Bosonization, Singularity Analysis, Nonlocal Symmetry Reductions and Exact Solutions of Supersymmetric KdV Equation

Xiao Nan Gao\textsuperscript{1}, S. Y. Lou\textsuperscript{2,3} and Xiao Yan Tang\textsuperscript{1}

\textsuperscript{1}Department of Physics, Shanghai Jiao Tong University, Shanghai, 200240, China
\textsuperscript{2}Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China
\textsuperscript{3}Faculty of Science, Ningbo University, Ningbo, 315211, China

(Dated: May 11, 2014)

Assuming that there exist at least two fermionic parameters, the classical $N = 1$ supersymmetric Korteweg-de Vries (SKdV) system can be transformed to some coupled bosonic systems. The boson fields in the bosonized SKdV (BSKdV) systems are defined on even Grassmann algebra. Due to the intrusion of other Grassmann parameters, the BSKdV systems are different from the usual non-supersymmetric integrable systems, and many more abundant solution structures can be unearthed. With the help of the singularity analysis, the Painlevé property of the BSKdV system is proved and a Bäcklund transformation (BT) is found. The BT related nonlocal symmetry, we call it as residual symmetry, is used to find symmetry reduction solutions of the BSKdV system. Hinted from the symmetry reduction solutions, a more generalized but much simpler method is established to find exact solutions of the BSKdV and then the SKdV systems, which actually can be applied to any fermionic systems.

PACS numbers: 11.30.Pb, 02.30.Ik, 11.10.Lm, 12.60.Jv, 04.65.+e, 02.30.Jr

\section{I. INTRODUCTION}

In quantum field theory, the bosonization approach is one of the powerful methods which simplifies the procedure to treat complex fermionic fields \cite{1}. It is difficult to find a proper bosonization procedure for both quantum and classical supersymmetric integrable models though the supersymmetric quantum mechanical problems can be successfully bosonized \cite{2}. To treat the integrable systems with fermions such as the super integrable systems \cite{3}, supersymmetric integrable systems \cite{4} and pure integrable fermionic systems \cite{5} is much more complicated than to study the integrable pure bosonic systems. Therefore, it is significant if one can establish a proper bosonization procedure to treat the supersymmetric systems even if in the classical level.

In our previous Letter \cite{6}, a bosonization approach with $N$ fermionic parameters to deal with SKdV system is developed such that the SKdV system can be solved by the usual KdV equation together with several linear differential equations without fermionic variables. Especially, some types of exact supersymmetric extensions of any solutions of the usual KdV equation can be obtained straightforwardly through the exact solutions of the KdV equation and the related symmetries.

The results of \cite{6} show us that for the SKdV equation there exist various kinds of localized excitations. In other words, in addition to the single supersymmetric traveling wave soliton solution (in the super space-time $\{x, \xi, \theta\}$) known in literature \cite{7}, there are infinitely many single traveling soliton extensions in the usual space-time $\{x, t\}$. The bosonization procedure and abundant properties of the soliton excitations of the classical SKdV system reveals some open problems in both classical and quantum theories. For instance, the fermionic fields take value on an infinite Grassmann algebra, that is on an algebra with infinite generators, however, in order to find some exact explicit solutions, we write down the solutions realized only on some finite dimensional Grassmann algebra \cite{6}. Hence, one of the important problems is how to obtain an extension to the case of infinite generators in the algebra. To modify this problem, in this paper, we extend the bosonization procedure of \cite{6} to a slightly generalized form by defining the bosonic fields on an infinite even Grassmann algebra $G_e$

\begin{equation}
G_e = \left\{ 1, \prod_{i=1}^{2n} \xi_i, \ n = 1, 2, \cdots, \infty \right\},
\end{equation}

where $\xi_i, \ i = 1, 2, \cdots, \infty$ are usual Grassmann parameters with the anti-commutation property $\xi_i \xi_j = -\xi_j \xi_i$. Applying the new method, many more abundant nonlinear excitations of the SKdV system can be discovered.

Although the bosonic fields are still defined on an infinite even Grassmann algebra $G_e$, one essential advantage of the method is that it can effectively avoid difficulties caused by intractable fermionic fields which are anticommuting. The $N = 1$ supersymmetric versions of the Korteweg-de Vries equation have been found for more than 20 years \cite{8-10}, which are the beginning of the field of supersymmetric integrable systems. The far-reaching significance lies in not only mathematics, but also the applications in various areas of modern theoretical physics especially in quantum field theory and cosmology such as superstring.
theory where it appears as a basic part of the string worldsheet physics or the theory of two-dimensional solvable lattice models, e.g., tricritical Ising models\cite{11,12}. Therefore, investigating their properties and searching for their exact solutions are of great importance and interest.

For the integrable SKdV system in the sense of possessing a Lax pair, many remarkable properties have been discovered, such as the Painlevé property\cite{13}, the bi-Hamiltonian structures\cite{14,15}, the Darboux transformation\cite{16}, the bilinear forms\cite{17,18}, the Bäcklund transformation (BT)\cite{19}, the Lax representation\cite{20} and the nonlocal conservation laws\cite{21}. Some types of multisoliton solutions are also known for the integrable SKdV system\cite{17–22}. However, because anticommutative fermionic fields bring some difficulties in dealing with supersymmetric equations, to get exact solutions of the supersymmetric systems is, especially, much more difficult than pure bosonic systems.

In Sec. II of this paper, we simply review the results of\cite{6} and then slightly extend it by defining the bosonic fields from the usual c-number space to the infinite Grassmann subalgebra, $G_e$. The remaining sections are devoted to the study of a special BSKdV system (BSKdV-2) which is a special realization of the SKdV on a Grassmann-2 algebra $G_2 \equiv \{1, \zeta_1, \zeta_2, \zeta_1\zeta_2\}$ in which the boson fields are still defined on the infinite even Grassmann algebra $G_e$. The similar results can be obtained for other special realizations of the SKdV (BSKdV-n) on a Grassmann-n algebra. In Sec. III, the Painlevé property and the Bäcklund transformation (BT) are studied by the standard singularity analysis. In Sec. IV, the infinitely many nonlocal symmetries are obtained starting from the BT related nonlocal symmetry, the residual symmetry (RS). The symmetry reductions related to both the local Lie point symmetries and the nonlocal RS are studied in Sec. V. According to the results of Sec. V, a more general but simpler method, the generalized tanh function expansion method, is proposed in Sec. VI to find exact solutions of the BSKdV-2 system. Some special explicit novel exact solutions of the BSKdV-2 and the SKdV are also investigated in Sec. VI. The last section includes a summary and some discussions.

II. REVIEW AND EXTENSION OF THE BOSONIZATION OF THE SKDV EQUATION

The $N = 1$ supersymmetric version of the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0, \quad (2)$$

is established by extending the classical spacetime $(x, t)$ to a super-spacetime $(\theta, x, t)$, where $\theta$ is a Grassmann variable, and the field $u$ to a fermionic superfield

$$\Phi(\theta, x, t) = \xi(x, t) + \theta u(x, t), \quad (3)$$

which leads to a nontrivial result\cite{3}

$$\Phi_t + 3(D\Phi_x)\Phi + 3(D\Phi)\Phi_x + \Phi_{xxx} = 0, \quad (4)$$

where $D = \partial_\theta + \theta \partial_x$ is the covariant derivative. The component version of Eq. (4) reads

$$u_t + u_{xxx} - 3\xi\xi_{xx} + 6uu_x = 0, \quad (5a)$$

$$\xi_t + \xi_{xxx} + 3u\xi + 3u\xi_x = 0, \quad (5b)$$

where $u$ and $\xi$ are bosonic and fermionic component fields, respectively. Vanishing $\xi$ in Eq. (5), only the usual classical KdV equation remains.

Previous studies of the SKdV system were directly based on Eq. (4) or (5). In this paper, similar to\cite{4}, we are only concentrated on bosonization of the SKdV equations by expanding the superfields with respect to the multi-fermionic parameters.

Firstly, we assume that for the solutions of the component fields $\xi$ and $u$ there exist at least two fermionic (Grassmann) parameters, say, $\zeta_1$ and $\zeta_2$. Thus, we can expand the fields as

$$\xi(x, t) = p\zeta_1 + q\zeta_2, \quad (6a)$$

$$u(x, t) = u_0 + u_1\zeta_1\zeta_2, \quad (6b)$$

where the coefficients $p \equiv p(x, t)$, $q \equiv q(x, t)$, $u_0 \equiv u_0(x, t)$ and $u_1 \equiv u_1(x, t)$ are four arbitrary functions with respect to the spacetime variables $x$ and $t$, then the SKdV system (5a–5b) is changed to the BSKdV-2 system

$$u_{0t} + u_{0xxx} + 6u_0u_{0x} = 0, \quad (7a)$$
from that of [6]. In this paper, all the boson fields \( p, q, u_0 \) and \( u_1 \) are defined on not only the usual \( c \)-number algebra (the algebra constituted by the usual complex numbers) but also the infinite even Grassmann algebra \( G_e \), while in [6] all the fields take values only on the usual \( c \)-number algebra. In other words, various solutions of the SKdV equation will be lost if the bosonic fields are not defined on \( G_e \). As a simplest example, the BSKdV (7a) possesses a solution

\[
 u_0 = \theta_1 \theta_2 J, \quad (8a)
\]

\[
 J + J_{xxx} = 0, \quad (8b)
\]

where \( \theta_1 \) and \( \theta_2 \) are Grassmann parameters. However, if \( u_0 \) is Grassmann parameter independent as in [6], the solution (8) will be lost.

Eq. (7a) has the same form of the usual KdV equation but with depending on the Grassmann parameters. Eqs. (7b) and (7c) are linear homogeneous in \( p \) and \( q \) respectively, and Eq. (7d) is linear nonhomogeneous in \( u_1 \).

Similarly, if there are at least \( N \geq 2 \) fermionic parameters \( \zeta_i \) (\( i = 1, 2, \ldots, N \)) in a special solution of the SKdV system, we can expand the component fields \( u \) and \( \xi \) in the form

\[
 u(x, t) = u_0 + \sum_{n=1}^{\lfloor \frac{N-1}{2} \rfloor} \sum_{1 \leq i_1 < \cdots < i_{2n} \leq N} u_{i_1 i_2 \cdots i_{2n-1}} \zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_{2n-1}}, \quad (9a)
\]

\[
 \xi(x, t) = \sum_{n=1}^{\lfloor \frac{N-1}{2} \rfloor} \sum_{1 \leq i_1 < \cdots < i_{2n-1} \leq N} v_{i_1 i_2 \cdots i_{2n-1}} \zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_{2n-1}}, \quad (9b)
\]

where the coefficients \( u_0 \equiv u_0(x, t), u_{i_1 i_2 \cdots i_{2n-1}} \equiv u_{i_1 i_2 \cdots i_{2n-1}}(x, t) \) (\( 1 \leq i_1 < i_2 < \cdots < i_{2n} \leq N \)) and \( v_{i_1 i_2 \cdots i_{2n-1}} \equiv v_{i_1 i_2 \cdots i_{2n-1}}(x, t) \) (\( 1 \leq i_1 < i_2 < \cdots < i_{2n-1} \leq N \)) are \( 2^N \) bosonic functions of classical spacetime variable \( x \) and \( t \) and also take values on the infinite even Grassmann algebra \( G_e \). Substituting Eq. (9) into the SKdV model (5), we obtain the following bosonic system of \( 2^N \) equations, the BSKdV-N system,

\[
 u_{0x} + u_{0xxx} + 6u_0 u_{0x} = 0, \quad (10a)
\]

\[
 L_0 v_{i_1 i_2 \cdots i_{2n-1}} = \begin{cases} 0 & \text{for } n = 1, \\ -3 \sum_{W_1} (-1)^{\tau(j_1, j_2, \ldots, j_N)} v_{j_1 j_2 \cdots j_{2n-1}} i_{j_1} i_{j_2} \cdots i_{j_{2n-1}} & \text{for } n = 2, 3, \ldots, \left\lfloor \frac{N+1}{2} \right\rfloor, \end{cases} \quad (10b)
\]

