Generalized holonomy in String-corrected Spacetimes

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Abstract. The quartic-curvature corrections derived from string theory have a specific impact on the geometry of target-space manifolds of special holonomy. In the cases of Calabi-Yau manifolds, $D = 7$ manifolds of $G_2$ holonomy and $D = 8$ manifolds of $\text{Spin}_7$ holonomy, string theory $\alpha'$ corrections conspire to preserve the unbroken supersymmetry of these backgrounds despite the fact that the $\alpha'$ corrections cause the Riemannian holonomy to lose its special character. We show how this supersymmetry preservation is expressed in the language of generalized holonomy for the Killing spinor operator.

1. Introduction
The effective action for the massless modes of a given string theory consists at the lowest order of the action for the corresponding supergravity theory, which is then modified by an infinite sequence of higher derivative corrections. These correction terms occur individually with finite coefficients but are of similar structure to those that occur with divergent coefficients in the corresponding quantized supergravity [1, 2, 3]. In this sense, the string theory may be viewed as a regularization of the corresponding supergravity. The string tension $\alpha'$ plays the rôle of the dimensionful cutoff parameter and gives the scale at which the microscopic physics of string theory begins to take over from the effective field theory of the massless modes. At the $(\alpha')^3$ level in type II theories, one encounters corrections quartic in Riemann curvatures; these are the first such corrections whose variations do not vanish subject to the leading order effective supergravity field equations, so they play a particularly important rôle in the onset of string-theory effects. This subject has had a revival of interest as appreciation has grown of the importance of non-compact Calabi-Yau spaces or $G_2$ manifolds as the underlying Ricci-flat geometries of brane spacetimes. In the compact cases, the relation between the quartic curvature corrections and Calabi-Yau compactifications has been studied in Ref. [4], while the non-compact case has been studied in [5].

This review, based on Refs [6, 7, 8, 9], summarizes the geometrical impact that the quartic corrections have on Calabi-Yau, $G_2$ and $\text{Spin}_7$ holonomy manifolds and on the preservation of supersymmetry in the face of the $\alpha'$ corrections, extending the earlier analysis given in Refs [10, 11]. It then proceeds to show how this supersymmetry preservation is expressed in the language of generalized structure groups and generalized holonomy for the $\alpha'$-corrected Killing spinor operator.
2. Quartic curvature corrections

The leading-order effective action for the massless modes of string theory always contains a bosonic NS sector, which when written in string frame takes the form

\[
I = \int d^{10}x \sqrt{-g} e^{-2\phi} \left( -R - \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} \right).
\] (1)

The string tree-level effective action can be derived either from string scattering calculations or from NSR formulation \(\sigma\)-model beta-functions. The leading-order effective action (1) acquires corrections in the \(\alpha'\) expansion. At \(\mathcal{O}(\alpha')\) in the heterotic string, for example, one has corrections

\[
\Delta_1 I = \alpha' \int \frac{1}{4} (F_{\mu \nu}^a F^{\mu \nu a} - R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}) .
\] (2)

For the type II strings, however, such \(\mathcal{O}(\alpha')\) corrections are absent. One finds the first important corrections instead at \(\mathcal{O}(\alpha'^3)\), where one encounters the first corrections whose variations do not vanish subject to the leading-order field equations that follow from (1). These corrections to the effective action start out quartic in curvatures, then are completed by superpartner terms. The full structure is difficult to express, owing to the absence of an off-shell superspace or tensor calculus formulation for the 32-supercharge type II theories. The leading structure in curvatures was found through string graviton scattering calculations \([12]\). Partial results on the supersymmetrization from a component-field viewpoint are given in Refs \([13, 14, 15]\), while an on-shell superspace attack on the problem (subjecting the invariant to the lowest-order field equations) is given in Ref. \([16]\). So far, unfortunately, these two directions of digging at the problem have not yet met, so the full structure of the quartic corrections still remains rather shrouded.

