On the convergence time of asynchronous distributed quantized averaging algorithms

Minghui Zhu and Sonia Martínez

Abstract

We come up with a class of distributed quantized averaging algorithms on asynchronous communication networks with fixed, switching and random topologies. The implementation of these algorithms is subject to the realistic constraint that the communication rate, the memory capacities of agents and the computation precision are finite. The focus of this paper is on the study of the convergence time of the proposed quantized averaging algorithms. By appealing to random walks on graphs, we derive polynomial bounds on the expected convergence time of the algorithms presented.

I. INTRODUCTION

Consider a network of (mobile or immobile) agents. The distributed consensus problem aims to design an algorithm that agents can utilize to asymptotically reach an agreement by communicating with nearest neighbors. This problem historically roots in parallel computation [2], and has attracted significant attention recently [3] [10] [14]. As a special case of the consensus problem, the distributed averaging problem requires that the consensus value be the average of individual initial states. The distributed averaging algorithm acts as the building block for many distributed tasks such as sensor fusion [16] and distributed estimation [12].

In real-world communication networks, the capacities of communication channels and the memory capacities of agents are finite. Furthermore, the computations can only be carried out with finite precision. From a practical point of view, real-valued averaging algorithms are not feasible and these realistic constraints motivate the problem of average consensus via quantized information. Another motivation for distributed quantized averaging is load balancing with
indivisible tasks. Prior work on distributed quantized averaging over fixed graphs includes [1], [6], [7], [11]. Recently, [13] examines quantization effects on distributed averaging algorithms over time-varying topologies. As in [11], we focus on quantized averaging algorithms preserving the sum of the state values at each iteration. This setup has the following properties of interest: the sum cannot be changed in some situations, such as load balancing; and the constant sum leads to a small steady-state error with respect to the average of individual initial states. This error is equal to either one quantization step size or zero (when the average of the initial states is located at one of the quantization levels) and thus is independent of \( N \). This is in contrast to the setup in [13] where the sum of the states is not maintained, resulting in a steady-state error of the order \( O(N^3 \log N) \).

The convergence time is typically utilized to quantify the performance of distributed averaging algorithms. The authors in [4], [15] study the convergence time of real-valued averaging, while [11], [13] discuss the case of quantized averaging. The polynomial bounds of the expected convergence time on fixed complete and linear graphs are derived in [11]. Recently, the authors in [13] give a polynomial bound on the convergence time of a class of quantized averaging algorithms over switching topologies. Among the papers aforementioned, [4], [13], [15] require global synchronization, and [11] needs some global information (e.g., a centralized entity or the total number of the edges) to explicitly bound the expected convergence times. However, real-world communication networks are inherently asynchronous environment and lack of centralized coordination.

**Statement of contributions.** The present paper proposes a class of distributed quantized averaging algorithms on asynchronous communication networks with fixed, switching and random topologies. The algorithms are shown to asymptotically reach quantized average consensus in probability. Furthermore, we utilize meeting times of two random walks on graphs as a unified approach to derive polynomial bounds on the expected convergence times of our presented algorithms. To the best of our knowledge, this note is the first step toward characterizing the expected convergence times of completely distributed quantized averaging algorithms over asynchronous communication networks. A preliminary conference version of this paper is in [17] where the convergence time of synchronous algorithms is also studied.
II. PRELIMINARIES AND PROBLEM STATEMENT

Here, we present the problem formulation along with some notation and terminology.

Asynchronous time model. In this note, we will employ the asynchronous time model proposed in [4]. More precisely, consider a network of $N$ nodes, labeled 1 through $N$. Each node has a clock which ticks according to a rate 1 Poisson process. Hence, the inter-tick times at each node are random variables with rate 1 exponential distribution, independent across nodes and independent over time. By the superposition theorem for Poisson processes, this setup is equivalent to a single global clock modeled as a rate $N$ Poisson process ticking at times $\{Z_k\}_{k \geq 0}$. By the orderliness property of Poisson processes, the clock ticks do not occur simultaneously. The inter-agent communication and the update of consensus states only occur at $\{Z_k\}_{k \geq 0}$. In the reminder of this paper, the time instant $t$ will be discretized according to $\{Z_k\}_{k \geq 0}$ and defined in terms of the number of clock ticks.

Network model. We will employ the undirected graph $G(t) = (V, E(t))$ to model the network. Here $V := \{1, \cdots, N\}$ is the vertex set, and an edge $(j, i) \in E(t)$ if and only if node $j$ can receive the message from node $i$ (e.g., node $j$ is within the communication range of node $i$) at time $t$. The neighbors of node $i$ at time $t$ are denoted by $\mathcal{N}_i(t) = \{j \in V \mid (j, i) \in E(t) \text{ and } j \neq i\}$. The state of node $i$ at time $t$ is denoted by $x_i(t) \in \mathbb{R}$ and the network state is denoted by $x(t) = (x_1(t), \cdots, x_N(t))^T$. Suppose the initial states $x_i(0) \in [U_{\min}, U_{\max}]$ for all $i \in V$ and some real numbers $U_{\min}$ and $U_{\max}$.

Quantization scheme. Let $R$ denote the number of bits per sample. The total number of quantization levels can be represented by $L = 2^R$ and the step size is $\Delta = (U_{\max} - U_{\min})/2^R$. The set of quantization levels, $\{\omega_1, \cdots, \omega_L\}$, is a strictly increasing sequence in $\mathbb{R}$ and the levels are uniformly spaced in the sense that $\omega_{i+1} - \omega_i = \Delta$. A quantizer $Q : [U_{\min}, U_{\max}] \to \{\omega_1, \cdots, \omega_L\}$ is adopted to quantize the message $u \in [U_{\min}, U_{\max}]$ in such a way that $Q(u) = \omega_i$ if $u \in [\omega_i, \omega_{i+1})$ for some $i \in \{1, \cdots, L - 1\}$. Assume that the initial states $x_i(0)$ for all $i \in V$ are multiples of $\Delta$.

Problem statement. The problem of interest in this paper is to design distributed averaging algorithms which the nodes can utilize to update their states by communicating with neighbors via quantized messages in an asynchronous setting. Ultimately, quantized average consensus is reached in probability; i.e., for any initial state $x(0)$, there holds that $\lim_{t \to \infty} \mathbb{P}(x(t) \in \cdots)$
\( \mathcal{W}(x(0)) = 1 \). The set \( \mathcal{W}(x(0)) \) is dependent on initial state \( x(0) \in \mathbb{R}^N \) and defined as follows. If \( \bar{x}(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0) \) is not a multiple of \( \Delta \), then \( \mathcal{W}(x(0)) = \{ x \in \mathbb{R}^N \mid x_i \in \{ \mathcal{Q}(\bar{x}(0)), \mathcal{Q}(\bar{x}(0)) + \Delta \} \} \); otherwise, \( \mathcal{W}(x(0)) = \{ x \in \mathbb{R}^N \mid x_i = \bar{x}(0) \} \). Now it is clear that the steady-state error is at most \( \Delta \) after quantized average consensus is reached.

**Notions of random walks on graphs.** In this paper, random walks on graphs play an important role in characterizing the convergence properties of our quantized averaging algorithms. The following definitions are generalized from those defined for fixed graphs in [5, 8].

