ON THE PROPERTIES OF SOLUTIONS SET FOR MEASURE DRIVEN DIFFERENTIAL INCLUSIONS

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Abstract. The aim of the paper is to present properties of solutions set for differential inclusions driven by a positive finite Borel measure. We provide for the most natural type of solution results concerning the continuity of the solution set with respect to the data similar to some already known results, available for different types of solutions. As consequence, the solution set is shown to be compact as a subset of the space of regulated functions. The results allow one (by taking the measure μ of a particular form) to obtain information on the solution set for continuous or discrete problems, as well as impulsive or retarded set-valued problems.

1. Introduction. We focus on the following problem:

\[ dx(t) \in G(t, x(t))d\mu(t), \quad x(0) = x_0, \]

where \( G : [0, 1] \times \mathbb{R}^d \rightarrow \mathcal{P}(\mathbb{R}^d) \) is a closed convex-valued multifunction and \( \mu \) is a positive regular Borel measure.

Such a kind of problems encompass as special cases differential and difference inclusions, impulsive and hybrid problems.

One of the important questions (e.g. when looking at numerical approximations) is to check the continuous dependence of solutions on the generating measures. Note that there are several different definitions of the notion of solution for measure differential problems and some of them are chosen in such a way to preserve this property ([21, 15]). In some other papers, like [21], the most natural definition of solution (e.g. [5]) was avoided for this reason: that a closure result was not available and, instead, a more complicated kind of solution was introduced.

Our goal here is to prove that for the natural definition of solution considered in [5] a result of continuity with respect to the data for measure driven inclusions is possible, but with different type of convergence for the sequence of measures.

In general there are two different approaches to the problem of continuous dependencies. One of them, proposed by Silva and Vinter [21] allows to prove, that if a sequence of measures \( (\mu_n)_n \) weakly* converges to \( \mu \), then the solutions \( x \) of the original problem are "close" (in some sense) to solutions \( x_n \) of the problem with \( \mu_n \) instead of \( \mu \). In this approach the convergence is not available in the points of discontinuity of the solution (or, in other words, in the atoms of the limit measure).

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Another approach is presented in [7] or [15, 10]. The solutions of the measure differential problems are uniformly approximated by solutions of some measure problems under uniform convergence assumptions for measures; instead of measures the authors consider, in fact, their primitives (or distribution functions) and a kind of convergence of these primitives is used.

We will prove continuous dependence results by considering a new type of convergence for sequences of measures, allowing us to obtain the convergence (on a subsequence) of the solutions towards a solution of the limit problem in the sense of measures and pointwise as well. Under stronger assumptions, we will be able to obtain even the uniform convergence. We also compare our result with earlier ones.

To make the paper self-contained, we collect some known facts for measures and their convergence. It looks necessary due to several different names used in different mathematical branches (functional analysis, probability theory or stochastic analysis) for the same ideas.

2. Notions and preliminary results. Consider the interval $[0,1]$ endowed with the borel $\sigma$-algebra. The classical Riesz Representation Theorem characterizes the finite regular Borel measures on a compact metrizable space as linear continuous functionals on the space of real continuous functions. This characterization is used by most of the authors studying measure driven equations, e.g. [21] and we will follow the same idea in most of our discussion. Sometimes, we will also use another approach, based on Lebesgue-Stieltjes integration.

2.1. Convergence of measures. Denote by $\mathcal{M}$ the set of all positive finite Borel measures over $[0,1]$. There are several topologies considered on this space, but only a few is interesting in the study of measure differential problems. Let us recall necessary definitions and then compare some kinds of convergence with respect to these topologies.

Definition 2.1. [2] A sequence $(\mu_n) \in \mathcal{M}$ is said to be weakly$^*$ convergent to $\mu \in \mathcal{M}$ if for each continuous function $f : [0,1] \to \mathbb{R}$ we have

$$\int_{[0,1]} f \, d\mu_n \to \int_{[0,1]} f \, d\mu \quad \text{as } n \to \infty.$$ 

This kind of convergence, denoted by $\mu_n \Rightarrow \mu$, is sometimes called "weak convergence" (cf. [2]), but we prefer the name used in functional analysis.

