Periods of generalized Tate curves

Takashi Ichikawa

Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502, Japan

MSC: 14H10, 14H15, 32G20, 11M32

Keywords: Generalized Tate curve, Period isomorphism, Gauss-Manin connection, Variation of Hodge structure, Monodromy weight filtration, Unipotent period, Multiple zeta value

ABSTRACT

A generalized Tate curve is a universal family of curves with fixed genus and degeneration data which becomes Schottky uniformized Riemann surfaces and Mumford curves by specializing parameters. For each generalized Tate curve, we give explicit formulas of the period isomorphism between its de Rham and Betti cohomology groups, and of the associated objects: Gauss-Manin connection, variation of Hodge structure and monodromy weight filtration. A remarkable fact is that similar formulas hold commonly for families of Riemann surfaces and of Mumford curves. Furthermore, we study the unipotent periods of a generalized Tate curve, and show their limits are described by multiple zeta values.

1. Introduction

The study of period isomorphisms between the de Rham and Betti cohomology groups of families of algebraic varieties is one of main subjects in algebraic geometry. Their behavior under degeneration of considering varieties is especially interesting, and the associated $p$-adic version, unipotent version and $p$-adic unipotent version are also studied. For a universal family of elliptic curves, these objects are described by hypergeometric functions and equations, however, there seems no similar description on curves of genus $> 1$. The aim of this paper is to give such explicit formulas for a universal family of general curves which are commonly applicable to complex and $p$-adic analytic cases using the theory of generalized Tate curves. Furthermore, we consider the unipotent version, and show the limits of unipotent periods are expressed by multiple zeta values.

A generalized Tate curve is introduced in [17] as a higher genus version of the Tate elliptic curve which gives a universal deformation of degenerate curves of fixed type. The base ring of a generalized Tate curve is taken to be primitive as far as possible, namely it consists of formal power series in deformation parameters whose
coefficients belong to the ring over \( \mathbb{Z} \) of moduli parameters of the degenerate curves. By specializing these parameters in \( \mathbb{C} \) (resp. a nonarchimedean complete valuation field), a generalized Tate curve gives families of Schottky uniformized Riemann surfaces (resp. Mumford curves).

In this paper, we give explicit formulas of abelian differentials defined on a generalized Tate curve called **universal differentials**, and show their variational formulas which imply an important fact that the universal differentials are stable, namely they have only logarithmic poles on the special fiber. By regarding the generalized Tate curve as a family of degenerating Riemann surfaces, we give the explicit description of the period isomorphism between the de Rham and Betti cohomology groups, and of the associated Gauss-Manin connection, variation of Hodge structure and monodromy weight filtration. This description is written by multiple moduli and deformation parameters in the category of formal geometry over \( \mathbb{Q} \), and is same to formulas for Mumford curves which is studied by Gerritzen [12] and de Shalit [7, 8] using Coleman integration [2, 3] in the special case of \( p \)-adic base fields.

As a further application of the above result, we study unipotent periods which give isomorphisms between the de Rham and Betti unipotent fundamental groups of curves. By the arithmeticity and stability of universal differentials, we give a \( \mathbb{Q} \)-structure and Hodge, weight filtrations on the unipotent de Rham bundle on a maximally degenerating family of curves. Furthermore, extending results of Deligne [5], Drinfel’d [9] and Hain [14] on the Knizhnik-Zamolodchikov-Bernard (KZB) equation in the case of genus 0 and 1, we show that the limits of the unipotent periods of a generalized Tate curve are expressed by multiple zeta values. These limits are concerned with the limit Hodge structure of the unipotent fundamental groups, and the \( p \)-adic unipotent version using Coleman integration in families of Mumford curves seems to give a higher genus extension of result of Furusho [10, 11].

2. Generalized Tate curve

2.1. Schottky uniformization

We review the theory of Schottky uniformization of Riemann surfaces. A Schottky group \( \Gamma \) of rank \( g \) is a free group with generators \( \gamma_i \in PGL_2(\mathbb{C}) \) (\( 1 \leq i \leq g \)) which map Jordan curves \( C_i \subset \mathbb{P}^1_\mathbb{C} = \mathbb{C} \cup \{\infty\} \) to other Jordan curves \( C_{-i} \subset \mathbb{P}^1_\mathbb{C} \) by reversing orientation, where \( C_{\pm 1}, \ldots, C_{\pm g} \) with their interiors are mutually disjoint. Each element \( \gamma \in \Gamma - \{1\} \) is conjugate to an element of \( PGL_2(\mathbb{C}) \) mapping \( z \) to \( \beta_\gamma z \) for some \( \beta_\gamma \in \mathbb{C}^\times \) with \( |\beta_\gamma| < 1 \) which is called the multiplier of \( \gamma \). Therefore, we have

\[
\frac{\gamma(z) - \alpha_\gamma}{z - \alpha_\gamma} = \beta_\gamma \frac{\gamma'(z) - \alpha'_\gamma}{z - \alpha'_\gamma}
\]

for some element \( \alpha_\gamma, \alpha'_\gamma \) of \( \mathbb{P}^1_\mathbb{C} \) called the attractive, repulsive fixed points of \( \gamma \) respectively. Then the discontinuity set \( \Omega_\Gamma \subset \mathbb{P}^1_\mathbb{C} \) under the action of \( \Gamma \) has a fundamental domain \( D_\Gamma \) given by the complement of the union of the interiors of \( C_{\pm i} \), and \( R_\Gamma = \Omega_\Gamma / \Gamma \).
is a (compact) Riemann surface of genus $g$ which is called Schottky uniformized by $\Gamma$ (cf. [23]). Furthermore, by a result of Koebe, every Riemann surface of genus $g$ can be represented in this manner.

2.2. Degenerate curve

We review a correspondence between certain graphs and degenerate pointed curves, where a (pointed) curve is called degenerate if it is a stable (pointed) curve and the normalization of its irreducible components are all projective (pointed) lines. A graph $\Delta = (V, E, T)$ means a collection of 3 finite sets $V$ of vertices, $E$ of edges, $T$ of tails and 2 boundary maps

$$b : T \to V, \quad b : E \to (V \cup \{\text{unordered pairs of elements of } V\})$$

such that the geometric realization of $\Delta$ is connected. A graph $\Delta$ is called stable if its each vertex has at least 3 branches. Then for a degenerate pointed curve, its dual graph $\Delta = (V, E, T)$ is given by the correspondence:

$$V \leftrightarrow \{\text{irreducible components of the curve}\},$$
$$E \leftrightarrow \{\text{singular points on the curve}\},$$
$$T \leftrightarrow \{\text{marked points on the curve}\}$$

such that an edge (resp. a tail) of $\Delta$ has a vertex as its boundary if the corresponding singular (resp. marked) point belongs to the corresponding component. For a a finite set $X$, let $\sharp X$ denote the number of elements of $X$. Then under fixing a bijection $\nu : T \to \{1, ..., \sharp T\}$, which we call a numbering of $T$, a stable graph $\Delta = (V, E, T)$ becomes the dual graph of a degenerate $\sharp T$-pointed curve of genus rank $\sharp \mathcal{H}_1(\Delta, \mathbb{Z})$, where each tail $h \in T$ corresponds to the $\nu(h)$th marked point. In particular, a stable graph without tail is the dual graph of a degenerate (unpointed) curve by this correspondence. If $\Delta$ is trivalent, i.e. any vertex of $\Delta$ has just 3 branches, then a degenerate $\sharp T$-pointed curve with dual graph $\Delta$ is maximally degenerate.

