KDV ON AN INCOMING TIDE

THIERRY LAURENS

Abstract. Given smooth step-like initial data $V(0,x)$ on the real line, we show that the Korteweg–de Vries equation is globally well-posed for initial data $u(0,x) \in V(0,x) + H^{-1}(\mathbb{R})$. The proof uses our general well-posedness result from [40].

As a prerequisite, we show that KdV is globally well-posed for $H^3(\mathbb{R})$ perturbations of step-like initial data. In the case $V \equiv 0$, we obtain a new proof of the Bona–Smith theorem [8] using the low-regularity methods that established the sharp well-posedness of KdV in $H^{-1}$ [35].

1. Introduction

The Korteweg–de Vries (KdV) equation

\[ \frac{d}{dt} u = -u''' + 6uu' \]

(where primes $u' = \partial_x u$ denote spatial differentiation) was proposed in [37] to describe the phenomena of solitary traveling waves (solitons) in shallow channels. Since its introduction over a century ago, the KdV equation has been thoroughly studied on the line $\mathbb{R}$ and the circle $\mathbb{R}/\mathbb{Z}$ and has been shown to exhibit a multitude of special features.

A fundamental direction of investigation for KdV has been well-posedness in the $L^2$-based Sobolev spaces $H^s(\mathbb{R})$ and $H^s(\mathbb{R}/\mathbb{Z})$. However, the derivative in the nonlinearity of KdV prevents straightforward contraction mapping arguments from closing, so preliminary results produced continuous dependence in a weaker norm than the space of initial data. One of the first results to overcome this loss of derivatives phenomenon was obtained by Bona and Smith [8] who established global well-posedness in $H^3(\mathbb{R})$. Numerous methods were developed in the following decades in the effort to lower the regularity $s$; see for example [7, 12, 15, 22, 28, 30, 36, 48, 50]. Recently, Killip and Vişan [35] introduced the method of commuting flows to demonstrate global well-posedness in $H^{-1}(\mathbb{R})$ and $H^{-1}(\mathbb{R}/\mathbb{Z})$, a result that is sharp in both topologies. In the $\mathbb{R}/\mathbb{Z}$ case, this result was already known [24].

Solutions in $H^s(\mathbb{R}/\mathbb{Z})$ spaces are spatially periodic and solutions in $H^s(\mathbb{R})$ spaces decay at infinity. However, there are other classes of initial data which are of physical interest. In particular, waveforms that are step-like—in the sense that $u(0,x)$ asymptotically approaches distinct constant values as $x \to \pm \infty$—arise naturally in the study of bore propagation and rarefaction waves. Such asymptotic behavior has real physical consequences. Indeed, we shall see below that the polynomial conservation laws are broken, and in the case of an incoming tide there is an infinite influx of energy into the system.
Our objective in this paper is to extend low-regularity methods for well-posedness to the regime of nonzero spatial asymptotics. We define the smooth step function
\[ W(x) = c_1 \tanh(x) + c_2 \] with \( c_1, c_2 \in \mathbb{R} \) fixed, which exponentially decays to its asymptotic values. As \(-u\) is proportional to the water wave height, \( W \) models an incoming tide if \( c_1 > 0 \) and an outgoing tide if \( c_1 < 0 \). In fact, we can always perform a boost to prescribe \( c_2 \) courtesy of the Galilean symmetries of KdV (1.1), but we will not make use of this.

A classical result in the study of step-like asymptotics is:

**Theorem 1.1.** Fix an integer \( s \geq 3 \). The KdV equation (1.1) with initial data \( u(0) \in W + H^s(\mathbb{R}) \) is globally well-posed in the following sense: \( u(t) = W + q(t) \) where \( q(t) \) is the global solution to
\[
\frac{d}{dt} q = -(q + W)'' + 6(q + W)(q + W')
\]
with initial data \( q(0) = u(0) - W \) in \( H^s(\mathbb{R}) \). Moreover, \( q(t) \) is in \( C_tH^s([-T,T] \times \mathbb{R}) \) for all \( T > 0 \), \( q(t) \) is unique in this class, and \( q(t) \) depends continuously upon the initial data \( q(0) \) in \( H^s(\mathbb{R}) \).

Theorem 1.1 is not new (as we will discuss below), but we will use its statement to formulate our main result. Applying Theorem 1.1 to the initial data \( q(0) \equiv 0 \), we conclude that given \( W \) there is a unique global solution \( V(t) = W + q(t) \) to KdV (1.1) with initial data \( W \), and \( t \mapsto V(t) - W \) is a continuous function into \( H^s(\mathbb{R}) \) for all \( s \geq 3 \). The main thrust of this work is to show that KdV is globally well-posed for \( H^{-1}(\mathbb{R}) \) perturbations of \( V(t) \):

**Theorem 1.2.** The KdV equation (1.1) with initial data \( u(0) \in W + H^{-1}(\mathbb{R}) \) is globally well-posed in the following sense: \( u(t) = V(t) + q(t) \) where \( V(t) \) solves KdV with initial data \( W \) and the equation
\[
\frac{d}{dt} q = -q''' + 6qq' + 6(Vq)'
\]
for \( q(t) \) with initial data in \( H^{-1}(\mathbb{R}) \) is globally well-posed.

Let us clarify the notion of well-posedness in Theorem 1.2. As we cannot make sense of the nonlinearity of KdV for arbitrary functions in \( H^{-1}(\mathbb{R}) \) (even in the distributional sense), the solutions in Theorem 1.2 are constructed as limits of solutions to a family of approximate equations. We then show that the data-to-solution map \( q(0) \mapsto q(t) \) is a jointly continuous function of \( t \in \mathbb{R} \) and \( q(0) \in H^{-1}(\mathbb{R}) \) into \( H^{-1}(\mathbb{R}) \). The notions of solution and uniqueness is that for the dense subset \( H^3(\mathbb{R}) \) of initial data \( q(0) \) the functions \( q(t) \) coincide with classical solutions (cf. [40] Th. 1.3) and the data-to-solution map is continuous.

The proof of Theorem 1.2 relies on our general well-posedness result [40], which proves that the equation (1.3) is well-posed in \( H^{-1}(\mathbb{R}) \) provided that the background wave \( V(t) \) satisfies certain criteria (which we will formulate below). Verifying that \( V(t) \) satisfies these criteria for the step-like initial data \( W \) will be accomplished by certain ingredients in the proof of Theorem 1.1, namely Corollary 3.6 and Propositions 6.1 and 6.2.

It is natural to ask whether KdV is also well-posed for \( H^{-1}(\mathbb{R}) \) perturbations of \( W \). Theorems 1.1 and 1.2 provide an affirmative answer to this question. By Theorem 1.2, there exists a solution \( u(t) = V(t) + q(t) \) to KdV (1.1) with initial data
\( u(0) = W + q(0) \) in \( W + H^{-1}(\mathbb{R}) \). Together with Theorem 1.1, we also obtain that 
\( t \mapsto u(t) - W \) is a continuous function into \( H^{-1}(\mathbb{R}) \) that depends continuously upon the initial data. For a precise statement of this well-posedness, see Corollary 6.4. We do not use this formulation in the statement of Theorem 1.2 because it does not reflect the reality of the proof.

Just as \( H^{-1}(\mathbb{R}) \) is the lowest regularity for which we can hope to have well-posedness in the case \( W \equiv 0 \) [41], we expect that Theorem 1.2 is sharp in the class of \( H^s(\mathbb{R}) \) spaces. There is a known technique [35, Cor. 5.3] for extending \( H^{-1}(\mathbb{R}) \) well-posedness to \( H^s(\mathbb{R}) \), \( s > -1 \), using equicontinuity, and so \( H^{-1}(\mathbb{R}) \) is the key space for establishing well-posedness.

Next we turn our attention to a discussion of prior work. In [3, §3], Benjamin, Bona, and Mahony discuss well-posedness for the (closely related) BBM equation with step-like initial data. In the case \( W \equiv 0 \), Bona and Smith [8] proved that KdV is well-posed in \( H^s(\mathbb{R}) \) for \( s \geq 3 \) by approximating KdV by a family of BBM equations. The formulation of Theorem 1.1 is inspired by [8]; indeed, Theorem 1.1 can be proved using their original argument. However, it is our proof of Theorem 1.1, not the formulation, that we need as an ingredient for Theorem 1.2. In contrast to the Bona–Smith approach, we approximate KdV by a family of commuting flows introduced by Killip and Vişan [35]. This has the advantage that the \textit{a priori} estimates are the same as those for KdV, and convergence can be demonstrated in a transparent way (by upgrading continuity in a lower regularity norm using equicontinuity).

Lower regularity than \( H^3(\mathbb{R}) \) has been obtained in the study of well-posedness for perturbations of a fixed step-like background wave. The first result was recorded in [25], who proved local well-posedness for perturbations in \( H^s(\mathbb{R}) \), \( s > \frac{3}{2} \), and global well-posedness for \( s \geq 2 \). Local well-posedness was then extended to \( s > 1 \) in [19] for the same family of background waves. Independently, local well-posedness for \( H^2(\mathbb{R}) \) perturbations was proved for gKdV in [51], along with global-in-time existence in the case of a kink solution background wave and initial data that is small in \( H^1(\mathbb{R}) \).

Subsequent to our work, a new result [44] for gKdV demonstrates local well-posedness for perturbations in \( H^s(\mathbb{R}) \), \( s > \frac{1}{2} \) and global well-posedness for \( s \geq 1 \). In addition to a larger class of equations, this work also applies to a wide variety of background waves, including both step-like and periodic asymptotics. In particular, the background wave is not assumed to be time-independent nor an exact solution, but rather is allowed to solve the equation modulo a localized error term.

The primary tool used in the literature to study step-like solutions of KdV has been the inverse scattering transform. In the case of a highly regular step-like background, existence for the Cauchy problem has been examined in [11, 13, 14, 17, 18, 26]. In order to employ the inverse scattering transform these results assume that \( u(0) - W \) is integrable against \( 1 + |x|^N \) for some \( N \geq 1 \), and consequently such methods are not suitable for \( H^s(\mathbb{R}) \) spaces. Nevertheless, as shown in [17], these methods do yield existence for Schwartz class perturbations. Classes of one-sided step-like initial data were treated in [21, 46, 47] and one-sided step-like elements of \( H^s_{loc}(\mathbb{R}) \) were treated in [20]. Despite the lack of assumptions at \(-\infty\) (the direction in which radiation propagates), these low-regularity arguments require rapid decay at \(+\infty\) and global boundedness from below. By comparison, our argument is symmetric in \( \pm x \) and in \( \pm u \).
The inverse scattering transform is also used to study the long-time behavior of such solutions; see for example [1, 2, 4–6, 16, 24, 31, 32, 38, 39, 42]. The asymptotics are spatially asymmetric and differ in the cases of tidal bores and rarefaction waves.

In this paper, we employ the method of commuting flows introduced in [35]. This method was used to prove both symplectic non-squeezing [43] and invariance of white noise [33] for KdV on the line. The method of commuting flows has also been adapted to other completely integrable systems, including the cubic NLS and mKdV equations [23], the fifth-order KdV equation [10], and the derivative NLS equation [34]. Together with [40], this is the first application of this method to exotic spatial asymptotics.

The presence of the background wave $W$ breaks the macroscopic conservation laws of KdV. A solution of KdV (1.1) must obey the microscopic conservation law

$$\frac{d}{dt}\left(\frac{1}{2} u^2\right) = \left[-uu'' + \frac{1}{2}(u')^2 + 2u^3\right]' .$$

For Schwartz solutions $u$ to KdV, integrating in space yields (macroscopic) conservation of the momentum

$$P(u) := \frac{1}{2} \int u(x)^2 \, dx .$$

However, if merely $u - W$ is Schwartz then we obtain

$$\frac{d}{dt} \int \frac{1}{2} \left[ u(t, x)^2 - u(0, x)^2 \right] \, dx = 2W(x)^3 \bigg|_{x=\pm\infty} .$$

In the case $c_1 > 0, c_2 = 0$ of an incoming tide, the RHS is equal to $4c_1^3 > 0$. The momentum’s growth is manifested in a dispersive shock that develops in the long-time asymptotics [16, Fig. 1].

Interpreting $W$ as an incoming or outgoing tide, we will refer to (1.2) as tidal KdV. To prove Theorem 1.1 we will show that tidal KdV is well-posed in $H^s(\mathbb{R})$ for $s \geq 3$. Computations similar to (1.5) show that the presence of $W$ in tidal KdV breaks all of the polynomial conservation laws of KdV. Despite this, we are able to adapt the method of commuting flows to tidal KdV because these conserved quantities do not blow up in finite time.

In order to introduce our methods, we will first present some notation. The KdV equation (1.1) is governed by the Hamiltonian functional

$$H_{\text{KdV}}(q) := \int \left( \frac{1}{2} q'(x)^2 + q(x)^3 \right) \, dx$$

via the Poisson structure

$$\{F,G\} = \int \frac{\delta F}{\delta q} (x) \left( \frac{\delta G}{\delta q} \right)' (x) \, dx .$$

Here we are using the notation

$$dF|_q(f) = \frac{d}{ds} \bigg|_{s=0} F(q + sf) = \int \frac{\delta F}{\delta q} (x) f(x) \, dx$$

for the derivative of the functional $F(q)$. This Poisson structure is the bracket associated to the almost complex structure $J := \partial_x$ and the $L^2$ pairing. In accordance with its name, the momentum functional (1.4) generates translations under this structure.
Our analysis will not rely upon these concepts, but we will borrow the convenient notations

\[ q(t) = e^{tJ\nabla H}q(0) \]

for the solution to \( \frac{dq}{dt} = \partial_x \delta H/\delta q \), and

\[ \frac{d}{dt} F \circ e^{tJ\nabla H} = \{ F, H \} \circ e^{tJ\nabla H} \]

for the quantity \( F(q) \) with \( q(t) = e^{tJ\nabla H}q(0) \).

In the case \( W \equiv 0 \), the authors of [35] introduced a family of commuting flows that approximate that of KdV. This approximation relies on the existence of a generating function \( \alpha(\kappa, q) \) for the KdV hierarchy of conserved quantities with the asymptotic expansion

\[ \alpha(\kappa, q) = \frac{1}{4\kappa^3} P(q) - \frac{1}{16\kappa^5} H_{\text{KdV}}(q) + O(\kappa^{-7}) \]

for Schwartz \( q \). Here \( P \) and \( H_{\text{KdV}} \) are the momentum and KdV energy functionals \[ \text{(1.3)} \] and \[ \text{(1.0)} \] respectively. The quantity \( \alpha(\kappa, q) \) is a renormalized logarithm of the transmission coefficient for the Schrödinger operator with potential \( q \) (i.e. perturbation determinant) at energy \( -\kappa^2 \), and is a real analytic functional of \( q \) in a neighborhood of the origin in \( H^{-1}(\mathbb{R}) \) for all \( \kappa \geq 1 \).

Rearranging the expansion \[ \text{(1.7)} \], the authors of [35] introduced the Hamiltonians

\[ H_\kappa(q) := -16\kappa^5 \alpha(\kappa, q) + 4\kappa^2 P(q) \]

and showed that their flow converges to that of KdV in \( H^{-1}(\mathbb{R}) \) as \( \kappa \to \infty \). The \( H_\kappa \) flows are easier to work with, as well-posedness follows from straightforward ODE arguments. Moreover, two flows with different energy parameters \( \kappa \) commute with one another, which facilitates the demonstration of convergence as \( \kappa \to \infty \).

Our general result \[ \text{(10)} \] is that the equation \[ \text{(1.3)} \] is well-posed in \( H^{-1}(\mathbb{R}) \) provided that for every \( T > 0 \) the background wave \( V : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfies the following:

(i) \( V \) solves KdV \[ \text{(1.1)} \] and is bounded in \( W^{2,\infty}(\mathbb{R}_x) \) uniformly for \( |t| \leq T \),

(ii) The solutions \( V_\kappa(t) \) to the \( H_\kappa \) flows with initial data \( V(0) \) are bounded in \( W^{4,\infty}(\mathbb{R}_x) \) uniformly for \( |t| \leq T \) and \( \kappa > 0 \) sufficiently large,

(iii) \( V_\kappa - V \to 0 \) in \( W^{2,\infty}(\mathbb{R}_x) \) as \( \kappa \to \infty \) uniformly for \( |t| \leq T \) and initial data in the set \( \{ V_\kappa : |t| \leq T, \ k \geq \kappa \} \).

To prove Theorem \[ \text{(1.2)} \] we need to study the \( H_\kappa \) flows \( V_\kappa(t) \) for step-like initial data \( W \). After subtracting the background profile \( W \), this is tantamount to showing that the method of commuting flows can be applied to tidal KdV \[ \text{(1.2)} \].

As the \( H_\kappa \) flows approximate KdV, we will need to construct analogous approximate equations for tidal KdV \[ \text{(1.2)} \]. Just as how we obtained tidal KdV from KdV, we subtract the background wave \( W \) from \( u \) to obtain the tidal \( H_\kappa \) flow for \( q = u - W \) with Hamiltonian \( H^W_\kappa \):

\[ e^{tJ\nabla H^W_\kappa} q = e^{tJ\nabla H_\kappa}(q + W) - W. \]

This tidal \( H_\kappa \) flow is indeed Hamiltonian, but we will not need the formula for the Hamiltonian; we only formally introduce \( H^W_\kappa \) so that we have a succinct notation for its flow. In proving Theorems \[ \text{(1.1)} \] and \[ \text{(1.2)} \] we will show that the \( H^W_\kappa \) flow is well-posed in \( H^s(\mathbb{R}) \) for \( s \geq 3 \), commutes with any other \( H^W_\kappa \) flow, and converges to tidal KdV in \( H^s(\mathbb{R}) \) as \( \kappa \to \infty \) uniformly on bounded time intervals.

This paper is organized as follows. In Section \[ \text{2} \] we define the diagonal Green’s function \( g \) for perturbations \( q \in H^{-1}(\mathbb{R}) \) of the background \( W \) which we will use
to formulate the tidal $H_\kappa$ flow. In Section 3 we prove a priori estimates and global well-posedness for the tidal $H_\kappa$ flow. As a stepping stone to convergence in $H^s$ norm, we prove in Section 4 that the tidal $H_\kappa$ flow converges in the weaker $H^{-2}$ norm. The entirety of Section 5 is dedicated to controlling the Fourier tail growth in time. We then combine the low-regularity convergence and Fourier tail control in Section 6 to obtain convergence in $H^s$ norm and conclude our main result.

Acknowledgments. I was supported in part by NSF grants DMS-1856755 and DMS-1763074. I would also like to thank my advisors, Rowan Killip and Monica Vişan, for their guidance.

2. Diagonal Green’s function

We begin by reviewing our notation and the necessary tools from [35], which can be consulted for further details.

For a Sobolev space $W^{k,p}(\mathbb{R})$ we use the spacetime norm
$$\|q\|_{C_tW^{k,p}(I \times \mathbb{R})} := \sup_{t \in I} \|q(t)\|_{W^{k,p}(\mathbb{R})}$$
for $I \subset \mathbb{R}$ an interval. In addition to the usual Sobolev spaces $W^{k,p}$ and $H^s$ we define the norm
$$\|f\|_{H^s(\mathbb{R})}^2 := \int_{\mathbb{R}} \left( \xi^2 + 4\kappa^2 \right)^s |\hat{f}(\xi)|^2 \, d\xi,$$
where our convention for the Fourier transform is
$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x) \, dx, \quad \|f\|_{L^2} = \|f\|_{L^2}.$$}

In analogy with the usual $H^s$ spaces, we have the elementary facts
$$\|wf\|_{H^s(\mathbb{R})} \lesssim \|w\|_{W^{1,\infty}} \|f\|_{H^s(\mathbb{R})}, \quad \|wf\|_{H^s(\mathbb{R})} \lesssim \|w\|_{H^1} \|f\|_{H^s(\mathbb{R})}$$
uniformly for $\kappa \geq 1$. We will exclusively use the $L^2$ pairing $\langle \cdot, \cdot \rangle$; the space $H^{-1}_{\kappa}$ is dual to $H^1_{\kappa}$ with respect to this pairing, and so the inequalities (2.2) for $H^{-1}_{\kappa}$ are implied by those for $H^1_{\kappa}$.

We write $\mathfrak{I}_p$ for the Schatten classes (also called trace ideals) of compact operators on the Hilbert space $L^2(\mathbb{R})$ whose singular values are $\ell^p$-summable. Of particular importance will be the Hilbert–Schmidt class $\mathfrak{I}_2$: recall that an operator $A$ on $L^2(\mathbb{R})$ is Hilbert–Schmidt if and only if it admits an integral kernel $a(x, y) \in L^2(\mathbb{R} \times \mathbb{R})$, and we have
$$\|A\|_{op} \leq \|A\|_{\mathfrak{I}_2} = \iint |a(x, y)|^2 \, dx \, dy.$$}

The product of two Hilbert–Schmidt operators $A$ and $B$ is of trace class $\mathfrak{I}_1$, the trace is cyclic:
$$\text{tr}(AB) := \iint a(x, y)b(y, x) \, dy \, dx = \text{tr}(BA),$$
and we have the estimate
$$|\text{tr}(AB)| \leq \|A\|_{\mathfrak{I}_2} \|B\|_{\mathfrak{I}_2}.$$}

Additionally, Hilbert–Schmidt operators form a two-sided ideal in the algebra of bounded operators, due to the inequality
$$\|BAC\|_{op} \leq \|B\|_{op} \|A\|_{\mathfrak{I}_2} \|C\|_{op}.$$
We denote the resolvent of the Schrödinger operator with zero potential by
\[ R_0(\kappa) := (-\partial_x^2 + \kappa^2)^{-1} \]
with integral kernel \( \langle \delta_x, R_0(\kappa) \delta_y \rangle = \frac{1}{2\pi} e^{-\kappa|x-y|} \).
The energy parameter \( \kappa \) will always be real and positive. Consequently, \( R_0(\kappa) \) will always be positive definite and so we may consider its positive definite square-root \( \sqrt{R_0(\kappa)} \).

