The Vacuum Einstein Equations via Holonomy around Closed
Loops on Characteristic Surfaces.

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Abstract

We reformulate the standard local equations of general relativity for asymptotically flat spacetimes in terms of two non-local quantities, the holonomy $H$ around certain closed null loops on characteristic surfaces and the light cone cut function $Z$, which describes the intersection of the future null cones from arbitrary spacetime points, with future null infinity.

We obtain a set of differential equations for $H$ and $Z$ equivalent to the vacuum Einstein equations. By finding an algebraic relation between $H$ and $Z$ this set of equations is reduced to just two coupled equations: an integro-differential equation for $Z$ which yields the conformal structure of the underlying spacetime and a linear differential equation for the “vacuum” conformal factor. These equations, which apply to all vacuum asymptotically flat spacetimes, are however lengthy and complicated and we do not yet know of any solution generating technique. They nevertheless are amenable to an attractive perturbative scheme which has Minkowski space as a zeroth order solution.
I. INTRODUCTION.

The vacuum Maxwell fields on Minkowski space can be expressed as Kirchhoff integrals taken over initial data, i.e., data given on a Cauchy surface. If we replace the spacelike surface by a characteristic, or null, surface, we get the analog to the Kirchhoff integral, the D’Adhemar integral [2]. Maxwell and Yang-Mills fields on both flat and asymptotically flat spacetimes have been given such D’Adhemar-like reformulations with considerable success [3–5]. The present work is an attempt to extend this technique to general relativity. We derive differential equations whose solutions would allow us to express the gravitational field at any given spacetime point as a series of D’Adhemar-like integral over free data given on a characteristic surface.

First consider the well understood analog to our problem, determining the Maxwell field at some interior point on Minkowski space. Free data is given on the null surface \( \mathcal{I} \), which is the three dimensional boundary of the spacetime manifold \( \mathcal{M} \). The future light cone emanating from a given interior point \( x^a \) would intersect the future half \( \mathcal{I}^+ \) of \( \mathcal{I} \), the intersection being a 2-surface. We refer to this 2-surface as the “light cone cut” of \( \mathcal{I}^+ \) associated with \( x^a \). The D’Adhemar formulation of the Maxwell equations involves integrals over the free data given on the light cone cut. It is crucial that the light cone structure of the underlying manifold be known, in order to perform this integration. Although for the Minkowski metric this is readily available, it takes considerable effort to obtain this information for an arbitrary background. A generalization of the D’Adhemar formulation of the Maxwell field in Minkowski space to the case when the background is curved is available [6], although the resulting equations become complicated. A further generalization of this technique to Yang-Mills fields on both flat and asymptotically flat spacetimes are also available [3]. In these (Yang-Mills) cases, in addition to the above mentioned difficulty in the determination of the light cones, the final equations are further complicated by the introduction of non-linearity. The D’Adhemar integrals (for the Yang-Mills case and the Maxwell case on a curved background) shows explicitly that propagation becomes non-Huygens–i.e., propagation is not
confined to the light cone.

A D’Adhemar-like formulation for general relativity might seem inaccessible since the gravitational field is not separable from the background and its characteristic surfaces: it requires *simultaneously* solving for the field and the null surfaces. It is nevertheless possible to give such a formulation by dividing the equations into two parts: one resembling the D’Adhemar integrals for a (self-dual or anti-self-dual) Yang-Mills field, and the other yielding the null surfaces that are needed to give meaning to the integrals in the first part. The two parts are strongly coupled.

The first set of equations is equivalent to the D’Adhemar version of the self-dual and the anti-self-dual Yang-Mills equations for the self-dual and anti-self-dual parts of an $O(3,1)$ connection, respectively, on a spacetime manifold $\mathcal{M}$ with an arbitrary asymptotically flat metric. The basic idea is to treat a special case of the $O(3,1)$ Yang-Mills connection so that it agrees with that of the background gravitational connection. This is done by imposing very restrictive conditions on the Yang-Mills data.

The second set of equations determines the light cones of the manifold $\mathcal{M}$. For a *given* metric, one could have obtained this information by integrating the null geodesic equations (see for example [7]). However, in our case where the metric is as yet undetermined, we derive differential equations for the characteristic surfaces that are coupled to the $O(3,1)$ Yang-Mills field equations.

The above two parts, which look hopelessly intertwined at first, nevertheless give results that are tractable and comprehensible. The exact equations, while complicated, are in principle solvable. An attractive feature of this approach is that the equations are amenable to an approximation scheme wherein the $n$-th order correction to the field is expressible as D’Adhemar integrals in terms of lower order.

As the material of this paper is not at all close to mainstream ideas, we believe that it might be appropriate first to give a broad perspective before going into the details.

There are two different sets of ideas that we try to weld together. The first is based on the use of characteristic surfaces on arbitrary asymptotically flat Lorentzian spacetimes. We
make essential use of the future light cones \( N_\times \) from arbitrary spacetime points \( x^a \) and their intersections, \( C_\times \), with future null infinity, \( \mathcal{I}^+ \), (referred to as the light cone cuts)—as well as the past light cones from points on \( \mathcal{I}^+ \). These characteristic surfaces are used to give the D’Adhemar-like reformulation of the Yang-Mills equations. Using Bondi coordinates \((u, \zeta, \bar{\zeta})\) on \( \mathcal{I}^+ \), \( C_\times \) is described by a function referred to as the light cone cut function on \( \mathcal{I}^+ \) of the form \( u = Z(x^a, \zeta, \bar{\zeta}) \), parametrized by the interior points \( x^a \) (see Figures 1 and 2). The cut function plays a dual role: for a fixed interior point \( x^a \) it describes the “cut” of \( \mathcal{I}^+ \), and for a fixed point \((u, \zeta, \bar{\zeta})\) on \( \mathcal{I}^+ \), it describes the past light cone of that point as \( x^a \) is varied. From the latter point of view the cut function becomes a three-parameter family of characteristic surfaces. The conformal structure of spacetime can be recovered from knowledge of the cut function, i.e., the conformal metric can be explicitly written in terms of derivatives of \( Z \). More specifically, the conformal metric can be completely expressed in terms of \( \Lambda \), defined from \( Z \) by \( \Lambda \equiv \bar{\partial}^2 Z \). The quantity \( \Lambda \), becomes one of our basic variables. These ideas are developed in Section II.

The second set of ideas of concern to us (see Section III) is based on the fact that, in some sense, the \( O(3,1) \) Yang-Mills equations contain, as a special class of solutions, the vacuum Einstein equations. The full Yang-Mills curvature, \( F \), can be broken into four pieces; self-dual and anti-self-dual on the internal indices as well as self-dual and anti-self-dual on the spacetime indices, so that

\[
F = +F^+ + F^- - F^+ - F^-, \tag{1}
\]

where the \(+, -\) symbols to the left refer to spacetime duality and those to the right refer to internal duality. The Einstein equations are almost encoded [11] into the algebraic statement that \( +F^- = -F^+ = 0 \) needing only an appropriate restriction of the data. The basic idea is to think of solving these equations (for the self-dual and the anti-self-dual fields \( +F^+ \) and \( -F^- \) and their respective connections) on a given asymptotically flat vacuum background metric, then introduce a soldering form and restrict the data so that it coincides with the background connection data. An important technical device is to give these equations a
D’Adhemar-like formulation which, as was mentioned earlier, can be done for Yang-Mills fields. The tool for this is the use of the holonomy operator $H$ (around special null paths) as the primitive variable for the Yang-Mills fields. The Bianchi identities, equivalent (because of the self- and anti-self-duality) to the Yang-Mills field equations become one set of our final equations. They are symbolically written as equations for $H$ as

$$ \mathcal{B}_\beta(H, A) = 0, $$

with $A$ being the restricted characteristic data.