An orientation of $\Delta = (V, E, T)$ means giving an orientation of each $e \in E$. Under an orientation of $\Delta$, denote by

$$\pm E = \{e, -e \mid e \in E\}$$

the set of oriented edges, and by $v_h$ the terminal vertex of $h \in E$ (resp. the boundary vertex $b(h)$ of $h \in T$). For each $h \in \pm E$, denote by let $|h| \in E$ be the edge $h$ without orientation.

2.3. Generalized Tate curve

Let $\Delta = (V, E, T)$ be a stable graph. Fix an orientation of $\Delta$, and take a subset $E$ of $\pm E \cup T$ whose complement $E_\infty$ satisfies the condition that

$$\pm E \cap E_\infty \cap \{-h \mid h \in E_\infty\} = \emptyset.$$
and that \( v_h \neq v_{h'} \) for any distinct \( h, h' \in \mathcal{E}_\infty \). We attach variables \( x_h \) for \( h \in \mathcal{E} \) and \( y_e = y_{-e} \) for \( e \in E \). Let \( A_0 \) be the \( \mathbb{Z} \)-algebra generated by \( x_h \ (h \in \mathcal{E}) \), \( 1/(x_e - x_{-e}) \) \((e, -e \in \mathcal{E}) \) and \( 1/(x_h - x_{h'}) \) \((h, h' \in \mathcal{E} \text{ with } h \neq h' \text{ and } v_h = v_{h'}) \), and put

\[
A_\Delta = A_0[[y_e \ (e \in E)]], \quad B_\Delta = A_\Delta \left[ \prod_{e \in E} y_e^{-1} \right].
\]

According to [17, Section 2], we construct the universal Schottky group \( \Gamma \) associated with oriented \( \Delta \) and \( \mathcal{E} \) as follows. For \( h \in \pm E \), let \( \phi_h \) be the element of \( PGL_2(B_\Delta) = GL_2(B_\Delta)/B_\Delta^\times \) given by

\[
\phi_h = \begin{pmatrix} x_h - x_{-h} y_h & -x_h x_{-h} (1 - y_h) \\ 1 - y_h & -x_{-h} + x_h y_h \end{pmatrix} \mod (B_\Delta^\times),
\]

where \( x_h \) (resp. \( x_{-h} \)) means \( \infty \) if \( h \) (resp. \( -h \)) belongs to \( \mathcal{E}_\infty \). Then

\[
\frac{\phi_h(z) - x_h}{z - x_h} = y_h \frac{\phi_h(z) - x_{-h}}{z - x_{-h}} \ (z \in \mathbb{P}^1),
\]

where \( PGL_2 \) acts on \( \mathbb{P}^1 \) by linear fractional transformation.

We review the following proposition which is shown in [16, Proposition 3.2] and [17, Lemma 1.2]:

**Proposition 2.1.** Let \( \phi \) be a product \( \phi_{h(1)} \cdots \phi_{h(l)} \) with \( v_{-h(i)} = v_{h(i+1)} \) \((1 \leq i \leq l-1) \) which is reduced in the sense that \( h(i) \neq -h(i + 1) \) \((1 \leq i \leq l - 1) \). If \( a \in A_\Delta \) satisfies \( a - x_{-h(l)} \in A_\Delta^\times \), then \( \phi(a) - x_{h(1)} \in I \). Furthermore, if \( a' - x_{-h(l)} \in A_\Delta^\times \), then \( \phi(a) - \phi(a') \in (a - a') y_\phi A_\Delta \), where \( y_\phi = y_{h(1)} \cdots y_{h(l)} \). Furthermore,

\[
\frac{d\phi(z)}{dz} \in y_\phi \left( A_\Delta \left[ \prod_{h \in \pm E} (z - x_h)^{-1} \right] \right).
\]

For any reduced path \( \rho = h(1) \cdot h(2) \cdots h(l) \) which is the product of oriented edges \( h(1), \ldots, h(l) \) such that \( v_{h(i)} = v_{-h(i+1)} \), there is an element \( \rho^* \) of \( PGL_2(B_\Delta) \) with reduced expression \( \phi_{h(l)} \phi_{h(l-1)} \cdots \phi_{h(1)} \). Fix a base vertex \( v_0 \) of \( V \), and consider the fundamental group \( \pi_1(\Delta, v_0) \) which is a free group of rank \( g = \text{rank}_\mathbb{Z} H_1(\Delta, \mathbb{Z}) \). Then the correspondence \( \rho \mapsto \rho^* \) gives an injective anti-homomorphism \( \pi_1(\Delta, v_0) \to PGL_2(B_\Delta) \). We denote this image by \( \Gamma_\Delta \).

It is shown in [17, Section 3] and [18, 1.4] (see also [19, Section 2] when \( \Delta \) is trivalent and has no loop) that for any stable graph \( \Delta \), there exists a stable pointed curve \( C_\Delta \) of genus \( g \) over \( A_\Delta \) which satisfies the following:

- The closed fiber \( C_\Delta \otimes_{A_\Delta} A_0 \) of \( C_\Delta \) obtained by substituting \( y_e = 0 \ (e \in E) \) becomes the degenerate pointed curve over \( A_0 \) with dual graph \( \Delta \) which is obtained from the collection of \( P_v := \mathbb{P}^1_{A_0} \ (v \in V) \) by identifying the points \( x_e \in P_{v_e} \) and \( x_{-e} \in P_{v_{-e}} \ (e \in E) \), where \( x_h \) denotes \( \infty \) if \( h \in \mathcal{E}_\infty \).
• $C_\Delta$ gives rise to a universal deformation of degenerate pointed curves with dual graph $\Delta$. More precisely, if $R$ is a noetherian and normal complete local ring with residue field $k$, and $C$ is a stable pointed curve over $R$ with nonsingular generic fiber such that the closed fiber $C \otimes_R k$ is a degenerate pointed curve with dual graph $\Delta$, in which all double points are $k$-rational, then there exists a ring homomorphism $A_\Delta \to R$ giving $C_\Delta \otimes_{A_\Delta} R \cong C$.

• $C_\Delta \otimes_{A_\Delta} B_\Delta$ is smooth over $B_\Delta$ and is Mumford uniformized (cf. [21]) by $\Gamma$.

• Take $x_h (h \in E)$ as complex numbers such that $x_e \neq x_{-e}$ and that $x_h \neq x_{h'}$ if $h \neq h'$ and $v_h = v_{h'}$, and take $y_e (e \in E)$ as sufficiently small nonzero complex numbers. Then $C_\Delta$ becomes a pointed Riemann surface which is Schottky uniformized by the Schottky group $\Gamma$ over $C$ obtained from $\Gamma_\Delta$.

We review the construction of $C_\Delta$ given in [17, Theorem 3.5]. Let $T_\Delta$ be the tree obtained as the universal cover of $\Delta$, and denote by $P_{T_\Delta}$ be the formal scheme as the union of $P^1_{A_\Delta}$’s indexed by vertices of $T_\Delta$ under the $B_\Delta$-isomorphism by $\phi_e (e \in E)$. Then it is shown in [17, Theorem 3.5] that $C_\Delta$ is the formal scheme theoretic quotient of $P_{T_\Delta}$ by $\Gamma_\Delta$.

