The Quantum Query Complexity of 0-1 Knapsack and Associated Claw Problems

V. Arvind Rainer Schuler
Institute of Mathematical Sciences Theoretische Informatik
C. I. T. Campus Universität Ulm
Chennai 600 113, India D-89069 Ulm, Germany
arvind@imsc.res.in rsc@informatik.uni-ulm.de

Abstract

We first give an \(\tilde{O}(2^{n/3})\) quantum algorithm for the 0-1 Knapsack problem with \(n\) variables. More generally, for 0-1 Integer Linear Programs with \(n\) variables and \(d\) inequalities we give an \(\tilde{O}(2^{n/3}n^d)\) quantum algorithm. For \(d = o(n \log n)\) this running time is bounded by \(\tilde{O}(2^{n(1/3+\epsilon)})\) for every \(\epsilon > 0\) and in particular it is better than the \(\tilde{O}(2^{n/2})\) upper bound for general quantum search.

To investigate whether better algorithms for these NP-hard problems are possible, we formulate a symmetric claw problem corresponding to 0-1 Knapsack and study its quantum query complexity. For the symmetric claw problem we establish a lower bound of \(\tilde{O}(2^{n/4})\) for its quantum query complexity. We have an \(\tilde{O}(2^{n/3})\) upper bound given by essentially the same quantum algorithm that works for Knapsack.

Additionally, we consider CNF satisfiability of CNF formulas \(F\) with no restrictions on clause size, but with the number of clauses in \(F\) bounded by \(cn\) for a constant \(c\), where \(n\) is the number of variables. We give a \(2^{1-\alpha}n^{2}\) quantum algorithm for satisfiability in this case, where \(\alpha\) is a constant depending on \(c\).

1 Introduction

The goal of the present paper is to explore the possibility of developing quantum algorithms for different NP-hard problems that are faster than what we get by a direct application of Grover’s search algorithm \([6]\). This can be seen as part of an active research theme in quantum computing: discovering tasks for which there are quantum algorithms that are significantly faster than classical algorithms (c.f. \([7, 8]\)). The difficulty in the area is that there are as yet only a few known techniques for designing quantum algorithms. The two central methods are essentially from Shor’s factoring algorithm \([11]\) and Grover’s search algorithm \([9]\). Other known algorithms are essentially based on these methods.

In this paper, we consider quantum algorithms for certain NP-hard problems which have a special divide-and-conquer structure that can be exploited to design significantly faster quantum algorithms than what a direct application of Grover’s search algorithm would yield. For example, we give an \(\tilde{O}(2^{n/3})\) quantum algorithm for the 0-1 Knapsack problem with \(n\) variables. This is a consequence of a more general result for 0-1 Integer Linear Programs with \(n\) variables and \(d\) inequalities for which we give an \(\tilde{O}(2^{n/3}n^d)\) quantum algorithm. For \(d = o(n / \log n)\) this running time is bounded by \(\tilde{O}(2^{n(1/3+\epsilon)})\) for every \(\epsilon > 0\) and in particular it is significantly better than the
\(O(2^{n/2})\) time bound if we directly apply Grover’s search algorithm. Our algorithms are, of course, based on Grover’s search algorithm and the powerful method of amplitude amplification \[6\] \[7\].

The next question we address is whether faster quantum algorithms for the above NP-hard problems are possible. To study this, we formulate a new claw-like problem (see \[4\] for claw problems) which we term as the symmetric claw problem and study its quantum query complexity. The symmetric claw problem essentially captures the structure of 0-1 Knapsack and we study its complexity in the quantum query model \[5\]. For the symmetric claw problem we are able to prove a lower bound of \(\tilde{O}(2^{n/4})\) in the quantum query model using Ambainis’ method \[4\]. The problem also has an \(\tilde{O}(2^{n/3})\) upper bound given by essentially the same quantum algorithm that works for 0-1 Knapsack.

Finally, we consider CNF satisfiability for the case of CNF formulas \(F\) with number of clauses linearly bounded in the number of variables. Note that we do not make any assumptions on the size of clauses in \(F\). More precisely, let \(CNF_c\) denote CNF formulas \(F\) such that the number of clauses in \(F\) is bounded by \(cn\) for the given constant \(c\), where \(n\) is the number of variables. We give a \(2^{(1-\alpha)n/2}\) quantum algorithm for satisfiability of inputs from \(CNF_c\), where \(\alpha\) is a constant depending on \(c\).

