C-VECTORS VIA $\tau$-TILTING THEORY

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ABSTRACT. Inspired by the tropical duality in cluster algebras, we introduce c-vectors for finite-dimensional algebras via $\tau$-tilting theory. Let $A$ be a finite-dimensional algebra over a field $k$. Each c-vector of $A$ can be realized as the (negative) dimension vector of certain indecomposable $A$-module and hence we establish the sign-coherence property of this kind of c-vectors. We then study the positive c-vectors for certain classes of finite-dimensional algebras. More precisely, we establish the equalities between the set of positive c-vectors and the set of dimension vectors of exceptional modules for quasitilted algebras and representation-directed algebras respectively. This generalizes the equalities of c-vectors for acyclic cluster algebras obtained by Chávez. To this end, a short proof for the sign-coherence of c-vectors for skew-symmetric cluster algebras has been given in the appendix.

1. INTRODUCTION

The c-vectors and g-vectors introduced by Fomin-Zelevinsky [15] are two kinds of integer vectors, which have played important roles in the theory of cluster algebras with coefficients. Both the vectors are conjectured to have a so-called sign-coherence property [15]. For skew-symmetric cluster algebras, Nakanishi [29] noticed a so-called tropical duality between c-vectors and g-vectors (cf. also [23, 28, 32]). With the assumption of sign-coherence of c-vectors, the tropical duality between c-vectors and g-vectors has been further generalized to skew-symmetrizable cluster algebras by Nakanishi and Zelevinsky [31]. Moreover, they showed that many properties/conjectures of cluster algebras follow from the tropical duality and hence follow from the sign-coherence of c-vectors. The sign-coherence conjecture is still open for general cluster algebras. But we do know that it holds for all the skew-symmetric cluster algebras due to the work of Derksen-Weyman-Zelevinsky [14], Plamondon [32] and Nagao [28]. All of the proofs involve the representation theory of algebras associated to quivers with potentials and no combinatorial proof is yet known.

On the other hand, c-vectors may be seen as a generalization of root systems. It follows from Nagao’s work [28] that each c-vector of a given skew-symmetric cluster algebra can be realized as the (negative) dimension vector of certain exceptional module for the associated Jacobian algebra. In particular, the set of c-vectors of
an acyclic cluster algebra is a subset of real Schur roots for the corresponding Kac-Moody algebra. In [11], Chávez showed the inverse inclusion is also true for acyclic cluster algebras (cf. also [33]). Moreover, he also proved in [12] that the set of positive c-vectors of a skew-symmetric cluster algebra of finite type coincides with the set of dimension vectors of all the exceptional modules over the corresponding representation-finite cluster-tilted algebra. Nakanishi and Stella [30] gave a diagrammatic description of c-vectors for cluster algebras of finite type. They proposed the root conjecture for any cluster algebras: for any skew-symmetrizable matrix $B$ any $c$-vector of the cluster algebra $A(B)$ is a root of the associated Kac-Moody algebra $g(A(B))$, where $A(B)$ is the Cartan counterpart of $B$.

In this paper, we pursue the representation-theoretic approach to study c-vectors. We introduce the notion of c-vectors for any finite-dimensional algebras via $\tau$-tilting theory introduced by Adachi-Iyama-Reiten [3]. Let $A$ be a finite-dimensional algebra over a field $k$. In [3], the authors showed that the indices $\text{ind}(M, Q)$ of a basic support $\tau$-tilting pair $(M, Q)$ form a $\mathbb{Z}$-basis for the Grothendieck groups $G_0(\text{per}A)$ of the preflect derived category $\text{per}A$. Let $D^b(\text{mod}A)$ be the bounded derived category of finitely generated right $A$-modules and $\langle -, - \rangle_A : G_0(\text{per}A) \times D^b(\text{mod}A) \to k$ the corresponding Euler bilinear form. We then define the c-vectors associated to $(M, Q)$ to be the dual basis of $\text{ind}(M, Q)$ in $G_0(D^b(\text{mod}A))$ with respect to the Euler bilinear form $\langle -, - \rangle_A$. Using the bijection between 2-term silting objects in $\text{per}A$ and the immediate $t$-structure on $D^b(\text{mod}A)$ in [26, 6], we show the sign-coherence property holds for this kind of c-vectors. When the algebra $A$ is a 2-Calabi-Yau tilted algebra, it follows from [16] and the tropical duality that the c-vectors we obtained here do coincide with the c-vectors for the corresponding cluster algebra. Let us mention here that, as a generalization of cluster algebras, Caldero-Chapoton algebras has been introduced in [10] for an arbitrary basic algebra, we hope that the c-vector we introduced here may provide new perspective in the investigation of Caldero-Chapoton algebras for finite-dimensional algebras.

The paper is organized as follows: After recall some definitions and basic properties related to $\tau$-tilting theory in Section 2, we introduce the definition of c-vectors for finite-dimensional algebras in Section 3. We show that each c-vector can be realized as the (negative) dimension vector of certain indecomposable module and establish the sigh-coherence property for c-vectors. Moreover, the relationship between positive c-vectors and negative c-vectors are also given. Section 4.1 is devoted
to study the c-vectors for quasitilted algebras. We show that the set \( cv^+(A) \) of positive c-vectors for a quasitilted algebra \( A \) coincides with the set \( \text{exdv}(A) \) of dimension vectors of exceptional \( A \)-modules. This generalizes the equalities for acyclic algebras established by Chávez [11]. These equalities also implies that an indecomposable \( A \)-module \( M \) of the quasitilted algebra \( A \) can be completed to a 2-term simple-minded collection of \( A \) if and only if \( M \) is an exceptional module (we refer to [6] for the definition of 2-term simple-minded collections). In Section 4.2 and 4.3, we also establish the equalities between the set \( cv^+(A) \) of positive c-vectors and the set \( \text{exdv}(A) \) of dimension vectors of exceptional modules for representation-directed algebras and cluster-tilted algebras of finite type respectively. A short proof for the sign-coherence of c-vectors for skew-symmetric cluster algebras has been given in Appendix A.

Notations. Throughout this paper, \( k \) denotes an algebraically closed field, \( A \) a finite-dimensional basic \( k \)-algebra. All modules are right modules. Let \( C \) be a category over \( k \), for an object \( M \in C \), denote by \( \text{add} \ M \) the full subcategory of \( C \) whose objects are direct summands of finite direct sum of \( M \).

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2. Recollection

In this section, we recall some definitions and basic properties of (support) \( \tau \)-tilting modules and (2-term) silting objects. We mainly follow [3, 11, 20].

2.1. Support \( \tau \)-tilting modules. Let \( A \) be a finite-dimensional algebra over \( k \) and \( \text{mod} \ A \) the category of finitely generated right \( A \)-modules. Let \( S_1, \cdots, S_n \) be all the pairwise non-isomorphic simple \( A \)-modules and \( P_1, \cdots, P_n \) the corresponding projective covers of \( S_1, \cdots, S_n \) respectively. Denote by \( \tau \) the Auslander-Reiten translation of \( \text{mod} \ A \).

An \( A \)-module \( M \) is called rigid if \( \text{Ext}^1_A(M, M) = 0 \). A module \( M \in \text{mod} \ A \) is called \( \tau \)-rigid provided \( \text{Hom}_A(M, \tau M) = 0 \). Let \( P_1^M \xrightarrow{f} P_0^M \rightarrow M \rightarrow 0 \) be a minimal projective resolution of \( M \), then \( M \) is \( \tau \)-rigid if and only if \( \text{Hom}_A(f, M) \) is surjective. Note that \( \tau \)-rigid implies rigid, but the converse is not true in general.

A \( \tau \)-rigid pair is \( (M, P) \) with \( M \in \text{mod} \ A \) and \( P \) a finitely generated \( A \)-modules, such that \( M \) is \( \tau \)-rigid and \( \text{Hom}_A(P, M) = 0 \). A \( \tau \)-rigid pair is called support \( \tau \)-tilting pair if \( |M| + |P| = n \), where \( |X| \) denotes the number of non-isomorphic
indecomposable direct summands of $X$. In this case, $M$ is a support $\tau$-tilting $A$-
module and $P$ is uniquely determined by $M$.

Recall that a full subcategory $\mathcal{T}$ of $\text{mod} \, A$ is a torsion class of $\text{mod} \, A$ provided that $\mathcal{T}$ is closed under quotients and extensions. An object $X \in \mathcal{T}$ is Ext-projective if $\text{Ext}^1_A(X, \mathcal{T}) = 0$. A torsion pair $(\mathcal{T}, \mathcal{F})$ is uniquely determined by its torsion class $\mathcal{T}$ in the sense that $\mathcal{F} = \perp \mathcal{T} := \{ N \in \text{mod} \, A| \text{Hom}_A(M, N) = 0 \text{ for all } M \in \mathcal{T} \}$. A torsion pair $(\mathcal{T}, \mathcal{F})$ is functorially finite provided that $\mathcal{T}$ is functorially finite, equivalently, there is an object $X \in \text{mod} \, A$ such that $\mathcal{T} = \text{Fac} \, X$, where $\text{Fac} \, X$ is the subcategory of $\text{mod} \, A$ formed by quotients of finite direct sum of $X$. Let $P(\mathcal{T})$ be the direct sum of one copy of each of the indecomposable Ext-projective objects in $\mathcal{T}$ up to isomorphism. It is well-known that $\mathcal{T} = \text{Fac} \, P(\mathcal{T})$. The following result due to [2] will be used frequently (cf. Theorem 5.10 in [2]).

**Proposition 2.1.** Let $A$ be a finite-dimensional algebra over $k$. If $M$ is a $\tau$-rigid $A$-module, then $\text{Fac} \, M$ is a functorially finite torsion class and $M \in \text{Fac} \, M$ is Ext-projective.

Let $f$-tors $A$ be the set of isomorphism classes of functorially finite torsion classes of $\text{mod} \, A$ and $s\tau$-tilt $A$ the set of isomorphism classes of basic support $\tau$-tilting $A$ modules. It has been shown [3] that the support $\tau$-tilting $A$-modules are closely related to the functorially finite torsion classes of $\text{mod} \, A$. Namely, we have the following bijection (cf. Theorem 2.7 of [3]).

