The geometric description of incompatible deformation of deep rock mass and its evolution

Chengzhi Qi¹, Wei Liu¹, Guzev Mikhail.², Kuan Jiang¹

¹Beijing Future Urban Design High-Tech Innovation Center and International Cooperation Base for Transportation Infrastructure Construction, Beijing University of Civil Engineering and Architecture, Beijing, 100044, China
²Institute for Applied Mathematics FEB RAS, Vladivostok 690041, Russia
qichengzhi65@163.com

Abstract. Rock mass has a complex internal structure. Under the action of external loads, incompatible deformation develops in rock mass that makes the concept of elementary volume and the Saint Venant's condition of compatible deformation problematic. The incompatibility of the deformation indicates the non-coincidence of the internal and the external metrics and the breaking of the Euclidean structure of the space. Therefore, the use of differential geometry to describe the incompatible deformation is natural. At present, the expression of non-Euclidean models is not concise and compact; the constitutive relation is not complete and the evolution of incompatibility parameters is absent. In this study, with the help of the orthogonal frame method, we used the generalized distortion tensor, torsion tensor, and Riemann tensor to describe the incompatible deformations. The used geometric parameters have clear physical meaning and can be used as thermodynamic variables. By constructing Helmholtz free energy and using irreversible thermodynamics, we obtained the constitutive equations. For completing the constitutive equations, the evolution equations of the used geometric parameters are derived. In this manner, the non-Euclidean description of the incompatible deformation extends the classical models of deformable bodies.

Key words: non-compatible deformation, non-Euclidean model, orthogonal frame method
1. Introduction

Rock mass has complex internal structure formed by geological structural fracture zones, fissures, joints, and cracks (commonly known as weakened planes), ranging from 10^6 to 10^{-6} m (from continental scale to mesoscopic and microscopic scales) [1–3]. The deformation of the rock mass is primarily concentrated on the weakened surfaces and the weakened zones of the structural blocks. When rock mass is subjected to the external force, rock mass deforms and fails at the largest structural level first because the strength at this level with the largest crack width is the least. When the external force continues to increase, the rock mass blocks at the smaller structural levels begin to deform and fail. That is, the structural levels of activated deformation and failure of rock mass gradually transform to the smaller levels. Therefore, the failure of the rock mass can be considered as a continuous process of symmetric breaking and the progressive symmetry localization. In this case, the displacement cannot be determined from the distortion tensors uniquely, that is, defects appear, and the compatibility condition cannot be satisfied:

\[ -R_{ijkl} = \frac{\partial^2 e_{ij}}{\partial x^k \partial x^l} \frac{\partial^2 e_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 e_{ij}}{\partial x^k \partial x^l} \frac{\partial^2 e_{kl}}{\partial x^i \partial x^j} \neq 0 \]  

where \( e_{ij} \) is the strain tensor, that is, the additional parameter \( R_{ijkl} \) for a description of the deformation of medium appears, which is nothing else, but the Riemann tensor.

In differential geometry, a manifold is a basic concept. A manifold is the extension of Euclidean space and is the result of sticking together small pieces of Euclidean space [4]. This situation determines that the concept of the manifold can be used to describe various defects in the medium. In this case, the description of deformation and failure of the medium requires the introduction of additional parameters to describe the internal defect structure of the medium and the usual parameters of continuum mechanics. Thus, there is a requirement to transform the Euclidean model to the non-Euclidean model of the medium.

In physical literature, Kondo and Bilby [5–10] reported the necessity of introducing non-Euclidean parameters. In fact, for the first time, they suggested using affine metrics to describe defect structures. Several theories deal with elastic continua with internal defects inside [11–28]. The common features of the different types of non-Euclidean models are that the torsion tensor is usually identified with dislocation, the Riemann tensor with disclination, and the sectional curvature with point defect. These tensors are then used to characterize the geometry of interactions between particles within a material. Any small perturbation of the structure, for example, the introduction of the above geometric tensors, leads to a non-Euclidean model. In general, this structure is stable to small changes of curvature tensor, torsion tensor, and non-metric tensor.

At present, the expression of non-Euclidean models is not concise and compact; moreover, in the constitutive relation, the evolution of incompatibility parameters is not directly related to the measurable parameters. Therefore, this study will focus on these problems.