Nonetheless, string theory itself gives an important part of the structure through the analysis of graviton and dilaton scattering. An expression for the purely gravitational quartic curvature part of the \(\mathcal{O}(\alpha'^3)\) corrections was given in Ref. \([17]\). The string tree-level \(\mathcal{O}(\alpha'^3)\) correction has the form

\[
\Delta I = \xi(\alpha')^3 \int d^{10}x e^{-2\phi} Y
\] (3)

where the tree-level coefficient \(\xi\) contains a characteristic transcendental factor \(\zeta(3)\), in agreement with the corresponding four-loop \(\sigma\)-model calculation \([18, 19]\). The structure of this counterterm and its consistency with supersymmetry was also discussed in Ref. \([20]\). In Ref. \([17]\), the integrand \(Y\) was given in light-cone gauge in a Grassmann integral form,

\[
Y = \int d^8\bar{\psi}_L d^8\psi_R \exp(R_{ijkl} \bar{\psi}_L \Gamma^{ij} \psi_L \bar{\psi}_R \Gamma^{kl} \psi_R).
\] (4)

Expanding the exponential and performing the Berezin integrations, the only terms that survive are quartic in curvatures, since each Berezin \(\psi\) integration must have one and only one corresponding \(\psi\) appearing in the integrand. Varying the correction term (3), one finds the corrections to the effective field equations. The purely gravitational \(\mathcal{O}(\alpha'^3)\) correction (4) will also be accompanied by a host of other terms involving the various fields of maximal string theories, both bosonic and fermionic, but the full structure remains unknown. Of particular interest would be to know the structure of terms involving the various antisymmetric tensor form-fields of the theory, including the 4-form \(G_{\mu \nu \rho \sigma}\) descending from \(D = 11\) M-theory. For the present, however, we will be content to consider corrections to purely gravitational configurations where the form fields are taken to vanish, at least initially (the \(\alpha'\) corrections can force them to be turned on, but this will happen at higher orders than we consider here).
Simplifying the Einstein equation by combining its trace with the dilaton equation, one finds the system
\begin{align}
R_{ij} + 2\nabla_i \nabla_j \phi - c (\alpha')^3 X_{ij} &= 0 \tag{5} \\
R + 4\nabla^2 \phi - 4(\partial \phi)^2 - c (\alpha')^3 Y &= 0, \tag{6}
\end{align}
where $X_{ij} = \delta Y / \delta g^{ij}$.

From the form of the quartic correction (4), one sees immediately that $Y = 0$ for supersymmetric spacetimes: expand the fermionic integration variables in a basis including the Killing spinors on the manifold and use this to recognize that the corresponding zero eigenvalues of $R_{ijkl} \Gamma^{kl}$ cause the Grassmann integral in (4) to vanish because some Grassmann integrations lack the corresponding integrands. This, however, does not kill off all the effects of this correction in the theory, as was pointed out in Ref. [21]. The variation of (3) does not in general vanish, since a varied curvature is not restricted to have zero eigenvalues. Nonetheless, the vanishing of $Y$ simplifies the discussion considerably because it removes an intrinsic $\mathcal{O}(\alpha')^3$ correction to the dilaton effective equation.

Varying $Y$ with respect to the metric to obtain the corrections to the Einstein equation, one finds in all cases field-equation corrections of the form
\begin{equation}
X_{ij} = \nabla^k \nabla^\ell X_{dij\ell} \tag{7}
\end{equation}
for some expression $X_{sk\ell}$ cubic in curvatures which is antisymmetric in the first pair and in the last pair of indices and symmetric under interchange of first and last pairs. Moreover, for the manifolds of initial special holonomy that we shall be considering, we also find that the single trace of $X_{ijkl}$ is automatically a double trace, $g^{ik} X_{ijkl} = g^{jl} Z$.

Since we consider solutions with initially constant dilaton $\phi$, we may write $\phi = \text{const} + \phi_1$, where $\phi_1$ is the $\mathcal{O}(\alpha')^3$ correction. Combining the corrected dilaton equation with the trace of the corrected Einstein equation, one finds
\begin{equation}
2\Box \phi_1 + (\alpha')^3 X = 0 \tag{8}
\end{equation}
where $X = g^{ij} X_{ij}$. Accordingly, since $X = \Box Z$, we can solve for $\phi_1$:
\begin{equation}
\phi_1 = -\frac{1}{2} (\alpha')^3 Z. \tag{9}
\end{equation}

Substituting the solution (9) into the corrected Einstein equation, one thus obtains a purely geometrical general form for the corrected Einstein equation,
\begin{equation}
R_{ij} = (\alpha')^3 \left( \nabla_i \nabla_j Z + \nabla^k \nabla^\ell X_{ij\ell} \right). \tag{10}
\end{equation}
The general form of $X_{ijkl}$ can be written out explicitly by varying the correction $Y$, but this general form is rather complicated and not very enlightening. It will take particular forms when evaluated in the various special holonomy cases that we consider, in which the various special invariant tensors that exist on the different manifolds will appear.