**Theorem 2.2.** Let $A$ be a finite-dimensional algebra over $k$. There is a bijection between $s\tau$-tilt $A$ and $f$-tors $A$ given by

$$M \in s\tau \text{-tilt } A \leftrightarrow \text{Fac} \, M \in f \text{-tors } A,$$

and its inverse is given by $\mathcal{T} \leftrightarrow P(\mathcal{T})$, where $\mathcal{T} \in f$-tors $A$.

2.2. **Silting objects.** Let $A$ be a finite-dimensional algebra over $k$ and $D^b(\text{mod} \, A)$ the bounded derived category of finitely generated right $A$-modules with suspension functor $\Sigma$. Recall that $n$ is the number of pairwise non-isomorphic simple $A$-modules. Let per $A$ be the perfect derived category of $A$, that is the smallest thick subcategory of $D^b(\text{mod} \, A)$ containing the object $A$. An object $Q \in \text{per} \, A$ is called presilting if $\text{Hom}_{\text{per} \, A}(Q, \Sigma^i Q) = 0$ for all $i > 0$. A presilting object $Q \in \text{per} \, A$ is called a silting object provided moreover $\text{thick}(Q) = \text{per} \, A$, where $\text{thick}(Q)$ is the smallest thick subcategory of $\text{per} \, A$ containing $Q$. It has been proved in [4] that each basic silting object has exactly $n$ indecomposable direct summands. A presilting object $P$ is called almost silting if the number of non-isomorphic indecomposable direct
summands of $P$ is $n - 1$. If there is an indecomposable object $X \in \text{per } A$ such that $P \oplus X$ is a silting object, then $X$ is called a complement of $P$. In general, an almost presilting object may have infinite complements.

Let $Q = X \oplus P$ be a basic silting object with $X$ indecomposable. Consider the triangle

$$X \xrightarrow{f} Q \rightarrow Y \rightarrow \Sigma X,$$

where $f$ is a minimal left $\text{add } P$-approximation of $X$. It has been shown in [4] that $Y \oplus P$ is a basic silting object and called the left mutation of $Q$ with respect to $X$. Dually, if we consider the triangle induced by a minimal right $\text{add } P$-approximation of $X$, we obtain the right mutation of $Q$ with respect to $X$.

A silting object $Q \in \text{per } A$ is 2-term silting if there is a triangle $P_1^Q \rightarrow P_0^Q \rightarrow Q \rightarrow \Sigma P_1^Q$, where $P_0^Q, P_1^Q \in \text{add } A$.

Denote by $\text{2-silt } A$ the set of isomorphism classes of 2-term silting objects of $\text{per } A$. The following has been established in [3].

**Theorem 2.3.** Let $A$ be a finite-dimensional algebra over $k$.

1. Let $P$ be an almost 2-term silting object in $\text{per } A$, there exists exactly two indecomposable objects $X, Y$ such that $P \oplus X$ and $P \oplus Y$ are 2-term silting objects in $\text{per } A$; Moreover, $P \oplus X$ and $P \oplus Y$ are related by a left or right mutation;
2. There is a bijection between $\tau$-tilt $A$ and $\text{2-silt } A$ given by

$$M \in \tau\text{-tilt } A \mapsto (P_1^M \oplus P \xrightarrow{(f,0)} P_0^M) \in \text{2-silt } A,$$

where $P_1^M \xrightarrow{f} P_0^M \rightarrow M$ is a minimal projective resolution of $M$ and $(M, P)$ is the support $\tau$-tilting pair.

### 2.3. t-structures on triangulated categories.

Let $\mathcal{D}$ be a triangulated category over $k$ with suspension functor $\Sigma$. A pair of full subcategory $(\mathcal{U}^{\leq 0}, \mathcal{V}^{\geq 0})$ of $\mathcal{D}$ is called a $t$-structure on $\mathcal{D}$ provided that

1. $\Sigma \mathcal{U}^{\leq 0} \subseteq \mathcal{U}^{\leq 0}$;
2. $\text{Hom}_\mathcal{D}(\mathcal{U}^{\leq 0}, \Sigma^{-1} \mathcal{V}^{\geq 0}) = 0$;
3. for each $X \in \mathcal{D}$, there is a triangle $U_X \rightarrow X \rightarrow V_X \rightarrow \Sigma U_X$ with $U_X \in \mathcal{U}^{\leq 0}$ and $U_X \in \Sigma^{-1} \mathcal{V}^{\geq 0}$.

A bounded $t$-structure on $\mathcal{D}$ is a $t$-structure $(\mathcal{U}^{\leq 0}, \mathcal{V}^{\geq 0})$ such that

$$\mathcal{D} = \bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{U}^{\leq 0} = \bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{V}^{\geq 0}.$$
For a given t-structure \((\mathcal{U}^{\leq 0}, \mathcal{V}^{\geq 0})\) on \(\mathcal{D}\), the subcategory \(\mathcal{A} = \mathcal{U}^{\leq 0} \cap \mathcal{V}^{\geq 0}\) of \(\mathcal{D}\) is called the heart of \((\mathcal{U}^{\leq 0}, \mathcal{V}^{\geq 0})\), which is an abelian category with the exact structure induced by the triangles of \(\mathcal{D}\). Moreover, for any \(X, Y \in \mathcal{A}\), we have \(\text{Hom}_\mathcal{A}(X, Y) = \text{Hom}_\mathcal{D}(X, Y)\) and \(\text{Ext}^1_\mathcal{A}(X, Y) = \text{Hom}_\mathcal{D}(X, \Sigma Y)\). The triangles in (3) are canonical and yield endofunctors \(\tau_{\leq 0}\) and \(\tau_{\geq 1}\) of \(\mathcal{D}\) such that \(\tau_{\leq 0}X = U_X\) and \(\tau_{\geq 1}X = V_X\). The functors \(\tau_{\leq 0}\) and \(\tau_{\geq 1}\) yield a family of cohomological functors \(H^i = \tau_{\leq i} \circ \tau_{\geq i} : \mathcal{D} \to \mathcal{A}\), where \(\tau_{\leq i} = \Sigma^{-i} \circ \tau_{\leq 0} \circ \Sigma^i\) and \(\tau_{\geq i} = \Sigma^{-i+1} \circ \tau_{\geq 1} \circ \Sigma^{-i}\).

Moreover, for each \(X \in \mathcal{D}\), we have a family of triangles
\[
\tau_{\leq i}X \to X \to \tau_{\geq i+1} \to \Sigma \tau_{\leq i}X, \quad \text{where } i \in \mathbb{Z}.
\]

The following is a consequence of the bijection between silting objects and bounded t-structures studied in [26].

**Lemma 2.4.** Let \(A\) be a finite-dimensional \(k\)-algebra and \(\mathcal{D}^b(\text{mod } A)\) the bounded derived category of finitely generated right \(A\)-modules. Let \((\mathcal{U}^{\leq 0}, \mathcal{V}^{\geq 0})\) be a bounded t-structure with length heart on \(\mathcal{D}^b(\text{mod } A)\) and \(\mathcal{A}\) the heart associated to \((\mathcal{U}^{\leq 0}, \mathcal{V}^{\geq 0})\). Then there is a silting object \(T \in \text{per } \mathcal{A}\) such that \(\text{mod } \text{End}_A(T) \cong \mathcal{A}\).

### 2.4. Negative dg algebra associated to a silting object.

Recall that \(A\) is finite-dimensional \(k\)-algebra. Let \(T = \bigoplus_{i=1}^n \in \text{per } \mathcal{A}\) be a silting object with indecomposable direct summands \(T_1, \ldots, T_n\) and \(\bar{T} = \text{RHom}_A(T, T)\) the dg endomorphism algebra of \(T\). By the definition of silting object, we know that the homology groups \(H^i(\bar{T})\) vanish for all \(i > 0\). Denote by \(\Gamma = \tau_{\leq 0}\bar{T}\) the truncation algebra of \(\bar{T}\). Let \(i : \Gamma \to \bar{T}\) be the canonical injective homomorphism of dg algebras. It is clear that \(i\) induces an equivalence of derived categories \(\mathcal{D}(\Gamma) \cong \mathcal{D}(\bar{T})\). On the other hand, we also have the surjective homomorphism \(\pi : \Gamma \to H^0(\Gamma) \cong \text{End}_A(T)\) of dg algebras, where \(\text{End}_A(T)\) is the endomorphism algebra of \(T\) in \(\text{per } \mathcal{A}\).

Let \(e_i = 1_{T_i} \in \text{Hom}_A(T_i, T_i), 1 \leq i \leq n\), be the primitive orthogonal idempotents in \(\text{End}_A(T)\), which will induce a decomposition of the identity of \(\Gamma\) into a sum of primitive idempotents. By abuse of notations, we still denote the corresponding primitive idempotents by \(e_1, \ldots, e_n\). Thus we have the decomposition of \(\Gamma = \bigoplus_{i=1}^n e_i \Gamma\) into indecomposable right \(\Gamma\)-modules. Moreover, the images \([e_1 \Gamma], \ldots, [e_n \Gamma]\) form a \(\mathbb{Z}\)-basis of the Grothendieck group \(G_0(\text{per } \Gamma)\) for the perfect derived category of \(\Gamma\). Let \(S_1^T, \ldots, S_n^T\) be pairwise non-isomorphic simple right \(\text{End}_A(T)\)-modules. Via the homomorphism \(\pi\), each simple \(\text{End}_A(T)\)-module \(S_i^T\) lifts to a simple dg \(\Gamma\)-module \(S_i^T\). Let \(\mathcal{D}_{fd}(\Gamma)\) be the finite-dimensional derived category of \(\Gamma\), that is the full triangulated subcategory of \(\mathcal{D}(\Gamma)\) formed by the dg \(\Gamma\)-modules whose homology has finite total dimension over \(k\). Similarly, the images \([S_1^T], \ldots, [S_n^T]\) form a \(\mathbb{Z}\)-basis of
the Grothendieck group $G_0(D_{fd}(\Gamma))$ of the finite-dimensional derived category $D_{fd}(\Gamma)$ of $\Gamma$. Let $\langle - , - \rangle_{\Gamma} : G_0(\text{per}\ \Gamma) \times G_0(D_{fd}(\Gamma)) \rightarrow k$ be the non-degenerate Euler bilinear form given by

$$\langle [P], [X] \rangle_{\Gamma} = \sum_{i \in \mathbb{Z}} \dim_k \text{Hom}_{\Gamma}(P, \Sigma^i X),$$

where $P \in \text{per}\ \Gamma$ and $X \in D_{fd}(\Gamma)$. For any $X \in D(\Gamma), t \in \mathbb{Z},$ we clearly have

$$\text{Hom}_{\Gamma}(e_t \Gamma, \Sigma^t X) = \begin{cases} k & t = 0; \\ 0 & \text{otherwise.} \end{cases}$$

if and only if $X \cong S^r_1$ in $D(\Gamma)$. In other words, $[e_1 \Gamma], \cdots, [e_n \Gamma]$ and $[S^r_1], \cdots, [S^n_1]$ are dual bases with respect to the Euler bilinear form $\langle - , - \rangle_{\Gamma}$. 

