Extra packing of mass of anisotropic interiors induced by MGD

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In this work we investigate the extra packing of mass within the framework of gravitational decoupling by means of Minimal Geometric Deformation approach. It is shown that, after a suitable set of the free parameters involved, the like–Tolman IV solution extended by Minimal Geometric Deformation not only acquire extra packing of mass but it corresponds to a stable configuration according to the adiabatic index criteria. Additionally, it is shown that the extra packing condition induce a lower bound on the compactness parameter of the seed isotropic solution and a stringent restriction on the decoupling parameter.

I. INTRODUCTION

Seminal Tolman’s work [1] about spherically symmetric and static fluid spheres driven by an isotropic matter distribution i.e, equal radial and transverse pressures ($p_r = p_\perp = p$), marked a stage in the seeking of analytic solutions of Einstein field equations describing exciting compact structures such as neutron stars, white dwarfs etc. All these objects corresponding to the final stage in the life of stars provide us real laboratories (impossible to recreate on Earth) to understand the behaviour in the strong gravitational field regime. From the astrophysical point of view some interesting features of these kind of configurations can be obtained. For example by measuring the surface gravitational red-shift $z_s$, we can infer macro-physical observables as the mass, $M$, and the radius, $R$, of a star which constitute the so called compactness parameter, $u \equiv M/R$. In this respect, Buchdahl [2] determined that for an isotropic matter distribution the ratio $M/R$ can not exceed the upper bound $M/R \leq 4/9$ which corresponds to a maximum gravitational surface red–shift $z_s = 2$.

However, as pointed out by Ruderman in his pioneering work [3], celestial bodies are not necessarily made of isotropic matter but they could contain local anisotropies at least in certain very high density ranges ($\rho > 10^{15} g/cm^3$), where the nuclear interactions must be treated relativistically. In this direction, the celebrated article by Bowers and Liang [4] laid the initial basis for the study of anisotropic structures within the framework of Einstein’s general relativity (GR from now on). Regarderingly, they found that the contributions coming from local anisotropies into the Tolman-Oppenheimer-Volkoff (TOV) equation [1,5] is of Newtonian origin.

The study of Buchdahl’s ratio in different frameworks involving self–gravitating compact structures entails intriguing queries, such as: i) How is this limit modified in the presence of additional fields? ii) How much do corrections to Einstein–Hilbert’s action influence mass–radius ratio? iii) Does the Buchdahl limit increase or decrease in the high dimension regime? Over the last two decades these question have been extensively studied [6–42] finding out interesting results. It can be shown that among all the mechanism leading to a modification in the Buchdahl’s limit, the simplest one corresponds to introduce local anisotropies within the matter distribution [43] which lead to $u > 4/9$, namely, stellar interiors with extra packing of mass. For example, in Ref. [42], it was shown that the introduction of a Kalb–Ramond field leads to a stable compact anisotropic configuration with extra packing of mass. It should be notice that the Kalb–Ramond field considered in [42] fill the whole space and as a consequence, the compact object is surrounded not by the Schwarzschild vacuum but by a background corresponding to an exterior solution of Einstein equations sourced by the Kalb-Ramond field. In this regard, we may wonder if it is possible to introduce local anisotropies to obtain extra packing of mass without modifying the Schwarzschild vacuum. In this concern, a simple and powerful tool, the so called Minimal Geometric Deformation (MGD) approach, has been developed to extend spherically symmetric and static solutions of the Einstein field equations sourced by a perfect fluid to anisotropic domains [10,62]. This methodology basically contains two main ingredients: i) An extra source $\theta_{\mu\nu}$ is added to the perfect fluid $T_{\mu\nu}$ and ii) a minimal geo-
metric deformation is introduced in the $g^{rr}$ component of the metric which allows to decouple the system in two equations, one for each source. It should be notice that as the system sourced by $T_{\mu\nu}$ is known, the rest of the method consist in to provide suitable extra constraints to solve for the $\theta$-sector. Regardly, a wide range of possibilities have been proposed, among which are: i) the so-called mimic constraint procedure [62] [67], ii) to impose an adequate decoupling function $f(r)$ meeting all the physical and mathematical requirements [68] [69], or iii) some anisotropy mechanism [70] [71]. Depending on the mechanism considered to close the $\theta$-sector the magnitude and sign of the dimensionless coupling constant $\alpha$ is determined in order to preserve a well defined anisotropy factor $\Delta \equiv p_\perp - p_r$ throughout the stellar medium. These last two years have seen a great growth in the development of gravitational decoupling by means of the MGD [72–99], what is more the inverse problem is determined in order to prescribe a well defined anisotropic behaviour of the stellar matter distribution comes from the $\theta$-sector. This procedure allows us to constraint the values of the decoupling parameter, $\alpha$, in order to describe a well behaved interior solution. What is more, the values obtained for $\alpha$ determine the size of the mass–radius ratio. As an example to illustrate the scheme we have used the deformed Tolman IV solution already obtained in [62] employing the mimic constraint grasp. It is shown that $\alpha$ can not be neither arbitrarily large or negative in order to describe a well posed stellar interior medium with extra packing mass.

