Conditional stability of unstable viscous shock waves in compressible gas dynamics and MHD

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Abstract

Extending our previous work in the strictly parabolic case, we show that a linearly unstable Lax-type viscous shock solution of a general quasilinear hyperbolic–parabolic system of conservation laws possesses a translation-invariant center stable manifold within which it is nonlinearly orbitally stable with respect to small $L^1 \cap H^3$ perturbations, converging time-asymptotically to a translate of the unperturbed wave. That is, for a shock with $p$ unstable eigenvalues, we establish conditional stability on a codimension-$p$ manifold of initial data, with sharp rates of decay in all $L^p$. For $p = 0$, we recover the result of unconditional stability obtained by Mascia and Zumbrun. The main new difficulty in the hyperbolic–parabolic case is to construct an invariant manifold in the absence of parabolic smoothing.

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1 Introduction

In this paper, extending our previous work in the semilinear and quasilinear parabolic case [Z5, Z6], we study for general quasilinear hyperbolic–parabolic systems of conservation laws including the equations of compressible gas dynamics and MHD the conditional stability and existence of a center stable manifold of a linearly unstable viscous Lax shock. This is part of a larger program initiated in [TZ1, TZ2, TZ3, TZ4, SS, BeSZ, Z5] going beyond simple stability analysis to study nontrivial dynamics and bifurcation, and associated physical phenomena, of perturbed viscous shock waves in the presence of linear instability. As discussed for example in [AMPZ, GZ, Z7], such conditionally stable shock waves can play an important role in asymptotic behavior as metastable states, and their center stable manifolds as separatrices bounding the basins of attraction of nearby stable solutions.

The main issue in the present case is to construct a (translation-invariant) center stable manifold about a standing viscous shock wave; once this is done, the conditional stability analysis follows in straightforward fashion by a combination of the semilinear argument of [Z5] and the unconditional stability analysis of [MaZ2, MaZ3, Z2] in the hyperbolic–parabolic case. As discussed in [Z6], the technical difficulty in constructing the center stable manifold for quasilinear equations is an apparent loss of regularity in the usual fixed point argument by which the center stable manifold is constructed. See also the related discussion of [LPS1, LPS2]. We accomplish this by the method introduced in [Z6] in the quasilinear parabolic case, combining an implicit...
fixed-point scheme with suitable time-weighted $H^s$-energy estimates. For related energy estimates, see [TZ3].

Consider a viscous shock solution

\begin{equation}
U(x, t) = \bar{U}(x - st), \quad \lim_{z \to \pm \infty} \bar{U}(z) = U_{\pm},
\end{equation}

of a hyperbolic–parabolic system of conservation laws

\begin{equation}
U_t + F(U)_x = (B(U)U_x)_x,
\end{equation}

$x \in \mathbb{R}$, $U \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times n}$. Profile $\bar{U}$ satisfies the traveling-wave ODE

\begin{equation}
B(U)U' = F(U) - F(U_-) - s(U - U_-).
\end{equation}

Denote $A(U) = F_U(U)$,

\begin{equation}
B_{\pm} := \lim_{z \to \pm \infty} B(z) = B(U_{\pm}), \quad A_{\pm} := \lim_{z \to \pm \infty} A(z) = F_U(U_{\pm}).
\end{equation}

Following [TZ3, Z2, Z3], we make the structural assumptions:

(A1) $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, $b$ nonsingular, where $U \in \mathbb{R}^n$, $U_1 \in \mathbb{R}^{n-r}$, $U_2 \in \mathbb{R}^r$, and $b \in \mathbb{R}^{r \times r}$; moreover, the $U_1$-coordinate $F_1(U)$ of $F$ is linear in $U$ (strong block structure).

(A2) There exists a smooth, positive definite matrix $A^0(U)$, without loss of generality block-diagonal, such that $A^0_{11}A_{11}$ is symmetric, $A^0_{22}b$ is positive definite but not necessarily symmetric, and $(A^0A)_{\pm}$ is symmetric.

(A3) No eigenvector of $A_{\pm}$ lies in $\text{Ker}B_{\pm}$ (genuine coupling [Kaw, KSh]).

To (A1)–(A3), we add the following more detailed hypotheses. Here and elsewhere, $\sigma(M)$ denotes the spectrum of a matrix or linear operator $M$.

(H0) $F, B \in C^k$, $k \geq 4$.

(H1) $\sigma(A_{11})$ real, constant multiplicity, with $\sigma(A_{11}) < s$ or $\sigma(A_{11}) > s$.

(H2) $\sigma(A_{\pm})$ real, simple, and different from $s$.

Conditions (A1)–(A3) are a slightly strengthened version of the corresponding hypotheses of [MaZ4, Z2, Z3] for general systems with “real”, or partially parabolic viscosity, the difference lying in the strengthened block structure condition (A1): in particular, the assumed linearity of the $U_1$ equation. Conditions (H0)–(H2) are the same as those in [MaZ4, Z2, Z3]. The
class of equations satisfying our assumptions, though not complete, is sufficiently broad to include many models of physical interest, in particular compressible Navier–Stokes equations and the equations of compressible magnetohydrodynamics (MHD), expressed in Lagrangian coordinates, with either ideal or “real” van der Waals-type equation of state as described in Appendix A. See [TZ3] for further discussion/generalization.

Remark 1.1. Conditions (H1)–(H2) imply that $U_{\pm}$ are nonhyperbolic rest points of ODE (1.3) expressed in terms of the $U_2$-coordinate, whence, by standard ODE theory,

$$|\partial_x^r (\bar{U} - U_{\pm})(x)| \leq Ce^{-\eta|x|}, \quad 0 \leq r \leq k + 1, \tag{1.5}$$

for $x \geq 0$, some $\eta$, $C > 0$; in particular, $|\bar{U}''(x)| \leq Ce^{-\eta|x|}$.

By the change of coordinates $x \to x - st$, we may assume without loss of generality $s = 0$ as we shall do from now on. Linearizing (1.2) about the (now stationary) solution $U \equiv \bar{U}$ yields linearized equations

$$U_t = LU := -(AU)_x + (BU)_x, \tag{1.6}$$

$$B(x) := B(\bar{U}(x)), \quad A(x)V := dF(\bar{U}(x))V - (dB(\bar{U}(x))V)\bar{U}_x, \tag{1.7}$$

for which the generator $L$ possesses [He, S, ZH] both a translational zero-eigenvalue and essential spectrum tangent at zero to the imaginary axis.

Our first main result is the existence of a translation-invariant center stable manifold about $\bar{U}$. Define the mixed norm

$$|U|_{H^{1,2}} := \sqrt{|U_1|_{H^1}^2 + |U_2|_{H^2}^2} \tag{1.8}$$

suggested by the hyperbolic–parabolic structure (A1).

**Theorem 1.2.** Under assumptions (A1)–(A3), (H0)–(H2), there exists in an $H^{1,2}$ neighborhood of the set of translates of $\bar{u}$ a codimension-$p$ translation invariant Lipschitz (with respect to $H^{1,2}$) center stable manifold $\mathcal{M}_{cs}$, tangent to quadratic order at $\bar{U}$ to the center stable subspace $\Sigma_{cs}$ of $L$ in the sense that

$$|\Pi_u(U - \bar{U})|_{H^{1,2}} \leq C|\Pi_{cs}(U - \bar{U})|^2_{H^{1,2}} \tag{1.9}$$
for \( U \in \mathcal{M}_{cs} \) where \( \Pi_{cs} \) and \( \Pi_u \) denote the center-stable and stable eigenprojections of \( L \), that is (locally) invariant under the forward time-evolution of (1.2) and contains all solutions that remain bounded and sufficiently close to a translate of \( \bar{U} \) in forward time, where \( p \) is the (necessarily finite) number of unstable, i.e., positive real part, eigenvalues of \( L \).

Next, specializing a bit further, we add to (H0)–(H3) the additional hypothesis that \( \bar{u} \) be a Lax-type shock:

\( \text{(H3)} \) The dimensions of the unstable subspace of \( A_- \) and the stable subspace of \( A_+ \) sum to \( n + 1 \).

We assume further the following spectral genericity conditions.

\( \text{(D1)} \) \( L \) has no nonzero imaginary eigenvalues.
\( \text{(D2)} \) The orbit \( \bar{U}(\cdot) \) is a transversal connection of the associated standing wave equation (1.3).
\( \text{(D3)} \) The associated inviscid shock \( (U_-, U_+) \) is hyperbolically stable, i.e.,

\[
\det(r_1^- , \ldots , r_{P-1}^- , r_{P+1}^+ , \ldots , r_n^+ , (U_+ - U_-)) \neq 0,
\]

where \( r_1^- , \ldots , r_{P-1}^- \) denote eigenvectors of \( A_- \) associated with negative eigenvalues and \( r_{P+1}^+ , \ldots , r_n^+ \) denote eigenvectors of \( A_+ \) associated with positive eigenvalues.

As discussed in [ZH, MaZ1], (D2)–(D3) correspond in the absence of a spectral gap to a generalized notion of simplicity of the embedded eigenvalue \( \lambda = 0 \) of \( L \). Thus, (D1)–(D3) together correspond to the assumption that there are no additional (usual or generalized) eigenvalues on the imaginary axis other than the transational eigenvalue at \( \lambda = 0 \); that is, the shock is not in transition between different degrees of stability, but has stability properties that are insensitive to small variations in parameters.

With these assumptions, we obtain our second main result characterizing the stability properties of \( \bar{u} \). In the case \( p = 0 \), this reduces to the nonlinear orbital stability result established in [MaZ1, MaZ2, MaZ3, Z2].

