INFINITE CONNECTED SUMS, $K$-AREA AND POSITIVE SCALAR CURVATURE

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ABSTRACT. Whyte [Why01] used the index theory of Dirac operators and Block-Weberer uniformly finite homology [BW92] to show that certain infinite connected sums do not carry a metric with nonnegative scalar curvature in their bounded geometry class. His proof uses a coarse version of the $A$-class to obstruct such metrics. In this note we prove a version of Whyte’s result where a variant of the notion of infinite $K$-area, originally due to Gromov [Gro96], is used to obstruct metrics with positive scalar curvature.

1. INTRODUCTION

We consider the category $BG_n$ of manifolds with bounded geometry. Thus, objects in $BG_n$ are complete $n$-dimensional Riemannian manifolds whose curvature tensor and covariant derivatives of all orders are uniformly bounded and whose injectivity radius is positive. The morphisms of $BG_n$ are diffeomorphisms with bounded distortion. This allows us to split the space of such metrics into classes, the so-called bounded geometry classes of metrics. Whenever we refer to a manifold, it should be understood that it is equipped with a complete metric varying within a fixed bounded geometry class. Also, all manifolds in this note will be spin, unless otherwise stated. Finally, in what follows we assume that $n \geq 4$.

We observe that in [BW92] a homology theory $H^{uf}_0$, named uniformly finite homology in degree zero, has been defined which is preserved by the above morphisms and therefore is a bounded geometry invariant. In particular, a subset $S \subset X$ defines a homology class $[S] \in H^{uf}_0(X)$ if it is locally uniformly finite in the sense that for each $r > 0$ there exists $C_r > 0$ such that the amount of points of $S$ inside any metric ball of radius $r$ is bounded from above by $C_r$ (for more on the functor $H^{uf}_0$ and its relation to bounded de Rham cohomology, see Section 3).

Now take $Y \in BG_n$, $M$ a closed manifold and $S \subset Y$ as above. Let $Y \#_S M$ be the manifold obtained by connected summing to $Y$ a copy of $M$ along a small neighborhood of each element of $S$. We remark that $BG_n$ is stable under such infinite connected sums, so that $Y \#_S M \in BG_n$ as well.

Whyte [Why01] used the Atiyah-Patodi-Singer index theory of Dirac operators to show that if $Y$ carries a metric of nonnegative scalar curvature (in a given bounded geometry class), $\tilde{A}(M) \neq 0$ and $[S] \neq 0$ then $Y \#_S M$ does not carry a metric of nonnegative scalar curvature (in the corresponding bounded geometry class).

Partially supported by CNPq/Brazil grant 312485/2018-2 and by FUNCAP/CNPq/PRONEX grant 00068.01.00/15.
Example 1.1. Let $S = \mathbb{Z}^n \subset \mathbb{R}^n$ be the standard integer lattice in flat Euclidean space. Then $[\mathbb{Z}^n] \neq 0$ and hence $\mathbb{R}^{4\mathbb{Z}^n}M^{4l}$ does not carry a metric with nonnegative scalar curvature if $\widehat{A}(M) \neq 0$ (for example we can take $M$ to be the product of $l$ K3 surfaces). More generally, we can take Whyte’s result applies to $Y\sharp M$ if $S \subset Y$ is an orbit under deck transformations and $\pi_1(Y_0)$ is amenable (for more on Whyte’s result, see Remarks 1.2 and 2.1).

The purpose of this note is to prove a version of Whyte’s result that uses a notion of $K$-area which is well suited to obstructing metrics with positive scalar curvature on these infinite connected sums. As explained in Section 4 below, this invariant, denoted $K^b_{\text{area}}(X, g)$, is indeed a Riemannian invariant of $(X, g) \in BG_{2k}$. However, the fact that it vanishes, is finite or infinite is an invariant of the bounded geometry class of the metric. This allows us to simply write $K^b_{\text{area}}(Y) = K^b_{\text{area}}(Y, g)$, with the proviso that $g$ is allowed to vary in its bounded geometry class.

For our purposes, it is convenient to decompose $BG_n = BG_n^A \sqcup BG_n^N$, where

$$BG_n^A = \left\{ X \in BG_n; H^0_0(X) \neq \{0\} \right\},$$

and $BG_n^N = BG_n \setminus BG_n^A$. Elements in $BG_n^A$ (respectively, $BG_n^N$) are called amenable (respectively, non-amenable); see Proposition 4.1 for a geometric characterization of this splitting of $BG_n$. With this terminology, our obstruction to positive scalar curvature goes as follows.

Theorem 1.1. Let $Y \in BG_{2k}^A$ be spin with $0 \leq K^b_{\text{area}}(Y) < +\infty$ and pick $H^0_0(Y) \ni [S] \neq 0$. Then $Y\sharp S M$ carries no metric with positive scalar curvature if $M$ is spin with $K_{\text{area}}(M) = +\infty$.

Here, $K_{\text{area}}(M)$ is defined as in [Gro96, Section 4]; see also Section 5 below for a review of this classical notion.

Theorem 1.1 follows immediately from the next two results.

Theorem 1.2. Let $Y \in BG_{2k}^A$ with $0 \leq K^b_{\text{area}}(Y) < +\infty$ and pick $H^0_0(Y) \ni [S] \neq 0$. Then $K^b_{\text{area}}(Y\sharp S M) = +\infty$ if $K_{\text{area}}(M) = +\infty$.

Theorem 1.3. If $X^{2k} \in BG_{2k}$ is spin with $K^b_{\text{area}}(X) = +\infty$ then it carries no metric (in the given bounded geometry class) whose scalar curvature has non-positive part concentrated outside arbitrarily large compact sets. In particular, it carries no metric with positive scalar curvature.

