The fracture of disordered media represents an important applied problem, with intriguing theoretical aspects. Statistical models have been successfully applied in the past to analyze fracture under quasistatic conditions, but the effect of cyclic loading is less explored [1]. Laboratory experiments reveal that fatigue failure under repeated loading is due to a combination of several mechanisms, among which damage growth, relaxation due to viscoelasticity, and healing of microcracks play an essential role [2, 3, 4]. Theoretical approaches have serious difficulties to capture all these mechanisms [2, 3, 4, 5, 6, 7] and fatigue life prediction is still very much an empirical science. Understanding this problem has crucial implications even for everyday applications. For example, fatigue failure occurring in roads due to repeated traffic loading cause main distress, limiting the lifetime of asphalt pavements.

In this Letter we present a detailed experimental and theoretical study of the fatigue performance of hot mix asphalt (HMA). We carried out fatigue life tests of specimens measuring the accumulation of deformation with the number of loading cycles and the lifetime of specimens varying the load amplitude. To obtain a theoretical understanding of the experimental findings, we worked out two novel modelling approaches for fatigue failure, namely, a fiber bundle model [8, 9] and a fuse model [10]. We then show that both descriptions capture the stochastic nature of the fracture process, the immediate breaking of material elements and the cumulative effect of the loading history. Two physical mechanisms are considered which limit the accumulation of damage: a finite activation threshold of crack nucleation below which the local load does not contribute to the ageing of the material and healing of microcracks under compression, which leads to damage recovery. The analytical and numerical results of the model calculations provide a good quantitative agreement with the experimental findings. We show that the competition of nucleation and healing of microcracks leads to novel type of scaling laws for fatigue fracture with universal scaling exponents.

In order to obtain a quantitative characterization of the process of fatigue failure, we carried out fatigue life tests of asphalt under cyclic diametric compression of cylindrical specimens at a constant external load \( \sigma_0 \) (see Fig. 1). HMA is the primary material used to construct and maintain pavements and roadways due to its good mechanical performance and high durability. From the structural point of view asphalt is a combination of aggregates (usually crushed stone and sand), filler (cement, hydrated lime or stone dust) and a bituminous binder. Cylindrical samples of HMA were produced using the Marshall...
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same load
occurs after a finite number of cycles
extreme, a lower threshold value of the external load
an immediate failure of the specimen. At the other ex-
loads 30% and 40% of the tensile strength
First, approaching the tensile strength of the material
reveals the existence of three distinct regimes.
results follow a power law known as Basquin law [2, 3, 4, 12]
FIG. 2: Deformation as a function of the number of loading
cycles. The continuous lines result from our theory. Inset: Load
on the fibers in FBM as function of time at different values of \( \sigma_0/\sigma_c \) for uniformly distributed threshold values setting \( \tau = \infty \) in Eq. 2.

method [11] and then loaded by a hydraulic device. Un-
der repeated loading at a constant amplitude \( \sigma_0 \), the de-
formation \( \varepsilon \) was monitored as a function of the number
of cycles \( N_{\text{cycle}} \). Furthermore, the total number of cycles
to complete failure \( N_f \) was measured varying \( \sigma_0 \). Figure
presents representative examples of \( \varepsilon (N_{\text{cycle}}) \) recorded
at loads 30% and 40% of the tensile strength \( \sigma_c \) of the
specimen. It can be observed that due to the gradual ac-
accumulation of damage, the deformation \( \varepsilon \) caused by the
same load \( \sigma_0 \) monotonically increases until catastrophic
failure occurs after a finite number of cycles \( N_f \). The
 derivative of \( \varepsilon (N_{\text{cycle}}) \) also shows a monotonous increase
and diverges when approaching the point of macroscopic
failure. Increasing the external load the functional form
of \( \varepsilon (N_{\text{cycle}}) \) remains the same, however, the lifetime of
the specimen \( N_f \) gets shorter. The fatigue lifetime \( N_f \)
measured at different fractions of the tensile strength \( \sigma_c \)
(Fig. 3) reveals the existence of three distinct regimes.

First, approaching the tensile strength of the material
\( \sigma_0/\sigma_c \rightarrow 1 \) the lifetime \( N_f \) rapidly decreases indicating
an immediate failure of the specimen. At the other ex-
treme, a lower threshold value of the external load \( \sigma_t \)
can be identified below which the specimen suffers only
partial damage giving rise to an infinite lifetime (fatigue
limit). In the intermediate regime the experimental re-
sults follow a power law known as Basquin law [2, 3, 4, 12]
\[
N_f \sim \left( \frac{\sigma_0}{\sigma_c} \right)^{-\alpha},
\]
where \( \alpha = 2.2 \pm 0.1 \), as shown in Fig. 3.

