The Yang–Baxter paradox

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Abstract
Consider the statement ‘Every Yang–Baxter integrable system is defined to be exactly-solvable’. To formalise this statement, definitions and axioms are introduced. Then, using a specific Yang–Baxter integrable bosonic system, it is shown that a paradox emerges. A generalisation for completely integrable bosonic systems is also developed.

Keywords: paradox, integrability, Yang–Baxter equation

1. Introduction
Integrable quantum systems provide many important insights into, and benchmarks for, many important physical phenomena, e.g. see [3]. These accomplishments in the application of integrable models predominantly stem from formulation through the Yang–Baxter equation [5, 6, 61], and associated exact solutions through Bethe ansatz techniques [11]. Despite this success, there is no consensus on an adequate definition for what constitutes integrability in quantum systems [14, 24, 34]. One of the difficulties in this regard relates to the notion of a ‘maximal’ set of mutually commuting operators. This is because every self-adjoint operator acting on a finite-dimensional vector space of dimension $D$ commutes with $D$ idempotent operators that project onto the elements of a basis of eigenstates. Likewise, the notion of an exact solution is imprecise. While the Bethe ansatz is universally recognised as an extremely powerful technique, it is applied in many guises besides the original co-ordinate formulation of Bethe [11]. These include algebraic [57], analytic [49], functional [27], thermodynamic [56], off-diagonal [13], double-row transfer matrix constructions [53], and by using separation of variables [52]. Moreover there are other techniques available to yield exact solutions, such as the Jordan–Wigner transform [36], those used in Kitaev-type models [33, 64], and those used in Rabi-type models [12, 31, 45, 65]. These somewhat confuse attempts to provide an unambiguous definition for what constitutes exact-solvability, and to identify its relationship to integrability. The (fermionic) Hubbard model serves as a classic example of how the defiant challenges can be. Using the co-ordinate Bethe ansatz, an exact solution was derived in

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1968 [37]. It took until 1986 for the first steps towards establishing Yang–Baxter integrability were completed, with the construction of a transfer matrix [51]. A clear understanding of the solution for the Yang–Baxter equation associated with the transfer matrix was delivered in 1995 [63]. Then, successful application of the algebraic Bethe ansatz was achieved in 1998 [43]. This was followed, in 2001, by the construction of the ladder operator to generate the conserved operators [41]. By contrast, a similar program for the Rabi model has so far not yielded any signs of Yang–Baxter integrability [4].

In the other direction, the title of Rodney Baxter’s famous book [7] reminds that there is also a distinction to be made between what is exactly-solved, and what is not yet exactly-solved but possibly exactly-solvable. It appears plausible to suggest that all Yang–Baxter integrable systems are, in principle, exactly-solvable. Some solutions are known. Others are unknown, and these serve as open problems for future research. However, the results presented below counter this perspective. For classical integrable systems that are generally analyzed by quadratures, it has been remarked that ‘integrability in classical mechanics does not require the solvability of the quadratures’ [24]. Likewise, it is argued here that Yang–Baxter integrability in quantum systems does not require exact-solvability. Without recourse to a concrete definition for what constitutes an exact solution, the result is instead deduced through inconsistencies between certain axioms and conclusions.

In section 2 the mathematical framework and notational conventions are established. This includes a brief discussion on the context of the problem via the one-dimensional Bose–Hubbard model, widely considered to not be exactly-solvable. In section 3, a specific Yang–Baxter integrable system is introduced. Then, in section 4, the main results are derived. This entails two instances where a set of axioms is prescribed to characterise exact-solvability. In both cases, the example from section 3 is used to show that defining solvability through integrability is inconsistent with the axioms. Concluding discussion is provided in section 5.

2. Preliminaries

All Hamiltonians below are formulated through sets of canonical boson operators \( B = \{ b_l, b_l^\dagger : l = 1, \ldots, L < \infty \} \) satisfying the commutation relations

\[
[b_k, b_l^\dagger] = \delta_{k,l} I, \\
[b_k, b_l] = [b_k^\dagger, b_l^\dagger] = 0,
\]

where \( I \) denotes the identity operator. Throughout, the field is \( \mathbb{C} \).

**Definition 1.** A self-adjoint operator \( \mathcal{H} \) expressed with non-trivial, polynomial-dependence in the operators from the universal enveloping algebra of \( B = \{ b_l, b_l^\dagger : l = 1, \ldots, L < \infty \} \) is said to be a bosonic Hamiltonian. If \( \mathcal{H} \) cannot be expressed in terms of the universal enveloping algebra of a proper subset of \( B \), then \( \mathcal{H} \) is said to have \( L \) degrees of freedom.

