ADHESION APPROXIMATION OF MULTI-DIMENSIONAL
ZERO-PRESSURE GAS DYNAMICS SYSTEM

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ABSTRACT. In this paper we find explicit formula for some physically interesting solutions of multidimensional zero pressure gas dynamics system and its adhesion approximation. This includes the spherically symmetric solutions. First we write down solution of the adhesion model and study its asymptotic behaviour. The radial components of the velocity and density satisfy a simpler equation. This enables us to get explicit formula for different types of domains and study its asymptotic behaviour. A class of solutions for the inviscid system with conditions on the mass instead of conditions at origin also is analyzed.

1. Introduction

One of the analytical models proposed to describe the large-scale structure of the universe is the adhesion model

\[ u_t + (u \cdot \nabla) u = \frac{\varepsilon}{2} \Delta u, \quad \rho_t + \nabla \cdot (\rho u) = 0. \quad (1.1) \]

Here \( u \) is the velocity and \( \rho \), the density of the particles (see [3] and references therein for the physical importance and analysis of solutions). This system and its inviscid counterpart, the multi-dimensional zero-pressure gas system

\[ u_t + (u \cdot \nabla) u = 0, \quad \rho_t + \nabla \cdot (\rho u) = 0 \quad (1.2) \]

have been an active field of research since they were introduced, see [1, 3, 4, 17, 18]. However there is no clear analytical understanding of the solutions of these equations. One question that remains is a well-posedness theory and large time behaviour of solution. In this paper we construct special solutions which are physically interesting and study its asymptotic behaviour.

The advantage of (1.2) over (1.1) is that it is simpler to describe the solution in the region where classical smooth solutions exist. It is well-known that smooth solutions for (1.2) does not exist globally in \( \{(x, t) : x \in \mathbb{R}^n, t > 0\} \), even when the initial data

\[ u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x) \quad (1.3) \]

are smooth with compact support. So the global solutions should be understood in a weak sense. Weak solutions are not unique and we need additional conditions to select the physical solution. One way of selecting the physical solution of (1.2) is by taking the limit of solution of (1.1) as \( \varepsilon \) goes to 0.

In the context of the large scale structure formation, the fastest growing mode in the linear theory has decaying vorticity. So it is natural to seek potential solutions of the model equations (1.1) and (1.2). Then the velocity can be represented in

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terms of a velocity potential \( \phi \), see \[18\]. In this case, \( u^\epsilon, \epsilon > 0 \) can be constructed using the standard Hopf-Cole transformation, see \[8, 18\]. Then the continuity equation for \( \rho^\epsilon \) is a linear equation with smooth bounded coefficients which can be solved using the method of characteristics. We use this method to construct weak asymptotic solution of \((1.1)\) and \((1.3)\).

For the inviscid system \((1.2)\), spherical solution of the form \( u = (x/r)q, \rho(x) = \rho(r), r = |x| \), was constructed for \( n = 3 \) in \[12\]. This is a special case of potential velocity. It was shown that the radial component of velocity and density satisfies

\[
q_t + qg_r = 0, \quad \rho_t + \frac{1}{r^2}(r^2 \rho q)_r = 0, \quad r > 0,
\]

where \( r = |x| \). By the change of variable \( p = r^2 \rho \), the equation \((1.4)\) takes the form

\[
q_t + qg_r = 0, \quad p_t + (qp)_r = 0, \quad r > 0.
\]

Classical theory of hyperbolic conservation laws of Lax \[13\] is not applicable due to formation of \( \delta \) waves. However the system \((1.5)\) has been well studied in \[7, 10, 11\]. Using explicit solution of this system, we obtain radial solution of \((1.2)\), with initial data at \( t = 0 \) and with different behaviors at the origin \( x = 0 \). It is clear that in the region of smoothness, \( u \) and \( \rho \) constructed using \((1.5)\) is a solution of the multidimensional system \((1.1)\). As the solution is not smooth, it is not obvious that the constructed distribution is a weak solution of the multidimensional system \((1.2)\).

We show that along the surface of discontinuity, the Rankine Hugoniot condition derived by Albeverio and Shelkovich \[1\] is satisfied.

In section 2, we construct weak asymptotic solution of \((1.2)\). In section 3, we derive equation for the radial case of \((1.1)\) and its linearization. In section 4, we solve boundary value problem for radial adhesion model and study its asymptotic behaviour for space dimensions \( n = 2 \) and \( n = 3 \). In the last section, we find explicit solution for the radial inviscid system, with conditions on the mass.

2. EXPLICIT WEAK ASYMPOTIC SOLUTION FOR MULTI-DIMENSIONAL CASE

In this section, we construct explicit weak asymptotic solution of the system \((1.2)\) with initial conditions \((1.3)\) using the adhesion approximation \((1.1)\), with an additional condition that \( u = \nabla_x \phi \).

First we recall the definition of weak asymptotic solution from \[1\] who studied the system \((1.2)\) without the condition \( u = \nabla_x \phi \). A family of smooth functions \((u^\epsilon, \rho^\epsilon)_{\epsilon > 0}\) is called a weak asymptotic solution of \((1.2)\) and \((1.3)\) if

\[
\begin{align*}
u^\epsilon_t + (u^\epsilon \cdot \nabla)u^\epsilon &= \mathcal{O}'(\mathcal{R}^n)(1), \\
\rho^\epsilon_t + \nabla \cdot (\rho^\epsilon u^\epsilon) &= \mathcal{O}'(\mathcal{R}^n)(1), \\
u^\epsilon(x,0) - u_0(x) &= \mathcal{O}'(\mathcal{R}^n)(1), \\
\rho^\epsilon(x,0) - \rho_0(x) &= \mathcal{O}'(\mathcal{R}^n)(1)
\end{align*}
\]

(2.1)

where \( < \mathcal{O}'(\mathcal{R}^n)(1), \eta > \) as \( \epsilon \to 0 \) uniformly in \( t > 0 \), for every \( \eta \in C_0^\infty(\mathcal{R}^n) \).

We consider the adhesion approximation \((1.1)\) with initial conditions which are regularizations of \((1.3)\). The velocity component \( u^\epsilon \) can be constructed using Hopf-Cole transformation, and then the continuity equation for \( \rho^\epsilon \) is solved using the method of characteristics. The resulting family \((u^\epsilon, \rho^\epsilon)_{\epsilon > 0}\), is shown to have necessary estimates to form weak asymptotic solution of \((1.2)\) and \((1.3)\).

