The enclosure method using a single point on the graph of the response operator for the Stokes system

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Abstract

An inverse obstacle problem governed by the Stokes system in the time domain is considered. Two types of extraction formulae about the geometry of an unknown obstacle are given by using the most recent version of the time domain enclosure method in a unified style. Each of the formulae employs only a single set of velocity and the Cauchy stress fields over a finite time interval on the surface of the region where a viscous incompressible fluid occupies.

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KEY WORDS: enclosure method, inverse obstacle problem, Stokes system, fluid

1 Introduction

The time domain enclosure method with a single input is one of analytical methods in inverse obstacle problems firstly initiated in [10] and developed in the series of articles [11, 13, 14] for an obstacle embedded in the whole space, and [15, 17, 19] for an obstacle embedded in a bounded domain. However the pursuit of the most recent version of the time domain enclosure method developed in [15, 19] is just beginning. There are a lot of important, typical and interesting inverse obstacle problems governed by various types of partial differential equations in the time domain. In this paper, we consider how this new method can be realized in an inverse obstacle problem governed by the Stokes system in the time domain. Furthermore, we add new knowledge which was not covered by the previous research [21] under the framework of the previous version of the enclosure method in the time domain [12].

First we formulate the problem. Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with $C^2$-boundary and $D$ a nonempty open subset of $\Omega$ with $C^2$-boundary such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. We denote by $n$ the unit outward normal to $\partial \Omega$.

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Let $0 < T < \infty$. Given $f$ on $\partial \Omega \times [0, T]$ let $u = u(x,t)$ and $p = p(x,t)$ solve

\[
\begin{cases}
\rho \frac{\partial u}{\partial t} - \text{div} \, \sigma(u, p) = 0, & (x,t) \in (\Omega \setminus \overline{D}) \times ]0, T[, \\
\nabla \cdot u = 0, & (x,t) \in (\Omega \setminus \overline{D}) \times ]0, T[, \\
u = 0, & (x,t) \in \partial D \times ]0, T[, \\
u = f, & (x,t) \in \partial \Omega \times ]0, T[, \\
u(x,0) = 0, & x \in \Omega \setminus \overline{D}.
\end{cases}
\]

(1.1)

Here $\rho$ denotes the density of the fluid occupying $\Omega \setminus \overline{D}$. The pair $(u, p)$ is that of the velocity and pressure of the fluid and $\sigma(u, p)$ denotes the Cauchy stress tensor

\[\sigma(u, p) = -pI_3 + 2\mu \text{Sym} \nabla u,\]

where $\mu$ denotes the viscosity of the fluid occupying $\Omega \setminus \overline{D}$. It is assumed that $\rho$ and $\mu$ are given by positive constants.

Note that to ensure the existence of the solution of (1.2) we always choose $f$ satisfying

\[
\int_{\partial \Omega} f(x,t) \cdot \nu(x) \, dS = 0, \quad 0 < t < T.
\]

(1.2)

We consider

\subsection*{Problem}

Fix $T$ and $f$ (to be specified later). Extract information about the geometry of $D$ from the pair of $f$ and the corresponding the Cauchy force $\sigma(u, p)\nu$ on $\partial \Omega$ over the time interval $[0, T]$. In other words, find a suitable $f$ in such a way that the corresponding $\sigma(u, p)\nu$ on $\partial \Omega$ over time interval $[0, T]$ yields some information about the location and shape of $D$.

In this paper, we present two types of the solutions to this problem in a unified style. It is an application of the idea introduced in [15] and the recent one in [19].

\section{1.1 Indicator function}

Let $v = v(x,t)$ and $q = q(x,t)$ satisfy

\[
\begin{cases}
\rho \frac{\partial v}{\partial t} - \text{div} \, \sigma(v, q) = 0, & (x,t) \in \mathbb{R}^3 \times ]0, T[, \\
\nabla \cdot v = 0, & (x,t) \in \mathbb{R}^3 \times ]0, T[.
\end{cases}
\]

(1.3)

Note that $v$ satisfies

\[
\int_{\partial \Omega} v \cdot \nu \, dS = \int_{\Omega} \nabla \cdot v \, dx = 0.
\]

Thus the vector field $f$ given by

\[
f = v, \quad (x,t) \in \partial \Omega \times ]0, T[.
\]

(1.4)
satisfies (1.2).

**Definition 1.1.** Let \((u, p)\) solve (1.1) with \(f\) given by (1.4). Let \(\tau > 0\). Define

\[
I_T(\tau; v, q) = \int_{\partial \Omega} (\sigma(w, \tilde{p}) n - \sigma(w_0, \tilde{q}) n) \cdot w_0 \, dS,
\]

where

\[
\begin{align*}
\mathbf{w}(x, \tau) &= \int_0^T e^{-\tau t} u(x, t) \, dt, \quad \mathbf{w}_0(x, \tau) = \int_0^T e^{-\tau t} v(x, t) \, dt, \\
\tilde{p}(x, \tau) &= \int_0^T e^{-\tau t} p(x, t) \, dt, \quad \tilde{q}(x, \tau) = \int_0^T e^{-\tau t} q(x, t) \, dt.
\end{align*}
\]

We call the function \(I_T(\tau; v, q)\) of independent variable \(\tau\) the *indicator function*.

In principle, the indicator function can be computed from the pair \((f, \sigma(u, p) \nu)\) on \(\partial \Omega\) over the time interval \([0, T]\) which is a point on the graph of the response operator \(f \mapsto -\sigma(u, p) \nu\).

We show that, for suitable two choices of the pair \((v, q)\) with \(q = 0\) the asymptotic behaviour of indicator function \(I_T(\tau; v, q)\) as \(\tau \to \infty\) yields some information about the geometry of obstacle \(D\). This is a solution to the problem mentioned above.

The point is: to generate the velocity and pressure of the fluid inside a given domain we prescribe a special velocity field on the boundary of the domain given by solving an evolution equation in the whole space.

The general outline of the method is as follows.
(i) Let \(\Phi = \Phi(x, t)\) with \((x, t) \in \mathbb{R}^3 \times [0, T]\) be a solution of the heat equation

\[
\rho \frac{\partial \Phi}{\partial t} - \mu \Delta \Phi = 0, \quad (x, t) \in \mathbb{R}^3 \times [0, T],
\]

such that \(K = \text{supp} \Phi(\cdot, 0)\) is compact and satisfies \(K \cap \Omega = \emptyset\).

(ii) Define the vector field \(\mathbf{v}\) by the formula

\[
\mathbf{v}(x, t) = \nabla \times \{\Phi(x, t) \mathbf{a}\}, \quad (x, t) \in \mathbb{R}^3 \times [0, T],
\]

where \(\mathbf{a}\) is an arbitrary constant unit vector.

We see that the pair of \(\mathbf{v}\) and \(q = 0\) solves (1.3).

