FREE RESOLUTIONS OF FUNCTION CLASSES VIA ORDER COMPLEXES

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ABSTRACT. Function classes are collections of Boolean functions on a finite set, which are fundamental objects of study in theoretical computer science. We study algebraic properties of ideals associated to function classes previously defined by the third author. We consider the broad family of intersection-closed function classes, and describe cellular free resolutions of their ideals by order complexes of the associated posets. For function classes arising from matroids, polyhedral cell complexes, and more generally interval Cohen-Macaulay posets, we show that the multigraded Betti numbers are pure, and are given combinatorially by the Möbius functions. We then apply our methods to derive bounds on the VC dimension of some important families of function classes in learning theory.

1. INTRODUCTION

For $n \in \mathbb{N}$, let $[n] := \{0, 1, \ldots, n-1\}$. A function class $C$ is a collection of Boolean functions on $[n]$, that is, $C \subseteq \{0, 1\}^n$.

A central question of learning theory is:

how much data is required to learn an unknown function $f^*$,

given that $f^*$ is in some known function class $C$?

Here, to learn $f^*$ means to identify some function $\hat{f} \in C$ such that $\hat{f}$ is identical to $f^*$ except on a small subset of $[n]$.

The classical answer to the question above is given by the VC dimension.

Definition 1.1. We say a subset $U \subseteq [n]$ is shattered by $C$ if every function on $U$ is a restriction of some function in $C$. The VC dimension (Vapnik-Chervonenkis dimension) $\text{dim}_{\text{VC}} C$ of $C$ is

$$\text{dim}_{\text{VC}} C := \max\{|U| \mid U \text{ is shattered by } C\}.$$  

For more than 40 years since its introduction, the VC dimension has occupied center-stage in learning theory and other analytically-flavored branches of computer science. It is a celebrated theorem in classical learning theory that the number of samples needed to learn an unknown function in $C$ is proportional to $\text{dim}_{\text{VC}} C$; we point to [KV94] for precise statements and more details on learning theory.

In this paper, we continue the study of the learning theoretic properties of $C$ using invariants of homological nature introduced by the third author [Yan17]. There is a natural simplicial complex $\hat{C}$ associated to a function class $C$, called the subplex of $C$. We consider the Stanley-Reisner ideal $I_C$ of $\hat{C}$ as well as its dual ideal $I_C^*$ — see §2 for details. One can then analyze the learning theoretic properties of $C$ by drawing upon the vast literature on squarefree monomial ideals.

Theorem 1.2. [Yan17, Theorem 3.11] Define the homological dimension $\text{dim}_h C$ of a function class $C$ as the projective dimension of $I_C^*$, i.e. $\text{dim}_h C := \text{projdim} I_C^*$. Then

$$\text{dim}_{\text{VC}} C \leq \text{dim}_h C.$$
The two quantities $\dim_{VC}$ and $\dim_h$ can be different, but they do coincide for many function classes of importance in computer science [Yan17, Section 3.1], such as the class of parity functions, the class of polynomial threshold functions, or the class of monotone conjunctions. Our goal in this paper is two-fold: (i) to investigate the multigraded Betti numbers of $I^*_C$, and (ii) to identify new large families of function classes for which $\dim_{VC}$ and $\dim_h$ coincide or approximately coincide.

Our main cases of interest are function classes with suitable semi-lattice structures. Let $\mathcal{P}$ be a subposet of the lattice of subsets of $[n]$ that is intersection-closed (see §4). We consider the function class $\mathcal{C}(\mathcal{P})$ defined by $\mathcal{P}$ by identifying subsets of $[n]$ with their indicator functions as in [HSW89]. Function classes arising in this way include:

- conjunctions (logical AND) of parity functions (i.e. conjunctions of linear functionals over $\mathbb{F}_2$), and more generally the lattice of flats of a matroid (see §5.1 and §6.3),
- downward-closed classes, and more generally the face poset of a polyhedral cell complex (see §5.2), and
- the class of $k$-CNFs (conjunctive normal forms) and the class of CSPs (constraint satisfaction problems) (see §6.2).

Our main result is a construction of an explicit free resolution of the ideal $I^*_C(\mathcal{P})$ via the order complex $\Delta_{\mathcal{P}}$ of $\mathcal{P}$. We refer to §3 and §4 for the relevant definitions and notation.

**Theorem 4.3.** Let $\mathcal{P} \subseteq 2^{[n]}$ be an intersection-closed poset, and $S$ the polynomial ring of the ideal $I^*_C(\mathcal{P})$. Denote by $\mathcal{F}(\Delta_{\mathcal{P}})$ the cellular chain complex of free $S$-modules arising from the order complex $\Delta_{\mathcal{P}}$ of $\mathcal{P}$, with vertices labelled by minimal generators of $I^*_C(\mathcal{P})$. Then $\mathcal{F}(\Delta_{\mathcal{P}})$ is acyclic and hence gives an $S$-free resolution of $S/I^*_C(\mathcal{P})$.

As a consequence, we obtain the multigraded Betti numbers of $I^*_C(\mathcal{P})$ purely in terms of the poset topology of $\mathcal{P}$.

**Theorem 4.4.** Let $\mathcal{P} \subseteq 2^{[n]}$ be an intersection-closed poset. Then the nonzero multigraded Betti numbers of $I^*_C(\mathcal{P})$ occur precisely in the degrees of certain monomials denoted $m(A, B)$ associated to each $A \leq B \in \mathcal{P}$ (see Definition 3.4), and in these degrees,

$$\beta_{i,m(A,B)}\left(I^*_C(\mathcal{P})\right) = \dim_{\mathbb{K}} H_{i-2}(\overline{\mathcal{C}}(A,B); \mathbb{K}), \quad \forall i \geq 1$$

where $\overline{\mathcal{C}}(A,B)$ denotes the truncated order complex of the interval $[A, B]$ (Definition 3.2).

**Corollary 4.6.** Let $\mathcal{P} \subseteq 2^{[n]}$ be an intersection-closed poset. Then $\dim_h \mathcal{C}(\mathcal{P}) \leq \text{rank}(\mathcal{P})$.

If furthermore all (open) intervals of $\mathcal{P}$ are Cohen-Macaulay (in which case we say $\mathcal{P}$ is interval Cohen-Macaulay), as is the case when $\mathcal{P}$ arises from a matroid or a polyhedral cell complex, then the multigraded Betti numbers of $I^*_C(\mathcal{P})$ are pure and can be described combinatorially by the Möbius function of $\mathcal{P}$ (see §5).

**Theorem 5.5.** Let $\mathcal{P} \subseteq 2^{[n]}$ be an intersection-closed poset that is interval Cohen-Macaulay, and $\mu_{\mathcal{P}}(\cdot, \cdot)$ the Möbius function of the poset $\mathcal{P}$. Then (with $m(A, B)$ as in Theorem 4.4),

$$\beta_{i,m(A,B)}\left(I^*_C(\mathcal{P})\right) = \begin{cases} |\mu_{\mathcal{P}}(A,B)| & \text{if } \text{rank}([A,B]) = i \\ 0 & \text{otherwise} \end{cases}$$

For example, when $\mathcal{P} = \mathcal{P}_M$ is the lattice of flats of a matroid $M$, the quantity $\mu_{\mathcal{P}}(F,G)$ for $F \subseteq G \in \mathcal{P}$ is known as the Möbius invariant of the matroid minor $M|G/F$. When $\mathcal{P} = \mathcal{P}_X$ is the face poset of a polyhedral cell complex $X$, then the quantity $\mu_{\mathcal{P}}(F,G)$ for $F \subseteq G \in \mathcal{P}$ is the reduced Euler characteristic of the boundary complex of a polytope, which is always $\pm 1$. 
In section §6, we apply our tools to give bounds for the VC dimension of various function classes of importance in learning theory, such as the class of $k$-CNFs, the class of d-CSPs, the class of conjunctions of parity functions, and more generally the class of conjunctions of polynomials over $\mathbb{F}_2$ (see §6 for definitions of these classes).

**Theorem 6.3.** Let $C$ be the class of $k$-CNFs and let $C^+$ be the class of monotone $k$-CNFs in $d$ variables. Then

$$\Omega(d^k) \leq \dim_{VC} C^+ \leq \dim_{b} C^+ \leq O(d^k)$$

$$\Omega(d^k) \leq \dim_{VC} C \leq \dim_{b} C \leq O(d^k),$$

where $\Omega$ and $O$ hide constants dependent on $k$ but independent of $d$.

