LEFT MULTIPLIERS OF REPRODUCING KERNEL HILBERT $C^*$-MODULES AND THE PAPADAKIS THEOREM

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ABSTRACT. We give a modified definition of a reproducing kernel Hilbert $C^*$-module (shortly, $RKHC^*M$) without using the condition of self-duality and discuss some related aspects; in particular, an interpolation theorem is presented. We investigate the exterior tensor product of $RKHC^*Ms$ and find their reproducing kernel. In addition, we deal with left multipliers of $RKHC^*Ms$. Under some mild conditions, it is shown that one can make a new $RKHC^*M$ via a left multiplier. Moreover, we introduce the Berezin transform of an operator in the context of $RKHC^*Ms$ and construct a unital subalgebra of the unital $C^*$-algebra consisting of adjointable maps on an $RKHC^*M$ and show that it is closed with respect to a certain topology. Finally, the Papadakis theorem is extended to the setting of $RKHC^*M$, and in order for the multiplication of two specific functions to be in the Papadakis $RKHC^*M$, some conditions are explored.

1. Introduction

Hilbert $C^*$-modules are generalization of Hilbert spaces by allowing the inner product to take its values in a $C^*$-algebra instead of the complex numbers. At the same time, they are extensions of $C^*$-algebras. Indeed, a $C^*$-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module if we define $\langle a, b \rangle = a^*b$ $(a, b \in \mathcal{A})$. For Hilbert $C^*$-modules $\mathcal{E}$ and $\mathcal{F}$, the set of all adjointable maps from $\mathcal{E}$ to $\mathcal{F}$ is denoted by $L(\mathcal{E}, \mathcal{F})$, and $L(\mathcal{E})$ stands for the unital $C^*$-algebra $L(\mathcal{E}, \mathcal{E})$. We assume that $\mathcal{A} \otimes_{alg} \mathcal{B}$ and $\mathcal{A} \otimes_\ast \mathcal{B}$ denote, respectively, the algebraic tensor product and an arbitrary fixed $C^*$-tensor product of the $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ with the corresponding $C^*$-tensor norm $\| \cdot \|_\ast$. For more details on the general theory of $C^*$-algebras, the reader is referred to [15]. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert $C^*$-modules over $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. The algebraic tensor product and the exterior tensor product of $\mathcal{E}_1$ and $\mathcal{E}_2$ are denoted by $\mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$ and $\mathcal{E}_1 \otimes \mathcal{E}_2$, respectively. Indeed, $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a Hilbert $\mathcal{A} \otimes_\ast \mathcal{B}$-module equipped with the following $\mathcal{A} \otimes_\ast \mathcal{B}$-valued inner product [12]:

$$\langle x \otimes y, z \otimes w \rangle = \langle x, z \rangle \otimes \langle y, w \rangle \quad (x, z \in \mathcal{E}_1, y, w \in \mathcal{E}_2).$$

Let $\mathcal{E}$ be a Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{A}$. The set of all bounded $\mathcal{A}$-linear maps from $\mathcal{E}$ to $\mathcal{A}$ is denoted by $\mathcal{E}'$. The space $\mathcal{E}$ can be embedded in $\mathcal{E}'$ via $\sim : \mathcal{E} \rightarrow \mathcal{E}'$ defined by $x \mapsto x$, where $\hat{x}(y) = \langle x, y \rangle$ $(y \in \mathcal{E})$. A Hilbert $C^*$-module $\mathcal{E}$ is called self-dual if $\mathcal{E} = \mathcal{E}'$. We refer the reader to [12, 20] for more details on the theory of Hilbert $C^*$-modules.

Throughout this article, $S$ and $X$ stand for nonempty sets. We denote $C^*$-algebras by $\mathcal{A}$ and $\mathcal{B}$. By $\mathcal{Z}(\mathcal{A})$ and $\mathcal{A}^+$ we mean the center of the $C^*$-algebra $\mathcal{A}$ and the set

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of all positive elements of $\mathfrak{A}$, respectively. When $\mathfrak{A}$ is unital, $\text{Inv}(\mathfrak{A})$ stands for the set of all invertible elements of $\mathfrak{A}$. The $C^*$-algebra of all $n \times n$ matrices with entries in $\mathfrak{A}$ is presented by $M_n(\mathfrak{A})$.

Aronszajn [3] defined the concept of reproducing kernel Hilbert space (shortly, \textit{RKHS}), and Schwartz [17] developed the concept. This theory has many applications in integral equations, complex analysis, and so on; see [11]. Indeed, an \textit{RKHS} $\mathcal{H}$ is a Hilbert space of $\mathbb{C}$-valued functions on a set $S$ such that, for all $s \in S$, the evaluation map $\delta_s : \mathcal{H} \to \mathbb{C}$ defined by $\delta_s(f) = f(s)$ is bounded. It follows from the Riesz representation theorem that, for every $s \in S$, there exists a unique element $k_s \in \mathcal{H}$ such that

$$\delta_s(f) = f(s) = \langle f, k_s \rangle \quad (f \in \mathcal{H}).$$

Furthermore, the two-variable function $K : S \times S \to \mathbb{C}$ defined by $K(s, t) = k_t(s) \ (s, t \in S)$ is called a reproducing kernel for $\mathcal{H}$. A theorem due to Moore [16, Theorem 2.14] states that for a scalar-valued positive definite kernel, there is a unique \textit{RKHS} such that $K$ is its reproducing kernel. Indeed, there is a two-sided relation between scalar-valued positive definite kernels and \textit{RKHS}s. For more information about reproducing kernel spaces we refer the interested reader to [2, 16, 4] and references therein.

The Papadakis theorem [16, Theorem 2.10] shows that $\{f_s : s \in S\}$ is a Parseval frame for an \textit{RKHS} if and only if $K(x, y) = \sum_{s \in S} f_s(x)f_s(y)$, where the series converges pointwise. In general, finding $k_s$ for every $s \in S$, and so $K$, is not easy, but the Papadakis theorem provides a useful benchmark.

Although Hilbert $C^*$-modules generalize Hilbert spaces, some fundamental properties of Hilbert spaces are no longer valid in Hilbert $C^*$-modules in their full generality. For instance, not every bounded $\mathfrak{A}$-linear operator is adjointable. Thus in the theory of Hilbert $C^*$-modules, it is interesting to ask which results, similar to those for Hilbert spaces, can be proved probably under some conditions. Inspiring by some ideas in the Hilbert space setting [16], we extend some significant classical results to the setting of Hilbert $C^*$-modules.

