Representable presheaves of groups on the homotopy category of cocommutative dg-coalgebras and Tannakian reconstruction

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Abstract. Motivated by rational homotopy theory, we study a representable presheaf of groups \( \mathcal{P} \) on the homotopy category of cocommutative differential graded coalgebras, its Lie algebraic counterpart and its linear representations. We prove a Tannaka type reconstruction theorem that \( \mathcal{P} \) can be recovered from the dg-category of its linear representations along with the forgetful dg-functor to the underlying dg-category of chain complexes.

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Contents

| Contents | Page |
|----------|------|
| 1. Introduction | 2 |
| 1.1. A selected history and motivation | 2 |
| 1.2. Results | 3 |
| 2. Notation and basic notions | 5 |
| 3. Representable presheaves of groups and presheaves of Lie algebras | 9 |
| 3.1. Representable presheaf of groups \( \mathcal{P}_{\Omega} : \text{ho} \text{ccdgC}(k) \rightarrow \text{Grp} \) | 10 |
| 3.2. Presheaf of Lie algebras \( T\mathcal{P}_{\Omega} : \text{ho} \text{ccdgC}(k) \rightarrow \text{Lie}(k) \) | 17 |
| 3.3. Complete ccdg-Hopf algebra and an isomorphism \( T\mathcal{P}_{\Omega} \cong \mathcal{P}_{\Omega} \) of presheaves | 21 |
| 4. Linear representation of a representable presheaf of groups | 28 |
| 4.1. Preliminary | 29 |
| 4.2. Linear representations and their dg-tensor category | 31 |
| 4.3. An isomorphism with the dg-tensor category of left dg-modules | 37 |
| 4.4. A ccdg-Hopf algebra versus the dg-category of its left dg-modules | 41 |
| 5. Tannakian reconstruction theorem | 43 |

References | 59 |

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1. Introduction

Throughout this paper $k$ is a fixed ground field of characteristic zero, and every differential graded (dg) object has homological $\mathbb{Z}$-grading, unless otherwise specified.

1.1. A selected history and motivation

There is a well-known picture connecting cocommutative Hopf algebras, abstract groups and Lie algebras.\(^1\) We obtain a group from a cocommutative Hopf algebra as the group of group-like elements and a cocommutative Hopf algebra from an abstract group $\Gamma$ as its group ring $k\Gamma$, which case $\Gamma$ is isomorphic to the group of group-like elements. The category $\text{Rep}(\Gamma)$ of linear representations of $\Gamma$ is isomorphic to the category of left modules over $k\Gamma$ as tensor categories. Cocommutative Hopf algebras have a similar correspondence with Lie algebras via the Lie algebra of primitive elements and the universal enveloping algebra. A group ring $k\Gamma$ can be completed by the powers of its augmentation ideal to a complete Hopf algebra $\hat{k}\Gamma$. For a complete Hopf algebra the group of group-like elements is determined by the Lie algebra of primitive elements, and vice versa\(^2\).

This classic picture can be naturally recast in the context of a representable presheaf of groups $P : \text{ccC}(k) \to \text{Grp}$ on the category $\text{ccC}(k)$ of cocommutative coalgebras.\(^3\) A representing object of $P$ is a cocommutative Hopf algebra $H$, where the group $P(C)$ for each cocommutative coalgebra $C$ is isomorphic to the group formed by the set of all coalgebra maps from $C$ to $H$ with the convolution product as the composition. In particular the group of group-like elements in $H$ is isomorphic to the group $P(k^\vee)$, where $k^\vee$ is the ground field as a cocommutative coalgebra. Conversely an abstract group $\Gamma$ determines a presheaf $P_{\Gamma}$ of groups on $\text{ccC}(k)$ represented by $k\Gamma$, and we have $P_{\Gamma}(k^\vee) \cong \Gamma$. We can also form the tensor category of linear representations of $P$, which is isomorphic to the tensor category of left modules over $H$, and reconstruct $P$ from the forgetful fiber functor. The usual description of the Lie algebra of primitive elements in $H$ can be also recast more functorially by associating a presheaf of Lie algebras $TP : \text{ccC}(k) \to \text{Lie}(k)$ on $\text{ccC}(k)$ to each representable presheaf of groups $P$ such that (a) the Lie algebra $TP(k^\vee)$ is isomorphic to the Lie algebra of primitive elements in $H$ representing $P$; (b) we have a natural isomorphism $TP \cong TP'$ whenever the presheaves of groups $P$ and $P'$ are isomorphic; and (c) we have a pair of natural isomorphisms $TP \cong \hat{P} : \text{cccC}(k) \to \text{Set}$, between the presheaves underlying $TP$ and $P$, whenever the presheaf of groups $P$ is pro-represented by a complete Hopf algebra.

The purpose of this paper is to expand the above picture one-step further by studying a representable presheaf of groups $\Psi : \text{hoccdgC}(k) \to \text{Grp}$ on the homotopy category $\text{hoccdgC}(k)$ of cocommutative dg coalgebras (ccdg-coalgebras), its Lie alge-

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1 We refer to [2] for a more extensive history.
2 A formal group is a representable presheaf of groups on $\text{ccC}(k)$ satisfying certain conditions [3].
braic counterpart, their linear representations and a Tannaka type reconstruction theorem.

In general, a representable presheaf of groups on a category can be regarded as a group object in the category. A fundamental example is the presheaf of groups $\mathcal{H}_X : h_0\mathbf{Top} \rightarrow \mathbf{Grp}$ on the homotopy category $h_0\mathbf{Top}$, of pointed topological spaces represented by the based loop space $\Omega X$, of a pointed space $X$, where the group $\mathcal{H}_X(Y)$ for every based space $Y$, is the group formed by the set $\{Y, \Omega X\}$ of homotopy types of all base point preserving continuous maps to $\Omega X$, so that we have $\mathcal{H}_X(S^n) \cong \pi_{n+1}(X)$ for $n \geq 0$. As it seems that a full understanding of $\mathcal{H}_X$ is out of reach, we may follow the ideas of rational homotopy theory of Quillen [7] and Sullivan [10] to replace $h_0\mathbf{Top}$ with the rational homotopy category $h_0\mathbf{Top}$, and consider suitable full subcategories. For example, Quillen has considered the full subcategory $h_0\mathbf{Top}[2]_\ast$, of 1-connected pointed spaces and constructed a full-embedding $\mathcal{S} : h_0\mathbf{Top}[2]_\ast \hookrightarrow \mathbf{hoccdgC}(\mathbb{Q})$, to the homotopy category $\mathbf{hoccdgC}(\mathbb{Q})$, of coaugmented ccdg-coalgebras over $\mathbb{Q}$. This gives us the motivation to develop a general theory of representable presheaves of groups on the homotopy category $\mathbf{hoccdgC}(k)$.

The categorial dual to a representable presheaf of groups $\mathcal{P} : h_0\mathbf{cdgC}(k) \rightarrow \mathbf{Grp}$ on $h_0\mathbf{cdgC}(k)$ is a representable functor $\mathcal{G} : h_0\mathbf{cdgA}(k) \rightarrow \mathbf{Grp}$ from the homotopy category $h_0\mathbf{cdgA}(k)$ of commutative dg-algebras (cdg-algebra) with cohomological grading. This is the dg-version of an affine group scheme $G$ over $k$, which we call an affine group dg-scheme. The study of affine group schemes and their linear representation is a classic subject in algebraic geometry, which has led to the theory of Tannakian categories [8,4]. A neutral Tannakian category is equivalent the category of finite dimensional linear representations of an affine group scheme along with the forgetful functor to the category of underlying finite dimensional vector spaces. We expect to have similar constructions for affine group dg-schemes, which is the main subject of a sequel to this paper [6]. Rational homotopy theory gives us additional motivation for our study, since Sullivan has constructed a contravariant full-embedding $\mathcal{F} : h_0\mathbf{Top}[1]_{\ast n} \rightarrow \mathbf{hoccdgA}(\mathbb{Q})_\ast$ of the rational homotopy category $h_0\mathbf{Top}[1]_{\ast n}$ of 0-connected nilpotent pointed spaces of finite types into the homotopy category $\mathbf{hoccdgA}(\mathbb{Q})_\ast$ of augmented cdg-algebras over $\mathbb{Q}$ with the cohomological grading [10].

1.2. Results

Let $\mathcal{P} : h_0\mathbf{cdgC}(k) \rightarrow \mathbf{Grp}$ be a representable presheaf of groups on the homotopy category $h_0\mathbf{cdgC}(k)$ of ccdg-coalgebras over $k$.

A representing object of $\mathcal{P}$ is a cocommutative dg Hopf algebra (ccdg-Hopf algebra) $\Omega$, and we use the notation $\mathcal{P}_{\Omega}$ for it. For each ccdg-coalgebra $C$ the group $\mathcal{P}_{\Omega}(C)$ is the group formed by the set of homotopy types of all morphisms $g : C \rightarrow \Omega$ of ccdg-coalgebras, and a homotopy equivalence $f : C \rightarrow C'$ of ccdg-coalgebras induces an isomorphism $\mathcal{P}_{\Omega}(C') \cong \mathcal{P}_{\Omega}(C)$ of groups. The functor category of repre-
sentable presheaves of groups on \( \text{hoccdgC}(k) \) is equivalent to the homotopy category \( \text{hoccdgH}(k) \) of ccdg-Hopf algebras. [Theorem 3.1]

The Lie theoretic counterpart of \( P^\Omega \) is a presheaf \( T \Psi^\Omega : \text{hoccdgC}(k) \to \text{Lie}(k) \) of Lie algebras over \( k \) on the homotopy category \( \text{hoccdgC}(k) \), defined so that we have a natural isomorphism \( T \Psi^\Omega \cong T \Psi^\Omega' \) whenever we have a natural isomorphism \( \Psi^\Omega \cong \Psi^\Omega' \) or, equivalently, whenever \( \Omega \) and \( \Omega' \) are homotopy equivalent as ccdg-Hopf algebras. [Theorem 3.2]

If \( \Omega \) is concentrated in degree zero, the group \( \Psi^\Omega(k') \) is isomorphic to the group of group-like elements in \( \Omega \) and the Lie algebra \( T \Psi^\Omega(k') \) is isomorphic to the Lie algebra of primitive elements in \( \Omega \).

A complete ccdg-Hopf algebra is the dg-version of Quillen's complete cc-Hopf algebra. If \( \Omega \) is a complete ccdg-Hopf algebra, we construct a natural isomorphism \( T \Psi^\Omega \cong \Psi^\Omega : \text{hoccdgC}(k) \to \text{Set} \) between the underlying presheaves, respectively, so that the representable presheaf \( \Psi^\Omega \) of groups can be recovered from the presheaf \( T \Psi^\Omega \) of Lie algebras by the Baker-Campbell-Hausdorff formula. [Theorem 3.3]

We use a chain model for dg-categories—categories enriched in the category of chain complexes over \( k \). We define a linear representation of \( \Psi^\Omega \) via a linear representation of the associated presheaf of groups \( \mathcal{P}_\Omega : \text{ccdgc}(k) \to \text{Grp} \) on the category \( \text{ccdgc}(k) \) of ccdg-coalgebras, which is represented by \( \Omega \) and induces \( \Psi^\Omega \) on the homotopy category \( \text{hoccdgC}(k) \). The linear representations of \( \mathcal{P}_\Omega \) form a dg-tensor category \( \text{Rep}(\mathcal{P}_\Omega) \), which is isomorphic to the dg-tensor category \( \text{dgMod}_L(\Omega) \) of left dg-modules over \( \Omega \). [Theorem 4.1]

We reconstruct \( \Psi^\Omega \) via the forgetful dg-functor \( \omega : \text{dgMod}_L(\Omega) \to \text{Ch}(k) \) to the underlying dg-category \( \text{Ch}(k) \) of chain complexes as follows.

We consider a dg-tensor functor \( C \otimes : \text{Ch}(k) \to \text{dgComod}_L^{\text{cofr}}(C) \), which sends each chain complex to the cofree left dg-comodule cogenerated by the chain complex over the ccdg-coalgebra \( C \). Composed with \( \omega \), we have a dg-tensor functor \( C \otimes \omega : \text{dgMod}_L(\Omega) \to \text{dgComod}_L^{\text{cofr}}(C) \). The set \( \text{End}(C \otimes \omega) \) of natural endomorphisms of the functor \( C \otimes \omega \) is naturally a dg-algebra over \( k \)—this is trivial.

By considering the subset \( Z_0 \text{End}^\omega(C \otimes \omega) \) of \( \text{End}(C \otimes \omega) \) consisting of tensorial natural endomorphisms belonging to the kernel of the natural differential, we construct a presheaf of groups \( \mathcal{P}_\omega : \text{ccdgc}(k) \to \text{Grp} \) on the category \( \text{ccdgc}(k) \) of ccdg-coalgebras and prove that it is represented by the ccdg-Hopf algebra \( \Omega \).

After introducing a notion of homotopy types of elements in \( Z_0 \text{End}^\omega(C \otimes \omega) \) analogous to the homotopy types of morphisms of ccdg-coalgebras, we show that the presheaf of groups \( \mathcal{P}_\omega \) on \( \text{ccdgc}(k) \) induces a presheaf of groups \( \Psi^\Omega \) on the homotopy category \( \text{hoccdgC}(k) \). We then, construct a natural isomorphism

\[
\Psi^\Omega \cong \Psi^\Omega : \text{hoccdgC}(k) \to \text{Grp}
\]
of presheaves of groups on the homotopy category $\text{hocc}dg\text{C}(k)$, which is our Tannakian reconstruction. [Theorem 5.1]

A typical example of a ccdg-Hopf algebra $\Omega$ is the complete tensor dg-Hopf algebra generated by a $\mathbb{Z}$-graded vector space via the cobar construction [1] of a ccdg-coalgebra or a $C_{\infty}$-coalgebra. If $X_*$ is a pointed 2-connected and pointed space, the cobar construction of the ccdg-coalgebra $\mathcal{C}(X_*)$, the rational Quillen model for $X_*$ is a rational Quillen model for the based loop space $\Omega X_*$. Let $\mathbb{Q}\Pi X_* : \text{QhoTop}(1) \to \text{Grp}$ be a presheaf of groups on $\mathbb{Q}\text{hoTop}(1)$, represented by $\Omega X_*$. Then we have an isomorphism $Q\Pi X_*(Y) \cong P_{\mathbb{Q}\mathcal{C}(X_*)}(\mathcal{C}(Y))$ of groups for every 1-connected pointed space $Y$ since Quillen’s functor $Q\mathcal{C}$ is a full-embedding.

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2. Notation and basic notions

This section is a collection of some basic notions mainly to set up notation.

Unadorned tensor product $\otimes$ is over the ground field $k$. By an element of a $\mathbb{Z}$-graded vector space we shall usually mean a homogeneous element $x$ whose degree will be denoted $|x|$. Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ and $W = \bigoplus_{n \in \mathbb{Z}} W_n$ be $\mathbb{Z}$-graded vector spaces. Then $V \otimes W = \bigoplus_{n \in \mathbb{Z}} (V \otimes W)_n$, where $(V \otimes W)_n = \bigoplus_{i+j=n \in \mathbb{Z}} V_i \otimes W_j$, is also a $\mathbb{Z}$-graded vector space. Denoted by $\text{Hom}(V,W) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(V,W)_n$ is the $\mathbb{Z}$-graded vector space of $k$-linear maps from $V$ to $W$, where $\text{Hom}(V,W)_n$ is the space of $k$-linear maps increasing the degrees by $n$.

A chain complex $(V, \partial_V)$ is often denoted by $V$ for simplicity. For example, the ground field $k$ is a chain complex with the zero differential. Let $V$ and $W$ be chain complexes. Then $V \otimes W$ and $\text{Hom}(V,W)$ are also chain complexes with the following differentials

$$\begin{align*}
\partial_{V \otimes W} &= \partial_V \otimes 1_W + 1_V \otimes \partial_W, \\
\partial_{V,W} f &= \partial_W \circ f - (-1)^{|f|} f \circ \partial_V \quad \forall f \in \text{Hom}(V,W)_{|f|}.
\end{align*}$$

For example, a chain map $f : (V, \partial_V) \to (W, \partial_W)$ is an $f \in \text{Hom}(V,W)_0$ satisfying $\partial_{V,W} f = \partial_V \circ f - f \circ \partial_W = 0$. Two chain maps $f$ and $\tilde{f}$ are homotopic, denoted by $f \sim \tilde{f}$, or have the same homotopy type, denoted by $[f] = [\tilde{f}]$, if there is a chain homotopy $\lambda \in \text{Hom}(V,W)_1$ such that $\tilde{f} - f = \partial_{V,W} \lambda$.

The set of morphisms from an object $C$ to another object $C'$ in a category $\mathcal{C}$ is denoted by $\text{Hom}_\mathcal{C}(C,C')$. We denote the set of natural transformations of functors $F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ by $\text{Nat}(F,G)$. For any functor $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is small, we use the notation $\hat{F} : \mathcal{C} \to \text{Set}$ for the underlying set valued functor obtained by composing with the forgetful functor $\text{Forget} : \mathcal{D} \to \text{Set}$. A presheaf of groups on a category $\mathcal{C}$ is a
functor $F : \mathcal{C} \rightarrow \mathcal{Grp}$ from the opposite category $\mathcal{C}$ of $C$ to the category $\mathcal{Grp}$ of groups and is called representable if $\tilde{F}$ is representable.

Every algebra and coalgebra are assumed to have a unit and a counit, respectively, and every dg-algebra and dg-coalgebra has homological grading. We will be pedantic about the canonical isomorphisms $k \otimes V \cong V$ and $V \otimes k \cong V$ by adopting notations $i_V : k \otimes V \rightarrow V$ and $i^1_V : V \rightarrow k \otimes V$ as well as $j_V : V \otimes k \rightarrow V$ and $j^1_V : V \rightarrow V \otimes k$ for those isomorphisms.

A dg-algebra on a $\mathbb{Z}$-graded vector space $A$ is a tuple $A = (A, u_A, m_A, \partial_A)$, which is both a chain complex $\{A, \partial_A\}$ and a $\mathbb{Z}$-graded associative algebra $\{A, u_A, m_A\}$ such that both the unit $u_A : k \rightarrow A$ and the product $m_A : A \otimes A \rightarrow A$ are chain maps:

\[
\left\{\begin{array}{l}
\partial_A \circ u_A = 0,
\end{array}\right. \quad \left\{\begin{array}{l}
m_A \circ (u_A \otimes I_A) = \partial_A \circ m_A,
\end{array}\right.
\]

\hspace{1cm}
\begin{equation}
\left\{\begin{array}{l}
m_A \circ (m_A \otimes I_A) = m_A \circ (I_A \otimes m_A).
\end{array}\right. \tag{2.2}
\end{equation}

A morphism $f : A \rightarrow A'$ of dg-algebras is simultaneously a chain map, $f \circ \partial_A = \partial_A \circ f$, and an unital algebra map, $f \circ u_A = u_A$, and $f \circ m_A = m_A \circ (f \otimes f)$. The dg-algebras form a category, denoted by $\mathbf{dgA}(k)$, where the composition of morphisms is the composition as linear maps.

A dg-coalgebra on $\mathbb{Z}$-graded vector space $C$ is a tuple $C = (C, \epsilon_C, \Delta_C, \partial_C)$, which is both a chain complex $\{C, \partial_C\}$ and a $\mathbb{Z}$-graded coassociative coalgebra $(C, \epsilon_C, \Delta_C)$ such that both the counit $\epsilon_C : C \rightarrow k$ and the coproduct $\Delta_C : C \rightarrow C \otimes C$ are chain maps:

\[
\left\{\begin{array}{l}
\epsilon_C \circ \partial_C = 0,
\end{array}\right. \quad \left\{\begin{array}{l}
(\epsilon_C \otimes I_C) \circ \Delta_C = \partial_C \circ (\epsilon_C \otimes \epsilon_C) \circ \Delta_C = J_C^{-1},
\end{array}\right.
\]

\hspace{1cm}
\begin{equation}
\left\{\begin{array}{l}
\Delta_C \circ \partial_C = \partial_C \otimes C \circ \Delta_C.
\end{array}\right. \tag{2.3}
\end{equation}

A morphism $f : C \rightarrow C'$ of dg-coalgebras is simultaneously a chain map, $f \circ \partial_C = \partial_{C'} \circ f$, and a counital coalgebra map, $\epsilon_{C'} \circ f = \epsilon_C$ and $\Delta_{C'} \circ f = (f \otimes f) \circ \Delta_C$. The dg-coalgebras form a category, denoted by $\mathbf{dgC}(k)$, where the composition of morphisms is the composition as linear maps.

Every dg-coalgebra $C$ in this paper is cocommutative that $\Delta_C = \tau \circ \Delta_C$, where $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x, \forall x, y \in C$. The full subcategory of cocommutative dg-coalgebras (ccdg-coalgebras) of $\mathbf{dgC}(k)$ is denoted by $\mathbf{ccdgC}(k)$, and we use the prefix "cc" for cocommutative.

Remark that the ground field $k$ is an algebra $k = (k, u_k, m_k)$, where $u_k = I_k$ and $m_k(a \otimes b) = a \cdot b$, and a coalgebra $k^\vee = (k, \epsilon_k, \Delta_k)$ with $\epsilon_k = I_k$ and $\Delta_k(1) = 1 \otimes 1$. A ccdg-bialgebra $\Omega = (\Omega, u_\Omega, m_\Omega, \epsilon_\Omega, \partial_\Omega)$ is simultaneously a dg-algebra $\{\Omega, u_\Omega, m_\Omega, \partial_\Omega\}$ and a ccdg-coalgebra $\{\Omega, \epsilon_\Omega, \Delta_\Omega, \partial_\Omega\}$ such that both the counit $\epsilon_\Omega : \Omega \rightarrow k$ and the coproduct $\Delta_\Omega : \Omega \rightarrow \Omega \otimes \Omega$ are morphisms of dg-algebras:

\[
\left\{\begin{array}{l}
\epsilon_\Omega \circ u_\Omega = u_k, \quad \epsilon_\Omega \circ m_\Omega = m_k \circ (\epsilon_\Omega \otimes \epsilon_\Omega),
\end{array}\right. \quad \left\{\begin{array}{l}
\Delta_\Omega \circ u_\Omega = (u_\Omega \otimes u_\Omega) \circ \Delta_k, \quad \Delta_\Omega \circ m_\Omega = m_{\Omega \otimes \Omega} \circ (\Delta_\Omega \otimes \Delta_\Omega),
\end{array}\right.
\]

\hspace{1cm}
\begin{equation}
\left\{\begin{array}{l}
\end{array}\right. \tag{2.4}
\end{equation}
or, equivalently, both the unit \( u_\Omega : k \to \Omega \) and the product \( m_\Omega : \Omega \otimes \Omega \to \Omega \) are morphisms of ccdg-coalgebras—remind that \( m_{\Omega \otimes \Omega} \circ (\Delta_\Omega \otimes \Delta_\Omega) = (m_\Omega \otimes m_\Omega) \circ (I_\Omega \otimes \tau \otimes I_\Omega) \circ (\Delta_\Omega \otimes \Delta_\Omega) = (m_\Omega \otimes m_\Omega) \circ \Delta_{\Omega \otimes \Omega} \).

A ccdg-Hopf algebra \( \Omega \) is a ccdg-bialgebra with an antipode \( \zeta_\Omega \), which is a linear map \( \zeta_\Omega : \Omega \to \Omega \) of degree zero satisfying the following axiom:

\[
m_\Omega \circ (\zeta_\Omega \otimes I_\Omega) \circ \Delta_\Omega = m_\Omega \circ (I_\Omega \otimes \zeta_\Omega) \circ \Delta_\Omega = u_\Omega \circ \epsilon_\Omega.
\]  

(2.5)

Then, \( \zeta_\Omega \) is automatically a chain map. Also, antipode \( \zeta_\Omega \) of ccdg-bialgebra is unique if exists and both an anti-algebra map and coalgebra map:

\[
\begin{align*}
\begin{cases}
\zeta_\Omega \circ u_\Omega = u_\Omega, \\
\zeta_\Omega \circ m_\Omega = m_\Omega \circ (\zeta_\Omega \otimes \zeta_\Omega) \circ \tau, \\
\epsilon_\Omega \circ \zeta_\Omega = \epsilon_\Omega, \\
\Delta_\Omega \circ \zeta_\Omega = (\zeta_\Omega \otimes \zeta_\Omega) \circ \Delta_\Omega.
\end{cases}
\end{align*}
\]  

(2.6)

A morphism \( f : \Omega \to \Omega' \) of ccdg-Hopf algebras is simultaneously a morphism of dg-algebras and a morphism of ccdg-coalgebras—it is, then, automatic that \( f \) commutes with the antipodes. The ccdg-Hopf algebras form a category \( \text{ccdgh}(k) \), where the composition of morphisms is the composition as linear maps.

We can also form the homotopy category of ccdg-Hopf algebras, for which we introduce the notion of homotopy type of ccdg-Hopf algebra morphisms.

**Definition 2.1.** A homotopy pair on \( \text{Hom}_{\text{ccdgh}}(k)(\Omega, \Omega') \) is a pair of 1-parameter families \( \{f(t), \xi(t)\} \in \text{Hom}(\Omega, \Omega')_0[t] \oplus \text{Hom}(\Omega, \Omega')_1[t] \), which is parametrized by time variable \( t \) with polynomial dependences and satisfies the homotopy flow equation

\[
ds dt f(t) = \partial_{\Omega, \Omega'} \xi(t) \text{ generated by } \xi(t), \text{ subject to the following two types of conditions:}
\]

- infinitesimal algebra map: \( f(0) \in \text{Hom}_{\text{dgA}}(k)(\Omega, \Omega') \) and

\[
\xi(t) \circ u_\Omega = 0, \quad \xi(t) \circ m_\Omega = m_{\Omega'} \circ (f(t) \otimes \xi(t) + \xi(t) \otimes f(t)).
\]

- infinitesimal coalgebra map: \( f(0) \in \text{Hom}_{\text{ccdgc}}(k)(\Omega, \Omega') \) and

\[
\epsilon_{\Omega'} \circ \xi(t) = 0, \quad \Delta_{\Omega'} \circ \xi(t) = (f(t) \otimes \xi(t) + \xi(t) \otimes f(t)) \circ \Delta_\Omega.
\]

Let \( \{f(t), \xi(t)\} \) be a homotopy pair on \( \text{Hom}_{\text{ccdgh}}(k)(\Omega, \Omega') \). By the homotopy flow equation, \( f(t) \) is determined uniquely by \( \xi(t) \) modulo an initial condition \( f(0) \) such that \( f(t) = f(0) + \partial_{\Omega, \Omega'} \int_0^t \xi(s)ds \), and we can check that \( f(t) \) is a family of morphisms of ccdg-Hopf algebras. We say \( f(1) \) is homotopic to \( f(0) \) by the homotopy \( \int_0^1 \xi(t)dt \) and denote \( f(0) \sim f(1) \), which is clearly an equivalence relation. In other words, two morphisms \( f \) and \( \tilde{f} \) of ccdg-Hopf algebras are homotopic, \( f \sim \tilde{f} \), if there is a homotopy pair connecting them (by the time 1 map). Then, we also say that \( f \) and \( \tilde{f} \) have the same homotopy type, denoted by \( [f] = [\tilde{f}] \). For any diagram

\[
\begin{array}{ccc}
\Omega & \xrightarrow{f} & \Omega' \\
\downarrow{f} & & \downarrow{f'} \\
\Omega'' & \xrightarrow{f''} & \Omega''
\end{array}
\]
the category $\mathit{ccdgH}(k)$, where $f \sim f'$ and $f' \sim f''$, it is straightforward to check that $f'' \circ f \sim f'' \circ f' \in \mathit{Hom}_{\mathit{ccdgH}(k)}(\Omega, \Omega')$ and the homotopy type of $f'' \circ f$ depends only on the homotopy types of $f$ and $f'$, so that we have the well-defined associative composition $[f''] \circ_h [f] := [f'' \circ f]$ of homotopy types. A morphism $\Omega \xrightarrow{f} \Omega'$ of ccdg-Hopf algebras is a \textit{homotopy equivalence} if there is a morphism $\Omega \xrightarrow{h} \Omega'$ of ccdg-Hopf algebras from the opposite direction such that $h \circ f \sim 1_\Omega$ and $f \circ h \sim 1_{\Omega'}$.

The homotopy category $\mathit{hoccdgH}(k)$ of ccdg-Hopf algebras over $k$ is defined such that the objects are ccdg-Hopf algebras and morphisms are homotopy types of morphisms of ccdg-Hopf algebras. A homotopy equivalence of ccdg-Hopf algebras is an isomorphism in the homotopy category $\mathit{hoccdgH}(k)$.

We define a homotopy pair of ccdg-coalgebras as the case of ccdg-Hopf algebras but without imposing the condition for infinitesimal algebra map. Then, we have corresponding notions for homotopy types of morphisms of ccdg-coalgebras and a homotopy equivalence of ccdg-coalgebras such that we can form the homotopy category $\mathit{hoccdgC}(k)$ of ccdg-coalgebras, whose morphisms are homotopy types of morphisms of ccdg-coalgebras.

Remark that our notion of homotopy category is the \textit{naive} one—all based on chain complex over field $k$ with explicitly defined homotopies of morphisms.$^3$

In this paper, a dg-category $\mathcal{C}$ over $k$ is a category enriched in the category $\mathbf{Ch}(k)$ of chain complexes over $k$—we refer to [5] for a review of dg-categories. Besides using the chain model we follows [9] for the notion of dg-tensor categories. The chain complex of morphisms from object $X$ to object $Y$ in a dg-category $\mathcal{C}$ by $\mathit{Hom}_\mathcal{C}(X, Y)$ with the differential $\partial_{\mathit{Hom}_\mathcal{C}(X, Y)}$. A morphism $f \in \mathit{Hom}_\mathcal{C}(X, Y)$ between two objects $X$ and $Y$ of $\mathcal{C}$ is an isomorphism if $f \in \mathit{Hom}_\mathcal{C}(X, Y)_0$ and satisfies $\partial_{\mathit{Hom}_\mathcal{C}(X, Y)} f = 0$ and there is an inverse $g \in \mathit{Hom}_\mathcal{C}(Y, X)_0$ satisfying $\partial_{\mathit{Hom}_\mathcal{C}(Y, X)} g = 0$.

A dg-functor $F : \mathcal{C} \to \mathcal{D}$ is a functor which induces chain maps between sets of morphisms. It follows that the set $\mathit{Nat}(F, G)$ of natural transformations of dg-functors is a chain complex $(\mathit{Nat}(F, G), \delta)$, where

- its degree $n$ element is a collection of morphisms $\eta = \{\eta_X : F(X) \to G(X) | X \in \text{Ob}(\mathcal{C})\}$ of degree $n$, where $\eta_X$ is called the component of $\eta$ at $X$, with the supercommuting naturalness condition, i.e. $G(f) \circ \eta_X = (-1)^{m_f} \eta_Y \circ F(f)$ for every morphism $f : X \to Y$ of degree $m$.