The following computation from [35, Prop. 2.1] lies at the heart of our analysis:

**Lemma 2.1.** For \( q \in H^{-1}(\mathbb{R}) \) we have
\[ \left\| \sqrt{R_0(\kappa)} q \sqrt{R_0(\kappa)} \right\|_{L^2}^2 = \frac{1}{\kappa} \int \frac{|q(\xi)|^2}{\xi^2 + 4\kappa^2} \, d\xi = \frac{1}{\kappa} \|q\|_{H^{-1}}^2 . \]

The identity (2.3) guarantees that the Neumann series for the resolvent of \(-\partial^2 + q\) with \( q \in H^{-1} \) converges for \( q \) sufficiently large. This construction also works for \( q \) perturbations of \( W \):

**Lemma 2.2 (Resolvents).** Given \( q \in H^{-1}(\mathbb{R}) \), there is a unique self-adjoint operator corresponding to \(-\partial^2 + W + q\) with domain \( H^1(\mathbb{R}) \). Moreover, given \( A > 0 \) there exists a constant \( \kappa_0 \) so that the series
\[ R(\kappa, W) = (-\partial^2 + W + \kappa^2)^{-1} = \sum_{\ell=0}^{\infty} (-1)^\ell \sqrt{R_0} (\sqrt{R_0} W \sqrt{R_0})^\ell \sqrt{R_0} \]
converges absolutely to a positive definite operator for \( \kappa \geq \kappa_0 \), and the series
\[ R(\kappa, W + q) = \sum_{\ell=0}^{\infty} (-1)^\ell \sqrt{R(\kappa, W)} (\sqrt{R(\kappa, W)} q \sqrt{R(\kappa, W)})^\ell \sqrt{R(\kappa, W)} \]
converges absolutely for all \( \|q\|_{H^{-1}} \leq A \) and \( \kappa \geq \kappa_0 \).

**Proof.** Initially we require that \( \kappa \geq 1 \). As \( W \in L^\infty \), we may define the operator \(-\partial^2 + W\) via the quadratic form
\[ \phi \mapsto \int (|\phi'(x)|^2 + W(x)|\phi(x)|^2) \, dx \]
equipped with the domain \( H^1(\mathbb{R}) \). Using the elementary estimates \( \|R_0\|_{op} \leq \kappa^{-2} \) and \( \|W\|_{op} \leq \|W\|_{L^\infty} \), it is clear that the series (2.4) for \( R(\kappa, V) \) is absolutely convergent for all \( \kappa^2 \geq 2 \|W\|_{L^\infty} \).

Expanding the series (2.4) and using the identity (2.3) we estimate
\[ \left\| \sqrt{R(\kappa, W)} q \sqrt{R(\kappa, W)} \right\|_{L^2}^2 \leq \text{tr} \{ R(\kappa, W) q R(\kappa, W)^* \} \]
\[ \leq \sum_{\ell, m=0}^{\infty} \left\| R_0 W \sqrt{R_0} \right\|_{op}^{\ell+m} \left\| \sqrt{R_0} q \sqrt{R_0} \right\|_{L^2}^2 \leq 4\kappa^{-1} \|q\|_{H^{-1}}^2 \]
for all \( \kappa^2 \geq 2 \|W\|_{L^\infty} \), and hence
\[ \left\| \sqrt{R(\kappa, W)} q \sqrt{R(\kappa, W)} \right\|_{L^2} \leq 2\kappa^{-1/2} \|q\|_{H^{-1}} . \]

Consequently, for \( \phi \in H^1(\mathbb{R}) \) we have
\[ \int q(x)|\phi(x)|^2 \, dx \leq \left\| \sqrt{R(\kappa, W)} q \sqrt{R(\kappa, W)} \right\|_{op} \int (|\phi'(x)|^2 + |W(x)||\phi(x)|^2) \, dx \]
unambiguously define $x$. This demonstrates that

$$\|g(x; \kappa, W + q) - g(x; \kappa, W)\|_{H^{1/2}_\kappa} \lesssim \kappa^{-1} \|q\|_{H^{1/2}_\kappa}$$

uniformly for $q \in B_A$ and $\kappa \geq \kappa_0$.

Proof. In Fourier variables, for $\kappa \geq 1$ we compute

$$\|\sqrt{R_0} \delta_x\|^2_{L^2} \lesssim \kappa^{-1}, \quad \|\sqrt{R_0} \delta_{x+h} - \sqrt{R_0} \delta_x\|^2_{L^2} \lesssim \int \frac{|e^{ih}\xi - 1|^2}{\xi^2 + 1} d\xi \lesssim |h|.$$ 

This demonstrates that $x \mapsto \sqrt{R_0} \delta_x$ is $\frac{1}{2}$-Hölder continuous as a mapping $\mathbb{R} \to L^2$. Therefore, from the series (2.4) we see that

$$\|\langle \delta_x, [R(\kappa, W) - R_0(\kappa)] \delta_y \rangle - \langle \delta_{x'}, [R(\kappa, W) - R_0(\kappa)] \delta_{y'} \rangle\| \lesssim \kappa^{-1/2} (|x - x'|^{1/2} + |y - y'|^{1/2}) \sum_{\ell=1}^\infty (\kappa^{-2} \|W\|_{L^\infty})^\ell.$$ 

The series converges provided that $\kappa \gg \|W\|_{L^\infty}^{1/2}$. Consequently, the Green’s function $G(x, y) = \langle \delta_x, R(\kappa, W) \delta_y \rangle$ is continuous in both $x$ and $y$, and so we may unambiguously define

$$g(x; \kappa, W) = \frac{1}{2\kappa} + \sum_{\ell=1}^\infty (-1)^\ell \langle \sqrt{R_0} \delta_x, (\sqrt{R_0 W \sqrt{R_0}})^\ell \sqrt{R_0} \delta_x \rangle.$$ 

The zeroth order term $\frac{1}{2\kappa}$ can be seen directly from the integral kernel for the free resolvent $R_0(\kappa)$.

Using the series (2.4) we obtain

$$\|\sqrt{R(\kappa, W)} \delta_x\|^2_{L^2} \leq \|\sqrt{R_0} \delta_x\|^2_{L^2} \sum_{\ell=1}^\infty (\kappa^{-2} \|W\|_{L^\infty})^\ell \lesssim \kappa^{-1},$$

$$\|\sqrt{R(\kappa, W)} \delta_{x+h} - \sqrt{R_0} \delta_x\|^2_{L^2} \leq \|\sqrt{R_0} \delta_{x+h} - \sqrt{R_0} \delta_x\|^2_{L^2} \sum_{\ell=1}^\infty (\kappa^{-2} \|W\|_{L^\infty})^\ell \lesssim |h|.$$
provided that $\kappa \gg \|W\|_{L^\infty}^{1/2}$. From the series (2.3) and the estimate (2.4) we then have

$$
|\langle \delta_x, [R(\kappa, W + q) - R(\kappa, W)] \delta_y \rangle - |\langle \delta_x', [R(\kappa, W + q) - R(\kappa, W)] \delta_y' \rangle| \leq \kappa^{-1/2} (|x - x'|^{1/2} + |y - y'|^{1/2}) \sum_{\ell=1}^{\infty} (2\kappa^{-1/2}A)^{\ell}
$$

for all $q \in B_A$. The series converges provided that $\kappa \gg A^2$. Therefore, the Green’s function $G(x, y; \kappa, W + q)$ is also a continuous function of $x$ and $y$ and we may define

$$
g(x; \kappa, W + q) = g(x; \kappa, W) + \sum_{\ell=1}^{\infty} (-1)^\ell \langle \sqrt{R}\delta_x, (\sqrt{R}q\sqrt{R})^\ell \sqrt{R}\delta_x \rangle,
$$

where $R = R(\kappa, W)$. This shows that $g(x; \kappa, W + q)$ is a real analytic functional $B_A \to H^1$.

Next, we check that $g(x; \kappa, W + q) - g(x; \kappa, W)$ is in $H^{s+2}_W$ by estimating the first $s+1 \geq 0$ derivatives in $H^1_W$ by duality. The Green’s function for a translated potential is the translation of the original Green’s function:

$$(2.10) \quad g(x; \kappa, q; h)) = g(x + h; \kappa, q) \quad \text{for all } h \in \mathbb{R}.
$$

Differentiating (2.10) at $h = 0$ and using the resolvent identity, we have

$$(2.11) \quad g^{(j)}(x; \kappa, W + q) = \sum_{\ell=0}^{\infty} (-1)^\ell \langle \delta_x, [\partial^j, R(\kappa, W)](qR(\kappa, W))^\ell \delta_x \rangle.
$$

Here, $[A, B] = AB - BA$ denotes the commutator and $\partial^j$ denotes $j$ spatial partial derivatives. Within the summand there are $\ell+1$ factors of $R(\kappa, W)$, and we expand each into the series (2.4) in powers of $W$ indexed by $m_i$. For $j = 0, \ldots, s+1$ and $f \in H^{-1}_{W_\kappa}$, this yields

$$
\left| \int f(x)[g(\kappa, W + q) - g(\kappa, W)]^{(j)}(x) \, dx \right| \leq \sum_{\ell=1}^{\infty} \sum_{m_0, \ldots, m_\ell = 0}^{\infty} \left| \text{tr} \left\{ f[\partial^j, R_0(WR_0)^{m_0}qR_0 \cdots qR_0(WR_0)^{m_\ell}] \right\} \right|.
$$

We distribute the derivatives $[\partial^j, \cdot]$ using the product rule. We use the operator estimate (2.3) for each factor of $\sqrt{R_0}q\sqrt{R_0}$ and estimate the remaining factors in operator norm. Given a multiindex $\sigma \in \mathbb{N}^\ell$ with $|\sigma| \leq j$, Hölder’s inequality in Fourier variables yields

$$
\prod_{i=1}^{\ell} \|q^{(\sigma_i)}\|_{H^{-1}_{W}} \leq \|q^{(|\sigma|)}\|_{H^{-1}_{W}} \|q\|_{H^{-1}_{\kappa}} \leq \|q\|_{H^{-1}_{\kappa}} \|q\|_{H^{-1}_{W}}.
$$

As $j \leq s+1$, we have

$$
\left| \int f(x)[g(\kappa, W + q) - g(\kappa, W)]^{(j)}(x) \, dx \right| \leq \sum_{\ell=1}^{\infty} \sum_{m_0, \ldots, m_\ell = 0}^{\infty} \|f\|_{H^{-1}_{W}} \|q\|_{H^{\ell}_{\kappa}} \left( \frac{\|q\|_{H^{-1}_{\kappa}}}{\kappa^{1/2}} \right)^{\ell-1} \left( \frac{\|W\|_{W^{s+1,\infty}}}{\kappa^{2}} \right)^{m_0 + \cdots + m_\ell}.
$$
First we perform the inner sum over \( m_0, \ldots, m_\ell \); re-indexing \( m = m_0 + \cdots + m_\ell \), we have
\[
\sum_{m_0, \ldots, m_\ell \geq 0} \left( \frac{\|W\|_{W^{1,\infty}}}{{\kappa}^2} \right)^{m_0 + \cdots + m_\ell} = \sum_{m=0}^{\infty} \frac{(\ell + m)!}{\ell! m!} \left( \frac{\|W\|_{W^{1,\infty}}}{{\kappa}^2} \right)^{m} \\
\lesssim \left( 1 - \frac{\|W\|_{W^{1,\infty}}}{{\kappa}^2} \right)^{\ell + 1} \leq 1
\]
uniformly in \( \ell \), provided that \( \kappa \gg \|W\|_{W^{1,\infty}}^{1/2} \). The sum over \( \ell \geq 1 \) then converges uniformly for \( \kappa \gg A^2 \), yielding
\[
\left| \int f[g(\kappa, W + q) - g(\kappa, W)]^{(j)} \, dx \right| \lesssim \kappa^{-1} \|f\|_{H_{\kappa}^{-1}} \|q\|_{H_{\kappa}^j} \quad \text{for } j = 0, \ldots, s + 1.
\]
Taking a supremum over \( \|f\|_{H_{\kappa}^{-1}} \leq 1 \), we obtain the estimate (2.13).

As an offspring of the resolvent \( R(\kappa, q) \), the diagonal Green’s function comes with some algebraic identities. In particular, in [35, Lem. 2.5–2.6] it is shown that for Schwartz \( q \) we have
\[
\int G(x, y; \kappa, q)G(y, x; \kappa, q) \, dy = g(x; \kappa, q)
\]
and
\[
\int G(x, y; \kappa, q) \left[ -f''' + 2qf' + 2(qf)' + 4\kappa^2 f' \right] (y)G(y, x; \kappa, q) \, dy
= 2f'(y)g(x; \kappa, q) - 2f(x)g'(x; \kappa, q)
\]
for all Schwartz \( f \). As is suggested by taking \( f = g(\kappa, q) \) in (2.14), multiplying by \( 1/2g(x; \kappa, q)^2 \), and integrating in \( x \), the diagonal Green’s function satisfies the ODE
\[
g'''(\kappa, q) = 2qg'(\kappa, q) + 2[qq(\kappa, q)]' + 4\kappa^2 g'(\kappa, q);
\]
see [35, Prop. 2.3] for a proof.

Ultimately, the convergence of the approximate flows will be dominated by the linear and quadratic terms of the series (2.1) for the diagonal Green’s function. Consequently, we will now record some useful operator identities for these two terms:

**Lemma 2.4.** For \( \kappa \geq 1 \) we have the operator identities
\[
16\kappa^5(\delta_x, R_0 f R_0 \delta_x) = 16\kappa^4 R_0(2\kappa) f = \left[ 4\kappa^2 + \partial^2 + R_0(2\kappa) \partial^4 \right] f,
\]
\[
16\kappa^5(\delta_x, R_0 f R_0 h R_0 \delta_x) = 3fh - 3[R_0(2\kappa) f''']^2[R_0(2\kappa) h''']
+ 4\kappa^2[R_0(2\kappa) f''']^2[R_0(2\kappa) h'''](-5 + R_0(2\kappa) \partial^2)
+ 4\kappa^2[R_0(2\kappa) f''']^2[R_0(2\kappa) h'''](5\partial^2 + 2R_0(2\kappa) \partial^4),
\]
where \( R_0 = R_0(\kappa) \).

**Proof.** From the integral kernel formula for \( R_0(\kappa) \) we see that \( \langle \delta_x, R_0 f R_0 \delta_x \rangle = \kappa^{-1} R_0(2\kappa) f \), which demonstrates the first equality of (2.16). The second equality follows from the symbol identity
\[
\frac{16\kappa^4}{\xi^2 + 4\kappa^2} = 4\kappa^2 - \xi^2 + \frac{\xi^4}{\xi^2 + 4\kappa^2}
\]
in Fourier variables.
Now we turn to the second identity (2.17). In [35] Appendix the Fourier transform of LHS (2.17) is found to be
\[
\mathcal{F}(\text{LHS (2.17)})(\xi) = \frac{8\kappa^4}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\xi^2 + (\xi - \eta)^2 + \eta^2 + 24\kappa^2}{(\xi^2 + 4\kappa^2)(\xi - \eta)^2 + 4\kappa^2)}(\eta^2 + 4\kappa^2) \, d\eta.
\]
The operator identity (2.17) then follows from the equality
\[
\frac{8\kappa^4}{(\xi^2 + 4\kappa^2)(\xi - \eta)^2 + 4\kappa^2)} = 3 - \frac{3\eta^2(\xi - \eta)^2}{(\xi - \eta)^2 + 4\kappa^2)} + \frac{4\kappa^2\xi^2}{(\xi^2 + 4\kappa^2)}.
\]

We will also need to know that after extracting the linear and quadratic terms from \(\kappa^5g(\kappa, q + W)\), the remainder tends to zero as \(\kappa \to \infty\):

**Lemma 2.5.** *Given an integer \(s \geq 1\) and \(A > 0\), we have*
\[
\kappa^5 \| \left\{ g(\kappa, q + W) + \langle \delta_x, R_0(q + W)R_0\delta_x \rangle - \langle \delta_x, R_0(q + W)R_0(q + W)R_0\delta_x \rangle \right\}^{(s+1)} \|_{L^2} \to 0 \quad \text{as} \quad \kappa \to \infty
\]
*uniformly for \(\|q\|_{H^s} \leq A\).*

**Proof.** We estimate the \(s\)th derivative in \(H^1\) by duality. Differentiating the translation identity (2.11) at \(h = 0\), we have
\[
g^{(s)}(x; \kappa, W + q) = \sum_{\ell = 0}^{\infty} (-1)^\ell \langle \delta_x, [\partial^s, R(\kappa, W)(qR(\kappa, W))^{\ell} \delta_x] \rangle.
\]
Within the summand there are \(\ell + 1\) factors of \(R(\kappa, W)\), and we expand each into the series (2.4) in powers of \(W\) indexed by \(m_i\). For \(f \in H^{-1}\) this yields
\[
\kappa^5 \left| \int f(x) \left\{ g(\kappa, q + W) + \langle \delta_x, R_0(q + W)R_0\delta_x \rangle - \langle \delta_x, R_0(q + W)R_0(q + W)R_0\delta_x \rangle \right\}^{(s)} \, dx \right|
\]
\[
\leq \kappa^5 \sum_{\ell \geq 0, m_0, \ldots, m_\ell \geq 0} \left| \text{tr} \left\{ f[\partial^s, R_0(WR_0)^{m_0}qR_0 \cdots qR_0(WR_0)^{m_\ell}] \right\} \right|.
\]
We distribute the derivatives \([\partial^s, \cdot]\) using the product rule. We then use the operator estimate (2.3) and the observation \(\|f\|_{H^{-1}} \lesssim \kappa^{-1} \|f\|_{L^2}\) to put the highest order \(q\) in \(L^2\). In the instance that there are no factors of \(q\), we put the highest order \(W\) term in \(L^2\) and use that \(W'\) is in \(H^{s-1}\). We then estimate all other terms in operator norm; the remaining factors of \(q\) have at most \(s - 1\) derivatives, and thus may be estimated in \(L^\infty\) via the embedding \(H^1 \hookrightarrow L^\infty\). This yields
\[
\text{RHS (2.19)} \lesssim \kappa^5 \sum_{\ell \geq 0, m_0, \ldots, m_\ell \geq 0} \kappa^{(s+1)/2} \max \left\{ \|q\|_{H^s}, \|W\|_{H^{s-1}} \right\} \kappa^{3/2} \times \left( \max \left\{ \|q\|_{H^s}, \|W\|_{W^{s,\infty}} \right\} \right)^{\ell + m_0 + \cdots + m_\ell - 1}.
\]
We re-index \(m = m_0 + \cdots + m_\ell\) and sum over \(\ell + m \geq 3\) as in (2.12). The sum converges provided \(\kappa \gg \|q\|_{H^s}^{1/2}\) and \(\kappa \gg \|W\|_{W^{s,\infty}}^{1/2}\). The condition \(\ell + m \geq 3\)
guarantees that when we sum over the parenthetical term we gain a factor \( \lesssim (\kappa^{-2})^2 \), and so we obtain

\[
\text{RHS}(2.19) \lesssim \kappa^{-1} \|f\|_{H^{-1}}.
\]

uniformly for \( \|q\|_{L^2} \leq A \) and \( \kappa \geq \kappa_0(A) \). The claim (2.18) follow by taking a supremum over \( \|f\|_{H^{-1}} \leq 1 \).

\[\square\]

3. Tidal \( H_\kappa \) Flow

The argument of [35] relies upon the Hamiltonians \( H_\kappa \) whose flows approximate that of KdV as \( \kappa \to \infty \). Specifically, in [35, Prop. 3.2] it is shown that the \( H_\kappa \) flow can be expressed in terms of the diagonal Green’s function as

\[
(3.1) \quad \frac{d}{dt} u = 16\kappa^5 g'(\kappa, u) + 4\kappa^2 u'.
\]

Moreover, the flows at any two energy parameters \( \kappa \) and \( \gamma \) commute:

\[
(3.2) \quad \{H_\kappa, H_\gamma\} = 0.
\]

We need an analogous approximate flow for step-like initial data. Mimicking how we obtained tidal KdV from KdV, we subtract the background \( W \) from the function \( u \) to obtain the tidal \( H_\kappa \) flow

\[
(3.3) \quad \frac{d}{dt} q = 16\kappa^5 g'(\kappa, q + W) + 4\kappa^2 (q + W)'
\]

for \( q := u - W \). The tidal \( H_\kappa \) flow is also Hamiltonian; however, we will not make use of its Hamiltonian.

