Distributed Nonconvex Constrained Optimization over Time-Varying Digraphs

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Abstract This paper considers nonconvex distributed constrained optimization over networks, modeled as directed (possibly time-varying) graphs. We introduce the first algorithmic framework for the minimization of the sum of a smooth nonconvex (nonseparable) function—the agent’s sum-utility—plus a Difference-of-Convex (DC) function (with nonsmooth convex part). This general formulation arises in many applications, from statistical machine learning to engineering. The proposed distributed method combines successive convex approximation techniques with a judiciously designed perturbed push-sum consensus mechanism that aims to track locally the gradient of the (smooth part of the) sum-utility. Sublinear convergence rate is proved when a fixed step-size (possibly different among the agents) is employed whereas asymptotic convergence to stationary solutions is proved using a diminishing step-size. Numerical results show that our algorithms compare favorably with current schemes on both convex and nonconvex problems.

1 Introduction

This paper focuses on the following (possibly) nonconvex multiagent composite optimization problem:

\[
\min_{x \in \mathcal{K}} U(x) \triangleq \sum_{i=1}^{I} f_i(x) + G^+(x) - G^-(x), \tag{P}
\]

where \( f_i : \mathbb{R}^m \to \mathbb{R} \) is the cost function of agent \( i \), assumed to be smooth (possibly) nonconvex; \( G : \mathbb{R}^m \to \mathbb{R} \) is a DC function, whose concave part \(-G^-\) is smooth; and \( \mathcal{K} \) is a closed convex subset of \( \mathbb{R}^m \). The function \( G \) is generally used to promote some extra structure on the solution, like sparsity. Note that, differently from most of the papers in the literature, we do not require the (sub)gradient of \( f_i \), \( G^- \) or \( G^+ \) to be (uniformly) bounded on \( \mathcal{K} \). Agents are connected through a communication network, modeled as a directed graph, possibly time-varying. Moreover, each agent \( i \) knows only its own function \( f_i \) (as well as \( G \) and \( \mathcal{K} \)). In this setting, the agents want to cooperatively solve Problem (P) leveraging local communications with their immediate neighbors.

Distributed nonconvex optimization in the form (P) has found a wide range of applications in several areas, including network information processing, telecommunications, multi-agent control, and machine learning. In particular, Problem (P) is a key enabler of many emerging nonconvex “big data” applications.
analytic tasks, including nonlinear least squares, dictionary learning, principal/canonical component analysis, low-rank approximation, and matrix completion [18], just to name a few. Moreover, the DC structure of $G$ allows to accommodate in an unified fashion convex and nonconvex sparsity-inducing surrogates of the $\ell_0$ cardinality function (cf. Sec. 2). Time-varying communications arise, for instance, in mobile wireless networks (e.g., ad-hoc networks), wherein nodes are mobile and/or communicate throughout fading channels. Moreover, since nodes generally transmit at different power and/or communication channels are not symmetric, directed links are a natural assumption.

In most of the above scenarios, data processing and optimization need to be performed in a distributed but collaborative manner by the agents within the network. For instance, this is the case in data-intensive (e.g., sensor-network) applications wherein the sheer volume and spatial/temporal disparity of scattered data render centralized processing and storage infeasible or inefficient.

While distributed methods for convex optimization have been widely studied in the literature, there are no such schemes for (P) (cf. Sec. 1.1). We propose the first family of distributed algorithms that converge to stationary solutions of (P) over time-varying (directed) graphs. Asymptotic convergence is proved, under the use of either constant uncoordinate step-sizes from the agents or diminishing ones. When a constant step-size is employed, the algorithms are showed to achieve sublinear convergence rate. Furthermore, the technical tools we introduce are of independent interest. Our analysis hinges on a descent technique valid for nonconvex, nonsmooth, constrained problems based on a novel Lyapunov-like function (see Sec. 1.2 for the list of contributions).

1.1 Related works

The design of distributed algorithms for (P) faces the following challenges: (i) $U$ is nonconvex and nonseparable; (ii) $G$ is nonsmooth; (iii) there are constraints; (iv) the graph is directed and time-varying, with no specific structure; and (v) the (sub)gradient of $U$ is not assumed to be bounded on $K$. We are not aware of any distributed design addressing (even a subset of) challenges (i)-(v), as documented next. Since the focus of this work is on distributed algorithms working on general network architectures, we omit to discuss the vast literature of schemes implementable on specific topologies, such as hierarchical networks (e.g., master-slave or shared memory systems); see, e.g., [6,15,16,32,36,37,47] and references therein for an entry point of this literature.

Distributed convex optimization: Although the focus of this paper is mainly on nonconvex optimization, we begin overviewing the much abundant literature of distributed algorithms for convex problems. We show in fact that, even in this simpler setting, some of the challenges (ii)-(v) remain unaddressed.

– Primal methods: While substantially different, primal methods can be generically abstracted as a combination of a local (sub)gradient-like step and a subsequent consensus-like update (or multiple consensus updates); examples include [23,27,31,38,39]. Algorithms for adaptation and learning tasks based on in-network diffusion techniques were proposed in [8,11,35]. Schemes in [8,11,23,31,35,38,99] are applicable only to undirected graphs; [8,31] require
the consensus matrices to be double-stochastic whereas [11, 35] use only row-stochastic matrices but are applicable only to strongly convex agents’ cost functions having a common minimizer. When the graph is directed, double-stochastic weight matrices compliant to the graph might not exist or are not easy to be constructed in a distributed way [20]. This requirement was removed in [27] where the authors combined the sub-gradient algorithm [31] with push-sum consensus [24]. Other schemes applicable to digraphs are [48, 49]. However, [31, 48, 49] cannot handle constraints. In fact, up until this work (and the associated conference papers [40, 41]) it was not clear how to leverage push-sum-like protocols to deal with constraints over digraphs. Finally, as far as challenge (v) is concerned, only recent proposals [30, 33, 38, 48, 49, 51] removed the assumption that the (sub-)gradient of $U$ has to be bounded; however, [30, 33, 38, 48, 49, 51] can handle only smooth and unconstrained problems while [33, 38, 39, 49, 51] are not implementable over digraphs.

- Dual-based methods: This class of algorithms is based on a different approach: slack variables are first introduced to decouple the sum-utility function while forcing consistency among these local copies by adding consensus equality constraints (compliant with the graph topology). Lagrangian dual variables are then introduced to deal with such coupling constraints. The resulting algorithms build on primal-dual updates, aiming at converging to a saddle point of the (augmented) Lagrangian function. Examples of such algorithms include ADMM-like methods [9, 22, 39, 45] as well as inexact primal-dual instances [10, 25, 26]. All these algorithms can handle only static and undirected graphs. Their extensions to time-varying graphs or digraphs seem not possible, because it is not clear how to enforce consensus via equality constraints over time-varying or directed networks. Furthermore, all the above schemes but [9, 39] require $U$ to be smooth and $P$ to be unconstrained.

In summary, even restricting to convex instances of (P), there exists no distributed algorithm in the literature that can deal with either constraints [issue (iii)] or nonsmooth $U$ [issue (ii)] with nonbounded (sub-)gradient [issue (v)] over (time-varying) digraphs. Also, it is not clear how to extend the convergence analysis developed in the above papers when $U$ is no longer convex.

**Distributed nonconvex optimization:** Distributed algorithms dealing with special instances of Problem (P) are scarce; they include primal methods [4, 12, 42, 44] and dual-based schemes [21, 53]. The key features of these algorithms are summarized in Table 1 and discussed next.

- **Primal methods:** The scheme in [4] combines the distributed stochastic projection algorithm, employing a diminishing step-size, with the random gossip protocol. It can handle smooth objective functions over undirected static graphs; no rate analysis of the scheme is known. In [12], the authors showed that the (randomly perturbed) push-sum gradient algorithm with diminishing (square summable) step-size, earlier proposed for convex objectives in [27], converges also when applied to nonconvex smooth unconstrained problems. Asymptotic convergence and a sublinear convergence rate were proved (the latter under the assumption that the set of stationary points of $U$ is finite). The first, to our knowledge, provably convergent distributed scheme for (P),
with $G^+ \neq 0$ and constraints $\mathcal{K}$, over time-varying graphs is NEXT [12]. The algorithm requires the consensus matrices to be doubly-stochastic. Asymptotic convergence was proved, when a diminishing step-size is employed: no rate analysis was provided. A special instance of NEXT was studied in [44], where the authors considered smooth (possibly nonconvex) $U$ over undirected static graphs. Under a diminishing step-size (and further technical assumptions on the set of stationary solutions), a sublinear convergence rate is proved. Finally, all the algorithms discussed above require that the (sub)gradient of $U$ is bounded on $\mathcal{K}$ (or $\mathbb{R}^m$). This is a key assumption to prove convergence: in the analysis of descent, it permits to treat the optimization and consensus steps separately, with the consensus error being a summable perturbation.

**Dual-based methods:** In [53] a distributed approximate dual subgradient algorithm, coupled with a consensus scheme (using double-stochastic weight matrices), is introduced to solve (P) over time-varying graphs. Assuming zero-duality gap, the algorithm is proved to asymptotically find a pair of primal-dual solutions of an auxiliary problem, which however might not be stationary for the original problem; also, consensus is not guaranteed. No rate analysis is provided. In [21], a proximal primal-dual algorithm is proposed to solve an unconstrained, smooth instance of (P) over undirected static graphs. The algorithm employs either a constant or increasing penalty parameter (which plays the role of the step-size); a global sublinear convergence rate is proved. The algorithm can also deal with nonsmooth convex regularizers and norm constraints when it is applied to some distributed matrix factorization problems.

**Gradient-tracking:** The proposed algorithmic framework leverages the idea of gradient tracking: each agent updates its own local variables along a direction that is a proxy of the sum-gradient $\sum_{i=1}^I \nabla f_i$ at the current iteration, an information that is not locally available. The idea of tracking the gradient averages through the use of consensus coupled with distributed optimization was independently introduced in [12,14] (NEXT framework) for constrained, nonsmooth, nonconvex instances of (P) over time-varying graphs and in [51] for the case of strongly convex, unconstrained, smooth optimization over static undirected graphs. This tracking protocol was extended to arbitrary (time-varying) digraphs (without requiring doubly-stochastic weight matrices) in our conference work [41]. A convergence rate analysis of the scheme in [51] was later developed in [30,33], with [30] considering (time-varying) directed graphs. We refer the reader to Sec. 3 for a more detailed discussion on this topic.

### Table 1

| DGM | NEXT | Push-sum | Prox-PDA | DeFW | SONATA |
|-----|------|----------|----------|------|--------|
| 4   | ✓    | ✓        | ✓        | ✓    |        |

- **nonsmooth $G^+$**
- **constraints**
- **unbounded gradient**
- **network topology**: time-varying
- **digraph**: restricted
- **step-size**: constant
- **complexity**
1.2 Summary of contributions

We summarize our major contributions as follows; see also Table 1.

1. **Novel algorithmic framework:** We propose the first provably convergent distributed algorithmic framework for the general class of Problem (P), addressing all challenges (i)-(iv). The proposed approach hinges on Successive Convex Approximation (SCA) techniques, coupled with a judiciously designed perturbed push-sum consensus mechanism that aims to track locally the gradient of $F$. Both communication and tracking protocols are implementable on arbitrary time-varying undirected or directed graphs, and in the latter case only column-stochasticity of the weight matrices is required. Also, feasibility of the iterates is preserved at each iteration. Either constant or diminishing step-size rules can be used in the same scheme, and convergence to stationary solutions of Problem (P) is established.

2. **Iteration complexity:** We prove that the proposed scheme has sublinear convergence rate as long as the positive step-size is smaller than an explicit upper bound; different step-sizes among the agents can also be used. To the best of our knowledge, this is the first convergence/complexity result of distributed algorithms employing a constant step-size for nonconvex (constrained) optimization over (time-varying) digraphs.

3. **New Lyapunov-like function and descent technique:** We improve upon existing convergence techniques and introduce new ones. Current analysis of distributed algorithms has trouble handling nonconvex, nonsmooth, constrained optimization. Moreover, in the presence of unbounded (sub-)gradients of the objective function, descent on the objective function while treating optimization and consensus errors separately no longer works. A new convergence analysis is introduced to overcome this difficulty based on a novel “Lyapunov”-like function that properly combines suitably defined weighted average dynamics, consensus and tracking disagreements.

4. **Broader class of problems and convergence results:** The proposed algorithmic framework and convergence results are applicable to a significantly larger class of (constrained) optimization problems and network topology than current distributed schemes, including several instances arising from machine learning, signal processing, and data analytic applications (cf. Sec. 2.1). Moreover, we contribute to the theory of distributed algorithms also for convex problems, being our schemes the first to provably deal with either constraints [issue (iii)] or nonsmooth $U$ [issue (ii)] with nonbounded (sub-)gradient [issue (v)] over (time-varying) digraphs. Finally, our algorithm contains as special cases several recently gradient-based algorithms whose convergence was proved under more restrictive assumptions on the optimization problem and network topology (cf. Sec. 5).

Finally, preliminary numerical results show that the proposed schemes compare favorably with state-of-the-art algorithms.

The rest of the paper is organized as follows. The problem setting is discussed in Sec. 2 along with some motivating applications. Some preliminary results, including a perturbed push-sum consensus scheme over time-varying
digraphs, are introduced in Sec. 3. Sec. 4 describes the proposed algorithmic framework along with its convergence properties, whose proofs are given in Sec. 6. Finally, some numerical results are presented in Sec. 7.

Notation. The set of nonnegative (resp. positive) natural number is denoted by $\mathbb{N}_+$ (resp. $\mathbb{N}_{++}$). A vector $x$ is viewed as a column vector; matrices are denoted by bold letters. We work with the space $\mathbb{R}^n$, equipped with the standard Euclidean norm, which is denoted by $\| \cdot \|_2$; when the argument of $\| \cdot \|$ is a matrix, the default norm is the spectral norm. When some other (vector or matrix) norms are used, such as $\ell_1$-norm, or infinity-norm, we will use the notation $\| \cdot \|_p$ with the corresponding value of $p$. The transpose of a vector $x$ is denoted by $x^\top$. The Kronecker product is denoted by $\otimes$. We use $1$ to denote a vector with all entries equal to 1, and $I$ to denote the identity matrix; With some abuse of notation, the dimensions of $1$ and $I$ will not be given explicitly but understood within the context. Given $I \in \mathbb{N}_{++}$, we define $[I] \triangleq \{1, \ldots, I\}$.

2 Problem Setup and Motivating Examples

We study Problem \[^{[P]}\] under the following assumptions.

Assumption A (On Problem \[^{[P]}\]) Given Problem \[^{[P]}\], suppose that

A.1 The set $\mathcal{K} \subseteq \mathbb{R}^n$ is (nonempty) closed and convex;

A.2 Each $f_i : \mathcal{O} \to \mathbb{R}$ is $C^1$, where $\mathcal{O} \supseteq \mathcal{K}$ is an open set, and $\nabla f_i$ is $L_i$-Lipschitz on $\mathcal{K}$;

A.3 $G^+ : \mathcal{K} \to \mathbb{R}$ is convex (possibly nonsmooth), and $G^- : \mathcal{O} \to \mathbb{R}$ is $C^1$ with $\nabla G^-$ being $L_G$-Lipschitz on $\mathcal{K}$;

A.4 $U$ is lower bounded on $\mathcal{K}$.

We also made the blanket assumption that each agent $i$ knows only its on function $f_i$ and the regularizer $G$ but not the functions of the other agents.

Assumptions A.1 A.2 and A.4 are quite standard and satisfied by several problems of practical interest. We remark that, as a major departure from most of the literature on distributed algorithms, we do not assume that the gradient of $F$ (and $G^-$) is bounded on the feasible set $\mathcal{K}$. This, together with the nonconvexity of $G$ as stated in A.3, opens the way to design for the first time distributed algorithms for a gamut of new applications, including several big-data problems in statistical learning; see Sec. 2.1 for details.

On the network topology: Agents communicate through a (possibly) time-varying network. Specifically, time is slotted with $n$ denoting the iteration index (time-slot); in each time-slot $n$, the communication network of agents is modeled as a (possibly) time-varying digraph $G^n = ([I], E^n)$, where $[I] = \{1, \ldots, I\}$ denotes the set of agents—the vertices of the graph—and the set of edges $E^n$ represents the agents’ communication links; we use $(i, j) \in E^n$ to indicate that the link is directed from node $i$ to node $j$. The in-neighborhood of agent $i$ at time $n$ is defined as $\mathcal{N}^\text{in}_i[n] = \{j \mid (j, i) \in E^n\} \cup \{i\}$ (we included in the set node $i$ itself, for notational simplicity); it represents the set of agents which node $i$ can receive information from. The out-neighborhood of agent $i$ is $\mathcal{N}^\text{out}_i[n] = \{j \mid (i, j) \in E^n\} \cup \{i\}$—the set of agents receiving information
from node \( i \) (including node \( i \) itself). The out-degree of agent \( i \) is defined as \( d^\text{out}_n \triangleq \left| \mathcal{N}^\text{out}_n \right| \). To let information propagate over the network, we assume that the graph sequence \( \{ G^n \}_{n \in \mathbb{N}_+} \) possesses some “long-term” connectivity property, as formally stated next.

**Assumption B (On graph connectivity)** The graph sequence \( \{ G^n \}_{n \in \mathbb{N}_+} \) is \( B \)-strongly connected, i.e., there exists a finite integer \( B > 0 \) such that the graph with edge set \( \bigcup_{t=k}^{k+B-1} E^t \) is strongly connected, for all \( k \geq 0 \).

We conclude this section discussing some instances of Problem (P) in the context of statistical learning.

### 2.1 Distributed sparse statistical learning

We consider two distributed nonconvex problems in statistical learning, namely: i) a nonconvex sparse linear regression problem; and ii) the sparse Principal Component Analysis (PCA) problem.

