CONCERNING A CONJECTURE OF TAKETOMI-TAMARU

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Abstract. We study the setting of 2-step nilpotent Lie groups in the particular case that its type \((p, q)\) is not exceptional. We demonstrate that, generically, the orbits of \(\mathbb{R}^+ \times \text{Aut}_0\) in \(GL(n)/O(n)\) are congruent even when a Ricci soliton metric does exist. In doing so, we provide a counterexample to the local version of a conjecture of Taketomi-Tamaru.

Among solvable and nilpotent groups, perhaps the most natural distinguished Riemannian metrics are those left-invariant metrics which are either Einstein or Ricci soliton. These metrics are known to minimize natural functionals [Heb98, Lau01], have maximal symmetry when compared to other left-invariant metrics [Jab11, Jab19, GJ19, GJ24], and constitute the full class of non-compact, homogeneous Einstein and Ricci soliton metrics [BL22]. The pursuit of algebraic and geometric criteria which guarantee or preclude the existence of these metrics has been a long standing avenue of investigation, with the lion’s share of the attention given to the algebraic side, see e.g. [Nik11].

Recently there have been several works approaching this question from the geometric side. Given a Lie group \(G\) with Lie algebra \(\mathfrak{g}\), one can study the left-invariant metrics on \(G\) by studying inner products on \(\mathfrak{g}\); thus, one considers the set of inner products which is naturally presented as the symmetric space \(GL(n)/O(n)\), where \(n = \dim \mathfrak{g}\).

The subgroup \(\mathbb{R}^+ \times \text{Aut}(\mathfrak{g}) \subset GL(n)\) acts on the space of inner products \(GL(n)/O(n)\). Any two inner products in the same \(\text{Aut}(\mathfrak{g})\)-orbit are isometric and so inner products in the same \(\mathbb{R}^+ \times \text{Aut}(\mathfrak{g})\)-orbit are isometric up to scaling. Note, in the nilpotent setting the \(\text{Aut}(\mathfrak{g})\)-orbits are precisely the isometry classes, but this is not necessarily true for solvable, non-nilpotent groups [GW88].

Given an inner product \(\langle \cdot, \cdot \rangle \in GL(n)/O(n)\), the orbit \(\mathbb{R}^+ \times \text{Aut}(\mathfrak{g}) \cdot \langle \cdot, \cdot \rangle\) has recently been dubbed the corresponding submanifold. We collect some recent results on the geometry of this corresponding submanifold.

\begin{itemize}
  \item Let \(G\) be a 3-dimensional solvable Lie group with left-invariant metric \(\langle \cdot, \cdot \rangle \in GL(3)/O(3)\). Then \((G, \langle \cdot, \cdot \rangle)\) is a Ricci soliton if and only if the corresponding submanifold \(\mathbb{R}^+ \times \text{Aut}(\mathfrak{g}) \cdot \langle \cdot, \cdot \rangle\) in \(GL(3)/O(3)\) is a minimal submanifold [HT17].
  \item Let \(G\) be a 4-dimensional nilpotent Lie group with left-invariant metric \(\langle \cdot, \cdot \rangle \in GL(4)/O(4)\). Then \((G, \langle \cdot, \cdot \rangle)\) is a Ricci soliton if and only if the corresponding submanifold \(\mathbb{R}^+ \times \text{Aut}(\mathfrak{g}) \cdot \langle \cdot, \cdot \rangle\) in \(GL(4)/O(4)\) is a minimal submanifold. Furthermore, it is shown that this result is false for some solvable groups in dimension 4 [Has14].
  \item Let \(G\) be an \(n\)-dimensional solvable Lie group. In [Tak18], a sufficient condition for a metric to be Ricci soliton is given in terms of the so-called slice representation of \(\mathbb{R}^+ \times \text{Aut}(\mathfrak{g})\) acting on \(GL(n)/O(n)\).
\end{itemize}
Let $G$ be an $n$-dimensional solvable Lie group. It is conjectured that if $\mathbb{R}^* \times \text{Aut}(g)$ does not act transitively on $GL(n)/O(n)$ and all the orbits are congruent, then $G$ does not admit a Ricci soliton \cite[Conj. 1.2]{TT18}. The authors go on to verify the conjecture for some classes of nilpotent groups in every dimension - the algebras there have large automorphism groups.

