Supersymmetric Extension of
the Non-Abelian Scalar-Tensor Duality

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The field theory dual to the Freedman-Townsend model of a non-Abelian anti-symmetric tensor field is a nonlinear sigma model on the group manifold $G$. This can be extended to the duality between the Freedman-Townsend model coupled to Yang-Mills fields and a nonlinear sigma model on a coset space $G/H$. We present the supersymmetric extension of this duality and find that the target space of this nonlinear sigma model is a complex coset space, $G^C/H^C$.

§1. Introduction

Electric-magnetic duality plays a crucial role in studies of non-perturbative dynamics of four-dimensional supersymmetric gauge theories.1) In four dimensions, there exists another duality between the scalar field and the anti-symmetric tensor (AST) field, called the scalar-tensor duality.2) The AST field $B(x) = \frac{1}{2} B_{\mu\nu}(x)dx^\mu \wedge dx^\nu$ is an important ingredient in supergravity and superstrings.3),4) The massless theory of the AST field possesses AST gauge invariance: $B(x) \rightarrow B(x) + d\xi(x)$, with $\xi(x)$ being a one-form gauge parameter. In this case, the dual field theory is a free scalar field theory.

The non-Abelian generalization of the theory of the AST field was made by Freedman and Townsend,5) and earlier by Ogievetsky and Polubarinov.6) It is called the ‘Freedman-Townsend (FT) model’. In this theory, the AST field $B_{\mu\nu} = B^i_{\mu\nu} T_i$ transforms as the adjoint representation of the group $G$, where $T_i$ are generators of $G$. This theory possesses a non-Abelian generalization of the AST gauge invariance.5),7) In this case, the dual scalar field theory is a nonlinear sigma model (NLSM) on the group manifold $G$, called the principal chiral model. Nonlinear sigma model in four dimensions have attracted interest as low-energy effective field theories of QCD. The NLSM on the coset spaces $G/H$ naturally appear when the global symmetry is spontaneously broken down to its subgroup $H$.8) The scalar-tensor duality can be extended to a duality between AST gauge theories coupled to the Yang-Mills (YM) gauge field and NLSM on coset spaces $G/H$.9)

Supersymmetry plays roles in realizing electric-magnetic duality and other types

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of dualities at the quantum level in gauge theories and string theories.\textsuperscript{1,4} In this paper, we investigate the supersymmetric extension of the scalar-tensor duality. The Abelian scalar-tensor duality in supersymmetric field theories has been studied extensively.\textsuperscript{10} -\textsuperscript{12} The non-Abelian generalization of scalar-tensor duality in supersymmetric field theories was first reported in Ref.\textsuperscript{13}. The dual field theories are the supersymmetric NLSM on the complex extension of the group manifold $G, G^C$. This is related to the fact that the target spaces of the supersymmetric NLSM must be K"{a}hler.\textsuperscript{14} We extended the previous study of scalar-tensor duality in supersymmetric theories from a group manifold $G^C$ to the coset spaces $G^C/H^C$.

This paper is organized as follows. In §2, we recapitulate the non-Abelian scalar-tensor duality in bosonic theories. In §3, we recall the scalar-tensor duality in supersymmetric field theories in the case of the complex extension of the group manifold as the target space of the NLSM.\textsuperscript{13} The relation between the (quasi-) Nambu-Goldstone bosons in the NLSM and the boson fields in the AST gauge theory is clarified. In §4 we construct the supersymmetric extension of the duality of the non-Abelian AST field theory and the NLSM on a coset space on the basis of the results in §§2 and 3. Section 5 is devoted to discussion.

§2. Non-Abelian scalar-tensor duality in bosonic theories

We begin by recapitulating the non-Abelian scalar-tensor duality\textsuperscript{5} and its $G/H$ generalization.\textsuperscript{9} We consider the non-Abelian AST field $B_{\mu\nu}$ with the group $G$. Our notation for its Lie algebra $G$ is

$$T_i \in \mathcal{G}, \quad i = 1, \ldots, \dim G,$$

$$[T_i, T_j] = if_{ij}^k T_k, \quad \text{tr}(T_i T_j) = c\delta_{ij}, \quad \text{(2.1)}$$

where the coefficients $f_{ij}^k$ are the structure constants, and $c$ is a positive constant. We use matrix notation for the AST field and the auxiliary vector field:

$$B_{\mu\nu}(x) \equiv B_{\mu\nu}^i(x)T_i, \quad A_{\mu}(x) \equiv A_{\mu}^i(x)T_i. \quad \text{(2.2)}$$

