Direct Solution of Second Order Ordinary Differential Equations Using a Class of Hybrid Block Methods

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Abstract: This research proposes the derivation of a class of hybrid methods for solution of second order initial value problems (IVPs) in block mode. Continuous linear multistep method of two cases with step number \( k = 4 \) is developed by interpolating the basis function at certain grid points and collocating the differential system at both grid and off-grid points. The basic properties of the method including order, error constant, zero stability, consistency and convergence were investigated. In order to examine accuracy of the methods, some differential problems of order two were solved and results generated show a better performance when comparison is made with some current methods.

Keywords: Block Method, Hybrid Points, Initial Value Problems, Power Series, Second Order

1 INTRODUCTION

Countless real-life problems in science and engineering are represented in differential equations which sometimes can be of the form:

\[
y''(x) = f(x, y, y'), y(x_0) = y_0, y'(x_0) = y'_0 \tag{1}
\]

Analytical solutions to the real-life problem in (1) do not exist in most cases, hence, the need for numerical method in providing approximate solution becomes essential. In the past, equation (1) is usually reduced to a system of first order ordinary differential equations (ODEs) and a suitable numerical algorithm for first order would be applied to solve the resulting equations. This approach has a lot of setbacks which include complexity in implementation, computational burden, and time wastage (Aree & Adeniyi, 2013; Kayode & Adeyeye, 2013; Awoyemi, 2001; Kayode & Adegboro, 2018). In overcoming these disadvantages, researchers like Ukpebor (2019), Omar & Kuboye (2018) and Kayode & Adeyeye (2013) amongst others developed numerical methods for solution of (1) without reducing it to a system of first order initial value problem (IVP). The accuracy of these methods in terms of error is not encouraging as thus can be improved by introducing hybrid points within the selected interval of integration.

Hence, this paper focuses at improving the solution of previous developed numerical methods with the use of hybrid points. The article is divided into six sections; Introduction of the work is presented in section 1, section 2 deals with the development of the schemes, section 3 presents the properties of the methods and section 4 discusses the implementation of the algorithms. Finally, section 5 and 6 consider discussion of results and conclusion.

2 METHODOLOGY

In this section, the derivations of the schemes are examined.

2.1 DERIVATION OF FIRST NUMERICAL SCHEME (FNS)

Powers series of the form:

\[
y(x) = \sum_{j=0}^{k+1} a_j x^j \tag{2}
\]

is considered as approximate solution to (1) where \( k = 4 \). Its second derivative is given as

\[
y''(x) = \sum_{j=2}^{k+1} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{3}
\]

Interpolating (2) at \( x_{n+i}, i = 1, 2 \) and collocating (3) at \( x_{n+i}, i = 1, 2, 3, 4 \) give a non-linear system of equations of the form

\[
CX = D \tag{4}
\]

where:

\[
C = \begin{bmatrix}
1 & x_{n+1} & x_{n+2}^2 & x_{n+3}^3 & x_{n+4}^4 & x_{n+5}^5 & x_{n+6}^6 \\
1 & x_{n+2} & x_{n+3}^2 & x_{n+4}^3 & x_{n+5}^4 & x_{n+6}^5 & x_{n+7}^6 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5
\end{bmatrix}
\]

\[
D = \left[ \begin{array}{c}
y_{n+1} \\
y_{n+2} \\
f_n \\
f_{n+1} \\
f_{n+1/2} \\
f_{n+1/2} \\
f_{n+3} \\
f_{n+4}
\end{array} \right]^T
\]