**Theorem 1.3.** Under (A1)–(A3), (H0)–(H3), and (D1)–(D3), \( \bar{u} \) is nonlinearly orbitally stable under sufficiently small perturbations in \( L^1 \cap H^3 \) lying on the codimension \( p \) center stable manifold \( \mathcal{M}_{cs} \) of \( \bar{u} \) and its translates, where \( p \) is the number of unstable eigenvalues of \( L \), in the sense that, for some \( \alpha(\cdot) \),
all $L^p$,

\[ |U(x, t) - \bar{U}(x - \alpha(t))|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1 + \frac{1}{p})} |U(x, 0) - \bar{U}(x)|_{L^1 \cap H^3},\]

\[ |U(x, t) - \bar{U}(x - \alpha(t))|_{H^3} \leq C(1 + t)^{-\frac{1}{4}} |U(x, 0) - \bar{U}(x)|_{L^1 \cap H^3},\]

\[ \dot{\alpha}(t) \leq C(1 + t)^{-\frac{1}{2}} |U(x, 0) - \bar{U}(x)|_{L^1 \cap H^3},\]

\[ \alpha(t) \leq C |U(x, 0) - \bar{U}(x)|_{L^1 \cap H^3}.\]

Moreover, it is orbitally unstable with respect to small $H^{1,2}$ perturbations not lying in $\mathcal{M}$, in the sense that the corresponding solution leaves a fixed-radius neighborhood of the set of translates of $\bar{U}$ in finite time.

**Remark 1.4.** It is straightforward, combining the pointwise argument of [RZ] with the observations of [Z6] in the strictly parabolic case, to extend Theorem 1.3 to the case of nonclassical under- or overcompressive shocks with (D1)–(D3) suitably modified as described in [Z6]. This gives at the same time new information even in the Lax case, including convergence of the phase $\alpha$ to a limit $\alpha(\pm \infty)$ at rate $(1 + t)^{-1/2}$, with $|\dot{\alpha}| \leq C(1 + t)^{-1}$.

### 1.1 Discussion and open problems

For results of a similar nature in the context of the nonlinear Schrödinger equation, but obtained by rather different techniques, we refer the reader to [Sc]. The method of [Sc] consists of constructing the stable manifold to a known center manifold using detailed linearized and nonlinear estimates similar to those used to prove stability in the spectrally stable case. That is, the existence and stability arguments are carried out at the same time.

This makes possible a somewhat finer analysis than the one we carry out here (in keeping with the somewhat weaker stability properties for the nonlinear Schrödinger equation, from which the linearized operator about the wave possesses essential spectrum on the whole of the imaginary axis lying outside a finite ball), but also a somewhat less general one; in particular, it is required that the linearized operator possess no pure imaginary eigenvalues other than zero. Note that we make no assumptions on the pure imaginary spectrum of the linearized operator in our construction of the center stable manifold, but only in the proof of conditional stability.

An interesting issue pointed out in [TZ3] is that the strong block structure assumptions (A1)–(A3) used to obtain the key variational energy estimates on which our arguments are based hold for the equations of gas dynamics.
or MHD written in Lagrangian coordinates, but not in Eulerian coordinates, and indeed the energy estimates themselves do not hold in that case; see Appendix A, [TZ3] for further discussion. On the other hand, the Lagrangian and Eulerian formulations are equivalent by a uniquely specified (up to translation, which may be factored out) transformation, and so the results do hold in Eulerian coordinates. It is an interesting open question whether the results of this paper hold for general symmetric hyperbolic–parabolic systems as defined in [MaZ4, Z2, Z3], which include among other examples both Lagrangian and Eulerian formulations.

2 Existence of Center Stable Manifold

Defining the perturbation variable \( V := U - \bar{U} \), we obtain after a brief computation the nonlinear perturbation equations

\[
V_t - LV = N(V),
\]

where \( L \) as in (1.6) denotes the linearized operator about the wave and \( N = \mathcal{N}(V)_x \) is a quadratic order residual, with (recalling (A1))

\[
\mathcal{N}(V) := \left( B(\bar{U} + V)(\bar{U} + V)_x - B(\bar{U})\bar{U}_x - B(\bar{U})V_x - dB(\bar{U})V_\bar{U}_x \right) - \left( F(\bar{U} + V) - F(\bar{U}) - dF(\bar{U})V \right)
\]

\[
= \left( O(|V|^2) \right. \left. O(|V|^{1/2} + |V_2|) \right).
\]

We seek to construct a Lipschitz \( H^{1,2} \)-local center stable manifold about the equilibrium \( v \equiv 0 \), that is, a locally invariant Lipschitz manifold tangent at \( v \equiv 0 \) to the center stable subspace \( \Sigma_{cs} \), that is quadratic-order tangent in the sense that \( |\Pi_u V|_{H^{1,2}} \leq C|\Pi_{cs} V|^2_{H^{1,2}} \).

2.1 Preliminary estimates

Conditions \((A2)\)–\((A3)\) imply that \( \Re \sigma (i\xi A - \xi^2 B)_\pm \leq -\frac{\theta_0 |\xi|^2}{1 + |\xi|^2} \) for some \( \theta_0 > 0 \), for all \( \xi \in \mathbb{R} \) [KSh], which in turn implies by standard spectral theory [He] that the essential spectrum of \( L \) lies entirely in the neutrally stable complex half-plane \( \Re \lambda \leq 0 \), and the unstable half-plane \( \Re \lambda > 0 \) either contains finitely
many eigenvalues of finite multiplicity, or else consists entirely of eigenvalues, as is easily shown (by energy estimates, for example) to be impossible. Thus, there is a well-defined unstable subspace $\Sigma_u$ consisting of the direct sum of the finitely many generalized eigenfunctions in the positive half-plane, and an associated unstable eigenprojection $\Pi_u$. Likewise, there is a complementary center–stable subspace $\Sigma_{cs}$ determined by the range of the complementary projection $\Pi_{cs} := \text{Id} - \Pi_u$.

**Proposition 2.1 ([MaZ3, Z2, Z3]).** Under assumptions (A1)–(A3), (H0)–(H2), $L$ generates a $C^0$ semigroup $e^{Lt}$ satisfying

\begin{align*}
|e^{tL}\Pi_{cs}|_{L^2 \to L^2} &\leq C_\omega e^{\omega t}, \\
|e^{-tL}\Pi_u|_{L^2 \to H^{1,2}} &\leq C_\omega e^{-\beta t},
\end{align*}

for some $\beta > 0$, and for all $\omega > 0$, for all $t \geq 0$.

**Proof.** Straightforward energy estimates yield resolvent bounds verifying the $C^0$-semigroup property; see [MaZ3, Z2] and especially [Z3], Prop. 3.6. The same estimates yield

$$|(\lambda - L)^{-1}|_{L^2 \to L^2} \leq C(\eta)$$

for $\lambda \geq \eta$ and $|\lambda|$ sufficiently large, $\eta > 0$ arbitrary, whereupon (2.3)(i) follows by Prüss’ theorem restricted to the center–stable subspace [Pr, ProK, KS]; see [Z3], Appendix A for further discussion, in particular Rmk. 6.23. Bound (2.3)(ii) follows by equivalence of norms for finite-dimensional spaces and standard finite-dimensional ODE estimates. \qed

Introducing a $C^\infty$ cutoff function

$$\rho(x) = \begin{cases} 1 & |x| \leq 1, \\ 0 & |x| \geq 2, \end{cases}$$

and recalling mixed-norm definition (1.8), let

$$N_\delta(V) := \rho\left(\frac{|V|_{H^{1,2}}}{\delta}\right)N(V).$$

**Lemma 2.2.** Assuming (A1)–(A3), (H0)–(H2), the map $N_\delta : H^{1,2} \to L^2$ is $C^{k+1}$ and its Lipschitz norm with respect to $V$ is $O(\delta)$ as $\delta \to 0$. Moreover,

\begin{equation}
|N_\delta(V)|_{L^2} \leq C|V|_{H^{1,2}}^2.
\end{equation}
Proof. The norm in $H^{1,2}$ is a quadratic form, hence the map

$$V \in H^{1,2} \mapsto \rho \left( \frac{|V| H^2}{\delta} \right) \in \mathbb{R}_+$$

is smooth, and $N^\delta$ is as regular as $N$. By Moser’s inequality,

$$|N(V)|_{L^2} \leq C(|V|_{H^1}|V|_{L^\infty} + |\partial_x V_2|_{H^1}|V|_{L^\infty} + |\partial_x V_2|_{L^\infty}|V|_{H^1}) \leq C|V|_{H^{1,2}}^2$$

and, similarly, $|N(V^1) - N(V^2)|_{L^2} \leq C|V^1 - V^2|_{H^{1,2}} \sup_{j=1,2} C|V_j|_{H^{1,2}}$ and thus $|dN|_{H^{1,2} \rightarrow L^2} \leq C|V|_{H^{1,2}}$ so long as $|V|_{H^{1,2}}$ remains bounded, in particular for $|V|_{H^{1,2}} \leq \delta$. Thus, $|N^\delta(V)|_{L^2} \leq |N(V)|_{L^2} \leq C|V|_{H^{1,2}}^2$ for $|V|_{H^{1,2}} \leq \delta$, while $N^\delta(V) = 0$ for $|V|_{H^{1,2}} \geq \delta$, verifying (2.4). The Lipschitz bound follows, likewise, by

$$|N^\delta(V^1) - N^\delta(V^2)|_{L^2} \leq \rho \left( \frac{|V^1|_{H^{1,2}}}{\delta} \right) - \rho \left( \frac{|V^2|_{H^{1,2}}}{\delta} \right) \left| L^\infty, |N(V^1)|_{L^2} \right.
\left. + \rho \left( \frac{|V^2|_{H^{1,2}}}{\delta} \right) \left| L^\infty, |N(V^1) - N(V^2)|_{L^2} \right.
\leq 3|V^1 - V^2|_{H^{1,2}}
\times \left( sup \frac{|N(V)|_{L^2}}{\delta} + sup |dN(V)|_{H^{1,2} \rightarrow L^2} \right).$$

