KAUFFMAN BRACKET SKEIN MODULE OF THE CONNECTED SUM OF TWO PROJECTIVE SPACES

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Abstract. Diagrams and Reidemeister moves for links in a twisted $S^1$-bundle over an unorientable surface are introduced. Using these diagrams, we compute the Kauffman Bracket Skein Module (KBSM) of $\mathbb{RP}^3 \# \mathbb{RP}^3$. In particular, we show that it has torsion. We also present a new computation of the KBSM of $S^1 \times S^2$ and the lens spaces $L(p, 1)$.

1. Introduction

Skein modules, which are invariants of 3-manifolds as well as of links in these manifolds, were introduced by J. Przytycki [5] and V. Turaev [6]. In this paper we compute the Kauffman bracket skein module (KBSM) of $\mathbb{RP}^3 \# \mathbb{RP}^3$ (Theorem 1). This is the first such computation for a closed non-prime manifold. Also, it is the second example of a fully computed KBSM with torsion for a closed manifold, after the case of $S^1 \times S^2$ [1]. Unlike the KBSM of $S^1 \times S^2$, the KBSM of $\mathbb{RP}^3 \# \mathbb{RP}^3$ does not split as a sum of cyclic modules (Proposition 7).

First, we recall the definition of this module. Throughout this paper $R$ will be the ring of Laurent polynomials in $A$, $R = \mathbb{Z}[A, A^{-1}]$. Let $M$ be an orientable 3-manifold. The Kauffman bracket skein module of $M$, or $S_{2, \infty}(M)$, is the $R$-module generated by isotopy classes of unoriented framed links in $M$ modulo local relations:

(K1) : $L_+ = AL_0 + A^{-1}L_\infty$
(K2) : $L \sqcup T = (-A^2 - A^{-2})L$

where $T$ is the trivial framed knot and the triple $L_+$, $L_0$ and $L_\infty$ is presented in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

It can be seen easily that $L^{(1)} = -A^3L$ in $S_{2, \infty}(M)$ where $L^{(1)}$ is obtained from $L$ by adding a positive twist. It is called the framing relation.

For example $S_{2, \infty}(S^3)$ is free cyclic, $S_{2, \infty}(F \times I)$ ($F$ orientable surface) is free, generated by isotopy classes of simple closed curves on $F$ without trivial components [5], $S_{2, \infty}(L(p, q))$ is free with $[p/2] + 1$ generators [2], whereas $S_{2, \infty}(S^1 \times S^2)$ has torsion [1].

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In the next section we introduce diagrams and Reidemeister moves for links in $N \times S^1$, where $N$ is an unorientable surface. In section 3 we recall some results about $S_{2,\infty}(S^1 \times B^2)$, $B^2$ a disk, from $[3]$. In section 4 we compute $S_{2,\infty}(\mathbb{R}P^3 \# \mathbb{R}P^3)$, we show that it contains torsion elements and does not split as a sum of cyclic $\mathbb{R}$-modules. Finally, in sections 5 and 6 we present a new way to compute $S_{2,\infty}(S^1 \times S^2)$ and $S_{2,\infty}(L(p,1))$.

2. Diagrams of links in $N \times S^1$ and Reidemeister moves

In $[4]$ the notion of diagrams of links in $F \times S^1$, where $F$ is an orientable surface, was introduced. Also, the Reidemeister moves for such diagrams were found. We recall these notions here, as they are essential for our construction.

To obtain a diagram of a link $L$ in $F \times S^1$, we cut $F \times S^1$ along $F \times \{1\}$, $1 \in S^1$, thus obtaining $F \times [0,1]$ in which $L$ becomes $L'$, a collection of arcs. Then, $L'$ is projected onto $F \times \{0\}$ $\approx F$ via a vertical projection yielding a set of closed curves in $F$ on which we keep some extra information: for each double point the usual information of over- and undercrossing depending on the relative height in $[0,1]$ of the projected points; for points coming from endpoints of arcs in $L'$ (which appeared after cutting $F \times S^1$) a dot with an arrow on it indicating the direction of increasing height in $[0,1]$ just before and just after the cut. In other words, travelling on $L$ in the direction of the arrow, and crossing it, one crosses the “roof” $F \times \{1\}$ and one emerges from the “floor” $F \times \{0\}$ in the cylinder $F \times [0,1]$.

The construction of a diagram in the case where $F$ is a disk with two holes is pictured in Figure 2.

![Figure 2](image)

Two links in $F \times S^1$ are isotopic if one can get from any diagram of one to any diagram of the other through a series of five Reidemeister moves, pictured in Figure 3. The interpretation of $\Omega_4$ and $\Omega_5$ is pictured in Figure 4.

We introduce now diagrams and Reidemeister moves for $N \times S^1$, where $N$ is an unorientable surface. Denote an unorientable surface of genus $k$ and $n-k$ boundary components by $N_{n,k}$. Such $N_{n,k}$ is constructed from a 2-sphere with $n$ holes, denoted $S_n^2$, by glueing Möbius bands to $k$ ($k \leq n$) of these holes, or equivalently, by glueing $k$ boundary components ($S^1$-s) of $S_n^2$, each component being glued to itself via the antipodal map. Denote these $k$ components by $C$. Denote the image after the glueing of $C$ in $N_{n,k}$ by $C'$.

Let $L$ be a link in $N \times S^1$ where $N = N_{n,k}$ for some $k \leq n$. Cutting $N$ along $C'$ gives $S_n^2$ and cutting $N \times S^1$ along $C' \times S^1$ gives $S_n^2 \times S^1$, in which $L$ becomes $L'$, a
collection of arcs with endpoints in \( \partial S^2_n \times S^1 \). For such \( L' \) in \( S^2_n \times S^1 \) a diagram is constructed in the same way as it was done for links in \( F \times S^1 \), with orientable \( F \). A diagram with double points (equipped with the information of over- and undercrossing) and dots with arrows is thus obtained. The difference is that now it may consist of arcs (and not only of closed curves), where the endpoints of arcs lie in \( C \) in antipodal pairs. An example of a diagram of a link in \( N_{3,2} \times S^1 \) is shown in Figure 5.

As before, there are five Reidemeister moves (three classical and two new ones involving the arrows). Moreover, there are extra moves obtained by considering the resolution of generic singularities for the diagrams. Three new generic singularities are possible: an arc can be tangent to \( C \), a double point can lie in \( C \) or a dot with an arrow can lie in \( C \). Resolving these singularities gives respectively \( \Omega_6 \), \( \Omega_7 \) and \( \Omega_8 \) moves pictured in Figure 6.
In $\Omega_7$, the upper branch on one side becomes the lower branch after going through $C$ as the $S^1$-bundle is twisted on $C$. For the same reason in $\Omega_8$ the arrow switches orientation after going through $C$.

The regular Reidemeister moves are all the Reidemeister moves except $\Omega_1$.

In the rest of this paper we will consider links and their diagrams in $N = N_{1,1} = \mathbb{R} P^2$, so that $N \times S^1$ is $\mathbb{R} P^3 \# \mathbb{R} P^3$.

3. The module $S_{2,\infty}(S^1 \times B^2)$

In this section we recall some results from [4] concerning $S_{2,\infty}(S^1 \times B^2)$, $B^2$ a disk, as they are used later. In the preceding section we described diagrams and Reidemeister moves for $F \times S^1$, where $F$ is any orientable surface, so in particular for $B^2 \times S^1$.

