Zero distribution for Angelesco Hermite–Padé polynomials

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Abstract. This paper considers the zero distribution of Hermite–Padé polynomials of the first kind associated with a vector function

\[ \vec{f} = (f_1, \ldots, f_s) \]

whose components \( f_k \) are functions with a finite number of branch points in the plane. The branch sets of component functions are assumed to be sufficiently well separated (which constitutes the Angelesco case). Under this condition, a theorem on the limit zero distribution for such polynomials is proved. The limit measures are defined in terms of a known vector equilibrium problem.

The proof of the theorem is based on methods developed by Stahl [59]–[63] and Gonchar and the author [27], [55]. These methods are generalized further in the paper in application to collections of polynomials defined by systems of complex orthogonality relations.

Together with the characterization of the limit zero distributions of Hermite–Padé polynomials in terms of a vector equilibrium problem, the paper considers an alternative characterization using a Riemann surface \( \mathcal{R}(\vec{f}) \) associated with \( \vec{f} \). In these terms, a more general conjecture (without the Angelesco condition) on the zero distribution of Hermite–Padé polynomials is presented.

Bibliography: 72 titles.

Keywords: rational approximations, Hermite–Padé polynomials, zero distribution, equilibrium problem, \( S \)-compact set.

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1. Introduction

1.1. Statement of the main theorems. Let \( s \in \mathbb{N} \) and let

\[ \vec{f} = (f_1, f_2, \ldots, f_s) \]

be a vector of analytic functions defined by their Laurent expansions at infinity

\[ f_k(z) = \sum_{m=0}^{\infty} \frac{f_{m,k}}{z^m}, \quad k = 1, 2, \ldots, s. \]  

(1)

For a natural number \( n \in \mathbb{N} \) the \( n \)th vector of first-kind Hermit–Padé polynomials \( q_{n,0}, q_{n,1}, \ldots, q_{n,s} \) is defined by the relation

\[ R_n(z) := (q_{n,0} + q_{n,1} f_1 + q_{n,2} f_2 + \cdots + q_{n,s} f_s)(z) = O\left( \frac{1}{z^{ns+s}} \right) \]  

(2)

as \( z \to \infty \) and by the condition \( q_{n,k} \in \mathbb{P}_n, k = 0, 1, \ldots, s \), where \( \mathbb{P}_n \) is the notation for the set of all polynomials of degree at most \( n \). The function \( R_n \) defined by (2) above is called the remainder.

The construction (1), (2) is classical. It was introduced in 1873 for the case \( f_k(z) = \exp\{k/z\} \) by Hermite, who used it to prove that the number \( e \) is transcendental. Hermite’s student Padé investigated in detail the case \( s = 1 \), which was later named after him. For further information we refer readers to [46], [49], [10].

The main result of this paper is related to the class \( A \) of functions \( f_k \) with a finite number of branch points. More precisely, for a fixed set \( e = \{a_1, \ldots, a_p\} \) of \( p \geq 2 \) distinct points we denote by \( \mathcal{A}_e \) the class of functions whose elements at infinity admit analytic continuation along any curve in the domain \( \Omega = \mathbb{C} \setminus e \) which begins at infinity. We shall assume that a function \( f \in \mathcal{A}_e \) is not single valued in \( \Omega \); some of the points in \( e \) may be regular or single-valued isolated singular points, but there are at least two branch points. Then we define

\[ \mathcal{A} = \bigcup \mathcal{A}_e, \]
where the union is taken over all finite sets $e$ of $p \geq 2$ distinct points in the plane. In other words, $\mathcal{A}$ is the space of all analytic functions in the plane with a finite number of branch points. In particular, $f \in \mathcal{A}$ means that there is a finite set $e = e(f) \subset \mathbb{C}$ such that $f \in \mathcal{A}_e$. We shall write $\vec{f} = (f_1, \ldots, f_s) \in \mathcal{A}$ if $f_j \in \mathcal{A}$ for all component functions $f_j$.

For a certain subclass of vector functions $\vec{f} \in \mathcal{A}$ we shall prove the weak convergence

$$\mu_{n,k} = \frac{1}{n} \sum_{q_{n,k}(x) = 0} \delta(x) \to \lambda_k \quad \text{as } n \to \infty,$$

of the zero-counting measures of the Hermite–Padé polynomials $q_{n,k}$ in (2) and characterize their limits $\lambda_k$. In application to measures $\to$ will always mean weak convergence.

The precise condition on $\vec{f}$ in our convergence theorem, which we shall call the Angelesco condition, is formulated in terms of a vector equilibrium problem associated with $\vec{f}$. We need to introduce some definitions first.

For a function $f \in \mathcal{A}$ we denote by $\mathcal{F}_0(f)$ the set of cuts that make $f$ single valued. Formally, $\mathcal{F}_0(f)$ is the set of compact sets $F \subset \mathbb{C}$ satisfying the condition $f \in H(\mathbb{C} \setminus F)$. By $H(\Omega)$ we denote, as usual, the space of holomorphic (analytic and single-valued) functions in a domain $\Omega$.

It is convenient to work with a more restricted class of cuts. We shall use the subclass $\mathcal{F}(f) \subset \mathcal{F}_0(f)$ such that any compact set $F \in \mathcal{F}(f)$ has a finite number of connected components not separating the plane and each connected component contains at least two branch points of $f$.

For a vector function $\vec{f} = (f_1, \ldots, f_s)$ whose components $f_k$ are in $\mathcal{A}$ we denote by $\mathcal{F}(\vec{f})$ the class of all vector compact sets $\vec{F} = (F_1, \ldots, F_s)$ such that $f_k \in \mathcal{F}(f_k)$ for $k = 1, \ldots, s$.

By $\mathcal{M}(F)$ we denote the set of all unit positive Borel measures on a compact set $F$. For a fixed vector compact set $\vec{F} = (F_1, \ldots, F_p) \in \mathcal{F}$ we define a family of vector measures

$$\vec{\mathcal{M}} = \vec{\mathcal{M}}(\vec{F}) = \{\vec{\mu} = (\mu_1, \ldots, \mu_s): \mu_j \in \mathcal{M}(F_j)\}.$$

The key component of any vector equilibrium problem is the interaction matrix. The matrix $A$ associated with the Angelesco equilibrium problem is the following:

$$A = \|a_{ij}\|_{i,j=1}^s, \quad \text{where } a_{ii} = 2 \text{ and } a_{ij} = 1 \text{ for } i \neq j, \ i, j = 1, \ldots, s. \quad (4)$$

Accordingly, the energy of a vector measure $\vec{\mu} \in \vec{\mathcal{M}}$ associated with matrix $A$ is defined by

$$\mathcal{E}(\vec{\mu}) = [A\vec{\mu}, \vec{\mu}] = \sum_{i,j=1}^s a_{ij} [\mu_i, \mu_j], \quad (5)$$

where

$$[\mu, \nu] = \int U^\nu d\mu$$

is the mutual energy of $\mu$ and $\nu$ and $U^\nu(z) = -\int \log |z-x| d\nu(x)$ is the logarithmic potential of the measure $\nu$. 


The next two lemmas assert the existence of solutions of two basic equilibrium problems associated with the Angelesco case. The first lemma is well known [13], [31] (see also the original papers [24], [26], [37], [28], [23], [36], [39]).

**Lemma 1.** For a fixed $\vec{F} \in \vec{F}(\vec{f})$ there exists a unique vector measure $\vec{\lambda} = \vec{\lambda}_\vec{F} \in \vec{M}(\vec{F})$ minimizing the energy

$$E(\vec{\lambda}) = \min_{\vec{\mu} \in \vec{M}(\vec{F})} E(\vec{\mu})$$

(vector equilibrium measure for $\vec{F}$).

We define the equilibrium energy functional $E[\vec{F}]$ on the class $\vec{F} = \vec{F}(\vec{f})$ by

$$E[\vec{F}] := \min_{\vec{\mu} \in \vec{M}(\vec{F})} E(\vec{\mu}) = E(\vec{\lambda}_\vec{F}), \quad \vec{F} \in \vec{F},$$

and assert the existence of a maximizing vector compact set $\vec{\Gamma} \in \vec{F}$.

**Lemma 2.** For a fixed $\vec{f} \in A$ there exists a vector compact set $\vec{\Gamma} \in \vec{F} = \vec{F}(\vec{f})$ such that

$$E[\vec{\Gamma}] = \max_{\vec{F} \in \vec{F}} E[\vec{F}].$$

The proof of Lemma 2 is presented in §4.1.1 below; it essentially follows from the results and methods developed in [55].

It is well known that in general the extremal vector compact set $\vec{\Gamma}$ is not unique. More precisely, the part $\vec{\Gamma}$ which does not belong to the support of the associated equilibrium measure is not unique. This part may vary in a domain determined by the equilibrium potential without changing the equilibrium energy.

For the part of $\vec{\Gamma}$ which carries the equilibrium measure we introduce the notation $\vec{\Gamma}^1 = (\Gamma_1^1, \ldots, \Gamma_s^1)$. Thus, the components of $\vec{\Gamma}^1$ are defined by

$$\Gamma_j^1 = \text{supp}(\lambda_j).$$

This part of $\vec{\Gamma}$ is unique, but the uniqueness is not used in the proofs.

The sets $\Gamma_j \setminus \Gamma_j^1$ are less important. They do not carry essential numbers of zeros of Hermite–Padé polynomials and they do not play an essential role in proofs. However, they also require some attention. They have to be selected in a certain way (the last inequality in (13) has to be satisfied). They may also contain singularities of components of $\vec{f}$ (when we modify contours of integration we have to avoid the singularities of $\vec{f}$). Finally, they are involved in the definition of the Angelesco condition on $\vec{\Gamma}$, which we introduce next.

In short, an important condition in our theorems on the zero distribution of Hermite–Padé polynomials is that the components of the extremal compact set $\vec{\Gamma}(\vec{f})$ are disjoint. More exactly, we introduce the following:

**Definition 1.** We say that a vector compact set $\vec{\Gamma}$ satisfies the (strict) Angelesco condition if it satisfies (8) and its components are disjoint:

$$\Gamma_i \cap \Gamma_j = \emptyset, \quad i \neq j.$$
Correspondingly, the vector function \( \tilde{f} = (f_1, \ldots, f_s) \in \tilde{A} \) is called an Angelesco vector function (system) if the associated vector compact set \( \tilde{\Gamma}(\tilde{f}) \) has the Angelesco property (8), (9).

Now we state the main result of the paper.

**Theorem 1.** Let the vector function

\[ \tilde{f} = (f_1, \ldots, f_s) \in \tilde{A}, \text{ where } f_k \in A, \ k = 1, \ldots, s, \]

satisfy the Angelesco condition (9). Then the sequences of zero-counting measures \( \mu_{n,k} \) of Hermite–Padé polynomials in (3) are weakly convergent as \( n \to \infty \),

\[ \mu_{n,k} \rightharpoonup \lambda_k, \quad k = 1, \ldots, s, \tag{10} \]

to the components \( \lambda_k \) of the vector equilibrium measure \( \lambda = \lambda_{\tilde{\Gamma}} \) of the extremal vector compact set \( \tilde{\Gamma} \) in the class \( \tilde{\mathcal{F}}(\tilde{f}) \) defined by (6).

It is important to add that Theorem 1 is not presented in the most general form. We choose to define the vector compact set \( \tilde{\Gamma} \in \tilde{\mathcal{F}}(\tilde{f}) \) (which is the key component of the theorem) as the solution of the ‘max-min’ energy problem (8) with the Angelesco condition (9). With this definition it is possible to prove the existence theorem by comparatively simple reduction to known results.

Actually, Theorem 1 is valid under the assumption that there is a vector compact set \( \Gamma \in \mathcal{F}(f) \) with the \( S \)-property (see the definition in (12)–(14) below) and the property (9) (which can be relaxed and replaced by the condition that \( \Gamma_i \cap \Gamma_j \) is a finite set for \( i \neq j \)).

Numerical experiments conducted by Suetin show that the existence of such a vector compact set is a more general condition than that stated in terms of (8). More precisely, the class of vector functions \( \tilde{f} \) such that a vector compact set \( \tilde{\Gamma} \in \tilde{\mathcal{F}}(\tilde{f}) \) with the \( S \)-property exists is essentially larger than the class in Theorem 1. It is more difficult to prove an existence theorem for such \( S \)-compact sets. For the moment the ‘max-min’ energy problem (8) seems to be the only efficient method for finding \( S \)-compact sets, but some \( S \)-compact sets do not satisfy (8).

**1.2. Remarks. Generalizations.**

1.2.1. **Asymptotics of the leading coefficient.** An important fact about first-kind Hermite–Padé polynomials is that their leading coefficients are involved in the definition of the polynomials (2). This relation determines the leading coefficients together with other coefficients of the polynomials. It turns out that the asymptotics of their magnitudes can be determined in terms of the vector equilibrium problem (6)–(8).

The conditions (2) determine the function \( R_n \) up to a normalizing multiplicative constant. It follows that the leading coefficient \( c_{n,k} \) of a polynomial \( q_{n,k}(x) = c_{n,k}Q_{n,k}(x) \) can be selected arbitrarily. In addition to Theorem 1 we have the following.
Theorem 2. If the normalizing constant in (2) is selected so that the sequence $|c_{n,k}|^{1/n}$ is convergent as $n \to \infty$ for a particular fixed $k$, then it is convergent for any $k$ and there is a constant $c > 0$, depending on the normalization, such that

$$\lim_{n \to \infty} |c_{n,k}|^{1/n} = ce^{w_k}, \quad k = 1, \ldots, s,$$

where $w_k$ is the $k$th equilibrium constant associated with the extremal compact set $\Gamma(\vec{f})$ (see (13) below).

The proofs of Theorems 1 and 2 are interconnected and are presented simultaneously (see §3).

1.2.2. Asymptotics of the remainder. An important complement to Theorems 1 and 2 is the $n$th root asymptotics for the remainder $R_n$.

Theorem 3. With a suitably selected normalizing constant in (2)

$$\frac{1}{n} \log |R_n(z)| \xrightarrow{\text{cap}} -U^\lambda(z), \quad z \in \mathbb{C} \setminus \Gamma,$$

where

$$\lambda = \sum_{k=1}^{s} \lambda_k \quad \text{and} \quad \Gamma = \bigcup_{k=1}^{s} \Gamma_k$$

(convergence in capacity on compact sets in $\mathbb{C} \setminus \Gamma$).

There are indications arising from semiclassical classes of vector functions $\vec{f}$ that this theorem can be significantly generalized in the following direction. The function $-U^\lambda(z)$, which is harmonic in $\Omega = \mathbb{C} \setminus \Gamma$, has a harmonic extension $g(z)$ from the domain $\Omega$ in the plane to an $(s+1)$-sheeted algebraic Riemann surface $\mathcal{R} = \mathcal{R}(\vec{f})$.

We shall go into certain details related to $\mathcal{R}$ in §4; here we mention that on the other $s$ domains $\Omega^{(k)} \subset \mathcal{R}$ whose projection to the plane is $\Omega$, the extended function is $g(z^{(k)}) = U^\lambda_k(z) + C$, where $C$ is a common constant, independent of $k$. The function $R_n(z)$ has a multivalued analytic continuation to $\mathcal{R}$. We conjecture that for any domain $\mathcal{D} \subset \mathcal{R}$ where a holomorphic branch of $R_n$ exists (for any $n$) we have

$$\frac{1}{n} \log |R_n(z)| \xrightarrow{\text{cap}} g(z), \quad z \in \mathcal{D}.$$

The conjecture is supported by the results in [44] related to a function $\vec{f}$ with components $f_k \in \mathcal{L}$ of the form

$$f_k(z) = \prod_{i=1}^{m_k} (z - a_{i,k})^{\alpha_{i,k}} \quad \text{with} \quad \sum_{i=1}^{m_k} \alpha_{i,k} = 0$$

(see also [33] and [68]). For functions in this class the weighted Hermite-Padé polynomials $q_{n,k}(z)f_k(z)$ and the remainders satisfy the same differential equation with polynomial coefficients of order $s+1$ (the polynomial coefficients depend on $n$ but their degrees are bounded by constants independent of $n$). Combining this fact with the results of our paper, we can prove the conjecture in the case when $\vec{f} \in \mathcal{L}$ and the Angelesco condition is satisfied.
1.2.3. Angelesco condition. The Angelesco condition (9) is crucial for Theorem 1; if it is essentially violated (say, two components of $\vec{\Gamma}^1$ have a common arc) then the zero distribution of the Hermite–Padé polynomials is actually determined by another $S$-compact set. At the same time, in such a case we do not have general tools for proving the existence of $S$-compact sets. So, we use the ‘max-min’ property to define the Angelesco condition. Next, we include a few short remarks on the Angelesco condition, meaning specifically condition (9) in Theorem 1.

It would be natural to suggest that the Angelesco condition depends only on the mutual position of the sets $e_k = e(f_k)$ of (actual) singular points of the component functions $f_k$, but in general that is not true. It is probably true in a case of ‘general position’. However, some sufficient conditions can be stated in terms of the vector set $\vec{e} = (e_1, \ldots, e_s)$.

In terms of branch sets, to deduce that (9) holds it is enough to assume that the component sets $e_j$ are well separated in a certain sense. In more precise terms, let $\hat{e}_k$ be the convex hull of $e_k$. It is possible to prove that (8) holds if the distances between the $\hat{e}_k$ are large enough compared to the sizes of those sets. However, it would not be an easy task to give constructive estimates of what is ‘enough’.

It is known that it is not enough to assume that the sets $\hat{e}_k$ are disjoint. The corresponding fact follows from the results in [5], where the case $s = 2$ was considered under the assumption that each of $e_1$ and $e_2$ consists of two points (and also under certain other assumptions; we shall mention more details of this case in §4). At the same time, the case of Markov-type functions show that the Angelesco systems can have overlapping sets $\hat{e}_k$. Anyway, characterization of the Angelesco condition in geometric terms is an interesting and difficult problem, which is not the main concern of this paper.

1.2.4. Second-kind Hermite–Padé polynomials. The methods of this paper (with modifications) can be used to study the zero-distribution theorem for second-kind Hermite–Padé polynomials under the same Angelesco condition. We introduce definitions in §2.1 below, where Markov-type functions are considered. However, theorems similar to Theorems 1–3 above can only be proved under a stronger assumption on the separation of the branch sets $e_k$; the distances between the $\hat{e}_k$ have to be large enough compared to the sizes of those sets.

1.2.5. More general assumptions on degrees of polynomials. The polynomials $q_{n,k}$ in (2) above were subject to the condition $q_{n,k} \in \mathbb{P}_n$, $k = 0, 1, \ldots, s$. This condition can be easily generalized. Consider an arbitrary sequence of vectors $\vec{d}_n = (d_{n,1}, \ldots, d_{n,s})$ with natural components. For any $n$ there exists a sequence of polynomials $q_{n,k}$, $k = 0, 1, \ldots, s$ such that $\deg q_{n,k} \leq d_{n,k}$ and (2) is valid with $O(1/z^{ns+s})$ replaced by $O(1/z^N)$, where $N = d_{n,1} + \cdots + d_{n,s} + s$.

The zero distribution of such polynomials is described by a vector equilibrium problem, which is a generalization of the problem in Lemmas 1 and 2 for the case when the total masses of the components are arbitrary positive numbers (prescribed in advance). The definitions are modified as follows.

Let $t > 0$ and let $\mathcal{M}^t(F)$ be the set of all positive Borel measures $\mu$ on the compact set $F$ with total mass $|\mu| = \mu(F) = t$. For a fixed vector compact set $\vec{F} = (F_1, \ldots, F_p) \in \mathcal{F}$ and a vector $\vec{t} = (t_1, \ldots, t_s)$ with positive components $t_j$ (the total masses of the components of the vector compact set) we define a family
of vector measures
\[ \mathcal{M}^\vec{t} = \mathcal{M}^\vec{t}(\vec{F}) = \{ \vec{\mu} = (\mu_1, \ldots, \mu_s) : \mu_j \in \mathcal{M}^F_j \} \].

Associated modifications of equilibrium problems are straightforward; the class of measures is modified, but the energy functional (4)–(6) remains the same.

The analogue of Lemma 1 is valid: there exists a unique vector measure \( \vec{\lambda}^F(\vec{t}) \) in the class \( \mathcal{M}^\vec{t}(\vec{F}) \) that minimizes the vector energy (6).

In the definition (7) of the equilibrium energy functional \( E[\vec{F}] \) the class \( \mathcal{M}(\vec{F}) \) has to be replaced by \( \mathcal{M}^\vec{t}(\vec{F}) \). Then the analogue of Lemma 2 is valid: there exists a vector compact set \( \vec{\Gamma} \in \vec{F} = \vec{F}(\vec{f}) \) which maximizes the equilibrium energy \( E(\vec{\lambda}^F(\vec{t})) \) over this class (this compact set will now depend on \( \vec{t} \)).

Now we have the following combined theorem.

**Theorem 4.** Let \( \vec{d}_n = (d_{n,1}, \ldots, d_{n,s}) \) be a given sequence of vectors with natural components and let \( q_{n,k} \) be the associated sequence of Hermite–Padé polynomials (2) \( \deg q_{n,k} \leq d_{n,k} \). If the following condition is satisfied:

\[ \frac{1}{n}d_{n,k} \to t_k > 0 \quad \text{as} \quad n \to \infty, \]

then the assertions of Theorem 1 (see (10)) and Theorem 3 are valid with the \( \lambda_k \) representing the components of the vector equilibrium measure \( \vec{\lambda}^F(\vec{t}) \). Moreover, the assertion of Theorem 2 is valid with the \( w_k \) standing for the equilibrium constants (see (13) below) associated with the vector equilibrium measure \( \vec{\lambda}^F(\vec{t}) \).

The proof of Theorem 4 is no different from the proofs of Theorems 1–3. We shall restrict ourself to proving the case \( t_k = 1 \) since this particular case is identical to the general one in all the essentials and allows us to simplify the notation significantly (the notation associated with arbitrary \( \vec{t} \) is rather cumbersome). At the same time we note that in §3 we use the generalized equilibrium problem as a technical tool in proving Theorems 1–3.

1.2.6. Possible generalizations of the class \( \mathcal{A} \) and the Angelesco conditions. It is also possible to generalize the classes \( \mathcal{A} \) of functions \( f \in \mathcal{A}(\overline{\mathbb{C}} \setminus e) \). Instead of finite sets \( e \) we can consider a set \( e \) of capacity zero. This would require some essential modifications in part related to properties of the extremal vector compact set \( \vec{\Gamma} \), and we shall not go into details. At the same time, in the ‘scalar’ situation associated with the GRS-theorem in [27], where more general settings on exceptional sets do not cause any additional problems, we preserve the assumptions of [27].

The Angelesco condition on \( \vec{\Gamma} \) can be relaxed. Essentially we need that each component of the support \( \Gamma^1_j = \text{supp}(\lambda_j) \) of the extremal vector measure \( \vec{\lambda}^F \) is disjoint from the other components of \( \vec{\Gamma} \). Moreover, the assertions of the theorem remain valid if we assume that each intersection \( \Gamma^1_i \cap \Gamma^1_j \) is a finite set for \( i \neq j \) (eventually, we can allow intersections of zero capacity). Such generalizations will make proofs more difficult and we do not attempt any of these.

Finally, it was mentioned above that the Angelesco conditions cannot be dropped without essential modification of the potential-theoretic problems corresponding to the zero-limit distributions of Hermite–Padé polynomials. It is believed that in
the general situation (no Angelesco condition) there is still an equilibrium problem describing such a zero distribution. The matrix $A$ and, in general, the conditions on total masses have to be changed. The general situation is essentially known for $s = 2$ for second-kind polynomials (see [5], [2], and [54] and also [57], [21], [69]) and for a Nikishin system. For $s > 2$ the form of the matrix is only known for some particular cases.