Theorem 1.3 has an independent interest as it shows that infinite $K^b$-area is an obstruction to the existence of metrics of positive scalar curvature in the bounded geometry setting. In fact, it provides a somewhat stronger result, namely, given an exhaustion $W_i \subset X$ by compact domains then there is no sequence of metrics $g_i$ in the given bounded geometry class with the non-positive part of the scalar curvature of $g_i$ contained in $X \setminus W_i$.

Remark 1.1. The assumption $\widehat{A}(M) \neq 0$ only makes sense if $\dim M$ (and hence $\dim Y$) is a multiple of 4. On the other hand, our result applies to certain manifolds in every even dimension and moreover the attached manifold $M$ can be chosen to be more familiar. It applies for example to $\mathbb{R}^{2k} \sharp \mathbb{Z}^{2k} \mathbb{T}^{2k}$, where $\mathbb{R}^{2k} \in BG_{2k}^A$ is the flat euclidean space, $\mathbb{Z}^{2k} \subset \mathbb{R}^{2k}$ is the standard integer lattice and $\mathbb{T}^{2k}$ is a torus (which has infinite $K$-area).
More generally, we can replace $\mathbb{T}^{2k}$ by any finitely enlargeable spin manifold \cite{GL80b}; see Remark 3.2. Also, as observed in \cite{Why01}, the class $[S] \in H^0_0(X), X \in BG_A^2$, lies in a non-Hausdorff homology group and hence standard obstructions based on C*-algebra techniques do not seem to work here.

Remark 1.2. It is shown in \cite{BW92} that if $Y^4, l \geq 2$, admits a metric with uniformly positive scalar curvature, $[S] = 0$ and $M$ is simply connected with $\hat{A}(M) \neq 0$ then $Y^S_M$ admits a positive scalar curvature metric. In view of this, Whyte’s result means that $[S] \neq 0$ is the only obstruction to the existence of such metrics on $Y^S_M$. The proof in \cite{BW92} uses a relative version of a surgery argument due to Gromov and Lawson \cite{GL80a}, which by its turn is based on Smale’s $h$-cobordism theory. In particular, simply connectedness of $M$ seems to be essential there (in order to kill handles with small index) and consequently the argument breaks down in our case because $K_{\text{area}}(M) = +\infty$ implies the non-triviality of $\pi_1(M)$ (see Remark 3.2).

Our proof of Theorem 1.1 follows Whyte’s approach with suitable modifications to account for the fact that we will be dealing with almost flat complex bundles over $X$. The argument makes use of the index theory of Atiyah-Patodi-Singer (APS) type boundary conditions (see Section 2). The classical notion of $K$-area is reviewed in Section 3 and our variant, which is well suited to the $BG_{2k}$ category, is introduced in Section 4. The proofs of Theorems 1.2 and 1.3 are presented in Section 5.

2. APS INDEX THEORY

Let $W$ be an oriented $n$-dimensional spin manifold with a fixed spin structure \cite{LM16}. In the presence of a Riemannian metric $g$, there exists over $W$ a canonical hermitian vector bundle $S_W$, the spinor bundle, which comes equipped with a Clifford product $c : \Gamma(TW) \to \Gamma(\text{End}(S_W))$ and a compatible connection $\nabla : \Gamma(S_W) \to \Gamma(T^*W \otimes S_W)$. Using these structures we can define the corresponding Dirac operator $\partial : \Gamma(S_W) \to \Gamma(S_W)$ acting on spinors,

$$\partial = \sum_{i=1}^n c(e_i)\nabla_{e_i},$$

where $\{e_i\}$ is a local orthonormal basis tangent to $W$. More generally, we can fix a hermitian vector bundle $E$ with compatible connection $\nabla$ and consider the twisted Dirac operator $\partial_E : \Gamma(S_W \otimes E) \to \Gamma(S_W \otimes E)$ acting on (twisted) spinors. The Weitzenböck decomposition for the corresponding Dirac Laplacian is

$$\partial_E^2 = \nabla^*\nabla + \frac{1}{4}\kappa + R[\nabla],$$

where $\nabla^*\nabla$ is the Bochner Laplacian of $S_W \otimes E$, $\kappa$ is the scalar curvature of $W$ and for $\psi \otimes \eta \in \Gamma(S_W \otimes E)$,

$$R[\nabla](\psi \otimes \eta) = \frac{1}{2} \sum_{ij} c(e_i)c(e_j)\psi \otimes R^\nabla_{e_i,e_j} \eta,$$

