Dynamics of charged particles and QPO of related disk models in the background of a distorted, deformed compact object- I. embedded in a uniform magnetic field

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This work presents the dynamic properties of charged test particles influenced by the gravitational and electromagnetic fields. Accordingly, we concentrate on the simplest static and axially symmetric metric in this work, which contains two quadrupole parameters. One relates to the central object, and another relates to the distortion parameter, which is linked to the external distribution of matter. This metric may associate the observable effects to these parameters as dynamical degrees of freedom. The astrophysical motivation for choosing such fields is the possibility to constitute a reasonable model for a real situation occurring in these objects’ vicinity. To test the role of large-scale magnetic fields in accretion processes, we start by analyzing different bound orbits of timelike orbits under the influence of the system’s different parameters. In particular, studying the influence of the strength of the magnetic field. This leads to the examine their stability concerning radial and/or vertical oscillations. The final goal is to discuss the oscillation modes’ resonant phenomena using different resonant models for disk-oscillation modes.

I. INTRODUCTION

There is no doubt on the role of the magnetic fields in the study of astrophysical systems. In the processes occurring in the vicinity of compact objects, the magnetic fields can manifest their fingerprint in many ways. For example, the local magnetic field in the thin accretion disk is assumed to be the source of viscosity in the accretion process in MRI simulations [1]. In fact, many black hole candidates are assumed to have an accretion disk forming from conducting plasma which dynamics can produce magnetic fields [2–4].

An external magnetic field at a large distance in a finite region can be approximated as a uniform magnetic field [5, 6]. Such a large-scale magnetic field could be initiated during the early phases of the expansion of the Universe [7–11]. Also, a compact object near the equatorial plane of a magnetar can be approximated to be in a uniform magnetic field if the magnetar is at a distance large enough [12, 13]. Besides, the motion in the gravitational field combined with an external electromagnetic field is a large variety of studies for example in [14–17]. Also, the energy from a collision of the charged particles in these systems can cause particles to accelerate [18, 20].

Besides, the fundamental frequencies of the motion of charged particles can model important astrophysical phenomena of twin peaks in the Fourier power spectra link to the pairs of the upper and lower frequencies (νU, νL) in high-frequency quasi-periodic oscillations (QPOs) observed in several the Low-Mass X-Ray Binaries (LMXBs) containing a black hole, a neutron star, or an active galactic nucleus [21–28]. Also, recently the source went through Burst Alert Telescope (BAT) onboard Swift [29, 30]. Since the peaks of high frequencies are close to the orbital frequency of the marginally stable circular orbit representing the inner edge of Keplerian disks.

Among many models serving to explain QPOs in the past years is the Relativistic Precession Model (RPM), which relates the twin-peak QPOs to the Keplerian and periastron precession frequency on an orbit located in the inner part of the accretion disk [31, 34].

Although, in general, the correlation between these frequencies is qualitatively well-fitted by the RPM model prediction, during these years, this model modified in many ways [35–36]. Regarding this modification, the high frequencies QPOs (two-picks) are considered as the resonances between oscillation modes of the well-known ratio 3 : 2 epicyclic resonance model, where identify the resonant frequencies with frequencies of radial and vertical epicyclic axisymmetric modes of disk oscillations [37, 38]. This is worth mentioning that the oscillations occur only in certain states of luminosity and hardness [22–28, 39], and this phenomenon is not universal [39]. For a review on this subject, see for example [41–42].

The properties of these fundamental frequencies have been extensively studied in particle motion in different backgrounds with a uniform magnetic field [6, 13, 43–51] among many others.

We explore these frequencies’ properties in the background of a distorted, deformed compact object with a relatively weak uniform magnetic field that does not affect the spacetime curvature in the vicinity of the compact object. This metric is the generalization of the so-called Q-metric up to quadrupole moments [52]. This metric has two parameters, aside from the mass of the central object, namely distortion parameter β and deformation parameter α, which are not independent of each other. We explain briefly about this metric in Section [1]. In this respect, the first static and axially symmetric solution with arbitrary quadrupole moment are described by [53]. Then [54] introduced a static solution with arbitrary quadrupole in prolate spheroidal coordinates. Later [55], and [56] found an equivalent transformation that leads to a simple solution and is known as Q-metric. This area of study has been discussed extensively in the literature [57–62], among many others.

Due to this approximation, the effects of the electromagnetic field on the stress-energy tensor are neglected in this work. Also, we assume the QPOs are caused by the fundamental epicyclic frequencies associated with the orbital motion of the matter in the accretion disk also their combinations that with the help of parameters in the metric and magnetic field one has more possibility to explain observed QPOs fre-
quencies with the same $3:2$ ratio [33, 35, 63]. Furthermore, using the Hamiltonian formalism of the charged particle dynamics, we examine bound orbits with the effective potential of gravitational field combined with the uniform magnetic field. We are particularly curious about the dynamic regime of motion, and it changes by the different combinations of parameters. Also, we have some freedom in defining these dynamical parameters; however, this work mostly focuses on studying the effect of the magnetic field’s parameter on this system.