**Theorem 2.1.** Assume \( u_0(x) = \nabla_x \phi_0 \) where \( \phi_0 \in W^{1,\infty}(\mathcal{R}^n) \) and \( \rho_0 \in L^\infty(\mathcal{R}^n) \). Let \( \phi^\epsilon_0 = \phi_0 \ast \eta^\epsilon, \nabla_x \phi^\epsilon_0 = \nabla_x \phi_0 \ast \eta^\epsilon \) and \( \rho^\epsilon_0 = \rho_0 \ast \eta^\epsilon \), where \( \eta^\epsilon \) is the usual Friedrichs mollifier in the space variable \( x \in \mathcal{R}^n \). Further let
 measurable, it follows that $u$ write the formula (2.7) for $\theta$ of (2.3) iff then $\rho(x, t, s) = \rho_0(x, t, 0) J(x, t, 0)$. Then $(u, \rho)$ is a weak asymptotic solution to (2.2) and (1.3).

Proof. Following [5, 8, 18], we note that if $\phi$ is a solution of

$$\phi_t + \frac{\nabla \phi}{2} = \frac{\epsilon}{2} \Delta \phi, \ \phi(x, 0) = \phi_0(x),$$

then $u = \nabla_x \phi$ is a solution of (1.1) with initial condition $u(x, 0) = \nabla_x \phi_0(x)$. Now using the Hopf-Cole transformation $\theta = e^{-\frac{\phi}{\epsilon}}$, it follows that $\phi$ is the solution of (2.3) if and only if $\theta$ is the solution of

$$\theta_t = \frac{\epsilon}{2} \Delta \theta, \ \theta(x, 0) = e^{-\frac{\phi_0(x)}{\epsilon}}.$$  

Solving (2.4), we get,

$$\theta(x, t) = \frac{1}{(2\pi\epsilon)^{n/2}} \int_{R^n} e^{-\frac{1}{\epsilon} \frac{|x-y|^2}{2} + \phi_0(y)} dy$$

Now

$$\theta(x, t) = \frac{1}{(2\pi\epsilon)^{n/2}} \int_{R^n} \partial_x e^{-\frac{1}{\epsilon} \frac{|x-y|^2}{2} + \phi_0(y)} dy$$

$$= -\frac{1}{(2\pi\epsilon)^{n/2}} \int_{R^n} \partial_y e^{-\frac{1}{\epsilon} \frac{|x-y|^2}{2} + \phi_0(y)} dy$$

$$= \frac{1}{(2\pi\epsilon)^{n/2}} \int_{R^n} \partial_y \phi_0(y) e^{-\frac{1}{\epsilon} \frac{|x-y|^2}{2} + \phi_0(y)} dy. \hspace{1cm} (2.6)$$

In the last line we used integration by parts with respect to the variable $y$. Since $\theta = e^{-\frac{\phi}{\epsilon}}$, we get from (2.3) and (2.6),

$$u(x, t) = \frac{1}{(2\pi\epsilon)^{n/2}} \int_{R^n} \nabla \phi_0(y) e^{-\frac{1}{\epsilon} \frac{|x-y|^2}{2} + \phi_0(y)} dy$$

From this formula it is clear that

$$|u|^L_{L^\infty(R^n \times (0, \infty))} \leq |\nabla \phi_0|_{L^\infty(R^n)}.$$  

Further as differentiation under the integral sign is justified if $\nabla \phi_0$ is bounded measurable, it follows that $u$ is a $C^\infty$ function in $R^n \times (0, \infty)$. Indeed, we can write the formula (2.7) for $u$ in the following form

$$u(x, t) = \frac{1}{(2\pi\epsilon)^{n/2}} \int_{R^n} \nabla \phi_0(x - \sqrt{(2t)} y) e^{-\frac{1}{\epsilon} \frac{|y|^2}{2} + \phi_0(x - \sqrt{(2t)} y)} dy.$$  

(2.8)
Differentiating (2.9) and using chain rule, it follows that

\[ |\partial_x^\alpha \partial_t^j u^\epsilon| \leq C_{\alpha,j} \epsilon^{|\alpha|+|j|}, \tag{2.10} \]

where \( C_{\alpha,j} \) depends only on \( |\nabla_x \phi_0^\epsilon|_{L^\infty} \). Next we consider the continuity equation with coefficient \( u^\epsilon \) and regularized initial condition,

\[ \rho_t + \nabla_x (u^\epsilon \rho) = 0, \quad \rho^\epsilon (x, 0) = \rho_0^\epsilon (x). \tag{2.11} \]

As \( u^\epsilon \) is smooth, we use the method of characteristics to find \( \rho \). Let \( \rho^\epsilon (x, t) \) be the solution of

\[ \frac{dX(s)}{ds} = u^\epsilon (X, s), \quad X(s = t) = x. \tag{2.12} \]

Since \( u^\epsilon \) satisfies the estimates (2.8) and (2.10), by the existence and uniqueness theory of ODE, there exists a unique solution \( X^\epsilon (s) \) to (2.12) for all \( 0 \leq s \leq t \). To make the dependence of \( x \) and \( t \) explicit, let us denote it by \( X^\epsilon (x, t, 0) \). This flow takes the point \( (x, t) \) to an initial point \( (X^\epsilon (x, t, 0), 0) \) and conversely.

Let \( J(X^\epsilon (x, t, s)) \) be the Jacobian determinant of \( X^\epsilon (x, t, s) \) with respect to \( x \).

Then

\[ \rho^\epsilon (x, t) = \rho_0^\epsilon (X^\epsilon (x, t, 0)) J(X^\epsilon (x, t, 0)) \tag{2.13} \]

is the solution of (2.11). The family \( (u^\epsilon, \rho^\epsilon)_{\epsilon > 0} \) given by (2.7) and (2.13) is a weak asymptotic solution. This follows easily as

\[ \epsilon \int_{\mathbb{R}^n} \Delta u^\epsilon \eta(x) dx = \epsilon \int_{\mathbb{R}^n} u^\epsilon \Delta \eta(x) dx = O(1) \epsilon \]

uniformly in \( t \) for every \( \eta \in C_0^\infty (\mathbb{R}^n) \) as \( u^\epsilon \) is bounded independent of \( \epsilon \) and \( (x, t) \) by the estimate (2.8). The solution \( (u^\epsilon, \rho^\epsilon)_{\epsilon > 0} \) satisfies the initial conditions because we have

\[ u^\epsilon (x, 0) = \nabla_x \phi_0^\epsilon (x) = u^\epsilon (x, 0) - \nabla_x \phi_0^\epsilon (x) + \nabla_x \phi_0^\epsilon (x) - \nabla_x \phi_0^\epsilon (x) \]

\[ = \nabla_x \phi_0^\epsilon (x) - \nabla_x \phi_0^\epsilon (x), \]

\[ \rho^\epsilon (x, 0) = \rho_0^\epsilon (x) - \rho_0^\epsilon (x) = \rho_0^\epsilon (x) - \rho_0^\epsilon (x) = \rho_0^\epsilon (x) - \rho_0^\epsilon (x) \]

where \( \nabla_x \phi_0^\epsilon (x) - \nabla_x \phi_0^\epsilon (x), \rho_0^\epsilon (x) - \rho_0^\epsilon (x) \) go to zero in distributions.