(iii) Prescribe \(f\) given by (1.4) as the velocity field on \(\partial \Omega\) over time interval \([0, T]\) and solve (1.1).

(iv) Compute the indicator function \(I_T(\tau; \mathbf{v}, 0)\). Then, the asymptotic behaviour of this indicator function as \(\tau \to \infty\) yields the quantity \(\text{dis}(D, K)\) for special \(K\) which depends on the initial data \(\Phi(\cdot, 0)\).

In what follows, we denote by \(B_R\) an open ball with radius \(R\). The symbol \(\chi_X\) denotes the characteristic function of a set \(X\).
1.2 Statement of the result

Now let us describe the result more precisely.

Let \( \eta > 0 \) and \( R_2 > R_1 > 0 \). Let \( m = 1, 2, \ldots \). In this paper, we choose two special initial data \( \Phi(\cdot, 0) = \Phi_{\text{ext}}(x), \Phi_{\text{int}}(x) \):

\[
\begin{align*}
\Phi_{\text{ext},m}(x) &= (\eta^2 - |x - p|^2)^m \chi_{B_\eta}(x), \\
\Phi_{\text{int},m}(x) &= (R_2^2 - |x - p|^2)^m (|x - p|^2 - R_1^2)^m \chi_{B_{R_2} \setminus B_{R_1}}(x),
\end{align*}
\]

where ball \( B_\eta \) and two concentric balls \( B_{R_1} \) and \( B_{R_2} \) are chosen in such a way that \( \overline{B_\eta} \cap \Omega = \emptyset \) and \( \Omega \subset B_{R_1} \). Here we made use of the common symbol \( p \) to denote the center of \( B_\eta \) in \( \Phi_{\text{ext},m} \) and of \( B_{R_1} \) and \( B_{R_2} \) in \( \Phi_{\text{int},m} \). Note that both \( \Phi_{\text{ext},m} \) and \( \Phi_{\text{int},m} \) belong to \( H^m(\mathbb{R}^3) \setminus H^{m+1}(\mathbb{R}^3) \).

Thus we have

\[
K = \begin{cases} 
B_\eta & \text{if } \Phi(\cdot, 0) = \Phi_{\text{ext},m}, \\
B_{R_2} \setminus B_{R_1} & \text{if } \Phi(\cdot, 0) = \Phi_{\text{int},m}
\end{cases}
\]

and \( K \) satisfies \( K \cap \Omega = \emptyset \).

Using the Fourier transform method, we see that there exists a unique \( \Phi \) in the class \( C([0, T]; H^m(\mathbb{R}^3)) \cap H^1(0, T; H^{m-1}(\mathbb{R}^3)) \) that satisfies (1.6) and the initial data \( \Phi(\cdot, 0) = \Phi_{\star,m} \), where \( \star = \text{ext} \) or \( \text{int} \). Then choose \( v \) given by (1.7) which belongs to \( H^1(0, T; H^{m-2}(\partial \Omega)) \) and thus \( f \) given by (1.4) belongs to \( H^1(0, T; H^{m-2-\frac{4}{3}}(\partial \Omega)) \).

We choose an arbitrary fixed \( m \geq 4 \). This implies \( f \in H^1(0, T; H^{\frac{3}{2}}(\partial \Omega)) \). Then, it is known that there exists a unique solution \( (u, p) \) of (1.1) such that \( u \in L^2(0, T; H^2(\Omega \setminus \overline{D})^3) \cap C([0, T]; H^1(\Omega \setminus \overline{D})^3) \cap H^1(0, T; L^2(\Omega \setminus \overline{D})^3) \) and \( p \in L^2(0, T; H^1(\Omega \setminus \overline{D})/\mathbb{R}) \). See Proposition 3.2 in [21].

Now we state the main result of this paper.

**Theorem 1.1.**

(i) Let \( T \) be an arbitrary fixed positive number. Then, there exists a positive number \( \tau_0 \) such that, for all \( \tau \geq \tau_0 \)

\[
\lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \log I_T(\tau; v, 0) = -2 \sqrt{\frac{\rho}{\mu}} \text{dist}(D, K). \tag{1.8}
\]

(ii) We have

\[
\lim_{\tau \to \infty} e^{\sqrt{\tau} T} I_T(\tau; v, 0) = \begin{cases} 
\infty & \text{if } T > 2 \sqrt{\frac{\rho}{\mu}} \text{dist}(D, K), \\
0 & \text{if } T < 2 \sqrt{\frac{\rho}{\mu}} \text{dist}(D, K).
\end{cases}
\]

(iii) If \( T = 2 \sqrt{\frac{\rho}{\mu}} \text{dist}(D, K) \), then we have, as \( \tau \to \infty \)

\[
e^{\sqrt{\tau} T} |I_T(\tau; v, 0)| = O(\tau^3). \tag{1.9}
\]
It is easy to see that
\[
\text{dist} (D, K) = \begin{cases} 
 d_D(z_1) - \eta, & \text{if } K = \overline{B_\eta}, \\
 R_1 - R_D(z_2), & \text{if } K = \overline{B_{R_2}} \setminus \overline{B_{R_1}},
\end{cases}
\]
where \(z_1\) denotes the center point of \(B_\eta\) and \(z_2\) the common center point of \(B_{R_1}\) and \(B_{R_2}\);
\[
d_D(z_1) = \text{dist} (\{z_1\}, D), \quad R_D(z_2) = \sup_{x \in D} |x - z_2|.
\]
Thus by Theorem 1.1 one can extract two quantities \(d_D(z_1)\) and \(R_D(z_2)\) which yield the largest/smallest sphere whose exterior/interior contains \(D\) from the indicator function according to the two choices of \(\Phi(\cdot, 0)\). See Figure 1 for an illustration of two spheres centered at two different points \(z_1\) and \(z_2\) with radii \(d_D(z_1)\) and \(R_D(z_2)\), respectively.

![Figure 1: An illustration of two large and small spheres centered at two different points \(z_1 \in \mathbb{R}^3 \setminus \overline{\Omega}\) and \(z_2 \in \Omega\) with radii \(d_D(z_1)\) and \(R_D(z_2)\), respectively.](image)

It should be emphasized that there is no restriction on \(T\). This indicates that the propagation speed of the signal governed by the Stokes system is not finite similar to that of the signal governed by the heat equation.

**Remark 1.1.** Since \(T\) in (i) is arbitrary, (ii) can be easily deduced from (i). Thus the point is to establish (i) and (iii).

**Remark 1.2.** Mathematically, from (1.8) we obtain \(\text{dist} (D, K)\) from the data over an arbitrary fixed time interval \([0, T]\). So this shows that the propagation speed of the signal governed by the Stokes system is infinite.

This paper is concerned with the reconstruction issue for an obstacle embedded in a bounded domain. For the uniqueness and stability issue see [1] including the stationary case. In [3] a stability estimate of log-log type in the stationary case has been established. For a numerical reconstruction method in the stationary case based on the method of fundamental solutions see [2].