In theories of gravity, the orthogonal frame method is very effective [29] because all gravity theories can be expressed in orthogonal frames in the form independent of the choice of the coordinates, i.e., in a general covariant form. In addition to purely geometric considerations, the idea of mapping all quantities to geometric quantities in a tangent space has physical causes, because only in a flat space can a geometric quantity be physically explained. In this method, all other geometric quantities can be associated with the primary vector frames in the tangent space.
The abovementioned characteristics of the orthogonal frame method determine that it can be effectively used in the establishing non-Euclidean models of media. Therefore, this study will discuss the application of the orthogonal frame method in the establishment of a non-Euclidean model of the medium. By incorporating geometric parameters into Helmholtz free energy and using irreversible thermodynamics, we obtained the constitutive equations. For completing the constitutive equations, we derived the evolution equations of the used geometric parameters. In this manner, the non-Euclidean description of the incompatible deformation extends the classical models of deformable bodies.

2. Description of non-compatible deformation of media by orthogonal frame

The modification of the Euclidean models describing the construction of defects is related to the change of the intrinsic geometric properties. In this study, the image of a continuous medium is assumed to be a point set M, and the mapping from such a point set to the Euclidean space R3 with coordinates $\zeta^\mu = (\mu = 1, 2, 3)$ is given. In geometry, the set endowed with differential homeomorphic coordinates is called a differentiable manifold, whereas $\zeta^\mu = (\mu = 1, 2, 3)$ is called a manifold point. We will consider the material as a 3D manifold M3 embedded in the 3D Euclidean space R3.

Suppose that at the initial time $t_0$, each point of the medium can be represented by coordinates $\zeta^\mu = (\mu = 1, 2, 3)$, where the coordinate base vector is $e^\mu$, and the metric tensor is $G_{\mu\nu} = e^\mu e^\nu = \delta_{\mu\nu}$. Then, the particle moves to $x^i$ at time $t$ after deformation, and the relation between the positions of particles at two moments can be expressed as follows:

$$x^i = x^i(\zeta^\mu, t)$$

and the velocity is

$$u^i = \frac{\partial x^i}{\partial t} \bigg|_{\zeta^\mu = \text{const}}$$

The trajectories of particles in Euclidean space are the integral curves of the following equation:

$$\frac{\partial x^i}{\partial t} = u^i$$

Along each such trajectory, it is clear that the initial coordinates are constant:

$$\frac{D\zeta^\mu}{Dt} = \frac{\partial \zeta^\mu}{\partial t} + \frac{\partial \zeta^\mu}{\partial x^k} \frac{dx^k}{dt} = \frac{\partial \zeta^\mu}{\partial t} + u^k \frac{\partial \zeta^\mu}{\partial x^k} = 0$$

where $D^j/Dt$ is the material derivative.

Taking the derivative of Eq. (5) for $x^i$ yields
\[
\frac{\partial (\zeta^\mu / \partial x^i)}{\partial t} + u^k \frac{\partial (\zeta^\mu / \partial x^i)}{\partial x_k} + \frac{\partial u_k}{\partial x^i} \frac{\partial \zeta^\mu}{\partial x^k} = 0
\]

(6)

\[
A = \left\| \zeta^\mu \right\| = \left\| \frac{\partial \zeta^\mu}{\partial x^i} \right\|
\]

where

\[
W = \left\| \frac{\partial u^i}{\partial x^i} \right\|
\]

If we introduce \( A \), then Eq. (6) becomes

\[
\frac{\partial A^\mu}{\partial t} + u^k \frac{\partial A^\mu}{\partial x^k} + W A = \frac{DA}{Dt} + W T A = 0
\]

(7)

which has the following form in components:

\[
\frac{\partial A^\mu}{\partial t} + u^k \frac{\partial A^\mu}{\partial x^k} + \frac{\partial u^k}{\partial x^i} A^\mu = 0
\]

(8)

The metric tensor can then be expressed in terms of distortion tensor in the following form:

\[
G = \left\| G_{ij} \right\| = A A^T = \frac{\partial \zeta^\mu}{\partial x_i} \frac{\partial \zeta^\mu}{\partial x_j}
\]