The mass dimensions of $B_{\mu\nu}$ and $A_{\mu}$ are 1 and 2, respectively. The Lagrangian of the FT model in first-order form is given by

$$\mathcal{L} = -\frac{1}{8c} \text{tr}(\epsilon^{\mu
u\rho\sigma} B_{\mu\nu} F_{\rho\sigma} + A_{\mu} A^{\mu}), \quad \text{(2.3)}$$

where $F_{\rho\sigma} \equiv \partial_\rho A_\sigma - \partial_\sigma A_\rho - i\lambda [A_\rho, A_\sigma]$, and $\lambda$ is the coupling constant, with dimension $-1$. This Lagrangian is invariant up to a total derivative under the AST gauge transformation, defined by

$$\delta B_{\mu\nu} = D_\mu(A)\xi_\nu - D_\nu(A)\xi_\mu, \quad \delta A_{\mu} = 0, \quad \text{(2.4)}$$

where $\xi_\mu(x) \equiv \xi_\mu^i(x)T_i$ is a vector field gauge parameter, and we have introduced the covariant derivative $D_\mu(A)\xi_\nu = \partial_\mu \xi_\nu - i\lambda [A_\mu, \xi_\nu]$. In addition, the Lagrangian (2.3) has the \textit{global} symmetry of the group $G$ under the transformation

$$B_{\mu\nu} \rightarrow B'_{\mu\nu} = g^{-1}B_{\mu\nu}g, \quad A_{\mu} \rightarrow A'_{\mu} = g^{-1}A_{\mu}g, \quad \text{(2.5)}$$
with \( g \in G \).

The second-order Lagrangian for \( B_{\mu
u}^i \) is obtained by eliminating \( A_{\mu}^i \) (see Ref. 5)). We obtain the dual formulation of the AST gauge theory by eliminating \( B_{\mu
u}^i \) instead. This gives the flatness condition for \( A_{\mu}, F_{\mu\nu} = 0 \). The solution of this equation is given by

\[
A_{\mu} = \frac{i}{\lambda} U^{-1} \partial_{\mu} U, \tag{2.6}
\]

where \( U(x) \in G \). Substituting (2.6) back into the Lagrangian (2.3), we obtain

\[
\mathcal{L} = \frac{1}{8c\lambda^2} \text{tr} \left( U^{-1} \partial_{\mu} U \right)^2. \tag{2.7}
\]

This is the NLSM on the group manifold \( G \), called the principal chiral model.

The generalization of the scalar-tensor duality to the case of a coset space \( G/H \) as the target space of the NLSM can be made by introducing the YM field for the subgroup \( H \),

\[
v_{\mu} \equiv v_{\mu}^a H_a \in \mathcal{H}. \tag{2.8}
\]

Here we have decomposed the Lie algebra \( G \) into the subalgebra \( H \) and the coset generators as

\[
T_i \in G, \quad H_a \in \mathcal{H}, \quad X_I \in G - \mathcal{H}, \quad \text{tr} (H_a X_I) = 0. \tag{2.9}
\]

The AST field and the auxiliary vector field are defined in the same way as in (2.2). The YM gauge transformations of the group \( H \) now read

\[
v_{\mu} \rightarrow v'_{\mu} = h^{-1}(x) \left( v_{\mu} + \frac{i}{e} \partial_{\mu} \right) h(x),
\]

\[
B_{\mu\nu} \rightarrow B'_{\mu\nu} = h^{-1}(x) B_{\mu\nu} h(x), \quad A_{\mu} \rightarrow A'_{\mu} = h^{-1}(x) A_{\mu} h(x), \tag{2.10}
\]

with \( h(x) \in H \). Here \( e \) is the YM coupling constant with dimension 0. The first-order Lagrangian invariant under this gauge transformation is given by

\[
\mathcal{L} = -\frac{1}{8c} e^{\mu\nu\rho\sigma} \text{tr} \left[ B_{\mu\nu} F_{\rho\sigma} \left( A + \frac{e}{\lambda} v \right) \right] - \frac{1}{8c} \text{tr} (A_{\mu} A^\mu), \tag{2.11}
\]

where \( F_{\mu\nu} \left( A + \frac{e}{\lambda} v \right) \equiv F_{\mu\nu} + \frac{e}{\lambda} (\partial_{\mu} v_{\nu} - \partial_{\nu} v_{\mu} - ie [v_{\mu}, v_{\nu}] - ie \left([A_{\mu}, v_{\nu}] + [v_{\mu}, A_{\nu}]\right)\).

The AST gauge transformation is modified from (2.4) after taking account of the coupling of \( B_{\mu\nu}^i \) with the YM field \( v_{\mu}^a \). One only has to replace the covariant derivative \( D_{\mu}(A) \) by \( D_{\mu}(A + \frac{e}{\lambda} v) \):

\[
\delta B_{\mu\nu} = D_{\mu} \left( A + \frac{e}{\lambda} v \right) \xi_{\nu} - D_{\nu} \left( A + \frac{e}{\lambda} v \right) \xi_{\mu},
\]

\[
\delta A_{\mu} = 0, \quad \delta v_{\mu} = 0. \tag{2.12}
\]

The case in which \( H \) is \( G \) itself is considered in Ref. 5). The model used in that work is different from ours in the sense that it contains the kinetic term of the YM field.
Note that the existence of the gauge field $v_\mu$ in the Lagrangian (2.11) explicitly breaks the original $G$ symmetry of the Lagrangian (2.3), since $v_\mu' = g^{-1}v_\mu g$ is not necessarily an element of $\mathcal{H}$.

