Computational Complexity of Arranging Music

William Moses and Erik Demaine
MIT
Domain & Motivation

Why is this problem interesting?
Arrangement

- Transcribe a piece written for instruments A to be played on instruments B
- Expand repertoire (Zelda Theme played by NSO, Canon in D on Guitar)
- Automated arrangement software
Musical Choreography

- Create a performance which fits with music
- Examples: Dancing, Cinema, Skating
- Similar to arrangement, representing moves by performer as notes on instrument
Problem Statement
Given a score consisting of $n$ instrumental parts, does there exist a valid arrangement of the piece for one instrument?
P vs NP (vs PSPACE ... )
Computational Complexity

- Decision Problems
  - How long will it take to compute?
  - $P :=$ Solvable in Polynomial time
  - $NP :=$ Nondeterministic Polynomial Time
Computational Complexity

- Reduction from A to B encodes A in B, means B at least as hard as A
- All problems in NP can be ‘reduced’ or encoded inside 3SAT
3SAT

- Does boolean formula have solution?
- Literal: Variable or not a variable
- Clause: 3 literals or’d together
- Formula: Series of clauses and’d together

\[(\neg x_1 \lor x_3 \lor x_4) \land (x_2 \lor \neg x_3 \lor x_4)\]

- Literal
- Literal
- Literal
- Clause
Summary Of Results
## Summary of Results

| NPC | P |
| --- | --- |
| **Required Percentage of Notes in Arrangement** |
| Consonance | ![Bar Graph](image1.png) |
| Finite Transition Speed | ![Bar Graph](image2.png) |
| Max \((j \geq 4)\)-note chord | ![Bar Graph](image3.png) |
| Max \((j=1)\)-note chord | ![Bar Graph](image4.png) |
Quantifying a “good” arrangement

How do we deem an arrangement acceptable?
Criteria for Valid Arrangement

- Must be possible to be played / performed
- Must reflect the original intent of the piece (recognizable)
- Must be pleasing to listen to / watch
Limitations on performance

- Transition speed
- Number of simultaneous notes / actions
Original Intent / Recognizable

- Must maintain entire melodies
- Must keep certain percentage of original notes
Pleasant Sounding

Consonance: Simultaneous notes (chords) allowed in certain intervals
Hardness of Consonance
# Summary of Results

| NPC | P |
|-----|---|

## Required Percentage of Notes in Arrangement

|                | 0%  | 33% | 50% | 100% |
|----------------|-----|-----|-----|------|
| Consonance     |     |     |     |      |
| Finite Transition Speed |     |     |     |      |
| Max \((j\geq4)\)-note chord |     |     |     |      |
| Max \((j=1)\)-note chord |     |     |     |      |
Consonance Requirements

- Parts are included or excluded in entirety (recognizable melody)
- Any simultaneous notes must be in consonance (pleasant sounding)
- At any given time, at least $n\%$ ($0 < n < 100$) of notes in the original song must be played (original intent)
Variable Gadgets

- Variables represented by the choice of one of two parts

- At most one part can be played from pleasant sounding requirement

- At least one part must be played from original intent requirement
True / False Literal

- We create parts which must be played (true) and must be omitted (false)

- A true literal can be created by simply having a note on its own -- which must be played in the arrangement

- A false literal can be created by creating a measure where it is in dissonance with a true literal
Clause Gadgets

- Clauses represented by a measure three variable parts and some true/false literals

- Sufficient true / false literals added to ensure that n% of notes being placed requires at least one variable to be played (50% depicted)
Entire 3SAT

\[(\neg x_1 \lor x_3 \lor x_4) \land (x_2 \lor \neg x_3 \lor x_4)\]

- 3SAT represented as a song, followed by a satisfying assignment
Finite Transition Speed
**Summary of Results**

| NPC | P |
|-----|---|
|     |   |

| Required Percentage of Notes in Arrangement |
|---------------------------------------------|
| 0%  | 33% | 50% | 100% |

- **Consonance**
- **Finite Transition Speed**
- **Max (j≥4)-note chord**
- **Max (j=1)-note chord**
Transition Requirements

- Parts are included or excluded in entirety (recognizable melody)

- Notes or chords cannot change more frequently than a half note (playable)

- At any given time, at least $n\%$ ($0 < n < 100$) of notes in the original song must be played (original intent)
Variable Gadgets

- By having the two parts to play notes offset by a quarter note, only one can be played without violating transition requirements.

- True / false literals used as padding to ensure at least n% notes played at any time.

- Again variables represented by the choice of one of two parts.
Variable Gadgets

- Three arrangements of the 3SAT variable selection, selecting all parts; true and X1; and true and NOT(X1).
Transition Conclusions

- Using the transition exclusion established in the variable gadget, we can create clauses.

