DIRECTIONAL ISOPERIMETRIC INEQUALITIES AND RATIONAL HOMOTOPY INVARIANTS

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ABSTRACT. We estimate the second order linking invariants of Lipschitz maps from an n-dimensional ellipse. The estimate uses a new directionally-dependent version of the isoperimetric inequality for cycles inside the ellipse. Using this work, we prove new lower bounds for the k-dilation of maps from one ellipse to another.

In this paper, we estimate a second-order rational homotopy invariant of a map in terms of the map’s Lipschitz constant. This problem turns out to be qualitatively harder than estimating a first-order rational homotopy invariant such as the Hopf invariant.

In [4] and [5], Gromov described a basic upper bound for the rational homotopy invariants of a Lipschitz map from a Riemannian manifold. In [2], I showed that when the domain is an n-dimensional ellipse, then the estimates for the Hopf invariant and the linking invariant are sharp up to a constant factor. (Recall that an n-dimensional ellipse $E$ with principal axes $E_0 \leq \ldots \leq E_n$ is the set $\{x \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} (x_i/E_i)^2 = 1\}$.)

We study a second-order linking invariant of maps from $S^n$ to a wedge of three spheres $S^{k_1} \vee S^{k_2} \vee S^{k_3}$. Gromov’s method gives an upper bound for this invariant in terms of a metric on the domain, a metric on the range, and the Lipschitz constant of the map. This upper bound, however, may be too large. As we will see, even if the domain is an ellipse and each sphere in the range has the unit sphere metric, the upper bound may be much too large. In this special case, we will give a better upper bound which is sharp up to a constant factor.

Before I state the results, I want to say something about the new method. Gromov’s method uses the isoperimetric inequality. The new idea in this paper is to replace ordinary isoperimetric inequalities by directional isoperimetric inequalities that separately keep track of the amount of volume of a surface pointing in different directions. If $J$ is an m-tuple of integers from 1 to n, let $P(J)$ denote the corresponding coordinate m-plane in $\mathbb{R}^n$. Now, if $C$ is an m-dimensional surface in $\mathbb{R}^n$, then we define $Vol_J(C)$ to be the volume of the projection of $C$ onto the corresponding m-plane. For example, suppose that $C$ is the long thin curve in the figure below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A long thin ellipse}
\end{figure}

Suppose the curve $C$ has $Vol_1(C) = 20$ and $Vol_2(C) = 4$. The total length of $C$ is slightly more than 20. In general, a plane curve of length 20 may bound a region of area 30, but because this curve has such small 2-volume, it bounds a significantly
smaller region. By keeping track of the volumes in different directions, we will be able to find more efficient fillings of certain cycles.

Loomis and Whitney proved a directional estimate in this spirit in [6].

**Theorem.** (Loomis, Whitney) Suppose that $U \subset \mathbb{R}^n$ is an open set. For any $(n-1)$-tuple $J$, let $A_J$ denote the area of the projection of $U$ onto $P(J)$. Then the volume of $U$ is bounded in terms of the areas of its projections as follows.

$$\Vol(U) \leq \left[ \prod_{J} A_J \right]^\frac{1}{n-1}.$$

In this formula, $J$ varies over the $n$ different $(n-1)$-tuples of integers from 1 to $n$.

The Loomis and Whitney theorem bounds the volume enclosed by a hypersurface in terms of the directional volumes of the hypersurface. In this paper we derive estimates in a similar spirit which apply to surfaces of any codimension.

Now we return to estimating rational homotopy invariants of Lipschitz maps. In the rest of the introduction, I will try to explain the answers to three questions.

1. How can we estimate rational homotopy invariants using isoperimetric inequalities?
2. Why are these estimates far from sharp for some second-order invariants?
3. How can we improve the estimates using directional isoperimetric inequalities?

The linking invariant is defined for a map $F$ from $S^n$ to the wedge of spheres $S^{k_1} \vee S^{k_2}$ provided that the dimensions obey $n + 1 = k_1 + k_2$ and $2 \leq k_1 \leq k_2$. We let $q_1$ denote a generic point in $S^{k_1}$ and $z_1$ denote the fiber $F^{-1}(q_1)$. The fiber $z_1$ is a closed oriented submanifold of $S^n$. We let $y_1$ be a chain in $S^n$ with $\partial y_1 = z_1$. We let $q_2$ be a generic point of $S^{k_2}$, and we define $z_2$ to be the intersection $F^{-1}(q_2) \cap y_1$. For generic $q_2$, this intersection will consist of finitely many points, each with an orientation. The signed number of points is the linking invariant of $F$, denoted $L(F)$. (By standard results in topology, $L(F)$ does not depend on the choices we made, and it is a homotopy invariant of $F$.)

Now suppose that we have a metric $g$ on $S^n$ and also a metric on the target $S^{k_1} \vee S^{k_2}$. Using these metrics, we can define the Lipschitz constant of the map $F$. Our goal is to bound the linking invariant of $F$ in terms of the Lipschitz constant and the metrics. In particular, I want to understand how the best bound depends on the metric $g$. In order to control the linking invariant of a map $F$, we will estimate the number of points in $z_2$, and in order to do that, we control the volumes of $z_1$ and $y_1$. Our estimate involves two geometric ingredients. The first ingredient is to find a fiber with small volume, using the coarea inequality.

**Lemma 1.** (Coarea inequality) Let $F : (M^n, g) \to (N^q, h)$ be a $C^\infty$ map with Lipschitz constant $L$. Then $F$ has a regular fiber with volume at most $L^q \Vol(M)/\Vol(N)$.

Using this lemma, we can bound the volume of $z_1$ in terms of the volume of $(S^n, g)$. The next step of the argument is to bound the volume of $y_1$ in terms of the volume of $z_1$. This step requires an isoperimetric inequality that holds in $(S^n, g)$. In this paper, we will focus on the case that $(S^n, g)$ is an ellipse. In that case, the relevant isoperimetric inequality is described by the following lemma.

**Lemma 2.** [2] Suppose that $E$ is an $n$-dimensional ellipse with principal axes $E_0 \leq \ldots \leq E_n$. Suppose that $z$ is an $m$-cycle in $E$. Then there is an $(m+1)$-chain $y$ in $E$ with $\partial y = z$ obeying the following estimate.
\[ \text{Vol}(y) \leq C(n)[E_{m+1} + E_{n-m} \text{Vol}(z)]. \]

Suppose that \( F \) is a map from the ellipse \( E \) to the wedge \( S^{k_1} \vee S^{k_2} \) equipped with its standard metric. We can bound the linking invariant of \( F \) in terms of its Lipschitz constant by combining the two lemmas above. First, we use Lemma 1 to bound the volume of \( z_1 \). Then we use Lemma 2 to bound the volume of \( y_1 \). Finally, we use Lemma 1 again to bound the volume of \( z_2 \). Since \( z_2 \) is a 0-cycle, its volume is equal to the number of points in it, and so the volume of \( z_2 \) controls the linking invariant. Putting together these steps, we get the following estimate.

\textbf{Proposition 1.} (\textsuperscript{[2]}) Let \( E \) be an \( n \)-dimensional ellipse with principal axes \( E_0 \leq \ldots \leq E_n \). Let \( 2 \leq k_1 \leq k_2 \) with \( n+1 = k_1 + k_2 \). Let \( F \) be a map from \( E \) to \( S^{k_1} \vee S^{k_2} \). Equip the target with its standard metric. Suppose that \( F \) has Lipschitz constant \( L \).

Then \( |L(F)| \leq C(n)E_{k_2} \text{Vol}(E)L^{n+1} \).

In \textsuperscript{[2]}, I showed that this estimate is sharp up to a factor \( C(n) \) for sufficiently large \( L \).

Now we turn to second order invariants. We will define a second-order linking invariant. It is closely analogous to the linking invariant we just considered, but it involves three fibers and two filling operations.

The second-order linking invariant is defined for a map \( F \) from \( S^n \) to the wedge of spheres \( S^{k_1} \vee S^{k_2} \vee S^{k_3} \) provided that the dimensions obey \( n+2 = k_1 + k_2 + k_3 \) and \( 2 \leq k_1 \leq k_2 \leq k_3 \). We let \( q_1 \) denote a generic point in \( S^{k_1} \) and \( z_1 \) denote the fiber \( F^{-1}(q_1) \). We let \( y_1 \) be a chain in \( S^n \) with \( \partial y_1 = z_1 \). We let \( q_2 \) be a generic point of \( S^{k_2} \), and we define \( z_2 \) to be the intersection \( F^{-1}(q_2) \cap y_1 \). The fiber \( z_2 \) is an integral cycle in \( S^n \). We let \( y_2 \) be a chain in \( S^n \) with \( \partial y_2 = z_2 \). Finally, we let \( q_3 \) be a generic point of \( S^{k_3} \), and we define \( z_3 \) to be the intersection \( F^{-1}(q_3) \cap y_2 \). This intersection will consist of finitely many points, each with an orientation. The signed number of points is the second-order linking invariant of \( F \), denoted \( L_2(F) \).

(Like the first-order linking invariant, \( L_2(F) \) does not depend on the choices we made, and it is a homotopy invariant of \( F \).)

We can use the same strategy to bound \( L_2(F) \) for a Lipschitz map from an ellipse. Lemma 1 bounds the volume of \( z_1 \). Then Lemma 2 bounds the volume of \( y_1 \). Then Lemma 1 bounds the volume of \( z_2 \). Then Lemma 2 bounds the volume of \( y_2 \). Finally, Lemma 1 bounds the volume of \( z_3 \) which bounds \( |L_2(F)| \). If we carry out the calculations, we get the following bounds.

\textbf{Proposition 2.} Suppose that \( 2 \leq k_1 \leq k_2 \leq k_3 \) and that \( n+2 = k_1 + k_2 + k_3 \). Let \( E \) be an \( n \)-dimensional ellipse with principal axes \( E_0 \leq \ldots \leq E_n \). Let \( F \) be a map from \( E \) to the wedge of unit spheres \( S^{k_1} \vee S^{k_2} \vee S^{k_3} \) with Lipschitz constant \( L \).