A survey in this background is of our interest because for several reasons. First, it is assumed that the Schwarzschild or Kerr metrics describe astrophysical compact objects in the relativistic astrophysical study. However, besides these setups, others can imitate a black hole’s properties, such as the electromagnetic signature [64]. Also, astrophysical observations may not be fitted, in general, within the general theory of relativity by using the Schwarzschild or Kerr metric [65, 66], like as the mentioned ratio of QPOs.

In addition, due to their strong gravitational field, considering astrophysical environments, compact objects are not necessarily isolated or possess spherical symmetry. In particular, it seems that the present understanding of astronomical phenomena mostly relies on studies of stationary and axially symmetric models. It has been shown [52] that the possible resonant oscillations can be directly observed when arising in the inner parts of accretion flow around a compact object, even if the source be steady and axisymmetric.

Moreover, this setup could constitute a reasonable model of a real situation that arises in this compact object’s vicinity with the possibility of analytic analysis via exercising parameters of models.

The paper’s organization is as follows: Section II presents the background object. The dynamics of charged particles in this background is presented in Section III. While Section IV explains epicyclic frequencies and stable circular orbits. The parametric resonances present in Section VI. Finally, the conclusions are summarized in Section VII.

Throughout this work, we use the signature $(-, +, +, +)$ and geometrized unit system $G = 1 = c$ (However, for an astrophysical application, we will use ST units. Latin indices run from 1 to 3, while Greek ones take values from 0 to 3).

II. SPACE-TIME OF DISTORTED DEFORMED COMPACT OBJECT

The first static and axially symmetric solution of Einstein’s field equation with arbitrary quadrupole moment were described in [53]. Then in [54] this presented in the prolate spheroidal coordinates. Later, Zipoy and Voorhees [55, 56] found an equivalent transformation that leads to a simple solution which can be treated analytically and known as $γ$-metric or $σ$-metric, and later on, with introducing a new parameter is known as q-metric [67].

In this paper, we choose to work on the generalized q-metric, which has q-metric as the seed metric, and considers the existence of a static and axially symmetric external distribution of matter in its vicinity up to quadrupole. In fact, due to their strong gravitational field, compact objects are not necessarily isolated or possess spherical symmetry considering astrophysical environments. Explicitly, it seems that the present understanding of astronomical phenomena mostly relies on studies of stationary and axially symmetric models. The metric has this form

$$ds^2 = -\left(\frac{x-1}{x+1}\right)^{(1+α)}e^{2β}dt^2 + M^2(x^2-1)e^{-2β}$$

$$\left(\frac{x+1}{x-1}\right)^{(1-α)}\left(\frac{x^2-1}{x^2-y^2}\right)^{α(2+α)}e^{2γ}$$

$$\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (1-y^2)dφ^2,$$ (1)

where $t \in (-∞, +∞), x \in (1, +∞), y \in [-1, 1]$, and $φ \in [0, 2π]$. The function $ψ$ plays the role of gravitational potential, and the function $γ$ is obtained by an integration of the explicit form of the function $ψ$. These are given by

$$ψ = -β\frac{y}{2}\left[-3x^3y^2 + x^3 + y^2 - 1\right],$$

$$γ = -2β(1 - y^2)$$

$$+ \frac{β^2}{4}(x^2 - 1)(1 - y^2)(9x^2y^2 + x^2 + y^2 - 1).$$ (3)

By its construction, this metric is valid locally [68, 69]. In fact, the metric and its circular geodesics are studied. In fact, this metric possesses three parameters; namely the total mass $M$, deformation parameter $α$, and distortion parameter $β$. These two parameters are chosen to be relatively small and connected to the q-metric and the surrounding external mass distribution, respectively. For vanishing $β$, we recover the q-metric, and in the case of $α = β = 0$, the Schwarzschild metric is retrieved. This metric may links the observable effects to the system due to taking these parameters as the new dynamical degrees of freedom. Also, as we have some freedom to define these dynamical variables, we try to minimize computational time and numerical errors. In addition, the circular geodesics in this background studied in [52].

The relation between the prolate spheroidal coordinates $(t, x, y, φ)$, and the Schwarzschild coordinates $(t, r, θ, φ)$ reads as

$$x = \frac{r}{M} - 1, \quad y = \cos θ.$$ (4)

In the rest of the work, we explore this background by analyzing the particle’s motion dynamics, combine with a uniform magnetic field.

III. DYNAMIC OF CHARGED PARTICLE IN A UNIFORM MAGNETIC FIELD

We consider a weak, external magnetic field because it does not influence the background spacetime. Explicitly, We are in-
interested in the dynamics that occur when a static, axisymmetric central compact object is embedded in a uniform magnetic field of strength \( B \) aligned with the central body’s symmetry axis. By comparing the compact object’s size with the typical length of varying the strength’s electric and magnetic fields, we can define a test particle.

Furthermore, the motion of neutral test particles is not influenced by magnetic fields satisfying this condition,

\[
B_G = 10^{19} \left( \frac{M_c}{M} \right) G
\]  

However, the charged test particles’ motion could be strongly influenced even by relatively weak test magnetic fields \( \mathbb{G} \).

