The most general axially symmetric electrovac spacetime admitting separable equations of motion

Naresh Dadhich$^a$, Z. Ya. Turakulov$^b$

Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India.

We obtain the most general solution of the Einstein electro-vacuum equation for the stationary axially symmetric spacetime in which the Hamilton-Jacobi and Klein-Gordon equations are separable. The most remarkable feature of the solution is its invariance under the duality transformation involving mass and NUT parameter, and the radial and angle coordinates. It is the general solution for a rotating (gravitational dyon) particle which is endowed with both gravoelectric and gravomagnetic charges, and there exists a duality transformation from one to the other. It also happens to be a transform of the Kerr-NUT solution. Like the Kerr family, it is also possible to make this solution radiating which asymptotically conforms to the Vaidya null radiation.

PACS numbers: 04.20.Jb, 04.70.Bw

I. INTRODUCTION

In spherical symmetry, vacuum spacetime is characterized by the Schwarzschild solution because the Einstein vacuum field equation can admit no other solution. The situation is however different for axial symmetry. First we have to assume stationarity. For the stationary axially symmetric spacetime, the Kerr solution is unique only under the assumptions of existence of regular horizon and asymptotic flatness. There exist asymptotically flat generalization of the Kerr solution in Tomimatsu-Sato series solutions [1,2] and asymptotically non-flat in the Kerr-NUT solution [3].

In this paper, we shall obtain the most general solution of the Einstein electro-vacuum equation for the stationary axially symmetric spacetime metric in which motion is integrable. That is, the Hamilton-Jacobi (HJ) equation for particle motion and the Klein-Gordon (KG) equation for scalar field propagation are separable. We shall henceforth term it as integrable spacetime (metric). The separability of the equations of motion was first considered long back by Carter [4]. In particular, the separability of HJ and KG equations was employed by one of us (ZYT) for obtaining the Kerr solution [5]. Integrability of motion is the primary requirement for physical understanding of spacetime. Under this physically motivated assumption, we shall seek the most general electrovac solution and show that it happens to be a transform of the Kerr-NUT solution with electric charge. The most general stationary axially symmetric electrovac spacetime with separable equations of motion is therefore characterized by the four parameters. They are the usual mass, spin and electric charge, and the NUT parameter. This spacetime is however not asymptotically flat, and the departure from it is the measure of the NUT parameter.

We first implement separability of the two equations in the most general stationary axially symmetric spacetime metric. This would lead to determining form and character of the metric coefficients. For this metric, we shall now solve the Einstein equation, which would readily give the most general solution for the electrovac spacetime. The unique solution so obtained could be transformed to the Kerr-NUT solution. If we now impose asymptotic flatness, it would reduce to the Kerr family. This is therefore a very direct and straightforward way of establishing the uniqueness of the Kerr family as well. This happens because our assumptions are more general than those of the Kerr family. It is well-known that motion is indeed integrable in the Kerr metric. Note that integrability of motion does imply existence of regular horizon but not asymptotic flatness. That is why the general family is the asymptotically non-flat Kerr-NUT family [3,6] which reduces to the Kerr family when asymptotic flatness is imposed.

One of the remarkable features of our derivation of the general solution is that it brings forth the duality between the mass $M$, and the NUT parameter, $l$. The solution is invariant under the duality transformation, $M \leftrightarrow il$, $r \leftrightarrow ia\lambda$, where $\lambda$ is an angle coordinate and $a$ is the rotation parameter. This is the duality between gravoelectric ($M$) and gravomagnetic ($l$) charges. This is therefore the most general solution for a localized source which is endowed with both gravitational electric and magnetic charges. When $l = 0$, it is the Kerr spacetime. On the other hand when $M = 0$, it is a vacuum spacetime which is dual to the Kerr spacetime [7]. One can go from one to the other by $M \leftrightarrow il$ and $r \leftrightarrow ia\lambda$; i.e. the Kerr solution goes over to the dual solution and the vice-versa. Though Kerr-NUT solution has been known for a long time, yet the dual to Kerr solution has not been considered earlier than [7]. Here the duality springs up quite naturally and obviously.
II. THE INTEGRABLE METRIC

Any coordinate system \( \{x^i\} \) in the spacetime specifies natural vector \( \{\partial_i\} \) and covector \( \{dx^i\} \) frames at each point. The metric in the covariant form is defined by the scalar product of the vectors, \( g_{ij} = \langle \partial_i, \partial_j \rangle \) while in the contravariant form by the scalar product of the covectors, \( g^{ij} = \langle dx^i, dx^j \rangle \). The stationarity and axial symmetric character of spacetime is characterized by existence of the two familiar Killing vectors, \( \partial_t \) and \( \partial_\varphi \). The intersections of \( t = \text{const.} \) and \( \varphi = \text{const.} \) are 2-dimensional surfaces which can be endowed with an orthogonal coordinate system \((u, v)\).

