Spectral Properties of Descent Algebra Elements

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Abstract
The descent algebra of finite Coxeter groups is studied by many famous mathematicians like Bergeron, Brown, Howlett, or Reutenauer. Blessenohl, Hohlweg, and Schocker, for example, proved a symmetry property of the descent algebra, when it is linked to the representation theory of its Coxeter group. The interest is particularly showed for the descent algebra of symmetric group. Thibon determined the eigenvalues and their multiplicities of the action on the group algebra of symmetric group of the descent algebra element, which is the sum over all permutations weighted by \(q^{\text{maj}}\). And even the author diagonalized the matrix of the action of the descent algebra element, which is the sum over all permutations weighted by the new introduced statistic \(\text{des}_X\). In this article, we give a more general result by determining the eigenvalues and their multiplicities of the action on the group algebra of finite Coxeter group of an element of its descent algebra.

1 Introduction
We keep the usual notations \(K\) for an algebraically closed field of characteristic 0, and \((W,S)\) for a finite Coxeter system, that is to say, \(W\) is a finite group generated by the elements of \(S\) subject to the defining relations
\[(sr)^{m_{sr}} = e, \text{ for all } s,r \in S\]
where \(e\) is the neutral element of \(W\), the \(m_{sr}\) are positive integers, and \(m_{ss} = 1\) for all \(s \in S\).

Let \(J \subseteq S\). We naturally use the notations \(W_J\) for the parabolic subgroup of \(W\) generated by the elements of \(J\), and \(c_J\) for a Coxeter element of \(W_J\) which is a product of the elements of \(J\) taken in some fixed order. We write \(c_J\) for the conjugacy class of \(c_J\).

Let \(J \subseteq S\). We write \(J\) for the set of subsets \(K \subseteq S\) such that the parabolic subgroups \(W_J\) and \(W_K\) are conjugate. Let \(\tilde{J}_1, \ldots, \tilde{J}_p\) be pairwise different such that
\[\{\tilde{J}_i\}_{i \in [p]} = \{\tilde{J}\}_{J \subseteq S}\]

Let \(J, K \subseteq S\). We write \(JW^K\) for the distinguished cross section for the double coset space \(W_J\backslash W/W_K\). If \(J = \{\varnothing\}\) resp. \(K = \{\varnothing\}\), we just write \(W^K\) resp. \(JW\).

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Let $J \subseteq S$. We write
\[ x_J := \sum_{w \in W^J} w. \]
Let $\Xi_W := \{ x_J \mid J \subseteq S \}$. We know that $\mathbb{K}[\Xi_W]$ is the descent algebra of $W$ which is a subalgebra of the group algebra $\mathbb{K}[W]$ multiplicatively equipped with [8] Theorem 1:
\[ x_J x_K = \sum_{L \subseteq K} a_{J KL} x_L \]
for $J, K \subseteq S$, with
\[ a_{J KL} := \left| \{ x \in \mathfrak{d}W^K \mid x^{-1}W_J \cap W_K = W_L \} \right|. \]
Let $u = \sum_{w \in W} \lambda_w w \in \mathbb{K}[W]$. We write $R_W(u) = (\lambda_{ww'}^{-1})_{w, w' \in W}$ for the matrix of the regular representation of the left-multiplication action of $u$ on $\mathbb{K}[W]$ relatively to the standard basis $\{ w \mid w \in W \}$. The purpose of this article is to prove the following theorem:

