We present a coordinate-free approach for constructing approximate first integrals of generalized slow-fast Hamiltonian systems, based on the global averaging method on parameter-dependent phase spaces with $S^1$-symmetry. Explicit global formulas for approximate second-order first integrals are derived. As examples, we analyze the case quadratic in the fast variables (in particular, the elastic pendulum), and the charged particle in a slowly-varying magnetic field.

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Adiabatic invariants of slow-fast systems

I. INTRODUCTION

Slow-fast Hamiltonian systems appear in the theory of adiabatic approximation. They are represented by equations of motion of the form

\[ \dot{y} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial y}, \]
\[ \dot{p} = -\varepsilon \frac{\partial H}{\partial q}, \quad \dot{q} = \varepsilon \frac{\partial H}{\partial p}, \]

where \( \varepsilon \) is a small perturbation parameter, and \((y, x) \in \mathbb{R}^{2r}, (p, q) \in \mathbb{R}^{2k}\) are said to be fast and slow variables, respectively. This system is Hamiltonian relative to the function \( H = H(p, q, y, x) \) and the rescaled canonical Poisson bracket on the product space \( \mathbb{R}^{2r} \times \mathbb{R}^{2k} \):

\[ \{f, g\} = \left( \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} \right) + \varepsilon \left( \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} \right). \]

Notice that the corresponding symplectic form

\[ \sigma = dy \wedge dx + \frac{1}{\varepsilon} dp \wedge dq, \]

has a singularity at \( \varepsilon = 0 \).

Usually, systems like (I.1) appear as a result of applying a scaling argument to an \( \varepsilon \)-dependent Hamiltonian on the standard phase space \( (\mathbb{R}^{2r+2k}, dY \wedge dX + dP \wedge dQ) \). Two common situations arise:

1. **Slowly varying Hamiltonians.** In this case the original Hamiltonian has the form

\[ H(\varepsilon^k P, \varepsilon^{1-k} Q, Y, X), \quad (0 \leq k \leq 1), \]

and the rescaling is implemented through the equations

\[ p = \varepsilon^k P, \quad q = \varepsilon^{1-k} Q, \quad y = Y, \quad x = X, \]

which lead to

\[ H(\varepsilon^k P, \varepsilon^{1-k} Q, Y, X) = H(p, q, y, x). \]

2. **Rapidly varying Hamiltonians.** Now the original Hamiltonian is

\[ H(P, Q, \frac{Y}{\varepsilon^k}, \frac{X}{\varepsilon^{1-k}}), \quad (0 \leq k \leq 1), \]
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and the rescaling
\[ p = P, \quad q = Q, \quad y = \frac{Y}{\varepsilon^k}, \quad x = \frac{X}{\varepsilon^{1-k}}, \]

leads to another one in the new variables:
\[ H(P, Q, Y, X) = \varepsilon H(p, q, y, x). \]

In both cases, the dynamics is described by (I.1).

These standard adiabatic models can be generalized in the following way\textsuperscript{20,28,29}. Suppose we start with two symplectic manifolds \((M_0, \sigma_0)\) and \((M_1, \sigma_1)\). Consider the product manifold \(M = M_0 \times M_1\) endowed with canonical projections \(\pi_0 : M \to M_0\) and \(\pi_1 : M \to M_1\), and an \(\varepsilon\)–dependent symplectic form
\[ \sigma = \pi_0^*\sigma_0 + \frac{1}{\varepsilon}\pi_1^*\sigma_1, \tag{I.2} \]
for \(\varepsilon \neq 0\). The Poisson bracket determined by \(\sigma\) is
\[ \{., .\} = \{., .\}_0 + \varepsilon\{., .\}_1, \tag{I.3} \]
where \(\{f, g\}_0 = \Psi_0(df, dg)\) and \(\{f, g\}_1 = \Psi_1(df, dg)\) are the Poisson brackets on \(M\) defined as the canonical lifts of the non-degenerate Poisson brackets on the symplectic factors \((M_0, \sigma_0)\) and \((M_1, \sigma_1)\), respectively. The Poisson bivector fields \(\Psi_0\) and \(\Psi_1\) on \(M\) are degenerate, as \(\text{rank } \Psi_0 = \text{dim } M_0\) and \(\text{rank } \Psi_1 = \text{dim } M_1\), and the corresponding symplectic leaves are given by the slices \(M_0 \times \{m_1\}\) and \(\{m_0\} \times M_1\). If \((p, q, y, x)\) denote a local coordinate system on \(M\) associated to the Darboux coordinates \((p, q)\) on \((M_1, \sigma_1)\) and \((y, x)\) on \((M_0, \sigma_0)\), then:
\[ \Psi_0 = \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}, \quad \text{and} \quad \Psi_1 = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}. \]
The Poisson brackets \(\{., .\}_0\) and \(\{., .\}_1\) will be called fast and slow, respectively.

From the viewpoint of deformation theory, the adiabatic-type Poisson bracket (I.3) is nontrivial in the following sense. The bivector field determined by (I.3) reads \(\Psi = \Psi_0 + \varepsilon\Psi_1\), so \(\Psi_1\) can be viewed as an infinitesimal deformation of \(\Psi_0\). One can show that this deformation is nontrivial, that is, the Poisson cohomology class of the 2-cocycle \(\Psi_1\) is nonzero.

For every function \(H \in C^\infty(M)\), denote by \(X_H\) the Hamiltonian vector field of \(H\) relative to \(\sigma\), \(i_{X_H}\sigma = -dH\). Then, \(X_H = X_H^{(0)} + \varepsilon X_H^{(1)}\), where \(X_H^{(0)}\) and \(X_H^{(1)}\) are the Hamiltonian
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vector fields of $H$ with respect to $\{.,0\}$ and $\{.,1\}$, respectively: $X^{(0)}_H = i_{dH}\Psi_0$ and $X^{(1)}_H = i_{dH}\Psi_1$. Also, denote by $d_0$ and $d_1$ the partial exterior derivatives on $M$ along $M_0$ and $M_1$, respectively. It is clear that $d = d_0 + d_1$ is the exterior derivative on $M$ and $d_0^2 = 0 = d_1^2$. Also, $d_0 \circ d_1 + d_1 \circ d_0 = 0$. Then, the relations $X^{(0)}_H = i_{d_0H}\Psi_0$, $X^{(1)}_H = i_{d_1H}\Psi_1$ hold.

By a generalized slow-fast Hamiltonian system we mean a Hamiltonian system of the form:

$$(M = M_0 \times M_1, \{.,\} = \{.,\}_0 + \varepsilon \{.,\}_1, H).$$

(I.4)

The corresponding Hamiltonian vector field $X_H = X^{(0)}_H + \varepsilon X^{(1)}_H$ gives rise to a perturbed dynamics, where the unperturbed vector field $X^{(0)}_H$ and the perturbation vector field $X^{(1)}_H$ are Hamiltonian with respect to different Poisson structures. As for any perturbed Hamiltonian system, it does make sense to search for adiabatic invariants, which are related to approximate first integrals of (I.4) as $\varepsilon \to 0$. We are interested in the existence of (additional to the Hamiltonian $H$) approximate first integrals of (I.4) for $\varepsilon \ll 1$, in the case when the flow of $X^{(0)}_H$ is periodic.