This condition comes from comparing the central body’s gravitational effect and the strength of the magnetic field \( B \) on its vicinity. For most astrophysical black holes, this condition is perfectly satisfied \( \mathbb{G} \). In fact, by using the fundamental variability plane, the magnitude of the magnetic field in the vicinity of a black hole estimates as

\[
B \sim \begin{cases} 
10^8 G & , \ M \sim M_\odot, \\
10^4 G & , \ M \sim 10^8 M_\odot.
\end{cases}
\]

Also, the Lorentz force is characterized by the specific charge of the particle by the order of

\[
b \sim 4.7 \times 10^7 \left( \frac{q}{e} \right) \left( \frac{m_p}{m} \right) \left( \frac{B}{10^8} \right) \left( \frac{M}{10M_\odot} \right),
\]

where \( m_p \) is the mass of a proton. These ratios in this characteristic suggest that the expression \( \mathbb{G} \) is relevant and cannot be neglected for the astrophysical scales.

We start our analysis by expressing the standard definition of electric-magnetic field definition,

\[
F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}.
\]

The external homogeneous magnetic field is chosen to be along the polar axis and described by the \( \phi \)-component of the vector potential \( \mathbb{G} \).

\[
A_{\phi} = \frac{1}{2}B(x^2 - 1)e^{-2\phi} \left( \frac{x + 1}{x - 1} \right)^{1+\alpha}.
\]

In addition, the Lorentz equation describes the charged test particle motion

\[
m \frac{du^\mu}{ds} = qF^\nu_{\phi}u^\nu,
\]

where \( u^\mu = \frac{dx^\mu}{dt} \) is the four-velocity of the particle with the mass \( m \) and charge \( q \). In the following part, we use Hamiltonian formalism to study the dynamics in this background and influenced by the magnetic field.

### A. Hamiltonian and effective potential on the equatorial plane

In this subsection, we use the general Hamiltonian formalism to describe effective potential and dynamics of a charged particle in the vicinity of a distorted, deformed compact object embedded in the external uniform magnetic field. The Hamiltonian for the charged particle motion is written as,

\[
H = \frac{1}{2} \left[ (p^\mu - qA^\mu)(\pi - qA_\mu) + m^2 \right],
\]

where the generalized canonical four-momentum is written in terms of four-momentum as follows

\[
\pi^\mu = p^\mu + qA^\mu.
\]

Considering the background structure is static and axisymmetric immersed in a homogeneous magnetic field, the conserved quantities specific energy and angular momentum of the particle can express as

\[
E = -\pi_t = \left( \frac{x - 1}{x + 1} \right)^{(1+\alpha)} e^{2\phi} \frac{dt}{ds},
\]

\[
L = \pi_\phi = (x^2 - 1)e^{-2\phi} \left( \frac{x + 1}{x - 1} \right)^{(1+\alpha)} \left( \frac{d\phi}{ds} + Q \right),
\]

where \( Q := \frac{aB}{m} \) is magnetic parameter. Then by using this approach, the effective potential is given by this relation

\[
V_{\text{eff}} = \left( \frac{x - 1}{x + 1} \right)^{(\alpha+1)} e^{2\phi} [\epsilon + \frac{Le^{2\phi}}{(x + 1)(1 - y^2)^2} \left( \frac{x - 1}{x + 1} \right)^{\alpha} - Q(x + 1)^2].
\]

The second term corresponds to the central force potential and electromagnetic potential energy. In general, we can discuss four different situations in terms of signs of \( L \) and \( Q \). However, because of the even power in the second term it is sufficient to consider only two situations:

1. \( LQ > 0 \), the Lorentz force pushes the particle away in the outward direction with respect to the central object.
2. \( LQ < 0 \), the Lorentz force pushes the particle in the direction of the \( z \)-axis towards the central object.

There is a shortcut in analysing the charged particle’s motion, by using the effective potential. In fact, this motion is determined by the energy’s boundaries given by \( E^2 = V_{\text{eff}} \).

The effective potential represented in Figure 1 for different values of parameters \( \alpha \) and \( \beta \).

In general, possible types of orbits, dependent on the parameters \( \epsilon, \mathcal{E}, L, \alpha \) and \( \beta \). As an analytical exploration, depending on the number of positive real zeros and the sign of \( \mathcal{E} - \epsilon \), one obtains different types of trajectories.
quency, aside from the magnitude, the influence of the sign of parameters one can obtain different trajectories. As one can see in the plots, only by a slight change in the value and sign of parameters one can obtain different trajectories. However, aside from the magnitude, the influence of the sign of magnetic parameter $Q$, combined with the sign’s of $\alpha$ is distinguished.

\begin{align}
\nu_x &= \frac{\Omega}{2}, \\
\nu_y &= \frac{\Omega}{2},
\end{align}

The vertical and keplerian frequencies are positive, but it is not the case for radial frequency. Therefore, the stable circular orbits are located at radial distances larger than the location of the ISCO at $x = 5$ in these coordinates. Also, for the Schwarzschild solution, we have $\omega_x^2 < \omega_y^2 = \Omega^2$. In fact, there exists a periapsis shift for bounded quasi-elliptic trajectory implying the effect of relativistic precession that changing the radius of the orbit \[^{[73]}\]. Indeed, this ordering between the frequencies contributes to the possible resonances in a given background. The behavior of the frequencies helps us to distinguish possible trajectories around a stable circular orbit.

For this purpose, radial and vertical motions around a circular equatorial plane are discussed in this section.