**Definition 2.** A linear operator \( K \) is a said to be a conserved operator of a bosonic Hamiltonian \( \mathcal{H} \) if \( K \) has non-trivial, polynomial-dependence on some of the operators from the universal enveloping algebra of \( B = \{ b_l, b_l^\dagger : l = 1, \ldots, L < \infty \} \) and satisfies

\[
[\mathcal{H}, K] = 0.
\]

**Remark 1.** By defining conserved operators in terms of polynomial functions, definition 2 excludes the possibility of identifying projection operators as conserved operators. For
example, the projection $P_0$ onto the vacuum $|0\rangle$ is formally given by

$$P_0 = |0\rangle \langle 0| = \prod_{k=1}^{\mathcal{L}} \prod_{l=1}^{L} (I - k^{-1}b_l^\dagger b_l)$$

and is consequently not considered to be a conserved operator under definition 2.

**Definition 3.** The total number operator $N$ is defined as

$$N = \sum_{j=1}^{\mathcal{L}} b_j^\dagger b_j.$$

If a bosonic Hamiltonian $\mathcal{H}$ satisfies

$$[\mathcal{H}, N] = 0,$$

then $\mathcal{H}$ is said to conserve total particle number.

**Remark 2.** A bosonic Hamiltonian $\mathcal{H}$ acts on an infinite-dimensional Fock space. If $\mathcal{H}$ conserves total particle number, then $\mathcal{H}$ admits a direct sum decomposition

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}(n)$$

where $\mathcal{H}(n)$ acts on a space $V(n)$ of finite dimension

$$\text{dim}(V(n)) = \frac{(n + \mathcal{L} - 1)!}{n!(\mathcal{L} - 1)!},$$

and $N$ acts on $V(n)$ as an $n$-fold multiple of the identity operator.

**Definition 4.** A set of linear operators $\{O_1, \ldots, O_q\}$ is said to be functionally-dependent if there exists a non-constant polynomial $f$ such that $f(O_1, \ldots, O_q) = 0$. If a set of linear operators is not functionally-dependent, then it is said to be functionally-independent.

**Definition 5.** A bosonic Hamiltonian $\mathcal{H}$ with $\mathcal{L}$ degrees of freedom is said to be completely integrable if there exists a functionally-independent set of conserved operators

$$\{\mathcal{H} = K_1, K_2, K_3, \ldots, K_{\mathcal{L}}\}$$

satisfying

$$[K_j, K_l] = 0, \quad \forall \ 1 \leq j, l \leq \mathcal{L}.$$
**Definition 6.** A bosonic Hamiltonian $H$ is said to be Yang–Baxter integrable if there exists an operator $t(u), u \in \mathbb{C}$, known as a transfer matrix, that has non-trivial, polynomial-dependence on some of the operators from the universal enveloping algebra of $\mathcal{B} = \{b_l, b^\dagger_l : l = 1, \ldots, \mathcal{L} < \infty\}$ and satisfies

$$[H, t(u)] = 0,$$

$$[t(u), t(v)] = 0, \quad \forall u, v \in \mathbb{C}.$$

**Remark 4.** Historically, a transfer matrix draws its name from models in lattice statistical mechanics. The above definition is, admittedly, vague in that it does not refer to a Yang–Baxter equation nor its relation to the transfer matrix. This is deliberate because there are numerous approaches that can be employed to construct commuting transfer matrices beyond the most familiar cases with periodic boundary conditions. These include, but are not limited to, the use of classical Yang–Baxter equations [9, 32], dynamical Yang–Baxter equation [26], star-triangle relations [1], tetrahedron equation [8], and through quantum determinants and Dunkl operators [10]. They may incorporate twisted boundary conditions [47], open boundary conditions [53], braided boundary conditions [39], or infinite chains [19]. However, these variants are not pertinent below because only one representative of Yang–Baxter integrability will be required to develop the arguments.

2.1. The one-dimensional Bose–Hubbard model

Next, to provide some background context, a brief discussion is given to the one-dimensional Bose–Hubbard model. Much of the detail is adapted from [48].