**Nonconvex Sparse Linear Regression.** Consider the problem of retrieving a sparse signal \( x \in \mathbb{R}^m \) from the observations \( \{ b_i \}_{i=1}^I \), where each \( b_i = A_ix \) is a linear measurement of the signal acquired by agent \( i \). A mainstream approach in the literature is to solve the following optimization problem

\[
\min_x \sum_{i=1}^I \left\| b_i - A_i x \right\|^2 + \lambda \cdot G(x),
\]

where the quadratic term measures the model fitness whereas the regularizer \( G \) is used to promote sparsity in the solution, and \( \lambda > 0 \) is chosen to balance the trade-off between the model fitness and solution sparsity. Problem (1) is clearly an instance of (P). Note that each agent knows only its own function \( f_i \) (since \( b_i \) is own only by agent \( i \)). Also, \( \nabla f_i \) is not bounded on \( \mathbb{R}^m \).

To promote sparsity on the solution, the ideal choice for \( G \) would be the cardinality of \( x \) (a.k.a. \( \ell_0 \) “norm” of \( x \)). However, its combinatorial nature makes the resulting optimization problem numerically intractable as the variable dimension \( m \) becomes large. Several convex and, more recently, also nonconvex surrogates of the \( \ell_0 \) function have been proposed in the literature. The structure of \( G \), as stated in Assumption A.3, captures either choices. For instance, one can choose as regularizer in (1), the \( \ell_2 \) or \( \ell_1 \) norm of \( x \) (and thus \( G^{-} = 0 \)), which leads to the ridge and LASSO regression problems, respectively. Moreover, a vast class of nonconvex surrogates can also be considered, including the SCAD \[17\], the “transformed” \( \ell_1 \) \[52\], the logarithmic \[46\], and the exponential \[7\]; see Table 2. It is well documented that nonconvex regularizers outperform the \( \ell_1 \) norm in enhancing solution sparsity. Quite interestingly, all the widely used nonconvex surrogates listed in Table 2 enjoy the following separable DC structure (see, e.g., [1,43] and references therein)

\[
G(x) = \sum_{i=1}^m g(x_i), \quad \text{with} \quad g(x_i) = \eta(\theta) |x_i| - \left( \eta(\theta) |x_i| - g(x_i) \right). \tag{2}
\]

where the expression of \( g : \mathbb{R} \to \mathbb{R} \) is given in Table 2 and \( \eta(\theta) \) is a fixed given function, defined in Table 3 for each of the surrogate \( g \) listed in Table 2.
Table 2 Examples of nonconvex surrogates of the \( \ell_0 \) function having a DC structure [cf. (2)]

| Penalty function | Expression |
|------------------|------------|
| Exp \[7\] | \( g_{\text{exp}}(x) = 1 - e^{-\theta|x|} \) |
| \( \ell_p(0 < p < 1) \) \[19\] | \( g_{\ell_p^+}(x) = (|x| + e)^{1/p} \) |
| \( \ell_p(p < 0) \) \[34\] | \( g_{\ell_p^-}(x) = 1 - (\theta|x| + 1)^p \) |
| SCAD \[17\] | \( g_{\text{scad}}(x) = \begin{cases} \frac{2\theta}{\theta + 1}|x|, & 0 \leq |x| \leq \frac{1}{\theta} \\ -\frac{\theta^2|x|^2 + 2\theta|x| - 1}{\alpha^2 - 1}, & \frac{1}{\theta} < |x| \leq \frac{\theta}{\alpha} \\ 1, & |x| > \frac{\theta}{\alpha} \end{cases} \) |
| Log \[46\] | \( g_{\log}(x) = \frac{\log(1 + \theta|x|)}{\log(1 + \theta)} \) |

Parameter \( \theta \) controls the tightness of the approximation of the \( \ell_0 \) function: in fact, it holds that \( \lim_{\theta \to +\infty} g(x_i) = 1 \) if \( x_i \neq 0 \), otherwise \( \lim_{\theta \to +\infty} g(x_i) = 0 \). Note that \( g^- \) is convex and has Lipschitz continuous first derivative \( dg^-/dx \) \[43\], whose closed form is given in Table 3.

It is not difficult to check that Problem (1), with any of the regularizers discussed above, is an instance of (P) and satisfies Assumption A. Also, note that the gradient of the smooth part is not bounded on \( \mathbb{R}^m \).

Sparse PCA. Consider finding the sparse principal component of a distributed data set given by the rows of a set of matrices \( D_i \)'s (each \( D_i \) is owned by agent \( i \)). The problem can be formulated as

\[
\max_{\|x\|_2 \leq 1} \sum_{i=1}^{l} \|D_i x\|^2 - \lambda \cdot G(x), \tag{3}
\]

where \( G \) can be any of the sparse-promoting regularizers discussed in the previous example. Clearly, Problem (3) is another (nonconvex) instance of Problem (P) (satisfying Assumption A).

3 Preliminaries: The perturbed condensed push-sum algorithm

The proposed algorithmic framework combines local optimization based on SCA with constrained consensus and tracking of gradient averages over digraphs. The consensus problem over graphs has been widely studied in the literature; a renowned distributed scheme solving this problem over (possibly time-varying) digraphs is the so-called push-sum algorithm [24]. A perturbed version of the push-sum scheme has been introduced in [27] to solve unconstrained optimization problems over (time-varying) digraphs. However, it is not clear how to leverage the push-sum update and extend these optimization schemes to deal with constraints. In this section, we introduce a reformulation of the perturbed push-sum protocol [27]—termed perturbed condensed push-sum—that is more suitable for the integration with constrained...
Table 3 Explicit expression of $\eta(\theta)$ and $dg^-/dx$ [cf. (2)]

| $g$      | $\eta(\theta)$ | $dg^-/dx$ |
|----------|-----------------|-----------|
| $g_{\text{exp}}$ | $\theta$ | $\text{sign}(x) \cdot \theta \cdot (1 - e^{-\theta|x|})$ |
| $g_{\text{cl}}$ | $\frac{1}{\theta} e^{1/\theta - 1}$ | $\frac{1}{\theta} \text{sign}(x) \cdot \left[ e^{\frac{1}{\theta} - 1} - (|x| + \varepsilon)^{\frac{1}{\theta} - 1} \right]$ |
| $g_{\text{cr}}$ | $-p \cdot \theta$ | $-\text{sign}(x) \cdot p \cdot \theta \cdot [1 - (1 + \theta|x|)^{p-1}]$ |
| $g_{\text{scad}}$ | $\frac{\theta}{\log(1+\theta)}$ | $\text{sign}(x) \cdot \theta$ |

optimization. This scheme will be then used to build the gradient tracking and constrained consensus mechanisms embedded in the proposed algorithmic framework (cf. Sec. 4).

Consider a network of $I$ agents, as introduced in Sec. 2, communicating over a time-varying digraph (cf. Assumption B). Each agent $i$ controls a vector of variables $x_i(t) \in \mathbb{R}^m$ as well as a scalar $\phi_i$ that are iteratively updated, based upon the information received from its immediate neighbors. Let $x_i^n$ and $\phi_i^n$ denote the values of $x_i(t)$ and $\phi_i$ at iteration $n \in \mathbb{N}_+$. We let agents’ updates be subject to a(n adversarial) perturbation; we denote by $\delta^n_i \in \mathbb{R}^m$ the perturbation injected in the update of agent $i$ at iteration $n$. Given $x_i^n$ and $\phi_i^n$, the perturbed condensed consensus algorithm reads:

$$\phi_i^{n+1} = \sum_{j=1}^{I} a_{ij}^n \phi_j^n,$$

$$x_i^{n+1} = \frac{1}{\phi_i^{n+1} + 1} \sum_{j=1}^{I} a_{ij}^n \phi_j^n x_j^n + \delta_i^{n+1},$$

for all $n \in \mathbb{N}_+$ and $i \in [I]$, where $x_i^n$ are arbitrarily chosen and $\phi_i^n$ are positive scalars such that $\sum_{i=1}^{I} \phi_i^n = I$; and $A^n_{(i)} = (a_{ij}^n)_{i,j=1}^{I}$ is a (possibly) time-varying matrix of weights whose nonzero pattern is compliant with the topology of the graph $G^n$, in the sense of the assumption below.

Assumption C (On the weight matrix $A^n$) Each $A^n_{(i)} = (a_{ij}^n)_{i,j=1}^{I}$ is compliant with $G^n$, that is,

C1. $a_{ii}^n \geq \kappa > 0$, for all $i \in [I]$;

C2. $a_{ij}^n \geq \kappa > 0$, if $(j,i) \in E^n$; and $a_{ij}^n = 0$ otherwise.

Under Assumption C the protocol (4) is implementable in a distributed fashion: each agent $i$ updates its own variables using only the information $\phi_j^n x_j^n$.
and $φ^n_i$ received from its current in-neighbors (and its own). We study convergence of (4) under the following further (standard) assumption on $A_n$.

**Assumption D (Column stochasticity)** Each matrix $A^n$ is column stochastic, that is, $1^T A^n = 1^T$.

The role of the extra variables $φ_i$ is to dynamically rebuild the row stochasticity of the equivalent weight matrix governing variables’ updates, which is a key condition to lock consensus. This can be easily seen rewriting the dynamics (4b) in terms of the equivalent weights $W^n ≜ (w^n_{ij})_{i,j=1}^I$:

$$x^{n+1}_{(i)} = \sum_{j=1}^I w^n_{ij} x^n_{(j)}, \quad w^n_{ij} ≜ \frac{a^n_{ij} φ^n_j φ^n_{i+1}}{φ^n_{i+1}}.$$ (5)

It is not difficult to check that, under Assumption D, $W^n$ is row-stochastic.

To state the main convergence result in compact form, we introduce the following notation. Let

$$x^n ≜ \begin{bmatrix} x^n_{(1)}^T, \ldots, x^n_{(I)}^T \end{bmatrix}^T,$$ (6a)

$$φ^n ≜ \begin{bmatrix} φ^n_1, \ldots, φ^n_I \end{bmatrix}^T,$$ (6b)

$$δ^n ≜ \begin{bmatrix} δ^n_1^T, \ldots, δ^n_I^T \end{bmatrix}^T.$$ (6c)

Noting that, in the absence of perturbation (i.e., $δ^n = 0$), the weighed sum $\sum_{i=1}^I φ^n_i x^n_{(i)}$ is an invariant of (4), that is, $\sum_{i=1}^I φ^n_{i+1} x^n_{(i)} = \cdots = \sum_{i=1}^I φ^n_i x^n_{(i)}$, we define the consensus disagreement at iteration $n$ as the deviation of each $x^n_{(i)}$ from the weighted average $(1/I) \sum_{i=1}^I φ^n_i x^n_{(i)}$:

$$e^n_x ≜ x^n - 1 \otimes \frac{1}{I} \sum_{i=1}^I φ^n_i x^n_{(i)}.$$ (7)

The dynamics of the error $e^n_x$ are studied in the following proposition (whose proof is postponed to Sec. 3.2).

**Proposition 1** Let $\{G^n\}_{n \in \mathbb{N}_+}$ be a sequence of digraphs satisfying Assumption B and let $\{(φ^n, x^n)\}_{n \in \mathbb{N}_+}$ be the sequence generated by the perturbed condensed push-sum protocol (7), for a given perturbation sequence $\{δ^n\}_{n \in \mathbb{N}_+}$ and weight matrices $\{A^n\}_{n \in \mathbb{N}_+}$ satisfying Assumptions C-D. Then, there hold:

(i) **[Bounded $\{φ^n\}_{n \in \mathbb{N}_+}$]:**

$$\inf_{n \in \mathbb{N}_+} \min_{i \in [I]} φ^n_i ≥ φ_{lb}, \quad φ_{lb} ≜ \kappa^2(I-1)B,$$

$$\sup_{n \in \mathbb{N}_+} \max_{i \in [I]} φ^n_i ≤ φ_{ub}, \quad φ_{ub} ≜ I - \kappa^2(I-1)B,$$ (8)

with $B ≥ 1$ and $κ ∈ (0,1)$ defined in Assumption B and Assumption C respectively;
(ii) **[Error decay]**: For all \( n, k \in \mathbb{N}_+, n \geq k, \)

\[
\|e_0^n\| \leq \lambda^k \|e_0^{n-k}\| + \lambda^k \sum_{i=0}^{k-1} \|\delta^{n-i}\|,
\]

where \( \lambda_1 \triangleq \min \left\{ \sqrt{2} I, 2c_0 I (\rho) \right\} \),

and \( c_0 \triangleq 2 \left( 1 + \tilde{\lambda}^{-1} (J-1)B \right) \), \( \rho \triangleq 1 - \tilde{\lambda} (J-1)B \), \( \tilde{\lambda} \triangleq \kappa^{2(J-1)B+1} / I \). (9)

Furthermore, there exists a finite \( B \in \mathbb{N}_+ \) such that \( \rho_B \triangleq 2c_0 I (\rho) \left( \frac{B}{2} \right) < 1 \). (10)

**Remark 1** The perturbed consensus algorithm \( \{\phi^n\} \) was mainly designed for digraphs. However, when the graph is undirected, one can choose the weight matrix \( A^n \) to be doubly stochastic and get rid of the auxiliary variables \( \phi^n \), just setting in \( \{\phi^n\} \). As a consequence, \( \phi^n \equiv 1 \) and \( W^n \equiv A^n \), for all \( n \in \mathbb{N}_+ \). In this case, using [29] Lemma 9, the expression of \( \lambda_1 \) in Proposition 1 can be tightened by letting \( \lambda_1 \triangleq \min \{1, \rho^{1/(J-1)B}\} \), with \( \rho \triangleq \sqrt{1 - \kappa / (2J^2)} \).

3.1 Discussion

Proposition 1 provides a unified set of convergence conditions of the perturbed consensus push-sum scheme that are applicable to any given perturbation sequence \( \{\delta^n\}_{n \in \mathbb{N}_+} \). We discuss next two special cases, namely: the plain average consensus problem and the distributed tracking of time-varying signals.

**1. (Weighted) average consensus:** Setting in \( \{\phi^n\} \) \( \delta^n = 0 \), for all \( n \in \mathbb{N}_+ \), \( \{\phi^n\} \) reduces to the plain (condensed) push-sum scheme, whose geometric convergence to the (weighted) average of the initial values, \( (1/I) \sum_{i=1}^{I} \phi^n_0 x^n_{(i)} \), follows readily from Proposition 1. More specifically, using \( \sum_{i=1}^{I} \phi^{n+1}_i x^{n+1}_{(i)} = \cdots = \sum_{i=1}^{I} \phi^0_i x^0_{(i)} \), \( \{\phi^n\} \) yields

\[
\left\| x^{n+1} - 1 \otimes \frac{1}{I} \sum_{i=1}^{I} \phi^0_i x^0_{(i)} \right\| \leq 2c_0 I (\rho) \left( \frac{B}{2} \right) \| e_0^0 \|, \quad n \in \mathbb{N}_+.
\]

Note that, since the weight matrix \( W^n \) in \( \{\phi^n\} \) is row stochastic, if the initial values \( x^0_{(i)} \) all belong to a common set \( \mathcal{K} \), then \( x^n_{(i)} \in \mathcal{K} \), for all \( n \in \mathbb{N}_+ \); that is feasibility of the iterates is preserved.

**2. Tracking of time-varying signals’ averages:** Consider the problem of tracking distributively the average of time-varying signals. At each iteration \( n \in \mathbb{N}_+ \), each agent \( i \) evaluates (or generates) a signal sample \( u^n_i \in \mathbb{R}^m \) from the (time-varying) sequence \( \{u^n_i\}_{n \in \mathbb{N}_+} \). The goal is to design a distributed algorithm obeying the communication structure of the graphs \( \mathcal{G}^n \) that tracks the average of the signals \( \{u^n_i\}_{n \in \mathbb{N}_+} \), that is,

\[
\lim_{n \to \infty} \|x^n - 1 \otimes u^n\| = 0, \quad u^n \triangleq \frac{1}{I} \sum_{i=1}^{I} u^n_i.
\]

(12)
The perturbed condensed push-sum algorithm (4) can be readily used to accomplish this task by setting
\[ \delta_{i}^{n+1} = \frac{1}{\phi_{i}^{n+1}} (u_{i}^{n+1} - u_{i}^{n}), \quad i \in [I], \ n \in \mathbb{N}_{+}, \] (13)
and \( x_{i}^{0} = u_{i}^{0} \), \( i \in [I] \). Convergence of this scheme is stated next.

Corollary 2 Let \( \{u_{i}^{n}\}_{n \in \mathbb{N}_{+}} \) be a given sequence such that \( \lim_{n \to \infty} \|u_{i}^{n+1} - u_{i}^{n}\| = 0 \), for all \( i \in [I] \). Consider the perturbed condensed push-sum protocol (4), under the assumptions of Proposition 1; and set \( \delta_{i}^{n+1} \) as in (13) and \( x_{i}^{0} = u_{i}^{0} \), for all \( i \in [I] \). Then, (12) holds.