**Theorem 1.1.** Let $d = \sum_{J \subseteq S} \lambda_J x_J \in \mathbb{K}[\Xi_W]$. Then the spectrum of $R_W(d)$ is
\[ \text{Sp}(R_W(d)) = \left\{ \Delta_j = \sum_{i=1}^p a_{J_i J_j} \left( \sum_{K_i \in J_i} \lambda_{K_i} \right) \right\}_{j \in [p]} \]
with corresponding multiplicities
\[ \{ m_{\Delta_j} = |c_{J_j}| \}_{j \in [p]}. \]
Let $w \in W$. Recall that the set of left resp. right descent set of $w$ is
\[ \text{DES}_L(w) := \{ s \in S \mid 1(sw) < 1(w) \} \text{ resp. } \text{DES}_R(w) := \{ s \in S \mid 1(ws) < 1(w) \}, \]
1 being the statistic which gives the length of the minimal expression of the elements of $W$ in terms of elements of $S$.
Let $J \subseteq S$. We write $D_J$ for the set of elements of $W$ with right descent set $J$. We have the disjoint union
\[ W^{S \setminus K} = \bigcup_{J \subseteq K} D_J. \]
Thus setting
\[ y_J := \sum_{w \in D_J} w, \]
we get
\[ x_{S \setminus K} = \sum_{J \subseteq K} y_J, \]
and hence by Möbius inversion formula
\[ y_J = \sum_{K \subseteq J \setminus K} (-1)^{|J \setminus K|} x_{S \setminus K}. \]
The descent algebra is a well studied object on the borderline of combinatorics and algebra ([1], [2], [3]). Especially the descent algebra relative to the Coxeter system of the symmetric group...
\((A_n, S_{A_n} = \{t_1, \ldots, t_n\})\), where \(t_i\) is the transposition \((i \ i + 1)\) \([7, 11]\). Thibon determined the eigenvalues and their multiplicities of the action of the element \([10, Theorem 56]\)

\[
\sum_{J \subseteq S_{A_n}} q^{\text{Maj}(J)} y_J \in \mathbb{R}(q)[\Xi_{A_n}],
\]

where

\[
\text{Maj}(J) := \sum_{j \in \{i \in [n] | t_i \in J\}} j.
\]

In \([17]\), Schocker worked on the descent algebra of the symmetric group. Using hyperplanes arrangement and random walk properties \([5, 6]\), Brown determined the condition of diagonalizability of the action of element of the descent algebra of finite Coxeter group, and gave a remarkable approach of the eigenvalues and corresponding multiplicities of its regular representation. In \([12]\), the regular representation of the element

\[
\sum_{J \subseteq S_{A_n}} \text{Des}_X(J) y_J \in \mathbb{R}(X_1, \ldots, X_n)[\Xi_{A_n}],
\]

where

\[
\text{Des}_X(J) := \sum_{j \in \{i \in [n] | t_i \in J\}} X_j,
\]

was diagonalized.

This paper is organized as follows. We begin with the calculation of the formula for the coefficients \(a_{JJK}\). Then, we determine the eigenvalues and their corresponding multiplicities of the regular representation of an element of the descent algebra of a finite Coxeter group. In the appendix part, we treat the Coxeter group \(F_4\) as complete example, and we give a counterexample of the formula in \([2, Theorem 6.5]\).

### 2 Special Coefficients of the Descent Algebra

In this section, we determine a formula for the values of \(a_{JJK}\). Through a slight modification, our formula corrects a mistake from the formula proposed in \([2, Theorem 6.5]\).

Let \(J \subseteq S\). We write \(N_J\) for the subgroup \([8, Corollary 3]\)

\[N_J := \{w \in W \mid w^{-1}W_J w = W_J\} \cap W^J.\]

Let \(w \in W\) and \(U, V\) be subgroups of \(W\). We write

\[U_w^V := \{x \in UwV \mid 1(x) \leq 1(wv), \forall u \in U, \forall v \in V\}\]

for the set of minimal double coset representatives of \(w\) relative to \(U\) and \(V\). If \(U = \{e\}\) resp. \(V = \{e\}\), we just write \(w^V\) resp. \(U_w\). For the case of parabolic subgroups, we just write \(W_J w W_K = J w K\) with \(J, K \subseteq S\). This lemma can be read off from \([18\, 1.Introduction]\).

**Lemma 2.1.** Let \(w \in W\) and \(J, K \subseteq S\). Then the set \(J w K\) contains a unique element. In this case, we consider \(J w K\) no more as a subset of \(W\) but as an element of \(W\).