- for every $\eta \in \mathit{Nat}(F, G)$ of degree $n$ we have $\delta \eta \in \mathit{Nat}(F, G)$ of degree $n-1$, whose component at $X$ is defined by $\delta \eta_X := \partial_{\mathit{Hom}_\mathcal{C}(F(X), G(X))} \eta_X$, and $\delta \circ \delta = 0$.

The dg-functors from $\mathcal{C}$ to $\mathcal{D}$ form a dg-category, with morphisms as the above natural transformations. In particular, the set $\mathit{End}(F)$ of natural endomorphisms has a canonical structure of dg-algebra.

$^3$ Our definition of the homotopy category of ccdg-coalgebras is equivalent to Quillen’s definition in [7] if we consider the full subcategory of 2-reduced ccdg-coalgebras.
A natural transformation \( \eta \) from a dg-functor \( F \) to another dg-functor \( G \) is often indicated by a diagram \( \eta : F \Rightarrow G : C \rightarrow D \). A natural transformation \( \eta \) is an (natural) isomorphism if the component morphism \( \eta_X : F(X) \rightarrow G(X) \) is an isomorphism in \( D \) for every object \( X \) of \( C \).

The notion of tensor category \([8, 4]\) has a natural generalization to dg-tensor category. For a dg-category \( C \) we have a new dg-category \( C \otimes C \), whose objects are pairs \( (X, Y) \) of objects of \( C \), and whose morphisms are the tensor products of Hom complexes of \( C \), i.e., \( \text{Hom}_{C \otimes C}(X \otimes Y, X' \otimes Y') = \text{Hom}_{C}(X, X') \otimes \text{Hom}_{C}(Y, Y') \) with the natural composition operation and differentials. Then we have a natural equivalence of dg-categories \( (C \otimes C) \otimes C \cong C \otimes (C \otimes C) \). A dg-category \( C \) is a dg-tensor category if we have a dg-functor \( \otimes : C \otimes C \rightarrow C \) and a unit object \( 1_C \) satisfying the associativity, the commutativity and the unit axioms. (See pp 40-41 in \([9]\) for the details.)

The fundamental example of dg-tensor category over \( k \) is the dg-category \( Ch(k) \) of chain complexes, whose set of morphisms \( \text{Hom}_{Ch(k)}(V, W) \) from a chain complex \( V \) to a chain complex \( W \) is the hom complex \( \text{Hom}(V, W) \) with the differential \( \partial_{\text{Hom}_{Ch(k)}(V, W)} = \partial_V \cdot \partial_W \). The dg-functor \( \otimes : Ch(k) \otimes Ch(k) \rightarrow Ch(k) \) sends \( (V, \partial_V) \otimes (W, \partial_W) \) to the chain complex \( (V \otimes W, \partial_V \otimes_W) \) and an unit object is the ground field \( k \) as a chain complex \((k, 0)\), where all coherence isomorphisms are strict.

A dg-tensor functor \( F : C \rightarrow D \) between dg-tensor categories is a dg-functor satisfying \( F(X \otimes Y) \cong F(X) \otimes F(Y) \) and \( F(1_C) \cong 1_D \). A tensor natural transformation \( \eta : F \Rightarrow G \) of dg-tensor functors is a natural transformation of degree 0 satisfying \( \eta_X \otimes \eta_Y \cong \eta_{X \otimes Y} \) and \( \eta_{1_C} \cong \eta_{1_D} \).

We use the notation \([a]\) for the homotopy type of a morphism \( a \) as well as for the homology class of a cycle \( a \), depending on the context.

### 3. Representable presheaves of groups and presheaves of Lie algebras

The purpose of this section is to develop basic theory of a representable presheaf of groups \( \mathfrak{P} : \text{hocc dgC}(k) \rightarrow \text{Grp} \) on homotopy category \( \text{hocc dgC}(k) \) and its Lie algebraic counterpart.

In Sect. 3.1, we check that a representing object of \( \mathfrak{P} \) is a ccdg-Hopf algebra \( \Omega \), and we use the notation \( \mathfrak{P}_{\Omega} \) for it. The group \( \mathfrak{P}_{\Omega}(C) \) for each ccdg-coalgebra \( C \) is the group formed by the set of homotopy types of all morphisms \( g : C \rightarrow \Omega \) of ccdg-coalgebras, and a homotopy equivalence \( f : C \rightarrow C' \) of ccdg-coalgebras induces an isomorphism \( \mathfrak{P}_{\Omega}(C') \cong \mathfrak{P}_{\Omega}(C) \) of groups. The category of representable presheaves of groups on \( \text{hocc dgC}(k) \) is equivalent to the homotopy category \( \text{hocc dgH}(k) \) of ccdg-Hopf algebras. We observe that the group \( \mathfrak{P}_{\Omega}(k') \), where \( k' \) is the dual coalgebra on \( k \), acts naturally on \( \mathfrak{P}_{\Omega}(C) \) so that the underlying presheaf \( \mathfrak{P}_{\Omega} : \text{hocc dgC}(k) \rightarrow \text{Set} \) is \( \mathfrak{P}_{\Omega}(k') \)-set valued. If \( \Omega \) is concentrated in degree zero, the group \( \mathfrak{P}_{\Omega}(k') \) is isomorphic to the group of group-like elements in \( \Omega \).
In Sect. 3.2, we define the Lie theoretic counterpart to \( \mathfrak{P}_\Omega \) by a presheaf of Lie algebras \( T\mathfrak{P}_\Omega : \text{hoccddgC}(k) \to \text{Lie}(k) \) on the homotopy category \( \text{hoccddgC}(k) \). The Lie algebra \( T\mathfrak{P}_\Omega(C) \) for each ccdg-coalgebra \( C \) is the Lie algebra formed by the set of homotopy types of all infinitesimal morphisms \( \nu : C \to \Omega \) of ccdg-coalgebras about the identity element of the group \( \mathfrak{P}_\Omega(C) \). We have a natural isomorphism \( T\mathfrak{P}_\Omega \cong T\mathfrak{P}_\Omega' \) whenever there is a natural isomorphism \( \mathfrak{P}_\Omega \cong \mathfrak{P}_\Omega' \) or, equivalently, whenever \( \Omega \) and \( \Omega' \) are homotopy equivalent as ccdg-Hopf algebras. If \( \Omega \) is concentrated in degree zero, the Lie algebra \( T\mathfrak{P}_\Omega(k^\nu) \) is isomorphic to the Lie algebra of primitive elements in \( \Omega \).

In Sect. 3.3, we consider a pro-representable presheaf of group \( \mathfrak{P}_\Omega \), whose representing object \( \Omega \) is a complete ccdg-Hopf algebra \( \Omega = \overline{\Omega} \). Complete ccdg-Hopf algebra is the dg-version of Quillen’s complete (cocommutative) Hopf algebra. We construct a natural isomorphism \( \mathfrak{P}_\Omega \xrightarrow{\ln} T\mathfrak{P}_\Omega : \text{hoccddgC}(k) \to \text{Set} \) between the underlying presheaves, so that the representable presheaf of groups \( \mathfrak{P}_\Omega \) can be recovered from the presheaf of Lie algebras \( T\mathfrak{P}_\Omega \) using the Baker-Campbell-Hausdorff formula.

### 3.1. Representable presheaf of groups \( \mathfrak{P}_\Omega : \text{hoccddgC}(k) \to \text{Grp} \)

The purpose of this subsection is to prove the following:

**Theorem 3.1 (Definition).** For each ccdg-Hopf algebra \( \Omega \), we have a representable presheaf of groups \( \mathfrak{P}_\Omega : \text{hoccddgC}(k) \to \text{Grp} \) on the homotopy category \( \text{hoccddgC}(k) \) of ccdg-coalgebras over \( k \), sending

- a ccdg-coalgebra \( C = (\mathcal{C}, \epsilon_C, \Delta_C, \delta_C) \) to the group
  \[
  \mathfrak{P}_\Omega(C) := \left\{ \text{Hom}_{\text{hoccddgC}(k)}(C, \Omega), \epsilon_C, \Delta_C, \delta_C \right\},
  \]
  with the group operation \( [g_1] \ast_{\mathcal{C}, \Omega} [g_2] := \left[ m_{\Omega} \circ (g_1 \otimes g_2) \circ \Delta_C \right] \), the identity element \( \epsilon_C, \Omega = \left[ u_{\Omega} \circ \epsilon_C \right] \), and the inverse \( g^{-1} := \left[ \zeta_{\Omega} \circ \overline{g} \right] \) of \( g \), where \( g_i \in \text{Hom}_{\text{ccdgc}(k)}(C, \Omega) \) is an arbitrary representative of the homotopy type \( [g_i] \in \text{Hom}_{\text{hoccddgC}(k)}(C, \Omega) \).

- a morphism \( [f] \in \text{Hom}_{\text{hoccddgC}(k)}(C, C') \) in the homotopy category of ccdg-coalgebras to a homomorphism \( \mathfrak{P}_\Omega([f]) : \mathfrak{P}_\Omega(C') \to \mathfrak{P}_\Omega(C) \) of groups defined by, \( \forall [g'] \in \text{Hom}_{\text{hoccddgC}(k)}(C', \Omega) \),
  \[
  \mathfrak{P}_\Omega([f])([g']) := [g' \circ f],
  \]
where \( f \in \text{Hom}_{\text{ccdgc}(k)}(C, C') \) and \( g' \in \text{Hom}_{\text{ccdgc}(k)}(C', \Omega) \) are arbitrary representatives of the homotopy types \( [f] \) and \( [g'] \), respectively.
such that $\mathcal{P}_\Omega([f])$ is an isomorphism of groups whenever $f : C \rightarrow C'$ is a homotopy equivalence of ccdg-coalgebras.

Representing object of a representable presheaf of groups $\mathfrak{P} : \text{hoccdgC}(k) \rightarrow \text{Grp}$ on $\text{hoccdgC}(k)$ is a ccdg-Hopf algebra $\Omega$ such that $\mathfrak{P} \cong \mathfrak{P}_\Omega$.

For each morphism $[\psi] \in \text{Hom}_{\text{hoccdgH}}(\Omega, \Omega')$ in the homotopy category of ccdg-Hopf algebras we have a natural transformation $\mathcal{N}_{[\psi]} : \mathfrak{P}_\Omega \Rightarrow \mathfrak{P}_{\Omega'} : \text{hoccdgC}(k) \rightarrow \text{Grp}$ defined such that for every ccdg-coalgebra $C$ and $[g] \in \text{Hom}_{\text{hoccdgC}}(C, \Omega)$ we have

$$\mathcal{N}_{[\psi]}([\alpha]) := [\psi \circ g],$$

where $\psi \in \text{Hom}_{\text{hoccdgH}}(\Omega, \Omega')$ and $g \in \text{Hom}_{\text{hoccgC}}(C, \Omega)$ are arbitrary representatives of the homotopy types $[\psi]$ and $[g]$, respectively.

We have an one-to-one correspondence $\text{Hom}_{\text{hoccdgH}}(\Omega, \Omega') \cong \text{Nat}(\mathfrak{P}_\Omega, \mathfrak{P}_{\Omega'})$ such that $\mathcal{N}_{[\psi]}$ is a natural isomorphism whenever $\psi : \Omega \rightarrow \Omega'$ is a homotopy equivalence of ccdg-Hopf algebras.

We divide the proof into few pieces. The main technical point is that the presheaf of groups $\mathfrak{P}_\Omega$ is defined via the associated presheaf of groups $\mathcal{P}_\Omega : \text{ccdgC}(k) \rightarrow \text{Grp}$ on the category $\text{ccdgC}(k)$ of ccdg-coalgebras, which is represented by $\Omega$ and induces $\mathfrak{P}_\Omega$ on the homotopy category $\text{hoccdgC}(k)$.

**Lemma 3.1 (Definition).** For every ccdg-Hopf algebra $\Omega$ we have a presheaf of dg-algebras $\mathcal{E}_\Omega : \text{ccdgC}(k) \rightarrow \text{dgA}(k)$ over $k$ on $\text{ccdgC}(k)$, sending each ccdg-coalgebra $C$ to the convolution dg-algebra $\mathcal{E}_\Omega(C) := (\text{Hom}(C, \Omega), u_\Omega \circ \epsilon_C, \ast_{C, \Omega}, \partial_{C, \Omega})$, where $[\alpha_1, \alpha_2] \in \text{Hom}(C, \Omega)$,

$$- u_\Omega \circ \epsilon_C : C \xrightarrow{\epsilon_C} k \xrightarrow{u_\Omega} \Omega$$

and

$$- \alpha_1 \ast_{C, \Omega} \alpha_2 := m_\Omega \circ (\alpha_1 \otimes \alpha_2) \circ \Delta_C : C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{\alpha_1 \otimes \alpha_2} \Omega \otimes \Omega \xrightarrow{m_\Omega} \Omega,$$

and each morphism $f : C \rightarrow C'$ of ccdg-coalgebras to a morphism $\mathcal{E}_\Omega(f) : \mathcal{E}_\Omega(C') \rightarrow \mathcal{E}_\Omega(C)$ of dg-algebras defined by $\mathcal{E}_\Omega(f)(\alpha') := \alpha' \circ f$ for all $\alpha' \in \text{Hom}(C', \Omega)$.

**Proof.** It is a standard fact that $\mathcal{E}_\Omega(C)$ is a dg-algebra: The convolution product $\ast_{C, \Omega}$ is associative due to the associativity of $m_\Omega$ and the coassociativity of $\Delta_C$ and $u_\Omega \circ \epsilon_C$ is the identity element for the convolution product due to the counity of $C \xrightarrow{\epsilon_C} k$ and the unity of $k \xrightarrow{u_\Omega} \Omega$. We have $\partial_{C, \Omega}(u_\Omega \circ \epsilon_C) = 0$ by $\epsilon_C \circ \partial_C = \partial_\Omega \circ u_\Omega = 0$ and $\partial_{C, \Omega}$ is a derivation of the convolution product since $\partial_C$ is a coderivation of $\Delta_C$ and $\partial_\Omega$ is a derivation of $m_\Omega$.

We check that $\mathcal{E}_\Omega(f)$ is a morphism of dg-algebras as follows: $\forall \alpha'_1, \alpha'_2 \in \text{Hom}(C', \Omega)$,

$$\mathcal{E}_\Omega(f)(u_\Omega \circ \epsilon_{C'}) = u_\Omega \circ \epsilon_{C'} \circ f = u_\Omega \circ \epsilon_C,$$
We show that

**Proof.**

**Lemma 3.2.** For every ccdg-Hopf algebra \( \Omega \) we have a representable presheaf of groups \( \mathcal{P}_\Omega : \text{ccdgC}(k) \to \text{Grp} \) on \( \text{ccdgC}(k) \), sending each ccdg-coalgebra \( C \) to a group

\[
\mathcal{P}_\Omega(C) := \{ \text{Hom}_{\text{ccdgC}(k)}(C, \Omega), u_\Omega \circ \epsilon_C, \star_{C, \Omega} \},
\]

where the inverse of \( g \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \) is \( g^{-1} := \zeta_\Omega \circ g \), and each morphism \( f : C \to C' \) of ccdg-coalgebras to a homomorphism \( \mathcal{P}_\Omega(f) : \mathcal{P}_\Omega(C') \to \mathcal{P}_\Omega(C) \) of groups defined by \( \mathcal{P}_\Omega(f)(g') := g' \circ f \) for all \( g' \in \text{Hom}(C', \Omega) \).

**Proof.** 1. We show that \( \mathcal{P}_\Omega(C) \) is a group for every ccdg-coalgebra \( C \) as follows. We remind that

\[
\text{Hom}_{\text{ccdgC}(k)}(C, \Omega) := \{ g \in \text{Hom}(C, \Omega) \mid \partial_{C, \Omega} g = 0, \epsilon_\Omega \circ g = \epsilon_C, \Delta_\Omega \circ g = (g \otimes g) \circ \Delta_C \},
\]

and check routinely that

- \( u_\Omega \circ \epsilon_C \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \), since \( C \xrightarrow{\epsilon_C} k \) and \( k \xrightarrow{u_\Omega} \Omega \) are morphisms of ccdg-coalgebras;

- \( g_1 \star_{C, \Omega} g_2 \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \) whenever \( g_1, g_2 \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \), since we have

\[
\begin{align*}
\partial_{C, \Omega}(g_1 \star_{C, \Omega} g_2) &= (\partial_{C, \Omega} g_1) \star_{C, \Omega} g_2 + g_1 \star_{C, \Omega} \partial_{C, \Omega} g_2 = 0, \\
\epsilon_\Omega \circ (g_1 \star_{C, \Omega} g_2) &= \epsilon_\Omega \circ m_\Omega \circ (g_1 \otimes g_2) \otimes \Delta_C = m_\Omega \circ (\epsilon_\Omega \circ g_1 \otimes \epsilon_\Omega \circ g_2) \circ \Delta_C \\
&= m_\Omega \circ (\epsilon_C \otimes \epsilon_\Omega) \circ \Delta_C = m_\Omega \circ (\epsilon_C \otimes 1_\Omega) \circ 1_C^{-1} = \epsilon_C,
\end{align*}
\]

and, by definition and by the cocommutativity of \( \Delta_C \),

\[
\begin{align*}
\Delta_\Omega \circ (g_1 \star_{C, \Omega} g_2) &= \Delta_\Omega \circ m_\Omega \circ (g_1 \otimes g_2) \circ \Delta_C \\
&= m_{\Omega \otimes \Omega} \circ (\Delta_\Omega \otimes \Delta_\Omega) \circ (g_1 \otimes g_2) \circ \Delta_C \\
&= (m_\Omega \otimes m_\Omega) \circ (1_\Omega \otimes 1_\Omega) \circ (g_1 \otimes g_1 \otimes g_2 \otimes g_2) \circ (\Delta_C \otimes \Delta_C) \circ \Delta_C \\
&= (m_\Omega \otimes m_\Omega) \circ (g_1 \otimes g_2 \otimes g_1 \otimes g_2) \circ (1_C \otimes 1_C) \circ (\Delta_C \otimes \Delta_C) \circ \Delta_C \\
&= (m_\Omega \otimes m_\Omega) \circ (g_1 \otimes g_2 \otimes g_1 \otimes g_2) \circ (\Delta_C \otimes \Delta_C) \circ \Delta_C \\
&= (m_\Omega \otimes m_\Omega) \circ (g_1 \otimes g_2) \circ (1_C \otimes 1_C) \circ \Delta_C \\
&= (g_1 \star_{C, \Omega} g_2) \circ (g_1 \star_{C, \Omega} g_2) \circ \Delta_C.
\end{align*}
\]

- \( g^{-1} := \zeta_\Omega \circ g \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \) whenever \( g \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \).
Then, by Lemma 3.1, $\mathcal{P}_\Omega(C) := \{\text{Hom}_{\text{ccdgC}[k]}(C, \Omega), u_\Omega \circ \epsilon_C, \epsilon_C \}$ is a monoid. On the other hand, by the antipode axiom eq. (2.5) we have, $\forall g \in \text{Hom}_{\text{ccdgC}[k]}(C, \Omega)$,

$$g \cdot_{C, \Omega} g^{-1} := m_{\Omega} \circ (g \otimes \zeta_\Omega) \circ g \circ \Delta_C = m_{\Omega} \circ ([u_\Omega \otimes \zeta_\Omega] \circ \Delta_\Omega \circ g = u_\Omega \circ \epsilon_\Omega \circ g = u_\Omega \circ \epsilon_C,$$

and, similarly, $g^{-1} \cdot_{C, \Omega} g = u_\Omega \circ \epsilon_C$. Hence, $\mathcal{P}_\Omega(C)$ is actually a group.

2. We have $\mathcal{P}_\Omega(f)(g') := g' \circ f \in \text{Hom}_{\text{ccdgC}[k]}(C, \Omega)$ whenever $g' \in \text{Hom}_{\text{ccdgC}[k]}(C', \Omega)$ since $C \xrightarrow{f} C'$ is a morphism of ccdg-coalgebras. Then, by Lemma 3.1, we have

- $\mathcal{P}_\Omega(f) : \mathcal{P}_\Omega(C') \rightarrow \mathcal{P}_\Omega(C)$ is a group homomorphism;
- $\mathcal{P}_\Omega(f' \circ f) = \mathcal{P}_\Omega(f) \circ \mathcal{P}_\Omega(f')$, $\forall f' \in \text{Hom}_{\text{ccdgC}[k]}(C', C^*)$.

Therefore $\mathcal{P}_\Omega : \text{ccdC}[k] \rightarrow \text{Grp}$ is a representable presheaf of groups on the category $\text{ccdC}[k]$ of ccdg-coalgebras. \(\Box\)

Remark 3.1. $\mathcal{P}_\Omega : \text{ccdC}[k] \rightarrow \text{Grp}$ is a presheaf of groups even if $\Omega$ is not a strictly co-commutative dg-Hopf algebra but, then, it is not a representable presheaf on $\text{ccdC}[k]$.

Lemma 3.3. Suppose $\mathcal{P}$ is a representable presheaf of groups $\text{ccdC}[k] \rightarrow \text{Grp}$ on $\text{ccdC}[k]$. Then $\mathcal{P} \cong \mathcal{P}_\Omega$ for some ccdg-Hopf algebra $\Omega$.

Proof. Since $\hat{\mathcal{P}} : \text{ccdC}[k] \rightarrow \text{Set}$ is representable, we have $\hat{\mathcal{P}} \cong \text{Hom}_{\text{ccdC}[k]}(-, \Omega)$ for a ccdg-coalgebra $\Omega$. We shall show that $\Omega$ carries a ccdg-Hopf algebra structure. We can restate the condition that $\hat{\mathcal{P}}$ factors through $\text{Grp}$ as follows:

- For each ccdg-coalgebra $C$, there is a structure group on $\hat{\mathcal{P}}(C)$. In other words, there are three structure functions $\mu^C : \hat{\mathcal{P}}(C) \times \hat{\mathcal{P}}(C) \rightarrow \hat{\mathcal{P}}(C)$, $\epsilon_C : \{\ast\} \rightarrow \hat{\mathcal{P}}(C)$ and $i^C : \hat{\mathcal{P}}(C) \rightarrow \hat{\mathcal{P}}(C)$ satisfying the group axioms.

- For each morphism $f : C \rightarrow C'$ of ccdg-coalgebras, the function $\hat{\mathcal{P}}(f) : \hat{\mathcal{P}}(C') \rightarrow \hat{\mathcal{P}}(C)$ is a group homomorphism.

This is equivalent to the existence of natural transformations $\mu : \hat{\mathcal{P}} \times \hat{\mathcal{P}} \rightarrow \hat{\mathcal{P}}$, $e : \{\ast\} \rightarrow \hat{\mathcal{P}}$, and $i : \hat{\mathcal{P}} \rightarrow \hat{\mathcal{P}}$ satisfying the group axioms. Here, $\{\ast\}$ is a functor $\text{ccdC}[k] \rightarrow \text{Set}$ sending every ccdg-coalgebra $C$ to a one-point set $\{\ast\}$.

Note that $k^\vee$ is a terminal object in the category $\text{ccdC}[k]$ since any morphism $C \rightarrow k^\vee$ of ccdg-coalgebras from a ccdg-coalgebra $C$ has to be the counit $\epsilon_C$ by the counit axiom. Let $\Omega \otimes \Omega$ be the ccdg-coalgebra obtained by the tensor product of the ccdg-coalgebra $\Omega$. We claim that there are natural isomorphisms of presheaves

$$\hat{\mathcal{P}} \times \hat{\mathcal{P}} \cong \text{Hom}_{\text{ccdC}[k]}(-, \Omega \otimes \Omega), \quad \{\ast\} \cong \text{Hom}_{\text{ccdC}[k]}(-, k^\vee). \quad (3.1)$$

Then, by the Yoneda lemma, the natural transformations $\mu$, $e$, and $i$ are completely determined by some morphisms $m_\Omega : \Omega \otimes \Omega \rightarrow \Omega$, $u_\Omega : k^\vee \rightarrow \Omega$ and $\zeta_\Omega : \Omega \rightarrow \Omega$ of ccdg-coalgebras, respectively. Applying the Yoneda lemma again, a plain calculation shows that
1. \( \mu \circ (\mu \times \mathbb{P}) = \mu \circ (\mathbb{P} \times \mu) \) implies the associativity of \( m_\Omega \).
2. \( \mu \circ (e \times \mathbb{P}) = \mu \circ (\mathbb{P} \times e) = \mathbb{P} \) implies the unit axiom of \( u_\Omega \).

3. inverse element axiom of \( i \) implies the antipode axiom of \( \zeta_\Omega \).

Therefore \( \Omega \) has a structure of ccdg-Hopf algebras.

Now we check the claimed isomorphisms in eq. (3.1). Let \( C \otimes C' \) be the ccdg-coalgebra obtained by the tensor product of ccdg-coalgebras \( C \) and \( C' \). Then we have the following projection morphisms of ccdg-coalgebras:

\[
\pi_C := C \otimes C' \xrightarrow{\mu \otimes \epsilon} C, \quad \pi_C' := C \otimes C' \xrightarrow{\epsilon \otimes \mu} C'.
\]

For each ccdg-coalgebra \( T \) we consider the function

\[
\text{Hom}_{\text{ccdgc}}(T, C \otimes C') \to \text{Hom}_{\text{ccdgc}}(T, C) \times \text{Hom}_{\text{ccdgc}}(T, C')
\]

defined by \( h \mapsto (\pi_C \circ h, \pi_C' \circ h) \). We show that the function is a bijection for every \( T \) by constructing its inverse. Given morphisms \( f : T \to C, g : T \to C' \) of ccdg-coalgebras, define \( \langle f, g \rangle := (\mu \otimes \epsilon) \circ \Delta_T : T \to C \otimes C' \). Then it is a morphism of ccdg-coalgebras since \( \Delta_T \) is cocommutative. It is obvious that \( \langle \pi_C \circ h, \pi_C' \circ h \rangle = h, \pi_C \circ \langle f, g \rangle = f \) and \( \pi_C' \circ \langle f, g \rangle = g \). Moreover, the above bijection is natural in \( T \in \text{ccdgc}(k) \). Therefore we have constructed the claimed isomorphisms in eq. (3.1).

Let \( x_1, x_2 \) be the elements in \( \mathcal{P}(C) \) that correspond to morphisms \( g_1, g_2 : C \to \Omega \) of ccdg-coalgebras via the bijection \( \mathcal{P}(C) \cong \mathcal{P}_{\Omega}(C) \). Then, by the Yoneda lemma, \( \mu_C(x_1, x_2) \in \mathcal{P}(C) \) corresponds to \( m_\Omega \circ \langle g_1, g_2 \rangle = g_1 \ast_{C, \Omega} g_2 \in \mathcal{P}(C) \), and \( \epsilon_C \in \mathcal{P}(C) \) corresponds to \( u_\Omega \circ \epsilon_C \in \mathcal{P}_{\Omega}(C) \). This shows that the bijection \( \mathcal{P}(C) \cong \mathcal{P}_{\Omega}(C) \) is an isomorphism of groups. Thus we have a natural isomorphism \( \mathcal{P} \cong \mathcal{P}_{\Omega} \) of the presheaves of groups on \( \text{ccdgc}(k) \).

**Lemma 3.4.** For every morphism \( \psi : \Omega \to \Omega' \) of ccdg-Hopf algebras we have a natural transformation \( \mathcal{N}^C_\psi : \mathcal{P}_{\Omega} \Rightarrow \mathcal{P}_{\Omega'} : \text{ccdgc}(k) \to \text{Grp} \) defined such that its component \( \mathcal{N}^C_\psi : \mathcal{P}_{\Omega}(C) \to \mathcal{P}_{\Omega'}(C) \) group homomorphism at each ccdg-coalgebra \( C \) is \( \mathcal{N}^C_\psi(\alpha) := \psi \circ \alpha \) for all \( \alpha \in \text{Hom}_{\text{ccdgc}}(k)(C, \Omega) \). We also have \( \text{Hom}_{\text{ccdgc}}(k)(\Omega, \Omega') \cong \text{Nat}(\mathcal{P}_{\Omega}, \mathcal{P}_{\Omega'}) \).

**Proof.** For every \( C, \mathcal{N}^C_\psi \) is a group homomorphism since, \( \forall g_1, g_2 \in \text{Hom}_{\text{ccdgc}}(k)(C, \Omega) \),

\[
\mathcal{N}^C_\psi(g_1 \ast_{C, \Omega} g_2) = \psi \circ m_{\Omega} \circ (g_1 \otimes g_2) \circ \Delta_C = m_{\Omega'} \circ (\psi \circ g_1 \otimes \psi \circ g_2) \circ \Delta_C \\
= \mathcal{N}^C_\psi(g_1) \ast_{C, \Omega'} \mathcal{N}^C_\psi(g_2), \\
\mathcal{N}^C_\psi(u_\Omega \circ \epsilon_C) = \psi \circ u_{\Omega'} \circ \epsilon_C = u_{\Omega'} \circ \epsilon_C.
\]
For every morphism \( f : C \to C' \) of ccdg-coalgebras we have, \( \forall g' \in \text{Hom}_{\text{cdgC}[k]}(C', \Omega) \),
\[
\mathcal{N}_\psi \circ \mathcal{P}_\Omega(f)(g') = \psi \circ (g' \circ f) = (\psi \circ g') \circ f = \mathcal{P}_{\Omega'}(f) \circ \mathcal{N}_\psi(g').
\]
Therefore, \( \mathcal{N}_\psi \in \text{Nat}(\mathcal{P}_\Omega, \mathcal{P}_{\Omega'}) \) whenever \( \psi \in \text{Hom}_{\text{cdgH}[k]}(\Omega, \Omega') \). Combined with the Yoneda lemma, we have \( \text{Nat}(\mathcal{P}_\Omega, \mathcal{P}_{\Omega'}) \cong \text{Hom}_{\text{cdgH}[k]}(\Omega, \Omega') \). \( \square \)

**Lemma 3.5.** Assume that
- \((g(t), \chi(t)), (g_1(t), \chi_1(t))\) and \((g_2(t), \chi_2(t))\) are homotopy pairs in \( \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \);
- \((f(t), \lambda(t))\) is a homotopy pair in \( \text{Hom}_{\text{cdgC}[k]}(C, C') \) and \((g'(t), \lambda'(t))\) is a homotopy pair in \( \text{Hom}_{\text{cdgC}[k]}(C', \Omega) \);
- \((\psi(t), \xi(t))\) is a homotopy pair in \( \text{Hom}_{\text{cdgH}[k]}(\Omega, \Omega') \).

Then we have the following homotopy pairs
(a) \((g_1(t) \ast_{C, \Omega} g_2(t)), \chi_1(t) \ast_{C, \Omega} \chi_2(t)\) on \( \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \).
(b) \((\xi_\Omega \circ g(t), \xi_\Omega \circ \lambda(t))\) on \( \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \).
(c) \((g'(t) \circ f(t), g'(t) \circ \lambda(t) + \chi'(t) \circ f(t))\) on \( \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \).
(d) \((\psi(t) \circ g(t), \psi(t) \circ \chi(t) + \xi(t) \circ g(t))\) on \( \text{Hom}_{\text{cdgC}[k]}(C, \Omega') \).

