.** Reference [6] consider the coefficient-free cluster algebra \( \mathcal{A}(Q_{KdV}^{N,M}, x) \). For any \( n, m \in \mathbb{Z} \), the cluster variables \( x_{n}^{m} \) satisfy the bilinear equation

\[
x_{n+1}^{m+1} x_{n-1}^{m} = x_{n-1}^{m+1} x_{n+1}^{m} + x_{n}^{m+1} x_{n}^{m}.
\]

(3.4)

This equation is nothing but the bilinear form of the discrete KdV equation [4]. The special cases of the dKdV quiver \( Q_{KdV}^{N,M} \) and proposition 3.1 \((N, M) = (1, 1), (2, 1), (3, 1)\) are given in [6].
Figure 4. A part of the dKdV quiver after the mutations $\mu_0', \mu_1', ..., \mu_i'$ and replacement of the vertices.

Figure 5. The HM quiver. (The vertices correspond to cluster variables.)
For any $N \geq 1$, let $Q^N_{HM}$ be the quiver as shown in figure 5 (the HM quiver). We denote by $I^N_{HM}$ the vertex set of $Q^N_{HM}$:

$$I^N_{HM} = \left\{ x_{i+j+k(N-1)} \mid i \in [0, N], j, k \in \mathbb{Z} \right\}. \quad (3.5)$$

Let $x$ be these cluster variables. We take $(Q^N_{HM}, x)$ as an initial seed and mutate it in the order $0, 1, \ldots, 0, 1, \ldots,$ where $\mu^i$ is the iteration of all mutations at the vertices

$$\left\{ x_{i+j+k(N-1)} \mid j, k \in \mathbb{Z} \right\} \subset I^N_{HM}. \quad (3.6)$$

In fact, $\mu^i$ is the iteration of all mutations at the vertices $i$ in figure 6. We denote by $x^{m+1,l-1}_{n+1}$ the new cluster variable obtained by mutation at $x^{m,l}_{n}$. Then we obtain the following proposition from the definition of mutation (2.4).

**Proposition 3.2.** Reference [6] consider the coefficient-free cluster algebra $A(Q^N_{HM}, x)$. For any $n, m, l \in \mathbb{Z}$, the cluster variables $x^{m,l}_{n}$ satisfy the bilinear equation

$$x^{m+1,l-1}_{n+1} = x^{m+1,l}_{n+1} + x^{m+1,l}_{n+1} + x^{m+1,l}_{n+1}. \quad (3.7)$$

This equation is the Hirota–Miwa equation [5]. The special case of the HM quiver $Q^N_{HM}$ and proposition 3.2 ($N = 2$) are given in [6].

The quiver of the discrete KdV equation can be obtained from a certain transformation of the quiver of the Hirota–Miwa equation. In particular, we show that the dKdV quiver $Q^1_{KdV}$ is obtained from the HM quiver $Q^2_{HM}$ by applying the following two operations successively on the quiver $Q^2_{HM}$. Here the vector $(n, m, l)$ denotes the vertex $x^{m,l}_{n}$.

1. Among the arrows from $(n, m, l) + k(1, 0, 1)$ to $(n', m', l') + k(1, 0, 1)$ ($k \in \mathbb{Z}$), we remove all the arrows with $k \neq 0$ in $Q^2_{HM}$ (see figure 7).
2. We superimpose the vertices $(n, m, l) + k(1, 0, 1)$ ($k \in \mathbb{Z}$) on the vertex $(n, m, l)$. (In figure 7, we superimpose the vertices with the same character.)

The dKdV quiver $Q^1_{KdV}$ is obtained from the above operation, which will be called a $(1, 0, 1)$-reduction of a quiver. In a similar way the $(a, b, c)$-reduction of a quiver is defined.
In fact, reduction of a quiver corresponds to reduction of a difference equation. In this case, the discrete KdV equation (3.4) is obtained from the Hirota–Miwa equation (3.7) by imposing the reduction condition $x_{n+1}^{m,l} = x_n^{m,l}$ and $x_n^m := x_n^m,0$. 

Figure 7. Reduction from the HM quiver to the dKdV quiver. (Superposition of vertices with same character.)

Figure 8. Reduction from the HM quiver to the dmKdV quiver.
3.2. The discrete mKdV equation and the discrete Toda equation

We construct cluster algebras whose cluster variables satisfy the discrete mKdV equation and the discrete Toda equation by the reduction of the quiver of the Hirota–Miwa equation. Let $Q_{mKdV}$ (the dmKdV quiver) be the quiver obtained from the $(0, 0, 2)$-reduction of the HM quiver $Q^1_{HM}$ (see figures 8, 9). We denote by $I_{mKdV}$ the vertex set of $Q_{mKdV}$.

Figure 9. The dmKdV quiver.

Figure 10. The dmKdV quiver. (The numbers at the vertices denote the order of mutations.)
Let $x$ be these cluster variables. We take $x_{Q(mKdV)}$ as an initial seed and mutate it in the order $0, 1, \ldots, \mu, \mu', \mu'', \ldots$, where $\mu'$ is the iteration of all mutations at the vertices $i$. (3.9)

In fact, $\mu'$ is the iteration of all mutations at the vertices $i$ in figure 10. We denote by $x_n^{m+1}$ and $w_n^{m+1}$ the new cluster variable obtained by mutation at $x_n^m$ and $w_n^m$ respectively. Then we obtain the following proposition by the definition of mutation (2.4).

