Bulk-Edge correspondence of entanglement spectrum in 2D spin ground states

Raul A. Santos

C.N. Yang Institute for Theoretical Physics,
Stony Brook University,
Stony Brook, NY 11794-3840, USA

General local spin S ground states, described by a Valence Bond Solid (VBS) on a two dimensional lattice are studied. The norm of these ground states is mapped to a classical O(3) model on the same lattice. Using this quantum-to-classical mapping we obtain the partial density matrix $\rho_A$ associated with a subsystem $A$ of the original ground state. We show that the entanglement spectrum of $\rho_A$ in a translation invariant lattice is given by the spectrum of a quantum spin chain at the boundary of region $A$, with local Heisenberg type interactions between spin 1/2 particles.

PACS numbers: 75.10.Kt, 75.10.Hk, 05.30.-d, 75.10.Jm

I. INTRODUCTION

Quantum entanglement, the spooky action at a distance that has been signaled as the characteristic of quantum mechanics, has received renewed attention recently, specially with the growth of quantum information science, and as a new tool to study properties of many-body systems. It has been found that entanglement is sensitive to topologically ordered states, quantum phase transitions, and even magnetic properties of solids.

A complete description of the entanglement properties of a bipartite pure state $|\Psi\rangle$ composed of subsystems $A$ and $B$ is given by the entanglement spectrum (ES) i.e. the eigenvalues of the reduced density matrix (RDM) of subsystem $A$ (or $B$). The RDM for the subsystem $A$, $\rho_A$, is obtained by tracing out the degrees of freedom belonging to $B$ from the density matrix $\rho = |\Psi\rangle\langle\Psi|$, which characterizes the state $|\Psi\rangle$. In general the density matrix of the subsystem $A$ may be written as $\rho_A = \exp(-\beta H_{\text{eff}})$, where $H_{\text{eff}}$ is an effective (also called entanglement) Hamiltonian. It has been shown that the entanglement Hamiltonian describes excitations living at the edge of partitions of the ground state of fractional quantum Hall states, one dimensional, and topological systems. In two dimensions, few analytical results on the entanglement spectrum in generic spin systems are known. Numerical studies of the ES have been performed in the 2D Affleck, Lieb, Kennedy and Tasaki (AKLT) model, which possess a known valence bond solid (VBS) ground state.

Lou et al. and Cirac et al. showed that the ES of a partition in the ground state of the AKLT model is related with the conformal XXX Heisenberg model on the boundary of the partition by using Monte Carlo and projected entangled pair states (PEPS) in finite size systems. In this paper we show that the ES of a partition of a whole class of ground states defined in translational invariant lattices, can be approximated by the thermal spectrum of a series of local Hamiltonians which are the conserved charges associated with the XXX Hamiltonian defined on the boundary of the partition.

In this paper we introduce the spin $S$ model in section 2 defined on a two dimensional lattice wrapped on a torus and construct its explicit VBS ground state following. Then, we derive an expression for the RDM (also called partial density matrix) $\rho_A$ in section 3. This operator is expressed in terms of classical variables in section 4. In this representation, the operator can be expanded in different graph contributions of the classical $O(3)$ model as presented in section 5. From this expression we identify the Heisenberg Hamiltonian for spin 1/2 particles in the boundary as the leading term in a sequence of boundary Hamiltonians. Evidence for the structure of the entanglement Hamiltonian is given in section 6 based on the analysis in the continuous limit. In the last section, we summarize the results and discuss further possible generalizations.

II. SPIN S VBS GROUND STATE ON A TWO DIMENSIONAL TORUS

As discussed in it is possible to construct a valence bond solid (VBS) ground state in a planar graph $\mathcal{G}$ (without edges starting and ending in the same site) in the following way: Given a planar graph $\mathcal{G}$, consisting of a set of vertices (sites) $V$ and edges $E$, with $z_i$ edges arriving to vertex $i$ (in graph theoretical language, $z_i$ is called coordination number), we place a local spin $S_i$ on the vertex with the condition $S_i = z_i/2$. The local spin state is constructed from the symmetric subspace of $z_i$ spins 1/2 (doing this we obtain a higher spin representation of the fundamental representations of $SU(2)$). Finally we antisymmetrize between nearest neighbors. Representing the spin 1/2 constituents of the spin $S_i$ at site $i$ as black dots, using a circle to indicate symmetrization and a bond between antisymmetric neighbors, we obtain a planar graph $\mathcal{G}'$ isomorphic to $\mathcal{G}$, see Fig 1.

