A FUNCTION FIELD ANALOGUE OF THE
RASMUSSEN-TAMAGAWA CONJECTURE

YOSHIAKI OKUMURA

Abstract. In the arithmetic of function fields, Drinfeld modules play role that
elliptic curves take on in the arithmetic of number fields. The aim of this paper
is to study a non-existence problem of Drinfeld modules with constrained torsion
points at places with large degree, which is motivated a conjecture of Christo-
pher Rasmussen and Akio Tamagawa related with abelian varieties over number
fields with some arithmetic constraints. We prove such non-existence of Drinfeld
modules in the case where the inseparable degree of base fields is not divided
by the rank of Drinfeld modules. In other case, we conversely give a example
of Drinfeld modules satisfying Rasmussen-Tamagawa-type conditions under some
assumptions.

1. Introduction

Let \( p \) be a prime number and fix some \( p \)-power \( q = p^n \). Write \( A := \mathbb{F}_q[t] \)
for the polynomial ring in one variable \( t \) over \( \mathbb{F}_q \) and set \( F := \mathbb{F}_q(t) \). Let \( K \) be a finite
extension of \( F \). In this article, we identify every monic irreducible element \( \pi \)
of \( A \) with the corresponding finite place of \( F \). Write \( \mathbb{F}_\pi = A/\pi A \)
for the residue field at \( \pi \).

Let \( r \) be a positive integer and \( \pi \) a monic irreducible element of \( A \). Define
\( \mathcal{D}(K, r, \pi) \) to be the set of isomorphism classes \([\phi]\) of rank-\( r \) Drinfeld modules over
\( K \) which satisfy the following two conditions:

1. \( \phi \) has good reduction at any finite places of \( K \) not lying above \( \pi \),
2. the mod \( \pi \) representation \( \overline{\rho}_{\phi, \pi} : G_K \to \text{GL}_r(\mathbb{F}_\pi) \) attached to \( \phi \) is of the
   form
   \[
   \overline{\rho}_{\phi, \pi} \simeq \begin{pmatrix}
   \chi_{\pi}^{i_1} & * & \cdots & * \\
   & \chi_{\pi}^{i_2} & \cdots & \vdots \\
   & & \ddots & * \\
   & & & \chi_{\pi}^{i_r}
   \end{pmatrix},
   \]
   where \( \chi_{\pi} \) is the mod \( \pi \) Carlitz character (see Example 2.7) and \( 0 \leq i_1, \ldots, i_r \leq q^{\deg(\pi)} - 1 \)
   are integers.

Consider the following.

**Question 1.1.** Does there exist a positive constant \( C > 0 \) depending only on \( K \) and
\( r \) which satisfies the following: if \( \deg(\pi) > C \), then the set \( \mathcal{D}(K, r, \pi) \) is empty ?

The motivation of this question is a non-existence conjecture on abelian varieties
stated by Rasmussen and Tamagawa [RT1]. Let \( k \) be a finite extension of \( \mathbb{Q} \) and \( g \)

2010 Mathematics Subject Classification: Primary 11G09; Secondary 11R58.
Keywords: Drinfeld modules; Rasmussen-Tamagawa conjecture; Galois representations.
a positive integer. For a prime number \( \ell \), denote by \( \tilde{k}_\ell \) the maximal pro-\( \ell \) extension of \( k(\mu_\ell) \) which is unramified outside \( \ell \), where \( \mu_\ell = \mu_\ell(k) \) is the set of \( \ell \)-th roots of unity. For an abelian variety \( X \) over \( k \), write \( k(\mathcal{X}[\ell]) := k(\bigcup_{n \geq 1} X[\ell^n]) \) for the field generated by all \( \ell \)-power torsion points of \( X \). Define \( A(k, g, \ell) \) to be the set of isomorphism classes \([X]\) of \( g \)-dimensional abelian varieties over \( k \) which satisfy the following equivalent conditions:

- (RT-1) \( k(X[\ell^\infty]) \subseteq \tilde{k}_\ell \),
- (RT-2) \( X \) has good reduction at any finite place of \( k \) not lying above \( \ell \) and \( k(X[\ell])/k(\mu_\ell) \) is an \( \ell \)-extension,
- (RT-3) \( X \) has good reduction at any finite place of \( k \) not lying above \( \ell \) and the mod \( \ell \) representation \( \bar{\rho}_{X,\ell} : G_k \rightarrow \text{GL}_{F_\ell}(X[\ell]) \cong \text{GL}_2(F_\ell) \) is of the form

\[
\begin{pmatrix}
\chi_1^{i_1} & * & \cdots & * \\
* & \chi_2^{i_2} & \cdots & * \\
\cdots & \cdots & \cdots & \cdots \\
* & * & \cdots & \chi_g^{i_g}
\end{pmatrix},
\]

where \( \chi_\ell \) is the mod \( \ell \) cyclotomic character.

These conditions come from the study of a question of Ihara [Ih] related with the kernel of the canonical outer Galois representation of the pro-\( \ell \) fundamental group of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \); see [RT1]. Rasmussen and Tamagawa conjectured the following.

**Conjecture 1.2** ([Rasmussen and Tamagawa [RT1, Conjecture 1]). The set \( \mathcal{A}(k, g, \ell) \) is empty for any \( \ell \) large enough.

Since the set \( \mathcal{A}(k, g, \ell) \) is always finite (see Section 5, or [RT1]), the conjecture is equivalent to say that the union \( \bigcup_{\ell} \mathcal{A}(k, g, \ell) \) is also finite. For example, the following cases are known:

- \( k = \mathbb{Q} \) and \( g = 1 \) [RT1, Theorem 2],
- \( k = \mathbb{Q} \) and \( g = 2, 3 \) [RT2, Theorem 7.1 and 7.2],
- for abelian varieties with everywhere semistable reduction [Oz1, Corollary 4.5] if \( k/\mathbb{Q} \) has odd degree, and [RT2, Theorem 3.6] in the general case,
- for abelian varieties with abelian Galois representations [Oz2, Corollary 1.3],
- for QM abelian surfaces over certain imaginary quadratic fields [Ar, Theorem 9.3].

We notice that, under the assumption of Generalized Riemann Hypothesis (GRH) for Dedekind zeta functions of number fields, the conjecture is true in general [RT2, Theorem 5.1]. The key tool of this proof is the effective version of Chebotarev density theorem for number fields, which holds under GRH. Rasmussen and Tamagawa also state the “uniform version” of the conjecture [RT2, Conjecture 2], which says that one can take a lower bound of \( \ell \) satisfying \( \mathcal{A}(k, g, \ell) = \emptyset \) depending only on the degree \( [k : \mathbb{Q}] \) and \( g \). For instance, the uniform version conjecture for CM abelian varieties is proved by Bourdon [Bo, Corollary 1] and Lombardo [L, Theorem 1.3]. Under GRH, the uniform version conjecture is true if \( [k : \mathbb{Q}] \) is odd [RT2, Theorem 5.2].

The arithmetic properties of Drinfeld modules are similar to that of elliptic curves over number fields. Under this analogy, the two conditions (D1) and (D2) can be regarded as a natural translation of the condition (RT-3). We will show that the
set $\mathcal{D}(K, r, \pi)$ is characterized by the three equivalent conditions same as $\mathcal{A}(k, g, \ell)$. Our main purpose of this paper is to prove the following theorem:

**Theorem 1.3** (Theorem 3.7 and Theorem 4.9). If $r$ does not divide the inseparable degree $[K : F]$ of $K/F$, then the answer to Question 1.1 is YES. Namely the set $\mathcal{D}(K, r, \pi)$ is empty for any $\pi$ whose degree is large enough.

The proof of Theorem 1.3 consists of the two cases: (i) $r = p^r$, and (ii) $r = r_0 \cdot p^r$ for some $r_0 > 1$ which is prime to $p$. In the case (ii), we use the effective version of Chebotarev density theorem for function fields proved by Kumar and Scherk [KS], which is a modification of the strategy in [RT2]. In this case, the uniform version result is also shown (Theorem 4.10), which is an analogue of [RT2] Theorem 5.2. However, the same argument does not work well in the case (i). The proof in the case (i) is provided by observations of the tame inertia weights of $\tilde{\rho}_{\phi, \pi}$ for any $[\phi] \in \mathcal{D}(K, r, \pi)$. This technique is used in [Oz1] and [RT2].

On the other hand, Drinfeld modules have many remarkable properties coming from the positive characteristic setting. In fact, although the Rasmussen-Tamagawa conjecture is true under GRH, the set $\mathcal{D}(K, r, \pi)$ must not be empty if $r$ divides $[K : F]$, and $r = q^\nu$ for some $\nu \geq 0$.

The organization of the paper is as follows. In Section 2, after reviewing several basic facts on Drinfeld modules, we study the ramification of Galois representations coming from Drinfeld modules, whose consequences are needed in the next section. In Section 3, for any $[\phi] \in \mathcal{D}(K, r, \pi)$, an important integer $e_\phi$ is introduced and we prove some non-trivial properties of it, which show the result in the case (i). The aim of Section 4 is to give the proof in the case (ii). For any $[\phi] \in \mathcal{D}(K, r, \pi)$, we introduce a character $\chi(m_\phi)$ and show the property that $\chi(m_\phi)$ never vanishes on the Frobenius elements of places with some conditions, which contradicts to a consequence of the effective Chebotarev density theorem if deg$(\pi)$ is sufficiently large. Finally, in Section 5, we construct a Drinfeld module satisfying both (D1) and (D2) for any $\pi$ under some assumptions. We also show that the set $\mathcal{D}(K, r, t)$ is infinite.

**Notation.** Let $p, q, A, F$, and $K$ be as above. Set $n_K := [K : F]$ and write $K_u$ for the separable closure of $F$ in $K$. For a finite place $u$ of $K$ above $\pi$, let $K_u$ be the completion of $K$ at $u$, $O_{K_u}$ its valuation ring, and $F_u$ its residue field. We use the same symbol $u$ for the normalized valuation of $K_u$. Set $q_u := \#F_u$. Denote by $I_{K_u}$ the inertia subgroup of $G_{K_u}$ at $u$ and write $\text{Frob}_u$ for the Frobenius element at $u$. Denote by $e_{u|\pi}$ the absolute ramification index of $u$ and set $f_{u|\pi} := [F_u : F_\pi]$. Let $F_\infty := F_q((1/t))$ be the completion of $F$ at the infinite place $\infty$ of $F$ and $\mathbb{C}_\infty$ the completion of a fixed algebraic closure of $F_\infty$. Denote by $| \cdot |$ the unique extension of the normalized absolute value of $F_\infty$ to $\mathbb{C}_\infty$. Regard every algebraic extension of $F$ as a subfield of $\mathbb{C}_\infty$ and restrict $| \cdot |$ to it. For any non-zero $a \in A$, we see that $|a| = \#(A/aA) = q^{\deg(a)}$.