**Proposition 3.3.** Consider the coefficient-free cluster algebra $A(Q_{mKdV}, x)$. For any $n, m \in \mathbb{Z}$, the cluster variables $w_n^m, x_n^m$ satisfy the bilinear equations:

\[
\begin{align*}
I_{mKdV} &= \left\{ w_{i,j}^{\mu_j}, x_{i,j}^{\mu_j} \mid i \in \{0, 1\}, j \in \mathbb{Z} \right\}. \\
\{ w_{i,j}^{\mu_j}, x_{i,j}^{\mu_j} \mid j \in \mathbb{Z} \} &\subset I_{mKdV}. \\
\end{align*}
\]

Note that nonautonomous bilinear equations can be obtained from cluster algebras with coefficients. Consider the cluster algebra with coefficients $A(Q_{mKdV}, x, y)$. Let the cluster variables $w_n^m, x_n^m$ be defined as in the coefficient-free case. By the definition of a mutation (2.2), the cluster variables $w_n^m, x_n^m$ satisfy the following bilinear equations:
Figure 12. Generalized dmKdV quiver. (The numbers at the vertices denote the order of mutations.)

\[
\begin{align*}
  w_{n+1}^m &= a_n^m x_n^m + b_n^m w_n^m + h_n^m x_{n+1}^m, \\
  x_{n+1}^m &= c_n^m w_n^m + d_n^m x_{n+1}^m + e_n^m w_n^m + f_n^m x_{n+1}^m.
\end{align*}
\]  

(3.11)

where \(a_n^m, b_n^m, c_n^m, d_n^m\) are rational functions of the coefficients of the initial seed for which \(a_n^m + b_n^m = c_n^m + d_n^m = 1\) holds. These equations are the bilinear form of the discrete mKdV equation.

The discrete mKdV equation (3.10) is obtained from the Hirota–Miwa equation (3.7) by imposing the reduction condition \(x_n^{m,i+2} = x_n^{m,i}\) and \(w_n^m = x_n^{m,0}, x_n^{m} = x_n^{m,1}\). This reduction of the difference equation corresponds to the \((0, 0, 2)\)-reduction of the HM quiver.

Let \(Q_{N,M}\) be the quiver as shown in figures 11, 12. This quiver is a generalization of the dmKdV quiver. In fact, \(Q_{N,M} = Q_{N,M}^{1,1}\). In figure 12, the numbers at vertices denote the order of mutations. Figure 12 shows the case where \(N \geq M\). The discrete mKdV equations (3.10) and (3.11) are obtained from the generalized dmKdV quiver in the same way.

Let \(Q_T\) (the dToda quiver) be the quiver obtained from the \((1, -1, 1)\)-reduction of the HM quiver \(Q_{1,1}\) (see figures 13, 14). We denote by \(I_T\) the vertex set of \(Q_T\):

\[
I_T = \left\{ x_{i,j}^m \mid i \in \{0, 1\}, j \in \mathbb{Z} \right\}.
\]  

(3.12)
**Figure 13.** Reduction from the HM quiver to the dToda quiver.

**Figure 14.** The dToda quiver.

**Figure 15.** The dToda quiver. (The numbers at the vertices denote the order of mutations.)
Let \( x \) be these cluster variables. We take \( (Q_T, x) \) as an initial seed and mutate it in the order 
\[
\mu_0, \mu_1, \mu_2, \ldots, \mu_i, \ldots
\]
where \( \mu_i \) is the iteration of all mutations at the vertices 
\[
\{ x_{i+1}^j \} / \in \mathbb{Z} \subset I_T. \tag{3.13}
\]
In fact, \( \mu_i \) is the iteration of all mutations at the vertices \( i \) in figure 15. We denote by \( x_{n+2}^m \) the new cluster variable obtained by mutation at \( x_n^m \). Then we obtain the following proposition by the definition of mutation (2.4).

**Proposition 3.4.** Consider the coefficient-free cluster algebra \( A(Q_T, x) \). For any \( n, m \in \mathbb{Z} \), the cluster variables \( x_n^m \) satisfy the bilinear equation
\[
x_{n+1}^{m+1} x_{n+1}^{m-1} = x_n^m x_n^m + (x_n^m)^2. \tag{3.14}
\]
This equation is the bilinear form of the discrete Toda equation. The discrete Toda equation (3.14) is obtained from the Hirota–Miwa equation (3.7) by imposing the reduction condition \( x_{n+1}^{m-1,i+1} = x_{n}^m \) and \( x_n^m := x_n^m(0) \). This reduction of the difference equation corresponds to the \((1, -1, 1)\)-reduction of the HM quiver.

### 4. \( q \)-discrete Painlevé equations satisfied by coefficients

In this section, we show that coefficients in cluster algebras can satisfy \( q \)-discrete Painlevé equations, if the initial seed includes suitable quivers with the mutation-period property. Quivers for the \( q \)-Painlevé I, II equations have been obtained in [6, 9]. In this paper, we introduce the quivers for the \( q \)-Painlevé III, VI equations in a similar way. We shall show that these quivers are obtained as reductions of the dKdV quiver and the dmKdV quiver.

#### 4.1. Mutation-periodic quivers

The quivers of \( q \)-discrete Painlevé equations have the property that mutations of their quivers are equal to permutation of the vertices. This is the so-called 'mutation-period' property [3]. We define a '\( \nu \)-period' introduced by Nakanishi [14] as a generalization of mutation-period as follows. Let \( Q \) be a quiver. For \( i = (i, i_2, \ldots, i_h) (i_j \in \{1, 2, \ldots, N\}) \), we define an iteration of mutations \( \mu_i \) as \( \mu_i(Q) = \mu_{i_h} \cdots \mu_{i_2} \mu_{i_1}(Q) \), where \( h \) is the number of applications of the mutation. For a permutation \( \nu \in S_N \), let \( \nu(Q) \) be the quiver in which we substituted vertices \( i \) for \( \nu(i) \) in \( Q \).

**Definition 4.1.** Reference [14] \( i \) is a \( \nu \)-period of \( Q \) if \( \mu_i(Q) = \nu(Q) \) holds. \( Q \) is said to be a mutation-periodic quiver if \( i \) and \( \nu \in S_N \), as defined above, exist.

In the case of \( h = 1 (i = (i)) \), all mutation-periodic quivers have already been obtained [3]. Quivers of \( q \)-discrete Painlevé equations are mutation-periodic quivers. These quivers arise from a reduction of the dKdV quiver and the dmKdV quiver. We consider both cluster algebras with coefficients and coefficient-free cluster algebras.

