Inertial Inchworm Crawling

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Abstract—Inchworm crawling allows for both quasistatic and dynamic gaits at a wide range of actuation frequencies. This locomotion mechanism is common in nonskeletal animals and exploited extensively in the bio-inspired field of soft robotics. In this work we develop and simulate the hybrid dynamic crawling of a three-link robot, with passive frictional contacts. We fabricate and experimentally test such robot under periodic inputs of joints’ angles, with good agreement to the theoretical predictions. This allows to comprehend and exploit the effects of inertia in order to find optimal performance in inputs’ parameters. A simple criterion of robustness to uncertainties in friction is proposed. Tuning the inputs according to this criterion improves the robustness of low-frequency actuation, while increasing the frequency allows for gaits with both high advancement velocity and robustness. Finally, the advantages of uneven mass distribution are studied. Time-scaling technique is introduced to shape inputs that achieve similar effect without reassembling the robot. A machine-learning based optimization is applied to these inputs to further improve the robot’s performance.

I. INTRODUCTION

In mobile robots, legged locomotion is advantageous in negotiating unstructured terrains, where wheeled and tracked vehicles have limited maneuverability or accessibility [1]. Bipedal legged robots are also widely studied due to their resemblance to humans and other mammals. Though this robotic configuration has relatively low dimensionality, it usually performs complicated dynamic locomotion, where the robot constantly undergoes unsteady motion of falling, followed by the replacement of the support foot in a cyclic pattern called gait.

Inchworm crawling is a type of legged locomotion which is characterized by multiple persistent ground contacts, which alternate stick-slip transitions. This allows for crawling gaits to be possible at a wide range of actuation frequencies. For rapid actuation, when the inertial effects are not negligible, dynamic inchworm crawling occurs (which will be the focus of this study). However, for slow actuation, the same locomotion mechanism also allows to remain in static balance while performing quasistatic movement – i.e. transitioning within a continuum of static equilibria – even on two contacts. Probably for that reason crawling is common in many nonskeletal animals, such as worms and caterpillars [2], octopi [3] and crawling cells [4]. Most of these creatures generate motion by manipulating fluids in their bodies [5] – which is a rather slow process, compared to skeletal animals with rapidly contracting muscles.

The actual inchworm, crawling cells and some robots actively manipulate the contact interaction by adhesion [6], [7] or sophisticated gripping spines [8]. Many crawling robots utilize directional friction [9]–[12], which do not allow to reverse the movement direction, while others perform crawling with passive frictional contacts [13], [14]. The latter, which will be the focus of this study, is a more complicated yet more general approach.

In recent years, crawling is exploited extensively [15] in the rapidly growing bio-inspired field of soft robotics [16]. In these robots, which often draw their inspiration from nonskeletal animals, continuous actuation and deformation are often similarly created by pressurizing fluids in internal cavities of their highly compliant structure [17]. Nonetheless, rapid dynamic movement of soft robots can also be achieved, with other actuation methods, such as magnetic fields [18].

In previous works [19] we have shown that a three-link model (Fig. 2) captures well the major phenomena of quasistatic inchworm crawling of a soft bipedal robot. This lumped model is a very basic form of multi-contact bipedal crawling, in analogy to McGeer’s biped [20] being a benchmark of bipedal walking. Yet, most “traditional” articulated legged robots are of higher complexity and often have more legs [21]–[23]. To the best of our knowledge, the minimal three-link robot has not been noticeably studied.

The purpose of this study is to model and analyze the dynamic bipedal frictional inchworm crawling, in order to comprehend and exploit the effects of inertia. First, we develop a model for the hybrid dynamic [24] crawling (i.e. with discrete
transitions between states) of a three-link robot with passive frictional contacts. The robot is actuated in open-loop by prescribing periodic inputs to its joints’ angles. We study the effects of frequency and other input parameters on the crawling gait, and find trends and optimal performance. We manufacture and experimentally test a three-link robot (Fig. 1) with good quantitative and excellent qualitative agreement with the theoretical predictions, which proves the applicability of our analysis (see supplementary video [25]). We also investigate the input shaping technique and apply machine-learning based optimization to improve the gaits’ performance.