The most important properties of this type of convergence are stated in the Portmanteau Theorem ([2, Theorem 2.1]), in particular: $\mu_n \Rightarrow \mu$ if and only if $\lim_{n \to \infty} \mu_n(A) = \mu(A)$ for any set $A$ of continuity points of $\mu$. This implies that we cannot expect the uniform (or even pointwise) convergence of approximated solution to the exact solution, but rather convergence $\mu$-a.e. (like in [21]).

A stronger kind of convergence is defined in the following way:

Definition 2.2. [2] A sequence $(\mu_n) \in \mathcal{M}$ is said to be uniformly convergent to the measure $\mu$ if $\lim_{n \to \infty} \sup_{A \text{ measurable}} |\mu_n(A) - \mu(A)| = 0$.

We will adapt this definition to the multivalued context in such a way to control the uniform convergence of approximated solutions (cf. also [13, 20] for a single-valued measure problem).

It is clear that uniform convergence implies the weak$^*$ convergence and the converse is not true, in general. But there are some conditions under which weak$^*$ convergence implies uniform convergence (see [17]).

2.2. Stieltjes integrals. First of all, recall that every finite Borel measure on $\mathbb{R}$ agrees with some Lebesgue-Stieltjes measure restricted to the class of Borel sets.
Theorem 2.3. ([3, Theorem 3.21]) Let \( \mu \) be a Borel measure on \( \mathbb{R} \) with \( \mu(B) < \infty \) for every bounded Borel set \( B \). Then there exists a nondecreasing, right-continuous function \( u : \mathbb{R} \to \mathbb{R} \) such that \( \mu(B) = \mu_a(B) \) for any Borel set \( B \). Here \( \mu_a \) denotes the Lebesgue-Stieltjes measure with distribution function \( u \).

This is the motivation for using, when necessary, instead of (1), of the form
\[
dx(t) \in G(t, x(t))du(t), x(0) = x_0
\]
and regarding the inclusion as a Stieltjes inclusion (in Lebesgue-Stieltjes or Kurzweil-Stieltjes approach).

Let us recall some basic properties of Lebesgue-Stieltjes and Kurzweil-Stieltjes integrals.

Definition 2.4. A function \( f : [0, 1] \to \mathbb{R}^d \) is said to be Kurzweil-Stieltjes-integrable with respect to \( u : [0, 1] \to \mathbb{R} \) on \([0, 1]\) (shortly, KS-integrable) if there exists an element denoted by (KS) \( \int_0^1 f(s)du(s) \in \mathbb{R}^d \) such that, for every \( \varepsilon > 0 \), there is a gauge \( \delta_\varepsilon \) on \([0, 1]\) with
\[
\left\| \sum_{i=1}^{p} f(\xi_i)(u(t_i) - u(t_{i-1})) - (\text{KS}) \int_0^1 f(s)du(s) \right\| < \varepsilon
\]
for every \( \delta_\varepsilon \)-fine partition \( \{[t_{i-1}, t_i], \xi_i : i = 1, ..., p\} \) of \([0, 1]\).

The KS-integrability is preserved on all sub-intervals of \([0, 1]\). The function \( t \mapsto (\text{KS}) \int_0^t f(s)du(s) \) is called the KS-primitive of \( f \) w.r.t. \( u \) on \([0, 1]\) (for the auxiliary definitions we refer to [23] or [19]).

Theorem VI.8.1 in [18] states that for right-continuous functions of bounded variation the Lebesgue-Stieltjes-integrability implies the KS-integrability.

For a general Banach space \((X, \| \cdot \|)\), a function \( u : [0, 1] \to X \) is said to be regulated if there exist the limits \( u(t+) \) and \( u(s-) \) for every points \( t \in [0, 1) \) and \( s \in (0, 1] \). It is well-known that the set of discontinuities of a regulated function is at most countable, that regulated functions are bounded and the space \( G([0, 1], X) \) of regulated functions is a Banach space when endowed with the sup-norm \( \|u\|_C = \sup_{t \in [0, 1]} \|u(t)\| \). A subspace of \( G([0, 1], X) \) \( G([0, 1], \mathbb{R}) \) will also be of interest: that of left-continuous regulated functions (known as left cádlág functions in probability theory), denoted by \( \mathcal{D}([0, 1]) \). For a function \( u : [0, 1] \to X \) the variation will be denoted by \( \text{var}(u) \) and if it is finite then \( u \) will be said to have bounded variation (or to be a bounded variation function). Any bounded variation function is regulated. The following property of the indefinite Kurzweil-Stieltjes integral implies that the solutions we will obtain are regulated functions.