Let \(K \subseteq S\) and \(K' \in \bar{K}\). We write

\[C_{K'K} := \{w \in W \mid w^{-1}W_{K'} w = W_K\}.\]
Lemma 2.2. Let $c \in C_{K'K}$, and $c_{K'K} \in \langle N_{K'}W_{K'}c_{N_{K}W_{K}} \rangle$. Then $c_{K'K} \in C_{K'K}$.

Proof. If $c = n'k'c_{K'K}nk$ with $n'k' \in N_{K'}W_{K'}$ and $nk \in N_{K}W_{K}$, then

\[
\begin{align*}
(n'k'c_{K'K}n k)^{-1}W_{K'} n'k'c_{K'K}n k &= W_{K},
(c_{K'K}n k)^{-1}W_{K'} c_{K'K}n k &= W_{K},
\end{align*}
\]

\[
\begin{align*}
\frac{c_{K'K}}{W_{K'}} &= c_{K'K},
\frac{c_{K'K}}{W_{K'}} &= \frac{nkW_{K}}{(nk)^{-1}},
\end{align*}
\]

Let $E \subseteq W$ and $U, V$ be subgroups of $W$. We write

\[
UEV := \bigcup_{w \in E} UwV.
\]

For the case of parabolic subgroups, we just write $WJE^{W_{K}} = J^{E^K}$, with $J, K \subseteq S$.

Lemma 2.3. Let $K \subseteq S$, $K' \subseteq \tilde{K}$, and $c_{K'K} \in \langle N_{K'}W_{K'}c_{N_{K}W_{K}} \rangle$. Then

\[
K'(c_{K'K}N_{K})^{K} = c_{K'K}N_{K} \text{ and } K'(N_{K'}c_{K'K})^{K} = N_{K'}c_{K'K}.
\]

Proof. It is clear that $(c_{K'K}N_{K})^{K} = c_{K'K}N_{K}$. Effectively, since $c_{K'K}$ is a left coset representative of the subgroup $N_{K}W_{K}$, and the elements of $N_{K}$ are left coset representatives of $W_{K}$, then the elements of $c_{K'K}N_{K}$ are left coset representatives of $W_{K}$.

We just then have to prove that $K'(c_{K'K}N_{K}) = c_{K'K}N_{K}$. Let $c_{K'K}n \in c_{K'K}N_{K}$ and let us suppose that $c_{K'K}n = k'b$, where $k' \in W_{K'}$ and $b \in K'W$. Then $1(c_{K'K}n) = 1(k'b) = 1(k') + 1(b)$ i.e.

\[
1(c_{K'K}n) \geq 1(b).
\]

On the other hand, we have $(k')^{-1}c_{K'K}n = c_{K'K}k_{1}n = c_{K'K}nk_{2} = b$ with $k_{1}, k_{2} \in W_{K}$. Then $1(c_{K'K}nk_{2}) = 1(c_{K'K}n) + 1(k_{2}) = 1(b)$, i.e.

\[
1(c_{K'K}n) \leq 1(b).
\]

The only possibility is then $k_{2} = k_{1} = k' = e$, so we get the result. The proof for $K'(N_{K'}c_{K'K})^{K} = N_{K'}c_{K'K}$ is analogous.

Lemma 2.4. Let $K \subseteq S$, $K' \subseteq \tilde{K}$, and $c_{K'K} \in \langle N_{K'}W_{K'}c_{N_{K}W_{K}} \rangle$. Then

\[
\{w \in K'W_{K} | w^{-1}W_{K'} w = W_{K}\} = c_{K'K}N_{K} = N_{K'}c_{K'K}.
\]

Proof. It is clear that \( \{w \in K'W_{K} | w^{-1}W_{K'} w = W_{K}\} = N_{K} \) for all $K \subseteq S$.

