Pair production in a strong magnetic field: the effect of a weak background gravitational field

Antonino Di Piazza*

Dipartimento di Fisica Teorica,
Strada Costiera 11, Trieste, I-34014, Italy
and
INFN, Sezione di Trieste, Italy

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Abstract

The production probability of an $e^- - e^+$ pair in the presence of a strong, uniform and slowly varying magnetic field is calculated by taking into account the presence of a background gravitational field. The curvature of the spacetime metric induced by the gravitational field not only changes the transition probabilities calculated in the Minkowski spacetime but also primes transitions that are strictly forbidden in absence of the gravitational field.
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*E-mail: dipiazza@ts.infn.it
1 Introduction

In view of the persistent interest in the high energy emissions from astrophysical compact objects [1], we present a continuation of the study on the electron-positron pairs production in the presence of strong ($\gg B_{cr} = m^2 c^3/(he) \simeq 4.4 \times 10^{13}$ gauss), time varying magnetic fields [2, 3, 4]. As we have pointed out in our previous papers, the physical situation we have in mind is the creation of $e^- - e^+$ pairs around astrophysical compact objects like neutron stars or black holes that can produce such strong magnetic fields [5, 6]. In this picture, the time variation of the magnetic field is a consequence of the rotation of the compact object or its gravitational collapse following a supernova explosion. Also, the creation of the electrons and of the positrons is only an intermediate step. In fact, our final goal is to give a possible theoretical interpretation of the so-called gamma-ray bursts (GRBs) that are thought to be originated as a consequence of a supernova explosion around such astrophysical compact objects [1]. In our formulation GRBs are pulses of photons produced by the electrons and the positrons created by the time dependent stellar magnetic field through their annihilation [7] and/or as synchrotron radiation [8]. Nevertheless, we are aware that the real physical situation is more complicated than our theoretical model for various reasons. In fact, on the one hand we treat just one process of pair production while, actually, other processes (the production of $e^- - e^+$ pairs by photons, see the review [9]) are present and can give dominant contributions. On the other hand, we stress much more “microscopic” features of the pair (photon) production such as selection rules that allow or prevent some particular elementary processes and so on by neglecting “macroscopic” aspects that would be very difficult to be treated analytically (the macroscopic structure of the compact object magnetic field or the presence of already created particles). These aspects are, instead, privileged in other papers where the microscopic details are not brought out (see e.g. [10, 11, 12, 13]). In these papers, inspired by the suggestion in [14] that nonthermal radio emissions can be accounted by the gravitational collapse of a massive star, a similar interpretation is given about the origin of a GRB as a consequence of a supernova explosion. In turn, the supernova explosion is thought to be caused by a combination of gravitational and magnetic factors and much more emphasis is given to the order of magnitude of the energy released or to other macroscopic features of the physical situation.

Instead, till now, we have neglected the effects of the gravitational field created by the neutron star or by the black hole by performing all the calculations in a flat spacetime. Even if there are situations in which this can be safely done [15, 16], it is interesting to study what happens if the effects
of the gravitational field are taken into account. Actually, since the early work by Hawking on black holes thermal emission \[17\], it has been widely demonstrated from a theoretical point of view that a gravitational field can be responsible by itself of particle creation (see Refs. \[18\] for particle creation in the presence of stellar gravitational fields and Refs. \[19\] for cosmological particle creation). The formalism used to deal with this subject is that of quantum field theory in curved spacetimes where the gravitational field is treated classically and the consequent curvature of the spacetime metric is assigned, while the matter fields (Klein-Gordon or Dirac fields) are quantized starting from a Lagrangian density written in a general covariant way \[20\] \[21\] \[22\]. In the same framework, the possibility of particle creation in the presence of gravitational and electromagnetic fields has also been investigated \[23\] \[24\].

In the cited papers the particle creation is due to the fact that the gravitational field is time-dependent and the Bogoliubov transformation technique is used to calculate the production rate. Instead, as we will explain in the following section, despite we start by using the same formalism, then we change our point of view. In fact, in our model the main process responsible for pair creation is still the magnetic field and the fact that it varies with time. The gravitational field is static and it enters the calculations because it modifies the one-particle states and energies of the electrons and positrons and consequently the production probabilities. In the present work, the structure of the gravitational field is assumed to be such that its effects can be calculated perturbatively.

The paper is structured as follows. In section 2 the physical assumptions and the consequent theoretical model is described: the gravitational effects are given starting from the Schwarzschild metric and then expanding it around a specific point, not too close to the event horizon and ending the expansion at the first derivative of the metric tensor. Then the magnetic field in this metric is given and the Hamiltonian of the second quantized electron field is built up. In section 3 the one-particle eigenstates and eigenenergies of the Dirac field in the given magnetic field with the perturbative effect of the gravity are displayed. In section 4 the pair production due to the time dependence of the magnetic field (in the constant gravitational field) are calculated with two explicit form of time variation. Some conclusions are presented in section 5. Finally, four appendices contain explicit calculations that would have made heavy the main text.

In what follows natural units ($\hbar = c = 1$) are used throughout and the signature of the Minkowski spacetime is assumed to be $+ - - -$. 

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2 Theoretical framework

As we have said in the Introduction, we want to calculate the probability of producing an electron-positron pair in the presence of a strong, time varying magnetic field and of a gravitational field created by a neutron star or by a black hole. Even if we assume that the spatial structure and the time evolution of both fields are given, the problem formulated in these terms is very difficult to be faced. In fact, to determine a realistic form of the gravitational field and of the (electro)magnetic field we should fix the physical properties of the source (its mass, its electric charge, its angular momentum and so on) and solve the system formed by the Einstein equations and the general covariant Maxwell equations. Clearly, solving the Einstein equations coupled with the general covariant Maxwell equations is a hopeless problem and a number of approximations have to be done. In particular, we first assume that the Einstein equations and the general covariant Maxwell equations are disentangled. This corresponds to neglect the gravitational field produced by the magnetic field and to assume that the spacetime metric is determined only by the stellar object. But, in order to fit our requests this stellar object should be capable to produce a time-varying magnetic field. In this sense, it could be a charged rotating black hole or a magnetized collapsing neutron star but the corresponding spacetime metrics would be still very difficult to deal with. The problem can be definitely simplified by assuming that the spacetime metric is actually that produced by an uncharged, non rotating star and that the corrections to this metric due to the charge or to the rotation of the star itself can be neglected. In this approximations, our starting point is the metric tensor corresponding to the field created by a spherical body of mass $M$ outside the body itself. If we call $t$, $X$, $Y$ and $Z$ the four coordinates, this metric tensor can be written as [25]

$$g_{\mu\nu}(X,Y,Z) = \text{diag} \left[ \frac{F^2(X,Y,Z)}{F^2_+(X,Y,Z)}, -F^4_+(X,Y,Z), -F^4_+(X,Y,Z), -F^4_+(X,Y,Z) \right]$$

(II.1)

where

$$F_\pm(X,Y,Z) = 1 \pm \frac{r_g}{4\sqrt{X^2 + Y^2 + Z^2}}$$

(II.2)

with $r_g = 2GM$ the gravitational radius of the body ($G$ is the gravitational constant). We have chosen the so-called isotropic metric instead of the usual (and equivalent) Schwarzschild metric, because from Eq. (II.1) we see that the spatial distance is proportional to its Euclidean expression and this will simplify our future calculations. We point out that in this metric the event horizon of the body is the spherical surface $\sqrt{X^2 + Y^2 + Z^2} = r_g/4$. 

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Now, an $e^- - e^+$ pair is created in a spatial volume with typical length given by the Compton length $\lambda = 1/m$ ($m$ is the electron mass). Since $\lambda$ is much smaller than the gravitational radius $r_g$ of a neutron star or of a black hole, we are allowed to make some simplifications on the metric tensor (II.1). In particular, we can always assume that the pair is created in a small neighborhood of the space point $P_0$ labeled by the coordinates $(X_0, 0, 0)$ with $X_0 > r_g/4 + \Delta$ and $\Delta > 0$. If $P = (X, Y, Z) = (X_0 + x, y, z)$ is a generic point in this neighborhood then $|x| \lesssim \lambda$, $|y| \lesssim \lambda$ and $|z| \lesssim \lambda$ and we can approximate the metric tensor $g_{\mu\nu}(P)$ calculated in $P$ with the metric tensor $g^{(1)}_{\mu\nu}(x, y, z) = g_{\mu\nu}(P_0) + \frac{\partial g_{\mu\nu}(P)}{\partial X} \bigg|_{P=P_0} x + \frac{\partial g_{\mu\nu}(P)}{\partial Y} \bigg|_{P=P_0} y + \frac{\partial g_{\mu\nu}(P)}{\partial Z} \bigg|_{P=P_0} z$ (II.3) in which only the terms up to first order in $x/X_0$, $y/X_0$ and $z/X_0$ have been kept. It can easily be seen that $g^{(1)}_{\mu\nu}(x, y, z)$ actually depends only on $x$ and that it can be written in the form

$$g^{(1)}_{\mu\nu}(x) = g^{(0)}_{\mu\nu} + h_{\mu\nu}(x)$$  

(II.4)

where

$$g^{(0)}_{\mu\nu} = g_{\mu\nu}(P_0) = \text{diag}(\phi_t, -\phi_s, -\phi_s, -\phi_s),$$  

(II.5)

$$h_{\mu\nu}(x) = \frac{\partial g_{\mu\nu}(P)}{\partial X} \bigg|_{P=P_0} x = \text{diag}(2g_t x, 2g_s x, 2g_s x, 2g_s x)$$  

(II.6)

with [see Eqs. (II.1) and (II.2)]

$$\phi_t = \left(1 - \frac{r_g}{4X_0}\right)^2 \left(1 + \frac{r_g}{4X_0}\right)^2, \quad \phi_s = \left(1 + \frac{r_g}{4X_0}\right)^4,$$  

(II.7)

$$g_t = \left(1 - \frac{r_g}{4X_0}\right) \frac{r_g}{2X_0^2} \left(1 + \frac{r_g}{4X_0}\right)^3, \quad g_s = \left(1 + \frac{r_g}{4X_0}\right)^3 \frac{r_g}{2X_0^2}.$$  

(II.8)

It is evident that, in order that $g^{(1)}_{\mu\nu}(x)$ is a good approximation of $g_{\mu\nu}(P)$, $X_0$ can not be chosen to be too close to the critical value $r_g/4$. Just to give an idea, it easy to see that, if $N \gg 1$ is a large pure number, then

$$\left| \frac{g_{\mu\nu}(P) - g^{(1)}_{\mu\nu}(x)}{g_{\mu\nu}(P)} \right| < \frac{1}{N}$$  

no sum  

(II.9)

with $\mu = 0, \ldots , 3$, if

$$X_0 > \frac{r_g}{4} + \sqrt{N} \lambda.$$  

(II.10)
This condition automatically implies that
\[ \left| \frac{h_{\mu\nu}(x)}{g_{\mu\nu}^{(0)}} \right| < \frac{1}{2\sqrt{N}} \] no sum \hspace{1cm} (II.11)

with \( \mu = 0, \ldots, 3 \) and then that \( h_{\mu\nu}(x) \) can be considered as a small correction of \( g_{\mu\nu}^{(0)} \). In what follows, we assume that the previous inequalities hold with sufficiently large \( N \).

Now, we pass to the description of the structure of the magnetic field that we will denote as \( \mathbf{B}(\mathbf{r}, t) \) and that, for simplicity, we will assume to lie in the \( y-z \) plane: \( \mathbf{B}(\mathbf{r}, t) = [0, B_y(\mathbf{r}, t), B_z(\mathbf{r}, t)] \). With similar arguments used in the case of the gravitational field we can assume that the magnetic field is uniform and slowly varying in time in the spacetime volume where an \( e^- - e^+ \) pair is created. By writing it as

\[
\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(t) = \begin{pmatrix} 0 \\ B_y(t) \\ B_z(t) \end{pmatrix} = B(t) \begin{pmatrix} 0 \\ \sin \vartheta(t) \\ \cos \vartheta(t) \end{pmatrix} \] \hspace{1cm} (II.12)

with

\[
B(t) = \sqrt{B_y^2(t) + B_z^2(t)}, \] \hspace{1cm} (II.13)

\[
\tan \vartheta(t) = \frac{B_y(t)}{B_z(t)} \] \hspace{1cm} (II.14)

and by also reminding that we work in the strong-field regime, the magnetic field is such that

\[
B(t) \gg B_{cr} = \frac{m^2}{e}, \] \hspace{1cm} (II.15)

\[
\frac{\dot{B}(t)}{B(t)} \ll \frac{1}{\lambda} = m \] \hspace{1cm} (II.16)

where \(-e < 0\) is the electron charge. Before going on we want to make two observations about the magnetic field. Firstly, we have shown that the magnetic field (II.12) does not satisfy the Maxwell equations in vacuum and in the spacetime with the metric (II.4), but an electric current is needed to produce it. Secondly, we point out that the assumptions that the gravitational field is static while \( \mathbf{B}(t) \) is time varying are not contradictory even if the two fields are produced by the same source. From a theoretical point of view, this circumstance happens, for example, for a spherical body which collapses keeping its spherical symmetry and without rotating. In this case, in fact,
the gravitational field of the body is static because of the Birkhoff theorem \[26\] while the magnetic field is found to be time-dependent from energy-conservation considerations. Obviously, the previous conclusion is correct if the gravitational field generated by the magnetic energy can be neglected.

Since the spacetime is curved and the metric tensor is not simply \(\eta_{\mu\nu}\), we must pay attention in defining the vector potential \(A_\mu(r, t)\) that gives rise to \(B(t)\). To this end, we assume that the three-dimensional components of the magnetic field \(B(t)\) define the spatial-spatial components of the full covariant electromagnetic tensor \(F_{\mu\nu}(r, t)\) that is

\[
F_{32}(t) = -F_{23}(t) \equiv 0, \quad (\text{II.17})
\]
\[
F_{13}(t) = -F_{31}(t) \equiv B_y(t), \quad (\text{II.18})
\]
\[
F_{21}(t) = -F_{12}(t) \equiv B_z(t) \quad (\text{II.19})
\]

while the mixed or the full contravariant components are built by using the metric tensor \(\eta_{\mu\nu}\). Now, by definition

\[
F_{\mu\nu}(r, t) \equiv A_\nu_{;\mu}(r; t) - A_{\mu\nu}(r, t) = A_{\nu\mu}(r, t) - A_{\mu\nu}(r, t), \quad (\text{II.20})
\]

then by means of Eqs. \(\text{II.17}-\text{II.19}\), we can choose the covariant vector \(A_\mu(r, t)\) as

\[
A_0(r, t) = 0, \quad (\text{II.21})
\]
\[
A_1(r, t) = \frac{1}{2} [r \times B(t)]_x, \quad (\text{II.22})
\]
\[
A_2(r, t) = \frac{1}{2} [r \times B(t)]_y, \quad (\text{II.23})
\]
\[
A_3(r, t) = \frac{1}{2} [r \times B(t)]_z. \quad (\text{II.24})
\]

These relations define a gauge analogous to the so-called “symmetric gauge” \[4\] in the Minkowski spacetime. Finally, it is convenient to define the three-dimensional vector potential \(A(r, t) = [A_x(r, t), A_y(r, t), A_z(r, t)]\) as

\[
A(r, t) = -\frac{1}{2} [r \times B(t)] \quad (\text{II.25})
\]

where the minus sign has been inserted to have \(\partial \times A(r, t) = B(t)\).

As we have done in our previous papers \[2, 3, 4\], we will calculate the probability to create the \(e^- - e^+\) pair from the vacuum by applying the adiabatic perturbation theory up to first order in the time derivative \(\dot{B}(t)\) of the magnetic field \[27\]. To do this, we have to build up the second quantized Hamiltonian of a Dirac field \(\Psi(r, t)\) in the presence of the slowly varying
magnetic field (II.12) and in the curved spacetime with the static metric tensor (II.4) and to determine its instantaneous eigenstates and eigenenergies [27]. The Lagrangian density of this system is given by [20]:

\[
\mathcal{L}(\Psi, \partial_\mu \Psi, \bar{\Psi}, \partial_\mu \bar{\Psi}, r, t) = \sqrt{-g^{(1)}(x)} \left\{ \frac{1}{2} \left[ \bar{\Psi} \gamma^{(1)\mu}(x) \left[ i \partial_\mu + i \Gamma_\mu(x) + eA_\mu(r, t) \right] \Psi - \bar{\Psi} \left[ i \partial_\mu - i \Gamma_\mu(x) - eA_\mu(r, t) \right] \gamma^{(1)\mu}(x) \Psi \right] - m \bar{\Psi} \Psi \right\}.
\] (II.26)

The presence of the gravitational field is represented in Eq. (II.26) by the following quantities,

\[
g^{(1)}(x) \equiv \det[g^{(1)}_{\mu\nu}(x)], \tag{II.27}
\]

\[
\gamma^{(1)\mu}(x) \equiv \gamma^\alpha e^{(1)\mu}_\alpha(x), \tag{II.28}
\]

\[
\Gamma_\mu(x) = -\frac{i}{4} \sigma^{\alpha\beta} e^{(1)\nu}_\alpha(x) e^{(1)}_{\beta\nu\mu}(x) \tag{II.29}
\]

where \(\gamma^\alpha\) are the usual Dirac matrices satisfying \(\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}\) (assumed in the Dirac representation), \(e^{(1)\mu}_\alpha(x)\) is a tetrad corresponding to the metric (II.4) [26, 20] and where \(\sigma^{\alpha\beta} = i[\gamma^\alpha, \gamma^\beta]/2\). We remind that in Eq. (II.26) the adjoint field \(\bar{\Psi}(r, t)\) is also defined as \(\bar{\Psi}(r, t) \equiv \Psi^\dagger(r, t) \gamma^0\) in a curved spacetime.

