GLOBAL GRAPH OF METRIC ENTROPY ON EXPANDING BLASCHKE PRODUCTS

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Abstract. We study the global picture of the metric entropy on the space of expanding Blaschke products. We first construct a smooth path in the space tending to a parabolic Blaschke product. We prove that the metric entropy on this path tends to 0 as the path tends to this parabolic Blaschke product. It turns out that the limiting parabolic Blaschke product on the unit circle is conjugate to the famous Boole map on the real line. Thus we can give a new explanation of Boole’s formula discovered more than one hundred and fifty years ago. We modify the first smooth path to get a second smooth path in the space of expanding Blaschke products. The second smooth path tends to a totally degenerate map. We see that the first and second smooth paths have completely different asymptotic behaviors near the boundary of the space of expanding Blaschke products. However, they represent the same smooth path in the space of all smooth conjugacy classes of expanding Blaschke products. We use this to give a complete description of the global graph of the metric entropy on the space of expanding Blaschke products. We prove that the global graph looks like a bell. It is the first result to show a global picture of the metric entropy on a space of hyperbolic dynamical systems. We apply our results to the measure-theoretic entropy of a quadratic polynomial with respect to its Gibbs measure on its Julia set. We prove that the measure-theoretic entropy on the main cardioid of the Mandelbrot set is a real analytic function and asymptotically zero near the boundary.

1. Introduction. The infimum of the metric entropy on the space of Anosov diffeomorphisms or area-preserving Anosov diffeomorphisms on a smooth Riemannian manifold has been studied in [6, 7]. In these papers, we constructed a smooth path starting from any point in the space such that it tends to the boundary of the space. We proved that the metric entropy on the path tends to zero as the path
tends to the boundary point. Thus the infimum of the metric entropy on the space of Anosov diffeomorphisms or area-preserving Anosov diffeomorphisms on a smooth Riemannian manifold is zero. It has been an interesting problem to see the asymptotical behavior of the metric entropy near the boundary of Anosov diffeomorphisms or area-preserving Anosov diffeomorphisms on a smooth Riemannian manifold. In this paper, we study this problem in the space of expanding Blaschke products.

The paper is organized as follows. In Section 2, we define the space of expanding Blaschke products of degree \( d \geq 2 \) and review several results about expanding Blaschke products. In Section 3, we construct our first smooth path in the space. The first path tends to a Blaschke product with a parabolic fixed point. Thus the limiting Blaschke product is not expanding. We call it a parabolic Blaschke product. In the same section we prove that the metric entropy on the first path tends to 0 as the path tends to the limiting parabolic Blaschke product (Theorem 3.1). It turns out that the limiting Blaschke product on the unit circle is smoothly conjugate to the famous Boole map on the real line. Therefore, in Section 4, we give a new explanation of Boole’s formula discovered more than one hundred and fifty years ago (Theorem B). In Section 5, we modify the first smooth path to obtain our second smooth path in the space. Every map in the second path preserves the Lebesgue measure. The second path tends to the identity, which we consider as totally degenerate in the view of Blaschke products of degree \( \geq 2 \). The metric entropy on the second path tends to 0 as the path tends to the totally degenerated map (Theorem 5.1). In Section 6, we note that even the first and second paths have completely different asymptotic behaviors near the boundary of the space of expanding Blaschke products but they represent the same smooth path in the space of smooth conjugacy classes of expanding Blaschke products. In the same section, we study the space of all smooth conjugacy classes of expanding Blaschke products and assign a metric on this space. Thus we have a metric space. When \( d = 2 \), this metric space is the unit disk, which is a manifold, and the boundary is the unit circle. We give a complete description of the global graph of the metric entropy on the unit disk (Theorem 6.1) and its asymptotical behavior near the unit circle. We prove that the global graph looks like a bell. When \( d > 2 \), the metric space is an orbifold, we describe the global graph of the metric entropy on this orbifold too. In Section 7, we apply Theorem 6.1 to the measure-theoretic entropy of a quadratic polynomial with respect to the Gibbs measure on its Julia set (the harmonic measure in the view of the basin of its attractive fixed point). We prove that the measure-theoretic entropy on the main cardioid of the Mandelbrot set is a real analytic function and asymptotically zero near the boundary (Corollary 2).

2. Expanding Blaschke products. Suppose \( \mathbb{C} \) is the complex plane. Let \( \Delta = \{ z \in \mathbb{C} \mid |z| < 1 \} \) be the unit disk and

\[
T = \partial \Delta = \{ z = e^{2\pi i\theta} \mid 0 \leq \theta < 1 \}
\]

be the unit circle. Consider the standard Borel \( \sigma \)-algebra \( \varpi \) on \( T \). The Lebesgue probability measure on \( T \) is the normalized arc-length measure

\[
m(A) = \int_A d\theta \quad \forall \ A \in \varpi.
\]

Consider a Blaschke product of degree \( d \geq 2 \),

\[
B(z) = e^{2\pi i\alpha} \prod_{n=1}^d \frac{z - a_n}{1 - \overline{a_n}z} \quad \text{where } |a_n| < 1 \text{ and } 0 \leq \alpha < 1.
\]
It preserves the unit disk $\Delta$, that is, $|B(z)| < 1$ for $z \in \Delta$, and the unit circle $T$, that is $|B(z)| = 1$ for $z \in T$. It is analytic on a neighborhood $U \supset \Delta$. It is said to be expanding (on $T$) if there are two constants $C > 0$ and $\lambda > 1$ such that

$$|(B^n)'(z)| \geq C\lambda^n \quad \forall z \in T \text{ and } \forall n \geq 1.$$ 

Let $\mathcal{B}(d)$ be the space of all expanding Blaschke products $B$ of degree $d$ in the form (1) with $B_0(z) = z^d$ as its basepoint. For simplicity, we use $B$ to denote $B(2)$.