- The map $\phi : \{w \in K'W_{K'} | w^{-1}W_{K'} w = W_{K}\} \to \{w \in K'W_{K} | w^{-1}W_{K'} w = W_{K}\}$

\[
\phi(n) := n c_{K'K}
\]

is clearly injective.
\begin{itemize}
\item The map \( \phi' : \{ w \in K'wK \mid w^{-1}wK \} \rightarrow \{ w \in K'wK' \mid w^{-1}wK' \} \),
\[ x \mapsto (c_{K'K}x^{-1})K' \]
is injective. Effectively, \( \phi'(x) = \phi'(y) \) means \( c_{K'K}x^{-1} = u_1c_{K'K}y^{-1}u_2 \) with \( u_1, u_2 \in W_{K'} \). Then \( c_{K'K}x^{-1} = c_{K'K}vy^{-1}u_2 \) and \( x^{-1} = vy^{-1}u_2 \) with \( v \in W_K \). The only possibility is \( v = u_2 = e \).
\end{itemize}

Then we deduce that \( \phi \) is bijective and \( \{ w \in K'wK \mid w^{-1}wK \} = N_{K'}c_{K'K} \).
The proof is analogous for \( \{ w \in K'wK \mid w^{-1}wK' \} = c_{K'K}N_K \).

Let \( J, K \subseteq S \), and, for all \( K' \in \tilde{K} \), let us fix an element \( c_{K'K} \in n_{K'wK'} \). \( C_{K'K} \).

We can now give the formula to determine \( a_{JK} \).

**Theorem 2.5.** Let \( J, K \subseteq S \). We have
\[
a_{JK} = \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}.
\]

**Proof.** Recall that
\[
a_{JK} := \left| \{ w \in JwK \mid W_J \cap wW_K w^{-1} = wW_K w^{-1} \} \right|,
\]
with \( wW_K w^{-1} = W_{K'} \) and \( K' \in \tilde{K} \cap 2^J \).

We have \( w \in c_{K'K}N_K = N_{K'}c_{K'K} \). But we must also have \( w = Jw \). That means:

- On the one hand, we must have \( c_{K'K} = Jc_{K'K} \). Otherwise \( J(c_{K'K}N_K) \cap C_{K'K} = \emptyset \).
- On the other hand, if \( c_{K'K} = Jc_{K'K} \), then \( w \in J(N_{K'}c_{K'K}) = (JN_{K'})c_{K'K} \).

Since
\[
|JN_{K'}| = \frac{|N_{K'}|}{|W_J \cap N_{K'}|},
\]
it follows that
\[
a_{JK} = \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_{K'}|}{|W_J \cap N_{K'}|} = \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|},
\]
since \( W_K \) and \( W_{K'} \) are conjugate.

Let \( J' \in \tilde{J} \) and \( K' \in \tilde{K} \). From Theorem 2.3, we deduce that
\[
a_{JK} = a_{J'K'K'}.
\]

This result can also be found in [2] Theorem 6.2].
3 Eigenvalues and Multiplicities

We are now able to determine the eigenvalues and their corresponding multiplicities. Let
\[ d = \sum_{J \subseteq S} \lambda_J x_J \in K[\Xi_W]. \]
We write \( v_{\Xi_W}(d) \) for the column vector of \( d \) relative to the basis \( \Xi_W \), and \( M_{\Xi_W}(d) \) for the matrix of the left-multiplication action of \( d \) on \( K[\Xi_W] \) relative the basis \( \Xi_W \) i.e.
\[
v_{\Xi_W}(d) = (\lambda_K)_{K \subseteq S} \quad \text{and} \quad M_{\Xi_W}(d) = \left( \sum_{J \subseteq S} \lambda_J a_{JKL} \right)_{K,L \subseteq S}.
\]

Let \( n \geq 2 \) and \((d_i)_{i \in [n]} \in K[\Xi_W]^n \). We have
\[
v_{\Xi_W}(\prod_{i \in [n]} d_i) = \left( \prod_{i \in [n-1]} M_{\Xi_W}(d_i) \right) \cdot v_{\Xi_W}(d_n).
\]

Recall that the noncommutative multiplication is defined in the following way:
\[
\prod_{i \in [n]} \lambda_i = \lambda_1 \lambda_2 \ldots \lambda_n.
\]

We write \( M_{\Xi_W}(d)_{i,K} \) for the column of \( M_{\Xi_W}(d) \) corresponding to the basis vector \( x_K \), i.e.
\[
M_{\Xi_W}(d)_{i,K} = \left( \sum_{J \subseteq S} \lambda_J a_{JKL} \right)_{L \subseteq S} = v_{\Xi_W}(d \cdot x_K).
\]

Lemma 3.1. Let \( d \in K[\Xi_W] \). Then \( R_W(d) \) and \( M_{\Xi_W}(d) \) have the same spectrum.