Let $\bigcirc$ be denoted by $x$. Thus, the diagram $x$ represents a knot that runs parallel to the $S^1$ core of $S^1 \times B^2$. Applying the framing relation (i.e. $\Omega_1$, $\Omega_5$, and another framing relation one gets the following easy Lemma ([4] Lemma 3.3):

**Lemma 1.** In $S_{2,\infty}(S^1 \times B^2)$, $\bigcirc = A^{-6} \bigcirc$

Let $D_n$ be the diagram with no crossings and one component with $n$ arrows on it, $n \in \mathbb{Z}$ (if $n > 0$ the arrows are counterclockwise and if $n < 0$ the arrows are clockwise). Figure 7 shows how to express it, as an element of $S_{2,\infty}(S^1 \times B^2)$, with $D_{n-1}$ and $D_{n-2}$. The last equality comes from Lemma [4]

![Figure 7.](image)

So, in the skein module we have the relation:

$$D_n = -A^{-2} x D_{n-1} - A^2 D_{n-2}$$

This suggests the following definition:

**Definition 1.** Let $P_n$, $n \in \mathbb{Z}$, be polynomials in $x$ with coefficients in the ring $R$ defined inductively by:

$P_0 = -A^2 - A^{-2}$
$P_1 = x$

$P_n = -A^{-2}xP_{n-1} - A^2P_{n-2}$

where the last relation is also used to define $P_n$ for all negative $n$.

**Proposition 1.** (Proposition 3.7) $S_2,\infty(S^1 \times B^2)$ is a free $R$-module, generated by \{x^n|n \in \mathbb{N}\}, where $x^n$ stands for $\bigcirc\bigcirc \ldots \bigcirc$ (n copies of $x$), $x^0 = \emptyset$.

The isomorphism in this Proposition was given by a refined Kauffman bracket, $<\rangle_r$, which to a diagram $D$ associates its unique linear expression in $x^n$-s. This bracket respects the Kauffman relation $(K1)$ and $(K2)$ and it was proven that it is invariant under regular Reidemeister moves. We will use $<\rangle_r$ in the next section.

Therefore, in what follows we will identify the polynomials $P_n$ with diagrams $D_n$, keeping in mind that they are polynomials in $x$ in $S_2,\infty(S^1 \times B^2)$.

**4. The module $S_2,\infty(\mathbb{R}P^3_2 \mathbb{R}P^3)$**

In this section, we compute the KBSM of $\mathbb{R}P^3_2 \mathbb{R}P^3$, using diagrams of links in a twisted $S^1$-bundle over $\mathbb{R}P^2$, which is the same as $\mathbb{R}P^3_2 \mathbb{R}P^3$. Thus, a diagram will consist of a family of arcs in a disc $B^2$, with endpoints lying on $\partial B^2$ in antipodal couples, information of under- and overcrossing for each double point, and with some arrows on the arcs outside the endpoints and outside the double points. An example of such a diagram is pictured in Figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure8.png}
\caption{Figure 8.}
\end{figure}

Let $D$ be a diagram of a link in $\mathbb{R}P^3_2 \mathbb{R}P^3$. In the same way as in the construction of the classical Kauffman bracket, each crossing of $D$ can be equipped with a positive or negative marker as is pictured in Figure 9.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure9.png}
\caption{Figure 9.}
\end{figure}

A state $s$ of $D$ is the choice of a marker for each crossing of $D$. Let $D(s)$ be the diagram obtained from $D$ by smoothing all crossings in $D$ according to the markers determined by $s$.

$D(s)$, viewed as a family of curves in $\mathbb{R}P^2$ (by identifying antipodal boundary points of the disk of the diagram to get $\mathbb{R}P^2$), consists of ovals and, at most, one projective line. The ovals may be nested (an oval is nested in another one, if it lies in
the disk that this second oval bounds). To each oval is associated its *arrow number*: it is the algebraic number of arrows on this oval with an arrow being *positive* (resp. *negative*), if it gives a counterclockwise (resp. clockwise) orientation to the disk the oval bounds. To the projective line is associated its *parity* $0$ or $1$ which is the parity of the number of arrows on it. An oval is *trivial* if its arrow number is equal to $0$ and if all ovals nested in it (there may be no such ovals) have arrow numbers equal to $0$.

Let $D'(s)$ be the diagram consisting of all non trivial ovals of $D(s)$ and the projective line (if there is one in $D(s)$), which are arranged in such a way that the projective line consists of one arc (with $0$ or $1$ arrow on it, equal to the parity of the projective line) and the ovals consist of simple closed curves in the disk of the diagram, nested in the same manner as in $D(s)$, each oval having $n$ arrows on it, where $n$ is the arrow number of this oval, arranged in a counterclockwise or clockwise way depending on the sign of $n$. Furthermore, all the ovals in $D'(s)$ lie on one side of the projective line and, if the parity of this line is $1$, they lie above the projective line on which the arrow points to the right.

An example of getting $D'(s)$ from $D(s)$ is pictured in Figure 10.

![Figure 10.](image_url)

Let $|s|$ be the number of trivial ovals of $D(s)$. Let $p(s)$ (resp. $n(s)$) be the number of crossings with positive (resp. negative) markers in $s$.

**Definition 2.** The *Kauffman bracket* of $D$ is given by the following sum taken over all states of $D$:

$$< D > = \sum_s A^{p(s)}(-A^2 - A^{-2})^{|s|} D'(s)$$

**Lemma 2.** The Kauffman bracket $<>$ is preserved under all regular Reidemeister moves with the exception of $\Omega_5$.

**Proof.** The proof of the invariance of $<>$ under $\Omega_2$ and $\Omega_3$-moves is analogous to the classical case (Kauffman bracket for classical diagrams [3]). For the moves $\Omega_4$, $\Omega_6$ and $\Omega_8$ it suffices to prove the invariance of $<>$ for diagrams without crossings (i.e. for each $D(s)$ separately). But then this invariance follows from the easy fact that these moves do not change the nesting of the ovals, the arrow numbers of the ovals and the parity of the arrows on the projective line. Finally, the invariance of $<>$ under $\Omega_7$ follows from the invariance under $\Omega_6$, which has to be applied twice as is pictured in Figure 11.
We will refine now the bracket $<>$. Let $D$ be a diagram, $s$ a Kauffman state of $D$. Then $D'(s)$ consists of ovals and, possibly, a projective line, as on the right of Figure 10. All ovals lie in some disk $B$, which lies itself in the interior of the disk of the diagram. Therefore, they may be viewed as a diagram of a link in $S^1 \times B$, for which a refined bracket $<>_r$ was defined in [4] using the isomorphism mentioned after Proposition 1. We use this bracket to define $< D'(s) >_r$ (outside of $B$ it leaves the projective line unchanged, if there is one present).

Definition 3. The refined Kauffman bracket of $D$ is given by the following sum taken over all states of $D$:

$$< D >_r = \sum_{s} A^{p(s) - n(s)} (-A^2 - A^{-2})^{\mid s \mid} < D'(s) >_r$$

Thus, for a diagram $D$, $< D >_r$ is a linear expression in diagrams pictured in Figure 12. Denote the first two by $E_{m,n}$, where $m \in \mathbb{Z}_2$ is the parity of the arrows on the projective line and $n$ is the number of $x$-s.

Let $D_r$, $D_l$, $D_u$ and $D_d$ be four diagrams without crossings which differ only locally as pictured in Figure 13. We assume also that the vertical strand present in these diagrams belongs to an oval (i.e. not to a projective line).