3. Universal differential

3.1. Universal differential

We define universal differentials on a generalized Tate curve. Let $\Delta$ be a stable graph, and let $\Gamma_\Delta = \text{Im} (\pi_1(\Delta, v_b) \to PGL_2(B_\Delta))$ be the universal Schottky group as above. Then it is shown in [17, Lemma 1.3] that each $\gamma \in \Gamma_\Delta - \{1\}$ has its attractive (resp. repulsive) fixed points $\alpha$ (resp. $\alpha'$) in $P^1_{B_\Delta}$ and its multiplier $\beta \in \sum_{e \in E} A_\Delta \cdot y_e$ which satisfy

$$\frac{\gamma(z) - \alpha}{z - \alpha} = \beta \frac{\gamma(z) - \alpha'}{z - \alpha'}.$$ 

Fix a set $\{\gamma_1, ..., \gamma_g\}$ of generators of $\Gamma_\Delta$, and for each $\gamma_i$, denote by $\alpha_i$ (resp. $\alpha_{-i}$) its attractive (resp. repulsive) fixed points, and by $\beta_i$ its multiplier. Then under the assumption that there is no element of $\pm E \cap \mathcal{E}_\infty$ with terminal vertex $v_b$, for each $1 \leq i \leq g$, we define the associated universal differential of the first kind as

$$\omega_i = \sum_{\gamma \in \Gamma_\Delta/\langle \gamma_i \rangle} \left( \frac{1}{z - \gamma(\alpha_i)} - \frac{1}{z - \gamma(\alpha_{-i})} \right) \, dz.$$ 

Assume that

$$\{h \in \pm E \cap \mathcal{E}_\infty \mid v_h = v_b\} = \emptyset, \quad \{t \in T \mid v_t = v_b\} \neq \emptyset.$$
Then for each \( t \in T \) with \( v_t = v_b \) and \( m > 1 \), we define the associated \textit{universal differential of the second kind} as

\[
\omega_{t,m} = \sum_{\gamma \in \Gamma \Delta} \frac{d\gamma(z)}{(\gamma(z) - x_t)^m}.
\]

Furthermore, put \( T_\infty = \{ t \in T \cap E_\infty \mid v_t = v_b \} \) whose cardinality is 0 or 1, and take a maximal subtree \( T_\Delta \) of \( \Delta \), and for each \( t \in T \), take the unique path \( \rho_t = h(1) \cdots h(l) \) in \( T_\Delta \) from \( v_t \) to \( v_b \), and put \( \phi_t = \phi_{h(l)} \cdots \phi_{h(1)} \). Then for each \( t_1, t_2 \in T \) with \( t_1 \neq t_2 \), we define the associated \textit{universal differential of the third kind} as

\[
\omega_{t_1,t_2} = \sum_{\gamma \in \Gamma \Delta} \left( \frac{d\gamma(z)}{\gamma(z) - \phi_{t_1}(x_{t_1})} - \frac{d\gamma(z)}{\gamma(z) - \phi_{t_2}(x_{t_2})} \right),
\]

where \( \phi_{t_1}(x_{t_1}) = \infty \) if \( t_1 \in T_\infty \).

**Theorem 3.1.**

1. For each \( 1 \leq i \leq g \), \( \omega_i \) is a regular differential on \( C_{\Delta} \otimes_{A_{\Delta}} B_{\Delta} \) (cf. [20, §3]).
2. For each \( t \in T \) with \( v_t = v_b \) and \( m > 1 \), \( \omega_{t,m} \) is a meromorphic differential on \( C_{\Delta} \otimes_{A_{\Delta}} B_{\Delta} \) which has only pole (of order \( m \)) at the point \( p_t \) corresponding to \( t \).
3. For each \( t_1, t_2 \in T \) such that \( t_1 \neq t_2 \), \( \omega_{t_1,t_2} \) is a meromorphic differential on \( C_{\Delta} \otimes_{A_{\Delta}} B_{\Delta} \) which has only (simple) poles at the points \( p_{t_1} \) (resp. \( p_{t_2} \)) corresponding to \( t_1 \) (resp \( t_2 \)) with residue 1 (resp. -1).
4. Take \( x_h \ (h \in \pm E \cup T) \) and \( y_e \ (e \in E) \) be complex numbers as in 2.3. Then \( \omega_i, \omega_{t,m}, \omega_{t_1,t_2} \) are abelian differentials on the Riemann surface \( R_\Gamma \), where \( \Gamma \) is the Schottky group obtained from \( \Gamma_\Delta \).

**Proof.** The assertions (1)–(3) follow from the construction of \( C_{\Delta} \) reviewed in 2.3 since \( \omega_i, \omega_{t,m}, \omega_{t_1,t_2} \) are differentials on \( P_{T_\Delta} \) by Proposition 2.1, and are \( \Gamma_\Delta \)-invariant. Then we prove (4). As is stated in 2.1, \( R_\Gamma \) is given by the quotient space \( \Omega_\Gamma / \Gamma \). Under the assumption on complex numbers \( x_h \) and \( y_e \), it is shown in [23] that \( \sum_{\gamma \in \Gamma} |\gamma(z)| \) is uniformly convergent on any compact subset in \( \Omega_\Gamma - \cup_{\gamma \in \Gamma} \gamma(\infty) \), and hence the assertion holds for \( \omega_{\alpha_x,m} \). If \( a \in \Omega_\Gamma - \cup_{\gamma \in \Gamma} \gamma(\infty) \), then \( \lim_{n \to \infty} \gamma_1^{\pm n}(a) = \alpha_{\pm,i} \), and hence

\[
d \left( \int_a^{\gamma(a)} \sum_{\gamma \in \Gamma} \frac{d\gamma(z)}{\gamma(z) - z} \right) = \sum_{\gamma \in \Gamma} \left( \frac{1}{z - (\gamma \gamma_1)(a)} - \frac{1}{z - \gamma(a)} \right) dz
\]

\[
= \sum_{\gamma \in \Gamma / (\gamma_1)} \lim_{n \to \infty} \left( \frac{1}{z - (\gamma \gamma_1^n)(a)} - \frac{1}{z - (\gamma \gamma_1^{-n})(a)} \right) dz
\]

\[
= \omega_i(z).
\]

Therefore, \( \omega_i \) is absolutely and uniformly convergent on any compact subset in \( \Omega_\Gamma \), and hence is an abelian differential on \( \Omega_\Gamma / \Gamma \). \( \square \)
3.2. Stability of universal differentials

For a vertex \( v \in V \), denote by \( C_v \) the corresponding irreducible component of \( C_\Delta \otimes A_\Delta A_0 \). Then \( P_v = \mathbb{P}^1_{A_0} \) is the normalization of \( C_v \).

**Theorem 3.2.**

(1) For each \( 1 \leq i \leq g \), let \( \phi_{h_i(1)} \cdots \phi_{h_i(l_i)} \) (\( h_i(j) \in \pm E \)) be the unique reduced product such that \( v_{-h_i(j)} = v_{h_i(j+1)} \) and \( h_i(1) \neq -h_i(l_i) \) which is conjugate to \( \gamma_i \). Then for each \( v \in V \), the pullback \( (\omega_i|_{C_v})^* \) of \( \omega_i|_{C_v} \) to \( P_v \) is given by

\[
\left( \sum_{v_{h_i(j)} = v} \frac{1}{z - x_{h_i(j)}} - \sum_{v_{-h_i(k)} = v} \frac{1}{z - x_{-h_i(k)}} \right) dz.
\]

(2) For each \( v \in V \), \( (\omega_{t,m}|_{C_v})^* \) is given by \( \frac{dz}{(z - x_t)^m} \) if \( v = v_t \), and is 0 otherwise.

