Quantum polarization transformations in anisotropic dispersive media

Yu I Bogdanov$^{1,5,6,7}$, A A Kalinkin$^{2,3}$, S P Kulik$^{4,7}$, E V Moreva$^{2,5}$ and V A Shershulin$^{1,2,6}$

1 Institute of Physics and Technology, Russian Academy of Sciences, 117218 Moscow, Russia
2 International Laser Center of Moscow State University, 119992 Moscow, Russia
3 Zavoisky Physical-Technical Institute, Russian Academy of Sciences, 420029 Kazan, Russia
4 Faculty of Physics, Moscow State University, 119992 Moscow, Russia
5 National Research Nuclear University ‘MEPHI’, 115409 Moscow, Russia
6 National Research University of Electronic Technology MIET, 124498 Moscow, Russia
E-mail: bogdanov_yuriy@inbox.ru and sergei.kulik@gmail.com

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Abstract. Based on the concept of $\chi$-matrix and Choi–Jamiólkowski states we develop the approach of quantum process reconstruction. The key part of the work is devoted to the adequacy of applied reconstruction models. The approach is tested with the statistical reconstruction of the polarization transformations in anisotropic and dispersive media realized by means of quartz plates and taking into account the spectral structure of input polarization states.

7 Authors to whom any correspondence should be addressed.

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1. Introduction

One of the most important trends in the development of quantum information technologies is associated with the development of a proper methodology for the control of quantum states and processes. This methodology is based on quantum measurements and is designed to provide an interface between the development of the hardware components of quantum computers and the practical implementation.

From a mathematical point of view, the basis of this methodology is the quantification of the statistical theory of quantum operations and measurements, which is based on probabilistic positive operator valued measures (resolution of identity) and ‘completely positive mapping of operator algebras in Hilbert space’ [1, 2]. From a technological point of view, such a mathematical theory has to support methods, algorithms and software that should be able to provide an adequate and complete assessment of the quality and effectiveness of specific quantum information systems [3, 4]. It is important to remember that the protocol is supposed to be informationally complete if the number of tomographically complementary projective measurements is equal to the number of parameters to be estimated. Adequacy means that the statistical data directly correspond to the physical density matrix (which has to be normalized, Hermitian and positive). However, generally for mixed states, it can be tested only if the protocol includes redundant measurements, i.e. when their total number exceeds $s^2$ (for the $s$-dimensional quantum system).

Formalism of quantum transformations is used to describe reduced dynamics of open quantum systems. The base of such dynamics is a concept of complete positivity, which was suggested and investigated in a number of works (Kraus [5], Lindblad [6], Gorini et al [7], Evans and Lewis [8] in statistical mechanics and Holevo [9] in quantum communication theory).

It is important to stress that the concept of complete positivity could be formalized through a variety of almost equivalent procedures based on: (i) extended unitary dynamics of an open quantum system interacting with an environment; (ii) Kraus operator product expansion; (iii) Choi–Jamiolkowski (CJ) isomorphism, as well as being based on the formalism of quantum Markovian semigroups dynamics.
It is worth noting that for the complete implementation of the problems of mathematical modelling of quantum information technology, it is not enough to use any one of these procedures listed above: one has to use all these approaches to describe their close relationship. Thus, the unitary representation of quantum operations in the extended space is needed to describe the relaxation processes that are parallel with the evolution of the Hamiltonian. In this case, all processes are considered from the general theoretical positions on the basis of a single Schrödinger equation. The formalism of the operator-sum allows one to visualize the amount of decomposed non-unitary evolution of the density matrix components, which are determined by the corresponding Kraus operators. Note, however, that the Kraus operators themselves are not uniquely defined (up to a very broad arbitrary unitary transformation) [10]. Calculation of the CJ states makes it obvious that the condition of complete positivity is satisfied. The relevant formalism is extremely important when analysing the quality of designed quantum gates. In fact, in this case, the analysis of the evolution of an infinite number of possible states can be replaced by the study of a single state, though set in a space of higher dimension (if the channel acts in an $s$-dimensional Hilbert space, then the corresponding CJ state is specified in the $s^2$-dimensional space). Finally the evolution matrix is a very important notion: it simply and visually links the input and output density matrixes. Note that the CJ $\chi$-matrix and the evolution matrix (presenting the corresponding operators) can easily be transformed one to another.

A powerful technique to obtain a positive estimator for the CJ operator is the maximum-likelihood estimation procedure [11]. Some tools were introduced in order to optimize the quantum process tomography (QPT) as well as quantum state tomography (QST). For example, since the number of linearly independent parameters of the process increases rapidly with the dimensions [12] then a complete characterization of this operator is problematic. In order to reduce the amount of measurement resources, the method of direct characterization of quantum dynamics (DCQD) [13] was established. For a family of informationally incomplete QPT, a more straightforward and conceptually different approach was suggested in order to obtain an estimator for the unknown quantum process to a fair degree of accuracy. This approach invokes the maximum-entropy principle [14]. The conventional maximum-entropy method [15] may be applied for simple qubit channels. It turns out that an appropriate concave channel entropy functional has a unique maximum in the CJ operator and is zero only when the quantum channel is unitary. The extended approach [16] establishes adaptive iterative algorithms to search for the estimator, which maximizes both the likelihood and entropy functionals using the channel entropy functional in [15].

The paper is organized as follows and is basically divided into two parts; the first one relates to the theory (sections 2–5) while the second one includes a description and analysis of the experiments (sections 6–8). Sections 2 and 3 introduce the general concept of the $\chi$-matrix, CJ states and, based on these cornerstones, QPT. Section 4 specifies the application of the described above concepts to the polarization transformations performed by retardant plates on polarization states of light taking into account their dispersive properties. In section 5 we discuss the numerical simulation results for QPT performed with several popular protocols and pay special attention to the adequacy of applied reconstruction models. Section 6 presents experimental techniques for reconstructing polarization transformations in anisotropic and dispersive objects (quartz plates), while section 7 describes the mixed state reconstruction as a sum over its quasi-pure components. Finally, section 8 is devoted to the influence of instrumental uncertainties due to the artificial optical anisotropy on the polarization reconstruction, which seems to be
important when looking at practical problems of state preparation and measurement. Sections 9 and 10 conclude the paper.

2. Quantum transformations and quantum noise

It is a well-known fact that the ideal quantum gate is described by a unitary transformation of the quantum state density matrix

$$\rho_{\text{out}} = U \rho_{\text{in}} U^+. \quad (1)$$

Real state evolution is never unitary. In realistic models it is necessary to take into account the unavoidable interaction of the quantum system with the environment, i.e. quantum noise. In the frame of theory of open quantum systems, state evolution is determined by the operator sum

$$E(\rho) = \sum_k E_k \rho E_k^+ \quad [1,2],$$

therefore the link between input and output states is given by

$$\rho_{\text{out}} = \sum_k E_k \rho_{\text{in}} E_k^+, \quad (2)$$

where $$E_k$$ are the transformation elements or so called Kraus operators.

If the Hilbert space has dimension $$s$$ then operators $$E_k$$ can be presented by $$s \times s$$-dimensional matrices. In the case of a unitary transformation the operator sum has only a single component, given by operator $$U$$.