\[
 L_n v_{i_1 i_2 \cdots i_{2n}} = \begin{cases} \sum_{W_2} (-1)^{\tau(j_1, j_2, \ldots, j_N)} v_{j_1 j_2 \cdots j_{2n}} i_{j_1} i_{j_2} & \text{for } n = 1, \\ 3 \sum_{W_2} (-1)^{\tau(j_1, j_2, \ldots, j_N)} v_{j_1 j_2 \cdots j_{2n}} i_{j_1} i_{j_2} & \text{for } n = 2, 3, \ldots, \left\lfloor \frac{N+1}{2} \right\rfloor, \end{cases} \quad (10c)
\]

where

\[
 \tau(j_1, j_2, \ldots, j_N) = \begin{cases} 0 & \text{for } j_1, j_2, \ldots, j_N \text{ is even permutation}, \\ 1 & \text{for } j_1, j_2, \ldots, j_N \text{ is odd permutation}, \end{cases}
\]

\[
 W_1 = \{(j_1, j_2, \ldots, j_{2n-1}) | l \leq j_1 < j_2 < \cdots < j_{2l} \leq 2n-1, 1 \leq j_{2l+1} < j_{2l+2} < \cdots < j_{2n-1} \leq 2n-1, 1 \leq l \leq n-1, j_{2l} \neq j_{2l}(h_1 \neq h_2) \},
\]

\[
 W_2 = \{(j_1, j_2, \ldots, j_{2n}) | l \leq j_1 < j_2 < \cdots < j_{2l+1} \leq 2n, 1 \leq j_{2l+1} < j_{2l+2} \cdots < j_{2n} \leq 2n, 1 \leq l \leq n, j_{2l} \neq j_{2l}(h_1 \neq h_2) \},
\]

\[
 W_3 = \{(j_1, j_2, \ldots, j_{2n}) | 1 \leq j_1 < j_2 < \cdots < j_{2l} \leq 2n, 1 \leq j_{2l+1} < j_{2l+2} \cdots < j_{2n} \leq 2n, 1 \leq l \leq n-1, j_{2l} \neq j_{2l}(h_1 \neq h_2) \}.
\]
and two operators read
\[ L_e(u_0) = \frac{\partial}{\partial x} + \frac{\partial}{\partial x^2} + 6u_0 \frac{\partial}{\partial x} + 6u_0 x, \]
\[ L_0(u_0) = \frac{\partial}{\partial x} + 3u_0 \frac{\partial}{\partial x} + 3u_0 x. \]

It is noted that the \( N = 2 \) case, though the forms of (9) and (10) are same as those in [6], however, the connotation or sense here is different because all the boson fields \( u_0, u_{i=2-1} \) and \( v_{i=2-1} \) are defined on \( G_r \) algebra while the boson fields of [6] are defined only on the usual c-number algebra.

In this paper, we just concentrate on the BSKdB-2 system. To study the integrability and exact solutions of the BSKdV-2 system [7], it is convenient to study its singularity.

### III. SINGULARITY ANALYSIS AND BÄCKLUND TRANSFORMATIONS OF THE BSKDV-2 SYSTEM.

Because of the difficulty to study the nonlinear systems, there is no method to find general solutions of any nontrivial nonlinear system except for C-integrable systems which can be directly transformed to linear ones or can be solved by direct integration. To find Laurent series solutions is an effective way to get the final general solution in linear case. In nonlinear case, the similar method is called singularity analysis or the Painlevé test. By means of the Painlevé analysis, various integrable properties such as the Bäcklund transformations, Lax pairs, infinitely many symmetries, bilinear forms, and so on, can be easily found if the studied model possesses Painlevé property, i.e., it is Painlevé integrable. In this section, we study the Painlevé property (PP) and the Bäcklund transformation of the BSKdV-2.

A nonlinear model is called Painlevé integrable, i.e., possessing Painlevé property, if all the movable singularities of its solutions are only poles. To prove the PP of the BSKdV-2 system, we expand the Boson fields \( u_0, p, q \) and \( u_1 \) as

\[ u_0 = \sum_{j=0}^{\infty} u_0 j \phi_j^{-\alpha_1}, \quad p = \sum_{j=0}^{\infty} p j \phi_j^{-\alpha_2}, \quad q = \sum_{j=0}^{\infty} q j \phi_j^{-\alpha_3}, \quad u_1 = \sum_{j=0}^{\infty} u_1 j \phi_j^{-\alpha_4}. \]  

BSKdV-2 system [7] possesses PP implies: (i) The constants \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4 \) are all positive integers such that all the non-pole singularities, algebraic and logarithmic branch points, are ruled out; (ii) There are twelve arbitrary functions in the series expansion (11) because four partial differential equations in [7] are third order. Furthermore, the function \( \phi \) should be arbitrary such that ALL the possible singularities are included.

After finishing the detailed calculations, it is not difficult to find that the BSKdV-2 is really Painlevé integrable because (11) possesses the form

\[ u_0 = \sum_{j=0}^{\infty} u_0 j \phi_j^{-2}, \quad p = \sum_{j=0}^{\infty} p j \phi_j^{-2}, \quad q = \sum_{j=0}^{\infty} q j \phi_j^{-2}, \quad u_1 = \sum_{j=0}^{\infty} u_1 j \phi_j^{-3}. \]  

where

\[ u_00 = -2\phi_x^2, \quad u_{01} = 2\phi_x x, \quad u_{02} = \frac{1}{6} \phi_x^2 (3\phi_{xx}^2 - \phi_x \phi_x - 4\phi_x \phi_{xxx}), \]
\[ p_1 = -(p_0 \phi_x^{-1})_x, \quad p_2 = \frac{1}{12} \phi_x^{-4} (p_0 \phi_x - 8p_0 \phi_x \phi_{xxx} + 21p_0 \phi_x^2 - 18p_0 \phi_x \phi_{xxx} + 6\phi_x^2 p_{0xx}), \]
\[ q_1 = -(q_0 \phi_x^{-1})_x, \quad q_2 = \frac{1}{12} \phi_x^{-4} (q_0 \phi_x - 8q_0 \phi_x \phi_{xxx} + 21q_0 \phi_x^2 - 18q_0 \phi_x \phi_{xxx} + 6\phi_x^2 q_{0xx}), \]
\[ u_{11} = \frac{1}{2} \phi_x^2 (3u_{10} \phi_x - 2\phi_x u_{10x}), \]
\[ u_{12} = -\frac{1}{2} \phi_x^4 (2u_{10} \phi_x \phi_{xxx} + 4u_{10} \phi_x \phi_{xx} - \phi_x^2 u_{10xx}^2 - 6u_{10} \phi_x^2), \]  

and other functions \( u_{0j}, p_j, q_j \) and \( u_{1j} \) are all determined by twelve arbitrary functions \( \phi, p_0, q_0, u_{10}, u_{04}, p_4, q_4, p_5, q_5, u_{15}, u_{06}, \) and \( u_{17} \).

Now, using the standard truncated Painlevé expansion

\[ u_0 = -2\phi_x^2 + 2\phi_x \phi_x^{-1} + u_2, \]
\[ p = p_0 \phi_x^{-1} - (p_0 \phi_x^{-1})_x \phi_x^{-1} + p_2, \]
\[ q = q_0 \phi_x^{-1} - (q_0 \phi_x^{-1})_x \phi_x^{-1} + q_2, \]
\[ u_1 = u_{10} \phi_x^{-3} + \frac{1}{2} \phi_x^2 (3u_{10} \phi_x - 2\phi_x u_{10x}) \phi_x^{-2} - \frac{1}{2} \phi_x^4 (2u_{10} \phi_x \phi_{xxx} + 4u_{10} \phi_x \phi_{xx} - \phi_x^2 u_{10xx}^2 - 6u_{10} \phi_x^2) \phi_x^{-1} + u_{13}, \]  

(14)
we have the following Bäcklund theorem:

**Theorem 1 (BT theorem).** If the fields $\phi$, $p_0$, $q_0$ and $u_{10}$ are the solutions of the following Schwarzian BSKdV-2 system,

\[
\begin{align*}
\phi_t + \phi_{xxx} - \frac{3}{2} \phi_x^2 + \lambda \phi_x &= 0, \\
p_0t + p_{0xx} + \frac{1}{2} \lambda p_{0x} + 9p_0 \phi_x^2 \phi_{xx} - \frac{33}{2} p_0 \phi_{x}^3 + \frac{1}{2} \lambda p_0 \phi_{xxx} - \frac{9}{2} p_0 \phi_{xx} - 6 \phi_x p_{0xx} \phi_{xx} + 63 \phi_x p_{0xx}^2 &= 0, \\
q_0t + q_{0xx} + \frac{1}{2} \lambda q_{0x} + 9q_0 \phi_x^2 \phi_{xx} - \frac{33}{2} q_0 \phi_{x}^3 + \frac{1}{2} \lambda q_0 \phi_{xxx} - \frac{9}{2} q_0 \phi_{xx} - 6 \phi_x q_{0xx} \phi_{xx} + 63 \phi_x q_{0xx}^2 &= 0, \\
u_{10t} + u_{10xx} - \frac{9}{2} u_{10x} \phi_{xx} - \frac{61}{2} u_{10x} \phi_{xxx} + \phi_{xxx} (6 \phi_{xxx} - \lambda \phi_x) - \frac{3}{2} (q_0 p_{00} - p_0 q_{00}) - 6 \phi_x (p_0 q_{0xx} - q_0 p_{0xx}) + 63 \phi_x (p_{0xx}^2 q_{0xx} - 2 \phi_x^2 + 3 \phi_x^3) &= 0,
\end{align*}
\]

then (14) is a BT between the solutions $\{u_0, p, q, u_1\}$ and $\{u_2, p_2, q_2, u_{13}\}$ while the latter solution is related to $\{\phi, p_0, q_0, u_{10}\}$ by

\[
\begin{align*}
u_2 &= \frac{1}{2} \phi_x^2 - \frac{1}{2} \phi_x^2 + \frac{1}{6}, \\
p_2 &= -3 \frac{p_0 \phi_{xxx}}{4 \phi_x^3} + 15 \frac{p_0 \phi_x^2}{8 \phi_x^3} - \frac{p_0}{12 \phi_x^3} - \frac{3 p_0 \phi_x^3}{2 \phi_x^3} + \frac{1}{2} \frac{p_{0xx} \phi_x^2}{2 \phi_x^3}, \\
q_2 &= -3 \frac{q_0 \phi_{xxx}}{4 \phi_x^3} + 15 \frac{q_0 \phi_x^2}{8 \phi_x^3} - \frac{q_0}{12 \phi_x^3} - \frac{3 q_0 \phi_x^3}{2 \phi_x^3} + \frac{1}{2} \frac{q_{0xx} \phi_x^2}{2 \phi_x^3}, \\
u_{13} &= \frac{\phi_i}{24 \phi_x^3} (3 q_{0x} - 3 p_{0x} + 3 \phi_x u_{10x} - 3 u_{10x} \phi_{xx}) + \frac{u_{10x}}{12 \phi_x^3} (14 \phi_x \phi_{xxx} - 51 \phi_x^2) - \frac{u_{10x} + 4 u_{10xx}}{24 \phi_x^3} \\
&\quad + \frac{p_{0xx} - q_{0xx} + 10 u_{10x} \phi_x + u_{10x} (\phi_x + 3 \phi_{xxx})}{8 \phi_x^3} - \frac{u_{10xx}}{8 \phi_x^3} (32 \phi_x \phi_{xxx} - 49 \phi_x^2),
\end{align*}
\]

It is obvious that the BT theorem 1 includes an auto-BT (14) and a nonauto-BT (16).