(9)

The evolution equation of \( G \) can then be obtained from Eq. (7):

\[
\frac{DG}{Dt} + GW + W T G = 0
\]

(10)

Equation (10) has the following form if it is expressed in terms of components:

\[
\frac{\partial G_{ij}}{\partial t} + u^l \frac{\partial G_{ij}}{\partial x^l} + G_{ik} \frac{\partial u^k}{\partial x^j} + \frac{\partial u^k}{\partial x^l} G_{lj} = 0
\]

(11)

For the distortion tensor \( A^\mu \), the deformation compatibility condition of the medium lies in the solvability of the equation \( A^\mu = \frac{\partial \zeta^\mu}{\partial x^i} \) relative to \( \zeta^\mu \). The necessary and sufficient condition for solvability relative to \( \zeta^\mu \) is

\[
\frac{\partial A^\mu}{\partial x^i} - \frac{\partial A^\mu}{\partial x^i} = 0
\]

(12)

The following tensors

\[
B^\mu_k = -\epsilon_{kl} \frac{\partial}{\partial x^l} A^\mu_j,
- B^\mu_k = \frac{\partial A^\mu}{\partial x^i} - \frac{\partial A^\mu}{\partial x^i},
- B^\mu_k = \frac{\partial A^\mu}{\partial x^i} - \frac{\partial A^\mu}{\partial x^i},
- B^\mu_k = \frac{\partial A^\mu}{\partial x^i} - \frac{\partial A^\mu}{\partial x^i}
\]

(13)

are called the Burgs tensor [30]:

\[
B = \left\| B^\mu_j \right\| = \left\| -\epsilon_{ij} \frac{\partial}{\partial x^i} A^\mu_j \right\|
\]

(14)
where $\varepsilon_{i,j}$ is the Levi-Civita symbol.

Thus, the necessary and sufficient conditions for the deformation compatibility of the medium is that Burgs tensor $B$ equals to 0 ($B = 0$).

The existence of defects in the medium ensures the Burgs tensor is no longer 0, and the deformation compatibility condition of the medium is no longer satisfied. Therefore, space is non-uniform. In differential geometry, orthogonal frames (quaternions) are often used to describe non-uniform spaces. In other words, a point of a manifold is considered as the common origin of two coordinate frames (affine frame and orthogonal frame):

$$\{O, e_\mu\}, \{O, h_i\}$$  \hspace{1cm} (15)

The affine frame is associated with an arbitrary coordinate system, whereas the orthogonal frame is associated with local orthogonal coordinates. The mixed scalar product of vectors belonging to different frames is the Lamé coefficients:

$$h_i^\mu = e^\mu \cdot h_i, \quad h_i' = e_\mu \cdot h_i'$$  \hspace{1cm} (16)

Let $h_i^\mu = h_i^\mu(x')$ be a non-degenerate transformation; therefore, the base vector of the observer space is as follows:

$$E_i(x) = h_i^\mu e_\mu (\zeta^\mu, t)$$  \hspace{1cm} (17)

Because of the existence of defects in the medium, this basis cannot be obtained using the Jacobian formula; thus, the basis is not coordinated. Therefore, the derived symmetric tensor from Eq. (17)

$$g = \|g_{ij}\| = E_i E_j = h_i^\mu h_j^\mu$$  \hspace{1cm} (18)

can be used as the internal metric tensor of medium with an internal defect structure. Note that $g$ is the external metric tensor of the medium. The Algebraic structure of the internal metric tensor $g$ is the same as that of the external metric tensor $G$.

Similar to the distortion tensor $A = \|A^\mu\| = \|\partial \zeta^\mu / \partial x^i\|$, $h_i^\mu = h_i^\mu(x')$ can be considered as the generalized distortion tensor of medium with defect structure. If $h_i^\mu$ is not the partial derivative for some three functions $f^\mu(x), (\mu = 1, 2, 3)$, that is, $h_i^\mu \neq \partial_i f^\mu$, then the frame is called non-holonomic. Geometrically, this indicates that there are no new local coordinates $y^\mu = f^\mu(x)$, making $\{E_i\}$ a coordinate frame relative to it. The formation of defects leads to irreversible deformation in the medium. However, because $h_i^\mu = h_i^\mu(x')$ is not a Jacobian transformation, the deviation from the Jacobi transformation is as follows:
Only when \( C_{ij}^\mu = 0 \) and \( h_{ij}^\mu = \delta_{ij}^\mu \), the internal metric tensor \( g \) can be consistent with the external metric tensor \( G \). The above quantities constitute non-holonomic quantities, which constitute general covariant tensors for indices \( \mu, \nu \).

