Modeling of nonlinear behaviour of three-dimensional bodies and shells of average thickness by finite element method

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Abstract. The work is devoted to the fundamentals of the technique of numerical investigation of the nonlinear behavior of three-dimensional bodies oriented to the application of the finite element method. The first section is devoted to the formulation of the problem, the introduction of finite deformations in the Lagrangian coordinate system, and the description of the Cauchy and Piola-Kirchhoff tensors. The second section describes the resolving equations obtained from the principle of virtual displacements. In the third section, we consider the test problem of the bending of a square plate, and compare the results with other authors.

1. Statement of the problem and kinematic relations
Among the many different procedures for solving nonlinear problems of mechanics of a deformed solid, the methods of stepwise loading are very widely used. From a physical point of view, we are talking about modeling the process of deformation in the form of a sequence of equilibrium states. At each step of loading, the distribution of external forces can change, which makes this approach even more attractive. Let us consider the main provisions of the step-by-step scheme for solving geometrically nonlinear problems in the theory of elasticity.

In some global Cartesian coordinate system $x_1, x_2, x_3$, with orms $\xi_1, \xi_2, \xi_3$, we will consider the deformation of a body loaded with mass $\tilde{Q}$ and surface $\tilde{R}$ forces. Let's divide this loading into several steps and determine for each of them the level of acting forces. Let $\tilde{Q}^{(m)}$ and $\tilde{R}^{(m)}$ - value of external loads at the loading step. The radius vector of the material point at each step will be denoted by $\tilde{R}^{(m)}$, where $\tilde{R}^{(0)}$ - the position of the material point in the initial state.

To describe the deformation of a rigid body, we will apply the Lagrange approach and introduce a "frozen" coordinate system $\xi_1, \xi_2, \xi_3$, relative to which the position of each point is invariant. The following is true:

$$\tilde{R}^{(m)}(\xi) = \sum_{i} x_1^{(m)}(\xi) \xi_1, x_2^{(m)}(\xi) \xi_2, x_3^{(m)}(\xi) \xi_3.$$

We define the displacement vector from the initial state to the current state

$$W^{(m)}(\xi) = \tilde{R}^{(m)} - \tilde{R}^{(0)} = \sum W_i^{(m)}(\xi) \xi_i,$$

and the displacement vector from the previous state to the current state

$$W_i^{(m)}(\xi) = \xi_i.$$
\[ V^{(m)}(\xi) = R^{(m)} - R^{(m-1)} = \sum V_i^{(m)}(\xi) e_i, \]

Deformations will be described with the help of the Green's deformation tensor, whose components in the vector form are written down

\[ E_i^{(m)} = \frac{1}{2} \left( \frac{\partial R_i^{(m)}}{\partial \xi_i} - \frac{\partial R_i^{(m-1)}}{\partial \xi_i} \right) \]

or

\[ e_i^{(m)} = \frac{1}{2} \left( \frac{\partial W_i^{(m)}}{\partial \eta_i} + \sum \frac{\partial W_i^{(m)}}{\partial X_j} \right) \]

In the first case, they determine the deformation of the medium with respect to the initial state, in the second - to the previous state.

If we identify the initial coordinates \( x_i^{(0)} \) with Lagrangian \( \xi_i \), then relation (1) can be written in the classical form

\[ E_i^{(m)} = \frac{1}{2} \left( \frac{\partial W_i^{(m)}}{\partial X_i} + \sum \frac{\partial W_i^{(m)}}{\partial X_j} \right) \]

where the physical components of deformations appear.

An alternative is the introduction of coordinate systems \( \eta_i^{(m)}, \eta_i^{(m)}, \eta_i^{(m)} \), which at each step are determined independently in such a way that \( \frac{\partial R_i^{(m)}}{\partial \eta_i} = \delta_i \), which is formally quite possible.