Proposition 1. [23, Proposition 2.3.16.] Let \( u : [0, 1] \to \mathbb{R} \) and \( g : [0, 1] \to \mathbb{R}^d \) be such that the Kurzweil-Stieltjes \( \int_0^t g(s)du(s) \) exists. If \( u \) is regulated, then so is the primitive \( h : [0, 1] \to \mathbb{R}, h(t) = \int_0^t g(s)du(s) \) and for every \( t \in [0, 1], \)
\[
h(t^+) - h(t) = g(t) [u(t^+) - u(t)] \quad \text{and} \quad h(t) - h(t^-) = g(t) [u(t) - u(t^-)].
\]
Here \( u(t^-) = \lim_{\tau \to t, \tau < t} u(\tau) \) and \( u(t^+) = \lim_{\tau \to t, \tau > t} u(\tau) \). Moreover, when \( u \) is of bounded variation and \( g \) is Lebesgue-Stieltjes-integrable w.r.t. the variation of \( u \), then \( h \) is also of bounded variation and so, the solutions are sometimes bounded variation functions.

Definition 2.5. ([8]) A set \( \mathcal{A} \subset G([0, 1], X) \) is said to be equi-regular if for every \( \varepsilon > 0 \) and every \( t_0 \in [0, 1] \) there exists \( \delta > 0 \) such that for all \( x \in \mathcal{A} \):

i) for any \( t_0 - \delta < t' < t_0; \) \( \|x(t') - x(t_0)\| < \varepsilon; \)

ii) for any \( t_0 < t'' < t_0 + \delta; \) \( \|x(t'') - x(t_0^+)\| < \varepsilon. \)

We will need the following result, that was proved in [8].

Lemma 2.6. A pointwise convergent sequence of functions which is equi-regular converges uniformly to its limit.
This is a consequence of an Ascoli-type result (which can be found in [8]):

**Lemma 2.7.** Let $\mathcal{A} \subset G([0,1], X)$ be equi-regulated and, for every $t \in [0,1]$, $\mathcal{A}(t) = \{x(t), x \in \mathcal{A}\}$ be relatively compact. Then $\mathcal{A}$ is relatively compact in $G([0,1], X)$.

We need as well some preliminary facts from set-valued analysis (see [4]).

The family of all nonempty bounded closed convex subsets of $\mathbb{R}^d$ will be denoted by $\mathcal{P}_{cb}(\mathbb{R}^d)$. A multifunction $\Gamma : [0,1] \to \mathcal{P}_{cb}(\mathbb{R}^d)$ is said to be (Hausdorff-) upper semi-continuous at a point $t_0 \in [0,1]$ if for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that the excess of $\Gamma(t)$ over $\Gamma(t_0)$ (in the sense of Hausdorff distance) is less than $\epsilon$ whenever $|t - t_0| < \delta_\epsilon$. Otherwise stated, $\Gamma(t) \subset \Gamma(t_0) + \epsilon B$, where $B$ is the closed unit ball of $\mathbb{R}^d$. In the obvious way we call a multifunction upper semi-continuous when it has this feature at any point $t_0 \in [0,1]$. By $S_\Gamma$ we mean the family of all measurable selections of $\Gamma$. For a closed subset $A$ of $\mathbb{R}^d$, denote by $||A|| = D(A, \{0\})$ where $D$ is the (Pompeiu-)Hausdorff distance.

3. **Solutions for measure differential inclusions.** In this section we remind of several definitions that were considered in literature for the notion of solution of a measure driven inclusion.

Among other papers (like [21, p. 731]), in [5] the following type of solution was considered:

**Definition 3.1.** A solution of the problem (1) is a function $x : [0,1] \to \mathbb{R}^d$ for which there exists a $\mu$-integrable function $g : [0,1] \to \mathbb{R}^d$ such that

$$x(t) = x_0 + \int_0^t g(s) \, d\mu(s), \quad \forall \ t \in [0,1]$$

and $g(t) \in G(t, x(t^-)) \ \mu$-a.e.

