Networks with Growth and Preferential Attachment: Modeling and Applications

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In this article we presented a brief study of the main network models with growth and preferential attachment. Such models are interesting because they present several characteristics of real systems. We started with the classical model proposed by Barabási and Albert\textsuperscript{1}: nodes are added to the network connecting preferably to other nodes that are more connected. We also presented models that consider more representative elements from social perspectives, such as the homophily between the vertices or the fitness that each node has to build connections\textsuperscript{2,3}. Furthermore, we showed a version of these models including the Euclidean distance between the nodes as a preferential attachment rule\textsuperscript{4}. Our objective is to investigate the basic properties of these networks as distribution of connectivity, degree correlation, shortest path, cluster coefficient and how these characteristics are affected by the preferential attachment rules. Finally, we also provided a comparison of these synthetic networks with real ones. We found that characteristics as homophily, fitness and geographic distance are significant preferential attachment rules to modeling real networks. These rules can change the degree distribution form of these synthetic network models and make them more suitable to model real networks.

I. INTRODUCTION

Complex systems has become a widely applied area of research because of everything around us can be described by a complex network, including social, technological or biological organisms. The growth and the preferential attachment considering that a node has higher probability to connect with a other node that already have many edges are famous ingredients\textsuperscript{1} to produce a power law degree distribution, frequently used topology to describe real systems.

In general, it has been shown that real networks present a power law degree distribution with $2 < \gamma < 3$\textsuperscript{5,13}. However, this is a controversial topic\textsuperscript{10,14,15}. In a recent study, Broido and Clauset\textsuperscript{14} investigated nearly 1000 of real networks using statistical tools. They showed evidences that power-law degree structured is not usual to be found in real-life. They evaluated social, biological, technological, transportation, and information networks. Their main conclusion is social networks are weakly scale-free while technological and biological networks are strongly scale-free. However, they also found that 51% of the real data set can be classified as some kind of scale-free category. Barabási also arguments\textsuperscript{1} that real networks, ruled by growth and preferential attachment, have power law with an exponential cutoff degree distribution.

In this paper, we investigated social and technological real networks and we found that they can be modeled by networks with growth and preferential attachment. To account for more realistic aspects, we considered other concepts in the preferential attachment as fitness\textsuperscript{2}, homophily\textsuperscript{3}, and Euclidean distance between nodes\textsuperscript{4}. Indeed, social systems often present these kind of feature’s connections\textsuperscript{16,17} and real-world systems in general are often embedded in Euclidean space\textsuperscript{18–21}. We investigated the phone calls\textsuperscript{11}, collaboration\textsuperscript{13} and e-mails networks\textsuperscript{12}. The first two are social networks because they describe family, friendship and/or professional interactions while the email network behaves as a technological network. We also found that the email network present a more “scale-free” behavior in its degree distribution while social networks are better described by a $q$-exponential degree distribution, according to the model proposed by Soares and collaborators\textsuperscript{4}.

The paper is divided as follows: The detailed description of networks models with growth and different rules of preferential attachments are found in section\textsuperscript{II} where we also studied some properties of these networks as degree distribution and assortativity. The main information and results about the networks are summarized in table\textsuperscript{I}. In section\textsuperscript{III} we provided a comparison of these synthetic networks with real ones. At last, we presented our final considerations in section\textsuperscript{IV}.

II. NETWORK MODELS WITH GROWTH AND PREFERENTIAL ATTACHMENT

A network model has properties similar to real systems. Networks are considered a powerful tools to represent patterns of connections between parts of systems such as Internet, power grid, food webs, social networks, etc\textsuperscript{5,22,23}. Some particular metrics properties, like degree distribution, shortest path length, and clustering coefficient have been attracted attention of physics com-

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Watts and Strogatz \cite{24} shows that real networks is characterized by average shortest path distance between two vertex and large clustering coefficient, describing these properties by small-world model.

Based on that, Barabási-Albert \cite{1} proposed two basics mechanisms that try to better characterize a real network: growth of system, adding new agents and preferential attachment, where a new agent connects preferentially with most connected nodes already on the network. The web expands with adding of new documents which links with older or well known sites, for instance. The probability that a new node will connect to a node with $k$ links is proportional to $k$, independently of geographic distance.

