Quantum mechanics with non-unitary symmetries

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Abstract
This article shows that one can consistently incorporate nonunitary representations of at least one group into the “ordinary” nonrelativistic quantum mechanics. This group turns out to be Lorentz group thus giving us an alternative approach to QFT for combining the quantum mechanics and special theory of relativity which keeps the concept of wave function (belonging to some representation of Lorentz group) through the whole theory. Scalar product has been redefined to take into the account the nonunitarity of representations of Lorentz group. Understanding parity symmetry turns out to be the key ingredient throughout the process. Instead of trying to guess an equation or a set of equations for some wave functions or fields (or equivalently trying to guess a Lagrangian for the same), one derives them based only on the superposition principle and properties of wave functions under Lorentz transformations and parity. The resulting model has striking similarities with the standard quantum field theory and yet has no negative energy states, no zitterbewegung effects, symmetric energy momentum tensor and angular momentum density tensor for all representations of Lorentz group (unifying the theoretical description of all particles), as well as clear physical interpretation. It also offers a possible interpretation why particles and antiparticles have opposite quantum numbers.

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In the last 70 years quantum field theory has been a great success in describing the microscopic world at high energies. Over the years general opinion has been that the field theory is what it is because there’s no other way to bind quantum mechanics with relativity. It has been generally believed that relativistic generalizations of single particle equations don’t seem to work and therefore we have to abandon the single particle interpretations and perform the second quantization to make the theory work. Thus the quantum field theory (QFT) was born. Historically, one would postulate the relativistically invariant Lagrangian, use it to derive and solve Euler-Lagrange equations of motion (or the other way around, postulate the equations of motion and derive the Lagrangian which reproduces them; both approaches are equivalent); since they wouldn’t work on single-particle level, one would perform the second quantization by introducing creation/destruction operators and reinterpreting the objects as acting in a Fock space of states, abandoning the idea of a wave function in the process.

The aim of this article is to there is another way. Let’s take a look at the problem from the different angle; we already have non-relativistic quantum mechanics and special theory of relativity that both seem to work well separately. Is there anything else we can do to make quantum mechanics relativistically invariant? The model developed in this article shows one can modify the scalar product instead. The result is relativistically invariant “ordinary” quantum mechanics. Understanding parity turns out to be essential for developing this model. The basic assumption of the model differs a little from the basic assumption of the QFT; never the less, the resulting theory, although (naturally) different from the QFT, has an amazing amount of similarities with it and yet none of the problems/peculiarities of the theory that were the reason for introducing the second quantization in the first place. The name used for it through the paper is relativistic wave-function model (RWFM).

The key concept of the article is the introduction of nonunitary symmetries in “standard” quantum mechanics. Since it can sometimes be hard to see the big picture from all the details, here is a short overview of a few steps that are being performed through the article:

1. start with the ordinary nonrelativistic quantum mechanics; in another words, states $|\psi\rangle$ are normalized $\langle\psi|\psi\rangle = 1$ and all symmetries are unitary $U^\dagger = U^{-1}$. To interpret the square of the wave function $|\psi(\vec{x}, t)|^2 = \rho$ as the probability, equation $\langle \psi(t)|\psi(t) \rangle \equiv 1$ should hold at all times. To
ensure that, $\rho$ must be the zeroth component of a conserved current

$$\partial_{\mu}j^{\mu} = \frac{\partial\rho(\vec{x}, t)}{\partial t} + \nabla \cdot \vec{j}(\vec{x}, t) = 0 \quad (1)$$

2. this is the key step: now introduce nonunitary symmetry; if we want to preserve as much of the nonrelativistic interpretation as possible, this symmetry should at least leave continuity equation (1) invariant. This turns out to be the $SO(1, 3)$ group. At this point we’ll assume it’s just a global, position–independent symmetry. Find finite-dimensional (and nonunitary) representations of the group. State vector $|\psi\rangle$ or wave function $\psi(\vec{x}, t) \equiv \langle \vec{x}|\psi\rangle$ has to belong to some representation of the group. It transforms as

$$|\psi\rangle \rightarrow |\psi'\rangle = S(\omega) |\psi\rangle = e^{-\frac{i}{2}\omega_{\mu\nu}J_{\mu\nu}^{\text{spin}}} |\psi\rangle \quad (2)$$

where matrices $J_{\mu\nu}^{\text{spin}}$ are constant matrices, generators of $SO(1, 3)$.

3. since the symmetry is nonunitary, scalar product has to be modified to compensate by introducing an operator $\mathcal{P}$ that will turn out to correspond to parity

$$\langle \psi|\phi\rangle \rightarrow \langle \psi|\mathcal{P}|\phi\rangle = \int \psi^\dagger(\vec{x}, t)\mathcal{P}\phi(\vec{x}, t) \, d^3x \quad (3)$$

4. if the space $\vec{x}$ and time $t$ didn’t transform under the $SO(1, 3)$ this would be the end of the story. Since they transform as four-vector $x^\mu = (t, \vec{x})$, we have to compensate for the change of $x^\mu$ by modifying the transformation law to

$$\psi(\vec{x}, t) \rightarrow \psi'(\vec{x}', t') = S'(\omega)\psi(\vec{x}, t) = e^{-\frac{i}{2}\omega_{\mu\nu}J_{\mu\nu}^{\text{spin}}} \psi(\vec{x}, t) \quad (4)$$

with

$$J_{\mu\nu}^{\text{spin}} = J_{\mu\nu}^{\text{spin}} + 1 \left( x^\mu \partial_{x^\nu} \mathbf{i} - x^\nu \partial_{x^\mu} \mathbf{i} \right) \quad (5)$$

where $1$ is the unit matrix in the space of matrices $J_{\mu\nu}^{\text{spin}}$. If we considered $x$ that belongs to some other representation of $SO(1, 3)$ group, differential term in equation (5) would be different; “spin” part doesn’t depend on the representation of the object $x$ and so remains the same.

5. Since we integrate over all space, we do recover translational invariance in space

$$\int \psi^\dagger(\vec{x}, t)\mathcal{P}\phi(\vec{x}, t) \, d^3x \neq \int \psi^\dagger(\vec{x} + \vec{b}, t)\mathcal{P}\phi(\vec{x} + \vec{b}, t) \, d^3x$$

but not in time. That’s not the worst thing; since the integration measure $d^3x$ isn’t $SO(1, 3)$ invariant, the whole scalar product is no longer invariant either. One could try to introduce the Lorentz invariant measure; it turns out this is not necessary. Expectation values of zeroth components of currents will be conserved and therefore translationally invariant in time. As long as theory deals with expectation values of conserved currents

$$Q^{\mu\ldots} = \langle \psi|j^{0,\mu\ldots}|\psi\rangle \quad (7)$$

or with transition amplitudes

$$\langle f|i \rangle = \langle \psi_f(t_f) |U(t_f, t_i) |\phi_i(t_i) \rangle$$

$$= \langle \psi_f | T \exp \int_{y_i}^{t_f} P^0 dt | \phi_i \rangle$$

$$\quad (8)$$
results will be invariant as long as matrix elements of hamiltonian are local

$$\langle \vec{x} | P^0 | \vec{y} \rangle = \delta^{(3)}(\vec{x} - \vec{y}) P^0(\vec{x}) .$$

so we end up with the theory that’s relativistically invariant.

One could ask a question why bother doing this at all since we already have the field theory that’s relativistically invariant. First section gives a brief overview of problems/peculiarities with single particle relativistic equations that lead to the second quantization as well as some problems left after the second quantization thus establishing a motivation. To make the comparison with QFT easier, Dirac’s spinor representation will be used as the example of RWFM approach and the results derived will be compared with the results of standard QFT. In the second section the scalar product of wave functions is redefined to be invariant to Lorentz transformations. Representations of spin 1/2 and 1 are treated in the third section. Wave functions are constructed as superpositions of momentum, spin and parity eigenstates. Based only on superposition principle and completeness, it is shown that all solutions in this approach must have positive energies. It is also shown that the parity symmetry in different Lorentz frames leads to both the Dirac equation for fermions and Maxwell equations for photons. If these equations are statements of symmetry already incorporated in the wave function, then it is reasonable to assume one need not enforce them as Euler-Lagrange equations of a given Lagrangian. Section four shows that the requirement of consistent Noether currents for translations, rotations and boosts determines the Lagrangian completely, regardless of the spin of considered representation. Section five addresses the question of the interpretation of solutions and the massless limit. Finally, appendixes list a few properties of Lorentz group and it’s representations, mainly to keep track of notation and conventions used through the article.

1 Overview of Dirac equation and second quantization

Dirac originally proposed the relativistic equation of the form

$$i \frac{\partial \psi}{\partial t} = \left( \frac{1}{i} \vec{\alpha} \cdot \vec{\nabla} + \beta m \right) \psi = H \psi$$

(10)

where $\alpha_i$ and $\beta$ are anticommuting matrices satisfying:

$$\alpha_i^2 = \beta^2 = 1$$

(11)

$$\{\alpha_i, \alpha_j\} = 0$$

(12)

$$\{\alpha_i, \beta\} = 0 .$$

(13)

One would then multiply the Dirac equation (10) by $\beta$ and rewrite it in a covariant notation

$$(i \gamma^\mu \partial_\mu - m) \psi = 0$$

(14)

where matrices $\gamma^\mu$ are defined as

$$\gamma^0 = \beta, \quad \gamma^i = \beta \alpha^i, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} .$$

(15)

By taking the Hermitian conjugate of the equation (14) we get the another one for the conjugate of $\psi$:

$$\partial_\mu \bar{\psi} i \gamma^\mu + \bar{\psi} m = 0$$

(16)

with $\bar{\psi} \equiv \psi^\dagger \gamma_0$. 

4
The next step would usually be to prove that this equation is relativistically invariant and then find the Lagrangian which reproduces this equation and use it to find the energy-momentum tensor, conserved current and angular momentum tensor. This Lagrangian is found to be\footnote{derivative operator is defined with the appropriate sign and factor to give (after partial integration) \( \int \bar{\psi} \frac{\partial}{\partial \phi} \psi = \int \bar{\psi} \frac{\partial}{\partial \phi} \psi = \int \bar{\psi} \frac{\partial}{\partial \phi} \psi \), or explicitly \( f(x) \frac{\partial}{\partial x} \psi = -\partial f(x) \bar{\psi} \) and \( f(x) \frac{\partial}{\partial g(x)} \psi = \{ f(x) \partial g(x)/\partial x - \partial f(x)/\partial x g(x) \}/2 \). Conjugation properties of these operators are \( (\partial)^\dagger = -\partial_x \), \( (\partial)^\dagger = -\partial_x \), \( (\partial)^\dagger = -\partial_x \).} 

\[ \mathcal{L}_D = \bar{\psi}(x) \left( i \frac{\partial}{\partial x} - m \right) \psi(x) \]  

(17) 

with \( \dot{p} \equiv \gamma \cdot p \). Going to the momentum space one finds the solutions of equations (14) and (16) to be

\[ \psi(x) = \int \frac{d^3q}{(2\pi)^{3/2} 2E_q} \left( e^{-i\gamma \cdot q \cdot x} \bar{b}_{qr} + e^{i\gamma \cdot q \cdot x} \bar{d}_{qr}^* \right) \] 

\[ \bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2} 2E_p} \left( e^{i\gamma \cdot p \cdot x} \bar{a}_{ps} + e^{-i\gamma \cdot p \cdot x} \bar{d}_{ps}^* \right) \]  

(18) 

Although at this point \( b_{ps} \) and \( d_{ps} \) are complex numbers, they will be treated as noncommuting quantities so that all results derived would still be valid after the second quantization. If we require the solution (18) to be invariant to the gauge transformation 

\[ \psi(x) \rightarrow \psi'(x) = e^{-i\alpha} \psi \quad \bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}e^{i\alpha} \]  

(19) 

or for infinitesimal \( \alpha \)

\[ \psi(x) \rightarrow (1 - i\alpha) \psi \quad \bar{\psi}(x) \rightarrow \bar{\psi}(1 + i\alpha) \]  

(20) 

we get the conserved current 

\[ j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x) \]  

(21) 

Requiring the translational invariance 

\[ \psi(x) \rightarrow \psi'(x') = \psi(x + a) \quad \bar{\psi}(x) \rightarrow \bar{\psi}'(x') = \bar{\psi}(x + a) \]  

(22) 

or for infinitesimal \( a \)

\[ \psi(x) \rightarrow (1 + a_\mu \partial^\mu) \psi(x) \quad \bar{\psi}(x) \rightarrow (1 + a_\mu \partial^\mu) \bar{\psi}(x) \]  

(23) 

one gets the energy momentum tensor 

\[ \theta^{\mu\nu}(x) = \bar{\psi}(x) \gamma^\mu \frac{\partial}{\partial \psi^\nu} \psi(x) \]  

(24) 

Following the same procedure for rotations and Lorentz boosts 

\[ \psi(x) \rightarrow \psi'(x') = S(\omega) \psi(x) \quad \bar{\psi}(x) \rightarrow \bar{\psi}'(x') = \bar{\psi}(x) \gamma^0 S^{\dagger}(\omega) \gamma^0 = \bar{\psi}(x) S^{-1}(\omega) \]  

(25) 

or for infinitesimal \( \omega \)

\[ \psi(x) \rightarrow \psi'(x') = \left\{ 1 - \frac{i}{2} \omega_{\mu\nu} \left( \sigma^{\mu\nu} + x^\mu i \partial^\nu - x^\nu i \partial^\mu \right) \right\} \psi(x) \] 

\[ \bar{\psi}(x) \rightarrow \bar{\psi}'(x') = \left\{ 1 + \frac{i}{2} \omega_{\mu\nu} \left( \sigma^{\mu\nu} - x^\mu i \partial^\nu - x^\nu i \partial^\mu \right) \right\} \bar{\psi}(x) \]  

(26) 

\[ \int \bar{\psi} \frac{\partial}{\partial \phi} \psi = \int \bar{\psi} \frac{\partial}{\partial \phi} \psi = \int \bar{\psi} \frac{\partial}{\partial \phi} \psi \], or explicitly \( f(x) \frac{\partial}{\partial x} \psi = -\partial f(x) \bar{\psi} \) and \( f(x) \frac{\partial}{\partial g(x)} \psi = \{ f(x) \partial g(x)/\partial x - \partial f(x)/\partial x g(x) \}/2 \). Conjugation properties of these operators are \( (\partial)^\dagger = -\partial_x \), \( (\partial)^\dagger = -\partial_x \), \( (\partial)^\dagger = -\partial_x \).
we get the generalized angular momentum density tensor

\[ J^{\mu,\alpha\beta} = \bar{\psi}(x) \left[ \gamma^\mu \left( x^\alpha \delta^\beta - x^\beta \delta^\alpha \right) + \frac{1}{2} \left( \gamma^\mu, \sigma^{\alpha\beta} \right) \right] \psi(x) \]

\[ = x^\alpha \partial^\mu - x^\beta \partial^\alpha + \frac{1}{2} \bar{\psi}(x) \left( \gamma^\mu, \sigma^{\alpha\beta} \right) \psi(x) \]  

(27)

While all of the above looks nice, there are a few problems with this formulation. It has been believed some of them are solved by second quantization, but some remain even after second quantization.

While the positive energy and negative energy momentum eigenstates of different spin

\[ \psi^+_{ps}(x) = \frac{e^{-ip \cdot x}}{(2\pi)^{3/2}} u_{ps} \]

\[ \psi^-_{ps}(x) = \frac{e^{ip \cdot x}}{(2\pi)^{3/2}} v_{ps} \]  

(28)

are separately orthogonal

\[ \int \bar{\psi}^+_{ps}(x) \psi^+_{ps}(x) \, d^3x = \delta^3(p - q) \delta_{r,s} \]

\[ \int \bar{\psi}^-_{ps}(x) \psi^-_{ps}(x) \, d^3x = -\delta^3(p - q) \delta_{r,s} \]  

(29)

scalar products of positive and negative energy momentum eigenstates aren’t vanishing, making the solutions non-orthogonal

\[ \int \bar{\psi}^+_{ps}(x) \psi^-_{ps}(x) \, d^3x = \delta^3(p + q) e^{2iE_p t} \bar{u}_{ps} u_{\tilde{p},r} \]

\[ \int \bar{\psi}^-_{ps}(x) \psi^+_{ps}(x) \, d^3x = \delta^3(p + q) e^{-2iE_p t} \bar{v}_{ps} v_{\tilde{p},r} \]  

(30)

with \( \tilde{p}^\mu = (p^0, -\vec{p}) \). The complete set of the solutions to the Dirac equation isn’t orthogonal so the decomposition

\[ \psi(x) = \int \frac{d^3p}{(2\pi)^3/2 2E_p} \left( e^{-ip \cdot x} b_{ps} d^*_p + e^{ip \cdot x} v_{ps} d^*_p \right) \]  

(31)

or

\[ \psi(x) = \int \frac{d^3p}{2E_p} \left( \psi^+_{ps}(x) b_{ps} + \psi^-_{ps}(x) d^*_p \right) \]  

(32)

isn’t a valid decomposition in a complete orthonormal set of functions. One deals with this by saying that we work with fields, not wave functions, and therefore there’s no need for orthogonality anyway.

Another “problem” is a consequence of the definition of eigenstates (28). Applying the momentum operator \( \vec{P} \equiv -i \vec{\nabla} \) to positive states produces the proper eigenvalue for the positive energy states

\[ -i \vec{\nabla} \psi^+_{ps} = \vec{p} \psi^+_{ps} \]  

(33)

but it gives us the wrong sign for the negative energy states

\[ -i \vec{\nabla} \psi^-_{ps} = -\vec{p} \psi^-_{ps} \]  

(34)

\(^2\)Note that the scalar product isn’t defined as \( \int \psi^\dagger \psi \, d^3x \) but as \( \int \psi^\dagger \gamma_0 \psi \, d^3x \). Reasons for this will be explained in section 2.1.

\(^3\)There is a minus sign for the norm of \( \psi^- \) solution that seems to contradict positiveness of the norm. That will also be discussed in section 2.1.
We can think of momentum eigenstates as states obtained by applying the Lorentz boost in the direction $\vec{p}$ and with the boost parameter $|\vec{p}|$ to the particle in its rest frame ($\vec{p} = 0$). After we boost the particle in one direction, negative energy particles appear to move in the opposite direction. To solve this problem Dirac proposed the hole interpretation saying that in physical vacuum all the negative energy states were filled and when an electron from this sea gets excited to positive energy it leaves a hole that we observe as anti-particle carrying the opposite charge. Stückelberg and a little later Feynman proposed the interpretation that “negative energy” solution of energy $-E$ and momentum $-\vec{p}$ represents a particle moving backwards in time which we observe as the particle of the opposite charge, energy and momentum moving forward in time.

