SKEW HOPF ALGEBRAS, IRREDUCIBLE EXTENSIONS AND
THE Π-METHOD

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Abstract. To a depth two extension $A|B$, we associate the dual bialgebroids $S := \text{End}_B A_B$ and $T := (A \otimes_B A)^R$ over the centralizer $R = C_A(B)$. In a setup which is quite common, where $R$ is a subalgebra of $B$, two nondegenerate pairings of $S$ and $T$ will define an anti-automorphism $\tau$ of the algebra $S$. Making use of a two-sided depth two structure, we show that $\tau$ is an antipode and $S$ is a Hopf algebroid of a type we call skew Hopf algebra. A final section discusses how $\tau$ and the nondegenerate pairings generalize to modules via the $\pi$-method for depth two.

1. Introduction

For reasons of symmetry in representation theory, given a bialgebra or bialgebroid one would like to expose the presence of an antipode. In the reconstruction of Hopf algebras and weak Hopf algebras in subfactor theory a key idea in the definition of antipode is to make use of the existence of two nondegenerate pairings of dual bialgebras. In terms of a depth two, finite index subfactor $N \subseteq M$ with trivial relative commutant, its basic construction $M \subseteq M_1$, and another one above, $M_1 \subseteq M_2$, there are conditional expectations $E_M : M_1 \rightarrow M$ and $E_{M_1} : M_2 \rightarrow M_1$. In addition to the Jones projections $e_1 \in M_1$ and $e_2 \in M_2$, the two relative commutants that are paired nondegenerately are in ordinary algebraic centralizer notation $C = C_{M_1}(N)$ and $V = C_{M_2}(M)$. The antipode $\tau : V \rightarrow V$ is then defined as the “difference” of two such pairings:

$$E_M E_{M_1}(ve_1e_2c) = E_M E_{M_1}(ce_2e_1\tau(v))$$

for $c \in C$ and $v \in V$: see for example, [8, 17, 19] for the details of why this formula works.

The duality method for defining antipode has been lying dormant in recent generalizations of depth two to algebras and rings and actions of bialgebroids on these. For example, antipodes have been defined recently in the case of a Frobenius extension $A|B$ as the restriction of a standard anti-isomorphism of the left and right endomorphism rings $\text{End}_B A \rightarrow \text{End}_B A$ to an anti-automorphism of the subring of bimodule endomorphisms $S = \text{End}_B A_B$: on depth two Frobenius extension this defines the antipode or its inverse in [2] in a dual way on both $S$ and $T$; it also necessitates a revision of the definition of the notion of Hopf algebroid using the notions of left and right bialgebroid. Antipodes have also been defined from geometric ideas of Lu [16] for H-separable extensions [10], extensions of Kanzaki separable algebras [12] and Hopf-Galois extensions [11], and from group theory in [14] for pseudo-Galois extensions.

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In this paper we define antipode as the difference of two nondegenerate pairings for a special extension $A|B$ where the centralizer $R$ is a subalgebra of the smaller algebra $B$, which we call irreducible extension. In this case, the two hom-groups, the left and right $R$-duals of the bialgebroid $T$, coincide. The plan to find an antipode from this identity and satisfying the several axioms of a Hopf algebroid works not so much because of any Frobenius structure (as assumed previously) but on a two-sided depth two structure as shown in sections 3 and 4 below. We recall that the dual bialgebroids $S$ and $T$ depend only a one-sided depth two structure [12], but at two stages in this paper (using both nondegenerate pairing and the proposition in section 3) we require a two-sided depth two structure. The Frobenius extension hypothesis avoided in this paper, makes one-sided depth two extensions two-sided [9, 6.4]. The Hopf algebroid structure we obtain on $S$ is very nearly a Hopf algebra which is finite projective over a commutative base ring: we discuss its properties after Theorem 4.1 and designate as skew Hopf algebras such a Hopf algebroid. We end with a discussion of how $\tau$ and the nondegenerate pairings generalize to modules via the $\pi$-method for depth two. A certain mapping between cochain complexes formed from the left- and right-handed $\pi$-methods is shown to be nullhomotopic.

2. Preliminaries on depth two extensions

Let $B$ be a unital subalgebra of $A$, an associative noncommutative algebra with unit over a commutative ground ring $K$. The algebra extension $A|B$ is depth two if there is a positive integer $N$ such that

$$A \otimes_B A \otimes * \cong A^N$$

as natural $B$-$A$ (left D2) and $A$-$B$-bimodules (right D2) [9]. For an $A$-$A$-bimodule $M$, denote the subgroup of $B$-central elements in $M$ by

$$M^B := \{ m \in M | \forall b \in B, bm = mb \}.$$

Equivalently, the algebra extension $A|B$ is depth two if there are elements

$$\beta_i \in S := \text{End}_{BA}, \ t_i \in T := (A \otimes_B A)^B$$

(called a left D2 quasibasis) such that all elements of $A \otimes_B A$ may be written as

$$(a, a' \in A)$$

$$a \otimes a' = \sum_{i=1}^{N} t_i \beta_i (a) a',$$

and similarly, there are elements (of a right D2 quasibasis) $\gamma_j \in S$, $u_j \in T$ such that

$$(a, a' \in A)$$

$$a \otimes a' = \sum_{j=1}^{N} a \gamma_j (a') u_j.$$
there are somewhat obvious isomorphisms $\text{Hom}(A, A \otimes_B A) \cong T$ and $\text{Hom}(A \otimes_B A, A) \cong S$ in either case of $A$- or $B$-$A$-bimodule homomorphisms. We fix the notations for both right and left $D_2$ quasibases throughout this paper.

For example, an $H$-separable extension $A|B$ is of depth two since the condition above on the tensor-square holds even more strongly as natural $A$-$A$-bimodules. Another example: $A$ a f.g. projective algebra over commutative ground ring $B$, since left or right $D_2$ quasibases are easily constructed from a dual basis. As a third class of examples, consider a Hopf-Galois extension $A|B$ with $n$-dimensional Hopf $k$-algebra $H \llbracket k \rrbracket$. Recall that $H$ acts from the left on $A$ with subalgebra of invariants $B$, induces a dual right coaction $A \to A \otimes_k H^*$, $a \mapsto a(0) \otimes a(1)$, and Galois isomorphism $\beta : A \otimes_B A \rightarrow A \otimes_k H^*$ given by $\beta(a \otimes a') = aa'_0 \otimes a'_{(1)}$, which is an $A$-$B$-bimodule, right $H^*$-comodule morphism. It follows that $A \otimes_B A \cong A$ as $A$-$B$-bimodules; as $B$-$A$-bimodules there is a similar isomorphism by making use of the alternative Galois isomorphism $\beta'$ given by $\beta'(a \otimes a') = a(0) a' \otimes a(1)$. The paper \cite{13} extends the definition above of depth two to include the case where the tensor-square of $A|B$ is isomorphic to any direct sum of $A$ with itself (not necessarily a finite direct sum as in eq. \cite{9}); thus any Hopf-Galois extension is depth two in this extended sense. However, this theory does not have a theory of dual bialgebroids congenial for the results in this paper, and we shall not make use of it.