1.1 Preliminaries

The set \(\{0, 1\}^n\) denotes the set of binary strings of length \(n\), and for a positive integer \(N\) we denote the set \(\{1, 2, \ldots, N\}\) by \([N]\).

We design new quantum algorithms for the following NP-hard problems.

0-1 Knapsack

Input: A list of positive integers \(c_1, c_2, \ldots, c_n\) and a positive integer \(K\).

Problem: Is there a subset \(S \subseteq [n]\) such that \(\sum_{i \in S} c_i = K\).

Actually, we consider is 0-1 Integer Linear Programs in general.

0-1 ILP

Input: Integers \(a_{ij}\) and \(b_i\), \(1 \leq j \leq n\) and \(1 \leq i \leq d\).

Problem: Is the following set of linear inequalities feasible?

\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad x_j \in \{0, 1\}, \quad 1 \leq j \leq n, \quad 1 \leq i \leq d.
\]

We use standard definitions from quantum computing from \[3\], \[4\]. In particular, we use the quantum query model (c.f. \[4\], \[5\]). In this model the input is a function \(f : [N] \rightarrow [M]\) and the values of \(f\) is accessed by oracle queries. The complexity of computing some property of \(f\) is measured as the number of oracle queries. A quantum computation with \(T\) queries can be seen as a sequence of unitary transformations:

\[
U_0 \rightarrow O \rightarrow U_1 \rightarrow \cdots O \rightarrow U_T,
\]

where the unitary transform \(O\) implements the oracle access to \(f\) and the \(U_i\) are arbitrary unitary transformations which do not depend on the input \(f\).

A central idea from quantum computing, which is a generalized form of Grover’s search algorithm that we use throughout the paper is amplitude amplification \[5\]: Essentially, if we have a
quantum algorithm $A$ with success probability $p$ then the success probability can be amplified to a constant by $O(\sqrt{1/p})$ calls to $A$ and $A^{-1}$.

A range tree (see § for details) is a data structure for storing a set $S$ of $N$ elements $\{x_1, x_2, \ldots, x_N\}$, each of which is a $d$-tuple of integers $x_i = (x_{i1}, x_{i2}, \ldots, x_{id})$. The data structure can be built in time $N \log^d N$ and needs space $N \log^d N$. The specific property of interest that we require is that we can process range queries on a range tree in time $\log^d N$ to retrieve one element, if it exists, of the set $S$ that satisfies the query bounds. More precisely, given a $d$-tuple of real numbers $(a_1, a_2, \ldots, a_d)$, in time $O(\log^d N)$ we can search for an element $x_i = (x_{i1}, x_{i2}, \ldots, x_{id})$ in the range tree such that $x_{ij} \leq a_j$, $1 \leq i \leq d$.

2 New Quantum Algorithms for 0-1 Integer Linear Programs

**Theorem 1** There is an $\tilde{O}(2^{n/3} n^d)$ quantum algorithm with constant error probability that solves 0-1 integer linear programs with $n$ variables and $d$ inequalities.

**Proof.** Let the input instance be

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad 1 \leq i \leq d, \quad x_j \in \{0, 1\}, \quad 1 \leq j \leq n.$$ 

The goal is to find a feasible solution. We give a stepwise description of the quantum algorithm.

Step 1. Partition the variables into two sets $A = \{x_1, \ldots, x_{n/3}\}$ and $B = \{x_{n/3+1}, \ldots, x_n\}$, where note that $|A| = n/3$ and $|B| = 2n/3$.

Step 2. For each of the $2^{n/3}$ 0-1 assignments $I$ to the variables in $A$ define the $d$-tuple $y_I = (y_I(1), \ldots, y_I(d))$ of integers, where

$$y_I(i) = \sum_{j=1}^{n/3} a_{ij} I_j,$$

where $I_j$ is the value of variable $x_j$ in assignment $I$.