II. PROBLEM FORMULATION

In order to investigate crawling at frequency range where inertial effects are significant, we have manufactured the three-link robot prototype in Fig. 1. The robot has a central link with length $l_0$, mass $m_0$ and moment of inertia $J_0$ and two distal links, with length $l_i$, mass $m_i$ and moment of inertia $J_i$, for $i = 1, 2$ (see Fig. 2). For most of this work we consider identical distal links $l_1 = l_2 \equiv l$, $m_1 = m_2 \equiv m$, $J_1 = J_2 \equiv J$, and in Section V we investigate the influence of asymmetric mass distribution. The experimental setup parameters are summarized in Table I. Two servomotors at the joints receive a sequence of angle commands from the microcontroller (in open-loop) and track it with internal closed-loop control.

Assuming planar motion and point-contacts, a corresponding three-link model is proposed in Fig. 2. The motion of the robot can be described by the generalized coordinates $\mathbf{q}(t) = [x \ y \ \theta \ \varphi_1 \ \varphi_2]^T$, where $(x, y, \theta)$ are the planar position and absolute orientation angle of the central link, and $(\varphi_1, \varphi_2)$ are the joint angles. Throughout this work we assume that the two joint angles $\mathbf{q}_i(t) = [\varphi_1(t) \ \varphi_2(t)]^T$ are prescribed directly as known periodic input functions (and hence their time-derivatives $\dot{\mathbf{q}}_i(t)$ are also known). This assumption, which is quite reasonable for a robot with joints controlled in closed-loop, significantly simplifies the analysis by effectively reducing the number of dynamically-evolving degrees-of-freedom (DoF).

For general prescribed joint angles $\mathbf{q}_i(t)$, at least one of the contacts is always constrained to slip (except for discrete times, where the relative velocity between feet vanishes). Therefore, there are only three possible combinations of the contacts’ states – stick-slip, slip-stick and slip-slip – which we now turn to model.

III. HYBRID DYNAMIC MODEL

In this section we introduce the derivation of a hybrid dynamic model which accounts for the different contact states of our three-link robot.

Denoting the position vectors of the contacts as $\mathbf{r}_i = [x_i \ y_i]^T$ (for $i = 1, 2$), the velocities of the $i$-th contact are

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = \mathbf{W}_i(q)\dot{q}_i, \quad (1)$$

with the Jacobian matrices

$$\mathbf{W}_i(q) = \frac{\partial \mathbf{r}_i(q)}{\partial q^T}. \quad (2)$$

The dynamic motion equations can be written in a standard matrix form [27]

$$\mathbf{M}(q)\ddot{\mathbf{q}} + \mathbf{B}(q, \dot{\mathbf{q}}) + \mathbf{G}(q) = \mathbf{E}\tau(t) + \mathbf{W}(q)^T\mathbf{F}(t), \quad (3)$$

where the terms of the matrices $\mathbf{M}, \mathbf{B}, \mathbf{G}, \mathbf{E}$ and the Jacobian $\mathbf{W} = [\mathbf{W}_1^T \ \mathbf{W}_2^T]^T$ are given in the Appendix, and $\tau(t) = [\tau_1(t) \ \tau_2(t)]^T$ are the internal input torques at the joints. The term $\mathbf{F}(t) = [f_1^x \ f_1^y \ f_2^x \ f_2^y]^T$ is the vector of generalized constraint forces, acting at the $i$-th contact in the normal (superscript $y$) and tangential (superscript $x$) directions (via the Jacobian $\mathbf{W}$). The contact forces vary with the different locomotion modes as we introduce next.

Though the constrained coordinates $\mathbf{q}_i(t)$ (and their time-derivatives) are known, we are still left with three unconstrained body coordinates $\mathbf{q}_b(t) = [x \ y]$, two actuation torques $\tau(t)$ and four contact forces $\mathbf{F}(t)$. This gives 9 unknowns with 5 motion equations, hence the motion is governed by additional constraints, as introduced next.