The AKLT Hamiltonian for which the VBS state constructed is a ground state is a sum over interactions on all edges $E$ of $\mathcal{G}$, $H = \sum_{(k,l)\in E} H_{kl}(S_k + S_l)$, where the
Hamiltonian density $H_{kl}$ is

$$H_{kl}(\vec{S}_k + \vec{S}_l) = \sum_{J=S_k+S_l+1-M_{kl}} A_{kl}^J \pi_{kl}^J(\vec{S}_k + \vec{S}_l),$$

where the coefficients $A_{kl}^J > 0$ are arbitrary and can depend on the edge $(k,l)$, while the positive number $M_{kl}$ is the number of bonds (edges) connecting the sites $k$ and $l$. The operator $\pi_{kl}^J(\vec{S}_k + \vec{S}_l)$ is a projector of the total spin $J_{kl} = \vec{S}_k + \vec{S}_l$ of the edge $(k,l)$ on the subspace of spin value $J$, its explicit form is

$$\pi_{kl}^J(J_{kl}) = \prod_{j=|S_k-S_l|, j \neq J} \frac{(J_{kl})^2 - J(j+1)}{J_{kl}(J_{kl}+1) - J(j+1)}.$$

The VBS state is the unique ground state of $H$. While this construction is totally general, in the rest of this discussion we will focus on graphs without boundaries, which can be embedded on a two dimensional torus, with $M_{ij} = 1$ for all edges.

III. PARTIAL DENSITY MATRIX AND SCHWINGER BOSON REPRESENTATION OF VBS GROUND STATE

In this section, we introduce a general way of writing the reduced density matrix of a pure system in terms of overlap matrices. These matrices have elements which correspond to overlap amplitudes between states spanning the ground space of Hamiltonians defined entirely in the subsystems. We apply these results to the VBS case introduced in the previous section.

Using the Schmidt decomposition, any ground state $|\Psi\rangle$ of a system can be written as

$$|\Psi\rangle = \sum_{\alpha} |A_\alpha\rangle \otimes |B_\alpha\rangle,$$

where the states $|A_\alpha\rangle$ and $|B_\alpha\rangle$ are related to the usual states appearing in the Schmidt decomposition by a scale factor. $|A_\alpha\rangle$ and $|B_\alpha\rangle$ are states defined in the subsystems $A$ and $B$, with associated Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The total system has a Hilbert space $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B$. The set of states $\{ |A_\alpha\rangle, |B_\alpha\rangle \}$ is a complete, linear independent but not orthonormal basis (in principle). The density matrix for this pure state is the projector onto the ground state $\rho = N^{-1} \langle \Psi | \Psi \rangle$. Tracing out the sites belonging to the subsystem $B$, we obtain the partial density matrix, which describe the system $A$, $\rho_A = \text{Tr}_B \rho$. Using (3) the partial density matrix becomes $\rho_A = N \sum_{\alpha \beta} |B_\beta\rangle |B_\alpha\rangle \langle A_\alpha| \langle A_\beta|$. Using standard algebraic techniques, the partial density matrix can be written as

$$(\rho_A)_{\alpha\beta} = \sum_\gamma \frac{\langle A_\beta | A_\alpha \rangle^* \langle B_\alpha | B_\gamma \rangle}{\langle \Psi | \Psi \rangle}.$$
\[ |\Psi_{\text{obs}}\rangle = \prod_{(i,j) \in A \cup B} (a_i^\dagger b_j - a_j^\dagger b_i^\dagger) \prod_{i \in \partial A, j \in \partial B} (a_i^\dagger a_j^\dagger + b_i^\dagger b_j^\dagger)|0\rangle, \]  

where we have applied a local basis transformation on the sites (vertices) in \( B \), \( a_i^\dagger \rightarrow -b_i^\dagger \) and \( b_j^\dagger \rightarrow a_j^\dagger \) just for later convenience. In the shared bonds, we can assign to an endpoint \( j \) of a bond, it’s partner in the other end of the bond to be \( \bar{j} \). Doing this we can expand (6) in the form\(^{23}\)

