Josephson vortices and the Meissner effect in stacked junctions and layered superconductors: Exact analytical results

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We present an exact mathematical description of Josephson vortices and of the Meissner effect in periodic thin-layer superconductor/insulator structures with an arbitrary number of identical junctions \( N - 1 \) \((2 \leq N < \infty\), where \( N \) is the number of superconducting layers\) in terms of localized solutions to a system of differential equations for phase differences. We establish a general criterion of the existence of localized solutions. We show that Meissner solutions are characterized by several Josephson lengths \( \lambda_{Ji} \) \((N \) lengths for even \( N \), and \((N - 1)/2 \) lengths for odd \( N \)). We derive an exact expression for the superheating field of the Meissner state, \( H_s \), as an explicit function of \( N \). For Josephson vortices, we find two basically different types of topological solutions: “vortex-plane” solutions and incoherent vortex solutions. Thermodynamically stable “vortex-plane” solutions represent a chain of \( N - 1 \) vortices (one vortex per each insulating layer). They are characterized by the same set of \( \lambda_{Ji} \) as the Meissner solutions. We obtain exact analytical expressions for their self-energy and for the lower critical field \( H_{c1} \). Incoherent vortex solutions comprise solutions with \( k < N - 1 \) vortices and different vortex-antivortex configurations. In contrast to the “vortex-plane” solutions, they prove to be thermodynamically unstable, and their spatial dependence is characterized, in general, by \( N - 1 \) length scales. As an illustration, we analyze 1-4-Josephson-junction stacks and investigate a transition to the layered superconductor limit \((N \to \infty)\).

I. INTRODUCTION

We present a rigorous mathematical examination of the problem of Josephson vortices and of the Meissner effect in thin-layer Josephson-junction stacks and layered superconductors, with a static external magnetic field \( \mathbf{H} \) applied parallel to the layers (along the \( z \) axis, see Fig. 1.) We consider periodic systems composed of an arbitrary number \( N - 1 \) of identical superconductor/insulator (S/I) junctions \((2 \leq N < \infty\), where \( N \) is the number of S-layers, with \( x \) being the layering axis). Our starting point is the microscopic Gibbs free-energy functional derived in Ref. [1]. Mathematical structure of this functional is analogous to that of the phenomenological Lawrence-Doniach model. [2] Thus, the treatment of our paper fully applies to the latter model as well.

Mathematically, both Josephson vortices in moderate fields \(|\mathbf{H}|\) and the inhomogeneous Meissner state are described by solutions of a system of nonlinear second-order differential equations for phase differences, \( \phi_n \), with square-integrable first-order derivatives. [3] (For brevity, we call here such solutions ”localized”). Our approach is substantially based on the observation of a nontrivial property of the differential equations for \( \phi_n \): We show that the problem of finding localized solutions for \( \phi_n \) can be reduced to solving a standard initial value problem. Using this key mathematical result, we establish an exact criterion of the existence of localized solutions. The existence criterion, in turn, allows us to obtain a complete classification of physical localized solutions. We have found three types of such solutions: Meissner solutions, topological ”vortex-plane” solutions, [4] and topological incoherent vortex solutions.

Meissner solutions are localized near the side boundaries \( y = -L \) and \( y = L \). In contrast to the well-known single-junction case, [5] the Meissner solutions in stacks with \( N \geq 4 \) turn out to be characterized by several Josephson lengths \( \lambda_{Ji} \) \((N \) lengths for even \( N \), and \((N - 1)/2 \) lengths for odd \( N \)). The Meissner solutions persist up to a certain superheating field of the Meissner phase, \( H_s \). We derive an exact expression for \( H_s \) as an explicit function of \( N \). We show that the field \( H_s \) simultaneously determines the penetration field for ”vortex planes”. (See below.)

Thermodynamically stable ”vortex-plane” solutions represent a chain of \( N - 1 \) Josephson vortices (one vortex per each I-layer), positioned in the symmetry plane \( y = 0 \). These solutions are uniquely determined by the vortex-penetration conditions at \(|\mathbf{H}| = H_s \). They are characterized by the same set of \( \lambda_{Ji} \) as the Meissner solutions. Such solutions were previously obtained for infinite \((N = \infty)\) layered superconductors. [5] Under the name of the ”coherent mode” or the ”in-phase mode” they are well-known in double-junction stacks. [6–7] For \( 4 \leq N < \infty \), the existence of the ”coherent mode” was predicted in Ref. [1]. (The authors of Ref. [1] specially emphasized the importance of this mode for practical applications.) Besides giving a proof of the existence and stability of the ”vortex planes” in the...
general case $2 \leq N < \infty$, we derive exact analytical expressions for their self-energy $E_v$ and for the lower critical field $H_{c1}$.

Incoherent vortex solutions comprise single-vortex solutions, vortex solutions with $2 \leq k < N - 1$ vortices in the plane $y = 0$, as well as different vortex-antivortex configurations. All such solutions satisfy the existence criterion. However, in contrast to the "vortex-plane" solutions, they prove to be thermodynamically unstable and do not meet the vortex-penetration conditions at any $|H| \neq 0$. It should be noted that the single-vortex solutions obtained in this paper have no resemblance to hypothetical Abrikosov-type vortices, introduced without proper mathematical justification in some previous publications. Besides being thermodynamically unstable, the actual single-vortex solutions are not uniquely determined by asymptotic boundary conditions. They are accompanied by singular phase-difference distribution in all $N - 1$ junctions, and their spatial dependence is characterized, in general, by $N - 1$ length scales.

Section II of the paper is devoted to exact mathematical formulation of the problem. In section III, we derive all major physical and mathematical results sketched above. The general consideration of this section is illustrated by several concrete examples in section IV. In particular, we analyze 1-4-junction stacks and investigate a transition to the layered-superconductor limit ($N \to \infty$). The obtained results are discussed in section V. Appendices A-C contain some additional mathematics, relevant to the subject of our study.

II. FORMULATION OF THE PROBLEM

We begin by writing down the microscopic Gibbs free-energy functional $\Omega$ of a periodic structure consisting of alternating $N$ superconducting (S) and $N - 1$ insulating (I) layers ($2 \leq N < \infty$):

\[
\Omega \left[ f_n, \phi_n, \frac{d\phi_n}{dy}, A_x, A_y; H \right] = \frac{H^2(T)}{4\pi} \alpha W_z \left[ \sum_{n=0}^{N-1} \int_{-L}^{L} dy \left[ -f_n^2(y) + \frac{1}{2} f_n^4(y) \right] + \zeta^2(T) \left( \frac{df_n(y)}{dy} \right)^2 + \zeta^2(T) \left[ \frac{d\phi_n(x,y)}{dy} - 2eA_y(np, y) \right]^2 f_n^2(y) \right]
\]

\[
+ \sum_{n=1}^{N-1} \int_{-L}^{L} dy \left[ \frac{r(T)}{2} \left[ f_{n-1}^2(y) + f_n^2(y) - 2f_n(y)f_{n-1}(y) \cos \left[ \phi_n(y) - 2e \int_{(n-1)p}^{np} dx A_x(x, y) \right] \right] \right]
\]

\[
+ \frac{4e^2\zeta^2(T)\lambda^2(T)}{a} \int_{(n-1)p}^{np} dx [H(x, y) - H]^2 \right],
\]

\[
r(T) = \frac{\zeta^2(T)\alpha}{a\xi_0},
\]

\[
\alpha = \frac{3\pi^2}{7\zeta(3)} \int_0^1 dt D(t) \ll 1,
\]

\[
\phi_n(y) = \varphi_n(y) - \varphi_{n-1}(y).
\]

Here $h = c = 1$; $a$ is the S-layer thickness; $p$ is the period, and $W_z$ is the length of the structure in the $z$ direction ($W_z \to \infty$); the length of the structure in the $y$ direction is $W_y = 2L$; $f_n(y) [0 \leq f_n(y) \leq 1]$ and $\varphi_n(y)$ are, respectively, the reduced modulus and the phase of the pair potential $\Delta_n(y)$ in the $n$th superconducting layer:
\[ \Delta_n(y) = \Delta(T) f_n(y) \exp \varphi_n(y), \]

with \( \Delta(T) \) being the microscopic gap at temperature \( T \); \( \xi_0 \) is the BCS coherence length; \( \zeta(T) \) and \( \lambda(T) \) are, respectively, the Ginzburg-Landau (GL) coherence length and the penetration depth; \( D(\cos \theta) \) is the incidence-angle-dependent tunneling probability of the I-layer between two successive S-layers; \( H_c(T) \) is the thermodynamic critical field; \( A = (A_x, A_y, 0) \) is the vector potential. The local magnetic field \( H(x, y) = [0, 0, H(x, y)] \) obeys the Maxwell equation

\[ H(x, y) = \frac{\partial A_y(x, y)}{\partial x} - \frac{\partial A_x(x, y)}{\partial y} \]

with boundary conditions

\[ H(0, y) = H((N - 1)p, y) = H, \quad y \in [-L, L], \]
\[ H(x, \pm L) = H, \quad x \in [0, (N - 1)p], \]

where \( H \) is a static external magnetic field applied along the z axis. (See Fig. 1.) The sum of the first three phase- and field-independent terms on the right-hand side of (1) represents the condensation energy. The fourth term is the kinetic energy of the intralayer currents. The last two terms are the Josephson energy and the field energy, respectively.

Expression (1) is valid under the conditions

\[ \frac{T_c - T}{T_c} \ll 1, \quad \xi_0 \ll a, \quad a \ll \min \{\zeta(T), \lambda(T), a^{-1} \xi_0\}, \quad a \ll p. \]

Conditions (2) (\( T_{c0} \) is the critical temperature of an isolated S-layer) and (3) ensure the applicability of the GL-type expansion within each S-layer. Condition (2) corresponds to the thin S-layer limit, whereas condition (3) is employed here for the sake of mathematical simplicity only. Being a first-order expansion in \( a/p \), equation (1) applies in fields \( |H| \ll H_{c2} \), where \( H_{c2} \) is the upper critical field.

Mathematical treatment of the functionals of the type (1) is described in full detail in Ref. [3], section III: One minimizes (1) with respect to \( f_n \) and \( A_x, A_y \), imposes the gauge \( A_x = 0 \) and eliminates \( A_y \) by integration. The result is a closed, complete set of coupled nonlinear mean-field equations for the reduced modulus of the pair potential \( f_n \) and the phase differences \( \phi_n \), together with relations for all physical quantities of interest. To simplify mathematical analysis of the mean-field equations, we introduce dimensionless units by

\[ \frac{x}{p} \to x, \]
\[ \frac{y}{\lambda_j \infty} \to y, \]
\[ \frac{H}{H_{s \infty}} \to H, \]

where the quantities on the left-hand side are dimensional, with \( \lambda_j \infty = (8\pi e j_0 p)^{-1/2} \) being the Josephson penetration depth (\( j_0 \) is the density of the Josephson current in a single junction with thick electrodes) and \( H_{s \infty} = (e p \lambda_j \infty)^{-1} \) being the superheating (penetration) field of the infinite layered superconductor. In our dimensionless units,
for example, the flux quantum is $\Phi_0 = \pi$, and the lower critical field of the infinite layered superconductor $H_{c1\infty} = \frac{2}{\pi}$.