#### 4.2. The \( q \)-Painlevé I equation

We construct the quiver of the \( q \)-Painlevé I equation \( Q_{PI} \) (the \( q \)-PI quiver) by the \((2, -1)\)-reduction of the dKdV quiver \( Q_{KdV}^{2,1} \). The triple of two vertices and a positive integer \( (x_n^m, x_n^{m'}, i) \) denotes \( i \) arrows from \( x_n^m \) to \( x_n^{m'} \). The dKdV quiver \( Q_{KdV}^{2,1} \) (figure 1) consists of the
vertices
\[ \{ x_{i+2}^{j} \mid i \in [0, 3], j \in \mathbb{Z} \} \]  
and the arrows
\[
\{(x_{i+2}^{j}, x_{i+2}^{j+1}; 1), (x_{i+2}^{j}, x_{i+2}^{j+1}; 2), (x_{i+2}^{j}, x_{i+2}^{j+1}; 1), (x_{i+2}^{j}, x_{i+2}^{j+1}; 1),
(x_{i+2}^{j}, x_{i+2}^{j+1}; 1), (x_{i+2}^{j}, x_{i+2}^{j+1}; 1), (x_{i+2}^{j}, x_{i+2}^{j+1}; 1) \mid j \in \mathbb{Z} \}.
\]

Among the arrows \((x_{i+2}^{m-j}, x_{i+2}^{m-j}; i) (j \in \mathbb{Z})\), we remove all the arrows with \( j \neq 0 \) in \( Q_{qPI}^{\mathbb{Z}} \). We obtain the quiver as shown in figure 16. Next, we superimpose the vertices \( x_{i+2}^{m-j} \) on the vertex \( x_{i+2}^{m} \). In figure 16, we superimpose the vertices \( x_{i+2}^{0}, x_{i+2}^{1} \) on the vertices \( x_{2}^{0}, x_{1}^{0} \) respectively. We obtain the \( q \)-PI quiver \( Q_{qPI} \) as shown in figure 17. Let \( \nu \in S_{4} \) be \( \nu: (1, 2, 3, 4) \mapsto (2, 3, 4, 1) \). \( \nu = (1) \) is a \( \nu \)-period of \( Q_{qPI} \). Note that each vertex \( i \) corresponds to a cluster variable \( x_{i} \) and a coefficient \( y_{i} \). We take \((Q_{qPI}, x, y)\) as an initial seed, where \( x = (x_{1}, x_{2}, x_{3}, x_{4}), y = (y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1}) \) and we mutate the initial seed in the order \( \mu_{1} = \mu_{1}, \nu \mu_{1}(i) = \mu_{2}, \mu_{2}, \mu_{2}, \nu \mu_{2}(i) = \mu_{3}, \ldots \). The new cluster variables and the new coefficients are denoted as \( x_{n} \mapsto x_{n+4}, y_{m,n} \mapsto y_{m,n+4} \). We put \( y_{n} := y_{n,n} (n \equiv m \pmod{4}) \) and obtain the following seeds:
\[
\ldots \xleftarrow{\mu_{1}} (Q_{PI}; x_{1}, x_{2}, x_{3}, x_{4}; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1})
\xleftarrow{\nu} (Q_{PI}; x_{5}, x_{6}, x_{7}, x_{8}; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1})
\xleftarrow{\mu_{2}} (Q_{PI}; x_{9}, x_{10}, x_{11}, x_{12}; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1})
\xleftarrow{\nu^{2}} (Q_{PI}; x_{13}, x_{14}, x_{15}, x_{16}; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1})
\xleftarrow{\mu_{3}} (Q_{PI}; x_{17}, x_{18}, x_{19}, x_{20}; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1})
\xleftarrow{\nu^{3}} (Q_{PI}; x_{21}, x_{22}, x_{23}, x_{24}; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1})
\xleftarrow{\mu_{4}} (Q_{PI}; x_{25}, x_{26}, x_{27}, x_{28}; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1})
\xleftarrow{\nu^{4}} (Q_{PI}; x_{29}, x_{30}, x_{31}, x_{32}; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1})
\xleftarrow{\mu_{5}} \ldots.
\]

The following proposition is obtained from the definition of mutation (2.4).
Proposition 4.2. Consider the coefficient-free cluster algebra \( A(Q_{Pl}, x) \). For any \( n \in \mathbb{Z} \), the cluster variables \( x_n \) satisfy the bilinear equation

\[
x_{n+4} x_n = x_{n+2}^2 + x_{n+3} x_{n+1}.
\]  

(4.4)

The bilinear equation (4.4) can be obtained from the discrete KdV equation (3.4) by imposing the reduction condition \( x_{n+2}^m = x_n^m \) and \( x_n = x_0 \). This reduction of the difference equation corresponds to the \((2, -1)\)-reduction of the dKdV quiver. It turns out that the corresponding coefficients satisfy the \(q\)-Painlevé I equation.

Theorem 4.3. Reference [6] Consider the cluster algebra with coefficient \( A(Q_{Pl}, x, y) \). For any \( n \in \mathbb{Z} \), the coefficients \( y_n \) satisfy the equation

\[
y_{n+1} y_{n-1} = c_2 c_1^n \frac{y_n^2 + 1}{y_n^n},
\]  

(4.5)

where \( c_1, c_2 \) are the conserved quantities

\[
c_1 = \frac{y_{n+3} (y_{n+1}^{-1} + 1)}{y_n (y_{n+2}^{-1} + 1)}, \quad c_2 = \frac{y_{n+2} y_{n+1}^2 y_{n-(n+1)}}{y_{n+1}^2 + 1},
\]  

(4.6)

and do not depend on \( n \).

