Null boundary controllability of a 1-dimensional heat equation with an internal point mass

Scott W Hansen * and Jose de Jesus Martinez *

Abstract
We consider a linear hybrid system composed by two rods of equal length connected by a point mass. We show that the system is null controllable with Dirichlet and Neumann controls. The results are based on a careful spectral analysis together with the moment method.

1 Introduction
In this article we prove the boundary null controllability of the temperature of a linear hybrid system consisting of two wires or rods connected by a point mass. More precisely, we consider the following system:

\[
\begin{align*}
\dot u - u'' &= 0, \quad t > 0, \quad x \in \omega_1 = (-1, 0) \\
\dot v - v'' &= 0, \quad t > 0, \quad x \in \omega_2 = (0, 1) \\
\dot z &= v'(t, 0) - u'(t, 0), \quad t > 0 \\
u(t, 0) &= v(t, 0) = z(t), \quad t > 0 \\
u(t, -1) &= 0,
\end{align*}
\]

(1)

with either Dirichlet control

\[v(t, 1) = f(t), \quad t > 0\]  

(2)

or Neumann control

\[v'(t, 1) = f(t), \quad t > 0.\]  

(3)

In the above and throughout this article, \(\dot{}\) denotes spatial derivatives and \(\dot{}\) denotes temporal derivatives. In addition, \(u = u(t, x)\) and \(v = v(t, x)\) denote the temperature on \(\omega_1\) and \(\omega_2\), and \(z = z(t)\) denotes the temperature of the point mass. The initial conditions at time \(t = 0\) are given by

\[
\begin{align*}
u^0(x) &= u(0, x), \quad x \in \omega_1 \\
v^0(x) &= v(0, x), \quad x \in \omega_2 \\
z^0 &= z(0),
\end{align*}
\]

where the triple \(\{u^0, v^0, z^0\}\) will be given in an appropriately defined function space.

System (1) with the homogenous boundary condition

\[v(t, 1) = 0, \quad t > 0\]  

(4)

can be viewed as the limit of the following “epsilon” system with unit density on \((-1, 1) \setminus (-\epsilon, \epsilon)\) and with density \(1/2\epsilon\) on \((-\epsilon, \epsilon)\):

\[
\begin{align*}
\dot u_\epsilon - u_\epsilon'' &= 0, \quad t > 0, \quad x \in (-1, -\epsilon) \\
\dot v_\epsilon - v_\epsilon'' &= 0, \quad t > 0, \quad x \in (\epsilon, 1) \\
\frac{1}{2\epsilon} \dot z_\epsilon - z_\epsilon'' &= 0, \quad t > 0, \quad x \in (-\epsilon, \epsilon)
\end{align*}
\]

(5)

*Department of Mathematics, Iowa State University, Ames, IA 50010, USA (shansen@iastate.edu) (jesusmtz@iastate.edu). Funding for this research was provided in part by the National Science Foundation under award number DMS-1312952.
where \( u_\epsilon, v_\epsilon \) and \( z_\epsilon \) satisfy the conditions

\[
\begin{align*}
  u_\epsilon(t, -\epsilon) &= z_\epsilon(t, -\epsilon), \quad z_\epsilon(t, -\epsilon) = v_\epsilon(t, -\epsilon), \\
  u_\epsilon'(t, -\epsilon) &= z_\epsilon'(t, -\epsilon), \quad z_\epsilon'(t, -\epsilon) = v_\epsilon'(t, -\epsilon), \\
  u_\epsilon(t, -1) &= v_\epsilon(t, 1) = 0,
\end{align*}
\]

for \( t > 0 \). In fact, in [8] the authors have shown that under appropriate assumptions of the initial data, solutions of [5] with [6] converge weakly to solutions of [1] and [4].

The hybrid system [1] is a variant of previously studied hybrid models for systems of strings and beams with interior point masses. Hansen and Zuazua used the method of characteristics in [10] to prove the boundary null controllability of an analogous string system with an interior point mass. In [13] Littman and Taylor use transform methods to prove boundary feedback stabilization of the string mass system. In [1] and [2], Castro and Zuazua used method of non-harmonic Fourier series to prove boundary controllability of systems of either Rayleigh or Euler-Bernoulli beams with interior point masses. We refer to [12], [14], [3], [18], [7] and [6] for related results on control and stabilization of systems of beams with end masses.

Our main results are the following.

**Proposition 1.0.1.** System [1] with either Dirichlet control [2] or Neumann control [3] is null controllable in any time \( T > 0 \). More precisely, given \( T > 0 \) there is a control \( f \in L^2(0,T) \) such that given initial data \( \{u^0, v^0, z^0\} \in L^2(\omega_1) \times L^2(\omega_2) \times \mathbb{R} \) we have that \( \{u(T,x), v(T,x), z(T)\} \) \( \in \{0,0,0\} \).

The solutions in Proposition 1.0.1 are defined by transposition in the spaces \( C(0,T;\mathcal{X}_{-1/2}) \) for the case of Dirichlet control and \( C(0,T;\mathcal{H}) \) for the case of Neumann control; see Section 3.

Our general approach is to reduce the control problem to a moment problem. We consider the case of Dirichlet control and Neumann control separately in Section 3.