Proof. It is clear that \( Sp(M_{\Xi_W}(d)) \subseteq Sp(R_W(d)) \).

We write \( 0_{2^{[s]}} \) for the matrix with entry 0 of \( K^{2^{[s]} \times 2^{[s]}} \), and \( I_{2^{[s]}} \) for the identity matrix of \( K^{2^{[s]} \times 2^{[s]}} \). Let \( \sum_{i=0}^{2^{[s]}} \mu_i t^i \) be the characteristic polynomial of \( M_{\Xi_W}(d) \) in the variable \( t \) with \( (\mu_i)_{i \in \{0\} \cup 2^{[s]}} \in K^{2^{[s]}+1} \). We have \( \sum_{i=0}^{2^{[s]}} \mu_i M_{\Xi_W}(d)^i = 0_{2^{[s]}} \), especially
\[
\mu_0 I_{2^{[s]}} + \sum_{i=1}^{2^{[s]}} \mu_i M_{\Xi_W}^{i-1}(d) \cdot M_{\Xi_W}(d)_{i,S} = 0_{2^{[s]}}+S.
\]

Since \( L_{2^{[s]}} + S = v_{\Xi_W}(e) \), and \( M_{\Xi_W}(d)_{i,S} = v_{\Xi_W}(d) \), then
\[
\mu_0 v_{\Xi_W}(e) + \sum_{i=1}^{2^{[s]}} \mu_i M_{\Xi_W}^{i-1}(d) \cdot v_{\Xi_W}(d) = v_{\Xi_W}(0).
\]

This means \( \mu_0 e + \sum_{i=1}^{2^{[s]}} \mu_i d^i = 0 \) and \( Sp(R_W(d)) \subseteq Sp(M_{\Xi_W}(d)) \). \( \square \)

For the rest of the section, we need a total order \( \succ \) on the subsets of \( S = \{ s_i \}_{i \in [s]} \) which was introduced by F. and N. Bergeron \[ \square \]. We define \( \min J := \min\{ i \in [s] \mid s_i \in J \} \), and assume that \( \min \emptyset = |S| + 1 \). Let \( J, K \subseteq S \) such that \( J \neq K \).

- If \( \min J > \min K \) then \( J \succ K \).
• Otherwise \( J \succ K \) if and only if \( J \setminus \{ s_{\min} J \} \succ K \setminus \{ s_{\min} K \} \).

We have already seen the definition of the set \( \{ \tilde{J}_i \}_{i \in [p]} \) in the introduction.

Let \( K_i \) be the element of \( \tilde{J}_i \) such that \( L_i \succ K_i \) for all \( L_i \in \tilde{J}_i \setminus \{ K_i \} \). We order the sets \( \{ \tilde{J}_i \}_{i \in [p]} \) such that \( K_i \succ K_j \) if \( i < j \).

**Proposition 3.2.** Let \( d = \sum_{J \subseteq S} \lambda_J x_J \in \mathbb{K}[\Xi_W] \). Then the spectrum of the matrix \( R_W(d) \) is

\[
\text{Sp}(R_W(d)) = \left\{ \sum_{i=1}^p a_{K_i, K_i}(\sum_{L_i \in \tilde{J}_i} \lambda_{L_i}) \right\}_{J \subseteq [p]}.
\]