In this section we will show that the tidal \( H_\kappa \) flow is globally well-posed in \( H^s \) for all integers \( s \geq 0 \). We restrict our attention to integer \( s \) since the result for non-integer \( s \geq 0 \) follows from interpolation. Once we obtain well-posedness, the commutativity (3.2) of the \( H_\kappa \) flows implies that any two tidal \( H_\kappa \) flows commute with each other.

We begin with local well-posedness. The \( H_\kappa \) flows are easier to work with because local well-posedness follows from a contraction mapping argument.

**Lemma 3.1.** Given an integer \( s \geq -1 \) and \( A > 0 \), there exists a constant \( \kappa_0 \) so that for \( \kappa \geq \kappa_0 \) the tidal \( H_\kappa \) flows (3.3) with initial data in the closed ball \( B_A \subset H^s(\mathbb{R}) \) of radius \( A \) are locally well-posed.

**Proof.** Fix an integer \( s \geq -1 \). The solution \( q(t) \) to the tidal \( H_\kappa \) flow satisfies the integral equation

\[
q(t) = e^{t4\kappa^2 \partial_x} q(0) + \int_0^t e^{(t-\tau)4\kappa^2 \partial_x} \left[ 16\kappa^5 g'(\kappa, q(\tau) + W) + 4\kappa^2 W' \right] d\tau.
\]

A contraction mapping argument proves local well-posedness, provided we have the Lipschitz estimate

\[
\|g'(\kappa, q + W) - g'(\kappa, \tilde{q} + W)\|_{H^s} \lesssim \|g(\kappa, q + W) - g(\kappa, \tilde{q} + W)\|_{H^s} + \|q - \tilde{q}\|_{H^s}
\]

uniformly on bounded subsets of \( H^s \).

Fix \( A > 0 \). It suffices to show that \( f \mapsto \|g(\kappa, \cdot + W)\|_{q}(f) \) is bounded \( H^s \to H^{s+2} \) uniformly for \( \|q\|_{H^s} \leq A \). Using the resolvent identity we calculate

\[
d[g(\kappa, \cdot + W)]_{q}(f) = -\langle \delta_x, R(\kappa, q + W) f R(\kappa, q + W) \delta_x \rangle.
\]
Just as we did for the single resolvent \( \langle \delta_x, R(\kappa, q + W)\delta_x \rangle \) in (2.8), we estimate the first \( s + 1 \) derivatives in \( H^1 \) by duality and expand each resolvent into a series. We conclude that there exists a constant \( \kappa_0 \) such that

\[
\|d[g(\kappa, \cdot + W)]\|_{H^{s+2}} \lesssim \|f\|_{H^s}
\]

uniformly for \( q \in B_A \) and \( \kappa \geq \kappa_0 \).

In order to obtain global well-posedness, we will prove \textit{a priori} estimates in \( H^s \) for all integers \( s \geq 0 \). Our energy arguments are inspired by those of Bona and Smith [8]. The family of BBM equations which Bona–Smith uses to approximate the KdV flow does not conserve the polynomial conserved quantities of KdV. One benefit of our method is that in the case \( W \equiv 0 \), the \( H_\kappa \) flows do conserve these quantities (as is suggested by the asymptotic expansion (1.7) and Poisson commutativity), and consequently the \textit{a priori} estimates are identical to that of KdV. In particular, in the case \( W \equiv 0 \) we obtain a new proof of the Bona–Smith theorem using the low-regularity methods from [35]. (This is not subsumed by [35, Cor. 5.3], which only addresses \( H^s(\mathbb{R}) \) for \( s \in [-1, 0) \).)

Our energy arguments are much simplified in the case \( \kappa = \infty \), where the tidal \( H_\kappa \) flow becomes tidal KdV. Our manipulations are motivated by the corresponding tidal KdV terms at \( \kappa = \infty \), where operations involving commutators and cycling the trace correspond to more elementary operations involving integration by parts. In particular, the reason for the restriction \( s \geq 3 \) is the same as in [8]: when estimating \( \frac{d}{dt}\|q^{(s)}(t)\|_{L^2}^2 \) under the KdV flow, \( s = 3 \) is the smallest integer for which the nonlinear contribution can be estimated in terms of \( \|q^{(s)}(t)\|_{L^2}^2 \) provided that we already control \( q(t) \) in \( H^{s-1} \).

We begin with \( s = 0 \):

**Proposition 3.2.** Given \( A, T > 0 \) there exist constants \( C \) and \( \kappa_0 \) such that solutions \( q_\kappa(t) \) to the tidal \( H_\kappa \) flow (3.3) obey

\[
\|q_\kappa(0)\|_{L^2} \leq A \quad \Rightarrow \quad \|q_\kappa(t)\|_{L^2} \leq C \quad \text{for all } |t| \leq T \text{ and } \kappa \geq \kappa_0.
\]

**Proof.** By approximation and local well-posedness we may assume that \( q(0) \in H^\infty \). Let

\[
E_0(t) := \frac{1}{2} \int q_\kappa(t, x)^2 \, dx.
\]

This is the first polynomial conserved quantity of the KdV hierarchy, and in the case \( W \equiv 0 \) one can directly show that \( \frac{d}{dt}E_0 = 0 \) under the \( H_\kappa \) flow using the ODE (2.11) satisfied by the diagonal Green’s function.

To counteract the factor of \( \kappa^5 \) in the tidal \( H_\kappa \) flow and obtain a bound for all \( \kappa \) large, we will extract the linear and quadratic terms. Using the translation identity (2.11), we write

\[
\frac{d}{dt}E_0 = \int q_\kappa \{ -16\kappa^5 \langle \delta_x, R_0q_\kappa R_0\delta_x \rangle + 4\kappa^2q_\kappa \} \, dx
\]

(3.4) \[+ \int q_\kappa \{ -16\kappa^5 \langle \delta_x, R_0W'R_0\delta_x \rangle + 4\kappa^2W' \} \, dx\]

(3.5) \[+ 16\kappa^5 \int q_\kappa \langle \delta_x, [\partial, R_0q_\kappa R_0q_\kappa R_0\delta_x] \rangle \, dx\]

(3.6)
Using the identity (2.3) and the observation
\[ \| \kappa W \| \lesssim \kappa, \|
\]
we will estimate the terms (3.4)–(3.9) separately.

The first linear contribution (3.5) from \( W \).
Using the operator identity (2.16) we write
\[ (3.4) = \int q_{\kappa} \left\{ -16\kappa^4 R_0 (2\kappa) q_{\kappa}^2 + 4\kappa^2 q_{\kappa}^2 \right\} dx. \]
This vanishes because the integrand is odd in Fourier variables, or equivalently the integrand is a total derivative.

Now we estimate the linear contribution (3.5) from \( W \). Using the operator identity (2.16) we write
\[ (3.5) = \int q_{\kappa} \left\{ - W'' - \left[ R_0 (2\kappa) W^{(5)} \right] \right\} dx \]
\[ \lesssim \| q_{\kappa} \|_{L^2} \left( \| W'' \|_{L^2} + \kappa^{-2} \| W^{(5)} \|_{L^2} \right) \lesssim E_0^{1/2} \lesssim E_0 + 1. \]
Note that \( W'' \) is Schwartz, and we allow our implicit constants to depend on the fixed function \( W \).

The first quadratic contribution (3.6) also vanishes. Distributing the derivative \( [\partial, \cdot] \) and noting that \( [\partial, R_0] = 0 \), we write
\[ (3.6) = 16\kappa^5 \left( \text{tr} \{ q_{\kappa} R_0 [\partial, q_{\kappa}] R_0 q_{\kappa} R_0 \} + \text{tr} \{ q_{\kappa} R_0 q_{\kappa} R_0 [\partial, q_{\kappa}] R_0 \} \right). \]
Both of these terms vanish by cycling the trace.

Next we turn to the second quadratic contribution (3.7). By linearity and cycling the trace, we can “integrate by parts” to write
\[ (3.7) = 16\kappa^5 \left( - \text{tr} \{ [\partial, q_{\kappa}] R_0 W R_0 q_{\kappa} R_0 \} + \text{tr} \{ q_{\kappa} [\partial, R_0 q_{\kappa}] R_0 W R_0 \} \right) \]
\[ = 16\kappa^5 \left( \text{tr} \{ q_{\kappa} R_0 q_{\kappa} [\partial, W] R_0 \} \right). \]
Using the estimate (2.23) and the observations \( \| \sqrt{R_0} \|_{op} \lesssim \kappa^{-1} \) and \( \| f \|_{H^{-1}} \lesssim \kappa^{-1} \| f \|_{L^2} \), we estimate
\[ (3.7) \lesssim \kappa^5 \left\| \sqrt{R_0 q_{\kappa}} \sqrt{R_0} \right\|_{L^2} \left\| \sqrt{R_0 W'} \sqrt{R_0} \right\|_{op} \lesssim \| W' \|_{L^\infty} E_0. \]

The quadratic \( W \) contribution (3.8) is easily estimated. We distribute the derivative and estimate
\[ (3.8) \lesssim \kappa^5 \left( \left\| \sqrt{R_0 q_{\kappa}} \sqrt{R_0} \right\|_{L^2} \left\| \sqrt{R_0 W'} \sqrt{R_0} \right\|_{op} \right). \]
Using the identity (2.23) and the observation \( \| f \|_{H^{-1}} \lesssim \kappa^{-1} \| f \|_{L^2} \), we obtain
\[ (3.8) \lesssim E_0^{1/2} \lesssim E_0 + 1. \]
For the series tail (3.9), we integrate by parts once to put the derivative on \( q_\kappa \) and we write
\[
|3.9| \leq 16\kappa^5 \sum_{\ell \geq 0, m_0 + \cdots + m_\ell \geq 0 \atop \ell + m_0 + \cdots + m_\ell \geq 3} \left| \text{tr} \left\{ q_\kappa R_0 (W R_0)^{m_0} q R_0 \cdots q R_0 (W R_0)^{m_\ell} \right\} \right|.
\]
Observe that the summand vanishes for \( m_0 + \cdots + m_\ell = 0 \) by writing \( q_\kappa = [\partial, q_\kappa] \) and cycling the trace, and so we may insert the condition \( m_0 + \cdots + m_\ell \geq 1 \) in the summation. We use the operator estimate (2.3) and the observation \( \lesssim \) factor converges provided \( \kappa \) from \( g \).

We split the sum into \( m \) and cycling the trace, and so we may insert the condition \( m_0 + \cdots + m_\ell \geq 1 \) in the summation. We use the operator estimate (2.3) and the observation \( \lesssim \) factor converges provided \( \kappa \) from \( g \).

Observe that the summand vanishes for \( m_0 + \cdots + m_\ell = 0 \) by writing \( q_\kappa = [\partial, q_\kappa] \) and cycling the trace, and so we may insert the condition \( m_0 + \cdots + m_\ell \geq 1 \) in the summation. We use the operator estimate (2.3) and the observation \( \lesssim \) factor converges provided \( \kappa \) from \( g \).

The sum converges provided \( \kappa \gg E_0^{1/3}(t) \) and \( \kappa \gg \|W\|_{L^\infty}^{1/2} \). The conditions \( m \geq 1 \) and \( \ell + m \geq 3 \) guarantee that when we sum over the two parenthetical terms we gain a factor \( \lesssim (\kappa^{-3/2})^2(\kappa^{-2}) \), and so we obtain
\[
\lesssim \kappa^{-1/2} \|q_\kappa\|_{L^2}
\]
for all \( \kappa \) large. Taking a supremum over \( \|f\|_{H^{-1}} \leq 1 \) and restricting to \( \kappa \) sufficiently large, we conclude there exists \( \kappa_0(E_0(t)) \) such that
\[
|3.9| \leq E_0^{1/2} \lesssim E_0 + 1 \quad \text{uniformly for } \kappa \geq \kappa_0(E_0(t)).
\]

Altogether, we have shown that there exist constants \( C \) and \( \kappa_0(E_0(t)) \) such that
\[
\left| \frac{d}{dt} E_0 \right| \leq C(E_0 + 1) \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(E_0(t)).
\]
Grönwall’s inequality then yields the bound
\[
E_0(t) \leq (E_0(0) + 1)e^{CT} - 1 \quad \text{uniformly for } |t| \leq T, \ \kappa \geq \kappa_0((E_0(0) + 1)e^{CT} - 1),
\]
which concludes the proof. \( \square \)

Next, we control the growth of the \( H^1 \) norm:

**Proposition 3.3.** Given \( A, T > 0 \) there exist constants \( C \) and \( \kappa_0 \) such that solutions \( q_\kappa(t) \) to the tidal \( H_\kappa \) flow (3.3) obey
\[
\|q(0)\|_{H^1} \leq A \quad \implies \quad \|q_\kappa(t)\|_{H^1} \leq C \quad \text{for all } |t| \leq T \text{ and } \kappa \geq \kappa_0.
\]

**Proof.** By approximation and local well-posedness we may assume that \( q(0) \in H^\infty \).

Let
\[
E_1(t) := \int \left\{ \frac{1}{2} (q_\kappa(t, x))^2 + q_\kappa(t, x)^3 \right\} \, dx
\]
denote the next polynomial conserved quantity of KdV.

We multiply the tidal \( H_\kappa \) flow (3.3) by \(-q_\kappa^3 + 3q_\kappa^2\) and integrate in space to obtain an expression for \( \frac{d}{dt} E_1 \). We then integrate by parts to remove the derivative from \( q_\kappa^3 + W - q_\kappa W \), expand both diagonal Green’s functions using the relation (2.13), and apply the identity (2.14) to obtain
Note that in the case \( W \equiv 0 \), all three integrals vanish and \( E_1 \) is conserved as expected. We will estimate the terms (3.10)–(3.12) separately.

We begin with the term (3.10). We integrate by parts once, expand \( g(\kappa, W) \) in a series, and extract the linear term:

\[
\begin{align*}
(3.10) &= \int q_\kappa'' [16\kappa^5 g'(\kappa, W) + 4\kappa^2 W''] \, dx \\
&+ 16\kappa^5 \sum_{m \geq 2} (-1)^m \text{tr} \{ q_\kappa' [\partial^2, R_0(W R_0)^m] \}.
\end{align*}
\]

For the first term we use the operator identity (2.16) to estimate

\[
\left| \int q_\kappa'' [-16\kappa^5 <\delta_x, R_0W''R_0\delta_x> + 4\kappa^2 W'''] \, dx \right|
\]

\[
= \left| \int q_\kappa'' [-W(4) - R_0(2\kappa)W(6)] \, dx \right| \lesssim \| q_\kappa'' \|_{L^2} (\| W'(4) \|_{L^2} + \kappa^{-2} \| W(6) \|).
\]

For the second term we distribute the two derivatives \([\partial^2, \cdot]\), use the estimate (2.28) and the observation \( \| f \|_{H^{-1}} \lesssim \kappa^{-1} \| f \|_{L^2} \) to put \( q_\kappa' \) and the highest order \( W \) term in \( L^2 \), and put the remaining terms in operator norm:

\[
16\kappa^5 \sum_{m \geq 2} \left| \text{tr} \{ q_\kappa' [\partial^2, R_0(W R_0)^m] \} \right|
\]

\[
\lesssim \kappa^5 \sum_{m \geq 2} m^2 \| q_\kappa'' \|_{L^2} \| W' \|_{H^1} \left( \frac{\| W \|_{W^{1,\infty}}}{\kappa^2} \right)^{m-1} \lesssim \| W' \|_{H^1} \| W \|_{W^{1,\infty}} \| q_\kappa'' \|_{L^2}
\]

uniformly for \( \kappa \gg \| W \|_{W^{1,\infty}}^{1/2} \). Altogether we conclude

\[
(3.10) \lesssim \| q_\kappa'' \|_{L^2}^2 + 1
\]

uniformly for \( \kappa \) large.

Next we turn to the term (3.11). Expanding \( g(\kappa, q_\kappa + W) \) and extracting the terms that are linear and quadratic in \( q_\kappa \) and \( W \), we write

\[
\begin{align*}
(3.11) &= -64\kappa^7 \int q_\kappa'' <\delta_x, R_0q_\kappa R_0\delta_x> \, dx \\
&+ 64\kappa^7 \int q_\kappa'' <\delta_x, R_0q_\kappa R_0q_\kappa R_0\delta_x> \, dx \\
&+ 4\kappa^2 \int \{ 16\kappa^5 q_\kappa'' (<\delta_x, R_0W R_0q_\kappa R_0\delta_x> + <\delta_x, R_0q_\kappa R_0W R_0\delta_x>) \} \, dx
\end{align*}
\]
Proposition 3.2, and so we obtain

\[ \ell \geq 1, m_0, \ldots, m_\ell \geq 0 \]

\[ \sum_{\ell \geq 1, m_0, \ldots, m_\ell \geq 0} (-1)^{\ell + m_0 + \cdots + m_\ell} \text{tr} \left\{ q'_\kappa \, R_0 (W R_0)^{m_0} q_\kappa R_0 \cdots q'_\kappa R_0 (W R_0)^{m_\ell} \right\} \cdot \]

The terms \((3.15)\) and \((3.14)\) vanish by cycling the trace:

\[ (3.15) = -64\kappa^7 \text{tr} \{ [\partial, q'_\kappa] R_0 q_\kappa R_0 \} = 0, \]

\[ (3.14) = 64\kappa^7 \text{tr} \{ [\partial, q'_\kappa] R_0 q_\kappa R_0 q_\kappa R_0 \} = 0. \]

For the term \((3.15)\), we integrate by parts to replace \(3W' q^2_\kappa\) by \(-6W q'_\kappa q_\kappa\). We then use the operator identity \((2.17)\) and the estimates \(\| R_0 (2\kappa) \partial_j \|_{\text{op}} \lesssim \kappa^{j-2} \) for \(j = 0, 1, 2\) (the estimate for \(j = 0\) is also true as an operator on \(L^\infty\) by the explicit kernel formula for \(R_0\) and Young’s inequality) to conclude

\[ \| (3.15) \| \lesssim \| q'_\kappa \|_{L^2}^2 + 1. \]

For the tail \((3.16)\) we estimate

\[ (3.16) \leq 64\kappa^7 \sum_{\ell \geq 1, m_0, \ldots, m_\ell \geq 0} \left| \text{tr} \left\{ q'_\kappa \, R_0 (W R_0)^{m_0} q_\kappa R_0 \cdots q'_\kappa R_0 (W R_0)^{m_\ell} \right\} \right|. \]

We put \(q'_\kappa\) and one other \(q_\kappa\) in \(L^2\) via the estimate \((2.3)\) and put the remaining terms in operator norm. We have \(\| q_\kappa \|_{L^2} \lesssim 1\) uniformly for \(|t| \leq T\) and \(\kappa\) large by Proposition 3.2 and so we obtain

\[ \lesssim \kappa^7 \sum_{\ell \geq 1, m_0, \ldots, m_\ell \geq 0} \frac{\| q'_\kappa \|_{L^2}}{\kappa^3} \left( \frac{\| q_\kappa \|_{L^\infty}}{\kappa^2} \right)^{\ell-1} \left( \frac{\| W \|_{L^\infty}}{\kappa^2} \right)^{m_0 + \cdots + m_\ell} \cdot \]

The condition \(\ell + m_0 + \cdots + m_\ell \geq 3\) yields a gain \(\lesssim (\kappa^{-2})^2\) when we sum over the two parenthetical terms, and so we obtain

\[ \lesssim \| q'_\kappa \|_{L^2} \lesssim \| q'_\kappa \|_{L^2}^2 + 1 \]

provided that \(\kappa \gg \| q_\kappa \|_{L^2}^{1/2}\) and \(\kappa \gg \| W \|_{L^\infty}^{1/2}\). From Proposition 3.2 we know that

\[ \| q_\kappa \|_{L^2} \lesssim 1, \quad \| q_\kappa \|_{L^\infty} \leq \| q_\kappa \|_{L^2}^{1/2} \| q'_\kappa \|_{L^2}^{1/2} \lesssim \| q'_\kappa \|_{L^2} \]

for \(\kappa \geq \kappa_0(T, \| q_0 \|_{L^2})\) sufficiently large, and so altogether we conclude

\[ \| (3.11) \| \lesssim \| q'_\kappa \|_{L^2}^2 + 1 \quad \text{uniformly for } \kappa \geq \kappa_0(\| q_\kappa \|_{H^1}). \]

It remains to estimate the term \((3.12)\). Expanding \(g(\kappa, q_\kappa + W) - g(\kappa, q_\kappa + W)\) and extracting the linear term, we write

\[ (3.12) \leq 32\kappa^5 \int \left[ W q'_\kappa + (W q_\kappa)' \right] \langle \delta_x, R_0 q_\kappa R_0 \delta_x \rangle \, dx \]

\[ + 32\kappa^5 \sum_{\ell \geq 1, m_0, \ldots, m_\ell \geq 0} \left| \text{tr} \left\{ [W q'_\kappa + (W q_\kappa)'] \right\} \right| \cdot \]