\hfill\Box

Corollary 2.3. Under assumptions (A1)–(A3), (H0)–(H2),

$$|e^{tL} \Pi_{cs} N^\delta|_{H^{1,2} \rightarrow L^2} \leq C_\omega \delta e^{\omega t},$$

$$|e^{-tL} \Pi_u N^\delta|_{H^{1,2} \rightarrow H^{1,2}} \leq C_\omega \delta e^{-\beta t},$$

for some $\beta > 0$, and for all $\omega > 0$, for all $t \geq 0$, with Lipschitz bounds

$$|e^{tL} \Pi_{cs} dN^\delta|_{H^{1,2} \rightarrow L^2} \leq C_\omega \delta e^{\omega t},$$

$$|e^{-tL} \Pi_u dN^\delta|_{H^{1,2} \rightarrow H^{1,2}} \leq C_\omega \delta e^{-\beta t}.$$

2.2 Fixed-point iteration scheme

Applying projections $\Pi_j$, $j = cs, u$ to the truncated equation

$$V_t - LV = N^\delta(V),$$

(2.7)
we obtain using the variation of constants formula equations

\[
\Pi_j V(t) = e^{L(t-t_{0,j})} \Pi_j V(t_{0,j}) + \int_{t_{0,j}}^{t} e^{L(t-s)} \Pi_j N^\delta(V(s)) \, ds,
\]

\( j = cs, u, \) so long as the solution \( V \) exists, with \( t_{0,j} \) arbitrary. Assuming growth of at most \( |V(t)|_{H^2} \leq Ce^{\tilde{\theta} t} \) in positive time, we find for \( j = u \) using bounds (2.3)(ii) and (2.6)(ii) that, as \( t_{0,u} \rightarrow +\infty \), the first term \( e^{L(t-t_{0,u})} \Pi_u V(t_{0,u}) \) converges to zero while the second, integral term converges to \( \int_{t}^{\infty} e^{L(t-s)} \Pi_u N^\delta(V(s)) \, ds \). So that, denoting \( w := \Pi_{cs} V, z := \Pi_u V, \) we have

\[
(2.8) \quad z(t) = \mathcal{T}(z,w)(t) := -\int_{t}^{\infty} e^{L(t-s)} \Pi_u N^\delta((w+z)(s)) \, ds.
\]

Likewise, choosing \( t_{0,cs} = 0 \), we have

\[
(2.9) \quad w(t) = e^{Lt} \Pi_{cs} w_0 + \int_{0}^{t} e^{L(t-s)} \Pi_{cs} N^\delta((w+z)(s)) \, ds,
\]

\( w_0 := \Pi_{cs} V(0) \). On the other hand, we find from the original differential equation projected onto the center stable component that \( w \) satisfies the Cauchy problem

\[
(2.10) \quad w_t - Lw = \Pi_{cs} N^\delta(w+z)
\]

with initial data \( w_0 = \Pi_{cs} V(0) \) given at \( t = 0 \). We shall use these two representations together to obtain optimal estimates, the first for decay, through standard linear semigroup estimates, and the second for regularity, through the nonlinear damping estimates (2.14)–(2.15) and (2.18)–(2.19) below.

Viewing (2.9), or alternatively (2.10), as determining \( w = \mathcal{W}(z,w_0) \) as a function of \( z \), we seek \( z \) as a solution of the fixed-point equation

\[
(2.11) \quad z = \tilde{T}(z,w_0) := \mathcal{T}(z,\mathcal{W}(z,w_0)).
\]

As compared to the standard ODE construction of, e.g., [B, VI, TZ1, Z5], in which (2.8)–(2.9) together are considered as a fixed-point equation for the joint variable \((w,z)\), this amounts to treating \( w \) implicitly. This is a standard device in situations of limited regularity; see, e.g., [CP, GMWZ, RZ].

It remains to show, first, that \( \mathcal{W} \), hence \( \tilde{T} \), is well-defined on a space of slowly-exponentially-growing functions and, second, that \( \tilde{T} \) is contractive on that space, determining a \( C^k \) solution \( z = z(w_0) \) similarly as in the usual ODE construction. We carry out these steps in the following subsections.
2.3 Nonlinear energy estimates

Define now the negatively-weighted sup norm

\[ \|f\|_{-\eta} := \sup_{t \geq 0} e^{-\eta t} |f(t)|_{H^1}, \]

noting that \( |f(t)|_{H^2} \leq e^{\tilde{\theta} t} \|f\|_{-\tilde{\theta}} \) for all \( t \geq 0 \), and denote by \( \mathcal{B}_{-\eta} \) the Banach space of functions bounded in \( \| \cdot \|_{-\eta} \) norm. Define also the auxiliary norm

\[ \|f\|_{L^2_{-\eta}} := \sup_{t \geq 0} e^{-\eta t} |f(t)|_{L^2}. \]

Lemma 2.4 ([Z5]). Under assumptions (A1)–(A3), (H0)–(H2), for all \( 1 \leq p \leq \infty, 0 \leq r \leq 4 \),

\[
\| \Pi_u \|_{L^p \to W^{r,p}}, \| \Pi_{cs} \|_{W^{r,p} \to W^{r,p}} \leq C.
\]

Proof. Recalling that \( L \) has at most finitely many unstable eigenvalues, we find that \( \Pi_u \) may be expressed as

\[ \Pi_u f = \sum_{j=1}^{p} \phi_j(x) \langle \tilde{\phi}_j, f \rangle, \]

where \( \phi_j, j = 1, \ldots, p \) are generalized right eigenfunctions of \( L \) associated with unstable eigenvalues \( \lambda_j \), satisfying the generalized eigenvalue equation \( (L - \lambda_j)^{r_j} \phi_j = 0, r_j \geq 1, \) and \( \tilde{\phi}_j \) are generalized left eigenfunctions. Noting that \( \phi_j, \phi_j \) and derivatives decay exponentially by standard theory [He, ZH, MaZ1], and estimating

\[ |\partial^j_x \Pi_u f|_{L^p} = \sum_j |\partial^j_x \phi_j \langle \tilde{\phi}_j f \rangle|_{L^p} \leq \sum_j |\partial^j_x \phi_j|_{L^p} |\tilde{\phi}_j|_{L^q} |f|_{L^p} \leq C |f|_{L^p} \]

for \( 1/p + 1/q = 1 \), we obtain the claimed bounds on \( \Pi_u \), from which the bounds on \( \Pi_{cs} = \text{Id} - \Pi_u \) follow immediately. \( \square \)

Proposition 2.5. Assuming (A1)–(A3), (H0)–(H2), denote \( A^0 := A^0(\bar{u}(x)) \) and take without loss of generality \( \sigma(A_{11}) < 0 \). Then, there exist a skew-symmetric matrix \( K(x) \) and constants \( C > 0, C_\sigma > 0 \) such that, defining the scalar weight \( \alpha \) by \( \alpha(0) = 1 \) and \( \alpha_x = C_\sigma |\bar{U}| \alpha \), the quadratic form

\[ \mathcal{E}(w) := \langle (C^2 A^0 + CK \partial_x - A^0 \partial_x^2) \alpha w, w \rangle \]
is equivalent to $|w|^2_{H^1}$ and, for $w_t - Lw = \begin{pmatrix} 0 \\ r \end{pmatrix}$,

$$\partial_t \mathcal{E}(w) \leq -\theta |w|_{H^1}^2 + C(|f|_{L^2} + |w|_{L^2}^2).$$

(2.13)

**Proof.** This follows by a linear version of the argument for Proposition 4.15, [Z3], somewhat simplified by the stronger block structure assumptions (A1)–(A3) made here. We first observe that, by symmetry of $(A^0 A)_{11}$ and $(A^0 A)_\pm$,

$$A^0 A(x) = \tilde{A}(x) + \begin{pmatrix} 0 & 0 \\ O(|\bar{U}_x|) & O(|\bar{U}_x|) \end{pmatrix}$$

where $\tilde{A}$ is the symmetric part of $A^0 A$. Next, we recall from [Kaw, KSh] that (A2)–(A3) imply existence of skew-symmetric $K_\pm$ such that $\Re(K A + B)_\pm > 0$. Defining $K(x)$ as a smooth interpolant between $K_\pm$ defined by arc length along the path of $\bar{U}$ between $U_\pm$, we obtain the result by a straightforward (if somewhat lengthy) computation for $C > 0$ and $C_\ast > C > 0$ sufficiently large, where “good” terms

$$-\langle \partial^2_x w, \alpha A^0 B \partial^2_x w \rangle \leq -\theta \langle \partial^2_x w_2, \alpha \partial^2_x w_2 \rangle,
- C^2 \langle \partial_x w, \alpha A^0 B \partial_x w \rangle \leq -C^2 \theta \langle \partial_x w_2, \alpha \partial_x w_2 \rangle,
- C \langle \partial_x w, \Re(K A + A^0 B) \alpha \partial_x w \rangle \leq -C \theta \langle \partial_x w, \alpha \partial_x w \rangle + C_2 \langle \partial_x w_1, |\bar{U}_x| \alpha \partial_x w_1 \rangle + C_2 \langle \partial_x w_2, \alpha \partial_x w_2 \rangle,$$

and

$$-\langle \partial_x w_1, (A^0 A)_{11} \partial^2_x w_1 \rangle = \langle \partial_x w_1, \partial_x (A^0 A)_{11} \cdot \partial_x w_1 \rangle \leq -\theta C_\ast \langle \partial_x w_1, |\bar{U}_x| \alpha \partial_x w_1 \rangle$$

resulting by rearrangement/integration by parts of the various components of $\partial_t \mathcal{E}(w)$, when summed together, after absorbing remaining terms, give a contribution of $-\theta |w|_{H^1}^2 + C(|w|_{L^2}^2 + |f|_{L^2}^2)$, verifying the result. See [MaZ4, Z2, Z3] for further details. \qed