In inchworm crawling, the legs maintain persistent contact with the ground, giving two kinematic constraints $y_1 = y_2 = 0$ (these assumptions are constantly verified by calculating the normal forces at the contacts and requiring $f_1^y > 0$). If one of the legs is in contact-sticking (stick-slip or slip-stick modes) we also get a third constraint $\dot{x}_1 = 0$ for the sticking contact, yet additional constitutive relations are required. Coulomb’s dry friction model dictates that the tangential forces $f_i^x$ must maintain

$$|f_i^x| \leq \mu f_i^y \quad \text{for a sticking contact} \quad (4a)$$

and

$$f_i^x = -\mu f_i^y \text{sign} \dot{x}_1, \quad \text{for a slipping contact}, \quad (4b)$$

where $\mu$ is Coulomb’s friction coefficient (for simplicity, we do not distinguish between static and kinetic friction coefficients). Thus the overall count gives 9 unknowns with 5 equations of motion, two constraints in the normal direction, and two tangential constrains – either $\dot{x}_i = 0$ for a sticking contact or $|\dot{x}_i| < \epsilon$ for slippage. This system is complete and allows for a closed solution.

We can now decompose matrices $\mathbf{M}$ and $\mathbf{W}$ into blocks corresponding to the constrained and unconstrained coordinates, and rearrange $\mathbf{F}$ and the constraints (using $\mathbf{M}$) such that all the unknowns are on one side of the equation (see Appendix for details). These motion equations can be solved...
simultaneously with the constraints for each contact state with a numerical solver (for our simulation we are using MATLAB® ode45 with event detection).

The conditions for transitions between contact states are as follows. A contact remains in slippage as long as its tangential velocity \( \dot{x}_i \neq 0 \). When \( \dot{x}_i = 0 \), the contact either turns to contact-sticking, or reverses slippage direction. Contact-sticking may only hold as long as the tangential friction force satisfies \( \mu f_x \geq 0 \), otherwise slippage occurs. The simulation detects such crossings and switches accordingly the constraints we solve for. Fig. 3 depicts a transition graph of all the contact states. It is of note, that due to simplifying assumptions of rigid bodies and Coulomb’s dry friction, Painlevé’s paradox may occur – giving either no- or multiple- consistent solutions of the contact state [28].

**IV. Analysis and Results**

In this section we investigate the influence of various parameters on the robot’s performance via the numerical simulation. Then, in Section V these results are compared to experiments (also see supplementary video [25]). We choose concrete periodic functions of the angles as follows

\[
\begin{align*}
\varphi_1(t) &= \varphi_0 + A \sin(\omega t + \psi/2), \\
\varphi_2(t) &= \varphi_0 + A \sin(\omega t - \psi/2),
\end{align*}
\]

where \( \varphi_0 \) is the nominal angle, \( A \) is the oscillation amplitude, \( \psi \) is the phase difference between the legs, and \( \omega \) is the frequency. The performance is measured via the net distance traveled per step (in steady state) \( S \equiv x_1(t) - x_1(t-T) \) and the average step velocity \( V \equiv S/T \), where \( T \equiv 2\pi/\omega \) is the period time.

Parametric investigation of the simulation shows some influence and coupling of all the parameters. The most significant effects are achieved by varying the actuation frequency \( \omega \), as shown in Fig. 4 (solid blue curves) for \( \omega = 180°, \varphi_0 = 110° \) and \( \psi = 20° \). To further comprehend these results we divide the frequencies into three ranges (marked by 1 to 4) in

| Parameter                        | Notation | Value | Units |
|----------------------------------|----------|-------|-------|
| Central link’s mass              | \( m_0 \) | 194   | gr    |
| Central link’s length            | \( l_0 \) | 187   | mm    |
| Central link’s moment of inertia | \( J_0 \) | 776.3 | kg mm²|
| Distal links’ mass               | \( m_1 = m_2 = m \) | 21    | gr    |
| Distal links’ length             | \( l_1 = l_2 = l \) | 170   | mm    |
| Distal links’ moment of inertia  | \( J_1 = J_2 = J \) | 98.7  | kg mm²|
| Friction coefficient – hard tips | \( \mu \) | 0.172 | –     |
| Friction coefficient – soft tips | \( \mu \) | 0.398 | –     |