The system satisfies definition 1, with the explicit Hamiltonian

$$H_{BH} = -t(b_1^\dagger b_\mathcal{L} + b_\mathcal{L}^\dagger b_1) - t \sum_{j=1}^{\mathcal{L}-1} (b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j) + U \sum_{j=1}^{\mathcal{L}} b_j^\dagger b_j b_j$$

from which it can be verified that definition 3 is met such that equation (1) is satisfied. It follows that for $\mathcal{L} = 2$, the so-called Bose–Hubbard dimer model, the system is completely integrable according to definition 5, with the set of conserved operators $\{H_{BH}, N\}$. For this case it is known that the system is Yang–Baxter integrable via a transfer matrix associated with either the classical Yang–Baxter equation [21] or the quantum Yang–Baxter equation [22, 23], and is exactly-solvable. See also [40]. The Hamiltonian (2) is again exactly-solvable in the limit $\mathcal{L} \to \infty$ [35, 44]. For other values of $\mathcal{L}$ with $n = 1$ the system can be solved by discrete Fourier transform, and for $n = 2$ by using centre-of-mass coordinates. But attempts to obtain solutions for higher values of $n$ have not succeeded [15]. The expectation that the model is not generally solvable is consistent with the characterisation of chaotic behavior found when $\mathcal{L} = 3$ [28].

Suppose it is accepted that (2) is not exactly-solvable. Then any definition for that characterisation needs to be able to identify, for some $\mathcal{L}$ and $n$, that the action of (2) on $V(n)$ is not exactly-solvable. That is to say, for bosonic Hamiltonians that conserve total particle number it is meaningful to refer to not exactly-solvable linear operators acting on finite-dimensional spaces, precisely those spaces within the decomposition (1). If this was not the case, and each linear operator obtained by restricting (2) to $V(n)$ was exactly-solvable, then the exact solution on the entire space would simply be obtained from the union of the exact solutions for each subspace.
3. A class of bosonic Yang–Baxter integrable systems

In this section a construction is described, via an explicit transfer matrix, for a class of bosonic Hamiltonians that are Yang–Baxter integrable. For notational convenience, the following sets of canonical boson operators are introduced, indexed by multiple labels:

\[ B_j = \{ a_{(j,\mu)}, a_{(j,\mu)}^\dagger, b_{(j,\mu)}, b_{(j,\mu)}^\dagger : \mu = 1, \ldots, m_j \}, \quad B = \bigcup_{j=1}^L B_j \]  

whereby

\[ 2 \sum_{j=1}^L m_j = L. \]  

Set the following notations for number operators:

\[ N_j = \sum_{\mu=1}^{m_j} \left( a_{(j,\mu)}^\dagger a_{(j,\mu)} + b_{(j,\mu)}^\dagger b_{(j,\mu)} \right), \]

\[ N_a = \sum_{j=1}^L \sum_{\mu=1}^{m_j} a_{(j,\mu)}^\dagger a_{(j,\mu)}, \]

\[ N_b = \sum_{j=1}^L \sum_{\mu=1}^{m_j} b_{(j,\mu)}^\dagger b_{(j,\mu)}, \]

\[ N = N_a + N_b = \sum_{j=1}^L N_j. \]

Let \( \{ \varepsilon_j : j = 1, \ldots, L \} \) denote a set of arbitrary, pairwise distinct, real parameters.

**Proposition 1.** The bosonic Hamiltonian

\[ H = U(N_a - N_b)^2 + \sum_{j=1}^L \varepsilon_j \sum_{\mu=1}^{m_j} \left( a_{(j,\mu)}^\dagger b_{(j,\mu)} + b_{(j,\mu)}^\dagger a_{(j,\mu)} \right), \quad U \in \mathbb{R}, \]  

is Yang–Baxter integrable.

**Proof.** This result is a generalisation of one found in [38], to which it reduces when \( m_j = 1 \) for all \( j = 1, \ldots, L = L/2 \). The key elements are outlined below.

For \( u, v \in \mathbb{C} \), let \( r(u, v) \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \). A form of classical Yang–Baxter equation reads [2, 29, 58]

\[ [r_{12}(u, v), r_{23}(v, w)] - [r_{21}(v, u), r_{33}(u, w)] + [r_{13}(u, w), r_{23}(v, w)] = 0 \]  

(6)
where the subscripts refer to the embedding of \( r(u, v) \) in \( \text{End}(C^2 \otimes C^2 \otimes C^2) \). A solution for (6) is given by [55]

\[
\begin{pmatrix}
\frac{2u^2}{u^2 - v^2} & 0 & 0 \\
0 & 0 & \frac{2uv}{u^2 - v^2} \\
- & - & - \\
0 & \frac{2uv}{u^2 - v^2} & 0 \\
0 & 0 & \frac{2u^2}{u^2 - v^2}
\end{pmatrix}
\]

(7)