**Corollary 2.6.** Under assumptions (A1)–(A3), (H0)–(H2), for $\|z\|_{L^\infty}$ bounded and $\delta$ sufficiently small, the solution $w$ of (2.10) exists for all $t \geq 0$ and, for any constant $\theta > 0$ and some $C = C(\theta)$,

$$\int_0^t e^{-\theta s} |w|_{H^1}^2(s) ds + e^{-\theta t} |w_t|_{H^1}^2 \leq C |w_0|_{H^1}^2 + C \int_0^t e^{-\theta s} (|w|_{L^2}^2 + |z|_{L^2}^2)(s) ds;$$

(2.14)
likewise, for $\theta > 2\eta$,
\begin{equation}
\int_{t}^{+\infty} e^{\theta(t-s)}|w|_{H^{1,2}}^{2}(s) \, ds \leq C|w|_{H^{1}}^{2}(t) + C\int_{t}^{+\infty} e^{\theta(t-s)}(|w|_{L^{2}}^{2} + |z|_{L^{2}}^{2})(s) \, ds \leq C e^{\theta t}|w_{0}|_{H^{1}}^{2} + C e^{2\eta t}(|w|_{L^{2}}^{2} + |z|_{L^{2}}^{2}).
\end{equation}

**Proof.** Observing that
\begin{equation}
|\Pi_{C_{N}^{\delta}}(w + z)|_{L^{2}} \leq \left( C\delta(|w|_{H^{1,2}} + |z|_{H^{1,2}}) \right),
\end{equation}
we obtain by Proposition 2.5 the inequality
\[
\partial_{t}\mathcal{E}(w) \leq -\tilde{\theta}|w|_{H^{1,2}} + C\delta(|w|_{H^{1,2}}^{2} + |z|_{H^{1,2}}^{2}) + C(|w|_{L^{2}}^{2} + |z|_{L^{2}}^{2})
\]
for some $\tilde{\theta} > 0$, which, for $\delta$ sufficiently small, gives
\[
\partial_{t}\mathcal{E}(w) \leq -\tilde{\theta}|w|_{H^{1,2}} + C_{2}(|w|_{L^{2}}^{2} + |z|_{L^{2}}^{2})
\]
by $|z|_{H^{1,2}} \leq C|z|_{L^{2}}$ (equivalence of finite-dimensional norms). From this inequality, global $H^{1}$ existence follows immediately and (2.14) and the first line of (2.15) follow readily by Gronwall’s inequality together with equivalence of $\mathcal{E}$ and $|w|_{H^{1}}^{2}$. For further discussion, see [Z6], proof of Proposition 2.6.

Bounding $|w(t)|_{H^{1}}^{2}$ in the second line of (2.15) using (2.14) now yields
\begin{equation}
\int_{t}^{+\infty} e^{\theta(t-s)}|w|_{H^{1,2}}^{2}(s) \, ds \leq C e^{\theta t} \left( |w_{0}|_{H^{1}}^{2} + \int_{0}^{+\infty} e^{-\theta s}(|w|_{L^{2}}^{2} + |z|_{L^{2}}^{2})(s) \, ds \right),
\end{equation}
whereupon the final line of (2.15) then follows by
\[
\int_{0}^{+\infty} e^{\theta(t-s)}(|\dot{w}|_{L^{2}}^{2} + |\dot{z}|_{L^{2}}^{2})(s) \, ds \leq \left( e^{\theta t} \int_{t}^{+\infty} e^{(2\eta-\theta)s} \, ds \right) \left( ||\dot{w}||_{L^{2}}^{2} + ||\dot{z}||_{L^{2}}^{2} \right) \leq C e^{2\eta t} (||\dot{w}||_{L^{2}}^{2} + ||\dot{z}||_{L^{2}}^{2}).
\]

**Corollary 2.7.** Under (A1)–(A3), (H0)–(H2), for $\|z_{1}\|_{L^{2}_{\eta}}$, $\|z_{2}\|_{L^{2}_{\eta}}$ bounded and $\delta$ sufficiently small, solutions $w_{1}, z_{1}$ and $w_{2}, z_{2}$ of (2.10) exist for all $t \geq 0$ and, for any constant $\theta > 0$ and some $C = C(\theta)$,
\begin{equation}
\int_{0}^{t} e^{-\theta s}|w_{1} - w_{2}|_{H^{1,2}}^{2}(s) \, ds \leq C|w_{0,1} - w_{0,2}|_{H^{1}}^{2} + C \int_{0}^{t} e^{-\theta s}(|w_{1} - w_{2}|_{L^{2}}^{2} + |z_{1} - z_{2}|_{L^{2}}^{2})(s) \, ds;
\end{equation}
and setting \( \parallel \cdot \parallel \) the integral representation (2.9), and applying (2.3)(i), (2.6)(i), and the definition \( \parallel \cdot \parallel \), solutions, denoting (\( \dot{z}, \dot{w} \)), \( w \) the unique solution \( w \). 

Thus, we need only verify the estimates.

Proof. Subtracting the equations for \( w_1, z_1 \) and \( w_2, z_2 \), we obtain, denoting \( \dot{w} := w_1 - w_2, \dot{z} := z_1 - z_2 \), the equation

\[
\dot{w} - L \dot{w} = \Pi_{c_s}(N^\delta(w_1 + z_1) - N^\delta(w_2 + z_2))
\]

with initial data \( \dot{w}_0 = w_{01} - w_{02} \) at \( t = 0 \), whence the result follows by the Lipshitz bound \( \delta \) on \( N^\delta : H^{1,2} \to L^2 \) and the same argument used in the proof of Corollary 2.6.

**Corollary 2.8.** Under assumptions \((A1)-(A3)\), \((H0)-(H2)\), for \( 3\omega < \eta < \beta \) and \( \delta > 0 \) and \( w_0 \in H^2 \) sufficiently small, for each \( z \in B_{-\eta} \), there exists a unique solution \( w := W(z, w_0) \in B_{-\eta} \) of (2.9), (2.10), with

\[
\|w\|_{L^2_{-\eta}} \leq C(\|w_0\|_{H^{1,2}} + \|z\|_{-\eta})
\]

and

\[
\|W(z_1, w_{01}) - W(z_2, w_{02})\|_{L^2_{-\eta}} \leq C\delta\|z_1 - z_2\|_{-\eta} + C\|w_{01} - w_{02}\|_{H^{1,2}}.
\]

Proof. We have already shown existence, while uniqueness follows from (2.22). Thus, we need only verify the estimates.

Consider a pair of data \( z_1, w_{01} \) and \( z_2, w_{02} \), and compare the resulting solutions, denoting (\( \dot{z}, \dot{w}, \dot{w}_0 \)) := \( (z_1 - z_2, w_1 - w_2, w_{01} - w_{02}) \). Using the integral representation (2.9), and applying (2.3)(i), (2.6)(i), and the definition of \( \parallel \cdot \parallel_{-\eta} \), we obtain for all \( t \geq 0 \)

\[
\|\dot{w}(t)\|_{L^2} \leq Ce^{\omega t}\|\dot{w}_0\|_{L^2} + C\delta \int_0^t e^{\omega (t-s)}(|\dot{w}|_{H^2} + |\dot{z}|_{H^2})(s) \, ds.
\]

Estimating

\[
C\delta \int_0^t e^{\omega (t-s)}|\dot{z}|_{H^2}(s) \, ds \leq C\delta\|\dot{z}\|_{-\eta} \int_0^t e^{\omega (t-s)}e^{\eta s} \, ds \leq C\delta\|\dot{z}\|_{-\eta}e^{\eta t}
\]
and, by (2.18) with $\theta = 3\omega$ together with the Cauchy–Schwarz inequality,

$$C\delta \int_0^t e^{\omega(t-s)}|\dot{w}|_{H^2}(s) \, ds \leq C\delta \left( \int_0^t e^{-\omega(t-s)} \right)^{1/2} \left( \int_0^t e^{3\omega(t-s)} |\dot{w}|^2_{H^2}(s) \, ds \right)^{1/2}$$

$$\leq C\delta \left( e^{3\omega t} |w_0|^2_{H^1} + \int_0^t e^{3\omega(t-s)} (|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2}(s)) \, ds \right)^{1/2},$$

we obtain, substituting in (2.23), (2.24)

$$|\dot{w}(t)|^2_{L^2} \leq C e^{6\omega |t|} |\dot{w}_0|^2_{H^1} + C \delta^2 \left( \|\ddot{z}\|^2_{-\eta} e^{2\eta t} + \int_0^t e^{3\omega(t-s)} (|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2}(s)) \, ds \right)$$

$$\leq e^{2\eta t} \left( C |\dot{w}_0|^2_{H^1} + C \delta^2 (\|\dot{w}\|^2_{L^2_{-\eta}} + \|\dot{z}\|^2_{-\eta}) \right).$$

