Outline

I  Introduction
II  Overview of methods
III (Example) Trap expansions
I. INTRODUCTION
“Adiabatic” in quantum mechanics

- A system remains in its instantaneous eigenstate
  - if $H(t)$ changes slowly
  - and if there is a gap between the eigenvalues.

(Born & Fock 1928)
• Adiabatic processes are ubiquitous and “robust”

• BUT:
  Typically they take too long
  ➔ Affected by noise and decoherence
  ➔ we may not have that time
  ➔ we may want to repeat the process many times

• We look for shortcuts
Fast Optimal Frictionless Atom Cooling in Harmonic Traps: Shortcut to Adiabaticity

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A method is proposed to cool down atoms in a harmonic trap without phase-space compression as in a perfectly slow adiabatic expansion, i.e., keeping the same populations of instantaneous levels in the initial and final traps, but in a much shorter time. This may require that the harmonic trap become transiently an explosive parabolic potential. The cooling times achieved are shorter than those obtained using optimal-control bang-bang methods and real frequencies.
Precedents

+related work by

R. Kosloff (quantum refrigerator & third principle),
C. Bender (brachistochrone),
D. Guéry-Odelin, D. Leibfried (transport),
T. Calarco (optimal control),
J. Schmiedmayer (splitting),
M. Deschamps (superadiabaticity)
N. Vitanov (fast preparation of internal states)
S. Guérin, H. R. Jauslin et al. (parallel adiabatic passage)

…
Fast trap expansions

Wave packet splitting
Fast transport
Accordion lattices
Population inversion & state preparation
Hum.... Microscopy of Quantum Correlations
Fast cooling of mechanical resonators

State control in Spintronics

Sympathetic cooling of cold atom mixtures
In optics:
- mode converters
- multiplexing and demultiplexing
Frequently the shortcuts can be optimized
II- Overview of methods

- 1 Inverse engineering
- 2 Invariants
- 3 Counterdiabatic approach
- 4 Finding alternatives by symmetry
- 5 Fast forward
- 6 Optimal Control Theory
II. 1 Inverse engineering methods

General idea:
Design the dynamics
Then deduce $H(t)$

$$U = \sum_n e^{i\alpha_n(t)} |\phi_n(t)\rangle \langle \phi_n(0)|$$

$$H(t) = i\hbar (\partial_t U)U^\dagger$$

Two basic routes

\begin{align*}
\begin{cases}
H_0(t)|\phi_n(t)\rangle = E_n^{(0)}(t)|\phi_n(t)\rangle & \text{“Transitionless tracking”} \\
I(t)|\phi_n(t)\rangle = \lambda_n|\phi_n(t)\rangle & \text{“counterdiabatic approach”}
\end{cases}
\end{align*}

Relation: Chen et al. PRA 83 (2011)

Third route: **FF approach** related to the others in Torrontegui et al. PRA 86 (2012)
II. 2 Dynamical invariants
\[
\frac{i\hbar}{\partial t} \Psi(t) = H(t)\Psi(t) \quad \frac{dI}{dt} \equiv \frac{\partial I(t)}{\partial t} + \frac{1}{i\hbar}[I(t), H(t)] = 0
\]

\[
\frac{d}{dt} \langle \psi(t)|I(t)|\psi(t) \rangle = 0
\]

\[
|\Psi(t)\rangle = \sum_n c_n |\psi_n(t)\rangle \quad \text{superposition of "dynamical modes"}
\]

\[
|\psi_n(t)\rangle = e^{i\alpha_n(t)} |\phi_n(t)\rangle
\]

\[
I(t) = \sum_n |\phi_n(t)\rangle \lambda_n \langle \phi_n(t)|
\]

\[
\alpha_n(t) = \frac{1}{\hbar} \int_0^t \langle \phi_n(t')|i\hbar \frac{\partial}{\partial t'} - H(t')|\phi_n(t') \rangle dt'
\]

Lewis-Riesenfeld phase
\[ H(t) = F(t) + i\hbar \sum_n |\partial_t \phi_n(t)\rangle \langle \phi_n(t)| \]

\[ F(t) = -\hbar \sum_n |\phi_n(t)\rangle \dot{\alpha}_n \langle \phi_n(t)| \]

**Boundary conditions**

\[
\begin{align*}
[I(0), H(0)] &= 0 \\
[I(t_f), H(t_f)] &= 0
\end{align*}
\]