Since \(g^{(1)}_{\mu\nu}(x)\) has been split as in Eq. (II.4) with the matrix \(h_{\mu\nu}(x)\) much smaller than the matrix \(g^{(0)}_{\mu\nu}\), we are allowed to keep in Eq. (II.26) only the first-order terms in \(h_{\mu\nu}(x)\). To do this, we need to calculate the quantities (II.27)-(II.29) up to first order. Concerning the determinant \(g^{(1)}(x)\), we have immediately

\[
g^{(1)}(x) \simeq g^{(0)}[1 + h(x)] \tag{II.30}
\]

where

\[
g^{(0)} \equiv \det(g^{(0)}_{\mu\nu}) = -\phi_t \phi_s^3, \tag{II.31}
\]

\[
h(x) \equiv h^\mu_\mu(x) = 2 \left( \frac{g_t}{\phi_t} - \frac{3g_s}{\phi_s} \right) x. \tag{II.32}
\]

Also, being the metric tensor \(g^{(1)}_{\mu\nu}(x)\) diagonal, we can choose a diagonal tetrad with

\[
e^{(1)0}_0(x) = \frac{1}{\sqrt{g^{(1)}_{00}(x)}} \simeq \frac{1}{\sqrt{\phi_t}} \left( 1 - \frac{g_t x}{\phi_t} \right), \tag{II.33}
\]

\[
e^{(1)i}_i(x) = \frac{1}{\sqrt{-g^{(1)}_{ii}(x)}} \simeq \frac{1}{\sqrt{\phi_s}} \left( 1 + \frac{g_s x}{\phi_s} \right) \quad \text{no sum} \tag{II.34}
\]
(note that while the Greek indices run from zero to three, the Latin ones run from one to three). By means of this tetrad it can be shown that the connections \( \Gamma_{\mu}^{(1)}(x) \) are already first-order quantities equal to

\[
\Gamma_{\mu}^{(1)}(x) = \frac{i}{4} \sigma^{1\beta} e_{\alpha}^{(0)} e_{\rho}^{(0)} dh_{\mu \rho}(x) \quad \text{(II.35)}
\]

where \( e_{\alpha}^{(0)\mu} \) is the diagonal zero-order tetrad with

\[
e_{0}^{(0)0} = \frac{1}{\sqrt{g_{00}^{(0)}}} = \frac{1}{\sqrt{\phi_t}}, \quad \text{(II.36)}
\]

\[
e_{i}^{(0)i} = \frac{1}{\sqrt{-g_{ii}^{(0)}}} = \frac{1}{\sqrt{\phi_s}} \quad \text{no sum.} \quad \text{(II.37)}
\]

Now, the procedure to calculate the Lagrangian density (II.26) up to first order in \( h_{\mu \nu}(x) \) is identical to that used in the weak-field approximation [28, 29] and we give only its final expression:

\[
\mathcal{L}^{(1)}(\Psi, \partial_{\mu} \Psi, \bar{\Psi}, \partial_{\mu} \bar{\Psi}, r, t) = \sqrt{\phi_t \phi_s^2} \left\{ \frac{(1 - g_{EX})}{2 \sqrt{\phi_t}} [\bar{\Psi} \gamma^0 (i \partial_0 \Psi) - (i \partial_0 \bar{\Psi}) \gamma^0 \Psi] + \frac{(1 - g_{PX})}{2 \sqrt{\phi_s}} [\bar{\Psi} \gamma^i [i \partial_i + eA_i(r, t)] \Psi - \bar{\Psi} [i \partial_i - eA_i(r, t)] \gamma^i \Psi] - (1 - g_M x) m \bar{\Psi} \Psi \right\} \quad \text{(II.38)}
\]

where we defined the three couplings\(^1\)

\[
g_E \equiv \frac{3g_s}{\phi_s} = \frac{3r_g}{2X_0^2} \left[ 1 + \frac{r_g}{4X_0} \right], \quad \text{(II.39)}
\]

\[
g_P \equiv \frac{2g_s}{\phi_s} - \frac{g_t}{\phi_t} = \frac{r_g}{2X_0^2} \left[ 2 - \left( 1 - \frac{r_g}{4X_0} \right)^{-1} \right] / \left( 1 + \frac{r_g}{4X_0} \right), \quad \text{(II.40)}
\]

\[
g_M \equiv \frac{3g_s}{\phi_s} - \frac{g_t}{\phi_t} = \frac{r_g}{2X_0^2} \left[ 3 - \left( 1 - \frac{r_g}{4X_0} \right)^{-1} \right] / \left( 1 + \frac{r_g}{4X_0} \right). \quad \text{(II.41)}
\]

\(^1\)We introduced three couplings for later convenience because, actually, only two of them are independent.
Note that the modifications induced by the metric tensor (II.4) in the Lagrangian density (II.38) are linear in $g_t$ and $g_s$ (obviously) but nonlinear in $\phi_t$ and $\phi_s$.

The definition of the Hamiltonian is a controversial operation in a curved spacetime \[30, 31, 32\]. We shall adopt the same definition given in \[33\] for a scalar field that is

$$\mathcal{H}^{(1)}(\Psi, \partial_i \Psi, \bar{\Psi}, \partial_i \bar{\Psi}, \Pi^{(1)}(r, t), \bar{\Pi}^{(1)}(r, t)) \equiv \bar{\Pi}^{(1)}(\partial_0 \Psi) + (\partial_0 \bar{\Psi}) \Pi^{(1)}(\Psi, \partial_\mu \Psi, \bar{\Psi}, \partial_\mu \bar{\Psi}, r, t)$$

(II.42)

where

$$\Pi^{(1)}(r, t) = \frac{\partial L^{(1)}}{\partial (\partial_0 \Psi)}$$

(II.43)

$$\bar{\Pi}^{(1)}(r, t) = \frac{\partial L^{(1)}}{\partial (\partial_0 \bar{\Psi})}$$

(II.44)

are the first-order conjugated momenta to the fields $\Psi(r, t)$ and $\bar{\Psi}(r, t)$ respectively. In our case,

$$\Pi^{(1)}(r, t) = \sqrt{\phi_s^3} \frac{i(1 - g_{EX})}{2} \Psi(r, t) \gamma^0,$$

(II.45)

$$\bar{\Pi}^{(1)}(r, t) = -\sqrt{\phi_s^3} \frac{i(1 - g_{EX})}{2} \gamma^0 \bar{\Psi}(r, t)$$

(II.46)

and then the Hamiltonian density (II.42) becomes

$$\mathcal{H}^{(1)}(\Psi, \partial_i \Psi, \bar{\Psi}, \partial_i \bar{\Psi}, r, t) = \sqrt{\phi_s^3} \{ -\frac{(1 - g_{MX})}{2\sqrt{\phi_s}} \left[ \bar{\Psi} \gamma^i [i \partial_i + e A_i(r, t)] \Psi - \bar{\Psi} \gamma^i \partial_i - e A_i(r, t) \gamma^i \Psi \right] + \}$$

(II.47)

and it results independent of $\Pi^{(1)}(r, t)$ and $\bar{\Pi}^{(1)}(r, t)$. Coherently, the total Hamiltonian is defined as \[33\]

$$H^{(1)}(t) \equiv \int d\mathbf{r} \mathcal{H}^{(1)}(\Psi, \partial_i \Psi, \bar{\Psi}, \bar{\Pi}^{(1)}(r, t)).$$

(II.48)

We point out that this definition can be shown to be equivalent in our case to the alternative definition which uses the energy-momentum tensor (see \[30\]}
for a detailed discussion about the relation between these two definitions). Now,

$$(1 - g_P x)(\partial_i \bar{\Psi}) \gamma^i \Psi = \partial_i [(1 - g_P x) \bar{\Psi} \gamma^i \Psi] - \bar{\Psi} \gamma^i \partial_i [(1 - g_P x) \Psi] \quad (\mathrm{II.49})$$

then the Hamiltonian density (\mathrm{II.47}) is equivalent apart from a derivative term to the asymmetric Hamiltonian density (we use the same symbol to indicate it)

$$\mathcal{H}^{(1)}(\Psi, \partial_i \Psi, \bar{\Psi}, \bar{\gamma}, \gamma, \bar{\gamma}, \bar{\gamma}, \gamma, \bar{\gamma}, \bar{\gamma}) = \sqrt{\phi_x \phi_s} \left\{ - \frac{1}{2 \sqrt{\phi_s}} [(1 - g_P x) \bar{\Psi} \gamma^i [i \partial_i + eA_i(r, t)] \Psi + + \bar{\Psi} \gamma^i [i \partial_i + eA_i(r, t)] [(1 - g_P x) \Psi] + + (1 - g_M x) m \bar{\Psi} \Psi \right\}. \quad (\mathrm{II.50})$$

This Hamiltonian density has the advantage that it can be written in the form

$$\mathcal{H}^{(1)}(\Psi, \partial_i \Psi, \bar{\Psi}, \bar{\gamma}, \gamma, \bar{\gamma}, \bar{\gamma}, \gamma, \bar{\gamma}, \bar{\gamma}) = \sqrt{\phi_x \phi_s} (1 - g_E x) \Psi^\dagger \mathcal{H}^{(1)}(\Psi, -i \bar{\gamma}, t) \Psi \quad (\mathrm{II.51})$$

where $\Psi^\dagger(r, t) = \Psi(r, t) \gamma^0$ and where we introduced the one-particle first-order Hamiltonian

$$\mathcal{H}^{(1)}(r, -i \bar{\gamma}, t) = \frac{\sqrt{\phi_t}}{1 - g_E x} \left\{ \frac{1}{2 \sqrt{\phi_s}} [(1 - g_P x) \alpha \cdot [-i \bar{\gamma} + eA(r, t)] + + \alpha \cdot [-i \bar{\gamma} + eA(r, t)] (1 - g_P x) \right\} + (1 - g_M x) \beta m \right\} \approx$$

$$\approx \sqrt{\phi_t} \left\{ \frac{1}{2 \sqrt{\phi_s}} [(1 - g_P x) \alpha \cdot [-i \bar{\gamma} + eA(r, t)] + + \alpha \cdot [-i \bar{\gamma} + eA(r, t)] (1 - g_P x) \right\} + (1 - g_M x) \beta m + + g_E x \left\{ \frac{1}{\sqrt{\phi_s}} \alpha \cdot [-i \bar{\gamma} + eA(r, t)] + \beta m \right\} \right\} \quad (\mathrm{II.52})$$

with $\beta = \gamma^0$ and $\alpha = (\alpha_x, \alpha_y, \alpha_z) = (\beta \gamma^1, \beta \gamma^2, \beta \gamma^3)$. Before explaining the reason of this apparently unusual definition we observe that the one-particle Hamiltonian (\mathrm{II.52}) can be written as the sum

$$\mathcal{H}^{(1)}(r, -i \bar{\gamma}, t) = \mathcal{H}^{(0)}(r, -i \bar{\gamma}, t) + \mathcal{I}(r, -i \bar{\gamma}, t) \quad (\mathrm{II.53})$$

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of the zero-order Hamiltonian
\[ H^{(0)}(r, -i\partial, t) = \sqrt{\phi_t} \left\{ \alpha \cdot [-i\partial + eA(r, t)] + \sqrt{\phi_s} \beta m \right\} \] (II.54)

and of the first-order interaction
\[ I(r, -i\partial, t) = \sqrt{\phi_t} \left\{ -\frac{g_P}{2\sqrt{\phi_s}} x\beta \cdot [-i\partial + eA(r, t)] \right\} - \frac{g_P}{2} x\gamma^{(0)}(r, -i\partial, t) + H^{(0)}(r, -i\partial, t) \]
\[ + g_E x H^{(0)}(r, -i\partial, t). \] (II.55)

In order to understand the definition (II.52) we have to introduce the scalar product for spinors in our curved spacetime. If \( \zeta_1(r, t) \) and \( \zeta_2(r, t) \) are two spinors it is defined as
\[ (\zeta_1, \zeta_2) \equiv \int_S dS \sqrt{-g^{(1)}(x)} \bar{\zeta}_1(r, t) \gamma^{(1)}(x) \zeta_2(r, t) \] (II.56)

where \( S \) is a Cauchy hyper-surface. We can choose \( S \) as the \( t = \text{const.} \) hyper-surface, then \[ dS = (dr, 0, 0, 0) \] (II.57)
and the scalar product (II.56) becomes
\[ (\zeta_1, \zeta_2) = \int d\mathbf{r} \sqrt{-g^{(1)}(x)} \bar{\zeta}_1(r, t) \gamma^{(1)}(x) \zeta_2(r, t) = \int d\mathbf{r} \sqrt{-g^{(1)}(x)} \bar{\zeta}_1(r, t) \zeta_2(r, t) \] (II.58)

where we used the definition (II.33). Finally, by exploiting Eqs. (II.30) - (II.32), the volume element can be written up to first order as
\[ d\mathbf{r} \sqrt{-g^{(1)}(x)} \approx \int d\mathbf{r} \sqrt{\phi_t^3 \frac{\phi_s^3}{\phi_t}} \left( 1 + \frac{g_t x}{\phi_t} - \frac{3g_s x}{\phi_s} - \frac{g_E x}{\phi_t} \right) = d\mathbf{r} \sqrt{\phi_s^3 (1 - g_E x)} \] (II.59)

and we obtain
\[ (\zeta_1, \zeta_2) = \int d\mathbf{r} \sqrt{\phi_s^3 (1 - g_E x)} \bar{\zeta}_1(r, t) \zeta_2(r, t). \] (II.60)
Now, with this definition of the scalar product we realize that the “asymmetric” Hamiltonian (II.52) is actually an Hermitian operator up to first-order terms. The fact that this definition of one-particle Hamiltonian is well posed is also corroborated by the form of the equation of motion of the field $\Psi(r, t)$. In fact, starting from the Lagrangian density (II.38), it can be seen that the equation of motion

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}^{(1)}}{\partial (\partial_0 \Psi)} + \partial \cdot \frac{\partial \mathcal{L}^{(1)}}{\partial (\partial \Psi)} = 0 \quad (\text{II.61})$$

can be written as

$$i \partial_0 \Psi = \mathcal{H}^{(1)}(r, -i \partial, t) \Psi. \quad (\text{II.62})$$

Finally, by using Eq. (II.51), the total Hamiltonian (II.48) becomes

$$H^{(1)}(t) = \int d\mathbf{r} \sqrt{\phi_3} (1 - gE) \bar{\Psi}(r, t) \mathcal{H}^{(1)}(r, -i \partial, t) \Psi(r, t). \quad (\text{II.63})$$

This Hamiltonian depends explicitly on time only through the dependence of $\mathcal{H}^{(1)}(r, -i \partial, t)$ on the magnetic field $\mathbf{B}(t)$, then because of the condition (II.16) it is a slowly varying quantity. In the next section we will determine its instantaneous eigenstates and eigenvalues that will be used to compute the pair creation probability by means of the adiabatic perturbation theory [27].

### 3 Determination of the instantaneous eigenstates and eigenenergies of the Hamiltonian $H^{(1)}(t)$

The first step to determine the instantaneous eigenstates and eigenenergies of the Hamiltonian (II.63) is to quantize and diagonalize the corresponding constant Hamiltonian. If we introduce the constant magnetic field

$$\mathbf{B} = \begin{pmatrix} 0 & B_y \\ B_y & B_z \end{pmatrix} = B \begin{pmatrix} 0 & \sin \vartheta \\ \cos \vartheta & \cos \vartheta \end{pmatrix} \quad (\text{III.1})$$

with

$$B = \sqrt{B_y^2 + B_z^2}, \quad (\text{III.2})$$

$$\tan \vartheta = \frac{B_y}{B_z} \quad (\text{III.3})$$

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and the vector potential in the symmetric gauge

\[ A(r) = -\frac{1}{2} (r \times B) \]  

then the Lagrangian density \((II.38)\), the one-particle Hamiltonian \((II.53)\) and the total Hamiltonian \((II.63)\) become respectively

\[
\mathcal{L}^{(1)}(\Psi, \partial_\mu \Psi, \bar{\Psi}, \partial_\mu \bar{\Psi}, r) = \sqrt{\phi_t} \phi_s \left\{ \frac{(1 - gE_x)}{2\sqrt{\phi_t}} \left[ \bar{\Psi} \gamma^0 (i\partial_0 \Psi) - (i\partial_0 \bar{\Psi}) \gamma^0 \Psi \right] + \\
+ \frac{(1 - gP)}{2\sqrt{\phi_s}} \left[ \bar{\Psi} \gamma^i [i\partial_i + eA_i(r)] \Psi - \\
- \bar{\Psi} [i\partial_i - eA_i(r)] \gamma^i \Psi \right] - \\
- (1 - gM_x) m\bar{\Psi} \Psi \right\},
\]

\((III.5)\)

\[
\mathcal{H}^{(1)}(r, -i\partial) = \mathcal{H}^{(0)}(r, -i\partial) + \mathcal{I}(r, -i\partial),
\]

\((III.6)\)

\[
H^{(1)} = \int \! d\mathbf{r} \sqrt{\phi_t} (1 - gE_x) \bar{\Psi}(r, t) \mathcal{H}^{(1)}(r, -i\partial) \Psi(r, t)
\]

\((III.7)\)

where [see Eqs. \((II.54)\) and \((II.55)\)]

\[
\mathcal{H}^{(0)}(r, -i\partial) = \sqrt{\phi_t} \left\{ \alpha \cdot [ -i\partial + eA_r(r)] + \sqrt{\phi_s} \beta m \right\},
\]

\((III.8)\)

\[
\mathcal{I}(r, -i\partial) = \sqrt{\phi_t} (g_P - g_M) \beta m x - \frac{gP}{2} \left[ x \mathcal{H}^{(0)}(r, -i\partial) + \mathcal{H}^{(0)}(r, -i\partial)x \right] + \\
+ gE_x \mathcal{H}^{(0)}(r, -i\partial).
\]

\((III.9)\)

Also, the equation of motion \((II.62)\) becomes

\[
i\partial_0 \Psi = \mathcal{H}^{(1)}(r, -i\partial) \Psi
\]

\((III.10)\)

then, if

\[
\Psi(r, t) = \Psi_{\pm, j}(r) \exp(\mp iw_j t)
\]

\((III.11)\)

with \(j\) a set of quantum numbers and \(\Psi_{\pm, j}(r)\) are a complete set of modes with energies \(w_j > 0\), then

\[
\mathcal{H}^{(1)}(r, -i\partial) \Psi_{\pm, j} = \pm w_j \Psi_{\pm, j}.
\]

\((III.12)\)
After applying the charge conjugation operator to the positive-energy solution \( \tilde{\Psi}^+ (r) \) of the equation (III.12) with \( e \) instead of \( -e \) in the one-particle Hamiltonian \( H^{(1)}(r, -i\partial) \) and calling the corresponding spinor \( V_j(r) \) (see [4] for details), Eq. (III.12) splits into two equations:

\[
H^{(1)}(r, -i\partial) U_j = w_j U_j, \tag{III.13}
\]

\[
H^{(1)}(r, -i\partial) V_j = -\tilde{w}_j V_j. \tag{III.14}
\]

where \( U_j(r) \equiv \Psi^+_j(r) \) and where we called \( \tilde{w}_j > 0 \) the energies of the positron states that, in the presence of a magnetic field, have, in general, a different dependence on the quantum numbers \( j \) [4]. The states \( U_j(r) \) and \( V_j(r) \) are assumed to be an orthonormal basis with respect to the scalar product (II.60), i.e.

\[
(U_j, U_{j'}) = (V_j, V_{j'}) = \delta_{j,j'}, \tag{III.15}
\]

\[
(U_j, V_{j'}) = (V_j, U_{j'}) = 0. \tag{III.16}
\]

Equations (III.13) and (III.14) with the orthonormalization relations (III.15) and (III.16) will be solved perturbatively in the next paragraph. At the moment, we observe that, in order to quantize and diagonalize the Hamiltonian (III.7), we have to expand the Dirac field \( \Psi(r, t) \) in the basis \( [U_j(r), V_j(r)] \) as

\[
\Psi(r, t) = \sum_j [c_j(t) U_j(r) + d^{(1)}_j(t) V_j(r)] \tag{III.17}
\]

with \( c_j(t) = c_j \exp(-iw_j t) \) and \( d^{(1)}_j(t) = d_j \exp(i\tilde{w}_j t) \) and we have to impose the usual anti-commutation rules among the coefficients \( c_j, c^{\dagger}_j, d_j \) and \( d^{(1)}_j \) that are now operators. In this way, by using the equation of motion (III.10), it can easily be shown that the Hamiltonian (III.7) assumes the usual diagonal form

\[
H^{(1)} = \int dr \sqrt{\phi^3_s} (1 - g_{EX}) \Psi^\dagger(r, t) H^{(1)}(r, -i\partial) \Psi(r, t) = \sum_j (w_j N_j + \tilde{w}_j \tilde{N}_j) \tag{III.18}
\]

where \( N_j = c^{\dagger}_j c_j, \tilde{N}_j = d^{(1)}_j d_j \) and where we neglected the vacuum energy. \(^2\) Obviously, if \( |0\rangle \) is the vacuum state the eigenstates of this Hamiltonian are the Fock states

\[
|\{n_j\}; \{\tilde{n}_j\}\rangle \equiv (c^{(1)}_{j_1})^{n_{j_1}} (c^{(1)}_{j_2})^{n_{j_2}} \cdots (d^{(1)}_{j_1})^{\tilde{n}_{j_1}} (d^{(1)}_{j_2})^{\tilde{n}_{j_2}} \cdots |0\rangle \tag{III.19}
\]

\(^2\)Note that to neglect the vacuum energy can be unsafe when we want to determine the time evolution of the gravitational field in the presence of quantum matter fields as its sources [20]. But in our case this fact does not cause any problems because the time evolution of the gravitational field is given (actually, in the present case the gravitational field is static).
with the corresponding eigenvalues
\[ E = \sum_l (w_l n_l + \tilde{w}_l \tilde{n}_l). \]  

(III.20)

In the next paragraph, we proceed by explicitly computing the one-particle spinors \( U_j(r) \) and \( V_j(r) \) and the energies \( w_j \) and \( \tilde{w}_j \) and we will come back to the problem of the instantaneous eigenstates and eigenvalues of the Hamiltonian (II.63) in the paragraph (3.3). By looking at the structure of the one-particle Hamiltonian (III.6), we see that the eigenvalue equations (III.13) and (III.14) can be solved perturbatively with respect to the interaction Hamiltonian \( I(r, -i\partial) \) by using the time-independent perturbation theory \([34]\). In particular, we will compute the eigenstates \( U_j(r) \) and \( V_j(r) \) and the eigenenergies \( w_j \) and \( \tilde{w}_j \) up to first order in the gravitational couplings \( g_E, g_P \) and \( g_M \).