Gaussian elimination technique is used to obtain the values of parameters \( a_j \) in (4) which are then substituted in (2) to produce a continuous scheme of the form:

\[
y(t) = \sum_{j=1}^{k} a_j y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 y_{n+1/2} f_{n+1/2} \tag{5}
\]

where:

\[
t = \left( \frac{x-x_{n+1}}{h} \right) \tag{6}
\]

\[
a_i(t) = (-t - 1) \tag{7}
\]

\[
a_i(t) = (t + 2) \tag{8}
\]
\[
\beta_0(t) = \left(\frac{(t^7)}{1512} - \frac{(t^6)}{2160} + \frac{(t^5)}{360} + \frac{(t^4)}{864} + \frac{(t^3)}{72} - \frac{(t^2)}{6048} + \frac{1}{240}\right)
\]
\[
\beta_1(t) = \left(\frac{(t^7)}{126} + \frac{(t^6)}{20} + \frac{(t^5)}{120} - \frac{(t^4)}{8} - \frac{(t^3)}{4} + \frac{(t^2)}{1260} + \frac{1}{10}\right)
\]
\[
\gamma_2(t) = \left(-\frac{(16t^7)}{945} - \frac{(16t^6)}{135} - \frac{(8t^5)}{45} + \frac{(8t^4)}{27} + \frac{(32t^3)}{45} - \frac{(64t^2)}{189} + \frac{97}{1}ight)
\]

In deriving the discrete schemes, Equation (5) is evaluated at the non-interpolating points, that is, \(x = x_{n+i}, \ i = 0, \frac{3}{2}, 3, 4\) while its derivative is computed by evaluating the derivatives of Equation (5) at all grid points, that is, \(x = x_{n+i}, \ i = 0, 1, 2, 3, 4\). These schemes are used to form the block method (8). The derivative of block method (8) is computed as shown in (9).

**2.2 DERIVATION OF SECOND NUMERICAL SCHEME (SNS)**

Equation (2) is interpolated at \(x_{n+i}, \ i = 2, 3\) and second derivative of (2) is collocated at \(x_{n+i}, \ i = 0, 1, \frac{5}{2}, 2, 3, 4\). The same step applied in developing FNS is also used in deriving SNS, hence this yields the block method (10). The derivative of block method (10) is computed as shown in (11).

\[
y_{n+1} = y_n + hy'_n + \frac{h^2}{30240} \left(1637f_{n+1} + 24180f_{n+2} - 27392f_{n+3} + 13338f_{n+4} - 1844f_{n+5} + 201f_{n+6}\right)
\]
\[
y_{n+2} = y_n + \frac{3}{2}hy'_n + \frac{h^2}{8960} \left(3243f_{n+1} + 15201f_{n+2} - 14688f_{n+3} + 7209f_{n+4} - 993f_{n+5} + 108f_{n+6}\right)
\]
\[
y_{n+2} = y_n + 2hy'_n + \frac{h^2}{1890} \left(953f_{n+1} + 4944f_{n+2} - 4096f_{n+3} + 2250f_{n+4} - 304f_{n+5} + 33f_{n+6}\right)
\]
\[
y_{n+3} = y_n + 3hy'_n + \frac{h^2}{1120} \left(879f_{n+1} + 5076f_{n+2} - 3840f_{n+3} + 3078f_{n+4} - 180f_{n+5} + 27f_{n+6}\right)
\]
\[
y_{n+4} = y_n + 4hy'_n + \frac{h^2}{945} \left(1016f_{n+1} + 5952f_{n+2} - 4096f_{n+3} + 3888f_{n+4} + 704f_{n+5} + 96f_{n+6}\right)
\]
\[
y'_{n+1} = y'_n + \frac{h}{360} \left(103f_{n+1} + 593f_{n+2} + 576f_{n+3} + 273f_{n+4} - 37f_{n+5} + 3f_{n+6}\right)
\]
\[
y'_{n+2} = y'_n + \frac{h}{2560} \left(727f_{n+1} + 4752f_{n+2} - 3200f_{n+3} + 1782f_{n+4} - 248f_{n+5} + 27f_{n+6}\right)
\]
\[
y'_{n+2} = y'_n + \frac{h}{270} \left(77f_{n+1} + 492f_{n+2} - 256f_{n+3} + 252f_{n+4} - 28f_{n+5} + 3f_{n+6}\right)
\]
\[
y'_{n+3} = y'_n + \frac{h}{40} \left(11f_{n+1} + 81f_{n+2} - 64f_{n+3} + 81f_{n+4} + 11f_{n+5}\right)
\]
\[
y'_{n+4} = y'_n + \frac{h}{45} \left(14f_{n+1} + 64f_{n+2} + 24f_{n+3} + 64f_{n+4} + 14f_{n+5}\right)
\]
3.1 ORDER OF THE BLOCK

