The homogeneous structure in a Cartan space

Edmundas Mazėtis

Lithuanian University of Education Sciences, Faculty of Science and Technology
Studentų g. 39, LT-08106 Vilnius
E-mail: edmundas.mazetis@leu.lt

Abstract. The homogeneous almost product structure on the Finsler space have Lievin Popescu studied. In this paper we study the integrability conditions for the homogeneous product structure in Cartan space with Miron connection.

Keywords: Cartan space, affine connection, homogeneous almost product structure, Nijenhuis tensor.

Let $(T^\ast M, \pi, M)$ be the contangent bundle, where $M$ is a differentiable, real $n$-dimensional manifold. If $(U, \varphi)$ is a local chart on $M$, then the coordinates of a point $u = (x, y) \in \pi^{-1}(U) \in T^\ast M$ will be denoted $(x^i, y_i), i, j, \ldots = 1, 2, \ldots, n$. The natural basis of the module $X(T^\ast 0)M$ is given by $(\partial_{x^i}, \partial_{y^i}) = (\partial_i, \partial^i)$.

Given $H(x, y)$ – the metrical function $T^\ast M \to \mathbb{R}$, the 2-homogeneous with respect to $y_i$, $g^{ij} = \frac{1}{2} \partial^j \partial^i H$ – the metrical tensor, then pair $(T^\ast M, g)$ is Cartan space. The linear connection are given by

$$L_{ij} = \gamma^{ik}_{ij} y_k - \frac{1}{2} \gamma^{kl}_{pq} \partial_{h} g^{pq} \partial_{k} g_{ij}$$  \hspace{1cm} (1)

respectively, such that $\delta_i = \partial_i - L_{ik} \partial^k$, and $(\delta_i, \partial^i)$ is a local basis of $X(T^\ast 0)M = X(T^\ast M) \setminus \{0\}$ which is called the adapted basis to $L_{ij}$. The vector fields $\delta_i$ and $\partial^i$ are 1 and 0 – homogeneous with respect to $y_i$. The tensor of curvature of linear connection $L$ are given by

$$R_{ijk} = \delta_{ij} L_{ik} - \delta_{ik} L_{ij}.$$  \hspace{1cm} (2)

Then differential geometric object $(L_{ij}, R_{ijk}, 0)$ will be called the Miron connection of Cartan spaces [1].

Let $L : (T^\ast M, g) \to R$ a differential function which is 1-homogeneous with respect to $y_i$, $r > 0$ is a constant. We define linear mapping $P : X(T^\ast 0)M \to X(T^\ast 0)M$ given by

$$P(\delta^i) = \frac{L}{r} g_{ik} \delta^k, \quad P(\partial^i) = \frac{r}{L} g^{ik} \delta_k.$$  \hspace{1cm} (3)

Proposition 1.

(a) $P$ is an almost product structure, $P^2 = I$,
(b) $P$ preserves the property of homogeneity of vector fields from $X(T^\ast 0)M$.

The proof is evident.
Proposition 2. Obtain such the identity \([\delta_i, \delta_j] = R_{kij}\partial^k, [\delta_i, \vartheta] = -\Gamma^{ij}_{ik}\partial^k, [\vartheta^i, \vartheta^j] = 0\).

Proof. Proof is follow from the definition of \(\delta_i\) and equality \([\delta_i, \delta_j]f = \delta_i(\delta_j f) - \delta_j(\delta_i f)\) for all functions \(f : T^*M \to \mathbb{R}\).

The almost product structure are integrable, if and only if is Nijenhuis tensor equal zero.

Theorem 1. The homogeneous almost product structure (3) is integrable, if and only if for the object linear connection \(L\) obtain the relations

\[ R_{kij} = \frac{L}{r^2}(g_{ih}g_{jk} - g_{jh}g_{ik})\partial^hL, \quad \text{(4)} \]
\[ \nabla_j g_{ik} - \nabla_i g_{jk} = \frac{1}{L}(g_{jk}\delta_i L - g_{ik}\delta_j L), \quad \text{(5)} \]

there \(\nabla\) - the operator of covariant derivative respect the Miron connection.

Proof. Let \(N\) be the Nijenhuis tensor of the homogeneous almost product structure \(P\)

\[ N(X,Y) = [PX, PY] - P[PX, Y] - P[X, PY] + P^2[X, Y]. \quad \text{(6)} \]

In the adapted basis we have

\[ N(\delta_i, \delta_j) = N^k_{ij}\delta_k + N^{n+k}_{ij}\partial^k, \quad \text{(7)} \]

where

\[ N^k_{ij} = g^{kp}(\nabla_j g_{kp} - \nabla_i g_{jp}) + L^{-1}(\delta_j L \delta^k_i - \delta_i L \delta^k_j), \]
\[ N^{n+k}_{ij} = \frac{L}{r^2}(g_{ih}g_{jk}\partial^hL - g_{ik}g_{jh}\partial^hL) - R_{kij}. \quad \text{(8)} \]

Analogous

\[ N(\delta_i, \vartheta^j) = -N(\vartheta^i, \delta_i) = N^k_{i+n+j}\delta_k + N^{n+k}_{i+n+j}\partial^k = -N^k_{n+j+i}\delta_k - N^{n+k}_{n+i+j}\partial^k, \quad \text{(9)} \]

where

\[ N^k_{i+n+j} = -N^k_{n+j+i} = \frac{\partial^k L}{L}g_{ih}g^{jk} - \frac{\partial^k L}{L}g_{ij} - \frac{r^2}{L^2}g^{ip}g^{bh}R_{hpi}, \]
\[ N^{n+k}_{j+n+i} = -N^{n+k}_{n+i+j} = g^{ih}\nabla_h g_{jk} + g_{pk}\nabla_j g^{ip} + L^{-1}g^{ih}g_{jk}\delta_h L - L^{-1}\delta^k_i L. \quad \text{(10)} \]