In the dimensionless form, the mean-field equations for $f_n$ and $\phi_n$ read

$$f_0(y) - f_0^3(y) = r(T) \left[ \frac{\epsilon^2}{2} \frac{d^2 f_0(y)}{dy^2} + \frac{2}{\epsilon^2 f_0'(y)} \right] + \frac{1}{2} \left[ f_0(y) - f_1(y) \cos \phi_1(y) \right],$$

$$f_n(y) - f_n^3(y) = r(T) \left[ \frac{\epsilon^2}{2} \frac{d^2 f_n(y)}{dy^2} + \frac{2}{\epsilon^2 f_n'(y)} \right] + \frac{1}{2} \left[ 2 f_n(y) - f_{n+1}(y) \cos \phi_{n+1}(y) - f_{n-1}(y) \cos \phi_{n-1}(y) \right], \quad 1 \leq n \leq N - 2,$$

$$f_{N-1}(y) - f_{N-1}^3(y) = r(T) \left[ \frac{\epsilon^2}{2} \frac{d^2 f_{N-1}(y)}{dy^2} + \frac{2}{\epsilon^2 f_{N-1}'(y)} \right] + \frac{1}{2} \left[ f_{N-1}(y) - f_{N-2}(y) \cos \phi_{N-1}(y) \right];$$

$$\frac{df_n}{dy}(\pm L) = 0, \quad 0 \leq n \leq N - 1;$$

$$\frac{1}{f_n^2(y)} [H_{n+1}(y) - H_n(y)] - \frac{1}{f_{n-1}^2(y)} [H_n(y) - H_{n-1}(y)] - \epsilon^2 H_n(y)$$

$$= -\frac{\epsilon^2}{2} \frac{d\phi_n(y)}{dy}, \quad 1 \leq n \leq N - 1,$$

$$H_0(y) = H_N(y) = H,$$

$$\frac{d\phi_n}{dy}(\pm L) = 2H, \quad 1 \leq n \leq N - 1,$$

where

$$\epsilon \equiv \sqrt{\frac{\alpha p}{\lambda}} < 1,$$

and the local magnetic field in the $n$th insulating layer ($n - 1 < x < n$) is given by

$$H_n(y) = \frac{1}{2} \int_{-L}^{y} du f_n(u) f_{n-1}(u) \sin \phi_n(u) + H$$

$$= \frac{1}{2} \int_{L}^{y} du f_n(u) f_{n-1}(u) \sin \phi_n(u) + H.$$
\( \phi_n(y) = -\phi_n(-y) + 0 \bmod 2\pi. \)

The dimensionless Gibbs free energy \( \Omega(H) \), normalized via the relation

\[
\frac{4\pi \Omega(H)}{H^2_s(T)\alpha \lambda_{\infty} W_z} \to \Omega(H),
\]

in terms of the mean-field quantities \( f_n, \phi_n \) and \( H_n(y) \) has the form

\[
\Omega(H) = \sum_{n=0}^{N-1} \int_{-L}^{L} dy \left[ -f_n^2(y) + \frac{1}{2} f_n^4(y) + \frac{r(T)\epsilon^2}{2} \left( \frac{df_n(y)}{dy} \right)^2 \right. \\
+ \left. \frac{2r(T)}{\epsilon^2 f_n^2(y)} [H_{n+1}(y) - H_n(y)]^2 + 2r(T) [H_n(y) - H_1]^2 \right] \\
+ \frac{r(T)}{2} \sum_{n=1}^{N-1} \int_{-L}^{L} dy \left[ f_n^2(y) + f_{n-1}^2(y) - 2f_n(y)f_{n-1}(y)\cos \phi_n(y) \right].
\]

(14)

Here, the two terms in the second line on the right-hand side are the kinetic energy of the intralayer currents and the field energy, respectively. The intralayer current in the \( n \)th S-layer \( J_n(y) \) (normalized to \( H_{s\infty} \)) and the density of the Josephson current between the \( n \)th and the \((n-1)\)th S-layers \( j_{n,n-1}(y) \) (normalized to \( j_0 \)) are given by

\[
J_n(y) = \frac{1}{4\pi} [H_n(y) - H_{n+1}(y)], \quad 0 \leq n \leq N - 1,
\]

(15)

and

\[
j_{n,n-1}(y) = 2 \frac{dH_n(y)}{dy} = f_n(y)f_{n-1}(y)\sin \phi_n(y), \quad 1 \leq n \leq N - 1,
\]

(16)

respectively. Relations (1)-(16) provide a complete, self-consistent description of the thin-layer periodic S/I structure in the temperature range (2) and in fields \(|H| \ll H_c\).

In this paper, we will be interested in physical solutions for \( \phi_n \) with a square-integrable first-order derivative, localized within a spatial range of order unity. (For brevity, we call such solutions "localized".) Therefore we assume the condition

\[
L \gg 1.
\]

(17)

Moreover, we assume that the temperature range satisfies the condition of the weak-coupling limit

\[
r(T) \ll 1.
\]

(18)

One can obtain a perturbative solution for \( f_n \) and \( \phi_n \) up to any desired order in \( r(T) \), starting from the zero-order solution to (1), (2),

\[
f_n = 1,
\]

(19)

and the zero-order equations for \( \phi_n \),

\[
H_{n+1}(y) - (2 + \epsilon^2) H_n(y) + H_{n-1}(y) = -\frac{\epsilon^2}{2} \frac{d\phi_n(y)}{dy}, \quad 1 \leq n \leq N - 1,
\]

(20)

where \( H_n(y) \) are given by (13) with \( f_n = 1 \) and satisfy the boundary conditions (1). For most applications, it is sufficient to consider expressions for physical quantities only in leading order in \( r(T) \). Thus, for example, substituting (19) and the solution of (20) into (14) immediately yields a first-order expansion for the Gibbs free energy, because first-order corrections to the condensation-energy term cancel out.
A detailed mathematical analysis of Eqs. (20) is the subject of section III. Here we point out that these equations can be transformed into a very useful for application form by solving for $H_n(y)$ (see Appendix A for mathematical details):

$$H_n(y) = h_n(y) + H_n,$$  

$$h_n(y) = \frac{\epsilon^2}{2} \sum_{m=1}^{N-1} G(n,m) \frac{d\phi_m(y)}{dy},$$

$$H_n = \frac{\mu^{-n} + \mu^{-N+n} - \mu^n - \mu^{N-n}}{\mu^{-N} - \mu^N},$$

where $G(n,m)$ are given by (A9), and $\mu$ is given by (A3). By (A8), and (10), (A13), expression (21) explicitly satisfies boundary conditions (10). Moreover, the $y$-independent quantities $H_n$ in (21) have clear physical meaning: Being solutions of (20) with $d\phi_m(y) dy \equiv 0$, they describe distribution of the local magnetic field within I-layers in the homogeneous Meissner state (see section III of Ref. [3]). Also note that $H_n = H_{N-n}$, which is a reflection of the symmetry of the problem. [By comparison, in an infinite layered superconductor $H_n \equiv 0$, and $N^{-1} \sum_{m=1}^{\infty} G(n,m) \rightarrow +\infty \sum_{m=-\infty}^{\infty} G_\infty(n,m) \ldots$, where $G_\infty(n,m)$ are defined by (A15).]

In addition, we point out that equations of the popular phenomenological Lawrence-Doniach model [2] also can be reduced to the dimensionless form (6)-(10), (12)-(16), with $r(T)$ being a phenomenological parameter, $\epsilon \equiv p / \lambda$, and $\Omega(H)$ normalized to $H^2_c(T) p / \lambda W / 4\pi$. (See Ref. [3] for more details.) Thus, all the consideration of this paper fully applies to the Lawrence-Doniach model as well.

### III. MAJOR RESULTS

#### A. The criterion of the existence of localized solutions

By differentiation with respect to $y$, integrodifferential equations (20) reduce to a system of $N-1$ ordinary nonlinear second-order differential equations

$$\frac{d^2 \phi_1(y)}{dy^2} = \frac{1}{\epsilon^2} \left[ (2 + \epsilon^2) \sin \phi_1(y) - \sin \phi_2(y) \right],$$

$$\frac{d^2 \phi_n(y)}{dy^2} = \frac{1}{\epsilon^2} \left[ (2 + \epsilon^2) \sin \phi_n(y) - \sin \phi_{n+1}(y) - \sin \phi_{n-1}(y) \right], \quad 2 \leq n \leq N - 2,$$

$$\frac{d^2 \phi_{N-1}(y)}{dy^2} = \frac{1}{\epsilon^2} \left[ (2 + \epsilon^2) \sin \phi_{N-1}(y) - \sin \phi_{N-2}(y) \right]$$

with boundary conditions (10).

Consider Eqs. (24) on the whole axis $-\infty < y < +\infty$. Two simple properties of (24) are quite obvious: If $\phi_n(y)$ ($1 \leq n \leq N - 1$) is a solution, the functions $\bar{\phi}_n(y)$ given by

$$\bar{\phi}_n(y) = \phi_n(y) + 2\pi k \quad (k \text{ is an integer}),$$

and

$$\tilde{\phi}_n(y) = \phi_n(y + c) \quad (c \text{ is an arbitrary constant})$$

are also solutions. [The latter is a result of the fact that $y$ does not enter explicitly the right-hand side of (24).] Our conclusions about the existence of localized solutions to (24) will be substantially based on another key property, which we formulate as a lemma:
Lemma. Consider an arbitrary interval $I = [L_1, L_2]$ and $y_0 \in I$. The initial value problem for Eqs. (24) with arbitrary initial conditions $\phi_n(y_0) = \alpha_n$, $\frac{d\phi_n}{dy}(y_0) = \beta_n$ has a unique solution in the whole interval $I$. This solution has continuous derivatives with respect to $y$ of arbitrary order and continuously depends on the initial data. (For the proof of the Lemma, see Appendix B.)

It is worth noting that the existence and uniqueness of a smooth solution to the initial value problem in the whole interval $I$ is rather nontrivial for nonlinear differential equations: For such equations, theorems of existence and uniqueness are usually valid only locally, in the neighborhood of initial data. In our case, global character of the solution and its infinite differentiability are ensured by the fact that $\phi_n$ enter the right-hand side of Eqs. (24) only as arguments of the sine. Note that because of the arbitrariness of the interval $\alpha$, the solution can be uniquely continued onto the whole axis $-\infty < y < +\infty$. Now we will show that the problem of finding localized solutions to (24) can be reduced to the standard initial value problem.

Differentiating (21) with respect to $y$ yields

$$\sin \phi_n(y) = \epsilon^2 \sum_{m=1}^{N-1} G(n,m) \frac{d^2 \phi_n(y)}{dy^2}.$$  

(27)

Multiplying (27) by $\frac{d\phi_n}{dy}$, summing over the layer index $n$ with the use of (A10) and performing integration, we arrive at the first integral of Eqs. (24):

$$C - \sum_{n=1}^{N-1} \cos \phi_n(y) = \epsilon^2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n,m) \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy},$$  

(28)

where $C$ is the constant of integration. Now let us choose an arbitrary point $y_0 \in [-L, L]$, where $L$ is sufficiently large [see the condition (17)]. We are looking for localized solutions of Eqs. (24) that in the region

$$\lambda_{\text{max}} \ll |y - y_0|,$$  

(29)

where $\lambda_{\text{max}}$ is determined by the maximum positive eigenvalue of the symmetric matrix $\tilde{G}(n,m)$ (see Appendix A), satisfy the asymptotic conditions

$$\phi_n(y) = \mod 2\pi + o(1),$$  

(30)

$$\frac{d\phi_n(y)}{dy} = o(1)$$  

(31)

for any $1 \leq n \leq N - 1$. By inserting (30) and (31) into (28), we establish the value of the constant of integration: $C = N - 1$. Thus Eq. (28) becomes

$$\sum_{n=1}^{N-1} \frac{3}{2} \phi_n(y) = \epsilon^2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n,m) \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy}.$$  

(32)

Substituting initial values $\phi_n(y_0) = \alpha_n$, $\frac{d\phi_n}{dy}(y_0) = \beta_n$ into (22), we obtain the general criterion of the existence of localized solutions to (24):

$$\sum_{n=1}^{N-1} \sin^2 \alpha_n = \epsilon^2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n,m) \beta_n \beta_m.$$  

(33)

Indeed, the Lemma guarantees the existence, uniqueness and differentiability of a solution for arbitrary $\alpha_n$ and $\beta_n$ in the whole interval $[-L, L]$. Owing to our special choice of the constant of integration in (32), the solution determined by $\alpha_n$ and $\beta_n$ obeying (33) will necessarily satisfy asymptotic conditions (30), (31) in the region (29). Moreover, this solution will automatically satisfy the conditions

$$\frac{d^k \phi_n(y)}{dy^k} = o(1)$$  

(34)

in the region (29) for any $1 \leq n \leq N - 1$ and $2 \leq k$, which can be verified by repeated differentiation of (27) and application of (30), (31). Below, we apply the criterion (33) to concrete physical situations.
B. Meissner solutions. The superheating (penetration) field $H_s$