**Corollary 6.8 & 6.9.** Homological dimension and VC dimension coincide for the class of conjunctions of parity functions. The same holds more generally for conjunctions of degree-bounded polynomials over $\mathbb{F}_2$.

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## 2. Subplexes and shatter complexes

Let $C \subseteq [2]^n$ be a class of Boolean functions on $[n]$. Following [Yan17], we define a simplicial complex from $C$ as follows:

**Definition 2.1.** The subplex $\diamond_C$ associated to $C$ is the simplicial complex with vertex set $[n] \times [2]$ corresponding to input-output pairs $(i, b)$ with $i \in [n], b \in [2]$, and has facets given by graphs of functions in $C$, i.e. facets($\diamond_C$) = $\{(i,f(i)) \mid i \in [n] \}$, $f \in C$.

**Remark 2.2.** It is possible that some points in $[n] \times [2]$ do not appear in the graph of any function of $C$. For example, if $f(i) = 0$ for every $f \in C$ for some $i \in [n]$, then no facet of $\diamond_C$ contains the vertex $(i, 1)$.

In this case, $\diamond_C$ is the cone over the subplex $\diamond_{C \mid [n] \setminus \{i\}}$ of the restriction of $C$ to $[n] \setminus \{i\}$, and passing to $[n] \setminus \{i\}$ loses no learning theoretic information about $C$.

We now define the main algebraic object of study:

**Definition 2.3.** The subplex ideal $I_C$ is the Stanley-Reisner ideal associated to the simplicial complex $\diamond_C$. Explicitly, in the polynomial ring $S := \mathbb{k}[x_{(i,b)} \mid (i,b) \in [n] \times [2]]$ over a fixed field $\mathbb{k}$, $I_C$ is a squarefree monomial ideal whose monomial minimal generators are the minimal nonfaces of $\diamond_C$.

We will often consider the Alexander dual of the subplex ideal, i.e. $I_C^* := \langle \prod_{i \in [n]} x_{i,1-f(i)} \mid f \in C \rangle$.

A partial function on $[n]$ is a function $f : A \to [2]$ defined on some subset $A \subseteq [n]$, and we denote by $\text{dom}(f) = A$ its domain. Henceforth, the term “function” (without the modifier “partial”) will always mean a complete function $[n] \to [2]$. Given $A \subseteq B \subseteq [n]$ and partial functions $f : A \to [2], g : B \to [2]$, we say that $f$ is a restriction of $g$, or equivalently $g$ is an extension of $f$, if $g|_A = f$. We can describe the monomial minimal generators of $I_C$, which come in two types, as follows:

Each such monomial, representing a minimal nonface of $\diamond_C$, encodes the fact that every function has to send $i \in [n]$ to a unique output in $[2]$, hence the name functional monomial.
Proposition 2.4 ([Yan17, Proposition 2.38]). The ideal $I_C$ is minimally generated by the functional monomials and monomials defined by extentures in the following way:

$$I_C = \left\langle x_{(i,0)} x_{(i,1)} \mid i \in [n] \right\rangle + \left\langle \prod_{i \in \text{dom}(f)} x_{(i,f(i))} \mid f \text{ extenture of } C \right\rangle.$$  

Having defined the ideals $I_C$ and $I_C^\ast$, one can then interpret algebraic properties of $I_C$ and $I_C^\ast$ in terms of the function class $C$, and vice versa. For instance:

Definition 2.5. [Yan17, Definition 11.6] The homological dimension of $C$, denoted $\dim_h C$, is defined to be the projective dimension of $I_C^\ast$, i.e.

$$\dim_h C := \text{projdim } I_C^\ast = \text{projdim } S/I_C^\ast - 1.$$  

Remark 2.6. By [MS05, Theorem 5.59], the projective dimension of a squarefree monomial ideal is related to the (Castelnuovo-Mumford) regularity of its Alexander dual in the following way:

$$\dim_h C = \text{projdim}(S/I_C^\ast) - 1 = \text{reg}(I_C) - 1 = \text{reg}(S/I_C).$$

Next, we interpret VC dimension algebraically. Note that if $U$ is shattered by $C$ (recall Definition 1.1), and $U' \subseteq U$, then $U'$ is also shattered by $C$. Thus the sets shattered by $C$ form a simplicial complex $SH_C$, called the shatter complex of $C$. In this way, the VC dimension of $C$ is one more than the dimension of $SH_C$.

Definition 2.7. Define the collapse map $\pi : S \rightarrow T := k[y_i \mid i \in [n]]$ sending $x_{(i,b)} \mapsto y_i$. This is a surjective map, with kernel given by $\langle x_{(i,0)} - x_{(i,1)} \mid i \in [n] \rangle$, generated by linear binomials naturally corresponding to the functional monomials.

The following observation is an easy consequence of Proposition 2.4:

Proposition 2.8. [Yan17, Theorem 3.3] Let $I_{SH_C}$ be the Stanley-Reisner ideal of $SH_C$ in the ring $T$. Then $\pi(I_C) = I_{SH_C} + \langle y_i^2 \mid i \in [n] \rangle$. Equivalently, $U \in SH_C$ if and only if $\prod_{i \in U} y_i \notin \pi(I_C)$.

This yields the following algebraic description of VC dimension:

Corollary 2.9. (With notation as in Definition 2.7) $\dim_{VC} C = \text{reg}(T/\pi(I_C))$.

Proof. Since $\langle y_i^2 \mid i \in [n] \rangle \subseteq \pi(I_C)$, the quotient ring $T/\pi(I_C)$ is Artinian and has a basis consisting of squarefree monomials. Hence, by [Eis05, Exercise 20.18], we have

$$\text{reg}(T/\pi(I_C)) = \max\{d \mid (T/\pi(I_C))_d \neq 0\} = \max\{|U| \mid \prod_{i \in U} y_i \notin \pi(I_C)\}.$$  

By Proposition 2.8, the last expression equals max\{|$|U| \mid U \in SH_C\} = \dim_{VC} C$. 

We now recall the following central result, relating the VC dimension and homological dimension of an arbitrary function class:

Theorem 2.10. [Yan17, Theorem 3.11] Let $C \subseteq [2][n]$ be a function class. Then $\dim_{VC} C \leq \dim_h C$.

A natural question to ask is when equality in Theorem 2.10 holds. In general, the difference between $\dim_h C$ and $\dim_{VC} C$ can be arbitrarily large:

Example 2.11. Consider the class $C = \{\delta_i \mid 1 \leq i \leq n\}$ of delta functions on $[n]$, where $\delta_i(j) = 1 \iff i = j$. Since the constant function 0 is an extenture of $C$ with domain of size $n$, the homological dimension $\dim_h C$ is at least $n - 1$ (in fact, $\dim_h C = n - 1$). However, $C$ cannot shatter any subset of size $> 1$, so $\dim_{VC} C = 1$.

This example also shows that $\dim_{VC} C$ cannot always be sandwiched between $\max\text{deg } I_C - 1$ (i.e., one less than the maximal size domain of an extenture) and $\dim_h C$.  

[MS05: M. Mustata, D. Shamov, Projective dimension of squarefree monomial ideals, arXiv preprint arXiv:0509196v1 [math.AC] 8 Sep 2005]
3. Posets and order complexes

In light of the inequality \( \dim_{VC} C \leq \dim_b C \), the importance of determining homological invariants — in particular a free resolution of \( I_C^2 \) becomes clear. To this end, we now bring additional combinatorics into the picture, by viewing function classes as arising from posets. In doing so we lose no generality, and at the same time gain methods and viewpoints to attack our motivating question of resolving \( I_C^2 \).

Let \( 2^{[n]} \) be the Boolean poset of all subsets of \([n]\), partially ordered by inclusion. We consider subposets \((\mathcal{P}, \leq)\) of \( 2^{[n]} \) which are compatible with the ambient Boolean poset, so that if \( A \leq B \) in \( \mathcal{P} \), then \( A \subseteq B \) as subsets of \([n]\). Let \( \mathcal{C}(\mathcal{P}) \subseteq [2]^{[n]} \) be the function class associated to \( \mathcal{P} \) by identifying subsets with their indicator functions: notationally, we distinguish between \( 2^{[n]} \) for sets and \([2]^{[n]}\) for functions.

**Remark 3.1.** There is a natural \((\mathbb{Z}/2\mathbb{Z})^n\) action on \([2]^{[n]}\) by flipping 0 and 1 in the outputs. The learning-theoretic properties considered in this paper, most notably \( \dim_{VC} C \), are invariant under this action. Thus, any results obtained for \( \mathcal{C}(\mathcal{P}) \) also apply to any class in the orbit of \( \mathcal{C}(\mathcal{P}) \) under the \((\mathbb{Z}/2\mathbb{Z})^n\)-action.