An alternative approach is to describe the weak asymptotics of the Hermite–Padé polynomials in terms of Abelian integrals on an algebraic Riemann surface. The main problem associated with this approach is that certain auxiliary parameters of this Riemann surface are in general not known. A conjecture on the determination of this Riemann surface is described briefly at the end of this Introduction. Further details related to this case can be found in §4.

1.3. Methods. The proof of Theorem 1 is contained in §3 of the paper. We outline briefly the methods used there. As usual, the starting point is the following system of complex (non-Hermitian) orthogonality relations

$$
\oint_{C} R_n(z) z^k \, dz = \oint_{C} (q_1 f_1 + \cdots + q_s f_s) z^k \, dz = 0, \quad k = 0, 1, \ldots, ns + s - 2, \quad (11)
$$

where $C$ is a Jordan contour separating $\bigcup e_j$ from infinity. Equations (11) are easily derived from the interpolation condition (2) using a standard procedure.

A general method for working with such relations is available in the case $s = 1$. In this case, the construction of Hermite–Padé approximations (1) reduces to diagonal Padé approximations. The polynomials $q_n = q_{n,1}$ are the denominators of diagonal Padé approximants to a (single) function element $f = f_1$ at infinity. The zero distribution for these polynomials was obtained by Stahl in his fundamental papers [59]–[63], where an original method of working with complex$^1$ orthogonality relations was created.

To this end we mention that there are a number of systems of so-called semiclassical Padé (and Hermite–Padé) polynomials which can be studied using generalizations of methods developed in the theory of classical orthogonal polynomials. In particular, differential equations were used in many instances (see [44] for further references). A general method based on the matrix Riemann–Hilbert method has been developed in the last two decades, which can be used to study the strong asymptotics of Padé polynomials of multivalued functions (see [9], [50], [53], [43]).

Stahl’s method was substantially generalized by Gonchar and this author in [27] in the case when $f(z)$ depends on $n$ in a certain way (the case of varying weight). Theorem 3 of [27], which we shall call the GRS-theorem, describes the zero distribution of complex orthogonal polynomials with varying weights. There are many applications of the theorem (see, for example, the reviews [3] and [56] and the recent papers [57], [17], [69]). In particular, after some modifications the GRS-theorem can be applied to study the Angelesco case of Hermite–Padé polynomials, and this is a main point of this paper. As an introduction to the complex case$^2$ we consider the Angelesco case for Markov-type functions in §2. We use this simpler situation

$^1$That is, non-Hermitian. — Russian translator’s note.

$^2$Determined in terms of non-Hermitian orthogonality relations. — Russian translator’s note.
to introduce some basic ideas of the method, in particular, a reduction of vector situations to weighted scalar ones.

1.3.1. Reduction of vector cases to scalar cases with external field. Markov-type functions. A reduction of asymptotics problems for Hermite–Padé polynomials to similar problems for scalar \((s = 1)\) orthogonal polynomials with varying weights was first applied in [24] to Markov-type functions (a matrix equilibrium problem was introduced in this connection). Such a reduction is based on a basic property of vector equilibrium distributions: here, each component is a scalar equilibrium measure in the external field generated by all the other components. In [24] this observation was used to study the zero distribution for the Angelesco Hermite–Padé polynomials associated with Markov-type functions. The systems of orthogonality relations for such polynomials are ordinary Hermitian orthogonality relations (see [71], [72], [66]), and this makes the Markov case essentially simpler than the case \(\vec{f} \in \mathcal{A}\). The associated equilibrium problems are also simpler in the Markov case. It is \textit{a priori} known that the measures characterizing the zero limit distribution are supported on the real line, and the max-min condition (8) holds automatically for the potential of a measure on the real line.

At the same time, formally speaking, the Markov case is not a particular case of the complex case and has its own independent interest. After the original paper [24] this case was studied in [28] in more general settings, including certain combinations of the Angelesco and Nikishin cases. In both papers, the second-kind polynomials were studied. We consider the Angelesco–Markov case in §2 for both first- and second-kind Hermite–Padé approximations. The results of our paper for first-kind approximations are probably new.

1.3.2. Reduction to the GRS-theorem in the complex case. In §3 we turn to the complex case which is significantly more complicated and the approach to the problem is at some points essentially different. We shall arrange for a reduction of the zero-distribution problem for Angelesco Hermite–Padé polynomials to a somewhat modified version of the GRS-theorem on the zero distribution of weighted (scalar) complex orthogonal polynomials. We revisit two basic lemmas in the proof of the GRS-theorem in [27]. Using these lemmas, we prove a lemma (Lemma 5 in §3 below) on the asymptotics of certain integrals of polynomials and this theorem is then used to reduce the vector case to a scalar case with external field. This part of the proof mainly uses techniques developed in [27].

The remaining part of the reduction process is related to equilibrium problems for vector potentials.

1.3.3. Equilibrium conditions. Vector \(S\)-equilibrium problems. For an arbitrary \(F \in \mathcal{F}\) the vector equilibrium measure \(\vec{\lambda} = \vec{\lambda}_F\) is uniquely defined by the following characteristic property

\[
W_k(z) = w_k = \min_{F_k} W_k \quad \text{for} \ z \in \text{supp}(\lambda_k),
\]

where the \(W_k\) are components of an associated vector potential, which can be represented in two equivalent ways as follows:
Zero distribution Hermite–Padé polynomials

\[ W_k(z) = \sum_{i=1}^{s} a_{i,k} U^{\lambda_i}(z) = U^{\lambda_k + \lambda}(z), \quad \text{where} \quad \lambda = \lambda_1 + \cdots + \lambda_s, \]

\[ W_k(z) = 2(U^{\lambda_k}(z) + \varphi_k(z)), \quad \text{where} \quad \varphi_k(z) = \frac{1}{2} \sum_{i \neq k}^{s} U^{\lambda_i}. \]  

(12)

Also, for an arbitrary \( \vec{F} \in \vec{F} \) the corresponding vector equilibrium measure \( \vec{\lambda} = \vec{\lambda}_{\vec{F}} \) is uniquely defined by the equilibrium conditions. We present these conditions in the case \( \vec{F} = \vec{\Gamma} \):

\[ W_k(z) = w_k, \quad z \in \Gamma^1_k = \text{supp}(\lambda_k); \quad W_k(z) \geq w_k, \quad z \in \Gamma_k. \]  

(13)

Thus, these equilibrium conditions are valid for any \( \vec{F} \in \vec{F} \) (with \( \vec{\lambda} = \vec{\lambda}_{\vec{F}} \)). The extremal vector compact set \( \vec{\Gamma} \) has in addition another important symmetry property.

The following vector \textit{S-property} is valid for the equilibrium potentials associated with the extremal compact set \( \vec{\Gamma} \) under the Angelesco condition. For any \( j = 1, \ldots, s \) there exists a finite set \( e_j \) such that \( \Gamma^\circ_j = \Gamma^1_j \setminus e_j \) is a disjoint finite union of open analytic arcs and

\[ \frac{\partial W_j}{\partial n_1}(\zeta) = \frac{\partial W_j}{\partial n_2}(\zeta), \quad \zeta \in \Gamma^\circ_j, \]  

(14)

where \( n_1 \) and \( n_2 \) are the opposite normals to \( \text{supp}(\lambda_j) \) at \( \zeta \). Actually, it is the \( S \)-property that we need in order to study the complex orthogonal polynomials. The max-min energy problem is a general way to prove the existence of such a compact set.

More precisely, solutions of asymptotics problems related to complex orthogonal polynomials are often reduced to an existence problem for a compact set with the \( S \)-property in a given class of compact sets (the \( S \)-equilibrium problem). In many cases such a problem can be solved by reduction to the problem of maximizing the equilibrium energy in a given class. This is exactly the method we use in our paper. As a remark we note that vector \( S \)-equilibrium problems in general settings are rather difficult to solve using this method. Under the Angelesco condition the reduction is simpler. We shall actually reduce the vector problem to a weighted scalar one and use methods developed in [55] to study weighted scalar problems (see also [42] and [20]).

To this end, we emphasize that an important difference between Markov (real) and general (complex) cases for Hermite–Padé polynomials lies precisely in the structure of the ‘electrostatics’ associated with the situation. For Markov-type functions the associated vector equilibrium problem does not include the second part which is related to the maximization of the equilibrium energy (8) in Lemma 2. Intervals of the real axis are precisely the ‘curves’ maximizing the equilibrium energy in classes of continua with fixed endpoints. Equivalently, the \( S \)-property (14) is valid for potentials of any measures on the real line, since the potentials of such measures are symmetric with respect to the real line. So, there is no existence problem for an \( S \)-compact set there.
In the complex case the $S$-property of the extremal compact set $\Gamma$ in (8) is a substitute for such a symmetry and is not available a priori. Finding compact sets with the $S$-property is then an important part of the problem. The $S$-property also plays an important role in the construction of a Riemann surface $\mathcal{R} = \mathcal{R}(\vec{f})$ which will be introduced in § 4 of this paper. In this section we first go into some technical detail regarding the properties of $\Gamma$. In particular, we prove the existence (Lemma 2) and some continuity properties of extremal compact sets. The methods used here were mostly developed in [55]. Then we turn to the Riemann surface, which is technically used to establish some properties of $\bar{\Gamma}$. At the same time it is useful in the discussion of the problem as a whole.

1.3.4. Riemann surface. There is an old observation going back to the 1970s and 1980s that for some classes of vector functions $\vec{f} = (f_1, \ldots, f_s) \in \mathcal{A}$ there is a Riemann surface which in a certain sense controls the asymptotics of the corresponding Hermite–Padé polynomials. The first instance of such a situation was reported in Kalyagin’s paper [32]. Riemann surfaces also play a central role in Nuttall’s important review [49] (see also [48]). Independently, for the Angelesco–Markov case the construction of such a surface from the solution of the vector equilibrium problem was presented by Aptekarev and Kalyagin in [4]. They also noticed that the connection goes both ways, and a vector equilibrium problem could be recovered from a known Riemann surface. Therefore, the Riemann surface can play a role similar to that played by the vector equilibrium measure $\vec{\lambda}(\vec{f})$; see also [29] and [30].

With time it became a commonly accepted conjecture that for arbitrary $\vec{f} = (f_1, \ldots, f_s) \in \mathcal{A}$ there exists a Riemann surface $\mathcal{R} = \mathcal{R}(\vec{f})$ with $s + 1$ sheets such that the asymptotics of the Hermite–Padé polynomials associated with $\vec{f}$ can be described in terms of special functions on this surface. This Riemann surface may present a general approach to the asymptotics of the Hermite–Padé polynomials and it is an alternative to the vector equilibrium problem. It is believed that the two ways are closely related and may be formally equivalent.

One of the main problems related to this conjecture is that in general $\mathcal{R}$ is not known. Another problem, not yet solved in general, is that of an exact way to go from the Riemann surface to the asymptotics. Note that the structure of the vector equilibrium problem is in general not known either. But the Riemann surface still has an advantage; it contains only a finite number of unknown parameters, while the vector equilibrium problem can have an unknown geometric structure.

To this end we present a conjecture (due to Martínez-Finkelshtein, Suetin, and the author) which has been formulated on a basis of partial results in [44] on the zero distribution of first-kind Hermite–Padé polynomials for functions $\vec{f} = (f_1, \ldots, f_s) \in \mathcal{L}$. Such polynomials satisfy a linear differential equation with polynomial coefficients, which gives a powerful approach to the asymptotics. We formulate a more constructive form of the conjecture on the existence of a Riemann surface $\mathcal{R} = \mathcal{R}(\vec{f})$, including a method which can help to answer the key question: how to determine its unknown parameters.

2. Markov-type functions

In this section we revisit the case of Markov-type functions which is a model situation where the Angelesco condition is most transparent.
The first result on the asymptotics of Hermite–Padé polynomials was obtained by Kalyagin [32] for the case of two Jacobi-type functions whose weights had non-overlapping but adjoint supports. He found a generating function for the second-kind Hermite–Padé polynomials and then used a generalization of the classical Darboux method to study the strong asymptotics of the polynomials; in this connection see also [71], [46], [8] and references therein. Kalyagin [32] was also the first to use an algebraic Riemann surface with three sheets to deduce an asymptotic formula for Hermite–Padé polynomials.

A potential-theoretic approach to the problem was developed in [24] by Gonchar and this author for a vector \( \mathbf{f} = (f_1, \ldots, f_s) \) of Markov-type functions. In the Angelesco situation a theorem on the zero distribution of the Hermite–Padé polynomials was proved in [24] and a vector equilibrium problem was used to characterize the limit zero distribution. The methods in the current paper originated in part from [24]. We shall recall some details related to this case.

One Markov-type function is the Cauchy transform of a positive measure on \( \mathbb{R} \) with compact support:

\[
    f(z) = \int \frac{d\sigma(t)}{z - t}, \quad z \in \mathbb{C} \setminus \text{supp}(\sigma).
\]

(15)

The Angelesco case for Markov-type functions is defined by the condition that supports of the associated measures are disjoint or, at least, non-overlapping. We do not pursue the maximal generality in this discussion, so we shall assume that the supports of the measures are disjoint intervals, the measures are absolutely continuous, and their densities are positive almost everywhere (a.e.) on corresponding intervals. Thus, we consider the vector function \( \mathbf{f} = (f_1, \ldots, f_s) \), where

\[
    f_k(z) = \int_{F_k} \frac{w_k(t)}{z - t} dt, \quad z \in \mathbb{C} \setminus F_k,
\]

(16)

and

\[
    d\sigma_k(t) = w_k(t) dt, \quad F_k = \text{supp}(\sigma_k), \quad k = 1, \ldots, s,
\]

and

\[
    w_k(x) > 0 \quad \text{a.e. on the interval } F_k, \quad k = 1, \ldots, s, \quad F_i \cap F_j = \emptyset, \quad i \neq j.
\]

(17)

We note that there is in a certain sense an opposite case when two Markov functions are generated by measures on the same interval \( F = \text{supp}(\sigma_1) = \text{supp}(\sigma_2) \). This case was introduced by Nikishin [45] (see also [46]) and was named after him.

2.1. Second-kind Hermite–Padé polynomials. The two types of Hermite–Padé approximation and polynomials are said to be of the first (or ‘Latin’) and second (or ‘German’) kind. Both kinds are associated with a vector \( \mathbf{f} = (f_1, \ldots, f_s) \) of \( s \in \mathbb{N} \) analytic functions defined by their Laurent expansion at infinity

\[
    f_k(z) = \sum_{m=0}^{\infty} \frac{f_{m,k}}{z^m}, \quad k = 1, 2, \ldots.
\]

(18)

First-kind polynomials \( Q_{n,k} \) were introduced in (2) above (in the case where the polynomials have equal degrees). We present a definition of the second-kind polynomials and approximations in the similar case when the order of approximation
is the same for each function. For a natural \( n \in \mathbb{N} \) the \( n \)th vector of second-kind \textit{Hermite-Padé approximations}

\[
\pi_{n,k}(z) = \frac{\hat{P}_{n,k}}{P_n}
\]

with a common denominator \( P_n(z) \in \mathbb{P}_{ns} \) is defined by the following relations

\[
P_n(z)f_k(z) - \hat{P}_{n,k} = O\left(\frac{1}{z^{n+1}}\right), \quad k = 1, \ldots, s.
\]  

Clearly, the degrees of all the numerator polynomials \( \hat{P}_{n,k} \) are at most \( ns \).

\text{Hermite–Padé polynomials of both kinds are equivalently defined by certain systems of orthogonality relations. In the Angelesco case for Markov-type functions the orthogonality involved is the usual (traditional) Hermitian orthogonality with positive weights, which is generalized in two distinct ways from the scalar case.}

The orthogonality conditions (11) for the remainder associated with the first-kind approximations in the Markov case (16), (17) take the form

\[
\int_C R_n(z) z^j \, dz = \int_C (q_{n,1} f_1 + \cdots + q_{n,s} f_s)(z) z^j \, dz = 2\pi i \left( \int_{F_1} q_{n,1}(x) w_1(x) x^j \, dx + \cdots + \int_{F_s} q_{n,s}(x) w_s(x) x^j \, dx \right) = 0
\]

for \( j = 0, 1, \ldots, ns + s - 2 \). This can be written equivalently as

\[
\int_{\mathbb{R}} r_n(x) g(x) \, dx = 0
\]

for any polynomial \( g \in \mathbb{P}_{ns+s-2} \), where the function \( r_n(x) = (q_{n,1} w_1 + \cdots + q_{n,s} w_s)(x) \) is defined on the whole of the real axis if we assume that \( w_k(x) = 0 \) in the complement of \( F_k \). It follows, in particular, that \( r_n \) has at least \( ns + s - 1 \) sign changes on \( \mathbb{R} \). For \( x \in F_k \) the function \( r_n(x) = q_{n,k} w_k \) changes sign at most \( n \) times. It follows that \( q_{n,k} \) has exactly \( n \) simple zeros in \( F_k \). In addition, \( r_n \) changes sign every time when we pass from \( F_k \) to \( F_{k+1} \). The orthogonality conditions for the second-kind polynomials with a common denominator \( P_n \in \mathbb{P}_{ns} \) in (19) are

\[
\int_{F_k} P_n w_k x^j \, dx = 0, \quad j = 0, 1, \ldots, n-1, \quad k = 1, \ldots, s.
\]  

In particular, it follows from here that the polynomial \( P_n \) in (19) has exactly \( n \) simple zeros on each interval \( F_k \) and therefore admits a factorization \( P_n = \prod_{k=1}^s P_{n,k} \), where all the polynomials \( P_{n,k} \) have degree \( n \) and the zeros of \( P_{n,k} \) belong to \( F_k \).

\textbf{2.2. Theorem on zero distribution.}

\textbf{Theorem 5.} As \( n \to \infty \),

\[
\begin{align*}
(A) \quad \nu_n := \frac{1}{n} \sum_{P_n(x) = 0 \atop P_n(x) = 0} \delta(x) & \xrightarrow{\ast} \lambda_1 + \lambda_2 + \cdots + \lambda_s, \\
(B) \quad \mu_{n,k} := \frac{1}{n} \sum_{q_{n,k}(x) = 0} \delta(x) & \xrightarrow{\ast} \lambda_k, \quad k = 1, \ldots, s.
\end{align*}
\]  

If $R_n$ is normalized by the conditions $c_{n,1} = 1$, then the other leading coefficients $c_{n,k}$ of the polynomials $q_{n,k}$ are positive and

$$\lim_{n \to \infty} \frac{1}{n} \log c_{n,k} = w_k - w_1,$$

(23)

where $\vec{\lambda} = \vec{\lambda}_\vec{F} \in \vec{\mathcal{M}}(\vec{F})$ is the vector equilibrium measure from Lemma 1 and the $w_k$ are the equilibrium constants in (12) and (13).

Thus, the zero distribution of Hermite–Padé polynomials of both the first and the second kind are represented by the same vector equilibrium measure $\vec{\lambda}$ (see Lemma 1). In other words, it follows from Theorem 5 that the modulus of the second-kind polynomial $P_n \in \mathbb{P}_{ns}$ has the same $n$th root asymptotics (up to a normalizing constant) as the modulus of the product $q_{n,1} \cdots q_{n,s}$ of first-kind polynomials.

We also note that in the case of Markov-type functions Lemma 2 is not involved in the definition of the limit measures $\lambda_k$. The reason is the symmetry with respect to $\mathbb{R}$ of the potentials of measures $\mu$ supported on $\mathbb{R}$: we have $U^\mu(\overline{z}) = U^\mu(\bar{z})$ for any such measure and for any $z$ not in the support of $\mu$. It follows that the $S$-property (14) is satisfied for $U^\mu$ at points where normal derivatives exist. Hence in the Markov case $\vec{\Gamma} = \vec{F}$.

An important difference between the first- and second-kind polynomials is that the leading coefficients of the first-kind polynomials are defined by the orthogonality conditions (20) like all the other coefficients. More precisely, the function $R_n$ is defined by (20) up to a multiplicative constant, so the leading coefficient $c_{n,k}$ of a polynomial $q_{n,k}(x) = c_{n,k}Q_{n,k}(x)$ can be selected arbitrarily as a normalization condition. Then the other coefficients are uniquely determined by the orthogonality conditions (20). The asymptotics of the leading coefficients presented in (23) is an important part of the solution of the problem. In particular, we need this asymptotics to obtain the asymptotics of $|q_{n,k}(z)|^{1/n}$ in $\mathbb{C} \setminus F$ as $n \to \infty$.

As we mentioned above, part (A) of Theorem 5 is well known (see [24] and [28]). Part (B) seems to be new. To this end we note that most papers in the literature on the asymptotics of Hermite–Padé polynomials have so far been devoted to second-kind approximations and polynomials.

In connection with the Markov case, see also some more recent results in [7] and [54], where a mixed case was studied for two Markov functions in the situation where one support is a subinterval of another. The Nikishin case was considered in [38] and [6] (see also [57], [68], [70]).

We shall present a proof Theorem 5 based on a standard reduction of a vector orthogonality to a scalar weighted orthogonality. In terms of the associated equilibrium problems this is equivalent to a possibility of defining a vector equilibrium in terms of a scalar weighted equilibrium. All these ideas are essentially well known, but we shall give a brief but connected presentation anyway. First, some of the results are new. Second, we use the opportunity to introduce in a comparatively simple situation certain arguments which will be generalized and used in a less trivial setting.

2.3. Equilibrium measure in an external field. We begin with necessary definitions.
Let $F \in \mathcal{F}$; recall that such compact sets are regular with respect to the Dirichlet problem. By an external field on $F$ we will understand in general a continuous real-valued function $\varphi(x)$ on $F$ unless explicitly stated otherwise. The class of external fields can be significantly generalized (see [58], for example) but we want to keep this exposition as short as possible.

As above, we use the notation $\mathcal{M}$ for all positive Borel measures in the plane. Let $t > 0$ and let $\mathcal{M}^t(F)$ be the set of $\mu \in \mathcal{M}$ on $F$ with total mass $\mu(F) = t$. By $\mathcal{E}_\varphi(\mu)$ we denote the (total) energy of a measure $\mu \in \mathcal{M}$ in the external field $\varphi$:

$$E_\varphi(\mu) = \iint \log \frac{1}{|x - y|} \, d\mu(x) \, d\mu(y) + 2 \int \varphi(x) \, d\mu(x).$$

(24)

The equilibrium measure $\lambda = \lambda_{\varphi,F}^t$ for a compact set $F$ in the external field $\varphi$ is defined by the minimization properties

$$\lambda \in \mathcal{M}^t(F), \quad \mathcal{E}_\varphi(\lambda) = \min_{\mu \in \mathcal{M}^t(F)} \mathcal{E}_\varphi(\mu).$$

(25)

Equivalently, the equilibrium measure $\lambda^t = \lambda_{\varphi,F}^t$ is defined by the following equilibrium conditions for the total potential:

$$(U^\lambda + \varphi)(x) \begin{cases} = w, & x \in \text{supp}(\lambda), \\ \geq w, & x \in F. \end{cases}$$

(26)

Equation (26) uniquely determines the pair consisting of the measure $\lambda \in \mathcal{M}(F)$ and the constant $w = w_{\varphi,F}$, the equilibrium constant. In the case $t = 1$ (the main case in this paper) we drop the index $t$ from our notation.