Further problems appear when one considers conserved currents of the theory. The current

$$\vec{j}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{2E_p (2\pi)^3} \left( e^{i(p-q) \cdot x} b_{p_\sigma}^{*} b_{q_\gamma} u_{p_\gamma} \gamma^\mu u_{q_\mu} + e^{-i(p-q) \cdot x} d_{p_\gamma} d_{q_\mu} \bar{v}_{p_\gamma} \gamma^\mu v_{q_\mu} + e^{i(p+q) \cdot x} b_{p_\sigma}^{*} d_{q_\mu} \bar{v}_{p_\gamma} \gamma^\mu v_{q_\mu} + e^{-i(p+q) \cdot x} b_{p_\sigma} d_{q_\gamma} \bar{v}_{p_\mu} \gamma^\mu u_{q_\mu} \right)$$

is formally conserved $\partial_\mu j^\mu(x) = 0$ which allow us to construct conserved charge by integrating over the whole volume $\mathbb{R}^3$

$$Q = \int j^0(x) d^3 x = \int \frac{d^3p}{2E_p} \left( b_{p_\sigma}^{*} b_{p_\sigma} + d_{p_\gamma} d_{p_\gamma} \right).$$

However, space part (with the help of Gordon identities from appendix C.3) of the total current

$$\vec{J}^i = \int \frac{d^3p}{2E_p} \frac{1}{E_p} \left( p^i \sum_s \left[ b_{p_\sigma}^{*} b_{p_\sigma} + d_{p_\gamma} d_{p_\gamma} \right] + \sum_{s,r} \left[ \frac{i e^{2iE_p t}}{2m} \bar{u}_{p_\sigma} \sigma^{ij} v_{p r} b_{p_\sigma} d_{p r}^{*} - \frac{i e^{-2iE_p t}}{2m} \bar{v}_{p_\sigma} \sigma^{ij} u_{p r} d_{p_\sigma} b_{p r}^{*} \right] \right)$$

(as well as current density) besides the group velocity term has a real oscillating term. This term of the order of magnitude $10^{-21} s$ traditionally called zitterbewegung has been without proper physical interpretation and is another reason physics community decided that single particle theories don’t work and should be discarded/reinterpreted, although it is present even after second quantization.

When integrating the zeroth component over the infinite volume, zitterbewegung terms are still present but don’t contribute since all the fields vanish at infinity. If one integrates over finite volume $V$,

$$\frac{d}{dt} \int_0^t \int \vec{j}(x) d^3 x = -\int_0^t \nabla \cdot \vec{j}(x) d^3 x = -\int_S \vec{n} \cdot \vec{j}(x) dS$$

where $\vec{n}$ is the unit normal vector to surface $S$, zitterbewegung terms on the right-hand side make it hard to interpret change of the charge contained in volume $V$ as the divergence of the flux of the current over the edge of the volume $S$.

After second quantization charge $Q$ doesn’t annihilate the vacuum, so one has to remove the divergent part “by hand” by introducing so called normal ordering. Even after normal ordering, zitterbewegung is still present; creation and annihilation operators from zitterbewegung part of the current come in combinations

$$\vec{j}(x) \sim b d + b^\dagger d^\dagger$$

which doesn’t annihilate the vacuum

$$\vec{j}(x) \ket{0} \neq 0$$
and mixes states with \( n \) particles with states with \( n + e^+ e^- \) pair states, for example vacuum and electron-positron pair
\[
\langle 0 \mid J^i \mid e^+ e^- \rangle \neq 0 .
\] (41)

After one introduces electromagnetic interactions through
\[
H_i = \int j^\mu(x) A_\mu(x) ,
\] (42)
those terms give infinite contributions to both higher order vacuum to vacuum transition matrix elements as well as infinite contributions to higher order (loop) diagrams.

Next set of conserved currents is energy-momentum tensor. First of all, energy momentum tensor isn’t symmetric. This is in itself enough of a problem and requires procedure for symmetrization [1] which is (again) introduced “by hand” and isn’t a consequence of any deeper principle. That set aside those terms give infinite contributions to both higher order vacuum to vacuum transition matrix elements and mixes states with states with parts that oscillate both like \( e^{\pm iEt} \) and \( e^{\pm i2Et} \), while the left hand side has only the \( e^{\pm iEt} \) oscillating part. The same holds for momentum components as well
\[
\frac{d}{d\tau_v} \int \theta^{00}(x) d^3x = - \int \frac{d}{dx_j} \theta^{j0}(x) d^3x = - \int \sum_{j=1}^{3} n^j \theta^{j0}(x) dS
\] (45)
which equals zero for infinite volume \( V \) and field vanishing at infinity. However, if the volume is finite, then the surface integral has the \( zitterbewegung \)-like behavior having parts that oscillate both like \( e^{\pm iEt} \) and \( e^{\pm i2Et} \), while the left hand side has only the \( e^{\pm iEt} \) oscillating part. The same holds for momentum components as well
\[
\frac{d}{d\tau_v} \int \theta^{0i}(x) d^3x = - \int \frac{d}{dx_j} \theta^{ji}(x) d^3x = - \int \sum_{j=1}^{3} n^j \theta^{ji}(x) dS
\] (46)
which makes it hard to interpret the \( \theta^{j0} \) components as the components of Poynting vector or \( \theta^{ji} \) components as the components of stress tensor.

Note that the conserved current and energy-momentum four vector aren’t proportional (both before and after the second quantization) so it’s impossible to interpret the current as probability density-flux current.

Things get even worst with angular momentum density tensor; again, tensor is formally conserved
\[
\partial_{\mu} J^{\mu,\alpha\beta} = 0 ,
\] which again allows us to construct the Lorentz transformation generators
\[
J^{\mu\nu} = \int J^{0,\mu\nu} d^3x = \int \frac{1}{2} \bar{\psi} \left( \gamma^\mu \partial^\nu - \gamma^\nu \partial^\mu \right) \gamma^0 + \left\{ \gamma^0, \frac{\sigma^{\mu\nu}}{2} \right\} \psi \bar{d}^3x .
\] (47)
it would lead to the vacuum with the preferred direction which would seem to contradict the experiment.

would either lead to or if one insists the other.

which would imply that what looks like electron-positron pair from one angle looks like vacuum from the other.

transforms every spinor component independently corresponding to the transformation \( \psi(x) \rightarrow \psi(\Lambda x) \). It’s behavior is more or less reasonable (aside from the problem of normal ordering) although derivative operators acting on \( b_{ps} \) and \( d^*_{ps} \) raise additional questions after second quantization. Spin part of generators

\[
J^{\mu\nu}_{spin} = \frac{1}{2} \int \bar{\psi} \left( \gamma^0, \frac{\sigma^{\mu\nu}}{2} \right) \psi \, d^3 x = \frac{1}{2} \int \bar{\psi} \left( \frac{\sigma^{\mu\nu}}{2} + \gamma^0 \frac{\sigma^{\mu\nu}}{2} \right) \psi
\]

doesn’t do so well. For space part we have \( \gamma^0 \sigma^{ij} \gamma^0 = \sigma^{ij} \) so we get rotation generators

\[
J_k \equiv \epsilon_{ijk} J^{ij} = \epsilon_{ijk} \int \bar{\psi} \frac{\sigma^{ij}}{2} \psi \, d^3 x
\]

\[
= \int \left( \frac{d^3 p}{(2\pi)^3 2E_p} \right) \sum_{s,r} \left( u_{ps}^\dagger \frac{\sigma^k}{2} u_{pr} b^*_s b_{pr} + v_{ps}^\dagger \frac{\sigma^k}{2} v_{pr} d_{ps} d^*_{pr} 
\right.
\]

\[
+ e^{2iE_p t} u_{ps}^\dagger \frac{\sigma^k}{2} v_{pr} b^*_s b_{pr} + e^{-2iE_p t} v_{ps}^\dagger \frac{\sigma^k}{2} u_{pr} d_{ps} d^*_{pr} \right)
\]

which again shows zitterbewegung-like behavior not only in the space parts of the currents but in zeroth components as well. Those zitterbewegung-like terms

\[
e^{2iE_p t} u_{ps}^\dagger \frac{\sigma^k}{2} v_{pr} b^*_s b_{pr} + e^{-2iE_p t} v_{ps}^\dagger \frac{\sigma^k}{2} u_{pr} d_{ps} d^*_{pr}
\]

mix positive and negative energy states (aside from making generators time dependent which is clearly a contradiction to the idea of a time conserved quantity). After second quantization these terms (and therefore the whole generators) no longer annihilate the vacuum so the finite rotations

\[
e^{-i\bar{\omega} \cdot \vec{J}} = 1 - i\bar{\omega} \cdot \vec{J} + (-i\bar{\omega} \cdot \vec{J})^2 + \ldots
\]

\[
\sim 1 + \left( b^j b + d^j d + d^j b^j + db \right) + \left( b^j b + d^j d + d^j b^j + db \right)^2 + \ldots
\]

would either lead to

\[
\langle 0 | e^{-i\bar{\omega} \cdot \vec{J}} | 0 \rangle \neq 1
\]

or if one insists \( \langle 0 | e^{-i\bar{\omega} \cdot \vec{J}} | 0 \rangle = 1 \) it would lead to an infinite series of integral constraints on operators \( b \) and \( d \) with the only solution being the trivial one. Spontaneous symmetry breaking doesn’t help since it would lead to the vacuum with the preferred direction which would seem to contradict the experiment.

Another consequence of these terms is that they mix states with \( n \) and \( n \pm 2 \) particles, for example vacuum and electron positron state

\[
\langle 0 | \vec{J} | e^+ e^- \rangle \neq 0 \quad \Rightarrow \quad \langle 0 | e^{-i\bar{\omega} \cdot \vec{J}} | e^+ e^- \rangle \neq 0
\]

which would imply that what looks like electron-positron pair from one angle looks like vacuum from the other.
For mixed space-time part of the tensor we have \( \gamma^0 \sigma^{0i} \gamma^0 = -\sigma^{0i} \), so the spin part of boost generator vanishes exactly:

\[
K_k \equiv J^{0k} = \frac{1}{2} \int \bar{\psi} \left( \gamma^0 \sigma^{0k} \gamma^0 \right) \psi \, d^3x = \frac{1}{4} \int \bar{\psi}(\sigma^{0k} + \gamma^0 \sigma^{0k} \gamma^0) \psi = 0 \quad (55)
\]

This would imply that every component of Dirac field transforms as scalar under boosts and as spinor under rotations which is a contradiction in definition.

Space part of the angular momentum density tensor \( J^{i,\alpha\beta} \) again shows all the above problems and some more. Writing them down in detail wouldn’t be particularly illuminating and so it will be skipped.

These problems are present in other representations as well. While a second quantization, some reinterpretation of symbols and some renormalization a bit later do offer possible solutions for some of these problems, they don’t solve all of them. Although the approach used in this article does call for reinterpretation as well, it doesn’t merely offer solutions for these problems/peculiarities; none of them are present in the theory in the first place.

## 2 Relativity and quantum mechanics

Non-relativistic quantum mechanics is based on a few simple postulates. First one says that physical states are represented by a complete set of normalized, complex vectors \( \psi \) in Hilbert space \( \mathcal{H} \):

\[
\bar{\psi} \cdot \psi \equiv 1 \quad (56)
\]

where \( \bar{\psi} \) is defined to be complex conjugate of the vector \( \psi \). We can write the vector \( \bar{\psi} \) (in some orthogonal basis) as a column matrix traditionally denoted as \( |\psi\rangle \). Equation (56) then becomes

\[
\langle \psi | \psi \rangle \equiv 1 \quad (57)
\]

where \( \langle \psi | \) is now defined to be Hermitean conjugate of the matrix \( |\psi\rangle \): \( \langle \psi | \equiv (|\psi\rangle)^\dagger \). If we are to interpret the square of the wave function as the probability, equation \( \langle \psi(t) | \psi(t) \rangle \equiv 1 \) should hold at all times. To ensure that, we require \( |\psi(\vec{x},t)\rangle^2 = \rho \) to be the zeroth component of a conserved current

\[
j^\mu = (\rho, \vec{j}) \quad (58)
\]

with \( \vec{j} = (-i\hbar/m) \bar{\psi} \psi \stackrel{\leftrightarrow}{\nabla} \psi \)

\[
\frac{\partial \rho(\vec{x},t)}{\partial t} + \nabla \cdot \vec{j}(\vec{x},t) = \partial_\mu j^\mu = 0
\]

If the system is in a state \( |\psi\rangle \), then the probability of finding it in another state \( |\phi\rangle \) is \( |\langle \phi | \psi \rangle|^2 \). Physical observables are described by Hermitean operators \( A = A^\dagger \) on the space \( \mathcal{H} \); the expectation value of operator \( A \) is defined to be \( \langle \psi | A | \psi \rangle \). If the system is invariant under certain symmetries, a theorem of Wigner states that such symmetries are represented by unitary (or antiunitary) operators \( U^{-1} = U^\dagger \). If that wasn’t the case, then the first postulate wouldn’t be valid since if \( |\phi\rangle = U |\psi\rangle \), then

\[
\langle \phi | \phi \rangle = \langle \psi | U^\dagger U | \psi \rangle \quad (59)
\]

is no longer invariant under the symmetry. Wave function \( \psi(x) \) can be represented as a superposition of orthogonal states \( \{ \phi_n \} \)

\[
\psi(\vec{x},t) = \sum_n c_n(t) \phi_n(\vec{x},t) \quad (60)
\]
where the set of functions \( \{ \phi_n \} \) is a complete set
\[
\sum_n \phi_n(\vec{x}, t) \phi^*_n(\vec{y}, t) = \delta(\vec{x} - \vec{y}) .
\] (61)

In fact, the wave function \( \psi(x, t) \) can be viewed as the factor in the decomposition of the abstract Hilbert space vector \( |\psi(t)\rangle \) in the orthonormal basis of vectors \( |\vec{x}\rangle \)
\[
|\psi(t)\rangle = \int |\vec{x}\rangle \langle \vec{x}| \psi(t) \rangle d^3x
\] (62)

where the vectors \( |\vec{x}\rangle \) are eigenvectors of the position operator \( \vec{x} \)
\[
\vec{x} |\vec{x}\rangle = |\vec{x}\rangle .
\] (63)

This is the basis for the interpretation that the value of \( \psi(x, t) \) is the amplitude (and therefore \( |\psi(x, t)|^2 \) the probability) for particle in the state \( |\psi(t)\rangle \) to be found at the position \( \vec{x} \). Position eigenstates can be eliminated completely from the equation (60) so it becomes
\[
|\psi(t)\rangle = \sum_n c_n(t) |\phi_n(t)\rangle
\] (64)

where the coefficients \( c_n \) are given by \( c_n = \langle \phi_n | \psi \rangle \). This principle of superposition is the most fundamental principle in quantum mechanics. It is only natural to keep it through the rest of the article.

On the other hand, special theory of relativity requires that the speed of light \( c \) is constant in all inertial frames, or mathematically the distance between two points in space-time
\[
c^2(t_2^2 - t_1^2) - (\vec{x}_2 - \vec{x}_1)^2
\] (65)

should be the same in all inertial frames. The group of transformations that obey this rule (Poincaré group) consists of all the translations in space-time, rotations in space as well as of Lorentz transformations (or boosts). Here we run into trouble; while translations in space-time are unitary, general Lorentz transformations aren’t. The group of proper Lorentz transformations (usually called simply Lorentz group) is \( SO(1, 3) \) which is known to be non-compact and therefore has no finite-dimensional unitary representations. Using infinite-dimensional representations would imply that (free) particle of given energy and momentum has infinite number of (degenerate) spin states. This doesn’t seem to be the case in nature. So we’re stuck with non-unitary representations of Lorentz group.

This brings us to the fundamental incompatibility; since the representations are non-unitary, expressions like \( \langle \psi' | \varphi' \rangle \equiv (|\psi'\rangle)^\dagger \cdot |\varphi'\rangle \) won’t be invariant under Lorentz boosts to another frame
\[
|\varphi\rangle \rightarrow |\varphi'\rangle = e^{-i\vec{\omega} \cdot \vec{K}} |\varphi\rangle = U |\varphi\rangle , \quad \langle \psi | \rightarrow \langle \psi' | = \langle \psi | e^{-i\vec{\omega} \cdot \vec{K}} \rangle^\dagger = \langle \psi | e^{+i\vec{\omega} \cdot (\vec{K})^\dagger} = \langle \psi | U
\] (66)

\[
\Rightarrow \quad \langle \psi' | \varphi' \rangle = \langle \psi | U^\dagger U | \varphi \rangle = \langle \psi | UU | \varphi \rangle \neq \langle \psi | \varphi \rangle .
\] (67)

The same holds for expectation values of operators \( \langle \psi | A | \psi \rangle \). On the other hand, rotation as well as translation generators are Hermitean which makes rotations and translations unitary and scalar product invariant; this must not be changed whatever we do. QFT deals with this problem by keeping the definition of scalar product, reinterpreting equations as equations for fields, not wave functions, and postulating that these fields act on Fock space and create or destroy states in that space. Commutation or anticommutation relations are postulated in such a way that will keep the scalar product positive definite. However, it turns out that merely assigning the name "scalar product" to a different, relativistically invariant quantity will allow us to keep the idea of wave function and construct consistent single-particle relativistically invariant theory, without negative energies, without zitterbewegung, with nice conserved current proportional with energy and momentum density proportional to the zeroth component of this current, as we had in the non-relativistic case which enabled the famous probabilistic interpretation.
2.1 Lorentz invariant scalar product

The “new” definition of scalar product turns out to be quite familiar. Let’s take another look at equation (65); in general, for complex four-vectors \( a \), we have the invariant (and real) quantity

\[
a_\mu^* a^\mu = a_0^* a_0 - a_1^* a_1 - a_2^* a_2 - a_3^* a_3 .
\]

If we use the Dirac’s bra-ket notation, four vector \( a^\mu \) can be written as

\[
a^\mu \equiv |a\rangle = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.
\]

Equation (68) can be written in matrix form as

\[
\langle a | P | a \rangle = a_0^* a_0 - a_1^* a_1 - a_2^* a_2 - a_3^* a_3 .
\]

where \( P \) is the parity matrix

\[
P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\]

Since this is the invariant quantity in the defining representation of Lorentz group, it should be no surprise that it works in all another representations as well. In general, we can redefine the scalar product by inserting another operator between bra and ket states; to make scalar product invariant, one has to have

\[
\langle \psi' | O | \varphi' \rangle = \langle \psi | U^\dagger O U | \varphi \rangle = \langle \psi | O | \varphi \rangle \quad \Rightarrow \quad O^{-1} U^\dagger O = U^{-1} .
\]

or in terms of generators, this operator must satisfy

\[
O^{-1} i \vec{J} O = i \vec{J} \quad O^{-1} i \vec{K} O = -i \vec{K} .
\]

If we take a look at the commutators of parity and Lorentz generators (appendix A)

\[
P \vec{J} \vec{P} = \vec{J} \quad P \vec{K} \vec{P} = -\vec{K} .
\]

and equation (70), it’s clear parity satisfies this condition. So far any coordinate dependence of states \( |\psi\rangle \), \( |\varphi\rangle \) or \( |a\rangle \) wasn’t mentioned. Operators above are in fact only spin parts of full generators so the operator \( O \) which makes the scalar product invariant is in fact only the spin part of full parity operator. “Orbital” parts of Lorentz generators as well as translation generators act on the coordinate part of vectors

\[
e^{-\frac{i}{\hbar} \sum_{\mu, \nu} P_{\mu,\nu} \phi_{x+\omega}(x)} = \phi_{x}(A(\omega)x + a)
\]

independently and don’t mix different spin components. Their products

\[
\phi^\dagger(x) P_{\text{spin}} \phi(x)
\]

are function of position \( x \) and as such in general are not Lorentz invariant for any representation. Their integrals, however, \textit{will} be invariant if they are integrated with the proper Lorentz-invariant measure. Lorentz invariant scalar product will then be

\[
\langle \psi | P | \varphi \rangle \equiv \langle \psi | \varphi \rangle
\]
where we define the “new” conjugate vector $\langle \overline{\psi} | \equiv \langle \psi | P$. This is in fact nothing new. We have already shown that contractions of covariant and contravariant vectors can be interpreted as parity operator sandwiched between two states. Another widely present example comes from Dirac representation; there Lorentz invariant product isn’t $\psi^\dagger(x)\psi(x)$ but $\bar{\psi}(x)\psi(x)$, where bar on $\psi$ means $\psi^\dagger \gamma_0$. Again, $\gamma_0$ is nothing but parity operator for $(1/2, 0) \oplus (0, 1/2)$ representation (see appendix B.1).