Let $H > 0$ for definiteness. Consider an intermediate length scale $R$ such that

$$1 \ll R < \frac{L}{2}. \quad (35)$$

(See Fig. 1.) The Meissner boundary value problem is specified by the boundary conditions (10) and

$$\phi_n(\pm(L - R)) = o(1), \quad \frac{d\phi_n}{dy}(\pm(L - R)) = o(1) \quad (36)$$

for any $1 \leq n \leq N - 1$. For $-L \leq y < -L + R$, we write using (12):

$$H_n(y) = \frac{1}{2} \int_{-L+R}^{y} du \sin \phi_n(u) + \frac{1}{2} \int_{-L}^{-L+R} du \sin \phi_n(u) + H. \quad (37)$$

According to (21), we make the identification

$$h_n(y) = \frac{1}{2} \int_{-L+R}^{y} du \sin \phi_n(u), \quad (38)$$

$$H_n = \frac{1}{2} \int_{-L}^{-L+R} dy \sin \phi_n(y) + H, \quad (39)$$

where $h_n(y)$ stands for the field penetrating through the $y = -L$ interface. Using the fact that $H_n < H$, and $H_n = H_{N-n}$, we establish the following important properties:

$$-\pi \leq \phi_n(y) < 0, \quad (40)$$

$$\phi_n(y) = \phi_{N-n}(y), \quad h_n(y) = h_{N-n}(y), \quad H_n(y) = H_{N-n}(y). \quad (41)$$

Relations (41) are a result of the symmetry of the problem. They imply that the number of independent equations describing the Meissner solution is $N$ for even $N$ and $\frac{N}{2}$ for odd $N$. The solution for $\phi_n(y)$ in the region $L - R < y \leq L$ can be obtained from the solution in the region $-L \leq y < -L + R$ using the property

$$\phi_n(y) = -\phi_n(-y), \quad (42)$$

resulting from the general relation (13). In the region $-R < y < R$, $\phi_n \equiv 0$, and we have

$$H_n(y) = H_n. \quad (43)$$

Now we apply relation (33) for $y_0 \equiv -L$. In view of the boundary conditions $\frac{d\phi_n}{dy}(-L) \equiv \beta_n = 2H$ [see (10)], we get

$$\frac{1}{N-1} \sum_{n=1}^{N-1} \sin \frac{\alpha_n}{2} = \frac{H^2}{H_s^2}, \quad (44)$$

where

$$H_s = \left[ 1 - \frac{\left( 2\sqrt{1 + \frac{\epsilon^2}{4} - \epsilon} \right) \left( 1 - \mu^{N-1} \right)}{\epsilon (N-1)(1 + \mu^{N-1})} \right]^{\frac{1}{2}}. \quad (45)$$
By a vortex-plane solution we understand a chain of \( N - 1 \) Josephson vortices (one vortex per each I-layer) positioned at \( y = 0 \). Analogously, an antivortex-plane solution is a chain of \( N - 1 \) Josephson antivortices (one antivortex per each I-layer). Such solutions are characterized by the symmetry

\[
\phi_n(y) = \pm 2\pi - \phi_n(-y)
\]

[see (46)] and asymptotic boundary conditions [31]

\[
\phi_n(-R) = o(1), \quad \phi_n(R) = \pm 2\pi + o(1),
\]

\[
\frac{d^k \phi_n(\pm R)}{dy^k} = o(1)
\]

for all \( 1 \leq n \leq N - 1 \) and any \( k \geq 1 \). [The "plus" sign in (46) and (47) corresponds to a vortex plane in fields \( H > 0 \), whereas the "minus" sign corresponds to an antivortex plane in fields \( H < 0 \).] The existence and uniqueness of these solutions follows immediately from the Lemma, the criterion (33) and the results of the previous subsection. Indeed, by (46), a vortex (antivortex) plane satisfies the conditions \( \alpha_n = \pm 2H_s \) on the right-hand side of (33), consistent with the vortex-penetration conditions (see the next paragraph). Hence the initial values
\[ \alpha_n \equiv \phi_n(0) = \pm \pi, \quad \beta_n \equiv \frac{d\phi_n}{dy}(0) = \pm 2H_s, \quad 1 \leq n \leq N - 1, \]  
(49)
determine a unique localized solution in the region \(-R < y < R\) that automatically meets the asymptotic boundary conditions \((47), (48)\).

Thermodynamic stability of vortex-plane (antivortex-plane) solutions is ensured by the fact that they satisfy the vortex-penetration conditions for \(H = \pm H_s\). Let \(H > 0\) for definiteness. (In what follows, we consider only vortex planes. The discussion of antivortex planes is quite analogous.) A vortex-plane solution can be constructed from the Meissner solution \(\phi^H_s\) in the region \(-L \leq y < -L + R\), discussed in the previous subsection. Indeed, using the properties \((25), (26)\), we obtain a solution
\[ \tilde{\phi}_n(y) = \phi^H_s(y - L) + 2\pi \]
in the region \(0 \leq y < R\) that satisfies the initial conditions \(\tilde{\phi}_n(0) = \pi, \frac{d\tilde{\phi}_n}{dy}(0) = 2H_s\). This solution can be continued \([12]\) into the region \(-R < y < 0\). By the uniqueness of a solution to the initial value problem, the obtained solution coincides with the vortex-plane solution in the interval \(-R < y < R\).

The total flux carried by a vortex plane is
\[ \Phi = \int_{-R}^{R} dy h_n(y) = \pi \left[ 1 - \frac{\mu^{-n} + \mu^{-N+n} - \mu_n - \mu^{N-n}}{\mu^{-N} - \mu^{N}} \right]. \]  
(51)
The total flux carried by a vortex plane is
\[ \Phi = \sum_{n=1}^{N-1} \Phi_n = \pi (N - 1) \left[ 1 - \frac{2\sqrt{1 + \epsilon^2} \epsilon - 1 - \mu_{N-1}^{-1}}{\epsilon (N - 1)} \right]. \]  
(52)
Note that in contrast to Josephson junctions with thick electrodes \([9]\) and infinite layered superconductors, \([1,3]\) the flux carried by a Josephson vortex in a finite thin-layer S/I structure is not quantized and is always smaller than the flux quantum \(\Phi_0 = \pi\). (This fact has been already pointed out in Ref. \([7]\).)

To determine the thermodynamic lower critical field \(H_{c1}\) at which the vortex-plane solutions become energetically favorable, we must calculate the difference between the Gibbs free energy in the presence of a single vortex plane, \(\Omega_v(H)\), and the Gibbs free energy of the homogeneous Meissner state, \(\Omega_M(H)\) [the sum of phase-independent terms in \((44)\)]. Substituting \((24)-(28)\) into \((44)\) and using \((A7), (A8)\), in first order in \(r(T) \ll 1\), we obtain:
\[ \Omega_v(H) - \Omega_M(H) = r(T) \left[ e^2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) \int_{-R}^{R} dy \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy} - 4\Phi H \right], \]  
(53)
where the total flux \(\Phi\) is given by \((52)\). The first term on the right-hand side of \((53)\) should be interpreted as the self-energy of the vortex plane:
\[ E_v = r(T) e^2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) \int_{-R}^{R} dy \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy} \]
\[
E_v = 4r(T) \int_{-R}^{R} \frac{N-1}{dy} \sum_{n=1}^{N-1} \sin \frac{\phi_n(y)}{2} \]

\[
= r(T) \sum_{n=1}^{N-1} \int_{-R}^{R} \left[ \frac{c^2}{2} \sum_{m=1}^{N-1} G(n, m) \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy} + 1 - \cos \phi_n(y) \right]. \tag{54}
\]

Note that \( E_v \) is exactly twice the energy of the Josephson currents in \([13]\). Besides formulas \([53] \) and \([54] \), with corresponding reinterpretation of \( \Omega_n(H) \), \( \Phi \) and \( E_v \), also hold for incoherent vortex solutions considered in the next subsection. They also apply to an infinite layered superconductor, taking account of the substitution \( \sum_{m,n=1}^{N-1} G(n, m) \ldots \rightarrow \sum_{m,n=-\infty}^{\infty} G_\infty(n, m) \ldots \), where \( G_\infty(n, m) \) are defined by \([A15]\). In this latter case, the self-energy should be additionally minimized with respect to the phases \( \varphi_n \), which immediately yields the exact solution \([13]\) with \( \phi_n(y) = \phi_{n+1}(y) \equiv \phi(y) \) and \( \lambda J_\infty = 1 \). [In the case \( N < \infty \), the minimization with respect to \( \varphi_n \) is not allowed by the boundary conditions \([3]\). \( \Phi \)]

From \([3]\), we get:

\[
H_{cl} = \frac{E_v}{4r(T)} \Phi. \tag{55}
\]

For thermodynamically stable solutions, we must necessarily have \( H_{cl} < H_s \). It is straightforward to verify that this condition is met by the vortex-plane solutions. Using \([27]\), \([44]\) (with the "plus" sign), the initial values \( \beta_n = 2H_s \) and integrating by parts, we convert \( E_v \) into the form

\[
E_v = 2r(T) \left[ \sum_{n=1}^{N-1} \int_{0}^{R} d\phi_n(y) \sin \phi_n(y) - 2H_s \Phi \right]. \tag{56}
\]

The first term on the right-hand side of \([56]\) is positive, because in the region \( 0 \leq y < R \) all \( \phi_n \) satisfy the relation \( \pi \leq \phi_n < 2\pi \). By the use of \([27]\) and \([A14]\), we obtain the following strict inequalities:

\[
2H_s \Phi < \sum_{n=1}^{N-1} \int_{0}^{R} d\phi_n(y) \sin \phi_n(y) < 4H_s \Phi,
\]

\[
0 < E_v < 4r(T)H_s \Phi.
\]

Hence,

\[
0 < H_{cl} < H_s,
\]

as anticipated. Note that in all special cases admitting exact analytical solutions (\( N = \infty, [13] \) and \( N = 2, 3, \) see section IV), \( H_{cl} = 2H_s \). Finally, we want to point out that in contrast to vortex-plane solutions in infinite layered superconductors, \([13]\) where \( H_n(y) = H_{n+1}(y) = H(y) \) and \( J_n = 0 \) for all \( n \), in finite structures the intralayer currents \( J_n \), in general, are not equal to zero, as can be easily seen from \([13]\). Only for even number of junctions (\( N \) is odd), in the central S-layer \( J_{N-1} = 0 \), by the symmetry \([11]\).

D. Single-vortex solutions and other localized incoherent vortex solutions

A single Josephson vortex positioned in the \( l \)th I-layer at \( y = 0 \) obeys symmetry relations

\[
\phi_l(y) = 2\pi - \phi_l(-y); \quad \phi_n(y) = -\phi_n(-y), \quad n \neq l, \tag{57}
\]

[see \([13]\)] and asymptotic boundary conditions \([13]\)

\[
\phi_l(-R) = o(1), \quad \phi_l(R) = 2\pi + o(1), \tag{58}
\]
\[ \phi_n(\pm R) = o(1), \quad n \neq l, \]  
\[ \frac{d^k \phi_n(\pm R)}{dy^k} = o(1), \quad \text{for all } 1 \leq n \leq N - 1, \text{ and any } k \geq 1. \]  
Moreover, \( \frac{dh_n(y)}{dy} > 0 \) in the region \(-R < y < 0\), and \( \frac{dh_n(y)}{dy} < 0 \) in the region \(0 < y < R\). Hence, \( \phi_n \) must satisfy the relations

\[ 0 < \phi_n(y) < \pi, \quad y \in (-R, 0); \quad -\pi < \phi_n(y) < 0, \quad y \in (0, R), \quad \text{for } n \neq l, \]  
[see (38) for \( y \in (-R, R) \)] and the initial conditions

\[ \alpha_l \equiv \phi_l(0) = \pi; \quad \alpha_n \equiv \phi_n(0) = 0, \quad n \neq l, \]  
\[ \beta_l \equiv \frac{d\phi_l(0)}{dy} > 0; \quad \beta_n \equiv \frac{d\phi_n(0)}{dy} < 0, \quad n \neq l. \]  
A necessary condition of the existence of such solutions is provided by the general criterion (33) and has the form

\[ \frac{c^2}{4} \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) \beta_n \beta_m = 1. \]  
Relation (64) imposes only one constraint on \( N - 1 \) quantities \( \beta_n \) and can be satisfied by different sets of \( \beta_n \). Therefore, in contrast to the vortex-plane problem, a solution to the single-vortex problem for any given vortex position \( l \) is not unique. The determination of the optimum set of \( \beta_n \) may require an additional examination of the vortex self-energy (see section III.B).