In what follows, we investigate the stability of circular motion in the presence of the homogeneous magnetic field. In this spacetime circular motion, geodesic equation and the relation between parameters studied in \[^{[52]}\]. The equation of motion for a particle with mass $\mu$ and electric charge $e$ is the geodesic equation with force in its right-hand side takes this standard form,

\begin{equation}
\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\eta \rho} \frac{dx^\eta}{ds} \frac{dx^\rho}{ds} = \frac{q}{m} F^\mu_\eta \frac{dx^\eta}{ds}
\end{equation}

To get this equation adapted for this mentioned background, we start with replacing $(x = x_0, y = 0)$ as we are in the equatorial plane. Also, replace all necessary Christoffel symbols we have on the left-hand side. For substitute the right hand side we use \[^{[8]}\] and \[^{[9]}\].

To describe the more general class of orbits slightly deviated from the circular orbits in the equatorial plane $x^\eta$, we can use the perturbation expansion, namely $x^\mu = x^\mu + \xi^\mu$. By substituting all these relations and into equation (18) and consider the external force as $f^\mu_\eta u^\eta (u^0)^{-1}$, also take into account only terms up to linear order in $\xi^\mu$, we have \[^{[43]}\]

\begin{equation}
\frac{d^2 \xi^\mu}{dt^2} + 2\gamma^\mu_\eta \frac{d\xi^\eta}{dt} + \xi^\rho \partial_\eta U^\mu = \frac{q}{mu^0} F^\mu_\eta ,
\end{equation}

where,

\begin{align}
\gamma^\mu_\eta &= \left[ 2\Gamma^\mu_{\eta \rho} u^\rho (u^0)^{-1} - \frac{q}{mu^0} F^\mu_\eta \right]_{t=0} , \\
U^\mu &= \left[ \gamma^\mu_\eta u^\eta (u^0)^{-1} - \frac{q}{mu^0} F^\mu_\eta u^0 (u^0)^{-1} \right]_{t=0} ,
\end{align}

Here the 4-velocity for the circular orbits in the equatorial plane is taken as $u^\eta = u^0(1, 0, 0, \Omega)$. Then the integration of equation (19) for the $t$ and $\phi$ components lead to

\begin{align}
\sigma &= 0.1, \beta = 1 e-06, \mathcal{L} = 20 \\
\sigma &= 0.4, \beta = 1 e-06, \mathcal{L} = 32
\end{align}
FIG. 2. Trajectories of particles for some choices of the parameters. In all plots $Q > 0$ and $\alpha > 0$. In the first raw, the initial radius is set to $r_0 = 9.5$, in the second one is $r_0 = 8$.

\[
\frac{d\xi^\eta}{dt} + \gamma^\eta \xi^\eta = \frac{q}{mu^3} \int f^\eta dt, \tag{22}
\]
\[
\frac{d^2\xi^x}{dt^2} + \omega_x^2 \xi^x = \frac{q}{mu^3} \left(f^x - \gamma^\eta \int f^\eta dt\right), \tag{23}
\]
\[
\frac{d^2\xi^y}{dt^2} + \omega_y^2 \xi^y = \frac{q}{mu^3} f^y. \tag{24}
\]

where here $\eta$ can be taken $t$, or $\phi$, and

\[
\omega_x^2 = \partial_x U^x - \gamma^x \gamma^\eta, \tag{25}
\]
\[
\omega_y^2 = \partial_y U^y. \tag{26}
\]

This system of equations describes radial phase and vertical oscillations of the charged particle around the circular orbits. For an alternative definition of the epicyclic harmonic motion, see [74]. The sign of frequencies determines the dynamic $\omega_x$ and $\omega_y$ so that we have stable circular orbits where they have
FIG. 3. Trajectories of particles for some choices of the parameters. In all plots $Q > 0$ and $\alpha < 0$. In the first raw, the initial radius is set to be $r_0 = 5$, in the second one is $r_0 = 4.5$.

A positive sign; otherwise, even a minimal perturbation can make a strong deviation from the unperturbed path. In the absence of the external force, these equations describe the free radial phase and vertical oscillations of particles around the circular orbits. Next, we analyze the behavior of these frequencies of test particles in this background.

A. Properties of epicyclic frequencies

These epicyclic frequencies in the background of a distorted, deformed compact object are written in the appendix A. In this case, as equations (A1) and (A2) suggest, we cannot write $\omega_x$ and $\omega_y$ as $\Omega$ with some coefficient, as a standard procedure in Schwarzschild and Kerr space-times. However, applying this procedure is also possible in this background when there is no magnetic field is presented.
FIG. 4. Trajectories of particles for some choices of the parameters. In all plots $Q < 0$ and $\alpha > 0$. In the plots, the initial radius is set to be $r_0 = 7.5$.

In Figure 6 and 7 we see the locations of maxima of the epicyclic frequencies $\omega_x$ and $\omega_y$. For any choice of parameters, the radial epicyclic frequency's extrema must be located above the marginally stable orbit. The extrema are dependent on the choice of parameters for the vertical frequency, but these are always a monotonic function of distance $x$.

The Figures 8 and 9 are depicted for the study of the existence and the stability of the timelike orbits with respect to vertical or radial oscillations. All the plots in Figure 8 are in the $(x,\beta)$–plane for two chosen values of the deformation parameter $\alpha$—one positive and one negative—and for different values of the magnetic parameter $Q$. While Figure 9 represents the region of existence and stability in the $(x,\alpha)$–plane for two chosen values of the distortion parameter $(\beta)$, and for the same values as in Figure 8 for the magnetic parameter.