For the first term \((3.18)\) we use the operator identity \((2.10)\) to write

\[ (3.18) = 8\kappa^2 \int [W q'_\kappa + (W q_\kappa)'] q_\kappa \, dx + \int [W q'_\kappa + (W q_\kappa)'] [q'_\kappa + R_0 (2\kappa) \partial^2 q'_\kappa] \, dx. \]
The first integral vanishes because the integrand is a total derivative. For the second integral, we integrate by parts to obtain
\[ \int [W q'_\kappa + (W q_\kappa)^\prime] [q''_\kappa + R_0(2\kappa) \partial^2 q''_\kappa] \, dx \]
\[ = - \int [2W' q'_\kappa + W'' q_\kappa] [q''_\kappa + R_0(2\kappa) \partial^2 q''_\kappa] \, dx \]
\[ + \int \{ W q'_\kappa [R_0(2\kappa) \partial^2 q''_\kappa] - W q''_\kappa [R_0(2\kappa) \partial^2 q''_\kappa] \} \, dx. \]
Those terms without \( q''_\kappa \) can be estimated using Cauchy–Schwarz and the observation \( \| R_0(2\kappa) \partial^2 \|_{\text{op}} \lesssim 1 \). For the remaining terms, we “integrate by parts” in Fourier variables:
\[ \left| \int \{ W q'_\kappa [R_0(2\kappa) \partial^2 q''_\kappa] - W q''_\kappa [R_0(2\kappa) \partial^2 q''_\kappa] \} \, dx \right| \]
\[ = (2\pi)^{-\frac{1}{2}} \left| \int \hat{W}(\xi - \eta) \hat{q}'_\kappa(\eta) \hat{q}_\kappa(\xi) \frac{i(\xi - \eta)\xi^2}{\xi^2 + 4\kappa^2} \, d\eta \right| \]
\[ \lesssim \int |\hat{W}'(\xi)| |\hat{q}_\kappa(\xi)| \, d\xi \lesssim \| \hat{W}' \|_{L^1} \| q'_\kappa \|_{L^2} \lesssim \| W' \|_{H^1} \| q'_\kappa \|_{L^2}. \]
In the last inequality, we used Cauchy–Schwarz to estimate
\[ \int |\hat{W}'(\xi)| \, d\xi \lesssim \left( \int \frac{d\xi}{\xi^2 + 1} \right)^\frac{1}{2} \left( \int (\xi^2 + 1) |\hat{W}'(\xi)|^2 \, d\xi \right)^\frac{1}{2}. \]
Together, we conclude
\[ \| q'_{\kappa} \|_{L^2} \lesssim \| q'_{\kappa} \|_{L^2} + 1. \]
For the tail (3.19) we put \( W q'_\kappa + (W q_\kappa)^\prime \) and one \( q_\kappa \) in \( L^2 \) using the estimate (2.3) and the observation \( \| f \|_{H^{-1}} \lesssim \kappa^{-1} \| f \|_{L^2} \), and we put all other terms in operator norm to obtain
\[ \| q_{\kappa} \|_{H^1} \lesssim \kappa^{-5} \sum_{\ell \geq 1, m_0, \ldots, m_\ell \geq 0} \frac{\| q_{\kappa} \|_{L^\infty}}{\kappa^2} \left( \frac{\| q_{\kappa} \|_{L^\infty}}{\kappa^2} \right)^{\ell-1} \left( \frac{\| W \|_{L^\infty}}{\kappa^2} \right)^{m_0 + \cdots + m_\ell} \lesssim \| q_{\kappa} \|_{H^1} \]
provided that \( \kappa \gg \| q_{\kappa} \|_{L^\infty}^{1/10} \) and \( \kappa \gg \| W \|_{L^\infty}^{1/10} \). Note the condition \( \ell + m_0 + \cdots + m_\ell \geq 2 \) yielded a gain \( \lesssim \kappa^{-2} \) when we summed over the parenthetical terms. Recalling our control (3.17) over the \( L^\infty \) norm of \( q_{\kappa} \), we conclude
\[ \| q_{\kappa} \|_{L^2} \lesssim \| q'_{\kappa} \|_{L^2} + 1 \quad \text{uniformly for } \kappa \geq \kappa_0(\| q'_{\kappa} \|_{L^2}). \]
Altogether we have obtained
\[ \left| \frac{d}{dt} E_1 \right| \lesssim \| q'_{\kappa} \|_{L^2}^2 + 1 \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(\| q'_{\kappa} \|_{L^2}). \]
We use \( E_1 \) and the estimates (3.17) to bound \( q'_{\kappa} \) in \( L^2 \):
\[ \| q'_{\kappa} \|_{L^2} \lesssim E_1 + \int q_{\kappa}^2 \, dx \lesssim E_1 + \| q_{\kappa} \|_{L^2}^1. \]
Together, we conclude that there exists a constant \( C = C(T, A) \) such that
\[ \| q'_{\kappa}(t) \|_{L^2}^2 \lesssim C + C \| q_{\kappa} \|_{L^2}^{1/2} + C \int_0^t \| q'_{\kappa}(s) \|_{L^2}^2 \, ds. \]
Proposition 3.4. Given \( q \) arguments are much simplified and the a priori proof.

**Proof.** Let \( \|q(t)\|_{L^2} \leq 1 \) for \( |t| \leq T \), and so we conclude

\[
\|q(t)\|_{L^2} \leq C(T, \|q(0)\|_{H^1}) \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(T, \|q(0)\|_{H^1}). \]

The last space for which we need to rely upon the corresponding polynomial conserved quantity to obtain an a priori estimate is \( H^2 \). Starting with \( H^3 \), the energy arguments are much simplified and the a priori estimates are proven inductively.

**Proposition 3.4.** Given \( A, T > 0 \) there exist constants \( C \) and \( \kappa_0 \) such that solutions \( q_\kappa(t) \) to the tidal \( H_\kappa \) flow (3.3) obey

\[
\|q(0)\|_{H^2} \leq A \implies \|q_\kappa(t)\|_{H^2} \leq C \quad \text{for all } |t| \leq T \text{ and } \kappa \geq \kappa_0.
\]

**Proof.** By approximation and local well-posedness we may assume that \( q(0) \in H^\infty \). Let

\[
E_2(t) := \int \left\{ \frac{1}{2}(q'_\kappa(t,x))^2 + 5q_\kappa(t,x)(q'_\kappa(t,x))^2 + \frac{5}{2}g_\kappa(t,x)^4 \right\} \, dx
\]

denote the third energy in the KdV hierarchy of conserved quantities.

We multiply the tidal \( H_\kappa \) flow (3.3) by \( q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q'' + 10q_\kappa^2 \) and integrate in space to obtain an expression for \( \frac{d}{dt} E_2 \). We then integrate by parts to remove the derivative from \( g(\kappa, q_\kappa + W) - g(\kappa, W) \), expand both diagonal Green’s functions using the relation (2.13), and apply the identity (2.14) to obtain

\[
\frac{d}{dt} E_2 = \int \left[ q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q'' + 10q_\kappa^2 \right] [16\kappa^5 g'(\kappa, W) + 4\kappa^2 W'] \, dx
\]

\[
+ 32\kappa^5 \int \left[ -q'_\kappa q'' - 2q_\kappa q''' + 15q_\kappa^2 q'_\kappa \right] g(\kappa, W) \, dx
\]

\[
+ 2\kappa^5 \int \left[ 2\kappa^2 (-q'''' + 6q_\kappa q'' - 2Wq'' + W'q'' + 12Wq_\kappa q_\kappa' + 3W'q_\kappa^2 \right]
\]

\[
\times \left[ g(\kappa, q_\kappa + W) - g(\kappa, W) \right] \, dx.
\]

Note that in the case \( W \equiv 0 \), all three integrals vanish and \( E_2 \) is conserved as expected.

In order to exhibit cancellation in the limit \( \kappa \to \infty \), we expand \( g(\kappa, q_\kappa + W) - g(\kappa, W) \) in powers of \( q_\kappa \) and \( W \) and regroup terms:

\[
\frac{d}{dt} E_2
\]

(3.20) \[ = \int \left[ q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q'' \right] [16\kappa^5 g'(\kappa, W) + 4\kappa^2 W'] \]

(3.21) \[ - 32\kappa^5 \int \left[ q_\kappa q' + 2q_\kappa q'' \right] \left[ g(\kappa, W) + \langle \delta_x, R_0 W R_0 \delta_x \rangle \right] \]

(3.22) \[ + 64\kappa^7 \int \left[ q_\kappa''' + 6q_\kappa q'' \right] \left[ g(\kappa, q_\kappa) - \frac{1}{2\kappa} \right] \]

(3.23) \[ + 32\kappa^5 \int \left\{ -2\kappa^2 q''' \left[ \langle \delta_x, R_0 W R_0 q_\kappa R_0 \delta_x \rangle + \langle \delta_x, R_0 q_\kappa R_0 W R_0 \delta_x \rangle \right] \right. \]

\[ + \left. [2Wq_\kappa'' + W'q_\kappa' \langle \delta_x, R_0 q_\kappa R_0 \delta_x \rangle + \langle q_\kappa q''' + 2q_\kappa q'' \rangle \langle \delta_x, R_0 W R_0 \delta_x \rangle \right\} \]

(3.24) \[ + 8\kappa^2 \int \left\{ 5Wq_\kappa^2 - 12\kappa^3 [4Wq_\kappa q_\kappa' + W'q_\kappa^2] \langle \delta_x, R_0 q_\kappa R_0 \delta_x \rangle \right. \]

\[ + \left. 48\kappa^5 q_\kappa q_\kappa' \left[ \langle \delta_x, R_0 W R_0 q_\kappa R_0 \delta_x \rangle + \langle \delta_x, R_0 q_\kappa R_0 W R_0 \delta_x \rangle \right] \right\} \]
is reflected in the condition

\[ m(3.26) \]

\[ \text{for the term } W \text{ terms and the highest order } \kappa \text{ we estimate the remaining terms in operator } \kappa \text{ separately.} \]

For the term \((3.20)\) we expand \(g(\kappa, W)\) in powers of \(W\):

\[ \|q_{n}\|_{H^{1}} \lesssim 1 \text{ by Proposition } 3.3 \text{ then Cauchy–Schwarz yields} \]

\[ \int [q_{n}(4) - 5(q_{n}')^{2} - 10q_{n}q_{n}''] \cdot (-\partial x, R_{0}W'R_{0}\delta x) + 4\kappa^{2}W' \]

\[ = \int [q_{n}(4) - 5(q_{n}')^{2} - 10q_{n}q_{n}''] \cdot -W'' - R_{0}(2\kappa)W^{(5)}. \]

For the term \(q_{n}^{(4)}\) we integrate by parts twice. As \(W'\) is Schwartz and \(\|q_{n}\|_{H^{1}} \lesssim 1\) by Proposition 3.3, then Cauchy–Schwarz yields

\[ \int [q_{n}(4) - 5(q_{n}')^{2} - 10q_{n}q_{n}''] \cdot -W'' - R_{0}(2\kappa)W^{(5)} \lesssim \|q_{n}''\|_{L^{2}} + 1 \lesssim \|q_{n}''\|_{L^{2}} + 1. \]

For the tail, we again integrate by parts twice for \(q_{n}^{(4)}\). We then estimate the \(q_{n}\) terms and the highest order \(W\) term in \(L^{2}\) using the estimate \((2.3)\) and \(\|f\|_{H_{-1}^{-1}} \lesssim \kappa^{-1} \|f\|_{L^{2}}\) and the remaining terms in \(L^{\infty}\). This yields

\[ 16\kappa^{5} \sum_{m \geq 2} \|q_{n}''\|_{L^{2}} \|W\|_{H^{2}} \left( \frac{\|W\|_{W^{1,\infty}}}{\kappa^{2}} \right)^{m-1} \lesssim \|q_{n}''\|_{L^{2}} \lesssim \|q_{n}''\|_{L^{2}} + 1 \]

provided that \(\kappa \gg \|W\|_{W^{1,\infty}}^{-1/2} \).

For the term \((3.21)\) we integrate by parts once to write

\[ (3.21) \leq 32\kappa^{5} \sum_{m \geq 2} \|q_{n}''\|_{L^{2}} \|q_{n}''\|_{L^{2}} \|q_{n}''\|_{L^{2}} \lesssim \|q_{n}''\|_{L^{2}} + 1. \]

We estimate the \(q_{n}\) terms and the one factor of \(W''\) in \(L^{2}\) using the estimate \((2.3)\) and the observation \(\|f\|_{H_{-1}^{-1}} \lesssim \kappa^{-1} \|f\|_{L^{2}}\), we estimate the remaining terms in operator norm. By Proposition 3.3, we have

\[ \|q_{n}\|_{L^{\infty}} \leq \|q_{n}''\|_{L^{2}} \|q_{n}''\|_{L^{2}} \lesssim \|q_{n}''\|_{L^{2}} + 1. \]
Together, we obtain
\[
|3.21| \lesssim \kappa^5 \sum_{m \geq 2} \left( \frac{\|q''_k\|_{L^2} + 1}{\kappa^3} \right) \frac{\|W\|_{L^\infty}}{\kappa^2} \left( \frac{\|W\|_{L^\infty}}{\kappa^2} \right)^{m-1} \lesssim \|q''_k\|_{L^2} \lesssim \|q''_k\|_{L^2}^2 + 1
\]
provided that \( \kappa \gg \|W\|_{L^2}^{-1} \).

The term \(3.22\) vanishes. Indeed, after integrating by parts and adding a total derivative we have
\[
3.22 = -64\kappa^7 \int (-q''_k + 3q'^2_k) q'(\kappa, q_k) \, dx
= -4\kappa^2 \int (-q''_k + 3q'^2_k) [16\kappa^5 g'(\kappa, q_k) + 4\kappa^2 q_k] \, dx.
\]
The integral on the RHS is \(\frac{d}{d\kappa} E_1\) in the case \( W \equiv 0 \), which we observed to vanish in Proposition 3.3.

For the term \(3.23\), we integrate by parts to write
\[
3.23 = 32\kappa^5 \int \left\{ 2\kappa^2 q''_k [\delta_x, R_0 W R_0 q_k R_0 \delta_x] + \langle \delta_x, R_0 q_k R_0 W R_0 \delta_x \rangle 
- 2W q''_k \langle \delta_x, R_0 q'_k R_0 \delta_x \rangle - q''_k \langle \delta_x, R_0 W R_0 \delta_x \rangle 
+ 2\kappa^2 q''_k [\delta_x, R_0 W R_0 q_k R_0 \delta_x] + \langle \delta_x, R_0 q_k R_0 W R_0 \delta_x \rangle 
- W' q''_k \langle \delta_x, R_0 q_k R_0 \delta_x \rangle - 2q_k q''_k \langle \delta_x, R_0 W R_0 \delta_x \rangle \right\}.
\]
We use the operator identities \(2.10\) and \(2.17\). Observe that the leading order contributions as \( \kappa \to \infty \) (i.e. \( 4\kappa^2 f \) in \(2.10\) and \( 3f h \) in \(2.17\)) cancel out. The remainder is easily estimated, yielding
\[
|3.23| \lesssim \|q''_k\|_{L^2} + 1.
\]

For the term \(3.24\), we write
\[
3.24 = 8\kappa^2 \int \left\{ + 48\kappa^5 q_k q'_k [\delta_x, R_0 W R_0 q_k R_0 \delta_x] + \langle \delta_x, R_0 q_k R_0 W R_0 \delta_x \rangle 
- 15W q''_k - 24\kappa^2 W q_k q'_k \langle \delta_x, R_0 q_k R_0 \delta_x \rangle - 12\kappa^3 W q'_k \langle \delta_x, R_0 q_k R_0 \delta_x \rangle + 2\kappa^2 q''_k [\delta_x, R_0 W R_0 q_k R_0 \delta_x] + \langle \delta_x, R_0 q_k R_0 W R_0 \delta_x \rangle \right\}.
\]
We use the operator identities \(2.10\) and \(2.17\). Observe that the leading order contributions as \( \kappa \to \infty \) (i.e. \( 4\kappa^2 f \) in \(2.10\) and \( 3f h \) in \(2.17\)) cancel out. The remainder is easily estimated, yielding
\[
|3.24| \lesssim \|q''_k\|_{L^2} + 1.
\]

For the tail \(3.25\), we integrate by parts once to obtain
\[
3.25 \lesssim \kappa^7 \sum_{\ell \geq 1, m_0 + \cdots + m_\ell \geq 1} \left| \text{tr} \left\{ (-q''_k + 3q'^2_k) \partial_x R_0 (W R_0)^m q_k R_0 \cdots q_k R_0 (W R_0)^m \right\} \right|.
\]
We put \(-q''_k + 3q'^2_k\) and the highest order \(q_k\) in \( L^2 \) using the identity \(2.23\) and the observation \( \|f\|_{H^{-1}} \lesssim \kappa^{-1} \|f\|_{L^2} \), and we estimate the remaining terms in operator norm:
\[
\lesssim \kappa^7 \sum_{\ell \geq 1, m_0 + \cdots + m_\ell \geq 1} \left( \frac{\|q_k\|_{H^1}}{\kappa^3} \right)^{\ell-1} \left( \frac{\|W\|_{L^\infty}}{\kappa^2} \right)^{m_0 + \cdots + m_\ell}.
\]
We re-index \( m = m_0 + \cdots + m_\ell \) and sum over \( \ell + m \) as in (2.12). The condition \( \ell + m_0 + \cdots + m_\ell \geq 3 \) guarantees a gain \( \lesssim (\kappa^{-2})^2 \) when we sum over the two parenthetical terms, and so we obtain an acceptable bound.

For the tail (3.26), we estimate
\[
|3.26| \lesssim \kappa^5 \sum_{1 \leq m_0, \ldots, m_\ell \geq 0} \left| \text{tr} \left\{ -4Wq'''_\kappa - 2W'q''_\kappa + 24Wq_\kappa q'_\kappa + 6W'q''_\kappa \right\} \times R_0(WR_0)^m q_\kappa R_0 \cdots q_\kappa R_0(WR_0)^m \right\}.
\]

For the term \( q'''_\kappa \) we integrate by parts once. We then put the square-bracketed term and the highest order factor of \( q_\kappa \) in \( L^2 \) using the identity (2.3) and the observation \( \|f\|_{H^{-1}} \lesssim \|f\|_{L^2} \), and we estimate the remaining terms in operator norm:
\[
|3.26| \lesssim \kappa^5 \sum_{1 \leq m_0, \ldots, m_\ell \geq 0} \left\| q''_\kappa \right\|_{L^2} + 1 \left\| q_\kappa \right\|_{H^1} \left( \frac{\left\| W \right\|_{W^{1,\infty}}}{\kappa^2} \right)^{m_0 + \cdots + m_\ell}.
\]

We re-index \( m = m_0 + \cdots + m_\ell \) and sum over \( \ell + m \) as in (2.12). The condition \( \ell + m_0 + \cdots + m_\ell \geq 2 \) guarantees a gain \( \lesssim \kappa^{-3/2} \kappa^{-2} \) when we sum over the two parenthetical terms, and so we conclude
\[
|3.26| \lesssim \| q''_\kappa \|_{L^2} + 1 \lesssim \| q''_\kappa \|_{L^2} + 1
\]
provided that \( \kappa \) is sufficiently large (independently of \( \| q''_\kappa \|_{L^2} \)).

Altogether, we have obtained
\[
\left| \frac{d}{dt} E_2 \right| \lesssim \| q''_\kappa \|_{L^2} + 1 \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0,
\]
where \( \kappa_0 \) depends only on \( T \) and \( \| q(0) \|_{H^1} \). Using Proposition 3.3 we can then bound
\[
\| q''_\kappa \|_{L^2} \lesssim E_2 + \left| \int q_\kappa q_\kappa' dx \right| + \left| \int q_\kappa^4 dx \right| \lesssim E_2 + 1.
\]

Together, we conclude that there exists a constant \( C = C(T, A) \) such that
\[
\| q''_\kappa(t) \|_{L^2}^2 \leq C + C \int_0^t \| q''_\kappa(s) \|_{L^2}^2 ds
\]
uniformly for \( |t| \leq T \text{ and } \kappa \geq \kappa_0 \). Grönwall’s inequality then yields
\[
\| q''_\kappa(t) \|_{L^2}^2 \leq C(T, \| q(0) \|_{H^2}) \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(T, \| q(0) \|_{H^1}),
\]
as desired.

