Abstract

Bell’s theorem states that quantum correlation function of two spins can not be represented as an expectation value of two classical random variables. Spin is described in Bell’s model by a single scalar random variable. We discuss another classical model of spin in which spin is described by a triple of classical random variables. It is shown that in this model the quantum correlation function can be represented as the expectation value of classical random variables. Implications of this result to the problem of local causality of quantum mechanics and relations with problems of moments are briefly mentioned.
1 Introduction

Bell’s theorem [1] states that there are quantum correlation functions that can not be represented as an average product of classical random observables. More specifically, the quantum correlation function of two spins can not be represented as the expectation value of two classical scalar random variables. There are many discussions of Bell’s theorem, see for example [2]-[24] for some recent references.

One often says that Bell deduced his inequality from realism and locality [7]. But in fact Bell uses not only realism and locality but also a special classical model of spin. Spin is described in Bell’s model by a single scalar classical random variable. However it is well known that spin operator has three components (three Pauli matrices). Therefore it is more natural to consider another classical model of spin in which spin is described by means of a triple of classical random variables.

The aim of this note is to show that if one uses the new classical model of spin then the quantum correlation function of two spins can be represented as the expectation value of local classical random variables. This should be contrasted with an interpretation of Bell’s theorem according to which one can not reproduce the quantum correlation function with classical probabilistic local model.

The paper is organized as follows. In the next section we remind the familiar derivation of Bell’s theorem. Then we describe the new classical model of spin and show that one can reproduce the quantum correlation function of two spins with classical local random variables. Finally, a relation of this result with the problem of moments and some its implications to the problem of local causality of quantum mechanics are briefly discussed.

2 Bell’s Theorem

Bell’s theorem says that the quantum-mechanical correlation function of two spins

\[ Q(a, b) = \langle \psi | \sigma \cdot a \otimes \sigma \cdot b | \psi \rangle = -a \cdot b \]  

(1)

can not be represented in the form

\[ Q(a, b) = \int f^{(1)}(a, \omega) f^{(2)}(b, \omega) dP(\omega) \]  

(2)
Here $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are unit 3-vectors, $\sigma \cdot a = \sigma_i a_i$ where $\sigma_i$ are Pauli matrices and

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1, -1\rangle - |-1, 1\rangle)$$

is a singlet state with spin 0. (We use as spin operators $\sigma_i$ instead of $\sigma_i/2$ just for simplicity of writing. One can make obvious rescaling to get the standard spin $1/2$ values.) Functions $f^{(1)}$ and $f^{(2)}$ should satisfy

$$|f^{(k)}(a, \omega)| \leq 1, \ k = 1, 2$$

and $dP(\omega)$ is a positive measure on some space $\Omega$ with $\int dP(\omega) = 1$ (such measure is called the probability measure). In quantum mechanics the space $\Omega$ is sometimes called the space of hidden variables.

In this approach quantum spin is represented by the random variable (field) $f(a, \omega)$ that takes values $\pm 1$. This description we call Bell’s model.

Let us remind that the familiar proof of Bell’s theorem uses the following inequality [2]

$$|C(a, b) - C(a, b') + C(a', b) + C(a', b')| \leq 2 \quad (3)$$

where

$$C(a, b) = \int f^{(1)}(a, \omega) f^{(2)}(b, \omega) dP(\omega)$$

is the classical correlation function. One can not set

$$C(a, b) = Q(a, b) = -ab$$

because there exist such vectors $(ab = a'b = a'b' = -ab' = \sqrt{2}/2)$ for which one has

$$|Q(a, b) - Q(a, b') + Q(a', b) + Q(a', b')| = 2\sqrt{2} \quad (4)$$

The last equality (4) contradicts to (3) and this proves Bell’s theorem.

### 3 Spin as a Random Variable

One concludes from Bell’s theorem that Bell’s model $f(a, \omega)$ of spin contradicts to quantum mechanics. In this section we describe another classical model of spin.
An interpretation of relation (2) is that one observer measures a projection of spin of a particle along vector $a$ while in a distant region of space a second observer measures the projection of spin of the second particle along vector $b$. Results of the measurements of the first observer are represented by a random variable $f^{(1)}(a, \omega)$ and results of the measurements of the second observer are represented by a random variable $f^{(2)}(a, \omega)$. The crucial restriction is that $|f^{(k)}| \leq 1$, $k = 1, 2$. Actually one can reduce problem to the case $f^{(k)} = \pm 1$, $k = 1, 2$.