Note that equation (77) has one fundamental consequence: if we interpret the scalar product of the state in Hilbert space with itself $\langle \psi | \psi \rangle$ as a norm of that state, there will have to be some states with negative norm in every nontrivial representation. From mathematical viewpoint, such a definition isn’t really a norm; however, since in physics one uses the phrases like “negative metric” to describe Minkowski metric tensor, a phrase “norm” will also be misused here in the same tradition to describe the relativistically invariant scalar product of a vector with itself. Since the “norm” has a parity operator in it’s definition, it need not be positive. Therefore one has to think twice before calling a zeroth component of the continuity current probability density.

This however isn’t inconsistent with the definition of the scalar product in nonrelativistic quantum mechanics. Since the parity operator isn’t uniquely defined by commutation relations (74), neither is the norm. If $P$ satisfies relations (74) so will $-P$; therefore, if we choose parity operator to be $-P$ instead of $P$, we have effectively multiplied the norm of all states with $-1$. In another words, for given state $|\psi\rangle$ we can always choose parity operator in such a way that the norm of that particular state is positive. As a consequence, in the non-relativistic limit where boosts no longer mix states of different spin

$$\psi(\vec{x}, t) \to \psi'(\vec{x}', t') = \psi(\vec{x} + \vec{v}t, t)$$

(78)

particle and antiparticle states transform separately under boosts and form a group of Galilean transformations. In this limit one can always choose the parity operator for different representations of Galilean group of transformations that will give positive definite norm for all spin states. Thus we can recover the proper non-relativistic scalar product as well as probabilistic interpretation.

What will finally fix parity operator is the requirement of it’s actions on particle states. For four-vector representations $(1/2, 1/2)$ we have “physical” requirement that spin 1 (vector states) part should have negative parity and spin 0 part (scalar state) positive. Same logic works for all $(j, j)$ representations which have states with spin $\ell \in \{0, 1, \ldots, 2j\}$ having parity $(-1)^\ell$.

### 3 Representations of Lorentz Group and parity equations

To have Lorentz invariant theory, besides the relativistically invariant scalar product, wave functions have to belong to various representations of Lorentz group. As it was mentioned before, any wave functions in a Hilbert space can always be decomposed in a complete set of functions. This fundamental property combined with the relativistic invariance will determine the behavior of wave functions for any given representation.

#### 3.1 Construction of the spinor representation in RWFM

Lorentz boost and rotation generators for spinor representation are nothing else but Dirac $\sigma$ matrices. Since their derivation is straightforward only the results in chiral representation are quoted here (derivation can be found in appendix B.1)

$$J_k = \frac{1}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = \frac{1}{4} \sum_{ij} \epsilon_{ijk} \sigma^{ij}$$

$$K_k = \frac{i}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} = \frac{1}{2} \sigma^{0k}$$

(79)
Finite transformations are then generated by

\[ S(\omega) = \exp \left( -\frac{i}{2} J^{\mu \nu} \omega_{\mu \nu} \right) = \exp \left( -\frac{i}{4} \sigma^{\mu \nu} \omega_{\mu \nu} \right). \quad (80) \]

Now, let’s construct Lorentz-invariant momentum and spin eigenfunctions; wave function will be a direct sum of \((1/2, 0)\) and \((0, 1/2)\) terms

\[ \psi(x) = \psi'^R(x) \oplus \psi'^L(x) = \begin{pmatrix} \psi'^R(x) \\ \psi'^L(x) \end{pmatrix}. \quad (81) \]

It is more convenient to add the null matrix to \(\psi'^R\) and \(\psi'^L\) matrices and make the sum in (81) normal instead of direct

\[ \psi'^R \rightarrow \psi_R(x) = \begin{pmatrix} \psi'^R(x) \\ 0 \end{pmatrix}, \quad \psi'^L \rightarrow \psi_L(x) = \begin{pmatrix} 0 \\ \psi'^L(x) \end{pmatrix}, \quad \psi(x) = \psi_R(x) + \psi_L(x) \quad (82) \]

Invariance conditions for translations and pure Lorentz transformations are

\[ \psi(x) \rightarrow \psi'(x') = \psi(x + a) \quad \psi(x) \rightarrow \psi'(x') = S(\omega)\psi(x). \quad (83) \]

Note that there is no \(S(a)\) term for translations; it implies that each spin component transforms separately under translations. This will make it possible to factor out the same translation-generator-eigenfunctions (momentum eigenfunctions) from the whole wave function \(\psi(x)\). At this point it’s convenient to (again) introduce Dirac’s bra ket notation. Wave function can then be written as

\[ \psi(x) = \langle \vec{x} | \psi(t) \rangle = \langle \vec{x} | \psi_R(t) \rangle + \langle \vec{x} | \psi_L(t) \rangle = \begin{pmatrix} \langle \vec{x} | \psi'^R_R(t) \rangle \\ \langle \vec{x} | \psi'^L_L(t) \rangle \end{pmatrix}. \quad (84) \]

We can insert a complete set of momentum and (time dependent) orthogonal spin eigenstates

\[ \int d^3 q \sum_{r,p} \frac{|\psi_{q,r,p} \rangle \langle \psi_{q,r,p}|}{\langle \psi_{q,r,p} | \psi_{q,r,p} \rangle} = \int d^3 q \sum_r (|\psi_{q,r,+} \rangle \langle \psi_{q,r,+}| - |\psi_{q,r,-} \rangle \langle \psi_{q,r,-}|) \equiv 1 \quad (85) \]

where we have acknowledged the fact that negative parity states have negative norm as well. Wave function then becomes

\[ \langle \vec{x} | \psi(t) \rangle = \int d^3 q \sum_{r,p} \frac{\langle \vec{x} | \psi_{q,r,p} \rangle \langle \psi_{q,r,p} | \psi \rangle}{\langle \psi_{q,r,p} | \psi_{q,r,p} \rangle} = \int d^3 q \sum_{r,p} (\langle \vec{x} | \psi_{q,r,+} \rangle \langle \psi_{q,r,+} | \psi \rangle - \langle \vec{x} | \psi_{q,r,-} \rangle \langle \psi_{q,r,-} | \psi \rangle) \quad (86) \]

Translational invariance tells us that spin and coordinate dependence can be factored

\[ \langle \vec{x} | \psi_{q,r,p}(t) \rangle = \frac{e^{i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} w_{q,r,p}(t) \quad (87) \]

where \(w_{q,r,p}\) is some matrix in spin space which in general can depend on momentum and parity of the state. Sign in the exponential is chosen to make the eigenvalue of momentum operator “positive” \(\vec{q}\)

\[ -i \nabla \langle \vec{x} | \psi_{q,r,p}(t) \rangle = \vec{q} \langle \vec{x} | \psi_{q,r,p}(t) \rangle. \quad (88) \]
For momentum and spin eigenstate $|\psi(t)\rangle = |\psi_{p,s}\rangle$ we have
\[
\langle \vec{x} | \psi_{p,s}(t) \rangle = \int d^3q \sum_{r,p'} \frac{e^{i\vec{p} \cdot \vec{x}}}{(2\pi)^{3/2}} w_{q,r,p'} |\psi_{q,r,p'}\rangle |\psi_{p,s}\rangle = \frac{e^{i\vec{p} \cdot \vec{x}}}{(2\pi)^{3/2}} w_{q,r,p}(t)
\] (89)
with the states normalized (at equal time) as
\[
\langle \psi_{q,r,p'} | \psi_{p,s}\rangle = \mathcal{P} \delta_{\vec{p},\vec{p}'} \delta_{r,s} \delta_{p',p} .
\] (90)
Since translations in space (and time) don’t mix different spin components, we can factor out the common $x$-dependent exponential. In $(1/2,0) \oplus (0,1/2)$ basis we have
\[
\langle \vec{x} | \psi_{p,s}(t) \rangle = \langle \vec{x} | \psi^R_{p,s}(t) \rangle + \langle \vec{x} | \psi^L_{p,s}(t) \rangle .
\] (91)
Now, parity will transform states with momentum $\vec{p}$ and spin $s$ to the state with momentum $-\vec{p}$ and spin $s'$; we can divide the coordinate and spin part $P = P_{\text{coord}}P_{\text{spin}}$ so that the coordinate part acts on $|\vec{x}\rangle$ while spin part acts on $|\psi_{p,s}\rangle$
\[
P\langle \vec{x} | \psi_{p,s}(t) \rangle = P_{\text{coord}} \frac{e^{i\vec{p} \cdot \vec{x}}}{(2\pi)^{3/2}} P_{\text{spin}} w_{q,r,p} = \frac{e^{-i\vec{p} \cdot \vec{x}}}{(2\pi)^{3/2}} P_{\text{spin}} w_{q,r,p}(t)
\] (92)
For massive particles there is always a nontrivial momentum eigenstate with $\vec{p} = 0$; this eigenstate has to be also parity eigenstate
\[
P\langle \vec{x} | \psi_{0,s}(t) \rangle = \pm \langle \vec{x} | \psi_{0,s}(t) \rangle
\] (93)
On the other hand, parity just exchanges $\psi_R$ and $\psi_L$ so obviously, parity eigenstates in the rest frame will be states with equal $\psi_R$ and $\psi_L$ with either the same or different relative sign
\[
\langle \vec{x} | \psi^+_{0,s}(t) \rangle = \langle \vec{x} | \psi^R_{0,s}(t) \rangle + \langle \vec{x} | \psi^L_{0,s}(t) \rangle = \frac{1}{(2\pi)^{3/2}} e^{-\vec{x} \cdot \vec{\theta}_1} u_{0,s}(t)
\]
\[
\langle \vec{x} | \psi^-_{0,s}(t) \rangle = \langle \vec{x} | \psi^R_{0,s}(t) \rangle - \langle \vec{x} | \psi^L_{0,s}(t) \rangle = \frac{1}{(2\pi)^{3/2}} e^{-\vec{x} \cdot \vec{\theta}_1} v_{0,s}(t)
\] (94)
where matrices $u$ and $v$ are defined to be
\[
u_{0,s}(t) = \begin{pmatrix} \chi_s(t) \\ \chi_s(t) \end{pmatrix} \quad v_{0,s}(t) = \begin{pmatrix} \chi_s(t) \\ -\chi_s(t) \end{pmatrix}
\] (95)
Translation invariance in time requires that the time dependence of matrices $\chi_s(t)$ can be factored to
\[
\chi_s(t) = e^{\pm i\kappa t} \chi_s
\] (96)
where $\kappa$ is a real positive number, and $\chi_s$ time independent bispinor. Parity eigenstates then become
\[
u_{0,s}(t) = e^{\pm i\kappa t} \begin{pmatrix} \chi_s \\ \chi_s \end{pmatrix} = e^{\pm i\kappa t} u_{0,s} \quad v_{0,s}(t) = e^{\pm i\kappa t} \begin{pmatrix} \chi_s \\ -\chi_s \end{pmatrix} = e^{\pm i\kappa t} v_{0,s} .
\] (97)
Spinors for finite momentum can be obtained from (97) by applying Lorentz boost in direction $\vec{\theta}$. Boost operator can again be decomposed into parts acting only on coordinates only and the part acting only on spin degrees of freedom.
\[
\langle \vec{x} | \psi^+_{p,s} \rangle = S(\vec{\theta}) \langle \vec{x} | \psi^+_{0,s} \rangle = S(\vec{\theta}) \left[ \frac{e^{\pm i\kappa t}}{(2\pi)^{3/2}} u_{0,s} \right] = S_{\text{coord}}(\vec{\theta}) \frac{e^{\pm i\kappa t}}{(2\pi)^{3/2}} S_{\text{spin}}(\vec{\theta}) u_{0,s} = S_{\text{coord}}(\vec{\theta}) \frac{e^{\pm i\kappa t} u_{ps}}{(2\pi)^{3/2}}
\] (98)
where we define spinors $u_{ps}$ and $v_{ps}$ to be the spinor obtained by boosting the rest frame spinors $u_{0,s}$ and $v_{0,s}$ to a frame where it has the momentum $\vec{p}$

$$u_{ps} \equiv S(\vec{\theta}) u_{0,s} , \quad v_{ps} \equiv S(\vec{\theta}) v_{0,s} .$$

If we require that the resulting state has the momentum $\vec{p}$ parallel to $\vec{\theta}$, we can again on the grounds of translational invariance in space-time conclude

$$\langle \vec{x}' | \psi_{ps}(t') \rangle = e^{\pm i \kappa'} \langle \vec{x}' | \psi^+_{ps} \rangle = \frac{e^{i \vec{p} \cdot \vec{x}'}}{(2\pi)^{3/2}} e^{\pm i \kappa' t'} u_{ps} .$$

Comparing equations (98) and (100) we get

$$S_{\text{coord}}(\vec{p}) e^{\pm i \kappa t} = e^{i \vec{p} \cdot \vec{x}} e^{\pm i \kappa' t'} .$$

To get the factor $e^{i \vec{p} \cdot \vec{x}}$ on the right hand side, sign in front of $\kappa$ must be positive and equal to particle energy in the rest frame, or in another hands $\kappa = m$. Then on the other side we have $\kappa' = E_{\vec{p}} = +\sqrt{p^2 + m^2}$.

$$\langle \vec{x} | \psi^+_{ps}(t) \rangle = \frac{e^{-i \vec{p} \cdot \vec{x}}}{(2\pi)^{3/2}} u_{ps}$$

This however determines the behavior of $\langle \vec{x} | \psi^-_{ps}(t) \rangle$ completely as well since they are both superpositions of same chiral wave functions

$$\langle \vec{x} | \psi^\pm_{ps}(t) \rangle = \langle \vec{x} | \psi^R_{ps}(t) \rangle \pm \langle \vec{x} | \psi^L_{ps}(t) \rangle$$

and therefore must have the same space-time dependent exponential

$$\langle \vec{x} | \psi^-_{ps}(t) \rangle = \frac{e^{-i \vec{p} \cdot \vec{x}}}{(2\pi)^{3/2}} v_{ps} .$$

The conclusion that both energies are positive is based on the requirement that the state boosted by $\vec{\theta}$ has the momentum parallel to $\vec{\theta}$. While this seems reasonable requirement, there’s nothing preventing us to require it to be antiparallel. That would lead to negative energies, and negative for all four solutions. But there’s no consistent way to have solutions with opposite energies as solutions of Dirac equation (18) do.

### 3.2 Parity equations for spinor representation

Zero momentum eigenfunctions satisfy

$$P \langle \vec{x} | \psi^+_{0,s}(t) \rangle = \langle \vec{x} | \psi^+_{0,s}(t) \rangle , \quad P \langle \vec{x} | \psi^-_{0,s}(t) \rangle = - \langle \vec{x} | \psi^-_{0,s}(t) \rangle$$

or for spinors

$$P_{\text{spin}} u_{0,s} = \gamma^0 u_{0,s} = u_{0,s} , \quad P_{\text{spin}} v_{0,s} = \gamma^0 v_{0,s} = - v_{0,s} .$$

In analogy with vector properties under parity, we’ll call solutions $u_{ps}$ axial spinors or pseudospinors, and solutions $v_{ps}$ polar spinors.

Now we may ask the question: in our frame these are zero momentum spinors; mirroring them to a fixed point reproduces themselves multiplied with $\pm 1$. In our frame this is parity operation; however, observer in a frame moving with some velocity will neither see zero momentum particle eigenstate nor
will mirroring that state to a point moving with the frame velocity look like parity to him. So how will that symmetry operation look to him?

At this point it’s convenient to apply unitary transformation to all matrices in spinor space which will make the parity operator diagonal and leave spin generators \(\vec{J}\) unchanged

\[
M_{ch} \to M_D = U^\dagger M_{ch} U \quad \psi_D = U^\dagger \psi_{ch} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

(107)

This gives us the original set of matrices used by Dirac to describe spinors. Spin eigenstates in rest frame are then

\[
\begin{align*}
  u_{0,s} &= \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}, \\
  v_{0,s} &= \begin{pmatrix} 0 \\ \chi_s \end{pmatrix}.
\end{align*}
\]

(108)

Boosting equation (106) to frame where particles have momentum \(\vec{p}\) we have

\[
S(\vec{p}) \gamma^0 S^{-1}(\vec{p}) S(\vec{p}) u_{0,s} = S(\vec{p}) u_{0,s} \quad S(\vec{p}) \gamma^0 S^{-1}(\vec{p}) S(\vec{p}) v_{0,s} = -S(\vec{p}) v_{0,s}.
\]

(109)

After evaluating

\[
S(\vec{p}) PS^{-1}(\vec{p}) = S(\vec{p}) \gamma^0 S^{-1}(\vec{p}) = \begin{pmatrix} E/m & -\vec{p} \cdot \vec{\sigma}/m \\ \vec{p} \cdot \vec{\sigma}/m & -E/m \end{pmatrix}
\]

(110)

In Dirac representation one can introduce \(\gamma\) matrices and express this in closed form as

\[
S(\vec{p}) PS^{-1}(\vec{p}) = \frac{\vec{p}}{m}
\]

(111)

so equation (109) becomes

\[
\frac{\vec{p}}{m} u_{ps} = u_{ps} \quad \frac{\vec{p}}{m} v_{ps} = -v_{ps}
\]

(112)

or

\[
(\vec{p} - m) u_{ps} = 0 \quad (\vec{p} + m) v_{ps} = 0.
\]

(113)

Making a superposition of these momentum eigenstates

\[
\begin{align*}
  \psi^+(x) &\equiv \int d^3 p \sum_s \langle \vec{x} | u_{ps} \rangle \langle u_{ps} | \psi^+ \rangle = \int \frac{d^3 p}{(2\pi)^{3/2} 2E_p} \sum_s e^{-ip \cdot x} u_{ps} b_{ps} = \begin{pmatrix} \varphi^+(x) \\ \chi^+(x) \end{pmatrix} \quad (\vec{p} - m) \\
  \psi^-(x) &\equiv -\int d^3 p \sum_s \langle \vec{x} | v_{ps} \rangle \langle v_{ps} | \psi^+ \rangle = \int \frac{d^3 p}{(2\pi)^{3/2} 2E_p} \sum_s e^{-ip \cdot x} v_{ps} d_{ps} = \begin{pmatrix} \varphi^-(x) \\ \chi^-(x) \end{pmatrix}
\end{align*}
\]

(114, 115)

one gets parity equations for spin 1/2

\[
(\vec{p} - m) \psi^+(x) = 0, \quad (\vec{p} + m) \psi^-(x) = 0.
\]

(116)

Quantity \(b_{ps}/2E_p = \langle u_{ps} | \psi^+ \rangle\) can be interpreted as amplitude for finding the state \(|\psi^+\rangle\) in momentum-spin eigenstate \(|u_{ps}\rangle\). Same holds for \(d_{ps}/2E_p = -\langle v_{ps} | \psi^- \rangle\).

First equation in (116) is what Dirac proposed as the relativistic equation of the first order which he hoped wouldn’t have negative energies. Since the spinor representation has 4 independent solutions, if polar spinors were to satisfy the Dirac equation, one artificially had to multiply the spin part of solution
for polar spinors with coordinate factor $\exp(+ip \cdot x)$ instead of $\exp(-ip \cdot x)$. In QFT this lead to solutions with negative energies which are clearly not present here.