For this notion of solution, an existence result was obtained:

**Theorem 3.2.** (Theorem 11 in [5]) Let $\mu \in M$ and let $G : [0,1] \times \mathbb{R}^d \to \mathcal{P}_{cb}(\mathbb{R}^d)$ satisfy the following hypothesis:

1) $G(\cdot, \cdot)$ is product Borel measurable,
2) $G(t, \cdot)$ is upper semi-continuous for every $t \in [0,1]$,
3) there exists a positive function $M \in L^1([0,1], \mu)$ and a constant $N > 0$ such that $G(t, y) \subset [M(t) + N ||y||] B$ for all $t \in [0,1]$ and $y \in \mathbb{R}^d$.

Then there exists at least one solution for the measure differential problem (1).

In a series of papers of Silva and his collaborators (e.g. [21]) another definition for solution was considered (going back to [6]). The main idea is to use reparametrization method for $\mu$ and in this way to transform the measure driven differential inclusions into usual differential inclusions.

**Definition 3.3.** A function $x : [0,1] \to \mathbb{R}^d$ is called a robust solution for (1) if $x(t) = x_0 + \int_0^t g(s) \, d\mu(s), \quad \forall t \in [0,1]$ for some $\mu$-integrable function $g$ such that $g(t) \in \tilde{G}(t, x(t^-)) = \tilde{G}(t, x(t^-)); \mu(\{t\}) \ \mu$-a.e., where $\tilde{G}$ is defined on $[0,1] \times \mathbb{R}^d \times (0, \infty)$ as follows:

If $\alpha > 0$, then

$$\tilde{G}(t, v, \alpha) = \left\{ \frac{y(\alpha) - v}{\alpha} : y \in AC^1([0,\alpha]), \ \dot{y}(\sigma) \in G(t, y(\sigma)) \text{a.e., } y(0) = v \right\}$$

and if $\alpha = 0$, then $\tilde{G}(t, v, 0) = G(t, v)$.

Theorem 4.1 in [21] reduces the matter of the existence of solutions in this sense to the existence of an usual differential inclusion and in Corollary 4.2 the existence of robust solutions is provided under Lipschitz continuity assumptions together with linear growth assumptions on the multifunction on the right-hand side.
Finally, let us recall another type of solution that was considered in the single-valued case by many authors (e.g. [20], [14] or [13]) for the measure driven equation seen as a Stieltjes inclusion:

**Definition 3.4.** (Definition 5.3.1 in [20]) A function \( x : [0, 1] \rightarrow \mathbb{R}^d \) is called an approximable solution if there exists a sequence \( \langle u_k \rangle \) of absolutely continuous functions pointwise convergent to \( u \) such that the sequence of corresponding solutions converge pointwise to \( x \) and if \( x \) does not depend on the choice of \( \langle u_k \rangle \).

Various related notions have also been taken into consideration, such as Definition 2.1 in [13] where the convergence of the approximating solutions towards \( x \) is required only at the continuity points of \( x \) or Definition 5.3.2 in [20], where the sequence \( \langle u_k \rangle \) is required to converge pointwise to \( u \) and such that the sequence of the variations of \( u_k \) converge as well.

4. **Main results.** Under the assumptions that \( G(\cdot, \cdot) \) has closed graph and the values of \( G \) are contained in some ball, Theorem 5.1 in [21] states that for the problem (1) the set of robust solutions is continuous with respect to data, in the sense that when a sequence of measures \( \langle \mu_k \rangle \) tends to \( \mu \) in the weak* topology, for any sequence \( \langle x_k \rangle \) of robust solutions corresponding to \( \mu_k \) there exists a robust solution \( x \) corresponding to \( \mu \) with the property that on a subsequence

\[ x_i \rightarrow x \text{ (weakly*)} \text{ and } x_i(t) \rightarrow x(t) \text{ except on the atoms of } \mu. \]

Our concept of solution does not offer this closure property (as stated in [21, p. 731]). However, some closure properties can be proved by taking different kind of convergence for the sequence of measures and this is what will be done in the sequel.

We start by introducing a new type of convergence for measures.

