The clock ambiguity: Implications and new developments

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Abstract. We consider the ambiguity associated with the choice of clock in time reparameterization invariant theories. This arbitrariness undermines the goal of prescribing a fixed set of physical laws, since a change of time variable can completely alter the predictions of the theory. We review the main features of the clock ambiguity and our earlier work on its implications for the emergence of physical laws in a statistical manner. We also present a number of new results: We show that (contrary to suggestions in our earlier work) time independent Hamiltonians may quite generally be assumed for laws of physics that emerge in this picture. We also further explore the degree to which the observed Universe can be well approximated by a random Hamiltonian. We discuss the possibility of predicting the dimensionality of space, and also relate the 2nd derivative of the density of states to the heat capacity of the Universe. This new work adds to the viability of our proposal that strong predictions for physical laws may emerge based on statistical arguments despite the clock ambiguity, but many open questions remain.

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1. INTRODUCTION

Every theory that is invariant under time reparameterization presents a problem the moment we attempt quantization. Quantization gives a preferential role to time (in the definition of canonical variables) that cannot be fulfilled in a theory that is unaltered by its reparameterization. A prominent example of such a theory is given by General Relativity and in this context there have been extensive discussions of the problem (see, for example, [2] for an early treatment or [3] for an comprehensive review). An approach often used in cosmology is to work in “superspace” finding time as an “internal” variable after quantization. The invariance is imposed on the quantum states of the superspace $|\psi\rangle_S$ as a physical condition involving the Hamiltonian constraint,

$$\mathcal{H}|\psi\rangle_S = 0.$$ (1)

In [4, 5], we argued that such an approach carries an intrinsic arbitrariness in the choice of “clock” subspace that leads in turn to an arbitrariness in the predictions of the theory; the clock ambiguity. We showed that its implications are so severe that we may need to see the laws of physics as we know them as an approximate emergent phenomenon.

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1 To appear in the proceedings of the The Origins of Time’s Arrow[1].
By taking the clock ambiguity seriously, we look for the emergence of physical properties derived from a Hamiltonian evolution chosen randomly, corresponding to an absolute ambiguity in the choice of clock. In [5] we singled out quasiseparability as a crucial feature of physical laws needed to sustain observers, and argued that quasiseparability is optimally achieved through locality (and thus through local field theory). In that context, we find our result from [5] that any sufficiently large random Hamiltonian can be interpreted (to a sufficiently good approximation) as a local field theory encouraging: It suggests that combining the randomness suggested by the clock ambiguity with the need for quasiseparability could yield local field theory as a prediction.

In this work, section 2 reviews the clock ambiguity and sketches the basic approach we advocated in [5] to seek predictive power based on a statistical analysis. Section 3 gives a new result that shows that one can quite generally take the physical laws that emerge in our analysis to have a time independent Hamiltonian (this result is in contrast to assumptions we made in our earlier work). Section 4 reviews our analysis from [5] showing that any sufficiently large random Hamiltonian can be interpreted, to a good approximation, as a local field theory. In section 4.2 we extend that work to discuss the possibility of predicting the dimensionality of space, and apply our analysis to a non-standard distribution of random Hamiltonians in section 4.3, with interesting implications for higher orders in our Taylor series comparison of random Hamiltonians with field theories. After reviewing our thinking about gravity in this picture in section 5, we extend our treatment to gravitating systems in section 6 by relating the derivatives of the density of states to appropriate thermodynamic quantities which can be estimated for gravitating systems. The result of this extension, while very crude, is encouraging. We present our conclusions in section 7.

2. SUMMARY OF THE CLOCK AMBIGUITY

The clock ambiguity arises from the treatment of time as “internal” in time reparameterization invariant theories. “Internal time” means that a subsystem of the universe is identified as the time parameter or “clock” and time evolution is revealed by examining correlations between the rest of the universe and the clock subsystem. In quantum theories this picture is typically expressed in “superspace”, of which the clock system is a subspace.

In previous work [4, 5] we pointed out that regardless of how careful one is to describe a universe as obeying specific physical laws, the same state in the same superspace can equally well describe a completely different physical world with completely different time evolution. One only has to identity a different clock subsystem to find this new description. This is the clock ambiguity. We have shown that the clock ambiguity is absolute, in the sense that all possible systems experiencing all possible time evolution can be extracted from the same superspace state by a suitable choice of clock.

We refer the reader to this earlier work for the details [4, 5]. Here we quote the main result. We assume a discrete formalism which allows us to write the state in superspace as

$$|\psi\rangle_S = \sum_{ij} \alpha_{ij} |t_i\rangle_C |j\rangle_R \equiv \sum_{i} |t_i\rangle_C |\phi_i\rangle_R.$$  \hspace{1cm} (2)
Here the subscripts $S$, $C$ and $R$ relate to the decomposition of superspace $S$ according to $S = C \otimes R$, and refer to the full superspace, the clock subspace and the “rest” of the superspace respectively. The bases $|i\rangle_C$ and $|j\rangle_R$ span the clock and “rest” subspaces. The second equality defines (by summing over $j$) $|\phi_i\rangle_R$, giving the wavefunctions of the “rest” subspace at times $t_i$.