**TABLE I: Summary of experimental setup parameters’ values**

Fig. 4 and plot in Fig. 5 the contacts’ velocities \( \dot{x}_i(t) \) for one representative frequency in each range. In the low frequencies (range 1), the inertial effects are minor and the robot performs gaits with almost ideal switching – where the contacts slip consecutively only in the desired advancement direction. This is illustrated by the contacts’ velocities \( \dot{x}_i(t) \) in Fig. 5(a) (dashed curves), and corresponds to results achieved with quasistatic analysis in [19]. In this frequency range, since the distance \( S \) is almost steady but the cycle time shortens as \( 1/\omega \), the average velocity \( V \) increases with frequency (see Fig. 4(b)). In range 2, as the frequency increases, a growing portion of the cycle exhibits slippage in the opposing direction, until the advancement is almost entirely canceled. The velocities \( \dot{x}_i(t) \) at a frequency with minimal advancement are depicted in Fig. 4(b). In this range, \( S \) shortens faster than the cycle time \( T \), and the average velocity decreases. Finally, for even higher frequencies (in range 3), though the robot continues to exhibit opposing slippage throughout the cycle, it also develops a rigid-body progression in the desired direction. This increases the net distance only slightly, but since the frequency is in a high range the average velocity rises rapidly. In Fig. 5(c) we notice that for a short portion of the cycle both contacts have positive slippage velocities \( \dot{x}_i > 0 \).

We can also find optimal phase difference \( \psi^* \) for each frequency. The optimum \( \psi^*(\omega) \) shifts (almost monotonically in \( \omega \)) from \( \psi^* \rightarrow 0 \) for low frequencies (with agreement to the quasistatic limit in [19]) towards \( \psi^* \approx 120° \) as the frequency rises. This is depicted by the dotted black curve over the \( \psi - \omega \) parametric space in Fig. 6. Moving average is applied to the curve in order to reduce numerical noise, especially for the frequency range where \( S \) is small.

Another interesting effect of inertia is observed with a variation of the nominal angle. At large enough \( \varphi_0 \) (i.e. “flatter” robot), by varying the input frequency \( \omega \) a direction reversal occurs, as shown in Fig. 7(a) in solid blue curve (for \( \varphi_0 = 145°, A = 18° \) and \( \psi = 20° \)). The same phenomenon is observed for a sweep of the nominal angle \( \varphi_0 \) at a frequency in the above range (\( \omega = 9 \) [rad/s]) in Fig. 7(b).

**V. Experiments**

In order to measure the performance of our robot in Fig. 1 we video it with a simple webcam. The video lens distortion
is then corrected with MATLAB® Computer Vision System Toolbox and post-processed with Kinovea software [29], which gives planar x-y position of selected points on the robot’s links. Since the robot is controlled in open-loop, as the frequency increases the servo struggles to follow the prescribed amplitude. Therefore we initially calibrate the amplitudes of inputs’ reference trajectories to achieve the desired angles in (5), and validate the other parameters (such as frequency).

In an additional preliminary calibration experiment, we measure the friction coefficient by tilting the plane with the robot rigidly standing on top, and finding (with image processing) the angle at which the robot starts to slip (see Table I).

At each set of parameters the robot performs five cycles, and the overall distance traveled is averaged. Each experiment is repeated four times, giving the black curve and error bars (representing the range of observations) in Fig. 4. The qualitative performance in frequency is very similar to the analytical prediction, giving multiple local extrema in average velocity $V$ in Fig. 4(b). Yet, the actual distance traveled at the low frequencies range is about two times smaller than the simulation predicts. The differences can be explained by sensitivity analysis to friction variations, which we study in the next section VI. We will show that robustness to friction uncertainties has major influence and must be considered in order to improve the predictive power of the model and simulations.

