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Modified analytical approach for generalized quadratic and cubic logistic models with Caputo-Fabrizio fractional derivative

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**Abstract** In this paper, a modified reproducing kernel algorithm is proposed to solve a class of quadratic and cubic logistic equations with Caputo-Fabrizio fractional derivative in Hilbert space. These equations are the generalization of Verhulst’s model which describes population growth taking in account that individuals will compete for limited resources. A novel reproducing kernel function is constructed to create an orthogonal system and to calculate the analytical and approximate solutions in the desirable Sobolev space. The stability, convergence, and complexity of the proposed approach are discussed. Furthermore, the effects of the Caputo-Fabrizio fractional derivatives are studied in solving the population growth model comparing with those of the classical Caputo derivatives. The main motivation for using the proposed technique is high accuracy and low computational cost compared to other existing methods especially when involving fractional differentiation operators. In this orientation, the effectiveness, applicability, and feasibility of this technique are verified by numerical examples. In a numerical viewpoint, the obtained results indicate that the suggested intelligent method has many advantages in accuracy and stability using the new Caputo-Fabrizio derivative.

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1. Introduction

Fractional calculus is the study of derivatives and integrals of non-integer order that provides an attractive mechanism to explain memory and hereditary characteristics of complex systems. Recently, it has been widely used in modelling real-world models due to its accuracy in providing and neglecting the influence of external forces as in physics, engineering, mechanics, biology, medicine and economics [1–6]. One model that has benefited from the development of fractional calculus is the logistic model, which is an attempt to describe several phenomena that possess growth data such as population growth, the spread of bacteria, infectious diseases and social

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media [2]. Broadly speaking, the importance of this model results from the fact that exponential growth can’t continue with limited resources. In most environments, when the number of individuals becomes large enough, resources will be depleted, slowing the growth rate. The maximum population size that these resources can support is called the carrying capacity. The logistic equation adds the carrying capacity as a moderating force in the growth rate. The actual formula of this model was first proposed by the Belgium mathematician Verhulst in 1835, who stated that the virtual increase of population is limited by the size and the fertility of the country. As a result, the population gets closer and closer to a steady state. The logistic equation adds the carrying capacity as a moderating force in the growth rate. The actual formula of this model was first proposed by the Belgium mathematician Verhulst in 1835, who stated that the virtual increase of population is limited by the size and the fertility of the country. As a result, the population gets closer and closer to a steady state.

\[
\frac{dQ}{dt} = \lambda Q(t) \left( 1 - \frac{Q(t)}{K} \right),
\]

where \( G, \lambda, \) and \( \delta \) are related to the population, growth rate parameter, and carrying capacity, respectively. This model, known as the Verhulst model, assumes that the growth rate \( G \) is linearly increasing and decreasing in terms of \( G \). Furthermore, by substituting \( Q(t) = G(t) / \delta \), the following standard logistic differential equation (SLDE) is obtained:

\[
\frac{dQ}{dt} = \lambda Q(t)(1 - Q(t)),
\]

which has the exact solution as follows

\[
Q(t) = \frac{Q_0}{Q_0 + (1 - Q_0)e^{-\lambda t}},
\]

where \( Q_0 = Q(0) \) is related to the initial population data.

Although the SLDE model includes many population growth factors, the classical quadratic logistic model is still not enough. This is because the classical view of population dynamics states that the growth rate of a population will decrease at a higher density and increase at a lower density due to competition for limited resources. However, the Allee effect, introduced in the 1950s, describes a situation when density is low and the population growth rate decreases due to the limited availability and impaired cooperative behaviours. As a simple mathematical example of the Allee effect, the following cubic growth model will be presented:

\[
\frac{dQ}{dt} = \lambda Q(t)(\kappa - Q(t)) \left( 1 - \frac{Q(t)}{\mu} \right),
\]

where \( Q \) is the population size, \( \Lambda \) is the intrinsic rate of increase, and \( \mu \) and \( \kappa \) are related to the carrying capacity and Allee threshold, respectively. However, to improve the efficiency of these nonlinear logistic models in describing growth and extracting sufficient and comprehensive information, the fractional versions of Verhulst and Allee models have been introduced to provide us with unlimited options for fractional orders in which the dynamic processes of the model can be better described.

Recently, fractional calculus has proven to be efficient not only in mathematics but also for many applications in mathematical physics, quantum mechanics, heat transfer engineering, bio-mathematics, fluid dynamics, traffic modeling, electrical circuits, nonlinear optics, etc. In the literature, a common agreement was reached based on the experimental results of several non-integer order models instead of an integer order, which is that the fractional model has a long memory effect compared to the integer-order model [7–10]. So, it is expected that the fractional logistic equation will give a more suitable description for the population growth. Other fractional models, including the generalized time-fractional diffusion equation, quadratic integro-differential equations, integral fractional differential–difference equations, fractional stochastic heat equation, fractional integro–partial differential equations, and fractional-order HIV/AIDS model, can be found in [11–16].

On the other hand, several concepts of fractional operators have been presented and developed in contrast to classic operators in calculus that have a clear concept and accurate engineering and physical interpretations. Among the well-known definitions widely employed in the context of fractional calculus are Riemann–Liouville, Caputo, Atangana–Baleanu, Riesz fractional operators [17–22]. Nevertheless, these definitions have several outstanding questions and some limitations. For example, the Riemann–Liouville derivative with respect to the constant is not equal to zero, where the initial conditions are expressed in terms of the fractional-order, which is impractical and lacks the physical meaning. In addition, both the definitions of Riemann–Liouville and Caputo include a weakness that specifically affects a portion of genetic memory, which is due to the presence of a singular kernel. This weakness affects the reality of models. Moreover, the solutions achieved by these operators have sometimes a very complicated formula or very difficult to obtain. To overcome such and other obstacles, to name a few, Caputo and Fabrizio suggested a new definition possessing a non-singular kernel [20]. This gentle new definition has the potential to describe the heterogeneities and formations of different scales, which cannot be well described by old versions of fractional derivatives. In [23], Losada and Nieto discussed the different properties of this new fractional derivative without having a singular kernel. To see more recent studies and results that have been conducted on this concept, the reader is asked to refer to [24–26].

