On the Jacobi Group and the Mapping Class Group of $S^3 \times S^3$

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Abstract

The paper contains a proof that the mapping class group of the manifold $S^3 \times S^3$ is isomorphic to a central extension of the (full) Jacobi group $\Gamma^J$ by the group of 7-dimensional homotopy spheres. Using a presentation of the group $\Gamma^J$ and the $\mu$-invariant of the homotopy spheres, we give a presentation of this mapping class group with generators and defining relations. We also compute cohomology of the group $\Gamma^J$ and determine a 2-cocycle that corresponds to the mapping class group of $S^3 \times S^3$.

1 Introduction

The central theme of this paper is the group of isotopy classes of orientation preserving diffeomorphisms on $S^3 \times S^3$. We will denote this group by $\pi_0\text{Diff}(S^3 \times S^3)$. In general, the group of isotopy classes of orientation preserving diffeomorphisms on a closed oriented smooth manifold $M$ will be denoted by $\pi_0\text{Diff}(M)$ and called the mapping class group of $M$ by analogy with the 2-dimensional case.

The article consists of two parts. Our goal in the first part will be to give a presentation of the mapping class group of $S^3 \times S^3$ with generators and defining relations. The main step in this direction is Theorem 1, where we prove that $\pi_0\text{Diff}(S^3 \times S^3)$ is a central extension of the (full) Jacobi group $\Gamma^J$. 


Γ^J by the group of 7-dimensional homotopy spheres Θ_7. The second part is
concerned with the cohomology group H^2(Γ^J, Z_{28}). We show that this group
is isomorphic to Z_{28} ⊕ Z_4 ⊕ Z_2 and determine a 2-cocycle that corresponds
to π₀Diff(S^3 × S^3).

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2 A presentation of the group π₀Diff(S^3 × S^3)

For a closed smooth oriented (k-1)-connected almost-parallelizable manifold
M^{2k}, k ≥ 2 the group π₀Diff(M) has been computed in terms of exact se-
quences by Kreck [13]. We will begin by recalling his results in the first
section. Next we will show that there is an isomorphism between the fac-
tor group π₀Diff(S^3 × S^3)/Θ and the full Jacobi group Γ^J. The group of
isotopy classes of diffeomorphisms that act trivially on homology of S^3 × S^3
is determined in §2.3. In the last section of this part we give a presentation
of π₀Diff(S^3 × S^3) with generators and defining relations (Theorem 3.). All
diffeomorphisms are assumed to be orientation preserving and integer coeffi-
cients are understood for all homology and cohomology groups.

2.1 Exact sequences of Kreck

Our focus in this paragraph is to review the results of Kreck [13]. First
we recall some of the definitions and notations. From now on by a mani-
fold M we will mean a closed oriented differentiable (k-1)-connected almost-
parallelizable 2k-manifold.

Denote by Aut H_k(M) the group of automorphisms of H_k(M, Z) pre-
serving the intersection form on M and (for k ≥ 3) commuting with the
function α : H_k(M) → π_{k-1}(SO(k)), which is defined as follows. Repre-
sent x ∈ H_k(M) by an embedded sphere S^k ↪ M. Then function α assigns
to x the classifying map of the corresponding normal bundle. Any diffeomor-
phism f ∈ Diff(M) induces map f_* which lies in Aut H_k(M). This gives a
homomorphism

κ : π₀Diff(M) → Aut H_k(M), [f] ↦ f_*
The kernel of $\kappa$ is denoted by $\pi_0 SDiff(M)$. For elements of $\pi_0 SDiff(M)$ Kreck defines the following invariant: Choose again a sphere $S^k \hookrightarrow M$ that represents an element $x \in H_k(M)$. Since $[f] \in \pi_0 SDiff(M)$ we can assume that $f|_{S^k} = \text{Id}$. The stable normal bundle $\nu(S^k) \oplus 1$ of $S^k$ in $M$ is trivial and we can choose some trivialization $\tau : \nu(S^k) \oplus 1 \rightarrow S^k \times \mathbb{R}^{k+1}$. Clearly, the differential of $f$ leaves the tangent bundle of $S^k$ invariant and hence induces an automorphism of the normal bundle $\nu(S^k) \oplus 1$. At each point $t \in S^k$ this automorphism gives (via trivialization $\tau$) an element $P_t$ of the group $SO(k+1)$ and hence we get an element $P \in \pi_k(SO(k+1))$. It is obvious that $P$ lies in the image of the map $S : \pi_k(SO(k)) \rightarrow \pi_k(SO(k+1))$ induced by the inclusion $SO(k) \hookrightarrow SO(k+1)$. It is a standard fact that element $P$ does not depend on trivialization $\tau$ of the normal bundle $\nu(S^k) \oplus 1$. This construction leads to a well defined homomorphism (cf. Lemmas 1,2 of [13])

$$\chi : \pi_0 SDiff(M) \rightarrow Hom(H_k(M), S\pi_k(SO(k)))$$

For $k \equiv 3 \pmod{4}$ the group $S\pi_k(SO(k))$ is isomorphic to the cyclic group $\mathbb{Z}$ and hence we can identify $Hom(H_k(M), S\pi_k(SO(k)))$ with the cohomology group $H^k(M)$. In this case one can describe $\chi(f)$ by the Pontrjagin class of the mapping torus $M_f$, but we will not use this description here so we omit the details. The following theorem is due to Kreck (cf. Theorem 2, [13]):

**Theorem:** $k \geq 3$. If $M^{2k}$ bounds a framed manifold, then the following sequences are exact:

$$0 \rightarrow \pi_0 SDiff(M) \rightarrow \pi_0 Diff(M) \xrightarrow{\kappa} Aut H_k(M) \rightarrow 0 \quad (1)$$

$$0 \rightarrow \Theta_{2k+1} \xrightarrow{\iota} \pi_0 SDiff(M) \xrightarrow{\chi} Hom(H_k(M), S\pi_k(SO(k))) \rightarrow 0 \quad (2)$$

**Remark:** If one considers $M^{2k}$ which does not bound a framed manifold then $Ker(\chi)$ will be a factor group $\Theta_{2k+1}/\Sigma_M$ instead of the whole group of homotopy spheres. Since $S^3 \times S^3$ is the boundary of $S^5 \times D^4$, we have $\Sigma_M = 0$ (cf. Lemma 3. of [13]).

Map $\iota$ is defined as follows. Present a homotopy sphere $\Sigma \in \Theta_{2k+1}$ as union $D^{2k+1} \cup_f D^{2k+1}$ and assume that $f = \text{Id}$ on a neighbourhood of the lower hemisphere $D^{2k} \subset S^{2k}$. Then $\iota(\Sigma)$ is the class of diffeomorphism on $M$ which is identity outside an embedded disk in $M$ and is equal to $f|_{D^{2k}}$ on this disk.
It follows from the second exact sequence that \( \pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7 \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \). Note also that \( \text{Aut} \ H_3(S^3 \times S^3) \cong SL_2(\mathbb{Z}) \). Indeed, the function \( \alpha : H_3(S^3 \times S^3) \to \pi_2(SO(3)) \) is zero map. Hence an element \( A \) of the group \( \text{Aut} \ H_3(S^3 \times S^3) \) will be any automorphism of \( \mathbb{Z} \oplus \mathbb{Z} \) which preserves the intersection form on \( S^3 \times S^3 \). It means that if we choose a basis for \( H_3(S^3 \times S^3) \) then \( A \in SL_2(\mathbb{Z}) \).

Exact sequences (1) and (2) induce the following short exact sequence

\[
0 \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7 \to SL_2(\mathbb{Z}) \to 0 \quad (3)
\]

In the next section we prove that this exact sequence splits and the group \( \pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7 \) is isomorphic to the Jacobi group \( \Gamma_J \).

### 2.2 Splitting of the Exact Sequence

The (full) Jacobi group \( \Gamma_J \) is a semidirect product of the modular group with the direct sum \( \mathbb{Z} \oplus \mathbb{Z} \). More precisely (cf. [8] §I.1.),

\[
\Gamma_J \overset{\text{def}}{=} SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2 = \text{set of pairs} \ (M, X) \text{ with } M \in SL_2(\mathbb{Z}), \ X \in \mathbb{Z} \oplus \mathbb{Z} \text{ and group law} \ (M, X) \cdot (M', X') = (MM', XM' + X') \text{ (notice that vectors are written as row vectors, i.e. } SL_2(\mathbb{Z}) \text{ acts on the right). It is interesting to note that } \Gamma_J \text{ first came up in the theory of Jacobi forms (see [8]). We will need}

\[\text{Lemma 1.} \quad \Gamma_J \text{ admits the following presentation:}
< y, u, a, b | yuy = uyu, \ (yuy)^4 = \text{id}, \ ab = ba, \ ay = yab, \ au = ua, \ by = yb, \ bu = uba^{-1} >\]

**Proof.** \( SL_2(\mathbb{Z}) \) has a presentation: \(< y, u \mid yuy = uyu, \ (yuy)^4 = \text{id} > \) (see for example [4]) where \( y \) and \( u \) correspond to matrices

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}
\]

respectively. \( (\text{It is a classical fact that } SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 \ast_{\mathbb{Z}_2} \mathbb{Z}_6). \text{ Hence } SL_2(\mathbb{Z}) \text{ has a presentation: } < x, z \mid x^4 = \text{id}, \ x^2 = z^3 >. \text{ One can use a map } f : f(z) = yu, \ f(x) = (yuy)^{-1} \text{ to show that these two presentations define isomorphic groups.} \) By definition of \( \Gamma_J \) the following sequence is exact.