We note that for more general classes of external fields $\varphi$ and/or more general classes of compact sets $F$ in the plane the equality in the first line in (26) holds away from a set of capacity zero (the inequality $\leq$ holds at any point). In connection with these definitions see the original papers [25], [27] and the book [58].

2.4. Orthogonal polynomials with varying weight on $\mathbb{R}$. Let $F$ be a finite union of intervals, let $\Phi_n(x)$ be a sequence of positive continuous real-valued functions on a set $F$ such that

$$\varphi_n(x) := \frac{1}{2n} \log \frac{1}{\Phi_n(x)} \to \varphi(x)$$

(27)

uniformly on $F$ (so that $\varphi$ is a continuous function on $F$), and let $f(x) > 0$ a.e. on $\Gamma$. Let the polynomials $Q_n(x) = x^n + \cdots$ be defined by the orthogonality relations

$$\int_F Q_n(x)x^k \Phi_n(x)f(x) \, dx = 0, \quad k = 0, 1, \ldots, n - 1.$$  

(28)

The sequence $Q_n$ presents a typical example of so-called orthogonal polynomials with varying weight; this usually means that the weight functions depend on the degree $n$ of the polynomial in such a way that their $n$th root asymptotics exist (it is clear that (27) remains essentially valid if we replace $\Phi_n(x)$ there with the total weight $\Phi_n(x)f(x)$). The following theorem by Gonchar and this author [25] was the first general result on the zero distribution of such orthogonal polynomials. We present a simplified version needed for our purposes.
Theorem 6. Under the assumptions on $F$, $\Phi_n$, and $f$ stated above,

$$\frac{1}{n} \mathcal{X}(Q_n) := \frac{1}{n} \sum_{Q_n(\zeta)=0} \delta(\zeta) \xrightarrow{*} \lambda,$$

where $\lambda = \lambda_\varphi$ is the equilibrium measure on $F$ in the external field $\varphi$.

This theorem has many applications; here we use it to reduce a vector zero distribution problem to a scalar one.

The original proof in [25] was based on the $L^2$ extremal property of the polynomials $Q_n$. We present a different proof based directly on the orthogonality conditions. This proof exhibits in a simple situation one of the basic elements of the method which we shall also use in the case of complex-valued weights.

Proof of Theorem 6. By contradiction, assume that the assertion of Theorem 6 is not true. Then it is possible, using the weak-star compactness of the space of unit measures (on the sphere), to select a weakly convergent subsequence $(1/n) \mathcal{X}(Q_n) \to \mu$ such that $\mu \neq \lambda$, where $n \in \Lambda = \{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$. Then a contradiction to the orthogonality relations (28) will be obtained by constructing a sequence of polynomials $P_n$, $n \in \Lambda$, of degree $< n$ for which the $n$th root asymptotics of the modulus of the expression

$$I_n(F) = \int_F Q_n(x)P_n(x)\Phi_n(x)f(x) \, dx$$

can be determined. This asymptotics will imply that $I_n(f) \neq 0$ for large enough $n$, which presents the desired contradiction.

We construct the desired polynomial $P_n$ by duplicating the polynomial $Q_n$, with a small modification. More precisely, we need that $\deg P_n < n$, so we have to drop at least one zero of $Q_n$. We shall drop two zeros selected as follows. Since $\mu \neq \lambda$ the equilibrium conditions (26) are not satisfied. It follows that there is a point $x_0 \in \text{supp} \mu$ such that $(U^\mu + \varphi)(x_0) > m = \min_{x \in F} (U^\mu + \varphi)(x)$. Taking into account the lower semicontinuity of $U^\mu$, we infer that there is an $r$-neighbourhood $\Delta = [x_0 - r, x_0 + r]$, $r > 0$, such that $(U^\mu + \varphi)(x) \geq m_1 > m$ for $x \in \Delta \cap F$. Since $x_0 \in \text{supp} \mu$, for large enough $n$ we can find two (not necessarily distinct) zeros $a_n$ and $b_n$ of $Q_n$ in the $r$-neighbourhood of $x_0$, and we define

$$P_n(x) = \frac{Q_n(x)}{(x-a_n)(x-b_n)}.
\tag{29}$$

With this $P_n$ we shall determine the asymptotics of $I_n(F)$ above. We have

$$\mathcal{X}_n = \frac{1}{n} \mathcal{X}(Q_nP_n) \xrightarrow{*} 2\mu.$$

Now we use well-known convergence properties of the potentials of an arbitrary weakly convergent sequence of measures, say, $\nu_n \to \nu$. This implies that $U^{\nu_n} \to U^\nu$ in linear Lebesgue measure on any rectifiable curve in the plane. The convergence of potentials is also semi-uniform from below. In particular, we have the convergence of minima $\min_F U^{\nu_n} \to \min_F U^\nu$ over any regular (for the Dirichlet problem)
compact set $F$ in the plane (see [35]). It follows that we have the convergence of (non-negative) functions

$$E_n(x) := \exp\{-U^{X_n}(x) + \varphi_n(x)\} \rightarrow E(x) := \exp\{-2(U^\mu(x) + \varphi(x))\}$$

on $F$ in measure and ‘semi-uniformly’ from above. Since $f > 0$ a.e. on $F$, it follows that

$$\lim_{n \to \infty} \left| I_n(F \setminus \Delta) \right|^{1/n} = \lim_{n \to \infty} \left( \int_{F \setminus \Delta} E_n^m(x) f(x) \, dx \right)^{1/n} = e^{-2m} = \max_{F \setminus \Delta} E(x) > 0$$

(note that $Q_n P_n \Phi_n f > 0$ a.e. on $F \setminus \Delta$). On the other hand, we have

$$\lim_{n \to \infty} \left| I_n(F \cap \Delta) \right|^{1/n} \leq \lim_{n \to \infty} \left( \int_{F \cap \Delta} E_n^m(x) f(x) \, dx \right)^{1/n} = e^{-2m_1} = \max_{F \cap \Delta} E(x) < e^{-2m}.$$

It follows that $\int_{F \setminus \Delta} Q_n P_n \Phi_n f \, dx \neq 0$ for large enough $n$, in contradiction to the orthogonality conditions for $Q_n$. □

2.5. Proof of Theorem 5. Part (A) of the theorem is well known, however, we present a proof. It is short and illustrates an important detail in the procedure of reduction of vector cases to weighted scalar ones.

**Proof of part (A).** We use the orthogonality relations (21). As we mention below, it follows from (21) that $P_n = \prod_{k=1}^s P_{n,k}$, where all the polynomials $P_{n,k}$ are monic of degree $n$, and the zeros of $P_{n,k}$ belong to $F_k$. We note that here we have important *a priori* information about the location of the zeros of the polynomials. This is crucial for the proof below (see the remarks in §2.6 on the case of complex-valued weights).

Select a subsequence $\Lambda = \{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that

$$\mu_{n,k} = \frac{1}{n} \chi(P_{n,k}) \rightarrow \mu_k \quad \text{as } n \to \infty, \ n \in \Lambda; \ k = 1, \ldots, s.$$ 

Now it is enough to prove that $\mu_k = \lambda_k$.

We shall show that for any $k$ the measure $\mu_k$ is the unit equilibrium measure in the external field $\varphi_k(x) = \frac{1}{2} \sum_{j \neq k} U^{\mu_j}(x)$. Indeed, the group of orthogonality conditions in (21) associated with the index $k$ defines $P_{n,k}$ as an orthogonal polynomial on $F_k$ with varying weight $\Phi_n(x) w_k(x)$, where $\Phi_n(x) = \prod_{j \neq k} P_{n,j}(x)$. For a fixed $k$ this function satisfies (28) with $\varphi(x) = \varphi_k(x)$, and the desired statement follows from Theorem 6. In terms of the energy functional $E(\vec{\mu})$ in Lemma 1 this statement means that the vector measure $\vec{\mu} = (\mu_1, \ldots, \mu_s)$ provides a componentwise minimum of this functional, that is, a minimum of $\mathcal{E}(\mu_1, \ldots, \mu, \ldots, \mu_s)$ over $\mu \in M(F_k)$ (the variable measure $\mu$ takes the place of $\mu_k$) is achieved when $\mu = \mu_k$.

The energy functional $\mathcal{E}(\vec{\mu})$ is convex (see [46], Chap. 5, Lemma 4.3) and therefore has no componentwise minima distinct from the global one. □
The main difference between the proof of part (B) of the theorem and the proof of part (A) above is that now the asymptotics (23) of the leading coefficients $c_{n,k}$ of the polynomials $q_{n,k}(x) = c_{n,k}x^n + \cdots$ are found simultaneously with the zero distribution.

Proof of part (B). We select a subsequence $\Lambda = \{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that

$$\mu_{n,k} = \frac{1}{n}X(q_{n,k}) \to \mu_k \quad \text{as} \quad n \to \infty, \quad n \in \Lambda; \quad k = 1, \ldots, s,$$

and also

$$|c_{n,k}|^{1/n} \to e^{-u_k} \quad \text{as} \quad n \to \infty, \quad n \in \Lambda; \quad k = 1, \ldots, s,$$

where $u_k \in [-\infty, +\infty]$. We have to take into account the possibility that some of the numbers $u_k$ may not be finite. In this connection it is convenient to normalize the function $r_n(x)$ so that $\max_k u_k = 0$ and therefore $u_k \geq 0$ for all $k$. Then all the numbers $e^{-u_k}$ are finite. The orthogonality conditions (20) are equivalent to the assertion that for any polynomial $P_n \in \mathbb{P}_{ns+s-2}$ we have

$$\sum_{j=1}^{s} \int_{F_j} q_{n,j}(x)P_n(x)w_k(x) \, dx = \int_{F} r_n(x)P_n(x) \, dx = 0,$$

where $r_n(x) = q_{n,k}(x)w_k(x)$ on $F_k$ and $r_n(x) = 0$ for $x \notin F = \bigcup F_k$.

Now we shall construct a polynomial $P_n \in \mathbb{P}_{ns+s-2}$ for which the above equation fails if the assertion of part (B) of the theorem is violated. A preliminary definition of $P_n$ contains a few parameters which will be determined later. We select an integer $k$ with $1 \leq k \leq s$ (let us call this index $k$ exceptional). Then we select any pair of zeros $a_n$ and $b_n$ of the polynomial $q_{n,k}$ and define

$$P_n(x) = \frac{h_n(x)}{(x-a_n)(x-b_n)} \prod c_{n,k}^{-1} q_{n,k}(x), \quad \text{(30)}$$

where $h_n(x)$ is a monic polynomial of degree $s - 1$ which has one simple zero on each gap between the intervals $F_k$. Thus, $P_n(x) \in \mathbb{P}_{ns+s-3}$ is a sequence of monic polynomials with limit zero distribution independent of the choices of the parameters $a_n$, $b_n$, and $h_n$. As $n \to \infty, \, n \in \Lambda$, we have

$$\frac{1}{n}X(P_n) \to \mu = \sum_{k=1}^{s} \mu_k,$$

and, therefore,

$$\frac{1}{n}X(q_{n,j}P_n) \to \mu_j + \mu, \quad j = 1, \ldots, s.$$

Next, we define

$$I_{n,j} = \int_{F_j} r_n(x)P_n(x) \, dx = \int_{F_j} q_{n,j}(x)w_j(x)P_n(x) \, dx \quad \text{for} \quad j = 1, \ldots, s.$$

It follows from the construction of the polynomials $P_n$ that the function $r_n(x)P_n(x)$ is non-negative on $F \setminus [a_n,b_n]$, and therefore, $r_n(x)P_n(x) = (q_{n,j}w_jP_n)(x) \geq 0$
on $F_j$ for any non-exceptional $j$; hence $I_{n,j} > 0$ for $j \neq k$. Finally, using the same convergence properties of potentials as in the proof of part A, we conclude that

$$I_{n,j}^{1/n} \to e^{-m_j} \quad \text{for } j \neq k \quad \text{and} \quad \lim I_{n,k}^{1/n} \leq e^{-m_k}$$

as $n \to \infty$, $n \in \Lambda$, where

$$m_j := u_j + \min_{x \in F_j} U^{\mu_j + \mu}(x), \quad j = 1, \ldots, s. \quad (32)$$

Now we prove that all the $m_j$, $j = 1, \ldots, s$, are equal. If it were not so, then the largest of them would be strictly greater than the smallest one, that is, there would exist a $k$ such that $m_k > m = \max_j m_j$. For this exceptional index $k$ we take $a_n$ and $b_n$ to be any two zeros of $P_n$ in $\mathcal{F}_k$ and choose the other parameters of $P_n$ arbitrarily. In terms of the constants $I_{n,j}$ the orthogonality of $r_n$ to this polynomial is written as

$$\sum_{j=1}^s I_{n,j} = 0, \quad \text{so that} \quad |I_{n,k}| = I_n := \sum_{j \neq k} I_{n,j}. \quad (31)$$

This would imply that $m_k = m_n$ in contradiction to the inequality $m_k > m$.

It remains to prove that $\bar{\mu} = \bar{\lambda}$. We again follow an approach used in the proof of part (A), that is, we observe that it is enough to prove that each component $\mu_k$ of $\bar{\mu}$ is the equilibrium measure in the field $\varphi_k(x) = \frac{1}{2} U^{\nu_k}(x)$, where $\nu_k = \mu - \mu_k = \sum_{j \neq k} \mu_j$. However, here we cannot directly invoke Theorem 6 as we did when proving part (A); instead we shall apply the method we used in the proof of that theorem.

Assume the contrary; one of the components, say $\mu_k$, is not an equilibrium measure in the field $\varphi_k(x)$. Without loss of generality we can assume that $k = 1$. Thus, the equilibrium condition is violated for $\mu_1$, which means that we can find a point $x_0 \in \text{supp } \mu_1$ such that

$$(U^{\mu_1} + \varphi_1)(x_0) > m_1 = \min_{x \in F} (U^{\mu_1} + \varphi_1)(x).$$

Then we can also find a neighbourhood $\Delta = [x_0 - r, x_0 + r]$, $r > 0$, such that $(U^{\mu_1} + \varphi_1)(x) \geq m_0 > m_1$ for $x \in \Delta \cap F$. It follows from $x_0 \in \text{supp } \mu_1$ that for $n$ large enough there exist two zeros $a_n$ and $b_n$ of $q_{n,1}$ in $\Delta$. Using these parameters, we define a polynomial $P_n(x)$ by (30). The completion of the proof is similar to the technique we used to prove the equality of the constants $m_j$. A small modification must be made: in place of the exceptional set $F_k$ (now $k = 1$) we single out a part of it, namely, the interval $\Delta_n = [a_n, b_n] \subset \Delta \subset F_1$. We have $r_n(x)P_n(x) \geq 0$ on $F \setminus \Delta_n$, and the orthogonality conditions (20) imply that

$$\int_{F \setminus \Delta_n} r_n P_n \, dx \leq \int_{F \setminus \Delta_n} r_n P_n \, dx = - \int_{\Delta_n} r_n P_n \, dx = \int_{\Delta_n} r_n P_n \, dx \leq \int |r_n P_n| \, dx.$$ 

Now we can raise both sides to the power $1/n$ and take the limit as $n \to \infty$, $n \in \Lambda$. As $m_k > m$, the limit on the left-hand side will be strictly larger than that on the right-hand side in contradiction to the inequality above. The end of the proof is identical to that of Theorem 6. $\square$
2.6. Remarks on a generalization for complex weights. In the next section, §3, we shall study the zero distributions of Hermite–Padé polynomials for functions \( \vec{f} \in A \). In equivalent terms this means that we shall be dealing with systems of complex orthogonality relations in place of Hermitian ones. This will require essential modifications in the method. At the same time, the method as a whole and certain technical elements will be similar to the approach we used above in the Markov case. To illustrate similarities and differences between the two cases in more specific terms we make a few remarks relating to the case of Markov-type functions with complex-valued weights. Such a situation can be considered as transitional between the two cases indicated above.

Let \( \vec{f} = (f_1, \ldots, f_s) \) be a vector function whose components are defined to be Markov-type functions

\[
f_k(z) = \int \frac{w_k(t)}{z - t} \, dt, \quad z \in \mathbb{C} \setminus F_k, \tag{33}
\]

\[
d\sigma_k(t) = w_k(t) \, dt, \quad F_k = \text{supp}(\sigma_k), \quad k = 1, \ldots, s,
\]

with complex-valued weights \( w_k(x) \) on disjoint real sets \( F_k \), each of which is a finite union of intervals, satisfying the following conditions:

\[
|w_k(x)| > 0 \quad \text{a.e. on the interval } F_k, \quad k = 1, \ldots, s; \quad F_i \cap F_j = \emptyset, \quad i \neq j, \tag{34}
\]

and there is a closed set \( e_k \subset F_k \) such that

\[
\arg w_k(x) \in C(F_k \setminus e_k), \quad \text{cap}(e_k) = 0, \quad k = 1, \ldots, s. \tag{35}
\]

It turns out that the generalizations of the two parts of Theorem 5 to this setting differ in status. More precisely, for Hermite–Padé polynomials of the first kind we have the following.

**Theorem 7.** Under the assumptions (34) and (35) on the functions \( w_k \) in (33) the assertions (22), (B), and (23) of Theorem 5 remain valid for the first-kind Hermite–Padé polynomials associated with \( \vec{f} \).

Theorem 7 is not a direct corollary of Theorem 1 but can be proved following the proof of Theorem 1 (see §3 below) step by step and even with significant simplifications. The situation for the second-kind Hermite–Padé polynomials is different; in this case we can only state a conjecture.

**Conjecture.** Under the assumptions (34) and (35) on the functions \( w_k \) in (33) the assertions (22) and (A) of Theorem 5 remain valid for the second-kind Hermite–Padé polynomials associated with \( \vec{f} \).

This conjecture looks very natural and it is supported by all known results. But at the moment it is not clear how it can be proved in its full generality. Neither can we prove in full generality the theorem similar to Theorem 1 for second-kind Hermite–Padé polynomials. A version of such a theorem can be proved but more restrictive assumptions on the separation of singularities of component functions will be needed. To take one step towards an explanation of the situation, we consider a plan of the proof.
Suppose we want to prove Theorem 7 (and/or the Conjecture) using the same method of reducing the vector case to a weighted scalar case. Then we have to begin with a complex version of Theorem 6. Such a theorem is known.

**Theorem 8.** Under all the assumptions on $F$, $\Phi_n$, and $Q_n$ in Theorem 6 (see (27) and (28)) and conditions similar to (34) and (35) on $f$

$$\frac{1}{n} \mathcal{X}(Q_n) = \frac{1}{n} \sum_{Q_n(\zeta) = 0} \delta(\zeta) \asto \lambda,$$

where $\lambda = \lambda_\varphi$ is the equilibrium measure on $F$ in the external field $\varphi$.

Again, Theorem 8 is not a direct corollary (nor a particular case) of the more general Theorem 9 which we discuss in §3, but it can be proved following the same path. We outline briefly some elements of this path which would, in particular, form an introduction to the settings and proof of Theorem 9.

The proof begins exactly like that for Theorem 6: we select a weakly convergent subsequence $(1/n)X(Q_n) \asto \mu$, where $n \in \Lambda = \{n_k\}_{k=1}^\infty \subset \mathbb{N}$. Now it is enough to show that $\mu = \lambda$.

Assuming the contrary ($\mu \neq \lambda$), we shall construct a sequence of polynomials $P_n$ which leads to a contradiction to the orthogonality conditions (28). In the proof of Theorem 6 we defined the required polynomials $P_n$ by a simple explicit formula (29). Then we determined the asymptotics of the integral involving these polynomials

$$I_n = \int_F Q_n(x)P_n(x)\Phi_n(x)f(x) \, dx,$$

and derived from this asymptotic behaviour that $I_n \neq 0$ for large enough $n$, in contradiction to (28).

The difference from the real case comes at this point. Assuming that $\mu \neq \lambda$, we have to construct polynomials $P_n$ in such a way that a lower bound for $|I_n|$ can be obtained. For complex weights $f$ it will not be possible to use the polynomials $P_n$ in (29) for this purpose (such a lower bound will not be valid for them). The method for constructing $P_n$ is essentially modified for complex $f$. Loosely speaking, we shall construct $P_n$ in such a way that the product $|Q_n(x)P_n(x)\Phi_n(x)|$ has a single sharp local maximum at a point $x_0$ where $f(x_0) > 0$ and $\arg f$ is continuous. This idea (it goes back to Stahl) is the central point of the method. A formal implementation of this plan is rather cumbersome; we postpone the discussion of the details until the next section.

It is an important detail in the hypothesis of Theorem 7 that the assumption (27) on the variable part $\Phi_n$ of the orthogonality weight remains the same as in Theorem 6; we assume that $\Phi_n(x) > 0$ (while the assumptions on $f$ are more general). The assertion of the theorem will not be valid for complex-valued $\Phi_n(x)$, and this would lead to a non-symmetric $\varphi$ in (27); the union of intervals $F \subset \mathbb{R}$ will not have the $S$-property in such an external field.

This also explains why the above conjecture on the asymptotics of second-kind Hermite–Padé polynomials is difficult to prove using the methods which were effective in the proof of Theorem 6. In the process of reducing vector orthogonal polynomials to weighted scalar ones we introduce the weights $\Phi_{n,k}(x) = \prod_{j \neq k} P_{n,j}(x)$. 
For complex-valued weights we lose any \textit{a priori} information on the location of the zeros of the polynomials $P_n = \prod_{j=1}^{s} P_{n,j}(x)$ and their possible limit distributions. Therefore, we cannot assert that $F \subset \mathbb{R}$ has the $S$-property in the field $\varphi$ defined in (27).

The first-kind Hermite–Padé polynomials turn to be technically simpler because of the specific structure of their orthogonality conditions (20). The asymptotics of the sum of the integrals is reduced to the asymptotics of the first of them. For each $k$ the integral $\int_{F_k}$ does not involve the other polynomials $q_{n,j}$ with $j \neq k$, in contrast to how they are involved in the case of second-kind polynomials. This would allow us in the case of complex weights to apply the same arguments we used in the real case.

In the case $\vec{f} \in A$ the situation is similar and we shall be able to prove a theorem on the zero distribution in the general Angelesco case for first-kind polynomials.

3. Proof of Theorems 1 and 2

Now we turn to the problem of the limit zero distribution for complex (non-Hermitian) orthogonal polynomials with analytic weights in the complex plane. One of the most general results in this direction is Theorem 3 in [27] (the so-called GRS-theorem). In this section we present a somewhat modified version of that theorem, which is then used to study Hermite–Padé polynomials. In the next subsection we discuss the original version of the theorem.

3.1. The GRS-theorem. The theorem in the original form characterizes the limit zero distribution as $n \to \infty$ of the sequence of polynomials $Q_n(z) \in \mathbb{P}_n$ defined by the following complex orthogonality relations

$$\int_{\Gamma} Q_n(z) z^k \Phi_n(z) f(z) \, dz = 0, \quad k = 0, 1, \ldots, n - 1,$$

with a varying (depending on $n$) weight function $w_n(z) = \Phi_n(z) f(z)$, where the integration in $\int_{\Gamma}$ is over a cycle enclosing the compact set $\Gamma$. The formal definition of $\int_{\Gamma}$ will be presented after we introduce conditions on $\Gamma$ and $f$ below.

The hypotheses of the GRS-theorem include several conditions on the compact set $\Gamma$, a sequence of analytic functions $\Phi_n(z)$ on $\Gamma$, and an analytic function $f$ in an open set surrounding $\Gamma$. These conditions remain unchanged in the new version of the theorem.