One can see that all the information about the state in the $R$ subspace and its time evolution is contained within the expansion coefficients $\alpha_{ij}$. In [4, 5] we show that arbitrary values $\alpha'_{ij}$ can result from expressing the same superspace state $|\psi\rangle_S$ according to suitable choices of the decomposition $S = C' \otimes R'$, or in other words, by making a suitable choice of clock. Thus any state evolving according to any Hamiltonian can be found, merely by choosing a new clock in the superspace.

One possible conclusion from the clock ambiguity is that the formalism that leads to this result must be wrong in some way (that in itself would have interesting implications). Otherwise, if we conclude that our fundamental theories really must have the clock ambiguity, the success of physics so far implies that it must be possible to come up with sharp predictions of specific physical laws, presumably based on some kind of statistical arguments, given that all possible physical laws are represented in the formalism.

In [5] we explored how one might go about formulating such a statistical analysis, and gave special emphasis to the quasiseparability of physical laws which seems so curial for our ability to survive and thrive as tiny observers. We noted that locality (as realized in the local field theories that describe the elementary particles and forces) is the ultimate origin of the quasiseparability we experience in our physical world. We also noted that in some sense local field theories give a maximal expression of quasi-locality. Thus we feel our result from [5], that any random Hamiltonian can yield a sufficiently good approximation to a local field theory is quite interesting. It suggests that the requirement of quasiseparability may universally lead to local field theories as one searches for emergent physical laws in theories with the clock ambiguity. We review and extend that result in section 4.

3. THE TIME INDEPENDENCE OF $H$

A randomly chosen clock leads to a randomly chosen set of $\alpha_{ij}$’s. Random $\alpha_{ij}$’s describe a randomly chosen state evolving under a random Hamiltonian. The lack of any a-priori reason to expect correlations between the $\alpha_{ij}$’s with different $i$ values suggests that in general the random Hamiltonian will be different for each time step (labeled by $i$). We discuss this issue in section III-B of [5].

However, in our earlier work we overlooked a rather simple point (kindly brought to our attention by Glenn Starkman [8]). The point is that $\alpha_{ij}$’s do not contain nearly enough information to specify a full Hamiltonian at each time. We can use this fact to add a requirement that the Hamiltonian is time independent without any loss of generality.

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2 We have recently learned that Chris Wetterich has considered very similar issues using the functional integral formalism[6, 7] .
assuming one does not take too many time steps. We show below that this constraint is very easy to meet.

A time step can be written as

$$\langle \psi(t_{i+1}) \rangle_R = [1 - i\hbar(\Delta t)\mathbf{H}(t_i)] \langle \psi(t_i) \rangle_R.$$  \hspace{1cm} (3)

By taking the inner product of this equation with $$\langle \psi(t_{i+1}) \rangle_R$$ one finds

$$1 = R \langle \psi(t_{i+1}) | 1 - i\hbar(\Delta t)\mathbf{H}(t_i) | \psi(t_i) \rangle_R.$$  \hspace{1cm} (4)

The inner product with $$\langle \psi^\perp(t_{i+1}) \rangle_R$$ gives

$$0 = R \langle \psi^\perp(t_{i+1}) | 1 - i\hbar(\Delta t)\mathbf{H}(t_i) | \psi(t_i) \rangle_R.$$  \hspace{1cm} (5)

where $$\langle \psi^\perp(t_{i+1}) \rangle_R$$ could be any one of $$N - 1$$ states orthogonal to $$\langle \psi(t_{i+1}) \rangle_R$$. As shown in Eqn. 2, the $$\alpha_{ij}$$ lead directly to the time evolving state vector $$\langle \psi(t_i) \rangle_R$$. One uses the information from the state vector at each time step to infer information about $$\mathbf{H}$$. Together Eqs. 4 and 5 give a total of $$N_R$$ complex (or $$2N_R$$ real) constraints on $$\mathbf{H}$$. Since a general $$N_R \times N_R$$ Hamiltonian has $$N_R^2$$ real degrees of freedom, the $$\alpha_{ij}$$’s do not contain enough information to define a full Hamiltonian at each time step. After all, the $$\alpha_{ij}$$’s only tell us about the evolution of a single state, whereas the Hamiltonian contains full information about the evolution of all possible states.

The fact that the Hamiltonian is highly underdetermined by a single time step can be exploited to add the condition that the Hamiltonian is time independent without loss of generality. As long as one is looking at no more than $$N_H/2$$ time steps, Eqs. 4 and 5 provide no more than $$N_H^2$$ real constraints which can be used to build at least one time independent Hamiltonian that describes the full time evolution. And to the extent that the $$\alpha_{ij}$$ are randomly generated, the Hamiltonians produced from the $$\alpha_{ij}$$’s should be randomly distributed as well. In fact, it seems reasonable to expect that the central limit theorem will give the distribution of Hamiltonians (generated by effectively inverting Eqs. 4 and 5) an enhanced degree of Gaussianity over whatever distribution generated the $$\alpha_{ij}$$’s.

