The conjunction fallacy and interference effects

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Abstract. In the present article we consider the conjunction fallacy, a well known cognitive heuristic experimentally tested in cognitive science, which occurs for intuitive judgments in situations of bounded rationality. We show that the quantum formalism can be used to describe in a very simple way this fallacy in terms of interference effect. We evidence that the quantum formalism leads quite naturally to violations of Bayes’ rule when considering the probability of the conjunction of two events. Thus we suggest that in cognitive science the formalism of quantum mechanics can be used to describe a quantum regime, the bounded-rationality regime, where the cognitive heuristics are valid.

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1. Introduction

This article addresses two main directions of research: the investigation of how the quantum formalism is compatible with Bayes’ rule of classic probability theory, and the attempt to describe with the quantum formalism systems and situations very different from the microscopic particles. A number of attempts has been done to apply the formalism of quantum mechanics to research fields different from quantum physics, for example in the study of rational ignorance [1] and of semantical analysis [2]. Quantum mechanics, for its counterintuitive predictions, seems to provide a good formalism to describe puzzling effects of contextuality. In the present article, we will try to describe within the quantum formalism an important heuristic of cognitive science, the conjunction fallacy [3]. This heuristic is valid in regime of bounded rationality, which is characterized by cognitive limitations of both knowledge and cognitive capacity. Bounded rationality [5] is a central theme in behavioral economics and it concerns with the ways in which the actual decision-making process influences agents’ decisions. A first attempt to describe this heuristic in terms of quantum formalism has been done in [6], without evidencing the importance of interference effects.

This article is organized in order to be readable both from quantum physicists and from experts of cognitive science. In section 2 we introduce the basic notation of quantum mechanics, and we show in 2.2 that the quantum formalism describing two non-commuting observables leads to violations of Bayes’ rule. In section 3 we describe the answers to a question in bounded-rationality regime in terms of vector state and density matrix of the quantum formalism. Finally, in section 4 we show how the quantum formalism can naturally describe the conjunction fallacy.

The main results of this article are: 1) tests on non-commuting observables lead to violations of Bayes’ rule; 2) the opinion-state of an agent for simple questions with only two possible answers can be represented, in bounded-rationality regime, by a qubit state; 3) the different questions in bounded-rationality regime can be formally written as operators acting on the qubit states; 4) the explicit answer of an agent to a question in regime of bounded rationality can be described as a collapse of the opinion state onto an eigenvector of the corresponding operator; 5) The probability relevant to a question \( A \), when analyzed in terms of the probability relevant to a second question \( B \) (corresponding to a non-commuting operator) evidences the violation of Bayes’ rule. The conjunction fallacy thus results as a consequence of this general fact.

In conclusion, we present a very general and abstract formalism which seems to describe the heuristic of conjunction fallacy. We think that a similar study can be done for other heuristics of cognitive science (this will be presented in new papers). Thus these heuristics could be simple applications of a general theory describing the bounded-rationality regime, which probably will lead to new interesting predictions. This could confirm the hypothesis that the processes of intuitive judgement could involve mechanisms at a quantum level in the brain.
2. Quantum basic formalism

We first introduce the standard bra-ket notation usually used in quantum mechanics, introduced by Dirac [7], and then the density matrix formalism. In particular, we focus our attention on the concept of qubit. In the simplest situation, a quantum state is defined by a ket $|s\rangle$, which is a vector in a complex separable Hilbert space $H$. If the dimension of $H$ is 2, the state describes a qubit, which is the unit of quantum information. Any quantum system prepared identically to $|s\rangle$ is described by the same ket $|s\rangle$.

In quantum mechanics, we call a measurable quantity an observable, mathematically described by an operator, for example $\hat{A}$, with the important requirement that it is hermitian: $\hat{A} = \hat{A}^\dagger$, where $\hat{A}^\dagger$ is the conjugate transpose. In the case of a single qubit, any observable $\hat{A}$ has two real eigenvalues $a_0$ and $a_1$ and two corresponding eigenvectors $|a_0\rangle$ and $|a_1\rangle$. Another property of hermitian operators is that its eigenvectors, if normalized, form an orthonormal basis, that is $\langle a_i | a_j \rangle = \delta(i,j)$, \hspace{1cm} (1)

where $i, j = 0, 1$ and $\delta(i,j)$ is the Kroneker delta, equal to 1 if $i = j$ and null otherwise. Given such a basis in the Hilbert space, we can write them in components as