We fix the following notation for a finite poset \( \mathcal{P} \):

- By \( A \leq B \in \mathcal{P} \) we mean “\( A, B \in \mathcal{P} \) with \( A \leq B \)”.
- For \( A \leq B \in \mathcal{P} \), we let \([A, B]\) (resp. \((A, B)\)) denote the closed (resp. open) interval \( [A, B] := \{C \in \mathcal{P} \mid A \leq C \leq B\} \), \((A, B) := \{C \in \mathcal{P} \mid A < C < B\} \).
- We denote by \( Ch_i(\mathcal{P}) \) the set of \( i \)-chains in \( \mathcal{P} \), i.e. \( Ch_i(\mathcal{P}) := \{C_0 < C_1 < \cdots < C_i \mid C_j \in \mathcal{P} \forall j = 0, \ldots, i\} \)
- and also \( Ch(\mathcal{P}) := \bigcup_{i \geq 1} Ch_i(\mathcal{P}) \), where \( Ch_{-1}(\mathcal{P}) := \{\emptyset\} \).
- The rank of a poset \( \mathcal{P} \) is \( \text{rank}(\mathcal{P}) := \max\{i \mid Ch_i(\mathcal{P}) \neq \emptyset\} \).
- The poset \( \mathcal{P} \) is bounded if it has a unique minimal element, denoted \( \hat{0} \), and a unique maximal element, denoted \( \hat{1} \).

**Definition 3.2.** Let \( \mathcal{P} \) be a poset. The order complex \( \Delta_{\mathcal{P}} \) associated to \( \mathcal{P} \) is the simplicial complex whose \( i \)-faces are the \( i \)-chains \( Ch_i(\mathcal{P}) \). In particular, the vertices of \( \Delta_{\mathcal{P}} \) are the elements of \( \mathcal{P} \), and the facets of \( \Delta_{\mathcal{P}} \) are maximal chains in \( \mathcal{P} \).

It is convenient to have the following variant of the order complex construction:

Let \( \mathcal{P} \) be a bounded poset. The truncated order complex \( \overline{\Delta}_{\mathcal{P}} \) is a simplicial complex whose \( i \)-faces are the \( i \)-chains in \( Ch_i(\mathcal{P}) \) that neither begin with \( \hat{0} \) nor end with \( \hat{1} \). In other words, \( \overline{\Delta}_{\mathcal{P}} := \begin{cases} \Delta_{\mathcal{P}\setminus\{\hat{0}, \hat{1}\}} & \text{if } \text{rank}(\mathcal{P}) \geq 2 \\ \{\emptyset\} & \text{if } \text{rank}(\mathcal{P}) = 1 \\ \emptyset & \text{if } \text{rank}(\mathcal{P}) = 0 \end{cases} \)

where \( \emptyset \) is the empty complex which has a single face (namely the empty set), and \( \emptyset \) is the null complex which has no faces.

**Remark 3.3.** Two key observations are in order. First, the empty complex \( \{\emptyset\} \) is distinguished by the following fact: the \((-1)\)-th reduced homology of a simplicial complex is nonzero if and only if the complex is the empty complex. Second, it is easily seen that the subcomplex of \( \Delta_{\mathcal{P}} \) consisting of chains that include \( \hat{0} \) or \( \hat{1} \), but not both, is the suspension of the truncated order complex \( \overline{\Delta}_{\mathcal{P}} \).

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\(^4\)Alternatively, for a bounded poset \( \mathcal{P} \), the truncated order complex \( \overline{\Delta}_{\mathcal{P}} \) is the order complex of the proper part \( \overline{\mathcal{P}} \) of \( \mathcal{P} \); see the survey \([\text{Wac07, §1.1}]\) for definitions and conventions.
Recall that our goal is to give a free resolution, as well as Betti numbers, of the dual ideal $I^*_c(P)$ in terms of the poset $P$. We prepare by fixing a convenient dictionary between monomials and (partial) functions, which will be used to describe the monomial minimal generators of $I^*_c$ and their least common multiples.

**Definition 3.4.** For subsets $A \subseteq B \subseteq [n]$, define corresponding partial functions and monomials

$$\delta(A, B)(i) := \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin B \end{cases} \quad \iff \quad m(A, B) := \prod_{i \in A} x_{(i,0)} \prod_{i \notin B} x_{(i,1)} \prod_{i \in B \setminus A} x_{(i,0)x_{(i,1)}}$$

where $\delta(A, B) : A \cup ([n] \setminus B) \to [2]$ is a partial function, and $m(A, B) \in S = \mathbb{k}[x_{(i,b)} \mid (i, b) \in [n] \times [2]]$ is a monomial of degree $|A| + (n - |B|) + 2(|B \setminus A|) = n + |B \setminus A|$.

**Remark 3.5.** We record some straightforward but useful observations relating subsets and partial functions:

1. For any function $f : [n] \to [2]$, one has $f = \delta(A, A)$ where $A = f^{-1}(1)$.
2. For $C \subseteq D \subseteq [n]$ and $A \subseteq B \subseteq [n]$, the partial function $\delta(C, D)$ extends $\delta(A, B)$ if and only if $A \subseteq C \subseteq D \subseteq B$. In this case, we write $\delta(C, D) \supseteq \delta(A, B)$.
3. For partial functions $f, g$ on $[n]$, define their intersection $f \cap g$ to be the partial function defined on $\{i \in \text{dom}(f) \cap \text{dom}(g) \mid f(i) = g(i)\}$ by $(f \cap g)(i) := f(i) = g(i)$. Then for $C \subseteq D \subseteq [n]$ and $A \subseteq B \subseteq [n]$,

$$\delta(C, D) \cap \delta(A, B) = \delta(C \cap A, D \cup B).$$

The next lemma collects more facts about the dictionary 3.4 relating partial functions with monomials which will be used in the sequel; we leave the easy verifications to the reader.

**Lemma 3.6.** Let $P \subseteq 2^{[n]}$ be a poset, and $C(P) \subseteq [2]^{[n]}$ the associated function class.

1. For two monomials $m, m'$, we write $m \preceq m'$ to mean that $m$ divides $m'$, and we let $\text{lcm}(m, m')$ denote the least common multiple of $m$ and $m'$. Then

$$\delta(C, D) \supseteq \delta(A, B) \iff \text{lcm}(m(C, D), m(A, B)) \leq \text{lcm}(m(C, D), m(A, B)).$$

2. The ideal $I^*_c(P)$ is minimally generated by the monomials

$$I^*_c(P) = \langle m(A, A) \mid A \in P \rangle.$$

3. The dictionary 3.4 gives an order-reversing isomorphism between the lcm-semilattice of monomial minimal generators of $I^*_c(P)$ and the semilattice of partial functions generated by intersections of functions in $C(P)$.

4. A squarefree monomial $m \in S$ is of the form $m(A, B)$ for some $A \subseteq B \subseteq [n]$ if and only if $x_{(i,0)}$ or $x_{(i,1)}$ divides $m$, for all $i \in [n]$.

**Proof.** Omitted.

4. Intersection-closed function classes

We now specialize to the main family of function classes under consideration.

**Definition 4.1.** A poset $P \subseteq 2^{[n]}$ is intersection-closed if $A, B \in P \implies A \cap B \in P$. In this case, we also say the associated function class $C(P)$ is intersection-closed. For any poset $P \subseteq 2^{[n]}$ (not necessarily intersection-closed) and any subset $A \subseteq [n]$, the closure of $A$ with respect to $P$ is defined as

$$\overline{A} := \bigcap_{B \in P : A \subseteq B} B.$$
Note that a poset $\mathcal{P} \subseteq 2^{[n]}$ is intersection-closed if and only if $\overline{A} \in \mathcal{P}$ for any $A \subseteq [n]$. As we shall see, besides forming a natural class of examples, intersection-closed function classes allow for rich interplay between algebra, combinatorics, and order theory. Intersection-closed function classes were also studied in [HSW89]. To help build intuition about these notions, we leave the proof of the following simple observation to the reader:

**Lemma 4.2.** Let $\mathcal{P} \subseteq 2^{[n]}$ be an intersection-closed poset, and $U \subseteq [n]$. Then $U$ is shattered by $\mathcal{C}(\mathcal{P})$ if and only if $\overline{A} \cap U = A$ for all $A \subseteq U$, or equivalently $\overline{A} \cap (U \setminus A) = \emptyset$ for all $A \subseteq U$.