For all this to work out, we need to constrain the number of time steps $$N_t$$ according to

$$N_t < N_H/2.$$  \hspace{1cm} (6)

We can estimate $$N_t$$ as the age of the Universe divided by the minimum time resolution $$\Delta t$$. Using arguments from section 4, $$\Delta t \equiv \Delta E$$ and the maximum value of $$\Delta E$$ (=? 10^{11} GeV) gives

$$N_t \approx \frac{\Delta E}{H_0} = \frac{10^{11} GeV}{10^{-42} GeV} = 10^{53}.$$  \hspace{1cm} (7)

By comparison, requiring a good match of the density of states to a field theory leads to Eqn. 11 giving

$$N_H \geq \frac{B E_M}{a E_0} \left[ 1 - \left( \frac{E_0 - E_S}{E_M} \right) \beta \right]^{-\gamma} \exp \left[ b \left( \frac{E_0}{\Delta k} \right)^{\alpha} \right]$$  \hspace{1cm} (8)

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The quantity $E_0/\Delta k$ in the exponent is the ratio of the energy of the Universe to the field theory $k$-space cutoff. Even choosing values from section 4 which minimize the bound on $N_H$ give exponentially large values for the exponent in Eqn. 8 and give lower bounds on $N_H$ which easily satisfy Eqn. 6 and validate the assumption of a time independent Hamiltonian\(^3\)

### 4. FIELD THEORY AND THE WIGNER SEMICIRCLE

#### 4.1. Our basic approach

The clock ambiguity implies that any random split of superspace into clock and rest subsystems should lead to a realization of “physical laws”. However, one expects that a random split would result in laws described by a random Hamiltonian. In [5] we discussed possible ways forward under those conditions. One thing we did was pose the question in the converse form to test this hypothesis. Namely, we evaluated the extent to which the known physical laws match to those derived from a random Hamiltonian evolution. In particular, we compared the spectrum of a free field theory, representing (approximately) the known physics, to the eigenvalue spectrum of random Hamiltonians.

Following [5], we do not undertake the project of specifically constructing field operators etc. in terms of the eigenstates of the Hamiltonian. This project is likely to be challenging, and is also likely to further involve a statistical analysis of different physical realizations consistent with the same eigenvalue spectrum and initial state $|\psi(t_1)\rangle_R$. We feel that our analysis at the level of the eigenvalue spectrum represents a first check of the viability of our line of reasoning, and we save the important question of defining field operators etc. for future stages of this work\(^4\).

The distribution of eigenvalues for a random Hamiltonian, represented as an $N_H \times N_H$ Hermitian matrix, follows under quite general assumptions [10] the Wigner semicircle rule in the large $N_H$ limit. Take, for example, the distribution of eigenvalues of a large Hermitian matrix with elements drawn from a Gaussian distribution depicted in Fig. 2.

On the other hand, the density of states for a free field theory grows, at large energies, like an exponential of a power of the energy. On the face of it, these two forms for $dN/dE$ are dramatically different. In order to press forward with the comparison we introduced a general parametrization for the random Hamiltonian and field theory spectral densities respectively:

$$
\frac{dN_R}{dE} = \begin{cases} 
    d^{N_H}_{E_M} \left(1 - \left(\frac{E-E_S}{E_M}\right)^\beta\right)^\gamma |E-E_S| < E_M , \\
    0 , \quad \text{otherwise} .
\end{cases}
$$

\(^3\) This argument appears to be very robust. For example, refining the time resolution to $\delta t = 1/M_P$ does not change the result at all.

\(^4\) When we presented this work at the *Origins of Time’s Arrow* meeting Lee Smolin drew our attention to work by Bennett *et al.* [9] which may offer a framework where specific symmetries and representations for elementary particles could be predicted in a scheme such as ours.
\[
\frac{dN_F}{dE} = \frac{B}{E} \exp \left\{ b \left( \frac{E}{\Delta k} \right)^\alpha \right\},
\]

where \( E_M \) and \( E_S \) represent the maximum eigenvalue of the random Hamiltonian and an offset energy between the two descriptions, \( \Delta k (\equiv 2\pi/L) \) is the resolution in \( k \)-space set by putting the field theory in a box of size \( L \) and \( B, b, \alpha \) and \( \gamma \) are dimensionless parameters.

Expanding both Eqns. 9 and 10 in a Taylor series around a given central energy \( E_0 = \rho R_H^3 = 10^{80} \text{GeV} \), corresponding to the current energy of the Universe, and trying to equate the results at each order in \( (E - E_0) \) we find the level of agreement between the two descriptions.