**Definition 2.1 (Random walks):** A random walk on the graph \( \mathcal{G}(t) \) under the transition matrix \( P(t) = (p_{ij}(t)) \), starting from node \( v \) at time \( s \), is a stochastic process \{ \( X(t) \) \}_t \geq s \) such that \( X(s) = v \) and \( \mathbb{P}(X(t+1) = j \mid X(t) = i) = p_{ij}(t) \). A random walk is said to be simple if for any \( i \in V \), \( p_{ii}(t) = 0 \) for all \( t \geq 0 \); otherwise, it is said to be natural.

**Definition 2.2 (Hitting time):** Consider a random walk on the graph \( \mathcal{G}(t) \), beginning from node \( i \) at time \( s \) and evolving under the transition matrix \( P(t) \). The hitting time from node \( i \) to the set \( \Lambda \subseteq V \), denoted as \( H_{\mathcal{G}(t),P(t),s}(i, \Lambda) \), is the expected time it takes this random walk to reach the set \( \Lambda \) for the first time. We denote \( H_{\mathcal{G}(t),P(t)}(\Lambda) = \sup_{s \geq 0} \max_{i \in V} H_{\mathcal{G}(t),P(t),s}(i, \Lambda) \) as the hitting time to reach the set \( \Lambda \). The hitting time of the pair \( i, j \), denoted as \( H_{\mathcal{G}(t),P(t),s}(i, j) \), is the expected time it takes this random walk to reach node \( j \) for the first time. Denote \( H_{\mathcal{G}(t),P(t)} = \sup_{s \geq 0} \max_{i,j \in V} H_{\mathcal{G}(t),P(t),s}(i, j) \) as the hitting time of going between any pair of nodes.

**Definition 2.3 (Meeting time):** Consider two random walks on the graph \( \mathcal{G}(t) \) under the transition matrix \( P(t) \), starting at time \( s \) from node \( i \) and node \( j \) respectively. The meeting time \( M_{\mathcal{G}(t),P(t),s}(i, j) \) of these two random walks is the expected time it takes them to meet at some node for the first time. The meeting time on the graph \( \mathcal{G}(t) \) is defined as \( M_{\mathcal{G}(t),P(t)} = \sup_{s \geq 0} \max_{i,j \in V} M_{\mathcal{G}(t),P(t),s}(i, j) \).

For the ease of notation, we will drop the subscript \( s \) in the hitting time and meeting time notions for fixed graphs. The following notion is only defined for fixed graphs.

**Definition 2.4 (Irreducibility and reversibility):** A random walk on the graph \( \mathcal{G} \) is irreducible if it is possible to get to any other node from any node. An irreducible random walk with stationary distribution \( \pi \) is called reversible if \( \pi_i p_{ij} = \pi_j p_{ji} \) for all \( i, j \in V \).

**Notations.** For \( \alpha \in \mathbb{R} \), define \( V_\alpha : \mathbb{R}^N \rightarrow \mathbb{R} \) as \( V_\alpha(x) = \sum_{i=1}^{N} (x_i - \alpha)^2 \). We define \( J : \mathbb{R}^N \rightarrow \mathbb{R} \) as \( J(x) = (\max_{i \in V} x_i - \min_{i \in V} x_i) / \Delta \). Denote the set \( \Theta = \{(k,k) \mid k \in V\} \). The distribution of a vector \( x \in \mathbb{R}^N \) is defined to be the list \{ \((q_1,m_1),(q_2,m_2),\ldots,(q_k,m_k)\) \} for some \( k \in V \).
where \( \sum_{k}^{m} m_{\ell} = N, q_i \neq q_j \) for \( i \neq j \) and \( m_{\ell} \) is the cardinality of the set \( \{ i \in V \mid x_i = q_\ell \} \). The cardinality of the set \( M \) is denoted by \( |M| \).

III. ASYNCHRONOUS DISTRIBUTED QUANTIZED AVERAGING ON FIXED GRAPHS

In this section, we propose and analyze an asynchronous distributed quantized averaging algorithm on the fixed and connected graph \( G \). Main references are [11] on quantized gossip algorithms and [5] on the meeting time of two simple random walks on fixed graphs.

A. Proposed algorithm

The asynchronous distributed quantized averaging algorithm on the fixed and connected graph \( G \) (AF, for short) is described as follows. Suppose node \( i \)'s clock ticks at time \( t \). Node \( i \) randomly chooses one of its neighbors, say node \( j \), with equal probability. Node \( i \) and \( j \) then execute the following local computation. If \( x_i(t) \geq x_j(t) \), then

\[
x_i(t+1) = x_i(t) - \delta, \quad x_j(t+1) = x_j(t) + \delta;
\]

otherwise,

\[
x_i(t+1) = x_i(t) + \delta, \quad x_j(t+1) = x_j(t) - \delta,
\]

where \( \delta = \frac{1}{2}(x_i(t) - x_j(t)) \) if \( \frac{x_i(t)-x_j(t)}{2\Delta} \) is an integer; otherwise, \( \delta = Q(\frac{1}{2}(x_i(t) - x_j(t))) + \Delta \). Every other node \( k \in V \setminus \{i, j\} \) preserves its current state; i.e., \( x_k(t+1) = x_k(t) \).

Remark 3.1: The precision \( \frac{\Delta}{2} \) is sufficient for the computation of \( \delta \) and thus the update laws (1) and (2). It is easy to verify that \( x_i(t) \in [U_{\min}, U_{\max}] \) and \( x_i(t) \) are multiples of \( \Delta \) for all \( i \in V \) and \( t \geq 0 \). Furthermore, the sum of the state values is preserved at each iteration.

If \( |x_i(t) - x_j(t)| = \Delta \), the update laws (1) and (2) become \( x_i(t+1) = x_j(t) \) and \( x_j(t+1) = x_i(t) \). Such update is referred to as a trivial average in [11]. If \( |x_i(t) - x_j(t)| > \Delta \), then (1) or (2) is referred to as a non-trivial average. Although it does not directly contribute to reaching quantized average consensus, trivial average helps the information flow over the network.

B. The meeting time of two natural random walks on the fixed graph \( G \)

To analyze the convergence properties of AF, we first study a variation of the problem in [8], namely, the meeting time of two natural random walks on the fixed graph \( G \). More precisely,
assume that the fixed graph \( G \) be undirected and connected. Initially, two tokens are placed on
the graph \( G \); at each time, one of the tokens is chosen with probability \( \frac{1}{N} \) and the chosen token
moves to one of the neighboring nodes with equal probability. What is the meeting time for
these two tokens?

The tokens move as two natural random walks with the transition matrix \( P_{AF} \) on the graph \( G \).
The matrix \( P_{AF} = (\tilde{p}_{ij}) \in \mathbb{R}^{N \times N} \) is given by \( \tilde{p}_{ii} = 1 - \frac{1}{N} \) for \( i \in V \), \( \tilde{p}_{ij} = \frac{1}{|N_j|} \) for \( (i, j) \in E \).
The meeting time of these two natural random walks is denoted as \( M_{(G,P_{AF})} \). Denote any of
these two natural random walks as \( X_N \). Correspondingly, we construct a simple random walk,
say \( X_S \), with the transition matrix \( P_{SF} \) on the graph \( G \) where the matrix \( P_{SF} = (p_{ij}) \in \mathbb{R}^{N \times N} \)
is given by \( p_{ii} = 0 \) and \( p_{ij} = \frac{1}{|N_i|} \) if \( (i, j) \in E \). The hitting times of the random walks \( X_S \) and
\( X_N \) are denoted as \( H_{(G,P_{SF})} \) and \( H_{(G,P_{AF})} \), respectively.