## 2 Preliminaries

We begin with a discussion of well-posedness of the system [1] with either homogeneous Dirichlet boundary condition [4] or Neumann boundary condition

\[
v'(t, 1) = 0, \quad t > 0.
\]

Given \( u, v \) and \( z \) defined on \( \omega_1, \omega_2 \) and \( \mathbb{R} \) respectively, define \( y = (u, v, z)^t \) where \(^t\) denotes transposition. Let

\[
\mathcal{H} = L^2(\omega_1) \times L^2(\omega_2) \times \mathbb{R}
\]

equipped with the norm

\[
\|y\|^2_{\mathcal{H}} = \|(u,v,z)\|^2_{\mathcal{H}} = \|u\|^2_{\omega_1} + \|v\|^2_{\omega_2} + |z|^2
\]

where \( \|\cdot\|_{\omega_i} \) is the usual norm in \( L^2(\omega_i) \) for \( i = 1, 2 \). In the Dirichlet case [4], let

\[
\begin{align*}
  \vartheta_{\omega_1} &= \{u \in H^1(\omega_1) \mid u(-1) = 0\} \\
  \vartheta_{\omega_2} &= \{v \in H^1(\omega_2) \mid v(1) = 0\} \\
  \vartheta &= \{(u,v) \in \vartheta_1 \times \vartheta_2 \mid u(0) = v(0)\}
\end{align*}
\]

equipped with the norms

\[
\begin{align*}
  \|u\|_{\vartheta_{\omega_i}}^2 &= \|u'\|^2_{L^2(\omega_i)}, \quad i = 1, 2 \\
  \|(u,v)\|_{\vartheta}^2 &= \|u\|_{\vartheta_{\omega_1}}^2 + \|v\|_{\vartheta_{\omega_2}}^2
\end{align*}
\]

One can see that \( \vartheta \) is algebraically and topologically equivalent to \( H^1_\Omega(\Omega) \) although it will be more convenient to think of \( \vartheta \) as a subspace of \( \vartheta_1 \times \vartheta_2 \). The space

\[
\mathcal{W} = \{(u,v,z) \in \vartheta \times \mathbb{R} \mid u(0) = v(0) = z\}
\]
is a closed subspace of \( \vartheta \times \mathbb{R} \) with norm \( \|(u,v,z)\|_W = \|(u,v)\|_0^2 \). In the Neumann case \( (7) \), replace the definition of \( \vartheta_{\omega_2} \) in \( (8) \) by

\[
\vartheta_{\omega_2} = H^1(\omega_2),
\]

and otherwise the space \( W \) is defined the same way. In either case, it is easy to show (see \( (8) \)) that the space \( W \) is densely and continuously embedded in the space \( \mathcal{H} \). Define the operator \( \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H} \) by

\[
\mathcal{A} = \begin{pmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \\ -\delta_0 d & \delta_0 d & 0 \end{pmatrix}
\]

where \( d \) denotes the (distributional) derivative operator, \( \delta_0 \) denotes the Dirac delta function with mass at \( x = 0 \), and the domain \( D(\mathcal{A}) \) of \( \mathcal{A} \) is given in the Dirichlet case \( (4) \) by

\[
D(\mathcal{A}) = \{ y \in W : u \in H^2(\omega_1), \ v \in H^2(\omega_2) \}.
\]

and in the Neumann case \( (7) \) by

\[
D(\mathcal{A}) = \{ y \in W : u \in H^2(\omega_1), \ v \in H^2(\omega_2), v'(1) = 0 \}.
\]

When \( D(\mathcal{A}) \) is endowed with the graph-norm topology

\[
\|y\|_{D(\mathcal{A})}^2 = \|y\|_W^2 + \|\mathcal{A}y\|_{\mathcal{H}}^2
\]

it becomes a Hilbert space with continuous embedding in \( \mathcal{H} \). We can therefore write the homogeneous point-mass systems \( (1), (4) \) and \( (1), (7) \) as

\[
y'(t) = \mathcal{A}y(t), \quad y(0) = y^0, \quad t > 0
\]

where \( y^0 = (u^0, v^0, z^0) \).

**Proposition 2.0.2.** The unbounded operator \( \mathcal{A} \) given by \( (10) \) in domain \( D(\mathcal{A}) \) as in \( (11) \) is a bijective, self-adjoint and dissipative operator with a compact inverse. Furthermore, \( \mathcal{A} \) is the infinitesimal generator of a strongly continuous, compact and analytic semigroup \( (T_t)_{t \geq 0} \).

Refer to \( (8) \) for a detailed proof of the above proposition for the Dirichlet case \( (1), (4) \). As a consequence of Proposition 2.0.2, given initial data \( y^0 \in \mathcal{H} \) there exists a unique solution

\[
y \in C([0, \infty); \mathcal{H})
\]

to the Cauchy problem \( (13) \). If in addition, \( y^0 \in D(\mathcal{A}) \) then \( y \in C([0, \infty), D(\mathcal{A})) \).