In terms of $x$-space bispinors $\varphi^\pm(x)$ and $\chi^\pm(x)$ parity equations give us a set of coupled first order equations

\begin{align}
    i \frac{\partial \varphi^\pm(x)}{\partial t} &= \pm m \varphi^\pm(x) - i \vec{\sigma} \cdot \nabla \chi^\pm(x) \\
    i \frac{\partial \chi^\pm(x)}{\partial t} &= \mp m \chi^\pm(x) - i \vec{\sigma} \cdot \nabla \varphi^\pm(x). 
\end{align}

Complete wave function for $(1/2, 0) \oplus (0, 1/2)$ will be the sum of polar and axial part

\begin{align}
    \psi(x) &= \langle \vec{x} | \psi \rangle = \langle \vec{x} | \left( \int d^3p \sum_{s,\vec{p}} \frac{|w_{p,s,\vec{p}}\rangle \langle w_{p,s,\vec{p}}|}{\langle w_{p,s,\vec{p}}| w_{p,s,\vec{p}} \rangle} \right) |\psi\rangle \\
    &= \int d^3p \sum_s \left( \langle \vec{x} | u_{p,s} \rangle \langle u_{p,s} | \psi \rangle - \langle \vec{x} | v_{p,s} \rangle \langle v_{p,s} | \psi \rangle \right) \\
    &= \int \frac{d^3p}{(2\pi)^{3/2}2E_p} \sum_s e^{-ip \cdot x} (b_{ps}u_{ps} + d_{ps}v_{ps}) = \begin{pmatrix} \varphi^+(x) + \varphi^-(x) \\ \chi^+(x) + \chi^-(x) \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix}
\end{align}

in the massless limit $m \to 0$ satisfies the set of equations

\begin{align}
    \frac{\partial \varphi(x)}{\partial t} &= -\vec{\sigma} \cdot \nabla \chi(x) \\
    \frac{\partial \chi(x)}{\partial t} &= -\vec{\sigma} \cdot \nabla \varphi(x).
\end{align}

Independent solutions to this system of equations satisfy $\varphi(x) = \pm \chi(x)$ and correspond to separate $(1/2, 0)$ and $(0, 1/2)$ transformations. Interpretation of this result is given in section 5.

This same argument about parity in different frames applies to all representations of Lorentz group. The fact that equations can be expressed in relativistically-covariant way and that the equations are linear in both energy and momentum (or alternatively both time and space derivatives) are unique property of Dirac representation since only the anticommutators of generators in Dirac representation satisfy Clifford algebra as well as usual $SU(2)$-generators algebra.

Another thing worth mentioning here is the fact that these differential equations are a consequence of the relativistic mixing of space and time. Applying the same arguments to representations of Galilean group will not have any time derivatives since the time is the same in all Galilean frames.

### 3.3 Parity equations for spin 1 representations

Now lets see what parity symmetry yields for higher spin representations. There are two representations that have eigenstates with spin 1, $(1, 0) \oplus (0, 1)$ and $(1/2, 1/2)$. For $(1, 0) \oplus (0, 1)$ representation transformation matrix $S(\theta)$ is given by

\begin{equation}
    S(\theta) = e^{-i\vec{\theta} \cdot \vec{S}} = \exp \left( \begin{array}{cc}
        -\vec{\theta} \cdot \vec{S} & 0 \\
        0 & -\vec{\theta} \cdot \vec{S} \end{array} \right)
\end{equation}

where matrices $S$ are given explicitly in (317) or (322). The exponential of $\pm \vec{\theta} \cdot \vec{S}$ in spin representation

\begin{equation}
    \pm \vec{\theta} \cdot \vec{S} = \pm \begin{pmatrix}
        \theta_0 & \theta_- & 0 \\
        \theta_+ & 0 & \theta_- \\
        0 & \theta_+ & -\theta_0
    \end{pmatrix} \quad \theta_\pm = \frac{\theta_1 \pm \theta_2}{\sqrt{2}} \quad \theta_0 = \theta_3
\end{equation}
or in coordinate representation
\[
\pm \vec{\theta} \cdot \vec{S} = \pm \begin{pmatrix}
0 & \theta_3 & -\theta_2 \\
-\theta_3 & 0 & \theta_1 \\
\theta_2 & -\theta_3 & 0
\end{pmatrix}
\]  

(123)
calculated explicitly yields
\[
e^{\pm \vec{\theta} \cdot \vec{S}} = 1 \pm \frac{\sinh \theta}{\theta} \vec{\theta} \cdot \vec{S} + \frac{\cosh \theta - 1}{\theta^2} \left( \vec{\theta} \cdot \vec{S} \right)^2
\]

(124)
or
\[
e^{-i\vec{\theta} \cdot \vec{K}} = 1 + \frac{\sinh \theta}{\theta} (-i\vec{\theta} \cdot \vec{K}) + \frac{\cosh \theta - 1}{\theta^2} \left(-i\vec{\theta} \cdot \vec{K}\right)^2
\]

(125)
where we have used the fact that \((\vec{\theta} \cdot \vec{S})^3 = \theta^2 \left(\vec{\theta} \cdot \vec{S}\right)\). Repeating the same procedure for \((1/2, 1/2)\) representation yields formally the same result
\[
S(\theta) = e^{-i\vec{\theta} \cdot \vec{K}} = 1 + \frac{\sinh \theta}{\theta} (-i\vec{\theta} \cdot \vec{K}) + \frac{\cosh \theta - 1}{\theta^2} \left(-i\vec{\theta} \cdot \vec{K}\right)^2
\]

(126)
but with different set of generators \(\vec{K}\). In spin representation one has
\[
-i\vec{\theta} \cdot \vec{K} = \begin{pmatrix}
0 & -\theta_+ & \theta_0 & \theta_-
\\
-\theta_- & \theta_0 & 0 & \theta_+
\\
\theta_0 & \theta_2 & 0 & \theta_3
\\
\theta_3 & \theta_0 & 0 & \theta_1
\end{pmatrix}
\]

(127)
while in coordinate representation this is
\[
-i\vec{\theta} \cdot \vec{K} = \begin{pmatrix}
0 & \theta_1 & \theta_2 & \theta_3
\\
\theta_1 & 0 & 0 & \theta_3
\\
\theta_2 & 0 & 0 & \theta_3
\\
\theta_3 & 0 & 0 & \theta_1
\end{pmatrix}
\]

(128)
Since
\[
\cosh \theta = \frac{E}{m} \quad \sinh \theta = \frac{|\vec{p}|}{m} \quad \frac{\vec{\theta} \cdot \vec{S}}{\theta} = \frac{\vec{p} \cdot \vec{S}}{|\vec{p}|}
\]
\[
\cosh \frac{\theta}{2} = \sqrt{\frac{E + m}{2m}} \quad \sinh \frac{\theta}{2} = \sqrt{\frac{E - m}{2m}}
\]

(129)
we can express both equations as
\[
S(\theta) = e^{-i\vec{\theta} \cdot \vec{K}} = 1 + \frac{1}{m} \left(-i\vec{p} \cdot \vec{K}\right) + \frac{1}{m(E + m)} \left(-i\vec{p} \cdot \vec{K}\right)^2
\]

(130)
with proper interpretation of generators \(\vec{K}\). Using this to transform parity we get
\[
S(\theta)PS(-\theta) = e^{-i\vec{\theta} \cdot \vec{K}}Pe^{i\vec{\theta} \cdot \vec{K}} = P \left(1 - \frac{2E}{m^2} \left(-i\vec{p} \cdot \vec{K}\right) + \frac{2E}{m^2} \left(-i\vec{p} \cdot \vec{K}\right)^2\right)
\]

(131)
where we have used the fact that \( P \vec{K} = -\vec{K} P \) and \((-i\vec{p} \cdot \vec{K})^3 = p^2 (-i\vec{p} \cdot \vec{K})\). Switching to parity basis for \((1, 0) \oplus (0, 1)\) representation, equation (131) in block form becomes

\[
S(\theta)PS(-\theta) = \begin{pmatrix}
1 + \frac{2}{m^2}(\vec{p} \cdot \vec{S})^2 & \frac{2E}{m^2}(\vec{p} \cdot \vec{S}) \\
-\frac{2E}{m^2}(\vec{p} \cdot \vec{S}) & -\left(1 + \frac{2}{m^2}(\vec{p} \cdot \vec{S})^2\right)
\end{pmatrix}
\]  

(132)

Using the same notation for parity eigenstates as for \((1/2, 0) \oplus (0, 1/2)\) representation

\[
u_{ps} \equiv \psi^+_{ps} = \begin{pmatrix} \varphi^+_{ps} \\ \chi^+_{ps} \end{pmatrix} \quad \psi_{ps} \equiv \psi^-_{ps} = \begin{pmatrix} \varphi^-_{ps} \\ \chi^-_{ps} \end{pmatrix}
\]

(133)

where \(\varphi\) and \(\chi\) are now matrices with three rows, parity condition for particle in the rest frame

\[
P u_{0,s} = u_{0,s}, \quad P v_{0,s} = -v_{0,s}
\]

(134)

boosted to a frame where particle has momentum \(\vec{p}^\prime\) becomes

\[
e^{-i\vec{p}^\prime \cdot \vec{K}} e^{-i\vec{p} \cdot \vec{K}} e^{-i\vec{p}^\prime \cdot \vec{K}} \psi^\pm_{0,s} = \pm e^{-i\vec{p} \cdot \vec{K}} \psi^\pm_{0,s}
\]

(135)

Since by definition

\[
\psi^+_{ps} \equiv u_{ps} = e^{-i\vec{p} \cdot \vec{K}} u_{0,s} = e^{-i\vec{p} \cdot \vec{K}} u_{0,s} \quad \psi^-_{ps} \equiv v_{ps} = e^{-i\vec{p} \cdot \vec{K}} v_{0,s} = e^{-i\vec{p} \cdot \vec{K}} v_{0,s}
\]

(136)

we get the set of equations

\[
\begin{align}
\left[\frac{m^2 + (\vec{p} \cdot \vec{S})^2}{2}\right] \varphi^\pm_{ps} + (\vec{p} \cdot \vec{S}) E \chi^\pm_{ps} &= 0 \\
- (\vec{p} \cdot \vec{S}) E \varphi^\pm_{ps} - \left[\frac{m^2 + (\vec{p} \cdot \vec{S})^2}{2}\right] \chi^\pm_{ps} &= 0
\end{align}

(137)

(138)

Following the same logic as in the case of Dirac representation, we can construct the wave function \(\psi^+\) and \(\psi^-\) of parts with positive and negative parity by making the superposition of a complete set of eigenstates of given parity

\[
\psi^+(x) \equiv \int d^3 p \sum_s (\vec{x} | u_{ps} \rangle \langle u_{ps} | \psi^+) = \int \frac{d^3 p}{(2\pi)^{3/2} 2E} \sum_s e^{-ip \cdot x} b_{ps} u_{ps} \equiv \begin{pmatrix} \varphi^+(x) \\ \chi^+(x) \end{pmatrix}
\]

(139)

\[
\psi^-(x) \equiv \int d^3 p \sum_s (\vec{x} | v_{ps} \rangle \langle v_{ps} | \psi^-) = \int \frac{d^3 p}{(2\pi)^{3/2} 2E} \sum_s e^{-ip \cdot x} d_{ps} v_{ps} \equiv \begin{pmatrix} \varphi^-(x) \\ \chi^-(x) \end{pmatrix}
\]

(140)

Since \(\vec{S}\) in coordinate basis can be expressed as \(S^i_{jk} = -ie^{ijk}\), identifying components of column-matrix \(f\) with components of vector

\[
\vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}
\]

(141)

product \((\vec{p} \cdot \vec{S})_{jk} = -ip^i e^{ijk}\) can be expressed as

\[
(\vec{p} \cdot \vec{S}) f = -ip^i e^{ijk} f^k = i(\vec{p} \times \vec{f})_j = (\nabla \times \vec{f})_j
\]

(142)
which gives us coupled differential equations for matrices $\varphi^\pm(x)$ and $\chi^\pm(x)$ representing regular 3-vectors $\vec{\varphi}^\pm(x)$ and $\vec{\chi}^\pm(x)$

\[
\frac{1}{2} m^2 \varphi^\pm(x) + \nabla \times \left( \nabla \times \varphi^\pm(x) \right) + \nabla \times \left( \frac{\partial \chi^\pm(x)}{\partial t} \right) = 0
\] \hspace{1cm} (143)

\[-\nabla \times \left( \frac{\partial \varphi^\pm(x)}{\partial t} \right) - \frac{1}{2} m^2 \chi^\pm(x) - \nabla \times \left( \nabla \times \varphi^\pm(x) \right) = 0 \hspace{1cm} (144)
\]

In the limit $m \to 0$ mass term becomes negligible, and the total wave function

\[
\psi(x) = \psi^+(x) + \psi^-(x) = \left( \begin{array}{c} \varphi^+(x) + \varphi^-(x) \\ \chi^+(x) + \chi^-(x) \end{array} \right) \equiv \left( \begin{array}{c} \vec{b}(x) \\ \vec{e}(x) \end{array} \right)
\] \hspace{1cm} (145)

satisfies equations

\[
i \nabla \times \left( \nabla \times \vec{b}(x) - \frac{\partial \vec{e}(x)}{\partial t} \right) = 0
\]

\[
\nabla \times \left( \nabla \times \vec{e}(x) + \frac{\partial \vec{b}(x)}{\partial t} \right) = 0 \hspace{1cm} (146)
\]

where we have changed the notation to emphasize the similarity with Maxwell equations in vacuum. For very small but non-vanishing mass $m$, right-hand side of equations (146) will be proportional to $m^2$ and so the corrections to (145) will be of order $(m/E)^2$. Taking this mass to be below the experimental limit for the mass of the photon gives corrections suppressed by about 40 orders of magnitude, far beyond any experimental detection.

For $(1/2, 1/2)$ representation, generators $\vec{K}$ to be inserted in (131) in coordinate representation are given by

\[
- i \vec{p} \cdot \vec{K} = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & 0 & \vec{p} \cdot \vec{p} & 0 \\ p_2 & \vec{p} \cdot \vec{p} & 0 & \vec{p} \cdot \psi_0(x) \\ p_3 & 0 & \vec{p} \cdot \psi_0(x) & -\vec{p} \cdot \vec{p} \end{pmatrix}
\] \hspace{1cm} (147)

Parity states

\[
\psi^\pm_{ps} \equiv e^{-i \vec{\theta} \cdot \vec{K}} \psi^\pm_{0,s}
\] \hspace{1cm} (148)

obtained by boosting the rest-frame eigenstates

\[
\psi^\pm_{0,s} = u_{0,s} = \begin{pmatrix} \psi^0_{0,s} \\ 0 \end{pmatrix}, \hspace{1cm} \psi^\pm_{0,s} = v_{0,s} = \begin{pmatrix} 0 \\ \overline{\psi^0_{0,s}} \end{pmatrix}
\] \hspace{1cm} (149)

can be identified (again in coordinate representation) with components of four vector $\psi^\mu = (\psi^0, \vec{\psi})$

\[
\psi^\pm_{ps} = \begin{pmatrix} \psi^0_{ps} \\ \vec{\rho} \cdot \psi^\pm_{0,s} \end{pmatrix}
\] \hspace{1cm} (150)

Action of generators $\vec{K}$ on these states in terms of “regular” components of four-vector $\psi^\mu$ can be expressed as

\[
- i \vec{p} \cdot \vec{K} \psi(x) = \begin{pmatrix} \vec{p} \cdot \vec{\psi}(x) \\ \vec{p} \psi_0(x) \end{pmatrix} = \begin{pmatrix} -i \nabla \cdot \vec{\psi}(x) \\ -i \nabla \psi_0(x) \end{pmatrix}
\] \hspace{1cm} (151)

\[
(- i \vec{p} \cdot \vec{K})^2 \psi(x) = \begin{pmatrix} \vec{p}^2 \psi_0(x) \\ \vec{p} \left( \vec{p} \cdot \vec{\psi}(x) \right) \end{pmatrix} = \begin{pmatrix} -\nabla^2 \psi_0(x) \\ -\nabla \left( \nabla \cdot \vec{\psi}(x) \right) \end{pmatrix}
\] \hspace{1cm} (152)
Wave function constructed as the superposition of eigenstates

\[ A^+(x) \equiv \int d^3p \sum_s \langle \vec{x}|u_{ps}\rangle \langle u_{ps}|\psi^+ \rangle = \int \frac{d^3p}{(2\pi)^3/2E_p} \sum_s e^{-ip\cdot x} b_{ps}u_{ps} \equiv \left( \frac{A_0^+(x)}{A^+(x)} \right) \]  
(153)

\[ A^-(x) \equiv \int d^3p \sum_s \langle \vec{x}|v_{ps}\rangle \langle v_{ps}|\psi^- \rangle = \int \frac{d^3p}{(2\pi)^3/2E_p} \sum_s e^{-ip\cdot x} d_{ps}v_{ps} \equiv \left( \frac{A_0^-(x)}{A^-(x)} \right). \]  
(154)

again satisfies a set of coupled differential equations

\[ \left( \frac{1\pm 1}{2} m^2 - \nabla^2 \right) A_0^\pm (x) - \frac{\partial}{\partial t} \nabla \cdot \vec{A}^\pm (x) = 0 \]  
(155)

\[-\frac{\partial}{\partial t} \nabla A_0^\pm (x) - \frac{1\pm 1}{2} m^2 \vec{A}^\pm (x) - \nabla \left( \nabla \cdot \vec{A}^\pm (x) \right) = 0 \]  
(156)

Looking again either at massless limit or ultra-relativistic regime, mass terms can be neglected compared do other terms and we end up with familiar equations for the total wave function \( A^\mu(x) \)

\[-\nabla \cdot \left( \frac{\partial \vec{A}(x)}{\partial t} + \nabla A_0(x) \right) = 0 \]  
(157)

\[\nabla \left( \frac{\partial A_0(x)}{\partial t} + \nabla \cdot \vec{A}(x) \right) = \nabla \left( \partial_\mu A^\mu(x) \right) = 0. \]  
(158)

If one tries to construct \((1, 0) \oplus (0, 1)\) representation from two \((1/2, 1/2)\) representations, \(k^\mu\) and \(A^\mu\), one gets electric and magnetic fields \(\vec{E} = -\partial \vec{A}/\partial t - \nabla A_0, \ \vec{B} = \nabla \times \vec{A}\). First equation in (157) then becomes

\[\nabla \cdot \vec{E}(x) = 0 \]  
(159)

while the last of Maxwell equations (in vacuum) \(\nabla \cdot \vec{B}(x) = 0\) is satisfied automatically. So one gets Maxwell equations plus the gauge condition \(\partial_\mu A^\mu(x) = 0\) as a consequence of parity symmetry!

### 4 Lagrangians and conserved currents in RWFM

To find the conserved currents for given representation one needs the Lagrangian. Since parity symmetry already incorporates Dirac’s equation or Maxwell equations in the theory, it seems reasonable not to “force” them on the system as Euler-Lagrange equations. So one would need some other guiding light for finding the proper Lagrangian.