(3) Denote by \( \rho_{j} = h_j(1) \cdots h_j(l_j) \) the unique path from \( v_{t_j} \) (\( t_j \in T \)) to \( v_h \) in \( T_\Delta \). Then for each \( v \in V \), \( (\omega_{t_1,t_2}|_{C_v})^* \) is given by

\[
\left( \sum_{v_{h} = v} \frac{1}{z - x_h} - \sum_{v_{-k} = v} \frac{1}{z - x_{-k}} \right) dz,
\]

where \( h, k \) runs through \( \{t_1, h_1(1), ..., h_1(l_1), -t_2, -h_2(1), ..., -h_2(l_2)\} \).

**Proof.** For the proof of (1), we may assume that \( \gamma_i = \phi_{h_i(1)} \cdots \phi_{h_i(l_i)} \). Let \( \gamma \) be an element of \( \Gamma_\Delta \). Then by Proposition 2.1, putting \( y_e = 0 \) (\( e \in E \)),

\[
\frac{1}{z - \gamma(\alpha_i)} - \frac{1}{z - \gamma(\alpha_{-i})} = \frac{\gamma(\alpha_i) - \gamma(\alpha_{-i})}{(z - \gamma(\alpha_i))(z - \gamma(\alpha_{-i}))}
\]

becomes

\[
\frac{1}{z - x_{h_i(j)}} - \frac{1}{z - x_{-h_i(j-1)}}
\]

if \( j \in \{1, ..., l_i\} \) (\( h_i(j-1) := h_i(l_i) \) when \( j = 1 \), \( v(h_i(j)) = v \), \( \phi_{h_i(j)} \phi_{h_i(j+1)} \cdots \phi_{h_i(l_i)} \) belongs to \( \gamma(\gamma_i) \), and becomes 0 otherwise. Therefore, the assertion follows from the definition of \( \omega_i \).

The assertion (2) immediately follows from Proposition 2.1, and (3) can be shown in the same way as above. \( \square \)

**Remark.** Let the notation be as above, and for a family of Schottky uniformized Riemann surfaces obtained from \( C_\Delta \) as in 2.3, let \( c_e \) be a cycle corresponding to \( e \in E - E' \) which is oriented by the right-hand rule for \( \gamma_j \). Then the restriction of \( \omega_j \) to an irreducible component of \( C_\Delta \) is characterized analytically by that its integral along \( c_e \) is \( 2\pi \sqrt{-1} \) if \( e = |h_{j,k}| \) for some integer \( 1 \leq k \leq l(j) \), and is 0 otherwise. Therefore, Theorem 3.2 can be extended for this family degenerating as
\[ y_e \to 0 \quad (e \in E'). \] Since Schottky uniformized Riemann surfaces make a nonempty open subset in the moduli space of Riemann surfaces, by the theorem of identity in the function theory of several variables, the extended version of Theorem 3.2 also holds for families of general Riemann surfaces. This modification gives a more explicit formula than [15, Corollary 4.6].

A **stable differential** on a stable curve is defined as (regular or meromorphic) section of the dualizing sheaf on this curve ([6]). By Theorems 3.1 and 3.2, we have:

**Theorem 3.3.**

1. For each \( 1 \leq i \leq g \), \( \omega_i \) is a regular stable differential on \( C_\Delta \).
2. For each \( t \in T \) with \( v_t = v_b \) and \( m > 1 \), \( \omega_{t,m} \) is a meromorphic stable differential on \( C_\Delta \) which has only pole (of order \( m \)) at the point \( p_t \) corresponding to \( t \).
3. For each \( t_1, t_2 \in T \) with \( t_1 \neq t_2 \), \( \omega_{t_1,t_2} \) is a meromorphic stable differential on \( C_\Delta \) which has only (simple) poles at the points \( p_{t_1} \) (resp. \( p_{t_2} \)) corresponding to \( t_1 \) (resp. \( t_2 \)) with residue 1 (resp. \( -1 \)).

4. Period map, Gauss-Manin connection and variation of Hodge structure

**4.1. Period map**

Let \( \Delta = (V, E, T) \) be a stable graph, and assume that there are a vertex \( v_b \in V \) satisfying

\[
\{ h \in \pm E \cap E_\infty \mid v_h = v_b \} = \emptyset, \quad \{ t \in T \mid v_t = v_b \} \neq \emptyset.
\]

Then there is an element of \( T \) with terminal vertex \( v_b \) which we denote by \( t_0 \).

**Theorem 4.1.** Let \( C_\Delta^\circ = C_\Delta - \{ p_t \mid t \in T \} \) be the open curve over \( A_\Delta \). Then

\[
\omega_i \quad (1 \leq i \leq g), \quad \omega_{t_0,m} \quad (2 \leq m \leq g + 1), \quad \omega_{t_0,t} \quad (t \in T - \{ t_0 \})
\]

make a basis of \( H^1_{dR} (C_\Delta^\circ \otimes A_\Delta (B_\Delta \wedge \mathbb{Q})) \).

**Proof.** To prove this theorem, we can regard \( C_\Delta^\circ \) as a family of open Riemann surfaces obtained from \( C_\Delta \) by removing sections associated with \( p_t \) \( (t \in T) \). For small (counterclockwise oriented) closed paths \( a_i \) \( (1 \leq i \leq g) \) in \( C_\Delta^\circ \) around \( \alpha_i \),

\[
\int_{a_i} \omega_j = 2\pi \sqrt{-1} \delta_{i,j}, \quad \int_{a_i} \omega_{t_0,m} = \int_{a_i} \omega_{t_0,t} = 0,
\]

where \( \delta_{i,j} \) denotes the Kronecker delta. Furthermore, for small closed paths \( a_t \) \( (t \in T) \) in \( C_\Delta^\circ \) around \( p_t \),

\[
\int_{a_t} \omega_{t_0,t'} = 2\pi \sqrt{-1} \delta_{t,t_0} = -2\pi \sqrt{-1} \delta_{t,t'}, \quad \int_{a_t} \omega_j = \int_{a_t} \omega_{t_0,m} = 0.
\]
Let $b_i \ (1 \leq i \leq g)$ be a closed path in $C_{\Delta}$ corresponding to a path from $p_{\infty}$ to $\gamma_i(p_{\infty})$. Then by Proposition 2.1, $\int_{b_i} \omega_{t_0,m} \ (m > 1)$ belongs to $A_{\Delta} \otimes \mathbb{Q}$ whose constant term is

$$\int_{\infty}^{\alpha_i} \frac{dz}{(z - x_{t_0})^m} = \frac{1}{(-m + 1)(\alpha_i - x_{t_0})^{m-1}}$$

By Proposition 2.1, the difference between 2 distinct elements of $\{\alpha_{\pm i} \ (1 \leq i \leq g)\}$ belongs to $A_{\Delta}^\times$ times a certain product of $y_e \ (e \in E)$. Therefore, the assertion follows from that the Vandermonde determinant $\det ((\alpha_i - x_{t_0})^j)_{1 \leq i,j \leq g}$ is an invertible element of $B_{\Delta}$. □

4.2. Gauss-Manin connection and variation of Hodge structure

Let $\nabla$ denote the Gauss-Manin connection for $C_{\Delta} \otimes A_{\Delta} \ (B_{\Delta} \hat{\otimes} \mathbb{Q})$ regarded as a family of curves over $B_{\Delta} \hat{\otimes} \mathbb{Q}$.