We are now ready for our main result: an explicit free resolution of $I_{\mathcal{C}(\mathcal{P})}^*$ from the combinatorial data of the intersection-closed poset $\mathcal{P}$. To be precise, we construct a cellular free resolution of $I_{\mathcal{C}(\mathcal{P})}^*$ on the order complex $\Delta_{\mathcal{P}}$ as follows: label each vertex $A$ of $\Delta_{\mathcal{P}}$ by the monomial

$$m(A,A) = \prod_{i \in A} x_{(i,0)} \prod_{i \notin A} x_{(i,1)}$$

(note that under the dictionary 3.4, these monomials correspond precisely to functions $f \in \mathcal{C}(\mathcal{P})$), and label each face of $\Delta_{\mathcal{P}}$ by the lcm of the monomials of its vertices. Such a labeling of $\Delta_{\mathcal{P}}$ defines a complex $\mathcal{F}(\Delta_{\mathcal{P}})$ of free $S$-modules:

$$\mathcal{F}(\Delta_{\mathcal{P}}): \ldots \to F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \ldots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \to 0$$

where $F_i = \oplus_m S(-m)^{\beta_{i,m}}$ is a free $S$-module with basis given by $i$-faces of $\Delta_{\mathcal{P}}$ and monomial shifts corresponding to labels; for details we point to [Eis05, Ch. 2] or [MS05, Ch. 4]. Note that $F_{-1} = S$, labeled by the monomial $1 \in S$ corresponding to the empty set, is the unique $(-1)$-dimensional face of $\Delta_{\mathcal{P}}$.

**Theorem 4.3.** Let $\mathcal{P} \subseteq 2^{[n]}$ be an intersection-closed poset. Then $\mathcal{F}(\Delta_{\mathcal{P}})$ is acyclic and hence gives an $S$-free resolution of $S/I_{\mathcal{C}(\mathcal{P})}^*$.

**Proof.** First, note that by Lemma 3.6(3), the monomials appearing as a label of a face in $\mathcal{F}(\Delta_{\mathcal{P}})$ are all of the form $m(A,B)$ for some $A \leq B \in \mathcal{P}$. For $\mathcal{F}(\Delta_{\mathcal{P}})$ to be acyclic, by [MS05, Proposition 4.5] we need to show that for any monomial $m$, the subcomplex $(\Delta_{\mathcal{P}})_{\leq m}$ is acyclic, where $(\Delta_{\mathcal{P}})_{\leq m}$ consists of all faces of $\Delta_{\mathcal{P}}$ labeled by monomials $\preceq m$.

Since all labels of $\Delta_{\mathcal{P}}$ are squarefree, it suffices to consider squarefree monomials $m$. If $m$ is squarefree but not of the form $m(A,B)$ for some $A \subseteq B \subseteq [n]$, then by Lemma 3.6(4) then there exists $i \in [n]$ such that neither $x_{(i,0)}$ nor $x_{(i,1)}$ divides $m$. In particular, no monomial of the form $m(A,B)$ can divide $m$, and so in this case $(\Delta_{\mathcal{P}})_{\leq m} = \emptyset$ is the null complex.

We are thus left with the case where $m = m(A,B)$ for some $A,B \subseteq [n]$. In this case, Lemma 3.6(1) implies that $(\Delta_{\mathcal{P}})_{\leq m}$ consists of chains $\{C_0 < \cdots < C_i\}$ with $A \subseteq C_0$ and $C_i \subseteq B$. Now since $\mathcal{P}$ is intersection-closed, $\overline{A} \in \mathcal{P}$, and moreover every vertex of $(\Delta_{\mathcal{P}})_{\leq m}$ is connected to the vertex $\{\overline{A}\} \in (\Delta_{\mathcal{P}})_{\leq m}$. In particular, $(\Delta_{\mathcal{P}})_{\leq m}$ is a cone over the subcomplex of $\Delta_{\mathcal{P}}$ consisting of chains in $\mathcal{P}$ starting from an element strictly greater than $\overline{A}$ and ending below $B$.

Putting the above reasoning together shows that $(\Delta_{\mathcal{P}})_{\leq m}$ is acyclic for any monomial $m$, and thus $\mathcal{F}(\Delta_{\mathcal{P}})$ is a free resolution. The image of the last map $\partial_0$ in $\mathcal{F}(\Delta_{\mathcal{P}})$ is the ideal in $S$ generated by the monomials $m(A,A)$ for $A \in \mathcal{P}$, which by Lemma 3.6(2) is exactly $I_{\mathcal{C}(\mathcal{P})}^*$.

Despite the non-minimality of the resolution $\mathcal{F}(\Delta_{\mathcal{P}})$, we can still describe all the multigraded Betti numbers of $I_{\mathcal{C}(\mathcal{P})}^*$: We remark that while the multigraded Betti numbers of $I_{\mathcal{C}(\mathcal{P})}^*$ could be computed as (reduced) homologies of links of the subplex $\hat{\mathcal{C}}(\mathcal{P})$ by Hochster’s formula [MS05, Corollary 1.40], the topology of links of $\hat{\mathcal{C}}(\mathcal{P})$ is somewhat unclear in general. The main point of the following theorem is that the Betti numbers are expressed as homologies of truncated order complexes of intervals.
Theorem 4.4. Let $\mathcal{P} \subseteq 2^{|n|}$ be an intersection-closed poset. Then the nonzero multigraded Betti numbers of $I^*_C(\mathcal{P})$ occur only in degrees $m(A, B)$ for some $A \leq B \in \mathcal{P}$, and in that case,

$$\beta_{i,m(A,B)}(I^*_C(\mathcal{P})) = \dim_k \bar{H}_{i-2}(\Delta_{[A,B]}; k), \quad \forall i \geq 1, \quad \text{and} \quad \beta_{0,m(A,A)}(I^*_C(\mathcal{P})) = 1.$$  

Proof. As in Theorem 4.3, the monomials appearing as a label of a face in $\mathcal{F}(\Delta \mathcal{P})$ are exactly \{m(A, B) | A \leq B \in \mathcal{P}\}. Since $\mathcal{F}(\Delta \mathcal{P})$ resolves $S/I^*_C(\mathcal{P})^*$, the multigraded Betti numbers only occur in such degrees, and by [MS05, Theorem 4.7], $\beta_{i,m(A,B)}(I^*_C(\mathcal{P})) = \dim_k \bar{H}_{i-1}((\Delta \mathcal{P})_{<m(A,B)}; k)$. Now the faces of $(\Delta \mathcal{P})_{<m(A,B)}$ are chains $\{C_0 < \ldots < C_t\}$ where $A \leq C_0$, $C_t \leq B$, and either $A \neq C_0$ or $C_t \neq B$. In other words, $(\Delta \mathcal{P})_{<m(A,B)}$ is the suspension of the truncated order complex $\Delta_{[A,B]}$ by Remark 3.3, so $\bar{H}_{i-1}((\Delta \mathcal{P})_{<m(A,B)}; k) \cong \bar{H}_{i-2}(\Delta_{[A,B]}; k)$. \hfill \Box

Corollary 4.5. Let $\mathcal{P} \subseteq 2^{|n|}$ be an intersection-closed poset. Then minimal generators of first syzygies of $I^*_C(\mathcal{P})$ are in bijection with cover relations in $\mathcal{P}$.

Proof. Taking $i = 1$ in Theorem 4.4, one has $\beta_{1,m(A,B)}(I^*_C(\mathcal{P})) = \dim_k \bar{H}_{-1}(\Delta_{[A,B]}; k)$ is nonzero $\iff \Delta_{[A,B]} = \emptyset$ $\iff \text{rank}([A,B]) = 1$ $\iff B$ covers $A$. \hfill \Box

Corollary 4.6. Let $\mathcal{P} \subseteq 2^{|n|}$ be an intersection-closed poset. Then there is an inequality

$$\dim_k \mathcal{C}(\mathcal{P}) \leq \text{rank}(\mathcal{P}),$$

and equality holds if and only if $\bar{H}_{\text{rank}(\mathcal{P})-2}(\Delta_{[0,\mathcal{P}]}; k) \neq 0$ for some maximal $B \in \mathcal{P}$.