When the soldering form (a tetrad field) is introduced, we can obtain a set of relationships between the spacetime variable $\Lambda \equiv \partial^2 Z$, and $H$, that we refer to as the $\Lambda(H)$-relations given symbolically by

$$ \Lambda = \Lambda(H). $$

Our third set of equations, referred to as the “field equations”, are simply the algebraic equations

$$ ^+ F^- = ^- F^+ = 0, $$

expressed as functions of the components of the holonomy operator, symbolically written as

$$ \mathcal{D}(H) = 0. $$

Our final task is two-fold: To

1. eliminate the holonomy variables from these three relationships and be left with equations only for the characteristic surfaces, i.e., equations for $Z$ (Section III.E.) These equations constitute the conformal Einstein equations and yield an equation that we refer to as the light cone cut equation (LCCE) which involves only $\Lambda$ and the free data, and

2. find an equation, derivable from equation (5), for the conformal factor that converts the conformal metric to a vacuum metric.

In Section IV we discuss the linearization of the theory and a perturbation scheme.
II. LIGHT CONE CUTS.

Let us consider a real, four-dimensional, asymptotically flat spacetime, \( M \), with a conformal metric \( g_{ab} \) and future null boundary \( I^+ \). We use Bondi coordinates \( (u, \zeta, \bar{\zeta}) \) to coordinatize \( I^+ \). A coordinate grid of the \( (u = \text{const.}) \) and the \( (\zeta, \bar{\zeta} = \text{const.}) \) surfaces on \( I^+ \) is shown in Figure 1.

Consider the future light cone \( N_x \) emanating from an internal point \( x^a \). A relation among the coordinates \( (u, \zeta, \bar{\zeta}) \) such as \( u = f(\zeta, \bar{\zeta}) \) describes locally a 2-surface which we refer to as a ‘cut’. The intersection of \( N_x \) with \( I^+ \) is a special cut, the ‘light cone cut’ \( C_x \) (Figure 2), given by

\[
 u = Z(x^a, \zeta, \bar{\zeta}). \tag{6}
\]

We refer to \( Z(x^a, \zeta, \bar{\zeta}) \) as the light cone cut function. The function \( Z(x^a, \zeta, \bar{\zeta}) \) is our basic variable. The conformal metric can be given as an explicit function of \( Z \) [8].

One could also interpret equation (6) as a description of the past light cone of the point \( (u, \zeta, \bar{\zeta}) \) on \( I^+ \) (also shown in Figure 2). That is, keeping \( u, \zeta \) and \( \bar{\zeta} \) fixed, if we vary \( x^a \), this equation is the locus of all spacetime points \( x^a \) that are null connected to the point \( (u, \zeta, \bar{\zeta}) \), which by definition is the past light cone of \( (u, \zeta, \bar{\zeta}) \) on \( I^+ \). From this important observation, we see that for fixed values of \( (u, \zeta, \bar{\zeta}) \), \( \nabla_a Z \) is a null covector, i.e.,

\[
g^{ab} Z_{,a}(x, \zeta, \bar{\zeta}) Z_{,b}(x, \zeta, \bar{\zeta}) = 0. \tag{7}
\]

Thus it follows that \( Z_{,a}(x, \zeta, \bar{\zeta}) \) at the point \( x^a \) sweeps out the null cone at that point as we vary \( \zeta \) and \( \bar{\zeta} \). Though it has been discussed elsewhere [8] and we will not go into the details, it is from equation (7), by taking several \( \zeta \) and \( \bar{\zeta} \) derivatives, the entire conformal metric can be reconstructed completely in terms of \( Z \) or more specifically \( \Lambda \equiv \partial^2 Z \).

The sphere of null directions at the point \( x^a \) is coordinatized by \( \zeta \) and \( \bar{\zeta} \). We use these coordinates for \( S^2 \) instead of the usual \( (\theta, \phi) \) because covariant differentiation on the sphere, which appears in many of the equations, takes on a particularly simple form in terms of \( (\zeta, \bar{\zeta}) \). We introduce the operators \( \bar{\partial} \) and \( \bar{\partial} \) [8].
\[ \delta \alpha_s = (1 + \zeta \bar{\zeta})^{1-s} \frac{\partial}{\partial \zeta} [(1 + \zeta \bar{\zeta})^s \alpha_s] \]  

(8)

and

\[ \bar{\delta} \alpha_s = (1 + \zeta \bar{\zeta})^{1+s} \frac{\partial}{\partial \bar{\zeta}} [(1 + \zeta \bar{\zeta})^{-s} \alpha_s], \]  

(9)

which operate on the quantity \( \alpha_s \) defined on the sphere, where \( s \), the spin weight, is assigned according to how \( \alpha \) transforms under a specific transformation. (See [10] for more about this transformation and properties of \( \delta \) and \( \bar{\delta} \).) It helps to think of \( \delta \) and \( \bar{\delta} \) loosely as differentiation with respect to \( \zeta \) and \( \bar{\zeta} \) respectively.

Assuming that we know the light cone cut function, and with the above definitions of \( \delta \) and \( \bar{\delta} \), the following set of quantities are well-defined:

\[ u = Z(x^a, \zeta, \bar{\zeta}), \]  

(10)

\[ \omega = \delta Z(x^a, \zeta, \bar{\zeta}), \]  

(11)

\[ \bar{\omega} = \bar{\delta} Z(x^a, \zeta, \bar{\zeta}), \]  

(12)

\[ R = \delta \bar{\delta} Z(x^a, \zeta, \bar{\zeta}), \]  

(13)

with \( (u, R, \omega, \bar{\omega}) \) having, respectively, spin-weights \( (0, 0, 1, -1) \). This set defines a sphere’s worth of coordinate transformations on the spacetime parametrized by \( (\zeta, \bar{\zeta}) \). Let

\[ (\theta^0, \theta^1, \theta^+, \theta^-) \equiv (u, R, \omega, \bar{\omega}). \]  

(14)

With this notation, the \( (\zeta, \bar{\zeta}) \)-dependent coordinate transformation can be written as

\[ \theta^i = \theta^i(x^a, \zeta, \bar{\zeta}), \]  

(15)

where the indices \( i, j \) will take on the values \( \{0, 1, +, -\} \).