Eigenstates coincide at 0 and \( t_f \)
II. 3 Counterdiabatic approach
(transitionless tracking)
Schrodinger & interaction Pictures

\[ |\psi_S\rangle = A|\psi_I\rangle \]

\[ i\hbar \partial_t |\psi_S\rangle = H |\psi_S\rangle \quad i\hbar \partial_t |\psi_I\rangle = H_I |\psi_I\rangle \]

\[ H = A H_I A^\dagger + K \quad H_I = A^\dagger (H - K) A \]

\[ K = i\hbar \dot{A} A^\dagger \]
Adiabatic basis IP

\[ H_0(t) |n_0(t)\rangle = E_n^{(0)}(t) |n_0(t)\rangle \]

\[ \langle n_0(t) | \dot{n}_0(t) \rangle = 0 \quad \text{Phase chosen for parallel transport} \]

IP based on

\[ |\psi_{I_1}\rangle = A_0^\dagger |\psi_S\rangle \]

\[ A_0(t) = \sum_n |n_0(t)\rangle \langle n_0(0)| \]

\[ H_1 = A_0^\dagger (H_0 - K_0) A_0 \]

\[ K_0 = i\hbar \dot{A}_0 A_0^\dagger \]
Adiabatic approximation

Neglect $K_0$ in

$$H_1 = A_0^\dagger (H_0 - K_0) A_0$$

$$i\hbar \partial_t |\psi_{I_1}\rangle = A_0^\dagger H_0 A_0 |\psi_{I_1}\rangle$$

Counterdiabatic (cd) term

It becomes EXACT if we

add $A_0^\dagger K_0 A_0$ to $H_1$

in the IP

equivalently $H_{cd}^{(0)} := K_0$ to $H_0$ in the SP

$\rightarrow H_0 + K_0$

This cd term may be a nuisance in the lab
Examples of cd-terms

Harmonic expansions/compressions:

\[ H_h = \frac{p^2}{2m} + m\tilde{\omega}^2 q^2 / 2 \]

\[ H_{cd}^{(0)} = -\frac{(pq + qp)}{4\tilde{\omega}} \hat{\omega} \]

Transport

\[ H_0 = \frac{p^2}{2m} + (q - q_0(t))^2 m\tilde{\omega}_0^2 / 2 \]

\[ H_{cd}^{(0)} = p\dot{q}_0 \]

Population inversion (2-level)

\[ H_0(t) = \begin{pmatrix} Z_0(t) & X_0(t) \\ X_0(t) & -Z_0(t) \end{pmatrix} \]

\[ H_{cd}^{(0)} = \hbar(\dot{\Theta}_0 / 2)\sigma_y \]
II. 4 Alternative routes by symmetry
Example: Trap expansions

\[ H_h = \frac{p^2}{(2m)} + m\tilde{\omega}^2 q^2, \]
\[ H_{cd}^{(0)} = -(pq + qp)\tilde{\omega}/(4\tilde{\omega}) \]
\[ H_S = H_h + H_{cd}^{(0)} \]
\[ A_q = \exp\left(i \frac{m\tilde{\omega}}{4\hbar\tilde{\omega}} q^2\right) \]
\[ H_I = A_q^\dagger (H_S - i\hbar \dot{A}_q A_q^\dagger) A_q = \frac{p^2}{(2m)} + m\tilde{\omega}'^2 q^2 / 2 \]
\[ \tilde{\omega}' = \left[ \tilde{\omega}^2 - \frac{3\tilde{\omega}^2}{4\tilde{\omega}^2} + \frac{\tilde{\omega}}{2\tilde{\omega}} \right]^{1/2} \]
\[ \dot{\tilde{\omega}}(t_f) = \tilde{\omega}(t_f) = 0 \]
\[ A_q(0) = A_q(t_f) = 1 \]
\[ \psi_I = \psi_S \text{ at } t = 0, t_f \]
II. 5 Fast Forward (FF)
Masuda, S. and Nakamura, K., 2010. Fast-forward of adiabatic dynamics in quantum mechanics. Proc. R. Soc. A 466, 1135.