### 2.1. Kähler manifolds

For initially Ricci-flat Kähler spaces with a constant dilaton, the $\mathcal{O}(\alpha')^3$ $Y$ variation $X_{ij}$ boils down to a simple form [10, 11]
\begin{align}
X_{ij} &= \nabla_i \nabla_j S_3 \tag{11} \\
Z &= S_3 = R_{ij}^{kl} R_{k\ell mn} R_{mn ij} - 2 R_{i \ell} R_{k\ell} R_{m n} R_{n i} \tag{12}
\end{align}
where the ‘hat’ notation indicates contraction with the complex structure $J_{ij}$: $V^i = J_{ij} V_j$. The complex structure is the invariant tensor possessed by a Kähler manifold that appears in the simplification of (10) in the Kähler case. In terms of the uncorrected Killing spinor $\eta$ on a Kähler manifold, the complex structure can be written $J_{ij} = -i \bar{\eta} \Gamma_{ij} \eta$.

In the Kähler case, the general form (10) of the corrected Einstein equation and the correction to the dilaton now take the specific form

\[
\begin{align*}
R_{ij} &= c (a')^3 \left( \nabla_i \nabla_j + \nabla_i \nabla_j \right) S_3 \\
\phi_1 &= -\frac{1}{2} c (a')^3 S_3.
\end{align*}
\]

On a Kähler manifold, one can use the fact that $R_{ij} = -R_{ji}$ to introduce the Ricci form $\rho = \frac{1}{2} R_{ij} dx^i \wedge dx^j$. Letting $df = \partial f dx^i$, $\bar{d}f = \partial f dx^i$ and going over to a Darboux basis of complex coordinates $(z^a, \bar{z}^{\bar{a}})$ where $\partial = \partial + \bar{\partial}$, $\bar{\partial} = i(\partial - \bar{\partial})$, the corrected Ricci condition can be written

\[
\rho = 2i c (a')^3 \partial \bar{\partial} S_3.
\]

This form of the corrected Ricci tensor for a Kähler manifold fits precisely with the requirements of the non-renormalization theorem for (2,2) supersymmetric nonlinear $\sigma$-models [22], as it shows that even though the Ricci tensor becomes nonvanishing, the corrected Ricci-form remains cohomologically trivial, since the scalar quantity $S_3$ is globally defined.

### 2.2. $G_2$ holonomy manifolds

The structure of Ricci corrections in the case of $D = 7$ manifolds of initial $G_2$ holonomy [7] works similarly to the Kähler case, except that, instead of a complex structure, the special tensor that appears in simplifying the $\mathcal{O}(a')^3$ corrections is now the invariant calibrating 3-form $c_{ijk}$ that exists for $G_2$ holonomy manifolds. There is now only one Killing spinor $\eta$, which is revealed in the decomposition of the 8-dimensional spinor of $\text{Spin}_7$, which decomposes under $G_2$ as $8 \to 7 \oplus 1$. When M-theory is reduced on a $G_2$ holonomy manifold, this Killing spinor yields $N = 1, D = 4$ surviving supersymmetry in the reduced theory, hence the high interest in such manifolds. A useful Fierz identity involving the Killing spinor $\eta$ (taken to be commuting) is

\[
\Gamma_i \eta \bar{\eta} \Gamma_i + \eta \bar{\eta} = 1.
\]

Thus, $\Gamma_i \eta \bar{\eta}$ provides a map between the spinor space orthogonal to $\eta$ and the $D = 7$ vector space. In terms of $\eta$, the invariant 3-form and its 4-form dual are given by

\[
c_{ijk} = i \bar{\eta} \Gamma_{ijk} \eta, \quad c_{ijk\ell} = \frac{1}{6} c_{ijkmn} c^{\ell mnp} = \bar{\eta} \Gamma_{ijk\ell} \eta.
\]