We introduce the coordinates \((t, u, v, \varphi)\) and write the most general stationary and axially symmetric metric in the contravariant \( g^{ij} \) form as follows:

\[
< d\varphi, d\varphi > = -A, \quad < dt, dt > = B, \quad < dt, d\varphi > = \Gamma, \quad < du, du > = -\bar{U}(u)^2/\Sigma, \quad < dv, dv > = -\bar{V}(v)^2/\Sigma.
\]

The contravariant form is chosen because this is how the metric appears in HJ equation, that is what we consider next. HJ equation for geodesics of the above metric is given by

\[
-A(\partial_\varphi S)^2 + B(\partial_t S)^2 + 2\Gamma \partial_t S \partial_\varphi S - \Sigma^{-1} (\partial_u S)^2 + (\partial_v S)^2 = m^2
\]

where \( m \) is the rest mass of the particle. As usual, the separability of the equation leads to

\[
S = Et + L\varphi + f(u) + g(v)
\]

where \( E \) and \( L \) are the two constants of motion corresponding to the two Killing vectors, \( \partial_t \) and \( \partial_\varphi \).

Substituting Eq. (3) into Eq. (2) would readily lead to

\[
A \Sigma = -U_{33}(u) + V_{33}(v), \quad B \Sigma = U_{00}(u) - V_{00}(v), \quad \Sigma = F(u) + G(v), \quad \Gamma \Sigma = U_{03}(u) + V_{03}(v).
\]

Here \( U_{ab}, \ V_{ab}, \ a, b = 0, 3 \) and \( F, \ G \) are arbitrary functions of their arguments. The determinant of the metric is given by

\[
-g = \Sigma^2(AB + \Gamma^2)^{-1}(\bar{U}\bar{V})^{-2}.
\]

Let us now turn to KG equation, which reads as

\[
\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{ab}\partial_a\phi)_{,b} = 0
\]

where comma denotes the partial derivative. Now separability of this equation leads to the separability of the block determinant, \( AB + \Gamma^2 \). That means

\[
\frac{(V_{33} - U_{33})(U_{00} - V_{00}) + (U_{03} + V_{03})^2}{(F + G)^2}
\]

must be separable; i.e. of the form \( (U(u)V(v))^{-2} \), say. We could hence write

\[
(U_{03}^2 - U_{00}U_{33}) + (V_{03}^2 - V_{00}V_{33}) + (U_{00}V_{33} + 2U_{03}V_{03} + V_{00}U_{33}) = [U(u)V(v)]^{-2}(F + G)^2.
\]

Note that \( \Sigma \) should have dimension of length square, which would mean the functions \( F(u), G(v) \) should be of this dimension. For \( F(u) \), it simply means that it is quadratic in \( u \), while \( G(v) \) must be given this dimension by
We have thus arrived at the integrable metric in which HJ and KG equations are separable, and it is given by

\[ A \Sigma = \frac{-a^2}{U^2} + \frac{1}{V^2}, \quad B \Sigma = \frac{F^2}{U^2} - \frac{G^2}{a^2 V^2}, \]
\[ \Gamma \Sigma = \frac{a F}{U^2} + \frac{G}{a V^2}, \quad \Sigma = F + G. \]

We have thus arrived at the integrable metric in which HJ and KG equations are separable, and it is given by

\[ <dt, dt> = \frac{F^2 U^{-2} - G^2 (a V)^{-2}}{F + G}, \quad <dt, d\varphi> = \frac{a U^{-2} F + (a V)^{-2} G}{F + G}, \]
\[ <d\varphi, d\varphi> = \frac{a^2 U^{-2} - V^{-2}}{F + G}, \quad <dr, dr> = \frac{-U^2}{F + G}, \quad <d\lambda, d\lambda> = -\frac{V^2}{F + G}. \]

that would read in the covariant form as follows:

\[ ds^2 = \frac{U^2}{F + G} dt^2 + 2a \frac{F V^2 + a^2 G U^2}{F + G} dt d\varphi - (F + G) \left( \frac{dr^2}{U^2} + \frac{d\lambda^2}{V^2} - \frac{F^2 V^2 - a^{-2} G^2 U^2}{F + G} d\varphi^2 \right) \]

where we have redefined the radial, \( du/\bar{U}(u) = dr/U(r) \) and the angle, \( dv/\bar{V}(v) = d\lambda/V(\lambda) \) coordinates. This is the integrable metric satisfying the condition of separability of HJ and KG equations. Alternatively, the separability of KG equation follows from the separability of HJ equation and the Einstein vacuum \( (T_{\theta\theta} = 0) \) equation [11].

Note that separability of the block determinant is also required for existence of horizon. The horizon is defined when the timelike corotating vector, \( \partial_t + \omega \partial_\varphi, \ \omega = -g_{t\varphi}/g_{\varphi\varphi}, \) turns null on the coordinate surface, \( r = \text{const}. \) It is easy to see that norm of this vector is proportional to the block determinant, and hence its separability would incorporate existence of horizon. Our integrable metric thus shares the property of existence of horizon with the Kerr family but it need not be asymptotically flat. This opens up the window for non asymptotically flat generalization of the Kerr family.

Alternatively, if we consider the null geodesics \( (m = 0) \), Eq.(2) would reduce to the following two equations,

\[ U_{00} E^2 + 2 U_{03} EL + U_{33} L^2 - f'^2 = C \]
\[ V_{00} E^2 - 2 V_{03} EL + V_{33} L^2 + g'^2 = C \]

where \( C \) is an arbitrary constant. It turns out that separability of KG equation could be traded off with the requirement that the quadratic expressions above are perfect squares.

It is known that black hole vacuum spacetimes admit event horizon which is a null surface with null geodesics as its generators. That means no null geodesics with its tangent vector lying on the horizon can leave it. It would be a coordinate surface, \( r = \text{const}. \) on which \( U = 0. \) Consider the null geodesic flow specified by the three integrals of motion \( E, \ L \) and \( C. \) The flow is defined in the region in which the momenta, \( p_u = f' \) and \( p_\lambda = g': \)

\[ f'^2 = (U_{00} E^2 + 2 U_{03} E L + U_{33} L^2) - C = (U_0 E + U_3 L)^2 - C \]
\[ g'^2 = C - (V_{00} E^2 - 2 V_{03} E L + V_{33} L^2) = C - (V_0 E - V_3 L)^2 \]

take real values. The integral of motion \( C \) cannot be taken zero because in that case \( g' \) becomes imaginary everywhere. Clearly, any change of this constant changes both components of the flow. In terms of \( \{ r, \lambda \} \) variables the components take the form

\[ p_r = \sqrt{(FE + aL)^2 - CU^2}, \quad p_\lambda = \sqrt{CV^2 - (GE - aL)^2}. \]

At the horizon both \( p_r, \ p'_r \) must vanish ensuring that null geodesics lying on it cannot leave it. The horizon is defined by \( U(r) = 0 \) and hence \( EF + aL = 0 \) at the horizon. That means for existence of horizon, the quadratic expressions above must be perfect squares. This would again lead to the integrable metric (8). Thus the requirement of existence of horizon is equivalent to the separability of the block determinant which is required for separability of KG equation.
III. THE GENERAL SOLUTION

The orthonormal tetrad frame for the metric (8) is given by
\[\nu^0 = U\sigma(dt + a^{-1}Gd\varphi), \quad \nu^1 = (U\sigma)^{-1}dr,\]
\[\nu^2 = (V\sigma)^{-1}d\lambda, \quad \nu^3 = V\sigma(-adt + Fd\varphi)\]
where
\[\sigma = (F + G)^{-1/2}.\]

We obtain the non-zero Ricci components in this tetrad frame and they are given by
\[R_{03} = -\frac{1}{2}(aUV\sigma^4)(F'' - a^{-2}G''),\]
\[R_{00} = -\frac{1}{2}(U^2 - a^2V^2)(F'^2 + a^{-2}G'^2)\sigma^6 - \left(\frac{U^2}{2}\right)''\sigma^2 + (U^2 + UU'F' - a^2V^2 - VV'G')\sigma^4,\]
\[R_{33} = \frac{1}{2}(U^2 - a^2V^2)(F'^2 + a^{-2}G'^2)\sigma^6 + \left(\frac{V^2}{2}\right)''\sigma^2 + (U^2 + UU'F' - a^2V^2 - VV'G')\sigma^4,\]
\[R_{11} = \left(\frac{U^2}{2}\right)''\sigma^2 + (U^2 - UU'F' - a^2V^2 + VV'G')\sigma^4,\]
\[R_{22} = -\frac{1}{2}U^2(F'^2 + a^{-2}G'^2)\sigma^6 + \left(\frac{V^2}{2}\right)''\sigma^2 + (U^2 + UU'F' + a^2V^2 - VV'G')\sigma^4.\]