It may be compactly expressed as

\[
r(u, v) = \left( \frac{u}{u - v} I + \frac{u}{u + v} \sigma_1 \sigma_2 \right) P
\]

where \( \sigma^z = \text{diag}(1, -1) \). The solution (7) satisfies [38]

\[
[\mathcal{J}_2(v), r_{12}(u, v)] - [\mathcal{J}_1(u), r_{21}(v, u)] = 0,
\]

where

\[
\mathcal{J}(u) = \begin{pmatrix} \alpha & u_\beta \\ u_\beta^* & -\alpha \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}.
\]

Next, consider the \( L \)-fold tensor product of universal enveloping algebras for \( \mathfrak{sl}(2) \) with generators \( \{ S^x_j, S^+_j, S^-_j : j = 1, \ldots, L \} \), satisfying the commutation relations

\[
[S^x_j, S^+_k] = \pm \delta_{jk} S^+_k, \quad [S^x_j, S^-_k] = 2\delta_{jk} S^-_k,
\]

(8)

and identify the Casimir invariants as \( C_j = 2(S^x_j)^2 + S^+_j S^-_j + S^-_j S^+_j \). Setting

\[
T^1_1(u) = \alpha I + \sum_{j=1}^L \frac{u^2}{u^2 - \varepsilon_j^2} (I + 2S^x_j),
\]

\[
T^2_1(u) = u\alpha I + \sum_{j=1}^L \frac{2u\varepsilon_j}{u^2 - \varepsilon_j^2} S^+_j,
\]

\[
T^2_2(u) = -\alpha I + \sum_{j=1}^L \frac{u^2}{u^2 - \varepsilon_j^2} (I - 2S^-_j),
\]

and

\[
T(u) = \begin{pmatrix} T^1_1(u) & T^2_1(u) \\ T^2_2(u) & T^1_2(u) \end{pmatrix}.
\]
it can be shown that
\[ [T_1(u), T_2(v)] = [T_2(v), r_{12}(u, v)] - [T_1(u), r_{21}(v, u)]. \]

Then the algebraic transfer matrix
\[ \tau(u) = \tr ((T(u))^2) = \sum_{j,k=1}^2 T_j^k(u)T_k^j(u) \]
(9)
satisfies
\[ [\tau(u), \tau(v)] = 0 \quad \forall \, u, v \in \mathbb{C}. \]

In detail,
\[ \tau(u) = 2(\alpha^2 + \beta^2 u^2)I + 2u^4 \left( \sum_{j=1}^L \frac{1}{u^2 - \epsilon_j^2} \right)^2 I + 4u^2 \sum_{j=1}^L \frac{\epsilon_j^2}{(u^2 - \epsilon_j^2)^2} C_j + 4 \sum_{j=1}^L \frac{u^2}{u^2 - \epsilon_j^2} T_j \]
where
\[ T_j = 2(S_j^+)^2 + 2\alpha S_j^+ + \beta \epsilon_j (S_j^+ + S_j^-) + \sum_{k \neq j} \theta_{jk}, \]
(10)
\[ \theta_{jk} = \frac{4\epsilon_j^2}{\epsilon_j^2 - \epsilon_k^2} S_j^- S_k^+ + \frac{2\epsilon_j \epsilon_k}{\epsilon_j^2 - \epsilon_k^2} (S_j^+ S_k^- + S_j^- S_k^+), \quad j \neq k. \]

By construction, the operators (10) satisfy
\[ [T_j, T_k] = 0, \quad 1 \leq j, k \leq L. \]
(11)
A direct verification of equation (11) is provided in the appendix. The operators (10) are recognised as generalisations of Gaudin operators [16, 30, 46, 52].

A representation \( \pi \) for the spin operators, in terms of the boson operators of (3), is afforded by the Jordan–Schwinger map
\[ \pi(S_j^-) = \frac{1}{2} \sum_{\mu=1}^{m_j} (a_{j,\mu}^+ a_{j,\mu} - b_{j,\mu}^+ b_{j,\mu}), \]
\[ \pi(S_j^+) = \sum_{\mu=1}^{m_j} a_{j,\mu}^+ b_{j,\mu}, \]
\[ \pi(S_j^0) = \sum_{\mu=1}^{m_j} b_{j,\mu}^+ a_{j,\mu}, \]
and extends to the entire \( L \)-fold universal enveloping algebra as an algebra homomorphism. Finally, set \( \alpha = 0, 2\beta = U^{-1} \), and
\[ \tau(u) = \pi(\tau(u)). \]
Then
\[ H = 2U \sum_{j=1}^{L} \pi(T_j) \]
and satisfies \([H, t(u)] = 0\), establishing the Yang–Baxter integrability of (5). \(\square\)

**Remark 5.** When \(L = \mathcal{L}/2 = 1\), the Hamiltonian (5) is equivalent to the dimer model limit of (2). Thus (5) can be viewed as an integrable, many-degrees of freedom, generalisation of the Bose–Hubbard dimer model, while (2) is a non-integrable generalisation of the same model.