In the next subsection we mention some articles using the idea of the enclosure method in the time domain and stationary cases.
1.3 Review of the related results

1.3.1 Time domain case

In [21] the response operator acting infinitely many \( f \) has been used. They are the separation of variables type having the form

\[
f(x, t) = \chi(t)v(x), \quad (x, t) \in \partial\Omega \times ]0, T[,
\]

where the function \( v \) defined in an open neighbourhood \( \tilde{\Omega} \) of \( \Omega \) and satisfies the following system depending on a large parameter \( \tau \) (in our notation) with an appropriate function \( q \):

\[
\begin{aligned}
\text{div} \sigma(v, q) - \tau \rho v &= 0, \quad x \in \tilde{\Omega} \\
\nabla \cdot v &= 0, \quad x \in \tilde{\Omega}.
\end{aligned}
\]

(1.11)

The function \( \chi(t) \) also depends \( \tau > 0 \) and satisfies \( \chi(0) = 0 \) and \( \chi(t) > 0 \) for all \( 0 < t \leq T \) with the normalized condition \( \int_0^T e^{-\tau t} \chi(t) \, dt = 1 \). This is just under the framework of an earlier version of the time domain enclosure method with infinitely many input proposed in [12].

In [21] they used two types of the solutions. One is the real exponential solution having the form

\[
v(x) = e^{\tilde{\tau}m \cdot x}, \quad x \in \mathbb{R}^3
\]

with \( q = 0 \) and \( \tilde{\tau} = \sqrt{\tau} \sqrt{\rho/\mu} \), where \( l \) and \( m \) are two unit constant vectors satisfying \( l \cdot m = 0 \).

Another one is a singular solution having its singularity on a line not intersecting with the convex hull of \( \Omega \). The solution has the following form. Let \( x_0 \) be an arbitrary point out side of the convex hull of \( \Omega \). Choose a unit vector \( c \) in such a way that the line \( l : x = x_0 + \lambda c, -\infty < \lambda < \infty \) lies in the outside of the convex hull of \( \Omega \). Then, the \( v \) takes the form

\[
v(x) = e^{-\tilde{\tau}|x-x_0|} \frac{c \times (x-x_0)}{|c \times (x-x_0)|^2}, \quad x \in \mathbb{R}^3 \setminus l
\]

(1.13)

and \( q = 0 \).

Using a spherical coordinates depending on \( c \) of \( \mathbb{R}^3 \), they showed that the \( v \) together with \( q = 0 \) satisfies (1.11) and essentially it’s energy integral locally outside the line \( l \) behaves like that of the function

\[
e^{-\tilde{\tau}|x-x_0|}.
\]

This means that the essential part is reduced to the heat equation case considered in [20].

Using \( v \) given by (1.12) and (1.13) with \( q = 0 \), they prescribe \( f \) given by (1.10) and gave extraction formulae of the quantities \( h_D(m) = \sup_{x \in D} x \cdot m \) and \( \text{dist}(x_0, D) \) provided \( x_0 \) is placed outside of the convex hull of \( \Omega \). In our method \( f \) is independent of \( \tau \) and yields \( \text{dist}(x_0, D) \) for all \( x_0 \in \mathbb{R}^3 \setminus \Omega \) and the quantity \( R_D(x_0) \) for all \( x_0 \in \mathbb{R}^3 \) which is not extracted in their paper.

1.3.2 Stationary case

In the article [6], the idea of the original enclosure method [8, 7] has been applied to the Stokes system in the stationary case (with variable viscosity). They state an extraction
procedure of the distance of an arbitrary point $x_0$ outside the convex hull of $\overline{\Omega}$ to an unknown obstacle $D$ from the associated Dirichlet-to-Neumann map acting on infinitely many special Dirichlet data. Their Dirichlet data are given by the trace onto $\partial \Omega$ of special solutions of the stationary Stokes system in an open neighbourhood of $\overline{\Omega}$ having several parameters. Note that their main small parameter $h$ plays the similar role as our large parameter $\tau$ via the relation $h \sim \tau^{-1}$. However, unlike the original enclosure method, they ensure only the existence of those solutions. This is because of their argument: a combination of a Carleman estimate and the Hahn-Banach theorem which is a well known argument and effective in uniqueness issue in various inverse problems. See Theorem 2.1 in [23] for a typical statement. On page 570 in [24] it is written that, in short, since the Runge type approximation theorem is not constructive, the probe method [9] introduced by the author has some problem to characterize the needed input data. However, it should be noted that their purpose is to cover a variable viscosity case. In that case the construction not the existence of the desired special solution itself is a problem.

Let us take a closer look at their results and comment on their extraction procedure based on Theorem 4.1 in [6]. Carefully reading the proof of Theorem 4.1 in [6], one sees that they have to tune the leading profiles of their solutions depending on whether the sphere $S_t(x_0) = \{x | |x - x_0| = t\}$ with radius $t > 0$ satisfies: (a) $S_t(x_0) \subset \mathbb{R}^3 \setminus \overline{D}$; (b) $S_t(x_0) \cap D \neq \emptyset$, $S_t(x_0) \cap \partial D \neq \emptyset$, $S_t(x_0) \cap \partial D$ consists of a single point; (c) $S_t(x_0) \cap D \neq \emptyset$. This means that a careful replacement of the reading profiles of their solutions and thus the solutions themselves is needed to obtain their desired results, especially in (b) and (c). See also (2) of Remark 4.1 on page 1977 in [23]. However $D$ itself is unknown. How should we tune? The author thinks that this problem has not yet been resolved in an exact sense. Compare the comments above with Remark 4.2 in [6], in particular, the last sentence “In real applications, we believe that the concerns in (b) and (c) can be ignored.”.

1.4 Outline of the paper

A brief outline of this paper is as follows. Theorems 1.1 is proved in Section 3. The proof is based on: a representation formula of the indicator function together with some energy estimates which are proved in Section 2; Proposition 3.1 which gives explicit formulae in some domains for the solutions of the modified Helmholtz equation with special inhomogeneous terms in the whole space and the proof is given in Section 4.

2 Preliminaries

In this section, the symbol $n$ denotes also the unit outward normal to $\partial D$.