In terms of methodology, numerical and approximate methods have been known as powerful mathematical tools for dealing with many complex natural models with local and non-local operators that arise in studies of physics, engineering, biology, chemistry, and other sciences. These advanced methods are used when all the classical analytical techniques accomplished fail. It is also being continuously improved to keep pace with the rapid developments taking place in the universe and the emergence of new global economic, financial, biological, chemical and astronomical models. However, with respect to the proposed model, some modern numerical approaches have been applied to solve the quadratic and cubic fractional Logistic equations, including the spectral Laguerre collocation method, variational iteration method, operational matrices of Bernstein polynomials, and finite difference method [27–29]. The motivation of the current study is to expand the applications of the reproducing kernel method (RKHS) in obtaining numerical approximate solutions to modified nonlinear fractional Logistic differential equations (NFLEs) with quadratic and cubic terms utilizing the new Caputo-Fabrizio concept. More specifically, consider the following forms of the NFLEs:

\[
\frac{D_0^\lambda Q(t)}{D_0^\lambda Q(t)} = \lambda Q(t)(1 - Q(t)), \quad t \geq 0, \quad Q(0) = Q_0,
\]
\[ CF_0^D Q(\tau) = \lambda Q(\tau) \left( 1 - \frac{Q(\tau)}{\mu} \right) (\kappa - Q(\tau)), \quad \tau \geq 0, \quad Q(0) = Q_0, \]

where \( 0 < \alpha \leq 1, \lambda, \mu, \kappa > 0 \) and \( CF_0^D \) indicates the Caputo-Fabrizio fractional derivatives, while \( Q(\tau) \) indicates smooth solution to be obtained in Hilbert space. These equations are generalized by applying the new fractional derivative rather than the classical integer order derivative to standard Logistic equations to improve model accuracy.

The RKHS technique is one of the most important numerical and approximate methods for several branches of numerical analysis which possesses many characteristics, features, and ability to apply, especially statistics, biology, machine learning, finance, etc. It has been successfully applied over the past few decades to solve different types of mathematical problems such as nonlinear boundary value problems, integral equations, integro-differential equations, singular perturbation differential equations, fuzzy differential equations, chaotic systems, partial differential equations, fractional models, and so on. Its tremendous capacity, features, and simplicity have made it the focus of recent researchers’ attention. For more details on the importance, uses and developments of the proposed method, please refer to [30–36]. For illustration without limitation, some of the main advantages of the RKHS method can be summarized as follows:

1. It is accurate and efficient with minimal effort to achieve results.
2. It is possible to choose any point in the integration interval.
3. It has attractive and practical implementation properties favorable to a wide variety of non-classic initial and boundary conditions.

This paper is organized as follows: Section 2 recalls some basic concepts of the Caputo-Fabrizio fractional derivatives. Desirable reproducing kernel Hilbert spaces are constructed in section 3. The structure of the analytic and approximate solutions is presented with some theories related to the convergence of the approximate solution to the exact one in section 4. In section 5, we investigate and analyze the stability of the proposed models using the RKHS approach. In section 6, illustrative examples are provided to demonstrate the efficiency and priority of the proposed method and the effect of different fractional operators on the curves of modified NFLDEs solutions. It is observed from the graphs and tabulated results that when \( \alpha \) increases, the growth rate decreases until it reaches the value of carrying capacity. Also, the growth curves which result depending on the classical Caputo derivative are not identical to those result using derivative with the exponential kernel. In the later operator, one can observe more curvature. Finally, a brief conclusion is outlined in Section 7.

2. Mathematical preliminaries

In this section, we present some known definitions of fractional calculus and investigate the description of the Caputo-Fabrizio derivative that will be used throughout the rest of this study.

**Definition 1.** [20] The Riemann–Liouville fractional integral (RLFI) of order \( \alpha > 0 \) is given as follows:

\[ J_0^\alpha Q(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - \omega)^{\alpha-1} Q(\omega) d\omega, \quad \tau > 0. \]

If \( \alpha = 0 \), then, \( J_0^\alpha Q(\tau) \) is the identity operator.

**Definition 2.** [20] The Riemann–Liouville fractional derivative (RLFD) of order \( \alpha > 0 \) is given as follows:

\[ D_0^\alpha Q(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau - \omega)^{n-\alpha-1} Q^{(n)}(\omega) d\omega, \quad \tau > 0, \quad n-1 < \alpha \leq n, n \in N. \]

**Definition 3.** [20] The Caputo fractional derivative (CFD) of order \( \alpha > 0 \) is given as follows:

\[ D_0^\alpha Q(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau - \omega)^{n-\alpha-1} Q^{(n)}(\omega) d\omega, \quad \tau > 0, \quad n-1 < \alpha \leq n, n \in N. \]

The CFD has advantages compared to RLFD as the Caputo derivative of the constant is zero and the initial conditions for the data, in Caputo sense, are expressed in terms of the integer-order derivatives that give a physical meaning to the applications as in the classic calculus. However, they both possess a kernel with singularity and since the kernel includes a memory effect, they cannot describe the full effect of memory. In this direction, the new Caputo-Fabrizio definition is introduced using a regular exponential kernel as follows:

**Definition 4.** [23] Let \( Q \) be a function in the usual Sobolev space over \([0, T]\). Then, the Caputo-Fabrizio fractional derivative (CFFD) of order \( z \) is given as follows:

\[ CF_0^D Q(\tau) = \frac{M(z)}{1 - z} \int_0^\tau Q(t) \exp \left[ -\frac{z(\tau - \omega)}{1 - z} \right] d\omega, \quad \tau \geq 0, \quad Q(0) = Q_0, \]

where \( 0 < \alpha < 1, M(z) \) is a normalization function such that \( M(0) = M(1) = 1, Q \in H^1[0, T], H^1(0, T) = \{ Q : Q, Q' \in L^2[0, T] \}, \) and

\[ L^2[0, T] = \left\{ Q : \int_0^T (Q(\tau))^2 d\tau < \infty \right\}. \]

In the present work, let \( M(z) = 1 \), so that the definition becomes

\[ CF_0^D Q(\tau) = \frac{1}{1 - z} \int_0^\tau Q'(\omega) \exp \left[ \frac{z(\tau - \omega)}{1 - z} \right] d\omega = \frac{1}{1 - z} Q'(\tau) * \exp \left[ \frac{z\tau}{1 - z} \right], \]

where the symbol \( * \) denotes the convolution.

Note that the CFFD of any constant is zero, that is, \( CF_0^D c = 0 \). Further, the Caputo-Fabrizio derivatives of some elementary functions have been derived in [23].