Torrontegui, E., Martínez-Garaot, S., Ruschhaupt, A., Muga, J. G., 2012. Shortcuts to adiabaticity: Fast-forward approach. Phys. Rev. A 86, 013601.

\[ i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = H(t)|\psi(t)\rangle \]

\[ H = T + V + G \]

\[ \langle x|T|\psi(t)\rangle = \frac{-\hbar^2}{2m} \nabla^2 \psi(x, t) \]

\[ \langle x|V(t)|x'\rangle = V(x, t)\delta(x - x') \]

\[ \langle x|G(t)|\psi(t)\rangle = g|\psi(x, t)|^2\psi(x, t) \]
\[ \langle \mathbf{x} | \psi(t) \rangle = r(\mathbf{x}, t)e^{i\phi(\mathbf{x}, t)}, \quad r(\mathbf{x}, t), \phi(\mathbf{x}, t) \in \mathbb{R} \]

\[ V(\mathbf{x}, t) = i\hbar \frac{\dot{r}}{r} - \hbar \dot{\phi} + \frac{\hbar^2}{2m} \left( \frac{2i \nabla \phi \cdot \nabla r}{r} + i \nabla^2 \phi - (\nabla \phi)^2 + \frac{\nabla^2 r}{r} \right) - gr^2 \]

\[ \text{Re}[V(\mathbf{x}, t)] = -\hbar \dot{\phi} + \frac{\hbar^2}{2m} \left( \frac{\nabla^2 r}{r} - (\nabla \phi)^2 \right) - gr^2 \]

\[ \text{Im}[V(\mathbf{x}, t)] = \hbar \frac{\dot{r}}{r} + \frac{\hbar^2}{2m} \left( \frac{2\nabla \phi \cdot \nabla r}{r} + \nabla^2 \phi \right). \]

\[ \text{Im}[V(\mathbf{x}, t)] = 0, \text{ i.e.} \]

\[ \frac{\dot{r}}{r} + \frac{\hbar}{2m} \left( \frac{2\nabla \phi \cdot \nabla r}{r} + \nabla^2 \phi \right) = 0 \]
FF: Comments

- Less ambitious but more flexible than Invariant-based engineering.
- Not always works (Why?)
- Applied so far to expansions, transport, wave-packet splitting
II 6 Optimal Control Theory

Pontryagin’s maximum principle

\[ \dot{x} = f(x(t), u) \]

Dynamical system

u = scalar control; x = state variables

\[ J(u) = \int_{0}^{t_f} g(x(t), u) \, dt \]

Cost function (minimize)

The extremal \( x(t) \) and the adjoint vector \( p(t) \) (Lagrange multipliers) obey Hamilton’s equations for the control Hamiltonian (which becomes maximal and constant)

\[ H_c = p_0 g(x(t), u) + p^T \cdot f(x(t), u) \]
OCT+ “invariant-based method” for transport, expansions...

- It helps to choose among the possible parametric paths according to physical criteria.

- It avoids the arbitrariness of cost functions that include fidelity AND some other variable

\[ J(u) = \int_0^{t_f} g(x(t), u) \, dt + C (1 - \text{fidelity}) \]

Arbitrary weight
III-TRAP EXPANSIONS

Chen et al PRL 104, 063002 (2010)
Motivation

- distribute atoms on a lattice
- reach very low T
- reduce $\Delta v$ in spectroscopy & metrology

Bottleneck in a “quantum refrigerator cycle” (Rezek et al 2009)
Sympathetic cooling

• Choi, Onofrio and Chundaram (2011, 2012): Deep degeneracy of Fermi gases by engineering the trapping frequency of a coolant

• Advantages
  Maximal heat capacity of coolant due to:
  conservation of atoms
  preservation of phase-space density

• Limits
  Transient excitation and spreading of the cooling cloud (reduces overlap)
Spin-squeezed many-body states

- Juliá-Díaz et al. (2012): coherent-spin-squeezed many-body state in bosonic Josephson junctions

- Bose-Hubbard $H$ mapped to one effective harmonic oscillator in Fock space with parabolic $V$. It requires control of the scattering length.
In general the state is a superposition of "expanding modes"

\[ \Psi_n(t, x) = \sum_n c_n e^{i\alpha_n(t)} \langle x | n(t) \rangle \]

\[ \psi(t, x) = \sum_n c_n e^{i\alpha_n(t)} \langle x | n(t) \rangle \]

\[ \alpha_n(t) = -(n + 1/2) \omega_0 \int_0^t dt' / b^2 \]
“Inverse engineering”

- Leave $\omega(t)$ undetermined at first
- Impose boundary conditions on $b$ so that
  $$|n(0)\rangle = |u_n(0)\rangle$$
  $$|n(t_f)\rangle = |u_n(t_f)\rangle$$

Formally

$$[H(0), I(0)] = 0$$
$$[H(t_f), I(t_f)] = 0$$

(no final vibrational excitation)