Let transformation (2) conserve the trace of the density matrix, then the following equality holds:

$$\text{Tr} (\rho_{\text{out}}) = \text{Tr} \left( \sum_k E_k \rho_{\text{in}} E_k^+ \right) = \text{Tr} \left( \left( \sum_k E_k^+ E_k \right) \rho_{\text{in}} \right) = 1. \quad (3)$$

Here we used invariance of the trace operation under cyclic permutation of its arguments. This equation holds for any input density matrices $$\rho_{\text{in}}$$. It is possible if transformation operators $$E_k$$ satisfy the following normalization condition:

$$\sum_k E_k^+ E_k = I, \quad (4)$$

where $$I$$ is an $$s \times s$$-dimensional identity matrix.

The $$\chi$$-matrix can easily be constructed using transformation elements $$E_k$$. These elements contain all the accessible information about the quantum process. The $$\chi$$-matrix plays a key role in QPT [10, 17, 18].

Let us take the matrix $$E_1$$ with dimension $$s \times s$$ and stretch it into the column $$e_1$$ with length $$s^2$$ putting the second column under the first one and so on. This column will serve as the first column of some matrix $$e$$. Similarly, matrix $$E_2$$ gives the second column of matrix $$e$$ and so forth.

Let us define $$s^2 \times s^2$$-dimensional matrix $$\chi$$ based on the matrix $$e$$:

$$\chi = ee^+. \quad (5)$$

It is important that the matrix $$\chi$$ can be interpreted as some density matrix in the $$s^2$$-dimensional space [19, 20].

Then any quantum transformation reduces to some state in the Hilbert space of a higher dimension, which is the so called CJ isomorphism [1]. The corresponding state can be regarded as a joint state of two subsystems (each of dimension $$s$$). To account for the normalization
condition (conservation of the trace) we must ensure that the reduced matrix \( \chi_A \) obtained by tracing out subsystem \( B \) should be equal to the \( s \times s \)-dimensional identity matrix

\[
\chi_A = \text{Tr}_B (\chi) = I. \quad (6)
\]

The \( \chi \)-matrix plays the same role as a density matrix in calculating the probabilities of possible measurement outcomes. Indeed, let the input state be a pure state given by the state-vector \( |c_{\text{in}}\rangle \) and corresponding density matrix \( \rho_{\text{in}} = |c_{\text{in}}\rangle \langle c_{\text{in}}| \). Note that at the same time the density matrix \( \rho_{\text{in}} \) serves as a projector since \( \rho_{\text{in}}^2 = \rho_{\text{in}} \). As a result of quantum transformation, a new state appears at the output \( \rho_{\text{out}} \). Let the measurement act as a projection \( \Pi = |c_m\rangle \langle c_m| \) of the output state onto the column-vector \( |c_m\rangle \). Then, as a consequence of the Born–von Neumann rule, the probability of a measurement outcome is

\[
P = \text{tr}(\rho_{\text{out}} \Pi). \quad (7)
\]

As a next step, let us consider projecting measurement onto an equivalent effective state, given by the tensor product of the complex conjugate of the input state and the state responsible for the measurement outcome

\[
|\tilde{c}_m\rangle = |c^*_{\text{in}}\rangle \otimes |c_m\rangle. \quad (8)
\]

The corresponding projector is \( \tilde{\Pi} = |\tilde{c}_m\rangle \langle \tilde{c}_m| \). If one considers the \( \chi \)-matrix as a density matrix then the probability for an equivalent effective measurement is

\[
\tilde{P} = \text{tr}(\chi \tilde{\Pi}). \quad (9)
\]

The straightforward calculation shows that both considered probabilities coincide: \( \tilde{P} = P \). Therefore, from a probabilistic point of view, the quantum process is completely described by the introduced \( \chi \)-matrix, which is completely equivalent to the assignment of transformation elements set \( E_k \).

This property can be expressed in another equivalent form using an auxiliary system (ancilla) and a so called ‘relative-state’ or CJ state \([1, 2, 18, 21–23]\). Let the quantum transformation \( E \) under consideration act on the \( s \)-dimensional system \( A \). Then let us add the auxiliary system \( B \) with the same dimension and consider the joined system \( AB \) in the Hilbert space \( H_B \otimes H_A \) which will be subject to the input of the maximally entangled state

\[
|\Phi\rangle = \frac{1}{\sqrt{s}} \sum_{j=1}^{s} |j\rangle \otimes |j\rangle. \quad (10)
\]

It is just the appropriate CJ state. Here the first factor in the tensor product relates to subsystem \( B \) while the second one relates to subsystem \( A \). Let the identity transformation \( I \) be applied to subsystem \( B \). Then the transformation \( (I \otimes E) \) will be fulfilled in the joined system \( AB \). It turns out that the normalized \( \chi \)-matrix automatically appears at the output of the system if the density matrix of the input state is \( |\Phi\rangle \langle \Phi| \):

\[
(I \otimes E)(|\Phi\rangle \langle \Phi|) = \rho_\chi, \quad (11)
\]

where \( \rho_\chi = \frac{1}{s} \chi \). The validity of (11) can be checked by a straightforward calculation

\[
\rho_\chi = \frac{1}{s} \sum_{j,j_1,k} |j\rangle \langle j_1| \otimes E_k |j\rangle \langle j_1| E_k^* = \frac{1}{s} \chi. \quad (12)
\]

Figure 1 clarifies the argument presented above.
In other words, the CJ isomorphism leads to a one-to-one correspondence between the trace preserving a completely positive map on $H_B$ and bipartite states $|\Phi\rangle$ on $H_B \otimes H_A$. In the following let us designate $\chi$ as any $\chi$-matrix independently of its normalization. Up to now we have considered the construction of the $\chi$-matrix based on the transformation elements $E_k$. However, it is easy to solve the inverse problem, namely finding $E_k$ knowing $\chi$. To do this, let us diagonalize the $\chi$-matrix

$$\chi = U_\chi D_\chi U_\chi^\dagger.$$  

(13)

Here $D_\chi$ is the diagonal matrix; its diagonal is formed by the eigenvalues of the $\chi$-matrix. All these eigenvalues are non-negative because of positive-definiteness of the $\chi$-matrix. Let us settle eigenvalues in descending order. Columns of matrix $U_\chi$ are eigenvectors of the $\chi$-matrix. Then matrix $e$ can be found by the formula

$$e = U_\chi D_\chi^{1/2}.$$  

(14)

Let rank $r$ of the quantum operation be the number of non-zero eigenvalues of the $\chi$-matrix. Obviously $1 \leq r \leq s^2$. Thus the arbitrary quantum operation can be reduced to the form including not more than $s^2$ matrices $E_k$. The first rank case corresponds to the unitary transformation.

Matrix $D_\chi$ can be easily reduced to the dimension $r \times r$ just by throwing away zero lines and columns. The matrix $U_\chi$ should be transformed to the dimension $s^2 \times r$ keeping first the $r$ columns and throwing away the others. Then matrix $e$ takes the dimension $s^2 \times r$. In this case equality $\chi = U_\chi D_\chi U_\chi^\dagger = ee^\dagger$ will continue to be carried out.