The paper is organized as follows. In the next section, we use some ideas of Szafraniec [19] to give a modified definition of a reproducing kernel Hilbert $C^*$-module (shortly, \textit{RKHC}*$M$) due to Heo [10] without using the condition of self-duality and discuss some related aspects. Such a lack of self-duality shows that our investigation is nontrivial and is not a straightforward generalization of the classical case of \textit{RKHS}s. In the same section, the exterior tensor product of \textit{RKHC}*$M$s is investigated and an interpolation theorem is presented. Section 3 deals with left multipliers of \textit{RKHC}*$M$s. Under some mild conditions, we show that one can make a new \textit{RKHC}*$M$ by a left multiplier. In addition, we introduce the Berezin transform of an operator in the context of \textit{RKHC}*$M$s and construct a unital subalgebra of the unital $C^*$-algebra consisting of adjointable maps on an \textit{RKHC}*$M$ and show that it is closed with respect to a certain topology. In section 4, we extend the Papadakis theorem to the setting of \textit{RKHC}*$M$ and find some conditions, in order for the multiplication of two specific functions to be in the Papadakis \textit{RKHC}*$M$.
2. A Modified Definition of $\text{RKHC}^*M$

We denote by $\mathcal{F}(S, \mathfrak{A})$ the set of all $\mathfrak{A}$-valued functions on $S$. It is clear that $\mathcal{F}(S, \mathfrak{A})$ is a right $\mathfrak{A}$-module equipped with the ordinary operations.

**Definition 2.1** (see [10]). By a kernel on $S$ we mean a map $K : S \times S \to \mathfrak{A}$. A kernel $K$ is called positive definite whenever the matrix $(K(s_i, s_j))_{i,j=1}^n \in \mathbb{M}_n(\mathfrak{A})$ is positive or, equivalently,

$$\sum_{i,j=1}^n a_i^*K(s_i, s_j)a_j \geq 0,$$

for all $n \in \mathbb{N}$, $s_1, s_2, \ldots, s_n \in S$, and $a_1, a_2, \ldots, a_n \in \mathfrak{A}$. Then, $K(s, t) = K(t, s)^*$ for all $s, t \in S$. We say that a kernel $K$ is strictly positive whenever $(K(s_i, s_j))_{i,j=1}^n$ is positive and invertible in $\mathbb{M}_n(\mathfrak{A})$.

The following definition of an $\text{RKHC}^*M$ differs from that of [10] and is inspired by [19].

**Definition 2.2.** A right $\mathfrak{A}$-submodule $\mathcal{E}$ of $\mathcal{F}(S, \mathfrak{A})$ is called a reproducing kernel Hilbert $C^*$-module if it satisfies the following conditions:

(i) $\mathcal{E}$ is a Hilbert $C^*$-module over $\mathfrak{A}$.

(ii) For every $s \in S$, there exists $k_s \in \mathcal{E}$ such that the evaluation map $\delta_s : \mathcal{E} \to \mathfrak{A}$ at $s \in S$ satisfies $\delta_s(f) = f(s) = \langle k_s, f \rangle$ for all $f \in \mathcal{E}$.

(iii) The $\mathfrak{A}$-linear span of $\{k_s : s \in S\}$ is dense in $\mathcal{E}$.

The element $k_s$ is called the reproducing kernel for the point $s \in S$.

The corresponding reproducing kernel $K : S \times S \to \mathfrak{A}$ is given by $K(s, t) = \langle k_s, k_t \rangle$ for every $s, t \in S$; see [5]. In what follows, we use the following theorems.

**Theorem 2.3.** [10, proposition 3.1] Let $S$, $\mathfrak{A}$, and $\mathcal{E}$ be as above. Then

(i) the kernel $K$ is positive definite;

(ii) $K(s, s) \in \mathfrak{A}^+$ for every $s \in S$;

(iii) $\|K(s, t)\|^2 \leq \|K(s, s)\|\|K(t, t)\|$ for all $s, t \in S$.

Now let $K : S \times S \to \mathfrak{A}$ be a positive definite kernel. For every $s \in S$, consider the function $k_s : S \to \mathfrak{A}$ by $k_s(t) = K(t, s)$. Assume that $\mathcal{E}_0$ is the right $\mathfrak{A}$-module of $\mathfrak{A}$-valued functions on $S$ generated by $\{k_s : s \in S\}$. Now, setting

$$\left\langle \sum_{i=1}^m k_{a_i}a_i, \sum_{j=1}^n k_{t_j}b_j \right\rangle := \sum_{i=1}^m \sum_{j=1}^n a_i^*K(s_i, t_j)b_j,$$

where $m, n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathfrak{A}$, and $s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n \in S$, we make $\mathcal{E}_0$ into a pre-Hilbert $\mathfrak{A}$-module. Suppose that $\mathcal{E}$ denotes its completion. In addition, $\mathcal{E}$ can be considered as a Hilbert $\mathfrak{A}$-module of $\mathfrak{A}$-valued functions on $S$, and evidently $\langle k_s, k_t \rangle = K(s, t)$ for all $s, t \in S$, which means that $K$ is its reproducing kernel. The above construction is given by Heo [10] and holds true for the definition 2.2. It entails the following result.

**Theorem 2.4.** [10, Theorem 3.2] If $K : S \times S \to \mathfrak{A}$ is positive definite, then there exists a unique Hilbert $\mathfrak{A}$-module consisting of $\mathfrak{A}$-valued functions on $S$ such that $K$ is its reproducing kernel.
Heo [10] introduced the concept of an \( \text{RKHC}^*M \) and transferred some of the classical theorems to the setting of \( \text{RKHC}^*M \)s. Regretfully, the requirement of self-duality in [10] is too strong to hold in nontrivial examples. For example, it is known that finitely generated Hilbert \( C^* \)-modules and Hilbert \( C^* \)-modules over finite-dimensional \( C^* \)-algebras are self-dual; see [6] and references therein.