**Definition 5.** [23] Let \( Q \in H^1[0, T] \) and \( 0 < \alpha \leq 1 \). Then, the Caputo-Fabrizio fractional integral of order \( \alpha \) is given by:
\( CF_{\delta}^\alpha Q(\tau) = \frac{2(1-z)}{(2-z)M(z)} Q(\tau) + \frac{2\tau}{(2-z)M(z) \int_0^\tau Q(\omega) \, d\omega } \)

(2.6)

Remark 1. [23] The relation between the Caputo-Fabrizio fractional derivative and the corresponding integral is given by:

\( (CF_{\delta}^\alpha) (CD_{\delta}^\alpha) Q(\tau) = Q(\tau) - Q(0) \).

(2.7)

3. Preliminaries of the RKHS Method

In this section, some of the essential facts of reproducing kernel theory are presented to construct two reproducing kernel classes in Hilbert space. During this study, \( AC[0, T] \) denotes the absolutely continuous real functions.

**Definition 6.** [30] A function \( B: \Omega \times \Omega \rightarrow C \) satisfies the two following condition: (i) \( B(\cdot, \xi) \in S, \forall \xi \in \Omega \), (ii) \( (z(\cdot), B(\cdot, \xi)) = z(\xi), z \in S \), is called reproducing-kernel function, whereas \( \Omega \) is a non empty abstract set and \( S \) is the Hilbert space. The property in (ii) is called "the reproducing property" because it indicates that the value of the function \( z \) at the point \( \xi \) is reproduced by the inner product of \( z \) with \( B(\cdot, \xi) \). This function possesses some important properties such as being unique, conjugate symmetric, and positive-definite.

**Remark 2.** Any Hilbert space \( S \) that possesses reproducing-kernel function is called the RKHS.

**Definition 7.** [30] The space of functions \( R^{1,2}[0, T] \), which is defined by

\[ R^{1,2}[0, T] = \{ Q(\tau), \; Q: [0, T] \rightarrow R, \; Q \in AC[0, T] \text{ and } Q' \in L^2[0, T] \}, \]

(3.1)

is a Hilbert space that possess the following inner product and norm

\[ \begin{align*}
\langle Q_1, Q_2 \rangle_{R^{1,2}} &= \langle Q_1(0), Q_2(0) \rangle \\
&+ \int_0^T Q_1(\tau)Q_2(\tau) \, d\tau, \; Q_1, Q_2 \in R^{1,2}[0, T], \\
\|Q\|_{R^{1,2}} &= \langle Q, Q \rangle^{1/2}, \; Q \in R^{1,2}[0, T].
\end{align*} \]

(3.2)

The unique representation of the reproducing-kernel function associated with the Hilbert space \( R^{1,2}[0, T] \) is given by

\[ S_{1}^{(1)}(v) = \frac{1}{2 \sinh(T)} [\cosh(\tau + v - T) + \cosh(\tau - v - T)]. \]

(3.3)

Moreover, the Hilbert space \( R^{1,2}[0, T] \) is complete reproducing kernel space.

**Definition 8.** ([33]) The space of functions \( R^{2,2}[0, T] \) is defined as follows:

\[ R^{2,2}[0, T] = \{ Q(\tau), \; Q: [0, T] \rightarrow R, \; Q, Q' \in AC[0, T], \; Q'' \in L^2[0, T] \}. \]

(3.4)

It has the following inner product and the norm:

\[ \begin{align*}
\langle Q_1, Q_2 \rangle_{R^{2,2}} &= \sum_{i=0}^{1} \langle Q_1(i), Q_2(i) \rangle \\
&+ \int_0^T Q_1(\tau)Q_2(\tau) \, d\tau, \; Q_1, Q_2 \in R^{2,2}[0, T], \\
\|Q\|_{R^{2,2}} &= \langle Q, Q \rangle^{1/2}, \; Q \in R^{2,2}[0, T].
\end{align*} \]

(3.5)

**Theorem 1.** [33] The unique representation of the reproducing-kernel function associated the \( R^{2,2}[0, T] \) can be written as

\[ S_{2}^{(2)}(v) = \begin{cases} 
\frac{1}{6} \tau(-v^2 + 3(\tau + v)), & 0 \leq v < \tau, \\
\frac{1}{6} \tau(-\tau^2 + 3(\tau + v)), & \tau < v \leq T.
\end{cases} \]

(3.6)

**Proof.** See [33].

**Theorem 2.** [30] If \( S_{1}^{(1)}(v) \) and \( S_{2}^{(2)}(v) \) are the reproducing kernel functions associated with the Hilbert spaces \( R^{1,2}[0, T] \) and \( R^{2,2}[0, T] \), respectively, then

\[ \| \frac{\partial S_{1}^{(1)}(v)}{\partial \tau} \|_{R^{1,2}}, \| \frac{\partial S_{2}^{(2)}(v)}{\partial \tau} \|_{R^{2,2}}, \text{ and } \| \frac{\partial^2 S_{2}^{(2)}(v)}{\partial \tau^2} \|_{R^{2,2}} \]

are continuous with respect to \( \tau \) on \([0, T]\).

4. Structure of Analytical Solution

In this section, the analytical solution of the quadratic and cubic FLDEs (1.1) and (1.2) are obtained in \( R^{2,2}[0, T] \). To perform this, we must homogenize the initial condition using the simple transformation \( Q(\tau) = Q(\tau) - Q_0 \). Consequently, the equivalent form of the quadratic FLDE (1.1) is given by

\[ CF_{\delta}^\alpha Q(\tau) = \lambda (Q(\tau) + Q_0)(1 - Q(\tau) - Q_0), \]

(4.1)

and the equivalent form of the cubic FLDE (1.2) is given by

\[ CF_{\delta}^\alpha Q(\tau) = \lambda (Q(\tau) + Q_0) \left(1 - \frac{Q(\tau) + Q_0}{\mu}(K - Q(\tau) - Q_0), \right) \]

(4.2)

subject to the initial condition

\[ Q(0) = 0. \]

(4.3)

After that, we characterize the differential linear operator as follows

\[ \begin{align*}
L: R^{2,2}[0, T] \rightarrow R^{1,2}[0, T]; \\
LQ(\tau) &= CF_{\delta}^\alpha Q(\tau).
\end{align*} \]

(4.4)