\[
0 \to \mathbb{Z} \oplus \mathbb{Z} \to \Gamma_J \to SL_2(\mathbb{Z}) \to 0
\]

Consider a homomorphism \( \alpha : SL_2(\mathbb{Z}) \to \Gamma_J \) defined by the formulas:

\[
\alpha(y) \overset{\text{def}}{=} (y, (0,0)), \quad \alpha(u) \overset{\text{def}}{=} (u, (0,0)). \text{ If we denote elements } (y, (0,0));
$(u,(0,0)); (id,(1,0)); (id,(0,1))$ by $y,u,a$ and $b$ respectively we see that these elements $y,u,a,b$ generate $\Gamma^J$ and the relations $yuy = uy$, $(yuy)^4 = id$, $ab = ba$ are satisfied. To find all defining relations for $\Gamma^J$ we need to find how $SL_2(\mathbb{Z})$ acts on the generators $a$ and $b$ of $\mathbb{Z}^2$ by conjugation. First note that $ab = (id,(1,0)) \cdot (id,(0,1)) = (id,(1,1))$ and $ba^{-1} = (id,(0,1)) \cdot (id,(-1,0)) = (id,(-1,1))$. Hence

$ay = (id,(1,0)) \cdot (y,(0,0)) = (y,(1,0) \cdot y + (0,0)) = (y,(1,1))$

$yab = (y,(0,0)) \cdot (id,(1,1)) = (y,(1,1)) \Rightarrow ay = yab$

$au = (id,(1,0)) \cdot (u,(0,0)) = (u,(1,0) \cdot u + (0,0)) = (u,(1,0))$

$ua = (u,(0,0)) \cdot (id,(1,0)) = (u,(1,0)) \Rightarrow au = ua$

$by = (id,(0,1)) \cdot (y,(0,0)) = (y,(0,1) \cdot y + (0,0)) = (y,(0,1))$

$yb = (y,(0,0)) \cdot (id,(0,1)) = (y,(0,1)) \Rightarrow by = yb$

$bu = (id,(0,1)) \cdot (u,(0,0)) = (u,(0,1) \cdot u + (0,0)) = (u,(-1,1))$

$uba^{-1} = (u,(0,0)) \cdot (id,(-1,1)) = (u,(-1,1)) \Rightarrow bu = uba^{-1}$

Remark: A different presentation of this group can be found in [6] (cf. also Lemma 6. below).

Consider now the standard sphere $S^3$ in Euclidean four-space $\mathbb{R}^4$, given by the equation: $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. This sphere can be identified with the special unitary group $SU(2)$ which is also known as group of unit quaternions. The group structure on $S^3$ induces the group structure on the product $S^3 \times S^3$. If we denote elements of the group $S^3$ by $s$ and $t$ we will write $(s,t)$ to denote the corresponding element of the group $S^3 \times S^3$. The product of two elements $(s,t)$ and $(s',t')$ will be the pair $(ss',tt')$ with quaternion multiplication understood.

**Theorem 1.** The factor group $\pi_0\text{Diff}(S^3 \times S^3)/\Theta_7$ is isomorphic to the Jacobi group $\Gamma^J$.

**Proof.** We will give a presentation of the factor group $\pi_0\text{Diff}(S^3 \times S^3)/\Theta_7$ that coincides with the above presentation of $\Gamma^J$. By $y$ and $u$ we denote the generators of $SL_2(\mathbb{Z})$ as above. Consider isotopy classes $[Y]$ and $[U]$ of the following diffeomorphisms of $S^3 \times S^3$ ($(s,t) \in S^3 \times S^3$):

$$Y : (s,t) \mapsto (s,st), \quad U : (s,t) \mapsto (t^{-1}s,t)$$

(4)

Define map $\beta : SL_2(\mathbb{Z}) \to \pi_0\text{Diff}(S^3 \times S^3)$ by the identities: $\beta(y) \overset{\text{def}}{=} [Y], \quad \beta(u) \overset{\text{def}}{=} [U]$ and extend it linearly to the whole group $SL_2(\mathbb{Z})$. We will show that $\beta$ is a well defined homomorphism from $SL_2(\mathbb{Z})$ to $\pi_0\text{Diff}(S^3 \times S^3)/\Theta_7$. 

\[5\]
First we check that $YUY = UYU$:

$$YUY : (s, t) \mapsto (s, st) \underbrace{\mapsto}_{Y} ((st)^{-1} s, st) = (t^{-1}, st) \underbrace{\mapsto}_{Y} (t^{-1}, t^{-1} st)$$

$$UYU : (s, t) \underbrace{\mapsto}_{U} (t^{-1} s, t) \underbrace{\mapsto}_{Y} (t^{-1} s, t^{-1} st) \underbrace{\mapsto}_{U} (t^{-1}, t^{-1} st)$$

Thus $YUY = UYU$ and hence $[Y][U][Y] = [U][Y][U]$. From now on we will denote a diffeomorphism and the isotopy class of it by the same capital letter, omitting the brackets. To prove the equality $(YUY)^4 = Id$ (Id stands for the identity diffeomorphism of $S^3 \times S^3$) we will need some auxiliary results.

Consider the following diffeomorphisms $A, B \in \text{Diff}(S^3 \times S^3)$:

$$A : (s, t) \mapsto (tst^{-1}, t) \quad B : (s, t) \mapsto (s, st s^{-1}) \quad (5)$$

If we choose spheres $S^3 \times 1$ and $1 \times S^3$ as generators of the group $H_3(S^3 \times S^3)$, it is obvious that diffeomorphisms $A$ and $B$ preserve these spheres and act trivially on homology of $S^3 \times S^3$.

**Lemma 2.** Isotopy classes of diffeomorphisms $A$ and $B$ generate the group $\text{Hom}(H_3(S^3 \times S^3), S\pi_3(SO(3))) \cong \mathbb{Z} \oplus \mathbb{Z}$

*Proof of the lemma.* Let us compute $\chi(B)$. Since $S^3 \times S^3$ is a parallelizable manifold and the normal bundle of $S^3_1 \overset{\text{def}}{=} S^3 \times 1$ in $S^3 \times S^3$ is trivial we need to find an element of the group $\pi_3(SO(3))$ that corresponds to the differential of $B$. Take a point $(s, 1) \in S^3_1$. We can identify the fiber of the normal bundle $\nu(S^3_1)$ at $(s, 1)$ with the fiber of the tangent bundle $\tau(s \times S^3)$ at this point. Furthermore, via the projection $\rho_2 : S^3 \times S^3 \rightarrow 1 \times S^3$ we can identify this tangent fiber at $(s, 1)$ with the tangent fiber at $(1, 1)$ (Lie algebra $\mathfrak{g}$ of the group $1 \times S^3$). Since $B_s(t) \overset{\text{def}}{=} B|_{s \times S^3}(s, t) = st s^{-1}$ the map $s \mapsto d_{(s, 1)} B_s$ will correspond to the *adjoint representation* $Ad : S^3_1 \rightarrow \text{Aut}(\mathfrak{g})$, and $\chi(B)(s, 1) = Ad_s \in SO(3)$. Thus $\chi(B)(S^3 \times 1) = Ad : S^3 \times 1 \rightarrow SO(3)$. If we choose an element $T \in \mathfrak{g}$ then it is well known that $Ad_s(T) = s T s^{-1}$. This map is a generator of the group $\pi_3(SO(3))$ (see [14], ch.I §2) and therefore the isotopy class of diffeomorphism $B$ is a generator of the group $H^3(S^3 \times S^3)$. In a similar way one can show that the isotopy class of diffeomorphism $A$ is the other generator (corresponding to the map $\chi(A)(1 \times S^3) : 1 \times S^3 \rightarrow SO(3)$ of the group $H^3(S^3 \times S^3)$).

Now we show that $AB = BA, \quad AY = YAB, \quad AU = UA, \quad BY = YB, \quad BU = UBA^{-1}$ in the factor group $\pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7$. The first equality follows from the results of Kreck, since as we just saw, $A$ and $B$ generate the abelian subgroup of the group $\pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7$. To prove the other equalities.
we use the group structure on $S^3 \times S^3$:

$AY : (s, t) \mapsto (s, st) \mapsto (stst^{-1}s^{-1}, st)$

$YAB : (s, t) \mapsto (s, st^{-1}s^{-1}) \mapsto (stst^{-1}s^{-1}, st) = (stst^{-1}s^{-1}, st) \Rightarrow AY = YAB,$

$AU : (s, t) \mapsto (t^{-1}s, t) \mapsto (tt^{-1}s^{-1}, t) = (st^{-1}, t)$

$UA : (s, t) \mapsto (st^{-1}, t) \mapsto (t^{-1}st^{-1}, t) = (st^{-1}, t) \Rightarrow AU = UA,$

$BY : (s, t) \mapsto (s, st) \mapsto (s, stst^{-1})$

$BU : (s, t) \mapsto (t^{-1}s, t) \mapsto (t^{-1}s, t^{-1}sts^{-1}t)$

$UBA^{-1} : (s, t) \mapsto (t^{-1}s, t^{-1}sts^{-1}t) = (t^{-1}s, t^{-1}sts^{-1}t) \Rightarrow BU = UBA^{-1}$

as required.