Besides failing to meet the requirement of uniqueness, single-vortex solutions, in general, break the symmetry (41), inherent to the original integrodifferential equations (8), (20) minimizing the Gibbs free-energy functional (1). (The only exclusion is a stack with an odd number of junctions \( N - 1 \) and \( l = \frac{N-1}{2} \).) Even more important is the fact that single-vortex configurations do not satisfy the vortex-penetration conditions for any \( H > 0 \). Indeed, starting from a single-vortex solution in the region \(-R < y < R\), we can construct a solution in the region \(-L \leq y < -L + R\) satisfying the Meissner boundary conditions (36). (Compare with the reverse procedure of the construction of a vortex-plane solution from the Meissner solution \( \phi_n^{H_s} \), described in the previous subsection.) However, the solution thus obtained will not represent any physics, because it cannot meet the physical boundary conditions \( \frac{dh_n(y)}{dy} \equiv \beta_n = 2H \) for any \( H > 0 \): According to (33), \( \beta_l \) and \( \beta_n \) with \( n \neq l \) have different signs. Physically, this means that isolated vortices cannot penetrate the periodic S/I structure at any static \( H > 0 \), and the penetration field for isolated vortices cannot be defined. On the basis of these observations, we conclude that, in contrast to vortex-plane solutions, single-vortex solutions are thermodynamically unstable and do not represent any solutions to (8), (20) for \( H > 0 \). This situation has a simple mathematical explanation. In contrast to differential equations (24), integrodifferential equations (8), (20) explicitly contain the external magnetic field \( H \). [See the explicit expressions for \( H_n(y) \), Eqs. (14).] In the case of Eqs. (24), the external field \( H \) enters only via the boundary conditions (11) at \( y = \pm R \), whereas single-vortex configurations are required to satisfy asymptotic boundary conditions (38)-(39) at \( y = \pm R \). It is therefore not surprising that Eqs. (24), restricted to the interval \((-R, R) \subset [-L, L]\), may possess redundant solutions that do not satisfy Eqs. (8), (20). [Note that the exact equations (8), valid for arbitrary \( r(T) \), do not, in general, reduce to any differential equations, if \( f_n(y) \neq \text{const.} \).]

The flux through the \( n \)th I-layer due to the vortex in the \( l \)th I-layer can be found using (22) and (38), (59):

\[ \Phi_n = \int_{-R}^{R} dy h_n(y) = \frac{\pi \epsilon}{2 \sqrt{1 + \epsilon^2}} \left[ \mu^{n_{-l}} - \frac{\mu^n (\mu^{-N} - \mu^{N-l}) + \mu^{N-n} (\mu^{-l} - \mu^l)}{\mu^{-N} - \mu^N} \right]. \]  
The total flux carried by the vortex in the \( l \)th layer is

\[ \Phi = \sum_{n=1}^{N-1} \Phi_n = \pi \left[ 1 - \frac{\mu^{-l} + \mu^{-N+l} - \mu^l - \mu^{N-l}}{\mu^{-N} - \mu^N} \right]. \]
For \( N < \infty \), the total flux \( \Phi \) is not quantized and is less than the flux quantum \( \Phi_0 = \pi \). (See the previous subsection.) Note that the total flux carried by a single vortex in the \( l \)th I-layer is exactly equal to the flux through the \( l \)th I-layer in the case of a vortex plane, given by Eq. (51) with \( n = l \).

The consideration of other localized incoherent vortex solutions to (24) (i.e., solutions with \( 2 \leq k < N - 1 \) vortices in the plane \( y = 0 \), and vortex-antivortex configurations) can be done along the same lines. All these solutions are thermodynamically unstable too. We want to underline that our general conclusions about the form of static single-vortex configurations completely agree with the results of numerical calculations of Ref. [4]. (See, in particular, Fig. 5 therein.) On the other hand, these single-vortex configurations have no resemblance to hypothetical Abrikosov-type vortices introduced without appropriate mathematical justification in Refs. [10,11]. Let alone thermodynamic instability, the actual single-vortex solutions to (24) are accompanied by singular phase-difference distribution in all \( N - 1 \) junctions, satisfying (61)-(63) and the existence condition (64). Their spatial dependence is characterized, in general, by \( N - 1 \) length scales, which is inherent to localized solutions of a system of coupled second-order differential equations. These intrinsic features of topological solutions in discrete periodic S/I structures cannot be reproduced by any imitation of Abrikosov vortices, typical of continuum type-II superconductors. It is also worth reminding that Abrikosov vortices in the London approximation are described by linear partial differential equations, whereas the ordinary differential equations (24) are essentially nonlinear. The principle of superposition of solutions is not valid for nonlinear equations. Unfortunately, this basic point is sometimes disregarded in literature.  

IV. PARTICULAR EXAMPLES

A. A single thin-layer junction \( (N = 2) \)

In this simplest case, the only nonzero element of the matrix \( G(n,m) \), given by (A9), is

\[
G(1,1) = \frac{1}{2 + \epsilon^2}.
\] (67)

By (27), a single phase difference \( \phi_1(y) = \phi(y) \) satisfies the usual static sine-Gordon equation

\[
\frac{d^2 \phi(y)}{dy^2} = \frac{1}{\lambda_J^2} \sin \phi(y),
\] (68)

with the Josephson length \[8\]

\[
\lambda_J = \frac{\epsilon}{\sqrt{2 + \epsilon^2}}.
\] (69)

Note that \( \lambda_J \), given by (69), for \( \epsilon \ll 1 \) is much smaller than the Josephson length of a single junction with thick electrodes, which in our dimensionless units is \( \lambda_{J0} = \sqrt{\frac{2}{\epsilon}} \). From (23) and (45), we get the local field in the homogeneous Meissner state

\[
H_1 = \frac{2H}{2 + \epsilon^2},
\] (70)

and the superheating (penetration) field

\[
H_s = \lambda_J^{-1} = \frac{\sqrt{2 + \epsilon^2}}{\epsilon},
\] (71)

respectively. For \( \epsilon \ll 1 \), the superheating (penetration) field \( H_s \), given by (71), is much higher than the corresponding field \[8\] of a single junction with thick electrodes \( H_{s0} = \lambda_{J0} \).

1. The Meissner solution

For the fields \( 0 \leq H \leq H_s = \frac{\sqrt{2 + \epsilon^2}}{\epsilon} \), the Meissner solution in the region \( -L \leq y < -L + R \) up to first order in \( r(T) \ll 1 \) is given by
\[
\phi(y) = -4 \arctan \frac{H \exp \left[ -\frac{(L+y)}{\lambda J} \right]}{H_s + \sqrt{H_s^2 - H^2}}, \quad (72)
\]

\[
H(y) \equiv H_1(y) = h(y) + H_1, \quad (73)
\]

\[
h(y) \equiv h_1(y) = \frac{2\lambda_J H}{H_s + \sqrt{H_s^2 - H^2}} \exp \left[ -\frac{(L+y)}{\lambda J} \right] \frac{2L+y}{H^2}, \quad (74)
\]

\[
j(y) \equiv j_1,0(y) = -4H \left[ H_s + \sqrt{H_s^2 - H^2} \right] \frac{\left[ H_s + \sqrt{H_s^2 - H^2} \right]^2 - H^2 \exp \left[ -\frac{2(L+y)}{\lambda J} \right] \exp \left[ -\frac{(L+y)}{\lambda J} \right]^2}{\left[ H_s + \sqrt{H_s^2 - H^2} \right]^2 + H^2 \exp \left[ -\frac{2(L+y)}{\lambda J} \right]^2}, \quad (75)
\]

\[
J(y) \equiv J_0(y) = J_1(y) = \frac{1}{4\pi} [H - H_1 - h(y)], \quad (76)
\]

\[
f(y) \equiv f_0(y) = f_1(y) = 1 - \frac{r(T)}{2} \left[ \lambda_J^{-2} h(y) + \frac{2}{\epsilon^2} [h(y) + H_1 - H] \right]^2. \quad (77)
\]

The Meissner solution in the region \(L - R < y \leq L\) can be obtained from (72)-(77) by means of the substitution \(y \to -y, \phi(y) \to -\phi(-y)\). In the region \(-R < y < R\), the solution is

\[
\phi(y) \equiv 0, \quad h(y) \equiv 0, \quad j(y) \equiv 0, \quad (78)
\]

\[
H(y) = H_1, \quad (79)
\]

\[
J(y) = \frac{1}{4\pi} [H - H_1], \quad (80)
\]

\[
f(y) = 1 - \frac{r(T)}{\epsilon^2} [H - H_1]^2. \quad (81)
\]

2. The vortex solution

In the region \(-R < y < R\), the vortex (antivortex) solution satisfies the initial conditions \(\alpha \equiv \phi(0) = \pi, \beta \equiv \frac{d\phi(0)}{dy} = \pm 2H_s\) [Eq. (49)] and has the form

\[
\phi(y) = \pm 4 \arctan \exp \left[ \frac{y}{\lambda_J} \right], \quad (82)
\]

\[
h(y) = \pm \lambda_J \cosh^{-1} \left[ \frac{y}{\lambda_J} \right], \quad (83)
\]

\[
j(y) = \mp 2 \cosh^{-2} \left[ \frac{y}{\lambda_J} \right] \sinh \left[ \frac{y}{\lambda_J} \right]. \quad (84)
\]
The quantities $H(y)$, $J(y)$ and $f(y)$ are given by (73), (76) and (77), respectively, with $h(y)$ taken from (83). Note that the field induced by a vortex at $y = 0$, in agreement with (50), is

$$h(0) = \lambda J = \frac{\epsilon^2 H_s}{2 + \epsilon^2},$$

and not $h(0) = H_s$, as in the case of a single junction with thick electrodes. By inserting (67) and (82) into (54), we obtain the vortex self-energy:

$$E_v = 8 r(T) \frac{\epsilon}{\sqrt{2 + \epsilon^2}}. \quad (85)$$

The vortex flux, according to (52), is

$$\Phi = \pi \frac{\epsilon^2}{2 + \epsilon^2},$$

and the lower critical field, by (55), is

$$H_{c1} = \frac{2}{\pi} H_s = \frac{2}{\pi} \sqrt{2 + \epsilon^2}. \quad (86)$$

Thus, for $\epsilon \ll 1$, the vortex flux $\Phi \ll \Phi_0 = \pi$, and the lower critical field (86) is much larger than the corresponding field of a single junction with thick electrodes $H_{c10} = \frac{2}{\pi} \sqrt{\frac{2}{2\pi}}$, in agreement with Ref. [7].