In the next subsection it is shown that \( \mathcal{A} \) has only negative eigenvalues, hence \( -\mathcal{A} \) is positive, self-adjoint it provides an isomorphism: \( D(\mathcal{A}) \to \mathcal{H} \). Moreover, fractional powers of \( -\mathcal{A} \) are well-defined. Let \( X_1 = D(\mathcal{A}) \) and for \( \alpha \in [0, 1] \), define \( X_\alpha = D((-\mathcal{A})^\alpha) \) and \( X_{-\alpha} = X'_\alpha \), the dual space relative to the pivot space \( \mathcal{H} = X_0 \) of \( X_\alpha \). Correspondingly, the semigroup \( \mathbb{T} \) remains an analytic semigroup on the invariant subspaces \( X_\alpha \), \( 0 \leq \alpha \leq 1 \), and extends continuously to an analytic semigroup on spaces \( X_\alpha \), \( -1 \leq \alpha \leq 0 \); see e.g., \( (10) \) for full explanation. The norm on \( X_\alpha \) is given by \( \|y\|_\alpha = \langle (-\mathcal{A})^\alpha y, (-\mathcal{A})^\alpha y \rangle_\mathcal{H} \). In particular, \( X_{1/2} \) is the completion of \( X_1 \) with respect to the norm

\[
\|y\|_{1/2}^2 = \langle -\mathcal{A}y, y \rangle_0.
\]

Integration by parts gives

\[
\|y\|_{1/2}^2 = \langle y, y \rangle_W.
\]

Thus, \( X_{1/2} \) is topologically equivalent to \( H_{0}^1(\Omega) \) in the Dirichlet case \( (4) \) and \( \{ f \in H^1(\Omega) : f(-1) = 0 \} \) in the Neumann case \( (7) \).
2.1 Spectral analysis for Dirichlet case \([1], [4]\)

By Proposition 2.0.2 the spectrum \(\sigma(A)\) of \(A\) is contained in the negative real axis and consists of eigenvalues \(\{\lambda_n\}\) tending to negative infinity with corresponding eigenvectors \(\{\varphi_n\}_{n \in \mathbb{N}}\) forming an orthogonal system for \(\mathcal{H}\).

**Proposition 2.1.1.** The eigenvalues \(\{\lambda_n\}_{n \in \mathbb{N}}\) of \(A\) in the Dirichlet case \([4]\) are distinct and given by

\[
\lambda_{2k} = -(k\pi)^2, \quad \lambda_{2k-1} = -\mu_k^2 \quad \text{for } k \in \mathbb{N}
\]

where \(\mu_k\) is the \(k\)-th positive root of the characteristic equation

\[
\mu = 2 \cot \mu.
\]  

(14)

The corresponding eigenvectors are given by

\[
\varphi_{2k}(x) = \begin{pmatrix} \sin(k\pi x) \\ \sin(k\pi x) \\ 0 \end{pmatrix}, \quad \varphi_{2k-1}(x) = \begin{pmatrix} \sin((1 + x)\mu_k) \\ \sin((1 - x)\mu_k) \\ \sin(\mu_k) \end{pmatrix}
\]

and \(\varphi_n \in D(A)\) for all \(n \in \mathbb{N}\).

**Proof.** Look for nontrivial functions \(\varphi_n = (U_n, V_n, Z_n)^t \in D(A)\) such that \(A\varphi_n = \lambda_n \varphi_n\). We use an even index in the case that \(Z_n = 0\) and an odd index when \(Z_n \neq 0\). The eigensystem corresponding to \(Z_{2k} = 0\) reduces to the problem of finding \((U_{2k}, V_{2k})\) such that

\[
\begin{align*}
U''_{2k}(x) &= \lambda_{2k} U_{2k}(x), \quad x \in \omega_1 \\
V''_{2k}(x) &= \lambda_{2k} V_{2k}(x), \quad x \in \omega_2 \\
U_{2k}(0) &= V_{2k}(0) \\
U_{2k}(0) &= V_{2k}(0) = 0 \\
U_{2k}(-1) &= V_{2k}(1) = 0.
\end{align*}
\]

It is easy to check that \(\varphi_{2k}\) satisfies the above with \(\lambda_{2k} = -(k\pi)^2\).

Now consider the case that \(Z_{2k-1} \neq 0\). The eigenvalue problem reduces to the problem of finding functions \((U_{2k-1}, V_{2k-1})\), and real value \(Z_{2k-1}\) such that

\[
\begin{align*}
U''_{2k-1}(x) &= -\mu_k^2 U_{2k-1}(x), \quad x \in \omega_1 \\
V''_{2k-1}(x) &= -\mu_k^2 V_{2k-1}(x), \quad x \in \omega_2 \\
V_{2k-1}(0) - U''_{2k-1}(0) &= -\mu_k^2 Z_{2k-1} \\
U_{2k-1}(0) &= V_{2k-1}(0) = Z_{2k-1} \\
U_{2k-1}(-1) &= V_{2k-1}(1) = 0.
\end{align*}
\]  

(15)

From the boundary condition \(U_{2k-1}(-1) = V_{2k-1}(1) = 0\), we have that the solution is of the form

\[
\begin{align*}
U_{2k-1}(x) &= \sin((x + 1)\mu_k) \\
V_{2k-1}(x) &= C \sin((x - 1)\mu_k)
\end{align*}
\]

for some constant \(C\) to be determined. The continuity condition \(U_{2k-1}(0) = V_{2k-1}(0) = Z_{2k-1}\) gives

\[
Z_{2k-1} = \sin(\mu_k) = -C \sin(\mu_k).
\]