Eliminating $A_\mu$ from the Lagrangian (2.11), we obtain the second-order Lagrangian for $B_{\mu\nu}$ coupled to the YM field $v_\mu$,

$$
\mathcal{L} = -\frac{1}{8} \left( \tilde{G}^\mu_i \tilde{K}^i_{\mu
u} + \frac{e}{\lambda} \epsilon^{\mu
u\rho\sigma} B^{i}_{\mu\nu} v^{i}_{\rho\sigma} \right),
$$

where we have defined the field strength $\tilde{G}^\mu_i \equiv \epsilon^{\mu\nu\rho\sigma} D_{\nu} B^{i}_{\rho\sigma} \equiv \epsilon^{\mu\nu\rho\sigma} (\partial_{\nu} B^{i}_{\rho\sigma} + e f^{ijk} v^{j}_{\rho\sigma} B^{k}_{\rho\sigma})$, and $v^{i}_{\mu\nu}$ is the YM field strength. Here, $\tilde{K}^{i}_{\mu\nu} \equiv g^{\mu\nu}\delta^{ij} - \lambda f^{ijk} \epsilon^{\mu\nu\rho\sigma} B^{k}_{\rho\sigma}$ through $\tilde{K}^{i}_{\mu
u} \tilde{K}^{ij} \delta_{\nu} = \delta^{i}_{\mu} \delta^{j\nu}$.

We now proceed to the derivation of the dual scalar field theory equivalent to the AST field theory (2.13). Elimination of $B^{i}_{\mu\nu}$ in the Lagrangian (2.11) yields the constraint

$$
F_{\mu\nu} (A + \frac{e}{\lambda} v) = 0.
$$

Its solution is

$$
A_\mu + \frac{e}{\lambda} v_\mu = \frac{i}{\lambda} U(x)^{-1} \partial_\mu U(x), \quad U(x) \in G.
$$

Substituting this expression back into Eq. (2.11), we obtain

$$
\mathcal{L} = \frac{1}{8c \lambda^2} \text{tr}(U^{-1} D_\mu U)^2,
$$

where we have introduced the covariant derivative, $D_\mu U \equiv \partial_\mu U + i e U v_\mu$. The Lagrangian (2.16) possesses the global $G_L \times$ local $H_R$ symmetry [where the subscript $L$ ($R$) refers to left (right) action], given by

$$
U(x) \rightarrow g U(x), \quad v_\mu(x) \rightarrow v_\mu(x),
$$

with $g \in G$, and

$$
U(x) \rightarrow U(x) h(x), \quad v_\mu \rightarrow h^{-1}(x) \left( v_\mu + \frac{i}{e} \partial_\mu \right) h(x),
$$

with $h(x) \in H$. The global $G$ transformation (2.17) is hidden in the AST gauge theory. Eliminating the auxiliary field $v_\mu$ by using the equation of motion, we obtain the usual form of the NLSM on $G/H$. 8,15

There is an alternative way of writing the first-order Lagrangian; we replace the auxiliary field $A_\mu$ as

$$
A_\mu \rightarrow A_\mu - \frac{e}{\lambda} v_\mu,
$$

by field redefinition, and the Lagrangian (2.11) then becomes

$$
\mathcal{L} = -\frac{1}{8c} \epsilon^{\mu\nu\rho\sigma} \text{tr} (B_{\mu\nu} F_{\rho\sigma}) - \frac{1}{8c} \text{tr} \left[ (A_\mu - \frac{e}{\lambda} v_\mu) (A_\mu - \frac{e}{\lambda} v_\mu) \right].
$$
This expression of the Lagrangian is obtained in Ref. 9). Note that it is invariant under the AST gauge transformation (2.4) with $\delta v_\mu = 0$, instead of (2.12). The YM gauge transformation of $A_\mu$ takes the usual form: $A_\mu \rightarrow A'_\mu = h^{-1}(x) \left( A_\mu + \frac{i}{\lambda} \partial_\mu \right) \times h(x)$.

§3. Supersymmetric extension of the non-Abelian scalar-tensor duality for group manifolds

In this section, after we recapitulate the non-Abelian scalar-tensor duality for the group manifold in supersymmetric field theories, we discuss the correspondence of fields in the two theories.

