On Algebraic Shift Equivalence of Matrices over Polynomial Rings

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Abstract

The paper studies algebraic shift equivalence of matrices over $n$-variable polynomial rings over a principal ideal domain $D(n \leq 2)$. It is proved that in the case $n = 1$, every non-nilpotent matrix over $D[x]$ is algebraically strong shift equivalent to a nonsingular matrix. In the case $n = 2$, an example of non-nilpotent matrix over $\mathbb{R}[x, y, z] = \mathbb{R}[x][y, z]$, which can not be algebraically shift equivalent to a nonsingular matrix, is given.

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1 Introduction

Let $A$ and $B$ be two square matrices of possibly different order over a semi-ring $R$ with 1. We say $A$ and $B$ are elementarily strong shift equivalent over $R$ if there exist matrices $U, V$ over $R$ such that

$$A = UV, VU = B$$

In this case we write $(U, V) : A \equiv B$ or simply $A \equiv B$.

We say $A$ and $B$ are algebraically strong shift equivalent over $R$ of lag $l$ and we write $A \approx B$(lag $l$) if there exists a sequence of $l$ elementary equivalence from $A$ to $B$:

$$(U_1, V_1) : A = A_0 \equiv A_1, (U_2, V_2) : A_1 \equiv A_2, \cdots, (U_l, V_l) : A_{l-1} \equiv A_l = B$$
Say that \( A \) is *algebraically strong shift equivalent* to \( B \) and write \( A \approx B \) if \( A \approx B(\text{lag } l) \) for some \( l \in \mathbb{N} \).

In symbolic dynamics, the case \( R = \mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty) \) is the most interesting (cf. \cite{12} or \cite{3}). However, the problem of the decidability of algebraic strong shift equivalence of matrices over \( R = \mathbb{Z}_+ \) is still open.

Let \( A \) and \( B \) be matrices over a semi-ring \( R \) with 1. If there exists a pair \( U \) and \( V \) of matrices over \( R \) such that

\[
AU = UB, \quad VA = BV, \quad A^l = UV, \quad B^l = VU
\]

then we say that \( A \) and \( B \) are *algebraically shift equivalent of lag \( l \) over \( R \). We denote this situation by \((U, V) : A \sim B(\text{lag } l)\). We say \( A \) is *algebraically shift equivalent* to \( B \) (and write \( A \sim B \)) if \( A \sim B(\text{lag } l) \) for some \( l \).

It is easy to check that *algebraically strong shift equivalence* implies *algebraically shift equivalence*. As is well-known, K.H. Kim and F.W. Roush showed in \cite{4} that the problem of algebraic shift equivalence of matrices over \( R = \mathbb{Z}_+ \) is decidable.

We say that a domain \( D \) has property \( \text{NSSEN} \) if every non-nilpotent matrix \( A \) over \( D \) is algebraically strong shift equivalent to some nonsingular matrix.

Similarly, we say that a domain \( D \) has property \( \text{NSEN} \) if every non-nilpotent matrix \( A \) over \( D \) is algebraically shift equivalent to some nonsingular matrix.

In \cite{2}, E.G. Effros showed that a principal ideal domain \( R \) has property \( \text{NSSEN} \). In \cite{1}, M. Boyle and D. Handelman showed that a commutative domain \( R \) admitting a non-free finitely generated projective module does not have property \( \text{NSEN} \). It is well-known that the Quillen-Suslin Theorem (cf. \cite{7} or \cite{8}) says that a projective module over the ring of \( n \)-variable polynomials over a principal ideal domain \( D \) is always free. So it is natural to ask the following problem.

**Does the ring of \( n \)-variable (\( n \in \mathbb{N} \)) polynomials over a principal ideal domain \( D \) has property \( \text{NSSEN} \) or property \( \text{NSEN} \)?**

This paper aims to answer the problem positively in the case \( n = 1 \) through proving the existence of full rank factorization for any matrix over the ring of univariate polynomials over \( D \), and answer the question negatively in the case \( n = 2 \) by a counterexample.

The main results of the paper are as follows.

**Theorem 1.** Let \( D \) be a principal ideal domain and \( A \) be a square matrix of order \( n \) over \( D[x] \). If \( A \) is not nilpotent and \( l = \min\{k \in \mathbb{N} | \text{rank}(A^k) = \text{rank}(A^{k+1})\} \), then there is an algebraic strong shift equivalence of lag \( l \) from \( A \) to a nonsingular matrix. In other words, \( D[x] \) has property \( \text{NSSEN} \).