The article is organized as follows. Sec. IV is dedicated to the presentation of the gravitational decoupling through MGD scheme. In Sec. IV the Buchdahl limit generalities and its extension to anisotropic domains are revisited. Sec. IV is devoted to introduce the extra packing of mass within the arena of MGD is and the constraints on the $\alpha$ parameter and the mass–radius ratios are determined for the minimally deformed Tolman IV space–time. Finally, Sec. V concludes the work.

Throughout the article we shall employ the mostly negative signature $(+, -, -, -)$.

II. ANISOTROPIC SOURCES:
GRAVITATIONAL DECOUPLING BY MGD

In this section we provide a short and comprehensive introduction to gravitational decoupling by MDG scheme. A full explanation can be found at references [58] [62] for example.

Let us consider the Einstein-Hilbert (E-H hereinafter) action describing the gravitational field coupled to a matter field through minimal coupling-matter principle given by

$$ S = S_{E-H} + S_M, \quad \text{(1)} $$

where the E-H action reads

$$ S_{E-H} = \frac{1}{2\kappa} \int \sqrt{-g}Rd^4x, \quad \text{(2)} $$

being $g \equiv det(g_{\mu\nu})$ the determinant of the metric tensor $g_{\mu\nu}$, $R$ the Ricci’s scalar and $\kappa = 8\pi Gc^{-4}$. From now on we shall employ relativistic geometrized units where Newton’s gravitational constant $G$ and the speed of light are taken to be the unit i.e, $G = c = 1$. The matter sector $S_M$ is given by the following general expression

$$ S_M = \int \sqrt{-g}\mathcal{L}_Md^4x, \quad \text{(3)} $$

where $\mathcal{L}_M$ stands for the Lagrangian-density matter. At this stage it is remarkable to note that the Lagrangian-density $\mathcal{L}_M$ describing the matter fields could contain new matter fields different from those which usually describe the material content i.e, isotropic, anisotropic or charged distributions among others. So, let us write $\mathcal{L}_M$ as

$$ \mathcal{L}_M = \mathcal{L}_M + \alpha \mathcal{L}_X, \quad \text{(4)} $$

and consider that the information on isotropic, anisotropic or charged fluids, among others are encoded in the Lagrangian-density $\mathcal{L}_M$ (throughout the text we will use barred quantities to refer the usual material content), while $\mathcal{L}_X$ encipher the new matter fields. In principle these incoming fields can be seen as corrections to general relativity [101]. So, putting all together and taking variations respect to the inverse metric tensor $\delta g^{\mu\nu}$ in [1], we arrive to the following field equations describing the gravitational-matter interaction

$$ \frac{\delta S}{\delta g^{\mu\nu}} = 0 \Rightarrow G_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \text{(5)} $$

where $G_{\mu\nu}$ is the Einstein’s tensor. The general expression for $T_{\mu\nu}$ after variations is given by

$$ T_{\mu\nu} = -\frac{1}{2} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M + \alpha \left( -\frac{2}{2} \frac{\delta \mathcal{L}_X}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_X \right). \quad \text{(6)} $$