A $C^\infty$-function on $M$,

$$F = F_0 + \varepsilon F_1 + \frac{\varepsilon^2}{2} F_2 + ... + \frac{\varepsilon^k}{k!} F_k + O(\varepsilon^{k+1}),$$

smoothly depending on the small parameter $\varepsilon \ll 1$, is said to be an approximate first integral of order $k \geq 0$ for $X_H = X^{(0)}_H + \varepsilon X^{(1)}_H$ if

$$\mathcal{L}_{X_H} F = O(\varepsilon^{k+1}).$$

An approximate first integral $F$ of order $k$ is an adiabatic invariant of order $k$ of the system (I.4), in the sense that

$$| F \circ F^{t}_{X_H}(p) - F(p) | = O(\varepsilon^k),$$

for a long time scale: $t \sim \frac{1}{\varepsilon}$ as $\varepsilon \to 0$ (here $F^{t}_{X_H}$ denotes the flow of the vector field $X_H$). For $k \geq 1$, the leading term $F_0$ of $F$ must be a common first integral of $X^{(0)}_H$ and the averaged perturbation vector field, and it is usually defined as the standard action (the classical adiabatic invariant) along the periodic trajectories of $X^{(0)}_H$ [3,13]. Thus, the problem is to find the further corrections $F_1, F_2, \ldots$. Under appropriate hypotheses for the unperturbed (fast) dynamics, we show that a slow-fast Hamiltonian system (I.4) admits an approximate first integral of arbitrary order, and derive global coordinate-free formulas for the first and
second order corrections $F_1$ and $F_2$. Our approach is based on the averaging method on
general phase spaces with $\mathbb{S}^1$—symmetry\textsuperscript{28,29} and results obtained elsewhere\textsuperscript{16,22,23}. The main idea is to apply a global $\mathbb{S}^1$—normalization to (I.4) in two stages. The first step is related to the $\mathbb{S}^1$—averaging of the original $\varepsilon$—dependent symplectic form (the Poisson bracket) and then, in the second step, to apply the canonical normalization to the deformed Hamiltonian on the “new” phase space with $\mathbb{S}^1$—symmetry. This allows us to avoid the traditional assumption on the existence of action-angle variables, and to work on domains where the $\mathbb{S}^1$—action is not necessarily free and trivial. We illustrate these results with a two cases of physical interest: Hamiltonians quadratic in the fast variables (with the elastic pendulum as a particular example), and the charged particle in a slowly varying magnetic field.

The paper is organized as follows. In the next section we state our hypotheses and main result. Then, in section III we show how to construct the desired approximate first integrals when the perturbed system $X_H = X_H^{(0)} + \varepsilon X_H^{(1)}$ is in normal form relative to the $\mathbb{S}^1$—action on $M$ induced by $X_H^{(0)}$. To achieve this, a set of homological equations must be solved. In section IV we prove that generalized slow-fast system satisfying our hypotheses also satisfy the conditions required to solve these homological equations. Section V is devoted to the proof of the main theorem, and the remaining sections analyze the examples.

II. HIGHER ORDER CORRECTIONS TO ADIABATIC INVARIANTS

Hypothesis 1 (Symmetry Hypothesis). We will assume that the flow of the unperturbed Hamiltonian vector field $X_H^{(0)}$, $\text{Fl}_{X_H^{(0)}}^t$, is periodic on $M$ with frequency function $\omega \in C^\infty(M)$, $\omega > 0$.

That means that $\text{Fl}_{X_H^{(0)}}^{t+T(m)} = \text{Fl}_{X_H^{(0)}}^t(m)$ for all $t \in \mathbb{R}$ and $m \in M$. Here $T = \frac{2\pi}{\omega}$ is the period function. Then, the flow of the vector field $\Upsilon := \frac{1}{\omega} X_H^{(0)}$ is $2\pi$—periodic and hence $\Upsilon$ is an infinitesimal generator of the $\mathbb{S}^1$—action on $M$. Let us associate to that $\mathbb{S}^1$—action the following operators acting on the space $\mathcal{T}_s^l(M)$ of all tensor fields on $M$ of type $(s, l)$: the averaging operator $\langle \cdot \rangle : \mathcal{T}_s^l(M) \to \mathcal{T}_s^l(M)$,

$$\langle A \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_{\Upsilon}^t)^* A \, dt,$$
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and the integrating operator \( \mathcal{S} : \mathcal{T}^{s}_{t}(M) \to \mathcal{T}^{s}_{t}(M) \),

\[
\mathcal{S}(A) := \frac{1}{2\pi} \int_{0}^{2\pi} (t - \pi)(\text{Fl}_{t}^{s})^{*}A \, dt.
\]

A tensor field \( A \in \mathcal{T}^{s}_{t}(M) \) is said to be invariant with respect to the \( S^{1} \)-action if \( (\text{Fl}_{t}^{s})^{*}A = A \) (for all \( t \in \mathbb{R} \)) or, equivalently, \( \mathcal{L}_{\Upsilon}A = 0 \). In terms of the \( S^{1} \)-average of \( A \) the \( S^{1} \)-invariance condition reads \( A = \langle A \rangle \). In particular,

\[
\mathcal{L}_{\Upsilon} \langle A \rangle = 0. \tag{II.1}
\]

The integrating operator gives solutions to the following homological equation,

\[
\mathcal{L}_{\Upsilon} \circ \mathcal{S}(A) = A - \langle A \rangle . \tag{II.2}
\]

It is clear that the frequency function \( \omega \) and the Hamiltonian \( H \) are \( S^{1} \)-invariant. Moreover, by the period-energy relation\(^{6,15} \) for periodic Hamiltonian flows, we have the equality

\[
d_{0}H \wedge d_{0}\omega = 0. \tag{II.3}
\]

This implies that the \( S^{1} \)-action is canonical with respect to the fast bracket \( \{ , \}_{0} \), \( \mathcal{L}_{\Upsilon}\Psi_{0} = 0 \). On the other hand, the \( S^{1} \)-action does not preserve the slow Poisson bracket \( \{ , \}_{1} \), in general.

Property \( \text{(II.3)} \) says that the 1-form \( \frac{1}{\omega}d_{0}H \) is \( d_{0} \)-closed. Our next assumption strengthens this condition.