In supersymmetric field theories, the AST field $B_{\mu\nu}(x)$ is a component of the (anti-)chiral spinor superfields $B_\alpha(x, \theta, \bar{\theta}) \left[ \bar{B}_\dot{\alpha}(x, \theta, \bar{\theta}) \right]$, satisfying the supersymmetric constraints

$$D_\alpha B_\beta(x, \theta, \bar{\theta}) = 0, \quad D_\bar{\alpha} \bar{B}_\dot{\beta}(x, \theta, \bar{\theta}) = 0. \quad (3.1)$$

These superfields can be expanded in terms of component fields as

$$B^\alpha(y, \theta) = \psi^\alpha(y) + \frac{1}{2} \theta^\alpha(C(y) + iD(y)) + \frac{1}{2} (\sigma^{\mu\nu})^{\alpha\beta} \theta_\beta B_{\mu\nu}(y) + \theta \lambda^\alpha(y),$$

$$\bar{B}_{\dot{\alpha}}(y^\dagger, \bar{\theta}) = \bar{\psi}_{\dot{\alpha}}(y^\dagger) + \frac{1}{2} \bar{\theta}_{\dot{\alpha}}(C(y^\dagger) - iD(y^\dagger)) + \frac{1}{2} (\sigma^{\mu\nu})^{\dot{\alpha}\dot{\beta}} \bar{\theta}^\dot{\beta} B_{\mu\nu}(y^\dagger) + \bar{\theta} \bar{\lambda}_{\dot{\alpha}}(y^\dagger), \quad (3.2)$$

where $y^\mu = x^\mu + i\theta \sigma^\mu \bar{\theta}$ and $D_\alpha = \partial / \partial \theta^\alpha$ (or $y^\dot{\mu} = x^\mu - i\theta \sigma^\mu \bar{\theta}$ and $D_{\dot{\alpha}} = -\partial / \partial \bar{\theta}^\dot{\alpha}$), and $(\sigma^{\mu\nu})^{\alpha\beta} = \frac{1}{4} (\sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu)^{\alpha\beta}$ (see Ref. 16). The mass dimension of $B_\alpha$ is $\frac{1}{2}$.

We consider the non-Abelian AST field $B_{\mu\nu}$ with the group $G$ [see (2.1)]. This field $B_{\mu\nu}$ is a component of $G$-valued (anti-)chiral spinor superfields $B_\alpha(x, \theta, \bar{\theta}) = B_\alpha(x, \theta, \bar{\theta})T_i \left[ \bar{B}_{\dot{\alpha}}(x, \theta, \bar{\theta}) = \bar{B}_{\dot{\alpha}}(x, \theta, \bar{\theta})T_i \right]$.

To construct the supersymmetric extension of the FT model, we introduce a $G$-valued auxiliary vector superfield $A(x, \theta, \bar{\theta}) = A^i(x, \theta, \bar{\theta})T_i$, satisfying the constraint $A^i = A^i$. Its field strengths are (anti-)chiral spinor superfields,

$$W_\alpha = -\frac{1}{4\lambda} \bar{D}D(e^{-\lambda A} D_\alpha e^{\lambda A}), \quad \bar{W}_{\dot{\alpha}} = \frac{1}{4\lambda} DD(e^{\lambda A} \bar{D}_{\dot{\alpha}} e^{-\lambda A}). \quad (3.3)$$

The first-order Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{2c} \left[ \int d^2 \theta \, \text{tr} \left( W^\alpha B_\alpha \right) + \int d^2 \bar{\theta} \, \text{tr} \left( \bar{W}_{\dot{\alpha}} \bar{B}_{\dot{\alpha}} \right) \right] + \frac{1}{4c} \int d^4 \theta \, \text{tr} \, A^2. \quad (3.4)$$

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**§5** See the discussion in §5 concerning variants of the Lagrangian (3.4).
This Lagrangian is verified by constructing the supersymmetric AST gauge transformation which leaves it invariant. To this end we first define the covariant spinor derivative $D_\alpha = D_\alpha + [e^{-\lambda A} D_\alpha e^{\lambda A}, \cdot]$. The AST gauge transformation is parametrized by a $G$-valued vector superfield $\Omega(x, \theta, \bar{\theta}) = \Omega^i(x, \theta, \bar{\theta}) T_i$ satisfying the constraint $\Omega^i \dagger = \Omega^i$.

$$\delta B_\alpha = -\frac{i}{4} \bar{D} D_\alpha (e^{-\lambda A} \Omega), \quad \delta \bar{B}_\dot{\alpha} = -\frac{i}{4} D D_{\dot{\alpha}} (\Omega e^{-\lambda A}),$$

$$\delta A = 0.$$  \hspace{1cm} (3.5)

This transformation is Abelian, though $\Omega$ is $G$-valued. The Lagrangian (3.4) is invariant under the global $G$ transformation

$$B_\alpha \to B'_\alpha = g^{-1} B_\alpha g, \quad \bar{B}_{\dot{\alpha}} \to \bar{B}'_{\dot{\alpha}} = g^{-1} \bar{B}_{\dot{\alpha}} g, \quad A \to A' = g^{-1} A g, \quad W_\alpha \to W'_\alpha = g^{-1} W_\alpha g,$$  \hspace{1cm} (3.6)

with $g \in G$.