So,

$$ T_{\mu\nu} = T_{\mu\nu} + \alpha \theta_{\mu\nu}, \quad \text{(7)} $$

being $\alpha$ a dimensionless parameter. As we are interested in the study of compact structures describing a spherically symmetric and static space-time, in order to solve the field equations [5] we supplement them with the most general line element

$$ ds^2 = e^\nu dt^2 - e^\eta dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad \text{(8)} $$
The staticity of this space-time (8) is ensured by considering \( \nu \) and \( \eta \) as functions of the radial coordinate \( r \) only. Putting together equations (5), (7) and (8) one obtains the following system of equations

\[
e^{-\eta} \left( \frac{\eta'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \tilde{\rho} + \alpha \theta^0_0, \tag{9}
\]

\[
e^{-\eta} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \tilde{\rho} - \alpha \theta^1_1, \tag{10}
\]

\[
\frac{e^{-\eta}}{4} \left( 2\nu'' + \nu'^2 + 2\frac{\nu' - \eta'}{r} - \nu' \eta' \right) = \tilde{\rho} - \alpha \theta^2_2. \tag{11}
\]

The corresponding Bianchi’s identity (conservation law) \( \nabla^\mu T_{\mu\nu} = 0 \) associated with the system (9)-(11) yields

\[
- \frac{d\bar{\rho}}{dr} - \alpha \left[ \frac{\nu'}{2} (\theta^0_1 - \theta^1_1) \right] - \frac{d\theta^1_1}{dr} + \frac{2}{r} (\theta^2_2 - \theta^1_1) \right] - \frac{\nu'}{2} (\tilde{\rho} + \bar{\rho}) = 0. \tag{12}
\]

Note that Eq. (12) represents the generalized Tolman-Oppenheimer-Volkoff (TOV) equation driven the hydrostatic equilibrium of the system. In this opportunity we have taken the input matter distribution to be an isotropic fluid configuration given by

\[
T_{\mu\nu} = (\tilde{\rho} + \bar{\rho}) u_\mu u_\nu - \bar{p} g_{\mu\nu}, \tag{13}
\]

where \( \tilde{\rho} \) and \( \bar{\rho} \) are the isotropic thermodynamic quantities representing the energy-density and the pressure, respectively. Moreover, \( u^\beta = e^{-\nu(r)/2} \delta^\beta_0 \) is the unit timelike four-velocity satisfying \( u^\beta u_\beta = 1 \). At this point one can identify the effective amounts as follows

\[
T^0_0 = \tilde{\rho} + \alpha \theta^0_1 \equiv \bar{\rho}, \tag{14}
\]

\[
T^1_1 = \tilde{\rho} - \alpha \theta^1_1 \equiv \tilde{\rho}_r, \tag{15}
\]

\[
T^2_2 = \tilde{\rho} - \alpha \theta^2_2 \equiv \tilde{\rho}_\perp. \tag{16}
\]

Note that, at this point, Eqs. (9), (10) and (11) correspond to a set of differential equations in which we have only separated the components of the matter sector. However, in the framework of MGD these equations can be successfully decoupled and at the end, the system becomes in two set of differential equations, one for each source. To this end, we need to introduce the so-called mimimal geometric deformation in the \( g^{\tau\tau} \) component of the metric given by the linear map

\[
e^{-\eta(r)} \mapsto e^{-\eta(r)} = \mu(r) + \alpha f(r). \tag{17}
\]