Then we have:
Lemma 3. The refined Kauffman bracket satisfies:

\begin{align}
(1) \quad < D_u >_r &= -A^{-2} < D_r >_r - A^2 < D_d >_r \\
(2) \quad < D_u >_r &= -A^{-4} < D_l >_r - A^{-2} < D_d >_r
\end{align}

Proof. Note that by definition of $< >_r$, $< D_u >_r = << D_u >>_r$, and the same is true for the other three diagrams. Thus, we may replace each of the four diagrams with its bracket (this reduces simply the number of endpoints of arcs on the boundary of the disk of the diagrams to 2, if there is a projective line, or 0 otherwise). The Lemma is now true because it is true for $< >_r$ defined on diagrams in $S^1 \times B^2$, as was proven in [4] (Lemma 3.6), and we use here the same $< >_r$. □

The following Lemma will be used to reduce all the cases of $\Omega_5$ moves to some standard ones:

Lemma 4. Suppose that $D_1$ is a diagram with one crossing and $D_2$ is obtained from $D_1$ with a single $\Omega_5$ move. Then there is a series of $\Omega_6$, $\Omega_7$ and $\Omega_8$ moves between $D_1$ and $D'_1$ and a similar series between $D_2$ and $D'_2$, where $D'_2$ is obtained from $D_2$ with a single $\Omega_5$ move and $D'_1$ is one of the four types presented in Figure 14. It is to be understood that in this Figure there are no other endpoints of arcs lying on the boundary of the disk, except, possibly, two such endpoints for types I and II corresponding to a projective line. There may be any configuration of ovals and arrows, including extra arrows on the arcs that are pictured.

![Figure 14](image-url)

Proof. The idea is to eliminate all arcs in $D_1$ and $D_2$ which do not contain the crossing and that have endpoints that are not antipodal. Denote such an arc by $a$. Then $a$ divides the disk of diagrams $D_1$ and $D_2$ into two regions $R$ and $R'$ of which exactly one, say $R$, does not contain antipodal points on the boundary of the disk.

Now $a$ can be pushed through $R$ and across the boundary of the disk with the required moves after, possibly, having to push any other arcs and ovals that lie in $R$. It may happen that the unique crossing lies in $R$ in which case it is pushed across the boundary (both in $D_1$ and in $D_2$) giving again diagrams that differ by a single $\Omega_5$ move. This process reduces the number of endpoints of arcs on the boundary of the disk by at least two. Repeating it, one eliminates all such arcs.

Now, if an endpoint belongs to an arc that does not contain the crossing, the other endpoint is antipodal to it and $D'_1$ is of type I or II. Otherwise all endpoints belong to arcs that contain the crossing: if there are four such endpoints $D'_1$ is of type III, if there are two such endpoints, it is of type IV. Finally if there are no endpoints of arcs, $D'_1$ is of type I or II. □
We will say that an $\Omega_5$ move is of type I, II, III or IV, if it is a move of the type described between diagrams $D_1'$ and $D_2'$ in the hypothesis of the preceding Lemma. From the remarks after Proposition 1, $<\cdot>_r$ is invariant under moves of type I and II for diagrams of links in $S^1 \times B^2$. As we use the same definition of $<\cdot>_r$ here, we also have:

**Lemma 5.** The refined Kauffman bracket, $<\cdot>_r$, is invariant under $\Omega_5$ moves of type I and II.

As $H_1(\mathbb{R}P^3 \sharp \mathbb{R}P^3; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and Kauffman relations (K1) and (K2) respect the $\mathbb{Z}_2$-homology, $S_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$ splits into four submodules corresponding to each homology class: $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$. We may therefore write:

$$S_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3) = S^0_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3) \oplus S^1_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$$

where $S^0_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$ is generated by elements of classes $(0, 0)$ and $(1, 1)$ and $S^1_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$ is generated by elements of classes $(1, 0)$ and $(0, 1)$.

There are two invariants for diagrams that do not change under Reidemeister moves and Kauffman relations: the parity of the number of arrows and the parity of half the number of the endpoints of arcs lying on the boundary of the disk.

If this second number is odd for $D$ (which is equivalent to the presence of a projective line in $D(s)$ for all Kauffman states $s$), then the corresponding link represents $(0, 1)$ or $(1, 0)$ in $H_1(\mathbb{R}P^3 \sharp \mathbb{R}P^3; \mathbb{Z}_2)$, the class depending on the parity of the arrows. For example, the projective line can be viewed as lying in the first copy of $\mathbb{R}P^3$ in $\mathbb{R}P^3 \sharp \mathbb{R}P^3$ and the projective line with an arrow on it can be viewed as lying in the second copy.

If half the number of the endpoints of arcs lying on the boundary of the disk is even (which is equivalent to the absence of the projective line in $D(s)$ for all Kauffman states $s$), then the corresponding link represents $(0, 0)$ or $(1, 1)$ in $H_1(\mathbb{R}P^3 \sharp \mathbb{R}P^3; \mathbb{Z}_2)$, the class depending on the parity of the arrows. For example, the trivial knot realizes $(0, 0)$, whereas a vertical $S^1$ in $\mathbb{R}P^2 \times S^1$ realizes $(1, 1)$.

4.1. **The submodule $S^1_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$**. A relation in $S^1_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$ is pictured in Figure 15.

![Figure 15](attachment:image.png)

We can express this relation using the notations introduced before (see Figure 12):

$$E_{m,n} = -A^{-2}E_{m-2,n} - A^{-4}E_{m-1,n+1}$$

Using that $m \in \mathbb{Z}_2$ and rearranging the terms gives:

$$E_{m+1,n+1} = (-A^4 - A^2)E_{m,n}$$

**Figure 15.**
It follows immediately that:

\[ E_{m,n} = (-A^4 - A^2)^n E_{m+n,0} \]

Notice that \( E_{m+n,0} \) is the projective line, or the projective line with an arrow on it.

We define a refinement of \( < >_r \) with the help of the preceding formula:

**Definition 4.** Let \( D \) be a diagram of a link in \( S^2_{1,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3) \). Then:

\[ < D >_r = \sum_i q_i E_{m_i,n_i} \]

for some \( q_i \in \mathbb{R} \).

The *refined projective Kauffman bracket* of \( D \), denoted \( < >_{rp} \), is given by the formula:

\[ < D >_{rp} = \sum_i q_i (-A^4 - A^2)^n E_{m_i+n_i,0} \]

We may extend now Lemma 3 to the case in which the vertical strand of the four diagrams of Figure 13 is in a projective line, if we replace \( < >_r \) with \( < >_{rp} \). We still assume that the diagrams have no crossings.

**Lemma 6.** The refined projective Kauffman bracket satisfies:

1. \( < D_u >_{rp} = -A^{-2} < D_r >_{rp} - A^2 < D_d >_{rp} \)
2. \( < D_u >_{rp} = -A^{-4} < D_l >_{rp} - A^{-2} < D_d >_{rp} \)

**Proof.** If the vertical strand belongs to an oval, this follows from Lemma 3 and the fact that \( < >_{rp} \) is a refinement of \( < >_r \).

Assume now, that the vertical strand is in a projective line. We have:

\[ < D_r >_{rp} = E_{m,n} \]

for some \( m \) and \( n \) and \( < D_u >_{rp} = E_{m-1,n-1} \). So:

\[ < D_r >_{rp} = (-A^4 - A^2)^n E_{m+n,0} \]
\[ < D_u >_{rp} = (-A^4 - A^2)^{n-1} E_{m+n,0} \]

Thus:

\[ < D_r >_{rp} = (-A^4 - A^2) < D_u >_{rp} = -A^4 < D_d >_{rp} - A^2 < D_u >_{rp} \]

Rearranging the terms gives (1).