Recall, two orbits being congruent means that there is an isometry $\phi \in GL(n)$ of the symmetric space $GL(n)/O(n)$ which sends one orbit to the other.

In the present work, we consider the connected components of the corresponding submanifolds, i.e. we will look at the orbits of the connected group $\mathbb{R}^0 \times \text{Aut}(g)_0$ in $GL(n)/O(n)$.

**Theorem A.** There exists a 9-dimensional nilpotent Lie group $G$ such that

(i) $G$ admits a Ricci soliton metric and

(ii) $\mathbb{R}^0 \times \text{Aut}(g)_0$ does not act transitively on $GL(9)/O(9)$ and the orbits are all congruent.

See Example 4.5 for details. This result provides a counterexample to the local version of the conjecture of Taketomi-Tamaru - i.e., on the connected components of the corresponding submanifolds. Whether or not the full conjecture holds remains an open and interesting question.

Even further, our result demonstrates that there is no criterion on the local geometry of the corresponding submanifolds which can completely determine the soliton condition, or likely any other distinguished condition. Whether or not the soliton condition can be characterized by the global action of $\mathbb{R}^* \times \text{Aut}(g)$ on $GL(n)/O(n)$ is an open and interesting question. For partial progress on this question, see \cite{Tak18}.

We note that we do not take up the remaining interesting question of whether or not it is possible that conditions such as the corresponding submanifold being minimal or not either guarantee or preclude the existence of a soliton metric. It would be interesting to know if the corresponding submanifolds appearing in Theorem A are indeed minimal submanifolds.

Finally, we point out that the phenomenon occurring in the theorem above is not special. In fact, this is the behavior seen in the generic setting, cf. Theorem 4.1.

1. **OUTLINE OF PROOF**

Our strategy is to work with Lie algebras whose algebra of derivations is very small. Recall, the Lie algebra of the automorphism group is the algebra of derivations.

Consider a 2-step nilpotent Lie algebra $n = v + z$, where $z = [n, n]$ is the commutator subalgebra and $v$ is a complement of $z$. As $n$ is 2-step nilpotent, $z = [n, n]$ is central, but might not be all of the center of $n$. The algebra $n$ comes equipped naturally with two kinds of derivations. First we have the (1,2)-derivation

\begin{equation}
D = \begin{bmatrix}
\text{Id}_v & 0 \\
0 & 2 \text{Id}_z
\end{bmatrix};
\end{equation}

then we have derivations which vanish on $z$ and map $v$ to $z$, i.e. ones of the form

\begin{equation}
\begin{bmatrix}
0 & 0 \\
\ast & 0
\end{bmatrix}.
\end{equation}

These are precisely the derivations valued in $z$ and they form an ideal in Der($n$). We denote this set of derivations by Der$_{v \to z}$. Combining the above, we have a subalgebra $\mathbb{R}(D) \ltimes \text{Der}_{v \to z}$.
of Der(n). In fact, one can argue that this is an ideal of Der(n), though we won’t need this fact. In general, the set of derivations is

\[(1.3) \quad \text{Der}(n) = \text{Der}(n) \cap (\mathfrak{gl}(v) \oplus \mathfrak{gl}(z)) \oplus \text{Der}_{v \rightarrow z};\]

and at the automorphism level one has

\[\text{Aut}(n) = (\text{Aut}(n) \cap GL(v) \times GL(z)) \exp(\text{Der}_{v \rightarrow z}),\]

where the subgroup exp(Der_{v \rightarrow z}) is a normal subgroup.

We are interested in the case when Der(n) is as small as possible; i.e., Der(n) = R(D) ⋉ Der_{v \rightarrow z}. For the rest of this section we make this assumption going forward. As we will see, this is the generic situation, cf. Theorem 2.2.

By Lie’s Theorem, we know there is some upper triangular matrix algebra \( s \supset \text{Der}(n) \).

When Der(n) is minimal, this is easily seen using a basis of \( v \) concatenated with a basis of \( z \). Observe, when Der(n) is minimal, it is then an ideal of \( s \). If \( S \) denotes the subgroup of \( GL(n, \mathbb{R}) \) with Lie algebra \( s \) then we have

- \( \mathbb{R}^{>0} \times \text{Aut}(n)_0 \) is a normal, proper subgroup of \( S \) and
- \( S \) acts transitively on \( GL(n, \mathbb{R})/O(n) \).