### B. A double-junction stack ($N = 3$)

In the double-junction case, the nonzero matrix elements (A9) of $G(n,m)$ are

$$G(1,1) = G(2,2) = \frac{2 + \epsilon^2}{(2 + \epsilon^2)^2 - 1}, \quad G(1,2) = G(2,1) = \frac{1}{(2 + \epsilon^2)^2 - 1}. \quad (87)$$

The corresponding $2 \times 2$ matrix $\tilde{G}(n,m)$ (see Appendix A) has two positive eigenvalues: $\frac{\lambda J_1^2}{2\pi}$ and $\frac{\lambda J_2^2}{2\pi}$, with the lengths

$$\lambda J_1 = \frac{\epsilon}{\sqrt{1 + \epsilon^2}}, \quad \lambda J_2 = \frac{\epsilon}{\sqrt{3 + \epsilon^2}}. \quad (88)$$

According to (15), the superheating (penetration) field is

$$H_s = \lambda J_1^{-1} = \frac{\sqrt{1 + \epsilon^2}}{\epsilon}, \quad (89)$$

which is smaller than the corresponding single-junction value (71), in agreement with Ref. [6]. The application of (23) yields the value of the local field in the homogeneous Meissner state:

$$H_1 = H_2 = \frac{H}{1 + \epsilon^2}. \quad (90)$$

#### 1. The Meissner solution

The Meissner solution in the fields $0 \leq H \leq H_s = \frac{\sqrt{1 + \epsilon^2}}{\epsilon}$ obeys the symmetry relations (41). The substitution of

$$\phi_1(y) = \phi_2(y) = \phi(y) \quad (91)$$

into (27), using (87), yields

$$\frac{d^2 \phi(y)}{dy^2} = \frac{1}{\lambda J_1^2} \sin \phi(y). \quad (92)$$
Thus, all the results of the single-junction case, Eqs. (72)-(81), apply if we substitute \( \lambda_J \rightarrow \lambda_{J1} \) (note that \( \lambda_J < \lambda_{J1} \)), make the identification

\[
\phi(y) \equiv \phi_1(y) = \phi_2(y), \quad H(y) \equiv H_1(y) = H_2(y), \quad h(y) \equiv h_1(y) = h_2(y),
\]

\[
j(y) \equiv j_{1,0}(y) = j_{2,1}(y), \quad J(y) \equiv J_0(y) = J_2(y), \quad f(y) \equiv f_0(y) = f_2(y),
\]

and take the values of \( H_s \) and \( H_1 \) from (89) and (90), respectively. Moreover,

\[
J_1(y) = 0,
\]

and

\[
f_1(y) = 1 - \frac{r(T)}{\lambda_{J1}^3} h(y).
\]

2. The vortex-plane solution

In the region \(-R < y < R\), the vortex-plane (antivortex-plane) solution describes two vortices (antivortices) [one vortex (antivortex) per I-layer] and satisfies the initial conditions \( \alpha_1 \equiv \phi_1(0) = \pi, \alpha_2 \equiv \phi_2(0) = \pi, \beta_1 \equiv \frac{d\phi_1(0)}{dy} = \pm 2H_s, \beta_2 \equiv \frac{d\phi_2(0)}{dy} = \pm 2H_s \) [Eq. (41)]. By (41), it obeys the symmetry (91) and Eq. (92), with

\[
\phi(y) = \pm 4 \arctan \exp \left[ \frac{y}{\lambda_{J1}} \right].
\]

Thus, explicit expressions for \( h_1(y) = h_2(y) \equiv h(y) \) and \( j_{1,0}(y) = j_{2,1}(y) \equiv j(y) \) can be obtained from single-junction Eqs. (83), (84), taking account of the substitution \( \lambda_J \rightarrow \lambda_{J1} \). The quantities \( H_1(y) = H_2(y) \equiv H(y), \) \( J_0(y) = J_2(y) \equiv J(y) \) and \( f_0(y) = f_2(y) \equiv f(y) \) are given by (73), (76) and (77), respectively, with \( H_1 \) taken from (90). For \( J_1(y) \) and \( f_1(y) \), we have (93) and (94), respectively.

The vortex-plane self-energy is

\[
E_v = 16r(T)\lambda_{J1} = 16r(T)\frac{\epsilon}{\sqrt{1 + \epsilon^2}},
\]

and the flux is

\[
\Phi = 2\pi \frac{\epsilon^2}{1 + \epsilon^2},
\]

which immediately leads to the lower critical field:

\[
H_{c1} = \frac{2}{\pi} H_s = \frac{2}{\pi} \frac{\sqrt{1 + \epsilon^2}}{\epsilon}.
\]

As can be seen by comparing (95) with the single-junction expression (85), the energy per vortex in the double-junction stack is higher. Finally, the field induced by the vortex plane at \( y = 0 \), according to (50), is

\[
h(0) \equiv h_1(0) = h_2(0) = \frac{\epsilon^2 H_s}{1 + \epsilon^2}.
\]
3. The vortex-antivortex solution

As can be easily seen, equations (24) with \( N = 3 \) admit in the region \( -R < y < R \) another exact topological solution, namely an incoherent vortex-antivortex solution

\[
\phi_1(y) = -\phi_2(y) \equiv \phi(y),
\]

where \( \phi(y) \) is given by

\[
\phi(y) = 4 \arctan \left[ \frac{y}{\lambda_2} \right].
\]

With \( \alpha_1 = \alpha_2 = \pi \) and \( \beta_1 = -\beta_2 = \pm 2\lambda_2^{-1} \), the vortex-antivortex solution explicitly satisfies the existence criterion (33). By (54), the self-energy of the vortex-antivortex solution is

\[
E_{va} = 16r(T)\lambda_2 = 16r(T) \frac{\epsilon}{\sqrt{3 + \epsilon^2}},
\]

which is lower than the self-energy of the vortex-plane solution (95).

However, the vortex-antivortex solution does not satisfy integrodifferential equations (20) with \( N = 3 \): These equations do not possess the symmetry (97) for any \( H \neq 0 \). The vortex-penetration conditions also cannot be met, because \( \beta_1 \beta_2 < 0 \). Moreover, the Gibbs free energy (14) of the vortex-antivortex pair is always positive with respect to the Gibbs free energy of the Meissner state: The flux \( \Phi \) carried by this pair is exactly equal to zero. Thus, the static vortex-antivortex solution is thermodynamically unstable, in agreement with the general consideration in section III.

As shown in Ref. [4], the vortex-antivortex solution can be realized in the dynamic regime, in the presence of an external current applied to the central S-layer.

4. Single-vortex solutions

Consider a configuration with a single vortex in one I-layer (say, with \( n = 2 \)) and no vortices in the other. The solution representing this configuration does not possess any symmetry. As shown in Appendix C, the self-energy of a single vortex, \( E_{sv} \), satisfies the exact inequality \( E_{sv} < E_{a} < E_{v} \), where \( E_{a} \) and \( E_{v} \) are given by (95) and (98), respectively. On the other hand, the total flux carried by a single vortex, according to (66) with \( N = 3 \) and \( l = 2 \), is

\[
\Phi = \pi \frac{\epsilon^2}{1 + \epsilon^2},
\]

which is exactly half the total flux of the vortex plane (96). Hence, the Gibbs free energy of a single vortex is positive with respect to that of the Meissner state for \( H \leq H_{c1} = \frac{2}{\sqrt{1 + \epsilon^2}} \). This example clearly illustrates thermodynamic instability of single-vortex solutions discussed in section III.

The two second-order differential equations describing the single-vortex configuration can be reduced [12] to one fourth-order equation

\[
\begin{align*}
\frac{d^4 \phi_2}{dy^4} + \frac{2 + \epsilon^2}{\epsilon^2} \sin \phi_2 \left( \frac{d \phi_2}{dy} \right)^2 - \frac{2 + \epsilon^2}{\epsilon^2} \cos \phi_2 \frac{d^2 \phi_2}{dy^2} \\
- \left[ \frac{2 + \epsilon^2}{\epsilon^2} \sin \phi_2 - \frac{d^2 \phi_2}{dy^2} \right] \left[ \left( 2 + \epsilon^2 \right) \cos \phi_2 \frac{d \phi_2}{dy} - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right]^2 \\
1 - \left( 2 + \epsilon^2 \right) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right] & = 0.
\end{align*}
\]

The phase difference \( \phi_1(y) \) can be found without any additional integration from the relations.
\[
\phi_1 = \arcsin \left[ (2 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right], \quad -\frac{\pi}{2} \leq \phi_1 \leq \frac{\pi}{2},
\]

\[
\phi_1 = -\arcsin \left[ (2 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right] + \pi, \quad \frac{\pi}{2} < \phi_1 < \pi,
\]

\[
\phi_1 = -\arcsin \left[ (2 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right] - \pi, \quad -\pi < \phi_1 < -\frac{\pi}{2}.
\]

The initial conditions for the vortex solution are given by the relations \(\alpha_1 \equiv \phi_1(0) = 0, \alpha_2 \equiv \phi_2(0) = \pi, \beta_1 \equiv \frac{d\phi_2(0)}{dy} < 0, \beta_2 \equiv \frac{d\phi_2(0)}{dy} > 0\) and must satisfy the existence criterion [14]:

\[
(2 + \epsilon^2) \left( \beta_1^2 + \beta_2^2 \right) - 2 |\beta_1| \beta_2 = \frac{4 \left[ (2 + \epsilon^2)^2 - 1 \right]}{\epsilon^2}.
\]

In view of the condition \(\beta_2 > 0\), the appropriate solution of [101] for \(\beta_2\) is

\[
\beta_2 \equiv \frac{d\phi_2(0)}{dy} = \frac{1}{2 + \epsilon^2} \left[ |\beta_1| + \frac{1}{\epsilon} \sqrt{(2 + \epsilon^2)^2 - 1} \left[ 4 \left( 2 + \epsilon^2 \right) - \epsilon^2 \beta_1^2 \right] \right],
\]

\[
0 < |\beta_1| \leq \frac{2\sqrt{2 + \epsilon^2}}{\epsilon}.
\]

By the symmetry [57], \(\frac{d^2 \phi_2(0)}{dy^2} = 0\). The initial condition on \(\frac{d^3 \phi_2}{dy^3}\) can be obtained from the relation

\[
\frac{d^3 \phi_2}{dy^3} = \frac{1}{\epsilon^3} \left[ (2 + \epsilon^2) \cos \phi_2(y) \frac{d^2 \phi_2(y)}{dy} - \cos \phi_1(y) \frac{d\phi_1(y)}{dy} \right]
\]

that follows directly from [24]:

\[
\frac{d^3 \phi_2(0)}{dy^3} = -\frac{1}{\epsilon^3} \sqrt{(2 + \epsilon^2)^2 - 1} \left[ 4 \left( 2 + \epsilon^2 \right) - \epsilon^2 \beta_1^2 \right].
\]

Thus, we have obtained a complete formulation of the single-vortex problem in terms of the standard initial value problem. In agreement with general consideration of section III, the problem admits a family of single-vortex solutions parameterized by \(|\beta_1|\).

Unfortunately, there are no general methods of analytical integration of nonlinear differential equations of order higher than two. However, numerical integration of [94] with the above-derived initial conditions should pose no problem. Moreover, it is not difficult to obtain asymptotics of the single-vortex solution in the region \(|y| \gg \lambda J_1\). For \(y \ll -\lambda J_1\), equations (99), (100) can be linearized:

\[
\frac{d^4 \phi_2}{dy^4} - \frac{2 \left( 2 + \epsilon^2 \right)}{\epsilon^2} \frac{d^2 \phi_2}{dy^2} + \frac{(2 + \epsilon^2)^2 - 1}{\epsilon^4} \phi_2 = 0,
\]

\[
\phi_1 = (2 + \epsilon^2) \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2}.
\]

The solution of (102), (103) obeying the vortex asymptotic conditions is straightforward:

\[
\phi_1(y) = C_1(\beta_1) \exp \left[ \frac{y}{\lambda_{J_1}} \right] - C_2(\beta_1) \exp \left[ \frac{y}{\lambda_2} \right],
\]

\[
\phi_2(y) = C_1(\beta_1) \exp \left[ \frac{y}{\lambda_{J_1}} \right] + C_2(\beta_1) \exp \left[ \frac{y}{\lambda_2} \right], \quad y \ll -\lambda_{J_1},
\]

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where $C_1(\beta_1), C_2(\beta_1) > 0$ are constants with respect to $y$, parameterized by the initial value $|\beta_1|$. The asymptotics for $y \gg \lambda_{J1}$ can be obtained from (104) by the use of the symmetry relations (57):

$$
\phi_1(y) = -C_1(\beta_1) \exp \left[ -\frac{y}{\lambda_{J1}} \right] + C_2(\beta_1) \exp \left[ -\frac{y}{\lambda_2} \right],
$$

$$
\phi_2(y) = 2\pi - C_1(\beta_1) \exp \left[ -\frac{y}{\lambda_{J1}} \right] - C_2(\beta_1) \exp \left[ -\frac{y}{\lambda_2} \right], \quad y \gg \lambda_{J1},
$$

Expressions (104), (105) illustrate a very important property of single-vortex solutions in multilayer structures: Their spatial dependence is characterized, in general, by $N-1$ different length scales, which agrees with the conclusions of Ref. [7].