Since \(Z_{2k-1}\) is nonzero we have that \(\mu_k\) is not a multiple of \(\pi\). Furthermore, we find that \(C = -1\). Then from the third equation in (15) we see that

\[
2 \cot(\mu_k) = \mu_k.
\]  

(16)
Hence the solution to the eigensystem \([15]\) is

\[
\begin{pmatrix}
U_{2k-1}(x) \\
V_{2k-1}(x) \\
Z_{2k-1}
\end{pmatrix} = \begin{pmatrix}
\sin((1 + x)\mu_k) \\
\sin((1 - x)\mu_k) \\
\sin(\mu_k)
\end{pmatrix}.
\]

Finally, note that since the function \(F(\mu) = 2 \cot \mu - \mu\) decreases monotonically from \(+\infty\) to \(-\infty\) over the interval \(((k - 1)\pi, k\pi)\) for all \(k \in \mathbb{N}\), there is exactly one root of \(F\) in each interval \(((k - 1)\pi, k\pi)\) for all \(k \in \mathbb{N}\). Hence the eigenvalues

\[
\{- (k\pi)^2\}_{k \in \mathbb{N}} \cup \{- \mu_k^2\}_{k \in \mathbb{N}}
\]

are distinct.

**Proposition 2.1.2.** The sequence \(\{\mu_k\}\) in the Dirichlet case \([4]\) satisfies the asymptotic estimate

\[
\mu_k = (k - 1)\pi + \frac{2}{k\pi} + O\left(\frac{1}{n^2}\right).
\]

Consequently, consecutive eigenvalues of \(A\) in \([13]\) satisfy the gap condition:

\[
|\lambda_{n+1} - \lambda_n| \geq 4 + O\left(\frac{1}{n}\right).
\]

Moreover, the eigenfunctions are asymptotically normalized in the sense that

\[
\lim_{n \to \infty} \|\varphi_n\| = 1.
\]

**Proof.** From the end of the previous proof, \(\mu_k = (k - 1)\pi + \epsilon_k\), where \(0 < \epsilon_k < \pi\). The characteristic equation \([14]\) can be rewritten as

\[
\frac{(k - 1)\pi + \epsilon_k}{2} = \cot \epsilon_k
\]

and thus by monotonicity,

\[(k - 1)\pi/2 < \cot \epsilon_k < k\pi/2.\]

Taking inverse cotangent of each term gives

\[
\arctan \frac{2}{k\pi} < \epsilon_k < \arctan \frac{2}{(k - 1)\pi}.
\]

Hence by Taylor’s formula we obtain \([17]\).

The estimate \([18]\) can be obtained from

\[
|\lambda_{2k+1} - \lambda_{2k}| = (\mu_{k+1} + k\pi)(\mu_{k+1} - k\pi)
\]

\[
= \left(2k\pi + O\left(\frac{1}{k}\right)\right) \left(\frac{2}{k\pi} + O\left(\frac{1}{k^2}\right)\right)
\]

\[
= 4 + O\left(\frac{1}{k}\right).
\]

Finally, it is easy to check that \(\|\varphi_{2k}\| = 1\) for all \(k \in \mathbb{N}\) and using estimate \([17]\) that \(\|\varphi_{2k-1}\|^2 = 1 + O(k^{-2})\).

\[\Box\]
2.2 Spectral analysis for Neumann case (1), (7)

As in Subsection 2.1, the eigenvalues of $A$ (denoted $\lambda_n$) form a discrete sequence of negative numbers tending to negative infinity with corresponding eigenvectors $\varphi_n$ which form an orthogonal system for $H$.

**Proposition 2.2.1.** The eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ of $A$ in the Neumann case (7) are given by $\lambda_n = -\mu_n^2$ where $\{\mu_n\}_{n \in \mathbb{N}}$ are the roots of the characteristic equation

$$\mu = 2 \cot 2\mu. \quad (19)$$

The corresponding eigenvectors are given by

$$\begin{align*}
\varphi_{2k-1}(x) &= \sqrt{2} \begin{pmatrix}
\sin(\mu_{2k-1}(x + 1)) \\
\tan \mu_{2k-1} \cos(\mu_{2k-1}(x - 1)) \\
\sin \mu_{2k-1}
\end{pmatrix} \\
\varphi_{2k}(x) &= \sqrt{2} \begin{pmatrix}
\cot \mu_{2k} \sin(\mu_{2k}(x + 1)) \\
\cos(\mu_{2k}(x - 1)) \\
\cos \mu_{2k}
\end{pmatrix}
\end{align*} \quad (20)$$

and $\varphi_n \in D(A)$ for all $n \in \mathbb{N}$.