**Proof.** We order the basis \( (x_J)_{J \subseteq S} \) of \( \mathbb{K}[\Xi_W] \) according to the total order \( \succ \) of F. and N. Bergeron that means we get a new ordered basis \( (x_{L_i})_{i \in [2^{|S|}]} \) such that \( L_i \succ L_j \) if \( i < j \). We assume that the rows and columns of the matrix \( M_{\Xi_W}(d) \) are ordered in increasing order by the order \( \succ \). We get \( d \cdot x_{L_j} \in \langle \{ x_{L_i} \}_{i \in [j]} \rangle \). Then the matrix of \( d \) on the basis \( (x_{L_i})_{i \in [2^{|S|}]} \) is an upper triangular matrix. The characteristic polynomial of this matrix in the variable \( t \) is

\[
\prod_{K \subseteq S} ((\sum_{J \subseteq S} \lambda_J a_{J, K K_i} - t),
\]

and

\[
\text{Sp}(M_{\Xi_W}(d)) = \left\{ \sum_{J \subseteq S} \lambda_J a_{J, K K_i} \right\}_{K \subseteq S}.
\]

Since \( a_{J, K K_i} = a_{J', K K_i} \) for all \( J, J' \in \tilde{J}_i \), and all \( K, K' \in \tilde{J}_i \), we get the result. \( \square \)

**Proposition 3.3.** Let \( d = \sum_{J \subseteq S} \lambda_J x_J \in \mathbb{K}[\Xi_W] \), and \( \Delta_j = \sum_{i=1}^p a_{K_i, K_i}(\sum_{L_i \in \tilde{J}_i} \lambda_{L_i}) \). Then the multiplicity of the eigenvalue \( \Delta_j \) of \( R_W(d) \) is

\[
m_{\Delta_j} = |\tilde{J}_j|.
\]

**Proof.** Let \( A = (a_{K_i, K_i})_{i,j \in [p]}, m = (m_{\Delta_j})_{j \in [p]}, u = (|W|)_{j \in [p]}, \) and \( c = (|\tilde{J}_j|)_{j \in [p]} \). At the end of the sixth section of [2], it is proved that \( A^{-1} u = c \).

Let \( tr \) be the trace map of square matrix. We have \( tr(R_W(d)) = |W| \sum_{J \subseteq S} \lambda_J \). Then,

\[
\sum_{i=1}^j a_{K_i, K_i} m_{\Delta_i} = |W| \lambda_{L_j} \text{ for } L_j \in \tilde{J}_j \text{ i.e.}
\]

\[
\sum_{i=1}^j a_{K_i, K_i} m_{\Delta_i} = |W|.
\]

In matrix form, we get \( Am = u \). Thus \( A^{-1} u = m \). \( \square \)

We note that with the matrix relation \( Am = u \), we can also get the cardinalities of the conjugacy classes of \( W \).
A Example of the Symmetry Group of 24-cell

Recall that the Coxeter system of the symmetry group of 24-cell with cardinality 1152 is \((F_4, \mathcal{S}_{F_4} = \{s_i\}_{i \in [4]})\), and its Coxeter graph is

\[ s_1 \longleftrightarrow s_2 \leftrightarrow s_3 \longleftrightarrow s_4 \]

Using Theorem 2.5 and the values of \(|N_K|\) in [8 page 74], we get the following values of \(a_{JKK}\) for the case of \(F_4\):

|   | \(\emptyset\) | \(\{s_1\}\) | \(\{s_4\}\) | \(\{s_1, s_2\}\) | \(\{s_2, s_3\}\) | \(\{s_1, s_4\}\) | \(\{s_1, s_2, s_3\}\) | \(\{s_2, s_3, s_4\}\) | \(\{s_1, s_3, s_4\}\) | \(\{s_1, s_2, s_4\}\) | \(\mathcal{S}_{F_4}\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \(\emptyset\) | 1152 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \(\{s_1\}\) | 576 | 48 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \(\{s_4\}\) | 576 | 0 | 48 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \(\{s_1, s_2\}\) | 192 | 48 | 0 | 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \(\{s_2, s_3\}\) | 144 | 24 | 24 | 0 | 8 | 0 | 0 | 0 | 0 | 0 | 0 |
| \(\{s_3, s_4\}\) | 192 | 0 | 48 | 0 | 0 | 12 | 0 | 0 | 0 | 0 | 0 |
| \(\{s_1, s_4\}\) | 288 | 24 | 24 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 |
| \(\{s_1, s_2, s_3\}\) | 24 | 24 | 6 | 12 | 4 | 0 | 0 | 2 | 0 | 0 | 0 |
| \(\{s_2, s_3, s_4\}\) | 24 | 6 | 24 | 0 | 4 | 12 | 0 | 0 | 2 | 0 | 0 |
| \(\{s_1, s_3, s_4\}\) | 96 | 8 | 24 | 0 | 0 | 6 | 4 | 0 | 0 | 2 | 0 |
| \(\{s_1, s_2, s_4\}\) | 96 | 24 | 8 | 6 | 0 | 0 | 4 | 0 | 0 | 0 | 2 |
| \(\mathcal{S}_{F_4}\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