Next, using (17) we obtain

\[
8\pi \bar{\rho} = \frac{1}{r^2} - \frac{\mu}{r^2} - \frac{\mu'}{r} \tag{18}
\]

\[
8\pi \tilde{\rho} = -\frac{1}{r^2} + \mu \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) \tag{19}
\]

\[
8\pi \theta^0_0 = \frac{\mu}{4} \left( 2\nu'' + \nu'^2 + 2\frac{\nu'}{r} \right) + \frac{\nu'}{4} \left( \nu' + \frac{2}{r} \right). \tag{20}
\]

along with the conservation equation

\[
\bar{p}' + \frac{\nu'}{2} (\tilde{\rho} + \bar{\rho}) = 0, \tag{21}
\]

for the isotropic sector. Similarly, we have the following equations for the \( \theta^- \) sector

\[
8\pi \theta^0_0 = f - \frac{f'}{r} \tag{22}
\]

\[
8\pi \theta^1_1 = -f \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) \tag{23}
\]

\[
8\pi \theta^2_2 = -f \left( \frac{2}{r^2} + 2\frac{\nu'}{r} - \frac{f'}{r} \right) - \frac{f'}{4} \left( \nu' + \frac{2}{r} \right). \tag{24}
\]

The corresponding conservation equation \( \nabla^\nu \theta^\nu_{\mu\nu} = 0 \) yields

\[
(\theta^1_1)' - \frac{\nu'}{2} (\theta^0_0 - \theta^1_1) - \frac{2}{r} (\theta^2_2 - \theta^1_1) = 0. \tag{25}
\]

At this point some comments are in order. First, Eqs. (22), (23) and (24) corresponds to the so-called quasi Einstein field equations in the sense that there is a missing \( \frac{1}{r} \). Remarkably, Eq. (25) corresponds to the TOV equation of the \( \theta^- \) sector. Second, it is clear that the interaction between the two sources is completely gravitational. This fact is reflected by equations (21) and (25), where both sectors are individually conserved. Finally, the deformation (17) can be generalized by considering the deformation of the \( g^{\tau\tau} \) component as reported in (101). However, this case is out of the scope of this work.

It worth noticing that, when an isotropic solution is considered, Eqs. (18), (19) and (20) are automatically satisfied. Even more, we use the metric function \( \nu \) to solve for the \( \theta^- \) sector. However, the \( \theta^- \) sector consists of four unknowns, namely \( (\theta^0_0, \theta^1_1, \theta^2_2, f) \) and only three equations (22)-(24). So, to close the \( \theta^- \) system it is necessary to prescribe some additional information. Regardingly, the so-called mimetic constraints have been broadly used. Indeed, the mimetic constraint for the pressure, namely

\[
\bar{p} = \theta^1_1, \tag{26}
\]

has been implemented to ensure the continuity of the second fundamental form. Remarkably, this condition allows to obtain an algebraic equation for the decoupling function \( f \). Otherwise, the mimetic constraint for the density,

\[
\tilde{\rho} = \theta^0_0, \tag{27}
\]

has been used also to extend the Tolman IV solution. As we shall see later, in this work we base our analysis of extra packing of mass in the Tolman IV solution extended by MGD using the mimetic constraint for the density.
III. BUCHDAHL’S LIMIT: ISOTROPIC AND ANISOTROPIC SOURCES REVISITING

Buchdahl’s limit states that for a spherically symmetric and static configuration describing an isotropic matter distribution, the maximum mass–radius ratio $u$ (also known as compactness factor) is given by

$$u \equiv \frac{M}{R} \leq \frac{4}{9},$$  \hspace{1cm} (28)

where $M$ and $R$ stand for the gravitational mass contained in the sphere and the radius of the star, respectively. As can be seen from (28), one has two options i) an equality and ii) an inequality. The first option holds by considering a constant energy density $\rho = \text{cte.}$, a degenerate metric i.e $g_{tt}(r = 0) = 0$ and a non well behaved pressure (divergent pressure at the center of the compact configuration). On the other hand, the inequality requires $\rho \geq 0$ (a positive defined energy density everywhere inside the star), $d\rho/dr \leq 0$ (monotonic decreasing energy density from the center to the boundary of the compact object) and a vanishing pressure $P$ at the surface of the structure. So, the value $4/9$ is an absolute upper limit for all static fluid spheres whose density does not increase outwards. As said before, condition (28) works in the case where $p_r = p_\perp$, that is in the isotropic case. Nevertheless, the isotropic condition does not represent a real astrophysical situation at all. This is so because celestial bodies are not necessarily made up by isotropic matter distributions. Relaxing the isotropic condition and allowing the presence of local anisotropies in the stellar interior new constraints arise on the compactness factor. Moreover, some modifications on relevant physical quantities such as the surface gravitational redshift $z_s$ are induced. In this respect Böhm and Harko [43] derived the corresponding upper bound for the mass–radius relation for an anisotropic matter distribution in presence of a cosmological constant $\Lambda$. They obtained the following general expression

