COUNTING INDEPENDENT SETS IN AMENABLE GRAPHS

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ABSTRACT. Given a locally finite graph \( \Gamma \), an amenable subgroup \( G \) of graph automorphisms acting freely and almost transitively on its vertices, and a \( G \)-invariant activity function \( \lambda \), consider the free energy \( f_G(\Gamma, \lambda) \) of the hardcore model defined on the set of independent sets in \( \Gamma \) weighted by \( \lambda \).

We define suitable ensembles of hardcore models and prove that, under some recursion-theoretic assumptions on \( G \), if \( \|\lambda\|_\infty < \lambda_c(\Delta) \), there exists an \( \varepsilon \)-additive approximation algorithm for \( f_G(\Gamma, \lambda) \) that runs in time \( \text{poly}\left( (1 + \varepsilon^{-1})|\Gamma(G)| \right) \) and, if \( \|\lambda\|_\infty > \lambda_c(\Delta) \), there is no such algorithm, unless \( \text{NP} = \text{RP} \), where \( \lambda_c(\Delta) \) denotes the critical activity on the \( \Delta \)-regular tree. This recovers the computational phase transition for the partition function of the hardcore model on finite graphs and provides an extension to the infinite setting.

As an application in symbolic dynamics, we use these results to develop efficient approximation algorithms for the topological entropy of subshifts of finite type with enough safe symbols, we obtain a representation formula of topological entropy in terms of random trees of self-avoiding walks, and we provide new conditions for the uniqueness of the measure of maximal entropy based on the connective constant of a particular associated graph.

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1. INTRODUCTION

Suppose that we are given a finite simple graph \( \Gamma = (V, E) \) and we are asked to count its number of independent sets. An independent set is a subset \( I \subseteq V \) such that \((v, v') \notin E\)
(i.e., \((v,v')\) is not an edge) for all \(v,v' \in I\). For example, if \(\Gamma\) is the 4-cycle \(C_4\) with \(V = \{v_1,v_2,v_3,v_4\}\) and \(E = \{(v_1,v_2), (v_2,v_3), (v_3,v_4), (v_4,v_1)\}\), it can be checked that there are exactly 7 different independent sets, namely \(\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1,v_3\}, \) and \(\{v_2,v_4\}\). A common generalization of this question is to ask for the “number” of weighted independent sets in \(\Gamma\): given an parameter \(\lambda > 0\) — usually called activity or fugacity —, we ask for the value of the summation

\[
Z_{\Gamma}(\lambda) := \sum_{I \in X(\Gamma)} \lambda^{|I|},
\]

where \(X(\Gamma)\) denotes the collection of all independent sets in \(\Gamma\) and \(|I|\), the cardinality of a given independent set \(I\). Notice that we recover the original problem — i.e., to compute \(|X(\Gamma)|\) — if we set \(\lambda = 1\) and that \(Z_{\Gamma}(\lambda) \geq \lambda^{|V|}\), by considering all the singletons \(\{\{v\}\}\). The sum \(Z_{\Gamma}(\lambda)\) corresponds to the normalization factor of the probability distribution \(P_{\Gamma}\) on \(X(\Gamma)\) that assigns to each \(I \in X(\Gamma)\) a probability proportional to \(\lambda^{|I|}\), i.e., the so-called partition function (also known as the independence polynomial) of the well-known hardcore model from statistical physics.

In general, it is not possible to compute exactly \(Z_{\Gamma}(\lambda)\) efficiently [29], even for the case \(\lambda = 1\) [50] (technically, to compute \(Z_{\Gamma}(\lambda)\) is an NP-hard problem and to compute \(|X(\Gamma)|\) is a \#P-complete problem). Therefore, one may attempt to at least finding ways to approximate \(Z_{\Gamma}(\lambda)\) efficiently.

In recent years, there has been a great deal of attention to the complexity of approximating partition functions of spin systems (e.g., see [4]), and the hardcore model, possibly together with the Ising model, occupies the most important place. One of the most noticeable results, due to Weitz [49], and then Sly [44] and Sly-Sun [45], is the existence of a computational phase transition for having a fully polynomial-time approximation scheme (FPTAS) for the approximation of \(Z_{\Gamma}(\lambda)\). In simple terms, Weitz developed an FPTAS, a particular kind of efficient deterministic approximation algorithm on the family of finite graphs with bounded degree \(\Delta\), provided \(\lambda < \lambda_c(\Delta)\), where \(\lambda_c(\Delta) := \frac{(\Delta-1)(\Delta-4)}{(\Delta-2)^2}\) denotes the critical activity for the hardcore model on the \(\Delta\)-regular tree \(T_{\Delta}\). Conversely, a couple of years later, Sly and Sun managed to prove that the existence of such an algorithm for \(\lambda > \lambda_c(\Delta)\) would imply that \(NP = RP\), the equivalence of two well-known computational complexity classes which are widely believed to be different [3].

The work of Weitz exploited a technique based on trees of self-avoiding walks and a special notion of correlation decay known as strong spatial mixing. Later, Sinclair et al. [42] studied refinements of Weitz’s result by considering families of finite graphs parameterized by their connective constant instead of their maximum degree, and established that there exists an FTPAS for \(Z_{\Gamma}(\lambda)\) for families of graphs with connective constant bounded by \(\mu\), whenever \(\lambda < \frac{\mu^\mu}{(\mu - 1)^{\mu - 1}}\).

Now, if \(\Gamma\) is an infinite graph, most of these concepts stop making sense. One way to deal with this issue is by choosing an appropriate normalization and by using the DLR formalism. The idea is roughly the following: suppose that we have a sequence \(\{\Gamma_n\}\) of finite subgraphs that “exhausts” \(\Gamma\) in some way. This sequence induces two other sequences: a sequence \(\{Z_{\Gamma_n}(\lambda)\}\) of partition functions and a sequence \(\{P_{\Gamma_n}\}\) of probability distributions. A way to extend the idea of “number of weighted independent sets (per site)” in \(\Gamma\) is by considering the sequence \(\{Z_{\Gamma_n}(\lambda)^{1/|\Gamma_n|}\}\) and hoping that it converges. Under the right assumptions on \(\Gamma\) and \(\{\Gamma_n\}\), this is exactly the case and something similar happens to \(\{P_{\Gamma_n}\}\). Moreover, there is an intimate connection between the properties of the limit measures and our ability to estimate the value of \(\lim_n |\Gamma_n|^{-1} \log Z_{\Gamma_n}(\lambda)\), i.e., to “approximately
count it. We denote such limit by $f(\Gamma, \lambda)$ and call it the free energy of the hardcore model $(\Gamma, \lambda)$, one of the most crucial quantities in statistical physics [5, 16, 41].

It can be checked that if $\Gamma$ is finite, to approximate the partition function $Z_\Gamma(\lambda)$ with a multiplicative error (in polynomial time) is equivalent to approximate the free energy $f(\Gamma, \lambda)$ with an additive error [28] (in polynomial time), so the problem of approximating $f(\Gamma, \lambda)$ recovers the problem of approximating the partition function in the finite case and, at the same time, extends the problem to the infinite setting.

The main goal of this paper is to establish a computational phase transition for the free energy on ensembles of —possibly infinite— hardcore models, i.e., we aim to prove the existence of an efficient additive approximation algorithm for the free energy when the activity is low and to establish that there is no efficient approximation algorithm for the free energy when the activity is high.

There have been many recent works related to the study of correlation decay properties and its relation to approximation algorithms for the free energy (and related quantities such as pressure, capacity, and entropy) in the infinite setting [15, 33, 7, 47, 31, 32]. In this work we put all these results in a single framework, which also encompasses the results from Weitz, Sly and Sun, and Sinclair et al., and at the same time generalizes them.

In 2009, Gamarnik and Katz [15] introduced what they called the sequential cavity method, which can be regarded as a sort of infinitary self-reducibility property [23]. Combining such method with Weitz’s results, they managed to prove that the free energy of the hardcore model in the Cayley graph of $\mathbb{Z}^d$ with canonical generators has an $\varepsilon$-additive approximation algorithm that runs in time polynomial in $\varepsilon^{-1}$ whenever $\lambda < \lambda_c(2d)$, where $2d$ is the maximum degree of the graph. Related results were also proven by Pavlov in [37], who developed an approximation algorithm for the hard square entropy, i.e., the free energy of the hardcore model in the Cayley graph of $\mathbb{Z}^2$ with canonical generators and activity $\lambda = 1$. Later, there were also some explorations due to Wang et al. [47] in Cayley graphs of $\mathbb{Z}^2$ with respect to other generators (e.g., the non-attacking kings system) in the context of information theory and algorithms for approximating capacities.

In this paper we prove that all these results fit and can be generalized to hardcore models $(\Gamma, \lambda)$ such that $\Gamma$ is a locally finite graph, $G \curvearrowright \Gamma$ is free and almost transitive for some countable amenable subgroup $G \leq \text{Aut}(\Gamma)$, and $\lambda : \mathcal{V} \rightarrow \mathbb{R}_{>0}$ is a —not necessarily constant— $G$-invariant activity function.

We denote the free energy by $f_G(\Gamma, \lambda)$ to emphasize the group $G$. First, based on results from [19, 8], we prove in Theorem 7.1 that $f_G(\Gamma, \lambda)$ can be obtained as the pointwise limit of a Shannon-McMillan-Breiman type ratio with regards to any Gibbs measure on $\mathcal{X}(\Gamma)$. Next, in Theorem 7.5, we prove that if $\lambda$ is such that $(\Gamma, \lambda)$ satisfies strong spatial mixing, then $f_G(\Gamma, \lambda)$ corresponds to the evaluation of a random information function, based on ideas about random invariant orders and the Kieffer-Pinsker formula for measure-theoretical entropy introduced in [2]. Then, in Theorem 7.6, using the previous representation theorem and the techniques from [49], we provide a formula for $f_G(\Gamma, \lambda)$ in terms of trees of self-avoiding walks in $\Gamma$. These first three theorems can be regarded as a preprocessing treatment of $f_G(\Gamma, \lambda)$ in order to obtain an arboreal representation of free energy to develop approximation techniques, but we believe that they are of independent interest.

Later, in Theorem 8.5, given a finitely generated amenable orderable group $G$ with a prescribed set of generators $S$ such that some algebraic past is decidable in exponential time (for example, $\mathbb{Z}^d$ with the lexicographic order or other groups such as the Heisenberg group $H_3(\mathbb{Z})$ with its usual order), a positive integer $\Delta$, and $\lambda_0 > 0$, we denote by $\mathcal{K}^\Delta_G(\lambda_0)$
the ensemble of hardcore models \((\Gamma, \lambda)\) such that \(G \rhd \Gamma\) is free and almost transitive, the maximum degree of \(\Gamma\) is bounded by \(\Delta\), the values of \(\lambda\) are bounded from above by \(\lambda_0\), and establish the following algorithmic implications: if \(\lambda_0 < \lambda_c(\Delta)\), there exists an \(\epsilon\)-additive approximation algorithm on \(\mathcal{H}_G^\Delta(\lambda_0)\) for \(f_G(\Gamma, \lambda)\) that runs in time \(\text{poly}((1 + \epsilon^{-1})|\Gamma/G|)\) and, if \(\lambda_0 > \lambda_c(\Delta)\), there is no such algorithm, unless \(\text{NP} = \text{RP}\), where \(\lambda_c(\Delta)\) denotes the critical activity on the \(\Delta\)-regular tree \(T_\Delta\).

Here, by an additive approximation algorithm, we mean a procedure that given \((\Gamma, \lambda)\) and \(\epsilon > 0\), outputs a number \(\hat{f}\) such that \(|f_G(\Gamma, \lambda) - \hat{f}| < \epsilon\) and by polynomial time, we mean that the procedure takes time polynomial in \(|\Gamma/G|\) and \(\epsilon^{-1}\), where \(|\Gamma/G|\) denotes the size of a (or any) fundamental domain of the action \(G \rhd \Gamma\), and therefore, all the information we need in order to construct \(\Gamma\). In particular, we show that if \(G\) is the trivial group, Theorem 8.5 directly recovers the results from Weitz, Sly, and Sun.

Finally, as an application in symbolic dynamics, we show how to use these results to establish representation formulas and efficient approximation algorithms for the topological entropy of nearest-neighbor subshifts of finite type with enough safe symbols and the pressure of single-site potentials with a vacuum state, which includes systems such as the Widom-Rowlinson model, some other weighted graph homomorphisms from \(\Gamma\) to any finite graph, among others. These results can also be regarded as an extension of the works by Marcus and Pavlov in \(\mathbb{Z}^d\) (see \([33, 32, 31]\)), who developed additive approximation algorithms for the entropy and free energy (or pressure) of general \(\mathbb{Z}^d\)-subshifts of finite type, with special emphasis in the \(d = 2\) case. We believe that these implications are relevant, especially in the light of results like the one from Hochman and Meyerovitch \([22]\), who proved that the set of topological entropies that a nearest-neighbor \(\mathbb{Z}^2\)-subshift of finite type can achieve coincides with the set of non-negative right-recursively enumerable real numbers, which includes numbers that are poorly computable or even not computable. In addition, we discuss the case of the monomer-dimer model and counting independent sets of line graphs, which is a special case that does not exhibit a phase transition. As a byproduct of our results, we also give sufficient conditions for the existence of a unique measure of maximal entropy for subshifts on arbitrary amenable groups.

The paper organized as follows: in Section 2, we introduce the basic concepts regarding graphs, homomorphisms, independent sets, group actions, Cayley graphs, and partition functions; in Section 3, we rigorously define free energy based on the notion of amenability and provide a way to reduce every almost free action \(G \rhd \Gamma\) to a free one; in Section 4, we define Gibbs measures and relevant spatial mixing properties; in Section 5, we develop the formalism based on tree of self-avoiding walks and discuss some of its properties; in Section 6, we present the formalism of invariant (deterministic and random) orders of a group; in Section 7, we prove Theorem 7.1, Theorem 7.5, and Theorem 7.6, which provide a randomized sequential cavity method that allows us to obtain an arboreal representation of free energy; in Section 8, we prove Theorem 8.5 and establish the algorithmic implications of our results; in Section 9, we provide reductions that allow us to translate the problem of approximating pressure of a single-site potential and the entropy of a subshift into the problem of counting independent sets and discuss other consequences that are implicit in our results.

2. Preliminaries

2.1. Graphs. A graph will be a pair \(\Gamma = (V, E)\) such that \(V\) is a countable set —the vertices— and \(E \subseteq V \times V\) is a symmetric relation —the edges—. Let \(\leftrightarrow\) be the equivalence relation generated by \(E\), i.e., \(v \leftrightarrow v'\) if and only if there exist \(n \in \mathbb{N}_0\) and \(\{v_i\}_{0 \leq i \leq \ell}\) such
that \( v = v_0, v' = v_n \), and \( (v_i, v_{i+1}) \in E \) for every \( 0 \leq i < \ell \). Denote by \( n(v, v') \) the smallest such \( n \). This induces a notion of distance in \( \Gamma \) given by

\[
\text{dist}_\Gamma(v, v') = \begin{cases} 
    n(v, v') & \text{if } v \leftrightarrow v', \\
    +\infty & \text{otherwise}.
\end{cases}
\]

Given a set \( U \subseteq V \), we define its boundary \( \partial U \) as the set \( \{ v \in V : \text{dist}_\Gamma(v, U) = 1 \} \), where \( \text{dist}_\Gamma(U, U') = \inf_{v \in U, v' \in U'} \text{dist}_\Gamma(v, v') \). In addition, given \( \ell \geq 0 \) and \( v \in V \), we define the ball centered at \( v \) with radius \( \ell \) as \( B_\Gamma(v; \ell) := \{ v' \in V : \text{dist}_\Gamma(v, v') \leq \ell \} \).

A graph \( \Gamma \) is

- **looseless**, if \( E \) is anti-reflexive (i.e., there is no vertex related to itself);
- **connected**, if \( v \leftrightarrow v' \) for every \( v, v' \in V \); and
- **locally finite**, if every vertex is related to only finitely many vertices.

Sometimes we will write \( V(\Gamma) \) and \( E(\Gamma) \) — instead of just \( V \) and \( E \) — to emphasize \( \Gamma \).

### 2.2. Homomorphisms.

Consider graphs \( \Gamma_1 \) and \( \Gamma_2 \). A **graph homomorphism** is a map \( g : V(\Gamma_1) \rightarrow V(\Gamma_2) \) such that

\[
(v, v') \in E(\Gamma_1) \implies (g(v), g(v')) \in E(\Gamma_2).
\]

We denote by \( \text{Hom}(\Gamma_1, \Gamma_2) \) the set of graph homomorphisms from \( \Gamma_1 \) to \( \Gamma_2 \).

A **graph isomorphism** is a bijective map \( g : V(\Gamma_1) \rightarrow V(\Gamma_2) \) such that

\[
(v, v') \in E(\Gamma_1) \iff (g(v), g(v')) \in E(\Gamma_2).
\]

If such a map exists, we say that \( \Gamma_1 \) and \( \Gamma_2 \) are **isomorphic**, denoted by \( \Gamma_1 \cong \Gamma_2 \).

A **graph automorphism** is a graph isomorphism from a graph \( \Gamma \) to itself. We denote by \( \text{Aut}(\Gamma) \) the set of graph automorphisms of \( \Gamma \). This set is a group when considering composition \( \circ \) as the group operation and the identity map \( \text{id}_\Gamma : V \rightarrow V \) as the identity group element \( 1_{\text{Aut}(\Gamma)} \). In this case, instead of writing \( g_1 \circ g_2 \), we will simply write \( g_1g_2 \) to emphasize the group structure.

### 2.3. Independent sets.

Given a subset \( U \subseteq V \), the **subgraph induced** by \( U \), denoted by \( \Gamma[U] \), is the graph with set of vertices \( U \) and set of edges \( E \cap (U \times U) \). A subset \( I \subseteq V \) is called an **independent set** if \( \Gamma[I] \) has no edges. We can also represent an independent set by its indicator function, i.e., by a map \( x : V \rightarrow \{0, 1\} \) so that

\[
[x(v) = 1 \text{ and } (v, v') \in E] \implies x(v') = 0.
\]

In addition, if we consider the finite graph \( H_0 := (\{0, 1\}, \{(0, 0), (0, 1), (1, 0)\}) \), then \( x \) can be also understood as a graph homomorphism from \( \Gamma \) to \( H_0 \) (see Figure 1).

\[\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node[shape=circle,draw=black] (A) at (0,0) {0};
  \node[shape=circle,draw=black] (B) at (1,0) {1};
  \draw (A) -- (B);
\end{tikzpicture}
\caption{The graph \( H_0 \).}
\end{figure}\]

We denote by \( X(\Gamma) \) the **set of independent sets** of \( \Gamma \). Notice that \( X(\Gamma) \subseteq \{0, 1\}^V \) can be identified with the set \( \text{Hom}(\Gamma, H_0) \) and that the empty independent set \( \emptyset^V \) always belongs to \( X(\Gamma) \). Sometimes we will denote such independent set by \( 0^V \).
2.4. **Group actions.** Let $G$ be a subgroup of $\text{Aut}(\Gamma)$. Given $g \in G$ and $v \in V$, the map $(g, v) \mapsto g \cdot v := g(v)$ is a (left) group action, this is to say, $1_G \cdot v = v$ and $(g g') \cdot v = g \cdot (g' \cdot v)$ for all $g' \in G$, where $1_G = 1_{\text{Aut}(\Gamma)}$. In this case, we say that $G$ acts on $\Gamma$ and denote this fact by $G \curvearrowright \Gamma$.

The group $G$ also acts on $\{0,1\}^V$ by precomposition. Given $g \in G$ and $x \in \{0,1\}^V$, consider the map $(g, x) \mapsto g \cdot x := x \circ g^{-1}$. A subset $X \subseteq \{0,1\}^V$ is called $G$-invariant if $g \cdot X = X$ for all $g \in G$, where $g \cdot X := \{g \cdot x : x \in X\}$. Clearly, if $x \in X(\Gamma)$, then $g \cdot x$ and $g^{-1} \cdot x$ also belong to $X(\Gamma)$, since $g \in \text{Aut}(\Gamma)$ and $x \in \text{Hom}(\Gamma, H_0)$. Therefore, $X(\Gamma)$ is $G$-invariant and the action $G \curvearrowright X(\Gamma)$ is well defined.

We will usually use the letter $v$ to denote vertices in $V$, the letter $g$ to denote graph automorphisms in $G$, and the letter $x$ to denote independent sets in $X(\Gamma)$. In order to distinguish the action of $G$ on $V$ from the action of $G$ on $X(\Gamma)$, we will write $g v$ instead of $g \cdot v$, without risk of ambiguity.

The action $G \curvearrowright \Gamma$ is always faithful, i.e., for all $g \in G \setminus \{1_G\}$, there exists $v \in V$ such that $g v \neq v$. The $G$-orbit of a vertex $v \in V$ is the set $G v := \{g v : g \in G\}$. The set of all $G$-orbits of $\Gamma$, denoted by $\Gamma/G$, is a partition of $V$ and it is called the quotient of the action.

We say that a subset $\emptyset \neq U \subseteq V$ is dynamically generating if $G U = V$, where $G F := \{g v : g \in F, v \in U\}$ for any $F \subseteq G$, and a fundamental domain if it is also minimal, i.e., if $U' \subseteq U$, then $G U' \subseteq V$. The action $G \curvearrowright \Gamma$ is almost transitive if $|\Gamma/G| < +\infty$ and transitive if $|\Gamma/G| = 1$. A graph $\Gamma$ is called almost transitive (resp. transitive) if $\text{Aut}(\Gamma) \curvearrowright \Gamma$ is almost transitive (resp. transitive).

The index of a subgroup $H \leq G$, denoted by $[G : H]$, is the cardinality of the set of cosets $\{Hg : g \in G\}$. We will usually consider subgroups of finite index. In this case, we have that $|\Gamma/G| = [\Gamma/G][G : H]$.