with $R^\nabla$ being the curvature tensor of $\nabla$. 
If $W$ is closed, which we assume for the moment, $\partial_E$ is a self-adjoint elliptic operator and $\ker \partial_E$, the space of harmonic spinors, has finite dimension. If $n = 2k$ one has a decomposition
\begin{equation}
S_W \otimes E = (S_W^+ \otimes E) \oplus (S_W^\mp \otimes E)
\end{equation}
into positive and negative spinors induced by the chirality operator and $\partial_E$ interchanges the factors, so we can decompose, according to (2.3),
\begin{equation}
\partial_E = \begin{pmatrix} 0 & \partial_E^- \\ \partial_E^+ & 0 \end{pmatrix},
\end{equation}
where
\begin{equation}
\partial_E^\pm = \partial_E|_{\Gamma(S_W^\pm \otimes E)} : \Gamma(S_W^\pm \otimes E) \to \Gamma(S_W^\mp \otimes E).
\end{equation}
It follows that $\partial_E^+$ and $\partial_E^-$ are adjoint to each other, so we get a well-defined index
\[ \text{ind} \partial_E^+ = \dim \ker \partial_E^+ - \dim \ker \partial_E^- . \]
The Atiyah-Singer index formula computes this integer as
\begin{equation}
\text{ind} \partial_E^+ = \int_W [\hat{A}(TW) \wedge \text{ch}(E)]_{2k},
\end{equation}
where $\hat{A}(TW) \in H^{4*}(W; \mathbb{Q})$ is the $\hat{A}$-class of $TW$, $\text{ch}(E) \in H^{2*}(W; \mathbb{Q})$ is the Chern character of $E$ and the notation $[\ ]_{2k}$ means that integration picks the element of degree $2k$ in the wedge product. Specializing to the case where $k = 2l$ and $E = W \times \mathbb{C}$ is the trivial line bundle (equipped with a flat connection) we get
\begin{equation}
\text{ind} \partial_E^+ = \hat{A}(W),
\end{equation}
where
\begin{equation}
\hat{A}(W) = \int_W [\hat{A}(TW)]_d
\end{equation}
is the $\hat{A}$-genus of $W$. Notice that in this case (2.1) reduces to
\begin{equation}
\partial^2 = \nabla^* \nabla + \frac{\kappa}{4}.
\end{equation}

Remark 2.1. The famous Lichnerowicz’s argument [LM16] is based on the fact that, since $\nabla^* \nabla$ is nonnegative, the positivity of $\kappa$ in (2.7) implies that $\partial$ is positive and hence invertible, which gives $\hat{A}(W) = 0$ by (2) and (2.5). So, $\hat{A}(W) \neq 0$ is a topological obstruction to the existence of metrics with $\kappa > 0$. Thus, the point of Whyte’s theorem is that if an obstructing $M$ (i.e. with $\hat{A}(M) \neq 0$) is “glued” to $Y$ along a nontrivial class $[S] \in H^{4l}_0(Y)$ then an obstruction to metrics with $\kappa \geq 0$ on $Y_S M$ arises, even if $Y$ originally carries such a metric. Our main result (Theorem 1.1) says that instead of nonzero $\hat{A}$-genus we can use infinite $K$-area as a “glued” obstruction to metrics of positive scalar curvature as long as $0 \leq K_{\text{area}}(Y) < +\infty$. 

We now consider the index theory for manifolds with boundary (see for example [APS75, BBW12, Gil93]). Assume from now on that $W$ is a compact spin manifold with dimension $n = 2k$ and nonempty smooth boundary $\Sigma \subset W$, and $\mathcal{E}$ is a hermitian bundle over $W$ with a compatible connection. Introduce Fermi coordinates $(x, u) \in \Sigma \times (0, \delta) \to \mathcal{U}$ in a collar neighborhood $\mathcal{U}$ of $\Sigma$ and set $\Sigma_u = \{(x, u) : x \in \Sigma\}$ so that $\Sigma_0 = \Sigma$. Then, restricted to $\mathcal{U}$,

$$\vartheta = \epsilon (\partial_u) \left( \partial_u + D - \frac{1}{2} H \right),$$

where $H$ is the mean curvature of the embeddings $\Sigma_u \subset \mathcal{U}$ (computed with respect to the inward unit vector field) and $D$ is a self-adjoint linear operator, the tangential Dirac operator, defined as follows. For each $u$, $S_W|\Sigma_u$ comes equipped with the Clifford product $c^u = -c(\partial/\partial u)c$, so if we consider the induced connection $\nabla^u = \nabla - \frac{1}{2} c^u(A)$, where $A$ is the shape operator of the embedding $\Sigma_u \subset \mathcal{U}$, then

$$D = \sum_{i=1}^{2k-1} c^u(e_i) \nabla^u e_i,$$

where $\{e_i\}$ is an orthonormal basis tangent to $\Sigma_u$. After twisting with $\mathcal{E}$, we obtain a first order self-adjoint elliptic operator $D\mathcal{E}$ acting on $\Gamma(S^+_\Sigma \otimes \mathcal{E}|\Sigma)$ and commuting with chirality, so we can decompose $S_W \otimes \mathcal{E}|\Sigma =: \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ and accordingly,

$$D\mathcal{E} = \left( \begin{array}{cc} D^+ & 0 \\ 0 & D^- \end{array} \right)$$

with $D^\pm$ self-adjoint. Under the natural identification $\mathcal{E}^+ = \mathcal{E}^-$ one has $D^+ = -D^-$ and hence $\text{Spec}(D\mathcal{E})$ is symmetric with respect to $0 \in \mathbb{R}$, but of course this does not need happen to the factors $D^\pm$. Thus, for $\text{Re } z \gg 0$ we define the eta function

$$\eta^\pm(z) = \sum_{0 \neq \lambda \in \text{Spec}(D^\pm)} \text{sign } \lambda |\lambda|^{-z} \dim E_\lambda(D^\pm),$$

where $E_\lambda(D^\pm)$ is the eigenspace of $D^\pm$ associated to $\lambda$. This extends meromorphically to the whole complex plane with the origin not being a pole and $\eta^+(0)$ is a well defined real number called the eta invariant of $D^\pm$. It measures the overall asymmetry of $\text{Spec}(D^\pm)$ with respect to the origin.