The Figure 10 is an extension of the Figure 9 for negative values of $Q$ close to zero. On the three Figures, the red line and the blue line correspond to $w_x^2 = 0$ and $w_y^2 = 0$, respec-
FIG. 5. Trajectories of particles for some choices of the parameters. In all plots $Q<0$ and $\alpha<0$. In the first row, the initial radius is set to be $r_0 = 5$, in the second one to $r_0 = 4.5$.

tively. The dark-green region bounded by those two lines represents the area where $w^2_x > 0$ and $w^2_y > 0$, which is the condition to have stability with respect to vertical and radial oscillations. Note that above the red line, where $w^2_x < 0$ and $w^2_y > 0$, orbits are stable with respect to vertical oscillations but unstable to the radial one. And on the contrary, below the blue line, where $w^2_x > 0$ and $w^2_y < 0$, the orbits are stable with respect to radial oscillations and unstable with respect to the vertical one. We start this analysis by exploring the existence of the timelike circular orbit for different parameters. In fact, the region of existence is clearly affected by the three parameters. In the $(x,\beta)$-plane, see Figure 11, we can see that switching $\alpha$ from negative to positive values tends to shrink the light-green region vertically, reducing the range of the distortion parameter $\beta$. The effect of the magnetic parameter is not monotonic; namely by decreasing $Q$, from positive to zero, the region is reduced in the vertical direction, but when $Q$ becomes negative, and we continue to decrease that parameter, the region
As for the existing area, here also the three parameters influence with different strength on the region of stability. We can see in Figure 9 for $Q > 0$, that the dark-green area remains very similar for $\beta$ positive or negative. Moreover, we can note that switching $\beta$ negative to positive values makes the area of stability shrink (see Figure 9). The opposite effect is visible when switching $\alpha$ from negative to positive (see Figure 11), and this true for all values of $Q$. However, the region of stability is hardly changing with the parameter $Q$. It starts to shrink when $Q$ decreases and continues by being pushed away from the central mass, to completely disappear for the magnetic parameter’s negative values. This due to the fact that, on the $(x, \beta)$–plane, the blue curve is coming up, and the red one is coming down, then the dark-green region is shrinking and at the end does not exist anymore. For larger negative values, a branch from above coming back in the physical range but keeps staying below the blue curve; stability conditions remain incompatible. On the $(x, \alpha)$–plane, this is due to the fact that the red curve going up and goes out from the physical range. This effect can be seen by combining the Figures 9 and 10.

This is mainly what is happening for negatives values of $Q$. A small area remains for values of the magnetic parameter close to zero, $Q = -0.01$. In this case, it means there is no stable orbit in any direction in the chosen physical range.

In Figures 11 and 12 different relations between the frequencies have been plotted. Similar to Figures 8 and 9, the red and the blue curves are representing $w_2^x = 0$ and $w_2^y = 0$, respectively. We add to this picture more lines that can help us analyze the order of the different epicyclic frequencies $w_2^x, w_2^y$ and $\Omega^2$. The orange line shows $w_2^x = \Omega^2$, which leads to the hatched line region $w_2^x > \Omega^2$. Also, $w_2^x = \Omega^2$ corresponds to the pink line related to the dark-green area where $w_2^x > \Omega^2$. Furthermore, $w_2^y = w_2^x$ is shown by the yellow line. The corresponding region $w_2^x > w_2^y$ is presented in the hatched dotted line. By analyzing both Figures, we can order the frequencies as a function of the magnetic parameter $Q$. We see that for $Q \geq 0$, the order’s behavior will depend on two crossing points. One when the orange line crossed the red line and another one when the orange, pink and yellow are crossing together.

For small values of $\beta$, the order is pretty steady (see plots with $Q \geq 0$ on the left in Figure 12), but by increasing $\beta$, the crossing points appear, then the behaviour of ordering becomes more complicated (see plots with $Q \geq 0$ on the left in the Figure 11). For instance, using the second plot of the first raw of Figure 11 in the stable region; namely, between the red and blue lines, the frequencies are ordered as follow:

1. Inside the hatched region (above the orange line) where $\beta$ is close to zero, and for both positive or negative $\alpha$:

   - from the red line to the pink one, we get $w_2^x < \Omega^2 < w_2^y$,
   - from the pink line to the yellow one, we have $\Omega^2 < w_2^x < w_2^y$,
   - above the yellow line, the order is $\Omega^2 < w_2^x < w_2^y$.

2. Outside the hatched region (below the orange line), where $\beta$ has larger negative values, for any $\alpha$, the order of the frequencies is different,

   - from the red line to the yellow one, we get $w_2^x < w_2^y < \Omega^2$,
   - from the yellow line to the pink one, we have $w_2^y < w_2^x < \Omega^2$.
FIG. 8. Stability of the timelike circular orbits of a charged particle in the (x, β)-plane. Timelike circular orbit exists in the green light area. The blue curve represents $w^2_x = 0$ and the red one $w^2_y = 0$. Timelike orbits are stable with respect to vertical and radial perturbations in the region where $w^2_x > 0$ and $w^2_y > 0$. This area is depicted by the green-gray region, which combined the conditions of existence and the condition of stability, $w^2_x > 0$ and $w^2_y > 0$. The analysis is done for two positive values of $Q$, two negatives values of $Q$, and the unmagnetized case $Q = 0$. Besides, we examine opposite signs of the deformation parameter $\alpha$.