The papers \cite{9, 10} defined dual bialgebroids with action and smash product structure within the endomorphism ring tower construction above a depth two ring extension $A|B$. In more detail, if $R$ denotes the centralizer of $B$ in $A$, a left $R$-bialgebroid structure on $S$ is given by the composition ring structure on $S$ with source and target mappings corresponding to the left regular representation $\rho : R^\text{op} \to S$, respectively. Since these commute ($\lambda_r \rho_s = \rho_s \lambda_r$ for every $r, s \in R$), we may induce an $R$-bimodule structure on $S$ solely from the left by

$$r \cdot \alpha \cdot s := \lambda_r \rho_s \alpha = ra(-)s.$$

Now an $R$-coring structure $(S, \Delta, \varepsilon)$ is given by

$$(5) \quad \Delta(\alpha) := \sum_i \alpha(-t_i^1)t_i^2 \otimes_R \beta_i$$

for every $\alpha \in S$, denoting $t_i = t_i^1 \otimes_B t_i^2 \in T$ by suppressing a possible summation, and

$$(6) \quad \varepsilon(\alpha) = \alpha(1)$$

satisfying the additional axioms of a bialgebroid (cf. appendix), such as multiplicativity of $\Delta$ and a condition that makes sense of this requirement. We have the equivalent formula for the coproduct \cite{9 Thm 4.1}:

$$(7) \quad \Delta(\alpha) := \sum_j \gamma_j \otimes_R u_j^1 \alpha(u_j^2 -)$$

For a depth two extension,

$$(8) \quad S \otimes_R S \xrightarrow{\Delta} \text{Hom}(B \otimes_B A_B, B \otimes_B A_B). \quad \alpha \otimes_R \beta \mapsto (x \otimes_B y \mapsto \alpha(x) \beta(y))$$

with inverse provided by $F \mapsto \sum_i F(- \otimes t_i^1)t_i^2 \otimes_R \beta_i$. Under this identification, the formula for the coproduct $\Delta : S \to S \otimes_R S$ becomes

$$(9) \quad \alpha(1)(x)\alpha(2)(y) = \alpha(xy)$$
using either equation.

The left action of $S$ on $A$ given by evaluation, $\alpha \triangleright a = \alpha(a)$, has invariant subalgebra (of elements $a \in A$ such that $\alpha \triangleright a = \varepsilon(\alpha)a$) equal precisely to $B$ if the natural module $A_B$ is balanced [9]. This action is measuring by eq. (7).

The smash product $A \rtimes S$, which is $A \otimes_R S$ as abelian groups with associative multiplication given by

\[ (x \times_\lambda a)(y \times_\mu b) = x(\lambda(1) \triangleright y) \times_\mu \lambda(2)b, \]

is isomorphic as rings to $\text{End}_A(B)$ [9].

In general $T = (A \otimes_B A)^B$ has a unital ring structure induced from $T \cong \text{End}_A(A \otimes_B A)$ via $F \mapsto F(1 \otimes 1)$, which is given by

\[ tu = u_1 t_1 \otimes t_2 u_2 \]

for each $t, u \in T$. There are obvious commuting homomorphisms of $R$ and $R^{\text{op}}$ into $T$ given by $r \mapsto 1 \otimes r$ and $s \mapsto s \otimes 1$, respectively. From the right, these two source and target mappings induce the $R$-$R$-bimodule structure $R_T$ given by

\[ r \cdot t \cdot s = (t^1 \otimes t^2)(r \otimes s) = rt^1 \otimes t^2 s, \]

the ordinary bimodule structure on a tensor product.

There is a right $R$-bialgebroid structure on $T$ with coring structure $(T, \Delta, \varepsilon)$ given by the two equivalent formulas:

\[ \Delta(t) = \sum_i t_i \otimes_R (\beta_i(t^1) \otimes_B t^2) = \sum_j (t^1 \otimes_B \gamma_j(t^2)) \otimes_R u_j \]

\[ \varepsilon(t) = t^1 t^2 \]

By [9] Thm 5.2] $\Delta$ is multiplicative and the other axioms of a right bialgebroid are satisfied.

As an example of $S$ and $T$, consider the Hopf-Galois extension $A|B$ of $k$-algebras introduced above. Since $\beta$ is an $A$-$B$-isomorphism, we may compute that $T \cong R \otimes_k H^*$ via $\beta$, which induces a smash product structure on $R \otimes H^*$ relative to the Miyashita-Ulbrich action of $H^*$ on $R$ from the right. The well-known isomorphism $\text{End}_A(B) \cong A \ltimes H$ via $a \ltimes h \mapsto \lambda(a)(h \triangleright \cdot)$ restricts to $S \cong R \ltimes H$, i.e., $S$ is a smash product of $R$ with $H$ via the restriction of the left action of $H$ to $R$. In both cases, the $R$-coring structures are the trivial ones induced from the coalgebras $H$ and $H^*$.

There is a right action of $T$ on $E := \text{End}_B A$ given by $f \cdot t = t^1 f(t^2 -)$ for $f \in E$. This is a measuring action by Eq. (3) since

\[ (f \cdot t(1)) \circ (g \cdot t(2)) = \sum_i t_i^1 f(t_i^2 \beta_i(t^1) g(t^2 -)) = f \cdot t. \]

The subring of invariants in $E$ is $\rho(A)$ [9]. Also, in analogy with $\text{End}_A(B) \cong A \ltimes S$, the smash product ring $T \rtimes E$ is isomorphic to $\text{End}_A(A \otimes_B A)$ via $\Psi$ given by

\[ \Psi(t \otimes f)(a \otimes a') = at^1 \otimes_B t^2 f(a'). \]

Sweedler [18] defines left and right $R$-dual rings of an $R$-coring. In the case of a left $R$-bialgebroid $H$ with $H_R$ and $rH$ finitely generated projective, such as $(S, \lambda, \rho, \Delta, \varepsilon)$ above, the left and right Sweedler $R$-dual rings are extended to right bialgebroids $H^*$ and $^*H$ in [9]. For example, $H^*$ has a natural nondegenerate pairing with $H$ denoted by $(h^*, h) \in R$ for $h^* \in H^*, h \in H$. Then the $R$-bimodule structure on $H^*$, multiplication, and comultiplication are given below, respectively,
where \( R \xrightarrow{s} H \xleftarrow{t} R^{op} \) denotes the commuting morphism set-up of the bialgebroid \( H \):

\[
\langle r \cdot h^* \cdot r', h \rangle := r \langle h^*, h(t(r')) \rangle \tag{15}
\]

\[
\langle h^* g^*, h \rangle := \langle g^*, \langle h^*, h(1) \rangle \cdot h(2) \rangle \tag{16}
\]

\[
\langle h^*, hh' \rangle := \langle h^*(1) \cdot \langle h^*(2), h' \rangle, h \rangle \tag{17}
\]

Of course, the unit of \( H^* \) is \( \varepsilon_H \) while the counit on \( H^* \) is \( \varepsilon(h^*) = \langle h^*, 1_H \rangle \). Eq. (16) is the formula for multiplication [18, 3.2(b)].