Here, we explain why \( \text{RKHC}^*M \)s are almost never self-dual; see [6]. For simplicity, we assume \( S = \mathbb{N} \). Then \( \xi = \sum_s k_s a_s \) can be viewed as a sequence \( \xi = (a_s)_{s \in \mathbb{N}} \), where \( a_s \in \mathfrak{A}, s \in \mathbb{N} \). The inner product on \( \mathcal{E} \) is given by

\[
\langle \xi, \eta \rangle = \sum_{s,t \in \mathbb{N}} a_s^* K(s,t) b_t.
\]

A sequence \( \xi \) lies in \( \mathcal{E} \) if and only if the series \( \sum_{s,t \in \mathbb{N}} a_s^* K(s,t) a_t \) is norm-convergent. Let \( f = (f_s) \) be a sequence such that all partial sums \( \sum_{s,t \in \mathbb{N}} f_s^* K(s,t) f_t \) are uniformly bounded. Then the formula \( f(\xi) = \sum_{s,t \in \mathbb{N}} f_s^* K(s,t) a_t \) is well-defined (i.e., the series is norm-convergent) and gives a bounded \( \mathfrak{A} \)-linear map from \( \mathcal{E} \) to \( \mathfrak{A} \), that is, \( f \in \mathcal{E}' \). It is easy to see that the two conditions, norm convergence and uniform boundedness, are different in most cases. For simplicity, assume that \( \mathfrak{A} \) is commutative, and so is of the form \( C_0(X) \) for a locally compact Hausdorff space \( X \). Then \( K(s,t) \)s are functions on \( X \). Suppose that \( \cup_{s \in \mathbb{N}} \text{supp } K(s,s) \) is infinite. Then one can find functions \( f_s \), \( s \in \mathbb{N} \), on \( X \) such that \( \|K(s,s) f_s\| = 1 \) (where \( \| \cdot \| \) is the sup-norm on \( X \)) and \( f_s f_t = 0 \) when \( s \neq t \). Then, obviously, \( (f_s)_{s \in \mathbb{N}} \in \mathcal{E}' \setminus \mathcal{E} \). Note that since \( K \) is positive definite, the condition that \( \cup_{s \in \mathbb{N}} \text{supp } K(s,s) \) is finite, implies that \( \text{supp } K(s,t) \) is finite as well.

While every closed subspace of a Hilbert space is orthogonally complemented, it is well known that submodules in Hilbert \( C^* \)-modules are often not orthogonally complemented. Furthermore, it is shown in [13] that if there exists a full Hilbert \( \mathfrak{A} \)-module in which every closed submodule is orthogonally complemented, then \( \mathfrak{A} \) is \( * \)-isomorphic to a \( C^* \)-algebra of (not necessarily all) compact operators on some Hilbert spaces; see [7] and references therein. The following theorem provides a class of orthogonally complemented submodules being \( \text{RKHC}^*M \)s. To achieve the next result, we need a lemma.

**Lemma 2.5.** Let \( \mathcal{E} \) be an \( \text{RKHC}^*M \) on a set \( S \) with the kernel \( K \). Then every orthogonally complemented submodule \( \mathcal{E}_0 \) of \( \mathcal{E} \) can be endowed with an \( \text{RKHC}^*M \) on \( S \).

**Proof.** Let \( P : \mathcal{E} \rightarrow \mathcal{E}_0 \) be the orthogonal projection onto \( \mathcal{E}_0 \). If \( k_s \) is the reproducing kernel at the point \( s \) in \( \mathcal{E} \), then \( P(k_s) \) evidently is the reproducing kernel for the point \( s \) in \( \mathcal{E}_0 \) satisfying the conditions of Definition 2.1. Note that

\[
f(s) = \langle k_s, f \rangle = \langle k_s, P(f) \rangle = \langle P(k_s), f \rangle,
\]

for all \( f \in \mathcal{E}_0 \). Thus \( \mathcal{E}_0 \) is an \( \text{RKHC}^*M \) with the reproducing kernel \( K_0(s,t) = \langle P(k_s), P(k_t) \rangle \). \qed

**Theorem 2.6.** Suppose that \( K(s_0,s_0) \) is invertible for some \( s_0 \in S \). Then the following properties hold:

(i) The submodule \( \mathcal{E}_0 = \{ f \in \mathcal{E} : f(s_0) = 0 \} \) is orthogonally complemented in \( \mathcal{E} \).
Proof. (i) Note that $k_{s_0}$ satisfying $\langle k_{s_0}, k_{s_0} \rangle = K(s_0, s_0)$ is invertible. Hence $P(f) = k_{s_0} \langle k_{s_0}, k_{s_0} \rangle^{-1} \langle k_{s_0}, f \rangle$ is the orthogonal projection onto the $\mathcal{A}$-linear span of $k_{s_0}$. Then $\mathcal{E} = k_{s_0} \mathcal{A} \oplus (k_{s_0} \mathcal{A})^\perp$. The condition $f \perp k_{s_0}$ can be written as $0 = \langle k_{s_0}, f \rangle = f(s_0)$. Hence $(k_{s_0} \mathcal{A})^\perp = \mathcal{E}_0$.

(ii) The reproducing kernel for $\mathcal{E}_0$ is given by
\[
\langle P(k_s), P(k_t) \rangle = \langle k_s, P(k_t) \rangle = \langle k_s, k_t - k_{s_0} \langle k_{s_0}, k_{s_0} \rangle^{-1} \langle k_{s_0}, k_t \rangle \rangle = K(s, t) - K(s, s_0)K(s_0, s_0)^{-1}K(s_0, t) = K_0(s, t).
\]

\[\square\]

The next result provides an interpolation theorem. We should notify that a finitely generated $\mathcal{A}$-submodule is not necessarily closed. For example, when $\mathcal{A} = C([0, 1]) = \mathcal{E}$, the submodule singly generated by the function $f(x) = x$ is not closed, and its closure is not finitely generated.

**Theorem 2.7.** Let $\mathcal{E}$ be an RKHC $\ast$-M on $S$ with the reproducing kernel $K$. Let $F = \{s_1, \ldots, s_n\}$ be a subset of distinct elements of $S$ such that the $\ast$-submodule generated by $k_{s_1}, \ldots, k_{s_n}$ is closed in $\mathcal{E}$, and let $a_1, \ldots, a_n \in \mathcal{A}$. Then there is a function $f \in \mathcal{E}$ of minimal norm such that $f(s_i) = a_i$ for all $1 \leq i \leq n$ if and only if $(a_1, \ldots, a_n)^\dagger$ is in the range of the matrix $(K(s_j, s_i)) \in M_n(\mathcal{A})$.