**IV. NONLOCAL SYMMETRIES OF THE BSKDV-2 FROM BT**

In the study of the nonlinear systems, the symmetry study is another powerful method. It is interesting that the truncated Painlevé expansion method can be used to find infinitely many nonlocal symmetries.

A symmetry of a nonlinear model is defined as a solution of its linearized system. For the BSKdV-2 system (7), its linearized system has the form

\[
\begin{align*}
\sigma^{\mu_0}_t + \sigma^{\mu_0}_{x+} + 6 \sigma^{\mu_0} u_0 + 6 u_0 \sigma^{\mu_0}_x &= 0, \\
\sigma^{\mu}_t + \sigma^{\mu}_{x+} + 3 \sigma^{\mu} p + 3 u_0 \sigma^p + 3 u_0 \sigma^\mu &= 0, \\
\sigma^p_t + \sigma^p_{x+} + 3 \sigma^p q + 3 u_0 \sigma^q + 3 u_0 \sigma^p &= 0, \\
\sigma^p_t + \sigma^p_{x+} + 6 (\sigma^\mu u_1)_x + 6 (u_0 \sigma^\mu)_x = 3 \sigma^p q + 3 p \sigma^p - 3 \sigma^p p - 3 \sigma^p q + 3 \sigma^p u_1 + 6 (u_0 \sigma^\mu)_x = 0
\end{align*}
\]

which means (7) is form invariant under the transformation

\[
\{u_0, p, q, u_1\} \rightarrow \{u_0, \sigma^\mu, \sigma^p, \sigma^q, \sigma^\mu\}
\]

with the infinitesimal parameter $\epsilon$.

For the integrable system there are infinitely many local and nonlocal symmetries. Recently, it is found that infinitely many nonlocal symmetries can be found from Darboux transformations [23, 24] or Bäcklund transformations [25]. From the truncated Painlevé expansion one can find not only BTs but also nonlocal symmetries.
For the BSKdV-2 system, a nonlocal symmetry

\[
\sigma_{RS} = \begin{pmatrix}
\sigma^{u0} \\
\sigma^p \\
\sigma^q \\
\sigma^m
\end{pmatrix} = \begin{pmatrix}
2\phi_{xx} \\
-(p_0\phi_{x}^{-1})_x \\
-(q_0\phi_{x}^{-1})_x \\
-\frac{1}{2\phi_x}(2u_{10}\phi_x\phi_{xxx} - 6u_{10}\phi_{xx}^2 + 4u_{10}\phi_x\phi_{xx} - u_{10}\phi_x^2)
\end{pmatrix}
\]  

(19)

can naturally read out from its truncated Painlevé expansion. In fact, the nonlocal symmetry (19) is the residual of the truncated Painlevé expansion, i.e., BT (14) with respect to the singularity manifold \( \phi \). Thus, we call this nonlocal symmetry as the residual symmetry (RS).

It is also interesting that RS (19) is just the infinitesimal form of the BT (14). To prove this conclusion, we have to solve the “initial value problem”

\[
\begin{align*}
\frac{d\bar{u}_0(e)}{de} &= 2\tilde{\phi}_x(e), \quad \bar{u}_0(0) = u_0, \\
\frac{d\bar{p}(e)}{de} &= -[\bar{p}_0(e)\tilde{\phi}_x(e)^{-1}]_x, \quad \bar{p}(0) = p, \\
\frac{d\bar{q}(e)}{de} &= -[\bar{q}_0(e)\tilde{\phi}_x(e)^{-1}]_x, \quad \bar{q}(0) = q, \\
\frac{d\bar{u}_1(e)}{de} &= -\frac{1}{2\phi(e)^2}[2\bar{u}_{10}(e)\tilde{\phi}_x(e)\tilde{\phi}_{xxx}(e) - 6\bar{u}_{10}(e)\tilde{\phi}_{xx}(e)^2 + 4\bar{u}_{10}(e)\tilde{\phi}_x(e)\tilde{\phi}_{xx}(e) - \bar{u}_{10}\phi_x(e^2)], \quad \bar{u}_1(0) = u_1.
\end{align*}
\]  

(20)

It is clear that to solve the initial value problem (20) is quite difficult due to the intrusion of the functions \( \phi, \ p_0, \ q_0, \ u_{10} \) and their differentiations.

To solve the initial value problem (20), we have to prolong the BSKdV-2 system (7) such that RS becomes a local Lie point symmetry for the prolonged system. Fortunately, the final result can be successfully obtained similar to the usual KdV equation and other integrable systems (23, 26):

\[
\begin{align*}
u_{01} + u_{0xxx} + 6u_0u_{0x} &= 0, \\
p_{t} + p_{xxx} + 3u_0p_x + 3u_0p &= 0, \\
q_{t} + q_{xxx} + 3u_0q_x + 3u_0q &= 0, \\
u_{1t} + u_{1xxx} + 6u_0u_{1x} + 6u_0u_1 &= 3(pq_{xx} - qp_{xx}). \\
\phi_{t} + \phi_{xxx} - \frac{3}{2} \phi_x^2 + \lambda \phi_x &= 0, \\
p_{0t} + p_{0xxx} + \frac{1}{2} \lambda p_{0x} + 9p_0 \phi_{xxx}^2 + \frac{33}{2} p_0 \phi_x^3 + \lambda p_0 \phi_{xxx}^2 - \frac{9}{2} p_0 \phi_{xxx} - 6p_{0xx} \phi_x + \frac{63}{4} p_0 \phi_x^2 &= 0, \\
q_{0t} + q_{0xxx} + \frac{1}{2} \lambda q_{0x} + 9q_0 \phi_{xxx}^2 + \frac{33}{2} q_0 \phi_x^3 + \lambda q_0 \phi_{xxx}^2 - \frac{9}{2} q_0 \phi_{xxx} - 6q_{0xx} \phi_x + \frac{63}{4} q_0 \phi_x^2 &= 0, \\
u_{10t} + u_{10xxx} - \frac{9}{\phi_x} u_{10xx} - \phi_{xxx}^2 - \frac{u_{10}}{\phi_x} + \frac{3}{\phi_x} (q_{0xx} p_{0xx} - p_{0xx} q_{0xx}) - \frac{6}{\phi_x} (p_0 q_{0x} - q_0 p_{0x}) \phi_x + \frac{63}{2} u_{10} \phi_x^2 + \frac{18}{\phi_x} u_{10} \phi_x \phi_{xxx}^2 + \frac{p_0 q_{0xx} - q_0 p_{0xx}}{4 \phi_x} (18 \phi_{xxx}^2 \phi_x - 2 \phi_x^2 + 33 \phi_{xx}^2) - \frac{42 u_{10} \phi_x^3}{\phi_x} &= 0.
\end{align*}
\]  

(21a)  

(21b)  

(21c)  

(21d)  

(21e)  

(21f)  

(21g)  

(21h)  

(21i)
\[
\phi_{1x} = \phi_2, \quad (21j)
\]
\[
\phi_{2x} = \phi_3, \quad (21k)
\]
\[
u_{10x} = v_1, \quad (21l)
\]
\[
u_{1x} = v_2, \quad (21m)
\]
\[
p_{0x} = p_1, \quad (21n)
\]
\[
q_{0x} = q_1. \quad (21o)
\]

Now it is not difficult to verify that the nonlocal symmetry, the RS symmetry [19], for the original BSKdV-2 system [7] becomes a local Lie point symmetry for the prolonged system [21]

\[
\sigma_{xx} = \begin{pmatrix}
\sigma^0 \\
\sigma^p \\
\sigma^q \\
\sigma^{u1} \\
\sigma^{p1} \\
\sigma^{q1} \\
\sigma^{v1} \\
\sigma^{v2}
\end{pmatrix} = \begin{pmatrix}
2\phi_2 \\
p_0\phi_2\phi_1^{-2} - p_1\phi_1^{-1} \\
q_0\phi_2\phi_1^{-2} - q_1\phi_1^{-1} \\
\frac{1}{2}(6u_{10}\phi_2^2 - 2u_{10}\phi_1\phi_3 - 4v_1\phi_1\phi_2 + v_2\phi_1^2)\phi_1^{-4} \\
-2\phi_1 \\
-6u_{10}\phi_1 \\
-2\phi_2 \\
-6\phi_1\phi_2 - 2\phi_3 \\
-4p_0\phi_1 - 4p_1 \\
-4q_0\phi_1 - 4q_1 \\
-6u_{10}\phi_1 - 6\phi v_1 \\
-6u_{10}\phi_2 - 12v_1\phi_2 - 6\phi v_2
\end{pmatrix}. \quad (22)
\]

Correspondingly, the initial value problem (20) is changed as

\[
\frac{d\tilde{u}_0(\epsilon)}{d\epsilon} = 2\tilde{\phi}_2(\epsilon), \quad \tilde{u}_0(0) = u_0, \quad (23a)
\]
\[
\frac{d\tilde{p}(\epsilon)}{d\epsilon} = \tilde{p}_0(\epsilon)\tilde{\phi}_2(\epsilon)\tilde{\phi}_1(\epsilon)^{-2} - \tilde{p}_1(\epsilon)\tilde{\phi}_1(\epsilon)^{-1}, \quad \tilde{p}(0) = p, \quad (23b)
\]
\[
\frac{d\tilde{q}(\epsilon)}{d\epsilon} = \tilde{q}_0(\epsilon)\tilde{\phi}_2(\epsilon)\tilde{\phi}_1(\epsilon)^{-2} - \tilde{q}_1(\epsilon)\tilde{\phi}_1(\epsilon)^{-1}, \quad \tilde{q}(0) = q, \quad (23c)
\]
\[
\frac{d\tilde{u}_1(\epsilon)}{d\epsilon} = -\frac{1}{2\phi_1(\epsilon)^2} [2\tilde{u}_{10}(\epsilon)\tilde{\phi}_1(\epsilon)\tilde{\phi}_3(\epsilon) - 6\tilde{u}_{10}(\epsilon)\tilde{\phi}_2(\epsilon)^2 + 4\tilde{v}_1(\epsilon)\tilde{\phi}_1(\epsilon)\tilde{\phi}_2(\epsilon) - \tilde{v}_2(\epsilon)\tilde{\phi}_1(\epsilon)^2], \quad \tilde{u}_1(0) = u_1, \quad (23d)
\]
\[
\frac{d\tilde{\phi}(\epsilon)}{d\epsilon} = -\tilde{\phi}(\epsilon)^2, \quad \tilde{\phi}(0) = \phi, \quad (23e)
\]
\[
\frac{d\tilde{p}_0(\epsilon)}{d\epsilon} = -4\tilde{p}_0(\epsilon)\tilde{\phi}(\epsilon), \quad \tilde{p}_0(0) = p_0, \quad (23f)
\]
\[
\frac{d\tilde{q}_0(\epsilon)}{d\epsilon} = -4\tilde{q}_0(\epsilon)\tilde{\phi}(\epsilon), \quad \tilde{q}_0(0) = q_0. \quad (23g)
\]
Theorem 2 (equivalent BT theorem). If \( \{u_0, p, q, u_1, \phi, p_0, q_0, u_{10}, \phi_1, \phi_2, \phi_3, p_1, q_1, v_1, v_2 \} \) is a solution of the prolonged BSKdV-2 system (21), so is \( \{\bar{u}_0, \bar{p}, \bar{q}, \bar{u}_1, \bar{\phi}, \bar{p}_0, \bar{q}_0, \bar{u}_{10}, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3, \bar{p}_1, \bar{q}_1, \bar{v}_1, \bar{v}_2 \} \) with

\[
\bar{u}_0 = u_0 + \frac{2e\phi_2}{\epsilon\phi + 1} - \frac{2e^2\phi_2^2}{(\epsilon\phi + 1)^2}, \quad (24a)
\]

\[
\bar{p} = p - \frac{\epsilon p_0[\epsilon\phi_2 + \phi_2(1 + \epsilon\phi)]}{\phi_1^2(1 + \epsilon\phi)^2} + \frac{\epsilon p_1}{\phi_1(1 + \epsilon\phi)}, \quad (24b)
\]