By differentiating the measured horizontal position of the contacts, and filtering with moving average, we get the contacts’ velocities $\dot{x}_1$ (blue) and $\dot{x}_2$ (black) – experiment results (solid curves) and simulation results (dashed curves). (a) $\omega = 0.3$ [rad/s] low frequency range $\text{I}$. (b) $\omega = 5.1$ [rad/s] frequency range $\text{II}$. (c) $\omega = 16$ [rad/s] high frequency range $\text{III}$. It is observed that at low frequencies (Fig. 5(a)) slippage occurs in the undesired direction, resulting in a smaller net traveling distance than predicted. This small but significant divergence from the analytical model can also be explained by friction uncertainties – see analysis in Section VI. For higher
frequencies (Fig. 3 b) and (c)), the experimental and analytical results match both qualitatively and quantitatively.

Finally, the phenomenon of direction reversal with variation of the nominal angle $\phi_i$ is also achieved in experiments (black curves in Fig. 7) – see supplementary video [25]. It is of note that this experiment required switching the robot’s tips to softer material, with higher friction coefficient, which exhibits more robust behavior at this frequency range. The friction coefficient was measured (see Table I) and simulated in the corresponding simulation predictions must be tested for sensitivity to inaccuracies.

VI. ROBUSTNESS TO FRICTION UNCERTAINTIES

Based on the results from our experiments and previous works [19], [26], practical considerations suggest that the simulation predictions must be tested for sensitivity to inaccuracies in friction. Modeling (or measuring) and simulating the actual distribution of the friction along various surfaces involves increased complexity. In this section we propose a deterministic and computationally-cheap criterion, which captures the effects of uneven friction, and allows for numerical optimization. We perform the sensitivity analysis by varying the friction coefficients at the two contacts such that $\mu_1/\mu_2 \in [1-\varepsilon, 1+\varepsilon]$ (where $\mu_i$ is the friction coefficient at contact $i$). In Fig. 4, the shaded blue area shows how the distance changes (decreases for the most part) as the inaccuracy in $\mu_1/\mu_2$ rises up to $\varepsilon = 0.1$ (which is close to the actual range measured in calibration experiments). A symmetric perturbation of the friction $\mu_1 = \mu_2 = (1\pm\varepsilon)\mu$ was found to insignificantly affect the distance $S$ for reasonable $\varepsilon$ (not shown). Assuming the friction varies along the gait and among the strides within this uncertainty range, we propose to average the maximal and minimal possible distances in the shaded area, i.e.

$$S_{\text{avg}} = \frac{1}{2} (S_{\text{min}} + S_{\text{max}}),$$

where

$$S_{\text{min/max}} = \min / \max \left\{ S_{\mu_1/\mu_2 \in [1-\varepsilon, 1+\varepsilon]} \right\}.$$
any optimization algorithm may be used here, including a straight-forward line search. Further optimization with more parameters, requires sophisticated algorithms and methods, as discussed in the next section.

VII. MASS ASYMMETRY AND TIME-SCALING INPUT SHAPING

In this section we investigate the influence of mass distribution on the step distance \( S \) and discover the effects of asymmetry. We introduce a time-scaling input shaping strategy, which achieves asymmetry without mechanical changes to the robot, and utilize machine-learning algorithm to optimize the parameters of such inputs.

Considering the ratio between the mass of the distal and center links \( \frac{m_1}{m_2} \), the distance \( S \) grows almost monotonically with the decrease of the distal links’ mass (see Appendix). A more divergent and influential parameter is the asymmetry of mass distribution among the distal links \( m_1/m_2 \), as depicted in Fig. 10 for various frequencies and constant ratio to the central link \( \frac{1}{2}(m_1 + m_2)/m_0 = 0.5 \). At the low frequencies range \( (\omega \lesssim 1 \text{ [rad/s]}\)\)), the gaits are highly sensitive to asymmetry in any direction, causing significant decrease in \( S \). As the frequency increases, the sensitivity is reduced and asymmetric mass distribution even becomes advantageous. The preference moves from heavier front link \( m_1 > m_2 \) at mid-range frequencies (where even direction reversal occurs) to heavier rear link \( m_2 > m_1 \) at high frequencies. Exploiting the advantages of asymmetric mass distribution requires mechanically reassembling the robot.
for each frequency. Therefore, we desire to achieve a similar improvement effect of the distance $S$ by shaping the input signals $\varphi_i(t)$.