Let \(\varphi(x)\) belong to some representation of Lorentz group. In QFT it represents a quantum field while in RWFM it represents a wave function. All known Lagrangian densities [3, 4] can be written in the form

\[ L = \varphi^\dagger_A(x) P_{AB} O_{BC} \varphi_C(x) \]  
(160)

where capital Latin indices represents spin and all other “internal” indices and operators \(O\) generally have some constants, some derivative operators, some spin matrices, and for theories with internal symmetries also some matrices in internal symmetry spaces. Action can be then written as\[^4\]

\[ I = \int L \, d^4x = \int_0^\infty \langle \varphi | O | \varphi \rangle \, dt \]  
(161)

\[^4\text{From now on it will be understood that } \langle \varphi | \text{ and } \bar{\varphi} \text{ mean } (|\varphi\rangle)^\dagger P \text{ and } \varphi^\dagger P \text{ for any representation}\]
Symmetries of wave functions will give us conserved currents which after integration over whole space \( \mathbb{R}^3 \) give us conserved quantities. Those conserved quantities will again have a form

\[
Q^{\mu \ldots} = \int j^{0,\mu \ldots} d^3x = \int \varphi_A(x) J^{0,\mu \ldots \mu} \varphi_B(x) d^3x = \langle \varphi | J^{0,\mu \ldots \mu} | \varphi \rangle
\]

which in RWFM suggests the interpretation of quantities \( Q^{\mu \ldots} \) as (conserved) expectation values of some operators \( J^{0,\mu \ldots \mu} \). The question arises which operator \( O \) should we choose for a particular representation of Lorentz group that will give us “good” conserved currents?

There are two fundamental conserved “currents” which every representation must reproduce properly: energy-momentum tensor and angular momentum density tensor. Their conserved “charges” will give matrix elements of generators of generators, energy, momentum, spin, etc. If there are no interactions, total energy should be additive, i.e. a sum of energies of all orthogonal modes, for all momentum, spin and parity eigenstates. General relativity puts even stronger restriction: energy-momentum tensor should be symmetric. Since symmetric energy-momentum tensor also gives us the proper total energy, the symmetry requirement will be adopted as another fundamental requirement of RWFM. This puts enough restrictions on operator \( O \) to determine it completely. Let’s take a look at the translational invariance requirement:

\[
\varphi(x) \rightarrow \varphi(x') = \varphi(x + a) \rightarrow (1 + a_\alpha \partial^\alpha) \varphi(x) \quad \text{for infinitesimal } a.
\]

For Lagrangians in the form (160) this means the change in action will be

\[
\delta I = \int (\delta \bar{\varphi}_A O_{AB} \varphi_B + \bar{\varphi}_A O_{AB} \delta \varphi_B).
\]

Substituting the constant infinitesimal parameter \( a_\alpha \) with function \( a_\alpha(x) \) (163) will give us the change in action proportional to \( \partial_\mu a_\alpha \) which is just the energy momentum tensor

\[
\delta I = \int \theta^{\mu \alpha} \partial_\mu a_\alpha d^4x
\]

with

\[
\theta^{\mu \nu} = \frac{\partial L}{\partial \dot{\varphi}^\mu A} \partial_\nu \varphi_A + \partial_\nu \bar{\varphi}_A \frac{\partial L}{\partial \dot{\bar{\varphi}}^A}.
\]

Effectively, what we are doing is substituting \( \partial_\mu \varphi_A \) with \( \partial_\nu \varphi_A \), for every spin component. We already know that one of the indices comes from derivative operator acting on field (or it’s conjugate); if energy-momentum tensor is to be symmetric, the other Lorentz index must also come from derivative operator, or in another words, indices of derivative operators must be coupled together. Contracting derivative operator with lets say Dirac’s gamma matrix wouldn’t give us symmetric energy momentum tensor. This condition narrows the form of Lagrangian to

\[
\mathcal{L} = \bar{\varphi}_A \overleftarrow{\partial}_\mu O'_{AB} \overrightarrow{\partial}_\mu \varphi_B + \bar{\varphi}_A O''_{AB} \varphi_B
\]

where operators \( O' \) and \( O'' \) don’t depend on any derivative operators.

Invariance to Lorentz transformations will put additional restriction on Lagrangian. For infinitesimal transformation

\[
\varphi_A(x) \rightarrow \varphi'_A(x') = S_{AB}(\omega) \varphi_B(x) \rightarrow \left\{ 1 - \frac{i}{2} \omega_{\mu \nu} \left[ J^{\mu \nu}_{AB} + (x^\mu \partial^\nu - x^\nu \partial^\mu) \delta_{AB} \right] \right\} \varphi_B(x)
\]

with

\[
S_{AB} = \mathbb{I} + \frac{i}{2} \omega_{\mu \nu} \left[ J^{\mu \nu}_{AB} + (x^\mu \partial^\nu - x^\nu \partial^\mu) \delta_{AB} \right] + \mathcal{O}(\partial^2).
\]
we again get the change in action to be proportional to the angular momentum density tensor $J^\alpha,\mu\nu$

$$\delta I = \int J^\alpha,\mu\nu \frac{\partial \omega_{\mu\nu}}{2} d^4x$$  (169)

Looking at the expression for the

$$J^\alpha,\mu\nu = \frac{\partial L}{\partial \varphi_A} L^\mu\nu_{AB} \varphi_B + \bar{\varphi}_A L^\mu\nu_{AB} \frac{\partial L}{\partial \varphi_B}$$

$$= \bar{\varphi}_A \bar{\varphi}^\alpha O'_{AB} L^\mu\nu_{BC} \varphi_C + \bar{\varphi}_A \bar{\varphi}^\mu O'_{AB} \varphi^\nu$$  (170)

where $L^\mu\nu_{AB} = P_{AC} L^\mu\nu_{CD} P_{DB}$. If the integral of zeroth component must give us expectation values of rotation and boost generators, operator $O'$ must be a constant.

To get the Lorentz invariant Lagrangian, the last factor in (167) must satisfy

$$e^{\frac{i}{2} \omega_{\mu\nu} J^\mu\nu} e^{-\frac{i}{2} \omega_{\mu\nu} J^\mu\nu} = O''$$  (171)

for all $\omega_{\mu\nu}$, or in another words it must be a scalar. Therefore it will be proportional to unit matrix in spin space, so it also has to be just a number. What we’re left with is

$$\mathcal{L} = \bar{\varphi}_A \left( c_1 \bar{\partial} \cdot \partial + c_2 \right) \varphi_A .$$  (172)

Values of $c_1$ and $c_2$ are finally fixed by the requirement of onshellness $p^2 = m^2$, or in another words, Euler-Lagrange equations should give us Klein-Gordon equation for every component of the wave function. This finally yields the Lagrangian

$$\mathcal{L} = \varphi_A \left( \bar{i} \partial \cdot \partial - m^2 \right) \varphi_A$$  (173)

which is almost identical to the Klein-Gordon Lagrangian for (complex) scalar field

$$\mathcal{L}_{KG} = \phi^* \left( \bar{i} \partial \cdot \partial - m^2 \right) \phi$$  (174)

with the field $\phi^*$ replaced with the proper field $\bar{\varphi}_A$ to make the Lagrangian relativistically invariant. Note that this derivation doesn’t depend on the representation of Lorentz group which gives us unified description of all representations.

### 4.1 Energy-momentum tensor

Let’s now derive all conserved currents for the spinor representation explicitly. Starting from Klein-Gordon-like Lagrangian (173)

$$\mathcal{L} = \bar{\psi} \left( \bar{i} \partial \cdot \partial - m^2 \right) \psi$$  (175)

and requiring the translational invariance in space-time we get the (obviously symmetric) energy-momentum tensor

$$\theta^\mu\nu(x) = \bar{\psi}(x) \left( \bar{i} \partial^\nu \bar{i} \partial^\mu + \bar{i} \partial^\mu \bar{i} \partial^\nu \right) \psi$$  (176)
leading to the energy-momentum four-vector

\[
P^\mu = \int d^3x \left( i \vec{\alpha} \cdot \vec{\beta} + i \vec{\beta} \cdot \vec{\alpha} \right) \langle \vec{x} | \psi \rangle
\]

\[
= \int d^3x \sum_{s,r,p,p'} \langle \psi | \psi_{s,p,r} \rangle \langle \psi_{s,p,r} | \vec{x} \rangle \left( i \vec{\alpha} \cdot \vec{\beta} + i \vec{\beta} \cdot \vec{\alpha} \right) \langle \psi_{s,p,r} | \psi \rangle
\]

\[
= \int d^3p \sum_{s,r,p} \langle \psi | \psi_{s,p} \rangle \left( 2p_0 p^\mu \vec{p} \right) \langle \psi_{s,p} | \psi \rangle = \int \frac{d^3p}{(2\pi)^3/2E_p} p^\mu \sum_s \left( b^*_s b_{ps} - d^*_s d_{ps} \right)
\]

There is a negative sign here which comes from negative norm of negative parity states \( v_{ps} \) states and has nothing to do with the energies of the solutions which are positive for all solutions.

States \( |\psi_{s,p}\rangle \) form a basis of Hilbert space which enables us to express energy-momentum four-vector operator through its matrix elements

\[
\hat{P}^\mu = |\psi_{s,p}\rangle \langle \psi_{s,p} | \hat{P}^\mu |\psi_{s,p}\rangle \langle \psi_{s,p} |
\]

Since the state \( |\psi\rangle \) is arbitrary, we can read the energy-momentum operator from its matrix element (177)

\[
\hat{P}^\mu = \int d^3p \sum_{s,p} |\psi_{s,p}\rangle \left( 2p_0 p^\mu \right) \langle \psi_{s,p} | \langle \psi_{s,p} | \psi \rangle
\]

\[
= \int d^3p \sum_s \left( 2p_0 p^\mu \right) \left( |\psi_{s,p,+}\rangle \langle \psi_{s,p,+} | - |\psi_{s,p,-}\rangle \langle \psi_{s,p,-} | \right)
\]

Note that both equations (180) and (177) have negative sign for both energy operator and expectation value. Never the less, energy-momentum operator always gives positive result for energy

\[
\hat{P}^\mu |\psi_{q,r,+}\rangle = \int d^3p \sum_s \left( 2p_0 p^\mu \right) \left( |\psi_{s,p,+}\rangle \langle \psi_{s,p,+} | |\psi_{q,r,+}\rangle - |\psi_{s,p,-}\rangle \langle \psi_{s,p,-} | |\psi_{q,r,+}\rangle \right)
\]

\[
= \left( 2q_0 q^\mu \right) |\psi_{q,r,+}\rangle
\]

\[
\hat{P}^\mu |\psi_{q,r,-}\rangle = \int d^3p \sum_s \left( 2p_0 p^\mu \right) \left( |\psi_{s,p,+}\rangle \langle \psi_{s,p,+} | |\psi_{q,r,-}\rangle - |\psi_{s,p,-}\rangle \langle \psi_{s,p,-} | |\psi_{q,r,-}\rangle \right)
\]

\[
= \left( 2q_0 q^\mu \right) |\psi_{q,r,-}\rangle
\]

Expectation values have negative part since norms of those states are negative, not energies. Dividing with \( \langle \psi | \psi \rangle \) we get the quantity

\[
E = \frac{\langle \psi | \hat{P}^0 | \psi \rangle}{\langle \psi | \psi \rangle} > 0
\]

which is by definition positive for all states in Hilbert space. Comparing equation (177)

\[
P^\mu = \int \frac{d^3p}{2E_p} p^\mu \sum_s \left( b^*_s b_{ps} - d^*_s d_{ps} \right)
\]
with the expression (44) from the Dirac Lagrangian

\[ P^\mu = \int \theta^{0\mu} d^3x = \int \frac{d^3p}{2E_p} p^\mu \left( b^*_p b_p - d^*_p d_p \right). \]  

(185)

we can see that they are the same aside from the ordering of the \(dd^*\) term. Primary reason for introducing anticommutators was to make the expectation value (185) positive definite; there’s no reason to do that here since the negative sign of EV doesn’t imply negative energy.

### 4.2 Angular momentum density tensor

Requirement for infinitesimal rotational and boost invariance for spinor of Lorentz group gives us

\[ \psi(x) \rightarrow \begin{cases} 
1 - \frac{i}{2} \omega_{\mu\nu} \left[ \frac{\sigma^{\mu\nu}}{2} + \left( x^\mu \vec{\partial}^\nu - x^\nu \vec{\partial}^\mu \right) \right] \psi(x) 
\end{cases} \]  

and

\[ \bar{\psi}(x) \rightarrow \bar{\psi}(x) \begin{cases} 
1 + \frac{i}{2} \omega_{\mu\nu} \left[ \frac{\sigma^{\mu\nu}}{2} + \left( x^\mu \vec{\partial}^\nu - x^\nu \vec{\partial}^\mu \right) \right] \end{cases} \]  

(186)

(187)

where matrix \(\gamma^0\) is parity matrix \(P\) in spinor space. This invariance gives us conserved currents

\[ J^{\alpha,\mu\nu} = \bar{\psi}(x) \left( \frac{\sigma^{\mu\nu}}{2} + \left( x^\mu \vec{\partial}^\nu - x^\nu \vec{\partial}^\mu \right) \right) \psi \]  

(189)

which lead to conserved quantities

\[ J^{\mu\nu} = \int J^{0,\mu\nu} d^3x = J^{\mu\nu}_{\text{coord}} + J^{\mu\nu}_{\text{spin}} \]  

(191)

where coordinate or orbital part is defined to be the part proportional to unit matrix in spin space

\[ J^{\mu\nu}_{\text{coord}} = \int \bar{\psi} \left( \left( x^\mu \vec{\partial}^\nu - x^\nu \vec{\partial}^\mu \right) i \vec{\partial}^0 + i \vec{\partial}^0 \left( x^\mu \vec{\partial}^\nu - x^\nu \vec{\partial}^\mu \right) \right) \psi d^3x \]  

(192)

and the spin part the rest

\[ J^{\mu\nu}_{\text{spin}} = \int \bar{\psi} \left( i \vec{\partial}^0 \sigma^{\mu\nu} + \sigma^{\mu\nu} i \vec{\partial}^0 \right) \psi d^3x. \]  

(193)

Both parts can be decomposed in momentum eigenfunctions \(|u_{ps}\) and \(|v_{ps}\)

\[ J^{\mu\nu} = \int \left[ b^*_p \left( p^\mu i \partial^\nu_p - p^\nu i \partial^\mu_p \right) b_p - d^*_p \left( p^\mu i \partial^\nu_p - p^\nu i \partial^\mu_p \right) d_p \right] \frac{d^3p}{2E_p} \]  

(194)
Spin part of generators in the momentum states is

\[ J_{\text{spin}}^{\mu\nu} = \int \bar{\psi} \left\{ i \partial_0 \sigma^{\mu\nu} \frac{1}{2} + \sigma^{\mu\nu} \frac{1}{2} i \partial_0 \right\} \psi \, d^3x \]

\[ = \int \frac{d^3p}{2E_p} \sum_{s,r} \left( b_{ps}^* b_{ps} \bar{u}_{ps} \sigma^{\mu\nu} \frac{1}{2} u_{pr} + d_{ps}^* d_{pr} \bar{v}_{ps} \sigma^{\mu\nu} \frac{1}{2} v_{pr} \right) + b_{ps}^* d_{pr} u_{ps} \sigma^{\mu\nu} \frac{1}{2} v_{pr} + d_{ps}^* b_{pr} \bar{v}_{ps} \sigma^{\mu\nu} \frac{1}{2} u_{pr} \right) . \tag{195} \]

Mixed elements are not going to vanish; for boost part it is what we expect, but for rotational part it means trouble. This makes it hard to interpret states \( u_{ps} \) as particles and states \( v_{ps} \) as anti-particles; it would be strange to have rotations mix particles and anti-particles (electrons and positrons for example). States \( u_{ps} \) and \( v_{ps} \) are (by definition) obtained from rest frame states by applying the boost operator

\[ |u_{ps}\rangle = e^{-i\vec{\omega} \cdot \vec{K}} |u_{0,s}\rangle \quad |v_{ps}\rangle = e^{-i\vec{\omega} \cdot \vec{K}} |v_{0,s}\rangle \] \tag{196} \]

Since boosts and rotations don’t commute, so if we started with spin eigenstates \( |u_{0,s}\rangle \) and \( |v_{0,s}\rangle \), final states \( |u_{ps}\rangle \) and \( |v_{ps}\rangle \) won’t be spin eigenstates which is another reason why they shouldn’t be used to describe particles. However, they \( do \) form a basis for given momentum \( \vec{p} \) so as long as we work with unpolarized states it doesn’t really matter which basis we use. The question of spine eigenstates is finally addressed in section 5.

It’s again instructive to compare these results with the generators obtained from Dirac Lagrangian. Coordinate part

\[ J_{\text{coord}}^{\mu\nu} = \int \frac{1}{2} \bar{\psi} \left( x^\mu \frac{\partial}{\partial \nu} - x^\nu \frac{\partial}{\partial \mu} \right) \psi \, d^3x \]

\[ = \int \frac{E_p}{m} \left( b_{ps}^* \left( p^\mu i \partial_\nu^p - p^\nu i \partial_\mu^p \right) b_{ps} - d_{ps} \left( p^\mu i \partial_\nu^\mu - p^\nu i \partial_\nu \right) d_{ps}^* \right) \frac{d^3p}{2E_p} . \tag{197} \]

is again (almost) the same, differing by a factor \( E/m \) and the ordering of \( dd^* \) terms. Spin part of rotation generators is again quite similar to the Dirac case

\[ J_k = \epsilon_{ijk} J^{ij} = \epsilon_{ijk} \int \bar{\psi} \frac{\sigma^{ij}}{2} \psi \, d^3x \]

\[ = \int \frac{d^3p}{2E_p} \sum_{s,r} \left( u_{ps}^* \frac{1}{2} \sigma^k u_{pr} b_{ps} b_{pr} + v_{ps}^* \frac{1}{2} \sigma^k v_{pr} d_{ps} d_{pr} \right) + \epsilon_{ijk} \left( u_{ps}^* \frac{1}{2} \sigma^k v_{pr} \bar{b}_{ps} b_{pr} + v_{ps}^* \frac{1}{2} \sigma^k u_{pr} \bar{d}_{ps} d_{pr} \right) \tag{198} \]

but it doesn’t have time dependent \( \text{zitterbewegung} \) exponentials. If one were to perform a second quantization it would annihilate the vacuum as it should and it wouldn’t mix states with different number of particles under rotation. Furthermore, boost part of generators doesn’t vanish as in Dirac case and it has the proper functional dependence similar to rotation generators.

### 4.3 “Probability” current

Finally, since wave functions for all representations are complex numbers, and since all observables are real numbers, changing the phase of wave function should leave all observables unchanged. In another words, symmetry transformation

\[ \psi(x) \rightarrow e^{-i\alpha} \psi(x) = (1 - i\alpha) \psi(x) \tag{199} \]
creates the current
\[
 j^\mu(x) = 2\bar{\psi}(x)i\gamma^\mu\psi(x)
\]
\[
 = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{d^3q}{(2\pi)^{3/2}} \sum_{s,p} \sum_{r,p'} e^{-i(p-q)\cdot x} \times (b^*_p \bar{u}_{ps} + d^*_p \bar{v}_{ps}) (p + q)^\mu (b_{qr}u_{qr} + d_{qr}v_{qr}) .
\]  

Conserved quantity will be the integral of zeroth component
\[
 Q = \int j^0(x) d^3x = \langle \varphi | \bar{\psi} | \varphi \rangle = \int \frac{d^3p}{2E_p} \sum_s (b^*_p b_p - d^*_p d_p) .
\]  

Note that there is no zitterbewegung terms due to the orthogonality of states. In non-relativistic quantum mechanics zero-th component of the four-current is positive definite so one can interpret it as the probability density and the current as probability flux. Since the current is no longer neither positive definite nor bound, one must think twice before calling it the probability current.

