Equigenerated Gorenstein ideals of codimension three

Dayane Lira1 · Zaqueu Ramos2 · Aron Simis3

Received: 11 June 2021 / Accepted: 5 May 2022 / Published online: 3 June 2022
© The Author(s), under exclusive licence to Universitat de Barcelona 2022

Abstract
We focus on the structure of a homogeneous Gorenstein ideal $I$ of codimension three in a standard polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n]$ over an infinite field $\mathbb{k}$, assuming that $I$ is generated in a fixed degree $d$. For such an ideal $I$, there is a simple formula relating this degree, the minimal number of generators of $I$, and the degree of the entries of the associated skew-symmetric matrix. We give an elementary characteristic-free argument to the effect that, for any such data linked by this formula, there exists a Gorenstein ideal $I$ of codimension three satisfying them. We conjecture that, for arbitrary $n \geq 2$, an ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$ generated by a general set of $r \geq n + 2$ forms of degree $d \geq 2$ is Gorenstein if and only if $d = 2$ and $r = \binom{n+1}{2} - 1$. We prove the ‘only if’ implication of this conjecture when $n = 3$. For arbitrary $n \geq 2$, we prove that if $d = 2$ and $r \geq (n+2)(n+1)/6$ then the ideal is Gorenstein if and only if $r = \binom{n+1}{2} - 1$, which settles the ‘if’ assertion of the conjecture for $n \leq 5$. We also elaborate around one of the questions of Fröberg–Lundqvist. In a different direction, we show a connection between the Macaulay inverse and the so-called Newton dual, a matter so far not brought out to our knowledge. Finally, we consider the question as to when the link $(\ell^m_1, \ldots, \ell^m_n) : \hat{f}$ is equigenerated, where $\ell_1, \ldots, \ell_n$ are independent linear forms and $\hat{f}$ is a form. We give a solution in some special cases.

Keywords Gorenstein ideal · Socle degree · Macaulay inverse · Newton dual · Space of parameters · General forms

Mathematics Subject Classification Primary 13H10 · 13D02 · Secondary 13A30 · 13C13 · 13C15 · 13C40

To the memory of Wolmer Vasconcelos.

D. Lira: Under a PhD fellowship from CAPES, Brazil (88882.440720/2019-01). Z. Ramos: Under a post-doc fellowship from INCTMAT/Brazil (88887.373066/2019-00). A. Simis: Partially supported by a CNPq Grant (302298/2014-2)

Dayane Lira
dayannematematica@gmail.com

Extended author information available on the last page of the article
1 Introduction

The literature on Gorenstein ideals is vast (for a tiny list, see [3, 7, 11, 15, 21–29, 36, 38, 42]), but perhaps not so much with an emphasis on the equigenerated case, even when the codimension is three. Largely, one expects this case to be easier to handle, by doing away with some of the technical difficulties of the arbitrary homogeneous case, where the numerical data are more involved. The question, of course, is as to whether there is a net gain in this restriction. We hope that the overall simplicity of the arguments in this work will justify doing it.

By and large we have been initially inspired by the classification ideas of [19, 26]. A fairly understood case is that of an ideal \( I \subset R = \mathbb{k}[x, y, z] \) of finite colength generated by quadrics. For example, one has:

**Theorem** ([19, Theorem 2.1 (i) and Theorem 3.6]) Let \( k \) be a field of characteristic \( \neq 2 \) and let \( I \subset R = \mathbb{k}[x, y, z] \) be an ideal of finite colength minimally generated by five quadrics. Then the following conditions are equivalent:

(i) \( I \) is a Gorenstein ideal.

(ii) \( I \) is syzygetic.

(iii) There exist 3 independent linear forms \( \{l_1, l_2, l_3\} \subset \mathbb{k}[x, y, z] \) such that

\[
I = (xl_2, xl_3, yl_3, xl_1 - yl_2, xl_1 - zl_3).
\]

Thus, in the case of quadric generators in dimension three, it gives characterizations other than the Pfaffian recipe of Buchsbaum–Eisenbud. Condition (ii) and (iii) may suggest that such Gorenstein ideals live on a dense Zariski open set in the parameter space of the coefficients of a set of generators. A set of forms in \( \mathbb{k}[x_1, \ldots, x_n] \) is said to be general if the corresponding parameter point in the parameter space belongs to a suitable dense Zariski open set. It turns out as we prove more generally (Theorem 2.14), that for arbitrary \( n \geq 2 \), an ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) generated by a general set of \( r \geq (n + 2)(n + 1)/6 \) quadrics is Gorenstein if and only if \( r = \binom{n+1}{2} - 1 \).

As a matter of fact, we conjecture an encompassing result to the effect that, for arbitrary \( n \geq 2 \), an ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) generated by a set of \( r \geq n + 2 \) general forms of degree \( d \geq 2 \) is Gorenstein if and only if \( d = 2 \) and \( r = \binom{n+1}{2} - 1 \). The previously mentioned theorem proves one implication of this conjecture. We are able to settle the reverse implication in dimension three, for which we give two different proofs. The first is partially based on previously existing material, but we state all facts ab initio in order to have a coherent exposé. Thus, one will recognize in the background some of the results by Hochster–Laksov [17], Diesel [7], Anick [1] and Migliore–Miró-Roig [32]. The second proof is achieved by making explicit that all equigenerated Gorenstein ideals of codimension 3, of given degree \( d \geq 3 \) and \( r \geq 5 \) number of generators, are parameterized by a proper closed subset of the space of parameters. For this, we estimate the socle degree, for which we needed to specify the graded free resolution. Due to the length and technicality of the involved arguments, we refer to [30, Appendix A] for the details.

We add a discussion around Fröberg’s Conjecture 2.7 (see [12]), more exactly, around [13, Question 2.5] that, in the presence of \( d \)-equigeneration, translates into the question of determining the coefficients of the power series \((1 - t^d)^r/(1 - t)^n \) for \( r \geq n \).
In this relation, we introduce the following notion: with \( R = \mathbb{k}[x_1, \ldots, x_n] \) as before, given nonnegative integers \( d, r, e \), a set (tuple) of forms \( \mathbf{f} \in (R_d)^r \) is \( (d, r, e) \)-extremal if \( \dim_k(R/I)_{d+e} = \max\{\dim_k R_{d+r} - r \dim_k R_e, 0\} \), where \( I = (\mathbf{f}) \subset R \). Proposition 2.8 translates [13, Question 2.5] in the \( d \)-equigenerated case into the quest for an explicit expression of a single integer \( j_0 \) in terms of \( n, d, r \). Moreover, then an affirmative answer to Conjecture 2.7 implies that a set of \( r \geq n \) general forms \( \mathbf{f} = \{f_1, \ldots, f_r\} \) of degree \( d \) is \( (d, r, e) \)-extremal for every \( 0 \leq e \leq \min\{d-1, j_0 - d\} \). A previous discussion in the equigenerated case, regarding the values of the Hilbert series in some intervals, has been given in [34].

In another direction, the minimal number \( \mu(I) \) of generators of a codimension 3 Gorenstein ideal \( I \subset \mathbb{k}[x, y, z] \) generated in degree \( d \) obeys a certain additional constraint, besides being an odd integer. A pair \( (d, r) \) of integers such that \( d \geq 2 \) and \( r \geq 3 \) will be called an equigenerated Gorenstein virtual datum in dimension \( n \geq 3 \) if \( (r - 1)/2 \) is a factor of \( d \). Such a datum will be said to be proper (in dimension \( n \)) if there exists a codimension 3 Gorenstein ideal in \( \mathbb{k}[x_1, \ldots, x_n] \) generated by \( r \) forms of degree \( d \), and moreover, the entries of a corresponding \( r \times r \) skew-symmetric matrix \( \Phi \) generate an ideal of codimension \( n \). Here, \( 2d/(r - 1) \) will be the degree of any entry of \( \Phi \). We show that any virtual datum in dimension \( n \geq 3 \) actually occurs (Theorem 1.4). Our proof stands over an infinite field of arbitrary characteristic. This question has been approached before in dimension 3 in [7] and by Conca–Valla ([5]) via an argument based upon a particular skew-symmetric matrix. In dimension 3 a related result, drawing upon the earlier example of Buchsbaum and Eisenbud [3, Proposition 6.2], is given in [30, Proposition 2.1.6].

Another subject dealt with in this paper is what we call the \((x_1^m, \ldots, x_n^m)\)-colon problem. It has long been known (see [2, Proposition 1.3]) that, in arbitrary characteristic, any homogeneous Gorenstein ideal of codimension \( n \) in \( \mathbb{k}[x_1, \ldots, x_n] \) can be obtained as a colon ideal \((x_1^m, \ldots, x_n^m) : \mathbf{f}) \) for some integer \( m \geq 1 \) and a form \( \mathbf{f} \). In this regard, two questions naturally come up: first, is there a more definite relation between \( \mathbf{f} \) and the ideal \( I \)? Second, is there a characterization as to when the resulting Gorenstein ideal is equigenerated in terms of the exponent \( m \) and the form \( \mathbf{f} \)?

We give an answer to the first question, at least in characteristic zero, in terms of the Macaulay inverse setup and the notion of Newton duality as introduced in [6, 8]. This is the content of Proposition 3.2 which says that Macaulay inverse to \( I \) is the (socle-like) Newton dual of the form \( \mathbf{f} \). The proposition also gives a complete characterization of \( \mathbf{f} \), and its degree, under the condition that none of its nonzero terms belongs to the ideal \((x_1^m, \ldots, x_n^m)\). The use of the Newton duality concept in these matters seem to be new.

The second problem above is more delicate. One usefullness of solving it is yet another simple way of producing equigenerated Gorenstein ideals. Obviously, merely controlling degrees in both terms of the colon operation does not lead too far—e. g., in characteristic zero the ideal \((x^d, y^d, z^d) : (x + y + z)^{d-1} \), for \( d \geq 2 \), is \( d \)-equigenerated, while \((x^d, y^3, z^3) : x^3 + y^3 + z^3 \) is not equigenerated. Yet, we solve this question in the case where \( I \) has linear resolution (Theorem 3.5), in terms of a degree constraint and show that, when the directrix \( \mathbf{f} \) is a power of \( x_1 + \cdots + x_n \), the solutions lie on a dense open set of the space of parameters (Proposition 3.7). The main part of the latter result requires characteristic zero because it invokes a result of Stanley, with algebraic proofs by Reid–Roberts–Roitman ([37] and Oesterlé in [4, Appendix A]).

In the way of considering the last problem, we introduce the notion of pure power gap, an integer measuring how far off from the socle degree is an exponent of the powers of independent linear forms lying in the ideal. We give the basic role of this invariant, hoping it will be useful in other contexts.
Finally, we look at some behavior of the usual algebras attached to an ideal, such as the Rees algebra and the fiber cone algebra. The relation between the depth of these algebras and the nature of the rational map defined by the linear space $I_d$ is discussed in Sect. 4 by completely elementary ways.

A list of the main results: Theorems 1.4, 2.10, 2.14, 3.5 and 4.2. Other significant results: Propositions 2.13, 3.2, 3.7 and 3.10.

2 Numerical data

2.1 The socle

Let $R = \mathbb{k}[x_1, \ldots, x_n]$ be a polynomial ring over an infinite field $\mathbb{k}$. Denote by $\mathfrak{m}$ the maximal homogeneous ideal of $R$. Given a homogeneous $\mathfrak{m}$-primary ideal $I \subset R$, consider the least integer $s$ such that $\mathfrak{m}^{s+1} \subset I$. Then, the graded $\mathbb{k}$-algebra $R/I$ can be written as

$$R/I = \mathbb{k} \oplus (R/I)_1 \oplus (R/I)_2 \oplus \cdots \oplus (R/I)_s$$

with $(R/I)_s \neq 0$. The socle of $R/I$, denoted $\text{Soc}(R/I)$, is the ideal $I : \mathfrak{m}/I \subset R/I$. Since $I$ and $\mathfrak{m}$ are homogeneous ideals, then $\text{Soc}(R/I)$ is a homogeneous ideal of $R/I$. In particular,

$$\text{Soc}(R/I) = \text{Soc}(R/I)_1 \oplus \cdots \oplus \text{Soc}(R/I)_s.$$

The integer $s$ is the socle degree of $R/I$. It is well known that $R/I$ is a Gorenstein ring if and only if $\text{Soc}(R/I) = \text{Soc}(R/I)_1$ and $\dim_\mathbb{k} \text{Soc}(R/I)_s = 1$.

Recall from the Buchsbaum-Eisenbud structure theorem [3, Theorem 2.1], that a homogeneous codimension 3 Gorenstein ideal $I \subset R = \mathbb{k}[x_1, \ldots, x_n]$ is generated by the $(r-1)$-Pfaffians of an $r \times r$ skew-symmetric matrix $\Phi$, with $r = \mu(I)$ odd. When $I$ is equigenerated, say, in degree $d \geq 1$, then the columns of $\Phi$ must be homogeneous of some standard degree $d_i$, $1 \leq i \leq r$. By the nature of each generator of $I$ as an $(r-1)$-Pfaffian of $\Phi$, it immediately follows that

$$2d = d_1 + d_2 + \cdots + d_{r-2} + d_{r-1} = d_1 + d_3 + \cdots + d_{r-1} + d_r = \cdots = d_2 + d_3 + \cdots + d_{r-1} + d_r.$$

By an elementary argument, $d_1 = d_2 = \cdots = d_r$.