From the Denjoy-Wolff Theorem in complex analysis, we have that

**Proposition 1.** The Blaschke product $B$ in (1) is expanding if and only if it has a fixed point $p \in \Delta$.

Suppose $\nu$ is a Borel measure on $T$. We say it is $B$-invariant if

$$\nu(B^{-1}(A)) = \nu(A) \quad \forall A \in \varpi.$$ 

If $m$ is $B$-invariant, we say $B$ preserves the Lebesgue measure.

**Proposition 2.** The map $B \in \mathcal{B}(d)$ preserves the Lebesgue measure if and only if $B(0) = 0$.

**Proof.** The proof uses harmonic theory. The map $B \in \mathcal{B}(d)$ preserves the Lebesgue measure if and only if

$$\int_T f(B(z))d\theta = \int_T f(z)d\theta \quad \text{for all continuous functions } f \text{ on } T.$$ 

Consider the harmonic extension $u$ of $f$ on $\Delta$. The mean value theorem says that

$$u(0) = \frac{1}{2\pi i} \int_T f(z)d\theta.$$

Since $B$ is analytic on $\Delta$, the composition $u \circ B$ is also a harmonic function on $\Delta$ and extends $f \circ B$ on $T$. So we have that

$$u(B(0)) = \frac{1}{2\pi i} \int_T f(B(z))d\theta.$$ 

Thus $u(0) = u(B(0))$ for all $f$ if and only if $B(0) = 0$. It completes the proof. \(\square\)

See [4] for invariant measures for inner functions. Furthermore, we have that

**Proposition 3.** Every $B \in \mathcal{B}(d)$ has a unique absolutely continuous ergodic $B$-invariant probability measure $\nu_B = (M^{-1})_*m$, where $M$ is a Möbius transformation mapping the fixed point of $B$ in $\Delta$ to $0$ and $(M^{-1})_*m$ means the push-forward measure, that is,

$$((M^{-1})_*m)(A) = m(M(A)) \quad \forall A \in \varpi.$$ 

The proof is straightforward and we leave it to the reader.

**Remark 1.** Suppose $p$ is the unique fixed point of $B$ in $\Delta$. The measure $\nu_B$ in Proposition 3 is the harmonic measure of $\Delta$ with pole at $p$, that is

$$\nu_B(A) = \int_A \frac{1 - |p|^2}{|z - p|^2}d\theta \quad \forall A \in \varpi.$$ 

For every $B \in \mathcal{B}(d)$, let $\nu = \nu_B$ in Proposition 3. Consider the measure-theoretic entropy $h_\nu(B|T)$ (refer to [10, 5]) and just denoted it as $h_\nu(B)$. We call the function

$$\mathcal{E}_d : \mathcal{B}(d) \rightarrow (0, \log d], \quad \mathcal{E}_d(B) = h_\nu(B), \quad B \in \mathcal{B}(d),$$

(2)
the metric entropy on \( \mathcal{B}(d) \). For simplicity, we use \( \mathcal{E} \) to denote \( \mathcal{E}_2 \). If we equip \( \mathcal{B}(d) \) with the maximum distance
\[
d(B, \tilde{B}) = \max_{z \in T} |B(z) - \tilde{B}(z)|,
\]
it is a smooth function (refer to [15]). The metric entropy \( \mathcal{E}_d \) takes the maximum value \( \log d \) at \( B_0 \) in \( \mathcal{B}(d) \). Note that \( \log d \) is the topological entropy of \( B/T \) for all \( B \in \mathcal{B}(d) \). It is important to know how the metric entropy varies on \( \mathcal{B}(d) \), in particular, near the boundary of \( \mathcal{B}(d) \).

3. The first smooth path. In this section, we construct our first smooth path in \( \mathcal{B} \) which tends to a parabolic Blaschke product and the metric entropy on this path is asymptotically zero.

Let \( a_\infty = 1/\sqrt{3} \). For any \( t \in [0, \infty) \), let \( a_t = a_\infty - 1/(t + \sqrt{3}) \). Define
\[
G_t(z) = \frac{z - a_t i}{1 + a_t i z}, \quad \frac{z + a_t i}{1 - a_t i z}
\]
Then \( G_t(z) \) has three fixed points:
\[
1, \quad 0 \leq p_t = \frac{b_t - \sqrt{(b_t)^2 - 4}}{2} < 1, \quad 1 < \tilde{p}_t = \frac{b_t + \sqrt{(b_t)^2 - 4}}{2} = \frac{1}{p_t} \leq \infty,
\]
where
\[
b_t = \frac{1}{a_t^2} - 1.
\]
For every \( t \), \( G_t \) has a unique critical point 0 in \( \mathbb{C} \). When \( t = 0 \), \( a_0 = 0 \), \( b_0 = \infty \), \( p_0 = 0 \), and \( \tilde{p}_0 = \infty \).

Our first smooth path is
\[
P_1 = \{ t \in [0, \infty) \rightarrow G_t(z) \in \mathcal{B} \} \tag{3}
\]
This path starts at the basepoint \( B_0 \) and has a limiting map as \( t \rightarrow \infty \),
\[
G_\infty(z) = \frac{z - a_\infty i}{1 + a_\infty i z}, \quad \frac{z + a_\infty i}{1 - a_\infty i z}.
\]
The map \( G_\infty(z) \) is again a Blaschke product of degree 2 preserving \( \Delta \). It has only one fixed point 1 in the extended complex plane. Since \( G_\infty'(1) = 1 \), it can not be expanding on \( T \). Therefore, it is on the boundary of \( \mathcal{B} \). We call it a parabolic Blaschke product.