One can again compare equation (201) with the equation for charge from Dirac’s equation (36) after the second integration and normal ordering
\[
 : Q : = \int : j^0(x) : d^3x = \int \frac{d^3p}{2E_p} \sum_s (b^*_p b_p - d^*_p d_p) .
\]

We get the same expression for charge, but the origin of the minus sign is different; here it comes from the negative norm, and there it comes from anticommutators. Space part of the total current
\[
 J^i = \int \frac{d^3p}{2E_p} \frac{1}{E_p} \left( p^i \sum_s [b^*_p b_p - d^*_p d_p] \right)
\]  

can be compared with the equivalent current from Dirac Lagrangian (37)
\[
 : J^i : = \int : j^i(x) : d^3x = \int \frac{d^3p}{2E_p} \frac{1}{E_p} \left( p^i \sum_s [b^*_p b_p + d_p d^*_p] + \sum_{s,r} \left[ \frac{ie^{2iE_p t}}{2m} \bar{u}_{ps} \sigma^0 v_{pr} b^*_p d^*_p - \frac{ie^{-2iE_p t}}{2m} \bar{v}_{ps} \sigma^0 u_{pr} d_p b^*_p \right] \right)
\]

Note that aside from zitterbewegung terms, current (203) is the same the normal ordered current from Dirac Lagrangian.

5 Matrix elements and particle interpretation

Let’s take another look at mixed matrix elements for rotation generators. It is most convenient to use Dirac’s representation; in this representation rotation generators and spinors are given by
\[
 J_i = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}
\]  

u_{ps} = \frac{1}{\sqrt{2m(E + m)}} \begin{pmatrix} (E + m)\chi_s \\ \bar{p} \cdot \bar{\sigma} \chi_s \end{pmatrix}, \quad v_{ps} = \frac{1}{\sqrt{2m(E + m)}} \begin{pmatrix} \bar{p} \cdot \bar{\sigma} \chi_s \\ (E + m)\chi_s \end{pmatrix} .
\]
Mixed matrix elements of rotation generators are then
\[
\bar{u}_{ps} \sigma_i \frac{1}{2} v_{pr} = \frac{1}{2m(E + m)} \left( (E + m)\chi_s - \chi_s \bar{p} \cdot \bar{\sigma} \right) \frac{1}{2} \left( \begin{array}{cc} \sigma_i & 0 \\ 0 & \sigma_i \end{array} \right) \left( \begin{array}{c} \bar{p} \cdot \bar{\sigma} \chi_r \\ (E + m) \chi_r \end{array} \right)
\]
\[
= \frac{1}{2m} \left( \chi_s \sigma_i \bar{p} \cdot \bar{\sigma} \chi_r - \chi_s \bar{p} \cdot \bar{\sigma} \sigma_i \chi_r \right) \frac{2}{m} \chi_s (\bar{p} \times \bar{\sigma})_i \chi_r
\]
\[
\bar{v}_{ps} \sigma_i \frac{1}{2} u_{pr} = \frac{-2}{m} \chi_s (\bar{p} \times \bar{\sigma})_i \chi_r
\]
which will always have at least one non-vanishing component. This would imply that for example applying rotation to pure electron state would give us mixture of electrons and positrons
\[
\langle \text{positron} | e^{-i\vec{\omega} \cdot \vec{J}} | \text{electron} \rangle \neq 0
\]
In the QFT this problem didn’t exist since the second quantization would just “sweep it under the carpet” by multiplying the non-vanishing elements with either two creation or two destruction operators (which of course produced another problems). This cannot be done here so the only thing we can do is revising our interpretation of states \( u_{ps} \) and \( v_{ps} \). As it was said earlier, since they are defined as states obtained by boosting the rest frame spin eigenstates, obviously they are not spin eigenstates at all. In non-relativistic quantum mechanics spin and momentum operators commute so proper spin eigenstates (in parity representation) are constructed as
\[
w^+_ps = e^{-ip \cdot x} \left( \begin{array}{c} \chi_s \\ 0 \end{array} \right), \quad w^-_{ps} = e^{-ip \cdot x} \left( \begin{array}{c} 0 \\ \chi_s \end{array} \right).
\]
We can think about them as states obtained by applying only coordinate part of the boost operator (which commutes with spin generators) to rest-frame spin eigenstates. Particle wave function can then be decomposed as
\[
\psi(x) = \langle \vec{x} | \psi \rangle = \langle \vec{x} | \left( \int d^3p \sum_{s,p} w_{ps,p} \langle w_{ps,p} | \psi \rangle \right) | \psi \rangle
\]
\[
= \int d^3p \sum_s \left( \langle \vec{x} | w^+_p,s \rangle \langle w^+_p,s | \psi \rangle - \langle \vec{x} | w^-_{p,s} \rangle \langle w^-_{p,s} | \psi \rangle \right)
\]
\[
= \int \frac{d^3p}{(2\pi)^{3/2} 2E_p} \sum_s e^{-ip \cdot x} \left( a^+_p w^+_p + a^-_p w^-_p \right)
\]
which is the same as the decomposition in \( u, v \) states with the substitution \( b \rightarrow a^+, d \rightarrow a^-, u \rightarrow w^+ \) and \( v \rightarrow w^- \). Their mixed matrix elements of rotation generators all vanish by their construction
\[
\langle w^+_p,s | \vec{J} | w^-_{p',r} \rangle \sim \left( \begin{array}{c} \chi_s \\ 0 \end{array} \right) \left( \begin{array}{cc} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{array} \right) \left( \begin{array}{c} 0 \\ \chi_r \end{array} \right) = 0
\]
\[
\langle w^-_{p,s} | \vec{J} | w^+_{p',r} \rangle \sim \left( \begin{array}{c} 0 \\ \chi_s \end{array} \right) \left( \begin{array}{cc} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{array} \right) \left( \begin{array}{c} \chi_r \\ 0 \end{array} \right) = 0.
\]
which implies
\[
\langle w_{ps}^\pm | e^{-i\vec{\omega} \cdot \vec{J}} | w_{pr}^\mp \rangle = 0
\]
as it should. Since boost generators aren’t diagonal in this representation, they will mix states of different parity

\[ \langle w^+_{ps} | \vec{K} | w^-_{pr} \rangle \sim (\chi_s \quad 0) \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} (0 \quad \chi_r) \neq 0 \]  

(215)

\[ \langle w^-_{ps} | \vec{K} | w^+_{pr} \rangle \sim (0 \quad \chi_s) \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} (\chi_r \quad 0) \neq 0 \]  

(216)

This implies that Lorentz boosts mix positive and negative parity and spin eigenstates

\[ \langle w^\pm_{ps} | e^{-i\vec{\omega} \cdot \vec{K}} | w^\mp_{pr} \rangle \neq 0 \]  

(217)

This phenomenon is already known from \((1, 0) \oplus (0, 1)\) representation where Lorentz boosts mix electric and magnetic fields. In fact, it’s a fundamental property of every \((j, 0) \oplus (0, j)\) representation which tells us that axial and polar states will exist and mix for every spin and can’t be separated.

When negative parity states are interpreted as wave functions of antiparticles, it also offers a possible and quite intriguing explanation why anti-particles have opposite quantum numbers from particles: all states (both polar and axial) will have the same eigenvalues for some “internal” symmetry generators like for example electric charge

\[ \hat{Q}_i = \int j^0(x) d^3x \]

(218)

but where that operator is coupled to whatever field through it’s (conserved) current

\[ \langle \mathcal{H}_I \rangle = \langle \psi^\pm | j^{\mu} \cdots | \psi^\pm \rangle \mathcal{O}_\mu (\text{some fields}) \sim q_i \left( \sum_{\vec{p}, s} |a^+_{ps}|^2 p^\mu \mathcal{O}_\mu - |a^-_{ps}|^2 p^\mu \mathcal{O}_\mu \right) \]  

(219)

**expectation values** of states with opposite parities will have relative minus sign which change the sign of the interaction Hamiltonian which we used to interpret as opposite quantum numbers. In the classical limit this relative minus sign leads to attractive or repulsive force. This is a property of states themselves, not the operators \(\hat{Q}_i\). Since every interaction of fermions in standard model is of the form (219), what we observe is the difference in sign of the energy and interpret it as opposite quantum number; in this model it comes from opposite parity instead.

### 5.1 Massless particles

Finally, a few words about massless (or equivalently ultra-relativistic) limit. In section 3 it was shown that in massless limit whole wave function satisfies a system of coupled differential equations. In the case of spinor representation these equations are

\[ i \frac{\partial}{\partial t} \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} = \begin{pmatrix} \vec{\sigma} \cdot (-i \nabla) & 0 \\ 0 & \vec{\sigma} \cdot (-i \nabla) \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} \]  

(220)

with solutions

\[ \psi_{R,L} = \begin{pmatrix} \varphi_{R,L}(x) \\ \pm \varphi_{R,L}(x) \end{pmatrix} \]  

(221)

corresponding to separate \((1/2, 0)\) and \((0, 1/2)\) transformations. Now, regardless of which one we choose, equations (220) become

\[ i \frac{\partial}{\partial t} \varphi_{R,L}(x) = \vec{\sigma} \cdot (-i \nabla) \varphi_{R,L}(x) \]  

(222)
With the interpretation of the previous subsection, we can decompose wave functions \( \varphi_{R,L}(x) \) in momentum and spin eigenstates in chiral basis

\[
\varphi_{R,L}(x) = \int \frac{d^3 p}{(2\pi)^{3/2} 2E_p} \sum_s e^{-ip \cdot x} a_{ps}^{R,L} w_{ps}^{R,L}
\]

which transforms the equation (220) to

\[
w_{ps}^{R,L}(x) = \frac{\vec{p} \cdot \vec{S}}{E_p} w_{ps}^{R,L}(x) = \Sigma w_{ps}^{R,L}(x).
\]

where the product of translation and rotation generators \( \Sigma \equiv \vec{p} \cdot \vec{S}/|\vec{p}| \) is nothing but helicity operator. Since all basis states are positive helicity eigenstates, the total wave function as a superposition of these states will also be a positive helicity eigenstate. However, just as in the case of the zeroth component of conserved currents, expectation values for particle and antiparticle states will have opposite signs due to the definition of scalar product

\[
\langle \psi | \Sigma | \psi \rangle \sim \left( \sum_{\vec{p},s} |a_{ps}^+|^2 - |a_{ps}^-|^2 \right)
\]

In nature neutrinos have been observed to have only positive helicity while antineutrinos have only negative helicities. This is just the matter of convention since we could have chosen to assign names “particle” and “antiparticle” in the reverse order and the parity operator is defined up to a sign anyway. What is physically important is that in the massless limit particles become helicity eigenstates with opposite expectation values.

Analog thing happens in \( (1,0) \oplus (0,1) \) representation as well; in massless limit electric and magnetic field satisfy equations (146). In terms of particle eigenstates this becomes

\[
\begin{align*}
\left( \hat{p} \cdot \vec{S} \right) \left[ \left( \hat{p} \cdot \vec{S} \right) \varphi_{ps}^{\pm} + E \chi_{ps}^{\pm} \right] &= 0 \\
\left( \hat{p} \cdot \vec{S} \right) \left[ \left( \hat{p} \cdot \vec{S} \right) \chi_{ps}^{\pm} - E \varphi_{ps}^{\pm} \right] &= 0
\end{align*}
\]

which again has solutions \( w_{ps}^{R,L} \) corresponding to \( \chi_{ps}^{\pm} = \pm \varphi_{ps}^{\pm} \) for separate \( (1,0) \) and \( (0,1) \) transformations. These solutions in terms of fields \( \vec{E} \) and \( \vec{B} \) are \( \vec{E} = \pm i \vec{B} \), consistent with the result familiar from the classical electrodynamics \( \vec{E} = i \vec{B} \). In this framework, one can interpret both the Dirac and Maxwell equations as a statement about parity properties of particles in the massless limit.

\section{Summary and Conclusions}

Symmetry treatment of non-relativistic quantum mechanics is generalized to include non-unitary representations of Lorentz group by redefining the scalar product of states in Hilbert space to make it relativistically invariant. This inevitably leads to states with negative norm. However, with this definition of scalar product, it is shown that superposition principle and orthogonality of quantum mechanics leads to the conclusion that all energies are positive. Furthermore, it is shown that the treatment of parity in different Lorentz frames leads to Dirac equation (for spinor representation) and to (sourceless) Maxwell equations for vector fields. It is demonstrated that if one requires “proper” behavior of Noether currents corresponding to translations, rotations and boosts, form of Lagrangian is determined completely for
any representation of Lorentz group. The resulting theory is just the “ordinary” single particle quantum mechanics, but now relativistically invariant. This theory doesn’t have negative energies, doesn’t show zitterbewegung-like effects, has clear transformation properties under Lorentz transformations as well as clear interpretation of physical particle states as momentum, spin and parity eigenstates. Continuity current density is proportional to the energy density and momentum density, just like in non-relativistic quantum mechanics, suggesting the possible probabilistic interpretation. One has to be careful and think twice before doing just that since norms are no longer positive definite which could potentially lead to trouble. While this might look a bit discouraging, one has to remember that states with negative norms are unavoidable in covariant formulation of QFT as well. Although that may seem to be the problem with the theory, it does also offer a nice interpretation why particles and antiparticles have opposite quantum numbers or why neutrinos always appear in nature with positive helicity and antineutrinos with negative. Due to the length of the subject, discussion of multiparticle states and the theory of interactions will be presented in a separate paper.

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A General properties of Lorentz group

Special theory of relativity requires that the speed of light $c$ remains the same in all inertial frames. Mathematically this means the 4-distance between 2 real 4-vectors

$$s = (t_2 - t_1)^2 - (\vec{x}_2 - \vec{x}_1)^2$$

must remain the same in all inertial frames. In quantum physics we have to deal with complex representations as well. If we write complex 4-vector $a^\mu$ in matrix notation as $|a\rangle$, with

$$|a\rangle \equiv \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

then the invariant quantity isn’t

$$\langle a | a \rangle = a_0^* a_0 + a_1^* a_1 + a_2^* a_2 + a_3^* a_3$$

but

$$a_\mu^* a^\mu = a_0^* a_0 - a_1^* a_1 - a_2^* a_2 - a_3^* a_3$$

where $a_\mu$ is defined to be $a^\mu$ multiplied by 4-tensor called metric tensor

$$g_{\mu\nu} = \text{diag}\{1, -1, -1, -1\}.$$
Pure rotations will mix only the space components of 4-vector

\[ |a'⟩ = S(ω) |a⟩ \]

so they can be represented in block form as

\[
S(ω) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = e^{-i\vec{ω} \cdot \vec{J}}
\]

with rotation generators

\[
J_x = i \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad
J_y = i \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad
J_z = i \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Pure Lorentz boosts \( S(ω) = e^{-i\vec{ω} \cdot \vec{K}} \) mix space and time coordinates

\[ |a'⟩ = S(ω) |a⟩ \]

and have generators

\[
K_x = i \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
K_y = i \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
K_y = i \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

They can be put together in 4D notation with \( J^{0i} \equiv K_i \) and \( J^{ij} \equiv \epsilon_{ijk} J_k \). Commutation relations

\[
[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k
\]

in 4D notation become

\[
[J^{\mu\nu}, J^{\alpha\beta}] = -i \left( g^{\mu\alpha} J^{\nu\beta} + g^{\nu\beta} J^{\mu\alpha} - g^{\mu\beta} J^{\nu\alpha} - g^{\nu\alpha} J^{\mu\beta} \right)
\]

and the finite Lorentz transformations can be written as

\[
S(ω) = e^{-iω_{μν} J^{μν}}.
\]

Up to this point it wasn’t important if vectors depend on coordinates or not. Generators (236) and (238) just mix different components of vectors. If those vectors do depend on coordinates, then those coordinates will have to be transformed as well. For state \( |a(x)⟩ \) we have

\[ |a'(x')⟩ = S(ω) |a(x)⟩ \]
which for infinitesimal \( \omega \) becomes

\[
|a'(x')\rangle = (1 - i\omega_{\mu\nu} J^{\mu\nu}) |a(x)\rangle = (1 - i\omega_{\mu\nu} \{ J^{\mu\nu} + x^{\mu} i \partial^{\nu} - x^{\nu} i \partial^{\mu}\}) |a(x)\rangle.
\] (243)

The first \( J^{\mu\nu} \) term will be called spin part of generators and the second coordinate part. It is trivial to see that spin and coordinate parts of generators commute and using that fact to explicitly show that the whole Lorentz transformation operator \( L^{\mu\nu} \) satisfies the same commutation relations (240).

Equation (229) will also be invariant to the translations in space-time

\[
x^{\mu} \rightarrow x^{\mu} + b^{\mu}
\] (244)

How should that affect vector \(|a\rangle\)? If it doesn’t depend on the coordinates \( x^{\mu} \) it should obviously be left unchanged by the translation of coordinate system. On the other hands, if it does depend on the coordinates, it should be just the same vector in new coordinates

\[
|a(x)\rangle \rightarrow |a'(x')\rangle = |a(x + b)\rangle.
\] (245)

This obviously transforms each component of the vector \(|a(x)\rangle\) independently. Expanding each component in Taylor series and keeping only the first term, we get the infinitesimal transformation

\[
|a(x)\rangle \rightarrow |a'(x')\rangle = e^{-ib^{\mu} P_{\mu}} |a(x)\rangle = (1 + b^{\mu} \partial_{\mu}) |a(x)\rangle
\] (246)

which give us the generator of transformations

\[
P^{\mu}(x) = i \partial^{\mu}.
\] (247)

It is important to notice that there is no spin part of this generator; it has only coordinate part.