- Interpolate $b(t)$
- Get $\omega(t)$ from Ermakov eq.
Boundary conditions

- $t=0$
  \[ b(0) = 1, \quad \dot{b}(0) = 0, \quad \ddot{b}(0) = 0 \]

- $t=t_f$
  \[ b(t_f) = \gamma = \left[ \omega_0 / \omega_f \right]^{1/2}, \quad \dot{b}(t_f) = 0, \quad \ddot{b}(t_f) = 0 \]
Example

Time Evolution:

\[ \omega_0 = 250 \times 2\pi \text{ Hz} \]
\[ \omega_f = 2.5 \times 2\pi \text{ Hz} \]
\[ t_f = 2 \text{ ms} \]
Example

Time Evolution:

\[ \omega_0 = 250 \times 2\pi \text{ Hz} \]
\[ \omega_f = 2.5 \times 2\pi \text{ Hz} \]
\[ t_f = 2 \text{ ms} \]
Interpolate between 0 and $t_f$ with an ansatz, e.g.

$$b(t) = \sum_{j=0}^{5} a_j t^j$$

Get $w(t)$ from Ermakov equation

$$\ddot{b} + \omega(t)^2 b = \frac{\omega_0^2}{b^3}$$

Energies and frequencies for different $t_f$ (polynomial b, ground state)

Expulsive for $t_f < 1/(2\omega_f) = 25$ ms
Compare with adiabatic trajectories

Adiabaticity condition

\[ \left| \sqrt{2} \dot{\omega} / (8 \omega^2) \right| \ll 1 \]

Linear ramp

\[ \omega(t) \rightarrow \omega_0 + (\omega_f - \omega_0) t / t_f \]

6 s to achieve a 1% relative error

Better strategy: solve

\[ \dot{\omega} / \omega^2 = c \]

\[ \omega(t) = \omega_0 / \left[ 1 - (\omega_f - \omega_0) t / (t_f \omega_f) \right] \]

45 ms for a 1% relative error
Nice experiment (G. Labeyrie et al. 2010): 87 Rb in magnetic trap

\[
H(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega_z^2(t)z^2 + mgz
\]

\[
\omega_{x,z} \approx \sqrt{\frac{\mu}{m}} \frac{B'(i_Q)}{\sqrt{B_0(i_Q, i_{B_0})}}
\]

\[
\omega_y = \sqrt{\frac{\mu}{m}} \sqrt{B''(i_Q)}.
\]

FIG. 1: Trapping geometry (figure in the horizontal plane). Ultracold $^{87}$Rb atoms are trapped in a Ioffe-Pritchard type magnetic trap created by current $i_Q$ running through the three QUIC coils 1, 2, and 3. An additional pair of coils (a and b) produces an homogeneous field along y, which allows an independent tuning of the trap minimum field $B_0$ via the current $i_{B_0}$.

- mgz: It needs generalized formalism (10 Boundary Cond. instead of 6)
- Number of atoms
N=10$^5$: $t_{\text{collision}}$=28 ms
- Initial temperature
T=1.63 mK -> Harmonic trap
Residual excitation attributed to:
- Approximate realization of $\omega(t)$
- Anharmonicities
Energy cost of shortcuts

\[ \Delta E \Delta t \geq \hbar / 2? \]

\[ \overline{E_n} \geq \frac{(2n + 1)\hbar}{4\omega_f t_f^2} \]
Third law (unattainability of T=0)

• W. Nernst [1906]

• Vanishing of cooling rate when pumping Q from a cold bath whose T -> 0 [e.g. Kosloff et al. 2009]
\[ t_f \geq \sqrt{\frac{(2n + 1) \hbar}{4 \omega_f E_n}} \]

\[ R \propto T_c^{3/2} \]
Condensates

\[ i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega(t)^2 x^2 + g|\psi|^2 \right] \psi \]

Use scaling methods instead of invariants

\[ \psi(\mathbf{r}, t) = b^{-d/2} e^{i(mr^2/2\hbar)(\hat{b}/\hat{b})} \phi(\mathbf{r}, t) \]

1 & 3D:
- gb constant (tuning g with Feschbach resonance)
- TF regime

2D:
No approximations or manipulation of g required

Ref: Muga et al. JPB 2010
SOME REFERENCES

• Chen et al. PRL 104 (2010): expansions
• Chen et al. PRL 105 (2010): pop. inversion, STIRAP
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• Torróntegui et al. Adv. At. Mol. Opt. Phys. (2013): review