We propose a time-scaling strategy, where the joint angles remain of the form $\varphi_i(t) = \psi_0 + A \sin(s(\omega t \pm \psi/2))$ as in [5], but a time-scaling function $s(\xi)$ is introduced as

$$s(\xi) = \xi + \alpha \sin(\xi + \Upsilon).$$

(8)

For $\alpha = 0$ these functions are reduced to the unscaled case of [5], but $\alpha \neq 0$ generates an uneven “internal” pace in $\varphi_i(s(t))$, while preserving the overall shape and the principal harmonic (as long as $\dot{s}(t) \geq 0$). This is illustrated in Fig.11(a) for optimal $\psi = \psi^*$ without time-scaling $\alpha = 0$ (dashed curves), compared to optimal choice of both ($\alpha, \Upsilon$) to maximize $S$ and same $\psi^*$ (solid curves), at $\omega = 3.2$ [rad/s]. Hereafter we also keep fixed amplitude $A = 18^\circ$ and nominal angle $\varphi_0 = 110^\circ$, in order to investigate only the effect of the scaling. Fig.11(b) shows how these time-scaled inputs significantly improve $S$ by both increasing $\dot{x}_i(t)$ in the advancement direction and reducing slippage in the opposing direction. At this frequency, the optimal scaled input achieves this improvement by extending the legs rapidly and retracting them slowly (see supplementary video [25]).

Since the system is highly nonlinear and non-smooth, induced by the hybrid contact-state transitions, finding optimal values of multiple parameters involves increased complexity. Gradient-based optimization algorithms (like Newton’s method or gradient descent) struggle and typically converge to a local minimum, which depends on the basin of attraction the starting point [30]. To search for global optimum, optimizations must be initialized from multiple points, which is time-consuming (we attempted to use MATLAB® fmincon with multistart). Contrarily, Bayesian optimization is a machine-learning approach which is based on fitting a Gaussian process [31]. This algorithm was shown to be very efficient for legged robotic locomotion [30], [32], [33], probably since such systems exhibit non-smooth behavior and coupling of parameters. We applied this machine-learning algorithm (via MATLAB® bayesopt) to our numerical simulation, described in Section VI and achieved rapid and stable convergence (compared to gradient-based methods). Fig.9 illustrates the machine-learning optimization of time-scaling parameters and phase difference ($\alpha, \Upsilon, \psi$) to maximize $S$ (solid purple curve) and $S_{avg}$ (dashed purple curve) for various frequencies. A significant improvement is achieved at the middle- and high-frequency ranges.

VIII. CONCLUDING DISCUSSION

This paper has addressed modeling of dynamic robotic inchworm-like crawling with passive frictional contacts. Such locomotion is sensitive to uncertainties in the friction, but once that is accounted for (with a minimal analysis we propose) the model shows good agreement with experimental results.

The model also allows for comprehension of the influence of inertia and other phenomena. For example it is observed that with low-frequency actuation, tuning the gaits’ parameters according to the criterion we proposed can significantly improve the robustness with respect to variations and unevenness in surface friction. At high frequencies the gaits demonstrate natural robustness with similar (and even higher) average velocities.

The quasistatic low-frequency range was also observed to be very sensitive to asymmetry. However, as the frequency rises asymmetry can actually be exploited to improve the gaits’ performance. Similar effect was also created by shaping the inputs with time-scaling technique, without mechanical changes to the robot.

The latter analysis was only investigated in simulation. Experimentally testing the optimizations proposed in Sections VI and VII on this and other legged robotic systems shall be the subject of future research. Also, a more complex friction model should be investigated.

The presented notions can be implemented in future into soft crawling robots, both quasistatic and dynamic. We also believe that crawling locomotion in articulated legged robots is advantageous and can be exploited more extensively. Finally, just as McGeer’s biped [20] was an inspiration to studies of simple and energetically efficient control, we hope that our three-link model will pave the way for future analytic and experimental works in inertial crawling locomotion.