**Claim 1.** $(B^{-1}YUY)^4 = Id, \ YUYB^{-1} = A^{-1}YUY, \ YUYA^{-1} = BYUY.$

**Proof of the claim:** $B^{-1}YUY : (s, t) \mapsto (t^{-1}, t^{-1}s) \mapsto (t^{-1}, tt^{-1}sts^{-1}t) = (t^{-1}, s) \Rightarrow (B^{-1}YUY)^4 = Id.$

Identities $AY = YAB, \ BU = UBA^{-1}, \ AB = BA$ and $AU = UA$ imply $YB^{-1}A^{-1} = A^{-1}Y, \ UB^{-1} = B^{-1}A^{-1}U.$ Then from the above equalities we get $YUYB^{-1} = YUYB^{-1}Y = YB^{-1}A^{-1}UY = A^{-1}YUY.$ Similarly we see that $YA^{-1} = BA^{-1}Y, \ UB = BUA,$ hence $YUYA^{-1} = YUBA^{-1}Y = YBAA^{-1}Y = YBYU = BYUY$ which proves the claim.

Using these identities we can show that $(YUY)^4 = Id, \ Id = (B^{-1}YUY)^4 = B^{-1}YUYB^{-1}YUYB^{-1}A^{-1}(YUY)^2 = B^{-1}YUYB^{-1}BYUYB^{-1}(YUY)^2 = B^{-1}YUYYUYB^{-1}(YUY)^2 = B^{-1}YUYA^{-1}(YUY)^3 = (YUY)^4.$ It implies that exact sequence (3) splits; the factor group $\pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7$ has four generators $Y, U, A, B$ and the following set of defining relations: $YUY = UYU, \ (YUY)^4 = Id, \ AB = BA, \ AY = YAB, \ AU = UA, \ BY = YB, \ BU = UBA^{-1}.$ In particular, we see that groups $\pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7$ and $\Gamma^j$ have the same presentations and therefore isomorphic.

Note that diffeomorphisms $A$ and $B$ have been considered by Browder (3, Theorem 6.) to give an example of diffeomorphisms of $S^3 \times S^3$ which are homotopic to the identity, but are not pseudo-isotopic to the identity.
2.3 Group $\pi_0 SDiff(S^3 \times S^3)$

In this paragraph we prove that group $\pi_0 SDiff(S^3 \times S^3)$ is isomorphic to the group $\mathcal{H}_{28}$ where

$$\mathcal{H}_m \overset{\text{def}}{=} \left\{ \begin{pmatrix} 1 & a & l \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{Z} \text{ and } l \in \mathbb{Z}_m \right\}.$$ 

Group $\mathcal{H}_m$ also can be described as the factor group of the group $\mathcal{H}$ (upper unitriangular $3 \times 3$ matrices with integer coefficients) by the cyclic subgroup generated by the matrix with $a = b = 0$ and $l = m$.

Idea of the proof is to compare presentations of two groups as we just did above. First we give a presentation of $\pi_0 SDiff(S^3 \times S^3)$. We know already that $H^3(S^3 \times S^3, \mathbb{Z})$ is generated by diffeomorphisms $A$ and $B$ (Lemma 2.). We also know from exact sequence (2) that $\pi_0 SDiff(S^3 \times S^3)$ is a central extension of the group $\mathbb{Z} \oplus \mathbb{Z}$ by $\Theta_7 \cong \mathbb{Z}_{28}$ (see [12]). If we denote by $\hat{\Sigma}$ the generator of $\Theta_7$ (by the generator we mean a homotopy 7-sphere which bounds a parallelizable manifold of signature 8) then we can choose $A, B$ and $\Sigma \overset{\text{def}}{=} \iota(\hat{\Sigma})$ as generators of $\pi_0 SDiff(S^3 \times S^3)$. The defining relations clearly will be: $A\Sigma = \Sigma A$, $B\Sigma = \Sigma B$, $ABA^{-1}B^{-1} = \Sigma^k$, $\Sigma^{28} = 1$ for some $k \in \mathbb{Z}_{28}$. So the goal is to figure out what this $k$ is.

We first define a map $\varsigma : \pi_0 SDiff(S^3 \times S^3) \longrightarrow \mathbb{Z}_{28}$ as follows. Take a representative $f \in Diff(S^3 \times S^3)$ of a class $[f] \in \pi_0 SDiff(S^3 \times S^3)$.

**Definition 1.**

$$\varsigma([f]) \overset{\text{def}}{=} D^4 \times S^3 \bigcup_f S^3 \times D^4$$

Where $\bigcup_f$ means identification of a point $(x, y) \in \partial(D^4 \times S^3)$ with the point $f(x, y) \in \partial(S^3 \times D^4)$.

Denote by $\Sigma_f$ the manifold obtained from $\varsigma([f])$ by smoothing the corners. It is clear that $\Sigma_f$ depends only on the isotopy class $[f]$ and not on a specific representative of this class. Note that $\Sigma_f$ is a homotopy sphere and $\varsigma$ is a well defined map from $\pi_0 SDiff(S^3 \times S^3)$ to the group $\Theta_7 \cong \mathbb{Z}_{28}$.

Remark: Map $\varsigma$ is analog of the Birman-Craggs homomorphism from the Torelli group of $S^2_g$ (2-dimensional surface of genus $g$) to $\mathbb{Z}_2$. See [11] for the details.
Lemma 3. The composition $\varsigma \circ \iota$ is the identity map of the group $\Theta_7$.

Proof. Take a sphere $\tilde{\Sigma}_\phi \in \Theta_7$. We will denote by $\phi$ the diffeomorphism of $S^3 \times S^3$ which is the identity outside an embedded disk $D^6 \subset S^3 \times S^3$ and corresponds to the element $\iota(\tilde{\Sigma}_\phi)$ of the mapping class group.

To show that $\varsigma \circ \iota = \text{Id}$ it is enough to show that $\tilde{\Sigma}_\phi$ is diffeomorphic to $\Sigma_\phi = \varsigma \circ \iota(\tilde{\Sigma}_\phi)$. We construct an h-cobordism between these two manifolds. Take the handlebody $D^4 \times S^3$ and remove from it an interior disk $D^7$. The resulting manifold is denoted by $D^4 \times S^3$. Boundary components of $D^4 \times S^3$ are $S^6$ and $S^3 \times S^3$. Take disks $D^6$ in these two components and connect them by a tube $D^6 \times I$ embedded into $D^4 \times S^3$. Next extend $\phi$ in an obvious way (by the identity outside the tube) to a diffeomorphism $\Phi$ of $D^4 \times S^3$. Consider now two manifolds: $D^5 \times S^3$ and $D^8$. Present the boundary of $D^5 \times S^3$ as the union:

$$\partial(D^5 \times S^3) = S^4 \times S^3 = D^4 \times S^3 \bigcup_{S^3 \times S^3} \bar{D^4 \times S^3} \bigcup_{S^6} D^7$$

and the boundary of $D^8$ as the union:

$$\partial(D^8) = S^7 = S^3 \times D^4 \bigcup_{S^3 \times S^3} \bar{D^4 \times S^3} \bigcup_{S^6} D^7$$

Using diffeomorphism $\Phi$ we can glue $D^5 \times S^3$ and $D^8$ together along the common submanifold $D^4 \times S^3$ to obtain a cobordism (after smoothing the corners) $W^8$ between $\tilde{\Sigma}_\phi$ and $\Sigma_\phi$. It is clear that $W^8$ is simply connected. Using Mayer-Vietoris exact sequence of the union $D^4 \times S^3 = \bar{D^4 \times S^3} \cup D^7$ we see that

$$H_*(\bar{D^4 \times S^3}) \cong \left\{ \begin{array}{cl} \mathbb{Z} & \text{if } * = 0, 3, 6 \\ 0 & \text{otherwise} \end{array} \right.$$  

In a similar way we can get homology groups of $W^8 = D^5 \times S^3 \cup D^8$:

$$H_*(W^8) \cong \left\{ \begin{array}{cl} \mathbb{Z} & \text{if } * = 0 \text{ or } 7 \\ 0 & \text{otherwise} \end{array} \right. \quad \text{and} \quad H_*(W^8, \Sigma_\phi) \cong 0$$

Thus, by the h-cobordism theorem two homotopy spheres $\tilde{\Sigma}_\phi$ and $\Sigma_\phi$ are diffeomorphic. \(\square\)
Theorem 2. The generators $A, B$ and $\Sigma$ of $\pi_0\text{SDiff}(S^3 \times S^3)$ satisfy the relation: $ABA^{-1}B^{-1} = \Sigma$

Proof. For the proof we construct a spin manifold $W^8$ bounded by the sphere $\Sigma_{ABA^{-1}B^{-1}}$ and compute the $\mu$-invariant $\mu(\Sigma_{ABA^{-1}B^{-1}})$ defined by Eells and Kuiper [7].