\[ |\Psi_{\text{obs}}\rangle = \sum_{\{\alpha\} \in \partial} \sum_{(i,j) \in A \cup B} (a_i^\dagger b_j^\dagger)^{\alpha_i} (a_j^\dagger b_i^\dagger)^{1-\alpha_i} \prod_{(i,j) \in A \cup B} (a_i^\dagger b_j^\dagger - a_j^\dagger b_i^\dagger)|0\rangle \]

\[ = \sum_{\{\alpha\}} |A_{\{\alpha\}}\rangle \otimes |B_{\{\alpha\}}\rangle, \]  

here \( \{\alpha\} = \{\alpha_1, \alpha_2, \ldots, \alpha_{2|\partial|}\} \) with \( \alpha_i = 0, 1 \), labels the different ground states of the subsystems, which span a Hilbert space of dimension \( 2^{|\partial|} \). The Hamiltonian in subsystem \( A \) is defined by \( H_A = \sum_{k,l} H_{kl} \) (and similarly for \( B \)), with \( H_{kl} \) given by (4). From (7), we can read off the form of the states \( |A_{\{\alpha\}}\rangle \)

\[ |A_{\{\alpha\}}\rangle = \prod_{(i,j) \in \partial} (a_i^\dagger b_j^\dagger - a_j^\dagger b_i^\dagger) \prod_{i \in \partial} (b_i^\dagger)^{1-\alpha_i} (a_i^\dagger)^{\alpha_i}|0\rangle, \]  

using (4) and (5), we can compute the density matrix \( \rho_A \) in terms of the overlap matrices \( M^A_{\{\alpha\},\{\beta\}} = \langle A_{\{\alpha\}}|A_{\{\beta\}}\rangle \). From eq. (4), the partial density matrix is constructed gluing together two of these overlap matrices, one for each subsystem, along the boundary of the partition, leaving one index free in each overlap matrix, obtaining a torus with a cut along the partition (see Fig. 2).

From this construction, we see that we can write \( \rho_A \) as a block diagonal operator, with a nontrivial block of dimension \( 2^{|\partial|} \times 2^{|\partial|} \), and a trivial block (full of zeros), of dimension \( (\dim \mathcal{H}_A - 2^{|\partial|}) \times (\dim \mathcal{H}_A - 2^{|\partial|}) \). This result can be understood from the properties of the VBS state. This state is annihilated at each and every site by the action of the Hamiltonian density \( H_{kl} \). After making the partition the states defined in the subsystems are still annihilated by the local Hamiltonians defined in each partition, but the states of the sites at the edges which cross from one subsystem to the other (in our notation, the edges belonging to the set \( \partial \) are free to have any possible state on them, as no local Hamiltonian defined in just one subsystem can act on this edges. This feature has been encountered before in the study of AKLT chains, where the dimension of the partial density matrix does not increase with the size of the system\(^{22}\).

The computation of this overlap matrix can be mapped to the computation of partition and correlation functions in an \( O(3) \) model, by means of the classical representation of the VBS state\(^{43}\) as we show in the next section.

### IV. QUANTUM TO CLASSICAL MAPPING

Introducing the spinor coordinates \( \phi_k^a = (u_k, v_k) = (e^{i\varphi_k/2} \cos \theta_k, e^{-i\varphi_k/2} \sin \theta_k) \) at site \( k \), with \( \varphi_k \in [0, 2\pi] \), \( \varphi_k \in [0, 2\pi] \), we can define the spin coherent state \( |\Omega_k\rangle \)

\[ |\Omega_k\rangle = \frac{(u_k a_k^\dagger + v_k b_k^\dagger)^{2S_k}}{\sqrt{(2S_k)!}}|0_k\rangle, \]  