For $G_2$ holonomy manifolds, instead of the “hat-flipping” relations of Kähler manifolds, one has the curvature relation $R_{ij\ell} c^{\ell mn} = 2 R_{ijmn}$; using the cyclic identity, the Ricci tensor for a $G_2$ holonomy manifold thus is required to vanish. This would seem to pose difficulties for having a perturbed $G_2$ holonomy manifold that preserves the supersymmetric structure of the uncorrected manifold, but “the usual supersymmetric miracles” once again come to the rescue, as we shall see.

Starting from the Einstein and dilaton equations (5,6), one finds for $G_2$ holonomy that the corrected system has

\[
X_{ij} = c_{ikm} c_{j\ell n} \nabla^k \nabla^\ell Z^{mn},
\]

where

\[
Z^{mn} = \frac{1}{32} \epsilon^{mn1 \cdots 7} \epsilon^{n1 \cdots 7} R_{i1i2j1j2} \cdots R_{i5i6j5j6}.
\]
Note that $Z^{ij} = Z^{ji}$, and that the Bianchi identity for $R_{ijk\ell}$ implies that $\nabla_i Z^{ij} = 0$. Accordingly, the corrected Ricci tensor (10) and dilaton satisfy

$$R_{ij} = c (\alpha')^3 (\nabla_j Z^{ij} + c_{ikm} c_{j\ell n} \nabla^k \nabla^\ell Z^{mn}) \quad (20)$$

$$\phi_1 = -\frac{1}{2} c (\alpha')^3 Z \quad (21)$$

$$Z = g_{ij} Z^{ij} \quad . \quad (22)$$

This system can be compared to (13) in the Kähler case. If one specializes the $G_2$ manifold to be of the form $K_6 \times S^1$ where $K_6$ is a six-dimensional Kähler manifold, then the curvature resides entirely in the directions $i, j = 1, \ldots, 6$ and the only non-vanishing component of $Z^{mn}$ is $Z^{77} = S$, while the non-vanishing components of the invariant 3-form provide the complex structure: $c_{ij7} = J_{ij}$.

3. Spin$_7$ Manifolds

In the case of $D = 8$ manifolds of initial Spin$_7$ holonomy, the special tensor that appears in simplifying the form of the $O(\alpha')^3$ corrections is now the calibrating 4-form on the Spin$_7$ manifold, which can be written in terms of the lowest-order Killing spinor as

$$c_{ijk\ell} = \bar{\eta} \Gamma_{ijk\ell} \eta . \quad (23)$$

The corrected system now has

$$X_{ij} = \frac{1}{2} \epsilon^{mnk}_{(i} \epsilon^{pq\ell}_{j)} \nabla_k \nabla_\ell Z_{mnpq} + \nabla^k \nabla^\ell Z_{mnk(1} \epsilon^{mn\ell)} \quad (24)$$

where

$$Z_{mnpq} = \frac{1}{64} \epsilon^{mni\ldots i_6} \epsilon^{pqj\ldots j_6} R_{i_1i_2j_1j_2} R_{i_3i_4j_3j_4} R_{i_5i_6j_5j_6} . \quad (25)$$