Step 3. Let $N = 2^{n/3}$. The set $X = \{y_I \mid I \text{ is a 0-1 assignment to } A\}$ is of size $N$. In time $\tilde{O}(N \log^{d-1} N)$ build a range tree (c.f. §, Theorem 2.11) to store the set $X$. The range tree has size $\tilde{O}(N \log^{d-1} N)$ and range queries can be processed in time $\log^d N$. This completes the preprocessing phase.

Step 4. Using the Hadamard transform prepare the uniform superposition over the set $T$ of 0-1 assignments to the $2n/3$ variables in $B$:

$$|\psi\rangle = \frac{1}{2^{n/3}} \sum_{a \in T} |a\rangle.$$ 

Step 5. Define a unitary transform $U$ as the standard reversible implementation of the following classical subroutine $f$ given $u \in T$ as input:

(i) Compute $z_j = \sum_{j=n/3+1}^{n} a_{ij} u_j$, $1 \leq i \leq d$. 

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(ii) Let \( \hat{z}_i = b_i - z_i \) for \( 1 \leq i \leq d \), giving a \( d \)-tuple of integers \( (\hat{z}_1, \ldots, \hat{z}_d) \).

(iii) Search in the range tree for a \( y_I \) such that \( y_I(1) \leq \hat{z}_1, \ldots, y_I(d) \leq \hat{z}_d \). Notice that this is a range query and it can be processed in time \( \log^d N \) in the range tree data structure \( \text{[9, Theorem 2.11]} \).

(iv) If such a tuple \( y_I \) is in \( X \) then \( f(u) = 1 \). Otherwise, \( f(u) = 0 \).

Step 6. With the initial state as the uniform superposition \( |\psi\rangle \) and using the unitary transform \( U \), apply Grover’s search algorithm to search for \( u \in T \) such that \( f(u) = 1 \).

It follows from the well-known analysis of Grover’s algorithm and amplitude amplification \( \text{[5, 7]} \) that the above algorithm has running time \( \tilde{O}(2^{n/3} \log^d N) = \tilde{O}(2^{n/3} n^d) \), with constant success probability.

**Corollary 2** There is an \( \tilde{O}(2^{n/3}) \) time quantum algorithms with constant success probability for 0-1 Knapsack. Additionally, any NP-hard optimization problem that takes the form of a 0-1 Integer Linear Program with constant number of constraints has an \( \tilde{O}(2^{n/3}) \) time quantum algorithms with constant success probability.

**Proof.** As an instance of 0-1 Knapsack consists of one equation, it can be expressed as a 0-1 integer linear program with two inequalities. Hence the algorithm of Theorem \( \text{[4]} \) yields the claimed quantum algorithm.

Consider any optimization problem that has a linear optimality function and the constraints can be expressed as a 0-1 integer linear program with \( d \) constraints. Using binary search we can reduce the optimization problem to feasibility of a 0-1 integer linear program with \( d + 1 \) constraints, which can be solved using the algorithm of Theorem \( \text{[4]} \) with running time \( \tilde{O}(2^{n/3}) \) for constant \( d \). Thus, the corresponding optimization problem can also be solved in time \( \tilde{O}(2^{n/3}) \) with constant success probability.

**Remark.** Interestingly, if the 0-1 ILP is of the following form:

\[
\begin{align*}
\text{minimize/maximize} & \quad \sum_i c_i x_i \\
\text{s.t.} & \quad \sum_{j=1}^n a_{ij} x_j = b_i, \quad 1 \leq i \leq d, \quad x_j \in \{0, 1\}, \quad 1 \leq j \leq n,
\end{align*}
\]

then it is easy to see that essentially the same quantum algorithm presented in Theorem \( \text{[4]} \) solves this optimization problem in time \( \tilde{O}(2^{n/3}) \), independent of the number of equations \( d \). The reason is that we do not have to maintain a range tree data structure as we do not have to process range queries. Just a sorted list would suffice as we only need to make exact queries in this case.

The above kind of 0-1 ILP is referred to as the 0-1 Group Problem in Nemhauser and Wolsey’s book \( \text{[8]} \). An interesting instance of this problem is the following NP-hard problem: given a CNF formula \( F \) (no restrictions on clause size), search for a satisfying assignment that satisfies exactly one variable in each clause. This is clearly an instance of the above 0-1 ILP. Thus we have the following corollary.