The comparison between Eqs. (19) and (13) shows that \( C_{ij}^\mu \) is similar to the Burgs tensor \( B = \begin{bmatrix} B_{ii}^\mu \end{bmatrix} \) and can be considered as the generalized Burgs tensor. Because of the appearance of defective structures in the medium, \( C_{ij}^\mu \neq 0 \), the deformation of the medium is no longer compatible.

For the quaternion coefficients \( h_{ij}^\mu \), the evolution equation is somewhat similar to Eq. (8). However, unlike Eq. (8), the source term appears on the right side of the equation.

\[
\frac{\partial h_{ij}^\mu}{\partial t} + u^k \frac{\partial h_{ij}^\mu}{\partial x^k} + \frac{\partial u^k}{\partial x^i} h_{ij}^\mu = -\psi_{ij}^\mu
\]

(20)

The appearance of the source term is because of the appearance of defects in the process of deformation, which leads to the incompatibility of deformation and the destruction of continuity. The breakdown of continuity results in either overlap or void within the medium.

Similarly, the evolution equation for the internal metric tensor is as follows:

\[
\frac{\partial g_{ij}}{\partial t} + u^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial u^k}{\partial x^j} + \frac{\partial u^k}{\partial x^i} g_{kj} = -\varphi_{ij}
\]

(21)

Using Eq. (18), we can obtain the relationship between \( \psi_{ij}^\mu \) and \( \varphi_{ij} \).

\[
\varphi_{ij} = \psi_{ij}^\mu h_{ij}^\mu + \psi_{ji}^\mu h_{ij}^\mu
\]

(22)

Differentiating Eq. (20) to \( x^i \), we obtain the transport equation of \( C_{ij}^\mu \).

\[
\frac{\partial}{\partial t} h_{ij}^\mu + u^j \frac{\partial h_{ij}^\mu}{\partial x^i} + \frac{\partial h_{ij}^\mu}{\partial x^j} + h_{ij}^\mu \frac{\partial u^i}{\partial x^j} + h_{ij}^\mu \frac{\partial u^j}{\partial x^i} = -h_{ij}^\mu
\]

(23)

Swapping the subscripts \( i \) and \( j \) and subtracting the resulting formula from Eq. (23) yield the transport equation:

\[
\frac{d}{dt} C_{ij}^\mu + C_{ij}^\mu \frac{\partial u^j}{\partial x^i} + \frac{\partial u^j}{\partial x^i} C_{ij}^\mu = -\frac{1}{2} \left( \frac{\partial \psi_{ij}^\mu}{\partial x^i} + \frac{\partial \psi_{ij}^\mu}{\partial x^j} \right), \quad C_{ij}^\mu|_{t=0} = 0
\]

(24)

Because Eq. (24) is linear, if its right side is 0, then we always have \( C_{ij}^\mu = 0 \).

3. Constitutive relations
The irreversible process of the medium can be expressed in terms of the second law of thermodynamics, which can be expressed as the Clausius–Duhem inequality [31]:

$$\rho_0 \dot{S}^{(i)} \geq 0$$  \hspace{1cm} (25)

where \( \rho_0 \) is the medium density, \( T \) is the absolute temperature scale, and \( \dot{S}^{(i)} \) is the irreversible part of the entropy generation. It can be expressed as follows:

$$\rho_0 \dot{S}^{(i)} = \sigma_{ij} \dot{\varepsilon}_{ij} - \rho_0 \psi - \rho_0 \dot{S} - \frac{\Delta T}{T} \cdot \tilde{q} \geq 0$$  \hspace{1cm} (26)

where \( \psi \) is the Helmholtz free energy, \( S \) is the entropy per unit mass, \( \Delta T \) is the temperature gradient, \( \tilde{q} \) is the heat flux, and \( \dot{\varepsilon}_{ij} \) is the deformation rate.