**Proof.** These can be checked by routine computations, which are omitted for the sake of space. \( \square \)

Now we are ready for the proof of Theorem 3.1.

**Proof (Theorem 3.1).** After Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5, we just need to check few things to finish the proof.

1. We check that the group \( \mathcal{P}_\Omega(C) \) is well-defined for every ccdg-coalgebra \( C \).
   - We have \( g_1 \ast_{C, \Omega} g_2 \sim \tilde{g}_1 \ast_{C, \Omega} \tilde{g}_2 \in \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \) whenever \( g_1 \sim \tilde{g}_1, g_2 \sim \tilde{g}_2 \in \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \); this follows from Lemma 3.5(a).
   - We have \( g^{-1} \sim \tilde{g}^{-1} \in \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \) whenever \( g \sim \tilde{g} \in \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \); this follows from Lemma 3.5(b).

    Also followed is that the homotopy type \([g_1 \ast_{C, \Omega} g_2]\) of \( g_1 \ast_{C, \Omega} g_2 \) depends only on the homotopy types \([g_1], [g_2] \in \text{Hom}_{\text{cdgC}[k]}(C, \Omega)\) of \( g_1 \) and \( g_2 \). Therefore the group \( \mathcal{P}_\Omega(C) \) is well-defined.

2. We check that the homomorphism \( \mathcal{P}_\Omega([f]) : \mathcal{P}_\Omega(C') \to \mathcal{P}_\Omega(C) \) of groups is well defined for every \([f] \in \text{Hom}_{\text{cdgC}[k]}(C, C') \). Let \( f \sim \tilde{f} \in \text{Hom}_{\text{cdgC}[k]}(C, C') \) and \( g' \sim \tilde{g}' \in \text{Hom}_{\text{cdgC}[k]}(C', \Omega) \). Then, by Lemma 3.5(c), we have \( g \circ f \sim g \circ \tilde{f} \sim \tilde{g} \circ f \sim \tilde{g}' \circ f \sim \tilde{g}' \circ \tilde{f} \sim g' \circ \tilde{f} \sim g' \circ f \sim g \circ f \)
$g \circ f \in \text{Hom}_{\text{cdgC}(k)}(C, \Omega)$ so that $\Psi_p([f])([g]) := [g \circ f]$ depends only on the homotopy types $[f]$ and $[g]$ of arbitrary representatives $f \in \text{Hom}_{\text{cdgC}(k)}(C, C')$ and $g' \in \text{Hom}_{\text{cdgC}(k)}(C', \Omega)$. Therefore $\Psi_p([f])$ is well defined group homomorphism. It is obvious that $\Psi_p([f])$ is an isomorphism of groups whenever $f : C \to C'$ is a homotopy equivalence of cdg-coalgebras.

3. We check that the natural transformation $\psi : \Psi_p \to \text{hoccdgC}(k) \to \text{Grp}$ is well-defined for every $[\psi] \in \text{Hom}_{\text{cdgH}(k)}(\Omega, \Omega')$. Let $\psi \sim \tilde{\psi} \in \text{Hom}_{\text{cdgH}(k)}(\Omega, \Omega')$ and $g, \tilde{g} \in \text{Hom}_{\text{cdgC}(k)}(C, \Omega)$. Then, by Lemma 3.5(d), we have

$$\psi \circ g \sim \psi \circ \tilde{g} \sim \tilde{\psi} \circ g \sim \tilde{\psi} \circ \tilde{g} \in \text{Hom}_{\text{cdgC}(k)}(C, \Omega')$$

so that $\mathcal{N}_{[\psi]}(g) := [\psi \circ g]$ for every cdg-coalgebra $C$ depends only on the homotopy types $[\psi]$ and $[g]$ of arbitrary representatives $\psi \in \text{Hom}_{\text{cdgH}(k)}(\Omega, \Omega')$ and $g \in \text{Hom}_{\text{cdgC}(k)}(C, \Omega)$. Therefore the natural transformation $\mathcal{N}_{[\psi]} : \Psi_p \Longrightarrow \Psi_p' \equiv \text{hoccdgC}(k)$ is well-defined such that $\mathcal{N}_{[\psi]} \in \text{Nat}(\Psi_p', \Psi_p')$ whenever $[\psi] \in \text{Hom}_{\text{cdgH}(k)}(\Omega, \Omega')$. Combined with the Yoneda lemma, we have $\text{Nat}(\Psi_p', \Psi_p') \equiv \text{Hom}_{\text{cdgH}(k)}(\Omega, \Omega')$, i.e., the category of representable presheaves of groups on $\text{hoccdgC}(k)$ over $k$ is equivalent to the homotopy category $\text{hoccdgH}(k)$ of cdg-Hopf algebras over $k$. It is obvious that $\mathcal{N}_{[\psi]} : \Psi_p(C) \longrightarrow \Psi_p'(C)$ is an isomorphism of groups for every cdg-coalgebra $C$ whenever $\psi : \Omega \to \Omega'$ is a homotopy equivalence of cdg-Hopf algebras. Therefore $\mathcal{N}_{[\psi]}$ is a natural isomorphism whenever $\psi$ is a homotopy equivalence of cdg-Hopf algebra. □

3.1.1. $\Psi(k')$ action on the presheaf $\tilde{\Psi} : \text{hoccdgC}(k) \to \text{Set}$. The ground field $k$, after forgetting the product, has a structure $k' = (k, \alpha, 0)$ of cdg-coalgebra with the counit $\varepsilon_k = 1$, the coproduct $\Delta_k = 1 \otimes 1$ and the zero differential, and is a terminal object in the category $\text{hoccdgC}(k)$. Therefore we have the group $\Psi_p(k')$, which plays a special role. The counit $\varepsilon_c : C \to k$ of every cdg-coalgebra $C$ is a morphism of cdg-coalgebra to $k'$. Therefore, there is a canonical group homomorphism $\Psi_p(\varepsilon_c) : \Psi_p(k') \to \Psi_p'(C)$. Let $\tilde{\Psi}_p : \text{hoccdgC}(k) \to \text{Set}$ be the representable presheaf of groups underlying $\Psi_p$.

**Lemma 3.6.** $\tilde{\Psi}_p : \text{hoccdgC}(k) \to \text{Set}$ is a representable presheaf of $\Psi_p(k')$-sets on the homotopy category of cdg-coalgebras.

**Proof.** It is trivial to check that $\tilde{\Psi}_p(C)$ is a $\Psi_p(k')$-set for every cdg-coalgebra $C$ with the action $\rho_C : \Psi_p(k') \times \tilde{\Psi}_p(C) \longrightarrow \tilde{\Psi}_p(C)$ defined by

$$\rho_C([g], [\alpha]) \rightarrow ([g \circ \epsilon_C] \ast_C \alpha) = [\epsilon_C \circ ((g \circ \epsilon_C) \otimes \alpha) \circ \Delta_C],$$

where $g \in \text{Hom}_{\text{cdgC}(k)}(k', \Omega)$ and $\alpha \in \text{Hom}_{\text{cdgC}(k)}(C, \Omega)$ are arbitrary representative of $[g]$ and $[\alpha]$ respectively, such that for every $[f] \in \text{Hom}_{\text{hoccdgC}(k)}(C, C')$ the following
diagram commutes

\[
\begin{array}{ccc}
\Psi_{\Omega}(k') \times \hat{\Psi}_{\Omega}(C') & \xrightarrow{\rho_{C'}} \hat{\Psi}_{\Omega}(C') \\
\downarrow_{1 \times \Psi_{\Omega}(f)} & \downarrow_{\Psi_{\Omega}(f)} \\
\Psi_{\Omega}(k') \times \hat{\Psi}_{\Omega}(C) & \xrightarrow{\rho_{C}} \hat{\Psi}_{\Omega}(C),
\end{array}
\]

i.e., \( \rho_{C} \circ (\mathbb{I} \times \hat{\Psi}_{\Omega}(f)) = \hat{\Psi}_{\Omega}(f) = \circ \rho_{C'} .
\]

\[\square\]

3.2. Presheaf of Lie algebras \( T\Psi_{\Omega} : \text{hocc}dgC(k) \to \text{Lie}(k) \)

For a ccdg-Hopf algebra \( \Omega \) we construct a presheaf \( T\Psi_{\Omega} : \text{hocc}dgC(k) \to \text{Lie}(k) \) of Lie algebras on the homotopy category of ccdg-coalgebras such that we have a natural isomorphism \( T\Psi_{\Omega} \cong T\Psi'_{\Omega} \) whenever we have a natural isomorphism \( \Psi_{\Omega} \cong \Psi'_{\Omega} \) or, equivalently, we have a homotopy equivalence \( \Omega \cong \Omega' \) of ccdg-Hopf algebras. The converse is not true in general.

Denoted by \( \text{THom}_{\text{ccd}gC}(k, \Omega) \), for each ccdg-coalgebra \( C \), is the set of all tangential ccdg-coalgebra maps about the identity \( e_{C,\Omega} := u_{\Omega} \circ \varepsilon_{C} : \)

\[
\text{THom}_{\text{ccd}gC}(k, \Omega) = \left\{ v \in \text{Hom}(C, \Omega)_{0} \middle| \begin{aligned}
\partial_{C,\Omega} v &= 0, \\
\epsilon_{\Omega} \circ v &= 0, \\
\Delta_{\Omega} \circ v &= (\epsilon_{C,\Omega} \otimes v + v \otimes \epsilon_{C,\Omega}) \circ \Delta_{C}
\end{aligned} \right\}. \tag{3.2}
\]

A homotopy pair \( (v(t), \sigma(t)) \) on \( \text{THom}_{\text{ccd}gC}(k, \Omega) \) is a pair of \( v(t) \in \text{Hom}(C, \Omega)_{0}[t] \)

and \( \lambda(t) \in \text{Hom}(C, \Omega)_{1}[t] \) satisfying the homotopy flow equation \( \frac{d}{dt} v(t) = \partial_{C,\Omega} \sigma(t) \)
generated by \( \sigma(t) \) subject to the following conditions

\[\begin{align*}
v(0) &\in \text{THom}_{\text{ccd}gC}(k, \Omega), \\
\epsilon_{\Omega} \circ \lambda(t) &= 0, \\
\Delta_{\Omega} \circ \lambda(t) &= (\epsilon_{C,\Omega} \otimes \sigma(t) + \sigma(t) \otimes \epsilon_{C,\Omega}) \circ \Delta_{C}.
\end{align*}\tag{3.3}\]

It is straightforward to check that \( v(t) \) is a family of tangential ccdg-coalgebra maps about the identity. We say \( v, \bar{v} \in \text{THom}_{\text{ccd}gC}(k, \Omega) \) are homotopic, \( v \sim \bar{v} \), or having the same homotopy type, \( [v] = [\bar{v}] \), if there is a homotopy flow connecting them. Denoted by \( \text{THom}_{\text{hocc}dgC}(k, \Omega) \) is the set of homotopy types of all infinitesimal ccdg-coalgebra maps about the identity.

**Theorem 3.2 (Definition).** For each ccdg-Hopf algebra \( \Omega \) we have a presheaf of Lie algebras \( T\Psi_{\Omega} : \text{hocc}dgC(k) \to \text{Lie}(k) \) on the homotopy category cocommutative dg-coalgebras, sending
– each ccdg-coalgebra $C$ to the Lie algebra $T^\mathfrak{P}_\Omega(C) := \left( \text{THom}_{\text{hoccdgC}}(C, \Omega), [-,-]_{\text{ccdg}} \right)$

with the Lie bracket defined by, $\forall [u_1], [u_2] \in (\text{THom}_{\text{hoccdgC}}(C, \Omega)$

$$[[u_1], [u_2]]_{\text{ho}} := \left[ m_\Omega \circ (u_1 \otimes u_2 - u_2 \otimes u_1) \circ \Delta_C \right] = [u_1]_{c,\Omega} [u_2] - [u_2]_{c,\Omega} [u_1],$$

where $u_1, u_2 \in \text{THom}_{\text{hoccdgC}}(C, \Omega)$ are arbitrary representatives of the homotopy types $[u_1], [u_2]$, respectively.

– each morphism $[f] \in \text{Hom}_{\text{hoccdgC}}(C, C')$ to the morphism $T^\mathfrak{P}_\Omega([f]) : T^\mathfrak{P}_\Omega(C') \to T^\mathfrak{P}_\Omega(C)$ of Lie algebras defined by, $\forall [u'] \in \text{THom}_{\text{hoccdgC}}(C', \Omega)$

$$T^\mathfrak{P}_\Omega([f])([u']) := [u' \circ f],$$

where $f$ and $u'$ are arbitrary representatives of the homotopy types $[f]$ and $[u']$, respectively, such that $T^\mathfrak{P}_\Omega([f])$ is an isomorphism of Lie algebra whenever $f : C \to C'$ is a homotopy equivalence ccdg-coalgebras.

For every morphism $[\psi] \in \text{Hom}_{\text{hoccgC}}(\Omega, \Omega')$ we have a natural transformation $T_\psi : T^\mathfrak{P}_\Omega \Rightarrow T^\mathfrak{P}_{\Omega'} : \text{hoccgC} \to \text{Lie}(k)$, whose component Lie algebra homomorphism $T_\psi = T^\mathfrak{P}_\Omega \circ \text{ho}^{c,\Omega} : T^\mathfrak{P}_{\Omega}(C) \to T^\mathfrak{P}_{\Omega'}(C)$ at each ccdg-coalgebra $C$ defined by, $\forall [v] \in \text{THom}_{\text{hoccgC}}(C, \Omega)$

$$T_\psi([v]) := \left[ \psi \circ v \right],$$

where $\psi \in \text{Hom}_{\text{hoccgC}}(\Omega, \Omega')$ and $v \in \text{THom}_{\text{hoccgC}}(C, \Omega)$ are arbitrary representatives of the homotopy types $[\psi]$ and $[v]$, respectively, such that $T_\psi : T^\mathfrak{P}_\Omega \Rightarrow T^\mathfrak{P}_{\Omega'} : \text{hoccgC} \to \text{Lie}(k)$ is a natural isomorphism whenever $\psi : \Omega \to \Omega'$ is a homotopy equivalence of ccdg-Hopf algebras.

For every ccdg-Hopf algebra $\Omega$ we define a presheaf of Lie algebras $T^\mathfrak{P}_\Omega : \text{ccdgC} \to \text{Lie}(k)$ on $\text{ccdgC}(k)$, which will induces $T^\mathfrak{P}_\Omega : \text{hoccdgC} \to \text{Lie}(k)$.

Lemma 3.7. For each ccdg-Hopf algebra $\Omega$ we have a presheaf of Lie algebras $T^\mathfrak{P}_\Omega : \text{ccdgC}(k) \to \text{Lie}(k)$, sending each ccdg-coalgebra $C$ to the Lie algebra $T^\mathfrak{P}_\Omega(C) := \left( \text{THom}_{\text{ccdgC}}(C, \Omega), [-,-]_{\text{ccdg}} \right)$, where $[v_1, v_2]_{c,\Omega} := v_1 c_{v_2} - v_2 c_{v_1} c_{v_2}$ for all $v_1, v_2 \in \text{THom}_{\text{ccdgC}}(C, \Omega)$, and each morphism $f : C \to C'$ of ccdg-coalgebras to a Lie algebra homomorphism $T^\mathfrak{P}_\Omega(f) : T^\mathfrak{P}_\Omega(C') \to T^\mathfrak{P}_\Omega(C)$ defined by $T^\mathfrak{P}_\Omega(f)(v') = v' \circ f$, $\forall v' \in \text{THom}_{\text{ccdgC}}(C', \Omega)$. 


Proof. 1. We show that $TP_{\Omega}(C)$ is a Lie algebra as follows. It is suffice to check that $[v_1, v_2]_{c, \Omega} \in \mathbf{THom}_{ccdgC[k]}(C, \Omega)$ for every $v_1, v_2 \in \mathbf{THom}_{ccdgC[k]}(C, \Omega)$ since the convolution product $\ast_{c, \Omega}$ is associative. We have

$$\partial_{C, \Omega}[v_1, v_2] = [\partial_{C, \Omega} v_1, v_2] + [v_1, \partial_{C, \Omega} v_2] = 0,$$

$$\epsilon_{\Omega} \circ [v_1, v_2] = \epsilon_{\Omega} \circ m_{\Omega} \circ ([v_1, v_2]_\circ) \circ \Delta_C = m_{\Omega} \circ (\epsilon_{\Omega} \otimes \epsilon_{\Omega}) \circ ([v_1, v_2]_\circ) \circ \Delta_C = 0,$$

$$\Delta_{\Omega} \circ [v_1, v_2] = \Delta_{\Omega} \circ m_{\Omega} \circ ([v_1, v_2]_\circ) \circ \Delta_C$$

$$= m_{\Omega} \otimes m_{\Omega} \circ ([v_1, v_2]_\circ \otimes e + e \otimes [v_1, v_2]_\circ) \circ \Delta_C$$

$$= (m_{\Omega} \otimes m_{\Omega}) \circ ([v_1, v_2]_\circ \otimes e + e \otimes [v_1, v_2]_\circ) \circ \Delta_C$$

$$= ([v_1, v_2]_\circ \otimes e + e \otimes [v_1, v_2]_\circ) \circ \Delta_C$$

$$= ([v_1, v_2]_\circ \otimes e + e \otimes [v_1, v_2]_\circ) \circ \Delta_C,$$

where we have used the short-hand notations $\ast_{C, \Omega} = \ast$, $u_{\Omega} \circ \epsilon_C = e$ and $[v_1, v_2]_\circ = v_1 \otimes v_2 - v_2 \otimes v_1$.

2. We show that $TP_{\Omega}(f) : TP_{\Omega}(C') \rightarrow TP_{\Omega}(C)$ is a Lie algebra homomorphism as follows. To begin with we need to check that $TP_{\Omega}(f)(v') = v' \circ f \in \mathbf{THom}_{ccdgC[k]}(C', \Omega)$ whenever $v' \in \mathbf{THom}_{ccdgC[k]}(C', \Omega)$:

$$\partial_{C, \Omega}(v' \circ f) = \partial_{C, \Omega} v' \circ f - v' \circ \partial_{C, \Omega} f = (\partial_{\Omega} \circ v' - v' \circ \partial_{C}) \circ f = 0,$$

$$\epsilon_{\Omega}(v' \circ f) = \epsilon_{\Omega}(v' \circ f) = (\epsilon_{\Omega} \circ v') \circ f = 0,$$

$$\Delta_{\Omega}(v' \circ f) = ([u_{\Omega} \circ \epsilon_{C'}] \otimes v' + v' \otimes (u_{\Omega} \circ \epsilon_{C'})) \circ \Delta_C \circ f$$

$$= ([u_{\Omega} \circ \epsilon_{C'}] \otimes v' + v' \otimes (u_{\Omega} \circ \epsilon_{C'})) \circ (f \otimes f) \circ \Delta_C$$

$$= ([u_{\Omega} \circ \epsilon_{C'}] \otimes v' + v' \otimes (u_{\Omega} \circ \epsilon_{C'})) \circ \Delta_C.$$

Then we check that $TP_{\Omega}(f)$ is a homomorphism of Lie algebras:

$$TP_{\Omega}(f)([v_1', v_2'])_{c, \Omega} = m_{\Omega} \circ [v_1', v_2']_\circ \circ \Delta_C \circ f = m_{\Omega} \circ [v_1', v_2']_\circ \circ (f \otimes f) \circ \Delta_C$$

$$= m_{\Omega} \circ [v_1', f \circ v_2', f \circ f]_\circ \circ \Delta_C = [TP_{\Omega}(f)(v_1'), TP_{\Omega}(f)(v_2')]_{c, \Omega}.$$

3. The functoriality of $TP_{\Omega}$ is obvious. □

Lemma 3.8. For each morphism $\psi : \Omega \rightarrow \Omega'$ of ccdg-Hopf algebras we have a natural transformation $T_{\Lambda} \psi : TP_{\Omega} \Rightarrow TP_{\Omega'} : ccdgC[k] \rightarrow Lie[k]$, whose component Lie algebra homomorphism $T_{\Lambda} \psi : TP_{\Omega}(C) \rightarrow TP_{\Omega'}(C)$ at every ccdg-coalgebra $C$ is defined by $T_{\Lambda} \psi(C)(v) = \psi \circ v$, $\forall v \in \mathbf{THom}_{ccdgC[k]}(C, \Omega)$. 
Proof. 1. We show that $T\mathcal{N}^C_\psi : T\mathcal{P}_\Omega(C) \to T\mathcal{P}_{\Omega'}(C)$ is a Lie algebra homomorphism for every ccdg-coalgebra $C$. We check that $T\mathcal{N}^C_\psi(v) := \psi \circ v \in \text{THom}_{cdg[k]}(C', \Omega)$ for all $v \in \text{THom}_{cdg[k]}(C, \Omega)$:

\[
\begin{align*}
\partial_{C, \Omega'}(\psi \circ v) &= \partial_{\Omega'} \circ (\psi \circ v) - \psi \circ v \circ \partial_C = \psi \circ (\partial_{\Omega'} \circ v - v \circ \partial_C) = 0, \\
\epsilon_{\Omega'} \circ (\psi \circ v) &= \epsilon_{\Omega} \circ v = 0, \\
\Delta_{\Omega'} \circ (\psi \circ v) &= (\psi \otimes \psi) \circ \Delta_{\Omega} \circ v = (\psi \otimes \psi) \circ ((u_{\Omega} \circ \epsilon_C) \otimes v + v \otimes (u_{\Omega} \circ \epsilon_C))
\end{align*}
\]

Now we check that $T\mathcal{N}^C_\psi : T\mathcal{P}_\Omega(C) \to T\mathcal{P}_{\Omega'}(C)$ is a Lie algebra homomorphism: for all $v_1, v_2 \in \text{THom}_{cdg[k]}(C, \Omega)$ we have

\[
T\mathcal{N}^C_\psi([v_1, v_2]_{C, \Omega}) = \psi \circ [v_1, v_2]_{\Omega} \circ \Delta_C = m_{\Omega'} \circ [\psi \circ v_1, \psi \circ v_2]_{\Omega} \circ \Delta_C = [T\mathcal{N}^C_\psi(v_1), T\mathcal{N}^C_\psi(v_2)]_{\Omega'}. 
\]

2. We show that $T\mathcal{N}^C_\psi : T\mathcal{P}_\Omega \to T\mathcal{P}_{\Omega'} : \text{cdgdG}[k] \to \text{Lie}[k]$ is a natural transformation: For every morphism $f : C \to C'$ of ccdg-coalgebras we have $T\mathcal{N}^C_\psi \circ T\mathcal{P}_\Omega(f) = T\mathcal{P}_{\Omega'}(f) \circ T\mathcal{N}^C_\psi$ since for all $v' \in \text{THom}_{cdg[k]}(C', \Omega)$ we obtain that

\[
T\mathcal{N}^C_\psi \circ T\mathcal{P}_\Omega(f)(v') = \psi \circ (v' \circ f) = (\psi \circ v') \circ f = T\mathcal{P}_{\Omega'}(f) \circ T\mathcal{N}^C_\psi(v').
\]

\[\square\]

Lemma 3.9. Assume that

- $(v_1(t), \sigma_1(t))$ and $(v_2(t), \sigma_2(t))$ are homotopy pairs on $\text{THom}_{cdg[k]}(C, \Omega)$;

- $(f(t), \lambda(t))$ is a homotopy pair on $\text{Hom}_{cdg[k]}(C, C')$ and $(v'(t), \sigma'(t))$ is a homotopy pair on $\text{THom}_{cdg[k]}(C', \Omega)$;

- $(\psi(t), \xi(t))$ is a homotopy pair on $\text{Hom}_{cdg[k]}(\Omega, \Omega')$.

Then we have following homotopy pairs

(a) $[[v_1(t), v_2(t)]_{C, \Omega}, [\sigma_1(t), \sigma_2(t)]_{C, \Omega}] + [v_1(t), \sigma_2(t)]_{C, \Omega}$ on $\text{THom}_{cdg[k]}(C, \Omega)$.

(b) $[v'(t) \circ f(t), v'(t) \circ \lambda(t) + \sigma'(t) \circ f(t)]$ on $\text{THom}_{cdg[k]}(C, \Omega)$.

(c) $[\psi(t) \circ v(t), \psi(t) \circ \sigma(t) + \xi(t) \circ v(t)]$ on $\text{Hom}_{cdg[k]}(C, \Omega')$.

Proof. These can be checked by routine computations, which are omitted for the sake of space. \[\square\]

Now we are ready for the proof of Theorem 3.2
**Proof (Theorem 3.2).** Based on Lemmas 3.7, 3.8 and 3.9, it is trivial to check that the Lie algebra \( TP_\Omega(C) \) is well-defined and \( TP_\Omega([f]) : TP_\Omega(C') \to TP_\Omega(C) \) is a well-defined Lie algebra homomorphism, depending only on the homotopy type \([f]\) of \( f \).

Therefore \( TP_\Omega \) is a presheaf of Lie algebras on the homotopy category \( hDgC(k) \) of ccdg-coalgebras as claimed, where the functoriality of \( TP_\Omega \) is obvious. It is also obvious that \( TP_\Omega([f]) \) is an isomorphism of Lie algebras whenever \( f : C \to C' \) is a homotopy equivalence of ccdg-coalgebras. It also trivial to check that the natural transformation \( T_N^\psi : TP_\Omega \Rightarrow TP_\Omega' : hDgC(k) \to Lie(k) \) is well-defined such that \( T_N^\psi \in \text{Nat}(TP_\Omega, TP_\Omega') \) whenever \( \psi \in \text{Hom}_{hDgC(k)}(\Omega, \Omega') \). Finally it is obvious that \( T_N^\psi \) is a natural isomorphism whenever \( \psi \) is a homotopy equivalence of ccdg-Hopf algebra. \( \square \)

Remind that the category of representable presheaves of groups on \( hDgC(k) \) is equivalent to the homotopy category \( hDgH(k) \) of ccdg-Hopf algebras—Theorem 3.1. Therefore, the assignments \( \Psi_\Omega \to TP_\Omega \) and \( \chi^\psi \to T_N^\psi \) define a functor \( T \) from the category of representable presheaves of groups on \( hDgC(k) \) to the category of presheaves of Lie algebras on \( hDgC(k) \) such that \( T_N^\psi \) is a natural isomorphism whenever \( \chi^\psi \) is a natural isomorphism.

The following is obvious by definitions.

**Corollary 3.1.** If \( \Omega \) is concentrated in degree zero, the group \( \Psi_\Omega(k^\vee) \) is the group of group-like elements in \( \Omega \), and the Lie algebra \( TP_\Omega(k^\vee) \) is the Lie algebra of primitive elements in \( \Omega \).

### 3.3. Complete ccdg-Hopf algebra and an isomorphism \( TP_\Omega \cong \Psi_\Omega \) of presheaves

Quillen has introduced the notion of complete cocommutative Hopf algebra to provide the Hopf algebra framework for the Malcev completion and groups defined by the Baker-Campbell-Hausdorff formula as well as for the rationalized homotopy groups of a pointed space \([7]\). For example, the \( k \)-rational group ring \( k\Gamma \) of an abstract group \( \Gamma \) can completed by the powers of its augmentation ideal \( \mathfrak{n} \) to a complete Hopf algebra \( k\hat{\Gamma} = \lim \downarrow k\Gamma / \mathfrak{n}^{n+1} \) such that the set of primitive elements form a Lie algebra \( \mathfrak{l}(k\hat{\Gamma}) \) and there is a bijective correspondence \( \exp \mathfrak{l}(k\hat{\Gamma}) \cong \text{G}(k\hat{\Gamma}) \) with the group
Consider a ccdg-Hopf algebra $\Omega = (\Omega, u_\Omega, m_\Omega, \epsilon_\Omega, \Delta_\Omega, \zeta_\Omega, \partial_\Omega)$.

We introduce some notations. Denote by $m_\Omega^{(n)} : \Omega \otimes \cdots \otimes \Omega \to \Omega$, $n \geq 1$, the $n$-fold iterated product generated by $m_\Omega : \Omega \otimes \Omega \to \Omega$ such that $m_\Omega^{(1)} = 1$ and $m_\Omega^{(n+1)} = m_\Omega \circ (m_\Omega^{(n)} \otimes I_\Omega) \equiv m_\Omega^{(n)} \otimes (\Omega \otimes \cdots \otimes \Omega)$. Denote by $m_{\Omega^\otimes \Omega}^{(n)} : (\Omega \otimes \Omega) \otimes \cdots \otimes (\Omega \otimes \Omega) \to (\Omega \otimes \cdots \otimes \Omega) \otimes \Omega$, $n \geq 1$, the $n$-fold iterated coproduct generated by $m_{\Omega^\otimes \Omega} = (m_\Omega \otimes m_\Omega) \circ (I_\Omega \otimes \cdots \otimes I_\Omega) : (\Omega \otimes \cdots \otimes \Omega) \otimes (\Omega \otimes \cdots \otimes \Omega) \to (\Omega \otimes \cdots \otimes \Omega) \otimes \Omega$. Then we have

$$\Delta_\Omega \circ m_\Omega^{(n)} = m_{\Omega^\otimes \Omega}^{(n)} \circ (\Delta_\Omega \otimes \cdots \otimes \Delta_\Omega).$$

The counit $\epsilon_\Omega : \Omega \to k$ is a canonical augmentation of $\Omega$, since we have $\epsilon_\Omega \circ u_\Omega = 1_k$, $\epsilon_\Omega \circ m_\Omega = m_k \circ (\epsilon_\Omega \otimes \epsilon_\Omega)$ and $\partial_\Omega = 0$. Let $J = \text{Ker} \epsilon_\Omega$ be the augmentation ideal. Then, we have a splitting $\Omega = k \cdot u_\Omega(1) \otimes J$ and a decreasing filtration

$$\Omega = J^0 \supset J^1 \supset J^2 \supset J^3 \supset \cdots,$$

where $J^n = m_{\Omega^\otimes J}^{(n)}(J \otimes \cdots \otimes J)$. It is straightforward to check that every structure of ccdg-Hopf algebra $\Omega$ is compatible with the above filtration. We can endow trivial filtration on $k$ and both $u_\Omega : k \to \Omega$ and $\epsilon_\Omega : \Omega \to k$ are filtration preserving map. It is obvious that $m_{\Omega^\otimes J} : J^i \otimes J^j \to J^{i+j}$. From $\partial_\Omega \circ u_\Omega = \epsilon_\Omega \circ \partial_\Omega = 0$, we have $\partial_\Omega : J \to J$ and $\partial_\Omega : J^n \subset J^n$ as $\partial_\Omega$ is a derivation of $m_\Omega$. From $\epsilon_\Omega \circ \zeta_\Omega = \epsilon_\Omega$ we have $\zeta_\Omega : J \to J$ and $\zeta_\Omega : J^n \to J^n$ since $\zeta_\Omega$ is anti-homomorphism of $m_\Omega$. Finally, we have $\Delta_\Omega : J^n \to \bigoplus_{i+j=n} J^i \otimes J^j$, which can be checked as follows: we have $\Delta_\Omega : J^0 \to J^0 \otimes J^0$ by definition and we can deduce that $\Delta_\Omega : J^0 \to J^0 \otimes J^1 \otimes J^1 \otimes J^0$ from the counit property, which is combined with the identity eq. (3.4) to show the lest. Then, $\hat{\Omega} = \lim_{\rightarrow}\left(\Omega/J^n+\right)$ is a complete augmented dg-algebra with the decreasing filtration

$$\hat{\Omega} = F^0(\Omega) \supset F^1(\hat{\Omega}) \supset F^2(\hat{\Omega}) \supset \cdots,$$

where $F^n(\hat{\Omega}) = \text{Ker}(\hat{\Omega} \to \Omega/J^{n+1})$. We also have extended coproduct

$$\hat{\Delta}_\Omega : \hat{\Omega} \to \hat{\Omega} \otimes \hat{\Omega} = \lim_{\rightarrow}\left(\Omega \otimes \Omega/F^{n+1}(\hat{\Omega} \otimes \hat{\Omega})\right),$$

where $F^n(\hat{\Omega} \otimes \hat{\Omega}) = \bigoplus_{i+j=n} F^i(\hat{\Omega}) \otimes F^j(\hat{\Omega})$, such that $\hat{\Omega} = (\hat{\Omega}, \hat{u}_\Omega, \hat{m}_\Omega, \hat{\epsilon}_\Omega, \hat{\Delta}_\Omega, \hat{\zeta}_\Omega, \hat{\partial}_\Omega)$ is a ccdg-Hopf algebra. A ccdg-Hopf algebra $\Omega$ complete if it is isomorphic to $\hat{\Omega}$.