REFERENCES

[1] P.-B. Wieber, R. Tedrake, S. Kuindersma, Modeling and control of legged robots, in: Springer handbook of robotics, Springer, 2016, pp. 1203–1234.
[2] L. I. v.Griethuisen, B. A. Trimmer, Caterpillar crawling over irregular terrain: anticipation and local sensing, J. of Comparative Physiology A 196 (6) (2010) 397–406.
[3] M. Calisti, F. Corucci, A. Arienti, C. Laschi, Bipedal walking of an octopus-inspired robot, in: Conference on Biomimetic and Biohybrid Systems, Springer, 2014, pp. 35–46.
[4] E. L. Barnhart, G. M. Allen, F. Jülicher, J. A. Theriot, Bipedal locomotion in crawling cells, Biophysical journal 98 (6) (2010) 933–942.
[5] S. Kim, C. Laschi, B. Trimmer, Soft robotics: a bioinspired evolution in robotics, Trends in Biotechnology 31 (5) (2013) 287–294.
Fig. 11: Gait with time-scaling (solid curves) and without time-scaling \( \alpha = 0 \) (dashed curves) for \( \omega = 3.2 \) [rad/s]. (a) Joint angles \( \varphi_1 \) (blue) and \( \varphi_2 \) (black). (b) Contacts’ velocities \( \dot{x}_1 \) (blue) and \( \dot{x}_2 \) (black).
Differentiating these constraints with respect to time gives

\[
\begin{bmatrix}
    \frac{\partial}{\partial t} (s_1 + c_2) \\
    \frac{\partial}{\partial t} (c_1 - c_2) \\
    \frac{\partial}{\partial t} (l_0 \cos \varphi_1 - l) - J \\
    \frac{\partial}{\partial t} (l_0 \cos \varphi_2 - l) - J
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    \frac{m_0 + 2m}{2} \\
    \frac{m_0 + 2m}{2} \\
    \frac{l m}{4} (l_0 \cos \varphi_1 - l) - J \\
    \frac{l m}{4} (l_0 \cos \varphi_2 - l) - J
\end{bmatrix}
\]  

\[
\begin{bmatrix}
    s_1 \\
    c_1 \\
    l_0 \cos \varphi_1 - l \\
    l_0 \cos \varphi_2 - l
\end{bmatrix}
\]

\[= 0.
\]

APPENDIX

Motion equations

The terms for the matrices in the dynamic motion equations (9) are as follows: the mass matrix \( M \) is given in (9), where

\[
M_{4,3} = 2J + J_0 + \frac{m_0}{2} (l_0^2 + \rho^2 - l_0l_0 \cos \varphi_1 - l_0l_0 \cos \varphi_2)
\]

and the abbreviations \( s_1 = \sin(\varphi_1 - \theta) \), \( s_2 = \sin(\varphi_2 + \theta) \), \( c_1 = \cos(\varphi_1 - \theta) \) and \( c_2 = \cos(\varphi_2 + \theta) \). Also

\[
B = \frac{1}{2lm} \begin{bmatrix}
    (\varphi_2 + \theta)^2 c_2 - (\varphi_1 - \theta)^2 c_1 \\
    (\varphi_2 + \theta)^2 s_2 - (\varphi_1 + \theta)^2 s_1 \\
    -\frac{1}{2} l_0 \theta^2 \sin \varphi_1 \\
    -\frac{1}{2} l_0 \theta^2 \sin \varphi_2
\end{bmatrix},
\]

where

\[
B_3 = \frac{1}{2} l_0 \left( (2\dot{\varphi}_1 - \varphi_1^2) \sin \varphi_1 + (\varphi_2^2 + 2\dot{\varphi}_2 \sin \varphi_2) \right),
\]

and

\[
G = g \begin{bmatrix}
    0 \\
    \frac{m_0 + 2m}{2} \\
    \frac{-l m c_1}{4} \\
    \frac{-l m c_2}{4}
\end{bmatrix}.
\]