Recall that $T$ is a silting object in $\text{per} A$, we have an equivalence $D(\Gamma) \cong D(\text{Mod } A)$ and hence an equivalence between $D(\Gamma)$ and $D(\text{Mod } A)$. Indeed, view $T$ as $\Gamma^{\text{op}} \otimes_k A$-module, the equivalence is given by $F := \otimes_k T : D(\Gamma) \rightarrow D(\text{Mod } A)$, which restricts to equivalences $\text{per } \Gamma \cong \text{per } A$ and $D_{fd}(\Gamma) \cong D^b(\text{mod } A)$ respectively.

Since $\Gamma$ is a finite-dimensional negative dg algebra, there is a standard t-structure $(D^{\leq 0}, D^{\geq 0})$ on $D(\Gamma)$ induced by the homology. More precisely,

$$D^{\leq 0} = \{X \in D(\Gamma) | \text{Hom}_{\Gamma}(\Gamma, \Sigma^i X) = 0 \text{ for all } i > 0\},$$

$$D^{\geq 0} = \{X \in D(\Gamma) | \text{Hom}_{\Gamma}(\Gamma, \Sigma^i X) = 0 \text{ for all } i < 0\}.$$

This t-structure restricts to the standard t-structure $(D^{\leq 0}_{fd}, D^{\geq 0}_{fd})$ on $D_{fd}(\Gamma)$, where $D^{\leq 0}_{fd} = D^{\leq 0} \cap D_{fd}(\Gamma)$ and $D^{\geq 0}_{fd} = D^{\geq 0} \cap D_{fd}(\Gamma)$. Moreover, the heart $D^{\leq 0}_{fd} \cap D^{\geq 0}_{fd}$ is equivalent to $\text{mod } \text{End}_A(T)$. The standard t-structure $(D^{\leq 0}_{fd}, D^{\geq 0}_{fd})$ induces a t-structure on $D^b(\text{mod } A)$ via the functor $F$. Denote by $(D^{\leq 0}_T, D^{\geq 0}_T)$ the resulting t-structure, that is

$$D^{\leq 0}_T = \{X \in D^b(\text{mod } A) | \text{Hom}_A(T, \Sigma^i X) = 0 \text{ for all } i > 0\},$$

$$D^{\geq 0}_T = \{X \in D^b(\text{mod } A) | \text{Hom}_A(T, \Sigma^i X) = 0 \text{ for all } i < 0\}.$$

Let $A = D^{\leq 0}_T \cap D^{\geq 0}_T$ be the heart of the t-structure $(D^{\leq 0}_T, D^{\geq 0}_T)$. It is clear that $F(S^r_1), \cdots, F(S^n_1)$ are all the simple objects of $A$. If $T$ is a 2-term silting object, by Theoerem 2.2 and 2.3, there is a support $\tau$-tilting module and a functorially finite torsion class corresponding to $T$. We have the following characterization of $A$ by torsion pair, which is a consequence of the bijections investigated in [26] (cf. also [5]), for completeness and later use, we include a proof.
Proposition 2.5. Keep the notations above. Assume that $T$ is a 2-term silting object and $M \in \text{mod} \ A$ is the associated support $\tau$-tilting $A$-module. Let $\mathcal{T}_M = \text{Fac} M$ be the functorially finite torsion class associated to $M$ and $\mathcal{F}_M = ^\perp \mathcal{T}_M$ the torsion free class. Then $(\Sigma \mathcal{F}_M, \mathcal{T}_M)$ is a torsion pair of $\mathcal{A}$. As a consequence, each simple object of $\mathcal{A}$ lies either in $\Sigma \mathcal{F}_M$ or in $\mathcal{T}_M$.

Proof. Let $P_1^M \xrightarrow{f} P_0^M \rightarrow M \rightarrow 0$ be a minimal projective resolution of $M$ and $(M, Q)$ the associated basic support $\tau$-tilting pair. We have $T = \cdots \rightarrow 0 \rightarrow P_i^M \oplus Q \xrightarrow{(f, 0)} P_0^M \rightarrow 0 \cdots$, where $P_0^M$ is in the zeroth component. Note that $X \in \mathcal{A}$ if and only if $\text{Hom}_A(T, \Sigma^i X) = 0$ for $i \neq 0$. A direct calculation shows that $\Sigma \mathcal{F}_M \subset \mathcal{A}$ and $\mathcal{T}_M \subset \mathcal{A}$. On the other hand, we clearly have $\text{Hom}_A(\Sigma \mathcal{F}_M, \mathcal{T}_M) = 0$. Note that the exact sequences of $\mathcal{A}$ are induced from the triangles of $\mathcal{D}^b(\text{mod} \ A)$. Thus to prove $(\Sigma \mathcal{F}_M, \mathcal{T}_M)$ is a torsion pair of $\mathcal{A}$, it remains to show that for each $X \in \mathcal{A}$, there is a triangle $\Sigma F_0 \rightarrow X \rightarrow T_0 \rightarrow \Sigma^2 F_0$ in $\mathcal{D}^b(\text{mod} \ A)$ with $T_0 \in \mathcal{T}_M$ and $F_0 \in \mathcal{F}_M$.

We claim that if $X \in \mathcal{A} \subset \mathcal{D}^b(\text{mod} \ A)$, then $H^i(X) = 0$ for $i \neq 0, -1$. Let $(C_{\leq 0}, C_{\geq 0})$ be the standard t-structure on $\mathcal{D}^b(\text{mod} \ A)$. For any $X \in \mathcal{A} \subset \mathcal{D}^b(\text{mod} \ A)$, consider the following triangle induced by the standard t-structure $(C_{\leq 0}, C_{\geq 0})$

$$\tau_{\leq -2} X \rightarrow X \rightarrow \tau_{\geq -1} X \rightarrow \Sigma \tau_{\leq -2} X.$$ 

Applying the functor $\text{Hom}_A(T, ?)$ yields the long exact sequence

$$\cdots \rightarrow \text{Hom}_A(T, \Sigma^i \tau_{\leq -2} X) \rightarrow \text{Hom}_A(T, \Sigma^i X) \rightarrow \text{Hom}_A(T, \Sigma^i \tau_{\geq -1} X) \rightarrow \text{Hom}_A(T, \Sigma^i \tau_{\leq -2} X) \rightarrow \cdots.$$ 

We have $\text{Hom}_A(T, \Sigma^i \tau_{\leq -2} X) = 0$ for all $i$ since $\text{Hom}_A(T, \Sigma^i X) = 0$ for all $i \neq 0$.

Recall that we have $\text{thick}(T) = \text{per} \ A$, which implies that $\tau_{\leq -2} X = 0$ in $\mathcal{D}^b(\text{mod} \ A)$.

As a consequence, $X \cong \tau_{\geq -1} X \in \Sigma C_{\geq 0}$. Now consider the triangle

$$\tau_{\leq 0} X \rightarrow X \rightarrow \tau_{\geq 1} X \rightarrow \Sigma \tau_{\leq 0} X,$$

applying the functor $\text{Hom}_A(T, ?)$ to the triangle yields a long exact sequence

$$\cdots \rightarrow \text{Hom}_A(T, \Sigma^i \tau_{\leq 0} X) \rightarrow \text{Hom}_A(T, \Sigma^i X) \rightarrow \text{Hom}_A(T, \Sigma \tau_{\geq 1} X) \rightarrow \text{Hom}_A(T, \Sigma^i \tau_{\leq 0} X) \rightarrow \cdots.$$ 

Again one can show that $\text{Hom}_A(T, \Sigma^i \tau_{\geq 1} X) = 0$ for all $i$, and hence $\tau_{\geq 1} X = 0$ in $\mathcal{D}^b(\text{mod} \ A)$. In particular, we have proved that $X \cong \tau_{\leq 0} \circ \tau_{\geq -1} X \in C_{\leq 0} \cap \Sigma C_{\geq 0}$, which implies that $H^i(X) = 0$ for $i \neq 0$ or $-1$.