It follows that $2d = (r-1)d'$, where $d'$ is the common values of the $d_i$’s, i.e.,

$$d = \frac{r-1}{2} d'.$$

Thus, an equigenerated Gorenstein ideal $I$ of codimension 3 complies with this relation, where $r$ is the number of generators, $d$ their common degree and $d'$ is the degree of any entry of the skew–symmetric matrix defining $I$.

We introduce the following terminology:

**Definition 1.1** A codimension 3 equigenerated Gorenstein virtual datum in dimension $n \geq 3$ is a pair $(d, r)$ of integers such that:

(i) $d \geq 2$ and $r \geq 3$.

(ii) $(r-1)/2$ is a factor of $d$. 

Springer
If no misunderstanding arises, we will mostly omit ‘codimension 3 equigenerated Gorenstein’ in the above terminology. Given such a datum, the integer $d' := 2d/(r - 1)$ will be called the skew-degree of the datum.

The datum is said to be proper (in dimension $n$) if there exists a codimension 3 Gorenstein ideal $I$ in $k[x_1, \ldots, x_p]$ with this datum, satisfying $\text{ht}_I(\Phi) = n$, where $\Phi$ is the skew-symmetric matrix whose $(r - 1)$-Pfaffians generate $I$. In this case the skew-degree is in fact the degree of any entry in the skew-symmetric matrix $\Phi$, perhaps justifying its designation.

Note that, since $n \geq 3$ then, rightfully, $r \leq 2d + 1$. An important case is when $d' = 1$, i.e., $r = 2d + 1$. We refer to it as the linear case. This happens, e.g., when $d$ is a prime number and $r \geq 4$.

**Lemma 1.2** Let $I \subset R = k[x, y, z]$ denote an equigenerated codimension 3 Gorenstein ideal with virtual datum $(d, (2d + d')/d')$, where $d' \geq 1$ is the skew degree of $I$ as introduced earlier. Then the socle degree of $RI$ is $2d + d' - 3$.

**Proof** It is well-known that, pretty generally, the socle degree of a graded Artinian Gorenstein quotient $RI$ of a standard graded polynomial ring $R = k[x_1, \ldots, x_p]$ is given by $D - n$, where $D$ is the last shift in the minimal graded $R$-resolution of $RI$ (see, e.g., [29, Lemma 1.3]). In the present case, the resolution has length $3$ and the first syzygies have shifted degree $d + d'$. Since the graded free resolution of a graded Gorenstein algebra is self-dual we have that $D = d + d' = 2d + d'$, hence the socle degree is as stated. □

### 2.2 Retrieval from numerical data

In this part we prove that, over an infinite field of arbitrary characteristic, any virtual datum in dimension $n \geq 3$ actually occurs, a question that has been solved before for $n = 3$ and homogeneous Gorenstein ideals which are not necessarily equigenerated [5, 7]. Our approach for the equigenerated case and arbitrary $n \geq 3$ is based on a simple reparametrization as follows.

Quite generally, for a standard graded polynomial ring $k[z_1, \ldots, z_N]$ over an infinite field $k$, and an integer $p \geq 1$, consider the following injective $k$-algebra map

$$\zeta_p : k[z_1, \ldots, z_N] \rightarrow k[z_1, \ldots, z_N], \quad z_i \mapsto z_i^p \quad (1 \leq i \leq N).$$

If $M$ is any matrix with entries in $k[z_1, \ldots, z_N]$, we denote by $\zeta_p(M)$ the matrix obtained by evaluating $\zeta_p$ at every entry of $M$. We focus on the case where $r$ is a given odd integer and $\Phi$ denotes the $r \times r$ generic skew-symmetric matrix. Let $B$ stand for the polynomial ring over $k$ in the nonzero entries of $\Phi$.

**Lemma 1.3** For any integer $p \geq 1$ the $(r - 1)$-Pfaffians of the skew-symmetric matrix $\zeta_p(\Phi)$ generate an ideal of $B$ of codimension 3.

**Proof** It is well-known that an $(r - 1)$-Pfaffian is a polynomial in the entries of the source skew-symmetric matrix (see, e.g., [33, Proposition 159]). Let $I \subset B$ denote the ideal of $(r - 1)$-Pfaffians of $\Phi$. Since $\zeta_p$ is a homomorphism of $k$-algebras, the ideal of $B$ generated by the $(r - 1)$-Pfaffians of $\zeta_p(\Phi)$ is the extended ideal $\zeta_p(I)B$, where $\zeta_p(I)$ is the image of $I$ by $\zeta_p$. Now, clearly, $\zeta_p(I) \subset \zeta_p(B)$ and $I \subset B$ are ideals of the same height. On the other hand, the extension $\zeta_p(B) \subset B$ is integral and $\zeta_p(B)$ is integrally closed in its field of
fractions because it is isomorphic to a polynomial ring over \( k \). Therefore, the going-down property of the extension \( \zeta_p(B) \subset B \) implies that \( \text{ht} \zeta_p(I)B = \text{ht} \zeta_p(I) = 3 \). 

**Theorem 1.4** Let \( n \geq 3 \) be an integer. Then every virtual datum \((d, r)\) in dimension \( n \) is proper.

**Proof** Denote by \( d' \) the skew-degree of the virtual datum \((d, r)\). Define \( u = \binom{r}{2} - n \leq \binom{r}{2} - 3 \).

We induct on \( u \).

If \( u = 0 \), by Lemma 1.3 the ideal generated by the \((r-1)\)-Pfaffians of the matrix \( \zeta_\alpha(\Phi) \) is a Gorenstein ideal with datum \((d, r)\).

Now, suppose that the result is true for a certain \( 1 \leq u < \binom{r}{2} - 3 \). Then there is an \( r \times r \) skew-symmetric matrix \( \Psi = (g_{ij}) \) whose entries \( g_{ij} \) are forms of degree \( d' \) in \( \mathbb{k}[x_1, \ldots, x_n] \), such that \( \text{ht} I_1(\Psi) = n \) and \( I = \text{Pf}_{r-1}(\Psi) \) is a codimension 3 Gorenstein ideal. Since one can assume that \( r \) is fixed, we may argue by descending induction on \( n \) instead. Thus, we are assuming that \( n > 3 \). Since \( I \) is an unmixed homogeneous ideal of codimension 3, then by a homogeneous version of the prime avoidance lemma, there is a linear form \( \ell \in R = \mathbb{k}[x_1, \ldots, x_n] \) that is \( R/\ell \)-regular. Without loss of generality, we can suppose that \( \ell = x_n - \sum_{i=1}^{n-1} \alpha_i x_i \). Now consider the following surjective \( \mathbb{k} \)-algebra homomorphism:

\[
\pi : \mathbb{k}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{k}[x_1, \ldots, x_{n-1}], \quad x_i \mapsto x_i \ (1 \leq i \leq n-1), \quad x_n \mapsto \sum_{i=1}^{n-1} \alpha_i x_i.
\]

Using again the fact that Pfaffians are polynomials in the entries of the source matrix, \( \pi \) induces a \( \mathbb{k} \)-algebra isomorphism

\[
\mathbb{k}[x_1, \ldots, x_n]/(\text{Pf}_{r-1}(\Psi), \ell) \cong \mathbb{k}[x_1, \ldots, x_{n-1}]/\text{Pf}_{r-1}(\tilde{\Psi})
\]

and

\[
\mathbb{k}[x_1, \ldots, x_n]/(I_1(\Psi), \ell) \cong \mathbb{k}[x_1, \ldots, x_{n-1}]/I_1(\tilde{\Psi})
\]

where \( \tilde{\Psi} = (\pi(g_{ij})) \). By the second isomorphism above, \( I_1(\tilde{\Psi}) \) has codimension \( n-1 \). Since \( \ell \) is regular on \( \mathbb{k}[x_1, \ldots, x_n]/I \) then \( \mathbb{k}[x_1, \ldots, x_{n-1}]/\text{Pf}_{r-1}(\tilde{\Psi}) \) is a Gorenstein ring of codimension

\[
(n - 1) - \dim \mathbb{k}[x_1, \ldots, x_{n-1}]/\text{Pf}_{r-1}(\tilde{\Psi}) = (n - 1) - (n-4) = 3.
\]

Thus, \( \text{Pf}_{r-1}(\tilde{\Psi}) \) is a codimension 3 Gorenstein ideal in \( \mathbb{k}[x_1, \ldots, x_{n-1}] \) with datum \((d, r)\), satisfying the condition \( \text{ht} I_1(\tilde{\Psi}) = n - 1 \). 

\[ \square \]

## 3 Parametrization

### 3.1 The span problem

Fix a ground field \( \mathbb{k} \) and integers \( r \geq 1, d \geq 1 \). Let \( g_i = \sum_{|\alpha|=d} \frac{Y^i}{\alpha} x^\alpha, 1 \leq t \leq r \) denote forms of degree \( d \), whose coefficients are mutually independent indeterminates over \( \mathbb{k} \), while \( x = \{x_1, \ldots, x_n\} \) is a set of independent variables over the polynomial ring.
A := \k[Y\sub{t}{a} | 1 \leq t \leq r, |\alpha| = d]. Note that A = A_1 \otimes \k \cdots \otimes \k A_r, where each A_i is up to variable names identified with the homogeneous coordinate ring of \mathbb{P}^N(\mathbb{k} = \binom{d+n-1}{d} - 1).

Consider the polynomial ring \mathcal{S} := F[x_1, \ldots, x_n], where F denote the fraction field of the domain A.

Fix yet another integer e \geq 0 and let D := \dim_F S_{d+e}. The F-vector subspace \mathcal{S}g_1 + \cdots + S_e g_r of \mathcal{S}_{d+e} is spanned by the set \{x^\beta g_i | |\beta| = e\} \cup \cdots \cup \{x^\beta g_r | |\beta| = e\}. The coefficient matrix of this set with respect to a canonical monomial basis of \mathcal{S}_{d+e} is a \(D \times (r \dim F S_e)\) matrix

\[
M_{d,e} := \begin{pmatrix} M_1 & \cdots & M_r \end{pmatrix},
\]

where \(M_i\) is the coefficient matrix of the set \(\{x^\beta g_i | |\beta| = e\}\) with respect to the same basis of \(\mathcal{S}_{d+e}\). In particular, the entries of the block \(M_i\) involve only the coefficients \(Y\sub{t}{d,0}, \ldots, Y\sub{n}{0,0,d}\) of the form \(g_i\).

**Definition 2.1** \(M_{d,e}\) could be called the parameter matrix of the e-span of r forms of degree d. A couple of variations of this construct will be discussed below (Sects. 2.2 and 3.2).

Of special interest is the specialization to ‘rational’ forms \(f\), i.e., having coefficients in \(\mathbb{k}\), via the canonical surjection \(A \twoheadrightarrow \mathbb{k}\). For these, let \(R = \mathbb{k}[x_1, \ldots, x_n]\) and consider the parameter map \(R_d \times \cdots \times R_d \rightarrow (\mathbb{P}^N)\mathbb{k}\) that associates to the vector \(f_1 \cdots f_r\) the point

\[
P_f := (\lambda\sub{1}{d,0,\ldots,0}^{(1)} : \cdots : \lambda\sub{r}{0,0,d,0}^{(1)} : \cdots : \lambda\sub{r}{0,0,d,0}^{(r)}).
\]

Here \(Y\sub{t}{a} \mapsto \lambda\sub{a}{t}\) and \(N = \dim_{\mathbb{k}} R_d - 1 = \binom{d+n-1}{d} - 1\).

We will freely denote \(M_{d,r}(P_f)\) the result of evaluating via \(A \twoheadrightarrow \mathbb{k}\).

The following simple example may help visualize the shape of \(M_{d,r,e}\).

**Example 2.2** Let \(r = d = n = 2\) and \(e = 1\). Then \(D = 4\) and one gets:

\[
M_{2,2,1} = \begin{pmatrix}
Y\sub{1}{2,0}^{(1)} & 0 & Y\sub{1}{2,0}^{(2)} & 0 \\
Y\sub{1}{1,1}^{(1)} & Y\sub{1}{2,0}^{(1)} & Y\sub{1}{2,0}^{(2)} & Y\sub{1}{2,0}^{(2)} \\
Y\sub{1}{0,2}^{(1)} & Y\sub{1}{1,1}^{(1)} & Y\sub{1}{1,1}^{(2)} & Y\sub{1}{1,1}^{(2)} \\
0 & Y\sub{0,2}^{(1)} & 0 & Y\sub{0,2}^{(2)}
\end{pmatrix}.
\]

In this example the matrix has maximal rank. The next example shows that the rank can drop in general. Let \(n = 3, d = 1, e = r = 2\). In this case, we have \(D = 10\) and rank \(M_{1,2,1} = 9\), where \(M_{1,2,1}\) is the following \(10 \times 12\) matrix:
Let now $f = \{f_1, \ldots, f_r\}$ denote forms of degree $d$ with coefficients in $k$. Set $R := \mathbb{k}[x_1, \ldots, x_n]$. If $R_{d+f_1} + \cdots + R_{d+f_r} = R_{d+e}$, we loosely say that the $d$-forms $f_1, \ldots, f_r$ span in degree $d + e$. A central question here asks when the matrix $M_{d,e}$ has maximal rank. This is tantamount to having that, for given $r, d, e$, ‘most’ tuples $(f_1, \ldots, f_r)$ of rational forms span.