**Hypothesis 2 (Parameterized Momentum Map).** We assume that

\[
\frac{1}{\omega}d_{0}H \quad \text{is} \quad d_{0}\text{-exact}, \tag{II.4}
\]

that is, there exists a smooth function \( J : M \to \mathbb{R} \) such that

\[
\frac{1}{\omega}d_{0}H = d_{0}J \tag{II.5}
\]

and hence

\[
\Upsilon = \mathbf{i}_{d_{0}J}\Psi_{0}. \tag{II.6}
\]

This means that the \( S^{1} \)-action associated to the periodic flow of \( X_{H}^{(0)} \) is Hamiltonian relative to \( \{ , \}_{0} \), with momentum map \( J \). Also, notice that condition \( \text{(II.4)} \) holds whenever \( M_{0} \) is simply connected.
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The momentum map \( J \) is uniquely determined by (II.6) up to a transformation of the form

\[
J \mapsto J + c \circ \pi_1,
\]

where \( c \) is an arbitrary smooth function on \( M_1 \). Pick a \( J \) and consider the 1–form \( d_1J \) on \( M \) and its \( S^1 \)–average \( \langle d_1J \rangle \).

Lemma II.1. There exists a closed 1–form \( \zeta \in \Omega^1(M_1) \) such that

\[
\langle d_1J \rangle = \pi_1^* \zeta.
\]

The De Rham cohomology class \([\zeta]\) is independent of the choice of the function \( c \) in (II.7).

Hypothesis 3 (Adiabatic Condition\(^{22,23}\)). We will assume that \([\zeta] = 0\), and hence, there exists a momentum map \( J \in C^\infty(M) \) in (II.6) satisfying the condition

\[
\langle d_1J \rangle = 0.
\]

(II.8)

Notice that the identities (II.5), (II.8), imply the relation

\[
\frac{1}{\omega} d_1 H - d_1 J = \frac{1}{\omega} \langle d_1 H \rangle.
\]

(II.9)

From here and (II.5), we conclude that sufficient conditions for the differentials \( dH \) and \( dJ \) to be linearly independent at a point \( m \in M \), are \( \langle d_1 H \rangle_m \neq 0 \) and \( (d_0 H)_m \neq 0 \).

It is clear that condition (II.8) holds in the case when \( M_1 \) is simply connected.

Remark. \(^{22,23}\) If the “fast” symplectic manifold \((M_0, \sigma_0)\) is exact, that is, \( \sigma_0 = d\eta \) for some \( \eta \in \Omega^1(M_0) \), then Hypothesis 1 implies Hypotheses 2 and 3. In this case, a momentum map \( J \) satisfying the adiabatic condition (II.8) is given by the formula

\[
J = \frac{1}{\omega} \iota_{X_H^{(0)}} \langle \pi_0^* \eta \rangle.
\]

In a domain where the \( S^1 \)–action is free, this formula gives the standard action\(^3\) along the periodic trajectories of \( X_H^{(0)} \).

Example II.2. Let \( M_0 = S^2 \subset \mathbb{R}^3 = \{ x = (x^1, x^2, x^3) \} \) is the unit sphere equipped with standard area form. Consider the Hamiltonian on \( M = S^2 \times M_1 \) of the form

\[
H = h + \omega n \cdot x
\]
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for some smooth functions \( h : M_1 \to \mathbb{R}, \omega : M_1 \to \mathbb{R} \) and \( n : M_1 \to \mathbb{S}^2 \). The corresponding fast Hamiltonian system satisfies all hypotheses above and the associated \( \mathbb{S}^1 \)-action is given by the rotations in \( \mathbb{R}^3 \) about the axis \(-n\). This action admits the momentum map \( J = n \cdot x \), which satisfies (II.8).

Now, we state our main result.

**Theorem II.3.** Consider a slow-fast Hamiltonian system (I.4) and assume that the unperturbed (fast) Hamiltonian vector field \( X_H^{(0)} \) satisfies the hypotheses 1–3 above. Let \( J \in C^\infty(M) \) be the momentum map of the associated \( \mathbb{S}^1 \)-action, as in (II.6), satisfying the adiabatic condition (II.8). Then, for any \( k \geq 0 \), the perturbed Hamiltonian vector field \( X_H = X_H^{(0)} + \varepsilon X_H^{(1)} \) admits an approximate first integral \( F \in C^\infty(M) \), of order \( k \), which is \( \varepsilon \)-close to \( J \). In particular, for \( k = 2 \), the formulae

\[
F = J + \varepsilon F_1 + \frac{\varepsilon^2}{2} F_2,
\]

where

\[
F_1 := -\frac{1}{\omega} \left( S\{\{H, J\}_1\} + \frac{1}{2} i_{dH} (i_{S(d_1 J)} \Psi_1) \right), \tag{II.11}
\]

\[
F_2 := \frac{2}{\omega} S \left( \{H, \frac{1}{\omega} S\{\{H, J\}_1\}_1 \right.
+ \frac{1}{2} \{H, i_{dH} (i_{S(d_1 J)} \Psi_1) \}_1 \right) \tag{II.12}
\]

give an approximate first integral of order 2,

\[
\mathcal{L}_{X_H} F = O(\varepsilon^3).
\]

Let us remark once again that the expressions (II.10), (II.11), (II.12) are intrinsic (coordinate-free) and global. They involve just the \( \mathbb{S}^1 \)-action induced by \( X_H^{(0)} \), the Hamiltonian \( H \), the “slow” Poisson bracket, and the averaging operators.

The following result follows from the remark above.

**Corollary II.4.** In the exact case, the assertions of Theorem II.3 remain true under just the symmetry hypothesis for \( X_H^{(0)} \).

Also, from II.3 we get the following result\textsuperscript{3,4}.

**Corollary II.5.** If \( \dim M_0 = 2 = \dim M_1 \), \( \langle d_1 H \rangle \neq 0 \) and \( d_0 H \neq 0 \) on \( M \), then, the 2–dimensional generalized slow-fast Hamiltonian system satisfying hypothesis \( I.5 \) is approximately integrable up to arbitrary order in \( \varepsilon \).
III. APPROXIMATE FIRST INTEGRALS VIA NORMAL FORMS

Suppose we are given, on a manifold $M$, a perturbed vector field of the form $A = A_0 + \varepsilon A_1$. Then, a $C^\infty$-function on $M$,

$$F = F_0 + \varepsilon F_1 + \frac{\varepsilon^2}{2} F_2 + \ldots + \frac{\varepsilon^k}{k!} F_k,$$

is an approximate first integral of order $\varepsilon^k$ for $A$ if and only if the functions $F_0, \ldots, F_k$ are solutions to the following homological equations:

$$(\text{III.1})$$

$$\mathcal{L}_{A_0} F_0 = 0,$$

$$\mathcal{L}_{A_0} F_1 = -\mathcal{L}_{A_1} F_0,$$

$$\mathcal{L}_{A_0} F_2 = -2 \mathcal{L}_{A_1} F_1,$$

$$\ldots$$

$$\mathcal{L}_{A_0} F_k = -k! \mathcal{L}_{A_1} F_{k-1}.$$ 

Assume that the flow of the unperturbed vector field $A_0$ is periodic with frequency function $\omega: M \to \mathbb{R}, \omega > 0$, and consider the $S^1$-action on $M$ with infinitesimal generator $\Upsilon := \frac{1}{\omega} A_0$.