**Proposition 3.1:** The meeting time of two natural random walks with transition matrices \( P_{AF} \)
on the fixed graph \( G \) satisfies that \( M_{(G,P_{AF})} \leq 2NH_{(G,P_{SF})} - N \).

*Proof:* Since the fixed graph \( G \) is undirected and connected, the random walks \( X_N \) and \( X_S \)
are irreducible. The reminder of the proof is based on the following claims:

(i) It holds that \( H_{(G,P_{AF})} \geq N \).

(ii) For any pair \( i, j \in V \) with \( i \neq j \), we have \( H_{(G,P_{AF})}(i, j) = NH_{(G,P_{SF})}(i, j) \).

(iii) For any \( i, j, k \in V \), the following equality holds:

\[
H_{(G,P_{AF})}(i, j) + H_{(G,P_{AF})}(j, k) + H_{(G,P_{AF})}(k, i) = H_{(G,P_{AF})}(i, k) + H_{(G,P_{AF})}(k, j) + H_{(G,P_{AF})}(j, i).
\]

(iv) There holds that \( M_{(G,P_{AF})} \leq 2H_{(G,P_{SF})} - N \).

Now, let us prove each of the above claims.

(i) The quantity \( H_{(G,P_{AF})}(i, j) \) reaches the minimum when \( N_i = \{j\} \). We now consider
the graph \( G \) with \( N_i = \{j\} \) and compute \( H_{(G,P_{AF})}(i, j) \). The probability that \( X_N \) stays up with node
\( i \) before time \( \ell \) and moves to node \( j \) at time \( \ell \) is \( \frac{1}{N} (1 - \frac{1}{N})^{\ell-1} \). Then, we have \( H_{(G,P_{AF})}(i, j) = \sum_{\ell=1}^{+\infty} \ell \frac{1}{N} (1 - \frac{1}{N})^{\ell-1} = N \) and Claim (i) holds.

(ii) For any pair \( i, j \in V \) with \( i \neq j \), it holds that \( H_{(G,P_{AF})}(i, j) = \sum_{k \in N_i} \frac{1}{|N_i|} H_{(G,P_{AF})}(k, j) + 1) + (1 - \frac{1}{N}) \sum_{k \in N_j} \frac{1}{|N_j|} H_{(G,P_{AF})}(k, j) \). Hence, we have that \( H_{(G,P_{AF})}(i, j) = N + \sum_{k \in N_i} \frac{1}{|N_i|} H_{(G,P_{AF})}(k, j) \).

Furthermore, \( H_{(G,P_{SF})}(i, j) = \sum_{k \in N_i} \frac{1}{|N_i|} (H_{(G,P_{SF})}(k, j) + 1) = 1 + \sum_{k \in N_i} \frac{1}{|N_i|} H_{(G,P_{SF})}(k, j) \).

Hence, Claim (ii) holds.

(iii) Denote by \( \pi_i = |N_i|/N_{max} \) and \( \pi = (\pi_1, \cdots, \pi_N)^T \) where \( N_{max} = \max_{i \in V} \{|N_i|\} \). Since
\( P_{AF}^{T} \pi = \pi \), then \( \pi \) is the stationary distribution of the random walk \( X_N \). Furthermore, for any
pair \( i, j \in V \), we have \( \pi_i \tilde{p}_{ij} = \frac{1}{N_i} \frac{1}{N_i} = \frac{1}{N_i} = \pi_j \tilde{p}_{ji} \) and thus the random walk \( X_N \) is reversible. From Lemma 2 of [3] it follows that Claim (iii) holds.

(iv) Claim (iv) is an extension of Theorem 2 in [3]. An immediate result of Claim (iii) gives a node-relation on \( V \); i.e., \( i \leq j \) if and only if \( H_{(G, P_{AF})}(i, j) \leq H_{(G, P_{AF})}(j, i) \). This relation is transitive and constitutes a pre-order on \( V \). Then there exists a node \( u \) satisfying \( H_{(G, P_{AF})}(v, u) \geq H_{(G, P_{AF})}(u, v) \) for any other node \( v \in V \). Such a node \( u \) is called hidden. As in [3], we define a potential function \( \Phi \) by \( \Phi(i, j) = H_{(G, P_{AF})}(i, j) + H_{(G, P_{AF})}(j, u) - H_{(G, P_{AF})}(u, j) \).

Define the functions \( \Phi(\tilde{i}, j) \) and \( M_{(G, P_{AF})}(\tilde{i}, j) \) below, the averages of the functions \( \Phi \) and \( M_{(G, P_{AF})} \) over the neighbors of node \( i \) and \( j \), respectively:

\[
\Phi(\tilde{i}, j) = \frac{1}{|N_i|} \sum_{k \in N_i} \Phi(k, j) = \frac{1}{|N_i|} \sum_{k \in N_i} H_{(G, P_{AF})}(k, j) + H_{(G, P_{AF})}(j, u) - H_{(G, P_{AF})}(u, j),
\]

\[
M_{(G, P_{AF})}(\tilde{i}, j) = \frac{1}{|N_i|} \sum_{k \in N_i} M_{(G, P_{AF})}(k, j).
\]

In Claim (ii), we have shown that \( H_{(G, P_{AF})}(i, j) = \sum_{k \in N_i} \frac{1}{|N_i|} H_{(G, P_{AF})}(k, j) + N. \) Thus, \( \Phi(\tilde{i}, j) + N = \Phi(i, j) \). Similarly, \( M_{(G, P_{AF})}(\tilde{i}, j) + N = M_{(G, P_{AF})}(i, j) \).

We are now in a position to show that for any pair \( i, j \in V \), it holds that

\[
M_{(G, P_{AF})}(i, j) \leq \Phi(i, j). \tag{3}
\]

Assume that (3) does not hold. Let \( \phi = \max_{w, v \in V} (M_{(G, P_{AF})}(w, v) - \Phi(w, v)) > 0 \). Choose a pair of \( i, j \) with minimum distance among the set \( \Xi = \{(w, v) \in V \times V \mid M_{(G, P_{AF})}(w, v) - \Phi(w, v) = \phi \} \). Toward this end, consider the following two cases:

1. \( j \in N_i \). Observe that \( \Phi(j, j) = H_{(G, P_{AF})}(j, j) + H_{(G, P_{AF})}(j, u) - H_{(G, P_{AF})}(u, j) \geq 0 = M_{(G, P_{AF})}(j, j) \). We have \( \Phi(\tilde{i}, j) + \phi > M_{(G, P_{AF})}(\tilde{i}, j) \) and thus

\[
M_{(G, P_{AF})}(i, j) = \Phi(i, j) + \phi = N + \Phi(\tilde{i}, j) + \phi > N + M_{(G, P_{AF})}(\tilde{i}, j) = M_{(G, P_{AF})}(i, j). \tag{4}
\]

2. \( j \notin N_i \). There exists node \( k \in N_i \) such that node \( k \) is closer to node \( j \) than node \( i \). Since the pair of \( i, j \) has the minimum distance in the set \( \Xi \), we have \( M_{(G, P_{AF})}(k, j) - \Phi(k, j) < \phi \). Hence, \( \Phi(\tilde{i}, j) + \phi > M_{(G, P_{AF})}(\tilde{i}, j), \) and thus (4) holds.