(\( |0_k\rangle \) being the vacuum state at site \( k \)), these states are complete but not orthogonal. Inserting the resolution of the identity

\[ 1_{2S_k+1} = \frac{2S_k + 1}{4\pi} \int d\Omega_k |\Omega_k\rangle \langle \Omega_k|, \]  

in \( M^A_{\{\alpha\},\{\beta\}} \), and using the result \( \langle 0|a_k^{\alpha_k} b_k^{\beta_k} |\Omega_k\rangle = \sqrt{(2S_k)!} U_{\alpha_k}^{\beta_k} v_k^{\alpha_k} u_k^{\beta_k} \), the following form of the overlap matrix is obtained (dropping overall constant factors)

\[ M^A_{\{\alpha\},\{\beta\}} = \int \prod_{i \in A} \frac{d\Omega_i}{4\pi} \prod_{(i,j) \in \partial A} \left(1 - \hat{\Omega}_i \cdot \hat{\Omega}_j \right) \times \prod_{k \in \partial A} (u_k)^{\alpha_k} (v_k)^{1-\alpha_k} (u_k^{\ast})^{\beta_k} (v_k^{\ast})^{1-\beta_k}, \]
here \( \hat{\Omega}_k = (\sin \theta_k \cos \varphi_k, \sin \theta_k \sin \varphi_k, \cos \theta_k) \) is the unit vector over the two dimensional sphere \( S^2 \) and \( u^* \) is the complex conjugate of \( u \). From (11) we see that the overlap matrix \( M^{[A]} \) is hermitian, so the partial density matrix \( \rho_A = \mathcal{N}(M^{[A]})^* M^{[B]} \) is also hermitian. Using now that \( (u_k)^{\dagger} a_k \phi^\alpha_k = \phi^\alpha_k \) (abusing notation, \( \alpha_k \) goes from being a power, to become a (supra)index, \( \phi^0 = u_k \), \( \phi^1 = v_k \)) and the identity

\[
2 \delta_{\alpha \beta} (\phi^\alpha)^\beta = \delta_{\alpha \beta} + \hat{\Omega}_k \cdot \vec{\sigma}_{\alpha \beta},
\]

where \( \delta_{\alpha \beta} \) is the Kronecker delta symbol, and \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) is a vector of Pauli matrices (no distinction is made between upper or lower greek indices); the expression for the overlap matrix can be written as

\[
M^{[A]}_{\{\alpha\} \{\beta\}} = \int \prod_{i \in A} \frac{d\Omega_i}{4\pi} \prod_{(i,j) \in E_A} (1 - \hat{\Omega}_i \cdot \hat{\Omega}_j) \times \prod_{k \in \partial A} ((\mathbb{I} + \hat{\Omega}_k \cdot \vec{\sigma})_{\alpha \beta}),
\]

combining this result with (4), the density matrix of the subsystem \( A \) becomes

\[
(\rho_A)_{\{\alpha\} \{\beta\}} = \frac{1}{Z} \int \prod_{k \in \partial} \frac{d\hat{\Omega}_k}{4\pi} \prod_{(i,j) \in E_A \cup E_B} (1 - \hat{\Omega}_i \cdot \hat{\Omega}_j) \times \prod_{(k,l) \in \partial} ((\mathbb{I} + \hat{\Omega}_k \cdot \vec{\sigma})(\mathbb{I} + \hat{\Omega}_l \cdot \vec{\sigma})_{\alpha \beta}),
\]

with \( Z \) the proper normalization factor to make \( \text{Tr} \rho_A = 1 \). We can expand the matrix product inside (13) using the product identity for Pauli matrices \( \sigma_\alpha \sigma_\beta = \delta_{\alpha \beta} \mathbb{I} + i \epsilon_{ijk} \sigma_k \) (repeated index implies sum) where \( \epsilon_{ijk} \) is the totally antisymmetric Levi-Civita tensor. The result of the term inside the square bracket in (13) is then

\[
[(\mathbb{I} + \hat{\Omega}_k \cdot \vec{\sigma})(\mathbb{I} + \hat{\Omega}_l \cdot \vec{\sigma})]_{\alpha \beta} = (1 + \hat{\Omega}_k \cdot \hat{\Omega}_l) \delta_{\alpha \beta} + (\hat{\Omega}_k + \hat{\Omega}_l + i(\hat{\Omega}_k \times \hat{\Omega}_l)) \cdot \vec{\sigma}_{\alpha \beta},
\]

where \( \hat{a} \times \hat{b} \) represent the cross product between vectors \( \hat{a} \) and \( \hat{b} \).