Notice that matrix $e$ and the corresponding matrices $E_k$ are ambiguously determined. Let matrix $e$ have $m$ columns and correspondingly its dimension be $s^2 \times m$ (with $m$ transformation elements $E_k$, $k = 1, 2, \ldots, m$). Obviously the $\chi$-matrix is not changed under the following transformation:

$$e \rightarrow e' = eU,$$  

(15)

where $U$ is the unitary $m \times m$-dimensional matrix.

New matrices $E'_k$, which are unitary, equivalent to the set of initial matrices $E_k$, will correspond to the new matrix $e'$. Notice that due to the optimization procedure (which simply reduces the number of transformation elements) the minimum $m = r$ can be reached. The initial number of transformation elements $m$ can be even greater than $s^2$ (in principle it can be arbitrarily large). It is important that the response of the quantum system can always be described with not more than $s^2$ transformation operators, independent of the number of elementary ‘noisy’ operators $E_k$. This property reflects the important feature of finite-dimensional quantum systems, namely their limited informational nature. The $\chi$-matrix can
be assigned different representations, which are determined by different sets of basic matrices. In fact, up to now we implicitly used the ‘natural’ representation. Let us describe this representation evidently.

Let \(| j \rangle\) be the ket-vector (column) with the \(j\)th element being equal to unity while the rest are zero. Analogously, let \(\langle k |\) be the bra-vector (row) with the \(k\)th element being equal to unity while the rest are zero. The matrix \(| j \rangle \langle k |\) has a unit element if it is settled on the cross of \(j\)th row and \(k\)th column. All the rest of the elements are zero. If the indices \(j\) and \(k\) take the values from 1 through \(s^2\) \((j, k = 1, 2, \ldots, s^2)\) then there are \(s^4\) such matrices. It is obvious that the \(\chi\)-matrix can be decomposed in the following form:

\[
\chi = \sum_{j,k} \chi_{jk} | j \rangle \langle k |. \tag{16}
\]

Here the set of \(s^4\) matrices \(| j \rangle \langle k |\) plays the role of the basis. Of course the decomposition coefficients change under transformation from matrices \(| j \rangle \langle k |\) to other basic sets of matrices. It corresponds to another representation of the \(\chi\)-matrix.

As an example let us describe the transformation from the ‘natural representation’ introduced above to another widely used representation, given by Pauli matrices.

The single-qubit basis set, determined by four \(2 \times 2\) matrices constructed by the unit matrix \(E\) and Pauli matrices \(X, Y, Z\), has the form

\[
I = E/\sqrt{2}, \quad X = \sigma_x/\sqrt{2} \quad Y = -i \sigma_y/\sqrt{2}, \quad Z = \sigma_z/\sqrt{2}. \tag{17}
\]

Then the two-qubit basic set is formed by tensor products of all possible pairs of those matrices (in total 16 operators), the three-qubit basic set is formed by 64 operators and so forth.

The transformation from matrix \(\chi\) in ‘natural’ basis to matrix \(\chi'\) in the basis given by Pauli matrices is determined by the following unitary transformation:

\[
\chi' = U_0^\dagger \chi U_0. \tag{18}
\]

Basic matrices (17) or their tensor products stretched into columns determine the columns of corresponding unitary transformation matrix \(U_0\).

It is worth mentioning that considered sets of basic matrices are orthonormal. In general, the set of basis matrix \((j = 1, \ldots, m)\) is orthonormal if

\[
\text{Tr}(a_j a_k^\dagger) = \delta_{jk}, \quad j, k = 1, \ldots, m. \tag{19}
\]

3. Quantum process tomography

QPT serves an important tool to characterize the operation of a particular quantum channel [24, 25]. Basically QPT is equivalent to a statistical reconstruction of the CJ state \(\rho_\chi\) (11). Performing quantum measurement protocol consisting of \(m\) rows [26] one registers \(m\) values of frequency of events \(k_j, j = 1, \ldots, m\). If the exposure time of the \(j\)th row of the protocol equals \(t_j\), then the registered experiment event number \(k_j\) is a random variable characterized by a Poisson distribution with mean value \(\lambda_j t_j\),

\[
P(k_j) = \frac{(\lambda_j t_j)^{k_j}}{k_j!} \exp\left(-\lambda_j t_j\right). \tag{20}
\]
Here the intensity of the event generation $\lambda_j$ (or the expected number of events while using the registration scheme measuring the event rate) is determined by corresponding intensity operator $\Lambda_j$.

$$\lambda_j = \text{Tr} \left( \Lambda_j \rho_\chi \right).$$  \hspace{1cm} (21)

The most convenient parameterization for the $\chi$-matrix and the corresponding density matrix $\rho_\chi$ is acquired by means of a purification procedure, which is determined by (5). Matrix $e$, after proper normalization, corresponds to the purified state-vector $c$. Notice that due to the unitary arbitrariness (15) the state vector, $c$ is ambiguously determined. However, all possible state-vectors correspond to the same density matrix $\rho_\chi$. Using a purified state-vector, formula (21) takes the form

$$\lambda_j = \langle c | \Lambda_j | c \rangle.$$

Each row of the protocol corresponds to a concrete set of measurement parameters, which links to the selection of a projection (8) of the quantum state at the input and output.

In this paper we used three different families of protocols introduced earlier in the works (‘R’(Singapore)-family: [27, 28]; ‘J’-family: [29, 30]; ‘BN’(Moscow)-family: [3, 4, 26]). The protocol J4, suggested in [29], performs projective measurements of (polarization) qubits upon fixed components of the Stokes vector $|H\rangle$, $|V\rangle$, $|{\scriptstyle-45^\circ}\rangle = \frac{1}{\sqrt{2}} \{ |H\rangle - |V\rangle \}$, and $|L\rangle = \frac{1}{\sqrt{2}} \{ |H\rangle - i |V\rangle \}$. In experiments these measurements are usually performed using two fixed retardant plates (half-wave and quarter-wave) and a polarization prism selecting the particular (vertical) polarization. If the measured qubits were projected on the states possessing tetrahedral symmetry then the protocol transforms to R4. The over-complete BN protocol exploits a single retardant plate and fixed polarization prism. The corresponding measurements are performed for $N$ orientations of the plate with a step of $180^\circ / N$. The optimization of such measurements is described in detail in [3, 4].