### C. A 3-junction stack ($N = 4$)

The consideration of all solutions to (24) in the case $N \geq 4$, including thermodynamically unstable incoherent vortex configurations, requires the use of $(N-1) \geq 3$ second-order nonlinear differential equations, or, equivalently, of one nonlinear equation of order $2 \ (N-1) \geq 6$. Therefore, for the sake of mathematical simplicity, from now on we concentrate only on Meissner solutions and topological vortex-plane solutions, both obeying the symmetry (51). (The analysis of incoherent vortex configurations for $N \geq 4$ can be done using the algorithm worked out in the previous subsection.)

For $N = 4$, the Meissner solutions and the vortex-plane solutions satisfy the relation $\phi_3(y) = \phi_1(y)$. The relevant two equations for $\phi_1(y)$ and $\phi_2(y)$ can be given the form

$$
\frac{d^2 \phi_2}{dy^2} + \frac{2 + \epsilon^2}{\epsilon^2} \sin \phi_2 \left( \frac{d\phi_2}{dy} \right)^2 - \frac{2 + \epsilon^2}{\epsilon^2} \cos \phi_2 \frac{d^2 \phi_2}{dy^2} = 0,
$$

$$
\frac{1}{4} \left[ \frac{2 + \epsilon^2}{\epsilon^2} \sin \phi_2 - \frac{d^2 \phi_2}{dy^2} \right] \left[ \left( 2 + \epsilon^2 \right) \cos \phi_2 \frac{d\phi_2}{dy} - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right] = 0,
$$

$$
\phi_1 = \arcsin \frac{1}{2} \left[ (2 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right], \quad -\frac{\pi}{2} \leq \phi_1 \leq \frac{\pi}{2},
$$

$$
\phi_1 = -\arcsin \frac{1}{2} \left[ (2 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right] + \pi, \quad \frac{\pi}{2} < \phi_1 < \pi,
$$

$$
\phi_1 = -\arcsin \frac{1}{2} \left[ (2 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right] - \pi, \quad -\pi < \phi_1 < -\frac{\pi}{2}.
$$

Note that Eqs. (106), (107) have the same mathematical structure as equations for single vortices in the case of a double-junction stack (99), (100). According to (41), the superheating (penetration) field for $N = 4$ is

$$
H_s = \sqrt{\frac{3 (2 + \epsilon^2)^2 - 2}{\epsilon \sqrt{10 + 3 \epsilon^2}}}.
$$

The local field in the homogeneous Meissner state, according to (28), is given by the relations

$$
H_1 = H_3 = \frac{(2 + \epsilon^2) H}{(2 + \epsilon^2)^2 - 2}, \quad H_2 = \frac{2H}{(2 + \epsilon^2)^2 - 2}.$$
1. The Meissner solution for $0 < H \ll H_s$

In fields $0 < H \ll H_s$, the general criterion of the existence of Meissner solutions (14) takes the form

$$\frac{1}{6} \left[ \alpha_1^2 + \frac{\alpha_2^2}{2} \right] = \frac{H^2}{H_s^2},$$

where $\alpha_1 \equiv \phi_1^2(-L) \ll 1$, $\alpha_2 \equiv \phi_2^2(-L) \ll 1$. Thus, equations (108), (109) can be linearized:

$$\frac{d^4 \phi_2}{dy^4} - \frac{2(2 + \epsilon^2)}{\epsilon^2} \frac{d^2 \phi_2}{dy^2} + \frac{(2 + \epsilon^2)^2 - 2}{\epsilon^4} \phi_2 = 0,$$

$$\phi_1 = \frac{1}{2} \left[ (2 + \epsilon^2) \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right].$$

The Meissner solution of (108), (110) in the region $-L \leq y < -L + R$, obeying the boundary conditions

$$\frac{d \phi_1}{dy} (-L) = \frac{d \phi_2}{dy} (-L) = 2H,$$

is

$$\phi_1(y) = -\frac{\epsilon H}{\sqrt{2}} \left[ \frac{(\sqrt{2} + 1)}{\sqrt{2 - \sqrt{2} + \epsilon^2}} \exp \left[ \frac{(y + L)}{\lambda_{J1}} \right] - \frac{(\sqrt{2} - 1)}{\sqrt{2 + \sqrt{2} + \epsilon^2}} \exp \left[ \frac{(y + L)}{\lambda_{J2}} \right] \right],$$

$$\phi_2(y) = -\epsilon H \left[ \frac{(\sqrt{2} + 1)}{\sqrt{2 - \sqrt{2} + \epsilon^2}} \exp \left[ \frac{(y + L)}{\lambda_{J1}} \right] + \frac{(\sqrt{2} - 1)}{\sqrt{2 + \sqrt{2} + \epsilon^2}} \exp \left[ \frac{(y + L)}{\lambda_{J2}} \right] \right],$$

where

$$\lambda_{J1} = \frac{\epsilon}{\sqrt{2 - \sqrt{2} + \epsilon^2}}, \quad \lambda_{J2} = \frac{\epsilon}{\sqrt{2 + \sqrt{2} + \epsilon^2}}.$$

We observe that spatial dependence of the solution (111) is characterized by two different Josephson lengths, $\lambda_{J1}$ and $\lambda_{J2}$, in agreement with the general consideration of section III. Moreover, $\phi_1(y), \phi_2(y) < 0$, and $|\phi_1(y)| < |\phi_2(y)|$. The values $\alpha_1 \equiv \phi_1(-L)$ and $\alpha_2 \equiv \phi_2(-L)$ meet the existence criterion (108), as they should. Using (111), we can derive explicit expressions for all physical quantities of interest from general formulas of section II. The Meissner solution in the region $L - R < y \leq L$ can be obtained from (111) by means of the substitution $\phi_{1,2}(y) \rightarrow -\phi_{1,2}(-y)$. In the region $-R < y < R$, we have $\phi_{1,2}(y) \equiv 0$, and $H_n(y) = H_n$, as usual.

2. The vortex-plane solution

The vortex-plane solution to (106), (107) in the region $-R < y < R$ obeys the initial conditions $\alpha_1 \equiv \phi_1(0) = \pi$, $\alpha_2 \equiv \phi_2(0) = \pi$ and $\beta_1 \equiv \frac{d \phi_1(0)}{dy} = 2H_s, \beta_2 \equiv \frac{d \phi_2(0)}{dy} = 2H_s$ [Eq. (109)]. By the symmetry (41), we also have $\frac{d^2 \phi_{2}(0)}{dy^2} = 0$. The initial condition on $\frac{d^2 \phi_2}{dy^2}$ is derived from the relation

$$\frac{d^3 \phi_2}{dy^3} = \frac{1}{\epsilon^2} \left[ (2 + \epsilon^2) \cos \phi_2 \frac{d \phi_2}{dy} - 2 \cos \phi_1 \frac{d \phi_1}{dy} \right]$$

that follows from (22):

$$\frac{d^3 \phi_2}{dy^3} = -2H_s.$$

In this way, we arrive at a complete formulation of the initial value problem for the vortex-plane configuration. In contrast to the single-vortex problem considered in the previous subsection, the above-derived initial conditions do
not contain any arbitrariness, in full agreement with our general conclusion in section III about the uniqueness of the vortex-plane solutions.

Although the vortex-plane problem admits only numerical integration, the asymptotics in the regions $|y| \gg \lambda J_1$ and $|y| \ll \lambda J_2$ can be readily obtained. Thus, on the basis of linearized Eqs. (109), (110), we have:

$$
\phi_1(y) = \frac{1}{\sqrt{2}} \left[ C_1 \exp \left( \frac{y}{\lambda J_1} \right) - C_2 \exp \left( \frac{y}{\lambda J_2} \right) \right],
$$

$$
\phi_2(y) = C_1 \exp \left( \frac{y}{\lambda J_1} \right) + C_2 \exp \left( \frac{y}{\lambda J_2} \right), \quad y \ll -\lambda J_1;
$$

$$
\phi_1(y) = 2\pi - \frac{1}{\sqrt{2}} \left[ C_1 \exp \left( -\frac{y}{\lambda J_1} \right) - C_2 \exp \left( -\frac{y}{\lambda J_2} \right) \right],
$$

$$
\phi_2(y) = 2\pi - C_1 \exp \left( -\frac{y}{\lambda J_1} \right) - C_2 \exp \left( -\frac{y}{\lambda J_2} \right), \quad y \gg \lambda J_1,
$$

where $C_1$ and $C_2$ are positive constants. The solution in the region $|y| \ll \lambda J_2$ is represented by Taylor series expansions:

$$
\phi_1(y) = \pi + 2\frac{H_s}{\epsilon} y - \frac{(1 + \epsilon^2)}{3\epsilon^2} \frac{H_s}{\epsilon} y^3 + \ldots,
$$

$$
\phi_2(y) = \pi + 2\frac{H_s}{\epsilon} y - \frac{1}{3} \frac{H_s}{\epsilon} y^3 + \ldots
$$

Finally, the total flux carried by the vortex plane, by (52), is

$$
\Phi = \frac{\epsilon^2}{2} \frac{(10 + 3\epsilon^2)}{(2 + \epsilon^2)^2 - 2},
$$

and the exact value of the induced field at $y = 0$, by (50), is

$$
h_1(0) = h_3(0) = \frac{\epsilon^2}{(2 + \epsilon^2)^2 - 2}, \quad h_2(0) = \frac{\epsilon^2}{2} \frac{(4 + \epsilon^2)}{(2 + \epsilon^2)^2 - 2}.
$$

(See Fig. 2.)

**D. A 4-junction stack ($N = 5$)**

For $N = 5$, the Meissner solutions and the vortex-plane solutions satisfy the relation $\phi_4(y) = \phi_1(y)$ and $\phi_3(y) = \phi_2(y)$. The equations for $\phi_1(y)$ and $\phi_2(y)$ reduce to the form

$$
\frac{d^4\phi_2}{dy^4} + \frac{1 + \epsilon^2}{\epsilon^2} \sin \phi_2 \left( \frac{d\phi_2}{dy} \right)^2 - \frac{1 + \epsilon^2}{\epsilon^2} \cos \phi_2 \frac{d^2\phi_2}{dy^2} + \frac{\left[ 1 + \frac{\epsilon^2}{\epsilon^2} \sin \phi_2 - \frac{d^2\phi_2}{dy^2} \right] \left[ (1 + \epsilon^2) \cos \phi_2 \frac{d\phi_2}{dy} - \epsilon^2 \frac{d^2\phi_2}{dy^2} \right]^2}{1 - \left[ (1 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2\phi_2}{dy^2} \right]^2} + \sqrt{1 - \left[ (1 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2\phi_2}{dy^2} \right]^2} \left[ \left( 2 + \epsilon^2 \right) \left( 1 + \epsilon^2 \right) - \frac{1}{\epsilon^4} \sin \phi_2 - \frac{2 + \epsilon^2}{\epsilon^2} \frac{d^2\phi_2}{dy^2} \right] = 0,
$$

(113)
\[ \phi_1 = \arcsin \left[ (1 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right], \quad -\frac{\pi}{2} \leq \phi_1 \leq \frac{\pi}{2}, \]

\[ \phi_1 = -\arcsin \left[ (1 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right] + \pi, \quad \frac{\pi}{2} < \phi_1 < \pi, \]

\[ \phi_1 = -\arcsin \left[ (1 + \epsilon^2) \sin \phi_2 - \epsilon^2 \frac{d^2 \phi_2}{dy^2} \right] - \pi, \quad -\pi < \phi_1 < -\frac{\pi}{2}. \]