The $G$-stabilizer of a vertex $v \in V$ is the subgroup $\text{Stab}_G(v) := \{g \in G : g v = v\}$. Notice that $|\text{Stab}_G(v)| = |\text{Stab}_G(v')|$ for all $v' \in Gv$. If $|\text{Stab}_G(v)| < \infty$ for all $v$, we say that the action is almost free, and if $|\text{Stab}_G(v)| = 1$ (i.e., if $\text{Stab}_G(v) = \{1_G\}$) for all $v$, we say that the action is free.

A relevant observation is that if we assume that $\Gamma$ is countable and $G \curvearrowright \Gamma$ is almost transitive and almost free, then $G$ must be a countable group. In this work, we will only consider almost free and almost transitive actions. In this case, there exists a finite fundamental domain $U_0 \subseteq V$ such that $|U_0| = |\Gamma/G|$ and, if $\Gamma$ is locally finite, then $\Gamma$ must have bounded degree, i.e., there is a uniform bound on the number of vertices that each vertex is related to. In such case, we denote by $\Delta(\Gamma)$ the maximum degree among all vertices of the graph $\Gamma$.

2.5. **Transitive case:** Cayley graphs. Consider a subset $S \subseteq G$ that we assume to be symmetric, i.e., $S = S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$. We define the (right) Cayley graph as $\text{Cay}(G, S) = (V, E)$, where $V = G$ and $E = \{(g, sg) : g \in G, s \in S\}$.

Cayley graphs are a natural construction used to represent groups in a geometric fashion. In this context, it is common to ask that $1_G \notin S$, $S$ to be finite, and $S$ to be generating, i.e., $G = \langle S \rangle$, where

$$\langle S \rangle := \{s_1 \cdots s_k : s_i \in S \text{ for all } 1 \leq i \leq k \text{ and } k \in \mathbb{N}\}.$$

Notice that if $1_G \notin S$, then $\text{Cay}(G, S)$ is loopless, if $S$ is finite, then $\text{Cay}(G, S)$ has bounded degree and, if $S$ is generating, then $\text{Cay}(G, S)$ is connected.
Now, suppose that $G \acts \Gamma$ is transitive (and free). Then, there exists a symmetric set $S \subseteq G$ such that

$$\Gamma \cong \text{Cay}(G,S).$$

Indeed, it suffices to take $S = \{g \in G : (v, gv) \in E\}$, where $v \in V$ is arbitrary (see [39]).

We will be interested in Cayley graphs $\Gamma = \text{Cay}(G,S)$ and their subgroup of automorphisms induced by group multiplication as a special and relevant case: given $g \in G$, we can define $f_g : \Gamma \rightarrow \Gamma$ as $f_g(g') = g'g$ and it is easy to check that $f_g \in \text{Aut}(\Gamma)$. Then $G$ acts (as a group, from the left) on $\Gamma$ so that $g \cdot g' = f_g(g') = g'g$ for all $g' \in G$ and $G \hookrightarrow \text{Aut}(\Gamma)$ by identifying $g$ with $f_{g^{-1}}$. In addition, via this identification, $G$ acts transitively on $\Gamma$ as a subgroup of graph automorphisms. In particular, every Cayley graph is transitive.

### 2.6. Partition functions

Given a graph $\Gamma = (V,E)$, let’s consider $\lambda : V \rightarrow \mathbb{R}_{>0}$, an **activity function**. We will say that the pair $(\Gamma, \lambda)$ is a **hardcore model**. We will say that a hardcore model $(\Gamma, \lambda)$ is **finite** if $\Gamma$ is finite. If $U \subseteq V$ is a finite subset, a fact that we denote by $U \Subset V$, and $x \in X(\Gamma)$ is an independent set, we define the $\lambda$-**weight** of $x$ on $U$ as

$$w_\lambda(x,U) := \prod_{v \in U} \lambda(v)^{z(v)}$$

and the $(\Gamma,U,\lambda)$-**partition function** as

$$Z_\Gamma(U,\lambda) := \sum_{x \in X(\Gamma,U)} w_\lambda(x,U) = \sum_{x \in X(\Gamma,U)} \prod_{v \in U} \lambda(v)^{z(v)},$$

where $X(\Gamma,U) := \{x \in X(\Gamma) : x(v) = 0 \text{ for all } v \notin U\}$ is the finite set corresponding to the subset of independent sets of $\Gamma$ **supported** on $U$. It is easy to check that there is an identification between $X(\Gamma,U)$ and $X(\Gamma[U])$. Then, the quantity $Z_\Gamma(U,\lambda)$ corresponds to the summation of independent sets of $\Gamma[U]$ weighted by $\lambda$. In the special case $\lambda = 1$, we have that $Z_\Gamma(U,1) = |X(\Gamma,U)| = |X(\Gamma[U])|$, i.e., the partition function is exactly the number of independent sets supported on $U$. If $(\Gamma,\lambda)$ is finite, we will simply write $Z_\Gamma(\lambda)$ instead of $Z_\Gamma(U,\lambda)$.

**Remark 2.1.** Notice that if $(v,v) \in E$ or $\lambda(v) = 0$, then $Z_\Gamma(U,\lambda) = Z_\Gamma(U \setminus \{v\},\lambda)$; due to this fact, we usually ask $\lambda$ to be strictly positive and that $\Gamma$ is loopless.

### 3. Free energy

Now, suppose that we have an increasing sequence $\{U_n\}_n$ of finite subsets of vertices exhausting $\Gamma$, i.e., $U_n \subseteq U_{n+1}$ and $\bigcup_n U_n = V$. Tentatively, we would like to define the exponential growth rate of $Z_\Gamma(V_n,\lambda)$ as

$$\lim_n \frac{\log Z_\Gamma(U_n,\lambda)}{|U_n|}.$$

In order to guarantee the existence of such limit, we will provide a self-contained argument based on the particular properties of the hardcore model and amenability. The reader that is familiar with this kind of arguments may skip the next part and go straight to Section 3.3.
3.1. Amenability. Let
\[ \mathcal{F}(G) := \{ F \subseteq G : 0 < |F| < \infty \} \]
be the set of finite nonempty subsets of \( G \). Given \( g \in G \) and \( K, F \subseteq G \), we denote \( Fg = \{ hg : h \in F \} \), \( gF = \{ gh : h \in F \} \), \( F^{-1} := \{ g^{-1} : g \in F \} \), and \( KF = \{ h : h \in K, g \in F \} \).

We say that \( \{ F_n \}_n \subseteq \mathcal{F}(G) \) is a right Følner sequence if
\[ \lim_{n} \frac{|F_n g \triangle F_n|}{|F_n|} = 0 \quad \text{for all } g \in G, \]
where \( \triangle \) denotes the symmetric difference. Similarly, \( \{ F_n \}_n \) is a left Følner sequence if
\[ \lim_{n} \frac{|gF_n \triangle F_n|}{|F_n|} = 0 \quad \text{for all } g \in G, \]
and \( \{ F_n \}_n \) is a two-sided Følner if it is both a left and a right Følner sequence. The group \( G \) is said to be amenable if it has a (right or left) Følner sequence. Notice that \( \{ F_n \}_n \) is left Følner if and only if \( \{ F_n^{-1} \}_n \) is right Følner. A Følner sequence \( \{ F_n \}_n \) is a Følner exhaustion if in addition \( F_n \subseteq F_{n+1} \) and \( \bigcup_n F_n = G \). Every countable amenable group has a two-sided Følner exhaustion (see [25, Theorem 4.10]).

3.2. Growth rate of independent sets. Given \( \emptyset \neq U \in V \), define \( \varphi_U : \mathcal{F}(G) \to \mathbb{R} \) as
\[ \varphi_U(F) := \log Z_{\Gamma}(FU, \lambda). \]

From now on, we will assume that \( \lambda : V \to \mathbb{R}_{>0} \) is \( G \)-invariant, this is to say,
\[ \lambda(gv) = \lambda(v) \quad \text{for all } g \in G. \]

In other words, \( \lambda \) is constant along the \( G \)-orbits, so it achieves at most \( |\Gamma/G| \) different values. We denote by \( \lambda_+ \) and \( \lambda_- \) the maximum and minimum among such values, respectively.

Now, let \( W \) be an abstract set, \( M \) a finite subset of \( W \), and \( k \in \mathbb{N} \). We will say that a finite collection \( \mathcal{X} \) of nonempty finite subsets of \( M \), with possible repetitions, is a \( k \)-cover of \( W \) if \( \sum_{K \in \mathcal{X}} I_k \geq k1_M \), where \( I_k : W \to \{0,1\} \) denotes the indicator function of a set \( A \subseteq W \). The following lemma is due to Downarowicz, Frej, and Romagnoli.

**Lemma 3.1 ([13]).** Let \( Y \) be a subset of \( A^n \), where \( A \) is a finite set and \( n \in \mathbb{N} \). Let \( \mathcal{K} \) be a \( k \)-cover of the set of coordinates \( M = \{1, \ldots, n\} \). For \( K \in \mathcal{K} \), let \( Y_K = \{ y_K : y \in Y \} \), where \( y_K \) is the restriction of \( y \) to \( K \). Then,
\[ |Y| \leq \prod_{K \in \mathcal{K}} |Y_K|^{\frac{1}{k}}. \]

Given \( \varphi : \mathcal{F}(G) \to \mathbb{R} \), we will say that \( \varphi \) satisfies **Shearer’s inequality** if
\[ \varphi(F) \leq \frac{1}{k} \sum_{K \in \mathcal{K}} \varphi(K) \]
for all \( F \in \mathcal{F}(G) \) and for all \( k \)-cover \( \mathcal{K} \) of \( F \). We have the following theorem.

**Theorem 3.2 ([25, Theorem 4.48]).** Given a countable amenable group \( G \), suppose that \( \varphi : \mathcal{F}(G) \to \mathbb{R} \) satisfies Shearer’s inequality and \( \varphi(Fg) = \varphi(F) \) for all \( F \in \mathcal{F}(G) \) and \( g \in G \). Then,
\[ \lim_{n} \frac{\varphi(F_n)}{|F_n|} = \inf_{F \in \mathcal{F}(G)} \frac{\varphi(F)}{|F|} \]
for any Følner sequence \( \{ F_n \}_n \).
Considering the two previous results, we obtain the next lemma.

**Lemma 3.3.** Given a fundamental domain \( U_0 \) of \( G \subset \Gamma \) and \( \lambda : V \to \mathbb{Q}_{>0} \) such that \( \lambda(v) = \frac{p_v}{q_v} \) with \( p_v, q_v \in \mathbb{N} \) for all \( v \in V \), we have that, for any Følner sequence \( \{F_n\}_n \),

\[
\lim_{n} \frac{\phi(F_n)}{|F_n|} = \inf_{F \in \mathcal{F}(G)} \frac{\phi(F)}{|F|},
\]

where \( \phi : \mathcal{F}(G) \to \mathbb{R} \) is given by \( \phi(F) = \log Z_\Gamma(FU_0, \lambda) + |F| \sum_{v \in U_0} q_v \).

**Proof.** Given \( F \in \mathcal{F}(G) \) and \( k \in \mathbb{N} \), let \( \mathcal{K} \) be a \( k \)-cover of \( F \). Notice that

\[
Z_\Gamma(FU_0, \lambda) = \sum_{x \in X(F, FU_0)} \prod_{v \in FU_0} \left( \frac{p_v}{q_v} \right)^{\lambda(v)} = \frac{1}{|FU_0|^k} \sum_{x \in X(F, FU_0)} \prod_{v \in FU_0} p_v^{\lambda(v)} q_v^{1-\lambda(v)}.
\]

Consider \( q := \max_v q_v, p := \max_v p_v, A := \{-q, \ldots, -1\} \cup \{1, \ldots, p\} \), and

\[
Y := \left\{ y \in A^{FU_0} : -q \leq y(v) \leq p \text{ and, for all } (v, v') \in E(\Gamma), y(v) \in \{1, \ldots, p\} \implies y(v') \in \{-q, \ldots, -1\} \right\}.
\]

Notice that

\[
|Y| = \sum_{x \in X(F, FU_0)} \prod_{v \in FU_0} p_v^{\lambda(v)} q_v^{1-\lambda(v)}.
\]

Therefore, by Lemma 3.1, and noticing that \( |Y_{KU_0}| \leq Z_\Gamma(KU_0, \lambda) \), we have that

\[
\prod_{v \in FU_0} q_v \cdot Z_\Gamma(FU_0, \lambda) = |Y| \leq \prod_{K \in \mathcal{K}} |Y_{KU_0}| \leq \prod_{K \in \mathcal{K}} \left( \prod_{v \in KU_0} q_v \cdot Z_\Gamma(KU_0, \lambda) \right)^{\frac{1}{k}},
\]

where we use that \( \{KU_0 : K \in \mathcal{K} \} \) is a \( k \)-cover of \( FU_0 \). Therefore, by \( G \)-invariance of \( \lambda \),

\[
\phi(F) = \log Z_\Gamma(FU_0, \lambda) + |F| \sum_{v \in U_0} q_v \leq \frac{1}{k} \sum_{K \in \mathcal{K}} \left( \log Z_\Gamma(KU_0, \lambda) + |K| \sum_{v \in U_0} q_v \right) \leq \frac{1}{k} \sum_{K \in \mathcal{K}} \phi(K),
\]

so \( \phi \) satisfies Shearer’s inequality. On the other hand, by \( G \)-invariance of \( X(\Gamma) \) and \( \lambda \), it follows that \( \phi(Fg) = \phi(F) \) for all \( F \in \mathcal{F}(G) \) and \( g \in G \). Therefore, by Theorem 3.2, we conclude. \( \square \)

**Proposition 3.4.** Given a fundamental domain \( U_0 \) of \( G \subset \Gamma \), we have that

\[
\lim_{n} \frac{\log Z_\Gamma(F_nU_0, \lambda)}{|F_n|} = \inf_{F \in \mathcal{F}(G)} \frac{\log Z_\Gamma(FU_0, \lambda)}{|F|},
\]

for any Følner sequence \( \{F_n\}_n \).
Proof. First, suppose that \( \lambda \) only takes rational values, i.e., \( \lambda : V \to \mathbb{Q}_{>0} \) so that \( \lambda(v) = \frac{p_v}{q_v} \) for all \( v \in V \). By Lemma 3.3, for \( \varphi(F) = \log Z_{\Gamma}(FU_0, \lambda) + |F| \sum_{v \in U_0} q_v \), we have that

\[
\lim_{n} \frac{\log Z_{\Gamma}(F_n U_0, \lambda)}{|F_n|} + \sum_{v \in U_0} q_v = \lim_{n} \frac{\varphi(F_n)}{|F_n|}
\]

\[
= \inf_{F \in \mathcal{F}(G)} \frac{\varphi(F)}{|F|}
\]

\[
= \inf_{F \in \mathcal{F}(G)} \frac{\log Z_{\Gamma}(FU_0, \lambda)}{|F|} + \sum_{v \in U_0} q_v,
\]

and, after cancelling out \( \sum_{v \in U_0} q_v \), we obtain that

\[
\lim_{n} \frac{\log Z_{\Gamma}(F_n U_0, \lambda)}{|F_n|} = \inf_{F \in \mathcal{F}(G)} \frac{\log Z_{\Gamma}(FU_0, \lambda)}{|F|}.
\]

Now, given a general \( \lambda \), we can always approximate it by some \( G \)-invariant \( \tilde{\lambda} : V \to \mathbb{Q}_{>0} \) arbitrarily close in the supremum norm. Given \( \varepsilon > 0 \), pick such \( \tilde{\lambda} \) so that \( \lambda(v) \leq \tilde{\lambda}(v) \leq (1 + \varepsilon)\tilde{\lambda}(v) \) for every \( v \). Then,

\[
\log Z_{\Gamma}(FU_0, \tilde{\lambda}) \leq \log Z_{\Gamma}(FU_0, \lambda)
\]

\[
\leq \log Z_{\Gamma}(FU_0, (1 + \varepsilon)\tilde{\lambda})
\]

\[
\leq |FU_0| \log (1 + \varepsilon) + \log Z_{\Gamma}(FU_0, \tilde{\lambda}),
\]

so,

\[
\frac{\log Z_{\Gamma}(FU_0, \tilde{\lambda})}{|F|} \leq \frac{\log Z_{\Gamma}(FU_0, \lambda)}{|F|} \leq |U_0| \log (1 + \varepsilon) + \frac{\log Z_{\Gamma}(FU_0, \tilde{\lambda})}{|F|}.
\]

Therefore,

\[
\liminf_{n} \frac{\log Z_{\Gamma}(F_n U_0, \lambda)}{|F_n|} \geq \inf_{F \in \mathcal{F}(G)} \frac{\log Z_{\Gamma}(FU_0, \tilde{\lambda})}{|F|}
\]

\[
= \inf_{F \in \mathcal{F}(G)} \frac{\log Z_{\Gamma}(F_n U_0, \tilde{\lambda})}{|F_n|}
\]

\[
\geq \limsup_{n} \frac{\log Z_{\Gamma}(F_n U_0, \lambda)}{|F_n|} - |U_0| \log (1 + \varepsilon),
\]

and since \( \varepsilon \) was arbitrary, we conclude. \( \square \)

In order to fully characterize \( \lim_{n} \frac{\log Z_{\Gamma}(U_n, \lambda)}{|F_n|} \), we have the following lemma.

**Lemma 3.5.** Let \( \{F_n\}_n \) be \( \text{Følner sequence} \) and \( U_0 \) a fundamental domain. Then, for any \( \text{Følner sequence} \ \{F_n\}_n \),

\[
\lim_{n} \frac{|F_n U_0|}{|F_n|} = \sum_{v \in U_0} |\text{Stab}(v)|^{-1}.
\]

**Proof.** First, pick \( v \in U_0 \). Since \( \text{Stab}(v) \) is finite, we have that \( \lim_{n} \frac{|F_n \text{Stab}(v)|}{|F_n|} = 1 \) due to amenability. On the other hand, \( F_n \text{Stab}(v) v = F_n v \) and for each \( v' \in F_n v \), there exist exactly \( |\text{Stab}(v)| \) different elements \( g \in F_n \text{Stab}(v) \) such that \( g v = v' \). In other words,

\[
|F_n \text{Stab}(v)| = |F_n v||\text{Stab}(v)|,
\]
so,
\[
\lim_{n} \frac{|F_n v|}{|F_n|} = \lim_{n} \frac{|F_n v|}{|F_n||\text{Stab}(v)|} = |\text{Stab}(v)|^{-1}.
\]

Therefore,
\[
\lim_{n} \frac{|F_n U_0|}{|F_n|} = \sum_{v \in U_0} \lim_{n} \frac{|F_n v|}{|F_n|} = \sum_{v \in U_0} |\text{Stab}(v)|^{-1}.
\]

\[\square\]

Now, given a fundamental domain \(U_0\), define
\[
f_G(\Gamma, U_0, \lambda) := \inf_{F \in \mathcal{F}(G)} \frac{\log Z_F(FU_0, \lambda)}{|FU_0|}.
\]
which, by Proposition 3.4 and Lemma 3.5, is equal to
\[
\sum_{v \in U_0} |\text{Stab}(v)|^{-1} \lim_{n} \frac{\log Z_F(F_n U_0, \lambda)}{|F_n|}
\]
for any Følner sequence \(\{F_n\}_n\) and, in particular, for any Følner exhaustion. Notice that, since \(GU_0 = V\), the sequence \(\{U_n\}_n\) defined as \(U_n = F_n U_0\) is an exhaustion of \(V\) in the sense that we were looking for. Now we will see that \(f_G(\Gamma, U_0, \lambda)\) is independent of \(U_0\).

**Proposition 3.6.** Given two fundamental domains \(U_0\) and \(U'_0\) of \(G \subset \Gamma\), we have that
\[
f_G(\Gamma, U_0, \lambda) = f_G(\Gamma, U'_0, \lambda).
\]

**Proof.** Since \(V = GU_0 = GU'_0\), there must exist \(K, K' \in \mathcal{F}(G)\) such that \(U'_0 \subseteq KU_0\) and \(U_0 \subseteq K'U'_0\). Then, for every \(F \in \mathcal{F}(G),\)
\[
F(U_0 \triangle U'_0) = (FU_0 \setminus FU'_0) \cup (FU'_0 \setminus FU_0)
\]
\[
\subseteq (FK'U'_0 \setminus FU'_0) \cup (FKU_0 \setminus FU_0)
\]
\[
= (FK' \setminus F)U'_0 \cup (FK \setminus F)U_0.
\]
Therefore, \(|FU_0 \triangle U'_0| \leq |FK' \setminus F| |U'_0| + |FK \setminus F| |U_0|\). Now, notice that for \(U, U' \subseteq V\), we always have that

1. \(Z_F(U \cup U', \lambda) \leq Z_F(U, \lambda) \cdot Z_F(U', \lambda),\) provided \(U \cap U' = \emptyset;\)
2. \(Z_F(U, \lambda) \leq Z_F(U', \lambda),\) provided \(U \subseteq U';\)
3. \(Z_F(U, \lambda) \leq (2\max\{1, \lambda_+\})^{|U'|}.,\)

Therefore,
\[
\log Z_F(FU_0, \lambda) \leq \log Z_F(FU_0 \cap FU'_0, \lambda) + \log Z_F(FU_0 \setminus FU'_0, \lambda)
\]
\[
\leq \log Z_F(FU'_0, \lambda) + \log Z_F(FU_0 \triangle FU'_0, \lambda)
\]
\[
\leq \log Z_F(FU'_0, \lambda) + |FU_0 \triangle FU'_0| \log(2\max\{1, \lambda_+\})
\]
\[
\leq \log Z_F(FU'_0, \lambda) + (|FK' \setminus F| |U'_0| + |FK \setminus F| |U_0|) \log(2\max\{1, \lambda_+\}).
\]
Finally, since \(|U_0| = |U_0'|\) and \(|FU_0| = |F|U_0|\), it follows by amenability that
\[
f_G(\Gamma, U_0, \lambda) = \lim_n \frac{\log Z_\Gamma(F_nU_0, \lambda)}{|F_nU_0|} \leq \lim_n \frac{\log Z_\Gamma(F_nU'_0, \lambda)}{|F_nU'_0|} + \lim_n \left(\frac{|F_nK' \setminus F_n|}{|F_n|} + \frac{|F_nK \setminus F_n|}{|F_n|}\right) \log(2\max\{1, \lambda_+\})
\]
and by symmetry of the argument, we conclude. \(\square\)

Then, we can consistently define the **Gibbs\( (\Gamma, \lambda)\)-free energy according to\( G\)** as
\[
f_G(\Gamma, \lambda) := f_G(\Gamma, U_0, \lambda),
\]
where \(U_0\) is an arbitrary fundamental domain of \(G \rtimes \Gamma\). In addition, it is easy to see that if \(G_1\) and \(G_2\) act almost transitively on \(\Gamma\) and the \(G_1\)-orbits and \(G_2\)-orbits coincide, i.e., \(G_1v = G_2v\) for all \(v \in V\), then
\[
f_{G_1}(\Gamma, \lambda) = f_{G_2}(\Gamma, \lambda).
\]

In particular, we have that \(f_G(\Gamma, \lambda)\) is equal for all \(G\) acting transitively on \(\Gamma\). Then, if there exists \(G \leq \text{Aut}(\Gamma)\) acting transitively on \(\Gamma\), it makes sense to define the **Gibbs\( (\Gamma, \lambda)\)-free energy** as
\[
f(\Gamma, \lambda) := \inf_{\emptyset \neq U \in V} \frac{\log Z_\Gamma(U, \lambda)}{|U|},
\]
which is a quantity that only depends on the graph \(\Gamma\) and the activity function \(\lambda\).