The goal of statistical reconstruction of a quantum process is finding the best (in some sense) reconstruction of the corresponding CJ state by proper processing of the experimental data. One of the best methods providing such a reconstruction is the Fisher maximal likelihood method. The task of this paper is finding such a purified state-vector, $c$, which gives the maximum likelihood function. In our case, this function corresponds to the product of Poisson probabilities over all rows of the protocol

$$L = \prod_{j=1}^{m} \frac{(\lambda_j t_j)^{k_j}}{k_j!} \exp \left( -\lambda_j t_j \right).$$

The necessary condition for an extremum of the function (23) leads to likelihood equation [26]

$$Ic = Jc,$$

where $I$ and $J$ are so called theoretical and empirical Hermitian Fisher information matrices

$$I = \sum_{j=1}^{m} t_j \Lambda_j, \quad J = \sum_{j=1}^{m} \frac{k_j}{\lambda_j} \Lambda_j.$$  \hspace{1cm} (25)

The Poissonian likelihood (23) is equivalent to a renormalized likelihood (Fermi extended likelihood), for which the extremal equation, equivalent to (24), was derived earlier in [31].
The normalization condition, which is automatically contained in likelihood equation (24), has the form
\[ \sum_{j=1}^{m} \lambda_j t_j = \sum_{j=1}^{m} k_j = n, \]  
(26)
where \( n \) is the total number of registered events.

Condition (26) links the total number of registered events \( n \) with the total (over all rows of the protocol) expected number of events. In the approach developed in this paper, condition (26) replaces the usually used identity normalization condition \( \langle c | c \rangle = 1 \). However, the state reconstructed by quantum tomography would be the true CJ state if matrix normalization conditions (4) and (6) were fulfilled. For the resulting quantum state to be the true CJ state, it must still satisfy the matrix normalization conditions (4) and (6).

The corresponding condition leads to the fact that a reduced (with respect to input A) CJ state has to be completely mixed and described by the density matrix
\[ \rho^{(A)} = \frac{I_s}{s}, \]  
(27)
where \( I_s \) is the \( s \)-dimensional identity matrix. The considered normalization condition can be taken into account by means of additional constraints imposed on likelihood function (23) \([23]\). However, in this paper we chose an alternative way, which consists of adding auxiliary statistics. The trivial auxiliary statistics corresponds to virtually measured states (27). Notice that when the completely mixed state \( \rho^{(A)} \) is projected onto any state vector \( |c^*_{in}\rangle \) one gets the same outcome probability \( \frac{1}{s} \):
\[ P\left( c^*_{in} \right) = \text{Tr}\left( |c^*_{in}\rangle \langle c^*_{in}| \rho^{(A)} \right) = \frac{1}{s}. \]  
(28)
The corresponding measurement operator in the CJ \( s^2 \)-dimensional space is a summation over all possible outcomes at the output (in subsystem \( B \))
\[ \Lambda\left( c^*_{in} \right) = |c^*_{in}\rangle \langle c^*_{in}| \otimes I_s. \]  
(29)
Here the CJ state \( \rho_\chi \) plays the role of a measurable state and
\[ \text{Tr}\left( \rho_\chi \Lambda\left( c^*_{in} \right) \right) = \frac{1}{s}. \]  
(30)
Relation (30) determines one of the auxiliary rows for each \( |c^*_{in}\rangle \). We also assume that the set of states \( |c^*_{in}\rangle \) is tomographically complete \([32]\). The accuracy of the maximal likelihood method is determined by a matrix of complete information, which is an analogue of the Fisher information matrix
\[ H = 2 \sum_j t_j \frac{(A_j c)(A_j c)^*}{\lambda_j} \]  
(31)
used for estimating the quantum state \([33]\). Matrix (31) is assigned in the real Euclidean space of a doubled dimension. To extract state-vector \( c \) in this representation, one needs to settle the imaginary part of the purified state vector right under its real part.
4. $\chi$-matrix for retardant dispersive plate

The ideal unitary transformation performed by a retardant plate on monochromatic light is a rotation on the Poincaré–Bloch sphere

$$ U = \exp (-i \delta \vec{n}) = I \cos \delta - i \vec{n} \sin \delta = \begin{pmatrix} \cos \delta - i n_z \sin \delta & -i(n_x + i n_y) \sin \delta \\ -i(n_x + i n_y) \sin \delta & \cos \delta + i n_z \sin \delta \end{pmatrix}. $$

(32)

This rotation can be written as follows:

$$ U = \exp (-i \delta \{n_\sigma \}) $$

(33)

where $\delta = \frac{\pi \Delta n h}{\lambda}$ is the optical thickness of the plate, $\Delta n = n_o - n_e$ is the birefringence of the plate material at a given wavelength, $\lambda$, $h$ is its geometric thickness, $n_x = \sin (2\alpha)$, $n_z = \cos (2\alpha)$ and $\alpha$ is the orientation angle between the optical axis of the phase plate and vertical direction $z$.

We assume that light propagates along axis $Y$ while the retardant plate and its optical axis lie in the plane $XZ$.

For the quartz plates we used in the experiment, the dispersion of the refractive indices is given by dispersion equations [34]

$$ n_o = \sqrt{1.30979 + \frac{1.04683 \cdot \lambda^2}{\lambda^2 - 0.01025} + \frac{1.20328 \cdot \lambda^2}{\lambda^2 - 108.584}}, $$

(34)

$$ n_e = \sqrt{1.32888 + \frac{1.05487 \cdot \lambda^2}{\lambda^2 - 0.01053} + \frac{0.97121 \cdot \lambda^2}{\lambda^2 - 84.261}}, $$

(35)

where the wavelength $\lambda$ is supposed to be given in micrometres.

Let us convert unitary transformation (32) to state-vector form. To do this we stretch the $2 \times 2$ matrix into the column with length 4 and normalize it

$$ |\Psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \delta - i n_z \sin \delta \\ -i(n_x + i n_y) \sin \delta \\ -i(n_x - i n_y) \sin \delta \\ \cos \delta + i n_z \sin \delta \end{pmatrix}. $$

(36)

This vector is defined by the single parameter $\delta$. Due to dispersion, different wavelengths $\lambda_j$ correspond to different optical thicknesses $\delta_j$ and rotation angles $\theta_j = 2\delta_j$ on the Poincaré–Bloch sphere, i.e. different state-vectors $|\Psi_j\rangle$, while the rotation axis $\vec{n} = \{n_x, n_y, n_z\}$ does not change.

It is worth pointing out that these states form an incoherent mixture with weights $P_j = P(\lambda_j) \Delta \lambda$, where $P(\lambda)$ is the spectral density and $\Delta \lambda$ is the wavelength quantization step. In order to reach as high an accuracy as possible, one needs to substitute the continuous wavelength interval with a large enough number of steps (knots). In this work we chose about 800 knots for the whole wavelength interval under numerical calculations and 7 knots under the experimental procedure (see below).

It is easy to notice that vector $|\Psi\rangle$ is in fact assigned in two-dimensional subspace

$$ |\Psi\rangle = a |\phi_1\rangle + b |\phi_2\rangle $$

(37)
with orthonormal basis states

\[ |\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} n_z \\ n_x + i n_y \\ n_x - i n_y \\ -n_z \end{pmatrix}. \]  

Hence the probability amplitudes take the following form:

\[ a = \cos(\delta), \quad b = -i \sin(\delta). \]  

Then the density matrix for the given wavelength is

\[ \rho = |\psi\rangle \langle \psi| = aa^* |\phi_1\rangle \langle \phi_1| + bb^* |\phi_2\rangle \langle \phi_2| + ab^* |\phi_1\rangle \langle \phi_2| + ba^* |\phi_2\rangle \langle \phi_1|. \]  

Therefore, the density matrix of the mixed states, corresponding to the \(\chi\)-matrix, is

\[ \rho_\chi = \sum_j |\psi_j\rangle \langle \psi_j| P_j. \]  

Finally we get the \(\chi\)-matrix

\[ \chi = 2\rho_\chi. \]  

The consideration above clearly shows why the \(\chi\)-matrix of a single waveplate has the rank \(r = 2\), while in the general case it can be \(r = 4\).