We now construct

\[ \bar{\delta}^2 Z \equiv \lambda(x^a, \zeta, \bar{\zeta}). \]  

(16)

The quantity \( \Lambda \) and its conjugate can be expressed as functions of the \( \theta^i \) by inverting the transformation (14) and eliminating the \( x^a \), to obtain
\[ \Lambda(\theta^i, \zeta, \bar{\zeta}) = \delta^2 Z. \] (17)

Later we show that the (conformal) Einstein equations can be encoded into the choice of \( \Lambda \).

With the \( \theta^i \) coordinate system, we have a one-form basis:

\[ d\theta^i = \theta^i_a dx^a, \quad \text{with} \quad \theta^i_a = \theta^i_a \] (18)

and the dual basis

\[ \frac{\partial}{\partial \theta^i} = \theta^a_i \frac{\partial}{\partial x^a}, \] (19)

with the relations

\[ \theta^i_a \theta^a_j = \delta^i_j, \] (20)
\[ \theta^i_a \theta^b_i = \delta^a_b. \] (21)

Any vector \( V_a \) can be written as \( V_a = V^i \theta^i_a \) and in particular we have

\[ \Lambda_a = \Lambda^i \theta^i_a = \Lambda^0 Z_a + \text{etc.}. \] (22)

The \( \Lambda_i \) play a basic role in what follows.

Note also that \( \bar{\delta} \theta^i_a \) can also be expressed as a linear combination of the \( \theta^i_a \), i.e., as

\[ \bar{\delta} \theta^i_a = T^i_j \theta^j_a \] (23)

From \( \theta^i_a \equiv \{ Z_a, \bar{\delta} \bar{\delta} Z_a, \bar{\delta} Z_a, \bar{\delta} Z_a \} \), we obtain, e.g., that

\[ \bar{\delta} \theta^0_a = \theta^+_a, \] (24)
\[ \bar{\delta} \theta^+_a = \Lambda_a = \Lambda^i \theta^i_a, \] (25)
\[ \bar{\delta} \theta^1_a = \theta^1_a, \] (26)
\[ \bar{\delta} \theta^1_a = T^1_i \theta^i_a \] (27)

with
\[ T^i_j = \frac{1}{q} \left[ \Lambda_1 (\bar{\partial} \Lambda_i + \bar{\Lambda}_0 \delta^+_i + \bar{\Lambda}_- \delta^-_i + \bar{\Lambda}_+ \Lambda_i - 2 \delta^-_i) 
\]
\[ + \bar{\partial} \Lambda_i + \Lambda_0 \delta^-_i + \Lambda_+ \delta^+_i + \Lambda_- \bar{\Lambda}_i - 2 \delta^+_i \right] , \]

and

\[ q = (1 - \Lambda_1 \bar{\Lambda}_1). \]  

(28)

Likewise,

\[ \bar{\partial} \theta^i_a = \bar{T}^i_j \theta^j_a \]  

(29)

with \( \bar{T}^i_j \) obtained from the complex conjugate coefficients of \( T^i_j \). The explicit form of these matrices with row and column indices \{0, 1, +, -\} are:

\[
T^i_j = \begin{pmatrix}
0 & 0 & 1 & 0 \\
T^1_0 & T^1_1 & T^1_+ & T^1_- \\
\Lambda_0 & \Lambda_1 & \Lambda_+ & \Lambda_- \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

(30)

and

\[
\bar{T}^i_j = \begin{pmatrix}
0 & 0 & 0 & 1 \\
\bar{T}^1_0 & \bar{T}^1_1 & \bar{T}^1_+ & \bar{T}^1_- \\
0 & 1 & 0 & 0 \\
\bar{\Lambda}_0 & \bar{\Lambda}_1 & \bar{\Lambda}_+ & \bar{\Lambda}_-
\end{pmatrix},
\]

(31)

III. HOLONOMY AND EINSTEIN EQUATIONS.

A. \( O(3,1) \) Yang-Mills Equations.

We begin with a real four-dimensional Lorentzian manifold \( M \) where we assume that the light cone structure of the previous Section is known and study an \( O(3,1) \) Yang-Mills field. The idea is to (roughly) think of the associated vector bundle as being the tangent bundle of the spacetime, though of course a soldering form is needed to make this precise.
The connection one-form, which is antisymmetric in the Lorentzian indices, \( \hat{i}, \hat{j}, \ldots \), can be decomposed into its self-dual and anti-self-dual parts,

\[
\gamma^\hat{i}_a j = \gamma^{+\hat{i}}_a j + \gamma^{-\hat{i}}_a j,
\]

where self-dual and anti-self-dual are defined by

\[
\gamma^{\pm\hat{i}}_a j = \frac{1}{2} \left( \gamma^\hat{i}_a j \mp i \gamma^s_{\hat{i}j} \right),
\]

and duality by

\[
\gamma^{sij}_a = \frac{1}{2} \epsilon^{\hat{i}\hat{j}\hat{k}\hat{l}} \gamma^\hat{k}\hat{l}_a,
\]

where \( \epsilon^{\hat{i}\hat{j}\hat{k}\hat{l}} \) is the alternating symbol with \( \epsilon_{0123} = -1 \). The curvature tensor can be similarly decomposed as,

\[
F_{ab}^\hat{i}_j = F_{ab}^{+\hat{i}}_j + F_{ab}^{-\hat{i}}_j,
\]

where the self-dual and anti-self-dual parts of \( F \) are constructed from the self-dual and anti-self-dual connections respectively, i.e.,

\[
F_{ab}^{\pm\hat{i}}_j = \nabla_{[a} \gamma^\pm_i j + \gamma_{\pm}^a, j^\pm_i j.
\]

With the above decompositions, the Bianchi identities and the field equations become

\[
\nabla_{[c} F_{ab]}^{\pm\hat{i}}_j + [\gamma_{[c}^{\pm}, F_{ab]}^{\pm\hat{i}} j = 0
\]

and

\[
\nabla^a F_{ab}^{\pm\hat{i}}_j + [\gamma^{\pm}_a, F_{ab]}^{\pm\hat{i}} j = 0,
\]

respectively. One is thus dealing with two independent Yang-Mills connections and fields.

It is possible to further decompose each of the two curvature tensors, now on the spacetime indices, into its spacetime self-dual and anti-self-dual parts, where we have used the existence of the Lorentzian metric. We will refer to spacetime duals as left duals and internal duals as right duals. The full curvature then has four terms:
1. the left and right self-dual part, $F_{ab}^{+}$;

2. the left anti-self-dual and right self-dual part, $F_{ab}^{-}$;

3. the left self-dual and right anti-self-dual part, $F_{ab}^{-}$;

4. the left anti-self-dual and right anti-self-dual part, $F_{ab}^{-}$.

Parts (1) and (2) are coupled as are parts (3) and (4), in the sense that they depend respectively on the $\gamma^+$ and $\gamma^-$.

We next assume that:

$$-F_{ab}^+ = F_{ab}^- = 0.$$  \hfill (39)

From this the field equations (38) are automatically satisfied via the Bianchi identities (37). Thus we have two independent Yang-Mills fields: a (left) self-dual field, $F_{ab}^+$ and a (left) anti-self-dual field, $F_{ab}^-$ satisfying the Bianchi identities (37) and the above condition (39). These two equations are rewritten in terms of a new variable, viz., the holonomy operator, in the next section.