**Remark 3.1.** In the almost transitive case, Equation (1) does not necessarily hold: consider the graph \(\Gamma\) obtained by taking the disjoint union of \(\Gamma_1 = \text{Cay}(\mathbb{Z}, \emptyset)\) and \(\Gamma_2 = \text{Cay}(\mathbb{Z}, \{1, -1\})\) and the constant activity function \(\lambda = 1\). Then, \(f_{\mathbb{Z}}(\Gamma_1, 1) = \log 2\) and \(f_{\mathbb{Z}}(\Gamma_2, 1) = \log \left(\frac{1 + \sqrt{5}}{2}\right)\), so
\[
f_{\mathbb{Z}}(\Gamma_1, 1) = \frac{1}{2}(f_{\mathbb{Z}}(\Gamma_1, 1) + f_{\mathbb{Z}}(\Gamma_2, 1)) > f_{\mathbb{Z}}(\Gamma_2, 1) \geq \inf_{\emptyset \neq U \in V} \frac{\log Z_\Gamma(U, 1)}{|U|}.
\]
The value of \(f_{\mathbb{Z}}(\Gamma_2, 1)\) corresponds to the topological entropy of the golden mean shift (see [26, Example 4.1.4] and Section 9).

The main theme of this paper will be to explore our ability to compute \(f_G(\Gamma, \lambda)\). The next subsection establishes that, without loss of generality for our purposes, we can assume that \(G \rtimes \Gamma\) is free.

### 3.3. From almost free to free
We proceed to show how to represent \(f_G(\Gamma, \lambda)\) for an almost free action \(G \rtimes \Gamma\) as \(f_G(\Gamma_b, \lambda_b)\) for a free action \(G \rtimes \Gamma_b\), where \(\Gamma_b\) and \(\lambda_b\) are some suitable auxiliary graph and activity function, respectively.

Given a graph \(\Gamma\) and \(b : V(\Gamma) \to \mathbb{N}\), let \(\Gamma_b\) be the new graph obtained by setting
\[
V(\Gamma_b) = \{(v, i) : v \in V(\Gamma), 1 \leq i \leq b(v)\},
\]
and
\[
E(\Gamma_b) = \bigcup_{v \in V(\Gamma)} \{(v, i, (v, j))\} \cup \bigcup_{(v, i) \in E(\Gamma)} \{(v, i, (v', j)) : 1 \leq i \leq b(v') \land i \neq j \land j \leq b(v')\}.
\]
In simple words, for each \( v \in V(\Gamma) \) there are \( b(v) \) copies in \( V(\Gamma_b) \) such that

1. the \( b(v) \) copies of \( v \) form a clique in \( \Gamma_b \);
2. for \( (v, v') \in E(\Gamma) \), each copy of \( v \) is connected to all copies of \( v' \) in \( \Gamma_b \).

The idea is to define \( \lambda_b : V(\Gamma_b) \to \mathbb{N} \) so that for every \( U \subseteq V(\Gamma) \) we have that

\[
Z_{\Gamma}(U, \lambda) = Z_{\Gamma_b}(U_b, \lambda_b),
\]

where \( U_b \) is the set of all copies of vertices in \( U \). Notice that each independent set in \( \Gamma_b[U_b] \) can be naturally identified with a unique independent set in \( \Gamma[U] \): for \( x' \in X(\Gamma_b) \), we can define \( x \in X(\Gamma) \) as \( x(v) = 1 \) if and only if there exists \( 1 \leq i \leq b(v) \) so that \( x'(v,i) = 1 \). Conversely, if \( \Gamma \) is finite, each independent set \( x \in X(\Gamma) \) can be identified with \( \prod_{v \in \Gamma} b(v)^{x(v)} \) copies in \( X(\Gamma_b) \). Therefore,

\[
Z_{\Gamma_b}(U_b, \lambda_b) = \sum_{x' \in X(\Gamma_b[U_b])} \prod_{v \in U_b} \lambda_b((v,i))^{x'(v,i)} \\
= \sum_{x \in X(\Gamma[U])} \prod_{v \in U} b(v)^{x(v)} \prod_{v \in U} \lambda_b(v)^{x(v)} \\
= \sum_{x \in X(\Gamma[U])} \prod_{v \in U} (b(v) \lambda_b(v))^{x(v)} \\
= Z_{\Gamma}(U, \lambda),
\]

where the last equality holds if we take \( \lambda_b \) so that \( (v,i) \mapsto \lambda_b(v,i) = \lambda(v)/b(v) \). Now, given an almost free action \( G \actson \Gamma \), we can define a free action \( G \actson \Gamma_b \) as follows. Take \( b(v) := |\text{Stab}_G(v)| \) and, for \( v \in V(\Gamma) \), label the elements of \( \text{Stab}_G(v) \) as \( \{g_{v,1}, \ldots, g_{v,b(v)}\} \). Then, set \( g(v,i) := (gv, gi) \), where

\[
g_{i} := \begin{cases} 
  i & \text{if } g \notin \text{Stab}_G(v), \\
  j & \text{if } g \in \text{Stab}_G(v), g = g_{v,k}, \text{ and } g_{v,k}g_{v,i} = g_{v,j}.
\end{cases}
\]

Clearly, \( G \actson \Gamma_b \) is free: if \( g(v,i) = (v,i) \), then \( gv = v \) and \( gi = i \). Since \( gv = v \), then \( g \in \text{Stab}_G(v) \), so \( g = g_{v,k} \) for some \( 1 \leq k \leq b(v) \). Since \( gi = i \), then \( g_{v,k}g_{v,i} = g_{v,j} \), so \( g_{v,k} = 1_G \).

Now, if \( G \actson \Gamma \) is almost transitive, then \( G \actson \Gamma_b \) is almost transitive, too, with \( |\Gamma_b/G| = |\Gamma/G| \). Given \( v' \in Gv \), we claim that \( (v', j) \in G(v,i) \) for any \( 1 \leq i, j \leq b(v) = b(v') \). First, since \( G \actson \Gamma \) is almost transitive, we can always find \( g_1 \in G \) such that \( g_1(v,i) = (v',i) \). In addition, we can find \( g_2 \in G \) such that \( g_2(v',i) = (v',j) \). In fact, it suffices to take \( g_2 = g_{v',j}g_{v,i}^{-1} \). Then, we have that \( g_2g_1(v,i) = (v',j) \), so \( G(v,i) = Gv \times \{1, \ldots, b(v)\} \). Moreover, if \( U_0 \) is a fundamental domain of \( G \actson \Gamma \), then \( U_0 \times \{1\} \) is a fundamental domain of \( G \actson \Gamma_b \).

4. Gibbs measures

Given a graph \( \Gamma = (V, E) \), consider the set \( \{0, 1\}^V \) endowed with the product topology and the set \( X(\Gamma) \), with the subspace topology. The set of independent sets \( X(\Gamma) \) is a compact and metrizable space. A base for the topology is given by the **cylinder sets**

\[
[x_U] := \{x' \in X(\Gamma) : x'_U = x_U\}
\]

for \( U \subseteq V \) and \( x \in X(\Gamma) \), where \( x_U \) denotes the *restriction* of \( x \) from \( V \) to \( U \). If \( U \) is a singleton \( \{v\} \), we will omit the brackets and simply write \( x_v \) and the same convention...
will hold in analogue instances. Given $W \subseteq V$, we denote by $\mathcal{B}_W$ the smallest $\sigma$-algebra generated by
\[ \{ [x_U] : U \subseteq W, x \in X(\Gamma) \}, \]
and by $\mathcal{B}_\Gamma$ the Borel $\sigma$-algebra, which corresponds to $\mathcal{B}_V$.

Let $\mathcal{M}(X(\Gamma))$ be the set of Borel probability measures $\mathbb{P}$ on $X(\Gamma)$. We say that $\mathbb{P}$ is $G$-invariant if $\mathbb{P}(A) = \mathbb{P}(g \cdot A)$ for all $A \in \mathcal{B}_\Gamma$ and $g \in G$, and $G$-ergodic if $g \cdot A = A$ for all $g \in G$ implies that $\mathbb{P}(A) \in \{0, 1\}$. We will denote by $\mathcal{M}_G(X(\Gamma))$ and $\mathcal{M}_G^{\text{erg}}(X(\Gamma))$ the set of $G$-invariant and the subset of $G$-ergodic measures that are $G$-invariant, respectively.

For $\mathbb{P} \in \mathcal{M}(X(\Gamma))$, define the support of $\mathbb{P}$ as
\[ \text{supp}(\mathbb{P}) := \{ x \in X(\Gamma) : \mathbb{P}([x_U]) > 0 \text{ for all } U \subseteq V \}. \]

Given $\emptyset \neq U \subseteq V$ and $y \in X(\Gamma)$, we define $\pi^y_U$ to be the probability distribution on $X(\Gamma, U)$ given by
\[ \pi^y_U(x) := w^y_\lambda(x, U)z^y_{\lambda}(U, \lambda)^{-1}, \]
where $w^y_\lambda(x, U) = w_\lambda(x, U)\mathbb{I}_{[y_U]}(x)$ and $z^y_{\lambda}(U, \lambda) = \sum_x w^y_\lambda(x, U)$. In other words, to each independent set $x$ supported on $U$, we associate a probability proportional to its $\lambda$-weight over $U$, $\prod_{v \in U} \lambda(v)^{x(v)}$, provided $x_U$ is compatible with $y_U$, in the sense that the element $z \in \{0, 1\}^V$ such that $z_U = x_U$ and $z_{U^c} = y_{U^c}$ is an independent set.

Now, given an activity function $\lambda : V \to \mathbb{R}_{>0}$, consider the hardcore model $(\Gamma, \lambda)$ and the collection $\pi_{(\Gamma, \lambda)} = \{ \pi^y_U : U \subseteq V, y \in X(\Gamma) \}$. We call $\pi_{(\Gamma, \lambda)}$ the Gibbs $(\Gamma, \lambda)$-specification. A measure $\mathbb{P} \in \mathcal{M}(X(\Gamma))$ is called a Gibbs measure (for $(\Gamma, \lambda)$) if for all $U \subseteq V$, $U' \subseteq U$, and $x \in X(\Gamma)$,
\[ \mathbb{P}([x_U]|\mathcal{B}_{U'}) (y) = \pi^y_U([x_U]) \quad \mathbb{P}\text{-a.s. in } y, \]
where $\pi^y_U([x_U])$ denotes the marginalization
\[ \pi^y_U([x_U]) = \sum_{x' \in X(\Gamma[U])|x'_U = x_U} \pi^y_{x'}(x') \]
and $\mathbb{P}(A|\mathcal{B}_U) = \mathbb{E}_\mathbb{P}(\mathbb{I}_A|\mathcal{B}_U)$ for $A \in \mathcal{B}_\Gamma$. We denote by $\mathcal{M}_G(\Gamma, \lambda)$ the set of Gibbs measures for $(\Gamma, \lambda)$.

An important question in statistical physics is whether the set of Gibbs measures is empty or not, and if not, whether there is a unique or multiple Gibbs measures [16].

4.1. The locally finite case. The model described in [16, Example 4.16] can be understood as an attempt to formalize the idea of a system where there is a single particle 1 (uniformly distributed) or none, i.e., 0 everywhere. There, it is proven that such model cannot be represented as a Gibbs measure. This example can be also viewed as a hardcore model in a countable graph that is complete (i.e., there is an edge between any pair of different vertices) and, in particular, in a non-locally finite graph. In other words, there exist examples of non-locally finite graphs $\Gamma$ such that the $(\Gamma, \lambda)$-specification $\pi_{(\Gamma, \lambda)}$ has no Gibbs measure.

From now on, we will always assume that $\Gamma$ is locally finite. In this case, the existence of Gibbs measures is guaranteed (see [9, 12]) and, moreover, Gibbs measures must be a Markov random field that is fully supported.

Indeed, it can be checked that $\pi_{(\Gamma, \lambda)}$ is an example of a Markovian specification (see [16, Example 8.24]). In this case, any Gibbs measure $\mathbb{P} \in \mathcal{M}_G(\Gamma, \lambda)$ satisfies the following local Markov property:
\[ \mathbb{P}([x_U]|\mathcal{B}_{U'}) (y) = \mathbb{P}([x_U]|\mathcal{B}_U) (y) \quad \mathbb{P}\text{-a.s. in } y, \]
for any $U \subseteq V$ and $x \in X(\Gamma)$. In other words, $\mathbb{P}$ is a Markov random field, so any event supported on a finite set conditioned to a specific value on its boundary is independent of events supported on the complement.

In addition, it can be checked that any such $\mathbb{P}$ must be fully supported, i.e., supp$(\mathbb{P}) = X(\Gamma)$. Indeed, it suffices to check that $X(\Gamma) \subseteq \text{supp}(\mathbb{P})$; the other direction follows directly from the definition of $\pi_{\Gamma, \lambda}$ and Gibbs measures. Now, given $x \in X(\Gamma)$ and $U \subseteq V$, we would like to check that $\mathbb{P}([x_U]) > 0$. To prove this, observe that given $x \in X(\Gamma)$, we have that $z \in \{0, 1\}^V$ defined as $z_U = x_U, z_{\partial U} \equiv 0$, and $z_{\partial V} = y_{\partial V}$, always belongs to $X(\Gamma)$ for any $y \in X(\Gamma)$, where $W = U \cup \partial U$. In particular, $\pi^*_W(z) > 0$ for any $y \in X(\Gamma)$. Then, considering that $\partial(W^c)$ is finite,

$$\mathbb{P}([x_U]) \geq \mathbb{P}([z_W]) = \sum_{y \in X(\Gamma), aw: \mathbb{P}([y_{\partial w}]) > 0} \mathbb{P}([y_{\partial w}])\mathbb{P}([y_{\partial w}]) = \sum_{y \in X(\Gamma), aw: \mathbb{P}([y_{\partial w}]) > 0} \pi^*_W(z)\mathbb{P}([y_{\partial w}]) \geq \pi^*_W(z)\mathbb{P}([y_{\partial w}]) > 0,$$

since $\mathbb{P}$ is a probability measure and there must exist $y^* \in X(\Gamma)$ such that $\mathbb{P}([y^*_{\partial w}]) > 0$. In other words, $X(\Gamma)$ satisfies the property (D*) introduced in [38, 1.14 Remark], which guarantees full support.

4.2. Spatial mixing and uniqueness. Given a Gibbs $(\Gamma, \lambda)$-specification $\pi_{\Gamma, \lambda}$, we define two spatial mixing properties fundamental to this work.

**Definition 4.1.** We say that a hardcore model $(\Gamma, \lambda)$ exhibits strong spatial mixing (SSM) if there exists a decay rate function $\delta : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $\lim_{\ell \to \infty} \delta(\ell) = 0$ and for all $U \subseteq V, v \in U$, and $y, z \in X(\Gamma)$,

$$|\pi^*_U([0^*]) - \pi^*_U([0^*])| \leq \delta(\text{dist}_y(v, D_U(y, z))),$$

where $[0^*]$ denotes the event that the vertex $v$ takes the value 0 and $D_U(y, z) := \{v' \in U^c : y(v') \neq z(v')\}$.

This definition is equivalent (see [32, Lemma 2.3]) to the —a priori— stronger following property: for all $U' \subseteq U \subseteq V$ and $x, y, z \in X(\Gamma)$,

$$|\pi^*_U([x_{U'}]) - \pi^*_U([x_{U'}])| \leq |U'|\delta(\text{dist}_y(U', D_U(y, z))).$$

Similarly, we say that $(\Gamma, \lambda)$ exhibits weak spatial mixing (WSM) if for all $U' \subseteq U \subseteq V$ and $x, y, z \in X(\Gamma)$,

$$|\pi^*_U([x_{U'}]) - \pi^*_U([x_{U'}])| \leq |U'|\delta(\text{dist}_y(U', U^c)).$$

Clearly, SSM implies WSM. Moreover, it is well-known that, in this context, WSM (and therefore, SSM) implies uniqueness of Gibbs measures [48]. In other words, $\mathcal{M}_{\text{Gibbs}}(\Gamma, \lambda) = \{\mathbb{P}_{\Gamma, \lambda}\}$, where $\mathbb{P}_{\Gamma, \lambda}$ denotes the unique Gibbs measure for $(\Gamma, \lambda)$. In this case, $\mathbb{P}_{\Gamma, \lambda}$ is always $\text{Aut}(\Gamma)$-invariant.

We say that $(\Gamma, \lambda)$ exhibits exponential SSM (resp. exponential WSM) if there exist constants $C, \alpha > 0$ such that $\pi_{\Gamma, \lambda}$ exhibits SSM (resp. WSM) with decay rate function $\delta(n) = C \cdot \exp(-\alpha \cdot n)$.

Given $U \subseteq V$, we denote by $\Gamma \setminus U$ the subgraph induced by $V \setminus U$, i.e., $\Gamma[V \setminus U]$. We have the following result due to Gamarnik and Katz.
**Proposition 4.1 ([15, Proposition 1]).** If a hardcore model \((\Gamma, \lambda)\) satisfies SSM, then so does the hardcore model \((\Gamma', \lambda)\) for any subgraph \(\Gamma'\) of \(\Gamma\). The same assertion applies to exponential SSM. Moreover, for every \(U \subseteq V\) and \(v \in V \setminus U\), the following identity holds:

\[
P_{\Gamma, \lambda}([0^v]) = P_{\Gamma \cup U, \lambda}([0^v]),
\]

where \(P_{\Gamma, \lambda}\) and \(P_{\Gamma \cup U, \lambda}\) are the unique Gibbs measures for \((\Gamma, \lambda)\) and \((\Gamma \cup U, \lambda)\), respectively, and \([0^U]\) denotes the event that all the vertices in \(U\) take the value 0. In particular, \(P_{\Gamma, \lambda}([0^v])\) is always well-defined, even if \(U\) is infinite.

**Remark 4.1.** Notice that any event of the form \([x_U]\) can be translated into an event of the form \([0^{U'}]\) for a suitable set \(U'\); it suffices to define \(U' = U \cup \partial\{v \in U : x_U(v) = 1\}\) since, deterministically, every neighbor of a vertex colored 1 must be 0, so Proposition 4.1 still holds for more general events. We also remark that in [15] it is assumed that \(\lambda\) is a constant function. Here we drop this assumption, but it is direct to check that the same proof of [15, Proposition 1] also applies to the more general non-constant case.

### 4.3. Families of hardcore models.

We introduce some notation regarding relevant families of hardcore models.

First, we will denote by \(\mathcal{H}\) the family of hardcore models \((\Gamma, \lambda)\) such that \(\Gamma\) is a countable locally finite graph and \(\lambda\) is any activity function \(\lambda : V(\Gamma) \to \mathbb{R}_{\geq 0}\).

Given a countable group \(G\), we will denote by \(\mathcal{H}_G\) the set of hardcore models \((\Gamma, \lambda)\) in \(\mathcal{H}\) for which \(G\) is isomorphic to some subgroup of \(\text{Aut}(\Gamma)\) such that \(G \lhd \Gamma\) is free and almost transitive and \(\lambda : V(\Gamma) \to \mathbb{R}_{\geq 0}\) is a \(G\)-invariant activity function.

Given a positive integer \(\Delta\), we will denote by \(\mathcal{H}_\Delta\) the set of hardcore models \((\Gamma, \lambda)\) in \(\mathcal{H}\) such that \(\Delta(\Gamma) \leq \Delta\). Notice that any hardcore model defined on the \(\Delta\)-regular (infinite) tree \(T_\Delta\) belongs to \(\mathcal{H}_\Delta\).

Given \(\lambda_0 > 0\), we will denote by \(\mathcal{H}(\lambda_0)\) the family of hardcore models \((\Gamma, \lambda)\) in \(\mathcal{H}\) such that \(\lambda_+ \leq \lambda_0\).