There are similar formulas for the right bialgebroid structure on the left \( R^* \)-dual \( \ast H \): see [9, 2.6]. In the particular case of the left bialgebroid \( S \) of a depth two ring extension, it turns out that \( S \) is isomorphic as \( R \)-bialgebroids to both \( R \)-duals, \( T^* \) and \( \ast T \) via two nondegenerate pairings, one of which is given by \( (\alpha \in S, t \in T) \):

\[
\langle \alpha, t \rangle = \alpha(t^1)t^2 \in R \tag{18}
\]

This induces an isomorphism of left \( R \)-modules \( S \to \text{Hom} \( T_R, R_R \) \) via \( \alpha \mapsto \langle \alpha, - \rangle \) with inverse

\[
\phi \mapsto \sum_i \phi(t_i) \beta_i.
\]

Significantly, there is another nondegenerate pairing of \( S \) and \( T \) for left and right D2 extensions given by

\[
[t, \alpha] = t^1 \alpha(t^2) \tag{19}
\]

[9, 5.3]. This induces an isomorphism of right \( R \)-modules \( S \to \text{Hom} \( R_T, R_R \) \) given by \( \alpha \mapsto [-, \alpha] \), with inverse given from a right D2 quasibasis by

\[
\psi \mapsto \sum_j \gamma_j(-)\psi(u_j).
\]

3. Irreducible extensions

We define a class of depth two extension where we may readily exploit the two nondegenerate pairings just given in eqs. (18) and (19). We say that an algebra extension \( A|B \) is irreducible if it is depth two and its centralizer \( C_A(B) = R \) is a subalgebra of \( B \), so \( R \subseteq B \). Then \( R \) is a commutative subalgebra, since \( rs = sr \) for all \( s, r \in R \) follows from noting for instance that \( r \in B \) and \( s \in A_B \).

This set-up is quite common. For example, irreducible depth two subfactors are irreducible in our sense since the centralizer is one-dimensional over the complex numbers [8, 17]. A second example: Taft’s Hopf algebras including Sweedler’s 4-dimensional Hopf algebra, which are generated by a grouplike element \( g \) and a skew-primitive element \( x \), over the commutative Frobenius subalgebra \( B \) generated by the element \( x \): this extension satisfies \( B = R \) and is depth two (in fact strongly graded, therefore Hopf-Galois) [6, 7]. A third example is the extension \( C \subset H \), the complex numbers as a subring in the real quaternions.

Another type of example of irreducible extension is the H-separable extension of full \( n \times n \) matrix algebra over the triangular matrix subalgebra, which of course has trivial centralizer. (If we pass to infinite dimensional matrices of finite type, a version of this example shows that Cuadra’s result for separable Hopf-Galois extension [4] does not extend to separable, depth two extensions; namely, an example
of a non-finitely generated $H$-separable extension.) Finally, note that an intermediate ring $B$ in an irreducible extension $A|C$ is irreducible if $A|B$ is D2, since $A^B \subseteq A^C \subseteq C \subseteq B$.

For an irreducible extension $A|B$ with the construction $T = (A \otimes_B A)^B$, we note that

$$\text{(20)} \quad \text{Hom}(T, R) = \text{Hom}(T, r)$$

This follows from $R \subseteq B$ and commutativity in $R$, for given $\phi \in \text{Hom}(T, R)$, $t \in T$, $r \in R$:

$$\phi(rt) = \phi(tr) = \phi(t)r = r\phi(t),$$

and a similar computation showing $^*T \subseteq T^*$ using left and right $R$-dual notation. In case $A|B$ is (two-sided) depth two and irreducible, the two nondegenerate pairings $(\alpha, \beta \in S)$

$$S \xrightarrow{\cong} \text{Hom}(T, R) = \alpha \mapsto \langle \alpha, - \rangle$$

and

$$S \xrightarrow{\cong} \text{Hom}(R, T) = \beta \mapsto [-, \beta]$$

induce a bijection $\tau$ of $S$ with itself completing a commutative triangle with these two mappings. Then define

$$\tau : S \to S, \quad \langle \alpha, t \rangle = [t, \tau(\alpha)]$$

for all $t \in T$ and $\alpha \in S$. We also make use of the notation $\alpha^\tau = \tau(\alpha)$, for which the last equation becomes

$$\text{(21)} \quad \alpha(t^1)^2 = t^1\alpha^\tau(t^2).$$

Notice that this approach will not work on the two nondegenerate pairings $T \to \text{Hom}(S, R)$, to define a self-bijection on $T$, unless we assume that $R$ coincides with the center of $A$.

**Lemma 3.1.** The mapping $\tau : S \to S$ is an anti-automorphism of $S$ satisfying $\tau(\rho_r) = \lambda_r$, $\tau(\lambda_r) = \rho_r$ for $r \in R$ and $\alpha^\tau(1) = \alpha(1)$.

**Proof.** It is clear that $\tau$ is linear and bijective. We note that for $t = t^1 \otimes_B t^2 \in T$, $\alpha, \beta \in S$,

$$t^1\beta^\tau\alpha^\tau(t^2) = \beta(t^1)\alpha^\tau(t^2) = \alpha\beta(t^1)t^2 = t^1(\alpha\beta)^\tau(t^2)$$

since $t^1 \otimes_B \alpha^\tau(t^2)$ and $\beta(t^1) \otimes_B t^2$ both are in $T$. Then $[t, \beta^\tau\alpha^\tau] = [t, (\alpha\beta)^\tau]$, so by nondegeneracy of this pairing, $\tau$ is an anti-automorphism of $S$.

We also check that

$$[t, \rho_r^\tau] = \langle \rho_r, t \rangle = t^1rt^2 = [t, \lambda_r]$$

whence $\rho_r^\tau = \lambda_r$. Since $R \subseteq B$ and $T = (A \otimes_B A)^B$, we note that

$$[t, \rho_r] = t^1t^2r = rt^1t^2 = \langle \lambda_r, t \rangle = [t, \lambda_r^\tau]$$

whence $\lambda_r^\tau = \rho_r$ for each $r \in R$.