For given data $(d, r, e)$ as above denote by $D_{d,r,e}$ the rank of the matrix $M_{d,r,e}$. Let $I_{D_{d,r,e}} \subset A$ denote the ideal of minors of $M_{d,r,e}$ of order $D_{d,r,e}$, and $V(I_{D_{d,r,e}}(M_{d,r,e})) \subset (\mathbb{P}^N)^r$ the corresponding projective subvariety. The following properties are immediately verified:

\begin{align*}
\text{(A) } D_{d,r,e} &\leq \min\{\dim_{\mathbb{k}} S_{d+e}, r \dim_{\mathbb{k}} S_e\} = \min\{\dim_{\mathbb{k}} R_{d+e}, r \dim_{\mathbb{k}} R_e\}. \\
\text{Equivalently, } & \quad \dim_{\mathbb{k}} R_{d+e} - D_{d,r,e} \geq \max\{\dim_{\mathbb{k}} R_{d+e} - r \dim_{\mathbb{k}} R_e, 0\}. \tag{3}
\end{align*}

\begin{align*}
\text{(B) } &\quad \text{For every set } f \in (R_d)^r, \dim_{\mathbb{k}}[f]_{d+e} = \text{rank } M_{d,r,e}(P_f) \leq D_{d,r,e}. \\
\text{Equivalently, } &\quad \dim_{\mathbb{k}}[R/(f)]_{d+e} \geq \dim_{\mathbb{k}} R_{d+e} - D_{d,r,e}. \tag{4}
\end{align*}

\begin{align*}
\text{(C) } &\quad \text{For every set } f \in (R_d)^r \text{ such that } P_f \in (\mathbb{P}^N)^r \setminus V(I_{D_{d,r,e}}(M_{d,r,e})), \\
&\quad \dim_{\mathbb{k}}[R/(f)]_{d+e} = \dim_{\mathbb{k}} R_{d+e} - D_{d,r,e}. \tag{5}
\end{align*}

**Definition 2.3** Let $R = \mathbb{k}[x_1, \ldots, x_n]$. Given $d, r, e$, a tuple of forms $f \in (R_d)^r$ is $(d, r, e)$-extremal if

\[ \dim_{\mathbb{k}}[R/I]_{d+e} = \max\{\dim_{\mathbb{k}} R_{d+e} - r \dim_{\mathbb{k}} R_e, 0\} \]

where $I = \langle f \rangle \subset R$.

For convenience, we isolate some basic facts in the following lemma.

**Lemma 2.4** Let $R = \mathbb{k}[x_1, \ldots, x_n]$. Given integer data $d, r, e$ as above, one has:

\[
\begin{pmatrix}
Y_1^{(1)} & 0 & 0 & 0 & 0 & 0 \\
Y_1^{(2)} & 0 & 0 & 0 & 0 & 0 \\
Y_2^{(1)} & 0 & 0 & 0 & 0 & 0 \\
Y_2^{(2)} & 0 & 0 & 0 & 0 & 0 \\
Y_3^{(1)} & 0 & 0 & 0 & 0 & 0 \\
Y_3^{(2)} & 0 & 0 & 0 & 0 & 0 \\
Y_4^{(1)} & 0 & 0 & 0 & 0 & 0 \\
Y_4^{(2)} & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
(i) If there is a \((d, r, e)\)-extremal tuple in \((R_d)^r\) then every tuple \(\mathbf{f} \in (R_d)^r\) such that \(P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{d,r,e}(M_{d,r,e}))\) is \((d, r, e)\)-extremal. In particular, for any such \(\mathbf{f}\) one has
\[
\dim_k R_{d+e} - D_{d,r,e} = \max\{\dim_k R_{d+e} - r \dim_k R_r, 0\}.
\]

(ii) For \(e = 0\), every tuple \(\mathbf{f} \in (R_d)^r\) such that \(P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{d,r,0}(M_{d,r,0}))\) is \((d, r, 0)\)-extremal.

(iii) For \(e = 1\), every tuple \(\mathbf{f} \in (R_d)^r\) such that \(P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{d,r,1}(M_{d,r,1}))\) is \((d, r, 1)\)-extremal.

(iv) If \(e > n(d-1)\) and \(r \geq n\) then every \(\mathbf{f} \in (R_d)^r\) such that \(P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{d,r,e}(M_{d,r,e}))\) is \((d, r, e)\)-extremal.

**Proof** (i) This follows from properties (A), (B) and (C) above.

(ii) This is just linear independence over \(k\).

(iii) This is the content of [17, Theorem 1].

(iv) In fact, for every \(e > n(d-1)\), \(\max\{\dim_k R_{d+e} - r \dim_k R_r, 0\} = 0\). For a sequence \(\mathbf{f}_0 \in (R_d)^r\) such that \(\{x_0^1, \ldots, x_0^n\} \subset \mathbf{f}_0\) we have \(\dim_k [R/(\mathbf{f}_0)]_{d+e} = 0\), that is, \(\mathbf{f}_0\) is \((d, r, e)\)-extremal. Thus, the claim follows by (i).

Recall that a set \(\mathbf{f} = \{f_1, \ldots, f_r\}\) of forms of the same degree is **general** when the corresponding parameter point \(P_{\mathbf{f}}\) belongs to a suitable dense Zariski open set of the parameter space.

Let \(\mathbf{f} = \{f_1, \ldots, f_r\}\) be a general set in a polynomial ring \(R = k[x, y, z]\) over an infinite field \(k\). In [1] the Hilbert series of \(R/(f_1, \ldots, f_r)\) is completely determined. In the next result we give an alternative argument in the case of an almost complete intersection \(R/(f_1,f_2,f_3,f_4)\). Although a consequence of [1], the actual argument reduces to computing the socle degree.

**Proposition 2.5** \((\text{char}(k) = 0)\) A general set of four forms of degree \(d \geq 1\) in \(R = k[x, y, z]\) is \((d, 4, e)\)-extremal, for arbitrary \(e \geq 1\) and generates an ideal of \(R\) with socle degree \(2d - 2\).

**Proof** By Lemma 2.4, it suffices to show the existence of one set of four forms of degree \(d\) satisfying the statement.

Let then \(J = (x^d, y^d, z^d) \subset R = k[x, y, z]\). The minimal graded free resolution of \(R/J\) is given by the Koszul complex
\[
0 \to R(-3d) \to R(-2d)^3 \to R(-d)^3 \to R \to R/J \to 0,
\]
from which one readily gets the Hilbert series of \(R/J\):
\[
H_{R/J}(t) = \frac{1 - 3t^d + 3t^{2d} - t^{3d}}{(1 - t)^3}.
\]
Therefore, the coefficients \(a_i\) of \(H_{R/J}(t)\) are:
\[
a_i = \begin{cases} 
\dim_k R_i & \text{if } 0 \leq i \leq d - 1 \\
\dim_k R_i - 3 \dim_k R_{i-d} & \text{if } d \leq i \leq 2d - 1 \\
\dim_k R_i - 3 \dim_k R_{i-d} + 3 \dim_k R_{i-2d} & \text{if } 2d \leq i \leq 3d - 3.
\end{cases}
\]
Now, by [40] (also [43, Corollary 3.5]), $R/J$ has the Strong Lefschetz property, that is, there is a linear form $L \in R$ such that for $f := L^d$ the multiplication map $[R/J]_{i-d} \to [R/J]_i$ by $f$ has maximal rank for all $i \geq d$. Consequently, setting $I := (x^d, y^d, z^d, f)$, since the image of this map is the vector space $(fR_{i-d}, J_i)/J_i = I_i/J_i$, then
\[
\dim_k I_i/J_i = \min\{\dim_k[R/J]_{i-d}, \dim_k[R/J]_i\},
\]
hence
\[
\dim_k[R/I]_i = \dim_k[R/J]_i - \min\{\dim_k[R/J]_{i-d}, \dim_k[R/J]_i\}
= \max\{\dim_k[R/J]_i - \dim_k[R/J]_{i-d}, 0\}
\]
for all $i \geq d$.

Thus, by (6) and an obvious calculation, we have:
\[
\dim_k[R/I]_i = \begin{cases} 
\dim_k R_j & \text{if } 0 \leq i \leq d - 1 \\
\max\{\dim_k R_i - 4 \dim_k R_{i-d}, 0\}, & \text{if } d \leq i \leq 2d - 1 \\
\max\{\dim_k R_i - 4 \dim_k R_{i-d} + 6 \dim_k R_{i-2d}, 0\}, & \text{if } 2d \leq i \leq 3d - 3.
\end{cases}
\]

Therefore, it follows that
\[
\dim_k[R/I]_i = \dim_k R_i - 4 \dim_k R_{i-d} > 0, \quad \text{for } d \leq i \leq 2d - 2
\]
and $\dim_k[R/I]_{2d-1} = 0$, thus showing that the socle degree is $2d - 2$. □

**Remark 2.6** The value of the socle degree has been previously established in [32, Lemma 2.5] for arbitrary codimension and not necessarily equigenerated ideal.

As a last issue in this subsection we highlight a connection, in the $d$-equigenerated case, between Fröberg’s conjecture [12] in arbitrary dimension and the concept of a $(d, r, e)$-extremal set of forms. Using Fröberg’s notation, in the $d$-equigenerated case $((1 - t^d)^e/(1 - t)^n)_+$ denotes the initial positive segment of the power series $(1 - t^d)^e/(1 - t)^n$.

Fröberg’s conjecture [12] for equigenerated ideals reads as follows:

**Conjecture 2.7** For a general set of forms $f = \{f_1, \ldots, f_n\}$ in $R = \mathbb{k}[x_1, \ldots, x_n]$ of degree $d \geq 1$, the Hilbert series of $R/(f_1, \ldots, f_n)$ is $((1 - t^d)^e/(1 - t)^n)_+$.

In [13, Question 2.5] it is asked what are the coefficients of the Hilbert series. In the equigenerated case, one has the following expression:

**Proposition 2.8** Let $R = \mathbb{k}[x_1, \ldots, x_n]$ and $r \geq n$ and $d \geq 1$ be integers. Then
\[
(1 - t^d)^e/(1 - t)^n)_+ = \sum_{j=0}^{j_0-1} \left(\sum_{i=0}^{r} (-1)^i \binom{r}{i} \dim_k R_{j-id}\right) t^j
\]
where $j_0$ is the least of the integers $d \leq j \leq dr - n$ satisfying $\sum_{i=0}^{r} (-1)^i \binom{r}{i} \dim_k R_{j-id} \leq 0$.

**Proof** The arbitrary term of the series $(1 - t^d)^e/(1 - t)^n$, is certainly well-known. We recall how to get its expression. Clearly, $1/(1 - t)^n = \sum_{n=0}^{\infty} \dim_k R_n t^n$. Thus,
\[
\frac{(1 - t^d)^r}{(1 - t)^n} = \left( \sum_{i=0}^r (-1)^i \binom{r}{i} t^{di} \right) \left( \sum_{\mu=0}^\infty \dim_{\mathbb{K}} R_\mu t^\mu \right)
= \sum_{j=0}^r \sum_{i=0}^\infty (-1)^i \binom{r}{i} \dim_{\mathbb{K}} R_\mu t^{\mu+id}
= \sum_{j=0}^\infty \left( \sum_{u \geq 0, 0 \leq i \leq r} (-1)^i \binom{r}{i} \dim_{\mathbb{K}} R_u \right) t^j
= \sum_{j=0}^\infty \left( \sum_{i=0}^r (-1)^i \binom{r}{i} \dim_{\mathbb{K}} R_{j-id} \right) t^j.
\]

From this and the definition of \(((1 - t^d)^r)/(1 - t)^n\)_+, the stated expression follows suit.

The above clearly translates [13, Question 2.5] in the \(d\)-equigenerated case into the quest for an explicit expression of the integer \(j_0\) in terms of \(n, d, r\). Also, we see that an affirmative answer to Conjecture 2.7 implies that a general set of \(r \geq n\) forms \(f = \{f_1, \ldots, f_r\}\) of degree \(d\) is \((d, r, e)\)-extremal for every \(0 \leq e \leq \min\{d - 1, j_0 - d\}\). A previous discussion in the equigenerated case, regarding the values of the Hilbert series in some intervals, has been given in [34].