Let the linear difference operator, \( L \), associated with the continuous multi-step method (5) be defined as

\[
L[y(x); h] = \sum_{j=0}^{\infty} a_j y(x_n + jh) - h^p y^{(p)}(x_n) \quad (12)
\]

\( y(x) \) represents an arbitrary test function that is continuously differentiable in the interval \([a, b]\). Expanding \( y(x_n + jh) \) and \( y'(x_n + jh) \) \( j = 0, 1, \ldots \) in taylor series about \( x_n \) and collecting like terms in \( h \) and \( y \) gives (13).

\[
\begin{align*}
\mathbf{y}'_{n+1} &= y_n + \frac{h}{360} \left( 112f_n + 413f_{n+1} - 537f_{n+2} + 576f_{n+3} - 217f_{n+4} \right) \\
\mathbf{y}'_{n+2} &= y_n + \frac{h}{270} \left( 81f_n + 412f_{n+1} - 108f_{n+2} + 256f_{n+3} - 108f_{n+4} \right) \\
\mathbf{y}'_{n+3} &= y_n + \frac{h}{40} \left( 12f_n + 61f_{n+1} - 9f_{n+2} + 64f_{n+3} - 9f_{n+4} \right) \\
\mathbf{y}'_{n+4} &= y_n + \frac{h}{45} \left( 14f_n + 64f_{n+1} + 24f_{n+2} + 64f_{n+3} + 14f_{n+4} \right)
\end{align*}
\]

3.2 ANALYSIS OF BASIC PROPERTIES OF THE BLOCK

Basic properties of the block methods are analysed to establish their validity in this section.

Definition 1: The difference operator \( L \) and the continuous multistep method (5) are said to be of order \( p \) if in (13), \( c_0 = c_1 = c_2 = \cdots = c_{p+1} = 0, c_{p+2} \neq 0 \) (Lambert, 1973).

Definition 2: The term \( c_{p+2} \) is called the error constant and it implies that the local truncation error (LTE) is given by

\[
t_{n+k} = c_{p+2} h^{p+2} y^{(p+2)}(x_n) = 0(h^{p+2})
\]

Rewriting the block form of Equation (8), in the form of (14), and writing (14) in explicit form gives (15). Using Taylor’s series expansion on (15), it yields (16).

\[
\begin{align*}
\mathbf{y}_{n+1} - y_n - hy'_{n} - h^2 y''_{n} &= \frac{1}{360} \left( 6637f_n + 3024f_{n+1} + 15201f_{n+2} - 537f_{n+3} - 461f_{n+4} \right) \\
\mathbf{y}_{n+3/2} - y_n - \frac{3}{2} hy'_{n} - h^2 y''_{n} &= \frac{1}{360} \left( 3243f_n + 15201f_{n+1} + 7209f_{n+2} + 993f_{n+3} + 224f_{n+4} \right) \\
\mathbf{y}_{n+2} - 2hy'_{n} - h^2 y''_{n} &= \frac{1}{360} \left( 953f_n + 1890f_{n+1} - 2048f_{n+2} + 152f_{n+3} + 11f_{n+4} \right) \\
\mathbf{y}_{n+3} - 3hy'_{n} - h^2 y''_{n} &= \frac{1}{360} \left( 879f_n + 1260f_{n+1} - 1539f_{n+2} + 11f_{n+3} - 32f_{n+4} \right) \\
\mathbf{y}_{n+4} - 4hy'_{n} - h^2 y''_{n} &= \frac{1}{360} \left( 1016f_n + 1984f_{n+1} - 144f_{n+2} - 32f_{n+3} - 32f_{n+4} \right)
\end{align*}
\]