5. Adequacy of the quantum operation (model) and numerical results

An important parameter in the statistical reconstruction of the quantum process and quantum state tomography is an adequate choice for the rank of the model. In the limiting case of infinitesimal quantum channel ‘contamination’, there is a situation where a pure process is reconstructed based on the model of a mixture. In this case, we have an explicitly inadequate model of QPT. However, this sort of model is widely used in both real and numerical experiments. In this section, we will see that using inadequate models of such a kind leads to a sharp decrease in the accuracy of the statistical reconstruction of quantum processes.

Let us consider a general case of an \(N\)-qubit system, so its dimension is \(s = 2^N\). The number of real parameters determining \(N\)-qubit operation with rank \(r\) is \(v = 2s^2r - r^2 - s^2\). For a completed rank of model \(r = s^2\) and the number of real parameters describing the model is \(v = s^4 - s^2\), whereas for an incomplete rank the corresponding number of degrees of freedom may be much less, \(r \ll s^2\). For example, for the quantum process with single qubits \((s = 2)\) the model of the completed rank has 8 real parameters and the model of the incomplete rank has 12 real parameters.

The accuracy of quantum state/process reconstruction is determined by a random value and its asymptotical distribution, which can be presented in the following form [35]:

\[ 1 - F = \sum_{j=1}^{j_{\text{max}}} d_j \xi_j^2, \]  

where \(d_j \geq 0\) are non-negative coefficients, \(\xi_j \sim N(0, 1), \quad j = 1, \ldots, j_{\text{max}}\) are independent normally distributed random values with zero mean and variance equal to unity, \(j_{\text{max}} = v - 1\).

It can be shown that in the asymptotic limit the loss of fidelity \((43)\) is proportional to \(1/\sqrt{n}\) for the inadequate model (then the pure quantum state/process is reconstructed like a
mixed one), while for the adequate model the loss of fidelity is proportional to $1/n$ [35]. Hence at the same sample size $n$, the reconstruction quality will be higher for the adequate model.

For numerical calculations, we used the process of polarization transformation of photons on the thick retardant plate with the following parameters: the central wavelength is $\lambda = 1.1509 \, \mu m$, the spectrum shape is given by the function $[\sin(\alpha x)]^2$, the full width half maximum (FWHM) is $\Delta \lambda = 0.008 \, \mu m$, the geometrical thickness of the retardant plate is $h = 5024 \, \mu m$ and the orientation angle is $\alpha = \pi/4$. In this case the density matrix for the CJ state to be reconstructed is

$$
\rho_X = \begin{pmatrix}
0.42099 & -0.0047933 i & -0.0047933 i & 0.42099 \\
0.0047933 i & 0.079005 & 0.079005 & 0.0047933 i \\
0.0047933 i & 0.079005 & 0.079005 & 0.0047933 i \\
0.42099 & -0.0047933 i & -0.0047933 i & 0.42099
\end{pmatrix}.
$$

This density matrix has the rank $r = 2$ and its non-zero eigenvalues are $\lambda_1 = 0.84212$ and $\lambda_2 = 0.15788$, thus the quantum operation with an incomplete rank and a reconstruction based on the model of full rank is clearly inadequate.

The numerical modelling results are shown in figure 2 for two protocols, R4 and J4, reconstructed by means of the adequate and inadequate models (with rank $r = 2$ and 4, correspondingly). Figures 2(a) and (b) correspond to quantum process reconstruction by protocols R4 and J4 by means of the adequate method when data generation and its reconstruction were done along the same rank $r = 2$ model. In this case, one can see good agreement between the numerical calculations (blue bars) and the exact theoretical prediction (green line) of the fidelity distribution. We chose $n = 10^4$ in each experiment, while the number of experiments was $N = 200$. 

**Figure 2.** Numerical calculations (blue bars) and theoretical predictions (green solid lines) for the fidelity distribution performed with two protocols R4 (a), (c) and J4 (b), (d), reconstructed by means of the adequate (a), (b) and inadequate (c), (d) models (with rank $r = 2$ and 4 correspondingly).
Figures 2(c) and (d) demonstrate the non-adequate reconstruction algorithm, when data generation is performed by means of the model with rank $r = 2$, while the reconstruction procedure was done by the model of rank $r = 4$. In this case, it turns out that the accuracy losses are much higher than in the corresponding adequate reconstruction method. As in the previous case, the reconstruction is performed by R4 and J4 protocols with $n = 10^4$ and $N = 50$. Detailed numerical calculations show that non-adequacy of the reconstruction model leads to an increase of losses more than six times for the R4 protocol and eight times for the J4 protocol.

6. Experimental set-up and protocol of quantum measurements

The experimental set-up is shown in figure 3(a). The halogen lamp serves as a source of wide-band light. The monochromator MDR-41 selects the radiation with a central frequency of 1.1509 $\mu$m and a variable spectral range from 0.8 up to 8 nm. A paraxial light beam is formed by the single-mode fibre SMF-28, supplied with micro objectives F240FC-1550. Simultaneously, such an optical system provides a fixed attenuation of the photon flux, so the registration system works in a proper linear regime. A Glan–Thompson prism is used for selecting vertical polarization $|V\rangle$, serving as the initial state for the following transformations performed by two retardant plates with geometrical thicknesses $h_1 = 19.5$ and $312.7$ $\mu$m in series.

These plates provide the set-up with three sets of given initial states. The first set ‘J4’ corresponds to polarization states being components of the Stokes vector $|H\rangle$, $|V\rangle$, $|-45^o\rangle = \frac{1}{\sqrt{2}} (|H\rangle - |V\rangle)$ and $|L\rangle = \frac{1}{\sqrt{2}} (|H\rangle - i |V\rangle)$ [29, 30]. In the second set, ‘R4’, the polarization states are chosen in a way to satisfy the following conditions:

$$\vec{a}_j \cdot \vec{a}_k = \frac{4}{3} \delta_{jk} - \frac{1}{3} = \begin{cases} \frac{1}{3} & \text{if } j = k, \\ \frac{-1}{3} & \text{if } j \neq k. \end{cases} \tag{44}$$

The four corresponding vectors are placed symmetrically on the Poincaré–Bloch sphere and form a tetrahedron [27].

In the third set, ‘B4’, the states under reconstruction are formed starting from the initial $|V\rangle$ polarization by following the rotation of the retardant plate $h_1 = 214 \mu$m at angles $\alpha = 0^o$, $15^o$, $30^o$, $45^o$ (see figure 3(b)).

The measurement part of the set-up consists of two retardant plates $h_3 = 19.5$ $\mu$m and $h_4 = 214$ $\mu$m, followed by an analyser (a Glan–Thompson prism transmitting vertical polarization $|V\rangle$) and a single-photon detector based on a GaAs-based avalanche photo-diode with a pigtailed input and an internal gate shaper [36].