However, there are other examples of real networks whose connectivity may depend on the geographic distance between the nodes, as a power grid. In addition to geographic distance, there may be other relevant ingredients to consider when connecting the elements of the system. Social interaction between people have intrinsic characteristics that should be taken into account as for example the influence one person has on another and the affinity between them, representing friendship, familiar or professional ties.

To model these features, some networks have been studied through over the years. We presented below some of them that consider preferential attachment rules according to the degree (Barabási-Albert model \cite{1}), or the fitness of the node to make connections \cite{2}, or the homophy between them \cite{3}, and finally, according to the euclidean distance between the nodes \cite{4}.

\subsection*{A. Barabási-Albert Network}

To explain in a simple way the behavior of technological networks, such as internet, Barabási and Albert \cite{1} proposed the following model:

- The system starts with $m_0$ nodes connected to each other.
- At each time step, a new node $j$ is entered on the network and it connects to a random node $i$ chosen at random with probability $\Pi(k_i|j)$ proportional to its degree ($k_i$), which means

$$\Pi(k_i|j) = \frac{k_i}{\sum_n k_n} \quad (1)$$

where the normalization $\sum_n k_n$ is the sum over all degree $k_n$ of each node $n$ already connected on the network.

These rules define what is know by Barabási-Albert (BA) model, and generate a network with a distribution of connectivity, say $P(k)$, that follows a power-law degree distribution of the form $P(k) \sim k^{-\gamma}$, with $\gamma = 3.0$, in the thermodynamic limit, which is independent of the value of $m_0$, as shown in figure \ref{fig:1}.

We can also calculate the clustering coefficient, say $(C)$, of the BA network \cite{25}. It is the tendency of the network to form fully connected sub-graphs in the neighborhood of a given vertex, and grows with the network size $N$ as:

$$\langle C \rangle \sim \frac{\ln(N)^2}{N} \quad (2)$$

We showed this behavior in figure \ref{fig:2}. The simulation data follow the same bias as given by equation \ref{eq:2}.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Distribution of the connectivity degree $P(k)$ of the BA network. Dots are the average over $10^3$ networks of size $N = 10^5$ and $m_0 = 3$. The dashed line has a slope $P(k) \sim k^{-3}$ and serves as a guide for the eyes.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{Clustering coefficient in function of the network size for BA network. The average was over 100 samples. The dots in the dashed line represents the theoretical value calculated from Eq. \ref{eq:2} and the dots in the continuous line is obtained from simulations.}
\end{figure}

Other important measure of networks is called shortest path length. The distance between two any nodes $i$ and $j$ is defined as the number of links in the shortest path that connects them, named $d_{ij}$. The measure that represents the average over all shortest paths that link all the possible pairs of vertices in the network is called the
average shortest path length \( (d) \) \[25\]. For BA network, it is given by \( \langle d \rangle \sim \frac{\log N}{\log(\log N)} \), confirming its small world property \[22\].

Other feature that should be analysed is the degree correlation. The nodes of a network can present a tendency to connect with other nodes that have a similar or dissimilar degree. When the first case happens one says the network is assortative correlated and if the second case occurs, the network is categorized as a disassortative correlated \[9\].

The simplest and most used way to quantify the degree correlation is given by the average degree of the nearest neighbors (nn) of a vertex \( i \) with degree \( k_i \) \[23\],

\[
k_{nn,i} = \frac{1}{k_i} \sum_{j \in \mathcal{N}(i)} k_j,
\]

where the sum runs over the nearest neighbors vertices of \( i \), represented by the set \( \mathcal{N}(i) \). The degree correlation is obtained by the average degree of the nearest neighbors, \( k_{nn}(k) \), for vertices of degree \( k \) \[26\]. That is,

\[
k_{nn}(k) = \frac{1}{N_k} \sum_{i|k_i=k} k_{nn,i},
\]

where \( N_k \) is the number of nodes of degree \( k \) and the sum runs over all vertices with the same degree \( k \). This quantity is related to the correlations between the degrees of connected nodes because in average it can be expressed as

\[
k_{nn}(k) = \sum_{k'} k' P(k'|k),
\]

where \( P(k'|k) \) is the probability of a node with degree \( k \) to have a neighbour node with degree \( k' \). If degrees of neighboring vertices are uncorrelated, \( P(k'|k) \) is just a function of \( k' \) and \( k_{nn}(k) \) is a constant. If \( k_{nn} \) increases with \( k \) then vertices with high degrees have a larger likelihood of being connected to each other. If \( k_{nn} \) decreases with \( k \), high degree vertices have larger probabilities of have neighbors with low degrees \[20\] \[27\].