The Jacobians in (11) are

\[
W_1 = \begin{bmatrix}
    \frac{w_1^r}{w_1^b} \\
    \frac{w_1^{y^r}}{w_1^{y^b}}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    \frac{1}{2} (2ls_1 + l_0 \sin \theta) & -ls_1 \\
    0 & 1 \\
    \frac{1}{2} (2lc_1 - l_0 \cos \theta) & -lc_1
\end{bmatrix}
\]

\[
W_2 = \begin{bmatrix}
    \frac{w_2^r}{w_2^b} \\
    \frac{w_2^{y^r}}{w_2^{y^b}}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    \frac{1}{2} (2ls_2 - l_0 \sin \theta) & ls_2 \\
    0 & 1 \\
    \frac{1}{2} (2lc_2 - l_0 \cos \theta) & lc_2
\end{bmatrix}.
\]

When one of the contacts is at slippage while the other maintains contact sticking – for concreteness let us consider stick-slip contact-state – from (11) we get three kinematic constraints: contact-sticking of one leg \( v_1 = W_1 q = 0 \) and no-detachment of the slipping leg \( y_2 = w_2 q = 0 \). Differentiating these constraints with respect to time gives

\[
\begin{bmatrix}
    W_1 \\
    w_2
\end{bmatrix} \dot{q} = - \begin{bmatrix}
    W_1 \\
    w_2
\end{bmatrix} \dot{q}.
\]

We can now formulate the dynamic equations (3) with the constraints (14) as

\[
\begin{bmatrix}
    M - W_1^T \Gamma & -W_2^T \Gamma \\
    W_2^b & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
    \dot{q} \\
    \dot{f}_1 \\
    \dot{f}_2
\end{bmatrix} = \begin{bmatrix}
    -B - G \Gamma + \tau \epsilon \\
    -W_1 \dot{q} \\
    -W_2 \dot{q}
\end{bmatrix},
\]

where

\[
\Gamma = \begin{bmatrix}
    -\mu \text{sign} \dot{x}_i \\
    0
\end{bmatrix}
\]

for the contact at slippage from (16).

Slip-slip contact state is formulated similarly to (15), though the contact forces of both contacts maintain (16) and the constraints are reduced to no-detachment only \( y_1 = y_2 = 0 \), giving

\[
\begin{bmatrix}
    M - W_1^T \Gamma & -W_2^T \Gamma \\
    W_2^b & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
    \dot{q} \\
    \dot{f}_1 \\
    \dot{f}_2
\end{bmatrix} = \begin{bmatrix}
    -B - G \Gamma + \tau \epsilon \\
    -W_1 \dot{q} \\
    -W_2 \dot{q}
\end{bmatrix}.
\]

The mass and Jacobian matrices can be decomposed into blocks corresponding to the constraint- and body-coordinates as \( M = [M_1, M_b] \) and \( W_i = [\{W_i\}_c, \{W_i\}_b] \). Then we can rearrange (15) and (17) such that all the unknowns are on one side as (only the formulation of stick-slip in (15) shown for brevity)

\[
\begin{bmatrix}
    \dot{q}_b \\
    \dot{f}_1 \\
    \dot{f}_2
\end{bmatrix} = \begin{bmatrix}
    -M_b \dot{q}_c - B - G \\
    -W_1 \dot{q} \\
    -W_2 \dot{q}
\end{bmatrix}.
\]

Finally, for a state-vector \( x(t) = [q_b^T, \dot{q}_b^T]^T \) and prescribed \( q_c(t), \dot{q}_c(t), \dot{q}_c(t) \), we get a series of complete first-order ODEs.

Mass distribution colormap

In Fig. 12 we present complete color maps of the step distance \( S \) the versus mass distribution in both parameters – ratio of distal to center links’ mass and asymmetry among the distal links. This completes the sections of mass asymmetry (for constant \( \frac{1}{2} (m_1 + m_2)/m_0 = 0.5 \) presented in the main paper (Fig. 10).
Fig. 12: Colormaps of step distance $S$ versus mass distribution for various frequencies – Distal-to-center links’ mass ratio $\frac{1}{2}(m_1 + m_2)/m_0$ versus asymmetry ratio $m_1/m_2$