Proof. The inequality follows from Theorem 4.3, as the order complex $\Delta \mathcal{P}$ has dimension $\text{rank}(\mathcal{P})$, so $\mathcal{F}(\Delta \mathcal{P})$ is a free resolution of $S/I^*_C(\mathcal{P})$ of length $\text{rank}(\mathcal{P}) + 1$. Equality is achieved precisely when $\beta_{r,m}(I^*_C(\mathcal{P})) = \bar{H}_{r-2}(\Delta_{[A,B]}; k) \neq 0$ for some monomial $m = m(A, B)$ with $r = \text{rank}([A,B]) = \text{rank}(\mathcal{P})$, which occurs only if $B$ is maximal in $\mathcal{P}$ and $A = \emptyset$. \hfill \Box

5. INTERVAL COHEN-MACAULAY POSETS

Although the free resolution of $I^*_C(\mathcal{P})$ for an intersection-closed poset $\mathcal{P}$ given in Theorem 4.3 is satisfactory from an algebraic viewpoint, it is natural to ask if the Betti numbers in Theorem 4.4 have some combinatorial meaning. Our goal in this section is to show that for certain combinatorial families of intersection-closed posets, the multigraded Betti numbers of $I^*_C(\mathcal{P})$ are given by the Möbius function of the poset.

Definition 5.1. Let $\mathcal{P}$ be a poset. The Möbius function $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ is recursively defined by

$$\mu(\emptyset, A) := 1 \text{ for any } A \in \mathcal{P} \quad \text{and} \quad \mu(A, B) := -\sum_{A \leq C < B} \mu(A, C) \text{ for any } A \leq B \in \mathcal{P}.$$

We often drop the subscript $\mathcal{P}$ when the poset is clear from the context, and when $\mathcal{P}$ is bounded we write $\mu(\mathcal{P}) := \mu(\emptyset, \mathcal{P})$. The following statement is well-known in the literature as the Philip Hall theorem; for a proof, see e.g. [Rot64, Proposition 3.6].

Proposition 5.2. Let $\mathcal{P}$ be a bounded poset. Then the reduced Euler characteristic of the truncated order complex $\check{\chi}(\Delta \mathcal{P})$ is given by the Möbius function, i.e.

$$\check{\chi}(\Delta \mathcal{P}) := \sum_{i \geq -1} (-1)^i \dim_k \bar{H}_i(\Delta \mathcal{P}; k) = \mu(\mathcal{P}).$$
In light of Theorem 4.4, these reduced Euler characteristics are alternating sums of multigraded Betti numbers. For intersection-closed posets satisfying a following variant of the Cohen-Macaulay property, all but one term in the alternating sum vanishes, and so the reduced Euler characteristics are in fact equal to the multigraded Betti numbers.

**Definition 5.3.** A simplicial complex \( \Delta \) is **Cohen-Macaulay** (over \( \mathbb{k} \)) if the Stanley-Reisner ideal \( I_\Delta \) is Cohen-Macaulay, i.e. the quotient ring \( S/ I_\Delta \) is a Cohen-Macaulay ring. We say that a poset \( \mathcal{P} \) is **Cohen-Macaulay** if the order complex \( \Delta_\mathcal{P} \) is Cohen-Macaulay, and that \( \mathcal{P} \) is **interval Cohen-Macaulay** if every open interval in \( \mathcal{P} \) is Cohen-Macaulay.

Note that the Cohen-Macaulay property depends only on the characteristic of the field \( \mathbb{k} \). A celebrated result of Reisner gives a characterization for a simplicial complex to be Cohen-Macaulay, in terms of homology of links. Recall that for a face \( F \) of a simplicial complex \( \Delta \), the **link** of \( F \) in \( \Delta \) is defined as \( \text{lk}_\Delta F := \{ G \in \Delta \mid F \cap G = \emptyset, \ F \cup G \in \Delta \} \).

**Theorem 5.4.** [Rei76, Theorem 1] A simplicial complex \( \Delta \) is Cohen-Macaulay if and only if \( \bar{H}_i(\text{lk}_\Delta F; \mathbb{k}) = 0 \) for all faces \( F \in \Delta \) and all \( i < \dim \text{lk}_\Delta F \).

For \( A \leq B \in \mathcal{P} \), the order complex \( \Delta_{(A,B)} \) of the open interval \( (A,B) \) is equal to the link \( \text{lk}_{\mathcal{P}} C \), where \( C \) is any chain in \( \mathcal{P} \) obtained by omitting all elements strictly between \( A \) and \( B \) from a maximal chain in \( \mathcal{P} \) containing \( A \) and \( B \). Consequently, the links of \( \Delta_{(A,B)} \) are links \( \text{lk}_{\mathcal{P}} F \) of \( \Delta_\mathcal{P} \) where \( F \) is a chain containing \( C \). It thus follows from Theorem 5.4 that if \( \mathcal{P} \) is Cohen-Macaulay, then \( \mathcal{P} \) is interval Cohen-Macaulay. Moreover, if \( \mathcal{P} \) is interval Cohen-Macaulay, then Theorem 5.4 and Proposition 5.2 implies \( \bar{\chi}(\Delta_{(A,B)}) = \mu_\mathcal{P}(A,B) \) for any \( A \leq B \in \mathcal{P} \) [Sta77, Theorem 10]. Combining this observation with Theorem 4.4 yields the following:

**Theorem 5.5.** Let \( \mathcal{P} \subseteq 2^n \) be an intersection-closed poset that is interval Cohen-Macaulay. Then the nonzero multigraded Betti numbers of \( I_{\mathcal{P}}^* \) occur only in degrees \( m(A,B) \) for \( A \leq B \in \mathcal{P} \), and

\[
\beta_{i,m(A,B)}(I_{\mathcal{P}}^*) = \begin{cases} 
|\mu_\mathcal{P}(A,B)| & \text{if } \text{rank}([A,B]) = i \\
0 & \text{otherwise.}
\end{cases}
\]

We conclude this section by discussing two distinguished families of interval Cohen-Macaulay intersection-closed posets: (i) the lattice of flats of a matroid, and (ii) the face poset of a polyhedral complex. In both cases, the property of being interval Cohen-Macaulay is established by shellability.

**Definition 5.6.** A simplicial complex on \( [n] \) is **shellable** if there is an ordering of its facets \( F_1, \ldots, F_s \) such that \( (\bigcup_{i<k} F_i) \cap F_k \) is pure of dimension \( (\text{dim} F_k - 1) \), for each \( k = 2, \ldots, s \). A poset \( \mathcal{P} \) is **shellable** if its order complex \( \Delta_\mathcal{P} \) is shellable.

The next proposition collects all results we need concerning shellability; for details and proofs, cf. the survey [Wac07] and the references therein such as [BM71, Sta77, Bjö80, BW96].

**Proposition 5.7.** Let \( \Delta \) be a simplicial complex, and \( \mathcal{P} \) a finite poset.

1. If \( \mathcal{P} \) is shellable, then so is any closed or open subinterval in \( \mathcal{P} \).
2. If \( \Delta \) is shellable, then it is Cohen-Macaulay.
3. A locally semimodular poset, in particular a geometric lattice (i.e. the lattice of flats of a matroid; see §5.1), is shellable.
4. The face poset of a polytope (or its boundary) is shellable.

We remark that while we only give detailed expositions for matroids and polyhedral cell complexes, there are other examples of interesting families of posets whose structure fits into the framework of this section (e.g. antimatroids, whose posets are upper-semimodular).
5.1. Matroids.

We give a brief overview of matroids — details for unproven claims may be found in [Oxl11].

**Definition 5.8.** A matroid $M = (E, \text{rk}_M)$ consists of a finite set $E$, called the ground set, and a rank function $\text{rk}_M : 2^E \to \mathbb{Z}_{\geq 0}$ such that

1. if $A \subseteq E$ then $\text{rk}_M(A) \leq |A|$,
2. if $A, B \subseteq E$ then $\text{rk}_M(A) \leq \text{rk}_M(B)$, and
3. if $A, B \subseteq E$ then $\text{rk}_M(A \cup B) + \text{rk}_M(A \cap B) \leq \text{rk}_M(A) + \text{rk}_M(B)$.

A subset $I \subseteq E$ is independent if $\text{rk}_M(I) = |I|$. The maximal independent sets are called the bases of $M$, and all have the same cardinality $\text{rank}(M) := \text{rk}_M(E)$, called the rank of $M$. The closure of a subset $A \subseteq E$ is $\text{cl}_M(A) := \{x \in E \mid \text{rk}_M(A) = \text{rk}_M(A \cup \{x\})\}$. A flat of $M$ is a closed subset, i.e. a subset $F \subseteq E$ such that $F = \text{cl}_M(F)$. The flats of a matroid, under inclusion, form an intersection-closed lattice $\mathcal{P}_M$ with $\text{rank}(\mathcal{P}_M) = \text{rank}(M)$. There is a unique flat of rank 0, whose elements are called loops. The Möbius invariant $\mu(M)$ of a matroid $M$ is the number $\mu_{\mathcal{P}_M}(0, 1)$, where 0, 1 are (respectively) the bottom, top elements of $\mathcal{P}_M$. The Möbius numbers of $\mathcal{P}_M$ are well-studied quantities of interest in combinatorics; see [Rot64] or [Zas87] for a survey.