3.1 Computation of the one-particle states \( U_j(r) \) and \( V_j(r) \) up to zero order and of the energies \( w_j \) and \( \tilde{w}_j \) up to first order

We first want to determine the zero-order solution of Eqs. (III.13) and (III.14). Up to this order those equations can be written as
\[
\sqrt{\phi_t} \left\{ \alpha \cdot [-i\partial + eA(r)] + \sqrt{\phi_s} \beta m \right\} U_j^{(0)} = w_j^{(0)} U_j^{(0)}, \tag{III.21}
\]
\[
\sqrt{\phi_t} \left\{ \alpha \cdot [-i\partial + eA(r)] + \sqrt{\phi_s} \beta m \right\} V_j^{(0)} = -\tilde{w}_j^{(0)} V_j^{(0)}. \tag{III.22}
\]

These equations are the eigenvalue equations of a particle with charge \(-e\) and mass \( \sqrt{\phi_s} m \) in the presence of the magnetic field (III.1) in the Minkowski spacetime, then we can write immediately their solutions \([35, 36]\). We have already studied these solutions in \([4]\) and we refer the reader to that paper for a detailed discussion about the meaning of the quantum numbers that will be introduced, their ranges of variation and so on.\(^3\) Here, we only quote the results that will be useful later. Firstly, we remind that a complete set of commuting observables of this physical system is made by the Hamiltonian \( \mathcal{H}^{(0)}(r, -i\partial) \), the longitudinal linear momentum \( \mathcal{P}_\parallel \), the longitudinal total angular momentum \( \mathcal{J}_\parallel = \mathcal{L}_\parallel + \mathcal{S}_\parallel \) and the transverse square distance \( R^2_\perp \) where

\(^3\)We warn the reader that some unavoidable changes of notation have been done with respect to Ref. \([4]\).
“longitudinal” and “transverse” are considered with respect to the direction of the magnetic field. For this reason, the eigenstates and the eigenenergies are characterized by four quantum numbers that are \( j \equiv \{n_d, k, \sigma, n_g\} \).

Actually, the eigenenergies of the electron do not depend on \( n_g \) and those of the positron do not depend on \( n_d \) [see Eqs. (15) and (34) in Ref. [4]]. This circumstance leads us to embody in \( r \) the quantum numbers \( \{n_d, k, \sigma\} \) and in \( q \) the quantum numbers \( \{n_g, k, \sigma\} \).

In conclusion, we have

\[
\begin{align*}
    j &\equiv \{n_d, k, \sigma, n_g\}, \\
    r &\equiv \{n_d, k, \sigma\}, \\
    q &\equiv \{n_g, k, \sigma\}
\end{align*}
\]

and, consequently

\[
\begin{align*}
    j &= \{r, n_g\} = \{q, n_d\}.
\end{align*}
\]

With these definitions the electron and positron eigenenergies \( w_j^{(0)} \) and \( \tilde{w}_j^{(0)} \) are given by the modified Landau levels

\[
\begin{align*}
    w_r^{(0)} &= \sqrt{\frac{\phi_t}{\phi_s}} \sqrt{\phi_s m^2 + k^2 + eB(2n_d + 1 + \sigma)} = \sqrt{\phi_t m^2 + \frac{\phi_t}{\phi_s} [k^2 + eB(2n_d + 1 + \sigma)]}, \\
    \tilde{w}_q^{(0)} &= \sqrt{\frac{\phi_t}{\phi_s}} \sqrt{\phi_s m^2 + k^2 + eB(2n_g + 1 - \sigma)} = \sqrt{\phi_t m^2 + \frac{\phi_t}{\phi_s} [k^2 + eB(2n_g + 1 - \sigma)]}.
\end{align*}
\]

The corresponding eigenstates will be indicated as \( u_j(\mathbf{r}) \) and \( v_j(\mathbf{r}) \) respectively and, since we have already studied them in [4], we will quote some of their properties in appendix A.

As it is evident from the expressions (III.27) and (III.28), the one-particle eigenenergies have two kinds of degenerations. The first one is due, as we have said, to the fact that the electron eigenenergies do not depend on the quantum number \( n_g \) and, symmetrically, the positron eigenenergies do not depend on the quantum number \( n_d \). The second one is due to the fact that the electron eigenstates with quantum numbers \( \{r_+, n_g\} = \{n_d, k, +1, n_g\} \) and \( \{r_, n_g\} = \{n_d + 1, k, -1, n_g\} \) have the same energy whatever \( n_g \) and, symmetrically, the positron eigenstates with quantum numbers \( \{\tilde{q}_+, n_d\} = \{n_d, k, +1, n_g + 1\} \) and \( \{\tilde{q}_-, n_d\} = \{n_d + 1, k, -1, n_g\} \) have the same energy whatever \( n_d \). This means, following the time-independent perturbation theory [21], that the eigenstates \( u_j(\mathbf{r}) \) and \( v_j(\mathbf{r}) \) will not represent, in general, the correct zero-order eigenfunctions \( U_j^{(0)}(\mathbf{r}) \) and \( V_j^{(0)}(\mathbf{r}) \). For this reason we indicated them by means of the symbols \( u_j(\mathbf{r}) \) and \( v_j(\mathbf{r}) \).
Now, we will compute explicitly only the zero-order electron eigenstates and the first-order electron eigenenergies, while the analogous results for the positron eigenstates and eigenenergies will be only quoted. Following the time-independent perturbation theory for degenerate states, we write the zero-order solutions of Eq. (III.13) with a given energy as linear combinations of all the degenerate eigenstates \( u_j(r) \) and \( v_j(r) \) with that energy. We will characterize the new states by means of the index \( x_0 \) and this choice will be understood later:

\[
U_{r_0, x_0}^{(0)}(r) = \sum_{n_g} P_{r_0, x_0; r_0, n_g}^{(0)} u_{r_0, n_g}(r),
\]

(III.29)

\[
U_{r-, x_0}^{(0)}(r) = \sum_{n_g} \left[ P_{r-, x_0; r-, n_g}^{(0)} u_{r-, n_g}(r) + P_{r-, x_0; r+, n_g}^{(0)} u_{r+, n_g}(r) \right],
\]

(III.30)

\[
U_{r+, x_0}^{(0)}(r) = \sum_{n_g} \left[ P_{r+, x_0; r-, n_g}^{(0)} u_{r-, n_g}(r) + P_{r+, x_0; r+, n_g}^{(0)} u_{r+, n_g}(r) \right].
\]

(III.31)

In the first of these equations we considered that the eigenenergies of the so-called electron transverse ground states \( \{r_0, n_g\} \) with \( r_0 = \{0, k, -1\} \), are degenerate only with respect to \( n_g \). Note that in Eqs. (III.30) and (III.31) the resulting spinors have been characterized, with an abuse of notation, by the quantum numbers \( r_- \) and \( r_+ \) respectively, even if, in general, they will be only eigenstate of the linear longitudinal momentum \( P_\| \) with eigenvalue \( k \). But, the coefficients \( P_{r_0, x_0; r_0, n_g}^{(0)}, P_{r_\pm, x_0; r_\pm, n_g}^{(0)} \), are the solutions of the secular equations [34]

\[
\sum_{n_g'} \left( I_{r_0, n_g; x_0, n_g'} - \epsilon_{r_0, x_0} \delta_{n_g, n_g'} \right) P_{r_0, x_0; r_0, n_g'}^{(0)} = 0,
\]

(III.32)

\[
\sum_{n_g'} \left[ \left( I_{r-, n_g; r-, n_g'} - \epsilon_{r-, x_0} \delta_{n_g, n_g'} \right) P_{r-, x_0; r-, n_g'}^{(0)} + \left( I_{r-, n_g; r+, n_g'} - \epsilon_{r-, x_0} \delta_{n_g, n_g'} \right) P_{r-, x_0; r+, n_g'}^{(0)} \right] = 0,
\]

(III.33)

\[
\sum_{n_g'} \left[ \left( I_{r+, n_g; r+, n_g'} - \epsilon_{r+, x_0} \delta_{n_g, n_g'} \right) P_{r+, x_0; r+, n_g'}^{(0)} + \left( I_{r+, n_g; r-, n_g'} - \epsilon_{r+, x_0} \delta_{n_g, n_g'} \right) P_{r+, x_0; r-, n_g'}^{(0)} \right] = 0,
\]

(III.34)

\[
\sum_{n_g'} \left[ \left( I_{r-, n_g; r-, n_g'} - \epsilon_{r+, x_0} \delta_{n_g, n_g'} \right) P_{r+, x_0; r-, n_g'}^{(0)} + \left( I_{r-, n_g; r+, n_g'} - \epsilon_{r-, x_0} \delta_{n_g, n_g'} \right) P_{r+, x_0; r+, n_g'}^{(0)} \right] = 0,
\]

(III.35)

\[
\sum_{n_g'} \left[ \left( I_{r+, n_g; r-, n_g'} - \epsilon_{r-, x_0} \delta_{n_g, n_g'} \right) P_{r-, x_0; r-, n_g'}^{(0)} + \left( I_{r+, n_g; r+, n_g'} - \epsilon_{r+, x_0} \delta_{n_g, n_g'} \right) P_{r-, x_0; r+, n_g'}^{(0)} \right] = 0.
\]

(III.36)
where, in general,
\[ I_{jj'} \equiv \int dr \sqrt{\phi^2} u^*_j(r) I(r, -i\partial) u_{j'}(r) \]  
(III.37)
and where \( \epsilon_{r_0,x_0} \), \( \epsilon_{r\pm,x_0} \) are the first-order corrections (to be determined) to the eigenenergy of the eigenstates \( U_{r_0,x_0}^{(0)}(r) \) and \( U_{r\pm,x_0}^{(0)}(r) \) respectively. In appendix B we show that the matrix elements \( I_{r\pm,n_g;r\pm,n_g'} \) vanish.\(^4\) This means that the perturbation \( I(r, -i\partial) \) does not remove the energy degeneracy of the eigenstates characterized by \( \{r-, n_g\} \) and \( \{r+, n_g\} \) and that all the coefficients \( P_{r\pm,x_{0};r\pm,n_g}^{(0)} \) can be put equal to zero. In other words, we have to diagonalize the perturbation \( I(r, -i\partial) \) inside every subspace labeled by the quantum numbers \( r = \{n_d, k, \sigma\} \), and the zero-order eigenstates are indeed characterized by the quantum numbers \( r \). Also, equations (III.29) - (III.31) become simply
\[ U_{r,x_0}^{(0)}(r) = \sum_{n_g} P_{r,x_0;r,n_g}^{(0)} u_{r,n_g}(r) \]  
(III.38)
while equations (III.32) - (III.36) can be written as the single equation
\[ \sum_{n_g'} \left( I_{r,n_g;r,n_g'} - \epsilon_{r,x_0} \delta_{n_g,n_g'} \right) P_{r,x_0;r,n_g'}^{(0)} = 0. \]  
(III.39)

Now, by means of the same technique used in appendix B we can show that the matrix elements \( I_{r,n_g;r,n_g'} \) can be written as
\[ I_{r,n_g;r,n_g'} = \left[ (g_P - g_M) \frac{\phi^2}{w_r^{(0)}} + (g_E - g_P) w_r^{(0)} \right] \int dx dy \theta_{n_d,n_g}^{(0)}(x,y) x_0(0) x_0(0) \theta_{n_d,n_g'}^{(0)}(x,y) \]  
(III.40)
where the operator \( x_0 \) and the functions \( \theta_{n_d,n_g}^{(0)}(x,y) \) have been defined in Eqs. (A.18) and (A.29) respectively. Now, from Eqs. (A.1), (A.6) and (A.26) we see that only the transverse functions \( \theta_{n_d,n_g}^{(0)}(x,y) \) in \( u_{j}(r) \) depend on \( n_g \), then we have to determine the coefficients \( P_{r,x_0;r,n_g}^{(0)} \) in such a way that the linear combination \( \sum_{n_g} P_{r,x_0;r,n_g}^{(0)} \theta_{n_d,n_g}^{(0)}(x,y) \) diagonalizes the operator \( x_0 \).\(^5\) This linear combination is given in \[35\] [it is Eq. (104) in Complement EVI]. The coefficients \( P_{r,x_0;r,n_g}^{(0)} \) result independent of the quantum numbers \( r \) and are given by
\[ P_{r,x_0;r,n_g}^{(0)} = P_{x_0,n_g}^{(0)} = \sqrt{\frac{eB}{\pi}} \frac{1}{\sqrt{2n_g n_g'}} H_{n_g}(\sqrt{eB} x_0) \exp \left( -\frac{eB x_0^2}{2} \right) \]  
(III.41)

\(^4\)Note that the matrix elements \( I_{r-,n_g;r+,n_g'}^{(0)} \) and \( I_{r+,n_g';r-,n_g}^{(0)} \) are not complex conjugated numbers because the operator \( I(r, -i\partial) \) is not Hermitian with respect to the scalar product implicitly used in the definition (III.37).

\(^5\)Now, it is clear why we called the additional index \( x_0 \).
where

\[ H_{n_g}(\sqrt{eB}x_0) = \frac{2^{n_g}}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds (\sqrt{eB}x_0 + is)^{n_g} \exp \left( -s^2 \right) \]  

(III.42)

is the \( n_g \)th-order Hermite polynomial \[37\]. In this way the spinors \( U_r(x,0) \) have the same form of the spinors \( u_j(r) \), but with the function \( \theta'_{n_d,n_g}(x,y) \) substituted by the function

\[ \Theta'_{n_d,x_0}(x,y) \equiv \sum_{n_g} P_{x_0,n_g}^{(0)} \theta'_{n_d,n_g}(x,y). \]  

(III.43)

By using the expression \([A32]\) of \( \theta'_{n_d,n_g}(x,y) \) we have

\[
\Theta'_{n_d,x_0}(x,y) = \frac{1}{\sqrt{n_d!}} (a_d^\dagger)^{n_d} \sqrt{\frac{eB}{\pi}} \sqrt{\frac{eB}{2}} \exp \left[ -\frac{eB}{2} \left( \frac{x_0^2}{2} + \frac{x^2 + y^2}{2} \right) \right] \frac{1}{\pi} \times \\
\times \int_{-\infty}^{\infty} ds \exp \left( -s^2 \right) \sum_{n_g} \frac{1}{n_g!} \left[ (\sqrt{eB}x_0 + is)\sqrt{eB}(x - iy) \right]^{n_g} = \\
= \frac{1}{\sqrt{n_d!}} (a_d^\dagger)^{n_d} \sqrt{\frac{eB}{\pi}} \sqrt{\frac{eB}{2}} \exp \left[ -\frac{eB}{2} \left( \frac{x_0^2}{2} + \frac{x^2 + y^2}{2} \right) \right] \frac{1}{\pi} \times \\
\times \exp \left[ eBx_0(x - yi) \right] \int_{-\infty}^{\infty} ds \exp \left[ -s^2 + i\sqrt{eB}(x - iy)s \right] = \\
= \frac{1}{\sqrt{n_d!}} (a_d^\dagger)^{n_d} \sqrt{\frac{eB}{\pi}} \sqrt{\frac{eB}{2\pi}} \exp \left\{ -\frac{eB}{2} \left[ (x - x_0)^2 - iy(x - 2x_0) \right] \right\} 
\]

(III.44)

where we used the formula \[37\]

\[ \int_{-\infty}^{\infty} ds \exp \left( -b_1^2s^2 + b_2s \right) = \frac{\sqrt{\pi}}{b_1} \exp \left( \frac{b_2^2}{4b_1^2} \right) \quad \text{Re}(b_1^2) > 0. \]  

(III.45)

From now on, the quantum number \( n_d \) completely disappears in our calculations because it is essentially substituted by the quantum number \( x_0 \). This circumstance allows us to simplify the notation. In fact, we can eliminate the subscript \( d \) from the remaining quantum number \( n_d \) and from the related operators, that is [see Eqs. \([A10]\) and \([A11]\)]

\[ a_d \rightarrow a \equiv \frac{1}{2} \left[ \sqrt{\frac{eB}{2}}(x - iy) + \sqrt{\frac{2}{eB}} \left( \frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \right], \]  

(III.46)

\[ a_d^\dagger \rightarrow a^\dagger \equiv \frac{1}{2} \left[ \sqrt{\frac{eB}{2}}(x + iy) - \sqrt{\frac{2}{eB}} \left( \frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right) \right], \]  

(III.47)

\[ n_d \rightarrow n. \]  

(III.48)
Also, later calculations will be simplified if we discretize the eigenvalues \( x_0 \). This can be done by imposing in the functions \( \Theta_{n,d,x_0}^\prime(x,y) \) the periodicity condition at \( x = 0 \)

\[
\exp \left( i \frac{eBx_0}{2} \right) = \exp \left( -i \frac{eBx_0}{2} \right) \quad (\text{III.49})
\]

where \( Y \) is the length of the quantization volume in the \( y \)-direction. In this way, the allowed eigenvalues are given by

\[
x_{0;\ell} = \frac{2\ell \pi}{eBY}, \quad \ell = 0, \pm 1, \ldots \quad (\text{III.50})
\]

We point out that if we imposed the periodicity condition at \( x_0 \neq 0 \) the allowed eigenvalues would change. Nevertheless, since at the end of the calculations we will perform the continuum limit \( Y \to \infty \), we are not interested in the exact values of the allowed eigenvalues but only in the eigenstate density \( \varrho(x_0) \) which is

\[
\varrho(x_0) \equiv \frac{d\ell}{dx_0} = \frac{eBY}{2\pi} \quad (\text{III.51})
\]

independently of \( x_0 \). For notational simplicity, we will still indicate the discrete eigenvalues \( x_{0;\ell} \) as \( x_0 \), then, taking into account the substitutions \( (\text{III.46})-(\text{III.48}) \), the function \( (\text{III.44}) \) becomes

\[
\Theta_{n,x_0}^\prime(x,y) = \frac{1}{\sqrt{n!}} (a^\dagger)^n \sqrt{\frac{eB}{\pi Y^2}} \exp \left\{ -\frac{eB}{2} \left[ (x-x_0)^2 - iy(x-2x_0) \right] \right\} .
\]  

(III.52)

The numerical factors have been chosen in such a way that if we define [analogously to the two-dimensional spinors \( \varphi_j^\prime(r) \) given in Eq. (A26)] the two-dimensional spinors

\[
\Phi_j^\prime(r) \equiv \frac{\exp(ikz)}{\sqrt{Z}} f_\sigma^\prime \Theta_{n,x_0}^\prime(x,y)
\]  

(III.53)

with

\[
J \equiv \{ n, k, \sigma, x_0 \},
\]

(III.54)

they result normalized as [see also Eq. (A35)]

\[
\int d\mathbf{r} \Phi_j^\dagger(r) \Phi_{j'}(r) = \delta_{J,J'}
\]  

(III.55)

with \( \delta_{J,J'} \equiv \delta_{n,n'} \delta_{k,k'} \delta_{\sigma,\sigma'} \delta_{x_0,x_0'} \). Finally, the zero order spinors \( U_j^{(0)}(r) \) are given by [see Eqs. (A1) and (A6)]

\[
U_j^{(0)}(r) = R_x^\dagger(\vartheta) U_j^{(0)}(r)
\]  

(III.56)
where

\[ U^{(0)}_{J}(r) = \frac{1}{\sqrt{\phi_s^3}} \sqrt{\frac{\phi_t + \sqrt{\phi_t} m}{2w_R^{(0)}}} \left( \frac{\Phi'_{J}(r)}{\sqrt{\phi_t \phi_s w_R^{(0)} + \sqrt{\phi_t} m}} \Phi'_{J}(r) \right) \]  \hspace{1cm} (III.57)

with

\[ R \equiv \{n, k, \sigma\} \]  \hspace{1cm} (III.58)

and are normalized as

\[ \int d\mathbf{r} \sqrt{\phi_s^3} U^{(0)\dagger}_{J}(\mathbf{r}) U^{(0)}_{J'}(\mathbf{r}) = \delta_{J,J'} . \]  \hspace{1cm} (III.59)