Each of the three sets of polarization states has been measured in the eigen basis. For instance, each of the states from the ‘J4’ basis was reconstructed by means of the tomographic protocol, based on projective measurements onto the corresponding states $|H\rangle$, $|V\rangle$, $|-45^o\rangle = \frac{1}{\sqrt{2}} (|H\rangle - |V\rangle)$ and $|L\rangle = \frac{1}{\sqrt{2}} (|H\rangle - i |V\rangle)$. Hence for each set we have measured 16 projections. Knowing the input and measured (output) states, one can reconstruct the $\chi$-matrix of the quantum process. In our experiments, the role of the quantum channel transforming the input states is played by the set of thick quartz plates, each with $5031$ $\mu$m thickness and optical axis oriented at $45^o$ with respect to the vertical direction. For each set of states ‘J4’, ‘R4’ and ‘B4’ two series of experiments have been performed. In the first series the quantum channel was loaded with quasi-monochromatic radiation (bandwidth is 0.8 nm). For this sort of state, the different spectral components forming its spectrum have almost the same polarization transformations because they practically coincide in frequency, so the state can be
considered as quasi-pure. In the second series the slit of the monochromator was completely open, so the broadband radiation passed through (FWHM is about 8 nm). Passing through the thick quartz plates, different spectral components get different polarization transformations due to the dispersion, so the output state was quasi-mixed in polarization. Using a single thick quartz plate provides 63% of the mixture in the output states, while using two plates gives 98% of the mixture.

Comparisons of $\chi$-matrices for quantum processes, reconstructed for each series of experiments with the $\chi$-matrices obtained by simulating the corresponding quantum processes, are shown in the table 1.

Generally, the spectral structure of the input polarization state is not considered in QPT, so the whole state is supposed to be transformed as a pure one. However, this is not

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**Figure 3.** (a) The experimental set-up for QPT by protocols R4 and J4. Thick quartz plates simulated an unknown quantum process by introducing polarization transformation at a given frequency component of the input state(s). (b) The experimental set-up for QPT by protocol B4. Thick quartz plates simulated an unknown quantum process by introducing polarization transformation at a given frequency component of the input state(s). (c) Experimental set-up for component-wise reconstruction of mixed polarization states.
true and sometimes the spectral structure of the state dramatically affects the reconstruction procedure [37]. For example, practically all used biphoton states possess an originally wide spectrum and without spectral post-selection it is not quite correct to consider such states as pure polarization states. Thus an analysis of the polarization reconstruction process, taking into account the spectral structure of the states, is of interest both for quantum state and quantum process tomography.

A convenient comparison of statistical reconstructions performed by different protocols can be performed based on the universal statistical distribution of quantum tomography accuracy [3]. Table 1 presents the fidelities measured with protocols J4, R4 and B4. Three different quantum processes for various spectral ranges 0.8 nm (quasi-pure) and 8 nm (mixed) for each protocol were reconstructed. The first row corresponds to an identity transformation, when there are no quartz plates between the preparation and measurement parts. In this case, the $\chi$-matrix of the quantum process is equal to the unitary matrix with accuracy depending on the quality of the state preparation/measurement procedure itself. The second and third rows correspond to the reconstruction of the quantum processes occurring in the single or double thick quartz plates. The reconstruction procedure depends strongly on the model rank. For a narrow enough spectrum of the input polarization states, the quality of reconstruction is higher when the simple model rank is used. For example, for protocol J4, the quantum process in the single thick quartz plate with quasi-pure states is reconstructed with fidelity $F = 0.9541$. If we reconstruct the same quantum process by the algorithm with higher rank, the fidelity is lower: $F = 0.9279$. The opposite situation is the case for protocol R4, when the quantum process in the single thick quartz plate with mixed states takes place. Here the adequate model, which takes into account the spectral width, gives a higher value for the fidelity $F = 0.9749$ (and $F = 0.8259$ for the non-adequate model).

### 7. Mixed state reconstruction as a sum over its quasi-pure components

Corresponding experiments have been performed in three stages. At the first stage we measured mixed states with different degrees of mixture.

An arbitrary mixed state is given by the density matrix

$$\rho = p_1 |H\rangle \langle H| + p_2 |V\rangle \langle V|,$$

(45)

where the basic vectors can be formed as an action of creation operators in vertical and horizontal polarization modes on the vacuum state

$$|H\rangle = a^\dagger |\text{vac}\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |V\rangle = b^\dagger |\text{vac}\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

(46)
At the same time, the corresponding commutator \([a, b] = 0\), and real coefficients satisfy the normalization condition \(\sum_{i=1}^{2} p_i = 1\). State (45) is often interpreted as an averaging over the classical probability distribution \(p = (p_1, 1 - p_1)\) of the vector state

\[
|\Psi\rangle = c_1 |H\rangle + c_2 |V\rangle
\]  

(47)

with random amplitudes \(c_1\) and \(c_2\) (\(|c_1|^2 = p_1, |c_2|^2 = p_2\)).

In an experiment, the mixed (or multimode) polarization light state with the given spectral width \(\Delta \omega = \frac{2\pi}{\tau_{\text{coh}}}\) can be formed by introducing a delay exceeding the coherence length \(l > l_{\text{coh}} = \frac{2\pi c}{\tau_{\text{coh}}}\) between ‘\(H\)’ and ‘\(V\)’ components of the initially pure polarization state (47).

In our experiments, as an element introducing the delay, we used birefringent material (quartz plate) oriented at 45° with respect to the basic polarization vectors \(|H\rangle\) and \(|V\rangle\) of the initial state (47).

The monochromator selects the central wavelength 1000 nm with spectral width 8 nm (see figure 3(c)).

In the first stage, we used the Glan–Thompson prism, selecting the vertical polarization component and a set of thick quartz plates, \(h_1 = 5031 \mu m\), oriented at 45° for preparing the initial polarization state to be reconstructed. As a result, an initially pure broadband state is transformed to mixed states, with a mixture degree depending on the spectral width of the prepared state. Indeed since the measuring part of the set-up possesses a finite spectral band, the polarization components were integrated over this band and a mixed polarization state was finally registered. For the statistical reconstruction of the state we used the B36 protocol to measure the event rate at each of the 36 orientations of the single retardant plate, \(h_2\), with thickness 312.7 \(\mu m\) introduced in the measurement part of the set-up. These measurements have been performed for several spectral components with varying transmission wavelengths of the monochromator. In our experiments we prepared two polarization qubits with different mixture degrees using single or doubled thick quartz plate(s) (see above). Then we applied the maximal likelihood method developed in [36] to the obtained data for the statistical reconstruction of input states with different mixture degrees.

Fidelity serves as a good quantitative measure for the correspondence between theoretical and reconstructed states. Generally, fidelity takes the form

\[
F = \left( \text{Tr} \sqrt{\rho_0^{1/2} \rho \rho_0^{1/2}} \right)^2,
\]  

(48)

where \(\rho_0\) and \(\rho\) are the theoretical and reconstructed density matrices, correspondingly.