The equation of motion for the auxiliary superfield $A$ has the complicated form

$$D^\alpha B_\alpha + e^{-\lambda A} D_{\dot{\alpha}} \bar{B}^\dot{\alpha} e^{\lambda A} = -A,$$  \hspace{1cm} (3.7)

where $D_\alpha B_\beta \equiv D_\alpha B_\beta + \{ e^{-\lambda A} D_\alpha e^{\lambda A}, B_\beta \}$. If we eliminate $A$ by solving Eq. (3.7), we obtain the second-order Lagrangian for $B_\alpha$, which we call the ‘supersymmetric theory of the FT model’. In practice it is difficult to solve Eq. (3.7) explicitly, and hence we do not write the Lagrangian for $B_\alpha$ explicitly.

On the other hand, if we eliminate $B_\alpha(x, \theta, \bar{\theta})$, we can obtain the dual NLSM as follows. The equation of motion for $B_\alpha(x, \theta, \bar{\theta})$ reads

$$-4\lambda W_\alpha(x, \theta, \bar{\theta}) = \bar{D} D (e^{-\lambda A} D_\alpha e^{\lambda A}) = 0,$$  \hspace{1cm} (3.8)

expressing the fact that $A$ is a pure gauge. The solution is written as

$$e^{\lambda A(x, \theta, \bar{\theta})} = e^{\phi^i(x, \theta, \bar{\theta})} e^{\phi(x, \theta, \bar{\theta})}, \quad \bar{D}_{\dot{\alpha}} \phi(x, \theta, \bar{\theta}) = 0.$$  \hspace{1cm} (3.9)

Here $\phi = \phi^i T_i$ is a $G$-valued chiral superfield. Taking account of Eq. (3.8) and substituting (3.9) back into the Lagrangian (3.4), we obtain the Lagrangian for $\phi$: \hspace{1cm} (3.10)

$$\mathcal{L} = \int d^4 \theta \ K(\phi, \phi^\dagger) = \int d^4 \theta \ \frac{1}{4c^2} \text{tr} \log^2 (e^{\phi^\dagger} e^{\phi}).$$

We have thus obtained the NLSM that is dual to the supersymmetric theory of the FT model.

In the Lagrangian (3.10), $K$ is the Kähler potential of the target space of the NLSM. It can be expanded in powers of $\phi^i$ and $\phi^i \dagger$. We have

$$\mathcal{L} = \int d^4 \theta \ \frac{1}{2\lambda^2} \left[ \phi^i \phi^i - \frac{1}{24} f_{mi}^j f_{jk}^l (\phi^m \dagger \phi^j \dagger \phi^k \phi^i + 2\phi^i \dagger \phi^m \dagger \phi^j \phi^k + 2\phi^k \dagger \phi^i \phi^j \phi^m) + \cdots \right].$$  \hspace{1cm} (3.11)
We find that the dual field theory is the free Wess-Zumino model if the group $G$ is Abelian.

Since the scalar components of $\phi^i$ are complex, the target space of this NLSM is $G^C$, the complex extension of $G$. The Lagrangian (3.10) is invariant under the global action of the group $G \times G$:

$$e^\phi \rightarrow e^{\phi'} = g_L e^\phi g_R, \quad (g_L, g_R) \in G \times G. \quad (3.12)$$

Note that the Lagrangian (3.10) is not invariant under $G^C \times G^C$, though the target space is $G^C$. We find from Eqs. (3.6) and (3.9) that the right action of Eq. (3.12) arises from the original global symmetry $G$ of the AST gauge theory. By contrast, the left action of Eq. (3.12) is a hidden global symmetry, preserving Eq. (3.9).

The Lagrangian (3.10) allows a simple interpretation from the viewpoint of the supersymmetric nonlinear realization of a global symmetry. In supersymmetric field theories, spontaneous symmetry breaking is caused by the superpotential. Since the superpotential is a holomorphic function of chiral superfields, its symmetry $G$ is promoted to $G^C$. Correspondingly, there appear quasi-Nambu-Goldstone (QNG) bosons in addition to ordinary Nambu-Goldstone (NG) bosons. The low-energy effective Lagrangian of these massless bosons and their fermionic superpartners form a supersymmetric NLSM whose target manifold is a Kähler manifold. This manifold is non-compact due to the existence of QNG bosons. Since the target space of the Lagrangian (3.10) is $G^C$, the NG and QNG bosons are equal in number for completely broken $G$.