Let us first implement the electrovac Einstein equation as vacuum would be contained in it. That is, \(R_{00} = -R_{11} = R_{22} = R_{33}\) and the rest being zero. Now \(R_{03} = 0\) gives
\[F'' = a^{-2}G'' = \text{const.}\] (9)
and \(R_{00} = R_{33}\) leads to
\[(U^2)'' = -(V^2)'' = \text{const.}\] (10)
Using them in \(R_{00} = -R_{11}\) yields
\[F'^2 + a^{-2}G'^2 = 4(F + G) = 4\sigma^{-2}.\] (11)
Then a remarkable reduction occurs and all the non-zero \(R_{ab}\) would collapse to
\[R_{00} = R_{33} = -R_{11} = R_{22} = \frac{-U^2 + UU'F' + a^2V^2 - VV'G' - F - G}{(F + G)^2}.\] (12)
Note that Eq. (11) would require that the constants in Eqs. (9) and (10) cannot be different. Further, linear term in the argument for \(F\) and \(G\) can be transformed away, and hence they will only contain the quadratic term. Then from Eqs (9-11), we thus obtain the general electrovac solution given by
\[U^2 = r^2 - 2Mr + p, \quad V^2 = -\lambda^2 + 2N\lambda + q, \quad F = r^2 + a^2, \quad G = a^2(\lambda^2 - 1)\] (13)
and then Eq.(12) would read as,
\[R_{00} = -R_{11} = R_{22} = R_{33} = \frac{-p + a^2q}{(F + G)^2} = \frac{Q^2}{(F + G^2)}\] (14)
which would reduce to vacuum when
\[ p = a^2 q, \quad Q = 0. \] (15)

It is clear that the constant \( Q^2 = -p + a^2 q \) would denote the electric charge on the source particle. The parameter \( a \) having the dimension of length was introduced while implementing the condition of separability of HJ equation. In the Kerr solution, it represents the usual rotation parameter. The remaining four parameters arise as constants of integration, of which \( M \) and \( Q \) represent mass and electric charge. The other two are dimensionless. It would turn out that the parameter \( q \) could be transformed away while \( l = aN \) would denote the NUT parameter.

We have thus obtained the general solution, hence unique, of the integrable metric given by (8), and it is given by

\[ ds^2 = \Lambda(dt + a d\varphi)^2 - (\Lambda)^{-1} \left[ (U^2 - a^2 V^2) \left( \frac{dr^2}{U^2} + \frac{d\lambda^2}{V^2} \right) + U^2 V^2 d\varphi^2 \right] \] (16)

where

\[ \Lambda = \frac{U^2 - a^2 V^2}{F + G}, \]
\[ \alpha = a \frac{(F - U^2)V^2 + (G a^{-2} + V^2)U^2}{U^2 - a^2 V^2}, \]
\[ F = r^2 + a^2, \quad G = a^2 (\lambda^2 - 1), \]
\[ U^2 = r^2 - 2Mr + p, \quad V^2 = -\lambda^2 + 2N\lambda + q. \] (17)

The most remarkable feature of this vacuum metric with \( Q = 0 \) is that it is invariant under the transformation \( M \leftrightarrow il, \quad r \leftrightarrow ia\lambda \) where \( l = aN \). For the electrovac case, we should in addition have \( p \leftrightarrow aq^2 \) which would imply \( Q^2 \leftrightarrow -Q^2 \). It is easy to verify that \( U^2 \leftrightarrow a^2 V^2, \quad F \leftrightarrow -G \). This indicates the duality between mass (gravoelectric charge) and the NUT parameter (gravomagnetic charge [12]). In this duality transformation the parameter \( a \) plays the crucial role. The transformation is defined only when it is non zero. This is therefore the most general solution for a localized source having both gravitational electric and magnetic charges. It is the magnetic charge that requires the solution to be asymptotic non flat. By the duality transformation, gravoelectric particle (Kerr) could be transformed into gravomagnetic (dual Kerr) particle.