There are two standard ways to construct a new matroid from $M = (E, \text{rk}_M)$ given a subset $T \subseteq E$: the restriction of $M$ to $T$ is the matroid $M|T = (T, T \setminus T)$ where $\text{rk}_M(T) := \text{rk}_M(A)$ for $A \subseteq T$, and the contraction of $M$ by $T$ is the matroid $M/T := (E \setminus T, \text{rk}_M/T)$ where $\text{rk}_M/T(A) := \text{rk}_M(A \cup T) - \text{rk}_M(T)$ for $A \subseteq E \setminus T$. A matroid minor of $M$ is a matroid that arises as $M/B/A$ for some $A \subseteq B \subseteq E$. When $F \subseteq G$ are flats of $M$, then the lattice of flats of $M|G/F$ is isomorphic to the interval $[F, G]$ in the lattice of flats of $M$.

**Example 5.9.** The prototypical example of a matroid is a set of vectors $E = \{v_0, \ldots, v_{n-1}\}$ in a vector space $V$, with rank function given by $\text{rk}_M(A) := \dim \text{span}(A)$ for $A \subseteq E$. In this case, the independent sets are exactly the subsets of $E$ which are linearly independent in $V$, and the flats are exactly $W \cap E$ for some vector subspace $W \subseteq V$, i.e. correspond to subspaces of $V$. Matroids arising in this way are called representable.

When $V$ is a vector space over the finite field $\mathbb{F}_2$, the function class of the lattice of flats is the set of conjuctions of parity functions (i.e. conjuctions of linear functionals over $\mathbb{F}_2$). We discuss this case further in §6.3.

Let $\mathcal{P}_M$ be the lattice of flats of a matroid $M$; lattices arising in this way are also known as geometric lattices. It follows from Proposition 5.7(3) that $\mathcal{P}_M$ is shellable and hence Cohen-Macaulay. Thus for matroids, Theorem 5.5 specializes to:

**Corollary 5.10.** Let $M$ be a matroid. Then for $F \subseteq G \in \mathcal{P}_M$ flats of $M$,

$$
\beta_{i, \text{rk}_M(F, G)}(I^*_{c(\mathcal{P}_M)}) = \begin{cases} 
\mu(M|G/F) & \text{if } \text{rank}(M|G/F) = i \\
0 & \text{otherwise.} 
\end{cases}
$$

In other words, the multigraded Betti numbers of $I^*_{c(\mathcal{P}_M)}$ are the Möbius invariants of the loopless matroid minors of $M$.

It is easy to check that if $M = (E, \text{rk}_M)$ is a matroid, then for any subset $A \subseteq E$, the closure $\text{cl}_M(A)$ of $A$ is equal to the intersection of all flats containing $A$. In particular, when $\mathcal{P} = \mathcal{P}_M$ is the lattice of flats of a matroid, the closure defined in Definition 4.1 agrees with the closure operation in the matroid. Now, if $B \subseteq E$ is a basis of $M$ and $I \subseteq B$, then the closure $\hat{T}$ of $I$ in $\mathcal{P}_M$ is disjoint from $B \setminus I$, and so it follows from Lemma 4.2 that any basis of $M$ is shattered by $c(\mathcal{P}_M)$. Hence for any basis $B$, we have $\dim_{\text{VC}} c(\mathcal{P}_M) \geq |B| = \text{rank}(M)$, and combining this with Corollary 4.6 and Theorem 2.10 yields:
**Corollary 5.11.** Let $\mathcal{P}_M$ be the lattice of flats of a matroid $M$, and $C(\mathcal{P}_M)$ the associated function class. Then
\[ \text{dim}_{VC} C(\mathcal{P}_M) = \text{dim}_h C(\mathcal{P}_M) = \text{rank}(M). \]

**Remark 5.12.** A major family of function classes with the property $\text{dim}_{VC} C = \text{dim}_h C$ given in [Yan17] is the case where $I_C$ is Cohen-Macaulay. This is true, for example, for downward-closed classes, i.e. classes $C$ such that $g \in C$ if $g^{-1}(1) \subseteq f^{-1}(1)$ and $f \in C$ [Yan17, Section 3.2]. We remark that while $\text{dim}_{VC} C(\mathcal{P}_M) = \text{dim}_h C(\mathcal{P}_M)$ for a matroid $M$, the suboplex ideal $I_C(\mathcal{P}_M)$ is almost never Cohen-Macaulay. Recall the Eagon-Reiner criterion [ER98] that $I_C$ is Cohen-Macaulay if and only if $I_C^*$ has a linear resolution. As $\deg m(F, G) = n + |F \setminus G|$, it follows from Theorem 5.5 that $I_C^*$ has a linear resolution if and only if $|F \setminus G| = \text{rank}([F, G])$ for all $F \subseteq G \in \mathcal{P}$. Only the matroids that are Boolean after removing loops — that is, matroids $M = (E, r_kM)$ such that $\text{rank}(M) = |E| - \#(\text{loops})$ — satisfy this condition.

We provide a matroidal example illustrating the theorems above.

**Example 5.13.** Let $C$ be the function class on $[4] = \{0, 1, 2, 3\}$ consisting of the 10 functions
\[ \{0000, 1000, 0100, 0010, 0001, 1100, 1010, 1001, 0111, 1111\}. \]

Here each binary string represents a function’s values on $[4]$. Under the correspondence between subsets of $[4]$ and their indicator functions, we have $C = C(\mathcal{P})$, where $\mathcal{P} = \mathcal{P}_M$ is the lattice of the flats of the matroid $M = U_{1,\{0\}} \oplus U_{2,(1,2,3)} \cong U_{1,1} \oplus U_{2,3}$. The graph whose cyclic matroid is $M$, the matrix over $\mathbb{F}_2$ whose columns represent $M$, and the lattice of flats of $M$ are drawn below:

The $\mathbb{Z}$-graded Betti table of $I_C^*$ is, in standard Macaulay2 [GS] format,
\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\text{total:} & 10 & 17 & 10 & 2 \\
4: & 10 & 11 & 3 & . \\
5: & . & 6 & 7 & 2 \\
\end{array}
\]

and in accordance with Theorem 5.5:

- There are 17 intervals of length 1 (i.e. cover relations) in $\mathcal{P}$, corresponding to the first total Betti number $\beta_1 = 17$. Note that there are 11 covering relations whose two sets differ by size 1, and 6 relations that differ by size 2.
- There are 8 intervals of length 2 in $\mathcal{P}$, all of which are Boolean except two: the intervals $[\emptyset, 123]$ and $[0, 0123]$, which are isomorphic as lattices. The Möbius invariant of a Boolean lattice is 1, and the Möbius invariant of $[\emptyset, 123]$ is 2. Note that $\beta_2 = 10 = 6(1) + 2(2)$.
- The top Betti number is $\beta_3 = 2$, as can be verified by computing the Möbius invariant of $M$. Recall that the Möbius invariant of $M$ is also the (reduced) Euler characteristic of the truncated order complex (see Definition 3.2 and Proposition 5.2), so we can also verify $\beta_3 = 2$ by noting that this complex, drawn below, is connected and has two-dimensional first (reduced, singular) homology.

The homological dimension of $C$ is 3, which is also the rank of the matroid $M$. Moreover, the VC dimension of $C$ is 3 as well; indeed, the whole set $\{0, 1, 2, 3\}$ is not shattered by $C$, but $\{0, 1, 2\}$ is.
5.2. Polyhedral cell complexes.

Definition 5.14. A (polyhedral) cell complex $X$ is a finite collection of convex polytopes (all living in a real vector space $\mathbb{R}^n$), called faces of $X$, satisfying two properties:

- If $F$ is a polytope in $X$ and $G$ is a face of $F$, then $G$ is in $X$.
- If $F$ and $G$ are in $X$, then $F \cap G$ is a face of both $F$ and $G$.

The vertex set $\text{Vert}(X)$ of $X$ is the set of 0-dimensional faces of $X$, and the facets of $X$ are the faces which are maximal with respect to inclusion.

Definition 5.15. The face poset $P_X$ of a cell complex $X$ is the subposet of $2^{\text{Vert}(X)}$ where each element of $P_X$ consists of the set of vertices of some face $F$ of $X$.

If $X$ is a cell complex, then the second property of definition 5.14 ensures that $P_X$ is a meet-semilattice, with meet given by set intersection. Note that if the facets of $X$ are simplices, then $P_X$ is downward-closed, i.e. is a simplicial complex.