**Corollary 3** There is an \( \tilde{O}(2^{n/3}) \) quantum algorithm that takes a CNF formula \( F \) with \( n \) variables as input (no restrictions on clause size or number of clauses) and searches, with constant success probability, for a satisfying assignment that satisfies exactly one variable in each clause.
3 The Symmetric Claw Problem

In order to study how far we can exploit this idea of dividing the input, we examine in this section a black-box version of the 0-1 Knapsack problem which we term as the Symmetric Claw problem.

For a pair of strings \( x, y \in \{0, 1\}^n \) let \( x|_y \) denote the substring of \( x \) obtained by projecting it on the positions where \( y \) is 1. Likewise, let \( x|_{\bar{y}} \) denote the substring of \( x \) obtained by projecting it on the positions where \( y \) is 0.

**Definition 4** The input to the symmetric claw problem is a function \( P : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}^m \times \{0, 1\}^m \) where we write \( P(x,y) = (P_1(x,y), P_2(x,y)) \) for every \( x,y \in \{0,1\}^n \). Additionally, the input function \( P \) is such that it fulfills the following promise symmetry conditions:

1. For \( x \in \{0, 1\}^n \), if \( P_1(x,y) = P_2(x,y) \) for some \( y \in \{0, 1\}^n \) then \( P_1(x,y) = P_2(x,y) \) for every \( y \in \{0, 1\}^n \).
2. For any \( x, x', y' \in \{0, 1\}^n \) if \( x|_y = x'|_y \) then \( P_1(x,y) = P_1(x',y) \). Likewise, if \( x|_{\bar{y}} = x'|_{\bar{y}} \) then \( P_2(x,y) = P_2(x',y) \).

Given such an input \( P \) the symmetric claw problem is to find an \( x \in \{0, 1\}^n \), if it exists, such that \( P_1(x,y) = P_2(x,y) \) for some \( y \in \{0, 1\}^n \), by querying \( P \) on \( (x,y) \in \{0, 1\}^n \times \{0, 1\}^n \).

Notice that the 0-1 Knapsack problem is an instance of the symmetric claw problem.

**Remark.** Consider a simpler version of the 0-1 Knapsack problem, where we seek a 0-1 assignment to variables \( x_1, \ldots, x_n \) such that \( \sum_{i=1}^n a_i x_i = 0 \), given integers \( a_1, \ldots, a_n \) as input.

For each \( y \in \{0, 1\}^n \) define a partition of the variable set \( A = \{ i \in [n] \mid y_i = 1 \} \) and \( B = \{ i \in [n] \mid y_i = 0 \} \). Define \( P_1(x,y) = \sum_{i \in A} a_i x_i \) and \( P_2(x,y) = -\sum_{i \in B} a_i x_i \).

It is easy to check that \( P_1 \) and \( P_2 \) fulfill conditions of Definition 4, making this version of 0-1 Knapsack an instance of the symmetric claw problem.

We first establish the upper bound result for symmetric claw problem. It is the same algorithm we described in the previous section.

**Theorem 5** There is an \( \tilde{O}(2^{n/3}) \) quantum algorithm with constant error probability for the symmetric claw problem in the quantum query model.

**Proof.** The algorithm is along exactly the same lines as the algorithm in Theorem 1 for 0-1 ILP. In fact it is an easier version of that algorithm as we do not require range trees here.

We fix \( y \in \{0, 1\}^n \) to be such that \( y_i = 1 \) for \( 1 \leq i \leq n/3 \) and \( y_i = 0 \) for \( n/3 + 1 \leq i \leq n \).

Let \( X_A = \{ x \in \{0, 1\}^n \mid x_i = 0 \text{ for } n/3 + 1 \leq i \leq n \} \) and \( X_B = \{ x \in \{0, 1\}^n \mid x_i = 0 \text{ for } 1 \leq i \leq n/3 \} \).

Compute and sort the list \( \{ P_1(x,y) \mid x \in X_A \} \) of size \( 2^{n/3} \) using a classical sorting algorithm in time \( \tilde{O}(2^{n/3}) \). This is the preprocessing phase.

Prepare a superposition of \( x \in X_B \) and search for \( P_2(x,y) \) in the sorted list using a classical binary search implemented by a unitary transform \( U \) defined as: For \( x \in X_B \), \( U|x \rangle = 1 \) if \( P_2(x,y) \) occurs in the sorted list and \( U|x \rangle = 0 \) otherwise.