Proof The proof follows readily from Proposition 1 and the following two facts: i) \( (1/|I|) \sum_{i=1}^{I} \phi_{i}^{n+1} x_{(i)}^{n+1} = \bar{u}_{n+1} \); and ii) \( \lim_{n \to \infty} \|\delta_{n}\| = 0 \) \( \Rightarrow \lim_{n \to \infty} \sum_{t=0}^{n-1} (\rho)^{t} \|\delta_{n-t}\| = 0 \).

\[ \square \]

3.2 Proof of Proposition 1

To prove Proposition 1, it is convenient to rewrite the perturbed consensus protocol (4) in a vector-matrix form. To do so, let us introduce the following quantities: given the weight matrix \( A_{n} \) compliant with \( G_{n} \) (cf. Assumption C) and \( W_{n} \) defined in (5), let \( D_{\phi_{n}} \triangleq \text{Diag}(\phi_{n}) \), \( \hat{D}_{\phi_{n}} \triangleq D_{\phi_{n}} \otimes I \), \( \hat{A}_{n} \triangleq A_{n} \otimes I \), \( \hat{W}_{n} \triangleq W_{n} \otimes I \), (14a)
(14b)
(14c)
(14d)
where \( \text{Diag}(\bullet) \) denotes a diagonal matrix whose diagonal entries are the elements of the vector argument, and \( I \) is the \( m \times m \) identity matrix. Under the column stochasticity of \( A_{n} \), it is not difficult to check that the following holds:

\[ W_{n} = (D_{\phi_{n+1}})^{-1} A_{n} D_{\phi_{n}} \quad \text{and} \quad \hat{W}_{n} = (\hat{D}_{\phi_{n+1}})^{-1} \hat{A}_{n} \hat{D}_{\phi_{n}}. \] (15)

Using the above notation and (6), the perturbed push-sum protocol (4) can be rewritten in matrix-vector form as

\[ \phi_{n+1}^{+} = A_{n} \phi_{n}^{+} \quad \text{and} \quad x_{n+1}^{+} = \hat{W}_{n} x_{n}^{+} + \delta_{n+1}^{+}. \] (16)

To study convergence of (16), it is convenient to introduce the following matrix products: given \( n, k \in \mathbb{N}_{+} \), with \( n \geq k \),

\[ A_{n:k} \triangleq \begin{cases} A_{n} A_{n-1} \ldots A_{k}, & \text{if } n > k, \\ A_{n}, & \text{if } n = k, \end{cases} \]
(17)
(17)
\[ W_{n:k} \triangleq \begin{cases} W_{n} W_{n-1} \ldots W_{k}, & \text{if } n > k, \\ W_{n}, & \text{if } n = k, \end{cases} \]
and

\[ \hat{A}^{n,k} \triangleq A^{n,k} \otimes I, \quad \hat{W}^{n,k} \triangleq W^{n,k} \otimes I. \quad (18) \]

Define the weight-averaging matrix

\[ J_{\phi^{n}} \triangleq \frac{1}{t} (1 (\phi^{n})^T) \otimes I, \quad (19) \]

so that \( J_{\phi^{n}} x^{n} = 1 \otimes \frac{1}{t} \sum_{i=1}^{t} \phi^{n}_{(i)} x^{n}_{(i)} \). Also, it is not difficult to check the following chain of equalities hold among \( J_{\phi^{n}}, \hat{W}^{n,t}, \) and \( \hat{A}^{n,t} \): for \( n,k \in \mathbb{N}_+, \)

\[ J_{\phi^{n+1}} \hat{W}^{n,k} (\otimes) J_{\phi^{k}} = J_{\phi^{k}} (\otimes) \hat{W}^{n,k} J_{\phi^{n}}, \quad (20) \]

where in (a) we used the definition of \( \hat{W}^{n} \) [cf. (15)], \( J_{\phi^{n+1}} \) [cd. (20)], and the column stochasticity of \( \hat{A}^{n} \); and (b) is due to the row stochasticity of \( W^{n,k} \).

To study the evolution of \( e^{n}_{k} \), we apply the \( x \)-update (16) recursively and obtain

\[ x^{n} = \hat{W}^{n-1:n-k} x^{n-k} + \sum_{t=1}^{k-1} \hat{W}^{n-1:n-t} \delta^{n-t} + \delta^{n}. \quad (21) \]

Using (20) and (21), the weighted average \( J_{\phi^{n}} x^{n} \) can be written as

\[ J_{\phi^{n}} x^{n} = J_{\phi^{n-k}} x^{n-k} + \sum_{t=1}^{k-1} J_{\phi^{n-t}} \delta^{n-t} + J_{\phi^{n}} \delta^{n}. \quad (22) \]

Subtracting (22) from (21) and using \( (\hat{W}^{n-1:n-k} - J_{\phi^{n-k}})J_{\phi^{n-k}} = 0 \) [cf. (20)], we can bound the consensus error \( e^{n+1}_{2} \) as

\[ \| e^{n}_{2} \| \leq \| \hat{W}^{n-1:n-k} - J_{\phi^{n-k}} \| \| e^{n-1}_{2} \| + \sum_{t=1}^{k-1} \| \hat{W}^{n-1:n-t} - J_{\phi^{n-t}} \| \| \delta^{n-t} \| + \| 1 - J_{\phi^{n}} \| \| \delta^{n} \|. \quad (23) \]

Convergence of the perturbed consensus protocol reduces to studying the dynamics of the matrix product \( \| \hat{W}^{n,k} - J_{\phi^{n}} \| \), as done in the lemma below.

**Lemma 3** Let \( \{G^{n}\}_{n \in \mathbb{N}_+} \) be a sequence of digraphs satisfying Assumption [D], let \( \{A^{n}\}_{n \in \mathbb{N}_+} \) be a sequence of weight matrices satisfying Assumptions [D] and let \( \{W^{n}\}_{n \in \mathbb{N}_+} \) be the sequence of row stochastic matrices related to \( \{A^{n}\}_{n \in \mathbb{N}_+} \) by (15). There holds:

\[ \| \hat{W}^{n,k} - J_{\phi^{k}} \| \leq \min \left\{ \sqrt{2} I, 2 c_{0} I (\rho) \left[ \frac{n+k+1}{(n-k+1)^{2}} \right] \right\}, \quad n,k \in \mathbb{N}_+, n \geq k, \quad (24) \]

where \( c_{0} \) and \( \rho \) are defined in Proposition [D].

**Proof** See Appendix [A]. □
The error decay law \([9]\) comes readily from \([23]\), Lemma \([3]\) and the following fact: \(\| \mathbf{I} - \mathbf{J}_{\phi^n} \| \leq \sqrt{2T} \leq \lambda^0 \triangleq \min\{2c_0I, \sqrt{2I}\} \), which is proved below. Let \( \mathbf{z} \in \mathbb{R}^{Jm} \) be an arbitrary vector; let us partition \( \mathbf{z} = [\mathbf{z}_1^T, \ldots, \mathbf{z}_J^T]^T \), with each \( \mathbf{z}_i \in \mathbb{R}^m \). Then,

\[
\| (\mathbf{I} - \mathbf{J}_{\phi^n}) \mathbf{z} \| \leq \| \mathbf{z} - \mathbf{J}_1 \mathbf{z} \| + \| \mathbf{J}_1 \mathbf{z} - \mathbf{J}_{\phi^n} \mathbf{z} \| \overset{(a)}{\leq} \| \mathbf{z} \| + \frac{\sqrt{T}}{I} \left \| \sum_{i=1}^{I} \mathbf{z}_i - \sum_{i=1}^{I} \phi_i^n \mathbf{z}_i \right \|
\]

\[
\leq \| \mathbf{z} \| + \frac{\sqrt{T}}{\sqrt{I^2 - I}} \| \mathbf{z} \| \leq \sqrt{2T} \| \mathbf{z} \| ,
\]

where in (a) we used \( \| \mathbf{I} - \mathbf{J}_1 \| = 1 \).

**4 Algorithmic Design**

We are ready to introduce the proposed distributed algorithm for Problem \([\mathcal{P}]\). To shed light on the core idea of the novel framework, we begin introducing an informal and constructive description of the algorithm (cf. Sec. \([4.1]\)), followed by its formal statement along with its convergence properties (cf. Sec. \([4.2]\)).

**4.1 SONATA at-a-glance**

Each agent \(i\) maintains and updates iteratively a local copy \(x_{(i)}\) of the global variable \(x\), along with an auxiliary variable \(y_{(i)} \in \mathbb{R}^m\); let \(x_{(i)}^n\) and \(y_{(i)}^n\) denote the values of \(x_{(i)}\) and \(y_{(i)}\) at iteration \(n\), respectively. Roughly speaking, the update of these variables is designed so that all the \(x_{(i)}^n\) will be asymptotically consensual, converging to a stationary solution of \([\mathcal{P}]\); and each \(y_{(i)}\) tracks locally the average of the gradients \((1/I) \cdot \sum_{i=1}^{I} \nabla f_i\), an information that is not available at the agent’s side. More specifically, the following two steps are performed iteratively and in parallel across the agents.

**Step 1: Local SCA.** The nonconvexity of \(f_i\) together with the lack of knowledge of \(\sum_{j \neq i} f_j\) in \(F\), prevent agent \(i\) to solve Problem \([\mathcal{P}]\) directly. To cope with these issues, we leverage SCA techniques: at each iteration \(n\), given the current iterate \(x_{(i)}^n\) and \(y_{(i)}^n\), agent \(i\) solves instead a convexification of \([\mathcal{P}]\), having the following form:

\[
\hat{x}_{(i)}^n \triangleq \arg \min_{x_{(i)} \in \mathcal{K}} \tilde{F}_i \left( x_{(i)}; x_{(i)}^n, y_{(i)}^n \right) + G^+ \left( x_{(i)} \right) ,
\]

and updates its \(x_{(i)}\) according to

\[
x_{(i)}^{n+1/2} = x_{(i)}^n + \alpha^n \left( \hat{x}_{(i)}^n - x_{(i)}^n \right) ,
\]

where \(\alpha^n \in (0, 1)\) is a step-size (to be properly chosen). In \([26]\), \(\tilde{F}_i(\bullet; x_{(i)}^n, y_{(i)}^n)\) is chosen as:

\[
\tilde{F}_i \left( x_{(i)}; x_{(i)}^n, y_{(i)}^n \right) \triangleq \tilde{F}_i \left( x_{(i)}; x_{(i)}^n \right) - \nabla G^+ \left( x_{(i)}^n \right) ^\top \left( x_{(i)} - x_{(i)}^n \right) + \left( I \cdot y_{(i)} - \nabla f_i \left( x_{(i)}^n \right) \right) ^\top \left( x_{(i)} - x_{(i)}^n \right) ,
\]
where \( \tilde{f}_i(x; x_n^{(i)}) \) is a strongly convex approximation of \( f_i \) at the current iterate \( x_n^{(i)} \) (see Assumption D below); the second term is the linearization of the smooth nonconvex function \(-G^-\); and \( y_{(i)}^n \), as anticipated, aims at tracking the gradient average \((1/I) \sum_{j=1}^I \nabla f_j(x_n^{(j)})\), that is, \( \lim_{n \to \infty} \| y_{(i)}^n - (1/I) \sum_{j=1}^I \nabla f_j(x_n^{(j)}) \| = 0 \). This sheds light on the role of the last term in (28): under the claimed tracking properties of \( y_{(i)}^n \), there would hold:

\[
\lim_{n \to \infty} \left\| (I \cdot y_{(i)}^n - \nabla f_i(x_n^{(i)})) - \sum_{j \neq i} \nabla f_j(x_n^{(j)}) \right\| = 0. \tag{29}
\]

Therefore, the last term in (28) can be seen as a proxy of the gradient sum \( \sum_{j \neq i} \nabla f_j(x_n^{(i)}) \), which is not available at agent \( i \)'s site. Building on the perturbed condensed push-sum protocol introduced in Sec. 3, we will show in Step 2 below how to update \( y_{(i)}^n \) so that (29) holds, using only local information.

The surrogate function \( \tilde{f}_i \) satisfies the following assumption.

**Assumption E (On surrogate function \( \tilde{f}_i \))** Let \( \tilde{f}_i : K \times K \to \mathbb{R} \) be a \( C^1 \) function with respect to its first argument, and such that

- **D1.** \( \nabla \tilde{f}_i (x; x) = \nabla f_i (x) \), for all \( x \in K \);
- **D2.** \( \tilde{f}_i (\bullet; y) \) is uniformly strongly convex on \( K \), with constant \( \tau_i \);
- **D3.** \( \nabla \tilde{f}_i (x; \bullet) \) is uniformly Lipschitz continuous on \( K \), with constant \( \tilde{L}_i \);

where \( \nabla \tilde{f}_i (x; y) \) denotes the partial gradient of \( \tilde{f}_i \) with respect to the first argument, evaluated at \((x, y)\).

Conditions D1-D3 are quite natural: \( \tilde{f}_i \) should be regarded as a (simple) convex, local, approximation of \( f_i \) at \( x \) that preserves the first order properties of \( f_i \). A gamut of choices for \( \tilde{f}_i \) satisfying Assumption E are available; some representative examples are discussed in Sec. 4.4.

**Step 2: Information mixing and gradient tracking.** To complete the description of the algorithm, we need to introduce a mechanism to ensure that i) the local estimates \( x_{(i)}^n \)'s asymptotically converge to a common value; and ii) each \( y_{(i)}^n \) tracks the gradient sum \( \sum_{j \neq i} \nabla f_j(x_n^{(j)}) \). To this end, we leverage the perturbed condensed push-sum protocol introduced in Sec. 3. Specifically, given \( x_{(i)}^{n+1/2} \)'s, each \( x_{(i)}^n \) is updated according to [cf. (4)]

\[
\phi_i^{n+1} = \sum_{j=1}^I a_i^{n} \phi_j^{n}, \quad x_{(i)}^{n+1} = \sum_{j=1}^I a_i^{n} x_{(j)}^{n+1/2}, \tag{30}
\]

where the \( a_i^{n} \) are chosen to satisfy Assumption C. Note that, the updates in (30) can be performed in a distributed way: each agent \( j \) only needs to select the set of weights \( \{a_i^{n}\}_j \) and send \( a_i^{n} \phi_j^n \) and \( a_i^{n} \phi_j^n x_{(j)}^{n+1/2} \) to its out-neighbors while summing up the information received from its in-neighbors.

To update the \( y_{(i)}^n \)'s we leverage again the perturbed condensed push-sum scheme [4], with \( \epsilon_i^{n+1} = (1/\phi_i^{n+1}) (\nabla f_i(x_{(i)}^{n+1}) - \nabla f_i(x_{(i)}^{n+1})) \) [cf. (13)]. The resulting gradient tracking mechanism reads...
\[
y^{n+1}_{(i)} = \frac{1}{\phi^i_{n+1}} \sum_{j=1}^I a_{ij}^{n} \phi^j_{n+1} y^{n}_{(j)} + \frac{1}{\phi^i_{n+1}} \left( \nabla f_i(x^{n+1}_{(i)}) - \nabla f_i(x^{n}_{(i)}) \right),
\]

with \( y^{0}_{(i)} = \nabla f_i(x^{0}_{(i)}) \). Note that the update of \( y^{n}_{(i)} \) can be performed locally by agent \( i \), with the same signaling as that of (30).

### 4.2 The SONATA algorithm

We can now formally introduce the proposed algorithm, SONATA, just combining steps (26), (27), (30), and (34)—see Algorithm 1.

**Algorithm 1 SONATA**

**Data**: \( x^0_{(i)} \in \mathcal{K} \), for all \( i \); \( \phi^0 = 1 \); \( y^0 = g^0 \). Set \( n = 0 \).

1. **(S.1)** If \( x^n \) satisfies termination criterion: STOP;
2. **(Distributed Local Optimization)** Each agent \( i \) compute locally \( \tilde{x}^n_{(i)} \) solving problem (26);
3. **(Information Mixing)** Each agent \( i \) compute
   \( \phi^{n+1}_{i} = \sum_{j=1}^I a_{ij}^{n} \phi^j_{n+1} \)
   \( x^{n+1}_{(i)} = \frac{1}{\phi^i_{n+1}} \sum_{j=1}^I a_{ij}^{n} \phi^j_{n+1} x^{n+1}_{(j)} \); (32)
   \( y^{n+1}_{(i)} = \frac{1}{\phi^i_{n+1}} \sum_{j=1}^I a_{ij}^{n} \phi^j_{n+1} y^{n}_{(j)} + \frac{1}{\phi^i_{n+1}} \left( \nabla f_i(x^{n+1}_{(i)}) - \nabla f_i(x^{n}_{(i)}) \right) \); (33)
   \( y^{n+1}_{(i)} = \frac{1}{\phi^i_{n+1}} \sum_{j=1}^I a_{ij}^{n} \phi^j_{n+1} y^{n}_{(j)} + \frac{1}{\phi^i_{n+1}} \left( \nabla f_i(x^{n+1}_{(i)}) - \nabla f_i(x^{n}_{(i)}) \right) \); (34)

4. **(S.4)** \( n \leftarrow n + 1 \), go to (S.1)

Note that the algorithm is distributed. Indeed, in Step 2, the optimization (26) is performed locally by each agent \( i \), computing its own \( \tilde{x}^n_{(i)} \). To do so, agent \( i \) needs to know the current \( x^n_{(i)} \) and \( y^n_{(i)} \), which are both available locally. There are then two consensus steps (Step 3) whereby agents transmit/receive information only to/from their out/in neighbors: one is on the optimization variables \( x^n_{(i)} \) (and the auxiliary scalars \( \phi^n_{i} \))—see (32)–(33)—and one is on the variables \( y^n_{(i)} \)—see (34).

### 4.3 Convergence and complexity analysis of SONATA

To prove convergence, in addition to Assumptions A-E, one needs some conditions on the step-size \( \alpha^n \). Since line-search methods are not practical in a
distributed environment, there are two other options, namely: i) a fixed (sufficiently small) step-size; and ii) a diminishing step-size. We prove convergence using either choices. Recalling the definition of the network parameters \( c_0, B, \rho_B, \phi_B, \) and \( \phi_{ub} \) as given in Proposition 3 [see also [10]] and introducing the problem parameters [cf. Assumptions A]

\[
L \triangleq \sum_{i=1}^{I} L_i, \quad L_{\text{max}} \triangleq \max_{1 \leq i \leq I} L_i + L_G, \quad L_{\text{max}} \triangleq \max_{1 \leq i \leq I} L_i,
\]

\[
c_r \triangleq \min_{1 \leq i \leq I} \tau_i, \quad c_L \triangleq \left( L\sqrt{I} + L_{\text{max}} + L_{\text{max}} \right) / I,
\]

the step-size can be chosen as follows.