The function $v$ is given by a solution of (1.3) with $\text{supp } v(\cdot, 0) \cap \Omega = \emptyset$ and $f$ given by (1.4). Note that the form of $v$ is not specified unlike (1.7).
2.1 Representation formula and its direct consequence

From (1.1) we have

\[
\begin{aligned}
\text{div} \sigma(w, \tilde{p}) - \tau \rho w &= e^{-\tau T} F, \quad x \in \Omega \setminus \overline{D}, \\
\nabla \cdot w &= 0, \quad x \in \Omega \setminus \overline{D}, \\
w &= 0, \quad x \in \partial D, \\
w &= w_0, \quad x \in \partial \Omega, \\
\end{aligned}
\]

(2.1)

where

\[F(x) = \rho u(x, T).\]

And also from (1.3) we have

\[
\begin{aligned}
\text{div} \sigma(w_0, \tilde{q}) - \tau \rho w_0 + v(x, 0) &= e^{-\tau T} F_0, \quad x \in \mathbb{R}^3, \\
\nabla \cdot w_0 &= 0, \quad x \in \mathbb{R}^3, \\
\end{aligned}
\]

(2.2)

where

\[F_0(x) = \rho v(x, T).\]

Noting that \(\text{supp} v(\cdot, 0) \cap \Omega = \emptyset\), from (2.1) and (2.2) we obtain

\[
\begin{aligned}
\int_{\partial \Omega} (\sigma(w, \tilde{p})n \cdot w_0 - \sigma(w_0, \tilde{q})n \cdot w) \, dS \\
= \int_{\partial D} \sigma(w, \tilde{p})n \cdot w_0 \, dS \\
+ \int_{\Omega \setminus \overline{D}} (\sigma(w, \tilde{p}) \cdot \nabla w_0 - \sigma(w_0, \tilde{q}) \cdot \nabla w) \, dx \\
+ e^{-\tau T} \int_{\Omega \setminus \overline{D}} (F \cdot w_0 - F_0 \cdot w) \, dx.
\end{aligned}
\]

(2.3)

Using \(\nabla \cdot w_0 = \nabla \cdot w = 0\) in \(\Omega \setminus \overline{D}\) and

\[\text{Sym} \nabla w_0 \cdot \nabla w = \text{Sym} \nabla w \cdot \nabla w_0,
\]

from (2.3) we obtain the first expression of the indicator function:

\[
I_T(\tau; v, q) = \int_{\partial D} \sigma(w, \tilde{p})n \cdot w_0 \, dS + e^{-\tau T} \int_{\Omega \setminus \overline{D}} (F \cdot w_0 - F_0 \cdot w) \, dx.
\]

(2.4)

Set

\[R = w - w_0.\]

We further rewrite formula (2.4).
Proposition 2.1. We have

\[ I_T(\tau; v, q) = J(\tau) + E(\tau) + e^{-\tau T} \mathcal{R}(\tau), \]  

(2.5)

where

\[ J(\tau) = \int_D (2\mu |\text{Sym} \nabla w_0|^2 + \tau \rho |w_0|^2) \, dx, \]  

(2.6)

\[ E(\tau) = \int_{\Omega \setminus \overline{D}} (2\mu |\text{Sym} \nabla R|^2 + \tau \rho |R|^2) \, dx \]  

(2.7)

and

\[ \mathcal{R}(\tau) = \int_D F_0 \cdot w_0 \, dx + \int_{\Omega \setminus \overline{D}} (F - 2F_0) \cdot R \, dx + \int_{\Omega \setminus \overline{D}} (F - F_0) \cdot w_0 \, dx. \]  

(2.8)

Proof. From (2.1) and (2.2) we have

\[
\begin{cases}
\text{div} \sigma(R, r) - \tau \rho R = e^{-\tau T}(F - F_0), & x \in \Omega \setminus \overline{D}, \\
\nabla \cdot R = 0, & x \in \Omega \setminus \overline{D}, \\
R = -w_0, & x \in \partial D, \\
R = 0, & x \in \partial \Omega,
\end{cases}
\]  

(2.9)

where

\[ r = \tilde{p} - \tilde{q}. \]

Write

\[
\int_{\partial D} \sigma(w, \tilde{p}) n \cdot w_0 \, dS = \int_{\partial D} \sigma(w_0, \tilde{q}) n \cdot w_0 \, dS + \int_{\partial D} \sigma(R, r) n \cdot w_0 \, dS.
\]

It follows from (2.2) in \( D \) that

\[
\int_{\partial D} \sigma(w_0, \tilde{q}) n \cdot w_0 \, dS = \int_D \left( \sigma(w_0, \tilde{q}) \cdot \nabla w_0 + \tau \rho |w_0|^2 \right) \, dx + e^{-\tau T} \int_D F_0 \cdot w_0 \, dx.
\]

It follows from (2.9) that

\[
\int_{\partial D} \sigma(R, r) n \cdot w_0 \, dS = -\int_{\partial D} \sigma(R, r) n \cdot R \, dS = \int_{\partial (\Omega \setminus \overline{D})} \sigma(R, r) n \cdot R \, dS = \int_{\Omega \setminus \overline{D}} (\sigma(R, r) \cdot \nabla R + \tau \rho |R|^2) \, dx + e^{-\tau T} \int_{\Omega \setminus \overline{D}} (F - F_0) \cdot R \, dx.
\]  

(2.10)
Thus we obtain
\[
\int_{\partial D} \sigma(w, \tilde{p}) n \cdot w_0 \, dS = \int_D (\sigma(w_0, \tilde{q}) \cdot \nabla w_0 + \tau \rho |w_0|^2) \, dx + \int_{\Omega \setminus \overline{D}} (\sigma(R, r) \cdot \nabla R + \tau \rho |R|^2) \, dx + e^{-\tau T} \left\{ \int_D F_0 \cdot w_0 \, dx + \int_{\Omega \setminus \overline{D}} (F - F_0) \cdot R \, dx \right\}.
\] (2.11)

Since we have \( \nabla \cdot w_0 = 0 \) in \( D \) and \( \nabla \cdot w = 0 \) in \( \Omega \setminus \overline{D} \), using the identity \( \text{Sym} A \cdot A = |\text{Sym} A|^2 \), we obtain the expression
\[
\begin{cases}
J(\tau) = \int_D (\sigma(w_0, \tilde{q}) \cdot \nabla w_0 + \tau \rho |w_0|^2) \, dx \\
E(\tau) = \int_{\Omega \setminus \overline{D}} (\sigma(R, r) \cdot \nabla R + \tau \rho |R|^2) \, dx.
\end{cases}
\] (2.12)

Moreover, one has
\[
\int_{\Omega \setminus \overline{D}} (F - F_0) \cdot R \, dx + \int_{\Omega \setminus \overline{D}} (F \cdot w_0 - F_0 \cdot w) \, dx = \int_{\Omega \setminus \overline{D}} (F - 2F_0) \cdot R \, dx + \int_{\Omega \setminus \overline{D}} (F - F_0) \cdot w_0 \, dx.
\]

Substituting (2.11) into (2.4) and using this together with (2.12), we obtain (2.5).

Now we state two estimates which tell us that the asymptotic behaviour of the terms \( E(\tau) \) and \( e^{-\tau T} R(\tau) \) as \( \tau \to \infty \) can be controlled from above by that of \( J(\tau) \) and \( \|w_0\|_{L^2(\Omega)} \).