**Corollary 1.** There exists a set of conserved operators \(C = \{H = K_1, K_2, K_3, \ldots, K_{2L}\}\) for \(H\), with elements given by
\[ K_j = \pi(T_j), \quad j = 2, \ldots, L, \]
\[ K_{L+j} = \pi(C_j), \quad j = 1, \ldots, L. \]

**Lemma 1.** If \(m_j = 1\) for all \(j = 1, \ldots, L = \mathcal{L}/2\), then \(C\) is functionally-independent provided \(\varepsilon_j \neq 0\) for all \(j = 1, \ldots, L\).

**Proof.** Each of the elements of \(C\) is a linear function in \(U\). It is seen that the limiting operators
\[ \bar{K}_1 = \lim_{U \to 0} K_1 = \sum_{j=1}^{L} \varepsilon_j \left( a_{j,1}^\dagger b_{j,1} + b_{j,1}^\dagger a_{j,1} \right), \]
\[ \bar{K}_j = \lim_{U \to 0} K_j = \varepsilon_j \left( a_{j,1}^\dagger b_{j,1} + b_{j,1}^\dagger a_{j,1} \right), \quad j = 2, \ldots, L, \]
\[ \bar{K}_{L+j} = \lim_{U \to 0} K_{L+j} = \frac{1}{2} N_j (N_j + 2I), \quad j = 2, \ldots, L, \]
form a functionally-independent set \(\bar{C} = \{\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_{2L}\}\). Now assume there exists a polynomial \(f\) such that
\[ f(K_1, \ldots, K_{2L}) = 0. \]
Let \(r \in \mathbb{R}\) denote the smallest number such that
\[ \bar{f} = \lim_{U \to 0} U^r f \]
is a non-zero polynomial. Then
\[ 0 = f(K_1, \ldots, K_{2L}) = \lim_{U \to 0} U^r f(K_1, \ldots, K_{2L}) = \bar{f}(K_1, \ldots, K_{2L}) \]
which is a contradiction since \(\bar{C}\) is functionally-independent. Hence, \(C\) must be functionally-independent. \(\square\)

**Corollary 2.** If \(m_j = 1\) for all \(j = 1, \ldots, L = \mathcal{L}/2\) and \(\varepsilon_j \neq 0\) for all \(j = 1, \ldots, L = \mathcal{L}/2\), then (5) is completely integrable according to definition 5.
Remark 6. It is not the case that all conserved operators of a Yang–Baxter integrable system are necessarily obtained from the transfer matrix. For example, the class of models in [62] have only two independent conserved operators originating from the transfer matrix. However those models are completely integrable in the sense of definition 5. If not all \( m_j = 1 \), corollary 2 does not assert that (5) is not completely integrable. However if it is completely integrable, alternative methods are required to establish this property.

4. Paradoxical statements on exact-solvability

The main results can now be formulated.

4.1. Yang–Baxter integrable systems

Axioms 1. A definition for exactly-solvable, self-adjoint, linear operators acting on complex vector spaces satisfies the following axioms:

(a) The set of operators acting on finite-dimensional spaces that are not exactly-solvable is non-empty;
(b) Given an arbitrary operator \( Y \) and an operator \( Z \) that is not exactly-solvable, the direct sum \( Y \oplus Z \) is not exactly-solvable.

Remark 7. The above itemised list is, in no way, intended to be exhaustive of suitable axioms to impose on a definition for exact-solvability. This is a minimal list for the purpose of establishing the results. Condition (a) is essential, otherwise there would be nothing to discuss. Without item (b) it would be possible for an operator to be exactly-solvable when its action on a subspace spanned by a subset of eigenstates is not. Such a scenario is difficult to justify.