Generally, it is assumed that the process of heat conduction does not depend on the local thermodynamic process; thus, we can split Eq. (26) into two independent inequalities:

(i) local entropy production inequality

$$\rho_0 \dot{S}^{(i)} = \sigma_{ij} \dot{\varepsilon}_{ij} - \rho_0 \psi - \rho_0 \dot{S} \geq 0$$  \hspace{1cm} (27)

(ii) heat conduction inequality

$$- \frac{\Delta T}{T} \cdot \tilde{q} \geq 0$$  \hspace{1cm} (28)

The strain rate is then divided into the sum of reversible and irreversible parts:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^r + \pi$$  \hspace{1cm} (29)

where \( \dot{\varepsilon}_{ij}^r \) is the reversible part of the deformation and \( \pi \) is the irreversible part of the deformation. The Helmholtz free energy is assumed to be a state function dependent on the reversible deformation, the internal variables \( \alpha_i \), and the temperature \( T \), that is,

$$\psi = \psi(e^r, \alpha_i, T)$$

From Eq. (26), we obtain

$$\left( \sigma_{ij} - \rho_0 \frac{\partial \psi}{\partial \varepsilon_{ij}} \right) \dot{\varepsilon}_{ij}^r + \sigma_{ij} \pi + \rho_0 \frac{\partial \psi}{\partial \alpha_i} \dot{\alpha}_i - \left( \rho_0 \frac{\partial \psi}{\partial T} + S \right) \frac{\Delta T}{T} \cdot \tilde{q} \geq 0$$  \hspace{1cm} (30)

For reversible processes, \( \alpha_i = 0 \) and \( \pi = 0 \), because the expression in parentheses in Eq. (30) is a state function, independent of \( e^r \) and \( S \), the following constitutive relation can be obtained:

$$\sigma_{ij} = \rho_0 \frac{\partial \psi}{\partial e_{ij}}$$

$$S = -\rho_0 \frac{\partial \psi}{\partial T}$$  \hspace{1cm} (31)

By introducing the generalized force conjugate to \( \alpha_i \)
we transform Eq. (30) into the following form:

$$\sigma_{ij} \dot{\alpha}_i + \rho_0 \dot{\psi} \frac{\partial \psi}{\partial \alpha_i} \dot{\alpha}_i - \frac{\Delta T}{T} \dot{\varphi} \geq 0$$  

(33)

Now, it is an established physical fact that the glide of dislocations is responsible for the critical plastic deformations that correspond to the production of Riemann–Christoffel curvature, but not necessarily of torsion tensor. Therefore, the criterion of yielding of the matter is mostly expressed in terms of Riemann–Christoffel curvature tensor [7]. Here, we considered the internal variable $\alpha_i = R$ and the following Helmholtz free energy function:

$$\rho_0 \psi = \frac{2}{2} \left( \varepsilon_{ik} \right)^2 + \mu \varepsilon_{ik} \varepsilon_{ij} - \omega \varepsilon_{ik} R - \frac{1}{2} \chi R^2$$  

(34)

where $\lambda$ and $\mu$ are the Lame's coefficients, and $\omega$ and $\chi$ are the material coefficients to be determined. Then, we can obtain the following expressions for the stresses

$$\sigma_{ij} = \delta_{ij} \left( \lambda \varepsilon_{ik} - \omega R \right) + 2 \mu \varepsilon_{ij}$$  

(35)

and for the generalized force conjugate to $R$

$$\beta_i = -\rho_0 \frac{\partial \psi}{\partial R} = \omega \varepsilon_{ik} + \chi R$$  

(36)

As shown by Kondo [5], the Riemann tensor can be expressed as follows:

$$R_{hijk} = \frac{1}{2} \left( \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{hk}}{\partial x^j \partial x^i} + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^h} - \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} \right)$$  

(37)