**Proposition 2.2.** We have, as \( \tau \to \infty \)
\[
E(\tau) = O(\tau J(\tau) + e^{-2\tau T})
\] (2.13)
and
\[
e^{-\tau T} |R(\tau)| = O(e^{-\tau T} \|w_0\|_{L^2(\Omega)} + e^{-\tau T} J(\tau)^{1/2} + \tau^{-1/2} e^{-2\tau T}).
\] (2.14)

**Proof.** First we give a proof of (2.13). From (2.10) and the second formula on (2.12), we have
\[
\int_{\Omega \setminus \overline{D}} \left( 2\mu |\text{Sym} \nabla R|^2 + \tau \rho \left| R + e^{-\tau T} \frac{F - F_0}{2\tau \rho} \right|^2 \right) \, dx = \int_{\partial D} \sigma(R, r) n \cdot w_0 \, dS + \frac{e^{-2\tau T}}{4\tau \rho} \int_{\Omega \setminus \overline{D}} |F - F_0|^2 \, dx.
\]
This together with (2.7) immediately yields
\[
E(\tau) \leq 2 \int_{\partial D} \sigma(R, r) n \cdot w_0 \, dS + O(\tau^{-1} e^{-2\tau T}).
\] (2.15)
Here we choose \( \tilde{w}_0 \in H^1(\Omega \setminus \overline{D})^3 \) in such a way that
\[
\begin{cases}
\text{div } \tilde{w}_0 = 0, & x \in \Omega \setminus \overline{D}, \\
\tilde{w}_0 = 0, & x \in \partial \Omega, \\
\tilde{w}_0 = w_0, & x \in \partial D
\end{cases}
\]
and
\[
\| \tilde{w}_0 \|_{H^1(\Omega \setminus \overline{D})} \leq C \| w_0 \|_{H^{1/2}(\partial D)} \leq C' \| w_0 \|_{H^1(D)}.
\]  
(2.16)

See the proof of Lemma 3.1 in [21] for this choice which is taken from [5]. Then, we can write
\[
\int_{\partial D} \sigma(\mathbf{R}, r) \mathbf{n} \cdot w_0 dS = \int_{\partial D} \sigma(\mathbf{R}, r) \mathbf{n} \cdot \tilde{w}_0 dS \quad (2.17)
\]
Integration by parts and the first equation on (2.9) give
\[
\int_{\partial(\Omega \setminus \overline{D})} \sigma(\mathbf{R}, r) \mathbf{n} \cdot \tilde{w}_0 dS = \int_{\Omega \setminus \overline{D}} \text{div} \sigma(\mathbf{R}, r) \cdot \tilde{w}_0 dx + \int_{\Omega \setminus \overline{D}} \sigma(\mathbf{R}, r) \cdot \nabla \tilde{w}_0 dx
\]
\[
= \int_{\Omega \setminus \overline{D}} \text{div} \sigma(\mathbf{R}, r) \cdot \tilde{w}_0 dx + \int_{\Omega \setminus \overline{D}} 2\mu \text{Sym} \nabla \mathbf{R} \cdot \text{Sym} \nabla \tilde{w}_0 dx
\]
\[
= \int_{\Omega \setminus \overline{D}} \tau \rho \mathbf{R} \cdot \tilde{w}_0 dx + e^{-\tau T} \int_{\Omega \setminus \overline{D}} (\mathbf{F} - \mathbf{F}_0) \cdot \tilde{w}_0 dx + \int_{\Omega \setminus \overline{D}} 2\mu \text{Sym} \nabla \mathbf{R} \cdot \text{Sym} \nabla \tilde{w}_0 dx.
\]

From (2.16) we see that this right-hand side has the bound
\[
C \| w_0 \|_{H^1(D)}(\tau \| \mathbf{R} \|_{L^2(\Omega \setminus \overline{D})} + \| \text{Sym} \nabla \mathbf{R} \|_{L^2(\Omega \setminus \overline{D})} + e^{-\tau T}).
\]
Applying this and (2.17) to the first term on (2.15), we obtain
\[
E(\tau) \leq C \| w_0 \|_{H^1(D)}(\tau \| \mathbf{R} \|_{L^2(\Omega \setminus \overline{D})} + \| \text{Sym} \nabla \mathbf{R} \|_{L^2(\Omega \setminus \overline{D})} + e^{-\tau T}) + O(\tau^{-1} e^{-2\tau T}).
\]
Moreover, from (2.7) we have, for all \( \tau \geq 1 \)
\[
\tau \| \mathbf{R} \|_{L^2(\Omega \setminus \overline{D})} + \| \text{Sym} \nabla \mathbf{R} \|_{L^2(\Omega \setminus \overline{D})} \leq C \tau^{1/2} E(\tau)^{1/2}.
\]  
(2.18)

From these we obtain
\[
E(\tau) \leq C \| w_0 \|_{H^1(D)}(\tau^{1/2} E(\tau)^{1/2} + e^{-\tau T}) + O(\tau^{-1} e^{-2\tau T}).
\]  
(2.19)
Here recall the Korn’s second inequality [4]
\[
\| w_0 \|_{H^1(D)} \leq C (\| w_0 \|_{L^2(D)}^2 + \| \text{Sym} \nabla w_0 \|_{L^2(D)}^2)^{1/2}.
\]
Since this right-hand side has the bound $J(\tau)^{1/2}$, from (2.19) we obtain
\[ E(\tau) \leq CJ(\tau)^{1/2}(\tau^{1/2}E(\tau)^{1/2} + e^{-\tau T}) + O(\tau^{-1}e^{-2\tau T}). \]
Then, a standard argument yields (2.13).

Next we give a proof of (2.14). From (2.8) we have
\[ e^{-\tau T}|R(\tau)| \leq C(e^{-\tau T}\|w_0\|_{L^2(\Omega)} + e^{-\tau T}\|R\|_{L^2(\Omega \setminus D)}). \]
Besides (2.18) yields the estimate
\[ \|R\|_{L^2(\Omega \setminus D)} \leq C\tau^{-1/2}E(\tau)^{1/2}. \]
Now from these together with (2.13) we obtain (2.14).

\[ \blacksquare \]

3 Proof Theorem 1.1

In this section the form of $v$ is specified as (1.7).

3.1 Propagation estimates

The following two lemmas are concerned with the asymptotic behaviour of $J(\tau)$ and $R(\tau)$ as $\tau \to \infty$.