---

4 The Malcev completion of $\Gamma$ is isomorphic to $\mathbb{G}(\mathbb{Q})$. If the rational Abelianization $\Gamma^{un} \otimes \mathbb{Q}$ of $\Gamma$ is finite dimensional one attach a pro-unipotent affine group scheme $\Gamma^{un} : \mathcal{A}(k) \to \text{Grp}$, which is pro-represented by the linearly compact dual commutative Hopf algebra to $\mathbb{Q}$. For $\Gamma = \pi_1(X)$, where $X$ is a $0$-connected pointed finite type space, $\Gamma^{un} = \pi_1^{un}(X)$ is called the pro-unipotent fundamental group (scheme) of the space, which was a starting point of the rational homotopy theory according to Sullivan—his paper [10] start with an algebraic model of unipotent local system (private communication).
Let $\Omega = \lim \{ \Omega^j \}^{j=1}_{j=n+1}$ be a complete ccdg-Hopf algebra and $\mathbf{P}_{\Omega} : hocc\text{-}cdgC(k) \to \mathbf{Grp}$ be the presheaf of groups pro-represented by $\Omega$. Let $T\mathbf{P}_{\Omega} : hocc\text{-}cdgC(k) \to \mathbf{Lie}(k)$ be the associated presheaf of Lie algebras. Let
\[
\hat{\mathbf{P}}_{\Omega} = \text{Hom}_{\text{cdgC}(k)}(-, \Omega) : hocc\text{-}cdgC(k) \to \mathbf{Set},
\]
\[
\hat{T}\mathbf{P}_{\Omega} = \text{THom}_{\text{cdgC}(k)}(-, \Omega) : hocc\text{-}cdgC(k) \to \mathbf{Set},
\]
be the underlying presheaves. The remaining part of this subsection is devoted to the proof of the following.

Consider a ccdg-coalgebra $C = (C, \epsilon_C, \Delta_C, \delta_C)$. Denote by $\Delta^{(n)}_C : C \to C^{\otimes n}, n \geq 1$, is the $n$-fold iterated coproduct generated by the coproduct $\Delta_C : C \to C \otimes C$ such that $\Delta^{(1)}_C = 1_C$ and $\Delta^{(n+1)}_C = (\Delta^{(n)}_C \otimes 1_C) \circ \Delta_C \equiv (\Gamma^{n-1}_C \otimes \Delta_C) \circ \Delta^{(n)}_C$.

**Theorem 3.3.** For every complete ccdg-Hopf algebra $\Omega$ we have natural isomorphism $\{\exp, \ln\}$ of the presheaves $\hat{T}\mathbf{P}_{\Omega} \cong \hat{\mathbf{P}}_{\Omega} : hocc\text{-}cdgC(k) \to \mathbf{Set}$, whose component $\{\exp^C, \ln^C\}$ at each ccdg-coalgebra $C$ are defined as follows: $\forall [g] \in \text{Hom}_{hocc\text{-}cdgC(k)}(C, \Omega)$ and $\forall [v] \in \text{THom}_{hocc\text{-}cdgC(k)}(C, \Omega)$,
\[
\exp^C([v]) := [\epsilon_C \circ u_{\Omega}] + \sum_{n=1}^{\infty} \frac{1}{n!} \left[ m^{(n)}_\Omega \circ (v \hat{\Phi} \ldots \hat{\Phi} v) \circ \Delta^{(n)}_C \right],
\]
\[
\ln^C([g]) := - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[ m^{(n)}_\Omega \circ (g \hat{\Phi} \ldots \hat{\Phi} g) \circ \Delta^{(n)}_C \right],
\]
where $g \in \text{Hom}_{hocc\text{-}cdgC(k)}(C, \Omega)$ and $v \in \text{THom}_{hocc\text{-}cdgC(k)}(C, \Omega)$ are arbitrary representatives of the homotopy types $[g]$ and $[v]$, respectively; and $\hat{g} := g - \epsilon_C \circ u_{\Omega}$.

**Remark 3.2.** The above theorem implies that the presheaf of groups $\mathbf{P}_{\Omega}$ can be recovered from the presheaf of Lie algebras $T\mathbf{P}_{\Omega}$ using the Baker-Campbell-Hausdorff formula. If $\mathfrak{g}/\mathfrak{g}^2$ is finite dimensional we can dualize $\mathbf{P}_{\Omega}$ to obtain a pro-unipotent affine group dg-scheme, which is a subject of the sequel to this paper.

We divide the proof into pieces.

**Lemma 3.10.** Let $C$ be a ccdg-coalgebra. Then we have

(a) $\alpha_1 \ast_{C, \Omega} \ldots \ast_{C, \Omega} \alpha_n = m^{(n)}_\Omega \circ (\alpha_1 \hat{\Phi} \ldots \hat{\Phi} \alpha_n) \circ \Delta^{(n)}_C$, $\forall \alpha_1, \ldots, \alpha_n \in \text{Hom}(C, \Omega)$ and $n \geq 1$;

(b) $\Delta^{(n)}_C \circ \Delta_C = (\Delta_C \otimes \ldots \otimes \Delta_C) \circ \Delta^{(n)}_C$ for all $n \geq 1$. 
Proof. The property (a) is trivial for \( n = 1 \), is the definition of the convolution product \(*_{C,\Omega}\) for \( n = 2 \) and the lest can be checked easily by an induction. The property (b) is trivial for \( n = 1 \), since \( \Delta_{C\otimes C} := \Delta_{C\otimes C} = (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_C) \circ (\Delta_C \otimes \Delta_C) \) and \( \Delta_{\Omega} := \mathbb{I}_{\Omega} \) becomes the identity \( (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_C) \circ (\Delta_C \otimes \Delta_C) = (\Delta_C \otimes \Delta_C) \circ \Delta_C \), which is valid due to the cocommutativity of \( \Delta_{\Omega} \). The lest can be checked easily by an induction. \( \square \)

**Lemma 3.11.** We have \( (\beta_1 \ast_{C,\Omega} \ldots \ast_{C,\Omega} \beta_n)(c) \in \mathcal{J}^n \), \( n \geq 1 \), for every \( c \in C \) whenever \( \beta_i \in \text{Hom}(C, \Omega) \) has the property \( \varepsilon_{\Omega} \circ \beta_i = 0 \) for all \( i = 1, \ldots, n \).

**Proof.** From \( \varepsilon_{\Omega} \circ \sum_{n=0}^{\infty} \frac{1}{n!} m_{\Omega}^{(n)} \circ (u \hat{\otimes} \ldots \hat{\otimes} v) \circ \Delta_{\Omega} \in \text{Hom}_{\text{cdgC[k]}(C, \Omega)} \), we obtain that \( \varepsilon_{\Omega} \circ (\beta_1 \ast_{C,\Omega} \ldots \ast_{C,\Omega} \beta_n) = \sum_{n=0}^{\infty} \frac{1}{n!} m_{\Omega}^{(n)} \circ (\beta_1 \hat{\otimes} \ldots \hat{\otimes} \beta_n) \circ \Delta_{\Omega} = 0 \). It follows that \( (\beta_1 \ast \ldots \ast \beta_n)(c) \in \mathcal{J}^n \) for all \( c \in C \) since \( (\beta_1 \ast \ldots \ast \beta_n)(c) \sum_{n=0}^{\infty} \frac{1}{n!} m_{\Omega}^{(n)} \circ (\beta_1(c) \hat{\otimes} \ldots \hat{\otimes} \beta_n(c)) \).

**Lemma 3.12.** Let \( \Omega \) be a complete cdg-Hopf-algebra. For any cdg-coalgebra \( C \), we have an isomorphism \( \text{THom}_{\text{cdgC[k]}(C, \Omega)} \xrightarrow{\exp^C = \text{exp}^C} \text{Hom}_{\text{cdgC[k]}(C, \Omega)} \), where

\[- \text{ for all } v \in \text{THom}_{\text{cdgC[k]}(C, \Omega)} \text{, we have } \exp^C(v) := u_\Omega \circ \varepsilon_C + \sum_{n=1}^{\infty} \frac{1}{n!} m_{\Omega}^{(n)} \circ (v \hat{\otimes} \ldots \hat{\otimes} v) \circ \Delta_{\Omega} \in \text{Hom}_{\text{cdgC[k]}(C, \Omega)} \]

\[- \text{ for all } g \in \text{Hom}_{\text{cdgC[k]}(C, \Omega)} \text{, we have } \ln^C(g) := - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} m_{\Omega}^{(n)} \circ (g \hat{\otimes} \ldots \hat{\otimes} g) \circ \Delta_{\Omega} \in \text{THom}_{\text{cdgC[k]}(C, \Omega)} \]

such that \( \exp^C(v) \sim \exp^C(\tilde{v}) \in \text{Hom}_{\text{cdgC[k]}(C, \Omega)} \) whenever \( v \sim \tilde{v} \in \text{THom}_{\text{cdgC[k]}(C, \Omega)} \), and \( \ln^C(g) \sim \ln^C(\tilde{g}) \in \text{THom}_{\text{cdgC[k]}(C, \Omega)} \) whenever \( g \sim \tilde{g} \in \text{Hom}_{\text{cdgC[k]}(C, \Omega)} \).

**Proof.** We use some shorthard notations. We set \( e = u_\Omega \circ \varepsilon_C \). We also set \( \ast = \ast_{C,\Omega} \), \( \alpha^0 = e \) and \( \alpha^n = \alpha \ast \ldots \ast \alpha \), \( n \geq 1 \), for all \( \alpha \in \text{Hom}(C, \Omega)^0 \). Then, by Lemma 3.10(a), we have

\[ \exp^C(v) = e + \sum_{n=1}^{\infty} \frac{1}{n!} v^n, \quad \ln^C(g) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \hat{g}^n. \quad (3.6) \]

Remind that \( e \in \text{Hom}_{\text{cdgC[k]}(C, \Omega)} \), i.e., \( \partial_{C,\Omega} e = 0 \), \( \varepsilon_{\Omega} \circ e = \varepsilon_C \) and \( \Delta_{\Omega} \circ e = (e \hat{\otimes} e) \circ \Delta_C \).

1. We have to justify that the infinite sums in the definition of \( \exp^C \) and \( \ln^C \) make sense. Note the \( \varepsilon_{\Omega} \circ v = 0 \) by definition. We also have \( \varepsilon_{\Omega} \circ e = \varepsilon_C \) and \( \Delta_{\Omega} \circ e = (e \hat{\otimes} e) \circ \Delta_C \).
\( \epsilon_C \) and \( g = g - e \). By Lemma 3.10, we have \( \nu^*(c), \tilde{g}^*(c) \in \mathcal{N} \) for all \( c \in C \) and \( n \geq 1 \). Therefore both \( \exp^C(v) \) and \( \ln^C(g) \) are well-defined.

2. We check that \( \exp^C(v) \in \text{Hom}_{\text{cdgC}(\mathbb{k})}(C, \Omega) \) for every \( v \in \text{THom}_{\text{cdgC}(\mathbb{k})}(C, \Omega) \):

\[
\partial_{C, \Omega} \exp^C(v) = 0, \quad \epsilon_{\Omega} \circ \exp^C(v) = \epsilon_C, \quad \Delta_{\Omega} \circ \exp^C(v) = (\exp^C(v) \otimes \exp^C(v)) \circ \Delta_C.
\]

The 1st relation is trivial since \( \partial_{C, \Omega} \) is a derivation of \( \bullet \) and \( \partial_{C, \Omega} e = \partial_{C, \Omega} t = 0 \). The 2nd relation is also trivial since \( \epsilon_{\Omega} \circ e = \epsilon_C \) and \( \epsilon_{\Omega} \circ \nu^* = (\epsilon_{\Omega} \circ \nu)^* = 0 \) for all \( n \geq 1 \). It remains to check the 3rd relation, which is equivalent to the following relations, \( \forall n \geq 0 \),

\[
\Delta_{\Omega} \circ \nu^* = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} (\nu^{*n-k} \otimes \nu^{*k}) \circ \Delta_C.
\]

(3.7)

For \( n = 0 \) the above becomes \( \Delta_{\Omega} \circ e = (e \circ \nu^*) \circ \Delta_C \), which is true.

For cases with \( n \geq 1 \), we adopt a new notation. Recall that \( \left( \Omega \otimes \Omega, u_{\Omega \otimes \Omega}, m_{\Omega \otimes \Omega} \right) \) is a \( \mathbb{Z} \)-graded associative algebra and \( \left( C \otimes C, \epsilon_{C \otimes C}, \Delta_{C \otimes C} \right) \) is a \( \mathbb{Z} \)-graded coassociative coalgebra. Therefore we have a \( \mathbb{Z} \)-graded associative algebra \( \left( \text{Hom}(C \otimes \Omega, \Omega \otimes \Omega), \ast, \circ \right) \), where \( \chi_1 \ast \chi_2 := m_{\Omega \otimes \Omega} \circ (\chi_1 \otimes \chi_2) \circ \Delta_{C \otimes C}, \forall \chi_1, \chi_2 \in \text{Hom}(C \otimes C, \Omega \otimes \Omega) \). We also have, for all \( n \geq 1 \) and \( \chi_1, \ldots, \chi_n \in \text{Hom}(C \otimes C, \Omega \otimes \Omega) \),

\[
\chi_1 \ast \cdots \ast \chi_n = m^{(n)}_{\Omega \otimes \Omega} \circ (\chi_1 \otimes \cdots \otimes \chi_n) \circ \Delta^{(n)}_{C \otimes C}.
\]

(3.8)

For example, consider \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \text{Hom}(C, \Omega) \) so that \( \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 \in \text{Hom}(C \otimes C, \Omega \otimes \Omega) \). Then we have \( (\alpha_1 \otimes \beta_1) \ast (\alpha_2 \otimes \beta_2) = (-1)^{|\beta_1||\alpha_2|} \alpha_1 \ast \beta_1 \ast \beta_2 \).

It follows that \( (v \circ \nu) \ast (e \circ \nu) = (e \circ \nu) \ast (v \circ \nu) \) since the both terms are \( v \circ \nu \). We also have \( (v \circ \nu)^{*n} = v^{*n} \circ e \) and \( (e \circ \nu)^{*n} = e \circ \nu^{*n} \). Combined with the binomial identity, we obtain that, \( \forall n \geq 1 \),

\[
(v \circ \nu + e \circ \nu)^{*n} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} (v^{*n-k} \otimes \nu^{*k}).
\]

Therefore the RHS of eq. (3.7) becomes

\[
\text{RHS} = (v \circ \nu + e \circ \nu)^{*n} \circ \Delta_C = m^{(n)}_{\Omega \otimes \Omega} \circ (v \circ \nu + e \circ \nu)^{*n} \circ \Delta^{(n)}_{C \otimes C} \circ \Delta_C
\]

\[= m^{(n)}_{\Omega \otimes \Omega} \circ (v \circ \nu + e \circ \nu)^{*n} \circ (\Delta_C \otimes \cdots \otimes \Delta_C) \circ \Delta^{(n)}_C, \]

where we use Lemma 3.10(b) for the last equality. Consider the LHS of eq. (3.7):

\[
\text{LHS} = \Delta_{\Omega} \circ \nu^* = \Delta_C \circ m^{(n)}_{\Omega \otimes \Omega} \circ (v \circ \nu \cdots \circ \nu) \circ \Delta^{(n)}_{C \otimes C}
\]

\[= m^{(n)}_{\Omega \otimes \Omega} \circ (v \circ \nu + e \circ \nu)^{*n} \circ (\Delta_C \circ \cdots \circ \Delta_C) \circ \Delta^{(n)}_C, \]

where we use eq. (3.4) for the 3rd equality and the property \( \Delta_{\Omega} \circ \nu = (v \circ \nu + e \circ \nu) \circ \Delta_C \) for the last equality. Therefore we conclude that \( \Delta_{\Omega} \circ \exp^C(v) = (\exp^C(v) \otimes \exp^C(v)) \circ \Delta_C \).
2. We check that \( \ln^C(g) \in \text{THom}_{\text{cdgC}[k]}(C, \Omega) \) for every \( g \in \text{Hom}_{\text{cdgC}[k]}(C, \Omega) \):
\[
\partial_{C, \Omega} \ln^C(g) = 0, \quad \epsilon_\Omega \circ \ln^C(g) = 0, \quad \Delta_\Omega \circ \ln^C(g) = (\ln^C(g) \hat{\otimes} e + e \hat{\otimes} \ln^C(g)) \circ \Delta_C.
\]
The 1st relation is obvious since \( \partial_{C, \Omega} (g) = \partial_{C, \Omega} g - \partial_{C, \Omega} e = 0 \) and \( \partial_{C, \Omega} \) is a derivation of \( \star \). The 2nd relation is also obvious since \( \epsilon_\Omega \circ \hat{\otimes} = (\epsilon_\Omega \circ \hat{\otimes})^n = 0 \) for all \( n \geq 1 \). Therefore it remains to check the 3rd relation.
Define \( \ln^C_\pi(\chi) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\chi - e \hat{\otimes} e)^{\star n} \) for all \( \chi \in \text{Hom}(C \otimes C, \Omega \hat{\otimes} \Omega) \) satisfying \( (\epsilon_\Omega \hat{\otimes} \epsilon_\Omega) \circ \chi = \epsilon_C \hat{\otimes} \epsilon_C \). Then, we have
\[
\ln^C_\pi (g \hat{\otimes} e) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (g \hat{\otimes} e - e \hat{\otimes} e)^{\star n} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\hat{\otimes} e)^{\star n} = \ln^C(g) \hat{\otimes} e,
\]
and, similarly, \( e \hat{\otimes} \ln^C(g) = \ln^C(g \hat{\otimes} e) \). Therefore, we have
\[
\ln^C(g) \hat{\otimes} e + e \hat{\otimes} \ln^C(g) = \ln^C_\pi (g \hat{\otimes} e) + \ln^C_\pi (e \hat{\otimes} g) = \ln^C_\pi [(g \hat{\otimes} e) \star (e \hat{\otimes} g)] = \ln^C_\pi (g \hat{\otimes} g).
\]
On the other hand, we have
\[
\Delta_\Omega \circ \hat{\otimes} = \Delta_\Omega \circ g \otimes e - e \hat{\otimes} e \circ \Delta_C = \left(\text{exp}^C(\epsilon) \circ g \hat{\otimes} e + e \hat{\otimes} \text{exp}^C(\epsilon) \circ g \hat{\otimes} e\right) \circ \Delta_C = \ln^C_\pi (g \hat{\otimes} e) \circ \Delta_C,
\]
where we have used \( \Delta_\Omega \circ g = \Delta_\Omega \circ g - \Delta_\Omega \circ e = (g \hat{\otimes} g - e \hat{\otimes} e) \circ \Delta_C \) for the 3rd equality. Therefore, we obtain that
\[
\Delta_\Omega \circ \ln^C(g) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (g \hat{\otimes} g - e \hat{\otimes} e)^{\star n} \circ \Delta_C = \ln^C_\pi (g \hat{\otimes} g) \circ \Delta_C
\]
\[
= \left(\text{exp}^C(\epsilon) \circ g \hat{\otimes} e + e \hat{\otimes} \text{exp}^C(\epsilon) \circ g \hat{\otimes} e\right) \circ \Delta_C.
\]
3. It is obvious now that \( \text{exp}^C(\text{exp}^C(\epsilon)) = \epsilon \) and \( \text{exp}^C(\ln^C) = g \). Hence \( \text{exp}^C, \ln^C \) is an isomorphism.

4. Let \( \nu \sim \nu \in \text{THom}_{\text{cdgC}[k]}(C, \Omega) \). Then we have a corresponding homotopy pair \( \langle \nu(t), \sigma(t) \rangle \) on \( \text{THom}_{\text{cdgC}[k]}(C, \Omega) \) such that \( \nu(0) = \nu \) and \( \nu(1) = \nu \). Let
\[
g(t) := \text{exp}^C(\nu(t)),
\]
\[
\lambda(t) := \sum_{n=1}^{\infty} \frac{1}{n!} \nu(t)^{n-1} \star \sigma(t) \star \nu(t)^{n-1}.
\]
Then it is trivial to check that \( \{ g(t), \lambda(t) \} \) is a homotopy pair on \( \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \), so that \( \exp^C(u) = g(0) \sim g(1) = \exp^C(\tilde{u}) \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \).

5. Let \( g \sim \tilde{g} \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \) and \( \{ g(t), \lambda(t) \} \) be the corresponding homotopy pair on \( \text{Hom}_{\text{ccdgC}(k)}(C, \Omega) \) such that \( g(0) = \tilde{g} \) and \( g(1) = \tilde{g} \). Let

\[
\nu(t) := \ln^C(g(t)), \\
\sigma(t) := -\sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{(-1)^{n}}{n} \tilde{g}(t)^{j-1} \ast \lambda(t) \ast \tilde{g}(t)^{n-j}.
\]

Then it is trivial to check that \( \{ \nu(t), \sigma(t) \} \) is a homotopy pair on \( \text{THom}_{\text{ccdgC}(k)}(C, \Omega) \), so that \( \ln^C(g) = \nu(0) \sim \nu(1) = \ln^C(\tilde{g}) \in \text{THom}_{\text{ccdgC}(k)}(C, \Omega) \).

**Lemma 3.13.** Let \( \Omega \) be a complete ccdg-Hopf-algebra. Then we have a natural isomorphism \( \begin{array}{ccc} T\mathcal{P}_{\Omega} & \overset{T\otimes}{\longrightarrow} & T\mathcal{P}_{\Omega} \\ \overset{\exp^C}{\longrightarrow} & & \overset{\ln^C}{\longrightarrow} \\ \mathcal{P}_{\Omega}(C) & \overset{\exp^C}{\longrightarrow} & \mathcal{P}_{\Omega}(C) \end{array} \) : \( \text{ccdgC}(k) \rightarrow \text{Set of presheaves on ccdgC(k)} \), whose component at each ccdg-coalgebra \( C \) is \( \{ \exp^C, \ln^C \} \) defined in Lemma 3.12.

**Proof.** We have shown that \( \begin{array}{ccc} T\mathcal{P}_{\Omega}(C) & \overset{T\otimes}{\longrightarrow} & T\mathcal{P}_{\Omega}(C) \\ \overset{\exp^C}{\longrightarrow} & & \overset{\ln^C}{\longrightarrow} \\ \mathcal{P}_{\Omega}(C) & \overset{\exp^C}{\longrightarrow} & \mathcal{P}_{\Omega}(C) \end{array} \) is an isomorphism for every ccdg-coalgebra \( C \). It remains to check the naturalness that for every morphism \( f : C \rightarrow C' \) of ccdg-coalgebras the diagrams are commutative:

\[
\begin{array}{ccc}
T\mathcal{P}_{\Omega}(C') & \overset{T\mathcal{P}_{\Omega}(f)}{\longrightarrow} & T\mathcal{P}_{\Omega}(C) \\
\exp^C & & \exp^C \\
\mathcal{P}_{\Omega}(C') & \overset{\mathcal{P}_{\Omega}(f)}{\longrightarrow} & \mathcal{P}_{\Omega}(C) \\
\ln^C & & \ln^C \\
\mathcal{P}_{\Omega}(f) & \overset{\mathcal{P}_{\Omega}(f)}{\longrightarrow} & \mathcal{P}_{\Omega}(f)
\end{array}
\]

That is, \( \mathcal{P}_{\Omega}(f) \circ \exp^C = \exp^C \circ T\mathcal{P}_{\Omega}(f) \) and \( \mathcal{P}_{\Omega}(f) \circ \ln^C = \ln^C \circ \mathcal{P}_{\Omega}(f) \). These are straightforward since for every \( u' \in T\mathcal{P}_{\Omega}(C') \) we have

\[
\begin{align*}
\mathcal{P}_{\Omega}(f)\left(\exp^C(u')\right) & = \exp^C(u') \circ f = u_{\Omega} \circ \epsilon_{C'} \circ f + \sum_{n=1}^{\infty} \frac{1}{n!} m_{\Omega}^{(n)}(u' \otimes \cdots \otimes u') \circ \Delta_{C'}^{(n)} \circ f \\
& = u_{\Omega} \circ \epsilon_{C} + \sum_{n=1}^{\infty} \frac{1}{n!} m_{\Omega}^{(n)}(u' \circ f \otimes \cdots \otimes u' \circ f) \circ \Delta_{C}^{(n)} = \exp^C(u' \circ f) \\
& = \exp^C \left( T\mathcal{P}_{\Omega}(f)(u') \right).
\end{align*}
\]

The naturalness of \( \ln \) can be checked similarly. \( \square \)
3. Linear representation of a representable presheaf of groups

Throughout this section we fix a ccdg-Hopf algebra \( \Omega = (\Omega, \mu_\Omega, m_\Omega, \epsilon_\Omega, \Delta_\Omega, \varepsilon_\Omega, \partial_\Omega) \). We define a linear representation of the presheaf of groups \( \Psi_\Omega : hccdgc(k) \rightarrow \text{Grp} \) on the homotopy category \( hccdgc(k) \) of ccdg-coalgebras via a linear representation of the associated presheaf of groups \( P_\Omega : ccdgc(k) \rightarrow \text{Grp} \) on the category \( ccdgc(k) \) of ccdg-coalgebras. Remind that \( P_\Omega \) is represented by \( \Omega \) and induces \( \Psi_\Omega \) on the homotopy category \( hccdgc(k) \).

The linear representations of \( P_\Omega \) form a dg-tensor category \( \text{Rep}(P_\Omega) \), which is isomorphic to the dg-tensor category \( dgmod_{\Omega}(\Omega) \) of left dg-modules over \( \Omega \). Working with the linear representations of \( P_\Omega \) instead of the linear representations of \( \Psi_\Omega \) will be a crucial step for a Tannakian reconstruction of \( \Psi_\Omega \).
4.1. Preliminary

Our main concern here is a dg-tensor category formed by cofree left dg-comodules over a cocommutative dg-coalgebra $C = (C, \epsilon_C, \Delta_C, \partial_C)$. We shall need the following basic lemma, which is due to the defining properties of dg-coalgebra $C$.

Lemma 4.1. For every pair $(M, N)$ of chain complexes we have an exact sequence of chain complexes

\[
0 \rightarrow \text{Hom}(C \otimes M, N) \xrightarrow{\hat{p}} \text{Hom}(C \otimes M, C \otimes N) \xrightarrow{\hat{q}} \text{Hom}(C \otimes M, C \otimes C \otimes N),
\]

where $\forall i + 1 \in \text{Hom}(C \otimes M, C \otimes i \otimes N)$, $i = 0, 1, 2$,

\[
\hat{p}(\alpha_i) := (I_C \otimes \alpha_i) \circ (\Delta_C \otimes I_M), \quad \hat{q}(\alpha_i) := (\Delta_C \otimes I_N) \circ \alpha_i - (I_C \otimes \alpha_i) \circ (\Delta_C \otimes I_M), \\
\hat{q}(\alpha_i) := \partial_N \circ (\epsilon_C \otimes I_N) \circ \alpha_i,
\]

such that

\[
\hat{r} \circ \hat{p} = 0, \quad \hat{q} \circ \hat{p} = \hat{r} = \text{Hom}(C \otimes M, N), \quad \hat{p} \circ \hat{q} + \hat{q} \circ \hat{r} = \text{Hom}(C \otimes M, C \otimes N).
\]

(4.1)

Remind that a left dg-comodule $(M, \rho_M)$ over a ccdg-coalgebra $C$ is a chain complex $M = (M, \Delta_M)$ together with a chain map $\rho_M : M \rightarrow C \otimes M$, called a coaction, making the following diagrams commute

\[
\begin{array}{ccc}
M & \xrightarrow{\rho_M} & C \otimes M \\
\downarrow{\rho_M} & & \downarrow{\rho_M} \\
C \otimes M & \xrightarrow{\Delta_C \otimes I_M} & C \otimes C \otimes M
\end{array}, \quad \begin{array}{ccc}
M & \xrightarrow{\rho_M} & C \otimes M \\
\downarrow{\rho_M} & & \downarrow{\rho_M} \\
C \otimes M & \xrightarrow{\Delta_C \otimes I_M} & C \otimes C \otimes M
\end{array}
\]

(4.3)

For every chain complex $M$ we have a cofree left dg-comodule $(C \otimes M, \Delta_C \otimes I_M)$ over $C$ with the cofree coaction $C \otimes M \xrightarrow{\Delta_C \otimes I_M} C \otimes C \otimes M$. We can form a dg-category $\text{dgComod}^{\text{cof}}(C)$ of cofree left dg-comodules over $C$, whose set of morphisms from $(C \otimes M, \Delta_C \otimes I_M)$ to $(C \otimes N, \Delta_C \otimes I_M)$ is $\text{Hom}_{\Delta_C}(C \otimes M, C \otimes N)$ with the differential $\partial_{\Delta_C \otimes I_M}$, where $\text{Hom}_{\Delta_C}(C \otimes M, C \otimes N)$ is the set of every $k$-linear map $\varphi : C \otimes M \rightarrow C \otimes N$ making the following diagrams commute

\[
\begin{array}{ccc}
C \otimes M & \xrightarrow{\Delta_C \otimes I_M} & C \otimes C \otimes M, \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
C \otimes N & \xrightarrow{\Delta_C \otimes I_N} & C \otimes C \otimes N
\end{array}
\]

i.e., $(\Delta_C \otimes I_N) \circ \varphi = (I_C \otimes \varphi) \circ (\Delta_C \otimes I_M) \iff \partial(\varphi) = 0$.
Corollary 4.1. There is a bijection \( \tilde{\rho} : \text{Hom}(C \otimes M, N) \rightarrow \text{Hom}_{\Delta_C} \left( C \otimes M, C \otimes N \right) : \tilde{q} \).