Equations (37) and (39) with a special choice of data (on $\mathcal{I}^+$) are equivalent to the Einstein equations.

**B. Holonomy and the Bianchi Identities.**

The variable that we have been (loosely) referring to as the holonomy operator and denoting by $H$, is more accurately the difference between the holonomy operator associated with an infinitesimal path and the identity. It should really be called the infinitesimal holonomy operator or the differential holonomy operator, though we will continue with the original name. There are two distinctly different sets of paths $\Delta x(\zeta, \bar{\zeta})$ and $\bar{\Delta} x(\zeta, \bar{\zeta})$ (with their own holonomies $H$ and $\bar{H}$) defined in the following manner: Consider an interior point $x^a$ of $\mathcal{M}$. For $\Delta x(\zeta, \bar{\zeta})$ we choose two null rays on the cone $N_x$ that are infinitesimally separated, namely, $\ell_x(\zeta, \bar{\zeta})$ and $\ell_x(\zeta + d\zeta, \bar{\zeta})$, extending from $x^a$ to $\mathcal{I}^+$ (see Figure 3). We
then form a closed loop by connecting the end points of these two rays on $\mathcal{I}^+$. At $\mathcal{I}^+$ the two-form constructed from this connecting vector and the tangent vector to the geodesics is self-dual. (In Minkowski space the entire path lies in a self-dual blade; it is the loss of this property in curved spacetimes that is the source of non-Huygens propagation for linear rest-mass zero fields). In a similar manner, one can choose the anti-self-dual triangle $\bar{\Delta}_x(\zeta, \bar{\zeta})$, which has $\ell_x(\zeta, \bar{\zeta})$ and $\ell_x(\zeta + d\zeta, \bar{\zeta})$ for its sides.

For our $O(3, 1)$ connection, the vectors that are being propagated around the closed loops ($\Delta_x$ and $\bar{\Delta}_x$) are thought of (using a soldering form $\lambda^a_i(x)$ introduced below) as being in the tangent (or cotangent) bundle. The effect of applying the operators $H$ or $\bar{H}$ to an arbitrary vector $V^\mu$ at $\mathcal{I}^+$ is, respectively,

$$V'^\mu = V^\nu \left( \delta^\mu \nu + \frac{H^\mu \nu(x^a, \zeta, \bar{\zeta}) d\zeta}{2P} \right) \tag{40}$$

and

$$V'^\mu = V^\nu \left( \delta^\mu \nu + \frac{\bar{H}^\mu \nu(x^a, \zeta, \bar{\zeta}) d\bar{\zeta}}{2P} \right), \tag{41}$$

where the $P = (1 + \zeta \bar{\zeta})$ is for notational purposes. $H$ takes us from a point on $\mathcal{I}^+$ along $\ell_x(\zeta, \bar{\zeta})$ down to $x^a$ then back to $\mathcal{I}^+$ along $\ell_x(\zeta + d\zeta, \bar{\zeta})$ and finally back to the starting point along a connecting vector on $\mathcal{I}^+$ (see Figure 3). For parallel transport around this loop one obtains the following:

$$H^\mu \nu(x^a, \zeta, \bar{\zeta}) = A^\mu \nu + (G^{-1})^b \nu b \partial G^\mu b, \tag{42}$$

where $G^\mu_a(x^a, \zeta, \bar{\zeta})$ is the parallel propagator that takes vectors from the point $x^a$ to $\mathcal{I}^+$ along the null geodesic $\ell_x(\zeta, \bar{\zeta})$, and $A^\mu \nu$ are the asymptotic values of the components of the Christoffel symbol in the direction of the connecting vector on $\mathcal{I}^+$. See [11] and [12] for a derivation of this result.

It will be convenient (see (59)) to express the above relation in terms of a given null tetrad $\lambda^a_i(x)$ defined on $\mathcal{M}$ and $\mathcal{I}^+$ with normalization

$$g_{ab} \lambda^a_i \lambda^b_j = \eta_{ij}, \tag{43}$$
with

\[ \eta_{ij} = \eta^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \] (44)

and \( H_{ij} = H^{\mu}_{\nu} \lambda^i_{\mu} \lambda^j_{\nu} \), etc. Tetrad indices are raised and lowered with \( \eta^{ij} \) and \( \eta_{ij} \), respectively. For the tetrad fields \( \lambda^\mu_i \) at \( I^+ \) we will use the Bondi tetrad \( \{ \ell_\mu, n_\mu, m_\mu, m_\mu \} \) associated with the Bondi coordinates.

The Bianchi Identities (37) can be written in terms of the holonomy operator as

\[ \bar{\partial}(H - A)^i_j - \bar{\partial}(H - A)^i_j + [H - A, \bar{H} - \bar{A}]^i_j - [H, \bar{A}]^i_j = 0. \] (45)

A derivation of this result, via an integration of (37) over the region \( V \) of Figure 3, can be found in [1]. Using \( \eta^{ij} \) to raise one of the indices, we obtain \( H^{ij} \), \( A^{ij} \), and their complex conjugates. Since these are skew in the \( \lambda^{\hat{i}\hat{j}} \) indices, they can be separated into self-dual and anti-self-dual parts, yielding

\[ H^{ij} = H^{(-)ij} + h^{(+)ij} \quad \text{and} \quad A^{ij} = A^{(+ij} + A^{(-)ij}. \] (46)

and

\[ \bar{H}^{ij} = \bar{H}^{(+ij}_j + \bar{h}^{(-)ij} \quad \text{and} \quad \bar{A}^{ij} = \bar{A}^{(-)ij} + \bar{A}^{(+ij}. \] (47)

We have three complex non-trivial components of \( H^{(-)} \) which we denote by \( H_\alpha \) and likewise three \( h_\alpha \), where \( \alpha = \{1, 2, 3\} \), as shown in Table 1. As for the characteristic data, the \( A \)'s, most of them are zeroes and ones: they are given by the following:

\[ A^{i_j} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\hat{\sigma}_B \\ -\hat{\sigma}_B & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}. \] (48)
and

\[
\tilde{A}_j^i = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -\mathbf{\sigma}_B & 1 \\
1 & -1 & 0 & 0 \\
-\mathbf{\sigma}_B & 0 & 0 & 0
\end{pmatrix}.
\] (49)

The self-dual and anti-self-dual parts of $H$ can be written out explicitly as well,

\[
H^{(-)ij} = \begin{pmatrix}
0 & \frac{1}{2}(H^{01} - H^{+-}) & H^{0+} & 0 \\
-\frac{1}{2}(H^{01} - H^{+-}) & 0 & 0 & H^{1-} \\
-H^{0+} & 0 & 0 & -\frac{1}{2}(H^{01} - H^{+-}) \\
0 & -H^{1-} & \frac{1}{2}(H^{01} - H^{+-}) & 0
\end{pmatrix}
\] (50)

and

\[
h^{(+)}ij = \begin{pmatrix}
0 & \frac{1}{2}(H^{01} + H^{+-}) & 0 & H^{0+} \\
-\frac{1}{2}(H^{01} + H^{+-}) & 0 & H^{1+} & 0 \\
0 & -H^{1+} & 0 & \frac{1}{2}(H^{01} + H^{+-}) \\
-H^{0-} & 0 & -\frac{1}{2}(H^{01} + H^{+-}) & 0
\end{pmatrix},
\] (51)

and likewise for $\overline{H}$.