**Lemma 3.1.** Let $U$ be an arbitrarily bounded open subset of $\mathbb{R}^3$. We have, as $\tau \to \infty$
\[ \sqrt{\tau}\|w_0\|_{L^2(U)} + \|\text{Sym} \nabla w_0\|_{L^2(U)} = O(\tau e^{-\sqrt{\tau} \sqrt{\rho/\mu} \text{dist}(U,K)} + \tau^{-1/2}e^{-\tau T}) \]  
(3.1)

**Proof.** Let $v_{00} \in H^1(\mathbb{R}^3)^3$ be the unique weak solution of
\[ \mu \Delta v - \tau \rho v + \Phi(x,0)a = 0, \quad x \in \mathbb{R}^3. \]  
(3.2)

We have the expression
\[ v_{00}(x) = \left( \frac{1}{4\pi\mu} \int_{\mathbb{R}^3} e^{-\sqrt{\tau} \sqrt{\rho/\mu} |x-y|} \Phi(\cdot,0) dy \right) a. \]  
(3.3)

Since $\Phi(\cdot,0) \in H^m(\mathbb{R}^3)$ we have $v_{00} \in H^{m+2}(\mathbb{R}^3)^2$.

Define
\[ w_{00} = \nabla \times v_{00} \in H^{m+1}(\mathbb{R}^3)^3. \]  
(3.4)

Then the pair $(w_{00}, \tilde{q})$ with $\tilde{q} = 0$ satisfies
\[ \begin{cases} \div \sigma(w_{00}, \tilde{q}) - \tau \rho w_{00} + \nabla \times (\Phi(x,0)a) = 0, & x \in \mathbb{R}^3, \\ \nabla \cdot w_{00} = 0, & x \in \mathbb{R}^3. \end{cases} \]  
(3.5)

From the expression (3.3) and (3.4) we have
\[ \sqrt{\tau}\|w_{00}\|_{L^2(U)} + \|\nabla w_{00}\|_{L^2(U)} \leq O(\tau e^{-\sqrt{\tau} \sqrt{\rho/\mu} \text{dist}(U,K)}) \]  
(3.6)
Set

\[ \epsilon_0 = e^{-\tau T}(w_0 - w_{00}). \]

We have

\[ w_0 = w_{00} + e^{-\tau T}\epsilon_0 \]

and it follows from (2.2), (3.2) and (3.5) that \( \epsilon_0 \) satisfies

\[
\begin{cases}
\text{div } \sigma(\epsilon_0, \tilde{q}) - \tau \rho \epsilon_0 = F_0, & x \in \mathbb{R}^3, \\
\nabla \cdot \epsilon_0 = 0, & x \in \mathbb{R}^3.
\end{cases}
\]

Since we have

\[
\int_{\mathbb{R}^3} (2\mu|\text{Sym } \nabla \epsilon_0|^2 + \tau \rho |\epsilon_0|^2 + F_0 \cdot \epsilon_0) \, dx = 0,
\]

one gets

\[
\int_{\mathbb{R}^3} (2\mu|\text{Sym } \nabla \epsilon_0|^2 + \tau \rho |\epsilon_0|^2) \, dx \leq 2 \times 2 \times \frac{\|F_0\|_{L^2(\mathbb{R}^3)}^2}{4\tau \rho} = O(\tau^{-1}).
\]

This gives

\[
\sqrt{\tau}\|\epsilon_0\|_{L^2(\mathbb{R}^3)} + \|\text{Sym } \nabla \epsilon_0\|_{L^2(\mathbb{R}^3)} = O(\tau^{-1/2}).
\]

Then, from (3.6) and (3.7) we obtain (3.1).

\[ \square \]

To obtain a lower estimate for \( J(\tau) \) we need a detailed expression of \( w_{00} = \nabla \times v_{00} \) in a neighbourhood of \( \mathcal{D} \).

Set

\[
v_{00} = \begin{cases} 
v_{\text{ext},m}, & \text{if } \Phi(\cdot, 0) = \Phi_{\text{ext},m}(\cdot, 0), \\
v_{\text{int},m}, & \text{if } \Phi(\cdot, 0) = \Phi_{\text{int},m}(\cdot, 0).
\end{cases}
\]

In what follows we write

\[ \tilde{\tau} = \sqrt{\tau} \cdot \frac{\sqrt{\rho}}{\mu}. \]

We have the following expression.

**Proposition 3.1.** (i) Let \( |x - p| > \eta \). We have

\[ v_{\text{ext},m}(x) = \frac{\eta^{2(1+m)}}{\mu} \cdot \frac{e^{-\tilde{\tau}|x-p|}}{|x-p|} a_m(\tilde{\tau}) \mathbf{a}, \]

where

\[ a_m(\tilde{\tau}) = \frac{1}{\tilde{\tau}} \int_0^1 s(1 - s^2)^m \sinh(\eta \tilde{\tau} s) \, ds. \]

(ii) Let \( |x - p| < R_1 \). We have

\[ v_{\text{int},m}(x) = \frac{2}{\mu} \cdot \frac{\sinh \tilde{\tau}|x-p|}{|x-p|} b_m(\tilde{\tau}) \mathbf{a}, \]

where

\[ b_m(\tilde{\tau}) = \frac{1}{\tilde{\tau}} \int_{R_1}^{R_2} s(R_2^2 - s^2)^m (s^2 - R_1^2)^m e^{-s \tilde{\tau}} \, ds. \]
The proof is given in the next section.

Concerning with the asymptotic behaviour of the coefficients \( a_m(\tilde{\tau}) \) and \( b_m(\tilde{\tau}) \), by Theorem 7.1 on p.81 in [22], we have, as \( \tilde{\tau} \to \infty \)

\[
a_m(\tilde{\tau}) \sim \eta 2^{m-1}m! \frac{e^{\tilde{\tau} \eta}}{\eta^m m!} \] \hfill (3.8)

and

\[
b_m(\tilde{\tau}) \sim 2^m m! R_{\eta}^{m+1} (R_2 - R_1^2) m \frac{e^{-R_1 \tilde{\tau}}}{\tilde{\tau}^{m+2}}. \] \hfill (3.9)

Now we are ready to prove the key lemma stated below.

**Lemma 3.2.** There exist positive numbers \( \tau_0 \) and \( \kappa \) such that, for all \( \tau \geq \tau_0 \)

\[
\tau^{\kappa} e^{2\sqrt{\tau} \sqrt{\rho/\mu} \text{dist}(D,K)} J(\tau) \geq C. \] \hfill (3.10)

**Proof.** It follows from (3.7) that

\[
J(\tau) \geq \frac{1}{2} \tau \rho \| w_{00} \|_{L^2(D)}^2 + O(\tau^{-1} e^{-2\tau T}). \] \hfill (3.11)

Thus it suffices to give a lower estimate for \( \| w_{00} \|_{L^2(D)}^2 \).