Finally, with $1_T = 1 \otimes_B 1$, the equality $\alpha(1) = \alpha^\tau(1)$ follows from $\langle \alpha, 1_T \rangle = [1_T, \alpha^\tau]$. $\square$

**Proposition 3.2.** Suppose $A|B$ is an irreducible extension with left D2 quasibasis $t_i \in T, \beta_i \in S$ and right D2 quasibasis $u_j \in T, \gamma_j \in S$. Then $t_i, \beta_i^\tau$ is a right D2 quasibasis and $u_j, \gamma_j^\tau$ is a left D2 quasibasis.
Proof. Note that for \( t \in T \), a special instance of eq. \( \mathbf{(3)} \) yields

\[
t = \sum_i t_i^1 \otimes_B t_i^2 \beta_i(t^1) t^2 = \sum_i t^1 \beta^+_i(t^2) t_i^1 \otimes_B t_i^2,
\]
since \( \beta_i(t^1) t^2 \in R \subseteq B \). Now recall that \( A \otimes_R T \cong A \otimes_B A \) via \( a \otimes_R t \mapsto at^1 \otimes_B t^2 \) since \( A|B \) is right \( D_2 \) (for an inverse is given by \( x \otimes_B y \mapsto \sum_j x \gamma_j(y) \otimes_R u_j \)). Note that

\[
A \otimes_B A \longrightarrow A \otimes_R T, \quad x \otimes_B y \longrightarrow \sum_i x \beta^+_i(y) \otimes_R t_i
\]
is a left inverse of \( A \otimes_R T \rightarrow A \otimes_B A \), \( a \otimes_R t \mapsto at^1 \otimes_B t^2 \) since

\[
\sum_i at^1 \beta^+_i(t^2) \otimes_R t_i = a \otimes_R \sum_i t^1 \beta^+_i(t^2) t_i = a \otimes_R t.
\]
Hence, it is also a right inverse, so

\[
(23) \quad x \otimes_B y = \sum_i x \beta^+_i(y) t_i
\]
for all \( x, y \in A \), which shows that \( t_i \in T, \beta^+_i \in S \) is a right \( D_2 \) quasibasis.

The argument that \( u_j, \gamma^+_j \) is a left \( D_2 \) quasibasis is very similar. \( \square \)

4. The Hopf Algebroid \( S \)

We are now in a position to show that the anti-automorphism \( \tau \) on \( S \), defined in eq. \( \mathbf{(22)} \), is an antipode satisfying the axioms of Böhm-Szlachányi \( \mathbf{[2]} \). In order for \( S \) to be a Hopf algebroid in the sense of Lu, we need one additional requirement, e.g. that \( R \) be a separable \( K \)-algebra, in order that we may find a section of the canonical epi \( S_K \otimes K S \rightarrow S \otimes_R S \).

Theorem 4.1. If \( A|B \) is an irreducible extension, then the anti-automorphism \( \tau \) on \( S = \text{End}_B A_B \) is an antipode and \( S \) is a Hopf algebroid.

Proof. We check that the axioms of a Hopf algebroid are satisfied, axioms given in for example \( \mathbf{[9, 8.7]} \) and repeated in an appendix below. Define a right bialgebroid structure on \( S \) over \( R^{op} = R \) by choosing target map \( t_R = \lambda \) and source map \( s_R = \rho \), whence the \( R \)-bimodule structure on \( S \) becomes

\[
r \cdot \alpha \cdot s = \alpha \rho \lambda, \quad \tau(a)(s) = ra(\tau(s)),
\]
the usual structure on \( S \) introduced above (since \( R \subseteq B \)). Then the left bialgebroid structure \( (S, R, \Delta, \varepsilon) \) introduced above in eqs. \( \mathbf{3}, \mathbf{6} \) and \( \mathbf{7} \) is also a right bialgebroid structure. In other words, the axioms (1) and (2) in \( \mathbf{[9, 8.7]} \) are satisfied by noting that \( s_L = t_R, t_L = s_R \), and taking \( \Delta_L = \Delta_R \) and \( \varepsilon_L = \varepsilon_R \). We check that also the axiom of a right bialgebroid, \( s_R(r)\alpha_1 \otimes_R \alpha_2 = \alpha_1 \otimes_R t_R(r)\alpha_2 \) is satisfied since \( \rho \cdot \alpha_1(x) = \alpha_1(x) \lambda, \alpha_2(y) = \alpha_2(y) \) \( (x, y \in A) \) in the identification \( S \otimes_R S \cong \text{Hom}_B(A_B \otimes_B A_B) \) in eq. \( \mathbf{8} \). In addition, the axiom

\[
\varepsilon(t_R(\varepsilon(\alpha))\beta) = \varepsilon(\alpha\beta) = \varepsilon(s_R(\varepsilon(\alpha)\beta) \quad (\alpha, \beta \in S)
\]
is satisfied since

\[
\varepsilon(\lambda \varepsilon(\alpha) \beta) = \alpha(\beta(1)) = \varepsilon(\alpha \beta)
\]
which equals \( \beta(1) = \varepsilon(\rho \varepsilon(\alpha) \beta) \), where we use \( \alpha(1) \in R \subseteq B \) and \( R \) is commutative.

We proceed to axiom (i):

\[
\tau(at_R(r)) = \tau(\lambda r) \tau(\alpha) = s_R(r) \tau(\alpha)
\]
by the lemma, and
\[ \tau(t_L(r)\alpha) = \tau(\alpha)\tau(\rho_r) = \tau(\alpha)s_L(r) \]
for all \( r \in R, \alpha \in S \).

Finally, axiom (ii) is satisfied since by eq. [5]
\[ \alpha(1) \triangleright \alpha(2) = \sum_{i} \alpha(\beta(2)(-)^{1}t_{i}^{2}) = \lambda_{\alpha(1)} = s_{L}(\varepsilon(\alpha)) \]
by the proposition, and by lemma,
\[ \tau(\alpha(1)) \triangleright \alpha(2) = \sum_{i} \tau(\rho_{i}^{2} \alpha \rho_{i}) \triangleright \beta_{i} = \sum_{i} \lambda_{i}^{1} \alpha^{\tau} \lambda_{i}^{2} \beta_{i} \]
\[ = \rho_{\alpha^{\tau}(1)} = s_{R}(\varepsilon(\alpha)). \]
Thus \( S \) and \( \tau \) form a Hopf algebroid. \( \square \)

For example, the Hopf algebroid structure on \( S \) coincides with that in [10] should the extension be H-separable as well as irreducible, since \( \tau \) exchanges \( \lambda_{r} \) and \( \rho_{r} \).

Note that Hopf algebroid \( (S, \tau) \) satisfies properties close to a Hopf algebra, among them:

(1) \( \tau(r \cdot \alpha) = \tau(\alpha) \cdot r \) and \( \tau(\alpha \cdot r) = r \cdot \tau(\alpha) \) for all \( r \in R, \alpha \in S \);
(2) \( \varepsilon(\alpha \beta) = \varepsilon(\alpha)\varepsilon(\beta) \) for all \( \alpha, \beta \in S \);
(3) \( \alpha(1) \triangleright \alpha(2) = \varepsilon(\alpha) \cdot 1_{S} \) for all \( \alpha \in S \);
(4) \( \tau(\alpha(1))\alpha(2) = 1_{S} \cdot \varepsilon(\alpha) \) for all \( \alpha \in S \);
(5) \( \varepsilon \tau = \varepsilon \);
(6) \( \tau \) is an “anti-coalgebra homomorphism.”