For \( H^s, s \geq 3 \) we proceed by induction:

**Proposition 3.5.** Given an integer \( s \geq 3 \) and \( A, T > 0 \) there exist constants \( C \) and \( \kappa_0 \) such that solutions \( q_\kappa(t) \) to the tidal \( H_\kappa \) flow (3.3) obey
\[
\| q(0) \|_{H^s} \leq A \quad \text{ implies } \quad \| q_\kappa(t) \|_{H^s} \leq C \quad \text{ for all } |t| \leq T \text{ and } \kappa \geq \kappa_0.
\]

**Proof.** We induct on \( s \), with the base case given by Proposition 3.4. Assume the result holds for \( s - 1 \).

By approximation and local well-posedness we may assume that \( q(0) \in H^\infty \). We define
\[
E_s(t) := \frac{1}{2} \int (q_\kappa^{(s)}(t, x))^2 dx.
\]
Expanding $g(\kappa, q_\kappa + W)$ in powers of $q_\kappa$ and $W$, we write

$$\frac{d}{dt} E_s = \int q_\kappa^{(s)} \left\{ -16\kappa^5 \langle \delta_x, R_0 q_\kappa^{(s+1)} R_0 \delta_x \rangle + 4\kappa^2 q_\kappa^{(s+1)} \right\} dx$$

(3.27)

$$+ \int q_\kappa^{(s)} \left\{ -16\kappa^5 \langle \delta_x, R_0 W^{(s+1)} R_0 \delta_x \rangle + 4\kappa^2 W^{(s+1)} \right\} dx$$

(3.28)

$$+ 16\kappa^5 \int q_\kappa^{(s)} \langle \delta_x, [\delta^{s+1}, R_0 q_\kappa, R_0 q_\kappa R_0] \delta_x \rangle dx$$

(3.29)

$$+ 16\kappa^5 \int q_\kappa^{(s)} \left\{ \langle \delta_x, [\delta^{s+1}, R_0 W R_0 q_\kappa, R_0] \delta_x \rangle + \langle \delta_x, [\delta^{s+1}, R_0 q_\kappa R_0 W R_0] \delta_x \rangle \right\} dx$$

(3.30)

$$+ 16\kappa^5 \int q_\kappa^{(s)} \langle \delta_x, [\delta^{s+1}, R_0 W R_0 W R_0] \delta_x \rangle dx$$

(3.31)

$$+ 16\kappa^5 \int q_\kappa^{(s)} \left\{ g(\kappa, q_\kappa + W) + \langle \delta_x, R_0 (q_\kappa + W) R_0 \delta_x \rangle + R_0 (q_\kappa + W) R_0 (q_\kappa + W) R_0 \delta_x \rangle \right\}^{(s+1)} dx.$$  

(3.32)

We will estimate the terms (3.27)–(3.32) separately.

The first linear contribution (3.27) vanishes. To see this, we use the first operator identity (2.10) to write

$$\int q_\kappa^{(s)} \left\{ -16\kappa^4 R_0 (2\kappa) q_\kappa^{(s+1)} + 4\kappa^2 q_\kappa^{(s+1)} \right\} dx = 0.$$  

In the last equality we noted that the integrand is odd in Fourier variables, or equivalently that the integrand of (3.27) is a total derivative.

Now we estimate the linear contribution (3.28) from $W$. Using the operator identity (2.10) and recalling that $W'$ is Schwartz, we estimate

$$\|q_\kappa^{(s)}\|_{L^2} \left( \|W^{(s+3)}\|_{L^2} + \kappa^{-2} \|W^{(s+5)}\|_{L^2} \right) \lesssim E_s^{1/2} \lesssim E_s + 1.$$  

In the first quadratic term (3.29) we distribute the derivatives $[\delta^{s+1}, \cdot]$. For the terms with $q_\kappa^{(s+1)}$, we “integrate by parts” to write

$$16\kappa^5 \left\{ \text{tr} \left\{ q_\kappa^{(s)} R_0 q_\kappa^{(s+1)} R_0 q_\kappa R_0 \right\} + \text{tr} \left\{ q_\kappa^{(s)} R_0 q_\kappa R_0 q_\kappa^{(s+1)} R_0 \right\} \right\}$$

$$= 16\kappa^5 \left\{ \langle \partial_x q_\kappa^{(s)}, R_0 q_\kappa R_0 \rangle \right\} - 16\kappa^5 \langle \text{tr} \left\{ q_\kappa^{(s)} R_0 q_\kappa^{(s+1)} R_0 \right\}, \delta_x, R_0 \rangle R_0 \delta_x \rangle.$$  

This leaves

$$\|q_\kappa^{(s)}\|_{L^2} \lesssim \kappa^5 \sum_{j=1}^s \left| \text{tr} \left\{ q_\kappa^{(s)} R_0 q_\kappa^{(j)} R_0 q_\kappa^{(s+1-j)} R_0 \right\} \right| + \left| \text{tr} \left\{ q_\kappa^{(s)} R_0 q_\kappa^{(s-1)} R_0 q_\kappa^{(s-1)} R_0 \right\} \right|.$$  

The last term only appears in the case $s = 3$, but we can see that it vanishes by writing $q_\kappa^{(s)} = [\partial_x, q_\kappa^{(s-1)}]$ and cycling the trace. Note that all copies of $q_\kappa$ now have at most $s$ derivatives. We put the two highest order factors of $q_\kappa$ in $L^2$ using the identity (2.23) and the observation $\|f\|_{H^{-1}} \lesssim \kappa^{-1} \|f\|_{L^2}$. As $s \geq 3$, the third
factor $q^{(j)}_\kappa$ has order $j \leq s - 2$ and may be estimated in operator norm because
\[ \|q^{(j)}_\kappa\|_L^2 \leq \|q_\kappa\|_{H^{-1}} \lesssim 1 \] by inductive hypothesis. This yields
\[ (3.29) \| q^{(s)}_\kappa \|_L^2 + \| q^{(s)}_\kappa \|_L^2 \lesssim E_s + 1. \]

The second quadratic contribution \[ (3.30) \] is similar. For the terms with $q^{(s+1)}_\kappa$ we "integrate by parts" to write
\[ 16\kappa^5 \left( \text{tr} \left\{ q^{(s)}_\kappa R_0 q^{(s+1)}_\kappa R_0 W R_0 \right\} + \text{tr} \left\{ q^{(s)}_\kappa R_0 W R_0 q^{(s+1)}_\kappa R_0 \right\} \right) \]
\[ = 16\kappa^5 \text{tr} \left\{ [\partial, q^{(s)}_\kappa R_0 q^{(s)}_\kappa R_0] W R_0 \right\} = -16\kappa^5 \text{tr} \left\{ q^{(s)}_\kappa R_0 q^{(s)}_\kappa R_0 [\partial, W] R_0 \right\}. \]

In all cases we put the two factors of $q_\kappa$ in $L^2$ using the identity \[ (2.3) \] and the observation \[ \| f \|_{H^{-1}} \lesssim \kappa^{-1} \| f \|_{L^2}, \] and the remaining factors in operator norm. This yields
\[ (3.31) \| q^{(s)}_\kappa \|_L^2 + \| q^{(s)}_\kappa \|_L^2 \lesssim E_s + 1. \]

The quadratic $W$ contribution \[ (3.31) \] is easily estimated. We put $q^{(s)}_\kappa$ and the higher order $W$ term in $L^2$ using the identity \[ (2.3) \] and the observation \[ \| f \|_{H^{-1}} \lesssim \kappa^{-1} \| f \|_{L^2}, \] and we put the remaining factor of $W$ in $L^\infty$. This yields
\[ (3.32) \| q^{(s)}_\kappa \|_L^2 \lesssim E_s + 1. \]

Next we turn to the series tail \[ (3.32). \] Applying the tail convergence \[ (2.18) \] to $q = q_\kappa$, we know there exists a constant $\kappa_0(E_s(t))$ so that
\[ 16\kappa^5 \left\| \{ g(\kappa, q_\kappa + W) \langle \delta_x, R_0(q_\kappa + W) R_0 \delta_x \rangle \right. \]
\[ - \langle \delta_x, R_0(q_\kappa + W) R_0(q_\kappa + W) R_0 \delta_x \rangle \} (s+1) \left\|_L^2 \leq 1 \right. \]
uniformly for $\kappa \geq \kappa_0(E_s(t))$. Therefore, by Cauchy–Schwarz we have
\[ (3.32) \leq (2E_s)^{1/2} \lesssim E_s + 1 \quad \text{uniformly for } \kappa \geq \kappa_0(E_s(t)). \]

Altogether, we have shown that there exists a constant $C = C(T,A)$ such that
\[ \left| \frac{d}{dt} E_s \right| \leq C(E_s + 1) \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(E_s(t)). \]
Grönwall’s inequality then yields the bound
\[ E_s(t) \leq (E_s(0) + 1)e^{CT} - 1 \quad \text{uniformly for } |t| \leq T, \ \kappa \geq \kappa_0((E_s(0) + 1)e^{CT} - 1), \]
which concludes the inductive step. \qed

As a consequence, we are able to upgrade local well-posedness to global well-posedness:

**Corollary 3.6.** Given an integer $s \geq 0$ and $A,T > 0$, there exists a constant $\kappa_0$ so that for $\kappa \geq \kappa_0$ the tidal $H_\kappa$ flows \[ (3.3) \] with initial data in the closed ball $B_A \subset H^s(\mathbb{R})$ of radius $A$ are globally well-posed.

**Proof.** Fix $A,T > 0$, let $C$ be the constant guaranteed by Propositions \[ 3.2 \] to \[ 3.5 \] and consider the closed ball $B_C \subset H^s$ of radius $C$. By local well-posedness (cf. Lemma \[ 3.1 \]) we know there exists $\delta > 0$ such that the integral equation is a contraction on $C_1 B_C([-\delta, \delta] \times \mathbb{R})$, and hence there exists a unique fixed point $q_\kappa$. However, by Propositions \[ 3.2 \] to \[ 3.5 \] we know that $q_\kappa(t)$ is in $B_C$ as long as $|t| \leq T$. Therefore, we may iterate the contraction argument to construct a unique solution in $C_1 H^s([-T, T] \times \mathbb{R})$ that depends continuously upon the initial data. \qed
4. Convergence at low regularity

Ultimately, we want to show that for initial data in $H^s$ with $s \geq 3$ the solutions $q_\kappa(t)$ to the tidal $H_\kappa$ flows converge in $H^s$. Although the linear and quadratic terms of the tidal $H_\kappa$ flow formally converge to tidal KdV as $\kappa \to \infty$, the first term in the error contains $q_\kappa^{(5)}$ (cf. (2.10)). Consequently, we will first demonstrate convergence in $H^{-2}$ so that we may absorb these five extra derivatives:

**Proposition 4.1.** Given $T > 0$ and a bounded set $Q \subset H^3$ of initial data, the corresponding solutions $q_\kappa(t)$ to the tidal $H_\kappa$ flows (4.3) are Cauchy in $C_t H^{-2}([-T,T] \times \mathbb{R})$ as $\kappa \to \infty$ uniformly for $q(0) \in Q$.

**Proof.** In the following all spacetime norms will be taken over the slab $[-T,T] \times \mathbb{R}$. Let $\kappa_0$ denote the constant from Corollary 3.6 for $s = 3$, so that for $\kappa \geq \kappa_0$ the solutions $q_\kappa(t)$ to the $H_\kappa$ flows exist in $C_t H^3$.

Consider the difference $q_\kappa - q_\sigma$ of two of these solutions with $\kappa \geq \sigma \geq \kappa_0$. Recall that the tidal $H_\kappa$ and tidal $H_\sigma$ flows commute (cf. (3.2)). Letting $H^W_\kappa$ denote the tidal $H_\kappa$ flow Hamiltonian, this allows us to write

$$q_\kappa(t) = e^{t J \nabla H^W_\kappa} q(0) = e^{t J \nabla (H^W_\kappa - H^W_\sigma)} e^{t J \nabla H^W_\sigma} q(0).$$

Consequently, we estimate

$$\|q_\kappa - q_\sigma\|_{C_t H^{-1}} \leq \sup_{q \in Q^*_T(\kappa)} \sup_{\kappa \geq \kappa_0} \|e^{t J \nabla (H^W_\kappa - H^W_\sigma)} q - q\|_{C_t H^{-1}},$$

for the set

$$Q^*_T(\kappa) := \{e^{t J \nabla H^W_\kappa} q(0) : |t| \leq T, q(0) \in Q\}$$

of tidal $H_\kappa$ flows. By the fundamental theorem of calculus, it suffices to show that under the difference flow $H^W_\kappa - H^W_\sigma$ we have

$$\sup_{q \in Q^*_T(\kappa)} \sup_{\kappa \geq \kappa_0} \left\| \frac{dq}{dt} \right\|_{C_t H^{-2}} \rightarrow 0 \quad \text{as} \quad \kappa \to \infty.$$

Note that $Q^*_T(\kappa)$ is a bounded subset of $H^3$ by the a priori estimate of Proposition 3.5.

Given initial data $q(0) \in Q^*_T(\kappa)$, let $q(t)$ denote the corresponding solution to the difference flow $H^W_\kappa - H^W_\sigma$. Then $q$ solves

$$\frac{d}{dt} q = 16 \xi^5 g'(\kappa, q + W) + 4 \xi^5 (q + W)' - 16 \kappa^5 g'(\xi, q + W) - 4 \kappa^2 (q + W)' \cdot$$

To exhibit cancellation in the limit $\kappa, \kappa \to \infty$, we expand $g'(\kappa, q + W)$ into a series in $\xi$ and $W$ and extract the linear and quadratic terms:

$$\frac{d}{dt} q = \left\{ -16 \xi^5 \langle \delta_x, R_0(\xi)(q + W)R_0(\xi) \rangle \delta_x + 4 \xi^5 (q + W) \right\}'$$

$$+ \left\{ 16 \kappa^5 \langle \delta_x, R_0(\kappa)(q + W)R_0(\kappa) \rangle \delta_x - 4 \kappa^2 (q + W) \right\}'$$

$$+ \left\{ 16 \xi^5 \langle \delta_x, R_0(\xi)(q + W)R_0(\xi) \rangle \delta_x ight\}'$$

$$+ \left\{ 16 \kappa^5 \langle \delta_x, R_0(\kappa)(q + W)R_0(\kappa) \rangle \delta_x \right\}'$$

$$+ \left\{ \text{terms with 3 or more } q \text{ or } W \right\}.$$

We will show that each of the terms (4.1)–(4.3) converge to zero.
For the linear term (1.1), we use the operator identity (2.16) to estimate
\[\|1.1\|_{H^{-2}} = \|[-R_0(2\varphi) + R_0(2\kappa)](q + W)^{(5)}\|_{H^{-2}}\]
\[\lesssim (\varphi^2 + \kappa^2)(\|q^{(5)}\|_{H^{-2}} + \|W^{(5)}\|_{H^{-2}}) \lesssim \kappa^{-2}(\|q\|_{H^3} + \|W''\|_{L^2})\]
uniformly for \(\varphi \geq \kappa\). As \(q \in Q^*_T(\kappa)\) is bounded in \(H^3\), we conclude that
\[\sup_{q \in Q^*_T(\kappa)} \|1.1\|_{C^1,H^{-2}} \to 0 \quad \text{as} \quad \kappa \to \infty.\]

For the quadratic term (1.2), we add and subtract the corresponding tidal KdV term \(6(q + W)(q + W)'\) and estimate
\[\|1.2\|_{H^{-2}} \lesssim \|16\varphi^2(\delta_x, R_0(\varphi)(q + W)R_0(\varphi)(q + W)R_0(\varphi)\delta_x) - 3(q + W)^2\|_{H^{-1}}\]
\[+ \|16\kappa^2(\delta_x, R_0(\kappa)(q + W)R_0(\kappa)(q + W)R_0(\kappa)\delta_x) - 3(q + W)^2\|_{H^{-1}}.\]
Using the operator identity (2.17) and the estimates \(\|R_0(2\kappa)\varphi\|_{\text{op}} \lesssim \kappa^{-2}\) for \(j = 0, 1, 2\) (the estimate for \(j = 0\) is also true as an operator on \(L^\infty\) by the explicit kernel formula for \(R_0\) and Young’s inequality), one can easily prove by duality that
\[\|16\kappa^2(\delta_x, R_0(\kappa)\varphi R_0(\kappa)\varphi R_0(\kappa)\delta_x) - 3\varphi g\|_{L^2} \lesssim \kappa^{-2}\|\varphi\|_{W^{2,\infty}} \|g\|_{H^2}.\]
Moreover, the roles of \(f\) and \(g\) can be exchanged since the identity (2.17) is symmetric in \(f\) and \(g\). Therefore, expanding the products \((q + W)(q + W)\) we have
\[\|1.2\|_{H^{-2}} \lesssim (\varphi^2 + \kappa^2)(\|q\|_{H^3}^2 + \|W\|_{W^{2,\infty}} \|q\|_{H^3} + \|W\|_{W^{2,\infty}} \|W'\|_{H^2}).\]
As \(q \in Q^*_T(\kappa)\) is bounded in \(H^3\), we conclude that
\[\sup_{q \in Q^*_T(\kappa)} \|1.2\|_{C^1,H^{-2}} \to 0 \quad \text{as} \quad \kappa \to \infty.\]

It only remains to show that the tails (1.3) converge to zero in \(C^1 H^{-2}\). In fact, by (2.18) we have convergence in the stronger \(C^1 L^2\) norm:
\[\sup_{q \in Q^*_T(\kappa)} \sup_{\kappa \geq \kappa'} \|1.3\|_{C^1,L^2} \to 0 \quad \text{as} \quad \kappa \to \infty.\]

5. Equicontinuity

We want to upgrade the \(H^{-2}\) convergence of the previous section to \(H^s, s \geq 3\).

This will be accomplished via the estimate
\[\|q(\varphi - \kappa)\|_{H^s} \lesssim (N + 1)^{s+2} \|q(\varphi - \kappa)\|_{H^{-2}} + \|q(\varphi - \kappa)\|_{H^{s+2}((\xi \geq N)}\]

In this section, we will show that we can pick \(N\) sufficiently large so that the second term on the RHS is arbitrarily small uniformly for \(\varphi, \kappa\) large. It then follows from Proposition 4.1 that the first term on the RHS converges to zero as \(\kappa, \varphi \to \infty\).

Uniform control over Fourier tails is called equicontinuity. Specifically, a set \(Q \subset H^s\) is equicontinuous in \(H^s\) if
\[\int_{\|\xi\| \geq N} (\xi^2 + 1)^s |\hat{q}(\xi)|^2 d\xi \to 0 \quad \text{as} \quad N \to \infty, \quad \text{uniformly for} \quad q \in Q.\]

This is equivalent to the notion of equicontinuity in the \(L^p\) precompactness theorem (cf. [35, Lem. 4.2]). In particular, precompact subsets of \(H^s\) are equicontinuous in \(H^s\).

It would suffice to show that the tidal \(H_\kappa\) flows \(\{q_\kappa(t) : \kappa \geq \kappa_0\}\) on bounded time intervals are equicontinuous. With the presence of the background wave \(W\)
in tidal KdV we expect the quantity \( \|q_\kappa(t)\|_{H^s} \) to grow, and so we must estimate this growth. Expanding the diagonal Green’s function in powers of \( q_\kappa \) and \( W \), we are able to control the linear and quadratic terms as we would for tidal KdV; however, it remains to control the higher order contributions which vanish in the limit \( \kappa \to \infty \). Consequently, instead of honest equicontinuity for the tidal \( H_\kappa \) flows \( q_\kappa(t) \), we will require \( \kappa \geq N \) in Proposition \( 5.3 \) so that \( O(\kappa^{-1}) \) contributions as \( \kappa \to \infty \) are also \( O(N^{-1}) \) as \( N \to \infty \).

In order to control the Fourier tail growth we will use a smooth Littlewood-Paley decomposition. We define Littlewood–Paley pieces via the following Paley decomposition. We define Littlewood–Paley pieces via the following

\[
\phi(\xi) = \begin{cases} 
1 & |\xi| \leq 1, \\
0 & |\xi| \geq 2.
\end{cases}
\]

Then the function

\[
\psi(\xi) := \sqrt{\phi(\xi) - \phi(2\xi)} \quad \text{satisfies} \quad \sum_{N \in \mathbb{Z}} \psi^2(\frac{\xi}{N}) = 1 \quad \text{for all} \ \xi \neq 0.
\]

Sums over capitalized indices will always be over the set \( 2^\mathbb{Z} := \{2^n : n \in \mathbb{Z}\} \). For Schwartz functions \( f \) we define

\[
P_N \hat{f}(\xi) = \psi\left(\frac{\xi}{N}\right) \hat{f}(\xi), \quad P_{\geq N} \hat{f}(\xi) = \sum_{K \geq N} \psi^2\left(\frac{\xi}{K}\right) \hat{f}(\xi), \quad P_{< N}^2 = 1 - P_{\geq N}^2.
\]

Our choice of partition of unity ensures that the square sum \( \sum P_N^2 f \) converges to \( f \) in \( L^p \) for \( p \in (1, \infty) \). We choose a square-sum decomposition because we will ultimately measure \( \|P_{\geq N} q_\kappa\|_{L^2}^2 \), which we may write as the \( L^2 \)-pairing of \( P_{\geq N} q_\kappa \) and \( q_\kappa \).

We remark that directly estimating the growth of \( \|P_{\geq N} q_\kappa\|_{L^2}^2 \) would fail due to the quadratic term of tidal KdV. Indeed, if we compute \( \frac{d}{dt} \|P_{\geq N} q_\kappa\|_{L^2}^2 \) under the tidal KdV flow, we obtain a term of the form

\[
\int \left( P_{\geq N} q_\kappa \right)^2 \left( 3q^2 \right)^{(s+1)} \ dx.
\]

Decomposing each factor of \( q = P_{\geq N} q + P_{< N} q \), the terms with at least one copy of \( P_{\geq N} q \) can be estimated by two factors of \( \|P_{\geq N} q_\kappa\|_{L^2} \). However, the high-low-low term

\[
\int \left( P_{\geq N} q_\kappa \right)^2 \left[ 3 \left( P_{\geq N} q \right) \left( P_{< N} q \right) \right]^{(s+1)} \ dx
\]

only contributes one factor of \( \|P_{\geq N} q_\kappa\|_{L^2} \), which does not guarantee that initially small Fourier tails remain small.