We will also combine the notation for these families in the natural way; for example, \(\mathcal{H}_G(\lambda_0)\) will denote the set of hardcore models \((\Gamma, \lambda)\) in \(\mathcal{H}\) such that \(G \lhd \Gamma\) is free and almost transitive, \(\lambda\) is \(G\)-invariant, \(\Delta(\Gamma) \leq \Delta\), and \(\lambda_+ \leq \lambda_0\).

### 5. Trees

Given a graph \(\Gamma\), a **trail** \(w\) in \(\Gamma\) is a finite sequence \(w = (v_1, \ldots, v_n)\) of vertices such that consecutive vertices are adjacent in \(\Gamma\) and the edges \((v_j, v_{j+1})\) involved are not repeated. For a fixed vertex \(v \in V(\Gamma)\), the **tree of self-avoiding walks starting from** \(v\), denoted by \(T_{\text{SAW}}(\Gamma, v)\), is defined as follows:

1. Consider the set \(W_0\) of trails starting from \(v\) that repeat no vertex and the set \(W_1\) of trails that repeat a single vertex exactly once and then stop (i.e., the set of non-backtracking walks that end immediately after performing a cycle). We define \(T_{\text{SAW}}(\Gamma, v)\) to be a rooted tree with root \(\rho = (v)\) such that the set of vertices \(V(T_{\text{SAW}}(\Gamma, v))\) is \(W_0 \cup W_1\) and the set of (undirected) edges \(E(T_{\text{SAW}}(\Gamma, v))\) corresponds to all the pairs \((w, w')\) such that \(w'\) is a one vertex extension of \(w\) or vice versa. In simple words, \(T_{\text{SAW}}(\Gamma, v)\) is a rooted tree that represents all self-avoiding walks in \(\Gamma\) that start from \(v\). It is easy to check that the set of leaves of \(T_{\text{SAW}}(\Gamma, v)\) contains \(W_1\), but they are not necessarily equal (e.g., see vertex \(b\) in Figure 2).

2. For \(u \in V(\Gamma)\), consider an arbitrary ordering of its neighbors \(\partial\{u\} = \{u_1, \ldots, u_d\}\).

Given \(w \in W_1\), we can represent such a walk as a sequence \(w = (v, \ldots, u, u_1, \ldots, u_j, u)\),
with \( u_i, u_j \in \partial \{ u \} \). Notice that \( i \neq j \), since we are not repeating edges. Considering this, we “tag” \( w \) with a 1 (occupied) if \( i < j \) and with a 0 (unoccupied) if \( i > j \).

Given a hardcore model \((\Gamma, \lambda)\), a vertex \( v \in V(\Gamma) \), a subset \( U \subseteq V(\Gamma) \), and an independent set \( x \in X(\Gamma) \), we are interested in computing the marginal probability that \( v \) is unoccupied in \( \Gamma \) given the partial configuration \( x_U \), i.e., \( P_{\Gamma, \lambda}(\square v | x_U) \). Notice that if \((\Gamma, \lambda)\) satisfies SSM (which includes the particular but relevant case of \( \Gamma \) being finite), then such probability is always well defined due to Proposition 4.1, even if \( U \) is infinite.

![Figure 2. A graph \( \Gamma \) and its corresponding (tagged) tree of self-avoiding walks \( T_{SAW}(\Gamma, v) \), where every trial/vertex is represented by the final vertex of the trial in \( \Gamma \) starting from \( v = a \).](image)

To understand better \( P_{\Gamma, \lambda}(\square v | x_U) \), we consider the hardcore model \((T_{SAW}(\Gamma, v), \overline{\lambda})\), where \( \overline{\lambda}(w) = \lambda(u) \) for every trial \( w \) ending in \( u \). In this context, a condition \( x_U \) in \((\Gamma, \overline{\lambda})\) is translated into the condition \( x_U \) in \( T_{SAW}(\Gamma, v) \), whose support is the set \( W(U) \) of trials \( w \) that end in \( u \) for some \( u \in U \), and \( \overline{x}(w) = x(u) \) for all such \( w \). We have the following result from [49], that we adapt to the more general non-constant \( \lambda \) case and we include its proof for completeness.

**Theorem 5.1 ([49, Theorem 3.1]).** For every finite hardcore model \((\Gamma, \lambda)\), every \( v \in V(\Gamma) \), and \( U \subseteq V(\Gamma) \),

\[
P_{\Gamma, \lambda}(\square v | x_U) = P_{T_{SAW}(\Gamma, v), \overline{\lambda}}(\square v | x_U).
\]

**Proof.** Instead of probabilities, we work with the ratios

\[
R_{\Gamma, \lambda}(v, x_U) := \frac{P_{\Gamma, \lambda}(\square v | x_U)}{P_{\Gamma, \lambda}(\square v | x_U)}.
\]

where if \( v \in U \) and \( x_U(v) \) is equal to 1 or 0, we let \( R_{\Gamma, \lambda}(v, x_U) \) to be \( \infty \) or 0, respectively. Notice that

\[
P_{\Gamma, \lambda}(\square v | x_U) = \frac{1}{1 + R_{\Gamma, \lambda}(v, x_U)} \quad \text{and} \quad P_{\Gamma, \lambda}(\square v | x_U) = \frac{R_{\Gamma, \lambda}(v, x_U)}{1 + R_{\Gamma, \lambda}(v, x_U)}.
\]
Given a finite tree $T$ rooted at $\rho$, let’s denote by $\{\rho_1, \ldots, \rho_d\}$ the set of neighbors $\partial\{\rho\}$ of $\rho$ and by $T_i$, for $i = 1, \ldots, d$, the corresponding subtrees starting from $\rho_i$, i.e., $V(T) = \{\rho\} \cup V(T_1) \cup \cdots \cup V(T_d)$. If we have a condition $x_U$ on $U$, we define $U_i = U \cap V(T_i)$ and $x_{U_i} = (x_U)_{|U_i}$. Considering this, we have that

$$R_{T, \lambda}(\rho, x_U) = \frac{\mathbb{P}_{T, \lambda}([1\rho][x_U])}{\mathbb{P}_{T, \lambda}([0\rho][x_U])} = \frac{\lambda(\rho) \cdot Z_T^{x_U}(\lambda)}{Z_T^{x_{\rho}}(\lambda)} \cdot \frac{Z_T^{x_{\rho}}(\lambda)}{Z_T^{x_U}(\lambda)}$$

$$= \lambda(\rho) \cdot \prod_{i=1}^d \frac{Z_{T_i}^{x_{\rho_i}}(\lambda)}{Z_{T_i}^{x_U}(\lambda)}$$

$$= \lambda(\rho) \cdot \prod_{i=1}^d \frac{\mathbb{P}_{T_i, \lambda}([0\rho][x_{U_i}])}{1 + R_{T_i, \lambda}(\rho_i, x_{U_i})},$$

where

$$Z_T^{x_U}(\lambda) := \sum_{y \in X(\Gamma), y \neq x_U} \prod_{v \in V(\Gamma)} \lambda(y)^{e(v)}.$$

Notice that this gives us a linear recursive procedure for computing $R_{T, \lambda}(\rho, x_U)$, and therefore $\mathbb{P}_{T, \lambda}([0\rho][x_U])$, with base cases: $R_{T, \lambda}(\rho, x_U) = 0$ or $+\infty$ if $\rho$ is fixed, and $R_{T, \lambda}(\rho, x_U) = \lambda(\rho)$ if $\rho$ is free and isolated.

Now, consider an arbitrary hardcore model $(\Gamma, \lambda)$ with neighbors $\partial\{v\} = \{u_1, \ldots, u_d\}$. We consider the auxiliary hardcore model $(\Gamma', \lambda')$, where

- $V(\Gamma') = V(\Gamma) \cup \{v_1, \ldots, v_d\}$,
- $E(\Gamma') = E(\Gamma) \cup \{(u_i, v_i)\}_{i=1}^d$,
- $\lambda'(v_i) = \lambda(v)^{1/d}$ for $i = 1, \ldots, d$, and $\lambda'(u) = \lambda(u)$, otherwise.

Notice that

$$R_{\Gamma', \lambda'}(v, x_U) = \frac{\mathbb{P}_{\Gamma', \lambda'}([1v][x_U])}{\mathbb{P}_{\Gamma', \lambda'}([0v][x_U])} = \frac{\mathbb{P}_{\Gamma', \lambda'}([1v_1, \ldots, v_d][x_U])}{\mathbb{P}_{\Gamma', \lambda'}([0v_1, \ldots, v_d][x_U])} = \prod_{i=1}^d \frac{\mathbb{P}_{\Gamma', \lambda'}([0v_1, \ldots, v_{i-1}, \{v_i+1, \ldots, v_d\}, [x_U]])}{\mathbb{P}_{\Gamma', \lambda'}([0v_1, \ldots, v_{i-1}, v_i, \{v_i+1, \ldots, v_d\}, [x_U]])} = \prod_{i=1}^d \frac{\mathbb{P}_{\Gamma', \lambda'}([1v_i][x_U \tau_i])}{\mathbb{P}_{\Gamma', \lambda'}([0v_i][x_U \tau_i])} = \prod_{i=1}^d R_{\Gamma', \lambda'}(v_i, x_U z_i),$$

where $z_i = 0^{v_i+1, v_{i+1}, v_d} \cdot 1_{v_i+1, v_d}$ and $x_U z_i$ is the concatenation of $x_U$ and $z_i$. Now, since $v_i$ is connected only to $u_i$, notice that

$$R_{\Gamma', \lambda'}(v_i, x_U z_i) = \frac{\lambda'(v_i) \cdot Z^{x_U z_i}_{\Gamma', \lambda'}(\lambda')}{{Z^{x_U z_i}_{\Gamma', \lambda'}}(\lambda')} = \frac{\lambda^{1/d}(v)}{1 + R_{\Gamma', \lambda'}(v_i, x_U z_i)}.$$
Therefore,
\[ R_{v,x}(v,x_U) = \prod_{i=1}^{d} \frac{\lambda^{1/d}(v)}{1 + R_{v,x}(u_i,x_U)} = \lambda(v) \cdot \prod_{i=1}^{d} \frac{1}{1 + R_{v,x}(u_i,x_U)}. \]

Notice that the previous recursion can increase the original number of vertices, but the number of free vertices always decrease, so the recursion ends. Then, we have that

(1) \[ R_{T',\lambda}(v,x_U) = \lambda(\rho) \cdot f(R_{T',\lambda}(\rho_1,x_{U_1}),\ldots,R_{T',\lambda}(\rho_d,x_{U_d})) \]

(2) \[ R_{T',\lambda}(v,x_U) = \lambda(v) \cdot f(R_{\Gamma',\lambda}(u_1,x_{U_1}),\ldots,R_{\Gamma',\lambda}(u_d,x_{U_d})), \]

where \( f(r_1,\ldots,r_d) = \prod_{i=1}^{d} \frac{1}{1 + r_i}. \) Now we proceed by induction in the number of free vertices. We can consider the base case where there are no free vertices (besides \( v \)) and the theorem is trivial. Then, if we know that the theorem is true when we have \( n \) free vertices, we prove it for \( n+1 \). Notice that if \( R_{\Gamma',\lambda}(v,x_U) \) involves \( n+1 \) free vertices, then \( R_{\Gamma',\lambda}(v,x_U) \) involves \( n \) free vertices, so by the induction hypothesis,

\[ R_{\Gamma',\lambda}(v,x_U) = R_{\Gamma_{SAW},\lambda}(\rho_v,x_{U'}). \]

Then, noticing that the rooted subtree \( (T_i,\rho_i) \) and the condition \( x_{U'} \) gives exactly the tree of self-avoiding walks of \( \Gamma' \) starting from \( u_i \) under the condition \( x_{U_i} \), we are done.

\[ \]

**Figure 3.** Condition on \( \Gamma \) and its representation on the tree of self-avoiding walks \( T_{SAW}(\Gamma,v) \) for \( v = a \).

**Remark 5.1.** The recursions presented in the proof of Theorem 5.1 give us a recursive procedure to compute the marginal probability of the root \( \rho \) of a tree \( T \) being occupied which requires linear time with respect to the size of the tree. On the other hand, if \( \Gamma \) is such that \( \Delta(\Gamma) \leq \Delta \), then \( T_{SAW}(\Gamma,v) \) is a subtree of \( T_\lambda \) and its size of \( T_{SAW}(\Gamma,v) \) can be (at most) exponential in the size of \( \Gamma \). Since hardcore models are Markov random fields and we are interested in the sensitivity of the root \( \rho \) associated to \( v \), we only need to consider the graph obtained after pruning all the subtrees below \( W(U) \) (see Figure 3).
Before stating the main results concerning hardcore models and strong spatial mixing, we will establish the following bounds.

**Lemma 5.2.** Given a finite hardcore model \((\Gamma, V) \in \mathcal{H}^\Delta\) and \(v \in V\), we have that

\[
0 < \frac{1}{1 + \lambda_+} \leq \mathbb{P}_{\Gamma, \lambda}([0^v]) \leq \frac{(1 + \lambda_+)^\Delta}{\lambda_- + (1 + \lambda_+)^\Delta} < 1
\]

and

\[
0 < \frac{\lambda_-}{\lambda_- + (1 + \lambda_+)^\Delta} \leq \mathbb{P}_{\Gamma, \lambda}([1^v]) \leq \frac{\lambda_+}{1 + \lambda_+} < 1.
\]

**Proof.** Notice that, since \(\mathbb{P}_{\Gamma, \lambda}\) is a Markov random field and a 1 at \(v\) forces 0s in \(\partial\{v\}\),

\[
\mathbb{P}_{\Gamma, \lambda}([1^v]) = \mathbb{P}_{\Gamma, \lambda}([1^v]|[0^{\partial(v)}])\mathbb{P}_{\Gamma, \lambda}([0^{\partial(v)}]) \leq \mathbb{P}_{\Gamma, \lambda}([1^v]|[0^{\partial(v)}]),
\]

so, considering that \(\frac{\lambda}{1 + \lambda}\) is increasing in \(\lambda > 0\), we obtain that

\[
\mathbb{P}_{\Gamma, \lambda}([1^v]) \leq \mathbb{P}_{\Gamma, \lambda}([1^v]|[0^{\partial(v)}]) = \frac{\lambda(v)}{1 + \lambda(v)} \leq \frac{\lambda_+}{1 + \lambda_+} < 1,
\]

and

\[
\mathbb{P}_{\Gamma, \lambda}([0^v]) = 1 - \mathbb{P}_{\Gamma, \lambda}([1^v]) \geq 1 - \frac{\lambda_+}{1 + \lambda_+} = \frac{1}{1 + \lambda_+} > 0.
\]

On the other hand, by Theorem 5.1, without loss of generality, we can suppose that \(\Gamma\) is a tree rooted at \(v\). Then, if \(\Gamma_i\) denotes the \(i\)th subtree of \(\Gamma\) rooted at \(v_i \in \partial\{v\}\),

\[
\frac{\mathbb{P}_{\Gamma, \lambda}([1^v])}{\mathbb{P}_{\Gamma, \lambda}([0^v])} = \lambda(v) \cdot \prod_{i=1}^d \frac{1}{1 + \frac{\mathbb{P}_{(\Gamma_i)}([1^v])}{\mathbb{P}_{(\Gamma_i)}([0^v])}} \geq \lambda_+ \cdot \prod_{i=1}^d \frac{1}{1 + \frac{\lambda_+}{1 + \lambda_+}} \geq \frac{\lambda_-}{(1 + \lambda_+)^\Delta}.
\]

Therefore, since \(\mathbb{P}_{\Gamma, \lambda}([0^v]) = 1 - \mathbb{P}_{\Gamma, \lambda}([1^v])\), we have that

\[
\mathbb{P}_{\Gamma, \lambda}([1^v]) \geq \frac{\lambda_-}{\lambda_- + (1 + \lambda_+)^\Delta} > 0 \quad \text{and} \quad \mathbb{P}_{\Gamma, \lambda}([0^v]) \leq \frac{(1 + \lambda_+)^\Delta}{\lambda_- + (1 + \lambda_+)^\Delta} < 1.
\]

\(\square\)

We define the **critical activity function** \(\lambda_c : [2, +\infty) \to (0, +\infty)\) as

\[
\lambda_c(t) := \frac{(t - 1)^{[t - 1]}}{(t - 2)^t}.
\]

From the works of Kelly [24] and Spitzer [46], we have the following result.

**Proposition 5.3.** For every \(\Delta \in \mathbb{N}\), the hardcore model \((\mathbb{T}_\Delta, \lambda_0)\) exhibits WSM if and only if \(\lambda_0 \leq \lambda_c(\Delta)\). If the inequality is strict, then \((\mathbb{T}_\Delta, \lambda_0)\) exhibits exponential WSM with a decay rate \(\delta\) involving constants that depend on \(\Delta\) and \(\lambda_0\).

We summarize in the following theorem the main results from [49], that relate the correlation decay in \((\mathbb{T}_\Delta, \lambda_0)\) with the correlation decay in \(\mathcal{H}^\Delta(\lambda_0)\). Here again, as in Theorem 5.1, the results in [49] are focused on the constant activity case. However, we can also adapt the results to the non-constant case by considering that the main tool used in [49] to prove them is [49, Theorem 4.1], which is based on hardcore models with non-constant activity functions.

**Theorem 5.4** ([49, Theorem 2.3 and Theorem 2.4]). Fix \(\Delta \in \mathbb{N}\) and \(\lambda_0 > 0\). Then,
(1) If \((T_\Delta, \lambda_0)\) exhibits WSM with decay rate \(\delta\), then \((T_\Delta, \lambda_0)\) exhibits SSM with rate
\[
(1 + \lambda_0)(\lambda_0 + (1 + \lambda_0)\delta)^{-1}\delta;
\]

(2) If \((T_\Delta, \lambda_0)\) exhibits SSM with decay rate \(\delta\), then \((\Gamma, \lambda)\) exhibits SSM with rate \(\delta\) for every \((\Gamma, \lambda) \in \mathcal{H}(\lambda_0)\).

Then, combining Proposition 5.3 and Theorem 5.4, we have that if \(\lambda_0 \leq \lambda_c(\Delta)\), then every hardcore model \((\Gamma, \lambda) \in \mathcal{H}(\lambda_0)\) exhibits SSM with the same decay rate \(\delta\), that would be exponential if the inequality is strict. In addition, observe that if \((\Gamma, \lambda)\) is a hardcore model such that \((T_\text{SAW}(\Gamma, \nu), \overline{\lambda})\) exhibits SSM with decay rate \(\delta\) for every \(\nu \in V(\Gamma)\), then \((\Gamma, \lambda)\) exhibits SSM with decay rate \(\delta\) as well. This follows from Theorem 5.1, since SSM is a property that depends on finitely supported events and the probabilities involved can be translated into probabilities defined on finite hardcore models which at the same time can be translated into events on finite subtrees of \(T_\text{SAW}(\Gamma, \nu)\). Considering this, we have the following theorem, which can be understood as a generalization of Theorem 5.1 to the infinite setting.

**Theorem 5.5.** Given a hardcore model \((\Gamma, \lambda)\) and \(\nu \in V(\Gamma)\) such that \((T_\text{SAW}(\Gamma, \nu), \overline{\lambda})\) exhibits SSM, then for every \(x \in X(\Gamma)\) and \(U \subseteq V(\Gamma)\),
\[
P_{\Gamma, \lambda}([0^0])([x_u]) = p_{T_\text{SAW}(\Gamma, \nu), \overline{\lambda}}([0^0])([\overline{x}]).
\]

**Proof.** Assume that \((T_\text{SAW}(\Gamma, \nu), \overline{\lambda})\) exhibits SSM with decay rate \(\delta\). Then, for every \(\ell \in \mathbb{N}\),
\[
P_{\Gamma, \lambda}([0^0])([x_u]) = \sum_{w \in \{0, 1\}^{\partial B_\ell(x, \nu) \setminus U}} p_{\Gamma, \lambda}([0^0])([x_u w])p_{\Gamma, \lambda}([w])([x_u])
\]
\[
\leq \sum_{w \in \{0, 1\}^{\partial B_\ell(x, \nu) \setminus U}} (p_{\Gamma, \lambda}([0^0])([x_u 0^\partial B_\ell(x, \nu) \setminus U])) + \delta(\ell)p_{\Gamma, \lambda}([w])([x_u])
\]
\[
=p_{\Gamma, \lambda}([0^0])([x_u 0^\partial B_\ell(x, \nu) \setminus U]) + \delta(\ell)
\]
and, similarly,
\[
p_{\Gamma, \lambda}([0^0])([x_u]) \geq p_{\Gamma, \lambda}([0^0])([x_u 0^\partial B_\ell(x, \nu) \setminus U]) - \delta(\ell).
\]
Therefore, since \(\lim_{\ell \to \infty} \delta(\ell) = 0\),
\[
p_{\Gamma, \lambda}([0^0])([x_u]) = \lim_{\ell \to \infty} p_{\Gamma, \lambda}([0^0])([x_u 0^\partial B_\ell(x, \nu) \setminus U]),
\]
and, by the same argument,
\[
p_{(T_\text{SAW}(\Gamma, \nu), \overline{\lambda})}([0^0])([\overline{x}]) = \lim_{\ell \to \infty} p_{(T_\text{SAW}(\Gamma, \nu), \overline{\lambda})}([0^0])([\overline{x} 0^\partial B_{(T_\text{SAW}(\Gamma, \nu), \overline{\lambda})}(\rho, \ell) \setminus W(U))].
\]

Considering this, the Markov random field property, and Proposition 4.1, we have that
\[
p_{\Gamma, \lambda}([0^0])([x_u]) = \lim_{\ell \to \infty} p_{(T_\text{SAW}(\Gamma, \nu), \overline{\lambda})}([0^0])([x_u 0^\partial B_{(T_\text{SAW}(\Gamma, \nu), \overline{\lambda})}(\rho, \ell) \setminus W(U))]
\]
\[
= \lim_{\ell \to \infty} p_{(T_\text{SAW}(\Gamma, \nu), \overline{\lambda})}([0^0])([x_u 0^\partial B_{T_\text{SAW}(\Gamma, \nu)}(\rho, \ell) \setminus W(U))]
\]
\[
= \lim_{\ell \to \infty} p_{(T_\text{SAW}(\Gamma, \nu), \overline{\lambda})}([0^0])([\overline{x} 0^\partial B_{T_\text{SAW}(\Gamma, \nu)}(\rho, \ell) \setminus W(U))]
\]
\[
= p^2_{T_\text{SAW}(\Gamma, \nu), \overline{\lambda}}([0^0])([\overline{x}]).
\]

\(\square\)
Notice that Theorem 5.5 requires that \((T_{\text{SAW}}(\Gamma, v), \lambda)\) exhibits SSM rather than the graph \((\Gamma, \lambda)\), since SSM on \(T_{\text{SAW}}(\Gamma, v)\) may be a stronger condition than SSM on \(\Gamma\). A key fact is that if \((T_\Lambda, \lambda_\Lambda)\) exhibits SSM, then \((T, \lambda_0)\) exhibits SSM for every subtree \(T\) of \(T_\Lambda\). Then, since for every \(\Gamma\) with \(\Delta(\Gamma) \leq \Delta\), we have that \(T_{\text{SAW}}(\Gamma, v)\) is a subtree of \(T_\Lambda\), it follows that \((T_{\text{SAW}}(\Gamma, v), \lambda)\), and therefore \((\Gamma, \lambda_0)\), exhibit SSM. Considering this, we have the following corollary.