The BA network is weakly disassortative as we showed in the figure 3. We observe that the preferential attachment interferes just in the connectivity of nodes recently added in the network. According to the rule of the model, these nodes connect preferably with hubs, creating a disassortative correlation for small values of \( k \). But, as long as the degree grows, the network becomes almost uncorrelated.

We also can use the Pearson coefficient, named \( c_P \), to quantify degree correlations, according to the expression \[27\]:

\[
c_P = \frac{\sum_e j_e k_e / E - [\sum_e (j_e + k_e) / (2E)]^2}{[\sum_e (j_e^2 + k_e^2) / (2E)] - [\sum_e (j_e + k_e) / (2E)]^2},
\]

where \( j_e \) and \( k_e \) are the degrees of the nodes that are in the beginning and in the end of the edge \( e \), and \( E \) is the total number of connections. This quantity ranges from \(-1\) to \(1\) meaning disassortative and assortative networks, respectively. It is a complementary information to the \( k_{nn}(k) \) measure. While the latter provides how the degree correlation can vary with \( k \), the Pearson coefficient \( (c_P) \) quantifies the degree correlation of the entire network according to a scale ranging from \(-1\) to \(1\). This measure was also used to complement the characterization of a topological phase transition on growth and preferential attachment model that consider the euclidean distance between the nodes, as we will see in section II D. In addition, it will be useful to compare the synthetic networks with real ones, in section III.

All the main information of the BA network is summarized in the table \[1\] as well the information about other networks that were also treated in this paper. In general, real networks present a power law degree distribution with \( 2 < \gamma < 3 \) \[4\] \[9\]. So, the BA model is restricted to describe a large set of them because its degree exponent is fixed \( \gamma \approx 3 \). Next, we show other features that can be added to the model to make it more realistic.

**B. Fitness Model: Bianconi-Barabási Network**

The original BA model produces a power-law network with the presence of sites that become privileged that is, with more connections over time. But this model does not taking into account the competitiveness, this means, the ability of younger nodes to acquire new neighbors. Facebook, for example, has become one of the most visited sites in a short period of time when compared to the Google, an older search website. Another example is the growth of corporations where some newer ones concen-
We observed that, when compared to the BA network, which makes the natural attribute, remains a power-law degree distribution but with an exponent more privileged sites and, consequently, more hubs than the BA network, which makes the gamma exponent smaller, that is, the network is more heterogeneous. In the inset of figure 4 we show the degree correlation measured through the nearest-neighbors degree for the Fitness model. The behavior is similar to that one we found for BA network. We also calculate the mean clustering coefficient \( \langle C \rangle \), the average shortest path length \( \langle d \rangle \), and the Pearson correlation coefficient \( c_p \). These informations are shown in table 1. We observed that, when compared to the BA model, this network presents a higher cluster coefficient and a lower Pearson correlation coefficient, but the average shortest path length pretty does not change.

### Table I: Table with all main informations of the networks that were investigated in this work.

| Network                        | \( \Pi(k_i|j) \) | \( P(k) \) | \( \gamma \) | \( \langle C \rangle \) | \( \langle d \rangle \) | \( c_p \) |
|-------------------------------|-----------------|-------------|-------------|----------------------|----------------------|-------|
| Barabási-Albert               | \( \frac{k_i}{\sum_n k_n} \) | ~ \( k^{-\gamma} \) | 3           | 0.0055(5)            | 4.3(1)              | -0.037(4) |
| Fitness Model (Bianconi-Barabási) | \( \frac{\eta_i k_i}{\sum_n \eta_i k_n} \) | ~ \( k^{-\gamma} \) | 2.25        | 0.028(7)             | 4.1(2)              | -0.09(1) |
| Homophilic Model              | \( \frac{(1-A_i) k_i}{\sum_n (1-A_n) k_n} \) | ~ \( k^{-\gamma} \) | 2.75        | 0.015(3)             | 4.2(2)              | -0.038(4) |
| Euclidean Distance Model      | \( \frac{k_i r_{ij}^{-\alpha} A}{\sum_n k_i r_{ij}^{-\alpha} A} \) | ~ \( c_q^{-k/\kappa} \) | -           | 0.0019(2)            | 4.7(1)              | 0.034(7) |
| Fitness Model with euclidean distance | \( \frac{\eta_i k_i r_{ij}^{-\alpha} A}{\sum_n \eta_i k_i r_{ij}^{-\alpha} A} \) | \( c_q^{-k/\kappa} \) | -           | 0.0034(4)            | 4.6(1)              | -0.046(8) |
| Homophilic Model with euclidean distance | \( \frac{(1-A_i) k_i r_{ij}^{-\alpha} A}{\sum_n (1-A_n) k_i r_{ij}^{-\alpha} A} \) | \( c_q^{-k/\kappa} \) | -           | 0.0020(2)            | 4.7(1)              | 0.028(7) |