There is a large class of interesting initial data which admit gradient type solutions as described in the theorem. One such case is when the initial data is of radial type.

\[ u(x, 0) = u_0 (x) = \frac{x}{r} q_0 (r), \tag{2.14} \]

Clearly it can be written as a gradient,

\[ u_0 (x) = \nabla_x \phi_0 (x) = \nabla_x (\int_0^{r} q_0 (s) ds). \]

For density we give the initial condition

\[ \rho (x, 0) = \rho_0 (x). \tag{2.15} \]
Theorem 2.2. The solution of (1.1), with initial conditions (2.14) and (2.15) with \( \int_0^\infty q_0(s) ds < \infty \) and \( \int_{R^n} \rho_0(x) dx < \infty \) has the following asymptotic behaviour. The velocity component goes to 0 as \( t \) tends to \( \infty \) uniformly on compact subsets of \( R^n \) and the mass \( \int_{R^n} \rho(x,t) dx \) is conserved.

Proof. From (2.16) and \( \theta = e^{-\frac{t}{2}} \), we get the following formula for the velocity

\[
u(x,t) = \int_{R^n} \frac{x-y}{\sqrt{4t}} e^{-\frac{|x-y|^2}{4t}} \left[ \frac{|x-y|^2}{2} + \int_0^y q_0(s) ds \right] dy.
\]

Making the change of variable \( y \to (x-y)/\sqrt{2t} \), we get,

\[
u(x,t) = \int_{R^n} ye^{-\frac{1}{2} |y|^2} \left[ |y|^2 + \int_0^y \sqrt{2t} q_0(s) ds \right] dy
\]

(2.16)

From (2.16), it follows that, as \( t \) goes to \( \infty \), \( u^r \) goes to 0 uniformly on compact subsets of \( R^n \). From the formula (2.13) we have

\[
\int_{R^n} \rho(x,t) dx = \int_{R^n} \rho_0(x(X^r(x,t,0)))J(X^r(x,t,0))
\]

\[
= \int_{R^n} \rho_0(x) dx,
\]

since for every \( t \), \( X^r(x,t,0) : R^n \to R^n \) is a diffeomorphism and the determinant of Jacobian, \( J(X^r(x,t,0)) \) is positive. So the mass \( \int_{R^n} \rho(x,t) dx \) is conserved. \( \square \)

3. Equations for radial case and their linearization

To study the radial solutions of (1.1) with prescribed initial and boundary conditions, we look for radial components of velocity and density of the form

\[
u(x,t) = \frac{x}{r} q(r,t), \quad \rho(x,t) = \rho(r,t), \quad r = |x|.
\]

(3.1)

We find the equations for \( q \) and \( \rho \) and show that the resulting system can be linearized using the Hopf-Cole transformation. More precisely, we have the following

Theorem 3.1. The equation (1.1), for radial components can be linearized by the transformation

\[
q(r,t) = -\epsilon \frac{ar}{a}, \quad \rho(r,t) = r^{-(n-1)} p(r,t)
\]

(3.2)

to the following system

\[
a_t = \frac{\epsilon}{2}(ar + \frac{(n-1)}{r} a_r), \quad p + (pq)_r = 0.
\]

(3.3)

Proof. A simple computation shows that

\[
(u_j)_{x_k} = \begin{cases} (1/r)q + (x_j^2/r^2)q_r - (x_j^2/r^3)q, & \text{if } k = j \\ (x_j x_k/r^2)q_r - (x_j x_k/r^3)q, & \text{if } k \neq j \end{cases}
\]

(3.4)

and

\[
(u_j)_{x_k x_l} = \begin{cases} (x_j^3/r^2)q_{rr} + (3x_j^3 - 3x_j^3/r^2)q_r + (3x_j^3/r - 2x_j^3/r^2)q, & \text{if } k = j \\ (x_j x_k^2/r^3)q_{rr} + (x_j^2 x_k/r^3 - 2x_j x_k^2/r^4)q_r + (x_j x_k^2/r^3 - x_j r/q)q, & \text{if } k \neq j \end{cases}
\]

(3.5)
From (3.1), (3.4) and (3.5), we have for \( j = 1, 2, \ldots, n \)
\[
(u_j)_t + \sum_{k=1}^{n} u_k(u_j)_x = (x_j/r) q_t + (x_j/r) q[(1/r) q + (x_j^2/r^2) q_r - (x_j^2/r^3) q]
+ q \sum_{k \neq j} \frac{x_k x_j}{r} \frac{x_j x_k}{r^2} q_r - \frac{x_j x_k}{r^3} q
= \frac{x_j}{r} [q_t + q q_r]
\]
and
\[
\frac{\epsilon}{2} \Delta u_j = \frac{\epsilon}{2} \frac{x_j^2}{r} \frac{x_j^2}{r^2} q_{rr} + \left( \frac{3}{r} - \frac{3 x_j^2}{r^3} \right) q_r + \left( \frac{3 x_j^2}{r^4} - \frac{3}{r^2} \right) q] \]
+ \sum_{k \neq j} \frac{x_k^2}{r^2} q_{rr} + \left( \frac{1}{r} - \frac{3 x_k^2}{r^3} \right) q_r + \left( \frac{3 x_k^2}{r^4} - \frac{1}{r^2} \right) q
= \frac{\epsilon}{2} \frac{x_j}{r} [q_{rr} + \frac{(n-1)}{r} q_r - \frac{(n-1)}{r^2} q].
\]
Using these in (1.1), we get
\[
\frac{x_j}{r} [q_t + q q_r] = \frac{\epsilon}{2} \frac{x_j}{r} [q_{rr} + \frac{(n-1)}{r} q_r - \frac{(n-1)}{r^2} q], \quad j = 1, 2, 3, \ldots, n, \quad (3.6)
\]
Also for the continuity equation, we have
\[
\rho_t + \sum_{k=1}^{n} (u_k \rho)_x = \rho_t + \sum_{k=1}^{n} \{(1/r) q + (x_k^2/r^2) q_r - (x_k^2/r^3) q \rho + (x_k^2/r^2) q \rho_r \}
= \rho_t + (n/r) q \rho + q \rho - (1/r) q \rho + q \rho_r
= \rho_t + ((n-1)/r) q \rho + (q \rho)_r
= \rho_t + r^{-(n-1)} (r^{(n-1)} \rho q)_r. \quad (3.7)
\]
From (3.6) and (3.7), it follows that \((u, \rho)\) of the form (3.1) is a solution of (1.1) iff
\[
q_t + q q_r = \frac{\epsilon}{2} \frac{x_j}{r} [q_{rr} + \frac{(n-1)}{r} q_r - \frac{(n-1)}{r^2} q],
\]
\[
\rho_t + \frac{1}{r^{(n-1)}} (r^{(n-1)} \rho q)_r = 0. \quad (3.8)
\]
Let \( p = r^{(n-1)} \rho \), then the above system, becomes
\[
q_t + q q_r = \frac{\epsilon}{2} \frac{x_j}{r} [q_{rr} + \frac{(n-1)}{r} q_r - \frac{(n-1)}{r^2} q],
\]
\[
\rho_t + (pq)_r = 0, \quad r > 0, \quad t > 0. \quad (3.9)
\]
The equation for velocity component can be linearized using Hopf-Cole transformation. For this first we note that if \( Q(r, t) \) is a solution of
\[
Q_t + \frac{1}{2} ((Q_r)^2) = \frac{\epsilon}{2} [Q_{rr} + \frac{(n-1)}{r} Q_r]. \quad (3.10)
\]
with initial and boundary conditions
\[
Q(r, 0) = \int_0^r q_0(s) ds, \quad Q_r(0, t) = q_B(t),
\]
then \( q(r, t) = Q_r(r, t) \) is a solution of the first equation of (3.9), with initial condition 
\( q(r, 0) = q_0(r) \) and boundary condition \( q(0, t) = q_B(t) \). Now let
\[
Q(r, t) = -\epsilon \log a^r.
\]