In general, the existence of a boundary implies that the space of harmonic spinors $\ker \partial_u = \ker D^+ \oplus \ker D^-$ is infinite dimensional and one has to impose suitable boundary conditions in order to restore finite dimensionality of kernels. Here we consider Atiyah-Patodi-Singer (APS) type boundary conditions and for this we need to introduce some notation. If $D$ is a self adjoint elliptic operator acting on sections of a bundle $\mathcal{F} \to \Sigma$, we denote by $\Pi_I(D) : L^2(\mathcal{F}) \to L^2(\mathcal{F})$ the spectral projection of $D$ associated to the interval $I \subset \mathbb{R}$. Also, if $\psi \in \Gamma(S_W \otimes \mathcal{E})$ we set $\varphi = \psi|\Sigma$. Now consider $\Gamma_{\geq 0}(S^+_W \otimes \mathcal{E}) = \{ \psi \in \Gamma(S^+_W \otimes \mathcal{E}) : \}$
\[ \Gamma(S^+_W \otimes \mathcal{E}); \Pi_{[0, +\infty)}(D_E)\varphi = 0 \} \text{ and } \Gamma_{>0}(S^-_W \otimes \mathcal{E}) = \{ \psi \in \Gamma(S^-_W \otimes \mathcal{E}); \Pi_{[0, +\infty)}(D_E)\varphi = 0 \}, \]

which are the domains of the operators

\[ \partial /\partial x^+ - E \big|_{\Gamma_{>0}(S^-_W \otimes \mathcal{E})} : \Gamma_{>0}(S^-_W \otimes \mathcal{E}) \to \Gamma(S^+_W \otimes \mathcal{E}) \]

and

\[ \partial /\partial x^- - E \big|_{\Gamma_{>0}(S^-_W \otimes \mathcal{E})} : \Gamma_{>0}(S^-_W \otimes \mathcal{E}) \to \Gamma(S^+_W \otimes \mathcal{E}), \]

respectively. These are adjoints to each other and moreover \( \partial /\partial x^+ - E \) is a Fredholm operator with a well defined index

\[ \text{ind} \partial /\partial x^+ - E = \text{dim ker} \partial /\partial x^+ - E - \text{dim ker} \partial /\partial x^- - E. \]

The following formula computes this index (see \[APS75\] or \[BBW12\] for the case where \( \mathcal{U} \) is an isometric product and \[Gil93\] for the general case):

\[ \text{(2.8)} \quad \text{ind} \partial /\partial x^+ - E = \int_W [\widehat{A}(TW) \wedge \text{ch}(\mathcal{E})]_{2k} + \int_{\Sigma} [T\widehat{A}(TW) \wedge \text{ch}(\mathcal{E})]_{2k-1} - \xi^E(0), \]

where \( T\widehat{A} \) is the transgression of \( \widehat{A} \) along \( \Sigma \), which is polynomial in the curvature, and

\[ \xi^E(0) = \frac{1}{2} \left( \eta^E(0) + \text{dim ker } D^+_E \right). \]

For further reference we present now the integral version of (2.1). Recalling that \( \varphi = \psi|_{\Sigma} \), the identity reads

\[ \text{(2.9)} \quad \int_X \left( \frac{\kappa}{4} |\psi|^2 + \langle R[\nabla] \psi, \psi \rangle + |\nabla \psi|^2 - |\partial \varphi|^2 \right) = \int_{\Sigma} \left( \langle D_E \varphi, \varphi \rangle - \frac{H}{2} |\varphi|^2 \right). \]

3. Gromov’s K-area

The concept of K-area was introduced by Gromov \[Gro96\] in order to quantify previous results on geometric-topological obstructions to the existence of metrics with positive scalar curvature. We now briefly review this material.

Let \( (X^{2k}, g) \) be a closed Riemannian manifold (not necessarily spin). By pulling back any of the half-spin bundles \( S^\pm_{S^{2k}} \to S^{2k} \), whose top Chern classes are non-trivial, under a degree one map we have that the set of complex vector bundles over \( X \) which are homologically non-trivial (i.e. which have at least a nonzero Chern number) is nonempty. Notice that by Chern-Weil theory, the Chern numbers, which are topological invariants of \( \mathcal{E} \), can be computed by integrating over \( X \) certain universal differential forms depending on the curvature tensor \( R^E \) of any compatible connection on \( \mathcal{E} \). Thus \( \mathcal{E} \) is homologically trivial (i.e. all Chern numbers vanish) if \( R^E = 0 \). Here and in what follows we abuse notation and denote simply by \( R^E \) the curvature of any compatible connection on \( \mathcal{E} \).

We then let \( (\mathcal{E}, \nabla) \) vary over the set of homologically non-trivial hermitian bundles (and compatible connections) over \( X \) and define the K-area of \( (X, g) \) by

\[ \text{(3.1)} \quad K_{\text{area}}(X, g) = \sup \frac{1}{\| R^E \|_g}, \]
where

\begin{equation}
\|R^{\mathcal{E}}\| = \sup_{v \wedge w \neq 0} \frac{\|R_{\mathcal{E},w}^{\mathcal{E}}\|_g}{\|v \wedge w\|_g}
\end{equation}

and \(\|v \wedge w\|_g^2 = g(v,v)g(w,w) - g(v,w)^2\). Clearly, the \(\mathcal{K}\)-area as defined above is a Riemannian invariant but the fact that it is finite or infinite is a topological property of \(X\). We note that the use of the operator norm in the definition allows us to conclude that if \((\mathcal{E}_1, \nabla_1)\) and \((\mathcal{E}_2, \nabla_2)\) then

\begin{equation}
\|R^{\nabla_1 \otimes \nabla_2}\|_g = \max \left\{ \|R^{\nabla_1}\|_g, \|R^{\nabla_2}\|_g \right\}; \quad \|R^{\nabla_1 \otimes \nabla_2}\|_g = \|R^{\nabla_1}\|_g + \|R^{\nabla_2}\|_g.
\end{equation}