$$|a_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |a_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$ \hspace{1cm} (2)

representing the quantum analogue to the two possible values 0 and 1 of a classical bit. An important difference is that in the quantum case a state can be in a linear superposition of 0 and 1, that is

$$|s\rangle = s_0|a_0\rangle + s_1|a_1\rangle,$$ \hspace{1cm} (3)

with $s_0$ and $s_1$ complex numbers. We also say that the state $|s\rangle$ is a superposition of the eigenstates $|a_i\rangle$. In the vector representation generated by formula (2), the ket $|s\rangle$ and its dual vector, the bra $\langle s|$, can be written respectively as

$$|s\rangle = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}, \quad \langle s| = \begin{pmatrix} s_0^* & s_1^* \end{pmatrix}.$$ \hspace{1cm} (4)

Another mathematical object, which is important in order to describe probabilities, is the inner product, also called braket. In general, the inner product of two kets $|s\rangle$ and $|s'\rangle$ can be written, in the basis of the observable $\hat{A}$, as $\langle s|s'\rangle = s_0s'_0^* + s_1s'_1^*$, where $s'_i$ are the components of $|s'\rangle$ in the same basis. Thus the inner product of $|s\rangle$ and its dual vector is $\langle s|s\rangle = |s_0|^2 + |s_1|^2$, and it is equal to 1 if the vector is normalized. Finally, $|s_i|^2 = |\langle a_i|s\rangle|^2$ is the probability $P(a_i)$ that the measure on the observable $\hat{A}$ has the outcome $i = 0, 1$. The state $|s\rangle$ is called a pure state, and describes a quantum state for which the preparation is complete: a preparation is complete when all the compatible observables have been defined (we will give further a precise definition of compatible observables).
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In the most general case, a quantum state is described by the density matrix $\hat{\rho}$, which is an hermitian operator acting on $H$. The density matrix $\hat{\rho}$ describes in general a mixed state, that is a state for which the preparation is not completely determined. For example, the state may be in the preparation $|s_1\rangle$ with a probability $P_1$, and in the preparation $|s_2\rangle$ with a probability $P_2$ (the two vectors may be not orthogonal). We also say that the mixed state is a (statistical) mixture of the two states (or of the two preparations):

$$\hat{\rho} = \sum_i P_i |s_i\rangle\langle s_i|.$$  \hspace{1cm} (5)

In the particular case where there is only one $P_i = 1$, we have $\hat{\rho} = |s\rangle\langle s|$, that is a pure state. The opposite situation is when the eigenvalues of the density matrix are all equal. In the single-qubit example, they are both $1/2$, and the resulting operator is the identity matrix acting on the Hilbert space $H$:

$$\hat{\rho} = \frac{1}{2} \hat{I} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (6)

The resulting state is called maximally mixed, and can be considered as the situation where the actual knowledge of the state is null. The elements of the single-qubit density matrix $\hat{\rho}$ can be expressed in the basis of $\hat{A}$, with $i, j = 0, 1$, as

$$\rho_{i,j} = \langle a_i|\hat{\rho}|a_j\rangle.$$  \hspace{1cm} (7)

where the diagonal elements $\rho_{i,i}$ represent the probability $P(a_i)$ to measure a certain value $a_i$ of the relevant observable. An equivalent expression of these probabilities can be written in terms of the trace-matrix operation:

$$P(a_i) = Tr(\hat{\rho}|a_i\rangle\langle a_i|) = \rho_{i,i}$$  \hspace{1cm} (8)

This formula is the most general expression of the probability to measure a value $a_i$ of an observable, when operating on quantum systems identically prepared in the state $\hat{\rho}$. The formalism of the density matrix helps us to write in the most general form the mean value of an observable $\hat{A}$ as

$$\langle \hat{A} \rangle = Tr(\hat{\rho}\hat{A}) ,$$  \hspace{1cm} (9)

which becomes, by using formula (7) and the basis vectors of $\hat{A}$, $\langle \hat{A} \rangle = \sum_i a_i P(a_i)$.