\[
\begin{align*}
\mathbf{y}_{n+1} &= y_n + \frac{h}{360} \left[ 1269f_n + 700f_{n+1} - 750f_{n+2} + 5760f_{n+3} - 2000f_{n+4} \right] \\
\mathbf{y}_{n+2} &= y_n + \frac{h}{270} \left[ 81f_n + 412f_{n+1} - 108f_{n+2} + 256f_{n+3} - 108f_{n+4} \right] \\
\mathbf{y}_{n+3} &= y_n + \frac{h}{40} \left[ 12f_n + 61f_{n+1} - 9f_{n+2} + 64f_{n+3} - 9f_{n+4} \right] \\
\mathbf{y}_{n+4} &= y_n + \frac{h}{45} \left[ 14f_n + 64f_{n+1} + 24f_{n+2} + 64f_{n+3} + 14f_{n+4} \right]
\end{align*}
\]
And collecting terms in \( h \) and \( y \) leads to the following:

\[
\begin{align*}
\mathcal{C}_B & = \begin{bmatrix}
\frac{1}{8!} (1)\lambda_1^8 & - \frac{1}{6!} \frac{403}{504} \lambda_1^6 & - \frac{856}{945} \lambda_1^6 & 3/2 & + \frac{247}{560} \lambda_2^6 & - \frac{461}{7560} (3)^6 & + \frac{67}{10080} (4)^6 \\
\frac{1}{6!} \frac{15201}{8960} (1)^6 & - \frac{1}{6!} \frac{280}{18960} (3)^6 & + \frac{7209}{9609} (2)^6 & - \frac{993}{8960} \lambda_1^6 & + \frac{27}{2240} (4)^6 \\
\frac{1}{6!} \frac{1}{315} (2)^6 & - \frac{1}{6!} \frac{824}{315} (1)^6 & - \frac{2048}{945} (3)^6 & + \frac{25}{21} (2)^6 & - \frac{152}{945} (3)^6 & + \frac{11}{630} (4)^6 \\
\frac{1}{6!} \frac{1}{315} (2)^6 & - \frac{1}{6!} \frac{1269}{315} (1)^6 & - \frac{24}{7} (3)^6 & + \frac{1539}{560} (2)^6 & - \frac{9}{56} (3)^6 & + \frac{27}{1120} (4)^6 \\
\frac{1}{6!} \frac{1}{315} (2)^6 & - \frac{1}{6!} \frac{1984}{315} (1)^6 & - \frac{4096}{945} (3)^6 & + \frac{144}{35} (2)^6 & + \frac{704}{945} (3)^6 & + \frac{32}{315} (4)^6
\end{bmatrix}
\end{align*}
\]

Hence, the block method (8) and (10) are both of order \( p = (6,6,6,6)^T \). Error constants of FNS and the corresponding error constants for SNS, after adopting same procedures used in FNS, are as follows;

\[
\begin{align*}
C_{F+2} & = [-0.0020337, -0.0036647, -0.0052910, -0.0080357, -0.0126984]^T \text{ for FNS} \\
C_{F+2} & = [-0.0038029, -0.0095238, -0.0121303, -0.0147321, -0.0211640]^T \text{ for SNS}
\end{align*}
\]

### 3.2 Zero Stability of the Block Method

**Definition 3:** The implicit block method (8) is said to be zero stable if the roots \( \lambda_s, s = 1, ..., n \) of the first characteristics polynomial \( \rho(\lambda) \), defined by

\[
\rho(\lambda) = \det(\lambda - A)
\]

satisfies \( |\lambda_s| \leq 1 \) and every root with \( |\lambda_s| = 1 \) has multiplicity not exceeding two in the limit as \( h \to 0 \).

From (8) using Definition 3 above as \( h \to 0 \)

\[
\rho(\lambda) = \det \lambda = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{vmatrix}
\]

\[
= \det \begin{vmatrix}
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 1 \\
\end{vmatrix}
\]

\[
\rho(\lambda) = \lambda^5 - \lambda^4
\]

\[
\rho(\lambda) = \lambda^4(\lambda - 1) = 0
\]

\[
\lambda^4 = 0, \lambda - 1 = 0
\]

\[
\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0, \quad \lambda_1 = 1
\]

Hence, the block method FNS is stable. Adopting the same procedures in finding zero stability of SNS, this was also found to be zero stable.