For any field $L$, denote by $G_L := \text{Gal}(L^{\text{sep}}/L)$ the absolute Galois group of $L$. The notation $C_j = C_j(x, y, \ldots, z)$ indicates a constant $C_j$ depending only on $x, y, \ldots$, and $z$. We use the notation $\rho^a$ for the semisimplification of a representation $\rho$.

**ACKNOWLEDGMENTS**

The author is grateful to his supervisor, Yuichiro Taguchi, for giving him useful advice about Drinfeld modules and for his guidance in preparing this paper. The
2. Drinfeld modules

2.1. Basic definitions. Let $L$ be an $A$-field equipped with an $\mathbb{F}_q$-algebra homomorphism $\iota: A \to L$. Such a pair $(L, \iota)$ is called an $A$-field. Let $\mathbb{G}_{a,L}$ be the additive group scheme defined over $L$. Denote by $\text{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L})$ the ring of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_{a,L}$, which is generated by the $q$-power Frobenius map. It is isomorphic to the non-commutative polynomial ring $L\{\tau\}$ in one variable $\tau$ satisfying $\tau c = c^q\tau$ for any $c \in L$. Let $r$ be a positive integer.

**Definition 2.1.** A Drinfeld module $\phi$ of rank $r$ defined over the $A$-field $L$ is an $\mathbb{F}_q$-algebra homomorphism

$$\phi: A \to L\{\tau\}; a \mapsto \phi_a$$

such that $\phi_t = \iota(t) + c_1\tau + \cdots + c_r\tau^r$, where $c_1, \ldots, c_r \in L$ and $c_r \neq 0$.

Note that $\phi$ is completely determined by the image $\phi_t$ of $t$. For two Drinfeld modules $\phi$ and $\psi$ over $L$, a morphism from $\phi$ to $\psi$ is an element $f \in L\{\tau\}$ such that $f\phi_a = \psi_a f$ for any $a \in A$. We say that $f$ is an isomorphism if there exists a morphism $g$ from $\psi$ to $\phi$ such that $fg = gf = 1$. It is easy to check that $f$ is an isomorphism if and only if $f \in L^\times$.

For any $a \in A$, its image $\phi_a$ is an endomorphism of $\mathbb{G}_{a,L}$, so that $\phi$ endows the additive group $\mathbb{G}_{a,L}(L^{\text{sep}}) \simeq L^{\text{sep}}$ with a new $A$-module structure defined by $a \cdot \lambda := \phi_a(\lambda)$. Denote this $A$-module by $\phi L^{\text{sep}}$. For any non-zero element $a \in A$, the set of $a$-torsion points

$$\phi[a] = \{\lambda \in \phi L^{\text{sep}}; a \cdot \lambda = \phi_a(\lambda) = 0\}$$

of $\phi$ is an $A$-submodule of $\phi L^{\text{sep}}$ on which $G_L$ acts. If $a$ is not contained in the kernel of $\iota$, then $\phi[a]$ is a free $A/aA$-module of rank $r$.

Let $K$ be a finite extension of $F$. From now on, unless otherwise stated, we regard $K$ as an $A$-field via the inclusion $A \hookrightarrow F \subset K$. Let $\phi$ be a rank-$r$ Drinfeld module over $K$. For any finite place $v$ of $K$, we can regard $\phi$ as a Drinfeld module over $K_v$ via $K\{\tau\} \hookrightarrow K_v\{\tau\}$.

**Definition 2.2.** (1) We say that $\phi$ has stable reduction at $v$ if there exists a Drinfeld module $\psi$ over $K_v$ such that $\psi$ is isomorphic to $\phi$ over $K_v$ and

$$\psi_t = t + c_1\tau + \cdots + c_r\tau^r$$

such that $c_1, \ldots, c_r \in \mathcal{O}_{K_v}$ and $c_{r'} \in \mathcal{O}_{K_v}^\times$ for some $1 \leq r' \leq r$.

(2) We say that $\phi$ has good reduction at $v$ if it has stable reduction at $v$ and $r' = r$.

**Proposition 2.3** (Drinfeld [Dri, Proposition 7.1]). Every Drinfeld module $\phi$ over $K$ has potentially stable reduction at any finite place $v$ of $K$.

**Proof.** Write $\phi_t = t + c_1\tau + \cdots + c_r\tau^r$ and set $R := \min_{1 \leq s \leq r'} \left\{ \frac{c_s}{\tau^s - 1} \right\}$. If $K'_v$ is a finite extension of $K_v$ such that $e(K'_v/K_v) \cdot R \in \mathbb{Z}$, then $\phi$ has stable reduction over $K'_v$. 

\[\square\]
Remark 2.4. In particular, we can take $K'_v/K_v$ as a tamely ramified finite separable extension whose ramification index $e(K'_v/K_v)$ divides $\prod_{i=1}^{\infty}(q^i-1)$. Every rank-one Drinfeld module clearly has potentially good reduction at any finite place. For any monic irreducible element $\pi \in A$, the set of $\pi$-torsion points $\phi[\pi]$ is a $G_K$-stable $r$-dimensional $\mathbb{F}_\pi$-vector space, so that the mod $\pi$ representation
\[ \rho_{\phi,\pi} : G_K \to \text{GL}_{\mathbb{F}_\pi}(\phi[\pi]) \simeq \text{GL}_r(\mathbb{F}_\pi) \]
attached to $\phi$ can be defined. Let $A_\pi := \lim A/\pi^n A$ be the $\pi$-adic completion of $A$. Considering the maps $\phi[\pi^{n+1}] \to \phi[\pi^n]$ defined by $x \mapsto \pi \cdot x$, one can define the $\pi$-adic Tate module $T_\pi(\phi) := \lim \phi[\pi^n]$, which is a free $A_\pi$-module of rank $r$ with continuous $G_K$-action.

Let $v$ be a finite place of $K$ above a finite place $\pi_0$ of $F$. The next proposition is an analogue of the Néron-Ogg-Shafarevich criterion for good reduction of abelian varieties (cf. [ST, Theorem 1]).

Proposition 2.5 (Takahashi [Tak] Theorem 1). The following are equivalent.
(1) $\phi$ has good reduction at $v$.
(2) $T_\pi(\phi)$ is unramified at $v$ for any monic irreducible element $\pi \neq \pi_0$.
(3) $T_\pi(\phi)$ is unramified at $v$ for some $\pi \neq \pi_0$.

If $\phi$ has good reduction at $v$, then the action of $\text{Frob}_v$ on $T_\pi(\phi)$ is well-defined, so that the characteristic polynomial of $\text{Frob}_v$
\[ P_v(T) := \det(T - \text{Frob}_v|T_\pi(\phi)) \in A_\pi[T] \]
can be defined.

Proposition 2.6 (Takahashi [Tak] Proposition 3 (ii)). The polynomial $P_v(T)$ has coefficients in $A$ which are independent of $\pi$, and any root $\alpha$ of $P_v(T)$ satisfies $|\alpha| = q_v^{1/r}$.

The following example gives a function field analogue of cyclotomic extensions of number fields.

Example 2.7 (cf. [Ros, Chapter 12]). The rank-one Drinfeld module $C : A \to F\{\tau\}$ determined by $C_t = t + \tau$ is called the Carlitz module. For any monic irreducible element $\pi \in A$, define
\[ \chi_\pi : G_F \to \text{GL}_{\mathbb{F}_\pi}(C[\pi]) \simeq \mathbb{F}_\pi^r, \]
which is called the mod $\pi$ Carlitz character. Since $C$ has good reduction at any finite place $\pi_0$ of $F$, the character $\chi_\pi$ is unramified at $\pi_0$ if $\pi_0 \neq \pi$. For any finite place $v$ of $K$ above $\pi_0 \neq \pi$, it is known that $\chi_\pi$ satisfies
\[ \chi_\pi(\text{Frob}_v) \equiv \pi_0^{f_v/\pi_0} \pmod{\pi}. \]
The mod $\pi$ Carlitz character induces an isomorphism $\text{Gal}(F(C[\pi])/F) \simeq \mathbb{F}_\pi^r$, so that $F(C[\pi])/F$ is a cyclic extension which is unramified outside $\pi$ and $\infty$. Moreover, it is known that $\pi$ is totally ramified in $F(C[\pi])$ and the ramification of the infinite place $\infty$ is as follows: there exists a subfield $F(C[\pi])_+$ with degree $[F(C[\pi]):F(C[\pi])_+] = q - 1$ such that $\infty$ is totally split in $F(C[\pi])_+$ and any place of $F(C[\pi])_+$ above $\infty$ is totally ramified in $F(C[\pi])$. 

The following example gives a function field analogue of cyclotomic extensions of number fields.
2.2. Tate uniformization. Let $u$ be a finite place of $K$ above $\pi$ and $\phi$ a rank-$r$ Drinfeld module over $K$. Suppose that $\phi$ has stable reduction at $u$. Then Drinfeld’s result on Tate uniformization gives an analytic description of $\phi$.

**Proposition 2.8** (Tate uniformization; Drinfeld [Dri] Section 7). There exist a unique Drinfeld module $\psi$ over $K_u$ with good reduction and a unique entire analytic epimorphism $e : \psi \to \phi$ such that $e$ is the identity on $\text{Lie}(\mathbb{G}_{a,K_u})$.

It is known that the rank $r'$ of $\psi$ satisfies $r' \leq r$ and the kernel $H := \text{Ker}(e)(K_u^{\text{sep}})$ is an $A$-lattice of rank $h := r - r'$ in $\psi(K_u^{\text{sep}})$, endowed with an action of a finite quotient of $G_{K_u}$. For any monic irreducible element $\pi_0 \in A$, the analytic morphism $e$ induces the short exact sequence

$$0 \to T_{\pi_0}(\psi) \to T_{\pi_0}(\phi) \to H \otimes_A A_{\pi_0} \to 0$$

of $A_{\pi_0}[G_{K_u}]$-modules. In the case where $\pi_0 \neq \pi$, the $I_{K_u}$-action on $T_{\pi_0}(\phi)$ is potentially unipotent since both $T_{\pi_0}(\psi)$ and $H \otimes_A A_{\pi_0}$ are potentially unramified at $u$.