First consider the case when \( K = \overline{B_\eta} \). It follows from (3.4) and (i) in Proposition 3.1 that \( w_{00} \) takes the form

\[
w_{00}(x) = \eta^{2(1+m)} \frac{\mu}{\mu} \cdot a_m(\tilde{\tau}) \nabla \left( \frac{e^{-\tilde{\tau}|x-p|}}{|x-p|} \right) \times a, \quad x \in D. \] \hfill (3.12)

Here we have

\[
\nabla \left( \frac{e^{-\tilde{\tau}|x-p|}}{|x-p|} \right) = -e^{-\tilde{\tau}|x-p|} \left( \frac{1}{|x-p|^2} + \frac{1}{|x-p|} \right) \frac{x-p}{|x-p|}. \]

Thus one gets

\[
\int_D \left| \nabla \left( \frac{e^{-\tilde{\tau}|x-p|}}{|x-p|} \right) \times a \right|^2 dx \geq C^2 \int_D e^{-2\tilde{\tau}|x-p|} \left| \frac{x-p}{|x-p|} \times a \right|^2 dx,
\]

where

\[
C = \frac{1}{\sup_{x \in D} |x-p|^2} + \frac{1}{\sup_{x \in D} |x-p|}.
\]

By Lemma A.3 in [16] there exist positive constants \( C_1, \tau_0 \) and \( \kappa' \) such that

\[
\int_D e^{-2\tilde{\tau}|x-p|} \left| \frac{x-p}{|x-p|} \times a \right|^2 dx \geq C_1 \tau^{2\eta \tilde{\tau}} \cdot \tau^{-\kappa' \tilde{\tau}} e^{-2\tau \text{dist}(D,K)}
\]

for all \( \tau \geq \tau_0 \). Thus from (3.12) one gets

\[
\| w_{00} \|_{L^2(D)}^2 \geq \left\{ \frac{\eta^{2(1+m)}}{\mu} \cdot a_m(\tilde{\tau}) \right\}^2 C^2 C_1 \tau^{2\eta \tilde{\tau}} \cdot \tau^{-\kappa' \tilde{\tau}} e^{-2\tau \text{dist}(D,K)}.
\]
Now from this together with (3.8) and (3.11) we see that (3.10) is valid by choosing \( \kappa = \frac{m+2}{2} + \kappa' - 1 \).

Next consider the case when \( K = \overline{B_{R_2}} \setminus B_{R_1} \). From (3.4) and (ii) in Proposition 3.1 we have

\[
\mathbf{w}_{00}(x) = \frac{2}{\mu} \cdot b_m(\bar{\tau}) \nabla \left( \frac{\sinh \bar{\tau} |x - p|}{|x - p|} \right) \times \mathbf{a}, \quad x \in D. \tag{3.13}
\]

By the proof of Lemma 4.3 in [19] we know that there exist positive constants \( C_2, \tau_1 \) and \( \kappa'' \) such that

\[
\int_D \left| \nabla \left( \frac{\sinh \bar{\tau} |x - p|}{|x - p|} \right) \times \mathbf{a} \right|^2 dx \geq C_2 \tau^{-\kappa''} e^{2\tau R_D(p)}
\]

for all \( \tau \geq \tau_1 \). This together with (3.13) yields

\[
\|\mathbf{w}_{00}\|_{L^2(D)}^2 \geq \left\{ \frac{2}{\mu} \cdot b_m(\bar{\tau}) \right\}^2 C_2 \tau^{-\kappa''} e^{2\tau R_D(p)}.
\]

Now from this together with (3.9) and (3.11) we see that (3.10) is valid by choosing \( \kappa = \frac{m+2}{2} + \kappa'' - 1 \). Note that we have made use of the relationship \( \text{dist} (\Omega, K) = R_1 - R_D(p) \).

\[\square\]

Remark 3.1. In contrast to the wave equation case, for example, (ii) of Lemma 2.4 in [15], there is no restriction on the lower bound for \( T \) in Lemma 3.2. See \( J(\tau) \) on (2.6), which comes from \( \mathbf{w}_0 \). The \( \mathbf{w}_0 \) comes from \( \mathbf{v} \) which solves (1.3) and is generated by the initial data supported on \( K \). The \( K \) is remote from \( D \). So Lemma 3.2 implicitly indicates that the signal governed by the Stokes system propagates with infinity speed.

3.2 Finishing the proof of Theorem 1.1

First we prepare three estimates. Letting \( U = \Omega \) in Lemma 3.1, we have

\[
\|\mathbf{w}_0\|_{L^2(\Omega)} = O(\sqrt{\tau} e^{-\sqrt{\tau} \sqrt{\rho/\mu} \text{dist}(\Omega, K) + \tau^{-1} e^{-\tau T}}). \tag{3.14}
\]

Besides a combination of (3.1) in the case \( U = D \) and (2.6) yields

\[
J(\tau) = O(\tau^2 e^{-2\sqrt{\tau} \sqrt{\rho/\mu} \text{dist}(D, K) + \tau^{-1} e^{-2\tau T}}). \tag{3.15}
\]

From this together with (2.13) we obtain

\[
E(\tau) = O(\tau^3 e^{-2\sqrt{\tau} \sqrt{\rho/\mu} \text{dist}(D, K) + e^{-2\tau T}}). \tag{3.16}
\]

Applying (3.14) and (3.15) to the right-hand side on (2.14), we obtain

\[
e^{-\tau T} \mathcal{R}(\tau) = O(e^{-2^{-1} \tau T}).
\]

Note that we just used (3.14) in which \( \text{dist}(\Omega, K) \) is replaced with 0. Then (2.5) gives

\[
I_T(\tau; \mathbf{v}, 0) = J(\tau) + E(\tau) + O(e^{-2^{-1} \tau T}). \tag{3.17}
\]
Since $E(\tau) \geq 0$, it follows from (3.10) that there exists a positive number $\tau_0$ such that, for all $\tau \geq \tau_0$
\[\tau^\kappa e^{2\sqrt{\tau} \sqrt{\rho/\mu \text{dist}(D,K)}} I_T(\tau; v, 0) \geq C/2.\] (3.18)

Applying (3.15) and (3.16) to the right-hand side on (3.17), we obtain
\[e^{2\sqrt{\tau} \sqrt{\rho/\mu \text{dist}(D,K)}} I_T(\tau; v, 0) = O(\tau^3).\] (3.19)

Then, from (3.18) and (3.19) we immediately obtain (1.8) and (1.9).
This completes the proof of Theorem 1.1.

4 Proof of Proposition 3.1

The computation presented here is a combination of the Parseval identity and the residue calculus as done in the proof of Proposition 3.1 in [18].