When \( a = 0 \), it reduces to the Reissner - Nordström (RN) solution of a charged black hole with a deficit angle. The dimensionless parameters attain non trivial physical meaning only in the presence of \( a \). It turns out that \( l = aN \) can represent the NUT parameter while the other one is redundant which could be transformed away. We shall now transform this solution to the Kerr - NUT form [3], which has only four parameters, \( M, \quad a, \quad Q, \quad l \) representing the NUT generalization of Kerr family [3]. The transformation that does this, proceeds as follows:

\[ q = (b^2 - l^2)/a^2, \quad N = l/a, \quad a^2 = d^2 + b^2 + l^2, \]
\[ \varphi = (a/b)\tilde{\varphi}, \quad t = t + (d^2/b)\tilde{\varphi}, \]
\[ \tilde{\alpha} = (a\alpha + d^2)/b. \]

This would lead to

\[ U^2 = r^2 - 2Mr - l^2 + b^2 - Q^2, \quad a^2 V^2 = b^2 \sin^2 \theta, \]
\[ a\lambda = l + b \cos \theta, \quad \alpha = 2\frac{b(2Mr + 2l^2 - Q^2) \sin^2 \theta + 2U^2 \cos \theta}{U^2 - b^2 \sin^2 \theta}. \]

Further note that all the metric functions now involve only the four parameters, \( M, \quad b, \quad Q, \quad l \), indicating removal of the redundant parameter. A straight forward calculation would, on dropping overhead bars and replacing \( b \) by \( a \), take the metric (16) to the form,

\[ ds^2 = \frac{U^2}{\rho^2} (dt - P d\varphi)^2 - \frac{\sin^2 \theta}{\rho^2} \left( (F + l^2) d\varphi - adt \right)^2 - \frac{\rho^2}{U^2} dr^2 - \rho^2 d\theta^2 \] (18)

where

\[ \rho^2 = r^2 + (l + a \cos \theta)^2, \quad F - U^2 = 2Mr + l^2 - Q^2, \quad P - a \sin^2 \theta = -2l \cos \theta, \quad F = r^2 + a^2. \] (19)

This is the charged Kerr - NUT solution where \( l \) is the NUT parameter representing the gravomagnetic monopole charge [3,12] and \( Q \) is the electric charge. The electromagnetic field 2-form would be given by

\[ F = Q \rho^{-4} (r^2 - (l + a \cos \theta)^2) \, dr \wedge (dt - P d\varphi) + 2Q \rho^{-4} ar \cos \theta \sin \theta d\theta \wedge ((F + l^2) d\varphi - adt). \]
It reduces to the Kerr - NUT solution [3] when $Q = 0$. The general electrovac solution we have found is therefore actually the Kerr - NUT spacetime with an electric charge. It is asymptotically non flat, and the NUT parameter is the measure of asymptotic non flatness. Now when we set $a = 0$, it reduces to NUT spacetime with an electric charge. In contrast, the metric (16) reduces to the charged black hole with a deficit angle. The metric (18) therefore fully encompasses the NUT character of spacetime.

We have thus found the general solution of the Einstein electrovac equation for the stationary axially symmetric spacetime having the integrable metric. For vacuum it reduces to the known Kerr - NUT solution. It is the general solution for a rotating gravitational dyon, particle which is endowed with both gravoelectric and gravomagnetic charges. There exists the duality relation between the two which is considered in the next section.

IV. DUAL AND RADIATING SPACETIMES

(a) Dual spacetime: In the metric (16), the solution is ultimately specified by the prescription of the two functions, one of the radial coordinate $r$ and the other of the angle coordinate $\lambda$. Let us denote these two functions as follows:

$$R(r) = F - U^2, \ a^2 \Lambda(\lambda) = G + a^2 V^2.$$  

The Kerr family is characterized by

$$R = 2Mr, \ \Lambda = 0.$$  \hfill (20)

On the other hand,

$$R = 0, \ a^2 \Lambda = 2l \lambda$$  \hfill (21)

would characterize the family dual to the Kerr [7]. Here we have considered the vacuum case with $Q = 0$, and $l = aN$. Eq. (17) characterizes Kerr solution when $l = 0$ and the dual Kerr solution when $M = 0$.