In both cases, we get to the contradiction \( M_{(G, P_{AF})}(i, j) > M_{(G, P_{AF})}(i, j), \) and thus (3) holds.

Combining Claims (i), (ii) and inequality (3) gives the desired result of \( M_{(G, P_{AF})} \leq 2NH_{(G, P_{AF})} - N. \)
C. Convergence analysis of AF

We now proceed to analyze the convergence properties of AF. The convergence time of AF is a random variable defined as follows: \( T_{\text{con}}(x(0)) = \inf \{t \mid x(t) \in \mathcal{W}(x(0)) \} \), where \( x(t) \) starts from \( x(0) \) and evolves under AF. Choose \( V_{\bar{x}(0)}(x) = \sum_{i=1}^{N} (x_i - \bar{x}(0))^2 \) as a Lyapunov function candidate for AF. One can readily see that \( V_{\bar{x}(0)}(x(t+1)) = V_{\bar{x}(0)}(x(t)) \) when a trivial average occurs and \( V_{\bar{x}(0)}(x) \) reduces at least \( 2\Delta^2 \) when a non-trivial average occurs. Hence, \( V_{\bar{x}(0)}(x) \) is non-increasing along the trajectories, and the number of non-trivial averages is at most \( \frac{1}{2\Delta^2} V_{\bar{x}(0)}(x(0)) \). Define the set \( \Psi = \{ x \in \mathbb{R}^N \mid \text{the distribution of } x \text{ is } \{(0,1),(\Delta,N-2),(2\Delta,1)\} \} \) and denote \( \mathbb{E}[T_\Psi] = \max_{x(0) \in \Psi} \mathbb{E}[T_{\text{con}}(x(0))] \). It is clear that the expected time between any two consecutive non-trivial averages is not larger than \( \mathbb{E}[T_\Psi] \). Then we have the following estimates on \( \mathbb{E}[T_{\text{con}}(x(0))] \):

\[
\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{1}{2\Delta^2} V_{\bar{x}(0)}(x(0)) \mathbb{E}[T_\Psi] \leq \frac{NJ(x(0))^2}{8} \mathbb{E}[T_\Psi],
\]

where the second inequality is a direct result of Lemma 4 in [11].

**Theorem 3.1:** For any \( x(0) \notin \mathcal{W}(x(0)) \), the expected convergence time \( \mathbb{E}[T_{\text{con}}(x(0))] \) of AF is upper bounded by \( \frac{N^2J(x(0))^2}{8}(\frac{8}{27}N^3 - 1) \).

**Proof:** By (5), it suffices to bound \( \mathbb{E}[T_\Psi] \). Assume that \( x(0) \in \Psi \). Before they meet for the first time, the values 0 and \( 2\Delta \) move as two natural random walks which are identical to \( X_N \) in Proposition 3.1. At their meeting for the first time, the values of 0 and \( 2\Delta \) average and quantized average consensus is reached. Hence, \( \mathbb{E}[T_\Psi] = M_{(\mathcal{G},P_{AF})} \) and thus inequality (5) becomes

\[
\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{NJ(x(0))^2}{8} M_{(\mathcal{G},P_{AF})} \leq \frac{NJ(x(0))^2}{8}(2NH_{(\mathcal{G},P_{SF})} - N),
\]

where we use Proposition 3.1 in the second inequality. By letting \( M = 0 \) in the theorem of Page 265 in [5], we can obtain the upper bound \( \frac{4}{27}N^3 \) on \( H_{(\mathcal{G},P_{SF})} \). Substituting this upper bound into inequality (6) gives the desired upper bound on \( \mathbb{E}[T_{\text{con}}(x(0))] \). \( \blacksquare \)

**Theorem 3.2:** Let \( x(0) \in \mathbb{R}^N \) and suppose \( x(0) \notin \mathcal{W}(x(0)) \). Under AF, almost any evolution \( x(t) \) starting from \( x(0) \) reaches quantized average consensus.

**Proof:** Denote \( \bar{T} = \frac{N^2J(x(0))^2}{4}(\frac{8}{27}N^3 - 1) \), and consider the first \( \bar{T} \) clock ticks of evolution of AF starting from \( x(0) \). It follows from Markov’s inequality that

\[
\mathbb{P}(T_{\text{con}}(x(0)) > \bar{T} \mid x(0) \notin \mathcal{W}(x(0))) \leq \frac{\mathbb{E}[T_{\text{con}}(x(0))] \bar{T}}{\mathbb{E}[T_{\text{con}}(x(0))]} \leq \frac{1}{2},
\]

February 10, 2010  DRAFT
that is, the probability that after $\tilde{T}$ clock ticks $AF$ has not reached quantized average consensus is less than $\frac{1}{2}$. Starting from $x(\tilde{T})$, let us consider the posterior evolution of $x(t)$ in the next $\tilde{T}$ clock ticks. We have

$$\mathbb{P}(T_{\text{con}}(x(\tilde{T})) > \tilde{T} \mid x(\tilde{T}) \notin W(x(0))) \leq \frac{\mathbb{E}[T_{\text{con}}(x(\tilde{T}))]}{\tilde{T}} \leq \frac{1}{2}. $$

That is, the probability that after $2\tilde{T}$ clock ticks $x(t)$ has not reached quantized average consensus is at most $(\frac{1}{2})^2$. By induction, it follows that after $n\tilde{T}$ clock ticks the probability $x(t)$ not reaching quantized average consensus is at most $(\frac{1}{2})^n$. Since the set $W(x(0))$ is absorbing, we have $\lim_{t \to \infty} \mathbb{P}(x(t) \notin W(x(0))) = 0$. This completes the proof.

IV. ASYNCHRONOUS DISTRIBUTED QUANTIZED AVERAGING ON SWITCHING GRAPHS

We now turn our attention to the more challenging scenario where the communication graphs are undirected but dynamically changing. We will propose and analyze an asynchronous distributed quantized averaging algorithm on switching graphs (AS, for short). The convergence rate of distributed real-valued averaging algorithms on switching graphs in [13] will be employed to characterize the hitting time of random walks on switching graphs.

A. Proposed algorithm

The main steps of AS can be summarized as follows. At time $t$, let node $i$’s clock tick. If $|\mathcal{N}_i(t)| \neq 0$, node $i$ randomly chooses one of its neighbors, say node $j$, with probability $\frac{1}{\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$. Then, node $i$ and $j$ execute the computation (1) or (2) and every other node $k \in V \setminus \{i, j\}$ preserves its current state. If $|\mathcal{N}_i(t)| = 0$, all the nodes do nothing at this time.

Here, we assume that the communication graph $G(t)$ be undirected and satisfies the following connectivity assumption also used in [3], [10], [13], [15].