**Theorem 2.** Let

\[
A = \begin{pmatrix}
0 & z & -y \\
-z & 0 & x \\
y & -x & 0
\end{pmatrix}
\]

Then \( A \) can not be algebraically shift equivalent to a nonsingular matrix over \( \mathbb{R}[x, y, z] = \mathbb{R}[x][y, z] \). In other words, \( \mathbb{R}[x, y, z] \) does not has property \( \text{NSEN} \).
The organization of the paper is as follows. In section 2 we prove the existence of full rank factorization of matrices over univariate polynomials. In section 3 we give the proof of the main results.

## 2 Existence of Full Rank Factorization

The aim of this section is to prove the following result, which will be used in section 3. The result is a slight generalization of the corresponding result in page 109 of [9] and our proof is based on Theorem 2 in [3] and some results in [9].

**Proposition 3.** Let \( D \) be a principal ideal domain and \( A \) be a matrix of size \( m \times n \) with rank \( r \) over \( D[x] \). Then there exists a full rank factorization of \( A \) over \( D[x] \):

\[
A = PQ
\]

where \( P \) and \( Q \) are matrices of size \( m \times r \) and \( r \times n \) over \( D[x] \) respectively.

To prove the proposition above, we first recall some facts about matrices over domains(cf. [6]), which are assumed to be commutative throughout the paper, and prove some lemmas.

Let \( D \) be a principal ideal domain. Throughout this section the matrices will be over \( D[x] \), which is a unique factorization domain (UFD). Similar to the case of fields, every non-zero matrix has a rank defined by its maximal order of non-zero minors. A square matrix \( A \) is invertible if and only if its determinant is a unit in \( D \). Let \( L \) and \( U \) be \( m \times r \) and \( r \times n \) matrices over \( D[x] \) respectively. Then the Binet-Cauchy theorem says that \( \Lambda^t(L) \Lambda^t(U) = \Lambda^t(LU) \), where \( \Lambda^t(L) \) and \( \Lambda^t(U) \) are the \( t \)-th compound matrix of \( L \) and \( U \) respectively for \( t \leq \min\{m, r, n\} \).

**Definition 4.** Let \( C \) be an \( m \times n \) matrix over \( D[x] \) with rank \( m \). We say that \( C \) is minor left prime (MLP) if 1 is the greatest common polynomial divisor (gcd) of all the \( m \times m \) minors. Say that \( C \) is factor left prime (FLP) if in any polynomial matrix factorization \( C = C_1 C_2 \), where \( C_1 \) is square, \( C_1 \) should be an invertible matrix.

**Lemma 5.** (cf. Theorem 2 in [3]) Let \( D \) be a principal ideal domain and \( A \) be an \( m \times n \) matrix (\( m \leq n \)) of rank \( m \) with entries in \( D[x] \). Let \( d(x) \) be the greatest common divisor of all \( m \)-th order minors of \( A \). Then \( A \) has a factorization as \( A = LU \) with \( \det L = d(x) \). If \( D \) is an Euclidean domain, then we have algorithm to find the factorization.

**Remark 6.** In [3], the lemma above is proved only for Euclidean domains. However, the authors of [3] remarked in page 656 that the result is also true for principal ideal domains.

**Lemma 7.** For a matrix \( A \) \( m \times n \) matrix (\( m \leq n \)) of rank \( m \) over \( D[x] \), \( A \) is MLP if and only if \( A \) is FLP.
Define at least one of them, say \( \Delta p \) where \( i \) of \( F \) over fields.

**Remark 8.** Lemma 7 is a slight generalization of Theorem 3 in [9], which says that MLP and FLP are equivalent for matrices over the ring of bivariate polynomials over fields.

**Lemma 9.** A matrix \( C \) of size \( m \times n \) with rank \( m \) over \( D[x] \) is MLP if and only if there exist \( d_j \in D \) and matrices \( Z_j \) such that

\[
CZ_j = d_j I_m
\]

where \( 0 \leq j \leq s \), \( s \in \mathbb{N} \cup \{0\} \) and \( \gcd(d_0, d_1, d_2, \ldots, d_s) = 1 \).

**Proof.** Denote by \( C_{i_1,i_2,\ldots,i_m} \) the \( m \times m \) submatrix from \( i_1, i_2, \ldots, i_m \) columns of \( C \), by \( C_{i_1,i_2,\ldots,i_m} \) the adjoint matrix of \( C_{i_1,i_2,\ldots,i_m} \), and by \( \Delta_{i_1,i_2,\ldots,i_m} \) the determinant of \( C_{i_1,i_2,\ldots,i_m} \). Then

\[
C_{i_1,i_2,\ldots,i_m} C^*_{i_1,i_2,\ldots,i_m} = \Delta_{i_1,i_2,\ldots,i_m} I_m
\]