\[
\bar{q} = q - \frac{\epsilon q_0[\epsilon\phi_2 + \phi_2(1 + \epsilon\phi)]}{\phi_1^2(1 + \epsilon\phi)^2} + \frac{\epsilon q_1}{\phi_1(1 + \epsilon\phi)}, \quad (24c)
\]

\[
\bar{u}_1 = u_1 + e u_{10} \left( \frac{3\phi_3^2 - \phi_1\phi_3}{\phi_1^2(1 + \epsilon\phi)} + \frac{3e\phi_2^2}{\phi_1^4(1 + \epsilon\phi)^2} + \frac{\epsilon^2}{(1 + \epsilon\phi)^3} \right) - \frac{e v_1}{\phi_1(1 + \epsilon\phi)} \left( \frac{2e\phi_2}{\phi_1^4(1 + \epsilon\phi)^2} + \frac{\epsilon}{1 + \epsilon\phi} \right) + \frac{e v_2}{2\phi_1^2(1 + \epsilon\phi)}, \quad (24d)
\]

\[
\bar{\phi} = \frac{\phi}{1 + \epsilon\phi}, \quad (24e)
\]

\[
\bar{p}_0 = \frac{p_0}{(1 + \epsilon\phi)^s}, \quad (24f)
\]

\[
\bar{q}_0 = \frac{q_0}{(1 + \epsilon\phi)^s}, \quad (24g)
\]

\[
\bar{u}_{10} = \frac{u_{10}}{(1 + \epsilon\phi)^s}, \quad (24h)
\]
\[ \tilde{\phi}_1 = \frac{\phi_1}{(1 + \varepsilon \phi)^2}, \]  
(24i)

\[ \tilde{\phi}_2 = \frac{\phi_2}{(1 + \varepsilon \phi)^2} - \frac{2\phi_1^2\varepsilon}{(1 + \varepsilon \phi)^3}, \]  
(24j)

\[ \tilde{\phi}_3 = \frac{\phi_3}{(1 + \varepsilon \phi)^2} - \frac{6\varepsilon \phi_1 \phi_2}{(1 + \varepsilon \phi)^3} + \frac{6\varepsilon^2 \phi_1^2}{(1 + \varepsilon \phi)^4} \]  
(24k)

\[ \tilde{p}_1 = \frac{p_1}{(1 + \varepsilon \phi)^4} - \frac{4\varepsilon \phi_1 p_0}{(1 + \varepsilon \phi)^5}, \]  
(24l)

\[ \tilde{q}_1 = \frac{q_1}{(1 + \varepsilon \phi)^4} - \frac{4\varepsilon \phi_1 q_0}{(1 + \varepsilon \phi)^5}, \]  
(24m)

\[ \tilde{v}_1 = \frac{v_1}{(1 + \varepsilon \phi)^6} - \frac{6\varepsilon \phi_1 u_{10}}{(1 + \varepsilon \phi)^7}, \]  
(24n)

\[ \tilde{v}_2 = \frac{v_2}{(1 + \varepsilon \phi)^6} - \frac{12\varepsilon \phi_1 v_1}{(1 + \varepsilon \phi)^7} - \frac{6\varepsilon u_{10}[\varepsilon \phi_2(1 + \varepsilon \phi) - 7\varepsilon \phi_1^2]}{(1 + \varepsilon \phi)^8}. \]  
(24o)

For the original system (7), it is not difficult to see that the BT theorems 1 and 2 are equivalent based on the trivial fact that the singularity manifold equation (15) is form invariant under the transformation

\[ 1 + \varepsilon \phi \rightarrow \phi, \ (\varepsilon \phi_1 \rightarrow \phi \varepsilon, \ \varepsilon \phi_2 \rightarrow \phi x), \]

The BT equivalent theorem 2 shows us an interesting result that the nonlocal RS of the truncated Painlevé expansion is just the infinitesimal form of the BT.

Notice that the BSKdV-2 is not explicitly \( \lambda \) dependent while the RS is explicitly \( \lambda \) dependent, then infinitely many nonlocal symmetries can be obtained in two ways, respectively,

\[ \sigma_{RS,i} = \sigma_{RS}(\lambda_i), \ i = 0, 1, 2, \cdots, \]  
(25)

and

\[ \sigma_{RS,n} = \frac{1}{n!} \frac{d^n}{d\varepsilon^n} \sigma_{RS}, \ n = 0, 1, 2, \cdots. \]  
(26)

For the usual KdV system, the parameter \( \lambda \) is just the spectral parameter, \( \sigma_{RS} \) is equivalent to the square spectral function symmetry [27], and the square spectral function symmetry form of \( \sigma_{RS,i} \) can be used to find algebraic geometry solutions with higher order genus [28]. \( \sigma_{RS,n} \) for the usual KdV equation can be equivalently obtained by applying the inverse recursion \( \Phi_{KdV} \) on \( \sigma_{RS} \) [29]:

\[ \sigma_{RS,n}(KdV) \sim \Phi_{KdV}^{-n} \sigma_{RS} \sim \Phi_{KdV}^{-n} \phi_{xx}. \]

V. SYMMETRY REDUCTIONS RELATED TO THE NONLOCAL RS

In Ref. [6], the symmetry reductions without nonlocal symmetry \( \sigma_{RS} \) have been given for the fields defined on the usual c-number algebra but not on \( G_c \). The symmetry reduction forms, if the boson fields are also defined on \( G_c \), have the same forms as those in [6], except that all the reduction functions and parameters are defined on \( G_c \). In this section, we are interested in studying the symmetry reductions of the BSKdV-2 system related to the nonlocal symmetry RS, \( \sigma_{RS} \).

Similar to the last section, because of the nonlocal property of \( \sigma_{RS} \), we have to study the symmetry reductions of the prolonged system [21] since the RS has been localized to the Lie point symmetry [22].
Using the standard Lie point symmetry approach to the prolonged system (21), we can find the general Lie point symmetry in the form

\[ \sigma_{pl} = \begin{pmatrix} \sigma^0 \\ \sigma^p \\ \sigma^q \\ \sigma^v_1 \\ \sigma^v_2 \\ \sigma^v_3 \end{pmatrix} = \begin{pmatrix} (cx + x_0)u_{0x} + (3ct + t_0)u_{0t} + 2cu_0 + 2C_0\phi_2 \\ (cx + x_0)p_1 + (3ct + t_0)p_1 + c_2p + C_0(p_0\phi_2\phi_1^{-2} - p_1\phi_1^{-1}) \\ (cx + x_0)q_1 + (3ct + t_0)q_1 + c_3q + C_0(q_0\phi_2\phi_1^{-2} - q_1\phi_1^{-1}) \\ (cx + x_0)u_{1x} + (3ct + t_0)u_{1t} + (c_2 + c_3 - c)u_1 + \frac{2c_2}{(6u_{10}\phi_1^2 - 2u_{10}\phi_1 - 4v_1\phi_1 + v_2\phi_1^2)}\phi_1^{-4} \\ (cx + x_0)p_{01} + (3ct + t_0)p_{01} + (c_2 + 2c_1 - 2c)p_0 + 4C_0p_0\phi \\ (cx + x_0)q_{01} + (3ct + t_0)q_{01} + (c_3 + 2c_1 - 2c)q_0 + 4C_0q_0\phi \\ (cx + x_0)u_{10x} + (3ct + t_0)u_{10t} + (c_3 + c_2 - 4c + 3c_1)u_{10} - 6C_0u_{10}\phi_1 \\ (cx + x_0)p_{101} + (3ct + t_0)p_{101} + (c_3 + c_2 - 4c + 3c_1)p_{1} - 4C_0p_{1}\phi_1 - 4C_0p_1 \\ (cx + x_0)q_{101} + (3ct + t_0)q_{101} + (c_3 + c_2 - 4c + 3c_1)q_{1} - 4C_0q_{1}\phi_1 - 4C_0q_1 \\ \end{pmatrix} \]

(27)

which includes the space-time scaling invariance (c-part of (27)), space, time and the fields \( \phi, p_1, q_1, v_1, v_2 \) translations \((x_0, t_0)\) and \((c_0, C_1, C_2, C_3, C_4)\) parts, respectively). RS \((C_0)\) part and the scaling transformations of the fields \( \phi, p \) and \( q \) \((c_1, c_2\) and \(c_3\) parts, respectively). Because we are concentrated on the RS related symmetry reductions in this paper, we fix \( C_0 = 1 \) without loss of generality for \( C_0 \neq 0 \). We also fix the trivial potential fields \((p_1, q_1, v_1, v_2)\) translations by taking \( C_1 = C_2 = C_3 = C_4 = 0 \) for simplicity. The last component of \( \sigma_{pl} \) shown in (27) indicates that to guarantee the space-time scaling invariance, the spectral parameter \( \lambda \) should also have a scaling like the field \( u_0 \).

To find symmetry reductions, i.e., to find group invariant solutions, implies to find solutions of both

\[ \sigma_{pl} = 0 \]

(28)

and the prolonged system (21). To solve the group invariant condition (28), we have to discuss two nontrivial cases for \( c \neq 0 \) and \( c = 0 \).

**Case I** \( c \neq 0 \). In this case the general solution of (28) with \( C_1 = C_2 = C_3 = C_4 = C_0 = 1 \) has the form

\[ \lambda = 0, \quad \xi = \frac{a_{1x} - 3b_{1x}a_1}{3T}, \quad T \equiv (a_{1t} - b_{1t}), \]

(29a)

\[ u_0 = T^{-\frac{1}{2}} \left[ U_0 + \frac{2}{a_1} \Psi_2 \tanh(W) - \frac{2}{a_1^2} \Psi_1^2 \tanh^2(W) \right], \quad W \equiv b_1 \ln \left( \frac{a_{1t} - b_{1t}}{b_1} \right) + \Psi(\xi), \]

(29b)

\[ p = T^{-\frac{c_0}{a_1}} \left[ P + \frac{P_0 \Psi_2 - P_1 \Psi_1}{a_1 \Psi_1^2} \tanh(W) - \frac{1}{a_1^2} P_0 \tanh^2(W) \right], \]

(29c)

\[ q = T^{-\frac{c_1}{a_1}} \left[ Q + \frac{Q_0 \Psi_2 - Q_1 \Psi_1}{a_1 \Psi_1^2} \tanh(W) - \frac{1}{a_1^2} P_0 \tanh^2(W) \right], \]

(29d)

\[ u_1 = T^{-\frac{n_{12}}{a_1}} \left[ U_1 - \frac{2U_{10} \Psi_1 \Psi_3 - 3\Psi_2^2}{2a_1 \Psi_1^4} \tanh(W) + \frac{3U_{10} \Psi_2 - 2V_1 \Psi_1}{a_1^2 \Psi_1^2} \tanh^2(W) + \frac{U_{10}}{a_1^3} \tanh^3(W) \right], \]

(29e)

\[ \Psi = a_0 + a_1 \tanh(W), \]

(29f)

\[ p_0 = T^{-\frac{c_0}{a_1}} P_0 \sech^4(W), \]

(29g)

\[ q_0 = T^{-\frac{c_1}{a_1}} Q_0 \sech^4(W), \]