**Remark 2.9.** By the theory on “analytic $\tau$-sheaves” (see [Ga1], [Ga2] and [Ga3]), the sequence (2.1) can be interpreted as follows. For any Drinfeld module $\phi$ over $K_u$, one can construct the analytic $\tau$-sheaf $\tilde{M}(\phi)$ associated with $\phi$, which is a locally free $\mathcal{O}_{\mathfrak{A}_{K_u}}$-module of finite rank on $\mathfrak{A}_{K_u}$ with some additional structure, where $\mathfrak{A}_{K_u}$ is the affine line $\mathfrak{A}_{K_u} = \text{Spec}A \times K_u$, seen as a rigid analytic space. Then the $\pi_0$-adic Tate module $T_{\pi_0}(\tilde{M}(\phi))$ of $\tilde{M}(\phi)$ can be defined and it is isomorphic to $T_{\pi_0}(\phi)$. The Tate uniformization implies that there exist an analytic $\tau$-sheaf $\tilde{N}$ which is potentially trivial and the exact sequence

$$0 \to \tilde{N} \to \tilde{M}(\phi) \to \tilde{M}(\psi) \to 0.$$ 

Since $\tilde{M} \mapsto T_{\pi_0}(\tilde{M})$ is a contravariant exact functor, we obtain

$$0 \to T_{\pi_0}(\tilde{M}(\psi)) \to T_{\pi_0}(\tilde{M}(\phi)) \to T_{\pi_0}(\tilde{N}) \to 0,$$

which coincides with the sequence (2.1) (for example, see [Ga4] Example 7.1]).

We would like to estimate the tame ramification of the lattice $H$. Suppose that $H$ is non-trivial, that is, $h \neq 0$ and consider the representation

$$\rho : I_{K_u} \to \text{GL}_A(H) \simeq \text{GL}_{h}(A).$$

For any finite subgroup $G$ of $\text{GL}_{h}(A)$, denote by $(\#G)_p$ the maximal prime-to-$p$ divisor of $\#G$.

**Lemma 2.10.** The integer $(\#G)_p$ divides $\prod_{s=1}^{h}(q^s - 1)$.

**Proof.** By the exact sequence

$$1 \to \text{SL}_h(A) \to \text{GL}_h(A) \to \mathbb{F}_q^* \to 1,$$

we only need to consider the order of $G' := G \cap \text{SL}_h(A)$. Applying the result of Soulé [So] Theorem 3 to $\text{SL}_h$ over $\mathbb{F}_q$, we see that there exists a standard parabolic subgroup $P = U \times L$ of $\text{SL}_h$ such that $G'$ is contained in $U(A) \times L(\mathbb{F}_q)$, where $L$ is the standard Levi subgroup of $P$ and $U$ is the unipotent radical of $P$. Since any element of $U(A)$ has $p$-power order, so that $(\#G')_p$ must divide $\#L(\mathbb{F}_q)$. Therefore $(\#G)_p$ divides $(q - 1) \cdot (\#\text{SL}_h(\mathbb{F}_q))_p = \prod_{s=1}^{h}(q^s - 1)$.

---

^1It is proved for more general algebraic groups by Margaux [Mar].
Proposition 2.11. There exists a finite separable extension $L/K_u$ such that

- the action of $I_L$ on $H$ is trivial,
- the ramification index $e(L_0/K_u)$ divides $\prod_{s=1}^{r-1}(q^s - 1)$, where $L_0$ is the maximal tamely ramified extension of $K_u$ in $L$.

Proof. Let $E/K_u$ be a finite Galois extension such that the action of $G_{K_u}$ on $H$ factors through $\text{Gal}(E/K_u)$. Now the image $I$ of $I_{K_u}$ by the natural restriction $G_{K_u} \to \text{Gal}(E/K_u)$ is the inertia subgroup of $\text{Gal}(E/K_u)$. For the representation

$$\rho : I \to \text{GL}_H(H) \simeq \text{GL}_H(A),$$

denote by $L$ the fixed field of $E$ by $J := \text{Ker}(\rho)$. By construction, the action of inertia subgroup $I_L$ of $G_L$ on $H$ is trivial. Since the image $\text{Im}(\rho)$ is a finite subgroup of $\text{GL}_A(A)$ of order $\#I/J = e(L_0/K_u)$ and $h \leq r - 1$, Lemma 2.10 implies that $\langle \#\text{Im}(\rho) \rangle_p = e(L_0/K_u)$ divides $\prod_{s=1}^{r-1}(q^s - 1)$. \[\square\]

Proposition 2.11 means that the Drinfeld module $\phi$ is “semistable” over $L$ in the following sense. By Remark 2.3, the analytic $\tau$-sheaf $\tilde{M}(\phi)$ is the extension of $\tilde{M}(\psi)$ by $\tilde{N}$ and both $\tilde{M}(\psi)$ and $\tilde{N}$ are “good” over $L$. Hence the analytic $\tau$-sheaf $\tilde{M}(\phi)$ is strongly semistable over $L$ in the sense of [Ga3, Definition 4.6].

3. Inertia action on torsion points

In this section, we show the non-existence result in $p$-power rank case. Let $\pi$ be a monic irreducible element of $A$.

3.1. Tame inertia weights. Let $u$ be a finite place of $K$ above $\pi$. For a fixed separable closure $K_u^{\text{sep}}$ of $K_u$ with residue field $\bar{F}_u$, denote by $K_u^{\text{ur}}$ (resp. $K_u^{\text{ur}}$) the maximal unramified (resp. maximal tamely ramified) extension of $K_u$ in $K_u^{\text{sep}}$, so that $I_{K_u}$ is isomorphic to $\text{Gal}(K_u^{\text{sep}}/K_u^{\text{ur}})$. Denote by $I^t := \text{Gal}(K_u^{\text{ur}}/K_u^{\text{ur}})$ the tame inertia subgroup of $I_{K_u}$. Let $d$ be a positive integer and $F$ the finite field with $q^d$ elements in $F_u$. Then $F$ is the finite extension of $F_\pi$ of degree $d$. Write $\mu_{q^d-1}(K_u^{\text{sep}})$ for the set of $(q^d-1)$-st roots of unity in $K_u^{\text{sep}}$ and fix the isomorphism $\mu_{q^d-1}(K_u^{\text{sep}}) \sim \mathbb{F}_u^\times$ coming from the reduction map $\mathcal{O}_{K_u^{\text{sep}}} \rightarrow \bar{F}_u$. For a uniformizer $\varpi$ of $K_u$, choose a solution $\eta \in K_u^{\text{sep}}$ to the equation $X^{q^d-1} - \varpi = 0$ and define

$$\omega_{d,K_u} : I_{K_u} \rightarrow \mu_{q^d-1}(K_u^{\text{sep}}) \sim \mathbb{F}_u^\times ; \sigma \mapsto \frac{\eta^\sigma}{\eta},$$

which is independent of the choices of $\varpi$ and $\eta$, and the character $\omega_{d,K_u}$ factors through $I^t$ (cf. [Se]). We call the $\text{Gal}(F/F_\pi)$-conjugates $(\omega_{d,K_u})^i$ for $0 \leq i \leq d - 1$ of $\omega_{d,K_u}$ the fundamental characters of level $d$. It is easy to check that

$$(\omega_{d,K_u})^{1+q+\cdots+q^{d-1}} = \omega_{1,K_u}$$

and $(\omega_{d,K_u})^{q^d-1} = 1$. For any finite separable extension $L$ of $K_u$, we see that $(\omega_{d,K_u})^{1+q+\cdots+q^{d-1}} = \omega_{1,L}$ by definition.

As an analogue of Serre’s classical result on the mod $\ell$ cyclotomic character ([Se, Proposition 8]), the following fact is known.

Proposition 3.1 (Kim [Ki] Proposition 9.4.3. (2)). The character $(\omega_{1,K_u})^{\varpi}$ coincides with the mod $\pi$ Lubin-Tate character restricted to $I_{K_u}$.
Remark 3.2. The mod $\pi$ Lubin-Tate character is the character coming from the Lubin-Tate formal group over $O_{K_u}$, which coincides with the mod $\pi$ Carlitz character $\chi_\pi$ restricted to $G_{K_u}$, so that $\chi_\pi = (\omega_{1,K_u})^{e_{\gamma_\pi}}$ on $I_{K_u}$.

Let $V$ be a $d$-dimensional irreducible $\mathbb{F}_q$-representation of $I_{K_u}$. Then the action of $I_{K_u}$ on $V$ factors through $I^1$, so that $V$ can be regarded as a representation of $I^1$. Using Schur’s Lemma, we see that $\text{End}_{I^1}(V)$ is the finite field of order $q^d$. Fix an isomorphism $f : \text{End}_{I^1}(V) \cong \mathbb{F}$ and regard $V$ as a one-dimensional $\mathbb{F}$-representation

$$\rho : I^1 \to \text{End}_{I^1}(V)^{\mathbb{F}} \cong \mathbb{F}$$

of $I^1$. Since $I^1$ is pro-cyclic and $\omega_{d,K_u}$ is surjective, there exists an integer $0 \leq j \leq q_d^0 - 2$ such that $\rho = (\omega_{d,K_u})^{j}$. If we decompose $j = n_0 + n_1 q_\pi + \cdots + n_{d-1} q_{\pi}^{d-1}$ with integers $0 \leq n_s \leq q_{\pi} - 1$, then the set $\{n_0, n_1, \ldots, n_{d-1}\}$ is independent of the choice of $f$.

Definition 3.3. These numbers $n_0, n_1, \ldots, n_{d-1}$ are called tame inertia weights of $V$. In general, for any $\mathbb{F}_q$-representation $\rho : G_{K_u} \to \text{GL}_d(\mathbb{F}_q)$, the tame inertia weights of $\rho$ are the tame inertia weights of all the Jordan-Hölder quotients of $V|_{I_u}$.

Denote by $\text{TI}_{K_u}(\rho)$ the set of tame inertia weights of $\rho : G_{K_u} \to \text{GL}_d(\mathbb{F})$.

3.2. Ramification of constrained torsion points. Let $\phi$ be a Drinfeld module over $K$ satisfying $[\phi] \in \mathcal{D}(K, r, \pi)$ and $u$ a finite place of $K$ above $\pi$. We can take a finite separable extension $K'_u$ of $K_u$ such that $\phi$ has stable reduction and $e(K'_u/K_u)$ divides $\prod_{s=1}^{n}(q^s - 1)$. By Tate uniformization, we obtain the exact sequence

$$0 \to \psi[\pi] \to \phi[\pi] \to H \otimes_A \mathbb{F}_q \to 0$$

of $\mathbb{F}_q[G_{K'_u}]$-modules. We also take a finite separable extension $L$ of $K'_u$ as in Proposition 3.1 and denote by $L_0$ the maximal tamely ramified extension of $K'_u$ in $L$. Set

$$C_1 = C_1(q, r) := (q^r - 1) \prod_{s=1}^{r-1}(q^s - 1)^2$$

and $e_u := e(L_0/K'_u) \cdot e(K'_u/\mathbb{F}_q)$. Then $e_u$ divides $e_{u|\pi} C_1(q, r)$.

Proposition 3.4. Every tame inertia weight of $\bar{\rho}_{\phi, \pi}|_{I_{L_0}}$ is between $0$ and $e_u$.