First we extend diffeomorphisms $A$ and $B$ to diffeomorphisms of the handlebodies $D^4 \times S^3$ and $S^3 \times D^4$ respectively. It can be done since in the definition (recall formulas (5)) we can assume that $s \in D^4$, $t \in S^3$ for diffeomorphism $A$ and $s \in S^3$, $t \in D^4$ for diffeomorphism $B$. These extensions we also denote by $A$ and $B$ respectively. Next we present $\Sigma_{ABA^{-1}B^{-1}}$ as the union of five manifolds: $D^4 \times S^3 \cup_{A} S^3 \times D^4 \cup_{B} S^3 \times S^3 \cup_{A^{-1}} I \cup_{B^{-1}} S^3 \times D^4$ where $A, B, A^{-1}$ and $B^{-1}$ belong to $\text{Diff}(S^3 \times S^3)$. Consider manifolds $D^8$, $D^5 \times S^3$ and $S^3 \times D^5$ with boundaries presented as the unions:

$$\partial(D^5 \times S^3) = S^4 \times S^3 = D^4 \times S^3 \cup D^4 \times S^3$$
$$\partial(S^3 \times D^5) = S^3 \times S^4 = S^3 \times D^4 \cup S^3 \times D^4$$
$$\partial(D^8) = S^7 = D^4 \times S^3 \cup S^3 \times S^3 \cup I \cup S^3 \times D^4.$$

Using extension diffeomorphisms $A$ and $B$ defined above, we now construct a manifold $W^8$ which will be used to compute $\mu(\Sigma_{ABA^{-1}B^{-1}})$.

Definition 2. Define $W^8$ to be the manifold obtained from the union

$D^5 \times S^3 \cup_A D^8 \cup_B D^8 \cup_{A^{-1}} D^8 \cup_{B^{-1}} S^3 \times D^5$

by smoothing the corners.

Claim 2. $D^5 \times S^3 \cup_A D^8 \sim D^8 \cup_{B^{-1}} S^3 \times D^5 \sim D^8$

Proof of the claim: Evidently, the union $D^5 \times S^3 \cup_{D^4 \times S^3} D^8$ is a simply connected manifold with simply connected boundary $D^4 \times S^3 \cup S^3 \times S^3 \times D^4$. Using exact sequence of Mayer-Vietoris it is easy to see that homology groups of $D^5 \times S^3 \cup_{D^4 \times S^3} D^8$ are trivial in all dimensions $>0$. Hence by the characterizations of the smooth $n$-disk $D^n$, $n \geq 6$ (see [10]) this union is diffeomorphic to the disk $D^8$. Same proof works in the second case.  

10
Thus we can write $W^8 = D^8 \cup_B D^8 \cup_A D^8$. Now note that $\partial(W^8) = M(f_B, f_A)$ where (using notations of Milnor [15]) by $M(f_B, f_A)$ we denote the boundary of the following union of three 8-disks:

$$(D^4 \times D^4)_1 \bigcup_{S^3 \times D^4} (D^4 \times D^4)_2 \bigcup_{D^4 \times S^3} (D^4 \times D^4)_3$$

The gluing maps are (cf. [15], §1): $(x_1, y_1) \xrightarrow{f_B} (x_2, y_2) \xrightarrow{f_A^{-1}} (x_3, y_3)$ where

$$y_3 = y_2 = f_B(x_1) \circ y_1, \quad x_3 = f_A(y_3)^{-1} \circ x_2 = f_A(y_3)^{-1} \circ x_1$$

and $f_B = f_A = f : S^3 \rightarrow SO(3) \xrightarrow{i_3} SO(4)$, defined by the formula: $f(x) \circ y = xyx^{-1}$ for $x \in S^3$ and $y \in SO(3)$. In particular, we see that our homotopy sphere $\Sigma_{ABA^{-1}B^{-1}}$ is diffeomorphic to the manifold $M(f_B, f_A)$. Using Mayer-Vietoris exact sequence for the manifold $W^8 \simeq D^8 \cup_B D^8 \cup_A D^8$ we see that $H^*(W^8, \mathbb{Z}) \simeq 0$, for $* = 1, 2$ or 3, and we can apply results of Eells and Kuiper ([3], §10) to the manifold $W^8$ to compute the $\mu$-invariant of $M(f_B, f_A) \simeq \Sigma_{ABA^{-1}B^{-1}}$. It is shown (see [3], page 109) that

$$\mu(M(f_B, f_A)) = \frac{B_1^2}{8(2!)^2} \left(1 + \frac{2}{2^3 - 1}\right) \left(\pm 2p_1(f_B)p_1(f_A)\right) = \pm \frac{p_1(f_B)p_1(f_A)}{448}$$

where $B_1 = 1/6$ is the first Bernoulli number and $p_1(f_B), p_1(f_A)$ are Pontrjagin numbers of the stable vector bundles over $S^4$ determined by the compositions (cf. [15], §3):

$$S^3 \xrightarrow{f_B} SO(4) \xrightarrow{i_3} SO(5), \quad S^3 \xrightarrow{f_A} SO(4) \xrightarrow{i_4} SO(5)$$

We show that $p_1(f_B) = p_1(f_A) = \pm 4$. It is well known how Pontrjagin numbers depend on the characteristic maps of the stable vector bundles over the spheres: $p_s[S^4] = \pm a_s \cdot \lambda \cdot (2s - 1)!$. Here $\lambda$ is the integer, corresponding to the characteristic map: $S^{4s-1} \rightarrow SO$ and $a_s = 1$ or 2 if $s$ is even or odd respectively. In our case $p_1(f_B) = \pm 2 \cdot [i_4 \circ f_B]$ and to find the integer $[i_4 \circ f_B]$ we need to find the homotopy class of the composition

$$S^3 \rightarrow SO(3) \xrightarrow{i_3} SO(4) \xrightarrow{i_4} SO(5).$$

It is a standard fact (which can be deduced from Theorem IV.1.12 of [3]) that inclusion $i = i_4 \circ i_3$ induces map $i_* : \pi_3(SO(3)) \rightarrow \pi_3(SO(5))$ which is
multiplication by 2. Hence $[i_4 \circ f_A] = 2$ and similarly $[i_4 \circ f_B] = 2$. Therefore
\[
\mu(\Sigma_{ABA^{-1}B^{-1}}) = \mu(M(f_B, f_A)) = \pm \frac{16}{448} = \pm \frac{1}{28}.
\]
For the generator $\hat{\Sigma}$ of $\mathbb{Z}_{28}$ we have (see \S 4 of \cite{7}) $\mu(\hat{\Sigma}) \equiv \frac{1}{27} \cdot 8 \pmod{1}$ or $\mu(\hat{\Sigma}) = \frac{1}{28}$. From the theorem of Eells and Kuiper (\cite{7}, p.103) we see (changing orientation of $\hat{\Sigma}$ if necessary) that $\Sigma_{ABA^{-1}B^{-1}} \simeq 1$. Since $ABA^{-1}B^{-1} = \Sigma^k = \iota(\hat{\Sigma}^k)$ we can apply Lemma 3. to get $\hat{\Sigma}^k = \iota \circ \iota(\hat{\Sigma}^k) = \iota(ABA^{-1}B^{-1}) = \hat{\Sigma}$. It shows that $k \equiv 1 \pmod{28}$ and finishes the proof.

As a corollary we get a presentation of the group $\pi_0 S\text{Diff}(S^3 \times S^3)$:
\[
< A, B, \Sigma \mid \Sigma^{28} = 1, A\Sigma = \Sigma A, B\Sigma = \Sigma B, ABA^{-1}B^{-1} = \Sigma >
\]
Consider now matrices:
\[
A' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad B' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \Sigma' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
It is easy to verify that these matrices $A'$, $B'$, $\Sigma'$ generate the group $\mathcal{H}$ and satisfy the defining relations: $< A', B', \Sigma' \mid A'\Sigma' = \Sigma' A', B'\Sigma' = \Sigma'B', A'B'A'^{-1}B'^{-1} = \Sigma' >$. Hence groups $\pi_0 S\text{Diff}(S^3 \times S^3)$ and $\mathcal{H}_{28}$ are isomorphic.

### 2.4 Generators and Relations

We have shown in \S 2.2 that the factor group $\pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7$ admits the following presentation: $< Y, U, A, B \mid YUY = UYU, (YUY)^4 = 1, AB = BA, AY = YAB, AU = UA, BY = YB, BU = UBA^{-1} >$. Furthermore, we have shown explicitly (recall the proof of Theorem 1.) that $YUY = UYU$, $AY = YAB$, $AU = UA$, $BY = YB$, $BU = UBA^{-1}$. Hence if we consider the group $\pi_0 \text{Diff}(S^3 \times S^3)$ as a central extension of $\pi_0 \text{Diff}(S^3 \times S^3)/\Theta_7$ by the group $\Theta_7$, we can choose $Y, U, A, B, \Sigma$ to be the generators. The defining relations will be: $YUY = UYU$, $AY = YAB$, $AU = UA$, $BY = YB$, $BU = UBA^{-1}$, $\Sigma^{28} = Id$, $\Sigma \equiv Y, U, A, B$, and $(YUY)^4 = \Sigma^m$, $ABA^{-1}B^{-1} = \Sigma^n$ for some $m, n \in \mathbb{Z}$. The symbol $\equiv$ means that element on the left commutes with any element on the right. By Theorem 2. we have $n = 1$.

**Claim 3.** The following identities are valid in the group $\pi_0 \text{Diff}(S^3 \times S^3)$:
\[
(B^{-1}YUY)^4 = Id, \quad YUYB^{-1} = A^{-1}\Sigma YUY, \quad YUYA^{-1} = BYUY.
\]
Proof. We know that $(B^{-1}YUY)^4 = Id$ (Claim 1.). Using identities: $AB = BA\Sigma$, $AY = YAB$, $UB = BAU$ we find that $YUYA^{-1} = YUA^{-1}B\Sigma Y = YA^{-1}UB\Sigma Y = Y\Sigma Y$. Similarly $YUYB^{-1} = YUB^{-1}Y = YA^{-1}B^{-1}UY = A^{-1}B\Sigma B^{-1}YUY = A^{-1}\Sigma YUY$.