**Theorem 4.2.**

(1) There exist elements $\eta_j \ (1 \leq j \leq g)$ of $H^1_{dR} (C_{\Delta} \otimes A_{\Delta} \ (B_{\Delta} \hat{\otimes} \mathbb{Q}))$ represented as $(B_{\Delta} \hat{\otimes} \mathbb{Q})$-linear sums of $\omega_{t_0,m} \ (2 \leq m \leq g + 1)$ which satisfy

$$\nabla(\omega_i) = \sum_{j=1}^{g} \eta_j \otimes (dP_{ij}/P_{ij}), \ \nabla(\eta_i) = 0 \ (1 \leq i \leq g),$$

where $P_{ij} \in B_{\Delta}$ denote the universal periods [17, Section 3] defined as $\exp \left( \int_{\gamma_i} \omega_j \right)$.

(2) The both sets $\{\eta_j \ | \ 1 \leq j \leq g\}$ and $\{\omega_{t_0,m} \ | \ 2 \leq m \leq g + 1\}$ give bases of the Hodge component $H^{0,1}$ of $H^1_{dR} (C_{\Delta} \otimes A_{\Delta} \ (B_{\Delta} \hat{\otimes} \mathbb{Q}))$.

**Proof.** By Theorem 4.1 and its proof, there are $(B_{\Delta} \hat{\otimes} \mathbb{Q})$-linear sums $\eta_j$ of $\omega_{t_0,m}$ such that $\int_{a_i} \eta_j = 0$ and $\int_{b_i} \eta_j = \delta_{i,j}$. Then the assertion (1) follows. Therefore, the Griffiths transversality implies that $\{\eta_j\}$ generates a sub $(B_{\Delta} \hat{\otimes} \mathbb{Q})$-module of $H^{0,1}$ with rank $g$ which is also generated by $\{\omega_{t_0,m}\}$. Then the assertion (2) follows. □

**Corollary 4.3.** Let $C/S$ be a family of Mumford curve over a $p$-adic field of characteristic 0 obtained from $C_{\Delta}$ as in 2.3. Then $\{\omega_i\}$ (resp. $\{\eta_i\}$) become $\{\alpha_i\}$ (resp. $\{\beta_i\}$) given in [12, Theorem 2] which are bases of the Hodge components $H^{1,0}(C)$ (resp. $H^{0,1}(C)$) given in [7, Theorems 2.8 and 2.9].

**Proof.** The universal differential $\omega_i$ becomes a differential of the first kind on $C$ whose residue (cf. [22]) for each edge in $E$ is constant, and $\eta_j$ becomes a differential of the second kind on $C$ whose Coleman integration (cf. [2, 3]) along $\gamma_i$ is $\delta_{i,j}$. Then the assertion follows from Theorems 2.8 and 2.9 of [7]. □

4.3. Monodromy weight filtration
Let $C_\Delta$ be the generalized Tate curves associated with a stable graph $\Delta = (V, E, T)$, and put $H = H^{1,0}_\text{DR}(C_\Delta \otimes_{\mathcal{A}_\Delta} (B_\Delta \otimes \mathbb{Q}))$. For a subset $E'$ of $E$, denote by

$$\{0\} \subset F^0_{E'} H \subset F^1_{E'} H \subset H, \quad N_{E'} : H/F^1_{E'} H \cong F^0_{E'} H$$

the monodromy weight filtration and operator with respect to the closed fiber of $C_\Delta$ obtained by $y_e$ ($e \in E'$).

**Theorem 4.4.**

(1) For each $e \in E$, let $\text{Res}_e$ be the linear map on $H$ defined by $\text{Res}_e (\omega_{t_0, m}) = 0$ and by

$$\text{Res}_e (\omega_i) = \hat{\{j \mid h_i(j) = e\} - \hat{\{k \mid h_i(k) = -e\}}},$$

where $\phi_{h_i(1)} \cdots \phi_{h_i(t)} (h_i(j) \in \pm E)$ denotes the reduced product with $v_{-h_i(j)} = v_{h_i(j+1)}$ which is conjugate to $\gamma_i$ ($1 \leq i \leq g$). Then we have

$$F^1_{E'} H = \text{Ker} \left( \bigoplus_{e \in E'} \text{Res}_e \right).$$

(2) Let $\Delta'$ be the graph obtained from $\Delta$ by shrinking each edge in $E - E'$ to one point. Then $\{\text{Res}_e\}_{e \in E'}$ gives rise to an isomorphism from $H/F^1_{E'} H$ onto the group $H^1(\Delta', B_\Delta \otimes \mathbb{Q})$ of singular cohomology classes. Furthermore, the line integral in $C_\Delta$ corresponding to cycles in $\Delta'$ gives an isomorphism

$$F^0_{E'} H \cong H^1(\Delta', B_\Delta \otimes \mathbb{Q}),$$

and $N_{E'}$ gives the identity map on $H^1(\Delta', B_\Delta \otimes \mathbb{Q})$ under the above isomorphisms.

**Proof.** The assertion (1) follows from that $\text{Res}_e$ gives the residue map for the family of Riemann surfaces associated with $C_\Delta$. Then by Theorem 4.2, $F^0_{E'} H = F^1_{E'} H$ has a basis $\{\omega_{t_0, m}\}$, and by Theorem 4.1, the line integrals from $p_\infty$ to $\gamma_i(p_\infty)$ ($1 \leq i \leq g$) gives an isomorphism $F^0_{E'} H \cong H^1(\Delta, B_\Delta \otimes \mathbb{Q})$. Furthermore, Theorem 4.2 implies that $N_E$ is the identity map on $H^1(\Delta, B_\Delta \otimes \mathbb{Q})$. Under this isomorphism, $F^0_{E'} H$ corresponds to $H^1(\Delta', B_\Delta \otimes \mathbb{Q}) \hookrightarrow H^1(\Delta, B_\Delta \otimes \mathbb{Q})$ which implies (2). 

4.4. Coleman integration in families of Mumford curves

We consider the problem of constructing Coleman integrals in families raised by Besser [1, 1.6] in the case of Mumford curves. Let $K$ be a $p$-adic field, namely a local field of characteristic $0$ whose residue field is of characteristic $p > 0$. Denote by $| \cdot |$ a multiplicative valuation on $K$, and by $\mathcal{S}_\Delta$ the $K$-analytic subspace of $K^E \times K^E$ consists of $(x_h, y_e)_{h \in E, e \in E}$ satisfying

$$|x_h| \leq 1, \quad |x_e - x_{-e}| = 1, \quad |x_h - x_{h'}| = 1 \quad (h, h' \in \mathcal{E} \text{ with } h \neq h' \text{ and } v_h = v_{h'}).$$

Then as is seen in 2.3, we have a family $C_\Delta/\mathcal{S}_\Delta$ of Mumford curves over $K$ from the generalized Tate curve $C_\Delta$ by specializing parameters. Therefore, the universal
differentials $\omega_i$ and $\omega_{t,m}$ in 3.1 give differentials on $C_{\Delta}/S_{\Delta}$ of the first and second kind respectively, and hence there exist their Coleman integrals given by [3] under fixing a logarithm homomorphism $\log : K^\times \to K$.