Proof. By (D2), the restriction $\bar{\rho}_{\phi, \pi}|_{I_{L_0}}$ is isomorphic to $(\omega_{1,L_0})^{j_1} \oplus \cdots \oplus (\omega_{1,L_0})^{j_r}$, where $\{j_1, \ldots, j_r\} \subset \text{TI}_{L_0}(\bar{\rho}_{\phi, \pi})$. Write $\bar{\rho} : G_{L_0} \to \text{GL}_d(\mathbb{F}_q)$ for the representation arising from $H \otimes_A \mathbb{F}_q$. Then the sequence $\Sigma$ implies $\bar{\rho}_{\phi, \pi} = \bar{\rho} \circ \pi$ on $G_{L_0}$, so that $\text{TI}_{L_0}(\bar{\rho}_{\phi, \pi}) = \text{TI}_{L_0}(\bar{\rho}) \cup \text{TI}_{L_0}(\bar{\rho})$. Let $M(\psi)$ and $\bar{N}$ be analytic $\tau$-sheaves on $\hat{K}^1_{L_0}$ attached to $\psi$ and $H$, respectively. Since $M(\psi)$ is good over $L_0$, we see that $\text{TI}_{L_0}(\bar{\rho}_{\phi, \pi}) \subset [0, e_u]$ by [Ga1], Theorem 2.14]. On the other hand, the analytic $\tau$-sheaf $\bar{N}$ is of dimension zero and good over $L$, so that every tame inertia weight of $\bar{\rho}|_{I_L}$ is zero by [Ga1], Theorem 2.14], which means that $\bar{\rho} \pi|_{I_L} = 1$. Hence we see that

$$(\omega_{1,L_0})^{j}|_{I_L} = (\omega_{1,L})^{e(L/L_0)j} = 1$$

for any $j \in \text{TI}_{L_0}(\bar{\rho})$, so that $e(L/L_0) \cdot j \equiv 0 \pmod{q_\pi - 1}$ holds. Since $e(L/L_0)$ is a $p$-power and $0 \leq j \leq q_\pi - 2$, we see that $j = 0$. \qed
The condition (D2) means that $\bar{\rho}_{\phi,\pi}$ is isomorphic to $\chi_1^i + \cdots + \chi_r^i$ for $0 \leq i_1, \ldots, i_r \leq q_\pi - 1$. By renumbering $\{j_1, \ldots, j_r\}$ if necessary, Proposition 3.1 and 3.2 means that $\chi_1^i|_{L_0} = (\omega_{1,L_0})^{i-e_u} = (\omega_{1,L_0})^{j_s}$ for any $1 \leq s \leq r$. Thus we obtain
\[(3.2) \quad i_s \cdot e_u \equiv j_s \pmod{q_\pi - 1}\]
for any $1 \leq s \leq r$.

For any finite place $v$ of $K$ not lying above $\pi$ and any integer $m$, denote by $P_{v,m}(T) = \det(T - \text{Frob}_v^m[T_{\pi}(\phi)]) \in A[T]$ the characteristic polynomial of $\text{Frob}_v^m$.

Set $C_2 = C_2(n_K, q, r) := r \cdot n_K^2 \cdot C_1(q, r)$.

Then we obtain the following important proposition.

**Proposition 3.5.** If $\deg(\pi) > C_2$, then $r$ divides $e_u$ and the congruence
\[(3.3) \quad i_s \cdot e_u \equiv j_s \pmod{q_\pi - 1}\]
holds for any $1 \leq s \leq r$.

**Proof.** Suppose that $\deg(\pi) > C_2$. Take a monic irreducible element $\pi_0 \in A$ with $\deg(\pi_0) = 1$ and a finite place $v$ of $K$ above $\pi_0$. Since $\phi$ has good reduction at $v$ by (D1), the polynomial $P_{v,e_u}(T)$ is well-defined. Now the roots of $P_{v,e_u}(T)$ are given by $\{\alpha_s^e_u\}_{s=1}^r$, where $\{\alpha_s\}_{s=1}^r$ are the roots of $P_v(T) = P_{v,1}(T)$. On the other hand, the condition (D2) implies that the roots of the polynomial $P_{v,e_u}(T) \pmod{\pi}$ in $\mathbb{F}_p[T]$ are given by $\{\chi_\pi(\text{Frob}_v)^{i_s-e_u}\}_{s=1}^r$. Set $\pi_v := \pi_0^{|e_u|}$. By the above relation (3.2), we see that $\chi_\pi(\text{Frob}_v)^{i_s-e_u} = \chi_\pi(\text{Frob}_v)^{j_s}$ for any $1 \leq s \leq r$ and
\[(3.3) \quad P_{v,e_u}(T) \equiv \prod_{s=1}^r (T - \chi_\pi(\text{Frob}_v)^{j_s}) \equiv \prod_{s=1}^r (T - \pi_v^{j_s}) \pmod{\pi}.

Denote by $S_k(x_1, \ldots, x_r)$ the fundamental symmetric polynomial of degree $k$ with $r$ variables $x_1, \ldots, x_r$ for $0 \leq k \leq r$. Then
\[
\prod_{s=1}^r (T - x_s) = \sum_{k=0}^r (-1)^k S_k(x_1, \ldots, x_r) T^{r-k}.
\]

Now $|\alpha_s^e_u| = q_v^{e_u/r}$ for any $1 \leq s \leq r$ by Proposition 2.6. For any $0 \leq k \leq r$, we obtain
\[
|S_k(\alpha_1^e_u, \ldots, \alpha_r^e_u) - S_k(\pi_v^{j_1}, \ldots, \pi_v^{j_r})| \leq \max_{1 \leq i_1 < \cdots < i_r \leq r} \left\{|k_{e_u^i}q_v^{e_u} \cdot j_1^{i_1} \cdots j_r^{i_r}|\right\} \leq q_v^{r \cdot e_u} = q_v^{r \cdot e_u} f_{e_u | n_K}
\]
since $j_s \leq e_u$ for each $s$ by Proposition 3.3. Since $e_u$ divides $e_u | n_K \cdot C_1(q, r)$ and both $e_u | n_K$ and $f_{e_u | n_K}$ are less than or equal to $n_K = [K : F]$, we see that
\[
q_v^{r - e_u} f_{e_u | n_K} \leq q^{C_2} < q^{\deg(\pi)} = |\pi|,
\]
which means that all absolute values of coefficients of $P_{v,e_u}(T) - \prod_{s=1}^r (T - \pi_v^{j_s})$ are smaller than $|\pi|$. Therefore the congruence (3.3) implies $P_{v,e_u}(T) = \prod_{s=1}^r (T - \pi_v^{j_s})$. Comparing the absolute values of the roots of $P_{v,e_u}(T)$ and $\prod_{s=1}^r (T - \pi_v^{j_s})$, we see that $e_u/r = j_s$ for any $1 \leq s \leq r$, which implies the conclusion.
Set $e_\phi := \gcd\{e_u; \ u|\pi\}$ and $S_r := \{s = (s_1, \ldots, s_r) \in \mathbb{Z}^r; 1 \leq s_k \leq r\}$.

**Lemma 3.6.** Suppose that $\deg(\pi) > C_2$.

1. $e_\phi[n_K \cdot C_1(q, r)]$. If $\pi$ is unramified in $K_s$, then $e_\phi[[K : F]] \cdot C_1(q, r)$.
2. For any $(s_1, \ldots, s_r) \in S_r$, the relation $e_\phi \cdot (i_{s_1} + \cdots + i_{s_r} - 1) \equiv 0 \pmod{q_{s_1} - 1}$ holds.

**Proof.** Immediately (1) follows from the construction of $e_u$. By Proposition 3.5 we see that $i_s \cdot e_\phi \equiv e_\phi (\pmod{q_{s_1} - 1})$. Adding this congruence for $s_1, \ldots, s_r$ gives

$$e_\phi \cdot (i_{s_1} + \cdots + i_{s_r}) \equiv e_\phi \pmod{q_{s_1} - 1},$$

which proves (2). \qed

There exist only finitely many places of $F$ which are ramified in $K_s$. Define $C_3 = C_3(K_s)$ to be the maximal degree of such places and set

$$C_4 = C_4(n_K, r, K_s) := \max\{C_2(n_K, q, r), C_3(K_s)\}.$$

**Theorem 3.7.** Suppose that $r = p^\nu > 1$ and $r$ does not divide $[K : F]_s$. If $\deg(\pi) > C_4$, then the set $\mathcal{D}(K, r, \pi)$ is empty.

**Proof.** Assume that $[\phi] \in \mathcal{D}(K, r, \pi)$. Now $\pi$ is unramified in $K_s$ by $\deg(\pi) > C_3$. Since every place of $K_s$ is totally ramified in $K$ if $K \neq K_s$, Proposition 3.5 and Lemma 3.6 imply that $r$ divides $[K : F]_s \cdot C_1(q, r)$. The integer $C_1(q, r)$ is prime to $p$ and so $r$ must divide $[K : F]_s$, which contradicts to the assumption on $r$. \qed

**Remark 3.8.** In Section 5, we see that if $r$ divides $[K : F]_s$ and $r = q^\nu$ for some integer $\nu > 0$, then $\mathcal{D}(K, r, \pi)$ must not be empty.

### 4. Observations at Places with Small Degree

**4.1. Effective Chebotarev density theorem.** Recall some basic facts on algebraic number theory in function fields. Let $L$ be an algebraic extension of $K$. The constant field $\mathbb{F}_L$ of $L$ is the algebraic closure of $\mathbb{F}_q$ in $L$. If $L = \mathbb{F}_L K$, then $L$ is called a constant field extension of $K$, which is unramified at any places ([Ros, Proposition 8.5]). If $\mathbb{F}_L = \mathbb{F}_K$, then $L$ is called a geometric extension of $K$. In general, the field $\mathbb{F}_L K$ is the maximal constant extension of $K$ in $L$ and the extension $L/\mathbb{F}_L K$ is geometric. Set $[L : K]_g := [L : \mathbb{F}_L K]$ if $L/K$ is finite, which is called the geometric extension degree of $L/K$. For example, for any $a \in A$, the field $F(C[a])$ arising from the Carlitz module is a geometric extension of $F$.

Denote by $\text{Div}(K)$ the divisor group of $K$, which is by definition the free abelian group generated by all places of $K$. We write divisors additively, so that a typical divisor is of the form $D = \sum_v n_v v$. The degree of a place $v$ of $K$ is defined by $\deg_K v := [\mathbb{F}_v : \mathbb{F}_K]$ and it is extended to any divisor $D = \sum_v n_v v$ by $\deg_K D = \sum_v n_v \deg_K v$. The degree $\deg_F \pi$ of a finite place $\pi$ of $F$ is exactly the degree $\deg(\pi)$ as a polynomial.