If \( k_3 < (n+1)/2 \), then \( L_2(F) \) is bounded as follows.

\[ |L_2(F)| \leq C(n)E_{n-k_3+1}E_{n-k_3+1} \text{Vol}(E)L^{n+2}. \]

If \( k_3 \geq (n+1)/2 \), then \( L_2(F) \) is bounded as follows.

\[ |L_2(F)| \leq C(n)E_{n-k_1+1}E_{k_3} \text{Vol}(E)L^{n+2}. \]
In the first case, \( k_3 < (n+1)/2 \), it turns out that this inequality is not sharp. The main result of this paper is a refined inequality which is sharp up to a constant factor.

In order to understand why this basic upper bound is not sharp, we have to understand a little bit about the isoperimetric inequality in ellipses given in Lemma 2 above. According to Lemma 2, an \( m \)-cycle \( z \) bounds a chain \( y \) with \( |y| \lesssim |E_{m+1} + E_{n-m}| \). This bound is sharp up to a constant factor. The way that the worst case cycle looks depends on the dimension \( m \). If \( m \geq (n-1)/2 \), then the smallest \( m \)-dimensional equator of \( E \) is the hardest \( m \)-cycle to fill (up to a constant factor).

On the other hand, if \( m < (n-1)/2 \), then the largest \( m \)-dimensional equator of \( E \) is the hardest \( m \)-cycle to fill.

If \( k_3 < (n+1)/2 \), then the cycle \( z_1 \) has dimension \( m_1 \) at least \((n-1)/2\), but the cycle \( z_2 \) has dimension \( m_2 \) less than \((n-1)/2\). Now we can informally describe why Proposition 2 is not sharp for \( k_3 < (n+1)/2 \). If Proposition 2 were sharp, it would mean that \( z_1 \) “looks like” the smallest \( m_1 \)-dimensional equator of \( E \) and \( z_2 \) “looks like” the largest \( m_2 \)-dimensional equator of \( E \). If \( z_1 \) actually were the smallest \( m_1 \)-dimensional equator of \( E \), then we could choose \( y_1 \) to be a hemisphere inside the smallest \( m_1 + 1 \)-dimensional equator of \( E \). In that case, \( z_2 \) would be an \( m_2 \)-cycle lying inside the smallest \( m_1 + 1 \)-dimensional equator of \( E \). Such a cycle looks very different from the largest \( m_2 \)-dimensional equator of \( E \), and in particular it can be filled much more efficiently. We can make this argument effective by keeping track of the directional volumes of \( y_i \) and \( z_i \).

Now we describe the argument in a bit more detail. We begin in the same way as the basic argument: we apply Lemma 1 to find a fiber \( z_1 \) with controlled volume. In the basic argument, we applied Lemma 2 to find a chain \( y_1 \) with volume at most \( \sim E_{m_1+1}|z_1| \). In our more refined argument, we construct a chain \( y_1 \) by a different method, which allows us to bound the directional volumes of \( y_1 \) in a useful way. In some directions, the volume of \( y_1 \) may be as large as \( E_{m_1+1}|z_1| \), but in most directions the directional volume of \( y_1 \) is much smaller. The directional volume of \( y_1 \) is concentrated in the directions where the ellipse \( E \) is small, such as \( I = [1, \ldots, m_1 + 1] \). Next we choose a fiber \( z_2 \subset y_1 \). In the basic argument, we applied Lemma 1 to find a fiber \( z_2 \) with controlled volume. In the refined argument, we want to control all the directional volumes of \( z_2 \) in terms of our bounds for the directional volumes of \( y_1 \). To do that, we use a small generalization of Lemma 1. At this stage, we have a bound for the total volume of \( z_2 \) which is the same as in the basic argument, but we have stronger bounds on many of the directional volumes of \( z_2 \) which show that \( z_2 \) is concentrated in the directions where \( E \) is small. Next, we again use a directional isoperimetric inequality to find a chain \( y_2 \) with boundary \( z_2 \). The basic isoperimetric inequality in Lemma 2 tells us that we can find \( y_2 \) with volume at most \( \sim E_{n-m_2}|z_2| \). But since the volume of \( z_2 \) is concentrated in the small directions, we can improve on that estimate and find \( y_2 \) with significantly smaller volume. Finally we use Lemma 1 to bound the volume of \( z_3 \) and thus bound \( |L_2(F)| \) as before.

**Theorem 1.** Suppose that \( 2 \leq k_1 \leq k_2 \leq k_3 \), and \( n + 2 = k_1 + k_2 + k_3 \). Let \( E \) be an \( n \)-dimensional ellipse with principal axes \( E_0 \leq \ldots \leq E_n \). Let \( F \) be a map from \( E \) to the wedge of unit spheres \( S^{k_1} \cup S^{k_2} \cup S^{k_3} \) with Lipschitz constant at most \( L \). Then \( L_2(F) \) is bounded as follows.
If \( k_3 < (n + 1)/2 \), then the upper-bound in Theorem 1 is better than the one in Proposition 2 by a factor \( E_{n-k_3+1}/E_{k_3} \), which may be arbitrarily large. On the other hand, the upper bound in Theorem 1 is sharp up to a constant factor.

**Theorem 2.** Suppose that \( 2 \leq k_1 \leq k_2 \leq k_3 \), and \( n+2 = k_1+k_2+k_3 \). Let \( E \) be an \( n \)-dimensional ellipse with principal axes \( E_0 \leq ... \leq E_n \). Suppose that \( L > C(n)E_{k_1}^{-1} \). Then there is a map \( F \) from \( E \) to the wedge of unit spheres \( S^{k_1} \lor S^{k_2} \lor S^{k_3} \) with Lipschitz constant \( L \) and \( L_2(F) \) bounded below as follows.

\[
L_2(F) \geq c(n)E_{n-k_1+1}E_{k_3}Vol(E)L^{n+2}.
\]

We pause here to make some comments. For a general Riemannian metric \((S^n,g)\), directional volumes are not even well-defined. The refined estimate in Theorem 1 is applicable only to ellipsoidal metrics, whereas the basic estimate is applicable to all metrics. But in its narrow range of applicability, the refined estimate outperforms the basic estimate and is sharp up to a constant factor.

I became interested in this question because I was trying to estimate the \( k \)-dilations of mappings from one ellipse to another. Recall that the \( k \)-dilation is a generalization of the Lipschitz constant that measures how much a mapping stretches \( k \)-dimensional areas. Gromov noticed that his upper bounds for maps with a given Lipschitz constant extend to maps with a given \( k \)-dilation for an appropriate value of \( k > 1 \) depending on the problem. Similarly, Theorem 1 can be extended to maps with a bound on the \( k \)-dilation. More precisely, if \( F \) has \( k_1 \)-dilation at most \( L^{k_1} \), then the conclusion of Theorem 1 still holds. Hence, we can estimate the largest value \( L_2(F) \) for a map \( F \) from \( E \) to a standard wedge of spheres with a given \( k_1 \)-dilation. This result implies some new estimates about the \( k \)-dilations of maps from one ellipse to another.

**Theorem 3.** Let \( E, E' \) be \( n \)-dimensional ellipses. Let \( E_0 \leq ... \leq E_n \) be the principal axes of \( E \). Let \( E'_0 \leq ... \leq E'_n \) be the principal axes of \( E' \). Let \( Q_i = E'_i/E_i \). Suppose that \( \Phi \) is a map from \( E \) to \( E' \) with degree \( D \). Suppose that \( 2 \leq k_1 \leq k_2 \leq k_3 \), \( n+2 = k_1+k_2+k_3 \) and \( k \leq k_1 \). Then the following inequality holds.

\[
Dil_k(\Phi) > c(n)[D|Q_{n-k_1+1}Q_{k_3}Q_1...Q_n|^{1/k}.
\]

The simplest example is the case \( n = 4 \) and \( k = k_1 = k_2 = k_3 = 2 \). In this case, the \( 2 \)-dilation of \( \Phi \) is at least \( \sim |D|^{1/3}Q_1^{1/3}Q_2^{1/3}Q_3^{1/3}Q_4^{1/3} \).

The problem of giving sharp lower bounds for the \( k \)-dilation of a degree 1 map from one \( n \)-dimensional ellipse to another looks very difficult. (To be clear, I would like an estimate which is sharp up to a constant factor \( C(n) \) independent of the principal axes of the ellipses.) The analogous problem for the \( 2 \)-dilation of mappings between \( 4 \)-dimensional rectangles was recently solved in [3], and the answer turned out to be complicated. The near-optimal mappings are far from linear. The possible pairs of rectangles are divided into several cases and in each case there is a rather different non-linear mapping. Also, it turns out that the smallest \( 2 \)-dilation of a degree 1 diffeomorphism may be larger than the smallest \( 2 \)-dilation of a degree 1 map by an arbitrary factor.
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1. BACKGROUND

We recall the definition of $L_2(F)$ and prove that it is a homotopy invariant. The invariant $L_2(F)$ is a special case of a rational homotopy invariant and so it fits into a general theory of rational homotopy invariants and differential forms developed by Sullivan [7]. (Historical question: who first defined the invariant $L_2$?) I don’t know a good reference in the literature, so we give a self-contained presentation along the lines of Bott and Tu (II). We define the invariant $L_2(F)$ and prove that it is a homotopy invariant.

Suppose that $F$ is a map from $S^n$ to the wedge of spheres $S^{k_1} \vee S^{k_2} \vee S^{k_3}$. We want to define $L_2(F)$ in the case that the dimensions obey the conditions $2 \leq k_1 \leq k_2 \leq k_3$ and $n + 2 = k_1 + k_2 + k_3$.

Let $q_i$ denote a point in $S^{k_i}$. First we consider the special case that $F$ is a $C^\infty$ map and that $q_i$ are regular values for $F$. (We mean here that $F$ is $C^\infty$ away from the inverse image of the basepoint of $S^{k_i}$.) In this case, we define $L_2(F)$ as follows. First we consider the fiber $z_1 = F^{-1}(q_1)$ in $S^n$. This fiber is an orientable manifold of dimension $n - k_1$. We choose an integral chain $y_1 \subset S^n$ with $\partial y_1 = z_1$. Next we consider the intersection of $y_1$ with the fiber $F^{-1}(q_2)$. After putting $y_1$ in general position, the intersection is a cycle $z_2$ of dimension $n - k_1 + 1 - k_2 = k_3 - 1$. We choose a chain $y_2$ in $S^n$ with $\partial y_2 = z_2$. The chain $y_2$ has dimension $n - k_1 - k_2 + 2 = k_3$. We let $\pi$ denote the retraction from $S^{k_1} \vee S^{k_2} \vee S^{k_3}$ to $S^{k_3}$. The composition $\pi \circ F$ maps $y_2$ to $S^{k_3}$ and maps the boundary of $y_2$ to the basepoint of $S^{k_3}$. Therefore, the composition has a well-defined degree, and the degree is defined to be $L_2(F)$.