Now, using Grover’s search we can find \( x \in X_B \) such that \( P_1(x',y) = P_2(x,y) \) for some \( x' \in X_A \). At this point, using the properties of \( P_1 \) and \( P_2 \) we can put together a complete 0-1 assignment \( x'' \)
from $x$ and $x'$ such that $P_1(x'',y) = P_2(x'',y)$. We can obtain $x''$ by concatenating the first $n/3$ bits from $x'$ and the last $2n/3$ bits from $x$.

We now turn to the question of lower bounds. Using the technique of Ambainis we establish a lower bound of $\Omega(2^{n/4})$ for the quantum query complexity of the symmetric claw problem. We first state the result in a form that we need for our setting.

**Theorem 6** Let $\mathcal{F}$ be the set of input functions $f$, where $f : [M] \rightarrow [M]$, and let $\phi : \mathcal{F} \rightarrow \mathbb{Z} \times \mathbb{Z}$ be the function which we wish to compute. Let $X,Y$ be two subsets of $\mathcal{F}$ such that $\phi(f) \neq \phi(g)$ for all $f \in X$ and $g \in Y$. Let $R \subseteq X \times Y$ such that:

(i) For every $f \in X$ there are at least $m$ different $g \in Y$ such that $(f,g) \in R$.

(ii) For every $g \in Y$ there are at least $m'$ different $f \in X$ such that $(f,g) \in R$.

(iii) For every $f \in X$ and $x \in [N]$, there are at most $l$ different $g \in Y$ such that $(f,g) \in R$ such that $f(x) \neq g(x)$.

(iv) For every $g \in Y$ and $x \in [N]$, there are at most $l'$ different $f \in X$ such that $(f,g) \in R$ such that $f(x) \neq g(x)$.

Any quantum algorithm that evaluates $\phi$ with constant success probability must make $\Omega(\sqrt{mm'})$ queries to the input function in the quantum query model.

Now we are ready to prove our lower bound result of this section.

**Theorem 7** Any quantum algorithm that solves the symmetric claw problem with constant success probability needs $\Omega(2^{n/4})$ queries in the quantum query model.

**Proof.** We will prove the claimed lower bound by applying Theorem 6. The inputs consist of the class $\mathcal{F}$ of functions $P : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^n \times \{0,1\}^n$, where we write $P(x,y) = (P_1(x,y), P_2(x,y))$, and every $P$ in $\mathcal{F}$ fulfills the conditions of the Definition 4. The goal of a quantum algorithm $\phi$ for the symmetric claw problem is to find $x \in \{0,1\}^n$ such that $P_1(x,y) = P_2(x,y)$ for some (and therefore, by the promise of Definition 4, every $y$). Let NO denote the subset of “no” instances of $\mathcal{F}$ and YES the subset of “yes” instances. Define $X \subseteq NO$ as:

$X = \{P \in \mathcal{F} \mid P_1(x,y) \neq P_2(x,y) \forall x,y \in \{0,1\}^n, P_1(x,y) \neq P_1(x',y) \text{ if } x|_y \neq x'|_y \text{ and } P_2(x,y) \neq P_2(x',y) \text{ if } x|_\overline{y} \neq x'|_\overline{y}\}$.

We will define the subset $Y$ of YES by first defining a relation $R \subseteq X \times YES$ and then setting

$Y = \{P' \in YES \mid (P,P') \in R \text{ for some } P \in X\}$. We now define the relation $R$.

Given $P \in X$, for every $x \in \{0,1\}^n$ we define a distinct $P' \in YES$ and include $(P,P')$ in $R$. Given $P \in X$ and $x \in \{0,1\}^n$, $P'$ is obtained from $P$ as follows:

- If $\sum_{i=1}^n y_i \geq n/2$ then define $P'_1(x,y) = P_1(x,y)$, and $P'_2(x,y) = P_1(x,y)$. Furthermore, for every $x' \in \{0,1\}^n$ such that $x'|_\overline{y} = x|_\overline{y}$ define $P'_2(x',y) = P_1(x,y)$. For such $x'$, notice that by definition of $X$ we have $P'_1(x',y) \neq P_1(x,y)$.