To overcome this, we introduce a more gradual high-frequency cutoff. Given an integer \( s \geq 3 \) and a Schwartz function \( f \), we define the Fourier multiplier

\[
\Pi_{\geq N} f(\xi) = m_{hi}(\xi) \hat{f}(\xi), \quad m_{hi}(\xi) = \sum_{K < 1} K^s \psi^2\left(\frac{\xi}{K}\right) + \sum_{K \geq 1} \psi^2\left(\frac{\xi}{K}\right).
\]

The power of \( s \) in the definition \( 5.1 \) will provide us with the replacement \( 5.6 \) for the Bernstein inequality satisfied by \( P_{\geq N}^2 \). We also define

\[
\Pi_{< N} f(\xi) = \sqrt{1 - m_{hi}^2\left(\frac{\xi}{N}\right)} \hat{f}(\xi) \quad \text{so that} \quad \Pi_{< N}^2 + \Pi_{\geq N}^2 = 1.
\]
For the Littlewood–Paley operators we have the familiar Bernstein inequalities
\[
\|P_N f^{(j)}\|_{L^p} \sim N^j \|P_N f\|_{L^p} \quad \text{for } p \in (1, \infty), \ j \in \mathbb{Z},
\]
(5.2) \quad \|P_N f^{(j)}\|_{L^\infty} \lesssim N^j \|P_N f\|_{L^\infty} \quad \text{for } j > 0.

Summing over \(N \in \mathbb{Z}\), we obtain the high and low frequency projection estimates
\[
\|P_{<N}^2 f^{(j)}\|_{L^p} \lesssim N^j \|P_{<2N}^2 f\|_{L^p} \quad \text{for } p \in [1, \infty), \ j > 0,
\]
(5.3) \quad \|P_{\geq N}^2 f^{(j)}\|_{L^p} \lesssim N^{-j} \|P_{\geq N}^2 f^{(j)}\|_{L^p} \quad \text{for } p \in (1, \infty), \ j > 0.

We will now obtain analogous Bernstein inequalities for our projection operators \(\Pi_{\geq N}\) and \(\Pi_{<N}^\circ\):

**Lemma 5.1.** Fix an integer \(s \geq 3\). Then the operators \(\Pi_{\geq N}\) defined in (5.1) are bounded on \(L^p\) for \(p \in [1, \infty]\) uniformly in \(N\), and we have the estimates
\[
\|\Pi_{<N} f\|_{L^p} = \|N^d \tilde{m}_{lo}(N^\cdot) * f\|_{L^p} \lesssim \|N^d \tilde{m}_{lo}(N^\cdot)\|_{L^1} \|f\|_{L^p} = \|m_{lo}\|_{L^1} \|f\|_{L^p}
\]
for any \(p \in [1, \infty]\).

For the inequality (5.3) we may now assume that \(q\) is Schwartz by approximation. We use the Bernstein inequality (5.2) to estimate
\[
\|\Pi_{<N}^2 q^{(s+j)}\|_{L^p} \leq \sum_{K < N} K^j \|P_{<2N}^2 q^{(s)}\|_{L^p}
\]
(5.4) \quad \lesssim \sum_{K < N} K^j \|P_{<2N}^2 q^{(s)}\|_{L^p} \lesssim N^j \|P_{<2N}^2 q^{(s)}\|_{L^p}.

Note that in the second line we inserted the operator \(P_{<2N}^2\) since \(P_{<2N}^2 P_{<2N}^2 = P_{K}^2\) for \(K < N\), and then used the boundedness of the operators \(P_{K}^2\).

For the inequality (5.6), we use the Bernstein inequalities (5.2) and (5.4) to estimate
\[
\|\Pi_{\geq N}^2 q^{(s-j)}\|_{L^p} \leq \sum_{K < N} K^{s-j} \|P_{K}^2 \Pi_{\geq N} q^{(s-j)}\|_{L^p} + \|P_{\geq N}^2 \Pi_{\geq N} q^{(s-j)}\|_{L^p}
\]
(5.5) \quad \lesssim \sum_{K < N} K^{s-j} \|\Pi_{\geq N} q^{(s)}\|_{L^p} + N^{-j} \|P_{\geq N}^2 \Pi_{\geq N} q^{(s)}\|_{L^p} \lesssim N^{-j} \|\Pi_{\geq N} q^{(s)}\|_{L^p}

for Schwartz \(q\). Note that in the second line we spent a factor of \(K^j\) to insert \(j\) derivatives on \(q\), and then used the boundedness of the operators \(P_{K}^2\).

Next, we will prove an estimate for a commutator involving \(\Pi_{\geq N}\) and \(\Pi_{<N}^\circ\):

**Lemma 5.2.** Let \(\tilde{P}^*_M = \sum_{K=M/4}^M P_{K}^2\) denote a fattened Littlewood–Paley projection. Then for all bounded functions \(w \in L^\infty(\mathbb{R}^2)\) and Schwartz functions \(f, g, h\)
we have

\[
\begin{align*}
\int & \int \left[ (P^2_M f)(\xi)(\Pi^2_N h)(\xi - \eta) \\
- & (P^2_M f)(\xi)(P_M \Pi^2_N f)(\xi - \eta)(\Pi^2_N h)(\xi - \eta) \right] (P^2_{< M} g)(\eta)w(\xi, \eta) \, d\xi \, d\eta \\
\lesssim & \|w\|_{L^\infty} \|P_M \Pi^2_N f\|_{L^2} \|P^2_{< M} g\|_{H^1} \left( \frac{M^2}{N} \|P^2_M \Pi^2_N h\|_{L^2} + \|P_M \Pi^2_N \Pi^2_N h\|_{L^2} \right)
\end{align*}
\]

uniformly for \( \kappa \) large.

**Proof.** Within the square brackets, we are interchanging a factor of \( P_M \Pi^2_N \) and \( \Pi^2_N \) between \( f \) and \( h \). We change to Fourier variables and break this maneuver into two steps, first moving \( P_M \Pi^2_N \) and then moving \( \Pi^2_N \):

\[
\begin{align*}
\int & \int \left[ (P^2_M f)(\xi)(\Pi^2_N h)(\xi - \eta) \\
- & (P^2_M f)(\xi)(P_M \Pi^2_N f)(\xi - \eta)(\Pi^2_N h)(\xi - \eta) \right] (P^2_{< M} g)(\eta)w(\xi, \eta) \, d\xi \, d\eta \\
= & \int \int (P_M \Pi^2_N f)(\xi) \left[ \psi(\frac{\xi}{M})m_{hi}(\frac{\xi}{M}) - \psi(\frac{\xi - \xi_N}{M})m_{hi}(\frac{\xi - \xi_N}{M}) \right] \\
& \times (\Pi^2_N h)(\xi - \eta)(P^2_{< M} g)(\eta)w(\xi, \eta) \, d\xi \, d\eta \\
& + \int \int (P_M \Pi^2_N f)(\xi) \left[ m_{lo}(\frac{\xi - \xi_N}{M}) - m_{lo}(\frac{\xi}{M}) \right] \\
& \times (P_M \Pi^2_N \Pi^2_N h)(\xi - \eta)(P^2_{< M} g)(\eta)w(\xi, \eta) \, d\xi \, d\eta,
\end{align*}
\]

(5.7)

(5.8)

where \( \psi \), \( m_{hi} \), and \( m_{lo} \) are the Fourier multipliers for the operators \( P_M \), \( \Pi^2_N \), and \( \Pi^2_N \) respectively. Observe that the RHS of the desired inequality vanishes for \( M \geq 8N \). Consequently, we will estimate the terms (5.7) and (5.8) for \( M \leq 4N \) and note that they vanish for \( M \geq 8N \).

Observe that the integrand of the first term (5.7) is supported in the region \( \frac{M}{4} \leq |\xi| \leq 2M, \ |\eta| \leq \frac{M}{4} \). On this region we have

\[
|\xi - \eta| \geq |\xi| - |\eta| \geq \frac{M}{4} - \frac{M}{8} \geq \frac{M}{8}, \quad |\xi - \eta| \leq |\xi| + |\eta| \leq 2M + \frac{M}{8} \leq 4M.
\]

Therefore we can insert \( \sum_{K=M/4}^{4M} \psi^2(\frac{\xi - \eta}{K}) \) into the integrand, which is the Fourier multiplier for the fattened Littlewood–Paley projection \( \hat{P}_M^2 = \sum_{K=M/4}^{4M} P_K^2 \) applied to \( h \). Now \( \hat{P}_M^2 \Pi^2_N h \) vanishes for \( M \geq 8N \), and so we may assume \( M \leq 4N \).

Next, we will estimate the first term (5.7). By the fundamental theorem of calculus,

\[
\left| \psi(\frac{\xi}{M})m_{hi}(\frac{\xi}{M}) - \psi(\frac{\xi - \xi_N}{M})m_{hi}(\frac{\xi - \xi_N}{M}) \right| \leq \int_0^1 s|\eta| \left| (\psi(\frac{\xi}{M})m_{hi}(\frac{\xi}{M}))'(\xi - s\eta) \right| \, ds \\
\lesssim & \ |\eta| \frac{M-1}{N^2} \text{ for } M \leq N.
\]

In the last inequality, we note that \( \psi(\frac{\xi}{M})m_{hi}(\frac{\xi}{M}) \) is a function with amplitude \( M^a/N^a \) supported in an annulus of width \( M \); indeed, for \( M \leq N \) we have

\[
\left| (\psi(\frac{\xi}{M})m_{hi}(\frac{\xi}{M}))' \right| \leq |\psi(\frac{\xi}{M})'m_{hi}(\frac{\xi}{M})| + |\psi(\frac{\xi}{M})m_{hi}(\frac{\xi}{M})'| \lesssim M^{-1} \cdot \frac{M^a}{N^a} + 1 \cdot \frac{M^{a-1}}{N^2}.
\]
This yields
\[
\|w\|_{L^\infty} \lesssim \|\mathcal{P}_m 1_{\geq N} f\|_{L^2} \|\mathcal{P}_m 1_{> N} h\|_{L^2} \approx \|\mathcal{P}_m 1_{\geq N} f\|_{L^2} \|\mathcal{P}_m 1_{> N} h\|_{L^2}.
\]

In the last inequality, we used Cauchy–Schwarz to estimate
\[
\int \left| (\mathcal{P}_m \hat{g})' (\xi) \right| d\xi \lesssim \left( \int \frac{d\xi}{\xi^2 + 1} \right)^{1/2} \left( \int (\xi^2 + 1) |(\mathcal{P}_m \hat{g})' (\xi)|^2 d\xi \right)^{1/2}.
\]

For the second term \((5.8)\), we note that the Fourier support of \(\Pi_{\geq 2N} \Pi_{< N} h\) is bounded by \(N\); in particular, \(\mathcal{P}_m \Pi_{\geq 2N} \Pi_{< N} h\) vanishes for \(M \geq 8N\). For \(M \leq 4N\) we estimate
\[
|m_{lo}(\frac{\xi}{N}) - m_{lo}(\frac{\xi}{N})| \leq \int_0^1 s|\eta| |(m_{lo}(\frac{\xi}{N}))(\xi - s\eta)| d\eta \leq |\eta| N^{-1}.
\]

This yields
\[
\|w\|_{L^\infty} \lesssim N^{-1} \|\mathcal{P}_m \Pi_{\geq 2N} f\|_{L^2} \|\mathcal{P}_m \Pi_{\geq 2N} \Pi_{< N} h\|_{L^2} \approx \|\mathcal{P}_m \Pi_{\geq 2N} f\|_{L^2} \|\mathcal{P}_m \Pi_{\geq 2N} \Pi_{< N} h\|_{L^2}.
\]

Combining this with the estimate of \((5.7)\), the claim follows. 

We are now equipped to prove our equicontinuity statement. Let \(Q(N) \subset H^s\) for \(N \in 2^i\) be bounded sets of initial data that satisfy
\[
(5.9) \quad Q(M) \supset Q(N) \text{ for } M \leq N, \quad \text{and } \lim_{N \to \infty} \sup_{q(0) \in Q(N)} \|\Pi_{\geq N} q(0)\|_{H^s} = 0.
\]

**Proposition 5.3.** Fix an integer \(s \geq 3\) and define the corresponding projection operator \((5.1)\). Given \(T > 0\) and bounded sets \(Q(N) \subset H^s\) of initial data satisfying \((5.9)\), the corresponding solutions \(q_k(t)\) to the tidal \(H_k\) flow \((5.3)\) obey
\[
\lim_{N \to \infty} \sup_{q(0) \in Q(N)} \sup_{\kappa \geq N} \|\Pi_{\geq N} q_k(t)\|_{C_t H^s([-T:T] \times \mathbb{R})} = 0.
\]

**Proof.** Expanding \(g(\kappa, q, + W)\) in powers of \(q_k\) and \(W\), we write
\[
\frac{d}{dt} \left( \|\Pi_{\geq N} q_k^{(s)}\|_{L^2}^2 \right)
\]
\[
= \int \left( \Pi_{\geq N} q_k^{(s)}(x) \right) \left\{ -16\kappa^2 \delta_x, R_0 q_k R_0 \delta_x \right\}^{(s+1)} dx
\]
\[
+ 16\kappa^5 \int \left( \Pi_{\geq N} q_k^{(s)}(x) \right) \left\{ \delta_x, R_0 W R_0 \delta_x \right\}^{(s+1)} dx
\]
\[
+ 16\kappa^5 \int \left( \Pi_{\geq N} q_k^{(s)}(x) \right) \left\{ \delta_x, R_0 R_0 q_k R_0 \delta_x \right\}^{(s+1)} dx
\]
\[
+ 16\kappa^5 \int \left( \Pi_{\geq N} q_k^{(s)}(x) \right) \left\{ \delta_x, R_0 W R_0 R_0 \delta_x \right\}^{(s+1)} dx
\]
\[
+ 16\kappa^5 \int \left( \Pi_{\geq N} q_k^{(s)}(x) \right) \left\{ \delta_x, R_0 q_k + W \right\}^{(s+1)} dx
\]
\[
+ 16\kappa^5 \int \left( \Pi_{\geq N} q_k^{(s)}(x) \right) \left\{ \delta_x, R_0 q_k + W \right\}^{(s+1)} dx.
\]
We will estimate the terms (5.10)–(5.13) separately.

The first linear term (5.10) vanishes. To see this, we use the first operator identity of (2.16) to write

\[
\text{(5.10)} = \int (\Pi_{\geq N} q^{(s)}_\kappa) \left\{ -16\kappa^4 R_0(2\kappa)q_\kappa + 4\kappa^2 q_\kappa \right\}^{(s+1)} dx = 0.
\]

In the last equality we note that the integrand is odd in Fourier variables, or equivalently that the integrand is a total derivative because differentiation commutes with the Fourier multipliers \(\Pi_{\geq N}\) and \(R_0\).

Now we estimate the linear contribution (5.11) from \(W\). Using the operator identity (2.17), we write

\[
\left| \text{(5.11)} \right| = \left| \int (\Pi_{\geq N} q^{(s)}_\kappa) \left\{ -W^{(s+3)} - R_0(2\kappa)W^{(s+5)} \right\} dx \right| \\
\lesssim \left\| \Pi_{\geq N} q^{(s)}_\kappa \right\|_{L^2} \left( \left\| \Pi_{\geq N} W^{(s+3)} \right\|_{L^2} + \kappa^{-2} \left\| W^{(s+5)} \right\|_{L^2} \right).
\]

Recalling that \(W'\) is Schwartz and \(\kappa \geq N\), we obtain

\[
\lesssim \left\| \Pi_{\geq N} q^{(s)}_\kappa \right\|_{L^2} \cdot N^{-2} \lesssim \left\| \Pi_{\geq N} q^{(s)}_\kappa \right\|_{L^2}^2 + N^{-4}.
\]

Next, we turn to the first quadratic contribution (5.12), which is nonvanishing due to the presence of the frequency cutoff \(\Pi_{\geq N}^2\). We write

\[
\text{(5.12)} = 16\kappa^5 \left| \frac{\text{tr}}{\Pi_{\geq N} q^{(s)}_\kappa} R_0 \left[ \partial^{s+1}, q_\kappa, R_0 q_\kappa, R_0 \right] \right| \\
\lesssim \sum_{j=0}^{s+1} \kappa^5 \left| \frac{\text{tr}}{\Pi_{\geq N} q^{(s)}_\kappa} R_0 q^{(j)}_\kappa R_0 q^{(s+1-j)}_\kappa R_0 \right|.
\]

Decomposing the highest order \(q_\kappa = \Pi_{\geq N} q_\kappa + \Pi_{\leq N} q_\kappa\) we have

\[
\left| \text{(5.12)} \right| \\
\lesssim \sum_{j=0}^{\frac{s+1}{2}} \kappa^5 \left| \frac{\text{tr}}{\Pi_{\geq N} q^{(s)}_\kappa} R_0 q^{(j)}_\kappa R_0 \left( \Pi_{\geq N} q^{(s+1-j)}_\kappa R_0 \right) \right| + \kappa^5 \left| \frac{\text{tr}}{\Pi_{\geq N} q^{(s)}_\kappa} R_0 q^{(j)}_\kappa R_0 \left( \Pi_{\leq N} q^{(s+1-j)}_\kappa R_0 \right) \right|
\]

\[
\left| \text{(5.12)} \right| \\
\lesssim \sum_{j=0}^{\frac{s+1}{2}} \kappa^5 \left| \frac{\text{tr}}{\Pi_{\geq N} q^{(s)}_\kappa} R_0 q^{(j)}_\kappa R_0 \left( \Pi_{\geq N} q^{(s+1-j)}_\kappa R_0 \right) \right| + \kappa^5 \left| \frac{\text{tr}}{\Pi_{\leq N} q^{(s)}_\kappa} R_0 q^{(j)}_\kappa R_0 \left( \Pi_{\leq N} q^{(s+1-j)}_\kappa R_0 \right) \right|.
\]

First we will estimate the high-frequency contribution (5.10). We can “integrate by parts” to eliminate the terms with \(q^{(s+1)}\). Specifically, by cycling the trace we have

\[
\text{tr} \left\{ (\Pi_{\geq N} q^{(s)}_\kappa) R_0 (\Pi_{\geq N} q^{(s+1)}_\kappa) R_0 q_\kappa R_0 \right\} + \text{tr} \left\{ (\Pi_{\geq N} q^{(s)}_\kappa) R_0 q_\kappa R_0 (\Pi_{\geq N} q^{(s+1)}_\kappa) R_0 \right\}
\]

\[
= \text{tr} \left\{ \left[ \partial, (\Pi_{\geq N} q^{(s)}_\kappa) R_0 (\Pi_{\geq N} q^{(s)}_\kappa) R_0 q_\kappa R_0 \right] q_\kappa R_0 \right\}
\]

\[
= - \text{tr} \left\{ (\Pi_{\geq N} q^{(s)}_\kappa) R_0 (\Pi_{\geq N} q^{(s)}_\kappa) R_0 q_\kappa R_0 \right\}.
\]

For the remaining terms we use the Hilbert–Schmidt norm estimate (2.3) and the observation \(\| f \|_{H^{s+1}} \lesssim k^{-1} \| f \|_{L^2}\) to put the two highest order terms in \(L^2\), and we
put the remaining terms in operator norm:

\[ \text{(5.16)} \lesssim \sum_{j=1}^{\lfloor \frac{s-1}{2} \rfloor} \| \Pi_{\geq N}^2 q_{\kappa}^{(s)} \|_{L^2} \| q_{\kappa}^{(j)} \|_{L^\infty} \| \Pi_{\geq N}^2 q_{\kappa}^{(s+1-j)} \|_{L^2}. \]

As \( s \geq 3 \) then the index \( j \) is at most \( s - 1 \), and so the term \( \| q_{\kappa}^{(s)} \|_{L^\infty} \) is uniformly bounded for \( |t| \leq T \) and \( \kappa \geq \kappa_0 \) by the embedding \( H^1 \hookrightarrow L^\infty \) and the a priori estimate of Proposition 3.3. The remaining term \( \| \Pi_{\geq N}^2 q_{\kappa}^{(s+1-j)} \|_{L^2} \) either matches the first factor \( \| \Pi_{\geq N}^2 q_{\kappa}^{(s)} \|_{L^2} \) or is \( \lesssim N^{-1} \) by the Bernstein inequality (5.6). Altogether we conclude

\[ \text{(5.17)} \lesssim \| \Pi_{\geq N}^2 q_{\kappa}^{(s)} \|_{L^2}^2 + \| \Pi_{\geq N}^2 q_{\kappa}^{(s)} \|_{L^2} \cdot N^{-1} \lesssim \| \Pi_{\geq N}^2 q_{\kappa}^{(s)} \|_{L^2}^2 + N^{-2}. \]

The low-frequency contribution \( \text{(5.17)} \) requires more manipulation. We will push one factor of \( \Pi_{\geq N} \) onto the low-frequency term and the resulting frequency cancellation will yield an acceptable contribution. As \( \Pi_{\geq N} \) is not a sharp frequency cutoff, we divide the first factor \( \Pi_{\geq N}^2 q_{\kappa}^{(s)} \) into its frequency scales:

\[ \text{(5.18)} \| \Pi_{\geq N}^2 q_{\kappa}^{(s)} \|_{L^2}^2 \lesssim \sum_M \sum_{j=0}^{\lfloor \frac{s-1}{2} \rfloor} \kappa^5 \left\{ \left| \left( \langle \Pi_{M}^{2} q_{\kappa}^{(s)} \rangle R_0 q_{\kappa}^{(j)} R_0 (\Pi_{\leq N}^2 q_{\kappa}^{(s+1-j)}) R_0 \right) \right| + \left| \left( \langle \Pi_{M}^{2} q_{\kappa}^{(s)} \rangle R_0 (\Pi_{\leq N}^2 q_{\kappa}^{(s+1-j)}) R_0 q_{\kappa}^{(j)} R_0 \right) \right| \right\}. \]