By the standard t-structure $(C_{\leq 0}, C_{\geq 0})$, for each $X \in \mathcal{A}$, we have the following triangle in $\mathcal{D}^b(\text{mod} \ A)$

$$\Sigma^{-1} H^{-1}(X) \rightarrow X \rightarrow H^0(X) \rightarrow \Sigma^2 H^1(X).$$
It remains to show that $H^0(X) \in \mathcal{T}_M$ and $H^{-1}(X) \in \mathcal{F}_M$ for $X \in \mathcal{A}$. It is easy to see that $\text{Hom}_A(T, \Sigma^i H^0(X)) = 0$ for all $i \neq 0$ and $\text{Hom}_A(T, \Sigma^i H^{-1}(X)) = 0$ for all $i \neq 1$. Consider the short exact sequence $0 \to T_{H^0(X)} \to H^0(X) \to F_{H^0(X)} \to 0$ in $\text{mod} \ A$ with $T_{H^0(X)} \in \mathcal{T}_M$ and $F^0_H(X) \in \mathcal{F}_M$. Applying $\text{Hom}_A(T, ?)$ to the exact sequence, one can show that $\text{Hom}_A(T, \Sigma F_{H^0(X)}) = 0$. Recall that we also have $\text{Hom}_A(T, \Sigma^i F_{H^0(X)}) = 0$ for all $i \neq 1$. Consequently, $F_{H^0(X)} = 0$ in $\mathcal{D}^b(\text{mod} \ A)$. In particular, we have $T_{H^0(X)} \cong H^0(X) \in \mathcal{T}_M$. Similarly, one can show that $H^{-1}(X) \in \mathcal{F}_M$. This finishes the proof. \hfill \square

3. C-vectors and its sign-coherence

3.1. Definition of c-vectors. Recall that $A$ is a finite-dimensional algebra over $k$ and $n$ is the number of non-isomorphic simple $A$-modules. Let $\mathcal{G}_0^{sp}(\text{add} \ A)$ be the split Grothendieck group of finitely generated projective $A$-modules. For a given $\tau$-rigid $A$-module $M$, let $P_i^M \xrightarrow{\ell} P_0^M \to M \to 0$ be a minimal projective resolution of $M$, the index of $M$ is defined to be $\text{ind}(M) = [P_0^M] - [P_1^M] \in \mathcal{G}_0^{sp}(\text{add} \ A)$. The g-vector of $M$ is $g(M) = (g_1, \cdots, g_n) \in \mathbb{Z}^n$, where $g_i = [\text{ind}(M) : P_i], 1 \leq i \leq n$. It has been proved in [3] that different $\tau$-rigid $A$-modules have different indices and hence different $g$-vectors.

For a given basic support $\tau$-tilting pair $(M, P)$ with decomposition of indecomposable modules $M = \bigoplus_{i=1}^n M_i, \ P = \bigoplus_{i=t+1}^n P_i^M$, we have the following $G$-matrix of $(M, P)$

$$G_{(M,P)} = (g(M_1), g(M_2), \cdots, g(M_t), -g(P_{t+1}^M), \cdots, -g(P_n^M)) \in M_n(\mathbb{Z}).$$

We know from [3] that for any basic support $\tau$-tilting pair $(M, P)$ the $G$-matrix $G_{(M,P)}$ is invertible over $\mathbb{Z}$. Inspired by the tropical duality between $g$-vectors and $c$-vectors in cluster algebras, we introduce the $C$-matrix of a basic support $\tau$-rigid pair $(M, P)$ to be the inverse of the transpose of the $G$-matrix $G_{(M,P)}$, i.e.

$$C_{(M,P)} := (G^T_{(M,P)})^{-1} \in M_n(\mathbb{Z}).$$

Each column vector of $C_{(M,P)}$ is called a c-vector of $A$ and denote by $cv(A)$ the set of all the $c$-vectors of $A$.

**Remark 3.1.** Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category with a cluster-tilting object $T$ which admits a cluster structure in the sense of [7]. Let $A = \text{End}_\mathcal{C}(T)$ be the endomorphism algebra of $T$. Let $cv_0(A)$ be the subset of $cv(A)$ formed by the c-vectors associated to support $\tau$-tilting $A$-modules which are connected with the support
By Proposition 6.2 of [10], we deduce that \( cv_0(A) \) coincides with the \( c \)-vectors of the cluster algebra associated to \( A \).

3.2. Sign-coherence of \( c \)-vectors. A vector \( c \) in \( \mathbb{Z}^n \) is called sign-coherence if \( c \) has either all entries nonnegative or all entries nonpositive. A non-zero vector in \( \mathbb{Z}^n \) is positive (resp. negative) if all components are nonnegative (resp. nonpositive). The sign-coherence phenomenon holds for this general setting.

**Theorem 3.2.** Let \( A \) be a finite-dimensional algebra over \( k \). Then each \( c \)-vector of \( A \) is sign-coherence.

**Proof.** Let \( S_1, \cdots, S_n \) be all the pairwise non-isomorphic simple \( A \)-modules and \( P_1, \cdots, P_n \) the corresponding projective covers of \( S_1, \cdots, S_n \) respectively. Let \( G_0(\text{per} \ A) \) and \( G_0(\mathcal{D}^b(\text{mod} \ A)) \) be the Grothendieck groups of \( \text{per} \ A \) and \( \mathcal{D}^b(\text{mod} \ A) \) respectively. Denote by \( \langle -,- \rangle_A : G_0(\text{per} \ A) \times G_0(\mathcal{D}^b(\text{mod} \ A)) \to k \) the Euler bilinear form given by \( \langle [P],[X] \rangle_A = \sum_{i \in \mathbb{Z}} \dim_k \text{Hom}_A(P, X^i) \) for any \( P \in \text{per} \ A \) and \( X \in \mathcal{D}^b(\text{mod} \ A) \).

It is clear that \([P_1],\cdots,[P_n]\) and \([S_1],\cdots,[S_n]\) are dual bases with respect to the Euler bilinear form \( \langle -,- \rangle_A \).

Let \((M,Q)\) be a basic support \( \tau \)-tilting pair of \( A \) and \( T \) the corresponding 2-term silting object in \( \text{per} \ A \). Let \( \Gamma = \bigoplus_{i=1}^n e_i \Gamma \) be the negative truncated dg algebra associated to \( T \) (cf. Section 2.4). Recall that we also have the Euler bilinear form \( \langle -,- \rangle_\Gamma : G_0(\text{per} \ \Gamma) \times G_0(\mathcal{D}^b(\text{mod} \ \Gamma)) \to k \) and there is an equivalence of triangulated categories \( F = \bigotimes \Gamma T_A : \mathcal{D}^b(\text{mod} \ A) \to \mathcal{D}^b(\text{mod} \ A) \). It is clear that the functor \( F \) induces an isomorphism of bilinear forms such that the following diagram is commutative

\[
\begin{array}{ccc}
\langle -,- \rangle_\Gamma : G_0(\text{per} \ \Gamma) \times G_0(\mathcal{D}^b(\text{mod} \ \Gamma)) & \to & k \\
\downarrow F & & \\
\langle -,- \rangle_A : G_0(\text{per} \ A) \times G_0(\mathcal{D}^b(\text{mod} \ A)) & \to & k.
\end{array}
\]

Note that the column vectors the \( G \)-matrix associated to \((M,Q)\) is the dimension vector of \( F(e_i \Gamma) \) in \( G_0(\text{per} \ A) \) with respect to the basis \([P_1],\cdots,[P_n]\). By the duality between \( G \)-matrix and \( C \)-matrix, we deduce that the \( c \)-vectors associated to \((M,Q)\) are the dimension vectors of \( F(S_i^\Gamma) \) for all the simple \( d \Gamma \)-module \( S_i^\Gamma \). Now the result follows from Proposition 2.5. \( \square \)

As a byproduct of the proof, we have the following criterion of \( c \)-vectors.

**Corollary 3.3.** Let \( A \) be a finite-dimensional algebra over \( k \). A vector \( c \in \mathbb{Z}^n \) is a \( c \)-vector of \( A \) if and only if there is a 2-term silting object \( T \in \text{per} \ A \) and an indecomposable \( A \)-module \( M \) satisfying one of the following conditions:
(1) \( \text{Hom}_A(T, \Sigma^i M) = \begin{cases} k & i = 0; \\ 0 & \text{otherwise.} \end{cases} \), i.e. \( c = \dim M \) is a positive c-vector.

(2) \( \text{Hom}_A(T, \Sigma^i M) = \begin{cases} k & i = 1; \\ 0 & \text{otherwise.} \end{cases} \), i.e. \( c = -\dim M \) is a negative c-vector.

3.3. Positive c-vectors and negative c-vectors. Let \( \text{cv}^+(A) \) be the set of positive c-vectors of \( A \) and \( \text{cv}^-(A) \) the set of negative c-vectors. The following result is a consequence of the bijection between 2-term silting objects and 2-term simple-minded collections investigated in [6]. In order to avoid more notations, we give a proof using the mutation of silting objects.

**Theorem 3.4.** Let \( A \) be a finite-dimensional algebra over \( k \). We have \( \text{cv}^-(A) = -\text{cv}^+(A) \). In particular, \( \text{cv}(A) = -\text{cv}^+(A) \cup \text{cv}^+(A) \).

**Proof.** We show the inclusion \( -\text{cv}^+(A) \subseteq \text{cv}^-(A) \). The inverse inclusion is similar. Let \( c \) be an arbitrary positive c-vector of \( A \). Then there is a 2-term silting object, say \( T \in \text{per} A \) and an indecomposable \( A \)-module \( M \) such that \( \text{Hom}_A(T, M) = k \) and \( \text{Hom}_A(T, \Sigma^i M) = 0 \) for all \( i \neq 0 \). We may rewrite \( T \) as \( T = T_M \oplus Q \) with \( T_M \) indecomposable such that \( \text{Hom}_A(T_M, M) = k, \text{Hom}_A(T_M, \Sigma^i M) = 0 \) for \( i \neq 0 \) and \( \text{Hom}_A(Q, \Sigma^i M) = 0 \) for all \( i \in \mathbb{Z} \). It is known that there is an indecomposable 2-term presilting, say \( T_N \), such that \( T' = T_N \oplus Q \) is a basic 2-term silting object in \( \text{per} A \). By (1) of Theorem 2.3 we know that \( T \) and \( T' \) are related by a left or right mutation. We claim that \( T' \) is the left mutation of \( T \). Otherwise, \( T \) is the left mutation of \( T' \) and we have the triangle \( T_N \to Q' \to T_M \to \Sigma T_N \) with \( Q' \in \text{add} Q \). Applying the functor \( \text{Hom}_A(?, M) \), we have a long exact sequence

\[
\cdots \text{Hom}_A(T_M, \Sigma^i M) \to \text{Hom}_A(Q', \Sigma^i M) \to \text{Hom}_A(T_N, \Sigma^i M) \to \text{Hom}_A(T_M, \Sigma^{i+1} M) \cdots,
\]

which implies that \( \text{Hom}_A(T', \Sigma^{-1} M) = k \) and \( \text{Hom}_A(T', \Sigma^i M) = 0 \) for all \( i \neq -1 \). Let \( \Gamma_{T'} \) be the negative dg algebra associated to \( T' \). The conditions \( \text{Hom}_A(T', \Sigma^{-1} M) = k \) and \( \text{Hom}_A(T', \Sigma^i M) = 0 \) for all \( i \neq -1 \) imply that \( R\text{Hom}_A(T', \Sigma^{-1} M) \) is a simple \( \Gamma_{T'} \)-module, which contradicts to Proposition 2.5.