Let \( A \) denote a self-adjoint, linear operator acting on a complex vector space of dimension \( D < \infty \), with matrix elements given by \( \{ A_{jk} : j, k = 1, \ldots, D \} \). Let \( X_{q(k,\nu)} \) denote the matrix elements of a unitary operator \( X \) that diagonalises \( A \), viz

\[
\sum_{p,q=1}^D X_{(j,\mu)p}^* A_{pq} X_{(k,\nu)q} = \varepsilon_j \delta_{jk} \delta_{\mu\nu},
\]

with \( \{ \varepsilon_j : j = 1, \ldots, L \} \) the spectrum of \( A \), each \( \varepsilon_j \) occurring with multiplicity \( m_j \), such that

\[
\sum_{j=1}^L m_j = D.
\]

Set \( \mathbb{E} = \{ e_i, e_i^\dagger, d_i, d_i^\dagger : i = 1, \ldots, D \} \) so \( \mathcal{L} = 2D \) and equation (4) holds. Define

\[
\mathbb{H} = U(N_c - N_d) - UI + \sum_{j,k=1}^D A_{jk}(e_j^\dagger d_k + d_j^\dagger e_k).
\]

(12)

where

\[
N_c = \sum_{j=1}^D e_j^\dagger e_j, \quad N_d = \sum_{j=1}^D d_j^\dagger d_j.
\]
Proposition 2. The Hamiltonian (12) is Yang–Baxter integrable.

Proof. Introducing operators
\[ a(k,\nu) = D \sum_{j=1}^{D} X_{j,k,\nu} c_j, \quad b(k,\nu) = D \sum_{j=1}^{D} X_{j,k,\nu} d_j, \]
leads to \( N_c = N_a, N_d = N_b \) and \( H + UI = H \) with \( H \) given by (5). Thus, (12) is Yang–Baxter integrable by proposition 1.

\[ \square \]

Remark 8. The class of Hamiltonians with the form (12) generalises a class of Hamiltonians introduced in [62]. The classes coincide when \( A \) has rank 1.

Proposition 3. Suppose that a specific \( A \) has been deemed to not be exactly-solvable, according a definition satisfying axioms 1. Then the corresponding Hamiltonian (12) is not exactly-solvable.

Proof. The Hamiltonian (12) commutes with the number operator
\[ N = N_a + N_b = N_c + N_d \]
and therefore (12) can be block-diagonalised as in (1). Now, \( \mathbb{H}(1) \) acts on a space of dimension \( 2D \), and can be represented as a tensor product of linear operators
\[ \mathbb{H}(1) \cong \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}. \]
Since \( A \) is not exactly-solvable, it then follows that \( \mathbb{H}(1) \) is not exactly-solvable, and so \( \mathbb{H} \) is not exactly-solvable, according to axioms 1 (b).

\[ \square \]

Corollary 3. The definition ‘A bosonic Hamiltonian (see definition 1) is exactly-solvable if it is Yang–Baxter integrable (see definition 6)’ violates propositions 2 and 3, and is therefore inconsistent with axioms 1.

4.2. Completely integrable systems

Axioms 2. A definition for exactly-solvable, self-adjoint, linear operators acting on complex vector spaces satisfies the following axioms:

(a) The set of operators acting on finite-dimensional spaces that are not exactly-solvable, and have multiplicity-free spectrum, is non-empty;
(b) Given an arbitrary operator \( Y \) and an operator \( Z \) that is not exactly-solvable, the direct sum \( Y \oplus Z \) is not exactly-solvable;
(c) Given an operator \( Z \) that is not exactly-solvable, then \( Z + \lambda I, \lambda \in \mathbb{R} \), where \( I \) is the identity operator, is not exactly-solvable.

Remark 9. The added restriction on item (a) is minimal. Any operator that has degenerate spectrum can have the degeneracy broken by an arbitrarily small perturbation. It is reasonable
to impose that operators that are not exactly-solvable remain so, generically, under arbitrarily small perturbation. Item (c) seems to be self-evident, but needs to be formally stated.

**Corollary 4.** Choose $\tilde{A}$ to have multiplicity-free spectrum and $\lambda \in \mathbb{R}$ such that $A = \tilde{A} + \lambda I$ has positive eigenvalues. Then the Hamiltonian (12) is completely integrable as a result of corollary 2, and the identification $\mathbb{H} + U I = H$ from the proof of proposition 2. If $A$ is deemed to not exactly-solvable, according to a definition satisfying axioms 2, then $A$ is not exactly-solvable and so (12) is not exactly-solvable by proposition 3.

**Corollary 5.** The definition ‘A bosonic Hamiltonian (see definition 1) is exactly-solvable if it is completely integrable (see definition 5)’ is inconsistent with axioms 2.

5. Discussion

Many paradoxes arise as a consequence of self-reference. Some are popularly known, such as the Liar paradox and the Barber paradox. Others, including Curry’s paradox and the Grelling–Nelson paradox, have had important impact in developing formal logic. The most prominent examples invoking self-reference in mathematics are arguably those associated with Russell and Gödel. See [17], for example.