(29h)
Substituting the group invariant solution (29) into the prolonged system (21), it is straightforward to get the symmetry reduction equations for the group invariant functions $U_0$, $P$, $Q$, $U_1$, $\Psi$, $P_0$, $Q_0$, $U_{10}$, $\Psi_1$, $\Psi_2$, $\Psi_3$, $P_1$, $Q_1$, $V_1$ and $V_2$: 

$$U_0 = \frac{1}{2\Psi_1^2} \left( \Psi_1^2 a_1^2 + \Psi_1^2 - \xi^2 \Psi_1^2 \right),$$  

$$\Psi_3 = \frac{1}{2\Psi_1^2} \left( 3\Psi_2^2 - 2a_1^2 \Psi_1^2 + 2\xi \Psi_1^2 \right),$$  

$$\Psi' = \frac{3}{a_1^2} \Psi_1,$$  

$$P'_0 = \frac{3}{a_1} P_1,$$  

$$Q'_0 = \frac{3}{a_1} Q_1,$$  

$$U'_{10} = \frac{3}{a_1} V_1,$$

where $U_0$, $P$, $Q$, $U_1$, $\Psi$, $P_0$, $Q_0$, $U_{10}$, $\Psi_1$, $\Psi_2$, $\Psi_3$, $P_1$, $Q_1$, $V_1$ and $V_2$ are all group invariant functions of $\xi$ and the constants $c$, $c_0$ and $c_1$ in (27) have been re-notated as

\[ c = -3a_1 \frac{b_1}{a_0^2 + a_1^2}, \quad c_0 = -a_0^2 + a_1^2, \quad c_1 = \frac{6b_1a_0 - a_1}{3b_1} \]

for notation simplicity.
\[ \Psi'_1 = \frac{3}{a_1} \Psi_2, \]  

\[ V'_1 = \frac{3}{a_1^2} (6U_{10} \Psi_1^2 + a_1^2 V_2), \]  

\[ \Psi'_2 = \frac{3}{2a_1^3 \Psi_1} (4\Psi_1^4 + 2a_1^2 \xi \Psi_1^2 + 3a_1^2 \Psi_2^2 - 2a_1^4 b_1 \Psi_1), \]  

\[ P' = \frac{3}{2a_1} \left( \frac{8}{a_1} \Psi_1^2 + \frac{3 \xi}{a_1} - \frac{3a_1 b_1}{a_1^2 \Psi_1} - \frac{3 \Psi_2^2}{a_1^2 \Psi_1} \right) P_0 + \frac{6}{a_1} \Psi^2 + \frac{9P_1 \Psi_2}{a_1 \Psi_1}, \]  

\[ Q'_1 = \frac{3}{2a_1} \left( \frac{8}{a_1} \Psi_1^2 + \frac{3 \xi}{a_1} - \frac{3a_1 b_1}{a_1^2 \Psi_1} - \frac{3 \Psi_2^2}{a_1^2 \Psi_1} \right) Q_0 + \frac{6}{a_1^3 \Psi_1^2} + \frac{9Q_1 \Psi_2}{a_1 \Psi_1}, \]  

\[ V_2 = 4 \left( \frac{2 \Psi_1 Q_0 - Q_1 \Psi_1^2}{\Psi_1^2} \right) P + 4 \left( \frac{P_1 \Psi_1^2}{\Psi_1^2} - 2P_0 \Psi_1 \right) Q + \left( \frac{2 \xi \Psi_1 - b_1 a_1^2}{\Psi_1^2} - \frac{5 \Psi_2^2}{2 \Psi_1^2} Q_1 \right) \frac{(a_1 + c_3 b_1 + c_2 b_1)}{a_1^2 \Psi_1} \]  

\[ U'_1 = -\frac{3V_2}{4a_1 \Psi_1^3} \left( \frac{5 \Psi_1^3 - a_1^2 b_1}{2a_1^2 \Psi_1} \right) + \frac{3V_1}{4a_1 \Psi_1^3} \left( \frac{5 \Psi_1^3}{\Psi_1^2} + \frac{5 \xi}{\Psi_1^2} + \frac{32 \Psi_1^3 - 11a_1^2 b_1}{4a_1^2 \Psi_1^2} \right) \Psi_2 + \frac{3a_1^2 + 2a_1^4 b_1}{a_1^2 \Psi_1^2} \]  

where, and in the latter of the paper, the primes ‘‘’’ on the functions (with only one independent variable) denote derivatives with respect to the corresponding independent variable.

Eliminating \( \Psi_2 \) in (30i) by using (30g), we get a single second order ordinary differential equation for the field \( \Psi_1 \)

\[ \Psi''_1 = \frac{9}{2a_1^3 \Psi_1} \left( 4\Psi_1^4 + 2a_1^2 \xi \Psi_1^2 + \frac{a_1^4}{3} \Psi_1^2 - 2a_1^4 b_1 \Psi_1 \right), \]  

which is an equivalent form of the Painlevé II equation. Thus, the group invariant solution is an interaction solution among a soliton and a Painlevé II wave.

**Case II c = 0.** The general solution of (28) with \( c = C_1 = C_2 = C_3 = C_4 = C_0 - 1 = 0 \) has the form

\[ \eta = x - Ct, \quad (x_0 \equiv C_{10}, \ t_0 \equiv -a_1/b_1, \ c_1 \equiv 2\Psi_0, \ c_0 \equiv a_1^2 - \Psi_0^2), \]  

(32a)
Substituting the group invariant solution (32) into the prolonged system (21), we get the second symmetry reduction:

\[ u_0 = U_0 - \frac{2}{a_1} \Psi_2 \tanh(W) - \frac{2}{a_1^2} \Psi_1^2 \tanh^2(W), \quad W \equiv b_1 t + \Psi, \]  

(32b)

\[ p = e^{\frac{b_1 t}{a_1}} \left[ P - \frac{P_1 \Psi_1 - P_0 \Psi_2}{a_1 \Psi_1^2} \tanh(W) + \frac{1}{a_1^2} P_0 \tanh^2(W) \right], \]  

(32c)

\[ q = e^{\frac{b_1 t}{a_1}} \left[ Q - \frac{Q_1 \Psi_1 - Q_0 \Psi_2}{a_1 \Psi_1^2} \tanh(W) + \frac{1}{a_1^2} Q_0 \tanh^2(W) \right], \]  

(32d)

\[ u_1 = e^{\frac{b_1 t}{a_1}} U_{10} \sech^6(W), \]  

(32e)

\[ \Psi = \Psi_0 + a_1 \tanh(W), \]  

(32f)

\[ p_0 = e^{\frac{b_1 t}{a_1}} P_0 \sech^4(W), \]  

(32g)

\[ q_0 = e^{\frac{b_1 t}{a_1}} Q_0 \sech^4(W), \]  

(32h)

\[ u_{10} = e^{\frac{b_1 t}{a_1}} U_{10} \sech^6(W), \]  

(32i)

\[ \Psi_1 = \Psi_1 \sech^2(W), \]  

(32j)

\[ \Psi_2 = \sech^2(W) \left[ \Psi_2 + \frac{2}{a_1} \Psi_1^2 \tanh(W) \right], \]  

(32k)

\[ \Psi_3 = \sech^2(W) \left[ \Psi_3 - \frac{6}{a_1} \Psi_1 \Psi_2 \tanh(W) + \frac{6}{a_1^2} \Psi_1^3 \tanh^2(W) \right], \]  

(32l)

\[ p_1 = e^{\frac{b_1 t}{a_1}} \sech^4(W) \left[ P_1 - \frac{4}{a_1} \Psi_1 P_0 \tanh(W) \right], \]  

(32m)

\[ q_1 = e^{\frac{b_1 t}{a_1}} \sech^4(W) \left[ Q_1 - \frac{4}{a_1} \Psi_1 Q_0 \tanh(W) \right], \]  

(32n)

\[ v_1 = e^{\frac{b_1 t}{a_1}} \sech^6(W) \left[ V_1 - \frac{6}{a_1} \Psi_1 u_{01} \tanh(W) \right], \]  

(32o)

\[ v_2 = e^{\frac{b_1 t}{a_1}} \sech^6(W) \left[ V_2 - \frac{6}{a_1} (\Psi_2 u_{01} + 2 \Psi_1 V_1) \tanh(W) + \frac{42}{a_1^2} U_{10} \Psi_1^2 \tanh^2(W) \right], \]  

(32p)

where \( U_0, P, Q, U_1, \Psi, P_0, Q_0, U_{10}, \Psi_1, \Psi_2, \Psi_3, P_1, Q_1, V_1 \) and \( V_2 \) are all group invariant solution of \( \eta \).

Substituting the group invariant solution (32) into the prolonged system (21), we get the second symmetry reduction:

\[ U_0 = \frac{-C + 2 \lambda + \frac{a_1 b_1}{2}}{2} \frac{\Psi_2^2}{2 \Psi_1^2}, \]  

(33a)
\( \Psi_3 = \frac{\Psi_2^2}{2\Psi_1} + \left( \frac{4}{3} - 2U_0 \right) \Psi_1, \)  
(33b)

\( \Psi' = \frac{\Psi_1}{a_1}, \)  
(33c)

\( P'_0 = P_1, \)  
(33d)

\( Q'_0 = Q_1, \)  
(33e)

\( U'_{10} = V_1, \)  
(33f)

\( \Psi'_1 = \Psi_2, \)  
(33g)

\[ V'_1 = \frac{1}{a'_1}(6U_{10}\Psi_1^2 + a_1^2 V_2), \]  
(33h)

\[ \Psi'_2 = (C - \lambda) \Psi_1 - a_1 b_1 + \frac{3\Psi_2^2}{2\Psi_1} + \frac{2\Psi_3}{a_1}, \]  
(33i)

\( P'_1 = \frac{2}{a'_1}(2P_0 + a_1^2 P) \Psi_1^2 - \frac{1}{3} P_0(9U_0 - 2\lambda) + \frac{3}{\Psi_1} \Psi_2 P_1 - \frac{3}{\Psi_1} \Psi_3 P_0, \)  
(33j)

\( Q'_1 = \frac{2}{a'_1}(2Q_0 + a_1^2 Q) \Psi_1^2 - \frac{1}{3} Q_0(9U_0 - 2\lambda) + \frac{3}{\Psi_1} \Psi_2 Q_1 - \frac{3}{\Psi_1} \Psi_3 Q_0, \)  
(33k)

\[ P' = \frac{\Psi_2}{\Psi_1} [P + P_0 \left( \frac{24U_0}{\Psi_1^2} - \frac{1}{a'_1} + \frac{45C - 62\lambda}{4\Psi_1} - \frac{12a_1 b_1}{\Psi_1^2} \right) + \frac{b_1 c_2 P_0}{2a_1 \Psi_1^2} + \left( \frac{1}{a'_1} + \frac{3C - 2\lambda}{6\Psi_1^2} \right) P_1 + \frac{12\Psi_2^2 P_0}{\Psi_1^3}], \]  
(33l)

\[ Q' = \frac{\Psi_2}{\Psi_1} [Q + Q_0 \left( \frac{24U_0}{\Psi_1^2} - \frac{1}{a'_1} + \frac{45C - 62\lambda}{4\Psi_1} - \frac{12a_1 b_1}{\Psi_1^2} \right) + \frac{b_1 c_2 Q_0}{2a_1 \Psi_1^2} + \left( \frac{1}{a'_1} + \frac{3C - 2\lambda}{6\Psi_1^2} \right) Q_1 + \frac{12\Psi_2^2 Q_0}{\Psi_1^3}], \]  
(33m)

\[ V_2 = 8\Psi_1(PQ_0 - QP_0) - \frac{45\Psi_2^2 U_{10}}{2\Psi_1^2} - \frac{45C - 62\lambda}{2} - \frac{31C}{2} - 35U_0 - \frac{33a_1 b_1}{2\Psi_1} + \frac{1}{\Psi_1} \left\{ \left[ 2Q_0 U_0 + 4Q \Psi_2^2 + \left( C - \frac{2\lambda}{3} + \frac{a_1 b_1}{2\Psi_1} \right) Q_0 \right] P_1 \right. \]

\[ - \left[ 2P_0 U_0 + 4P \Psi_1^2 + \left( C - \frac{2\lambda}{3} + \frac{a_1 b_1}{2\Psi_1} \right) P_0 \right] Q_1 \right. \]

\[ - \left[ 10\Psi_1 U_0 + 5 \left( C - \frac{4\lambda}{3} \right) \Psi_1 + \frac{9}{2} a_1 b_1 \right] V_1 \]

\[ - \frac{c_2 + c_3}{2a_1} b_1 \Psi_1 U_{10} + \frac{c_2 - c_3}{2a_1} b_1 Q_0 P_0 - 4U_1 \Psi_1^2 \right\}, \]  
(33n)

\[ U'_1 = \frac{V'_2}{\Psi_1 \Psi_1} \left( \frac{U_0 - a_1 b_1}{2} - \frac{\Psi_2^2}{2a_1^2} \right) + \frac{\lambda}{12} \left[ \frac{405\Psi_2^2}{16\Psi_1^3} + \frac{425}{8} U_0 - \frac{205}{6} \Lambda - \frac{2\Psi_1^2}{a_1^2} \right. \]

\[ + \frac{405}{16} C - \frac{207}{8} \Psi_1 a_1 b_1 \right] - \frac{b_1}{a_1} (c_2 + c_3) \right] \]  
(33o)
This reduction system can be readily solved because (33b) is only an autonomous second order ordinary differential equation by using (33g).

\[ \psi''_1 = (C - \lambda) \psi_1 - b_1 + \frac{3}{2} \psi_1^2 + 2 \psi_1, \quad \psi_1 \equiv a_1 \Psi_1. \]  

(34)

The general solution of (34) can be expressed by the following incomplete elliptic integral

\[ \int \frac{d\psi_1}{\sqrt{4\psi_1^2 + C_0 \psi_1^2 - 2(C - \lambda) \psi_1^2 + b_1 \psi_1}} = \eta - \eta_0. \]  

(35)

We do not discuss more about the solution of (33) here but come back in the next section.