Figure 4: Distribution of connectivity for Bianconi-Barabási model with \( m_0 = 3, N = 10^4 \) based on 1000 network realizations. The graph is on the log-log scale. The dashed line, with slope \( P(k) \sim k^{-2.25} \), is a guide for the eyes. This network has more privileged sites and, consequently, more hubs than the BA network, which explains its smaller value of \( \gamma \). Inset: Degree correlation measured through the nearest-neighbors degree for the same set of networks. The behavior is similar to that one we found for BA network.

### C. Homophilic Model

In a social network, people tend to relate to others who share common characteristics such as musical taste, football team, religion, and work. To take this tendency
into account in social network models, we can include a connection parameter called homophily.

Almeida and colleagues [3] proposed a model introducing this parameter through an intrinsic property value of each node, called \( \eta_i \), similar to the previous model. The homophily between any two nodes \( i \) and \( j \), say \( A_{ij} \), is defined as the module of the difference between \( \eta_i \) and \( \eta_j \), that is, \( A_{ij} = |\eta_i - \eta_j| \). The lower is \( A_{ij} \), the greater the affinity between both and, consequently, the greater the probability to connect with each other. The proposed algorithm is as follows:

- It starts with \( m_0 \) sites connected to each other, in the same way as in the BA network, but introducing a characteristic \( \eta_i \) for each node, chosen randomly in a uniform distribution \( p(\eta) \) in the interval \([0, 1]\).

- For each time step, add a node \( j \) that links to other \( m_0 \) nodes already on the network. Each \( j \) node connects preferably to a node \( i \) according to the probability

\[
\Pi(k_i | j) = \frac{(1 - A_{ij})k_i}{\sum_n (1 - A_{in})k_n} \tag{8}
\]

- The procedure of the second item is repeated until the network reaches a previously established size \( N \).

The competition between the degree of connectivity and the affinity between the nodes generates a network with a power law degree distribution, but with \( \gamma \approx 2.75 \), as we shown in figure 5. This value is lesser than the exponent obtained in the BA model (\( \gamma = 3.0 \)) but it is greater than the value obtained in the Fitness network (\( \gamma = 2.25 \)). This difference is explained because in the BA network, only the degree of connectivity is considered to make links, which generates the phenomenon “rich gets richer”. In the Fitness model, nodes newly inserted in the network can become hubs more easily as long as they have high fitness. That is, there is a democratization because a node can become hub regardless of its age, as is the case of facebook and google network, for example. In the homophilic model, a node \( j \) can assume a value of \( \eta_j = 0.5 \), for example. When this happens, if it tries to connect to a node \( i \) that has \( \eta_i = 0.3 \) or another node \( k \) which has \( \eta_k = 0.7 \), the affinities between both pairs \( ij \) and \( jk \) are the same. So, in this case, according to the expression \( \Pi \) which dictates the preference in the connection is the degree of the node \( \eta \).

In the inset of figure 5 we show the degree correlation measured through the nearest-neighbors degree for the Homophilic model. The behavior is similar to that one we found for the other networks. We also calculate the mean clustering coefficient \( \langle C \rangle \), the average shortest path length \( \langle d \rangle \), and the Pearson correlation coefficient \( c_{pr} \). These informations are shown in table 1. We observed that, when compared to BA model, this network presents almost the same \( \langle d \rangle \) and \( c_{pr} \) but a slightly higher clustering coefficient.