Therefore \( T' \) has to be the left mutation of \( T \) and there is a triangle

\[
T_M \to Q_1 \to T_N \to \Sigma T_M, \text{where } Q_1 \in \text{add} Q.
\]

Applying the functor \( \text{Hom}_A(?, M) \), we obtain a long exact sequence

\[
\cdots \text{Hom}_A(T_N, \Sigma^i M) \to \text{Hom}_A(Q_1, \Sigma^i M) \to \text{Hom}_A(T_M, \Sigma^i M) \to \text{Hom}_A(T_N, \Sigma^{i+1} M) \cdots.
\]
We have \( \text{Hom}_A(T_N, \Sigma M) = k \) and \( \text{Hom}_A(T^i_N, \Sigma^i M) = 0 \) for all \( i \neq 1 \). As a consequence, \( \text{Hom}_A(T', \Sigma M) = k \) and \( \text{Hom}_A(T', \Sigma^i M) = 0 \) for all \( i \neq 1 \). In particular, \( -\dim M \) is a negative c-vector by Corollary 3.3 and and we have \( -\text{cv}^+(A) \subseteq \text{cv}^-(A) \). □

3.4. The left-right symmetry of c-vectors. Let \( A^{\text{op}} \) be the opposite \( k \)-algebra of \( A \). We have the dualities

\[
D = \text{Hom}_k(?, k) : \text{mod} A \to \text{mod} A^{\text{op}} \quad \text{and} \quad (-)^* = \text{Hom}_A(?, A) : \text{add} A \to \text{add} A^{\text{op}}.
\]

For any \( X \in \text{mod} A \), let

\[
P_1^d \xrightarrow{d} P_0 \to X \to 0
\]

be a minimal projective resolution of \( X \), its transpose \( \text{Tr} X \in \text{mod} A^{\text{op}} \) is defined by the following exact sequence

\[
P_0^* \xrightarrow{d^*} P_1^* \to \text{Tr} X \to 0.
\]

For any \( M \in \text{mod} A \), we can decompose \( M \) as \( M = M_{pr} \oplus M_{np} \), where \( M_{pr} \) is a maximal projective direct summand of \( M \). The following left-right symmetry of \( \tau \)-rigid modules has been established in [3] (cf. Theorem 2.14 of [3]).

**Theorem 3.5.** Let \( A \) be a finite-dimensional \( k \)-algebra. There is a bijection \( (-)^\circ \) between \( \tau \)-tilt \( A \) and \( \tau \)-tilt \( A^{\text{op}} \) given by \( (M, Q)\circ = (\text{Tr} M_{np} \oplus Q^*, M_{pr}^*) \), where \( (M, Q) \) is a support \( \tau \)-tilting pair of \( A \). Moreover, \( (-)^{\circ \circ} = \text{id} \).

Let \( M \) be an indecomposable non-projective \( \tau \)-rigid \( A \)-modules. By Theorem 3.5 we infer that \( \text{Tr} M \) is also \( \tau \)-rigid as \( A^{\text{op}} \)-module. Moreover, we clearly have \( g(M) = -g(\text{Tr} M) \). On the other hand, for any indecomposable projective \( A \)-module \( P \), we also have \( g(P) = -g(P^*) \). Now the following result is an immediate consequence of the definition of c-vectors and Theorem 3.5, Theorem 3.2 and Theorem 3.4.

**Proposition 3.6.** Let \( A \) be a finite-dimensional \( k \)-algebra and \( A^{\text{op}} \) its opposite algebra. Then we have \( \text{cv}(A) = -\text{cv}(A^{\text{op}}) \) and \( \text{cv}^+(A) = \text{cv}^+(A^{\text{op}}) \).

4. C-VECTORS AND DIMENSION VECTORS

For an algebra \( A \), let \( \text{dv}(A) \) be the set of dimension vectors of indecomposable \( A \)-modules. By Corollary 3.3 we know that each positive c-vector can be realized as the dimension vector of an indecomposable \( A \)-module, that is, \( \text{cv}^+(A) \subseteq \text{dv}(A) \). However, the inverse inclusion is not true in general. The aim of this section is to study the positive c-vectors for quasitilted algebras, representation-directed algebras and cluster-tilted algebras of finite type.
4.1. c-vectors of quasitilted algebras.

4.1.1. Hereditary abelian categories. We follow [27]. Throughout this section, let \( \mathcal{H} \) be a hereditary abelian \( k \)-category with finite-dimensional morphism and extension spaces. As a consequence of finite-dimensional morphism space, \( \mathcal{H} \) is a Krull-Schmidt category, i.e. each object of \( \mathcal{H} \) is a finite direct sum of indecomposable objects with local endomorphism ring. We refer to [27] for examples and basic properties of hereditary categories.

Let \( D^b(\mathcal{H}) \) be the bounded derived category of \( \mathcal{H} \) with the suspension functor \( \Sigma \). An object \( X \) in \( D^b(\mathcal{H}) \) is called rigid provided that \( \text{Hom}_{D^b(\mathcal{H})}(X, \Sigma X) = 0 \). A rigid object \( X \) is exceptional if \( \text{dim}_k \text{Hom}_{D^b(\mathcal{H})}(X, X) = 1 \). In particular, an exceptional object has to be indecomposable. The following fundamental result is due to Happel-Ringel [20] (cf. also [1, 27]).

Lemma 4.1. Let \( E \) and \( F \) be indecomposable objects in \( \mathcal{H} \) such that \( \text{Hom}_{D^b(\mathcal{H})}(F, \Sigma E) = 0 \). Then all non-zero homomorphism \( f : E \to F \) is a monomorphism or epimorphism. In particular, each indecomposable \( E \) without self-extensions is exceptional.

Let \( \mathcal{C} \) be a full subcategory of \( D^b(\mathcal{H}) \) and \( M \) an indecomposable object in \( \mathcal{C} \). A path in \( \mathcal{C} \) from \( M \) to itself is called a cycle in \( \mathcal{C} \), that is a sequence of non-zero non-isomorphism between indecomposable objects in \( \mathcal{C} \) of the form

\[
M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \cdots \xrightarrow{f_r} M_r = M.
\]

The following result is a consequence of Lemma 4.1 which is crucial for our investigation of c-vectors for quasitilted algebras.

Lemma 4.2. Let \( T \) be an object in \( D^b(\mathcal{H}) \) such that \( \text{Hom}_{D^b(\mathcal{H})}(T, \Sigma T) = 0 \). Then the subcategory \( \text{add} T \) has no cycle.

Proof. Suppose that there is a cycle in \( \text{add} T \), say \( M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} M_r = M \), where \( M_1, \cdots, M_r \) are indecomposable objects in \( \text{add} T \). We may assume that \( M_0 \in \mathcal{H} \). Note that \( M_0 \) is exceptional and \( f_1 \) is non-zero non-isomorphism, which imply that \( M_1 \not\cong M_0 \). We claim that some of \( M_1, \cdots, M_{r-1} \) are not in \( \mathcal{H} \). Otherwise, by Lemma 4.1 each \( f_i \) is either monomorphism or epimorphism. If there is an epimorphism \( f_i \) followed by a monomorphism \( f_{i+1} \), then \( f_{i+1} \circ f_i : M_{i-1} \to M_{i+1} \) is non-zero and is neither a monomorphism nor an epimorphism, which contradicts to Lemma 4.1. On the other hand, if there is a monomorphism \( f_i \) followed by an epimorphism \( f_{i+1} \), we may consider the cycle \( M_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_r} M_r \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{r-1}} M_{i-1} \xrightarrow{f_i} M_i \). This cycle turns out to admit an epimorphism followed by a monomorphism, a contradiction. Hence all of the \( f_i \) are either epimorphisms or monomorphisms.
Remark 4.4. In the above Corollary, if $f$ is a positive $c$-vector of $A$ and all $f_i$ are non-zero, we deduce that some of $M_1,\cdots, M_{r-1}$ belong to $\Sigma^t \mathcal{H}$ for certain $t > 0$. In particular, $M_{r-1}$ belongs to $\Sigma^m \mathcal{H}$ for some $m > 0$. Consequently, $\text{Hom}_D(M_{r-1}, M_0) = 0$, which contradicts to $f_r \neq 0$.

**Corollary 4.3.** Let $A$ be a finite-dimensional $k$-algebra such that $\text{mod} A$ is equivalent to the heart of a bounded $t$-structure on $\mathcal{D}^b(\mathcal{H})$. Then the Gabriel quiver $Q_A$ has no cycle.

**Proof.** Let $\mathcal{A}$ be the heart of a bounded $t$-structure on $\mathcal{D}^b(\mathcal{H})$ which is equivalent to $\text{mod} A$. We consider $A$ as an object in $\mathcal{D}^b(\mathcal{H})$ via the equivalence. We clearly have $\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(A, \Sigma A) = \text{Ext}_A^1(A, A) = 0$. Note that any cycle in the Gabriel quiver $Q_A$ induces an oriented cycle in $\text{add} A$. Now the result follows from Lemma 4.2.

Remark 4.4. In the above Corollary, if $\mathcal{H}$ is the category of finitely generated right $H$-modules for certain hereditary algebra $H$, then by Lemma 2.4 there is a silting object $T \in \mathcal{D}^b(\mathcal{H})$ such that $A \cong \text{End}_{\mathcal{D}^b(\mathcal{H})}(T)$.

For any finite-dimensional $k$-algebra $A$, let $\tau \text{dv}(A)$ be the set of dimension vectors of indecomposable $\tau$-rigid $A$-modules. The following result gives a lower bound of $c$-vector for algebras related to the hearts of bounded $t$-structures on $\mathcal{D}^b(\mathcal{H})$.

**Proposition 4.5.** Let $A$ be a finite-dimensional $k$-algebra such that $\text{mod} A$ is equivalent to the heart of a bounded $t$-structure on $\mathcal{D}^b(\mathcal{H})$. Then we have $\tau \text{dv}(A) \subseteq \text{cv}^+(A)$.