**Proof.** The eigenvalue problem $A \varphi_n = \lambda_n \varphi_n$ with $\varphi_n = (U_n, V_n, Z_n)' \in D(A)$ is the following system:

$$\begin{align*}
U_n''(x) &= \lambda_n U_n(x), \quad x \in \omega_1 \\
V_n''(x) &= \lambda_n V_n(x), \quad x \in \omega_2 \\
V_n(0) - U_n(0) &= \lambda_n Z_n \\
U_n(0) &= V_n(0) = Z_n \\
U_n(-1) &= V_n'(1) = 0.
\end{align*} \quad (21)$$

First note that the possibility of $Z_n = 0$ leads to the trivial solution. Hence $Z_n \neq 0$ for all $n \in \mathbb{N}$. Then from the first two equations and the boundary conditions we find that

$$\begin{align*}
U_n(x) &= \sin(\mu_n(x + 1)) \\
V_n(x) &= C \cos(\mu_n(x - 1))
\end{align*}$$

for some nonzero constant $C$ to be determined. The continuity condition $U_n(0) = V_n(0)$ gives

$$\sin \mu_n = C \cos \mu_n$$

and since $Z_n$ is nonzero for all $n \in \mathbb{N}$ we have that $C = \tan \mu_n$. Then from the third equation in (21) we see that

$$\mu_n = -\tan \mu_n + \cot \mu_n$$

which is equivalent to the characteristic equation (19). Hence the corresponding sequence of eigenvectors is

$$\varphi_n(x) = \begin{pmatrix}
\sin(\mu_n(1 + x)) \\
\tan \mu_n \cos(\mu_n(x - 1)) \\
\sin \mu_n
\end{pmatrix} \mu_n$$

which agrees with (20) after multiplying by normalizing factors $\sqrt{2}$ for $n = 2k - 1$ and $\sqrt{2} \cot \mu_n$ for $n = 2k$.

Following the ideas of Proposition 2.1.2 one can prove the following result.
Proposition 2.2.2. The sequence \( \{\mu_k\} \) in the Neumann case (7) satisfies the asymptotic estimate
\[
\mu_k = \frac{(k-1)\pi}{2} + \frac{1}{k\pi} + O\left(k^{-2}\right).
\]

Consequently, consecutive eigenvalues of \( A \) in (13) satisfy the gap condition:
\[
|\lambda_{n+1} - \lambda_n| \geq \frac{n\pi^2}{2} + O\left(1\right).
\]

Moreover, the eigenfunctions are asymptotically normalized in the sense that
\[
\lim_{n \to \infty} \|\varphi_n\| = 1.
\]

3 Proof of Controllability results

We begin with the case of Neumann control: (1), (3).

3.1 Neumann control

The dual observation problem to (1), (3) is
\[
\begin{align*}
-\ddot{u} - \dddot{u} &= 0, \quad t > 0, \ x \in \omega_1 \\
-\ddot{v} - \dddot{v} &= 0, \quad t > 0, \ x \in \omega_2 \\
-\ddot{z} &= \ddot{v}(t, 0) - \dddot{u}(t, 0), \quad t > 0 \\
\ddot{u}(t, 0) &= \ddot{v}(t, 0) = \ddot{z}(t), \quad t > 0 \\
\ddot{u}(t, -1) &= \ddot{v}(t, 1) = 0, \quad t > 0
\end{align*}
\]

with terminal data at \( t = T \) given by
\[
\begin{align*}
\ddot{u}^T(x) &= \ddot{u}(T, x), \quad x \in \omega_1 \\
\ddot{v}^T(x) &= \ddot{v}(T, x), \quad x \in \omega_2 \\
\ddot{z}^T &= \ddot{z}(T)
\end{align*}
\]

By letting \( \tilde{y} = (\tilde{u}, \tilde{v}, \tilde{z})^t \), the above problem can be written as
\[
-\ddot{\tilde{y}} = \mathcal{A}\tilde{y}, \quad \tilde{y}(T) = \tilde{y}^T \in \mathcal{H}, \quad t > 0.
\]

Then \( \tilde{y} \in C([0, T], \mathcal{H}) \) and is given by
\[
\tilde{y}(t) = T(T - t)\tilde{y}^T; \quad 0 \leq t \leq T.
\]

Let \( y \) be a smooth solution of the control problem with smooth \( f \in L^2(0, T) \). Formal integration by parts then shows
\[
0 = \int_0^T \int_{-1}^0 (\ddot{u} - \dddot{u})\ddot{u} \, dx \, dt + \int_0^T \int_0^1 (\ddot{v} - \dddot{v})\ddot{v} \, dx \, dt
\]
\[
= \langle y(T), \tilde{y}^T \rangle_{\mathcal{H}} - \langle y^0, \tilde{y}(0) \rangle_{\mathcal{H}} - \int_0^T f(t)\tilde{v}(t, 1) \, dt.
\]

Equivalently,
\[
\langle y(T), \tilde{y}^T \rangle_{\mathcal{H}} = \langle y^0, T\tilde{y}^T \rangle_{\mathcal{H}} + \int_0^T f(t)\tilde{v}(t, 1) \, dt.
\]
Since the functional $\ell(\tilde{y}) := \tilde{v}(1)$ is continuous on $X_{1/2} = \mathcal{W}$ it follows from Propositions 5.1.3 and 10.2.1 in [16] that for solutions of (24) there exists $C > 0$ for which
\[
\|\tilde{v}(t, 1)\|_{L^2(0, T)} \leq C\|\tilde{y}^T\|_{\mathcal{H}} \quad \forall \tilde{y}^T \in \mathcal{H}.
\] (27)
Hence, equation (26) uniquely defines $y(T)$ as an element of $\mathcal{H}$. Applying this definition for $s \in [0, T]$ we see
\[
y \in C([0, T], \mathcal{H})
\] (28)
and there exists $C > 0$ for which
\[
\|y\|_{L^\infty(0, T; \mathcal{H})} \leq C(\|y^0\|_{\mathcal{H}} + \|f\|_{L^2(0, T)}).
\] (29)
As before we have the following lemma.