Also, \( S \) is finite projective over the base ring \( R \), which is commutative. However, such basic algebraic properties of a Hopf algebra as \( r \cdot 1 = 1 \cdot r \) and \( (\alpha \cdot r) \beta = \alpha (r \cdot \beta) \) are suspended for \( S \) (unless \( R \) coincides with the center of \( A \)). We propose to call a Hopf algebroid with equal right and left bialgebroid structures over a commutative base ring, possessing an anti-automorphism exchanging source and target, both mappings with image in the center, and satisfying the properties enumerated directly above, a skew Hopf algebra. Given the general nature of the example \( S \), we would expect that skew Hopf algebras are quite common occurrences.

Corollary 4.2. Suppose \( A|B \) is an irreducible extension and \( R \) is a separable \( K \)-algebra. Then \((S, \tau)\) is a Hopf algebroid in the sense of Lu.

Proof. Let \( e = e^1 \otimes_{K} e^2 \) be a separability element for \( R \). Define a section \( \eta : S \otimes_{R} S \to S \otimes_{K} S \) of the canonical epi \( S \otimes_{K} S \to S \otimes_{R} S \) by \( \eta(\alpha \otimes_{R} \beta) = \alpha \cdot e^1 \otimes_{K} e^2 \cdot \beta \), since \( e^1 e^2 = 1 \) and \( re = er \) for \( r \in R \). Then the axiom \( \mu(\text{id} \otimes \tau)\eta \Delta = s_{L} \circ \varepsilon \) follows from
\[ [\mu(\text{id} \otimes \tau)\eta \Delta(\alpha), u] = \sum_{i} u^1 \alpha(\beta(2)(u^2 e^2 \cdot t_{i}^{1}) t_{i}^{2}) = u^1 \alpha(1)u^2 = [\lambda_{\alpha(1)}, u] \]
by the proposition and since \( R \subseteq B \), \( e^2 e^1 = 1 \). The other axioms follow as in the proof of the theorem. \( \square \)

Of course, like the inverse in group theory, antipodes are important to the representation theory of a bialgebroid. For example, we can now define a right \( S \)-module algebra structure on \( A \) from the left structure by \( a \ll a = a^{\tau} \triangleright a \), which satisfies the measuring rule \( (xy) \ll a = (x \ll a(2))(y \ll a(1)) \) for \( x, y \in A \) and \( \alpha \in S \).
The antipode on $S$ will not dualize readily to an antipode on $T$ without the
duality properties discussed in [3], such as $S$ possessing a nondegenerate integral
element (in $\text{Hom}_{(B \otimes A_B, B_B)}$) such as a Frobenius homomorphism). However, if
$R = Z(A)$, as mentioned above an antipode $\tau_T$ is definable in the same way as
$\tau = \tau_S$.

The computations in this section expose the hypotheses that are necessary for
the antipode defined in [8, 4.4]. The correspondence of the theory in this section
with that in [8] is discussed in [9, 8.9], and depends on the equation for a depth
two Frobenius extension $A \mid B$ with trivial one-dimensional centralizer:
\begin{equation}
[t, \alpha] = E_M E_M (\psi(t)e_1e_2\phi(\alpha))
\end{equation}
for certain anti-isomorphisms $\psi : T \to C_{M_1}(M)$ defined in [9, 8.2] and $\phi : S \to C_{M_1}(N)$ defined in [9, 8.4].

5. The $\pi$-method for depth two extensions

In this section we extend Doi and Takeuchi’s $\pi$-method for Hopf-Galois exten-
sions [5] to D2 extensions. Then we extend the antipode in the previous section to
a certain bimodule hom-group for irreducible extensions. We point out that the $\pi$-
method yields a nullhomotopic mapping between relative Hochschild cochains with
coefficients for the irreducible extension $A \mid B$ and a certain Hochschild cohomology
theory with coefficients for the $R$-coring $T$.

Suppose $A \mid B$ is an rD2 extension, and $A_B$ is a module. Again let $T = (A \otimes_B
A)^B$, the right bialgebroid over the centralizer $R = A^B$. Let $u_j \in T$ and $\gamma_j \in S$ be
rD2 quasibases. Recall from [12] the coaction $\rho_T : A \to A \otimes_R T$ which makes $A$
into a right comodule algebra: in Sweedler notation this is given by
\begin{equation}
a_{(0)} \otimes_R a_{(1)} = \sum_j \gamma_j(a) \otimes_R u_j
\end{equation}
Clearly, $b_{(0)} \otimes b_{(1)} = b \otimes 1_T$ if $b \in B$.

**Proposition 5.1.** The mapping $\pi_L : \text{Hom}(R_T, R_M) \to \text{Hom}(B_A, B_M)$ given by
\begin{equation}
\pi_L(f)(a) = a_{(0)} f(a_{(1)})
\end{equation}
is a left $B$-linear, $R$-linear isomorphism.

**Proof.** The inverse to $\pi_L$ is given by
\begin{equation}
\pi_L^{-1}(g)(t) = t^1 g(t^2)
\end{equation}
for $g \in \text{Hom}(B_A, B_M)$. We note that
\begin{equation}
t^1 \pi_L(f)(t^2) = \sum_j t^1 \gamma_j(t^2) f(u_j) = f(t)
\end{equation}
for $f \in \text{Hom}(R_T, R_M), t \in T$, since $t^1 \gamma_j(t^2) \in A^B = R$ and $\sum_j t^1 \gamma_j(t^2) u_j = t$. We
also note that
\begin{equation}
\sum_j \gamma_j(a) u_j g(u_j^2) = g(a)
\end{equation}
for $a \in A, g \in \text{Hom}(B_A, B_M)$ by eq. [11]. Hence, $\pi_L^{-1}$ is indeed the inverse of $\pi_L$.