This is a rather flexible concept which can be easily adapted to other settings. For instance, if \(X \in B G_{2k}\) we may retain the definition (3.1) but restricting to bundles which are trivial at infinity (i.e. in a neighborhood of the point at infinity in the one-point compactification of \(X\)). The allowable connections are required to be flat at infinity so that characteristic numbers related to \(\mathcal{E}\) are obtained by integrating over \(X\) characteristic differentials forms with compact support. By the pull back construction above, we always have that \(K_{\text{area}}(X,g) > 0\). In any case, with this definition, the fact that the \(\mathcal{K}\)-area is finite or infinite is obviously a bounded geometry type invariant of \(X \in B G_{2k}\), so we may simply write \(K_{\text{area}}(X) = K_{\text{area}}(X,g)\) whenever no confusion arises. It turns out, however, that this notion of \(\mathcal{K}\)-area in \(B G_{2k}\) is not suitable for the purposes we have in mind; see Remark 4.1. In any case, the following useful characterizations are readily derived from the definitions.

**Proposition 3.1.** i) \(K_{\text{area}}(X,g) = +\infty\) if and only if for any \(\epsilon > 0\) there exists a homologically non-trivial admissible \(\mathcal{E}\) over \(X\) with \(\|R^{\mathcal{E}}\|_g \leq \epsilon\); ii) \(K_{\text{area}}(X,g) < +\infty\) if and only if there exists \(\epsilon_{X,g} > 0\) such that if \(\mathcal{E}\) over \(X\) is admissible and \(\|R^{\mathcal{E}}\|_g \leq \epsilon_{X,g}\) then \(\mathcal{E}\) is homologically trivial.

**Remark 3.1.** It is crucial here to let \(\text{rank} \mathcal{E} \to +\infty\) as \(\epsilon \to 0\) in case \(\text{vol}_{2k}(X) < +\infty\). In fact, by Chern-Weil theory we have \(1 \leq \|R^{\mathcal{E}}\|_g^k \text{vol}_{2k}(X)P(\text{rank} \mathcal{E}, k)\) for some polynomial \(P\) in case \(\mathcal{E}\) is homologically non-trivial.

**Remark 3.2.** Examples of manifolds with infinite \(\mathcal{K}\)-area in the closed category include tori and, more generally, finitely enlargeable manifolds \([GL80b]\). This includes solvable manifolds and non-positively curved manifolds whose fundamental group is residually finite. On the other hand, simply connected closed manifolds always have finite \(\mathcal{K}\)-area. In the bounded geometry category, let us take a Riemannian manifold \(X^{2k}\) which is hyper-euclidean in the sense that for some \(\epsilon > 0\) there exists a nonzero degree, proper map \(f : X \to \mathbb{R}^{2k}\) satisfying \(\|f_*\omega\| \leq \epsilon \|\omega\|\) for any 2-form \(\omega\) over \(X\). This gives \(K_{\text{area}}(X) \geq \epsilon^{-1}K_{\text{area}}(\mathbb{R}^{2k})\) and sending \(\epsilon \to 0\) (which can be accomplished by scaling) we get \(K_{\text{area}}(X) = +\infty\). As a simple example we can take \(X\) to be any Hadamard manifold within its \(B G_{2k}\) class. In particular, \(K_{\text{area}}(\mathbb{R}^{2k}) = +\infty\).

4. **Uniformly finite homology, bounded de Rham cohomology and \(K_{\text{area}}^b\)**

As mentioned in the Introduction, Block and Weinberger \([BW92]\) have defined a bounded geometry (in fact, coarse) invariant homology \(H^u_0\), the so-called uniformly finite homology in degree zero. If \(Y \in B G_n\) then \(S \subset Y\) defines a class \([S] \in H^u_0(Y)\) if \(S\) is locally
uniformly finite. Our aim here is to extend Whyte’s obstruction to $Y♯S$ in case the attached manifold satisfies $K_{\text{area}}(M) = +\infty$. For this we need the appropriate notion of $K$-area for objects in $BG_{2k}$.

Now, a key property of $H^{uf}_0$ is that it is naturally dual to $H^n_b$, the bounded de Rham cohomology in degree $n = \dim X$ [Why01, Lemma 2.2]. If $Y \in H^n_b(X)$ we denote by $Y^{uf}$ the corresponding homology class in $H^{uf}_0(X)$. Regarding this identification, the following is well-known; see [BW92, Theorem 3.1].

**Proposition 4.1.** $X \in BG^n_N$ if and only $X$ is open at infinity, which means by definition that large compact domains in $Y$ satisfy a linear isoperimetric inequality. Equivalently, any bounded $n$-form is the differential of a bounded $(n - 1)$-form.

The next result goes one step further and provides a useful criterion to decide when a given class in $H^n_b(X) \simeq H^{uf}_0(X)$ is trivial; see [Why01, Lemma 2.4].

**Proposition 4.2.** $Y \in H^n_b(X)$ vanishes if and only if for all sufficiently large compact domain $W \subset X$ with boundary $\Sigma$ there holds
\[
\left| \int_W Y \right| \leq C \text{vol}_{2k-1}(\Sigma),
\]
where $C > 0$ depends on curvature and second fundamental form bounds (and possibly on $Y$).

It is also proved in [Why01, Lemma 2.1] that any characteristic polynomial $\mathcal{Q}$ gives, after evaluation on the curvature, a characteristic number $\mathcal{Q}^b \in H^n_b(X)$. A closer look at the proof of the Poincaré duality $H^{uf}_0 \cong H^n_b$ in [Why01] reveals that this construction behaves quite well under the infinite connected sum operations we are dealing with.