FIG. 9. Stability of the timelike circular orbits in the (x, β)-plane. Timelike circular orbit exists in the green light area. The blue curve represents $w^2_y = 0$ and the red one $w^2_x = 0$. Timelike orbits are stable with respect to vertical and radial perturbations in the region where $w^2_x > 0$ and $w^2_y > 0$. The analysis is for two positive values of $Q$, two negatives values of $Q$, also for the unmagnetized case $Q = 0$. Besides, we examine opposite signs of the deformation parameter $\alpha$. 
to finding the desired combination of resonances that links them to the observations.

Even so, in thick and thin accretion disks, the oscillations’ frequency is related to the Keplerian frequency or the radial and vertical epicyclic frequencies of the test particle circular motion [78, 86]. However, by evolution of the system, in time the eigenfrequencies of oscillations start to deviate from the initial ones [87–89]. In fact, the standard resonance model assumes the non-linearity between oscillation modes in the accretion disk [75, 80, 91]. Mostly in this section, our main focus is on the thin disk oscillation. In what follows, we explain these two resonance models briefly.

A. Parametric resonance

This resonance is governed by Mathieu equation.

\[
\frac{d^2 \xi^y}{dt^2} + \omega^2_y \left[ 1 + \omega^2 h \cos(\omega_t^y t) \right] \xi^y = 0. \tag{27}
\]

considering the vertical and radial epicyclic oscillations with the frequencies \( v_y = \frac{\omega_y}{2\pi} \) and \( v_r = \frac{\omega_r}{2\pi} \), the Mathieu equation implies that a parametric resonance is excited strongly for the lowest possible value of integer number in the ratio of these two [92].

In the case of a black hole, because in its vicinity \( v_r < v_y \) satisfies, one can say that this lowest possible value is \( n = 3 \), which means that \( 3v_r = 2v_y \). This explains most of the observed \( 3 : 2 \) ratio in high frequencies QPOs.

B. Non-linear resonance

In a more realistic physical model, small deviations from the planar circular motion are necessary to consider. Therefore, the related Models are based on the forced non-linear oscillator when the equations govern the behavior of these fundamental frequencies read as

\[
\frac{d^2 \xi^y}{dt^2} + \omega^2_y \xi^y + \text{[non-linear terms in } \xi^y] = h_1 \cos(\omega^y_0 t), \tag{28}
\]

\[
\frac{d^2 \xi^x}{dt^2} + \omega^2_x \xi^x + \text{[non-linear terms in } \xi^x] = h_2 \cos(\omega^x_0 t), \tag{29}
\]

where \( \omega_0 \) is the frequency of the external force. Also, \( \frac{\omega_y}{\omega_x} = \frac{k}{l} \).
FIG. 11. Order of the different epicyclic frequencies in the $(x, \beta)$-plane. Timelike circular orbit exists in the green light area. The blue curve represents $w^2_x = 0$ and the red one $w^2_\beta = 0$. The pink, orange and yellow depict $w^2_x = \Omega^2$, $w^2_\beta = \Omega^2$ and $w^2_x = w^2_\beta$ respectively. The green-gray area shows the region where $w^2_x > \Omega^2$. The hatched line region represents the area where $w^2_\beta > \Omega^2$. Finally, the hatched dotted area is where $w^2_x > w^2_\beta$. The analysis is for different values of $Q$. Besides, we test two different values of the deformation parameter $\alpha$, one positive and one negative.

FIG. 12. Order of the different epicyclic frequencies in the $(x, \alpha)$-plane. Timelike circular orbit exists in the green light area. The blue curve represents $w^2_\alpha = 0$ and the red one $w^2_x = 0$. The pink, orange and yellow depict $w^2_\alpha = \Omega^2$, $w^2_x = \Omega^2$ and $w^2_\alpha = w^2_x$ respectively. The green-gray area shows the region where $w^2_\alpha > \Omega^2$. The hatched line region represents the area where $w^2_x > \Omega^2$. Finally, the hatched dotted area is where $w^2_\alpha > w^2_x$. The analysis is for different signs of $Q$. In addition, we test two different values of the deformation parameter $\beta$, one positive and one negative.
where \( k, l \) are small positive integers. The non-linear terms are responsible for beat frequencies in the resonant for \( \xi(t) \) and \( \xi(t) \). As it may guess, these equations are related to dissipative processes because of the nature of non-linearity in them and the change in their amplitude on the contrary to the non-linear equation above \([92]\).

Since the details of dissipative processes in accretion disks are not known in general, one can not find the corresponding \( h_1 \) and \( h_2 \) exactly in this more realistic model. One choice is to solve them mathematically. Then, try to find the best match between them and observed QPOs \([75]\).