### 3.2 Gorenstein ideals and general sets of forms

We now draw some consequences in dimension three. First, one has the following constraint for Gorenstein ideals in this dimension.

**Proposition 2.9** Let \(r \geq 5\) be an odd integer and \(d = (r-1)/2\). If there is a Gorenstein ideal \(I \subset \mathbb{K}[x, y, z]\) generated by a general set of \(r\) forms of degree \(d\), then \(d = 2\), i.e., \(r = 5\).

**Proof** Let \(I\) denote such an ideal. By the symmetry of the Hilbert function of \(R/I\) we have:

\[
h_{R/I}(d+1) = h_{R/I}(d-3) = \binom{d-1}{2}.
\]

Then, since \(I\) is generated by a general set of forms, drawing upon Lemma 2.4 (iii), we get

\[
\binom{d-1}{2} = h_{R/I}(d+1) = \max \left\{ \frac{d+3}{2} - 3r, 0 \right\} = \max \left\{ \frac{d+3}{2} - 3(2d+1), 0 \right\}.
\]

A direct calculation shows that the only possibility for \(d\) a positive integer is that \(\binom{d-1}{2} = 0\), hence \(d = 2\). □

The main theorem is now a simple consequence:
Theorem 2.10 (char(\k) = 0) Let I \subset R = \mathbb{k}[x, y, z] be a Gorenstein ideal generated by a general set of r \geq 5 forms of degree d \geq 2. Then r = 5 and d = 2.

Proof Let J \subset I be generated by n + 1 = 4 of the forms. By Proposition 2.5, the socle degree of R/J is 2d − 2. Then the socle degree of R/I is at most 2d − 2. But, since I is Gorenstein, the socle degree of R/I is 2d + d' − 3 (Lemma 1.2). Therefore, d' = 1. Now apply Proposition 2.9.

Remark 2.11 An alternative proof of the above result, based on explicit calculations, is available in [30, Appendix A].

More generally, we state the following conjecture.

Conjecture 2.12 Let I \subset \mathbb{k}[x_1, \ldots, x_n] be an ideal generated by a general set of r \geq n + 2 forms of degree d \geq 2. Then I is Gorenstein if and only if d = 2 and r = \binom{n+1}{2} − 1.

We next proceed towards the converse implication of this conjecture, by focusing on quadric generators. To carry on, we elaborate on some further notation. Given integers n \geq 3, d \geq 2 and r \geq 5, consider the following \binom{d+n-1}{d} \times r generic matrix

\[
Y := Y_{n,d,r} = \begin{pmatrix}
Y^{(1)}_{d,0} & \cdots & Y^{(r)}_{d,0} \\
\vdots & \ddots & \vdots \\
Y^{(1)}_{0,d} & \cdots & Y^{(r)}_{0,d}
\end{pmatrix},
\]

whose entries along the columns can be thought of as the respective coordinates of r copies of \mathbb{P}^{(d+n-1)−1}.

One way of thinking about r forms \mathbf{f} = \{f_1, \ldots, f_r\} \subset R = \mathbb{k}[x] = \mathbb{k}[x_1, \ldots, x_n] of degree d \geq 1 is as the matrix product \[x_d] \cdot Y(P_\mathbf{f}), where \[x_d] is the list of monomials of degree d in x and Y(P_\mathbf{f}) denotes the \binom{d+n-1}{d} \times r matrix Y evaluated at \mathbf{f} (hence, with entries in \mathbb{k}).

Throughout, set \mathbf{m} = (x). Our main technical result is as follows.

Proposition 2.13 Let I \subset R = \mathbb{k}[x_1, \ldots, x_n] be an ideal generated by a general set of r quadrics, where n \leq r \leq N − 1 and N = \binom{n+1}{2}. Then the socle of R/I has no nonzero linear forms.

Proof A quadric in \mathbb{k}[x_1, \ldots, x_n] depends on N = \binom{n+1}{2} coefficients. We will accordingly denote the corresponding indices by \{1, 1\}, \{1, 2\}, \ldots, \{n − 1, n\}, \{n, n\}. Thus, the matrix \[Y_{n,2,r}\] has the shape

\[
\begin{pmatrix}
Y^{(1)}_{1,1} & \cdots & Y^{(r)}_{1,1} \\
\vdots & \ddots & \vdots \\
Y^{(1)}_{n,n} & \cdots & Y^{(r)}_{n,n}
\end{pmatrix},
\]

where delimiters have been omitted. Let \mathbf{f} = \{f_1, \ldots, f_r\} \subset R be a general set of quadrics generating I. For simplicity, set \[P := P_\mathbf{f}\]. As explained above, one has the following matrix equality

\[\mathbf{f} = [x_2] \cdot Y_{n,2,r}(P).\]
The $N \times r$ matrix $Y_{n,2,r}$ can be decomposed in two vertical blocks as
\[
Y_{n,2,r} = \begin{pmatrix} B \\ L \end{pmatrix}
\]
where
\[
B = \begin{pmatrix} Y^{(1)}_{1,1} & \cdots & Y^{(r)}_{1,1} \\ \vdots & \ddots & \vdots \\ Y^{(1)}_{u,v} & \cdots & Y^{(r)}_{u,v} \end{pmatrix}_{r \times r}
\quad \text{and} \quad
L = \begin{pmatrix} Y^{(1)}_{n,n} & \cdots & Y^{(r)}_{n,n} \\ \vdots & \ddots & \vdots \\ Y^{(1)}_{n,v} & \cdots & Y^{(r)}_{n,v} \end{pmatrix}_{(N-r) \times r}
\]
for suitable $1 \leq u \leq v \leq n$. Thus,
\[
[f] \cdot \text{cof}(B)(P) = [x_2] \cdot \begin{pmatrix} \Delta(P) \mathbb{I}_r \\ L(P) \cdot \text{cof}(B)(P) \end{pmatrix}
\]
where $\text{cof}(B)$ is the matrix of cofactors of $B$, $\mathbb{I}_r$ is the identity matrix of order $r$ and $\Delta = \det B$. Write, say,
\[
L \cdot \text{cof}(B) = \begin{pmatrix} g^{u',v'}_{1,1} & \cdots & g^{u',v'}_{u,v} \\ \vdots & \ddots & \vdots \\ g^{u,n}_{1,1} & \cdots & g^{u,n}_{u,v} \end{pmatrix}
\]
Since $f$ is a general set, then $\Delta(P) \neq 0$. Thus, by (8):
\[
I = \left(\Delta(P)x_ix_j + \sum_{(u',v') \leq (i,j) \leq (u,n)} g^{u',v'}_{i,j}(P)x_{u'}x_{v'} \mid (1, 1) \leq (i, j) \leq (u, v)\right).
\]
Let $\ell' = a_1x_1 + \cdots + a_nx_n$ be a linear form in the socle of $R/I$. For any given $1 \leq k \leq n$, one has
\[
a_1x_1x_k + \cdots + a_nx_nx_k = \sum_{(1,1) \leq (i,j) \leq (u,v)} \alpha_{i,j} \left(\Delta(P)x_ix_j + \sum_{(u',v') \leq (i,j) \leq (u,n)} g^{u',v'}_{i,j}(P)x_{u'}x_{v'}\right).
\]
Comparing coefficients we get a linear system
\[
A_r(P) \cdot a = 0
\]
where $a$ is the transpose of the matrix $(a_1 \cdots a_n)$ and $A_r(P)$ is an $(N-r)n \times n$ matrix whose entries belong to the set
\[
\{0\} \cup \{g^{u',v'}_{i,j}(P) : (1, 1) \leq (i,j) \leq (u,v), (u',v') \leq (s,t) \leq (n,n)\} \cup \{\Delta(P)\}.
\]
We claim that rank $(A(P)_r) = n$, which says that the assumed linear form is zero. We divide the proof in two cases:

**Case 1:** $r = N - 1$.

In this case, the relation (10) has the following format for $1 \leq t \leq n$:
\[
\begin{align*}
& a_1 x_1 x_i + \cdots + a_n x_n x_i = \\
& \quad \sum_{1 \leq i \leq j \leq n} a_{ij} (\Delta(P)x_i x_j - g_{ij}^{n,n}(P)x_n^2), \\
\end{align*}
\]

Comparing coefficients yields

\[
\mathcal{A}_{N-1}(P) = 
\begin{pmatrix}
\begin{array}{ccc}
g_{1,1}^{n,n}(P) & \cdots & g_{1,n-1}^{n,n}(P) \\
\vdots & \ddots & \vdots \\
g_{1,n-1}^{n,n}(P) & \cdots & g_{1,n}^{n,n}(P) \\
g_{n,1}^{n,n}(P) & \cdots & g_{n,n-1}^{n,n}(P) \\
g_{n,n}^{n,n}(P) & \cdots & \Delta(P)
\end{array}
\end{pmatrix}
\]

Since the entries of \(Y_{n,2,N-1}\) are mutually independent indeterminates, there is a \(\mathbb{k}\)-homomorphism \(\phi : \mathbb{k}[Y_{n,2,N-1}] \rightarrow \mathbb{k}[L]\) mapping \(B\) to the identity matrix and fixing the entries of \(L\). Thus, by (9)

\[
L = [\phi(g_{1,1}^{n,n}) \cdots \phi(g_{1,n}^{n,n}) \phi(g_{2,1}^{n,n}) \cdots \phi(g_{2,n}^{n,n}) \cdots \phi(g_{n-1,n}^{n,n}) \phi(g_{n,n}^{n,n})].
\]

In particular, the determinant

\[
\mathcal{D} := \det \begin{pmatrix}
g_{1,1}^{n,n} & \cdots & g_{1,n-1}^{n,n} & g_{1,n}^{n,n} \\
\vdots & \ddots & \vdots & \vdots \\
g_{1,n-1}^{n,n} & \cdots & g_{1,n}^{n,n} & \Delta
\end{pmatrix}
\]

specializes to

\[
\phi(\mathcal{D}) = \det \begin{pmatrix}
Y_{n,n}^{(1)} & \cdots & Y_{n,n}^{(n-1)} & Y_{n,n}^{(n)} \\
\vdots & \ddots & \vdots & \vdots \\
Y_{n,n}^{(n-1)} & \cdots & Y_{n,n}^{(N-1)} & Y_{n,n}^{(N)} \\
Y_{n,n}^{(n)} & \cdots & Y_{n,n}^{(N)} & 1
\end{pmatrix}
\]

The latter does not vanish as it is the sum of two forms in two different degrees, none of which vanishes. Hence, \(\mathcal{D} \neq 0\) as well.

**Case 2:** \(r < N - 1\).

It is enough to find a point \(P'\) such that \(\text{rank} (\mathcal{A}_r(P')) = n\). For this, consider a general set of quadrics \(f' = \{f'_1, \ldots, f'_{N-1}\}\). By the previous case, as applied to \(J := (f'_1, \ldots, f'_{N-1})\), the socle of \(R/J\) has no linear forms. Since \(r \geq n\) and \(\{f'_1, \ldots, f'_{N-1}\}\) is general, we can assume that the \(r\) first elements \(f'_1, \ldots, f'_r\) of \(f'\) are such that \(J' = (f'_1, \ldots, f'_r)\) is \(m\)-primary and \(\Delta(P') \neq 0\), where \(P' \in (\mathbb{P}^{N-1})'\) is the point corresponding to the set \(\{f'_1, \ldots, f'_r\}\).

If \(\text{rank} (\mathcal{A}_r(P')) < n\) then \(\mathcal{A}_r(P') \cdot a = 0\) has a nonzero solution \(a\), so \(\text{Soc}(R/J')_1 \neq 0\). But, this is nonsense since \(\text{Soc}(R/J')_1 \subset \text{Soc}(R/J)_1\) because \(J' \subset J\) lives in degree 2.

The next result follows suit.

**Theorem 2.14** Let \(I \subset R = \mathbb{k}[x_1, \ldots, x_n]\) be an ideal generated by a general set of \(r\) quadrics, where \(\frac{(n+2)(n+1)}{6} \leq r \leq N - 1\) and \(N = \binom{n+1}{2}\). Then \(\text{Soc}(R/I) = \text{Soc}(R/I)_2\) and \(\dim_\mathbb{k} \text{Soc}(R/I)_2 = N - r\). In particular, \(R/I\) is Gorenstein if and only if \(r = N - 1\).

**Proof** We have
If and only if
\[ R / I = \mathbb{k} \oplus (R/I)_1 \oplus (R/I)_2 \] as a \( \mathbb{k} \)-vector space, where \( \dim_{\mathbb{k}} (R/I)_1 = n \) and \( \dim_{\mathbb{k}} (R/I)_2 = N - r \). Hence, by Proposition 2.13,
\[ \text{Soc}(R/I) = \text{Soc}(R/I)_2 \text{ and } \dim_{\mathbb{k}} \text{Soc}(R/I)_2 = N - r. \]

In particular, \( \dim_{\mathbb{k}} \text{Soc}(R/I) = 1 \) if and only if \( r = N - 1 \), that is, \( R/I \) is an Artinian Gorenstein algebra if and only if \( r = N - 1 \). \( \square \)

4 On the \((x_1^m, \ldots, x_n^m)\)-colon problem

It is known (see [2, Proposition 1.3]) that any homogeneous Gorenstein ideal of codimension \( n \) in \( \mathbb{k}[x] = \mathbb{k}[x_1, \ldots, x_n] \) can be obtained as a colon ideal \((x_1^m, \ldots, x_n^m) : \mathfrak{f}\), for some integer \( m \geq 1 \) and some form \( \mathfrak{f} \). In this section we deal with some of the main questions regarding this representation.