In order to raise and lower the $i, j$ indices, we have to use the null-coordinate version of
the Minkowski metric $\eta_{ij}$ and $\eta^{ij}$. For example,

\[
H^{0}_{\ 1} = H^{00} = 0, \text{ by skew symmetry ,} \quad (52)
\]

\[
H^{0\ +} = -H^{0\ -}. \quad (53)
\]

Also note that the complex conjugate of a quantity, say, $\overline{H^{0\ -}}$ gives us $\overline{H^{0\ +}}$: the 0 and
1 indices are insensitive to complex conjugation, while $+$ and $-$ are interchanged. The
following table displays our notation.
With this notation the holonomy Bianchi identities (45) become:

\[
\begin{align*}
\bar{\partial}h_1 - \bar{\partial}H_1 + 2h_1H_1 - 2H_1h_2 + 2H_2 &= 0, \\
\bar{\partial}h_2 - \bar{\partial}H_2 + h_3H_1 - H_3(h_1 + 1) + h_1 &= -\hat{\sigma}_bH_1, \\
\bar{\partial}h_3 - \bar{\partial}H_3 + 2h_2(H_3 - 1) - 2H_2h_3 &= -\delta\hat{\sigma}_b + 2\hat{\sigma}_bH_2,
\end{align*}
\]  

(54)  

(55)  

(56)

where we have the data or “driving” terms (i.e., those that involve $\sigma_b$) on the right. Equation (45) actually contains this triplet as well as their conjugates. They are the result of taking the self-dual and anti-self-dual parts of equation (45). This is the first set of equations that was given symbolically as equation (2) in Section I. (Note that the first equation is algebraic in $h_2$ and $H_2$, and the second is algebraic in $h_3$ and $H_3$. We will exploit this structure in Section III.D where we study these equations in more detail.

### C. The $\Lambda(H)$-relations.

In the previous Section we defined the parallel propagator, $G^\mu_a(x^a, \zeta, \bar{\zeta})$, which takes vectors from a point on $I^+$ to an interior point $x^a$ along a null geodesic $\ell_x(\zeta, \bar{\zeta})$. In particular,
the tetrad fields \{\lambda^i_\mu, \lambda^\mu_i\} that are defined on \(I^+\) can be parallelly propagated in this manner to the point \(x^a\). The parallelly propagated tetrad fields at \(x^a\) are given by

\[
e^i_a = \lambda^i_\mu G^\mu_a, \tag{57}\]

where \(\lambda^i_\mu\) represents the tetrad \{\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu\} on \(I^+\). We use the following notation for the parallelly propagated tetrad at \(x^a\):

\[
e^i_a \equiv \{e^0_a, e^1_a, e^+_a, e^-_a\} \equiv \{\ell_a, n_a, m_a, \overline{m}_a\}. \tag{58}\]

If we take two sets of tetrads on \(I^+\), one at \((\zeta, \bar{\zeta})\) and the other (on the same cut, associated with \(x^a\)) at \((\zeta + d\zeta, \bar{\zeta})\), and parallelly propagate both to the point \(x^a\), the difference in terms of the holonomy operator \(H\) and the free data \(A\) is given by

\[
e^a_j \partial e^j_a = H^i_j - A^i_j , \tag{59}\]

\[
e^a_j \tilde{\partial} \tilde{e}^j_a = \overline{H}^i_j - \overline{A}^i_j. \tag{60}\]

We have introduced three different tetrad bases: \(\lambda^i_a, \lambda^i_\mu, \) and \(e^i_a\). The first of these, \(\lambda^i_a\), is an arbitrary tetrad given at \(x^a\) and therefore independent of \(\zeta\) and \(\bar{\zeta}\). The second, \(\lambda^i_\mu\), is a \((\zeta, \bar{\zeta})\)-dependent Bondi tetrad given on \(I^+\). And the third, \(e^i_a\), is the result of parallelly propagating the \(\lambda^i_\mu\) in from \(I^+\) to \(x^a\) along \(\ell_x(\zeta, \bar{\zeta})\), and therefore is also \((\zeta, \bar{\zeta})\)-dependent.

From Section III, we have at our disposal the \(\theta^i\) coordinates and the associated bases \(\theta^i_a\) and \(\theta^a_i\) at \(x^a\). Since any vector at \(x^a\) can be expressed as a linear combination of either of the two sets \{\(\theta^i_a\)\} and \{\(e^i_a\)\}, we can go from one to the other, using the invertible transformation

\[
\theta^i_a = \Sigma^i_\ell e^\ell_a. \tag{61}\]

We choose one of the ‘legs’ of the parallelly propagated tetrad vectors, \(e^0_a\), equal to the vector \(\nabla_a Z\), i.e.,

\[
e^0_a \equiv \ell_a = Z_{,a} \equiv \theta^0_a. \tag{62}\]
Using this assumption and equations (59) and (60) with our notation from Table 1, the transformation matrix $\Sigma^i \equiv \Sigma$ and its inverse $\Sigma^{-1} \equiv \Sigma^i$ can be calculated (by application of $\bar{\sigma}$ and $\bar{\tau}$ to (59) and (60)) and written explicitly as functions of the $H$’s and $h$’s [1]. $\Sigma$ is given by

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
\Sigma^1_0 & \Sigma^1_1 & \Sigma^1_+ & \Sigma^1_- \\
-(H_2 + h_2) & 0 & (1 + h_1) & H_1 \\
-\bar{H}_2 + h_2 & 0 & \bar{H}_1 (1 + h_1) & 
\end{pmatrix}$$

(63)

with

$$\begin{align*}
\Sigma^1_0 &= -\bar{\sigma}(\bar{H}_2 + h_2) + (\bar{H}_2 + h_2)(H_2 + \bar{H}_2) + \bar{H}_1(\dot{\sigma}_B + \bar{\tau}_3) + (1 + h_1)(H_3 - 1) \\
\Sigma^1_1 &= (1 + h_1)(1 + \bar{H}_1) + H_1\bar{H}_1 \\
\Sigma^1_+ &= \bar{\sigma}\bar{H}_1 - (\bar{H}_2 + h_2)(1 + \bar{H}_1) - \bar{H}_1(H_2 - \bar{H}_2) \\
\Sigma^1_- &= \bar{\sigma}\bar{H}_1 + (H_2 - \bar{H}_2)(1 + h_1) - H_1(\bar{H}_2 + h_2)
\end{align*}$$