**VI. GENERALIZED TANH FUNCTION EXPANSION METHOD OF BSKDV-2 SYSTEM**

**A. Tanh Function Expansion Method**

According to the first four equations of the symmetry reductions of (30) and (33), we can develop a simple method, the tanh function expansion method, to get more general solutions of the BSKdV-2 system. The usual tanh function expansion method is used only to find single traveling soliton or solitary wave solutions of nonlinear systems. The result of the last section implies that the tanh function expansion method can be extended to get many more exact solutions.

The generalized expansion solution for the both reductions (30) and (33) has the form

\[ u_0 = u_{00} + u_{01} \tanh(w) + u_{02} \tanh^2(w), \]  

(36a)

\[ p = p_0 + p_1 \tanh(w) + p_2 \tanh^2(w), \]  

(36b)

\[ q = q_0 + q_1 \tanh(w) + q_2 \tanh^2(w), \]  

(36c)

\[ u_1 = u_{10} + u_{11} \tanh(w) + u_{12} \tanh^2(w) + u_{13} \tanh^3(w), \]  

(36d)

where \( w, u_{00}, u_{01}, u_{02}, u_{10}, u_{11}, u_{12}, u_{13}, p_0, p_1, p_2, q_0, q_1, \) and \( q_2 \) are functions of \( \{x, t\} \) and should be determined later.

After some direct calculations by substituting (36) into the BSKdV-2 system (7), we can prove the following nonauto-Bäcklund transformation theorem:

**Theorem 3 (Nonauto-BT theorem).** If \([w, f, g, h]\) is a solution of

\[ w_t + w_{xxx} - \frac{3}{2} w_x w_{x}^{-1} - 2 w_x^3 + \lambda w_x = 0, \]  

(37a)

\[ f_t + f_{xxx} - \lambda f_x - \frac{3}{2} f_x w_x w_{x}^{-1} = 0, \]  

(37b)

\[ g_t + g_{xxx} - \lambda g_x - \frac{3}{2} g_x w_x w_{x}^{-1} = 0, \]  

(37c)

\[ h_t + h_{xxx} - 2 \lambda h_x + \lambda_1 + (\lambda f - 3 f_{xx}) g_x - (\lambda g - 3 g_{xx}) f_x - \frac{3}{2} (2 h_x + g f_x - f g_x) w_x w_{x}^{-1} = 0, \]  

(37d)

then \([u_0, p, q, u_1]\) with

\[ u_0 = -2 w_x^2 \tanh^2(w) + 2 w_{xx} \tanh(w) + \frac{1}{2} w_x w_x^{-1} - \frac{1}{2} w_x^2 w_x^{-2} + \frac{2}{3} \lambda, \]  

(38a)

\[ p = f w_x^2 \tanh^2(w) - (f w_x)_x \tanh(w) + \frac{1}{2} f_x w_x w_x^{-1} + \frac{f}{4} w_x^2 w_x^{-2} - \frac{f}{4} w_x w_x^{-1} - \frac{f}{3} \lambda + \frac{1}{2} f_{xx}, \]  

(38b)
\[ q = gw^2 \tanh^2(w) - (gw_a) \tanh(w) + \frac{1}{2} g_w w_x w_x^{-1} + \frac{g}{4} w_x^2 w_x^{-2} - \frac{g}{4} w_x w_x^{-1} - \frac{g}{4} + \frac{1}{2} g_{xx}, \]  
\[ (38c) \]

\[ u_1 = hw^3 \tanh^2(w) - \frac{w_x}{2} (2h_x w_x + 3hw_{xx}) \tanh^2(w) + \frac{1}{4} (4w_x h_x + 3hw_x^2 w_x^{-1} - 2hw_x - 2h_x + 2w_x h_x) \tanh(w) \]
\[ + \frac{\lambda_1 - \frac{w_x}{8w_x} [w_x (h_x + g f_x - f g_x) - h_x w_x] - \frac{1}{6} f_x (\lambda g - 3g_x) + \frac{1}{6} g_x (\lambda f - 3f_x) + \frac{1}{8} (h_x + f g_x - g f_x) }{8w_x} (hw_{xx} - 2h_x w_x + 4hw_{xx}) - \frac{w_x^2}{8w_x} (hw_{xx} + 2h_x w_x), \]
\[ (38d) \]

is a solution of the BSKdV-2 system (7).

**B. Exact solutions**

At first glance, to find solutions of (37) is still difficult. However, it is interesting that we can find that some nontrivial solutions of the BSKdV-2 from some quite trivial solutions of (37). Here are some interesting examples.

**Example 1 The Single soliton.** A quite trivial straight-line solution of (37) has the form

\[ w = k x + \omega t + x_0, \quad p = k_1 x + \omega_1 t + x_1, \quad q = k_2 x + \omega_2 t + x_2, \quad h = k_3 x + \omega_3 t + x_3, \]
\[ (39a) \]

\[ \omega = k (2 k^2 - \lambda), \quad \omega_1 = \frac{k_1}{2} (6k^2 - \lambda), \quad \omega_2 = \frac{k_2}{2} (6k^2 - \lambda), \quad \omega_3 = \frac{2 k_3 + k_1 x_2 - k_2 x_2 (6k^2 - \lambda) - \lambda_1}{2}, \]
\[ (39b) \]

where all the free constants \( k, k_1, k_2, k_3, \lambda, \lambda_1, x_0, x_1, x_2 \) and \( x_3 \) can all be defined on not only the usual c-number algebra but also the even Grassmann algebra \( G_c \). Substituting the line solution (40) into the nonauto-BT theorem yields the following soliton solution of the BSKdV and then the SKdV system

\[ u_0 = 2 k^2 \sech^2(w) - k^2 - \frac{\lambda}{6}, \quad w \equiv k x + (2k^3 - k\lambda) t + x_0, \quad h \equiv k_3 x + \omega_3 t + x_3, \]
\[ (40a) \]

\[ p = f ( k^2 \tanh^2(w) - \frac{k^2}{2} + \frac{7\lambda}{12} ) - k k_1 \tanh(w), \quad f \equiv k_1 x + \omega_1 t + x_1, \]
\[ (40b) \]

\[ q = g ( k^2 \tanh^2(w) - \frac{k^2}{2} + \frac{7\lambda}{12} ) - k k_2 \tanh(w), \quad g \equiv k_2 x + \omega_2 t + x_2, \]
\[ (40c) \]

\[ u_1 = - k k^3 \sech^2(w) \tanh(w) + \frac{1}{2} k^2 k^3 (2 \sech^2(w) - 1) + \frac{(30k^2 - \lambda)(2k f - k_1 g)}{48} + \frac{\lambda_1}{24} - \frac{(6k^2 - \lambda)(k_2 x_1 - k_1 x_2)}{16}. \]
\[ (40d) \]

Though the soliton solution (40) is a traveling wave in the \((x, t)\) space-time for the boson field \( u_0 \), it is not a traveling wave for other boson fields \( p, q \) and \( u_1 \), and then the superfield \( \Phi \) of SkdV is not a traveling wave except for the case of \( f, g, \) and \( h \) being constants, i.e., \( k_1 = k_2 = k_3 = \lambda_1 = 0 \).

This example reveals that the nonauto-BT theorem is an straightforward transformation of the BSKdV-2 which straightens the single soliton to a straight-line solution.

**Example 2 Painlevé II extensions.** It is known that for the usual KdV system, there exists a Painlevé II reduction if one uses the scaling symmetry. In the same way, applying the scaling invariance (and space time translations) to the field \( w \), we have the scaling group invariant solution

\[ w = c_1 \ln(t - t_0) + \Psi(\xi), \quad \xi \equiv \frac{x - x_0}{(t - t_0)^{2/3}}, \]
\[ (41) \]

with the equivalent Painlevé II reduction

\[ \Psi'' = \frac{3}{2} \Psi^2 \Psi^{-1} + 3 \Psi' + \frac{1}{3} \xi \Psi' - c_1, \]
\[ (42) \]

where \( c_1, \ x_0 \) and \( t_0 \) are constants defined on \( G_c \).

To find the corresponding solution for other fields \( f, g \) and \( h \), we have to solve

\[ (t - t_0) f_{tt} + f_{t\xi\xi} + \frac{1}{6} \Psi^{-1} f_{\xi}(\xi \Psi_1 - 9c_1) = 0, \quad f \equiv f_1(\xi, t), \quad g \equiv g_1(\xi, t). \]
\[ (43a) \]
The solution of the BSKdV-2 system (7) and then the SKdV equation (4) can be obtained from the nonauto-BT theorem 3,

\[
(t - t_0)g_{1t} + g_{1zzzz} + \frac{1}{6}\Psi_1^{-1}g_{1t}(\xi\Psi_1 - 9c_1) = 0, \quad h = h_1(\xi, t) - \lambda_1(t - t_0),
\]

(43b)

\[
(t - t_0)h_{1t} + h_{1zzzz} + \frac{1}{3}\Psi_1^{-1}h_{1t}(2\xi\Psi_1 - 9c_1) + \frac{1}{2}f_1[6g_{1zzzz} + g_1\Psi_1^{-1}(\xi\Psi_1 - 3c_1)] - \frac{1}{2}g_{1t}[6f_{1zzzz} + f_1\Psi_1^{-1}(\xi\Psi_1 - 3c_1)] = 0.
\]

(43c)

A special solution of (43) has the form

\[
f_1 = \sum_{n=1}^{N} P_n(\xi)(t - t_0)^{\alpha_n}, \quad g_1 = \sum_{m=1}^{M} Q_m(\xi)(t - t_0)^{\beta_m},
\]

\[
h_1 = \sum_{n=1}^{N} \sum_{m=1}^{M} U_{nm}(\xi)(t - t_0)^{\alpha_n+\beta_m},
\]

(44a)

\[
P'''' = -\alpha_n P_n - \frac{P_n}{6\Psi_1^2}(\xi\Psi_1 - 9c_1),
\]

(44b)

\[
Q'''' = -\beta_n Q_m - \frac{Q_m}{6\Psi_1^2}(\xi\Psi_1 - 9c_1),
\]

(44c)

\[
U'''' = -(\alpha_n + \beta_m)U_{nm} - \frac{U_{nm}}{\Psi_1^2}(2\xi\Psi_1 - 9c_1) - \frac{P_n}{2}[6Q_m^{''} - Q_m(\xi\Psi_1 - 3c_1)] + \frac{Q_m}{2}[6P_n^{''} - P_n(\xi - 3c_1)]
\]

(44d)

where \( N \) and \( M \) are arbitrary integers and \( \alpha_n, n = 1, 2, \cdots, N \) and \( \beta_m, m = 1, 2, \cdots, M \) are arbitrary constants defined on the Grassmann algebra.