Assumption 4.1 (Periodical connectivity): There exists some $B \in \mathbb{N}_{>0}$ such that, for all $t \geq 0$, the undirected graph $(V, E(t) \cup E(t+1) \cup \cdots \cup E(t+B-1))$ is connected.

Remark 4.1: In the AS, the probability that node $i$ chooses a neighbor $j$ is $\frac{1}{\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$. Thus, this information should be available to node $i$. In this way, the matrix $P_{AS}(t)$ defined later is symmetric and double stochastic.

February 10, 2010 DRAFT
B. The meeting time of two natural random walks on the time-varying graph $\mathcal{G}(t)$

Before analyzing $\mathcal{A}S$, we consider the following problem which generalizes the problem in Section III-B to the case of dynamically changing graphs.

The meeting time of two natural random walks on the time-varying graph $\mathcal{G}(t)$. Assume that $\mathcal{G}(t)$ be undirected and satisfies Assumption 4.1. Initially, two tokens are placed on $\mathcal{G}(0)$. At each time, one of the tokens is chosen with probability $\frac{1}{N}$. The chosen token at some node, say $i$, moves to one of the neighbors, say node $j$, with probability $\frac{1}{\max(|N_i(t)|,|N_j(t)|)}$ if $|N_i(t)| \neq 0$; otherwise, it will stay up with node $i$. What is the meeting time for these two tokens?

Clearly, the movements of two tokens are two natural random walks, say $X_1$ and $X_2$, on the switching graph $\mathcal{G}(t)$. Their meeting time is denoted as $M_{\mathcal{G}(t),P_{\mathcal{A}S}(t)}$ where the transition matrix $P_{\mathcal{A}S}(t) = (\bar{p}_{ij}(t))$ is given as follows: if $|N_i(t)| \neq 0$, then $\bar{p}_{ij}(t) = \frac{1}{\max(|N_i(t)|,|N_j(t)|)}$ for $(i,j) \in E(t)$ and $\bar{p}_{ii}(t) = 1 - \sum_{(i,j) \in E(t)} \frac{1}{\max(|N_i(t)|,|N_j(t)|)}$; if $|N_i(t)| = 0$, then $\bar{p}_{ii}(t) = 1$. One can easily verify that the matrix $P_{\mathcal{A}S}(t)$ is symmetric and doubly stochastic. The natural random walks $X_1$ and $X_2$ on the graph $\mathcal{G}(t)$ are equivalent to a single natural random walk, say $X_M$, on the product graph $\mathcal{G}(t) \times \mathcal{G}(t)$. That is, $X_M$ moving from node $(i_1, i_2) \in V \times V$ to node $(j_1, j_2) \in V \times V$ on the graph $\mathcal{G}(t) \times \mathcal{G}(t)$ at time $t$, is equivalent to $X_1$ moving from $i_1$ to $j_1$ and $X_2$ moving from $i_2$ to $j_2$ on the graph $\mathcal{G}(t)$ at time $t$. Denote the transition matrix of the random walk $X_M$ as $Q(t) = (q_{(i_1,i_2),(j_1,j_2)}(t)) \in \mathbb{R}^{N^2 \times N^2}$.

In the following lemma, we will consider the random walk $X_M$ on the graph $\mathcal{G}(t) \times \mathcal{G}(t)$ with the absorbing set $\Theta$ and the transition matrix $\bar{Q}(t) \in \mathbb{R}^{N^2 \times N^2}$. Denote $e_{(\ell_1,\ell_2)}$ by the row corresponding to $(\ell_1, \ell_2) \in V \times V$ in a $N^2 \times N^2$ identity matrix. The transition matrix $\bar{Q}(t)$ is defined by replacing the row associated with the absorbing state $(\ell_1, \ell_2) \in \Theta$ in $Q(t)$ with $e_{(\ell_1,\ell_2)}$. Define $\vartheta_{(\ell_1,\ell_2)}(t) = \mathbb{P}(X_M(t) = (\ell_1, \ell_2))$, $\vartheta(t) = \text{col}\{\vartheta_{(\ell_1,\ell_2)}(t)\} \in \mathbb{R}^{N^2}$, $\vartheta_\Theta(t) = \sum_{(\ell_1,\ell_2) \in \Theta} \vartheta_{(\ell_1,\ell_2)}(t)$ for the random walk $X_M$, and $\bar{\vartheta}_{(\ell_1,\ell_2)}(t) = \mathbb{P}(X_M(t) = (\ell_1, \ell_2))$, $\bar{\vartheta}(t) = \text{col}\{\bar{\vartheta}_{(\ell_1,\ell_2)}(t)\} \in \mathbb{R}^{N^2}$, $\bar{\vartheta}_\Theta(t) = \sum_{(\ell_1,\ell_2) \in \Theta} \bar{\vartheta}_{(\ell_1,\ell_2)}(t)$ for the random walk $\bar{X}_M$.

**Lemma 4.1:** Consider a network of $N$ nodes whose communication graph $\mathcal{G}(t)$ be undirected and satisfies Assumption 4.1. Let $(i_1, i_2) \in V \times V$ be a given node and suppose that the random walks $X_M$ and $\bar{X}_M$ start from node $(i_1, i_2)$ at time 0. Then it holds that $\bar{\vartheta}_\Theta(t) \geq \vartheta_\Theta(t) \geq \frac{1}{2N}$ for $t \geq t_1$ where $t_1$ is the smallest integer which is larger than $B(8N^6 \log(\sqrt{2N} + 1))$.

**Proof:** It is not difficult to check that $\mathcal{G}(t) \times \mathcal{G}(t)$ is undirected and satisfies Assumption 4.1.
with period $B$. The minimum of nonzero entries in $Q(t)$ is lower bounded by $\frac{1}{N(N-1)}$, and $Q(t)$ is symmetric. Observe that for any $(i_1, i_2) \in V \times V$ and any $t \geq 0$, $\sum_{(j_1, j_2) \in V \times V} q_{(i_1, i_2)(j_1, j_2)}(t) = \sum_{(j_1, j_2) \in V \times V} \tilde{p}_{i_1j_1}(t)\tilde{p}_{i_2j_2}(t) = \sum_{j_1 \in V} \tilde{p}_{i_1j_1}(t) \times \sum_{j_2 \in V} \tilde{p}_{i_2j_2}(t) = 1$ where we use the fact that the matrix $P_{AS}(t)$ is doubly stochastic. Hence, the matrix $Q(t)$ is doubly stochastic.