**Theorem 3.** The fractional operator \( L \) from \( R^{2,2}[0, T] \) into \( R^{1,2}[0, T] \) is bounded and linear.

**Proof.** We need to prove the presence of a positive constant \( K \) such that

\[ \|LQ(\tau)\|_{R^{1,2}} \leq K\|Q(\tau)\|_{R^{2,2}}. \]
To do this, the reproducing property are applied together with Schwartz inequality, for \(i = 0, 1\), such that

\[
\left| (LQ)^{(i)}(\tau) \right| = \left| \frac{\partial^{(i+1)} D_{1}^{2} Q(\tau)}{\partial \tau} \right| = \left\langle Q(\tau), \frac{\partial^{(i+1)} D_{1}^{2} S_{i}^{(0)}(\tau)}{\partial \tau} \right\rangle
\]

\[
\leq \left\| \frac{\partial^{(i+1)} D_{1}^{2} S_{i}^{(0)}(\tau)}{\partial \tau} \right\|_{\mathcal{R}^{2}} \left\| Q(\tau) \right\|_{\mathcal{R}^{2}} \leq \gamma \left\| Q(\tau) \right\|_{\mathcal{R}^{2}}.
\]

Here \(\gamma = \left\| \frac{\partial^{(i+1)} D_{1}^{2} S_{i}^{(0)}(\tau)}{\partial \tau} \right\|_{\mathcal{R}^{2}}\), \(i = 0, 1\). From Theorem 3.6 and the definition of CFFDD, we can simply deduce that \(\partial^{(i+1)} D_{1}^{2} S_{i}^{(0)}(\tau)\) is uniformly bounded about \(\tau\). Now, by using the norm over the space \(R^{2}\), we have

\[
\left\| LQ(\tau) \right\|_{\mathcal{R}^{2}}^{2} = \left\| [LQ(0)]^{2} + \int_{0}^{\theta} \left[ \left( \frac{\partial^{(i+1)} D_{1}^{2} S_{i}^{(0)}(\tau)}{\partial \tau} \right)^{2} \right] d\tau \right\|_{\mathcal{R}^{2}}^{2}
\]

\[
\leq \left( \left\| \theta^{(i+1)} \right\|_{\mathcal{R}^{2}}^{2} + \int_{0}^{\theta} \left[ B_{1} \right]^{2} d\tau \right) \left\| Q(\tau) \right\|_{\mathcal{R}^{2}}^{2} = \left( \left\| \theta^{(i+1)} \right\|_{\mathcal{R}^{2}}^{2} + \int_{0}^{\theta} \left[ B_{1} \right]^{2} d\tau \right) \left\| Q(\tau) \right\|_{\mathcal{R}^{2}}^{2}.
\]

Taking \(K = \left( \left\| \theta^{(i+1)} \right\|_{\mathcal{R}^{2}}^{2} + \int_{0}^{\theta} \left[ B_{1} \right]^{2} d\tau \right)\), we can write \(\left\| LQ(\tau) \right\|_{\mathcal{R}^{2}}^{2} \leq K\left\| Q(\tau) \right\|_{\mathcal{R}^{2}}^{2}\). Hence, \(L\) is bounded while the linearity is obvious.

Thus, let the sequence \(\left\{ \tau_{i} \right\}_{i=1}^{\infty}\) be dense in \([0, T]\), put \(\varphi_{i}(\tau) = S_{i}^{(1)}(\tau)\) and \(\psi_{i}(\tau) = L^{*} \varphi_{i}(\tau)\), where \(L^{*}\) is the adjoint operator of \(L\). The orthonormal function system \(\left\{ \overline{\psi_{i}(\tau)} \right\}_{i=1}^{\infty}\) of the space \(R^{2}[0, T]\) can be derive using the well-known Gram-Schmidt orthogonalization process of \(\left\{ \tau_{i} \right\}_{i=1}^{\infty}\) such that

\[
\overline{\psi_{i}(\tau)} = \sum_{k=1}^{\infty} \alpha_{ik} \overline{\psi_{k}(\tau)}, \quad i = 1, 2, \ldots
\]

where \(\alpha_{ik}\) are the orthonormalization coefficient such that \(\alpha_{ik} > 0\).

**Theorem 4.** If \(\left\{ \tau_{i} \right\}_{i=1}^{\infty}\) is dense on \([0, T]\), then the sequence \(\left\{ \overline{\psi_{i}(\tau)} \right\}_{i=1}^{\infty}\) is complete function system in \(R^{2}[0, T]\), and \(\overline{\psi_{i}(\tau)} = L_{1} S_{i}^{(1)}(\tau)\left|_{\tau_{i}} \right.\).

**Proof.** Note that

\[
\overline{\psi_{i}(\tau)} = L^{*} \varphi_{i}(\tau) = \left\langle L^{*} \varphi_{i}(\tau), S_{i}^{(1)}(\tau) \right\rangle_{\mathcal{R}^{2}} = \left\langle L S_{i}^{(1)}(\tau), \varphi_{i}(\tau) \right\rangle_{\mathcal{R}^{2}} = L S_{i}^{(1)}(\tau)\left|_{\tau_{i}} \right.
\]

To show the completeness, let \(\left\langle Q(\tau), \overline{\psi_{i}(\tau)} \right\rangle = 0\). This mean that

\[
\left\langle Q(\tau), L^{*} \varphi_{i}(\tau) \right\rangle = \left\langle L Q(\tau), \varphi_{i}(\tau) \right\rangle = \left\langle L Q(\tau), S_{i}^{(1)}(\tau) \right\rangle = \left\langle L Q(\tau), S_{i}^{(1)}(\tau) \right\rangle
\]

By the density of the sequence \(\left\{ \tau_{i} \right\}_{i=1}^{\infty}\) on \([0, T]\), we have \(L Q(\tau) = 0\) and from the existence of the inverse operator \(L^{*}\) for the bounded linear operator \(L\), we conclude that the set \(\left\{ \overline{\psi_{i}(\tau)} \right\}_{i=1}^{\infty}\) is complete.

**Theorem 5.** Let \(Q \in R^{2}[0, T]\) be a unique solution of quadratic FLDEs (4.1) and (4.3). Then, the analytical solution has the form

\[
Q(\tau) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{ik} (Q(\tau) + Q_{0})(1 - Q(\tau) - Q_{0}) \overline{\psi_{i}(\tau)}.
\]

where \(\alpha_{ik}\) are the orthonormalization coefficients.