Obviously, from Eq. (III.40) the eigenvalue corresponding to this eigenfunction is, up to first order,

\[ w^{(1)}_{J} = w^{(0)}_{R} + \left[ (g_P - g_M) \frac{\phi_t m^2}{w_R^{(0)}} + (g_E - g_P) w^{(0)}_{R} \right] x_0 = \]

\[ = \sqrt{\phi_t m^2 + \frac{\phi_t}{\phi_s} \left[ k^2 + eB(2n + 1 + \sigma) \right]} \left\{ 1 + \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{k^2 + eB(2n + 1 + \sigma)}{\phi_s m^2 + k^2 + eB(2n + 1 + \sigma)} \right\} x_0 \]  \hspace{1cm} (III.60)

where we used the definitions (II.39)-(II.41) of the coefficients \( g_E, g_P \) and \( g_M \). This result has a clear interpretation in classical terms. In fact, the classical Lagrangian of a particle in the given metric (II.4) is [25]

\[ L(\mathbf{r}, \mathbf{v}) = -m \sqrt{g_{00}^{(1)}(x) + g_{ii}^{(1)}(x)v^2} \]  \hspace{1cm} (III.61)

with

\[ v^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \]  \hspace{1cm} (III.62)

or, up to first order,

\[ L^{(1)}(\mathbf{r}, \mathbf{v}) = -m \sqrt{\phi_t - \phi_s v^2} \left( 1 + \frac{g_t + g_s v^2}{\phi_t - \phi_s v^2} x \right) . \]  \hspace{1cm} (III.63)

From this expression and by defining the first-order gravitational force on the particle as [25]

\[ f^{(1)}(\mathbf{r}, \mathbf{v}) \equiv \frac{\partial L^{(1)}(\mathbf{r}, \mathbf{v})}{\partial \mathbf{r}} \]  \hspace{1cm} (III.64)

22
we easily obtain that the gravitational force lies along the $x$-axis and its $x$-component is equal to

$$f_x^{(1)}(v) = -\frac{m (g_t + g_s v^2)}{\sqrt{\phi_t - \phi_s v^2}}. \quad (III.65)$$

Finally, if we express $v^2$ in terms of the momentum $p$ defined as

$$p \equiv \frac{\partial L(r, v)}{\partial v} = -\frac{mg_i^{(1)}(x)v}{\sqrt{g_{00}^{(1)}(x) + g_i^{(1)}(x)v^2}}$$

we have

$$f_x^{(1)}(p) = f_x^{(1)}[v(p)] = -\sqrt{\phi_t m^2 + \frac{\phi_t}{\phi_s} p^2} \left( \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{p^2}{\phi_s m^2 + p^2} \right) \quad (III.67)$$

where, for simplicity, we used the same symbol to indicate the force as a function of $v$ and of $p$. By performing the obvious identification

$$p^2 \sim k^2 + eB(2n + 1 + \sigma) \quad (III.68)$$

and by comparing Eqs. (III.67) and (III.60), we conclude that the first-order correction to the eigenenergies is nothing but the gravitational potential energy of the electron in the presence of the constant and uniform gravitational force (III.67).

In an analogous way we can write the positron eigenenergies up to first order as

$$\tilde{\omega}_J^{(1)} = \sqrt{\phi_t m^2 + \frac{\phi_t}{\phi_s} [k^2 + eB(2n + 1 - \sigma)] \left\{ 1 + \left[ \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{k^2 + eB(2n + 1 - \sigma)}{\phi_s m^2 + k^2 + eB(2n + 1 - \sigma)} \right] x_0 \right\} \quad (III.69)$$

We observe that, analogously to the case of the electron eigenstates and eigenenergies, the quantum number $n_d$ is here substituted by $x_0$, and it is pointless to keep the subscript $g$ in the remaining quantum number $n_g$ because there is no more possibility of ambiguities.

The zero-order positron eigenstates are given by [see Eqs. (A2) and (A7)]

$$V_J^{(0)}(r) = R_x^0(\vartheta) V_J^{(0)}(r) \quad (III.70)$$

with

$$V_J^{(0)}(r) = \frac{\sigma}{\sqrt{\phi_s}} \sqrt{\frac{\tilde{\omega}_R^{(0)} + \sqrt{\phi_s m}}{2 \tilde{\omega}_R^{(0)}}} \left( -\sqrt{\frac{\phi_t}{\phi_s} \frac{V'(r, -i\vartheta)}{\sqrt{\phi_t m^2 + \phi_s m}} X'_J(r) \right) \quad (III.71)$$

23
and [see Eq. (A27)]

\[ X_j'(r) \equiv \frac{\exp(-ikz)}{\sqrt{Z}} f'_{-\sigma} \Theta'_{n,x_0}(x, y). \] (III.72)

Obviously, these states are normalized as

\[ \int dr \sqrt{\phi^3} V^{(0)}_j(r) V^{(0)}_{j'}(r) = \delta_{J,J}. \] (III.73)

and are orthogonal to the eigenstates \( U^{(0)}_J(r) \) up to zero-order terms:

\[ \int dr \sqrt{\phi^3} V^{(0)*}_j(r) U^{(0)}_{J'}(r) = \int dr \sqrt{\phi^3} U^{(0)*}_j(r) V^{(0)}_{J'}(r) = 0. \] (III.74)

### 3.2 Computation of the one-particle states \( U_j(r) \) and \( V_j(r) \) up to first order

We can pass now to the determination of the one-particle states \( U_j(r) \) and \( V_j(r) \) up to first order in the perturbation \( I(r, -i\partial) \). From Eqs. (III.60) and (III.69) we see that the first-order energies of the transverse ground states that are given by

\[ \epsilon^{(1)}_{k,x_0} \equiv w^{(1)}_{0,k,-1,x_0} = \langle 0^{(1)}_{0,k,-1,x_0} = \sqrt{\phi_t m^2 + \frac{\phi_t}{\phi_s} k^2 \left[ 1 + \left( \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{k^2}{m^2 + k^2} \right) x_0 \right] \] (III.75)

are independent of the magnetic field. As we have seen in [4], this fact gives the transverse ground states a particular relevance in the case of strong magnetic fields because their energies are much smaller than that of the excited Landau levels that depend on \( B \). In particular, we have shown that in this case the probability that a pair is created with both the electron and the positron in a transverse ground state is much larger than the other probabilities [4]. For this reason we will only calculate this probability and then we need to compute only the one-particle transverse ground states corrected up to first order. As in the previous paragraph, we will present the calculations of the first-order corrections to the electron transverse ground states and we will quote the analogous results for the positron transverse ground states.

The first order corrections to a given zero-order electron transverse ground state come from the coupling of this state to the following classes of states:

1. the zero-order electron transverse ground states with the same energy (all but the state to be corrected);
2. the zero-order electron transverse ground states with different energy;
3. the zero-order electron states which are not transverse ground states;
4. the zero-order positron states.

Suppose, now, that we want to calculate the first-order corrections to the state labeled by $J_0 \equiv \{0, k, -1, x_0\} = \{R_0, x_0\}$. The states in the first class are labeled by $\{R_0, x'_0\}$ with $x'_0 \neq x_0$ because they have the same energy of the $J_0$-state. But, all the contributions vanish because the perturbation (III.9) can not couple two eigenstates of $H^{(0)}(r, -i\partial)$ with the same $n_d$ and two different $x_0$ and $x'_0$. Similarly, the states in the second class are characterized by $J'_0 \equiv \{0, k', -1, x'_0\}$ with $k' \neq k$ and, since $[I(r, -i\partial), P_\parallel] = 0$ they do not give any contributions. Instead, the contributions from the states of the remaining two classes are, in general, different from zero then we write the state $U^{(1)}_{J_0}(r)$ as

$$U^{(1)}_{J_0}(r) = U^{(0)}_{J_0}(r) + \sum' J' P^{(1)}_{J_0, J'} U^{(0)}_{J'}(r) + \sum J' Q^{(1)}_{J_0, J'} V^{(0)}_{J'}(r) \quad (III.76)$$

where the primed sum does not include the transverse ground states and where

$$P^{(1)}_{J_0, J'} = \frac{1}{\varepsilon^{(0)}_k - w^{(0)}_{R'}} \int dr \sqrt{\phi^{(0)}_s} U^{(0)\dagger}_{J'}(r) I(r, -i\partial) U^{(0)}_{J_0}(r), \quad (III.77)$$

$$Q^{(1)}_{J_0, J'} = \frac{1}{\varepsilon^{(0)}_k + w^{(0)}_{R'}} \int dr \sqrt{\phi^{(0)}_s} V^{(0)\dagger}_{J'}(r) I(r, -i\partial) U^{(0)}_{J_0}(r). \quad (III.78)$$

In these equations we introduced the zero-order energies of the transverse ground states $\varepsilon^{(0)}_k$ defined as [see Eqs. (III.27) and (III.28)]

$$\varepsilon^{(0)}_k \equiv w^{(0)}_{R_0} = \tilde{w}^{(0)}_{\tilde{R}_0} = \sqrt{\phi t m^2 + \phi t^2 k^2} \quad (III.79)$$

with $\tilde{R}_0 = \{0, k, +1\}$. We start by calculating the coefficients $P^{(1)}_{J_0, J'}$. From the expression (III.9) of the interaction Hamiltonian and from Eqs. (III.56)
and \((\text{III.57})\) we have

\[
P^{(1)}_{J_0,J'} = \frac{1}{\varepsilon_k^{(0)} - w_R^{(0)}} \left[ \frac{w_R^{(0)} + \sqrt{\phi_m}}{2 w_R^{(0)}} \right] \left[ \frac{\varepsilon_k^{(0)} + \sqrt{\phi_m}}{2 \varepsilon_k^{(0)}} \right] \times \left\{ \left[ \sqrt{\Phi_t} (g_P - g_M) m - \frac{g_P}{2} \left( w_R^{(0)} + \varepsilon_k^{(0)} \right) + g_E \varepsilon_k^{(0)} \right] \int dr \Phi_{n',x_0}^t (r) \psi_{0,x_0} (r) + \left[ -\sqrt{\Phi_t} (g_P - g_M) m - \frac{g_P}{2} \left( w_R^{(0)} + \varepsilon_k^{(0)} \right) + g_E \varepsilon_k^{(0)} \right] \frac{\phi_t}{\phi_s} \times \right. \\
\times \frac{1}{\varepsilon_k^{(0)} + \sqrt{\phi_m} w_R^{(0)} + \sqrt{\phi_m}} \int dr \Phi_{n',x_0}^t (r) \psi' (r, -i \partial) x \psi' (r, -i \partial) \Phi_{0,x_0}^t (r) \left. \left\} \right\}.
\]

\((\text{III.80})\)

By using the orthonormal properties of the functions \(\Phi_{n,x_0}^t (r)\) [see Eq. \((\text{III.55})\)] and Eq. \((\text{B5})\) we obtain

\[
P^{(1)}_{J_0,J'} = \frac{1}{\varepsilon_k^{(0)} - w_R^{(0)}} \left[ \frac{w_R^{(0)} + \sqrt{\phi_m}}{2 w_R^{(0)}} \right] \left[ \frac{\varepsilon_k^{(0)} + \sqrt{\phi_m}}{2 \varepsilon_k^{(0)}} \right] \times \left\{ \left[ \sqrt{\Phi_t} (g_P - g_M) m - \frac{g_P}{2} \left( w_R^{(0)} + \varepsilon_k^{(0)} \right) + g_E \varepsilon_k^{(0)} \right] \frac{1}{\sqrt{2eB}} \delta_{n',+1} \delta_{\sigma',-1} + \left[ -\sqrt{\Phi_t} (g_P - g_M) m - \frac{g_P}{2} \left( w_R^{(0)} + \varepsilon_k^{(0)} \right) + g_E \varepsilon_k^{(0)} \right] \frac{\phi_t}{\phi_s} \times \right. \\
\times \frac{1}{\varepsilon_k^{(0)} + \sqrt{\phi_m} w_R^{(0)} + \sqrt{\phi_m}} \left[ ik \delta_{n',0} \delta_{\sigma',+1} + \frac{k^2}{\sqrt{2eB}} \delta_{n',+1} \delta_{\sigma',-1} \right] \delta_{k',k} \delta_{x_0',x_0} = \left( B^{(1)}_{k,x_0} \delta_{n',+1} \delta_{\sigma',-1} - i C^{(1)}_{k,x_0} \delta_{n',0} \delta_{\sigma',+1} \right) \delta_{k',k} \delta_{x_0',x_0}
\]

\((\text{III.81})\)
where we defined the coefficients

\[
B_{k,x_0}^{(1)} \equiv \frac{1}{\mathcal{E}_k^{(0)} - \varepsilon_k^{(0)}} \left[ \frac{\mathcal{E}_k^{(0)} + \sqrt{\phi_t m}}{2\mathcal{E}_k^{(0)}} \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_t m}}{2\varepsilon_k^{(0)}}} \right. \\
\times \left\{ \sqrt{\phi_t (g_M - g_P)} \frac{m}{\sqrt{2eB}} \left[ 1 - \frac{\phi_t}{\phi_s} \frac{k^2}{(\mathcal{E}_k^{(0)} + \sqrt{\phi_t m})(\varepsilon_k^{(0)} + \sqrt{\phi_t m})} \right] + \right. \\
\left. + \frac{1}{\sqrt{2eB}} \left[ \frac{g_P}{2} (\mathcal{E}_k^{(0)} + \varepsilon_k^{(0)}) - g_E \varepsilon_k^{(0)} \right] \left[ 1 + \frac{\phi_t}{\phi_s} \frac{k^2}{(\mathcal{E}_k^{(0)} + \sqrt{\phi_t m})(\varepsilon_k^{(0)} + \sqrt{\phi_t m})} \right] \right\} ,
\]

(III.82)

\[
C_{k,x_0}^{(1)} \equiv \frac{1}{\mathcal{E}_k^{(0)} - \varepsilon_k^{(0)}} \left[ \frac{\mathcal{E}_k^{(0)} + \sqrt{\phi_t m}}{2\mathcal{E}_k^{(0)}} \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_t m}}{2\varepsilon_k^{(0)}}} \right. \\
\times \left[ \sqrt{\phi_t (g_M - g_P)} m - \frac{g_P}{2} (\mathcal{E}_k^{(0)} + \varepsilon_k^{(0)}) + g_E \varepsilon_k^{(0)} \right] \frac{\phi_t}{\phi_s} \frac{k}{(\mathcal{E}_k^{(0)} + \sqrt{\phi_t m})(\varepsilon_k^{(0)} + \sqrt{\phi_t m})} 
\]

(III.83)

with [see Eqs. (III.27) and (III.28)]

\[
\mathcal{E}_k^{(0)} \equiv w_{0,k,+1}^{(0)} = w_{1,k,-1}^{(0)} = \tilde{w}_{0,k,-1}^{(0)} = \tilde{w}_{1,k,+1}^{(0)} = \sqrt{\phi_t m^2 + \frac{\phi_t}{\phi_s} (k^2 + 2eB)}
\]

(III.84)

the zero-order energy of the first excited Landau level. In the same way we
can write the coefficients $Q_{j_0,j'}^{(1)}$ as

\[
Q_{j_0,j'}^{(1)} = \frac{\sigma'}{\varepsilon_k^{(0)} + \bar{\omega}_R^{(0)}} \left( \frac{\bar{\varepsilon}_k^{(0)} + \sqrt{\phi_{j_0}}}{2\bar{\varepsilon}_k^{(0)}} \right) \times \left\{ \left[ \sqrt{\phi_{j_0}} (g_P - g_M) m - \frac{g_P}{2} (\varepsilon_k^{(0)} - \bar{\omega}_R^{(0)}) + g_E \varepsilon_k^{(0)} \right] (1) \sqrt{\frac{\phi_{j_0}}{\phi_s} \bar{\omega}_R^{(0)} + \sqrt{\phi_{j_0}} m} \right. \\
\times \left[ -i \delta_{\eta',0} \delta_{\sigma',-1} - k x_0 \delta_{\eta',0} \delta_{\sigma',+1} - \frac{k}{\sqrt{2eB}} \delta_{\eta',+1} \delta_{\sigma',+1} \right] + \\
+ \left[ -\sqrt{\phi_{j_0}} (g_P - g_M) m - \frac{g_P}{2} (\varepsilon_k^{(0)} - \bar{\omega}_R^{(0)}) + g_E \varepsilon_k^{(0)} \right] \sqrt{\frac{\phi_{j_0}}{\phi_s} \varepsilon_k^{(0)} + \sqrt{\phi_{j_0}} m} \times \left[ -k x_0 \delta_{\eta',0} \delta_{\sigma',+1} - \frac{k}{\sqrt{2eB}} \delta_{\eta',+1} \delta_{\sigma',+1} \right] \delta_{\eta',-k} \delta_{\eta',-x_0} = \\
\left( -D_{k,x_0}^{(1)} \delta_{\eta',0} \delta_{\sigma',+1} + iE_{k,x_0}^{(1)} \delta_{\eta',0} \delta_{\sigma',-1} - F_{k,x_0}^{(1)} \delta_{\eta',+1} \delta_{\sigma',+1} \right) \delta_{\eta',-k} \delta_{\eta',-x_0} \\
(III.85)
\]

with

\[
D_{k,x_0}^{(1)} = \frac{1}{2} \left( \frac{\phi_{j_0}}{\phi_s} \right) k x_0, \\
(III.86)
\]

\[
E_{k,x_0}^{(1)} = \frac{1}{\varepsilon_k^{(0)} + \bar{\omega}_R^{(0)}} \left( \frac{\varepsilon_k^{(0)} + \sqrt{\phi_{j_0}}}{2\varepsilon_k^{(0)}} \right) \times \left\{ \sqrt{\phi_{j_0}} (g_M - g_P) m - \frac{g_P}{2} (\varepsilon_k^{(0)} - \bar{\omega}_R^{(0)}) - g_E \varepsilon_k^{(0)} \right\} \sqrt{\frac{\phi_{j_0}}{\phi_s} \varepsilon_k^{(0)} + \sqrt{\phi_{j_0}} m}, \\
(III.87)
\]

\[
F_{k,x_0}^{(1)} = \frac{1}{\varepsilon_k^{(0)} + \bar{\omega}_R^{(0)}} \left( \frac{\varepsilon_k^{(0)} + \sqrt{\phi_{j_0}}}{2\varepsilon_k^{(0)}} \right) \times \left\{ \sqrt{\phi_{j_0}} (g_M - g_P) m \left( \frac{1}{\varepsilon_k^{(0)} + \sqrt{\phi_{j_0}} m} + \frac{1}{\varepsilon_k^{(0)} + \sqrt{\phi_{j_0}} m} \right) + \\
+ \left[ \frac{g_P}{2} (\varepsilon_k^{(0)} - \bar{\omega}_R^{(0)}) + g_E \varepsilon_k^{(0)} \right] \left( \frac{1}{\varepsilon_k^{(0)} + \sqrt{\phi_{j_0}} m} - \frac{1}{\varepsilon_k^{(0)} + \sqrt{\phi_{j_0}} m} \right) \right\}. \\
(III.88)
\]
If we also define the coefficients $A_{k,x_0}^{(1)}$ as

$$A_{k,x_0}^{(1)} = \frac{gE}{2}x_0 \quad (\text{III.89})$$

then, the first order transverse ground state $U_{0,k,-1,x_0}^{(1)}(r)$ can be written simply as

$$U_{0,k,-1,x_0}^{(1)}(r) = \left(1 + A_{k,x_0}^{(1)}\right)U_{0,k,-1,x_0}^{(0)}(r) + B_{k,x_0}^{(1)}U_{1,k,-1,x_0}^{(0)}(r) - iC_{k,x_0}^{(1)}U_{0,k,+1,x_0}^{(0)}(r) - D_{k,x_0}^{(1)}V_{0,-k,+1,x_0}^{(0)}(r) + iE_{k,x_0}^{(1)}V_{0,-k,-1,x_0}^{(0)}(r) - F_{k,x_0}^{(1)}V_{1,-k,+1,x_0}^{(0)}(r). \quad (\text{III.90})$$

The term $A_{k,x_0}^{(1)}U_{0,k,-1,x_0}^{(0)}(r) = gE x_0 U_{0,k,-1,x_0}^{(0)}(r)/2$ has been added to compensate for the factor $(1 - gE x_0)$ in the scalar product $\langle \text{III.60} \rangle$ and then to have the states correctly normalized up to first order, as

$$\langle U_{j_0}^{(1)}, U_{j_0'}^{(1)} \rangle = \delta_{j_0,j_0'}. \quad (\text{III.91})$$

We observe that, even if $A_{k,x_0}^{(1)}, \ldots, F_{k,x_0}^{(1)}$ are all first order-quantities in the couplings $g_E, g_P$ and $g_M$, the coefficients $B_{k,x_0}^{(1)}, C_{k,x_0}^{(1)}, E_{k,x_0}^{(1)}$, and $F_{k,x_0}^{(1)}$ go, for strong magnetic fields, as $1/\sqrt{2eB}$ and this circumstance makes them, in general, much smaller than $A_{k,x_0}^{(1)}$ and $D_{k,x_0}^{(1)}$. We will exploit this observation in the next paragraph.