**Assumption F** The step-size \( \{\alpha^n\}_{n \in \mathbb{N}_+} \) satisfies either one of the following conditions:

\( F1. \) (diminishing): \( \{0, 1\} \supseteq \alpha^n \downarrow 0 \) and \( \sum_{n=0}^{\infty} \alpha^n = \infty; \)

\( F2. \) (fixed): \( \alpha^n \equiv \alpha, \) for all \( n \in \mathbb{N}_+, \) with

\[
\alpha \leq \min \left\{ \frac{(1-\rho_B)\sigma}{\sqrt{2}cB}, \frac{2c_r \phi_B}{I \phi_{ub}} \left( \frac{L + L_G}{I} + \frac{2c_l B c}{1 - \rho_B} \sqrt{\frac{2}{1 - \sigma^2}} + \frac{12L_{\text{max}} \phi_B^{-1} B^2 c^2}{(1 - \rho_B)^2} \sqrt{\frac{1}{1 - \sigma^2}} \right)^{-1} \right\}, \tag{36}
\]

where \( \sigma \) is an arbitrary constant \( \sigma \in (0, 1) \) and \( c = I \sqrt{2I}. \)

In addition, if all \( \mathbf{A}^x \) are double stochastic, the upper bound in (36) holds with \( c = 1, B = B, \phi_B = \phi_{ub} = 1, \) and \( \rho_B = \left( 1 - \kappa / (2I^2) \right)^{1/2}. \)

We can now state the convergence results of the proposed algorithm, postponing all the proofs to Sec. 6. Given \( \{\mathbf{x}^n \triangleq (x^n_i)_{i=1}^{I}\}_{n \in \mathbb{N}_+} \) generated by Algorithm 1, convergence is stated measuring the distance of the average sequence \( \bar{x}^n \) from optimality and well as the consensus disagreement among the local variables \( x^n_i \)'s. Distance from stationarity is measured by the following function:

\[
J(\bar{x}^n) \triangleq \left\| \bar{x}^n - \arg\min_{\mathbf{x} \in \mathbb{X}} \left\{ \left( \nabla F(\bar{x}^n) - \nabla G^{-1}(\bar{x}^n) \right)^\top (\mathbf{z} - \bar{x}^n) + \frac{1}{2} \| \mathbf{z} - \bar{x}^n \|^2 + G(\mathbf{z}) \right\} \right\|.
\tag{37}
\]

Note that \( J \) is a valid measure of stationarity because it is continuous and \( J(\bar{x}^\infty) = 0 \) if and only if \( \bar{x}^\infty \) is a \( d \)-stationary solution of Problem P [16].

The consensus disagreement at iteration \( n \) is defined as

\[
D(\mathbf{x}^n) \triangleq \| \mathbf{x}^n - \mathbf{1}_I \otimes \bar{x}^n \|.
\]

Note that \( D \) is equal to 0 if and only if all the \( x^n_i \)'s are consensual. We combine the metrics \( J \) and \( D \) in a single merit function, defined as

\[
M(\mathbf{x}^n) \triangleq \max \{ J(\mathbf{x}^n)^2, D(\mathbf{x}^n)^2 \}.
\]

We are now ready to state the main convergence results for Algorithm 1.
**Theorem 4 (asymptotic convergence)** Given Problem (P) and Algorithm 1, suppose that Assumptions A-F are satisfied; and let \( \{x^n\}_{n \in \mathbb{N}_+} \) be the sequence generated by the algorithm. Then, there holds \( \lim_{n \to \infty} M(x^n) = 0 \).

Under a constant step-size (Assumption F.2), the next theorem provides an upper bound on the number of iterations needed to decrease \( M(x^n) \) below a given accuracy \( \epsilon > 0 \).

**Theorem 5 (complexity)** Suppose that Assumptions A-E are satisfied; and let \( \{x^n\}_{n \in \mathbb{N}_+} \) be the sequence generated by Algorithm 1, with a constant step-size \( \alpha_n = \alpha \), satisfying Assumption F.2. Given \( \epsilon > 0 \), let \( T_\epsilon \) be the first iteration \( n \) such that \( M(x^n) \leq \epsilon \). Then \( T_\epsilon = O(1/\epsilon^2) \).

**Remark 6 (generalizations)** Theorems 4 and 5 can be established with minor modifications under the setting wherein each agent \( i \) uses different constant step-size \( \alpha_i \). Also, the assumption on the strongly convexity of the surrogate function \( \tilde{f}_i \) (Assumption E.2) can be weakened to just convexity, if the feasible set \( K \) is compact. With mild additional assumptions on \( G^- \)–see [12]–we can extend convergence results in Theorem 4 to the case wherein agents solve their subproblems (26) inexactly. We omit further details because of space limitation.

### 4.4 Discussion

Theorem 4 (resp. Theorem 5) provides the first convergence (resp. complexity) result of distributed algorithms for **constrained** and/or **composite** optimization problems over time-varying (undirected or directed) graphs, which significantly enlarges the class of convex and nonconvex problems which distributed algorithms can be applied to with convergence guarantees.

SONATA represents a gamut of algorithms, each of them corresponding to a specific choice of the surrogate function \( \tilde{f}_i \), step-size \( \alpha^n \), and matrices \( A^n \). Convergence is guaranteed under several choices of the free parameters of the algorithms, some of which are briefly discussed next.

- **On the choice of \( \tilde{f}_i \)**. Examples of \( \tilde{f}_i \) satisfying Assumption E are
  - **Linearization**: Linearize \( f_i \) and add a proximal regularization (to make \( \tilde{f}_i \) strongly convex), which leads to
    \[
    \tilde{f}_i(x_{(i)}; x^n_{(i)}) = f_i(x^n_{(i)}) + \nabla f_i(x^n_{(i)})(x_{(i)} - x^n_{(i)}) + \frac{\tau_i}{2} \|x_{(i)} - x^n_{(i)}\|^2;
    \]
  - **Partial Linearization**: Consider the case where \( f_i \) can be decomposed as \( f_i(x_{(i)}) = f_i^{(1)}(x_{(i)}) + f_i^{(2)}(x_{(i)}) \), where \( f_i^{(1)} \) is convex and \( f_i^{(2)} \) is nonconvex with Lipschitz continuous gradient. Preserving the convex part of \( f_i \) while linearizing \( f_i^{(2)} \) leads to the following valid surrogate
    \[
    \tilde{f}_i(x_{(i)}; x^n_{(i)}) = f_i^{(1)}(x_{(i)}) + f_i^{(2)}(x^n_{(i)}) + \frac{\tau_i}{2} \|x_i - x^n_{(i)}\|^2 \]
    \[
    + \nabla f_i^{(2)}(x^n_{(i)})(x_i - x^n_{(i)});
    \]
– *Partial Convexification:* Consider the case where \(x_{(i)}\) is partitioned as \((x_{(i,1)}, x_{(i,2)})\), and \(f_i\) is convex in \(x_{(i,1)}\) but not in \(x_{(i,2)}\). Then, one can convexify only the nonconvex part of \(f_i\), which leads to the surrogate:

\[
\tilde{f}_i(x_{(i)}; x_{(i)}^n) = f_i(x_{(i,1)}, x_{(i,2)}^n) + \frac{\tau_i}{2} \|x_{(i,2)} - x_{(i,2)}^n\|^2 + \nabla^{(2)} f_i(x_{(i,2)}^n)^\top (x_{(i,2)} - x_{(i,2)}^n),
\]

where \(\nabla^{(2)} f_i\) denotes the gradient of \(f_i\) with respect to \(x_{(i,2)}\). Other choices of surrogates can be obtained hinging on \([15,16,36]\).

• **On the choice of the step-size.** Several options are possible for the step-size sequence \(\{\alpha^n\}_n\) satisfying the diminishing-rule in Assumption 1; see, e.g., \([2]\). Two instances we found to be effective in our experiments are: i) \(\alpha_n = \alpha_0/(n+1)\beta\), with \(\alpha_0 > 0\) and \(0.5 < \beta \leq 1\); and ii) \(\alpha_n = \alpha^{n-1}(1 - \mu \alpha^{n-1})\), with \(\alpha_0 \in (0, 1]\), and \(\mu \in (0, 1]\).

• **On the choice of matrix \(A^n\).** When dealing with digraphs, the key requirement of Assumption \([1]\) is that each \(A^n\) is column stochastic. Such matrices can be built locally by the agents: each agent \(j\) can simply choose weight \(a^n_{ij}\) for \(i \in \mathcal{N}_j^{\text{out}}[n]\) so that \(\sum_{i \in \mathcal{N}_j^{\text{out}}[n]} a^n_{ij} = 1\). As a special case, \(A^n\) can be set to be the following push-sum matrix \([24]\): \(a^n_{ij} = 1/d^n_i\), if \((j, i) \in \mathcal{E}^n\); and \(a^n_{ij} = 0\), otherwise; where \(d^n_i\) is the out-degree of agent \(i\). In this case, the information mixing process in Step 2 becomes a broadcasting protocol, which requires from each agent only the knowledge of its out-degree.

When the digraphs \(G^n\) admit a double-stochastic matrix (e.g., they are undirected), as already observed in Sec. 3 (cf. Remark \([1]\)), one can choose \(A^n\) as double-stochastic; and the consensus and tracking protocols in Step 3 reduce respectively to

\[
\begin{align*}
x_{(i)}^{n+1} &= \sum_{j=1}^I a^n_{ij} (x_{(j)}^n + \alpha^n (x_{(j)}^n - x_{(j)}^n)) \\
y_{(i)}^{n+1} &= \sum_{j=1}^I a^n_{ij} y_{(j)}^n + \nabla f_i(x_{(i)}^n) - \nabla f_i(x_{(i)}^n).
\end{align*}
\]

Several choices have been proposed in the literature to build in a distributed way a double stochastic matrix \(A^n\), including: the Laplacian, Metropolis-Hastings, and maximum-degree weights; see, e.g., \([50]\).

• **ATC/CAA updates.** In the case of unconstrained optimization, the information mixing step in Algorithm 1 can be performed following two alternative protocols, namely: i) the *Adapt-Then-Combine-based* (ATC) scheme; and ii) the *Combine-And-Adapt-based* (CAA) approach (termed “consensus strategy” in \([35]\). The former is the one used in \([30]\): each agent \(i\) first updates its local copy \(x_{(i)}^n\) along the direction \(x_{(i)}^n - x_{(i)}^n\), and then combines its new update with that of its neighbors via consensus. Alternatively, in the CAA update, agent \(i\) first mixes its own local copy \(x_{(i)}^n\) with that of its neighbors via consensus,
and then performs its local optimization-based update using $\tilde{x}_{(i)}^n - x_{(i)}^n$, that is

\[ x_{(i)}^{n+1} = \frac{1}{\phi_{n+1}^n} \sum_{j=1}^l a_{ij} \phi_{j}^n x_{(j)}^n + \frac{\phi_n}{\phi_{n+1}^n} \cdot \alpha^n (\tilde{x}_{(i)}^n - x_{(i)}^n). \]

It is not difficult to check that SONATA based on CAA updates converges under the same conditions as in Theorem 4.

5 SONATA and special cases

In this section, we contrast SONATA with related algorithms proposed in the literature \cite{12,14,51} and very recent proposals \cite{30,33,49} for special instances of Problem (P). We show that algorithms in \cite{30,33,49} are all special cases of SONATA and NEXT, proposed in our earlier works \cite{12,14,41}.

We preliminarily rewrite Algorithm 1 in a matrix-vector form. Similarly to $x^n$, define the concatenated vectors

\[ \tilde{x}^n \triangleq [\tilde{x}_1^n, \ldots, \tilde{x}_I^n]^\top, \]
\[ y^n \triangleq [y_1^n, \ldots, y_I^n]^\top, \]
\[ g^n \triangleq [g_1^n, \ldots, g_I^n]^\top, \quad g^n_i \triangleq \nabla f_i(x_i^n), \]
\[ \Delta x^n \triangleq \tilde{x}^n - x^n, \]

where $\tilde{x}_i^n$ and $y_i^n$ are defined in \cite{26} and \cite{34}, respectively. Using the above notation and the matrices introduced in \cite{14a}, SONATA [cf. \cite{32,34}] can be written in compact form as

\[ \phi^{n+1} = A^n \phi^n \]
\[ x^{n+1} = \tilde{W}^n (x^n + \alpha^n \Delta x^n) \]
\[ y^{n+1} = \tilde{W}^n y^n + (\tilde{D}_{\phi^{n+1}})^{-1} (g^{n+1} - g^n). \]

5.1 Preliminaries: SONATA-NEXT and SONATA-L

Since \cite{30,33,49,51} are applicable only to unconstrained ($\mathcal{K} = \mathbb{R}^m$), smooth ($G = 0$) and convex (each $f_i$ is convex) multiagent problems, in the following, we consider only such an instance of Problem (P). Choose each $\tilde{f}_i$ as first order approximation of $f_i$ plus a proximal term, that is,

\[ \tilde{f}_i(x_i^n; x_{(i)}^n) = f_i(x_{(i)}^n) + \nabla f_i(x_i^n)^\top (x_{(i)}^n - x_{(i)}^n) + \frac{\tau_i}{2} \| x_{(i)}^n - x_{(i)}^n \|^2, \]

and set $\tau_i = I$. Then, $\tilde{x}_{(i)}^n$ can be computed in closed form [cf. \cite{26}]:

\[ \tilde{x}_{(i)}^n = \arg\min_{x_{(i)}} (I \cdot y_{(i)}^n)^\top (x_{(i)} - x_{(i)}^n) + \frac{I}{2} \| x_{(i)} - x_{(i)}^n \|^2 \]
\[ = \arg\min_{x_{(i)}} \frac{I}{2} \| x_{(i)} - x_{(i)}^n + y_{(i)}^n \|^2 = x_{(i)}^n - y_{(i)}^n. \]

Therefore, $\Delta x_{(i)}^n = \tilde{x}_{(i)}^n - x_{(i)}^n = y_{(i)}^n$. 
Substituting (46) into (44) and using either ATC or CAA mixing protocols, Algorithm 1 reduces to
\[
\begin{align*}
\phi^{n+1} &= A^n \phi^n \\
x^{n+1} &= \begin{cases} \hat{W}^n (x^n - \alpha^n y^n) & \text{(ATC-based update)} \\ \hat{W}^n x^n - \alpha^n (\hat{D}_{\phi^{n+1}})^{-1} \hat{D}_{\phi^n} y^n & \text{(CAA-based update)} \end{cases} \\
y^{n+1} &= \hat{W}^n y^n + \left(\hat{D}_{\phi^{n+1}}\right)^{-1} \left(g^{n+1} - g^n\right)
\end{align*}
\] (47)
which we will refer to as (ATC/CAA-)SONATA-L (L stands for “linearized”).

When the digraph $G^n$ admits a double-stochastic matrix $A^n$, and $A^n$ in (43) is chosen so, the iterates (47) can be further simplified as reduces to
\[
\begin{align*}
x^{n+1} &= \begin{cases} \hat{W}^n (x^n - \alpha^n y^n) & \text{(ATC-based update)} \\ \hat{W}^n x^n - \alpha^n y^n & \text{(CAA-based update)} \end{cases} \\
y^{n+1} &= \hat{W}^n y^n + g^{n+1} - g^n
\end{align*}
\] (48)
where $W^n = A^n$ and thus $\hat{W}^n = W^n \otimes I_m$. The ATC-based updates coincide with our previous algorithm NEXT [based on the surrogate (46)], introduced in [12–14]. We will refer to (48) as (ATC/CAA-)NEXT-L.

5.2 Connection with current algorithms

We can now show that the algorithms recently studied in [30, 33, 49, 51] are all special cases of SONATA and NEXT, earlier proposed in [12–14].

Aug-DGM [51] and Algorithm in [33]. Introduced in [51] for undirected, time-invariant graphs, the Aug-DGM algorithm reads
\[
\begin{align*}
x^{n+1} &= \hat{W} (x^n - \alpha^n y^n) \\
y^{n+1} &= \hat{W} (y^n + g^{n+1} - g^n)
\end{align*}
\] (49)
where $\hat{W} \triangleq W \otimes I_m$; $W$ is a double stochastic matrix satisfying Assumption [C] and $\alpha$ is the vector of agents’ step-sizes $\alpha_i$’s.

A similar algorithm was proposed independently in [33] (in the same networking setting of [51]), which reads
\[
\begin{align*}
x^{n+1} &= \hat{W} (x^n - \alpha y^n) \\
y^{n+1} &= \hat{W} y^n + g^{n+1} - g^n
\end{align*}
\] (50)
Clearly Aug-DGM [51] in (49) with the $\alpha_i$’s equal, and Algorithm [33] in (50) coincide with (ATC)-NEXT-L [cf. (48)].

(Push-)DIGing [30]. Appeared in [30] and applicable to $B$-strongly connected undirected graphs, the DIGing Algorithm reads
\[
\begin{align*}
x^{n+1} &= \hat{W}^n x^n - \alpha y^n \\
y^{n+1} &= \hat{W}^n y^n + g^{n+1} - g^n
\end{align*}
\] (51)
where $W^n$ is a double-stochastic matrix satisfying Assumption [C]. Clearly, DIGing coincides with (CAA-)NEXT-L [12,14] cf. (18). The push-DIGing algorithm, studied in the same paper [30], extends DIGing to $B$-strongly connected digraphs. It turns out that push-DIGing coincides with (ATC-)SONATA-L cf. Eq. (47) when $a^n_{ij} = 1/d^n_{ij}$.