This equation (4.5) is the \(q\)-Painlevé I equation [10]. The \(q\)-Painlevé I equation (4.5) is obtain from the bilinear equation (4.4) by the following transformation of variables:

\[
y_n = \frac{x_{n+2} x_n}{x_{n+1}^2}.
\]  

(4.7)

Proof. By the definition of mutation (2.1), the coefficients \( y_{n,m} \) satisfy

\[
y_n = y_{n,n-1} (y_{n-1} + 1),
\]

\[
y_{n,n-1} = y_{n,n-2} (y_{n-2}^{-1} + 1)^{-2},
\]

\[
y_{n,n-2} = y_{n,n-3} (y_{n-3}^{-1} + 1),
\]

\[
y_{n,n-3} = y_{n-4},
\]  

(4.8)

where we consider the index \( n \) of coefficients \( y_{n,m} \) as \( n \in \mathbb{Z}/4\mathbb{Z} \). We then obtain an equation only for \( y_n \):

\[
y_{n+4} = \frac{(y_{n+3} + 1)(y_{n+1} + 1)}{(y_{n+2}^{-1} + 1)^2 y_n}.
\]  

(4.9)

We put

\[
u_n := \frac{y_{n+3} (y_{n+1}^{-1} + 1)}{y_n (y_{n+2}^{-1} + 1)}, \quad v_n := \frac{y_{n+2} y_{n+1}^2 y_n}{y_{n+1}^2 + 1}.
\]  

(4.10)

and find \( u_{n+1} = u_n \) from (4.9). Hence we obtain the conserved quantity \( u_n = c_1 \). Similarly, we obtain \( v_{n+1} = c_1 v_n \) from \( u_n = c_1 \) and we obtain \( v_n = c_2 c_1^{n+1} \), where \( c_2 = v_n c_1^{-n} \) is also a conserved quantity. We obtain the \(q\)-Painlevé I equation (4.5) from \( v_n = c_2 c_1^{n+1} \). \(\square\)
4.3. The q-Painlevé II equation

We construct the quiver of the q-Painlevé II equation $Q_{\text{PII}}$ (the q-PII quiver) by the $(3, -1)$-reduction of the dKdV quiver $Q_{\text{dKdV}}$. The dKdV quiver $Q_{\text{dKdV}}$ (figure 1) consists of the vertices 
\[ \{ x_{0j} \mid j \in \{0, 4\} \} \] and the arrows 
\[ \{ (x_{0j}, x_{0j+1}; 1), (x_{0j}, x_{0j+2}; 2), (x_{0j}, x_{0j+3}; 1), (x_{0j}, x_{0j+4}; 1) \mid j \in \{0, 4\} \}. \] (4.11)

Among the arrows $(x_{m+3j}, x_{m+3j}^-; i)$ $(j \in \mathbb{Z})$, we remove all the arrows with $j \neq 0$ in $Q_{\text{dKdV}}$. We obtain the quiver as shown in figure 18. Next, we superimpose the vertices $x_{0j}$ on the vertices $x_{0j}$ respectively. We obtain the q-PII quiver $Q_{\text{PII}}$ as shown in figure 19. Let $\nu \in S_5$ be 
\[ \nu: (1, 2, 3, 4, 5) \mapsto (2, 3, 4, 5, 1). \] $\iota = (1)$ is a $\nu$-period of $Q_{\text{PII}}$. Note that each vertex $i$ corresponds to a cluster variable $x_i$ and a coefficient $y_i$. We take $(Q_{\text{PII}}, x, y)$ as an initial seed, where $x = (x_1, x_2, x_3, x_4, x_5), y = (y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1}, y_{5,1})$ and mutate the initial seed in the order $\mu_\iota, \mu_{\nu(1)}, \mu_{\nu(2)}, \mu_{\nu^2(4)}, \ldots$. The new cluster variables and the new coefficients are denoted as $x_n \rightarrow x_{n+5}, y_{n,m} \rightarrow y_{n+5,m}$. We put $y_n := y_{m,n} (n \equiv m \pmod{5})$ and obtain the following seeds:
\[ \ldots \xrightarrow{\mu_\iota} (Q_{\text{PII}}; x_1, x_2, x_3, x_4, x_5; y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1}, y_{5,1}) \]
\[ \xrightarrow{\mu_\iota} (\nu(Q_{\text{PII}}); x_6, x_7, x_8, x_9; y_{1,2}, y_{2,2}, y_{3,2}, y_{4,2}, y_{5,2}) \]
\[ \xrightarrow{\mu_\iota} (\nu^2(Q_{\text{PII}}); x_1, x_2, x_3, x_4, x_5; y_{1,3}, y_{2,3}, y_{3,3}, y_{4,3}, y_{5,3}) \xrightarrow{\mu_\iota} \ldots. \] (4.13)

We obtain the following proposition from the definition of mutation (2.4).

**Proposition 4.4.** Consider the coefficient-free cluster algebra $A(Q_{\text{PII}}, x)$. For any $n \in \mathbb{Z}$, the cluster variables $x_n$ satisfy the bilinear equation
The bilinear equation (4.14) is obtained from the discrete KdV equation (3.4) by imposing the reduction condition \( x_{n+3}^{-1} = x_{n+2}^{-1} = x_n^{-1} \). This reduction of the difference equation corresponds to the \((3, -1)\)-reduction of the dKdV quiver. Its coefficients satisfy the \(q\)-Painlevé II equation.

**Theorem 4.5.** Reference [6] consider the cluster algebra with coefficient \( A(Q_{\text{II}}, x, y) \). For any \( n \in \mathbb{Z} \), the coefficients \( y_n \) satisfy the equation

\[
y_{n+1} y_{n-1} = c_2 c_3 \left( y_n - 1 \right) c^3_1 \frac{y_n + 1}{y_n},
\]

where \( c_1, c_2, c_3 \) are the conserved quantities

\[
c_1^2 = \frac{y_{n+1} \left( y_{n+1}^{-1} + 1 \right)}{y_{n+3} \left( y_{n+3}^{-1} + 1 \right)}, \quad c_2^2 = \frac{y_{n+2}^2 y_{n+2} y_{n+1}^2 y_{n+1}}{(y_{n+2}^2 + 1)(y_{n+1}^2 + 1)} c_1^{-4n+3},
\]

\[
c_3^2 = \frac{y_{n+1} \left( y_{n+1}^{-1} + 1 \right)}{y_{n+2} \left( y_{n+2}^{-1} + 1 \right)} c_1^{-1}
\]

and do not depend on \( n \).