**Proposition 4.3.** If $Y \in H^n_b(Y)$ is given by a characteristic form via Chern-Weil theory, $[S] \in H^{uf}_0(Y)$ and $M$ is closed then
\[
Y^{uf}(Y♯S) = Y^{uf}(Y) + Y(M)[S], \tag{4.1}
\]
where $Y(M) = \int_M Y$ is the corresponding characteristic number computed over $M$.

The following result is proved in [Why01, Theorem 2.3].

**Theorem 4.1.** If $X \in BG_n$ is spin and carries a metric with $\kappa \geq 0$ then $\hat{A}^b(X) = 0$.

**Remark 4.1.** This last result is combined with (4.1) in [Why01] to yield the obstruction mentioned in Remark 1.1. Thus, it is tempting to try to extend Theorem 4.1 by replacing $\hat{A}^b(X) \neq 0$ with infinite $K$-area as defined in Section 3. However, counterexamples to the corresponding statement are easily found. For instance, the bounded geometry class of the standard flat metric in $\mathbb{R}^{2k}$ carries a scalar flat metric but has infinite $K$-area by Remark 3.2.

The counterexample in the previous remark asks for the proper modification of the notion of $K$-area in $BG_{2k}$. With this goal in mind, we consider Hermitean bundles $(E, \nabla)$ over $X \in BG_{2k}$ such that:

- the curvature $R^{\nabla}$ is uniformly bounded over $X$, so the corresponding characteristic numbers lie in $H^n_b(X)$. 

Notice that we do not require that the bundle $E$ is trivial in a neighborhood of infinity. Thus, our choice here departs a bit from the general philosophy in [Gro96], where the allowable bundles in the open case are tied to compactly supported cohomology; see Section 3. In any case, with this notion of admissible bundle at hand, the idea is to retain the definition (3.1) but restricting to Hermitian bundles $(E, \nabla)$ which are $b$-homologically non-trivial in the sense that at least one bounded Chern number $H^{2k}_b(X) \ni c^l_1(E) \neq 0$, $|I| = k$. We denote the corresponding geometric invariant by $K^b_{\text{area}}(X)$, the $K^b$-area of $X$.

It might well happen that the set of $b$-homological non-trivial bundles is empty, in which case the bounded $K$-area vanishes for trivial reasons. By Proposition 4.4, this happens if $X$ is non-amenable. Thus, it only possibly holds that $0 < K^b_{\text{area}}(X) \leq +\infty$ if $X$ is amenable. In any case, the fact that this invariant vanishes, is finite or infinite is a bounded geometry invariant property of $X$.

The next property helps to detect an important class of manifolds with null $K^b_{\text{area}}$.

**Definition 4.1.** We say that $X \in \mathcal{BG}_n$ is large at infinity if there exists an exhaustion of $X$ by regular compact domains $W$ and $c > 0$ such that $\operatorname{vol}_{n-1}(\Sigma = \partial W) \geq c$ as $W$ sweeps out $X$.

**Proposition 4.4.** If $X \in \mathcal{BG}_{2k}$ is large at infinity and there exists a neighborhood $U$ of infinity with the homotopy type of the sphere $S^{2k-1}$ then $K^b_{\text{area}}(X) = 0$. In particular, $K^b_{\text{area}}(\mathbb{R}^{2k}) = 0$.

**Proof.** By Bott periodicity, any (admissible) bundle $E$ over $U$ is stably trivial, that is, there exists $l$ such that $E' = E \oplus \Theta^l$ is trivial over $U$ (here and in the following, $\Theta^l$ is the trivial bundle with typical fiber $\mathbb{C}^l$ and varying base space; this trivial bundle is always assumed to be endowed with a flat connection). We note that $l$ may be chosen to depend only on $k$ and $r$. Hence, each Chern number $c^l_1(E')$ is induced by a compactly supported form. Since both the curvature and the rank of $E'$ are controlled, we have

$$\frac{\int_W c^l_1(E')}{\operatorname{vol}_{2k-1}(\Sigma)} \leq C \frac{\operatorname{vol}_{2k} (\text{supp } c^1_1(E'))}{c},$$

so $c^l_1(E')$ vanishes in $H^{2k}_b(X)$ by Proposition 4.2. Since $E$ and $E'$ have the same bounded Chern numbers, we see that any such $E$ is $b$-homologically trivial. \qed

5. The proofs of Theorems 1.2 and 1.3

Proposition 4.4 is a bit discouraging as it shows that elements in a large class of amenable manifolds have null $K^b$-area for trivial reasons (no $b$-homological non-trivial bundle is available). Fortunately, starting with amenable manifolds with null or finite $K^b$-area, the infinite connected sum construction above allows us to display a large supply of manifolds with infinite $K^b$-area. This is precisely the content of Theorem 1.2 whose proof we now present.

5.1. The proof of Theorem 1.2. Let $X = Y \#_S M$ and assume by contradiction that $0 \leq K^b_{\text{area}}(X) < +\infty$. Thus, if $g$ is a metric in the given bounded geometry class, by the analogue of Proposition 3.1 there exists $\epsilon > 0$ such that if $(E, \nabla)$ admissible over $X$ satisfies $\|R^\nabla\| \leq \epsilon$ then $E$ is $b$-homologically trivial.
Let \((\mathcal{F}, \nabla_\mathcal{F})\) an admissible bundle over \(Y\) with \(\|R^\nabla_{\mathcal{F}}\| \leq \epsilon' < \epsilon\). If \(\mathcal{V} \subset M\) is a compact tubular neighborhood of the sphere \(\mathbb{S}^{2k-1}\) over which the connected sum operation leading to \(X\) was carried out, then \(\mathcal{V}\) has the same homotopy type as \(\mathbb{S}^{2k-1}\). Hence, by Bott periodicity, there exists \(l\) such that \(\mathcal{F}' = \mathcal{F} \oplus \Theta^l\) is trivial over \(\mathcal{V}\). In particular, \(\mathcal{F}'\) may be extended to \(M\) (as a trivial bundle) and hence to \(X\).