\textbf{Condition 1.} The first condition is the convergence of the (analytic) functions $\Phi_n(z)$ in the following sense. Let $\Phi_n \in H(\mathcal{O})$, where $\mathcal{O}$ is an open subset of the disk $\{z : |z| < 1/2\}$ and assume that for the sequence of holomorphic functions $\Phi_n(z)$ in $\mathcal{O}$ we have the convergence

$$\frac{1}{2n} \log \frac{1}{|\Phi_n(z)|} \to \varphi(z),$$

uniformly on compact subsets of $\mathcal{O}$, to a real function $\varphi(z)$ which is harmonic in each connected component of $\mathcal{O}$. We note that the condition $\mathcal{O} \subset \{z : |z| < 1/2\}$ is technical. It can easily be replaced by $\mathcal{O} \subset \{z : |z| < 1\}$, but if it is entirely omitted, then we would need to discuss more carefully the $n$th root asymptotics.
of the spherical normalizations of polynomials in terms of spherically normalized potentials.

**Condition 2.** The compact set $\Gamma$ in (36) belongs to $\mathcal{O}$ and has the S-property in the external field $\varphi$ in (37) (which is explained in Definition 2 below). Moreover, the complement of the support of the equilibrium measure $\lambda = \lambda_{\varphi, \Gamma}$ is connected.

We note that in the case of multipoint Padé approximants the complement of the extremal compact set is not necessarily a domain, that is, it can consist of several domains (see [20], [16], [17]). In the case of the Hermite–Padé polynomials for general complex Nikishin systems the situation is very similar to multipoint Padé approximants (see [43], [57], [68]).

**Definition 2.** Let $\mathcal{O}$ be an open set and $\varphi(z)$ a real function which is harmonic in each connected component of $\mathcal{O}$. We say that a compact set $\Gamma \in \mathcal{F}$ has an $S$-property relative to the external field $\varphi$ if there exists a set $e \subset \Gamma$ of zero capacity such that for any $\zeta \in \Gamma \setminus e$ there is a neighbourhood $D = D(\zeta)$ of $\zeta$ such that $\text{supp}(\lambda) \cap D$ is an analytic arc and, furthermore,

$$\frac{\partial}{\partial n_1}(U^\lambda + \varphi)(\zeta) = \frac{\partial}{\partial n_2}(U^\lambda + \varphi)(\zeta), \quad \zeta \in \Gamma^o = \text{supp}(\lambda) \setminus e,$$

where $\lambda = \lambda_{\varphi, \Gamma}$ is the equilibrium measure for $\Gamma$ in $\varphi$ and $n_1$ and $n_2$ are the two oppositely directed normals to $\Gamma$ at $\zeta \in \Gamma$ (we can actually allow $\varphi$ to have a small singular set included in $e$).

We shall call the set $\Gamma^o = \text{supp}(\lambda) \setminus e$ the regular part of $\Gamma^1 = \text{supp}(\lambda)$. It is clear that the regular part is a finite or countable union of disjoint open analytic arcs.

**Condition 3** on the function $f$ will be stated as $f \in \mathcal{H}_0(\mathcal{O} \setminus \Gamma)$, where we denote by $\mathcal{H}_0(\mathcal{O} \setminus \Gamma)$ the class of all functions $f \in H(\mathcal{O} \setminus \Gamma)$ which also have the following property. There exists a set $e_0 \subset \Gamma$ of zero capacity such that for any arc $\gamma \subset \Gamma \setminus (e \cup e_0)$ the function $f$ has a continuous extension to $\gamma$ from both sides of the arc and the difference of the boundary values of $f$ has no zeros on $\gamma$.

Thus, $\Gamma$ is the set of singularities of $f$, and by $\int_{\Gamma}$ in (36) we understand integration over a union $C = \bigcup C_j$ of mutually exterior piecewise smooth Jordan contours $C_j$ containing all the singularities of $f$ in the union of their interiors, $\Gamma \subset \bigcup \text{int} C_j$.

In what follows we say that a polynomial $Q$ is spherically normalized if

$$Q(z) = \prod_{\zeta: Q(\zeta) = 0 \atop |\zeta| \leq 1} (z - \zeta) \cdot \prod_{\zeta: Q(\zeta) = 0 \atop |\zeta| > 1} \left(1 - \frac{z}{\zeta}\right).$$

The concept of the spherically normalized potential of a measure $\mu$, $\text{supp}(\mu) \subset \overline{\mathbb{C}}$, is introduced similarly:

$$U^\mu_*(z) = \int_{|\zeta| \leq 1} \log \frac{1}{|z - \zeta|} d\mu(\zeta) + \int_{|\zeta| > 1} \log \frac{1}{|1 - z/\zeta|} d\mu(\zeta).$$

We now state the GRS-theorem (Theorem 3 in [27]).
Theorem 9. Let the polynomials $Q_n(z) \in \mathbb{P}_n$ be defined by the orthogonality relations (36) and assume that Conditions 1, 2, and 3 on $\Phi_n(z)$, $\Gamma$, and $f$, respectively, are satisfied. Then the following assertions hold.

(i) As $n \to \infty$,
\[
\frac{1}{n} \mathcal{X}(Q_n) \rightarrow^* \lambda,
\]
where $\lambda = \lambda_{\varphi, \Gamma}$.

(ii) If the $Q_n$ are spherically normalized, then the convergence in capacity
\[
\left| \oint_{\Gamma} \frac{Q_n^2(z)}{\zeta - z} \Phi_n(z)f(z) \, dz \right|^{1/n} \rightarrow e^{-2w_{\varphi}}
\]
holds, where $w_{\varphi}$ is the equilibrium constant in (13) for $\Gamma$ and $\varphi$ for the spherically normalized total potential of $\lambda$.

3.2. Outline of the proof of Theorem 9. Basic lemmas. We are interested here in the proof of part (i). It begins with the selection of a weakly convergent subsequence $(1/n)\mathcal{X}(Q_n) \to \mu$ as $n \to \infty$, $n \in \Lambda = \{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$. Now it is enough to prove that $\mu = \lambda$. Assume the contrary: $\mu \neq \lambda$. Under this assumption we shall construct a sequence of polynomials $P_n$, $\deg P_n < n$, $n \in \Lambda$, such that
\[
\left| \oint_{\Gamma} Q_n(z)P_n(z)\Phi_n(z)f(z) \, dz \right|^{1/n} \rightarrow c > 0, \quad n \to \infty, \quad n \in \Lambda,
\]
in contradiction to the orthogonality relations.

The same approach was used in the proof of Theorem 6 above. The construction of the sequence $P_n$ now is different. The procedure is rather long and involves several steps.

First, starting with a given measure $\mu$ (representing a hypothetical zero distribution of orthogonal polynomials) we construct a measure $\sigma$ which will later play the role of the limit zero distribution for the sequence $P_n$. The properties of this measure $\sigma$ are listed in Lemma 3 below, which asserts the existence of a measure with the required properties.

To state the lemma we need to mention one important property of compact sets with the $S$-property in a harmonic field, namely, the existence of a reflection function $z \mapsto z^*$ in a neighbourhood $U(z_0)$ of any regular point $z_0 \in \Gamma^\circ$. In fact, to assert the existence of such a function we only need the local analyticity of $\Gamma^\circ$. It follows from the analyticity of $\Gamma$ at $z_0$ that a neighbourhood $U(z_0)$ of this point can be selected so that there is a conformal map $\psi: U(z_0) \to D_1 = \{z: |z| < 1\}$ with $\psi(z_0) = 0$ and $\psi(\Gamma^\circ \cap U(z_0)) = (-1, 1)$. Then for $z \in U(z_0)$ we define $z^* = \psi^{-1}(\overline{\psi(z)})$.

Thus, $z \mapsto z^*$ is an anticonformal map similar to complex conjugation. In particular, $z^* = z$ for points on $\Gamma^\circ$. By $U = U(z_0)$ we shall denote $\ast$-symmetric neighbourhoods of the points $z_0 \in \Gamma^\circ$. For a set $E \subset U$ we write $E^* = \{z^*: z \in E\}$. For any measure $\nu$ on $U$ we denote by $\nu^*$ the reflected measure defined by $\nu^*(E) = \nu(E^*)$, $E \subset U$. 
Lemma 3. Let $\varphi$ be a harmonic function in an open set $\mathcal{O} \subset \mathbb{C}$, and assume that a compact set $\Gamma \subset \mathcal{O}$ has the $S$-property in the field $\varphi$ and that the complement of the support of $\lambda = \lambda_{\varphi, \Gamma}$ is connected (that is, Condition 2 for $\Gamma$ is satisfied).

Suppose $\mu$ is a measure in the plane with $|\mu| \leq 1$ and $\mu \neq \lambda$. Then there exists an $\bar{r} = \bar{r}(\mu) \in (0, 1)$ such that, for any positive $r \leq \bar{r}$, any positive measure $\eta$ with $|\eta| \leq 1 - r$ and with support in $\mathbb{C} \setminus \mathcal{O}$, any $\varepsilon \in (0, 1 - r)$, and any compact set $e \subset \Gamma$ of capacity zero, there exist a point $z_0 \in \Gamma \setminus e$, a $*$-symmetric neighbourhood $^3 \mathcal{U} = \mathcal{U}(z_0) \subset D_\varepsilon(z_0)$ of it, and a positive measure $\sigma = \sigma(\mu, \eta)$ with the following properties:

(i) $|\sigma| = r + |\eta| + \varepsilon < 1$;
(ii) $\sigma|_\mathcal{U} = (\mu|_\mathcal{U})^*$;
(iii) $U^\sigma(z) = +\infty$ for $z \in e$;
(iv) $(U^{\mu+\sigma} + 2\varphi)(z^*) = (U^{\mu+\sigma} + 2\varphi)(z)$ for $z \in \mathcal{U}$;
(v) $(U^{\mu+\sigma} + 2\varphi)(z_0) < (U^{\mu+\sigma} + 2\varphi)(z)$ for $z \in \Gamma \setminus \{z_0\}$;
(vi) the total variation of the signed measure $\sigma - (r\lambda + \eta)$ is at most $\varepsilon$.

Proof. The hypotheses and assertions (i)–(v) of Lemma 3 are in essence identical to those of Lemma 9, p. 340 in [27] and the construction of the proof does not require significant modifications. Still, technical changes are to be made. To place them in a proper context we make a few comments on the structure of the proof.

We mention a few differences in notation first. The external field was denoted by $\psi$ in Lemma 9 and the $S$-compact set in this field was denoted by $F$ (the neighbourhood of it where $\psi$ is defined was denoted by $U$); $\Gamma$ was the support of the equilibrium measure $\lambda$. The notation for the potential of a measure $\mu$ was $V^{\mu}$. Here we use our notation ($\varphi$ for the field, $\Gamma$ for the $S$-compact set, $\mathcal{O}$ for its neighbourhood, and so on). There are a few more minor differences, which should not cause problems.

The main modification in Lemma 3 is the assertion (vi), meaning that an arbitrary sufficiently small positive measure $\eta$ can be included as a part of $\sigma$. The measure $\sigma$ is explicitly constructed in the process of the proof of Lemma 9 in [27] so that only an arbitrarily small part of the constructed measure $\sigma$ is not known explicitly. The new property (vi) of the measure $\sigma$ means that this measure can be constructed in such a way that it has the form

$$\sigma = r\lambda + \eta + \nu,$$

where $\eta$ is an arbitrary positive measure with $|\eta| < 1 - r$ and $r \leq \bar{r}(\mu) < 1$. The additional (signed) measure $\nu$ can be made arbitrarily small in variation. We note that $\sigma$ is a positive measure, but $\nu$ is not in general. The measure $\nu$ is essentially constructive too, but it plays a technical role and its form is not really important. In Lemma 5 below we shall show that it is possible to eliminate this part by passing to a limit.

To explain the way the measure $\eta$ is incorporated in $\sigma$ we reproduce the beginning of the proof of Lemma 9 in [27]. The parameters in the proof of Lemma 9 appear in the following order. First, given a measure $\mu$, a function $W$ is introduced by the formula

$$W(z) = W(z; t, \eta) = U^{\mu+t\eta}(z) + (1 + 2t)\varphi(z) - (1 - 2t)(U^\lambda + \varphi)(z),$$

$^3D_\varepsilon(z_0)$ is the open disk with radius $\varepsilon$ and centre $z_0$. 
where $\eta$ is an arbitrary unit measure and $t \in [0, 1/2]$. Then two associated functions are defined,

$$w_0(t, \eta) = \inf_{\Gamma^1} W(z; t, \eta) \quad \text{and} \quad w_1(t, \eta) = \inf_{\partial \mathcal{O}} W(z; t, \eta).$$

Since $w_0(0, \eta) < w_1(0, \eta)$ (note that these quantities are independent of $\eta$), it is concluded that there is a $t \in (0, 1/2)$ such that $w_0(t, \eta) < w_1(t, \eta)$ for any measure $\eta$ with support in $\mathcal{O}$ ($t$ can be made smaller). The value of $t$ is fixed from this point on.

Then the procedure for selecting

$$\eta = \frac{1}{2}(\eta_1 + \eta_2)$$

is described as follows. A unit measure $\eta_1$ is selected on $e$ so that $U^{\nu_1} = +\infty$ on $e$. Then a point $z_0 \in \Gamma^0$ is selected where $U^{\mu + (t/2)\eta_1} + \varphi$ assumes the minimum value on $\Gamma$. Since $U^{\mu + (t/2)\eta_1} + \varphi = +\infty$ on $e$, $z_0$ is an interior point of one of the open analytic arcs in $\Gamma^0$. Then a unit measure $\eta_2$ is selected whose potential $U^{\nu_2}(z)$ has a strict minimum over $\Gamma$ at the point $z_0$. As part of the proof of the lemma the construction of $\eta$ now has to be modified as follows. First, we define $\tau = 1 - t$ with the above choice of $t$. Then we select an arbitrary unit measure with compact support in $\mathbb{C} \setminus \mathcal{O}$. From this point on we follow the procedure of the proof of Lemma 9 in [27]. At this point it would be convenient to modify our notation. The parameter $t$ is already renamed as $1 - \tau$ and this number is fixed and we write $\eta$ in place of $\eta_1$. The total mass of the other components, $\eta$ and $\nu$, of $\sigma = r\lambda + \eta + \nu$ which we denote by $\nu$ just needs to be small. We fix a positive $\varepsilon < (1 - \tau)/4$.

In total, we will have four components of the measure $\nu$. Two of them, $(t/2)(\eta_1 + \eta_2)$ selected above, have to be renormalized so that their total masses are $\leqslant \varepsilon$. The third component (denoted by $\nu$ in Lemma 9 in [27]) is defined as $\mu^*_{\mathcal{O}}$ (see (ii)). The choice of $\mathcal{U}$ allows us to make it small. Finally, the last component is also supported in an arbitrarily small neighbourhood of $z_0$ and can also be made arbitrarily small. \(\square\)

Next, we state another result in [27] (Lemma 8 on p.337) as Lemma 4. We use some different notation here. In particular, our $\mathcal{O}$, $\Phi_n(z)$, $\varphi$, $G_n$, and $g_n$ are denoted by $\mathcal{U}$, $\Psi_n$, $\psi/2$, $P_n$, and $q_n$, respectively, in [27], Lemma 8. The content is not changed and the proof in [27] remains valid.

**Lemma 4.** Let $\mathcal{O}$ be a domain, let $L \subset \mathcal{O}$ be a simple closed analytic arc, and let the functions $\Phi_n(z) \in H(\mathcal{O})$ satisfy Condition 1 (see (37)). Furthermore, let $w(z)$ be a continuous function without zeros on $L$ and let $G_n(z)$ be a sequence of polynomials such that

$$\nu_n := \frac{1}{n} \mathcal{X}(G_n) \xrightarrow{*} \nu \quad \text{as} \quad n \to \infty, \quad n \in \Lambda = \{n_k\}_k^{\infty}.$$  \hspace{1cm} (39)

Suppose that the following conditions are satisfied in some $*$-symmetric neighbourhood $\mathcal{U}$ of an interior point $z_0 \in L$:

(i) $\nu_n|_{\mathcal{U}} = (\nu_n|_{\mathcal{U}})^*$ and all zeros of $G_n$ on $L \cap \mathcal{U}$ are of even multiplicity;

(ii) $U^{\mu + \sigma}(z^*) = U^{\mu + \sigma}(z)$ for $z \in \mathcal{U}$;
(iii) \( U^{\mu+\sigma}(z_0) < U^{\mu+\sigma}(z) \) for \( z \in L \setminus \{z_0\} \).

Then there exists a sequence of polynomials \( g_n(z) \) with zeros in \( U \) and with 
\[
\frac{(\text{deg } g_n)}{n} \to 0 \quad \text{as } n \to \infty
\]
such that 
\[
\left| \int_L g_n(z)G_n(z)\Phi_n(z)w(z) \, dz \right|^{1/n} \to \exp\{-\left(U^{\nu} + 2\varphi\right)(z_0)\}
\]
(40)

(the potential and polynomials are spherically normalized).

The original proof of the GRS-theorem in [27] was a combination of Lemmas 8 and 9 and another rather simple Lemma 7 in [27], which asserts the inequalities (44) and (45) below. Here it is convenient for our purposes to single out a further intermediate lemma, combining Lemmas 3 and 4.

**Lemma 5.** Let Conditions 1, 2 and 3 on \( \Phi_n(z) \), \( \Gamma \), and \( f \), respectively, be satisfied and assume that a sequence of spherically normalized polynomials \( Q_n(z) \in \mathbb{P}_n \) has a limit zero distribution
\[
\mu_n := \frac{1}{n} \mathcal{X}(Q_n) \overset{\ast}{\to} \mu \quad \text{as } n \to \infty, \quad n \in \Lambda = \{n_k\}_{k=1}^{\infty},
\]
(41)
along a subsequence \( \Lambda \) represented by a measure \( \mu \) with total mass \( \mu(\mathbb{C}) \leq 1 \) and different from the equilibrium measure \( \lambda = \lambda_{\varphi,\Gamma} \).

Let \( \eta \) be a measure with compact support in \( \mathbb{C} \setminus \mathcal{O} \) and \(|\eta| \leq 1 - r \), \( 0 < r \leq \bar{r} = \bar{\tau}(\mu) < 1 \). Let 
\[
\sigma = \sigma(\mu, \eta) = r\lambda + \eta + \nu
\]
be the measure associated with \( \mu \) and \( \eta \) whose existence is asserted in Lemma 3 above; in particular, \(|\nu| < 1 - r\). Then there exists a spherically normalized sequence of polynomials \( P_n \) satisfying the conditions \( \text{deg } P_n < n \),
\[
\mu_n := \frac{1}{n} \mathcal{X}(P_n) \overset{\ast}{\to} \sigma \quad \text{as } n \to \infty, \quad n \in \Lambda = \{n_k\}_{k=1}^{\infty},
\]
(42)
and 
\[
\left| \int_{\Gamma} Q_n(z)P_n(z)\Phi_n(z)f(z) \, dz \right|^{1/n} \to e^{-m},
\]
(43)
where
\[
m = m(\mu + \sigma) = \min_{z \in \Gamma^{\circ}}(U^{\mu+\sigma} + 2\varphi)(z) = (U^{\mu+\sigma} + \varphi)(z_0)
\]
(the potential of \( \mu + \sigma \) is spherically normalized).

**Proof.** As we mentioned, Lemma 5 is essentially a combination of Lemmas 3 and 4. A reduction to Lemmas 3 and 4 just requires some comments on the connections between the two lemmas.

First, the exceptional set \( e \) which appears in the hypotheses of Lemma 3 will be a union of three sets. The first, \( \Gamma^{\circ} \setminus \Gamma^{1} \), consists of the end points of analytic arcs in \( \text{supp } \lambda \). Here the points where the equilibrium conditions (13) are violated (if any) are also included. The second is \( e(f) \) (the singularities of \( f \)). On any arc in \( \Gamma^{\circ} \setminus e(f) \) the function \( f \) has continuous boundary values \( f^\pm(z) \) and the third part of the exceptional set consists of the zeros of \( w(z) = (f^+ - f^-)(z) \) on \( \Gamma^{\circ} \setminus e(f) \).
Next, the conditions of Lemma 3 are satisfied, and the lemma asserts the existence of a point \( z_0 \in \Gamma^o \setminus e \) and a \( * \)-symmetric neighbourhood \( \mathcal{U} = \mathcal{U}(z_0) \subset \Gamma^o \setminus e \) of it and a measure \( \sigma = \sigma(\mu) \) with the properties (i)–(vi).

We define an analytic arc \( L = \Gamma^o \cap \mathcal{U} \) and \( w(z) = (f^+ - f^-)(z) \) on \( L \). Then we construct a sequence of polynomials \( \widetilde{P}_n \) with

\[
\frac{1}{n} \mathcal{X}(\widetilde{P}_n) \to \sigma.
\]

This can be done with a significant degree of freedom, and the choice of \( \widetilde{P}_n \) can be made subject to certain extra conditions.

Namely, using (ii) of Lemma 3, we can select the zeros of \( \widetilde{P}_n \) in \( \mathcal{U} \) so that they are \( * \)-symmetric to zeros of \( Q_n \) in \( \mathcal{U} \). In particular, the zeros of \( \widetilde{P}_n Q_n \) on \( L = \Gamma \cup \mathcal{U} \) have even multiplicity. Thus we deduce condition (i) of Lemma 4 for the measure \( \nu = \mu + \sigma \). The other two conditions (ii) and (iii) on the potential of \( \nu \) in Lemma 4 are satisfied since they are identical to assertions (iv) and (v) of Lemma 3.

For \( G_n = \widetilde{P}_n Q_n \) the conditions of Lemma 4 are satisfied and the lemma asserts the existence of polynomials \( g_n \) of degree \( o(n) \) with property (40). We set \( P_n = g_n \widetilde{P}_n \). For these polynomials the assertion (43) in Lemma 5 with \( \int_{\Gamma} \) in (43) replaced by \( \int_{L} \) (and \( w \) in place of \( f \)) follows from Lemma 4.

Finally, to pass to \( \int_{\Gamma} \) in (43) we use the inequality

\[
\lim_{n \to \infty} \left| \int_{\Gamma \setminus L} Q_n(z) P_n(z) \Phi_n(z) f(z) \, dz \right|^{1/n} \leq e^{-m'},
\]

where

\[
m' = \min_{\Gamma \setminus L}(U^{\mu+\sigma} + 2\varphi) > m,
\]

and the contour \( \Gamma \) is modified so that both copies of \( L \) belong to \( \Gamma \). This assertion follows actually from the quite general properties of the convergence of potentials of weakly convergent measures (see, for example, Lemma 7 in [27] and comments there).

### 3.3. A modification of Lemma 5

After a suitable generalization Lemma 5 may be of independent interest. We mention briefly a more general problem connected with this lemma.

Consider the setting of Lemma 5 again: given \( \Phi_n(z) \), \( \Gamma \), and \( f \) with Conditions 1, 2 and 3, respectively, and two sequences of spherically normalized polynomials \( Q_n \) and \( P_n \) satisfying conditions (41) and (42), respectively. Now return to the inequality (44), which we write in application to the whole of \( \Gamma \):

\[
\lim_{n \to \infty} \left| \int_{\Gamma} Q_n(z) P_n(z) \Phi_n(z) f(z) \, dz \right|^{1/n} \leq e^{-m},
\]

where

\[
m = \min_{\Gamma}(U^{\mu+\sigma} + 2\varphi)
\]
(see Lemma 7 in [27]). As we mentioned above, this inequality is rather general; it follows from
\[ \frac{1}{n} \mathcal{X}(P_n Q_n) \xrightarrow{*} \mu + \sigma \]
and some mild assumptions on \( \varphi, f, \) and \( \Gamma \) (no need for the \( S \)-property) and the convergence in (37).