The experiments show that the fatigue crack growth is
localized to a narrow region between the loading plates
(see Fig. 1b) where locally a tensile stress emerges per-
pendicular to the external load [12]. To give a theoretical
description of the failure process, we focus on this region
and discretize it by a fiber bundle model (FBM) as illus-
trated in Fig. 1: a, b, c. We consider a bundle of parallel
linear elastic fibers with the same Young modulus \( E \). Un-
der diametrical compression of the disc-shaped specimen,
the fibers experience a tensile loading and gradually fail
due to immediate breaking or to the ageing of material
elements [2]. More precisely, the following two mecha-
nisms are considered: (I) fiber \( i \) \( (i = 1, \ldots, N) \) breaks
instantaneously at time \( t \) when its local load \( p_i(t) \) ex-
ceeds the tensile strength \( p_{th}^i \) of the fiber. (II) All intact
fibers undergo a damage accumulation process due to the
load they have experienced. The amount of damage \( \Delta c_i \)
ocurred under the load \( p_i(t) \) in a time interval \( \Delta t \) is
assumed to have the form \( \Delta c_i = ap_i(t)^{\gamma} \Delta t \) [12], hence,
the total accumulated damage \( c_i(t) \) until time \( t \) can be
obtained by integrating over the entire loading history [12].
The exponent \( \gamma > 0 \) controls the damage accumu-
lation rate and \( a > 0 \) is a scale parameter. The fibers
can only tolerate a finite amount of damage and break
when \( c_i(t) \) exceeds a threshold value \( c_{th}^i \). The two break-
ning thresholds \( p_{th}^i \) and \( c_{th}^i \) are random variables with a
joint probability density function \( h(p_{th}, c_{th}) \). Assuming
independence of the two breaking modes, \( h \) can be fac-
torized into a product \( h(p_{th}, c_{th}) = f(c_{th})g(p_{th}) \), where
\( f(c_{th}) \) and \( g(p_{th}) \) are the probability densities and \( F(c_{th}) \)
and \( G(p_{th}) \) the cumulative distributions of the breaking
thresholds \( p_{th} \) and \( c_{th} \), respectively. For simplicity, af-
fter each breaking event the load of the broken fiber is
equally redistributed over the intact ones irrespective of
their distance from the failure point (global load sharing)
Under a constant tensile load $\sigma_0$, the load on a single fiber $p_0$ is initially determined by the quasi-static constitutive equation of FBM $\sigma_0 = [1 - G(p_0)] p_0$. The external load $\sigma_0$ must fall below the tensile strength of the bundle $\sigma_0 < \sigma_c$, otherwise the bundle will fail immediately. As time elapses, the fibers accumulate damage and break due to their finite damage tolerance. These breakings, however, increase the load on the remaining intact fibers which in turn induce again immediate breakings. This way, in spite of the independence of the threshold values $p_{th}$ and $c_{th}$, the two breaking modes are dynamically coupled, gradually driving the system to macroscopic failure in a finite time $t_f$ at any load values $\sigma_0$. Healing of microcracks can be captured in the model by introducing a finite range $\tau$ for the memory, over which the loading history contributes to the accumulated damage $\overline{\delta}$. Finally, the evolution equation of the system can be cast in the form

$$\sigma_0 = [1 - F(a \int_0^t \exp \left( \frac{-t'}{\tau} \right) p(t')^\gamma dt')] [1 - G(p(t))] p(t),$$

where the integral in the argument of $F$ provides the accumulated damaged at time $t$ taking into account the finite range of memory by the exponential term $\overline{\delta}$. In principle, the range of memory $\tau$ can take any positive value $\tau > 0$ such that during the time evolution of the bundle the damage accumulated during the time interval $t' < t - \tau$ heals. Equation (2) is an integral equation which has to be solved for the load $p(t)$ on the intact fibers at a given external load $\sigma_0$ with the initial condition $p(t = 0) = p_0$. The product in Eq. (2) arises due to the independence of the two breaking thresholds. We note that Eq. (2) recovers the usual constitutive behavior of FBM [8] when damage accumulation is suppressed either by increasing the exponent $\gamma$ or decreasing the range of memory $\tau \to 0$.

The inset of Fig. 2 presents examples of the solution $p(t)$ of Eq. (2) obtained for breaking thresholds uniformly distributed in the interval $[0, 1]$ at different ratios $\sigma_0/\sigma_c$ setting $\tau \to \infty$. Since $p(t)$ is simply related to the macroscopic deformation $\varepsilon$ of the bundle $p(t) = E_\varepsilon(t)$, these results can directly be compared to the experimental findings. The agreement seen in Fig. 2 is obtained using Weibull distributions $P(x) = 1 - \exp \left[ - (x/\lambda)^m \right]$ for the two breaking thresholds in Eq. (2) with the same Weibull exponent $m$ and scale parameter $\lambda$. Our calculations also reveal that approaching macroscopic failure, the derivative of $p(t)$ has a power law divergence as a function of the time to failure $t_f \sim (t_f - t)^{-\beta}$. Fig. 4 shows that the exponent $\beta$ solely depends on the type of disorder. Specifically, for breaking thresholds distributed over a finite and infinite range, we obtain the exponents $\beta = 1.5 \pm 0.02$ and $\beta = 1.8 \pm 0.06$, respectively, defining two different universality classes of the fatigue failure.