**Proof.** We need to prove that for any indecomposable $\tau$-rigid $A$-module $M$, $\dim k M \in \text{cv}^+(A)$. Since $M$ is $\tau$-rigid, the subcategory $\text{Fac} M$ is a functorially finite torsion class. Let $P = P(\text{Fac} M)$ be the one copy of each indecomposable $\text{Ext}$-projective object in $\text{Fac} M$. We may write $P = M \oplus M_1 \oplus \cdots \oplus M_r$. Note that $P$ is a support $\tau$-tilting $A$-module, hence $\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(P, \Sigma P) = \text{Ext}_A^1(P, P) = 0$. On the other hand, by the definition of $\text{Fac} M$, we deduce that $\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(M_i, M_i) = \text{Hom}_A(M_i, M_i) \neq 0$ for any $i$. Now Lemma 4.2 implies that $\text{Hom}_D(M_i, M) = 0$ for all $i$. Let $T$ be the 2-term silting object in $\mathcal{D}^b(\text{mod} A)$ corresponding to $P$. A direct calculation shows that $\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, M) = k$ and $\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, \Sigma M) = 0$ for all $i \neq 0$. Hence $\dim M$ is a positive $c$-vector of $A$ by Corollary 3.3. In particular, we have proved that $\tau \text{dv}(A) \subseteq \text{cv}^+(A)$. 

□
4.1.2. Piecewise hereditary algebras. Recall that $\mathcal{H}$ is a hereditary abelian category with finite-dimensional morphism and extension spaces, $\mathcal{D}^b(\mathcal{H})$ is the bounded derived category of $\mathcal{H}$. An object $T \in \mathcal{D}^b(\mathcal{H})$ is a tilting complex if

1. $\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, \Sigma^n T) = 0$ for all $0 \neq n \in \mathbb{Z}$;
2. for each $X \in \mathcal{D}^b(\mathcal{H})$, the condition $\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, \Sigma^n X) = 0$ for all $n \in \mathbb{Z}$ implies that $X = 0$ in $\mathcal{D}^b(\mathcal{H})$.

A tilting complex $T$ of $\mathcal{D}^b(\mathcal{H})$ is called a tilting object of $\mathcal{H}$ if $T \in \mathcal{H}$. A finite-dimensional $k$-algebra $A$ is called piecewise hereditary if $A$ is isomorphic to the endomorphism algebra of a tilting complex in $\mathcal{D}^b(\mathcal{H})$. It is called quasi-tilted if moreover $T$ is a tilting object in $\mathcal{H}$. A tilting complex $T$ induces an equivalence of triangulated categories $\mathcal{L} \otimes \text{End}_{\mathcal{D}^b(\mathcal{H})}(T) : \mathcal{D}^b(\mathcal{H}) \rightarrow \mathcal{D}^b(\mathcal{H})$. By Happel’s theorem [19], if $\mathcal{H}$ is a connected hereditary abelian $k$-category with finite-dimensional morphism and extension spaces which admits a tilting complex, then $\mathcal{H}$ is derived equivalent to the category $\text{mod} \ H$ for certain finite-dimensional hereditary $k$-algebra or to the category $\text{coh} X$ of coherent sheaves over a weighted projective line [17].

Note that when $\mathcal{H}$ is the category $\text{mod} \ A$ for some finite-dimensional hereditary $k$-algebra, the endomorphism algebra of a tilting module in $\text{mod} \ A$ is called a tilted algebra [5, 20]. In particular, tilted algebras are quasi-tilted.

Let $A$ be a finite-dimensional $k$-algebra which is derived equivalent to $\mathcal{H}$. The algebra $A$ turns out to be piecewise hereditary. Indeed, let $K : \mathcal{D}^b(\text{mod} \ A) \rightarrow \mathcal{D}^b(\mathcal{H})$ be the equivalent functor. It is clear that $K(A)$ is a tilting complex of $\mathcal{D}^b(\mathcal{H})$ and $A \cong \text{End}_{\mathcal{D}^b(\mathcal{H})}(K(A))$. For a finite-dimensional $k$-algebra $A$, a module $M$ is called exceptional provided $\dim_k \text{Hom}_A(M, M) = 1$ and $\text{Ext}^1_A(M, M) = 0$. We define

$$\text{exdv}(A) = \{ \dim M | M \in \text{mod} A \text{ is exceptional} \}.$$ 

Our next result gives a upper bounded of positive $c$-vectors by exceptional modules for piecewise hereditary algebras.

**Theorem 4.6.** Let $A$ be a piecewise hereditary $k$-algebra. Then we have $\tau \text{dv}(A) \subseteq \text{cv}^+(A) \subseteq \text{exdv}(A)$.

**Proof.** The first inclusion $\tau \text{dv}(A) \subseteq \text{cv}^+(A)$ follows from Proposition [45] directly.

Let $c$ be a positive $c$-vector, by Corollary 3.2, there is a 2-term silting object $T \in \text{per} A$ and an indecomposable $A$-module $M$ with $\dim M = c$ such that $\text{Hom}_A(T, M) = k$ and $\text{Hom}_A(T, \Sigma^i M) = 0$ for all $i \neq 0$. Let $\Gamma$ be the negative dg algebra associated to $T$ and $F := L \otimes_T T : \mathcal{D}^b(\Gamma) \rightarrow \mathcal{D}^b(\text{mod} A)$ the equivalent functor. The condition $\text{Hom}_A(T, M) = k$ and $\text{Hom}_A(T, \Sigma^i M) = 0$ for all $i \neq 0$ implies that there is a simple
Let $H$ be the hereditary abelian category $H$ such that $D^b(\text{mod } A) \cong D^b(H)$, then we have $D^b(\text{mod } A) \cong D^b(H)$. Therefore $A$ is equivalent to the heart of a bounded t-structure on $D^b(H)$. On the other hand, we have $A \cong \text{mod } \text{End}_A(T)$. Hence $\text{Hom}_A(M, M) = \text{Hom}_A(S, S) = \text{Hom}_A(S, S) = k$. Moreover, by Corollary 4.3, we deduce that $0 = \text{Hom}_A(S, \Sigma S) = \text{Hom}_A(S, \Sigma S) = \text{Ext}_A^1(M, M) = 0$. In particular, $c \in \text{exdv}(A)$. This completes the proof.

If $A$ is a finite-dimensional hereditary algebra over $k$, then we clearly have $\tau \text{dv}(A) = \text{exdv}(A)$. Hence we obtain the equalities for acyclic cluster algebras established by Chávez in [11].

Corollary 4.7. Let $A$ be a finite-dimensional hereditary algebra over $k$. Then we have $\text{cv}^+(A) = \text{exdv}(A)$.

4.1.3. Quasitilted algebras. Let $A$ be a quasitilted algebra. By definition, there is a hereditary abelian $k$-category $H$ with a tilting object $T \in H$ such that $A \cong \text{End}_H(T)$. In this case, the category $\text{mod } A$ of finitely generated right $A$-modules has a nice interpretation via torsion theory of $H$.

Let $\mathcal{T}$ (resp. $\mathcal{F}$) be the full subcategory of $H$ consisting of all objects $X$ (resp. $Y$) of $H$ satisfying $\text{Ext}_H^1(T, X) = 0$ (resp. $\text{Hom}_H(T, Y) = 0$). Let $F : D^b(\text{mod } A) \to D^b(H)$ be the triangle equivalent functor. Then the standard t-structure $(D^{\leq 0}, D^{\geq 0})$ on $D^b(\text{mod } A)$ induces a t-structure $(D^{\leq 0}, D^{\geq 0})$ via the functor $F$ (cf. Section 2.4). Moreover, $\text{mod } A$ is equivalent to the heart $A$ of $(D^{\leq 0}, D^{\geq 0})$ via the functor $F$. The following results are well-known, see [5, 20, 27].

Lemma 4.8.

1. $(\mathcal{T}, \mathcal{F})$ is a torsion pair over $H$;
2. $(\Sigma \mathcal{F}, \mathcal{T})$ is a splitting torsion pair over $A$;
3. Under the identification of $\text{mod } A$ with $A$, we have $\text{pd } M_A \leq 1$ for $M \in \mathcal{T}$ and $\text{id } N_A \leq 1$ for $N \in \Sigma \mathcal{F}$.

Theorem 4.9. Let $A$ be a quasitilted algebra over $k$, then we have $\text{cv}^+(A) = \text{exdv}(A)$.

Proof. We need to show that for each exceptional $A$-module $M$, the dimension vector $\text{dim } M$ is a positive $c$-vector of $A$. We identify $\text{mod } A$ with $A$ as above. Let $M$ be an exceptional $A$-module. Since $(\Sigma \mathcal{F}, \mathcal{T})$ is splitting, $M$ lies either in $\mathcal{T}$ or in $\Sigma \mathcal{F}$. If $M \in \mathcal{T}$, then by Lemma 4.8 we have $\text{pd } M \leq 1$. As a consequence, $M$ is an
indecomposable $\tau$-rigid $A$-module. By Proposition 4.5, we deduce that $\dim M$ is a positive $c$-vector of $A$.

Now suppose that $M \in \Sigma \mathcal{F}$, then $\text{id} M \leq 1$. Recall that we have the usual duality $D = \text{Hom}_k(?, k) : \text{mod} A \to \text{mod} A^{\text{op}}$. It is clear that $D(M) \in \text{mod} A^{\text{op}}$ is an exceptional $A^{\text{op}}$-module and has projective dimension at most one. Therefore $D(M)$ is an indecomposable $\tau$-rigid $A^{\text{op}}$-module. On the other hand, we clearly have $D^b(\text{mod} A^{\text{op}}) \cong D^b(\mathcal{H}^{\text{op}})$, where $\mathcal{H}^{\text{op}}$ is the opposite category of $\mathcal{H}$, which is a hereditary abelian category. By Proposition 4.5 again, we deduce that $\dim D(M)$ is a positive $c$-vector of $A^{\text{op}}$. Note that $\dim D(M) = \dim M \in \text{cv}^+(A)$. Now the result follows from Theorem 4.6. □

4.2. $c$-vectors of representation-directed algebras. A finite-dimensional $k$-algebra is called representation-directed if there is no cycle in $\text{mod} A$. Let $A$ be a finite-dimensional representation-directed $k$-algebra. By the definition, we know that every indecomposable $A$-module is $\tau$-rigid and also exceptional. This section is devoted to study the $c$-vectors for representation-directed algebras. Namely, we have the following equality.