Suppose that $L$ is a finite separable extension of $K$. Then the conorm map $i_{L/K} : \text{Div}(K) \to \text{Div}(L)$ is defined to be the linear extension of

$$i_{L/K} v = \sum_{w|v} e_{w|v} w,$$

where $v$ is a place $K$. The following is known (cf. [Ros, Proposition 7.7]).
Lemma 4.1. Let \( w \) be a place of \( L \) above a place \( v \) of \( K \) and \( D \in \text{Div}(K) \). Then

\[
\deg_L i_{L/K} D = \left[ L : K \right]_G \deg_K D \quad \text{and} \quad \deg_L w = \frac{f_{w/v}}{[\mathbb{F}_L : \mathbb{F}_K]} \deg_K v.
\]

Define \( \Sigma_{L/K} \) to be the divisor of \( K \) which is the sum of all ramified places of \( K \) in \( L \). For any place \( w \) of \( L \) above a place \( v \) of \( K \), denote by \( p_w \) the maximal ideal of \( \mathcal{O}_{L_w} \) and let \( \delta_w \) the exact power of \( p_w \) dividing the different of \( \mathcal{O}_{L_w} \) over \( \mathcal{O}_{K_v} \). Then it satisfies \( \delta_v \geq e_w/v - 1 \) with equality holding if and only if \( p \) does not divide \( e_w/v \). Define the ramification divisor of \( L/K \) by \( D_{L/K} = \sum w \delta_w w \). For any intermediate field \( K'/L/K \), we see that

\[
D_{L/K} = D_{L/K'} + i_{L/K'} D_{K'/K}
\]

(for example, see [Se, Chapter III 4]). Hence \( D_{L/K'} \leq D_{L/K} \) holds. In addition, the following holds (cf. [CL, Lemma 2.6]).

Lemma 4.2. Let \( L/K \) and \( L'/K \) be finite separable extensions. Then

\[
D_{LL'/K} \leq i_{LL'/L} D_{L/K} + i_{LL'/L} D_{L'/K}.
\]

Now let \( E \) be a finite Galois extension of \( K \) and \( v \) a place of \( K \) unramified in \( E \). For any place \( w \) of \( E \) above \( v \), denote by \( \text{Fr}_{w|v} \) the Frobenius element in \( \text{Gal}(E/K) \). These elements consist a conjugacy class

\[
\left[ \frac{E/K}{v} \right] := \{ \text{Fr}_{w|v} ; w|v \}
\]

in \( \text{Gal}(E/K) \). As a consequence of the effective version of Chebotarev density theorem [KS, Theorem 1], the following holds.

Proposition 4.3 (Chen and Lee [CL, Corollary 3.4]). Let \( E/K \) be a finite Galois extension and \( \Sigma \) a divisor of \( K \) such that \( \Sigma \geq \Sigma_{E/K} \). Set \( d_0 := [\mathbb{F}_K : \mathbb{F}_q] \) and \( d := [\mathbb{F}_E : \mathbb{F}_K] \). Define the constant \( B = B(E/K, \Sigma) \) by

\[
B = \max \{ \deg_K \Sigma, \deg_E D_{E/\mathbb{F}_E K}, 2[\mathbb{F}_E : \mathbb{F}_K] - 2, 1 \}.
\]

Then for any nonempty conjugacy class \( \mathcal{C} \) in \( \text{Gal}(E/K) \), there exists a place \( v \) of \( K \) with \( v \notin \Sigma \) such that

- \( \mathcal{C} = \left[ \frac{E/K}{v} \right] \),
- \( \deg_K v \leq \frac{1}{d_0} \log_q \left( \frac{1}{3} (B + 3g_K + 3) + d \right) \),

where \( g_K \) is the genus of \( K \).

Let \( \pi \) be a monic irreducible element of \( A \) and \( m \geq 1 \) an integer divides \( #\mathbb{F}_q^x = q_\pi - 1 \). A monic irreducible element \( \pi_0 \) distinct to \( \pi \) is called an \( m \)-th power residue modulo \( \pi \) if \( (\pi_0 \mod \pi) \in \mathbb{F}_q^{x,m} \). As an application of Proposition 4.3 we would like to show that one can find an \( m \)-th power residue modulo \( \pi \) whose degree is smaller than \( \deg(\pi) \) if \( \deg(\pi) \) is sufficiently large. Denote by \( F_m \) the unique subfield of \( F(C[\pi]) \) with \( [F_m : F] = m \) and consider the character \( \chi(m) : G_F \xrightarrow{\chi} \mathbb{F}_q^x \rightarrow \mathbb{F}_q^{x,m} \).

Lemma 4.4. The following are equivalent.

- \( \pi_0 \) is an \( m \)-th power residue modulo \( \pi \).
- \( \chi(m)(\text{Frob}_{\pi_0}) = 1 \).
- \( \text{Frob}_{\pi_0}|_{F_m} = \text{id} \).
Proof. It is trivial when $m = 1$. If not, then this lemma immediately follows from that $\chi_\pi(\text{Frob}_{\pi_0}) \equiv \pi_0 \pmod{\pi}$ and $F_m$ is the fixed field of $F^{\text{sep}}$ by $\text{Ker}(\chi(m))$. \hfill \square

Denote by $\tilde{K}$ the Galois closure of $K_s$ over $F$ and set $E := \tilde{K}F_m$, which is also a Galois extension of $F$. Consider the divisor $\Sigma := \Sigma_{E/F} + \pi + \infty \in \text{Div}(F)$. For the constant $B = B(\Sigma) = \max\{\deg_F \Sigma, \deg_E \text{Div}(\Sigma_{E/F}) \},$ we obtain the following estimate.

**Lemma 4.5.** Let $n$ be a positive integer. Then there exists a constant $C_5 = C_5(K_s, n_K, m, n) > 0$ such that for any $\pi$ satisfying $\deg(\pi) > C_5$, the inequality

$$4 \log_q \frac{2}{3}(B + 3) + [F_\pi : F_q] < \frac{1}{n} \deg(\pi)$$

holds.

Proof. We first compute an upper bound of $B$. We may assume that $\pi$ is unramified in $K_s$. Since the degree $[K_s : F]$ is less than or equal to $n_K!$, we see that $[F_\pi : F_q] \leq n_K!$ and $[E : F_q] \leq m \cdot n_K!$. By example 2.7, the infinite place $\infty$ of $F$ is split at most $m$ places in $F_m$ whose ramification indexes divide $q - 1$ and $\pi$ is totally ramified or unramified (if $m = 1$) in $F_m$. Thus we see that

$$\deg_F \Sigma \leq \deg_F (\Sigma_{F_m/F} + \Sigma_{E/F} + \pi + \infty) \leq 2 \deg(\pi) + 2 + \deg_F \Sigma_{E/F}.$$

Now $\text{Div}(\Sigma_{E/F}) \leq \text{Div}(\Sigma_{E/F})$ holds. Lemma 4.1 and 4.2 imply

$$\deg_E \text{Div}(\Sigma_{E/F}) \leq \deg_E (\Sigma_{F_m/F} + \Sigma_{E/F} + \pi + \infty) \leq \deg_E \Sigma_{E/F} \leq \deg_E \Sigma_{F_m/F}.$$  

Hence there exist positive constants $B_1$ and $B_2$ depending only on $K_s, n_K,$ and $m$ such that $B \leq B_1 \cdot \deg(\pi) + B_2$ holds. Therefore if $\deg(\pi)$ is sufficiently large, then $4 \log_q \frac{2}{3}(B + 3) + [F_\pi : F_q] < \frac{1}{n} \deg(\pi)$ holds. \hfill \square

Proposition 4.3 and Lemma 4.5 imply the following.

**Proposition 4.6.** Let $n$ be a positive integer. If $\deg(\pi) > C_5$, then there exist a monic irreducible element $\pi_0 \in A$ and a place $v$ of $K$ above $\pi_0$ such that

- $\pi_0$ is an $m$-th power residue modulo $\pi$,
- $\deg(\pi_0) < \frac{1}{n} \deg(\pi)$,
- $f_{v|\pi_0} = 1$.

Proof. We may assume that $K = K_s$ since the extension $K/K_s$ is totally ramified at any places if $K \neq K_s$. Let $\tilde{K}$ and $E = \tilde{K}F_m$ be as above and fix an element $\sigma \in \text{Gal}(E/F)$ such that $\sigma|_{K/F} = \text{id}$. For the conjugacy class $C$ of $\sigma$ in $\text{Gal}(E/F)$, by Proposition 4.3 and Lemma 4.5 there exists a place $\pi_0$ of $F$ with $\pi_0 \notin \Sigma$ (hence it is a finite place) such that $[E/F_{\pi_0}] = C$ and $\deg(\pi_0) < \frac{1}{n} \deg(\pi)$, so that $\sigma = \text{Frob}_{w|\pi_0}$ for some place $w$ of $E$. Then the decomposition group $Z_w$ of $w$ over $\pi_0$ is generated by $\sigma$ and it is a subgroup of $\text{Gal}(E/F_{\pi_0})$. Denote by $K'$ the fixed field of $E$ by $Z_w$. Then the place $v'$ of $K'$ below $w$ satisfies $e_{v'|\pi_0} = 1$ and $f_{v'|\pi_0} = 1$. Hence $f_{v|\pi_0} = 1$, where $v$ is the place of $K$ below $v'$. By construction, we see that $\text{Frob}_{v|F_m} = \text{id}$. Lemma 4.3 means that $\pi_0$ is an $m$-th power residue modulo $\pi$. \hfill \square
4.2. Not $p$-power rank case. Let $\phi$ be a rank-$r$ Drinfeld module over $K$ satisfying $[\phi] \in \mathcal{D}(K,r,\pi)$. In this subsection, we always assume that $r = r_0 \cdot p^\nu$ for some $r_0 > 1$ which is prime to $p$. Now let $i_1, \ldots, i_r$ be the positive integers satisfying $\overline{\rho_{\phi,\pi}}^{ss} \simeq \chi_\pi^{i_1} \oplus \cdots \oplus \chi_\pi^{i_r}$ by (D2). For any $s = (s_1, \ldots, s_r) \in S_r$, set $\varepsilon_s := \chi_\pi^{is_1 + \cdots + is_r - 1}$ and define

$$
\varepsilon := (\varepsilon_s)_{s \in S_r} : G_F \to (\mathbb{F}_\pi^\times)^{\oplus r}.
$$

Set $m_\phi := \#\varepsilon(G_F)$, which is the least common multiple of the orders of $\varepsilon_s$. Since $\varepsilon$ factors through $\mathbb{F}_\pi^\times$, the image $\varepsilon(G_F)$ is cyclic and $m_\phi (q_\pi - 1)$. Then we obtain the following commutative diagram

Hence a monic irreducible element $\pi_0$ is an $m_\phi$-th power residue modulo $\pi$ if and only if $\varepsilon_s(Frob_{\pi_0}) = 1$ for any $s \in S_r$.