An easy calculation shows that \( Q \) satisfies (3.10) iff \( a^r \) satisfies
\[
a_t = \frac{\epsilon}{2}[a_{rr} + \frac{(n-1)}{r} a_r], \tag{3.11}
\]

We use these results in the next section to construct solutions of the radial system 
in domains with boundary.

4. Radial solutions of adhesion model with boundary conditions

Here we are interested in getting explicit solutions for (1.1) in some spherically
symmetric domains of the form \( \Omega = D \times (0, \infty) \) for 2 and 3 space dimensions. First
we consider the domain \( \Omega = \{(x, t) : |x| = r < R, t > 0 \} \). We take initial conditions,
\[
u(x, 0) = \frac{x}{|x|} q_0(r), \quad \rho(x, 0) = \rho_0(r), |x| < R, \tag{4.1}
\]
and boundary conditions
\[
\lim_{|x| \to R} \frac{x}{|x|} u(x, t) = q_B, \tag{4.2}
\]
where \( q_B \) is constant. For \( \rho \) we need boundary condition only if \( q_B \) is negative.
\[
\lim_{|x| \to R} \rho(x, t) = \rho_B(t), \text{ if } q_B < 0. \tag{4.3}
\]
We also assume first order consistency condition of the initial and boundary data
at \( \{(x, t) : |x| = R, t = 0 \} \).

Let \( \Gamma = \{(x, t) \in \Omega \colon \text{either } |x| = R \text{ or } t = 0 \} \). Define \( p_T \) on \( \Gamma \) by
\[
p_T(r, t) = \begin{cases} 
\rho_0(r), & \text{if } t = 0, \\
\rho_B(t), & \text{if } r = R. 
\end{cases} \tag{4.4}
\]
As \( q \) is smooth, the characteristic of \( p_t + (qq)_r \), \( \beta = 0 \) passing through \( (r, t) \), \( \beta(r, t, s) \)
e exists for all \( s < t \), till it meets \( \Gamma \), at time \( t = t_0(r, t) \geq 0 \).
Let \( J_k \), for \( k = 0, 1 \), denote the Bessel functions of first kind and order \( k \). With
these notations, we have the following theorem.

**Theorem 4.1.** Explicit solution for (1.1), with initial and boundary conditions
(4.1) - (4.3) is given by
\[
u(x, t) = -\frac{x}{|x|} \int_0^R \partial_r G(r, \xi, t) e^{-\frac{\epsilon}{2} \int_0^\xi q_0(s) ds} d\xi \tag{4.5}
\]
\[
\rho(x, t) = r^{-(n-1)} p_T(\beta(r, t, t_0(r, t)), t_0(r, t)) e^{-\int_{t_0(r, t)}^t \beta(r, t, s) ds},
\]
where \( G(r, \xi, t) \) is described below. For \( n = 2 \), \( G \) has the form
\[
G(r, \xi, t) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 \xi}{((q_B/\epsilon)^2 R^2 + \mu_n^2)} J_0^2(\mu_n R) J_0(\frac{\mu_n \xi}{R}) e^{-\frac{\epsilon \mu_n^2 t}{R^2}}, \tag{4.6}
\]
where \( \mu_n \) are positive solutions of the transcendental equation \( \mu J_1(\mu) - kR J_0(\mu) = 0 \)
and for \( n = 3 \), \( G \) takes the form
notations, we have the following theorem.

As \( q \)

\[
G(r, \xi, t) = \frac{2\epsilon}{Rr} \sum_{n=1}^{\infty} \frac{\mu_n^2 + ((qB/\epsilon)R - 1)^2}{\mu_n^2 + ((qB/\epsilon)R - 1)^2} \sin \left( \frac{\mu_n r}{R} \right) \sin \left( \frac{\mu_n \xi}{R} \right) e^{-\frac{\epsilon^2}{R^2} \frac{t^2}{r^2}},
\]

(4.7)

where \( \mu_n \) are positive solutions of the transcendental equation \( \mu \cot(\mu) + (qB/\epsilon)R - 1 = 0 \).

Proof. By Theorem 3.1, it follows that the velocity \( u \) is given by \( u(x, t) = -e^{-\frac{\epsilon}{|x|}} \frac{\partial}{\partial r} \), where \( a \) satisfies the linear problem

\[
a_t = \frac{\epsilon}{2} \left[ a_{rr} + \frac{(n-1)}{r} a_r \right], \quad r < R, \quad t > 0
\]

\[
a(r, 0) = e^{-\frac{\epsilon}{r}} \frac{a(0)}{r}, \quad r < R
\]

\[
e a_r(R, t) + qB a(R, t) = 0, \quad t > 0.
\]

(4.8)

Then \( G \) given by (4.6) and (4.7) are just the Green’s functions for the boundary value problem (4.8), for \( n = 2 \) and \( n = 3 \) respectively, see [16]. The formula (4.5) for \( u \) follows.