**Proof.** Since \(Q\) is integrable function, then there exists an associated generalized Fourier series such that

\[
Q(\tau) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{ik} (Q(\tau) + Q_{0})(1 - Q(\tau) - Q_{0}) \overline{\psi_{i}(\tau)}.
\]

The approximate solution \(Q_{N}(\tau)\) of quadratic FLDEs (4.1) and (4.3) can be obtained by taking the first \(N\)-terms of the series in (4.9) as follows

\[
Q_{N}(\tau) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{ik} (Q(\tau) + Q_{0})(1 - Q(\tau) - Q_{0}) \overline{\psi_{i}(\tau)}.
\]

**Theorem 6.** Let \(Q(\tau) \in R^{2}[0, T]\) be a unique solution of cubic FLDEs (4.2) and (4.3). Then, the analytical solution has the form

\[
Q(\tau) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{ik} (Q(\tau) + Q_{0})(1 - \frac{1}{\mu} (Q(\tau) + Q_{0}))
\]

\[
\times (\kappa - Q(\tau) - Q_{0}) \overline{\psi_{i}(\tau)}.\]

Similarly, the approximate solution of cubic FLDEs (4.2) and (4.3) can be obtained by

\[
Q_{N}(\tau) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{ik} (Q(\tau) + Q_{0})(1 - \frac{1}{\mu} (Q(\tau) + Q_{0}))
\]

\[
\times (\kappa - Q(\tau) - Q_{0}) \overline{\psi_{i}(\tau)}.\]

Next, we show that the approximate solution in (4.11) and its derivative are uniformly convergent to the analytical solution (4.9) and its derivative, respectively. The same discussion can be carried out for the cubic FLDE. Prior to convergence analysis, we also show that the approximation errors are a monotone decreasing sequence that converges to zero.

**Lemma 1.** Let \(e_{N}(\tau) = Q(\tau) - Q_{N}(\tau)\) for all \(t \in [0, T]\), where \(Q(\tau)\) and \(Q_{N}(\tau)\) are given by (4.9) and (4.11), respectively. Then, \(\left\| e_{N}(\tau) \right\|_{\mathcal{R}^{2}} \) forms a monotone decreasing sequence such that \(\left\| e_{N}(\tau) \right\| \to 0\) as \(N \to \infty\).

**Proof.** First, we observe that
In this section, the stability of the RKHS method for the solution of quadratic and cubic FLDEs is investigated and discussed.

**Lemma 2.** [37] If $Q(\tau) \in R^{2,2}[0, T]$, then there exists a constant $M > 0$ such that

$$\|Q^{(k)}\|_{\infty} \leq M \|Q\|_{\mathbb{E}^2}, \; k = 0, 1,$$

(5.1)

where $\|Q\|_{\infty} = sup\{|Q(\tau)| \tau \in [0, T]\}$.

**Theorem 8.** Let $L$ be the operator defined in (4.4) and $\{\tau_i\}_{i=1}^{\infty}$ be a dense set in $[0, T]$. If the quadratic FLDE

$$LQ(\tau) = \lambda (Q(\tau) + Q_0)(1 - Q(\tau) - Q_0)$$

(5.2)

has a unique solution, then the approximate solution in (4.11) achieved by the RKHS method is stable.

**Proof.** First, since the RKHS approximate solution $Q_N(\tau)$ to the quadratic FLDE (5.2) converges uniformly to $Q(\tau)$. So,

$$\forall \epsilon_1, \exists N_1 > 0 \; \text{such that} \; \|Q_{N_1} - Q\|_{\mathbb{E}^2} < \epsilon_1, \; \forall N > N_1.$$

Also, let $Q_N(\tau)$ be the RKHS approximate solution of the following equation

$$LQ(\tau) = \lambda (Q(\tau) + Q_0)(1 - Q(\tau) - Q_0) + \zeta(\tau),$$

(5.3)

for small bounded perturbation $\zeta(\tau) > 0$ such that

$$Q_N(\tau) = \sum_{j=1}^{\infty} \sigma_j (\lambda (Q(\tau) + Q_0)(1 - Q(\tau) - Q_0) + \zeta(\tau)) \psi_j(\tau).$$

(5.4)

So,

$$\forall \epsilon_2, \exists N_2 > 0 \; \text{such that} \; \|Q_N - Q\|_{\mathbb{E}^2} < \epsilon_2, \; \forall N > N_2,$$

where $Q^*$ is the analytical solution of Eq. (5.3).

Now, let $\zeta = sup\{|\zeta(\tau)| \tau \in [0, T]\}, \; N^* = max\{N_1, N_2\}$, and take $\epsilon_1 = \epsilon_2 = \frac{\delta}{3}$. Hence, we have

$$\|Q_N - Q\|_{\mathbb{E}^2} < \frac{\delta}{3} \; \text{and} \; \|Q_N - Q^*\|_{\mathbb{E}^2} < \frac{\delta}{3} \; \forall N > N^*.$$

Moreover, since the quadratic FLDE has a unique solution and $L$ is a bounded linear operator. Therefore, $L(Q - Q^*) = \zeta(\tau)$, which yields that $(Q - Q^*)(\tau) = (L^{-1}\zeta)(\tau)$ and

$$\|Q - Q^*\|_{\mathbb{E}^2} = \|L^{-1}\zeta\|_{\mathbb{E}^2} \leq \|L^{-1}\| \zeta \leq c_N,$$

where $c = \|L^{-1}\|$. From the above discussion, we have

$$\|Q_N - Q_N\|_{\mathbb{E}^2} = \|Q_N - Q^* + Q^* - Q + Q - Q_N\|_{\mathbb{E}^2} \leq \|Q_N - Q^*\|_{\mathbb{E}^2} + \|Q^* - Q\|_{\mathbb{E}^2} + \|Q - Q_N\|_{\mathbb{E}^2} \leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta, \; \forall N > N^*.$$

But from Lemma 5.1, it follows that

$$\lim_{\tau \to 0} \|Q_N - Q_N\|_{\mathbb{E}^2} = 0.$$

**Theorem 9.** Let $L : R^{2,2}[0, T] \to R^{2,1}[0, T]$ be the operator defined in (4.4) and $\{\tau_i\}_{i=1}^{\infty}$ be a dense set in $[0, T]$. If the cubic FLDE

$$LQ(\tau) = \lambda (Q(\tau) + Q_0)(1 - \frac{1}{\mu} Q(\tau) + Q_0) (\kappa - Q(\tau) - Q_0)$$

has a unique solution, then the approximate solution in (4.13) achieved by the RKHS method is stable.