Equating the zeroth order sets the size of the space of the random Hamiltonian to be exponentially large:

\[
N_H = \frac{BE_M}{a E_0} \left[ 1 - \left( \frac{E_0 - E_S}{E_M} \right)^\beta \right]^{-\gamma} \exp \left[ b \left( \frac{E_0}{\Delta k} \right)^\alpha \right].
\]

Strictly speaking, this expression only gives a lower bound on \( N_H \), since we only really know upper bounds on \( \Delta k \).

Equating the first order (as well assuming equality at zeroth order) sets the offset energy \( E_S \) in terms of the energy of the Universe \( E_0 \) by the following implicit expression:

\[
-\beta \gamma \frac{E_0}{E_0 - E_S} \left( \frac{E_0 - E_S}{E_M} \right)^\beta = \alpha b \left( \frac{E_0}{\Delta k} \right)^\alpha.
\]

Assuming equality and 0th and 1st order, the relative difference at the second order is fixed and given by

\[
\Delta_2 \equiv \frac{\Delta dN}{dE} \bigg|_{E_0} \approx \frac{\alpha^2 b^2}{\gamma} \left( \frac{E_0}{\Delta k} \right)^{2\alpha} \frac{(E - E_0)^2}{E_0^2}.
\]

Table 1 shows the value of \( \Delta_2 \) for different values of the exponent \( \alpha \) in Eqn. 10, the field theory \( k \)-space lattice spacing \( \Delta k \) and the range of validity of the field theoretical description

\[
\Delta E = E - E_0
\]

which can be thought of in terms of a minimum timescale on which field theory is valid, given by \( \delta t \sim 1/\Delta E \). The idea is to check if the disagreement between the density of states of a random Hamiltonian and a free field theory at 2nd order, \( \Delta_2 \) can be “sufficiently small” for realistic parameters. We find that the parameter most critical to this analysis is \( \alpha \), and we discuss its value in the next section.
FIGURE 1. A plot of the density of eigenvalues for a random Hamiltonian using Eqn. 9 and a field theory using Eqn. 10 matching the zeroth and first order terms in a Taylor expansion around $E_0$ (the vertical line).

TABLE 1. Value of $\Delta_2$ for different choices of $\alpha$, $\Delta k$ and $\Delta E$. As in [5], values for $\Delta E$ are set by accelerator ($10^3 GeV$) or cosmic ray ($10^{11} GeV$) bounds. Values for $\Delta k$ are set by the photon mass bound ($10^{-25} GeV$) or the size of the Universe ($10^{-42} GeV$).

| $\alpha$ | $\Delta k (GeV)$ | $\Delta E (GeV)$ | $\Delta_2$ |
|----------|------------------|------------------|------------|
| $1/2$    | $10^{-25}$       | $10^3$           | $10^{-24.5}$ |
| $1/2$    | $10^{-25}$       | $10^{11}$        | $10^{-16.5}$ |
| $1/2$    | $10^{-42}$       | $10^3$           | $10^{-16}$  |
| $1/2$    | $10^{-42}$       | $10^{11}$        | $10^{-8}$   |
| $3/4$    | $10^{-25}$       | $10^3$           | $10^{1.8}$  |
| $3/4$    | $10^{-25}$       | $10^{11}$        | $10^{9.8}$  |
| $3/4$    | $10^{-42}$       | $10^3$           | $10^{14.5}$ |
| $3/4$    | $10^{-42}$       | $10^{11}$        | $10^{22.5}$ |
| $1$      | $10^{-25}$       | $10^3$           | $10^{28}$   |
| $1$      | $10^{-25}$       | $10^{11}$        | $10^{36}$   |
| $1$      | $10^{-42}$       | $10^3$           | $10^{45}$   |
| $1$      | $10^{-42}$       | $10^{11}$        | $10^{53}$   |

4.2. The value of $\alpha$ and the dimensions of space

The results for the density of states of a field theory in $1+1$ dimensions for bosons and fermions can be derived from different instances of the Cardy formula for conformal field theories in 2d [11]. This formula relates the entropy of the field theory to its energy
\( E \) and central charge \( c \) in the following way

\[
S = \log N(E) = \frac{1}{2\pi} \sqrt{\frac{c}{6}} (E - \frac{c}{24} ),
\]

and implies Eqn. 10 with exponent \( \alpha = 1/2 \). The asymptotic density of states can also be found for a conformal field theory in higher number of dimensions [12] and grows as 

\[ e^{F_E^{(d-1)/d} \text{ where } E_E \text{ is the extensive energy}} \]

where the Casimir energy \( E_C \) is taken into account the total energy \( E = E_E + E_C \) is sub-extensive and the dependence of the entropy on energy changes. Verlinde [13], based on holographic arguments, proposed that the Cardy formula is satisfied also in the case of higher dimensional field theories.