### 3.3 Consistency and Convergence

In establishing the consistency and convergence of the methods, the approach proposed by Lambert (1973) is adopted. This confirms the consistency of the methods because its order is greater than one; that is, \( p = 6 > 1 \). Hence, it also validates the convergence of the scheme since it is zero stable and consistent.

### 4 Implementation of the Methods

Three second order initial value problems of ODEs are considered to examine the accuracy of the newly developed methods.

**4.1 Test Problem 1:** Source (Awari, 2013)

\[
y'' = 100y, y(0) = 1, y'(0) = -10; \quad h = 0.01;
\]

Exact solution \( y(x) = e^{-10x} \)

**4.2 Test Problem 2:** Source (Kuboye, et al., 2018)

\[
y'' = x(y')^2, y(0) = 1, \quad y'(0) = \frac{1}{2}, h = \frac{1}{32}
\]

Exact solution \( y''(x) = 1 + \frac{1}{2} \left( \frac{2x^2}{x^2} \right) \)

**4.3 Test Problem 3:** Source (Kayode & Adegboro, 2018)

\[
y'' = y', y(0) = 0, \quad y'(0) = -1, h = 0.1
\]

Exact solution \( y''(x) = 1 - \exp(x) \)

### 5 Results

Tables 1-3 show the error values obtained from our implementation.

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engineering.fuoye.edu.ng/journal
Table 1. Comparison of the developed Block Methods with Omar & Kuboye (2018) and Kayode & Adegboro (2018) for solving Test Problem 1

| h     | Error in new method \( k = 4, h = 0.01, s = \frac{3}{2} \) (FNS) | Error in new method \( k = 4, h = 0.01, s = \frac{5}{2} \) (SNS) | Error in previous method (Omar & Kuboye, 2018), \( k = 5, h = 0.01 \) | Error in previous method (Kayode & Adegboro, 2018), \( k = 4, h = 0.01 \) |
|-------|---------------------------------------------------------------|---------------------------------------------------------------|---------------------------------------------------------------|---------------------------------------------------------------|
| 0.01  | 1.7239432e-11                                               | 3.1779135e-11                                               | 2.452483e-13                                                  | 8.1951512e-09                                                 |
| 0.02  | 4.5028758e-11                                               | 7.9988238e-11                                               | 6.21087e-11                                                  | 9.3132251e-09                                                 |
| 0.03  | 6.9175443e-11                                               | 1.2497792e-10                                               | 1.574575e-09                                                  | 1.02185382e-08                                                |
| 0.04  | 1.1053702e-10                                               | 1.8214896e-10                                               | 1.55945e-08                                                  | 1.20304247e-08                                                |
| 0.05  | 1.7659828e-09                                               | 1.8476993e-09                                               | 9.175628e-08                                                  | 1.32911134e-08                                                |
| 0.06  | 3.4460591e-09                                               | 3.5425421e-09                                               | 2.382931e-07                                                  | 1.44797260e-08                                                |
| 0.07  | 5.1578813e-09                                               | 5.2701455e-09                                               | 3.872521e-07                                                  | 1.59095202e-08                                                |
| 0.08  | 6.9324004e-09                                               | 7.0578210e-09                                               | 5.40966e-07                                                  | 1.70934742e-08                                                |
| 0.09  | 9.8575126e-09                                               | 9.9930404e-09                                               | 7.076502e-07                                                  | 1.87699845e-08                                                |
| 0.10  | 1.2885945e-08                                               | 1.3035513e-08                                               | 9.190554e-07                                                  | 2.04968695e-08                                                |