**Proof.** Assume that $P_F$ is the orthogonal projection onto the orthogonally complemented submodule $\mathcal{E}_F$ finitely generated by $\{k_s : s \in F\}$. So that $\mathcal{E} = \mathcal{E}_F \oplus \mathcal{E}_F^\perp$, see [14, Lemma 2.3.7]. Let $f \in \mathcal{E}$ and let $P_F(f) = \sum_{j=1}^n k_{s_j} b_j \in \mathcal{E}_F$, where $b_j$'s are in $\mathcal{A}$. Then $P_F(f)(s) = f(s)$ for all $s \in F$, since $f(s) = \langle k_s, f \rangle = 0$ for all $s \in F$ if and only if $f \in \mathcal{E}_F^\perp$.

If there is a function $f \in \mathcal{E}$ such that $f(s_i) = a_i$ for all $1 \leq i \leq n$, then
\[
a_i = f(s_i) = P_F(f)(s_i) = \langle k_{s_i}, P_F(f) \rangle = \left\langle k_{s_i}, \sum_{j=1}^n k_{s_j} b_j \right\rangle = \sum_{j=1}^n K(s_i, s_j) b_j.
\]

Thus $(a_1, \ldots, a_n)^\dagger = (K(s_j, s_j)) (b_1, \ldots, b_n)^\dagger$ is in the range of the matrix $(K(s_j, s_i))$. In addition, if $g \in \mathcal{E}$ interpolates these points, then $(g - f)(s) = 0$ for all $s \in S$. Hence $g - f \in \mathcal{E}_F^\perp$, whence $g = f + h$ with $h \in \mathcal{E}_F^\perp$. Therefore,
\[
\|P_F(f)\| = \|P_F(f + h)\| \leq \|f + h\| = \|g\|.
\]

Thus, $P_F(f)$ is the unique function of minimum norm that interpolates these values.

Conversely, if $(a_1, \ldots, a_n)^\dagger$ is in the range of $(K(s_j, s_i))$ and $(a_1, \ldots, a_n)^\dagger = (K(s_i, s_j)) (b_1, \ldots, b_n)^\dagger$ for some $b_1, \ldots, b_n \in \mathcal{A}$, then
\[
a_i = \sum_{j=1}^n K(s_i, s_j) b_j = \left\langle k_{s_i}, \sum_{j=1}^n k_{s_j} b_j \right\rangle.
\]

Putting $f := \sum_{j=1}^n k_s b_j \in \mathcal{E}_F$, we get $f = P_F(f)$ and $a_i = \langle k_{s_i}, f \rangle = f(s_i)$ for $1 \leq i \leq n$. \[\square\]
Remark 2.8. If $\mathbf{a} = (a_1, \ldots, a_n)^t$ and $\mathbf{b} = (b_1, \ldots, b_n)^t$ are in the Hilbert $C^*$-module $A^n$ with its natural inner product (see [14]) and $(a_1, \ldots, a_n)^t = (K(s_i, s_j))(b_1, \ldots, b_n)^t$, then we can choose $f$ such that $\|f\| = \|\langle \mathbf{a}, \mathbf{b} \rangle\|^{1/2}$. In fact, if $f := \sum_{j=1}^n k_s b_j$, then

$$
\|f\|^2 = \left\| \sum_{i=1}^n k_s b_i, \sum_{j=1}^n k_s b_j \right\| = \left\| \sum_{1 \leq i, j \leq n} b_i^* K(s_i, s_j) b_j \right\|
$$

$$
= \left\| \left( b_i \right)_i, \left( \sum_{j=1}^n K(s_i, s_j) b_j \right)_i \right\| = \| \langle \mathbf{b}, \mathbf{a} \rangle \|.
$$

If $K$ is strictly positive, then $\mathbf{b}$ is uniquely defined by $\mathbf{a}$. Thus, from the arguments at the first part of the proof of the above theorem, there is a unique $f \in E_F$ satisfying the conditions of Theorem 2.7.

Now, we investigate the exterior tensor product of $RKHC^*Ms$. Let $K_1 : X \times X \to A$ and $K_2 : S \times S \to B$ be positive definite kernels on sets $X$ and $S$, respectively. It follows from Theorem 2.4 that there are $RKHC^*M E_1$ and $E_2$ over $A$ and $B$ consisting of $A$-valued functions on $X$ and $B$-valued functions on $S$, respectively. We define $K : (X \times S) \times (X \times S) \to A \otimes_s B$, where $\otimes_s$ denotes a fixed $C^*$-tensor product with the $C^*$-cross-norm $\| \cdot \|_*$ by

$$
K((x, s), (y, t)) = K_1(x, y) \otimes K_2(s, t), \quad (x, s), (y, t) \in X \times S.
$$

Let $\xi_i = \sum_{k=1}^n a_i^k \otimes b_i^k \in A \otimes_{alg} B$, $i = 1, \ldots, m$. Then

$$
\sum_{i, j=1}^n \xi_i^* K((x_i, s_i), (x_j, s_j)) \xi_j
$$

$$
= \sum_{k, l=1}^n \left( \sum_{i, j=1}^n a_i^k K_1(x_i, x_j)a_j^l \right) \otimes \left( \sum_{i, j=1}^n b_i^k K_2(s_i, s_j)b_j^l \right).
$$

Set

$$
\alpha_{kl} := \sum_{i, j=1}^n (a_i^k)^* K_1(x_i, x_j)a_j^l \in A, \quad \beta_{kl} := \sum_{i, j=1}^n (b_i^k)^* K_2(s_i, s_j)b_j^l \in B.
$$

Then the matrices $(\alpha_{kl})_{k, l=1}^n$ and $(\beta_{kl})_{k, l=1}^n$ are positive elements of $M_n(A)$ and of $M_n(B)$, respectively. Then, by Lemma 4.3 of [12],

$$
\sum_{i, j=1}^n \xi_i^* K((x_i, s_i), (x_j, s_j)) \xi_j = \sum_{k, l=1}^n \alpha_{kl} \otimes \beta_{kl} \geq 0. \quad (2.1)
$$

Since the set of all positive elements in a $C^*$-algebra is closed, we conclude that (2.1) is also valid for every choice of elements $\xi$ in $A \otimes_s B$. Thus $K$ is a positive definite kernel. Again, in virtue of Theorem 2.4, there exists a Hilbert $A \otimes_s B$-module $E$ of $A \otimes_s B$-valued functions on $X \times S$ such that $K$ is its reproducing kernel. Now this question raises: What relations are there between $E$, $E_1$, and $E_2$, where $E_1$ and $E_2$ are $RKHC^*Ms$ with kernels $K_1$ and $K_2$, respectively?