It is first established under which condition the form \( \mathfrak{f} \) is uniquely determined and what is its degree in terms of \( m \) and the socle degree of \( I \). Then we prove that \( \mathfrak{f} \) can be retrieved from \( I \) by taking the so-called (socle-like) Newton dual of a minimal generator of the Macaulay inverse of \( I \).

Then we give conditions under which the Gorenstein ideal \( I \) is equigenerated in terms of the exponent \( m \) and the form \( \mathfrak{f} \). We solve this problem in the case where \( I \) has linear resolution
\[ 0 \to R(-2d - n + 2) \to R(-d - n + 2)^{b_{d-1}} \to \cdots \to R(-d - 1)^{b_2} \to R(-d)^{b_1} \to R \]
where \( b_1 = \mu(I) \).

These questions will be subsumed under the designation the colon problem, to avoid ‘link’ which has already many uses. We call \( \mathfrak{f} \) a directrix form (of \( I \)) associated to the regular sequence \([x_1^m, \ldots, x_n^m]\).

4.1 Macaulay inverse system versus Newton duality

We briefly recall some main features of the Macaulay inverse system. For recent accounts of this classical theme see, e. g., [10, 11, 26].

Let \( V \) be a vector space of dimension \( n \) over a field \( \mathbb{k} \) and let \( x_1, \ldots, x_n \) be a basis for \( V \). Let \( R = \text{Sym}_{\mathbb{k}}(V) = \mathbb{k}[x_1, \ldots, x_n] \) be the standard graded polynomial ring in \( n \) variables over \( \mathbb{k} \). Set \( y_1, \ldots, y_n \) for the dual basis on \( V^* = \text{Hom}_{\mathbb{k}}(V, \mathbb{k}) \) and consider the divided power ring
\[ D_\mathbb{k}(V^*) = \bigoplus_{i \geq 0} \text{Hom}(R_i, \mathbb{k}) = \mathbb{k}[y_1, \ldots, y_n]. \]

In particular, \( \{y^{[\alpha]} \mid \alpha \in \mathbb{N}^n \text{ and } |\alpha| = j \} \) is the dual basis of \( \{x^\alpha \mid \alpha \in \mathbb{N}^n \text{ and } |\alpha| = j \} \) on \( D_\mathbb{k}(V^*) = \text{Hom}(R_j, \mathbb{k}) \). If \( \alpha \in \mathbb{Z}^n \) then we set \( y^{[\alpha]} = 0 \) if some component of \( \alpha \) is negative. Make \( D_\mathbb{k}(V^*) \) into a module over \( R \) through the following action
\[ R \times D_{\mathbb{k}}(V^*) \rightarrow D_{\mathbb{k}}(V^*), \quad (f = \sum_m a_m \mathbf{x}^m, F = \sum_\beta b_\beta \mathbf{y}^\beta) \mapsto fF = \sum_{a,\beta} a_m b_\beta \mathbf{y}^{\beta - a}. \]

For a homogeneous ideal \( I \subset R \) and an \( R \)-submodule \( M \subset D_{\mathbb{k}}(V^*) \) one defines:

\[ \text{Ann}(I) := \{ g \in D_{\mathbb{k}}(V^*) \mid I g = 0 \} \quad \text{and} \quad \text{Ann}(M) := \{ f \in R \mid \beta M = 0 \}. \]

Then \( \text{Ann}(I) \) is an \( R \)-submodule of \( D_{\mathbb{k}}(V^*) \), while \( \text{Ann}(M) \) is an ideal of \( R \). The \( R \)-module \( \text{Ann}(I) \) is called the Macaulay inverse (system) of \( I \).

The main basic result regarding this construction is due to Macaulay [31]. In the present language it can be stated in the following form:

**Theorem 3.1 (Macaulay Duality, [26, Theorem 1.4])** There exists a one-to-one correspondence between the set of nonzero homogeneous codimension \( n \) Gorenstein ideals of \( R \) and the set of nonzero homogeneous cyclic submodules of \( D_{\mathbb{k}}(V^*) \) given by \( I \mapsto \text{Ann}(I) \) with inverse \( M \mapsto \text{Ann}(M) \). Moreover, the socle degree of \( RI \) is equal to the degree of a homogeneous generator of \( \text{Ann}(I) \).

The Macaulay–Matlis duality meets yet another version in terms of the Newton polyhedron nature of the homogeneous forms involved so far.

For this, recall the notion of the Newton (complementary) dual of a form \( f \in \mathbb{k}[\mathbf{x}] = \mathbb{k}[x_1, \ldots, x_n] \) in a polynomial ring over a field \( \mathbb{k} \), as introduced in [6, 8]. Namely, start out with the log matrix \( A \) of the constituent monomials of \( f \) (i.e., the nonzero terms of \( f \)). This is the matrix whose columns are the exponents vectors of the nonzero terms of \( f \) in, say, the lexicographic ordering. It is denoted \( \mathcal{N}(f) \). Then, the Newton dual log matrix (or simply the Newton dual matrix) of the Newton log matrix \( \mathcal{N}(f) = (a_{ij}) \) is the matrix \( \mathcal{N}^*(f) = (a_{ij} - a_{ij}) \), where \( a_i = \max_j \{ a_{ij} \} \), with \( 1 \leq i \leq n \) and \( j \) indexes the set of all nonzero terms of \( f \).

In other words, denoting \( \mathbf{a} := (a_1 \cdots a_n)' \), one has

\[ \mathcal{N}^*(f) = [ \mathbf{a} \cdots a_1 ]_{(n+1) \times r} - \mathcal{N}(f), \]

where \( r \) denotes the number of nonzero terms of \( f \). The vector \( \mathbf{a} \) is called the directrix vector of \( \mathcal{N}(f) \) (or of \( f \) by abuse).

We note that taking the Newton dual is a true duality upon forms not admitting monomial factor, in the sense that, for such a form \( f \), \( \mathcal{N}(f) = \mathcal{N}(f) \) holds.

We define the Newton dual of \( f \) to be the form \( \hat{f} \) whose terms are the ordered monomials obtained form \( \mathcal{N}(f) \) affected by the same coefficients as in \( f \).

Our next result asserts that directrix forms and Macaulay inverse generators obey a duality in terms of the above Newton dual. Given a directrix form \( \hat{f} \) associated to the regular sequence \( \{ x_1^m, \ldots, x_n^m \} \) – i.e., \( \{ x_1^m, \ldots, x_n^m \} : \hat{f} = I \) – it will typically admit monomial terms belonging to the ideal \( \langle x_1^m, \ldots, x_n^m \rangle \). In order to fix this inconvenience, we redefine the socle-like Newton dual of such directrix form by taking as directrix vector \( v := (m - 1 \cdots m - 1)' \).

**Proposition 3.2** Let \( I \subset R = \mathbb{k}[\mathbf{x}] = \mathbb{k}[x_1, \ldots, x_n] \) be a homogeneous codimension \( n \) Gorenstein ideal with socle degree \( s \). Given an integer \( m \geq 1 \), suppose that \( I \) admits a directrix form \( \hat{f} \) associated to the regular sequence \( \{ x_1^m, \ldots, x_n^m \} \). Then:
(i) \( \hat{J} \) is a degree \( n(m - 1) - s \) form uniquely determined, up to a scalar coefficient, by the condition that no nonzero term of \( \hat{J} \) belongs to the ideal \( (x_1^m, \ldots, x_n^m) \).

(ii) The socle-like Newton dual of \( \hat{J} \) is a minimal generator of the Macaulay inverse to \( I \) (having dual degree \( s \)), and its socle-like Newton dual retrieves \( \hat{J} \).

**Proof** Suppose \( \hat{J} = \sum_{|\alpha| = \deg \hat{J}} a_{\alpha} x^{\alpha} \). Then, the socle-like Newton dual of \( \hat{J} \) is \( \hat{I} = \sum a_{\alpha} y^{\alpha} \), where \( \hat{a} := v - a \) (in particular, \( \deg \hat{I} = n(m - 1) - \deg \hat{J} \)). Given a homogeneous polynomial \( h = \sum_{|\beta| = \deg h} b_{\beta} x^{\beta} \in R \) one has:

\[
h \hat{J} = \sum_{|\gamma| = \deg \hat{J} + \deg h} \left( \sum a_{\alpha} b_{\beta} \right) x^{\gamma}, \quad \text{and} \quad h \hat{I} = \sum_{|\gamma| = \deg \hat{I} + \deg h} \left( \sum a_{\alpha} b_{\beta} \right) y^{\gamma}
\]

with \( \hat{\gamma} = v - \gamma \). In particular, for every \( \gamma \), the coefficient of \( x^{\gamma} \) as a term in \( h \hat{J} \) is equal to the coefficient of \( y^{\hat{\gamma}} \) as a term in \( h \hat{I} \). Moreover, the \( i \)th coordinate of \( \gamma \) is larger than \( m \) if and only if the \( i \)th coordinate of \( \hat{\gamma} \) is negative. Thus,

\[
h \in (x_1^m, \ldots, x_n^m) : \hat{J} \iff \text{for every } \sum_{a+\beta=\gamma} a_{\alpha} b_{\beta} \neq 0, \gamma \text{ has a coordinate larger than } m
\]

\[\iff \text{for every } \sum_{a-\beta=\hat{\gamma}} a_{\alpha} b_{\beta} \neq 0, \hat{\gamma} \text{ has a negative coordinate}
\]

\[\iff h \hat{I} = 0 \iff h \in \operatorname{Ann}(\hat{J}). \]  \( \quad (13) \)

Therefore, \( I = \operatorname{Ann}(\hat{J}) \), that is, \( \hat{J} \) is a minimal generator of the Macaulay inverse to \( I \). By construction, one has \( \deg \hat{J} = n(m - 1) - \deg J \). On the other hand, it is well known that the degree of a minimal generator of the Macaulay inverse to \( I \) is the socle degree of \( I \), i.e., \( \deg \hat{J} = s \). Therefore, \( \deg \hat{J} = n(m - 1) - s \). Since \( \hat{J} \) is uniquely determined, up to a scalar coefficient, the form \( \hat{J} \) is uniquely determined as well, up to a scalar coefficient, by the condition that no nonzero term of \( \hat{J} \) belongs to the ideal \( (x_1^m, \ldots, x_n^m) \). Thus, assertion (i) follows.

Assertion (ii) follows from the above. \( \square \)

**Remark 3.3** Item (i) of Proposition 3.2 is stable under a change of coordinates. In other words, it holds true replacing the sequence \( \{x_1^m, \ldots, x_n^m\} \) by a sequence \( \{\ell_1^m, \ldots, \ell_n^m\} \), where \( \{\ell_1, \ldots, \ell_n\} \) are independent linear forms. Thus, if \( I \) is a homogeneous codimension \( n \) Gorenstein ideal such that \( (\ell_1^m, \ldots, \ell_n^m) : \hat{J} = I \), for some form \( \hat{J} \in R \), then \( \hat{J} \) is uniquely determined, up to a scalar coefficient, by the condition that no nonzero term of \( \hat{J} \), written as a polynomial in \( \ell_1, \ldots, \ell_n \), belongs to the ideal \( (\ell_1^m, \ldots, \ell_n^m) \). \( \square \)

### 4.2 The case of a linear resolution

In this subsection we characterize when \( I \) is a codimension \( n \) equigenerated Gorenstein ideal with linear resolution in terms of the exponent \( m \) and the form \( \hat{J} \in R = \mathbb{k}[x] = \mathbb{k}[x_1, \ldots, x_n] \). The preliminaries remain valid in arbitrary characteristic, but characteristic zero is called upon in item (ii) of Proposition 3.7 below.

Let \( e, e' \) be positive integers and let \( \hat{f} = \sum_{|\alpha| = e} a_{\alpha} x^{\alpha} \in R_e \) and \( g = \sum_{|\beta| = e'} b_{\beta} x^{\beta} \in R_{e'} \) be forms. Given an integer \( m \geq 1 \), write...
\[ g\hat{\mathbf{f}} = \sum_{\mathbf{x}' \in (x_1'', \ldots, x_m'')} \left( \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) \mathbf{x}' + \sum_{\mathbf{x}' \in (x_1'', \ldots, x_m'')} \left( \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) \mathbf{x}', \quad (14) \]

where \( \gamma \in \mathbb{N}^n \) is a running \( n \)-tuple. To this writing associate a matrix \( \mathcal{M}_{\ell,e',m} \) whose rows are indexed by the \( n \)-tuples \( \gamma \) such that \( |\gamma| = e + e' \) and whose columns are indexed by the \( n \)-tuples \( \beta \) such that \( |\beta| = e' \). The entries of the matrix are specified as follows:

\[
\text{the } (\gamma, \beta)\text{-entry of } \mathcal{M}_{\ell,e',m} = \begin{cases} 
0, & \text{if some coordinate of } \gamma - \beta \text{ is } < 0 \\
\alpha, & \text{if each coordinate of } \alpha = \gamma - \beta \text{ is } \geq 0.
\end{cases}
\]

In addition, let \( \chi \) denote the row matrix \( [\mathbf{x}'] \) with the monomial entries \( \mathbf{x}' \not\in (x_1'', \ldots, x_m'') \), and let \( \mathbf{b} \) stand for the column matrix whose entries are the coefficients \( b_\beta \) of \( g \).