Finally, with analogous calculations it can be shown that the first-order positron transverse ground state $V_{0,k,+1,x_0}^{(1)}(r)$ can be written as

$$V_{0,k,+1,x_0}^{(1)}(r) = \left(1 + A_{k,x_0}^{(1)}\right)V_{0,k,+1,x_0}^{(0)}(r) + B_{k,x_0}^{(1)}V_{1,k,+1,x_0}^{(0)}(r) - iC_{k,x_0}^{(1)}V_{0,k,-1,x_0}^{(0)}(r) - D_{k,x_0}^{(1)}U_{0,-k,+1,x_0}^{(0)}(r) + iE_{k,x_0}^{(1)}U_{0,-k,-1,x_0}^{(0)}(r) - F_{k,x_0}^{(1)}U_{1,-k,+1,x_0}^{(0)}(r). \quad (\text{III.92})$$

The states $V_{0,k,+1,x_0}^{(1)}(r)$ are also normalized as

$$\langle V_{j_0}^{(1)}, V_{j_0'}^{(1)} \rangle = \delta_{j_0,j_0'}. \quad (\text{III.93})$$

and they are orthogonal up to first-order to the states $U_{j_0}^{(1)}(r)$:

$$\langle V_{j_0}^{(1)}, U_{j_0'}^{(1)} \rangle = \langle U_{j_0}^{(1)}, V_{j_0'}^{(1)} \rangle = 0. \quad (\text{III.94})$$

Before continuing our computation of the instantaneous eigenstates of the Hamiltonian $H^{(1)}(t)$ we want to show a nice property about the mean
value of the velocity operator $\alpha$ in the first-order eigenstates (III.90) and (III.92). In particular, we will calculate the mean value of the $x$-component of the velocity given by $v_x \equiv \alpha_x$ and of the component of the velocity in the axis orthogonal to the $x$-axis and to the direction of $B$ given by $v_\perp \equiv \alpha_y \cos \vartheta - \alpha_z \sin \vartheta$ [see Eq. (III.3)] in the eigenstates $U_{j_0}^{(1)}(r) = U_{0,k,-1,x_0}^{(1)}(r)$ and $V_{j_0}^{(1)}(r) = V_{0,k,+1,x_0}^{(1)}(r)$

\[
\langle U_{j_0}^{(1)} | v_x | U_{j_0}^{(1)} \rangle = \int \, dr \sqrt{2}(1 - g_{Ex})U_{0,k,-1,x_0}^{(1)}(r)\alpha_x U_{0,k,-1,x_0}^{(1)}(r), \quad (III.95)
\]

\[
\langle U_{j_0}^{(1)} | v_\perp | U_{j_0}^{(1)} \rangle = \int \, dr \sqrt{2}(1 - g_{Ex})U_{0,k,-1,x_0}^{(1)}(r)(\alpha_y \cos \vartheta - \alpha_z \sin \vartheta)U_{0,k,-1,x_0}^{(1)}(r) \quad (III.96)
\]

and

\[
\langle V_{j_0}^{(1)} | v_x | V_{j_0}^{(1)} \rangle = \int \, dr \sqrt{2}(1 - g_{Ex})V_{0,k,+1,x_0}^{(1)}(r)\alpha_x V_{0,k,+1,x_0}^{(1)}(r), \quad (III.97)
\]

\[
\langle V_{j_0}^{(1)} | v_\perp | V_{j_0}^{(1)} \rangle = \int \, dr \sqrt{2}(1 - g_{Ex})V_{0,k,+1,x_0}^{(1)}(r)(\alpha_y \cos \vartheta - \alpha_z \sin \vartheta)V_{0,k,+1,x_0}^{(1)}(r). \quad (III.98)
\]

Now, we proceed by considering only the first two mean values because that concerning the positrons can be calculated in an analogous way. By using the transformation properties (A.43)-(A.45) and the definitions $\alpha_{\pm} = (\alpha_x \pm i\alpha_y)/2$ we have

\[
\langle U_{j_0}^{(1)} | v_x | U_{j_0}^{(1)} \rangle = \int \, dr \sqrt{2}(1 - g_{Ex})U_{0,k,-1,x_0}^{(1)}(r)\alpha_x U_{0,k,-1,x_0}^{(1)}(r) = \langle U_{j_0}^{(1)} | \alpha_- | U_{j_0}^{(1)} \rangle + \langle U_{j_0}^{(1)} | \alpha_+ | U_{j_0}^{(1)} \rangle, \quad (III.99)
\]

\[
\langle U_{j_0}^{(1)} | v_\perp | U_{j_0}^{(1)} \rangle = \int \, dr \sqrt{2}(1 - g_{Ex})U_{0,k,-1,x_0}^{(1)}(r)(\alpha_y \cos \vartheta - \alpha_z \sin \vartheta)U_{0,k,-1,x_0}^{(1)}(r) = i \left( \langle U_{j_0}^{(1)} | \alpha_- | U_{j_0}^{(1)} \rangle - \langle U_{j_0}^{(1)} | \alpha_+ | U_{j_0}^{(1)} \rangle \right) \quad (III.100)
\]
where, by keeping only the terms up to first order

\[
\langle U^r_{J_0} | \alpha_- | U^r_{J_0} \rangle = \int dr \sqrt{\phi_s} U^r_{0,k,-1,x_0}(r) \alpha_- \left[ \mathcal{B}^{(1)}_{k,x_0} U^r_{1,k,-1,x_0}(r) - i \mathcal{C}^{(1)}_{k,x_0} U^r_{0,k,+1,x_0}(r) + i \mathcal{E}^{(1)}_{k,x_0} V^r_{0,-k,-1,x_0}(r) - F^{(1)}_{k,x_0} V^r_{1,-k,+1,x_0}(r) \right],
\]

(III.101)

\[
\langle U^r_{J_0} | \alpha_+ | U^r_{J_0} \rangle = \int dr \sqrt{\phi_s} \left[ \mathcal{B}^{(1)}_{k,x_0} U^r_{1,k,-1,x_0}(r) + i \mathcal{C}^{(1)}_{k,x_0} U^r_{0,k,+1,x_0}(r) - i \mathcal{E}^{(1)}_{k,x_0} V^r_{0,-k,-1,x_0}(r) - F^{(1)}_{k,x_0} V^r_{1,-k,+1,x_0}(r) \right] \alpha_+ U^r_{0,k,-1,x_0}(r).
\]

(III.102)

Now, in appendix D we indicate the technique to calculate these matrix elements and the results are

\[
\langle U^r_{J_0} | \alpha_- | U^r_{J_0} \rangle - \langle U^r_{J_0} | \alpha_+ | U^r_{J_0} \rangle = i \frac{\sqrt{\phi_s m^2 + k^2}}{2eB} \left( \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{k^2}{\phi_s m^2 + k^2} \right),
\]

(III.103)

then, from Eqs. (III.99) and (III.100) we obtain

\[
\langle U^r_{J_0} | v_x | U^r_{J_0} \rangle = 0,
\]

(III.104)

\[
\langle U^r_{J_0} | v_\perp | U^r_{J_0} \rangle = -\frac{1}{eB} \sqrt{\phi_s} \left( \phi_t m^2 + \phi_t k^2 \right) \left( \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{k^2}{\phi_s m^2 + k^2} \right),
\]

(III.105)

and, analogously,

\[
\langle V^r_{J_0} | v_x | V^r_{J_0} \rangle = 0,
\]

(III.106)

\[
\langle V^r_{J_0} | v_\perp | V^r_{J_0} \rangle = \frac{1}{eB} \sqrt{\phi_s} \left( \phi_t m^2 + \phi_t k^2 \right) \left( \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{k^2}{\phi_s m^2 + k^2} \right).
\]

(III.107)

This results are just the quantum counterpart of the so-called \((E \times B)\)-drift-velocity effect typical of the motion of a charged particle in the presence of a uniform and constant electromagnetic field \((E, B)\) \[25\]. In our case, obviously, the electric force is substituted by the gravitational force [see Eq. (III.67)] and the sign of the drift velocity depends on the sign of the charge of the particle.
### 3.3 Instantaneous first-order eigenstates and eigenvalues of the Hamiltonian $H^{(1)}(t)$

The results obtained in the previous paragraphs will be used here to solve our initial problem: the determination of the instantaneous eigenstates and eigenenergies of the slowly varying Hamiltonian

$$H^{(1)}(t) = \int dr \sqrt{\phi^3_s(1 - gE_x)} \Psi^+(r, t) \mathcal{H}^{(1)}(r, -i\partial_t) \Psi(r, t). \tag{III.108}$$

In fact, we have only to substitute everywhere in the results just obtained the constant magnetic field $B$ with the time-dependent magnetic field $B(t)$ given in Eq. (II.12). In this way, the one-particle eigenstates both at zero and first order will depend explicitly on time such as the eigenenergies apart from those of the transverse ground states that do not depend on the magnetic field.

With this prescription if we expand the field $\Psi(r, t)$ with respect to the first-order instantaneous basis $[U^{(1)}_J(r, t), V^{(1)}_J(r, t)]$ as

$$\Psi(r, t) = \sum_J \left[ c^{(1)}_J(t) U^{(1)}_J(r, t) + d^{(1)\dagger}_J(t) V^{(1)}_J(r, t) \right]$$

the instantaneous eigenstates of the Hamiltonian (II.63) are given by the Fock states

$$|\{n_J(t); \tilde{n}_{J'}(t)\}⟩ \equiv \left[ c^{(1)\dagger}_{J_1}(t) \right]^{n_{J_1}} \left[ c^{(1)\dagger}_{J_2}(t) \right]^{n_{J_2}} \cdots \left[ d^{(1)\dagger}_J(t) \right]^{\tilde{n}_J} \left[ d^{(1)\dagger}_{J'}(t) \right]^{\tilde{n}_{J'}} |0(t)⟩ \tag{III.110}$$

with $|0(t)⟩$ the instantaneous vacuum state, while the instantaneous eigenenergies are

$$E^{(1)}(t) = \sum_J \left[ w^{(1)}_{J_1}(t)n_J + w^{(1)}_{J_2}(t)\tilde{n}_{J'} \right]. \tag{III.111}$$

In this framework the creation of a pair at time $t$ with the electron in the state labeled by $J$ and the positron in the state labeled by $\tilde{J}'$ is the transition from the vacuum state $|0(t)⟩$ to the pair state $|1_J(t); \tilde{1}_{J'}(t)⟩ \equiv c^{\dagger}_J(t)d^{\dagger}_{J'}(t) |0(t)⟩$.

### 4 Calculation of the production probabilities

As we have said in the previous section we can calculate the probability of producing an $e^- - e^+$ pair in the presence of the slowly varying magnetic field (II.12) and of the static gravitational field described by the metric tensor (II.3) by means of the adiabatic perturbation theory up to first order in the
time-derivative of the magnetic field. In order to avoid the possible confusion between the “first order” relative to the adiabatic perturbation theory and the “first order” relative to the gravitational couplings $g_E, g_P$ and $g_M$, in what follows we will always refer to the second one. In particular, the symbol $(1)$ indicates quantities that are first-order in the gravitational couplings.

From now on, the gravitational field will not play any further role: we took into account its presence by correcting the one-particle electron and positron eigenstates and eigenenergies. Now, following the adiabatic perturbation theory, the operator responsible of the pair production is the time derivative of the second-quantized Hamiltonian (II.63) that, by using Eq. (II.52), can be written as

$$\dot{H}^{(1)}(t) = \int d\mathbf{r} \sqrt{\phi_s^3} \langle 1 - g_E x \rangle \Psi^\dagger(\mathbf{r}, t) \partial_\mathbf{B} \mathcal{H}^{(1)}(\mathbf{r}, -i\partial_t, t) \cdot \dot{\mathbf{B}}(t) \Psi(\mathbf{r}, t) =$$

$$= \int d\mathbf{r} \sqrt{\phi_s^3} \langle 1 - g_E x \rangle \Psi^\dagger(\mathbf{r}, t) \sqrt{\frac{\phi_s}{\phi_e}} \left[ 1 - (g_P - g_E)x \right] \frac{e}{2} (\mathbf{r} \times \alpha) \cdot \dot{\mathbf{B}}(t) \Psi(\mathbf{r}, t) \simeq$$

$$\simeq \sqrt{\frac{\phi_s e \dot{\mathbf{B}}(t)}{2}} \cdot \int d\mathbf{r} \sqrt{\phi_s^3} \langle 1 - g_P x \rangle \Psi^\dagger(\mathbf{r}, t) (\mathbf{r} \times \alpha) \Psi(\mathbf{r}, t).$$

Starting from the second line of this equation the physical meaning of the result can be understood. In fact, the vector $\alpha$ can be interpreted as the one-particle relativistic operator corresponding to the velocity of the electron, then the quantity $(\mathbf{r} \times \alpha) \cdot \dot{\mathbf{B}}(t) = -[\mathbf{r} \times \dot{\mathbf{B}}(t)] \cdot \alpha$ is proportional to the scalar product of the external electric field $-\partial \mathbf{A}(\mathbf{r}, t)/\partial t$ [see Eq. (II.25)] and of the electron velocity that is to the work per unit time done by the induced electric field itself. On this respect, we want to do a couple of observations that can also be referred to our previous papers [2, 3, 4]. Firstly, the induced electric field $\mathbf{E}(\mathbf{r}, t) = -\partial \mathbf{A}(\mathbf{r}, t)/\partial t$ is always perpendicular to the magnetic field $\mathbf{B}(t)$. Secondly, due to the presence of $\mathbf{E}(\mathbf{r}, t)$ we may conclude that this last field is the responsible for the production according to the well-known mechanism proposed by Schwinger in [38]. There are however two differences to be stressed:

1. because of the presence of the time depending magnetic field $\mathbf{B}(t)$ the electric field $\mathbf{E}(\mathbf{r}, t)$ is rotational, so it does not admit a scalar potential and the interpretation of pair production as tunnel effect is not straightforward;

2. the one-particle states of the produced electron-positron pair are states in a magnetic field and they very different from the one-particle states in
an electric field (for example, the transverse motion here is quantized and the energy levels with different \( n_d \) are well separated from each other).

Now, we pointed out in the previous section that the probability that a pair is created with both the electron and positron in a transverse ground state is much larger than the others probabilities. If the electron is assumed to be created in the transverse ground state \( U^{(1)}_{J_0}(r,t) \) with \( J_0 = \{0, k, -1, x_0\} \) and the positron in the transverse ground state \( V^{(1)}_{\tilde{J}_0}(r,t) \) with \( \tilde{J}_0 = \{0, k', +1, x'_0\} \), then the matrix element of the transition is given by

\[
\hat{H}^{(1)}_{J_0\tilde{J}_0}(t) \equiv |1_{J_0}(t); \tilde{1}_{\tilde{J}_0}(t)| \hat{H}^{(1)}(t)|0(t)\rangle = \\
\sqrt{\frac{\phi_s}{\phi_t}} \epsilon_B(t) \cdot \int d\mathbf{r} \sqrt{\phi_s(1 - g_p x)} U^{(1)\dagger}_{J_0}(r,t) (\mathbf{r} \times \alpha) V^{(1)}_{\tilde{J}_0}(r,t) 
\]

(IV.2)

where \(|0(t)\rangle\) and \(|1_{J_0}(t); \tilde{1}_{\tilde{J}_0}(t)\rangle\) are the vacuum and the pair state at time \( t \) respectively. The creation amplitude at time \( t \) can be calculated from this matrix element as \cite{27}

\[
\gamma^{(1)}_{J_0\tilde{J}_0}(t) = \frac{1}{\varepsilon^{(1)}_{k,x_0} + \varepsilon^{(1)}_{k',x'_0}} \int_0^t dt' \hat{H}^{(1)}_{J_0\tilde{J}_0}(t') \exp \left[ i \left( \varepsilon^{(1)}_{k,x_0} + \varepsilon^{(1)}_{k',x'_0} \right) t' \right] 
\]

(IV.3)

where we used the fact that the first-order energies \( \varepsilon^{(1)}_{k,x_0} \) of the transverse ground states do not depend on \( \mathbf{B}(t) \) and then on time [see Eq. (III.75)].

In the following we will distinguish two different time evolutions of the magnetic field:

1. rotating magnetic field: \( \mathbf{B}(t) = \mathbf{B}_\wedge(t) = B_\wedge(0, \sin \omega t, \cos \omega t) \);

2. magnetic field varying only in strength: \( \mathbf{B}(t) = \mathbf{B}_\uparrow(t) = (0, 0, B_\uparrow(t)) \)

with

\[
B_\uparrow(t) = B_f + (B_i - B_f) \exp \left( -\frac{t}{\tau} \right) 
\]

(IV.4)

and \( B_i < B_f \). Obviously, in both cases the magnetic field is assumed to be strong and slowly varying that is [see Eqs. (II.13) and (II.16)]

\[
B_\wedge \gg \frac{m^2}{e}, \hspace{1cm} (IV.5) \\
\omega \ll m \hspace{1cm} (IV.6)
\]

34
and

\[ B_i \gg \frac{m^2}{e}, \quad (IV.7) \]

\[ \frac{B_f - B_i}{B_i} \tau \ll m. \quad (IV.8) \]

The first configuration describes better that case in which the magnetic field is produced by a rotating body while the second one concerns a situation in which it is generated by a collapsing body.