Recall that the exterior tensor product $E_1 \otimes E_2$ of Hilbert $C^*$-modules $E_1$ over $A$ and $E_2$ over $B$ is defined as the Hilbert $C^*$-module over $A \otimes_s B$ obtained by
completion of $\mathcal{E}_1 \otimes_{\mathfrak{alg}} \mathcal{E}_2$ with respect to the norm

$$
\|u\|^2 = \left\| \sum_{i,j=1}^{n} \langle f_i, f_j \rangle \otimes \langle g_i, g_j \rangle \right\|_2,
$$

where $u = \sum_{i=1}^{n} f_i \otimes g_i \in \mathcal{E}_1 \otimes_{\mathfrak{alg}} \mathcal{E}_2$; see [12].

Set $k^1_i(x) := K_1(x, y), k^2_s(s) := K_2(s, t)$, and $k_{(y,t)}(x, s) := K((x, s), (y, t))$. Clearly, $k_{(y,t)}(x, s) = k^1_y(x) \otimes k^2_t(s)$. By the assumption, the $\mathfrak{A}$-linear spans of $\{k^1_x : x \in X\}$, $\{k^2_s : s \in S\}$, and $\{k_{(x,s)} : (x, s) \in X \times S\}$ are dense in $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}$, respectively.

We claim that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is unitarily equivalent to $\mathcal{E}$. Let $f_i = \sum_{x \in X} k^1_x a^i_x$ and $g_i = \sum_{s \in S} k^2_s b^i_s$, where $a^i_x \in \mathfrak{A}$, $b^i_s \in \mathfrak{B}$, and both sums have a finite number of nonzero summands. Set

$$
\Phi \left( \sum_{i=1}^{n} \sum_{x \in X,s \in S} k^1_x a^i_x \otimes k^2_s b^i_s \right) := \sum_{x \in X, s \in S} k_{(x,s)} \sum_{i=1}^{n} a^i_x \otimes b^i_s.
$$

(2.2)

For $u \in \mathcal{E}_1 \otimes_{\mathfrak{alg}} \mathcal{E}_2$, define $\hat{u} \in \mathbb{F}(X \times S, \mathfrak{A} \otimes_{\ast} \mathfrak{B})$ by

$$
\hat{u}(x, s) = \langle k^1_x \otimes k^2_s, u \rangle, \quad (x, s) \in X \times S.
$$

Let $u = \sum_{i=1}^{n} f_i \otimes g_i$, where $f_i = \sum_{x \in X} k^1_x a^i_x$, $g_i = \sum_{s \in S} k^2_s b^i_s$, where $a^i_x \in \mathfrak{A}$, $b^i_s \in \mathfrak{B}$, and both sums have a finite number of nonzero summands. Then

$$
\hat{u}(y, t) = \sum_{i=1}^{n} \sum_{x \in X,s \in S} k^1_x(y) a^i_x \otimes k^2_s(t) b^i_s
$$

$$
= \sum_{x \in X,s \in S} k_{(y,t)}(y, t) \sum_{i=1}^{n} a^i_x \otimes b^i_s = \Phi(u)(y, t),
$$

which shows that the map $\Phi$, defined in (2.2), is well-defined and that $\hat{u} \in \mathcal{E}$.

It is clear that $\Phi$ is an isometry between dense subspaces of $\mathcal{E}_1 \otimes \mathcal{E}_2$ and of $\mathcal{E}$, hence it extends to a surjective isometry $\Phi : \mathcal{E}_1 \otimes \mathcal{E}_2 \to \mathcal{E}$.

**Definition 2.9.** Suppose that $K_1 : X \times X \to \mathfrak{A}$ and $K_2 : S \times S \to \mathfrak{B}$ are kernels. We call the map $K : (X \times S) \times (X \times S) \to \mathfrak{A} \otimes_{\ast} \mathfrak{B}$ defined by

$$
K((x, s), (y, t)) = K_1(x, s) \otimes K_2(y, t), \quad (x, s), (y, t) \in X \times S
$$

the tensor product of the kernels $K_1$ and $K_2$ and denote it by $K_1 \otimes K_2$.

We summarize the above results in the following theorem.

**Theorem 2.10.** Let $K_1$ and $K_2$ be positive definite kernels and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be their associated Hilbert $C^\ast$-modules. Then $K_1 \otimes K_2$ is a positive definite kernel, and its associated Hilbert $C^\ast$-module is unitarily equivalent to the exterior tensor product of $\mathcal{E}_1$ and $\mathcal{E}_2$.

3. Left multipliers of RKHC $\ast$-Ms

If $F_1$ and $F_2$ are submodules of $\mathbb{F}(S, \mathfrak{A})$, then a function $f \in \mathbb{F}(S, \mathfrak{A})$ for which $fF_1 \subseteq F_2$ is called a left multiplier of $F_1$ into $F_2$. Note that

$$
fF_1 = \{ fh : h \in F_1 \},
$$
where \( fh \) is the multiplication of \( f \) and \( h \). The set of all left multipliers of \( F_1 \) into \( F_2 \) is denoted by \( \mathcal{M}(F_1, F_2) \). Clearly, \( \mathcal{M}(F_1, F_2) \) is a linear space. Moreover, \( \mathcal{M}(F) \) stands for \( \mathcal{M}(F, F) \) being an algebra. For every \( f \in \mathcal{M}(F_1, F_2) \), there is a linear map \( M_f : F_1 \to F_2 \) that is defined by \( M_f(h) = fh \) for all \( h \in F_1 \).

The following lemma is a useful property of RKHC*Ms.

**Lemma 3.1.** Suppose that \( \mathcal{E} \) is an RKHC*M on a set \( S \) with the kernel \( K \). If a sequence \( (h_n) \) in \( \mathcal{E} \) converges to \( h \), then \( \lim_n h_n(s) = h(s) \) for each \( s \in S \).

**Proof.** It is easily concluded from

\[
\|h_n(s) - h(s)\| = \|\langle k_s, h_n \rangle - \langle k_s, h \rangle\| \leq \|k_s\| \|h_n - h\|.
\]

\( \square \)

Let \( \mathcal{E} \) be an RKHC*M on \( S \) and let \( g : S \to \mathfrak{A} \) be a function. Put

\[
\mathcal{E}_g = \{ gh : h \in \mathcal{E} \}.
\]

In the next theorem, we endow the right \( \mathfrak{A} \)-submodule \( \mathcal{E}_g \) of \( \mathbb{F}(S, \mathfrak{A}) \) with an RKHC*M structure.