Then equality (14) can be rewritten as

\[
g\hat{\mathbf{f}} = \chi \cdot \mathcal{M}_{\ell,e',m} \cdot \mathbf{b} + \sum_{\mathbf{x}' \in (x_1'', \ldots, x_m'')} \left( \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) \mathbf{x}'.
\]

(15)

It is important to observe that the matrix \( \mathcal{M}_{\ell,e',m} \) depends only on the integers \( e, e' \) and \( m \), and not on the details of \( g \).

From this, it follows immediately:

**Lemma 3.4** \( g \in I = (x_1'', \ldots, x_m'') : \hat{\mathbf{f}} \) if and only if \( \mathcal{M}_{\ell,e',m} \cdot \mathbf{b} = 0 \). In particular, \( I_{e'} = \{0\} \) if and only if \( \text{rank } \mathcal{M}_{\ell,e',m} = \binom{e + n - 1}{n - 1} \).

To tie up the ends, consider the parameter map

\[
R_e \to \mathbb{P}^{\binom{e + n - 1}{n - 1}} , \quad \hat{\mathbf{f}} = \sum_{|\alpha| = e} a_\alpha \mathbf{x}^\alpha \mapsto P_\hat{\mathbf{f}} = (a_{(e, \ldots, 0)} : \cdots : a_{(0, \ldots, e)})
\]

in the notation of Sect. 2.1. Let \( \{Y_{e,0, \ldots, 0}, \ldots, Y_{0, \ldots, e}\} \) denote the coordinates of \( \mathbb{P}^{\binom{e + n - 1}{n - 1}} \) and let \( \mathcal{M}_G_{\ell,e',m} \) stand for the matrix whose entries are obtained by replacing each \( a_\alpha \) in \( \mathcal{M}_{\ell,e',m} \) by the corresponding \( \chi_y \).

**Theorem 3.5** Let \( m \geq 1 \) be an integer and let \( \hat{\mathbf{f}} \in R = \mathbb{K}[x_1, \ldots, x_n] \) be a form. The following are equivalent:

(i) \( I = (x_1'', \ldots, x_m'') : \hat{\mathbf{f}} \) is a codimension \( n \) equigenerated Gorenstein ideal with linear resolution.

(ii) The integer \( s := n(m - 1) - \deg \hat{\mathbf{f}} \) is even and \( \text{rank } \mathcal{M}_{\deg \hat{\mathbf{f}}, s/2, m} = \binom{s/2 + n - 1}{n - 1} \).

(iii) The integer \( s := n(m - 1) - \deg \hat{\mathbf{f}} \) is even and \( P_\hat{\mathbf{f}} \) is a point in the Zariski open set \( \mathbb{P}^{\binom{e + n - 1}{n - 1}} \setminus V(I_k(\mathcal{M}_G_{e,s/2, m})) \), with \( k := \binom{s/2 + n - 1}{n - 1} \) and \( e = \deg \hat{\mathbf{f}} \).

**Proof** (i) \( \Rightarrow \) (ii) Suppose that \( I \) is equigenerated in degree \( d \). Since \( I \) has linear resolution then the socle degree of \( I \) is \( 2d - 2 \). Thus, by the Proposition 3.2, \( s = 2d - 2 \). In particular, \( s \) is an even integer. On the other hand, since \( I \) is generated in degree \( d \) then \( I_{s/2} = I_{d-1} = \{0\} \). Hence, by the Lemma 3.4, \( \text{rank } \mathcal{M}_{\deg \hat{\mathbf{f}}, s/2, m} = \binom{s/2 + n - 1}{n - 1} \).
(ii)$\Rightarrow$(i) We claim that $I$ is codimension $n$ Gorenstein ideal generated in degree $t = s/2 + 1$. The ideal $I$ is Gorenstein of codimension $n$ because it is the link of the homogeneous almost complete intersection $J = (x_1^m, \ldots, x_n^m, f)$ with respect to the complete intersection of pure powers $(x_1^m, \ldots, x_n^m)$. By Proposition 3.2, the socle degree of $I$ is $s$. Thus, $(R/I)_{2s-1} = (R/I)_{s+1} = \{0\}$. On the other hand, since rank $M_{\deg f, s/2, m} = \binom{s/2+n-1}{n-1}$, then $I_{s-1} = (R/I)_{s/2} = \{0\}$ by Lemma 3.4. Since $I$ is a codimension $n$ Gorenstein ideal and $(R/I)_{2n-1} = \{0\}$ and $I_{s-1} = 0$ it follows from [26, Proposition 1.8] that $I$ is generated in degree $t$ and has linear resolution.

Remark 3.6 The key point for proving the implication (i) $\Rightarrow$ (ii) is the use of [26, Proposition 1.8], which characterizes the $m$-primary Gorenstein ideals with linear resolution through estimates for the initial degree and the socle degree. For other classes of equigenerated Gorenstein ideals the examples show that a similar characterization must take into account not only the initial degree and the socle degree. For example, the ideals $(x^5, y^5, z^5) : (x+y+z)^5$ and $(x^5, y^5, z^5) : x^3y^2 + y^3z^2 + x^2z^3$ have the same initial degree and the same socle degree. However, the first ideal is equigenerated in degree 4 while the second is minimally generated in degree 4 and 5. Extending Theorem 3.5 to other ideals should include additional conditions.

The question remains as to when the Zariski open set $\mathbb{P}(s_{e+1})^{-1} \setminus V(I_k(\mathcal{M}_{e,s/2,m}))$ is nonempty, where $k := (s/2+n-1)$ and $e = \deg f$. The next result determines all pair of integers $m, e \geq 1$, with even $s = n(m-1) - e$, for this to be the case when $f = (x_1 + \cdots + x_n)^e$.

Proposition 3.7 (char$(k) = 0$) Let $m, e \geq 1$ integers such that $s = n(m-1) - e$ is even. Set $d := s/2 + 1$.

(i) If $m < d$ then $\mathbb{P}(s_{e+1})^{-1} \setminus V(I_k(\mathcal{M}_{e,s/2,m})) = \emptyset$.

(ii) If $m \geq d$ then $I = (x_1^m, \ldots, x_n^m) : (x_1 + \cdots + x_n)^e$ is a codimension $n$ Gorenstein ideal generated by forms of degree $d$ with linear resolution. In particular, $\mathbb{P}(s_{e+1})^{-1} \setminus V(I_k(\mathcal{M}_{e,s/2,m}))$ is a dense open set.

Proof (i) We claim that there is no form $f$ of degree $e$ such that $I = (x_1^m, \ldots, x_n^m) : f$ is an equigenerated codimension $n$ Gorenstein ideal with linear resolution. In fact, otherwise $I$ would be an ideal generated in degree $d$ with $(x_1^m, \ldots, x_n^m) \subset I$, which is absurd. Hence, by Theorem 3.5, $\mathbb{P}(s_{e+1})^{-1} \setminus V(I_k(\mathcal{M}_{e,s/2,m})) = \emptyset$.

(ii) We mimic the argument of [26, Proposition 7.24]. Namely, by applying [26, Proposition 1.8], it is sufficient to show that $R_{2d-1} \subset I$ and $I_{d-1} = \{0\}$. Clearly,

$$R_{n(m-1)+1} \subset (x_1^m, \ldots, x_n^m).$$

Moreover,

$$(x_1 + \cdots + x_n)^e R_{2d-1} \subset R_{2d-1+e} = R_{n(m-1)+1}.$$

Hence, $R_{2d-1} \subset I$. On the other hand, the initial degree of $I/(x_1^m, \ldots, x_n^m)$ is at least $d$ as a consequence of the Lefschetz like result of R. Stanley, as proved in [37, Theorem 5]. Since $m \geq d$ by assumption then the initial degree of $I$ is at least $d$. Therefore,
The pure power gap

Let \( \mathcal{E} = \{ \ell_1, \ldots, \ell_n \} \subset R = \mathbb{K}[x_1, \ldots, x_n] \) be a regular sequence of linear forms and let \( I \subset R \) be a homogeneous codimension \( n \) Gorenstein ideal with socle degree \( s \). Denote by \( m(I, \mathcal{E}) \) the least index \( m \) such that \( \{ \ell_1^m, \ldots, \ell_n^m \} \subset I \). Since \( R_{s+1} = I_{s+1} \), then \( m(I, \mathcal{E}) \leq s + 1 \). The pure power gap of \( I \) with respect to the regular sequence \( \mathcal{E} \) is \( g(I, \mathcal{E}) := s + 1 - m(I, \mathcal{E}) \). The absolute pure power gap of \( I \) (or simply, the pure power gap of \( I \)) is \( g(I) := s + 1 - \min_{\mathcal{E}} m(I, \mathcal{E}) \).

To start we have the following basic ring-theoretic result:

**Lemma 3.9** Let \( m_1, \ldots, m_n \geq 1 \) be integers and \( \mathfrak{f} \) a form in \( R = \mathbb{K}[x_1, \ldots, x_n] \). Then

\[
(e_1^{m_i}, \ldots, e_n^{m_i}): \mathfrak{f} = (e_1^{m_i+1}, \ldots, e_i^{m_i+1}, \ldots, e_n^{m_i}): e_i^{1} \mathfrak{f}
\]

for every \( 1 \leq i \leq n \). In particular,

\[
(e_1^{m_i}, \ldots, e_n^{m_i}): \mathfrak{f} = (e_1^{m_i+k}, \ldots, e_i^{m_i+k}, \ldots, e_n^{m_i+k}): (e_1 \cdots e_n)^k \mathfrak{f}
\]

for each \( k \geq 0 \).

**Proof** One can assume that \( i = 1 \). The inclusion \( (e_1^{m_1}, \ldots, e_n^{m_n}): \mathfrak{f} \subset (e_1^{m_1+1}, \ldots, e_n^{m_n}): e_1^{1} \mathfrak{f} \) is immediate. Thus, consider \( h \in (e_1^{m_1+1}, \ldots, e_n^{m_n}): e_1^{1} \mathfrak{f} \). Then,

\[
e_1^{1} \mathfrak{f} h = p_1 e_1^{m_1+1} + \cdots + p_n e_n^{m_n}
\]

for certain \( p_1, \ldots, p_n \in R \). In particular, \( e_1 \) divides \( p_2 e_2^{m_2} + \cdots + p_n e_n^{m_n} \). We can write

\[
p_i = e_1 q_i + r_i, \quad \text{for each } 2 \leq i \leq n,
\]

where \( r_2, \ldots, r_n \) are polynomials in \( \mathbb{K}[e_2, \ldots, e_n] \). Thus,

\[
p_2 e_2^{m_2} + \cdots + p_n e_n^{m_n} = q_2 e_1 e_2^{m_2} + \cdots + q_n e_1 e_n^{m_n} + r_2 e_2^{m_2} + \cdots + r_n e_n^{m_n}.
\]

Since \( e_1 \) divides \( p_2 e_2^{m_2} + \cdots + p_n e_n^{m_n} \) and \( r_2 e_2^{m_2} + \cdots + r_n e_n^{m_n} \in \mathbb{K}[e_2, \ldots, e_n] \) then

\[
p_2 e_2^{m_2} + \cdots + p_n e_n^{m_n} = q_2 e_1 e_2^{m_2} + \cdots + q_n e_1 e_n^{m_n}.
\]

Thus,

\[
\mathfrak{f} h = p_1 e_1^{m_1} + q_2 e_2^{m_2} + \cdots + q_n e_n^{m_n},
\]

that is, \( h \in (e_1^{m_1}, \ldots, e_n^{m_n}): \mathfrak{f} \). Therefore, \( (e_1^{m_1}, \ldots, e_n^{m_n}): \mathfrak{f} = (e_1^{m_1+1}, \ldots, e_n^{m_n}): e_1^{1} \mathfrak{f} \) as stated.

\[\square\]
To see an application, recall from Remark 3.3 that if $I$ is a homogeneous codimension $n$ Gorenstein ideal such that $(\ell_1^m, \ldots, \ell_n^m) : \mathfrak{f} = I$, where $\ell_1, \ldots, \ell_n$ are linear forms, then $\mathfrak{f}$ is uniquely determined, up to a scalar coefficient, by the condition that no nonzero term of $\mathfrak{f}$, written as a polynomial in $\ell_1, \ldots, \ell_n$, belongs to the ideal $(\ell_1^m, \ldots, \ell_n^m)$.