ADD-OPT [49]. Finally, we mention the ADD-OPT algorithm, proposed in [49] for strongly connected static digraphs, which takes the following form:

$$z^{n+1} = \hat{A}z^n - \alpha \tilde{y}^n$$

$$\phi^{n+1} = A \phi^n$$

$$x^{n+1} = (\hat{D}_{\phi^{n+1}})^{-1} z^{n+1}$$

$$\tilde{y}^{n+1} = \hat{A} \tilde{y}^n + g^{n+1} - g^n,$$

where $A$ is a column stochastic matrix satisfying Assumption [C] and $\hat{A} = A \otimes I_m$. Defining $y^n = (\hat{D}_{\phi^{n+1}})^{-1} \tilde{y}^n$, it is not difficult to check that (52) can be rewritten as

$$\phi^{n+1} = A \phi^n, \quad W = (\hat{D}_{\phi^{n+1}})^{-1} \hat{A} \hat{D}_{\phi^n}$$

$$x^{n+1} = \hat{W}x^n - \alpha (\hat{D}_{\phi^{n+1}})^{-1} \hat{D}_{\phi^n} y^n$$

$$y^{n+1} = \hat{W}y^n + (\hat{D}_{\phi^{n+1}})^{-1} (g^{n+1} - g^n).$$

Comparing Eq. (47) and (53), one can see that ADD-OPT coincides with (CAA-)SONATA-L.

We summarize the connections between the different versions of SONATA(-NEXT) and its special cases in Table 4.

6 Convergence Proof of SONATA

In this section, we prove convergence of SONATA; because of space limitation we prove only Theorem 4. The proof consists in studying the dynamics of a suitably chosen Lyapunov function along the weighted average of the agents’ local copies, and of the consensus disagreement and tracking errors. We begin introducing some convenient notation along with some preliminary results. For the sake of simplicity, all the results of the forthcoming subsections are stated under the blanket Assumptions A-F.

6.1 Notations and preliminaries

The weighted average and associated consensus disagreement are denoted by

$$\bar{x}_{\phi^n} \triangleq \frac{1}{T} \left( \phi^{nT} \otimes I_m \right) x^n \quad \text{and} \quad e^n_x \triangleq x^n - J_{\phi^n} x^n,$$

respectively. Similar quantities are defined for the tracking variables $y^n_{(i)}$:

$$\bar{y}_{\phi^n} \triangleq \frac{1}{T} \left( \phi^{nT} \otimes I_m \right) y^n \quad \text{and} \quad e^n_y \triangleq y^n - J_{\phi^n} y^n.$$
Table 4 Connection of SONATA with current algorithms

| Algorithms   | Connection with SONATA | Instance of Problem (P) | Graph topology/Weight matrix |
|--------------|-------------------------|-------------------------|----------------------------|
| NEXT         | special case of SONATA  | $F$ nonconvex $G \neq 0$   | time-varying digraph/doubly-stochastic weights |
| Aug-DGM      | ATC-NEXT-L ($\alpha = \alpha I_f$) | $F$ convex $G = 0$   | static undirected graph/doubly-stochastic weights |
| DIGing       | CAA-NEXT-L | $F$ convex $G = 0$ | time-varying digraph/doubly-stochastic weights |
| push-DIGing  | ATC-SONATA-L | $F$ convex $G = 0$ | time-varying digraph/column-stochastic weights |
| ADD-OPT      | ATC-SONATA-L | $F$ convex $G = 0$ | static digraph/column-stochastic weights |

Recalling (39), define the deviation of the local solution $\hat{x}_{n}^{i}$ of each agent from the weighted average as

$$\Delta \hat{x}_{n}^{i}, \phi \triangleq \hat{x}_{n}^{i} - \bar{x}^{\phi, n}$$  \hspace{1cm} (56)

and the associated stacked vector

$$\Delta \hat{x}_{n}^{\phi} \triangleq \hat{x}^{n} - J^{\phi}x^{n}$$  \hspace{1cm} (57)

Note that $\Delta x^{n}$ [cf. (39)] can be rewritten as

$$\Delta x^{n} = \Delta \hat{x}_{n}^{\phi} - e_{n}^{n}$$  \hspace{1cm} (58)

Using the above notation, the dynamics of $\hat{x}_{\phi^{n}}$ and $\hat{y}_{\phi^{n}}$ generated by Algorithm 1 are given by [cf. (14) and (45)]:

$$\hat{x}_{\phi^n+1} = \hat{x}_{\phi^n} + \frac{e_{n}^{n}}{I} (\phi_{i}^{n})^{T} \otimes I_{m} \, \Delta \hat{x}_{\phi}^{n}$$  \hspace{1cm} (59a)

$$\hat{y}_{\phi^{n+1}} = \hat{y}_{\phi^{n}} + g_{n+1}^{n} - g_{n}^{n}$$  \hspace{1cm} (59b)

Note that, since $y_{0} = g_{0}^{0}$ and $\phi_{0}^{0} = 1$, we have $\hat{y}_{\phi^{n}} = \hat{g}_{n}^{n}$, for all $n \in \mathbb{N}_{+}$.

Finally, we introduce the error-free local solution map of each agent $i$, denoted by $\hat{x}_{(i)} : \mathcal{K} \to \mathcal{K}$: Given $z \in \mathcal{K}$ and $i = 1, \ldots, I$, let
Each Lemma 7 of the next lemma follows similar steps as in [16, Prop. 8] and thus is omitted.

It is not difficult to check that \( \tilde{x}_{(i)}(\bullet) \) enjoys the following properties (the proof of the next lemma follows similar steps as in [16] Prop. 8 and thus is omitted).

**Lemma 7** Each \( \tilde{x}_{(i)}(\bullet) \) satisfies:

i) **Lipschitz continuity**: \( \tilde{x}_{(i)}(\bullet) \) is \( \hat{L} \)-Lipschitz continuous on \( \mathcal{K} \), that is, there exists a finite \( L > 0 \) such that

\[
\| \tilde{x}_{(i)}(z) - \tilde{x}_{(i)}(w) \| \leq \hat{L} \| z - w \|, \quad \forall z, w \in \mathcal{K};
\]

ii) **Fixed-points**: The set of fixed points of \( \tilde{x}_{(i)}(\bullet) \) coincides with the set of \( d \)-stationary solutions of Problem \( \mathcal{P} \).

The next result shows that, as expected, the disagreement between agent \( i \)'s solution \( \tilde{x}_{(i)}^n \) and its error-free counterpart \( \hat{x}_{(i)}(\tilde{x}_{(i)}^n) \) asymptotically vanishes if both the consensus error \( e_x^n \) and the tracking error \( e_y^n \) do so.

**Lemma 8** \( \tilde{x}_{(i)}^n \) [cf. (26)] and \( \hat{x}_{(i)}(\tilde{x}_{(i)}^n) \) [cf. (60)] satisfy:

\[
\left\| \tilde{x}_i(x_{(i)}^n) - \tilde{x}_{(i)}^n \right\| \leq \frac{f}{\tau_i} \| e_y^n \| + \frac{2 L \hat{L}}{\tau_i} \| e_x^n \|. \tag{62}
\]

Therefore, \( \| e_x^n \|, \| e_y^n \| \xrightarrow{n \to \infty} 0 \Rightarrow \| \tilde{x}_i(x_{(i)}^n) - \tilde{x}_{(i)}^n \| \xrightarrow{n \to \infty} 0. \)

The last result of this section is a standard martingale-like result; the proof follows similar to that of [3] Lemma 1 and thus is omitted.

**Lemma 9** Let \( \{X^n_n\}_{n \in \mathbb{N}_+}, \{Y^n_n\}_{n \in \mathbb{N}_+} \) and \( \{Z^n_n\}_{n \in \mathbb{N}_+} \) be three sequences such that \( X^n \) and \( Z^n \) are nonnegative, for all \( n \in \mathbb{N}_+ \). Suppose that

\[
\sum_{k=0}^{B-1} Y^{n+k} \leq \sum_{k=0}^{B-1} Y^{n+k} - \sum_{k=0}^{B-1} X^{n+k} + \sum_{k=0}^{B-1} Z^{n+k}, \quad n = 0, 1, \ldots, \tag{63}
\]

and that \( \sum_{n=0}^{\infty} Z^n < +\infty \). Then, either \( \sum_{k=0}^{B-1} Y^{n+k} \to -\infty \), or else \( \sum_{k=0}^{B-1} Y^{n+k} \) converges to a finite value and \( \sum_{n=0}^{\infty} X^n < +\infty. \)

### 6.2 Average descent

We begin our analysis studying the dynamics of \( U \) along the trajectory of \( x_0^n \).

We define the total energy of the optimization input \( \alpha^n \Delta \tilde{x}_\phi^n \) and consensus errors \( e_x^n \) and \( e_y^n \) in \( B \) consecutive iterations [\( B \) is defined in Lemma 3]:

\[
E_{\Delta \tilde{x}} \triangleq \sum_{t=0}^{B-1} (\alpha^{n+t})^2 \left\| \Delta \tilde{x}_\phi^{n+t} \right\|^2, \quad E_x^n \triangleq \sum_{t=0}^{B-1} \| e_x^{n+t} \|^2, \quad E_y^n \triangleq \sum_{t=0}^{B-1} \| e_y^{n+t} \|^2. \tag{64}
\]

Recalling the definitions of \( c_r, \phi_{ib}, \) and \( \phi_{ub} \) [see (5) and (35)], we have the following.
Lemma 10 Let \( \{ (x^n, y^n) \}_{n \in \mathbb{N}_+} \) be the sequence generated by Algorithm 7. Then, there holds

\[
\begin{align*}
\sum_{k=0}^{B-1} U \left( \bar{x}^{n+k} \right) & \leq \sum_{k=0}^{B-1} U \left( \bar{x}^{n+k} \right) - \frac{c_r \phi_{lb}}{I} \cdot \sum_{k=0}^{B-1} \sum_{t=0}^{B-1} \alpha^n \| \Delta \bar{x}^{n+k+t} \|^2 \\
& + \frac{\phi_{ub}}{2} \left( L + L_G \right) \cdot \phi_{ub} + c_L \epsilon_x + \epsilon_y \sum_{k=0}^{B-1} E_{\Delta \bar{x}}^{n+k} + \frac{\phi_{ub}}{2} \sum_{k=0}^{B-1} \left( c_L \epsilon_x^{-1} E_{\Delta \bar{x}}^{n+k} + \epsilon_y E_{\Delta \bar{x}}^{n+k} \right),
\end{align*}
\]

where \( \epsilon_x > 0 \) and \( \epsilon_y > 0 \) are arbitrary, finite constants.

Proof Denote for simplicity \( \bar{F} \triangleq F - G^- \). Since \( \bar{f}_i \) is strongly convex and \( G^+ \) is convex, by the first order optimality of \( \bar{x}^{(i)}(\phi_n) \), we have

\[
\begin{align*}
\left( \Delta \bar{x}^{(i)}(\phi_n) \right)^T \left( I : y^{(i)}(\phi_n) + \nabla \bar{f}_i(\bar{x}^{n}(\phi_n)) - \nabla G^-(\bar{x}^{n}(\phi_n)) - \nabla f_i(\bar{x}^{n}(\phi_n)) \right) & + G^+(\bar{x}^{n}(\phi_n)) - G^+(\bar{x}^{n}(\phi_n)) \\
& \leq -\tau_i \| \Delta \bar{x}^{n}(\phi_n) \|^2.
\end{align*}
\]

Since \( \nabla f_i \) and \( \nabla G^- \) are \( L_i \) and \( L_G \)-Lipschitz, respectively, \( \nabla F \) is \( (L + L_G) \)-Lipschitz, where \( L \triangleq \sum_{i=1}^{I} L_i \) [cf. def. (65)]. Applying the descent lemma to \( \bar{F} \) and using (69) yields

\[
\begin{align*}
F(\bar{x}^{n+1}) & \leq F(\bar{x}^{n}) + \frac{\alpha^n}{I} \nabla F(\bar{x}^{n})^T \left( (\phi_n)^T \otimes I_m \right) \Delta \bar{x}^{n}(\phi_n) \\
& + \frac{L + L_G}{2} \cdot \frac{(\alpha^n)^2}{I} \left\| (\phi_n)^T \otimes I_m \right\| \Delta \bar{x}^{n}(\phi_n) \| \Delta \bar{x}^{n}(\phi_n) \|^2 \tag{a} \\
& \leq F(\bar{x}^{n}) + \frac{L + L_G}{2} \cdot \frac{(\alpha^n)^2}{I} \phi_{ub} \| \Delta \bar{x}^{n}(\phi_n) \|^2 \\
& - \frac{\alpha^n}{I} \sum_{i=1}^{I} \phi_{i}^n \left( \tau_i \| \Delta \bar{x}^{n}(\phi_n) \|^2 + G^+(\bar{x}^{n}(\phi_n)) - G^+(\bar{x}^{n}(\phi_n)) \right) \\
& + \frac{\alpha^n}{I} \sum_{i=1}^{I} \phi_{i}^n \left( \nabla F(\bar{x}^{n}(\phi_n)) + \nabla G^-(\bar{x}^{n}(\phi_n)) - I : y^{n}(\phi_n) + I : y^{n}(\phi_n) \right)^T \Delta \bar{x}^{n}(\phi_n) \\
& + \frac{\alpha^n}{I} \sum_{i=1}^{I} \phi_{i}^n \left( \nabla f_i(\bar{x}^{n}(\phi_n)) - \nabla f_i(\bar{x}^{n}(\phi_n)) + \nabla \bar{f}_i(\bar{x}^{n}(\phi_n)) - \nabla \bar{f}_i(\bar{x}^{n}(\phi_n)) \right)^T \Delta \bar{x}^{n}(\phi_n) \tag{b} \\
& \leq F(\bar{x}^{n}) + \frac{L + L_G}{2} \cdot \frac{(\alpha^n)^2}{I} \phi_{ub} \| \Delta \bar{x}^{n}(\phi_n) \|^2 \\
& - \frac{\alpha^n}{I} \sum_{i=1}^{I} \phi_{i}^n \left( \tau_i \| \Delta \bar{x}^{n}(\phi_n) \|^2 + G^+(\bar{x}^{n}(\phi_n)) - G^+(\bar{x}^{n}(\phi_n)) \right)
\end{align*}
\]
where in (a) we used (66), Assumption E.1, and the bound (107) (along with some basic manipulations); in (b) we used $y_{\phi^n} = \mathbf{g}^n$ [cf. (59b)]; (c) follows from the $L_\ell$-Lipschitz continuity of $\nabla f_i$, $L_G$-Lipschitz continuity of $\nabla G^-$, and the uniformly $L_\ell$-Lipschitz continuity of $\nabla f_i(x; \bullet)$; and in (d) we used the inequality $\|x\| \leq \sqrt{n}\|x\|$, and the definition of $c_L$ [cf. (35)].

Invoking the convexity of $G^+$ and using (59a), we can write

$$G^+(\bar{x}_{\phi^{n+1}}) \leq (1 - \alpha^n)G^+(\bar{x}_{\phi^n}) + \frac{\alpha^n}{T} \sum_{i=1}^{I} \phi_i^n G^+(\bar{x}_{(i)})$$

which combined with (67) yields

$$U(\bar{x}_{\phi^{n+1}}) \leq U(\bar{x}_{\phi^n}) - \frac{\alpha^n}{T} \psi_{ib} c_{\tau} \|\Delta \bar{x}_{\phi}^n\|^2 + \frac{L + L_G}{2} \|\Delta \bar{x}_{\phi}^n\|^2 \cdot \frac{(\alpha^n)^2}{I} \phi_{ab}^2 \|\Delta \bar{x}_{\phi}^n\|^2$$

(67)
Lemma 11

The disagreements

We first study the dynamics of these quantities, we put forth a new analysis, based on the following steps:

and tracking error descent-based techniques used in the literature of distributed gradient-based in Lemma 3, yields \( \epsilon > 0 \) and \( \epsilon_y > 0 \). Applying the above inequality recursively for \( B \) steps, with \( B \) defined in Lemma 3 yields

\[
U(\tilde{x}_{\phi^n + B}) \leq U(\tilde{x}_{\phi^n}) - \frac{\epsilon_r}{\rho_B} \sum_{t=0}^{B-1} \alpha^{n+t} \| \Delta x_{\phi}^{n+t} \|^2 + \frac{\epsilon_y}{2} \Delta x_{\phi}^{n+1} \| \epsilon_y \| + \frac{\epsilon_y}{2} \Delta x_{\phi}^{n+1} \| \epsilon_y \|,
\]

where the last inequality follows from the Young’s inequality, with \( \epsilon_x > 0 \) and \( \epsilon_y > 0 \). Applying the above inequality recursively for \( B \) steps, with \( \tilde{B} \) defined in Lemma 3 yields

\[
U(\tilde{x}_{\phi^n + B}) \leq U(\tilde{x}_{\phi^n}) - \frac{\epsilon_r}{\rho_B} \sum_{t=0}^{B-1} \alpha^{n+t} \| \Delta x_{\phi}^{n+t} \|^2 + \frac{\epsilon_y}{2} \Delta x_{\phi}^{n+1} \| \epsilon_y \| + \frac{\epsilon_y}{2} \Delta x_{\phi}^{n+1} \| \epsilon_y \|,
\]

Summing up over \( \tilde{B} \) consecutive iterations leads to the desired result. □

Since, for sufficiently small \( \alpha^n \), the negative term on the RHS of (68) dominates the positive third term, to prove convergence of \( \{U(\tilde{x}_{\phi^n + B})\}_{n \in N} \), descent-based techniques used in the literature of distributed gradient-based algorithms would call for the summability of the consensus error \( \{E_{x,n}^n\}_{n \in N} \) and tracking error \( \{E_{y,n}^n\}_{n \in N} \) sequences. However, under constant step-size or unbounded (sub-)gradient of \( U \), it seems not possible to infer such a result by just studying the dynamics of \( \{E_{x,n}^n\}_{n \in N} \) and \( \{E_{y,n}^n\}_{n \in N} \) independently from the optimization error \( \Delta x_{\phi}^{n} \). Therefore, exploring the interplay between these quantities, we put forth a new analysis, based on the following steps:

- **Step 1**: We first bound \( E_{x,l}^n \) and \( E_{y,l}^n \) [specifically, term iv in (68)] as a function of \( E_{x,l}^n \) (and thus \( \Delta x_{\phi}^{n} \))—see Proposition 12 [cf. Sec. 6.3.1]. Using Proposition 12, we then prove that \( \{E_{x,l}^n\}_{n \in N} \) and \( \{E_{y,l}^n\}_{n \in N} \) are summable, if \( \{E_{x,l}^n\}_{n \in N} \) is so—see Proposition 14 [cf. Sec. 6.3.2].