Also:

\[ < D_l >_{rp} = -A^4 < D_u >_{rp} - A^2 < D_d >_{rp} \]

Rearranging the terms gives (2). \( \square \)

We can now prove:

**Proposition 2.** The refined projective Kauffman bracket, \( < >_{rp} \), is invariant under all regular Reidemeister moves.

Thus, \( S^1_{2,\infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3) \) is isomorphic with \( R \oplus R \) and it is generated by the projective line and the projective line with one arrow on it.

**Proof.** All that remains is the invariance of \( < >_{rp} \) under \( \Omega_5 \) moves. It is sufficient to prove it for diagrams with a single crossing, the one in the move. From Lemma 3 and Lemma 4 it follows that one has only to check the invariance of \( < >_{rp} \) for a move that is of type IV in Figure 14. One may assume that there are only \( x \)-s inside
and outside the loop containing the only crossing, as all ovals are expressed with $x$-s already with $<>$. Using Lemma 6, it is possible to push all $x$-s inside the loop of the only crossing into the loop and all $x$-s outside the loop into the loop as well. After applying some $\Omega_8$ moves to push all arrows on one part of the projective line, we obtain the move pictured in Figure 16.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure16.png}
\caption{Figure 16.}
\end{figure}

Then, using Lemma 6 again, it is possible to reduce $m$ to 0 or 1 by pushing some $x$-s out of the loop and into the left part of the projective line in Figure 16.

For $m = 0$:
\[
< D' >_{rp} = AE_{n-1,0} + A^{-1} < E_{n,1} >_{rp} = AE_{n+1,0} + A^{-1}(-A^4 - A^2)E_{n+1,0} = -A^3E_{n+1,0} = < D >_{rp}
\]

the last equality from the framing relation.

For $m = 1$:
\[
< D >_{rp} = A < E_{n+1,1} >_{rp} + A^{-1}E_{n,0} = A(-A^4 - A^2)E_{n+2,0} + A^{-1}E_{n,0} = (-A^5 - A^3 + A^{-1})E_{n,0}
\]
\[
< D' >_{rp} = AE_{n-2,0} + A^{-1}(-A^2 - 1)(-A^2) + (A^4 + 1)E_{n,0}
\]

because a circle with two arrows on it is $P_2$ and we have (see Definition 1):
\[
< P_2 >_r = -A^{-2}x < P_1 >_r - A^2 < P_0 >_r = -A^{-2}x^2 + A^4 + 1
\]

So:
\[
< D' >_{rp} = AE_{n,0} + A^{-1}(-A^{-2}(-A^4 - A^2)^2E_{n,0} + (A^4 + 1)E_{n,0}) = (A - A^{-3}(A^8 - 2A^6 + A^4) + A^3 + A^{-1})E_{n,0} = (-A^5 - A^3 + A^{-1})E_{n,0} = < D >_{rp}
\]

\[\square\]

4.2. The submodule $S_{2,\infty}^0(\mathbb{RP}^3 \sharp \mathbb{RP}^3)$.

**Proposition 3.** $S_{2,\infty}^0(\mathbb{RP}^3 \sharp \mathbb{RP}^3)$ is the quotient of $R[x]$ modulo relations:
\[
(R_n): \ AP_{n-2} + A^{-1}P_{n+2} = AP_n + A^{-1}P_n, \ n \geq 2, \ n \in \mathbb{N}
\]

where $P_n$ are polynomials in $R[x]$ from Definition 1

Proof. For any diagram $D$ of a link in $S_{2,\infty}^0(\mathbb{RP}^3 \sharp \mathbb{RP}^3)$, $< D >_r \in R[x]$. From Lemma 4 and Lemma 5 $< >_r$ is invariant under all regular Reidemeister moves except the $\Omega_8$ moves of type III in Figure 14. Thus, $S_{2,\infty}^0(\mathbb{RP}^3 \sharp \mathbb{RP}^3)$ is the quotient of $R[x]$ modulo relations coming from $\Omega_8$ moves of type III.

Using $\Omega_8$ moves and Lemma 3 all arrows can be pushed on two strands, as in Figure 17.
Applying Kauffman relation \((K1)\), followed by \(\Omega_6\) and \(\Omega_8\) moves, now gives relations:

\[
AP_{n-m-1} + A^{-1}P_{m+1-n} = AP_{m-n-1} + A^{-1}P_{n+1-m}, \quad m, n \in \mathbb{Z}
\]

Clearly, one can take \(m = 1\) without omitting any relation, which gives \((R_n)\), \(n \in \mathbb{Z}\).

Finally, noticing that \((R_n)\) is the same as \((R_{2-n})\) it is sufficient to take \((R_n)\) with \(n \geq 2\) \((R_1)\) is a trivial relation so it can be disregarded. \(\square\)

In the rest of this subsection a simplification of \((R_n)\)-s, \(n \geq 2\), is presented.

A change of variable is useful for \(P_n\)-s: let \(t = -A^{-3}x\) (as a diagram \(t\) is \(x\) with a negative kink added).

**Lemma 7.** For all \(n \in \mathbb{Z}\), \(P_{-n}(t, A) = P_n(t, A^{-1})\)

**Proof.** We have:

\[
P_0 = -A^2 - A^{-2}, \quad P_1 = x = -A^3t, \quad P_{-1} = A^{-6}x = -A^{-3}t
\]

Thus:

\[
P_0(t, A) = P_0(t, A^{-1}), \quad P_{-1}(t, A) = P_1(t, A^{-1})
\]

The recurrence relation for \(P_n\) in \(t\) is:

\[
P_n = -A^{-2}xP_{n-1} - A^2P_{n-2} = AtP_{n-1} - A^2P_{n-2}
\]

From the last relation one gets:

\[
P_{n-2} = A^{-1}tP_{n-1} - A^{-2}P_n
\]

Replacing \(n\) with \(2 - n\):

\[
P_{-n} = A^{-1}tP_{-n+1} - A^{-2}P_{-n+2}
\]

Suppose now, by induction, that \(P_{-i}(t, A) = P_i(t, A^{-1})\) for \(0 \leq i < n\).

Then:

\[
P_{-n}(t, A) = A^{-1}tP_{-n+1}(t, A) - A^{-2}P_{-n+2}(t, A)
\]

\[
= A^{-1}tP_{-n-1}(t, A^{-1}) - A^{-2}P_{n-2}(t, A^{-1})
\]

\[
= P_n(t, A^{-1})
\]

Therefore:

Let \(Q_n, n \in \mathbb{N} \cup \{-2, -1\}\) be defined by:

\[
Q_{-2} = -1, \quad Q_{-1} = 0 \quad \text{and} \quad Q_n = tQ_{n-1} - Q_{n-2}
\]

Thus \(Q_0 = 1\) and \(Q_1 = t\).
Lemma 8. For \( n \geq 0 \) we have:
\[
P_n = -A^{n+2}Q_n + A^{n-2}Q_{n-2}, \quad P_{-n} = -A^{-n-2}Q_n + A^{-n+2}Q_{n-2}
\]
Proof. Let:
\[
P' = -A^2, \quad P' = -A^3t, \quad P' = AtP_{n-1} - A^2P_{n-2}
\]
\[
P'' = -A^{-2}, \quad P'' = 0, \quad P'' = AtP_{n-1} - A^2P_{n-2}
\]
Then \( P_n = P'_n + P''_n \).