(114)

Mathematical structure of Eqs. (113), (114) is analogous to that of Eqs. (106), (107) of the 3-junction stack, which allows us to skip some details in what follows. According to (45), the superheating (penetration) field for \( N = 5 \) is

\[ H_s = \frac{\sqrt{2} \left[ (2 + \epsilon^2) (1 + \epsilon^2) - 1 \right]}{\epsilon \sqrt{5 + 2 \epsilon^2}}. \]

The local field in the homogeneous Meissner state, according to (23), is

\[ H_1 = H_4 = \frac{(1 + \epsilon^2) H}{(2 + \epsilon^2) (1 + \epsilon^2) - 1}, \quad H_2 = H_3 = \frac{H}{(2 + \epsilon^2) (1 + \epsilon^2) - 1}. \]

1. The Meissner solution for \( 0 < H \ll H_s \)

For the fields \( 0 < H \ll H_s \), the Meissner solution in the region \(-L \leq y < -L + R\) has the form

\[ \phi_1(y) = -\frac{\epsilon H}{2 \sqrt{5}} \left[ \frac{\left( \sqrt{3} - 1 \right) (3 + \sqrt{5})}{\sqrt{\frac{3 - \sqrt{5}}{2} + \epsilon^2}} \exp \left[ \frac{y + L}{\lambda_{j1}} \right] + \frac{\left( \sqrt{5} + 1 \right) (3 - \sqrt{5})}{\sqrt{\frac{3 + \sqrt{5}}{2} + \epsilon^2}} \exp \left[ \frac{-(y + L)}{\lambda_{j2}} \right] \right], \]

\[ \phi_2(y) = -\frac{\epsilon H}{\sqrt{5}} \left[ \frac{3 + \sqrt{5}}{\sqrt{\frac{3 - \sqrt{5}}{2} + \epsilon^2}} \exp \left[ \frac{y + L}{\lambda_{j1}} \right] - \frac{3 - \sqrt{5}}{\sqrt{\frac{3 + \sqrt{5}}{2} + \epsilon^2}} \exp \left[ \frac{-(y + L)}{\lambda_{j2}} \right] \right], \]

where

\[ \lambda_{j1} = \frac{\epsilon}{\sqrt{\frac{3 - \sqrt{5}}{2} + \epsilon^2}}, \quad \lambda_{j2} = \frac{\epsilon}{\sqrt{\frac{3 + \sqrt{5}}{2} + \epsilon^2}}. \]

(115)

By comparison with \( \lambda_{j1}, \lambda_{j2} \) of the 3-junction case, Eq. (113), the Josephson length given by (115) are larger, which illustrates a general tendency: Josephson lengths increase with increasing \( N \). As in the 3-junction case, \( \phi_1(y), \phi_2(y) < 0 \), and \( |\phi_1(y)| < |\phi_2(y)| \). The values \( \alpha_1 \equiv \phi_1(-L) \) and \( \alpha_2 \equiv \phi_2(-L) \) satisfy the existence criterion (14), as expected.

2. The vortex-plane solution

The vortex-plane solution to (113), (114) in the region \(-R < y < R\) is characterized by the initial conditions \( \alpha_1 \equiv \phi_1(0) = \pi, \alpha_2 \equiv \phi_2(0) = \pi \) and \( \beta_1 \equiv \frac{d\phi_1(0)}{dy} = 2H_s, \beta_2 \equiv \frac{d\phi_2(0)}{dy} = 2H_s \) [Eq. (15)]. In addition, we have \( \frac{d^2 \phi_1(0)}{dy^2} = 0 \) and \( \frac{d^2 \phi_2(0)}{dy^2} = -2H_s \), as in the 3-junction case.

The asymptotics of the vortex-plane solution in the region \( |y| \gg \lambda_{j1} \) are

\[ \phi_1(y) = \frac{1}{2} \left[ \left( \sqrt{5} - 1 \right) C_1 \exp \left[ \frac{y}{\lambda_{j1}} \right] - \left( \sqrt{5} + 1 \right) C_2 \exp \left[ \frac{y}{\lambda_{j2}} \right] \right], \]

\[ \phi_2(y) = \frac{1}{2} \left[ \left( \sqrt{5} + 1 \right) C_1 \exp \left[ \frac{y}{\lambda_{j1}} \right] + \left( \sqrt{5} - 1 \right) C_2 \exp \left[ \frac{y}{\lambda_{j2}} \right] \right], \]
\[ \phi_2(y) = C_1 \exp \left[ \frac{y}{\lambda_{J_1}} \right] + C_2 \exp \left[ \frac{y}{\lambda_{J_2}} \right], \quad y \ll -\lambda_{J_1}; \]
\[ \phi_1(y) = 2\pi - \frac{1}{2} \left( (\sqrt{5} - 1) C_1 \exp \left[ -\frac{y}{\lambda_{J_1}} \right] - (\sqrt{5} + 1) C_2 \exp \left[ -\frac{y}{\lambda_{J_2}} \right] \right), \]
\[ \phi_2(y) = 2\pi - C_1 \exp \left[ -\frac{y}{\lambda_{J_1}} \right] - C_2 \exp \left[ -\frac{y}{\lambda_{J_2}} \right], \quad y \gg \lambda_{J_1}, \]

where \( C_1, C_2 > 0 \). In the region \(|y| \ll \lambda_{J_2}\), we have:
\[ \phi_1(y) = \pi + 2H_{s}y - \frac{(1 + \epsilon^2) H_{s}}{3\epsilon^2} y^3 + \ldots, \]
\[ \phi_2(y) = \pi + 2H_{s}y - \frac{1}{3} H_{s} y^3 + \ldots \]

The total flux carried by the vortex plane is
\[ \Phi = 2\pi \frac{\epsilon^2 (5 + 2\epsilon^2)}{(2 + \epsilon^2) (1 + \epsilon^2) - 1}, \]
and the exact value of the induced field at \( y = 0 \) is
\[ h_1(0) = h_4(0) = \frac{\epsilon^2 (2 + \epsilon^2) H_{s}}{(2 + \epsilon^2) (1 + \epsilon^2) - 1}, \quad h_2(0) = h_3(0) = \frac{\epsilon^2 (3 + \epsilon^2) H_{s}}{(2 + \epsilon^2) (1 + \epsilon^2) - 1}. \]

(See Fig. 2.)

E. The layered-superconductor limit \((N - 1 \gg 2 \left\lfloor \frac{1}{\epsilon} \right\rfloor)\)

The limit \( N - 1 \gg 2 \left\lfloor \frac{1}{\epsilon} \right\rfloor \) stands for the integer part of \( \frac{1}{\epsilon} \) corresponds to the situation when a periodic thin-layer S/I structure can be regarded as a "layered superconductor" rather than merely a \((N - 1)\)-Josephson-junction stack. Indeed, in this limit the superheating (penetration) field, Eq. (23), becomes
\[ H_{s} = 1 + \frac{\sqrt{1 + \epsilon^2 - \frac{\epsilon}{2}}}{\epsilon (N - 1)} \]
and for \((N - 1) \rightarrow \infty\) tends to the limiting value of an infinite layered superconductor \( H_{s \infty} = 1 \). \( \text{(We remind that the lower critical field of an infinite layered superconductor, according to Refs. [13], is } H_{c1 \infty} = \frac{\epsilon}{2}. \text{)} \)

The total flux carried by a vortex plane, by (23), in the considered limit is
\[ \Phi = \pi(N - 1) \left[ 1 - \frac{2\sqrt{1 + \epsilon^2 - \epsilon}}{\epsilon (N - 1)} \right], \]

which for \((N - 1) \rightarrow \infty\) tends to the limiting value of one flux quantum \( \Phi_0 = \pi \) per vortex, as expected for an infinite layered superconductor. \( \text{[13]} \)

Moreover, according to (23), for I-layers whose index \( n \) satisfies the condition \( \left[ \frac{1}{\epsilon} \right] \ll n \ll N - 1 - \left[ \frac{1}{\epsilon} \right] \), we have
\[ |H_n| = |H| (\mu^n + \mu^{N-n}) \ll |H|. \]

In other words, for \(|H| \leq H_{s \infty} = 1\), the inside junctions exhibit the complete Meissner effect in the region \(-R < y < R\)

Thus, the integer \( \left[ \frac{1}{\epsilon} \right] \) determines the number of junctions near the boundaries \( x = 0 \) and \( x = N - 1 \) that "feel" the influence of the superconductor/vacuum interfaces. Far from the boundaries, i.e. in the region \( \left[ \frac{1}{\epsilon} \right] \ll x \ll N - 1 - \left[ \frac{1}{\epsilon} \right] \), one can apply all the results of the theory of infinite layered superconductors. \( \text{[13]} \)
Within the framework of standard methods of the theory of ordinary differential equations, we have obtained a complete mathematical description of Josephson vortices and of the Meissner effect in periodic thin-layer S/I structures. A rigorous examination of the properties of Eqs. (23) allowed us to establish the general existence criterion (13), which formed a solid basis for our subsequent physical and mathematical conclusions. One of the most striking physical consequences is that the derivation of the exact expression for the vortex-penetration field $H_s$, Eq. (13), did not require any explicit solution of (23). Although explicit analytical solutions to Eqs. (24) proved to be possible only in a limited number of cases discussed in section IV, numerical integration of these equations should pose no problem owing to the algorithm worked out in the paper.

All the three types of localized solutions obtained in the paper possess a number of interesting physical and mathematical properties. For example, the Meissner solutions are characterized by several different Josephson lengths $\lambda_{ji}$ ($\frac{N}{2}$ lengths for even $N$, and $\frac{N-1}{2}$ lengths for odd $N$). Unfortunately, this important fact was not noticed in previous publications. (4) We think that our result may prove to be useful in view of the current experimental efforts (10,11) to verify the interlayer tunneling model of high-$T_c$ superconductivity (12,22) by measuring the $c$-axis penetration depth. [The penetration of the parallel magnetic field with a distribution of length scales has been recently observed (23) in the organic layered superconductor $\kappa$-(BEDT-TTF)$_2$Cu(NCS)$_2$.]

Mathematically, Josephson vortices, represented by both the vortex-plane solutions and incoherent vortex solutions, are static sine-Gordon-type solitons. They satisfy the standard (13) asymptotic boundary conditions, Eqs. (17), (18), and Eqs. (23)-(26). The expression for their self-energy (54) is a direct generalization of the well-known expression for a single junction. (3) The thermodynamically stable vortex-plane solutions demonstrate a profound difference between Josephson-vortex formation in weakly-coupled multilayer structures and Abrikosov-vortex formation in continuum type-II superconductors (isotropic or not). The proof of the existence of such solutions in the general case of $(N-1)$-junction stacks establishes relationship to the well-know "coherent mode" (alias the "in-phase" mode) in a double-junction stack (17) and the recently obtained (13) vortex-plane solutions in infinite $(N = \infty)$ layered superconductors. However, in contrast to the latter two cases, the vortex-plane solutions for $4 \leq N < \infty$ are characterized by several $\lambda_{ji}$, as the Meissner solutions. Moreover, in contrast to the case $N = \infty$, the intralayer currents, $J_n$, in the presence of a vortex plane are not equal to zero, with the exception of $J_{N-1}$, in the case of odd $N$.

The single-vortex solutions are not uniquely determined by the asymptotic boundary conditions (28)-(31), as follows from the existence criterion (14). Their spatial dependence is characterized, in general, by $N-1$ length scales. In contrast to the vortex planes, isolated Josephson vortices cannot penetrate the periodic S/I structure at any $|H| \neq 0$ and do not satisfy the original integrodifferential equations (6) minimizing the Gibbs free-energy functional (6). Although the self-energy of incoherent vortex solutions may be lower than the self-energy of a vortex plane, their Gibbs free energy is positive with respect to the Gibbs free energy of the Meissner state for $|H| \leq H_{c1}$ ($H_{c1}$ is the lower critical field for the vortex-plane solution), as is illustrated by our analysis of a double-junction stack in section IV. However, incoherent vortex solutions can be realized in the dynamic regime. (4) In layered superconductors, isolated vortices may also emerge as metastable topological entities owing to pinning by extended defects. We emphasize that our general conclusions about the form of single-vortex configurations stand in full agreement with numerical results of Ref. (4).