Using normality of \( \mathbb{R}^{>0} \times \text{Aut}(n)_0 \) and transitivity of \( S \) above, we immediately see that all the \( \mathbb{R}^{>0} \times \text{Aut}(n)_0 \)-orbits in \( GL(n, \mathbb{R})/O(n) \) are congruent; furthermore, as \( \mathbb{R}^{>0} \times \text{Aut}(n)_0 \) is a proper subgroup of \( S \), it does not act transitively on \( GL(n, \mathbb{R})/O(n) \).

Below we detail for which types of 2-step nilpotent Lie groups the above arguments come together. In a generic sense, many algebras simultaneously

- admit a soliton metric and
- have smallest possible derivation algebra.

These two facts combined give the existence of a counterexample. However, putting your finger on a counterexample is significantly more challenging than one might example - see Example 4.5.

2. The \( j \)-map and algebras of type \((p, q)\)

Consider a 2-step nilpotent algebra \( n = v + z \) where \( z = [n, n] \). We say \( n \) is of type \((p, q)\) if \( \dim z = p \) and \( \dim v = q \). Given an algebra of type \((p, q)\), with an inner product \( \langle \cdot, \cdot \rangle \), we may consider the so-called \( j \)-map studied by Eberlein and others [Ebe94]:

\[ j : z \to \mathfrak{so}(v), \]

defined by

\[ \langle j(z)v, w \rangle = \langle [v, w], z \rangle. \]

Taking an orthonormal basis \( \{z_1, \ldots, z_p\} \) of \( z \), one may associate to \( n \) a \( p \)-tuple of skew-symmetric matrices

\[ C = (C_1, \ldots, C_p) \in \mathfrak{so}(q)^p = \mathfrak{so}(q) \otimes \mathbb{R}^p \]

via \( C_i = j(z_i) \in \mathfrak{so}(v) \). The set of 2-step nilpotent algebras of type \((p, q)\) forms a Zariski open set \( V_{pq}^0 \) in \( \mathfrak{so}(q) \otimes \mathbb{R}^p \), being those \( p \)-tuples whose entries are linearly independent. We note that the constraint of linear independence forces us to have \( 1 \leq p \leq \frac{1}{2}q(q - 1) = \dim \mathfrak{so}(q) \).
Interestingly, both the automorphism group and the isomorphism classes of 2-step nilpotent algebras of type \((p, q)\) can be read off from a natural \(GL(q) \times GL(p)\) action on \(\mathfrak{so}(q) \otimes \mathbb{R}^p\). This action is given as follows. For \((g, h) \in GL(q) \times GL(p)\) and \(M \otimes v \in \mathfrak{so}(q) \otimes \mathbb{R}^p\),

\[
(g, h) \cdot M \otimes v = gMq^t \otimes hv,
\]

where the \(GL(p)\) action on \(\mathbb{R}^p\) is the standard one; of course, one extends linearly. We note that there is an induced Lie algebra action of \(\mathfrak{gl}(q) \oplus \mathfrak{gl}(p)\) given by

\[
(X, Y) \cdot M \otimes v = (XM + MX^t) \otimes v + M \otimes Yv,
\]

for \(X \in \mathfrak{gl}(q)\) and \(Y \in \mathfrak{gl}(p)\). This action and its relationship to nilpotent geometry have been explored in depth by Eberlein [Ebe03]. We record some useful facts here, cf. Equation 1.3.

- Two algebras of type \((p, q)\) are isomorphic if and only if their corresponding \(p\)-tuples lie in the same \(GL(q) \times GL(p)\) orbit.
- If \(n\) corresponds to \(C \in \mathfrak{so}(q) \otimes \mathbb{R}^p\), then \(\text{Der}(n) \cap (\mathfrak{gl}(q) \oplus \mathfrak{gl}(p)) \simeq (\mathfrak{gl}(q) \oplus \mathfrak{gl}(p))_C\), where the right-hand side is the stabilizer of the Lie algebra action at \(C\).