- **Step 2**: Using Propositions 12 and 14, we build a new Lyapunov function [cf. Sec. 6.4], whose convergence implies the summability of \( \{E_{x,l}^n\}_{n \in N} \) and thus convergence of all error sequences [cf. Sec. 6.5], as stated in Theorem 4.

6.3 Interplay among \( E_{x,l}^n \), \( E_{y,l}^n \), and \( E_{x,l}^n \)

6.3.1 Bounding \( E_{x,l}^n \) and \( E_{y,l}^n \)

We first study the dynamics of \( \| e_{x,l}^n \| \) and \( \| e_{y,l}^n \| \).

**Lemma 11** The disagreements \( \| e_{x,l}^n \| \) and \( \| e_{y,l}^n \| \) satisfy

\[
\| e_{x,l}^{n+B} \| \leq \rho_B \| e_{x,l}^n \| + c \sum_{t=0}^{B-1} \alpha^{n+t} \| \Delta x^{n+t} \|,
\]

(69)

\[
\| e_{y,l}^{n+B} \| \leq \rho_B \| e_{y,l}^n \| + c L_{\text{mx}} \phi_{\text{lb}}^{-1} \sum_{t=0}^{B-1} (2 \| e_{y,l}^{n+t} \| + \alpha^{n+t} \| \Delta x^{n+t} \|),
\]

(70)
where \( c = 1 \sqrt{27} \). Furthermore, if all \( A^n \) are double stochastic, then \((69)\) and \((70)\) hold with \( B = B, \rho_B = \sqrt{1 - \kappa/(27)} \) and \( c = 1 \).

Proof See Appendix [3].

Using Lemma [11] we now study the dynamics of the weighted sum of the disagreements \( \|e^x_n\| \) and \( \|e^y_n\| \) over \( B \) consecutive iterations.

**Proposition 12** The sequences \( \{\|e^x_n\|^2\}_{n \in \mathbb{N}^+} \) and \( \{\|e^y_n\|^2\}_{n \in \mathbb{N}^+} \) satisfy

\[
\sum_{k=0}^{B-1} \frac{k + 1 + (B - k - 1)\tilde{\rho}}{1 - \tilde{\rho}} \|e^x_{n+k} + \bar{B}\| \leq \sum_{k=0}^{B-1} \frac{k + 1 + (B - k - 1)\tilde{\rho}}{1 - \tilde{\rho}} \|e^x_{n+k}\|^2
\]

\[
- \sum_{k=0}^{B-1} \frac{\alpha_n^k}{\mu_n} \leq \sum_{k=0}^{B-1} \frac{k + 1 + (B - k - 1)\tilde{\rho}}{1 - \tilde{\rho}} \|e^y_{n+k}\|^2
\]

\[
\sum_{k=0}^{B-1} \frac{k + 1 + (B - k - 1)\tilde{\rho}}{1 - \tilde{\rho}} \|e^y_{n+k}\|^2 
\]

\[
- \sum_{k=0}^{B-1} \frac{\alpha_n^k}{\mu_n} \leq \sum_{k=0}^{B-1} \frac{k + 1 + (B - k - 1)\tilde{\rho}}{1 - \tilde{\rho}} \|e^y_{n+k}\|^2
\]

where \( \alpha_n^k \equiv \max_{k=0,\ldots,2B-2} \alpha_{n+k}^k; \tilde{\rho} \equiv \rho_B^2 (1 + B\epsilon); \) and \( \epsilon > 0 \) is any constant such that \( \tilde{\rho} < 1 \).

Proof We prove only \((71)\); \((72)\) can be proved using similar steps. Squaring both sides of the inequality \((69)\) leads to

\[
\|e^x_{n+B}\|^2 
\]

\[
\leq \rho_B^2 \|e^y_n\|^2 + \left( c \cdot \sum_{t=0}^{B-1} \alpha_{n+t}^k \|\Delta x_{n+t}\| \right)^2 + 2 \sum_{t=0}^{B-1} \rho_B \alpha_{n+t}^k \|e^y_n\| \|\Delta x_{n+t}\|
\]

\[
(a) \leq \rho_B^2 (1 + B\epsilon) \|e^y_n\|^2 + \sum_{t=0}^{B-1} \left( \frac{1}{\epsilon} + B\epsilon \right) c^2 \left( \alpha_{n+t}^k \right)^2 \|\Delta x_{n+t}\|^2
\]

\[
(b) \leq \tilde{\rho} \|e^y_n\|^2 + \sum_{t=0}^{B-1} \left( \frac{1}{\epsilon} + B\epsilon \right) 2 c^2 \left( \alpha_{n+t}^k \right)^2 \left( \|\Delta x_{n+t}\|^2 + \|e^x_{n+t}\|^2 \right)
\]

\[(73)\]
where (a) follows from the Young’s inequality, with \( \epsilon > 0 \), and the Jensen’s inequality; and in (b) we used [58]. Note that, since \( \rho_B < 1, \rho = \rho_B^2 (1 + B \epsilon) < 1 \), for all \( \epsilon \in (0, (1 - \rho_B^2) / (\rho_B^2 B)) \).

Denote \( \tilde{\alpha}_{\text{mx}}^n \triangleq \max_{k=0, \ldots, \tilde{B} - 1} \alpha^{n+k} \). Multiplying (73) by \( 1 / (1 - \rho) \) [resp. \( \tilde{\rho} / (1 - \tilde{\rho}) \)], adding \( \| e^n_k \|^2 \) (resp. \( \| e^{n+B}_x \|^2 \)) to both sides, and using the definitions of \( E^n_{\Delta x} \) and \( E^{n}_{x_\perp} \) [cf. (64)], yield

\[
\frac{1}{1 - \rho} \| e^{n+B}_x \|^2 + \| e^n_x \|^2 
\leq \frac{\tilde{\rho}}{1 - \tilde{\rho}} \| e^n_x \|^2 + \| e^n_x \|^2 + \frac{2 \epsilon^2}{1 - \tilde{\rho}} \left( \frac{1}{\epsilon} + \tilde{B} \right) \left( E^n_{\Delta x} + \tilde{\alpha}_{\text{mx}}^n E^n_{x_\perp} \right)
\]

\[
\frac{1}{1 - \rho} \| e^{n+B}_x \|^2 = \frac{1}{1 - \rho} \| e^{n+B}_x \|^2
\]

\[
\leq \frac{\tilde{\rho}}{1 - \tilde{\rho}} \| e^n_x \|^2 + \frac{2 \epsilon^2}{1 - \tilde{\rho}} \left( \frac{1}{\epsilon} + \tilde{B} \right) \left( E^n_{\Delta x} + \tilde{\alpha}_{\text{mx}}^n E^n_{x_\perp} \right),
\]

respectively.

We write now \( \sum_{k=0}^{\tilde{B} - 1} E^{n+k}_{x_\perp} \) as

\[
\sum_{k=0}^{\tilde{B} - 1} E^{n+k}_{x_\perp} = \left( \| e^{n+2B-2}_x \|^2 + 2 \| e^{n+2B-3}_x \|^2 + \cdots + (\tilde{B} - 1) \| e^{n+B}_x \|^2 \right)
\]

\[
+ (\tilde{B} - 1) \| e^{n+B-1}_x \|^2 + (\tilde{B} - 1) \| e^{n+B-2}_x \|^2 + \cdots + \| e^n_x \|^2
\]

Using (74) and (75) on the two terms in (76), we obtain the following bounds:

\[
\frac{\tilde{\rho}}{1 - \tilde{\rho}} \left( \| e^{n+2B-2}_x \|^2 + 2 \| e^{n+2B-3}_x \|^2 + \cdots + (\tilde{B} - 1) \| e^{n+B}_x \|^2 \right)
\]

\[
+ \left( \| e^{n+2B-2}_x \|^2 + 2 \| e^{n+2B-3}_x \|^2 + \cdots + (\tilde{B} - 1) \| e^{n+B}_x \|^2 \right)
\]

\[
\leq \frac{\tilde{\rho}}{1 - \tilde{\rho}} \left( \| e^{n+B-2}_x \|^2 + 2 \| e^{n+B-3}_x \|^2 + \cdots + (\tilde{B} - 1) \| e^n_x \|^2 \right)
\]

\[
+ \frac{2 \epsilon^2}{1 - \tilde{\rho}} \left( \frac{1}{\epsilon} + \tilde{B} \right) \left[ \left( E^n_{\Delta x} + (\tilde{\alpha}_{\text{mx}}^n)^2 E^n_{x_\perp} \right) + \cdots + (\tilde{B} - 1) \left( E^n_{\Delta x} + (\tilde{\alpha}_{\text{mx}}^n)^2 E^n_{x_\perp} \right) \right],
\]

and

\[
\frac{1}{1 - \rho} \left( \tilde{B} \| e^{n+2B-1}_x \|^2 + (\tilde{B} - 1) \| e^{n+2B-2}_x \|^2 + \cdots + \| e^{n+B}_x \|^2 \right)
\]
\[
\sum_{n=0}^{\infty} \left( E^n_{\Delta X} + \left( \tilde{\alpha}_n \right)^2 E^n_{x} \right) \leq \left( B \left( 1 + \tilde{\alpha}_n \right) \right)^2 + \left( B - 1 \right) \left( \left( e^2 + B - 2 \right)^2 + \cdots + \left( e^n \right)^2 \right)
\]

(78)

Summing (77) and (78) and rearranging terms while using (76), it is not difficult to check that

\[
\sum_{k=0}^{B-1} \frac{k+1 + \left( B - k - 1 \right) \tilde{\rho}}{1 - \tilde{\rho}} \left( \sum_{k=0}^{B-1} E^n_{x} \right) \leq \sum_{k=0}^{B-1} \frac{k+1 + \left( B - k - 1 \right) \tilde{\rho}}{1 - \tilde{\rho}} \left( \sum_{k=0}^{B-1} E^n_{x} \right)^2
\]

(79)

which leads to the desired result (71).\]

We use now Proposition [12] in conjunction with Lemma [9] to prove the summability of \( \{ E^n_{x} \}_{n \in \mathbb{N}_+} \) and \( \{ E^n_{y} \}_{n \in \mathbb{N}_+} \), under that of \( \{ E^n_{\Delta X} \}_{n \in \mathbb{N}_+} \). Let

\[
\alpha_{mx} \triangleq \sigma \sqrt{\frac{1 - \tilde{\rho}}{2B \left( B + \epsilon^{-1} \right) c^2}}
\]

(80)

with \( \sigma \in (0, 1) \). This implies [recall the definition of \( \mu^n \) in (71)]

\[
\mu^n \geq \mu_{min} \triangleq \left( 1 - \left( \epsilon^{-1} + B \right) \frac{2Bc^2}{1 - \tilde{\rho}} \alpha_{mx}^2 \right) = 1 - \sigma^2 > 0, \quad \forall \alpha_{mx} \leq \alpha_{mx}.
\]

(81)

**Proposition 13** Suppose that i) \( \sum_{n=0}^{\infty} \left( \alpha_n \right)^2 \left( \Delta X^n_{x} \right)^2 < \infty \); and ii) \( \alpha_n \leq \alpha_{mx} \), for all but finite \( n \in \mathbb{N}_+ \). Then, the consensus and tracking disagreements satisfy \( \sum_{n=0}^{\infty} \left( e^n_{x} \right)^2 < \infty \) and \( \sum_{n=0}^{\infty} \left( e^n_{y} \right)^2 < \infty \), respectively.

**Proof** It follows from (64) that it is sufficient to prove \( \sum_{n=0}^{\infty} E^n_{\Delta X} < \infty \) (for \( \sum_{n=0}^{\infty} \left( e^n_{x} \right)^2 < \infty \) and \( \sum_{n=0}^{\infty} \left( e^n_{y} \right)^2 < \infty \)). We prove next only the former result.

By Assumption [1] and (81), there exists a sufficiently large \( n \), say \( \bar{n} \), such that \( \mu^n \geq \mu_{min} > 0 \), for all \( n \geq \bar{n} \). We assume, without loss of generality, that \( \bar{n} = 0 \). Applying Lemma [9] to (71) [cf. Proposition [12]] we have \( \sum_{n=0}^{\infty} E^n_{\Delta X} < +\infty \Rightarrow \sum_{n=0}^{\infty} E^n_{x} < +\infty \). It is then sufficient to prove that
\[
\sum_{n=0}^{\infty} (\alpha^n)^2 \| \Delta x^n \|^2 < \infty \quad \Rightarrow \quad \sum_{n=0}^{\infty} E_{\Delta x}^n < +\infty. 
\]
This comes readily from the following chain of inequalities:
\[
\sum_{k=0}^{n} E_{\Delta x}^k = \sum_{k=0}^{n} \sum_{t=0}^{B-1} (\alpha^{k+t})^2 \| \Delta x^{k+t} \|^2 \leq B \sum_{k=0}^{n+B-1} (\alpha^k)^2 \| \Delta x^k \|^2. 
\]

\[\square\]

6.3.2 Bounding term iv in 65

We are now ready to bound term iv in 65, as stated next.

**Proposition 14** Suppose that \( \alpha^n \leq \alpha_{mx} \), then
\[
\epsilon_y^{-1} \sum_{k=0}^{B-1} \frac{k + 1 + (\bar{B} - k - 1) \bar{\rho}}{1 - \bar{\rho}} \| e_{y}^{n+B+k} \|^2
\]
\[
+ \frac{1}{\mu_{\text{min}}} \left( c_L \epsilon_x^{-1} + \epsilon_y^{-1} \epsilon_{\perp} (2 + \alpha_{mx}^2) \right) \sum_{k=0}^{B-1} \frac{k + 1 + (\bar{B} - k - 1) \bar{\rho}}{1 - \bar{\rho}} \| e_{x}^{n+B+k} \|^2
\]
\[
- \epsilon_y^{-1} \left( \epsilon_y^{-1} \epsilon_{\perp} (2 + \alpha_{mx}^2) + c_L \epsilon_x^{-1} \right) \frac{c_{\Delta}}{\mu_{\text{min}}} \sum_{k=0}^{B-1} E_{\Delta x}^{n+k}, \tag{82}
\]

**Proof** Multiplying (72) by \( \epsilon_y^{-1} \) on both sides we have
\[
\epsilon_y^{-1} \sum_{k=0}^{B-1} \frac{k + 1 + (\bar{B} - k - 1) \bar{\rho}}{1 - \bar{\rho}} \| e_{y}^{n+B+k} \|^2
\]
\[
\leq \epsilon_y^{-1} \sum_{k=0}^{B-1} \frac{k + 1 + (\bar{B} - k - 1) \bar{\rho}}{1 - \bar{\rho}} \| e_{y}^{n+k} \|^2
\]
\[
- \epsilon_y^{-1} \sum_{k=0}^{B-1} E_{x,\perp}^{n+k} + \epsilon_y^{-1} \epsilon_{\perp} (2 + \alpha_{mx}^2) \sum_{k=0}^{B-1} E_{x,\perp}^{n+k} + \epsilon_y^{-1} \epsilon_{\perp} \sum_{k=0}^{B-1} E_{\Delta x}^{n+k}
\]
\[
= \epsilon_y^{-1} \sum_{k=0}^{B-1} \frac{k + 1 + (\bar{B} - k - 1) \bar{\rho}}{1 - \bar{\rho}} \| e_{y}^{n+k} \|^2 \tag{83}
\]
\[
- \epsilon_y^{-1} \sum_{k=0}^{B-1} E_{y,\perp}^{n+k} - c_L \epsilon_x^{-1} \sum_{k=0}^{B-1} E_{x,\perp}^{n+k}
\]
\[ + (\epsilon_y^{-1} c_L (2 + \alpha_{\text{max}})^2 + c_L \epsilon_x^{-1}) \sum_{k=0}^{B-1} E_{x_k}^{n+k} + \epsilon_y^{-1} y \sum_{k=0}^{B-1} E_{y_k}^{n+k}. \]

Since \( \alpha^n \leq \alpha_{\text{mx}} \), we have \( \alpha^n_{\text{mix}} \leq \alpha_{\text{mx}} \) and \( \mu^n \geq \mu_{\text{min}} \). Eq. (71) then implies

\[
\sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{y_k}^{n+b+k}\|^2 \leq \sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{y_k}^{n+k}\|^2 - \mu_{\text{min}} \sum_{k=0}^{B-1} E_{x_k}^{n+k} + c_\Delta \sum_{k=0}^{B-1} E_{\Delta x_k}^{n+k}. \]

Multiplying both sides of the above inequality by \((\epsilon_y^{-1} c_L (2 + \alpha_{\text{mx}})^2 + c_L \epsilon_x^{-1})/\mu_{\text{min}}\) and using the fact that \( \alpha^n_{\text{mix}} \leq \alpha_{\text{mx}} \), we have

\[
\frac{1}{\mu_{\text{min}}} \left( c_L \epsilon_x^{-1} + \epsilon_y^{-1} c_L (2 + \alpha_{\text{mx}})^2 \right) \sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{y_k}^{n+b+k}\|^2 \leq \frac{1}{\mu_{\text{min}}} \left( c_L \epsilon_x^{-1} + \epsilon_y^{-1} c_L (2 + \alpha_{\text{mx}})^2 \right) \sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{y_k}^{n+k}\|^2 - \left( c_L \epsilon_x^{-1} + \epsilon_y^{-1} c_L (2 + \alpha_{\text{mx}})^2 \right) \sum_{k=0}^{B-1} E_{x_k}^{n+k} + \left( c_L \epsilon_x^{-1} + \epsilon_y^{-1} c_L (2 + \alpha_{\text{mx}})^2 \right) c_\Delta \mu_{\text{min}} \sum_{k=0}^{B-1} E_{\Delta x_k}^{n+k} \quad (84) \]

Adding (84) to (83) leads to the desired result. \( \square \)

### 6.4 Lyapunov-like function and its descent properties

We are now in the position to construct a function whose descent properties (every \( B \) iterations) will used to prove Theorem 4. Because of that, we will refer to such a function as Lyapunov-like function.