The mapping $\pi_L$ is left $B$-linear, since
\begin{equation}
\pi_L(bf)(a) = a_{(0)} bf(a_{(1)}) = \pi(f)(ab).
\end{equation}
Note that $\pi_L^{-1}$ is left $R$-linear since
$$\pi_L^{-1}(rg)(t) = t^1(rg)(t^2) = t^1 g(t^2 r) = \pi_L^{-1}(g)(tr). \quad \square$$
Similarly, if $A | B$ is $\ell D2$, $N_A$ is a module and $\beta_i \in S$, $t_i \in T$ are $\ell D2$ quasibases, then we have the right module dual of the proposition:
$$\pi_R : \text{Hom}(T_R, N_R) \xrightarrow{\cong} \text{Hom}(A_B, N_B), \quad \pi_R(h)(a) = \sum_i h(t_i)\beta_i(a),$$
which has inverse mapping given for $g \in \text{Hom}(A_B, N_B)$, $t \in T$ by
$$\pi_R^{-1}(g)(t) = g(t^1)t^2$$
Suppose that $A_P A$ is a bimodule. We let $P^B$ denote the elements $p \in P$ satisfying $pb = bp$ for all $b \in B$. Note that $P^B$ is a natural $(R, R)$-bimodule, since $R$ and $B$ commute.
First note that $\pi_L : \text{Hom}(\tau T, R_P) \xrightarrow{\cong} \text{Hom}(B A_B, P_B)$ restricts to
$$\pi_L : \text{Hom}(\tau T, R^P_R) \xrightarrow{\cong} \text{Hom}(B A_B, B P_B)$$
Similarly $\pi_R$ above restricts to
$$\pi_R : \text{Hom}(T_R, P^B_R) \xrightarrow{\cong} \text{Hom}(B A_B, B P_B).$$
Note that the inverses are given by
$$\pi_R^{-1}(h)(u) = h(u^1)u^2, \quad \pi_L^{-1}(h)(u) = u^1h(u^2)$$
for $h \in \text{Hom}(B A_B, B P_B), u \in T$. Of course if $P = A$ these restricted mappings recover the isomorphisms $\text{Hom}(T_R, R_R) \cong S \cong \text{Hom}(\tau T, R_P)$ below eq. \[13\].
Now suppose that the ring extension $A | B$ is irreducible. The antipode in the previous sections then extends to a bijection of the hom-group $\text{Hom}(B A_B, B P_B)$ onto itself as follows. Since $R \subseteq B$, it follows that $\text{Hom}(T_R, P^B_R) = \text{Hom}(\tau T, R^P_R)$. Then define the mapping
$$\sigma_P = \pi_L \circ \pi_R^{-1} : \text{Hom}(B A_B, B P_B) \xrightarrow{\cong} \text{Hom}(B A_B, B P_B)$$
which is then given by
$$\sigma_P(\alpha)(a) = \pi_L(\pi_R^{-1}(\alpha))(a) = \sum_j \gamma_j(a)\alpha(u^1_j)u^2_j$$
for $\alpha \in \text{Hom}(B A_B, B P_B)$. Now let $\alpha^\sigma = \sigma_P(\alpha)$. Then for all $t \in T$,
$$\alpha(t^1)t^2 = t^1 \alpha^\sigma(t^2),$$
which follows from the following short computation:
$$t^1 \sigma_P(\alpha)(t^2) = \sum_j t^1 \gamma_j(t^2)\alpha(u^1_j)u^2_j = \alpha(t^1)t^2,$$
since $t^1 \gamma_j(t^2) \in R \subseteq B$ and eq. \[14\]. A glance at eq. \[22\] shows that $\sigma_A = \tau$, the antipode of $S$ defined in previous sections. The proof of the proposition is similar to previously and therefore omitted.

**Proposition 5.2.** Suppose $A | B$ is an irreducible extension and $A_P A$ is a bimodule. Define two pairings of $\text{Hom}(B A_B, B P_B)$ and $T$ with values in $P^B$ by $\langle \alpha, t \rangle = \alpha(t^1)t^2$ and $|u, \beta| = u^1\beta(u^2)$. Then the two pairings are non-degenerate w.r.t. $\alpha, \beta \in \text{Hom}(B A_B, B P_B)$. Moreover, the mapping $\alpha \mapsto \alpha^\sigma$ defined by $\langle \alpha, t \rangle = [t, \alpha^\sigma]$ is a bijection satisfying $\sigma^{+1}(\lambda_p) = \rho_p$ for each $p \in P$, and $\alpha^\sigma(1_A) = \alpha(1_A)$. 

\[10\] LARS KADISON
5.1. Remark on relative Hochschild cohomology with coefficients. We remark below on how the $\pi$-method leads to a nullhomotopic mapping between certain Hochschild cohomology theories. Continuing the notation just above, note that $\text{Hom}(BA, BP_B)$ is the first relative Hochschild cochain group of $A \mid B$ with coefficients in a bimodule $A_P$, denoted by $C^1(A, B; P)$. The zero’th group is $P^B$ with differential $d^0 : P^B \to \text{Hom}(BA, BP_B)$ given by $d^0(p) = \rho_p - \lambda_p$ for $p \in P^B$.

The second relative Hochschild cochain group is $C^2(A, B; P) = \text{Hom}(BA \otimes_A AB, BP_B)$. The differential at this level is given by $d^1 : \text{Hom}(BA, BP_B) \to \text{Hom}(BA \otimes_A AB, BP_B)$ defined by

$$dh(x \otimes_B y) = xf(y) - f(xy) + f(x)$$

The cohomology groups are denoted by $\text{HH}^n(A, B; P)$ for $n \geq 0$; recall or note that $\text{HH}^1(A, B; P)$ is isomorphic to the group of derivations killing $B$ modulo the group of inner derivations w.r.t. elements in $P^B$. For the sake of brevity we refer the reader to textbooks on homological algebra for the details of higher order cochain groups and differentials.

Note that applications of the hom-tensor relation to the left- and right-handed $\pi$-methods with $A_M = \text{Hom}(A_B, P_B)$ yields two isomorphisms

$$\text{Hom}(BA \otimes_A A_B, BP_B) \overset{\approx}{\longrightarrow} \text{Hom}(\text{HH}RT, \text{HH}^2P_B)$$

denoted by $\pi_L^2$ and $\pi_R^2$ given by

$$\pi_L^2(h)(u \otimes_R t) = u^1 h(u^2 t^1 \otimes_B t^2)$$

$$\pi_R^2(h)(u \otimes_R t) = h(u^1 \otimes_B u^2 t^1 t^2)$$

The inverses are given by

$$\pi_L^{-2}(g)(x \otimes_B y) = \sum_{j,k} \gamma_k(x\gamma_j(y))g(u_k \otimes_R u_j)$$

$$\pi_R^{-2}(g)(x \otimes_B y) = \sum_{i,j} g(t_i \otimes_R t_j)\beta_j(\beta_i(x)y)$$

Likewise we define the obvious generalized isomorphisms $\pi_L^n$ and $\pi_R^n$ on the $n$-cochains $C^n(A, B; P)$.