2.1. Collapse of the state vector

One of the axioms of quantum mechanics states that, given an initial mixed state $\hat{\rho}$ and an observable $\hat{A}$ acting on a discrete Hilbert space, we can define from the eigenvectors $\{|a_i\rangle\}$ of $\hat{A}$ the projection operators $\{|a_i\rangle\langle a_i|\}$, where for a single qubit $i = 0, 1$. Thus if the measure of the observable $\hat{A}$ is the eigenvalue $a_i$, the state updates as

$$\hat{\rho} \rightarrow |a_i\rangle\langle a_i|$$  \hspace{1cm} (10)
This general formula is valid for an orthonormal basis \(\{a_i\}\), and defines the collapse of the initial state onto the state vector \(|a_i\rangle\). At first the collapse of the state may seem quite obvious. For example, given the initial probability distribution \(P(a_j)\) given by equation (8) corresponding to the initial state \(\hat{\rho}\), we have from simple calculations that

\[
P(a_j) \rightarrow P(a_j) = \delta(a_i, a_j).
\]

This means that, after measuring a certain value \(a_i\) of \(\hat{A}\), the probability that the observable actually has another value \(a_j \neq a_i\) is null. This fact is valid also in classic probability theory. Nevertheless, we will show in the next subsection that the collapse leads to violation of classic laws of probability theory when considering more than one observable.

Finally we note that, in the study of rational ignorance presented in [1], the collapse admits a very simple interpretation. Given an initial opinion state, described by \(\hat{\rho}\) and a question \(\hat{A}\), after a subject has given an answer \(a_i\), the probability that the repetition of the same question \(\hat{A}\) in the same conditions gives a different answer is null.

2.2. Non-commuting operators and Bayes’ rule

In quantum mechanics the operators associated to the observables may not commute: for example, given two operators \(\hat{A}\) and \(\hat{B}\), acting on the same Hilbert space \(H\), the ordered product \(\hat{A}\hat{B}\) can be different from \(\hat{B}\hat{A}\): in this case the two operators do not commute and we write \([\hat{A}, \hat{B}] \neq 0\), where \([\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}\) is the commutator of the two observables. The consequences of this fact are very important, and lead to violation of Bayes’ rule. Let us consider for simplicity a single-quit system and the eigenvectors \(\{a_i\}\) and \(\{b_i\}\) of \(\hat{A}\) and \(\hat{B}\) respectively (with \(i = 0, 1\)). The probability of measuring the value \(a_i\) or \(b_i\) for the observables \(\hat{A}\) or \(\hat{B}\) respectively is given by equation (8), that is:

\[
P(a_i) = \text{Tr}(\hat{\rho}|a_i\rangle\langle a_i|); \quad P(b_i) = \text{Tr}(\hat{\rho}|b_i\rangle\langle b_i|).
\]

We now consider the conditional probability \(P(b_j|a_i)\), defined as the probability to measure the observable \(\hat{B}\) with value \(b_j\), given the occurrence of a measurement of \(\hat{A}\) with value \(a_i\). In quantum mechanics, the occurrence of a measurement of \(\hat{A}\) with result \(a_i\) means that the actual state is \(|a_i\rangle\), independently from the initial state before the measurement. This is a consequence of the quantum collapse, and leads to many differences form the classic case. In quantum mechanics thus we have that

\[
P(b_j|a_i) = |\langle b_j|a_i\rangle|^2 = P(a_i|b_j).
\]

Let us now consider the Bayes’ rule, which defines the joint probability to measure contemporarily the values \(a_i\) and \(b_j\) for observables \(A\) and \(B\) respectively:

\[
P(a_i)P(b_j|a_i) = P(b_j)P(a_i|b_j) = P(a_i, b_j).
\]

This equation is very important in classical probability theory, since it links the joint probabilities relevant to \(\hat{A}\) and \(\hat{B}\) to the conditional probabilities. In quantum mechanics one can not measure contemporarily two commuting operators. From a formal point of
view, this impossibility is evidenced from the fact that, by using the equations (12) and (13), we have in general that

\[ P(a_i)P(b_j|a_i) \neq P(b_j)P(a_i|b_j) \]

(15)

This means that the joint probability \( P(a_i, b_j) \) can not be univocally defined. What we can rigorously define is

\[ P(a_i \rightarrow b_j) = P(a_i)P(b_j|a_i), \]

(16)

where \( P(a_i \rightarrow b_j) \) is the probability to measure \( a_i \) for the observable \( \hat{A} \) and then the answer \( b_j \) for the observable \( \hat{B} \). Form the previous observation, we have that \( P(a_i \rightarrow b_j) \neq P(b_j \rightarrow a_i) \). We call \( P(a_i \rightarrow b_j) \) the consecutive probability to measure \( a_i \) and then \( b_j \). Equation (15) evidences that in quantum mechanics the Bayes’ rule is violated. Many of the paradoxical results of quantum mechanics are due to this violation. In the present article, we will focus our attention on the conjunction fallacy, which we will study in the next sections.