By Lemma 4.1, we can check that \( \text{Hom}_{\Delta_{C}} \left( C \otimes M, C \otimes M' \right) \) is a chain complex with the differential \( \partial_{\text{Cmod},C_{\text{mod}}} \) and \( \varphi' \circ \varphi \in \text{Hom}_{\Delta_{C}} \left( C \otimes M, C \otimes M'^{\prime} \right) \) whenever \( \varphi \in \text{Hom}_{\Delta_{C}} \left( C \otimes M, C \otimes M' \right) \) and \( \varphi' \in \text{Hom}_{\Delta_{C}} \left( C \otimes M', C \otimes M'^{\prime} \right) \). It is obvious that differentials are derivations of the composition operation. Therefore, \( \text{dgComod}_{L}^{\text{cofr}}(C) \) is a dg-category.

Lemma 4.2 (Definition). The dg-category \( \text{dgComod}_{L}^{\text{cofr}}(C) \) is a dg-tensor category with the following tensor structure.

1. The tensor product of cofree left dg-comodules \( (C \otimes M, \Delta_{C} \otimes \mathbb{I}_{M}) \) and \( (C \otimes M', \Delta_{C} \otimes \mathbb{I}_{M'}) \) over \( C \) is the cofree left dg-comodule \( (C \otimes M \otimes M', \Delta_{C} \otimes \mathbb{I}_{M \otimes M'}) \) over \( C \), and \( (C \otimes k, \Delta_{C} \otimes \mathbb{I}_{k}) \) is the unit object.

2. The tensor product of morphisms \( \varphi \in \text{Hom}_{\Delta_{C}} \left( C \otimes M, C \otimes N \right) \) and \( \varphi' \in \text{Hom}_{\Delta_{C}} \left( C \otimes M', C \otimes N' \right) \) is the morphism \( \varphi \otimes_{\Delta_{C}} \varphi' \in \text{Hom}_{\Delta_{C}} \left( C \otimes M \otimes M', C \otimes N \otimes N' \right) \), where

\[
\varphi \otimes_{\Delta_{C}} \varphi' := \left( \varphi \otimes \tilde{q}(\varphi') \right) \circ \left( \mathbb{I}_{C} \otimes \tau \otimes \mathbb{I}_{M'} \right) \circ \left( \Delta_{C} \otimes \mathbb{I}_{M} \otimes \mathbb{I}_{M'} \right)
\]

Equivalently, \( \varphi \otimes_{\Delta_{C}} \varphi' \) is determined by the following equality:

\[
\tilde{q}(\varphi \otimes_{\Delta_{C}} \varphi') = \left( \tilde{q}(\varphi) \otimes \tilde{q}(\varphi') \right) \circ \left( \mathbb{I}_{C} \otimes \tau \otimes \mathbb{I}_{M'} \right) \circ \left( \Delta_{C} \otimes \mathbb{I}_{M} \otimes \mathbb{I}_{M'} \right) ; C \otimes M \otimes M' \rightarrow N \otimes N'.
\]

3. \( \partial_{C \otimes M \otimes M', C \otimes N \otimes N'}(\varphi \otimes_{\Delta_{C}} \varphi') = \partial_{C \otimes M, C \otimes N}(\varphi) \otimes_{\Delta_{C}} \varphi' + (-1)^{|\varphi|} \varphi \otimes_{\Delta_{C}} \partial_{C \otimes M', C \otimes N'} \varphi' \).

Proof. Property 1 is obvious. For property 2, it is straightforward to check that \( \partial(\varphi \otimes_{\Delta_{C}} \varphi') = 0 \) whenever \( \partial(\varphi) = \partial(\varphi') = 0 \). Property 3 can be checked by a straightforward computation. \( \square \)

Lemma 4.3. We have a dg-tensor functor \( C \otimes : \text{Ch}(k) \rightarrow \text{dgComod}^{\text{cofr}}_{L}(C) \) for every ccdg-coalgebra \( C \)

\[
C \otimes \left( M \xrightarrow{\psi} M' \right) \rightarrow \left( (C \otimes M, \Delta_{C} \otimes \mathbb{I}_{M}) \xrightarrow{L_{C} \otimes \psi} (C \otimes M', \Delta_{C} \otimes \mathbb{I}_{M'}) \right).
\]

Proof. It is obvious that \( C \otimes \) is a dg-functor whose tensor property follows from the easy identity \( \left( \mathbb{I}_{C} \otimes \psi \right) \otimes_{\Delta_{C}} \left( \mathbb{I}_{C} \otimes \psi' \right) = \mathbb{I}_{C} \otimes \psi \otimes \psi' : C \otimes M \otimes N \rightarrow C \otimes M' \otimes N' \) for all linear maps \( \psi : M \rightarrow N \) and \( \psi' : M' \rightarrow N' \). \( \square \)
4.2. Linear representations and their dg-tensor category

For every ccdg-coalgebra \( C \), we have the following constructions.

1. Let \( \text{End}_{\Delta_C}(C \otimes M) := \text{Hom}_{\Delta_C}(C \otimes M, C \otimes M) \), which is the \( \mathbb{Z} \)-graded vector space of linear maps \( \varphi : C \otimes M \rightarrow C \otimes M \) satisfying \( (\Delta_C \otimes I_M) \circ \varphi = (I_C \otimes \varphi) \circ (\Delta_C \otimes I_M) \). Then we have a dg-algebra

\[
\mathcal{E}_M(C) = \{ \text{End}_{\Delta_C}(C \otimes M), I_{C \otimes M}, \circ, \partial_{C \otimes M, C \otimes M} \}. \tag{4.4}
\]

2. Let \( \mathcal{G}_M(C) := \{ \text{End}_{\Delta_C}(C \otimes M), I_{C \otimes M}, \circ \} \).

3. Let \( \text{H} \text{o}_{\Delta_C}(C \otimes M) \) be the set of homology classes of elements in \( \text{Z}_0 \text{Aut}_{\Delta_C}(C \otimes M) \) that is every degree zero element \( \varphi \) has a composition inverse \( \varphi^{-1} \) and satisfies \( \partial_{C \otimes M, C \otimes M} \circ \varphi = 0 \). Then we have a group

\[
\mathcal{H}_M(C) := \{ \text{H} \text{o}_{\Delta_C}(C \otimes M), I_{C \otimes M}, \circ \}. \tag{4.6}
\]

The above constructions are functorial.

**Lemma 4.4.** For every chain complex \( M \) we have a functor \( \mathcal{E}_M : \text{ccdgc}(k) \rightarrow \text{dgA}(k) \), sending each ccdg-coalgebra \( C \) to the dg-algebra \( \mathcal{E}_M(C) \), and each morphism \( f : C \rightarrow C' \) of dg-coalgebras to a morphism \( \mathcal{E}_M(f) : \mathcal{E}_M(C') \rightarrow \mathcal{E}_M(C) \) of dg-algebra defined by, \forall \varphi' \in \mathcal{E}_M(C') = \text{End}_{\Delta_C}(C' \otimes M),

\[
\mathcal{E}_M(f)(\varphi') := \mathcal{E}_M(f) \circ \varphi' \in \mathcal{E}_M(C) = \text{End}_{\Delta_C}(C \otimes M), \tag{a}
\]

\[
(b) \quad \mathcal{E}_M(f)(\varphi' \circ \varphi'_2) = \mathcal{E}_M(f)(\varphi'_1) \circ \mathcal{E}_M(f)(\varphi'_2); \tag{b}
\]

\[
(c) \quad \mathcal{E}_M(f) \circ \partial_{C' \otimes M, C' \otimes M} = \partial_{C \otimes M, C \otimes M} \circ \mathcal{E}_M(f). \tag{c}
\]
Proof. We already know that $\mathcal{E}_M(C)$ is a dg-algebra eq. (4.4). We check that $\mathcal{E}_M(f)(\varphi') \in \text{End}_{\Delta C}(C \otimes M)$, i.e., $\bar{\epsilon}(\mathcal{E}_M(f)(\varphi')) = 0$. We have

$$
\bar{\epsilon}(\mathcal{E}_M(f)(\varphi')) := \Delta_C \otimes \mathbb{I}_M) \circ (I_C \otimes \bar{q}(\varphi')) \circ (I_C \otimes f \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \\
- (I_C \otimes I_C \otimes \bar{q}(\varphi')) \circ (I_C \otimes I_C \otimes f \otimes \mathbb{I}_M) \circ (I_C \otimes \Delta_C \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \\
= 0,
$$

where we used the coassociativity of $\Delta_C$. It remains to show that $\mathcal{E}_M(f)$ is a morphism of dg-algebras—the properties $(a)$, $(b)$ and $(c)$. Remind that $\bar{q}(\varphi') := \iota_M \circ (\epsilon_C \otimes 1_M) \circ \varphi'$.

For the property $(a)$, we have

$$
\mathcal{E}_M(f)(I_C \otimes \mathbb{I}_M) := (I_C \otimes I_M \circ (\epsilon_C \otimes \mathbb{I}_M)) \circ (I_C \otimes f \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) = I_C \otimes \mathbb{I}_M,
$$

where we have used $\epsilon_C \circ f = \epsilon_C$ and the counit property of $\Delta_C$. For the property $(b)$, we have

$$
\mathcal{E}_M(f)(\varphi'_1) \circ \mathcal{E}_M(f)(\varphi'_2) := (I_C \otimes I_M \circ (\epsilon_C \otimes \mathbb{I}_M) \circ \varphi'_1) \circ (I_C \otimes f \otimes \mathbb{I}_M) \circ (I_C \otimes I_C \circ f \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \\
- (I_C \otimes I_C \circ f \otimes \mathbb{I}_M) \circ ((\Delta_C \otimes I_C) \circ (\Delta_C \otimes \mathbb{I}_M)) \\
= (I_C \otimes I_M \circ (\epsilon_C \otimes \mathbb{I}_M) \circ \varphi'_1) \circ (I_C \otimes I_C \circ f \otimes \mathbb{I}_M) \circ (I_C \otimes I_C \circ \varphi'_2) \\
= (I_C \otimes I_C \circ f \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \\
= \mathcal{E}_M(f)(\varphi'_1 \circ \varphi'_2),
$$

where we have used the coassociativity of $\Delta_C$ for the 3rd equality, $f$ being a coalgebra map for the 4th equality, $\varphi'_2 \in \text{End}_{\Delta C}(C \otimes M)$ that $(I_C \otimes \varphi'_2) \circ (\Delta_C \otimes \mathbb{I}_M) = (\Delta_C \otimes I_M) \circ \varphi'_2$ for the 5th equality and the counit property of $\Delta_C$ for the 6th equality, while all the other moves are plain. For the property $(c)$, we have

$$
\partial_{\mathbb{C} \otimes \mathbb{M}} \left( \mathcal{E}_M(f)(\varphi') \right) := \partial_{\mathbb{C} \otimes \mathbb{M}} \circ (I_C \otimes I_M \circ (\epsilon_C \otimes \mathbb{I}_M) \circ \varphi') \circ (I_C \otimes f \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \\
- (\epsilon_C \otimes \mathbb{I}_M) \circ (I_C \otimes I_M \circ (\epsilon_C \otimes \mathbb{I}_M) \circ \varphi') \circ (I_C \otimes f \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \circ \partial_{\mathbb{C} \otimes \mathbb{M}} \\
= \mathcal{E}_M(f)(\partial_{\mathbb{C} \otimes \mathbb{M}} \circ \varphi') - (\epsilon_C \otimes \mathbb{I}_M) \circ (I_C \otimes f \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \circ \partial_{\mathbb{C} \otimes \mathbb{M}} \circ \varphi',
$$

where we have used properties that $\partial_C$ is a coderivation of $\Delta_C$ and $f : C \to C'$ is a chain map, together with some obvious moves and cancellations. \(\square\)
Lemma 4.5. For every chain complex $M$ we have a presheaf of groups

$$\mathcal{G}_M : \text{ccdgc}(k) \rightarrow \text{Grp},$$

sending each ccdg-coalgebra $C$ to the group $\mathcal{G}_M(C)$ and each morphism $f : C \rightarrow C'$ of ccdg-coalgebras to a homomorphism $\mathcal{G}_M(f) : \mathcal{G}_M(C') \rightarrow \mathcal{G}_M(C)$ of groups defined by $\mathcal{G}_M(f) := E_M(f)$, such that

(a) $\mathcal{G}_M(f)(\varphi') \sim \mathcal{G}_M(f)(\varphi') \in Z_0\text{Aut}_\Delta C(C \otimes M)$ for all $\varphi' \in Z_0\text{Aut}_\Delta C(C' \otimes M)$ whenever $f \sim \bar{f} \in \text{Hom}_{\text{ccdgc}(k)}(C, C')$, and

(b) $\mathcal{G}_M(f)(\varphi') \sim \mathcal{G}_M(f)(\varphi') \in Z_0\text{Aut}_\Delta C(C \otimes M)$ for all $f \in \text{Hom}_{\text{ccdgc}(k)}(C, C')$ whenever $\varphi' \sim \bar{\varphi}' \in Z_0\text{Aut}_\Delta C(C' \otimes M)$.

Proof. We already know that $\mathcal{G}_M(C)$ is a group, eq. (4.5). We check that $\mathcal{G}_M(f)$ is a group homomorphism as follows: Due to properties (a) and (b) in Lemma 4.4, it is suffice to check that $\partial_{C \otimes M, C \otimes M}(\mathcal{G}_M(f)(\varphi')) = 0$ for every $\varphi' \in \mathcal{G}_M(C') = Z_0\text{Aut}_\Delta C(C' \otimes M)$. This is obvious by property (c) in Lemma 4.4, since $\mathcal{G}_M(f)(\varphi') := E_M(f)(\varphi')$ and $\partial_{C \otimes M, C \otimes M} \varphi' = 0$ by definitions. Therefore $\mathcal{G}_M$ is a presheaf of groups on $\text{ccdgc}(k)$.

Property (a) is checked as follows. From the condition $f \sim \bar{f} \in \text{Hom}_{\text{ccdgc}(k)}(C, C')$, there is a homotopy pair $(f(t), \lambda(t))$ on $\text{Hom}_{\text{ccdgc}(k)}(C, C')$ such that $f(0) = f$, $f(1) = \bar{f}$ and $\bar{f} = f + \partial_{C, C'} \chi$, where $\chi := \int_0^1 \chi(t) dt \in \text{Hom}(C, C')$. Then, for every $\varphi' \in Z_0\text{Aut}_\Delta C(C \otimes M)$ we have

$$\mathcal{G}_M(\bar{f})(\varphi') - \mathcal{G}_M(f)(\varphi') = \hat{p} \left[ \hat{q}(\varphi') \circ (\partial_{C, C'} \chi \otimes \mathbb{I}_M) \right] = \partial_{C \otimes M, C \otimes M} \hat{p} \left[ \hat{q}(\varphi') \circ (\chi \otimes \mathbb{I}_M) \right],$$

since both $\hat{p}$ and $\hat{q}$ are chain maps and $\partial_{C' \otimes M, C' \otimes M} \varphi' = 0$.

Property (b) is checked as follows. From the condition $\varphi' \sim \bar{\varphi}' \in Z_0\text{Aut}_\Delta C(C' \otimes M)$, we have $\bar{\varphi}' - \varphi' = \partial_{C' \otimes M, C' \otimes M} \lambda$ for some $\lambda \in \text{End}_\Delta C(C' \otimes M)$. Then, for every $f \in \text{Hom}_{\text{ccdgc}(k)}(C, C')$ we have

$$\mathcal{G}_M(f)(\bar{\varphi}') - \mathcal{G}_M(f)(\varphi') = \hat{p} \left[ \hat{q}(\partial_{C' \otimes M, C' \otimes M} \lambda) \circ (f \otimes \mathbb{I}_M) \right] = \partial_{C \otimes M, C \otimes M} \hat{p} \left[ \hat{q}(\lambda) \circ (f \otimes \mathbb{I}_M) \right],$$

since both $\hat{p}$ and $\hat{q}$ are chain maps and $\partial_{C, C'} f = 0$. □

Lemma 4.6. For every chain complex $M$ we have a presheaf of groups

$$\mathfrak{G}_M : \text{hoccgc}(k) \rightarrow \text{Grp}.$$
on $\text{hoccdgC}(k)$, sending each ccdg-coalgebra $C$ to the group $\mathfrak{G}t_M(C)$, and each $[f] \in \text{Hom}_{\text{hoccdgC}(k)}(C, C')$ to a group homomorphism $\mathfrak{G}t_M([f]) : \mathfrak{G}t_M(C') \to \mathfrak{G}t_M(C)$ defined by, for all $[\varphi'] \in H_0 \text{Aut}_{\Delta_c}(C' \otimes M) = \mathfrak{G}t_M(C)$,

$$\mathfrak{G}t_M([f])([\varphi']) := [\mathfrak{G}t_M(f)(\varphi')],$$

where $f \in \text{Hom}_{\text{ccdgc}(k)}(C, C')$ and $\varphi' \in Z_0 \text{Aut}_{\Delta_c}(C' \otimes M)$ are arbitrary representatives of $[f]$ and $[\varphi']$, respectively.

**Proof.** We already know that $\mathfrak{G}t_M(C)$ is a group, eq. (4.6). Due to Lemma 4.5, it remains to check that the homology class $[\mathfrak{G}t_M(f)(\varphi')]$ of $\mathfrak{G}t_M(f)(\varphi')$ depends only on the homology class $[\varphi']$ of $\varphi'$ and the homotopy type $[f]$ of $f$, which are evident since $\mathcal{E}_M(f) = \mathfrak{G}t_M(f)$ is a chain map—property (c) in Lemma 4.4, and the homology class of $\mathfrak{G}t_M(f)(\varphi')$ depends only on $[f]$ and $[g']$ due to properties (a) and (b) in Lemma 4.5. $\square$

We are ready to define a linear representation of a representable presheaf of groups $\mathcal{P}_\Omega : \text{ccdgc}(k) \to \text{Grp}$.

**Definition 4.1.** A linear presentation of a representable presheaf of groups $\mathcal{P}_\Omega$ on the category $\text{ccdgc}(k)$ of ccdg-coalgebras is a pair $(M, \rho_M)$, where $M$ is a chain complex and $\rho_M : \mathcal{P}_\Omega \Rightarrow \mathfrak{G}t_M : \text{ccdgc}(k) \to \text{Grp}$ is a natural transformation of the presheaves.

**Remark 4.1.** Note that $\rho_M : \mathcal{P}_\Omega \Rightarrow \mathfrak{G}t_M$ is a natural transformation of contravariant functors:

- $\rho^C_M : \mathcal{P}_\Omega(C) \to \mathfrak{G}t_M(C)$ is a homomorphism of groups for every ccdg-coalgebra $C$ so that we have $\rho^C_M(g) \in Z_0 \text{Aut}_{\Delta_c}(C \otimes M)$ for every $g \in \text{Hom}_{\text{ccdgc}(k)}(C, \Omega)$, and
- for every morphism $f : C \to C'$ of ccdg-coalgebras the diagram commutes

\[
\begin{array}{ccc}
\mathcal{P}_\Omega(C') & \xrightarrow{\mathcal{P}_\Omega(f)} & \mathcal{P}_\Omega(C) \\
\rho^C_M \downarrow & & \downarrow \rho^C_M \\
\mathfrak{G}t_M(C') & \xrightarrow{\mathfrak{G}t_M(f)} & \mathfrak{G}t_M(C)
\end{array}
\]

\text{ie.}, $\rho^C_M \circ \mathcal{P}_\Omega(f) = \mathfrak{G}t_M(f) \circ \rho^C_M$ .

(4.7)

Since $\mathcal{P}_\Omega$ is representable, the Yoneda lemma implies that a natural transformation $\rho_M : \mathcal{P}_\Omega \Rightarrow \mathfrak{G}t_M$ is completely determined by the universal element $\rho^0_M(\mathbb{I}_\Omega) : \Omega \otimes M \to \Omega \otimes M$. Indeed, the naturalness of $\rho_M$ impose that $\rho^C_M(g) = \rho^C_M(\mathcal{P}_\Omega(g)(\mathbb{I}_\Omega)) = \mathfrak{G}t_M(g)(\rho^C_M(\mathbb{I}_\Omega))$ for every morphism $C \xrightarrow{g} \Omega$ of ccdg-coalgebras. Explicitly, we have

\[
\rho^C_M(g) = \tilde{\rho}(\tilde{\rho}(\rho^C_M(\mathbb{I}_\Omega)) \circ (g \otimes \mathbb{I}_M)) \quad \Longleftrightarrow \quad \tilde{\rho} \left( \rho^C_M(g) \right) = \tilde{\rho}(\rho^C_M(\mathbb{I}_\Omega)) \circ (g \otimes \mathbb{I}_M),
\]

(4.8)

where $\tilde{\rho}$ and $\tilde{\rho}$ are defined in Lemma 4.1.
Lemma 4.7. A linear representation $\rho_M : \mathcal{P}_\Omega \to \mathcal{G}_M$ of $\mathcal{P}_\Omega$ induces a natural transformation $\mathbf{g}_M : \mathbf{P}_\Omega \to \mathbf{G}_M : \text{hoccdgC}(k) \to \text{Grp}$, whose component $\mathbf{g}_M^C$ at each ccdg-coalgebra $C$ is the homomorphism $g_M^C : \mathbf{P}_\Omega(C) \to \mathbf{G}_M(C)$ of groups defined by $g_M^C([g]) = [\rho_M^C(g)] \in H_0 \text{Aut}_C(C \otimes M)$ for all $[g] \in \text{Hom}_{\text{hoccdgC}}(k)(C, \Omega)$, where $g \in \text{Hom}_{\text{ccdgc}}(C, \Omega)$ is an arbitrary representative of $[g]$.

Proof. Let $g \sim \tilde{g} \in \text{Hom}_{\text{ccdgc}}(k)(C, \Omega)$. Then there is a homotopy pair $(g(t), \chi(t))$ on $\text{Hom}_{\text{ccdgc}}(k)(C, \Omega)$ such that we have a family $g(t) = g + \partial_{\Omega, t} \int_0^1 \chi(s) ds$ of morphisms of ccdg-coalgebras satisfying $g(0) = g$ and $g(1) = \tilde{g}$. From eq. (4.8) it follows that $\rho_M^C(g) \sim \rho_M^C(\tilde{g}) \in Z_0 \text{Aut}_C(C \otimes M)$, since both $\rho$ and $\tilde{\rho}$ are chain maps and $\rho^C_0(1_B) \in Z_0 \text{Aut}_C(C \otimes M)$. Therefore we have $[\rho_M^C(g)] = [\rho_M^C(\tilde{g})] \in H_0 \text{Aut}_C(C \otimes M)$, so that $g_M^C : \mathbf{P}_\Omega(C) \to \mathbf{G}_M(C)$ is well-defined homomorphism of groups for every $C$. The naturalness of $\mathbf{g}_M^C$, i.e., for every $[f] \in \text{Hom}_{\text{hoccdgC}}(k)(C, C')$ we have $g_M^C \circ \mathbf{P}_\Omega([f]) = \mathbf{g}_M^C([f]) \circ g_{M'}^C$ due to the naturalness eq. (4.7) of $\rho_M^C$ and by the definitions of $g_M^C$, $\mathbf{P}_\Omega([f])$ and $\mathbf{g}_M^C([f])$. □

Definition 4.2. A linear representation of the presheaf of groups $\mathbf{P}_\Omega$ on the homotopy category $\text{hoccdgC}(k)$ is a pair $(M, g_M)$ of chain complex $M$ and a natural transformation $\mathbf{g}_M : \mathbf{P}_\Omega \to \mathbf{G}_M : \text{hoccdgC}(k) \to \text{Grp}$, which is induced from a linear representation $\rho_M : \mathcal{P}_\Omega \to \mathcal{G}_M : \text{ccdgc}(k) \to \text{Grp}$ of the presheaf of groups $\mathcal{P}_\Omega$ on $\text{ccdgc}(k)$.

Remark 4.2. Despite of the above definition we will work with linear representations of $\mathcal{P}_\Omega$ rather than those of $\mathbf{P}_\Omega$. Working with the dg-tensor category of linear representations of $\mathcal{P}_\Omega$ will be a crucial step for Tannakian reconstructions of both $\mathcal{P}_\Omega$ and $\mathbf{P}_\Omega$ in the next section. The linear representations of $\mathcal{P}_\Omega$ form a dg-tensor category $\text{Rep}(\mathcal{P}_\Omega)$. We may regard $\text{Rep}(\mathcal{P}_\Omega)$ as "the dg-tensor category" of linear representations of $\mathbf{P}_\Omega$, where the category of linear representations of $\mathbf{P}_\Omega$ can be defined as the homotopy category of $\text{Rep}(\mathcal{P}_\Omega)$. We will not elaborate this point as we will never use it.

Here are two basic examples of linear representations of $\mathcal{P}_\Omega$.

Example 4.1 (The trivial representation). The ground field $k$ as a chain complex $k = (k, 0)$ with zero differential defines the trivial representation $(k, \rho_k)$, where the component $\rho_k^C$ of $\rho_k : \mathcal{P}_\Omega \to \mathcal{G}_k$ at every ccdg-coalgebra $C$ is the homomorphism $\rho_k^C : \mathcal{P}_\Omega(C) \to \mathcal{G}_k(C)$ of groups such that, $\forall g \in \text{Hom}_{\text{ccdgc}}(k)(C, \Omega)$,

$$\rho_k^C(g) := \mathbb{I}_C \otimes \mathbb{I}_k : C \otimes k \to C \otimes k. \quad (4.9)$$

Example 4.2 (The regular representation). Associated to the ccdg-Hopf algebra $\Omega$ as a chain complex we have the regular representation $(\Omega, \rho_\Omega)$, where the component $\rho_\Omega^C$
of $\rho^C_\Omega: \mathcal{P}_\Omega \to \mathcal{G}_\Omega$ at every ccdg-coalgebra $C$ is the homomorphism $\rho^C_\Omega: \mathcal{P}_\Omega(\Omega) \to \mathcal{G}_\Omega$ of groups defined by, $\forall g \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega)$,

$$
\rho^C_\Omega(g) = (I_C \otimes m_\Omega) \circ (I_C \otimes g \otimes I_\Omega) \circ (\Delta_C \otimes I_\Omega) = \bar{p}(m_\Omega \circ (g \otimes I_\Omega))
$$

(4.10)

We can check that $(\Omega, \rho_\Omega)$ is a linear representation as follows:

- We have $\partial g \circ \rho_C^C = \bar{p}(m_\Omega \circ (\partial g \otimes I_\Omega)) = 0$ for all $g \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega)$, since both $\bar{p}$ and $m_\Omega$ are chain maps;

- We have $\rho_\Omega(\Omega \circ \rho_C^C) = (I_C \otimes m_\Omega) \circ (I_C \otimes (\rho_C^C \circ \rho_C^C) \otimes I_\Omega) \circ (\Delta_C \otimes I_\Omega) = I_C \otimes \Omega$.

- For all $g_1, g_2 \in \text{Hom}_{\text{ccdgC}(k)}(C, \Omega)$:

$$
\rho^C_\Omega(g_1 \circ \rho_C^C \circ \rho_C^C(g_2)) := (I_C \otimes m_\Omega) \circ (I_C \otimes (\rho_C^C \circ (g_1 \otimes g_2) \otimes \Delta_C) \otimes I_\Omega) \circ (\Delta_C \otimes I_\Omega) = (I_C \otimes m_\Omega) \circ (I_C \otimes (g_1 \otimes I_\Omega) \circ (\Delta_C \otimes I_\Omega) \circ (I_C \otimes m_\Omega) \circ (I_C \otimes g_2 \otimes I_\Omega) \circ (\Delta_C \otimes I_\Omega)) = \rho^C_\Omega(g_1) \circ \rho^C_\Omega(g_2).
$$

The 2nd equality is due to coassociativity of $\Delta_C$ and the associativity of $m_\Omega$. $\square$

**Definition 4.3 (Lemma).** Linear representations of the representable presheaf of groups $\mathcal{P}_\Omega$ form a dg-tensor category $\text{Rep}(\mathcal{P}_\Omega)$ defined as follows.

(a) An object is a linear presentation $(M, \rho_M)$ of $\mathcal{P}_\Omega$.

(b) A morphism $\psi: (M, \rho_M) \to (M', \rho_{M'})$ of linear representations of $\mathcal{P}_\Omega$ is a linear map $\psi: M \to M'$ making the following diagram commutative for every ccdg-coalgebra $C$ and every $g \in \mathcal{P}_\Omega(C)$

$$
\begin{array}{ccc}
C \otimes M & \xrightarrow{I_C \otimes \psi} & C \otimes M' \\
\rho^C_M(g) & \downarrow & \rho^C_{M'}(g) \\
C \otimes M & \xrightarrow{I_C \otimes \psi} & C \otimes M',
\end{array}
$$

i.e., $(I_C \otimes \psi) \circ \rho^C_M(g) = \rho^C_{M'}(g) \circ (I_C \otimes \psi)$,

where $\rho^C_M: \mathcal{P}(C) \to \mathcal{G}(C)$ is the component of the natural transformation $\rho_M$ at $C$.

(c) The differential of a morphism $\psi: (M, \rho_M) \to (M', \rho_{M'})$ of linear representations is the morphism $\partial \circ \rho_C \circ \psi: (M, \rho_M) \to (M', \rho_{M'})$ of linear representations.

(d) The tensor product $(M, \rho_M) \otimes (M', \rho_{M'})$ of two objects is the linear representation $(M \otimes M', \rho_{M \otimes M'})$, where $M \otimes M' = (M \otimes M', \partial_{M \otimes M'})$ is the tensor product of chain
complexes and \( \rho_{M \otimes M'} : \mathcal{P}_\Omega \Rightarrow \mathcal{G}_{M \otimes M'} \) is the natural transformation whose component \( \rho^C_{M \otimes M'} : \mathcal{P}_{\Omega}(C) \rightarrow \mathcal{G}_{M \otimes M'}(C) \) at every ccdg-coalgebra \( C \) is the group homomorphism defined by, \( \forall g \in \text{Hom}_{\text{ccdgc}}(\Omega, \Omega) \),

\[
\rho^C_{M \otimes M'}(g) := \rho^C_M(g) \otimes_{\Delta_C} \rho^C_{M'}(g).
\]

The unit object for the tensor product is the trivial representation \((k, \rho_k)\) in Example 4.1.