This yields

$$\|\dot{w}(t)\|_{L^2_{-\eta}} \leq C |\dot{w}_0|_{-\eta} + C \delta (\|\dot{w}\|_{L^2_{-\eta}} + \|\dot{z}\|_{-\eta}),$$

from which (2.22) follows by smallness of $\delta$. The bound (2.21) follows similarly.

\[\square\]

### 2.4 Basic existence result

**Proof of Proposition 1.2 without translational independence. (i) (Contraction mapping argument)** We find using (2.22), (2.6)(ii), and Lemma 2.2 that

$$\|\ddot{T}(z_1, w_{0,1}) - \ddot{T}(z_2, w_{0,2})\|_{-\eta}$$

(2.25)

$$\leq \sup_t C\delta e^{-\eta t} \int_t^{+\infty} e^{\beta(t-s)} (|\dot{w}|_{H^2} + |\dot{z}|_{H^2}(s)) \, ds$$

$$\leq \sup_t C_1 \delta \left( \|\ddot{z}\|_{-\eta} + e^{-\eta t} \int_t^{+\infty} e^{\beta(t-s)} |\dot{w}|_{H^2}(s) \, ds \right).$$

Using the Cauchy–Schwarz inequality and (2.19) with $\theta = \beta$ to estimate

$$\int_t^{+\infty} e^{\beta(t-s)} |\dot{w}|_{H^2}(s) \, ds \leq \left( \int_t^{+\infty} e^{\beta(t-s)} \, ds \right)^{1/2} \left( \int_t^{+\infty} e^{\beta(t-s)} |\dot{w}|^2_{H^2}(s) \, ds \right)^{1/2}$$

$$\leq C_3 \left( |\dot{w}_0|^2_{H^1} + \int_0^t e^{\beta(t-s)} (|\dot{w}|^2_{L^2} + |\dot{z}|^2_{L^2}(s)) \, ds \right)^{1/2}$$

$$\leq C_3 e^{\eta t} \left( |\dot{w}_0|^2_{H^1} + \|\dot{w}\|^2_{L^2_{-\eta}} + \|\dot{z}\|_{L^2_{-\eta}} \right)$$

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and applying (2.22), we obtain
\[(2.26) \quad \|\hat{T}(z_1, w_{0,1}) - \hat{T}(z_2, w_{0,2})\|_{\gamma} \leq C\delta(\|\dot{w}_0\|_{H^{1,2}} + \|\dot{z}\|_{L^2_{\gamma}}).\]
Parallel estimates yield
\[
\|\hat{T}(z, w_0)(t)\|_{\gamma} \leq C\delta(\|w_0\|_{H^{1,2}} + \delta\|z\|_{\gamma}),
\]
so that, taking \(\delta\) and \(|w_0|_{H^{1,2}}\) sufficiently small, that \(\hat{T}(\cdot, w_0)\) maps the ball \(B(0, r) \subset B_{-\eta}\) to itself, for \(r > 0\) arbitrarily small but fixed.

This yields at once contractivity on \(B(0, r)\), hence existence of a unique fixed point \(z = Z(w_0)\), and Lipshitz continuity of \(Z\) from \(\Sigma_{cs}\) to \(B_{-\eta}\), by the Banach Fixed-Point Theorem with Lipschitz dependence on parameter \(w_0\).

(ii) (Existence of a Lipschitz invariant manifold) Defining
\[
(2.27) \quad \Phi(w_0) := Z(w_0)|_{t=0} = -\int_0^{+\infty} e^{-Ls} \Pi_u N^\delta(v(s)) \, ds,
\]
we obtain a Lipschitz function from \(\Sigma_{cs} \to \Sigma_u\), whose graph over \(B(0, r)\) is the invariant manifold of solutions of (2.7) growing at exponential rate \(|v(t)| \leq Ce^{\eta t}\) in forward time. From the latter characterization, we obtain evidently invariance in forward time. Since the truncated equations (2.7) agree with the original PDE so long as solutions remain small in \(H^{1,2}\), this gives local invariance with respect to (1.2) as well. By uniqueness of fixed point solutions, we have \(Z(0) = 0\) and thus \(\Phi_{cs}(0) = 0\), so that the invariant manifold passes through the origin. Likewise, any bounded, sufficiently small solution of (2.1) in \(H^{1,2}\) is a bounded small solution of (2.7) as well, so by uniqueness is contained in the center stable manifold.

(iii) (Quadratic-order tangency) By (2.27), (2.3), (2.4), (2.15), amd (2.21),
\[
|\Phi(w_{cs})|_{H^{1,2}} = \left| \int_0^{+\infty} e^{-Ls} \Pi_u N^\delta(w + z)(s) \, ds \right|_{H^{1,2}} \\
\leq C \int_0^{+\infty} e^{-\beta s}(\|N^\delta(v)\|_{H^{1,2}}^2 + |Z|_{H^{1,2}}(s)^2) \, ds \\
\leq C_2(\|w_{cs}\|_{H^{1,2}}^2 + \|Z\|_{\gamma}^2).
\]
By \(Z(0) = 0\) and Lipshitz continuity of \(Z\), we have \(\|Z\|_{\gamma} \leq C|w_{cs}|_{H^{1,2}}\), whence
\[
(2.28) \quad |\Phi(w_{cs})|_{H^{1,2}} \leq C_3 |w_{cs}|_{H^{1,2}}^2,
\]
verifying quadratic-order tangency at the origin.
2.5 Translation-invariance

We conclude by indicating briefly how to recover translation-invariance of the center stable manifold, following [Z5, TZ1]. Differentiating with respect to $x$ the traveling-wave ODE, we recover the standard fact that $\phi := \bar{U}_x$ is an $L^2$ zero eigenfunction of $L$, by the decay of $\bar{U}_x$ noted in Remark 1.5.

Define orthogonal projections

\begin{equation}
\Pi_2 := \frac{\phi \langle \phi, \cdot \rangle}{|\phi|^2_{L^2}}, \quad \Pi_1 := \text{Id} - \Pi_2,
\end{equation}

onto the range of right zero-eigenfunction $\phi := \bar{U}_x$ of $L$ and its orthogonal complement $\phi^\perp$ in $L^2$, where $\langle \cdot, \cdot \rangle$ denotes standard $L^2$ inner product.

**Lemma 2.9.** Under the assumed regularity $h \in C^{k+1}$, $k \geq 2$, $\Pi_j$, $j = 1, 2$ are bounded as operators from $H^s$ to itself for $0 \leq s \leq k + 2$.

**Proof.** Immediate, by the assumed decay of $\phi = \bar{u}_x$ and derivatives. \qed

**Proof of Proposition 1.2, with translational independence.** Introducing the shifted perturbation variable

\begin{equation}
V(x, t) := U(x + \alpha(t), t) - \bar{U}(x)
\end{equation}

we obtain the modified nonlinear perturbation equation

\begin{equation}
\partial_t V = LV + \mathcal{N}(V)_x - \partial_t \alpha \phi + \partial_x V,
\end{equation}

where $L := \frac{\partial F}{\partial U}(\bar{U})$ and $\mathcal{N}$ as in (2.2) is a quadratic-order Taylor remainder.

Choosing $\partial_t \alpha$ so as to cancel $\Pi_2$ of the righthand side of (2.31), we obtain the reduced equations

\begin{equation}
\partial_t V = \Pi_1(LV + \mathcal{N}(V))
\end{equation}

and

\begin{equation}
\partial_t \alpha = \frac{\pi_2(LV + \mathcal{N}(V))}{1 + \pi_2(\partial_x V)}
\end{equation}

$V \in \phi^\perp$, $\pi_2 V := \langle \tilde{\phi}, V \rangle |\phi|^2_{L^2}$, of the same regularity as the original equations. $V(0) \in \phi^\perp$, or

\[ \langle \phi, U_0(x + \alpha) - \bar{U}(x) \rangle = 0. \]
Assuming that $U_0$ lies in a sufficiently small tube about the set of translates of $\bar{U}$, or $U_0(x) = (\bar{U} + W)(x - \beta)$ with $|W|_{H^{1,2}}$ sufficiently small, this can be done in a unique way such that $\hat{\alpha} := \alpha - \beta$ is small, as determined implicitly by

$$0 = \mathcal{G}(W, \hat{\alpha}) := \langle \phi, \bar{U}(\cdot + \hat{\alpha}) - \bar{U}(\cdot) \rangle + \langle \phi, W(\cdot + \alpha) \rangle,$$

an application of the Implicit Function Theorem noting that

$$\partial_\alpha \mathcal{G}(0, 0) = \langle \phi, \partial_x \bar{U}(\cdot) \rangle = |\phi|^2_{L^2} \neq 0.$$

With this choice, translation invariance under our construction is clear, with translation corresponding to a constant shift in $\alpha$ that is preserved for all time.

Clearly, (2.33) is well-defined so long as $|\partial_x V|_{L^2} \leq C|V|_{H^{1,2}}$ remains small, hence we may solve the $v$ equation independently of $\alpha$, determining $\alpha$-behavior afterward to determine the full solution

$$U(x, t) = \bar{U}(x - \alpha(t)) + V(x - \alpha(t), t).$$

Moreover, it is easily seen (by the block-triangular structure of $L$ with respect to this decomposition) that the linear part $\Pi_1 L = \Pi_1 L \Pi_1$ of the $v$-equation possesses all spectrum of $L$ apart from the zero eigenvalue associated with eigenfunction $\phi$. Thus, we have effectively projected out this zero-eigenfunction, and with it the group symmetry of translation.