Also

\[ N(\vartheta^i, \vartheta^j) = N^k_{n+i+n+j}\delta_k + N^{n+k}_{n+i+n+j}\partial^k, \quad \text{(11)} \]

where

\[ N^k_{n+i+n+j} = r^2L^{-3}(g^{ik}g^{jh}\delta_h L - g^{ih}g^{jk}\delta_i L) + r^2L^{-2}(g^{ih}\nabla_h g^{jk} - g^{ih}\nabla_j g^{ik}), \]
\[ N^{n+k}_{n+i+n+j} = L^{-1}(\partial^i L \delta^k_j - \partial^j L \delta^k_i) + r^2L^{-2}g^{ip}g^{ib}R_{kph}. \quad \text{(12)} \]

Let \(N = 0\), from (8) follow (4) and (5). The calculation give, as from (4) and (5) follow Eqs. (10) and (12).
The homogeneous structure in a Cartan space

Then the following class of Riemannian metrics may be considered on $X(T_0^*M)$

$$G = g_{ij} \, dx^i \, dx^j + g^{ij} \, Dy_i \, Dy_j,$$

(13)

where $Dy_i = dy_i - L_{ik} \, dx^k$. We have

$$G(\delta_i, \delta_j) = g_{ij}, \quad G(\varepsilon^i, \varepsilon^j) = g^{ij}, \quad G(\delta_i, \varepsilon^j) = G(\varepsilon^i, \delta_j) = 0.$$  

(14)

We define the two form $\Phi$ by the class of almost product structure $P$ [2]:

$$\Phi(X, Y) = G(X, PY)$$

(15)

for all vector fields $X, Y$ on $X(T_0^*M)$.

**Proposition 3.** The expression of the 2-form $\Phi$ in a local adapted frame $(\delta_i, \partial_j)$ on $X(T_0^*M)$ is given by

$$\Phi(\delta_i, \delta_j) = 0, \quad \Phi(\delta_i, \partial_j) = \frac{r}{L} \delta^j_i, \quad \Phi(\partial_i, \delta_j) = \frac{L}{r} \delta^j_i, \quad \Phi(\partial_i, \partial_j) = 0.$$  

(16)

The proof follow from Eqs. (3), (14) and (15).

**Theorem 2.** The class of almost product structures $P$ is Kähler, if and only if, $L$ is constant.

**Proof.** From the (16) we have local expression the 2-form $\Phi$:

$$\Phi = \frac{L}{r} \, Dy_i \wedge dx^i + \frac{r}{L} \, dx^i \wedge Dy_i = \left(\frac{L}{r} - \frac{r}{L}\right)Dy_i \wedge dx^i.$$  

(17)

As

$$D \, Dy_i = -dL_{ik} \wedge dx^k = R_{ijk} \, dx^k \wedge dx^h + \Gamma^h_{ik} \, dx^k \wedge Dy_h,$$

(18)

then

$$D \Phi = \left(\frac{\delta_k L}{r} + \frac{r \delta_k L}{L^2}\right) dx^k \wedge Dy_i \wedge dx^i + \left(\frac{\delta^k L}{r} + \frac{r \delta^k L}{L^2}\right) Dy_k \wedge Dy_i \wedge dx^i$$

$$+ \left(\frac{L}{r} - \frac{r}{L}\right) R_{ijk} \, dx^k \wedge dx^h \wedge dx^i + \left(\frac{L}{r} - \frac{r}{L}\right) \Gamma^h_{ik} \, dx_k \wedge Dy_h \wedge dx^i.$$  

(19)

From the Ricci identity follows $R_{ijk} \, dx^k \wedge dx^h \wedge dx^i = 0$, from Eqs. (1) and (2) follows $\Gamma^h_{ik} = \Gamma^h_{ki}$, and $\Gamma^h \, dx^k \wedge Dy_h \wedge dx^i = 0$. Then

$$D \Phi = \left(\frac{1}{r} + \frac{r}{L^2}\right) \left(\delta^k L \, dx^k \wedge Dy_i \wedge dx^i + \delta^k L \, Dy_k \wedge dx_i \wedge dx^i\right).$$  

(20)

As $r > 0$, then $\frac{1}{r} + \frac{r}{L^2} \neq 0$. There for $D \Phi = 0$, if and only if, $\delta_k L = 0$, $\partial^k L = 0$.

**Corollary 1.** If the almost product structure $P$ is Kähler, it is integrable.
References

[1] L. Popesku. Integrability conditions for the homogeneous almost product structure. *Diff. Geom. Dyn. Syst.*, 8:210–214, 2006.

[2] D.D. Porosniuc. A class of locally symmetric Kähler Einstein structures on the nonzero cotangent bundle of a space form. *Balkan J. Geom. Appl.*, 9:68–81, 2004.

REZIUMĖ

**Kartano erdvių homogeninės struktūros**

*Ed. Mazėtis*

Darbe nagrinėjamos Kartano erdvių su Mirono afiniaja sietimis homogeninės sandaugos struktūros, surastos jų integruojamumo sąlygos.

**Raktiniai žodžiai:** Kartano erdvės, afiniosios sietys, homogeninės sandaugos struktūros, Nijenhuiso tenzorius.