The paradoxes of section 4 also arise through self-reference. The Hamiltonian (12) is defined in terms of an operator $A$ that is assumed to possess a particular property i.e. $A$ is not exactly-solvable. However, the class of integrable systems to which (12) belongs is used to define the negation of that same property. The surprise is not so much that a paradox occurs. It is, rather, the existence of the Yang–Baxter integrable system (12) that is sufficiently general to expedite such a self-referential construction.

In an attempt to resolve the paradoxes, consideration can be given to changing relevant axioms and definitions. For example, a simple resolution is obtained by declaring that all self-adjoint operators on finite-dimensional spaces are exactly-solvable. It is not an unreasonable position to take, because the spectrum of each operator is determined by its characteristic polynomial. But it does not help to understand why some systems are amenable to a Bethe ansatz solution and others are not. Another route might be to revise definition 5. Rather than define complete integrability in terms of the existence of conserved operators, it could be weakened to only include cases where the explicit form of the conserved operators is known. This feature does not currently apply to (12) when $A$ is not exactly-solvable. Expressing the conserved operators of corollary 1 in terms of the generating set $\mathbb{B}$ that defines (12) leads to expressions with explicit dependence on the unknown quantities $X_{q(k,\nu)}$ and $\epsilon_j$. However the drawback here is that this system, now classified as non-integrable, has exactly the same energy-level statistics as an integrable system. Integrable systems are generally expected to display a signature Poisson distribution for the energy gaps, as has been observed in bosonic systems [18] analogous to those discussed above. By defining integrability in a basis-dependent manner, it would subsequently render all studies relating integrability to energy-level statistics as meaningless.

The above results were formulated around a solution of the classical Yang–Baxter equation (6). This is not of a standard form, with the corresponding solution (7) not possessing the skew-symmetry property $r_{12}(u,v) = -r_{21}(v,u)$. The construction of the transfer matrix from this solution can be understood as the quasi-classical limit of a double-row transfer matrix built around reflection equations [42, 54], rather than the conventional Yang–Baxter equation. This observation suggests that it may be possible to lift the construction presented here into that double-row transfer matrix setting, to extend the paradox to a wider class of Yang–Baxter integrable systems.
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Data availability statement

No new data were created or analysed in this study.

Appendix

Here it is shown that equation (11) holds, viz

\[
[T_j, T_k] = 0, \quad 1 \leq j, k \leq L,
\]

by direct use of the commutation relations (8). Recall that

\[
T_j = 2(S_j^+)^2 + 2\alpha S_j^z + 2\beta \varepsilon_j S_j^- + 2\varepsilon_j S_j^z + \sigma_j \varepsilon_j^2 \sum_{k \neq j} \theta_{jk},
\]

\[
\theta_{jk} = \frac{4\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \frac{2\varepsilon_j \varepsilon_k}{\varepsilon_j^2 - \varepsilon_k^2} (S_j^+ S_k^- + S_j^- S_k^+), \quad j \neq k.
\]

First note that for \(j \neq p\)

\[
[\theta_{jk}, \theta_{pk}] = \frac{8\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} \varepsilon_p \varepsilon_{pk} \frac{S_j^z S_k^z - S_p^z S_k^z}{\varepsilon_j^2 - \varepsilon_k^2} + \frac{8\varepsilon_p^2}{\varepsilon_j^2 - \varepsilon_k^2} \varepsilon_{jk} \varepsilon_k \frac{S_p^z S_j^z - S_p^z S_j^z}{\varepsilon_j^2 - \varepsilon_k^2} + \frac{8\varepsilon_k^2}{\varepsilon_j^2 - \varepsilon_k^2} \varepsilon_j \varepsilon_p \frac{S_k^z S_j^z - S_p^z S_j^z}{\varepsilon_j^2 - \varepsilon_k^2},
\]

yielding

\[
[\theta_{jk}, [\theta_{jk}, \theta_{pk}]] + [\theta_{jk}, [\theta_{jk}, \theta_{pj}]] + [\theta_{jk}, [\theta_{jk}, \theta_{kp}]] = 0.
\]