Accordingly, in $D = 8$ we have the corrected system of equations

$$R_{ij} = c (\alpha')^3 (X_{ij} + \nabla_i \nabla_j Z) \quad (26)$$

$$\phi_1 = -\frac{1}{2} c (\alpha')^3 Z \quad (27)$$

$$Z = Z^{mn} \quad . \quad (28)$$

4. Preservation of Supersymmetry

The corrected field equations, at least those corresponding to tree-level string amplitudes in light-cone gauge, can equally well be derived from supersymmetric nonlinear $\sigma$-models, as was done in the classic papers [18, 19]. Since such calculations preserve all the world-sheet supersymmetries that are linearly realized in string light-cone gauge, and since one expects to be able to relate light-cone and covariant amplitudes, one expects that Killing spinors of the leading order compactifying manifold should be preserved in the face of the $\alpha'$ corrections. The way this happens in all cases is that the corrected effective field equations can be interpreted as integrability conditions for corrected Killing spinor equations. How this occurs in practice involves specific details of the geometry of the special holonomy manifold. Another way, in principle, to derive the corrected Killing spinor equations, would be to carry out the full covariant $D = 10$ or $D = 11$ supersymmetrization of the quartic corrections and thus derive the corrected form of the spinor field supersymmetry transformations. The conditions for setting the spinor fields and their variations to zero would then give the corrected Killing equations. This subject has been partly studied in Refs [13, 14, 15], but the results are so far incomplete.

A more direct way to study the preservation of supersymmetry is to search for ways in which the corrected Einstein-dilaton equations can be interpreted as integrability conditions for a corrected Killing spinor equation.
How this can work is illustrated by the Kähler cases [10, 11]. Any Kähler manifold admits a
gauge covariantly constant spinor $\eta$ satisfying
\[ D_i \eta =: \nabla_i \eta - \frac{1}{2} \iota V_i \eta = 0 , \tag{29} \]
where $V_i$ is a vector field on the manifold satisfying $\partial_i V_j - \partial_j V_i = \rho_{ij}$, where $\rho_{ij}$ is, as above, the Ricci form. If the first Chern class vanishes, then $V_i$ is globally defined; otherwise not. The integrability condition following from (29) gives
\[ \Gamma^j [D_i, D_j] \eta = - \frac{1}{2} R^j_{ikl} \Gamma^k \eta - \frac{1}{2} i \rho_{ij} \Gamma^j \eta = 0 , \tag{30} \]
so the Killing spinor condition required to generate the corrected field equation is simply given by taking
\[ V_i = - c(\alpha')^3 \nabla i S_3 . \tag{31} \]

In the $G_2$ case, the task looks more difficult because $G_2$ holonomy implies Ricci flatness, as we have seen, so the corrected manifold cannot be any more a $G_2$ manifold (at least with respect to the original Levi-Civita connection). Nonetheless, supersymmetry and the integrability-condition structure are preserved. Let $D_i = \nabla i + Q_i$ for a proposed corrected Killing spinor condition $D_i \eta = 0$ in analogy to (29). The integrability condition $[D_i, D_j] \eta = 0$ then implies
\[ \frac{1}{4} R_{ijkl} \Gamma^{kl} \eta + Q_{ij} \eta = 0 , \tag{33} \]
where $Q_{ij} = \nabla i Q_j - \nabla j Q_i$. Using the Fierz identity (16), one finds
\[ R_{ijkl} c^{kl} m + 4 i \eta \Gamma_m Q_{ij} \eta = 0 \tag{34} \]
and $\bar{\eta} Q_{ij} \eta = 0$. Multiplying (33) by $\Gamma_i$ and using (34) yields the corrected Ricci condition $R_{ij} + 2 \bar{\eta} \Gamma_j k Q_l^k \eta = 0$. One thus finds that the corrected Killing spinor condition for $G_3$ holonomy manifolds is given by the Clifford-algebra-valued result
\[ Q_i = - \frac{1}{8} c(\alpha')^3 c_{ijk} \nabla^j Z^{kl} \Gamma_\ell . \tag{35} \]

In the case of the $D = 8$ Spin$_7$ manifolds, we find the Killing spinor operator correction
\[ Q_i = \frac{1}{4} c_{ijkl} \nabla^j Z^{kl} \Gamma_m \Gamma_{mn} . \tag{36} \]

Using corresponding Fierz identities, the integrability condition for the corrected $D = 8$ Killing spinor equation once again reproduces the corresponding corrected Einstein equation (26).