Proposition III.1. Assume that there exists a smooth function $J \in C^\infty(M)$ which is a common first integral of the unperturbed vector field $A_0$ and the averaged perturbation vector field $\langle A_1 \rangle$, that is, $\mathcal{L}_{A_0} J = 0$ and $\mathcal{L}_{\langle A_1 \rangle} J = 0$. Then:

(a) For an arbitrary $C_1 \in C^\infty(M)$, the function

$$F = J + \varepsilon (F_1^0 - \langle C_1 \rangle),$$

where $F_1^0 := -\frac{1}{\omega} \mathcal{L}_{S(A_1)} J$ satisfies $\langle F_1^0 \rangle = 0$, is an approximate second order first integral of $A$.

(b) If $C_1$ can be chosen so that the normalization condition of second order holds:

$$\langle \mathcal{L}_{A_1} F_1^0 \rangle = \langle \mathcal{L}_{\langle A_1 \rangle} C_1 \rangle,$$  

(III.2)

then, $A$ admits an approximate first integral $F$, of order 2, of the form

$$F = J + \varepsilon (F_1^0 - \langle C_1 \rangle) + \frac{\varepsilon^2}{2} (F_2^0 - \langle C_2 \rangle),$$
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where

\[ F_2^0 := \frac{2}{\omega} \left( S \circ L_{A_1} \left( \frac{1}{\omega} L_{S(A_1)} J \right) - L_{S(A_1)} \langle C_1 \rangle \right), \]

satisfies \( \langle F_2^0 \rangle = 0 \), and \( C_2 \in C^\infty(M) \) is an arbitrary smooth function.

Proof. The statement follows from the solvability condition for homological equations \( \text{(III.1)} \).

Let us remark that the direct approach for finding approximate first integrals \( \text{(III.2)} \) leads to the verification of normalization condition \( \text{(III.2)} \), which is not necessarily satisfied in general. It is precisely here that our hypotheses come into play. As we will see, they will allow us first, to construct a normal form for the perturbed Hamiltonian vector field \( X_H = X_H^0 + \varepsilon X_H^1 \), and then, from a first integral of the truncated normal form (which, in turn, is built out of the normalization transformation and the momentum map), an approximate first integral of the original system.

Proposition III.2. Under the symmetry hypothesis for \( A_0 \), for any \( k > 1 \), the perturbed vector field \( A = A_0 + \varepsilon A_1 \) admits an \( S^1 \)-invariant global normal form of order \( k \),

\[ T_\varepsilon(A_0 + \varepsilon A_1) = A_0 + \varepsilon \langle A_1 \rangle + \frac{\varepsilon^2}{2} \bar{A}_2 + \ldots + \frac{\varepsilon^k}{k!} \bar{A}_k + O(\varepsilon^{k+1}), \]

where \( \langle \bar{A}_i \rangle = \bar{A}_i \), for \( i \in \{2, \ldots, k\} \).

The near-identity transformation \( T_\varepsilon \) is given by the time-\( \varepsilon \) flow of an \( \varepsilon \)-dependent vector field on \( M \). Therefore, the normalization transformation \( T_\varepsilon \) is well-defined on any relatively compact open subset in \( M \) for small enough \( \varepsilon \). The \( S^1 \)-invariant vector fields \( \bar{A}_2, \ldots, \bar{A}_k \) are well-defined on the whole \( M \), and they determine the truncated normal form \( \text{(III.2)} \) of order \( k \) of \( A \).

This leads to the following criterion.

Corollary III.3. Suppose that an \( S^1 \)-invariant smooth function \( J : M \to \mathbb{R} \) is a first integral of the truncated \( S^1 \)-invariant normal form of order \( k \) for \( A \),

\[ \mathcal{L}_{\langle A_1 \rangle} J = \mathcal{L}_{\bar{A}_2} J = \ldots = \mathcal{L}_{\bar{A}_k} J = 0. \quad \text{(III.3)} \]

Then, the function

\[ F = J \circ T_\varepsilon^{-1} = J + \varepsilon F_1 + \frac{\varepsilon^2}{2} F_2 + \ldots + \frac{\varepsilon^k}{k!} F_k + O(\varepsilon^{k+1}), \]

where, for \( s \in \{1, \ldots, k\} \), \( F_s = \frac{d^s}{d\varepsilon^s} \big|_{\varepsilon=0} (J \circ T_\varepsilon^{-1}) \), is an approximate first integral on \( M \) of order \( k \) for \( A \).
IV. NORMALIZATION OF SLOW-FAST HAMILTONIAN SYSTEMS

In this section, we show that conditions (III.3) are satisfied by generalized slow-fast Hamiltonian systems (I.4). To this end, we will assume that the unperturbed Hamiltonian vector field \( X_{H}^{(0)} \) satisfies the hypotheses 1, 2, 3. We also suppose that a momentum map \( J \) as in (II.6) is given, and that it satisfies the adiabatic condition (II.8). As mentioned above, generally the symplectic form \( \sigma \) (I.2) and the Poisson bracket (I.3) are not invariant with respect to the \( \mathbb{S}^1 \)-action on \( M \) associated to the periodic flow of \( X_{H}^{(0)} \). This raises the question of the normalization of the perturbed Hamiltonian vector field \( X_{H} = X_{H}^{(0)} + \varepsilon X_{H}^{(1)} \) relative to the \( \mathbb{S}^1 \)-action. To get \( \mathbb{S}^1 \)-invariant global normal forms, one can apply to \( X_{H} \) a non-canonical (non-Hamiltonian) Lie transform method. But our point is to maintain the Hamiltonian setting. For that purpose, we proceed the normalization procedure in two steps. In the first stage, we correct the drawbacks of our \( \varepsilon \)-dependent phase space by constructing an \( \mathbb{S}^1 \)-invariant Poisson bracket \( \{ , \}_{\text{inv}} \) (a symplectic form) which is \( \varepsilon \)-close to the original one \( \{ , \} \). In a second stage, we normalize the deformed Hamiltonian \( H \circ \Phi_{\varepsilon} \) up to desired order in \( \varepsilon \) by applying a near-identity transformation which will be defined as the Hamiltonian flow relative to \( \{ , \}_{\text{inv}} \). Here \( \Phi_{\varepsilon} \) is a Poisson isomorphism between \( \{ , \}_{\text{inv}} \) and \( \{ , \} \). Thus, this second step is basically a modification of the Deprit algorithm for \( \varepsilon \)-dependent phase spaces.

Let us associate to the momentum map \( J \) the 1–form (on \( M \)) \( \Theta := S(d_{1}J) \), which has the property \( \langle \Theta \rangle = 0 \).