- If $\sum_{i=1}^n y_i < n/2$ then define $P'_2(x,y) = P_2(x,y)$, and $P'_1(x,y) = P_2(x,y)$. Furthermore, for every $x' \in \{0,1\}^n$ such that $x'|_y = x|_y$ define $P'_1(x',y) = P_2(x,y)$. For such $x'$, notice that by definition of $X$ we have $P_2(x',y) \neq P_2(x,y)$.
• For all other pairs \((u,v) \in \{0,1\}^n \times \{0,1\}^n\) define \(P'_1(u,v) = P_1(u,v)\) and \(P'_2(u,v) = P_2(u,v)\).

Let \(P'\) be defined as above for a given \(P \in X\) and \(x \in \{0,1\}^n\). Notice that by construction \(P'\) is in YES and, furthermore, \(x\) is the unique element of \(\{0,1\}^n\) such that \(P'_1(x,y) = P'_2(x,y)\) for \(y \in \{0,1\}^n\). Thus we define \(2^n\) distinct elements \(P' \in YES\), one for every \(x \in \{0,1\}^n\), such that \((P, P') \in R\). Therefore, in the terminology of Theorem 8, \(m = 2^n\). For the remainder of the proof, we denote by \(P_x'\) the element \(P' \in YES\) defined as above from an \(x \in \{0,1\}^n\) and a given \(P \in X\).

We next show that for any \(P \in X\) and \((x,y) \in \{0,1\}^n \times \{0,1\}^n\) there are at most \(2^{n/2}\) elements \(P' \in Y\) such that \(P(x,y) \neq P'(x,y)\) and \((P, P') \in R\). In terms of Theorem 8 this implies \(l \leq 2^{n/2}\).

We argue this in two cases:

Case 1. Suppose \(\sum^n_{i=1} y_i \geq n/2\). Then notice from the definition of \(R\) that \(P'_x\) differs from \(P\) at \((x,y)\) exactly when \(x|_y = z|_y\). Thus, there are at most \(2^{n/2}\) such elements \(P'_x\).

Case 2. Suppose \(\sum^n_{i=1} y_i < n/2\). Again, notice from definition of \(R\) that \(P'_x\) differs from \(P\) at \((x,y)\) exactly when \(x|_\overline{y} = z|_\overline{y}\). Thus, again, there can be at most \(2^{n/2}\) such elements \(P'_x\).

Finally, notice that \(l' \leq m'\) always holds.

We can now apply the lower bound result Theorem 8. Since

\[
\frac{m m'}{ll'} \geq \frac{m}{l} \geq \frac{2^n}{2^{n/2}},
\]

The lower bound of \(\Omega(\sqrt{\frac{m m'}{l l'}})\) given by Theorem 8 yields \(\Omega(2^{n/4})\). This completes the proof.

4 Quantum Algorithm for CNF-SAT

In this section we use ideas from Section 2 to give a simple quantum search algorithm for CNF-SAT when the number of clauses is linearly bounded in the number of variables. More precisely, for any constant \(c > 0\) we define the following class of CNF formulas:

\(\text{CNF}_c = \{ F \mid F \text{ is in conjunctive normal form with } n \text{ variables and } F = C_1 \land C_2 \land \ldots \land C_m \text{ where } m \leq cn\}\).

**Theorem 8** For any constant \(c > 0\) there is an \(\tilde{O}(2^{(1-\alpha)n/2})\) quantum algorithm for satisfiability for inputs \(F \in \text{CNF}_c\), where \(\alpha\) is a constant such that \(\alpha < 1/6\) and \(H(\alpha) \leq \frac{1}{4c}\). Here, \(H(\alpha)\) is the entropy function defined as \(-\alpha \log_e \alpha - (1-\alpha) \log_e (1-\alpha)\).