A.1 Parity

We can easily see that operator with the property

\[
P \vec{a} = -\vec{a}
\] or \[
P \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}
\] (248)

leave (229) invariant as well. In matrix form, it can be written as

\[
P = \mathcal{P}_{\mu}^{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\] (249)

By direct multiplication we can show that

\[
P \vec{J} P = \vec{J} \quad P \vec{K} P = -\vec{K}
\] (250)

or in 4D notation

\[
P J^{\mu\nu} P = \mathcal{P}_{\alpha}^{\mu} \mathcal{P}_{\beta}^{\nu} J^{\alpha\beta}
\] (251)
In mathematical terms, six generators of Lorentz group can be decomposed as a direct product of two $SU(2)$ groups. Vectors will then belong to a direct product space of these two $SU(2)$ groups. If we use

\[ \vec{J} = \vec{A} \otimes 1 + 1 \otimes \vec{B} \quad \vec{K} = i \left( \vec{A} \otimes 1 - 1 \otimes \vec{B} \right) \]  

(252)

we can re-express (239) as

\[
\begin{align*}
[J_i, J_j] &= [A_i \otimes 1, A_j \otimes 1] + [1 \otimes B_i, A_j \otimes 1] + [A_i \otimes 1, 1 \otimes B_j] + [1 \otimes B_i, 1 \otimes B_j] \\
&= [A_i \otimes 1, A_j \otimes 1] + [1 \otimes B_i, 1 \otimes B_j] \\
&= [A_i, A_j] \otimes 1 + 1 \otimes [B_i, B_j] \\
[K_i, K_j] &= i^2 ([A_i \otimes 1, A_j \otimes 1] - [1 \otimes B_i, A_j \otimes 1] - [A_i \otimes 1, 1 \otimes B_j] \\
&+ [1 \otimes B_i, 1 \otimes B_j]) = i^2 ([A_i \otimes 1, A_j \otimes 1] + [1 \otimes B_i, 1 \otimes B_j]) \\
&= -([A_i, A_j] \otimes 1 + 1 \otimes [B_i, B_j]) \\
[J_i, K_j] &= i ([A_i \otimes 1, A_j \otimes 1] - [1 \otimes B_i, A_j \otimes 1] + [A_i \otimes 1, 1 \otimes B_j] \\
&= [1 \otimes B_i, 1 \otimes B_j]) = i ([A_i \otimes 1, A_j \otimes 1] - [1 \otimes B_i, 1 \otimes B_j]) \\
&= i ([A_i, A_j] \otimes 1 - 1 \otimes [B_i, B_j]) .
\end{align*}
\]

(253)

(254)

(255)

We can easily see that equations (239) are satisfied if both $A$ and $B$ satisfy $SU(2)$ algebra

\[ [A_i, A_j] = i\epsilon_{ijk} A_k \quad \text{and} \quad [B_i, B_j] = i\epsilon_{ijk} B_k . \]

(256)

General representation of Lorentz group will be labeled by two "spin" degrees of freedom $(j, j')$. Note that the first equation in (252) is identical to rules for addition of spin in non-relativistic quantum mechanics. However, this "direct product" isn’t a real direct product since parity can’t be expressed as a direct product of two $SU(2)$ operators. To show that, let’s assume the contrary. Parity would then be

\[ P = P_1 \otimes P_2 . \]

(257)

Since $P^\dagger = P^{-1} = P$, we have to have

\[ P_1 P_1 = P_2 P_2 = 1 . \]

(258)

Let’s assume operators $\vec{J}$ and $\vec{K}$ are irreducible. Inserting (257) into (250)

\[
\begin{align*}
P \vec{J} P &= (P_1 \otimes P_2)(\vec{A} \otimes 1 + 1 \otimes \vec{B})(P_1 \otimes P_2) \\
P \vec{K} P &= (P_1 \otimes P_2)i(\vec{A} \otimes 1 - 1 \otimes \vec{B})(P_1 \otimes P_2)
\end{align*}
\]

(259)

(260)

we get the system of equations

\[
\begin{align*}
(P_1 \vec{A} P_1) \otimes 1 + 1 \otimes (P_2 \vec{B} P_2) &= \vec{A} \otimes 1 + 1 \otimes \vec{B} \\
(P_1 \vec{A} P_1) \otimes 1 - 1 \otimes (P_2 \vec{B} P_2) &= -\vec{A} \otimes 1 + 1 \otimes \vec{B}
\end{align*}
\]

(261)

(262)

which would imply

\[
\begin{align*}
(P_1 \vec{A} P_1) \otimes 1 &= 1 \otimes \vec{B} \quad \text{and} \quad 1 \otimes (P_2 \vec{B} P_2) = \vec{A} \otimes 1
\end{align*}
\]

(263)

which is clearly impossible. So, inserting decomposition (252) into commutation relations (250) we get

\[ P(\vec{A} \otimes 1)P = 1 \otimes \vec{B} \quad P(1 \otimes \vec{B})P = \vec{A} \otimes 1 . \]

(264)
Since parity exchanges $A$ and $B$ eigenstates in $(a, b)$ representations, only combinations $(j, j)$ will be invariant under parity. However, even if operators $\vec{J}$ and $\vec{K}$ can be decomposed in a direct sum of operators

\[ \vec{J} = \vec{J}_1 \oplus \vec{J}_2 = \begin{pmatrix} \vec{J}_1 & 0 \\ 0 & \vec{J}_2 \end{pmatrix}, \quad \vec{K} = \vec{K}_1 \oplus \vec{K}_2 = \begin{pmatrix} \vec{K}_1 & 0 \\ 0 & \vec{K}_2 \end{pmatrix}, \]

representation of Lorentz group will be irreducible as long as at least one operator can’t be decomposed in a direct sum. If operators $\vec{J}_1$ and $\vec{K}_1$ belong to the $(a_1, b_1)$ representation and $\vec{J}_2$ and $\vec{K}_2$ belong to the $(a_2, b_2)$, then their direct sums will also satisfy (239). In block-diagonal form requirement $P^\dagger = P$ gives us

\[ P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^\dagger & P_{22} \end{pmatrix} \]

with matrices $P_{11}$ and $P_{22}$ Hermitean. Adding the condition $PP = 1$ gives us

\[ PP = \begin{pmatrix} P_{11}^2 + P_{12}P_{22}^\dagger + P_{12}^\dagger P_{22} & P_{11}P_{12} + P_{12}^\dagger P_{22} \\ P_{12}^\dagger P_{11} + P_{22}P_{12} & P_{12}^\dagger P_{12} + P_{22}^2 \end{pmatrix} = 1. \]

This has two obvious solutions; first one is $P_{12} = 0$, $P_{11}^2 = 1$, $P_{22}^2 = 1$. In this case the parity is a direct sum of parities, so this solution doesn’t yield irreducible representation of $SO(1, 3)$. The other solution is $P_{12}P_{12}^\dagger = P_{12}^\dagger P_{12} = 1$, $P_{11} = P_{22} = 0$. After reinserting this back into (250) we have

\[ \begin{pmatrix} 0 & P_{12} \\ P_{12}^\dagger & 0 \end{pmatrix} \begin{pmatrix} \vec{J}_1 & 0 \\ 0 & \vec{J}_2 \end{pmatrix} \begin{pmatrix} 0 & P_{12} \\ P_{12}^\dagger & 0 \end{pmatrix} = \begin{pmatrix} P_{12}\vec{J}_2P_{12}^{-1} & 0 \\ 0 & P_{12}^{-1}\vec{J}_1P_{12} \end{pmatrix} = \vec{J} \]

(268)

\[ \begin{pmatrix} 0 & P_{12} \\ P_{12}^\dagger & 0 \end{pmatrix} \begin{pmatrix} \vec{K}_1 & 0 \\ 0 & \vec{K}_2 \end{pmatrix} \begin{pmatrix} 0 & P_{12} \\ P_{12}^\dagger & 0 \end{pmatrix} = \begin{pmatrix} P_{12}\vec{K}_2P_{12}^{-1} & 0 \\ 0 & P_{12}^{-1}\vec{K}_1P_{12} \end{pmatrix} = -\vec{K}. \]

(269)

From this we can see that representations $(a_1, b_1)$ and $(a_2, b_2)$ have to be of a same dimension. After decomposing $\vec{J}_{1,2}$ and $\vec{K}_{1,2}$ into $SU(2)$ products, we get

\[ P_{12}^{-1}(\vec{A}_1 \otimes 1_2)P_{12} = 1_2 \otimes \vec{B}_2 \quad P_{12}(1_2 \otimes \vec{B}_2)P_{12}^{-1} = \vec{A}_1 \otimes 1_1 \]

(270)

\[ P_{12}(\vec{A}_2 \otimes 1_2)P_{12}^{-1} = 1_1 \otimes \vec{B}_1 \quad P_{12}^{-1}(1_1 \otimes \vec{B}_1)P_{12} = \vec{A}_2 \otimes 1_2 \]

(271)

which will be parity-invariant only if $a_1 = b_2$ and $b_1 = a_2$. Direct sums of three or more $SU(2) \otimes SU(2)$ representations will produce only representations of $SO(1, 3)$ reducible to direct sum of $(j, j)$ and $(j, j') \oplus (j', j)$ representations.

## B Representations of Lorentz group

Since all representations of the same type have some similarities, they will also have some similar properties which can be derived and studied jointly.

### B.1 $(j, 0) \oplus (0, j)$ representations

Starting point for these representations is equation (252)

\[ \vec{J} = \vec{A} \otimes 1 + 1 \otimes \vec{B} \quad \vec{K} = i\vec{A} \otimes 1 - i1 \otimes \vec{B}. \]

(272)
By definition, all finite transformations belonging to spin 0 representation must map to identity operator

\[ e^{-i\vec{\omega} \cdot \vec{B}} |0 0\rangle = |0 0\rangle \]  

(273)

which implies that generators \( \vec{B} \) must annihilate \( |0 0\rangle \) state.

\[ \vec{B} |0 0\rangle \equiv 0 . \]  

(274)

Surviving part of generators will then be

\[ \vec{J} = \vec{A} \otimes 1 , \quad \vec{K} = i\vec{A} \otimes 1 \]  

(275)

acting on states \( |jm\rangle \otimes |0 0\rangle \). Vector \( |0 0\rangle \) belongs to scalar representation of Lorentz group which is 1-dimensional, so the direct product \( |jm\rangle \otimes |0 0\rangle \) is essentially the same as the vector \( |jm\rangle \) and so we can disregard it

\[ |jm\rangle \otimes |0 0\rangle \rightarrow |jm\rangle . \]  

(276)

Therefore, we can write the matrices for this representation as

\[ \vec{J} \rightarrow \vec{S} , \quad \vec{K} \rightarrow i\vec{S} \]  

(277)

where matrices \( \vec{S} \) are \( 2j + 1 \)-dimensional generator matrices of Lorentz group from appendix C.1.

Similar holds for \((0, j)\) representation only here \( \vec{A} \) annihilates the states and \( \vec{B} \) gives us the generators. From equation (272) we can see that the only difference will be in the sign of \( \vec{K} \) generator

\[ \vec{J} \rightarrow \vec{S} , \quad \vec{K} \rightarrow -i\vec{S} , \quad |0 0\rangle \otimes |jm\rangle \rightarrow |jm\rangle \]  

(278)

Complete generators in block-matrix form will then be

\[ \vec{J} = \begin{pmatrix} \vec{S} & 0 \\ 0 & \vec{S} \end{pmatrix} , \quad \vec{K} = i \begin{pmatrix} \vec{S} & 0 \\ 0 & -\vec{S} \end{pmatrix} \]  

(279)

From (270) and (271) we can see that the parity operator must then be

\[ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

(280)

Parity eigenstates (for zero momentum) in spin space are

\[ u_{0s} = \begin{pmatrix} |jm\rangle \\ |jm\rangle \end{pmatrix} , \quad v_{0s} = \begin{pmatrix} |jm\rangle \\ -|jm\rangle \end{pmatrix} . \]  

(281)

For finite momentum we get spinors \( u_{ps} \) and \( v_{ps} \) by applying Lorentz boost to rest-frame spinors

\[ u_{ps} \equiv S(\vec{\omega})u_{0s} , \quad v_{ps} \equiv S(\vec{\omega})v_{0s} \]  

(282)

where \( \vec{\omega} \) points in the direction of \( \vec{p} \) and has the appropriate magnitude. Note however that they are not spin eigenstates. For zero momentum spinors \( u_{0,s} \) and \( v_{0,s} \) are spin eigenfunctions

\[ u_{0,s} = \begin{pmatrix} \chi_s \\ \chi_s \end{pmatrix} , \quad v_{0,s} = \begin{pmatrix} \chi_s \\ -\chi_s \end{pmatrix} \]  

(283)
is \( \chi_s \) are spin eigenstates for spin \( j \) representation of \( SU(2) \) group. Now consider any spinor \( \varphi_{ps} \) which is spin eigenstate in direction \( \hat{n} \) with some value \( \lambda \)

\[
\left( \hat{n} \cdot \vec{J} \right) \varphi_{ps} = \lambda \varphi_{ps}.
\]  

(284)

Applying boost operator to state \( \varphi_{ps} \) gives us (by definition) state \( \varphi'_{p',s'} \)

\[
\varphi'_{p',s'} = e^{-i\vec{\omega} \cdot \vec{K}} \varphi_{ps}.
\]  

(285)

If that state were also spin eigenstate

\[
\left( \hat{n} \cdot \vec{J} \right) \varphi'_{p',s'} = \left( \hat{n} \cdot \vec{J} \right) e^{-i\vec{\omega} \cdot \vec{K}} \varphi_{ps} = e^{-i\vec{\omega} \cdot \vec{K}} \left( \hat{n} \cdot \vec{J} \right) \varphi_{ps} - \left( \hat{n} \cdot \vec{J} \right) e^{-i\vec{\omega} \cdot \vec{K}} \varphi_{ps}
\]  

(286)

\[
= e^{-i\vec{\omega} \cdot \vec{K}} \lambda \varphi_{ps} - \left( \hat{n} \cdot \vec{J} \right) e^{-i\vec{\omega} \cdot \vec{K}} \varphi_{ps}
\]  

(287)

\[
= \lambda \varphi'_{p',s'} - \left( \hat{n} \cdot \vec{J}, e^{-i\vec{\omega} \cdot \vec{K}} \right) \varphi_{ps}
\]  

(288)

it would imply that all components of \( J \) commute with all components of \( K \) which we know isn’t true.

Since coordinate (“orbital”) part of boost generators commute with spin part of rotation generators, the proper way to create momentum and spin eigenstates is to apply only the coordinate part of boost generators to rest frame momentum and spin eigenstates

\[
e^{-i\vec{\omega} \cdot \vec{K}_{\text{coord}}} \psi_{0,s}(x) = e^{-i\vec{\omega} \cdot \vec{K}_{\text{coord}}} e^{-imt} \psi_{0,s} = e^{-ip \cdot x} \frac{e^{-imt}}{(2\pi)^{3/2}} u_{0,s} = \psi_{p,s}(x)
\]  

(289)

We will call those states \( w_{ps}^\pm \) (although they don’t depend on momentum)

\[
\psi_{p,s}(x) = e^{-i\vec{\omega} \cdot \vec{K}_{\text{coord}}} e^{-imt} (2\pi)^{3/2} u_{0,s} = e^{-ip \cdot x} (2\pi)^{3/2} w_{p,s}.
\]  

(290)

Note that momentum and spin dependence factor as they should since translation generator commutes with spin part of rotation generator. This is by no means specific to \( (j,0) \oplus (0,j) \) representations. Since it’s a consequence of commutation relations for operators, it will be valid for all representations. It is generally possible to make the transformation of operators

\[
M_{ch} \rightarrow M_P = U^\dagger M_{ch} U \quad \varphi_P = U^\dagger \varphi_{ch}
\]  

(291)

in which the parity operator is diagonal. This is achieved with the unitary matrix

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]  

(292)

where 1 is unit matrix of appropriate dimension, which diagonalizes parity and leave rotation generators unchanged

\[
P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{J} = \begin{pmatrix} \vec{S} & 0 \\ 0 & \vec{S} \end{pmatrix}
\]  

(293)

but makes boost operators non-diagonal

\[
\vec{K} = i \begin{pmatrix} 0 & \vec{S} \\ \vec{S} & 0 \end{pmatrix}.
\]  

(294)

Momentum-parity-spin eigenstates in this representation decompose to a direct sum of positive and negative parity part

\[
w_{ps}^+ = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}, \quad w_{ps}^- = \begin{pmatrix} 0 \\ \chi_s \end{pmatrix}.
\]  

(295)
**B.2 \((j,j)\) representations**

Let’s take a look at the first equation in (272). A look at any quantum mechanics textbook (i.e. [2], section 3.7) will show that it’s identical to the equation for addition of spin in non-relativistic quantum mechanics. So the spin eigenstates for \((j,j)\) representation will then be a sum of \(2j, 2j-1, \ldots, 0\) irreducible spin representations. They are related to spin \(j\) states through Clebsch-Gordon coefficients

\[
|j', m\rangle = \sum_{m_1, m_2} |jj' m j j; m_1 m_2\rangle |j, m_1\rangle \otimes |j, m_2\rangle \quad j' = 0, 1, \ldots, 2j
\]  

(296)

We can in general choose such a basis where rotation generator is reducible

\[
\vec{J} = \begin{pmatrix}
\vec{J}^{(0)} & 0 & 0 & \cdots \\
0 & \vec{J}^{(1)} & 0 & \cdots \\
0 & 0 & \vec{J}^{(2)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

(297)

This doesn’t give irreducible representations of whole Lorentz group since boost generators \(K\) won’t be diagonal. A look at the second equation in (272) shows that operators \(K^\pm = K_1 \pm iK_2\) can change values \((a,b)\) only by 1. Operator \(K_3\) multiplies each \((a,b)\) state with some factor and so changes the relative sign for sums of states. \(K\) operators are the sum of the same operators as \(J\) operators, so the action of both operators will give the same states with different coefficients

\[
J |jm\rangle \equiv J |ja; jb\rangle = |jm'\rangle \equiv c_1 |ja'; jb\rangle + c_2 |ja; jb'\rangle \\
K |jm\rangle \equiv K |ja; jb\rangle = |jm'\rangle \equiv d_1 |ja'; jb\rangle + d_2 |ja; jb'\rangle
\]

(298)

Since those states are orthogonal, they cannot have the same \(j\) value; since \(J\) doesn’t change the \(j\) value, \(K\) must. Since the values of \(a\) and \(b\) have been changed by 1 or 0, \(K\) can only raise or lower the \(j\) value by 1, or in another words, \(K\) will connect only states with spin \(j'\) and \(j' \pm 1\). In the basis where \(\vec{J}\) is diagonal this can be represented in block-matrix form as

\[
\vec{K} = \begin{pmatrix}
0 & \vec{K}_{01} & 0 & \cdots \\
\vec{K}_{10} & 0 & \vec{K}_{12} & \cdots \\
0 & \vec{K}_{21} & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(299)

Since boost generators anti-commutes with \(K\), action of \(K\) will change parity of the state

\[
P |jm\rangle = \lambda |jm\rangle , \quad PK |jm\rangle = -\lambda P |jm\rangle
\]

(300)

Therefore, states with spin \(j'\) and \(j' \pm 1\) will have opposite parity. This also implies there will never be the same number of states with opposite parities.

Again, we’ll call the states obtained by boosts \(u_{ps}\) and \(v_{ps}\)

\[
u_{ps} = e^{-i\tilde{\phi} K} u_{0s} \quad , \quad v_{ps} = e^{-i\tilde{\phi} K} v_{0s}
\]

(301)

and spin eigenstates \(w_{ps}^\pm\). Wave function will then be a superposition of those eigenfunctions

\[
\varphi^{(j,j)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{-ip \cdot x} \left( \sum_s b_{ps} u_{ps} + \sum_{s'} d_{ps'} v_{ps'} \right) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{-ip \cdot x} \sum_{s,ps} a_{ps} w_{ps}^P
\]

(302)
with some coefficients which are in general complex functions which depend on momentum, spin and parity of the state. It is important to emphasize that all $(j, j)$ wave functions on the quantum level are complex numbers which cannot be restricted to be real functions.

**B.3** $(j, j') \oplus (j', j)$ representations

This class of representations mixes the properties of former two. Like in the $(j, 0) \oplus (0, j)$ representations, rotation and boost generators will be the direct sum of two irreducible parts of equal dimensions $(j, j')$; similar to $(j, j)$ representation, rotation generators for each $(j, j')$ part will be a direct sum of generators for different spin, and again, boost generators will be irreducible. Rotations will act separately on states with different spin and parity while the boost will mix them.

**B.4** Negative definite scalar products and norms

Let’s show that states with negative “norms” are unavoidable. Consider the action of parity on states belonging to representation $(j, j)|jm\rangle \otimes |jm'\rangle$:

\[
P|jm\rangle \otimes |jm'\rangle = |jm'\rangle \otimes |jm\rangle
\]

(303)

States with $m = m'$ will obviously be invariant; for states with $m \neq m'$ there are two linear combinations which are parity eigenstates

\[
P \frac{1}{\sqrt{2}} \left( |jm\rangle \otimes |jm'\rangle \pm |jm'\rangle \otimes |jm\rangle \right) = \pm \frac{1}{\sqrt{2}} \left( |jm\rangle \otimes |jm'\rangle \pm |jm'\rangle \otimes |jm\rangle \right).
\]

(304)

For $(j, j') \oplus (j', j)$ representation states $(|jm\rangle \otimes |j'm'\rangle \oplus |j'n\rangle \otimes |jn\rangle)$ parity exchanges $m \leftrightarrow n$ and $m' \leftrightarrow n'$. Again, parity invariant combinations will be

\[
P \left( |jm\rangle \otimes |j'm'\rangle \oplus |j'n\rangle \otimes |jn\rangle \pm |jn\rangle \otimes |j'n\rangle \oplus |j'm'\rangle \otimes |jm\rangle \right)
\]

\[
= \pm \left( |jm\rangle \otimes |j'm'\rangle \oplus |j'n\rangle \otimes |jn\rangle \pm |jn\rangle \otimes |j'n\rangle \oplus |j'm'\rangle \otimes |jm\rangle \right).
\]

(305)

In both cases, we have explicitly constructed states of both negative and positive parity. This construction shows that it’s impossible to have states with either parity without states with opposite parity. Now take the state with negative parity $\varphi$ and take it’s norm

\[
\langle \overline{\varphi} | \varphi \rangle = \sum_A \varphi_A^I P_{AB} \varphi_B = - \sum_A \varphi_A^I \varphi_A
\]

(306)

which is by definition negative number q.e.d.