V. GRAPH EXPANSION OF THE DENSITY MATRIX

In this section we derive the structure of the entanglement Hamiltonian as a sequence of spin 1/2 Hamiltonians with increasing interaction length, using the quantum to classical correspondence introduced in the previous section.

From (15), two types of expressions can be assigned to each edge on \( \partial \). We draw a straight line between \( k \) and \( l \) whenever in that bond we have the expression \( (1 + \Omega_k \cdot \Omega_l) \delta_{\alpha \beta} \), while we put a wiggly line for \( (\Omega_k + \Omega_l + i(\Omega_k \times \Omega_l)) \cdot \vec{\sigma}_{\alpha \beta} \). Expanding the product over the boundary in (14), we obtain a sum where each term has either a wiggly or straight line corresponding to \( (k,l) \in \partial \). All the other bonds that don’t belong to \( \partial \) have a straight line associated with them.

In general, for a planar graph \( \mathcal{L} \), the expression

\[
Z_{O(N)} = \int \prod_{k \in \mathcal{L}} \frac{d\hat{\Omega}_k}{S_N} \prod_{(i,j) \in \mathcal{L}} (1 + x \hat{\Omega}_i \cdot \hat{\Omega}_j)
\]

where \( \hat{\Omega} \) is an \( N \) dimensional unit vector; corresponds to the partition function over \( \mathcal{L} \) of the \( O(N) \) model which is analogous to a model of overlapping loops. To see this, we use that

\[
\int \frac{d\hat{\Omega}_k}{S_N} \hat{\Omega}_k \cdot \hat{\Omega}_k = 1, \quad \int \frac{d\hat{\Omega}_k}{S_N} \prod_{i=1}^{\text{odd}} (\hat{\Omega}_i \cdot \hat{\Omega}_k) = 0,
\]

where \( S_N \) is the area of the \( S^{N-1} \) sphere and the second property follows form the invariance of the integration measure under change \( \hat{\Omega}_k \rightarrow -\hat{\Omega}_k \). As the only terms that contribute to \( Z_{O(N)} \) are the ones with a product of even \( (\hat{\Omega}_i \cdot \hat{\Omega}_k) \) terms at each site of the graph, the whole partition function can be written as

\[
Z_{O(N)} = \sum_C w(\zeta, \mathcal{C}) x^{\Gamma(\mathcal{C})}
\]

with \( \mathcal{C} \) a particular configuration of loops of total length \( \Gamma(\mathcal{C}) \) that can be embedded in the graph \( \mathcal{L} \), and \( w(l, \mathcal{C}) \) being the corresponding weight associated with a loop \( \zeta \) and with the particular configuration of loops \( \mathcal{C} \). For example, for the hexagonal lattice (coordination number \( z_i = 3 \)) each site has associated just two bonds, and each integration of a site gives a factor of \( \frac{1}{N} \), except the last integration which closes the loop. The partition function is then

\[
Z_{O(N)} = \sum_C w(\zeta) \Gamma(\mathcal{C}) N^{n(\mathcal{C})}, \text{with } n(\mathcal{C}) \text{ the number of loops in the configuration } \mathcal{C}.
\]

The computation of spin correlations \( \langle \Omega_m \cdot \Omega_k \rangle \) corresponds then to the computation of \( Z_{O(N)} \), with configurations that allow loops and open paths that begin at site \( m \) and end at site \( k \). From (13), expanding the product over the partition’s boundary we get a sum over different configurations of loops and open strands in the \( O(N) \) model, over the graph \( \mathcal{L} \) with defects (wiggly lines). In the present case, \( x = -1 \) and \( N = 3 \) for the classical partition function of the VBS ground state.