**Theorem 4.10.** Let $A$ be a representation-directed algebra over $k$. We have $\text{cv}^+(A) = \text{exdv}(A)$.

**Proof.** We need to prove the dimension vector of each indecomposable $A$-module is a positive $c$-vector. Let $M$ be an arbitrary indecomposable $A$-modules. Consider the torsion class $\text{Fac} M$ generated by $M$, which is a functorially finite torsion class. Let $N := P(\text{Fac} M)$ be the direct sum of one copy of each of the indecomposable $\text{Ext}$-projective objects in $\text{Fac} M$. We clearly have $M \in \text{add} N$. By Theorem 2.2, we deduce that $N$ is a support $\tau$-tilting $A$-module. Equivalently, there is a projective $A$-module $P$ such that $(N, P)$ is a support $\tau$-tilting pair. Let $T$ be the corresponding 2-term silting complex and $\Gamma$ the negative dg algebra of $T$. Let $G := \text{RHom}_A(T, ?) : D^b(\text{mod} A) \to D^b(\Gamma)$ be the inverse of the functor $F := ? \otimes_{\Gamma} T$. A direct calculation shows that $H^0(G(M)) = k$ and $H^i(G(M)) = 0$ for $i \neq 0$. In particular, the dimension vector $\dim M$ is a $c$-vector of $A$ by Corollary 3.3. This finishes the proof. □

Note that an algebra derived equivalent to a representation-finite hereditary algebra has to be a representation-direct algebra. We have the following special case of the above theorem.

**Corollary 4.11.** Let $A$ be finite-dimensional algebra over $k$. If $A$ is derived equivalent to a representation-finite hereditary algebra, then $\text{cv}^+(A) = \text{exdv}(A)$.
4.3. c-vectors of cluster-tilted algebras. Let $H$ be a finite-dimensional hereditary $k$-algebra and $\mathcal{D}^b(H)$ the bounded derived category of finitely generated right $H$-modules. Denote by $\Sigma$ the suspension functor of $\mathcal{D}^b(H)$ and $\tau$ the Auslander-Reiten translation functor. The cluster category $\mathcal{C}_H$ has been introduced in [9] as the orbit category $\mathcal{D}^b(H)/\tau^{-1} \circ \Sigma$ of $\mathcal{D}^b(H)$. It admits a canonical triangle structure such that the projection $\pi_H : \mathcal{D}^b(H) \to \mathcal{D}^b(H)/\tau^{-1} \circ \Sigma$ is a triangle functor [21]. An object $T$ in $\mathcal{C}_H$ is called a cluster-tilting object provided that

- $\circ \operatorname{Hom}_{\mathcal{C}_H}(T, \Sigma T) = 0$;
- $\circ$ if $X \in \mathcal{C}_H$ such that $\operatorname{Hom}_{\mathcal{C}_H}(T, \Sigma X) = 0$, then $X \in \operatorname{add} T$.

Let $n$ be the number of pairwise non-isomorphic simple $H$-modules. Each basic cluster-tilting object in $\mathcal{C}_H$ has exactly $n$ indecomposable direct summands. The endomorphism algebra $\operatorname{End}_{\mathcal{C}_H}(T)$ of a cluster-tilting object $T \in \mathcal{C}_H$ is a cluster-tilted algebra [8]. It is known that cluster-tilted algebras are 1-Gorenstein algebras. Moreover, the functor $\operatorname{Hom}_{\mathcal{C}_H}(T, ?) : \mathcal{C}_H \to \operatorname{mod} \operatorname{End}_{\mathcal{C}_H}(T)$ yeilds an equivalence $\mathcal{C}_H/\Sigma T \cong \operatorname{mod} \operatorname{End}_{\mathcal{C}_H}(T)$, where $\mathcal{C}_H/\Sigma T$ is the additive quotient of $\mathcal{C}_H$ by the morphism factorizing through $\Sigma T$ (cf. [8, 24]).

Proposition 4.12. Let $H$ be a finite-dimensional hereditary $k$-algebra and $\mathcal{C}_H$ the corresponding cluster category. Let $T$ be a cluster-tilting object and $A$ the endomorphism algebra of $T$. Let $M$ be an indecomposable preprojective or preinjective $H$-module such that $\operatorname{Hom}_{\mathcal{C}_H}(T, M) \neq 0$, then the dimension vector $\dim \operatorname{Hom}_{\mathcal{C}_H}(T, M)$ of $A$-module is a positive c-vector of $A$.

Proof. It is easy to see that there is a cluster tilting object $T_M = M \oplus M_1 \cdots \oplus M_{n-1}$ such that $\operatorname{Hom}_{\mathcal{C}_H}(M_i, M) = 0$ for all $1 \leq i \leq n-1$. Applying the functor $\operatorname{Hom}_{\mathcal{C}_H}(T, ?)$, we deduce that $N_A := \operatorname{Hom}_{\mathcal{C}_H}(T, T_M)$ is a support $\tau$-tilting $A$-module. Let $P$ be the 2-term silting object corresponding to $N_A$. A direct calculation shows that

$$\operatorname{Hom}_A(P, \Sigma^i \operatorname{Hom}_{\mathcal{C}_H}(T, M)) = \begin{cases} k & i = 0; \\ 0 & \text{else} \end{cases}$$

In particular, $\dim \operatorname{Hom}_{\mathcal{C}_H}(T, M)$ is a positive c-vector of $A$ by Corollary 3.3. \qed

Note that for a representation-finite hereditary algebra $H$, each $H$-module is a preprojective module. As a consequence, we recover the following equality of c-vectors for skew-symmetric cluster algebras of finite type in [12].

Corollary 4.13. Let $A$ be a cluster-tilted algebra of representation-finite type. We have $\operatorname{cv}^+(A) = \operatorname{dv}(A)$. 

A. Cluster algebras with principal coefficients and c-vectors. We follow [15, 16]. For an integer \( x \), we set \( [x]_+ = \max\{x, 0\} \) and \( \text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \). Let \( 1 \leq n \leq m \in \mathbb{N} \). Let \( \mathbb{QP} \) be the algebra of Laurent polynomials in the variables \( x_{n+1}, \ldots, x_m \) and \( \mathcal{F} \) the field of fractions of the ring of polynomials with coefficients in \( \mathbb{QP} \) in \( n \) indeterminates. A seed in \( \mathcal{F} \) is a pair \( (\tilde{B}, \mathbf{x}) \) consisting of an \( m \times n \) integer matrix \( \tilde{B} \) whose principal part (that is the submatrix formed by the first \( n \) rows) is skew-symmetrizable and a free generating set \( \mathbf{x} = \{x_1, x_2, \ldots, x_n\} \) of the field \( \mathcal{F} \). The matrix \( \tilde{B} \) is the exchange matrix and \( \mathbf{x} \) is the cluster of the seed \( (\tilde{B}, \mathbf{x}) \).

Elements of the cluster \( \mathbf{x} \) are cluster variables of the seed \( (\tilde{B}, \mathbf{x}) \).

For any \( 1 \leq k \leq n \), the seed mutation of \( (\tilde{B}, \mathbf{x}) \) in the direction \( k \) transforms \( (\tilde{B}, \mathbf{x}) \) into a new seed \( \mu_k(\tilde{B}, \mathbf{x}) = (\tilde{B}', \mathbf{x}') \), where

- the entries \( b'_{ij} \) of \( \tilde{B}' \) are given by
  \[
  b'_{ij} = \begin{cases} 
  -b_{ij} & \text{if } i = k \text{ or } j = k, \\
  b_{ij} + \text{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{else}.
  \end{cases}
  \]

- the cluster \( \mathbf{x}' = \{x'_1, \ldots, x'_n\} \) is given by \( x'_j = x_j \) for \( j \neq k \) and \( x'_k \in \mathcal{F} \) is determined by the exchange relation
  \[
  x'_k x_k = \prod_{i=1}^{m} x^{|b_{ik}|}_i + \prod_{i=1}^{m} x^{|-b_{ik}|}_i.
  \]

It is clear that mutation in a fixed direction is an involution. The cluster algebra with coefficients \( \mathcal{A}(\tilde{B}) = \mathcal{A}(\tilde{B}, \mathbf{x}) \) is the subalgebra of \( \mathcal{F} \) generated by all the cluster variables which can be obtained from the initial seed \( (\tilde{B}, \mathbf{x}) \) by iterated mutations.

If \( m = 2n \) and the coefficient part of \( \tilde{B} \) (that is the submatrix formed by the last \( m - n \) rows) is the identity matrix \( E_n \), then \( \mathcal{A}(\tilde{B}) \) is a cluster algebra with principal coefficients.

Let \( T_n \) be the \( n \)-regular tree, whose edges are labeled by the numbers \( 1, 2, \ldots, n \) so that the \( n \) edges emanating from each vertex carry different labels. A cluster pattern is the assignment of a seed \( (\tilde{B}_t, \mathbf{x}_t) \) to each vertex \( t \) of \( T_n \) such that the seeds assigned to vertices \( t \) and \( t' \) linked by an edge labeled \( k \) are obtained from each other
by the seed mutation \( \mu_k \). A cluster pattern is uniquely determined by an assignment of the initial seed \((B, x)\) to any vertex \( t_0 \in T_n \).

Let \( A(B) \) be a cluster algebra with principal coefficients. We fix a cluster pattern of \( A(B) \). For any \( t \in T_n \), let \( C_t \) be the coefficient part of the matrix \( B_t \). Each column vector of \( C_t \) is called a c-vector of \( A(B) \). It has been conjectured that each c-vector of \( A(B) \) is sign-coherence. The following positive answer has been established in [14] for skew-symmetric cluster algebras. We refer to [28, 32] for two alternative proofs.

**Theorem A.1.** Let \( B \) be a \( 2n \times n \) integer matrix whose principal part is skew-symmetric and whose coefficient part is the identity matrix \( E_n \). Then each c-vector of the cluster algebra \( A(B) \) is sign-coherence.