**Definition 4.1.** Let \( \mu, \mu_n \in \mathcal{M} \). We say that \( \langle \mu_n \rangle \) is H-weakly* convergent to \( \mu \) if for every continuous function \( f : [0, 1] \rightarrow \mathbb{R} \), \( \int_0^1 f(s) \text{Bd}(\mu_n - \mu)(s) \rightarrow 0 \) in the Hausdorff topology, for every \( t \in [0, 1] \).

In fact, this is a variant of the weak* convergence of measures more appropriate to the set-valued case, as the following result shows.

**Proposition 2.** The sequence \( \langle \mu_n \rangle \) is H-weakly* convergent to \( \mu \) if and only if \( \int_0^1 f(s) \text{d}(\mu_n - \mu)(s) \rightarrow 0 \) in the Hausdorff topology for every Hausdorff-continuous multifunction \( F : [0, 1] \rightarrow \mathcal{P}_{cb}(\mathbb{R}^d) \).

**Proof.** As multifunctions of the form \( f(s)B \) are continuous whenever \( f \) is a continuous function, one implication is obvious. For the other implication, let \( F : [0, 1] \rightarrow \mathcal{P}_{cb}(\mathbb{R}^d) \) be a continuous multifunction. Then the real function \( f(s) = \max\{|x| : x \in F(s)\} \) is continuous (Theorem 1.4.16 in [1]) and the assertion comes from the fact that \( F(s) \subset f(s)B \), whence

\[ \left\| \int_0^t F(s) \text{d}(\mu_n - \mu)(s) \right\| \leq \left\| \int_0^t f(s) \text{Bd}(\mu_n - \mu)(s) \right\| \]

which tends to 0 for every \( t \in [0, 1] \) in the Hausdorff topology. \( \square \)

**Remark 1.** Obviously, if \( \mu_n \) H-weakly* converges to \( \mu \), then it converges in the usual weak* sense (since we can consider, in particular, single-valued functions in the definition and it is known that the Hausdorff distance between points is exactly the usual norm distance).

**Example 1.** Denote by \( \delta_x \) the delta Dirac measure concentrated at the point \( x \). Consider the following measure \( \mu(A) = \lambda(A) + \delta_{\frac{x}{2}} \), where \( \lambda \) denotes the Lebesgue measure on \([0, 1]\). Let \( \mu_n \) be a sequence of measures defined as \( \mu_n = \delta_{\frac{x}{2}} + \sum_{k=1}^n \frac{1}{n} \cdot \delta_{\frac{k}{n^2}} \). Since \( \int_{[0, 1]} gd\mu = \int_{[0, 1]} gd\lambda + g \left( \frac{1}{2} \right) \) and \( \int_{[0, 1]} gd\mu_n = g \left( \frac{1}{2} \right) + \sum_{k=1}^n \frac{1}{n^2} \cdot g \left( \frac{k}{n^2} \right) \), for any function
therefore (continuous) selection of a multifunction $f$.

Let $\mu \in \mathcal{M}$ and $\mathcal{S} = \mathcal{R}^d \to \mathcal{P}_{cb}(\mathcal{R}^d)$ satisfy the hypothesis of Theorem 3.2 with $M$ continuous and $N = 0$. If $(\mu_n)_n$ converges $H$-weakly* to $\mu$, then for every $x_n \in \mathcal{S}_n$ one can find an element $x \in \mathcal{S}$ and a subsequence pointwisely convergent to $x$ such that $dx_{x_n} \to dx$ weakly*

Proof. Take $x_n \in \mathcal{S}_n$ and $g_n(t) = G(t, x_n(t^-))$ for all $t \in [0, 1]$. On the remaining $\mu_n$-null-measure part of $[0, 1]$ we can take any selection of $G(t, x_n(t^-))$ and so, without restricting the generality, we suppose that $g_n(t) \in G(t, x_n(t^-))$ for all $t \in [0, 1]$. It follows that $||g_n(t)|| \leq M(t)$, for all $t \in [0, 1]$ and $n \in \mathbb{N}$.