**Lemma IV.1.** The average \( \langle \sigma \rangle \) of the symplectic form (I.2) has the representation \( \langle \sigma \rangle = \sigma - d\Theta \). Moreover, for any \( \mathbb{S}^1 \)-invariant, relatively compact \( N \subset M \), and small enough \( \varepsilon \), the averaged 2-form \( \langle \sigma \rangle \) is non-degenerate on \( N \), and there exists a near-identity transformation \( \Phi_{\varepsilon} : N \to M \), \( (\Phi_{0} = \text{id}) \) which is a symplectomorphism between \( \langle \sigma \rangle \) and \( \sigma \), \( \Phi_{\varepsilon}^{*}\sigma = \langle \sigma \rangle \).

**Proof.** The proof of this statement is based on a parametric version of the Moser homotopy method, where the symplectomorphism \( \Phi_{\varepsilon} \) is constructed as follows. Fix an \( \mathbb{S}^1 \)-invariant, relatively compact subset \( N \subset M \). Then, define the family of 2-forms on \( M \) depending on the parameter \( \lambda \)

\[
\delta_{\lambda} := d_{1}\Theta + \frac{(1 - \lambda)}{2}\{\Theta \wedge \Theta\}_{0},
\]
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where $\frac{1}{2}\{\Theta \wedge \Theta\}_0$ denotes the 2-form defined by its action on a pair of vector fields $X, Y \in \mathcal{X}(M)$ by

$$\frac{1}{2}\{\Theta \wedge \Theta\}_0(X, Y) := \{\Theta(X), \Theta(Y)\}_0,$$

and consider the time-dependent vector field $Z_\lambda$ on $N$, depending on $\epsilon$ as a parameter, and uniquely determined by the relations

$$i_{Z_\lambda} (\pi^*_1 \sigma_1 - \epsilon (1 - \lambda) \delta \lambda) = -\Theta, \quad (IV.1)$$
$$i_{Z_\lambda} \pi^*_0 \sigma_0 = 0. \quad (IV.2)$$

It follows that the flow of $\epsilon Z_\lambda$ is well-defined on $N$, for small enough $\epsilon$ and all $\lambda \in [0, 1]$. Then, it suffices to take

$$\Phi_\epsilon = F_\epsilon^{\lambda} \big|_{\lambda=1} \quad (IV.3)$$

Now, denote by $\{f, g\}^{\text{inv}} = \Psi^{\text{inv}}(df, dg)$ the non-degenerate Poisson bracket on $N$ associated to the averaged symplectic form $\langle \sigma \rangle$. Then, its Poisson bivector field $\Psi^{\text{inv}}$ is $S^1$-invariant, and it has the representation

$$\Psi^{\text{inv}} = \Psi_0 + \epsilon \left( \langle \Psi_1 \rangle + \mathcal{L}(V) \Psi_0 \right) + O(\epsilon^2), \quad (IV.4)$$

where

$$V := \frac{1}{2} i_{\Theta} \Psi_1. \quad (IV.5)$$

The following observation shows the role of adiabatic condition (II.8).

**Lemma IV.2.** Let $J$ be the momentum satisfying adiabatic condition (II.8). Then,

$$\Upsilon = X_j^{(0)} = i_{dJ} \Psi^{\text{inv}}, \quad (IV.6)$$

that is, the $S^1$-action is canonical on $(N, \{\cdot, \cdot\}^{\text{inv}})$ with momentum map $J$.

**Remark.** The adiabatic condition (II.8) was introduced in the context of the theory of Hannay-Berry connections on fibred phase spaces with symmetry. In our case, the 1-form $\Theta = S(d_1 J)$ appearing in formulas (II.11) and (II.12), just represents the Hamiltonian form of the Hannay-Berry connection on the trivial bundle $M_0 \times M_1 \to M_1$ over the “slow” base $M_1$ with the “fast” fiber $M_0$ endowed with an $S^1$-action associated to the periodic Hamiltonian flow of $X_H^{(0)}$. If $\Theta$ is closed, then by Lemma IV.1 the original symplectic form (I.2) is $S^1$-invariant.
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We arrive at the following normalization result.

**Theorem IV.3.** Suppose that the unperturbed Hamiltonian vector field \( X_H^{(0)} \) satisfies the hypotheses \[ H \] \[ 2 \] \[ 3 \]. Then, for any \( m \geq 1 \) and small enough \( \varepsilon \), there exists a near-identity transformation \( T_{\varepsilon} : N \to M \) which takes the original slow-fast Hamiltonian system (I.4) into an \( S^1 \)–invariant normal form of order \( m \) of the form \( (N, \{ , \}^{\text{inv}}, H \circ T_{\varepsilon}) \), where

\[
H \circ T_{\varepsilon} = H + \sum_{s=1}^{m} \frac{\varepsilon^s}{s!} \langle K_s \rangle + O(\varepsilon^{m+1}),
\]

for some smooth functions \( K_1, \ldots, K_m \) on \( M \). In particular,

\[
K_1 := \frac{1}{2} i dH i\Theta \Psi_1.
\]

**Proof.** In the first step, after applying the near-identity transformation \( \Phi_{\varepsilon} \) [IV.3] to the original system (I.4), we get a new perturbed Hamiltonian system relative to the \( S^1 \)–invariant Poisson bracket, \( (N, \{ , \}^{\text{inv}}, H \circ \Phi_{\varepsilon}) \), and a deformed Hamiltonian,

\[
H \circ \Phi_{\varepsilon} = H + \sum_{s=1}^{m} \varepsilon^s \tilde{H}_s + O(\varepsilon^{m+1}).
\]

Here the functions

\[
\tilde{H}_s = \frac{d^s}{d\varepsilon^s} \big|_{\varepsilon=0} (H \circ \Phi_{\varepsilon}), \quad s \in \{1, \ldots, m\},
\]

are not necessarily \( S^1 \)–invariant. By applying Lemmas [IV.4] and [IV.2] a direct computation gives the result

\[
\tilde{H}_1 = \frac{1}{2} i dH i\Theta \left( \langle \Psi_1 \rangle + L_{\langle \Psi \rangle} \Psi_0 + \Psi_1 \right),
\]

where \( K_1 \) is given by (IV.8). In a second step, we normalize the truncated Taylor series (the \( m \)–th jet) of the deformed Hamiltonian (IV.9) by using the Hamiltonian flows relative to the \( S^1 \)–invariant Poisson bracket \( \{ , \}^{\text{inv}} \). Thus, we are looking for some smooth functions on \( M, G_0, \ldots, G_{m-1} \), such that the time–\( \varepsilon \) flow of the Hamiltonian vector field \( i_{dG}^{\text{inv}} \Psi \) of the function

\[
G = G_0 + \sum_{i=1}^{m-1} \varepsilon^i \frac{i}{i!} G_i,
\]

gives us the desired normalization transformation,

\[
\left( H + \sum_{s=1}^{m} \frac{\varepsilon^s}{s!} \tilde{H}_s \right) \circ F^\varepsilon_{i dG}^{\text{inv}} = H + \sum_{s=1}^{m} \frac{\varepsilon^s}{s!} \langle K_s \rangle + O(\varepsilon^{m+1}).
\]
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By a standard Lie transform argument, we conclude that $G_0, \ldots, G_{m-1}$, must satisfy a set of homological equations on $M$:

$$
\{H, G_0\}_0 = \mathcal{R}_1 - \langle K_1 \rangle,
\{H, G_1\}_0 = \mathcal{R}_2 - \langle K_2 \rangle, \quad (IV.11)
$$

\[
\ldots
\]

$$
\{H, G_{m-1}\}_0 = \mathcal{R}_m - \langle K_m \rangle,
$$

where the functions $\mathcal{R}_2, \ldots, \mathcal{R}_m$ are defined by the recursive procedure given by the modified Deprit diagram. The corrections to the standard Deprit diagram come from the Taylor expansion of the Poisson tensor $\Psi^{\text{inv}}$ in $\varepsilon$ at $\varepsilon = 0$. In particular, by using $(IV.4), (IV.5)$, one can show that

$$
\mathcal{R}_1 = \tilde{H}_1, \quad (IV.12)
$$

and

$$
\mathcal{R}_2 = \tilde{H}_2 + \mathcal{L}^2_{i_dG_0} \Psi_0 H + 2 \mathcal{L}_{i_dG_0} \Psi_0 \tilde{H}_1 + i_dH i_dG_0 (\langle \Psi_1 \rangle + \mathcal{L}_V \Psi_0). \quad (IV.13)
$$

To assure the solvability of the homological equations $(IV.11)$, we choose the functions $K_1, \ldots, K_m$ in such a way that

$$
\langle \mathcal{R}_1 \rangle = \langle K_1 \rangle, \ldots, \langle \mathcal{R}_m \rangle = \langle K_m \rangle.
$$

In particular, taking into account that $H$ and $\Psi_0$ are $S^1$-invariant, and the property $\langle \Theta \rangle = 0$, we deduce from $(IV.10), (IV.12)$ and $(IV.13)$, that $K_1$ is just given by $(IV.8)$, and one can put

$$
K_2 = \tilde{H}_2 + \{S(\frac{\tilde{H}_1}{\omega}), \tilde{H}_1\}_0.
$$

The global solutions to the homological equations $(IV.11)$ are given by the formulae,

$$
G_0 = \frac{1}{\omega} S(\mathcal{R}_1), \ldots, G_{m-1} = \frac{1}{\omega} S(\mathcal{R}_m).
$$

Finally, the normalization transformation in $(IV.7)$ is defined by

$$
\mathcal{T}_\varepsilon = \Phi_\varepsilon \circ F_{i_dG_0}^{\varepsilon} \Psi^{\text{inv}}.
$$

□
Corollary IV.4. Let $\mathcal{T}_\varepsilon$ be the normalization transformation in (IV.7) and $\tilde{H} = H \circ \mathcal{T}_\varepsilon$. Then, the momentum map $J$ is an approximate first integral of order $m$ for the Hamiltonian vector field $i_{\tilde{H}} \Psi^{\text{inv}}$,

$$\mathcal{L}_{\tilde{H}} \Psi^{\text{inv}} J = O(\varepsilon^{m+1}).$$

Proof. By using properties (IV.6) and (IV.7), we get

$$\mathcal{L}_{\tilde{H}} \Psi^{\text{inv}} J = \{\tilde{H}, J\}^{\text{inv}} = \{\tilde{H}, J\}_0 = -\mathcal{L}_\mathcal{T} \tilde{H}$$

$$= -\mathcal{L}_\mathcal{T} H - \sum_{i=1}^m \frac{\varepsilon^i}{i!} \mathcal{L}_\mathcal{T} \langle K_i \rangle + O(\varepsilon^{m+1})$$

$$= O(\varepsilon^{m+1}),$$

where, in the last equation, we have used the $\mathbb{S}^1$-invariance of $H$, and the property (II.1). □

V. PROOF OF THE MAIN RESULT

Now, we have all the elements required to give a proof of Theorem II.3. Notice that

$$X_H = (\mathcal{T}_\varepsilon^{-1})^* (i_{\tilde{H}} \Psi^{\text{inv}}),$$

and hence by Corollary IV.4, the function $J \circ \mathcal{T}_\varepsilon^{-1}$ is an approximate first integral of order $m$ for $X_H$. This proves the first assertion of Theorem II.3. In particular, by taking the Taylor expansion of second order

$$J \circ \mathcal{T}_\varepsilon^{-1} = J + \varepsilon F_1 + \frac{\varepsilon^2}{2} F_2 + O(\varepsilon^3),$$

(V.1)

we deduce from (III.1) that the functions $F_1$ and $F_2$ must satisfy the homological equations

$$\mathcal{L}_\mathcal{T} F_1 = -\frac{1}{\omega} \{H, J\}_1,$$  \hspace{1cm} (V.2)

and

$$\mathcal{L}_\mathcal{T} F_2 = -\frac{1}{\omega} \{H, F_1\}_1$$  \hspace{1cm} (V.3)

A general solution to the first equation is given by

$$F_1 = -\frac{1}{\omega} \mathcal{S}(\{H, J\}_1) - \langle C_1 \rangle,$$
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for a function $C_1 \in C^\infty(M)$ which has to satisfy the solvability condition for the second equation

$$\frac{1}{\omega} \langle \{H, C_1\} \rangle_1 = - \langle \{H, \frac{1}{\omega} S(\{H, J\}) \} \rangle_1 .$$

It is difficult to find $C_1$ from this equation. Instead of following this approach, we will derive an explicit formula for $F_1$ by using the definition of $T_\varepsilon$. Firstly, one can verify by a direct computation that the second term in the right-hand side of (V.1) is given by

$$F_1 = - \{G_0, J\}_0 - \frac{1}{2} i d i \Theta \Psi_0,$$

where $G_0$ is a solution to the homological equation (IV.11). From this fact, and properties (II.5), (II.9), we get the following result.

**Lemma V.1.** The second term in the Taylor expansion (V.1) is represented as follows

$$F_1 = - \frac{1}{\omega} S(\{H, J\})_1 - \frac{1}{\omega} (K_1),$$

where $K_1$ is given by (IV.8).

Therefore, one can put $C_1 = \frac{1}{\omega} K_1$. Finally, a particular solution to (V.1) is given by the formula $F_2 = \frac{1}{\omega} S(\{H, F_1\})_1$, which leads to the representation (II.12). This ends the proof of Theorem II.3.