**Theorem 4.5.** The Coleman integrals of $\omega_i$ and $\omega_{t,m}$ (determined unique up to constants) on $C_{\Delta}/S_{\Delta}$ are given by

$$
\log \left( \prod_{\gamma \in \Gamma/(\gamma_i)} \frac{z - \gamma(\alpha_i)}{z - \gamma(\alpha_{-i})} \right)
$$

and

$$
\sum_{\gamma \in \Gamma} \frac{1}{1 - m} \left( \frac{1}{(\gamma(z) - x_i)^{m-1}} - \frac{1}{(\gamma(z_0) - x_i)^{m-1}} \right),
$$

where $\Gamma$ denotes Schottky groups over $K$ associated with $\Gamma_{\Delta}$, and $z_0$ is a point on $\mathbb{P}^1_K$ outside the limit set of $\Gamma$.

**Proof.** We use the description given by de Shalit [8] of Coleman integrals on semistable curves. Let $C_K$ be a Mumford curve over $K$ as a member of $C_{\Delta}$. Then it is shown in [8, 0.4 and 1.1] that there exists a cover $\tilde{C}_K$ of $C_K$ as a $K$-analytic space with action of $\Gamma$ such that $\tilde{C}_K/\Gamma \cong C_K$, and the Coleman integrals on $C_K$ are functions on $\tilde{C}_K$ obtained by anti-differentiating. Therefore, the assertion follows from the explicit formulas of $\omega_i$ and $\omega_{t,m}$. \hfill $\square$

**Corollary 4.6.** Let $C_K$ be a Mumford curve over a $p$-adic field $K$ obtained from $C_{\Delta}$ by the base change corresponding to a ring homomorphism $\varphi$ from $A_{\Delta}$ to the valuation ring of $K$, and put $E' = \{ e \in E \mid \varphi(y_e) = 0 \}$. Then the monodromy weight filtration and monodromy operator

$$
\{0\} \subset F^0_{E'} H \subset F^1_{E'} H \subset H, \quad N_{E'} = H/F^1_{E'} H \to F^0_{E'} H.
$$

given in Theorem 4.4 become the $p$-adic monodromy weight filtration and monodromy operator for $C_K$ (cf. [4]).

**Proof.** The assertion follows from Theorem 4.5 and [8, Theorem 0.1]. \hfill $\square$

5. Unipotent period

5.1. Unipotent fundamental group and period

Assume that $n$ is an integer $> 1$, and let $C^o$ be an algebraic curve over a subfield $K$ of $\mathbb{C}$ which is obtained from a proper smooth curve $C$ of genus $g$ by removing $n$ points. Since $C^o$ is not complete, $H^1_{\text{dR}}(C^o)$ has a basis $B_{C^o}$ consisting of $2g + n - 1$ meromorphic differentials on $C/K$ of the first or second kind which may have poles outside $C^o$. For $w_1, \ldots, w_m \in B_{C^o}$, we define a $\mathcal{D}$-module $\mathcal{D}(w_1, \ldots, w_m)$ on $C^o$ given
by the trivial bundle $K^{m+1} \times C^\circ$ with connection form $d - \sum_{i=1}^m e_{i,i+1}w_i$, where $e_{i,j}$ is the square matrix of degree $m + 1$ whose $(k,l)$-entry is $\delta_{i,k} \cdot \delta_{j,l}$. We consider the tannakian subcategory of $D$-modules on $C^\circ$ generated by $\mathcal{D}(w_1, \ldots, w_m) \ (w_i \in B_{C^\circ})$. Since these underlying bundles are trivial, for each $K$-rational point $x$ on $C^\circ$, one can define the fiber functor on this category by taking the (trivial) fibers over $x$. Denote by $\pi^1_{\text{an}}(C^\circ; x)$ the tannakian fundamental group of this category which becomes a profinite algebraic group over $K$, and by $\mathcal{A}^{\text{dR}}(C^\circ; x)$ the enveloping algebra of the Lie algebra $\text{Lie}(\pi^1_{\text{an}}(C^\circ; x))$.

Let $(C^\circ)^\text{an}$ be a Riemann surface associated with $C^\circ \otimes_K \mathbb{C}$, and for $K$-rational points $x, y$ on $C^\circ$, denote by $\pi_1((C^\circ)^\text{an}; x,y)$ the set of homotopy classes of paths from $x$ to $y$ in $(C^\circ)^\text{an}$. When $x = y$, $\pi_1((C^\circ)^\text{an}; x,x)$ becomes the fundamental group $\pi_1((C^\circ)^\text{an}; x)$ of $(C^\circ)^\text{an}$ with base point $x$. We consider the tannakian category of unipotent local systems on $(C^\circ)^\text{an}$ with fiber functor obtained from taking the fiber over $x$. Then it is shown in [5] that its tannakian fundamental group $\pi^1_{\text{Be}}((C^\circ)^\text{an}; x)$ is a pro-algebraic group over $\mathbb{Q}$, and the associated enveloping algebra $\mathcal{A}^\text{Be}((C^\circ)^\text{an}; x)$ of $\text{Lie}(\pi^1_{\text{Be}}((C^\circ)^\text{an}; x))$ is isomorphic to

$$\lim_{m \to \infty} \mathbb{Q}[\pi_1((C^\circ)^\text{an}; x)]/I^m,$$

where $I$ denotes the augmentation ideal of the group ring $\mathbb{Q}[\pi_1((C^\circ)^\text{an}; x)]$. Since $\pi_1((C^\circ)^\text{an}; x)$ is a free group of rank $2g + n - 1$, $\mathcal{A}^\text{Be}((C^\circ)^\text{an}; x)$ becomes the ring of noncommutative formal power series over $\mathbb{Q}$ in $2g + n - 1$ variables.

**Theorem 5.1.** There exists a canonical isomorphism

$$\pi^1_{\text{dR}}(C^\circ; x, y) \otimes_K \mathbb{C} \cong \pi^1_{\text{Be}}((C^\circ)^\text{an}; x, y) \otimes_{\mathbb{Q}} \mathbb{C}.$$

**Proof.** We may show the assertion in the case when $x = y$. For each $\mathcal{D}$-module $\mathcal{D}(w_1, \ldots, w_m)$, there exists the associate local system on $(C^\circ)^\text{an}$ described by the iterated integrals $\int w_1' \cdots w_r'$ for $r \leq m$ and $w'_i \in \{w_1, \ldots, w_m\}$. By this association, we have a group homomorphism

$$\pi^1_{\text{Be}}((C^\circ)^\text{an}; x) \otimes_{\mathbb{Q}} \mathbb{C} \to \pi^1_{\text{dR}}(C^\circ; x) \otimes_K \mathbb{C}$$

which gives a ring homomorphism

$$\varphi : \mathcal{A}^\text{Be}((C^\circ)^\text{an}; x) \otimes_{\mathbb{Q}} \mathbb{C} \to \mathcal{A}^\text{dR}(C^\circ; x) \otimes_K \mathbb{C}.$$