Now from

\[
\theta_{jk} + \theta_{kj} = 4S_j^z S_k^z,
\]

\[
[S_j^z S_k^z, \theta_{jk}] = \frac{2\varepsilon_j \varepsilon_k}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z - S_k^z S_j^z = [S_j^z S_k^z, \theta_{jk}], \quad j \neq p,
\]

it follows that

\[
[\theta_{jk}, \theta_{kp}] + [\theta_{kp}, \theta_{kj}] + [\theta_{kp}, \theta_{pk}] = [\theta_{jk}, \theta_{jk}] + [\theta_{kj}, \theta_{kj}] + [\theta_{kp}, \theta_{kp}] = 0,
\]
leading to

\[\sum_{p \neq j} \sum_{l \neq k} \left[ \theta_{jp}, \theta_{kl} \right]\]

\[= \left[ \theta_{jk}, \theta_{kj} \right] + \sum_{p \neq j,k} \left[ \theta_{jp}, \theta_{kp} \right] + \sum_{p \neq j,k} \left[ \theta_{jp}, \theta_{kj} \right] + \sum_{p \neq j,k} \left[ \theta_{jk}, \theta_{kp} \right]\]

\[= 4 \left( S^+_k S^-_k, \theta_{kj} \right) \]

\[= \frac{8 \varepsilon_k \varepsilon_j}{\varepsilon^2_k - \varepsilon^2_j} \left( S^+_j S^-_k - S^-_k S^+_j \right) + \left( S^+_j S^-_k - S^-_j S^+_k \right) S^+_k S^-_k \].

Also,

\[\left[ S^+_j, \theta_{kj} \right] = \frac{2 \varepsilon_k \varepsilon_j}{\varepsilon^2_k - \varepsilon^2_j} \left( S^+_j S^-_j - S^-_j S^+_j \right)\]

\[= \left[ S^+_k, \theta_{jk} \right]\]

and

\[\varepsilon_k \left[ S^+_j + S^-_j, \theta_{kj} \right] = \varepsilon_k \left( \frac{4 \varepsilon^2_j}{\varepsilon^2_j - \varepsilon^2_k} S^+_k S^-_j - S^-_j S^+_j \right) + \frac{4 \varepsilon_k \varepsilon_j}{\varepsilon^2_k - \varepsilon^2_j} S^+_j S^-_k \]

\[= \varepsilon_k \left[ S^+_k + S^-_k, \theta_{jk} \right].\]

Then, for \(j \neq k\),

\[\left[ T_j, T_k \right] = \sum_{l \neq k} \left[ 2 \left( S^+_l \right)^2 + 2 \alpha S^+_l + \beta \varepsilon_j S^+_l + S^-_j \right], \theta_{kl} \]

\[+ \sum_{p \neq j} \left[ \theta_{jp}, 2 \left( S^+_k \right)^2 + 2 \alpha S^+_k + \beta \varepsilon_k \left( S^+_k + S^-_k \right) \right] + \sum_{p \neq j} \sum_{l \neq k} \left[ \theta_{jp}, \theta_{kl} \right]\]

\[= 2 S^+_j \left[ S^+_j, \theta_{kj} \right] + 2 \left[ S^+_j, \theta_{kj} \right] S^+_j + 2 S^+_j \left[ \theta_{jk}, S^+_k \right] + 2 \left[ \theta_{jk}, S^+_j \right] S^+_k\]

\[+ \frac{8 \varepsilon_k \varepsilon_j}{\varepsilon^2_k - \varepsilon^2_j} \left( S^+_j S^-_k - S^-_k S^+_j \right) + \left( S^+_j S^-_k - S^-_j S^+_k \right) S^+_k S^-_k \]

\[= 2 \left( S^+_j - S^-_j \right) \left[ S^+_j, \theta_{kj} \right] + 2 \left[ S^+_j, \theta_{kj} \right] \left( S^+_j - S^-_j \right)\]

\[+ \frac{8 \varepsilon_k \varepsilon_j}{\varepsilon^2_k - \varepsilon^2_j} \left( S^+_j S^-_k - S^-_k S^+_j \right) + \left( S^+_j S^-_k - S^-_j S^+_k \right) S^+_k S^-_k \]

\[= \frac{4 \varepsilon_j \varepsilon_k}{\varepsilon^2_k - \varepsilon^2_j} \left( \left( S^+_j - S^-_j \right) \left( S^+_j S^-_k - S^-_j S^+_k \right) + \left( S^+_j S^-_k - S^-_j S^+_k \right) \left( S^+_k - S^-_k \right) \right)\]

\[+ \frac{8 \varepsilon_j \varepsilon_k}{\varepsilon^2_k - \varepsilon^2_j} \left( S^+_j S^-_k - S^-_k S^+_j \right) + \left( S^+_j S^-_k - S^-_j S^+_k \right) S^+_k S^-_k \].
\[
\frac{4 \varepsilon_k \varepsilon_j}{\varepsilon_k - \varepsilon_j} \left[ (S^+_k S^-_k - S^-_j S^+_k) \cdot (S^+_j + S^-_k) \right] = 0.
\]

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