Remark 4.3. The explicit form of \( \rho_{M \otimes M'}^C(g) := \rho^C_M(g) \otimes_{\Delta_C} \rho^C_{M'}(g) \) is

\[
\rho^C_{M \otimes M'}(g) = \left( \rho^C_M(g) \otimes \delta(\rho^C_{M'}(g)) \right) \circ (\Delta_C \otimes \text{id}_M \otimes \text{id}_{M'}) \circ (\Delta_C \otimes \text{id}_M \otimes \text{id}_{M'})
\]

Equivalently, \( \rho^C_{M \otimes M'}(g) \) is determined by the following equality:

\[
\delta(\rho^C_{M \otimes M'}(g)) = \left( \delta(\rho^C_M(g)) \otimes \delta(\rho^C_{M'}(g)) \right) \circ (\Delta_C \otimes \text{id}_M \otimes \text{id}_{M'}) \circ (\Delta_C \otimes \text{id}_M \otimes \text{id}_{M'}). \tag{4.11}
\]

Proof. It is trivial to check that \( \partial_{M,M'} \psi \) is a linear representation whenever \( \psi \) is a linear representation. Then it becomes obvious that \( \text{Rep}(\mathcal{P}_\Omega) \) is a dg-category. It is also trivial to check that the tensor product and the unit object in (d) endow \( \text{Rep}(\mathcal{P}_\Omega) \) with a structure of dg-tensor category. \( \square \)

4.3. An isomorphism with the dg-tensor category of left dg-modules

A right dg-module over ccdg-Hopf algebra \( \Omega \), as a dg-algebra \((\Omega, u_\Omega, m_\Omega, \partial_\Omega)\), is a tuple \((M, \gamma_M)\) where \( M = (M, \partial_M) \) is a chain complex and \( \gamma_M : \Omega \otimes M \rightarrow M \) is a chain map making the diagrams commute

\[
\begin{array}{ccc}
\Omega \otimes M & \xrightarrow{\gamma_M} & M \\
\downarrow{u_\Omega \otimes 1_M} & & \downarrow{1_M} \\
k \otimes M & \xrightarrow{\partial_M} & M
\end{array}
\]

\[
\begin{array}{ccc}
\Omega \otimes \Omega \otimes M & \xrightarrow{m_\Omega \otimes 1_M} & \Omega \otimes M \\
\downarrow{1_\Omega \otimes \gamma_M} & & \downarrow{\gamma_M} \\
\Omega \otimes M & \xrightarrow{\gamma_M} & M
\end{array}
\]

That is

\[
\gamma_M \circ \partial_M = \partial_M \circ \gamma_M,
\]

\[
\begin{aligned}
\gamma_M \circ (u_\Omega \otimes 1_M) &= 1_M, \\
\gamma_M \circ (1_\Omega \otimes \gamma_M) &= \gamma_M \circ (m_\Omega \otimes 1_M).
\end{aligned}
\]

A morphism \((M, \gamma_M) \xrightarrow{\psi} (M', \gamma_{M'})\) of left dg-modules over \( \Omega \) is a linear map \( \psi : M \rightarrow M' \) making the following diagram commutes

\[
\begin{array}{ccc}
\Omega \otimes M & \xrightarrow{\gamma_M} & M \\
\downarrow{1_\Omega \otimes \psi} & & \downarrow{\psi} \\
\Omega \otimes M' & \xrightarrow{\gamma_{M'}} & M'
\end{array}
\]

i.e.,

\[
\psi \circ \gamma_M = \gamma_{M'} \circ (1_\Omega \otimes \psi).
\]
It is trivial to check that \( \partial_{M,M'} \psi : (M, \gamma_M) \to (M', \gamma_{M'}) \) is a morphism of left dg-modules over \( \Omega \) whenever \( \psi \) is so, and we have \( \partial_{M,M'} \circ \partial_{M,M'} = 0 \). For every consecutive morphism \( \psi'' : (M'', \gamma_{M''}) \to (M', \gamma_{M'}) \) of left dg-modules over \( \Omega \) we also have \( \partial_{M,M'}(\psi'' \circ \psi') = \partial_{M,M'} \psi'' \circ \psi' + (-1)^{\deg \psi'} \partial_{M,M'} \psi \). Therefore, we have a dg-category \( \text{dgMod}_L(\Omega) \) of left dg-modules over \( \Omega \).

The tensor product \((M, \gamma_M) \otimes_{\Delta_{\text{pres}}}(M', \gamma_{M'})\) of left dg-modules \((M, \gamma_M)\) and \((M', \gamma_{M'})\) over \( \Omega \) is the left dg-module \((M \otimes M', \gamma_{M \otimes M'})\) over \( \Omega \), where \( M \otimes M' \) is the chain complex with the differential \( \partial \)

\[
\gamma_{M \otimes M'} := \gamma_M \otimes \gamma_{M'} \circ (\Delta_{\Omega} \otimes I_{M'}) \circ (\Delta_{\Omega} \otimes I_{M}).
\] (4.12)

In diagram \( \gamma_{M \otimes M'} : \Omega \otimes M \otimes M' \to M \otimes M' \) as

\[
\Omega \otimes M \otimes M' \xrightarrow{\Delta_{\Omega} \otimes I_{M'} \otimes I_{M'}} \Omega \otimes \Omega \otimes M \otimes M' \xrightarrow{I_{\Omega} \otimes \tau \otimes I_{M'}} \Omega \otimes \Omega \otimes M \otimes M' \xrightarrow{\gamma_{M} \otimes \gamma_{M'}} M \otimes M'.
\]

Then, given left dg-modules \((M, \gamma_M), (M', \gamma_{M'})\) and \((M'', \gamma_{M''})\) over \( \Omega \), the isomorphism \( (M \otimes M') \otimes M'' \cong M \otimes (M' \otimes M'') \) of chain complexes becomes an isomorphism

\[
\left( (M, \gamma_M) \otimes_{\Delta_{\Omega}} (M', \gamma_{M'}) \right) \otimes_{\Delta_{\Omega}} (M'', \gamma_{M''}) \cong (M, \gamma_M) \otimes_{\Delta_{\Omega}} (M', \gamma_{M'}) \otimes_{\Delta_{\Omega}} (M'', \gamma_{M''})
\]

of left dg-modules over \( \Omega \), since \( \Delta_{\Omega} \) is coassociative.

The ground field \( k \) has the canonical structure \((k, \gamma_k)\) of left dg-module \( \Omega \otimes k \xrightarrow{\gamma_k} k \) over \( \Omega \), where \( \gamma_k := m_k \circ (\epsilon_k \otimes I_k) \). Then, for every left dg-module \((M, \gamma_M)\) over \( \Omega \), the isomorphism \( M \otimes k \cong k \otimes M \) of chain complexes becomes an isomorphism

\[
(M, \gamma_M) \otimes_{\Delta_{\Omega}} (k, \gamma_k) \cong (k, \gamma_k) \otimes_{\Delta_{\Omega}} (M, \gamma_M)
\]

of left dg-modules over \( \Omega \), due to the counit axiom \( \epsilon_C \circ (\epsilon_C \otimes I_C) \circ \Delta_C = I_C \circ (I_C \otimes \epsilon_C) \circ \Delta_C = I_C \). Therefore, the dg-category of left dg-modules over \( \Omega \) is a dg-tensor category \( \text{dgMod}_L(\Omega) \).

**Theorem 4.1.** The dg-category \( \text{Rep}(\mathcal{P}_\Omega) \) of linear representations of a representable presheaf of groups \( \mathcal{P}_\Omega \) on \( \text{cdgc}(k) \) is isomorphic to the dg-category of \( \text{dgMod}_L(\Omega) \) of left dg-modules over \( \Omega \) as dg-tensor categories. Explicitly, we have an isomorphism of dg-tensor categories \( X : \text{Rep}(\mathcal{P}_\Omega) \xrightarrow{\sim} \text{dgMod}_L(\Omega) : Y \) defined as follows.

- \( X \) sends each representation \( \{M, \rho_M\} \) to the left dg-module \( (M, \gamma_M) \), where

\[
\gamma_M := \mathcal{q}(\rho_M^\Omega(\mathcal{I}_\Omega)) = \mathcal{J}_M \circ (\epsilon_{\Omega} \otimes \mathcal{I}_M) \circ \rho_M^\Omega(\mathcal{I}_\Omega) : \Omega \otimes M \to M,
\] (4.13)

and each morphism \( \psi : \{M, \rho_M\} \to \{M', \rho_{M'}\} \) of representations to the morphism \( \psi : (M, \gamma_M) \to (M', \gamma_{M'}) \) of left dg-modules.
- \( \gamma \) sends each left dg-module \( (M, \gamma_M) \) to the representation \( (M, \tilde{\rho}_M) \), where the component \( \tilde{\rho}_M \) of \( \tilde{\rho} \) at a ccdg-coalgebra \( C \) is defined by, \( \forall g \in \text{Hom}_{\text{ccdgc}}(k)(C, \Omega) \),

\[
\tilde{\rho}_M(g) := \tilde{\rho}(\gamma_M \circ (g \otimes 1_M)) = (I_C \otimes \gamma_M) \circ (I_C \otimes g \otimes I_M) \circ (\Delta_C \otimes I_M) : C \otimes M \to C \otimes M,
\]

and each morphism \( \psi : (M, \gamma_M) \to (M', \gamma_M') \) of left dg-modules to the morphism \( \psi : (M, \tilde{\rho}_M) \to (M', \tilde{\rho}_M') \) of representations.

\textbf{Proof.} We need to check that both \( X \) and \( Y \) are dg-tensor functors and show that they are inverse to each other.

1. We check that \( (M, \tilde{\gamma}_M) = X(M, \rho_M) \) is a left dg-module over \( \Omega \) as follows.

From eq. (4.8) in Remark 4.1 and the definition eq. (4.13), we have the following relation for every morphism \( g : C \to \Omega \) of ccdg-coalgebras:

\[
\tilde{\gamma}(\rho_M(g)) = \tilde{\gamma}_M \circ (g \otimes 1_M) : C \otimes M \to M.
\]

The component \( \rho_M \) of \( \rho \) at every ccdg-coalgebra \( C \), by definition, is a morphism \( \rho_M : \mathcal{P}(C) \to \mathcal{G}(C) \) of groups, i.e., for every pair of morphisms \( g_1, g_2 : C \to \Omega \) of ccdg-coalgebras, we have

\[
\rho_M(g_1 \cdot g_2) = \rho_M(g_1) \cdot \rho_M(g_2).
\]

Applying \( \tilde{\gamma} \) on the 1st equality of eq. (4.16) and using eq. (4.15), we have

\[
\tilde{\gamma}_M \circ (u_{\Omega} \otimes 1_M) \circ (\epsilon_C \otimes 1_M) = \epsilon_C \otimes 1_M.
\]

By putting \( C = k^\vee \), we obtain that \( \tilde{\gamma}_M \circ (u_{\Omega} \otimes 1_M) = 1_M \).

Applying \( \tilde{\gamma} \) on the 2nd equality of eq. (4.16), we have

\[
\tilde{\gamma}_M \circ (m_{\Omega} \otimes 1_M) \circ (g_1 \otimes g_2 \otimes I_M) \circ (\Delta_C \otimes I_M) = \tilde{\gamma}_M \circ (I_{\Omega} \otimes \tilde{\gamma}_M) \circ (g_1 \otimes g_2 \otimes I_M) \circ (\Delta_C \otimes I_M).
\]

Consider the dg-coalgebra \( \Omega \otimes \Omega \) and the projection maps \( \pi_1, \pi_2 : \Omega \otimes \Omega \to \Omega \)

\[
\pi_1 := \Omega \otimes \Omega \xrightarrow{\text{Id} \otimes \pi_2} \Omega \otimes k \xrightarrow{\pi_1} \Omega, \quad \pi_2 := \Omega \otimes \Omega \xrightarrow{\pi_1 \otimes \text{Id}_k} k \otimes \Omega \xrightarrow{\text{Id}_k \otimes 1_\Omega} \Omega,
\]

which are morphisms of dg-coalgebras. We can check that \( (\pi_1 \otimes \pi_2) \circ (\Delta_{\Omega} \otimes 1_\Omega) = 1_{\Omega \otimes \Omega} \) from an elementary calculation. Let \( C = \Omega \otimes \Omega \). By substituting \( g_1 = \pi_1, g_2 = \pi_2 \) in eq. (4.18) we obtain that \( \tilde{\gamma}_M \circ (m_{\Omega} \otimes 1_M) = \tilde{\gamma}_M \circ (I_{\Omega} \otimes \tilde{\gamma}_M) \).

Finally we check that \( \tilde{\gamma}_M : \Omega \otimes M \to M \) is a chain map:

\[
\partial_{\Omega \otimes M, \Omega, \tilde{\gamma}_M} = \partial_{\Omega \otimes M, \Omega} \tilde{\gamma}(\rho_M(I_{\Omega})) = \tilde{\gamma}(\partial_{C \otimes M, C \otimes M} \rho_M(I_{\Omega})) = 0,
\]

where we have used the facts that \( \tilde{\gamma} \) is a chain map defined in Lemma 4.1 and \( \rho_M(I_{\Omega}) \in \text{ZpAut}_{\Delta_0}(\Omega \otimes M) \).
2. We show $X$ is a dg-tensor functor. Given a morphism $\psi : (M, \rho_M) \to (M', \rho_{M'})$ of representations, $X(\psi) = \psi : (M, \tilde{\gamma}_M) \to (M', \tilde{\gamma}_{M'})$ is a morphism of left dg-modules over $\Omega$, since the following diagram commutes

\[
\begin{array}{cccccc}
\Omega \otimes M & \xrightarrow{\rho_M^{\Omega}(\Omega)} & \Omega \otimes M & \xrightarrow{\rho_{M'}^{\Omega}(\Omega)} & k \otimes M & \xrightarrow{i_M} & M \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega \otimes M' & \xrightarrow{\rho_{M'}^{\Omega}(\Omega)} & \Omega \otimes M' & \xrightarrow{\rho_{M'}^{\Omega}(\Omega)} & k \otimes M' & \xrightarrow{i_{M'}} & M'
\end{array}
\]

where the very left square commutes since $\psi$ is a morphism of representations and the commutativity of the other squares are obvious—note that the horizontal compositions are exactly $\tilde{\gamma}_M, \tilde{\gamma}_{M'}$.

It is obvious that $X(\tilde{\delta}_{M,M'}^\Omega(\psi)) = \tilde{\delta}_{M,M'}^\Omega(X(\psi))$. Therefore $X$ is a dg-functor. The tensor property of $X$ is checked as follows:

- From Example 4.1 we have $X([k, \rho_k]) = ([k, \gamma_k])$.

- For two representations $\{M, \rho_M\}, \{M', \rho_{M'}\}$ of $\mathcal{P}_\Omega$, we have

\[
\gamma X(M, \rho_M) \otimes (M', \rho_{M'}) = \left(\gamma X(M, \rho_M) \otimes \gamma X(M', \rho_{M'})\right) \circ (\Delta_{\Omega} \otimes \Delta_{\rho_M \rho_{M'}})
\]

The 1st equality is from eq. (4.12), and the 2nd equality is from eq. (4.11). We conclude that $X(\{M, \rho_M\} \otimes \{M', \rho_{M'}\}) = X(M, \rho_M) \otimes \Delta_{\Omega} X(M', \rho_{M'})$.

3. We show that $\{M, \tilde{\rho}_M\} = \Psi [M, \gamma_M]$ is a representation of $\mathcal{P}_\Omega$ as follows. We first show that $\tilde{\rho}_M^C : \mathcal{P}_\Omega(C) \to \mathcal{G}_M(C)$ is a homomorphism of groups for every $C$:

- We have $\rho_M^C(u_C \circ \rho_C) = (I_C \otimes \gamma_M) \circ (I_C \otimes (u_C \circ \rho_C) \otimes I_M) \circ (\Delta_C \otimes I_M) = I_{C \otimes M}$, where we have used the counity of $\Delta_C$ and the property $\gamma_M \circ (u_C \otimes I_M) = i_M$.

- For all $g_1, g_2 \in \text{Hom}_{\text{cdgC}}(k, C)$:

\[
\rho_M^C(g_1, g_2) := [I_C \otimes \gamma_M] \circ [I_C \otimes (g_1 \otimes g_2) \circ \Delta_C] \circ I_M
\]

where the 2nd equality is due to coassociativity of $\Delta_C$ and the property $\gamma_M \circ (m_C \otimes I_M) = \gamma_M \circ (I_C \otimes \gamma_M)$.

- We have $\tilde{\rho}_M^C \circ (\tilde{\rho}_C \otimes \rho_M) = 0$ for all $g \in \text{Hom}_{\text{cdgC}}(k, C)$, since both $\tilde{\rho}$ and $\gamma_M$ are chain maps.

Combining all the above, we conclude that $\tilde{\rho}_M^C(g) \in Z_0 \text{Aut}_{\Delta_C}(C \otimes M)$ for all $g \in \text{Hom}_{\text{cdgC}}(k, C)$, and $\tilde{\rho}_M^C$ is a homomorphism of groups. We check the naturalness
of $\tilde{\rho}_M$ that for every morphism $f : C \rightarrow C'$ of ccdg-coalgebras we have $\rho^C_M \circ \mathcal{P}_f(f) = \mathcal{G}_f(M) \circ \tilde{\rho}_M$ as follows: for all $g' \in \textbf{Hom}_{\text{cddg}(k)}(C', \Omega)$ we have

$$\rho^C_M(\mathcal{P}_f(f)(g')) = \tilde{\rho}_M(g' \circ f) = \tilde{\rho}_M(\gamma_M \circ (g' \circ f \circ I_M)),$$

$$\mathcal{G}_f(M)(\rho^C_M(g')) = \tilde{\rho}_M(\gamma_M \circ (g' \circ f \circ I_M)), $$

$$\mathcal{G}_f(M)(\rho^C_M(g')) = \tilde{\rho}_M(\gamma_M \circ (g' \circ f \circ I_M)).$$

where we have used $\tilde{\rho}_M = \textbf{I}_{\text{Hom}(C' \otimes M, N)}$. 

4. We show $\mathbf{Y}$ is a dg-tensor functor. Given a morphism $\psi : (M, \gamma_M) \rightarrow (M', \gamma_M')$ of left dg-modules over $\Omega$, $\mathbf{Y}(\psi) = \psi : (M, \tilde{\rho}_M) \rightarrow (M', \tilde{\rho}_M)$ is a morphism of representations, since the following diagram commutes for every ccdg-coalgebra map $g : C \rightarrow \Omega$:

$$\begin{array}{c}
C \otimes M \xrightarrow{\Delta_C \otimes I_M} C \otimes C \otimes M \xrightarrow{\mathcal{I}_C \otimes g \otimes I_M} C \otimes \Omega \otimes M \xrightarrow{\mathcal{I}_C \otimes \gamma_M} C \otimes M \\
\downarrow{\mathcal{I}_C \otimes \psi} \quad \downarrow{\mathcal{I}_C \otimes \psi} \quad \downarrow{\mathcal{I}_C \otimes \psi} \quad \downarrow{\mathcal{I}_C \otimes \psi} \\
C \otimes M' \xrightarrow{\Delta_C \otimes I_M'} C \otimes C \otimes M' \xrightarrow{\mathcal{I}_C \otimes g \otimes I_M'} C \otimes \Omega \otimes M' \xrightarrow{\mathcal{I}_C \otimes \gamma_M'} C \otimes M'.
\end{array}$$

It is obvious that $\mathbf{Y}(\partial_{M', \gamma_M} \psi) = \partial_{M, \gamma_M} \mathbf{Y}(\psi) \equiv \partial_{M, \gamma_M}(\mathbf{Y}(\psi))$. Therefore $\mathbf{Y}$ is a dg-functor. The tensor property of $\mathbf{Y}$ is checked as follows:

- From Example 4.1, we have $\mathbf{Y}(k, \gamma_k) = (k, p_k)$.

- Let $(M, \gamma_M, M', \gamma_M')$ be left dg-modules over $\Omega$, and $g : C \rightarrow \Omega$ be a morphism of ccdg-coalgebras. Then by eq. (4.11), we have

$$\tilde{\alpha}_M(\mathbf{Y}(M, \gamma_M) \otimes \mathbf{Y}(M', \gamma_M'))(g) = (\gamma_M \otimes \gamma_M') \circ (I_C \otimes \tau \otimes I_M) \circ ((g \otimes g) \circ \Delta_C) \circ (g \otimes I_M) \circ \tilde{\rho}_M(g) \circ \gamma_M' \circ (g' \otimes I_M) \circ \tilde{\rho}_M(g') \circ \gamma_M'.

\text{Therefore we have } \mathbf{Y}((M, \gamma_M) \otimes (M', \gamma_M')) = \mathbf{Y}(M, \gamma_M) \otimes \mathbf{Y}(M', \gamma_M').

5. It is immediate from the constructions that $X$ and $Y$ are inverse to each other. \qed

4.4. A ccdg-Hopf algebra versus the dg-category of its left dg-modules

Key properties of ccdg-Hopf algebra $\Omega$ are reflected in the dg-category $\text{dgMod}_\Omega(\Omega) \cong \text{Rep}(\mathcal{P}_\Omega)$ which phenomena will make our Tannakian reconstruction possible.

To begin with, the $\Omega$ as a chain complex has the structures $(\Omega, m_\Omega)$ of both left and right dg-modules over $\Omega$, and these properties are equivalent to $\Omega$ being a dg-algebra. The following lemma reflects the ccdg-bialgebra properties of $\Omega$. 

Lemma 4.8. Associated to $(\Omega, m_\Omega)$, we have following morphisms of left $\Omega$ dg-modules:

(a) $(\Omega, m_\Omega) \xrightarrow{\Delta_\Omega} (\Omega \otimes \Omega, \gamma_{\Omega \otimes \Omega})$ by the coproduct $\Delta_\Omega : \Omega \to \Omega \otimes \Omega$;

(b) $(\Omega, m_\Omega) \xrightarrow{\epsilon_\Omega} (k, \gamma_k)$ by the counit $\epsilon_\Omega : \Omega \to k$;

(c) $(\Omega \otimes M, m_\Omega \otimes I_M) \xrightarrow{\gamma_M} (M, \gamma_M)$ by the action $\gamma_M : \Omega \otimes M \to M$ for every left dg-module $(M, \gamma_M)$ over $\Omega$.

Proof. (a) follows from $\Delta_\Omega$ being a morphism of dg-algebras: $\gamma_{\Omega \otimes \Omega} : (\Omega \otimes \Omega) \xrightarrow{\Delta_\Omega} \Omega \otimes \Omega \otimes \Omega$.

(b) follows from $\epsilon_\Omega$ being a morphism of dg-algebras: $\epsilon_k : \Omega \otimes k \to k$.

(c) follows from the module axiom that $\gamma_M$ satisfies:

Since the antipode $\zeta_\Omega : \Omega \to \Omega$ is an anti-homomorphism of dg-algebras, we have a left dg-module $(\Omega^*, \gamma_{\Omega^*})$ over $\Omega$, where $\Omega^* = \Omega$ as chain complex but

$\gamma_{\Omega^*} := m_\Omega \circ (I_\Omega \otimes \zeta_\Omega) \circ \tau : \Omega \otimes \Omega^* \to \Omega^*$,

i.e., $y \otimes z \mapsto (-1)^{|y||z|} m_\Omega(z \otimes \zeta_\Omega(y))$.

Using the counit $\epsilon_\Omega : \Omega \to k$, we can associate every left dg-module $(M, \gamma_M)$ over $\Omega$ to a left dg-module $(M, \gamma_M)$ over $\Omega$, where $M = M$ as chain complexes while the left action is $\gamma_M := m_M \circ (\epsilon_\Omega \otimes I_M) : \Omega \otimes M \xrightarrow{\epsilon_\Omega \otimes I_M} k \otimes M \xrightarrow{\epsilon_M} M$. Then, the following lemma reflects the properties of the antipode $\zeta_\Omega$.

Lemma 4.9. Associated to $(\Omega^*, \gamma_{\Omega^*})$, we have following morphisms of left $\Omega$ dg-modules:

(a) $(\Omega^*, \gamma_{\Omega^*}) \xrightarrow{\Delta_\Omega} (\Omega^* \otimes \Omega^*, \gamma_{\Omega^* \otimes \Omega^*})$ by the coproduct $\Delta_\Omega : \Omega \to \Omega \otimes \Omega$;

(b) $(\Omega^*, \gamma_{\Omega^*}) \xrightarrow{\epsilon_\Omega} (k, \gamma_k)$ by the counit $\epsilon_\Omega : \Omega \to k$;

(c) $(\Omega^* \otimes M, \gamma_{\Omega^*} \otimes \gamma_M) \xrightarrow{\gamma_M} (M, \gamma_M)$ by the action $\gamma_M : \Omega \otimes M \to M$ for every left dg-module $(M, \gamma_M)$ over $\Omega$.

Proof. We shall see that (a) and (b) follows from $\zeta_\Omega$ being a morphism of dg-coalgebras and (c) follows from the antipode axiom for $\zeta_\Omega$.

(a) $\gamma_{\Omega^* \otimes \Omega^*} \circ (I_\Omega \otimes \Delta_\Omega) = \Delta_\Omega \circ \gamma_{\Omega^*}$: We have

$\gamma_{\Omega^* \otimes \Omega^*} \circ (I_\Omega \otimes \Delta_\Omega) := (m_\Omega \otimes m_\Omega) \circ (I_\Omega \otimes \zeta_\Omega \otimes I_\Omega) \circ (\tau \otimes \tau) \circ (I_\Omega \otimes \tau \otimes I_\Omega) \circ (\Delta_\Omega \otimes \Delta_\Omega)$

$= (m_\Omega \otimes m_\Omega) \circ (\zeta_\Omega \circ \zeta_\Omega \otimes \tau) \circ (\Delta_\Omega \otimes \Delta_\Omega),$

$\Delta_\Omega \circ \gamma_{\Omega^*} := (m_\Omega \otimes m_\Omega) \circ (I_\Omega \otimes \tau \otimes I_\Omega) \circ (\Delta_\Omega \otimes \Delta_\Omega) \circ (I_\Omega \otimes \tau)$

$= (m_\Omega \otimes m_\Omega) \circ (\Delta_\Omega \otimes \Delta_\Omega) \circ (\zeta_\Omega \otimes I_\Omega),$
where $\gamma : (\tau \otimes \tau) \circ (\Omega \otimes \Omega) : \Omega^{\otimes 4} \to \Omega^{\otimes 4}$. From the property $(\zeta \otimes \zeta) \circ \Delta = \Delta \circ \zeta$, we obtain that $\gamma \circ \Delta = \Delta \circ \gamma$.

(b) $\gamma \circ \gamma = \gamma \circ (\Omega \otimes \Omega)$: We have

$$\gamma \circ \gamma = \gamma \circ (\Omega \otimes \Omega) = \gamma \circ (\Omega \otimes \Omega),$$

where we used the property $\gamma = \gamma \circ \zeta$ and commutativity of $m_k$.

(c) $\gamma \circ \gamma \circ \gamma = \gamma \circ (\Omega \otimes \gamma)$: We have

$$\gamma \circ \gamma \circ \gamma = \gamma \circ (\Omega \otimes \gamma) = \gamma \circ (\Omega \otimes \gamma) = \gamma \circ (\Omega \otimes \gamma).$$

In the above we have used $\gamma \circ (\Omega \otimes \gamma) = \gamma \circ (\Omega \otimes \gamma)$ for the 3rd and the 6th equalities, the associativity of $m_k$ for the 4th equality, and the antipode axiom $m_k \circ (\zeta \otimes \Omega) \circ \Delta = \Omega \otimes \epsilon$ for the 5th equality. The rest equalities are straightforward.

The morphisms of left dg-modules over $\Omega$ in Lemmas 4.8 and 4.9 are crucially used in the next section.

5. Tannakian reconstruction theorem

Let $\Omega = (\Omega, \epsilon, \zeta, \Delta)$ be a ccdg-Hopf algebra. Consider the forgetful functor $\omega : \text{dgMod}_{\Omega} \to \text{Ch} (k)$ from the dg-category of left dg-modules over $\Omega$ to the dg-category of chain complexes over $k$. The functor $\omega$ sends a left dg-module $(M, \gamma_M)$ over $\Omega$ to its underlying chain complex $M$, and a morphism $\psi : (M, \gamma_M) \to (M', \gamma_M')$ of left dg-modules over $\Omega$ to the underlying $k$-linear map $\psi : M \to M'$. Out of $\omega$, 
we will construct three presheaves $\mathcal{E}_{\omega}$, $\mathcal{P}_{\omega}^\circ$, and $\Psi^\circ_{\omega}$, and establish natural isomorphisms:

$$\mathcal{E}_{\omega} \cong \mathcal{E}_{\Omega} : \mathit{ccdgC}(k) \rightarrow \mathit{dgA}(k),$$

$$\mathcal{P}_{\omega}^\circ \cong \mathcal{P}_{\Omega} : \mathit{ccdgC}(k) \rightarrow \mathit{Grp},$$

which constitute our reconstruction theorem.

The forgetful functor $\omega$ is a dg-tensor functor sending

- $(k, \gamma_k)$ to $k$, and
- the tensor product $(M, \gamma_M) \otimes_{\Delta_{\Omega}} (M', \gamma_{M'})$ of left dg-modules over $\Omega$ to the tensor product $M \otimes M'$ of the underlying chain complexes.

Furthermore, $\omega$ sends the following isomorphisms of left dg-modules over $\Omega$

$$((M, \gamma_M) \otimes_{\Delta_{\Omega}} (M', \gamma_{M'})) \otimes_{\Delta_{\Omega}} (M''', \gamma_{M'''}) \cong (M, \gamma_M) \otimes_{\Delta_{\Omega}} ((M', \gamma_{M'}) \otimes_{\Delta_{\Omega}} (M'', \gamma_{M''})),$$

$$(M, \gamma_M) \otimes_{\Delta_{\Omega}} (k, \gamma_k) \cong (M, \gamma_M) \otimes_{\Delta_{\Omega}} (k, \gamma_k) \otimes_{\Delta_{\Omega}} (M, \gamma_M)$$

to the corresponding isomorphisms $(M \otimes M') \otimes M'' \cong M \otimes (M' \otimes M'')$ and $M \otimes k \cong M \cong k \otimes M$ of the underlying chain complexes.

In Lemma 4.3, we defined the dg-tensor functor $C \otimes - : \mathit{Ch}(k) \rightarrow \mathit{dgComod}^{\text{cofr}}_L(C)$ for each ccdg-coalgebra $C$. By composing it with $\omega$, we get a dg-tensor functor

$$C \otimes \omega : (C \otimes -) \circ \omega : \mathit{dgMod}_L(\Omega) \rightarrow \mathit{dgComod}^{\text{cofr}}_L(C),$$

sending

- each left dg-module $(M, \gamma_M)$ over $\Omega$ to a cofree left dg-comodule $(C \otimes M, \Delta_C \otimes \Delta_M)$ over $C$, and
- each morphism $\psi : (M, \gamma_M) \rightarrow (M', \gamma_{M'})$ of left dg-modules over $\Omega$ to a morphism $1_C \otimes \psi : (C \otimes M, \Delta_C \otimes \Delta_M) \rightarrow (C \otimes M', \Delta_C \otimes \Delta_{M'})$ of left dg-comodules over $C$.