3. Bounded rationality and Hilbert spaces

The bounded rationality [5] is a property of an agent (a person which makes decisions) that behaves in a manner that is nearly optimal with respect to its goals and resources. In general, an agent acts in bounded-rationality regime when there is a limited time in which to make decisions, or when he is also limited by schemas and other decisional limitations. As a result, the decisions are not fully thought through and they are rational only within limits such as time and cognitive capability. There are two major causes of bounded rationality, the limitations of the human mind, and the structure within which the mind operates. This impacts decision models that assume us to be fully rational: for example when calculating expected utility, it happens that people do not make the best choices. Since the effects of bounded rationality are counterintuitive and may violate the classical probability theory (and the Bayes’ rule), we will often speak of bounded-rationality regime as the set of situations where the bounded rationality is an actual property.

We will show that some typical behaviors of the bounded-rationality regime can be described in a very effective way by the quantum formalism. We will study from a statistical point of view the opinion state of agents having the same initial information. In particular, we will assume that the opinion state of an agent can be represented as a qubit state, that is in terms of a density matrix \( \hat{\rho} \) or, in simple cases, of a ket \( |s\rangle \) in a Hilbert space \( H \) of dimension 2. As in quantum mechanics experiments, it is important to define carefully the preparation of the opinion state. Every previous information given to an agent before performing a test can be considered as the preparation of the opinion state. When we repeat a test on more agents, it is important that their opinion state is (at least in theory) identically prepared. We note here that it is not easy to prepare the opinion state of a number of people in an identical state. Nevertheless, the quantum formalism can help us with the concept of mixed state.
The basic test in the context of bounded rationality is a question. We consider a question $A$ for which the possible answers can only be 0 or 1 (false or true), and we associate it to an operator $\hat{A}$ acting on the Hilbert space $H$. Like in quantum mechanics, the question $A$ is an observable, in the sense that we can observe an answer: thus, when speaking of questions, we will consider directly the associated operator $\hat{A}$.

The answers 0 and 1 are associated to the eigenvalues $a_0 = 0$ and $a_1 = 1$ of $\hat{A}$, while the eigenvectors $|a_0\rangle$ and $|a_1\rangle$ correspond to the opinion states relevant to the answers 0 and 1 respectively:

$$\hat{A}|a_0\rangle = 0|a_0\rangle = 0; \quad \hat{A}|a_1\rangle = |a_1\rangle.$$  \hspace{1cm} (17)

The eigenvectors $|a_0\rangle$ and $|a_1\rangle$ have a very precise meaning: if the opinion state of an agent can be described for example by $|a_1\rangle$, this means that the answer to the question $\hat{A}$ is 1 with certainty. If we repeat the same question to many agents in the same opinion state (thus identically prepared), each agent will give the same answer 1. If instead the opinion state about the question is definitely 0, then we have the eigenvector $|a_0\rangle$.

Any observable can be written in the basis of its eigenvectors as $\hat{A} = \sum_i a_i|a_i\rangle\langle a_i|$. A superposition of the opinion states $|a_i\rangle$ about question $\hat{A}$ is, like in equation (3), $|s\rangle = s_0|a_0\rangle + s_1|a_1\rangle$, where $|s\rangle^2 = |\langle a_i|s\rangle|^2$ is the probability $P(a_i)$ that the agent gives an answer $i = 0, 1$ to the question $\hat{A}$. In the most general case, the opinion state can be represented as a density matrix $\hat{\rho}$. The probability that the answer to the question $\hat{A}$ is $a_i$ is given by equation (8): $P(a_i) = Tr(\rho|a_i\rangle\langle a_i|)$.

We note that the formalism introduced is the same used to describe questions in regime of rational ignorance [1], where people choose to remain uninformed about a question $\hat{A}$. In fact, the bounded rationality can be considered as a more general than the rational ignorance, where the question is preceded by some additional information.

4. The conjunction fallacy

The conjunction fallacy is a well known cognitive heuristic which occurs in bounded rationality when some specific conditions are assumed to be more probable than the general ones. More precisely, many people tend to ascribe higher probabilities to the conjunction of two events than to one of the single events. The most often-cited example of this fallacy originated with Amos Tversky and Daniel Kahneman [3] is the case of Linda, which we will consider carefully in this article. The conjunction fallacy has been later studied in a detailed way [4], in order to show that the fallacy does not depend by other factors: for example, the interpretation of expressions like probability and and.