For a general map $F$, we homotope $F$ to a map $F_{nice}$ which is $C^\infty$ and for which $q_i$ are regular values. Then we define $L_2(F)$ to be $L_2(F_{nice})$.

We have to prove that $L_2(F)$ is well-defined, and does not depend on the choices that we made above. In particular, it does not depend on the choice of $y_1$, it does not depend on the choice of $q_i$, and it does not depend on the choice of $F_{nice}$. Also, we prove that $L_2$ is a homotopy invariant of $F$.

**Proposition 1.1.** The quantity $L_2(F)$ is independent of the choices. First, if $F$ is $C^\infty$ and $q_i$ are regular values of $F$, then $L_2(F)$ is independent of the choices of $y_1$ and $y_2$. Second, $L_2(F)$ is independent of the choice of $F_{nice}$. Third, $L_2$ is a homotopy invariant. Finally, $L_2(F)$ is independent of the choice of $q_i$.

**Proof.** We assume that $F$ is $C^\infty$ and that $q_i$ are regular values of $F$. A priori, $L_2$ depends on $F$, $y_1$, and $y_2$, so we write it as $L_2(F, y_1, y_2)$. We prove that $L_2(F)$ is independent of $y_1$ and $y_2$.

First we show that $L_2$ is independent of the choice of $y_2$. Suppose we chose a different cycle $y_2'$. Let $\Sigma$ denote $y_2 - y_2'$. Note that $\Sigma$ is a $k_3$-cycle in $S^n$. The difference $L_2(F, y_1, y_2) - L_2(F, y_1, y_2')$ is given by the degree of the cycle $\pi \circ F(\Sigma)$ in $S^{k_3}$. But the cycle $\Sigma$ is exact, so this degree is zero. Since $L_2$ does not depend on the choice of $y_2$, we may write it as $L_2(F, y_1)$.

Next we show that $L_2$ is independent of the choice of $y_1$. We study $L_2(F, y_1) - L_2(F, y_1')$. The difference $y_1 - y_1'$ is a cycle in $S^n$ that bounds a chain $\partial$. Now we define $z_2$ to be the intersection of $y_1$ with $F^{-1}(q_2)$ and $z_2'$ to be the intersection of $y_1'$ with $F^{-1}(q_2)$. We define $\partial$ to be the intersection of $\partial$ with $F^{-1}(q_2)$. The
boundary of $B$ is $z_2 - z'_2$. Next we pick a chain $y_2$ with boundary $z_2$, and we define $y'_2$ to be $y_2 - B$. Note that the boundary of $y'_2$ is $z_2 - z_2 + z'_2 = z'_2$. Recall that $L_2(F, y_1)$ is the degree of $\pi \circ F(y_2)$ and $L_2(F, y'_1)$ is the degree of $\pi \circ F(y'_2)$. Therefore the difference $L_2(F, y_1) - L_2(F, y'_1)$ is the degree of $\pi \circ F(B)$. But $B$ lies in the fiber $F^{-1}(q_2)$ and $q_2$ is a point in $S^{k_3}$, so $\pi \circ F(B)$ is the basepoint $\ast$. Since $L_2$ does not depend on the choice of $y_1$, we may write it as $L_2(F)$.

Now we suppose that $F$ is a homotopy from $F_0$ to $F_1$. We assume that $q_i$ are regular values for $F$, $F_0$, and $F_1$. Under these hypotheses, we prove that $L_2(F_0) = L_2(F_1)$.

Suppose that $F : S^n \times [0, 1] \rightarrow S^{k_1} \vee S^{k_2} \vee S^{k_3}$ is a homotopy from $F_0$ to $F_1$. We have to check that $L_2(F_0) = L_2(F_1)$. First we consider the fiber $F^{-1}(q_1)$. This fiber is a homology from $F_0^{-1}(q_1)$ to $F_1^{-1}(q_1)$. We let $y_{1.0}$ be a chain filling $F_0^{-1}(q_1)$ in $S^n \times \{0\}$, and we let $y_{1.1}$ be a chain filling $F_1^{-1}(q_1)$ in $S^n \times \{1\}$. The sum $y_{1.1} + F^{-1}(q_1) - y_{1.0}$ defines a cycle in $S^n \times [0, 1]$, and we defined $y_1$ to be a chain filling this cycle. The dimension of $F^{-1}(q_1)$ is $n + 1 - k_1$, and so the dimension of $y_1$ is $n + 2 - k_1$.

Next we intersect $y_1$ with the fiber $F^{-1}(q_2)$. We make the following definitions.

$z_{2.0} := y_{1.0} \cap F^{-1}(q_2) \subset S^n \times \{0\}$.
$z_{2.1} := y_{1.1} \cap F^{-1}(q_2) \subset S^n \times \{1\}$.
$z_2 := y_1 \cap F^{-1}(q_2) \subset S^n \times \{0, 1\}$.

Here $z_{2.0}$ and $z_{2.1}$ are cycles, and $z_2$ is a chain with boundary $z_{2.1} - z_{2.0}$.

Next we fill $z_2$. We let $y_{2.0}$ be a chain in $S^n \times \{0\}$ with boundary $z_{2.0}$. We let $y_{2.1}$ be a chain in $S^n \times \{1\}$ with boundary $z_{2.1}$. And we let $y_2$ be a chain in $S^n \times [0, 1]$ with boundary $y_{2.1} + z_2 - y_{2.0}$.

Finally, we consider the map $\pi \circ F$ from $y_2$ to $S^{k_3}$. The map $\pi \circ F$ takes $z_2$ to the basepoint. The image $\pi \circ F(y_{2.0})$ is a $k_3$-cycle in $S^{k_3}$ of degree $L_2(F_0)$. The image $\pi \circ F(y_{2.1})$ is a $k_3$-cycle in $S^{k_3}$ of degree $L_2(F_1)$. The image $\pi \circ F(y_2)$ is a homotopy from $\pi \circ F(y_{2.0})$ to $\pi \circ F(y_{2.1})$. Hence $L_2(F_0) = L_2(F_1)$.

Now for any map $F$, we can homotope $F$ to a $C^\infty$ map $F_{nice}$ for which $q_i$ are regular values. We define $L_2(F)$ to be $L_2(F_{nice})$. Because of the homotopy result we proved above, the value of $L_2(F)$ does not depend on how we choose $F_{nice}$. It follows that $L_2$ is a homotopy invariant of $F$.

Finally, we check that $L_2$ does not depend on the choice of $q_i$ as long as $q_i \in S^{k_i}$ and $q_i$ is not the base point. Let $\tilde{q}_i \in S^{k_i}$ be some other points, and let $L_2$ be the linking invariant defined using $\tilde{q}_i$ in place of $q_i$. Let $G$ be a diffeomorphism of $S^{k_1} \vee S^{k_2} \vee S^{k_3}$, homotopic to the identity, taking $\tilde{q}_i$ to $q_i$. Let $F$ be any map from $S^n$ to $S^{k_1} \vee S^{k_2} \vee S^{k_3}$ so that $\tilde{q}_i$ are regular values. Then $G \circ F$ has $q_i$ as regular values, and $L_2(G \circ F) = L_2(F)$. But since $L_2$ is a homotopy invariant and $G$ is homotopic to the identity, $L_2(G \circ F) = L_2(F)$.

Next we consider an example to show that the $L_2$ invariant can be non-trivial. Suppose that $f : S^n \rightarrow S^{k_1} \vee S^{k_2} \vee S^{k_3}$ is a continuous map. Suppose that $g : S^{n-k_1+1} \rightarrow S^{k_2} \vee S^{k_3}$ is a continuous map. We assume as usual that $n + 2 = k_1 + k_2 + k_3$ and $2 \leq k_1 \leq k_2 \leq k_3$, and therefore the linking invariants of $f$ and $g$ are each defined. We let $g^+$ denote the map from $S^{k_1} \vee S^{n-k_1+1}$ to $S^{k_1} \vee S^{k_2} \vee S^{k_3}$ which is equal to the identity on $S^{k_1}$ and is equal to $g$ on $S^{n-k_1+1}$. Then the composition $g^+ \circ f$ maps $S^n$ to $S^{k_1} \vee S^{k_2} \vee S^{k_3}$. The second-order linking invariant $L_2(g^+ \circ f)$ is equal to the product $L(g)L(f)$, which may be non-zero.
For completeness, we calculate $L_2(g^+o f)$. Let $F = g^+o f$. We let $z_1 = F^{-1}(q_1) = f^{-1}(q_1)$. Then we choose a chain $y_1$ with $\partial y_1 = z_1$. Next we let $z_2 = y_1 \cap f^{-1}(q_2) = y_1 \cap f^{-1}[g^{-1}(q_2)]$. Now we let $w$ be $g^{-1}(q_2)$, which is a cycle in $S^{n-k_1+1}$. We choose a chain $v$ with $\partial v = w$. Next we have to choose a chain $y_2$ with $\partial y_2 = z_2$. The trick in this calculation is that we choose $y_2 = y_1 \cap f^{-1}(v)$. Finally, we define $z_3 = y_2 \cap F^{-1}(q_3)$. Expanding this formula, we see $z_3 = y_1 \cap f^{-1}(v) \cap f^{-1}[g^{-1}(q_3)]$, which we rewrite as $z_3 = y_1 \cap f^{-1}[v \cap g^{-1}(q_3)]$. But $v \cap g^{-1}(q_3)$ is a finite collection of points. If we add them with multiplicity we get $L(g)$ points. On the other hand, for each such point $p$ in $S^{n-k_1+1}$, $y_1 \cap f^{-1}(p)$ is a finite collection of points with total multiplicity $L_2(F) = L(f)L(g)$.

In the rest of this section, we prove the two propositions from the introduction. These propositions are weaker than the main theorem, and we include their proofs mostly for background. The main point of the paper is the improvement between Proposition 1.3 and Theorem 1. The proofs of these propositions are easy variations on the material in [2]. The proof of Proposition 1.2 is essentially due to Gromov.