The face poset of a cell complex is interval Cohen-Macaulay, as follows from combining Proposition 5.7 with the following lemma:

Lemma 5.16. [Zie95, Thm 2.17(ii)] If $F \subseteq G$ are two faces of a polytope $X$, then the interval $[F, G] \subseteq P_X$ is the face poset of another polytope of dimension $\dim G - \dim F + 1$.

As the (reduced) Euler characteristic of the boundary of a polytope is $\pm 1$, Theorem 5.5 can be rephrased in this context as follows.

Corollary 5.17. For $F \leq G \in P_X$ faces of a cell complex $X$, we have

$$\beta_{i,m}(F, G)(I^*_C(P_X)) = \begin{cases} 1 & \text{if } i = \dim G - \dim F \\ 0 & \text{otherwise.} \end{cases}$$

Since the rank of $P_X$ is one more than the dimension of $X$, we get the first inequality of the following corollary.

Corollary 5.18. Let $P_X$ be the face poset of a polyhedral cell complex $X$ of dimension $\dim X$, and $C(P_X)$ the associated function class. Then

$$\dim_{VC} C(P_X) \leq \dim_{h} C(P_X) = \dim X + 1.$$ 

Equality holds iff $X$ has a simplex of full dimension ($= \dim X$). If any polytope in $X$ of maximal dimension has a simplex as a facet or a simple vertex (i.e. a vertex incident on exactly $\dim X$ edges), then

$$\dim_{h} C(P_X) - 1 \leq \dim_{VC} C(P_X) \leq \dim_{h} C(P_X).$$

This is always the case if $\dim X \leq 3$.

Proof. As mentioned before, Corollary 5.17 (or just Corollary 4.6) shows the first inequality.

If $X$ has a $(\dim X)$-dimensional simplex, then the vertices of this simplex is shattered by the functions corresponding to the faces. Conversely, $\dim_{VC} C(P_X) = \dim_{h} C(P_X)$ implies that there is a rank $(\dim X + 1)$ Boolean sublattice. This sublattice, being maximal, must correspond to the
face lattice of a maximal polyhedral cell. But any polytope with a Boolean face lattice has to be a simplex, so this yields the claim.

Similarly, if a \((\dim X)\)-dimensional polytope in \(X\) has a simplicial facet, then the \((\dim X)\) vertices of this facet are shattered, so that

\[
\dim X \leq \dim\VC(P_X).
\]

Likewise, when a \((\dim X)\)-dimensional polytope has a simple vertex, then its \((\dim X)\) neighbor vertices are shattered by the faces of this polytope, and the same inequality holds.

Finally, we consider the case when \(X\) has dimension at most 3. Every edge shatters the two points it contains, so \(\dim\VC(P_X) = \dim\VC(P_X) = 2\) when \(\dim X = 1\), and \(2 \leq \dim\VC(P_X) \leq \dim\VC(P_X) \leq 3\) when \(\dim X = 2\).

Now suppose \(\dim X = 3\) (we may assume \(X\) is a polytope). We show that \(X\) has to either have a triangular facet or a vertex with 3 neighbors. Suppose not, and let \(v, e, f\) respectively denote the number of vertices, edges, and 2-faces of \(X\). Since every vertex is incident on at least 4 edges, we have \(2e \geq 4v\). Moreover, since every face has at least 4 edges, we get \(2e \geq 4f\). But by Euler’s formula, this means

\[
2 = v - e + f \leq \frac{1}{2}e - e + \frac{1}{2}e = 0,
\]

a contradiction as desired.

\[\square\]

**Remark 5.19.** In dimension \(\geq 4\), it is no longer true that every polytope has either a simplex facet or a simple vertex. For instance, the 24-cell is a 4-dimensional polytope in which every facet is an octahedron and each vertex is incident on 6 edges.

**Remark 5.20.** The assumption of having either a simple vertex or a simplex facet holds generically, in the sense that the convex hull of a set of points in general position is a simplicial polytope (i.e. has all facets being simplices), and the intersection of a collection of half-spaces in general position is a simple polytope (i.e. all of whose vertices are simple) [Zie95].

### 6. Applications to computer science

In this section, we apply our new tools to various function classes in computer science. We first review some terminology: fix \(d \in \mathbb{N}\), and set \(n := 2^d\).

We consider function classes on \([n]\) consisting of Boolean formulas, as follows: identify \([n]\) with the set \([2]^d\) of binary strings \((s_0s_1\ldots s_{d-1})\) of length \(d\), and let \(x_0, \ldots, x_{d-1}\) be Boolean variables, representing Boolean functions \(x_i : [2]^d \to [2]\) that send \((s_0 \ldots s_{d-1}) \mapsto 1\) if \(s_i = 1\) and is 0 otherwise. The negation \(\neg x_i : [2]^d \to [2]\) sends \((s_0 \ldots s_{d-1}) \mapsto 1\) if \(s_i = 0\) and is 0 otherwise. A **literal** is a variable \(x_i\) or its negation \(\neg x_i\). The *conjunction* of a set of literals is their logical AND (denoted with \&). The *disjunction* of a set of literals is their logical OR (denoted with \lor).

A **Boolean formula** is any expression that can be built up by conjunctions and disjunctions of literals, and naturally represents a function \([2]^d \to [2]\) mapping a binary string to its evaluation under the formula.

**Example 6.1.** The conjunction of \(\{x_1, \neg x_3, x_7\}\) is a Boolean function \(x_1 \land \neg x_3 \land x_7 : [2]^d \to [2]\) such that \((s_0 \ldots s_{d-1}) \mapsto 1\) if \(s_1 = 1, s_3 = 0,\) and \(s_7 = 1\), and is 0 otherwise. Similarly, \(x_1 \lor \neg x_3 \lor x_7\) is the Boolean function that is 0 iff \(s_1 = 0, s_3 = 1,\) and \(s_7 = 0\). The expression \((x_1 \land \neg x_3) \lor (x_7 \land \neg x_1)\) is an example of a Boolean formula.

### 6.1. Application of results on polyhedral complexes

The class of conjunctions arises as the indicator functions of the faces of a cube, where an empty conjunction (the constant function 1) corresponds to the entire cube, the contradictory conjunction (e.g. \(x_1 \land \neg x_1\)) corresponds to the empty set, and a nonrepeating conjunction of length \(k\) (e.g. \(x_1 \land \cdots \land x_k\)) corresponds to a face
of codimension \( k \). Corollary 5.17 then recovers the Betti numbers of the class of conjunctions [Yan17, Section 2.3.4] by taking \( X \) to be the cell complex of a cube \([0, 1]^d\).

### 6.2. Applications of the rank bound

In the following, we use our rank bound Corollary 4.6 to show that homological dimension of the class of \( k \)-CNFs is equal to its VC dimension, up to constant multiplicative factors. We first recall the definition of \( k \)-CNF.

**Definition 6.2.** A \( k \)-CNF (Conjunctive Normal Form) is a boolean formula that is a conjunction (AND) of a number of clauses

\[
C_1 \land \cdots \land C_m \quad \text{ (for example, when } k = 2, \quad (x_1 \lor \neg x_3) \land (x_2 \lor x_7) \land \neg x_2)
\]

where each clause \( C_i \) is a disjunction (OR) of at most \( k \) literals. A monotone \( k \)-CNF is a \( k \)-CNF without any negations appearing. The class of (monotone) \( k \)-CNFs in \( d \) variables is the class of functions in \([2]^d \rightarrow [2]\) consisting of functions corresponding to all (monotone) \( k \)-CNFs.

**Theorem 6.3.** Let \( C \) be the class of \( k \)-CNFs and let \( C^+ \) be the class of monotone \( k \)-CNFs in \( d \) variables. Then, with \( e \) denoting Euler’s constant,

\[
\binom{d}{k} \leq \dim_{VC} C^+ \leq \sum_{i=0}^k \binom{d}{i} \leq (ed/k)^k
\]

so that

\[
\Omega(d^k) \leq \dim_{VC} C^+ \leq \dim_{h} C^+ \leq O(d^k)
\]

\[
\Omega(d^k) \leq \dim_{VC} C \leq \dim_{h} C \leq 2^k \binom{d}{k}
\]

so that

\[
\Omega(d^k) \leq \dim_{VC} C^+ \leq \dim_{h} C^+ \leq O(d^k)
\]

where \( \Omega \) and \( O \) hide constants dependent on \( k \) but independent of \( d \).