In this case, the expressions for the relative deformations (2) are simplified, which for physical components will have the form

\[ e_i^{(m)} = \frac{1}{2} \left( \frac{\partial V_i^{(m)}}{\partial X_i} + \sum \frac{\partial V_i^{(m)}}{\partial X_j} \right) \]

this follows from

\[ \sum \frac{\partial x_i^{(m-1)}}{\partial \eta_j} = \delta_i; \quad \frac{\partial x_i^{(m-1)}}{\partial \eta_j} = \delta_i; \quad \frac{\partial V_i^{(m)}}{\partial \eta_j} = \sum \frac{\partial V_i^{(m)}}{\partial x_i^{(m-1)}} \frac{\partial x_i^{(m-1)}}{\partial \eta_j} = \frac{\partial V_i^{(m)}}{\partial x_i^{(m-1)}} . \]

The physical relationships (3) describe the deformation of the unit volume extracted in the initial state and oriented with respect to the global axes \( x_i \), and the ratio (4) of the unit volume extracted in the deformed state at the (m-1) loading step, but also oriented with respect to the global axes \( x_i \).

To describe the stress state, we will use two types of tensors: the Cauchy stress tensor with components \( \sigma_i \) oriented relative to the deformed faces and related to their areas and the second Piola-Kirchhoff stress tensor with components \( S_i^{(m)} \), also oriented with respect to deformed faces, but related to their areas in an undeformed state.

Since in the future the solution of problems is guided by the application of the finite element method within the framework of the isoparametric approach, it is sufficient to restrict ourselves to introducing the physical components of these tensors.

Let the unit volume be chosen as the base volume in the undeformed state, the deformation of which at each step is described by the relations (1). Then the stress state will be characterized by the components of the Kirchhoff stress tensor \( \bar{S}_i^{(m)} \) related to the undeformed volume. In this case, the problem reduces to determining their increments at the loading step.

If the unit volume is chosen as the base volume at the previous (m-1) -th loading step, then its stress state will be characterized by the sum \( \sigma_i^{(m-1)} + \Delta S_i^{(m)} \), where \( \sigma_i^{(m-1)} \) - the components of the
accumulated true stresses arising on the faces of the allocated volume (oriented with respect to global axes $x_i$) and stress increments $\Delta S^{(m)}_{ij}$, which are Kirchhoff stresses with respect to the (m-1) state.

2. The resolving equation and the physical relationships

The resolving equations will be obtained from the principle of virtual displacements; therefore, it is necessary to specify strictly with respect to what volume all the integrals should be assigned.

Let's consider two variants, the first one is when the integration is carried out over the initial volume (in this case deformations (3) and stresses $S^{(m)}_{ij}$ are used; the second one is when the integration is over the volume of the previous (m-1) state, which is known (in this case deformations and the stress $\sigma^{(m)}_{ij} + \Delta S^{(m)}_{ij}$ are used (4)).

To obtain the resolving equations in the first variant, it is necessary to determine what functions will be unknown at the loading step: either general displacements $W^{(m)}$ or their increments $\Delta W^{(m)} = W^{(m)} - W^{(m-1)}$.

In the first case, the equation of equality of the work will have the form

$$\int_{\Omega} S^{(m)}_{ij} \delta E^{(m)}_{ij} d\Omega = \int_{\Omega} Q^{(m)} \delta \omega d\Omega + \int_{S_{c}} R^{(m)} \delta \omega dS$$

(5)

where $S_{c}$ is the part of the surface where the surface forces are given. An analysis of this equation shows that we have a direct problem of determining the stress and strain in the m-th state, taking into account the entire complexity of the nonlinearity of this problem. It is possible to use the fact that the solution is known in the (m-1) – th state only for the interpolation of the first approximation for the iterative procedure for solving Eq. (5).