Now it is easy to see that $m = -1$. Indeed, $Id = (B^{-1}YUY)^2B^{-1}YUYB^{-1}YUY = (B^{-1}YUY)^2B^{-1}A^{-1}\Sigma(YUY)^2 = B^{-1}YUYB^{-1}YUYA^{-1}B^{-1}(YUY)^2 = B^{-1}YUYB^{-1}BYUYB^{-1}(YUY)^2 = B^{-1}YUYA^{-1}\Sigma(YUY)^3 = B^{-1}B\Sigma(YUY)^4 = \Sigma(YUY)^4$ Hence $(YUY)^4 = \Sigma^{-1}$.

Let us now collect all the information we obtained so far and state the main theorem of this paper.

**Theorem 3.** The mapping class group of $S^3 \times S^3$ admits the following presentation:

\[
\begin{pmatrix}
Y & U & YUY = UYU, \quad (YUY)^4 = \Sigma^{-1}, \quad \Sigma \Rightarrow Y, U, A, B \\
A & B & BU = UBA^{-1}, \quad AY = YAB, \quad AB = BA\Sigma, \\
\Sigma & & BY = YB, \quad AU = UA
\end{pmatrix}
\]

with $Y = (y, 0), \ U = (u, 0), \ A = (a, 0), \ B = (b, 0), \ \Sigma = (id, 1) \in \Gamma^J \times \mathbb{Z}_{28}$.

**Remark:** It is well known that the mapping class group of the 2-torus is generated by two Dehn twists. Wajnryb has shown that for an orientable surface $S^2_g$ of genus $g \geq 2$ the mapping class group can be generated by two elements that are not Dehn twists in general. It follows from Theorem 3. and the work of Choie (see [3], Theorem 2.1.) that $\pi_0 \text{Diff}(S^3 \times S^3)$ can also be generated by two elements.

3 Cohomology of the Jacobi Group $\Gamma^J$

It is usually difficult to obtain information about a group having just generators and defining relations. The aim of this part is to give an alternative description of the mapping class group $\pi_0 \text{Diff}(S^3 \times S^3)$ using the cohomology theory of groups. Since $\pi_0 \text{Diff}(S^3 \times S^3)$ is a central extension of the Jacobi group $\Gamma^J$ by $\mathbb{Z}_{28}$ it is natural to ask what element of the group $H^2(\Gamma^J, \mathbb{Z}_{28})$ corresponds to this extension. First we classify all central extensions of $SL_2(\mathbb{Z})$ by $\mathbb{Z}$ and determine a 2-cocycle that generates $H^2(SL_2(\mathbb{Z}))$. 13
In the second section we show that $\Gamma^J$ is isomorphic to an amalgamated product. Using a Mayer-Vietoris exact sequence of this amalgamated product we compute the cohomology groups of $\Gamma^J$. Finally, we specify an element of $H^2(\Gamma^J, \mathbb{Z}_{28})$ that corresponds to the mapping class group $\pi_0 \text{Diff}(S^3 \times S^3)$.

### 3.1 Central extensions of $SL_2(\mathbb{Z})$ and $\mathbb{Z} \oplus \mathbb{Z}$

There are several ways to find cohomology groups of $SL_2(\mathbb{Z})$ with trivial $\mathbb{Z}$-coefficients. All of these groups are of course well known. The reason why we make some calculations here is that in the next section these calculations will be generalized to find the group $H^2(\Gamma^J)$ and its generators.

Consider the following exact sequence: $0 \to \mathbb{Z} \to E \to G \to 0$. Group $E$ is called an extension of $G$ by $\mathbb{Z}$. If the normal subgroup $\mathbb{Z}$ lies in the center of $E$, this extension is called central. The equivalence classes of such extensions are in 1-1 correspondence with the elements of the second cohomology group $H^2(G, \mathbb{Z})$. We will usually denote this group by $H^2(G)$ forgetting the coefficients (only if $\mathbb{Z}$ is the trivial $G$-module).

Recall that $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 \ast \mathbb{Z}_2 \mathbb{Z}_6$ and we can consider a Mayer-Vietoris exact sequence:

$$
\to H^n(SL_2(\mathbb{Z})) \to H^n(\mathbb{Z}_4) \oplus H^n(\mathbb{Z}_6) \to H^n(\mathbb{Z}_2) \to H^{n+1}(SL_2(\mathbb{Z})) \to \quad (7)
$$

Cohomology groups of $\mathbb{Z}_m$ are known: $H^{2k}(\mathbb{Z}_m) \cong \mathbb{Z}_m$, $H^{2k-1}(\mathbb{Z}_m) \cong 0$ for $k \geq 1$. Hence $H^1(SL_2(\mathbb{Z})) \cong 0$. We also have the following fragments:

$$
0 \to H^{2n}(SL_2(\mathbb{Z})) \to H^{2n}(\mathbb{Z}_4) \oplus H^{2n}(\mathbb{Z}_6) \xrightarrow{i^*} H^{2n}(\mathbb{Z}_2) \to H^{2n+1}(SL_2(\mathbb{Z})) \to 0
$$

with $j^* = i^*_4 + i^*_6$ and $i_k : \mathbb{Z}_2 \to \mathbb{Z}_k$ - multiplication by $k/2$. If we denote by $t$, $z$ and $z'$ generators of groups $H^2(\mathbb{Z}_2)$, $H^2(\mathbb{Z}_4)$ and $H^2(\mathbb{Z}_6)$ respectively, then it is easy to see that $i^*_4(z) = i^*_6(z') = t$ i.e. $j^*(n, m) = n + m$. Hence $\text{Im}(j^*) = H^2(\mathbb{Z}_2)$, $\text{Ker}(j^*)$ is generated by the element $(1, 5) \in \mathbb{Z}_4 \oplus \mathbb{Z}_6$ and isomorphic to $\mathbb{Z}_{12}$. Thus $H^3(SL_2(\mathbb{Z})) \cong 0$ and $H^2(SL_2(\mathbb{Z})) \cong \mathbb{Z}_{12}$. It follows from the properties of the norm map ([5], ch.III) that $H^{2k-1}(SL_2(\mathbb{Z})) \cong 0$ and $H^{2k}(SL_2(\mathbb{Z})) \cong \mathbb{Z}_{12}$ for any $k \geq 1$.

Now we write down an explicit function $f : SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \to \mathbb{Z}$ that generates the group $H^2(SL_2(\mathbb{Z}))$. This function will be defined in terms of a generator of the group $H^2(\mathbb{Z}_{12})$ which we describe first. Consider the group $\mathbb{Z}_m$ as the multiplicative group on elements $\{z^1, \ldots, z^p, \ldots, z^{m-1}, id\}$ and define a function $f_m : \mathbb{Z}_m \times \mathbb{Z}_m \to \mathbb{Z}$ by the formula:
Definition 3.

\[ f_m(z^p, z^q) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \bar{p} + \bar{q} \geq m \\ 0 & \text{if } \bar{p} + \bar{q} < m \end{cases} \quad (8) \]

where \( \bar{p}, \bar{q} \in \{0, 1, 2, \ldots, m - 1\} \) and \( \bar{p} \equiv p \pmod{m}, \ \bar{q} \equiv q \pmod{m} \)

Lemma 4. Function \( f_m \) is a generator of the group \( H^2(\mathbb{Z}_m) \cong \mathbb{Z}_m \).

Proof. Equality \( f_m(z^p, z^q) + f_m(z^{p+q}, z^r) = f_m(z^q, z^r) + f_m(z^p, z^{q+r}) \) shows that \( f_m \) is a 2-cocycle. Verification of this equality is straightforward and left as an exercise. Let \( < c > \) and \( < z \mid z^m = id > \) be presentations of groups \( \mathbb{Z} \) and \( \mathbb{Z}_m \) respectively. Then a central extension of \( \mathbb{Z}_m \) by \( \mathbb{Z} \) will have a presentation:

\[ < Z, C \mid ZC = CZ, Z^m = C^k > \]

with \( Z = (z, 0), C = (id, 1) \in \mathbb{Z}_m \times \mathbb{Z} \) and some \( k \in \mathbb{Z} \). It follows from formula (8) that \( k = 1 \) for the function \( f_m \), and that the cocycle \( t \cdot f_m \) defines the extension with \( k = t \). We denote this extension by \( E_t \). If \( m \nmid s - t \) then cocycles \( s \cdot f_m \) and \( t \cdot f_m \) define non equivalent extensions. Indeed, suppose that \( E_s \) and \( E_t \) are equivalent. Then there exists an isomorphism \( \iota : E_s \longrightarrow E_t \) that makes the following diagram commute:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \overset{i_s}{\longrightarrow} & E_s & \overset{\rho_s}{\longrightarrow} & \mathbb{Z}_m & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \iota & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \overset{i_t}{\longrightarrow} & E_t & \overset{\rho_t}{\longrightarrow} & \mathbb{Z}_m & \longrightarrow & 0
\end{array}
\]

Using the above presentation of groups \( E_s \) and \( E_t \) one can see from this diagram that \( s - t = lm \) for some \( l \in \mathbb{Z} \). Hence \( f_m \) is a generator of the group \( H^2(\mathbb{Z}_m) \).