So far we have developed our ideas for general planar graphs with no loops and no more than one bond shared between neighbors (\( M_{ij} = 1 \)), but from now on we will focus the discussion on translation invariant lattices with the previous restrictions. The discussion will remain general for lattices subject to the mentioned restrictions.
that can be embedded on a torus. Using translation symmetry, we can expand the product over the boundary in different contributions of translational invariant Hamiltonians along the boundary, with increasing number of non-trivial operators (Pauli matrices) acting on the local Hilbert space associated with a bond. The first term of the expansion correspond to the identity in the $2^{(2j)}$-dimensional Hilbert space of the boundary. The second term, which is proportional to a constant external magnetic field acting on the boundary chain, vanish. This follows from the observation that in this term, we have just one wiggly bond placed in the boundary - let’s say at bond $k$ with endpoints $k$ and $k'$ - and the rest are just straight lines, which after integration will generate all the configurations of loops, and open lines that start at $k$, travel through the lattice and end at site $k'$ (for this type of bonds we will use dashed lines, to indicate the corresponding connection on the lattice). So we will have a term which is proportional to the spin correlation between $k$ and $k'$, and an integral of the form (see fig 3.a)

$$
\int \frac{d\Omega_k}{4\pi} \frac{d\tilde{\Omega}_k}{4\pi} (\tilde{\Omega}_i \cdot \Omega_k)^m (\tilde{\Omega}_k + \tilde{\Omega}_i + i(\Omega_k \times \tilde{\Omega}_i)) \cdot \vec{\sigma}_{\alpha \beta},
$$

with $m$ odd, which vanish trivially. The next terms in the expansion have two Pauli matrices acting on the different bonds. These terms are proportional to the only SU(2) invariants that can be constructed with two vectors (of Pauli matrices), namely $\vec{\sigma}_i \cdot \vec{\sigma}_j$ (see fig 3.b). Depending on the separation between the wiggly bonds along the boundary, we have different contributions for which the numerical factor should decay exponentially with this distance, given that the VBS model is expected to have a mass gap (fact that is proven for linear and hexagonal lattices), result which is in agreement with the $O(N)$ model being noncritical for $N > 2$ at $x = -\frac{45}{4}$.

With the previous results, we can write the following expansion for the density matrix $\rho_A$

$$\rho_A = \frac{I}{2^d} + \sum_{r,i} A_r \vec{\sigma}_i \cdot \vec{\sigma}_{i+r} + \sum_{ijk} A_{ijk} \vec{\sigma}_i \cdot (\vec{\sigma}_j \times \vec{\sigma}_k) + \ldots \quad (19)$$

where $I$ is the $2^d \times 2^d$ identity operator and the coefficients $A_r$ and $A_{ijk}$ are related to the correlation functions of the $O(N)$ on the lattice $L$, with some sites and bonds erased along the boundary. Specifically for the first coefficient $A_r$ we have

$$A_r \sim (\tilde{\Omega}_k \cdot \tilde{\Omega}_{k+r}) (\tilde{\Omega}_k \cdot \tilde{\Omega}_{k+r}) L_k - (\tilde{\Omega}_k \cdot \tilde{\Omega}_{k+r}) (\tilde{\Omega}_k \cdot \tilde{\Omega}_{k+r}) L_k, \quad (20)$$

Here the correlation function is computed over the lattice $L_k$ which is the same lattice $L$ but with the bonds $(k,k)$ and $(k+r,k+r)$ erased. This relation is exact for hexagonal lattices, while for other lattices with coordination number greater than 3, all the other possible contractions between even number of legs at the boundary sites have to be included. As usual with gapped systems, we expect that this correlation decays exponentially with the separation of the spins, then we have $A_r \sim \exp(-r/\xi_1)$. Numerical studies for two-leg VBS ladders have been performed being in agreement with this general result. For $r = 1$, the second term in (20) vanishes in the thermodynamic limit when minimum distance paths joining the sites are cycles who travel the lattice. Also taking the limit of infinite size of the $A$ and $B$ subsystems, the interaction between the two boundary chains along different cuts of the partition vanishes. Then the total density matrix is the tensor product of matrices with the expansion (19), for each cut.

It is clear that the first nontrivial term in the expansion (19) is the XXX Heisenberg Hamiltonian. We can also determine whether this interaction is ferro or antiferromagnetic in the simplest hexagonal lattice, from the loop expansion. The structure of the lattice determines the sign of the interaction through the number of bonds.
that define the allowed paths between site \( k \) and site \( k + 1 \) (each bond has an associated \( x = -1 \)). An overall minus sign comes from the contraction of two wiggly lines. Then it is easy to show that for the hexagonal lattice with a partition like the one in Fig. 4a, all the paths connecting the boundary sites have even number of bonds, then the sign of the boundary XXX Hamiltonian is \(-1\), so the boundary chain interaction is ferromagnetic. Numerical results\(^{23,24}\) in finite size square lattices for a partition like Fig. 4b, indicate that in the square grid the interaction is anti-ferromagnetic.