It seems to us that it is not natural to describe quantum spin by means of the classical scalar random function $f(a, \omega)$. There are three components of the spin operator (Pauli matrices $\sigma_1$, $\sigma_2$ and $\sigma_3$) and therefore one tends to describe spin by using three classical random variables $\xi^{(1)}_i(\omega)$ and $\xi^{(2)}_i(\omega)$ that take values $\pm 1$. Let us show that there are random variables $\xi^{(1)}_i(\omega)$ and $\xi^{(2)}_i(\omega)$ such that

$$\int_0^1 \xi^{(1)}_i(\omega)\xi^{(2)}_j(\omega)dP(\omega) = -\delta_{ij} = \langle \psi | \sigma_i \otimes \sigma_j | \psi \rangle$$  \hspace{1cm} (5)

We choose the segment $[0, 1]$ as the space $\Omega$ with the measure $dP(\omega) = d\omega$ and set

$$\xi_1(\omega) = 1,$$

$$\xi_2(\omega) = \begin{cases} 
1, & \omega \in (\frac{1}{4}, \frac{1}{2}) \text{ or } \omega \in (\frac{3}{4}, 1) \\
-1, & \text{otherwise}
\end{cases}$$

$$\xi_3(\omega) = \begin{cases} 
1, & \omega \in (0, \frac{1}{2}) \\
-1, & \text{otherwise}
\end{cases}$$

Then one has

$$\int_0^1 \xi_i(\omega)\xi_j(\omega)d\omega = \delta_{ij}$$

Now if we take

$$\xi^{(1)}_i(\omega) = \xi_i(\omega), \quad \xi^{(2)}_i(\omega) = -\xi_i(\omega)$$

then we obtain (5).

Having random variables $\xi^{(1)}_i$ and $\xi^{(2)}_i$ one can represent the quantum correlation function as the expectation value of classical random variables

$$\langle \psi | \sigma_i a_i \otimes \sigma_j b_j | \psi \rangle = -a_i b_i = \int_0^1 \xi^{(1)}_i(\omega)a_i\xi^{(2)}_j(\omega)b_j d\omega$$  \hspace{1cm} (6)
Let us stress that we do not interpret $\xi_i(\omega)a_i$ as describing results of individual measurements along an arbitrary vector $a$. In this model individual measurements are described by random variables $\xi_i(\omega)$ which correspond to a fixed system of coordinates. One uses $\xi_i(\omega)a_i$ only to compute the expectation value of spins along the vector $a$.

4 Discussions and Conclusions

The essence of Bell’s theorem is that the following problem of moments has no solution (one takes $ab = \cos(\alpha - \beta)$ and $f^{(1)} = -f^{(2)}$ in (2))

$$\cos(\alpha - \beta) = \int f(\alpha, \omega)f(\beta, \omega)dP(\omega)$$

(7)

where one assumes

$$|f(\alpha, \omega)| \leq 1$$

(8)

Here the last condition (8) is crucial. It means that one describes spin by a single classical random variable which transforms as scalar under rotations in space. However it is well known that spin operator is not a scalar and it transforms as a vector under rotations. Therefore if we want to describe spin by means of classical random variables then it seems more natural to use not a single scalar random variable as J.S. Bell did but to use a triple of random variables as it was done in the previous section.

If we relax the condition (8) then one can solve the problem of moments. For example the following problem of moments

$$\cos(\alpha - \beta) = 2\int f(\alpha, \omega)f(\beta, \omega)dP(\omega)$$

(9)

has a solution

$$\cos(\alpha - \beta) = 2\int_{0}^{2\pi} \cos(\alpha - \omega)\cos(\beta - \omega)d\omega$$

There is no contradiction between representation (8) and Bell’s theorem. Indeed if we take $f^{(1)}(a, \omega) = \xi^{(1)}_i(\omega)a_i$ and $f^{(2)}(b, \omega) = \xi^{(2)}_i(\omega)b_i$ then the representation (8) has the form of (2) but now the condition (8) is not satisfied.

The spectral theorem of von Neumann [25] is relevant in this discussion. It states that if $A_i$, $i = 1, \ldots, n$ is a set of commuting observables (Hermitian
operators in a Hilbert space $\mathcal{H}$) then for any unit vector $\psi \in \mathcal{H}$ there exists a representation

$$\langle \psi | A_1 \cdots A_n | \psi \rangle = \int f_1(\omega) \cdots f_n(\omega) dP(\omega)$$

This representation is local in the sense that every $f_i(\omega)$ is a real function depending only on $A_i$ and on $\omega$. In particular one can apply this theorem in the case when the condition of locality means that $A_i$ is an operator in a Hilbert space $\mathcal{H}_i$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Therefore quantum correlation function of commutative observables can be always represented as an expectation value of local classical random variables.

Let us summarize the main properties of the classical models of spin discussed in this paper. In quantum mechanics spin is represented by three Pauli matrices $\sigma_i$ with standard vector transformation rules under rotations. In Bell’s model spin is represented by the random field $f(a, \omega) = \pm 1$. In this paper spin is represented by three random variables $\xi_i(\omega)$ with appropriate transformation rules under rotations. Bell’s model of spin can not reproduce the quantum correlation functions of two spins while the model with $\xi_i(\omega)$ can do it. Although the model with $\xi_i(\omega)$ is classical it does not provide a complete description of reality because it refuses to describe results of individual measurements along an arbitrary vector $a$. The model describes only results of individual measurements along the three preferred orthogonal vectors and also expectation values along an arbitrary vector.

To conclude, in this paper the new classical model of spin is discussed which perhaps can help in further considerations of problems of locality, reality, and causality in quantum mechanics. In particular it would be interesting to reconsider from this point of view the Einstein, Podolsky and Rosen paradox which in its original form is not equivalent to its Bohm’s spin formulation used in Bell’s consideration.

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