Define an \( l \) by \( m \) matrix \( Z_{i_1,i_2,\ldots,i_m} \) as follows: its \( i_1, i_2, \ldots, i_m \)-th rows are the \( 1, 2, \ldots, m \)-th rows of \( C^*_{i_1,i_2,\ldots,i_m} \) and all other rows are 0. Then we have

\[
CZ_{i_1,i_2,\ldots,i_m} = \Delta_{i_1,i_2,\ldots,i_m} I_m
\]

Suppose that \( C \) be MLP. Let \( F \) be the quotient field of \( D \). Then \( D[x] \) is a subring of \( F[x] \), which is a principal ideal domain. Since 1 is the \( \gcd \) of \( \{ \Delta_{i_1,i_2,\ldots,i_m} \} \) \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n \), there exist \( a_{0,i_1,i_2,\ldots,i_m} \in D[x] \) such that

\[
\sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} a_{0,i_1,i_2,\ldots,i_m} \Delta_{i_1,i_2,\ldots,i_m} = d_0 \in D \setminus \{0\}
\]

If \( d_0 = 1 \), then let \( s = 0 \). Otherwise, suppose that

\[
d_0 = \prod_{j=1}^{s} p_j^{t_j}
\]

where \( p_j \) are distinct prime factors of \( d_0 \) in \( D \), \( t_j \in \mathbb{N} \), and \( s \in \mathbb{N} \). Since the \( \gcd \) of \( \{ \Delta_{i_1,i_2,\ldots,i_m} \} \) \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n \) is 1, for any \( j \in \{1, 2, \ldots, s\} \), there exists at least one of them, say \( \Delta_{i_1,i_2,\ldots,i_m} \), is not divisible by \( p_j \). Let \( d_0 + \Delta_{j_1,j_2,\ldots,j_m} = d_j \).

Define

\[
a_{j_1,j_2,\ldots,j_m} = \begin{cases} 
  a_{0,i_1,i_2,\ldots,i_m} & \text{if } i_1,i_2,\ldots,i_m \neq j_1,j_2,\ldots,j_m \\
  a_{0,i_1,i_2,\ldots,i_m} + 1 & \text{if } i_1,i_2,\ldots,i_m = j_1,j_2,\ldots,j_m 
\end{cases}
\]

4
Let
\[ Z_j = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} a_{j,i_1 i_2 \cdots i_m} Z_{i_1 i_2 \cdots i_m} \]

Then
\[ CZ_j = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} a_{j,i_1 i_2 \cdots i_m} \Delta_{i_1 i_2 \cdots i_m} I_m = (d_0 + \Delta_{j_1 j_2 \cdots j_m}) I_m = d_j I_m \]

Obviously \( \gcd(d_0, d_1, d_2, \ldots, d_s) = 1 \). The proof is complete.

Remark 10. The proof of Lemma 9 is adapted from that of Theorem 2 in [9].

Lemma 11. Suppose that \( A_{11}, A_{12} \) and \( A_{21} \) are three matrices over \( D[x] \) such that \( A_{21} A_{11}^{-1} A_{12} \) is a matrix over \( D[x] \). If \( C = (A_{11}, A_{12}) \) is MLP, then \( A_{21} A_{11}^{-1} \) is a matrix over \( D[x] \).

Proof. By Lemma 9, we can prove the result using almost the same method in the proof of the corollary to Theorem 2 in [9]. The detail is omitted.

Now we can prove Proposition 3.

Proof. It suffices to consider the case \( A \) being an \( m \times n \) polynomial matrix with rank \( r < \min(m, q) \). By an appropriate interchange of rows and columns we can always assume that
\[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \]

where \( A_{11} \) is a nonsingular matrix of order \( r \). By Lemma 5 and Lemma 7, we can suppose that \( (A_{11}, A_{12}) = L(U_{11}, U_{12}) \), where \( (U_{11}, U_{12}) \) is MLP. Then
\[ A = \begin{pmatrix} L & 0 \\ 0 & I_{m-r} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ A_{21} & A_{22} \end{pmatrix} \]

Note that
\[ A_{22} = A_{21} U_{11}^{-1} U_{12} \]

By Lemma 11, \( A_{21} U_{11}^{-1} \) is a matrix over \( D[x] \). Obviously, we have
\[ \begin{pmatrix} U_{11} & U_{12} \\ A_{21} & A_{22} \end{pmatrix} = A_1 A_2 \]

where
\[ A_1 = \begin{pmatrix} I_r \\ A_{21} U_{11}^{-1} \end{pmatrix}, \quad A_2 = (U_{11}, U_{12}) \]

Let
\[ A_3 = \begin{pmatrix} L \\ A_{21} U_{11}^{-1} \end{pmatrix} \]
Then
\[ A = A_3 A_2 \]
is a full rank factorization. \hfill \Box

Remark 12. Although the proof above is almost the same as that in page 108-109 in [9], it is included here for the readers' convenience.