Adding (65) and (82) (multiplied by \( \phi_{ub}/2 \)), yields

\[
\sum_{k=0}^{B-1} U (x_{\phi^n+k}) + \phi_{ub}/2 \sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{y_k}^{n+b+k}\|^2 \leq \sum_{k=0}^{B-1} U (x_{\phi^n+k}) + \phi_{ub}/2 \sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{y_k}^{n+k}\|^2 \]

\[
+ \phi_{ub}/2 \mu_{\text{min}} \left( c_L \epsilon_x^{-1} + \epsilon_y^{-1} c_L (2 + \alpha_{\text{mx}})^2 \right) \sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{x_k}^{n+b+k}\|^2 \]

\[
\leq \sum_{k=0}^{B-1} U (x_{\phi^n+k}) + \phi_{ub}/2 \sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{y_k}^{n+k}\|^2 \]

\[
+ \phi_{ub}/2 \mu_{\text{min}} \left( c_L \epsilon_x^{-1} + \epsilon_y^{-1} c_L (2 + \alpha_{\text{mx}})^2 \right) \sum_{k=0}^{B-1} k + 1 + (B - k - 1) \tilde{\rho} \|e_{x_k}^{n+k}\|^2. \]

\[ + \frac{\phi_{ub}}{2\mu_{\min}} \left( c_L e^{-1}_x + e_y^{-1} e^\perp \right) \sum_{k=0}^{B-1} \frac{k+1 + (\bar{B} - k - 1)\bar{\rho}}{1 - \bar{\rho}} \|e^{n+k}_x\|^2 \]

\[ - \frac{c_T}{I} \phi_{lb} \sum_{k=0}^{B-1} \sum_{t=0}^{B-1} \alpha^{n+k+t} \|\Delta x^{n+k+t}_\phi\|^2 \]

\[ + \frac{\phi_{ab}}{2} \left( L + \frac{L_G}{I} \right) \cdot \phi_{ub} + c_L e_x + e_y + e_y^{-1} e \sum_{k=0}^{B-1} E^{n+k}_x \]

\[ + \frac{\phi_{ab}}{2\mu_{\min}} \left( c_L e^{-1}_x + e_y^{-1} e^\perp \right) \sum_{k=0}^{B-1} \sum_{t=0}^{B-1} \beta^{n+k+t} \|\Delta x^{n+k+t}_\phi\|^2. \] (85)

Define

\[ V^n = \sum_{k=0}^{B-1} U (\bar{x}^{\phi,n+k}) + \phi_{ab} \sum_{k=0}^{B-1} \frac{k+1 + (\bar{B} - k - 1)\bar{\rho}}{1 - \bar{\rho}} \|e^{n+k}_x\|^2 \]

\[ + \frac{\phi_{ab}}{2\mu_{\min}} \left( c_L e^{-1}_x + e_y^{-1} e^\perp \right) \sum_{k=0}^{B-1} \sum_{t=0}^{B-1} \beta^{n+k+t} \|\Delta x^{n+k+t}_\phi\|^2, \] (86)

and

\[ \beta^n = \frac{c_T}{I} \phi_{lb} - \frac{\phi_{ab}}{2} \alpha^n \left( L + \frac{L_G}{I} \right) \cdot \phi_{ub} + c_L e_x + e_y + e_y^{-1} e \]

\[ + \frac{c_{\Delta}}{\mu_{\min}} \left( c_L e^{-1}_x + e_y^{-1} e^\perp \right) \left( 2 + \alpha_{mx} \right)^2. \] (87)

Substituting (86) and (87) in (85), we obtain the desired descent property of \( V^n \): for sufficiently large \( n \), it holds

\[ V^{n+\bar{B}} \leq V^n - \sum_{k=0}^{B-1} \sum_{t=0}^{B-1} \beta^{n+k+t} \alpha^{n+k+t} \|\Delta x^{n+k+t}_\phi\|^2. \] (88)

6.5 Proof of Theorem

The proof consists in two steps, namely:

- **Step 1**: Leveraging the descent property of the Lyapunov-like function, we first show that \( \lim_{n \to \infty} \|\Delta x^n_\phi\| = 0 \), either using a diminishing or constant step-size \( \alpha^n \) (satisfying Assumption F); and

- **Step 2**: Using the results in Step 1, we conclude the proof showing that i) \( \lim_{n \to \infty} D(x^n) = 0 \) and ii) \( \lim_{n \to \infty} J(x^n) = 0 \).
6.5.1 Step 1: \( \lim_{n \to \infty} \| \Delta \bar{x}_n \| = 0 \)

Let us distinguish the two choices of step-size, namely: \( \alpha^n \) is constant (satisfying Assumption F.1); or \( \alpha^n \) is diminishing (satisfying Assumption F.2).

**Case 1: constant step-size.** Set \( \alpha^n = \alpha \) for all \( n \in \mathbb{N} \). To obtain the desired descent on \( V^n \) [cf. (88)], \( \alpha \) has to be chosen so that \( \beta^n = \beta > 0 \) [cf. (87)]. We show next that if \( \alpha \) satisfies (36) [cf. Assumption F.2], then \( \beta > 0 \).

Recall that (88) holds under the assumption that \( \alpha \leq \alpha_{\text{mx}} \), with \( \alpha_{\text{mx}} \) defined in (80). Substituting the expressions of \( \alpha_{\text{mx}} \) and \( \mu_{\text{min}} = 1 - \sigma^2 \) [cf. (81)] in (87) and using the definitions of \( c_\Delta \) and \( c_\perp \) [cf. Proposition 12], one can check that \( \beta^n = \beta > 0 \) [cf. (87)] if, in addition to \( \alpha \leq \alpha_{\text{mx}} \), \( \alpha \) satisfies also

\[
\alpha \leq \frac{2 c_r \phi_{lb}}{I \phi_{ub}} \left( \frac{L + L_G}{I} \cdot \phi_{ub} + c_L \epsilon_x + \epsilon_y + \frac{c_\Delta}{1 - \sigma^2} \left( c_L \epsilon_x^{-1} + 9 c_\perp \epsilon_y^{-1} \right) \right)^{-1},
\]

where \( \epsilon_x, \epsilon_y > 0 \) are free parameters. The above upperbound is maximized by

\[
\epsilon_x = \sqrt{\frac{c_\Delta}{1 - \sigma^2}} = \sqrt{\frac{2B \left( \epsilon^{-1} + \bar{B} \right)}{(1 - \bar{\rho})(1 - \sigma^2)}},
\]

\[
\epsilon_y = \frac{9 c_\perp c_\Delta}{1 - \sigma^2} = 6 L_{\text{mx}} \phi_{lb}^{-1} \left( \epsilon^{-1} + \bar{B} \right) \frac{\bar{B} c_r^2}{1 - \bar{\rho}} \sqrt{\frac{1}{1 - \sigma^2}}.
\]

Combining \( \alpha \leq \alpha_{\text{mx}} \) and (83), we get the following bound for \( \alpha \):

\[
\alpha \leq \min \left\{ \sigma \sqrt{\frac{1 - \bar{\rho}}{2c_r^2 B (B + \epsilon^{-1}) \cdot \phi_{ub}}} \left( \frac{L + L_G}{I} \cdot \phi_{ub} + 2c_r c_L \right) \frac{2B \left( \epsilon^{-1} + \bar{B} \right)}{(1 - \bar{\rho})(1 - \sigma^2)} \right\},
\]

where recall that \( \epsilon < (1 - \rho_B^2)/({\rho_B^2 \bar{B}}) \) [cf. Proposition 12]. Since \( (1 - \bar{\rho})/(\epsilon^{-1} + \bar{B}) \) is maximized by \( \epsilon = (1 - \rho_B)/({\rho_B \cdot \bar{B}}) \) with the corresponding value being \( (1 - \rho_B)^2/\bar{B} \), we obtain from (90) the final bound (36).

Under (86), using (88) and Lemma 4 (recall that \( \liminf_{n \to \infty} V^n > -\infty \), since \( U \) is bounded from below on \( K \)) we get \( \lim_{n \to \infty} \| \Delta \bar{x}_n \| = 0 \) and, by Proposition 13

\[
\lim_{n \to \infty} \| e_n^0 \| = 0 \quad \text{and} \quad \lim_{n \to \infty} \| e_n^0 \| = 0.
\]

**Case 2: diminishing step-size.** Since \( \alpha^n \) is diminishing, there exists a sufficiently large \( n_2 \) so that \( \beta^n \geq \beta > 0 \) for all \( n \geq n_2 \), implying

\[
\sum_{n=0}^{\infty} \sum_{t=0}^{B-1} \alpha^{n+t} \left\| \Delta \bar{x}_{n+t} \right\|^2 < \infty,
\]

(91)
which together with $\sum_{n=0}^{\infty} \alpha^n = \infty$ and Proposition 13 yield

$$\lim_{n \to \infty} \| \Delta \tilde{x}_n^\phi \| = 0;$$

$$\lim_{n \to \infty} \| e_n^a \| = 0 \quad \text{and} \quad \lim_{n \to \infty} \| e_n^b \| = 0. \tag{93}$$

We prove next that $\limsup_{n \to \infty} \| \Delta \tilde{x}_n^\phi \| = 0$, which together with \[92\] implies $\lim_{n \to \infty} \| \Delta \tilde{x}_n^\phi \| = 0$. Suppose that $\limsup_{n \to \infty} \| \Delta \tilde{x}_n^\phi \| > 0$. This, together with $\liminf_{n \to \infty} \| \Delta \tilde{x}_n^\phi \| = 0$, implies that there exists an infinite set of indices $N$ such that for all $n \in N$, one can find an integer $i_n > n$ such that:

$$\| \Delta \tilde{x}_n^\phi \| < \eta, \quad \| \Delta \tilde{x}_{i_n}^\phi \| > 2\eta$$

$$\eta \leq \| \Delta \tilde{x}_n^\phi \| \leq 2\eta, \quad n < j < i_n. \tag{94}$$

Denote $\hat{x}^a_{(i)} \triangleq \hat{x}^a_i(x^a_{(i)})$ and $\tilde{x}^a \triangleq [\hat{x}^a_{(1)} \ldots \hat{x}^a_{(t)}]^T$. We have:

$$\eta \leq \| \Delta \tilde{x}_n^\phi \| - \| \Delta \hat{x}_n^\phi \| \leq \| \Delta \tilde{x}_n^\phi - \Delta \hat{x}_n^\phi \|$$

$$\leq \| \tilde{x}^n_n - \tilde{x}^a \| + \| J_{\phi^a} x^n_n - J_{\phi^a} x^n \|$$

$$\leq \| \tilde{x}^n_n - \tilde{x}^a \| + \| \tilde{x}^a - \tilde{x}^n_n \| + \| \tilde{x}^a - x^n \| + \| J_{\phi^a} x^n_n - J_{\phi^a} x^n \| \tag{95}$$

$$\leq \hat{L} \| \tilde{x}^n_n - \tilde{x}^a \| + \| J_{\phi^a} x^n_n - J_{\phi^a} x^n \| + e_1^n$$

$$\leq \hat{L} \left( \| \tilde{x}^a - J_{\phi^a} x^n \| + \| x^n - J_{\phi^a} x^n \| + \sqrt{1} \| \tilde{x}^a - x^n \| \right)$$

$$+ \sqrt{1} \| \tilde{x}^a - x^n \| + e_1^n$$

$$\leq \left( \hat{L} + 1 \right) \sqrt{1} \left( \| \tilde{x}^a - x^n \| + \| e_1^n \| \right) + e_1^n$$

$$\leq \left( \hat{L} + 1 \right) \sqrt{1} \left( \| \tilde{x}^a - x^n \| + \| e_1^n \| \right) + e_1^n$$

$$\leq \left( \hat{L} + 1 \right) \sqrt{1} \left( \| \tilde{x}^a - x^n \| + \| e_1^n \| \right) + e_1^n$$

$$\leq \left( \hat{L} + 1 \right) \sqrt{1} \eta \leq \sum_{n=0}^{\infty} \eta^n \| \Delta \tilde{x}_n^\phi \|^2 + e_1^n + e_2^n + e_3^n. \tag{96}$$

where in (a) we used \[91\] [cf. Lemma 7]; (b) follows from \[59a\]; and in (c) we used the lower bound in \(95\).

Since i) $\lim_{n \to \infty} \| e_2^n \| = 0$ and $\lim_{n \to \infty} \| e_3^n \| = 0$ [cf. \(93\)]; ii) $\lim_{n \to \infty} \| \tilde{x}^a_n - \tilde{x}^a \| = 0$ [cf. \(91\)]; and iii) $\sum_{n=0}^{\infty} \alpha^{n+1} \| \Delta \tilde{x}_{n+1}^\phi \|^2 < \infty$ [cf. \(91\)], there exists a sufficiently large $n_3$ such that the right-hand-side of \(96\) is less than $\eta$, for all $n > n_3$, which leads to a contradiction. Therefore, $\limsup_{n \to \infty} \| \Delta \tilde{x}_n^\phi \| = 0$. 


6.5.2 Step 2: \( \lim_{n \to \infty} M(x^n) = 0 \)

Recall that in the previous subsection we proved that i) \( \lim_{n \to \infty} \|\Delta \phi^n\| = 0 \); ii) 
\( \lim_{n \to \infty} \|e^n\| = 0 \); and iii) \( \lim_{n \to \infty} \|e^n\| = 0 \), using either a constant step-size \( \alpha^n \equiv \alpha \), with \( \alpha \) satisfying (36), or a diminishing one. The statement \( \lim_{n \to \infty} D(x^n) = 0 \) follows readily from point ii) and

\[
\lim_{n \to \infty} \|x^n(i) - \hat{x}^n\| \leq \lim_{n \to \infty} \|x^n(i) - \hat{x}^n\| + \lim_{n \to \infty} \|\hat{x} - \hat{x}^n\| \\
\leq \lim_{n \to \infty} \|x^n(i) - \hat{x}^n\| + \lim_{n \to \infty} \frac{1}{T} \sum_{j=1}^{T} \|x^n(j) - \hat{x}^n\| = 0. \quad (97)
\]

Next we show \( \lim_{n \to \infty} J(\bar{x}) = 0 \). Recall the definition \( J(\bar{x}) \triangleq \|\bar{x}(\bar{x}) - \bar{x}^n\| \), where for notation simplicity, we set

\[
\bar{x}(\bar{x}) \triangleq \arg\min_{x \in \mathcal{X}} \left\{ \left( \nabla F(\bar{x}) - \nabla G^{-}(\bar{x}) \right) (z - \bar{x}) + \frac{1}{2} \|z - \bar{x}\|^2 + G(z)^\top \right\}. \quad (98)
\]

Since

\[
J(\bar{x}) \leq \|\bar{x}(\bar{x}) - \bar{x}^n\| + \|\bar{x}(\bar{x}) - \bar{x}(\bar{x})\|, \quad (99)
\]

it is sufficient to show that the two terms on the right hand side are asymptotically vanishing, which is proved below.

- \( \lim_{n \to \infty} \|\bar{x}(\bar{x}) - \bar{x}^n\| = 0 \). We bound \( \|\bar{x}(\bar{x}) - \bar{x}^n\| \) as

\[
\|\bar{x}(\bar{x}) - \bar{x}^n\| \leq \|\bar{x}(\bar{x}) - \bar{x}^n\| + \|\bar{x}^n - \bar{x}^n\| + \|\bar{x}(\bar{x}) - \bar{x}(\bar{x})\| \equiv (a) \|\bar{x}(\bar{x}) - \bar{x}^n\| + (1 + \bar{L})\|\bar{x}^n - \bar{x}^n\|, \quad (100)
\]

where (a) follows from Lemma 7. From (97) we know \( \lim_{n \to \infty} \|\bar{x}(\bar{x}) - \bar{x}^n\| = 0 \).

To show \( \|\bar{x}(\bar{x}) - \bar{x}^n\| \) is asymptotically vanishing, we bound it as

\[
\|\bar{x}(\bar{x}) - \bar{x}^n\| \leq \|\bar{x}(\bar{x}) - \bar{x}(\bar{x})\| + \|\bar{x}(\bar{x}) - \bar{x}(\bar{x})\| + \|\bar{x}(\bar{x}) - \bar{x}(\bar{x})\| + \|\bar{x}(\bar{x}) - \bar{x}(\bar{x})\|. \quad (101)
\]

The result \( \lim_{n \to \infty} \|\bar{x}(\bar{x}) - \bar{x}^n\| = 0 \) follows from Lemma 7, Lemma 8 and points i)-iii).

From (100) and (101) we conclude

\[
\lim_{n \to \infty} \|\bar{x}(\bar{x}) - \bar{x}^n\| = 0. \quad (102)
\]
• We prove \( \lim_{n \to \infty} \| \bar{x}(x^n) - \tilde{x}_i(x^n) \| = 0 \). Using the first order optimality conditions of \( x(x^n) \) and \( \tilde{x}_i(x^n) \), we can bound their difference as
\[
\| \bar{x}(x^n) - \tilde{x}_i(x^n) \| \leq \| \nabla \tilde{f}_i(\tilde{x}_i(x^n); x^n) - \nabla f_i(x^n) - \tilde{x}_i(x^n) + x^n \|
\leq \| \nabla \tilde{f}_i(\tilde{x}_i(x^n); x^n) - \nabla f_i(\tilde{x}_i(x^n)) \| + \| \nabla f_i(x^n) - \tilde{x}_i(x^n) - x^n \| \\
(103)
\]
Using (102) we have
\[
\lim_{n \to \infty} \| \bar{x}(x^n) - \tilde{x}_i(x^n) \| = 0. \quad (104)
\]
The proof is completed just combining (99), (102) and (104).