### 4.1 Rotating magnetic field

In the case of rotating magnetic fields we know that in Minkowski spacetime it is already possible to create a pair with both the electron and the positron in a transverse ground state \[4\]. Our task here is to calculate the corrections due to the presence of the gravitational field to this “flat” creation probability. For this reason and reminding that for strong magnetic fields the coefficients \( A^{(1)}_{k,x_0} \) and \( D^{(1)}_{k,x_0} \) are much larger than \( B^{(1)}_{k,x_0}, C^{(1)}_{k,x_0}, E^{(1)}_{k,x_0} \) and \( F^{(1)}_{k,x_0} \), we can keep for simplicity only the terms proportional to \( A^{(1)}_{k,x_0} \) and \( D^{(1)}_{k,x_0} \) (apart from the zero-order term) in the expressions of the first-order transverse ground states.\(^6\) With this simplification the matrix element (IV.2) can be written as

\[ \text{Note that in the case of purely rotating magnetic field the coefficients } A^{(1)}_{k,x_0}, \ldots, F^{(1)}_{k,x_0} \text{ are time-independent because they depend on the strength of the magnetic field. The same thing can be said about the electron and positron eigenenergies and about the rotated eigenstates } U^{(1)}_{J}(r) \text{ and } V^{(1)}_{J}(r). \text{ Instead, the eigenstates } U^{(1)}_{J}(r,t) \text{ and } V^{(1)}_{J}(r,t) \text{ depend on time through the rotation operator } \mathcal{R}_z[\theta(t)] = \mathcal{R}_z(\omega t). \]
\[ \dot{H}_{\sim}(J_0^0, t) = \sqrt{\frac{\phi_t}{\phi_s}} e^{B_{\sim}(t)} \cdot \int dr \sqrt{\phi_s^3 (1 - g_P x)} U_{J_0}^{(1)\dagger}(r) R_x(\omega t) (r \times \alpha) R_x^\dagger(\omega t) V_{J_0}^{(1)}(r) = \sqrt{\frac{\phi_t}{\phi_s}} e^{B_{\sim}} \int dr \sqrt{\phi_s^3 (1 - g_P x)} U_{J_0}^{(1)\dagger}(r) (r \times \alpha)_y V_{J_0}^{(1)}(r) \simeq \sqrt{\frac{\phi_t}{\phi_s}} e^{B_{\sim}} \int dr \sqrt{\phi_s^3 (1 - g_P x)} \times \left[ (1 + A_{k,x_0}^{(1)}) U_{0,k,-1,x_0}^{(0)\dagger}(r) - D_{0,-k,+1,x_0}^{(1)} V_{0,k,-1,x_0}^{(0)}(r) \right] x \alpha \times \left[ (1 + A_{k',x_0}^{(1)}) V_{0,k',+1,x'_0}^{(0)}(r) - D_{0,-k',-1,x'_0}^{(1)} U_{0,k',-1,x'_0}^{(0)}(r) \right] \right]. \] (IV.9)

The matrix element does not depend on time and, if we keep only the first order terms in \( g_E, g_P \) and \( g_M \), we have

\[ \dot{H}_{\sim}(J_0^0, t) \simeq \sqrt{\frac{\phi_t}{\phi_s}} e^{B_{\sim}} \left\{ \int dr \sqrt{\phi_s^3} U_{0,k,-1,x_0}^{(0)\dagger}(r) (1 - g_P x) x \alpha_z V_{0,k',+1,x'_0}^{(0)}(r) + \int dr \sqrt{\phi_s^3} \left[ A_{k,x_0}^{(1)} U_{0,k,-1,x_0}^{(0)\dagger}(r) - D_{0,-k,+1,x_0}^{(1)} V_{0,k,-1,x_0}^{(0)}(r) \right] x \alpha_z V_{0,k'+1,x'_0}^{(0)}(r) + \right. \left. \int dr \sqrt{\phi_s^3} U_{0,k,-1,x_0}^{(0)\dagger}(r) x \alpha_z \left[ A_{k',x_0}^{(1)} V_{0,k',+1,x'_0}^{(0)}(r) - D_{0,-k',-1,x'_0}^{(1)} U_{0,k',-1,x'_0}^{(0)}(r) \right] \right\}. \] (IV.10)

Now, from Eq. (A14) we see that the matrix elements of the operator \((1 - g_P x) x\) between two transverse ground states that have both \( n = 0 \) are equal to that of the operator \((1 - g_P x_0) x_0 + 1/(2eB)\) or, by coherently neglecting the last term, to that of \((1 - g_P x_0) x_0\). In this way, the matrix element (IV.10)
becomes

\[ \hat{H}^{(1)}_{\wedge_\gamma,(J_0,J'_0)} \simeq \sqrt{\frac{\phi_t e\omega B_{\wedge}}{\phi_s}} \left\{ [1 - (g_P - g_E)x_0]x_0 \times \right. \\
\left. \int dr \sqrt{\phi_s^2 U^{(0)\dagger}_{0,k,-1,x_0}(r) \alpha_z V^{(0)\dagger}_{0,k',+1,x'_0}(r)} - \\
-D_{k',x'_0}^{(1)} x_0 \int dr \sqrt{\phi_s^2 V^{(0)}_{0,-k,+1,x_0}(r) \alpha_z V^{(0)}_{0,k',+1,x'_0}(r)} - \\
-D_{k,x_0}^{(1)} x_0 \int dr \sqrt{\phi_s^2 U^{(0)}_{0,k,-1,x_0}(r) \alpha_z U^{(0)}_{0,k',-1,x'_0}(r)} \right\}. \]  

(IV.11)

By using the explicit expressions of the transverse ground states given in appendix C it can be easily seen that

\[ \int dr \sqrt{\phi_s^2 U^{(0)\dagger}_{0,k,-1,x_0}(r) \alpha_z V^{(0)\dagger}_{0,k',+1,x'_0}(r)} = -\sqrt{\frac{\phi_t m}{\varepsilon_{k}^{(0)}}} \delta_{k,-k'} \delta_{x_0,x'_0}. \]  

(IV.12)

\[ \int dr \sqrt{\phi_s^2 V^{(0)\dagger}_{0,-k,+1,x_0}(r) \alpha_z V^{(0)\dagger}_{0,k',+1,x'_0}(r)} = \sqrt{\frac{\phi_t m}{\varepsilon_{k}^{(0)}}} \delta_{k,-k'} \delta_{x_0,x'_0}. \]  

(IV.13)

\[ \int dr \sqrt{\phi_s^2 U^{(0)\dagger}_{0,k,-1,x_0}(r) \alpha_z U^{(0)\dagger}_{0,k',-1,x'_0}(r)} = \sqrt{\frac{\phi_t m}{\varepsilon_{k}^{(0)}}} \delta_{k,-k} \delta_{x_0,x'_0}. \]  

(IV.14)

By substituting these results and the expressions (III.86) of the coefficients \( D^{(1)}_{k,x_0} \) in Eq. (IV.11), we obtain

\[ \hat{H}^{(1)}_{\wedge_\gamma,(J_0,J'_0)} = -\sqrt{\frac{\phi_t e\omega B_{\wedge}}{\phi_s}} \frac{\sqrt{\phi_t m x_0}}{\sqrt{\phi_s m^2 + k^2}} \left[ 1 + \left( \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{\phi_s m^2}{\phi_s m^2 + k^2} \right) x_0 \right] \delta_{k,-k'} \delta_{x_0,x'_0}. \]  

(IV.15)

where we used the definitions (III.33)-(III.41) of \( g_E, g_P \) and \( g_M \). Because of the presence of the Kronecker delta functions, the only transition amplitude different from zero is given by [see Eq. (IV.3)]

\[ \gamma^{(1)}_{\wedge_\gamma,(0,k,-1,x_0,0,-k,+1,x_0)}(t) = \frac{1}{2\varepsilon_{k,x_0}^{(1)}} \frac{\hat{H}^{(1)}_{\wedge_\gamma,(J_0,J'_0)}}{2i\varepsilon_{k,x_0}^{(1)}} \exp \left( i2\varepsilon_{k,x_0}^{(1)} t \right) - 1. \]  

(IV.16)

If we square the modulus of this expression and multiply it by the number of states [see Eq. (III.31)]

\[ d\mathcal{N}_{\wedge_\gamma} = \frac{eB_{\wedge}Y}{2\pi} dx_0 \times \frac{Z}{2\pi} dk. \]  

(IV.17)
we obtain the differential probability that a pair is present at time \( t \) with the electron (positron) between \( x_0 \) and \( x_0 + dx_0 \) and with longitudinal momentum between \( k \) and \( k + dk \) (\( -k \) and \( -k - dk \)):

\[
dP^{(1)}(x_0, k; t) = \frac{eB\phi_s}{16\phi_t} \left( \frac{e\omega B}{2\pi} \right)^2 \frac{\phi_s m^2 x_0^2}{(\phi_s m^2 + k^2)^3} \left[ 1 - \left( \frac{2g_t}{\phi_t} + \frac{g_s k^2 - 5\phi_s m^2}{\phi_s \phi_s m^2 + k^2} \right) x_0 \right] \times \sin^2 \left( \epsilon_{k,x_0}^{(1)} \right) dV_{(1)} dk
\]

(IV.18)

where the continuum limits \( Y \to \infty \) and \( Z \to \infty \) have been tacitly performed and where the “physical” first-order volume \( dV^{(1)} = \phi_s^{3/2} (1 - 3g_s x_0/\phi_s) (Y Z dx_0) \) has been introduced. In conclusion, we obtain our final result by averaging Eq. (IV.18) with respect to time and by dividing it by the phase-space volume \( dV^{(1)} \)

\[
\left\langle \frac{dP^{(1)}(x_0, k; t)}{dV dk} \right\rangle = \frac{1}{2\pi^2 \phi_t} \left( \frac{eB}{4(\phi_s m^2 + k^2)} \right)^3 \left( \sqrt{\phi_s m \omega x_0} \right)^2 \times \left[ 1 - \left( \frac{2g_t}{\phi_t} + \frac{g_s k^2 - 5\phi_s m^2}{\phi_s \phi_s m^2 + k^2} \right) x_0 \right].
\]

(IV.19)

This result is to be compared to the “flat” probability that can be obtained by putting \( g_t = g_s = 0 \) and \( \phi_t = \phi_s = 1 \) [see Eqs. (II.7) and (II.8)] and that is identical to the result (1) in [7]. We note that, since in our approximations \( g_t/\phi_t \) can be assumed to be much larger than \( g_s/\phi_s \), the creation probability increases for negative values of \( x_0 \). Analogously to the sign of the first-order correction to the eigenenergies [see discussion below Eq. (III.60)], this fact can be understood in classical terms by observing that the gravitational force (III.64) associated to the metric (II.4) has only the negative \( x \)-component (III.67) and then, the electron and the positron are more likely created at \( x_0 < 0 \). Nevertheless, we point out that the first-order correction of the probability is not proportional to the gravitational force but this is due to the fact that the operator responsible of the pair creation is not the Hamiltonian but its time-derivative which has a completely different structure. From this point of view, it is worth noting that the physical origin of the presence of the length \( x_0 \) in Eq. (IV.19) is twofold. In fact, since, as we have said below Eq. (IV.1), the matrix element of the time-derivative of the Hamiltonian is proportional to the matrix element of the power supplied by the external electric field which grows linearly with the transverse spatial coordinates, then the probability (IV.19) results proportional to \( x_0^2 \). Instead, the first order correction due to the gravitational field is proportional to \( x_0 \) because the gravitational force is uniform [see Eq. (III.63)]. In this sense,
the dependence of our final results on the quantity $x_0$ is unavoidable since it is a direct consequence of the (approximated) model we used that is of the assumed uniformity both of the magnetic and of the gravitational field at a microscopic scale. A more realistic model should take into account that these fields are not actually uniform in the whole space and that they vanish far from their source but the calculations would be too complicated. Concerning this point, we want to stress that from the derivation itself of Eq. (IV.19), a clear physical meaning can be assigned to the quantity $x_0$. In fact, it can be interpreted as a typical length scale in which the magnetic (gravitational) field produced by the astrophysical compact object can be assumed to be uniform and this fact gives the possibility to do also a quantitative prediction from Eq. (IV.19).

A final note concerns the presence in Eq. (IV.19) of an overall factor $\phi_s/\phi_t > 1$ that increases the creation probability with respect to its “flat” value.

### 4.2 Magnetic field varying only in strength

We have seen in [4] that, if the magnetic field varies only in strength it is impossible in the Minkowski spacetime to create a pair with both the electron and the positron in a transverse ground state. Instead, we want to show here that this process is allowed in the spacetime metric (II.4) because of the corrections induced by the gravitational field on the transverse ground states of the electron and of the positron. In fact, if we choose the same electron and positron states used in the previous paragraph, the transition matrix element (IV.2) in this case becomes

$$\hat{H}_{t;\tilde{J}_0 J_0}^{(1)}(t) = \sqrt{\frac{\phi_t e B_+(t)}{2}} \int dr \sqrt{\phi_s^3} (1 - g_F x) U_{\tilde{J}_0 J_0}^{(1)\dagger}(r, t) (\mathbf{r} \times \alpha)_z V_{\tilde{J}_0 J_0}^{(1)}(r, t)$$

(IV.20)

where we used the fact that, since the magnetic field lies in the $z$-direction for every $t$ then $R_x[\theta(t)] \equiv I$.

Now, the selection rule (50) in [4] concerning the spin of the transverse ground states allows us to conclude that for the zero-order transverse ground
states the following equalities hold:

\[
\int dr U_{j_0}^{(0)\dag}(r,t) (r \times \alpha)_{\times} V_{j_0}^{(0)}(r,t) = 0, \quad (\text{IV.21})
\]

\[
\int dr U_{j_0}^{(0)\dag}(r,t) (r \times \alpha)_{\times} U_{j_0}^{(0)}(r,t) = 0, \quad (\text{IV.22})
\]

\[
\int dr V_{j_0}^{(0)}(r,t) (r \times \alpha)_{\times} V_{j_0}^{(0)}(r,t) = 0 \quad (\text{IV.23})
\]

where \( J_{j_0}^{(0)} \equiv \{ 0, k, -1, x_0' \} \) and \( \bar{J}_{j_0}^{(0)} \equiv \{ 0, k, +1, x_0' \} \). By exploiting these equations and by keeping only the terms up to first order we can write Eq. (IV.20) as

\[
\dot{H}_{\uparrow; (J_0 \bar{J}_0)}^{(1)}(t) \sim \frac{\sqrt{\phi_s e \bar{B}_\uparrow(t)}}{2} \int dr \sqrt{\phi_s^2 U_{0,k,-1,x_0}^{(0)\dag}(r,t)} (r \times \alpha)_{\times} \times \]

\[
\left[ B_{k,x_0}^{(1)}(t)V_{1,k',+1,x_0'}^{(0)}(r,t) - i C_{k',x_0'}^{(1)}(t)V_{1,k',-1,x_0'}^{(0)}(r,t) + i E_{k',x_0'}^{(1)}(t)U_{0,-k',-1,x_0'}^{(0)}(r,t) - F_{k',x_0'}^{(1)}(t)U_{0,-k',+1,x_0'}^{(0)}(r,t) \right] \times \]

\[
+ \frac{\sqrt{\phi_s e \bar{B}_\uparrow(t)}}{2} \int dr \sqrt{\phi_s^2 B_{k,x_0}^{(1)}(t)U_{1,k,-1,x_0}^{(0)\dag}(r,t) + i C_{k,x_0}^{(1)}(t)U_{1,k,1,x_0}^{(0)\dag}(r,t) - i E_{k,x_0}^{(1)}(t)V_{0,-k,1,x_0}^{(0)\dag}(r,t) - F_{k,x_0}^{(1)}(t)V_{0,-k,-1,x_0}^{(0)\dag}(r,t) \times \]

\[
(r \times \alpha)_{\times} V_{0,k',+1,x_0'}^{(0)}(r,t). \quad (\text{IV.24})
\]

This expression can be further simplified if we use the definitions (A14) and (A16) of the operators \( x \) and \( y \). In fact, by using the substitutions (III.46)- (III.47) the operator \((r \times \alpha)_{\times}\) can be written as

\[
(r \times \alpha)_{\times} = x \alpha_y - y \alpha_x = \left[ x_0 + \frac{1}{\sqrt{2 e B}} (a + a^\dagger) \right] i (\alpha_- - \alpha_+) -
\]

\[
- \left[ y_0 + \frac{i}{\sqrt{2 e B}} (a - a^\dagger) \right] (\alpha_- + \alpha_+) =
\]

\[
i \alpha_- \left( x_0 + iy_0 + \sqrt{\frac{2}{e B}} a^\dagger \right) - i \alpha_+ \left( x_0 - iy_0 + \sqrt{\frac{2}{e B}} a \right). \quad (\text{IV.25})
\]

Now, from the expressions (C1) and (C2) of the transverse ground states we conclude that only the operator \( i \alpha_- (x_0 + iy_0) \) gives a non-vanishing contribution in the first integral of Eq. (IV.24) and that, analogously, only the
operator \(-i\alpha_+(x_0 - iy_0)\) gives a non-vanishing contribution in the second one, then the matrix element (IV.24) can be written as

\[
\hat{H}^{(1)}_{\uparrow;(J_0 J_0^-)}(t) = i \sqrt{\frac{\phi_+ e \hat{B}_\uparrow(t)}{2}} \left[ (x_0 + iy_0) \left< U^{(1)}_{J_0}(t) | \alpha_- | V^{(1)}_{J_0}(t) \right> - (x_0 - iy_0) \left< U^{(1)}_{J_0}(t) | \alpha_+ | V^{(1)}_{J_0}(t) \right> \right] =
\]

\[
i \sqrt{\frac{\phi_+ e \hat{B}_\uparrow(t)}{2}} \left\{ \begin{array}{c}
\left[ x_0 + \frac{1}{eB_\uparrow(t)} \frac{\partial}{\partial x_0} \right] \left< U^{(1)}_{J_0}(t) | \alpha_- | V^{(1)}_{J_0}(t) \right> - \\
\left[ x_0 - \frac{1}{eB_\uparrow(t)} \frac{\partial}{\partial x_0} \right] \left< U^{(1)}_{J_0}(t) | \alpha_+ | V^{(1)}_{J_0}(t) \right>
\end{array} \right\}
\]

(IV.26)

where

\[
\left< U^{(1)}_{J_0}(t) | \alpha_- | V^{(1)}_{J_0}(t) \right> = \int d\mathbf{r} \sqrt{\phi_+} U^{(0)\uparrow}_{0,k,-1,x_0}(\mathbf{r}, t) \alpha_- \times
\]

\[
\times \left[ B_{k',x_0'}^{(1)}(t) V^{(0)}_{1,k',+1,x_0'}(\mathbf{r}, t) - iC_{k',x_0'}^{(1)}(t) V^{(0)}_{0,k',-1,x_0'}(\mathbf{r}, t) + \\
iE_{k',x_0'}^{(1)}(t) U^{(0)\uparrow}_{0,-k',+1,x_0'}(\mathbf{r}, t) - F_{k',x_0'}^{(1)}(t) U^{(0)\uparrow}_{1,-k',-1,x_0'}(\mathbf{r}, t) \right],
\]

(IV.27)

\[
\left< U^{(1)}_{J_0}(t) | \alpha_+ | V^{(1)}_{J_0}(t) \right> = \int d\mathbf{r} \sqrt{\phi_+} \left[ B_{k,x_0}^{(1)}(t) U^{(0)\uparrow}_{1,k,-1,x_0}(\mathbf{r}, t) + iC_{k,x_0}^{(1)}(t) U^{(0)\uparrow}_{0,k,1,x_0}(\mathbf{r}, t) - \\
iE_{k,x_0}^{(1)}(t) V^{(0)\uparrow}_{0,-k,-1,x_0}(\mathbf{r}, t) - F_{k,x_0}^{(1)}(t) V^{(0)\uparrow}_{1,-k,1,x_0}(\mathbf{r}, t) \right] \times
\]

\[
\alpha_+ V^{(0)}_{0,k',+1,x_0'}(\mathbf{r}, t)
\]

(IV.28)

and where we used Eq. (4.23). The calculation of these matrix elements (IV.27) and (IV.28) is quite tedious and the technique to perform it is sketched in appendix D. In particular, it can be shown that

\[
\left< U^{(1)}_{J_0}(t) | \alpha_- | V^{(1)}_{J_0}(t) \right> = - \left< U^{(1)}_{J_0}(t) | \alpha_+ | V^{(1)}_{J_0}(t) \right> = -i \frac{1}{2eB_\uparrow(t)} \sqrt{\phi_+} (g_M - g_R) mk \delta_{k,-k'} \delta_{x_0,x_0'}
\]

(IV.29)

and, for this reason the terms with the derivative with respect to \(x_0\) cancel.
each other and the final expression of $\dot{H}^{(1)}_{\uparrow(J_0,J'_0)}(t)$ is

$$
\dot{H}^{(1)}_{\uparrow(J_0,J'_0)}(t) = \sqrt{\frac{\phi_t}{\phi_s}} \frac{1}{B_\uparrow(t)} \frac{1}{2\varepsilon_k} \sqrt{\phi_t(g_M - g_P)mkx_0\delta_{k,-k}\delta_{x_0,x'_0}}.
$$

(IV.30)

We want to point out that the disappearance of the electron charge $-e$ is only a consequence of the strong field approximation $B_\uparrow(t) \gg B_{cr}$. In particular, it would be incoherent to conclude that this matrix element does not vanish in the limit $e \to 0$ because in this limit $B_{cr} = m^2/e \to \infty$ and the strong field condition can not be satisfied. Nevertheless, we will see that the creation probability which is physically meaningful will contain $e$.