The isomorphism \(\Psi : \text{Der}(n) \cap (\mathfrak{gl}(q) \oplus \mathfrak{gl}(p)) \rightarrow (\mathfrak{gl}(q) \oplus \mathfrak{gl}(p))_C\) above is given by

\[
\Psi(X, Y) = (-X^t, Y).
\]

At the group level, we have an isomorphism \(\text{Aut}(n) \cap (GL(q) \times GL(p)) \rightarrow (GL(q) \times GL(p))_C\) given by \(\Psi(g, h) = \left((g^t)^{-1}, h\right)\).

**Definition 2.1.** We say that \((p, q)\) is exceptional if either \((p, q)\) or \((\frac{1}{2}q(q - 1) - p, q)\) appears in the list below. Note, \(\frac{1}{2}q(q - 1) = \dim \mathfrak{so}(q)\).

\[
\begin{align*}
(1, q) & \text{ for } q \geq 2 \\
(\frac{1}{2}q(q - 1), q) & \text{ for } q \geq 2 \\
(2, k) & \text{ for } k \geq 3 \\
(3, k) & \text{ for } 4 \leq k \leq 6
\end{align*}
\]

As we will see below, the exceptional types \((p, q)\) are when the generic derivation algebras are larger than the minimal possible one appearing in the next theorem. Interestingly, most of the go-to examples that people work with fall into the exceptional cases above and so do not reflect the nature of generic 2-step nilpotent geometry.

**Theorem 2.2.** For non-exceptional types, a generic algebra has as its derivation algebra the minimal possible derivation algebra, i.e.

\[
\text{Der} = \mathbb{R}(D) \oplus \text{Der}_{u \rightarrow z},
\]

where \(D\) is the \((1, 2)\)-derivation given in Eqn. 1.1. More precisely, there exists a Zariski open set in \(\mathfrak{so}(q)^p\) with the property above.

**Lemma 2.3.** Take \(C \in \mathfrak{so}(q)^p\). If \(SL(q) \times SL(p) \cdot C\) is closed, then the stabilizer of the \(\mathfrak{gl}(q) \oplus \mathfrak{gl}(p)\) action at \(C\) is \(\mathbb{R}(D) \oplus (\mathfrak{sl}(q) \oplus \mathfrak{sl}(p))_C\).
Remark 2.4. Conditions are needed on the orbit to ensure the result in the lemma. For example, for type $(2, 2k + 1)$ it is known that there is one generic orbit of the $GL(2k + 1) \times GL(2)$ action and this orbit is open. As such, one can see that the stabilizer of $\mathfrak{gl}(2k + 1) \oplus \mathfrak{gl}(2)$ is bigger than the stabilizer of $\mathfrak{sl}(2k + 1) \oplus \mathfrak{sl}(2)$ extended by the $(1,2)$ derivation given in Eqn. 1.1.

Proof of the lemma. We write $\mathfrak{gl}(q) \oplus \mathfrak{gl}(p)$ as $\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{sl}(q) \oplus \mathfrak{sl}(p)$. Each $\mathbb{R}$ factor acts by scaling on $C$ (cf. Eqn. 2.1). Given $X \in (\mathfrak{gl}(q) \oplus \mathfrak{gl}(p))_C$, we see that it may be written as $X = X_1 + X_2$ with $X_1$ acting by scaling and $X_2 \in \mathfrak{sl}(q) \oplus \mathfrak{sl}(p)$. Thus

$$X_2 \cdot C = rC,$$

for some $r \in \mathbb{R}$. This implies

$$\exp(tX_2) \cdot C = e^{rt}C,$$

for $t \in \mathbb{R}$. However, since $SL(q) \times SL(p) \cdot C$ is closed it cannot contain the origin in its boundary, thence we see that $r = 0$ and $X_2$ stabilizes $C$. In turn, $X_1$ must stabilize $C$ and so is a multiple of the $(1,2)$-derivation $D$.

The proof of the theorem above now follows immediately from the lemma combined with some general theory on certain representations of $SL(q) \times SL(p)$ which we present in the following section.

3. The action of $SL_p \times SL_q$

For a detailed discussion on the following facts and their relationship to nilpotent geometry, we refer the interested reader to [Jab08, Chapter 7].