Now it is easy to check by induction that:
\[
P'_n = -A^{n+2}Q_n, \quad n \geq 0
\]
\[
P''_n = A^{n-2}Q_{n-2}, \quad n \geq 2
\]
Thus for \( n \geq 2 \):
\[
P_n = -A^{n+2}Q_n + A^{n-2}Q_{n-2}
\]
For \( n = 0 \) and \( n = 1 \), \(-A^{n+2}Q_n + A^{n-2}Q_{n-2}\) become respectively:
\[
-A^2Q_0 + A^{-2}Q_{-2} = -A^2 - A^{-2} = P_0
\]
\[
-A^3Q_1 + A^{-1}Q_{-1} = -A^t = P_1
\]

The second part of the assertion follows from Lemma 7 and the fact that \( Q_n \) are polynomials in \( t \) only. \( \square \)

Recall that \((R_n), \, n \geq 2\), is:
\[
AP_{n-2} + A^{-1}P_{n-2} = AP_n + A^{-1}P_n
\]
Using the preceding Lemma, it can be rewritten as:
\[
A(-A^nQ_{n-2} + A^{n-4}Q_{n-4}) + A^{-1}(-A^{-n}Q_{n-2} + A^{-n+4}Q_{n-4}) =
\]
\[
A(-A^{-n-2}Q_n + A^{-n+2}Q_{n-2}) + A^{-1}(-A^{n+2}Q_n + A^{n-2}Q_{n-2})
\]
Which, after rearranging the terms, becomes for \( n \geq 2 \):
\[
(R'_n) : (A^{n+1} + A^{-n-1})(Q_n - Q_{n-2}) = (A^{n+3} + A^{-n+3})(Q_n - Q_{n-2}),
\]
Let \( q_{n,i} = \sum_{k=1}^{i} (A^{n-4k+1} + A^{-n+4k-1}) \)

Proposition 4. For \( 1 \leq i \leq \frac{n}{2} \), \((R'_n)\) can be rewritten using \((R'_k)\), \( k < n \), as:
\[
(A^{n+1} + A^{-n-1})(Q_n - Q_{n-2i}) = q_{n,i}(Q_{n-2i} - Q_{n-2i-2})
\]
Proof. The proof is by induction on \( i \). For \( i = 1 \) this is \((R'_n)\). Now suppose that the formula is true for \( i \leq \frac{n}{2} - 1 \). We have:
\[
(A^{n+1} + A^{-n-1})(Q_n - Q_{n-2i}) =
\]
\[
(A^{n+1} + A^{-n-1})(Q_n - Q_{n-2i} + Q_{n-2i} - Q_{n-2i-2}) =
\]
\[
(q_{n,i} + A^{n+1} + A^{-n-1})(Q_{n-2i} - Q_{n-2i-2})
\]
the last equality by induction hypothesis.
\[
(q_{n,i} + A^{n+1} + A^{-n-1})(Q_{n-2i} - Q_{n-2i-2}) = (A^{n+1} + A^{-n-1} + A^{n-1} + A^{-n+3} + .. + A^{n+4i-1})(Q_{n-2i} - Q_{n-2i-2})
\]
Notice that:
\[
(A^{n+1} + A^{-n+4i-1}) = (A^{n-2i+1} + A^{-n+2i-1})A^{2i}
\]
\[(A^{n-3} + A^{-n+4i-3}) = (A^{n-2i+1} + A^{-n+2i-1})A^{2i-4}\]

...\[
(A^{n-4i+1} + A^{-n-1}) = (A^{n-2i+1} + A^{-n+2i-1})A^{2i-4i}\]

Thus:

\[\left(q_{n,i} + A^{n+1} + A^{-n-1}\right)(Q_{n-2i} - Q_{n-2i-2}) =\]

\[\left(A^{n-2i+1} + A^{-n+2i-1}\right)(Q_{n-2i} - Q_{n-2i-2})(A^{2i} + A^{2i-4} + \ldots + A^{2i-4i}) =\]

\[\left(A^{n-2i-3} + A^{-n+2i+3}\right)(Q_{n-2i-2} - Q_{n-2i-4})(A^{2i} + A^{2i-4} + \ldots + A^{2i-4i})\]

where the last equality comes from \((R'_{n-2i})\), which can be applied as \(n - 2i \geq 2\).

So:

\[\left(A^{n+1} + A^{-n-1}\right)(Q_n - Q_{n-2i} - 2) =\]

\[\left(A^{n-2i-3} + A^{-n+2i+3}\right)(A^{2i} + A^{2i-4} + \ldots + A^{2i-4i})(Q_{n-2i-2} - Q_{n-2i-4}) =\]

\[\left(A^{n-3} + A^{-7} + \ldots + A^{-4i-3} + A^{-n+3} + A^{-n+7} + \ldots + A^{-n+4i+3}\right)(Q_{n-2i-2} - Q_{n-2i-4}) = q_{n,i+1}(Q_{n-2i-2} - Q_{n-2i-4})\]

\[\square\]

**Corollary 1.** \((R'_{n})\) can be rewritten as:

\[(A^{n+1} + A^{-n-1})(Q_n - 1) = 2(A + A^{-1})\sum_{k=1}^{\frac{n}{2}} A^{n+2-4k}, \text{ for } n \text{ even,}\]

\[(A^{n+1} + A^{-n-1})(Q_n - t) = 2t\sum_{k=1}^{\frac{n+1}{2}} A^{n+1-4k}, \text{ for } n \text{ odd.}\]

**Proof.** If \(n\) is even, one can take \(i = \frac{n}{2}\) in the preceding Proposition. Then:

\[Q_{n-2i} = Q_0 = 1, \; Q_0 - Q_{-2} = 1 - (-1) = 2\]

\[q_{n, \frac{n}{2}} = A^{n-3} + A^{n-7} + \ldots + A^{-n+1} + A^{-n+3} + A^{-n+7} + \ldots + A^{-n-1} =\]

\[(A + A^{-1})(A^{n-2} + A^{n-6} + \ldots + A^{-n+2}) = (A + A^{-1})\sum_{k=1}^{\frac{n}{2}} A^{n+2-4k}\]

If \(n\) is odd, one can take \(i = \frac{n-1}{2}\) in the preceding Proposition. Then:

\[Q_{n-2i} = Q_1 = t, \; Q_1 - Q_{-1} = t - 0 = t\]

\[q_{n, \frac{n-1}{2}} = A^{n-3} + A^{n-7} + \ldots + A^{-n+3} + A^{-n+3} + A^{-n+7} + \ldots + A^{-n-3} =\]

\[2\sum_{k=1}^{\frac{n-1}{2}} A^{n+1-4k}\]

\[\square\]

We can summarize the results of the two preceding subsections in the main theorem (note that we only need \(Q_n\) with \(n \geq 2\) to state this theorem):
Theorem 1. $\mathcal{S}_{2,\infty}(\mathbb{RP}^3 \mathbb{P}^3) = R \oplus R \oplus R[t]/S$, where $R = \mathbb{Z}[A, A^{-1}]$ and $S$ is the submodule of $R[t]$ generated by:

\[(A^{n+1} + A^{-n-1})(Q_n - 1) - 2(A + A^{-1}) \sum_{k=1}^{\frac{n}{2}} A^{n+2-4k}, \text{ for } n \geq 2 \text{ even,}\]

\[(A^{n+1} + A^{-n-1})(Q_n - t) - 2t \sum_{k=1}^{\frac{n+1}{2}} A^{n+1-4k}, \text{ for } n \geq 3 \text{ odd,}\]

where $Q_0 = 1$, $Q_1 = t$ and $Q_n = tQ_{n-1} - Q_{n-2}$.