Taking the extensive energy expression for a field theory in \( 3 + 1 \) dimensions would fix the constant \( \alpha = 3/4 \) in our parametrization of the density of states Eqn. 10. A first assessment of table 1 indicates that the agreement between the field theory and random Hamiltonian would be poor (with \( \alpha = 3/4, \Delta_2 \gg 1 \) for all entries). An alternative interpretation might be to note that the transition from \( \alpha = 1/2 \) to \( \alpha = 3/4 \) in our table occurs roughly right at the point where \( \Delta_2 \) shifts from small to large values. Given that all our estimates are very rough at this stage, there may be a hint here of a way in which our methods could predict the three dimensions of space, as the maximum value consistent with a random Hamiltonian.

On the other hand, if we assume Verlinde is correct and use the universal Cardy formula, that implies \( \alpha = 1/2 \) for any \( d \). Then the difference \( \Delta_2 \) is negligible and random Hamiltonians give a density of states that appears strongly consistent with the field theoretical one, at the expense of any apparent preference for the value of \( d \).

### 4.3. Wigner’s tail

It may appear disturbing that we are attempting to match expressions Eqns. 9 and 10, the latter having positive second derivative everywhere while the former in the case of the Wigner semicircle is negative definite; the case depicted in Fig. 1. As discussed above, it may simply be the case that this discrepancy is negligible, and is not a problem.

One might also wonder if this may change if the perfectly Gaussian probability distribution is altered, for example, if the width of the distribution of eigenvalues is different in different energy ranges\(^5\). To be concrete, one may consider the distribution containing a small cubic piece. In such a case (studied in [14]) the exponent \( \gamma \) in the density of states may be changed from 1/2 (Wigner semicircle) to 3/2 which has regions of positive second derivative near the tails of the distribution as depicted in Fig. 2. This possibility is included in our parametrization given in Eqn. 9. The corresponding improvement in matching can be inferred from Eqn. 13; an increase in \( \gamma \) leads to a smaller relative difference \( \Delta_2 \).

Let us point out, as a curiosity, that a distribution highly distorted from Gaussianity might lead to a perfect matching with the field theory distribution. In fact, letting \( \gamma \) grow

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\(^5\) We thank Jaume Garriga for suggesting this direction of investigation.
FIGURE 2. A plot of the density of eigenvalues for a random Hamiltonian \((E_M = 1, N_H = 1000)\) in the cases of: a Gaussian distribution (black) giving rise to the Wigner Semicircle and Gaussian plus a cubic "interaction" term (gray) with concave tails.

makes the generalized random density of states (Eqn. 9) approach an exponential of the form of the field theory one (Eqn. 10). In order to see this we may take \(E_M \gg E_0 - E_S\) in Eqn. 12 to find

\[
-\gamma \left( \frac{E_0 - E_S}{E_M} \right)^\beta \approx \left[ \frac{\alpha}{\beta} \left( 1 - \frac{E_S}{E_0} \right) \right] b \left( \frac{E_0}{\Delta k} \right)^\alpha,
\]

and choose parameters such that the coefficient in brackets is approximately one. Therefore, we have that the random density of states has the form

\[
\frac{dN_R}{dE} = a \frac{N}{E_M} \left( 1 + \frac{x}{\gamma} \right)^\gamma,
\]

where, \(x \equiv -\gamma(E - E_S/E_M)^\beta \approx b(E/\Delta k)^\alpha\) for \(E \approx E_0\), that in the limit of large \(\gamma\) reproduces the exponential behavior of the field theory density of states. However, we don’t think that such a distortion of the distribution could be the outcome of a truly statistical averaging procedure. Furthermore, it seems contradictory to the spirit of this work to seek out an exotic distribution. That would appear to undermine the hope that the our methods could one day offer some real predictive power.

5. INCLUDING GRAVITY

In this and previous work we have not discussed gravity at length. In [5] we suggested that gravity could naturally emerge when a more general metric is allowed when interpreting a random Hamiltonian as a local field theory (vs. the Minkowski metric implicit in the discussion in section 4). In such a picture we do not expect a full consistent theory describing arbitrary spacetimes to emerge. It would be enough to get a theory of space-time that would be consistent for the actual state of the Universe and similar states. It
is not even clear, for example, that the full number of states associated with black hole entropy would need to be part of the spectrum in such a picture, since the microscopic properties of black holes do not really appear to be part of our physical world. It seems reasonable to proceed carefully with this issue, and avoid jumping to conclusions about gravity in this picture until some of these ideas have been worked out more systematically.

In the next section we will try a different approach. Specifically, we will relate the curvature of the Wigner semicircle to the specific heat of the Universe. In estimating the specific heat we use standard notions of the heat capacity of gravitating systems, and thereby implicitly introduce gravity into our analysis. We do this with the caveat that this approach may take us even further out on a limb than the other (admittedly speculative) ideas discussed elsewhere in this paper. Interestingly, the analysis in the next section yields intriguing results even when the more exotic forms of gravitational entropy (black hole and De Sitter entropy) are ignored. Thus the analysis of Section 6 seems to apply even in the context of the more conservative ideas about gravity reviewed in this section.