We now examine the relation between the fields in the AST gauge theory and the NG and QNG bosons in the dual NLSM. This relation can be worked out explicitly in the Abelian case as follows. The AST gauge transformation of $B_{\mu\nu}$ is given by $\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$, where $\xi_\mu$ is the vector component of the parameter vector superfield $\Omega$. Using the AST gauge transformation of the other fields, we can take the Wess-Zumino gauge, in which $\psi_\alpha = D_\alpha = 0$. The second-order action for $B_{\alpha}$, obtained by eliminating $A$ in the Lagrangian (3.4) for the Abelian case, is given by

$$S = -\int d^4x \int d^4\theta \frac{1}{2} G^2 = \int d^4x \left( -\frac{1}{2} \partial_\mu C \partial^\mu C - i \bar{\lambda} \sigma^\mu \partial_\mu \lambda + \frac{1}{4} G^\mu G_\mu \right). \quad (3.13)$$

Here $G(x, \theta, \bar{\theta})$ is the field strength of $B_\alpha$ defined by $G \equiv \frac{1}{2} (D_\alpha B_\alpha + \bar{D}_{\bar{\alpha}} \bar{B}^{\bar{\alpha}}) = \theta \sigma^\mu \partial_\mu G_\mu + \cdots$, where $G_\mu \equiv \varepsilon^{\mu\nu\rho\sigma} \partial_\nu B_\rho$. We find from this action that $B_{\mu\nu}$ corresponds to the NG boson and $C$ to the QNG boson. For the non-Abelian case, the second-order Lagrangian is an interacting field theory, and the relation between NG bosons and $B_{\mu\nu}$ and that between QNG bosons and $C$ are less straightforward.

§4. Supersymmetric extension of the non-Abelian scalar-tensor duality for coset spaces

4.1. Anti-symmetric tensor gauge theory coupled to Yang-Mills fields

We now generalize the scalar-tensor duality to the case of coset space. In supersymmetric theories, we introduce the YM vector superfield $V(x, \theta, \bar{\theta}) = V^a(x, \theta, \bar{\theta}) H_a,$
where the $H_a$ are generators of a subgroup $H$ of $G$. We decompose the Lie algebra as in Eq. (2.9).

As mentioned above, there are two expressions we can use to generalize the FT model to the case of coset space. We use the second expression (2.20). In this case, we do not need to modify the AST gauge transformation and the field strength.

In analogy to the YM gauge transformation of $A_\mu$ in the bosonic case, the transformation of $A$ and the YM gauge field $V$ are closely related, with $\lambda A$ and $eV$ transforming identically. The YM gauge transformation is

$$
B_\alpha \to B'_\alpha = e^{-iA}B_\alpha e^{iA}, \quad \bar{B}_{\dot{\alpha}} \to \bar{B}'_{\dot{\alpha}} = e^{-iA^\dagger} \bar{B}_{\dot{\alpha}} e^{iA^\dagger},
$$

$$
e^{eV} \to e^{eV'} = e^{-iA^\dagger} e^{eV} e^{iA}, \quad e^{\lambda A} \to e^{\lambda A'} = e^{-iA^\dagger} e^{\lambda A} e^{iA},
$$

where the gauge parameter is an $H$-valued chiral superfield $\Lambda(x, \theta, \bar{\theta}) = \Lambda^a(x, \theta, \bar{\theta}) H_a$, satisfying $\bar{D}_{\dot{\alpha}} A(x, \theta, \bar{\theta}) = 0$. The field strengths of $A$ are the same as those in (3.3). They transform as

$$
W_\alpha \to W'_\alpha = e^{-iA} W_\alpha e^{iA}, \quad \bar{W}_{\dot{\alpha}} \to \bar{W}'_{\dot{\alpha}} = e^{-iA^\dagger} \bar{W}_{\dot{\alpha}} e^{iA^\dagger},
$$

under the YM gauge transformation (4.1).

In Eq. (3.4) we have already given the Lagrangian in first-order form in the case in which there is no YM gauge field. A generalization of the Lagrangian to that in which $G$ is fully gauged by supersymmetric YM field, is given in Ref. 13). The model investigated there contains the kinetic term of the YM field. The dual field theory is a massive YM field theory.

We construct the model in which the subgroup $H$ of $G$ is gauged by the YM field. As in the bosonic case, the model we consider here does not contain the kinetic term of the YM field. The Lagrangian for this model is

$$
\mathcal{L} = -\frac{1}{2c} \left[ \int d^2 \theta \, \text{tr} \left( W^\alpha B_\alpha \right) + \int d^2 \bar{\theta} \, \text{tr} \left( \bar{W}_{\dot{\alpha}} \bar{B}^{\dot{\alpha}} \right) \right] + \frac{1}{4c\lambda^2} \int d^\theta \, \text{tr} \, \log^2 (e^{-eV} e^{\lambda A}).
$$

If we set $V = 0$, we recover the Lagrangian (3.4). We should note that the global action of $G$, Eq. (3.6), is explicitly broken by the partial gauging of the subgroup $H$ of $G$; the $H$-valuedness of $V$ is not maintained under the $G$-action $V \to V' = g^{-1} V g$.

The equation of motion of the auxiliary field $A$ is difficult to solve, and we do not write the second-order Lagrangian for $B_\alpha$.