**Corollary 5.6.** Fix \(\Delta \in \mathbb{N}\). Then, every \((\Gamma, \lambda) \in \mathcal{H}^\Delta(\lambda_\Lambda(\Delta))\) exhibits SSM and for every \(v \in V(\Gamma), x \in X(\Gamma), \) and \(U \subseteq V(\Gamma), \) 

\[
P_{\Gamma, \lambda}(\{0^\ell\} | [x_U]) = P_{T_{\text{SAW}}(\Gamma, v), \lambda}(\{0^\ell\} | [x_U]).
\]

Since we are ultimately interested in studying the interplay between the SSM property on \(T_{\text{SAW}}(\Gamma, v)\) and \(\Gamma\), we may wonder whether is really necessary to have control over the full \(\Delta\)-regular tree \(T_\Delta\). In [42], it was proven a refinement of this fact by considering the connective constant of the graphs involved.

### 5.1. Connective constant

Following [42], given a family of finite graphs \(\mathcal{G}\), we define the connective constant of \(\mathcal{G}\) as the infimum over all \(\mu > 0\) for which there exist \(a, c > 0\) such that for any \(\Gamma \in \mathcal{G}\) and any \(v \in V(\Gamma)\), it holds that \(\sum_{k=1}^{\ell} N_{\Gamma}(v, k) \leq c \mu^\ell\) for all \(\ell \geq a \log |V(\Gamma)|\), where \(N_{\Gamma}(v, k)\) denotes the number of self-avoiding walks in \(\Gamma\) of length \(k\) starting from \(v\). We denote by \(\mu(\mathcal{G})\) the connective constant of \(\mathcal{G}\). This definition extends the more usual definition of connective constant for a single infinite almost transitive graph \(\Gamma\), which is given by

\[
\mu(\Gamma) := \max_{v \in V(\Gamma)} \lim_{\ell \to \infty} \frac{N_{\Gamma}(v, \ell)}{\ell^\ell}.
\]

Indeed, if \(\Gamma\) is almost transitive, then \(\mu(\Gamma) = \mu(\mathcal{G}(\Gamma))\), where \(\mathcal{G}(\Gamma)\) denotes the family of finite subgraphs of \(\Gamma\). Notice that \(\mu(\Gamma)\) exists due to Fekete’s lemma and that, if \(\Gamma\) is connected, then \(\mu(\Gamma) = \lim_{\ell \to \infty} N_{\Gamma}(v, \ell)^{1/\ell}\) for arbitrary \(v\). Roughly, the connective constant measures the growth rate of the number of self-avoiding walks according to their length or, equivalently, the branching of \(T_{\text{SAW}}(\Gamma, v)\). In general, it is not an easy task to compute \(\mu(\Gamma)\) (e.g., see [14]).

Considering this, we extend the definition of strong spatial mixing to families of graphs as follows: Given a family of graphs \(\mathcal{G}\) and a family of activity functions \(\Lambda = \{\lambda_\Gamma\}_{\Gamma \in \mathcal{G}}\) with \(\lambda_\Gamma : V(\Gamma) \to \mathbb{R}_{>0}\), we say that \((\mathcal{G}, \Lambda)\) satisfies strong spatial mixing if there exists a decay rate function \(\delta : \mathbb{N} \to \mathbb{R}_{>0}\) such that \(\lim_{\ell \to \infty} \delta(\ell) = 0\) and for all \(\Gamma \in \mathcal{G}\), for all \(U \subseteq V(\Gamma), v \in U,\) and \(y, z \in X(\Gamma),\)

\[
|\pi_{\Gamma, U}^v([0^\ell]) - \pi_{\Gamma, U}^z([0^\ell])| \leq \delta(\text{dist}_\Gamma(v, D_U(y, z))),
\]

where \(\pi_{\Gamma, U}^v\) denotes the specification element corresponding to the hardcore model \((\Gamma, \lambda_\Gamma)\). We translate into this language the following result from [42].

**Theorem 5.7** ([42]). Let \(\mathcal{G}\) be a family of almost transitive locally finite graphs and \(\Lambda = \{\lambda_\Gamma\}_{\Gamma \in \mathcal{G}}\) a set of activity functions such that

\[
\sup_{\Gamma \in \mathcal{G}} \lambda_\Gamma < \lambda_\Lambda(\mu(\mathcal{G}) + 1).
\]

Then, \((\mathcal{G}, \Lambda)\) exhibits exponential SSM.

Notice that if a graph has maximum degree \(\Delta\), then \(\mu(\Gamma) \leq \Delta - 1\). In addition, observe that \(N_{\Gamma}(v, \ell) = N_{\text{SAW}}(\Gamma, v)(\rho, \ell)\). We have the following corollary.
Corollary 5.8. If $(\Gamma, \lambda)$ is a hardcore model such that
\[ \lambda_+ < \lambda_c(\mu(\Gamma) + 1), \]
then $(T_{SAW}(\Gamma, v), \lambda)$ exhibits (exponential) SSM for every $v \in V(\Gamma)$. In particular, $(\Gamma, \lambda)$ exhibits (exponential) SSM and for every $v \in V(\Gamma)$, $x \in X(\Gamma)$, and $U \subseteq V(\Gamma)$,
\[ P_{\Gamma, \lambda}(\{0^p\}|\{x_U\}) = P_{T_{SAW}(\Gamma, v), \lambda}(\{0^p\}|\{x_U\}). \]

6. Orders

We have already explored the main combinatorial and measure-theoretical tools that we require to establish the main results. In this section, we present some concepts of a more group-theoretical nature, namely, our ability to order a given group.

6.1. Orderable groups. Let $\prec$ be a strict total order on $G$. We say that $\prec$ is an invariant (right) order if, for all $h_1, h_2, g \in G$,
\[ h_1 \prec h_2 \implies h_1 g \prec h_2 g. \]

If $\prec$ is an invariant order, then the associated algebraic past $\Phi_\prec := \{g \in G : g \prec 1_G\}$ is a semigroup such that
\[ G = \Phi_\prec \cup \{1_G\} \cup \Phi_\prec^{-1}. \]

Notice that $h \prec g \iff hg^{-1} \in \Phi_\prec$, so $\Phi_\prec$ fully determines $\prec$ and vice versa.

A group $G$ will be called orderable if it admits an invariant order and almost orderable if there exists an orderable subgroup $H \leq G$ such that $[G : H] < \infty$. Notice that if $G$ is almost orderable and $G \curvearrowright \Gamma$ is almost transitive and free with fundamental domain $U_0$, then $H \curvearrowright \Gamma$ is almost also transitive and free with fundamental domain $KU_0$, where $K \in \mathcal{F}(G)$ is any finite set of representatives. In particular, $|\Gamma/H| = |\Gamma/G|[G : H]$. For this reason, since we are interested in almost transitive actions, there is no loss of generality if, given an almost orderable group, we assume that is just orderable.

Orderability is a local property, that is, if every finitely-generated subgroup of a given group is orderable, then the whole group is orderable. Given a finitely generated group $G$, a generating set $S$, and its corresponding Cayley graph $\Gamma = \text{Cay}(G, S)$, we define the volume growth function as $g_\Gamma(n) = |B_{\Gamma}(1_G, n)|$. We say that $G$ has polynomial growth if $g_\Gamma(n) \leq p(n)$ for some polynomial $p$. It is well-known that groups with polynomial growth are amenable and a classic result due to Gromov asserts that they are virtually nilpotent [18]. Without further detail, from Schreier’s Lemma, it is also well-known that finite index subgroups of finitely generated groups are also finitely generated [30, Proposition 4.2] and finitely generated nilpotent groups have a torsion-free nilpotent subgroup with finite index [40, Proposition 2]. From this, and since torsion-free nilpotent groups are orderable [36, p.37], it follows that any finitely generated group $G$ with polynomial growth is amenable and almost orderable. In particular, all our results that apply to amenable and almost orderable groups will hold for groups of polynomial growth, but they will also hold in groups of super-polynomial —namely, exponential— growth, like solvable groups that are not virtually nilpotent [35] and, more concretely, cases like the Baumslag-Solitar groups $BS(1, n)$, that can also be ordered. On the other hand, not every amenable groups is almost orderable. In order to address this issue, we introduce a randomized generalization of invariant orders.
6.2. **Random orders.** Consider now the set of relations \( \{0,1\}^{G \times G} \) endowed with the product topology and the closed subset \( \text{Ord}(G) \) of strict total orders \( \prec \) on \( G \). We will consider the action \( G \acts \text{Ord}(G) \) given by

\[ h_1(g \cdot \prec) h_2 \iff (h_1 g) \prec (h_2 g) \]

for \( h_1, h_2, g \in G \) and \( \cdot \prec \in \text{Ord}(G) \). An **invariant random order** on \( G \) is a \( G \)-invariant Borel probability measure on \( \text{Ord}(G) \). Notice that a fixed point for the action \( G \acts \text{Ord}(G) \) corresponds to a (deterministic) invariant order on \( G \). The space of invariant random orders will be denoted by \( \mathcal{M}_G(\text{Ord}(G)) \).

Invariant random orders were introduced in [2] in order to answer problems about predictability in topological dynamics through what they called the Kieffer-Pinsker formula for the Kolmogorov-Sinai entropy of a group action.

Now, as in the deterministic case, we can also define a notion of past for the group. An **invariant random past** on \( G \) is a random function \( \Phi : G \to \{0,1\}^G \) or, equivalently, a Borel probability measure on \( \{\{0,1\}^G\}^G \) that satisfies, for almost every instance of \( \Phi \), the following properties:

1. for all \( g \in G \), the condition \( g \notin \Phi(g) \) holds;
2. for all \( g, h \in G \), if \( g \in \Phi(h) \), then \( \Phi(g) \subseteq \Phi(h) \);
3. if \( g \neq h \), then either \( g \in \Phi(h) \) or \( h \in \Phi(g) \); and
4. for all \( g \in G \), the random subsets \( \Phi(g) \) and \( \Phi(1_G)g \) have the same distribution.

Notice that if \( \prec \) is an invariant random total order, then the random function \( g \mapsto \{h \in G : h \prec g\} \) defines an invariant random past.

In contrast to deterministic invariant orders, every countable group \( G \) admits at least one invariant random total order. Namely, consider the random process \( (\chi_g)_{g \in G} \) of independent random variables such that each \( \chi_g \) has uniform distribution on \([0,1] \). This process is invariant and each realization of it induces an order on \( G \) almost surely.

7. **Counting**

From now on, given \( (\Gamma, \lambda) \in \mathcal{H}_G \), we always assume that there is some (or any) fixed fundamental domain \( U_0 \) for \( G \acts \Gamma \) and we introduce the auxiliary function \( \phi_\lambda : X(\Gamma) \to \mathbb{R} \) given by

\[ \phi_\lambda(x) = \frac{1}{|\Gamma/G|} \sum_{v \in U_0} x(v) \log \lambda(v). \]

7.1. **A pointwise Shannon-McMillan-Breiman type theorem.** The next theorem establishes a pointwise Shannon-McMillan-Breiman type theorem for Gibbs measures (related results can be found in [19] and [8]). In order to prove it we use the Pointwise Ergodic Theorem [27], which requires Følner sequence \( \{F_n\}_n \) to be **tempered**, a technical condition that is satisfied by every Følner sequence up to a subsequence and that we will assume without further detail.

**Theorem 7.1.** Let \( G \) be a countable amenable group. For every \( (\Gamma, \lambda) \in \mathcal{H}_G \) and every \( \mathbb{P} \in \mathcal{M}_{\text{Gibbs}}(\Gamma, \lambda) \),

\[ \lim_{n} \left[ -\frac{1}{|F_n U_0|} \log \mathbb{P}(x_{F_n U_0}) \right] = - \int \phi_\lambda d\mathbb{Q} + f_G(\Gamma, \lambda) \quad \mathbb{Q}(x)\text{-a.s. in } x, \]

for any tempered Følner sequence \( \{F_n\}_n \) and any \( \mathbb{Q} \in \mathcal{M}_{\text{erg}}(X(\Gamma)) \).
Proof. Consider the sets $U_n = F_n U_0$ and $M_n = U_n \cup \partial U_n$. Notice that, by amenability, $\lim_{n \to \infty} \frac{|M_n|}{|U_n|} = 1$. Indeed, define $K = \{ g \in G : \text{dist}_G(U_0, gU_0) \leq 1 \}$. Then, $1_G \in K$ and $U_0 \cup \partial U_0 \subseteq K U_0$. Since $\Gamma$ is locally finite and the action is free, $K$ is finite. In addition, $U_n \cup \partial U_n \subseteq F_n K U_0$. Therefore, by amenability,

$$1 \leq \lim_{n} \frac{|U_n \cup \partial U_n|}{|U_n|} \leq \lim_{n} \frac{|F_n K U_0|}{|F_n U_0|} = 1,$$

so $\lim_{n} \frac{\partial U_n}{|U_n|} = 0$. Fix independent sets $x \in X(\Gamma[U_n])$, $z_1, z_2 \in X(\Gamma[M_n \setminus U_n])$, and $y \in X(\Gamma)$ such that $x z_2 y \in X(\Gamma)$ for $i = 1, 2$. Then,

$$\frac{\pi_M^x (x z_1)}{\pi_M^x (x z_2)} = \frac{w_\lambda^x (x z_1, M_n) Z^x_\Gamma (M_n, \lambda)^{-1}}{w_\lambda^x (x z_2, M_n) Z^x_\Gamma (M_n, \lambda)^{-1}} = \frac{w_\lambda (x z_1, M_n) 1_{[\lambda M_n]}(x z_1)}{w_\lambda (x z_2, M_n) 1_{[\lambda M_n]}(x z_2)} = \prod_{v \in M_n \setminus U_n} \lambda (v)^{z_1(v)} = \prod_{v \in M_n \setminus U_n} \lambda (v)^{z_1(v) - z_2(v)}.

Therefore, $\frac{\pi_M^x (x z_1)}{\pi_M^x (x z_2)} \leq \max \{ 1, \lambda_+ \} |M_n \setminus U_n|$. Taking $z_2 = 0^\Gamma$ and adding over all possible $z_1$, we obtain that

$$1 \leq \frac{\pi_M^x ([x U_n])}{\pi_M^x ([x U_n] M_n \setminus U_n)} = \sum_{z \in \{ 0, 1 \}^{M_n \setminus U_n} : x z \in X(\Gamma[M_n])} \frac{\pi_M^x (x U_n z)}{\pi_M^x (x U_n z M_n \setminus U_n)} \leq \left( \max \{ 1, \lambda_+ \} \right) |M_n \setminus U_n|.$$

On the other hand, we have that

$$\frac{\pi_M^x ([x U_n] M_n \setminus U_n)}{\pi_M^x ([x U_n] M_n \setminus U_n)} = \frac{w_\lambda^x (x U_n M_n \setminus U_n, M_n) Z^x_\Gamma (M_n, \lambda)^{-1}}{w_\lambda^x (x U_n M_n \setminus U_n, M_n) Z^x_\Gamma (M_n, \lambda)^{-1}} = \frac{Z^x_\Gamma (M_n, \lambda)}{Z^x_\Gamma (M_n, \lambda)},$$

since

$$w_\lambda^x (x U_n M_n \setminus U_n, M_n) = w_\lambda^0 (x U_n M_n \setminus U_n, M_n) = w_\lambda (x U_n, U_n)$$

and $Z^x_\Gamma (M_n, \lambda) = Z_\Gamma (M_n, \lambda)$. In addition,

$$1 \leq \frac{Z^x_\Gamma (M_n, \lambda)}{Z_\Gamma (M_n, \lambda)} \leq \left( \max \{ 1, \lambda_+ \} \right) |M_n \setminus U_n|.$$
\[
\begin{align*}
Z'_T(M_n, \lambda) & \leq Z_T(M_n, \lambda) \\
& = \sum_{x \in X(\Gamma[U_n])} \sum_{z \in X(\Gamma[M_n \cup U_n])} \prod_{v \in U_n} \lambda(v)^{x(v)} \prod_{v \in M_n \setminus U_n} \lambda(v)^{z(v)} \\
& \leq \sum_{x \in X(\Gamma[U_n])} \sum_{z \in X(\Gamma[M_n \cup U_n])} \prod_{v \in U_n} \lambda(v)^{x(v)} \max\{1, \lambda_+\}^{\left|M_n \setminus U_n\right|} \\
& \leq \max\{1, \lambda_+\}^{\left|M_n \setminus U_n\right|} \sum_{x \in X(\Gamma[U])} 2^{|M_n \setminus U_n|} \prod_{v \in U_n} \lambda(v)^{x(v)} \\
& = (2 \max\{1, \lambda_+\})^{\left|M_n \setminus U_n\right|} Z_T(U_n, \lambda) \\
& \leq (2 \max\{1, \lambda_+\})^{\left|M_n \setminus U_n\right|} Z'_T(M_n, \lambda).
\end{align*}
\]

Therefore,

\[
1 \leq \frac{\pi^y_{M_n, n}(x_{U_n})}{\pi^y_{M_n, n}(x_{U_n}, 0^{M_n \cup U_n})} = \frac{\pi^y_{M_n, n}(x_{U_n})}{\pi^y_{M_n, n}(x_{U_n}, 0^{M_n \cup U_n})} \frac{\pi^y_{M_n, n}(x_{U_n}, 0^{M_n \cup U_n})}{\pi^y_{M_n, n}(x_{U_n}, 0^{M_n \cup U_n})} \leq (2 \max\{1, \lambda_+\})^{\left|M_n \setminus U_n\right|}.
\]

In particular, since \(\mathbb{P}(x_{F_n U_0}) = \mathbb{E}_F(\mathbb{P}(x_{U_n} | \mathcal{B}_{M_n})(y)) = \mathbb{E}_F(\pi^y_{M_n, n}(x_{U_n}))\), we have that

\[
1 \leq \frac{\mathbb{P}(x_{F_n U_0})}{\mathbb{P}(x_{U_n}, 0^{M_n \cup U_n})} \leq (2 \max\{1, \lambda_+\})^{\left|M_n \setminus U_n\right|},
\]

so

\[
\log \mathbb{P}(x_{F_n U_0}) - \log \pi^y_{M_n, n}(x_{U_n}, 0^{M_n \cup U_n}) \leq 2 |M_n \setminus U_n| \log (2 \max\{1, \lambda_+\}).
\]

Now, since \(w^y_\lambda(x_{M_n}, M_n) = w_\lambda(x_{M_n}, M_n)\) for every \(x\), we have that

\[
\pi^y_{M_n, n}(x_{U_n}, 0^{M_n \cup U_n}) = w_\lambda(x_{U_n}, 0^{M_n \cup U_n}, M_n) Z_T(M_n, \lambda)^{-1} = w_\lambda(x_{U_n}, U_n) Z_T(M_n, \lambda)^{-1}.
\]

Therefore,

\[
\log \mathbb{P}(x_{F_n U_0}) - \log w_\lambda(x_{U_n}, U_n) - \log Z_T(M_n, \lambda) \leq 2 |M_n \setminus U_n| \log (2 \max\{1, \lambda_+\}),
\]

so

\[
\lim_{n \to \infty} \frac{-\log \mathbb{P}(x_{F_n U_0})}{|U_n|} + \left( \frac{\log w_\lambda(x_{U_n}, U_n)}{|U_n|} - \frac{\log Z_T(M_n, \lambda)}{|U_n|} \right) \leq \lim_{n} \frac{|M_n \setminus U_n|}{|U_n|} (2 \max\{1, \lambda_+\}) = 0,
\]

and we conclude that

\[
-\lim_{n} \frac{\log \mathbb{P}(x_{F_n U_0})}{|F_n U_0|} = -\lim_{n} \frac{\log w_\lambda(x_{U_n}, U_n)}{|U_n|} + \frac{\log Z_T(M_n, \lambda)}{|U_n|}
\]

\[
= -\lim_{n} \left( \frac{1}{|F_n|} \sum_{x \in F_n} \frac{1}{|U_0|} \sum_{v \in \mathcal{G}_0} x(v) \log \lambda(v) \right) + f_G(\Gamma, \lambda),
\]
where we have used that $Z_T(U_n, \lambda) \leq Z_T(M_n, \lambda) \leq (2 \max\{1, \lambda_+\})^{M_n \setminus U_n} Z_T(U_n, \lambda)$. Finally, notice that
\[
\frac{1}{|F_n|} \sum_{g \in F_n} \frac{1}{|U_0|} \sum_{v \in E[U_0]} x(v) \log \lambda(v) = \frac{1}{|F_n|} \sum_{g \in F_n} \phi(x \cdot x)
\]
and by the Pointwise Ergodic Theorem, we obtain that
\[
Z_1 \text{ for some } u \text{ and } \lambda.
\]

and we conclude the proof.

\[
\pi^I_x([x_v]) \geq C.
\]

We have the following lemma.