We shall give a direct proof basing on Proposition 6.10 of [16] in the end of this section.

**A.2. Quivers with potentials and mutations.** We follow [13, 22]. Let \( Q = (Q_0, Q_1) \) be a finite quiver, where \( Q_0 \) is the set of vertices and \( Q_1 \) is the set of arrows. Let \( kQ \) be the path algebra of \( Q \) and \( \widehat{kQ} \) its completion with respect to path length. Thus, \( \widehat{kQ} \) is a topological algebra and the paths of \( Q \) form a topological basis. The continuous zeroth Hochschild homology of \( \widehat{kQ} \) is the vector space \( HH_0(\widehat{kQ}) \) obtained as the quotient of \( \widehat{kQ} \) by the closure of the subspace \([kQ, \widehat{kQ}]\) of all commutators. It has a topological basis formed by the classes of cyclic paths of \( Q \). For each arrow \( a \) of \( Q \), the cyclic derivative with respect to \( a \) is the unique continuous map

\[
\partial_a : HH_0(\widehat{kQ}) \rightarrow \widehat{kQ}
\]

which takes the class of a path \( p \) to the sum

\[
\sum_{p=uvu} vu
\]

taken over all decompositions of \( p \) as a concatenation of path \( u, a, v \), where \( u, v \) are of length \( \geq 0 \). A potential on \( Q \) is an element \( W \) of \( HH_0(\widehat{kQ}) \) which does not involve cycles of length \( \leq 2 \).

Let \( (Q, W) \) be a quiver with potential and \( k \) a vertex of \( Q \). With certain mild condition for \((Q, W)\) at the vertex \( k \), Derksen-Weyman-Zelevinsky [13] introduced the mutation of \((Q, W)\) in the direction \( k \) which transforms \((Q, W)\) into a new quiver with potential \( \mu_k(Q, W) = (Q', W') \) (for the precisely definition, we refer to [13]). In this case, we call the quiver with potential \((Q, W)\) is mutable at vertex \( k \). The quiver \( Q \) and \( Q' \) have the same vertices but different arrows. In general, the resulting quiver with potential \( \mu_k(Q, W) \) may not be mutable at certain vertices. But if the quiver
$Q$ has no loops nor 2-cycles, there exists a non-degenerate potential $W$ on $Q$ such that we can indefinitely mutate the quiver with potential $(Q, W)$. Moreover, each quiver with potential obtained from $(Q, W)$ by iterated mutations has no loops nor 2-cycles.

Let $Q$ be a finite quiver without loops nor 2-cycles with vertex set $Q_0 = \{1, 2, \cdots, m\}$. We define an $m \times m$ integer matrix $B(Q)$ associated to $Q$ such that

$$b_{ij} = |\{\text{arrows from vertex } i \text{ to vertex } j\}| - |\{\text{arrows from vertex } j \text{ to vertex } i\}|.$$ 

Conversely, for a given integer skew-symmetric matrix $B$, there is a unique quiver $Q$ without loops nor 2-cycles such that $B(Q) = B$. Let $W$ be a non-degenerate potential on $Q$. We may assign each vertex $t \in \mathbb{T}_m$ a quiver with potential $(Q_t, W_t)$ which can be obtained from $(Q, W)$ by iterated mutations such that the quivers with potentials assigned to $t$ and $t'$ linked by an edge labeled $k$ are obtained from each other by the mutation $\mu_k$. By Proposition 7.1 in [13], if $(Q_t, W_t)$ and $(Q_{t'}, W_{t'})$ are linked by an edge $k$, then we have $B(Q_t) = \mu_k(B(Q_{t'}))$.

### A.3. Ginzburg dg algebras and derived equivalences

Let $Q$ be a finite quiver and $W$ a potential on $Q$. The Ginzburg dg algebra $\Gamma(Q_0, W)$ of $(Q, W)$ introduced by Ginzburg [18] is constructed as follows: Let $\overrightarrow{Q}$ be the graded quiver with the same vertices as $Q$ and whose arrows are

- the arrow of $Q$, which are of degree 0;
- an arrow $a^* : j \to i$ of degree $-1$ for each arrow $a : i \to j$ of $Q$;
- a loop $t_i : i \to i$ of degree $-2$ for each vertex $i$ of $Q$.

The underlying graded algebra of $\Gamma(Q, W)$ is the completion of the graded path algebra $k\overrightarrow{Q}$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $\overrightarrow{Q}$. In particular, the $n$-component of $\Gamma(Q, W)$ consisting of elements of the form $\sum_{p} \lambda_p p$, where $p$ runs over all paths of degree $n$. The differential $d$ of $\Gamma(Q, W)$ is the unique continuous linear endomorphism homogenous of degree 1 which satisfies the Leibniz rule

$$d(uv) = (du)v + (-1)^p uv,$$

for all homogeneous $u$ of degree $p$ and all $v$, and takes the following values on the arrows of $\overrightarrow{Q}$:

- $da = 0$ for each arrow $a$ of $Q$;
- $d(a^*) = \partial_a W$ for each arrow $a$ of $Q$;
- $d(t_i) = e_i(\sum_{a}[a, a^*])e_i$ for each vertex $i$ of $Q$, where $e_i$ is the lazy path at $i$ and the sum runs over the set of arrows of $Q$. 

Let $Q$ be a finite quiver without loops nor 2-cycles with vertex set $\{1, 2, \cdots, m\}$ and $W$ a non-degenerate potential on $Q$. Denote by $\Gamma_{(Q,W)}$ the Ginzburg dg algebra associated to $(Q,W)$. Let $k$ be a vertex of $Q$ and $\Gamma_{\mu_k(Q,W)}$ the Ginzburg dg algebra associated to $\mu_k(Q,W)$. Let $e_1, \cdots, e_m$ be the idempotents of $\Gamma_{(Q,W)}$ and $\Gamma_{\mu_k(Q,W)}$ associated to the vertices of $(Q,W)$ and $\mu_k(Q,W)$. Let $\mathcal{D}(\Gamma_{(Q,W)})$ and $\mathcal{D}(\Gamma_{\mu_k(Q,W)})$ be the derived categories of $\Gamma_{(Q,W)}$ and $\Gamma_{\mu_k(Q,W)}$ respectively. The following result is due to Keller-Yang [25].

**Theorem A.2.** There is a triangle equivalence

$$\Phi : \mathcal{D}(\Gamma_{\mu_k(Q,W)}) \rightarrow \mathcal{D}(\Gamma_{(Q,W)})$$

which sends the $e_i\Gamma_{\mu_k(Q,W)}$ to $e_i\Gamma_{(Q,W)}$ for $i \neq k$ and to the mapping cone of the morphism $e_k\Gamma_{\mu_k(Q,W)} \rightarrow \oplus_{k \rightarrow j} e_j\Gamma_{(Q,W)}$ for $i = k$, where the sum is taken over the arrows in $Q$.

**A.4. Proof of Theorem [A.1]** We fix a cluster pattern of $\mathcal{A}(\tilde{B})$ by assigning the initial seed $(\tilde{B}, x)$ to the vertex $t_0 \in \mathbb{T}_n$.

Let $Q = (Q_0, Q_1)$ be a finite quiver without loops nor 2-cycles with vertex set $Q_0 = \{1, 2, \cdots, n\}$ such that $B(Q)$ is the principal part of the initial matrix $\tilde{B}$. We define a new quiver $\tilde{Q}$ such that the set of vertices $\tilde{Q}_0 = Q_0 \cup \{1 + n, 2 + n, \cdots, 2n\}$ and the set of arrows $\tilde{Q}_1 = Q_1 \cup \{i + n \rightarrow i | i \in Q_0\}$. Let $W$ be a non-degenerate potential on $\tilde{Q}$. We may assign each vertex $t \in \mathbb{T}_n$ a quiver with potential $(\tilde{Q}_t, W_t)$ which can be obtained from $(\tilde{Q}, W)$ by iterated mutations of $\mu_k$ for $1 \leq k \leq n$ such that the quivers with potentials assigned to $t$ and $t'$ linked by an edge labeled $k$ are obtained from each other by the mutation $\mu_k$. Let $(\tilde{Q}_0, W_0)$ be the quiver with potential $(\tilde{Q}, W)$. For each quiver with potential $(\tilde{Q}_t, W_t)$, let $B(\tilde{Q}_t)$ be the corresponding skew-symmetric matrix and $B(\tilde{Q}_t)^\circ$ the submatrix of $B(\tilde{Q}_t)$ formed by the first $n$ columns. Recall that for each vertex $t \in \mathbb{T}_n$, we have a seed $(\tilde{B}_t, x_t)$ by the fixed cluster pattern. By Proposition 7.1 in [13], we deduce that $B(\tilde{Q}_t)^\circ = \tilde{B}_t$ for all $t \in \mathbb{T}_n$. Let $C_t = (c_{ij}^t) \in M_n(\mathbb{Z})$ be the coefficient part of $B_t$, we clearly have $c_{ij}^t = |\{\text{arrows from vertex } i + n \text{ to vertex } j\}| - |\{\text{arrows from vertex } j \text{ to vertex } i + n\}|$.

Note that we have $\text{Hom}_{\mathcal{D}(\Gamma_{(Q,W)})}(e_{i+n}\Gamma_{(Q,W)}, e_{j+n}\Gamma_{(Q,W)}) = 0$ for any $1 \leq i \neq j \leq n$. It follows from Theorem [A.2] that there is no arrow between vertex $i + n$ and $j + n$ in the quiver $Q_t$ for any $t \in \mathbb{T}_n$. Suppose that there is a vertex $t \in \mathbb{T}_n$ such that the $k$th column vector of $C_t$ is not sign-coherence. Hence there exist vertices $i + n$ and $j + n$ for $1 \leq i \neq j \leq n$ such that $c_{ik}^t > 0$ and $c_{jk}^t < 0$. Now consider the mutation at
vertex $k$, we obtain that in the quiver with potential $\mu_k(\tilde{Q}_t, W_t)$ there are $c_{ik}^t \times c_{jk}^t$ arrows from $i + n$ to $j + n$, a contradiction.

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