Using $\Sigma$, its conjugate, the equations (23), i.e.,

$$\bar{\sigma}\theta^i_a = T^i_j \theta^j_a$$

(64)

and their conjugates, the $T^i_j$’s and $\bar{T}^i_j$’s of Section II can be expressed directly in terms of the $H$’s and $h$’s via the $\Sigma^i_j$ by

$$T = (\bar{\sigma}\Sigma) \cdot \Sigma^{-1} + \Sigma \cdot (H - A) \cdot \Sigma^{-1}$$

(65)

where the boldface represents the corresponding matrices [1]. Since $T^i_j$ was shown in equation (30) to have the form

$$T \equiv (T^i_j) = \left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
T^1_0 & T^1_1 & T^1_+ & T^1_- \\
\Lambda_0 & \Lambda_1 & \Lambda_+ & \Lambda_- \\
0 & 1 & 0 & 0
\end{array}\right)$$

(66)
we obtain from (65) and (66) the \( \Lambda(H) \)-relations:

\[
\Lambda_1 \equiv \Lambda_1 = \frac{2(\bar{H}_1 + 1)H_1}{(\bar{h}_1 + 1)(h_1 + 1) + \bar{H}_1 H_1},
\]

\[
\Lambda_\pm = W - \frac{1}{2} \left( \Lambda_1 \bar{W} + \bar{\partial} \Lambda_1 - \partial \ln q \right),
\]

\[
\Lambda_- = \frac{1}{2} \left[ \bar{\partial} \Lambda_1 - \Lambda_1 \frac{\bar{\partial}(H_1 H)}{H_1 H_1} - \Lambda_1 \bar{\partial} \ln \left( \frac{q}{1 - \sqrt{q}} \right) \right],
\]

\[
\Lambda_0 = \bar{\sigma}_B + \frac{1}{2} \Lambda_1 - \frac{1}{4} \bar{\partial} \Lambda_1 + O(H^2),
\]

where \( W \) is defined by

\[
W \equiv \frac{(\bar{\partial} \Lambda_1 - 2\Lambda_-)}{\Lambda_1} - \bar{\partial} \ln q,
\]

with

\[
q \equiv (1 - \Lambda_1 \bar{\Lambda}_1).
\]

See [1] and [4] for a more complete derivation of these equations. The precise form of equation (70) turns out to be rather lengthy and is given in Appendix A of [1].

**D. The Holonomy Field Equations.**

In this section, we describe an important result obtained earlier [11], wherein the vacuum Einstein equations in the form

\[
^{+}F_{ab} = 0 \quad (73)
\]

were imposed on the holonomy operator. The goal is to obtain field equations for the holonomy operator, and therefore we first need a relationship between the curvature tensor and the holonomy operator.

It can been shown [11], using a non-Abelian version of Stokes’ theorem, that the holonomy operator satisfies

\[
H = \int_{s_0}^{\infty} (F_{ab}^+ + F_{ab}^-) \ell^a M^b ds = h^{(+)} + H^{(-)} \quad (74)
\]
and

$$\overline{H} = \int_{s_0}^{\infty} (F^+_{ab} + F^-_{ab}) \ell^a \overline{M}^b ds = H^{(+)} + h^{(-)}, \quad (75)$$

where the $H$ and $h$ with the plus and minus signs were defined in Section III.B, $s$ is an affine parameter along the generators of the light cone at $x^a$ with $s = s_0$ at $x^a$. $M^a$ and $\overline{M}^a$ are connecting vectors between, respectively the long legs of $\Delta_x$ and $\overline{\Delta}_x$. See [11] for more details.

The two equations (74) and (75) can be inverted to obtain $F^-_{ab}$ and $F^+_{ab}$ explicitly in terms of the components of the holonomy operator. Specifically $F^-_{ab}$ and $F^+_{ab}$ can be expressed as two derivatives of the $H$’s with respect to $\theta^1 = R$. Using this expression for the curvature in $^+F^-_{ab} = 0$ one obtains a differential relationship between the different components of $H$ and $\overline{H}$, given by

$$[q^{-1}h_{a,R}]_R + \delta[q^{-1}\Lambda_1 h_{a,R}]_R = [q^{-1}\overline{\Lambda}_1 H_{a,R}]_R + \delta[q^{-1}H_{a,R}]_R, \quad (76)$$

where

$$\delta = \sqrt{q^{-1} - 1} \Lambda_1. \quad (77)$$

Note that the equations (76), which we refer to as the field equations, are three linear relations between the $H$ and $h$ with coefficients depending $Z$ via $\Lambda_1$. They are our final set of equations. In the next section, we will use these with the Bianchi identities and the $\Lambda(H)$-relations to simplify the overall structure of the theory.

E. The Full Theory.

We have at this point three sets of equations:

- the three holonomy Bianchi identities,

$$\partial h_1 - \bar{\partial} H_1 + 2h_1 H_1 - 2H_1 h_2 + 2H_2 = 0, \quad (78)$$

$$\partial h_2 - \bar{\partial} H_2 + h_3 H_1 - H_3(h_1 + 1) + h_1 = -\tilde{\sigma}_b H_1, \quad (79)$$

$$\partial h_3 - \bar{\partial} H_3 + 2h_2(H_3 - 1) - 2H_2 h_3 = -\tilde{\sigma}_b + 2\tilde{\sigma}_b H_2, \quad (80)$$

the holonomy field equations,

\[ [q^{-1}h_{\alpha,R}]_R + \delta[q^{-1}\Lambda_1 h_{\alpha,R}]_R = [q^{-1}\tilde{\Lambda}_R h_{\alpha,R}]_R + \delta[q^{-1}H_{\alpha,R}]_R, \quad (81) \]

- the \( \Lambda(H) \)-relations,

\[
\begin{align*}
\Lambda_1 &\equiv \Lambda_1 = \frac{2(h_1 + 1)H_1}{(h_1 + 1)(h_1 + 1) + H_1 H_1}, \\
\Lambda_+ &\equiv W - \frac{1}{2} \left( \Lambda_1 W + \bar{\delta} \Lambda_1 - \bar{\delta} \ln q \right), \\
\Lambda_- &\equiv \frac{1}{2} \left[ \delta \Lambda_1 - \Lambda_1 \frac{\bar{\delta}(H_1 \bar{H}_1)}{H_1 \bar{H}_1} - \Lambda_1 \bar{\delta} \ln \left( \frac{q}{1 - \sqrt{q}} \right) \right], \\
\Lambda_0 &\equiv \dot{\sigma}_B + \frac{1}{2} \Lambda_1 - \frac{1}{4} \delta \bar{\delta} \Lambda_1 + O(H^2),
\end{align*}
\]

and their conjugates. The first set involves the \( H \)'s and \( h \)'s and the data only, while the other two involve the \( H \)'s and \( h \)'s and the \( \Lambda \)_i. We can think of the \( H \)'s and \( h \)'s as describing the “field” and the \( \Lambda \)_i, the “background”. In this Section we study the above equations and show how they can be reduced to a smaller and simpler set.