Now consider the canonical projection \( ab : SL_2(\mathbb{Z}) \longrightarrow (SL_2(\mathbb{Z})_{ab} \cong \mathbb{Z}_{12} \). We define the function \( f : SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \longrightarrow \mathbb{Z} \) by the formula

Definition 4. \( f(M, N) \overset{\text{def}}{=} f_{12}(M_{ab}, N_{ab}) \) where \( M_{ab} = ab(M) \in \mathbb{Z}_{12} \)

Lemma 5. Function \( f(M, N) = f_{12}(M_{ab}, N_{ab}) \) is a generator of the group \( H^2(SL_2(\mathbb{Z})) \).
Proof. Since \((M \cdot N)_{ab} = M_{ab} \cdot N_{ab}\) it follows from the previous lemma that \(f\) is a 2-cocycle. Let \(\langle c \rangle\) and \(\langle y, u \mid yuy = uyu, (yuy)^4 = id \rangle\) be presentations of groups \(Z\) and \(SL_2(Z)\). Then a central extension of \(SL_2(Z)\) has a presentation:

\[
\langle Y, U, C \mid YU = YUC^a, (YUY)^4 = C^b, CY = YC, CU = UC \rangle
\]

with \(Y = (y, 0), U = (u, 0), C = (id, 1)\) and some \(a, b \in Z\). Using the group law: \((g, k) \cdot (h, l) = (gh, k + l + f(g, h))\) one can easily find that function \(f\) defines the extension with \(a = 0\) and \(b = 1\). Cocycle \(t \cdot f\) defines the extension with \(a = 0\) and \(b = t\). As in the proof of Lemma 4. one can show that if \(12 \nmid s - t\) then cocycles \(s \cdot f\) and \(t \cdot f\) define non equivalent extensions. Therefore function \(f\) is indeed a generator of \(H^2(SL_2(Z))\).

Remark: Consider another function \(\varphi : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{Z}\) defined as follows. For \(v = (\lambda, \mu), \) and \(w = (\lambda', \mu'), \) \(\varphi(v, w) \overset{\text{def}}{=} \begin{vmatrix} \lambda & \mu \\ \lambda' & \mu' \end{vmatrix} = \lambda \mu' - \lambda' \mu.\) It can be easily verified that \(\varphi\) is a cocycle cohomologous to \(2g.\) Note that \(\varphi\) defines the extension which is isomorphic to the Heisenberg group \(H(Z).\) This later one is \(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\) as a set and with multiplication:

\[
(a, b, c) \cdot (a', b', c') \overset{\text{def}}{=} (a + a', b + b', c + c' + a \cdot b' - b \cdot a')
\]

Here one can consider so called (real) Jacobi group \(G^J(Z)\) (cf. [1]) defined as the semidirect product of \(SL_2(Z)\) and \(H(Z): G^J(Z) = SL_2(Z) \ltimes H(Z).\)
3.2 Group $\Gamma^J$ as an amalgamated product

In this paragraph we will compute the cohomology groups of $\Gamma^J$. First we present $\Gamma^J$ as an amalgamated product and then use a Mayer-Vietoris exact sequence of this product to find $H^2(\Gamma^J)$.

Consider for $m = 2, 4, 6$ the cyclic groups $\mathbb{Z}_m$ generated by the matrices \[
\begin{pmatrix}
-1 & 0 \\
0 & -1 
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0 
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 \\
-1 & 1 
\end{pmatrix}
\] respectively. These are subgroups of $SL_2(\mathbb{Z})$ and they act on the elements of $\mathbb{Z} \oplus \mathbb{Z}$ in the same natural way as $SL_2(\mathbb{Z})$ does. With respect to this action we define the semidirect products $G_m \overset{\text{def}}{=} \mathbb{Z}_m \rtimes \mathbb{Z}^2$.

**Lemma 6.** Group $\Gamma^J$ is isomorphic to the amalgamated product $G_4 \ast_{G_2} G_6$.

**Proof.** First we give presentations of groups $G_m$ with generators and defining relations. Denote the generators of $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ by $\alpha$, $\beta$, $\gamma$ and elements $(id, (1, 0))$, $(id, (0, 1))$ of the group $G_m$ by $A$ and $B$ respectively.

**Note:** To avoid cumbersome notations we use the same letters $A$ and $B$ to denote elements of different groups. We hope it will not cause any confusion. Using group law $(M, X) \cdot (M', X') = (MM', XM' + X)$ one can easily show (cf. proof of Lemma 1. above) that $A\alpha = \alpha A^{-1}$, $B\alpha = \alpha B^{-1}$, $B\beta = \beta A^{-1}$, $A\beta = \beta B$ and $A\gamma = \gamma B$, $B\gamma = A\gamma A^{-1}$. Hence we get presentations:

\[
\begin{align*}
G_2 &\simeq <A, B, \alpha \mid AB = BA, \alpha^2 = id, A\alpha = \alpha B^{-1}, A\alpha = \alpha A^{-1} > \\
G_4 &\simeq <A, B, \beta \mid AB = BA, \beta^4 = id, B\beta = \beta A^{-1}, A\beta = \beta B > \\
G_6 &\simeq <A, B, \gamma \mid AB = BA, \gamma^6 = id, B\gamma = A\gamma A^{-1}, A\gamma = \gamma B >
\end{align*}
\]

Next define maps $\iota_4 : G_2 \rightarrow G_4$ and $\iota_6 : G_2 \rightarrow G_6$ by the formulas: $\iota_4(A) = A$, $\iota_4(B) = B$, $\iota_4(\alpha) = \beta^2$ and $\iota_6(A) = A$, $\iota_6(B) = B$, $\iota_6(\alpha) = \gamma^3$. Map $\iota_4$ induces commutative diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & G_2 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \\
\downarrow & \simeq & \downarrow & \iota_4 & \downarrow & \times 2 & & \\
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & G_4 & \longrightarrow & \mathbb{Z}_4 & \longrightarrow & 0 \\
\end{array}
\]

from which it follows that $\iota_4$ is a monomorphism. Similarly one proves that $\iota_6$ is a monomorphism. Since $\alpha$ is identified with $\beta^2$ and $\gamma^3$ we obtain the
we find that

Lemma 1.  

The following presentation of $G_4 \ast G_2 G_6$ with generators and defining relations:

\[
\begin{pmatrix}
\beta & \gamma \\
A & B
\end{pmatrix}
\begin{align*}
AB &= BA, & A\beta &= \beta B, & A\gamma &= \gamma B, & B\gamma &= A\gamma A^{-1} \\
B\beta &= \beta A^{-1}, & \beta^2 &= \gamma^3, & \beta^4 &= id
\end{align*}
\]

Consider now two elements: $U \overset{def}{=} \beta\gamma^{-1}$ and $Y \overset{def}{=} \gamma^2\beta^{-1}$. Obviously, $UY = \gamma$ and $UYU = \beta$. One can easily show that the above presentation of the group $G_4 \ast G_2 G_6$ is equivalent to one of the Jacobi group $\Gamma^J$ obtained in Lemma 1.

To find the cohomology of $G_m$ we will use the Lyndon-Hochschild-Serre (LHS) spectral sequence (see [10] or [1], §7.2) of the split extension that defines $G_m$: $E_2^{p,q} = H^p(\mathbb{Z}_m, H^q(\mathbb{Z}_2, \mathbb{Z})) \Rightarrow H^{p+q}(G_m, \mathbb{Z})$. We need to know how $\mathbb{Z}_m$ acts on $H^q(\mathbb{Z}_2)$. Suppose $f \in H^q(\mathbb{Z}_2)$, that is $f : \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \to \mathbb{Z}$. Then we have $\mathcal{M} \circ f(\sigma_1, \ldots, \sigma_q) = f(\mathcal{M}^{-1}\sigma_1\mathcal{M}, \ldots, \mathcal{M}^{-1}\sigma_q\mathcal{M})$ (since $\mathbb{Z}$ is a trivial $\mathbb{Z}_m$-module) where $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; (0,0)$ are the corresponding elements of the group $G_m$ (cf. [10], p.117). If we denote by $M$ the matrix of the element $\mathcal{M}$ and by $\sigma_i$ the vector of the element $\sigma_i$, we find that $M \circ f(\sigma_1, \ldots, \sigma_q) = f(\sigma_1 \cdot M, \ldots, \sigma_q \cdot M)$ where on the right we mean multiplication of the row vector by a matrix. Groups $H^q(\mathbb{Z}_2)$ are nonzero only for $q = 0, 1, 2$.

It is easy to verify that $\mathbb{Z}_m$ acts trivially on groups $H^0(\mathbb{Z}_2) \cong \mathbb{Z}$ and $H^2(\mathbb{Z}_2) \cong \mathbb{Z}$. If we denote by $(n, k)$ an element of $H^1(\mathbb{Z}_2) \cong \mathbb{Z} \oplus \mathbb{Z}$, it can be shown that $M \circ (n, k) = (n, k) \cdot M^T$.