As $\mu_n$ and $\mu$ are Borel measures, the measurability with respect to all measures $(\mu_n)_n$ and $\mu$ is the same. So, the functions $(g_n)_n$ are $\mu$-measurable and bounded by an integrable function, therefore, by a classical result (e.g. Theorem T247C in [9]), there exists a subsequence $(g_{n_k})_k$ and a $\mu$-integrable function $g : [0, 1] \to \mathcal{R}^d$ such that $g_{n_k} \to g$ with respect to the weak-$L^1(\mu)$ topology. In particular,

$$\int_0^t g_{n_k}(s)d\mu(s) \to \int_0^t g(s)d\mu(s), \forall t \in [0, 1].$$

For every $\varepsilon > 0$ and $t \in [0, 1]$ there exists $k_{\varepsilon, t} \in \mathbb{N}$ satisfying

$$\left| \left| \int_0^t g_{n_k}(s)d\mu(s) - \int_0^t g(s)d\mu(s) \right| \right| < \frac{\varepsilon}{2}, \forall k \geq k_{\varepsilon, t}.$$

Afterwards, using the $H$-weak* convergence of $(\mu_n)_n$ we get the existence of $k_{\varepsilon, t} \in \mathbb{N}$:

$$\left| \left| \int_0^t M(s)Bd(\mu_{n_k} - \mu)(s) \right| \right| < \frac{\varepsilon}{2}, \forall k \geq k_{\varepsilon, t},$$

therefore

$$\left| \left| \int_0^t g_{n_k}(s)d\mu_{n_k}(s) - \int_0^t g(s)d\mu(s) \right| \right| \leq \left| \left| \int_0^t g_{n_k}(s)d(\mu_{n_k} - \mu)(s) \right| \right| + \left| \left| \int_0^t g_{n_k}(s)d\mu(s) - \int_0^t g(s)d\mu(s) \right| \right| < \varepsilon,$$

for all $k$ greater than $\max(k_{\varepsilon, t}, k_{\varepsilon, t})$. Otherwise said, $x_{n_k}(t) = x_0 + \int_0^t g_{n_k}(s)d\mu_{n_k}(s)$ tends to $x_0 + \int_0^t g(s)d\mu(s)$ pointwise.

It remains to prove that $x(t) = x_0 + \int_0^t g(s)d\mu(s) \in \mathcal{S}$, in other words to prove that $g(t) \in G(t, x(t^-))$, $\mu$-a.e. This comes from the fact that one can find a sequence $g_{n_k}$ of convex combinations of $\{g_{n_m}, m \geq k\}$ which strongly $L^1$-converges to $g$ (Mazur’s Lemma) and so, a subsequence of it converges $\mu$-a.e. to $g$. Now take into account that the multifunction $G(t, \cdot)$ is upper semicontinuous, whence for each $t \in [0, 1]$ $G(t, x_{n_k}(t^-)) \in G(t, x(t^-)) + \varepsilon B$, for all $k$ greater than some $k_{\varepsilon, t}$ and the condition is thus proved.
Remark 2. Comparing to the results in [21] (Theorem 5.1), we showed that for the more
of solutions:

\[\quad \text{measures tends to a measure in the sense of Definition 4.1 we obtain that on a subsequence}
\]

\[\quad \text{natural definition of solution that we considered, under the assumption that the sequence of}
\]

\[\quad \text{f: } [0, 1] \rightarrow \mathbb{R}, \text{ for any } t \in [0, 1]
\]

\[\quad \left\| \int_0^t h(s)dx_{n_k}(s) - \int_0^t h(s)dx(s) \right\| = \left\| \int_0^t h(s)g_{n_k}(s)d\mu_{n_k}(s) - \int_0^t h(s)g(s)d\mu(s) \right\|
\]

\[\quad \leq \left\| \int_0^t h(s)g_{n_k}(s)d(\mu_{n_k} - \mu)(s) \right\| + \left\| \int_0^t h(s)g_{n_k}(s)d\mu(s) - \int_0^t h(s)g(s)d\mu(s) \right\|
\]

whence the sum is arbitrarily small when \( k \to \infty \) and so, \( dx_{n_k} \to dx \) weakly*. We applied for
the first equality the substitution Theorem 2.3.19 in [23] for KS-integrals (with the remark
that the involved integrals exist as Lebesgue-Stieltjes integrals, therefore as KS-integrals as
well).

\[\square\]

Remark 2. Comparing to the results in [21] (Theorem 5.1), we showed that for the more
natural definition of solution that we considered, under the assumption that the sequence of
measures tends to a measure in the sense of Definition 4.1 we obtain that on a subsequence
of solutions:

\[\quad x_{n_k}(t) \to x(t), \quad \forall t \in [0, 1] \quad \text{and} \quad dx_{n_k} \to dx \text{ weakly*},
\]

so the pointwise convergence of a sequence of solutions holds in the atoms of \( \mu \) too.