**Lemma 3.1.1.** The control problem (1), (3) is null controllable in time $T > 0$ if and only if, for any $y^0 \in \mathcal{H}$ there is $f \in L^2(0, T)$ such that
\[
\langle y^0, T_T\tilde{y}^T \rangle_{\mathcal{H}} = -\int_0^T f(t)\tilde{v}(t, 1)dt
\] (30)
holds for all $\tilde{y}^T \in \mathcal{H}$, where $\tilde{y}$ is the solution to the observation problem (24).

**Proof.** First assume that (30) holds for all $\tilde{y}^T \in \mathcal{H}$. Then by (26), $y(T) = 0$. Conversely, if $f$ is a control for which $y(T) = 0$, then (30) follows from equation (26). $\square$

We are now ready to reduce the control problem (1), (3) to a moment problem. Any initial data $y^0 = (u^0, v^0, z^0)^t$ in $\mathcal{H}$ for the control problem can be expressed in terms of the eigenfunctions as
\[
y^0 = \sum_{n \in \mathbb{N}} y^0_n \varphi_n
\] (31)
where the Fourier coefficients $\{y^0_n\}_{n \in \mathbb{N}}$ belong to $\ell^2$. Let $\tilde{y}_n = (\tilde{u}_n, \tilde{v}_n, \tilde{z}_n)^t$ be the eigensolution of (24) given by
\[
\tilde{y}_n(t, x) = e^{\lambda_n(T-t)}\varphi_n(x).
\] (32)
In particular, note that
\[
\tilde{v}_n(t, 1) = \begin{cases} 
\sqrt{2}e^{\lambda_n(T-t)}\tan \mu_n, & n \text{ odd} \\
\sqrt{2}e^{\lambda_n(T-t)}, & n \text{ even}.
\end{cases}
\]
Applying these solutions to equation (30) we obtain the following moment problem:
\[
\frac{a_n}{b_n}e^{\lambda_n T} = \int_0^T f(T-\tau)e^{\lambda_n \tau}d\tau, \quad n \in \mathbb{N}
\] (33)
where
\[
b_n = \begin{cases} 
-\tan \mu_n, & n \text{ is odd} \\
-1, & n \text{ is even}
\end{cases}
\] (34)
and by Proposition 2.2.2 $a_n = \|\varphi_n\|^2y^0_n \in \ell^2$. In particular note that for $n = 2k - 1$
\[
\tan \mu_{2k-1} = \tan \left(\frac{1}{k\pi} + O(k^{-2})\right) = \frac{1}{k\pi} + O(k^{-2})
\]
and furthermore since $\tan \mu_{2k-1} \neq 0$ for all $k \in \mathbb{N}$, there exists $\epsilon > 0$ such that

$$|b_n| \geq \frac{\epsilon}{n}, \quad \forall n \in \mathbb{N}.$$  

From our estimates of $\mu_n$, $\lambda_n$, $b_n$ and $a_n$, it is easy to show that there are constants $K, \delta > 0$ such that

$$\left| \frac{a_n e^{\lambda_n T}}{b_n} \right| \leq Ke^{-\delta n^2}, \quad n \in \mathbb{N}. \quad (35)$$

From equations (22) and (23) we see that the series $\sum 1/\lambda_n$ converges, and that there exists a constant $\rho > 0$ such that $|\lambda_{k+1} - \lambda_k| > \rho$ for all $k \in \mathbb{N}$. This implies the existence of a biorthogonal sequence $\{\theta_j(\tau)\}_{j \in \mathbb{N}}$ (see [15], [5]) such that

$$\int_0^T \theta_j(\tau)e^{\lambda_n \tau} d\tau = \delta_{j,n} = \begin{cases} 1, & j = n \\ 0, & j \neq n. \end{cases} \quad (36)$$

By the method of Russell and Fattorini in [15] we have that there are $M_1, M_2 > 0$ such that

$$\|\theta_j\| \leq M_1 e^{M_2 j}. \quad (37)$$

It is easy to see that the above implies the convergence of

$$f(T - \tau) = \sum_{j \in \mathbb{N}} \frac{\partial_j}{\zeta_j} e^{\lambda_j T} \theta_j(\tau)$$

which provides a solution to the moment problem (47). The proof of Proposition 1.0.1 for the case of Neuman control (3), is a direct consequence of Lemma 3.1.1 and the existence of the biorthogonal sequence $\{\theta_j(\tau)\}_{j \in \mathbb{N}}$.