Consider the first summand of RHS \( \text{(5.18)} \). We split \( q_{\kappa}^{(j)} = P_{<\frac{M}{2}} q_{\kappa}^{(j)} + P_{\frac{M}{2}} q_{\kappa}^{(j)} \) into high and low frequencies; the high-frequency contribution can be estimated directly, and for the low-frequency term we trade factors of \( P_{M} \Pi_{\leq N} \) and \( \Pi_{\geq N} \) between \( q_{\kappa}^{(s)} \) and \( q_{\kappa}^{(s+1-j)} \) to create a commutator:

\[ \kappa^5 \left\{ \left| \left( \langle \Pi_{M}^{2} q_{\kappa}^{(s)} \rangle R_0 q_{\kappa}^{(j)} R_0 (\Pi_{\leq N}^2 q_{\kappa}^{(s+1-j)}) R_0 \right) \right| + \left| \left( \langle \Pi_{M}^{2} q_{\kappa}^{(s)} \rangle R_0 (\Pi_{\leq N}^2 q_{\kappa}^{(s+1-j)}) R_0 q_{\kappa}^{(j)} R_0 \right) \right| \right\}. \]

For the term \( \text{(5.19)} \) we put the two highest order terms in \( L^2 \) and the lowest order term in \( L^\infty \). This yields

\[ \text{(5.19)} \lesssim \begin{cases} \min \{ \frac{M}{N}, 1 \} \| \Pi_{M}^{2} q_{\kappa}^{(s)} \|_{L^2} \cdot M^{-2} \cdot N & \text{if } j = 0, \\ \min \{ \frac{M}{N}, 1 \} \| \Pi_{M}^{2} q_{\kappa}^{(s)} \|_{L^2} \cdot M^{-1} \cdot 1 & \text{if } j \geq 1. \end{cases} \]

For the term \( \text{(5.20)} \), we can now integrate by parts for the \( j = 0 \) case:

\[ \text{tr} \left\{ \left( \langle \Pi_{M}^{2} \Pi_{\leq N} q_{\kappa}^{(s)} \rangle R_0 (P_{<\frac{M}{2}} q_{\kappa}) R_0 (P_{M} \Pi_{\leq N} q_{\kappa}^{(s+1)}) R_0 \right) \right\} + \text{tr} \left\{ \left( \langle \Pi_{M}^{2} \Pi_{\leq N} q_{\kappa}^{(s)} \rangle R_0 (P_{\frac{M}{2}} q_{\kappa}) R_0 (P_{M} \Pi_{\leq N} q_{\kappa}^{(s+1)}) R_0 \right) \right\} = \text{tr} \left\{ \left( \langle \Pi_{M}^{2} \Pi_{\leq N} q_{\kappa}^{(s)} \rangle R_0 (P_{M} q_{\kappa}^{(s+1)}) R_0 \right) \right\} = -\text{tr} \left\{ \left( \langle \Pi_{M}^{2} \Pi_{\leq N} q_{\kappa}^{(s)} \rangle R_0 (P_{M} q_{\kappa}^{(s+1)}) R_0 \right) \right\}. \]
which is now the summand for \( j = 1 \). For \( j \geq 1 \) we put the two highest order terms in \( L^2 \) and the lowest order term in \( L^\infty \) to obtain

\[
\| M \Pi_{\geq N} q^{(s)}_\kappa \|_{L^2} \leq \begin{cases} \| P_M \Pi_{\geq N} q^{(s)}_\kappa \|_{L^2} \cdot 1 & \text{if } j = 1, \\
\| P_M \Pi_{\geq N} q^{(s)}_\kappa \|_{L^2} \cdot 1 \cdot N^{-1} \min\{ \frac{N^s}{N^s}, 1 \} & \text{if } j \geq 2.
\end{cases}
\]

For the commutator term \((5.21)\) we will apply the estimate of Lemma 5.2 to the functions \( f = q^{(s)}_\kappa, g = q^{(j)}_\kappa \), and \( h = q^{(s+1-j)}_\kappa \). Writing the trace as an iterated integral and changing to Fourier variables, we have

\[
(5.21) = \kappa^5 \text{tr} \left\{ \left[ (\Pi_{<N}^2 h) R_0 (P_{M}^2 \Pi_{<N}^2 f) R_0 \right. \right.
- \left. \left( P_M \Pi_{\geq N} \Pi_{<N} h \right) R_0 \left( P_M \Pi_{\geq N} \Pi_{<N} f \right) R_0 \right\}. \]

Changing variables \( \eta_1 = \xi_2 - \xi_1, \eta_2 = \xi_3 - \xi_2, \eta_3 = \xi_3 \), this becomes

\[
\frac{\kappa^5}{(2\pi)^{\frac{3}{2}}} \left\{ \int \left[ (\Pi_{<N}^2 h) (-\eta_1 - \eta_2) (P_{M}^2 \Pi_{<N}^2 f) (\eta_2) \right. \right.
- \left. \left( P_M \Pi_{\geq N} \Pi_{<N} h \right) (-\eta_1 - \eta_2) (P_M \Pi_{\geq N} \Pi_{<N} f) (\eta_2) \right]\]

\[
\times \left\{ (\eta_1^2 + \eta_2^2)((\eta_3 - \eta_2)^2 + \kappa^2)((\eta_3 - \eta_1 - \eta_2)^2 + \kappa^2) \right\}.
\]

The functions \( f, g, h \) are now independent of \( \eta_3 \), and so we may evaluate the \( \eta_3 \) integral using residue calculus:

\[
\frac{\kappa^4}{2(2\pi)^{\frac{3}{2}}} \left\{ \int \left[ (\Pi_{<N}^2 h) (-\eta_1 - \eta_2) (P_{M}^2 \Pi_{<N}^2 f) (\eta_2) \right. \right.
- \left. \left( P_M \Pi_{\geq N} \Pi_{<N} h \right) (-\eta_1 - \eta_2) (P_M \Pi_{\geq N} \Pi_{<N} f) (\eta_2) \right]\]

\[
\times \left\{ (\eta_1^2 + \eta_2^2)((\eta_3 - \eta_2)^2 + \kappa^2)((\eta_3 - \eta_1 - \eta_2)^2 + \kappa^2) \right\}.
\]

This is now of the form of Lemma 5.2 for the multiplier

\[
w(\xi, \eta) = \kappa^4 (2\pi)^{\frac{1}{2}} \frac{2(2\pi)^{\frac{3}{2}} (\eta_1^2 + \eta_2^2 + \eta_3^2)}{\eta_1^2 + \eta_2^2 + \eta_3^2}.
\]

Moreover, this multiplier is bounded uniformly in \( \kappa \):

\[
\| w \|_{L^\infty} = \frac{3}{16} (2\pi)^{-\frac{3}{2}} \text{ for all } \kappa > 0.
\]

Therefore, by Lemma 5.2 and the Bernstein inequalities \((5.6)\) and \((5.5)\) we have

\[
\| P_M \Pi_{\geq N} q^{(s)}_\kappa \|_{L^2} \leq \begin{cases} \| P_M \Pi_{\geq N} q^{(s)}_\kappa \|_{L^2} + \| P_M \Pi_{\geq N} q^{(s)}_\kappa \|_{L^2}^2 & \text{if } j = 0, \\
\| P_M \Pi_{\geq N} q^{(s)}_\kappa \|_{L^2} + N^{-1} \| P_M \Pi_{\geq N} q^{(s)}_\kappa \|_{L^2}^2 & \text{if } j \geq 1,
\end{cases}
\]

for \( M \leq 4N \).
We repeat the decomposition (5.19)–(5.21) for the second term in the summand of RHS (5.18). At each step we obtain the same estimates; indeed, although we cannot commute the operators within the trace, we still obtain the same integral because $w$ was symmetric in $\xi$ and $\eta$.

Altogether, we obtain the following estimate of the low-frequency quadratic contribution (5.17):

$$|5.17| \lesssim \sum_M \| \hat{F} M \Pi_{\geq N} q_k^{(s)} \|_{L^2}^2 + \sum_{M \leq 4N} \frac{N}{M} + \sum_{M \geq N} \frac{1}{M} \lesssim \| \Pi_{\geq N} q_k^{(s)} \|_{L^2}^2 + N^{-1}.$$ 

In the last inequality, we noted that the sum of the multipliers in Fourier variables is bounded.

For the quadratic term (5.13) involving $q_k$ and $W$, we can repeat the decomposition (5.19)–(5.21). Previously we put $q_k^{(0)}$ in $L^\infty$ and not $L^2$ since it was the lowest order term, and consequently the same estimates apply because $W \in L^\infty$ and $W'$ is Schwartz.

The quadratic term (5.14) for $W$ can be estimated directly. Extracting the leading term as $\kappa \to \infty$, we write

$$5.14$$

\begin{align*}
\int (\Pi_{\geq N} q_k^{(s)}) (3W^2)^{(s+1)} dx \\
+ \int (\Pi_{\geq N} q_k^{(s)}) \left\{ 16\kappa^5 (\delta_x, R_0 W R_0 W R_0 \delta_x) - 3W^2 \right\}^{(s+1)} dx.
\end{align*}

For (5.22) we distribute the $s + 1$ derivatives and move one $\Pi_{\geq N}$ off of $q_k$:

$$5.22$$

$$\lesssim \sum_{j=0}^{s+1} \int (\Pi_{\geq N} q_k^{(s)}) \Pi_{\geq N} (W^{(j)} W^j) dx \lesssim \| \Pi_{\geq N} q_k^{(s)} \|_{L^2} \cdot N^{-1} \lesssim \| \Pi_{\geq N} q_k^{(s)} \|_{L^2}^2 + N^{-2}.$$ 

In the second line we noted that $W^{(j)} W^j$ is Schwartz since $W'$ is Schwartz and $W \in L^\infty$ is smooth. For (5.23) we use the operator identity (5.17) and the estimates $\| R_0 (2\kappa) \partial^j \|_{\text{op}} \lesssim \kappa^{j-2}$ for $j = 0, 1, 2$ (the estimate for $j = 0$ is also true as an operator on $L^\infty$ by the explicit kernel formula for $R_0$ and Young’s inequality) to prove by duality that

$$\| 16\kappa^5 (\delta_x, R_0 (\kappa) f R_0 (\kappa) h R_0 (\kappa) \delta_x) - 3 f g \|_{L^2} \lesssim \kappa^{-2} \| f \|_{W_{2,\infty}} \| h \|_{H^2}.$$ 

Moreover, the roles of $f$ and $h$ can be exchanged since the identity (2.17) is symmetric in $f$ and $h$. Distributing the $s + 1$ derivatives and recalling $\kappa \geq N$, we estimate

$$5.23$$

$$\lesssim \| \Pi_{\geq N} q_k^{(s)} \|_{L^2} \| W \|_{H^{s+3}} \| W' \|_{H^{s+3}} \lesssim \| \Pi_{\geq N} q_k^{(s)} \|_{L^2}^2 + N^{-4}.$$ 

Finally, we estimate the tail (5.15) using Cauchy–Schwarz and (2.18):

$$5.15$$

\begin{align*}
\Pi_{\geq N} q_k^{(s)} \|_{L^2} \cdot o(1) \lesssim \| \Pi_{\geq N} q_k^{(s)} \|_{L^2}^2 + o(1)
\end{align*}

uniformly for $\kappa \geq N$ as $N \to \infty$. Note that $o(1)$ as $\kappa \to \infty$ implies $o(1)$ as $N \to \infty$ due to the restriction $\kappa \geq N$.

Altogether, we have shown there exists a constant $C$ such that

$$\left\| \frac{d}{dt} \Pi_{\geq N} q_k^{(s)} (t) \right\|_{L^2} \leq C \| \Pi_{\geq N} q_k^{(s)} (t) \|_{L^2}^2 + o(1) \quad \text{as } N \to \infty,$$
uniformly for $|t| \leq T$, $\kappa \geq N$, and $q(0) \in Q(N)$. By Grönwall’s inequality, we then have
\[
\|\Pi_{\geq N} q_\kappa^{(s)}(t)\|_{L^2}^2 \leq e^{CT} \|\Pi_{\geq N} q_\kappa^{(s)}(0)\|_{L^2}^2 + o(1) \quad \text{as } N \to \infty,
\]
uniformly for $|t| \leq T$, $\kappa \geq N$, and $q(0) \in Q(N)$. By (5.3), the term $\|\Pi_{\geq N} q_\kappa^{(s)}(0)\|_{L^2}$ converges to zero as $N \to \infty$ uniformly for $q(0) \in Q(N)$. Therefore we conclude
\[
\sup_{q(0) \in Q(N)} \sup_{\kappa \geq N} \|\Pi_{\geq N} q_\kappa(t)\|_{C^1_t H^s([-T,T] \times \mathbb{R})} \to 0 \quad \text{as } N \to \infty,
\]
as desired. \hfill \Box

6. Well-posedness

The goal of this section is to prove our two main results, Theorems 1.1 and 1.2. The first step is to show that the tidal $H_\kappa$ flows converge in $H^s$ as $\kappa \to \infty$ by combining the low-regularity convergence of Proposition 5.1 and the uniform Fourier tail control from Proposition 5.3.

**Proposition 6.1.** Fix an integer $s \geq 3$ and $T > 0$. Given bounded sets $Q(\kappa) \subset H^s$ of initial data satisfying (5.3), the corresponding tidal $H_\kappa$ solutions $q_\kappa(t)$ are Cauchy in $C^1_t H^s([-T,T] \times \mathbb{R})$ as $\kappa \to \infty$ uniformly for $q(0) \in Q(\kappa)$.

**Proof.** In the following all spacetime norms will be over the slab $[-T,T] \times \mathbb{R}$. Splitting at a large frequency $N$ to be chosen, we estimate
\[
\|q_\kappa - q_\kappa\|_{C^1_t H^s(\xi \geq N)} \leq (N+1)^{s+2} \|q_\kappa - q_\kappa\|_{C^1_t H^{-2}} + \|q_\kappa - q_\kappa\|_{C^1_t H^s(\xi \geq N)}.
\]
For the second term we estimate
\[
\|q_\kappa - q_\kappa\|_{C^1_t H^s(\xi \geq N)} \leq 2 \left( \|\Pi_{\geq N} q_\kappa\|_{C^1_t H^s} + \|\Pi_{\geq N} q_\kappa\|_{C^1_t H^s} \right).
\]
Fix $\epsilon > 0$. First, by Proposition 5.3 we take $N = N_0$ sufficiently large to ensure that $\Pi_{\geq N} q_\kappa$ is bounded by $\epsilon/2$ for all $\kappa$. With $N_0$ fixed, we then use Proposition 5.1 to pick $\kappa_0 \geq N_0$ so that the first term of $\Pi_{\geq N} q_\kappa$ is bounded by $\epsilon/2$ for all $\kappa \geq \kappa_0$. Together, we conclude that $\|q_\kappa - q_\kappa\|_{C^1_t H^s} \leq \epsilon$ for all $\kappa \geq \kappa_0$. \hfill \Box

Next, we show that the limits guaranteed by Proposition 6.1 solve tidal KdV:

**Proposition 6.2.** Fix an integer $s \geq 3$ and $T > 0$. Given initial data $q(0) \in H^s(\mathbb{R})$, there exists a corresponding solution $q(t)$ to tidal KdV (1.2) in $C^1_t H^s \cap C^1_t H^{s-3}([-T,T] \times \mathbb{R})$.

**Proof.** In the following all spacetime norms will be taken over the slab $[-T,T] \times \mathbb{R}$. Applying Proposition 6.1 to the single function $Q = \{q(0)\}$, we define $q(t)$ to be $\lim_{\kappa \to \infty} q_\kappa(t)$ which we know exists in $C^1_t H^s$. It remains to show that $\frac{d}{dt} q$ is in $C^1_t H^{s-3}$ and is equal to the RHS of tidal KdV (1.2). We already know that the RHS (1.2) is in $C^1_t H^{s-3}$, so it suffices to show that $\frac{d}{dt} q_\kappa$ converges to $\text{RHS} (1.2)$ in the lower regularity norm $C^1_t H^{-1}$.

We will extract the linear and quadratic terms of the tidal $H_\kappa$ flow to witness its convergence to tidal KdV. Using the translation identity (2.11), we write
\[
\frac{d}{dt} q_\kappa \quad (6.3) \quad = -16\kappa^5 \partial_x (R_0 q_\kappa^2 R_0 \delta_x) + 4\kappa^2 \frac{d}{dt} q_\kappa,
\]
\[
\frac{d}{dt} q_\kappa \quad (6.4) \quad = -16\kappa^5 \partial_x (R_0 W' R_0 \delta_x) + 4\kappa^2 W'
\]
We will show that the first five terms (6.3)–(6.7) converge in $C_t H^{-1}$ to the terms of tidal KdV (1.2), and the tail (6.8) converges to zero as $\kappa \to \infty$.

For each term on the RHS, we put two terms in $C_t H^{-1}$. The last term converges to zero since the operator $R_0(2\kappa)\partial^2$ is readily seen via Fourier variables to converge strongly to zero as $\kappa \to \infty$. As the regularity $s - 3 \geq 0$ is greater than $-1$, we conclude

$$\text{(6.3)} \to -q''' \quad \text{in} \quad C_t H^{-1} \quad \text{as} \quad \kappa \to \infty.$$  

For the linear contribution (6.4) from $W$, we again use the operator identity (2.16) to obtain

$$\text{(6.4)} \to -W''' \quad \text{in} \quad C_t H^{-1} \quad \text{as} \quad \kappa \to \infty.$$  

Next we turn to the first quadratic term (6.5). We write

$$\text{(6.5)} = -6q''q + \{16\kappa^3 \delta_x, [\partial, R_0q_R, R_0q_R, R_0\delta_x] - 6q''q\}.$$  

As $q \to q$ in $C_t H^s$, then the first term of the RHS above converges to $6qq'$ in $C_t H^{s-1}$ and hence in $C_t H^{-1}$ as well. For the second term we estimate in $H^{-1}$ by duality. For $\phi \in H^1$ we distribute the derivative $[\partial, \cdot]$ using the product rule and use the operator identity (2.17) to obtain

$$\int \{16\kappa^3 \langle \delta_x, [\partial, R_0q_R, R_0q_R, R_0\delta_x] - 6q''q \rangle \phi \} \, dx$$

$$= \int \{-6[R_0(2\kappa)q'''] - 6q''q''] \phi + 8\kappa^2[R_0(2\kappa)q'] [R_0(2\kappa)q'''](-5\phi + R_0(2\kappa)\partial^2\phi)$$

$$+ 8\kappa^2[R_0(2\kappa)q'] [R_0(2\kappa)q'''](5\phi'' + 2R_0(2\kappa)\partial^2\phi''') \} \, dx.$$  

For each term on the RHS, we put two terms in $L^2$ and the remaining term in $L^\infty$. For those terms with $\phi'''$ we integrate by parts once, we put all factors of $\phi'$ in $L^2$, and we put $\phi$ in $L^\infty \supset H^1$. We put the highest order $q_R$ term in $L^2$ and the lower order term in $L^2$ or $L^\infty$ as needed. Using $\|R_0(2\kappa)\partial^j\|_{op} \lesssim \kappa^{j-2}$ for $j = 0, 1, 2$ (the estimate for $j = 0$ is also true as an operator on $L^\infty$ by the explicit kernel formula for $R_0$ and Young’s inequality), we obtain

$$\left| \int \{16\kappa^3 \langle \delta_x, [\partial, R_0q_R, R_0q_R, R_0\delta_x] - 6q''q \rangle \phi \} \, dx \right| \lesssim \kappa^{-2} \|\phi\|_{H^1} \|q_R\|_{H^s}^2.$$  


Taking a supremum over $\|\phi\|_{H^1} \leq 1$, we conclude

$$\text{(6.5)} \to 6q' \quad \text{in } C_t H^{-1} \text{ as } \kappa \to \infty.$$  

The second quadratic term (6.6) is similar, but now we must put $W$ in $L^\infty$. First we write

$$\text{(6.6)} = 6(Wq_5)' + \{16\kappa^5 \langle \delta_x, [\partial, R_0WR_0q_5R_0] \rangle + 16\kappa^5 \langle \delta_x, [\partial, R_0q_5R_0WR_0] \rangle - 6(Wq_5)' \}.$$  

As $q_5 \to q$ in $C_t H^s$, the first term of the RHS above converges to $6(Wq)'$ in $C_t H^{s-1}$ and hence in $C_t H^{-1}$ as well. For the second term we estimate in $H^{-1}$ by duality. For $\phi \in H^1$ we distribute the derivatives $[\partial, \cdot]$ fusing the product rule and use the operator identity (2.17). For the term $\langle \delta_x, R_0WR_0q_5R_0\delta_x \rangle$ this yields

$$\int \{16\kappa^5 \langle \delta_x, R_0WR_0q_5R_0\delta_x \rangle - 3Wq_5' \} \phi \, dx$$

$$= \int \{-3[R_0(2\kappa)W'][R_0(2\kappa)q_5']\phi + 4\kappa^2[R_0(2\kappa)W][R_0(2\kappa)q_5'(-5\phi + R_0(2\kappa)\partial^2\phi) + 4\kappa^2[R_0(2\kappa)W][R_0(2\kappa)q_5'(5\phi'' + 2R_0(2\kappa)\partial^2\phi')] \} \, dx.$$  

