Cooperative behavior in a spatial model of “commons”

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Abstract

We study a lattice model of “commons”, where a resource is shared locally among the agents of various cooperative tendency. The payoff function of an agent is proportional to the fraction of his operation rate and the net output of the resource. After each time step a site is occupied by the neighbor of maximum profit or by its owner himself. In steady state the model is dominated by “altruist” agents with a small minority of selfish agents, forming a complex pattern. The dynamics selects cooperative levels in a way that the model becomes critical. We study the critical behavior of the model in case of moderate mutation rate and find the power spectrum of fluctuation of activity shows a $1/f^\alpha$ behavior with $\alpha \sim 1.30$. In case of very slow mutation rate the steady state has slow fluctuations which helps the evolution of higher cooperative tendency.

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1 Introduction

Selfishness is regarded as the rational behavior of agents in various social and economic studies. However, there are cases where altruistic or cooperative tendency can survive. The game of “prisoners dilemma” is one such case where cooperative behavior is necessary to maximize one’s payoff [1]. Axelrod and Hamilton have shown that selfish agents with finite memory will eventually learn to cooperate if the game is played many times [1]. In a later work, May and Nowak showed that in a spatially extended population, altruistic or cooperative tendency can survive by forming spatial territories [2]. A single altruist is not strong but it becomes strong in a group because of its cooperative tendency. Thus once a domain of altruists is formed in space it can survive against the selfish members of the species. In some biological systems altruistic behavior is observed instead of purely selfish behavior [3].

The problem of “tragedy of commons” is the other such case where the selfish tendency to maximize one’s gain leads to a degeneration of the common resource, and thus reduces everyones gain [4]. We study a spatial version of this problem in which we assume a resource spread over all the sites of the lattice. If an uncontrolled exploitation by a few agents leads to a local degeneration of the resource; its negative effect is felt only in an immediate neighborhood. Similarly, if the resource is maintained in a good condition, its positive effect is also felt locally. Thus, the resource is shared only locally. This is the situation observed in case of most natural resources.
We model this situation with evolution rules similar to those of May and Nowak. In our model, instead of only two types of agents, there are many types of agents whose cooperative tendencies are labeled by a real variable between 0 and 1. The agents of type 1 are totally cooperative and the agents of type 0 are totally selfish, and the agents of type between 0 and 1, represent a mixed strategy. In the steady state, the density profile of agents develops two peaks, one near 0 representing the defector band and the other near 1, representing the cooperator band. Agents from these two bands form a complex spatial pattern (see Fig. 1). Members of the defector band form thin lines whereas members of the cooperator band form compact domains, which we will also call colonies. It should be emphasized that the colonies are formed only by cooperators, demonstrating the advantage of cooperative over selfish behavior. This pattern is similar to the one found by May and Nowak in certain range of the parameter used in their model.

The complex pattern formed in this model has some interesting statistical properties. If we allow mutation, i.e., a change of strategy, of agents at a moderate rate, then we find that the model self-organizes to a critical steady state. The power spectrum of fluctuation of the activity has a $1/f^\alpha$ form with $\alpha = 1.30$. The critical behavior is lost for very slow mutation rate. Thus this case of self-organized criticality is different from that of the sandpile model which is critical at very slow drive rate [5].

The moderate mutation rate helps evolution of cooperation in the initial stage. But, the large critical fluctuations restrict the cooperative tendency to a maximum of $8/13$. At very slow mutation rate an interesting process of evolution starts. A more stable steady state helps the growth of cooperative tendency higher than 8/13.