We put \( f_n := y_{2n}, g_n := y_{2n+1} \) and obtain

\[
f_{n+1} f_n = c_2 c_3 c_1^{-1} c_1^{2n+1} \frac{g_n + 1}{g_n},
\]

\[
g_n g_{n-1} = c_2 c_3 c_1^{2n} \frac{f_n + 1}{f_n}
\]

from (4.15). This equation is the \(q\)-Painlevé II equation [11]. The \(q\)-Painlevé II equation (4.15) is obtained from the bilinear equation (4.14) by the following transformation of variables:

\[
y_n = \frac{x_{n+3} x_n}{x_{n+2} x_{n+1}}
\]

**Proof.** By the definition of mutation (2.1), the coefficients \( y_{n,m} \) satisfy

\[
y_n = y_{n-1} \left( y_{n-1}^{-1} + 1 \right), \quad y_{n,n-1} = y_{n,n-2} \left( y_{n-2}^{-1} + 1 \right)^{-1}, \quad y_{n,n-2} = y_{n,n-3} \left( y_{n-3}^{-1} + 1 \right)^{-1},
\]

\[
y_{n,n-3} = y_{n,n-4} \left( y_{n-4}^{-1} + 1 \right), \quad y_{n,n-4} = y_{n-5}^{-1},
\]

where we think of the index \( n \) of the coefficients \( y_{n,m} \) as \( n \in \mathbb{Z}/5\mathbb{Z} \). We obtain an equation only for \( y_n \).
We put
\[ u_n := \frac{y_{n+1} (y_{n+1} - 1)}{y_n (y_n - 1)}, \quad v_n := \frac{y_{n+2} y_n}{y_{n+1} + 1}, \] (4.21)
and find \( u_{n+1} = u_n \) from (4.20). Hence, we obtain the conserved quantity
\[ u_n = \frac{y_{n+1} (y_{n+1} - 1)}{y_n (y_n - 1)}. \] Similarly, we obtain
\[ v_n = \frac{y_{n+2} y_n}{y_{n+1} + 1}, \]
and find \( v_{n+1} = v_n \) from (4.20). Hence, we obtain the conserved quantity
\[ v_n = \frac{y_{n+2} y_n}{y_{n+1} + 1}. \] We then obtain the \( q \)-Painlevé II equation (4.15) from
\[ v_n = c_2 c_3^{(-1)^{n+1}} c_1^{n+1}. \]
□

4.4. The \( q \)-Painlevé III equation

We construct the quiver of the \( q \)-Painlevé III equation \( Q_{PIII} \) (the \( q \)-PIII quiver) by the \( (2, -1) \)-reduction of the \( dmKdV \) quiver \( Q^{2,1}_{mKdV} \). The \( dmKdV \) quiver \( Q^{2,1}_{mKdV} \) (figure 11) consists of the vertices
\[ \{ w_i^{+2j}, x_i^{-2j} \mid i \in [0, 2], j \in \mathbb{Z} \} \] (4.22)
and the arrows
\[ \{ (w_i^{+2j}, w_j^{+2j}; 1), (w_i^{-2j}, w_j^{-2j}; 1), (w_i^{+2j}, x_j^{+2j}; 1), (x_i^{+2j}, x_j^{+2j}; 1), (w_i^{+2j}, x_j^{-2j}; 1), (x_i^{+2j}, w_j^{-2j}; 1), \} \] (4.23)
Among the arrows \((w_i^{m+2j}, w_i^{m-2j}; i), (x_i^{m+2j}, x_i^{m-2j}; i) \) \( j \in \mathbb{Z} \), we remove all the arrows with \( j \neq 0 \) in \( Q^{2,1}_{mKdV} \). We obtain the quiver as shown in figure 20. Next, we superimpose the vertices \( w_i^{m+2j}, x_i^{m+2j} \) on the vertex \( w_i^{m}, x_i^{m} \) respectively. In figure 20, we superimpose the vertices \( w_0^{+}, x_0^{+} \) on the vertices \( w_0^{-}, x_0^{-} \) respectively. We obtain the \( q \)-PIII quiver \( Q_{qPIII} \) as shown in figure 21. Let \( \nu \in S_6 \) be \( \nu : (1, 2, 3, 4, 5, 6) \mapsto (3, 4, 5, 6, 2, 1) \), \( i = (1, 2) \) is a \( \nu \)-period of \( Q_{PIII} \). Note that vertices \((1, 2, 3, 4, 5, 6)\) correspond to cluster variables.
We take \((Q_{\text{PIII}}, x, y)\) as an initial seed and mutate it in the order \(\mu_i = \mu_2 \mu_1, \mu_{i(j)} = \mu_i \mu_j, \mu_{i(j)} = \mu_6 \mu_5\ldots\). The new cluster variables are denoted as \(w_n \to x_{n+3}, x_n \to w_{n+3}\) and the new coefficients are denoted as \(y_{n,m} \to z_{n,m+1}, z_{n,m} \to y_{n,m+1}\) (if the coefficient corresponds to the vertex to which the mutation is applied) or \(y_{n,m} \to y_{n,m+1}, z_{n,m} \to z_{n,m+1}\) (otherwise). We put \(y_n = y_{n,m}, z_n = z_{n,m} (n \equiv m \text{ (mod 3)})\) and obtain the following seeds:

\[
\cdots \leftrightarrow (Q_{\text{PIII}}; w_1, x_1, w_2, x_2, w_3, x_3; y_1, z_1, y_2, z_2, y_3, z_3, y_4, z_4) \\
\mu_2 \mu_1 \leftrightarrow (\nu (Q_{\text{PIII}}); x_4, w_4, w_2, x_2, w_3, x_3; z_4, y_2, y_2, y_3, z_3, x_3) \\
\mu_6 \mu_5 \leftrightarrow (\nu^2 (Q_{\text{PIII}}); x_4, w_4, w_5, x_5, z_3, x_3; \nu, y_3, z_2, z_3, y_3, z_3) \\
\mu_4 \mu_1 \leftrightarrow (Q_{\text{PIII}}; x_4, w_4, w_5, x_5, w_6, z_4, y_4, z_2, z_3, y_3, z_4, \nu) \\
\mu_4 \mu_2 \leftrightarrow (\nu (Q_{\text{PIII}}); w_7, x_7, x_5, w_6, x_6, y_1, z_5, z_5, y_5, z_5, y_5, z_5). \\
\] (4.24)

We obtain the following proposition from the definition of mutation (2.4).