### 3.2 Dirichlet Control

The dual observation problem to (1), (2) is

$$\begin{cases} -\dot{\tilde{u}} - \tilde{u}'' = 0, & t > 0, \quad x \in \omega_1 \\ -\dot{\tilde{v}} - \tilde{v}'' = 0, & t > 0, \quad x \in \omega_2 \\ -\dot{\tilde{z}} = \tilde{v}'(t,0) - \tilde{u}'(t,0), & t > 0 \\ \tilde{u}(t,0) = \tilde{v}(t,0) = \tilde{z}(t), & t > 0 \\ \tilde{u}(t,-1) = \tilde{v}(t,1) = 0, & t > 0 \end{cases} \quad (38)$$

with terminal data at $t = T$ given by

$$\begin{cases} \tilde{u}^T(x) = \tilde{u}(T,x), & x \in \omega_1 \\ \tilde{v}^T(x) = \tilde{v}(T,x), & x \in \omega_2 \\ \tilde{z}^T = z(T) \end{cases} \quad (39)$$

and observation $Y(t) = \tilde{v}'(t,1)$. By letting $\tilde{y} = (\tilde{u}, \tilde{v}, \tilde{z})^t$, the above problem can be written as a Cauchy problem as

$$-\dot{\tilde{y}} = A\tilde{y}, \quad \tilde{y}(T) = \tilde{y}^T, \quad t > 0. \quad (40)$$

If $\tilde{y}^T \in X_{1/2} = W$ then $\tilde{y} \in C([0,T], X_{1/2})$ is given by

$$\tilde{y}(t) = T(T-t)\tilde{y}^T; \quad 0 \leq t \leq T.$$
Let $y$ be a smooth solution of the control problem with smooth $f \in L^2(0, T)$ and let $\tilde{y}$ be solution of the dual problem \eqref{eq:dual}. Integration by parts as earlier results in the identity
\[
\langle y(T), \tilde{y}^T \rangle = \langle y^0, T_T \tilde{y}^T \rangle_{\mathcal{H}} - \int_0^T f(t) \tilde{v}'(t, 1) \, dt
\] (41)
where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $X_{-1/2} \times X_{1/2}$.

In the case of the heat equation
\[
\begin{cases}
\dot{q} = q'' & 0 < x < 1, \ t > 0 \\
q(t, 0) = q(t, 1) = 0 & t > 0 \\
q(0, x) = q^0 \in H^1_0(0, 1) & 0 < x < 1
\end{cases}
\]

it is well known (e.g. \cite{4}) that for each $T > 0$ there exists $C > 0$ for which
\[
\|q'(\cdot, 1)\|_{L^2(0, T)} \leq C \|q^0\|_{H^1_0(0, 1)}.
\]

One can verify that the same estimate holds for solutions of \eqref{eq:dual} in the sense that there exists $C > 0$ for which
\[
\|\tilde{v}'(t, 1)\|_{L^2(0, T)} \leq C \|\tilde{y}^T\|_{1/2} \quad \forall \tilde{y}^T \in X_{1/2}.
\] (42)

Since the semigroup $T$ is strongly continuous on $X_{1/2} = H^1_0(\Omega)$ it follows that the identity \eqref{eq:identity} defines the value $y(s)$ for all $s \in [0, T]$ as an element of $X_{-1/2}$ for which there exists $C > 0$ such that
\[
\|y\|_{L^\infty(0, T; X_{-1/2})} \leq C(\|y^0\|_{\mathcal{H}} + \|f\|_{L^2(0, T)})
\]
and moreover
\[
y \in C([0, T], X_{-1/2}).
\] (43)

The above estimate \eqref{eq:estimate} is sometimes referred to as admissibility of the boundary control operator corresponding to Dirichlet control, and can also be derived in the framework of “well posed boundary control systems”; see of \cite{16}, Prop. 10.7.1.

Analogous to Lemma \ref{lem:boundary} the following lemma characterizes the problem of null controllability of (1), (2) in terms of the solution $\tilde{y}$ of the observation problem \eqref{eq:dual}.

**Lemma 3.2.1.** The control problem (1), (2) is null controllable in time $T > 0$ if and only if, for any $y^0 \in \mathcal{H}$ there is $f \in L^2(0, T)$ such that
\[
\langle y^0, T_T \tilde{y}^T \rangle_{\mathcal{H}} = \int_0^T f(t) \tilde{v}'(t, 1) \, dt
\] (44)
holds for all $\tilde{y}^T \in \mathcal{H}$, where $\tilde{y} = (\tilde{u}, \tilde{v}, \tilde{z})^t$ is a solution of \eqref{eq:dual}.

We are now ready to reduce the control problem (1), (2) to a moment problem. Any initial data $y^0 = (u^0, v^0, z^0)^t$ in $\mathcal{H}$ for the control problem can be expressed in terms of the eigenfunctions as
\[
y^0 = \sum_{n \in \mathbb{N}} y^0_n \varphi_n
\] (45)
where the Fourier coefficients $\{y^0_n\}_{n \in \mathbb{N}}$ belong to $l^2$. Let $\tilde{y}_n = (\tilde{u}_n, \tilde{v}_n, \tilde{z}_n)^t$ be the eigensolution of \eqref{eq:dual} given by
\[
\tilde{y}_n(t, x) = e^{\lambda_n(T-t)} \varphi_n(x).
\] (46)

In particular, note that
\[
\tilde{v}_n(t, 1) = \begin{cases}
e^{\lambda_{2k}(T-t)k\pi(-1)^k}, & n = 2k \\
-e^{\lambda_{2k-1}(T-t)} \mu_k, & n = 2k - 1.
\end{cases}
\]
We plug these solutions into equation (44) to obtain the corresponding moment problem

\[ a_n e^{\lambda_n T} = b_n \int_0^T f(T - \tau)e^{\lambda_n \tau} d\tau \]  