**Proposition 3.10** Let $I \subset R$ be a homogeneous codimension $n$ Gorenstein ideal with socle degree $s$. Suppose that as above, $\ell_1, \ldots, \ell_n$ are linear forms such that $I = (\ell_1^{s+1}, \ldots, \ell_n^{s+1}) : \mathfrak{f}$ with $\mathfrak{f}$ uniquely determined, up to a scalar coefficient, by the condition that no nonzero term of $\mathfrak{f}$ belongs to the ideal $(\ell_1^{s+1}, \ldots, \ell_n^{s+1})$. Then, $g(I, \mathfrak{e})$ is the largest index such that $(\ell_1 \cdots \ell_n)^g(I, \mathfrak{e})$ divides $\mathfrak{f}$.

**Proof** Denote $m_0 := m(I, \mathfrak{e})$ and $g := g(I, \mathfrak{e})$. Then $(\ell_1^{m_0}, \ldots, \ell_n^{m_0}) : I$ is an almost complete intersection $J = (\ell_1^{m_0}, \ldots, \ell_n^{m_0}, \mathfrak{f}_0)$, for some form $\mathfrak{f}_0 \in R$. Since $R = \mathbb{k}[\ell_1, \ldots, \ell_n]$, we can write $\mathfrak{f}_0$ as a polynomial in these linear forms and get rid of the terms belonging to the ideal $(\ell_1^{m_0}, \ldots, \ell_n^{m_0})$. This way, the latter is part of a minimal set of generators of $J$. Therefore, $(\ell_1^{m_0}, \ldots, \ell_n^{m_0}) : J = (\ell_1^{m_0}, \ldots, \ell_n^{m_0}) : \mathfrak{f}_0$ is Gorenstein and $I = (\ell_1^{m_0}, \ldots, \ell_n^{m_0}) : \mathfrak{f}_0$.

By Lemma 3.9 one has

$$I = (\ell_1^{m_0}, \ldots, \ell_n^{m_0}) : \mathfrak{f}_0 = (\ell_1^{m_0+g}, \ldots, \ell_n^{m_0+g}) = (\ell_1 \cdots \ell_n)^g \mathfrak{f}_0 = (\ell_1^{s+1}, \ldots, \ell_n^{s+1}) : (\ell_1 \cdots \ell_n)^g \mathfrak{f}_0. \tag{16}$$

Consider

$$\mathfrak{f}_0 = \sum_{|\alpha| = \deg \mathfrak{f}_0} a_\alpha \ell_1^{a_1} \cdots \ell_n^{a_n}.$$

Then,

$$(\ell_1 \cdots \ell_n)^g \mathfrak{f}_0 = \sum_{|\alpha| = \deg \mathfrak{f}_0} a_\alpha \ell_1^{a_1+g} \cdots \ell_n^{a_n+g}.$$

For each nonzero $a_\alpha$, one has $a_\alpha \leq m_0 - 1$ for each $1 \leq i \leq n$. Hence, $a_\alpha + g \leq m_0 + g - 1 = s$ for each $1 \leq i \leq n$. Thus, no nonzero term of $(\ell_1 \cdots \ell_n)^g \mathfrak{f}_0$ belongs to the ideal $(\ell_1^{s+1}, \ldots, \ell_n^{s+1})$. Then, since $\mathfrak{f}$ is uniquely determined, up to a scalar coefficient, by $I = (\ell_1^{s+1}, \ldots, \ell_n^{s+1}) : \mathfrak{f}$ and the condition that no nonzero term of $\mathfrak{f}$ belongs to the ideal $(\ell_1^{s+1}, \ldots, \ell_n^{s+1})$, one has $\mathfrak{f} = \lambda(\ell_1 \cdots \ell_n)^g \mathfrak{f}_0$ for some nonzero $\lambda \in \mathbb{k}$. Hence, $(\ell_1 \cdots \ell_n)^g$ divides $\mathfrak{f}$.

Finally, we assert that $g$ is the largest index with this property, a claim that is obvious if $m_0 = 1$, because in this case $\mathfrak{f}_0$ is a nonzero scalar. Thus, suppose $m_0 \geq 2$. If $g$ is not the largest index such that $(\ell_1 \cdots \ell_n)^g$ divides $\mathfrak{f}$, then $(\ell_1 \cdots \ell_n)^{g'}$ divides $\mathfrak{f}_0$. Hence, by Lemma 3.9,

$$I = (\ell_1^{m_0-1}, \ldots, \ell_n^{m_0-1}) : \mathfrak{f}_0 = (\ell_1^{m_0-1}, \ldots, \ell_n^{m_0-1}) : \mathfrak{f}_0,$$

so, $(\ell_1^{m_0-1}, \ldots, \ell_n^{m_0-1}) \subset I$, contradicting that $m_0$ is least such that $(\ell_1^{m_0}, \ldots, \ell_n^{m_0}) \subset I$. \[\Box\]
5 The associated Rees algebra

5.1 Equigenerated ideals of finite colength

Let \( I \subset R = \mathbb{k}[x,y,z] \) be an equigenerated ideal of finite colength. In this section we focus on the Rees algebra \( \mathcal{R}(I) \cong R[It] \subset R[t] \), the associated graded ring \( \text{gr}_t(R) = \mathcal{R}(I)/I\mathcal{R}(I) \) and the fiber cone algebra \( \mathcal{F}(I) = \mathcal{R}(I)/m\mathcal{R}(I) \), where \( m := (x,y,z) \). The eventual goal is an application to the case where \( I \) is Gorenstein. The nature of the associated graded ring for Artinian Gorenstein rings in any dimension has been considered earlier by Iarrobino [22].

Note that the so-called condition \( G_3 \) is automatic since the ideal is \( m \)-primary. Some features in this section might have appeared elsewhere coming from a different angle. Yet, it may be useful to have elementary proofs of the results below, where \( G_3 \) is not directly used.

**Proposition 4.1** Let \( R = \mathbb{k}[x_1, \ldots, x_n] \) be a standard graded polynomial ring over a field and let \( m \) be its maximal homogeneous ideal. Let \( I \) be a homogeneous \( d \)-equigenerated \( m \)-primary ideal.

(a) If \( I^{m_0} = \mathfrak{m}^{md_0} \) for some \( m_0 \geq 1 \), then \( I^m = \mathfrak{m}^{md} \) for \( m \geq m_0 \).

(b) Let \( m = m_0 \) be minimal possible in (a). Then the reduction number of \( I \) is at most \( \max\{m_0, r(\mathfrak{m}^d)\} \), where \( r(\mathfrak{m}^d) \) denotes the reduction number of \( \mathfrak{m}^d \).

(c) The (regular) rational map \( \mathfrak{F} : \mathbb{P}^{n-1} \to \mathbb{P}^{d(I)-1} \) defined by a set of forms spanning \( [I]_d \) is birational onto the image.

(d) The Rees algebra \( \mathcal{R}(I) \) satisfies the condition \( R_1 \) of Serre.

(e) \( \text{depth} \text{gr}_t(R) = 0 \).

**Proof** (a) Now, \( I^{m_0} \subset \mathfrak{m}^d I^{m_0-1} \subset \mathfrak{m}^{md} \mathfrak{m}^{(m_0-1)d} = I^{m_0} \), hence, \( I^{m_0} = \mathfrak{m}^d I^{m_0-1} \). Then, for \( m \geq m_0 \),

\[
I^{m+1} = I^{m+1-m_0} I^{m_0} = I^{m+1-m_0} \mathfrak{m}^d I^{m_0-1} = I^m \mathfrak{m}^d = \mathfrak{m}^{md} \mathfrak{m}^{(m+1)d},
\]

and so on.

(b) Let \( J \subset I \) be a homogeneous minimal reduction. Since \( \mathfrak{m}^d \) is the integral closure of \( I \), then \( J \) is also a minimal reduction of \( \mathfrak{m}^d \). But, as is well-known, the latter has reduction number at most \( n-1 \) for any minimal reduction (see, e.g., [9, Corollary 7.12]). Setting \( N = \max\{m_0, r(\mathfrak{m}^d)\} \), one has:

\[
I^{N+1} = (\mathfrak{m}^d)^{N+1} \quad \text{by (a)} \\
= J(\mathfrak{m}^d)^N \quad \text{because } J \text{ is a minimal reduction of } \mathfrak{m}^d \\
= J^N \quad \text{by (a)}.
\]

(c) By (a), the Hilbert polynomial \( HP(\mathcal{F}(I), \mathfrak{m}) \) of the fiber cone \( \mathcal{F}(I) \) is

\[
HP(\mathcal{F}(I), \mathfrak{m}) = \binom{md + n - 1}{n - 1} = \frac{d^{n-1}}{(n-1)!} \mathfrak{m}^{n-1} + \text{lower degree terms of } \mathfrak{m}.
\]

Hence, the multiplicity \( e(\mathcal{F}(I)) \) of \( \mathcal{F}(I) \) is \( d^{n-1} \). On the other hand, by [39, Theorem 6.6 (a)] the degree \( \deg(\mathfrak{F}) \) of the rational map \( \mathfrak{F} \) is
\[
\deg(\mathcal{F}) = \frac{d^{n-1}}{e(\mathcal{F}(I))}.
\]

Thus, \(\deg(\mathcal{F}) = 1\), as asserted.

(d) Consider the Hilbert-Samuel polynomial \((m \gg 0)\)

\[
\lambda(R/I^{m+1}) = e_0(I)\binom{n+m}{n} - e_1(I)\binom{n+m-1}{n-1} + \text{lower degree terms of } m
\]

and the Hilbert polynomial

\[
\lambda(R/\overline{I}^{m+1}) = \overline{e}_0(I)\binom{n+m}{n} - \overline{e}_1(I)\binom{n+m-1}{n-1} + \text{lower degree terms of } m
\]

where \(\overline{I}^{m+1}\) denotes the integral closure of \(I^{m+1}\). By (a), \(I^m = \overline{I}^m\) for every \(m \geq m_0\). Thus, in particular, \(e_1(I) = \overline{e}_1(I)\). Hence, by [18, Proposition 3.2], \(R(I)\) satisfies the condition \(R_1\) of Serre.

(e) By (a), one has an exact sequence

\[
0 \to R(I) \to R(m^d) \to C \to 0,
\]

with \(C\) a module of finite length. In particular, depth \(C = 0\). Since \(R(m^d)\) is Cohen–Macaulay, then depth \(R(I) = \text{depth } C + 1 = 1\).

Now, clearly \(\text{depth } \text{gr}_I(R) \leq \text{depth } R(I) = 1\). Supposing that \(\text{depth } \text{gr}_I(R) > 0\), let \(a \in I \setminus I^2\) be such that its image in \(I/I^2 \subseteq \text{gr}_I(R)\) is a regular element. Then one has an exact sequence

\[
0 \to \text{gr}_I(R)(-1) \to R(I)/aR(I) \to R_{R/(a)}(I/(a)) \to 0
\]

(see [41, Proposition 5.1.11]). Since \(a\) is regular on \(R(I)\) then the middle term has depth zero, while the rightmost term – being a Rees algebra over a Cohen–Macaulay ring of dimension \(\geq 1\) – has depth \(\geq 1\). It follows that \(\text{gr}_I(R) \simeq \text{gr}_I(R)(-1)\) has depth zero; a contradiction. \(\square\)

5.2 Syzygetic ideals

For the main result in this part recall the notion of a syzygetic ideal \(I \subset R\) as being one such that the natural surjection \(S^*_R(I) \to R_R(I)\) is an isomorphism in degree \(\leq 2\), where \(S^*_R(I)\) is the symmetric algebra of \(I\). In particular, for such an ideal, \(I^2\) coincides with the second symmetric power of \(I\), hence the minimal number of generators of \(I^2\) is given by \(\binom{\mu(I)+1}{2}\), where \(\mu(I)\) stands for the minimal number of generators of \(I\).

In the ternary case we can bring over the fiber cone.

**Theorem 4.2** Let \(I \subset R = \mathbb{k}[x, y, z]\) denote a codimension 3 homogeneous \(d\)-equigenerated Gorenstein ideal minimally generated by \(2d + 1\) forms. Assume that \(d \geq 2\).

(a) \(I^m = m^{nd}\) for every \(m \geq 2\).
(b) The reduction number of \(I\) is 2.
(c) The (regular) rational map \( \mathcal{F} : \mathbb{P}^2 \rightarrow \mathbb{P}^{2d} \) defined by a set of forms spanning \( |I|_d \) is birational onto the image.

(d) The Rees algebra \( \mathcal{R}(I) \) satisfies the condition \( R_1 \) of Serre.

(e) \( \text{depth gr}_I(R) = 0 \).

(f) The fiber cone \( \mathcal{F}(I) \) is not Cohen-Macaulay.

Recall that \( I \) is syzygetic (see [16, Proposition 2.10] for the case where \( \text{char}(\Bbbk) \neq 2 \) and [35, Theorem 1] for any \( \Bbbk \)).