In general, we consider two dichotomic questions $A$ and $B$, with possible answers $a_0, a_1$ and $b_0, b_1$. The typical experiments of [3] and [4] consist in a preparation of the opinion state, which provides some information to the agent, and the following question: what is more probable or frequent between $a_1$ (or $b_1$) and $a_1$-and-$b_1$. The agents manifest in all these experiments a strict preference for the answer $a_1$-and-$b_1$: this evidences the
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correlation fallacy, since the Bayes’ rule (13) entails that
\[ P(b_1) = P(a_0)P(b_1|a_0) + P(a_1)P(b_1|a_1) \geq P(a_1)P(b_1|a_1). \] (18)
In other words, the conjunction of two events \( a_1 \) and \( b_1 \) is always less probable than
of one of two events. Nonetheless, the agents often consider more likely \( a_1 \)-and-\( b_1 \) than
\( a_1 \) (or \( b_1 \)). We stress that the experimental results of [3] and [4] should be considered
carefully: they give a direct information of how many agents consider the difference
\( P(a_1, b_1) - P(a_1) \) positive, not of the probabilities relevant to \( a_1 \)-and-\( b_1 \) and \( a_1 \) (or \( b_1 \)).
The probability \( P(a_1, b_1) \) can be obtained with a different test, where it is asked to
the agents if they consider \( a_1 \)-and-\( b_1 \) true or false, with the information provided in the
preparation of the test.

We now introduce the quantum formalism in order to show that the conjunction
fallacy can be described and interpreted in such a formalism. In particular, we consider
two operators \( \hat{A} \) and \( \hat{B} \), associated respectively to the questions \( A \) and \( B \). Since \( A \) and
\( B \) are dichotomic questions, we can describe the opinion state of agents as vectors in a
two-dimensional Hilbert space. Both the eigenvectors of \( \hat{A} \) and \( \hat{B} \), defined by equation
(17), form two orthonormal bases of the Hilbert space \( H \). Thus we can express the
eigenvectors of \( \hat{A} \) in the basis of the eigenvectors of \( \hat{B} \), obtaining the general equations:
\[ |a_0\rangle = \cos(\theta)|b_0\rangle + \sin(\theta)e^{i\phi}|b_1\rangle \] (19)
\[ |a_1\rangle = -\sin(\theta)e^{-i\phi}|b_0\rangle + \cos(\theta)|b_1\rangle, \]
and vice-versa
\[ |b_0\rangle = \cos(\theta)|a_0\rangle - \sin(\theta)e^{i\phi}|a_1\rangle \] (20)
\[ |b_1\rangle = \sin(\theta)e^{-i\phi}|a_0\rangle + \cos(\theta)|a_1\rangle \]
The transformations above are a change of basis, which can be described in terms of a
unitary operator \( \hat{U} \) (element of \( SU(2) \) group) such that \( \sum_i U_{ij}|a_i\rangle = |b_j\rangle \). Moreover,
they are useful to compute the conditional probabilities \( P(a_1|b_1) = P(a_0|b_0) = \cos^2(\theta) \)
and \( P(a_1|b_0) = P(a_0|b_1) = \sin^2(\theta) \). It is important to note that, as evidenced in
[6], in quantum mechanics we can not consider simultaneously the two events \( a_1 \) and
\( b_1 \); what we can consider are the conditional probabilities \( P(a_1|b_1) \), remembering that
\( P(a_1|b_1)P(a_1) \) could be different from \( P(b_1|a_1)P(b_1) \). In other words, the elements of
the two basis of \( A \) and \( B \) should be handled carefully.

First of all, we consider a mixed state, that is an incoherent mixture of states \( |a_i\rangle \)
with probabilities \( |\alpha_i|^2 \):
\[ \hat{\rho} = |\alpha_0|^2|a_0\rangle\langle a_0| + |\alpha_1|^2|a_1\rangle\langle a_1|. \] (21)
We show that for this state the conjunction fallacy is not allowed: if we compute the
probability \( P(b_1) = \langle b_1|\hat{\rho}|b_1\rangle \), one obtains by using equation (19) the classical formula
\[ P(b_1) = P(a_0)P(b_1|a_0) + P(a_1)P(b_1|a_1). \] (22)
This equation is consistent with formula (18), and evidences that \( P(b_1) \) can not be lower
than \( P(a_1)P(b_1|a_1) \). Mixed states thus exhibit a behavior similar to the classic situation,
without conjunction fallacy.
Let us now consider as the initial state the following superposition