**Theorem 3.2.** Suppose that \( \mathcal{E} \) is an RKHC*M on a set \( S \) with the kernel \( K \) and that \( g : S \to \mathfrak{A} \) is an arbitrary function. Then the following assertions hold:

(i) \( \mathcal{E}_0 := \{ h \in \mathcal{E} : gh = 0 \} \) is closed.

(ii) If \( \mathcal{E}_0 \) is orthogonally complemented, then \( \mathcal{E}_g \) is an RKHC*M with kernel \( K'(s, t) = g(s)K(s, t)g(t)^* \).

**Proof.** (i) It follows from Lemma 3.1 that \( \mathcal{E}_0 \) is closed.

(ii) It follows from the assumption that \( \mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_0^\perp \). Therefore

\[
\mathcal{E}_g = \{ gh + gh^\# : \tilde{h} \in \mathcal{E}_0, h^\# \in \mathcal{E}_0^\perp \} = \{ gh : h \in \mathcal{E}_0^\perp \}.
\]

We define an \( \mathfrak{A} \)-valued inner product on \( \mathcal{E}_g \) by

\[
\langle gh_1, gh_2 \rangle = \langle h_1, h_2 \rangle
\]

for all \( h_1, h_2 \in \mathcal{E}_0^\perp \). This is well-defined, since if \( gh = gh' \) for \( h, h' \in \mathcal{E}_0^\perp \), then \( h - h' \in \mathcal{E}_0 \cap \mathcal{E}_0^\perp = \{0\} \). From the inner product on \( \mathcal{E}_g \), it is clear that \( \varphi_g : \mathcal{E}_0^\perp \to \mathcal{E}_g \) by \( \varphi_g(h) = gh \) is a surjective linear isometry. Hence \( \mathcal{E}_g \) is a Hilbert \( C^* \)-module isomorphic with \( \mathcal{E}_0^\perp \). Thus the reproducing kernel structure of \( \mathcal{E}_0^\perp \) constructed in Lemma 2.5 can be transferred onto \( \mathcal{E}_g \). More precisely, for each \( h \in \mathcal{E}_0^\perp \), we have

\[
(gh)(s) = g(s)h(s) = g(s)\langle k_s, h \rangle = g(s)\langle k^\#_s, h \rangle = g(s)\langle gk^\#_s, gh \rangle = g(s)\langle gk_s, gh \rangle = \langle gk_s g(s)^*, gh \rangle
\]
for some \( k_s = \bar{k}_s + k_s^\# \in \mathcal{E}_0 \oplus \mathcal{E}_Q^+ \). Hence the evaluation map \( \delta_s \) can be represented by \( \langle gk_s, g(s)^*, \cdot \rangle \) with \( k_s = gk_s g(s)^* \in \mathcal{E}_g \). In addition, the corresponding reproducing kernel is
\[
K'(s,t) = \langle k'_s, k'_t \rangle = \langle gk_s g(s)^*, gk_t g(t)^* \rangle = g(s) \langle gk_s, g_k^0(t)^* \rangle + g(s) \langle k_s^\#, k_t^\# g(t)^* \rangle + 0
\]
\[
= g(s) \langle k_s^\#, k_t^\# g(t)^* + g(s) \bar{k}_t(s) g(t)^* \rangle = g(s) \langle k_s^\#, k_t g(t)^* \rangle = g(s) K(s,t) g(t)^*
\]
for every \( s, t \in S \).

In the following theorem, \( k_s^1 \) and \( k_s^2 \) are the reproducing kernels at the point \( s \in S \) for RKHC*Ms \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), respectively.

**Proposition 3.3.** Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be RKHC*Ms on a nonempty set \( S \). If \( f \in \mathcal{M}(\mathcal{E}_1, \mathcal{E}_2) \), then \( M_f \in L(\mathcal{E}_1, \mathcal{E}_2) \) and \( M_f^* (k^2_s) = k^1_s f(s)^* \) for all \( s \in S \).

**Proof.** For every \( h \in \mathcal{E}_1, s_1, \ldots, s_n \in S \), and \( a_1, \ldots, a_n \in \mathfrak{A} \), we have
\[
\left\langle \sum_{i=1}^{n} k_{s_i}^2 a_i, M_f(h) \right\rangle = \sum_{i=1}^{n} a_i^* \langle k_{s_i}^2, f h \rangle = \sum_{i=1}^{n} a_i^* f(s_i) h(s_i)
\]
\[
= \sum_{i=1}^{n} a_i^* f(s_i) \langle k_{s_i}^1, h \rangle = \left\langle \sum_{i=1}^{n} k_{s_i}^1 f(s_i)^* a_i, h \right\rangle.
\]
Hence \( M_f^* (\sum_{i=1}^{n} k_{s_i}^2 a_i) = \sum_{i=1}^{n} k_{s_i}^1 f(s_i)^* a_i \). In particular, \( M_f^* (k^2_s) = k^1_s f(s)^* \) for all \( s \in S \).

Thus, if \( f \in \mathcal{M}(\mathcal{E}) \) and \( \mathfrak{A} \) is a unital C*-algebra, then
\[
f(s) = \langle k_s, f k_s \rangle K(s,s)^{-1} = \langle k_s, M_f(k_s) \rangle K(s,s)^{-1},
\]
for every point \( s \in S \) for which \( K(s,s) \in \text{Inv}(\mathfrak{A}) \). Thus, we can present the following definition in the same manner as in the classical case [16] and transfer some known facts in the theory of RKHSs to context of RKHC*M.

**Definition 3.4.** Let \( \mathcal{E} \) be an RKHC*M on \( S \) over a unital C*-algebra \( \mathfrak{A} \). Let \( K \) be its associated kernel and let \( T \in L(\mathcal{E}) \) be arbitrary. Then the function
\[
B_T : \{ s \in S : K(s,s) \text{ is invertible} \} \rightarrow \mathfrak{A}
\]
defined by \( B_T(s) = \langle k_s, T(k_s) \rangle K(s,s)^{-1} \) is called the Berezin transform of \( T \) associated by \( \mathfrak{A} \).

**Theorem 3.5.** Let \( \mathcal{E} \) be an RKHC*M on \( S \) with the reproducing kernel \( K \) over a unital C*-algebra \( \mathfrak{A} \). Let
\[
L = \{ M_f : f \in \mathcal{M}(\mathcal{E}) \text{ and } f(s) = 0 \text{ whenever } K(s,s) \text{ is not invertible} \}.
\]
Then \( L \) is a unital subalgebra of \( L(\mathcal{E}) \).