To obtain the resolving equation in the second case, we introduce the increment of deformations

$$\Delta E^{(m)}_{ij} = E^{(m)}_{ij} - E^{(m-1)}_{ij} = \frac{1}{2} \left[ \frac{\partial \Delta W^{(m)}_{ij}}{\partial x_i} + \frac{\partial \Delta W^{(m)}_{ij}}{\partial x_j} + \sum_{k=1}^{n} \left( \frac{\partial \Delta W^{(m)}_{ik}}{\partial x_i} \right) \frac{\partial \Delta W^{(m)}_{jk}}{\partial x_j} \right]$$

In this notation the variational equation has the form

$$\int_{\Omega} \left[ (S^{(m)}_{ij} + \Delta S^{(m)}_{ij}) \delta \Delta E^{(m)}_{ij} d\Omega = \int_{\Omega} Q^{(m)} \delta \omega d\Omega + \int_{S_{c}} R^{(m)} \delta \omega dS \right]$$

(6)

In contrast to relation (5), expression (6) admits linearization, under the assumption that the gradient of the increments of displacements is much less than unity, that is,

$$\frac{\partial \Delta W^{(m)}_{ij}}{\partial x_i} \ll 1$$

(7)

The next step is to define the linear part of $\Delta E^{(m)}_{ij}$ in the form

$$\Delta E^{(m)}_{ij} = \frac{1}{2} \delta_{ij} + \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial \Delta W^{(m)}_{ik}}{\partial x_i} \right) \frac{\partial \Delta W^{(m)}_{jk}}{\partial x_j}$$

(8)

With this notation, after neglecting small second orders, we obtain the linear equation

$$\int_{\Omega} \left[ \Delta S^{(m)}_{ij} \delta E^{(m)}_{ij} + S^{(m)}_{ij} \delta \left( \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial \Delta W^{(m)}_{ik}}{\partial x_i} \right) \frac{\partial \Delta W^{(m)}_{jk}}{\partial x_j} \right) \right] d\Omega = \int_{\Omega} Q^{(m)} \delta \omega d\Omega + \int_{S_{c}} R^{(m)} \delta \omega dS - \int_{S_{c}} S^{(m)}_{ij} \delta E^{(m)}_{ij} d\Omega$$

(9)

The second variant assumes as unknown functions a vector $\tilde{V}^{(m)}$, which is a vector of increment of displacements when passing from the previous state to the current state, that is

$$\tilde{V}^{(m)} = \Delta \tilde{W}^{(m)}$$

(10)

The variational equation of the principle of possible displacements analogous to (5), (6) will be written in the form
\[
\int_{\Omega} \left( \sigma^{(m)} + \Delta \sigma^{(m)} \right) \delta \varepsilon^{(m)} \, d\Omega = \int_{\Omega} \mathbf{Q}^{(m)} \delta \varepsilon^{(m)} \, d\Omega + \int_{S_{\text{e}}} \mathbf{R}^{(m)} \delta \mathbf{n} \, dS
\]  
(11)

Under the assumption of validity of (7) with allowance for (10), this equation also admits linearization. If we introduce the "classical" linear and nonlinear deformations
\[
\varepsilon^{(n)} = \frac{1}{2} \left( \frac{\partial V^{(m)}}{\partial X^{(m)}} + \frac{\partial V^{(m)}}{\partial X^{(m)}} \right),
\]
(12)
\[
\eta^{(n)} = \frac{1}{2} \sum \left( \frac{\partial V^{(m)}}{\partial X^{(n)}} - \frac{\partial V^{(m)}}{\partial X^{(m)}} \right)
\]
(13)
then after the linearization (11), the resulting linear equation has the form
\[
\int_{\Omega} \left( \Delta \varepsilon^{(n)} \delta \varepsilon^{(n)} + \varepsilon^{(n)} \delta \eta^{(n)} \right) \, d\Omega = \int_{\Omega} \mathbf{Q}^{(n)} \delta \varepsilon^{(n)} \, d\Omega + \int_{S_{\text{e}}} \mathbf{R}^{(n)} \delta \mathbf{n} \, dS - \int_{\Omega} \varepsilon^{(n)} \delta \varepsilon^{(n)} \, d\Omega
\]
(14)

The difference between equation (5) and analogous (9) in a simpler relation for linear deformations, since relation (12) is simpler (8).