This simplification procedure begins with the following important observation: equation (82) and its complex conjugate are two algebraic relations between the six quantities, viz., \( \Lambda_1 \) and \( \bar{\Lambda}_1 \), \( H_1 \), \( \bar{H}_1 \), \( h_1 \) and \( \bar{h}_1 \). Solving for \( h_1 \) and \( \bar{h}_1 \) algebraically yields,

\[
\begin{align*}
h_1 &= -\frac{H_1}{\delta} - 1, \\
\bar{h}_1 &= -\frac{\bar{H}_1}{\delta} - 1,
\end{align*}
\]

where

\[
\delta = \frac{\sqrt{q} - 1}{\Lambda_1} \quad \text{and} \quad q = 1 - \Lambda_1 \bar{\Lambda}_1,
\]

Thus \( h_1 \) and \( \bar{h}_1 \) are not independent quantities and equations (86) and (87) allow us to eliminate them completely. This algebraic structure extends to the other \( h \)'s. We are able to solve, from equations (78) and (79), for all the remaining \( h \)'s and completely eliminate them from the remainder of the analysis. The algebraic relationships between \( H_\alpha \) and \( h_\alpha \) for all \( \alpha = \{1, 2, 3\} \) takes the form
\[ h_\alpha = -\frac{H_\alpha}{\delta} + G_\alpha, \]  

(89)

where \( G_\alpha \) is a function of \( \Lambda_1 \) and the preceding \( H \)'s. See [1] for explicit expressions for the \( G_\alpha \). The important consequence of the above structure is that after eliminating the \( h \)'s from the holonomy field equations (81) we obtain a very attractive set of three differential equations for \( H_\alpha \):

\[
\frac{\partial^2 H_\alpha}{\partial R^2} + X \frac{\partial H_\alpha}{\partial R} + Y H_\alpha = \mu \frac{\partial^2 G_\alpha}{\partial R^2} + \nu \frac{\partial G_\alpha}{\partial R},
\]

(90)

where \( X, Y, \mu \) and \( \nu \) are functions only of \( \Lambda_1 \) and \( \bar{\Lambda}_1 \).

These three equations are linear with the only difference among them being in the homogeneous terms: the first of them, the equation for \( H_1 \), is homogeneous with the coefficients depending only on \( \Lambda_1 \) (a function of \( R \)), while for the next pair the inhomogeneous terms are driven by the solutions of the previous ones. From this structure, if the homogeneous equation could be solved, the remaining ones could be solved by quadratures.

To summarize, we have so far used the information contained in two (of the three) Bianchi identities, one of the \( (\Lambda, H) \)-equation, and all the field equations. The third Bianchi identity has not yet played any role. Assuming that we could solve (90) for \( H_1 \) as a function (functional) of \( \Lambda_1 \) we would have all the \( H_\alpha = H_\alpha(\Lambda_1, \bar{\Lambda}_1) \) and their complex conjugates. These expressions, which would be explicit functions of \( \Lambda_1 \) and \( \bar{\Lambda}_1 \), could then be substituted in the third Bianchi identity. The resulting equation would involve only \( \Lambda, \bar{\Lambda} \), and the data. The single resulting equation, involving only \( \Lambda_1 \) and \( \bar{\Lambda}_1 \), i.e.,

\[
\bar{\delta}(H_3/\delta + G_3) - \bar{\delta}H_3 + 2(H_2/\delta + G_2)(H_3 - 1) - 2H_2(H_3/\delta + G_3)
\]

\[= -\bar{\delta}\sigma_\beta \bar{\sigma}_\beta + 2\bar{\delta} H_2, \]

(91)

would be our sought for equation for the determination of the characteristic surfaces and therefore the conformal metric of the Einstein spacetime.

The equation, however, can be manipulated, from the leading terms (but not explicitly) of the remaining \( \Lambda(H) \)-relations, into a simpler and more attractive equation that is referred
to as the light cone cut equation (LCCE) whose solution directly yields the light cone cut function.

Using the solution to equation (90) in the form

\[ H_1 = \frac{1}{2} \Lambda_1 + O(\Lambda^2), \]  

we have from substitution into equation (91)

\[ \bar{\partial}^3 \Lambda_1 = -4 \bar{\partial} \sigma_B + \text{higher order terms}, \]  

(93)

where \( \sigma_B \) is free data given on \( \mathcal{I}^+ \). Manipulating this and using the equation (85) of the \( \Lambda(H) \)-relations we obtain,

\[ \bar{\partial}^2 \Lambda_0 = \bar{\partial}^2 \sigma_B + \bar{\partial}^2 \dot{\sigma}_B + \text{higher order terms}, \]  

(94)

or, from the definition \( \bar{\partial}^2 Z_0 = \Lambda_0 \),

\[ \bar{\partial}^2 \bar{\partial}^2 Z = \bar{\partial}^2 \sigma_B + \bar{\partial}^2 \sigma_B + \text{higher order terms} \]  

(95)

as our final equation for the light cone cut function \( Z \). Although the higher order terms have not been worked out explicitly, they can be obtained via a perturbation scheme (see next Section). The solutions to equation (95) yield the conformal structure of asymptotically flat vacuum spacetime.

In the above analysis it was assumed that the solution to the equation for \( H_1(\Lambda) \), namely,

\[ \frac{\partial^2 H_1}{\partial R^2} + X \frac{\partial H_1}{\partial R} + Y H_1 = 0, \]  

(96)

was known. Since it is a linear second order o.d.e. solutions for arbitrary \( \Lambda(R) \) must exist, though it is not clear—nor is it likely—that we will be able to solve it explicitly. It will however, always be of the form

\[ H_1 = \frac{1}{2} \Lambda_1 + O(\Lambda^2) \]  

(97)

with the higher order terms computable perturbatively.
Since the LCCE, (95), determines the conformal structure of vacuum spacetimes, the only remaining quantity to be determined is the conformal factor needed to convert the conformal metric into a vacuum metric. In the $\theta^i$ coordinate system the relation between the conformal metric $g$ and the full metric $\hat{g}$ is given by

\[
\hat{g}^{ij} = \Omega^2 g^{ij}
\]  

(98)

It can be shown [1] that the conformal factor can be expressed as a simple function of $H_1$ and $\Pi_1$, namely,

\[
\Omega^2 = (1 + h_1)(1 + \overline{h}_1) + H_1 \overline{H}_1.
\]  

(99)

[As an aside we remark that from equation (96), it is possible to show that the conformal factor (99) satisfies the equation

\[
\frac{d^2 \Omega}{dR^2} = Q(\Lambda)\Omega,
\]

where $Q = Q(\Lambda)$ is a simple known function of $\Lambda_1$. The above equation is known in the literature [13] as the Einstein bundle equation.]

IV. AN ITERATIVE SCHEME.

Much of the understanding of the structure and meaning of our final equations has come from the application of an iterative scheme to the formal theory and looking at the leading behavior. In this section, we present the results obtained in linear order.