Here we discuss the field redefinition corresponding to (2.19) in the bosonic case. Using the second alternative, corresponding to (2.20), the AST gauge transformation and the field strengths are not modified. The Lagrangian for the first alternative, corresponding to (2.11), is obtained from the Lagrangian (4.3) by the field redefinition of $A$

$$
e^{\lambda A} \to e^{eV} e^{\lambda A}.
$$

Using the Hausdorff formula, this can be rewritten as

$$
A \to A + \frac{e}{\lambda} V + \frac{e}{2} [A, V] + \cdots.
$$
We obtain the Lagrangian as

\[ \mathcal{L} = -\frac{1}{2c} \left[ \int d^2 \theta \, \text{tr} (\tilde{W}^\alpha B_\alpha) + \int d^2 \bar{\theta} \, \text{tr} (\tilde{W}_\dot{\alpha} \tilde{B}^\dot{\alpha}) \right] + \frac{1}{4c} \int d^4 \theta \, \text{tr} A^2. \]  

(4.6)

Here \( \tilde{W}^\alpha \) and \( \tilde{W}_\dot{\alpha} \) are defined by

\[ \tilde{W}_\alpha = -\frac{1}{4\lambda} \bar{D} \bar{D} \left[ e^{-\lambda A} e^{-eV} D_\alpha (e^{eV} e^{\lambda A}) \right], \]

\[ \tilde{W}_\dot{\alpha} = \frac{1}{4\lambda} DD \left[ e^{eV} e^{\lambda A} D_\dot{\alpha} (e^{-\lambda A} e^{-eV}) \right], \]

(4.7)

which are obtained from (3.3) using (4.4). The AST gauge transformation (3.5) is also modified by (4.4) to

\[ \delta B_\alpha = -\frac{i}{4} \bar{D} \bar{D} D_\alpha (e^{-\lambda A} e^{-eV} \Omega), \quad \delta \bar{B}^\dot{\alpha} = -\frac{i}{4} DD \bar{D} \dot{\alpha} (\Omega e^{-\lambda A} e^{-eV}), \]

\[ \delta A = 0, \quad \delta V = 0, \]

(4.8)

with \( \bar{D} = D_\alpha + [e^{-\lambda A} e^{-eV} D_\alpha (e^{eV} e^{\lambda A}), \cdot] \). The YM gauge transformation of \( A \) becomes

\[ A \rightarrow A' = e^{-iA} Ae^{iA}. \]

(4.9)

At first sight, it might appear that this is inconsistent with the reality condition, but this is not the case, as \( e^{-eV} e^{\lambda A} \) and \( V \) satisfy the reality condition under the redefinition (4.4).

4.2. Dual nonlinear sigma model on complex coset spaces

To obtain the dual NLSM, we return to the Lagrangian (4.3). Elimination of \( B_\alpha(x, \theta, \bar{\theta}) \) through its equation of motion again gives

\[ e^{\lambda A(x, \theta, \bar{\theta})} = e^{\phi \dagger(x, \theta, \bar{\theta})} e^{\phi(x, \theta, \bar{\theta})}, \quad \bar{D} \phi(x, \theta, \bar{\theta}) = 0, \]

(4.10)

where \( \phi \) is a \( G \)-valued chiral superfield. Proceeding in the same way as in §3, we obtain the NLSM whose Kähler potential is

\[ K(\phi, \phi \dagger, V) = \frac{1}{4c\lambda^2} \text{tr} \log^2(e^{-eV} e^{\phi \dagger} e^{\phi}). \]

(4.11)

As in the bosonic case, the NLSM (4.11) is invariant under the global transformation of \( G \)

\[ e^{\phi} \rightarrow e^{\phi'} = ge^{\phi}, \quad g \in G \]

(4.12)

and the gauge transformation of \( H \)

\[ e^{eV} \rightarrow e^{eV'} = h^{\dagger} e^{eV} h, \quad e^{\phi} \rightarrow e^{\phi'} = e^{\phi} h, \]

\[ h = e^{iA^a(x, \theta, \bar{\theta}) H_a}, \quad \bar{D} \phi^a(x, \theta, \bar{\theta}) = 0. \]

(4.13)
The global action (4.12) of $G$ corresponds to the left action in Eq. (3.12), which does not correspond to any symmetry in the AST gauge theory. The local action (4.13) of $H$ corresponds to the right action in Eq. (3.12), and its origin is, of course, the gauge transformation (4.1). Equation (4.11) can be interpreted as the Lagrangian of the dual NLSM formulated in terms of the hidden local symmetry.\(^{15}\)

The physical degrees of freedom can be found as follows. First, decompose $\phi(x, \theta, \bar{\theta})$ into its $H$-valued part and the rest as

$$e^{\phi(x, \theta, \bar{\theta})} = \xi(x, \theta, \bar{\theta})h(x, \theta, \bar{\theta}),$$

$$\xi = e^{i\varphi^I(x, \theta, \bar{\theta})X_I}, \quad h = e^{i\alpha^a(x, \theta, \bar{\theta})H_a},$$