(47)

for all \( n \in \mathbb{N} \) where

\[ b_n = \tilde{\nu}'_n(T, 1) = \begin{cases} (-1)^k k\pi, & n = 2k \\ -\mu_k, & n = 2k - 1 \end{cases} \]  

(48)

and by Proposition 2.2, \( a_n = \| \varphi_n \|^2 y_n^0 \in \ell^2 \). Again, it is easy to show that there exists constants \( K, \delta > 0 \) such that (35) holds. From equations (17) and (18) we see that the series \( \sum 1/\lambda_n \) converges, and that there exists a constant \( \rho > 0 \) such that \( |\lambda_{k+1} - \lambda_k| > \rho \) for all \( k \in \mathbb{N} \). This implies the existence of a biorthogonal sequence \( \{\theta_j(\tau)\}_{j \in \mathbb{N}} \) such that there are constants \( M_1, M_2 > 0 \) such that

\[ \|\theta_j\| \leq M_1 e^{M_2 j}. \]

Hence, as earlier,

\[ f(T - \tau) = \sum_{j \in \mathbb{N}} \frac{a_j}{b_j} e^{\lambda_j T} \theta_j(\tau) \]

converges and provides a solution to the moment problem (47). The proof of Proposition 1.0.1 for the case of Dirichlet control (2), is a direct consequence of Lemma 3.2.1.

**Remark 3.2.1.** The numbers \( b_k \) in (48) and (34), are called control input coefficients and can be viewed as Fourier coefficients of an element \( b \) of \( X_{-1} \) for which the control problem (1) with either (2) or (3) can be formulated as

\[ \dot{y} = Ay + bf, \quad y(0) = y_0. \]

In the Neumann case, the input element is admissible on the state space \( X_0 = H \), or equivalently that (28) and (29) hold. In the Dirichlet case, \( b \) is admissible on the state space \( X_{-1/2} \). Both of these spaces are slightly suboptimal in the sense that the Carleson measure criterion due to Ho and Russell [11] and Weiss [17] can be used as in [9] to show admissibility holds in the spaces \( X_{1/4} \) and \( X_{-1/4} \) respectively for the Neumann and Dirichlet control problems.

**References**

1. C. Castro and E. Zuazua, *Boundary controllability of a hybrid system consisting in two flexible beams connected by a point mass*, SIAM J. Control Optim., 36:1576-1595, (1998).

2. C. Castro and E. Zuazua, *Exact boundary controllability of two Euler-Bernoulli beams connected by a point mass*, Math. Comput. Modeling, 32 (2000), pp. 955-969.

3. F. Conrad and O. Morgul, *On the stabilization of a flexible beam with a tip mass*, SIAM, vol. 96, No. 6, pp. 1962-1986, (1998).

4. L. C. Evans, *Partial Differential Equations*, Vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2010.

5. E. Fernández-Cara, M. González-Burgos and L. de Teresa, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010), no. 7, 1720-1758.

6. B. Z. Guo, *Riesz basis approach to the stabilization of a flexible beam with a tip mass*, SIAM J. Control Optim., 39 (2001), pp. 1736-1747.
[7] B. Z. Guo and S. A. Ivanov, *Boundary Controllability and Observability of a One-Dimensional Nonuniform SCOEL System*, J. Optim. Theory Appl., 127 (2005), pp. 89-108.

[8] S. Hansen and J. Martinez, *Modeling of a heat equation with Dirac density*, arXiv:1506.07936

[9] S. Hansen and B. Y. Zhang, *Boundary control of a linear thermoelectric beam*, J. Math. Anal. Appl., 210 (1997), pp. 182-205.

[10] S. Hansen and E. Zuazua, *Exact controllability and stabilization of a vibrating string with an interior point mass*, SIAM J. Cont. Optim. 33 (5), (1995) 1357–1391.

[11] L. F. Ho and D. Russell, *Admissible input elements for systems in Hilbert space and a Carleson measure criterion*, SIAM J. Control Optim. 21 (1983), 614-640.

[12] W. Littman and L. Markus, *Exact boundary controllability of a hybrid system of elasticity*, Arch. Rational Mech. Anal., 103 (1988), pp. 193-236.

[13] W. Littman and S. W. Taylor, *Boundary feedback stabilization of a vibrating string with an interior point mass*, Nonlinear Problems in Mathematical Physics and Related Topics I, in: Int. Math. Ser., vol 1, 2002, pp. 271287.

[14] O. Morgul, B. P. Rao and F. Conrad, *On the stabilization of a cable with a tip mass*, IEEE Trans. Automat. Control. 39 (1994), pp. 2140-2145.

[15] D. Russell and H. O. Fattorini, *Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations*, Quart. Appl. Math. 43 (1974), 45-69.

[16] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Advanced Texts, Birkhäuser, Basel, Switzerland, 2009.

[17] G. Weiss, *Admissibility of input elements for diagonal semigroup on l^2*, Systems Control Lett. 10 (1998), 79-82.

[18] X. Zhao and G. Weiss, *Well-posedness, regularity and exact controllability of the SCOEL model*, Math. Control Signals Syst., 22, (2010), pp. 91-127.