**Proof.** The proof is similar to proof of Theorem 5.2.

6. Computational simulations

In this section, the efficiency and accuracy of the RKHS method are demonstrated by including some numerical examples for modified FLDEs. The quadratic FLDE model is studied in Examples 6.1 and 6.2, while the cubic FLDE model is...
studied in Example 6.3 and investigated in different ways. Similar examples have been discussed in [38] as models to describe the growth of cancer tumors. In fact, replacing the ordinary derivatives by fractional derivatives refines the solution given by the classical logistic equation and extends its range of applications in the study of tumor dynamics. The results obtained in the first two examples are compared to the exact solution at the integer-order \( z = 1 \), and with each other at different values of fractional-order \( z \). Meanwhile, solution curves are compared using both Caputo-Fabrizio and Caputo derivatives to see the effect of these derivatives on the behavior of RKHS solutions. Anyhow, since there is no exact solution to the last example, the recurrence errors are calculated instead of absolute errors. Also, the approximate solutions of cubic FLDEs are obtained using the RKHS and successive substitution (SS) method based on Caputo-Fabrizio fractional derivative. In this regard, the RKHS results show that the method provides a convenient methodology for controlling convergence (SS) technique to solve such a cubic FLDE using both CFFD and CFD. In practice, the behavior of the RKHM solutions are presented using the CFFD and CFD with different values of \( z \) such that \( z \in \{1, 0.0, 0.95, 0.85, 0.75, 0.65, 0.55\} \).

**Example 6.1.** Consider the following quadratic FLDE with Caputo-Fabrizio fractional derivative:

\[
\begin{align*}
\text{CFD} D_0^\alpha Q(\tau) &= \frac{1}{2} Q(\tau)(1 - Q(\tau)), \quad \tau \geq 0, \ 0 < \alpha \leq 1, \\
Q(0) &= \frac{1}{2}.
\end{align*}
\tag{6.1}
\]

In particular, the exact solution at \( z = 1 \) is \( Q(\tau) = \frac{\tau^2}{e^{\tau^2}} \).

By taking \( \tau_i = \frac{1}{i}, i = 1, \ldots, N \), and \( N = 30 \), the numerical outcomes of the proposed method are listed in the form of tables and graph representations as follows: Numerical approximations for Example 6.1 compared with the exact solutions at \( z = 1 \) are given in Table 1 over the interval \([0, 1]\) with step size \( h = 0.1 \). Fig. 1 exhibits a comparison among the behavior curves of exact and approximate solutions at \( z = 1 \). While in Table 2, the numerical results at different values of fractional order \( z \) are summarized using both CFFD and CFD. In Fig. 2, the behavior of the RKHM solutions are presented using the CFFD and CFD with different values of \( z \) such that \( z \in \{1, 0.0, 0.95, 0.85, 0.75, 0.65, 0.55\} \).

**Example 6.2.** Consider the following quadratic FLDE with Caputo-Fabrizio fractional derivative:

\[
\begin{align*}
\text{CFD} D_0^\alpha Q(\tau) &= \frac{1}{2} Q(\tau)(1 - Q(\tau)), \quad \tau \geq 0, \ 0 < \alpha \leq 1, \\
Q(0) &= \frac{1}{2}.
\end{align*}
\tag{6.2}
\]

In particular, the exact solution at \( z = 1 \) is \( Q(\tau) = \frac{\tau^2}{e^{\tau^2}} \).

By taking \( \tau_i = \frac{i}{N}, i = 1, \ldots, N \), and \( N = 30 \), the numerical approximations of the solution using the proposed method are summarized in the form of tables and graph representations as follows: The absolute errors of Example 6.2 at \( z = 1 \) are shown in Table 3. Table 4 shows the numerical results at different values of fractional order \( z \) using both CFFD and CFD concepts. The RKHS solutions for different values of \( z \) using both fractional definitions are presented in Fig. 3.

**Example 6.3.** Consider the following cubic FLDE with Caputo-Fabrizio fractional derivative:

\[
\begin{align*}
\text{CFD} D_0^\alpha Q(\tau) &= \frac{1}{2} Q(\tau)(1 - Q(\tau) - 1), \quad \tau \geq 0, \ 0 < \alpha \leq 1, \\
Q(0) &= \frac{1}{2}.
\end{align*}
\tag{6.3}
\]

This model has no exact solution. So, we first implement the successive substitution (SS) technique to solve such a cubic FLDE in which Caputo-Fabrizio integrals are applied on both sides of cubic FLDE (6.3) to obtain the Volterra integral equation shown as follows (see Fig. 4):

\[
\begin{align*}
Q(\tau) &= Q(0) + \frac{1 - \tau}{1 - \tau} Q(\tau) \left(1 - \frac{Q(\tau) - 1}{\tau} \right) \\
&\quad + \frac{\tau}{(1 - \tau)} \int_0^\tau Q(\omega) \left(1 - \frac{Q(\omega) - 1}{\tau} \right) d\omega.
\end{align*}
\tag{6.4}
\]

Consequently, the following iteration relation formula will be defined:

\[
\begin{align*}
Q_{n+1}^{\alpha} (\tau) &= Q(0) + \frac{1 - \tau}{1 - \tau} Q_n(\tau) \left(1 - \frac{Q_n(\tau) - 1}{\tau} \right) \\
&\quad + \frac{\tau}{(1 - \tau)} \int_0^\tau Q_n(\omega) \left(1 - \frac{Q_n(\omega) - 1}{\tau} \right) d\omega,
\end{align*}
\tag{6.5}
\]

for \( N = 0, 1, 2, \ldots \), where the initial iteration is \( Q(0) = \frac{1}{2} \). Hence, the analytical solution can be achieved as follow:

\[
Q(\tau) = \lim_{N \to \infty} Q_n(\tau).
\tag{6.6}
\]