**C** Groups and Dirac matrices

**C.1** Representations of $SU(2)$ generators

$SU(2)$ generators satisfy commutation relations

\[
[S_i, S_j] = i \epsilon_{ijk} S_k
\]

(307)
It is sometimes convenient to define raising and lowering operators \( S_+ \) and \( S_- \) as

\[
S_\pm = S_1 \pm i S_2
\]  

(308)

We choose the basis in which \( S_3 \) is diagonal matrix

\[
(S_3)_{kl} = \delta_{k,l}(j - k + 1) \quad \text{or} \quad S_3 = \begin{pmatrix}
 j & 0 & \cdots & 0 \\
 0 & j - 1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & -j
\end{pmatrix}.
\]  

(309)

From commutation relations for \( S_\pm \) we get

\[
S_+ |j, m\rangle = \sqrt{j(j + 1) - m(m + 1)} |j, m + 1\rangle
\]  

(310)

from which we can construct raising operator with only off-diagonal elements

\[
(S^+)_{kl} = \delta_{k,l-1}\sqrt{j(j + 1) - l(l - 1)} \quad \text{or} \quad S_3 = \begin{pmatrix}
 0 & \sqrt{2j} & 0 & \cdots \\
 0 & 0 & \sqrt{2(j - 1)} & \cdots \\
 0 & 0 & 0 & \ddots \\
 \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]  

(311)

\( S^- \) will be Hermitean conjugate of \( S^+ \), from which we can calculate \( S_1 \) and \( S_2 \)

\[
S_1 = \frac{S_+ + S_-}{2} \quad S_2 = \frac{S_+ - S_-}{2i}
\]  

(312)

Lowest-dimensional representation of \( SU(2) \) generators is spin 1/2 representation where generators are Pauli matrices multiplied by factor 1/2

\[
S_i = \frac{\sigma_i}{2}
\]  

(313)

where Pauli matrices are given by

\[
\sigma^1 = \begin{pmatrix}
 0 & 1 \\
 1 & 0
\end{pmatrix} \quad \sigma^2 = \begin{pmatrix}
 0 & -i \\
 i & 0
\end{pmatrix} \quad \sigma^3 = \begin{pmatrix}
 1 & 0 \\
 0 & -1
\end{pmatrix}
\]  

(314)

\[
S_+ = \frac{\sigma_1 + i \sigma_2}{2} = \begin{pmatrix}
 0 & 1 \\
 0 & 0
\end{pmatrix} \quad S_- = \frac{\sigma_1 - i \sigma_2}{2} = \begin{pmatrix}
 0 & 0 \\
 1 & 0
\end{pmatrix}.
\]  

(315)

Spin eigenstates \(|jm\rangle \) in this form are

\[
|\frac{1}{2} \frac{1}{2}\rangle = \begin{pmatrix}
 1 \\
 0
\end{pmatrix} \quad |\frac{1}{2} \frac{1}{2}\rangle = \begin{pmatrix}
 0 \\
 1
\end{pmatrix}
\]  

(316)

Next representation is spin 1 representation; generators for it are given by

\[
S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
 0 & 1 & 0 \\
 1 & 0 & 1 \\
 0 & 1 & 0
\end{pmatrix} \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
 0 & -i & 0 \\
 i & 0 & -i \\
 0 & i & 0
\end{pmatrix} \quad S_3 = \begin{pmatrix}
 1 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & -1
\end{pmatrix}.
\]  

(317)
Raising and lowering operators are
\[
S_+ = \begin{pmatrix}
0 & \frac{\sqrt{2}}{2} & 0 \\
0 & 0 & \frac{\sqrt{2}}{2} \\
0 & 0 & 0
\end{pmatrix}, \quad
S_- = \begin{pmatrix}
0 & 0 & 0 \\
\frac{\sqrt{2}}{2} & 0 & 0 \\
0 & \frac{\sqrt{2}}{2} & 0
\end{pmatrix},
\]
while spin eigenstates are
\[
|1\ 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad
|1\ 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad
|1\ -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]
Spin 1 eigenstates are can be also thought of as components of vector in spherical coordinates. Generators in spherical coordinates are related to the Cartesian coordinates through
\[
S_i^\text{spher} = V S_i^\text{cart} V^\dagger
\]
where \( V \) is given by
\[
V = \begin{pmatrix}
-\frac{1}{\sqrt{2}} & i & 0 \\
\frac{1}{\sqrt{2}} & 0 & 1 \\
0 & i & 0
\end{pmatrix}
\]
where rotation generators in Cartesian coordinates are given by
\[
S_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad
S_2 = i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad
S_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Vectors in spherical coordinates are related to Cartesian coordinates through
\[
\begin{pmatrix} r_{-1} \\ r_0 \\ r_{+1} \end{pmatrix} = V^* \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} -(r_x + i r_y) / \sqrt{2} \\ r_z \\ (r_x - i r_y) / \sqrt{2} \end{pmatrix}
\]
For spin 3/2 generators are given by
\[
S_1 = \frac{1}{2} \begin{pmatrix}
0 & \sqrt{3} & 0 & 0 \\
\sqrt{3} & 0 & 2 & 0 \\
0 & 2 & 0 & \sqrt{3} \\
0 & 0 & \sqrt{3} & 0
\end{pmatrix}, \quad
S_2 = \frac{1}{2} \begin{pmatrix}
0 & -i \sqrt{3} & 0 & 0 \\
i \sqrt{3} & 0 & -2i & 0 \\
0 & 2i & 0 & -i \sqrt{3} \\
0 & 0 & i \sqrt{3} & 0
\end{pmatrix}, \quad
S_3 = \frac{1}{2} \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3
\end{pmatrix},
\]
\[
S_+ = \begin{pmatrix}
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{3} \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
S_- = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0
\end{pmatrix}.
\]
and spin eigenstates are

\[
\begin{pmatrix}
3 \\
2
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \quad \begin{pmatrix}
3 \\
1 \\
2
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

etc. (326)

Finally, for spin 2 we have

\[
S_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & \sqrt{3}/2 & 0 & 0 \\
0 & \sqrt{3}/2 & 0 & \sqrt{3}/2 & 0 \\
0 & 0 & \sqrt{3}/2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} \quad S_2 = \begin{pmatrix}
0 & -i & 0 & 0 & 0 \\
i & 0 & -\sqrt{3}/2i & 0 & 0 \\
0 & \sqrt{3}/2i & 0 & -\sqrt{3}/2i & 0 \\
0 & 0 & \sqrt{3}/2i & 0 & -i \\
0 & 0 & 0 & i & 0
\end{pmatrix}
\]

\[
S_3 = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -2
\end{pmatrix}
\]

\[
S_+ = \begin{pmatrix}
0 & 2 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & 0 \\
0 & 0 & 0 & \sqrt{6} & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \quad S_- = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
0 & \sqrt{6} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{6} & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
2 \\
2
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \quad \begin{pmatrix}
2 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

etc. (329)

C.2 4D Dirac matrices and spinors

Four-dimensional \(\gamma\) matrices \(\gamma^\mu\) (with \(\mu = 0, 1, 2, 3\)) and the matrix \(\gamma^5\)

\[
\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta
\]

satisfy the anti-commutator relations

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^\mu\nu, \quad g^\mu\nu = \text{diag}(1, -1, -1, -1, 1).
\]

If we expand this set with the commutator

\[
\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]
\]

we get the 5 dimensional Clifford algebra (with \(\mu, \nu = 0, 1, 2, 3, 5\)) \(\gamma\) and \(\sigma\) matrices satisfy commutation relations (in 5D)

\[
[\gamma^\mu, \sigma^{\alpha\beta}] = 2i\left(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha\right).
\]
Hermitean conjugates of $\gamma^\mu$ are given by

$$(\gamma^0)\dagger = \gamma^0$$  
$$(\gamma^5)\dagger = \gamma^5$$  
$$(\gamma^i)\dagger = -\gamma^i$$, \quad i = 1, 2, 3 \quad (334)$$

From this we can calculate the Hermitean conjugates for $\sigma$ matrices as well:

$$(\sigma^{ij})\dagger = \sigma^{ij}$$  
$$(\sigma^{0i})\dagger = -\sigma^{0i}$$  
$$(\sigma^{50})\dagger = \sigma^{50}$$  
$$(\sigma^{5i})\dagger = -\sigma^{5i}$$ \quad (335)$$

We can see that either 4D subset of these matrices $\sigma^{\mu\nu} = \{\sigma^{0i}, \sigma^{ij}\}$ (with $\gamma^\mu = \{\gamma^0, \gamma^i\}$) or $\sigma^{\mu\nu} = \{\sigma^{5i}, \sigma^{ij} (\gamma^\mu = \{\gamma^5, \gamma^i\})$ will satisfy the appropriate commutation relations so both can be identified with generators of Lorentz transformations. They correspond to different representations of 4D Dirac matrices.

### C.2.1 Dirac or Parity representation of $\gamma$ matrices

Commutation relations don’t determine the $\gamma$ and $\sigma$ matrices completely. If $\{\gamma^\mu, \sigma^{\mu\nu}\}$ is a set of matrices satisfying 4D Clifford algebra, any set related to this one by unitary transformation $\{U^{-1}\gamma^\mu U, U^{-1}\sigma^{\mu\nu} U\}$ will satisfy the same relations. Dirac originally proposed the representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  
$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$  
$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (336)$$

where $\sigma^i$ are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$  
$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$  
$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \quad (337)$$

$\sigma$ matrices in this representation are

$$\sigma^{0i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} = i\alpha^i$$  
$$\sigma^{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\sigma^{50} = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\gamma^0\gamma^5$$  
$$\sigma^{5i} = i \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = -i\gamma^i\gamma^5 \quad . \quad (338)$$

Lorentz boost matrices in this representation are

$$S_D(\omega) = \exp\left(-\frac{i}{2}\sigma^{\mu\nu}\frac{\omega^{\mu\nu}}{2}\right) \quad (339)$$

$$= \exp\left\{\frac{1}{2} \begin{pmatrix} 0 & \vec{\omega} \cdot \vec{\sigma} \\ \vec{\omega} \cdot \vec{\sigma} & 0 \end{pmatrix}\right\} = \begin{pmatrix} \cosh\frac{\omega}{2} & \hat{\omega} \cdot \vec{\sigma} \sinh\frac{\omega}{2} \\ \hat{\omega} \cdot \vec{\sigma} \sinh\frac{\omega}{2} & \cosh\frac{\omega}{2} \end{pmatrix} \quad (340)$$

where $\hat{\omega}$ is unit vector in the direction of $\vec{\omega}$, $\cosh(\omega/2) = \sqrt{(E + m)/2m}$, $\sinh(\omega/2) = \sqrt{(E - m)/2m}$ and $\hat{\omega} \cdot \vec{\sigma} = \vec{p} \cdot \vec{\sigma}/\sqrt{E^2 - m^2}$. Expressed through $p^\mu$ we get

$$S_D(\omega) = \frac{1}{\sqrt{2m(E + m)}} \begin{pmatrix} E + m & \vec{p} \cdot \vec{\sigma} \\ \vec{p} \cdot \vec{\sigma} & E + m \end{pmatrix} \quad (341)$$

In this representation parity matrix $\gamma^0$ is diagonal, but helicity operator $\gamma^5$ and Lorentz boosts $S_D(\omega)$ aren’t.
C.2.2 Weyl or chiral representation of $\gamma$ matrices

If we make the change $\gamma^0 \rightarrow \gamma^5$ and $\gamma^5 \rightarrow -\gamma^0$ we get the chiral or Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tilde{\gamma} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(342)

$$\sigma^{0i} = i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = i\alpha^i \quad \sigma^{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\sigma^{50} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\gamma^0 \gamma^5 \quad \sigma^{5i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} = -i\gamma^i \gamma^5$$

(343)

In chiral representation we get Lorentz boost matrices

$$S_{Ch}(\omega) = \exp \left( -i \frac{\sigma^{\mu\nu}}{2} \omega_{\mu\nu} \right) = \exp \left\{ \frac{1}{2} \begin{pmatrix} -\omega \cdot \sigma & 0 \\ 0 & \omega \cdot \sigma \end{pmatrix} \right\}$$

$$= \begin{pmatrix} \cosh \frac{\omega}{2} - \hat{\omega} \cdot \hat{\sigma} \sinh \frac{\omega}{2} & 0 \\ 0 & \cosh \frac{\omega}{2} + \hat{\omega} \cdot \hat{\sigma} \sinh \frac{\omega}{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2m(E + m)}} \begin{pmatrix} E + m - \vec{p} \cdot \hat{\sigma} & 0 \\ 0 & E + m + \vec{p} \cdot \hat{\sigma} \end{pmatrix}$$

(344)

This representation has the advantage that all Lorentz transformations $S_{Ch}$ as well as chirality operator $\gamma^5$ are diagonal, but parity operator $\gamma^0$ isn’t. The connection between chiral and Dirac $\gamma$ matrices is

$$\gamma_{ch} = U \gamma_D U^\dagger \quad \phi_{ch} = U \phi_D \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(345)

or in particular

$$u_D(p, s) = \begin{pmatrix} u_R(p, s) + u_L(p, s) \\ u_R(p, s) - u_L(p, s) \end{pmatrix} \quad v_D(p, s) = \begin{pmatrix} u_R(p, s) - u_L(p, s) \\ u_R(p, s) + u_L(p, s) \end{pmatrix}$$

(346)

where $u_L(p, s)$ and $u_R(p, s)$ are chiral (bi)spinors belonging to $(1/2, 0)$ and $(0, 1/2)$ representations of Lorentz group.

C.3 Gordon identities

Using the fact that spinor satisfy equations

$$(\hat{p} - m) u_{p, s} = 0 \quad \bar{u}_{p, s} (\hat{p} - m) = 0 \quad (\hat{p} + m) v_{p, s} = 0 \quad \bar{v}_{p, s} (\hat{p} + m) = 0$$

(347)

multiplying the first equation in (347) by $\phi$ (where $a$ is arbitrary four vector) from the left, and second from the right and adding them we get

$$\bar{u}_{p, s} [a - (\hat{p} - m) \phi + \phi (\hat{p} - m)] u_{q, r} = 0$$

(348)
which can be rewritten as

$$-2m \vec{a}_{p,s} \hat{a} \psi_{q,r} + \vec{b}_{p,s} \left( \frac{\hat{p} + \hat{q}}{2}, \hat{a} \right) + \left( \frac{\hat{p} - \hat{q}}{2}, \hat{a} \right) \right) \psi_{q,r} = 0 \quad (349)$$

Evaluating the commutators and anti-commutators to

$$\{ \hat{a}, \hat{b} \} = 2a \cdot b \quad [\hat{a}, \hat{b}] = -2i \sigma^{\mu\nu} a_\mu b_\nu \quad (350)$$

adding or subtracting the proper combinations of equations (347) and after differentiation with respect to $a_\mu$ we get Gordon identities:

$$\vec{a}_{p,s} \gamma^\mu \psi_{q,r} = \frac{1}{2m} \vec{a}_{p,s} \left[ (p + q)\gamma^\mu + i \sigma^{\mu\nu} (p - q)_\nu \right] \psi_{q,r} \quad (351)$$

$$\vec{b}_{p,s} \gamma^\mu \psi_{q,r} = -\frac{1}{2m} \vec{b}_{p,s} \left[ (p + q)\gamma^\mu + i \sigma^{\mu\nu} (p - q)_\nu \right] \psi_{q,r} \quad (352)$$

$$\vec{a}_{p,s} \gamma^\mu \psi_{q,r} = \frac{1}{2m} \vec{a}_{p,s} \left[ (p - q)\gamma^\mu + i \sigma^{\mu\nu} (p + q)_\nu \right] \psi_{q,r} \quad (353)$$

$$\vec{b}_{p,s} \gamma^\mu \psi_{q,r} = -\frac{1}{2m} \vec{b}_{p,s} \left[ (p - q)\gamma^\mu + i \sigma^{\mu\nu} (p + q)_\nu \right] \psi_{q,r} \quad (354)$$

C.4 Direct product of representation and operators

If Operator $\mathcal{A}$ is an operator acting an a Hilbert space of dimension $n$ spanned by a complete set of vectors $|\psi\rangle$ and $\mathcal{B}$ is an operator acting on a Hilbert space of dimension $m$ spanned by a complete set of vectors $|\phi\rangle$, then the direct product of operators $\mathcal{A}$ and $\mathcal{B}$ is defined to be

$$(\mathcal{A} \otimes \mathcal{B})_{ik,jl} \equiv \mathcal{A}_{ij} \otimes \mathcal{B}_{kl} \quad (356)$$

which acts on the direct product of spaces

$$|\psi \otimes \phi\rangle_{ij} \equiv |\psi\rangle_i \otimes |\phi\rangle_j \quad (357)$$

of dimension $m \times n$ like

$$(\mathcal{A} \otimes \mathcal{B})_{ik,jl} |\psi \otimes \phi\rangle_{ij} = (A_{ij} \otimes B_{kl}) |\psi\rangle_i \otimes |\phi\rangle_j \equiv (A_{ij} |\psi\rangle_j) \otimes (B_{kl} |\phi\rangle_l) \quad (358)$$

For single space operators (operators in one space multiplied by identity operator in another space) we have

$$[A \otimes 1, B \otimes 1]_{ik,jl} = (A \otimes 1)_{ik,mn} (B \otimes 1)_{mn,jl} - (B \otimes 1)_{ik,mn} (A \otimes 1)_{mn,jl} = A_{im} \otimes B_{mj} \otimes 1_{nl} - B_{im} \otimes A_{mj} \otimes 1_{nl} = (A_{im} B_{mj}) \otimes (1_{kn} 1_{nl}) - (B_{im} A_{mj}) \otimes (1_{kn} 1_{nl}) = [A, B]_{ij} \otimes 1_{kl} = (A, B)_{ik,jl} \quad (359)$$

Following the same procedure, operators in different spaces commute:

$$[A \otimes 1, 1 \otimes B]_{ik,jl} = (A \otimes 1)_{ik,mn} (1 \otimes B)_{mn,jl} - (1 \otimes B)_{ik,mn} (A \otimes 1)_{mn,jl} = A_{im} \otimes 1_{kn} B_{mj} \otimes 1_{nl} - 1_{im} \otimes B_{kn} A_{mj} \otimes 1_{nl} = (A_{im} 1_{mj}) \otimes (1_{kn} B_{nl}) - (1_{im} A_{mj}) \otimes (B_{kn} 1_{nl}) = A_{ij} \otimes B_{kl} - A_{ij} \otimes B_{kl} = 0 \quad (360)$$
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