**Proof.** It was established in [KV94] that \( \dim_{VC} C, \dim_{VC} C^+ \geq \binom{d}{k} \) by noticing that the set of inputs

\[
\{ x \in [2]^d \mid \sum_i x_i = d - k \}
\]

is shattered by \( C^+ \) (and thus also by \( C \)). So it suffices to establish the upper bounds. We start with the class of \( k \)-CNFs, and then deal with the monotone case.

**k-CNF:** We start with the upper bound of \( \dim_{h} C \),

\[
\dim_{h} C \leq 2^k \binom{d}{k}.
\]

We prove this via the rank bound on homological dimension (Corollary 4.6) and by showing that the rank of \( C \), as a poset in the natural partial order \( f \leq g \iff f^{-1}(1) \subseteq g^{-1}(1) \), is bounded by the right hand side of (1).

Consider a chain of functions \( 0 < f_1 < \cdots < f_m < 1 \) in \( C \), where \( 0 \) (resp. \( 1 \)) denotes the constant function \( 0 \) (resp. \( 1 \)). Each \( f_i \) is a conjunction of disjunctive clauses,

\[
f_i = \bigwedge_j C_{ij}, \quad \text{each } C_{ij} \text{ is a disjunction of at most } k \text{ literals}.
\]

Because \( f_i < f_{i+1} < \cdots \leq f_m, \) we have

\[
f_i = \bigwedge_{i' = i}^m f_{i'} = \bigwedge_{i' = i}^m \bigwedge_j C_{ij}.
\]

Therefore we may assume that the clauses of the functions are in (strict) inclusion order

\[
\{C_{ij}\}_j \supset \cdots \supset \{C_{mj}\}_j \supset \emptyset.
\]
Furthermore, we can assume that the clauses all have exactly \( k \) literals, as any size-\( k' \) clause, \( k' \leq k \), can be written as a conjunction of such clauses. For example,

\[
x_1 \lor \cdots \lor x_k = x_1 \lor \cdots \lor x_k \lor (x_{k+1} \land \neg x_{k+1}) \lor \cdots \lor (x_d \land \neg x_d)
\]

by the distributivity of \( \land \) and \( \lor \). There are only \( 2^k \binom{d}{k} \) unique clauses with exactly \( k \) literals (choose the \( k \) variables first, and then decide whether to negate each of them). Therefore, the chain above can be at most \( 2^k \binom{d}{k} \) long.

By Corollary 4.6, this proves the desired upper bound on homological dimension.

**Monotone k-CNF:** The upper bound for \( \mathbb{C}^+ \) can be proved similarly, except here we cannot express a size-\( k' \) clause, \( k' \leq k \), as a conjunction of size-\( k' \) monotone disjunctions. The bound is then established by noting that there are \( \sum_{i=0}^{k'} \binom{d}{i} \) unique clauses of size \( \leq k \).

**Remark 6.4.** When \( k = 1 \), the class of (resp. monotone) \( k \)-CNFs is just the class of (resp. monotone) conjunctions. According to [Yan17, Section 3.1], the homological dimension of (resp. monotone) conjunctions in \( d \) Boolean variables is \( d + 1 \) (resp. \( d \)). At the same time, Theorem 6.3 only says that the homological dimension is between \( d \) and \( 2d \) (resp. \( d \) and \( d + 1 \)), so the upper bound of Theorem 6.3 is not tight in this case.

**Remark 6.5.** The logic of Theorem 6.3 can be applied straightforwardly to bound the homological dimension of CSP classes, which we discuss now.

In general, given a collection of Boolean functions \( f_i : [2]^d \to [2] \), their conjunction \( \bigwedge_i f_i \) is the function that sends \( v \in [2]^d \) to 1 iff \( f_i(v) = 1 \) for all \( i \). Likewise, their disjunction \( \bigvee_i f_i \) is the function that sends \( v \in [2]^d \) to 0 iff \( f_i(v) = 0 \) for all \( i \). These definitions generalize the notions of conjunction and disjunction introduced earlier for literals.

**Definition 6.6.** Let \( \mathbb{D} \) be a set of Boolean functions on \( [2]^d \). The class of \( \mathbb{D} \)-CSPs (Constraint Satisfaction Problems) is the conjunction closure of \( \mathbb{D} \), i.e. it contains all functions of the form \( \bigwedge_i f_i \) where each \( f_i \in \mathbb{D} \).

For example, if we let \( \mathbb{D} \) be the set of all (resp. monotone) disjunctions of size at most \( k \), then \( \mathbb{D} \)-CSPs are just (resp. monotone) \( k \)-CNFs.

**Theorem 6.7.** The class \( \mathbb{C} \) of \( \mathbb{D} \)-CSPs satisfies

\[
\dim_{\mathbb{V}C} \mathbb{C} \leq \dim_{\mathbb{H}} \mathbb{C} \leq |\mathbb{D}|.
\]

**Proof.** We consider the natural semilattice structure of \( \mathbb{D} \) induced from the conjunction closure of \( \mathbb{C} \). By the same reasoning as in the proof of theorem 6.3, any chain in this semilattice \( \mathbf{0} < f_1 < \cdots f_m < \mathbf{1} \) must correspond to a chain of reverse inclusions

\[
\mathbb{D} \supset \{g_{lj}\}_k \supset \cdots \supset \{g_{mj}\}_j \supset \emptyset
\]

of collections of functions \( g_{ij} \in \mathbb{D} \), in such a way that \( f_i = \bigwedge g_{ij} \). This chain can be at most \( |\mathbb{D}| \) long since each strict inclusion must differ by some new function in \( \mathbb{D} \). By Corollary 4.6, this yields the upper bound on homological dimension. \( \square \)

Note that this last bound is in general far from sharp: it follows from Corollary 6.8 below that the class of parity functions in \( d \) Boolean variables and its conjunction closure have the same VC-dimension \( d \), but the size of the class of parity functions is \( 2^d \).
6.3. Applications of results on matroids. We conclude with some applications of Corollary 5.11.

**Corollary 6.8.** For the class $C$ of conjunctions of parity functions in $d$ variables,
\[ \dim_{VC} C = \dim_h C = d. \]

**Proof.** Consider the representable rank-$d$ matroid given by all vectors in $\mathbb{F}_2^d$. The flats of this matroid are the subspaces of $\mathbb{F}_2^d$, and the rank function is the vector space dimension over $\mathbb{F}_2$.

Recall that a parity function $[2]^d \to [2]$ is an $\mathbb{F}_2$-linear functional under the isomorphism $[2] \cong \mathbb{F}_2$.

Every parity function is the indicator function of a hyperplane in $\mathbb{F}_2^d$, and every conjunction of parity functions is the indicator function of a subspace of $\mathbb{F}_2^d$, which is an intersection of hyperplanes. Thus the class of conjunctions of parity functions is exactly the function class associated to the matroid above. Corollary 5.11 then yields the result.

By considering a matroid over $\mathbb{F}_2$ in a higher dimensional space, we can also generalize this result to higher degree polynomials over $\mathbb{F}_2$.

**Corollary 6.9.** For the class $C \subseteq [2][2]^d$ of conjunctions of polynomials over $\mathbb{F}_2$ of degree $\leq k$,
\[ \dim_{VC} C = \dim_h C = \sum_{i=0}^{k} \binom{d}{i}. \]

**Proof.** Let $D := \sum_{i=0}^{k} \binom{d}{i}$. Consider the embedding
\[ \phi : \mathbb{F}_2^d \to \mathbb{F}_2^D, \quad x = (x_1, \ldots, x_d) \mapsto \left( \prod_{i \in U} x_i \right)_{U \subseteq [d], |U| \leq k} \]
sending $(x_i)$ to the vector of all monomials in $x_i$ with degree $\leq k$.

The linear matroid given by the image of $\phi$ has rank $D$. Any flat of this matroid is the zero set of a system of $\mathbb{F}_2$-polynomial equations $\{p_i(x)\}_i$ of degree at most $k$. The indicator function of such a set is the conjunction of the Boolean functions $\{p_i(x) - 1\}_i$:
\[ p_i(x) = 0 \quad \forall i \iff x \text{ satisfies } \bigwedge_i (p_i(x) - 1). \]
Thus the class $C$ under consideration is exactly the class of conjunctions of $\mathbb{F}_2$-polynomials with degree at most $k$, and the claim follows from Corollary 5.11.

**Remark 6.10.** Since the VC dimension and homological dimension of parity functions (resp. $F_2$-polynomials with degree at most $k$) are both $d$ (resp. $\sum_{i=0}^{k} \binom{d}{i}$) as well [Yan17, Section 3.1], the above results show that adding the operation of conjunction does not increase the “complexity” of these classes, from both a learning-theoretic and a homological point of view.

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