3 Proof of The Main Results

We first prove Theorem 1.

Proof. Let \( A \in M_n(D[x]) \), Suppose that

\[ A = B_1 C_1, C_1 B_1 = B_2 C_2, C_2 B_2 = B_3 C_3, \cdots \]
is a sequence of full rank factorizations, i.e., \( B_i C_i \) are full rank factorizations of \( C_{i-1} B_{i-1} \), for \( i = 2, 3, \ldots \). If \( C_i B_i \) is \( p \times p \) and has rank \( q < p \), then the size of \( C_{i+1} B_{i+1} \) will be \( q \times q \). That is, the size of \( C_{i+1} B_{i+1} \) must be strictly smaller than that of \( C_i B_i \) when \( C_i B_i \) is singular. It follows that there eventually must be a pair of factors \( B_k \) and \( C_k \), such that \( C_k B_k \) is either nonsingular or is zero. Let \( l \) be the first integer for which this occurs. Write

\[ A^l = (B_1 C_1)^l = B_1 (C_1 B_1)^{l-1} C_1 = \cdots = B_1 B_2 \cdots B_{l-1} (B_l C_l) C_{l-1} C_{l-2} \cdots C_1 \]
and

\[ A^{l+1} = B_1 B_2 \cdots B_{l-1} B_l (C_l B_l) C_{l-1} C_{l-2} \cdots C_1 \]

If \( C_l B_l = 0 \), then \( A^{l+1} = 0 \). So \( A \) is algebraically strong shift equivalent to a nonsingular matrix \( C_l B_l \) if \( A \) is not nilpotent. Assume that \( C_l B_l \) is nonsingular. If \( B_l = p \times r \) and \( B_l = r \times p \), then rank \( B_l C_l = r \). Since \( C_l B_l \) is \( r \times r \) and nonsingular, it follows that \( rank(C_l B_l) = r = rank(B_l C_l) \). Note that \( B_i \) and \( C_i \) are of full column rank and full row rank respectively for \( i = 1, 2, 3, \cdots \). It follows that

\[ rank(A^{l+1}) = rank(C_l B_l) = rank(B_l C_l) = rank(A^l) \]

Finally, it is easy to check that \( l = min\{k \in \mathbb{N}|rank(A^k) = rank(A^{k+1})\} \). The proof is now complete. \hfill \Box

Remark 13. The proof above is adapted from that of a theorem about the existence of Drazin inverse of matrices over fields in [10].

To prove Theorem 2, we need the following two lemmas.

Lemma 14. (cf. [II]) Let \( R \) be a commutative domain admitting a finitely generated projective module \( P \) that is not free. From a module \( Q \) such that \( P \oplus Q \cong R^n \), let \( e : R^n \to R^n \) be the endomorphism that projects onto \( P \). Then any matrix representation for \( e \) is not algebraically shift equivalent to a nonsingular matrix over \( R \).
Lemma 15. (cf. Example 1.2.2 in [11]) Let \( R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1) \) and \( \sigma \) be the following homomorphism induced by the unimodular row \( \alpha = (x, y, z) \):

\[
\sigma : R^3 \to R : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto ax + by + cz
\]

Then \( Q \oplus R \cong R^3 \) and \( Q = \ker(\sigma) \) is a stably free (projective) module which is not free.

Now we can give the proof of Theorem 2.

Proof. Let \( \varphi : R = \mathbb{R}[x, y, z] \to \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1) \) be the natural surjective ring homomorphism. Note that

\[
A^3 = -(x^2 + y^2 + z^2)A
\]

We have

\[
\varphi(A^4) = \varphi(-A^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix}
\]

and thus \( \varphi(A^4) \) is an idempotent matrix over \( \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1) \).

Note that \( \text{Im}\varphi(A^4) \cong \ker(\sigma) \), where \( \sigma \) be the module homomorphism induced by the unimodular row \( \alpha = (x, y, z) \) in Lemma 15. Suppose that \( A \) is algebraically shift equivalent to a nonsingular 2 \( \times \) 2 matrix, say \( B \). Then, we have an algebraic shift equivalence between \( \varphi(A^4) \) and \( \varphi(B^4) \) and thus \( \det(I_2 - t\varphi(B^4)) = \det(I_3 - t\varphi(A^4)) = (1 - t)^2 \). So \( \varphi(B^4) \) is a nonsingular 2 \( \times \) 2 matrix over \( \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1) \). A contradiction to Lemma 15 arises. The proof is complete.

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