**Proof** (a) Since \( I \) is syzygetic then
\[
\mu(I^2) = \left( \frac{\mu(I) + 1}{2} \right) = \left( \frac{2d + 2}{2} \right) = \mu(m^{2d}).
\]
Thus, \( I^2 \subset m^{2d} \) is an inclusion of homogeneous ideals in the same degree \( 2d \), having the same minimal number of homogeneous generators. Hence, \( I^2 = m^{2d} \) and, by Proposition 4.1, \( I^m = m^{md} \) for every \( m \geq 2 \).

Items (c) through (e) are follow directly from the analogous statements in Proposition 4.1. It remains to deal with (b) and (f).

(b) By (a) and Proposition 4.1, one has \( r(I) \leq 2 \). On the other hand, since \( I \) is syzygetic one has \( 2 \leq r(I) \). Hence, \( r(I) = 2 \).

(f) Suppose that the fiber cone is Cohen–Macaulay. Then the reduction number \( r(I) \) is the Castelnuovo-Mumford regularity \( \text{reg}(\mathcal{F}(I)) \) of \( \mathcal{F}(I) \) (see, e.g., [14, Proposition 1.2]). By (b), the latter is 2. But since \( I \) is syzygetic, the defining ideal of \( \mathcal{F}(I) \) over \( S := \Bbbk[T_1, \ldots, T_{2d+1}] \) admits no forms of degree 2, hence is generated in the single degree 3. Together these imply that the minimal graded free resolution of \( \mathcal{F}(I) \) over \( S \) is linear. Moreover, the length of the resolution is \( 2d + 1 - e(I) = 2d - 2 \). Again, since \( \mathcal{F}(I) \) is Cohen–Macaulay, by [20, Theorem 1.2] the multiplicity of the fiber cone \( \mathcal{F}(I) \) is
\[
eq \left( \frac{e(\mathcal{F}(I))}{2d - 2} \right) = \left( \frac{2d^2}{2} \right) = d(2d - 1).
\]
Now consider the rational map \( \mathcal{F} : \mathbb{P}^2 \rightarrow \mathbb{P}^{2d} \) defined by the given generators of \( I \) in degree \( d \), and let \( \text{deg}(\mathcal{F}) \) denote the degree of \( \mathcal{F} \). Since \( I \) is equigenerated then \( \mathcal{F}(I) \) is isomorphic to the \( \Bbbk \)-subalgebra \( \Bbbk[|I|_d] \subset R \), while the latter is up to degree normalization the homogeneous defining ideal of the image of \( \mathcal{F} \). Then, since \( \mathcal{F} \) is birational, by [39, Theorem 6.6 (a)] one has \( e(\mathcal{F}(I)) = d^2 \), i.e., \( 2d - 1 = d \), which is absurd for \( d \geq 2 \).

\[ \square \]

5.3 Application to the Gorenstein case

**Corollary 4.3** (char(\( \Bbbk \)) \( \neq 2 \)) Let \( I \) denote a codimension 3 homogeneous Gorenstein ideal in \( \Bbbk[x, y, z] \) with datum \( (d, 2d + 1) \), where \( d \geq 2 \). Then all assertions of Theorem 4.2 hold true.

**Proof** Since char(\( \Bbbk \)) \( \neq 2 \), then \( I \) is syzygetic [16, Proposition 2.8]. \( \square \)

For the non-linear case, we have the following:
Proposition 4.4 Let $I \subset R = \mathbb{k}[x, y, z]$ be a codimension 3 Gorenstein ideal with datum $(d, r)$ and skew-degree $d'$. Let $\mathcal{F} : \mathbb{P}^2 \to \mathbb{P}^{r-1}$ be the rational map defined by the linear system $|I|_d$. If the reduction number of $I$ is at most 2 and $\mathcal{F}(I)$ is Cohen-Macaulay then:

(a) $(r - 2)$ divides $d'^2$.
(b) If $r \geq 5$ then $\mathcal{F}$ is not birational onto the image.

Proof (a) Since $I$ is syzygetic, the assumption implies that the reduction number of $I$ is exactly 2. Since $\mathcal{F}(I)$ is Cohen-Macaulay then the same argument as in the proof of Theorem 4.2 (g) yields $e(\mathcal{F}(I)) = \binom{r-1}{2}$. Again, by [39, Theorem 6.6], $\binom{r-1}{2} \deg(\mathcal{F}) = d'^2$. By definition, $d = (r - 1)d'/2$, hence

$$2(r - 2) \deg(\mathcal{F}) = (r - 1)d'^2.$$  \hfill (17)

Since gcd{$(r - 2), (r - 1)$} = 1 then $(r - 2)$ divides $d'^2$, as desired.

(b) Since $(r - 1)/2 > 1$ then (17) forces $\deg(\mathcal{F}) > 1$. Hence, $\mathcal{F}$ is not birational. \hfill $\square$

Acknowledgements Upon posting a first version of this work on the arXiv, it has been brought to our knowledge by A. Iarrobino that some of our results have been considered before. We thank him for pointing out the missing references, which are now included. Since our approach has often different features from the previously existent literature, we decided to keep it as a possible new angle of consideration. We also thank R. Fröberg for useful conversations around his celebrated conjecture and related issues, where he pointed out some inaccuracies.

References

1. Anick, D.J.: Thin algebras of embedding dimension three. J. Algebra 100, 235–259 (1986)
2. Buchsbaum, D., Eisenbud, D.: Remarks on ideals and resolutions. Istituto Nazionale di Alta Matematica, Symposia Mathematica, Volume XI, Bologna (1973)
3. Buchsbaum, D., Eisenbud, D.: Algebraic structures for finite free resolutions, and some structure theorems for ideals of codimension 3. Am. J. Math. 99, 447–485 (1977)
4. Busè, L., Chardin, M., Simis, A.: Elimination and nonlinear equations of Rees algebras. J. Algebra 324, 1314–1333 (2010)
5. Conca, A., Valla, G.: Betti numbers and liftings of Gorenstein codimension three ideals. Commun. Algebra 28, 1371–1386 (2000)
6. Costa, B., Simis, A.: New constructions of Cremona maps. Math. Res. Lett. 20, 629–645 (2013)
7. Diesel, S.J.: Irreducibility and dimension theorems for families of height 3 Gorenstein algebras. Pac. J. Math. 172, 365–397 (1996)
8. Doria, A., Simis, A.: The Newton complementary dual revisited. J. Algebra Appl. 17, 1850004-1–16 (2018)
9. Eisenbud, D., Huneke, C., Ulrich, B.: The regularity of Tor and graded Betti numbers. Am. J. Math. 128, 573–605 (2006)
10. Elias, J.: Singular library for computing Macaulay’s inverse systems. arXiv: 1501.01786v1 [math.AC] 8 Jan 2015
11. Elias, J., Rossi, M.E.: The structure of the inverse system of Gorenstein $\mathfrak{k}$-algebras. Adv. Math. 314, 306–327 (2017)
12. Fröberg, R.: An inequality for Hilbert series of graded algebras. Math. Scand. 56, 117–144 (1985)
13. Fröberg, R., Lundqvist, S.: Questions and conjectures on extremal Hilbert series. Revista de la Unión Matemática Argentina 59, 415–429 (2018)
14. Garrousian, M., Simis, A., Tohaneanu, S.O.: A blowup algebra for hyperplane arrangements. Algebra Number Theory 12, 1401–1429 (2018)
15. Harima, T.: A note on Artinian Gorenstein algebras of codimension three. J. Pure Appl. Algebra 135, 45–56 (1999)
16. Herzog, J., Simis, A., Vasconcelos, W.V.: Koszul homology and blowing-up rings. In: Proceedings of the Trento Conference in Communication Algebra, Lecture Notes in Pure and Applied Mathematics, vol. 84, Marcel Dekker, New York, pp. 79–169 (1983)
17. Hochster, M., Laksov, D.: The linear syzygies of generic forms. Commun. Algebra 15, 227–239 (1987)
18. Hong, J., Simis, A., Vasconcelos, W.V.: On the equations of almost complete intersections. Bull. Braz. Math. Soc. 43, 171–199 (2012)
19. Hong, J., Simis, A., Vasconcelos, W.V.: Ideals generated by quadrics. J. Algebra 423, 177–189 (2015)
20. Huneke, C., Miller, M.: A note on the multiplicity of Cohen-Macaulay algebras with pure resolutions. Can. J. Math. 37, 1149–1162 (1985)
21. Iarrobino, A., Emsalem, J.: Some zero-dimensional generic singularities; finite algebras having small tangent space. Compos. Math. 36, 145–188 (1978)
22. Iarrobino, A.: Associated graded algebra of a Gorenstein Artin algebra. Mem. Amer. Math. Soc. 107, No. 514, Amer. Math. Soc. Providence (1994)
23. Iarrobino, A., Kanev, V.: Power sums, Gorenstein algebras, and determinantal varieties. Appendix by A. Iarrobino and Steven L. Kleiman The Gotzmann Theorems and the Hilbert scheme, Lecture Notes in Mathematics, 1721. Springer, Berlin (1999). xxxii+345 pp. ISBN: 3-540-66766-0
24. Iarrobino, A., Srinivasan, H.: Some Gorenstein Artin algebras of embedding dimension four: components of $PGOR(H)$ for $H = (1; 4; 7; \ldots ; 1)$. J. Pure Appl. Algebra 201, 62–96 (2005)
25. Jelisiejew, J.: Classifying local Artinian Gorenstein algebras. Collect. Math. 68, 101–127 (2017)
26. El Khouri, S., Kustin, A.: Artinian Gorenstein algebras with linear resolutions. J. Algebra 420, 402–474 (2014)
27. Kleppe, J.O.: Maximal families of Gorenstein algebras. Trans. Am. Math. Soc. 358, 3133–3167 (2006)
28. Kustin, A.R., Polini, C., Ulrich, B.: The equations defining blowup algebras of height three Gorenstein ideals. Algebra Number theory 11, 1489–1525 (2017)
29. Kustin, A., Ulrich, B.: If the socle fits. J. Algebra 147, 63–80 (1992)
30. Lira, D.S.: Equigenerated Gorenstein ideals of codimension 3. With a chapter on general forms, Ph.D. Thesis, Universidade Federal da Paraíba, João Pessoa, Brazil (2022)
31. Macaulay, F.: The Algebraic Theory of Modular Systems, Reissued with an Introduction by P. Roberts in 1994. Cambridge University Press, Cambridge (1916)
32. Migliore, J., Miró-Roig, R.M.: On the minimal free resolution of $n + 1$ general forms. Trans. Am. Math. Soc. 355, 1–36 (2003)
33. Muir, T.: A Treatise on the Theory of Determinants. Macmillan and Co., London (1882)
34. Nenashev, G.: A note on Fröberg’s conjecture for forms of equal degrees. C. R. Math. Acad. Sci. Paris 355(3), 272–276 (2017)
35. Planas-Vilanova, F.: Regular Local Rings of Dimension Four and Gorenstein syzygetic prime ideals. J. Algebra 601, 105–114 (2022)
36. Ragusa, A., Zappalà, G.: Properties of 3-codimensional Gorenstein schemes. Commun. Algebra 29, 303–318 (2001)
37. Reid, L., Roberts, L.G., Roitman, M.: On complete intersections and their Hilbert functions. Can. Math. Bull. 34, 525–535 (1991)
38. Miró-Roig, R. M., Hoa Tran, Q.: The weak Lefschetz property for Artinian Gorenstein algebras of codimension three. J. Pure Appl. Algebra 224, 106305 (2020)
39. Simis, A., Ulrich, B., Vasconcelos, W.V.: Codimension, multiplicity and integral extensions. Math. Proc. Camb. Philos. Soc. 140, 237–257 (2001)
40. Stanley, R.: Weyl groups, the hard Lefschetz theorem, and the Sperner property. SIAM J. Algebr. Discrete Methods 1, 168–184 (1980)
41. Vasconcelos, W.V.: Arithmetic of Blowup Algebras, London Math. Soc., Lecture Note Series 195. Cambridge University Press, Cambridge (1994)
42. Watanabe, J.: A note on Gorenstein rings of embedding dimension three. Nagoya Math. J. 50, 227–232 (1973)
43. Watanabe, J.: The Dilworth Number of Artinian Rings and Finite Posets with Rank Function. Commutative Algebra and Combinatorics, Advanced Studies in Pure Mathematics, vol. 11, pp. 303–312. Kinokuniya Co., North Holland, Amsterdam (1987)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
Authors and Affiliations

Dayane Lira¹ · Zaqueu Ramos² · Aron Simis³

Zaqueu Ramos
zaqueu@mat.ufs.br

Aron Simis
aron@dmat.ufpe.br

¹ Departamento de Matemática, CCEN, Universidade Federal da Paraíba, João Pessoa, PB 58051-900, Brazil

² Departamento de Matemática, CCET, Universidade Federal de Sergipe, São Cristovão, SE 49100-000, Brazil

³ Departamento de Matemática, CCEN, Universidade Federal de Pernambuco, Recife, PE 50740-560, Brazil