**Proposition 1.2.** Suppose that $2 \leq k_1 \leq k_2 \leq k_3$ and $n = k_1 + k_2 + k_3 - 2$. Let $F$ be a map from $(S^n, g)$ to $(S^{k_2} \vee S^{k_3}, h_1 \vee h_2 \vee h_3)$ with $k_1$-dilation at most $L^{k_1}$. (For instance, $F$ may have Lipschitz constant $L$.) Then $L_2(F)$ is bounded as follows.

\[ |L_2(F)| \leq \text{Iso}_{n-k_1}(g)\text{Iso}_{k_3-1}(g)\text{Vol}(g)L^{n+2}\text{Vol}(h_1)^{-1}\text{Vol}(h_2)^{-1}\text{Vol}(h_3)^{-1}. \]

Proof. By the coarea formula, we can choose $q_1$ so that $z_1 = F^{-1}(q_1)$ has volume at most $L^{k_1}\text{Vol}(g)\text{Vol}(h_1)^{-1}$. (See [2] for more details.)

Then we can choose $y_1$ with volume at most $\text{Iso}_{n-k_1}(g)L^{k_1}\text{Vol}(g)\text{Vol}(h_1)^{-1}$.

Using the coarea formula again, we choose $q_2 \in S^{k_2}$ so that the volume of $z_2 = y_1 \cap F^{-1}(q_2)$ is at most $\text{Iso}_{n-k_1}(g)\text{Vol}(g)L^{k_1+k_2}\text{Vol}(h_1)^{-1}\text{Vol}(h_2)^{-1}$.

Then we can choose $y_2$ with volume at most $\text{Iso}_{n-k_1}(g)\text{Iso}_{k_3-1}(g)\text{Vol}(g)\text{Vol}(h_1)^{-1}\text{Vol}(h_2)^{-1}\text{Vol}(h_3)^{-1}$.

But the degree of $\pi \circ F$ on $y_2$ is at most $L^{k_3}\text{Vol}(y_2)\text{Vol}(h_3)^{-1}$. Filling in our bound for the volume of $y_2$ finishes the proof. \qed

In [2], we estimated the isoperimetric constants of ellipses.

**Proposition.** ([2]) Let $E$ be an $n$-dimensional ellipse with principal axes $E_0 \leq \ldots \leq E_n$. Up to a constant factor $C(n)$, $\text{Iso}(E) \sim E_{k+1} + E_{n-k}$.

Plugging this estimate into the last proposition, we immediately get the following estimate.

**Proposition 1.3.** Suppose that $2 \leq k_1 \leq k_2 \leq k_3$ and that $n + 2 = k_1 + k_2 + k_3$. Let $E$ be an $n$-dimensional ellipse with principal axes $E_0 \leq \ldots \leq E_n$. Let $F$ be a map from $E$ to the wedge of unit spheres $S^{k_1} \vee S^{k_2} \vee S^{k_3}$ with $k_1$-dilation at most $L^{k_1}$.

If $k_3 < (n+1)/2$, then $L_2(F)$ is bounded as follows.

\[ |L_2(F)| \leq C(n)E_{n-k_1+1}E_{n-k_3+1}\text{Vol}(E)L^{n+2}. \]

If $k_3 \geq (n+1)/2$, then $L_2(F)$ is bounded as follows.
\[ |L_2(F)| \leq C(n)E_{n-k_1+1}E_{k_3}Vol(E)L^{n+2}. \]

In this paper, we will study how sharp this estimate is and improve it in the first case, \( k_3 < (n + 1)/2 \). In Section 5, we will construct maps with large \( L_2 \) invariant, proving the following theorem.

\textbf{Theorem 2.} Suppose that \( 2 \leq k_1 \leq k_2 \leq k_3 \), and \( n+2 = k_1 + k_2 + k_3 \). Let \( E \) be an \( n \)-dimensional ellipse with principal axes \( E_0 \leq ... \leq E_n \). Suppose that \( C(n)E_1^{-1} \). Then there is a map \( F \) from \( E \) to the wedge of unit spheres \( S^{k_1} \lor S^{k_2} \lor S^{k_3} \) with Lipschitz constant \( L \) and \( L_2(F) \) bounded below as follows.

\[ L_2(F) \geq c(n)E_{n-k_1+1}E_{k_3}Vol(E)L^{n+2}. \]

This theorem shows that our proposition is sharp up to a constant factor in the case \( k_3 \geq (n+1)/2 \). In the other case, it turns out that the proposition is not sharp up to a constant factor. We will improve it in the next two sections.

2. Directionally-dependent isoperimetric inequalities

We begin by defining directional volume. Let \( C \) be an integral Lipschitz m-chain in \( \mathbb{R}^n \). Suppose that \( J \) is an m-tuple of distinct integers between 1 and \( n \). Let \( P(J) \) denote the m-plane with coordinates \( x_i, i \in J \). We define the \( J \)-volume of \( C \) to be the volume of the projection of \( C \) to \( P(J) \), counted with geometric multiplicity. For example, if \( J \) is any \((n-1)\)-tuple of numbers from 1 to \( n \), and \( C \) is the unit \((n-1)\)-sphere in \( \mathbb{R}^n \), then the \( J \)-volume of \( C \) is equal to twice the volume of the unit \((n-1)\)-ball.

Here’s another way of defining \( J \)-volume. Let \( TC_x \) denote the tangent plane to \( C \) at \( x \). For Lipschitz chains, \( TC_x \) is defined for almost every \( x \) in \( C \). By an abuse of notation, we write \( TC_x \cdot P(J) \) to denote the inner product of the unit k-vector corresponding to \( TC_x \) and the unit k-vector corresponding to \( P(J) \).

\[ Vol_J(C) := \int_C |TC_x \cdot P(J)| dvol(x). \]

The total volume of \( C \) is roughly equal to the sum of the volumes in different directions.

\[ Vol(C) \leq \sum_J Vol_J(C) \leq \binom{n}{m} Vol(C). \]

In [6], Loomis and Whitney proved a directional estimate for the volumes of open sets in \( \mathbb{R}^n \). Their original estimate was written in terms of the projections of a set to coordinate planes, but an immediate corollary is the following estimate.

\textbf{Theorem.} (Loomis, Whitney) Suppose that \( H \) is a closed embedded hypersurface in \( \mathbb{R}^n \). Let \( V \) denote the volume of the region enclosed by \( H \). This volume is bounded in terms of the directional volumes of \( H \) by the following formula.

\[ V \leq \prod_J Vol_J(H)^{\frac{1}{n-1}}. \]

Here the product is taken over the \((n-1)\)-tuples \( J \) of numbers from 1 to \( n \).
We are interested in estimates that hold for cycles of any codimension. The fundamental isoperimetric inequality for cycles of any codimension was proven by Federer and Fleming.

**Theorem.** (Federer, Fleming) Suppose that \( z \) is a closed \( k \)-cycle in \( \mathbb{R}^n \). Then there is a \((k+1)\)-chain \( y \) with \( \partial y = z \) obeying the following volume bound.

\[
|y| \leq C(n)|z|^{\frac{k+1}{k+2}}.
\]

There is a natural conjecture that generalizes the Loomis-Whitney theorem to cycles of any codimension, which we include here for reference.

**Conjecture.** Suppose that \( z \) is a closed \( k \)-cycle in \( \mathbb{R}^n \). Then there is a \((k+1)\)-chain \( y \) with \( \partial y = z \) so that for every \((k+1)\)-tuple \( I \), the \( I \)-volume of \( y \) is bounded in terms of the directional volumes of \( z \) as follows.

\[
\text{Vol}_I(y) \leq C(n)\left[ \prod_{J \subset I} \text{Vol}_J(z) \right]^{\frac{1}{|I|}}.
\]

(The product is taken over all \( k \)-tuples \( J \) contained in \( I \). For each \( I \), there are \((k+1)\) such \( k \)-tuples \( J \).

In this paper, we need directional isoperimetric estimates for cycles in an ellipse. The estimate that we prove will depend on the principal axes of the ellipse. In fact, our goal is to understand how the directional isoperimetric estimates depend on the principal axes.

In the introduction to this paper, we mentioned an isoperimetric estimate for cycles in an ellipse from [2].

**Isoperimetric Inequality in an Ellipse.** ([2]) Suppose that \( E \) is an \( n \)-dimensional ellipse with principal axes \( E_0 \leq \ldots \leq E_n \). Suppose that \( z \) is an integral \( m \)-cycle in \( E \). Then there is an \((m+1)\)-chain \( y \) with \( \partial y = z \) obeying the following estimate.

\[
|y| \leq C(n)|E_{n-m} + E_{m+1}|z|.
\]

We have defined the directional volumes for chains in Euclidean space. We extend the definition to chains in \( E \) in the following way. The ellipse \( E \) is \( C(n) \)-bilipschitz to the double of a rectangle \( R \) with dimensions \( E_1 \leq \ldots \leq E_n \). We fix a particular bilipschitz equivalence. Now given an \( m \)-chain \( z \) in the double of \( R \), we let \( z_N \) be the intersection of \( z \) with the Northern hemisphere and we let \( z_S \) be the intersection of \( z \) with the Southern hemisphere. We view \( z_N \) and \( z_S \) as chains in the rectangle \( R \), and so we know how to define their directional volumes. Then we define the \( J \)-volume of \( z \) to be \( \text{Vol}_J(z_N) + \text{Vol}_J(z_S) \).

Now we refine the isoperimetric inequality above, taking into account the directional volumes of \( y \) and \( z \).

**Directional Isoperimetric Inequality in an Ellipse.** Let \( z \) be an integral \( m \)-cycle in \( E \). Then \( z \) bounds an \((m+1)\)-chain \( y \) with the following bounds on directional volumes. For each \((m+1)\)-tuple \( I \), we let \( i \) denote the smallest number in \( I \) and \( e \) denote the smallest number not in \( I \).

\[
\text{Vol}_I(y) \leq C(n) \left[ E_i\text{Vol}_{I-i}(z) + \sum_{d=1}^{e-1} E_d\text{Vol}_{I-d}(z) \right].
\]
Since $i \leq n - m$ and $d \leq e - 1 \leq m + 1$, we see that the total volume of $y$ is bounded by $C(n)[E_{n-m} + E_{m+1}]Vol(z)$ recovering the standard isoperimetric inequality in $E$.

Our directional isoperimetric inequality improves on the standard one in two ways. First, if we input a cycle $z$ with only a bound on the total volume of $z$, then we get out a chain $y$ whose total volume obeys the standard bound, but which has smaller directional volumes in most directions. Second, if we input a cycle $z$ with some control on the directional volumes, then we may be able to output a chain $y$ with smaller total volume then the standard isoperimetric inequality can deliver.