Moreover:

Remark 3. If \((\text{var}(\mu_n - \mu))_n\) converges weakly* to the zero measure, then in Definition
4.1 \( \int_0^t f(s)Bd(\mu_{n_k} - \mu)(s) \to \{0\}\) in the Hausdorff topology for every continuous function
\( f: [0, 1] \rightarrow \mathbb{R} \), uniformly in \( t \in [0, 1] \). This is because

\[\quad \left\| \int_0^t f(s)Bd(\mu_{n_k} - \mu)(s) \right\| \leq \int_0^1 |f(s)|d\text{var}(\mu_{n_k} - \mu)(s).
\]

In particular, this holds when the difference of measures \( \mu_n \) and \( \mu \) is a positive measure (or,
more general, it has constant sign) convergent weakly* to the zero measure.

Bearing this in mind, we can offer an alternative closure-type result.

Theorem 4.3. Let \( \mu, \mu_n \in \mathcal{M} \) and \( G: [0, 1] \times \mathbb{R}^d \rightarrow \mathcal{P}_{cb}(\mathbb{R}^d) \) satisfy the hypothesis of
Theorem 3.2 with \( M \) continuous and \( N = 0 \). If \((\mu_n)_n\) converges \( H \)-weakly* to \( \mu \) such that
in the definition the condition is uniform in \( t \in [0, 1] \), then for every \( x_n \in \mathcal{S}_n \) one can find
\( x \in \mathcal{S} \) and a subsequence uniformly convergent to \( x \).

Proof. As in the proof of Theorem 4.2, taking \( x_n \in \mathcal{S}_n \) and \( g_n(t) \in G(t, x_n(t) -) \) \( \mu_n \)-a.e such that \( x_n(t) = x_0 + \int_0^t g_n(s)d\mu_n(s) \), one can extract a subsequence \((g_{n_k})_k\) and a \( \mu \)-integrable
function \( g: [0, 1] \rightarrow \mathbb{R}^d \) such that \( g_{n_k} \to g \) with respect to the weak-\( L^1(\mu) \) topology, so

\[\quad \int_0^t g_{n_k}(s)d\mu(s) \to \int_0^t g(s)d\mu(s), \quad \forall t \in [0, 1].
\]

On the other hand, the sequence \((g_{n_k})_k\) is dominated by the function \( M \) which is integrable
w.r.t. \( \mu \). Therefore, the sequence of their primitives \( \left( \int_0^t g_{n_k}(s)d\mu(s) \right)_k \) is equi-regulated.
Indeed, by Lemma 2.1 in [8] is suffices to prove that for every $\varepsilon > 0$ there exists $0 = t_0 < ... < t_{n_\varepsilon} = 1$ such that for any $[t', t''] \subset (t_i, t_{i+1})$, for every $i = 0, ..., n_\varepsilon - 1$:

$$\left\| \int_0^{t'} g_{n_k}(s) d\mu(s) - \int_0^{t''} g_{n_k}(s) d\mu(s) \right\| < \varepsilon, \forall k.$$  

As $\int_0^t M(s) d\mu(s)$ is regulated, for any $\varepsilon > 0$ there exists $0 = t_0 < ... < t_{n_\varepsilon} = 1$ such that for any $[t', t''] \subset (t_i, t_{i+1})$: $\left| \int_0^t M(s) d\mu(s) - \int_0^{t''} M(s) d\mu(s) \right| < \varepsilon$, for every $i = 0, ..., n_\varepsilon - 1$. But this implies that $\left\| \int_0^{t'} g_{n_k}(s) d\mu(s) - \int_0^{t''} g_{n_k}(s) d\mu(s) \right\| \leq \int_0^{t''} M(s) d\mu(s) < \varepsilon$ and the equi-regulatedness is proved.

Lemma 2.6 implies now that $\int_0^t g_{n_k}(s) d\mu(s) \to \int_0^t g(s) d\mu(s)$ uniformly.