2 “Tragedy of commons”

Consider a resource shared by a group of $N$ agents. When the exploitation rate $X$ is very slow compared to the regeneration rate of the resource, the yield $P(X)$ is proportional to $X$. However, when $X$ becomes approximately equal to the regeneration rate the yield $P(X)$ reaches a saturation point. If the resource is exploited at an even faster rate, it degenerates, and $P(X)$ decreases. Thus the yield $P(X)$, at the exploitation rate $X$ has the form shown in Fig. 2. $P(X)$ is maximum at $X_c$, and goes to zero at $X_m$ (the resource is fully destroyed at $X_m$). For definiteness, we assume the form $P(X) = X(2 - \frac{X}{N})$. So $X_c = N$, and $X_m = 2N$. The income of the $i$th agent $P_i(x_i)$ is proportional to his exploitation rate $x_i$.

$$P_i(x_i) = \frac{x_i}{X} P(X) = x_i(2 - \frac{X}{N}) ,$$

where $\sum x_i = X$.

Consider the case when all the agents operate at a rate $x = 1$, the income of each of them is 1, and net income of the group is maximum. Now, let the $j$th agent increases his exploitation rate from 1 to 2. Then $X = N + 1$, $P_i = 1 - 1/N$, for $i \neq j$, and $P_j = 2(1 - 1/N)$. Thus the $j$th agent almost doubles his income, while the other members incur a small loss of order $1/N$. The net income of the group also decreases by a fraction of order $1/N^2$. Therefore, if the agents try to maximize their personal income, the system is pushed beyond the optimum exploitation rate $X_c$, towards $X_m$, where the income of all agents goes to zero, and the “tragedy of commons” occurs [4].

One observes that in this case, there is a conflict between individual interest and global interest (and eventually the individual interest has conflict with itself). Thus if we try to maximize the income of each agent by the set of equations of form $\frac{dx_i}{dt} = \Gamma \frac{dP(x_i)}{dx_i} + \eta$, we find that this set of equations,
Figure 1: Typical spatial pattern for lattice size $L = 95$, and mutation rate $p = 0.001$ (above). The lighter gray shades represent the higher cooperative levels and the darker gray shades represent the lower cooperative levels. In case of the slow mutation rate, i.e., $p = 0.000001$ (below) the cooperative band is broad and contains many levels shown by the different gray shades.
instead of maximizing the $P_i$’s, lead to a state of lower $P_i$’s. Hence, the separate optimization of the individual incomes, does not work.

In the physical systems this kind of a situation can be seen in case of granular media. If sand is allowed to flow in presence of the gravitational field, the individual tendency of grains to minimize their potential energy may lead to a jammed configuration. Thus individual tendency has conflict with itself.

Here, we try to understand that how cooperation can evolve so that the “tragedy of commons” can be averted.

3 Model of locally shared resource (LSR)
We study a model of locally shared resource (LSR). This is defined on a square lattice, with a real variable $s_i \in [0, 1]$, called the cooperative level of the occupant of the $i$th site. This represents the cooperative tendency of an agent. The exploitation rate depends on level $s_i$ through, $x_i = 2 - s_i$ ( clearly $x_i \in [1, 2]$). The higher $s_i$ implies higher cooperative tendency and lower $x_i$. The agents of $s < \frac{1}{2}$, will be called “defectors” or “selfish”, and those of $s \geq \frac{1}{2}$ will be called “cooperators” or “altruists”. The resource near the site $i$ is shared in a neighborhood $D(i)$, which consists of the occupants of the $i$th site and its nearest and next nearest neighbors. Thus $N = 9$ in two dimensions. The payoff of the occupant of the $i$th site can be expressed in terms of $s_i$ using (1) and the relation $x_i = 2 - s_i$.

$$P_i = \frac{1}{N}(2 - s_i) \sum_{j \in D(i)} s_j .$$

The evolution rule consists of two parts:

1. Mutual struggle: After one time step, the site $i$ will be occupied by the neighbor of highest payoff, $i.e.$, $s_i \rightarrow s_h$, where $h \in D(i)$ has highest income $P_h$.

2. Mutation: The levels of a small fraction $p$ of agents are changed to randomly chosen levels.
We study this model with both periodic and fixed boundary conditions. In case of fixed boundary condition, the set $D(i)$ has only four elements for the sites on the corners and it has six elements for the sites on the edges. In case of periodic boundary condition, $D(i)$ contains nine elements for all $i$.