$$|s\rangle = \alpha_0|a_0\rangle + \alpha_1|a_1\rangle$$

(23)

where $\alpha_i$ are in general complex parameters such that $P(a_i) = |\alpha_i|^2$, reproducing the same statistical predictions for $\hat{A}$ of (24). By using equation (19), we can express this state in the basis of $\hat{B}$, obtaining

$$|s\rangle = [\alpha_0\cos(\theta) - \alpha_1\sin(\theta)e^{-i\phi}]|b_0\rangle + [\alpha_0\sin(\theta)e^{i\phi} + \alpha_1\cos(\theta)]|b_1\rangle.$$  

(24)

We now consider the probabilities $P(a_1)$, $P(b_1)$ and the conditional probability $P(a_1|b_1)$: from equation (24), we have that $P(b_1) = |\alpha_0\sin(\theta)e^{i\phi} + \alpha_1\cos(\theta)|^2$, obtaining

$$P(b_1) = P(a_0)P(b_1|a_0) + P(a_1)P(b_1|a_1) + Re[\alpha_0\alpha_1^*\sin(2\theta)e^{i\phi}]$$

(25)

The presence of the last term, known as the interference term $I(s, A)$ can produce conjunction fallacy effects: in fact, if we impose that $P(a_0)P(b_1|a_0) + I(s, A) < 0$, we have $P(b_1) < P(a_1)P(b_1|a_1)$. Thus the sign of the interference term can determine the conjunction fallacy, while the parameter $\phi$ can give to this effect more or less strength. A positive interference term enhances the prevalence of $P(b_1)$ on $P(a_1, b_1)$, which can be considered a reverse conjunction fallacy. The conjunction fallacy can appear also for $P(a_1)$, if we write the same initial state in the basis of $\hat{B}$

$$|s\rangle = \beta_0|b_0\rangle + \beta_1|b_1\rangle$$

(26)

The probability $P(a_1)$ can be written, with similar calculations, as

$$P(a_1) = P(b_0)P(a_1|b_0) + P(b_1)P(a_1|b_1) - Re[\beta_0\beta_1^*\sin(2\theta)e^{i\phi}]$$

(27)

evidencing once again an interference term $I(s, B)$. If we want the presence of conjunction fallacy $P(a_1) < P(b_1)P(a_1|b_1)$, we impose $P(b_0)P(a_1|b_0) + I(s, B) < 0$.

We consider now the results presented in [10], where several probability combination models for conjunction errors are presented: we want to show that the use of quantum formalism allows us to explain the experimental data (and in particular the results of table III of [11]) in a more complete way. We consider for simplicity real superposition coefficients $\alpha_i, \beta_j$, and $\phi = 0$: interference effects can occur also without complex numbers. Moreover, since $|\alpha_0|^2 + |\alpha_1|^2 = 1$, we can write $\alpha_0 = acos(\theta_0), \alpha_1 = asin(\theta_0)$, with a positive number. Thus the basis transformation (19) leads to the simple relations $\beta_0 = acos(\theta_a - \theta)$ and $\beta_1 = asin(\theta_a - \theta)$. It is evident that the angle $\theta$ controls the correlations between the questions $A$ and $B$: for $\theta \simeq 0$ the answers $a_1$ and $b_1$ are strictly correlated, for $\theta \simeq \pm \pi/4$ they are uncorrelated, while for $\theta \simeq \pm \pi/2$ they are anti-correlated. Similarly, $\theta_a$ controls the probabilities $P(a_i)$: if $\theta_a \simeq 0$, then $P(a_1) \simeq 1$, and if $\theta_a \simeq \pm \pi/2$, then $P(a_1) \simeq 0$. The presence of conjunction fallacy for $P(b_1)$ and $P(a_1)$ entails, with such assumptions, respectively

$$1 + 2\tan(\theta_a)\cotan(\theta) < 0$$

(28)

$$1 - 2\tan(\theta_a - \theta)\cotan(\theta) < 0$$

where $\cotan(x) = tan(x)^{-1}$. These two formulas can take into account from a qualitative point of view the experimental results of [11]: for correlated questions ($\theta \simeq 0$) the
two inequalities are simultaneously satisfied for a range of \( \theta_a \) such that \( \theta_a \approx \pi/2 \) and \( \theta_a - \theta > \pi/2 \), which means a configuration of probability \( P(a_1)/P(b_1) \) high/high. This configuration also allows a range of \( \theta_a \) zero conjunction errors, for \( \theta \) positive. For anti-correlated questions (\( \theta \approx \pm \pi/2 \)) the two inequalities can not simultaneously satisfied; only one inequality can be satisfied when \( |\theta_a| \approx \pm \pi/2 \), which means a configuration of probability high/low or low/high. Finally, for uncorrelated questions (\( \theta \approx \pm \pi/4 \)) the two inequalities are simultaneously satisfied when \( \theta_a \approx \pm \pi/2 \), which means a configuration of probability high/high.