We may therefore construct the center stable manifold for the reduced equation (2.32), automatically obtaining translation-invariance when we extend to the full evolution using (2.33). See [TZ1, Z5] for further details.

3 Conditional stability analysis

Define similarly as in Section 2.5 the perturbation variable

$$(3.1) \quad V(x, t) := U(x + \alpha(t), t) - \bar{U}(x)$$

for $U$ a solution of (1.2), where $\alpha$ is to be specified later. Subtracting the equations for $U(x + \alpha(t), t)$ and $\bar{U}(x)$, we obtain the nonlinear perturbation equation

$$(3.2) \quad V_t - LV = \mathcal{N}(V) - \hat{\alpha}(U_x + V_x),$$
where $L := -A\partial_x + \partial_x B\partial_x$ as in (1.6) denotes the linearized operator about $\bar{U}$ and $\mathcal{N}$ as in (2.2) is a nonlinear residual, satisfying

$$\mathcal{N}(V) = O(|V|^2 + |V||V_x|),$$
$$\partial_x \mathcal{N}(V) = O(|V||\partial_x V| + |V|^2 + |V||V_{xx}|)$$

so long as $|V|_{H^1}$ (hence $|V|_{L^\infty}$ and $|U|_{L^\infty}$) remains bounded.

### 3.1 Projector bounds

Let $\Pi_u$ denote the eigenprojection of $L$ onto its unstable subspace $\Sigma_u$, and $\Pi_{cs} = \text{Id} - \Pi_u$ the eigenprojection onto its center stable subspace $\Sigma_{cs}$.

**Lemma 3.1.** Assuming (H0)–(H1),

$$\Pi_j \partial_x = \partial_x \tilde{\Pi}_j$$

for $j = u, cs$ and, for all $1 \leq p \leq \infty$, $0 \leq r \leq 4$,

$$|\Pi_u|_{W^{r,p} \rightarrow W^{r,p}} , |\tilde{\Pi}_u|_{W^{r,p} \rightarrow W^{r,p}} \leq C,$$

$$|\Pi_{cs}|_{W^{r,p} \rightarrow W^{r,p}} , |\tilde{\Pi}_{cs}|_{W^{r,p} \rightarrow W^{r,p}} \leq C.$$

**Proof.** Recalling (see the proof of Theorem 1.2) that $L$ has at most finitely many unstable eigenvalues, we find that $\Pi_u$ may be expressed as

$$\Pi_u f = \sum_{j=1}^{p} \phi_j(x) \langle \tilde{\phi}_j, f \rangle,$$

where $\phi_j$, $j = 1, \ldots, p$ are generalized right eigenfunctions of $L$ associated with unstable eigenvalues $\lambda_j$, satisfying the generalized eigenvalue equation $(L - \lambda_j)^{r_j} \phi_j = 0$, $r_j \geq 1$, and $\tilde{\phi}_j$ are generalized left eigenfunctions. Noting that $L$ is divergence form, and that $\lambda_j \neq 0$, we may integrate $(L - \lambda_j)^{r_j} \phi_j = 0$ over $\mathbb{R}$ to obtain $\lambda_j^{r_j} \int \phi_j dx = 0$ and thus $\int \phi_j dx = 0$. Noting that $\phi_j$, $\tilde{\phi}_j$ and derivatives decay exponentially by standard theory [He, ZH, MaZ1], we find that

$$\phi_j = \partial_x \Phi_j$$

with $\Phi_j$ and derivatives exponentially decaying, hence

$$\tilde{\Pi}_u f = \sum_j \Phi_j \langle \partial_x \tilde{\phi}_j, f \rangle.$$
Estimating

\[ |\partial_j^i \Pi_u f|_{L^p} = \sum_j |\partial_j^i \phi_j \langle \hat{\phi}_j f \rangle|_{L^p} \leq \sum_j |\partial_j^i \phi_j|_{L^p} |\hat{\phi}_j| f|_{L^p} \leq C|f|_{L^p} \]

for \(1/p + 1/q = 1\) and similarly for \(\partial_r^i \tilde{\Pi}_u f\), we obtain the claimed bounds on \(\Pi_u\) and \(\tilde{\Pi}_u\), from which the bounds on \(\Pi_{cs} = \text{Id} - \Pi_u\) and \(\tilde{\Pi}_{cs} = \text{Id} - \tilde{\Pi}_u\) follow immediately.

### 3.2 Linear estimates

Let \(G_{cs}(x, t; y) := \Pi_{cs} e^{Lt} \delta_y(x)\) denote the Green kernel of the linearized solution operator on the center stable subspace \(\Sigma_{cs}\). Then, we have the following detailed pointwise bounds established in [TZ2, MaZ1].

**Proposition 3.2** ([TZ2, MaZ1]). Under (A0)–(A3), (H0)–(H3), (D1)–(D3), the center stable Green function may be decomposed as

\[ G_{cs} = E + H + \tilde{G}, \]

where

\[ E(x, t; y) = \partial_x \bar{U}(x)e_j(y, t), \]

\[ e(y, t) = \sum_{a_k > 0} \left( \text{erf}(y + (a_k - t)/\sqrt{4t}) - \text{erf}(y - a_k t) \right) l_k^-(y) \]

for \(y \leq 0\) and symmetrically for \(y \geq 0\), \(l_k^- \in \mathbb{R}^n\) constant,

\[ H(x, t; y) = \sum_{j=1}^J O(e^{-\theta_0 t}) \delta_{x - a_j^- t}(-y), \quad \theta > 0, \]

and

\[ |\tilde{G}(x, t; y)| \leq Ce^{-\eta(|x-y| + t)} + \sum_{k=1}^n t^{-1/2} e^{-x-y-a^-_k t^2} e^{-\eta x^+} \]

\[ + \sum_{a^-_k > 0, a^-_j < 0} \chi_{\{|a^-_k t| \geq |y|\}} t^{-1/2} e^{-x-a^-_j (t-|y|/a^-_k))} e^{-\eta x^+} \]

\[ + \sum_{a^-_k > 0, a^-_j > 0} \chi_{\{|a^-_k t| \geq |y|\}} t^{-1/2} e^{-x-a^-_j (t-|y|/a^-_k))} e^{-\eta x^-}, \]
\[
\partial_y \tilde{G}(x, t; y) \leq Ce^{-\eta(|x-y|+t)} + Ct^{-1/2} \left( \sum_{k=1}^{n} t^{-1/2} e^{-((x-y-a_k^+ t)/2)/M t} e^{-\eta x^+} \right) + \sum_{a_k^+ > 0, a_j^- < 0} \chi_{\{|a_k^+ t| \geq |y|\}} t^{-1/2} e^{-((x-a_j^- (t-|y|/a_k^-))/2)/M t} e^{-\eta x^-}
\]

for \( y \leq 0 \) and symmetrically for \( y \geq 0 \), for some \( \eta, C, M > 0 \), where \( a_j^\pm \) are the eigenvalues of \( A^\pm = dF(U^\pm) \), \( a_j^*, j = 1, \ldots, J \) are the eigenvalues of \( A_{11} \), \( x^\pm \) denotes the positive/negative part of \( x \), and indicator function \( \chi_{\{|a_k^+ t| \geq |y|\}} \) is 1 for \( |a_k^+ t| \geq |y| \) and 0 otherwise.

**Proof.** As observed in [TZ2], it is equivalent to establish decomposition

\[
G = G_u + E + H + \tilde{G}
\]

for the full Green function \( G(x, t; y) := e^{Lt} \delta_y(x) \), where

\[
G_u(x, t; y) := \Pi_u e^{Lt} \delta_y(x) = e^{\gamma t} \sum_{j=1}^{p} \phi_j(x) \tilde{\phi}_j(y) t
\]

for some constant matrix \( M \in \mathbb{C}^{p \times p} \) denotes the Green kernel of the linearized solution operator on \( \Sigma_u \), \( \phi_j \) and \( \tilde{\phi}_j \) right and left generalized eigenfunctions associated with unstable eigenvalues \( \lambda_j, j = 1, \ldots, p \).

The problem of describing the full Green function has been treated in [ZH, MaZ3], starting with the Inverse Laplace Transform representation

\[
G(x, t; y) = e^{Lt} \delta_y(x) = \oint_{\Gamma} e^{L(\lambda - L(\varepsilon))^{-1} \delta_y(x)} d\lambda,
\]

where

\[
\Gamma := \partial \{ \lambda : \Re \lambda \leq \eta_1 - \eta_2 |\Re \lambda| \}
\]

is an appropriate sectorial contour, \( \eta_1, \eta_2 > 0 \); estimating the resolvent kernel \( G^\pm(x, y) := (\lambda - L(\varepsilon))^{-1} \delta_y(x) \) using Taylor expansion in \( \lambda \), asymptotic ODE techniques in \( x, y \), and judicious decomposition into various scattering, excited, and residual modes; then, finally, estimating the contribution of various
modes to (3.12) by Riemann saddlepoint (Stationary Phase) method, moving contour $\Gamma$ to a optimal, “minimax” positions for each mode, depending on the values of $(x, y, t)$.

In the present case, we may first move $\Gamma$ to a contour $\Gamma'$ enclosing (to the left) all spectra of $L$ except for the $p$ unstable eigenvalues $\lambda_j$, $j = 1, \ldots, p$, to obtain

$$G(x, t; y) = \oint_{\Gamma'} e^{\lambda t}(\lambda - L)^{-1}\delta_y(x),$$

where $\text{Residue}_{\lambda_j(e)}(e^{\lambda t}(\lambda - L)^{-1}\delta_y(x)) = G_u(x, t; y)$, then estimate the remaining term $\oint_{\Gamma'} e^{\lambda t}(\lambda - L)^{-1}\delta_y(x)$ on minimax contours as just described. See the proof of Proposition 7.1, [MaZ3], for a detailed discussion of minimax estimates $E + G$ and of Proposition 7.7, [MaZ3] for a complementary discussion of residues incurred at eigenvalues in $\{\Re \lambda \geq 0\} \setminus \{0\}$. See also [TZ1].