VI. CONTINUOUS LIMIT AND ENTANGLEMENT HAMILTONIAN

In order to unveil the structure of the entanglement Hamiltonian, we can analyze the partial density matrix\(^{14}\) taking the lattice spacing in the original discrete model to zero, obtaining a continuous version of the model. In this limit we can show the locality of the model to zero, obtaining a continuous version of the Hamiltonian, we can analyze the partial density matrix

\[
\rho_A[\sigma] = \frac{1}{Z} \int \mathcal{D}\Omega \mathcal{D}\alpha \exp \left[-\frac{i\pi}{2} \int d^2x \, \bar{\Omega} \cdot \bar{\sigma} \right]
\]

with \( \sigma^k(x_1, x_2) = \sigma^k(x_1)(\delta(x_2) + \delta(x_2 - L_A)), \; (k = 1..3) \) an spin 1/2 field defined at the boundary of \( A \), which we have placed conveniently at \( x_2 = 0 \) and \( x_2 = L_A \). The action \( S[\Omega, \alpha] \) is given by

\[
S[\Omega, \alpha] = \frac{1}{2g_0} \int d^2x \left\{ (\nabla \bar{\Omega})^2 + i\alpha(x)(\bar{\Omega}x^2 - 1) \right\}
\]

where we have introduced a bare coupling \( g_0 \). As the discussion is essentially the same for any number of components of the \( \Omega \) field, we now consider the more general \( N \) component case with the corresponding \( O(N) \) global symmetry. We can integrate out the field \( \Omega \), as the action in this field is quadratic, obtaining

\[
\rho_A[\sigma] = \frac{1}{Z} \int \mathcal{D}\alpha \exp \left(-\pi^2g_0^2 \int dx \, dy \, \sigma^k(x) \Delta^{-1}(x - y) \sigma^k(y) + \frac{i}{2g_0} \int d^2x \, \alpha(x) - \frac{N}{2} \text{tr} \ln \Delta \right),
\]

where \( \Delta(x) = -\nabla^2 + i\alpha(x) \).
limit, keeping $N g_0^2$ fixed. In this limit, we can evaluate the integral \[ \frac{1}{(2\pi)^2} \frac{k^2 + m^2}{k^2 + m^2} = \lim_{\Lambda \to \infty} \frac{1}{2\pi} \ln \left( \frac{\Lambda}{m} \right) = \frac{1}{N g_0^2}. \] (25)

This equation is divergent, but can it be made finite by renormalizing the bare coupling $g_0$ at an arbitrary renormalization scale $\Lambda$ as $\frac{1}{g^2} = \frac{1}{9} + \frac{N}{2\pi} \ln \left( \frac{\Lambda}{M} \right)$. 

Inserting this equation back in \[ \int \frac{d^2k}{(2\pi)^2} \rho \] , we get the following expression for $m$ in terms of the physical coupling $g$, the renormalization scale $M$ and the number of components $N$ of the original $\Omega$ field,

\[ m = M \exp \left[ -\frac{2\pi}{g^2 N} \right]. \] (26)

In this large $N$ limit, we can compute the Entanglement Hamiltonian as the logarithm of the reduced density matrix, obtaining

\[ H_{\text{ent}} = \ln \rho_A[\sigma] = (\pi g)^2 \int dx dy \sigma^k(x) \Delta^{-1}(x-y) \sigma^k(y), \] (27)

where $\Delta^{-1}(x) = K_0(m|x|)/2\pi$ is the zeroth order modified Bessel function. The exponential decay of $K_0(m|x|)$ for large $x$ is what defines a local interaction at the boundary of $A$. Although this result is obtained in the large $N$ limit, the general features of the $N = 3$ model are believed to be captured in this limit.\[ \] (28)