According to Proposition 3, every \( G_t \), \( 0 \leq t < \infty \), has a unique absolutely continuous ergodic \( G_t \)-invariant probability measure \( \nu_t = (M_t^{-1})_* m \) where
\[
M_t(z) = \frac{z - p_t}{1 - p_t z} \tag{4}
\]
The density of \( \nu_t \) (with respect to \( m \)) is
\[
d\nu_t = \frac{1 - p_t^2}{1 + p_t^2 - p_t(z + \overline{z})} d\theta = \frac{1 - p_t^2}{1 + p_t^2 - 2p_t \cos(2\pi \theta)} d\theta.
\]
In other words, every measure \( \nu_t \) is given by the Poisson integral,
\[
\nu_t(A) = \int_A \frac{1 - p_t^2}{1 + p_t^2 - 2p_t \cos(2\pi \theta)} d\theta \quad \forall A \in \mathcal{W}.
\]
The first main result in this paper is that

**Theorem 3.1.** The metric entropy \( \mathcal{E}|P_1 \) tends to 0 as \( t \rightarrow \infty \).
Proof. Let $z = e^{2\pi i\theta}$ on $T$ with $0 \leq \theta \leq 1$ and consider

$$G_t(z) = e^{2\pi i\Theta_t(z)}.$$ 

Then

$$\Theta_t'(\theta) = \frac{G_t'(z)z}{G_t(z)} = \frac{2(1-a_t^2)(1+a_t^2)}{|z-a_t|^2(z+a_t)} = \frac{2(1-a_t^2)(1+a_t^2)}{(1+a_t^2)^2 + a_t^2(z-\pi)^2} =$$

$$\frac{2(1-a_t^2)}{1+a_t^2} \cdot \frac{1}{1 + \frac{a_t^2}{(1+a_t^2)}(z-\pi)^2} = \frac{2(1-a_t^2)}{1+a_t^2} \cdot \frac{1}{1 - \frac{2a_t^2\sin^2(2\pi\theta)}{(1+a_t^2)}}.$$ 

From Rohlin’s formula, we have that

$$\mathcal{E}(G_t) = h_{\nu_t}(G_t) = \int_0^1 \log \Theta_t'(\theta) d\nu_t =$$

$$= \int_0^1 \log \left( \frac{2(1-a_t^2)}{1+a_t^2} \cdot \frac{1}{1 - \frac{2a_t^2\sin^2(2\pi\theta)}{(1+a_t^2)}} \right) \cdot \frac{1 - p_t^2}{1 + p_t^2 - 2p_t\cos(2\pi\theta)} d\theta.$$ 

Since $\nu_t$ is a probability measure on $T$, we have

$$\int_0^1 \frac{1 - p_t^2}{1 + p_t^2 - 2p_t\cos(2\pi\theta)} d\theta = 1, \quad \forall \ t \in [0, \infty).$$

As $t \to \infty$, $p_t \to 1$, $a_t^2 \to 1/3$,

$$\frac{2(1-a_t^2)}{1+a_t^2} \cdot \frac{1}{1 - \frac{2a_t^2\sin^2(2\pi\theta)}{(1+a_t^2)^2}} \to \frac{1}{1 - \frac{3\sin^2(2\pi\theta)}{8}};$$

and

$$\frac{1}{1 + p_t^2 - 2p_t\cos(2\pi\theta)} \to \frac{1}{2(1 - \cos(2\pi\theta))}.$$ 

Thus for any $\epsilon > 0$, we have a $\delta > 0$ and a $t_1 > 0$ such that for any $0 \leq \theta \leq \delta$ and $1 - \delta \leq \theta \leq 1$ and $t > t_1$, we have

$$\left| \log \left( \frac{2(1-a_t^2)}{1+a_t^2} \cdot \frac{1}{1 - \frac{2a_t^2\sin^2(2\pi\theta)}{(1+a_t^2)^2}} \right) \right| < \frac{\epsilon}{3}.$$ 

This implies that for any $t > t_1$,

$$\left| \int_0^\delta \log \left( \frac{2(1-a_t^2)}{1+a_t^2} \cdot \frac{1}{1 - \frac{2a_t^2\sin^2(2\pi\theta)}{(1+a_t^2)^2}} \right) \cdot \frac{1 - p_t^2}{1 + p_t^2 - 2p_t\cos(2\pi\theta)} d\theta \right| < \frac{\epsilon}{3}$$

and

$$\left| \int_\delta^1 \log \left( \frac{2(1-a_t^2)}{1+a_t^2} \cdot \frac{1}{1 - \frac{2a_t^2\sin^2(2\pi\theta)}{(1+a_t^2)^2}} \right) \cdot \frac{1 - p_t^2}{1 + p_t^2 - 2p_t\cos(2\pi\theta)} d\theta \right| < \frac{\epsilon}{3}.$$ 

Let

$$E = [0, 1] \setminus \left( [0, \delta] \cup [1 - \delta, 1] \right)$$

and

$$M = \sup_{t_1 \leq t < \infty} \sup_{\theta \in E} \left\{ \left| \frac{2(1-a_t^2)}{1+a_t^2} \cdot \frac{1}{1 - \frac{2a_t^2\sin^2(2\pi\theta)}{(1+a_t^2)^2}} \cdot \frac{1 - p_t^2}{1 + p_t^2 - 2p_t\cos(2\pi\theta)} \right| \right\} < \infty.$$ 