**Proof.** Let \(\alpha < 1/6\) be a constant to be fixed later, as claimed in the statement. Let the input instance from \(\text{CNF}_c\) be \(F = C_1 \land C_2 \land \ldots \land C_m\) from \(\text{CNF}_c\) with variables \(x_1, x_2, \ldots, x_n\). Let \(\frac{1}{\alpha} = k\), where we assume for simplicity of analysis that \(k\) is an integer. Partition the variable set \(\{x_1, x_2, \ldots, x_n\}\) into \(k\) equal size sets \(A_1, A_2, \ldots, A_k\), where \(|A_i| = \alpha n\), \(1 \leq i \leq k\). Suppose \(F\) is satisfiable and \(a^*\) is some fixed satisfying assignment for \(F\). Denote by \(A_i\) the set of variables \(\{x_1, \ldots, x_n\} \setminus A_i\) for each \(i\). Denote by \(b^*_i\) the partial assignment given by \(a^*\) when restricted to variables in \(A_i\). Similarly, let \(c^*_i\) denote the partial assignment given by \(a^*\) when restricted to variables in \(A_i\).

**Claim.** There is an \(i\) : \(1 \leq i \leq k\) such that the partial assignment \(c^*_i\) satisfies at least \((1-\alpha)m\) clauses of \(F\).
Suppose the claim is false. Then \( \overline{A_i} \) satisfies at most \((1 - \alpha)m\) clauses of \( F\) for each \( i\). Since \( a^* \) is a satisfying assignment, for each \( i \) there is a set \( S_i \) of more than \( \alpha m \) clauses satisfied by the partial assignment \( b_i^* \) such that \( S_i \cap S_j = \emptyset \) for \( i \neq j \). This is impossible since there are only \( m \) clauses in \( F \). The claim follows.

We now give a step-wise description of the algorithm. The algorithm is a loop with index \( i : 1 \leq i \leq k \), where in the \( i \)th iteration it considers the partition of the variable set into \( A_i \) and \( \overline{A_i} \). The algorithm succeeds when it considers a partition \( A_i \) and \( \overline{A_i} \) such that \( c_i^* \) satisfies at least \((1 - \alpha)m\) clauses of the input \( F \) (such an index exists by the claim).

**Step 1.** Partition the variable set into \( A_i \) and \( \overline{A_i} \).

**Step 2.** For \( 1 \leq l \leq \alpha m \) build a sorted table \( T_i \) consisting of the following set of pairs \((u, b)\): \( u \in \{0, 1\}^m \) is an \( m \)-bit string with exactly \( l \) 1’s and \( b \) is a truth assignment to variables in \( A_i \) such that \( b \) satisfies each clause \( C_j \) for which \( u_j = 1 \), where \( 1 \leq j \leq m \). All other pairs \((u, b)\) are discarded.

Then, using a classical sorting algorithm and treating \( u \) in \((u, b)\) as an \( m \)-bit integer key, sort the pairs in \( T_i \) in increasing order. This entire step can be done in time \( \tilde{O}((\frac{m}{\alpha})2^\alpha n) \).

**Step 3.** Using the Hadamard transform prepare a uniform superposition of the truth assignments \( v \) to variables in \( \overline{A_i} \). Let \( S \) be the set of all such truth assignments. Notice that \(|S| = 2^{(1 - \alpha)m}\).

The superposition is

\[
|\psi\rangle = \sqrt{\frac{1}{2^{(1 - \alpha)m}}} \sum_{v \in S} |v\rangle.
\]

**Step 4.** Define a unitary transform \( U \) as the standard reversible implementation of the following classical subroutine \( f \):

(i) Compute the vector \( u \in \{0, 1\}^m \) where \( u_j = 1 \) if and only if \( C_j \) is not satisfied by the partial assignment \( v \).

(ii) If \( u \) has more than \( \alpha m \) 1’s then \( f(v) \) returns value 0 and stop.

(iii) Otherwise, if \( u \) has \( l\) 1’s in it, \( 1 \leq l \leq \alpha m \), then do a binary search for \( u \) in the table \( T_i \).

If \((u, b)\) is found in \( T_i \) for some \( b \) then \( f(v) = 1 \) else \( f(v) = 0 \). Notice that if \( f(v) = 1 \) then \( v \) and \( b \) together give a satisfying assignment for \( F \).

**Step 5.** Now, starting with initial state as \( |\psi\rangle \) and using the unitary transform \( U \) to implement \( f \), apply Grover search to find \( v \in S \) such that \( f(v) = 1 \).

**Step 6.** Output the satisfying assignment \( (b, v) \).

To argue correctness it suffices to notice by our earlier claim that for some \( i \), \( A_i \) and \( \overline{A_i} \) is a partition such that in Step 3, \( f(v) = 1 \) for at least the partial assignment \( v = c_i^* \) obtained from \( a^* \).