The first rule of evolution in our model is same as that in the model of May and Nowak. However in their model, there are only two types of agents: One who always cooperates ($s = 1$) and the other who always defects ($s = 0$). The game of prisoners’ dilemma is played between two adjacent neighbors. If both cooperate then the payoff of each of them is 1. If one defects while the other cooperates then the payoff of the defector is $b > 1$, while the payoff of the cooperator is 0. If both defect then the payoff of each of them is 0. If agent of the $i$th site plays this game with all his eight neighbors, then his payoff is

$$P_i = \left[ b - (b - 1)s_i \right] \sum_{j \in D(i)-i} s_j . \quad (3)$$

In the LSR model, there are infinitely many types of agents labeled by a continuous variable $s$. If we consider a population of only two levels $s_1$ and $s_2$, then this model is similar to the model of May and Nowak. Note that the two models are not identical because the summation in (2) for the LSR model includes $i$. The effective $b$ can be defined as the ratio of payoffs of $s_1$ and $s_2$ in identical neighboring configurations. Ignoring $O(1/N)$ correction, we can write $b(s_1, s_2) \sim (2 - s_1)/(2 - s_2)$. However, it is important to point out that $b$ alone does not determine the behavior of the model.

## 4 Density profile in LSR model

We simulate this model for different mutation rates on lattices of various sizes. The statistical behavior of the model does not depend on the boundary condition. However, it depends on the mutation rate. For moderate mutation, i.e. $p > 0.0005$ approximately, the model shows critical

![Figure 3: The density profile for the moderate mutation rate (left). There is a huge peak at level $s = 8/13$. In case of the slow mutation rate (right) the levels $s > 8/13$ have also got populated.](image)
Figure 4: The cumulative density showing the fraction of population in cooperative levels below level $s$. In case of a moderate mutation rate ($p = 0.001$), there is a huge jump at $s = 8/13$. Approximately 71% of population lies in this level. In case of a very slow mutation rate levels $s > 8/13$ also get populated.

behavior. In this case, the average cooperative level $\langle s \rangle \simeq 0.50$. In case of very slow mutation rate, i.e. $p < 0.0005$ approximately, the model slowly develops higher cooperative regions, and the average cooperative level rises to $\langle s \rangle \simeq 0.53$.

In both cases the highest cooperative members (i.e. $s = 1$) do not survive, but a little practical agents from a band of lower cooperative levels, i.e. $\frac{8}{13} < s < \frac{5}{7}$, dominate in the steady state (see the second peak in the density plot, Fig. 3, and second step in the cumulative plot, Fig. 4). We call it the cooperator band (C-band). This band constitutes around 71% of the population. In case of a moderate mutation rate this band shrinks to just one level $s = \frac{8}{13}$. In case of a slow mutation rate the higher levels also form colonies. As a result the population of cooperators is evenly distributed in range $\frac{8}{13} \leq s \leq \frac{5}{7}$.

Around 25% of the population come from the defector band (D-band), corresponding to levels, $0.1 < s < 0.3$ approximately (see the first peak in density plot, Fig. 3, and the first step in the cumulative plot, Fig. 4). The level lying between these two bands, i.e. $0.3 < s < 0.6$ approximately, are almost absent from the population. Thus we find that in steady state the density profile of cooperative levels develops two peaks.

The C-band is very sharp, while the D-band is wide and contains more levels. This is because the spatially close levels have very high mutual competition. As a result the levels of C-band eliminate each other and in the end only a few levels survive. The levels of D-band have low density. They exist as scattered separated lines. A defector of one level hardly encounters a defector of another level. Therefore, they are not able to eliminate each other.

A good deal of understanding about the density profile of agents of various levels can be obtained by considering the stability of some of the special cases.