Universality of the above results, in the sense of being capable of being written without using special tensors belonging to specific special-holonomy manifolds, is demonstrated by the fact that in all the cases considered – Kähler, $G_2$ and Spin$_7$ – the corrected Killing spinor conditions $D_i \eta = 0$, with correction operators (32),(35), or (36), can also be rewritten in a $D = 10$ Lorentz-covariant form $(i,j = 0, \ldots, 9)$ [10]:
\[ \nabla_i \eta - \frac{3}{4} c(\alpha')^3 \nabla_s R_{ijkl} R^s_{TMN} R^{TJ} R_{PQ} \Gamma^{KLMNPQ} = 0 . \tag{37} \]
5. Generalized Structure Groups and Generalized Holonomy

Despite the fact that string corrections change the original special holonomy, the corrected forms of the Killing spinor equation show that supersymmetry can nonetheless be preserved. This happens because the differential operator appearing in the corrected Killing spinor equation (37) includes terms with a Gamma-matrix structure going beyond the ordinary context of Riemannian holonomy. In order to understand the impact of this generalized holonomy, we need first to understand what might be the generalized structure group into which the generalized holonomy group could be embedded.

Consider this algebraically within the D=11 M-theory Clifford algebra for \( SO(1,10) \), restricting attention, as above, to purely gravitational situations where the 4-form field \( G_{\mu\nu\rho\sigma} \) is taken to vanish. The ordinary Riemannian structure group is the \( D = 11 \) Lorentz group \( SO(1,10) \), which is generated by the 55 \( \Gamma[2] \) matrices \( \Gamma_{MN} \). Since the corrected Killing spinor operator in (37) contains also \( \Gamma[6] \) matrices, one expects the corresponding generalized structure group to have general \( \Gamma[6] \) generators as well. This adds 462 additional generators. The \( \Gamma[2] \) and the \( \Gamma[6] \) do not form a closed algebra, however. Closure dictates the inclusion also of the 11 single \( \Gamma[1] \) matrices \( \Gamma_M \). Then for the total of 528 generators \( (\Gamma[2], \Gamma[6], \Gamma[1]) \), one finds closure on the Lie algebra of the group \( Sp(32) \). If one extends consideration to more general backgrounds where the 4-form field \( G_{\mu\nu\rho\sigma} \) is also turned on, the additional Gamma matrix structures in the Killing spinor operator cause the generalized structure group to be extended to \( SL(32,\mathbb{R}) \) [23].

The generalized structure group \( Sp(32) \) for the purely gravitational context is very large, considerably larger than what is actually relevant for our present discussion where the curvature is assumed to be confined to just the transverse portion of the spacetime, \( i.e. \) where we do not consider curvature deformations of the \( d \)-dimensional Poincaré invariant subspace (which could be either a lower-dimensional Minkowski vacuum or a brane worldvolume). Restricting the indices on the \( (\Gamma[2], \Gamma[6], \Gamma[1]) \) matrices to take their values only in the \( n = D - d \) transverse dimensions, one finds more manageable generalized transverse structure groups:

| Transverse Dimension | Transverse Structure Group |
|----------------------|---------------------------|
| \( n = 6 \)          | \( SO(6) \times U(1) \)   |
| \( n = 7 \)          | \( SO(8) \)               |
| \( n = 8 \)          | \( SO(8)_+ \times SO(8)_- \) |

Within these generalized transverse structure groups, the Killing spinor operator (37) will generate corresponding generalized holonomy subgroups. As with ordinary Riemannian holonomy, a necessary condition for some supersymmetry to be preserved by a given set of background fields is that the decomposition of the representation carried by the supersymmetry parameter with respect to the generalized transverse structure group must decompose in a way that includes one or more singlets when the representation is restricted to the generalized holonomy group. The number of such singlets, taken together with the dimensionality of the spinor representation on the \( d \)-dimensional worldvolume, determines the degree of supersymmetry that is left unbroken in a given corrected background. One difference with respect to the uncorrected spaces, however, is that although a decomposition of the supersymmetry parameter representation that includes a singlet is a necessary condition for supersymmetry preservation, it is not sufficient. There can be yet higher order integrability conditions for the Killing spinor equation that need to be checked as well [24].