Since $p_t \to 1$ as $t \to \infty$, we have a $t_2 \geq t_1$ such that for any $t > t_2$,

$$|1 - p_t^2| < \frac{\epsilon}{3M}.$$
This implies that
\[ \left| \int_{E} \log \left( \frac{2(1 - a^2)}{1 + a^2} - \frac{1}{1 + 2a^2 \sin^2(2\pi \theta)} \right) \cdot \frac{1 - p^2}{1 + p^2 - 2p \cos(2\pi \theta)} \, d\theta \right| < \frac{\epsilon}{3}. \]

Finally, we get
\[ E(G_t) = \int_{1}^{t} \log \left( \frac{2(1 - a^2)}{1 + a^2} - \frac{1}{1 + 2a^2 \sin^2(2\pi \theta)} \right) \cdot \frac{1 - p^2}{1 + p^2 - 2p \cos(2\pi \theta)} \, d\theta < \epsilon \]
for any \( t > t_2 \). This completes the proof of the theorem.

**Corollary 1.** The infimum of the metric entropy \( E \) on \( B \) is zero.

**Remark 2.** One visible proof of Theorem 3.1 without a detailed calculation is that \( \nu_t \) tends to the Dirac measure \( \delta_1 \) on \( T \), which is the probability measure with the density 1 at 1 \( \in T \) and the density 0 at all other points in \( T \), and \( \Theta_t'(0) \) tends to 1 as \( t \to \infty \).

4. **Application to the Boole map.** More than one hundred and fifty years ago, G. Boole [3] discovered the surprising formula
\[ \int_{-\infty}^{\infty} \phi(x) \, dx = \int_{-\infty}^{\infty} \phi \left( x - \frac{1}{x} \right) \, dx \]  \hspace{1cm} (5)
for any integrable function \( \phi \) on the real line. In modern words, this says that the dynamical system generated by the Boole map
\[ F(x) = x - \frac{1}{x} \]
preserves the Lebesgue measure \( m_0(A) = \int_A \, dx \) on the real line \( \mathbb{R} \) with the standard Borel \( \sigma \)-algebra \( \mathcal{B} \). More precisely,
\[ m_0(F^{-1}(A)) = m_0(A) \quad \forall \, A \in \mathcal{B}. \]  \hspace{1cm} (6)
The Boole map plays an important role in the infinite ergodic theory (refer to [1]). In this section, as a byproduct of the previous section, we give a new proof of Boole’s formula (5) in the form (6) as follows.

**Theorem B.** The Boole map \( F \) is on the boundary of \( B \) and the Lebesgue measure \( m_0 \) on the real line \( \mathbb{R} \) is a \( \sigma \)-finite \( F \)-invariant measure.

**Proof.** Consider the Möbius transformation
\[ z = N(x) = \frac{x - i}{x + i} : \mathbb{R} \to T. \]
Then we have that
\[ N : -\infty, -1, 0, 1, \infty \mapsto 1, i, -1, -i, 1. \]

And
\[ x = N^{-1}(z) = -i \frac{z + 1}{z - 1}. \]
Consider the conjugate map of the Boole map \( F \),
\[ G_\infty(z) = N \circ F \circ N^{-1}(z) = \frac{z - a_\infty i}{1 + a_\infty i z} \cdot \frac{z + a_\infty i}{1 - a_\infty i z}. \]
It is the parabolic Blaschke product as the limit of the path \( P_1 \). So the Boole map can be thought of a map on the boundary of \( B \).
Recall that \( m(A) = \int_A d\theta \) is the Lebesgue measure on \( T \). Consider the Poisson integral
\[
\mu(A) = \int_A \frac{4\pi}{2(1 - \cos(2\pi \theta))} d\theta \quad \forall \ A \in \varpi.
\]
Then \( \mu \) is a \( \sigma \)-finite absolutely continuous measure on \( T \) with the density
\[
d\mu = \frac{4\pi}{2(1 - \cos(2\pi \theta))} d\theta.
\]

Since \( x = N^{-1}(z) \), we have that
\[
dx = |(N^{-1})'(z)dz| = \frac{4\pi}{|z - 1|^2} d\theta = \frac{4\pi}{2 - (z + \bar{z})} d\theta = \frac{4\pi}{2(1 - \cos(2\pi \theta))} d\theta = d\mu.
\]
This says that the Lebesgue measure \( m_0 \) on the real line \( \mathbb{R} \) is the push-forward measure \((N^{-1})_*\mu\).

For each \( G_t, \nu_t \) with the density
\[
d\nu_t = \frac{1 - p^2_t}{1 + p^2_t - 2p_t \cos(2\pi \theta)} d\theta.
\]
is the unique absolutely continuous \( G_t \)-invariant probability measure. Consider the Poisson integral
\[
\mu_t(A) = \int_A \frac{4\pi}{1 + p^2_t - 2p_t \cos(2\pi \theta)} d\theta \quad \forall \ A \in \varpi.
\]
Then
\[
\mu_t = \frac{4\pi}{1 - p^2_t} \nu_t
\]
is a Borel measure on \( T \) (but not a probability measure) and a \( G_t \)-invariant measure too.

Since \( G_t \) tends to \( G_\infty \) uniformly on \( T \), the Poisson integral \( \mu_t(A) \) tends to the Poisson integral \( \mu(A) \) as \( t \to \infty \) for all \( A \in \varpi \). This implies that \( \mu_t \to \mu \) in the weak \( * \)-topology as \( t \to \infty \). We get \( \mu \) is a \( G_\infty \)-invariant measure. Thus \( m_0 \) is an \( F \)-invariant measure, which means that (6) as well as (5) holds.