However, this simplification is accompanied by the appearance of an additional stress calculation \( \sigma^{(n)} \) for the transition to the next loading step. For this it is necessary to use the relations connecting the stress tensors of Cauchy and Kirchhoff, which in our case have the form
\[
\sigma^{(n)} = \frac{1}{\det J} \sum \frac{\partial X^{(m)}}{\partial X^{(n)}} \frac{\partial X^{(m)}}{\partial X^{(n)}} \left( \sigma^{(m)} + \Delta \sigma^{(m)} \right)
\]
(15)
where
\[
\left[ J \right] = \left[ \begin{array}{cccc}
\frac{\partial X^{(m)}}{\partial X^{(n)}} & \frac{\partial X^{(m)}}{\partial X^{(n)}} & \frac{\partial X^{(m)}}{\partial X^{(n)}} \\
\frac{\partial X^{(m)}}{\partial X^{(n)}} & \frac{\partial X^{(m)}}{\partial X^{(n)}} & \frac{\partial X^{(m)}}{\partial X^{(n)}} \\
\frac{\partial X^{(m)}}{\partial X^{(n)}} & \frac{\partial X^{(m)}}{\partial X^{(n)}} & \frac{\partial X^{(m)}}{\partial X^{(n)}}
\end{array} \right]
\]
(16)

Let us analyze the several solutions considered for solving the problems of mechanics of deformable bodies with allowance for geometric nonlinearity. Each of them has a "right to life" and there are numerous examples demonstrating their working capacity. Considering the effectiveness of the application of the finite element method for the numerical discretization of variational equations, it is necessary to single out the latter approach. In this case, the integrands will be the simplest (in the sense of the degree of the polynomial for deformations \( \varepsilon_{(n)} \)) and to calculate the integrals we can use the quadratures that are usually used in problems of the linear theory of elasticity. Moreover, the principal part of the operator, defined by the integral
\[
\int_{\Omega} \Delta \varepsilon^{(n)} \delta \varepsilon^{(n)} \, d\Omega
\]
(17)
exactly coincides with the operator of the linear theory of elasticity and determines the corresponding stiffness matrix. For these problems, a large theoretical and experimental material has been accumulated on the study of the properties of certain approximations and a wide range of methods for improving the rate of convergence has been developed. Therefore, it is supposed to be possible to use these developments to solve nonlinear problems.

Based on these considerations, the last approach is taken as the basis for further research. It is known in the literature as the "modernized incremental theory of Lagrange" or "Update Lagrangian formulation".
3. The problem of plate bending
The problem of the bending of rectangular plates in a geometrically nonlinear formulation for various boundary conditions and transverse loads was considered in [3-4]. To verify the proposed method, as an example, a square (40x40x1) plate was used, having a hinged support on all edges, under uniform transverse pressure (Fig. 1).

![Figure 1](image)

In the calculation, the ¼ part was modeled, splitting it into a grid with 8 node finite elements of 10x10 size, that is, each element had dimensions 2x2x1. The Poisson's ratio was assumed to be equal 0.3. The results of the calculations are shown in Fig. 2 and Fig. 3. Fig. 2 contains load-deflection curves between dimensionless parameters

\[ \hat{q} = \frac{q a^4}{E h^3}, \quad \frac{w}{h} = \frac{w}{w} \]

(18)

The graph shows the following solutions: line ar, given in [2]; from works [3-4]; obtained according to the presented technique in increments \( q = 10 \). In Fig. 3 shows the results of calculating the stresses, in the form of a dimensionless parameter \( \hat{\sigma} = \frac{\sigma a^2}{4Eh} \), at the center of the plate (at this point \( \sigma_{xx} = \sigma_{yy} \) because of the symmetry of the problem). The curve corresponds to the presented calculation. The results of [3] are marked by triangles.

![Figure 2](image)

![Figure 3](image)

It should be noted here that the obtained results of the problem in a geometrically nonlinear setting for deflections exceeding the thickness differ significantly from the linear solution. The difference between the three-dimensional solution in the theory of elasticity and the results of calculation according to the theory of thin plates is insignificant, and for maximum stresses it is practically absent.

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