The evolution of $\vartheta(t)$ is governed by the equation $\vartheta(t + 1) = Q^T(t)\vartheta(t)$ with initial state $\vartheta(0) = e_{(i_1, i_2)}^T$. Consider the Lyapunov function $V_{\frac{1}{N^2}}(\vartheta) = \sum_{i=1}^{N^2} (\vartheta_i - \frac{1}{N^2})^2$ with $V_{\frac{1}{N^2}}(\vartheta(0)) = 1 - \frac{1}{N^2}$. It follows from Lemma 5 in [13] that

$$V_{\frac{1}{N^2}}(\vartheta((k+1)B)) \leq (1 - \frac{1}{2N^5(N-1)})V_{\frac{1}{N^2}}(\vartheta(kB))$$

for $k \in \mathbb{N}_0$. Denote $1 \in \mathbb{R}^{N^2}$ as the vector of $N^2$ ones and note that

$$V_{\frac{1}{N^2}}(\vartheta(t)) - V_{\frac{1}{N^2}}(\vartheta(t + 1)) = (\vartheta(t) - \frac{1}{N^2}1)^T(I - Q(t)Q^T(t))(\vartheta(t) - \frac{1}{N^2}1).$$

Since $Q(t)$ is doubly stochastic, so is $Q(t)Q^T(t)$. Hence, the diagonal entries of the matrix $\Gamma(t) = I - Q(t)Q^T(t) = (\gamma_{ij}(t)) \in \mathbb{R}^{N^2 \times N^2}$ are dominant in the sense of $\gamma_{ii}(t) = \sum_{j \neq i} \gamma_{ij}(t)$. According to Gershgorin theorem in [9], all eigenvalues of $\Gamma(t)$ lie in a closed disk centered at $\max_{i \in \{1, \ldots, N^2\}} \gamma_{ii}(t)$ with a radius $\max_{i \in \{1, \ldots, N^2\}} \gamma_{ii}(t)$. Hence, $\Gamma(t)$ is positive-semidefinite. Consequently, $V_{\frac{1}{N^2}}(\vartheta(t)) - V_{\frac{1}{N^2}}(\vartheta(t + 1)) \geq 0$ and thus $V_{\frac{1}{N^2}}(\vartheta(t))$ is non-increasing along the trajectory of $\vartheta(t)$. Combining (7) with the non-increasing property of $V_{\frac{1}{N^2}}(\vartheta(t))$ gives that

$$V_{\frac{1}{N^2}}(\vartheta(t)) \leq V_{\frac{1}{N^2}}(\vartheta(0))(1 - \frac{1}{2N^5(N-1)})^{\frac{t}{B} - 1} = \frac{N^2 - 1}{N^2} - \frac{1}{2N^5(N-1)})^{\frac{t}{B} - 1}. (8)$$

Since $\vartheta(t)^T1 = 1$, then $\vartheta_{\min}(t) := \min((t_1, t_2) \in V \times V) \vartheta(t_1, t_2)(t) \leq \frac{1}{N^2}$. Since $V_{\frac{1}{N^2}}(\vartheta(t)) \geq (\vartheta_{\min}(t) - \frac{1}{N^2})^2$, inequality (8) gives that $\vartheta_{\min}(t) \geq \frac{1}{N^2} - (\frac{2^2 - 1}{N^2} - \frac{1}{2N^5(N-1)})^{\frac{t}{B} - 1}$. Therefore, it holds that $\vartheta_{\min}(t) \geq \frac{1}{2N^2}$ for $t \geq B(\frac{\log(4N^2(N^2-1))}{\log(1 - \frac{1}{2N^5(N-1)})} + 1)$. Since $\log x \leq x - 1$, there holds $-\log(1 - \frac{1}{2N^5(N-1)}) \leq 2N^5(N-1) \leq 2N^6$. Hence, we have that $\vartheta_{\min}(t) \geq \frac{1}{2N^2}$ and thus $\vartheta(t) \geq \frac{1}{2N}$ for $t \geq t_1$.

Note that the evolution of $\bar{\vartheta}(t)$ is governed by the equation $\bar{\vartheta}(t + 1) = \bar{Q}(t)^T\bar{\vartheta}(t)$ with $\bar{\vartheta}(0) = e_{(i_1, i_2)}$. Since the set $\Theta$ is absorbing, $\bar{\vartheta}(t) \geq \vartheta(t)$ for all $t \geq 0$ and thus the desired result follows.

**Proposition 4.1:** The meeting time of two natural random walks with transition matrix $P_{AS}(t)$ on the time-varying graph $G(t)$ satisfies that $M(G(t), P_{AS}(t)) \leq 4Nt_1$.

**Proof:** Denote by $H(G(t) \times G(t), Q(t))(\Theta)$ the hitting time of the random walk $X_M$ to reach the set of $\Theta$. Observe that $M(G(t), P_{AS}(t)) = H(G(t) \times G(t), Q(t))(\Theta)$. To find an upper bound on
from $(i_1, i_2)$ at time 0 with $i_1 \neq i_2$ and the set $\Theta$ is the absorbing set of $X_{M}^{(i_1, i_2)}$. The transition matrix of $X_{M}^{(i_1, i_2)}$ is $\tilde{Q}(t)$ defined before Lemma 4.1. Define
\[ \vartheta_{(\ell_1, \ell_2)}(t) = \mathbb{P}(X_M^{(i_1, i_2)}(t) = (\ell_1, \ell_2)), \]
and
\[ \vartheta_{(i_1, i_2)}(t) = \text{col}\{\vartheta_{(\ell_1, \ell_2)}(t)\} \in \mathbb{R}^{N^2}. \]
The dynamics of $\vartheta_{(i_1, i_2)}(t)$ is given by $\vartheta_{(i_1, i_2)}(t + 1) = \tilde{Q}(t)^T \vartheta_{(i_1, i_2)}(t)$ with the initial state $\vartheta_{(i_1, i_2)}(0) = e_{(i_1, i_2)}^T$.

Define the function $\mu_{(\ell_1, \ell_2)}^{(i_1, i_2)} : \mathbb{N} \rightarrow \{0, 1\}$ in such a way that $\mu_{(\ell_1, \ell_2)}^{(i_1, i_2)} = 1$ if $X_{M}^{(i_1, i_2)}(t) = (\ell_1, \ell_2)$; otherwise, $\mu_{(\ell_1, \ell_2)}^{(i_1, i_2)}(t) = 0$. Define $n_{(\ell_1, \ell_2)}^{(i_1, i_2)} = \sum_{\tau=0}^{+\infty} \mu_{(\ell_1, \ell_2)}^{(i_1, i_2)}(\tau)$ which is the total times that the random walk $X_{M}^{(i_1, i_2)}$ is at node $(\ell_1, \ell_2)$. Then, the hitting time $H_{(g(t) \times g(t), Q(t), 0)}((i_1, i_2), \Theta)$ of $X_{M}^{(i_1, i_2)}$ equals the expected time that $X_{M}^{(i_1, i_2)}$ stays up with the nodes in $V \times V \setminus \Theta$, that is, 
\[
H_{(g(t) \times g(t), Q(t), 0)}((i_1, i_2), \Theta) = \sum_{(\ell_1, \ell_2) \notin \Theta} \mathbb{E}[n_{(\ell_1, \ell_2)}^{(i_1, i_2)}] = \sum_{(\ell_1, \ell_2) \notin \Theta} \mathbb{E}[\sum_{\tau=0}^{+\infty} \mu_{(\ell_1, \ell_2)}^{(i_1, i_2)}(\tau)]
= \sum_{(\ell_1, \ell_2) \notin \Theta} \sum_{\tau=0}^{+\infty} \mathbb{E}[\mu_{(\ell_1, \ell_2)}^{(i_1, i_2)}(\tau)] = \sum_{\tau=0}^{+\infty} \sum_{(\ell_1, \ell_2) \notin \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(\tau).
\] (9)