For numerical calculations we divide the initial spectrum of the input state into relatively small segments, while the polarization state is the sum of the states corresponding to the different segments

\[
|\Psi\rangle = \sum_{k} a_k |\Phi(\omega_k)\rangle,
\]  

(49)

where \(|\Phi(\omega_k)\rangle = c_1(\omega_k) |H\rangle + c_2(\omega_k) |V\rangle\) and amplitudes \(a_k\) are determined by the spectral shape of the light passing through the monochromator.

Before passing transforming polarization elements (retardant plates), the density matrix takes the form

\[
\rho^{\text{in}} = |\Psi\rangle \langle \Psi| = \sum_{k,j} a^*_k a_j |\Phi(\omega_k)\rangle \langle \Phi(\omega_j)|.
\]  

(50)
As was noted before, the thin retardant plates transform the state as a whole, i.e. all spectral components get the same polarization transformation, whereas the thick retardant plate transforms the initial density matrix as

\[ \rho' = G(\omega_k) \rho^{in} G^*(\omega_k) , \]

where the matrix

\[ G(\omega_k) = \begin{pmatrix} t_k & r_k \\ -r_k^* & t_k^* \end{pmatrix} \]

depends on complex coefficients \( r_k = i \sin(\delta_k) \sin(2\alpha) \), \( t_k = \cos(\delta_k) + i \sin(\delta_k) \cos(2\alpha) \). Here index \( k \) denotes the chosen frequency component.

Using (51) let us link each input and output frequency component and find the reduced density matrix taking the trace over frequencies

\[ \rho = \sum_k |a_k|^2 G(\omega_k) |\Phi(\omega_k)\rangle \langle \Phi(\omega_k)| G^*(\omega_k). \]

As a result, we got fidelity \( F_1 = 0.997 \) for the first mixed state (63% mixture) and \( F_2 = 0.9986 \) for the second one (98% mixture).

At the second stage, each of the two mixed-state components were measured. To do this we divided the whole spectrum, having the shape of \( \left[ \frac{\sin(x)}{x} \right]^2 \), into seven ranges with central wavelengths \( \lambda_i = 994, 996, 998, 1000, 1002, 1004, 1006 \) nm as shown in figure 4.

The experimental set-up was the same. However, at this stage the monochromator selected relatively narrowband radiation, 0.8 nm, which indicated the above central wavelengths in the

**Figure 4.** The solid line shows the spectral shape of the radiance passing through the monochromator (open slit). Numbered points designate the selected spectral components used for the spectrum digitization and corresponding spectral shape for the narrow slit of the monochromator.
Table 2. Reconstructed quasi-pure components for two mixed polarization states.

| Number of spectral component, $i$ | Fidelity, $F \left( \rho^\text{pure}_{\text{theor}}, \rho^\text{exp}_{i} \right)$ | Single plate (63% mixture) | Double plate (98% mixture) | Central wavelength (nm) | Weight of the component, $a_i$ |
|-----------------------------------|-------------------------------------------------|-----------------------------|----------------------------|-------------------------|----------------------------|
| 1                                 | 0.9970                                          | 0.9983                      | 994                        | 0.1690                  |
| 2                                 | 0.9995                                          | 0.9975                      | 996                        | 0.4955                  |
| 3                                 | 0.9994                                          | 0.9947                      | 998                        | 0.8470                  |
| 4                                 | 0.9984                                          | 0.9972                      | 1000                       | 1                       |
| 5                                 | 0.9965                                          | 0.9940                      | 1002                       | 0.8470                  |
| 6                                 | 0.9958                                          | 0.9961                      | 1004                       | 0.4955                  |
| 7                                 | 0.9966                                          | 0.9972                      | 1006                       | 0.1690                  |

Table 3. Mixed state reconstruction.

| Contribution of components | Entropy, $S$ | Fidelity, $F \left( \rho^\text{mixed}_{\text{theor}}, \rho^\text{comp}_{i} \right)$ |
|----------------------------|--------------|-------------------------------------------------|
|                           | Single plate (63% mixture) | Double plate (98% mixture) | Single plate (63% mixture) | Double plate (98% mixture) |
| 1–7                       | 0.6344   | 0.9835                                          | 0.9974                      | 0.9990                  |
| 2–6                       | 0.5310   | 0.9562                                          | 0.9934                      | 0.9975                  |
| 3, 4, 5                   | 0.2906   | 0.7150                                          | 0.9617                      | 0.9402                  |
| 2, 4, 6                   | 0.6202   | 0.9871                                          | 0.9973                      | 0.9993                  |
| 1–4                       | 0.3820   | 0.7870                                          | 0.9537                      | 0.9479                  |
| 2, 3, 7                   | 0.5214   | 0.4521                                          | 0.8785                      | 0.7860                  |

The polarization states at each of the seven spectral components have been reconstructed by means of the B36 protocol. The corresponding wavelengths and calculated fidelities of the (pure state) density matrix reconstruction are provided in table 2.

In order to reconstruct the mixed state as a sum of its quasi-pure components, we used the following formula:

$$
\rho^\text{comp}_{\text{exp}} = \sum_{i=1}^{7} a_i \rho_i,
$$

where $\rho^\text{comp}_{\text{exp}}$ is the density matrix of the mixed state, $a_i$ are the weights of its components (see table 1), which are calculated using the transmission function of the monochromator, shown in figure 7, and $\rho_i$ are the density matrices of the $i$th quasi-pure states.

Table 3 contains the results of the statistical reconstruction with the different contributions of the different spectral components. The first line shows the total fidelity if all spectral components contributed correspondingly.

At the third stage of the experiments we compare the accuracy of the reconstruction using formula (53), depending on the number of selected quasi-pure components of the state/spectrum. The second and lower lines show the value of fidelity if partial components are used for the reconstruction procedure. Notice that the fidelity of the reconstruction takes relatively high values ($F > 0.99$) even if a small number of frequency components are involved. We have
estimated the entropy of the resulting state as well as the fidelities between the theoretical density matrix $\rho_{\text{mix}}^{\text{theor}}$ and the experimental ones calculated using (53) for the different sets of quasi-pure components (see column 1) and both polarization qubits, with different mixture degrees. The last value can be extracted using a simple formula for the entropy

$$S = - \sum_{n=1}^{2} \lambda_n \log \lambda_n,$$

where $\lambda_n$ are the eigenvalues of the density matrix.

Table 3 clearly demonstrates that even three components (for instance components 2, 4 and 6) chosen symmetrically and uniformly with respect to the central wavelength of the spectral profile are sufficient for a mixed state reconstruction with fidelity $F > 0.99$. For an irregular choice of components the quality of the reconstruction falls and leads to increasing purity of the reconstructed state, which is more or less the predictable result following from qualitative considerations.

8. Influence of instrumental uncertainties due to mechanically induced optical anisotropy on the polarization reconstruction

One of the possible uncertainties arising at the stages of preparation, transformation and measuring polarization states is caused by the piezoelectric effect (or ‘photoelasticity’) [38]. So in this section we apply the technique developed for characterizing the unwanted polarization transformation introduced by the external mechanical force applied to the glass substrate used as part of the regular mirror.