(4.14)

where the $X_I$ are the coset generators and $\varphi^I(x, \theta, \bar{\theta})$ and $\alpha^a(x, \theta, \bar{\theta})$ are chiral superfields. By fixing the gauge as

$$e^{eV} = h^\dagger e^{V_0} h, \quad h = e^{i\alpha^a(x, \theta, \bar{\theta})H_a},$$

(4.15)

we obtain the Kähler potential,

$$K(\phi, \phi^\dagger, V_0) = \frac{1}{4c\lambda^2} \text{tr} \log^2(e^{-eV_0}\xi^\dagger\xi).$$

(4.16)

This Lagrangian is invariant under the global $G$ transformation, defined by

$$\xi \to \xi' = g\xi h(g, \xi),$$

$$e^{eV_0} \to e^{eV_0'} = h^\dagger(g, \xi)e^{V_0}h(g, \xi), \quad g \in G,$$

(4.17)

where the $H$-valued chiral superfield $h(g, \xi(x, \theta, \bar{\theta}))$ is a compensator needed to recover the decomposition in Eq. (4.14).

The superfield $\xi$ in (4.17) is a coset representative of $G^C/H^C$ and the bosonic parts of $\varphi^I(x, \theta, \bar{\theta})$ parameterize $G^C/H^C$. The superfield $\alpha^a(x, \theta, \bar{\theta})$ in Eq. (4.14) has been absorbed into $V^a$ as a result of the supersymmetric Higgs mechanism. Each chiral superfield $\varphi^I$ in this NLSM contains the same number of NG and QNG bosons. The relation to the general theory of supersymmetric nonlinear realizations\(^{19}\) is, however, obscure at this stage, since we are unable to eliminate $V$ explicitly.

§5. Discussion

We have constructed a supersymmetric extension of the duality between AST gauge theories and the NLSM on coset spaces. The two models have apparently different manifest symmetries.

It is useful to clarify the relations between the symmetries of the two models. We do this explicitly in the bosonic case and simply note that the situation is the same in the supersymmetric case. The relations between symmetries of the AST gauge theories and the NLSM are listed in Table I for both the FT model and the $HR$-gauged FT model.

This table deserves several comments:

i) The AST gauge symmetry is hidden in the gauged NLSM.
Table I. Relations between the symmetries of the AST gauge theories and the NLSM. The upper table corresponds to the FT scalar-tensor duality. The $G/H$ scalar-tensor duality, corresponds to the lower table, is obtained by gauging the FT model.

| Model          | FT model                  | NLSM on $G$      |
|----------------|---------------------------|------------------|
| Symmetry       | global $G_R \times$ AST gauge | global $G_L \times$ global $G_R$ |
| Model          | $H_R$-gauged FT model     | $H_R$-gauged NLSM on $G$ | $G/H$ NLSM |
| Symmetry       | local $H_R \times$ AST gauge | global $G_L \times$ local $H_R$ | global $G_L$ |

ii) The global symmetry $G_R$, Eq. (2.5) [(3.6) in the SUSY case], of the FT model is broken in the gauged FT model.

iii) The global symmetry $G_L$, Eq. (2.17) [(4.12) in the SUSY case] of the gauged NLSM is hidden in the gauged FT model.

iv) The local symmetry $H_R$, Eq. (2.18) [(4.13) in the SUSY case], of the gauged NLSM is hidden in the NLSM on $G/H$.

i) and iii) are also true in the non-gauged models.

In this paper, we consider the specific form (3.4) of the supersymmetric Lagrangian. The last term of (3.4) can be generalized to

$$\int d^4\theta F(A) = \int d^4\theta f(\text{tr} e^{\lambda A}, \text{tr} e^{2\lambda A}, \text{tr} e^{3\lambda A}, \ldots),$$

where $F(A)$ is a general $G$-invariant function of $A$.\(^{20}\) In this case, we obtain the NLSM with the general form of the Kähler potential

$$K(\phi, \phi^\dagger) = f(\text{tr} (e^{\phi^\dagger} e^\phi), \text{tr} (e^{\phi^\dagger} e^\phi)^2, \text{tr} (e^{\phi^\dagger} e^\phi)^3, \ldots),$$

instead of (3.10). The arbitrariness of the Kähler potential reflects the fact that the metric of the target space $G^C$ is not completely determined by the symmetry $G \times G$, and it is related to the existence of QNG bosons.

The existence of solitons plays a role in the duality in supersymmetric gauge theories.\(^1\) This suggests that perhaps solitons also play a role in the scalar-tensor duality. From this point of view it is of interest to extend the scalar-tensor duality by including higher-order derivative terms in the NLSM and the supersymmetric NLSM.\(^21\)

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