Anyhow, by converting the cubic FLDE (6.3) into a homogeneous initial value problem (6.4), the RKHS method can be applied by taking \( \tau_i = \frac{i}{N}, i = 0, 1, \ldots, N \), and \( N = 10 \). The numerical computations of Example 6.3 are listed in form of

| \( \tau \) | Exact Solution | RKHS Solution | Absolute Error |
|---|---|---|---|
| 0.0 | 0.25000 | 0.25000 | 0 |
| 0.1 | 0.25949 | 0.25949 | 4.4123 \times 10^{-7} |
| 0.2 | 0.26921 | 0.26921 | 9.9065 \times 10^{-7} |
| 0.3 | 0.27916 | 0.27916 | 1.5257 \times 10^{-6} |
| 0.4 | 0.28934 | 0.28934 | 2.0637 \times 10^{-6} |
| 0.5 | 0.29972 | 0.29972 | 2.5853 \times 10^{-6} |
| 0.6 | 0.31032 | 0.31032 | 3.0952 \times 10^{-6} |
| 0.7 | 0.32112 | 0.32112 | 3.6007 \times 10^{-6} |
| 0.8 | 0.33212 | 0.33212 | 4.0470 \times 10^{-6} |
| 0.9 | 0.34330 | 0.34330 | 4.5491 \times 10^{-6} |
| 1.0 | 0.35466 | 0.35466 | 4.9951 \times 10^{-6} |
tables and graphs as follows: Table 5 is dedicated to represent the numerical results of the RKHS method compared with the SS iterations. From this table, it can be observed that despite the high accuracy of the SS approach, while the efficacy of the RKHS method is felt when calculating the difference between the results of these two methods. Fig. 5 exhibits a comparison among the behavior of the RKHS solution and SS solution at $\alpha = 1$. While Table 6 shows the RKHS solutions within the Caputo-Fabrizio concept for different values of fractional order $\alpha$ such that $\alpha \in \{0.55, 0.65, 0.75, 0.85\}$. On the other hand, the recurrence errors of RKHS solution are calculated within Caputo-Fabrizio concept instead the absolute error and tabulated them in Table 7. Here, it can noted

### Table 2 RKHS solutions using CFD and CFFD of Example 6.1.

| $\tau$ | $\alpha = 1$ | $\alpha = 0.95$ | $\alpha = 0.75$ | $\alpha = 0.55$ |
|--------|---------------|-----------------|-----------------|-----------------|
| 0.0    | 0.250000000   | 0.250000000     | 0.250000000     | 0.250000000     |
| 0.1    | 0.259491352   | 0.262766027     | 0.281471847     | 0.301955613     |
| 0.2    | 0.269213658   | 0.272919722     | 0.289548690     | 0.308409349     |
| 0.3    | 0.279162926   | 0.281810741     | 0.297732805     | 0.314660207     |
| 0.4    | 0.289334330   | 0.291628713     | 0.306035827     | 0.321364781     |
| 0.5    | 0.299722259   | 0.301452814     | 0.314528143     | 0.327920822     |
| 0.6    | 0.310320309   | 0.311838517     | 0.322978810     | 0.334523388     |
| 0.7    | 0.321121278   | 0.32271706      | 0.331608532     | 0.341171912     |
| 0.8    | 0.332117169   | 0.334851400     | 0.349156846     | 0.354593203     |
| 0.9    | 0.343299199   | 0.354357483     | 0.361360156     | 0.361360156     |

### Table 3 Absolute errors of Example 6.2 at $\alpha = 1$.

| $\tau$ | Exact Solution | RKHS Solution | Absolute Error |
|--------|----------------|---------------|----------------|
| 0.2    | 0.5249791875  | 0.5249788716  | 3.15903910 x 10^{-7} |
| 0.3    | 0.5374298453  | 0.5374293722  | 4.73191098 x 10^{-7} |
| 0.4    | 0.5498339973  | 0.5498336494  | 6.32423051 x 10^{-7} |
| 0.5    | 0.5621765009  | 0.5621757056  | 7.95317434 x 10^{-7} |
| 0.6    | 0.5744425168  | 0.5744415533  | 9.63465269 x 10^{-7} |
| 0.7    | 0.5866175789  | 0.5866164406  | 1.13828138 x 10^{-6} |
| 0.8    | 0.5986876601  | 0.5986863392  | 1.32095553 x 10^{-6} |
| 0.9    | 0.6106392339  | 0.6106377215  | 1.51241820 x 10^{-6} |
| 1.0    | 0.6224593312  | 0.6224576179  | 1.71330940 x 10^{-6} |
Table 4 Numerical results for Example 2 for $\tau \in [0,1]$ using the RKHS method.

| $\tau$ | $\alpha = 0.9$ | $\alpha = 0.8$ | $\alpha = 0.7$ |
|--------|----------------|----------------|----------------|
| 0.0    | 0.5            | 0.5            | 0.5            |
| 0.1    | 0.5222589      | 0.5335800      | 0.5448084      |
| 0.2    | 0.5334473      | 0.5434630      | 0.5533589      |
| 0.3    | 0.5445848      | 0.5532853      | 0.5618552      |
| 0.4    | 0.5556656      | 0.5630464      | 0.5702924      |
| 0.5    | 0.5666793      | 0.5727389      | 0.5786665      |
| 0.6    | 0.5776156      | 0.5823556      | 0.5869731      |
| 0.7    | 0.5884647      | 0.5918897      | 0.5952082      |
| 0.8    | 0.5992169      | 0.6013348      | 0.6033679      |
| 0.9    | 0.6098632      | 0.6106847      | 0.6114484      |
| 1.0    | 0.6203948      | 0.6199334      | 0.6194464      |

Table 5 RKHS and SS solutions of Example 6.3 at $\alpha = 1$.

| $\tau$ | SS solution | RKHS solution | $|RKHS - SS|$ |
|--------|-------------|---------------|-------------|
| 0.0    | 0.5         | 0.5           | 0.0         |
| 0.1    | 0.488120    | 0.488194      | 0.00007426  |
| 0.2    | 0.476238    | 0.476526      | 0.00028807  |
| 0.3    | 0.464369    | 0.464997      | 0.00062796  |
| 0.4    | 0.452525    | 0.453605      | 0.00108045  |
| 0.5    | 0.440720    | 0.442352      | 0.00163213  |
| 0.6    | 0.428967    | 0.431237      | 0.00226973  |
| 0.7    | 0.417279    | 0.420259      | 0.00298015  |
| 0.8    | 0.405670    | 0.409421      | 0.00375055  |
| 0.9    | 0.394151    | 0.398720      | 0.00456838  |
| 1.0    | 0.382736    | 0.388157      | 0.00542147  |

Fig. 3 Solution behavior of Example 6.2 for $\alpha = 1$ and $\tau \in [0,1]$: Exact is green line and RKHS is orange line.