$$\frac{2M}{R} \leq \left( 1 - \frac{8\pi}{3} \Lambda R^2 \right) \left[ 1 - \frac{1}{9} \left( 1 - \frac{2\chi(\rho)}{(1 - \frac{8\pi}{3} \Lambda R^2) (1 + F)^2} \right) \right],$$  \hspace{1cm} (29)

being $\langle \rho \rangle$ the mean energy density and $F$ is a function proportional to the anisotropy factor $\Delta = p_\perp - p_r$, which is given by

$$F = 2 \frac{\Delta(R)}{\langle \rho \rangle} \left[ \frac{\text{arcsin} \left( \sqrt{\frac{2 M \chi(R)}{R}} \right)}{\sqrt{\frac{2 M \chi(R)}{R}}} - 1 \right],$$  \hspace{1cm} (30)

where $\chi$ stands for

$$\chi(r) \equiv 1 + \frac{4\pi}{3} \frac{\Lambda r^3}{m(r)}.$$  \hspace{1cm} (31)

As we are interested in studying space-time without cosmological constant from now on we will set off $\Lambda$. Then the upper bound (29) becomes

$$2u \leq 1 - \frac{1}{9(1 + F)^2},$$  \hspace{1cm} (32)

and $F$ turns

$$F = \frac{R^2 \Delta(u)}{4u} \left( \frac{\sin^{-1}(\sqrt{2u})}{\sqrt{2u}} - 1 \right),$$  \hspace{1cm} (33)

since eliminating $\Lambda$ then $\chi(r) = 1$. At this point a couple of comments are in order. First, note that $F$ is a positive quantity. Otherwise, the condition $\Delta \geq 0$ could be violated. As a consequence, non–vanishing values of $F$, Eq. (32) allows extra packing of mass in compact stellar structures. Second, from Eq. (33), it is straightforward to show that the bounds of the compactness parameter of the anisotropic distribution, $u$, is defined in the interval

$$\frac{4}{9} \leq u < \frac{1}{2}.$$  \hspace{1cm} (34)

In the next section we will discuss in details how the local anisotropies introduced in the stellar interior by gravitational decoupling through MGD approach contributes on the maximum mass-radius ratio value allowable for relativistic anisotropic fluid spheres in the arena of GR.

IV. LOCAL ANISOTROPY INDUCED BY MGD

In the context of MGD, the anisotropy of the total matter sector seeded by a perfect fluid, can be written as

$$\Delta = p_\perp - p_r = \alpha(-\theta_2^2 - (-\theta_1^2)).$$  \hspace{1cm} (35)

Note that, in order to satisfy the requirement $\Delta > 0$, we have to impose extra constraints, for example, $-\theta_2^2 - (-\theta_1^2) > 0$ and $\alpha \geq 0$. Now, as commented at the end of the previous section, the introduction of local anisotropies could lead to extra packing of mass in the interior of a compact object. In this sense, given the link between the anisotropy and the decoupler matter content (see Eq. (35)), the gravitational decoupling of sources by MGD can be thought as a kind of mechanism that allows to introduce such an extra packing.