Brzeziński and Wisbauer define a Hochschild cohomology of an $R$-coring with coefficients in an $(R, R)$-bimodule $[1, 30.15]$. For the $R$-coring $T$ and $(R, R)$-bimodule $P^B$, this specializes to the zero’th cochain group $\text{Hom}(\text{HH}RT, \text{HH}^1P^B)$ (which is equal to both one-sided $R$-linear hom-groups considered above since $A \mid B$ is irreducible), and first cochain group $\text{Hom}(\text{HH}RT \otimes_R T, \text{HH}^2P^B)$. Let $\varepsilon : T \to R$ be the counit of $T$ given by $\varepsilon(t) = t^1 t^2$, the multiplication mapping $A \otimes_A A \to A$ restricted to $T$. The differential is given by $(h \in \text{Hom}(\text{HH}RT, \text{HH}^2P^B))$

$$\delta^0(h)(u \otimes_R t) = \varepsilon(u)h(t) - h(u)\varepsilon(t)$$

$$\delta^1(g)(v \otimes_R t) = \varepsilon(u)g(v \otimes_R t) - g(u \varepsilon(v) \otimes_R t) + g(u \otimes_R v)\varepsilon(t)$$

for $g \in \text{Hom}(\text{HH}RT \otimes_R T, \text{HH}^2P^B)$. For example, $\delta^0 \delta^0 = 0$ by a short computation. The higher cochain groups and differentials are defined similarly and we refer to $[1, 30.15]$ for the details. (This cochain complex recovers Hochschild relative cochains in case the $A$-coring $A \otimes_B A$ with counit $\varepsilon'(x \otimes y) = xy$ takes the place of $R$, $T$ and $\varepsilon$.) Denote the cochain groups in this complex by $C^n(T, R; P^B)$, and call it the Hochschild coring complex.
Next we define a cochain homomorphism \( \Phi_n : C^n(A, B; P) \to C^{n-1}(T, R; PB) \), for \( n \geq 1 \), noting the shift of one downwards in degree in the Hochschild coring complex. Define \( \Phi_1 : \text{Hom} (BA, BP) \to \text{Hom} (RT, RPB) \) as
\[
\Phi_1 = \pi_L^{-1} + \pi_R^{-1}, \quad \Phi_1(h)(u) = u^1h(u^2) + h(u^1)u^2
\]
for \( h \in \text{Hom} (BA, BP) \). Note that \( \Phi_1 \) kills \( B \)-linear derivations.

Define \( \Phi_2 : \text{Hom} (BA \otimes_B AB, BP) \to \text{Hom} (RT \otimes_R RT, RPB) \) by \( \Phi_2 = \pi_L^2 - \pi_R^2 \), or in more detail,
\[
(\Phi_2g)(u \otimes_R t) = u^1g(u^2t^1 \otimes_B t^2) - g(u^1 \otimes_B u^2t^1)t^2
\]

The \( n \)th mapping \( \Phi_n \) is easily defined from two obvious generalized mappings, \( \pi_L^n, \pi_R^n : \text{Hom} (BA \otimes_B \cdots \otimes_B AB, BP) \to \text{Hom} (RT \otimes_R \cdots \otimes_R RT, RPB) \); as \( \Phi_n = \pi_L^n + (-1)^{n+1}\pi_R^n \).

We compute for \( f \in \text{Hom} (BA, BP) \),
\[
\delta^0 \Phi_1 f(u \otimes_R t) = (\Phi_1 f)(u^1u^2t^2) - (\Phi_1 f)(u^1t^1)^2
\]
\[
= u^1u^2t^1f(t^2) + f(u^1u^2t^1)t^2 - f(u^1u^2t^1)t^2 - u^1f(u^2t^1)^2
\]
\[
= u^1(df)(u^2t^1 \otimes_B t^2) - (df)(u^1 \otimes_B u^2t^1)t^2 = (\Phi_2 df)(u \otimes_R t)
\]

after two middle terms cancel.

Moreover, for \( g \in \text{Hom} (BA \otimes_B AB, BP) \),
\[
\Phi_4(dg)(v \otimes_R u \otimes_R t) = v^1(dg)(v^2u^1 \otimes_B u^2t^1 \otimes_B t^2) + (dg)(v^1 \otimes_B v^2u^1 \otimes_B u^2t^1)^2
\]
\[
= (\Phi_2g)(\varepsilon(v)u \otimes_R t) - (\Phi_2g)(g\varepsilon(u) \otimes_R t) + (\Phi_2g)(v \otimes_R u \varepsilon(t)) = (\delta^3 \Phi_2g)(v \otimes_R u \otimes_R t)
\]

after cancellation of the pair of middle terms \( \pm v^1g(v^2u^1 \otimes_B u^2t^1)^2 \).

We omit the tedious but similar computation in degree \( n \) which establishes that \( \Phi \) is a cochain mapping.

**Proposition 5.3.** Suppose \( A \mid B \) is an irreducible extension and \( A PA \) is a bimodule. Let \( R = AB \) and \( T = (A \otimes_B A)^B \). Then the mapping of cochain groups \( \Phi_n : C^n(A, B; P) \to C^{n-1}(T, R; PB) \) is nullhomotopic.

**Proof.** We define a homotopy \( s_n : C^{n+2}(A, B; P) \to C^n(T, R; PB) \) first in degree zero by \( s_0(f)(t) = f(t^1 \otimes_B t^2) \) where \( f \in \text{Hom} (BA \otimes_B AB, BP) \), \( t \in T \subseteq A \otimes_B A \), so \( f(t) \in PB \). We claim there is a natural inclusion \( \iota_n \) of \( T \otimes_R \cdots \otimes_R T \) (\( n \) times \( T \)) into \( A \otimes_B \cdots \otimes_B A \) \((n + 1)\) times \( A \) given by
\[
\iota_n(u_1 \otimes_R \cdots \otimes_R u_n) = u_1^1 \otimes_B u_1^2 \otimes_B \cdots \otimes_B u_{n-1}^1 \otimes_B u_n^2
\]

In fact, \( \iota_n \) is an isomorphism onto \((A \otimes_B \cdots \otimes_B A)^B \) which follows from showing that for any ring \( C \) and \((A, C)\)-bimodule \( M \)
\[
T \otimes_R M \xrightarrow{\iota_n} A \otimes_B M
\]
via \( t \otimes_R m \mapsto t^1 \otimes_B t^2 m \). This is a \((B, C)\)-bimodule isomorphism with inverse given by \( a \otimes_B m \mapsto \sum_i t_i \otimes_R \beta_i(a)m \) using left \( D2 \) quasibases \( \beta_i \in S \), \( t_i \in T \). Now we may apply this with \( C = A, M = A \otimes_B A, A \otimes_B A \otimes_B A, \ldots \), then restrict to \((-)B \), substitute and iterate to prove the claim.

Now define \( s_n(g) = g \circ \iota_{n+1} \) for \( g \in C^{n+2}(A, B; P) \). We note that \( \delta^n s_n + s_{n+1} \delta^{n+2} = \Phi_{n+2} \); e.g., for \( f \in C^2(A, B; P) \), \( u, t \in T \),
\[
\delta^n(s_0f)(u \otimes_R t) + s_1(d^2f)(u \otimes_R t) = \varepsilon(u)f(t) - f(u)\varepsilon(t) + (df)(u^1 \otimes_B u^2t^1 \otimes_B t^2)
\]
\[
= u^1f(u^2t^1 \otimes_B t^2) - f(u^1 \otimes_B u^2t^1)^2 = (\Phi_2f)(u \otimes_R t)
after cancellation of the middle terms in $(df)(u^1 \otimes_B u^2 t^1 \otimes_B t^2)$. The general case is computed similarly. □