On the other hand, since \(K_{\text{area}}(M) = +\infty\), there exists a homologically non-trivial bundle \((\mathcal{G}, \nabla_{\mathcal{G}})\) over \(M\) with \(\|R^\nabla_{\mathcal{G}}\| \leq \epsilon'\). Again, there exists \(m\) such that \(\mathcal{G}' = \mathcal{G} \oplus \Theta^m\) is trivial if restricted to \(\mathcal{V}\), so \(\mathcal{G}'\) may be extended to \(Y\) (as a trivial bundle) and hence to \(X\). As before, \(l\) and \(m\) may be chosen to depend only on \(k\) and \(r\).

We next consider the bundle \(\mathcal{H} = \mathcal{F}' \oplus \mathcal{G}'\). Since we assume that \(0 \leq K_{\text{area}}^b(X) < +\infty\), \(\mathcal{H}\) is \(b\)-homologically trivial over \(X\) and by duality we find that \(c_{J}(\mathcal{H})^{uf}(X) = 0\) for any bounded Chern number class \(c^b_J(\mathcal{H})\). However, we may find \(J\) such that \(c_{J}(\mathcal{H})(M) = c_{J}(\mathcal{G}')\) is non-zero. By Proposition 4.3, \(c_{J}(\mathcal{H})^{uf}(Y) = c_{J}(\mathcal{H})(M)[S] \neq 0\), so \(\mathcal{H}|_\Sigma\) is \(b\)-homologically non-trivial. Since \(\mathcal{H}|_\Sigma\) and \(\mathcal{F}\) have the same bounded Chern numbers, \(\mathcal{F}\) is \(b\)-homologically non-trivial as well. But this means that \(K_{\text{area}}^b(Y) = +\infty\), a contradiction that completes the proof of Theorem 1.2.

5.2. The proof of Theorem 1.3 We fix \(X \in B\mathbb{S}^{2k}\) spin with \(K_{\text{area}}^b(X) = +\infty\) and consider \(W \subset X\) a compact regular domain with \(\Sigma = \partial W\). By passing to another metric in the bounded geometry class if needed, we may assume that \(\kappa > 0\) on \(W\). Also, we may assume that the second fundamental form of \(\Sigma\) is uniformly bounded. Note that the existence of arbitrarily large domains satisfying this latter condition with bounds depending on the underlying geometry follows from the chopping construction in [CG90] combined with the bounded geometry assumption.

We first observe that applying index theory to \(W\) and arguing as in the proof of [Why01, Theorem 2.3], we obtain

\[
\int_W [\hat{A}(TX)]_{2k} \leq C\text{vol}_{2k-1}(\Sigma),
\]

which means that \(\hat{A}^b(X) = 0\) by Proposition 4.2. Henceforth, \(C > 0\) is a constant which only depends on the dimension and the curvature bounds (in case it further depends on additional parameters, we explicitly indicate this by using them as subscripts). Our aim is to adapt the twisted case the argument leading to \((5.1)\).

Let \(\kappa_W := \inf_W \kappa_g > 0\). By the analogue of Proposition 4.1 for each \(\epsilon > 0\) there exists an admissible \(b\)-homologically non-trivial bundle \((E, \nabla)\) over \(X\) with \(\|R^\nabla\|_{\kappa_m} \leq \epsilon\), or equivalently, \(\|R^\nabla\|_g \leq \epsilon\kappa_W\). In view of \((2.2)\) this gives \(\|\mathcal{R}^\nabla\|_g \leq \epsilon C_k \kappa_W\) for some \(C_k > 0\) depending only on \(k\). It follows, that restricted to \(W\),

\[
\mathcal{R}^\nabla + \frac{\kappa}{4} \geq \left(-\epsilon C_k + \frac{1}{4}\right) \kappa_W,
\]

so if \(\epsilon = \epsilon_{k,W} \leq \min\{1/8C_k, \kappa_W^{-1}\}\) we obtain not only the curvature bound \(\|R^\nabla\|_g \leq 1\), but also the pointwise estimate

\[
\mathcal{R}^\nabla + \frac{\kappa}{4} \geq \frac{\kappa_W}{8} > 0
\]
on \(W\).
As in the proof of (5.1) we next appeal to the index formula (2.8). For simplicity we set \( s = \text{vol}_{2k-1}(\Sigma) \) and \( r = \text{rank} \mathcal{E} \). It is well-known that \( |\xi^+_{\Sigma}(0)| \leq Crs \) [Ram93 Theorem 3.1.1]. Moreover, from the curvature bounds at our disposal we clearly have

\[
\left| \int_{\Sigma} [\mathcal{T}\hat{A}(TX) \wedge \text{ch}(\mathcal{E})]_{2k-1} \right| \leq CP(r)s,
\]

where \( P \) is a certain polynomial; compare with Remark 3.1.