The proof of this directional isoperimetric inequality is a more complicated version of the proof of the standard isoperimetric inequality in [2]. Since the proof below is somewhat involved, it might help the reader to look at the proof in [2] first.

We build up to the result we need in three steps. First we prove an estimate for absolute cycles in a rectangle. Second we prove an estimate for relative cycles in a rectangle. Third, we combine these results to get an estimate for cycles in an ellipse.

**Isoperimetric inequality for absolute cycles in a rectangle**

**Proposition 2.1.** Suppose that $z$ is an $m$-dimensional cycle in the rectangle $R$ with dimensions $R_1 \leq \ldots \leq R_n$. Then there is an $(m+1)$-chain $y$ in $R$ with $\partial y = z$ obeying the following bounds.

For each $(m+1)$-tuple $I$, let $i$ denote the smallest element of $I$. Let $I - i$ denote the $m$-tuple formed by removing $i$ from $I$.

$$Vol_I(y) \leq R_i Vol_{I-i}(z).$$

**Proof.** We proceed by induction on the dimension $n$. The result is vacuous when the dimension is zero.

Let $\pi$ denote the projection from $\mathbb{R}^n$ onto the plane $x_1 = 0$. There is a homology from $z$ to $\pi(z)$ consisting of a union of lines. (It lies in the cylinder $\pi(z) \times [0, 1]$.) This homology has $(1 \cup J)$-volume at most $R_i Vol_J(z)$ for each $J$ that does not contain $1$. It has $I$-volume 0 for any $I$ that does not contain 1.

By induction, $\pi(z)$ bounds a chain $y' \subset \{0\} \times [0, R_2] \times \ldots \times [0, R_n]$, where the $I$-volume of $y'$ is bounded by $R_i Vol_{I-i}(\pi(z)) \leq R_i Vol_{I-i}(z)$, where $I$ is any $(m+1)$-tuple of $2..n$. Assembling the first homology with the filling $y'$ finishes the proof. □

Remark: For each $I$, we have bounded the $I$-volume of $y$ in terms of only the $I - i$ volume of $z$. If $J$ is an $m$-tuple containing 1, then $J$ is not equal to $I - i$ for any $(m+1)$-tuple $I$. Therefore, we can bound the total volume of $y$ using only some of the directional volumes of $z$. We will need this observation in the proof of our isoperimetric inequality for ellipses.

**Isoperimetric inequality for relative cycles in a rectangle**

Next we study relative cycles in a rectangle. We think of a relative cycle as a chain $z$ with $\partial z$ contained in the boundary of the rectangle. We would like to “push” $z$ into the boundary. In other words, we want to find an $(m+1)$-chain $y$ with $\partial y = z + B$ where $B$ is contained in $\partial R$. For our purposes, we need estimates for both the size of $y$ and the size of $B$. 
Proposition 2.2. Suppose that $z$ is an $m$-dimensional relative integral cycle in the rectangle $R$. Then there is an $(m+1)$-chain $y$ with $\partial y = z + B$ and $B$ contained in $\partial R$ obeying the following estimates.

Let $I$ be an $(m+1)$-tuple. Let $e$ denote the smallest number not in $I$.

\[
Vol_I(y) \lesssim e^{-1} \sum_{d=1}^{e-1} R_d Vol_{I-d}(z).
\]  

(*)

If $J$ is an $m$-tuple that does not include 1, and if $e$ denotes the smallest number in $[2..n]$ which does not lie in $J$, then the $J$-volume of $B$ obeys the following estimate.

\[
Vol_J(B) \lesssim Vol_J(z) + e^{-1} \sum_{d=1}^{e-1} R_d Vol_{1\cup J-d}(z).
\]

(**)

(We are only able to bound some of the directional volumes of $B$. If $J$ includes 1, then we do not prove any upper bound on $Vol_J(B)$.)

Proof. We begin by proving a lemma that covers a special case. The special case occurs when the boundary of $z$ lies only in the bottom and sides of $\partial R$ and does not touch the top of $\partial R$. The following figure illustrates an example of a relative cycle $z$ in this special case.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{cycle.png}
\caption{A cycle that doesn’t touch the top of the rectangle $R$.}
\end{figure}

Lemma 2.1. Suppose that $z$ is an $m$-chain in $R$ with $\partial z$ lying in $\partial R$. Let $R'$ be the $(n-1)$-dimensional rectangle $[0, R_2] \times \ldots \times [0, R_n]$ so that $R = [0, R_1] \times R'$. Suppose that the boundary of $z$ does not intersect $\{R_1\} \times R'$. Then there is an $(m+1)$-chain $y$ with $\partial y = z + B$ where $B$ is an $m$-chain in $\partial R$ obeying the following inequalities.

1. If $I$ is an $(m+1)$-tuple containing 1, then $Vol_I(y) \leq R_1 Vol_{I-1}(z)$.
2. If $I$ is an $(m+1)$-tuple that does not contain 1, then $Vol_I(y) = 0$.
3. If $J$ is an $m$-tuple that does not contain 1, then $Vol_J(B) \leq Vol_J(z)$.

Proof. Let $I : z \rightarrow R$ be the identity embedding, and let $I_1, \ldots I_n$ be its $n$ coordinates. Now we construct a map $f : z \times [0,1] \rightarrow \hat{R}$ with coordinate functions defined as follows: $f_I(x, t) = tI_I(x)$ and for all $i \neq 1$, $f_i(x, t) = I_i(x)$. We define the filling $y$ to be $f(z \times [0,1])$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{filling.png}
\caption{The filling of a cycle that doesn’t touch the top of $R$.}
\end{figure}

First we check that $y$ is a filling of $z$. The boundary of $y$ is equal to $f(\partial z \times [0,1]) - f(z \times \{0\}) + f(z \times \{1\})$. Let $x$ be a point in $\partial z$. If $x$ lies in a "side" of $R$.
(i.e. in \([0, R_1] \times \partial R'\)), then \(f(x, t)\) lies in that "side" of \(R\) for all \(t\). If \(x\) lies in the "bottom" of \(R\) (i.e. in \([0, R_1] \times R'\)), then \(f(x, t)\) lies in the bottom of \(R\) for all \(t\). By assumption, \(\partial z\) lies in the sides and bottom of \(R\). Therefore, \(f(\partial z \times [0, 1])\) lies in the boundary of \(R\). Also \(f(z \times \{0\})\) lies in the bottom of \(R\). Finally \(f\) restricted to \(z \times \{1\}\) is the identity, and so \(f(z \times \{1\})\) is just \(z\). Hence the boundary of \(y\) is equal to \(z\) plus a chain lying in \(\partial R\). We call this chain \(B\).

Now it remains to prove our estimates for \(y\) and \(B\). Let \(\pi\) denote the projection from \(R\) onto \(R'\). Then \(y\) lies in \([0, R_1] \times \pi(z)\). That proves estimates 1 and 2. According to the last paragraph, \(B\) is made up of two pieces: \(-f(\partial z \times [0, 1])\) and \(-f(z \times \{0\})\). The first piece has \(J\)-volume equal to zero unless \(1\) lies in \(J\). The last piece is just the projection \(\pi(z)\). Since \(Vol_J(\pi(z)) \leq Vol_J(z)\) for any \(J\), we get the last inequality. \(\square\)

To prove Proposition 2.2, we reduce our situation to the case of the lemma by cutting an arbitrary cycle \(z\) into pieces in such a way that each piece can be filled either by using Lemma 2.1 or by induction on the dimension. By induction, we can assume that Proposition 2.2 holds for rectangles of dimension \(n-1\).

We consider the slices \(z_h := z \cap \{x| x_1 = h\}\) for various heights \(h\). By the coarea inequality, we can choose \(h\) so that the following inequality holds for each \((m-1)\)-tuple \(K\) in \(2\ldots n\).

\[
Vol_K(z_h) \lesssim R_1^{-1} Vol_{1\cup K}(z).
\]

(On the other hand, if \(K\) contains 1, then the \(K\)-volume of \(z_h\) is zero.)

The following picture shows an example of \(z_h\).

**Figure 4. Intersecting a cycle with a plane.**

In this figure, the solid oriented curve denotes the cycle \(z\). The dotted line denotes the plane \(x_1 = h\). The three dark points denote their intersection, \(z_h\).

We now decompose \(z\) into two pieces as follows.

\[
z = (z - [0, R_1] \times z_h) + [0, R_1] \times z_h = z_1 + z_2.
\]

We deal with the first piece by decomposing it into an upper and lower half: \(z_1 = z_+ + z_-\) where \(z_+\) is the part of \(z_1\) lying above \(x_1 = h\) and \(z_-\) is the part of \(z_1\) lying below \(x_1 = h\). The chains \(z_+\) and \(z_-\) are each relative cycles in \(R\). The following figure shows \(z_+\) and \(z_-\) in the example from the last figure.

**Figure 5. Dividing a cycle into a top piece and a bottom piece**
In this figure, the dotted curves denote the cycle $z_+$ and the solid curves denote the cycle $z_-$. As in the previous figure, the three dark points denote $z_h$.

The cycle $z_+$ avoids the bottom of $R$ and $z_-$ avoids the top of $R$, and so we can fill them both using Lemma 2.1. Let $y_+$ be the filling of $z_+$ and $y_-$ the filling of $z_-$. According to Lemma 2.1, the directional volumes of $y_+$ are bounded in terms of the directional volumes of $z^+$. But $\text{Vol}_J(z^+) \leq \text{Vol}_J(z) + \text{Vol}_J(z_h \times [0, R_1])$. We chose $h$ so that $\text{Vol}_K(z_h) \lesssim R_1^{-1} \text{Vol}_{1 \cup K}(z)$ for each (m-1)-tuple $K$, and so $\text{Vol}_J(z_h \times [0, R_1]) \lesssim \text{Vol}_J(z)$ for every $J$. Then the conclusion of Lemma 2.1 shows that $y_+$ obeys $(\ast)$ and $B_+$ obeys $(\ast\ast)$. The same holds for $z_-$ and its filling $y_-$. Finally we define $y_1 = y_+ + y_-$ and $B_1 = B_+ + B_-$. We have seen that $\partial y_1 = z_1 + B_1$, and that $B_1$ lies in the boundary of $R$, and that $y_1$ and $B_1$ obey the directional volume estimates $(\ast)$ and $(\ast\ast)$.