The Kerr solution goes over to the dual solution under the transformation

$$M \rightarrow il, \ r \leftrightarrow i\alpha \lambda$$  \hfill (22)

and the vice-versa. This is the duality relation between the gravoelectric and gravomagnetic charges. It takes the field of the one to that of the other.

(b) Radiating spacetime: One of the ways to make a static or stationary spacetime to radiate is to transform it into the Eddington retarded coordinates and make mass $M$ function of the retarded time. By this method, Kerr solution could be transformed into a radiating spacetime having trace free stresses which asymptotically conform to the Vaidya radial null flux [10]. The same procedure could be applied to turn the Kerr - NUT family (18) radiating. This is what we shall demonstrate now.

Under the transformation,

$$dt \rightarrow dt - \frac{r^2 + a^2 + l^2}{U^2} dr, \ d\varphi \rightarrow d\varphi - \frac{a}{U^2} dr$$

the metric (18) for the case $Q = 0$ transforms into the Eddington retarded coordinates to read as,

$$ds^2 = \frac{U^2 - a^2 \sin^2 \theta}{\rho^2} dt^2 - 2 dt dr + \frac{4a(Mr + l^2) \sin^2 \theta + U^2 \cos \theta}{\rho^2} dtd\varphi$$

$$+ 2(a \sin^2 \theta - 2l \cos \theta) drd\varphi - \rho^2 d\theta^2$$

$$- \frac{1}{\rho^2} \left( (r^2 + a^2 + l^2)^2 \sin^2 \theta - (a \sin^2 \theta - 2l \cos \theta)^2 U^2 \right) d\varphi^2,$$  \hfill (23)

where $M = M(t)$ is an arbitrary function of the retarded Eddington time $t$ as defined above. This is the Kerr - NUT radiating spacetime with trace free stresses which asymptotically go over to the Vaidya null radiation flux. When $l = 0$, it reduces to the Kerr radiating spacetime [10]. The non zero stresses are given in the Appendix A, and it can be easily seen that $R = 0$, and they asymptotically reduce to

$$R^t_t = -\frac{M}{r^2}, \ R^r_\varphi = -\frac{M}{r^2}(4l \cos \theta - 2a \sin^2 \theta)$$

which conform to the Vaidya null radiation [8,9]. When $a = l = 0$, only $R^t_t$ survives, and it is then the Vaidya radiating star solution. The asymptotic behavior is similar when $a$ and/or $l$ are non zero.
V. DISCUSSION

From the standpoint of physical understanding and application, it is important that motion is integrable in the spacetime metric which we wish to seek as a solution of the Einstein equation. By integrability of motion, we mean separability of HJ and KG equations which respectively govern motion of particles and scalar field. Taking this as the general guiding principle and an over riding concern, it therefore becomes imperative to tailor the metric a priori for integrability. As in [5], we have thus first considered separability of HJ and KG equations, which led to the form of the metric as given in Eq. (8). Then the non zero Ricci components assume very simple form which readily yields to give the general solution given by Eqs (13, 16-19). It is the most general solution of the Einstein electrovac equation for the integrable axially symmetric stationary spacetime. Further it can be transformed to the Kerr - NUT solution thereby establishing uniqueness of the Kerr - NUT family for the integrable metric. The NUT parameter is the cause of asymptotic non flatness. When we impose asymptotic flatness, the NUT parameter vanishes and it reduces to the Kerr family. Like the Kerr solution, the Kerr - NUT solution could also be made radiating by letting the mass parameter be an arbitrary function of the retarded Eddington time. The spacetime is non empty but has trace free stresses which asymptotically conform to the Vaidya null radiation as is the case for the Kerr solution [10].

The assumption of integrability incorporates existence of regular horizon, which is required by separability of KG equation, but not of asymptotic flatness. That is why it leaves room open for asymptotically non flat generalization of the Kerr family. It is interesting that this generalization is unique and is given by the Kerr - NUT family. Since the Kerr metric is integrable, any metric which would contain it would also have to be integrable. For the integrable metric, the Kerr - NUT family is the most general solution. In other way, we have identified integrability of the metric as the uniqueness condition for the Kerr - NUT family. If we ask, what is the most general electrovac spacetime which includes the Kerr family? The answer is that it is the Kerr - NUT family and it is unique. The separability of KG equation can be traded off for existence of regular horizon. That is, the requirement of HJ separability and regular horizon is equivalent to the metric integrability. In the Kerr uniqueness conditions if we trade off asymptotic flatness for HJ separability, we arrive at the Kerr - NUT uniqueness conditions. On the other hand HJ separability and the vacuum equation imply KG separability [11]. In the the asymptotic limit all other metric coefficients go over to the Minkowski form but for $g_{t\phi}$ which is the cause for asymptotic non flatness. Its source is the gravomagnetic charge, NUT parameter $l$. As asymptotic flatness is the characteristic of the field of the gravoelectric charge (mass), the asymptotic non flatness is similarly the characteristic of the gravomagnetic charge (NUT parameter).