Write
\[\int_{\mathbb{R}^3} e^{-ix \cdot \xi} \Phi_{ext,m}(x) \, dx = \int_{B_\eta} e^{-ix \cdot \xi} (\eta^2 - |x - p|^2)^m \, dx\]
\[= e^{-ip \cdot \xi} \int_0^\eta r^2 (\eta^2 - r^2)^m \, dr \int_{S^2} e^{-ir \omega \cdot \xi} \, d\omega.\]

Since we have
\[\int_{S^2} e^{-ir \omega \cdot \xi} \, d\omega = 4\pi \frac{\sin r|\xi|}{r|\xi|},\]
one gets
\[\hat{\Phi}_{ext,m}(\xi) = 4\pi e^{-ip \cdot \xi} \int_0^\eta r^2 (\eta^2 - r^2)^m \sin r|\xi| \frac{r|\xi|}{|\xi|} \, dr\]
\[= 4\pi \eta^{2(1+m)} e^{-ip \cdot \xi} |\xi|^{-1} A(|\xi|),\] (4.1)

where
\[A_m(z) = \int_0^1 s(1 - s^2)^m \sin (\eta z s) \, ds.\]

Similary we have
\[\hat{\Phi}_{int,m}(\xi) = 4\pi e^{-ip \cdot \xi} |\xi|^{-1} B_m(|\xi|),\] (4.2)
where
\[B_m(z) = \int_{S_1}^{S_2} s(R_2^2 - s^2)^m (s^2 - R_1^2)^m \sin zs \, ds.\]
Using the Parseval identity, (3.3) and (4.1) we obtain

\[ v_{\text{ext},m}(x) = \frac{1}{\mu(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix\cdot\xi}}{|\xi|^2 + \tau^2} \cdot 4\pi \eta^{2(1+m)} e^{-ip\cdot|\xi|}^{-1} A_m(|\xi|) \, d\xi \, d\alpha \]

\[ = \frac{2\eta^{2(1+m)}}{\mu(2\pi)^2} \int_{\mathbb{R}^3} \frac{e^{ix-p\cdot\xi}}{|\xi|(|\xi|^2 + \tau^2)} A_m(|\xi|) \, d\xi \, d\alpha \]

\[ = \eta^{2(1+m)} \cdot \frac{1}{\mu\pi} \int_0^\infty \frac{1}{r^2 + \tau^2} A_m(r) \, dr \int_0^\pi \sin \phi \, e^{i|x-p|r \cos \phi} \, d\phi \, d\alpha \]

\[ = \eta^{2(1+m)} \cdot \frac{1}{\mu\pi} \int_0^\infty \frac{1}{r^2 + \tau^2} A_m(r) \, dr \, d\alpha \]

\[ = \eta^{2(1+m)} \cdot \frac{1}{\mu\pi} \int_0^\infty \frac{1}{r^2 + \tau^2} A_m(r) \sin |x-p| \, r \, dr \, d\alpha. \]  

(4.3)

Note that, at the final step we made use of the evenness of the integrand with respect to $r$.

Let $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$. We have

\[ |A_m(z)| \leq Ce^{\eta R \sin \theta} \]

and

\[ |e^{i|x-p|z}| = e^{-|x-p|R \sin \theta}. \]

A standard residue calculus yields: if $|x-p| > \eta$, then we have

\[ \int_{-\infty}^{\infty} \frac{1}{r^2 + \tau^2} A_m(r) e^{i|x-p|r} \, dr \]

\[ = 2\pi i \frac{1}{2i\tau} A_m(i\tau) e^{i|x-p|i\tau} \]

\[ = \frac{\pi}{\tau} \int_0^1 s(1-s^2)^m \sin \eta i\tau \, s \, ds \, e^{-|x-p|i\tau} \]

\[ = \frac{i \pi}{\tau} \int_0^1 s(1-s^2)^m \sinh(\eta \tau) \, s \, ds \, e^{-|x-p|i\tau} \]

and thus

\[ \int_{-\infty}^\infty \frac{1}{r^2 + \tau^2} A_m(r) \sin |x-p| \, r \, dr = \pi a_m(\tau) \, e^{-|x-p|i\tau}. \]

Now from (4.3) we obtain (i).
For (ii), using the Parseval identity, (3.3) and (4.2) we obtain

\[ v_{int,m}(x) = \frac{1}{\mu(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix \xi}}{||\xi||^2 + \tau^2} \cdot 4\pi \, |\xi|^{-1} e^{-ip \xi} \, B_m(||\xi||) d\xi a \]

\[ = \frac{4\pi}{\mu(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix - p |\xi|}}{||\xi||(|\xi|^2 + \tau^2)} \, B_m(||\xi||) d\xi a \]

\[ = \frac{2}{\mu \pi} \int_0^\infty \frac{r}{r^2 + \tau^2} \cdot B_m(r) \, dr \int_0^\pi e^{ix - p r \cos \phi} \sin \phi \, d\phi \, a \]

\[ = -i \frac{2}{\mu \pi |x - p|} \int_0^\infty \frac{1}{r^2 + \tau^2} \cdot B_m(r) (e^{ix - p r} - e^{-ix - p r}) \, dr \, a \]

\[ = \frac{4}{\mu \pi |x - p|} \int_0^\infty \frac{1}{r^2 + \tau^2} \cdot B_m(r) \sin |x - p| r \, dr \, a \]

\[ = \frac{2}{\mu \pi |x - p|} \int_{-\infty}^\infty \frac{1}{r^2 + \tau^2} \cdot B_m(r) \sin |x - p| r \, dr \, a. \]

Here one can write

\[ \int_{-\infty}^\infty \frac{1}{r^2 + \tau^2} \cdot B_m(r) \sin |x - p| r \, dr = \text{Im} \int_{-\infty}^\infty \frac{C_m(r)}{r^2 + \tau^2} \sin |x - p| r \, dr, \]

where

\[ C_m(z) = \int_{R_1}^{R_2} s(R_2^2 - s^2)^m(s^2 - R_1^2)^m \, e^{isz} \, ds, \quad m = 0, 1, \ldots. \]

Let \( z = Re^{i\theta}, 0 \leq \theta \leq \pi \). We have

\[ |C_m(z)| \leq C(R_1, R_2) e^{-RR_1 \sin \theta} \]

and

\[ |\sin |x - p| z| \leq e^{R|x - p| \sin \theta}. \]

Using the residue theorem, we obtain: if \( |x - p| < R_1 \), then

\[ \int_{-\infty}^\infty \frac{C_m(r)}{r^2 + \tau^2} \sin |x - p| r \, dr = \pi \tilde{\tau}^{-1} C_m(i\tilde{\tau}) \sin (|x - p| i\tilde{\tau}). \]

Since

\[ C_m(i\tilde{\tau}) = \int_{R_1}^{R_2} s(R_2^2 - s^2)^m(s^2 - R_1^2)^m e^{-s \tilde{\tau}} \, ds \]

and

\[ \sin (|x - p| i\tilde{\tau}) = \frac{e^{-i\tilde{\tau}|x - p|} - e^{i\tilde{\tau}|x - p|}}{2i} = i \frac{1}{2} (e^{i\tilde{\tau}|x - p|} - e^{-i\tilde{\tau}|x - p|}) = i \sinh \tilde{\tau}|x - p|. \]

Thus one gets

\[ \text{Im} \int_{-\infty}^\infty \frac{C_m(r)}{r^2 + \tau^2} \sin |x - p| r \, dr = \pi b_m(\tilde{\tau}) \sinh \tilde{\tau}|x - p|. \]
Now from (4.4) we obtain (ii).
This completes the proof of Proposition 3.1.

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