**Remark 3.** R. Adler and B. Weiss [2] also proved that the Lebesgue measure \( m_0 \) is ergodic with respect to the Boole map \( F \). This can be seen from \( G_\infty \) as well. First \( G_\infty \) is an almost expanding circle endomorphism in the following sense that it is analytic, it has only one parabolic fixed point 1, and all other periodic points on \( T \) are expanding. Then \( \mu \) is the unique absolutely continuous \( \sigma \)-finite ergodic \( G_\infty \)-invariant measure on \( T \) (refer to [11]). Since \( F = N^{-1} \circ G_\infty \circ N \) and \( m_0 = (N^{-1})_*\mu \). Therefore, \( m_0 \) is the unique absolutely continuous \( \sigma \)-finite ergodic \( F \)-invariant measure on the real line.

5. **The second smooth path.** We will modify \( P_1 \) in (3) to get our second smooth path \( P_2 \) such that its limit is the identity, which is totally degenerate in the view of Blaschke products of degree \( \geq 2 \).

Let \( p_t \) be the unique fixed point of \( G_t \) in \( \Delta \). Note that \( G_t \) fixes 1 too. Let \( M_t \) be the Möbius transformation in (4) mapping \( p_t \) to 0 and fixing 1. Consider the conjugating Blaschke product
\[
H_t(z) = M_t \circ G_t \circ M_t^{-1} \in \mathcal{B}.
\]
Since all $M_t(z)$, $G_t(z)$, and $p_t$ depend on $t \in [0, \infty)$ smoothly, our second smooth path is

$$P_2 = \{ t \in [0, \infty) \to H_t(z) \in B \}$$

(7)

where

$$H_t(z) = \frac{z - r_t}{1 - r_t z} \quad \text{where} \quad r_t = \frac{-2p_t}{1 + p_t^2}.$$

Since $H_t(0) = 0$ for all $t \geq 0$, Proposition 2 says that every map in $P_2$ preserves the Lebesgue measure. The second main result in this paper is that

**Theorem 5.1.** For any $z \neq -1 \in T$, $H_t(z) \to z$ as $t \to \infty$. Moreover, the metric entropy $\mathcal{E}|P_2$ tends to 0 as $t$ goes to $\infty$.

**Proof.** For any $z \neq -1$, $H_t(z) \to z$ since $r_t \to -1$ as $t \to \infty$.

Since $G_t(z)$ and $H_t(z)$ are conjugated by $M_t$,

$$h_m(H_t) = h_{\nu_t}(G_t),$$

we have that $h_m(H_t) \to 0$ as $t \to \infty$. This completes the proof. \qed

**Remark 4.** Since $H_t(-1) = 1$ for all $t$, $H_t(-1) \not\to -1$ as $t \to \infty$.

6. The global graph of the metric entropy. For any $B$ and $\tilde{B}$ in $\mathcal{B}(d)$, $B|T$ and $\tilde{B}|T$ are topologically conjugate, that is, there is an orientation-preserving homeomorphism $h : T \to T$ such that

$$h \circ B = \tilde{B} \circ h$$

(8)

(refer to [16, 9]). Furthermore, $h$ is actually a quasisymmetric circle homeomorphism (refer to [9]).

We say $B$ and $\tilde{B}$ are smoothly conjugate if $h$ in (8) is a diffeomorphism of $T$ and denote $B \sim_s B$. Note that in this case, $h$ in (8) is a diffeomorphism (or a symmetric homeomorphism) of $T$ if and only if it is a Möbius transformation (refer to [17, 8]). For every $B \in \mathcal{B}(d)$, we use $[B]$ to denote its smooth conjugacy class, that is,

$$[B] = \{ \tilde{B} \in \mathcal{B}(d) \mid \tilde{B} \sim_s B \}.$$

We use $\mathcal{S} \mathcal{B}(d)$ to denote the space of all smooth conjugacy classes, that is,

$$\mathcal{S} \mathcal{B}(d) = \{ \tau = [B] \mid B \in \mathcal{B}(d) \}.$$

Since the metric entropy is a smooth conjugacy invariant, the smooth function defined in (2) induces a smooth function which we still denote as

$$\mathcal{E}_d : \mathcal{S} \mathcal{B}(d) \to (0, \log d], \quad \mathcal{E}_d(\tau) = h_{\nu_t}(B), \quad B \in \tau,$$

and call it the metric entropy on $\mathcal{S} \mathcal{B}(d)$. For simplicity, we use $\mathcal{E}$ to denote $\mathcal{E}_2$.

**Remark 5.** In a general study of dynamical systems, a smooth conjugacy class is usually smaller than the Lipschitz conjugacy class containing it. However, every smooth conjugacy class in $\mathcal{B}(d)$ equals to the Lipschitz conjugacy class containing it in $\mathcal{B}(d)$. Furthermore, we also proved that in [8], every smooth conjugacy class in $\mathcal{B}(d)$ equals to the symmetric conjugacy class containing it in $\mathcal{B}(d)$. 
are conjugated by $h$. Two Blaschke products in the form (9) are smoothly conjugated if and only if they are conjugated by $h(z) = az$ for some $\alpha \in \mathcal{G}$. In addition, $(c_1, \cdots, c_{d-1})$ and $(c_{\beta(1)}, \cdots, c_{\beta(d-1)})$ represent the same map in the form (9) for every $\beta \in \mathcal{P}$. Thus elements in $SB(d)$ are mapped bijectively to a point in the space