Now we find a formula for \( \rho \). Notice that \( \rho(x, t) = \frac{1}{r a_r} p \) where \( p \) satisfies

\[
q_t + p q_r = -p_r q.
\]

Method of characteristics says along the curve \( \frac{dr}{ds} = p, \frac{d\xi}{ds} = -p_r q \). Let us consider the characteristic curve \( (\beta(r, t, s), s) \) passing through \( (x, t) \). Integrating the equation for \( q \), along this characteristic, we get \( q(x, t) = q(\beta(r, t, t_0)) + \int_{t_0}^t e^{-\int_{t_0}^s p(\beta(r, t, s))} ds \), where \( t_0 = t_0(r, t)(< t) \) is the time the curve touches the boundary \( \Gamma \).

Now we consider the equation (4.11) in the domain \( \Omega = \{(x, t) : R_1 < |x| < R_2, \quad t > 0\} \) with initial conditions

\[
u(x, 0) = \frac{x}{|x|} q_0(r), \quad \rho(x, 0) = \rho_0(r), R_1 < |x| < R_2
\]

(4.9)

and boundary conditions

\[
\lim_{|x| \to R_i} \frac{x}{|x|} u(x, t) = q_i, \quad i = 1, 2.
\]

(4.10)

where \( q_i \) are constants. For \( \rho \) we need boundary condition on \( (|x| = R_i) \) only if \((-1)^{i+1} q_i \) is positive.

\[
\lim_{|x| \to R_i} \rho(x, t) = \rho_i(t), \quad i f, \quad (-1)^{i+1} q_i > 0.
\]

(4.11)

We also assume first order consistency condition of the initial and boundary data at \( \{(x, t) : |x| = R_, \quad t = 0\} \). Let \( \Gamma = \{ (x, t) \in \Omega : either \quad |x| = R_i \quad or, \quad t = 0\} \). Define \( p_r \) on \( \Gamma \) by

\[
p_r(r, t) = \begin{cases} p_0(r), & \text{if } t = 0, \\
p_1(t), & \text{if } r = R_i. \end{cases}
\]

(4.12)

As \( q \) is smooth, let \( \beta(r, t, s) \) be the characteristic of \( p_t + (qp)_r = 0 \) passing through \( (r, t) \) and let this characteristic meet the boundary point at time \( t = t_0(r, t) \).

Let \( J_k, \quad k = 0, 1 \), denote the Bessel functions of first kind and order \( k \) and \( Y_k, \quad k = 0, 1 \), denote the Bessel functions of second kind and order \( k \). With these notations, we have the following theorem.
Theorem 4.2.

\[
u(x, t) = -\frac{x}{|x|} \int_{R_1}^{R_2} \partial_r G(r, \xi, t)e^{\frac{1}{\lambda} \int_0^t q_0(s)ds} d\xi
\]
\[
\rho(x, t) = r^{-(n-1)} p_0(\beta(r, t, t_0(r, t)), t_0(r, t)) e^{-\int_{t_0(r, t)}^t q_0(\beta(r, s, t))ds},
\]
where \(G(r, \xi, t)\) is described below. For \(n = 2\), \(G\) has the form

\[
G(r, \xi, t) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{B_n} [(q_2/e) J_0(\lambda_n r_2) - \lambda_n J_1(\lambda_n r_2)]^2 \xi H_n(r, t)H_0(\xi) e^{-\frac{\lambda_n^2}{2} t}
\]

where

\[
B_n = (\lambda_n^2 + (q_2/e)^2)[-(k_1/e) J_0(\lambda_n R_1) + \lambda_n J_1(\lambda_n R_1)]^2 - (\lambda_n^2 + (k_1/e)^2) (q_2/e) J_0(\lambda_n R_2) - \lambda_n J_1(\lambda_n R_2)]^2,
\]

\[
H_n(r) = [-(q_1/e) Y_0(\lambda_n R_1) + \lambda_n Y_1(\lambda_n R_1)] J_0(\lambda_n r) - [-(q_1/e) Y_0(\lambda_n R_1) + \lambda_n Y_1(\lambda_n R_1)] Y_0(\lambda_n r)
\]

and the \(\lambda_n\) are positive roots of the equation

\[
[-(q_1/e) J_0(\lambda R_1) + \lambda J_1(\lambda R_1)] [(q_2/e) Y_0(\lambda R_2) - \lambda Y_1(\lambda R_2)] - [(q_2/e) J_0(\lambda R_2) - \lambda J_1(\lambda R_2)] [-(q_1/e) Y_0(\lambda R_1) + \lambda Y_1(\lambda R_1)] = 0.
\]

For \(n = 3\), \(G\) has the form

\[
G(r, \xi, t) = \frac{2\xi}{r} \sum_{n=1}^{\infty} \frac{(b_1^2 + R_2^2 \lambda_n^2) \Psi_n(r) \Psi_n(\xi) e^{-\frac{\lambda_n^2}{2} t}}{R_2 - R_1 (b_1^2 + R_1^2 \lambda_n^2)(b_2^2 + R_2^2 \lambda_n^2) + (b_1 R_2 + b_2 R_1)(b_1 b_2 + R_1 R_2 \lambda_n^2)}
\]

where

\[
\Psi_n(r) = b_1 \sin[\lambda_n(r - R_1)] + R_1 \lambda_n \cos[\lambda_n(r - R_1)]
\]

\[
b_1 = -(q_1/e) R_1 + 1, \quad b_2 = (q_2/e) R_2 - 1,
\]

and \(\lambda_n\) are positive solutions of the transcendental equation

\[
(b_1 b_2 - R_1 R_2 \lambda^2) \sin[\lambda(R_2 - R_1)] + \lambda(R_1 b_1 + R_2 b_2) \cos[\lambda(R_2 - R_1)] = 0.
\]

Proof. From the theorem (3.1), we get \(u = -e^{\frac{\phi}{a}}\) where \(a\) satisfies

\[
a_t = \frac{\epsilon}{2} [a_{rr} + \frac{(n-1)}{r} a_r], \quad R_1 < r < R_2, \quad t > 0
\]

\[
a(r, 0) = e^{\frac{1}{2} \int_0^t q_0(s)ds}, \quad R_1 < r < R_2
\]

\[
a_{r_1}(R_1, t) + q_1 a(R_1, t) = 0, \quad i = 1, 2, \quad t > 0.
\]

Then \(G\) given by \([4.14]-[4.16]\) and \([4.17]-[4.19]\) are just the Green’s functions for the boundary value problem \([4.20]\) for \(n = 2\) and \(n = 3\) respectively, see \([16]\).