**Lemma 7.2.** Given $\Delta \in \mathbb{N}$ and $(\Gamma, \lambda) \in \mathcal{H}^\Delta \lambda$, there exists a constant $C = C(\Delta, \lambda_0^+ \lambda_0^-) > 0$ such that for every $U \in V(\Gamma)$, $x \in X(\Gamma)$, and $v \in U$,
\[
\pi^I_x([x_v]) \geq C.
\]

**Proof.** Fix $(\Gamma, \lambda) \in \mathcal{H}^\Delta \lambda$, $U \in V(\Gamma)$, $x \in X(\Gamma)$, and $v \in U$. Notice that if $U^c \cap \partial \{v\} \neq \emptyset$ and $u \in U^c \cap \partial \{v\}$ is such that $x(u) = 1$, then necessarily $x_v = 0^v$ and $\pi^I_x([x_v]) = 1$. On the other hand, if $U \cap \partial \{v\} = \emptyset$, then $\pi^I_x([x_v]) = \mathbb{P}(U^c \cap \lambda \{x_v\})$ for $U' = U \setminus \{u \in U : x(u') = 1 \text{ for some } u' \in U^c \cap \partial \{u\}\}$, so, by Lemma 5.2,
\[
\pi^I_x([x_v]) = \min\left\{\frac{1}{1 + \lambda_+}, \frac{\lambda_-}{\lambda_- + (1 + \lambda_+)^\Delta}\right\} \geq \frac{\min\{1, \lambda_+\}}{\lambda_- + (1 + \lambda_+)^\Delta} > 0,
\]

Therefore, by taking $C = \min\left\{1, \frac{\min\{1, \lambda_+\}}{\lambda_- + (1 + \lambda_+)^\Delta}\right\} = \frac{\min\{1, \lambda_+\}}{\lambda_- + (1 + \lambda_+)^\Delta}$, we conclude. \qed

7.2. A randomized sequential cavity method. Suppose now that $(\Gamma, \lambda) \in \mathcal{H}^\Delta \lambda$ is such that the Gibbs $(\Gamma, \lambda)$-specification satisfies SSM and let $\mathbb{P}$ be the unique Gibbs measure. Considering this, we define the function $I_P : X(\Gamma) \times (\{0, 1\})^G \to \mathbb{R}$ given by
\[
I_P(x, \Phi) := \lim_{n \to \infty} I_{P,n}(x, \Phi),
\]

where
\[
I_{P,n}(x, \Phi) := -\log \mathbb{P}(\{x_{U_0}\} | x(\{0, 1\}^c \cap F_n U_0))
\]

and $\{F_n\}_n$ is any exhaustion of $G$ (not necessarily Følner).

**Lemma 7.3.** If $(\Gamma, \lambda) \in \mathcal{H}^\Delta \lambda$ is such that the Gibbs $(\Gamma, \lambda)$-specification satisfies SSM and $\mathbb{P}$ is the unique Gibbs measure, then the function $I_P$ is measurable, non-negative, defined everywhere, and bounded.

**Proof.** Since $\mathbb{P}(\{x_{U_0}\} | x(\{0, 1\}^c \cap F_n U_0))$ depends on finitely many coordinates in both $X(\Gamma)$ and $(\{0, 1\})^G$, $0 < \mathbb{P}(\{x_{U_0}\} | x(\{0, 1\}^c \cap F_n U_0)) < 1$, and $-\log(\cdot)$ is a continuous function, $I_{P,n}$ is measurable and since $I_P$ is a limit superior, it is measurable as well.

By SSM and Proposition 4.1, $\lim_{n \to \infty} \mathbb{P}(\{x_{U_0}\} | x(\{0, 1\}^c \cap F_n U_0)) = \mathbb{P}(\{x_{U_0}\} | x(\{0, 1\}^c \cap F_n U_0))$ is always a well-defined limit. By Lemma 7.2, and since $(\Gamma, \lambda) \in \mathcal{H}^\Delta \lambda$ for some $\Delta$, there exists a constant $C > 0$ such that for every $v \in V$, $U \in V \setminus \{v\}$, and $x \in X(\Gamma)$,
\[
\pi^I_x([x_v]) \geq C.
\]
Now, combined with the SSM property, this implies that for every \( v \in V, U \subseteq V, \) and \( x \in X(\Gamma) \), we have that
\[
P([x_v] | [x_U]) \geq C.
\]
Indeed, if \( v \in U \), this is direct since \( P([x_v] | [x_U]) = 1 \). On the other hand, if \( v \notin U \), by SSM,
\[
P([x_v] | [x_U]) = \lim_{\ell \to \infty} P([x_v] | [x_{U \cup B_{\Gamma}(v, \ell'} \cap \ell]) = \lim_{\ell \to \infty} \pi_x \mid_{U \cup B_{\Gamma}(v, \ell')} \geq \lim_{\ell \to \infty} C = C.
\]
Therefore, by conditioning and iterating, we obtain that
\[
1 \geq P([x_{U_0}] | [x_{\Phi(U_0)}]) = \prod_{i=1}^{U_0} P([x_{\Phi(U_0)}] | [x_{\Phi(U_0 \cup \{v_1, \ldots, v_{i-1}\})}]) \geq \prod_{i=1}^{U_0} C = C^{|\Gamma|/|\Gamma'|},
\]
so
\[
0 \leq I_\mathbb{P}(x, \Phi) = -\log P([x_{U_0}] | [x_{\Phi(U_0)}]) \leq -|\Gamma|/|\Gamma'| \log C < +\infty,
\]
i.e., \( I_\mathbb{P}(x, \Phi) \) is bounded (and, in particular, integrable).

Following [2], given an invariant random past \( \tilde{\Phi} : G \to \{0,1\}^G \) on \( G \) with law \( \tilde{\nu} \in \mathcal{M}_G((\{0,1\}^G)^G) \), we denote
\[
\mathbb{E}_{\tilde{\Phi}} f(\Phi) = \int f(\Phi) d\tilde{\nu}(\Phi),
\]
for \( f \in L^1(\tilde{\nu}) \). Now, since \( I_\mathbb{P} \) is measurable, non-negative, and bounded, we have that for every \( \mathcal{Q} \in \mathcal{A}_G(X(\Gamma)) \), the function \( I_\mathbb{P} \) is integrable with respect to \( \mathcal{Q} \times \tilde{\nu} \) by Fubini’s theorem, the function \( \mathbb{E}_{\tilde{\Phi}} I_\mathbb{P} : X(\Gamma) \to \mathbb{R} \) is integrable, defined \( \mathcal{Q} \)-almost everywhere, and satisfies that
\[
\int \mathbb{E}_{\tilde{\Phi}} I_\mathbb{P}(x, \tilde{\Phi}) d\mathcal{Q}(x) = \mathbb{E}_{\tilde{\Phi}} \int I_\mathbb{P}(x, \tilde{\Phi}) d\mathcal{Q}(x).
\]
We call \( \mathbb{E}_{\tilde{\Phi}} I_\mathbb{P} \) the random \( \mathbb{P} \)-information function (with respect to \( \tilde{\Phi} \)).

**Lemma 7.4.** For every \( \mathcal{Q} \in \mathcal{A}_G(X(\Gamma)) \) and for \( \mathcal{Q} \)-almost every \( x \),
\[
\lim_{n} \left[ \frac{-1}{|F_n U_0|} \log P([x_{F_n U_0}] | [x_{F_n U_0}]) - \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{E}_{\tilde{\Phi}} I_\mathbb{P}(g \cdot x, \tilde{\Phi}) \right] = 0.
\]

**Proof.** Fix a (tempered) Følner sequence \( \{F_n\}_n \). By the properties of \( \tilde{\Phi} \), for \( \tilde{\nu} \)-almost every instance \( \Phi \), every \( F_n \) can be ordered as \( g_1, \ldots, g_{|F_n|} \) so that \( \Phi(g_1) \cap F_n = \{g_1, \ldots, g_{|F_n| - 1}\} \).

Then,
\[
P([x_{F_n U_0}] | [x_{F_n U_0}]) = \prod_{g \in F_n} P([x_{g F_n U_0}] | [x_{F_n U_0}]),
\]
Given \( \ell > 0 \), let \( K_\ell = \{g \in G : \text{dist}_G(U_0, g U_0) \leq \ell \} \).

If \( g \in \text{Int}_{K_\ell}(F_n) = \{g \in G : K_\ell g \subseteq F_n\} \), then
\[
|P([x_{g F_n U_0}] | [x_{(g F_n U_0) \cap K_\ell U_0}]) - P([x_{F_n U_0}] | [x_{(F_n U_0) \cap K_\ell U_0}])| \leq |U_0| \delta(\ell)
\]
and
\[
|P([x_{g F_n U_0}] | [x_{(g F_n U_0) \cap K_\ell U_0}]) - P([x_{F_n U_0}] | [x_{(F_n U_0) \cap K_\ell U_0}])| \leq |U_0| \delta(\ell),
\]
so
\[
|P([x_{g F_n U_0}] | [x_{F_n U_0}]) - P([x_{F_n U_0}] | [x_{F_n U_0}])| \leq 2|U_0| \delta(\ell).
\]
On the other hand, by Lemma 7.2 and the discussion after it, for every \( g \in G \) and \( U \subseteq V \),
\[
P([x_{g F_n U_0}] | [x_{U_0}]) \geq C^{1/|G'|} > 0.
\]
Therefore, by the Mean Value Theorem,

\[
|\log \mathbb{P}([x_F U_0]| [x_{\Phi(g)} U_0]) - \log \mathbb{P}([x_F U_0]| [x_{\Phi(g)} U_0])| \leq \frac{2|U_0|}{C|\Gamma|} \delta(\ell).
\]

Notice that

\[
\log \mathbb{P}([x_F U_0]) = \sum_{g \in \text{Int}_{K_G}(F_n)} \log \mathbb{P}([x_{\Phi(g)} U_0])
\]

so

\[
|\log \mathbb{P}([x_F U_0]) - \sum_{g \in F_n} \log \mathbb{P}([x_{\Phi(g)} U_0])| \leq |\text{Int}_{K_G}(F_n)| \frac{2|U_0|}{C|\Gamma|} \delta(\ell)
\]

Now, given \(\varepsilon > 0\), there exists \(\ell\) and \(n_0\) such that for every \(n \geq n_0\),

\[
\left| \frac{\log \mathbb{P}([x_F U_0])}{|F_n|} - \frac{1}{|F_n|} \sum_{g \in F_n} \log \mathbb{P}([x_{\Phi(g)} U_0]) \right| \leq \varepsilon.
\]

By \(G\)-invariance of \(\mathbb{P}\),

\[
\mathbb{P}([x_{\Phi(g)} U_0]| [x_{\Phi(g)} U_0]) = \mathbb{P}(g^{-1} \cdot \{g \cdot x\} U_0| [x_{\Phi(g)} U_0])
\]

\[
= \mathbb{P}([g \cdot x U_0]| [x_{\Phi(g)} U_0])
\]

\[
= \mathbb{P}([g \cdot x U_0]| [g \cdot x_{\Phi(g)} U_0]),
\]

and combining this fact with the previous estimate, we obtain that

\[
\left| \frac{\log \mathbb{P}([x_F U_0])}{|F_n|} - \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{E}_{\Phi} \log \mathbb{P}([g \cdot x U_0]| [g \cdot x_{\Phi(g)} U_0]) \right| \leq \varepsilon.
\]

Integrating against \(\tilde{v}\), we obtain that, for \(Q\)-almost every \(x\),

\[
\left| \frac{\log \mathbb{P}([x_F U_0])}{|F_n|} - \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{E}_{\Phi} \log \mathbb{P}([g \cdot x U_0]| [g \cdot x_{\Phi(g)} U_0]) \right| \leq \varepsilon,
\]

and since \(\tilde{\Phi}(g)\) has the same distribution as \(\tilde{\Phi}(1_G)\), we get that

\[
\mathbb{E}_{\Phi} \log \mathbb{P}([g \cdot x U_0]| [g \cdot x_{\Phi(g)} U_0]) = \mathbb{E}_{\Phi} \log \mathbb{P}([g \cdot x U_0]| [g \cdot x_{\Phi(1_G)} U_0])
\]

\[
= \mathbb{E}_{\Phi} \log \mathbb{P}([g \cdot x U_0]| [g \cdot x_{\Phi(1_G)} U_0])
\]

\[
= \mathbb{E}_{\Phi} f_{P}(g \cdot x, \Phi),
\]

so

\[
\left| - \frac{\log \mathbb{P}([x_F U_0])}{|F_n|} - \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{E}_{\Phi} f_{P}(g \cdot x, \Phi) \right| \leq \varepsilon,
\]

and since \(\varepsilon\) is arbitrary and the limit exists \(Q\)-almost surely, we conclude. \(\square\)

We have the following representation theorem for free energy, which can be regarded as a randomized generalization of the results in [15, 33, 8] tailored for the specific case of the hardcore model, but that holds in every amenable group (not necessarily \(\mathbb{Z}^d\) or an almost orderable group).
Theorem 7.5. Let \((\Gamma, \lambda) \in \mathcal{H}_G\) such that the Gibbs \((\Gamma, \lambda)\)-specification satisfies SSM and \(P\) is the unique Gibbs measure. Then,

\[
f_G(\Gamma, \lambda) = \int (E_{\tilde{\Phi}} I_P + \phi_\lambda) dQ.
\]

for any \(Q \in \mathcal{M}_G(X(\Gamma))\) and for any invariant random past \(\tilde{\Phi}\) of \(G\). In particular,

\[
f_G(\Gamma, \lambda) = I_{\tilde{\Phi} \lambda}(0^\Gamma).
\]

Proof. First, notice that if the statement holds for every \(Q \in \mathcal{M}_G(X(\Gamma))\), then it holds for every \(Q \in \mathcal{M}_G(X(\Gamma))\) by the Ergodic Decomposition Theorem. Then, without loss of generality, we can assume that \(Q\) is \(G\)-ergodic. Considering this, by Theorem 7.1, for \(Q\)-almost every \(x\),

\[
limit_{n} \left[ \frac{1}{|F_n|} \log P(|x F_n|) \right] = - \int \phi_\lambda dQ + f_G(\Gamma, \lambda)
\]

By Lemma 7.4, for \(Q\)-almost every \(x\),

\[
limit_{n} \left[ \frac{1}{|F_n|} \log P(|x F_n|) \right] = \limit_{n} \frac{1}{|F_n|} \sum_{g \in F_n} E_{\tilde{\Phi}} I_P(g \cdot x, \tilde{\Phi}).
\]

Therefore, for \(Q\)-almost every \(x\),

\[
f_G(\Gamma, \lambda) - \int \phi_\lambda dQ = \limit_{n} \frac{1}{|F_n|} \sum_{g \in F_n} E_{\tilde{\Phi}} I_P(g \cdot x, \tilde{\Phi}).
\]

Integrating against \(Q\), we obtain that

\[
\limit_{n} \frac{1}{|F_n|} \sum_{g \in F_n} E_{\tilde{\Phi}} I_P(g \cdot x, \tilde{\Phi}) dQ = \limit_{n} \frac{1}{|F_n|} \sum_{g \in F_n} E_{\tilde{\Phi}} I_P(g \cdot x, \tilde{\Phi}) dQ
\]

\[
= \limit_{n} \frac{1}{|F_n|} \sum_{g \in F_n} E_{\tilde{\Phi}} I_P(g \cdot x, \tilde{\Phi}) dQ
\]

\[
= \limit_{n} \frac{1}{|F_n|} \sum_{g \in F_n} E_{\tilde{\Phi}} I_P(g \cdot x, \tilde{\Phi}) dQ
\]

\[
= \int E_{\tilde{\Phi}} I_P(x, \tilde{\Phi}) dQ
\]

\[
= \int E_{\tilde{\Phi}} I_P(x, \tilde{\Phi}) dQ,
\]

where the first equality is due to the Dominated Convergence Theorem, the second and last equalities are due to Tonelli’s Theorem, and the third equality is due to the \(G\)-invariance of \(Q\). We conclude that

\[
f_G(\Gamma, \lambda) = \int (E_{\tilde{\Phi}} I_P + \phi_\lambda) dQ.
\]

In particular, if \(Q = \delta_{0^\Gamma} \in \mathcal{M}_G(X(\Gamma))\), the Dirac measure supported on \(0^\Gamma\), then

\[
f_G(\Gamma, \lambda) = E_{\tilde{\Phi}} I_P(0^\Gamma, \tilde{\Phi}) + \phi_\lambda(0^\Gamma) = E_{\tilde{\Phi}} I_P(0^\Gamma, \tilde{\Phi}).
\]

\(\square\)
7.3. An arboreal representation of free energy. The following theorem tell us that, under some special conditions, $f_G(\Gamma, \lambda)$ can be expressed using $|\Gamma/G|$ terms that depend on the probability that the roots of some particular trees are unoccupied.

**Theorem 7.6.** Let $(\Gamma, \lambda) \in \mathcal{H}_G$ such that the Gibbs $(T_{\mathrm{SAW}}(\Gamma, v), \lambda)$-specification satisfies SSM for every $v \in V(\Gamma)$ and let $v_1, \ldots, v_{|\Gamma/G|}$ be an arbitrary ordering of a fundamental domain $U_0$. Given an invariant random order $\tilde{\Phi}$ of $G$, denote by $\Gamma_i(\tilde{\Phi})$ the random graph given by $\Gamma \setminus (\tilde{\Phi}(1_G) U_0 \cup \{v_1, \ldots, v_i\})$. Then,

$$f_G(\Gamma, \lambda) = -\sum_{i=1}^{\lceil \Gamma/G \rceil} \mathbb{E}_{\tilde{\Phi}} \log \mathbb{P}_{T_{\mathrm{SAW}}(\Gamma(\tilde{\Phi}), v_i)}([0^{\rho_i}]),$$

where $\rho_i$ denotes the root of $T_{\mathrm{SAW}}(\Gamma(\tilde{\Phi}), v_i)$. In particular, if $\prec$ is a deterministic invariant order of $G$,

$$f_G(\Gamma, \lambda) = -\sum_{i=1}^{\lceil \Gamma/G \rceil} \log \mathbb{P}_{T_{\mathrm{SAW}}(\Gamma(\tilde{\Phi}), v_i)}([0^{\rho_i}]).$$

**Proof.** By Theorem 7.5, we know that

$$f_G(\Gamma, \lambda) = \mathbb{E}_{\tilde{\Phi}} \mathbb{I}_{\tilde{\Phi}}(\Gamma, \lambda) \mathbb{P}(\tilde{\Phi}(1_G)) = \mathbb{E}_{\tilde{\Phi}}(-\log \mathbb{P}_{T_{\mathrm{SAW}}(\Gamma(\tilde{\Phi}), v_i)}([0^{\rho_i}])).$$

By iterating conditional probabilities, linearity of expectation, and Proposition 4.1 (see Figure 4),

$$-\mathbb{E}_{\tilde{\Phi}} \log \mathbb{P}_{T_{\mathrm{SAW}}(\Gamma(\tilde{\Phi}), v_i)}([0^{\rho_i}]) = \sum_{i=1}^{\lceil \Gamma/G \rceil} \mathbb{E}_{\tilde{\Phi}} \log \mathbb{P}_{T_{\mathrm{SAW}}(\Gamma(\tilde{\Phi}), v_i)}([0^{\rho_i}][0^{\Phi(1_G) U_0}])$$

$$= -\sum_{i=1}^{\lceil \Gamma/G \rceil} \mathbb{E}_{\tilde{\Phi}} \log \mathbb{P}_{T_{\mathrm{SAW}}(\Gamma(\tilde{\Phi}), v_i)}([0^{\rho_i}][0^{\Phi(1_G) U_0 \cup \{v_1, \ldots, v_i-1\}}])$$

$$= -\sum_{i=1}^{\lceil \Gamma/G \rceil} \mathbb{E}_{\tilde{\Phi}} \log \mathbb{P}_{T_{\mathrm{SAW}}(\Gamma(\tilde{\Phi}), v_i)}([0^{\rho_i}]).$$

**Figure 4.** The graph $\Gamma \setminus \Phi \_ U_0$ and the corresponding graphs $\Gamma_i(\Phi \_)$ for $G = \mathbb{Z}^2$ and the lexicographic order $\prec$.
Finally, by Theorem 5.5, we obtain

\[- \sum_{i=1}^{\left\lfloor \frac{|\Gamma|}{G} \right\rfloor} E_{\Phi} \log P_{T_{\text{SAW}}(\Gamma_i(\Phi), \lambda)}([0^\rho]) = - \sum_{i=1}^{\left\lfloor \frac{|\Gamma|}{G} \right\rfloor} E_{\Phi} \log P_{T_{\text{SAW}}(\Gamma_i(\Phi), \lambda)}([0^\rho]).\]

In particular, if \(\prec\) is a deterministic invariant order on \(G\) (see Figure 5), we have that

\[f_G(\Gamma, \lambda) = - \sum_{i=1}^{\left\lfloor \frac{|\Gamma|}{G} \right\rfloor} \log P_{T_{\text{SAW}}(\Gamma_i(\Phi \prec), \lambda)}([0^\rho]).\]

\[\square\]

8. A COMPUTATIONAL PHASE TRANSITION IN THE THERMODYNAMIC LIMIT

Given an amenable countable group \(G\), we are interested in having an algorithm to efficiently approximate \(f_G(\Gamma, \lambda)\) in some uniform way over \(\mathcal{H}_G\).