Since groups $H^*(\mathbb{Z}_m)$ are well known we only need to compute $H^*(\mathbb{Z}_m, \mathbb{Z}_2)$ with respect to the action described above. $H^*(\mathbb{Z}_m, \mathbb{Z}_2)$ can be found using the norm map $\tilde{N} : M_G \to M^G$ for group $G = \mathbb{Z}_m$ and the module $M = \mathbb{Z}_2$, since $H^{2k}(G, M) \cong \ker \tilde{N}$ and $H^{2k-1}(G, M) \cong \coker \tilde{N}$ for any $k \geq 1$ (see [3], ch.III).

1) For $\mathbb{Z}_2$ we have $M_G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the generators $(1,0),(0,1)$. It is obvious that $M^G \cong 0$ and we get:

\[H^{2k}(\mathbb{Z}_2, \mathbb{Z}_2) \cong 0, \quad H^{2k+1}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{for any } k \geq 0\]

2) For $\mathbb{Z}_4$ we have $M_G \cong \mathbb{Z}_2$ with the generator $(1,0)$. Here $M^G \cong 0$ also, and we get: $H^{2k}(\mathbb{Z}_4, \mathbb{Z}_2) \cong 0, \quad H^{2k+1}(\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \text{for any } k \geq 0$.

3) In the case of $\mathbb{Z}_6$, $M_G \cong M^G \cong 0$ and $H^k(\mathbb{Z}_6, \mathbb{Z}_2) \cong 0 \quad \text{for any } k \geq 0$.

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Lemma 7. For any \( k \geq 0 \) and \( m \in \{2, 4, 6\} \), \( H^{2k+1}(G_m) \cong 0 \). Furthermore, \( H^2(G_2) \cong H^0(\mathbb{Z}_2) \oplus H^1(\mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}) \oplus H^2(\mathbb{Z}_2) \cong \mathbb{Z} \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2 \), \( H^2(G_4) \cong H^0(\mathbb{Z}_4) \oplus H^1(\mathbb{Z}_4, \mathbb{Z} \oplus \mathbb{Z}) \oplus H^2(\mathbb{Z}_4) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \), \( H^2(G_6) \cong H^0(\mathbb{Z}_6) \oplus H^2(\mathbb{Z}_6) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \).

Proof. We prove this lemma only for \( m = 2 \). In the other cases the proof is similar. Consider the first quadrant of the LHS spectral sequence \( E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(\mathbb{Z}^2)) \), for \( p, q \geq 0 \).

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & \ldots \\
0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \ldots \\
\mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & \ldots \\
\end{array}
\]

From this \( E_2 \)-term we see that \( E_2^{*,*} \cong E_3^{*,*} \cong \ldots \cong E_\infty^{*,*} \). Hence for any \( k \geq 0 \), \( H^{2k+1}(G_2) \cong 0 \). Since \( H_1(G_2) \cong (G_2)_{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), by the Universal Coefficient Theorem \( H^2(G_2) \cong \text{Hom}(H_2(G_2), \mathbb{Z}) \oplus \text{Ext}(H_1(G_2), \mathbb{Z}) \cong \text{Hom}(H_2(G_2), \mathbb{Z}) \oplus \mathbb{Z}_2^3 \), and therefore \( H^2(G_2) \cong H^0(\mathbb{Z}_2) \oplus H^1(\mathbb{Z}_2, \mathbb{Z}^2) \oplus H^2(\mathbb{Z}_2) \cong \mathbb{Z} \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2 \).

\[\square\]

Theorem 4. \( \Gamma^J \) has the following homology and cohomology groups:
\( H_1(\Gamma^J) \cong \mathbb{Z}_{12}, \quad H_2(\Gamma^J) \cong \mathbb{Z} \oplus \mathbb{Z}_2, \quad H^1(\Gamma^J) \cong 0, \quad H^2(\Gamma^J) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}, \quad H^3(\Gamma^J) \cong \mathbb{Z}_2 \).

Proof. First note that it follows from the presentation of the group \( \Gamma^J \) that \( H_1(\Gamma^J) \cong (\Gamma^J)_{ab} \cong \mathbb{Z}_{12} \). To compute the cohomology groups, we use a Mayer-Vietoris exact sequence of the amalgamated product \( G_1 \ast_{G_2} G_6 \). By the previous lemma we get the following fragment of this exact sequence:

\[ 0 \rightarrow H^2(\Gamma^J) \rightarrow H^2(G_4) \oplus H^2(G_6) \rightarrow H^2(G_2) \rightarrow H^3(\Gamma^J) \rightarrow 0 \]

where \( j = \iota_4^* + \iota_6^* \) and \( \iota_m^* \) is induced by the inclusion \( \iota_m : G_2 \hookrightarrow G_m \) (cf. proof of Lemma 6.). Commutative diagram (10) together with the functorial dependence (cf. [1], §7.2) induces the functorial map \( H^p(\mathbb{Z}_4, H^q(\mathbb{Z}^2)) \rightarrow H^p(\mathbb{Z}_2, H^q(\mathbb{Z}^2)) \). Hence \( H^2(G_4) \xrightarrow{\iota_4^*} H^2(G_2) \) is the map induced by multiplication: \( \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \). If we denote by \( x, y, z \) generators of groups \( H^0(\mathbb{Z}_4) \cong \mathbb{Z}, \quad H^1(\mathbb{Z}_4, \mathbb{Z}^2) \cong \mathbb{Z}_2, \quad H^2(\mathbb{Z}_4) \cong \mathbb{Z}_4 \) respectively then we see that \( \iota_4^*(x, y, z) = \ldots \)
(x', y', 0, z') where x', y', z' are generators of the groups \( \mathbb{Z} \cong H^0(\mathbb{Z}_2) \), \( \mathbb{Z}_2 \subset H^1(\mathbb{Z}_2, \mathbb{Z}^2) \), \( \mathbb{Z}_2 \cong H^2(\mathbb{Z}_2) \). Similarly one can show that \( \iota_0^a(s, t) = (x', 0, 0, z') \) where by \( s \) and \( t \) we denoted the generators of groups \( H^0(\mathbb{Z}_6) \) and \( H^2(\mathbb{Z}_6) \).

Hence \( H^3(\Gamma^J) \cong \text{Coker}(j) \cong \mathbb{Z}_2 \) and \( H^2(\Gamma^J) \cong \text{Ker}(j) \cong \mathbb{Z} \oplus \mathbb{Z}_{12} \). It is clear that \( H^1(\Gamma^J) \cong 0 \). We can assume that \( H_2(\Gamma^J) \cong \mathbb{Z}^n_2 \oplus \text{Tor}_2 \) and \( H_3(\Gamma^J) \cong \mathbb{Z}^n_3 \oplus \text{Tor}_3 \) where by \( \text{Tor}_k \) we denote the elements of finite order. Then by the Universal Coefficient Theorem \( \mathbb{Z} \oplus \mathbb{Z}_{12} \cong H^2(\Gamma^J) \cong \text{Ext}(\mathbb{Z}_{12}, \mathbb{Z}) \oplus \text{Hom}(\mathbb{Z}^n_2 \oplus \text{Tor}_2, \mathbb{Z}) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}^n_2 \), that is \( n_2 = 1 \). Similarly we have \( \mathbb{Z}_2 \cong H^3(\Gamma^J) \cong \text{Ext}(\mathbb{Z} \oplus \text{Tor}_2, \mathbb{Z}) \oplus \text{Hom}(\mathbb{Z}^n_3 \oplus \text{Tor}_3, \mathbb{Z}) \cong \text{Ext}(\text{Tor}_2, \mathbb{Z}) \oplus \mathbb{Z}^n_3 \). Therefore \( n_3 = 0 \) and \( \text{Tor}_2 \cong \mathbb{Z}_2 \) as required.

\[\text{Remark:}\] To find the other cohomology groups of \( \Gamma^J \) one could use the LHS spectral sequence of the defining extension of \( \Gamma^J \): \( H^p(\text{SL}_2(\mathbb{Z}), H^q(\mathbb{Z}^2)) \Rightarrow H^{p+q}(\Gamma^J) \). It can be seen that \( H^*(\text{SL}_2(\mathbb{Z}), H^2(\mathbb{Z}^2)) \cong H^*(\text{SL}_2(\mathbb{Z}), H^0(\mathbb{Z}^2)) \cong H^*(\mathbb{Z}_{12}); H^{2k}(\text{SL}_2(\mathbb{Z}), H^1(\mathbb{Z}^2)) \cong \mathbb{Z}_2 \) and \( H^{2k-1}(SL_2(\mathbb{Z}), H^1(\mathbb{Z}^2)) \cong 0 \) for any \( k \geq 1 \). The previous theorem implies that \( d_2^{k, 2} \) of this spectral sequence is the zero map. Using the cup product we deduce that \( d_2^{1, 2} \) is zero, hence \( d_1^{1, 2} \) is zero and so on. Applying Proposition 7.3.2. of [4] and Theorem 4. of [4] we see that \( H^{2k+1}(\Gamma^J) \cong \mathbb{Z}_2 \) and \( H^{2k+2}(\Gamma^J) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \) for any \( k \geq 1 \).