Now we have reduced matters to a cycle $z_2$ of the special form $[0, R_1] \times z_h$. By induction on the dimension, there is an m-chain $y_h$ in $[0, R_2] \times \ldots \times [0, R_n]$ with $\partial y_h = z_h + B_h$ obeying the estimate $(\ast)$. If we spell out $(\ast)$ we get the following.

Let $J$ be any m-tuple in $[2..n]$. Suppose that $e$ denotes the smallest number in $[2..n]$ but not in $J$.

$$\text{Vol}_J(y_h) \lesssim \sum_{d=2}^{e-1} R_d \text{Vol}_{J-d}(z_h) \lesssim \sum_{d=2}^{e-1} R_1^{-1} R_d \text{Vol}_{1 \cup J-d}(z). \quad (1)$$

Now we define $y_2 = [0, R_1] \times y_h$. Suppose that $I$ includes 1. Let $J$ denote $I - 1$. Then let $e$ denote the smallest number not in $I$, which is the same as the smallest number in $[2..n]$ but not in $J$.

$$\text{Vol}_I(y_2) = R_1 \text{Vol}_J(y_h) \lesssim \sum_{d=2}^{e-1} R_d \text{Vol}_{I-d}(z).$$

On the other hand, if $I$ does not include 1, then the $I$-volume of $y_2$ is zero. So $y_2$ obeys inequality $(\ast)$.

Next we define $B_2$ by setting $\partial y_2 = z_2 + B_2$. Since $y_2 = [0, R_1] \times y_h$, the boundary $\partial y_2$ is equal to $[0, R_1] \times z_h + [0, R_1] \times B_h - \{0\} \times y_h + \{1\} \times y_h$. The first term $[0, R_1] \times z_h$ is $z_2$, and so the remaining terms are equal to $B_2$. The three terms making up $B_2$ each lie in $\partial R$.

Finally, we have to bound (some of) the directional volumes of $B_2$. Suppose that $J$ is an m-tuple which does not contain 1. The $J$-volume of $[0, R_1] \times B_h$ is zero. Each of the other two terms has J-volume equal to that of $y_h$. Applying (1), we get the following estimate for any m-tuple $J$ which does not contain 1.

$$\text{Vol}_J(B_2) \lesssim \sum_{d=2}^{e-1} R_1^{-1} R_d \text{Vol}_{1 \cup J-d}(z).$$

This equation shows that $B_2$ obeys $(\ast\ast)$.

Finally, we set $y = y_1 + y_2$ and $B = B_1 + B_2$. Now we have $\partial y = z + B$, where $B$ lies in $\partial R$, and $y$ and $B$ obey inequalities $(\ast)$ and $(\ast\ast)$. \[\square\]

Isoperimetric inequality for cycles in an ellipse

Let $E$ be the n-dimensional ellipse with principal axes $E_0 \leq \ldots \leq E_n$. We recall the definition of the directional volume for chains in $E$. The ellipse $E$ is
C(n)-bilipschitz to the double of a rectangle \( R = [0, E_1] \times ... \times [0, E_n] \). We fix a bilipschitz equivalence to use throughout the paper. Each copy of \( R \) in the bilipschitz equivalence is a hemisphere of \( E \). Now given an \( m \)-chain \( z \) in the double of \( R \), we let \( z_N \) be the intersection of \( z \) with the Northern hemisphere and we let \( z_S \) be the intersection of \( z \) with the Southern hemisphere. We view \( z_N \) and \( z_S \) as chains in the rectangle \( R \), and so we know how to define their directional volumes. If \( J \) is any \( m \)-tuple of the numbers from 1 to \( n \), we define the \( J \)-volume of \( z \) to be \( Vol_J(z_N) + Vol_J(z_S) \).

With this definition of directional volume, we can state our directional isoperimetric inequality.

**Proposition 2.3.** Let \( z \) be an \( m \)-dimensional integral cycle in \( E \). Then \( z \) bounds an \( (m+1) \)-chain \( y \) with the following bounds on directional volumes. For each \((m+1)\)-tuple \( I \), we let \( i \) denote the smallest number in \( I \) and \( e \) denote the smallest number not in \( I \).

\[
Vol_I(y) \lesssim E_iVol_{I-\{i\}}(z) + \sum_{d=1}^{e-1} E_dVol_{I-\{d\}}(z).
\]

Proof. This proposition follows by combining the previous two. As above, we let \( z_S \) be the intersection of \( z \) with the Southern hemisphere. The chain \( z_S \) is a relative cycle. We apply Proposition 2.2, which tells us that there is a chain \( y_S \) in the Southern hemisphere with \( \partial y_S = z_S + B \) and \( B \subset \partial R \) obeying the following estimates.

1. Let \( I \) be an \((m+1)\)-tuple and \( e \) the smallest number not in \( I \). Then \( Vol_I(y_S) \lesssim \sum_{d=1}^{e-1} E_dVol_{I-\{d\}}(z_S) \).

2. Let \( J \) be an \( m \)-tuple not containing 1 and \( f \) the smallest number in \([2..n] \) not in \( J \). Then \( Vol_J(B) \lesssim Vol_J(z_S) + \sum_{d=1}^{f-1} E_1^{-1} E_dVol_{1\cup J-\{d\}}(z_S) \).

Now \( z_N - B \) is an absolute \( m \)-cycle in the Northern hemisphere. We apply the directional isoperimetric inequality for absolute cycles to fill it. This inequality tells us that there is a chain \( y_N \) in the Northern hemisphere with \( \partial y_N = z_N - B \) obeying the following estimate.

3. Let \( I \) be an \((m+1)\)-tuple and let \( i \) be the smallest number in \( I \). Then \( Vol_I(y_N) \lesssim E_i[Vol_{I-\{i\}}(z_N) + Vol_{I-\{i\}}(B)] \).

Since \( i \) is the smallest number in \( I \), \( I - i \) does not contain 1, and so we can use 2 to bound \( Vol_{I-\{i\}}(B) \). We do this in two cases. First we consider the case \( i > 1 \). In this case \( I - i \) does not contain 2. Hence \( f = 2 \) in inequality 2, and we conclude that \( Vol_{I-\{i\}}(B) \lesssim Vol_{I-\{i\}}(z_S) \). Plugging this inequality into 3, we get the following estimate.

4a. If \( i > 1 \), then \( Vol_I(y_N) \lesssim E_iVol_{I-\{i\}}(z) \).

Next we consider the case \( i = 1 \). We recall that \( e \) is the smallest number not in \( I \). Hence \( e \) is also the smallest number in \([2..n] \) which is not in \( I - 1 \). In this case, inequality 2 tells us that \( Vol_{I-\{1\}}(B) \lesssim E_1^{-1} \sum_{d=1}^{e-1} E_dVol_{I-\{d\}}(z) \). Plugging this estimate into inequality 3, we get the following.

4b. If \( i = 1 \), then \( Vol_I(y_N) \lesssim \sum_{d=1}^{e-1} E_dVol_{I-\{d\}}(z) \).

Finally, we let \( y = y_N + y_S \). The boundary \( \partial y = z \). Combining estimates 1, 4a, and 4b, we see that the directional volumes of \( y \) obey the conclusion of the proposition. □
3. A coarea inequality for directional volumes

In order to bound the second-order linking invariant of a Lipschitz map, we also need a directional version of the coarea inequality. This inequality is only a minor variation on the standard one. The proof combines the general coarea formula with some calculations in exterior algebra.

Proposition 3.1. Let \( y \) be a \( C^\infty \) \( m \)-chain in \( \mathbb{R}^n \), and let \( F \) be a \( C^\infty \) map from \( y \) to \( (N^k, h) \) with \( q \)-dilation at most \( \Lambda \). Then \( F \) has a fiber \( z = F^{-1}(n) \) for some \( n \in N \) obeying the following estimates for the directional volumes. Let \( k = m - q \) be the dimension of \( z \) and let \( J \) be any \( k \)-tuple

\[
Vol_J(z) \leq C(n) \Lambda Vol(N)^{-1} \sum_{J \subseteq I} Vol_I(y).
\]

Proof. First we write down the general coarea formula, which holds for any function \( G \) on \( y \).

\[
\int_y \text{Jac}[dF(x)] G(x) dvol(x) = \int_N \left( \int_{F^{-1}(n)} G dvol_{F^{-1}(n)} \right) dvol_h(n).
\]

At points where \( \text{Jac}[dF(x)] \neq 0 \), the kernel of \( dF \) is a \( k \)-plane. We write \( V(F) \) to denote the unit \( k \)-vector parallel to this \( k \)-plane. (We should specify a choice of orientation, but the orientations won’t matter because we will always take absolute values.) Then we take \( G = |J \cdot V(F)| \). With this choice, the integral over the fiber \( \int_{F^{-1}(n)} G \) is exactly the \( J \)-volume of \( F^{-1}(n) \). Therefore, we have the following formula.

\[
\int_N Vol_J[F^{-1}(n)] dvol_h(n) = \int_y \text{Jac}[dF(x)] |J \cdot V(F)| dvol \leq \Lambda \int_y |J \cdot V(F)| dvol.
\]

We don’t know in which direction the plane \( V(F) \) points, except that it is a subplane of the tangent plane to \( y \). Let \( P \) denote the tangent space to \( y \) at a given point. We are led to estimate \( \sup_{Q^k \subseteq P^m} |J \cdot Q| \), the largest possible value of the term \( |J \cdot V(F)| \). This is a problem about exterior algebra which turns out to have a clean answer.

Lemma 3.1. Let \( P^m \) denote an \( m \)-dimensional plane in \( \mathbb{R}^n \). Let \( J \) be a \( k \)-tuple. By abuse of notation, we also let \( J \) denote the unit \( k \)-vector corresponding to the \( k \)-tuple \( J \).

\[
\sup_{Q^k \subseteq P^m} |J \cdot Q| = \left[ \sum_{J \subseteq I} |I \cdot P|^2 \right]^{1/2}.
\]

Proof. Both sides are invariant if we rotate the plane \( P \) that leaves the plane spanned by \( J \) invariant. By using such a rotation, we can arrange that \( P \cap J^\perp \) is in standard position. We let \( K \) be an \((m-k)\)-tuple disjoint from \( J \). Because of the rotational symmetry, we can assume without loss of generality that \( P \cap J^\perp = K \). We let \( Q_0 = P \cap K^\perp \). Hence \( P = Q_0 \oplus K \).