We consider the element

\[
d = \sum_{J \subseteq \mathcal{S}_{F_4}} \lambda_J x_J \in \mathbb{K}[\mathcal{S}_{F_4}].
\]

The eigenvalues of \(R_{\mathcal{F}_4}(u)\) are

\[
\Delta_1 = 1152\lambda_5 + 576(\lambda_{s_1} + \lambda_{s_2} + \lambda_{s_3} + \lambda_{s_4}) + 192(\lambda_{s_1, s_2} + \lambda_{s_1, s_3} + \lambda_{s_1, s_4}) + 144\lambda_{s_2, s_3} + 288(\lambda_{s_1, s_2} + \lambda_{s_1, s_3} + \lambda_{s_1, s_4}) + 24(\lambda_{s_1, s_2, s_3} + \lambda_{s_1, s_2, s_4}) + 96(\lambda_{s_1, s_3, s_4} + \lambda_{s_1, s_2, s_4}) + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_2 = 48(\lambda_{s_1} + \lambda_{s_2} + \lambda_{s_3} + \lambda_{s_4}) + 24(\lambda_{s_2, s_3} + \lambda_{s_2, s_4} + \lambda_{s_3, s_4} + \lambda_{s_1, s_2, s_3} + \lambda_{s_1, s_2, s_4} + \lambda_{s_1, s_3, s_4}) + 6\lambda_{s_2, s_3, s_4} + 8\lambda_{s_1, s_2, s_4} + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_3 = 48(\lambda_{s_1} + \lambda_{s_2} + \lambda_{s_3} + \lambda_{s_4}) + 24(\lambda_{s_2, s_3} + \lambda_{s_2, s_4} + \lambda_{s_3, s_4} + \lambda_{s_1, s_2, s_3} + \lambda_{s_1, s_2, s_4} + \lambda_{s_1, s_3, s_4}) + 6\lambda_{s_1, s_2, s_3} + 8\lambda_{s_1, s_2, s_4} + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_4 = 12\lambda_{s_1, s_2} + 12\lambda_{s_1, s_2, s_3} + 6\lambda_{s_1, s_2, s_4} + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_5 = 8\lambda_{s_2, s_3} + 4(\lambda_{s_1, s_2, s_3} + \lambda_{s_2, s_3, s_4}) + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_6 = 12(\lambda_{s_1, s_4} + \lambda_{s_2, s_3, s_4}) + 6\lambda_{s_1, s_2, s_4} + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_7 = 4(\lambda_{s_1, s_3} + \lambda_{s_1, s_4} + \lambda_{s_2, s_4} + \lambda_{s_1, s_2, s_3} + \lambda_{s_1, s_2, s_4}) + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_8 = 2\lambda_{s_1, s_2, s_3} + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_9 = 2\lambda_{s_2, s_3, s_4} + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_{10} = 2\lambda_{s_1, s_2, s_4} + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_{11} = 2\lambda_{s_1, s_3, s_4} + \lambda_{\mathcal{S}_{F_4}}
\]

\[
\Delta_{12} = \lambda_{\mathcal{S}_{F_4}}
\]
with corresponding multiplicities

\[
m_{\Delta_1} = 1 = |\mathcal{F}|
m_{\Delta_2} = 12 = |s_1|
m_{\Delta_3} = 12 = |s_4|
m_{\Delta_4} = 32 = |s_1s_2|
m_{\Delta_5} = 54 = |s_2s_3|
m_{\Delta_6} = 32 = |s_3s_4|
m_{\Delta_7} = 72 = |s_4|
m_{\Delta_8} = 84 = |s_1s_2s_3|
m_{\Delta_9} = 84 = |s_2s_3s_4|
m_{\Delta_{10}} = 96 = |s_1s_3|
m_{\Delta_{11}} = 96 = |s_2s_4|
m_{\Delta_{12}} = 577 = |s_1s_2s_3s_4|
\]