Our general solution is therefore the most general axially symmetric stationary solution for the spacetime having integrable metric. It is thus the unique solution for a rotating gravitational dyon.

The vacuum spacetime describing a non radiating source would in general be stationary and axially symmetric. Further the metric should be integrable if it has to describe a physically meaningful and understandable situation. We have obtained the most general solution under these physically motivated conditions. It would therefore describe the most general source which could in general have both gravitational electric and magnetic charge. This is a separate matter that it happens to be a transform of the Kerr - NUT solution. Our general electrovac solution thus describes the most general particle in general relativity. For the integrable spacetime, it is unique which implies that there cannot exist any other electro-vacuum solution.

No sooner gravitational electric and magnetic charges are admitted, the question of their duality becomes pertinent. The most remarkable feature of the general solution is its invariance under the duality transformation $M \leftrightarrow il$, $r \leftrightarrow ia\lambda$ exhibiting duality between the gravoelectric charge, mass $M$ and the gravomagnetic charge, NUT parameter, $l$. The parameter $a$ plays the critical role in this duality transformation, which makes sense only when $a$ is non zero. This duality naturally leads to dual of the Kerr solution which is also a non trivial vacuum spacetime. This would however not lead to dual to the Schwarzschild solution. The dual to the Kerr solution has naturally sprung up once the duality which includes the Kerr family? The answer is that it is the Kerr - NUT family and it is unique. The separability of KG equation can be traded off for existence of regular horizon. That is, the requirement of HJ separability and the vacuum equation imply KG separability [11]. In the the asymptotic limit all other metric coefficients go over to the Minkowski form but for $g_{t\phi}$ which is the cause for asymptotic non flatness. Its source is the gravomagnetic charge, NUT parameter $l$. As asymptotic flatness is the characteristic of the field of the gravoelectric charge (mass), the asymptotic non flatness is similarly the characteristic of the gravomagnetic charge (NUT parameter).

Our general solution is therefore the most general axially symmetric stationary solution for the spacetime having integrable metric. It is thus the unique solution for a rotating gravitational dyon.

The vacuum spacetime describing a non radiating source would in general be stationary and axially symmetric. Further the metric should be integrable if it has to describe a physically meaningful and understandable situation. We have obtained the most general solution under these physically motivated conditions. It would therefore describe the most general source which could in general have both gravitational electric and magnetic charge. This is a separate matter that it happens to be a transform of the Kerr - NUT solution. Our general electrovac solution thus describes the most general particle in general relativity. For the integrable spacetime, it is unique which implies that there cannot exist any other electro-vacuum solution.

No sooner gravitational electric and magnetic charges are admitted, the question of their duality becomes pertinent. The most remarkable feature of the general solution is its invariance under the duality transformation $M \leftrightarrow il$, $r \leftrightarrow ia\lambda$ exhibiting duality between the gravoelectric charge, mass $M$ and the gravomagnetic charge, NUT parameter, $l$. The parameter $a$ plays the critical role in this duality transformation, which makes sense only when $a$ is non zero. This duality naturally leads to dual of the Kerr solution which is also a non trivial vacuum spacetime. This would however not lead to dual to the Schwarzschild solution. The dual to the Kerr solution has naturally sprung up once the duality which includes the Kerr family? The answer is that it is the Kerr - NUT family and it is unique. The separability of KG equation can be traded off for existence of regular horizon. That is, the requirement of HJ separability and the vacuum equation imply KG separability [11]. In the the asymptotic limit all other metric coefficients go over to the Minkowski form but for $g_{t\phi}$ which is the cause for asymptotic non flatness. Its source is the gravomagnetic charge, NUT parameter $l$. As asymptotic flatness is the characteristic of the field of the gravoelectric charge (mass), the asymptotic non flatness is similarly the characteristic of the gravomagnetic charge (NUT parameter).