### D. Networks with Euclidean distance

The models presented above do not take into account the spatial distance between the agents that compose the network. But in many real systems, that variable plays an important role. For example, in the city model proposed by Ribeiro and colaborators [28], the authors observed how the Euclidean distance influences the potential of cities and in scale’s law to measure socio-economic and infrastructure indicators. There are other works that showed the relation between social interaction and spatial properties [20] [21] [29]. For example, in the paper [20], the authors analyzed online social networks and they found that spatial distance restricts who interacts with whom and denser connected groups tend to arise at shorter spatial distances. In the following subsections we reconstructed the standard models shown previously including euclidean distance between the nodes as an attachment ingredient.

#### 1. Model proposed by Soares et. al

Soares and colleagues [4] built a model in which the preferred connection dynamics happens according to the degree of connectivity but also considers the Euclidean distance between the nodes. To build the model, we consider that each site is inserted on a continuous plane.
The first node is added at an arbitrary distance from the origin and the others are isotropically distributed with a probability \( P_G(r) \propto r^{-(2+\alpha_G)} \), which depends on the distance \( r \) from the center of mass, which is positioned at \( r_{cm} \) from the origin and is re-calculate at each time step. The exponent \( \alpha_G \) (G refers to the growth) is responsible for the network growth, that is, defines how close or distant the nodes will be placed. The calculation of the position of the center of mass is given by \( r_{cm} = \frac{1}{M} \sum_{n=1}^N m_n r_n \) where \( m_n \) is the mass of the node \( n \), and \( r_n \) is the vector-distance of this node to the origin, and \( M = \sum_{n=1}^N m_n \) is the total mass. The network has a total of \( N \) nodes, and we can consider each node with mass \( m_n = 1 \), so we have \( r_{cm} = \frac{1}{N} \sum_{n=1}^N r_n \). Each new site \( j \) connects to a pre-existing node \( i \) following a rule of preferential connection that depends on the distance between them, \( r_{ij} \) and the degree of connectivity of the node \( i \), that is,

\[
\Pi(k_i|j) = \frac{k_i r_{ij}^{-\alpha_A}}{\sum_n k_n r_{in}^{-\alpha_A}}.
\]

The \( \alpha_A \) exponent (A refers to the word attachment) controls the influence of spatial distance between sites in the preferential attachment. If \( \alpha_A = 0 \), we recover the BA network that does not take into account the spatial distance between the nodes. This model preserves the preferential attachment according to the degree of the nodes, but also taking into account the geographical distance as a criterion to dispute for links. In figure 6 we show a plot of an example of a network generated according to this algorithm.

Numerical results show that the parameter \( \alpha_G \) does not affect the behavior of the connectivity distribution \( P(k) \) of the network (see Figure 7(b)). This parameter refers just to the distance distribution in relation to the center of mass, and acts only on size scale but not on the structure of the network, and consequently it does not impact on the preferential attachment rules. On the other hand, as \( \alpha_A \) increases, the connectivity distribution changes (see figure 7(a)). Soares et. al. [4] showed that the degree distributions of networks generated according to their model are very well fitted with the form

\[
P(k) = P(0)e_q^{-k/k_0}
\]

where \( k_0 > 0 \) is the characteristic number of connections, \( P(0) \) is a constant to be normalized, \( q \) is the entropic index and \( e_q \) is the \( q \)-exponential defined by

\[
e_q \equiv [1+(1-q)x]^{1/(1-q)},
\]

where the natural exponential function is a particular case: \( e^x = e_{x=1} \).

The authors [4] showed that both \( k_0 \) and \( q \) are functions of \( \alpha_A \). So, as \( \alpha_A \) increases, a topological phase transition occurs in the connectivity distribution [11][30]. The network changes from a completely heterogeneous network (\( \alpha_A = 0 \)) to an increasingly homogeneous network as \( \alpha_A \) tends to infinity. Such phase transition also appears in the degree correlation of the nodes, as we show in the calculation of \( k_{nn}(k) \) for different values of \( \alpha_A \) (see figure 3(a)). The transition is clearer in the graph [5(b)] in which we show the calculation of Pearson’s coefficient as a function of \( \alpha_A \). Close to \( \alpha_A = 2 \), Pearson’s coefficient changes from a negative value, which characterizes a disassociative network, to a positive value, which characterizes an associative network.