Let \(\mathcal{M} \subseteq \mathcal{H}_G\) be a family of hardcore models. We will say that \(\mathcal{M}\) admits a fully polynomial-time additive approximation (FPTAA) for \(f_G(\Gamma, \lambda)\) if there is an algorithm such that, given an input \((\Gamma, \lambda) \in \mathcal{M}\) and \(\varepsilon > 0\), outputs \(\hat{f}\) with

\[|f_G(\Gamma, \lambda) - \hat{f}| \leq \varepsilon,\]

in polynomial time in \(|(\Gamma, \lambda)|\) and \(\varepsilon^{-1}\), where \(|(\Gamma, \lambda)|\) denotes the length of any reasonable representation \((\Gamma, \lambda)\) of \((\Gamma, \lambda)\). A FPTAA will be what we will regard as an efficient and uniform approximation algorithm for \(f_G(\Gamma, \lambda)\).

**Remark 8.1.** In order to not having to deal with numerical details about the representation of \(\lambda\), we will always implicitly assume that the values taken by \(\lambda\) have a bounded number of digits uniformly on \(\mathcal{M}\).

---

**Figure 5.** A representation of \(T_{\text{SAW}}(\Gamma_i(\Phi \prec), \lambda)\) and a logarithmic depth truncation for \(G = \mathbb{Z}^2\) and \(\Gamma = \text{Cay}(\mathbb{Z}^2, \{\pm(1,0), \pm(0,1)\})\).
8.1. FPTAS, Weitz’s algorithm, and a computational phase transition. Notice that if $G$ is the trivial group $\{1\}$, then $\mathcal{H}_{\{1\}}$ is exactly the family of finite hardcore models. In such case, we have that

$$f_G(\Gamma, \lambda) = \frac{\log Z_\Gamma(\lambda)}{|V(\Gamma)|},$$

and we can translate an approximation of $Z_\Gamma(\lambda)$ into an approximation of $f_G(\Gamma, \lambda)$ and vice versa.

In this finite context, it is common to consider a fully polynomial-time approximation scheme. Given a family $\mathcal{M} \subseteq \mathcal{H}_{\{1\}}$ of finite hardcore models, we will say that $\mathcal{M}$ admits a fully polynomial-time approximation scheme (FPTAS) for $Z_\Gamma(\lambda)$ if there is an algorithm such that, given an input $(\Gamma, \lambda) \in \mathcal{M}$ and $\varepsilon > 0$, outputs $\hat{Z}$ with

$$Z_\Gamma(\lambda)e^{-\varepsilon} \leq \hat{Z} \leq Z_\Gamma(\lambda)e^{\varepsilon},$$

in polynomial time in $|V(\Gamma)|$ and $\varepsilon^{-1}$. An FPTAS is regarded as an efficient and uniform approximation algorithm for $Z_\Gamma(\lambda)$.

Notice that if we take logarithms and divide by $|V(\Gamma)|$ in the previous equation, we obtain that

$$|f_G(\Gamma, \lambda) - \hat{f}| = \left| \frac{\log Z_\Gamma}{|V(\Gamma)|} - \frac{\log \hat{Z}}{|V(\Gamma)|} \right| \leq \frac{\varepsilon}{|V(\Gamma)|},$$

where $\hat{f} = \frac{\log \hat{Z}}{|V(\Gamma)|}$, so an FPTAS for $Z_\Gamma(\lambda)$ is equivalent to an FPTAA for $f_G(\Gamma, \lambda)$, since a polynomial in $|V(\Gamma)|$ and $\varepsilon^{-1}$ is also a polynomial in $|\langle \Gamma, \lambda \rangle|$ and $|V(\Gamma)|\varepsilon^{-1}$ and vice versa.

We will fix a positive integer $\Delta$ and $\lambda_0 > 0$. Given such parameters, we aim to develop a fully polynomial-time additive approximation on $\mathcal{M} = \mathcal{H}_G^\Delta(\lambda_0)$.

The main theorem in [49] was the development of an FPTAS for $Z_\Gamma(\lambda_0)$ on $\mathcal{H}_G^\Delta(\lambda_0)$ for $\lambda_0 < \lambda_c(\Delta)$. It is not difficult to see that the theorem extends to non-constant activity functions $\lambda$. Then, and also translated into the language of free energy, we have the following result.

**Theorem 8.1 ([49])**. For every $\Delta \in \mathbb{N}$ and $0 < \lambda_0 < \lambda_c(\Delta)$, there exists an FPTAS (resp., an FPTAA) on $\mathcal{H}_{\{1\}}^\Delta(\lambda_0)$ for $Z_\Gamma(\lambda)$ (resp., for $G(\Gamma, \lambda)$).

This theorem was subsequently refined in [42] by considering connective constants instead of maximum degree $\Delta$. A very interesting fact is that when classifying graphs according to their maximum degree, then Theorem 8.1 is in some sense optimal due to the following theorem.

**Theorem 8.2 ([44, 45])**. For every $\Delta \geq 3$ and $\lambda_0 > \lambda_c(\Delta)$, there does not exist an FPTAS (resp., an FPTAA) on $\mathcal{H}_{\{1\}}^\Delta(\lambda_0)$ for $Z_\Gamma(\lambda_0)$ (resp., for $f_G(\Gamma, \lambda_0)$), unless NP = RP.

**Remark 8.2.** In [44, 45], it is considered a stronger version of Theorem 8.2, where instead of the lack of existence of an FPTAS, it is proven the lack of existence of an FPRAS, which is a randomized (and therefore, weaker) version of an FPTAS.

The combination of Theorem 8.1 and Theorem 8.2 is what is regarded as a computational phase transition. We aim to extend these theorems to the infinite setting. A theoretical advantage about considering $f_G(\Gamma, \lambda)$ instead of $Z_\Gamma(\lambda)$ is that the free energy still makes sense in infinite graphs and at the same time recovers the theory for $Z_\Gamma(\lambda)$ in the finite case.
8.2. A computational phase transition in the infinite setting. For algorithmic purposes, in this section we only consider finitely generated orderable groups $G$ with fixed symmetric set of generators $S$ and deterministic invariant order $\prec$. Recall that we are not losing generality if we assume $G$ to be orderable instead of just almost orderable, so in particular our results apply to every group of polynomial growth—including the very relevant case $G = \mathbb{Z}^d$—but are not restricted to them, i.e., our results also apply to some groups of exponential growth (e.g., $BS(1,n)$).

Notice that if $(\Gamma, \lambda) \in \mathcal{H}^2_{G,\lambda}$, then it suffices to know $\Gamma[U_0]$ for some fundamental domain $U_0$ and $L = \{(v_1, s, v_2) \in U_0 \times S \times U_0 : (v_1, sv_2) \in E(\Gamma)\}$ in order to fully reconstruct the graph $\Gamma$. In particular, the size of the necessary information to reconstruct $\Gamma$ is bounded by a polynomial in $\Delta|\Gamma/G|$. In addition, given a $G$-invariant activity function $\lambda : V(\Gamma) \to \mathbb{Q}_{>0}$, we only need to know $\lambda|U_0$ to recover $\lambda$, i.e., just $|\Gamma/G|$ many rational numbers. Therefore, in this context, the length $|\langle \Gamma, \lambda \rangle|$ of the representation $\langle \Gamma, \lambda \rangle$ of a hardcore model $\langle \Gamma, \lambda \rangle$ will be polynomial in $\Delta|\Gamma/G|$.

We are interested in being able to generate in an effective way balls of arbitrary radius in Cay($G,S$) and to recognize the set $\Phi_\prec U_0$. To do this, assume that the algebraic past of $G$ is decidable in exponential time, i.e., given an input word $w \in S^*$, we can decide whether $\pi(w) \in \Phi_\prec$ or not in time $\exp(O(|w|))$, where $|w|$ denotes the length of $w$ and $\pi : S^* \to G$ is the usual evaluation map. Notice that this implies that the word problem is solvable in exponential time, since $\pi(w) = 1_G$ if and only if $\pi(w) \notin \Phi_\prec$ and $\pi(w^{-1}) \notin \Phi_\prec$. Now, given $\ell > 0$, if we wish to solve the word problem in exponential time, then Cay($G,S$) is constructible in exponential time as well (see [34, Theorem 5.10]), this is to say, we can generate $B_{\text{Cay}(G,S)}(1_G, \ell)$ in time $\exp(O(\ell))$, where the $O$-notation regards $|S|$ and $\Delta$ as constants. Now, having $B_{\text{Cay}(G,S)}(1_G, \ell)$ it is possible to construct $\Gamma[B_{\text{Cay}(G,S)}(1_G, \ell)U_0]$ in time $O(\exp(|\Gamma/G|) \exp(O(\ell)))$ by identifying each $g \in B_{\text{Cay}(G,S)}(1_G, \ell)$ with a copy $\Gamma[gU_0]$ of $\Gamma[U_0]$ and by connecting it to other adjacent copies according to $L$. Finally, we can remove all the copies $\Gamma[gU_0]$ with $g \in \Phi_\prec$, and this can be done in time $|B_{\text{Cay}(G,S)}(1_G, \ell)| \exp(O(\ell)) = \exp(O(\ell))$, since $|B_{\text{Cay}(G,S)}(1_G, \ell)| \leq |S|^{\ell}$. Therefore, if $\Phi_\prec$ is decidable in exponential time, then $\Gamma[B_{\text{Cay}(G,S)}(1_G, \ell) \setminus \Phi_\prec]U_0$ can be constructed in time $O(\exp(|\Gamma/G|) \exp(O(\ell)))$ and from this we can obtain directly the graphs $T_{\text{SAW}}(\Gamma, (\Phi_\prec), v_i)$ truncated at depth $\ell$, which are the ones that we are ultimately interested in.

Proposition 8.3. Let $G$ be a finitely generated amenable orderable group such that its algebraic past is decidable in exponential time. Then, for every $\Delta \in \mathbb{N}$ and $0 < \lambda_0 < \lambda_\prec(\Delta)$, there exists an FPTAA on $\mathcal{H}^2_{G,\lambda}(\lambda_0)$ for $f_G(\Gamma, \lambda)$.

Proof. Pick $(\Gamma, \lambda)$ as in the statement and enumerate as $U_0 = \{v_1, \ldots, v_n\}$ the fundamental domain of $G \cap \Gamma$. Denote $n = |\Gamma/G|$. Then, by Theorem 7.6,

$$f_G(\Gamma, \lambda) = -\sum_{i=1}^n \log p_i,$$

where $p_i = \mathbb{P}_{T_{\text{SAW}}(\Gamma, (\Phi_\prec), v_i)}(\{0^n\})$. Given $\varepsilon > 0$, our goal is generate numbers $\hat{q}_i$ and $\hat{r}_i$ such that

$$\hat{q}_i \leq q_i \leq \hat{r}_i \leq \hat{q}_i \left(1 + \frac{\varepsilon}{(1+\varepsilon)n}\right)$$

for every $i = 1, \ldots, n$. If we manage to compute such approximations, we have that

$$\prod_{i=1}^n \hat{q}_i \leq \prod_{i=1}^n \hat{r}_i \leq \prod_{i=1}^n 
\left(\hat{q}_i \left(1 + \frac{\varepsilon}{(1+\varepsilon)n}\right)\right)
\leq \left(1 + \frac{\varepsilon}{(1+\varepsilon)n}\right)^n \prod_{i=1}^n \hat{q}_i.$$
Therefore, it is possible to generate a pair of numbers $\hat{Z}_1 = (\prod_{\ell=1}^n \hat{r}_\ell)^{-1}$ and $\hat{Z}_2 = (\prod_{\ell=1}^n \hat{r}_\ell)^{-1}$ such that

$$\hat{Z}_2 \geq Z \geq \hat{Z}_1 \geq \hat{Z}_2 \left(1 + \frac{\varepsilon}{(1 + \varepsilon)n}\right)^{-n} \geq \hat{Z}_2 \left(1 - \frac{\varepsilon}{(1 + \varepsilon)}\right) = \hat{Z}_2 \frac{1}{(1 + \varepsilon)},$$

so $\hat{Z}_1 \leq Z \leq \hat{Z}_2 \leq (1 + \varepsilon)\hat{Z}_1$, where $Z = (\prod_{\ell=1}^n p_\ell)^{-1}$. Therefore,

$$\log \hat{Z}_1 \leq - \sum_{i=1}^n \log p_i = f_G(\Gamma, \lambda) \leq \log \hat{Z}_1 + \log(1 + \varepsilon) \geq \log \hat{Z}_1 + \varepsilon,$$

so $\hat{f} = \log \hat{Z}_1$ would be the required approximation. Now, in order to estimate $p_i$, i.e., the probability that $\rho_i$ is unoccupied in $T_i = T_{\text{SAW}}(\Gamma_i(\Phi_\omega), v_i)$, we denote

$$\hat{q}_i = \min \{ \mathbb{P}_{T_i, \mathcal{X}}(\{0^\rho\}) \}, \mathbb{P}_{T_i, \mathcal{X}}(\{0^\rho\})\} \rho$$

and

$$\hat{r}_i = \max \{ \mathbb{P}_{T_i, \mathcal{X}}(\{0^\rho\}) \}, \mathbb{P}_{T_i, \mathcal{X}}(\{0^\rho\})\} \rho$$

for $\ell > 0$ to be defined. It is known that the “all 0” and “all 1” conditions are the conditions that maximize and minimize the probability that $\rho_i$ is unoccupied among all conditions at distance $\ell$ and whether the “all 0” (resp. “all 1”) condition maximizes or minimizes such probability depends on the parity of $\ell$. In particular, $\hat{q}_i \leq p_i \leq \hat{r}_i$ and, by SSM,

$$0 \leq \hat{r}_i - \hat{q}_i \leq C \exp(-\alpha \ell).$$

We want that $\hat{q}_i \leq \hat{r}_i \leq \hat{q}_i \left(1 + \frac{\varepsilon}{(1 + \varepsilon)n}\right)$. This is equivalent to have

$$0 \leq \hat{r}_i - \hat{q}_i \leq \hat{q}_i \frac{\varepsilon}{(1 + \varepsilon)n}.$$  

Now, by Lemma 5.2, $\hat{q}_i \geq \frac{1}{1 + \lambda_+}$ for every $i$, so it is enough to have that $\hat{r}_i - \hat{q}_i \leq \frac{\varepsilon}{1 + \lambda_+ (1 + \varepsilon)n}$. In consequence, it suffices to take $\ell > 0$ such that

$$C \exp(-\alpha \ell) \leq \frac{\varepsilon}{1 + \lambda_+ (1 + \varepsilon)n},$$

so we can pick

$$\ell^* = \left\lceil \frac{1}{\alpha} \log \left(C(1 + \lambda_0)(1 + \varepsilon^{-1})n\right) \right\rceil.$$

For every $i = 1, \ldots, n$, the size of the ball $B_{T_i}(\rho_i, \ell)$ in $T_i$ is bounded by

$$|B_{T_i}(\rho_i, \ell^*)| \leq \Delta^* \leq (C(1 + \lambda_0)(1 + \varepsilon^{-1})n)^{\frac{\log \Delta}{\log \Delta - n}},$$

which is also a bound for the order of time required for computing $\hat{r}_i$, because $T_i$ is a tree (see Figure 5). Notice that to construct each $B_{T_i}(\rho_i, \ell^*)$ it suffices to be able to construct $B_{\Gamma_i(\Phi_\omega)}(v_i, \ell^*)$, which takes time $\text{poly}(n) \exp(O(1)) = \text{poly}(\varepsilon^{-1})n$. Finally, since we require to do this procedure $n$ times (one for each $i$), we have that the total order of the algorithm is still $\text{poly}(1 + \varepsilon^{-1})n$, i.e., a polynomial in $n$ and $\varepsilon^{-1}$ where the constants involved depend only on $\Delta$, $\lambda_0$, and $|S|$.

\begin{remark}
\textbf{Remark 8.3.} Notice that Proposition 8.3 holds for groups of exponential growth, despite it involves a polynomial time algorithm. In addition, in virtue of Theorem 5.7, the families of graphs in Proposition 8.3 could be parameterized according to their connective constant instead of the maximum degree.
\end{remark}
Next, we reduce the problem of approximating the partition function of a finite hardcore model to the problem of approximating the free energy of a hardcore model in $\mathcal{H}_G$.

**Proposition 8.4.** Let $G$ be an amenable group. Then, for every $\Delta \geq 3$ and $\lambda_0 > \lambda_c(\Delta)$, there does not exist an FPTAA on $\mathcal{H}_G^{\Delta}(\lambda_0)$ for $f_G(\Gamma, \lambda_0)$, unless $\text{NP} = \text{RP}$.

**Proof.** Suppose that we have a FPTAA on $\mathcal{H}_G^{\Delta}(\lambda_0)$ for $f_G(\Gamma, \lambda_0)$ for some amenable group $G$. Then, we claim that we would have an FPTAS on $\mathcal{H}_G^{\Delta}(\lambda_0)$ for $Z_\Gamma(\lambda_0)$, contradicting Theorem 8.2. Indeed, given the input $(\Gamma, \lambda) \in \mathcal{H}_G^{\Delta}(\lambda_0)$, it suffices to consider the graph $\Gamma^G$ made out of copies $\{\Gamma_g\}_{g \in G}$ of $\Gamma$ indexed by $g \in G$, where $\Gamma_g$ denotes the copy in $\Gamma_g$ of the vertex $v$ in $\Gamma$. Then, there is a natural action $G \acts \Gamma^G$ consisting on just translating copies of vertices, i.e., $g \cdot v_g = v_{g^{-1}}$, and a fundamental domain of the action is $U_0 = V(\Gamma_{1_G})$. Therefore, since $|\Gamma^G|/G = |V(\Gamma)|$, if we could $\varepsilon$-approximate in an additive way $f_G(\Gamma, \lambda_0)$ in polynomial time in $|\Gamma^G|/G$ and $\varepsilon^{-1}$, then we would be able to $\varepsilon$-approximate in a multiplicative way $Z_\Gamma(\lambda_0)$ in polynomial time in $|V(\Gamma)|$ and $\varepsilon^{-1}$, because

$$f_G(\Gamma, \lambda_0) = \inf_{F \in \mathcal{F}(G)} \frac{\log Z_{\Gamma^G}(FU_0, \lambda_0)}{|FU_0|}$$

$$= \inf_{F \in \mathcal{F}(G)} \frac{\log Z_{\Gamma^G}(U_0, \lambda_0)|F|}{|F||U_0|}$$

$$= \frac{\log Z_{\Gamma^G}(U_0, \lambda_0)}{|U_0|}$$

$$= \frac{\log Z_\Gamma(\lambda_0)}{|V(\Gamma)|},$$

but this contradicts Theorem 8.2. \qed

Considering Proposition 8.3 and Proposition 8.4, we have the following theorem.

**Theorem 8.5.** Let $G$ be a finitely generated amenable orderable group such that its algebraic past is decidable in exponential time. Then, for every $\Delta \geq 3$ and $\lambda_0 > 0$, if $\lambda_0 < \lambda_c(\Delta)$, there exists an FPTAA on $\mathcal{H}_G^{\Delta}(\lambda_0)$ for $f_G(\Gamma, \lambda_0)$, and, if $\lambda_0 > \lambda_c(\Delta)$, there is no such FPTAA, unless $\text{NP} = \text{RP}$.

**Remark 8.4.** Notice that Theorem 8.5 still holds for $\Delta = 1$ and $\Delta = 2$. The first case is trivial and in the second case, there is no phase transition and the conditions for the existence of an FPTAA hold for every $\lambda_0$.

9. Reductions

In this section we provide a set of reductions to relate the results already obtained for hardcore models with other systems.

9.1. $G$-subshifts and conjugacies. Given a countable group $G$ and a finite set $\Sigma$ endowed with the discrete topology, the full shift is the set $\Sigma^G$ of maps $\omega : G \to \Sigma$ endowed with the product topology. We define the $G$-shift as the group action $G \times \Sigma^G \to \Sigma^G$ given by $(g, \omega) \mapsto g \cdot \omega$, where $(g \cdot \omega)(h) = \omega(hg)$ for all $h \in G$. A $G$-subshift $\Omega$ is a $G$-invariant closed subset of $\Sigma^G$.

Given two $G$-subshifts $\Omega_1$ and $\Omega_2$, we say that a map $\varphi : \Omega_1 \to \Omega_2$ is a conjugacy if it is bijective, continuous, and $G$-equivariant, i.e., $g \cdot \varphi(x) = \varphi(g \cdot x)$ for every $\omega \in \Omega_1$ and $g \in G$. In this context, such maps are characterized as sliding block codes (e.g., see [26, 11]) and provide a notion of isomorphism between $G$-subshifts.
Any $G$-subshift $\Omega$ is characterized by the existence of a family of forbidden patterns $\mathcal{F} \subseteq \bigcup_{F \in \mathcal{F}(G)} \Sigma^F$ such that $\Omega = \mathcal{X}_{\mathcal{F}}$, where

$$X_{\mathcal{F}} = \{ \omega \in \Sigma^G : (g \cdot x)_F \notin \mathcal{F} \text{ for all } g \in G \}.$$ 

If the family $\mathcal{F}$ can be chosen to be finite, we say that $\Omega$ is a $G$-subshift of finite type (G-SFT). Given a finite set $S \subseteq G$, we can consider a family of $|\Sigma| \times |\Sigma|$ binary matrices $M = \{M_s\}_{s \in S}$ with rows and columns indexed by the elements of $\Sigma$, and define the set

$$\Omega_M = \{ \omega \in \Sigma^G : M_s(\omega(g), \omega(sg)) = 1 \text{ for all } g \in G, s \in S \}.$$ 

The set $\Omega_M$ is a special kind of $G$-SFT known as nearest neighbor (n.n.) G-SFT. It is known that for every $G$-SFT there exists a conjugacy to a n.n. $G$-SFT, so we are not losing much generality by considering n.n. $G$-SFTs instead of general $G$-SFTs.