Because of irreversible deformation, we consider the following internal metric tensor $g_{ij}$:

$$g_{ij} = \delta_{ij} - 2 \pi_{ij}$$  

(38)

where $\delta_{ij}$ is the Kronecker delta. As shown in [26], the above internal metric tensor has a non-Euclidean geometric structure. Substituting Eq. (38) into Eq. (37) yields

$$R_{hijk} = -\frac{\partial^2 \pi_{hk}}{\partial x^i \partial x^j} - \frac{\partial^2 \pi_{ij}}{\partial x^h \partial x^k} + \frac{\partial^2 \pi_{ij}}{\partial x^h \partial x^k} + \frac{\partial^2 \pi_{ij}}{\partial x^h \partial x^k}$$  

(39)

and the scalar curvature

$$R = g^{h0}g^{ik}R_{hijk}$$  

(40)

Therefore, the curvature may be considered as isotropic damage parameter, and $R$ in Eq (35) leads to softening on the occurrence of incompatibility of deformation, which agrees with the tests. To form a complete set of equations describing the irreversible thermodynamic behavior of matter, in addition to Eqs. (35) and (36), we should supplement the evolution equation of internal variables.
4. Evolution of incompatible deformations

At first, we use Euler’s viewpoint in describing the deformation of continuum motion. The complete deformation of the continuum can be described by the Almansi tensor $A_{ij}$. The complete deformation tensor $A_{ij}$ can be decomposed into reversible components $e_{ij}$ and irreversible components $\tau_{ij}$, as in Eq. (29).

As reported by Kondo [7], elastic deformations are compatible deformations confined in the ordinary three-space, thus preserving the topological invariance of the whole manifold. The irreversible plastic deformations are geometrical changes into the “bad” (imperfect) condition of a crystal. The theory of elasticity satisfies the compatibility condition. The violation of compatibility is named non-elasticity or a type of plasticity that induces a non-homeomorphism of the matter-manifold in the Euclidean three-space.

Under small values of deformations, we can consider the following simplified evolution equation of internal metric tensor $g_{ij}$ by neglecting the higher-order terms smallness in the left-hand side of Eq. (21) with consideration of Eq. (38):

$$\frac{\partial g_{ij}}{\partial t} = -\varphi_{ij} = 2\frac{\partial \tau_{ij}}{\partial t} = p_{ij}$$

(41)

which can be looked at as Ricci’s flow [32], and the theory there can be used.

Under the subcritical condition, the irreversible deformation of the matter is thermally activated, which can be expressed as follows:

$$p_{ij} = 2\hat{e}_{ij0} \exp \left[ -\frac{U(\sigma_{ij})}{kT} \right]$$

(42)

where $\hat{e}_{ij0}$ is a constant, $U(\sigma_{ij})$ is the activation energy, $k$ is the Boltzmann constant, and $T$ is the absolute temperature.

In the following operation, we will consider the case in which $\nabla_k g_{ij} = 0$.

From $g^{ik} g_{kl} = \delta^i_j$, we have

$$\left( \frac{\partial g^{ik}}{\partial t} \right) g_{kl} + g^{ik} g_{kl} = 0$$

; therefore,

$$\frac{\partial g^{ik}}{\partial t} = \left( \frac{\partial g^{ik}}{\partial t} \right) g_{kl} g^{lm} = -g^{ik} g^{jm} p_{kl}$$

(43)

Because in the local coordinate $\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$, where $\partial_i = \partial / \partial x_i$, the evolution equation of $\Gamma^k_{ij}$ is

$$\frac{\partial \Gamma^k_{ij}}{\partial t} = \frac{1}{2} \frac{\partial g^{ik}}{\partial t} (\partial_j g_{il} + \partial_l g_{ij} - \partial_i g_{jl}) + \frac{1}{2} g^{kl} \left( \frac{\partial g^{i\ell}}{\partial t} + \frac{\partial g_{i\ell}}{\partial t} \right) - \partial_i \left( \frac{\partial \Gamma^k_{ij}}{\partial t} \right)$$

(44)
If we select the geodesic coordinate, then for any point \( A \in M^3 \), we have \( \Gamma^1_{0j}(A) = 0 \). Hence, for any tensor \( T \), we have \( \partial_t T_{jk} = \nabla_j T_{jk} \). It follows that \( \partial_t g_{jk}(A) = 0 \) for all \( i, j, k \). Therefore, from the above equation, we obtain

\[
\frac{\partial \Gamma^i_{jk}}{\partial t}(A) = \frac{1}{2} g^{ik} \left( \nabla_j \left( \frac{\partial g_{kl}}{\partial t} \right) + \nabla_k \left( \frac{\partial g_{jl}}{\partial t} \right) - \nabla_l \left( \frac{\partial g_{jk}}{\partial t} \right) \right) = \frac{1}{2} g^{ik} \left( \nabla_j \left( P_{jl} + \nabla_j P_{jl} - \nabla_i P_{ij} \right) \right) \tag{45}
\]

Equation (45) holds in any coordinate system because both sides are tensor components.