It is possible to recover the Basquin law Eq. (1) from Eq. (2), i.e. it can be shown analytically that for $\sigma_0/\sigma_c << 1$ and $\tau \to \infty$ the lifetime of the system has a power law dependence on the external load $t_f \sim \left( \frac{\sigma_0}{\sigma_c} \right)^{-\gamma}$, where $\gamma$ is the damage accumulation exponent, independent on the type of disorder. Figure 4 shows that the numerical results are in excellent agreement with the above analytic prediction.

Due to Eq. (2), without healing ($\tau \to \infty$) the cumulative effect of the loading history gives rise to a macroscopic failure of the system at any load. However, our experiments revealed that damage recovery caused by healing of microcracks results in a finite fatigue limit $\sigma_f$, below which the sample does not break. Since healing takes place in the polymer binder, it can be controlled by changing the temperature [9]. In our FBM the healing of microcracks is captured by the finite range of memory $\tau$. This is illustrated in Fig. 5 presenting the lifetime $t_f$ for a fixed load varying the value of $\tau$ over a broad range. It can be seen that by decreasing $\tau$, the lifetime of the system increases and, due to the competition between nucleation of new microcracks and healing of the existing ones, a finite critical value $\tau_c$ emerges below which the system only suffers a partial damage and has an infinite lifetime $t_f \to \infty$. Plotting the lifetime $t_f$ as function of the distance from the critical point $\tau_c$ (inset of Fig. 5) we find a power law dependence of $t_f$

$$t_f \sim (\tau - \tau_c)^{-\delta},$$

where $\delta = 0.5 \pm 0.01$ is an universal exponent, independent of the type of disorder and of the damage accumulation exponent $\gamma$. Consequently, at a given temperature where the system is parameterized by a fixed value of $\tau$, a finite fatigue limit $\sigma_f$ emerges at which (Fig. 6) the lifetime diverges [9].
limits damage accumulation is a finite activation threshold of microcrack nucleation. In order to study this effect we consider the random fuse model (RFM) of fracture and extend it by introducing a history dependent ageing variable of fuses. We construct an $L \times L$ tilted square lattice of initially fully intact bonds with identical conductance but random failure thresholds $i_c$. The threshold values $i_c$ are uniformly distributed between a small current value $i_0$ and 1 ($i_0 << 1$). For a given value of current $I$ applied between two bus bars of the lattice, the local current through each bond is determined by solving numerically the Kirchhoff equations. Fuses burn out irreversibly when the current exceeds the local failure thresholds. This process is then followed by the recalculation of the current values. In order to capture fatigue cracking in the model, intact fuses are assumed to undergo an ageing process, modeled by a variable $A(t) = \sum_{i' \neq i} a(i(t') - i_0)$. A fuse fails due to fatigue when $A(t) > A_{\text{max}}$, where $A_{\text{max}}$ is a failure threshold uniformly distributed between $1 - b$ and $1 + b$ with $b = 0.1$. Comparing the two modelling approaches, the ageing variable $A(t)$ of RFM is analogous to the accumulated damage $c(t)$ of FBM, however, only current values above $i_0$ contribute to $A(t)$, which captures the finite activation threshold of microcrack nucleation. The inset of Fig. 4 demonstrates that RFM of ageing fuses provides qualitatively the same behavior as FBM, i.e. rapid failure at high current values $I$, a Basquin regime Eq. 1 at intermediate currents with an exponent equal to $\gamma$ and a finite fatigue limit $\sigma_I$ determined by the threshold current $i_0$.

In summary, our experiments on the fatigue failure of asphalt under cyclic compression revealed three regimes of the failure process depending on the load amplitude: instantaneous breaking, a Basquin regime of a power law decrease of lifetime and the existence of a fatigue limit below which no failure occurs. We introduced two novel modeling approaches both capturing the essential ingredients of the fatigue failure of bituminous materials. These models provide a comprehensive description of the experimental findings and additionally revealed novel scaling laws of fatigue fracture: approaching the macroscopic failure, the process of fatigue fracture accelerates and is characterized by a finite time power law singularity of the deformation rate. The exponent is different for bounded and unbounded disorder distributions defining two universality classes of fatigue fracture. Due to the interplay between damage and healing, at each load level a critical value of the range of memory emerges where the lifetime of the system has a universal power law divergence.

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\[ \sigma \propto \tau^{\gamma} \]

**FIG. 5**: Lifetime $t_f$ of FBM at a constant load $\sigma_0/\sigma_c = 0.7$ varying $\tau$. Approaching $\tau_c$ the lifetime diverges. Inset: $t_f$ as a function of the distance from the critical range of memory $\tau - \tau_c$ for different disorder distributions and $\gamma$ exponents. The horizontal lines indicate the value of $t_f(\tau = \infty)$.

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