It follows from Lemma 4.1 that $\vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(t) \geq \frac{1}{2N}$ for $t \geq t_1$. With that, the fact of $\vartheta_{(i_1, i_2)}^{(i_1, i_2)}(t)^T \mathbf{1} = 1$ implies that
\[
\sum_{(\ell_1, \ell_2) \notin \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(t_1) \leq 1 - \frac{1}{2N}. \tag{10}
\]

For each $(k_1, k_2) \notin \Theta$, we construct the random walk $\tilde{X}_{M}^{(k_1, k_2)}$ in such a way that $\tilde{X}_{M}^{(k_1, k_2)}$ starts from $(k_1, k_2)$ at time $t_1$ and the set $\Theta$ is the absorbing set of $\tilde{X}_{M}^{(k_1, k_2)}$. The transition matrix of $\tilde{X}_{M}^{(k_1, k_2)}$ is $\tilde{Q}(t)$. Define $\tilde{\vartheta}_{(\ell_1, \ell_2)}^{(k_1, k_2)}(t) = \mathbb{P}(\tilde{X}_{M}^{(k_1, k_2)}(t) = (\ell_1, \ell_2))$. Following the foregoing arguments for $X_{M}^{(i_1, i_2)}$, we have
\[
\sum_{(\ell_1, \ell_2) \notin \Theta} \tilde{\vartheta}_{(\ell_1, \ell_2)}^{(k_1, k_2)}(2t_1) \leq 1 - \frac{1}{2N}. \tag{11}
\]

Combining (10) and (11) gives that
\[
\sum_{(\ell_1, \ell_2) \notin \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(2t_1) = \sum_{(\ell_1, \ell_2) \notin \Theta} \sum_{(k_1, k_2) \notin \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(t_1) \tilde{\vartheta}_{(\ell_1, \ell_2)}^{(k_1, k_2)}(2t_1)
= \sum_{(k_1, k_2) \notin \Theta} \sum_{(\ell_1, \ell_2) \notin \Theta} \tilde{\vartheta}_{(\ell_1, \ell_2)}^{(k_1, k_2)}(t_1) \tilde{\vartheta}_{(\ell_1, \ell_2)}^{(i_1, i_2)}(2t_1) \leq (1 - \frac{1}{2N})^2. \tag{12}
\]

By induction, we have $\sum_{(\ell_1, \ell_2) \notin \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(nt_1) \leq (1 - \frac{1}{2N})^n$ and then obtain a strictly decreasing sequence $\sum_{(\ell_1, \ell_2) \notin \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(nt_1)$ with respect to $n \in \mathbb{Z}_0$. Since the set $\Theta$ is absorbing, then
$\sum_{(\ell_1, \ell_2) \not\in \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(t)$ is non-increasing with respect to $t \geq 0$. Therefore, we have the following estimate

$$\sum_{(\ell_1, \ell_2) \not\in \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(t) \leq \sum_{(\ell_1, \ell_2) \not\in \Theta} \vartheta_{(\ell_1, \ell_2)}^{(i_1, i_2)}(0)(1 - \frac{1}{2N})^{\tau_1 - 1} = (1 - \frac{1}{2N})^{\tau_1 - 1}.$$  \hfill (13)

Substituting (13) into (9) gives that

$$H_{(G(t) \times G(t), Q(t), 0)}((i_1, i_2), \Theta) \leq \sum_{\tau=0}^{+\infty} (1 - \frac{1}{2N})^{\tau_1 - 1} = (1 - \frac{1}{2N})^{\tau_1 - 1} \cdot \frac{1}{1 - (1 - \frac{1}{2N})^{\tau_1}}. \hfill (14)$$

Since $t_1 > 1$, it holds that $(1 - \frac{1}{2N})^{\tau_1} \leq 2^{\frac{1}{\tau_1}} < 2$. It follows from Bernoulli’s inequality that $(1 - \frac{1}{2N})^{\tau_1} \leq 1 - \frac{1}{2Nt_1}$, and thus $\frac{1}{1 - (1 - \frac{1}{2N})^{\tau_1}} \leq 2Nt_1$. Inequality (14) becomes

$$H_{(G(t) \times G(t), Q(t), 0)}((i_1, i_2), \Theta) \leq 4Nt_1. \hfill (15)$$

Actually, inequality (15) holds for any starting time, any starting node $(i_1, i_2)$. Thus it holds that $M_{(G(t), P_{AS}(t))} = H_{(G(t) \times G(t), Q(t), \cdot)}(\Theta) \leq 4Nt_1$. This completes the proof. \hfill $\blacksquare$

C. Convergence analysis of AS

We are now in the position to characterize the convergence properties of AS. The quantities $T_{\text{con}}(x(0))$ and $T_\Psi$ for AS are defined in a similar way to those in Section III.

**Theorem 4.1:** Let $x(0) \in \mathbb{R}^N$ and suppose $x(0) \notin W(x(0))$. Assume that $G(t)$ be undirected and satisfies Assumption 4.1. Under AS, almost any evolution $x(t)$ starting from $x(0)$ reaches quantized average consensus. Furthermore, $\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{1}{2}BJ(x(0))^2N^2(16N^7 + 1)$.

**Proof:** Note that inequality (5) also hold for AS. Similar to Theorem 3.1 we have $\mathbb{E}[T_{\Psi}] = M_{(G(t), P_{AS}(t))}$. Then, the following estimate on $\mathbb{E}[T_{\text{con}}(x(0))]$ holds:

$$\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{NJ(x(0))^2}{8}M_{(G(t), P_{AS}(t))}. \hfill (16)$$

Substituting the upper bound on $M_{(G(t), P_{AS}(t))}$ in Proposition 4.1 into (16) and using $\log(\sqrt{2N}) \leq 2N$ gives the desired upper bound on $\mathbb{E}[T_{\text{con}}(x(0))]$ of AS. The reminder of the proof on the convergence to quantized average consensus is analogous to Theorem 3.2 and thus omitted. \hfill $\blacksquare$
V. Discussion

A. Asynchronous distributed quantized averaging on random graphs

Random graphs have been widely used to model real-world networks such as Internet, transportation networks, communication networks, biological networks and social networks. The Erdős - Rényi model \( \mathcal{G}(N, p) \) is the most commonly studied one, and constructed by randomly placing an edge between any two of \( N \) nodes with probability \( p \).

For any given time, the probability that the (directed) edge \((i, j)\) is selected is

\[
p_0 := \frac{1}{N} \sum_{m=0}^{N-2} \frac{p}{m+1} C_m^{N-2} p^m (1-p)^{N-2-m},
\]

that is, node \( i \) is active, the edge \((i, j)\) with other \( m \in \{0, \ldots, N-2\} \) edges connecting node \( i \) are placed, and the edge \((i, j)\) is selected by node \( i \).