Finally, other two more evidence that the topological phase transition can be discussed. We can measure the level of heterogeneity of a network using the quantity \( \kappa = \langle k^2 \rangle / \langle k \rangle \), where \( \langle k^p \rangle \) is the \( p \)-th moment of the degree distribution. If \( \kappa / \langle k \rangle > 1 \) the network is considered heterogeneous because it means that the second moment of the degree distribution can diverge when \( N \rightarrow \infty \) while for homogeneous networks \( \kappa / \langle k \rangle \approx 1 \) [27]. As can be seen in figure 3 the network becomes more homogeneous as \( \alpha_A \) increases because \( \kappa / \langle k \rangle \) decreases and approaches to one. We also calculated the average shortest path length. When the network becomes more homogeneous, the average shortest path length increases because the number of hubs decreases and consequently the path between the nodes increases. This measure, also shown in figure 9, reinforces the topological phase transition.

2. Variations of the model proposed by Soares et. al.

It is possible to include euclidian distance in the homophilic and the fitness models, as investigated by Nunes and collaborators [30]. For example, when we study the social interaction in a city [28], the parameter \( \eta_i \) can represent the influence of different places localized in the city. So it is possible to use the fitness model with Euclidean distance to try to explain, for example, why some places in a city is more attractiveness than others to open
a store, coffee shop or gas station. We also can use the homophilic model including spatial distance to study the influence of the topology in a formation of neighborhoods, since people tend to cluster with people that have a similar social class, religion or workplace [17, 31, 32].

The algorithms used to construct both models were already shown in previous sections. Now, we just have to include the metric, using the function $P_G \sim r^{-(\alpha_G+2)}$ to distribute the nodes in a continuous plane and change the preferential attachment rules that become,

$$\Pi(k_i|j) = \frac{\eta_i k_i r_{ij}^{-\alpha_A}}{\sum_n \eta_n k_n r_{in}^{-\alpha_A}} \quad \text{and} \quad \Pi(k_i|j) = \frac{(1 - A_{ij})k_i r_{ij}^{-\alpha_A}}{\sum_n (1 - A_{in})k_n r_{in}^{-\alpha_A}},$$

for fitness and homophilic models, respectively. Nunes [30] also shown a topological phase transition, as $\alpha_A$ increases for fitness model and no influence of the parameter $\alpha_G$ in the pattern of the connectivity distribution. We obtained the same results for homophilic networks. The data are not shown because they are very similar to the results shown in figure [7].
networks with size topological phase transition. We performed 1000 samples of
tween the nodes increases. This measure also reinforces the
distance (number of links that connects any pair of nodes) be-
average path increases, as the hubs disappear and then the
When the network becomes more homogeneous, the shortest
plot the calculation of the shortest average path (red line).
\[ \langle k \rangle \]
\[ \frac{\kappa}{\langle k \rangle} \]
\[ \alpha \]
\[ A = 5 \]

The networks are:
• Phone Calls: nodes represent cell phone users and the edges exist if they have called each other at least once during the investigated period. Data are from [11].
• Collaboration network: each node represents an author in a scientific collaboration and the edges between them represent a co-authored at least one paper in the period from January 1993 to April 2003. The data are obtained from arXiv preprint Condense Matter Physics [13].
• Email: nodes are email address and a directed link from one node to another represents at least one email sent. The data are collected during 112 days in the University of Kiel (Germany) [12].

According to the table [11] the Pearson’s coefficient of phone calls and collaboration networks are positive while for email network this coefficient is negative. The first two are social networks and they are basically related to family/friendship and professional interactions, respectively. In reference [9], Newman found similar results for biology and mathematics coauthorship. However, the email network, although it also describes some social interaction, behaves more as a technological network. In the reference cited above, the author also found similar value for World-Wide-Web.

Table II: Size \( N \) and Pearson’s correlation coefficient \( r \) for different real networks. We compared the values with the Pearson’s coefficient calculated for synthetic networks with the same size. For the phone calls network, we used the homophilic network including euclidean distance \( (\alpha_A = 5) \). For the collaboration network, we used the BA network including euclidean distance \( (\alpha_A = 5) \) and finally for the email network, we used the fitness network including euclidean distance \( (\alpha_A = 1) \).

| Real Network | \( N \)  | \( c_P \) | \( c_P \) (synthetic network) |
|--------------|--------|---------|-----------------------------|
| Phone Calls  | 36594  | 0.282   | 0.120                       |
| Collaboration| 23132  | 0.134   | 0.112                       |
| Email        | 57194  | -0.075  | -0.078                      |

Now, we can compare this real systems with our investigated models. In the case of phone calls network, we used the homophilic model and we investigated how the euclidean distance between the nodes of the system affects the network’s degree distribution. Homophilic model was chosen because it is reasonable to assume that telephone calls happen between people who have a certain affinity with each other, whether for personal, family or professional reasons. This hypothesis is corroborated in recent works [16, 17].