The formula \([4.13]\) for \(u\) follows. The formula for \(\rho\) is obtained by the method of characteristics the same way as in the previous theorem and details are omitted. \(\square\)

Remarks on large time behaviour:

It is easy to get the large time behaviour of the velocity from the formulas. First we consider the problem in the domain \(\{(x, t) : |x| < R, t > 0\}\). Then \(u\) given by \([4.3]-[4.7]\) has the following asymptotic form :
\[
\lim_{t \to \infty} u(x, t) = -\epsilon \frac{x}{|x|} \int_0^R \frac{\partial_r G(r, \xi) e^{-\frac{1}{2} \int_0^\xi \frac{q_0(s)}{d\xi} d\xi}}{\int_0^R G(r, \xi) e^\frac{1}{2} \int_0^\xi \frac{q_0(s)}{d\xi} d\xi} \] 

where for \( n = 2, \)

\[
G(r, \xi) = \xi J_0(\frac{\mu_1 \xi}{R}) J_0(\frac{\mu_1 r}{R})
\]

\( \mu_1 \) being the first positive solution of the equation \( \mu J_1(\mu) - kRJ_0(\mu) = 0. \) For \( n = 3, \)

\[
G(r, \xi) = \frac{\xi}{r} \sin(\frac{\mu_1 r}{R}) \sin(\frac{\mu_1 \xi}{R}),
\]

\( \mu_1 \) being the first positive solution of the equation \( \mu \cot(\mu) + \frac{\mu_1}{\epsilon} - 1 = 0. \)

Now consider the problem in the domain \( \{ (x, t) : R_1 < |x| < R_2, t > 0 \}. \) Then \( u \) given by (4.13)- (4.19) has the following asymptotic form:

\[
\lim_{t \to \infty} u(x, t) = -\epsilon \frac{x}{|x|} \int_{R_1}^{R_2} \frac{\partial_r G(r, \xi) e^{-\frac{1}{2} \int_0^\xi \frac{q_0(s)}{d\xi} d\xi}}{\int_{R_1}^{R_2} G(r, \xi) e^\frac{1}{2} \int_0^\xi \frac{q_0(s)}{d\xi} d\xi} \] 

where for \( n = 2, \)

\[
G(r, \xi) = \xi H_1(r) H_1(\xi)
\]

where \( H_1 \) is given by (4.16) with \( \lambda_1 \) being the first positive root of the equation (4.18). For \( n = 3, \)

\[
G(r, \xi) = \frac{\xi}{r} \Psi_1(r) \Psi_1(\xi)
\]

where \( \Psi_1(r) = b_1 \sin[r_1 (r - R_1)] + R_1 \lambda_1 \cos[r_1 (r - R_1)], b_1 = -(q_1/\epsilon)R_1 + 1, \]

\( b_2 = (q_2/\epsilon)R_2 - 1, \) and \( \lambda_1 \) is the first positive solution of the equation (4.18).

Asymptotic behaviour of \( \rho \) is more complex and depends on whether mass flows out or flows in. Consider the mass \( m(t) = \int_D \rho(x, t)dx. \) When \( \rho \) is spherically symmetric and \( D = \{ x : |x| < R \}, \)

\[
m(t) = \omega_{n-1} \int_0^R r^{n-1} \rho(r, t)dr.
\]

From the second equation of (3.8), we have

\[
\frac{dm(t)}{dt} = -q(R, t)p(R, t)
\]

and when \( \rho \) is spherically symmetric and \( D = R_1 < |x| < R_2, \)

\[
\frac{dm(t)}{dt} = -(q(R_2, t)p(R_2, t) - q(R_1, t)p(R_1, t)).
\]

This shows that mass is conserved if velocity is zero on the space boundary.

5. Spherically symmetric solution for the inviscid case

The aim of this section is to find explicit global radial solution of

\[
u_t + (u, \nabla) u = 0, \quad \rho_t + \nabla.(\rho u) = 0, \tag{5.1}
\]

with initial conditions

\[
u(x, 0) = q_0(r), \quad \rho(x, 0) = \rho_0(r) = r^{-(n-1)}p_0(r), \quad r = |x|, \tag{5.2}
\]

a condition on the mass

\[
\int_{R^n} \rho(x, t)dx = p_B(t) \tag{5.3}
\]
and prescribed normal velocity at the origin,

\[ \lim_{x \to 0} \frac{u(x,t)}{r} = q_B(t). \quad (5.4) \]

For initial data \( u_0(x) \), in the space of bounded variation, the vanishing viscosity limit of the velocity component \( u \) remains in the space of bounded variation, see [8]. The density \( \rho \) is generally a measure. In an important paper, local study of the Cauchy problem for (5.1) and propagation of delta wave front was carried out by Albeverio and Shelkovich [1] in the frame work of the weak asymptotic method. We summarize the definitions and results in [1] relevant to our work.

A distribution \((u, \rho)\) is called a generalized \( \delta \)-shock wave type solution if it is the distributional limit of a weak asymptotic solution \((u^\epsilon, \rho^\epsilon)\), as \( \epsilon \) goes to 0. They started with an ansatz of the form

\[ u(x,t) = u_+ H(S(x,t) > 0) + u_- H(S(x,t) < 0), \quad \rho = \bar{\rho}(x,t) + \epsilon(t) \delta_{S(x,t)=0} \quad (5.5) \]

where \( u_+, u_- \) and \( \bar{\rho} \) are smooth functions away from the surface \( S(x,t) = 0 \). They considered a smooth approximation \((u^\epsilon, \rho^\epsilon)\) of (5.5) by a suitable regularization of the Heaviside function \( H \) and \( \delta \) measure. They showed that if this regularization is a weak asymptotic solution in the sense (2.1), then \((u, \rho)\) should satisfy the equation (5.1) in the region of smoothness and along the surface of discontinuity \( S(x,t) = 0 \), the Rankine-Hugoniot condition,

\[ S_t + u_\delta \nabla_x S|_{\Gamma} = 0, \quad \frac{\delta \epsilon}{\delta t} + \nabla_{\Gamma}(\hat{\epsilon} u_\delta) = ([\bar{\rho}u] - [\bar{\rho}] u_\delta) \nabla_x S|_{\Gamma}, \quad (5.6) \]

is satisfied. Here \( u_\delta = \frac{u_+ + u_-}{2} \), \( u_+, u_- \) being the velocities behind and ahead of the surface of discontinuity, respectively and \( \Gamma_t = \{x : S(x,t) = 0\} \).

Conversely, if there exists a distribution \((u, \rho)\) and a surface \( S(x,t) = 0 \) satisfying the above properties, then there exists a weak asymptotic solution \((u^\epsilon, \rho^\epsilon)\) of (5.1) whose limit is \((u, \rho)\).