**Lemma 4.7.** If $\deg(\pi) > C_2(n_K, q, r)$, then $m_\phi$ divides the greatest common divisor $(e_\phi, q_\pi - 1)$. In particular, it divides $n_K C_1(q, r)$.

**Proof.** It follows from Lemma 3.6(2). \hfill $\Box$

**Proposition 4.8.** If there exist a monic irreducible element $\pi_0$ and a finite place $v$ of $K$ above $\pi_0$ such that $\deg(\pi) > f_{v|\pi_0} \deg(\pi_0)$ and $r_0$ does not divide $f_{v|\pi_0}$, then $m_\phi > 1$ and $\chi(m_\phi)(Frob_v) \neq 1$.

**Proof.** Assume that either $m_\phi = 1$ or $\chi(m_\phi)(Frob_v) = 1$ holds. Then $\varepsilon_s(Frob_v) = 1$ for any $s \in S_r$. Denote by $a_{v,p^\nu} \in A$ the coefficient of $T^{r-p^\nu}$ in the characteristic polynomial $P_v(T)$ of $Frob_v$ on $T_\pi(\phi)$. It is given by $a_{v,p^\nu} = (-1)^{p^\nu} S_{p^\nu}(\alpha_1, \ldots, \alpha_r)$, where $\alpha_1, \ldots, \alpha_r$ are the roots of $P_v(T)$ and $S_{p^\nu}(x_1, \ldots, x_r)$ is the fundamental symmetric polynomial of degree $p^\nu$ with $r$ variables. Consider the subset $S_{p^\nu}^0 := \{(s_1, \ldots, s_{p^\nu}) : 1 \leq s_1 < \cdots < s_{p^\nu} \leq r\}$ of $\mathbb{Z}^{p^\nu}$. Then the product $S_{p^\nu}^0$ can be regarded as a subset of $S_r$. Since $S_{p^\nu}(x_1, \ldots, x_r)$ is the sum of $\binom{r}{p^\nu}$ monomials of degree $p^\nu$, we obtain that

$$(a_{v,p^\nu})^{r_0} = (-1)^r S_{p^\nu}(\alpha_1, \ldots, \alpha_r)^{r_0} = (-1)^r \sum_{(s_1, \ldots, s_{p^\nu}) \in S_{p^\nu}} \chi_\pi^{i_{s_1} + \cdots + i_{s_{p^\nu}}}(Frob_v)^{r_0} \equiv (-1)^r \sum_{s \in S_{p^\nu}^0} \varepsilon_s(Frob_v) \chi_\pi(Frob_v) \equiv (-1)^r \sum_{s \in S_{p^\nu}^0} \chi_\pi(Frob_v) = (-1)^r \left( \binom{r}{p^\nu} \right)^{r_0} f_{v|\pi_0}^{r_0} \neq 0 \pmod{\pi}$$
since \((p_r^r)\) is not divided by \(p\). Now we see that
\[
|a_{v,p^r}|^{r_0} \leq q_v = q_{f_{v|\pi_0}^{\deg(\pi_0)}} < |\pi| \quad \text{and} \quad \left|(-1)^r \left( \frac{r}{p^r} \right)_{\pi_0}^{f_{v|\pi_0}} \right| = |\pi_{f_{v|\pi_0}}| = q_v < |\pi|.
\]
Hence the above congruence implies \((a_{v,p^r})^{r_0} = (-1)^r \left( \frac{r}{p^r} \right)_{\pi_0}^{f_{v|\pi_0}}\). Comparing the \(\pi_0\)-adic valuation of both sides, we obtain \(r_0|f_{v|\pi_0}\), which is contradiction. \(\square\)

Set
\[
C_6(K, n, K, r) := \max\{C_5(K, n, K, m, 1); m|n_KC_1(q, r)\}
\]
\[
C_7(K, n, K, r) := \max\{C_2(n, K, q, r), C_6(K, n, K, r)\}.
\]
Then we have the following theorem.

**Theorem 4.9.** Suppose that \(r = r_0p^r\) is as above and \(\deg(\pi) > C_7\). Then the set \(\mathcal{D}(K, r, \pi)\) is empty.

**Proof.** Assume that \(\mathcal{D}(K, r, \pi)\) is not empty and \([\phi] \in \mathcal{D}(K, r, \pi)\). By Proposition 4.6, there exist a monic irreducible element \(\pi_0\) and a place \(v\) of \(K\) above \(\pi_0\) with \(f_{v|\pi_0} = 1\) and \(\deg(\pi_0) < \deg(\pi)\) such that \(\pi_0\) is an \(m_\phi\)-th power residue modulo \(\pi\), so that \(\chi(m_\phi)(\text{Frob}_{\pi_0}) = 1\). However, since \(\pi_0\) and \(v\) satisfy the assumption of Proposition 4.8, we see that \(\chi(m_\phi)(\text{Frob}_v) = \chi(m_\phi)(\text{Frob}_{\pi_0}) \neq 1\). \(\square\)

By the same argument, we can also prove an uniform version results. For a fixed finite separable extension \(K_0\) of \(F\) with degree \(n_0 := [K_0 : F]\) and a positive integer \(n\), set
\[
C_8 = C_8(K_0, r, n) := \max\{C_2(n_0, q, r), \max\{C_5(K_0, m, n); m|n_KC_1(q, r)\}\}.
\]

**Theorem 4.10.** Let \(r = r_0p^r\), \(K_0\), and \(n\) be as above. Suppose that \(r_0\) does not divide \(n\) and \(\deg(\pi) > C_8\). Then the set \(\mathcal{D}(K, r, \pi)\) is empty for any finite extension \(K\) of \(K_0\) satisfying \([K : K_0] = n\).

**Proof.** Let \(K\) be a finite extension of \(K_0\) with \([K : K_0] = n\) and assume that \([\phi] \in \mathcal{D}(K, r, \pi)\). Applying Proposition 4.6 to \(K_0\), we can find a monic irreducible element \(\pi_0\) and a finite place \(v_0\) of \(K_0\) above \(\pi_0\) with \(f_{v_0|\pi_0} = 1\) such that \(n\ deg(\pi_0) < \deg(\pi)\) and \(\pi_0\) is an \(m_\phi\)-th power residue modulo \(\pi\), so that \(\chi(m_\phi)(\text{Frob}_{\pi_0}) = 1\). Now we can take a place \(v\) of \(K\) above \(v_0\) such that \(f_{v|\pi_0}(= f_{v|\pi_0})\) is not divided by \(r_0\). Indeed, if not, then \(r_0\) must divide \(n = \sum_{v|\pi_0} c_{v|\pi_0} f_{v|\pi_0}\). Since \(f_{v|\pi_0} \deg(\pi_0) < n \deg(\pi_0) < \deg(\pi)\), by Proposition 4.8, we see that \(\chi(m_\phi)(\text{Frob}_v) = \chi(m_\phi)(\text{Frob}_{\pi_0})^{f_{v|\pi_0}} \neq 1\). It is contradiction. \(\square\)

5. **Comparison with number field case**

In the last section, we focus on the difference between the Rasmussen-Tamagawa conjecture and its Drinfeld module analogue. Let \(\pi \in A\) be a monic irreducible element.

5.1. **Characterizing conditions of \(\mathcal{D}(K, r, \pi)\).** We would like to characterize the set \(\mathcal{D}(K, r, \pi)\) same as \(\mathcal{A}(k, g, \ell)\). In number field case, the set \(\mathcal{A}(k, g, \ell)\) is characterized by the equivalent conditions (RT-1), (RT-2), and (RT-3) in Section 1. The equivalence of them follows from the criterion of Néron-Ogg-Shafarevich and the next group theoretic lemma.
Lemma 5.1 (Rasmussen and Tamagawa \[RT2\], Lemma 3.4). Let \( \mathbb{F} \) be a finite field of characteristic \( \ell \). Suppose \( G \) is a profinite group, \( N \subset G \) is a pro-\( \ell \) open normal subgroup, and \( C = G/N \) is a finite cyclic subgroup with \( \#C \mid \#\mathbb{F}^\times \). Let \( V \) be an \( \mathbb{F} \)-vector space of dimension \( r \) on which \( G \) acts continuously. Fix a group homomorphism \( \chi : G \to \mathbb{F}^\times \) with \( \text{Ker}(\chi_0) = N \). Then there exists a filtration

\[
0 = V_0 \subset V_1 \subset \cdots \subset V_r = V
\]
such that each \( V_s \) is \( G \)-stable and \( \dim \mathbb{F} V_s = s \) for any \( 0 \leq s \leq r \). Moreover, for each \( 1 \leq s \leq r \), the \( G \)-action on each quotient \( V_s/V_{s-1} \) is given by \( \chi_0^{i_s} \) for some integer \( i_s \) satisfying \( 0 \leq i_s < \#C \).

Remark 5.2. In \[RT2\], this lemma is proved when \( \mathbb{F} = \mathbb{F}_\ell \), and the general case can be proved in the same way.

Let \( \phi \) be a rank-\( r \) Drinfeld module over \( K \). Consider the field \( K(\phi[\pi^\infty]) := K(\bigcup_{n \geq 1}\phi([\pi^n])) \) generated by all \( \pi \)-power torsion points of \( \phi \), so that it coincides with the fixed field of \( K^{\text{sep}} \) by the kernel of \( \rho_{\phi,\pi} : G_K \to \text{GL}_r(A_\pi) \). Recall that the mod \( \pi \) Carlitz character \( \chi_{\pi} : G_K \to \text{GL}_{F_{\pi}}(\mathcal{C}[\pi]) \simeq \mathbb{F}_\ell^\times \) is an analogue of the mod \( \ell \) cyclotomic character. For the field \( K_{\phi,\pi} := K(\phi[\pi]) \cap K(\mathcal{C}[\pi]) \), we can prove the next proposition in the same way as the abelian variety case.

Proposition 5.3. Then the following conditions are equivalent.

- (DR-1) \( K(\phi[\pi^\infty])/K_{\phi,\pi} \) is a pro-\( p \)-extension which is unramified at any finite place of \( L \) not lying above \( \pi \),
- (DR-2) \( \phi \) has good reduction at any finite place of \( K \) not lying above \( \pi \) and \( K(\phi[\pi])/K_{\phi,\pi} \) is a \( p \)-extension,
- (DR-3) \( \phi \) satisfies (D1) and (D2).

Remark 5.4. Unlike the abelian variety case, the field \( K(\phi[\pi]) \) may not contain \( K(\mathcal{C}[\pi]) \). For example, for \( x \in \mathbb{F}_q^\times \setminus \{1\} \), consider the rank-one Drinfeld module \( \phi \) over \( F \) determined by \( \phi_t = t + x\tau \) and suppose \( q \neq 2 \). Then the fields \( F(\phi[t]) \) and \( F(\mathcal{C}[t]) \) are generated by the roots of \( t + x\tau^{q-1} \) and \( t + T^{q-1} \), respectively. By Kummer theory, we see that \( F(\phi[t]) \neq F(\mathcal{C}[t]) \), so that \( F(\phi[t]) \not\supset F(\mathcal{C}[t]) \).