Table 2. Comparison of the developed Block Methods (FNS and SNS) with Ukpebor (2019) for solving Test Problem 2

| h     | Error in new method \( k = 4, h = 0.1, s = \frac{3}{2} \) (FNS) | Error in new method \( k = 4, h = 0.1, s = \frac{5}{2} \) (SNS) | Error in (Ukpebor, 2019) \( k = 4, h = 0.1 \) |
|-------|---------------------------------------------------------------|---------------------------------------------------------------|---------------------------------------------------------------|
| 0.1   | 2.2204460e-16                                               | 8.6751961e-10                                               | 1.689804e-11                                                  |
| 0.2   | 2.2204460e-16                                               | 7.4824766e-09                                               | 8.055976e-10                                                  |
| 0.3   | 4.4408921e-16                                               | 2.6335182e-08                                               | 8.615662e-09                                                  |
| 0.4   | 1.9984014e-15                                               | 6.5082084e-08                                               | 4.024075e-08                                                  |
| 0.5   | 5.9952043e-15                                               | 1.3337141e-07                                               | 1.406582e-07                                                  |
| 0.6   | 1.376666e-14                                                | 2.4401810e-07                                               | 3.816098e-07                                                  |
| 0.7   | 3.1308289e-14                                               | 4.1478648e-07                                               | 9.264709e-07                                                  |
| 0.8   | 6.7945649e-14                                               | 6.7123324e-07                                               | 2.002368e-06                                                  |
| 0.9   | 1.4455104e-13                                               | 1.0514474e-06                                               | 4.118350e-06                                                  |
| 1.0   | 3.0708769e-13                                               | 1.6143170e-06                                               | 7.978886e-06                                                  |

Table 3. Comparison of the developed Block Methods (FNS and SNS) with Kayode & Adeyeye (2013) and Ukpebor (2019) for solving Test Problem 3

| h     | Error in new method \( k = 4, h = 0.1, s = \frac{3}{2} \) (FNS) | Error in new method \( k = 4, h = 0.1, s = \frac{5}{2} \) (SNS) | Error in (Ukpebor, 2019) |
|-------|---------------------------------------------------------------|---------------------------------------------------------------|---------------------------------------------------------------|
| 0.1   | 2.8033131e-15                                               | 7.6094235e-08                                               | 3.35488e-09                                                  |
| 0.2   | 1.4460655e-14                                               | 1.6743205e-07                                               | 3.304564e-08                                                  |
| 0.3   | 3.8025139e-14                                               | 2.6037378e-07                                               | 1.233789e-07                                                  |
| 0.4   | 7.4662948e-14                                               | 3.7193407e-07                                               | 3.155858e-07                                                  |
| 0.5   | 1.2800871e-13                                               | 4.8543388e-07                                               | 6.588997e-07                                                  |
| 0.6   | 1.9995177e-13                                               | 6.2171345e-07                                               | 1.14380e-05                                                  |
| 0.7   | 2.9531952e-13                                               | 7.6036627e-07                                               | 1.211782e-06                                                  |
| 0.8   | 4.1633363e-13                                               | 9.2679437e-07                                               | 2.043321e-06                                                  |
| 0.9   | 5.6954441e-13                                               | 1.0961457e-06                                               | 3.234818e-06                                                  |
| 1.0   | 7.5828233e-13                                               | 1.2944219e-06                                               | 7.095032e-06                                                  |
|       |                                                               |                                                               | 6.72615e-05                                                  |
It is apparent from Tables 1, 2 and 3 that results of new hybrid block methods outperform the current methods (Kayode & Adeyeye, 2013; Kayode & Adegboro, 2018; Omar & Kuboye, 2018; Ukpebor, 2019) for solving problems 1 and 3 despite the high step length k considered. Furthermore, in Table 2 results of the new block methods are also better when compared with existing methods for solving problem 2. The performance of the methods FNS and SNS using three test problems in comparison with existing methods are shown in Tables 1-3.

6 CONCLUSION

The derivation of block methods with hybrid points using interpolation and collocation approach for solving second order initial value problems of ODEs has been examined in this paper. The derived schemes proved to be efficient in solving second order IVPs as it compared favourably with existing methods in terms of accuracy. Furthermore, it is obvious in Tables 1-3 that FNS is more effective in solving second order ODEs than SNS. Hence, the two developed numerical schemes are resourceful for solving second order ODEs directly.

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