6. Appendix: Axioms for Hopf Algebroids

In this appendix we review the definitions of left bialgebroid, Lu's Hopf algebroid, right bialgebroid and the definition of Hopf algebroid by Böhm-Szlachányi. First, for the definition of a left bialgebroid $(H,R,s_L,t_L,\Delta,\varepsilon)$, $H$ and $R$ are $K$-algebras and all maps are $K$-linear. First, recall from [10] that the source and target maps $s_L$ and $t_L$ are algebra homomorphism and anti-homomorphism, respectively, of $R$ into $H$ such that $s_L(r)t_L(s) = t_L(s)s_L(r)$ for all $r,s \in R$. This induces an $R$-$R$-bimodule structure on $H$ (from the left in this case) by $r \cdot h \cdot s = s_L(r)t_L(s)h$ ($h \in H$). With respect to this bimodule structure, $(H,\Delta,\varepsilon)$ is an $R$-coring (cf. [13]), i.e. with coassociative coproduct and $R$-$R$-bimodule map $\Delta : H \to H \otimes_R H$ and counit $\varepsilon : H \to R$ (also an $R$-bimodule mapping). The image of $\Delta$, written in Sweedler notation, is required to satisfy

$$a(1)t_L(r) \otimes a(2) = a(1) \otimes a(2)s_L(r)$$

for all $a \in H, r \in R$. It then makes sense to require that $\Delta$ be homomorphic:

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(1) = 1 \otimes 1$$

for all $a,b \in H$. The counit must satisfy the following modified augmentation law:

$$\varepsilon(ab) = \varepsilon(as(\varepsilon(b))) = \varepsilon(at(\varepsilon(b))), \quad \varepsilon(1_H) = 1_R.$$

The axioms of a right bialgebroid $H'$ are opposite those of a left bialgebroid in the sense that $H'$ obtains its $R$-bimodule structure from the right via its source and target maps and, from the left bialgebroid $H$ above, we have that $(H'^{op}, R, t'^{op}_L, s'^{op}_L, \Delta, \varepsilon)$ (in that precise order) is a right bialgebroid: for the explicit axioms, see [9] Section 2.

In addition, the left $R$-bialgebroid $H$ is a Hopf algebroid in the sense of Lu $(H,R,\tau)$ if (antipode) $\tau : H \to H$ is an algebra anti-automorphism such that

1. $\tau t_L = s_L$;
2. $\tau(a(1))a(2) = t_L(\varepsilon(\tau(a)))$ for every $a \in A$;
3. there is a linear section $\eta : H \otimes_R H \to H \otimes_K H$ to the natural projection $H \otimes_K H \to H \otimes_R H$ such that:

$$\mu(H \otimes \tau)\eta \Delta = s_L \varepsilon.$$

The following is one of several equivalent definitions of Böhm-Szlachányi’s Hopf algebroid [2], excerpted from [9] 8.7.

**Definition 6.1.** We call $H$ a Hopf algebroid if there are left and right bialgebroid structures $(H,R,s_L,t_L,\Delta_L,\varepsilon_L)$ and $(H,R^{op},s_R,t_R,\Delta_R,\varepsilon_R)$ such that

1. $\text{Im } s_R = \text{Im } t_L$ and $\text{Im } t_R = \text{Im } s_L$,
2. $(1 \otimes \Delta_L)\Delta_R = (\Delta_R \otimes 1)\Delta_L$ and $(1 \otimes \Delta_R)\Delta_L = (\Delta_L \otimes 1)\Delta_R$

with anti-automorphism $\tau : H \to H$ (called an antipode) such that

1. $\tau(at_R(r)) = s_R(r)\tau(a)$ and $\tau(t_L(r)a) = \tau(a)s_L(r)$ for $r \in R, a \in H$,
2. $a^{(1)}\tau(a^{(2)}) = s_L(\varepsilon_L(a))$ and $\tau(a^{(1)})a^{(2)} = s_R(\varepsilon_R(a))$

where $\Delta_R(a) = a^{(1)} \otimes a^{(2)}$ and $\Delta_L(a) = a^{(1)} \otimes a^{(2)}$. 


The relationship between Lu’s Hopf algebroid and this alternative Hopf algebroid with more pleasant tensor categorical properties is discussed in [2, 3] and other papers by Böhm and Szlachányi.

References

[1] T. Brzeziński and R. Wisbauer, *Corings and Comodules*, Cambridge Univ. Press, 2003.
[2] G. Böhm and K. Szlachányi, Hopf algebroids with bijective antipodes: axioms, integrals and duals, *J. Algebra* **274** (2004), 708–750.
[3] G. Böhm, Integral theory for Hopf algebroids, *Alg. Rep. Theory* **8** (2005), 563–599.
[4] J. Cuadra, A Hopf algebra having a separable Galois extension is finite dimensional, *Proc. A.M.S.*, to appear.
[5] Y. Doi and M. Takeuchi, Hopf-Galois extensions of algebras, the Miyashita-Ulbrich action, and Azumaya algebras, *J. Algebra* **121** (1989), 488-516.
[6] M. Graña, J.A. Guccione and J.J. Guccione, Decomposition of some pointed Hopf algebras given by the canonical Nakayama automorphism, *J. Pure Appl. Alg.* **210** (2007), 493–500.
[7] L. Kadison, *New examples of Frobenius extensions*, University Lecture Series **14**, Amer. Math. Soc., Providence, 1999.
[8] L. Kadison and D. Nikshych, Hopf algebra actions on strongly separable extensions of depth two, *Adv. in Math.* **163** (2001), 258–286.
[9] L. Kadison and K. Szlachányi, Bialgebroid actions on depth two extensions and duality, *Adv. in Math.* **179** (2003), 75–121.
[10] L. Kadison, Hopf algebroids and H-separable extensions, *Proc. A.M.S.* **131** (2003), 2993–3002.
[11] L. Kadison, Hopf algebroids and Galois extensions, *Bull. Belg. Math. Soc.-Simon Stevin* **12** (2005), 275–293.
[12] L. Kadison, The endomorphism ring theorem for Galois and depth two extensions, *J. Algebra* **305** (2006), 163–184.
[13] L. Kadison, Hopf algebroids and pseudo-Galois extensions, in: Birkhäuser Trends in Math, to appear, QA/0508411.
[14] L. Kadison, Infinite index subalgebras of depth two, *Proc. A.M.S.*, to appear, QA/0607350.
[15] H. Kreimer and M. Takeuchi, Hopf algebras and Galois extensions of an algebra, *Indiana Univ. Math. J.* **30** (1981), 675–692.
[16] J.-H. Lu, Hopf algebroids and quantum groupoids, *Int. J. Math.* **7** (1996), 47–70.
[17] D. Nikshych and L. Vainerman, A characterization of depth 2 subfactors of II_1 factors, *J. Func. Analysis* **171** (2000), 278–307.
[18] M.E. Sweedler, The predual theorem to the Jacobson-Bourbaki theorem, *Trans. A.M.S.* **213** (1975), 391–406.
[19] W. Szymański, Finite index subfactors and Hopf algebra cross products, *Proc. A.M.S.* **120** (1994), 519–528.

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