From the work of Knop-Littelmann [KL87], we know that in the case of non-exceptional types $(p, q)$, the generic stabilizers of the the $SL(q) \times SL(p)$ representation above are finite (i.e. they have trivial connected component). From [PV94], we then have that generic orbits of $SL(q) \times SL(p)$ are closed. We summarize this information below.

**Theorem 3.1.** For non-exceptional types $(p, q)$, there is a Zariski open set in $\mathfrak{so}(q)^p$ whose elements $C$ satisfy

(i) $SL(q) \times SL(p) \cdot C$ is closed, and

(ii) $(\mathfrak{sl}(q) \times \mathfrak{sl}(p))_C = \{0\}$

Combining part (ii) of this theorem with the lemma above, the proof of Theorem 2.2 is complete.

Determining when a particular orbit is closed is non-trivial as closedness is a global property and these orbits are non-compact. As we will need to do this to obtain an explicit example for Theorem A, we lay out some criteria from the general theory below.

Consider the maps $m_1 : \mathfrak{so}(q)^p \to \text{symm}(q)$ and $m_2 : \mathfrak{so}(q)^p \to \text{symm}(p)$ defined by

$$m_1(C) = \sum_{i=1}^{q} (C_i)^2,$$

$$(m_2(C))_{ij} = \text{trace}(C_i C_j), \text{ for } 1 \leq i, j \leq p.$$
Proposition 3.3. Take the action $\text{SL}(q) \times \text{SL}(p)$ on $\mathfrak{so}(q)^p$ is given by

$$m(C) = \left( m_1(C) - \frac{\text{trace } m_1(C)}{q} \text{Id}, m_2(C) - \frac{\text{trace } m_2(C)}{p} \text{Id} \right).$$

The elements in $m^{-1}(0)$ are called the minimal vectors of the representation and we see that

$$m^{-1}(0) = \{ C \in \mathfrak{so}(q)^p \mid m_1(C) = \alpha \text{Id} \text{ and } m_2(C) = \beta \text{Id}, \text{ for some } \alpha, \beta \in \mathbb{R} \}.$$  

From the general theory [RS90], we have the following result.

**Proposition 3.2.** An orbit $\text{SL}(q) \times \text{SL}(p) \cdot C$ is closed if and only if $\text{SL}(q) \times \text{SL}(p) \cdot C \cap m^{-1}(0) \neq \emptyset$.

Even further, in the setting of 2-step, nilpotent geometry, these minimal vectors are precisely the metric structure constants for soliton metrics satisfying the extra geometric condition that their Ricci tensor is invariant under the geodesic flow [Ebe08].

For clarity in the following proposition, we use the following notation.

\[ G = \text{GL}(q) \times \text{GL}(p) \]
\[ \mathfrak{g} = \text{gl}(q) \times \text{gl}(p) \]
\[ K = \{ g \in G \mid g^t = g^{-1} \} = \text{O}(q) \times \text{O}(p) \]
\[ \mathfrak{p} = \{ X \in \mathfrak{g} \mid X^t = X \} = \text{symm}(q) \times \text{symm}(p) \]

**Proposition 3.3.** Take $C \in \mathfrak{so}(q)^p$. If $m_1(C) = \alpha \text{Id}$ and $m_2(C) = \beta \text{Id}$ for some $\alpha, \beta \in \mathbb{R}$, then

(i) $C$ is a minimal point of the $\text{SL}(q) \times \text{SL}(p)$ action.
(ii) $\text{SL}(q) \times \text{SL}(p) \cdot C$ is closed;
(iii) The stabilizer subgroup of $\text{GL}(q) \times \text{GL}(p)$ at $C$ is the product $G_C = K_C \exp(\mathfrak{g}_C \cap \mathfrak{p})$.
(iv) When $\mathfrak{g}_C$ has minimal dimension, i.e. $\mathfrak{g}_C = \mathbb{R}^\text{span}(\Psi(D))$, one has $K_C$ finite and $G_C = K_C \exp(\mathbb{R}^\text{span}(\Psi(D)))$.