Finally, our exact results clearly show that the phase-difference equations (24) do not admit any solutions in the form of Abrikosov-type vortices, suggested in Refs. (14,15). As we have already pointed out, (13) the hypothesis of Abrikosov-type solutions in infinite layered superconductors leads to incorrect estimates of the value of the lower critical field $H_{c1\infty}$. Unfortunately, it seems that some other theoretical predictions, based on the assumption of the existence of Abrikosov-type solutions, should also be revised.

**APPENDIX A: THE SOLUTION OF THE FINITE DIFFERENCE EQUATION FOR $H_n(Y)$**

Equations (20) can be regarded as a nonhomogeneous finite difference equation for $H_n(y)$ with respect to the layer index $n$, subject to boundary conditions (9). According to general theory of such equations, (22) its solution for $\varepsilon < 1$ can be represented in the form

$$H_n(y) = h_n(y) + H_{n},$$

(A1)

where
\[ h_n(y) = \frac{\epsilon^2}{2} \sum_{m=1}^{N-1} G(n, m) \frac{d\phi_m(y)}{dy} \]  

is the particular solution of (20) satisfying the boundary conditions

\[ h_0(y) = h_N(y) = 0, \]  

and

\[ H_n = H \left( \mu^{-n} + \mu^{-N+n} - \mu^n - \mu^{N-n} \right), \]

\[ \mu = 1 + \frac{\epsilon^2}{2} - \epsilon \sqrt{1 + \frac{\epsilon^2}{4}} < 1 \]

is the solution of the homogeneous form of (20) (with the zero right-hand side) meeting the boundary conditions

\[ H_0 = H_N = H. \]

The quantities \( G(n, m) \) in (A2) are matrix elements of \((N+1) \times (N+1)\) matrix Green’s function \( G(0 \leq n, m \leq N) \). They obey the nonhomogeneous finite difference equation

\[ G(n+1, m) - (2 + \epsilon^2) G(n, m) + G(n-1, m) = -\delta_{n,m} \]  

(\( \delta_{n,m} \) is the Kronecker index) with the boundary conditions

\[ G(0, m) = G(N, m) = 0. \]

The explicit form of \( G(n, m) \) is

\[ G(n, m) = \frac{1}{2\epsilon \sqrt{1 + \frac{\epsilon^2}{4}}} \left[ \mu^{n-m} - \mu^n \left( \mu^{m-N} + \mu^{N-n} \right) \right]. \]

The following properties of \( G(n, m) \) can be easily verified using (A7) and (A9):

\[ G(n, m) = G(m, n), \]  

\[ G(n, N - m) = G(N - n, m), \]  

\[ G(n, m) > 0 \text{ for any } 1 \leq n, m \leq N - 1, \]  

\[ \sum_{m=1}^{N-1} G(n, m) = \frac{1}{\epsilon^2} \left[ 1 - G(n, 1) - G(n, N - 1) \right] \]

\[ = \frac{1}{\epsilon^2} \left[ 1 - \mu^{-n} + \mu^{-N+n} - \mu^n - \mu^{N-n} \right], \quad 1 \leq n \leq N - 1, \]

\[ \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) = \frac{1}{\epsilon^2} \left[ N - 1 - \frac{2\sqrt{1 + \frac{\epsilon^2}{4}} - \epsilon - 1 - \mu^{N-1}}{1 + \mu^{N-1}} \right]. \]

Note that matrix Green’s function for an infinite layered superconductor, \( G_\infty(n, m) \), is determined by the matrix elements
\[ G_{\infty}(n, m) = \frac{f^{[n-m]}}{2\epsilon \sqrt{1 + \epsilon^2}}, \quad -\infty < n, m < +\infty, \quad (A15) \]

that satisfy the summation rule

\[ \sum_{m=1}^{N-1} G_{\infty}(n, m) = \frac{1}{\epsilon^2}. \]

Consider now a \((N-1) \times (N-1)\) matrix \(\tilde{G}(n, m)\), whose matrix elements are given by the right-hand side of \((A19)\) with \(1 \leq n, m \leq N-1\). Of special physical importance are positive eigenvalues of \(\tilde{G}(n, m)\): They determine characteristic length scales of localized solutions to \((24)\). Indeed, in the asymptotic region where \(\frac{d\phi_n(y)}{dy} = o(1)\) and \(\phi_n(y) = o(1)\), equations \((27)\) can be linearized with the result

\[ \sum_{m=1}^{N-1} G(n, m) \frac{d^2\phi_m(y)}{dy^2} = \frac{1}{\epsilon^2} \phi_n(y), \quad 1 \leq n \leq N-1. \]

The substitution \(\phi_n(y) \propto \exp\left[\pm \frac{y}{\lambda}\right]\) yields

\[ \sum_{m=1}^{N-1} G(n, m) \phi_m(y) = \frac{\lambda^2}{\epsilon^2} \phi_n(y), \quad 1 \leq n \leq N-1, \]

which is exactly an eigenvalue equation for \(\tilde{G}(n, m)\), with \(\frac{\lambda^2}{\epsilon^2}\) being a positive eigenvalue.

**APPENDIX B: PROOF OF THE LEMMA**

By introducing new functions

\[ \psi_1(y) = \phi_1(y), \psi_2(y) = \phi_2(y), \ldots, \psi_{N-1}(y) = \phi_{N-1}(y), \]

\[ \psi_N(y) = \frac{d\phi_1(y)}{dy}, \psi_{N+1}(y) = \frac{d\phi_2(y)}{dy}, \ldots, \psi_{2N-2}(y) = \frac{d\phi_{N-1}(y)}{dy}, \quad (B1) \]

we convert \((24)\) into an equivalent normal system of \(2N-2\) first-order equations

\[ \frac{d\psi_i(y)}{dy} = F_i(\psi_1, \psi_2, \ldots, \psi_{2N-2}), \quad 1 \leq i \leq 2N-2, \quad (B2) \]

\[ F_i(\psi_1, \psi_2, \ldots, \psi_{2N-2}) \equiv \psi_{i+N-1}, \quad 1 \leq i \leq N-1, \]

\[ F_N(\psi_1, \psi_2, \ldots, \psi_{2N-2}) \equiv \frac{1}{\epsilon^2} \left[ (2 + \epsilon^2) \sin \psi_1 - \sin \psi_2 \right], \]

\[ F_i(\psi_1, \psi_2, \ldots, \psi_{2N-2}) \equiv \frac{1}{\epsilon^2} \left[ (2 + \epsilon^2) \sin \psi_i - \sin \psi_{i-1} - \sin \psi_{i+1} \right], \quad N+1 \leq i \leq N-3, \]

\[ F_{2N-2}(\psi_1, \psi_2, \ldots, \psi_{2N-2}) \equiv \frac{1}{\epsilon^2} \left[ (2 + \epsilon^2) \sin \psi_{N-1} - \sin \psi_{N-2} \right], \]

subject to initial conditions

\[ \psi_i(y_0) = \alpha_i, \quad 1 \leq i \leq N-1, \]
\[ \psi_i(y_0) = \beta_{i-N+1}, \quad N \leq i \leq 2N-2. \] (B3)

To prove the statement of the Lemma, it is sufficient to observe that all \( F_i(\psi_1, \psi_2, \ldots, \psi_{2N-2}) \) are continuous functions of their arguments for \( y \in (-\infty, +\infty) \) and \( \psi_k \in (-\infty, +\infty) \) (1 \( \leq k \leq 2N-2 \)). Moreover, their partial derivatives with respect to \( \psi_k \) satisfy the relation

\[ \left| \frac{\partial F_i(\psi_1, \psi_2, \ldots, \psi_{2N-2})}{\partial \psi_k} \right| \leq \frac{4 + \epsilon^2}{\epsilon^2} \] (B4)

for \( y \in (-\infty, +\infty) \) and \( \psi_k \in (-\infty, +\infty) \) (1 \( \leq i, k \leq 2N-2 \)). Thus, the Lipschitz conditions with respect to \( \psi_k \) are met for \( y \in (-\infty, +\infty) \) and \( \psi_k \in (-\infty, +\infty) \) (1 \( \leq k \leq 2N-2 \)), which immediately guarantees [12] the existence and uniqueness of a solution to (B2), satisfying arbitrary initial conditions (B3), in an arbitrary interval \( I = [L_1, L_2] \) such that \( y_0 \in I \). Continuous dependence of the solution on initial data is a result of continuous dependence of \( F_i(\psi_1, \psi_2, \ldots, \psi_{2N-2}) \) on their arguments and of the condition (B4). Infinite differentiability of the solution automatically follows from infinite differentiability of \( F_i(\psi_1, \psi_2, \ldots, \psi_{2N-2}) \) with respect to their arguments.

**APPENDIX C: THE UPPER AND THE LOWER BOUNDS FOR THE SELF-ENERGY OF SINGLE-VORTEX SOLUTIONS IN A DOUBLE-JUNCTION STACK**

To determine the upper and the lower bounds of the self-energy of single-vortex solutions in a double-junction stack, we must find the extrema of the right-hand side of (54) with \( N = 3 \) under the normalization condition

\[ \sum_{n=1}^{2} \int_{-R}^{R} dy \left( \frac{d\phi_n(y)}{dy} \right)^2 = \text{const} < \infty. \]

This leads to the variational principle

\[ \frac{\delta}{\delta \phi_n(y)} \left[ \epsilon^2 \sum_{k=1}^{2} \sum_{m=1}^{2} \int_{-R}^{R} du G(k, m) \frac{d\phi_k(u)}{du} \frac{d\phi_m(u)}{du} - \lambda^2 \sum_{k=1}^{2} \int_{-R}^{R} du \left( \frac{d\phi_k(u)}{du} \right)^2 \right] = 0, \]

where \( \lambda^2 \) is a Lagrange multiplier. Performing the variation with the use of the boundary conditions \( \frac{d\phi_n(\pm R)}{dy} \to 0 \), we arrive at the eigenvalue problem for the \( 2 \times 2 \) matrix \( \tilde{G}(n, m) \), determined by the matrix elements (87):

\[ \sum_{m=1}^{2} G(n, m) \frac{d^2\phi_m(y)}{dy^2} = \frac{\lambda^2}{\epsilon^2} \frac{d^2\phi_n(y)}{dy^2}. \]

The only two eigenvalues of \( \tilde{G}(n, m) \) are \( \lambda_{J1}^2 \) and \( \lambda_{J2}^2 \), with \( \lambda_{J1} \) and \( \lambda_{J2} \) given by (88). The larger eigenvalue, \( \frac{\lambda_{J1}^2}{\epsilon^2} \), corresponds to the vortex-plane solution with \( \phi_1 = \phi_2 \) and determines the upper bound (92) for the self-energy. The smaller eigenvalue, \( \frac{\lambda_{J2}^2}{\epsilon^2} \), corresponds to the vortex-antivortex solution with \( \phi_1 = -\phi_2 \) and determines the lower bound (98) for the self-energy. Thus, for the self-energy of the single-vortex solutions, \( E_{sv} \), we necessarily have

\[ E_{va} < E_{sv} < E_{v}. \]
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FIGURE CAPTIONS

1. Fig. 1. The geometry of the problem (schematically). Here $N = 12$; $\lambda_{J,\infty} << R < L/2$, $\lambda_{J,\infty}^{-2} = 8\pi\varepsilon j_0 p$; $H > 0$.

2. Fig. 2. The distribution of the self-induced field of a vortex plane $h(x, y)$ at $y = 0$, for $\epsilon = 0.5$: a) a 3-junction stack, $H_s = 1.849$; b) a 4-junction stack, $H_s = 1.662$. 