To study the convergence properties of \( \mathcal{G}(N, p) \), it is equivalent to study AF on complete graphs with the transition matrix \( P_{\text{AR}} = (\hat{p}_{ij}) \in \mathbb{R}^{N \times N} \) where \( \hat{p}_{ij} = p_0 \) and \( \hat{p}_{ii} = 1 - (N-1)p_0 \). The meeting time is denoted as \( M(\mathcal{G}(N, p), P_{\text{AR}}) \). The probability that the two tokens meet for the first time at time \( t \) is \( 2p_0 \), that is, one of the tokens is chosen and simultaneously the edge between the two tokens is chosen. Hence, we have

\[
M(\mathcal{G}(N, p), P_{\text{AR}}) = \sum_{\ell=1}^{\infty} \ell 2p_0(1-2p_0)^{\ell-1} = \frac{1}{2p_0}.
\]

Observe that \( p_0 = \frac{p}{N} \sum_{m=0}^{N-2} \frac{1}{m+1} C_m^{N-2} p^m (1-p)^{N-2-m} \geq \frac{2p}{N(N-1)} \sum_{m=0}^{N-2} C_m^{N-2} p^m (1-p)^{N-2-m} = \frac{2p}{N(N-1)} \frac{N!}{16p_0} \leq \frac{N^2(N-1)}{32p}.
\]

Like Theorem 3.1, we have

\[
\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{NJ(x(0))^2}{8} \mathbb{E}[T_0] = \frac{NJ(x(0))^2}{8} M(\mathcal{G}(N, p), P_{\text{AR}}) = \frac{NJ(x(0))^2}{8} \cdot \frac{1}{2p_0}.
\]

B. Discussion on the bounds obtained

Consider a fixed graph \( L_N^m \) with \( N \) vertices consists of a clique on \( m \) vertices, including vertex \( i \), and a path of length \( N - m \) with one end connected to one vertex \( k \neq i \) of the clique, and the other end of the path being \( j \). It was shown in [5] that \( H_{(L_N^m, P_{\text{SF}})}(t) = O(N^3) \) where \( m_0 = \lfloor \frac{2N+1}{3} \rfloor \).

Let us consider the case that the algorithm AF is implemented on the graph \( L_N^m \) and initial states \( x_i(0) = 0, x_j(0) = 2 \) and \( x_k(0) = 1 \) for all \( k \neq i, j \). Observe that \( \mathbb{E}[T_{\text{con}}(x(0))] = M(\mathcal{G}(L_N^m, P_{\text{AF}})) \).

From Proposition 3.1 we have that \( \mathbb{E}[T_{\text{con}}(x(0))] = O(N^4) \), that is one order less than the bound in Theorem 3.1.

Consider switching graphs \( \mathcal{G}(t) \) where \( \mathcal{G}(t) \) is the graph \( L_N^m \) defined above when \( t \) is a multiple of \( B \); otherwise, all the vertices in \( \mathcal{G}(t) \) are isolated. Random walks on \( \mathcal{G}(t) \) can be viewed as time-scaled versions of those on \( L_N^m \), that is, random walks on \( \mathcal{G}(t) \) only make the movements when \( t \) is a multiple of \( B \). Let us consider the case that the algorithm AS is implemented on the
graph $L_N^{m_0}$ and initial states $x_i(0) = 0$, $x_j(0) = 2$ and $x_k(0) = 1$ for all $k \neq i, j$. Following the same lines above, we have that the bound on $\mathbb{E}[T_{\text{con}}(x(0))]$ is $O(BN^4)$ which is $N^4 \log N$-order less than that in Theorem 3.2.

It can be directly computed that $H(\mathcal{G}_{\text{com}}, P_{SF})$ is $O(N^2)$ where $\mathcal{G}_{\text{com}}$ is a complete graph with $N$ vertices. Following the same lines in Theorem 3.1 we have that $\mathbb{E}[T_{\text{con}}(x(0))]$ is $O(N^3)$ when the algorithm $\text{AF}$ is implemented on the graph $\mathcal{G}_{\text{com}}$. It implies that the convergence of $\text{AF}$ on $\mathcal{G}_{\text{com}}$ is as fast as that on $\mathcal{G}(N, p)$ when $p$ is independent of $N$. This is consistent with the fact that the underlying graph of $\mathcal{G}(N, p)$ is $\mathcal{G}_{\text{com}}$.

REFERENCES

[1] T.C. Aysal, M.J. Coates and M. Rabbat, Distributed average consensus using probabilistic quantization, Proceedings of IEEE Workshop on Statistical Signal Processing, Madison, USA, pp. 640 - 644, August 2007.
[2] D. P. Bertsekas and J. N. Tsitsiklis, Parallel and distributed computation: numerical methods, Prentice Hall, 1989.
[3] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, Convergence in multiagent coordination, consensus, and flocking, Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC’05), Seville, Spain, pp. 2996 - 3000, December 2005.
[4] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, Randomized gossip algorithms, IEEE Transactions on Information Theory, Special issue of IEEE Transactions on Information Theory and IEEE ACM Transactions on Networking, vol. 52, No. 6, pp. 2508 - 2530, June 2006.
[5] G. Brightwell and P. Winkler, Maximum hitting time for random walks on graphs, Random Structures and Algorithms, vol. 1, No. 3, pp. 263 - 276, 1990.
[6] R. Carli, F. Bullo, and S. Zampieri, Quantized average consensus via dynamic coding/decoding schemes, Proceedings of IEEE Conference on Decision and Control, Cancun, Mexico, December 2008, To appear.
[7] R. Carli, F. Fagnani, P. Frasca, T. Taylor and S. Zampieri, Communication constraints in the state agreement problem, Automatica, to appear.
[8] D. Coppersmith, P. Tetali and P. Winkler, Collisions among random walks on a graph, SIAM Journal on Discrete Mathematics, vol. 6, No. 3, pp. 363 - 374, 1993.
[9] R.A. Horn and C.R. Johnson, Matrix analysis, Cambridge, U.K.: Cambridge University Press, 1987.
[10] A. Jadbabaie, J. Lin, and A. S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Transactions on Automatic Control, vol. 48, No. 6, pp. 988 - 1001, June 2003.
[11] A. Kashyap, T. Basar and R. Srikant, Quantized consensus, Automatica, vol. 43, pp. 1192 - 1203, July 2007.
[12] S. Martínez, Distributed representation of spatial fields through adaptive interpolation schemes, Proceedings of the 2007 American Control Conference (ACC’07), New York, USA, pp. 2750 - 2755, July 2007.
[13] A. Nedic, A. Olshevsky, A. Ozdaglar and J. N. Tsitsiklis, On distributed averaging algorithms and quantization effects, MIT LIDS Report 2778, November 2007, https://netfiles.uiuc.edu/angelia/www/nedich.html.
[14] R. Olfati-Saber, J. A. Fax, and R. M. Murray, Consensus and cooperation in networked multi-Agent systems, Proceedings of the IEEE, vol. 95, No. 1, pp. 215-233, January 2007.
[15] A. Olshevsky and J.N. Tsitsiklis, *Convergence speed in distributed consensus and averaging*, SIAM Journal on Control and Optimization, to appear.

[16] L. Xiao, S. Boyd and S. Lall, *A scheme for robust distributed sensor fusion based on average consensus*, International Conference on Information Processing in Sensor Networks, pp. 63 - 70, Los Angeles, 2005.

[17] M. Zhu and S. Martínez, *On the convergence time of distributed quantized averaging algorithms*, Proceedings of the 47th IEEE Conference on Decision and Control, Cancún, Mexico, to appear 2008