6. HEAT CAPACITY AND $N'''$

Here we return to curvature of the $dN/dE$ vs $E$ curve, i.e., the third derivative of $N(E)$, and estimate its value from a thermodynamic perspective. We will use the fact that the heat capacity (or its intensive counterpart, the specific heat) is a thermodynamic quantity related to $N'''$. As discussed in the previous section, we will incorporate gravity by considering thermodynamic quantities defined for gravitating systems such as black holes.

Our starting point is the standard canonical ensemble expression for the entropy of a system with energy in a range $\Delta E$ around a central energy $E_0$:

$$S = \log \left( \frac{dN(E_0)}{dE} \Delta E \right)$$  \hspace{1cm} (18)

This leads to

$$\frac{1}{T} \equiv \frac{dS}{dE} = \frac{d(\log(dN/dE) \Delta E))}{dE} = \frac{N'''}{N''},$$  \hspace{1cm} (19)

and using $C \equiv dE/dT$

$$\frac{1}{C} = \frac{d}{dE} \left( \frac{N'''}{N''} \right) = 1 - \frac{N'N'''}{N''^2}.$$  \hspace{1cm} (20)

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6 We find it intriguing that this picture bears some resemblance to approaches that explicitly reject a full “third quantized” superspace formalism, such as that discussed in [15].

7 This section differs significantly from the 1st version posted on the ArXiv. An error in Eqn. 25 of V1 (which is clearly dimensionally wrong) propagated to a number of places in that section. In this version we have corrected the error and subsequent discussion.
When discussing these thermodynamic quantities one must generally be careful to state what is being varied and what is being held fixed when differentiating. We will return to that question a bit later in this section.

Plugging the generalized Wigner form (Eqn. 9) for the density of states into Eqn. 20 gives
\[
\frac{1}{C} = 1 - \frac{(\gamma - 1)Q - (\beta - 1)}{2\beta^2 \gamma Q}.
\] (21)

Here \(1/Q = ((E_0 - E_S)/E_M)^\beta - 1\) is an exponentially small quantity if the order by order matching described in section 4.1 is performed. Thus, to an excellent approximation we have
\[
C = \left(1 + \frac{1 - \gamma}{2\beta^2 \gamma}\right)^{-1}.
\] (22)

Taking parameters around the Winger case (\(\gamma = 1/2\) and \(\beta = 2\)) gives \(C = 9/8\).

Originally, the motivation for this line of investigation was the following: The second derivative of the density of states of the Wigner semicircle has the opposite sign to that of the field theory density of states (as can be seen by inspecting Fig. 1). The heat capacity is related to the second derivative of the density of states, and is negative for strongly gravitating systems. Strongly gravitating systems dominate the entropy of the Universe, so perhaps the negative specific heat of strongly gravitating systems in the Universe allows one to more fully reconcile the density of states of real matter with the Wigner semicircle at second order. This idea is not realized however, because the the second derivative of the density of states is not related to the specific heat in a sufficiently simple way. For the cases we consider, the second derivative of the density of states remains positive, even when the specific heat is negative. Forced to abandon this simple idea, we none the less move forward with the comparison with thermodynamic quantities which still turns out to give interesting results.

Due to the additivity of the entropy, it will be convenient to work with the derivatives of entropy with respect to \(E\)
\[
d^nS/dE^n.
\] (23)

These quantities can be constructed by differentiating \(S(E)\) directly, or they can be constructed from other thermodynamic quantities. For example, the \(n = 2\) case can be related to the heat capacity using
\[
\frac{1}{T^2 C} = -\frac{d^2 S}{dE^2}
\] (24)

(which can be derived from Eqns. 19 and 20).

If we write the entropy of the Universe as a sum over different components (such as radiation and black holes) labeled by \(i\) one has
\[
\frac{d^2 S_{\text{tot}}}{dE^2} = \frac{d^2}{dE^2} \sum_i S_i = \sum_i S''_i = -\sum_i \frac{1}{T_i^2 C_i}.
\] (25)
The Wigner density of states (Eqn. 9) gives

\[ -\frac{d^2S}{dE^2} = \frac{N'' - N'N'''}{N'} = \frac{1 + \beta Q}{E - E_S} \frac{dS}{dE} = \gamma \beta \frac{(1 + Q)(1 + \beta Q)}{(E - E_S)^2}. \]  

(26)

We wish to compare Eqn. 26 with Eqn. 25. To do so we will either estimate \( T_i \) and \( C_i \) or \( S_i'' \) directly for the various components of the Universe. We consider four main contributions coming from radiation (R), black holes (BH), dark matter (DM) and dark energy (DE).

**Radiation:** To compute the radiation component we take a gas of photons with energy \( E_R = \rho_R H^3 = T_R^4 R \) and temperature \( T_R = 10^{-13} \text{GeV} \). Keeping the volume \( H^{-3} = (10^{-42} \text{GeV})^{-3} \) fixed we obtain

\[ C_R = 2 \times 10^{88}, \quad \frac{1}{T_R^2 C_R} = 10^{-62} \text{GeV}^{-2}, \]  

(27)

and entropy \( S_R = 4C_R/3 \sim 10^{88} \).