Fig. 4 Graphical results for Example 6.2 using CFFD and CFD with respect to different fractional order: Yellow line $\alpha = 0.7$; Green line $\alpha = 0.8$; Red line $\alpha = 0.9$; Brown line $\alpha = 1$; and Black line $\alpha = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
displays the effect of changing values of initial conditions on changes over the set \{0.2, 0.4, 0.6, 0.8, 1\}. Finally, Figs. 6 and 7 for Example 6.3 using the RKHS method with initial condition.

that the value of the recurrence error is so small that it is negligible, and consequently confirms to us the integrity and accuracy of the calculations. Table 8 shows the approximate results for Example 6.3 using the RKHS method with initial condition changes over the set \{0.2, 0.4, 0.6, 0.8, 1\}. Finally, Figs. 6 and 7 displays the effect of changing values of initial conditions on the RKHS solution trajectories for Example 6.3 at $x = 0.5$.

7. Conclusion

In this paper, the RKHS method has been successfully applied to obtaining the approximate solutions for a class of nonlinear FLDEs in Hilbert space, including the modified quadratic and cubic models. The fractional derivative has been described in the Caputo-Fabrizio sense. A discussion for the stability of the proposed method and its convergence analysis have been given. Several examples have been carried out to show the efficiency of the proposed method. By comparing our results with the exact solution for integer-order derivative, it can be observe that the proposed method yields accurate approximations. The nonlinear FLDEs for different values of fractional order have been solved to see the effects of the fractional derivative on the logistic curve. Evidently from the tabulated results and graphical approximation for the population growth curves, the effect of different fractional derivatives is remarkable and consistent. In future studies, the effects of some other

| $\tau$ | $Q_0 = 0.2$ | $Q_0 = 0.4$ | $Q_0 = 0.6$ | $Q_0 = 0.8$ |
|-------|-------------|-------------|-------------|-------------|
| 0.0   | 0.2         | 0.4         | 0.6         | 0.8         |
| 0.1   | 0.197351    | 0.396263    | 0.596486    | 0.797793    |
| 0.2   | 0.194455    | 0.392178    | 0.592643    | 0.795378    |
| 0.3   | 0.191319    | 0.387750    | 0.588476    | 0.792759    |
| 0.4   | 0.187950    | 0.382989    | 0.583992    | 0.789937    |
| 0.5   | 0.184356    | 0.377905    | 0.579197    | 0.786917    |
| 0.6   | 0.180545    | 0.372508    | 0.574101    | 0.783703    |
| 0.7   | 0.176529    | 0.368611    | 0.568772    | 0.780999    |
| 0.8   | 0.172320    | 0.360828    | 0.563042    | 0.776711    |
| 0.9   | 0.167929    | 0.354573    | 0.557102    | 0.772943    |
| 1.0   | 0.163372    | 0.348066    | 0.550905    | 0.769003    |

Table 7 The recurrence errors $| Q_0(\tau) - Q_0(\tau) |$ by RKHS method of Example 6.3.

| $\tau$ | $\alpha = 1$ | $\alpha = 0.95$ | $\alpha = 0.85$ | $\alpha = 0.75$ | $\alpha = 0.65$ |
|-------|-------------|-------------|-------------|-------------|-------------|
| 0.0   | 0.0         | 0.0         | 0.0         | 0.0         | 0.0         |
| 0.1   | 9.8523 $\times 10^{-9}$ | 2.8645 $\times 10^{-7}$ | 2.9164 $\times 10^{-7}$ | 3.0797 $\times 10^{-7}$ | 3.3692 $\times 10^{-7}$ |
| 0.2   | 3.9409 $\times 10^{-8}$ | 6.0017 $\times 10^{-7}$ | 6.1105 $\times 10^{-7}$ | 6.4508 $\times 10^{-7}$ | 7.0501 $\times 10^{-7}$ |
| 0.3   | 8.8670 $\times 10^{-8}$ | 9.4118 $\times 10^{-7}$ | 9.5823 $\times 10^{-7}$ | 1.0111 $\times 10^{-6}$ | 1.1031 $\times 10^{-6}$ |
| 0.4   | 1.5764 $\times 10^{-7}$ | 1.3095 $\times 10^{-6}$ | 3.6752 $\times 10^{-6}$ | 1.4055 $\times 10^{-6}$ | 1.5299 $\times 10^{-6}$ |
| 0.5   | 2.4631 $\times 10^{-7}$ | 1.7050 $\times 10^{-6}$ | 1.7358 $\times 10^{-6}$ | 1.8281 $\times 10^{-6}$ | 1.9839 $\times 10^{-6}$ |
| 0.6   | 3.5468 $\times 10^{-7}$ | 2.1279 $\times 10^{-6}$ | 2.1662 $\times 10^{-6}$ | 2.2779 $\times 10^{-6}$ | 2.4628 $\times 10^{-6}$ |
| 0.7   | 4.8276 $\times 10^{-7}$ | 2.5780 $\times 10^{-6}$ | 2.6243 $\times 10^{-6}$ | 2.7544 $\times 10^{-6}$ | 2.9647 $\times 10^{-6}$ |
| 0.8   | 6.3054 $\times 10^{-7}$ | 3.0554 $\times 10^{-6}$ | 3.1099 $\times 10^{-6}$ | 3.2561 $\times 10^{-6}$ | 3.4865 $\times 10^{-6}$ |
| 0.9   | 7.9803 $\times 10^{-7}$ | 3.5601 $\times 10^{-6}$ | 3.6229 $\times 10^{-6}$ | 3.7815 $\times 10^{-6}$ | 4.0248 $\times 10^{-6}$ |
| 1.0   | 9.8523 $\times 10^{-7}$ | 4.0921 $\times 10^{-6}$ | 4.1632 $\times 10^{-6}$ | 4.3285 $\times 10^{-6}$ | 4.5756 $\times 10^{-6}$ |
fractional operators such as the Atangana-Baleanu and conformable fractional derivatives to population growth models, including the generalized Logistic model and Gompertz model, can be discussed and examined. By working on various fractional operators, we can determine which is more suitable for predicting growth curves using data collected from real populations. On a more intuitive note, an important issue that can be studied using the fractional logistic model is the way in which Coronavirus (COVID-19) spreads compared to data published from the World Health Organization.

Declaration Competing Interest

The authors declare that they have no conflicts of interest.

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Fig. 7 RKHS trajectories of Example 6.3 at z = 0.5 with different values of initial conditions: Black $Q_0 = 0.2$, gray $Q_0 = 0.4$, red $Q_0 = 0.6$, green $Q_0 = 0.8$, and Brown $Q_0 = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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