The creation amplitudes at time $t$ can be calculated by means of Eq. (IV.3) and the only one different from zero is equal to

$$
\gamma^{(1)}_{\uparrow(0,k,-1,x_0,0,-k,+1,x_0)}(t) = \sqrt{\frac{\phi_t}{\phi_s}} \frac{k}{4(\varepsilon_k^{(0)})^2} \sqrt{\phi_t(g_M - g_P)mx_0} \int_0^t dt' \frac{\dot{B}_\uparrow(t')}{B_\uparrow(t')} \exp \left( 2i\varepsilon_k^{(0)}t' \right)
$$

(IV.31)

where only the first-order terms in the gravitational couplings have been kept. Now, by substituting the expression (IV.4) of the magnetic field $B_\uparrow(t)$ and by calculating its time derivative

$$
\dot{B}_\uparrow(t) = \frac{B_f - B_i}{\tau} \exp \left( -\frac{t}{\tau} \right)
$$

(IV.32)

the creation amplitude can be written as

$$
\gamma^{(1)}_{\uparrow(0,k,-1,x_0,0,-k,+1,x_0)}(t) = \sqrt{\frac{\phi_t}{\phi_s}} \frac{k}{4(\varepsilon_k^{(0)})^2} \sqrt{\phi_t(g_M - g_P)mx_0} \int_0^{t/\tau} ds' \frac{\exp \left[ - \left( 1 - 2i\varepsilon_k^{(0)}t/\tau \right) s' \right]}{\exp(-s') - \frac{B_f - B_i}{B_f - B_i}}
$$

(IV.33)

with $s' = t'/\tau$ a dimensionless variable. Now, in our model $\tau$ is a macroscopic time connected to the typical evolution times of a neutron star or of a black hole, then we can safely assume that $\varepsilon_k^{(0)} \tau \gg 1$. This allows us to give an asymptotic estimate of the remaining integral in the limit $t \to \infty$. The result is

$$
\gamma^{(1)}_{\uparrow(0,k,-1,x_0,0,-k,+1,x_0)}(t \to \infty) \sim \sqrt{\frac{\phi_t}{\phi_s}} \frac{k}{4(\varepsilon_k^{(0)})^2} \sqrt{\phi_t(g_M - g_P)mx_0} \frac{1}{2i\varepsilon_k^{(0)}\tau} \frac{B_f - B_i}{B_i}.
$$

(IV.34)
Finally, by squaring the modulus of this expression and by multiplying it by the number of states at $t \to \infty$

$$dN_t(t \to \infty) = \frac{eB_t(t \to \infty)\varphi_s^2}{2\pi}dx_0 \times \frac{Z}{2\pi}dk = \frac{eB_f}{(2\pi)^2}dV^{(0)}dk \quad \text{(IV.35)}$$

with $dV^{(0)} = \sqrt{\varphi_s^3}YZdx_0$ the “physical” quantization volume up to zero order [see Eq. (IV.5)], we obtain the differential probability that a pair is created with the electron (positron) between $x_0$ and $x_0 + dx_0$ and with longitudinal momentum between $k$ and $k + dk$ ($-k$ and $-k - dk$)

$$dP^{(1)}_t(x_0, k; t \to \infty) \sim \frac{\phi_s}{\phi_t} \sqrt{\frac{\varphi_s}{\varphi_t}} \frac{\phi_s m^2 k^2}{\varphi_t (\phi_s m^2 + k^2)^3} eB_f \left( \frac{B_f - B_t}{\tau B_t} \right)^2 \left( \frac{g_s x_0}{16\pi \varphi_s} \right)^2 dV^{(0)}dk.$$ 

(IV.36)

In this equation the continuum limits $Y \to \infty$ and $Z \to \infty$ have been performed and, since the probability is already proportional to $g_s$ and our calculations are exact up to first order in $g_s$ and $g_t$, it is enough to use the zero-order “physical” volume $dV^{(0)}$. Finally, the corresponding probability per unit volume and unit longitudinal momentum is given by

$$dP^{(1)}_t(x_0, k; t \to \infty) \sim \left( \frac{g_s}{16\pi \varphi_s} \right)^2 \frac{\phi_s}{\phi_t} \frac{eB_f}{\varphi_t (\phi_s m^2 + k^2)^3} \left( \frac{B_f - B_t}{\tau B_t} \right)^2 \left( \sqrt{\varphi_s} m x_0 k \right)^2 dV^{(0)}dk.$$ 

(IV.37)

As in the case of a rotating magnetic field we observe the presence of the overall factors $\phi_s/\phi_t$ and $x_0^2$. Also, in [2] it was calculated the total probability that a pair is created in a slowly varying magnetic field with fixed direction but in the Minkowski spacetime [see Eq. (20) in this paper]. We remind that in this case either the electron or the positron must be created in a state which is not a transverse ground state. In order to have a quantity to be compared to the total probability given in Eq. (20) in [2], we have to integrate Eq. (IV.36) with respect to $x_0$ and to $k$. After these integrations it can easily be seen that the order of magnitude of the ratio between our probability and that given in [2] is

$$\eta \sim \frac{\phi_s}{\phi_t} \frac{1}{\sqrt{\varphi_s} m} \left( \frac{g_s}{\varphi_s} \right)^2 \frac{1}{\sqrt{eB_f}}.$$ 

(IV.38)

If we substitute the expressions (II.7) and (II.8) of $\varphi_t$, $\varphi_s$ and $g_s$ we see that because of the inequality (II.10) and of the strong-field condition $eB_f \gg m^2$

7We have to be satisfied of an order-of-magnitude comparison because the time evolution of the magnetic field used in [2] is different from that used here.
then $\eta$ results much less than one and then the gravitational effect is in our approximations small. Nevertheless, the effect is there and it is reasonable to imagine that it can be amplified in the presence of a real gravitational field which is not restricted by our assumptions. For this reason we decided to consider also a particular case slightly different from the one at hand and where the gravitational field can be treated without approximations and this is the subject of the next paper. However, it must be pointed out that the perturbative approach gives the possibility to treat more general magnetic and gravitational field configurations while the strong field case the two fields must be necessarily parallel.

5 Conclusions

In this paper we have modified our previous results given in [2, 3, 4] about the production of $e^- - e^+$ pairs in the presence of strong, uniform and slowly varying magnetic fields taking into account the presence of the gravitational field represented by the metric tensor (11.4). We point out that this metric tensor resulted from some approximations that we have done in section (2) and that we resume for the sake of clarity: we have neglected the gravitational field due to the magnetic field energy, we have neglected the gravitational effects of the angular momentum of the stellar object and we have assumed to be not to close to the event horizon of the body.

We have treated the gravitational field perturbatively up to first-order in the couplings $g_E$, $g_P$ and $g_M$ and we have seen how its presence modifies the one-particle eigenstates and eigenenergies of the electron and of the positron. This circumstance, obviously, changes the pair production probabilities we calculated in absence of the gravitational field, i.e. in the Minkowski space-time.

In particular, we have reexamined the case of a purely rotating magnetic field [4] and of a magnetic field varying only in strength [2]. Firstly, in both cases we have found that the production probabilities contain an amplification factor $\phi_s/\phi_t > 1$ [see Eqs. (IV.19) and (IV.37)]. Also, in the case of the purely rotating magnetic field we have seen that the presence of the gravitational field in the plane orthogonal to the magnetic field brakes the rotational symmetry in this plane and makes more likely that a pair is created at $x_0 < 0$ [see Eq. (IV.19)]. Instead, in the case of a magnetic field with fixed direction we obtained a qualitative new result: in the presence of the gravitational field it is possible to create a pair with both the electron and the positron in a transverse ground state. Nevertheless, we have seen at the end of the previous section that this probability is a small quantity with respect to the
total probability that a pair is created in the Minkowski spacetime. But, this result is a consequence of the fact that the gravitational field has been treated perturbatively and this treatment could not fit what it happens, for example, near the event horizon of a black hole. From this point of view, the information we have gained is that in the presence of a gravitational field this new effect is there. In the next paper we analyze a particular case in which the gravitational field can be treated without approximations and we will show that in that case the effect is relevant.

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Appendix A

In this appendix we want to quote some properties of the eigenstates $u_j(r)$ and $v_j(r)$ of the zero-order Hamiltonian (III.8). Firstly, we remind that $j \equiv \{n_d, k, \sigma, n_g\}$ embodies all the quantum numbers and that $u_j(r)$ and $v_j(r)$ can be written in the form [4]

$$u_j(r) = R_x^\dagger(\vartheta)u'_j(r), \quad (A1)$$

$$v_j(r) = R_x^\dagger(\vartheta)v'_j(r) \quad (A2)$$

where

$$R_x(\vartheta) = \exp(-i\vartheta J_x) \quad (A3)$$

with $J_x = L_x + S_x$ the $x$-component of the one-particle total angular momentum operator and $\vartheta$ is defined in Eq. (III.3). The rotated spinors $u'_j(r)$ and $v'_j(r)$ are the solutions of Eqs. (III.21) and (III.22) with the vector potential

$$A'(r) = -\frac{1}{2}(r \times B') \quad (A4)$$

and

$$B' = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{B_y^2 + B_z^2} \end{pmatrix} \quad (A5)$$
[see Eqs. (III.1)-(III.4)], and are given by \[4\]

\[
u_j'(r) = \frac{1}{4 \sqrt{\phi_s}} \sqrt{\frac{w_r^{(0)} + \sqrt{\phi_m} m}{2 w_r^{(0)}}} \left( \sqrt{\frac{\phi_t}{\phi_s w_r^{(0)}} + \sqrt{\phi_m}} \varphi_j'(r) \right),
\]

(A6)

\[
u_j'(r) = \frac{1}{4 \sqrt{\phi_s}} \sqrt{\frac{\tilde{w}_q^{(0)} + \sqrt{\phi_m} m}{2 \tilde{w}_q^{(0)}}} \left( -\sqrt{\frac{\phi_t}{\phi_s \tilde{w}_q^{(0)}} + \sqrt{\phi_m}} \chi_j'(r) \right),
\]

(A7)

where the indices \(r\) and \(q\) and the energies \(w_r^{(0)}\) and \(\tilde{w}_q^{(0)}\) have been defined in Eqs. (III.24) and (III.25) and in Eqs. (III.27) and (III.28) respectively and where the numerical factor \(1/4 \sqrt{\phi_s}\) has been inserted to make the spinors correctly normalized with respect to the scalar product (II.60) at zero order:

\[
\int \! dr \sqrt{\phi_s} u_j^\dagger(r) u_{j'}(r) = \int \! dr \sqrt{\phi_s} u_{j'}^\dagger(r) u_j(r) = \delta_{jj'},
\]

(A8)

\[
\int \! dr \sqrt{\phi_s} v_j^\dagger(r) v_{j'}(r) = \int \! dr \sqrt{\phi_s} v_{j'}^\dagger(r) v_j(r) = \delta_{jj'}
\]

(A9)

with \(\delta_{jj'} \equiv \delta_{n_d,n'_d} \delta_{k,k'} \delta_{\sigma,\sigma'} \delta_{n_g,n'_g}\). Now, we give a more explicit expression of the operator \(\mathcal{V}'(r, -i \partial)\) and of the two-dimensional spinors \(\varphi_j'(r)\) and \(\chi_j'(r)\) appearing in Eqs. (A6) and (A7). To do this, we need to express the operators \(x, -i \partial/\partial x, y\) and \(-i \partial/\partial y\) with respect to the destruction and creation operators (that we indicate as \(a_d, a_d^\dagger, a_g\) and \(a_g^\dagger\) respectively) corresponding to the quantum numbers \(n_d\) and \(n_g\) and vice versa. These expressions can be found in \([35]\) and are given by

\[
a_d = \frac{1}{2} \left[ \sqrt{\frac{eB}{2}} (x - iy) + \sqrt{\frac{2}{eB}} \left( \frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right) \right],
\]

(A10)

\[
a_d^\dagger = \frac{1}{2} \left[ \sqrt{\frac{eB}{2}} (x + iy) - \sqrt{\frac{2}{eB}} \left( \frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \right],
\]

(A11)

\[
a_g = \frac{1}{2} \left[ \sqrt{\frac{eB}{2}} (x + iy) + \sqrt{\frac{2}{eB}} \left( \frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \right],
\]

(A12)

\[
a_g^\dagger = \frac{1}{2} \left[ \sqrt{\frac{eB}{2}} (x - iy) - \sqrt{\frac{2}{eB}} \left( \frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right) \right]
\]

(A13)
and by

\[ x = \frac{1}{2} \sqrt{\frac{2}{eB}} (a_g + a_g^\dagger + a_d + a_d^\dagger) = x_0 + \frac{1}{\sqrt{2eB}} (a_d + a_d^\dagger), \quad (A14) \]

\[ \frac{1}{i} \frac{\partial}{\partial x} = \frac{1}{2i} \sqrt{\frac{eB}{2}} (a_g - a_g^\dagger + a_d - a_d^\dagger) = \frac{eB}{2} y_0 + \frac{1}{2i} \sqrt{\frac{eB}{2}} (a_d - a_d^\dagger), \quad (A15) \]

\[ y = \frac{1}{2i} \sqrt{\frac{2}{eB}} (a_g - a_g^\dagger - a_d + a_d^\dagger) = y_0 - \frac{1}{i} \sqrt{\frac{2}{eB}} (a_d - a_d^\dagger), \quad (A16) \]

\[ \frac{1}{i} \frac{\partial}{\partial y} = -\frac{1}{2} \sqrt{\frac{eB}{2}} (a_g + a_g^\dagger - a_d - a_d^\dagger) = -\frac{eB}{2} x_0 + \frac{1}{2} \sqrt{\frac{eB}{2}} (a_d + a_d^\dagger) \quad (A17) \]

where we introduced the operators

\[ x_0 = \frac{1}{2} \sqrt{\frac{2}{eB}} (a_g + a_g^\dagger), \quad (A18) \]

\[ y_0 = \frac{1}{2i} \sqrt{\frac{2}{eB}} (a_g - a_g^\dagger). \quad (A19) \]

Starting from the well-known commutation relations among the operators \( x, -i \frac{\partial}{\partial x}, y \) and \(-i \frac{\partial}{\partial y} \) it is easy to see that

\[ [a_g, a_g^\dagger] = [a_d, a_d^\dagger] = 1, \quad (A20) \]

\[ [a_g, a_d] = [a_g, a_d^\dagger] = 0 \quad (A21) \]

and that

\[ [x_0, y_0] = \frac{i}{eB}. \quad (A22) \]

In particular, this last commutator allows us to put

\[ y_0 = \frac{1}{i} \frac{\partial}{\partial x_0}. \quad (A23) \]

Now, the operator \( \mathcal{V}'(\mathbf{r}, -i \partial) \) in Eqs. (A6) and (A7) is defined as [see Eq. (21) in [1]]

\[ \mathcal{V}'(\mathbf{r}, -i \partial) \equiv \mathbf{\sigma} \cdot [-i \partial + e \mathbf{A}'(\mathbf{r})] \quad (A24) \]

then by substituting the expressions (A14)-(A17) it can easily be shown that

\[ \mathcal{V}'(\mathbf{r}, -i \partial) = i \sqrt{2eB} (a_d^\dagger \sigma_+ - a_d \sigma_-) + \sigma_z \frac{1}{i} \frac{\partial}{\partial z} \quad (A25) \]

where \( \sigma_\pm = (\sigma_x \pm i \sigma_y)/2 \). It is also convenient here to express the two-dimensional spinors \( \varphi_\rho'(\mathbf{r}) \) and \( \chi_\rho'(\mathbf{r}) \) in Cartesian coordinates instead of in
cylindrical coordinates as in Eq. (22) of Ref. [4]:

\[
\varphi_j'(r) = \frac{\exp(ikz)}{\sqrt{Z}} f'_d \theta'_{n_d,n_g}(x,y), \quad (A26)
\]

\[
\chi_j'(r) = \frac{\exp(-ikz)}{\sqrt{Z}} f'_{-\sigma} \theta'_{n_g,n_d}(x,y). \quad (A27)
\]

In these expressions \(Z\) is the length of the quantization volume in the \(z\)-direction,

\[
f'_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f'_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (A28)
\]

and where the scalar functions

\[
\theta'_{l_1,l_2}(x,y) = \sqrt{\frac{eB}{2\pi l_1! l_2!}} (a_d^\dagger)^{l_1} (a_g^\dagger)^{l_2} \exp \left[ -\frac{eB(x^2 + y^2)}{4} \right] \quad (A29)
\]

depend only on the transverse coordinates. In Eqs. (A26) and (A27) the operators \(a_d^\dagger\) and \(a_g^\dagger\) are supposed to be expressed as in Eqs. (A11) and (A13) and from those equations it can be shown that

\[
(a_g^\dagger)^{n_g} \exp \left[ -\frac{eB(x^2 + y^2)}{4} \right] = \sqrt{\left( \frac{eB}{2} \right)^{n_g}} (x - iy)^{n_g} \exp \left[ -\frac{eB(x^2 + y^2)}{4} \right] ,
\]

\[
(a_d^\dagger)^{n_d} \exp \left[ -\frac{eB(x^2 + y^2)}{4} \right] = \sqrt{\left( \frac{eB}{2} \right)^{n_d}} (x + iy)^{n_d} \exp \left[ -\frac{eB(x^2 + y^2)}{4} \right] ,
\]

\[
(a_g^\dagger)^{n_g} \exp \left[ -\frac{eB(x^2 + y^2)}{4} \right] = \sqrt{\left( \frac{eB}{2} \right)^{n_g}} (x - iy)^{n_g} \exp \left[ -\frac{eB(x^2 + y^2)}{4} \right] ,
\]

and then that the transverse functions (A29) can be written as

\[
\theta'_{l_1,l_2}(x,y) = \frac{1}{\sqrt{l_1! l_2!}} (a_d^\dagger)^{l_1} \sqrt{\left( \frac{eB}{2} \right)^{l_2+1}} \frac{1}{\pi l_2!} (x - iy)^{l_2} \exp \left[ -\frac{eB(x^2 + y^2)}{4} \right] .
\]

Finally, we want to derive some orthonormalization properties of the two-dimensional spinors \(\varphi_j'(r)\) and \(\chi_j'(r)\) and of the spinors \(u_j(r)\) and \(v_j(r)\). From the commutation relations (A20) it can easily be shown that, given two quantum numbers \(n_d\) and \(n'_d\) with \(n'_d \geq n_d\), then

\[
(a_d)^{n_d}(a_d^\dagger)^{n'_d} = (a_d a_d^\dagger)^{n_d}(a_d^\dagger)^{n'_d-n_d} = (1+a_d a_d^\dagger)^{n_d}(a_d^\dagger)^{n'_d-n_d} = \sum_{l=0}^{n_d} \binom{n_d}{l} (a_d^\dagger)^l (a_d^\dagger)^{n'_d-n_d} \quad (A33)
\]
and, symmetrically,

\[(a_g)^{n_g}(a_g^\dagger)^{n_g'} = (a_g a_g^\dagger)^{n_g} (a_g^\dagger)^{n_g'} = (1 + a_g^\dagger a_g)^{n_g} (a_g^\dagger)^{n_g'} = \left( \sum_{l=0}^{n_g} \binom{n_g}{l} (a_g^\dagger a_g)^l \right) (a_g^\dagger)^{n_g'} \]

(A34)

if \( n_g' \geq n_g \) (if \( n_g' \leq n_g \) analogous relations are obtained). With the help of these two equations and by reminding that \([a_g, a_d] = 0\) it can be seen that the following orthonormalization relations hold:

\[
\int dr \varphi_j^\dagger(r) \varphi_{j'}(r) = \delta_{j,j'}, \quad (A35)
\]

\[
\int dr \chi_{j'}^\dagger(r) \chi_{j'}(r) = \delta_{j,j'}. \quad (A36)
\]