**Proposition 4.6.** Consider the coefficient-free cluster algebra \(A(Q_{\text{PIII}}, x)\). For any \(n \in \mathbb{Z}\), the cluster variables \(w_n, x_n\) satisfy the bilinear equations:

\[
\begin{align*}
w_{n+3}x_n &= x_{n+2}w_{n+1} + w_{n+2}x_{n+1}, \\
x_{n+3}w_n &= w_{n+2}x_{n+1} + x_{n+2}w_{n+1}
\end{align*}
\] (4.25)

The bilinear equation (4.25) are obtained from the discrete mKdV equation (3.10) by imposing the reduction condition \(w_m = w_0, x_{m-1} = x_m\) and \(w_n = w_n, x_n = x_0\). This reduction of the difference equation corresponds to the \((2, -1)\)-reduction of the dmKdV quiver. In this case, the coefficients satisfy the \(q\)-Painlevé III equation.

**Theorem 4.7.** Consider the cluster algebra with coefficient \(A(Q_{\text{PIII}}, x, y)\). For any \(n \in \mathbb{Z}\), the coefficients \(y_n, z_n\) satisfy the equations:

\[
\begin{align*}
y_{n+1}y_{n-1} &= c_2 c_3^2 c_1^{2n} \frac{y_n + 1}{y_n + c_3 c_1^{-1}} \left(\frac{y_n + c_3 c_1^{-1}}{c_1^n}\right). \\
z_{n+1}z_{n-1} &= c_2^{-1} c_3^2 c_1^{2n} \frac{z_n + 1}{z_n + c_3 c_1^{-1}} \left(\frac{z_n + c_3 c_1^{-1}}{c_1^n}\right).
\end{align*}
\] (4.26)
where $c_1$, $c_2$, $c_3$, $c_4$ are conserved quantities

$$
c_1^2 = \frac{y_{n+2} z_{n+2}^2}{y_n z_n},
$$
$$
c_2^2 = \frac{y_{n+2} y_n (y_{n+1}^{-1} + 1)^2}{z_{n+2} z_n (y_{n+1}^{-1} + 1)^2},
$$
$$
c_3^2 = y_{2n+1} z_{2n+1} y_{2n} z_{2n+1} c_1^{-(4n+1)},
$$
$$
c_4^2 = \frac{y_{2n} z_{2n}}{y_{2n+1} z_{2n+1}}
$$

and do not depend on $n$.

If we put $f_n := y_{2n}$, $g_n := y_{2n+1}$, we obtain

$$
f_n f_n + g_n c_3 y_{n+1}^{4n+2} = \frac{g_n + 1}{g_n (g_n + c_3 c_4 c_1 z_{n+1}^{-1})},
$$

$$
g_n g_{n-1} = c_2 c_3 c_1^{4n} \frac{f_n + 1}{f_n (f_n + c_3 c_4 z_{n}^{-1})}
$$

from (4.26). These equations are the $q$-Painlevé III equation [12].

**Proof.** By the definition of mutation (2.1), the coefficients $y_{n,m}$, $z_{n,m}$ satisfy

$$
y_n = y_{n-1} (y_{n-1}^{-1} + 1)^{-1},
$$
$$
y_{n,n-1} = y_{n-2} (y_{n-2}^{-1} + 1)^{-1} (y_{n-2} + 1),
$$
$$
y_{n,n-2} = z_{n-3},
$$
$$\frac{z_n}{z_{n-1}} = y_{n-1} (y_{n-1}^{-1} + 1)^{-1},
$$
$$
y_{n,n-1} = y_{n-2} (y_{n-2}^{-1} + 1)^{-1} (y_{n-2} + 1),
$$
$$
$$

$$
(4.29)
$$

where we consider the index $n$ of the coefficients $y_{n,m}$, $z_{n,m}$ as $n \in \mathbb{Z}/3\mathbb{Z}$. The equations, only for $y_n$, $z_n$ are

$$
y_{n+3} = \frac{(y_{n+2} + 1)(y_{n+1}^{-1} + 1)}{(y_{n+2}^{-1} + 1)(y_{n+1}^{-1} + 1)},
$$

$$
z_{n+3} = \frac{(z_{n+2} + 1)(y_{n+1}^{-1} + 1)}{(y_{n+2}^{-1} + 1)(z_{n+1}^{-1} + 1)}.
$$

(4.30)

If we put

$$
u_n := \frac{y_{n+2} (y_{n+1}^{-1} + 1)}{z_n (y_{n+1}^{-1} + 1)},
$$

$$
v_n := \frac{z_{n+2} (y_{n+1}^{-1} + 1)}{y_n (z_{n+1}^{-1} + 1)},
$$

$$
t_n := y_n z_n,
$$

(4.31)

we find $u_{n+1} = u_n$, $y_{n+1} = y_n$ from (4.30), and obtain conserved quantities $u_n = c_1 c_2$, $y_n = c_3 c_1^{-1}$. We also obtain $t_{n+2} = c_3^2 t_n$ from $u_n v_n = c_2^2$, and $t_n = c_3 c_4^{-(4n+1)}$. $c_1^2 = t_{2n+1}^{-1} t_{2n+1} c_1^{(4n+1)}$ and $c_2^2 = t_{2n+1}^{-1} t_{2n+1} c_2$ are conserved quantities. We obtain the $q$-Painlevé III equation (4.26) from $u_n = c_1 c_2$ and $t_n = c_3 c_4^{-(4n+1)}$ by the elimination of $z_n$. The equation for $y_n$ is the same as that for $y_n$. 