In [10] other possible models to explain these data are presented; for example, the probability combination models, where level of ratings of probability \( R(A), R(B) \) are connected to the belief strength \( S(A), S(B) \) through a function \( M \) nonlinear. A modified version of this model introduces the additional term \( s_0 \), which can be interpreted as the initial impression. In [11] has been purposed a signed summation of belief strength, reproducing some of the the conjunction effects, but allowing for presence of self contradictory conjunction. Moreover, in [10] some arguments against the representativeness interpretation of conjunction fallacy are presented: the unrelated case, in fact, seems to be unexplained by representativeness arguments.

Finally, we note that the experiments evidencing the conjunction fallacy show the rates of agents which have considered the conjunction of the two events more probable than the single events: in other words, the experiments show how many agents have considered \( P(a_1,b_1) \) higher than \( P(b_1) \) for example. Equation (25) entail that if all the agents are in the same state (23), then the rate of agents which exhibit conjunction fallacy is 100%. To solve this difference from experimental data, we note that we have used the hypothesis that all the agents are in the same state (23). However, the quantum formalism allows us to prepare the opinion state in a more general way: for example, we can prepare agents in the state (23) with a probability \( P_1 \), and in a state (21) which does not exhibit conjunction fallacy with a probability \( 1 - P_1 \), thus reproducing the experimental predictions. At the moment, the experimental data are not enough to determine completely the initial state of the system: in fact, we should measure not only the frequency of agents for which \( P(a_1) \) is lower than \( P(a_1,b_1) \), but also \( P(a_1), P(b_1) \) and the conditional probabilities \( P(a_i,b_j) \).

4.1. Other quantum approaches

We note that a recent paper [12] contains a different attempt to describe a similar fallacy in terms of quantum formalism. In particular, the experimental results of Hampton [13] are considered: given two concepts \( A \) and \( B \) and an item \( X \), the membership weights relevant to \( A \) and \( B \) (\( \mu(A) \) and \( \mu(B) \)) are compared with \( \mu(A or B) \). The experiment of [13] evidences that in many cases \( \mu(A or B) < \mu(A) \) and \( \mu(A or B) < \mu(B) \). This effect is called underextension of the two concepts \( A \) and \( B \). The attempt of [12] is to deduce this effect from the description of concept membership in terms of quantum formalism. In particular, the situation of complete membership respect to the concept
The rich mathematical formalism of quantum mechanics and the interference effects allow us to predict or explain other fallacies in the bounded-rationality regime. For example, 1) the ordering effects, 2) the disjunction effect, 3) the conditional probability fallacy, 4) the framing effect and 5) the uncertainty effect. We give in this final section a brief description of these effects, which will be described in other articles.

1) In the bounded-rationality regime, the order with which we consider two questions $A$ and $B$ is important (see also the case of rational ignorance [11]). Similarly to the repeated Stern-Gerlach experiment, we ask a question relevant to the operator $\hat{A}$, then a second question relevant to the non-commuting operator $\hat{B}$ and finally again $\hat{A}$. Equation (16) defines the probability that the result of the second question is $b_j$, given $a_i$. But the third measure leads to a non-null probability $P(a_k \rightarrow b_j)$ for $k \neq i$: in other words the third question can give a result for the observable $\hat{A}$ different from the first. The bounded rationality situation has manifested an irrational behavior of the agent.
The question \( \hat{A} \) has been asked two times, but what has been changed is the context. The opinion state of the agent in the test has evidenced a contextuality effect. Thus it has great importance, in bounded rationality, also the temporal order of the different questions.