**Edge stability:** Consider the domains of two types of agents $s$ and $s'$, with $s > s'$, separated
Figure 5: The subconfigurations showing (a) edge, (b) corner and (c) step. The gray site denoted \( s' \), and white site denotes \( s \). The full configuration can be obtained by extending this pattern.

along a straight line (see Fig. 5). The agents of level \( s \) is represented by white color, and the agents of level \( s' \) is represented by gray color. The agent 1 is among the strongest white agent can have influence on the domain boundary. The agent 2 is the strongest white agent near the domain boundary. And, the agent 3 is the strongest gray agent near the domain boundary. The payoffs of these three agents are, \( P_1 = (2 - s)s \), \( P_2 = \frac{1}{9}(2 - s)(6s + 3s') \), and \( P_3 = \frac{1}{9}(2 - s')(3s + 6s') \). Now there are three possibilities:

- \( P_2 > P_3 \), i.e. \( s' < 1 - s \). In this case the white agent 2 can take over the gray agent 3, and the edge will move towards the domain of the level \( s' \). Hence, the level \( s \) will be called edge-strong against the level \( s' \). The levels \( s < \frac{1}{2} \) are edge-strong against all the lower levels.

- \( P_1 < P_3 \), i.e. \( 2 - \frac{3}{2}s < s' < s \). In this case the gray agent 3 can take over his neighboring white agents including agent 2, and the edge will move towards the domain of the level \( s \), so the level \( s \) is edge-weak against the level \( s' \). The levels \( s < \frac{4}{9} \) are never edge-weak against any of the lower levels.

- \( P_2 < P_3 \), but \( P_1 > P_3 \), i.e. \( 1 - s < s' < 2 - \frac{3}{2}s \). In this case the edge will be stable, so the level \( s \) will be called edge-stable against the level \( s' \).

**Corner stability:** Consider now the case in which a domain of level \( s' \) forms a corner in a domain of level \( s \), and \( s > s' \) (see Fig. 5). Here again we select three agents using the same criterion as in the previous case. The payoffs of these three agents are, \( P_1 = (2 - s)s \), \( P_2 = \frac{1}{9}(2 - s)(8s + s') \), and \( P_3 = \frac{1}{9}(2 - s')(5s + 4s') \). Again there are three possibilities:

- \( P_2 > P_3 \), i.e. \( s' < \frac{1}{2}(3 - 4s) \). In this case the corner of \( s' \) will be decimated. Hence, the level \( s \) will be called corner-strong against the level \( s' \). The levels \( s < \frac{1}{2} \) are corner-strong against all the lower levels, and the levels \( s > \frac{3}{4} \) are never corner-strong against any of the lower levels.

- \( P_1 < P_3 \), i.e. \( 2 - \frac{9}{4}s < s' < s \). The domain of \( s' \) will grow near the corner, so the level \( s \) is corner-weak against the level \( s' \). The levels \( s < \frac{8}{13} \) are never corner-weak against any of the lower level.

- \( P_2 < P_3 \), but \( P_1 > P_3 \), i.e. \( \frac{1}{2}(3 - 4s) < s' < 2 - \frac{9}{4}s \). In this case corner will be stable, so the level \( s \) will be called corner-stable against the level \( s' \).
**Step stability**: Consider now the case in which a domain of level $s'$ forms a step in a domain of level $s$, and $s > s'$. The payoffs of the three agents marked in Fig. 5 are, $P_1 = (2 - s)s$, $P_2 = \frac{1}{9}(2 - s)(8s + s')$, and $P_3 = \frac{1}{9}(2 - s')(4s + 5s')$. Again there are three possibilities:

- $P_2 > P_3$, i.e. $s' < \frac{5}{9}(1 - s)$. In this case the agent 3 will be taken over by the agent 2. Thus the domain of level $s$ will grow along the step. The level $s$ will be called *step-strong* against the level $s'$. The levels $s < \frac{5}{13}$ are step-strong against all the lower levels.