Corollary 2. An element in $\mathcal{S}_{2,\infty}(\mathbb{RP}^3 \mathbb{P}^3)$ can be written in a unique way in the following form (called canonical):

\[(r_1, r_2, p_0 + p_1 t + \sum_{k=1}^{n} p_{2k}(Q_{2k} - 1) + \sum_{k=1}^{m} p_{2k+1}(Q_{2k+1} - t))\]

where $n, m \in \mathbb{N}$, $r_1, r_2, p_0, p_1 \in R$ and, for $k \geq 2$, $p_k \in \mathbb{Z}[A]$, $\deg(p_k) \leq 2k + 1$.

Proof. Clearly, an element in $\mathcal{S}_{2,\infty}(\mathbb{RP}^3 \mathbb{P}^3)$ can be written in the canonical form, using relations $(\mathcal{R}^*_n)$. Now, if two elements $m_1$ and $m_2$ are in the canonical form and represent the same element in $\mathcal{S}_{2,\infty}(\mathbb{RP}^3 \mathbb{P}^3)$, then

$m_1 - m_2 = (r_1, r_2, p_0 + p_1 t + \sum_{k=1}^{n} p_{2k}(Q_{2k} - 1) + \sum_{k=1}^{m} p_{2k+1}(Q_{2k+1} - t))$

is in the canonical form and belongs to the submodule $S$. Therefore $r_1 = r_2 = p_0 = p_1 = 0$. Now, as $m_1 - m_2 \in S$, $p_k$ must be of the form $(A^{k+1} + A^{-k-1})p'_k$, for some $p'_k \in R$. As $\deg(p_k) \leq 2k + 1$, this is possible only if $p'_k = 0$. Thus, all $p_k$-s are zero and $m_1 = m_2$.

We show now that there are torsion elements in $\mathcal{S}_{2,\infty}(\mathbb{RP}^3 \mathbb{P}^3)$.

Proposition 5. Let $n$ be even or $n = 1 \pmod{4}$, $n \geq 2$. Then, there exists a torsion element $m \in \mathcal{S}_{2,\infty}(\mathbb{RP}^3 \mathbb{P}^3)$, such that $\deg_t(m) = n$, where $m$ is in the canonical form.

Proof. Suppose that $n$ is even. Then $A^{n+1} + A^{-n-1} = (A + A^{-1})r_n$, where $r_n = (A^n - A^{-n-2} + \ldots - A^{n+2} + A^{-n})$. Let:

\[m = A^n(r_n(Q_n - 1) - 2 \sum_{k=1}^{\frac{n}{2}} A^{n+2-4k})\]

Then $A^n r_n$ is in $\mathbb{Z}[A]$ of degree $2n$, so $m$ is in canonical form and, from Corollary 2, $m$ is not 0 in $\mathcal{S}_{2,\infty}(\mathbb{RP}^3 \mathbb{P}^3)$. However:

\[(A + A^{-1})m = A^n((A^{n+1} + A^{-n-1})(Q_n - 1) - 2(A + A^{-1}) \sum_{k=1}^{\frac{n}{2}} A^{n+2-4k}) = 0\]

so $m$ is a torsion element.

Suppose now that $n = 1 \pmod{4}$. Then $A^{n+1} + A^{-n-1} = (A^2 + A^{-2})r_n$, where $r_n = A^n - A^{n-5} + \ldots - A^{-n+5} + A^{-n+1}$. Also:

\[\sum_{k=1}^{\frac{n+1}{2}} A^{n+1-4k} = (A^{n-3} + A^{n-7} + \ldots + A^{-n+3}) = (A^2 + A^{-2})(A^{n-5} + A^{n-13} + \ldots)\]
+A^{-n+5} = (A^2 + A^{-2}) \sum_{k=1}^{n-1} A^{n-3-8k}

Let:

\[ m = A^{n-1}(Q_n - t) - 2t \sum_{k=1}^{n-1} A^{n+3-8k} \]

Then \( A^{n-1}r_n \) is in \( \mathbb{Z}[A] \) of degree \( 2n - 2 \), so \( m \) is in canonical form and, from Corollary 2, \( m \) is not 0 in \( S_{2,\infty}(\mathbb{R}P^3|\mathbb{R}P^3) \). However:

\[(A^2 + A^{-2})m = A^{n-1}((A^{n+1} + A^{-n-1})(Q_n - t) - 2t(A^2 + A^{-2}) \sum_{k=1}^{n-1} A^{n+3-8k}) = 0 \]

so \( m \) is a torsion element.

Now, we show that there are not always torsion elements in all degrees in the canonical form.

Notice first that \( R \) is a UFD. Indeed, the invertible elements in \( R \) are of the form \( \pm A^n \), and two different decompositions of an element in \( R \) into irreducible elements would give, after multiplication by some invertible element, two different decompositions into irreducible elements in \( \mathbb{Z}[A] \), a UFD. Also, an irreducible polynomial in \( \mathbb{Z}[A] \) is irreducible in \( R \).

**Proposition 6.** Let \( m \in S_{2,\infty}(\mathbb{R}P^3|\mathbb{R}P^3) \) be written in the canonical form. Suppose that \( \deg(m) = 3 \). Then \( m \) is not a torsion element.

Proof. The polynomial \( A^8 + 1 \) is irreducible in \( \mathbb{Z}[A] \), which can be checked by replacing \( A \) with \( A + 1 \) and applying the Eisenstein irreducibility criterion. Therefore, \( A^4 + A^{-4} \) is irreducible in \( R \).

Now, the odd and even powers of \( t \) are independent in \( S_{2,\infty}(\mathbb{R}P^3|\mathbb{R}P^3) \) (they correspond to different homology classes in \( H_1(\mathbb{R}P^3|\mathbb{R}P^3; \mathbb{Z}_2) \)), so we may assume that \( m = p_3(Q_3 - t) + p_1 t \), with \( p_3 \in \mathbb{Z}[A] \), \( \deg(p_3) \leq 7 \), \( p_3 \neq 0 \) and \( p_1 \in R \).

Suppose that for some \( r \in R \), \( rm = 0 \). Then \( rp_3 = (A^4 + A^{-4})r' \), for some \( r' \in R \). Now, as \( (A^4 + A^{-4}) \) is irreducible in \( R \), it divides \( r \) or \( p_3 \). But it cannot divide \( p_3 \) as \( \deg(p_3) \leq 7 \). Thus \( r = (A^4 + A^{-4})w \), for some \( w \in R \). We have then:

\[ rm = (A^4 + A^{-4})w(p_3(Q_3 - t) + p_1 t) = w(2 + (A^4 + A^{-4})p_1) t = 0 \]

Thus, \( w(2 + (A^4 + A^{-4})p_1) = 0 \) and, as \( R \) is a domain, \( w = 0 \) or \( 2 + (A^4 + A^{-4})p_1 = 0 \).

The second case is ruled out by the fact that the lowest and highest powers in \( A \) of \( (A^4 + A^{-4})p_1 \) are different if \( p_1 \neq 0 \). Thus \( w = 0 \), so \( r = 0 \) and \( m \) is not a torsion element. \( \square \)

Finally, unlike the case of \( S_{2,\infty}(S^1 \times S^2) \) (see Theorem 2), we have:

**Proposition 7.** The module \( S_{2,\infty}(\mathbb{R}P^3|\mathbb{R}P^3) \) does not split as a sum of cyclic \( R \)-modules.