Thus, if \( F \) is satisfiable the algorithm will find a satisfying assignment with constant success probability in time \( \tilde{O}((\frac{m}{\alpha})2^\alpha n) + \tilde{O}(2^{(1 - \alpha)n/2}) \), where the first term is the preprocessing time in Step 2, and the second term is the time for the Grover search.

Since \( m \leq cn \), an easy analysis yields \( (\frac{m}{\alpha}) \leq 2H(\alpha)c + \alpha \) for large \( n \). Thus, if we choose \( \alpha < 1/6 \) such that \((1 - \alpha)/2 \geq H(\alpha)c + \alpha \), then the overall time taken is \( \tilde{O}(2^{(1 - \alpha)n/2}) \). It suffices to choose \( \alpha \) such that \( H(\alpha) < \frac{1}{2c} \). ■
5 Discussion

The question that arises is whether we can close the gap between the upper and lower bounds for the symmetric claw problem. Also, for the 0-1 ILP problem with \( n \) variables and \( d \) inequalities we do not have a better upper bound than \( \tilde{O}(2^{n/3})n^d \). To study this issue in the quantum query model we define the simultaneous claw problem.

**Definition 9** Let \((f_i, g_i), 1 \leq i \leq d\) be functions where \( f_i : [N] \to X \) and \( g_i : [N] \to X \), where \( X \) is some set. The simultaneous claw problem is to find an \( x \in [N] \) such that \( f_i(x) = g_i(x) \) for each \( i = 1, 2, \ldots, d \).

We can also define a symmetric version of the simultaneous claw problem similar to Definition 4 but for simplicity we focus on this definition. It is known \(^4\) that for \( d = 1 \) there is an \( \tilde{O}(N^{3/4}) \) upper bound and the recent new techniques of Aaronson \(^1\) followed by Shi’s sharpened results \(^11\) imply an \( \Omega(N^{2/3}) \) lower bound for the problem.

We show here that the simultaneous claw problem has a quantum upper bound of \( \tilde{O}(N^{3/4} \log^d N) \). On the other hand, we are not able to strengthen the lower bound of \( \Omega(N^{2/3}) \) which already holds for a single claw pair \((f, g)\). The following theorem is easy to prove.

**Theorem 10** There is a quantum algorithm that takes as input a collection \((f_i, g_i), 1 \leq i \leq d\) of functions where \( f_i : [N] \to X \) and \( g_i : [N] \to X \), where \( X \) is some set, which makes \( \tilde{O}(N^{3/4} \log^d N) \) queries in the quantum query model and outputs a simultaneous claw \( x \in [N] \), if it exists, such that \( f_i(x) = g_i(x) \) for each \( i = 1, 2, \ldots, d \).

We recall the \( r \)-to-1 collision problem: given as input a function \( f : [N] \to X \) with the promise that \( f \) is either 1-1 or \( r \)-to-1, the problem is to find a pair \( x \neq y \in [N] \) such that \( f(x) = f(y) \). The recent results of Aaronson \(^1\) followed by Shi \(^11\) imply an \( \Omega(N^{1/3}) \) lower bound for the 2-to-1 collision problem.

Analogous to the simultaneous claw problem we can define the simultaneous 2-to-1 collision problem which is given as input functions \( f_i : [N] \to X \), \( 1 \leq i \leq d \), with the promise that either all the \( f_i \) are 1-1 or all are 2-to-1, find a pair \( x \neq y \in [N] \) if it exists such that \( f_i(x) = f_i(y) \) for each \( i \). The \( \Omega(N^{1/3}) \) lower bound of Shi \(^11\) clearly holds but we have nothing better than that. On the other hand, we have a straightforward algorithm that gives an upper bound for the problem.

**Theorem 11** There is a quantum algorithm that takes as input a collection \( f_i, 1 \leq i \leq d \) of functions \( f_i : [N] \to X \) where \( X \) is some set, makes \( \tilde{O}(N^{3/4} \log^d N) \) queries in the quantum query model and outputs a simultaneous collision \( x \neq y \in [N] \) such that \( f_i(x) = f_i(y) \) for each \( i = 1, 2, \ldots, d \), if it exists.

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