Another aspect that deserves to be pointed out, is that the connection between the anisotropy induced by the gravitational decoupling and the compactness parameter given by Eqs. (32) leads to

$$2u \leq 1 - \frac{1}{9 \left( 1 + \frac{16\pi R^2 (\theta_1^2 - \theta_2^2) (\sin^{-1}(\sqrt{2u})}{\sqrt{2u}} - 1) \right)^2}.$$  \hspace{1cm} (36)

Note that when $\alpha \to 0$ we recover the Buchdahl’s limit for the isotropic case, namely, $u \leq \frac{4}{9}$, as expected. However, expression (36) must be considered as a formal expression because matching conditions namely, the continuity of the first and the second fundamental form of the total
solution leads to non-linear equations involving $\alpha$ and $u$. In this sense, obtaining an analytical expression of the bound of the compactness parameter of the total solution, $u$, in terms of the decoupler parameter is not possible. Instead, we can try to find the connection between $u$ and $\alpha$ in an alternative manner. To this end, we shall consider the MGD–extended Tolman IV solution previously reported in [62] as follows. The seed solution which solve the isotropic sector parametrized by $(\nu, \mu, \rho, p)$ is given by

\begin{align}
\rho & = \frac{\alpha r^2}{8\pi(A^2 + r^2)^2}, \quad (42) \\
\Delta & = \frac{\alpha r^2}{8\pi(A^2 + r^2)^2}, \quad (42)
\end{align}

form where $\alpha > 0$, necessarily. Furthermore, extended solution is sourced by

\begin{align}
\tilde{\rho} & = (\alpha + 1)\rho \\
\tilde{\rho}_r & = \tilde{\rho} - \frac{\alpha ((A^2 + 3r^2)(A^2 + C^2 + r^2))}{8\pi C^2 (A^2 + r^2)^2} \\
\tilde{\rho}_\perp & = \tilde{\rho} + \frac{\alpha r^2}{8\pi(A^2 + r^2)^2}.
\end{align}

Finally, it can be demonstrated that the compactness parameters of the total sector, $u = \frac{M}{R}$, and $u_0 = \frac{M_0}{R}$ for Tolman IV are related by

\begin{equation}
2u = 2u_0 + \alpha \frac{R^2}{C^2} \left( \frac{A^2 + C^2 + R^2}{A^2 + 2R^2} \right), \quad (46)
\end{equation}

It is remarkable that from the above expression, when $u_0 = \frac{4}{9}$, the anisotropic solution acquire an extra packing in the sense that

\begin{equation}
u \geq \frac{4}{9}, \quad (47)
\end{equation}

given that all the parameters involved are positive numbers. However, note that the above corresponds to the critical situation in which $u_0$ acquires its maximum allowed value. In this sense, we may wonder how the parameters should be set to ensure extra packing of mass in more realistic situations where $u_0 < 4/9$. It is worth mentioning that such a set is not trivial as we shall see in what follows.

After imposing continuity of the first and the second fundamental form it is obtained (see Ref. [62])

\begin{align}
B & = \sqrt{1 - 2u_0 - \alpha \frac{R^2}{C^2} \left( \frac{A^2 + C^2 + R^2}{A^2 + 2R^2} \right)} \quad (48) \\
C & = \sqrt{\frac{(1 + \alpha) (A^2 + R^2)}{(A^2 + 3R^2) - \alpha (A^2 + 3R^2)}}. \quad (49)
\end{align}

Now, using (49) in (48) we obtain

\begin{equation}
A = \sqrt{\frac{\alpha - 3u(1 + \alpha) + 3u_0(1 + \alpha)}{(u - u_0)(1 + \alpha)}}, \quad (50)
\end{equation}

At this point, it is worth noticing that from Eqs. (48), (49) and (50) the parameters $A, B, C$ are functions on \{u, u_0, \alpha, R\} so we demand

\begin{align}
A & > 0 \quad (51) \\
B & > 0 \quad (52) \\
C & > 0. \quad (53)
\end{align}

for some set \{u, u_0, \alpha, R\} such that

\begin{align}
u_0 & < \frac{4}{9} \quad (54) \\
\frac{4}{9} & < u < \frac{1}{2} \quad (55) \\
\alpha & > 0 \quad (56) \\
R & > 0. \quad (57)
\end{align}