Furthermore, if \( \{ M_{f_\alpha} \}_{\alpha \in I} \) is a net in \( L \) such that \( \langle M_{f_\alpha} h_1, h_2 \rangle \rightarrow \langle Th_1, h_2 \rangle \) \( (h_1, h_2 \in \mathcal{E}) \) for some \( T \in L(\mathcal{E}) \), then \( T = M_f \) for some \( f \in \mathcal{F}(S, \mathfrak{A}) \).
Proof. Since
\[ \lambda M_f + M_g = M_{\alpha f + g}, \quad M_f \circ M_g = M_{fg} \quad (f, g \in \mathcal{M}(\mathcal{E}), \lambda \in \mathbb{C}), \]
\(L\) is an algebra. Moreover, \(M_1\) is the unit of \(L\), where \(1 \in \mathcal{M}(\mathcal{E})\) is the constant function onto the unit of \(\mathfrak{A}\).

Next, we show that \(T = M_f\) for some \(f \in \mathcal{M}(\mathcal{E})\). We have
\[ \lim_{a} f_{a}(s) = \lim_{a} \langle k_{s}, M_{f_a}(k_{s}) \rangle K(s, s)^{-1} = \langle k_{s}, T(k_{s}) \rangle K(s, s)^{-1} = B_T(s), \]
for every \(s \in S\) for which \(K(s, s)\) is invertible. Set \(f(s) := B_T(s)\) whenever \(K(s, s)\) is invertible and \(f(s) := 0\) otherwise. To complete the proof, we shall show that \(T = M_f\). We have
\[
\left\langle \sum_{i=1}^{n} k_{s_i}a_i, Th \right\rangle = \lim_{a} \left\langle \sum_{i=1}^{n} k_{s_i}a_i, M_{f_a}h \right\rangle = \lim_{a} \left\langle \sum_{i=1}^{n} k_{s_i}a_i, f_{a}h \right\rangle \\
= \lim_{a} \sum_{i=1}^{n} a_{i}^{*}f_{a}(s_{i})h(s_{i}) = \sum_{i=1}^{n} a_{i}^{*}f(s_{i})h(s_{i}) \\
= \sum_{i=1}^{n} a_{i}^{*}\langle k_{s_i}, fh \rangle = \left\langle \sum_{i=1}^{n} k_{s_i}a_i, M_fh \right\rangle ,
\]
for every \(h \in \mathcal{E}, s_1, \ldots, s_n \in S\), and \(a_1, \ldots, a_n \in \mathfrak{A}\). Thus \(T = M_f\). \(\square\)

4. Papadakis theorem for RKHC*-Ms

We recall the following definitions from [8].

**Definition 4.1.** Let \(J\) be an arbitrary subset of \(\mathbb{N}\) and let \(\mathcal{E}\) be a Hilbert \(C^*\)-module over a unital \(C^*\)-algebra \(\mathfrak{A}\). A sequence \((x_j)_{j \in J}\) in \(\mathcal{E}\) is said to be a (standard) frame if there are real numbers \(C, D > 0\) such that
\[ C\langle x, x \rangle \leq \sum_{j \in J} \langle x, x_j \rangle \langle x_j, x \rangle \leq D\langle x, x \rangle \quad (4.1) \]
for every \(x \in \mathcal{E}\) in which the sum in the middle of inequality (4.1) converges in norm.

The sharp numbers (i.e., maximal for \(C\) and minimal for \(D\)) are called frame bounds. A frame \(\{x_j : j \in J\}\) is said to be a tight frame if \(C = D\), and normalized if \(C = D = 1\). Therefore, a set \(\{x_j : j \in J\}\) is a normalized tight frame whenever the equality
\[ \langle x, x \rangle = \sum_{j \in J} \langle x, x_j \rangle \langle x_j, x \rangle \quad (4.2) \]
is valid for every \(x \in \mathcal{E}\).

Now, we extend the Papadakis theorem to RKHC*-Ms.

**Theorem 4.2.** Let \(\mathcal{E}\) be an RKHC*-\(M\) on a set \(S\) over a unital \(C^*\)-algebra \(\mathfrak{A}\) and let \(K\) be its corresponding reproducing kernel. Then \(\{f_j : j \in J\} \subseteq \mathcal{E}\) is a normalized tight frame for \(\mathcal{E}\) if and only if
\[ K(s, t) = \sum_{j \in J} f_j(s)^{*}f_j(t) \quad (s, t \in S), \quad (4.3) \]
where the sum is convergent in norm.
Proof. Suppose that \( \{f_j : j \in J\} \) is a normalized tight frame for \( \mathcal{E} \). It follows from (4.2) that
\[
\langle f, f \rangle = \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle,
\]
for every \( f \in \mathcal{E} \). Therefore, by the polarization identity, we can write
\[
K(s, t) = \langle k_s, k_t \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle k_t + i^k k_s, k_t + i^k k_s \rangle
\]
\[
= \frac{1}{4} \sum_{k=0}^{3} i^k \sum_{j \in J} (f_j(t) + i^k f_j(s))^* (f_j(t) + i^k f_j(s))
\]
\[
= \sum_{j \in J} \frac{1}{4} \sum_{k=0}^{3} i^k (f_j(t) + i^k f_j(s), f_j(t) + i^k f_j(s))
\]
\[
= \sum_{j \in J} \langle f_j(s), f_j(t) \rangle = \sum_{j \in J} f_j(s)^* f_j(t)
\]
for all \( s, t \in S \).

Conversely, let (4.3) hold for some family \( \{f_j : j \in J\} \) and let the sum in (4.3) converge in the norm topology. Then
\[
\langle k_s, k_s \rangle = K(s, s) = \sum_{j \in J} f_j(s)^* f_j(s) = \sum_{j \in J} \langle k_s, f_j \rangle \langle f_j, k_s \rangle
\]
for every \( s \in S \). Hence, by the density of \( \mathcal{A} \)-linear span of \( \{k_s : s \in S\} \) in \( \mathcal{E} \) and the joint continuity of inner product, we derive
\[
\langle f, f \rangle = \sum_{j \in J} \langle f, f_j \rangle \langle f_j, f \rangle
\]
for all \( f \in \mathcal{E} \). It follows from (4.2) that \( \{f_j : j \in J\} \) is a normalized tight frame. \( \square \)

As we already mentioned in the introduction, \( RKHC^*M \)s are rarely self-dual. Recall that \( \mathcal{E}' \) denotes the dual module of \( \mathcal{E} \).