B Counterexample on the Special Coefficients

The following formula is proposed in [2, Theorem 6.5]: For \( J, K \subseteq S \),

\[
a_{JKK} = \sum_{K' \in K \cap 2J} \frac{|N_K|}{W_J \cap N_K'}.
\]

Recall that the Coxeter system of the symmetry group of the dodecahedron with cardinality 120 is \( (H_3, S_{H_3} = \{s_i\}_{i \in [3]}) \), and its Coxeter graph is

\[
\begin{array}{c}
s_1 \\ s_1 \leftrightarrow s_2 \\ s_2 \leftrightarrow s_3
\end{array}
\]

We have the values of \(|N_K|\) for \( H_3 \) in [8, page 79]. If we use the formula in [2, Theorem 6.5], we get the following values of \( a_{JKK} \) for \( H_3 \):

\[
\begin{array}{c|cccccc}
\emptyset & \{s_1\} & \{s_1, s_2\} & \{s_2, s_3\} & \{s_1, s_3\} & S_{H_3} \\
\hline
\emptyset & 120 & 0 & 0 & 0 & 0 & 0 \\
\{s_1\} & 60 & 4 & 0 & 0 & 0 & 0 \\
\{s_1, s_2\} & 12 & 8 & 2 & 0 & 0 & 0 \\
\{s_2, s_3\} & 20 & 8 & 0 & 2 & 0 & 0 \\
\{s_1, s_3\} & 30 & 4 & 0 & 0 & 2 & 0 \\
S_{H_3} & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Let \( A = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ 60 & 4 & 0 & 0 & 0 & 0 \\ 12 & 8 & 2 & 0 & 0 & 0 \\ 20 & 8 & 0 & 2 & 0 & 0 \\ 30 & 4 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \). We know from Proposition 3.3 that \( A^{-1} \begin{pmatrix} 120 \\ 120 \\ 120 \\ 120 \\ 120 \end{pmatrix} \) gives

9
the cardinalities of the conjugacy classes of $H_3$. However, we get

$$A^{-1} = \begin{pmatrix} 120 & 15 \\ 120 & -6 \\ 120 & -10 \\ 120 & 15 \\ 120 & 105 \end{pmatrix}$$

which is absurd.

But if we use Theorem 2.5, the values of $a_{JKK}$ calculated for $H_3$ are:

|         | $\emptyset$ | $\{s_1\}$ | $\{s_1, s_2\}$ | $\{s_2, s_3\}$ | $\{s_1, s_3\}$ | $S_{H_3}$ |
|---------|-------------|-------------|-----------------|-----------------|----------------|-----------|
| $\emptyset$ | 120        | 0           | 0               | 0               | 0              | 0         |
| $\{s_1\}$ | 60         | 4           | 0               | 0               | 0              | 0         |
| $\{s_1, s_2\}$ | 12        | 4           | 2               | 0               | 0              | 0         |
| $\{s_2, s_3\}$ | 20        | 4           | 0               | 2               | 0              | 0         |
| $\{s_1, s_3\}$ | 30        | 4           | 0               | 0               | 2              | 0         |
| $S_{H_3}$ | 1          | 1           | 1               | 1               | 1              | 1         |

Let $B = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 \\ 60 & 4 & 0 & 0 & 0 \\ 12 & 4 & 2 & 0 & 0 \\ 20 & 4 & 0 & 2 & 0 \\ 30 & 4 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. Then we get the following cardinalities of the conjugacy classes of $H_3$

$$B^{-1} = \begin{pmatrix} 120 & 15 \\ 120 & 24 \\ 120 & 20 \\ 120 & 15 \\ 120 & 45 \end{pmatrix}$$

which are the correct values.

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