\section{VII. CONCLUSIONS}

Given the structure of the VBS ground state, it is possible to define on any planar graph, without loops, a VBS state, where the local spin at site $i$ is given by $z_i/2$, with $z_i$ the coordination number at site $i$. Using the Schwinger boson representation of the VBS ground state and the classical variable representation of this state, an expression for the partial density matrix $\rho_A$, which describe the physical subsystem $A$, obtained by partitioning the whole unique ground state, can be written. This expression for $\rho_A$ decomposes into a classical loop expansion of the $O(3)$ model in the gapped phase. Analyzing the different loop contributions, and assuming translation invariance, we have shown that the partial density matrix that describes a subsystem of the VBS ground state can be expressed as a sum over different rotation-invariant quantum operators, where the Heisenberg interaction between nearest neighbors gives the largest nontrivial contribution to the expansion. This quantum operators act on a spin $1/2$ chain in the boundary of the partition. The translational invariance assures us that the different contributions along the boundary are equally weighted, so the boundary operator is given by the XXX Heisenberg Hamiltonian. Here we discuss the case of translation invariant lattices which can be embedded in a torus, but for other lattices with different topologies we expect similar results.

For non translational invariant lattices, the first nontrivial local interaction term is expected to be also of the type $\sigma_i \sigma_{i+1}$ but the Hamiltonian along the boundary will have different numerical prefactors for each local Heisenberg interactions, generating a non invariant Heisenberg Hamiltonian in the boundary.

In the continuous limit, we show that the entanglement Hamiltonian for this model is actually a local Hamiltonian, where the Hamiltonian density corresponds to a Heisenberg interaction of spin $1/2$ particles.

The analysis shown in this paper should be useful for studying other dimensions $d > 2$ or other two dimensional lattices with more than one bond between a pair of sites. In that case, the local dimension of the spin operators in the boundary Hamiltonian should increase, having then boundary chains with higher representations of $SU(2)$ per site, but still with $SU(2)$ invariant local interactions.

\section{Acknowledgments}

R. S. wish to thank F.N.C. Paraan for discussions and Dr. Tzu-Chieh Wei and Dr. Vladimir Korepin for careful reading this manuscript and for useful comments. R.S. particularly thanks Dr. Ignacio Cirac for discussions and the Max Planck Institute for Quantum Optics for hospitality. R.S. is supported by a Fulbright-Conicyt Fellowship.
We can write the operator $\rho_A$ as a matrix using the basis $|i\rangle = \sum_\alpha U_{i\alpha}|A_\alpha\rangle$ and its dual $|j\rangle = \sum_\gamma V_{j\gamma}\langle A_\gamma|$. We have $(\rho_A)|i\rangle = (\rho_A)_{ji} = N\sum_{\alpha\beta\gamma\delta}V_{j\gamma}\langle A_\gamma|A_\alpha\rangle U_{i\alpha}\langle A_\beta|A_\delta\rangle$. However from the orthonormality condition $\langle j|i\rangle = \delta_{ij}$ follows the relation $\sum_\gamma V_{j\gamma}\langle A_\gamma|A_\alpha\rangle = (U^{-1})_{\alpha j}$, which inserted back in the expression for $(\rho_A)_{ji}$ simplifies it to $\langle j|\rho_A|i\rangle = N\sum_{\alpha\beta\gamma}U_{i\alpha}\langle A_\alpha|A_\beta\rangle^*\langle B_\beta|A_\gamma\rangle(U^{-1})_{\alpha j}$.

D. F. Pollmann, A. Auerbach, and F. D. M. Haldane, Phys. Rev. Lett. 93, 227203 (2004)

B. Nienhuis, in Phase Transitions and Critical Phenomena, Vol. 11, edited by C. Domb and J. Lebowitz (Academic Press, London, 1987).

B. Nienhuis, Phys. Rev. Lett. 49, 1062 (1982)

W. Guo, H. W. J. Blöte, and F. Y. Wu, Phys. Rev. Lett. 85, 3874 (2000)

W. A. Bardeen, B. W. Lee, and R. E. Shrock, Phys. Rev. D 14, 985 (1976)

E. Brézin, J. Zinn-Justin, and J. C. Le Guillou, Phys. Rev. D 14, 2615 (1976)

M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Westview Press, 1995) USA

A. B. Zamolodchikov and A. B. Zamolodchikov, Ann. of Phys. 210, 253 (1991)