Now the natural questions to ask are the following. Suppose a sequence \( Q_n \) of polynomials with (41) is given. What are the conditions (in particular on measures \( \mu \) and \( \sigma \)) which would imply that there is a sequence \( P_n \) satisfying (42) such that equality holds in (45)? Another question is what conditions will guarantee that the limit exists instead of just the upper limit.

Here we are not going to investigate these problems in the general settings. For the purposes of application to Hermite–Padé polynomials we need to prove one particular proposition which is a corollary of Lemmas 3–5. Loosely speaking, we shall show that in the setting of Lemma 5 equality holds in (45) for any \( \mu \neq \lambda \) if \( \sigma - \lambda \) is small enough. We do not ask if \( \lim \) exists, so we are ready to pass to a subsequence of \( \Lambda \).

**Lemma 6.** Let \( \mathcal{O} \subset D_{1/2} \) be an open set and assume that the sequence of functions \( \Phi_n \in H(\mathcal{O}) \), the compact set \( \Gamma \subset \mathcal{O} \), and the function \( f \in H_0(\mathcal{O} \setminus \Gamma) \) satisfy Conditions 1, 2, and 3. Let \( Q_n(z) \) be a sequence of polynomials satisfying (41) with a measure \( \mu \) such that \( \mu \neq \lambda \) with \( |\mu| \leq 1 \). Then there exists an \( \bar{r} \in (0, 1) \) such that, for any \( r \in [\bar{r}, 1] \) and any measure \( \eta \) with compact support in \( \mathbb{C} \setminus \mathcal{O} \) and with \( |\eta| = 1 - r \), there is a sequence of polynomials \( P_n, n \in \Lambda \), such that
\[ \frac{1}{n} \mathcal{X}(P_n) \xrightarrow{*} r \lambda + \eta \quad \text{as } n \to \infty, \ n \in \Lambda, \]
\[ \text{deg } P_n < n, \]
all the zeros of \( P_n \) belong to \( \Gamma \cup \mathcal{O} \), and
\[ \left| \int_{\Gamma} Q_n(z) P_n(z) \Phi_n(z) f(z) \, dz \right|^{1/n} \to \exp \left\{ - \min_{\Gamma} (U^{\mu + \sigma} + 2\varphi) \right\}, \] (46)
where \( \sigma = r \lambda + \eta \) (the potentials and polynomials are spherically normalized).

**Proof.** We fix an exceptional set \( e \) as in the proof of Lemma 5. According to Lemma 3 there exist an \( \bar{r} = \bar{r}(\mu) \in (0, 1) \) and a measure \( \eta \) with \( |\eta| \leq 1 - r \) such that for any \( \varepsilon \in (0, (1 - r)/4) \) there is a point \( z_0 = z_0(\varepsilon) \in \Gamma^{\circ} \setminus e \), a \( *\)-symmetric neighbourhood \( U = U(z_0) \) of it, and a positive measure \( \sigma = \sigma(\mu, \varepsilon) \) with the properties (i)–(vi). We pass to the limit as \( \varepsilon \to 0 \).

Fix an \( r \in [\bar{r}, 1] \) and a measure \( \eta \) with \( |\eta| < 1 - r \), and consider the sequence \( \varepsilon = 1/N \) with \( N \in \mathbb{N} \). Denote the associated sequence of measures \( \sigma \) by
\[ \sigma_N = \sigma(\mu, \frac{1}{N}) = r \lambda + \eta + \nu_N. \]

By Lemma 5 for any \( \sigma_N \) we have a sequence of polynomials \( P_{n,N} \) with
\[ \frac{1}{n} \mathcal{X}(P_{n,N}) \xrightarrow{*} \sigma_N \]
and
\[ M_{n,N} := \left| \int_{\Gamma} (Q_n P_n \Phi_n f)(z) \, dz \right|^{1/n} \rightarrow M_N := \exp \left\{ -\min_{\Gamma} (U^\mu + \sigma_N + 2\varphi) \right\} \] (47)
as \( n \rightarrow \infty \), \( n \in \Lambda = \{ n_k \}_{k=1}^{\infty} \). Note that the sequence \( \Lambda \) does not depend on \( N \) (it is the subsequence for which the sequence of the counting measures for \( Q_n \) converges; see (41)).

By (vi) in Lemma 3 we have the convergence \( \sigma_N \rightarrow r\lambda \) (in total variation and therefore weak-*). It follows that
\[ M_N \rightarrow M = \exp \left\{ -\min_{\Gamma} (U^\mu + r\lambda + 2\varphi) \right\} . \]
Using the standard procedure, we can select a diagonal subsequence in the table \( M_{n,N} \) which converges to \( M \). This gives a subsequence \( \Lambda_1 \subset \Lambda \) and a sequence \( N_n, n \in \Lambda_1 \), such that
\[ N_n \rightarrow \infty \quad \text{and} \quad M_{n,m_n} \rightarrow M \quad \text{as} \quad n \rightarrow \infty, \ n \in \Lambda_1. \]
Since \( \sigma_N \rightarrow r\lambda \) in variation, we also have \( \sigma_{N_n} \rightarrow r\lambda \).

Finally, a selection of the zeros of \( P_n \) can be made so that
\[ \frac{1}{n} X(P_n) \overset{*}{\rightharpoonup} r\lambda \quad \text{as} \quad n \rightarrow \infty, \ n \in \Lambda_1 \]
and all the zeros of \( P_n \) are in \( \Gamma \cup \mathcal{O} \). Thus, the proof of the lemma is completed. \( \square \)

Next we prove a lemma concerning the case \( \mu = \lambda \), which is not contained in Lemma 6. This case is indeed exceptional. First, the case is exceptional in Lemma 5, too, so we cannot directly invoke the lemma. Second, we cannot expect to construct polynomials \( P_n \) with (46) and \( \deg P_n < n \) as in Lemma 6. Such polynomials should satisfy \( \deg P_n \geq n \) (otherwise orthogonal polynomials will provide us with a counterexample to the assertion of the lemma).

This suggests that we will need to use a sequence of measures \( \sigma_N \rightarrow \lambda \) approximating \( \lambda \) from above in the sense that \( |\sigma_N| > 1 \). In the proof of the next statement, Lemma 7, we implement such an approximation. Actually, for future reference we need the method of the proof of Lemma 7 rather than its assertion.

**Lemma 7.** Let \( \mathcal{O} \subset D_{1/2} \) be an open set and assume that the sequence of functions \( \Phi_n \in H(\mathcal{O}) \), the compact set \( \Gamma \subset \mathcal{O} \), and the function \( f \in H_0(\mathcal{O} \setminus \Gamma) \) satisfy Conditions 1, 2 and 3. Let \( Q_n(z) \) be a sequence of polynomials satisfying (41) with \( |\mu| \leq 1 \). Then there exists a sequence of polynomials \( P_n, n \in \Lambda_1 \subset \Lambda \), such that
\[ \frac{1}{n} X(P_n) \overset{*}{\rightharpoonup} \lambda \quad \text{as} \quad n \rightarrow \infty, \ n \in \Lambda_1, \]
and (46) holds with \( \sigma = \lambda \).

**Proof.** To make it possible to apply Lemma 3 formally to a sequence of polynomials \( Q_n \) with \( (1/n)X(Q_n) \overset{*}{\rightharpoonup} \lambda \), we argue as follows. Fix \( R > 1 \) and consider the new sequence of positive integers
\[ n' = [Rn] \in \Lambda', \quad n \in \Lambda, \]
where \([x]\) stands for the integral part of \(x\). Next, we perform a substitution in the index (variable) \(n\) of a polynomial (the index is interpreted here as an independent variable of the function \(n \mapsto Q_n\)). We shall consider the polynomial \(Q_n\) as an element of \(\mathbb{P}_{n'}\), that is, the sequence \(Q_n, \ n \in \Lambda\), is now interpreted as the sequence \(Q_{n'}, \ n' \in \Lambda'\). We have

\[
\frac{1}{n'} \mathcal{X}(Q_{n'}) \overset{\ast}{\to} r\lambda, \quad \text{where} \quad r = \frac{1}{R} < 1.
\]

So, the limit measure is now different from \(\lambda\).

Next, we need to make the same substitution in the index \(n\) of the varying weights \(\Phi_n(z)\). This would result in a change of the function \(\varphi\) representing the asymptotics of \(\Phi_n(z)\). In place of (37) we have to write

\[
\frac{1}{2n'} \log \frac{1}{|\Phi_n(z)|} \to \psi(z) = r\varphi(z). \quad (48)
\]

Finally, the compact set \(\Gamma\) has the \(S\)-property in the field \(\varphi\) but not in the field \(\psi\), and we have to modify \(\Gamma\) in order to use Lemma 5.

More precisely, it would be enough to prove that for sufficiently small \(R - 1 > 0\) there is a compact set \(\Gamma' = \Gamma(R)\) homotopic to \(\Gamma\) and with the \(S\)-property in the field \(\psi\) such that the integral \(\int_{\Gamma'}\) in (46) can be replaced by \(\int_{\Gamma'}\) and such that as \(R \to 1\) we have the convergence of compact sets

\[
\Gamma(R) \to \Gamma
\]

in the Hausdorff metric and also the weak convergence

\[
\lambda' = \lambda(R) \to \lambda
\]

of their equilibrium measures in the associated fields \(\psi = r\varphi\). We shall prove such a lemma (Lemma 8) in §4.

For fixed \(R\) and \(\Gamma' = \Gamma(R)\) we now obtain from Lemma 5 that there exists a sequence of polynomials \(P_{n'}, \ n' \in \Lambda' \subset \Lambda\), such that

\[
\frac{1}{n'} \mathcal{X}(P_{n'}) \overset{\ast}{\to} \lambda' \quad \text{as} \quad n' \to \infty, \ n' \in \Lambda',
\]

and hence

\[
\left| \int_{\Gamma'} Q_{n'}(z)P_{n'}(z)\Phi_{n'}(z)f(z) \, dz \right|^{1/n'} \to e^{-m}, \quad n' \to \infty, \quad (49)
\]

where

\[
n' = [nR] \in \Lambda', \quad m = m(R) = R \min_{z \in \Gamma'}((1 + r)U\lambda' + 2\psi)(z).
\]

The constant \(m(R)\) in the exponent has been obtained as follows. First, using the expressions in Lemma 5, we write this relation with \(n'\) in place of \(n\). Next, we perform the substitution \(n' = [nR]\), which changes the exponent \(1/n'\) to \((1/n)r\). Finally, we raise both sides of the equality to the power \(R\) and, returning to the
original notation \((Q_n \text{ in place of } Q_n' \text{ and so on}),\) obtain the factor \(R\) in the exponent on the right-hand side.

To conclude the proof we consider the sequence
\[
R_N = 1 + \frac{1}{N} \to 1
\]
and then use a diagonal process to find a desired sequence of polynomials. This part is similar to what we did in the proof of part (ii) of Theorem 3 in [27]. By a suitable selection of a sequence \(N_n\) we can obtain a sequence of polynomials \(P_n = P_{n,N_n}\) such that \((1/n)\mathcal{X}(P_n) \to \lambda\) along some subsequence of \(\Lambda\). Since
\[
m(R) \to m_0 = \min_{z \in \Gamma}(2U^\lambda + 2\varphi)(z) \quad \text{as } R \to 1,
\]
the left-hand side of (49) will converge to \(e^{-m_0}\). □

3.4. Proof of Theorems 1 and 2. Let
\[
q_{n,k}(z) = c_{n,k}Q_{n,k}(z)
\]
be the polynomials defined in (2) and let the coefficients \(c_{n,k}\) be determined by the condition that the \(Q_{n,k} \in P_n\) are spherically normalized.

Let \(\tilde{\Gamma}\) be the extremal compact set associated with \(\tilde{f}\) and let \(\tilde{\lambda} = \lambda_{\tilde{\Gamma}}\) be its vector equilibrium measure with unit components \(\lambda_k\); we let \(\lambda = \sum_{i=1}^s \lambda_i\) and furthermore,
\[
w_k = \min_{z \in \Gamma_k} W_k(z), \quad \text{where } W_k(z) = U^\lambda_k + \lambda(z)
\]
(see (12) and (13)).

The beginning of the proof of Theorem 1 is similar to the proof in the case of Markov functions in §2. First we select a sequence \(\Lambda \subset \mathbb{N}\) such that, as \(n \to \infty\), \(n \in \Lambda\), we have
\[
\frac{1}{n}\mathcal{X}(Q_{n,k}) \to \mu_k \quad \text{and} \quad |c_{n,k}|^{1/n} \to e^{-u_k}, \quad k = 1, \ldots, s,
\]
where \(u_k \in [-\infty, +\infty]\). The remainder \(R_n\) in (2) can be normalized in such a way that \(u_k \geq 0\) and \(\min u_k = 0\). Then we have \(u_k \in [0, +\infty]\), and infinite values of \(u_k\) are not excluded a priori.

Now we have to prove that \(\mu_k = \lambda_k\) and all the numbers \(\tilde{m}_k = u_k + w_k, \quad k = 1, \ldots, s\), are equal. We shall do this in two steps. First, we define
\[
m_k = u_k + \min_{z \in \Gamma_k} U^\mu_k + \lambda(z) = u_k + w_k + \min_{z \in \Gamma_k} U^\mu_k(z), \quad k = 1, \ldots, s,
\]
and prove that \(m_1 = \cdots = m_s\). Then we prove that \(\mu_k = \lambda_k\), which also implies that \(m_k = \tilde{m}_k\), thus completing the process.

In both steps we proceed by contradiction: assuming that the desired assertion is not valid, we come to a contradiction to the orthogonality relations (11), which can be equivalently written as
\[
\sum_{k=1}^s I_{n,k} = 0, \quad \text{where} \quad I_{n,k} = I_{n,k}(G_n) = \int_{\Gamma_k} q_{n,k}(z)G_n(z)f_k(z)\,dz,
\]
and \(G_n \in \mathbb{P}_{ns}\). For the convenience of readers we shall first present details of the proof in the case \(s = 2\).
3.4.1. Proof of Theorems 1 and 2 in the case $s = 2$. In this section we have $k = 1, 2$. To prove that $m_1 = m_2$ for the constants $m_k$ in (51) we assume the contrary: $m_1 \neq m_2$. Without loss of generality we can assume that $m_1 < m_2$ (changing the indices if necessary).

We select a spherically normalized sequence of polynomials $\Phi_n$ with

$$\frac{1}{n}X(\Phi_n) \to \lambda_2$$

and zeros on $\Gamma_2$. For this sequence

$$\frac{1}{2n} \log \frac{1}{|\Phi_n(z)|} \to \varphi(z) = \frac{1}{2}U^{\lambda_2}(z)$$

(53)

uniformly in a neighbourhood of $\Gamma_1$. Now the conditions of Lemma 6 are satisfied with $\Gamma = \Gamma_1$, $Q_n = c_{n,k}^{-1}q_{n,k}$, and $\mu = \mu_1$ from (50) and $\sigma = \lambda_1$. It follows from Lemma 6 that there is a (sub)sequence of polynomials $P_n$ with $n \in \Lambda_1 \subset \Lambda$ and $(1/n)X(P_n) \to \lambda_1$ such that

$$\left| \oint_{\Gamma_1} Q_n(z)P_n(z)\Phi_n(z)f(z) \, dz \right|^{1/n} \to \exp\left\{ -\min_{\Gamma}(U^{\mu+\lambda_1} + 2\varphi) \right\} = \exp\left\{ -\min_{\Gamma} U^{\mu+\lambda} \right\},$$

(54)

where $\lambda = \lambda_1 + \lambda_2$. We multiply (54) by $|c_{n,k}|^{1/n}$ and write it in terms of the integrals

$$I_{n,k} = I_{n,k}(G_n) = \oint_{\Gamma_k} q_{n,k}(z)G_n(z)f_k(z) \, dz = c_{n,k} \oint_{\Gamma_k} Q_{n,k}(z)G_n(z)f_k(z) \, dz,$$

(55)

with $G_n = P_n\Phi_n$ (cf. (52) above). Taking (51) into account, we have

$$|I_{n,1}|^{1/n} \to e^{-m_1}, \quad \text{where} \quad m_1 = u_1 + \min_{\Gamma_1} U^{\mu_1+\lambda}$$

($m_1$ is the same constant as in (51) for $k = 1$). For the second integral of $q_{n,2}$ over $\Gamma_2$ with the same $G_n = P_n\Phi_n$ we only need the upper bound

$$\overline{\lim}|I_{n,2}|^{1/n} \leq e^{-m_2}, \quad \text{where} \quad m_2 = u_2 + \min_{\Gamma_2} U^{\mu_2+\lambda}.$$ 

Since $m_1 < m_2$, the last two relations combined prove that $I_{n,1} + I_{n,2} \neq 0$ for large enough $n \in \Lambda_1$. This will contradict the orthogonality relations if $\deg G_n \leq 2n$. This inequality can certainly be satisfied by our construction of the polynomials $P_n$ and $\Phi_n$ if $\mu_1 \neq \lambda_1$. In this case we come to a contradiction, showing that $m_1 = m_2$. The exceptional case $\mu = \lambda_1$ requires significant additional efforts (for the moment it is not clear how to avoid complications). The problem is that we cannot use the assertion of Lemma 7, for the following reason. Suppose that as in the case for $\mu \neq \lambda_1$ we have selected a spherically normalized sequence of polynomials $\Phi_n$ with $(1/n)X(\Phi_n) \to \lambda_2$ and with zeros on $\Gamma_2$. For this sequence we have (53) uniformly in a neighbourhood of $\Gamma_1$, and by Lemma 7 we can select a sequence of polynomials
Let $P_n$, $n \in \Lambda_1 \subset \Lambda$, such that $(1/n)\mathcal{X}(P_n) \to \lambda_1$ as $n \to \infty$, $n \in \Lambda_1$, and (46) holds with $\sigma = \lambda_1$. Then we define $G_n = P_n\Phi_n$, but we cannot use this polynomial to obtain a contradiction to the orthogonality conditions. The polynomial $P_n$, whose exact degree is out of control, is selected after $\Phi_n$ and we cannot prove that $\deg G_n \leq 2n$.

For this reason, to prove that $m_1 = m_2$ without any restriction on $\tilde{\mu}$ we shall use the proof of Lemma 7. More precisely, we shall use the method of the proof of Lemma 7 combined with certain modifications of the basic equilibrium problem.

We consider a more general vector equilibrium problem which assigns different total masses to components of the vector equilibrium measure. The settings associated with an arbitrary vector $\vec{t}$ were outlined in the Introduction. Here we need a one-parameter family of equilibrium problems. Let $t > 0$ be small enough; we introduce a vector $\vec{t} = (1 + t, 1 - t)$ whose components will represent the total masses $t_k = \lambda_k(F_k)$ in the equilibrium problem associated with the class of vector measures

$$\mathcal{M}^t = \mathcal{M}^t(\vec{F}) = \{ \tilde{\mu} = (\mu_1, \mu_2); \mu_j \in \mathcal{M}^{t_j}(F_j) \},$$

$$t_1 = 1 + t, \quad t_2 = 1 - t.$$

All definitions of parameters associated with the equilibrium problems (4)–(8) are modified in a clear way for this more general equilibrium problem and become functions of $t$. Let $\overline{\Gamma}(t) = \overline{\Gamma}(t, \vec{f})$ be the extremal compact set associated with the max-min energy problem (see (4)–(8)) in the class $\mathcal{M}^t$ and let $\tilde{\lambda}^t = (\lambda_1^t, \lambda_2^t)$. Actually, the vector equilibrium measure and compact set depend analytically on $t$. For the moment we only need to know that the dependences $\overline{\Gamma}(t)$ and $\tilde{\lambda}^t$ are continuous at $t = 0$. In particular, if the Angelesco condition is satisfied for the extremal vector compact set $\Gamma = \Gamma(0)$, then there is a $\delta > 0$ such that $\Gamma(\vec{t})$ depends on $\vec{t}$ continuously for $|t| < \delta$ (the Hausdorff metric is considered in the space of vector compact sets; see the definition in §4.1.1 for details).

It follows that the constants $m_k = m_k(t)$ in (51) (defined for the same vector measure $\tilde{\mu}$ in (50)) are continuous as functions of $t$. Thus, we can select $t \in (0, 1)$ such that the inequality $m_1(t) < m_2(t)$ is still valid. We fix this value of $t$.

Now we can use the same approach as in the proof for $\mu \neq \lambda_1$. The situation here is quite similar to the proof of Lemma 7 and there is no need to repeat all the details. In short, we are using the following difference between the cases $t = 0$ and $t > 0$: the unit measure $\mu = \lambda_1$ (which remains the same) cannot be equal to the (new) equilibrium measure $\lambda$ since the normalization $|\lambda| = 1 + t > 1$ is different. Thus, we are indeed back in the case $\mu \neq \lambda_1$, which was treated in some detail above. With this, the proof of the equality $m_1 = m_2$ is complete.

It remains to prove that $\tilde{\mu} = \tilde{\lambda}$ if we know that $m_1 = m_2$. If this is not true, then we can assume without loss of generality that $\mu_1 \neq \lambda_1$. From this assumption we shall come to a contradiction.

We follow a procedure similar to that used above to prove that $m_1 = m_2$. The first step is the same; we select a spherically normalized sequence of polynomials $\Phi_n$ with $(1/n)\mathcal{X}(\Phi_n) \to \lambda_2$ and with zeros on $\Gamma_2$. For this sequence (53) is valid uniformly in a neighbourhood of $\Gamma_1$. 


A change in the selection of $\sigma$ is made at the next step. We apply Lemma 6 with $\Gamma = \Gamma_1$, $Q_n = c_{n,k}^{-1} q_{n,k}$, and $\mu = \mu_1$ from (50) and

$$\sigma = (1 - t)\lambda_1 + t\eta,$$

where $\eta$ is the balayage of $\lambda_1$ to $\Gamma_2$ and $t > 0$ is small enough so that the assumptions of Lemma 6 are satisfied. It follows from the lemma that there is a (sub)sequence of polynomials $P_n$ with $n \in \Lambda_1 \subset \Lambda$ and $(1/n)\mathcal{X}(P_n) \to \sigma$ such that

$$\left| \oint_{\Gamma_1} Q_n(z) P_n(z) \Phi_n(z) f(z) \, dz \right|^{1/n} \to \exp\left\{ - \min_{\Gamma} (U^{\mu+\sigma} + 2\varphi) \right\} = \exp\left\{ - \min_{\Gamma} (U^{\mu+\lambda+tv}) \right\},$$

where $\lambda = \lambda_1 + \lambda_2$ and $\nu = \eta - \lambda_1$. We multiply this relation by $|c_{n,k}|^{1/n}$ and write it in terms of the integrals $I_{n,k}$ defined in (55) with $G_n = P_n \Phi_n$. Using the same arguments as above (following (55)), we get that

$$|I_{n,1}|^{1/n} \to e^{-m_1(t)} \quad \text{and} \quad \lim_{n \to \infty} |I_{n,2}|^{1/n} \leq e^{-m_2(t)}, \quad (57)$$

where

$$m_1(t) = u_1 + \min_{\Gamma_1} U^{\mu_1+\lambda+tv} \quad \text{and} \quad m_2(t) = u_2 + \min_{\Gamma_2} U^{\mu_2+\lambda+tv}.$$

According to the definition of $\nu$ and the properties of the balayage we have

$$U^{\nu}(z) = c = \text{const} \quad \text{on} \quad \Gamma_2 \quad \text{and} \quad U^{\nu}(z) = c - G^{\eta}(z) \quad \text{on} \quad \Gamma_1,$$

where $G^{\eta}(z)$ is the Green’s potential of the measure $\eta$ with respect to the domain $\Omega = \overline{\mathbb{C}} \setminus \Gamma_2$. We have $G^{\eta}(z) > 0$ in $\Omega$ and, therefore, $U^{\nu}(z) < c$ on $\Gamma_1$. It follows that

$$m_1(t) = u_1 + \min_{\Gamma_1} (U^{\mu_1+\lambda} + tU^{\nu}) < tc + u_1 + \min_{\Gamma_1} U^{\mu_1+\lambda} = tc + m_1.$$

At the same time

$$m_2(t) = u_1 + \min_{\Gamma_1} (U^{\mu_1+\lambda} + tU^{\nu}) = tc + u_1 + \min_{\Gamma_1} U^{\mu_1+\lambda} = tc + m_2,$$

Since $m_1 = m_2$, we have $m_1(t) < m_2(t)$ for $t > 0$. Now (57) contradicts the orthogonality relations and the proof is complete.