Our general solution is therefore the most general axially symmetric stationary solution for the spacetime having integrable metric. It is thus the unique solution for a rotating gravitational dyon.

As the Kerr solution describes the gravitational field of a rotating gravoelectric monopole, the dual solution describes the gravitational field of a rotating gravomagnetic monopole. Even though the dual solution is not asymptotically flat, it would be interesting to do all that which has been done for the Kerr geometry in the dual Kerr geometry so as to gauge the physical effects of asymptotic non flatness. To begin with, we would in a separate paper study motion,
which is a priori integrable, in the Kerr - NUT spacetime with particular reference to the dual solution [14].

Acknowledgment: We thank the anonymous referee for pointing out some relevant references. ZT thanks ICTP for a travel grant under its BIPTUN program and IUCAA for warm hospitality which made this work possible.

APPENDIX A:

The non zero components of Ricci tensor for the metric (25) are given by

\[
R\,_{tt} = \frac{-4\left(a^2 + 2l^2 - 2r^2 + 4al \cos \theta + a^2 \cos 2\theta\right)}{X^3} \left(-2l \cot \theta + a \sin \theta\right)^2 \dot{M}
\]  
(A1)

\[
R\,_{\varphi \varphi} = \frac{4\left(a^2 + 2l^2 - 2r^2 + 4al \cos \theta + a^2 \cos 2\theta\right)}{X^3} \csc \theta^2 \left(-2l \cos \theta + a \sin \theta\right)^3 \dot{M}
\]  
(A2)

\[
R\,_{\theta \theta} = \frac{-2r \left(-\left(a^2 - 8l^2\right) \cos \theta + a\left(2l + 6l \cos 2\theta + a \cos 3\theta\right)\right)}{X^2} \csc \theta \dot{M}
\]  
(A3)

\[
R\,_{\theta t} = \frac{-4r \left(-\left(a^2 - 8l^2\right) \cos \theta + a\left(2l + 6l \cos 2\theta + a \cos 3\theta\right)\right)}{X^3} \csc \theta \dot{M}
\]  
(A4)

\[
R\,_{\varphi \theta} = \frac{-16r\left(1 + a \cos \theta\right) \csc \theta \left(-2l \cos \theta + a \sin \theta\right)^2 \dot{M}}{X^3}
\]  
(A5)

\[
R\,_{\varphi t} = \frac{-4\left(a^2 + 2\left(l^2 - r^2\right) + 4al \cos \theta + a^2 \cos 2\theta\right)}{X^3} \csc \theta^2 \left(-2l \cos \theta + a \sin \theta\right) \dot{M}
\]  
(A6)

\[
R\,_{t \varphi} = \frac{4\left(a^2 + 2\left(l^2 - r^2\right) + 4al \cos \theta + a^2 \cos 2\theta\right)}{X^3} \left(-2l \cos \theta + a \sin \theta\right) \dot{M}
\]  
(A7)

\[
R\,_{t t} = \frac{-\left(-a^4 + (12a^2 + 16r^2)(r^2 + l^2) + 8al\left(a^2 + 2(l^2 + r^2)\right) \cos \theta + 4a^2(5l^2 + r^2) \cos 2\theta + 8a^3l \cos 3\theta + a^4 \cos 4\theta\right)}{X^3} \dot{M}
\]  
(A8)

\[
R\,_{\varphi \varphi} = \frac{4r}{\sin^2 \theta X^2} \left(-2l \cos \theta + a \sin \theta\right)^2 \dot{M}
\]  
(A9)

where

\[
X = \left(a^2 + 2l^2 + 2r^2 + 4bl \cos \theta + b^2 \cos^2 2\theta\right).
\]

Clearly \(R = R\,_{tt} + R\,_{\varphi \varphi} = 0\).
[8] P C Vaidya, L K Patel, Phys. Rev. D7, 3590 (1973).
[9] P C Vaidya, L K Patel and P V Bhatt, Gen. Relativ. Grav. 7, 701 (1976).
[10] M Carmeli and M Kaye, Ann. Phys. 103, 97 (1977).
[11] S Bonanos, Comm. Math. Phys. 49, 53 (1976).
[12] D Lynden-Bell and M Nouri-Zonoz, Rev. Mod. Phys. 70, 427 (1998).
[13] S Chandrasekhar, The Mathematical Theory of Black Holes, Clarendon Press, Oxford, 1983.
[14] N Dadhich, S Joshi and Z Y Turakulov, to be submitted.