2) The disjunction effect is an intriguing phenomenon discovered by Tversky and Shafrir [9] with important consequences in modelling the interactions between inference and decision. This effect, like the conjunction fallacy, considers the probabilities relevant to two events which can be associated to two non-commuting operators. A first attempt to give an explanation of the effect within the quantum formalism has been given by [14], by considering the two questions relevant to two different Hilbert spaces. The consequences of this approach are that we obtain an entangled state, but also that the evolution of the initial state can lead to a state which contradicts the initial information given to the agent. A new explanation of the disjunction effect will be given in a separate paper: here we only observe that the conjunction fallacy can be applied to show how the perceived probability \( P(b_1) \) (for example) can be lower than \( P(a_0)P(b_1|a_0) + P(a_1)P(b_1|a_1) \), because the interference terms like in equation (25) may appear.

3) The framing effects may have a similar explanation of the conjunction and disjunction effect: the interference terms in fact are able to lower the probability \( P(b_j) \) when the initial information lead to an opportune pure state in a basis relevant to a non-commuting operator.

4) The conditional probability fallacy is the assumption that \( P(a_j|b_i) = P(b_i|a_j) \). In classical probability theory this is not valid in general, while in quantum mechanics it is always true, as can be seen from equation (13).

5) Finally we make the hypothesis of the existence of an uncertainty effect, which is a consequence of the well-known uncertainty principle in quantum mechanics. The experimental data of [4] show that a not-null percentage of agents in the tests fails in the implication \( X - and - Y \models X, Y \). We argue that this percentage may change, depending on the commutator of the associated operators \( \hat{X}, \hat{Y} \). The uncertainty principle in Hilbert spaces with discrete dimension has been formulated in a more general form [15]: given the uncertainty of an observable \( \hat{X} \), defined as the statistical variance of the randomly fluctuating measurement outcomes \( \delta^2(\hat{X}) = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 \), the local uncertainty relations [15] state that, for any set \( \{\hat{X}, \hat{Y}, \hat{Z}, \ldots \} \) of non-commuting operators, there exist a non-trivial limit \( U \) such that \( \delta^2(\hat{X}) + \delta^2(\hat{Y}) + \delta^2(\hat{Z}) + \ldots \geq U \). This new form of the uncertainty principle may apply to bounded-rationality regime, leading to predictions similar to those cited of [4]: even if we try to prepare the opinion state of agents such that the uncertainty of \( X \) and \( Y \) is null, the sum \( \delta^2(\hat{X}) + \delta^2(\hat{Y}) \) can not be null.

All these paradoxical effects in general are due to the usual belief that we can assign pre-defined elements of reality to individual observables also in regime of bounded rationality. In a classical situation, if we ask to an agent the two questions associated to the observables \( \hat{A}, \hat{B} \), we can consider simultaneously the two answers \( (a_i, b_j) \), and we
can study the joint probability $P(a_i, b_j)$. In a bounded rationality regime, instead, this is not possible if the related observables are non-commuting. We say that the answers to these questions can not be known contemporarily, thus giving an important limit to the complete knowledge of the opinion state of an agent in bounded-rationality regime.

This effect in microscopic world is called quantum contextuality [8], and evidences, for any measurement, the influence of other non-commuting observables previously considered.

6. Conclusions

This article, addressed both to quantum physicists and to experts of cognitive science, evidences the incompatibility of quantum formalism with Bayes’ rule of classic probability theory, by deriving the violation of equation (18) in bounded-rationality regime. In particular, we use mathematical objects like vector state and density matrix to describe the opinion state of agents, and hermitian operators for the questions: in section 4 we show that the conjunction fallacy can be explained as an interference effect when two different questions (relevant to two non-commuting operators) are considered.

This seems to confirm the comment of [4], for which the conjunction fallacy seems to involve failure to coordinate the logical structure of events with first impressions about chance. The first impressions about chance may be encoded in the quantum phase, which leads to interference effects. In fact, we have seen that states (23) and (21) do not differ for the statistical predictions of $A$, but for the presence of a phase, which gives us the information of how the same superposition of states is considered initially by the agent.

Thus we conclude that the conjunction fallacy can be considered as a natural consequence of the quantum formalism used to describe the bounded-rationality regime. By the way, the formalism introduced does not only give an explanation of the fallacy, but also has a predictive character: in fact, we have predicted a reverse conjunction fallacy, for which the probability of the conjunction of two events is much less than the probability assigned to a single event. Moreover, in section 5 we propose other effects which are consequences of the formalism introduced.

As evidenced in the conjunction fallacy, the experimental test evidencing these effects are very difficult to be performed, because they must avoid collateral effects and the preparation of the opinion states must be considered carefully.

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