This equality also holds for the second term $\langle \delta_x, R_0q_5'R_0WR_0\delta_x \rangle$ because the identity (2.17) is symmetric in $f$ and $h$. For those terms with $\phi''$ we integrate by parts once to obtain $\phi'$ which we put in $L^2$, we put all factors of $W$ in $L^\infty$, and we put the remaining terms in $L^2$. This yields

$$\left| \int \{16\kappa^5 \langle \delta_x, R_0WR_0q_5'R_0\delta_x \rangle - 3Wq_5' \} \phi \, dx \right| \lesssim \kappa^{-2} \|\phi\|_{H^1} \|q_5\|_{H^s},$$

and similarly for the term $\langle \delta_x, R_0q_5'R_0WR_0\delta_x \rangle$. The remaining two contributions from $\langle \delta_x, R_0WR_0q_5R_0\delta_x \rangle$ and $\langle \delta_x, R_0q_5R_0WR_0\delta_x \rangle$ are even easier, since $W'$ is Schwartz and $q_5$ has one less derivative. Taking a supremum over $\|\phi\|_{H^1} \leq 1$, we conclude

$$\text{(6.6)} \to 6(Wq)' \quad \text{in } C_t H^{-1} \text{ as } \kappa \to \infty.$$  

The third quadratic term (6.7) is similar. We write

$$\text{(6.7)} = 6WW' + \{16\kappa^5 \langle \delta_x, [\partial, R_0WR_0WWR_0] \rangle - 6WW' \}.$$  

We easily estimate the second term above using the operator identity (2.17) and noting that $W \in L^\infty$ and $W'$ is Schwartz. This yields

$$\text{(6.7)} \to 6WW' \quad \text{in } C_t H^{-1} \text{ as } \kappa \to \infty.$$  

Lastly, we show that the tail (6.8) converges to zero in $C_t H^{-1}$. We will estimate in $H^{-1}$ by duality. For $\phi \in H^1$ we write

$$\left| \int \phi \cdot \text{(6.8)} \, dx \right|$$

$$\leq 16\kappa^5 \sum_{\ell \geq 0, \, m_0, \ldots, m_\ell \geq 0} | \text{tr} \left\{ \phi [\partial, R_0(WR_0)^{m_0}q_5R_0 \cdots q_5R_0(WR_0)^{m_\ell}] \right\} |.$$
Recall that we first expanded $g(\kappa, q_\kappa + W)$ in powers of $q_\kappa$, the $\ell$th term having $\ell$-many factors of $q_\kappa R(\kappa, W)$, and then expanded each $R(\kappa, W)$ into a series in $W$ indexed by $m$. The condition $\ell + m_0 + \cdots + m_\ell \geq 3$ reflects that we have already accounted for all of the summands with one and two $q_\kappa$ or $W$. We distribute the derivative $[\partial, \cdot]$, use the estimate (2.3) and the observation $\|f\|_{L^2_x} \lesssim \kappa^{-1} \|f\|_{L^2}$ to put $\phi$ and all copies of $q_\kappa$ in $L^2$, and then estimate $W$ in operator norm to obtain

$$\lesssim \kappa^{5} \sum_{\ell \geq 0, m_0, \ldots, m_\ell \geq 0} \frac{\|\phi\|_{L^2} \left( \frac{\|q_\kappa\|_{H^{1}}}{\kappa^{3/2}} \right)^{\ell} \left( \frac{\|W\|_{W^{1,\infty}}}{\kappa^{2}} \right)^{m_0 + \cdots + m_\ell}}{\ell + m_0 + \cdots + m_\ell \geq 3}.$$ 

We first sum over the indices $m_0, \ldots, m_\ell \geq 0$ as we did in (2.12) using that $W \in W^{1,\infty}$, and then we sum over $\ell \geq 1$ since $q_\kappa$ is bounded in $C_t H^{2}$ for all $\kappa$ large. In doing so, the condition $\ell + m_0 + \cdots + m_\ell \geq 3$ guarantees that summing over the two pararenthetical terms yields a gain $\lesssim (\kappa^{-3/2})^3$, from which we conclude

$$\lesssim \kappa^{-1} \|\phi\|_{H^{1}}.$$ 

Taking a supremum over $\|\phi\|_{H^{1}} \leq 1$ we obtain

$$(6.8) \rightarrow 0 \quad \text{in} \quad C_t H^{-1} \quad \text{as} \quad \kappa \rightarrow \infty. \quad \square$$

We now use a classical $L^2$ energy argument to show that we have unconditional uniqueness for initial data in $H^s$, $s \geq 3$:

**Lemma 6.3.** Fix $T > 0$. Given an initial data $q(0) \in H^3$, there is at most one corresponding solution to tidal KdV (1.12) in $(C_t H^{3} \cap C^1_t L^2)([0,T] \times \mathbb{R})$.

**Proof.** Suppose $q(t)$ and $\tilde{q}(t)$ are both in $(C_t H^{3} \cap C^1_t L^2)([0,T] \times \mathbb{R})$, solve tidal KdV, and have the same initial data $q(0) = \tilde{q}(0)$. From the differential equation, we see that the difference obeys

$$\left| \frac{d}{dt} \int \frac{1}{2}(q - \tilde{q})^2 \, dx \right| = \left| \int (q - \tilde{q})\{ -(q - \tilde{q})''' + 3(q^2 - \tilde{q}^2)' + [6W(q - \tilde{q})]' \} \, dx \right|.$$ 

The first term $(q - \tilde{q})'''$ contributes a total derivative and vanishes, while the remaining terms can be integrated by parts to obtain

$$= \left| \int (q - \tilde{q})^2 \{ \frac{3}{2}(q + \tilde{q})' + 3W' \}(t, x) \, dx \right| \leq (\frac{3}{2} \|q\|_{L^\infty} + \frac{3}{2} \|\tilde{q}\|_{L^\infty} + 3 \|W'\|_{L^\infty}) \|q - \tilde{q}\|_{L^2}^2.$$ 

Estimating $\|q'\|_{L^\infty} \lesssim \|q\|_{H^2}$, $\|\tilde{q}'\|_{L^\infty} \lesssim \|\tilde{q}\|_{H^2}$ and noting that $W'$ is Schwartz, we conclude that there exists a constant $C$ depending on $W$ and the norm of $q$ and $\tilde{q}$ in $C_t H^{3}([0,T] \times \mathbb{R})$ such that

$$\left| \frac{d}{dt} \|q(t) - \tilde{q}(t)\|_{L^2}^2 \right| \leq C \|q(t) - \tilde{q}(t)\|_{L^2}^2.$$ 

Grönwall’s inequality then yields

$$\|q(t) - \tilde{q}(t)\|_{L^2}^2 \leq \|q(0) - \tilde{q}(0)\|_{L^2}^2 e^{CT}$$ 

uniformly for $|t| \leq T$. As the RHS vanishes by premise, we conclude that $\tilde{q}(t) = q(t)$ for all $|t| \leq T$. \quad \square

We are now ready to prove our two main results. First, we complete the proof of continuous dependence upon initial data in $H^s$, $s \geq 3$:
Proof of Theorem 1.1. Fix an integer \( s \geq 3 \). We want to show that tidal KdV (1.2) is globally well-posed for initial data \( q(0) \in H^s(\mathbb{R}) \).

Fix \( T > 0 \) and a convergent sequence \( q_n(0) \) of initial data in \( H^s(\mathbb{R}) \). It suffices to show that the corresponding solutions \( q_n(t) \) of tidal KdV (1.2) constructed in Proposition 6.2 are Cauchy in \( C_t H^s([-T,T] \times \mathbb{R}) \) as \( n \to \infty \).

Consider the set \( Q := \{ q_n(0) : n \in \mathbb{N} \} \) of initial data, which is bounded and equicontinuous in \( H^s \) since it is convergent in \( H^s \). Let \( H^s_{\kappa} \) denote the Hamiltonian for the tidal \( H_{\kappa} \) flow. We estimate

\[
\|q_n(t) - q_m(t)\|_{C_t H^s} \leq \|e^{t J \nabla H^s_{\kappa}} q_n(0) - e^{t J \nabla H^s_{\kappa}} q_m(0)\|_{C_t H^s} + 2 \sup_{q \in Q} \sup_{\kappa \geq \kappa_0} \|e^{t J \nabla H^s_{\kappa}} q - e^{t J \nabla H^s_{\kappa}} q\|_{C_t H^s},
\]

(6.9)

where the spacetime norms are over the slab \([-T, T] \times \mathbb{R} \). By Proposition 6.1, the second term of RHS (6.9) can be made arbitrarily small uniformly in \( n, m \) by picking \( \kappa \) sufficiently large. The first term of RHS (6.9) then converges to zero as \( n, m \to \infty \) due to the well-posedness of the tidal \( H_{\kappa} \) flow (cf. Corollary 3.6). \( \square \)

From our understanding of tidal KdV at high-regularity, we are now able to conclude that KdV is well-posed for \( H^{-1}(\mathbb{R}) \) perturbations of step-like solutions:

Proof of Theorem 1.2. Let \( V(t) = W + q(t) \) be the solution to KdV (1.1) corresponding to the tidal KdV solution with initial data \( q(0) \equiv 0 \). We want to show that KdV (1.1) is globally well-posed for initial data \( u(0) \in V(0) + H^{-1}(\mathbb{R}) \). By our general result [40], it suffices to show that for every \( T > 0 \) the following conditions are satisfied:

(i) \( V \) solves KdV and is bounded in \( W^{2,\infty}(\mathbb{R}_x) \) uniformly for \( |t| \leq T \),

(ii) The solutions \( V_{\kappa}(t) \) to the \( H_{\kappa} \) flows with initial data \( V(0) \) are bounded in \( W^{4,\infty}(\mathbb{R}_x) \) uniformly for \( |t| \leq T \) and \( \kappa \) sufficiently large,

(iii) \( V_{\kappa} - V \to 0 \) in \( W^{2,\infty}(\mathbb{R}_x) \) as \( \kappa \to \infty \) uniformly for \( |t| \leq T \) and initial data in the set \( \{ V_{\kappa}(t) : |t| \leq T, \kappa \geq \kappa_0 \} \).

Fix \( T > 0 \). As \( q(0) \equiv 0 \) is in \( H^3 \), the \( a \ priori \) estimate of Proposition 3.6 guarantees that the tidal \( H_{\kappa} \) flows \( q_{\kappa}(t) \) are bounded in \( C_t H^5([-T,T] \times \mathbb{R}) \) uniformly for \( \kappa \) large. By definition of the tidal \( H_{\kappa} \) flow we have that \( V_{\kappa}(t) = W + q_{\kappa}(t) \) solves the \( H_{\kappa} \) flow. Combined with the embedding \( H^1 \hookrightarrow L^\infty \), this shows that (ii) is satisfied.

By Proposition 5.3, we know that the sets \( Q(\kappa) := \{ q_{\kappa}(t) : |t| \leq T, \kappa \geq \kappa_0 \} \) obey (6.9). Therefore, by Proposition 6.1, we know that \( q_{\kappa} \to q \) in \( C_t H^5([-T,T] \times \mathbb{R}) \) as \( \kappa \to \infty \) uniformly for initial data in \( Q(\kappa) \). Consequently \( V_{\kappa}(t) \) converges to \( V(t) = W + q(t) \) in \( C_t W^{4,\infty}([-T,T] \times \mathbb{R}) \), which shows that (iii) is satisfied.

Finally, by Proposition 6.2, we know \( q(t) \) is in \( C_t H^5([-T,T] \times \mathbb{R}) \) and solves tidal KdV. Therefore \( V(t) \) solves KdV and is in \( C_t W^{4,\infty}([-T,T] \times \mathbb{R}) \), which shows that (i) is satisfied. \( \square \)

Lastly, we record the following reformulation of well-posedness for \( H^{-1}(\mathbb{R}) \) perturbations of \( W \):

Corollary 6.4. Fix a sequence of initial data \( u_n(0) \in W + H^3(\mathbb{R}) \) with \( u_n(0) \to W \) convergent in \( H^{-1}(\mathbb{R}) \) as \( n \to \infty \), and let \( u_n(t) \) denote the corresponding solutions to KdV (1.1) guaranteed by Theorem 1.1. Then there exists a continuous function
$u : \mathbb{R}_+ \rightarrow W + H^{-1}(\mathbb{R})$ so that $u_n(t) - u(t) \rightarrow 0$ in $H^{-1}(\mathbb{R})$ as $n \rightarrow \infty$ uniformly on bounded time intervals.

References

[1] K. Andreiev, I. Egorova, T. L. Lange, and G. Teschl, Rarefaction waves of the Korteweg-de Vries equation via nonlinear steepest descent, J. Differential Equations 261 (2016), no. 10, 5371–5410. MR3548255

[2] V. B. Baranetski and V. P. Kotlyarov, Asymptotic behavior in a back front domain of the solution of the KdV equation with a “step type” initial condition, Teoret. Mat. Fiz. 126 (2001), no. 2, 214–227. MR1863082

[3] T. B. Benjamin, J. L. Bona, and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, Philos. Trans. Roy. Soc. London Ser. A 272 (1972), no. 1220, 47–78. MR427868

[4] R. F. Bikbaev, Structure of a shock wave in the theory of the Korteweg-de Vries equation, Phys. Lett. A 141 (1989), no. 5-6, 289–293. MR1025275

[5] R. F. Bikbaev, Time asymptotics of the solution of the nonlinear Schrödinger equation with boundary conditions of “step-like” type, Teoret. Mat. Fiz. 81 (1989), no. 1, 3–11. MR1025306

[6] R. F. Bikbaev and R. A. Sharipov, The asymptotic behavior, as $t \to \infty$, of the solution of the Cauchy problem for the Korteweg-de Vries equation in a class of potentials with finite-gap behavior as $x \to \pm \infty$, Teoret. Mat. Fiz. 78 (1989), no. 3, 345–356. MR996221

[7] J. Bona and R. Scott, Solutions of the Korteweg-de Vries equation in fractional order Sobolev spaces, Duke Math. J. 43 (1976), no. 1, 87–99. MR393887

[8] J. L. Bona and R. Smith, The initial-value problem for the Korteweg-de Vries equation, Philos. Trans. Roy. Soc. London Ser. A 278 (1975), no. 1287, 555–601. MR385355

[9] I. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, Geom. Funct. Anal. 3 (1993), no. 3, 209–262. MR1215780

[10] J. Bourgain, Time asymptotics of the solution of the nonlinear Schrödinger equation with boundary conditions of “step-like” type, Teoret. Mat. Fiz. 81 (1989), no. 1, 3–11. MR1025306

[11] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, J. Amer. Math. Soc. 16 (2003), no. 3, 705–749. MR1969209

[12] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, J. Amer. Math. Soc. 16 (2003), no. 3, 705–749. MR1969209

[13] A. Cohen, Solutions of the Korteweg-de Vries equation with steplike initial profile, Comm. Partial Differential Equations 9 (1984), no. 8, 751–806. MR748367

[14] A. Cohen and T. Kappeler, Solutions to the Korteweg-de Vries equation with initial profile in $L^1_1(\mathbb{R}) \cap L^\infty_1(\mathbb{R}^+)$, SIAM J. Math. Anal. 18 (1987), no. 4, 991–1025. MR982486

[15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, J. Amer. Math. Soc. 16 (2003), no. 3, 705–749. MR1969209

[16] I. Egorova, Z. Gladka, V. Kotlyarov, and G. Teschl, Long-time asymptotics for the Korteweg-de Vries equation with step-like initial data, Nonlinearity 25 (2013), no. 7, 1839–1864. MR3071444

[17] I. Egorova, K. Grunert, and G. Teschl, On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data. I. Schwartz-type perturbations, Nonlinearity 22 (2009), no. 6, 1431–1457. MR2507328

[18] I. Egorova and G. Teschl, On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data II. Perturbations with finite moments, J. Anal. Math. 115 (2011), 71–101. MR2855034

[19] C. Gallo, Korteweg-de Vries and Benjamin-Ono equations on Zhidkov spaces, Adv. Differential Equations 10 (2005), no. 3, 277–308. MR2123133

[20] S. Grudsky, C. Remling, and A. Rybkin, The inverse scattering transform for the KdV equation with step-like singular Miura initial profiles, J. Math. Phys. 56 (2015), no. 9, 091505, 14. MR3395045

[21] S. Grudsky and A. Rybkin, On positive type initial profiles for the KDV equation, Proc. Amer. Math. Soc. 142 (2014), no. 6, 2079–2086. MR3182026
[22] Z. Guo, Global well-posedness of Korteweg-de Vries equation in $H^{-3/4}(\mathbb{R})$, J. Math. Pures Appl. (9) 91 (2009), no. 6, 583–597. MR2531556
[23] B. Harrop-Griffiths, R. Killip, and M. Visan, Sharp well-posedness for the cubic NLS and mKdV in $H^s(\mathbb{R})$, 2020. Preprint arXiv:2003.05011.
[24] ¯E. Ja. Hruslov, Asymptotic behavior of the solution of the Cauchy problem for the Korteweg-de Vries equation with steplike initial data, Mat. Sb. (N.S.) 99(141) (1976), no. 2, 261–281, 296. MR0487088
[25] R. Iorio, F. Linares, and M. Scialom, KdV and BO equations with bore-like data, Differential Integral Equations 11 (1998), no. 6, 895–915. MR1659252
[26] T. Kappeler, Solutions of the Korteweg-de Vries equation with steplike initial data, J. Differential Equations 63 (1986), no. 3, 306–331. MR848272
[27] T. Kappeler and P. Topalov, Global wellposedness of KdV in $H^{-1}(T, \mathbb{R})$, Duke Math. J. 135 (2006), no. 2, 327–360. MR2267286
[28] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, Spectral theory and differential equations (Proc. Sympos., Dundee, 1974; dedicated to Konrad Jörgens), 1975, pp. 25–70. Lecture Notes in Math., Vol. 448. MR0407477
[29] C. E. Kenig, G. Ponce, and L. Vega, Well-posedness of the initial value problem for the Korteweg-de Vries equation, J. Amer. Math. Soc. 4 (1991), no. 2, 323–347. MR1086966
[30] R Killip, J. Murphy, and M. Visan, Invariance of white noise for KdV on the line, Invent. Math. 222 (2020), no. 1, 203–282. MR4145790
[31] R Killip, M. Ntekoume, and M. Visan, On the well-posedness problem for the derivative nonlinear Schrödinger equation, 2021. Preprint arXiv:2101.12274.
[32] KdV is well-posed in $H^{-1}$, Ann. of Math. (2) 190 (2019), no. 1, 249–305. MR3990604
[33] N. Kishimoto, Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity, Differential Integral Equations 22 (2009), no. 5-6, 447–464. MR2501679
[34] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philos. Mag. (5) 39 (1895), no. 240, 422–443. MR3363408
[35] V. P. Kotlyarov and E. Ya. Khruslov, Solitons of the nonlinear Schrödinger equation, which are generated by the continuous spectrum, Teoret. Mat. Fiz. 68 (1986), no. 2, 172–186. MR871046
[36] V. Yu. Novokshenov, Time asymptotics for soliton equations in problems with step initial conditions, Sovrem. Mat. Prilozh. 5, Asimptot. Metody Funkts. Anal. (2003), 138–168. MR2152933
[37] M. Ntekoume, Symplectic non-squeezing for the KdV flow on the line, 2019. Preprint arXiv:1911.11355.
[38] J. M. Palacios, Local well-posedness for the gKdV equation on the background of a bounded function, 2021. Preprint arXiv:2104.15126.
[39] M. Reed and B. Simon, Methods of modern mathematical physics. II. Fourier analysis, self-adjointness, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR0493420
[46] A. Rybkin, The Hirota τ-function and well-posedness of the KdV equation with an arbitrary step-like initial profile decaying on the right half line, Nonlinearity 24 (2011), no. 10, 2953–2990. MR2842104

[47] , KdV equation beyond standard assumptions on initial data, Phys. D 365 (2018), 1–11. MR3739918

[48] J. C. Saut and R. Temam, Remarks on the Korteweg-de Vries equation, Israel J. Math. 24 (1976), no. 1, 78–87. MR454425

[49] R. Temam, Sur un problème non linéaire, J. Math. Pures Appl. (9) 48 (1969), 159–172. MR261183

[50] M. Taniuti and T. Mukasa, Parabolic regularizations for the generalized Korteweg-de Vries equation, Funkcial. Ekvac. 14 (1971), 89–110. MR312097

[51] P. Zhidkov, Korteweg-de Vries and nonlinear Schrödinger equations: qualitative theory, Lecture Notes in Mathematics, vol. 1756, Springer-Verlag, Berlin, 2001. MR1831831

Thierry Laurens, Department of Mathematics, University of California, Los Angeles, CA 90095, USA

Email address: laurenst@math.ucla.edu