3.4.2. Proof of Theorems 1 and 2 (the general case). A generalization of the proof from the case $s = 2$ to the general case $s \geq 2$ is rather straightforward. In short, all the arguments remain valid with some, mostly obvious, modifications.

First, in the case $s > 2$ we have to use a more general vector equilibrium problem which assigns total masses to components of the vector equilibrium measure according to the components of a vector $\vec{t} = (t_1, \ldots, t_s)$. So we are going to have several parameters instead of one. We shall consider vectors $\vec{t} = (t_1, \ldots, t_s)$ such that

$$t_1 + \cdots + t_s = s.$$
near the point \( \vec{t}_0 = (1, \ldots, 1) \). Thus, the class (56) of vector measures generalizes to
\[
\mathcal{M}^\vec{t} = \overline{\mathcal{M}^\vec{t}(F)} = \{ \vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_s) : \mu_j \in \mathcal{M}^t(F_j) \}.
\] (58)
The definitions of \( \vec{\Gamma}(\vec{t}) = \vec{\Gamma}(\vec{t}, \vec{f}) \), \( \vec{\lambda}(\vec{t}) = (\lambda_1(\vec{t}), \ldots, \lambda_s(\vec{t})) \) (and other parameters) associated with the equilibrium problems (4)–(8) are modified according to this more general equilibrium problem and become functions of \( \vec{t} \). In particular, after a sequence \( \Lambda \subset \mathbb{N} \) is selected so that (50) is satisfied, we define the functions
\[
m_k = m_k(\vec{t}) = u_k + \min_{\Gamma_k} U^{\mu_k + \lambda}, \quad \lambda = \lambda_1 + \cdots + \lambda_s, \quad k = 1, \ldots, s,
\] (59)
where \( \Gamma_k \) and \( \lambda_k \) are functions of \( \vec{t} \).

As in the case \( s = 2 \) we prove first that at \( \vec{t} = \vec{t}_0 \) all the numbers \( m_k(\vec{t}) \) are equal. The proof follows the same path as in the case \( s = 2 \). We assume the contrary, and then by playing with components of \( \vec{t} \) (subject to the condition \( t_1 + \cdots + t_s = s \)) in a neighbourhood of \( t_0 \) we can find a particular \( \vec{t} \) for which one of the numbers \( m_k(\vec{t}) \) will be strictly larger than the others. Then we come to a contradiction to the orthogonality relations as in the case \( s = 2 \).

Next, under the assumption
\[
m_1(\vec{t}_0) = \cdots = m_s(\vec{t}_0),
\]
we have to prove that \( \vec{\mu} = \vec{\lambda}(\vec{t}_0) \). Again, we assume the contrary, which means that there is an index, say, \( k = 1 \) such that \( \mu_1 \neq \lambda_1 \). From here we need to come to a contradiction. This part of the proof requires a single modification in the choice of the measure \( \eta \). Instead of the balayage of \( \lambda_1 \) onto \( \Gamma_2 \) in the case \( s = 2 \), we define \( \eta \) as the balayage of \( \lambda_1 \) to the union \( \bigcup_{k=2}^s \Gamma_k \). With this the proofs of Theorems 1 and 2 are complete.

4. Extremal compact set \( \vec{\Gamma}(\vec{f}) \) and associated Riemann surface \( \mathcal{R}(\vec{f}) \)

First we shall prove Lemma 2 on the existence of an extremal compact set \( \vec{\Gamma}(\vec{f}) \) maximizing the energy equilibrium functional \( E[\vec{F}] \) (see (4)–(8)) in the class \( \vec{F}(\vec{f}) \) associated with \( \vec{f} \). Then we prove that \( \vec{\Gamma}(\vec{f}) \) has the \( S \)-property (14).

As we noted in the Introduction, we in fact need the \( S \)-property of \( \vec{\Gamma}(\vec{f}) \), and Lemma 2 is just a convenient way of defining \( \vec{\Gamma}(\vec{f}) \). It was also noted that in the general situation this extremal compact set in the max-min energy problem will not have the \( S \)-property induced by the matrix \( A \) in (4). However, under the Angelesco condition this property holds, and it is a direct corollary of known results. We refer to [55], where the necessary references can be found.

The assertion of Lemma 2 requires some comment. Actually, the lemma is also a corollary of known facts and techniques but no single reference can be applied. In § 4.2 we include a few remarks explaining the reduction of the vector problem to a scalar one.

4.1. Proof of Lemma 2 and \( S \)-property of \( \vec{\Gamma}(\vec{f}) \). The key point in the proof of Lemma 2 is the continuity of the energy functional \( E[F] \) in the Hausdorff metric.
4.1.1. Hausdorff metric in a space of vector compact sets. To introduce a version of the vector Hausdorff metric on the set $\vec{F} = \vec{F}(f)$ of vector compact sets $\vec{F} = (F_1, \ldots, F_s)$ we employ the usual scalar Hausdorff metric. For two compact sets $F_1, F_2 \subset \mathbb{C}$ their Hausdorff distance $\delta_H$ is defined as

$$\delta_H(F_1, F_2) = \inf \{ \delta > 0 : F_1 \subset (F_2)_\delta, \ F_2 \subset (F_1)_\delta \},$$

where $(F)_\delta = \{ z \in \mathbb{C} : \min_{\zeta \in F} |z - \zeta| < \delta \}$ is the $\delta$-neighbourhood of $F$. An associated distance $d_H$ between two vector compact sets $\vec{F}^1$ and $\vec{F}^2$ with $s$ components is defined as follows:

$$d_H(\vec{F}^1, \vec{F}^2) = \sum_{k=1}^{s} \delta_H(\vec{F}^1_k, \vec{F}^2_k).$$

The properties of the metric space of vector compact sets are essentially the same as in the scalar case. In particular, the set of all vector compact sets in a closed disk $\overline{D}_R = \{ z : |z| \leq R \}$ of radius $R > 0$ is compact. The same is true for the class $\vec{F} \in \vec{F}_R$ which consists of vector compact sets $\vec{F} \in \vec{F}$ whose components are in $\overline{D}_R$.

4.1.2. Continuity of the vector equilibrium energy in the Hausdorff metric. We need only ‘pointwise’ continuity as stated in the following lemma.

**Lemma 8.** For any $\vec{\Gamma} \in \vec{F}$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $\vec{F} \in \vec{F}$ with $d_H(\vec{\Gamma}, \vec{F}) < \delta$

$$|\mathcal{E}[\vec{\Gamma}] - \mathcal{E}[\vec{F}]| < \varepsilon.$$

**Proof.** The assertion of the lemma is similar to Theorem 9.8 in [55]. More precisely, Theorem 9.8 is a scalar weighted version of Lemma 8. We have no external field in our case. Generalization of Theorem 9.8 to the vector situation is rather straightforward. We need to prove vector versions of Lemmas 9.4–9.6 in §9.4 in [55]. To do that we can, for instance, use componentwise balayage for vector measures as follows.

Let $\lambda$ and $\mu$ be the equilibrium measures of $\vec{\Gamma}$ and $\vec{F}$, respectively. Let $\mu'_k$ be the balayage of $\lambda_k$ to $F_k$ and let $\lambda'_k$ be the balayage of $\mu_k$ to $\Gamma_k$, $k = 1, \ldots, s$. Then we have the estimates

$$\mathcal{E}[\vec{\Gamma}] \leq \mathcal{E}(\vec{\lambda}) \leq \mathcal{E}[\vec{F}] + \varepsilon/2 \quad \text{and} \quad \mathcal{E}[\vec{F}] \leq \mathcal{E}(\vec{\mu}) \leq \mathcal{E}[\vec{\Gamma}] + \varepsilon/2.$$

The first inequality in each pair is the extremal property of the equilibrium measure. The second inequality in each pair is a vector version of Lemmas 9.4–9.6 in §9.4 in [55], which is obtained by taking the sum over the components.

4.1.3. Proof of Lemma 2. To complete the proof of Lemma 2 we consider a maximizing sequence $\vec{\Gamma}_n$ in the extremal problem (6), that is,

$$\mathcal{E}[\vec{\Gamma}_n] \to \mathcal{E} = \sup_{\vec{F} \in \vec{F}} \mathcal{E}[\vec{F}].$$
First we prove that any such sequence is bounded. It is enough to prove that the sequence of corresponding supports $\overline{\Gamma}_n$ remains bounded. We note that actually a maximizing sequence can be selected in the convex hull of $e = \bigcup e(f_k)$. We need only show that there is a finite positive $R$ such that

$$\overline{\Gamma}_n^1 \subset D_R = \{z : |z| \leq R\} \quad \text{for large enough } n,$$

assuming that $e = \bigcup e_k \subset D_1$.

For an arbitrary $\bar{\mathbf{F}} \in \overline{\mathcal{F}}$ we denote, as usual, by $\bar{\lambda} = \bar{\lambda}_{\bar{\mathbf{F}}}$ the extremal (equilibrium) measure in (6) and, further,

$$W_k(z) = U^{\lambda_k + \lambda}(z), \quad \lambda = \lambda_1 + \cdots + \lambda_s; \quad w_k = \min_{z \in F_k} W_k(z)$$

(see (12) and (13)).

The measure $\mu = \lambda_k + \lambda$ is a positive Borel measure in the plane with total mass $s + 1$. If $\mathcal{M}$ is the set of all such measures $\mu$, then the maximal value of $\min_{z \in F_k} U^\mu(z)$ on the space $\mathcal{M}$ is equal to $(s + 1) \log(1/\text{cap}(F))$ (and it is attained for $\mu = (s + 1)\omega$, where $\omega$ is the Robin measure of $F$). It follows that

$$w_k \leq (s + 1) \log \frac{1}{\text{cap}(F)} \leq C,$$

where $C$ is a constant depending on $e$ and independent of $\bar{\mathbf{F}} \in \overline{\mathcal{F}}$. Let $\bar{\mathbf{F}} = \overline{\Gamma}_n \in \overline{\mathcal{F}}$ be a member of a maximizing sequence. By the definition of $\overline{\mathcal{F}}$ each component of $\overline{\Gamma}_n^1$ consists of a finite number of continua, each containing at least two points of the corresponding $e_k$. Now if (for some $k$) $\Gamma \subset (\overline{\Gamma}_n^1)_k$ is such a continuum and there is a point $z \in \Gamma$ with $|z| \geq R$, then

$$\text{cap}(\Gamma) \geq \frac{R - 1}{4} \quad \text{and} \quad w_k \leq (s + 1) \log \frac{4}{R - 1}.$$

On the other hand, it follows from the definitions that for any $\bar{\mathbf{F}} \in \overline{\mathcal{F}}$ we have

$$\mathcal{E}[\bar{\mathbf{F}}] = w_1 + \cdots + w_s.$$

Thus, if the sequence $\Gamma_n$ is not bounded, then we have $\mathcal{E}[\overline{\Gamma}_n] \to -\infty$ (along some subsequence), and therefore $\mathcal{E} = \sup_{\bar{\mathbf{F}} \in \overline{\mathcal{F}}} \mathcal{E}[\bar{\mathbf{F}}] = -\infty$, which is a contradiction since $\mathcal{E}$ is evidently finite.

Finally, combining the assertions made above, we conclude that some subsequence of the minimizing sequence $\overline{\Gamma}_n$ converges in the Hausdorff metric to a vector compact set $\overline{\Gamma} \in \overline{\mathcal{F}}$ and by the continuity of the energy

$$\mathcal{E}(\overline{\Gamma}) = \mathcal{E}$$

(see (62)). This completes the proof of Lemma 2.

4.1.4. $S$-property of $\overline{\Gamma}$. Each component $\Gamma_k$ of the extremal compact set $\overline{\Gamma}$ defined by the extremal problem (8) is a solution of a scalar weighted problem if we assume
that the other components are fixed. For instance, with respect to variations of the first component \((k = 1)\) we have the following extremal property of \(\Gamma_1\):

\[
\mathcal{E}[\tilde{\Gamma}] = \mathcal{E}[(\Gamma_1, \Gamma_2, \ldots, \Gamma_s)] = \max_{F_1 \in \mathcal{F}(f_1)} \mathcal{E}[(F_1, \Gamma_2, \ldots, \Gamma_s)];
\]

and the same is true for \(k = 2, \ldots, s\) (follows directly from definitions). Also, the \(k\)th component \(\lambda_k\) of the vector equilibrium measure \(\tilde{\lambda}\) provides the minimum for the total energy in the class \(\mathcal{M}(\Gamma_k)\) when the other components are fixed. The equilibrium energy \(\mathcal{E}(\tilde{\lambda})\) as a function of the \(k\)th component \(\lambda_k\) can be represented as

\[
\mathcal{E}(\tilde{\lambda}) = \mathcal{E}_{\varphi_k}(\lambda_k) + C_k,
\]

where \(\varphi_k(z) = \frac{1}{2} \sum_{i \neq k} U^{i\lambda_i}\), and \(C_k\) does not depend on \(\lambda_k\). Under the Angelesco condition the external field \(\varphi_k\) is harmonic in a neighbourhood of \(\Gamma_k\) and the \(S\)-property of \(\tilde{\Gamma}\) with respect to the \(k\)th coordinate follows by Theorem 3.4 in [55].

### 4.2. Extremal compact set in the scalar case \(s = 1\).

A representation of \(\tilde{\Gamma}(\vec{f})\) as a whole or its components is an interesting and complicated problem. It is connected with many other problems, and taken together they constitute an area of research in classical complex analysis. We shall review a few particular results in this area which are related to the study of \(\tilde{\Gamma}(\vec{f})\) in the Angelesco situation. This case is much simpler than the general one and can essentially be reduced to the weighted scalar case \(s = 1\). Indeed, each component \(\Gamma_k\) is the scalar extremal compact set in the harmonic external field \(\varphi_k\) generated by the potentials of equilibrium measures of the other components.

We shall go into some details related to the case, beginning with the non-weighted situation, when Hermite–Padé polynomials become Padé polynomials, that is, the numerators and denominators of diagonal Padé approximants at infinity for a single functional element \(f = f_1 \in \mathcal{A}\) are complex orthogonal polynomials with weight independent of \(n\).

#### 4.2.1. Stahl’s theorem. Extremal compact set \(\Gamma(f)\).

In the case \(s = 1\) the definition (2) with \(p_n = -q_{n,0}\) and \(q_n = q_{n,1}\) becomes

\[
R_n(z) := (q_n f - p_n)(z) = O\left(\frac{1}{z^{n+1}}\right).
\]

The rational function \(\pi_n = p_n / q_n\) is the (diagonal) Padé approximant of order \(n\) to \(f\) at infinity.

One of the main problems in the theory of Padé approximants in the 1960s–70s was the convergence problem for functions \(f \in \mathcal{A}\). The problem is essentially equivalent to the problem of the zero distribution of the denominators \(q_n\) and it can be viewed as a particular case of the zero distribution problem for Hermite–Padé polynomials. The first results in this direction were obtained by Nuttall (for functions with quadratic branch points), who also made a general conjecture (see [47] and [49]): for any \(f \in \mathcal{A}\)

\[
\pi_n \xrightarrow{\mathrm{cap}} f, \quad z \in \mathbb{C} \setminus \Gamma(f), \quad \text{where} \quad \mathrm{cap}(\Gamma(f)) = \min_{F \in \mathcal{F}} \mathrm{cap}(F), \quad \Gamma(f) \in \mathcal{F},
\]
where the class $\mathcal{F}(f)$ of admissible cuts is same as above (see the Introduction) and $\overset{\text{cap}}{\longrightarrow}$ stands for convergence in capacity (on compact sets in the indicated domain).

A general theorem on the convergence of Padé approximants, including Nuttall’s conjecture, was proved by Stahl [59]–[63] (some of the methods used in this paper originated there). In particular, he proved the zero distribution formula

$$\frac{1}{n} \lambda(q_n) \rightarrow \lambda,$$

where $\lambda = \lambda_f$ is the Robin (equilibrium) measure of the extremal compact set $\Gamma(f)$. The (negative) equilibrium potential is (up to a constant) the Green’s function $g(z)$ for the complement of the extremal compact set:

$$w - U^\lambda(z) = g(z), \quad z \in \Omega = \overline{\mathbb{C}} \setminus \Gamma(f).$$

Stahl also found the rate of convergence of $\pi_n \overset{\text{cap}}{\longrightarrow} f$.

Now we consider in some detail the geometric component of the theorem. In other words, we are interested in the geometric structure of the minimal capacity compact set $\Gamma$. We restrict ourself to the case of finite sets $e$, as we generally do in this paper. The assumption of Stahl’s original theorem (see [62] and [63]) was that $\text{cap} e = 0$, which is essentially more general and accordingly less constructive. A characterization in terms of a quadratic differential (see below) is still valid, but becomes more complicated (the differential is not rational). The associated Riemann surface can also be introduced but will not be closed. Here we do not discuss the general case.

The minimum capacity property of $\Gamma(f)$ is equivalent to the maximal equilibrium energy, so that $\Gamma(f)$ is the exact scalar analogue of our vector compact set $\vec{\Gamma}$. Hence the study of $\Gamma(f)$ would be the first step in the study of the geometry of the vector compact set $\vec{\Gamma}(\vec{f})$. In the scalar case $s = 1$ without an external field there is a well-developed theory.

Many results of this theory can be generalized, in one way or another, to the weighted case, and then to the vector case $s > 1$ (for the weighted case see [27] and also [11], [12], [16], [18], [19]). However, generalizations are not always obvious and very often exist only as conjectures (see [64] and [65]). A theory for the vector case does not yet exist; there are several separated fragments. The case $s = 1$ may serve as a good introduction to the matter.

4.2.2. Extremal compact set $\Gamma(f)$ in terms of a quadratic differential. The minimal capacity problem in the class $\mathcal{F}(f)$ for finite $e(f)$ is a direct generalization of the classical Chebotarev problem of the continuum of minimum capacity containing a set $e$. The solution of the problem and its characterization in terms of critical trajectories of a quadratic differential has been well known since the 1930s (see references and details in [67]). The solution in the class $\mathcal{F}(f)$ is essentially similar and is presented in the following lemma.

**Lemma 9.** Let $f \in \mathcal{A}$, $e(f) = \{a_1, a_2, \ldots, a_p\}$, and

$$A(z) = (z - a_1)(z - a_2) \cdots (z - a_p).$$
Then there exists a polynomial
\[ V_f(z) = (z - v_1)(z - v_2) \cdots (z - v_{p-2}), \quad \text{where } v_j = v_j(f), \]
of degree \( p - 2 \) and depending on \( f \) such that the extremal compact set \( \Gamma(f) \) is a union of certain critical trajectories of the quadratic differential \( -(V/A)(dz)^2 \), where \( V = V_f \).

Moreover, \( -(V/A)(dz)^2 \) is a quadratic differential with closed trajectories.

For a proof see [59]; see also [67]. An alternative proof based on a ‘max-min’ energy problem was presented in [52]; see the review [55] for further details.

The connection of extremal compact sets with quadratic differentials is fundamental. In particular, it allows us to introduce a rich differential geometric context for the potential-theoretic max-min energy problem. From there we can obtain a number of equivalent reformulations of the problem (for instance, in terms of the moduli of families of curves) and this is a large source of methods (see [22], [34], [67], [40], [55] for a general discussion and for further references).

We shall make a few short remarks extending the assertions of Lemma 9 and showing broader content to some extent.

First we comment on a quadratic differential with closed trajectories. A trajectory is a (maximal) curve \( \gamma \) such that
\[ -\frac{V(z)}{A(z)} (dz)^2 > 0, \quad \text{or} \quad \text{Re} \int_a^z \sqrt{\frac{V(t)}{A(t)}} \, dt = \text{const}, \quad \text{on } \gamma. \]

Trajectories are also certain particular geodesics of the metric \(|V/A||dz|^2\) in the plane. All such curves are analytic. Critical trajectories are analytic arcs connecting two points in the set of zeros of \( AV \). For a quadratic differential with closed trajectories each non-critical trajectory is closed. In this case there is a finite signed (real) measure \( \lambda \) such that
\[ U^\lambda(z) = -\text{Re}\int_a^z \sqrt{\frac{V(t)}{A(t)}} \, dt \quad \text{or} \quad C^\lambda(z) := \int_\gamma \frac{d\lambda(x)}{z - x} = \sqrt{\frac{V(z)}{A(z)}}. \]

Any such measure \( \lambda \) is a critical point of the logarithmic energy functional with respect to local variations with fixed set \( e \) (a possibly signed critical measure). Also, the potential of any such measure is a constant on all connected components of the support.

For a given \( A \) there is a large set of polynomials \( V \) such that \( -(V/A)(dz)^2 \) is a quadratic differential with closed trajectories; such polynomials \( V \) are dense in \( \mathbb{P}_{p-2} \). Consequently, there is a large set of signed critical measures. For our current purposes positive critical measures are more important.

Given \( A \), there is a \((p - 2)\)-parameter family of polynomials \( V \) such that \( -(V/A)(dz)^2 \) is a quadratic differential with closed trajectories and the associated measure \( \lambda \) is positive. By using the zeros \( v_i \) as parameters, this family can be represented as a union of \( 3^{p-2} \) analytic bordered manifolds (cells). The polynomials \( V \) associated with Stahl’s compact sets \( \Gamma(f) \) are included in this
family (as corner points of cells). In this case the measure \( \lambda \) is the Robin measure of \( \Gamma(f) \) (see [40] for further details).

Next we consider some details related to the set of \( V \)-polynomials produced by extremal compact sets \( \Gamma(f) \) of all functions \( f \in \mathcal{A} \) with the same branch set \( e \) and hence associated with the same polynomial \( A \).

4.2.3. Chebotarev continuum. Set \( \widehat{V}_e \). Lemma 9 would serve as a constructive characterization of \( \Gamma(f) \) if we can determine the corresponding polynomial \( V(f) \). Here we briefly discuss a combinatorial component of the problem.

The polynomial \( V(f) \) depends, first of all, on the branch set \( e \) of the function \( f \in \mathcal{A} \). It also depends on the branch type of the function. We say that two functional elements at infinity \( f, g \in \mathcal{A}(\overline{\mathbb{C}} \setminus e) \) have the same branch type if after analytic continuation along any loop in \( \overline{\mathbb{C}} \setminus e \) either both elements \( f \) and \( g \) remain unchanged or both change. If \( f \) and \( g \) have the same branch type, then \( \Gamma(f) = \Gamma(g) \). The converse is not true, functions with different branch type can have the same extremal compact set, since not all the loops turn out to be important for the solution of the extremal problem.