On the one hand, \( \sup_{Q^k \subseteq P^m} |J \cdot Q| = |J \cdot Q_0| \). To see this, let \( Q \) be any plane in \( P \) and write its fundamental \( k \)-vector as a wedge of unit vectors \( v_1 \wedge \ldots \wedge v_k \) with \( v_i \) in \( P \). Decompose each vector \( v_i \) into a piece in \( Q_0 \) and a piece in \( K \), \( v_i = u_i + w_i \).
Expanding the wedge product, we get a sum of terms. Each term involving any $w_i$ vanishes when we take the inner product with $K$. The other term is equal to $cQ_0$ for some $c$ with $|c| \leq 1$.

On the other hand, the right-hand side is also equal to $|J \cdot Q_0|$. The right-hand side vanishes unless $I = J \cup K$, and so the right hand side is $|J \wedge K \cdot Q_0 \wedge K| = |J \cdot Q_0|$. □

Applying Lemma 3.1 we get the following,

$$\int_N \text{Vol}_I(F^{-1}(n))dvol(n) \leq \Lambda \int \left| \sum_{J \subseteq I} |J \cdot Ty|^2 \right|^{1/2}dvol \leq \Lambda \sum_{J \subseteq I} \text{Vol}_I(y).$$

This formula holds for each choice of $J$. Therefore, we may choose $n \in \mathbb{N}$ so that for every $k$-tuple $J$, the following holds.

$$\text{Vol}_I(F^{-1}(n)) \leq \left(\frac{n}{k}\right)^{\Lambda \text{Vol}(N)^{-1}} \sum_{J \subseteq I} \text{Vol}_I(y).$$ □

4. Dilations and second-order linking invariants

In this section we prove Theorem 1.

**Theorem 1.** Suppose that $2 \leq k_1 \leq k_2 \leq k_3$, that $n + 2 = k_1 + k_2 + k_3$. Let $E$ be an $n$-dimensional ellipse with principal axes $E_0 \leq \ldots \leq E_n$. Let $F$ be a map from $E$ to the wedge of unit spheres $S^{k_1} \vee S^{k_2} \vee S^{k_3}$ with $k_1$-dilation at most $L^{k_1}$. Then $L_2(F)$ is bounded as follows.

$$|L_2(F)| \leq C(n)E_{n-k_1+1}E_{k_3} \text{Vol}(E)L^{n+2}.$$  

The idea of the proof is to imitate the argument in the proof of Proposition 1.3 but to substitute the directional isoperimetric for the standard isoperimetric inequality.

**Proof.** We begin by choosing a point $q_1$ in $S^{k_1}$ and looking at the fiber $z_1 = F^{-1}(q_1)$. For generic $q_1$, the inverse image is a manifold of dimension $n - k_1$. By the coarea formula, we can choose $q_1$ so that the fiber $F^{-1}(q_1)$ has volume at most $\text{Vol}(E)L^{k_1}/\text{Vol}(S^{k_1}) \leq C(n)\text{Vol}(E)L^{k_1}$. In particular, each $J$-volume of $z_1 = F^{-1}(q_1)$ is at most $C(n)V\text{ol}(E)L^{k_1}$.

Next we choose a chain $y_1$ with boundary $z_1$, using the directional isoperimetric inequality Proposition 2.3. The chain $y_1$ will obey the following directional volume bounds.

Let $I$ be a $(n - k_1 + 1)$-tuple. Let $i$ denote the smallest element in $I$ and let $e$ denote the smallest element not in $I$.

$$\text{Vol}_I(y_1) \leq C(n)\text{Vol}(E)L^{k_1}[E_i + E_{e-1}].$$ (1)

Next we choose a point $q_2$ in $S^{k_2}$ and look at the intersection $z_2 = y_1 \cap F^{-1}(q_2)$. By using the directional coarea inequality, we can bound the directional volumes of $z_2$.

The cycle $z_2$ has dimension $n - k_1 + 1 - k_2$. Let $J$ be a tuple of that dimension, and let $I$ be an $(n - k_1 + 1)$-tuple containing $J$. Let $i$ be the smallest element in $I$.
and let $e$ be the smallest element not in $I$. Because of the cardinality of $I$, $i \leq k_1$. The tuple $I$ is formed by adding $k_2$ elements to the tuple $J$. Let $f$ denote the $(k_2 + 1)^{st}$ smallest element which is not in $J$. (In other words, we list the elements not in $J$ from smallest to largest, and let $f$ be the $(k_2 + 1)^{st}$ element in this list.) Then $e \leq f$. Therefore we get the following estimate for the $J$-volumes of $z_2$.

$$Vol_J(z_2) \leq C(n)Vol(E) L^{k_1+k_2}[E_{k_1} + E_{f-1}]. \quad (2)$$

The third step of the proof is to apply the directional isoperimetric inequality again to estimate the size of a filling $y_2$ of $z_2$.

The chain $y_2$ has dimension $n - k_1 - k_2 + 2 = k_3$. Let $K$ denote a tuple of that dimension. Let $k$ denote the smallest element in $K$ and let $g$ denote the smallest element not in $K$. Proposition 2.3 gives the following estimate.

$$Vol_K(y_2) \leq C(n)[E_k Vol_{K-k}(z_2) + \sum_{c=1}^{g-1} E_c Vol_{K-c}(z_2)].$$

Next we plug in the estimate for the $J$-volume of $z_2$ from equation 2. We use this estimate to substitute for $Vol_{K-k}(z_2)$ and $Vol_{K-c}(z_2)$.

$$Vol_K(y_2) \lesssim Vol(E) L^{k_1+k_2} \left[ E_k (E_{k_1} + E_{f(K-k)-1}) + \sum_{c=1}^{g-1} E_c (E_{k_1} + E_{f(K-c)-1}) \right].$$

We will check that the bracketed expression is bounded by $E_{n-k_1+1}E_{k_3}$.

First we deal with the term $E_k E_{k_1}$. Recall that $k$ is the smallest element in $K$. Because the cardinality of $K$ is $k_3$, $k \leq n - k_3 + 1$. Hence $E_k E_{k_1} \leq E_{n-k_3+1}E_{k_1} \leq E_{n-k_1+1}E_{k_3}$.

Second we deal with the term $E_k E_{f(K-k)-1}$. Recall that $f(K-k)$ is the $(k_2+1)^{st}$ smallest element not in $K-k$. The cardinality of $K-k$ is $n - k_1 - k_2 + 1$, and so $f \leq n - k_1 + 2$. Hence if $k \leq k_3$, then $E_k E_{f(K-k)-1} \leq E_{n-k_1+1}E_{k_3}$. On the other hand, if $k \geq k_3 + 1 \geq k_2 + 1$, then the numbers $1, ..., k_2 + 1$ are all not in $K-k$, and so $f(K-k) = k_2 + 1$. In this case $E_k E_{f(K-k)-1} \leq E_{n-k_3+1}E_{k_2} \leq E_{n-k_1+1}E_{k_3}$.

Third we deal with the term $E_{k_1} E_c$. The cardinality of $K$ is $k_3$ and $g$ is the smallest element not in $K$, so $g \leq k_3 + 1$. Since $c \leq g - 1$, it follows that $c \leq k_3$. Hence $E_c E_{k_1}$ is bounded by $E_{n-k_3+1}E_{k_3}$, which is bounded by $E_{n-k_1+1}E_{k_3}$.

Finally, we deal with the term $E_c E_{f(K-c)-1}$. Recall that $f(K-c)$ is the $(k_2+1)^{st}$ smallest element not in $K-c$. Since $K-c$ has $k_3 - 1$ elements, $f(K-c) \leq k_2 + k_3 = n - k_1 + 2$. Therefore, $E_c E_{f(K-c)-1} \leq E_{n-k_1+1}E_k$. In the last paragraph, we saw that $c \leq k_3$. So the product $E_c E_{f(K-c)-1} \leq E_{n-k_1+1}E_{k_3}$.

Putting together the different terms, we have bounded the total volume of $y_2$ as follows.

$$Vol(y_2) \leq C(n) L^{k_1+k_2} Vol(E) E_{n-k_1+1}E_{k_3}.$$
In this section we prove Theorem 2.

Theorem 2. Suppose that \(2 \leq k_1 \leq k_2 \leq k_3\), that \(n + 2 = k_1 + k_2 + k_3\). Let \(E\) be an \(n\)-dimensional ellipse with principal axes \(E_0 \leq \ldots \leq E_n\). Suppose that \(L > C(n)E_1^{-1}\). Then there is a map \(\Phi\) from \(E\) to the wedge of unit spheres \(S^{k_1} \lor S^{k_2} \lor S^{k_3}\) with Lipschitz constant \(L\) and \(L_2(\Phi)\) bounded below as follows.

\[
L_2(\Phi) \geq c(n)E_{n-k_1+1}E_{k_3}Vol(E)L^{n+2}.
\]

Proof. We begin by constructing a map with a large linking invariant. During the proof, we will use the map twice. In order for the proof to fit together, we need to carefully choose the range of the map.

We will use the following vocabulary. For any dimension \(d \leq n\), we let \(E[d]\) denote the \(d\)-dimensional ellipse with principal axes \(E_0 \leq \ldots \leq E_d\). Note that \(E[d]\) is \(C(n)\)-bilipschitz to the double of the rectangle \([0, E_1] \times \ldots \times [0, E_d]\).

The domain of the map is the ellipse \(E\). We will have to keep track of the “tips” of the ellipse \(E\). If \(E\) is given by the equation \(\sum_{i=0}^{d}(x_i/E_i)^2 = 1\), then the tips of \(E\) are the two points \((0, \ldots, 0, \pm E_n)\).

The topology of the range is as follows. Let \(d, e\) be integers at least 2 so that \(n + 1 = d + e\). Let \(p, q\) be antipodal points on the sphere \(S^d\). Let \(*\) be a basepoint of the sphere \(S^e\). The range of our map is the space \(X\) given by taking the union of \(S^d\) and \(S^e\) and then identifying the points \(p, q\), and \(*\). This identified point is the basepoint of \(X\).

Next we define a metric on the space \(X\), which just means picking a metric on \(S^d\) and a metric on \(S^e\). The metric on \(S^d\) is the ellipsoidal metric \(E[d]\). The two points \(p, q\) are the tips of the ellipse. (If the ellipse is given by the equation \(\sum_{i=0}^{d}(x_i/E_i)^2 = 1\) in \(\mathbb{R}^{d+1}\), then the points \(p, q\) are the points \((0, \ldots, 0, \pm E_d)\).)