We now examine the index term in (2.8). We take a harmonic spinor \( \psi \in \ker \mathcal{D}_{E,>0} \cup \ker \mathcal{D}_{E,>0} \). By (5.2) and (2.9),

\[
\int_{\Sigma} \langle D_{\mathcal{E}} \varphi, \varphi \rangle \geq a \int_{\Sigma} |\varphi|^2, \quad \varphi = \psi|_{\Sigma},
\]

where we may assume that \( a = \inf_{\Sigma} H/2 < 0 \). We now Fourier expand \( \varphi = \sum_{\lambda \leq 0} c_{\lambda} \phi_{\lambda} \),

where \( D_{\mathcal{E}} \phi_{\lambda} = \lambda \phi_{\lambda} \). The left-hand side then becomes \( \sum_{\lambda \leq 0} |c_{\lambda}|^2 \) and we see at once that at least one \( \lambda \) lies in the interval \([a,0]\) if \( \psi \neq 0 \). In other words, the linear map

\[
\Xi : \ker \mathcal{D}_{E,>0} \cup \ker \mathcal{D}_{E,>0} \rightarrow \Pi_{[a,0]}(D_{\mathcal{E}}(L^2(\mathcal{E}|_{\Sigma}))), \quad \Xi(\psi) = \sum_{\lambda \in [a,0]} c_{\lambda} \phi_{\lambda},
\]

is well defined and injective. The conclusion is that

\[
\dim \ker \mathcal{D}_{E,>0} + \dim \ker \mathcal{D}_{E,>0} \leq N(a^2),
\]

where \( N(a^2) \) is the number of eigenvalues of the Dirac Laplacian \( D_{\mathcal{E}} \) in the interval \([0,a^2]\). A standard argument [Why01] shows that \( |N(a^2)| \leq Crs \). As a consequence, \( |\text{ind} \mathcal{D}_{E,>0}| \leq Crs \).

Remark 5.1. The reason why the estimates on \( |\xi^+_{\Sigma}(0)| \) and \( |\text{ind} \mathcal{D}_{E,>0}| \) are linear in \( r \) is that the corresponding arguments are spectral in nature and rely on the curvature bounds to compare the spectra of \( D_{\mathcal{E}} \) and \( D_{\Theta^r} \), where \( \Theta^r = \Sigma \times \mathbb{C}^r \) is the trivial bundle endowed with a flat connection, and then using that this latter Dirac operator is just a sum of copies of \( D \). On the other hand, since we are not assuming that \( \mathcal{E}|_{\Sigma} \) is trivial, the estimate in (5.3) is polynomial in principle.

If we now combine the estimates above with (2.8), the conclusion is that

\[
\left| \int_W [\hat{A}(TX) \wedge \text{ch}(\mathcal{E})]_{2k} \right| = \left| \int_W [\hat{A}(TW) \wedge \text{ch}(\mathcal{E})]_{2k} - r \int_W [\hat{A}(TX)]_{2k} \right| \leq Crs.
\]

Here, the reduced Chern character of \( \mathcal{E} \) is

\[
\text{ch}(\mathcal{E}) = \text{ch}(\mathcal{E}) - r = \text{ch}_1(\mathcal{E}) + \text{ch}_2(\mathcal{E}) + \ldots,
\]

with \( \text{ch}_i(\mathcal{E}) \in \Gamma(\wedge^{2i}TX) \) defined by a universal homogeneous characteristic polynomial of degree \( i \) in \( R^\infty \).

We now follow [Gro96] and bring the mechanism of Adams operations to our discussion. Recall that this is a rule that to each \( \mu \in \mathbb{N} \) and \( \mathcal{E} \) as above associates a (virtual)
bundle $\Psi_{\mu}E$ which is a universal expression in terms of tensor products of exterior powers of $E$. It is compatible with the Chern character map in the sense that

$$\text{ch}(\Psi_{\mu}E) = \sum_{j \geq 0} \text{ch}_j(E) \mu^j.$$  

In particular, $\text{rank } \Psi_{\mu}E = r = \text{rank } E$. Moreover, for each $\nu \in \mathbb{N}$ and a multi-index $\mu_{(\nu)} = (\mu_1, \ldots, \mu_\nu)$, one has that

$$\Psi_{\mu_{(\nu)}} E := \Psi_{\mu_1} E \otimes \cdots \otimes \Psi_{\mu_\nu} E$$

satisfies $\text{rank } \Psi_{\mu_{(\nu)}} E = r^\nu$ and

$$\text{ch}_j \Psi_{\mu_{(\nu)}} E = \sum_{i_1 + \cdots + i_\nu = j} \mu_1^{i_1} \cdots \mu_\nu^{i_\nu} \text{ch}_{i_1}(E) \wedge \cdots \wedge \text{ch}_{i_\nu}(E).$$

Also, by (3.3) the curvature bounds are “stable” under Adams operations, so the argument leading to (5.4) works fine to yield

$$\left| \int_W [\hat{A}(TX) \wedge \hat{\text{ch}}(\Psi_{\mu_{(\nu)}} E)]_{2k} \right| \leq C_{r,\nu}.$$  

In particular, if we take $\nu = k$ then Proposition 4.2 implies that $\mathcal{D}^b(\mu_{(k)}) = 0$, where

$$\mathcal{D}(\mu_{(k)}) = [\hat{A}(TX) \wedge \hat{\text{ch}}(\Psi_{\mu_{(k)}} E)]_{2k}$$

$$= \sum_{i_1 + \cdots + i_k = k} \mu_1^{i_1} \cdots \mu_k^{i_k} \text{ch}_{i_1}(E) \wedge \cdots \wedge \text{ch}_{i_k}(E) + \ldots,$$

with the dots corresponding to “lower order terms”. Thus, each bounded Chern character class $(\text{ch}_{i_1}(E) \wedge \cdots \wedge \text{ch}_{i_k}(E))^b$ vanishes. Since each bounded Chern number $c^b_l(E)$ is a universal rational linear combination of such classes, we see that each $c^b_l(E)$ vanishes as well. But this means that $E$ is $b$-homologically trivial and this contradiction completes the proof of Theorem 1.3.

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