Using then the fact that in the definition of $H$-weak* convergence of $(\mu_n)_n$ we have uniformity in $t \in [0, 1]$ we get that $\int_0^t g_{n_k}(s) d\mu_n(s) \to \int_0^t g(s) d\mu(s)$ uniformly in $t \in [0, 1]$ and the assertion is thus proved.

In particular, when $\mu_n = \mu$, we get:

**Corollary 1.** Under the hypothesis of Theorem 3.2 with $M$ continuous and $N = 0$, the solution set of measure inclusion (1) is compact in $G([0,1], \mathbb{R}^d)$ equipped with the topology of uniform convergence.

When considering the Kuratowski notion of the superior limit for sequences of sets, namely

$Limsup_n A_n = \{ x; \exists$ increasing sequence $(n_k)_k \in \mathbb{N} : x = \lim_{k \to \infty} x_{n_k}, x_{n_k} \in A_{n_k} \}$,

the preceding Theorem 4.2 yields that if in Theorem 3.2 $N = 0$, then the $H$-weak* convergence of $(\mu_n)_n$ to $\mu$ (in the uniform way, as in Remark 3) implies that

$Limsup_{n \in \mathbb{N}} S_n \subset S$

as subsets of $G([0,1], \mathbb{R}^d)$.

In fact, we can prove that this is true without imposing the condition $N = 0$. For this purpose, we need to make a change in the notion of convergence of the sequence of measures, a change that, as Remark 4 below shows, is very natural.

**Definition 4.4.** We say that $(\mu_n)_n$ is càdlàg-$H$-weakly* convergent to $\mu$ if for every left-càdlàg function $f : [0,1] \to \mathbb{R}$, $\int_0^t f(s) B d(\mu_n - \mu)(s) \to \{0\}$ in the Hausdorff topology for every $t \in [0,1]$.

**Remark 4.** Theorem 1 in [16] implies that in the single-valued case putting in the definition of weak* convergence of a sequence of measures càdlàg (or left-càdlàg) functions instead of continuous functions does not change proofs. The reason consists in the fact that the topological dual spaces for $C([0,1])$ and $\mathcal{D}([0,1])$ coincide when the two spaces are endowed with the uniform topology.

**Theorem 4.5.** Let $\mu, \mu_n$ and $G : [0,1] \times \mathbb{R}^d \to \mathcal{P}_{cb}(\mathbb{R}^d)$ satisfy the hypothesis of Theorem 3.2 with $M$ continuous and suppose that $(\mu_n)_n$ converges càdlàg-$H$-weakly* to $\mu$ in the uniform way, as in Remark 3. Then $Limsup_{n \in \mathbb{N}} S_n \subset S$.

**Proof.** Let $x \in Limsup_{n \in \mathbb{N}} S_n$ and a sequence $(x_{n_k})_k \in S_{n_k}$ uniformly convergent to $x$. For every $\varepsilon > 0$ there exists $k_\varepsilon$ such that

$$\|x_{n_k}(t) - x(t)\| < \varepsilon, \forall k \geq k_\varepsilon, \forall t \in [0,1]$$

which has as consequence the fact that $\|x_{n_k}(t^+ - x(t^-))\| < \varepsilon, \forall k \geq k_\varepsilon, \forall t \in [0,1]$. 

For every $k \in \mathbb{N}$ one can find $g_{n_k}(t) \in G(t, x_{n_k}(t^-)) \forall t \in [0,1]$ with

$$x_{n_k}(t) = x_0 + \int_0^t g_{n_k}(s) d\mu_{n_k}(s).$$

It follows that

$$\|g_{n_k}(t)\| \leq M(t) + N\|x_{n_k}(t^-)\| \leq M(t) + N\|x(t^-)\| + 1, \forall k \geq k_1, \forall t \in [0,1]$$

and from this point the proof goes as in Theorem 4.3 since $M(t) + N\|x(t^-)\| + 1$ is a left-càdlàg function (as the sum of a continuous function with a left-càdlàg one).

By applying [11, Theorem 3] one more property can be obtained.

**Corollary 2.** Under the assumptions of Theorem 4.5, $\limsup_{n \to \infty} \mu_n(S_n) \leq \mu(S)$.

**Remark 5.** Theorem 4.2 allows one to obtain the solutions of difficult measure differential problems by considering simpler measures (e.g. discrete ones) convergent to the original measure and looking at the accumulation points of sequences of corresponding solutions.

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