Proof of Proposition 5.3. Since the kernel of \( \text{GL}_r(A_\pi) \to \text{GL}_r(\mathbb{F}_\pi) \) is a pro-\( p \) group, the extension \( K(\phi[\pi^\infty])/K(\phi[\pi]) \) is always pro-\( p \). The extension \( K(\mathcal{C}[\pi])/K \) is unramified at any finite place of \( K \) not lying above \( \pi \) (Example 2.7). Hence the conditions (DR-1) and (DR-2) are equivalent by Proposition 2.5. Suppose that (DR-2) holds. Then the condition (DR-3) immediately follows from Lemma 5.4 for \( G = \text{Gal}(K(\phi[\pi])/K), \chi_0 = \chi_{\pi}|_G, N = \text{Ker}(\chi_{\pi}|_G) = \text{Gal}(K(\phi[\pi])/K_{\phi,\pi}), \) and \( V = \phi[\pi] \). Conversely, if (DR-3) holds, then the image \( \bar{\rho}_{\phi,\pi}(N) \) of \( N = \text{Gal}(K(\phi[\pi])/K_{\phi,\pi}) \) is contained in

\[
\begin{pmatrix}
1 \\
\vdots \\
* \\
1
\end{pmatrix} \in \text{GL}_r(\mathbb{F}_\pi),
\]

which is a Sylow \( p \)-subgroup of \( \text{GL}_r(\mathbb{F}_\pi) \). Since \( \bar{\rho}_{\phi,\pi}|_G \) is injective, we see that \( K(\phi[\pi])/K_{\phi,\pi} \) is a \( p \)-extension. \( \square \)
5.2. Non-emptiness of \( \mathcal{D}(K, r, \pi) \). In this subsection, we assume that the rank \( r \) divides \([K : F]\), and \( r = q^n \) for some \( n \geq 0 \). Namely we assume that the \( r \)-power map \( \mathbb{F}_q \rightarrow \mathbb{F}_q \) is identity. Under these assumptions, we would like to show that the set \( \mathcal{D}(K, r, \pi) \) must not be empty. If \( r = 1 \), then \( \mathcal{D}(K, 1, \pi) \) is not empty since the Carlitz module \( C \) always satisfies (D1) and (D2) by definition. Consider the case \( r > 1 \).

Lemma 5.5. \( K_s = K^{p^M} \).

Proof. Set \([K : F] = p^M\) for some \( M > 0 \). Since \( K \) is a purely inseparable extension of \( K_s \) of degree \( p^M \), the field \( K^{p^M} \) is contained in \( K_s \). Consider the sequence of fields \( K \supset K^p \supset \cdots \supset K^{p^M} \). [Ros Proposition 7.4] implies that each extension \( K^{p^M} / K^{p^{M+1}} \) is of degree \( p \). Hence \([K : K^{p^M}] = p^M = [K : K_s]\), which means that \( K_s = K^{p^M} \).

Since \( r \) divides \([K : F]\), Lemma 5.5 implies that \( K \) contains the field \( F^{1/r} \). In particular, the \( r \)-th root \( t^{1/r} \) of \( t \in A \) is contained in \( K \). Now we consider a new \( A \)-field structure \( \iota \) of \( K \) by \( \iota(t) = t^{1/r} \) and the rank-one Drinfeld module \( C' \) over the \( A \)-field \((K, \iota)\) determined by \( C'_t = \iota(t) + \iota = t^{1/r} + \iota \).

Lemma 5.6. The \( G_K \)-action on \( C'[\pi] \) is given by the mod \( \pi \) Carlitz character \( \chi_\pi \).

Proof. It suffices to show that \( C'[\pi] \) and \( C[\pi] \) are isomorphic as \( \mathbb{F}_\pi[G_K] \)-modules, where \( C \) is the Carlitz module. For any polynomial \( \mu = \sum c_i \tau^i \in K\{\tau\} \), set \( (\cdot)^r \mu := \sum c_i^r \tau^i \). Then the map \( K\{\tau\} \rightarrow K\{\tau\}; \mu \mapsto (\cdot)^r \mu \) is an \( \mathbb{F}_\pi \)-algebra homomorphism and \( (\cdot)^r C'_a = C_a \) holds for any \( a \in A \). For any \( \lambda \in \pi \cdot K^{\text{sep}} \), we see that \( (C'_a(\lambda))^r = (\cdot)^r C'_a(\lambda^r) = C_a(\lambda) \). This means that \( (\cdot)^r : \pi \cdot K^{\text{sep}} \rightarrow \pi \cdot K^{\text{sep}} \) is \( A \)-linear and it is trivially \( G_K \)-equivariant. Moreover if \( \lambda \in C'[\pi] \), then \( \lambda^r \) is contained in \( C[\pi] \) since \( C_\pi(\lambda^r) = (C'_a(\lambda))^r = 0 \). Hence the map induces the isomorphism \( (\cdot)^r : C'[\pi] \rightarrow C[\pi] \) of \( \mathbb{F}_\pi \)-vector spaces. Indeed it is trivially injective and both \( C'[\pi] \) and \( C[\pi] \) are of dimension one over \( \mathbb{F}_\pi \).

Now the \( r \)-power map \( (\cdot)^r : A \rightarrow A \) is \( \mathbb{F}_\pi \)-linear and the composite \( \iota \circ (\cdot)^r : A \rightarrow K \) coincides with the inclusion \( A \hookrightarrow F \subset K \). Then the composite \( \Phi : A \xrightarrow{(\cdot)^r} A \xrightarrow{C'} K\{\tau\} \) is a rank-\( r \) Drinfeld module over \( K \) determined by \( \Phi_t = C'_t = (t^{1/r} + \iota)^r \).

For any \( 1 \leq s \leq r \), set \( V_s := \{ \lambda \in \Phi K^{\text{sep}} ; C'_{a^s}(\lambda) = 0 \} \), which is a \( G_K \)-stable \( \mathbb{F}_\pi \)-subspace of \( \Phi[\pi] \) and \( V_r = \Phi[\pi] \) by the definition of \( \Phi \). Note that \( V_s = C'[\pi^s] \) as an abelian group, but \( A \)-module structures are different.

Lemma 5.7. Let \( i \) be the positive integer such that \( ir \equiv 1 \mod q_n - 1 \) and \( i < q_n - 1 \). Then the \( G_K \)-action on \( V_1 \) is given by \( \chi_i^\times \).

Proof. For any \( \lambda \in V_1 \) and \( \sigma \in G_K \), it is enough to show that \( \sigma(\lambda) = \chi_i(\sigma)^i \cdot \lambda \). Now \( V_1 = C'[\pi] \) as a set. Take an element \( a_\sigma \in A \) such that \( \bar{a}_\sigma = \sum a_\sigma \in A / (A/\pi A)^\times = \mathbb{F}_\pi^\times \). Since \( a_\sigma^{q_n} \equiv a_\sigma \mod \pi \) by the assumption on \( i \), we see that \( \chi_i(\sigma)^i \cdot \lambda = \bar{a}_\sigma \cdot \lambda = \Phi_{a_\sigma}^i(\lambda) = C'_{a^i_\sigma}(\lambda) = C'_a(\lambda) \).
On the other hand, regarding $\lambda$ as an element of $\mathcal{O}'[\pi]$, we obtain
\[ \sigma(\lambda) = \chi_{\pi}(\sigma) \cdot \lambda = C'_\alpha(\lambda) \]
by Lemma 5.6. Hence $\sigma(\lambda) = \chi_{\pi}(\sigma)^i \cdot \lambda$. \hfill \qed

**Proposition 5.8.** The Drinfeld module $\Phi$ satisfies (D1) and (D2).

**Proof.** For any finite place $v$ of $K$, clearly $v(t^{1/r}) \geq 0$, so that $\Phi_t$ has coefficients in $\mathcal{O}_{K_v}$ and its leading coefficient is $1 \in \mathcal{O}_{K_v}^\times$. Hence it has good reduction at $v$. Consider the filtration
\[ 0 \subset V_1 \subset \cdots \subset V_s = \Phi[\pi] \]
of $\Phi[\pi]$ defined as above. For any $1 < s \leq r$, the map $V_s \to V_1; \lambda \mapsto C'_{\pi^s_{v}}(\lambda)$ is a $G_K$-equivariant surjective $F$-linear map and its kernel is $V_{s-1}$. This implies that $\tilde{\rho}_{\Phi,\pi}$ is of the form
\[ \tilde{\rho}_{\Phi,\pi} \simeq \begin{pmatrix} \chi_{\pi}^i & \ast \\ \ast & \chi_{\pi}^i \end{pmatrix} \]
by Lemma 5.7. Hence $\Phi$ satisfies (D2). \hfill \qed

5.3. **Infiniteness of $\mathcal{D}(K, r, t)$**. Finally, we construct an infinite subset of $\mathcal{D}(K, r, t)$. In number field case, the set $\mathcal{D}(k, g, \ell)$ is always finite because of the Shafarevich conjecture proved by Faltings [Fal], which states that there exist only finitely many isomorphism classes of abelian varieties over fixed $k$ with fixed dimension $g$ which have good reduction outside a finite set of finite places of $k$. However, the Drinfeld module analogue of it does not hold:

**Example 5.9.** For any $a \in A$, consider the rank-2 Drinfeld module $\phi^{(a)} : A \to F\{\tau\}$ given by $\phi_{t}^{(a)} = t + a \tau + \tau^2$. We can easily see that $\phi^{(a)}$ has good reduction at any finite place of $F$. If a Drinfeld module $\psi$ over $F$ is isomorphic to $\phi^{(a)}$, then there exists an element $c \in F^\times$ such that $c\phi_{b}^{(a)} = \psi_{c}c$ for any $b \in A$. Hence $\psi$ is determined by
\[ \psi_t = t + c^{a-1}a \tau + c^{a-2}\tau^2. \]
Since $\psi$ also must have good reduction at any finite place of $F$, we obtain $c \in F_q^\times$. Hence, for two distinct monic elements $a, a' \in A$, we see that $\phi^{(a)}$ is not isomorphic to $\phi^{(a')}$. Therefore the set of isomorphism classes $\{[\phi^{(a)}]; a \in A\}$ is infinite.