Proof. Suppose that \( S_{2,\infty}(\mathbb{R}P^3|\mathbb{R}P^3) = \bigoplus_i R \oplus \bigoplus_j R/I_j \), for some ideals \( I_j \) in \( R \).

We can write:

\[ t = t_1 + .. + t_n + t_{tor} \]
where $t_1, \ldots, t_n$ are non zero elements in different free $R$ summands of $\mathcal{S}_{2, \infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$ and $t_{tor}$ is in $\bigoplus_j R/I_j$ (as $t$ is not a torsion element $n$ must be at least 1).

As $(A^4 + A^{-4})(Q_3 - t) = 2t$, $2t = (A^4 + A^{-4})r_i$, for some $r_i \in R$, $1 \leq i \leq n$. As $(A^4 + A^{-4})$ is irreducible in $R$ (see the proof of Proposition 10) and does not divide 2, it divides $t_i$, so $t_i = (A^4 + A^{-4})s_i$, for some $s_i \in R$. Thus, for $s = s_1 + \ldots + s_n$:

$$t = (A^4 + A^{-4})s + t_{tor}$$

From Corollary 2 it follows that for $m \in \mathcal{S}_{2, \infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$, $m \neq 0$ implies $2m \neq 0$.

Thus, if for some $r \in R$, $2rm = 0$, then $rm = 0$. Let $r \in R$, $r \neq 0$, be such that $rt_{tor} = 0$. From the preceding remark, we may assume that $r \ (mod \ 2) \neq 0$. Then:

$$rt = (A^4 + A^{-4})rs$$

If $s$, as a polynomial in $t$, contains some even powers of $t$, they must be killed by $(A^4 + A^{-4})r$. Therefore, without changing the last equation, we may assume that $s$ contains only odd powers of $t$. So $s$ has the form:

$$s = p_1 t + p_3(Q_3 - t) + \ldots + p_k(Q_k - t), \ p_i \in R$$

and:

$$rt = (A^4 + A^{-4})(rp_1 t + rp_3(Q_3 - t) + \ldots + rp_k(Q_k - t))$$

For $i \geq 3$, all $(A^4 + A^{-4})rp_i(Q_i - t)$ must be in $Rt$, so $(A^4 + A^{-4})rp_i(Q_i - t) - r'it \in S$, for some $r' \in R$ ($S$ is the submodule in Theorem 1). This implies that all $r'_i$ are divisible by 2. So, for some $q \in R$:

$$rt = (A^4 + A^{-4})rp_1 t + 2qt = (A^4 + A^{-4})rp_1 t \ (mod \ 2)$$

This is impossible from considerations of the highest and lowest power of $A$ in $r \ (mod \ 2)$ and $(A^4 + A^{-4})rp_1 \ (mod \ 2)$ and the fact that $r \ (mod \ 2) \neq 0$. \hfill \Box

5. The module $\mathcal{S}_{2, \infty}(S^1 \times S^2)$

In 11 the skein module $\mathcal{S}_{2, \infty}(S^1 \times S^2)$ has been computed. We present in this section an alternative computation of this skein module, similar to the computation of $\mathcal{S}_{2, \infty}(\mathbb{R}P^3 \sharp \mathbb{R}P^3)$.

$S^1 \times S^2$ is $S^1 \times B^2 / \partial B^2$. Therefore diagrams of links in $S^1 \times S^2$ are the same as diagrams of links in $B^2 \times S^1$. The additional generic singularity occurs when an arc contains the infinity point (i.e. $\partial B^2$ in $B^2 / \partial B^2$). Resolving it gives an additional $\Omega_{\infty}$ move, presented in Figure 18.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure18.png}
\caption{Figure 18.}
\end{figure}
One more time, it is possible to define a refined Kauffman bracket, $<\rangle_r$, on diagrams of links in $S^1 \times S^2$. Here it is exactly the $<\rangle_r$ that was used to prove Proposition 1. In particular it is invariant under all regular Reidemeister moves except $\Omega_{\infty}$ and it satisfies Lemma 3.

**Proposition 8.** $S_{2,\infty}(S^1 \times S^2)$ is the quotient of $R[x]$ modulo relations:

$$(S_n): P_n = P_{-n}, \ n \geq 1, \ n \in \mathbb{N}$$

**Proof.** From the remarks before the Proposition it follows that $S_{2,\infty}(S^1 \times S^2)$ is the quotient of $R[x]$ modulo all relations coming from $\Omega_{\infty}$ moves. As before, we may consider such moves only on diagrams without crossings. Consider such a move, presented on the left of Figure 19.

**Figure 19.**

We may apply $<\rangle_r$ inside and outside the oval that is pushed through infinity, to reduce the situation to the middle of Figure 19. Then, using Lemma 3 all $x$-s are pushed inside this oval. Thus, the situation is simplified to the case presented on the right of Figure 19 in which a $P_n$ is pushed through infinity, becoming $P_{-n}$, $n \in \mathbb{Z}$. Thus, the relations needed to get $S_{2,\infty}(S^1 \times S^2)$ from $S_{2,\infty}(S^1 \times B^2)$ are $P_n = P_{-n}$, $n \in \mathbb{Z}$. Obviously, it suffices to take $n \geq 1$. □

From Lemma 8 the relations $(S_n)$ can be expressed as:

$$-A^{n+2}Q_n + A^{n-2}Q_{n-2} = -A^{-n-2}Q_n + A^{-n+2}Q_{n-2}$$

which can be rearranged becoming, for $n \geq 1$, the relations:

$$(S'_n): (-A^{n+2} + A^{-n-2})Q_n = (-A^{-n-2} + A^{-n+2})Q_{n-2}$$

Let $q_{n,i} = \sum_{k=1}^{i} (-A^{-n-4k+2} + A^{-n+4k+2})$
Let \( Q_{n,i} = \sum_{k=0}^{i} Q_{n-2k} \)

**Proposition 9.** For \( 1 \leq i \leq \frac{n+1}{2} \), \((S'_{n})\) can be rewritten using \((S'_{k})\), \( k < n \), as:

\[
(-A^{n+2} + A^{-n-2})Q_{n,i-1} = \hat{q}_{n,i}Q_{n-2i}
\]

Proof. The proof is by induction on \( i \). For \( i = 1 \) this is \((S'_{n})\). Now suppose that the formula is true for \( i \leq \frac{n-1}{2} \). We have:

\[
(-A^{n+2} + A^{-n-2})Q_{n,i} = (-A^{n+2} + A^{-n-2})(Q_{n,i-1} + Q_{n-2i}) = \\
(\hat{q}_{n,i} - A^{n+2} + A^{-n-2})Q_{n-2i}
\]

the last equality by induction hypothesis.

\[
(\hat{q}_{n,i} - A^{n+2} + A^{-n-2})Q_{n-2i} = \\
(-A^{n+2} - A^{-n-2} - A^{-n-6} - A^{-n-4i+2} + A^{-n-2} + A^{-n-2} + A^{-n+2} + A^{-n+4i-2})Q_{n-2i}
\]

Notice that:

\[
(-A^{n+2} + A^{-n+4i-2}) = (-A^{n-2i+2} + A^{-n+2i-2})A^{2i} \\
(-A^{n-2} + A^{-n+4i-6}) = (-A^{n-2i+2} + A^{-n+2i-2})A^{2i-4} \\
\vdots \\
(-A^{n-4i+2} + A^{-n-2}) = (-A^{n-2i+2} + A^{-n+2i-2})A^{2i-4i}
\]

Thus:

\[
(\hat{q}_{n,i} - A^{n+2} + A^{-n-2})Q_{n-2i} = \\
(-A^{n-2i+2} + A^{-n+2i-2})Q_{n-2i}(A^{2i} + A^{2i-4} + \ldots + A^{2i-4i}) = \\
(-A^{n-2i+2} + A^{-n+2i+2})Q_{n-2i-2}(A^{2i} + A^{2i-4} + \ldots + A^{2i-4i})
\]

where the last equality comes from \((S'_{n-2i})\), which can be applied as \( n - 2i \geq 1 \).