**Black Holes:** We use the total black hole entropy estimate of [16]:

\[ S_{BH}^{tot} = \sum_{gal} 4\pi \frac{M_{BH}^2(gal)}{m_{pl}^2} \sim 3.2 \times 10^{10} \frac{E_{BH}^{105 \text{GeV}}}{10^{10} M_\odot}, \]  

(28)

where the sum is over galaxies \( (N_{gal} \sim 10^{11}) \) within the volume \( H^{-3} \) and \( M_{BH}(gal) \) is the distribution of masses of supermassive black holes at the galactic cores, which we approximate here as being peaked at \( M = 10^7 M_\odot \). Using \( E_{BH} = N_{gal} M = 10^{11} 10^7 M_\odot = 10^{-5} E_0 \) and \( M_\odot = 10^{57} \text{GeV} \) (i.e., \( T_{BH} \sim 10^{64} \text{GeV} \)) we obtain

\[ C_{BH} = -2.1 \times 10^{91} \left( \frac{M}{10^7 M_\odot} \right)^2, \quad \frac{1}{T_{BH}^2 C_{BH}} \sim -10^{-38} \text{GeV}^{-2}. \]  

(29)

**Dark Matter:** We infer the dark matter temperature by equating the dark matter kinetic energy with thermal energy:

\[ T_{DM} \sim \left( \frac{v}{100 \text{km/s}} \right)^2 \frac{m_{DM}}{100 \text{GeV}} 10^{-4} \text{GeV} \sim 10^{-4} \text{GeV}, \]  

(30)

with \( m_{DM} \) being the mass of the dark matter particle. We consider that only a fraction of the energy differential \( dE \), of order \( v^2/c^2 \sim 10^{-3} \), goes into thermal energy. These leads to a dark matter heat capacity of order

\[ C_{DM} \sim \pm 10^{-6}, \quad \frac{1}{T_{DM}^2 C_{DM}} \sim \pm 10^{-2} \text{GeV}^{-2}. \]  

(31)

In virialized bound systems there would be a negative contribution coming from the gravitational energy twice as large as the kinetic component leading to a negative heat capacity (and the negative sign in Eqn. 31). Non-bound dark matter would contribute
with a positive sign. We allow both signs in Eqn. 31 because our analysis is not detailed enough to consider which effect dominates.

**Dark Energy:** We use the de Sitter entropy $S_{DE} = E^2/m_{Pl}^2 \sim 10^{120}$ (with $E \equiv \rho_{DE}H^{-3}$) giving

$$d^2S_{DE}/dE^2 \sim 2/m_{Pl}^2 \sim 10^{-40} \text{GeV}^{-2}$$  \hspace{1cm} (32)

with a temperature of order $T_{DE} \sim H_0 \sim 10^{-42} \text{GeV}$.

**Total for the Universe:** Because the Universe is comprised of different components which are not in equilibrium, we work with Eqn. 24 which is easy to treat as a sum of independent components. Plugging all four components into Eqn. 24 (with $i = \{R, BH, DM, DE\}$) leads to an expression of the form

$$\frac{1}{T^2C} = -\sum_i d^2S_i/dE^2,$$  \hspace{1cm} (33)

to be compared with $(1 + \beta Q)dS/(E_0 - E_i)dE$ (from Eqn. 26) for the random Hamiltonian.

We notice that the ratios of $S_i$ to $dS_i/dE = T_i^{-1}$ and of $dS_i/dE$ to $d^2S_i/dE^2 = T_i^{-2}C_i^{-1}$ for each component in the above estimates are all of order $E_0$. The regularity of these ratios makes it possible to reconcile the two descriptions if the following relation holds:

$$\frac{1 + \beta Q}{E_S - E_0} \sim \frac{1}{E_0},$$  \hspace{1cm} (34)

which at this point of our analysis does not lead to any inconsistency with our previous results since the parameter $E_S$ was still unconstrained.

Indeed, the generalized distributions we proposed, Eqns. (9) and (10), have more free parameters than constraining equations, Eqns.(11)-(13), even after setting $\alpha = 1/2$. Therefore, it appears that demanding consistency as we have done above does not produce onerous constrains on the system. A caveat to this conclusion could come from any insights that suggest that the properties of ratios of derivatives scaling as $E_0^{-1}$ is non-trivial for the actual Universe, but on the face of it this seems to be a straightforward result that obtains for a great variety of functional forms for $S(E)$.