**VI. HAMILTONIANS QUADRATIC IN THE FAST VARIABLES**

Let us particularize the previous developments in the case of a slow-fast Hamiltonian system $(\mathbb{R}_2^{(y,x)} \times \mathbb{R}_k^{(p,q)}, \frac{1}{\varepsilon} dp \wedge dq + dy \wedge dx, H)$ where the Hamiltonian $H$ is a quadratic function in the “fast” variables $z = (y, x)$. To this end, let us associate to every matrix-valued function $A \in \mathfrak{sl}(2, \mathbb{R}) \otimes C^\infty(\mathbb{R}_k^{(p,q)})$ the function $Q_A = - \frac{1}{2} J A z \cdot z$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the dot denotes the euclidian scalar product. The Hamiltonian vector field relative to the “fast” Poisson bracket $\{ , \}_0$ is given by $X^{(0)}_{Q_A} = A z \cdot \frac{\partial}{\partial z}$. Consider a Hamiltonian of the form $H = h + \omega Q_A$, for some smooth functions $h = h(p, q)$ and $\omega = \omega(p, q) > 0$. We assume that $\det A = 1$ on an open domain in $\mathbb{R}_k^{(p,q)}$. This implies that $X^{(0)}_H$ has periodic flow with frequency function $\omega$, hence the infinitesimal generator of the $\mathbb{S}^1$–action is $X^{(0)}_{Q_A}$, and the associated $\mathbb{S}^1$–action is given by the linear flow $F^t_A = \cos t I + \sin t A$. The corresponding momentum map is $J = Q_A$. 

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It is easy to see that hypotheses 1–3 hold in this case. For an arbitrary \( S \in \mathfrak{sl}(2, \mathbb{R}) \otimes C^\infty(\mathbb{R}^{2k}) \), we have the following identities,

\[
\langle Q_S \rangle = \frac{1}{2} Q_{S - ASA},
\]

\[
S(Q_S) = \frac{1}{4} Q_{[A,S]}.
\]

The Hamiltonian vector field \( X_H = X_H^{(0)} + \varepsilon X_H^{(1)} \) admits an approximate first integral \( F \) of second order, \( \mathcal{L}_{X_H} F = O(\varepsilon^3) \), of the form \( F = J + \varepsilon F_1 + \frac{\varepsilon^2}{2} F_2 + O(\varepsilon^3) \), where \( J \) is the momentum map and \( F_1, F_2 \) are given in (II.11), (II.12). In the quadratic case these formulae are reduced to:

\[
F_1 = -\frac{1}{4\omega} (Q_{[A,B]} + Q_A Q_{[A,C]}) + \frac{\omega}{4} \sum_{i=1}^{k} \left( Q_A \frac{\partial A}{\partial p_i} Q_A \frac{\partial A}{\partial q_i} - Q_A \frac{\partial A}{\partial q_i} Q_A \frac{\partial A}{\partial p_i} \right),
\]

\[
= -Q \hat{Q}(A),
\]

and,

\[
F_2 = \frac{1}{2\omega} Q_{[h, \frac{\hat{Q}(A)}{\hat{Q}(A)} + Q_A [A,C] - \frac{\hat{Q}(A)}{\hat{Q}(A)}]},
\]

where we have introduced the notations \( B := \{h,A\}, C := \{\omega, A\} \) (the bracket being computed separately for each coefficient of the matrix \( A \)),

\[
\hat{Q}(A) := \sum_{i=1}^{k} \left( Q Q_A \frac{\partial A}{\partial p_i} \frac{\partial A}{\partial q_i} - Q A \frac{\partial A}{\partial q_i} Q A \frac{\partial A}{\partial p_i} \right).
\]

Example VI.1. We start from the Breitenberger-Mueller model for the Hamiltonian of the elastic pendulum:

\[
\tilde{H} = \frac{1}{2} (p_x^2 + \omega_p^2 x^2) + \frac{1}{2} (p_y^2 + \omega_y^2 y^2 + \gamma x^2 y).
\]

With the rescaling \( x = \frac{X}{\omega_p}, y = \omega_p^2 Y \), we get,

\[
\tilde{H} = \frac{1}{2} (p_x^2 + (\frac{\omega_p}{\omega_y})^2 X^2) + \frac{1}{2} (p_y^2 + (\omega_y^2)^2 y^2 + \gamma (\frac{\omega_p}{\omega_y})^2 X^2 Y).
\]

Next, we introduce the parameter \( \varepsilon = \frac{\omega_p}{\omega_y} \), which according to the physical meaning of the problem, can be considered small: \( \varepsilon \ll 1 \). In terms of this parameter the Hamiltonian reads,

\[
\tilde{H} = \frac{1}{2} (p_x^2 + \varepsilon^2 X^2) + \frac{1}{2} (p_y^2 + (\omega_y^2)^2 y^2 + \gamma \varepsilon^2 X^2 Y),
\]
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which can be written as \( \tilde{H} = H(p_x, \varepsilon X, p_y, Y) \). A further rescaling \( p_x = p, q = \varepsilon X, y = p_Y \) and \( x = Y \), leads to the slow-fast Hamiltonian system:

\[
H = \frac{1}{2}(p^2 + q^2) + \frac{1}{2}(y^2 + \Omega^2 x^2 + \gamma q^2 x),
\]

where \( \Omega = \omega_s \omega_p^2 \), and the \( \varepsilon \)-dependent Poisson bracket \( \{,\} = \{,\}_0 + \varepsilon \{,\}_1 \) on \( \mathbb{R}^2(p, q) \times \mathbb{R}^2(y, x) \) (both brackets, \( \{,\}_0 \) and \( \{,\}_1 \), are the canonical one on \( \mathbb{R}^2 \)). The Hamiltonian vector field \( X_H = X_H^{(0)} + \varepsilon X_H^{(1)} \) is readily computed,

\[
X_H^{(0)} = -(\Omega^2 x + \frac{1}{2} \gamma q^2) \frac{\partial}{\partial y} + y \frac{\partial}{\partial x},
\]

\[
X_H^{(1)} = -(1 + \gamma x) q \frac{\partial}{\partial p} + p \frac{\partial}{\partial q}.
\]

The flow of \( X_H^{(0)} \) is periodic with constant frequency function \( \Omega \). The momentum map of the \( S^1 \)-action \( J \) in given by

\[
J(p, q, y, x) = \frac{\Omega}{2}(x + \frac{\gamma}{2 \Omega^2} q^2)^2 + \frac{1}{2 \Omega} y^2,
\]

and, finally, the approximate first integral of second order for \( X_H \) has the form

\[
F = J + \varepsilon \frac{\gamma}{\Omega^3} pqy + \varepsilon^2 \frac{\gamma}{4 \Omega^3} \left( \gamma q^2 (x + \frac{\gamma}{2 \Omega^2} q^2) (x - \frac{3 \gamma}{2 \Omega^2} q^2) + 4(q^2 - p^2)(x + \frac{\gamma}{2 \Omega^2} q^2) - \frac{\gamma}{\Omega^2} q^2 y^2 \right).
\]

VII. CHARGED PARTICLE IN A SLOWLY VARYING MAGNETIC FIELD

The adiabatic invariants appearing in the motion of charged particles in a slowly varying magnetic field are well-known in plasma physics since long ago \cite{bohm, stephenson, morel}. Indeed, for this problem explicit expressions for corrections of the classical first and second adiabatic invariants have been given up to first order \cite{salamin, gurarie, maia}. Here, we will consider a cylindrically symmetric configuration for the magnetic field (in cylindrical coordinates)

\[
B = B_r \mathbf{u}_r + B_z \mathbf{u}_z = B \mathbf{u}_r - \frac{\mathbf{z}}{r} \mathbf{u}_z.
\]
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Notice that $\frac{\partial B}{\partial r} = 0$, so, rather than magnetic traps, we will study a force-free contracting plasmoid.