$$N^{d-1} = \Delta^{d-1}/(\mathcal{G} \cup \mathcal{P}).$$

The distance

$$d_{\Delta^{d-1}}(c, c') = \sum_{k=1}^{d-1} |c_k - c_k'|$$

on $\Delta^{d-1}$ induces a distance

$$d_{N^{d-1}}([c], [c']) = \min_{\alpha \in \mathcal{G}, \beta \in \mathcal{P}} \sum_{k=1}^{d-1} |c_k - \alpha c'_{\beta(k)}|.$$  

on $N^{d-1}$. It further induces a distance on $SB(d)$, denoted as $d_{SB(d)}(\cdot, \cdot)$. In this way, we can view $(SB(d), d_{SB(d)}(\cdot, \cdot))$ as a metric space. The basepoint is $\tau_0 = [B_0]$. When $d = 2$, $SB = \Delta$ with 0 as the basis point, which is a manifold, and the boundary $\partial(SB)$ is the unit circle $T$. For $d > 2$, $SB(d)$ is an orbifold with $[0] = [(0, \cdots, 0)]$ as the basis point.

The two paths $P_1$ and $P_2$ in (3) and (7), which we constructed in Section 3 and Section 5, have completely different asymptotic behaviors near the boundary of $B$, that is, $P_1$ tends to a parabolic Blaschke product and $P_2$ tends to a totally degenerate map. However, they represent the same path in $SB$. This path is our third smooth path

$$P_3 = \{ t \in [0, \infty) \to \tau_t = [G_t] = [H_t] \in SB \}.$$  

The third main result in this paper is that

**Theorem 6.1.** The metric entropy $\mathcal{E} : SB = \Delta \to [0, \log 2]$ is a real analytic function with level curves $T_r = \{ c \in \mathbb{C} \mid |c| = r \}$ for $0 \leq r < 1$. It is a strictly decreasing function (with respect to the level curves) and takes the maximum value $\log 2$ at its unique critical point $\tau_0$, that is, $\mathcal{E}'(0) = 0$ and $\mathcal{E}(0) = \log 2$ is the global maximum value of $\mathcal{E}$ on $\ Delta$. Moreover,  

$$\mathcal{E}(c) \to 0 \text{ as } c \to T = \partial(SB)$$

and along every radius $c = re^{2\pi i \theta}$, $0 \leq r < 1$,

$$\mathcal{E}'(c) \to -\infty \text{ as } c \to T = \partial(SB).$$

The global graph of $\mathcal{E}$ looks like a bell as it is illustrated in Figure 1.
Remark 6. Theorem 6.1 is the first result to have a global picture about the metric entropy on a space of hyperbolic dynamical systems, in particular, the asymptotic behavior near the boundary of the space. An answer to the space of smooth conjugacy classes of Anosov diffeomorphisms on a general Riemannian manifold is still a challenger problem.

Proof of Theorem 6.1. Given any \( \tau \in S_B \), there is a unique representation \( B_\tau \in \tau \) preserving the Lebesgue measure \( m \) and in the form

\[
B_\tau(z) = z \frac{z - c_\tau}{1 - c_\tau z}, \quad c_\tau \in \Delta.
\]

This says that \( S_B \) is the unit disk \( \Delta \), that is, \( \tau = c_\tau \), and \( d_{SB}(\tau, \tau') = |c_\tau - c_{\tau'}| \).

The boundary \( \partial(S_B) \) is the unit circle \( T \).

Without cause of the confusion, we suppress the subscript \( \tau \). Write \( z = e^{2\pi i \theta} \), \( 0 \leq \theta < 1 \), and the representation in (10) as \( B(z) = e^{2\pi i \Theta(\theta)} \). Then

\[
\Theta'(\theta) = B'(z)z = \frac{z}{z - c} + \frac{\overline{z}}{\overline{z} - \overline{c}} = 2\Re \left( \frac{z}{z - c} \right).
\]

Suppose \( c = re^{2\pi i \theta_0} \). Then we have that

\[
\Re \left( \frac{z}{z - c} \right) = \frac{1 - r \cos(2\pi \theta - \theta_0)}{1 - 2r \cos(2\pi \theta - \theta_0) + r^2}.
\]

Thus we have that

\[
\Theta'(\theta) = 2 \frac{1 - r \cos(2\pi \theta - \theta_0)}{1 - 2r \cos(2\pi \theta - \theta_0) + r^2}.
\]

From Rohlin’s formula, we have that

\[
\mathcal{E}(c) = \int_0^1 \log \Theta'(\theta) d\theta = \int_0^1 \log \left( \frac{2 - 1 - r \cos(2\pi \theta)}{1 - 2r \cos(2\pi \theta) + r^2} \right) d\theta
\]

\[
= \int_0^1 \log \left( \frac{2 - 1 - r \cos(2\pi \theta)}{1 - 2r \cos(2\pi \theta) + r^2} \right) d\theta.
\]

We see that \( \mathcal{E}(c) \) takes the same value on the circle of radius \( 0 \leq r < 1 \) centered 0. So, we can write \( \mathcal{E}(c) = \mathcal{E}(r) \). The function \( \mathcal{E}(r) \) is a real analytic function of
0 \leq r < 1 \text{ due to the formula for } \Theta'(\theta) \text{ above providing an appropriate estimate from above, uniform with respect to } \theta.