Anyway, the dependence of \( \Gamma(f) \) on the branch type of \( f \) is reduced to a finite number of options. To make it formal, for a fixed \( e \) consider the set of all functions \( f \in \mathcal{A}(\overline{\mathbb{C}} \setminus e) \). Then the set of all associated extremal compact sets \( \Gamma(f) \) is finite, and the set

\[
\widehat{V}_e = \{ V(f) : f \in \mathcal{A}(\overline{\mathbb{C}} \setminus e) \}
\]

of corresponding polynomials \( V(f) \) is also finite (this is not entirely obvious, but is still a simple corollary of known results). As a remark, the number of elements in this set depends on the location of the points \( a \in e \). It is easy to calculate the maximum number \( m_p \) (for fixed \( p \)) when \( p = \#(e) \) is small; we have

\[
m_2 = m_3 = 1, \quad m_4 = 2, \quad \text{and} \quad m_5 = 3
\]

(in the case of general position \( m_p \) is equal to the actual number of elements in \( \widehat{V}_e \)). Starting with \( p = 6 \), this counting becomes more complicated. A general approach to this combinatorics can be based on an analysis of the Chebotarev continuum associated with the set \( e \).

For a (finite) set \( e \) the Chebotarev continuum \( \Gamma_e \) is the continuum of minimal capacity in the class of continua containing \( e \). The existence and characterization problem for such a continuum is known in geometric function theory as Chebotarev’s problem. It was solved independently by Groetsch and Lavrentiev in the 1930s (see [67] for details).

In particular, Lemma 9 has long been known for the solution of Chebotarev’s problem. The polynomial \( V = V_e \) corresponding to \( \Gamma_e \) is a particularly important element of the set \( \widehat{V}(e) \), since it can be used for the construction of the other Stahl compact sets associated with the same set \( e \).

The structure of the Chebotarev continuum \( \Gamma_e \) depends on the configuration of the points in \( e \) and in general also has a non-trivial combinatorial component. It is not our purpose here to present a complete analysis of the situation. Assume for simplicity that the polynomial \( V_e \) has simple zeros. This constitutes a case
of ‘general position’ for the points \( a_j \in c \) (configurations \( \{a_k\} \) not satisfying this condition have positive codimension). Then the continuum \( \Gamma_e \) is a union of \( 2p - 3 \) analytic arcs \( \gamma_k \):

\[
\Gamma_e = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{2p-3}.
\]

The interiors of the arcs \( \gamma_k \) are disjoint, and their endpoints belong to the set of roots of the polynomial \( A(z)V_e(z) \). Each \( a \in e \) is an endpoint of a single arc, and each root \( v \) of the polynomial \( V_e \) is a common endpoint of three arcs. We shall say that \( \gamma_i \) is an \( a-v \)-arc if one of its endpoints belongs to \( e \) and the other is a zero of \( V \). If both endpoints of an arc are zeros of \( V \), then we call it a \( v-v \)-arc (there are no \( a-a \)-arcs in a Chebotarev continuum). In total, \( \Gamma_e \) consists of \( p \) arcs of type \( a-v \) and \( p - 3 \) arcs of type \( v-v \). The graph (tree) generated by this collection of arcs can serve as a definition of the combinatorial structure of \( \Gamma_e \). If this structure is known, then the points \( v_i \) are uniquely defined by the system of equations

\[
\text{Re} \int_{\gamma_k} \sqrt{\frac{V(t)}{A(t)}} \, dt = 0, \quad k = 1, 2, \ldots, 2p - 4 \tag{63}
\]

(with a suitable choice of the orientation, the sum of the integrals over all \( 2p - 3 \) arcs is equal to \( \pi i \), so that the equality for \( k = 2p - 3 \) is a corollary of the equality for the other cases).

All the other elements of the set \( \hat{V}_e \) different from \( \Gamma_e \) can be obtained using the following procedure of ‘fusion’ of connected zeros of \( V_e \). We select any \( v-v \)-arc in the Chebotarev continuum and eliminate this arc from \( \Gamma_e \). This will divide \( \Gamma_e \) into two disjoint connected components and, as a corollary, we obtain a partition of \( e \) into two subsets, \( e = e_1 \cup e_2 \). Using this partition, we introduce a modified minimal capacity problem in the class of compact sets \( \mathcal{F} = F_1 \cup F_2 \), where \( F_i \) is a continuum containing \( e_i, \, i = 1, 2 \). The solution \( \bar{\Gamma} \) of this problem will have the form \( \bar{\Gamma} = \Gamma_1 \cup \Gamma_2 \) and it will be one of the Stahl compact sets associated with \( e \). The corresponding polynomial \( \hat{V} \in \hat{V}_e \) will have (at least) one double zero replacing the original \( v-v \)-pair. The other zeros of \( V \) can be put in correspondence with the remaining zeros of \( V_e \) (any polynomial \( V \in \hat{V}_e \) different from \( \hat{V}_e \) has multiple zeros). Then this operation can be repeated until no \( v-v \)-arcs remain.

It is also possible to describe the modification of the system (63) needed to pass to the corresponding system of equations defining the zeros of \( \hat{V} \in \hat{V}_e \); we mention only that the number of \( v \) parameters is now \( p - 3 \) (one of them is marked as a double root). Consequently we will have fewer equations by one.

A fusion of a pair \( v-v \) into a double zero can be carried out continuously, using ‘intermediate’ critical measures. More precisely, the finite set of Robin measures associated with Stahl’s compact sets is transformed into a finite-dimensional variety of \( e \)-critical measures (this method was presented in [41]). The zeros of the extended family of polynomials \( V \) play the role of coordinates on the family of critical measures, and the Chebotarev continuum \( \Gamma_e \) can be viewed as the origin of this coordinate system.

The system of equations (63) shows that, locally, the real quantities \( \text{Re} \, v_i \) and \( \text{Im} \, v_i \) are real-analytic functions of \( \text{Re} \, a_k \) and \( \text{Im} \, a_k \). It is known that the dependence \( v_i(a_k) \) is globally continuous.
4.2.4. Green’s function for the domain $\Omega = \mathbb{C} \setminus \Gamma(f)$. Next we review basic facts related to an alternative characterization of the compact sets $\Gamma(f)$ for $f \in \mathcal{A}$ in terms of $g$-functions of a certain family of hyperelliptic Riemann surfaces. A $g$-function can be introduced as a harmonic continuation of the Green’s function of the complement of $\Gamma(f)$ (with a pole at infinity). In turn, the Green’s function $g(z)$ is reduced to the equilibrium potential as follows:

$$g(z) = w - U^\lambda(z), \quad z \in \Omega = \mathbb{C} \setminus \Gamma(f), \quad (64)$$

where $w$ is the equilibrium constant such that $g = 0$ on $\Gamma$. The function $g$ is harmonic in the finite part of $\Omega$ and $g(z) = \log |z| + O(1)$ at infinity. These are the characteristic properties of the Green’s function of an arbitrary regular domain containing infinity.

For the Green’s function $g(z)$ of the complement of an extremal compact set $\Gamma(f)$ and also for the associated complex Green’s function $G(z) = g(z) + i\tilde{g}(z)$ explicit formulae follow from Lemma 9. If $V$ is the associated polynomial, then $G'(z) = \sqrt{V(z)/A(z)}$. Hence

$$g(z) = \text{Re} G(z) \quad \text{and} \quad G(z) = \int_a^z \sqrt{\frac{V(t)}{A(t)}} \, dt \quad (a \in \mathbb{C}) \quad (65)$$

(the branch of the root is such that $g(z) = \log |z| + O(1)$ at infinity). In particular, it follows that the extremal compact set $\Gamma$ has the $S$-property. In terms of the Green’s function $g$ this is stated as follows:

$$\frac{\partial g}{\partial n_1}(\zeta) = \frac{\partial g}{\partial n_2}(\zeta), \quad \zeta \in \Gamma^o. \quad (66)$$

Since $g(z) = w - U^\lambda(z)$, the same can be written for the equilibrium potential $U^\lambda$ (cf. (14)).

The $S$-property (66) plays an important role in Stahl’s approach to the asymptotics of complex orthogonal polynomials (Padé denominators). In terms of this property Stahl defined the extremal compact set $\Gamma(f)$. The compact set $\Gamma(f) \in \mathcal{F}(f)$ is the unique compact set with the $S$-property such that on any analytic arc in $\Gamma(f)$ the jump satisfies $(f^+ - f^-)(\zeta) \neq 0$.

Thus, in this case the definition in terms of the $S$-property is equivalent to the definition in terms of the max-min energy problem. The same is true in many other cases. A general approach to the existence problem for $S$-curves in the given class based on this equivalence is presented in [55]. We note, however, that the $S$-property is more general; it characterizes rather arbitrary critical points of the energy functional and not only (local) maxima of equilibrium energy.

4.2.5. Riemann surface $\mathcal{R}(f)$. Now from the domains $\Omega$ and their Green’s functions we move to Riemann surfaces and their $g$-functions. A characterization of $S$-compact sets in terms of $g$-functions was used by Nuttall [51], [49] in the particular case of functions $f$ with quadratic branch points. The case when the branch points are real goes back to Akhiezer [1]. A more general approach was outlined in [56]. Here we make a few remarks following the presentation in [56].
The Green’s function $g$ of the domain $\Omega$ has a harmonic continuation to the hyperelliptic Riemann surface $\mathcal{R} = \mathcal{R}(f)$ of the function $\sqrt{AV}$. We interpret $\mathcal{R}$ in the standard way as a (two-sheeted) branched covering over $\overline{\mathbb{C}}$ with canonical projection $\pi: \mathcal{R} \to \overline{\mathbb{C}}$. This fact follows directly from (65).

At the same time the assertion on the extension of $g$ from $\Omega$ to $\mathcal{R}$ can be derived from the $S$-property (66) combined with the boundary condition $g(z) = 0$ for $z \in \Gamma$. The first step of the proof is the construction of $\mathcal{R}$ based on a standard procedure of gluing $\mathcal{R}$ from two copies $\Omega_1$ and $\Omega_2$ of $\Omega$ closed by adding the set of accessible boundary points. Formally, such a closure can be defined by introducing the inner metric (the same in each copy)

$$\text{dist}(z, \zeta) = \inf \{ \ell(\gamma) : \gamma \subset \Omega_k, \ z, \zeta \in \gamma \}, \quad (67)$$

where $k = 1, 2$ and $\ell(\gamma)$ is the length of a curve $\gamma$. The closure of $\Omega_k$ in this metric,

$$\overline{\Omega}_k = \Omega_k \cup \Gamma_k^+ \cup \Gamma_k^- \quad (68)$$

contains two identical copies of the topological boundary $\partial \Omega_k = \Gamma_k$.

Next, we introduce the equivalence relation $\sim$ in the topological sum $\Sigma = \overline{\Omega}_1 \sqcup \overline{\Omega}_2$ by identifying $\Gamma_1^+$ with $\Gamma_2^-$ and $\Gamma_2^+$ with $\Gamma_1^-$ (each interior point is equivalent only to itself). Then we define $\mathcal{R} = \Sigma/\sim$ as the quotient space with respect to this equivalence. Local coordinates and the canonical projection $\pi(\zeta)$ are defined in the standard way. Thus, the construction of $\mathcal{R}$ is complete.

Next, starting with the Green’s function $g_\Omega(z)$ of $\Omega$ we define the $g$-function on $\mathcal{R}$:

$$g(\zeta) = g_\Omega(\pi(\zeta)), \quad z \in \Omega_1 \quad \text{and} \quad g(\zeta) = -g_\Omega(\pi(\zeta)), \quad z \in \Omega_2, \quad (69)$$

and $g(\zeta) = 0$ for any $\zeta \in \mathcal{R}$ with $\pi(\zeta) \in \Gamma$.

The continuity of $g$ on the finite part of $\mathcal{R}$ follows by the continuity of $g_\Omega(z)$ in $\mathbb{C}$. In addition to this, the $S$-property (66) combined with the definition of $g$ in (69) shows that the gradient of $g$ is continuous at finite points of $\mathcal{R}$ whose projections are not in $e$. It follows that $g$ is harmonic in $\mathcal{R} \setminus \pi^{-1}(\infty)$.

Independently of the Green’s function of a domain $\Omega$, the $g$-function of an arbitrary hyperelliptic Riemann surface $\mathcal{R}$ (not branched at infinity) is defined as a unique real-valued harmonic function $g(\zeta): \mathcal{R} \setminus \pi^{-1}(\infty) \to \mathbb{R}$ on the finite part of $\mathcal{R}$ with the following behaviour at infinity

$$g(\zeta) = \log |z| + O(1), \quad \zeta \to \infty^{(1)}, \quad z = \pi(\zeta), \quad (70)$$

$$g(\zeta) = -\log |z| + O(1), \quad \zeta \to \infty^{(2)},$$

and with normalization

$$g(z^{(1)}) + g(z^{(2)}) \equiv 0$$

(note that $g(z^{(1)}) + g(z^{(2)}) \equiv \text{const}$ according to (70)). The corresponding complex $G$-function is a third-kind Abelian integral with two marked points $\zeta_1 = \infty^{(1)}$ and $\zeta_2 = \infty^{(2)}$ and the divisor indicated in (70). The differential $dG(\zeta)$ is the associated third-kind Abelian differential.
For the two-sheeted Riemann surface associated with an extremal compact set \( \Gamma(f) \) the compact set itself is the projection of the zero level \( \{ \zeta : g(\zeta) = 0 \} \subset \mathcal{R} \) of the \( g \)-function on the plane \( \overline{\mathbb{C}} \). The associated complex function \( G(z) \) is a multivalued analytic function of \( \mathcal{R} \) with real part \( g(z) \). Formulae (65) remain valid.

Now we observe that the form of the \( g \)-function for \( \mathcal{R}(f) \) in (65) is different from the generic form of the \( g \)-function of a hyperelliptic Riemann surface. Indeed, consider the \( g \)-function associated with a generic hyperelliptic Riemann surface \( \mathcal{R} \).

Let the branch points of \( \mathcal{R} \) be the (distinct) zeros of a polynomial

\[
X(z) = \prod_{i=1}^{2p}(z - x_i)
\]

\((\deg X \text{ is even, so that } \infty^{(1,2)} \in \mathcal{R} \text{ are not branch points})\). The \( g \)-function of \( \mathcal{R} \) has the representation

\[
g(z) = \text{Re} G(z), \quad G(z) = \int_{x_1}^{z} \frac{Y(t)}{\sqrt{X(t)}} dt,
\]

(71)

where the polynomial

\[
Y(z) = \prod_{i=1}^{p-1}(z - y_i)
\]

is uniquely determined by \( X(z) \) (at this point we assume that the zeros of \( X \) are simple).

Using formula (65), we can define a map \( X \to Y \) from \( \mathbb{P}_{2p} \) to \( \mathbb{P}_{p-1} \). More precisely, this formula defines a map \( X \to Y \) for a polynomial \( X \) with simple zeros. Then this map can be extended by continuity to polynomials with multiple zeros. It is easy to verify that this extension has the following property. If \( x_0 \) is a zero of \( X \) of multiplicity \( 2m \) or \( 2m+1 \), where \( m \in \mathbb{N} \), then \( Y(z) \) has a zero of multiplicity \( m \) at \( x_0 \). The continuity of the map \( X \to Y \) is preserved in a neighbourhood of a polynomial \( X \in \mathbb{P}_{2p} \) with multiple zeros.

Next, we apply the map defined above to \( X = AV \), where \( A = z^p + \cdots \) is a fixed polynomial with simple zeros (which come from \( e(f) \)) and \( V(z) = \prod_{i=1}^{p-2}(z - v_i) \) is variable. This defines another map

\[
T: \mathbb{P}_{p-2} \ni V \to Y \in \mathbb{P}_{p-2}.
\]

(72)

Now we characterize \( \widehat{V}(e) \) in terms of \( T \).

**Theorem 10.** The set \( \widehat{V}(e) \) coincides with the set of fixed points of the map \( T \).

**Proof.** The fact that for any \( V \in \widehat{V}(e) \) we have \( T(V) = V \) follows directly from comparing (71) and (63).

Conversely, for any \( V(z) = z^{p-2} + \cdots \) with \( T(V) = V \) there exists an \( f \in \mathcal{A}(\overline{\mathbb{C}} \setminus e) \) with \( V = V(f) \). For instance, we can construct such an \( f \) as follows.

We know that the \( g \)-function of the Riemann surface of the function \( \sqrt{AV} \) has the form (65). Let \( \Gamma \) be the projection of the zero level of \( g \) on the plane. The set \( \Gamma \)
is compact, has the $S$-property, and contains $e' \subset e$ (there can be cancellations, but $e'$ contains at least two points). $\Gamma$ has a finite number of connected components which define a partition of $e'$ into groups of points belonging to the same connected component of $\Gamma$. The partition of $e'$ defines a factorization of the corresponding polynomial $A$ into a product of polynomials $A_i(z)$ whose zeros belong to the same connected component of $\Gamma$.

In turn, using this factorization of $A$, we define a function $f \in \mathcal{A}(\mathbb{C} \setminus e)$ by

$$f(z) = z^m \prod_{i=1}^{m} A_i^{-1/d_i}(z), \quad \text{where } d_i = \deg A_i.$$ 

The Stahl compact set $\Gamma(f)$ of the function $f$ is $\Gamma$. $\square$

In an equivalent form the assertion of Theorem 10 can be stated as follows. Given $A$ and any $V \in \tilde{V}(e)$, the corresponding Riemann surface $\mathcal{R}$ satisfies the following property. The derivative $G'(z)$ of the (complex) $G$-function for the Riemann surface $\mathcal{R}$ of the function $\sqrt{V/A}$ can only have poles in the set $A$. Conversely, any such polynomials $V$ belong to $\tilde{V}(e)$.

We can give a summary of the above considerations related to the case $s = 1$ as follows. Given a function $f \in \mathcal{A}(\mathbb{C} \setminus e)$, there exists a unique extremal compact set $\Gamma(f)$ whose Robin measure represents the limit zero distribution of the Padé polynomials for $f$.

By Lemma 9 this compact set is determined by a unique pair of polynomials $A(z)$ (representing $e(f)$) and $V = V(f) \in \tilde{V}(e)$; different $V(f)$ represent different branch types of functions $f$ with the same branch sets. Finally, the Riemann surface $\mathcal{R} = \mathcal{R}(f)$ of the function $\sqrt{V/A}$ and the corresponding $G$-function are also uniquely defined by $f$ (by way of the pair $A$, $V$).

This establishes a one-to-one correspondence between any two of the following three sets associated with a given set $e = \{a_1, \ldots, a_p\}$ of distinct points in the plane:

1) the set of $S$-compact sets $\Gamma(f)$ for all functions $f \in \mathcal{A}(\mathbb{C} \setminus e)$,
2) the set of polynomials $\tilde{V}(e)$,
3) the set of Riemann surfaces $\tilde{\mathcal{R}}(e) = \{\mathcal{R}(f): f \in \mathcal{A}(\mathbb{C} \setminus e)\}$.

In the last section, §4.4, of the paper we present a conjecture which generalizes these facts to the vector case $s = 1$ to a certain extent. It turns out that in this case it is more convenient to analyze the situation in terms of the set of Riemann surfaces generalizing the set $\tilde{\mathcal{R}}(e)$. For the Angelesco case the conjecture is supported by the main result of this paper.

4.2.6. Weighted case $s = 1$. Extremal compact set in an external field. As we have mentioned above, each component of the extremal vector compact set $\tilde{\Gamma}(\tilde{f})$ is itself a scalar extremal compact set in the external field $\varphi$ induced by the equilibrium measures of all other components. Thus, we can pass to the vector case using a weighted version of the scalar problem.

Generally speaking, the presence of the external field may create a new and significantly more complicated situation. However, in the study of the vector Angelesco case we only meet such a weighted situation when the effect of the presence of the external field is rather mild. In essence, all we have said above with regard
to the non-weighted case remains valid, possibly in a somewhat modified form. We briefly discuss what needs to be changed.

The setting for the weighted problem is the following. Together with functions \( f \in \mathcal{A}(\mathbb{C} \setminus e) \), where \( e = \{a_1, \ldots, a_p\} \), we consider a simply connected domain \( D \) containing \( e \) and a harmonic function \( \varphi(z) \) in \( D \).

For any (unit) measure in \( D \) we define the weighted energy \( \mathcal{E}_\varphi(\mu) \) in accordance with (24). For any compact set \( F \in \mathcal{F}(f) \) lying in \( D \) we define the (unit, weighted) equilibrium measure \( \lambda_F \) by (25).

Our primary assumption now is that for any \( f \in \mathcal{A}(\mathbb{C} \setminus e) \) there exists a compact set \( \Gamma = \Gamma(f, \varphi) \in \mathcal{F}(f) \) with

\[
\mathcal{E}_\varphi[\Gamma] = \max_{F(f) \in \mathcal{F}} \mathcal{E}_\varphi[F], \quad \text{where} \quad \mathcal{E}_\varphi[F] = \mathcal{E}_\varphi(\lambda_F). \tag{73}
\]

We shall eventually be interested in the case when \( \Gamma \) is one of the components of the Angelesco vector compact set \( \vec{\Gamma}(\vec{f}) \) and the field \( \varphi(z) \) is the potential of the other components of the vector equilibrium measure. In this case our assumption on the existence of an extremal compact set in (73) will become a corollary of the Angelesco condition. In general weighted settings such a compact set need not exist and is not necessarily unique if it exists (see [55] for details).

**Lemma 10.** Under the assumptions above on the extremal compact set \( \Gamma = \Gamma(f, \varphi) \in \mathcal{F}(f) \) the equilibrium measure \( \lambda = \lambda_\Gamma \) satisfies the condition

\[
B(z) := A(z) \left( \int \frac{d\lambda(x)}{x - z} + \Phi'(z) \right)^2 \in H(D), \tag{74}
\]

where

\[
\Phi(z) = \varphi(z) + i\tilde{\varphi}(z) \in H(D)
\]

is an analytic function in \( D \) with real part \( \varphi(z) \). Moreover, the extremal compact set \( \Gamma(f) \) is a union of certain critical trajectories of the quadratic differential \( -(B/A) (dz)^2 \).

For the proof see [40], where the representation of the lemma is obtained for critical (so-called \((A, \varphi)\)-critical) measures. The fact that the equilibrium measure of an extremal compact set \( \Gamma(f, \varphi) \) is critical is explained in [55].

The formal investigation of the combinatorial structure of \( \Gamma(f, \varphi) \) under general assumptions on \( \varphi \) is not the purpose of this paper. We are interested in the case related to the Angelesco vector equilibrium problem. Informally speaking, the Angelesco condition implies that the external field is harmonic in a neighbourhood of \( \Gamma \) and that it is small enough.