The Pearson correlation coefficient of the investigated synthetic network is not very similar to the value obtained for the real network (see table II). However we appreciated how accurate the degree distribution of this synthetic network is when compared to the real one. As shown in figure [10] when we used the attachment parameter \( \alpha_A = 5 \), the fits works extremely well, emphasizing the importance of considering geographic distance between the elements of the system when modeling real social networks. Indeed, a lot of work have followed this line [20, 21, 22].

The same analysis can be done for the collaboration network. The Pearson correlation coefficient of the synthetic network is similar to the one calculated for the real system. In this study, the only change was the synthetic network investigated. We chose the traditional Barabási-Albert model but also including the Euclidean distance and, as we showed in figure [11], the fit using \( \alpha_A = 5 \) is accurate as well. In networks of scientific co-authorship more distinguished researchers, such as university professors, tend to publish works with less famous researchers, such as their graduate students. This support both assumptions: the BA preferential attachment rule according to the degree of the node and the influence of the distance between the elements of the system. For the collaboration network, the fitness and homophilic models also showed reasonable results. As long as we in-
networks than in social ones. This can also be related to the geographical distance of the nodes in technological distribution of real data. It shows a smaller impact of $\alpha$ synthetic network considering the euclidean distante. son correlation coefficient with compared to the fitness lines are related to synthetic networks.

![Figure 11: Degree distribution of a collaboration network compared with the distinct degree distributions of synthetic networks with the same size generated according to the Barabasi-Albert model including Euclidean distance.](image)

Increased the value of $\alpha_A$, the preferential connection rule involving the Euclidean distance prevails in relation to the others.

![Figure 10: Degree distribution of a phone calls network compared with the distinct degree distributions of synthetic networks with the same size generated according to the homophlic model including Euclidean distance.](image)

Finally, the email network presents a very similar Pearson correlation coefficient with compared to the fitness synthetic network considering the euclidean distance. But here the parameter $\alpha_A = 1$ fits better the degree distribution of real data. It shows a smaller impact of the geographical distance of the nodes in technological networks than in social ones. This can also be related to the fact that this real network has directed links. As this email network was obtained from a university, the fitness model was chosen based on the fact that, in academia, students tend to send more emails to teachers than the otherwise. So the message sent depends on how influential (greater fitness) the reciever is.

![Figure 12: Degree distribution of an email network compared with the distinct degree distributions of synthetic networks with the same size generated according to the fitness model including Euclidean distance.](image)

In this work we have studied network models with growth and different rules of preferential attachment. We reviewed some important algorithms such as the Barabasi-Albert model and others that includes fitness, homophily and/or Euclidean distance as strategies to make connections between nodes. From an applicable perspective, these models are useful to model real-world networks because they present characteristics found in social sytems and also in technological ones, as we showed in the last section.

Our results corroborated with evidences that power-law degree structured is not very common in real systems. We evaluated two social and one technological network and we compared the degree distribution of these networks to degree distributions generated by growth and preferential attachment models. Our main conclusion is that the real networks analysed are better fitted with models which consider traits as fitness, homophily and euclidean distance between nodes. We observed that geographic distance between nodes seems to be an important factor to model specially real social systems. This feature changes the form of the degree distribution of a power law to a q-exponential according to the model proposed by Soares and collaborators [4]. Our results...
are in agreement with recent studies involving real networks [11, 15, 20, 21, 29, 33].

We also supplemented the characterization of these synthetic networks investigating measures as clustering, average shortest path length, degree distribution and assortativity.

Finally, it is important to mention that many dynamical processes as epidemics, rumor propagation and synchronization were extensively investigated in scale-free topologies as the Barabási-Albert network. However the study of these dynamics in substrates where the distance between the elements of the system is taken into account needs to further advance, since this element has already been shown to be very important. Even on online social networks [16, 17, 19, 21, 29], it seems to influence the connection between the nodes, as well as fitness and homophily.

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