- Just like in the consonance problem these clauses can be used to create any 3SAT, showing the transition problem to be NP-hard.
Max j-note Chord
### Summary of Results

| NPC | P |
| --- | --- |

| Required Percentage of Notes in Arrangement |
| --- |
| 0% | 33% | 50% | 100% |

- **Consonance**
- **Finite Transition Speed**
- **Max (j≥4)-note chord**
- **Max (j=1)-note chord**
Max j-note Chord

- Parts are included or excluded in entirety (recognizable melody)
- Only up to j notes can be played simultaneously (playable)
- At any given time, at least $n\% \ (0 < n < 100)$ of notes in the original song must be played (original intent)
- Using reduction from X-3SAT (only 1 variable is true)
Variable Gadgets ($j \geq 1$)

- Again variables represented by the choice of one of two parts

- For $j = 1$, the same variable gadget from consonance can be used (since only one can be played).

- For higher $j$, pad the measure with additional true / false literals to ensure only one part can be played
### Summary of Results

|                  | NPC | P       | Required Percentage of Notes in Arrangement |
|------------------|-----|---------|--------------------------------------------|
| Consonance       |     |         | ![Graph](image1)                            |
| Finite Transition Speed |     |         | ![Graph](image2)                            |
| Max (j≥4)-note chord |     |         | ![Graph](image3)                            |
| Max (j=1)-note chord |     |         | ![Graph](image4)                            |
Clause Gadgets ($j \geq 4$)

- Clause represented by measure with many variable / literal tracks playing simultaneously, but must choose subset of at most $j$

- Pad the measure with additional true / false literals to ensure only one variable can be played and still meets minimum number of note requirement
### Summary of Results

| NPC | P |
|-----|---|
|     |   |

#### Required Percentage of Notes in Arrangement

|                      | 0% | 33% | 50% | 100% |
|----------------------|----|-----|-----|------|
| Consonance           | ![0%](image) | ![33%](image) | ![50%](image) | ![100%](image) |
| Finite Transition Speed | ![0%](image) | ![33%](image) | ![50%](image) | ![100%](image) |
| Max \((j\geq4)\)-note chord | ![0%](image) | ![33%](image) | ![50%](image) | ![100%](image) |
| Max \((j=1)\)-note chord | ![0%](image) | ![33%](image) | ![50%](image) | ![100%](image) |
Clause Gadgets ($j = 1$), $\frac{1}{3}$ notes

- For case where at least $n\%$ ($0 < n \leq 33.3$) of notes must be played:
  - Clause represented by measure with 3 variable tracks playing simultaneously, but can only play 1
  - Must play 1 from original intent requirement
  - Still NP-hard
## Summary of Results

| NPC | P | Required Percentage of Notes in Arrangement |
|-----|---|--------------------------------------------|
|     |   | 0%  | 33% | 50% | 100% |
| Consonance | | ![Gray](#) | ![Gray](#) | ![Gray](#) | ![Gray](#) |
| Finite Transition Speed | | ![Gray](#) | ![Gray](#) | ![Gray](#) | ![Gray](#) |
| Max \((j\geq4)\)-note chord | | ![Gray](#) | ![Gray](#) | ![Gray](#) | ![Gray](#) |
| Max \((j=1)\)-note chord | | ![Gray](#) | ![Gray](#) | ![Gray](#) | ![Gray](#) | ![Blue](#) |
$j = 1, n > \frac{1}{3}$ notes (P)

- Solvable in polynomial time via 2-coloring, with two colors: played and not played
Edge Cases
Summary of Results

| NPC | Required Percentage of Notes in Arrangement |
|-----|------------------------------------------|
| P   | 0%                             | 33%             | 50% | 100% |

- Consonance
- Finite Transition Speed
- Max (j≥4)-note chord
- Max (j=1)-note chord
Edge Cases

- Allowing $n=0$ for the requirement that $n\%$ of notes to be played makes the problem $P$ since you can simply play no notes.

- Likewise allowing $n=100$ makes the problem $P$ since you must select all tracks and thus can check that no other conditions are violated in a linear scan through the notes played.
Fun Applications
Applications

- Creating rhythm game (Rock Band, Dance Dance Revolution) tracks is thus NP-hard
- Choreographing dance and fight scenes in music is thus NP-hard.
**Questions?**

| NPC | P |
|-----|---|

|               | 0%    | 33%    | 50%    | 100%  |
|---------------|-------|--------|--------|-------|
| Consonance    | ![Graph](chart1.png) | ![Graph](chart2.png) | ![Graph](chart3.png) | ![Graph](chart4.png) |
| Finite Transition Speed | ![Graph](chart1.png) | ![Graph](chart2.png) | ![Graph](chart3.png) | ![Graph](chart4.png) |
| Max (j≥4)-note chord | ![Graph](chart1.png) | ![Graph](chart2.png) | ![Graph](chart3.png) | ![Graph](chart4.png) |
| Max (j=1)-note chord | ![Graph](chart1.png) | ![Graph](chart2.png) | ![Graph](chart3.png) | ![Graph](chart4.png) |