So:

\[
(-A^{n+2} + A^{-n-2})Q_{n,i} = \\
(-A^{n-2i-2} + A^{-n+2i+2})(A^{2i} + A^{2i-4} + \ldots + A^{2i-4i})Q_{n-2i-2} = \\
(-A^{n-2} - A^{n-6} - A^{-n-4i+2} + A^{-n+2} + A^{-n+6} + \ldots + A^{-n+4i+2})Q_{n-2i-2} = \\
\hat{q}_{n,i+1}Q_{n-2i-2}
\]

\( \square \)

Let \( \hat{Q}_n = Q_{n,\frac{n}{2}} = Q_n + Q_{n-2} + \ldots + Q_0 \), for \( n \) even

\( \hat{Q}_n = Q_{n,\frac{n-1}{2}} = Q_n + Q_{n-2} + \ldots + Q_1 \), for \( n \) odd.

**Theorem 2.** \( S_{2,\infty}(S^1 \times S^2) = R \oplus \bigoplus_{n=1}^{\infty} R/\{1 - A^{2n+4}\} \)
Proof. If \( n \) is even, one can take \( i = \frac{n}{2} \) in the preceding Proposition. Then:

\[
\hat{q}_n = -A^{n-2} - A^{n-6} - \ldots - A^{-n+2} + A^{-n+6} + \ldots + A^{n-2} = 0
\]

Thus:

\[
(-A^{n+2} + A^{-n-2})\hat{Q}_n = 0
\]

If \( n \) is odd, one can take \( i = \frac{n+1}{2} \) in the preceding Proposition. Then:

\[
(-A^{n+2} + A^{-n-2})\hat{Q}_n = \hat{q}_{n+1} Q_{-1} = 0
\]

because \( Q_{-1} = 0 \).

Multiplying these relations by invertible elements \( A^{n+2} \), one gets:

\[
(1 - A^{2n+4})\hat{Q}_n = 0
\]

It suffices now to show that \( \{\emptyset\} \cup \{\hat{Q}_n, n \geq 1\} \) is a basis of \( S_{2,\infty}(S^1 \times B^2) \). As \( \{\emptyset\} \cup \{t^n, n \geq 1\} \) is such a basis, so \( \{\emptyset\} \cup \{t^n, n \geq 1\} \) is also such a basis. Now, \( \text{deg}_t(\hat{Q}_n) = n \) and \( \hat{Q}_n \) are monic polynomials in \( t \), so they form, together with \( \emptyset \), a basis of \( S_{2,\infty}(S^1 \times B^2) \).

\[ \square \]

6. The modules \( S_{2,\infty}(L(p, 1)) \)

In [2] the skew modules \( S_{2,\infty}(L(p, q)) \), \( L(p, q) \) lens spaces, were computed. We present here a new computation of \( S_{2,\infty}(L(p, 1)) \), similar to the computation of \( S_{2,\infty}(S^1 \times S^2) \) of the previous section, though much simpler.

\( L(p, 1) \) is obtained from \( S^1 \times B^2 \) by attaching a disk \( B \) to it, where \( \partial B \) is glued to a curve of type \( (p, 1) \) in \( S^1 \times \partial B^2 \), then attaching a 3-ball. This last operation does not change the skein module, so we only need to consider the attaching of the disk.

A link in \( S^1 \times B^2 \cup B \) can be pushed along \( B \) so that it lies in \( S^1 \times B^2 \). Therefore, diagrams of links in \( L(p, 1) \) are the same as diagrams of links in \( S^1 \times B^2 \). The extra Reidemeister move, \( \Omega_{\infty, p} \), comes from the sliding of an arc along \( B \), so that it becomes a \((p, 1)\) curve with a small segment removed. In a diagram, such a curve goes around the rest of the diagram and has \( p \) arrows on it, see Figure 20.

\[ \text{Figure 20.} \]

It is possible to define a refined Kauffman bracket, \( <\cdot, \cdot>_{r} \), on diagrams of links in \( L(p, 1) \), exactly like in the previous section. So, again, it is invariant under all regular Reidemeister moves except \( \Omega_{\infty, p} \) and it satisfies Lemma 3.

**Proposition 10.** \( S_{2,\infty}(L(p, 1)) \) is the quotient of \( R[x] \) modulo relations:

\[
(T_n) : P_n = P_{p-n}, \ n > |p/2|, \ n \in \mathbb{N}
\]
Proof. The proof follows the proof of Proposition 8. Like before, \( S_{2,\infty}(L(p, 1)) \) is the quotient of \( R[x] \) modulo relations coming from \( \Omega_{\infty,p} \), and these relations can be reduced to the cases in which a \( P_n \) is slid through the attached disk, becoming \( P_{p-n} \), see Figure 21.

![Figure 21.](image)

Note, that for \( p = 0 \) this is just Proposition 8, as \( L(0, 1) = S^1 \times S^2 \).

Theorem 3. Let \( p \geq 1 \). Then \( S_{2,\infty}(L(p, 1)) = R[x]/(x^{\lfloor p/2 \rfloor} + 1) \).

Proof. From Definition it follows that, for \( n > 0 \), \( deg_x(P_n) = n \) and the leading coefficient of \( P_n \) is an invertible element in \( R \). From Lemma 7, for \( n < 0 \), \( P_n(t, A) = P_{-n}(t, A^{-1}) \), where \( t = -A^{-3}x \). Thus \( deg_x(P_n) = deg_t(P_n) = deg_x(P_{-n}) = deg_x(P_{-n}) = -n \). Also, as \( t \) is equal to \( x \) up to an invertible element in \( R \), the leading coefficient of \( P_n \) is an invertible element in \( R \). Thus, for all \( n \in \mathbb{Z}, n \neq 0 \), \( P_n = a_n x^n + \hat{P}_n \), for some \( a_n \) invertible in \( R \) and some \( \hat{P}_n \) satisfying \( deg_x(\hat{P}_n) < |n| \). Relations \((T_n)\), \( n > \lfloor p/2 \rfloor \), become:

\[
a_n x^n + \hat{P}_n = a_{p-n} x^{p-n} + \hat{P}_{p-n}
\]

Or:

\[
x^n = a_n^{-1}(\hat{P}_n + a_{p-n} x^{p-n} + \hat{P}_{p-n})
\]

As, \( p \geq 1 \), \( |p - n| < n \), so the degree in \( x \) of the right hand side of the preceding relation is smaller than \( n \).

Thus, these relations allow to express all \( x^n \), \( n > |p/2| \), with some lower degree polynomials. Therefore, as there are no other relations, \( S_{2,\infty}(L(p, 1)) \) is free, generated by \( 0, x, \ldots, x^{p/2} \).

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