An interesting feature of the above discussion is that it applies to a variety of different cases: The entropy and its various derivatives calculated above are clearly dominated by the contributions from $S_\Lambda$. But one could “conservatively” argue that $S_\Lambda$ is quite abstractly defined, and should not be allowed to contribute to comparisons with the Wigner density of states. Perhaps the Wigner density of states should only be equated with degrees of freedom that are more physically observable. Removing $S_\Lambda$ from the computation would allow $S_{BH}$ to dominate. Since ratios of derivatives of $S_{BH}$ have the same properties, the comparison with Wigner goes through unchanged. Similar arguments might cause one to leave out $S_{BH}$ as well. Then $S_{DM}$ dominates and again the analysis goes through.

Interestingly, if one considers the dark matter to be dominant, one can consider integrating the discussion here with the comparison of Wigner with field theory in Minkowski space in Sections 4.1 and 4.2. The possibility that most of the dark matter
entropy is in states that are only linearly perturbed gravitationally is consistent with current observations, and under those conditions it may be reasonable to combine the constraints presented here with those from Section 4. The value of $E_S$ needed to satisfy Eqn. (34) together with the field theory requirements is exponentially close to $-E_M$, half the width of the Wigner distribution, with $E_0 \ll E_M$. 

What are we to make of this comparison? We are trying to learn if the Wigner semicircle gives a sufficiently good approximation to the density of states of the Universe. Our current analysis assumes that it is possible to take the Wigner semicircle density of states in the vicinity of some energy $E_0$ and set up a correspondence with eigenstates of a Hamiltonian that describes the Universe more or less as we know it. In this section we assume this correspondence allows us to use the thermodynamic quantities as estimated above. Specifically, the differentiation with respect to $E$ should reflect the differences between the thermodynamic quantities calculated at $E_0$, and for a similar cosmological interpretation of the Wigner density of states an energy $dE$ away. A careful understanding of how the black holes, radiation, etc. change as one shifts by $dE$ and reinterprets the density of states cosmologically would be required to give our calculations more rigor (of the sort commonly expressed, for example, by holding specified properties fixed when differentiating thermodynamic quantities). In the absence of such rigor, we hope that the simple differentiations performed in this section give a reasonable approximation to the desired result.

The crudeness of our methods warrant a great deal of caution, but we still find it a curiosity, perhaps even an encouraging curiosity that our comparison yields results that are comparable within an order of magnitude, and possibly even with the right sign.

### 7. SUMMARY AND CONCLUSIONS

The clock ambiguity suggest that we must view physical laws as emergent from a random ensemble of all possible laws. We started this article with a review of our earlier work showing the origin of the clock ambiguity. We then outlined and expanded upon our earlier ideas about the central role of quasiseparability in such a statistical analysis, and discussed how this could lead to a prediction that local field theory should provide the basic form for physical laws. We have shown that (contrary our earlier assumptions) one can quite generally assume physical laws that emerge in this picture will have a time independent Hamiltonian. We reviewed our earlier work that shows how the density of states of a free field theory can be well approximated by a random Hamiltonian, and extended this work to include a possible predictive link to the number of dimensions of space. We also explored a higher order analysis that (favorably) compares the curvature of the density of states of a random Hamiltonian with that of the observed Universe using estimates of the specific heat of the various components of the Universe.

While most of our discussion here is rather heuristic, our new results all add to the case that a statistical approach to physical laws may indeed be viable. In the case of the time independence of the Hamiltonian, we feel we have presented a very solid result which gives a significant improvement over our earlier discussions. All in all, while many open questions remain that could ultimately undermine our approach, we feel that a statistical
approach to the emergence of physical laws remains an interesting possibility which has accumulated additional support from the work presented here.

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REFERENCES

1. A. Albrecht, and A. Iglesias, “The clock ambiguity: Implications and new developments,” in The Origins of Time’s Arrow, edited by J. Khoury, and L. Mersini-Houghton, New York Academy of Sciences Press, New York, 2008.
2. R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 117, 1595–1602 (1960).
3. C. J. Isham (1992), gr-qc/9210011.
4. A. Albrecht, Lect. Notes Phys. 455, 323–332 (1995), gr-qc/9408023.
5. A. Albrecht, and A. Iglesias, Phys. Rev. D77, 063506 (2008), arXiv:0708.2743[hep-th].
6. C. Wetterich, Nucl. Phys. B314, 40 (1989).
7. C. Wetterich, Nucl. Phys. B397, 299–338 (1993).
8. G. Starkman (2007), private communication.
9. D. Bennett, N. Brene, and H. Nielsen, Physica Scripta. T15, 158 (1987).
10. M. L. Mehta, Random Matrices, Academic Press, 1991.
11. J. L. Cardy, Nucl. Phys. B270, 186–204 (1986).
12. T. Banks (1999), hep-th/9911068.
13. E. P. Verlinde (2000), hep-th/0008140.
14. E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber, Commun. Math. Phys. 59, 35 (1978).
15. T. Banks, W. Fischler, and S. Paban, JHEP 12, 062 (2002), hep-th/0210160.
16. T. W. Kephart, and Y. J. Ng, JCAP 0311, 011 (2003), gr-qc/0204081.