- $P_1 < P_3$, i.e. $2 - \frac{9}{5}s < s' < s$. The domain of $s'$ will grow, so the level $s$ is *step-weak* against the level $s'$. The levels $s < \frac{5}{7}$ are never step-weak against any of the lower level.

- $P_2 < P_3$, but $P_1 > P_3$, i.e. $\frac{1}{2}(3 - 4s) < s' < 2 - \frac{9}{7}s$. In this case step will be stable, so the level $s$ will be called *step-stable* against the level $s'$.

From this simplistic analysis of corner edge and step stability, one can understand the density profile shown in Fig. 4.

The levels $s < \frac{1}{2}$ do not have any threat from the lower levels, but they are edge-weak and corner-weak against the higher levels. Therefore they cannot form colonies and survive just as thin lines.

The levels $\frac{1}{2} < s < \frac{8}{13}$ are not edge-weak or corner-weak against any level. Therefore they form colonies in the initial stage. However, because of their step-weakness, the colonies are slowly decimated by the higher levels.

The level $s = \frac{8}{13}$ is not weak against any level. Therefore, in case of moderate mutation rate there is a huge jump in cumulative plot at this level (see Fig. 4). It constitutes 71% of the whole population.

The levels $\frac{8}{13} < s < \frac{5}{7}$ are not edge-weak or step-weak against any level. But they are corner-weak against some of the lower levels. In case of moderate mutation rate these levels are decimated by the level $s = \frac{8}{13}$ (so there is a huge peak at $s = \frac{8}{13}$). However, in case of very slow mutation rate these levels can form some colonies by forming a protective coat of defectors near the corners. Therefore, in this case the population of cooperators is evenly distributed in the whole range (see Fig. 4, the second step is broadened).

The levels $s > \frac{5}{7}$ are very weak against the lower levels. They are not even stable as thin lines. Therefore they are completely wiped out from the population. Their high cooperative tendency works against their survival.

### 5 Moderate mutation rate

The behavior of this model depends on the mutation rates. For moderate mutation rate, i.e., $p > 0.0005$ approximately, the model shows critical fluctuation in time. We study the power spectrum of the number of active sites (sites changing their levels) with both fixed and periodic boundary conditions. We find that the power spectrum does not depend on the boundary condition and it shows $1/f^\alpha$ behavior, with $\alpha \sim 1.30$ (see Fig. 4 and 7).

The mutation of the occupant of a site creates an avalanche. In case of the moderate mutation rate, the system does not get enough time to relax, and these avalanches overlap with each other leading to a bigger avalanche. The overlap of these avalanches cannot be regarded as uncorrelated superposition of smaller avalanches because of the temporal correlation. These big avalanches are responsible for the nontrivial behavior of the power spectrum. In case of the slow mutation rate, i.e.,...
Figure 6: Log-log plot of the power spectrum of activity for the lattice size $L = 45, 95, \text{and } 190$ and a mutation rate $p = 0.001$ in case of fixed boundary condition.

Figure 7: Log-log plot of the power spectrum of activity for the lattice size $L = 95$ and a mutation rate $p = 0.001$ in case of periodic boundary condition.
\( p < 0.0005 \), the system gets enough time to relax, therefore the avalanches do not overlap. In this case, the power spectrum has \( 1/f^2 \) behavior. We note that self-organized criticality observed in this model, in case of moderate mutation rate, is different from that of the sandpile model, in which the avalanches do not overlap because of the slow drive rate 3.

The “\( 1/f \)” form of the power spectrum alone does not imply that the model has long range temporal correlation. This kind of behavior is seen in models with no translational invariance 5, 6. In this case the autocorrelation may have exponential form with different relaxation time at different point in space. So the power spectrum at each point in space has the form \( 1/f^2 \), i.e., the power spectrum of a random walk. Averaging over space, one obtains a power spectrum of form \( 1/f^0 \), with a nontrivial value of \( \alpha \), which is less than 2. By this mechanism even a noncritical models can show a \( 1/f \) noise 3. Thus \( 0 < \alpha < 2 \), does not imply long rang autocorrelation.