In order to find the generalized holonomies obtained in the various \( \alpha' \)-corrected cases, concrete examples are needed. In [9], this was done using a series of explicit non-compact special holonomy spaces of the various types taken from Refs [6, 7, 8]. For these cases, the effect of the \( \alpha' \) corrections on the curvature and hence the generalized holonomy can be explicitly calculated.
In the $n = 6$ case (the classic case of Calabi-Yau 3-folds), the leading order Riemannian geometry $SU(3)$ is enlarged to $U(3) = SU(3) \times U(1)$ because the Ricci tensor correction (13) turns on the $U(1)$ factor corresponding to the Ricci form (15). The space remains Kählerian despite this. The generalized holonomy is also $SU(3) \times U(1)$. Supersymmetry remains unbroken because the supersymmetry spinor parameter which transforms under the generalized structure group $SU(4) \times U(1)$ as a $(4, 1)$ representation decomposes under $SU(3) \times U(1)$ into $(3, 1) \oplus (1, 1)$. Since this decomposition contains a $(1, 1)$ singlet, supersymmetry is preserved. Starting from an $(N = 1)_{10}$ (i.e. 16-supercharge) theory in $D=10$, such compactifications preserve 1/4 supersymmetry, corresponding to minimal $(N = 1)_{1}$ supersymmetry in $D = 4$. In some more detail, the net effect of including the higher-order corrections is to introduce $\Gamma_{(6)}$ terms that are dual to $i \Gamma_{7}$ in the transverse space $K_{6}$, and so the leading-order $SU(3)$ special holonomy of the Calabi-Yau background gets enlarged to $SU(3) \times U(1)$. This extra $U(1)$ factor generated by $i \Gamma_{7}$ is cancelled in the Killing spinor equation, however, by a $U(1)$ contribution proportional to $J^{ij} \Gamma_{ij}$ coming from the spin connection of the deformed background. The $U(1)$ factor in the generalized holonomy is thus a mixture of the Riemannian $U(1)$ and the generalized holonomy $i \Gamma_{7}$ generator.

In the $n = 7$ case, one has an initial manifold of $G_{2}$ special holonomy. As we have seen above, the Ricci tensor corrections render this Riemannian holonomy completely general for $n = 7$ dimensions, i.e. it becomes $SO(7)$. The generalized holonomy group also turns out to be $SO(7)$. The generalized transverse structure group as shown in (39) is now, however, $SO(8)$. The supersymmetry parameter transforms in the $8$ spinor representation of $SO(8)$, and under the decomposition into $SO(7)$, this breaks up into $7 \oplus 1$ representations. Thus, despite the loss of the special Riemannian holonomy, the generalized holonomy remains special and gives rise to singlets in the decomposition. So supersymmetry can also be preserved in this case as well.

In the $n = 8$ case, one initially has a $Spin_{7}$ manifold of dimension 8. The $\alpha'$ corrections to the metric deform the ordinary Riemannian geometry into a general $SO(8)$ holonomy space. As in the $n = 7$, $G_{2}$ case, one can see this from the fact that the integrability condition for $Spin_{7}$ holonomy requires vanishing of the Ricci tensor. Nonetheless, supersymmetry manages to be preserved in the face of the string-theory corrections because of the contributions of the $\Gamma_{[6]}$ terms in the Killing spinor operator (37). The generalized transverse holonomy shown in (40) is now a more complicated structure than in the lower dimensional cases: for $n = 8$ we have $SO(8)_{+} \times SO(8)_{-}$. Making use of explicit noncompact $Spin_{7}$ manifolds and evaluating the corresponding generalized holonomy, however, shows that it has grown to $SO(8)_{+} \times Spin_{7}$. The supersymmetry parameter transforms as an $(8, 1) \oplus (1, 8)$ under the generalized transverse structure group $SO(8)_{+} \times SO(8)_{-}$. Under decomposition into the generalized holonomy group, this breaks apart into $(8, 1) \oplus (1, 7) \oplus (1, 1)$. Once again, the presence of the singlet in the generalized holonomy decomposition is the key for allowing supersymmetry to be preserved despite the ordinary Riemannian holonomy having become completely general.

It seems clear that these remarkable adjustments which allow for supersymmetry to be preserved in the face of the string corrections should have a further expression in the language of $G$-structures for the various cases. A puzzling aspect of this situation, however, is that although the generalized structure groups and generalized holonomies clearly seem to be the correct way to describe the integrability conditions for the Killing spinor operator (37), these groups are not themselves actual symmetries of the initial uncorrected supergravity equations. This would seem to point to their possible origin in extended quantum symmetries of string and M-theory beyond the classical field theory level.
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