**Corollary 3.3 ([MaZ1]).** Assuming (A1)–(A3), (H0)–(H3), (D1)–(D3),

$$|\int_{-\infty}^{+\infty} H(\cdot, t; y)f(y)dy|_{L^p} \leq Ce^{-\theta t}|f|_{L^p},$$

$$|\int_{-\infty}^{+\infty} \tilde{G}(\cdot, t; y)f(y)dy|_{L^p} \leq C(1 + t)^{-\frac{1}{2} + \frac{1}{q} - \frac{1}{p}}|f|_{L^q},$$

$$|\int_{-\infty}^{+\infty} \tilde{G}_y(\cdot, t; y)f(y)dy|_{L^p} \leq C(1 + t)^{-\frac{1}{2} + \frac{1}{q} - \frac{1}{p} - \frac{1}{2}|f|_{L^q}},$$

for all $t \geq 0$, some $C > 0$, for any $1 \leq q \leq p$ (equivalently, $1 \leq r \leq p$) and $f \in L^q$, where $1/r + 1/q = 1 + 1/p$.

**Proof.** Standard convolution inequalities together with bounds (3.8)–(3.10); see [MaZ1, MaZ2, MaZ3, Z2] for further details.

**Corollary 3.4 ([Z4]).** The kernel $e$ satisfies

$$|e_y(\cdot, t)|_{L^p}, |e_t(\cdot, t)|_{L^p} \leq Ct^{-\frac{1}{2}(1 - 1/p)},$$

$$|e_{y_t}(\cdot, t)|_{L^p} \leq Ct^{-\frac{1}{2}(1 - 1/p) - 1/2},$$
for all $t > 0$. Moreover, for $y \leq 0$ we have the pointwise bounds
\[
|e_{uy}(y, t)|, |e_{ty}(y, t)| \leq Ct^{-\frac{1}{2}} \sum_{\alpha_k > 0} \left( e^{-\frac{(y+a_k^- t)^2}{Mt}} + e^{-\frac{(y-a_k^- t)^2}{Mt}} \right),
\]
for $M > 0$ sufficiently large, and symmetrically for $y \geq 0$.

**Proof.** Direct computation using definition (3.7); see [Z4, MaZ2, MaZ3] or [Z5], Appendix A. □

### 3.3 Reduced equations II

Recalling that $\partial_x \bar{U}$ is a stationary solution of the linearized equations $U_t = LU$, so that $L \partial_x \bar{U} = 0$, or
\[
\int_{-\infty}^{\infty} G(x, t; y) \bar{U}_x(y) dy = e^{Lt} \bar{U}_x(x) = \partial_x \bar{U}(x),
\]
we have, applying Duhamel’s principle to (3.2),
\[
V(x, t) = \int_{-\infty}^{\infty} G(x, t; y)V_0(y) dy - \int_{t}^{t} \int_{-\infty}^{\infty} G_y(x, t-s; y)(\mathcal{N}(V) + \dot{\alpha}V)(y, s) dy ds + \alpha(t) \partial_x \bar{u}(x).
\]

Defining
\[
\alpha(t) = -\int_{-\infty}^{t} e(y, t)V_0(y) dy + \int_{0}^{t} \int_{-\infty}^{+\infty} e_y(y, t-s)(\mathcal{N}(V) + \dot{\alpha}V)(y, s) dy ds,
\]
following [ZH, Z4, MaZ2, MaZ3], where $e$ is as in (3.7), and recalling the decomposition $G = E + H + G_u + \bar{G}$ of (3.11), we obtain the reduced equations
\[
V(x, t) = \int_{-\infty}^{\infty} (G_u + H + \bar{G})(x, t; y)V_0(y) dy - \int_{0}^{t} \int_{-\infty}^{\infty} (G_u + \bar{G})_y(x, t-s; y)(\mathcal{N}(V) + \dot{\alpha}V)(y, s) dy ds + \int_{0}^{t} \int_{-\infty}^{\infty} H(x, t-s; y)(\mathcal{N}(V) + \dot{\alpha}V)_y(y, s) dy ds,
\]

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and, differentiating (3.16) with respect to \( t \), and observing that \( e_y(y,s) \to 0 \) as \( s \to 0 \), as the difference of approaching heat kernels,

\[
\dot{\alpha}(t) = -\int_{-\infty}^{\infty} e_t(y,t)V_0(y)\,dy \tag{3.18}
\]

\[
+ \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)(N(V) + \dot{\alpha}V)(y,s)\,dy\,ds.
\]

Note that this (nonlocal in time) choice of \( \alpha \) and the resulting reduced equations are different from those of Section 2.5. As discussed further in [Go, Z4, MaZ2, MaZ3, Z2], \( \alpha \) may be considered in the present context as defining a notion of approximate shock location.

### 3.4 Nonlinear damping estimate

**Proposition 3.5** ([MaZ3]). Assuming (A0)-(A3), (H0)-(H2), let \( V_0 \in H^3 \), and suppose that for \( 0 \leq t \leq T \), the \( H^3 \) norm of \( V \) remains bounded by a sufficiently small constant, for \( V \) as in (3.1) and \( u \) a solution of (1.2). Then, for some constants \( \theta_{1,2} > 0 \), for all \( 0 \leq t \leq T \),

\[
\|V(t)\|_{H^3}^2 \leq C e^{-\theta_1 t}\|V(0)\|_{H^3}^2 + C \int_0^t e^{-\theta_2(t-s)}(\|V\|_L^2 + |\dot{\alpha}|^2)\,(s)\,ds.
\]

**Proof.** Essentially identical to the proof of Proposition 2.5 and corollaries, but using Moser’s inequality to bound the nonlinear term instead of the Lipshitz bound imposed by truncation. See [MaZ4, Z2, Z3] for detailed proofs of more general results. \( \square \)

### 3.5 Proof of nonlinear stability

Decompose now the nonlinear perturbation \( V \) as

\[
V(x,t) = w(x,t) + z(x,t), \tag{3.20}
\]

where

\[
w := \Pi_{cs}V, \quad z := \Pi_{s}V. \tag{3.21}
\]
Applying $\Pi_{cs}$ to (3.17) and recalling commutator relation (3.4), we obtain an equation

\begin{equation}
 w(x, t) = \int_{-\infty}^{\infty} \tilde{G}(x, t; y) w_0(y) \, dy
 - \int_0^t \int_{-\infty}^{\infty} \tilde{G}_y(x, t - s; y) \Pi_{cs}(N(V) + \dot{\alpha}V)(y, s) \, dy \, ds
 + \int_0^t \int_{-\infty}^{\infty} H(x, t - s; y) \Pi_{cs}(N(V) + \dot{\alpha}V)(y, s) \, dy \, ds
\end{equation}

for the flow along the center stable manifold, parametrized by $w \in \Sigma_{cs}$.

**Lemma 3.6.** Assuming (A1)–(A3), (H0)–(H2), for $V$ lying initially on the center stable manifold $M_{cs}$,

\begin{equation}
 |z|_{W^{r,p}} \leq C |w|_{H^2}^2
\end{equation}

for some $C > 0$, for all $1 \leq p \leq \infty$ and $0 \leq r \leq 4$, so long as $|w|_{H^2}$ remains sufficiently small.

**Proof.** By (1.9), we have immediately $|z|_{H^{1,2}} \leq C |w|_{H^{1,2}}^2$, whence (3.23) follows by equivalence of norms for finite-dimensional vector spaces, applied to the $p$-dimensional subspace $\Sigma_u$. (Alternatively, we may see this by direct computation using the explicit description of $\Pi_u V$ afforded by Lemma 3.1.)

**Proof of Theorem 1.3.** Recalling by Theorem 1.2 that solutions remaining for all time in a sufficiently small radius neighborhood $N$ of the set of translates of $\bar{u}$ lie in the center stable manifold $M_{cs}$, we obtain trivially that solutions not originating in $M_{cs}$ must exit $N$ in finite time, verifying the final assertion of orbital instability with respect to perturbations not in $M_{cs}$.

Consider now a solution $V \in M_{cs}$, or, equivalently, a solution $w \in \Sigma_{cs}$ of (3.22) with $z = \Phi_{cs}(w) \in \Sigma_u$. Define

\begin{equation}
 \zeta(t) := \sup_{0 \leq s \leq t} \left( |w|_{H^3}(1 + s)^{\frac{1}{2}} + (|w|_{L^\infty} + |\dot{\alpha}(s)|)(1 + s)^{\frac{1}{2}} \right).
\end{equation}

We shall establish:

**Claim.** For all $t \geq 0$ for which a solution exists with $\zeta$ uniformly bounded by some fixed, sufficiently small constant, there holds

\begin{equation}
 \zeta(t) \leq C_2 (E_0 + \zeta(t)^2) \quad \text{for} \quad E_0 := |V_0|_{L^1 \cap H^3}.
\end{equation}

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From this result, provided $E_0 < 1/4C_2^2$, we have that $\zeta(t) \leq 2C_2E_0$ implies $\zeta(t) < 2C_2E_0$, and so we may conclude by continuous induction that

$$\zeta(t) < 2C_2E_0$$

for all $t \geq 0$, from which we readily obtain the stated bounds. (By standard short-time $H^s$ existence theory, $V \in H^3$ exists and $\zeta$ remains continuous so long as $\zeta$ remains bounded by some uniform constant, hence (3.26) is an open condition.)