Here $D$ is the $(1,2)$-derivation defined in Equation 1.1. Recall, the stabilizer subalgebra and stabilizer subgroup are, respectively, isomorphic to the derivations and automorphisms that preserve the subspaces $\mathfrak{v} = \mathbb{R}^q$ and $\mathfrak{z} = \mathbb{R}^p$; see Eqn. 2.2 for the definition of the isomorphism $\Psi$.

**Proof.** The first two items were observed by Eberlein [Ebe08]; we include a short proof for completeness.

When $m_1(C)$ and $m_2(C)$ are multiples of the identity, their traceless parts are zero and we immediately have that $m(C) = 0$, cf. Equation 3.2, that is, $C$ is a minimal point of the action $\text{SL}(q) \times \text{SL}(p)$. The orbit $\text{SL}(q) \times \text{SL}(p) \cdot C$ is closed by the previous proposition.

Understanding the stabilizer is more subtle. If we were considering the stabilizer of the group $\text{SL}(q) \times \text{SL}(p)$, then we would have a decomposition as above from the standard theory using minimal vectors. In the case of the $\text{GL}(q) \times \text{GL}(p)$ action, the moment map is $m_{\text{GL}}(C) = (m_1(C), m_2(C))$ and $C$ being a minimal point for the $\text{SL}(q) \times \text{SL}(p)$ action implies

$$m_{\text{GL}}(C) \cdot C = \lambda C,$$
for some \( \lambda \in \mathbb{R} \). That is, \( C \) is a critical point of the function \( F = ||m_{GL}||^2 \). From the general theory [Kir84], one knows that the stabilizer subgroup decomposes as a semi-direct product

\[
G_C = (G^{\Psi(D)})_C \exp \left( (u^{\Psi(D)})_C \right),
\]

where \( G^{\Psi(D)} \) is the collection of elements of \( G \) which commute with \( \Psi(D) \), while \( (u^{\Psi(D)}) \) is the nilpotent algebra \( \{ X \in \mathfrak{g} \mid \lim_{t \to \infty} \exp(t\Psi(D))X\exp(t\Psi(D))^{-1} = 0 \} \). See [Kir84, Section 6] for more details on the critical points of \( F \), the stratification of the representation space, and the associated parabolic groups at the critical points of \( F \).

As \( \Psi(D) \) lies in the center of \( G \), we see that \( u^{\Psi(D)} \) is trivial. As \( \Psi(D) \) is symmetric, we have \( G_C = (G^{\Psi(D)})_C = K_C \exp(\mathfrak{g}_C \cap \mathfrak{p}) \), as desired.

The final claim follows immediately from the third as \( K_C \) being compact with dimension 0 implies finite.

\[ \square \]

4. Application to nilsolitons

It was recognized in [Ebe08] that if the orbit \( SL(q) \times SL(p) \cdot C \) is closed, then the associated 2-step nilpotent Lie group of type \((p, q)\) admits a soliton metric. Applying the above facts to nilpotent geometry, we now have the following.

**Theorem 4.1.** Consider a non-exceptional type \((p, q)\). There exists a Zariski open set of \( \mathfrak{so}(q)^p \) such that the corresponding 2-step nilpotent Lie groups \( N \) satisfy

(i) \( N \) admits a nilsoliton metric and

(ii) the derivation algebra of \( \mathfrak{n} = \text{Lie } N \) is the minimal one described in Theorem 2.2.

Using any of these generic algebras, we have all the necessary conditions to carry out the recipe in Section 1 for building a counterexample to the local version of the conjecture by Taketomi-Tamaru. Part of the challenge with building examples is that, in practice, one tends adds extra symmetries to make solving a geometry problem easier. Here we are removing symmetries to have a generic object. So, we have the classic problem of finding the hay in the haystack.

The smallest dimension where Theorem 4.1 applies is dimension 9, and this is for algebras of type \((4, 5)\). Below, we give an explicit example of a 9-dimensional, 2-step nilpotent Lie algebra of type \((4, 5)\) which satisfies Theorem A. Although a generic algebra of type \((4, 5)\) would suffice, in practice it is very hard to know if one is holding a generic algebra. As we explain in Remark 4.4 we still do not know if our example is truly generic.