From the last formula for } \mathcal{E}(r), \text{ we have

\[ \mathcal{E}'(r) = -\int_0^1 \frac{\cos(2\pi \theta)}{1 - r \cos(2\pi \theta)} \, d\theta - 2 \int_0^1 \frac{r - \cos(2\pi \theta)}{1 - 2r \cos(2\pi \theta) + r^2} \, d\theta \]

\[ = -\frac{1}{2\pi} \left( \int_0^{2\pi} \frac{\cos \theta}{1 - r \cos \theta} \, d\theta + 2 \int_0^{2\pi} \frac{r - \cos \theta}{1 - 2r \cos \theta + r^2} \, d\theta \right) = -\frac{1}{2\pi} \left( I(r) + 2II(r) \right). \]

The first integral

\[ I(r) = \int_0^{2\pi} \frac{\cos \theta}{1 - r \cos \theta} \, d\theta = 2r \int_0^{\pi} \left( \frac{\cos^2 \theta}{1 - r^2 \cos^2 \theta} + \frac{\sin^2 \theta}{1 - r^2 \sin^2 \theta} \right) \, d\theta > 0 \]

for all 0 < r < 1 and I(0) = 0. The second integral

\[ II(r) = \int_0^{2\pi} \frac{r - \cos \theta}{1 - 2r \cos \theta + r^2} \, d\theta \]

\[ = 2r \int_0^{\pi} \left( \frac{(1 + r^2) - 2 \cos \theta}{(1 + r^2)^2 - 4r^2 \cos^2 \theta} + \frac{(1 + r^2) - 2 \sin \theta}{(1 + r^2)^2 - 4r^2 \sin^2 \theta} \right) \, d\theta \]

\[ = 2r \int_0^{\pi} \left( \frac{r^2 - \cos(2\theta)}{(1 + r^4) - 2r^2 \cos(2\theta)} + \frac{r^2 + \cos(2\theta)}{(1 + r^4) + 2r^2 \cos(2\theta)} \right) \, d\theta. \]

Using the identity

\[ \int_0^{\pi} \left( \frac{a - \cos(2\theta)}{(1 + a^2) - 2a \cos(2\theta)} + \frac{a + \cos(2\theta)}{(1 + a^2) + 2a \cos(2\theta)} \right) \, d\theta = a \int_0^{\pi} \left( \frac{a^2 - \cos(2\theta)}{(1 + (a^2)^2) - 2a^2 \cos(2\theta)} + \frac{a^2 + \cos(2\theta)}{(1 + (a^2)^2) + 2a^2 \cos(2\theta)} \right) \, d\theta \]

where a is any constant, we have that II(r) = 0 for any 0 \leq r < 1. This implies that

\[ \mathcal{E}'(r) = -I(r) < 0 \text{ for all } 0 < r < 1 \text{ and } \mathcal{E}'(0) = 0. \]

Thus } \mathcal{E}(r) \text{ is a strictly decreasing function on } [0, 1). \text{ Moreover, } \mathcal{E} \text{ has a unique critical point at } c = 0 \text{ and } \mathcal{E}(0) = \log 2 \text{ is the global maximum value of } \mathcal{E} \text{ on } \Delta.

Consider maps in our second path } P_2 \text{ in (7) and following Theorem 5.1, we have that } \mathcal{E}(-r_t) \to 0 \text{ as } t \to \infty. \text{ This implies that } \mathcal{E}(c) \to 0 \text{ as } c \to T. \text{ We completed the proof.} \]

When } d > 2, \text{ } S\mathcal{B}(d) \text{ is an orbifold. The boundary } \partial(S\mathcal{B}(d)) \text{ has two types of boundary points. One type is those points } \{(b_1, \ldots, b_{k-1}, b_{k}, \ldots, b_{d-1})\} \text{ such that for some } 2 \leq k \leq d-1, b_i \in \Delta \text{ for } 1 \leq i \leq k - 1 \text{ and } |b_j| = 1 \text{ for } k \leq j \leq d - 1. \text{ We call them partial boundary points. When } \tau = [e] \in S\mathcal{B}(d) \text{ tends to a partial boundary point } [b] = [(b_1, \ldots, b_{k-1}, b_{k}, \ldots, b_{d-1})], \text{ we have that for } z, \text{ except for finite number of points, in } T,

\[ B_{c}(z) \to B_{b}(z) = a \frac{z - b_1}{1 - b_1 z} \ldots \frac{z - b_{k-1}}{1 - b_{k-1} z}, \ |a| = 1. \]
The map $B_b(z)$ is an expanding Blaschke product of degree $k$. Thus $[b] \in B(k)$. Moreover,
\begin{equation}
\mathcal{E}_d([c]) \to \mathcal{E}_k([b]) \in (0, \log k) \text{ as } c \to b.
\end{equation}

The other type is those points $[(b_1, \cdots, b_i, \cdots, b_{d-1})]$ such that $|b_i| = 1$ for all $1 \leq i \leq d - 1$. We call them total boundary points. When $[c] \in SB(d)$ tends to a total boundary point $[b] = [(b_1, \cdots, b_{d-1})]$, we have that for $z$, except for finite number of points, in $T$,
\begin{equation}
B_c(z) \to B_b(z) = a\cdot z \quad \text{with } |a| = 1.
\end{equation}

The map $B_b(z)$ is a totally degenerate map and we have that
\begin{equation}
\mathcal{E}_d([c]) \to 0 \text{ as } c \to b.
\end{equation}

Similar to Theorem 6.1, we have

**Theorem 6.2.** For $d > 2$, the metric entropy $\mathcal{E}_d : SB(d) = N^{d-1} \to (0, \log d)$ has a positive limit if $[c]$ tends to a partial boundary point and the zero limit if $[c]$ tends to a total boundary point. The global graph of $\mathcal{E}_d$ is illustrated as in Figure 2.