In our model there is a spatial pattern (Fig. 1) which could lead to spatially varying relaxation time. However, this pattern slowly changes in time because of the random mutations. To check for the spatially varying relaxation time, we study the waiting time (i.e., the time taken between two consecutive changes of level), with both types of boundary conditions. For fixed boundary condition we select three different sites on lattice. The first site is near the center, the second is near a boundary but away from a corner, and the third is near the a corner. The frequency distribution of the waiting time in the first two cases show a fairly good power law behavior in two decades (see Fig. 8). It can be fitted to the form: \( n(t) \sim t^{-\beta} \), with \( \beta = 1.25 \). At the site near a corner, the waiting time is large and does not fit well with a power law. However, the contribution to the power spectrum from such a sites is negligible. For periodic boundary condition, the waiting time distribution is similar to the one obtained for the site in the bulk with fixed boundary condition (Fig. 3). Thus in both cases, the relaxation time does not significantly depend on the position. Thus the “\( 1/f \)” behavior of the power spectrum is due to the temporal correlation of the steady state dynamics.

We have found similar behavior in the power spectrum of the average cooperative level \( \langle S \rangle \). Thus the exponent \( \alpha \) of the power spectrum appears to be universally in this model.

Our model has many periodically changing subconfigurations in the steady state. For this reason, analysis in terms of avalanches produced by small local perturbations is forbiddingly difficult. Therefore we look for a different method to investigate the dynamical behavior. The analysis of the waiting time gives an alternative way to explore the critical dynamics of the model.

6 Slow mutation rate

In case of a slow mutation rate, i.e., \( p < 0.0005 \) approximately, the model slowly evolves colonies of higher cooperation level, i.e., \( s > 8/13 \) (see Fig. 3 and 4). The defectors play an important role in protecting these colonies against the raid of the level \( s = 8/13 \). The concave part (corner) in the boundary of the colony of higher cooperators is weak against the immediate lower levels. However, one defector at the corner makes the colony of higher cooperators stable.

The higher cooperative levels are not able to grow in the initial stage. They are mainly threatened by the immediate lower levels, rather than the defectors. Slowly a small domain of higher level is formed by mutation near the boundary of the colony of lower cooperators. This process is normally very slow and may take hundreds of time steps. The defectors in the neighborhood of these cooperators become strong and may advance toward the colony of \( s = 8/13 \), whenever there is a fluctuation at the boundary because of mutation. Behind the line of these defectors the colony of higher cooperators grows. The growth is a very slow process and takes tens of thousands of time steps.
Figure 8: Log-log plot of the waiting time distribution for (i) a site near the center (top), (ii) a site near the boundary but away from corner (middle), and (iii) a site near the corner (bottom). The top and the middle graphs have been shifted up by multiplying by 100 and 10 respectively.

Figure 9: Log-log plot of the waiting time distribution for periodic boundary condition.
The steady state in this case is not critical. The power spectrum of fluctuation of activity has a $1/f^2$ form, which is same as the power spectrum of a random walk.

7 Summary

We have explored the possibility of survival of cooperative tendency in a model of locally shared common resources. The spatial extension of the model allows the cooperators to form stable colonies (domains). The selfish agents can only survive in the neighborhood of cooperators, therefore they are found on the boundaries between these colonies. Thus, natural selection has provision for coexistence of both type of behaviors.

In addition, this model shows interesting statistical behavior. A moderate mutation rate makes the model critical with fluctuation of all time scales. The critical fluctuations are helpful in evolution of cooperative behavior in the initial stage. However, it work against the evolution of higher cooperative level. In case of slow mutation rate the system becomes noncritical and an interesting process of evolution starts. The higher cooperators form a coat of defectors and slowly advance in the territory of lower cooperators. Thus the average cooperative level slowly increases.

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