Furthermore, for the Riemann curvature tensor \( R^i_{jk} = \partial_j \Gamma^l_{ik} - \partial_j \Gamma^l_{ik} + \Gamma^l_{jk} \Gamma^i_{lp} - \Gamma^l_{lp} \Gamma^i_{jk} \), its evolution equation is

\[
\frac{\partial R^i_{jk}}{\partial t} = \partial_j \left( \frac{\partial \Gamma^l_{ik}}{\partial t} \right) - \partial_j \left( \frac{\partial \Gamma^l_{ik}}{\partial t} \right) + \Gamma^l_{jk} \frac{\partial \Gamma^i_{lp}}{\partial t} + \Gamma^l_{lp} \frac{\partial \Gamma^i_{jk}}{\partial t} - \Gamma^l_{ik} \frac{\partial \Gamma^i_{lp}}{\partial t} - \Gamma^l_{lp} \frac{\partial \Gamma^i_{jk}}{\partial t} \tag{46}
\]

In the geodesic coordinate centered at any point \( A \in M^3 \), form the above equation we have

\[
\frac{\partial R^i_{jk}}{\partial t}(A) = \nabla_j \left( \frac{\partial \Gamma^l_{ik}}{\partial t} \right)_A - \nabla_j \left( \frac{\partial \Gamma^l_{ik}}{\partial t} \right)_A \tag{47}
\]

Substituting Eq. (47) into the above equation yields

\[
\frac{\partial R^i_{jk}}{\partial t}(A) = \frac{1}{2} g^{ik} \left( \nabla_j \left( \nabla_j P_{kp} + \nabla_k \nabla_j P_{kp} - \nabla_k P_{jk} - \nabla_j \nabla_j P_{kp} \right) \right)
\]

By contracting Eq. (48) on \( i = l \), we obtain the evolution equation of Ricci tensor \( R^i_{jk} \):

\[
\frac{\partial R_{jk}}{\partial t} = \frac{1}{2} g^{pq} \left( \nabla_q \nabla_j P_{kp} + \nabla_q \nabla_k P_{jp} - \nabla_q P_{jk} - \nabla_j \nabla_k P_{qp} \right) \tag{49}
\]

The scalar curvature function \( R \) is defined by \( R = g^{jk} R_{jk} \); therefore, its evolution is

\[
\frac{\partial R}{\partial t} = \frac{\partial g^{jk}}{\partial t} R_{jk} + g^{jk} \frac{\partial R_{jk}}{\partial t} = g^{jk} \left( \nabla_j \nabla_k P_{pq} - \nabla_k \nabla_j P_{pq} + \frac{\partial}{\partial t} \left( P_{jk} R_{ml} \right) \right) \tag{50}
\]

In this manner, the constitutive description of the irreversible thermodynamic behavior of matter is completed.

5. Conclusions

The rock mass is a system with complex structural hierarchy. Under the action of external loads, the change of mechanical property of rock mass is a progressive process of symmetry localization in which the incompatibility of rock deformation occurs. Thus, it is natural to use the concept of the manifold to describe this process. The incompatibility of deformation indicates that the external and internal metrics do not coincide, and the Euclidean space structure of the medium is broken. In the case of incompatible deformation, the orthogonal frame method of differential geometry can be used to completely describe the internal structure changes of a medium. The geometric parameters used in this process have a clear physical meaning. They are thermodynamic quantities that describe incompatible deformation. Thus, in this study, we used the orthogonal frame method to
describe the incompatible deformation of rock mass surrounding a deep level tunnel. The basic equations for the description of incompatible deformation of media are established. The constitutive equation is obtained by irreversible thermodynamics, and the evolution equation of the internal parameter is obtained.

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