As an example, we show photographs (see figure 5) of an initially isotropic glass plate (1 inch diameter substrate for the regular dielectric mirror) settled between crossed polarization prisms (see figure 6).

The 1 inch diameter glass plate is settled in a regular holder (Thorlabs, KM100). The figure demonstrates clearly the appearance of artificial anisotropy caused by an unwanted polarization transformation in the plate. An analogous effect can easily be observed in the reflected light if instead of the plate the regular dielectric mirror were used. If the effect is linear with respect to
Figure 6. Set-up for observing polarization transformations caused by induced anisotropy. The fused-silica plate $Q$ to which a mechanical stress $F$ is applied is placed between crossed polarizing prisms $P$ and $A$.

the applied mechanical stress, $\sigma = \frac{F}{S}$, then the following relation holds:

$$\Delta n = n_e - n_o = K_1 \sigma,$$

(55)

where $K_1$ is the Brewster constant with a typical value about $K_1 \approx 10^{-12} - 10^{-11}$ m$^2$ N$^{-1}$.

If the stress is applied uniformly then the optical element becomes birefringent and behaves as a uniaxial crystal with an optical axis parallel to the mechanical force. In other words, the refractive index takes the uncertainty in its meaning, which depends on the external stress.

To increase the quality of the state reconstruction and preparation, these kinds of uncertainties have to be taken into account. However, to do this one needs to know which particular transformation is performed by a given element. In this particular case, a direct calculation of the opto-mechanical effect meets some trouble and seems to be ineffective. That is why we used the QPT method to reconstruct the transformation matrix.

We used the ‘R4’ set (see above) as the set of input states, forming a tetrahedron on the Poincaré–Bloch sphere. Each state subjected to a polarization transformation induced by mechanical stress was measured according to the R4 protocol. As a result, the $\chi$-matrix (which completely describes the quantum process) was reconstructed.

The experimental set-up is completely analogous to that shown in figure 6; however, instead of thick quartz plates we used a 1 inch glass plate placed in the standard Thorlabs holder. Using small pinholes to limit the beam, we selected a particular region on the plate where a homogeneous distribution of the mechanical stress (or colour in figure 5) was present. In our experiment, we selected the central region of the plate. This simple technique guarantees the spatial homogeneity of the selected region, which allows a fixed transformation matrix to be measured. We prepared the set of initial states using two plates of thickness 824 and 356 $\mu$m and initial vertically polarized light. Then we performed an analogous measurement with protocol B36 using the 312.7 $\mu$m plate. The reconstructed CJ state has the following form:

$$\rho_X = \begin{pmatrix}
0.50211 & 0.01813 + 0.00096273i & -0.009349 - 0.015417i & 0.19478 + 0.45978i \\
0.01813 - 0.00096273i & 0.00065645 & -0.00036712 - 0.00053873i & 0.0009146 + 0.016228i \\
-0.009349 + 0.015417i & -0.00036712 + 0.00053873i & 0.00064743 & -0.017744 - 0.00258031i \\
0.19478 - 0.45978i & 0.0009146 - 0.016228i & -0.017744 + 0.00258031i & 0.49659
\end{pmatrix}.$$  

Figure 7 illustrates the results of the statistical reconstruction of the process. One can see good agreement between the statistical model and the experimental data.

To check the homogeneity of the induced anisotropy, we reconstructed the density matrix for the R4 set and calculated entropy

$$S = -\sum_{n=1}^{2} \lambda_n \log_2 \lambda_n$$

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with \( \lambda \) being the eigen values of the density matrix. The degree of mixture did not exceed 0.1\%, which confirms the adequacy of our method.

The reconstructed \( \chi \)-matrix clearly shows that the glass plate subjected to induced anisotropy transforms each of the states from the R4 set uniformly, i.e. it can be presented as a standard SU2 matrix

\[
G = \left[ \begin{array}{cc} t & r \\ -r^* & t^* \end{array} \right],
\]

where \( t = \cos \delta + i \sin \delta \cos (2\alpha), r = i \sin \delta \sin (2\alpha) \) and \( \delta = \frac{\pi \Delta n L}{\lambda} \) is optical thickness.

Knowing both crucial parameters, the wavelength, \( \lambda \), and the geometrical thickness of the plate, \( L \), one can reconstruct the orientation of the induced optical axis, \( \alpha \), and birefringence, \( \Delta n \). In our case, for the particular mechanical stress applied to the glass plate, we reconstructed the following: \( \alpha = 91^\circ \) and \( \Delta n = 2.2 \times 10^{-3} \).

We would like to stress that the considered distortion of the polarization states becomes essential if even a relatively small mechanical stress is applied to the initially isotropic optical elements (like beamsplitters, substrates of dielectric mirrors, glass filters, etc). Usually it happens when the optical elements mentioned above are settled in the holders.

Practically avoiding this sort of effect is problematic, since even tiny mechanical pressure causes unwanted anisotropy, which introduces unpredictable changes in the polarization state passing through such an element. Moreover, these effects become crucial at the stage quantum polarization states are prepared or measured. That is why developed QPT methods for testing different parts of the corresponding apparatus seem to be effective and practically required in quantum computation/communication systems.

9. Conclusions

In this work we present a common approach to the statistical reconstruction of the quantum process. The approach is based on the \( \chi \)-matrix and CJ states and includes well-developed
methods of quantum state tomography applied to the CJ state(s). The above study shows that the use of inadequate models leads to a dramatic decrease of the accuracy of the quantum process statistical reconstruction. In fact, it turns out that asymptotically, with increasing sample volume $n$, the loss of accuracy behaves like $1/\sqrt{n}$ for an inadequate model. At the same time, for an adequate model, the corresponding dependence is $1/n$ [37]. This result is clearly demonstrated by the numerical experiments in section 5.

Accounting for the effects of dispersion on polarization transformations is fundamental both to quantum optics and quantum information technology. A complete and rigorous description of this phenomenon is only possible through the consistent application of the theory of quantum operations, which has been carried out in this work. Of course, a description of some of the particular transformations can be done through the traditional methods of optics. However, the approach based on CJ states implemented in our work allows one to achieve much better results. In a sense, the corresponding CJ state gives a complete characterization of the quantum optical element and, in particular, for any input state it is easy to find the corresponding output state.

The key point of this paper is using the concept of adequacy, completeness and accuracy of quantum measurements that we have developed earlier [4, 26–28, 33]. This methodology is designed to make the quantum measurement protocols maximally effective, which is very useful for practical applications. With this approach, we were able to solve the problem of constructing the corresponding CJ state for particular polarization transformations in dispersive media with dispersion and test this approach in numerical and physical experiments.

The experiments, as well as the numerical calculations, are completely consistent with the theoretical distributions for the loss of fidelity and clearly demonstrate the adequacy of the suggested QPT method to analyse the polarization transformations performed in anisotropic and dispersive media, such as the polarization mode dispersion in optical fibres [39]. We hope that the relevant formalism will prove to be extremely important to analyse the quality of designed quantum gates.

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