Let $W$ be a $G_K$-stable one-dimensional $F_q$-vector space contained in $K^\text{sep}$ and write $\kappa_W : G_K \to \mathbb{F}_q^\times$ for the character attached to $W$. Set $P_W(T) := \prod_{\lambda \in W}(T - \lambda)$, which is an $F_q$-linear polynomial of the form
\[ P_W(T) = T^n + c_W T, \quad c_W := \left( \prod_{\lambda \in W\setminus\{0\}} \lambda \right) \in K^\times \]
by [Go, Corollary 1.2.2]. For any $c \in K^\times$, denote by $\bar{c} \in K^\times/(K^\times)^{q-1}$ the class of $c$ and by $\kappa_{(c)} : G_K \to \mathbb{F}_q^\times$ the character corresponding to $\bar{c}$ by the map $K^\times/(K^\times)^{q-1} \to \text{Hom}(G_K, \mathbb{F}_q)$ of Kummer theory.

**Lemma 5.10.** For the above element $c_W \in K^\times$, the character $\kappa_{(-c_W)}$ coincides with $\kappa_W$.

**Proof.** Since $\lambda^{q-1} = -c_W$ for any $\lambda \in W\setminus\{0\}$, the character $\kappa_{(-c_W)}$ is given by $\kappa_{(-c_W)}(\sigma) = \sigma(\lambda)/\lambda = \kappa_W(\sigma)$ for any $\sigma \in G_K$. \hfill \qed
Identify $\mathbb{F}_t = A/tA = \mathbb{F}_q$. Then $C[t]$ is a one-dimensional $\mathbb{F}_q$-subspace of $K^{\text{sep}}$ and $P_{C[t]}(T) = T^2 + tT$ by the definition of $C$. By Lemma 5.10, we see that $\chi_t = \kappa_{(-t)}$. Note that $\chi_t^i = \kappa_{((-t)^{r})}$ for any integer $i$.

Take $r$ elements $c_1, \ldots, c_r \in K^\times$. For any $1 \leq s \leq r$, define $f_s(\tau) := (\tau + c_s)(\tau + c_{s-1}) \cdots (\tau + c_1) \in K\{\tau\}$ and set $W_s := \text{Ker}(f_s : K^{\text{sep}} \to K^{\text{sep}})$, which is a $G_K$-stable $s$-dimensional $\mathbb{F}_q$-subspace of $K^{\text{sep}}$. Thus we obtain the filtration

$$0 = W_0 \subset W_1 \subset \cdots \subset W_r$$

of $\mathbb{F}_q[G_K]$-modules.

**Lemma 5.11.** The $\mathbb{F}_q$-linear representation $\bar{\rho} : G_K \to \text{GL}_{\mathbb{F}_q}(W_r) \simeq \text{GL}_r(\mathbb{F}_q)$ is of the form

$$\bar{\rho} \simeq \begin{pmatrix} \kappa_{(-c_1)} & \ast & \cdots & \ast \\ \ast & \kappa_{(-c_2)} & \ddots & \vdots \\ \vdots & \ddots & \ast \\ \kappa_{(-c_r)} & \cdots & \ast & \kappa_{(-c_r)} \end{pmatrix}.$$ 

**Proof.** For any $1 \leq s \leq r$, the quotient $W_s/W_{s-1}$ is isomorphic to $\text{Ker}(\tau + c_s : K^{\text{sep}} \to K^{\text{sep}})$ as an $\mathbb{F}_q[G_K]$-module. Hence each $W_s/W_{s-1}$ is embedded into $K^{\text{sep}}$. By Lemma 5.10 the action of $G_K$ on $W_s/W_{s-1}$ is given by $\kappa_{(-c_s)}$. \hfill $\square$

Fix $r$ integers $i_1, \ldots, i_r$ satisfying $\sum_{s=1}^r i_s = 1$. For any $m = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ such that $\sum_{s=1}^r m_s = 0$, consider the $\mathbb{F}_q$-algebra homomorphism $\phi^m : A \to K\{\tau\}$ given by

$$\phi^m_t = (-1)^{r-1} \prod_{s=1}^r (\tau - (-t)^{k_s}),$$

where $k_s = i_s + m_s(q-1)$ for any $1 \leq s \leq r$. Since $\sum_{s=1}^r k_s = 1$, the constant term of $\phi^m_t$ is $(-1)^{r-1} \prod_{s=1}^r (-(-t)^{k_s}) = (-1)^{2r} t = t$, so that $\phi^m$ is a rank-$r$ Drinfeld module over $K$.

**Proposition 5.12.** The isomorphism class $[\phi^m]$ is contained in $\mathcal{D}(K, r, t)$. Moreover, the mod $t$ representation attached to $\phi^m$ is of the form

$$\bar{\rho}_{\phi^m,t} \simeq \begin{pmatrix} \chi_t^{i_1} & \ast & \cdots & \ast \\ \ast & \chi_t^{i_2} & \ddots & \vdots \\ \vdots & \ddots & \ast \\ \chi_t^{i_r} & \cdots & \ast & \chi_t^{i_r} \end{pmatrix},$$

where $i_1, \ldots, i_r$ are above fixed integers.

**Proof.** For any finite place $v$ of $K$ not lying above $t$, since $-t \in \mathcal{O}_{K_v}$ and the leading coefficient of $\phi^m_t$ is $(-1)^{r-1}$, we see that $\phi^m$ has good reduction at $v$. Now $\phi^m[t]$ coincides with the kernel of $\prod_{s=1}^r (\tau - (-t)^{k_s})$. Applying Lemma 5.11 to $f_s = (\tau - (-t)^{k_s}) \cdots (\tau - (-t)^{k_1})$, we see that $\bar{\rho}_{\phi^m,t}$ is given as above since $\kappa_{((-t)^{k_s})} = \chi_t^{k_s} = \chi_t^{i_s}$ for any $1 \leq s \leq r$. \hfill $\square$

We can see that the set $\mathcal{D}(K, r, t)$ is infinite as follows. For any positive integer $m$, consider $(-m, 0, \ldots, 0, m) \in \mathbb{Z}^r$ and set $\phi^{m} := \phi^{(-m, 0, \ldots, 0, m)}$. Then $\phi^{m}_t = t + c_1 \tau + \cdots + c_{r-1} \tau^{r-1} + (-1)^{r-1} \tau^r$ for some $c_1, \ldots, c_{r-1} \in K$ and $c_{r-1} = (-t)^{m(q-1)} +$
\[ \sum_{s=2}^{r-1} (-t)^s + (-t)^{s+m(q-1)} \]  For any finite place \( u \) of \( K \) above \( t \), if \( m \) is sufficiently large, then
\[ u(c_{r-1}) = (i_1 - m(q - 1))u(-t) < 0, \]
hence we see that \( u(c_{r-1}) \to -\infty \) (as \( m \to \infty \)). On the other hand, by the same argument of example 5.9, any Drinfeld module \( \psi \) which is isomorphic to \( \phi^m \) must be of the form \( \psi_i = x^{-i} \phi_{i}^m x \) for some \( x \in \mathbb{F}_K^\times \). These facts imply that, for sufficiently large distinct positive integers \( m_1 \) and \( m_2 \), two Drinfeld modules \( \phi^m_1 \) and \( \phi^m_2 \) are not isomorphic. Therefore the subset \( \{ \phi^m; m \in \mathbb{Z}_{>0} \} \) of \( \mathcal{D}(K, r, t) \) is infinite.

**References**

[Ar] K. Arai, *Algebraic points on Shimura curves of \( \Gamma_0(p) \)-type*, J. reine angew. Math. 690 (2014), 179–202

[Bo] A. Bourdon, *A uniform version of a finiteness conjecture for CM elliptic curves*, Math. Res. Lett. 22, (2015), 403–416

[CL] I. Chen and Y. Lee, *Explicit isogeny theorem for Drinfeld modules*, Ph.D. thesis, Universiteit Gent (2001)

[Dri] V. G. Drinfeld, *Elliptic modules*, Math. USSR Sub. 23 (1974), 561–592

[Fal] G. Faltings, *Finiteness theorems for abelian varieties over number fields*, in: “Arithmetic Geometry”, G. Cornell, J. H. Silverman (eds.), Springer-Verlag (1986), 9–27

[Go] D. Goss, *Basic structures of function field arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete Volume 35, Springer-Verlag, Berlin (1996)

[Gal1] F. Gardeyn, *\( t \)-motives and Galois representations*, Ph.D. thesis, Universiteit Gent (2001)

[Gal2] F. Gardeyn, *A Galois criterion for good reduction of \( \tau \)-sheaves*, J. Number Theory 97 (2002), 447–471

[Gal3] F. Gardeyn, *The structure of analytic \( \tau \)-sheaves*, J. Number Theory 100 (2003), 332–362

[Gal4] F. Gardeyn, *Analytic morphisms of \( t \)-motives*, Math. Ann. 325 (2003), 795–828

[Ih] Y. Ihara, *Profinite braid groups, Galois representations and complex multiplications*, Ann. of Math. 123 (1986), 43–106

[Ki] W. Kim, *Galois deformation theory for norm fields and its arithmetic applications*, Thesis, The University of Michigan (2009)

[KS] V. Kumer and J. Scherk, *Effective versions of the Chebotarev density theorem for function fields*, C. R. Acad. Paris Sér. I Math. 319, no.6 (1994), 523–528

[L] D. Lombardo, *On the uniform Rasmussen-Tamagawa conjecture in the CM case*, arXiv:1511.09019

[Mar] B. Margaux, *The structure of the group \( G(k[t]) \): Variations on a theme of Soulé*, Algebra Number Theory 3, no.4 (2009), 393–409

[Oz1] Y. Ozeki, *Non-existence of certain Galois representations with a uniform tame inertia weight*, Int. Math. Res. Not. (2011), 2377–2395

[Oz2] Y. Ozeki, *Non-existence of certain CM abelian varieties with prime power torsion*, Tohoku Math. J. 65 (2013), 357–371

[Ros] M. Rosen, *Number Theory in Function Fields*, Graduate Texts in Mathematics 210, Springer-Verlag, New York (2002)

[RT1] C. Rasmussen and A. Tamagawa, *A finiteness conjecture on abelian varieties with constrained prime power torsion*, Math. Res. Lett. 15 (2008), 1223–1231

[RT2] C. Rasmussen and A. Tamagawa, *Arithmetic of abelian varieties with constrained torsion*, Trans. Amer. Math. Soc. 369 (2017), 2395–2424

[Se] J.-P. Serre, *Propriétés galoisiennes des points d’ordre fini des courbes elliptiques*, Invent. Math. 15 (1972), 259–331

[Se] J.-P. Serre, *Local Fields*, Graduate Texts in Mathematics 67, Springer, New York (1979)

[ST] J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. 88 (1968), 492–517

[Sou] C. Soulé, *Chevalley groups over polynomial rings*, in: “Homological group theory”, edited by C. T. C. Wall, London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press (1979), 359–367
[Tak] T. Takahashi, *Good reduction of elliptic modules*, J. Math. Soc. Japan **34**, no.3 (1982), 475–487

Department of Mathematics, Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

*E-mail address*: okumura.y.ab@m.titech.ac.jp