Further starting from an initial discontinuous wave front satisfying entropy condition, they constructed \((u, \rho)\) and a surface \( S(x,t) = 0 \) satisfying (5.1) in the region of smoothness and the Rankine Hugoniot condition (5.6) along surface of discontinuity \( S(x,t) = 0 \), using the work of Majda [14] [15] on existence and stability of multidimensional shock fronts. This construction is only local.

For the radial case we give a global construction of \((u, \rho)\). We also do not require entropy condition on the initial data for \( u \), as the equation is reduced to one dimensional Burgers equation and hence we use the well-known one dimensional theory.

As a first step of construction of \((u, \rho)\), we derive the equation for the radial case. An easy calculation as in section 3, shows that \( q \) and \( p = r^{(n - 1)} \rho \) satisfy the equations

\[ q_t + q q_r = 0, \quad p_t + (pq)_r = 0, \quad (5.7) \]

with initial conditions

\[ q(x,0) = q_0(r), \quad p(x,0) = p_0(r), \quad r = |x|, \quad (5.8) \]

and boundary condition for \( q \) and integral condition on \( p \):

\[ q(0,t) = q_B(t), \quad \omega_{n-1} \int_0^\infty p(r,t)dr = p_B(t) \quad (5.9) \]
which is to be understood in a weak sense [2]. Weak formulation for the boundary condition for \( q \) is

\[ \begin{align*}
\text{either } q(0^+, t) &= q_B(t) \\
\text{or } q(0^+, t) &\leq 0 \text{ and } q^2(0^+, t) \leq q_B^+(t)^2
\end{align*} \]

(5.10)

and for \( p \) is

\[ \text{if } q(0^+, t) > 0 \text{ then } \omega_{n-1} \int_0^\infty p(r, t) dr = p_B(t). \]

(5.11)

To describe the solution, we follow [6, 9, 10]. We introduce a class of paths in the quarter plane \( D = \{(z, s) : z \geq 0, s \geq 0\} \). For each fixed \((r, r_0, t), r \geq 0, r_0 \geq 0, t > 0, C(r, r_0, t)\) denotes the following class of paths \( \beta \). Each path is connected from the point \((r_0, 0)\) to \((r, t)\) and is of the form \( z = \beta(s) \), where \( \beta \) is a piecewise linear function of maximum three lines. On \( C(r, r_0, t) \), we define a functional

\[ J(\beta) = \frac{1}{2} \int_{\{s; \beta(s) = 0\}} (q_B(s)^+)^2 ds + \frac{1}{2} \int_{\{s; \beta(s) \neq 0\}} \frac{d\beta(s)^2}{ds} ds. \]

(5.12)

We call \( \beta_0 \) the straight line path connecting \((r_0, 0)\) and \((r, t)\) which does not touch the space boundary \( x = 0 \), namely \( \{(0, t), t > 0\} \). Then let

\[ A(r, r_0, t) = J(\beta_0) = \frac{(r-r_0)^2}{t}. \]

(5.13)

Any \( \beta \in C^*(r, r_0, t) = C(r, r_0, t) - \beta_0 \) is made up of three pieces, namely lines connecting \((r_0, 0)\) to \((0, t_1)\) in the interior and \((0, t_1)\) to \((0, t_2)\) on the boundary and \((0, t_2)\) to \((r, t)\) in the interior. For such curves, it can be easily seen from (5.12) that

\[ J(\beta) = J(r, r_0, t, t_1, t_2) = - \int_{t_1}^{t_2} \frac{(q_B(s)^+)^2}{2} ds + \frac{r_0^2}{2t_1} + \frac{r^2}{2(t-t_2)}. \]

(5.14)

For curves \( \beta \in C^*(r, r_0, t) \) made up of two straight lines with one piece lying on the boundary \( r = 0 \), we can write down a similar expression as in the earlier case.

It was proved in [10] that there exists \( \beta^* \in C^*(r, r_0, t) \) and corresponding \( t_1(r, r_0, t), t_2(r, r_0, t) \) so that

\[ B(r, r_0, t) = \min \{ J(\beta) : \beta \in C^*(r, r_0, t) \} \]

\[ = \min \{ J(r, r_0, t, t_1, t_2) : 0 \leq t_1 < t_2 < t \} \]

(5.15)

\[ = J(r, r_0, t, t_1(r, r_0, t), t_2(r, r_0, t)) \]

is Lipschitz continuous. Further

\[ m(r, r_0, t) = \min \{ J(\beta) : \beta \in C(r, r_0, t) \} \]

\[ = \min \{ A(r, r_0, t), B(r, r_0, t) \} \]

(5.16)

and

\[ Q(r, t) = \min \{ m(r, r_0, t) + \int_0^r q_0(s) ds, 0 \leq r_0 < \infty \} \]

(5.17)

are Lipschitz continuous function in their variables. Further, minimum in (5.17) is attained at some value of \( r_0 \geq 0 \), which depends on \((r, t)\); we call it \( r_0(r, t) \). If \( A(r, r_0(r, t), t) \leq B(r, r_0(r, t), t) \),

\[ Q(r, t) = \frac{(r-r_0(r, t))^2}{2t} + \int_0^{r_0(r, t)} q_0(s) ds \]

(5.18)
and if \( A(r, r_0(r, t), t) > B(r, r_0(r, t), t) \),

\[
Q(x, t) = J(r, r_0(r, t), t, t_1(r, r_0(r, t), t), t_2(r, r_0(r, t), t)) + \int_0^{r_0(r, t)} q_0(s)ds. \tag{5.19}
\]

Here and henceforth \( r_0(r, t) \) is a minimizer in (5.17) and in the case of (5.19), \( t_2(r, t) = t_2(r, r_0(r, t), t) \) and \( t_1(r, t) = t_1(r, r_0(r, t), t) \). With these notations, we have the following result.

**Theorem 5.1.** With \( r_0(r, t), A(r, r_0(r, t), t), B(r, r_0(r, t), t), t_1(r, t), t_2(r, t) \) as defined above, define

\[
u(x, t) = \begin{cases} \frac{r-r_0(r, t)}{r-t_1(x, t)}, & \text{if } A(r, r_0(r, t), t) < B(r, r_0(r, t), t), \
\frac{r}{r-t_1(x, t)}, & \text{if } A(r, r_0(r, t), t) > B(r, r_0(r, t), t), \end{cases}
\tag{5.20}
\]

and

\[
P(r, t) = \begin{cases} -\int_{r_0(r, t)}^{\infty} p_0(z)dz, & \text{if } A(r, r_0(r, t), t) < B(r, r_0(r, t), t), \\
\omega_{n-1}p_B(t_2(x, t)), & \text{if } A(r, r_0(r, t), t) > B(r, r_0(r, t), t). \end{cases}
\tag{5.21}
\]

and set

\[
\rho(x, t) = \frac{\partial_r (P(r, t))}{r^{n-1}}. \tag{5.22}
\]

Then the distribution \((u(x, t), \rho(x, t))\) given by (5.20)-(5.22) satisfies (5.1) in the region of smoothness and (5.6), along a discontinuity surface. Further it satisfies the initial conditions (5.2), mass conditions (5.3) and normal velocity at the origin (5.4) in the weak sense (5.8)-(5.11).