The metric on \(S^e\) is the one-point compactification of a rectangle with dimensions \(E_{n-e+1} \times \ldots \times E_n\). In other words, the metric is given by taking the Euclidean rectangle \([0, E_{n-e+1}] \times \ldots \times [0, E_n]\) and collapsing the boundary to a point. The basepoint \(*\) is the point we added to do the compactification, or in other words the point corresponding to the boundary. (Our metric is singular at the base point, but the singularity does not create any problems.) We write \((X, h)\) to refer to the space \(X\) equipped with this metric.

A map \(F : S^n \to X\) has a linking invariant defined in the same way as for a map from \(S^n\) to \(S^d \lor S^e\). Namely, let \(q_1\) be a generic point of \(S^d \subset X\) and \(q_2\) a generic point of \(S^e \subset X\), and look at the linking number of the two disjoint cycles \(F^{-1}(q_1)\) and \(F^{-1}(q_2)\).

Lemma 5.1. There is a map \(F : E \to (X, h)\) with linking invariant 1 and with Lipschitz constant at most \(C(n)\). Moreover, this map takes the tips of \(E\) to the basepoint of \(X\).

Proof. We begin by writing down two open sets inside of \(E\). Geometrically, the open sets are thick linked spheres. In order to write them down, we think of \(E\) as the double of the rectangle with dimensions \(E_1 \times \ldots \times E_n\).

The first set \(U\) has the form \(S^{n-e} \times B^e\). It is the double of the following product:

\[
\prod_{i=1}^{n-e}[0, E_i] \times \prod_{i=n-e+1}^{n}[(1/3)E_i, (2/3)E_i].
\]

A core sphere \(S^{n-e}\) in \(U\) is given by the double of \(\prod_{i=1}^{n-e}[0, E_i]\) times a point.
We define the map $F$ on $U$ as follows. We let $\pi_U$ denote the projection from $U$ to the rectangle $R(U) = \prod_{i=n-k_1+1}^{n} [(1/3)E_i, (2/3)E_i]$. There is a degree one map $\psi_U$ from $R(U)$ to $(S^n, h)$ taking the boundary of $R(U)$ to the basepoint and with Lipschitz constant 3. The map $F$ on $U$ is given by the composition $\psi_U \circ \pi_U$. It maps the boundary of $U$ to the basepoint of $S^n$. Since $S^n \subset X$, we can think of $F$ as a map from $(U, \partial U)$ to $(X, \ast)$.

We let $V_0$ be the rectangle $\prod_{i=n-k_1+1}^{n} [(1/10)E_i, (9/10)E_i]$ minus the interior of $R(U)$. The set $V_0$ is homeomorphic to $S^{n-1} \times [0, 1]$. Now the set $V$ is the double of the product $\prod_{i=1}^{n} (0, E_i) \times V_0$. Therefore $V$ is homeomorphic to $S^{n-1} \times S^{n-k_1-1} \times [0, 1]$. We call a copy of $S^{n-1}$ times a point a core sphere of $V$. Note that a core sphere of $V$ and a core sphere of $U$ are linked with linking number 1.

Recall that $n - e = d - 1$. Topologically $V$ has the form $S^{n-1} \times S^{d-1} \times [0, 1]$. Up to a $C(n)$-bilipschitz equivalence, the set $V$ is bilipschitz to a Riemannian product of the following form: $Core \times E[d - 1] \times [0, E_d]$. Here $Core$ is a copy of $S^{n-1}$ equipped with an ellipsoidal metric with principal axes $E_{n-k_1+1} \leq \ldots \leq E_n$.

We define the map $F$ on $V$ as follows. We let $\pi_V$ be the projection from $V$ to $E[d - 1] \times [0, E_d]$. Next, there is a map $\psi_V$ from $E[d - 1] \times [0, E_d]$ to $E[d]$, taking the two boundary components of the domain to the two tips of the range. The map has degree 1 and Lipschitz constant at most $C(n)$. We define $F$ on $V$ to be the composition $\psi_V \circ \pi_V$. The map $\pi_V$ takes the boundary of $V$ to the boundary of $E[d - 1] \times [0, E_d]$, and so the map $F$ takes the boundary of $V$ to the tips of $E[d]$. Now, the identification map $E[d] \rightarrow X$ takes the tips of $E[d]$ to the basepoint of $X$. Therefore, we can think of $F$ as a map from $V$ to $X$ taking the boundary of $V$ to the basepoint of $X$.

We have now defined $F$ on $U$ and on $V$. The sets $U$ and $V$ are disjoint, and $F$ maps their boundaries to the basepoint of $X$. We extend $F$ to all of $E$ by mapping the rest of $E$ to the basepoint of $X$. The tips of $E$ are in the complement of $U \cup V$, and so they get mapped to the basepoint of $X$ as claimed. The map $F$ has Lipschitz constant at most $C(n)$.

The last step is to check that the linking invariant of $F$ is equal to 1. We let $q_1$ be a generic point in $S^d \subset X$ and $q_2$ a generic point in $S^n \subset X$. The preimage $F^{-1}(q_1)$ is a core sphere of $V$. The preimage $F^{-1}(q_2)$ is a core sphere of $U$. These two core spheres have linking number 1.

Now we return to the proof of Theorem 2. First we apply the lemma with $e = k_1$. We get a map from $E$ to $X$. Recall that $X$ is formed by gluing together $S^{n-k_1+1}$ and $S^{k_1}$. The metric on $S^{n-k_1+1}$ is $E[n-k_1+1]$.

The next step of the proof is to apply the lemma again with domain $E[n-k_1+1]$. This time, we choose $e = k_2$. The lemma gives us a map $F_2$ from $E[n-k_1+1]$ to a space $X'$. The space $X'$ is formed from $S^{k_2} \cup S^{k_2}$ by identifying the basepoint of $S^{k_2}$ and two antipodal points of $S^{k_2}$. The map of $F_2$ sends the tips of $E[n-k_1+1]$ to the basepoint of $X'$. Therefore, $F_2$ extends to a map from $X$ to $X' \cup S^{k_2}$, taking the copy of $S^{k_2}$ in $X$ identically to the copy of $S^{k_2}$ in $X' \cup S^{k_2}$. By composing $F_2 \circ F$, we get a map from $E$ to $X' \cup S^{k_2}$. We call this map $\Phi_1$, and we abbreviate $Y = X' \cup S^{k_2}$.

The second order linking invariant is defined for a map $\Phi$ from $S^n$ to $Y$ in the usual way. Namely, let $q_i$ be a generic point in $S^{k_i} \subset Y$, and repeat the usual procedure with the fibers $\Phi^{-1}(q_i)$. The map $\Phi_1$ has $L_2(\Phi_1) = 1$. (This calculation is essentially the same as the calculation of $L_2(g \circ f)$ from Section 1.)
The space $Y$ is equipped with a metric $g$, and with respect to this metric the map $\Phi_1$ has Lipschitz constant at most $C(n)$. The metric on $S^{k_1} \subset Y$ is the one-point compactification of the rectangle with dimensions $E_{n-k_1+1} \times \ldots \times E_n$. The metric on $S^{k_2} \subset Y$ is the one-point compactification of the rectangle with dimensions $E_{n-k_1-k_2+2} \times \ldots \times E_{n-k_1+1}$. The metric on $S^{k_3} \subset Y$ is the ellipsoidal metric $E[k_3]$.

Next we construct a map $\alpha$ from $Y$ to $S^{k_1} \cup S^{k_2} \cup S^{k_3}$, which takes $S^{k_1} \subset Y$ to $S^{k_1}$. We put the standard unit sphere metric on each sphere in the range. For large $L$, we can find $\alpha$ with Lipschitz constant $L$ and with degree $D_1$ at least $c(n)E_{n-k_1+1} \cdots E_nL^{k_1}$ on $S^{k_1}$, with degree $D_2$ at least $c(n)E_{k_3} \cdots E_{n-k_1+1}L^{k_2}$ on $S^{k_2}$, and with degree $D_3$ at least $c(n)E_1 \cdots E_{k_3}L^{k_3}$ on $S^{k_3}$.

The map $\Phi$ is $\alpha \circ \Phi_1$. It has Lipschitz constant at most $C(n)L$ and $L_2(\Phi) = D_1D_2D_3$, which is at least $c(n)E_{k_3}E_{n-k_1+1}Vol(E)L^{n+2}$.

6. APPLICATION TO K-DILATION OF DEGREE NON-ZERO MAPS

Our two theorems immediately imply a new lower bound on the k-dilation of a map from one ellipse to another.

**Theorem 3.** Let $E, E'$ be $n$-dimensional ellipses. Let $E_0 \leq \ldots \leq E_n$ be the principal axes of $E$. Let $E'_0 \leq \ldots \leq E'_n$ be the principal axes of $E'$. Let $Q_i = E'_i/E_i$. Suppose that $\Phi$ is a map from $E$ to $E'$ with degree $D$. Suppose that $2 \leq k_1 \leq k_2 \leq k_3$, $n + 2 = k_1 + k_2 + k_3$ and $k \leq k_1$. Then the following inequality holds.

$$\text{Dil}_k(\Phi) > c(n)[|D|Q_{n-k_1+1}Q_{k_3}Q_1 \cdots Q_n]^{1/k}.$$  

**Proof.** By Theorem 2, we can find a map $F$ from $E'$ to $S^{k_1} \cup S^{k_2} \cup S^{k_3}$ with Lipschitz constant $L$ large and $L_2(F) \geq c(n)E_{n-k_1+1}E'_{k_3}E'_1 \cdots E'_nL^{n+2}$. Therefore, $|L_2(F \circ \Phi)|$ is at least $c(n)|D|E_{n-k_1+1}E'_{k_3}E'_1 \cdots E'_nL^{n+2}$. On the other hand, the map $F \circ \Phi$ has $k_1$-dilation at most $\text{Dil}_k(\Phi)E^{k_1/k}L^{k_1}$. By Theorem 1, the norm of $L_2(F \circ \Phi)$ must be at most $C(n)E_{n-k_1+1}E_{k_3}E_1 \cdots E_nL^{n+2} \text{Dil}_k(\Phi)(n+2)/k$. Comparing the upper and lower bounds we get the estimate.  

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