The solution of the BSKdV-2 system (7) and then the SKdV equation (4) can be obtained from the nonauto-BT theorem 3,

\[
u_0 = \left(\Psi_1^2 - \frac{\sqrt{\Psi_1}}{\sqrt{\Psi_1}} + 2\Psi_1\tanh(w) - 2\Psi_1^2 \tanh^2(w)\right)(t - t_0)^{-2/3}.
\]

(45a)

\[
p = \left(-\Psi_1 f_1 \text{sech}^2(w) - \Psi_1 f_1 \tanh(w) + \frac{1}{2}[\Psi_1^{-1}(\Psi_1 f_1)_{\xi}]_\xi - \frac{f_1}{12\Psi_1}(\xi\Psi_1 - 3c_1)\right)(t - t_0)^{-2/3}.
\]

(45b)

\[
q = \left(-\Psi_1 g_1 \text{sech}^2(w) - \Psi_1 g_1 \tanh(w) + \frac{1}{2}[\Psi_1^{-1}(\Psi_1 g_1)_{\xi}]_\xi - \frac{g_1}{12\Psi_1}(\xi\Psi_1 - 3c_1)\right)(t - t_0)^{-2/3}.
\]

(45c)

\[
u_1 = \Psi_1^2 \left(\frac{h_1}{t - t_0} - \lambda_1\right) \tanh^2(w) + \Psi_1 \left(3\lambda_1 \Psi_1 \Psi_1' \right) \left\{ \frac{3h_1 \Psi_1' + 2\Psi_1 h_1'\xi}{t - t_0} \right\} \tanh^2(w) + \frac{h_1}{12} \frac{9\Psi_1^2 \xi + 2\Psi_1 (\xi\Psi_1 - 3c_1)}{t - t_0}
\]

\[
+ \frac{2\Psi_1 h_1\xi + \Psi_1 h_{1\xi\xi}}{2(t - t_0)} - \frac{\alpha_1}{12\Psi_1^2} \left(9\Psi_1^2 \xi + 2\Psi_1 (\xi\Psi_1 - 3c_1)\right) \tanh(w) + \frac{\alpha_1}{24\Psi_1^2} \left(2\Psi_1^2 - 3c_1 \Psi_1 \Psi_1' + 2\Psi_1^2 \Psi_1' + 3\Psi_1^3 \right)
\]

\[
+ \frac{g_1}{8} \left( f_1 \left(2\Psi_1^2 \Psi_1 - 3c_1\right) \right) - f_1 \left( \frac{g_1^2}{3\Psi_1'} \right) - \frac{g_1}{8} \left( \frac{g_1^2 (2\Psi_1^2 - 3c_1)}{3\Psi_1'} \right) - \frac{h_1 (3\Psi_1^2 \xi + 3c_1 \Psi_1 \Psi_1' + 2\Psi_1^2 \Psi_1')}{24\Psi_1^2 (t - t_0)}
\]

\[
+ \frac{2(\xi\Psi_1') \left( f_1 \left(2\Psi_1^2 \Psi_1 - 3c_1\right) \right) - f_1 \left( \frac{g_1^2}{3\Psi_1'} \right) - \frac{g_1}{8} \left( \frac{g_1^2 (2\Psi_1^2 - 3c_1)}{3\Psi_1'} \right) - \frac{h_1 (3\Psi_1^2 \xi + 3c_1 \Psi_1 \Psi_1' + 2\Psi_1^2 \Psi_1')}{24\Psi_1^2 (t - t_0)} - 1
\]

(45d)

It is clear that the symmetry reduction (29) of the last section is only equivalent to a special case of (45) with (44) and \( N = M = 1, \alpha_1 = \beta_1 \).

When \( N = M = 1, \alpha_1 = \beta_1, h_1 = 0 \) and \( f_1 \) and \( g_1 \) are constants defined on \( G_x \), the solution (45) becomes much simpler, especially for the field \( u_1 \),

\[
u_0 = \left(\Psi_1^2 - \frac{\sqrt{\Psi_1}}{\sqrt{\Psi_1}} + 2\Psi_1\tanh(w) - 2\Psi_1^2 \tanh^2(w)\right)(t - t_0)^{-2/3},
\]

(46a)

\[
p = f_1 \left[-\Psi_1 \text{sech}^2(w) - \Psi_1 \tanh(w) + \frac{1}{2}(\Psi_1^{-1})' - \frac{1}{12\Psi_1}(\xi\Psi_1 - 3c_1)\right](t - t_0)^{-2/3},
\]

(46b)
\[ q = g_1 \left[ -\Psi_1^2 \text{sech}^2(w) - \Psi_1' \tanh(w) + \frac{1}{2} \left( \Psi_1^{-1} \Psi_1' \right)' - \frac{1}{12 \Psi_1} (\xi \Psi_1 - 3 \xi_1) \right] (t - t_0)^{-2/3}, \] (46c)

\[ u_1 = \lambda_1 \left\{ -\Psi_1 \tanh(w) + \frac{3}{2} \Psi_1^2 \tanh^2(w) - \frac{1}{12 \Psi_1} \left[ 9\Psi_1^2 + 2 \Psi_1 (\xi \Psi_1 - 3 \xi_1) \right] \tanh(w) \right. \]
\[ \left. + \frac{1}{24 \Psi_1} \left( 2 \Psi_1^3 - 3 \xi_1 \Psi_1' + 2 \xi \Psi_1^2 \Psi_1' + 3 \Psi_1^3 \right) \right\}. \] (46d)

**Example 3 Soliton-Cnoidal waves.** In nonlinear systems, the solitons (or solitary waves) and the periodic cnoidal waves are two types of typical excitations. To find the interaction solutions between solitons and cnoidal periodic waves is quite difficult. However, it is quite simple to find the soliton-cnoidal interaction solutions by using the method proposed in this section, because one of the solitons has been straightened for the \( w \) field. Thus we can look for the solutions with one straight line \((k_1 k_0 x - \omega_0 t)\) plus an undetermined traveling wave \((\Psi(k_1 x - \omega_0 t))\) for the \( w \) field

\[ w = k_1 k_0 x - \omega_0 t + \Psi(\eta), \quad \eta \equiv k_1 x - \omega_0 t. \] (47)

Substituting (47) into the nonauto-BT, we have

\[ \Psi_1 \Psi_1'' - \frac{3}{2} \Psi_1^2 - 2 \Psi_1^4 + \frac{1}{2} \gamma_2 \Psi_1^2 + \gamma_1 \Psi_1, \quad \omega_1 = k_1 \lambda - \frac{1}{2} k_1^2 \gamma_2, \quad \omega_0 = -k_1^3 \gamma_1 + k_1 k_0 \lambda - \frac{1}{2} \gamma_2 k_0 k_1^3, \] (48a)

\[ \Psi_1 \left( k_1^3 f_2 t + f_2 qq \right) - \frac{1}{4} f_2 \left( \left( \gamma_2 + 2 \lambda k_1^2 \right) \Psi_1 - 6 \gamma_1 \right) = 0, \quad f \equiv f_2(q, t), \quad g \equiv g_2(q, t), \] (48b)

\[ \Psi_1 \left( k_1^3 g_2 t + g_2 qq \right) - \frac{1}{4} g_2 \left( \left( \gamma_2 + 2 \lambda k_1^2 \right) \Psi_1 - 6 \gamma_1 \right) = 0, \quad h \equiv h_2(q, t) - \lambda \lambda t, \] (48c)

\[ \Psi_1(k_1^3 h_2 + h_2 qq) - h_2(\gamma \Psi_1 + 3 \gamma) + \frac{1}{4} f_2 \left[ 12 \Psi_1 g_2 qq + g_2 \left( (2 \lambda k_1^2 - 3 \gamma_2) \Psi_1 - 6 \gamma_1 \right) \right] \]
\[ + \frac{1}{4} g_2 \left[ 12 \Psi_1 f_2 qq + f_2 \left( (2 \lambda k_1^2 - 3 \gamma_2) \Psi_1 - 6 \gamma_1 \right) \right] = 0, \] (48d)

with arbitrary constants \( \lambda, k_1, k_0, \gamma_1 \) and \( \gamma_2 \) defined on the \( G \) algebra.

The general solution of (48a) reads

\[ \int \frac{d \Psi_1}{W(\Psi_1)} = \eta - \eta_0, \quad W \equiv W(\Psi_1) = \sqrt{4 \Psi_1^4 + \gamma_3 \Psi_1^3 + \gamma_2 \Psi_1^2 + \gamma_1 \Psi_1} \] (49)

with two further arbitrary constants \( \gamma_3 \) and \( \eta_0 \).

If the parameters \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) are rewritten as

\[ \gamma_1 = 4 w_0 \mu^2 v, \quad \gamma_2 = 4 \mu^2 (3 v - 1 - m^2), \quad \gamma_3 = -\frac{4 \mu [2 v (1 + m^2) - 3 v^2 - m^2]}{v w_0}, \quad w_0 = \pm \sqrt{\frac{(m^2 - v)(1 - v)}{v}}, \] (50)

with arbitrary constants \( \mu, v \) and \( m \) defined on \( G \), the solution (49) can be expressed by the Jacobi elliptic functions, say,

\[ \Psi_1 = \frac{\mu w_0}{1 - v \sin^2(\mu \eta, m)} - w_0 \mu, \quad (k_0 = -w_0 \mu), \] (51)

and then

\[ \Psi = w_0 E_\pi (\sin^2(\mu \eta, m), v, m), \] (52)

where the function \( E_\pi(\zeta, v, m) \) is the third kind of incomplete elliptic integral

\[ E_\pi(\zeta, v, m) = \int_0^\zeta \frac{dt}{(1 - v t^2) \sqrt{(1 - t)(1 - m^2 t)}} \] (53)
Some types of special solutions of (53) for the fields $f_2$, $g_2$ and $h_2$ can be obtained by the variable separation approach and the linear superpositions:

$$f_2 = \sum_{n=1}^{N} P_n(\Psi_1)e^{\alpha_n t}, \quad g_2 = \sum_{m=1}^{M} Q_m(\Psi_1)e^{\beta_m t}, \quad h_2 = \sum_{n=1}^{N} \sum_{m=1}^{M} U_{nm}(\Psi_1)e^{(\alpha_n + \beta_m) t},$$

(54a)

$$\Psi_1 W^2 P_n'' + 3 \Psi_1 WW' P_n'' + \left(24 \Psi_1^3 + 3 \gamma_3 \Psi_1^2 + \frac{3}{4} \gamma_2 \Psi_1 - \frac{\lambda \Psi_1}{2k_1} - \frac{3 \gamma_1}{2} \right) P_n' + \frac{\alpha_n \Psi_1 P_n}{k_1^3 W},$$

(54b)

$$\Psi_1 W^2 Q_m'' + 3 \Psi_1 WW' Q_m'' + \left(24 \Psi_1^3 + 3 \gamma_3 \Psi_1^2 + \frac{3}{4} \gamma_2 \Psi_1 - \frac{\lambda \Psi_1}{2k_1} - \frac{3 \gamma_1}{2} \right) Q_m' + \frac{\beta_m \Psi_1 Q_m}{k_1^3 W},$$

(54c)

$$\Psi_1 W^2 U_m'' + 3 \Psi_1 WW' U_m'' + \left[3 \Psi_1 W^2 - 2 \Psi_1 W W' - \gamma_2 \right] U_m' + \frac{(\alpha_n + \beta_m) \Psi_1 U_m}{k_1^3 W}$$

$$+ \left(24 \Psi_1^3 + 3 \gamma_3 \Psi_1^2 + \frac{3}{4} \gamma_2 \Psi_1 - \frac{\lambda \Psi_1}{2k_1} - \frac{3 \gamma_1}{2} \right) \frac{\lambda}{2} \Psi_1 + \frac{(\alpha_n + \beta_m) \Psi_1 U_m}{k_1^3 W} \right) P = 0,$$