We expand all quantities in powers of a small parameter $\epsilon$ which measures the deviation from flatness. From the assumption that $\epsilon$ enters as a multiplicative factor of the Bondi shear, i.e. via $\epsilon \sigma_B$, it becomes clear that the expansions have the form

\[
Z = Z^{(0)} + \epsilon Z^{(1)} + \epsilon^2 Z^{(2)} + ..., \quad (100)
\]

\[
e_a^i = e_a^i + \epsilon e_a^i + \epsilon^2 e_a^i + ..., \quad (101)
\]

\[
\Lambda = \Lambda^{(1)} + \epsilon^2 \Lambda^{(2)} + \epsilon^3 \Lambda^{(3)} + ..., \quad (102)
\]
\[ H = \epsilon^{(1)} H + \epsilon^2 H^{(2)} + \epsilon^3 H^{(3)} \ldots, \]  
\[ h = \epsilon^2 h^{(2)} + \epsilon^3 h^{(3)} + \ldots. \]  

There is no zeroth order contribution to \( \Lambda \) or the holonomy operator since these quantities are zero for Minkowski space. Furthermore, a direct calculation (using equations (86), (88), and (124)) shows that \( h \) begins at second order.

Since \( \Lambda \) starts with order one in the perturbation expansion, it follows from the relationship between \( Z \) and \( \Lambda \) that \( Z^{(0)} \) satisfies

\[ \varpi^2 Z_a^{(0)} = 0, \]  
whose solution is the Minkowski space light cone cut function

\[ Z^{(0)} = x^a \ell_a^{(0)}, \]

with

\[ \ell_a^{(0)}(\zeta, \bar{\zeta}) = \frac{1}{\sqrt{2}P} (1 + \zeta \bar{\zeta}, \zeta + \bar{\zeta}, i(\zeta - \bar{\zeta}), -1 + \zeta \bar{\zeta}); \quad P = (1 + \zeta \bar{\zeta}). \]

**A. Linearized Gravity.**

We begin the linearization with the three holonomy Bianchi identities, which to first order are:

\[ H_2^{(1)} = \frac{1}{2} \bar{\varpi} H_1^{(1)}, \]  
\[ H_3^{(1)} = -\bar{\varpi} H_2^{(1)}, \]  
\[ \bar{\varpi} H_3^{(1)} = \bar{\varpi} \bar{\varpi} H_B^{(1)}, \]

where the \( h \)'s have disappeared since they have no first order contribution. The next step in the process of elimination would naturally be to solve the holonomy field equations obtaining the \( H \)'s in terms of the \( \Lambda \). However, to first (and second order) we can circumvent this step and consider the first of the \( (\Lambda, H) \)-equations directly. It leads to an expression for \( H_1 \),
which with equations (106) and (107), yields the first order $H_\alpha$ that automatically satisfy the field equations (90) to first order. (For third and higher order we would have to solve the field equations before using the $\Lambda(H)$-relations for the simplification procedure.)

We thus have

$$H_1^{(1)} = \frac{1}{2} \Lambda_1,$$  \hspace{1cm} (109)

$$H_2^{(1)} = \frac{1}{4} \bar{\partial} \Lambda_1^{(1)},$$  \hspace{1cm} (110)

$$H_3^{(1)} = -\frac{1}{4} \bar{\partial}^2 \Lambda_1^{(1)}.$$  \hspace{1cm} (111)

Our final equation for the cut function is the linearized third Bianchi identity, which, after substituting for $H_3$, is

$$\bar{\partial}^3 \Lambda_1 = -4 \bar{\partial} \sigma_B.$$  \hspace{1cm} (112)

(Although we did not have to solve the holonomy field equations, it is a straightforward but slightly tedious process to verify that the expressions (109)-(111) for the $H$’s do satisfy the field equations. A minor subtlety in the calculation is that the linearization of the field equations yields identically vanishing expressions, so that one must look at the equations at their first non-vanishing order. Performing the calculation, nevertheless, serves as a consistency check between the different sets of equations.)

At this point we have used all the equations except the three remaining $\Lambda(H)$-relations,

$$\Lambda_- = \frac{1}{2} \partial \Lambda_1,$$  \hspace{1cm} (113)

$$\Lambda_+ = -\frac{1}{2} \partial \Lambda_1,$$  \hspace{1cm} (114)

$$\Lambda_0 = \dot{\sigma}_B + \frac{1}{2} \Lambda_1 - \frac{1}{4} \bar{\partial} \Lambda_1.$$  \hspace{1cm} (115)

These relations can be used to write (112) in the more symmetric form as the linearized version of the light cone cut equation:

$$\bar{\partial}^2 \partial Z = \partial^2 \sigma_B(Z) + \bar{\partial}^2 \sigma_B(Z),$$  \hspace{1cm} (116)
which is the light cone cut equation accurate to first order. In other words, this equation
is equivalent to the the linearized conformal Einstein equations. (The above equation for
linear theory has been derived from the vanishing of the Bach tensor, by Lionel Mason [14].)

In linear theory (see (99)) the conformal factor equals 1, and therefore in this approxima-
tion we have not only the conformal Einstein equations but the Einstein equations them-
selves.

Equation (116), is to be thought of as an equation for $Z$, whose solution can be written
as a sphere integral using the Green function $G$ of the operator $\bar{\delta}^2\delta^2$. See Appendix C of [1]
for a derivation of the Green function of this operator. In other words we can write

$$Z(x, \zeta, \bar{\zeta}) = \int G(\zeta, \zeta; \eta, \bar{\eta})[\bar{\delta}^2\sigma_B(0)(x, \eta, \bar{\eta})] + \bar{\delta}^2\sigma_B(Z(0)(x, \eta, \bar{\eta}))dS_\eta, \quad (117)$$

with $dS_\eta$ the sphere volume element in $(\eta, \bar{\eta})$ coordinates, as the general solution to the
asymptotically flat linearized vacuum Einstein equations. In principle, though we have not
yet done so in practice, the higher order terms could be calculated successively by similar
integrals but with the integrands depending on terms to lower order in the perturbation
scheme.

V. SUMMARY AND CONCLUSIONS.

We have obtained three sets of coupled equations, the holonomy Bianchi Identities,
the $\Lambda(H)$-relations and the “field equations” for the holonomy operator $H$ and the light
cone cut function $Z$, which are equivalent to the full vacuum Einstein equations. These
equations, which already have built into them the free choice of Bondi data $\sigma_B(u, \zeta, \bar{\zeta})$, can
be manipulated and simplified (in structure) to one (complicated) equation, the LCCE, and
a simple one for the conformal factor. On analysis, they yield perturbatively, a D’Adhemar-
like formulation of general relativity.

On the negative side these equations are quite unusual and are based on unfamiliar
ideas and variables, and unfortunately are quite complicated. They nevertheless have some
features of considerable attractiveness: new insights often can be gained from the use of new variables; the perturbative solutions, from given data, are essentially unique; as the perturbation calculation proceeds to higher order the formalism yields the corrected light cone structure from the preceding order, in contrast to the usual perturbation theory which uses, at all orders, the Minkowski light cone structure.

As a final comment, we remark that the work reported on here is the direct antecedent of another approach to general relativity (the Null Surface Theory of GR, to be reported elsewhere) that is based solely on families of characteristic surfaces as the basic variable of the theory—without any mention of fields or holonomies. The new view, which has certain similarities to the present work and will probably yield considerable simplifications over it, could not have been developed without the current approach.

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