**Proof of Claim.** By Lemma 3.6,

$$|w_0|_{L^1 \cap H^3} \leq |V_0|_{L^1 \cap H^3} + |z_0|_{L^1 \cap H^3} \leq |V_0|_{L^1 \cap H^3} + C|w_0|_{H^3}^2,$$

whence

$$|w_0|_{L^1 \cap H^3} \leq CE_0.$$

Likewise, by Lemma 3.6, (3.24), (3.3), and Lemma 3.1, for $0 \leq s \leq t$ and $2 \leq p \leq \infty$,

$$|\Pi_{cs}(N(V) + \dot{\alpha}V)(y, s)|_{L^2} \leq C\zeta(t)^2(1 + s)^{-\frac{3}{4}},$$

$$|\Pi_{cs}(N(V) + \dot{\alpha}V)_y(y, s)|_{L^p} \leq C\zeta(t)^2(1 + s)^{-\frac{1}{2}}.$$

Combining the latter bounds with representations (3.22)–(3.18) and applying Corollary 3.3, we obtain

$$|w(x, t)|_{L^p} \leq \left| \int_{-\infty}^\infty G(x, t; y)w_0(y)\, dy \right|_{L^p}$$

$$+ \left| \int_0^t \int_{-\infty}^\infty \tilde{G}_y(x, t - s; y)\Pi_{cs}(N(V) + \dot{\alpha}V)(y, s)\, dy\, ds \right|_{L^p}$$

$$+ \left| \int_0^t \int_{-\infty}^\infty H(x, t - s; y)\Pi_{cs}(N(V) + \dot{\alpha}V)_y(y, s)\, dy\, ds \right|_{L^p}$$

$$\leq E_0(1 + t)^{-\frac{1}{2}(1 - \frac{1}{p})} + C\zeta(t)^2 \int_0^t (t - s)^{-\frac{3}{4} + \frac{1}{2p}}(1 + s)^{-\frac{3}{4}}\, dy\, ds$$

$$+ C\zeta(t)^2 \int_0^t e^{-\theta(t-s)}(1 + s)^{-\frac{1}{2}}\, dy\, ds$$

$$\leq C(E_0 + \zeta(t)^2)(1 + t)^{-\frac{1}{2}(1 - \frac{1}{p})}$$
and, similarly, using Hölder’s inequality and applying Corollary 3.4, (3.29)

\[ |\dot{\alpha}(t)| \leq \int_{-\infty}^{\infty} |e_t(y, t)||V_0(y)| dy \\
+ \int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y, t - s)||\tilde{\Pi}_{cs}(N(V) + \dot{\alpha}V)(y, s)| dy ds \\
\leq |e_t|_{L^\infty}|V_0|_{L^1} + C\zeta(t)^2 \int_0^t |e_{yt}|_{L^2}(t - s)|\tilde{\Pi}_{cs}(N(V) + \dot{\alpha}V)|_{L^2}(s) ds \\
\leq E_0(1 + t)^{-\frac{1}{2}} + C\zeta(t)^2 \int_0^t (t - s)^{-\frac{3}{4}}(1 + s)^{-\frac{3}{4}} ds \\
\leq C(E_0 + \zeta(t)^2)(1 + t)^{-\frac{1}{2}}.

Applying Lemma 3.5 and using (3.28) and (3.29), we obtain, finally,

(3.30) \[ |w|_{H^3}(t) \leq C(E_0 + \zeta(t)^2)(1 + t)^{-\frac{1}{4}}. \]

Combining (3.28), (3.29), and (3.30), we obtain (3.25) as claimed.

As discussed earlier, from (3.25), we obtain by continuous induction (3.26), or \( \zeta \leq 2C_2|V_0|_{L^1 \cap H^2} \), whereupon the claimed bounds on \( |V|_{L^p} \) and \( |V|_{H^3} \) follow by (3.28) and (3.30), and on \( |\dot{\alpha}| \) by (3.29). Finally, a computation parallel to (3.29) (see, e.g., [Maz3, Z2]) yields \( |\alpha(t)| \leq C(E_0 + \zeta(t)^2) \), from which we obtain the last remaining bound on \( |\alpha(t)| \).

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A Example systems

1. The general Navier–Stokes equations of compressible gas dynamics, written in Lagrangian coordinates, appear as

\[
\begin{align*}
\left\{ \begin{array}{l}
v_t - u_x = 0, \\
u_t + p_x = ((\nu/v)u_x)_x, \\
(e + u^2/2)_t + (pu)_x = ((\kappa/v)T_x + (\mu/v)uu_x)_x,
\end{array} \right.
\]

where \( \nu > 0 \) denotes specific volume, \( u \) velocity, \( e > 0 \) internal energy, \( T = T(v, e) > 0 \) temperature, \( p = p(v, e) \) pressure, and \( \mu > 0 \) and \( \kappa > 0 \) are
coefficients of viscosity and heat conduction, respectively. Defining \( U_1 := (v) \), \( U_2 := (u, e + |u|^2/2) \), we find that conditions (A1)–(A3) and (H1)–(H3) are thus satisfied under the mild assumptions of ideal temperature dependence

\[
(A.2) \quad T = T(e)
\]

monotone temperature-dependence

\[
(A.3) \quad T_\varepsilon > 0,
\]

and thermodynamic stability of the endstates,

\[
(A.4) \quad (p_v)_\pm < 0, (T_\varepsilon)_\pm > 0;
\]

see, e.g., [MaZ4, Z2, Z3, TZ3] for further discussion. Notably, this allows the interesting case of a van der Waals-type equation of state, with \( p_v > 0 \) for some values of \( v \) along the connecting profile.

2. The equations of MHD in Lagrangian coordinates are

\[
(A.5) \quad \begin{aligned}
&v_t - u_{1x} = 0, \\
v_{1t} + ((p + (1/2\mu_0)B_2^2 + B_3^2))_x = ((v/v)u_{1x})_x, \\
v_{2t} - ((1/\mu_0)B_1^*B_2)_x = ((v/v)u_{2x})_x, \\
v_{3t} - ((1/\mu_0)B_1^*B_3)_x = ((v/v)u_{3x})_x, \\
(vB_2)_t - (B_1^*u_2)_x = ((1/\sigma\mu_0v)B_{2x})_x, \\
(vB_3)_t - (B_1^*u_3)_x = ((1/\sigma\mu_0v)B_{3x})_x,
\end{aligned}
\]

\[
\begin{aligned}
&((e + (1/2)(u_1^2 + u_2^2 + u_3^2) + (1/2\mu_0)v(B_2^2 + B_3^2)))_x, \\
&\quad +[(p + (1/2\mu_0)B_2^2 + B_3^2))u_1 - (1/\mu_0)B_1^*(B_2u_2 + B_3u_3)]_x \\
&\quad = [(v/v)u_{1x}u_{1x} + (\mu/v)(u_{2xu_{2x} + u_{3xu_{3x})} \\
&\quad + (\kappa/v)T_x + (1/\sigma\mu_0^2v)(B_2B_{2x} + B_3B_{3x})]_x, \\
\end{aligned}
\]

where \( v \) denotes specific volume, \( u = (u_1, u_2, u_3) \) velocity, \( p = P(v,e) \) pressure, \( B = (B_1^*, B_2, B_3) \) magnetic induction, \( B_1^* \) constant, \( e \) internal energy, \( T = T(v,e) > 0 \) temperature, and \( \mu > 0 \) and \( \nu > 0 \) the two coefficients of viscosity, \( \kappa > 0 \) the coefficient of heat conduction, \( \mu_0 > 0 \) the magnetic permeability, and \( \sigma > 0 \) the electrical resistivity. Under (A.2)–(A.4), conditions (A1)–(A3) are again satisfied, and conditions (H1)–(H3) are satisfied (see [MaZ4]) under the generically satisfied assumptions that the endstates \( U^\pm_\varepsilon \) be strictly hyperbolic (i.e., have simple eigenvalues), and the speed \( s \) be nonzero, i.e., the shock move with nonzero speed relative to the background.
fluid velocity, with $U_1 := (v)$, $U_2 := (u, B, e + |u|^2/2 + v|B|^2/2\mu_0)$. (For gas dynamics, only Lax-type shocks and nonzero speeds can occur, and all points $U$ are strictly hyperbolic.)

3. (MHD with infinite resistivity/permeability) An interesting variation of (A.5) that is of interest in certain astrophysical parameter regimes is the limit in which either electrical resistivity $\sigma$, magnetic permeability $\mu_0$, or both, go to infinity, in which case the right-hand sides of the fifth and sixth equations of (A.5) go to zero and there is a three-dimensional set of hyperbolic modes $(v, vB_2, vB_3)$ instead of the usual one. By inspection, the associated equations are still linear in the conservative variables. Likewise, (A1)–(A3), (H1)–(H3) hold under (A.2)–(A.4) for nonzero speed shocks with strictly hyperbolic endstates.

4. (multi-species gas dynamics or MHD) Another simple example for which the hyperbolic modes are vectorial is the case of miscible, multi-species flow, neglecting species diffusion, in either gas dynamics or magnetohydrodynamics. In this case, the hyperbolic modes consist of $k$ copies of the hyperbolic modes for a single species, where $k$ is the number of total species. Again, (A1)–(A3), (H1)–(H3) hold for nonzero speed shocks with strictly hyperbolic endstates under assumptions (A.2)–(A.4).

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