Now, if we calculate the square of the operator \( \mathcal{V}'(r, -i\partial) \) as given in Eq. \( (A25) \) we have

\[
\mathcal{V}'^2(r, -i\partial) = -\frac{\partial^2}{\partial z^2} + 2eB \left[ a_g^\dagger a_d(\sigma_- \sigma_+ + \sigma_+ \sigma_-) \right] = -\frac{\partial^2}{\partial z^2} + eB \left( 2a_g^\dagger a_d + 1 + \sigma_z \right), \quad (A37)
\]

then, from the expressions (III.27) and (III.28) of the zero-order eigenenergies, we obtain

\[
\frac{\phi_t}{\phi_s} \int dr \varphi_j^\dagger(r) \mathcal{V}'(r, -i\partial) \varphi_{j'}(r) = \frac{w^{(0)}_r - \sqrt{\phi_t m}}{w^{(0)}_r + \sqrt{\phi_t m}} \delta_{j,j'}, \quad (A38)
\]

\[
\frac{\phi_t}{\phi_s} \int dr \chi_{j'}^\dagger(r) \mathcal{V}'(r, -i\partial) \chi_{j'}(r) = \frac{\tilde{w}^{(0)}_q - \sqrt{\phi_t m}}{\tilde{w}^{(0)}_q + \sqrt{\phi_t m}} \delta_{j,j'}. \quad (A39)
\]

Finally, from these relations and from Eqs. (A11), (A2), (A6) and (A7) we derive immediately the equalities

\[
\int dr \sqrt{\phi_s^2} u_j^\dagger(r) \beta u_{j'}(r) = \int dr \sqrt{\phi_s^2} u_j^\dagger(r) \beta u_{j'}(r) = \frac{\sqrt{\phi_t m}}{w^{(0)}_r} \delta_{j,j'}, \quad (A40)
\]

\[
\int dr \sqrt{\phi_s^2} v_j^\dagger(r) \beta v_{j'}(r) = \int dr \sqrt{\phi_s^2} v_j^\dagger(r) \beta v_{j'}(r) = -\frac{\sqrt{\phi_t m}}{\tilde{w}^{(0)}_q} \delta_{j,j'}. \quad (A41)
\]

where we used the fact that [see Eq. (A3)]

\[
\mathcal{R}_z(\vartheta) \beta \mathcal{R}_z^\dagger(\vartheta) = \beta \quad (A42)
\]

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By the way, since they are often used in the main text, we also remind the following transformation properties of the matrices $\alpha$

$$
\mathcal{R}_x(\vartheta)\alpha_x\mathcal{R}_x^\dagger(\vartheta) = \alpha_x,
$$

(A43)

$$
\mathcal{R}_x(\vartheta)\alpha_y\mathcal{R}_x^\dagger(\vartheta) = \alpha_y \cos \vartheta + \alpha_z \sin \vartheta,
$$

(A44)

$$
\mathcal{R}_x(\vartheta)\alpha_z\mathcal{R}_x^\dagger(\vartheta) = -\alpha_y \sin \vartheta + \alpha_z \cos \vartheta
$$

(A45)

and of the position operator $r$

$$
\mathcal{R}_x(\vartheta)x\mathcal{R}_x^\dagger(\vartheta) = x,
$$

(A46)

$$
\mathcal{R}_x(\vartheta)y\mathcal{R}_x^\dagger(\vartheta) = y \cos \vartheta + z \sin \vartheta,
$$

(A47)

$$
\mathcal{R}_x(\vartheta)z\mathcal{R}_x^\dagger(\vartheta) = -y \sin \vartheta + z \cos \vartheta.
$$

(A48)
Appendix B

We want to show explicitly here that

\[ \mathcal{I}_{r_- n_g r_+ n'_g} \equiv \int dr \sqrt{\phi_s^2 u_{r_-, n_g}^\dagger(r)} \mathcal{I}(r, -i \partial) u_{r_+, n'_g}(r) = 0, \quad (B1) \]

\[ \mathcal{I}_{r_+ n_g r_- n'_g} \equiv \int dr \sqrt{\phi_s^2 u_{r_+, n_g}^\dagger(r)} \mathcal{I}(r, -i \partial) u_{r_-, n'_g}(r) = 0 \quad (B2) \]

where \( r_- = \{n_d + 1, k, -1\} \) and \( r_+ = \{n_d, k, +1\} \). We will prove only the first of these equalities being the other analogous. From Eq. (III.9) we have

\[ \mathcal{I}_{r_- n_g r_+ n'_g} = \int dr \sqrt{\phi_s^2 u_{r_-, n_g}^\dagger(r)} \left[ \sqrt{\phi_t (g_P - g_M)} m - (g_P - g_E) w_{r_-}^{(0)} \right] w_{r_+}^{(0)}(r) \quad (B3) \]

where we used Eqs. (A43) and (A46) and the fact that \( w_{r_-}^{(0)} = w_{r_+}^{(0)} \). This equation can be rewritten in the form [see Eq. (A6)]

\[ \mathcal{I}_{r_- n_g r_+ n'_g} = \frac{w_{r_-}^{(0)} + \sqrt{\phi_t m}}{2w_{r_-}^{(0)}} \left\{ \sqrt{\phi_t (g_P - g_M)} m - (g_P - g_E) w_{r_-}^{(0)} \right\} \int dr \phi_{r_- n_g}^\dagger(r) \phi_{r_+ n'_g} \frac{\mathcal{V}'(r, -i \partial)}{w_{r_-}^{(0)} + \sqrt{\phi_t m}} x \mathcal{V}'(r, -i \partial) \mathcal{V}'(r, -i \partial) \phi_{r_+ n'_g}(r) \right\} \]

\[ \mathcal{I}_{r_- n_g r_+ n'_g} \equiv \frac{w_{r_-}^{(0)} + \sqrt{\phi_t m}}{2w_{r_-}^{(0)}} \left\{ \sqrt{\phi_t (g_P - g_M)} m - (g_P - g_E) w_{r_-}^{(0)} \right\} \int dr \phi_{r_- n_g}^\dagger(r) \mathcal{V}'(r, -i \partial) \mathcal{V}'(r, -i \partial) \phi_{r_+ n'_g}(r) \}

\[ (B4) \]

Now, by observing that the spin of the two states is different we realize that the first integral vanishes because of the orthonormalization relation (A35). Also, from the definition (A24) of the operator \( \mathcal{V}'(r, -i \partial) \) and from its expression (A25) we have that

\[ \mathcal{V}'(r, -i \partial) \mathcal{V}'(r, -i \partial) = \mathcal{V}'(r, -i \partial), x] \mathcal{V}'(r, -i \partial) + x \mathcal{V}'(r, -i \partial) = \]

\[ = -i \sigma_x \mathcal{V}'(r, -i \partial) + x \mathcal{V}'(r, -i \partial) = \]

\[ = \sqrt{2eB} \left( a_d \sigma_x \sigma_+ - a_d \sigma_x \sigma_+ \right) - \sigma_x \sigma_z \frac{\partial}{\partial z} + \]

\[ + x \left[ \frac{\partial^2}{\partial z^2} + eB \left( 2a_d \sigma_x + 1 + \sigma_z \right) \right] \quad \]

\[ (B5) \]

where we used Eq. (A37). The only term of this operator that can change the spin of the electron is the second one, but it can not change the value of the quantum number \( n_d \). This means that the second integral in Eq. (B4) also vanishes and this completes the prove of Eq. (B1).
Appendix C

We want to give here the explicit expression of the zero-order transverse ground states and of the zero-order states corresponding to the first-excited Landau levels both of the electron and of the positron. We remind that we used them to calculate the transition matrix elements (IV.11), (III.101), (III.102), (IV.27) and (IV.28). These states can be easily obtained by substituting Eq. (A25) in Eqs. (III.57) and (III.71). We give only the final results.
for the rotated states because, actually, we used only them

\[
U^{r(0)}_{0,k,-1,x_0}(r) = \frac{1}{\sqrt{\phi_0^3}} \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_0 m}}{2\varepsilon_k^{(0)}}} \begin{pmatrix} 0 \\ -\frac{\varepsilon_k^{(0)}}{\phi_0 \varepsilon_k^{(0)} + \sqrt{\phi_0 m}} k\Theta'_{0,x_0}(x, y) \end{pmatrix} \frac{\exp(ikz)}{\sqrt{Z}},
\]

(C1)

\[
V^{r(0)}_{0,k,+1,x_0}(r) = \frac{1}{\sqrt{\phi_0^3}} \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_0 m}}{2\varepsilon_k^{(0)}}} \begin{pmatrix} 0 \\ \Theta'_{0,x_0}(x, y) \end{pmatrix} \frac{\exp(-ikz)}{\sqrt{Z}},
\]

(C2)

\[
U^{r(0)}_{0,k,+1,x_0}(r) = \frac{1}{\sqrt{\phi_0^3}} \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_0 m}}{2\varepsilon_k^{(0)}}} \begin{pmatrix} \Theta'_{0,x_0}(x, y) \\ -\frac{\varepsilon_k^{(0)}}{\phi_0 \varepsilon_k^{(0)} + \sqrt{\phi_0 m}} k\Theta'_{0,x_0}(x, y) \end{pmatrix} \frac{\exp(ikz)}{\sqrt{Z}},
\]

(C3)

\[
U^{r(0)}_{1,k,-1,x_0}(r) = \frac{1}{\sqrt{\phi_0^3}} \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_0 m}}{2\varepsilon_k^{(0)}}} \begin{pmatrix} \Theta'_{1,x_0}(x, y) \\ -\frac{\varepsilon_k^{(0)}}{\phi_0 \varepsilon_k^{(0)} + \sqrt{\phi_0 m}} i\sqrt{2eB} \Theta'_{0,x_0}(x, y) \end{pmatrix} \frac{\exp(ikz)}{\sqrt{Z}},
\]

(C4)

\[
V^{r(0)}_{0,k,-1,x_0}(r) = \frac{1}{\sqrt{\phi_0^3}} \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_0 m}}{2\varepsilon_k^{(0)}}} \begin{pmatrix} \Theta'_{1,x_0}(x, y) \\ -\frac{\varepsilon_k^{(0)}}{\phi_0 \varepsilon_k^{(0)} + \sqrt{\phi_0 m}} -i\sqrt{2eB} \Theta'_{0,x_0}(x, y) \end{pmatrix} \frac{\exp(-ikz)}{\sqrt{Z}},
\]

(C5)

\[
V^{r(0)}_{1,k,+1,x_0}(r) = \frac{1}{\sqrt{\phi_0^3}} \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_0 m}}{2\varepsilon_k^{(0)}}} \begin{pmatrix} \Theta'_{1,x_0}(x, y) \\ -\frac{\varepsilon_k^{(0)}}{\phi_0 \varepsilon_k^{(0)} + \sqrt{\phi_0 m}} -i\sqrt{2eB} \Theta'_{0,x_0}(x, y) \end{pmatrix} \frac{\exp(-ikz)}{\sqrt{Z}}.
\]

(C6)

In these expressions we have used the definitions (III.53) and (III.72) for the two-dimensional spinors \(\Phi'(r)\) and \(X'(r)\) and the definitions (III.79) and (III.81) for the energies \(\varepsilon_k^{(0)}\) and \(\varepsilon_k^{(0)}\).
Appendix D

In this appendix we will give an example which explains how to calculate the matrix elements \((III.101)\), \((III.102)\) and \((IV.27)\) and \((IV.28)\). In particular, we will calculate the integral

\[
\left\langle U^\prime_0 \left| \alpha_- \right| U^\prime_0 \right\rangle = \int d\mathbf{r} \sqrt{\frac{\phi_s}{\phi_t}} U^\prime_0 (\mathbf{r}) \alpha_- \left[ B^{(1)}_{k,x_0} U^{(0)}_{1,k,-1,x_0} (\mathbf{r}) - i C^{(1)}_{k,x_0} U^{(0)}_{1,k,1,x_0} (\mathbf{r}) + \right.
\]

\[
+i E^{(1)}_{k,x_0} V^{(0)}_{1,-k,-1,x_0} (\mathbf{r}) - F^{(1)}_{k,x_0} V^{(0)}_{1,-k,1,x_0} (\mathbf{r}) \right].
\]

We first observe that, given a generic spinor \(S\),

\[
S = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix}
\]

then

\[
\alpha_- S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} = \begin{pmatrix} 0 \\ S_3 \\ 0 \\ S_1 \end{pmatrix}.
\]

In this way, from the expressions \((C3)-(C6)\) of the zero-order electron and positron eigenstates we see that

\[
\alpha_- \left[ B^{(1)}_{k,x_0} U^{(0)}_{1,k,-1,x_0} (\mathbf{r}) - i C^{(1)}_{k,x_0} U^{(0)}_{1,k,1,x_0} (\mathbf{r}) + i E^{(1)}_{k,x_0} V^{(0)}_{1,-k,-1,x_0} (\mathbf{r}) - F^{(1)}_{k,x_0} V^{(0)}_{1,-k,1,x_0} (\mathbf{r}) \right] =
\]

\[
\left[ \begin{array}{c} \frac{\sqrt{\phi_t} E^{(0)}_{k,x_0} + \sqrt{\phi_t} m}{\mathcal{E}^{(0)}_k + \sqrt{\phi_t} m} \\ 0 \\ 0 \\ \frac{\sqrt{\phi_t} E^{(0)}_{k,x_0} - \sqrt{\phi_t} m}{\mathcal{E}^{(0)}_k + \sqrt{\phi_t} m} \end{array} \right] \Theta'_{0,x_0} (x,y) \frac{\exp(ikz)}{\sqrt{Z}}.
\]

\[(D1)\]
By using the expression (C1) for $U_{0,k-1,0}(r)$ we obtain

$$\langle U_{j_0}^{(1)}|\alpha_-|U_{j_0}^{(1)}\rangle = -i \sqrt{\frac{\varepsilon_k^{(0)} + \sqrt{\phi_k m}}{2\varepsilon_k^{(0)}}} \sqrt{\frac{E_k^{(0)} + \sqrt{\phi_k m}}{2E_k^{(0)}}} \times$$

$$\times \left[ \frac{\phi_t \sqrt{2eB_{k,x_0}^{(1)} + kC_{k,x_0}^{(1)}}}{\phi_s \varepsilon_k^{(0)} + \sqrt{\phi_k m}} + E_{k,x_0}^{(1)} - \frac{\phi_t k}{\phi_s \varepsilon_k^{(0)} + \sqrt{\phi_k m}} \left( C_{k,x_0}^{(1)} - \frac{\phi_t kE_{k,x_0}^{(1)} - \sqrt{2eB_{k,x_0}^{(1)}}}{\phi_s E_k^{(0)} + \sqrt{\phi_k m}} \right) \right].$$

(D5)

At this point we have to substitute the coefficients $B_{k,x_0}^{(1)}, C_{k,x_0}^{(1)}, E_{k,x_0}^{(1)}$ and $F_{k,x_0}^{(1)}$ by means of Eqs. (III.82), (III.83), (III.87) and (III.88) and many terms cancel each other. After some algebra we obtain

$$\langle U_{j_0}^{(1)}|\alpha_-|U_{j_0}^{(1)}\rangle = -i \frac{\varepsilon_k^{(0)} + \sqrt{\phi_k m} \varepsilon_k^{(0)} + \sqrt{\phi_k m}}{2\varepsilon_k^{(0)} 2E_k^{(0)}} \times$$

$$\times \left\{ \frac{1}{\varepsilon_k^{(0)} - \varepsilon_k^{(0)}} \left[ \sqrt{\phi_t (g_M - g_P)m + \frac{g_P}{2} \left( \varepsilon_k^{(0)} + \varepsilon_k^{(0)} \right) - g_E \varepsilon_k^{(0)}} \right] \right. \times$$

$$\left. \times \left[ \sqrt{\phi_t \varepsilon_k^{(0)} + \sqrt{\phi_k m}} + \frac{1}{\sqrt{\phi_s \varepsilon_k^{(0)} + \sqrt{\phi_k m}}} \right] \right.$$

$$+ \left. \frac{1}{\varepsilon_k^{(0)} + \varepsilon_k^{(0)}} \left[ \sqrt{\phi_t (g_M - g_P)m - \frac{g_P}{2} \left( \varepsilon_k^{(0)} - \varepsilon_k^{(0)} \right) - g_E \varepsilon_k^{(0)}} \right] \times$$

$$\times \left[ \sqrt{\phi_t \varepsilon_k^{(0)} + \sqrt{\phi_k m}} + \frac{1}{\sqrt{\phi_s \varepsilon_k^{(0)} + \sqrt{\phi_k m}}} \right] \right.$$

$$- \left. \frac{1}{\varepsilon_k^{(0)} - \varepsilon_k^{(0)}} \left[ \sqrt{\phi_t (g_M - g_P)m - \frac{g_P}{2} \left( \varepsilon_k^{(0)} + \varepsilon_k^{(0)} \right) + g_E \varepsilon_k^{(0)}} \right] \times$$

$$\times \left[ \sqrt{\phi_t \varepsilon_k^{(0)} + \sqrt{\phi_k m}} \phi_t \phi_s \varepsilon_k^{(0)} + \sqrt{\phi_k m} \left( \varepsilon_k^{(0)} + \sqrt{\phi_k m} \right) \right. -$$

$$\left. - \frac{1}{\varepsilon_k^{(0)} + \varepsilon_k^{(0)}} \left[ \sqrt{\phi_t (g_M - g_P)m + \frac{g_P}{2} \left( \varepsilon_k^{(0)} - \varepsilon_k^{(0)} \right) + g_E \varepsilon_k^{(0)}} \right] \times$$

$$\times \left[ \sqrt{\phi_t \varepsilon_k^{(0)} + \sqrt{\phi_k m}} \phi_t \phi_s \varepsilon_k^{(0)} + \sqrt{\phi_k m} \left( \varepsilon_k^{(0)} + \sqrt{\phi_k m} \right) \right] \right\}. \quad \text{(D6)}$$
Finally, if we use the equalities

\[
1 - \frac{\phi_t}{\phi_s} \frac{k^2}{\epsilon_k^{(0)} + \sqrt{\phi_t m}} = \frac{2 \sqrt{\phi_t m}}{\epsilon_k^{(0)} + \sqrt{\phi_t m}}, \\
1 + \frac{\phi_t}{\phi_s} \frac{k^2}{\epsilon_k^{(0)} + \sqrt{\phi_t m}} = \frac{2 \epsilon_k^{(0)}}{\epsilon_k^{(0)} + \sqrt{\phi_t m}}
\]

(D7)
(D8)

we have

\[
\langle U'^{(1)}_{J_0} | \alpha_- | U'^{(1)}_{J_0} \rangle = -i \sqrt{\frac{\phi_t}{\phi_s}} \frac{1}{2 \epsilon_k^{(0)} \epsilon_k^{(0)}} \times \left\{ \frac{1}{\epsilon_k^{(0)} - \epsilon_k^{(0)}} \times \right. \\
\left. \times \left[ \sqrt{\phi_t m (g_M - g_P)} \sqrt{\phi_t m} + \left[ \frac{g_P}{2} \left( \epsilon_k^{(0)} + \epsilon_k^{(0)} \right) - g_E \epsilon_k^{(0)} \right] \epsilon_k^{(0)} \right] + \\
+ \frac{1}{\epsilon_k^{(0)} + \epsilon_k^{(0)}} \times \\
\times \left[ \sqrt{\phi_t m (g_M - g_P)} \sqrt{\phi_t m} - \left[ \frac{g_P}{2} \left( \epsilon_k^{(0)} - \epsilon_k^{(0)} \right) + g_E \epsilon_k^{(0)} \right] \epsilon_k^{(0)} \right] \right\} = \\
= i \frac{1}{2 e B} \sqrt{\frac{\phi_s}{\phi_t}} \left[ (g_M - g_P) \frac{\phi_t m^2}{\epsilon_k^{(0)}} + (g_P - g_E) \epsilon_k^{(0)} \right] = \\
= i \frac{\sqrt{\phi_s m^2 + k^2}}{2 e B} \left( \frac{g_t}{\phi_t} + \frac{g_s}{\phi_s} \frac{k^2}{\phi_s m^2 + k^2} \right)
\]

(D9)

that is one of the equalities in Eq. (III.103). We say again that all the other matrix elements can be calculated with the same technique.
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