**Proof.** Explicit formula for the entropy weak solution \( q \) satisfying (5.8)-(5.10) is given in [6, 9]. This formula involves only a finite dimensional minimization, namely (5.15)-(5.17) in three variables \((t_1, t_2, y)\). The formula takes the form

\[
q(r, t) = \begin{cases} \frac{r-r_0(r, t)}{t-t_1(x, t)}, & \text{if } A(r, r_0(r, t), t) < B(r, r_0(r, t), t), \\
\frac{r}{t-t_1(x, t)}, & \text{if } A(r, r_0(r, t), t) > B(r, r_0(r, t), t). \end{cases}
\tag{5.23}
\]

To get the component \( p \) we consider the problem for

\[
P(r, t) = -\int_r^{\infty} p(s, t)ds \tag{5.24}
\]

so that

\[
p(r, t) = \partial_r (P(r, t)). \tag{5.25}
\]

It is easy to see from (5.7), (5.8) and (5.11) that \( P \) must satisfy

\[
P_1 + qP_z = 0,
\]

with initial condition

\[
P(r, 0) = -\int_r^{\infty} p_0(s)ds
\]

and boundary condition

\[
\text{if } q(0+, t) > 0 \text{ then } P(r, t)dr = \frac{1}{\omega_{n-1}}p_B(t).
\]

An explicit formula for the solution of this problem is given in [9] and has the form (5.21). The formula (5.20)-(5.22) is then obtained from the transformation (5.4) and (5.23)-(5.25).
Now we show that \((u, \rho)\) given by (5.20)-(5.22) is a solution of (5.1) in the region of smoothness. This fact follows easily as \((q, p)\) satisfies the system (5.7) and the weak form of initial and boundary conditions (5.8)-(5.11) (see [9]). Also it was shown in [9], that \((q, p)\) is a weak solution of (5.7) and that the surface of discontinuity of \((u, \rho)\) and \((q, p)\) are same. In order to show that \((u, \rho)\) satisfies the Rankine-Hugoniot conditions, we use the corresponding conditions for the system (5.7). This follows from [11] and we give the details here.

Consider a solution \((q(r, t), p(r, t))\) of the system (5.7) with a discontinuity on the surface \(S(r, t) = 0\), with \(S(r, t) = r - s(t)\). In a neighbourhood of this surface, we assume \((q, p)\) has the form

\[
q(r, t) = \bar{q}(x, t), \quad p(r, t) = \bar{p}(r, t) + e(t)\delta_{r=s(t)},
\]

where \(\bar{q}, \bar{p}\) are smooth except on the surface \(r = s(t)\), and \(e(t)\) a differentiable function of \(t\). The Rankine-Hugoniot condition for \((q, p)\) takes the form

\[
\frac{ds(t)}{dt} = \frac{q_+ - q_-}{2}, \quad \frac{de}{dt} = [qp] - [p] \frac{ds}{dt}.
\]

We show that the distributions (5.20)-(5.22) satisfy (5.6). As \(S\) the surface of discontinuity, we use the corresponding conditions for the system (5.7). An easy computation gives

\[
S_t = -\frac{ds}{dt}, \quad \nabla_x S = \frac{x}{r}.
\]

The first equation of (5.6) becomes

\[
S_t + u_3 \nabla_x S|_{\Gamma_1} = -\frac{ds}{dt} + \frac{q_+ + q_-}{2} \frac{x}{r} \frac{x}{r} = -\frac{ds}{dt} + \frac{q_+ + q_-}{2} = 0,
\]

where we used the first equation of (5.27). To verify the second equation in (5.6), we compute each terms in radial components.

\[
\frac{\delta e}{\delta t} + \frac{\partial e}{\partial t} + G \frac{\partial e}{\partial r} = \frac{\partial e}{\partial r} - S_r \frac{\partial e}{\partial r} = \frac{de}{dt}.
\]

Also we know from [11],

\[
\nabla_{\Gamma_1}(\dot{e}u_3) = -2KG\dot{e},
\]

where \(K\) is the mean curvature of the surface of discontinuity, \(K = -\frac{1}{2} \nabla_\nu, \nu = \frac{\dot{e}}{K}\) and \(G = -\frac{\dot{S}_r}{S_r}\). An easy computation gives

\[
K = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial u_3}{\partial x_i} = -\frac{1}{2} \frac{1}{r} + \frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2}{r^3} = -\frac{n-1}{2r}.
\]

Using (5.32) in (5.31), we get

\[
\nabla_{\Gamma_1}(\dot{e}u_3) = \frac{n-1}{r} \left( -\frac{S_t}{S_r} \right) \dot{e} = \frac{n-1}{r} \frac{dr}{dt} \dot{e}.
\]

Finally,

\[
([\rho u] - [\rho u_3]) \nabla_x S|_{\Gamma_1} = \left( \frac{1}{r^{(n-1)}} - \frac{1}{r^{(n-1)}} [pq] - \frac{1}{r^{(n-1)}} \frac{q_+ + q_-}{2} \right) S_r \frac{x}{r} = \frac{S_r}{r^{(n-1)}} ([pq] - [p] \frac{ds}{dt}).
\]
Since \( S_r = 1 \), we get from (5.30), (5.33) and (5.34),
\[
\begin{align*}
\delta \dot{e} + \nabla | \dot{\Gamma}(\dot{e}u_\delta) - (|p u| - |p| u_\delta), \Delta S |_{\Gamma} = & \frac{\delta \dot{e}}{\delta t} + \frac{n-1}{r} \frac{dr}{dt} \dot{e} - \frac{S_r}{r^{(n-1)}} (|p q| - |p| \frac{dr}{dt}) \\
= & \frac{1}{r^{(n-1)}} \frac{d (r^{(n-1)} \dot{e})}{dt} - [p q] - [p] \frac{dr}{dt} \\
= & \frac{1}{r^{(n-1)}} \frac{d c_e}{dt} - [p q] - [p] \frac{dr}{dt} = 0,
\end{align*}
\]
where in the last equality we used the second equation of (5.27). This completes the proof of the theorem. \( \square \)

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