Under these conditions the combinatorial structure of \( \Gamma(f, \varphi) \) remains similar to the structure of \( \Gamma(f) = \Gamma(f, 0) \) and, in particular, exactly \( p - 2 \) zeros of \( B \) are involved in this structure. To make this more explicit assume that \( f \) has a generic branch type, that is, \( \Gamma(f, \varphi) \) is a continuum. Then there exists a neighbourhood of this compact set where \( B \) has exactly \( p - 2 \) zeros and each of them is an endpoint of one of the analytic arcs constituting \( \Gamma(f, \varphi) \). The number of arcs is at most \( 2p - 3 \) and it is \( 2p - 3 \) if all these zeros of \( B \) are simple.
For a formal proof of the last statement it is more convenient to use Riemann surfaces and their $G$-functions (in particular, the quadratic differential $(dG)^2$ is the appropriate analogue of $(V/A)(dz)^2$). Here we mention briefly a homotopic argument which can also be used for formal proofs. Assume again that $\Gamma(f, \varphi)$ is a continuum. We introduce a parameter $t \geq 0$ and consider the family of vector fields $t\varphi(z)$ and the corresponding compact sets $\Gamma_t = \Gamma(f, t\varphi)$. Let $B_t(z)$ be the corresponding $B$-function in (74). For $t = 0$ we are back in the non-weighted case and know that
\[ B_0(z) = V(z) \in \mathbb{P}_{p-2}. \]

Thus, $B_0(z)$ has exactly $p - 2$ zeros in a simply connected neighbourhood $D$ of $\Gamma$. If for any $\tau \in [0, t]$ the compact set $\Gamma_\tau$ belongs to the domain of harmonicity of the external field, then the zeros of $B$ are still defined by the system of equations (63) with $V$ replaced by $B$. It follows from these equations that short trajectories of the quadratic differential $-(B/A)(dz)^2$ constituting $\Gamma_\tau$ are preserved and change continuously as functions of $\tau \in [0, t]$. Thus, $\Gamma_\tau$ and, in particular, the zeros of $B_\tau(z)$ depend on $\tau$ continuously (actually this dependence is real-analytic). Note that the combinatorial structure of $\Gamma_\tau$ can change (bifurcation points are values of $\tau$ where $B$ has multiple zeros) but the number of parameters involved in the structure remains constant.

Using the above arguments, we can calculate the number of parameters involved in the structure of the vector compact set $\vec{\Gamma}(\vec{f})$.

4.3. Riemann surface $R(\vec{f})$. The asymptotics of the Angelesco Hermite–Padé polynomials associated with the vector function $\vec{f} \in A$ in (1) can be described in terms of the $g$-function of an algebraic Riemann surface $R = R(\vec{f})$. First, we introduce the $g$-function associated with a generic Riemann surface defined as a branched covering of the sphere.

4.3.1. $G$-function of a Riemann surface. Let $R$ be an arbitrary algebraic Riemann surface defined as a branched $(s+1)$-sheeted covering of $\mathbb{C}$ with canonical projection $\pi: R \to \mathbb{C}$. Assume that the elements of the set
\[ \pi^{-1}(\infty) = \{\infty^{(i)}, \ i = 0, 1, \ldots, s\} \]
are distinct. We fix one of them and denote it by $\infty^{(0)}$. Then there exists a unique function $g(\zeta): R \to \mathbb{R}$ with the following properties. The function $g$ is harmonic on the finite part of $R$, we have
\[ g(\zeta) = -s \log |z| + O(1), \quad \zeta \to \infty^{(0)}, \]
\[ g(\zeta) = \log |z| + O(1), \quad \zeta \to \infty^{(i)}, \quad i = 1, \ldots, s, \]
where $z = \pi(\zeta)$, and finally, $g$ is normalized by the condition
\[ \sum g(z^{(i)}) = 0. \]

We call $g$ the (real) $g$-function of $R$ (with one marked point $\infty^{(0)}$).

Together with the real $g$-function we define the complex one $G = g + i\tilde{g}$ so that $g = \text{Re} G$. The function $G(z)$ is a multivalued analytic function on $R$; equivalently,
$G$ is a third-kind Abelian integral with poles at infinities and the divisor indicated in (75) above.

In many instances it is convenient to identify the coordinate $\zeta \in \mathcal{R}$ with its projection $z = \pi(\zeta)$. We shall use this identification when it cannot lead to ambiguities. The notation $z^{(k)}$ (for certain elements of $\pi^{-1}(z)$) can be used if a numbering of sheets is already defined or irrelevant.

The derivative $G'(z)$ of the complex $G$-function is a meromorphic (rational) function on $\mathbb{R}$. In other words $G'$ is an algebraic function. Recall that in the case $s = 1$ we have $G' = \sqrt{V/A}$, so that $w = G'$ is a solution of the quadratic equation $Aw^2 - V = 0$, where $A$ is a polynomial with roots at branch points and $V$ is another polynomial which can be determined.

Not much is known so far for the case $s > 1$. There are a few isolated results showing the situation is much more complicated. Next we discuss the Angelesco case, which is in many ways simpler than the general one.

4.3.2. Riemann surface: existence theorem. In the Angelesco case the Riemann surface $\mathcal{R}$ associated with the extremal vector compact set $\vec{\Gamma} = \vec{\Gamma}(\vec{f})$ can be constructed for a given vector compact set $\vec{\Gamma}$ with the $S$-property by using a procedure quite similar to the construction of the hyperelliptic Riemann surface $\mathcal{R}(f)$ described in §4.2.5 above (in place of the Green’s functions we have to use equilibrium potentials). We state the final result of this procedure as a theorem.

**Theorem 11.** Let $\vec{\Gamma} = \vec{\Gamma}(\vec{f})$ be the extremal vector compact set associated with the vector function $\vec{f} \in \mathcal{A}$ and assume that the Angelesco condition is satisfied for $\vec{f}$. Let

$$
\lambda = \lambda_1 + \cdots + \lambda_s, \quad \text{where} \quad \vec{\lambda}_\Gamma = (\lambda_1, \ldots, \lambda_s) \quad \text{and} \quad \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s.
$$

Then the total equilibrium potential $U^\lambda(z)$ has a harmonic continuation from the domain $\Omega = \overline{\mathbb{C}} \setminus \Gamma$ to a Riemann surface $\mathcal{R}$ which is an $(s + 1)$-sheeted branched covering of the sphere with canonical projection $\pi: \mathcal{R} \to \overline{\mathbb{C}}$. Moreover, if

$$
\pi^{-1}(\Omega) = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_s,
$$

then

$$
U^{\lambda}(\pi(\zeta)) = g(\zeta) + C, \quad \zeta \in \Omega_0; \quad w_k - U^{\lambda_k}(\pi(\zeta)) = g(\zeta) + C, \quad \zeta \in \Omega_k,
$$

where $k = 1, \ldots, s$ and the (disjoint) domains $\Omega_k \subset \mathcal{R}$ are numbered so that the projection of $\partial \Omega_k \subset \partial \Omega_0$ is $\Gamma_k^1 = \text{supp}(\lambda_k)$ ($\Omega_0$ and $\Omega_k$ are connected through $\Gamma_k^1$).

**Proof.** The construction of $\mathcal{R}$ is based on the equilibrium conditions and the $S$-properties of the extremal compact set (see (12) and (13)) and uses the same standard procedure of gluing $\mathcal{R}$ from the (closures of the) $s + 1$ plane domains

$$
\Omega_k = \overline{\mathbb{C}} \setminus \Gamma_k^1, \quad k = 1, \ldots, s, \quad \text{and} \quad \Omega_0 = \overline{\mathbb{C}} \setminus \bigcup \Gamma_k^1
$$

(the domain $\Omega_0$ was used in §4.2.5, in the case $s = 1$). The closure $\overline{\Omega}_k$ is defined on the basis of the inner metric in $\Omega_k$ in same way as in (67) and (68), where in place of $\Gamma_k$ we use $\Gamma_k^1$ for positive $k$ and $\Gamma$ for $k = 0$. 

Then, again as in the case $s = 1$ we introduce the equivalence relation $\sim$ in the topological sum

$$
\Sigma = \prod_{k=0}^{s} \overline{\Omega}_k,
$$

joining the copy of $(\Gamma_k^1)^+$ on the zeroth sheet with the copy of $(\Gamma_k^1)^-$ on the $k$th sheet and vice versa in the standard way. Finally, we define $\mathcal{R} = \Sigma/\sim$ as the quotient space with respect to this equivalence. Local coordinates and the structure of a branched covering over $\mathbb{C}$ are introduced on $\mathcal{R}$ in a standard way.

To prove (76) we use the equilibrium conditions (13) and the $S$-property (14) for $\vec{\Gamma}$:

$$
W_k(z) := U_{\lambda_k}^+(z) = w_k, \quad z \in \Gamma_k^1, \quad k = 1, \ldots, s,
$$

$$
\frac{\partial W_k(z)}{\partial n_1} = \frac{\partial W_k(z)}{\partial n_2}, \quad z \in (\Gamma_k^1)^c, \quad k = 1, \ldots, s
$$

(the second equality holds at the interior points of arcs in $\Gamma_k^1$). Equivalently, we have

$$
U_{\lambda}(z) = w_k - U_{\lambda_k}(z), \quad z \in \Gamma_k^1, \quad k = 1, \ldots, s,
$$

$$
\frac{\partial}{\partial n_{1,2}} U_{\lambda}(z) = \frac{\partial}{\partial n_{2,1}} (w_k - U_{\lambda_k}(z)), \quad z \in (\Gamma_k^1)^c, \quad k = 1, \ldots, s.
$$

It follows, first, that the function $U(z)$ defined by

$$
U(z) = U_{\lambda}(z), \quad z \in \Omega_0, \quad \text{and} \quad U(z) = w_k - U_{\lambda_k}(z), \quad z \in \Omega_k,
$$

has a continuous extension to the whole of the Riemann surface $\mathcal{R}$. Moreover, $U(z)$ is continuously differentiable on the open analytic arcs in $\Gamma_k^1$. Hence $U$ is harmonic in the finite part of $\mathcal{R}$. Finally, the asymptotics of $U$ at the infinities is the same as that of $g$ in (75). Therefore $g(z) = U(z) + C$ on $\mathcal{R}$. The theorem is proved.

In terms of the $G$-function the main Theorem 1 of the paper can be stated as follows.

**Theorem 12.** Under the assumptions of Theorem 1 the following convergence is valid for $k = 1, \ldots, s$ as $n \to \infty$:

$$
C_{n,k}(z) = \frac{1}{n} \frac{Q'_{n,k}(z)}{Q_{n,k}(z)} = \frac{d\mu_{n,k}(t)}{z - t} \to C_{\lambda_k}(z) = \frac{d\lambda_k(t)}{z - t} = G'(z), \quad z \in \Omega_k,
$$

in the plane Lebesgue measure $m_2$ and in capacity on compact sets in $\Omega_k = \mathbb{C} \setminus \Gamma_k^1$. (77)

The support $\Gamma_k^1$ of the measure $\lambda_k$ is a finite union of analytic arcs. In the complementary domains $\Omega_k = \mathbb{C} \setminus \Gamma_k^1$ each function $w = C_{\lambda_k}(z)$ is a branch of the derivative $G'$ of the $G$-function for $\mathcal{R}$.

The next theorem is also related to Theorem 1, but it is not an immediate corollary of this theorem.
Theorem 13. Under the assumptions of Theorem 1, for any branch of the remainder \( R_n \) in any simply connected domain \( D \subset \mathbb{R} \),
\[
\frac{1}{n} \frac{R_n'(ζ)}{R_n(ζ)} \xrightarrow{m_2} G'(ζ) \quad \text{and} \quad \frac{1}{n} \log |R_n(ζ)| \xrightarrow{\text{cap}} g(ζ), \quad z \in D,
\] (78)
as \( n \to \infty \) in the plane Lebesgue measure \( m_2 \) and in capacity on compact sets in \( Ω_k = \overline{C} \setminus Γ_k^1 \).

A proof of this theorem requires at least a generalization of part (ii) of Theorem 9 to the case of Hermite–Padé polynomials. Actually, Theorem 13 contains more than that, since part (ii) of Theorem 9 asserts convergence only on the main sheet of the Riemann surface. The complete proof of Theorem 13 will take significant additional effort and we do not present it in this paper.

4.3.3. Riemann surface: algebraic equation. As a corollary of Theorem 12, we derive an algebraic equation whose solution is the derivative \( G' \) of the complex \( G \)-function. We state a theorem for the case \( s = 2 \) since it contains all the important details and at the same time it is much less cumbersome.

Theorem 14. Let \( \vec{f} = (f_1, f_2) \in \mathcal{A} \) be a vector function and assume that the Angelesco condition is satisfied for \( \vec{f} \). Let \( G(z) = G(z; \vec{f}) \) be the \( G \)-function for the associated Riemann surface \( \mathcal{R}(\vec{f}) \). For \( j = 1, 2 \) let \( A_j(z) = z^{p_j} + \cdots \) be polynomials with roots at the branch points of \( f_j \) (so that \( p_j = \#(e_j) \) and \( p = p_1 + p_2 \)).

Then there exist polynomials
\[
E(z) = z^{p-2} + \cdots \quad \text{and} \quad F(z) = z^{p-3} + \cdots
\]
such that the derivative \( w = G'(z) \) of \( G(z; \vec{f}) \) is defined by
\[
A(z)w(z)^3 - 3E(z)w(z) + 2F(z) = 0.
\] (79)

Moreover, all zeros of the polynomial \( F \) are zeros of \( G' \), and this polynomial can be represented in the form
\[
F(z) = V_1(z)V_2(z)B(z),
\]
where
\[
\deg V_1 = p_1 - 2, \quad \deg V_2 = p_2 - 2, \quad \text{and} \quad B(z) = z - b.
\]

Zeros of \( V_1 \) and \( V_2 \) are identified with zeros of the quadratic differential whose trajectories constitute the components of the extremal compact set \( \vec{Γ} \).

In the particular case when \( p_1 = p_2 = 2 \) the equation (79) was presented in [5] and plays a fundamental role in this paper.

Proof. Let \( \vec{Γ}(\vec{f}) = (Γ_1, Γ_2) \) be the extremal vector compact set for \( \vec{f} \) and let \( \vec{λ} = (λ_1, λ_2) \) be the corresponding equilibrium measure. By
\[
C^μ = \int (z - x)^{-1} \, dμ(x)
\]
we denote the Cauchy transform of the measure $\mu$. The Function $C'$ is also the derivative of the complex potential of $\mu$. By Theorem 12 the three branches of

$$G'(z) = C^\lambda_k(z), \quad z \in \Omega_k = \overline{\mathbb{C}} \setminus \Gamma_k,$$

$$G'(z) = -C^\lambda(z), \quad z \in \Omega = \overline{\mathbb{C}} \setminus \Gamma,$$

(80)

where $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\lambda = \lambda_1 + \lambda_2$.

Let

$$P(w, z) = w^3 + r_1(z)w(z) + r_0(z)$$

be the polynomial in $w$ with rational coefficients $r_j(z)$ which is associated with the algebraic function $G'$, that is, $P(G'(z), z) = 0$. It follows from (80) that

$$P(w, z) = (w - C^\lambda_1(z))(w - C^\lambda_2(z))(w + C^\lambda(z))$$

(81)

and therefore

$$r_0(z) = C^\lambda_1(z)C^\lambda_2(z)C^\lambda(z) \quad \text{and} \quad r_1(z) = -C^\lambda_1(z)C^\lambda_2(z) + (C^\lambda(z))^2. \quad (82)$$

Each function $C^\lambda_k(z)$ has a singularity of the form $c(z)(z-a)^{-1/2}$ at any $z = a \in e_k$, where $c(z)$ is analytic and $\neq 0$ at $z = a$; the same is true for $C^\lambda(z)$ and any $a \in e_1 \cup e_2$. It follows that each function $r_j(z)$ has at most a simple pole at any $a \in e_1 \cup e_2$. On the other hand, as $z \to \infty$ we have

$$zC^\lambda_k(z) \to 1, \quad k = 1, 2, \quad \text{and} \quad zC^\lambda(z) \to 2.$$

Hence

$$r_0(z) = 2 \frac{2}{z^3} + O\left(\frac{1}{z^4}\right) \quad \text{and} \quad r_1(z) = \frac{3}{z^2} + O\left(\frac{1}{z^3}\right),$$

and therefore

$$r_0 = \frac{2F}{A} \quad \text{and} \quad r_1 = \frac{3E}{A},$$

where $F$ and $E$ are some monic polynomials of degrees $p - 3$ and $p - 2$, respectively. Thus, after multiplication by $A$ the polynomial $P(w, z)$ can be reduced to a polynomial in the two variables $w$ and $z$ of the form indicated in (79).

It follows from (79) that the projections of zeros of the algebraic function $w = G'$ coincide with zeros of the polynomials $F(z) = z^{p-3} + \cdots$. The function $G'(\zeta)$ is a rational function of order $p$ on $\zeta \in R$ (assuming that there are no cancellations), and therefore should have $p$ zeros on $R$. Of these, it has $p_k - 2$ zeros at zeros of the polynomial $V_k$ associated with $\Gamma_k$, where $k = 1, 2$. It has a simple zero at each copy of $\infty$ and this accounts for $p - 1$ zeros. One more zero remains, whose projection on the plane we have denoted by $b$. □

4.4. Subclass $\mathcal{L} \subset \mathcal{A}$. Laguerre-type equation. Conjecture. There are several subclasses of the class $\mathcal{A}$ which can be seen as generalizations of classical weights (or their Cauchy transforms). The corresponding Padé and Hermite–Padé polynomials are often called semiclassical, since certain properties of classical orthogonal polynomials are preserved for them. An important example is the class $\mathcal{L}$
defined as follows. For a fixed set \( e = \{a_1, \ldots, a_p\} \) of \( p \geq 2 \) distinct points we denote by \( \mathcal{L}_e \) the class of functions \( f \) of the form

\[
\mathcal{L}_e = \left\{ f(z) = f(z; \alpha) := \prod_{j=1}^{p} (z - a_j)^{\alpha_j} : \alpha_j \in \mathbb{C} \setminus \mathbb{Z} \right\}. \tag{83}
\]

We also assume that \( \sum_{j=1}^{p} \alpha_j = 0 \) and the branch at infinity is selected by the normalization \( f(\infty) = 1 \). It is clear that \( \mathcal{L}_e \subset \mathcal{A}(\mathbb{C} \setminus e) \). Let

\[
\mathcal{L} = \bigcup \mathcal{L}_e
\]

be the union of the classes \( \mathcal{L}_e \) for all sets \( e \). We have \( \mathcal{L} \subset \mathcal{A} \). Functions \( f \in \mathcal{L} \) are sometimes called generalized Jacobi functions.

An important property of this class is that for a fixed \( e \) it contains functions with an arbitrary branch type, so that the family of extremal cuts associated with functions \( f \in \mathcal{L}_e \) is the same as the corresponding family for functions \( f \in \mathcal{A}_e \).

Another important fact is that the corresponding Hermite–Padé polynomials satisfy a Laguerre-type differential equation with polynomial coefficients. More precisely, let

\[
\vec{f} = (f_1, \ldots, f_s), \quad f_k \in \mathcal{L},
\]

and let \( q_{k,n}, k = 0, 1, \ldots, s, \) be the associated Hermite–Padé polynomials. Then for any \( n \in \mathbb{N} \) each of the \( s + 1 \) functions

\[
w = q_{0,n}, q_{1,n}f_1, \ldots, q_{s,n}f_s
\]

satisfies a linear differential equation

\[
\Pi_{s+1}(z)w^{(s+1)} + \Pi_s(z)w^{(s)} + \cdots + \Pi_1(z)w' + \Pi_0(z)w = 0, \tag{84}
\]

where the coefficients \( \Pi_k(z) = \Pi_{k,n}(z) \) are polynomials depending on \( n \) whose degrees are uniformly bounded by a number depending only on the numbers of branch points of the component functions \( f_k \).

This equation was derived in [44], where some corollaries on asymptotics were obtained for simple particular functions \( f \in \mathcal{L} \) with a few branch points. After [44] was published, a general conjecture was developed by this author, which we present here in a somewhat abbreviated form. The two assertions below, if proved, would create a general basis for the investigation of the asymptotics of Hermite–Padé polynomials.

**Conjecture.** Let \( \vec{f} = (f_1, \ldots, f_s) \) and \( f_k \in \mathcal{A} \). Let \( A_k(z) \) be the monic polynomial with zeros at the branch points of \( f_k \) and let

\[
A(z) = \prod_{k=1}^{s} A_k(z).
\]

Then there exists a finite set \( \hat{\mathcal{R}} = \hat{\mathcal{R}}(A) \) of algebraic Riemann surfaces \( \mathcal{R} \) depending only on \( A \) which satisfy the following two conditions:
1) the projections of branch points of a Riemann surface $R$ belong to the set of zeros of the polynomial $AV$, where $V(z)$ is a polynomial of degree $\deg A - 2$;  
2) the projections of the poles of the derivative $G'$ of the complex $G$-function of $R$ belong to the set of zeros of $A$.

Moreover, given $\vec{f} = (f_1, \ldots, f_s)$, there is a unique Riemann surface $R \in \hat{\mathcal{R}}$ such that for any (properly normalized) branch of the remainder $R_n$ associated with $\vec{f}$ the sequence of functions $(1/n) \log |R_n(z)|$ converges in capacity to the $g$-function of $R \in \hat{\mathcal{R}}$ in any domain $\mathcal{D} \subset R$, provided that $R_n$ has a single-valued branch in $\mathcal{D}$.

In the Angelesco case a part of this conjecture is proved in this paper. It follows from the results above that in the Angelesco case we can go further and give a complete description of $R(\vec{f})$. In the general case such a complete description is only known for a number of particular cases, and it contains many details which we shall not discuss here (see [5]).

The convergence of the sequence of functions $(1/n) \log |R_n(z)|$ in capacity in the Angelesco case is stated above as Theorem 13. As mentioned above, Theorem 13 can be proved by methods used in this paper, but a complete proof would require considerable additional efforts and go beyond our framework.

In the case of general assumptions on the configuration of branch points the conjecture is not yet proved, even for functions in $\mathcal{L}$ (see (83)). However, it seems that all the necessary tools are available if the methods of [44] are combined with the method presented in [56] and the current paper. We include a few comments, which outline the main steps of the method which can be used to prove the conjecture for the class $\mathcal{L}$.

First, we make the Riccati substitution

$$u_n = \frac{w'_n}{nw_n}$$

in the equation (84), which reduces this equation to a non-linear differential equation with rational coefficients depending on $n$. The equation obtained can be normalized in such a way that the families of coefficient functions become compact sets with respect to uniform convergence in the spherical metric. This makes the family of solutions a compact set with respect to convergence in the plane measure $m_2$. In particular, any sequence of functions $R'_n/(nR_n)$ has an $m_2$-convergent subsequence, and the same is true for the Cauchy transforms $C_{n,k}$ of the zero-counting measures for the Hermite–Padé polynomials in (77).

It is comparatively easy to prove that any limit equation is algebraic. The original differential equation was obtained as an equality to zero of a certain Wronskian. The Riccati substitution reduces this Wronskian to a Vandermond determinant whose elements are convergent in the measure $m_2$. This proves that the $m_2$-limit $C_k(z)$ of any convergent subsequence of the sequences $C_{n,k}$ has the following property: there exists a polynomial in two variables $P(w, z)$ (independent of $k$) such that

$$P(z, C_k(z)) = 0 \quad \text{for } m_2\text{-almost all } z \text{ in the plane.}$$

There is a well-known and partially proved conjecture (see [14] and [15]) that the equality $P(z, C^\mu(z)) = 0$ for $m_2$-almost all $z$ in the plane for a Cauchy transform of
a positive measure $\mu$ implies that this measure is supported on a finite number of analytic arcs. If proved, this would allow us to assert that any convergent subsequence of the sequence $R_n/(nR_n)$ converges in the plane Lebesgue measure to an algebraic function $F$ which is meromorphic on the algebraic Riemann surface $\mathcal{R}$ and such that $F = G'$, where $G$ is the complex $G$-function for $\mathcal{R}$.

To complete the proof of the conjecture for the class $\mathcal{L}$ we need to prove that $\mathcal{R}$ does not depend on the subsequence. This will be a corollary of the uniqueness of the Riemann surface $\mathcal{R}$ introduced in the conjecture. This part of the problem may be more difficult since it is related to non-trivial combinatorics, which we shall not discuss here.

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