We build our example by employing the tools from Section 3. Consider tuple \( C = (C_1, \ldots, C_4) \in \mathfrak{so}(5)^4 \) given by
Lemma 4.2. The point $C \in \mathfrak{so}(5)^4$, above, is a minimal vector for the $SL(5) \times SL(4)$-orbit and so $SL(5) \times SL(4) \cdot C$ is closed. Further, the stabilizer subgroup has minimal dimension and satisfies $(GL(5) \times GL(4))_C \supset (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \exp(\mathbb{R}\text{-span}(\Psi(D)))$.

Remark 4.3. Calculating the stabilizer subalgebra is a linear algebra problem. However, finding the stabilizer subgroup is challenging. Even with the help of a computer algebra system, we are not able to compute the full stabilizer subgroup.
Proof. Recall the definitions of \( m_1(C) \) and \( m_2(C) \) given in Equation 3.1. Computing, we have

\[
m_1(C) = \begin{bmatrix}
-2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & -2
\end{bmatrix}
\]

\[
m_2(C) = \begin{bmatrix}
-\frac{5}{2} & 0 & 0 & 0 \\
0 & -\frac{5}{2} & 0 & 0 \\
0 & 0 & -\frac{5}{2} & 0 \\
0 & 0 & 0 & -\frac{5}{2}
\end{bmatrix}
\]

By Proposition 3.3, we have that \( C \) is a minimal vector and the orbit \( SL(5) \times SL(4) \cdot C \) is closed.

Computing the stabilizer subalgebra \( g_C \) is a straightforward linear algebra problem that can be done by hand; however, we use Maple to double check our calculations and verify that

\[
g_C = \mathbb{R} \cdot \text{span}(\Psi(D)).
\]

This shows that the stabilizer subalgebra and stabilizer subgroup have minimal dimension. Lastly, we show that the stabilizer \( K_C = G_C \cap (O(5) \times O(4)) \) contains a subgroup of the form \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Observe, for \( \epsilon = \pm 1 \), we have \[
\epsilon \begin{bmatrix}
\epsilon \\
-\epsilon \\
\epsilon \\
-\epsilon \\
\epsilon
\end{bmatrix}
\begin{bmatrix}
\epsilon \\
-\epsilon \\
\epsilon \\
-\epsilon \\
\epsilon
\end{bmatrix} \in GL(5) \times GL(4)
\]
lies in the kernel of the action and so lies in the stabilizer of every point. Further, one can quickly check that

\[
\begin{bmatrix}
\epsilon & -\epsilon & \epsilon & -\epsilon \\
\epsilon & -\epsilon & \epsilon & -\epsilon \\
\epsilon & -\epsilon & \epsilon & -\epsilon \\
\epsilon & -\epsilon & \epsilon & -\epsilon
\end{bmatrix}
\begin{bmatrix}
-1 & -1 & 1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & 1 & -1
\end{bmatrix}
\]
lies in stabilizer. Clearly, these matrices all commute and we have a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

\[\square\]

**Remark 4.4.** If one assumes that $(g, h) \in K_C$ has diagonal $h$, then $g$ must be diagonal and $(g, h)$ in subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ constructed above. However, we do not know if $h$ must always be diagonal. It would be interesting to know the full stabilizer subgroup of $C$. Even further, we do not know the stabilizer in general position. Certainly, it contains $\mathbb{Z}_2$ from kernel of the action. It would be interesting to know if the stabilizer in general position must be larger. If the generic stabilizer were only $\mathbb{Z}_2$, then our example would not be a generic point. Other examples we have constructed also contain $\mathbb{Z}_2 \times \mathbb{Z}_2$ in their stabilizers.

**Example 4.5.** The 9-dimensional, 2-step nilpotent Lie group associated to $C$ above satisfies the conditions of Theorem A.

**Proof.** From the lemma, we know that $SL(5) \times SL(4) \cdot C$ is closed and so the 2-step nilpotent Lie algebra admits a soliton metric \[Ebe08\].

The derivation algebra is minimal as the stabilizer subalgebra of the $\mathfrak{sl}(5) \times \mathfrak{sl}(4)$-action has minimal dimension. We now satisfy all the conditions of the recipe given in Section [1] and we have our example.

Lastly, we point out that since $q = 5$ is odd, our automorphism group must be disconnected as it contains the element \[
\begin{bmatrix}
-\text{Id}_5 \\
\text{Id}_4
\end{bmatrix}
\] \[\in GL(5) \times GL(4)\], which has negative determinant.