![Figure 2. A graph of the metric entropy $\mathcal{E}_d$ on $SB(d)$ for $d > 2$.](image)

### 7. Application to the Mandelbrot set.

Consider the family of quadratic polynomials $q_c(z) = z^2 + c$, $c \in \mathbb{C}$. The Mandelbrot set is
\begin{equation}
\mathcal{M} = \{c \in \mathbb{C} \mid q^n_c(0) \not\to \infty \text{ as } n \to \infty\},
\end{equation}
which is a compact and connected subset of $\mathbb{C}$.

A point $p$ is called a periodic point of $q_c$ of period $n \geq 1$ if $q^n_c(p) \neq p$ but $q^n_c(p) = p$. The number $\lambda = (q^n_c)'(p)$ is called the multiplier of $q_c$ at a period point $p$ of period $n$. In particular, when $n = 1$, we call $p$ a fixed point of $q_c$. We can classify all periodic points into the following categories:
- attractive if $|\lambda| < 1$;
- repelling if $|\lambda| > 1$;
- indifferent if $|\lambda| = 1$.

The main cardioid $\mathcal{M}_0$ is, by definition,
\begin{equation}
\mathcal{M}_0 = \{c \in \mathcal{M} \mid q^n_c(0) \to \text{ the attractive fixed point } p_c \text{ of } q_c \text{ as } n \to \infty\}
\end{equation}

We know that $\mathcal{M}_0$ is a simply connected domain containing 0. Thus we have a Riemann map $\varphi : \mathcal{M}_0 \to \Delta$ with $\varphi(0) = 0$. 

For $c \in \mathcal{M}_0$, let $p_c = (1 - \sqrt{1 - 4c})/2$ be the attractive fixed point of $q_c$ in $\mathbb{C}$ and let

$$D_c = \{ z \in \mathbb{C} \mid q_c^n(z) \to p_c \text{ as } n \to \infty \}$$

be the basin of $p_c$. Then $D_c$ is a simply connected domain containing 0 and the boundary $J_c = \partial D_c$ is called the Julia set which is a Jordan curve (actually, a quasicircle). Let $\phi_c : D_c \to \Delta$ be a Riemann map with $\phi_c(p_c) = 0$. It can be extended to a homeomorphism from $D_c \to \overline{\mathbb{D}}$, which we still denote as $\phi_c$. By using the Schwarz reflection principle, we know that $\phi_c \circ q_c \circ \phi_c^{-1}$ is an expanding Blaschke product of degree 2 fixing 0. By choose an appropriate $\phi_c$, we assume in term of (10)

$$B_{-\lambda(c)}(z) = \phi_c \circ q_c \circ \phi_c^{-1}(z) = z \frac{z + \lambda(c)}{1 + \lambda(c)z}$$

where

$$\lambda(c) = B'_{-\lambda(c)}(0) = q'_c(p_c) = 1 - \sqrt{1 - 4c}$$

is the multiplier of $B_{-\lambda(c)}$ at its fixed point 0 (the multiplier of $q_c$ at its fixed point $p_c$) and a holomorphic function of $c \in \mathcal{M}_0$.

Let $\mu_c$ be the the harmonic (probability) measure of the domain $D_c$ with pole at $p_c$. The Lebesgue measure $m$ is the harmonic measure of $\Delta$ with pole at 0. We have that $\mu_c = (\phi_c^{-1})_*m$ is the push-forward measure of $m$ by $\phi_c^{-1}$. The measure $\mu_c$ supported on $J_c$ (m supported on $T$ resp.) satisfies the Gibbs property for the dynamical system $q_c$ ($B_{-\lambda(c)}$ resp.) and the potential $-\log |B'_{-\lambda(c)}| \circ \phi_c$ ($-\log |B'_{-\lambda(c)}| \text{ resp.}$) (refer to [9, 12, 13, 14]). We call $\mu_c$ (m resp.) the Gibbs measure for $q_c$ ($B_{-\lambda(c)}$ resp.). Consider the measure-theoretic entropy

$$h_{\mu_c}(q_c|J_c) = h_m(B_{-\lambda(c)}|T).$$

It defines a function

$$E_q : \mathcal{M}_0 \to (0, \log 2], \quad E_q(c) = h_{\mu_c}(q_c|J_c).$$

One application of Theorem 6.1 is that

**Corollary 2.** The function $E_q : \mathcal{M}_0 \to (0, \log 2]$ is a real analytic function with level curves $M_r = \{ c \in \mathcal{M}_0 \mid |\lambda(c)| = r \}$ for $0 \leq r < 1$. It is a strictly decreasing function (with respect to the level curves) and takes the maximum value $\log 2$ at its unique critical point 0, that is, $E'_q(0) = 0$ and $E_q(0) = \log 2$ is the global maximum value of $E_q$ on $\mathcal{M}_0$. Moreover,

$$E_q(c) \to 0 \quad \text{as} \quad c \to \partial \mathcal{M}_0$$

and along gradients (curves perpendicular to the level curves),

$$E'_q(c) \to -\infty \quad \text{as} \quad c \to \partial \mathcal{M}_0.$$

The global graph of $E_q$ looks like a distorted bell (refer to Figure 1).

**Proof.** Since the derivative $|\lambda(c)|' = 2/\sqrt{1 - 4c} > 0$ on $(-3/4, 1/4) = \mathcal{M}_0 \cap \mathbb{R}$ and since $E_q(c) = E(-\lambda(c)) = E(|\lambda(c)|)$, the corollary is a direct consequence of Theorem 6.1.

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