**Theorem 5.** The split extension \( 0 \rightarrow \mathbb{Z}^2 \xrightarrow{i} \Gamma^J \xrightarrow{\rho} \text{SL}_2(\mathbb{Z}) \rightarrow 0 \), induces the following split exact sequence (compare with the five-term exact sequence [3], §7.2):

\[
0 \rightarrow H^2(\text{SL}_2(\mathbb{Z})) \xrightarrow{i^*} H^2(\Gamma^J) \xrightarrow{\rho^*} H^2(\mathbb{Z}^2) \rightarrow 0
\] (11)

**Proof:** Choose the presentation of \( \Gamma^J \) found in Lemma 1. Then any central extension \( E: 0 \rightarrow \mathbb{Z} \rightarrow E \xrightarrow{\pi} \Gamma^J \rightarrow 0 \) has the following presentation:

\[
\begin{pmatrix}
Y & U \\
A & B \\
\Sigma
\end{pmatrix}
\begin{pmatrix}
YU = UYUS_{k_0} & (YU)^4 = \Sigma^{k_1} \& AB = BAS_{k_2} \\
BU = UBA^{-1}S_{k_3} & AY = YAB\Sigma^{k_4} \& AU = UA\Sigma^{k_5} \\
BY = YBS_{k_6} & \Sigma \ni Y, U, A, B
\end{pmatrix}
\]

with some \( k_0, \ldots, k_6 \in \mathbb{Z} \) and \( Y = (y, 0) \), \( U = (u, 0) \), \( A = (a, 0) \), \( B = (b, 0) \) and \( \Sigma = (id, 1) \). Consider element \( U \overset{\text{def}}{=} U\Sigma^{k_0} \). Then from equality \( YU = UYUS_{k_0} \) we get \( YU = YU \). Instead of \( (YU)^4 = \Sigma^{k_1} \) we get
\((YUY)^4 = \Sigma^{k_1+4k_0}\) and so on. Thus, by changing \(U\) to \(U\Sigma^{k_0}\) we can make \(k_0 = 0\). Similarly, changing \(A\) to \(A\Sigma^{-k_3}\) and \(B\) to \(B\Sigma^{k_4}\) we can eliminate \(k_3\) and \(k_4\) and assume that \(k_0 = k_3 = k_4 = 0\) in the above presentation of \(E\). Now one can easily obtain the following equalities: \(BYUY = YUYA^{-1}\Sigma^{2k_6}\), \(BUYU = UYUA^{-1}\Sigma^{k_6-k_5}\). Therefore \(k_6 = -k_5\). Analogously, if we compare \(AYUY\) with \(AUYU\) we find that \(k_5 = k_6\). Hence \(k_5 = k_6 = 0\). It means that we can choose set-theoretic cross-sections of \(\pi\) (i.e., functions \(s_{m,n} : \Gamma^J \to E\) so that \(\pi \circ s_{m,n} = id\)) in such a way that any central extension of \(\Gamma^J\) by \(Z\) has a presentation:

\[
E_{m,n} \overset{\text{def}}{=} \langle YU, A B \Sigma \mid YUY = UYU, (YUY)^4 = \Sigma^n, AB = BA\Sigma^m, BU = UBA^{-1}, AY = YAB, AU = UA, BY = YB, \Sigma \ni Y, U, A, B \rangle
\]

and any element of \(H^2(\Gamma^J)\) is cohomologous to a cocycle, defined by one of these \(s_{m,n}\).

Choose a 2-cocycle \(\omega_1\) that defines the extension \(E_{1,0}\). Evidently, \(i^*(\omega_1)\) defines the extension \((\mathbb{Z}^2)\) of \(\mathbb{Z}^2\) by \(\mathbb{Z}\) and therefore generates \(H^2(\mathbb{Z}^2)\). Hence \(i^*\) is onto and \(\omega_1\) generates the direct summand \(\mathbb{Z}\) of \(H^2(\Gamma^J)\). As for the other generator \(\omega_2\) of \(\mathbb{Z}_{12} \subset H^2(\Gamma^J)\), we can choose \(\rho^*(f)\) (recall Def. 4 above). It can be verified that \(\omega_2 = \rho^*(f)\) defines the extension \(E_{0,1}\) and the subgroup generated by \(\omega_2\) is the kernel of \(i^*\). This proves the exactness. Proof of the asserted splitting is left to the reader (cf. Proposition 7.3.2. of [9]). \(\square\)

### 3.3 On groups \(H^2(\Gamma^J, \mathbb{Z}_{28})\) and \(\pi_0\text{Diff}(S^3 \times S^3)\)

In this short paragraph we determine an element of \(H^2(\Gamma^J, \mathbb{Z}_{28})\) that corresponds to the mapping class group of \(S^3 \times S^3\).

Consider a short exact sequence \(0 \to A \overset{\mu}{\to} B \overset{\nu}{\to} C \to 0\) of \(G\)-modules. Then we know from homological algebra that there is a natural map \(\delta : H^n(G, C) \to H^{n+1}(G, A)\) such that the sequence

\[
\ldots \to H^n(G, A) \overset{\mu_*}{\to} H^n(G, B) \overset{\nu_*}{\to} H^n(G, C) \overset{\delta}{\to} H^{n+1}(G, A) \to \ldots
\]

is exact. Therefore the short exact sequence \(0 \to \mathbb{Z} \overset{x_{28}}{\to} \mathbb{Z} \to \mathbb{Z}_{28} \to 0\) of trivial \(\Gamma^J\)-modules gives the long exact sequence

\[
\to H^2(\Gamma^J) \overset{\mu_2}{\to} H^2(\Gamma^J) \overset{\nu_2}{\to} H^2(\Gamma^J, \mathbb{Z}_{28}) \overset{\delta}{\to} H^3(\Gamma^J) \overset{\mu_3}{\to} H^3(\Gamma^J) \to \ldots
\]
Map $\mu_n$ is multiplication by 28 and we have (from Theorem 4.) $H^2(\Gamma^J, \mathbb{Z}_{28}) \cong \text{Im}(\nu_2) \oplus \text{Ker}(\mu_3) \cong \mathbb{Z}_{28} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$. It follows from the presentation of $\pi_0\text{Diff}(S^3 \times S^3)$ given in Theorem 3. that this mapping class group is the factor group of the central extension of $\Gamma^J$ that corresponds to cocycle $\omega_1 - \omega_2$ of $H^2(\Gamma^J)$. It is clear that $\nu_2(\omega_1 - \omega_2)$ will be a cocycle that corresponds to the group $\pi_0\text{Diff}(S^3 \times S^3)$. If we denote a generator of the summand $\mathbb{Z}_2$ by $\tilde{\omega}_3$, and generators of $\mathbb{Z}_{28}$, $\mathbb{Z}_4$ by $\tilde{\omega}_1 \overset{\text{def}}{=} \nu_2(\omega_1)$ and $\tilde{\omega}_2 \overset{\text{def}}{=} \nu_2(\omega_2)$ respectively, then these three cocycles $\tilde{\omega}_i$, $i \in \{1, 2, 3\}$ generate group $H^2(\Gamma^J, \mathbb{Z}_{28})$ and $\tilde{\omega}_1 - \tilde{\omega}_2$ will be a cocycle that defines an extension isomorphic to $\pi_0\text{Diff}(S^3 \times S^3)$.

**Remark:** One can deduce from the proof of Theorem 5. that cocycle $\tilde{\omega}_3$ defines the extension $E$ with $k_0 = k_1 = k_2 = k_3 = k_4 = 0$ and $k_5 = k_6 = 1$.

**References**

[1] R. Berndt and R. Schmidt: *Elements of the Representation Theory of the Jacobi Group*, Birkhäuser, Boston 1998

[2] J. Birman: *On Siegel’s modular group*, Math. Ann. 191 (1971), 59-68

[3] W. Browder: *Diffeomorphisms Of 1-Connected Manifolds*, Trans. Amer. Math. Soc. 128 (1967), 155-163

[4] W. Browder: *Surgery on simply-connected manifolds*, Springer-Verlag, New York, 1972

[5] K. Brown: *Cohomology of Groups*, Grad. Texts in Math. vol.87, Springer-Verlag, New York, 1982

[6] Y. Choie: *A short note on the full Jacobi group*, Proc. Amer. Math. Soc. 123 (1995), 2625-2627

[7] J. Eells and N. Kuiper: *An invariant for certain smooth manifolds*, Annali di Math. 60 (1962), 93-110

[8] M. Eichler and D. Zagier: *The Theory of Jacobi Forms*, Birkhäuser, Boston 1985

[9] L. Evens: *The Cohomology of Groups*, Oxford Univ. Press, 1991
[10] G. Hochschild and J-P. Serre: *Cohomology of Group Extensions*, Trans. Amer. Math. Soc. 74 (1953), 110-134

[11] D. Johnson: *A Survey of the Torelli Group*, Contemporary Math. 20 (1983), 165-179

[12] M. Kervaire and J. Milnor: *Groups of homotopy spheres*, Annals of Math. 77 (1963), 504-537

[13] M. Kreck: *Isotopy classes of diffeomorphisms of \((k - 1)\)-connected \(2k\)-manifolds*, Algebraic topology, Aarhus 1978, 643-663, Lecture Notes in Math., vol.763, Springer, Berlin, 1979

[14] B. Lawson and M. Michelson: *Spin geometry*, Princeton Math. Series, vol. 39, 1989

[15] J. Milnor: *Differentiable structures on spheres*, American J. Math. 81 (1959), 962-972

[16] J. Milnor: *Lectures on the h-Cobordism Theorem*, notes by L. Siebenmann and J. Sondow, Princeton University Press, 1965

[17] B. Wajnryb: *Mapping class group of a surface is generated by two elements*, Topology 35, No. 2 (1996), 377-383