Without loss of generality we shall set $R = 1$ [102] in what follows, so that Eqs. (48) and (49) read

\begin{align}
B & = \sqrt{\frac{(2u - 1)(-\alpha + 3(\alpha + 1)u - 3(\alpha + 1)u_0)}{\alpha - 2(\alpha + 1)u + 2(\alpha + 1)u_0}} \quad (58) \\
C & = \sqrt{\frac{\alpha(-\alpha + 2(\alpha + 1)u - 2(\alpha + 1)u_0)}{(u - u_0)((\alpha - 1)\alpha + 2(\alpha + 1)u - 2(\alpha + 1)u_0)}}. \quad (59)
\end{align}

Now, Eqs. (51), (52), (53), constrained by (54), (55) and (57) reduce to following extra conditions

\begin{align}
3(\alpha + 1)u & < \alpha + 3(\alpha + 1)u_0 \quad (60) \\
2(\alpha + 1)u & < \alpha + 2(\alpha + 1)u_0 \quad (61) \\
\alpha^2 + 2(\alpha + 1)u & < \alpha + 2(\alpha + 1)u_0, \quad (62)
\end{align}
from where $u_0$ and $\alpha$ acquire extra constraints, namely

$$u_0 > 0.38889$$
$$0.2 < \alpha < 0.333$$

(63)
(64)

At this point some comments are in order. First, it is worth noticing that (63) states a lower bound for the compactness parameter of the perfect fluid solution. Even more, for values out the interval $0.38889 < u_0 < \frac{4}{9}$, (65)

the anisotropic compactness $u$ can not surpass the Buchdahl limit for isotropic solutions and and the extra packing of mass is not possible in this case. Second, the extra packing condition allows to restrict the appropriate values for the decoupling parameter as shown in (64). Finally, it is worth mentioning that constraints given by Eqs. (63) and (64) correspond to a particular case we obtained when the system (60), (61) and (62) is reduced. However, we considered the the most simplest case in order to illustrate the how the process works.

As a particular case we shall study the behaviour of a solution with extra packing of mass considering $u = 0.45$, $u_0 = 0.39$ and $\alpha = 0.3$. In figure 1 it is shown the behavior of the density, and the effective pressures of the anisotropic solution. Note that all the quantities shown in the profiles are monotonously decreasing and reach their maximum value at the center of the star as expected. Besides, the radial pressure vanishes a the surface of the star.

![FIG. 1: Decreasing behavior of density and pressures of the anisotropic solution.](image)

Now we analyse the stability of the solution using the adiabatic index criteria. As shown in figure 2 the behaviour of the adiabatic index, $\Gamma$, reveals that for $u = 0.45$, we obtain a stable interior configuration from the anisotropic solution ($\Gamma > 4/3$).

![FIG. 2: Adiabatic index showing stability for the anisotropic interior solution case (greater than 4/3).](image)

In this sense, we conclude that MGD method not only allow to extend isotropic solutions to anisotropic domains but it can be used to obtain a physical acceptable interior with extra packing of mass. Furtermore, the extra packing condition leads to modifications in the interval of the compactness parameter of the isotropic solution. More precisely, given that $0.38889 < u_0 < \frac{4}{9}$, not any arbitrary isotropic solution allows being extended with extra packing but only those with a high compactness comparable with observational data of neutron stars. Finally, it should be emphasize that the extra packing condition leads also to a stringent restriction in the decoupling parameter $\alpha$ as shown in Eq. (64).

V. CONCLUDING REMARKS

In this work we implemented the Minimal Geometric Deformation approach to induce extra packing of mass in stellar interiors. As anisotropic solution we considered the Tolman IV extended by using the mimetic constraint to the density. We obtained that although the induced anisotropy enters naturally in the Bömer and Harko criteria to induce extra packing, obtaining an analytical relationship between the decoupling parameters and the compactness of the star is not possible. However, we were able to set the space of parameters in an appropriate manner to obtain a well posed anisotropic solution with extra packing. We conclude that the Minimal Geometric Deformation approach allows to extend isotropic solutions to anisotropic interiors with extra packing of mass. Besides, the extra packing condition induce a lower bound on the compactness parameter of the isotropic system used as a seed and the decoupling parameter get a restriction within a well defined interval.
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