Let $B_m((C^\circ)^\text{an})$ be the $\mathbb{C}$-vector space of iterated integrals spanned by

$$\int w_1 \cdots w_r \ (w_i \in B_{C^\circ}, \ r \leq m),$$

and $H^0(B_m((C^\circ)^\text{an}); x)$ be the space consisting of elements of $B_m((C^\circ)^\text{an})$ whose restriction to $\{\text{loops in } (C^\circ)^\text{an} \text{ based at } x\}$ is homotopy functional. Then it is shown in
[13, (5.3)] that $H^0(B_m((C^o)^{an}); x)$ is the dual space of $\mathbb{C}[\pi_1((C^o)^{an}; x)]/I^{m+1}$, and that by taking leading terms of iterated integrals, we have an exact sequence

$$0 \to H^0(B_{m-1}((C^o)^{an}); x) \to H^0(B_m((C^o)^{an}); x) \to H^1_{\text{dR}}((C^o)^{an})^\otimes m.$$ 

Denote by $(I^m/I^{m+1})^\vee$ the dual space of $I^m/I^{m+1}$. Then there exists the associated injective linear map

$$(I^m/I^{m+1})^\vee \otimes_{\mathbb{Q}} \mathbb{C} \to H^1_{\text{dR}}((C^o)^{an})^\otimes m \cong H^1_{\text{dR}}(C^o)^{\otimes m} \otimes_K \mathbb{C}$$

which is the inverse of the map

$$H^1_{\text{dR}}(C^o)^{\otimes m} \otimes_K \mathbb{C} \to (I^m/I^{m+1})^\vee \otimes_{\mathbb{Q}} \mathbb{C}$$

induced from the above homomorphism $\phi$. This implies that $\phi$ is an isomorphism. The remaining assertions follow from that the tannakian fundamental group $\pi_1^1 (\sharp = \text{dR}, \text{Be})$ and its Lie algebra are given as the subsets of $\mathcal{A}^\sharp$ which consist of grouplike elements and primitive elements in $\mathcal{A}^\sharp$ respectively. 

Following [13], we define the Hodge (resp. weight) filtrations $F^\bullet$ (resp. $W_\bullet$) on $\pi_1^{\text{dR}}(C^o; x, y)$ as follows. First, $F^p$ is spanned by $\mathcal{D}$-modules $\mathcal{D}(w_1, \ldots, w_r)$ for at least $p$ elements of $B_{C^o}$ which are holomorphic on $C$ or having only simple poles at $D = C - C^o$. Second, $W_\bullet$ is spanned by $\mathcal{D}(w_1, \ldots, w_r)$ for at most $l$ elements of $B_{C^o}$ which have only simple poles at $D$. Take a $K$-basis $\{w_1, \ldots, w_{2g+n-1}\}$ of $H^1_{\text{dR}}(C^o)$ consisting of meromorphic differentials on $C$, and a point $x$ on $D$ such that $w_i$ ($1 \leq i \leq 2g + n - 1$) are regular or having simple poles at $x$. Then for a point $y$ on $C^o$, Theorem 5.1 also holds by replacing $x$ with a tangential point on $C^o$ starting at $x$, where the associated iterated integrals are

$$\int_x^y w_1 \cdots w_m \quad (w_1: \text{regular at } x).$$

Let $x, y$ are points or tangential points on $C^o$. Then by Theorem 5.1, for a topological path $t_{xy}$ and the de Rham path $d_{xy}$ from $x$ to $y$, $d_{xy}^{-1} t_{xy}$ gives an element of $\pi_1^{\text{dR}}(C^o; x) \otimes_K \mathbb{C}$. If we fix a $K$-basis $\{w_i\}$ of $H^1_{\text{dR}}(C^o)$ and denote by $W_i$ the symbols corresponding to $w_i$, then we have an injection from $\pi_1^{\text{dR}}(C^o; x)$ into the ring $\mathbb{C}\langle\langle W_i\rangle\rangle$ of noncommutative formal power series of $W_i$ which sends the dual of $w_i$ to $e^{W_i}$. The unipotent periods are coefficients of

$$d_{xy}^{-1} t_{xy} \in \pi_1^{\text{dR}}(C^o; x) \otimes_K \mathbb{C} \hookrightarrow \mathbb{C}\langle\langle W_i\rangle\rangle$$

which are given by iterated integrals

$$\int_{t_{xy}} w_1 \cdots w_m,$$

where if $x$ (resp. $y$) are tangential points, then $w_1$ (resp. $w_m$) are taken to be regular at $x$ (resp. $y$).
5.2. Limits of unipotent periods

Let the notation be as in 4.1, and assume that $\Delta$ is trivalent, namely $C_\Delta$ is maximally degenerate. Then one can take
\[ \{x_h \mid h \in \pm E \cup T, \ v_h = v\} = \{0, 1, \infty\}, \]
hence $A_0 = \mathbb{Z}$, and each points $p_{x_h}$ on $C_0 = C_\Delta \otimes_{A_\Delta} A_0$ has two $\mathbb{Z}$-rational tangential points over $\mathbb{Z}$. Let $\omega_i (1 \leq i \leq g)$, $\eta_j (1 \leq j \leq g)$, $\omega_t = \omega_{t_0}, t$ ($t \in T - \{t_0\}$) be as in Theorems 4.1 and 4.2.

For $h_1, h_2 \in \pm E \cup T$ such that $p_{h_1} \neq p_{t_0}$, take a tangential point $\tilde{p}_{h_i}$ over $\mathbb{Z}$ starting at $p_{h_i}$ for each $i = 1, 2$. Then a unipotent period for $C_\Delta$ from $\tilde{p}_{h_1}$ to $\tilde{p}_{h_2}$ is defined as an iterated integral
\[ \int_{\tilde{p}_{h_1}}^{\tilde{p}_{h_2}} w_1 \cdots w_m, \]
where $w_k \in \{\omega_i, \eta_j, \omega_t\}$ such that $w_1|_{C_v}$ (resp. $w_m|_{C_v}$) have no simple pole at $p_{h_1}$ (resp. $p_{h_2}$). Since $\{\omega_i, \eta_j, \omega_t\}$ is a $(B_\Delta \otimes \mathbb{Q})$-basis of $H^1_{\text{dR}}(C_\Delta \otimes_{A_\Delta} (B_\Delta \otimes \mathbb{Q}))$, the $(B_\Delta \otimes \mathbb{Q})$-module generated by $D$-modules $D(w_1, \ldots, w_m)$ ($w_k \in \{\omega_i, \eta_j, \omega_t\}$) gives rise to a natural $\mathbb{Q}$-structure of the vector bundle associated with $\pi_{\Delta}^{\text{dR}}$ on a deformation of $C_0$. Furthermore, $F^p$ and $W_l$ are defined as its subbundles such that $F^p$ is generated by $D(w_1, \ldots, w_m)$ for at least $p$ elements of $\{\omega_i, \omega_t\}$, and $W_l$ is generated by $D(w_1, \ldots, w_m)$ for at most $l$ elements of $\{\omega_t\}$.

**Theorem 5.2.** The above unipotent period is expressed as a Laurent power series in $y_e$ ($e \in E$) whose minimal terms are generated over $\mathbb{Q}$ by $\pi \sqrt{-1}$ and multiple zeta values.