If Aut had only two components, then it would be a normal subgroup of $S$, cf. discussion in Section [2]. A nilpotent group with such an automorphism group would then give a full counterexample to the conjecture of Taketomi-Tamaru. It would be interesting to know if such exists or if the automorphism group must always have more than two components - in which case it would not be a normal subgroup of $S$.

**References**

[BL22] Christoph Böhm and Ramiro A. Lafuente, *Homogeneous Einstein metrics on Euclidean spaces are Einstein solvmanifolds*, Geom. Topol. 26 (2022), no. 2, 899–936. MR 4444271

[Ebe94] Patrick Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. École Norm. Sup.(4) 27 (1994), no. 5, 611–660.

[Ebe03] , *The moduli space of 2-step nilpotent Lie algebras of type $(p, q)$*, Explorations in complex and Riemannian geometry, Contemporary Mathematics, American Mathematical Society 332 (2003), 37–72.

[Ebe08] , *Riemannian 2-step nilmanifolds with prescribed Ricci tensor*, Geometric and probabilistic structures in dynamics, Contemp. Math., vol. 469, Amer. Math. Soc., Providence, RI, 2008, pp. 167–195.

[GJ19] Carolyn Gordon and Michael Jablonski, *Einstein solvmanifolds have maximal symmetry*, Journal of Differential Geometry 111 (2019), no. 1, 1–38.

[GJ24] , *Ricci soliton solvmanifolds have infinitesimal maximal symmetry*, to appear in Transactions of the American Mathematical Society (2024).

[GW88] Carolyn S. Gordon and Edward N. Wilson, *Isometry groups of Riemannian solvmanifolds*, Trans. Amer. Math. Soc. 307 (1988), no. 1, 245–269.

[Has14] Takahiro Hashinaga, *On the minimality of the corresponding submanifolds to four-dimensional solv-solitons*, Hiroshima Math. J. 44 (2014), no. 2, 173–191. MR 3251821
[Heb98] Jens Heber, *Noncompact homogeneous Einstein spaces*, Invent. Math. **133** (1998), no. 2, 279–352.

[HT17] Takahiro Hashinaga and Hiroshi Tamaru, *Three-dimensional solvsolitons and the minimality of the corresponding submanifolds*, Internat. J. Math. **28** (2017), no. 6, 1750048, 31. MR 3663797

[Jab08] Michael Jablonski, *Real geometric invariant theory and Ricci soliton metrics on two-step nilmanifolds*, Thesis (May 2008).

[Jab11] ______, *Concerning the existence of Einstein and Ricci soliton metrics on solvable lie groups*, Geometry & Topology **15** (2011), no. 2, 735–764.

[Jab19] ______, *Maximal symmetry and unimodular solvmanifolds*, Pacific J. Math. **298** (2019), no. 2, 417–427. MR 3936023

[Kir84] Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes 31, Princeton University Press, Princeton, New Jersey, 1984.

[KL87] Friedrich Knop and Peter Littelman, *Der grad erzeugender funktionen von invariantenringen. (German)* [The degree of generating functions of rings of invariants], Math. Z. **196** (1987), no. 2, 211–229.

[Lau01] Jorge Lauret, *Ricci soliton homogeneous nilmanifolds*, Math. Ann. **319** (2001), no. 4, 715–733.

[Nik11] Y. Nikolayevsky, *Einstein solvmanifolds and the pre-Einstein derivation*, Trans. Amer. Math. Soc. **363** (2011), 3935–3958.

[PV94] V.L. Popov and E.B. Vinberg, *Algebraic geometry IV: II. invariant theory*, Springer-Verlag, Berlin Heidelberg, 1994.

[RS90] R.W. Richardson and P.J. Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math. Soc. **42** (1990), 409–429.

[Tak18] Yuichiro Taketomi, *On a Riemannian submanifold whose slice representation has no nonzero fixed points*, Hiroshima Math. J. **48** (2018), no. 1, 1–20. MR 3771997

[TT18] Y. Taketomi and H. Tamaru, *On the nonexistence of left-invariant Ricci solitons—a conjecture and examples*, Transform. Groups **23** (2018), no. 1, 257–270. MR 3763948