7 Numerical results

7.1 Sparse regression

In this section, we test the performance of SONATA on the sparse linear regression problem \( (1) \) [cf. Sec. 2.1]. We generated the data set as follows. The ground truth signal \( \bar{x}^* \in \mathbb{R}^{500} \) is built by first drawing randomly a vector from the normal distribution \( \mathcal{N}(0, 1) \), then thresholding the smallest 80% of its elements to zero. The underlying linear model is \( \mathbf{b}_i = \mathbf{A}_i \bar{x}^* + \mathbf{n}_i \), where the observation matrix \( \mathbf{A}_i \in \mathbb{R}^{20 \times 500} \) is generated by first drawing i.i.d. elements from the distribution \( \mathcal{N}(0, 1) \), and then normalizing the rows to unit norm; and \( \mathbf{n}_i \) is the additive noise, with i.i.d. entries from \( \mathcal{N}(0, 0.1) \). We simulated 100 Monte Carlo trials, generating in each trial new \( \mathbf{A}_i \)'s and \( \mathbf{n}_i \)'s. We considered a time-varying digraph, composed of \( I = 30 \) agents. In every time slot, a new digraph is generated according to the following procedure: each agent \( i \) has two out-neighbors, one of them belonging to a chain connecting all the agents and the other one picked uniformly at random. To promote sparsity we use the (nonconvex) log function \( G(x) = \lambda \cdot \sum_i \log(1 + \theta/|x_i|)/\log(1 + \theta) \), where the parameter \( \theta \) controls the tightness of the approximation of the \( \ell_0 \) function.

We set \( \lambda = 0.1 \) and \( \theta = 2 \). It is convenient to rewrite \( G(x) \) in the DC form \( G(x) = G^+(x) - G^-(x) \), with \( G^+(x) = \| x \|_1 \cdot (\theta/\log(1 + \theta)) \). It is not difficult to check that such \( G^+ \) and \( G^- \) satisfy Assumption [A3]; see, e.g., [1].

We run SONATA considering two alternative choices of \( \tilde{f}_i \), namely:

• **SONATA-PL** (PL stands for partial linearization): Since \( f_i = \| \mathbf{b}_i - \mathbf{A}_i x \|^2 \) is convex, one can keep \( \tilde{f}_i \) unaltered and set in (28) \( \tilde{f}_i(x^{(i)}) = f_i(x^{(i)}) + \tau_{PL}/2 \cdot \| x^{(i)} - x^{(i)}_0 \|^2 \). We set \( \tau_{PL} = 1.5 \). The unique solution \( \tilde{x}_i^{(n)} \) of the resulting subproblem (29) is computed using the FLEXA algorithm, with the following tuning (see [11] for details): the initial point is selected randomly; the proximal parameter in the subproblems solved by FLEXA is set to be 2; and the step-size of FLEXA is chosen according to the diminishing rule \( \gamma^+ = \gamma^{-1} \left( 1 - \mu \gamma^{-1} \right) \), with \( \gamma^0 = 0.5 \) and \( \mu = 0.01 \), with \( r \) denoting the
The solution that is, $x \cdot x$ operator (intended to be applied to $x$ where $\| \cdot \|$ iteration, and (inner) iteration index. We terminate FLEXA when $J^r_{i(1)} \leq 10^{-8}$, with $J^r_{i(1)} \triangleq \| \tilde{x}^{n,r}_{i(i)} - S_{\alpha} \langle \tilde{x}^{n,r}_{i(i)} - 2 A_{i}^\top (A_i x_{i(i)} - b_i) - \tau_{PL} (\tilde{x}^{n,r}_{i(i)} - x_{i(i)}) + \tilde{\pi}^n_i + \lambda \nabla G^- (x_{i(i)}) \|_\infty$, where $x_{i(i)}^{n,r}$ denotes the value of $x_{i(i)}$ at the $n$-th outer and the $r$-th inner iteration, and $S_{\alpha}(x) \triangleq \text{sign}(x) \cdot \max(\|x\| - \alpha 1, 0)$ is the soft-thresholding operator (intended to be applied to $x$ component-wise).

**SONATA-L** (L stands for linearization): To obtain a closed form expression for $\tilde{x}^{n}_{i(i)}$ in (28), one can choose $\tilde{f}_i$ as linearization of $f_i$ (plus the proximal term), that is, $f_i(x_{i(i)}) = 2 A_{i}^\top (A_i x_{i(i)} - b_i) + (\tau_{L}/2) \cdot \| x_{i(i)} - x_{i(i)}^n \|^2$. We set $\tau_L = 1.5$. The solution $\tilde{x}^{n}_{i(i)}$ of the resulting subproblem (28) has the following closed form expression $\tilde{x}^{n}_{i(i)} = S_{\alpha \lambda/\tau_L} (\tilde{x}^{n}_{i(i)} - \frac{1}{\tau_L} (2 A_{i}^\top (A_i x_{i(i)} - b_i) + \tilde{\pi}^n_i - \lambda \nabla G^- (x_{i(i)}))).$

As benchmark, we also simulated the subgradient-push algorithm [27] with diminishing step-size. Note that there is no proof of convergence for such a scheme, when applied to the nonconvex, nonsmooth problem (1). For all the algorithms, we use the same step-size rule: $\alpha^n = \alpha^{n-1} (1 - \mu \alpha^{n-2})$, with $\alpha^0 = 0.5$ and $\mu = 0.01$. Also, for all algorithms, we set $x_{0,i} = 0$, for all $i$.

We monitor the progresses of the algorithms towards stationarity and consensus using respectively the following two functions: i) $J^n \triangleq \| \tilde{x}^n - S_{\alpha L} (\tilde{x}^n - 2 \sum A_{i}^\top (A_i x^n - b_i) + \lambda \nabla G^- (x^n)) \|_\infty$; and ii) $D^n \triangleq \| \tilde{x}^n - J x^n \|_\infty$. It is not difficult to check that $J^n$ is a valid distance of the average iterates $\tilde{J} x^n$ from stationarity: it is continuous and zero if and only if its argument is a stationary solution of (1). We also use the normalized mean squared error (NMSE), defined as $\text{NMSE}^n \triangleq \| x^n - (1 \otimes I) x^n \|^2 / (I : \| x^n \|^2)$.

In Fig. 4 we plot $\log_{10} J^n$ and $\log_{10} D^n$ (subplot (a)) and the NMSE (subplot (b)) versus the number of agents’ message exchanges, averaged over 100 Monte-Carlo trials (we applied the $\log_{10}$ transform to $J^n$ and $D^n$ so that their distribution is closer to the normal one). The figures show that both versions of SONATA are much faster than the distributed gradient algorithm. This seems mainly due to the gradient tracking mechanism put forth by the proposed scheme. Under the same tuning, SONATA-PL converges faster than SONATA-L. According to our intensive simulations (not reported here), SONATA-PL becomes up to one order of magnitude faster than SONATA-L when $\tau_{PL}$ is reduced whereas reducing $\tau_L$ slows down SONATA-L.

### 7.2 Distributed PCA

Our second application is the distributed PCA problem

$$\min_{\|x\|_2 \leq 1} \quad F(x) \triangleq -\sum_{i=1}^{I} \| D_i x \|^2,$$

with $I = 30$.

Each agent $i$ locally owns a data matrix $D_i \in \mathbb{R}^{d_i \times m}$ and communicate via a time-varying digraph generated in the same way as the previous sparse regression example (cf. Sec. 7.1).

Since $f_i(x) \triangleq -\| D_i x \|^2$ is concave, to apply SONATA we construct $\tilde{f}_i$, by linearizing $f_i$, which leads to $\tilde{F}_i(x_{i(i)}; x^n_{i(i)}) = (I \cdot y^n_{i(i)})^\top (x_{i(i)} - x^n_{i(i)}) + (\tau/2) \cdot$.
Fig. 1 Sparse regression problem with log regularizer: SONATA-PL, SONATA-L, and subgradient-push; average of $\log_{10} J_n$ and $\log_{10} D_n$ vs. agent’s message exchange [subplot (a)]; average of NMSE vs. agent’s message exchange [subplot (b)].

The solution $\tilde{x}_n^{(i)}$ of the resulting subproblem has the closed form solution

$$
\tilde{x}_n^{(i)} = P_{\|x_n^{(i)}\| \leq 1} \left( x_n^{(i)} - \frac{I \cdot y_n^{(i)}}{\tau} \right),
$$

where $P$ denotes the Euclidean projection onto the set $\{x_n^{(i)} : \|x_n^{(i)}\| \leq 1\}$. As benchmark, we implemented also the gradient projection algorithm [4], adapted to time-varying network. Note that there is no formal proof of this algorithm in the simulated setting. The performance of the algorithms is tested on both synthetic and real data sets, as detailed next.

7.2.1 Synthetic data

Each agent $i$ locally owns a data matrix $D_i \in \mathbb{R}^{30 \times 500}$, whose rows are i.i.d., drawn by the $\mathcal{N}(0, \Sigma)$. The covariance matrix $\Sigma$, whose eigendecomposition is $\Sigma = U \Lambda U^T$, is generated as follows: we synthesize $U$ by first generating a square matrix whose entries follow the i.i.d. standard normal distribution, then perform the QR decomposition to obtain its orthonormal basis; and the eigenvalues $\text{diag}(\Lambda)$ are i.i.d. uniformly distributed in $[0, 1]$.

The algorithms are tuned as follows: $x_0^{(i)}$ is generated with i.i.d elements drawn by the standard Normal distribution. The step-size $\alpha_n$ is chosen according to the diminishing rule used in the previous example, where we set $\alpha^0 = 1$ and $\mu = 10^{-3}$ for SONATA and $\alpha^0 = 1$ and $\mu = 10^{-2}$ for the gradient algorithm. The proximal parameter $\tau$ for SONATA is set to be 1. The distance of $\bar{x}_n$ from stationarity is measured by $J_n \triangleq \|x_n - P_{\|x_n\| \leq 1} (x_n - \nabla F(x_n))\|_{\infty}$, while the consensus disagreement $D_n$ and the NMSE $n$ are defined as in the previous example; in the definition of NMSE $n$ the ground truth signal $x^\star$ is now the leading eigenvector of matrix $\sum_{i=1}^{I} D_i^T D_i$.

In Fig. 2, we plot $\log_{10} J_n$ and $\log_{10} D_n$ [subplot (a)] and the NMSE [subplot (b)] versus the number of agents’ message exchanges, averaged over 100 Monte-Carlo trials. In each trial, $\Sigma$ is fixed while the $D_i$’s are randomly generated. Fig. 2(a) clearly shows that SONATA can find a stationary point efficiently while the gradient algorithm progresses very slowly. More interestingly, Fig. 2(b) shows that SONATA always find the leading eigenvector whereas the gradient algorithm fails to achieve a small NMSE value.
7.2.2 Gene expression data

This second experiment tests SONATA on a real-world data set. Specifically, we used the breast cancer gene expression data set [5], which consists of \( d = 158 \) samples and \( m = 12625 \) genes per sample. We first uniformly randomly permute the order the samples and then equally divided the samples among the \( I = 30 \) agents. To avoid the issue that \( d \) is not divisible by \( I \), we let the first \( I - 1 \) agents owning \( d_i = \lfloor d/I \rfloor \) samples each, while the \( I \)-th agent owning the remaining samples. The samples are preprocessed by subtracting the mean from all of them. Note that this can be achieved distributively by running an average consensus algorithm beforehand.

The rest of the setting and tuning of the algorithms are the same as those described in Sec. 7.2.1. In Fig. 3, we plot \( \log_{10} J^n \) and \( \log_{10} D^n \) [subplot (a)] and the NMSE [subplot (b)] versus the number of agents’ message exchanges, averaged over 100 Monte-Carlo trials. In each trial, samples are randomly partitioned among the agents. From the figure we can see that the behavior of the algorithms on the gene expression data set is similar to that on synthetic
data set. Moreover, SONATA converges quite fast even though the variable dimension of the real data set we adopted is massive.

8 Appendix

A Proof of Lemma 3

We begin introducing the following intermediate result.

Lemma 15 In the setting of Lemma 3, the following hold:
(i) The elements of $A^{n,0}$, $n \in \mathbb{N}_+$, can be bounded as
\[
\inf_{t \in \mathbb{N}_+} \left( \min_{1 \leq i \leq t} (A^{i,0}1)_i \right) \geq \phi_n, \tag{106}
\]
\[
\sup_{t \in \mathbb{N}_+} \left( \max_{1 \leq i \leq t} (A^{i,0}1)_i \right) \leq \phi_{\ast n}, \tag{107}
\]
where $\phi_n$ and $\phi_{\ast n}$ are defined in (3).

(ii) For any given $n, k \in \mathbb{N}_+$, $n \geq k$, there exists a stochastic vector $\xi^k = [\xi_1^k, \ldots, \xi_n^k]^T$ (i.e., $\xi^k > 0$ and $1^T \xi^k = 1$) such that
\[
\|W_{ij}^n - \xi_i^k\| \leq c_0 (\rho) \frac{n-k+1}{\sqrt{m+n}}, \quad \forall i, j \in [L], \tag{108}
\]
where $c_0$ and $\rho$ are defined in (7).

The proof Lemma 15 follows similar steps as those in [31, Lemma 2, Lemma 4] and thus is omitted, although the results in [31] are established under a stronger condition on $G^n$ than Assumption 5.

We prove now Lemma 3. Let $z \in \mathbb{R}^m$ be an arbitrary vector. For each $\ell = 1, \ldots, m$, define $z_{\ell} = (I \otimes e_\ell^0)z$, where $e_\ell^0$ is the $\ell$-th canonical vector; we denote by $z_{i,j}$ the $j$-th component of $z_{\ell}$, with $j \in [L]$. We have
\[
\left\| (\tilde{W}_{n,k} - J_{\phi^k}) z \right\|_2 \leq \sqrt{I \cdot \sum_{\ell=1}^m \left\| (W_{n,k} - \frac{1}{I} 1_1 (\phi^k)^\top) z_{\ell} \right\|_\infty}^2. \tag{109}
\]

We bound next the above term. Given $\xi^k$ as in Lemma 15 [cf. (108)], define $E_{n,k}^\ell = W_{n,k} - I (\xi^k)^\top$, whose $ij$-th element is denoted by $E_{i,j}^{n,k}$. We have
\[
\left\| (W_{n,k} - \frac{1}{I} 1_1 (\phi^k)^\top) z_{\ell} \right\|_\infty \leq \left\| (W_{n,k} - \frac{1}{I} 1_1 (\phi^{n+1})^\top W_{n,k}) z_{\ell} \right\|_\infty
\]
\[
= \left\| (I - \frac{1}{I} I_1 (\phi^{n+1})^\top) E_{n,k}^\ell z_{\ell} \right\|_\infty \leq \max_{1 \leq i \leq t} \left( \frac{1 - \phi^{n+1}}{I} \sum_{j=1}^I |E_{i,j}^{n,k}| |z_{i,j}| + \sum_{j' \neq i} \frac{\phi^{n+1}}{I} \sum_{j=1}^I |E_{j,j'}^{n,k}| |z_{i,j'}| \right)
\]
\[
\leq 2 c_0 (\rho) \frac{n-k+1}{\sqrt{m+n}} \left\| z_{\ell} \right\|_1 \leq 2 c_0 (\rho) \frac{n-k+1}{\sqrt{m+n}} \sqrt{I} \left\| z_{\ell} \right\|_2. \tag{110}
\]
Combining (109) and (110) we obtain
\[
\left\| \tilde{W}_{n,k} - J_{\phi^k} \right\|_2 \leq 2 c_0 I (\rho) \frac{n-k+1}{\sqrt{m+n}}. \tag{111}
\]
Moreover, the matrix difference above can be alternatively uniformly bounded as follows:
\[
\left\| \tilde{W}_{n,k} - J_{\phi^k} \right\| = \|(I - J_{\phi^{n+1}}) \tilde{W}_{n,k} \| \leq \| I - J_{\phi^{n+1}} \| \| \tilde{W}_{n,k} \|^{(a)} \leq \sqrt{I} \cdot \sqrt{I},
\]
where (a) follows from (25) and $\| \tilde{W}_{n,k} \| \leq \sqrt{I}$. This completes the proof. \qed
B Proof of Lemma 11

Recall the SONATA update written in vector-matrix form in (43–45). Note that the x-update therein is a special case of the perturbed condensed push-sum algorithm (16), with perturbation $\Delta x^{n+1} = \alpha_n \bar{W} x^n$. We can then apply Proposition 1 and readily obtain (69).

To prove (70), we follow a similar approach: noticing that the y-update in (45) is a special case of the perturbed condensed push-sum algorithm (16), with perturbation $\Delta y^{n+1} = (\bar{D}_{\phi_{n+1}})^{-1} (g^{n+1} - g^n)$, we can write

$$ \|e^{n+1}_{\bar{D}}\| \leq \rho_B \|e_n^y\| + \sqrt{2} I \sum_{t=0}^{B-1} \|\bar{D}_{\phi_{n+t+1}}^{-1} (g^{n+t+1} - g^{n+t})\| $$

$$ \leq \rho_B \|e_n^y\| + \sqrt{2} I \sum_{t=0}^{B-1} \|\bar{W}^{n+t}(x^{n+t} + \alpha^{n+t} \Delta x^{n+t}) - x^{n+t}\| $$

$$ \leq \rho_B \|e_n^y\| + \sqrt{2} I \sum_{t=0}^{B-1} \left( \|\bar{W}^{n+t} e^{n+t}_x\| + \|e^{n+t}_x\| + \alpha^{n+t} \|\bar{W}^{n+t} \Delta x^{n+t}\| \right) $$

$$ \leq \rho_B \|e_n^y\| + \sqrt{2} I \sum_{t=0}^{B-1} \left( \sqrt{I} + 1 \right) \|e^{n+t}_x\| + \alpha^{n+t} \sqrt{I} \|\Delta x^{n+t}\| $$

This completes the proof.

References

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