Let $\mathrm{End}(C \otimes \omega) := \mathrm{Nat}(C \otimes \omega, C \otimes \omega)$ be the set of natural endomorphisms of the functor $C \otimes \omega$. We write an element in $\mathrm{End}(C \otimes \omega)$ as $\eta^C_M$, and denote $\eta^C_M$ as its component at a left dg-module $(M, \gamma_M)$ over $\Omega$. The component at the tensor product $(M, \gamma_M) \otimes_{\Delta_{\Omega}} (M', \gamma_{M'})$ is denoted by $\eta^C_{M \otimes_{\Delta_{\Omega}} M'}$. Be aware that for a chain complex $M$, the component at the free left dg-module $(\Omega \otimes M, m_\Omega \otimes \Delta_M)$ over $\Omega$ is denoted by $\eta^C_{\Omega \otimes M}$. We have the following structure of dg-algebra on $\mathrm{End}(C \otimes \omega)$:

$$\mathcal{E}_{\omega}(C) := (\mathrm{End}(C \otimes \omega), \imath^C, \circ, \delta^C),$$

where $\imath^C := \imath_0 \otimes \omega$ is the identity natural transformation, $\circ$ is the composition and $\delta^C$ is the differential given by $(\delta^C \eta^C)_M := \partial_{C \otimes M, C \otimes M}^\omega \eta^C_M$. 

\[\text{Jaehyeok Lee, Jae-Suk Park}\]
Lemma 5.1. We have a presheaf of dg-algebras $E_\omega: \mathbf{ccdgC(k)} \to \mathbf{dgA(k)}$, on $\mathbf{ccdgC(k)}$, sending

- each ccdg-coalgebra $C$ to the dg-algebra $E_\omega(C)$, and
- each morphism $f: C \to C'$ of ccdg-coalgebras to a morphism $E_\omega(f): E_\omega(C) \to E_\omega(C')$ of dg-algebras, where the image of $\eta^{C'} \in \operatorname{End}(C' \otimes \omega)$ is defined by

$$E_\omega(f)(\eta^{C'})_M := \hat{\eta}_M(f \otimes \mathbb{1}_M)$$

$$= (I_C, \otimes \hat{\eta}(\eta^{C'})_M) \circ (I_C, \otimes f \otimes \mathbb{1}_M) \circ (\Delta_C \otimes \mathbb{1}_M)$$

$$= (I_C, \otimes (I_M \circ (\epsilon_C, \otimes \mathbb{1}_M) \circ \eta^{C'})_M) \circ (I_C, \otimes f \otimes \mathbb{1}_M) \circ (\Delta_C \otimes \mathbb{1}_M)$$

for every left dg-module $(M, \gamma_M)$ over $\Omega$.

Proof. Given $\eta^{C'} \in \operatorname{End}(C' \otimes \omega)_n$, and a morphism $f: C \to C'$ of ccdg-coalgebras, we first show that $E_\omega(f)(\eta^{C'}) \in \operatorname{End}(C' \otimes \omega)_n$. For a degree $m$ morphism $\psi: (M, \gamma_M) \to (M', \gamma_{M'})$ of left dg-modules over $\Omega$, the following diagram commutes due to the naturality of $\eta^{C'}$:

$$
\begin{array}{ccc}
C' \otimes M & \xrightarrow{I_C \otimes \psi} & C' \otimes M' \\
\eta^{C'}_M & \downarrow & \eta^{C'}_{M'} \\
C' \otimes M & \xrightarrow{I_C \otimes \psi} & C' \otimes M'
\end{array}
$$

Therefore we have

$$E_\omega(f)(\eta^{C'})_M \circ (I_C \otimes \psi) = (I_C, \otimes (I_M \circ (\epsilon_C, \otimes \mathbb{1}_M) \circ \eta^{C'})_M \circ (f \otimes \psi)) \circ (\Delta_C \otimes \mathbb{1}_M)$$

$$= (-1)^{nm} (I_C, \otimes (I_M \circ (\epsilon_C, \otimes \mathbb{1}_M) \circ \eta^{C'})_M \circ (f \otimes \psi)) \circ (\Delta_C \otimes \mathbb{1}_M)$$

$$= (-1)^{nm} \eta^{C'}_M \circ (I_C \otimes \psi) \circ E_\omega(f)(\eta^{C'})_M,$$

which implies that $E_\omega(f)(\eta^{C'}) \in \operatorname{End}(C' \otimes \omega)_n$. It remains to show that

- $E_\omega(f)$ is a morphism of dg-algebras,
- $E_\omega(g \circ f) = E_\omega(f) \circ E_\omega(g)$ for another ccdg-coalgebra map $g: C' \to C''$, and
- $E_\omega(I_C) = \mathbb{1}_C$.

These follow from the analogous properties of $E_M$ for chain complexes $M = \omega(M, \gamma_M)$, as stated in Lemma 4.4. □

For $\eta^C \in \operatorname{End}(C \otimes \omega)$, we write

$(a)$ $\eta^C \in Z_0 \operatorname{End}(C \otimes \omega)$ if $\eta^C$ is of degree 0 and $\delta^C \eta^C = 0$. 

(45)
Proof. For each module \( M \), \( \eta^C \in \text{End}^\Omega(C \otimes \omega) \) if \( \eta^C \) is of degree \( 0 \), \( \eta^C_k = \mathbb{I}_k = \mathbb{I}_{C_k} \) and

\[
\eta^C_{M \otimes M'} = \eta^C_M \otimes_{\Delta_C} \eta^C_{M'} := (\eta^C_M \otimes (\eta^C_M)') \circ (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_{M'}) \circ (\Delta_C \otimes \mathbb{I}_{M \otimes M'}) \quad (5.2)
\]

holds for every pair of left dg-modules \((M, \gamma_M), (M', \gamma_{M'})\) over \( \Omega \).

An element \( \eta^C \) in \( \text{End}^\Omega(C \otimes \omega) \) is called a tensor natural transformation.

**Lemma 5.2.** If \( \eta^C \in \text{End}^\Omega(C \otimes \omega) \), then for every pair of left dg-modules \((M, \gamma_M), (M', \gamma_{M'})\) over \( \Omega \), we have

\[
(\delta^C \eta^C)_{M \otimes M'} = (\delta^C \eta^C)_M \otimes_{\Delta_C} \eta^C_{M'} + \eta^C_M \otimes_{\Delta_C} (\delta^C \eta^C)_{M'}.
\]

**Proof.** Since \( \eta^C \) is a tensor natural transformation, we have

\[
(\delta^C \eta^C)_{M \otimes M'} = \partial_{C \otimes M' \circ \gamma_M \circ M} \eta^C_{M \otimes M'}
\]

\[
= \partial_{C \otimes M' \circ \gamma_M \circ M} \eta^C_{M' \otimes \Delta_C} \eta^C_M
\]

\[
= (\partial_{C \otimes M' \circ \gamma_M \circ M}) \otimes_{\Delta_C} \eta^C_{M'} + (-1)^{|\eta^C|} \eta^C_M \otimes_{\Delta_C} (\partial_{C \otimes M' \circ \gamma_M \circ M}')
\]

\[
= (\delta^C \eta^C)_M \otimes_{\Delta_C} \eta^C_{M'} + \eta^C_M \otimes_{\Delta_C} (\delta^C \eta^C)_{M'}.
\]

\( \square \)

We define \( Z_0 \text{End}^\Omega(C \otimes \omega) := Z_0 \text{End}(C \otimes \omega) \cap \text{End}^\Omega(C \otimes \omega) \). Clearly, \( Z_0 \text{End}^\Omega(C \otimes \omega) \) is closed under composition and contains \( \mathbb{I}^C = \mathbb{I}_{C \otimes \omega} \). Thus \( Z_0 \text{End}^\Omega(C \otimes \omega), \mathbb{I}^C, \circ \) is a monoid. We shall show that this is in fact a group. We begin with a technical lemma.

**Lemma 5.3.** For every \( \eta^C \in \text{End}(C \otimes \omega) \) its component \( \eta^C_{\mathbb{I} \otimes M} \) at the free left \( \Omega \) dg-module \((\Omega \otimes M, m_\Omega \otimes \mathbb{I}_M)\) generated by a chain complex \( M \) is \( \eta^C_{\mathbb{I} \otimes M} = \eta^C_{\mathbb{I}} \otimes \mathbb{I}_M \).

**Proof.** For each \( z \in M \), define a linear map \( f_z : \Omega \rightarrow \Omega \otimes M \) of degree \( |z| \) by \( f_z(a) := (-1)^{|a||z|} a \otimes z \) for all \( a \in \Omega \). Then \( f_z : (\Omega, m_\Omega) \rightarrow (\Omega \otimes M, m_\Omega \otimes \mathbb{I}_M) \) is a morphism of left dg-modules over \( \Omega \). Since \( \eta^C \) is a natural transformation, the following diagram commutes

\[
\begin{array}{ccc}
C \otimes \Omega & \xrightarrow{\mathbb{I}_C \otimes f_z} & C \otimes \Omega \otimes M \\
\eta^C_{\mathbb{I}} \downarrow & & \downarrow \eta^C_{\mathbb{I} \otimes M} \\
C \otimes \Omega & \xrightarrow{\mathbb{I}_C \otimes f_z} & C \otimes \Omega \otimes M
\end{array}
\]

i.e., \( \eta^C_{\mathbb{I} \otimes M} \circ (\mathbb{I}_C \otimes f_z) = (-1)^{|\eta^C||z|} (\mathbb{I}_C \otimes f_z) \circ \eta^C_{\mathbb{I}}. \)
For every $c \in C$ and $a \in \Omega$ we can write $\eta^C_{M}(c \otimes a)$ as a finite sum $\eta^C_{M}(c \otimes a) = \sum_i c_i \otimes a_i$ for some $c_i \in C$ and $a_i \in \Omega$, where $|c_i| + |a_i| = |c| + |a| + |\eta^C_i|$. Then we obtain that

$$
\eta^C_{M}(c \otimes a \otimes z) = (-1)^{|a||c|} \eta^C_{M} \circ (I_C \otimes f_z)(c \otimes a)
$$

$$
= (-1)^{|a||c|+|\eta^C_i|} \eta^C_{M}(c \otimes a)
$$

$$
= (-1)^{|a||c|+|\eta^C_i|} \sum_i (I_C \otimes f_z)(c_i \otimes a_i)
$$

$$
= \sum_i c_i \otimes a_i \otimes z = (\eta^C_{M} \otimes I_M)(c \otimes a \otimes z).
$$

It follows that $\eta^C_{M} = \eta^C_M \otimes I_M$ since the above equality holds for all $c$, $a$, and $z$. □

**Proposition 5.1.** $\mathcal{P}_\omega^\Omega(C) := (Z_0 \text{End}^\Omega(C \otimes \omega), I^C, \circ)$ is a group for every ccdg-coalgebra $C$.

**Proof.** Associated with each $\eta^C \in Z_0 \text{End}^\Omega(C \otimes \omega)$, we introduce a natural endomorphism $\zeta(\eta^C) \in \text{End}(C \otimes \omega)$ whose component $\zeta(\eta^C)_M$ at each left dg-module $(M, \gamma_M)$ over $\Omega$ is defined by

$$
\zeta(\eta^C)_M := (I_C \otimes \gamma_M) \circ (I_M^C \otimes I_M) \circ (I_C \otimes u_M \otimes I_M) \circ (I_C \otimes \iota_M^M) : C \otimes M \to C \otimes M.
$$

We verify that $\zeta(\eta^C)$ is a natural transformation, since for every morphism $\psi : (M', \gamma_{M'}) \to (M, \gamma_M)$ of left dg-modules over $\Omega$, the following diagram commutes

$$
\begin{array}{cccccc}
C \otimes M & \xrightarrow{I_C \otimes \iota_M^M} & C \otimes M & \xrightarrow{I_C \otimes u_M \otimes I_M} & C \otimes M & \xrightarrow{I_C \otimes \gamma_M} & C \otimes M \\
\downarrow{I_C \otimes \psi} & & \downarrow{I_C \otimes \psi} & & \downarrow{I_C \otimes \psi} & & \downarrow{I_C \otimes \psi} \\
C \otimes M' & \xrightarrow{I_C \otimes \iota_{M'}^M} & C \otimes M' & \xrightarrow{I_C \otimes u_M \otimes I_M} & C \otimes M' & \xrightarrow{I_C \otimes \gamma_{M'}} & C \otimes M'.
\end{array}
$$

We claim that $\zeta(\eta^C)$ is also in $Z_0 \text{End}^\Omega(C \otimes \omega)$. First, note that $\zeta(\eta^C)$ is in $Z_0 \text{End}(C \otimes \omega)$. This is because for each left dg-module $(M, \gamma_M)$ over $\Omega$, all the maps $I_C \otimes \gamma_M, I_M^C \otimes I_M, I_C \otimes u_M \otimes I_M, I_C \otimes \iota_M^M$ are of degree 0 and in the kernels of differentials. Next, we show $\zeta(\eta^C)$ is a tensor natural transformation. From Lemma 4.9(a), the coproduct $\Delta_M : (\Omega^*, \gamma_M) \to (\Omega^* \otimes \Omega^*, \gamma_{M} \otimes \gamma_M)$ is a morphism of left dg-modules over $\Omega$. Since $\eta^C$ is a tensor natural transformation, we have

$$
\begin{array}{c}
\eta^C_{M} \otimes \Delta_M : C \otimes M \otimes \Omega \\
\eta^C_{M} \otimes \Delta_M
\end{array}
\begin{array}{c}
\eta^C_{M} \otimes \Delta_M \circ (I_C \otimes \Delta_M) = (I_C \otimes \Delta_M) \circ \eta^C_{M} \\
\eta^C_{M} \otimes \Delta_M
\end{array}
\begin{array}{c}
\eta^C_{M} \otimes \Delta_M
\end{array}
\begin{array}{c}
\eta^C_{M} \otimes \Delta_M
\end{array}
\begin{array}{c}
\eta^C_{M} \otimes \Delta_M
\end{array}
\begin{array}{c}
i.e., \quad (\eta^C_{M} \otimes \Delta_M \circ (I_C \otimes \Delta_M)) = (I_C \otimes \Delta_M) \circ \eta^C_{M}.
\end{array}
Thus for left dg-modules \((M, \gamma_M)\) and \((M’, \gamma_{M’})\) over \(\Omega\), we have

\[
\zeta(\eta^C)_M \otimes_{\Delta_C} \zeta(\eta^C)_{M’} = (\iota_C \otimes \gamma_M \otimes \gamma_{M’}) \circ (\iota_{C \otimes \Omega} \otimes \tau \otimes \iota_{M’})
\]

\[
\circ \left( (\eta^C_{\Omega} \otimes_{\Delta_C} \eta^C_{M_1}) \circ (\iota_C \otimes \Delta_M) \circ (\iota_C \otimes u_M) \right) \circ \iota_{M \otimes M’}
\]  

\[
= (\iota_C \otimes \gamma_M \otimes \gamma_{M’}) \circ (\iota_{C \otimes \Omega} \otimes \tau \otimes \iota_{M’})
\]

\[
\circ \left( (\iota_C \otimes \Delta_M) \circ \eta^C_{\Omega} \circ (\iota_C \otimes u_M) \right) \circ \iota_{M \otimes M’}
\]

\[
= (\iota_C \otimes \gamma_M \otimes \gamma_{M’}) \circ \left( (\eta^C_{\Omega} \circ (\iota_C \otimes u_M)) \otimes \iota_{M \otimes M’} \right) \circ (\iota_C \otimes \iota_M \otimes \iota_{M’})
\]

\[
= \zeta(\eta^C)_{M \otimes_{\Delta_C} M’}.
\]

Moreover, from Lemma 4.9(b), the counit \(\epsilon_{\Omega} : (\Omega^*, \gamma_{\Omega’}) \rightarrow (k, \gamma_k)\) is also a morphism of left dg-modules over \(\Omega\). Since \(\eta^C\) is a tensor natural transformation, we have

\[
\eta^{C}_{\Omega} 
\]

\[
\eta^{C}_{\Omega} \circ \iota_C \otimes \epsilon_{\Omega} = (\iota_C \otimes \epsilon_{\Omega}) \circ \eta^C_{\Omega}. 
\]

Therefore we have \(\zeta(\eta^C)_k = (\iota_C \otimes \epsilon_{\Omega}) \circ \eta^C_{\Omega} \circ (\iota_C \otimes u_M) = (\iota_C \otimes \epsilon_{\Omega}) \circ (\iota_C \otimes u_M) = \eta^C_{\Omega} \circ (\iota_C \otimes u_M)\). This shows \(\zeta(\eta^C) \in Z_{\Omega}(\omega)\).

Finally, we show that \(\zeta(\eta^C)\) is the left inverse of \(\eta^C\). Lemma 4.9(c) states that for each left dg-module \((M, \gamma_M)\) over \(\Omega\), the action map \(\gamma_{M} : (\Omega^* \otimes M, \gamma_{\Omega’ \otimes \Delta_C} M) \rightarrow (M_*, \gamma_M)\) is a morphism of left dg-modules over \(\Omega\). Since \(\eta^C\) is a tensor natural transformation, we have

\[
\eta^{C}_{\Omega} \circ (\iota_C \otimes \gamma_M) \circ (\eta^C_{\Omega} \otimes_{\Delta_C} \eta^C_{M}) = \eta^C_{M} \circ (\iota_C \otimes \gamma_M).
\]

Note that \(\eta^C_{M} = \eta^C_{\Omega}\). Indeed, by Lemma 4.8, the action map \(\gamma_{M} : (\Omega \otimes M, m_{\Omega} \otimes 1_M) \rightarrow (M_*, \gamma_M)\) and the counit \(\epsilon_{\Omega} : (\Omega, m_{\Omega}) \rightarrow (k, \gamma_k)\) are morphisms of left dg-modules over \(\Omega\). Since \(\eta^C\) is a tensor natural transformation, the following diagrams commute:

\[
\begin{align*}
\begin{array}{ccc}
C \otimes \Omega & \xrightarrow{\iota_C \otimes \gamma_M} & C \otimes M \\
\eta^C_{\Omega} & = & \eta^C_{\Omega} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
\eta^C_{\Omega} & = & \eta^C_{\Omega} \\
\eta^C_{\Omega} & = & \eta^C_{\Omega} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
C \otimes \Omega & \xrightarrow{\iota_C \otimes \epsilon_{\Omega}} & C \otimes k \\
\eta^C_{\Omega} & = & \eta^C_{\Omega} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
\eta^C_{\Omega} & = & \eta^C_{\Omega} \\
\eta^C_{\Omega} & = & \eta^C_{\Omega} \\
\end{array}
\end{align*}
\]
The equality on the left diagram is due to Lemma 5.3. Therefore, we have

\[\eta^C_M = \eta^C_M \circ (I_C \otimes \gamma_M) \circ (I_C \otimes u_{M} \otimes I_M) \circ (I_C \otimes i_{M}^{-1})\]

\[= (I_C \otimes \gamma_M) \circ (I_C \otimes u_{M} \otimes I_M) \circ (I_C \otimes i_{M}^{-1})\]

\[= (I_C \otimes i_{M}) \circ (I_C \otimes \epsilon_{M} \otimes I_M) \circ (I_C \otimes u_{M} \otimes I_M) \circ (I_C \otimes i_{M}^{-1})\]

\[= (I_C \otimes i_{M}) \circ (I_C \otimes \epsilon_{M} \otimes I_M) \circ (I_C \otimes i_{M}^{-1}) = \eta^C_M.\]

Using eq. (5.2), we finally prove that \(\zeta(\eta^C) \circ \eta^C = I^C:\)

\[\zeta(\eta^C)_M \circ \eta^C_M = \zeta(\eta^C)_M \circ \tilde{\eta}(\eta^C_M)\]

\[= \zeta(\eta^C)_M \circ (I_C \otimes \tilde{\eta}(\eta^C_M)) \circ (\Delta_C \otimes I_M)\]

\[= (I_C \otimes \gamma_M) \circ (I_C \otimes \eta^C_M) \circ (I_C \otimes u_{M} \otimes I_M) \circ (I_C \otimes i_{M}^{-1})\]

\[= (I_C \otimes \gamma_M) \circ (I_C \otimes i_{M}) \circ (I_C \otimes \epsilon_{M} \otimes I_M) \circ (I_C \otimes i_{M}^{-1})\]

\[= (I_C \otimes \gamma_M) \circ (I_C \otimes i_{M}) = \eta^C_M.\]

We conclude that \(\mathcal{P}_w^\text{Aut}(C) = Z_0 \text{End}^\omega(C \otimes \omega) = Z_0 \text{Aut}^\omega(C \otimes \omega)\) is a group, since every monoid with all left inverses is a group. \(\square\)

The following lemma shows that the above construction is functorial.

**Lemma 5.4.** We have a presheaf of groups \(\mathcal{P}_w^\omega : \text{cdcgC}(k) \to \text{Grp}\) on the category \(\text{cdcgC}(k)\) of cdg-coalgebras, sending

- each cdg-coalgebra \(C\) to the group \(\mathcal{P}_w^\omega(C)\), and
- each morphism \(f : C \to C'\) of cdg-coalgebras to a homomorphism \(\mathcal{P}_w^\omega(f) : \mathcal{P}_w^\omega(C') \to \mathcal{P}_w^\omega(C)\) of groups defined by \(\mathcal{P}_w^\omega(f) := E_w(f)\).

**Proof.** It suffices to check that for every morphism \(f : C \to C'\) of cdg-coalgebras, we have \(E_w(f)(\eta^C) \in Z_0 \text{End}^\omega(C \otimes \omega)\) whenever \(\eta^C \in Z_0 \text{End}^\omega(C' \otimes \omega)\), i.e.,

\[E_w(f)(\eta^C) \in Z_0 \text{End}^\omega(C \otimes \omega)\];

\[E_w(f)(\eta^C)_k = I^C_k;\]

\[E_w(f)(\eta^C)[M_{i,j}, M'] = E_w(f)(\eta^C)_M \otimes \Delta_C E_w(f)(\eta^C)_{M'} \text{ for all left dg-modules (}M, \gamma_M\) and (}M', \gamma_{M'}\) over \(\Omega).\]

Property (1) is obvious since \(E_w(f)\) is a chain map. Property (2) follows from \(\eta^C_k = I^C_k\) and \(\epsilon_C \circ f = \epsilon_C\), since we have

\[E_w(f)(\eta^C)_k = (I_C \otimes m_k) \circ (I_C \otimes (\epsilon_C \otimes I_k) \circ \eta^C_k \circ (f \otimes I_k)) \circ (\Delta_C \otimes I_k) = I^C_k.\]
Note that Property (3) is equivalent to the condition

\[ \tilde{\eta}(\mathcal{E}_\omega(f)(\eta^C))_{M_{\otimes_{\Delta^C}}M'} = \tilde{\eta}(\mathcal{E}_\omega(f)(\eta^C))_{M_{\otimes_{\Delta^C}}} \mathcal{E}_\omega(f)(\eta^C)_{M'}. \]

which can be checked as follows:

\[ \tilde{\eta}(\mathcal{E}_\omega(f)(\eta^C))_{M_{\otimes_{\Delta^C}}} \mathcal{E}_\omega(f)(\eta^C)_{M'} \]

\[ = \left( \tilde{\eta}(\mathcal{E}_\omega(f)(\eta^C))_{M_{\otimes_{\Delta^C}}} \mathcal{E}_\omega(f)(\eta^C)_{M'} \right) \circ (I_C \otimes \tau \otimes I_{M'} \otimes (\Delta_C \otimes \mathbb{I}_{M_{\otimes_{\Delta^C}}M'}) ) \]

\[ = \left( \tilde{\eta}(\mathcal{E}_\omega(f)(\eta^C))_{M_{\otimes_{\Delta^C}}} \mathcal{E}_\omega(f)(\eta^C)_{M'} \right) \circ (f \otimes \mathbb{I}_M \otimes \tau \otimes I_{M'} \otimes (\Delta_C \otimes \mathbb{I}_{M_{\otimes_{\Delta^C}}M'}) ) \]

\[ = \left( \tilde{\eta}(\mathcal{E}_\omega(f)(\eta^C))_{M_{\otimes_{\Delta^C}}} \mathcal{E}_\omega(f)(\eta^C)_{M'} \right) \circ (f \otimes I_{M_{\otimes_{\Delta^C}}M'} ) \]

\[ = \tilde{\eta}(\mathcal{E}_\omega(f)(\eta^C))_{M_{\otimes_{\Delta^C}}M'}, \]

where we have used \((f \otimes \tau) \circ \Delta_C = \Delta_C \circ f\) on the 3rd equality. \( \square \)

Recall that \(\text{Hom}(C, \Omega)\) has a structure of dg-algebra. We shall construct an isomorphism \(\text{Hom}(C, \Omega) \cong \text{End}(C \otimes \omega)\) of dg-algebras that is natural in \(C \in \text{cdcdgC}(k)\). Recall the set \(\text{Hom}_{\text{cdcdgC}(k)}(C, \Omega)\) form the group \(P_{\Omega}(C)\). We shall construct an isomorphism \(P_{\Omega}(C) \cong P^\otimes_{\omega}(C)\) of groups functorially for every cdcdg-coalgebra \(C\). Then, we shall construct a presheaf of groups \(P^\otimes_{\omega} : \text{cdcdgC}(k) \rightarrow \text{Grp}\), which is isomorphic to the representable presheaf \(P_{\Omega} : \text{cdcdgC}(k) \rightarrow \text{Grp}\) of groups.

Remind that \(P_{\Omega}(C)\) is the group formed by the set \(\text{Hom}_{\text{cdcdgC}(k)}(C, \Omega)\) of homotopy types of elements in \(\text{Hom}_{\text{cdcdgC}(k)}(C, \Omega)\). Likewise, we need to define homotopy types of elements in \(Z_0\text{End}^\otimes(C \otimes \omega)\)-taking homology classes is not compatible with the tensor condition eq. (5.2): let \(\eta^C \in Z_0\text{End}^\otimes(C \otimes \omega)\) and \(\tilde{\eta}^C = \eta^C + \delta^C \lambda^C\) for some \(\lambda^C \in \text{End}(C \otimes \omega)\) of degree 1, then \(\eta^C\) and \(\tilde{\eta}^C\) belong to the same homology class but \(\tilde{\eta}^C\), in general, is not a tensor natural transformation.

**Definition 5.1.** A homotopy pair on \(Z_0\text{End}^\otimes(C \otimes \omega)\) is a pair \((\eta(t)^C, \lambda(t)^C)\) of \(\eta^C(t) \in \text{End}(C \otimes \omega)\), \(t\), and \(\lambda(t)^C \in \text{End}(C \otimes \omega)\), \(t\), where \(t\) is a polynomial time variable of degree 0, satisfying the homotopy flow equation \(\frac{d}{dt} \eta(t)^C = \delta^C \lambda(t)^C\) generated by \(\eta(t)^C\) subject to the following conditions:

\[
\eta(0)^C \in Z_0\text{End}^\otimes(C \otimes \omega), \quad \left\{ \begin{array}{l}
\lambda(t)^C = 0, \\
\lambda(t)^C \otimes_{\Delta^C} \eta(t)^C_{M} = \lambda(t)^C \otimes_{\Delta^C} \eta(t)^C_{M} + \eta(t)^C_{M} \otimes_{\Delta^C} \lambda(t)^C_{M}'.
\end{array} \right.
\]

Let \((\eta(t)^C, \lambda(t)^C)\) be a homotopy pair on \(Z_0\text{End}^\otimes(C \otimes \omega)\). Then, we have \(\eta(t)^C = \eta(0)^C + \delta^C \int_0^t \lambda(s)^C ds\), and \(\delta^C \eta(t)^C = 0\) since \(\delta^C \eta(0)^C = 0\). By applying Lemma 5.2,
we also have
\[
\frac{d}{dt}\left(\eta(t)_{\mathcal{M}^t_{\mathcal{M}^t}} - \eta(t)_{\mathcal{M}^t_{\mathcal{M}^t}} \otimes_{\mathcal{C}} \eta(t)_{\mathcal{M}^t_{\mathcal{M}^t}}\right) = \delta^C \left(\lambda(t)_{\mathcal{M}^t_{\mathcal{M}^t}} - \lambda(t)_{\mathcal{M}^t_{\mathcal{M}^t}} \otimes_{\mathcal{C}} \eta(t)_{\mathcal{M}^t_{\mathcal{M}^t}}\right) - \eta(t)_{\mathcal{M}^t_{\mathcal{M}^t}} \otimes_{\mathcal{C}} \lambda(t)_{\mathcal{M}^t_{\mathcal{M}^t}} = 0,
\]
so that \(\eta(t)_{\mathcal{M}^t_{\mathcal{M}^t}} = \eta(0)_{\mathcal{M}^t_{\mathcal{M}^t}} \otimes_{\mathcal{C}} \eta(t)_{\mathcal{M}^t_{\mathcal{M}^t}}\) for all \(t\) since \(\eta(0)_{\mathcal{M}^t_{\mathcal{M}^t}} = \eta(0)_{\mathcal{M}^t_{\mathcal{M}^t}} \otimes_{\mathcal{C}} \eta(0)_{\mathcal{M}^t_{\mathcal{M}^t}}\). Therefore \(\eta(t)^C\) is a family of elements in \(Z_0 \text{End}^\circ([C \otimes \mathcal{O}])\). Then, we declare that \(\eta(1)^C\) is homotopic to \(\eta(0)^C\) by the homotopy \(\int_0^1 \lambda(t)^C dt\), and denote \(\eta(0)^C \sim \eta(1)^C\), which is clearly an equivalence relation. In other words, two elements \(\eta^C\) and \(\tilde{\eta}^C\) in the set \(Z_0 \text{End}^\circ([C \otimes \mathcal{O}])\) are homotopic, \(\eta^C \sim \tilde{\eta}^C\), if there is a homotopy pair connecting them (by the time 1 map). Then, we also say that \(\eta^C\) and \(\tilde{\eta}^C\) have the same homotopy type, and denote it as \([\eta^C] = [\tilde{\eta}^C]\).

Let \(hoZ_0 \text{End}^\circ([C \otimes \mathcal{O}])\) be the set of homotopy types of elements in \(Z_0 \text{End}^\circ([C \otimes \mathcal{O}])\). It is a routine check that \(\eta'^C \circ \eta^C \sim \eta'^C \circ \tilde{\eta}^C \in Z_0 \text{End}^\circ([C \otimes \mathcal{O}])\) whenever \(\eta'^C \sim \eta'^C\), \(\eta^C \sim \tilde{\eta}^C \in Z_0 \text{End}^\circ([C \otimes \mathcal{O}])\) and the homotopy type of \(\eta'^C \circ \eta^C\) depends only on the homotopy types of \(\eta^C\) and \(\tilde{\eta}^C\). Therefore we have well-defined associative composition \([\eta^C] \circ \tilde{\eta}^C\) := \([\eta'^C \circ \eta^C]\). This shows that \((hoZ_0 \text{End}^\circ([C \otimes \mathcal{O}]), [\cdot^C], \circ)\) is a group.