(54d)

with new arbitrary constants $\alpha_n, n = 1, 2, \cdots, N$ and $\beta_m, m = 1, 2, \cdots, M$ defined on $G_e$. The corresponding solution of the BSKdV-2 system reads

$$u_0 = k_1^3 (2 \Psi_1^2 \tanh^2(w) + 2 W \tanh(w) - \Psi_1^2) + \frac{\lambda}{6} - \frac{k_1^3 W}{4 \Psi_1^2} (2 \Psi_1 W W' - W),$$

(55a)

$$p = f_2 k_1^3 \Psi_1 \tanh^2(w) - k_1^3 (f_2 \Psi_1) \tanh(w) + \frac{f_2}{24 \Psi_1^2} \left[6 k_1^3 W^2 - 6 \Psi_1 k_1^2 \gamma_1 - \Psi_1^2 (2 \lambda - 3 k_1^2 \gamma_2) \right] + \frac{k_1^3}{2} g_2 (f_2 \Psi_1) \tanh(w),$$

(55b)

$$q = g_2 k_1^3 \Psi_1 \tanh^2(w) - k_1^3 (g_2 \Psi_1) \tanh(w) + \frac{g_2}{24 \Psi_1^2} \left[6 k_1^3 W^2 - 6 \Psi_1 k_1^2 \gamma_1 - \Psi_1^2 (2 \lambda - 3 k_1^2 \gamma_2) \right] + \frac{k_1^3}{2} (g_2 \Psi_1) \tanh(w),$$

(55c)

$$u_1 = k_1^3 (h_2 - \lambda_1 t) \Psi_1 \tanh^2(w) - \frac{k_1^3}{2} \left[3 W_2 \Psi_1 + 3 W (h_2 - \lambda_1 t) \tanh^2(w) + k_1^3 \left(\frac{1}{4} (\lambda_1 \gamma_2 + 2 h_2 \gamma_1) \Psi_1 - \frac{3 \lambda_1}{4 \Psi_1} W^2 W' + W h_2 \right) \right]$$

$$+ \frac{1}{4} \lambda_1 \gamma_1 t - \frac{1}{4} \left(2 \gamma_1 + 3 \Psi_1 - \frac{3 W^2}{\Psi_1} \right) \tanh(w) + \frac{k_1^3}{2} \left[ \frac{g_2}{2} - \frac{1}{2} \left(\frac{\lambda}{3 k_1^2} - \frac{2}{2} \gamma_1 \Psi_1 \right) \right] + \frac{k_1^3}{2} \left(\frac{f_2}{2} - \frac{1}{2} \left(\frac{\lambda}{3 k_1^2} - \frac{2}{2} \gamma_1 \Psi_1 \right) \right) \tanh(w)$$

$$- \frac{1}{2} \left(\frac{\lambda}{3 k_1^2} - \frac{2}{2} \gamma_1 \Psi_1 \right) f_1 \right) \tanh(w)$$

$$- \frac{k_1^3}{8 \Psi_1^2} \left(2 W h_2 \gamma_1 + \frac{1}{4} \lambda_1 \gamma_1 \right) + \frac{1}{4} \lambda_1 + \frac{3 h_2}{24},$$

(55d)

It is clear that if we take $N = M = 1, \beta_1 = \alpha_1, \lambda_1 = 0, f_2, g_2, h_2, \text{and } \alpha_2$ are constants defined on $G_e$, then the solution (55) is reduced back to the special symmetry reductions (32)–(35) discussed in the last section.

VII. CONCLUSIONS

In summary, the simple bosonization approach proposed in the previous Letter [6] is extended such that all the bosonized fields are defined on even Grassmann algebra $G_e$. With $N$ fermionic parameters, the bosonization procedure of the supersymmetric systems has been successfully applied to the SKdV equation in detail. Such an integrable nonlinear system is simplified to the KdV equation together with several linear differential equations. Though the supersymmetric KdV is bosonized to BSKdV-N systems for arbitrary $N$, all the boson fields are different to the traditional non-supersymmetric boson systems, because they are defined on $G_e$ algebra.
The BSKdV-2 system is proved to be Painlevé integrable, which means that the model possesses Painlevé property. Starting from the standard truncated Painlevé expansion, a Bäcklund transformation with a free spectral parameter is found. Furthermore, we find that the residual of the BT, i.e., the truncated Painlevé expansion, is a nonlocal symmetry of the BSKdV-2 and the symmetry is defined as RS (residual symmetry). It is proved that the RS is just equivalent to the infinitesimal form, the generator, of the BT. For sake of the free spectral parameter in RS, infinitely many nonlocal symmetries can be found in two ways. Especially, the higher order nonlocal symmetries can be obtained simply by differentiating the RS any times with respect to the spectral parameter.

The nonlocal symmetry RS is nonlocal for the original BSKdV-2 system, however, it can be successfully localized by enlarging it to the prolonged BSKdV-2 system [21]. Thanks to the localization processing, the nonlocal symmetry RS is used to find possible symmetry reductions. Especially, the usual Painlevé II reductions and the cnoidal wave reduction solutions can be extended an additional soliton. This fact implies that starting from any seed solution of an integrable model, the BT will introduce an additional soliton to the original seed, because the RS is just the infinitesimal form of BT. To manifest the correctness of this conjecture, a much simpler method, the generalized tanh function expansion method, is proposed. The generalized tanh function expansion method for the BSKdV-2 system leads to a non-auto-BT theorem, which strengthens a single soliton to a straight-line solution. Hereafter, to add a soliton to any seed wave becomes a quite simple work: simply plus a straight line solution on a general solution in the non-auto-BT leads to the interaction solution between soliton and arbitrary other seed waves for the original BSKdV-2 system. Using the non-auto-BT theorem, various exact explicit solutions of the BSKdV-2 system are obtained.

In this paper, we only investigate the properties and exact solutions of the BSKdV-2 system, nonetheless, all the results are similar for arbitrary BSKdV-N systems. For instance, one can prove that the BSKdV-N system [10] is Painlevé integrable because of the existence of the Painlevé expansion

$$u_0 = \sum_{j=0}^{\infty} u_{0j}\phi^{-2}, \quad v_{i_1i_2\cdots i_{2N-1}} = \sum_{j=0}^{\infty} v_{i_1i_2\cdots i_{2N-1},j}\phi^{-2}, \quad u_{i_1i_2\cdots i_{2N}} = \sum_{j=0}^{\infty} u_{i_1i_2\cdots i_{2N},j}\phi^{-3},$$

where the expansion coefficients $u_{0j}$, $v_{i_1i_2\cdots i_{2N-1}}$ and $u_{i_1i_2\cdots i_{2N}}$ are determined by $3 \times 2^N$ arbitrary functions $\phi$, $u_{0j}$, $v_{i_1i_2\cdots i_{2N-1},0}$, $v_{i_1i_2\cdots i_{2N-1},2}$, $v_{i_1i_2\cdots i_{2N-1},5}$, $u_{i_1i_2\cdots i_{2N-1},0}$, $u_{i_1i_2\cdots i_{2N-1},2}$, and $u_{i_1i_2\cdots i_{2N}}$.

The bosonization approach can be applied to not only the supersymmetric integrable systems but also all the models with fermion fields no matter whether they are integrable or not. It should be emphasized that the solutions obtained via the bosonization procedure are completely different from those obtained via other methods such as the bilinear approach [7].

The results of our previous Letter [6] and this paper show that for the SkdV equation there exist various kinds of localized excitations and the interaction solution between different types of waves. For instance, the solutions [47], [55] demonstrate that for one excitation of the usual non-supersymmetric KdV model, there exist infinitely many possible excitations for SkdV system.

The richness of the soliton structure and other types of excitations of the classical SkdV reveal some open problems in both classical and quantum theories. For instance, generically the fermionic fields take value on an infinite Grassmann subalgebra, the even Grassmann algebra $G_2$. Hence, one of the important problems is how to obtain an extension to the case of infinite full Grassmann algebra. In the quantum level, one of the most important topics may be how to reflect the richness of the localized excitations in the usual quantization procedure of the supersymmetry models [12].

A. Acknowledgement

The work is sponsored by the National Natural Science Foundation of China (No. 11175092 and No. 11275123), Shanghai Knowledge Service Platform for Trustworthy Internet of Things (No. ZF1213), and K. C. Wong Magna Fund in Ningbo University.
[7] Y. C. Hon and E. G. Fan, Theoret. Mat. Fiz. 166 (2011) 366-387; E. G. Fan and Y. C. Hon, Stud. Appl. Math. 125 (2010) 343-373.
[8] Yu. I. Maniu and A. O. Radul, Commun. Math. Phys. 98 (1985) 65-67.
[9] P. Mathieu, J. Math. Phys. 28 (1988) 2499-2506.
[10] P. Mathieu, Phys. Lett. B 203 (1988) 287-291.
[11] D. J. Gross and A. Migdal, Nucl. Phys. B 34 (1990) 333; M. Dognlass, Phys. Lett. B 238 (1990) 176; R. Dijkgraaf and E. Witten, Nucl. Phys. B 342 (1990) 486.
[12] P. P. Kulish, A. M. Zeitlin, Phys. Lett. B 597 (2004) 229; Nucl. Phys. B 709 (2005) 578; Nucl. Phys. B 720 (2005) 289.
[13] P. Mathieu, Phys. Lett. A 128 (1988) 169.
[14] W. Oevel and Z. Popowicz, Commun. Math. Phys. 139 (1991) 441-460.
[15] J. M. Figueroa-OFarrill, J. Mas, Rev. Math. Phys. 3 (1993) 479.
[16] Q. P. Liu, Lett. Math. Phys. 35 (1995) 115.
[17] A. S. Carstea, Nonlinearity 13 (2000) 1645-1656.
[18] A. S. Carstea, A. Ramani and B. Grammaticos, Nonlinearity 14 (2001) 1419-1423.
[19] Q. P. Liu and Y. F. Xie, Phys. Lett. A 325 (2004) 139.
[20] Q. P. Liu and X. B. Hu, J. Phys. A: Math. Gen. 38 (2005) 6371-6378.
[21] S. Andrea, A. Restuccia and A. Sotomayor, J. Math. Phys. 46, 103517 (2005).
[22] Q. P. Liu, M. Manas, Phys. Lett. B 396 (1997) 133-140.
[23] S. Y. Lou and X. B. Hu, Chin. Phys. Lett. 10 (1993) 577; S. Y. Lou and X. B. Hu, J. Math. Phys. 38 (1997) 6401; S. Y. Lou and X. B. Hu, J. Phys. A. 30 (1997) L95.
[24] X. P. Cheng, C. L. Chen and S. Y. Lou, Interactions among different types of nonlinear waves described by the Kadomtsev-Petviashvili Equation arXiv:1208.3259.
[25] S. Y. Lou, X. R. Hu and Y. Chen, arXiv:1201.3409 (2012); J. Phys. A: Math. Theor. 45 (2012) 155209; X. R. Hu, S. Y. Lou and Y. Chen, Phys. Rev. E 85 (2012) 056607.
[26] S. Y. Lou, X. R. Hu and Y. Chen, Interactions Between Solitons and Other Nonlinear Schrödinger Waves arXiv:1208.5314.
[27] X. Y. Lou, J. Phys. A: Math. Phys. 30 (1997) 4803.
[28] C. W. Cao, Henan Sci. 5 (1987) 1; Chin. Q. J. Math. 3 (1988) 90; Sci. China A 33 (1990) 528; C. W. Cao and X. G. Geng, J. Phys. A 23 (1990) 4117; J. Math. Phys. 32 (1991) 2323.
[29] S. Y. Lou, Physica Scripta 57 (1998) 481; X. B. Hu, S. Y. Lou and X. M. Qian, Stud. Appl. Math. 122 (2009) 305.