□
4.5. The q-Painlevé VI equation

We now construct the quiver of the q-Painlevé VI equation \( Q_{PV} \) (the q-PVI quiver) by the \((2, 2)\)-reduction of the dmKdV quiver \( Q_{mKdV} \). The dmKdV quiver \( Q_{mKdV} \) (figure 9) consists of the vertices \( \{w_{i,j}, x_{i,j} | i \in [0, 1], j \in \mathbb{Z} \} \) and the arrows

\[
\{ (w_{i,j}^{-2}, w_{i+1,j}^{-2}; 1), (w_{i,j}^{-2}, w_{i,j}^{1-2}; 1), (w_{i-1,j}^{-2}, w_{i-1,j+1}^{-2}; 1), (x_{i,j}^{-2}, x_{i,j}^{1-2}; 1), (x_{i,j}^{1-2}, x_{i,j}^{1-2}; 1), (x_{i,j}^{1-2}, x_{i,j+1}^{1-2}; 1), (x_{i+1,j}^{1-2}, x_{i+1,j+1}^{1-2}; 1), \}
\]

(4.32)

and the arrows

\[
\{ (w_{i,j}^{-2}, w_{i+1,j+1}^{-2}; 1), (w_{i,j}^{-2}, w_{i+1,j+1}^{1-2}; 1), (x_{i,j}^{-2}, x_{i+1,j}^{1-2}; 1), (x_{i,j+1}^{1-2}, x_{i+1,j+1}^{1-2}; 1), \}
\]

(4.33)

Among the arrows \( (w_{m+2,j}^{-2}, x_{n+2,j}^{-2}; i), (w_{n,j}^{-2}, x_{m,j}^{1-2}; i) \) \((j \in \mathbb{Z})\), we remove all the arrows with \( j \neq 0 \) in \( Q_{mKdV}^{1,1} \). We obtain the quiver as shown in figure 22. Next, we superimpose the vertices \( w_{i,j}, x_{i,j} \) on the vertices \( w_{0,j}, x_{0,j} \) respectively. In figure 22, we superimpose the vertices \( w_{m+2,j}, x_{n+2,j} \) on the vertices \( w_{0,j}, x_{0,j} \) respectively. We obtain the q-PVI quiver \( Q_{PV} \) as shown in figure 23. Let \( \nu \in S_8 \) be \( \nu: (1, 2, 3, 4, 5, 6, 7, 8) \mapsto (5, 6, 7, 8, 4, 3, 2, 1) \). We take \( (Q_{PV}, x, y) \) as an initial seed and mutate it in the order \( \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \). The new cluster variables are denoted as \( w_{n,m}, x_{n,m}, y_{n,m+1}, z_{n,m}, y_{n,m-1}, x_{n,m}, z_{n,m+1} \) (in case the coefficient corresponds to the vertex to which the mutation is applied) or \( y_{n,m}, z_{n,m+1}, y_{n,m}, z_{n,m+1}, x_{n,m}, Z_{n,m} \) (otherwise). We put
We obtain the following proposition from the definition of mutation (2.4).

**Proposition 4.8.** Consider the coefficient-free cluster algebra \( A(Q_{\text{PV1}}, x) \). For any \( n \in \mathbb{Z} \), the cluster variables \( w_n, x_n, W_n, X_n \) satisfy the bilinear equations:

\[
\begin{align*}
    w_{n+2}X_n &= x_{n+1}W_{n+1} + w_{n+1}X_{n+1}, \\
    x_{n+2}W_n &= w_{n+1}X_{n+1} + x_{n+1}W_{n+1}, \\
    W_{n+2}x_n &= X_{n+1}w_{n+1} + W_{n+1}X_{n+1}, \\
    X_{n+2}w_n &= W_{n+1}X_{n+1} + W_{n+1}x_{n+1}.
\end{align*}
\]  
\( (4.35) \)

The bilinear equation (4.35) are obtained from the discrete mKdV equation (3.10) by imposing the reduction condition \( w_{n+2} = w_n, x_{n+2} = x_n \) and \( w_n = w_0, x_n = x_0 \), \( W_n = w_{n-1}, X_n = x_{n-1} \). This reduction of the difference equation corresponds to the \((2, -2)\) -reduction of the dmKdV quiver. Moreover, the coefficients satisfy the \( q \)-Painlevé VI equation.

**Theorem 4.9.** Consider the cluster algebra with coefficient \( A(Q_{\text{PV1}}, x, y) \). For any \( n \in \mathbb{Z} \), the coefficients \( y_n, z_n, Y_n, Z_n \) satisfy the equations:
Proof. By the definition of mutation (2.1), the coefficients \(y_{n,m}, z_{n,m}, Y_{n,m}, Z_{n,m}\) satisfy
\[
y_{n+1} y_{n-1} = c_1^{-1} c_2^2 c_3^2 c_1^{2n} \left( y_{n} + 1 \right) \left( c_2 c_5^{-1} c_4^{-1} c_3^{-1} c_6^{-1} c_{n+1} + 1 \right),
\]
\[
z_{n+1} z_{n-1} = c_2 c_3^2 c_5^{-2} c_1^{2n} \left( z_{n} + 1 \right) \left( c_3 c_4^{-1} c_5^{-1} c_6^{-1} z_{n+1} + 1 \right),
\]
\[
y_{n+1} Y_{n-1} = c_3 c_2^{-2} c_5^2 c_1^{2n} \left( y_{n} + 1 \right) \left( c_3 c_4^{-1} c_5^{-1} c_6^{-1} y_{n+1} + 1 \right),
\]
\[
Z_{n+1} Z_{n-1} = c_3^3 c_2^2 c_5^{-2} c_1^{2n} \left( Z_{n} + 1 \right) \left( c_3 c_4^{-1} c_5^{-1} c_6^{-1} Z_{n+1} + 1 \right),
\]
where \(c_1, c_2, \ldots, c_6\) are conserved quantities
\[
c_1^2 = \frac{y_{n+1} z_{n+1} Y_{n+1} Z_{n+1}}{y_{n} z_{n} Y_{n} Z_{n}}, \quad c_2 = y_{n} z_{n} Y_{n} Z_{n} c_1^{-2n},
\]
\[
c_3^2 = y_{2n+1} z_{2n+1} Y_{2n+1} Z_{2n+1} c_1^{-(4n+1)}, \quad c_4^2 = \frac{y_{2n} z_{2n}}{y_{2n+1} z_{2n+1}} c_1,
\]
\[
c_5^2 = y_{2n+1} Y_{2n+1} y_{2n} Z_{2n} c_1^{-(4n+1)}, \quad c_6^2 = \frac{y_{2n} Y_{2n}}{y_{2n+1} Y_{2n+1}} c_1.
\]