**Lemma 5.5.** We have a presheaf of groups \(\mathcal{P}^\circ_\omega : hccdg_{C(k)} \to \text{Grp}\) on the homotopy category \(hccdg_{C(k)}\) of ccdg-coalgebras, sending

- each ccdg-coalgebra \(C\) to the group \(\mathcal{P}^\circ_\omega(C) := hoZ_0 \text{End}^\circ([C \otimes \mathcal{O}]), [\cdot^C], \circ\), and

- each morphism \(f : C \to C'\) of dg-coalgebras to the group homomorphism \(\mathcal{P}^\circ_\omega([f]) : \mathcal{P}^\circ_\omega(C') \to \mathcal{P}^\circ_\omega(C)\) defined by

\[
\mathcal{P}^\circ_\omega([f])(\eta^C) := \left[\mathcal{P}^\circ_\omega(f)(\eta^C)\right].
\]

**Proof.** All we need to show is that \(\mathcal{P}^\circ_\omega(f)(\eta^C) \sim \mathcal{P}^\circ_\omega(f)(\tilde{\eta}^C) \in Z_0 \text{End}^\circ([C \otimes \mathcal{O}])\) whenever \(f \sim \tilde{f} \in \text{Hom}_{ccdg_{C(k)}}(C, C')\) and \(\eta^C \sim \tilde{\eta}^C \in Z_0 \text{End}^\circ([C' \otimes \mathcal{O}])\). It suffices to show the following statement: Let \((f(t), s(t))\) be a homotopy pair on \(\text{Hom}_{ccdg_{C(k)}}(C, C')\) and \((\eta(t)^C, \lambda(t)^C)\) be a homotopy pair on \(Z_0 \text{End}^\circ([C' \otimes \mathcal{O}])\). Then the pair

\[
\left(\xi(t)^C := \mathcal{E}_\omega(f(t))([\eta(t)^C]), \chi(t)^C := \mathcal{E}_\omega(s(t))(\lambda(t)^C) + \mathcal{E}_\omega(f(t))(\eta(t)^C)\right)
\]

is a homotopy pair on \(Z_0 \text{End}^\circ([C \otimes \mathcal{O}])\) that the pair \((\xi(t)^C, \chi(t)^C)\) has the following properties:

1. \(\frac{d}{dt}\xi(t)^C = \delta^C \chi(t)^C\);
2. \(\xi(0)^C \in Z_0 \text{End}^\circ([C \otimes \mathcal{O}]);\)
(3) $\chi(t)_C^C = 0$;

(4) $\chi(t)_{M \oplus \gamma M'} = \chi(t)_M \otimes_{\Delta C} \xi(t)_M + \chi(t)_{M'} \otimes_{\Delta C} \chi(t)_{M'}^C$ for all left dg-modules $(M, \gamma_M)$ and $(M', \gamma_{M'})$ over $\Omega$.

For Property (1), let $\xi(t)_M^C$ be the component of $\xi(t)^C$ at a left dg-module $(M, \gamma_M)$ over $\Omega$. Then we have

$$\frac{d}{dt} \xi(t)_M^C = \frac{d}{dt} \tilde{\varphi} \left( \iota_M \circ (\epsilon_{C'} \otimes \mathbb{I}_M) \circ \eta(t)_M^C \circ (f(t) \otimes \mathbb{I}_M) \right)$$

$$= \tilde{\varphi} \left( \iota_M \circ (\epsilon_{C'} \otimes \mathbb{I}_M) \circ \partial_{C' \otimes M, C' \otimes M} (\lambda(t)_M^C \circ (f(t) \otimes \mathbb{I}_M)) \right)$$

$$+ \tilde{\varphi} \left( \iota_M \circ (\epsilon_{C'} \otimes \mathbb{I}_M) \circ \eta(t)_M^C \circ (\partial_{C, C'} s(t) \otimes \mathbb{I}_M) \right)$$

$$= (\delta^C \chi(t)^C)_M',$$

where we have used $\partial_{C, C'} f(t) = 0$ and $\partial_{C' \otimes M, C' \otimes M} \eta_0 = 0$ for the 3rd equality. Property (2) is obvious since $f(0) : C \rightarrow C'$ is a morphism of ccdg-coalgebras and $\eta_0^C$ is in $Z_0 \text{End}^\otimes(C' \otimes \mathbb{I})$. Property (3) follows from $\lambda(t)_k^C = 0$, $\eta(t)_k^C = \mathbb{I}_k^C$ and $\epsilon_{C'} \circ s(t) = 0$, since we have

$$\chi(t)_k^C = \tilde{\varphi} \left( m_k \circ (\epsilon_{C'} \otimes \mathbb{I}_k) \circ \lambda(t)_k^C \circ (f(t) \otimes \mathbb{I}_k) + m_k \circ (\epsilon_{C'} \otimes \mathbb{I}_k) \circ \eta(t)_k^C \circ (s(t) \otimes \mathbb{I}_k) \right)$$

$$= \tilde{\varphi} \left( m_k \circ (\epsilon_{C'} \otimes \mathbb{I}_k) \circ (s(t) \otimes \mathbb{I}_k) \right) = 0.$$

We note that Property (4) is equivalent to the condition

$$\tilde{\varphi} \left( \chi(t)_M^C \otimes_{\Delta C} \xi(t)_M^C + \xi(t)_M^C \otimes_{\Delta C} \chi(t)_M^C \right),$$

which can be checked as follows. We consider the 1st term in the RHS of eq. (5.3):

$$\tilde{\varphi} \left( \chi(t)_M^C \otimes_{\Delta C} \xi(t)_M^C \right) = \tilde{\varphi} \left( \chi(t)_M^C \otimes \tilde{\varphi} \left( \xi(t)_M^C \right) \right) \circ (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_{M'}) \circ (\Delta_C \otimes \mathbb{I}_{M \oplus M'})$$

$$= \tilde{\varphi} \left( \chi(t)_M^C \right) \circ \tilde{\varphi} \left( \xi(t)_M^C \right) \circ (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_{M'}) \circ (\Delta_C \otimes \mathbb{I}_{M \oplus M'})$$

$$+ \tilde{\varphi} \left( \lambda(t)_M^C \right) \circ \tilde{\varphi} \left( \eta(t)_M^C \right) \circ (f(t) \otimes \mathbb{I}_{M'}) \circ (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_{M'}) \circ (\Delta_C \otimes \mathbb{I}_{M \oplus M'})$$

$$+ \tilde{\varphi} \left( \partial_{C', C} \gamma(t)_M^C \right) \circ \tilde{\varphi} \left( \eta(t)_M^C \right) \circ (f(t) \otimes \mathbb{I}_{M'}) \circ (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_{M'}) \circ (\Delta_C \otimes \mathbb{I}_{M \oplus M'}).$$
Combining with the similar calculation for the 2nd term in the RHS of eq. (5.3), we obtain that
\[
\begin{aligned}
&\bar{q}\left(\chi(t)^C_\Omega \otimes_{\Delta_C} \xi(t)^C_\Omega + \xi(t)^C_\Omega \otimes_{\Delta_C} \chi(t)^C_\Omega\right) \\
&= \left(\bar{q}\left(\lambda(t)^C_\Omega \otimes \eta(t)^C_\Omega\right) + \bar{q}\left(\eta(t)^C_\Omega \otimes \lambda(t)^C_\Omega\right)\right) \circ (f(t) \otimes I_M \otimes f(t) \otimes I_M) \\
&\circ (\Delta_C \otimes I_{M'}) \circ (\Delta_C \otimes I_{M \otimes M'}) \\
&+ \left(\bar{q}\left(\eta(t)^C_\Omega \otimes \lambda(t)^C_\Omega\right)\right) \circ (s(t) \otimes I_M \otimes f(t) \otimes I_{M'} + f(t) \otimes I_M \otimes s(t) \otimes I_{M'}) \\
&\circ (\Delta_C \otimes I_{M \otimes M'}) \\
&= \left(\bar{q}\left(\lambda(t)^C_\Omega \otimes_{\Delta_C} \eta(t)^C_\Omega + \eta(t)^C_\Omega \otimes_{\Delta_C} \lambda(t)^C_\Omega\right)\right) \circ (f(t) \otimes I_{M \otimes M'}) \\
&\circ (\Delta_C \otimes I_{M \otimes M'}) \circ (s(t) \otimes I_{M \otimes M'}) \\
&= \bar{q}\left(\lambda(t)^C_\Omega \otimes_{\Delta_C} \eta(t)^C_\Omega\right) \circ (f(t) \otimes I_{M \otimes M'}) + \bar{q}\left(\eta(t)^C_\Omega \otimes_{\Delta_C} \lambda(t)^C_\Omega\right) \circ (s(t) \otimes I_{M \otimes M'}) \\
&= \bar{q}\left(\chi(t)^C_\Omega \otimes_{\Delta_C} \eta(t)^C_\Omega\right).
\end{aligned}
\]

In the above, we used \((s(t) \otimes f(t) + f(t) \otimes s(t)) \circ \Delta_C = \Delta_{C'} \circ s(t)\) and \((f(t) \otimes f(t)) \circ \Delta_C = f(t) \otimes f(t) \circ \Delta_C = \Delta_{C'} \circ f(t)\) on the 2nd equality, and used \(\lambda(t)^C_\Omega \otimes_{\Delta_C} \eta(t)^C_\Omega = \eta(t)^C_\Omega \otimes_{\Delta_C} \lambda(t)^C_\Omega\) and \(\eta(t)^C_\Omega \otimes_{\Delta_C} \eta(t)^C_\Omega = \eta(t)^C_\Omega \otimes_{\Delta_{C'}} \eta(t)^C_\Omega\) on the 4th equality. \(\square\)

Now we are ready to state the main theorem in this section.

**Theorem 5.1.** We have a natural isomorphism of presheaves of groups \(\mathcal{P}_\omega^\otimes \cong \mathcal{P}_\Omega : \text{hocc\,cdgC}(k) \to \text{Grp}\) on the homotopy category of cocommutative dg-coalgebras. Equivalently, the presheaf of groups \(\mathcal{P}_\omega^\otimes\) on \(\text{hocc\,cdgC}(k)\) is representable and represented by the cdg-Hopf algebra \(\Omega\).

The remaining part of this section is devoted to the proof of the above theorem, which is divided into several pieces.

**Proposition 5.2.** We have natural isomorphisms of presheaves
\[
\begin{aligned}
\mathcal{E}_\omega^\otimes \cong \mathcal{E}_\Omega : \text{cdgC}(k) \to \text{dgA}(k), \\
\mathcal{P}_\omega^\otimes \cong \mathcal{P}_\Omega : \text{cdgC}(k) \to \text{Grp}.
\end{aligned}
\]
In particular the presheaf of groups \(\mathcal{P}_\omega^\otimes\) on \(\text{cdgC}(k)\) is representable and represented by the cdg-Hopf algebra \(\Omega\).

The proof of this proposition is based on the forthcoming two lemmas. Remind that in Lemma 3.1, we defined the dg-algebra \(\mathcal{E}_\Omega(C) = \{\text{Hom}(C, \Omega), u_\Omega \circ \epsilon_C, \ast_{C, \Omega}, \partial_{C, \Omega}\}\) for every cdg-coalgebra \(C\).
Lemma 5.6. We have an isomorphism \( \tilde{\eta}^C : \mathcal{E}_\Omega(C) \rightarrow \mathcal{E}_\omega(C) \) of dg-algebras for every ccdg-coalgebra \( C \), where

- for each \( \alpha \in \text{Hom}(C, \Omega) \), the component of \( \tilde{\eta}^C(\alpha) \in \text{End}(C \otimes \omega) \) at a left dg-module \( (M, \gamma_M) \) over \( \Omega \) is defined by
  \[
  \tilde{\eta}^C(\alpha)_M := \tilde{p}(\gamma_M \circ (\alpha \otimes 1_M)) = (1_C \otimes \gamma_M) \circ (1_C \otimes \alpha \otimes 1_M) \circ (\Delta_C \otimes 1_M) : C \otimes M \rightarrow C \otimes M.
  \]

- for each \( \eta^C \in \text{End}(C \otimes \omega) \), the linear map \( \tilde{g}^C(\eta^C) \in \text{Hom}(C, \Omega) \) is defined by
  \[
  \tilde{g}^C(\eta^C) := \hat{q}(\eta^C_{1\Omega}) \circ (1_C \otimes u_\Omega) \circ j_C^{-1} = I_\Omega \circ (\epsilon_C \otimes 1_\Omega) \circ \eta^C_{1\Omega} \circ (1_C \otimes u_\Omega) \circ j_C^{-1} : C \rightarrow \Omega.
  \]

Proof. The map \( \tilde{g}^C \) is well-defined, since \( \tilde{g}^C(\eta^C) \) is obviously a \( k \)-linear map. The map \( \tilde{\eta}^C \) is also well-defined. This is because for every morphism \( \psi : (M, \gamma_M) \rightarrow (M', \gamma'_{M'}) \) of left dg-modules over \( \Omega \), the following commutative diagram

\[
\begin{array}{ccc}
C \otimes M & \xrightarrow{\Delta_C \otimes 1_M} & C \otimes C \otimes M \\
| I_C \otimes \psi & & | I_C \otimes \psi \\
C \otimes M' & \xrightarrow{\Delta_C \otimes 1_{M'}} & C \otimes C \otimes M'
\end{array}
\]

implies that \( \tilde{\eta}^C(\alpha) \) is a natural transformation. We first prove that \( \tilde{g}^C \) and \( \tilde{\eta}^C \) are inverse to each other.

- \( \tilde{g}^C(\tilde{\eta}^C(\alpha)) = \alpha \) holds for all \( \alpha \in \text{Hom}(C, \Omega) \):
  \[
  \tilde{g}^C(\tilde{\eta}^C(\alpha)) = \hat{q}(\tilde{\eta}^C(\alpha)_{1\Omega}) \circ (1_C \otimes u_\Omega) \circ j_C^{-1} = \hat{p}(\tilde{\eta}^C(\alpha)_{1\Omega} \circ (1_C \otimes u_\Omega)) \circ (1_C \otimes u_\Omega) \circ j_C^{-1} = m_\Omega \circ (\alpha \otimes 1_\Omega) \circ (1_C \otimes u_\Omega) \circ j_C^{-1} = \alpha.
  \]

- \( \tilde{\eta}^C(\tilde{g}^C(\eta^C)) = \eta^C \) holds for all \( \eta^C \in \text{End}(C \otimes \omega) \): Let \( (M, \gamma_M) \) be a left dg-module over \( \Omega \). Lemma 4.8(a) states that \( \gamma_M : (\Omega \otimes M, m_\Omega \otimes 1_M) \rightarrow (M, \gamma_M) \) is a morphism of left dg-modules over \( \Omega \). Since \( \eta^C \) is a natural transformation, the following diagram commutes

\[
\begin{array}{ccc}
C \otimes \Omega \otimes M & \xrightarrow{1_C \otimes \gamma_M} & C \otimes M \\
\downarrow \eta^C_{1M} \circ 1_M = \hat{q} \circ 1_M & & \downarrow \eta^C_M \circ 1_M = \eta^C_M \\
C \otimes \Omega \otimes M & \xrightarrow{1_C \otimes \gamma_M} & C \otimes M
\end{array}
\]

i.e., \( (1_C \otimes \gamma_M) \circ (\eta^C_{1\Omega} \circ 1_M) = (\eta^C_M \circ (1_C \otimes \gamma_M)) \).

Jaehyeok Lee, Jae-Suk Park
The equality on the diagram is by Lemma 5.3. Thus we have

\[ \tilde{\eta}^C(\tilde{g}^C(\eta^C))_M = (\mathbb{I}_C \otimes \gamma_M) \circ (\mathbb{I}_C \otimes (\mathbb{I}_C \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M)) \circ (\Delta_C \otimes \mathbb{I}_M) \]

\[ = (\mathbb{I}_C \otimes \gamma_M) \circ (\mathbb{I}_C \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \]

\[ = (\mathbb{I}_C \otimes \gamma_M) \circ (\mathbb{I}_C \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \]

\[ = \mathbb{I}_M \circ \eta^C \circ (\mathbb{I}_C \otimes u_C) \circ (\Delta_C \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) = \eta^C. \]

We are left to show that \( \tilde{\eta}^C \) and \( \tilde{g}^C \) are morphisms of dg-algebras. Since they are inverse to each other, it suffices to show that \( \tilde{\eta}^C \) is a morphism of dg-algebras. Clearly, \( \tilde{\eta}^C \) is a \( k \)-linear map of degree 0. Let \( (M, \gamma_M) \) be a left dg-module over \( \Omega \).

- \( \tilde{\eta}^C \) is a chain map, i.e. \( \delta C \circ \tilde{\eta}^C = \tilde{\eta}^C \circ \partial_{C, \Omega} \). Indeed, for \( a \in \text{Hom}(C, \Omega) \),

\[ \delta_C(\tilde{\eta}^C(a))_M = \partial_{C \otimes M, C \otimes M}((\mathbb{I}_C \otimes \gamma_M) \circ (\mathbb{I}_C \otimes a \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M)) \]

\[ = (\mathbb{I}_C \otimes \gamma_M) \circ (\mathbb{I}_C \otimes \partial_{C \otimes \Omega} \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \]

\[ = \tilde{\eta}^C(\partial_{C \otimes \Omega} \mathbb{I}_M). \]

The 2nd equality follows from the properties \( \partial_{R \otimes M, M} \gamma_M = 0 \) and \( \partial_{C \otimes \Omega} \Delta_C = 0 \).

- \( \tilde{\eta}^C \) sends the identity to the identity, i.e. \( \tilde{\eta}^C(u_C \circ \epsilon_C) = \mathbb{I}_M \):

\[ \tilde{\eta}^C(u_C \circ \epsilon_C)_M := (\mathbb{I}_C \otimes \gamma_M) \circ (\mathbb{I}_C \otimes (u_C \circ \epsilon_C) \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) = \mathbb{I}_C \otimes \mathbb{I}_M = \mathbb{I}_M. \]

- \( \tilde{\eta}^C \) preserves the binary operations, i.e. \( \tilde{\eta}^C(a_1 \ast_{C, \Omega} a_2) = \tilde{\eta}^C(a_1) \circ \tilde{\eta}^C(a_2) \) for all \( a_1, a_2 \in \text{Hom}(C, \Omega) \):

\[ \tilde{\eta}^C(a_1 \ast_{C, \Omega} a_2)_M := (\mathbb{I}_C \otimes \gamma_M) \circ (\mathbb{I}_C \otimes (m_C \circ (a_1 \circ a_2) \otimes \Delta_C) \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \]

\[ = (\mathbb{I}_C \otimes \gamma_M) \circ (\mathbb{I}_C \otimes \partial_{C \otimes \Omega} \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \circ (\mathbb{I}_C \otimes \partial_{C \otimes \Omega} \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M) \]

\[ = \tilde{\eta}^C(a_1)_M \circ \tilde{\eta}^C(a_2)_M. \]

The 2nd equality is due to the coassociativity of \( \Delta_C \) and the action axiom of \( \gamma_M \).

\[ \Box \]

In Lemma 3.2, we showed that \( P_{\Omega}(C) = \{ \text{Hom}_{cddg}(C, \Omega), u_{\Omega} \circ \epsilon_C, \ast_{C, \Omega} \} \) is a group for every cddg-coalgebra \( C \). The inverse of \( g \in \text{Hom}_{cddg}(C, \Omega) \) is given by \( g^{-1} := \epsilon_{\Omega} \circ g \). Remind that \( \text{Hom}_{cddg}(C, \Omega) \) is the subset of Hom\((C, \Omega)\) consisting the morphisms of cddg-coalgebras:

\[ \text{Hom}_{cddg}(C, \Omega) = \{ g \in \text{Hom}(C, \Omega) \mid \partial_{C, \Omega} g = 0, \Delta_{\Omega} g = (g \otimes g) \circ \Delta_C,\ \epsilon_{\Omega} \circ g = \epsilon_C \}. \]
Lemma 5.7. For every ccdg-coalgebra \( C \), the isomorphism in Lemma 5.6 gives an isomorphism \( \tilde{\eta}^C : \mathcal{P}_C(C) \rightarrow \mathcal{P}_{\mathcal{C}g}(C) : \tilde{g}^C \) of groups.

Proof. We only need to show two things: \( \tilde{g}^C \in Z_0 \text{End}^\otimes(C \otimes \omega) \subset \text{Hom}_{\text{ccdg}C[k]}(C, \Omega) \) and \( \tilde{\eta}^C \in \text{Hom}_{\text{ccdg}C[k]}(C, \Omega) \subset Z_0 \text{End}^\otimes(C \otimes \omega) \).

1. For \( \eta^C \in Z_0 \text{End}^\otimes(C \otimes \omega) \) we have \( \tilde{g}^C(\eta^C) \in \text{Hom}_{\text{ccdg}C[k]}(C, \Omega) \).

- \( \tilde{g}^C(\eta^C) \) is of degree 0 and satisfies \( \tilde{\delta}^C(\eta^C) = 0 \): This is immediate since \( \eta^C \) is of degree 0 with \( \tilde{\delta}^C \eta^C = 0 \), and \( \tilde{g}^C \) is a chain map by Lemma 5.6.

- \( \epsilon^\Omega \circ \tilde{g}^C(\eta^C) = \epsilon^C \): Lemma 4.8(b) states that \( \epsilon^\Omega : (\Omega, m_\Omega) \rightarrow (k, \gamma_k) \) is a morphism of left dg-modules over \( \Omega \). Since \( \eta^C \) is a tensor natural transformation, the following diagram commutes:

\[
\begin{array}{ccc}
C \otimes \Omega & \xrightarrow{\eta^C} & C \otimes k \\
\downarrow{\eta^C_\Omega} & & \downarrow{\eta^C \otimes \epsilon_k} \\
C \otimes \Omega & \xrightarrow{1_C \otimes \epsilon_\Omega} & C \otimes k
\end{array}
\]

i.e., \( (1_C \otimes \epsilon_\Omega) \circ \eta^C_\Omega = \eta^C \circ \epsilon_\Omega \).

Therefore we have
\[
\epsilon^\Omega \circ \tilde{g}^C(\eta^C) = m_k \circ (\epsilon^C \otimes \epsilon_k) \circ (\eta^C \otimes \epsilon_\Omega) \circ \eta^C_\Omega \circ (1_C \otimes u_\Omega) \circ J^{-1}_C
\]
\[
= m_k \circ (\epsilon^C \otimes \epsilon_k) \circ (\eta^C \otimes \epsilon_\Omega) \circ (1_C \otimes u_\Omega) \circ J^{-1}_C
\]

\( \Delta_\Omega \circ \tilde{g}^C(\eta^C) \) is a morphism of left dg-modules over \( \Omega \). Since \( \eta^C \) is a tensor natural transformation, the following diagram commutes:

\[
\begin{array}{ccc}
C \otimes \Omega & \xrightarrow{\eta^C_\Omega} & C \otimes \Omega \otimes \Omega \\
\downarrow{\eta^C_\Omega} & & \downarrow{\eta^C_\Omega \otimes \epsilon_\Omega} \\
C \otimes \Omega & \xrightarrow{1_C \otimes \Delta_\Omega} & C \otimes \Omega \otimes \Omega
\end{array}
\]

i.e., \( (1_C \otimes \epsilon_\Omega) \circ \eta^C_\Omega = \eta^C \otimes \epsilon_\Omega \otimes \epsilon_\Omega \circ \eta^C_\Omega \circ (1_C \otimes u_\Omega) \circ J^{-1}_C \).

Therefore we get
\[
\Delta_\Omega \circ \tilde{g}^C(\eta^C) = 1_{\Omega \otimes \Omega} \circ (\epsilon^C \otimes 1_{\Omega \otimes \Omega}) \circ (\eta^C_\Omega \otimes \eta^C_\Omega) \circ (1_C \otimes u_\Omega) \circ J^{-1}_C
\]
\[
= 1_{\Omega \otimes \Omega} \circ (\epsilon^C \otimes 1_{\Omega \otimes \Omega}) \circ (\eta^C_\Omega \otimes \eta^C_\Omega) \circ (1_C \otimes u_\Omega) \circ J^{-1}_C
\]

- \( \tilde{\eta}^C(g) \) is of degree 0 and satisfies \( \tilde{\delta}^C \tilde{\eta}^C(g) = 0 \): This is immediate, since \( g \) is of degree 0 with \( \tilde{\delta}^C g = 0 \), and \( \tilde{\eta}^C \) is a chain map by Lemma 5.6.
Let $\tilde{\eta}^C : \mathcal{E}_\Omega(C) \to \mathcal{E}_\omega(C)$ be the natural transformation, which is the inverse of $\eta^C$. Let $g : C' \to C$ be a morphism of cdg-coalgebras. We need to show that the following diagram commutes,

\[
\begin{array}{ccc}
\mathcal{E}_\Omega(C') & \xrightarrow{\eta^C} & \mathcal{E}_\omega(C') \\
\mathcal{E}_\Omega(f) \downarrow & & \downarrow \mathcal{E}_\Omega(f) \\
\mathcal{E}_\Omega(C) & \xrightarrow{\tilde{\eta}^C} & \mathcal{E}_\omega(C),
\end{array}
\]

i.e., $\mathcal{E}_\omega(f) \circ \tilde{\eta}^C = \eta^C \circ \mathcal{E}_\Omega(f)$.

Let $g : C' \to \Omega$ be a linear map and $(M, \gamma_M)$ be a left dg-module over $\Omega$. Then

\[
\tilde{\eta}(\mathcal{E}_\omega(f)(\tilde{\eta}^C(g)_M)) = \tilde{\eta}(\mathcal{E}_\omega(g \circ f)_M) = \mathcal{E}_\omega(g \circ f)_M = \gamma_M \circ (g \circ f)_M = \gamma_M \circ (g \circ f)_M.
\]

Therefore $(\mathcal{E}_\omega(f) \circ \tilde{\eta}^C)(g) = (\eta^C \circ \mathcal{E}_\Omega(f))(g)$ holds for all $g : C' \to \Omega$. \qed
We end this paper with the proof of Theorem 5.1.

**Proof (Theorem 5.1).** By Proposition 5.2 and the definitions of $\mathcal{P}_\Omega$ and $\mathcal{P}_\omega^\circ$, it is sufficient to show that for every ccdg-coalgebra $C$

(a) $\overline{\eta}^C$ sends a homotopy pair $\{g(t), \chi(t)\}$ on $\text{Hom}_{\text{ccdC}}[k](C, \Omega)$ to a homotopy pair $\overline{\eta}^C(g(t)), \overline{\eta}^C(\chi(t))$ on $\mathbb{Z}_0\text{End}^\circ(C \otimes \omega)$, and

(b) $\overline{g}^C$ sends a homotopy pair $\{\eta(t)^C, \lambda(t)^C\}$ on $\mathbb{Z}_0\text{End}^\circ(C \otimes \omega)$ to a homotopy pair $\overline{g}^C(\eta(t)^C), \overline{g}^C(\lambda(t)^C)$ on $\text{Hom}_{\text{ccdC}}[k](C, \Omega)$.

Then $\overline{\eta}^C$ and $\overline{g}^C$ will give an isomorphism of groups $\mathcal{P}_\omega^\circ(C) \cong \mathcal{P}_\Omega(C)$. Moreover, this isomorphism is natural in $C \in \text{ccdC}[k]$ by Proposition 5.2 and Lemma 5.5.

We will prove (a) only since the proof of (b) is similar. We need to check the pair $\overline{\eta}^C(g(t)), \overline{\eta}^C(\chi(t))$ has the following properties.

1. $\frac{d}{dt}\overline{\eta}^C(g(t)) = \delta^C \overline{\eta}^C(\chi(t))$,
2. $\overline{\eta}^C(\chi(t)) = 0,$ and
3. $\overline{\eta}^C(\chi(t)) = 0,$ and
4. $\overline{\eta}^C(\chi(t)) = 0,$ and

where the last equality should hold for every left dg-modules $(M, \gamma_M)$ and $(M', \gamma_{M'})$ over $\Omega$. Property (1) follows from the condition $\frac{d}{dt} \overline{\eta}^C(g(t)) = \delta^C \overline{\eta}^C(\chi(t))$, since $\frac{d}{dt} \overline{\eta}^C(g(t)) = \overline{\eta}^C(\delta^C \overline{\eta}^C(\chi(t)))$. Property (2) follows from the condition $\overline{g}(0) \in \text{Hom}_{\text{ccdC}}[k](C, \Omega)$. Property (3) follows from the condition $\epsilon_\Omega \circ \chi(t) = 0$, since we have $\overline{\eta}^C(\chi(t)) = (\mathbb{I}_C \otimes m_k) \circ (\mathbb{I}_C \otimes (\epsilon_\Omega \circ \chi(t)) \otimes \mathbb{I}_k) \circ (\Delta_C \otimes \mathbb{I}_k)$. Finally we check that Property (4) is a consequence of the condition $\overline{\eta}^C(\chi(t) \otimes g(t) + g(t) \otimes \chi(t)) \circ \delta^C = 0$. We note that Property (4) is equivalent to the identity

$$\overline{\eta}^C(\chi(t)) \otimes \Delta_C \overline{\eta}^C(g(t)) = \overline{\eta}^C(\chi(t)) \otimes \Delta_C \overline{\eta}^C(g(t)),$$

which can be checked as follows. We begin with the 1st term in the LHS of eq. (5.4):

$$\overline{\eta}^C(\chi(t)) \otimes \Delta_C \overline{\eta}^C(g(t))$$

$$= \overline{\eta}^C(\chi(t)) \otimes \Delta_C \overline{\eta}^C(g(t)) \circ (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M \otimes M')$$

$$= \overline{\eta}^C(\chi(t)) \otimes \Delta_C \overline{\eta}^C(g(t)) \circ (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M \otimes M')$$

$$= \overline{\eta}^C(\chi(t)) \otimes \Delta_C \overline{\eta}^C(g(t)) \circ (\mathbb{I}_C \otimes \tau \otimes \mathbb{I}_M) \circ (\Delta_C \otimes \mathbb{I}_M \otimes M').$$
After the similar calculation for the 2nd term in the LHS of eq. (5.4), we obtain that
\[ \tilde{\eta}^C \left( \chi(t) \right)_M \otimes_{\Lambda_C} \eta^C \left( g(t) \right)_{M'} + \tilde{\eta}^C \left( g(t) \right)_M \otimes_{\Lambda_C} \eta^C \left( \chi(t) \right)_{M'} \]
\[ = (\gamma_M \otimes \gamma_{M'}) \circ (\Omega \otimes \tau \otimes I_{M'}) \circ (\chi(t) \otimes g(t) \otimes \chi(t) \otimes \Omega_{M' \otimes M'}) \circ (\Delta_C \otimes I_{M' \otimes M'}) \]
\[ = (\gamma_M \otimes \gamma_{M'}) \circ (\Omega \otimes \tau \otimes I_{M'}) \circ (\Delta_C \otimes I_{M' \otimes M'}) \circ (\chi(t) \otimes I_{M' \otimes M'}) \]
\[ = \gamma_{M' \otimes M} \circ (\chi(t) \otimes I_{M' \otimes M'}) = \tilde{\eta}^C \left( \chi(t) \right)_{M' \otimes M} \]
\[ \square \]

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