VI. QPO MODELS

In fact, there are a variety of possibilities in the combination of resonances. In the parametric and forced resonances, normally, people try to connect these combinations to the frequency ratio of oscillations; however, physical details vary \([92]\). Here we concentrate on analyzing these frequency ratio resonant models in this background and leave the physical details for the next step. Also, in the disk’s oscillation, we assume the resonant occurs at a radius of the disk where frequencies of the twin oscillatory modes are measurable. However, the existing QPO models are incomplete tasks. Since the oscillations predicted by these models are usually not seen in the same way in the MHD simulations. Furthermore, none of these models has a good match, especially with the full QPOs amplitudes, and the visibility on the source spectral data, in the LMXBs.

Among these various models of the resonance’s models between some modes of accretion disk oscillations, we consider the group of QPO models considered in \([93]\) and examine them in this background. For a detailed discussion on these models see also \([75, 76, 94, 95]\).

1. RPM

The Relativistic Precession Model (RPM) is one of the first attempts to model QPOs, proposed by \([96, 97]\). In RPM the upper frequency is defined as the Keplerian frequency \( \nu_U = \Omega \) and the lower frequency is defined as the periastron frequency i.e. \( \nu_p := \nu_L = \Omega - \nu_x \). Their correlations are obtained by varying the radius of the associated circular orbit. Within this framework, it is usually assumed that the variable component of the observed X-ray are in a bright localized spot or blob orbiting the compact object on a slightly eccentric orbit. Therefore due to the relativistic effects, the observed radiation is supposed to be periodically modulated.

2. TDM

Another one is the Tidal Disruption Model (TDM) presented in \([98, 99]\). This follows also very similar approach as the RPM. In this model, the QPOs are assumed as a result of tidal disruption of large accreting inhomogeneities. In TDM the upper frequency is defined as \( \nu_U = \Omega + \omega_x \) and the lower frequency is defined as \( \nu_L = \Omega \).

3. WDM

The Warped Disk Model (WDM) introduced in \([84]\), which is related to oscillatory modes in a warped accretion disk. In this model, the HF QPOs are assigned to a non-linear resonance between two accretion disk oscillations modes. In WDM the upper frequency is defined as \( \nu_U = 2\Omega - \omega_x \) and the lower frequency is \( \nu_L = 2\Omega - 2\omega_x \). Also, resonant phenomena could be relevant. In more realistic versions of this model, the higher harmonic oscillations are also considered up to the third order, then frequencies like \( 3\Omega - \omega_x \) possible to consider.

4. EpM-KpM

The Epicyclic resonance Model (EpM) \([75]\) is the simplest variant. It is about considering radial and vertical epicyclic oscillations and relates them to the resonance of axisymmetric disk-oscillation modes. The Keplerian resonance Model (KpM) considers a resonance between the orbital Keplerian and the radial epicyclic oscillations. In EpM the upper frequency is defined as \( \nu_U = \omega_x \) and the lower frequency is \( \nu_L = \omega_x \). In KpM the upper frequency is defined as \( \nu_U = \Omega \) and the lower frequency is \( \nu_L = \omega_x \).

5. RP1M-RP2M

The RP1 model by Bursa in 2005 and the RP2 model \([100]\), both consider different combinations of non-axisymmetric disk-oscillation modes. In RP1M the upper frequency is defined as the Keplerian frequency \( \nu_U = \omega_x \) and the lower frequency is \( \nu_L = \Omega - \omega_x \). In RP2M the upper frequency is defined as \( \nu_U = 2\Omega - \omega_x \) and the lower frequency is \( \nu_L = \Omega - \omega_x \).

TABLE I. Frequency relations corresponding to individual QPO models

| Model | \( \nu_U \) | \( \nu_L \) |
|-------|------------|------------|
| RPM   | \( \Omega \) | \( \Omega - \omega_x \) |
| Kp    | \( \Omega \) | \( \omega_x \) |
| Ep    | \( \omega_x \) | \( \omega_x \) |
| TD    | \( \Omega + \omega_x \) | \( \Omega \) |
| WD    | \( 2\Omega - \omega_x \) | \( 2\Omega - 2\omega_x \) |
| RP1   | \( \omega_x \) | \( \Omega - \omega_x \) |
| RP2   | \( 2\Omega - \omega_x \) | \( \Omega - \omega_x \) |

The behaviour of these models is illustrated in Figures \([13, 14]\) and \([13, 14]\). In these Figures, the radius of the \( 3:2 \) frequencies for the different models (WD, TD, RP, Ep, Kp, RP1, RP2) is plotted, with respect to the deformation parameter \( \beta \).
This radius depends on all parameters, $\alpha$, $\beta$, and $Q$. However, on both Figures, our main focus is on analyzing the radius for different values of the parameter $Q$. As it is seen in Section V A where we discussed the existence and stability of the timelike orbits, only small negatives values of $Q$ allow the orbits’ stability with respect to vertical and radial oscillations together.