We say that a n.n. $G$-SFT $\Omega_M$ has a safe symbol if there exists $a \in \Sigma$ such that $a$ can be adjacent to any other symbol $b \in \Sigma$. Formally, this means that, for all $s \in S$ and $b \in \Sigma$, $M_s(a, b) = M_s(b, a) = 1$.

### 9.2. Entropy and potentials

Given a $G$-subshift $\Omega$, we define its topological entropy as

$$h_G(\Omega) := \lim_n \frac{\log |\Omega_{F_n}|}{|F_n|},$$

where $\{F_n\}_n$ is a Følner sequence and $\Omega_F = \{ \omega_F : \omega \in \Omega \}$ is the set of restrictions of points in $\Omega$ to the set $F \subseteq G$. It is known that the definition of $h_G(\Omega)$ is independent of the choice of Følner sequence and is also a conjugacy invariant, i.e., if $\phi : \Omega_1 \rightarrow \Omega_2$ is a conjugacy, then $h_G(\Omega_1) = h_G(\Omega_2)$.

A potential is any continuous function $\phi : \Omega \rightarrow \mathbb{R}$. Given a potential, we define the pressure as

$$p_G(\phi) := \lim_n \frac{\log |Z_{F_n}(\phi)|}{|F_n|},$$

where $Z_{F_n}(\phi) = \sum_{\omega \in \Omega_{F_n}} \sup_{\omega \in [\omega]} \exp \left( \sum_{g \in F} \phi(g \cdot \omega) \right)$. Notice that $p_G(1) = h_G(\Omega)$.

A single-site potential is any potential that only depends on the value of $\omega$ at 1, i.e., $\omega_{1G}$. In other words, and without risk of ambiguity, we can think that a single-site potential is just a function $\phi : \Sigma \rightarrow \mathbb{R}$. In this case, $Z_{F_n}(\phi)$ has the following simpler expression:

$$Z_{F_n}(\phi) = \sum_{w \in \Omega_{F_n}} \prod_{g \in F} \exp(\phi(w(g))).$$

In this context, we will say that a symbol $a \in \Sigma$ is a vacuum state if $a$ is a safe symbol and $\phi(a) = 0$.

### 9.3. Reduction 1: from a hardcore model to a n.n. $G$-SFT with a vacuum state

Let $(\Gamma, \lambda)$ be a hardcore model in $\mathcal{H}_G$. If $G \cong \Gamma$ is transitive, then $\Gamma = \text{Cay}(G, S)$ for some finite symmetric set $S \subseteq G$. Then, it is easy to see that if $\Sigma = \{0, 1\}$ and, for all $s \in S$,

$$M_s = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

then $\Omega_M$ coincides with the set $X(\Gamma)$ and 0 is a safe symbol. In addition, there is a natural relationship between the activity function $\lambda$ and the single-site potential given by $\phi(0) = 0$ and $\phi(1) = \log \lambda(v)$, where $v$ is some (or any) vertex $v$. In other words, if $G \cong \Gamma$ is transitive, then $(\Gamma, \lambda)$ corresponds to a n.n. $G$-SFT with a vacuum state.

More generally, if $G \cong \Gamma$ is almost transitive, then $(\Gamma, \lambda)$ can also be interpreted as a n.n. $G$-SFT with a vacuum state. Indeed, consider the set $\Sigma = X(\Gamma[U_0])$, i.e., the set of independent sets of the subgraph $\Gamma[U_0]$ induced by some fundamental domain $U_0$. Since $\Gamma$
is locally finite and $G \curvearrowright \Gamma$ is free, there must exist a finite set $S \subseteq G \setminus \{1_G\}$ such that $SU_0$ contains all the vertices adjacent to $U_0$. Considering this, we define a collection of matrices $M_\Gamma = \{M_x\}_{x \in S}$, where

$$M_x(x, x') = \begin{cases} 1 & \text{if } xx' \in X(\Gamma[U_0 \cup sU_0]), \\ 0 & \text{otherwise}, \end{cases}$$

and $xx'$ denotes the concatenation of the independent set $x$ of $\Gamma[U_0]$ and the independent set $x'$ of $\Gamma[sU_0]$. In other words, $M_x(x, x') = 1$ if and only if the union of the independent set $x$ and the independent set $x'$ is also an independent set of $\Gamma[U_0 \cup sU_0]$.

Then, there is a natural identification between $\Omega_{M_\Gamma} \subseteq \Sigma_\Gamma^G$ and $\chi(\Gamma)$. In particular, the symbol $0^{\cup \rho} \in X(\Gamma[U_0])$ plays the role of a safe symbol in $\Omega_{M_\Gamma}$. Moreover, we can define the single-site potential $\phi_\lambda : \Omega_{M_\Gamma} \to \mathbb{R}$ given by $\phi_\lambda(\omega) = \sum_{v \in U_0} \omega_v \log \lambda(v)$. Then, for every $F \in \mathcal{F}(G)$,

$$Z_F(\phi_\lambda) = \sum_{\omega \in \Omega_F} \prod_{g \in F} \exp \left( \sum_{v \in U_0} \omega_v \log \lambda(v) \right)$$

$$= \sum_{\omega \in X(\Gamma[U_0])} \prod_{g \in F} \exp \left( \sum_{v \in U_0} \omega_v \log \lambda(v) \right)$$

$$= \sum_{\omega \in X(\Gamma[U_0])} \prod_{g \in F} \prod_{v \in U_0} \exp(\omega_v \log \lambda(v))$$

$$= \sum_{\omega \in X(\Gamma[U_0])} \prod_{v \in U_0} \lambda(v)^{\omega(v)}$$

$$= Z_{\Gamma}(\rho_{U_0}, \lambda).$$

Therefore, $p_G(\Omega_\Gamma, \phi_\lambda) = f_G(\Gamma, \lambda)$. In the language of dynamics, for every almost transitive and locally finite graph $\Gamma$, there exists a n.n. $G$-SFT with a safe symbol $\Omega_\Gamma$ such that $G \curvearrowright X(\Gamma)$ and $G \curvearrowright \Omega_\Gamma$ are conjugated. Moreover, this gives us a way to identify any hardcore model $(\Gamma, \lambda) \in \mathcal{H}_G$ with the corresponding $G$-SFT $\Omega_\Gamma$ and the single-site potential $\phi_\lambda$.

9.4. **Reduction 2: from a n.n. $G$-SFT with a vacuum state to a hardcore model.** Conversely, given a n.n. $G$-SFT $\Omega$ and a potential with a vacuum state, we can translate such scenario into a hardcore model. Indeed, consider the graph $\Gamma_\Omega$ defined as follows:

- for every $g \in G$, consider a finite graph $\Gamma_g$ isomorphic to $K_{|\Sigma|}$, the complete graph with $|\Sigma|$ vertices. In other words, for each $g \in G$ and for each $a \in \Sigma$ there will be a vertex $v_{g,a} \in V(\Gamma_g)$ and for every $a \neq b$, the edge $(v_{g,a}, v_{g,b})$ will belong to $E(\Gamma_g)$;
- the graph $\Gamma_\Omega$ will be the union of all the finite graphs $\Gamma_g$ plus some extra edges;
- for every $s \in S$ and $a, b \in \Sigma$, we add the edge $(v_{1_G,a}, v_{s,b})$ if and only if $M_s(a, b) = 0$;
- we define $\lambda_\phi : \Gamma_\Omega \to \mathbb{R}_{>0}$ as $\lambda_\phi(v_{g,a}) = \exp(\phi(a))$ for every $g \in G$ and $a \in \Sigma$.

Then, $G$ acts on $\Gamma_\Omega$ in the natural way and $V(\Gamma_1)$ corresponds to a fundamental domain of the action $G \curvearrowright \Gamma_\Omega$. In the language of dynamics, for every n.n. $G$-SFT with a safe symbol $\Omega$, there exists an almost transitive and locally finite graph $\Gamma_\Omega$ such that $G \curvearrowright \Omega$ and $G \curvearrowright X(\Gamma_\Omega)$ are conjugated. Moreover, it is clear that

$$f_G(\Gamma_\Omega, \lambda_\phi) = p_G(\Omega, \phi),$$

so in particular, all the representation and approximation theorems for free energy of hardcore models can be used to represent and approximate the pressure of n.n. $G$-SFTs $\Omega$ and potentials $\phi$ with a vacuum state, provided $(\Gamma_\Omega, \lambda_\phi)$ satisfies the corresponding hypotheses.
Relevant cases like the Widom-Rowlinson model [17] and graph homomorphisms from $\Gamma$ to any finite graph with some vertex (which plays the role of a safe symbol) connected to every other vertex fall in this category.

![Figure 6](image-url)

**Figure 6.** On the left, a sample of a configuration in the n.n. SFT $\Omega$ corresponding to proper 3-colorings of Cay($\mathbb{Z}^2$, $\{\pm(1, 0), \pm(0, 1)\}$) plus a safe symbol 0, where each square corresponds to an element of $\mathbb{Z}^2$. On the right, the independent set in the graph $\Gamma_{\Omega}$ representing the configuration in $\Omega$.

### 9.5 Topological entropy and constrainedness of n.n. $G$-SFTs with safe symbols.

Let $\Omega \subseteq \Sigma^G$ be a n.n. $G$-SFT with $|\Sigma| = n_s + n_u$, where $n_s$ denotes the number of safe symbols in $\Sigma$ and $n_u$ denotes the number of symbols that are not safe symbols (unsafe). Consider the n.n. $G$-SFT $\Omega_{n_s} \subseteq \Sigma_{n_s}^G$ obtained after collapsing all the safe symbols in $\Sigma$ into a single one, so that the $|\Sigma_{n_s}| = 1 + n_u$, and construct the graph $\Gamma_{\Omega_{n_s}}$. Then, given $F \in \mathcal{F}(G)$, we have that

$$|\Omega_F| = \sum_{x \in X(\Gamma_{\Omega_{n_s}}, FU_0)} \prod_{v \in FU_0} 1^{x(v)} n_s^{1-x(v)}$$

$$= \sum_{x \in X(\Gamma_{\Omega_{n_s}}, FU_0)} \prod_{v \in FU_0} \left( \frac{1}{n_s} \right)^{x(v)} n_s$$

$$= n_s^{\sum_{x \in X(\Gamma_{\Omega_{n_s}}, FU_0) \in FU_0} x(v)} \prod_{v \in FU_0} \left( \frac{1}{n_s} \right)^{x(v)}$$

$$= n_s^{\sum_{x \in X(\Gamma_{\Omega_{n_s}}, FU_0)} x(v)} \prod_{v \in FU_0} \left( \frac{1}{n_s} \right)^{x(v)}$$

$$= n_s^{\sum_{x \in X(\Gamma_{\Omega_{n_s}}, FU_0)} x(v)} \prod_{v \in FU_0} \left( \frac{1}{n_s} \right)^{x(v)}$$
so, considering that \( n_u = |U_0| = |\Gamma_{\Omega_{nu}}/G| \),

\[
h_G(\Omega) = \lim_n \frac{\log |\Omega_{F_n}|}{|F_n|} = |U_0| \log n_s + \lim_n \frac{Z_{\Omega_{nu}}(FU_0,1/n_s)}{|F_n|} = |U_0| \log n_s + |U_0|f_G(\Gamma_{\Omega_{nu}},1/n_s) = n_u \left( \log n_s + f_G(\Gamma_{\Omega_{nu}},1/n_s) \right).
\]

Therefore, to understand and approximate \( h_G(\Omega) \) reduces to study the hardcore model on \( \Gamma_{\Omega_{nu}} \) with constant activity \( \frac{1}{n_s} \). In particular, if

\[
\frac{1}{n_s} < \lambda_c(\mu(\Gamma_{\Omega_{nu}})),
\]

the hardcore model \( (\Gamma_{\Omega_{nu}},1/n_s) \) satisfies exponential SSM and the theory developed in the previous sections applies. This motivates the definition of the \textit{constraintedness} of a n.n. \( G \)-SFT \( \Omega \) as the connective constant of \( \Gamma_{\Omega_{nu}} \), i.e.,

\[
\mu(\Omega) := \mu(\Gamma_{\Omega_{nu}}),
\]

which can be regarded as a measure of how much constrained is \( \Omega \) (the higher \( \mu(\Omega) \), the more constrained it is). Notice that if

\[
\frac{1}{n_s} < \lambda_c(\mu(\Omega) + 1),
\]

then \( (\Gamma_{\Omega_{nu}},1/n_s) \) satisfies exponential SSM. In particular, \( \Omega_{nu} \) has a unique \textit{measure of maximal entropy} and therefore, also \( \Omega \) has unique measure of maximal entropy, namely, the pushforward measure (see [10, 20]). Moreover, the topological entropy of \( \Omega_{nu} \) has an arboreal representation and can be approximated efficiently. Since \( \mu(\Omega) \leq \Delta(\Gamma_{\Omega_{nu}}) - 1 \), we have that it suffices that

\[
\frac{1}{n_s} < \lambda_c(\Delta(\Gamma_{\Omega_{nu}})),
\]

For example, the n.n. \( G \)-SFT \( \Omega \) represented in Figure 6 satisfies that \( \Delta(\Gamma_{\Omega_{nu}}) = 6 \) and \( \lambda_c(6) = \frac{5}{\sqrt[5]{3}} = \frac{3125}{3846} \); then, if \( n_s > \frac{4096}{3125} = 1.31072 \), we see that it suffices to have 2 copies of the safe symbol 0 in order to have exponential SSM.

In general, since each vertex of the fundamental domain is connected to \( n_u - 1 \) vertices in the clique and to at most \( n_u \) vertices for each element \( s \) in the generating set \( S \), we see that each vertex in \( \Gamma_{\Omega_{nu}} \) is connected to at most \( (n_u - 1) + |S| n_u \) other vertices. Then, we can estimate that

\[
\Delta(\Gamma_{\Omega_{nu}}) \leq (|S| + 1)n_u - 1,
\]

so, in particular, if

\[
\frac{1}{n_s} < \lambda_c((|S| + 1)n_u - 1),
\]

exponential SSM holds (and therefore, again, uniqueness of measure of maximal entropy). This last equation and its relationship with the constraintedness of \( \Omega \) has a similar flavor to the relationship between the percolation threshold \( p_c(\mathbb{Z}^d) \) of the \( \mathbb{Z}^d \) lattice and the concept of \textit{generosity} for \( \mathbb{Z}^d \)-SFTs introduced in [20] by Häggström.
Remark 9.1. It may be the case that a n.n. $G$-SFT $\Omega \subseteq \Sigma^G$ with a safe symbol could be represented by a graph $\Gamma$ which is better in terms of connectedness or maximum degree compared with the canonical representation $\Gamma_\Omega$, since we could encode $\Sigma$ using other fundamental domains, with a lower connectivity than the complete graph. For example, the n.n. $\mathbb{Z}^2$-SFT $\Omega_\Gamma$ corresponding to the graph $\Gamma$ on the left in Figure 7 has 7 symbols (the 7 independent sets of the 4-cycle), including a safe one. However, the canonical graph representation of $\Omega_\Gamma$, i.e., the graph $\Gamma_\Omega$, has a fundamental domain consisting of a clique with 6 vertices, without considering extra connections. In particular, we see that both, $\Gamma$ and $\Gamma_\Omega$, represent $\Omega$, but $\Delta(\Gamma) = 3 < 6 \leq \Delta(\Gamma_\Omega)$. This motivates a finer notion of constrainedness, namely,

$$\tilde{\mu}(\Omega) = \inf\{\mu(\Gamma) : \Gamma \text{ represents } \Omega\},$$

and the aforementioned results would still hold if we replace $\mu(\Omega)$ by $\tilde{\mu}(\Omega)$. Notice that a fundamental domain $U_0$ has at least $|U_0| + 1$ independent sets (the empty one and all the singletons). In particular, this implies that $\tilde{\mu}(\Omega)$ is a minimum, since we only need to optimize over graphs $\Gamma$ with a fundamental domain $U_0$ such that $X(\Gamma[U_0]) = |\Sigma_n|.$

9.6. The monomer-dimer model and line graphs. Given a graph $\Gamma = (V, E)$, we say that two different edges $e_1, e_2 \in E$ are incident if they have one vertex in common. A matching in $\Gamma$ is a subset $M$ of $E$ without incident edges. In a total parallel with the hardcore model case, we can represent a matching with an indicator function $m : E \to \{0, 1\}$, denote the set of matchings of $\Gamma$ by $X^c(\Gamma)$, and define the associated partition function for some activity function $\lambda : E \to \mathbb{R}_{>0}$ as

$$Z^\lambda_f(\lambda) = \sum_{m \in X^c(\Gamma)} \prod_{e \in E} \lambda(e)^{m(e)}.$$

The pair $(\Gamma, \lambda)$ is called the monomer-dimer model and, as for the case of the hardcore model, we can define its associated free energy and Gibbs measures for a Gibbs specification adapted to this case.

An important feature of the monomer-dimer model is that, despite all its similarities with the hardcore model, it exhibits the SSM property for all values of $\lambda$ [6] and, in particular, there is no phase transition [21].
Considering this, most of the results presented in this paper, in particular the ones related to representation and approximation, can be adapted to counting matchings (see [15] for a particular case), and there will not be a phase transition. One way to see this is through the line graph $L(\Gamma)$ of the given graph $\Gamma$. Indeed, if we define $L(\Gamma)$ as the graph with set of vertices $E$ and set of edges containing all the adjacent edges in $E$, it is direct to see that there is a correspondence between matchings in $\Gamma$ and independent sets in $L(\Gamma)$, i.e.,

$$Z^\varepsilon(\lambda) = Z_{L(\Gamma)}(\lambda)$$

In particular, this tell us that all the results in our paper that involve some restriction on $\lambda$, apply to every graph that can be obtained as a line graph of another one without restriction on $\lambda$. For example, the graph $\Gamma$ represented on the right in Figure 7 corresponds to the line graph of the Cayley graph of $\mathbb{Z}^2$ with canonical generators, i.e.,

$$\Gamma = L(\text{Cay}(\mathbb{Z}^2, \{\pm(1,0), \pm(0,1)\})).$$

Then, this observation implies that we can represent and approximate $f_{\mathbb{Z}^2}(\Gamma, \lambda)$ for every $\mathbb{Z}^2$-invariant activity function $\lambda$ on $\Gamma$.

### 9.7. Spectral radius of matrices and occupation probabilities on trees.

A curious consequence of the hardcore model representation of a n.n. $G$-SFT with a safe symbol is that when $G = \mathbb{Z}$ and $S = \{1\}$, then $h_{\mathbb{Z}}(\Omega)$ has a well known characterization in terms of the transition matrix $M = M_1$ [26].

If $M$ is irreducible and aperiodic, there is always a unique stationary Markov chain $P_M$ associated to $M$ such that $\log \lambda_M = h_{\mathbb{Z}}(\Omega_M)$, where $\lambda_M$ denotes the Perron eigenvalue of $M$ and we consider the natural invariant order in $\mathbb{Z}$.

Now, if $M$ is a matrix such that the $i$th row and the $i$th column have no zeros, then $M$ is irreducible and aperiodic, and in fact, the $i$th symbol, let’s call it $a$, is a safe symbol. In such case, we have that

$$\log \lambda_M = h_{\mathbb{Z}}(\Omega_M) = -\log P_M([a^0]|[a^{-N}]) = -\log P_M([a^0]|[a^{-1}]),$$

Therefore, $\lambda_M = \frac{1}{P_M(a^0|a^{-1})}$ and to compute the spectral radius of $M$ reduces to compute $P_M(a^0|a^{-1})$. For example, consider the following matrix

$$M = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 \\
\end{pmatrix},$$

where $a$ is the symbol associated to the first row. Given such matrix $M$, we can always construct a graph representation $\Gamma_{\Omega_M}$ of $\Omega_M$ as in Figure 8.

![Figure 8. A graph representation of the 0-1 matrix $M$.](image-url)
Now, it is known that $|\mathbb{P}_M(0^n|0^{(n-1)}) - \mathbb{P}_M(0^n|0^{(n-1)})|$ goes to zero exponentially fast as $n$ goes to infinity and, since every Markov chain is a Markov random field, we have that

$$
\mathbb{P}_M(a_0|a_{(n-1)}) = \mathbb{P}_\Gamma \Omega M \left[ {0,\ldots,n-1} \cup \{0\} \right] (0^n)
$$

This gives us an arboreal representation and a method to compute the spectral radius of any such matrix $M$, that we believe could be of independent interest.

9.8. **Final comments.** We have established a computational phase transition for computing the free energy of hardcore models on almost transitive amenable graphs. To do this, the techniques presented here could be broken down in mainly two aspects: representation and approximation.

By representation, we mean all the preprocessing done to $f_G(\Gamma, \lambda)$ in Theorem 7.1, Theorem 7.5, and Theorem 7.6. The first theorem does not use much more than the definition of Gibbs measure and it is possible to prove that holds in other contexts (for example, see [19]). The second theorem requires SSM to hold, a property that other models exhibit and it is possible to extend it to such other cases (see [8]). On the other hand, in [1] it was proven that analogue formulas could be obtained in the non-uniqueness regime for the hardcore model and other classical ones. The third theorem is very particular to the hardcore model, but the representation of marginal probabilities using trees of self-avoiding walks can be extended to any 2-spin model with isotropic nearest-neighbor constraints and interactions; in particular, this includes the Ising model case [43].

Later, in Theorem 8.5, that involves the approximation aspects of our work, we gave an approximation algorithm for $f_G(\Gamma, \lambda)$ based on computing the occupation probability on a special tree satisfying an exponential decay of correlations. We believe that other techniques could be explored such as the ones developed by Barvinok [4] or the transfer matrix method [32] adapted to other groups besides $\mathbb{Z}^d$. In this sense, it does not seem unlikely that our results could be extended to other values of $\lambda$ outside of the interval $[0, \lambda_c(\Delta)]$.

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