Let us write the equations of motion of a charged ($e = 1$) particle in this field in such a way that the slow and fast components are apparent. In terms of the Clebsch potentials

$$
\begin{align*}
\alpha(\varepsilon r, \theta, \varepsilon z) &= -B\varepsilon^2 rz, \\
\beta(\varepsilon r, \theta, \varepsilon z) &= \theta,
\end{align*}
$$

we have $B = \frac{1}{\varepsilon^2} \nabla \alpha \times \nabla \beta$; also, we can write $B = \nabla \times A$, where $A = \frac{1}{\varepsilon^2} \alpha \nabla \beta$. The Hamiltonian is given by $H = \frac{1}{2} ||\tilde{p} - A||^2$, where $\tilde{p}$ is the kinematical momentum. Let us apply the canonical transformation with generating function

$$
F = \frac{S}{\varepsilon} p_2 + \frac{\beta}{\varepsilon} p_1 + \frac{\alpha}{\varepsilon} p_3 - p_1 p_3,
$$

where we have introduced another variable, $S$, such that $\frac{S}{\varepsilon}$ is the arc-length along along field lines so, in the far-field regime $r \gg 1$ (which will be the one of interest for us)

$$
\frac{S}{\varepsilon} \approx r.
$$

Notice that this transformation, being canonical, does not alter the (canonical) Poisson brackets. From (VII.2), we get

$$
\begin{align*}
\beta &= \varepsilon q_1 + \varepsilon q_3 \\
S &= \varepsilon q_2 \\
\alpha &= \varepsilon q_3 + \varepsilon p_1
\end{align*}
$$

Now, $\tilde{p} = \nabla F = \frac{\partial F}{\partial S} \nabla S + \frac{\partial F}{\partial \alpha} \nabla \alpha + \frac{\partial F}{\partial \beta} \nabla \beta$. So $\tilde{p} - A = \frac{\partial F}{\partial S} \nabla S + \frac{\partial F}{\partial \alpha} \nabla \alpha - \frac{1}{\varepsilon^2} \alpha \nabla \beta$.

Substituting (VII.4):

$$
\tilde{p} - A = \frac{p_2}{\varepsilon} \nabla S + \frac{p_3}{\varepsilon} \nabla \alpha - \frac{q_3}{\varepsilon} \nabla \beta,
$$

Thus

$$
||\tilde{p} - A||^2 = p_2^2 + \varepsilon^2 B_2 p_3^2 (r^2 + z^2) + \frac{q_3^2}{\varepsilon^2 r^2} - 2\varepsilon B z p_2 p_3.
$$

Now, we expand this expression in powers of $(p_3, q_3)$; to this end, we use $\alpha = \varepsilon p_1$, $\beta = \varepsilon q_1$, $S = \varepsilon q_2$, which, together with (VII.1), (VII.3), gives $z = -\frac{p_1}{\varepsilon B q_2}$ and $r = q_2$. Thus, we get a Hamiltonian explicitly displaying slow and fast variables:

$$
H(q_1, \varepsilon p_1, \varepsilon q_2, p_2, q_3, p_3) = 
\frac{1}{2} \left[ p_2^2 + p_3^2 \left( B^2 (\varepsilon q_2)^2 + \frac{(\varepsilon p_1)^2}{(\varepsilon q_2)^2} \right) + 2 \frac{\varepsilon p_1^2 p_2}{(\varepsilon q_2)^2} p_3 + \frac{q_3^2}{(\varepsilon q_2)^2} \right].
$$
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After the obvious rescaling \( \tilde{p}_1 = \varepsilon p_1, \tilde{q}_2 = \varepsilon q_2 \), dropping the tildes for simplicity, and noticing that \( q_1 \) is a cyclic variable (so we can take \( p_1 = \lambda \), a parameter) we obtain the Hamiltonian system

\[
H(q, p, \lambda) = \frac{1}{2} \left[ \frac{p_2^2 + p_3^2}{q_2^2} \left( B^2 q_2^2 + \lambda^2 + 2\lambda p_2 p_3 + \frac{q_3^2}{q_2^2} \right) \right].
\]

The unperturbed Hamiltonian vector field (“fast” variables) is given by

\[
X_H^{(0)} = -\frac{q_3}{q_2} \frac{\partial}{\partial p_3} + \left( \frac{B^2 q_2^4 + \lambda^2}{q_2^2} p_3 + \lambda \frac{p_2}{q_2} \right) \frac{\partial}{\partial q_3},
\]

and it has periodic flow with frequency function

\[
\omega = \sqrt{\frac{B^2 q_2^4 + \lambda^2}{q_2^2}} > 0.
\]

The flow is of the infinitesimal generator of the \( S^1 \)-action \( \Upsilon = \frac{1}{\omega} X_H^{(0)} \) is given by

\[
\mathcal{F}_t^\Upsilon \left( \begin{array}{c} p_2 \\ q_2 \\ p_3 \end{array} \right) = \left( \begin{array}{c} p_2 \\ q_2 \\ p_3 + \lambda \frac{p_2}{\omega q_2} \end{array} \right) \cos(t) - \frac{q_3}{q_2} \omega \sin(t) - \lambda \frac{p_2}{\omega q_2} \sin(t) + q_3 \cos(t)
\]

and the momentum map reads

\[
J(p_2, q_2, p_3, q_3) := \frac{1}{2\omega} \left( \frac{q_3^2}{q_2^2} + \left( \omega q_2 p_3 + \lambda \frac{p_2}{\omega q_2} \right)^2 \right).
\]

A straightforward computation shows that, indeed, \( J = \frac{v^2}{L} \), where \( v_\perp \) is the transverse momentum. From our previous results, we see that the Hamiltonian vector field \( X_H = X_H^{(0)} + \varepsilon X_H^{(1)} \) admits an approximate first integral \( F \) to any arbitrary order in \( \varepsilon \). In particular, to second order it has the form \( F = J + \varepsilon J_1 + \frac{\varepsilon^2}{2} J_2 + O(\varepsilon^3) \), where \( J \) is the momentum map, and

\[
J_1(p_2, q_2, p_3, q_3) = -\frac{q_3}{q_2^2 \omega} \left( p_2 p_3 q_2^2 B^4 + \lambda q_2^4 (\lambda^2 p_3^2 - 2p_2^2 q_2^2) B^2 + \lambda^3 (q_3^2 + (p_2 q_2 + \lambda p_3)^2) \right).
\]

The explicit expression for \( J_2 \) is too long to be displayed here (it can be easily computed with the aid of a computer algebra system such as Maxima).
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