The magnetic parameter values have been chosen as follows: one negative value $Q = -0.001$, the unmagnetized case $Q = 0$, and two positives values $Q = 0.01$ and $Q = 0.1$. About the effect of the magnetic parameter: firstly, we can see that for negatives values of $Q$, two radii can satisfy the 3 : 2 QPOs. This phenomenon appears for all models. Secondly, increasing $Q$ tends to move the upper part of the curve to the right and flatten all the lines to inverse the curve’s slope. Thus, increasing $Q$, allows to choose more values of $\alpha$ where the radius of the 3 : 2 QPOs exists. Furthermore, increasing the magnetic parameter causes to push away this radius closer to the central mass. About the different models, we can see that the 3 : 2 ratio of the RP, RP1, and RP2 models are occurring at the radius similar to each other. The same effect is seen for the Ep and Kp models. However, models seem to deviate from each other in the case of $Q < 0$ in the upper part of the curves. About the analysis of negative $\beta$ depicted in the Figure 14 generally the behavior is the same as for positive $\beta$ discussed earlier. However, slight differences occur on the curves’ upper branch for the negative value of $Q$. We can see that in this case, the Kp and Ep diverge from each other. Also, the same effect is seen for RP1, RP2, and RP. This is worth to mention that in this background, with different combinations of parameters $\alpha$ and $\beta$, also this is possible to have other ratios which can be relevant in other observed data like in other twin frequencies observed in the Microquasar GRS 1915 + 105 (see for example [101]). Nevertheless, rotation modify the radial profiles of the vertical and radial frequencies, which is the subject of works in progress.

### Appendix A: Epicyclic frequencies in the uniform magnetic field

The following functions give epicyclic frequencies of particles’ circular motion in the background of a distorted, deformed compact object immersed in a uniform magnetic field. The vertical frequency is given by

\[
\omega_y^2 = e^{-2\gamma}(\frac{x^2 - 1}{x^2})^{-\alpha(2+\alpha)}\left\{\Omega^2 \frac{xf_1(x, \beta) + S}{S} + (1 + f_1(x, \beta))\Omega\omega_B \right\}, \tag{A1}
\]

And radial frequency is given by

\[
\omega_r^2 = \Omega^2 e^{-2\gamma}(1 - \frac{1}{x^2})^{-\alpha(2+\alpha)} \left\{ g_1(x, \beta, \alpha) \frac{x - S}{S} + g_2(x, \beta, \alpha) \right\} + e^{-2\gamma}(1 - \frac{1}{x^2})^{-\alpha(2+\alpha)} \left\{ -\omega_B^2 x(S - x)^2 + \Omega\omega_B g_2(x, \beta, \alpha) \right\}, \tag{A2}
\]

### VII. SUMMARY AND CONCLUSION

In this paper, we studied the dynamics of test charged particles in the presents of a uniform magnetic field. Also, study different QPOs models considered in [23]. This study explores these tasks in the vicinity of a deformed compact object up to the quadrupole. This background is a generalization of the q-metric by considering a distribution of matter in its vicinity. This metric is static and axisymmetric and contains two parameters: distortion parameter $\beta$ and deformation parameter $\alpha$. This background was briefly explained in Section II. These two parameters’ dependency reflects into motion and epicyclic frequencies of particles. However, our main goal in this work was to analyze the influence of magnetic parameters on this system. In fact, the result shows a strong deviation from the nonmagnetic case in general.

Our results show that even the regular orbits for some combinations of parameters of metric turn to behave chaotically with a magnetic field. However, further, inspection revealed that magnetic parameter brings a profound perturbation in the dynamics and oscillation modes. We explore different QPOs models that are related to disk-oscillation modes. However, the physical details of these resonance models are different. This also depends explicitly on the time evolution of the desired system. The next step of this work would be to consider these physical details to obtain a comprehensive understanding of this phenomenon. This is also possible to explore more about this background by considering different input systems, which is the subject of the next work. Also, considering rotation definitely helps to model a more realistic complex system of real astronomical objects. Another future work can be a discussion on including the strong magnetic field, which is influenced on the metric itself.
FIG. 13. The radius of the 3:2 frequency ratio for different models. The radius is the function of the $\alpha$ parameter for different values of the magnetic term $Q$. On all the plots, $\beta = 0.000001$.

FIG. 14. The radius of the 3:2 frequency ratio for different models. The radius is the function of the $\alpha$ parameter for different values of the magnetic term $Q$. On all the plots, $\beta = -0.000005$. 
\[ S = 1 + \alpha + \beta x - \beta^2, \]
\[ f_1(x, \beta) = \beta(-1 + 3x^2), \]
\[ g_1(x, \beta, \alpha) = 2\alpha^2 + \alpha^2 \left(6 - 2\beta x(-1 + x^2)\right) + 2\alpha \left(2 + x \left(x + \beta \left(-1 + x^2\right)^3 - 4 + \beta x(-1 + x^2)\right)\right) + x \left(2x - \beta \left(-1 + x^2\right) \left(5 + x \left(-x + 2\beta \left(-1 + x^2\right)^3 - 3 + \beta x(-1 + x^2)\right)\right)\right), \]
\[ g_2(x, \beta, \alpha) = 2\alpha^2 - 4x \left(1 + \alpha + (-1 + \beta)x - \beta x^3\right)^2 - 2\alpha^2 \left(-3 + x - \beta x + \beta x^3\right) + 2\alpha \left(2 + x \left(-2 + \beta \left(-1 + x^2\right)^3 - 4 + \beta x(-1 + x^2)\right)\right) - x \left(-1 + x^2\right) \left(1 + \beta \left(-4 - 3x + 2\beta \left(-1 + x^2\right)^3 - 3 + \beta x + \beta x^3\right)\right). \]

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