**Proof.** We may show the assertion in the case that $v_{h_1} = v_{h_2}$, and hence we may assume that $\tilde{p}_{h_2}$ is the half rotation of $\tilde{p}_{h_1}$ or that $(\tilde{p}_{h_1}, \tilde{p}_{h_2})$ is identified with the interval $(0, 1)$. By the universal expression of $\omega_i, \eta_j, \omega_t$ (Theorems 3.1 and 4.2), and the stability of $\omega_i, \omega_t$ (Theorem 3.2), in the former case, the minimal terms are positive powers of $\pi \sqrt{-1}$, and in the latter case, the minimal terms are $\mathbb{Q}$-linear sums of multiple zeta values by the following lemma. □

**Lemma 5.3.** Let $w_1, \ldots, w_m$ be elements of
\[ \left\{ z^k dz, (z - 1)^l dz \mid k, l \in \mathbb{Z} \right\}. \]
Then the iterated integral
\[ \int_{\gamma} w_1 \cdots w_m \]
along \( \gamma \in \pi_1 (\mathbb{C} - \{0, 1\}; a, z) \) becomes a linear sum over the ring
\[
R = \mathbb{Q} [z^{\pm 1}, (z - 1)^{\pm 1}]
\]
of the multiple polylogarithms
\[
J (\alpha_{j_1} - a, \ldots, \alpha_{j_k} - a; 0, z - a) = \int_0^{z-a} \frac{dz}{z - (\alpha_{j_1} - a)} \cdots \frac{dz}{z - (\alpha_{j_k} - a)},
\]
where \( \alpha_{j_i} = 0, 1 \), and the iterated integrals is taken along the path \( \{t - a \mid t \in \gamma \} \).

**Proof.** Since
\[
J (\alpha_{j_1}, \ldots, \alpha_{j_k}; a, z) = J (\alpha_{j_1} - a, \ldots, \alpha_{j_k} - a; 0, z - a),
\]
it is enough to show the assertion that for all integers \( n_i \),
\[
\int_{\gamma} J (\alpha_{j_1}, \ldots, \alpha_{j_k}; a, z) f(z) dz; \ f(z) = \prod_{i=1}^{s} (z - \alpha_{l_i})^{n_i}
\]
is an \( R \)-linear sums of \( J (\alpha_{j_1}, \ldots, \alpha_{j_m}; a, z) \). Since
\[
\frac{1}{(z - \alpha_j)^s(z - \alpha_l)^t} = \frac{1}{\alpha_j - \alpha_l} \left( \frac{1}{(z - \alpha_j)^s(z - \alpha_l)^{t-1}} - \frac{1}{(z - \alpha_j)^{s-1}(z - \alpha_l)^t} \right)
\]
for \( j \neq l \) and \( s, t > 0 \), we may assume that \( f(z) \) is an integral power of \( z - \alpha_l \). When \( f(z) = (z - \alpha_l)^{-1} \), we have
\[
\int_{\gamma} J (\alpha_{j_1}, \ldots, \alpha_{j_k}; a, z) \frac{dz}{z - \alpha_l} = J (\alpha_{j_1}, \ldots, \alpha_{j_k}, \alpha_l; a, z),
\]
and hence the assertion follows. When \( f(z) = (z - \alpha_l)^n \) with \( n \neq -1 \), we have
\[
\int_{\gamma} J (\alpha_{j_1}, \ldots, \alpha_{j_k}; a, z) f(z) dz
\]
\[= J (\alpha_{j_1}, \ldots, \alpha_{j_k}; a, z) \frac{(z - \alpha_l)^{n+1}}{n+1}
\]
\[- \int_{\gamma} J (\alpha_{j_1}, \ldots, \alpha_{j_{k-1}}; a, z) \left( \frac{1}{z - \alpha_{j_k}} \frac{(z - \alpha_l)^{n+1}}{n+1} \right) dz
\]
which implies the assertion by the induction on \( k \). \( \square \)

**Acknowledgments**

This work is partially supported by the JSPS Grant-in-Aid for Scientific Research No. 17K05179.
References

[1] A. Besser, Heidelberg lectures on Coleman integration, in: J. Stix (Ed.), The Arithmetic of Fundamental Groups, Vol. 2 of Contributions in Mathematical and Computational Sciences, Springer Berlin Heidelberg, 2012, pp. 3–52.

[2] R. Coleman, Torsion points on curves and $p$-adic abelian integrals, Ann. of Math. 121 (1985) 111–168.

[3] R. Coleman and E. de Shalit, $P$-adic regulators on curves and special values of $p$-adic $L$-functions, Invent. Math. 93 (1988) 239–266.

[4] R. Coleman and A. Iovita, The Frobenius structure and monodromy operators for curves and abelian varieties, Duke Math. J. 97 (1999) 171–215.

[5] P. Deligne, Le groupe fondamental de la droite projective moins trois points, in Galois groups over $\mathbb{Q}$, Publ. MSRI, vol. 16, Springer, 1989, pp. 79–298.

[6] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, Publ. Math. IHES 36 (1969) 75–109.

[7] E. de Shalit, Differentials of the second kind on Mumford curves, Israel J. of Math. 71 (1990) 1–16.

[8] E. de Shalit, Coleman integration versus Schneider integration on semistable curves, Doc. Math. Extra Volume Coates (2006) 325–334.

[9] V. G. Drinfel’d, On quasitriangular quasi-Hopf algebras and a group closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. 2 (1991) 829–860.

[10] H. Furusho, $p$-adic multiple zeta values I – $p$-adic multiple polylogarithms and the $p$-adic KZ equation, Invent. Math. 155 (2004) 258–286.

[11] H. Furusho, $p$-adic multiple zeta values II – Tannakian interpretations. Amer. J. Math. 129 (2007) 1105–1144.

[12] L. Gerritzen, Periods and Gauss-Manin connection for families of $p$-adic Schottky groups, Math. Ann. 275 (1986) 425–453.

[13] R. Hain, The geometry of the mixed Hodge structure on the fundamental group, in Algebraic Geometry-Bowdoin 1985, Part 2, Proc. Symp. Pure Math., vol. 46.2, Amer. Math. Soc., 1987, pp. 247–282.

[14] R. Hain, Notes on the universal elliptic KZB equation, arXiv: 1309.0580.

[15] X. Hu and C. Norton, General variational formulas for abelian differentials. to appear in IMRN.
[16] T. Ichikawa, $p$-adic theta functions and solutions of the KP hierarchy. Comm. Math. Phys. 176 (1996) 383–399.

[17] T. Ichikawa, Generalized Tate curve and integral Teichmüller modular forms, Amer. J. Math. 122 (2000) 1139–1174.

[18] T. Ichikawa, Teichmüller groupoids and Galois action, J. reine angew. Math. 559 (2003) 95–114.

[19] Y. Ihara, H. Nakamura, On deformation of maximally degenerate stable marked curves and Oda’s problem, J. reine angew. Math. 487 (1997) 125–151.

[20] Yu. I. Manin, V. Drinfeld, Periods of $p$-adic Schottky groups. J. Reine Angew. Math. 262/263 (1972) 239–247.

[21] D. Mumford, An analytic construction of degenerating curves over complete local rings, Compos. Math. 24 (1972) 129–174.

[22] P. Schneider, Rigid analytic $L$-transforms, Lecture Notes in Math. vol. 1068, Springer-Verlag, Berlin, 1984, pp. 216–230.

[23] F. Schottky, Über eine spezielle Function, welche bei einer bestimmten linearen Transformation ihres Arguments unverändert bleibt, J. reine angew. Math. 101 (1887) 227–272.