If we put \(f_n = y_{2n}, g_n = y_{2n+1}\), we obtain
\[
f_{n+1} f_n = c_2 c_3^2 c_5^2 c_1^{2n+2} \left( f_{n} + 1 \right) \left( c_2 c_5^{-1} c_4^{-1} c_3^{-1} c_6 g_n + 1 \right),
\]
\[
g_n g_{n-1} = c_2 c_3^2 c_5^2 c_1^{2n} \left( f_{n} + 1 \right) \left( c_2 c_5^{-1} c_4^{-1} c_3^{-1} f_{n-1} + 1 \right),
\]
from (4.26). These equations are nothing but the \(q\)-Painlevé VI equation [13].
where we consider the index $n$ of the coefficients $y_{n,m}$, $z_{n,m}$, $Y_{n,m}$, $Z_{n,m}$ as $n \in \mathbb{Z}/2\mathbb{Z}$. The equations only for $y_{n}$, $z_{n}$, $Y_{n}$, $Z_{n}$ we obtain, are:

\[ y_{n+2} = \frac{(y_{n+1} + 1)(Z_{n+1} + 1)}{(z^{-1}_{n+1} + 1)}Y_{n}, \]  
(4.40)

\[ z_{n+2} = \frac{(z_{n+1} + 1)(Y_{n+1} + 1)}{(y^{-1}_{n+1} + 1)}Z_{n}, \]  
(4.41)

\[ Y_{n+2} = \frac{(Z_{n+1} + 1)(z_{n+1} + 1)}{(y^{-1}_{n+1} + 1)}Y_{n}, \]  
(4.42)

\[ Z_{n+2} = \frac{(Z_{n+1} + 1)(Y_{n+1} + 1)}{(z^{-1}_{n+1} + 1)}Z_{n}. \]  
(4.43)

We have

\[ \frac{y_{n+2}z_{n+2}Y_{n+2}Z_{n+2}}{y_{n+1}z_{n+1}Y_{n+1}Z_{n+1}} = \frac{y_{n+1}z_{n+1}Y_{n+1}Z_{n+1}}{y_{n}z_{n}Y_{n}Z_{n}}, \]  
(4.44)

\[ y_{n+2}z_{n+2}Y_{n}Z_{n} = y_{n+1}z_{n+1}Y_{n+1}Z_{n+1}, \]  
(4.45)

\[ y_{n+2}Y_{n+2}Z_{n} = y_{n+1}z_{n+1}Y_{n+1}Z_{n+1}. \]  
(4.46)

from (4.40) $\times$ (4.41) $\times$ (4.42) $\times$ (4.43), (4.40) $\times$ (4.41), and (4.40) $\times$ (4.42) respectively, and we obtain

\[ \frac{y_{n+1}z_{n+1}Y_{n+1}Z_{n+1}}{y_{n}z_{n}Y_{n}Z_{n}} = c_{1}^{2}, \]  
(4.47)

from (4.44), where $c_{1}$ is a constant. In fact

\[ y_{n}z_{n}Y_{n}Z_{n} = c_{2}c_{1}^{2n}, \]  
(4.48)

from $y_{n+1}z_{n+1}Y_{n+1}Z_{n+1} = c_{2}^{2}y_{n}z_{n}Y_{n}Z_{n}$, where $c_{2}$ is a constant. Eliminating $Y_{n}$, $Z_{n}$ from (4.45) and (4.48) we obtain $y_{n+2}z_{n+2} = c_{1}^{2}y_{n}z_{n}$. If we eliminate $z_{n}$, $Z_{n}$ from (4.46) and (4.48) we obtain $y_{n+2}Y_{n+2} = c_{1}^{2}y_{n}Y_{n}$. Therefore,

\[ y_{n}z_{n} = c_{3}c_{4}^{(-1)^{n}}c_{1}^{n}, \quad Y_{n} = c_{5}c_{6}^{(-1)^{n}}c_{1}^{n}, \]  
(4.49)

where $c_{3}$, $c_{4}$, $c_{5}$, $c_{6}$ are constants. Eliminating $z_{n}$, $Y_{n}$ from (4.48) and (4.49) we obtain

\[ y_{n} = c_{2}^{2}c_{3}^{(-1)^{n}}c_{4}^{(-1)^{n}}c_{1}^{n}, \]  
(4.50)

Finally, eliminating $z_{n}$, $Y_{n}$, $Z_{n}$ from (4.49), (4.50), and (4.40) we obtain the $q$-Painlevé VI equation (4.36). The equations for $z_{n}$, $Y_{n}$, $Z_{n}$ are the same as that for $y_{n}$.

\[ \blacksquare \]

5. Conclusion

We have shown that cluster variables can satisfy the discrete KdV equation, the Hirota–Miwa equation, the discrete mKdV equation, and the discrete Toda equation, if we take appropriate quivers of initial seeds. We have also shown that the coefficients of certain cluster algebras satisfy the $q$-Painlevé I–II, VI equations. These cluster algebras are obtained from a reduction
of the quivers for some integrable partial difference equations. The $q$-Painlevé III equation (4.26) and the $q$-Painlevé VI equation (4.36) are classified as type $(A_2 + A_1)^{(1)}$ and $D_5^{(1)}$ in the classification of root systems [8]. So far we have not obtained the $q$-discrete Painlevé equations of type $A_1^{(1)}$ and $E_6^{(1)}$ from cluster algebras. In the future, we wish to clarify the relations between these equations and cluster algebras. Quivers of higher order analogue of $q$-Painlevé I, II equations are obtained in [9]. To obtain quivers of higher order analogues of the $q$-Painlevé III, VI equations is also a problem we wish to address in the future.

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