**Lemma 4.3.** Elements of \( \mathcal{E}' \) can be thought of as functions on \( S \), that is, there is an inclusion \( \mathcal{E}' \subset C(S, \mathcal{A}) \) that extends the inclusion \( \mathcal{E} \subset C(S, \mathcal{A}) \).

**Proof.** Suppose that \( F \in \mathcal{E}' \). Set \( F(s) := F(k_s) \). This gives us a map \( \mathcal{E}' \to C(S, \mathcal{A}) \). To show that this map is faithful, suppose that \( F(s) = 0 \) for any \( s \in S \). Then \( F \) vanishes on a dense subset of \( \mathcal{E} \), and hence is zero. \( \square \)

It is clear that if \( K \) is a kernel of the form (4.3), then it is a positive definite kernel; see [9] for the Kolmogorov decomposition at the setting of Hilbert \( C^* \)-modules. Hence, employing Theorem 2.4, there exists a Hilbert \( \mathcal{A} \)-module consisting of \( \mathcal{A} \)-valued functions on \( S \) such that \( K \) is its reproducing kernel. This is a motivation for the following definition.
**Definition 4.4.** Let \( K : S \times S \to \mathcal{A} \) be the positive definite kernel defined by
\[
K(s, t) = \sum_{\alpha \in I} e_{\alpha}(s)^* e_{\alpha}(t) \quad (s, t \in S),
\]
where \( \{e_{\alpha}\}_{\alpha \in I} \) is a family in \( \mathbb{F}(S, \mathcal{A}) \) with the property that \( \sum_{\alpha \in I} e_{\alpha}(s)^* e_{\alpha}(s) \) converges in \( \mathcal{A} \). Then \( K \) is called the Papadakis kernel, and the Hilbert \( \mathcal{A} \)-module consisting of \( \mathcal{A} \)-valued functions on \( S \), given by Theorem 2.4, is called the Papadakis Hilbert \( \mathcal{A} \)-module.

Let \( K \) be a Papadakis kernel on \( S \) for some family \( \{e_{\alpha}\}_{\alpha \in I} \subseteq \mathbb{F}(S, \mathcal{A}) \) and let \( \mathcal{E} \) be the associated Papadakis Hilbert \( C^* \)-module. The following theorem shows that the multiplication of an element of \( \mathbb{F}(S, \mathcal{A}) \) satisfying suitable conditions and that \( e_{\alpha} (\alpha \in I) \) is an element of \( \mathcal{E} \). Note that \( e_{\alpha}^* : S \to \mathcal{A} \) is defined by \( e_{\alpha}^*(s) = e_{\alpha}(s)^* \) for all \( \alpha \in I \) and \( s \in S \). To achieve our next result, we mimic some ideas of [18].

**Definition 4.5.** A subset \( P \) of \( S \) is said to be a set of uniqueness of \( \mathcal{E} \subseteq \mathbb{F}(S, \mathcal{A}) \) if the \( \mathcal{A} \)-linear span of \( k_p, p \in P \), is dense in \( \mathcal{E} \). In this case, we write \( P \in \mathbb{U}(\mathcal{E}) \).

Note that if \( f, g \in \mathcal{E} \) with \( f(p) = g(p) \) for any \( p \in P \), then \( f = \sum_{p \in P} k_p a_p \) and \( g = \sum_{p \in P} k_p b_p \), where \( a_p, b_p \in \mathcal{A} \). Then \( f = g \).

**Theorem 4.6.** Let \( K \) be the Papadakis kernel for some family \( \{e_{\alpha}\}_{\alpha \in I} \subseteq \mathbb{F}(S, \mathcal{A}) \) and let \( \mathcal{E} \) be the associated Papadakis Hilbert \( C^* \)-module. Let \( e_{\alpha} \in \mathcal{M}(\mathcal{E}', \mathcal{E}) \) and let \( X \in \mathbb{U}(\mathcal{E}) \). Assume that \( \psi : X \to \mathcal{Z}(\mathcal{A}) \) is a function and that \( c > 0 \) is such that
\[
\sum_{i,j=1}^{n} a_i^* K(x_i, x_j) (c^2 - \psi(x_i)^* \psi(x_j)) a_j \geq 0, \tag{4.4}
\]
for all \( n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X \), and \( a_1, a_2, \ldots, a_n \in \mathcal{A} \). Then for every \( \alpha \in I \), there is a unique function \( \varphi_{\alpha} \in \mathcal{E} \) such that
\[
\varphi_{\alpha}(x) = e_{\alpha}(x) \psi(x), \quad x \in X
\]
or, equivalently,
\[
\varphi_{\alpha} = e_{\alpha} \psi
\]
and
\[
e_{\alpha} \varphi_{\beta} = e_{\beta} \varphi_{\alpha},
\]
for all \( \alpha, \beta \in I \). Furthermore, if \( \text{ran}(e_{\alpha}) \subseteq \mathcal{Z}(\mathcal{A}) \) and \( K(s, s) \) is invertible for every \( s \in S \), then
\[
|\varphi_{\alpha}(s)| \leq c |e_{\alpha}(s)|, (s \in S).
\]

**Proof.** Inequality (4.4) can be restated as follows:
\[
c^2 \left( \sum_{i=1}^{n} k_i a_i, \sum_{j=1}^{n} k_j a_j \right) \geq \sum_{\alpha \in I} \left( \sum_{i=1}^{n} e_{\alpha}(x_i) a_i \psi(x_i) \right)^* \left( \sum_{j=1}^{n} e_{\alpha}(x_j) a_j \psi(x_j) \right)
\]
\[
= \sum_{\alpha \in I} \left| \sum_{i=1}^{n} e_{\alpha}(x_i) a_i \psi(x_i) \right|^2 \tag{4.5}
\]
for every \( n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X \) and \( a_1, a_2, \ldots, a_n \in \mathfrak{A} \). Put
\[
D = \left\{ \sum_{i=1}^{n} k_{x_i} a_i : n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X, a_1, a_2, \ldots, a_n \in \mathfrak{A} \right\}.
\]

For every \( \alpha \in I \), we define \( \varphi_{\alpha} : D \to \mathfrak{A} \) by
\[
\varphi_{\alpha} \left( \sum_{i=1}^{n} k_{x_i} a_i \right) = \sum_{i=1}^{n} e_{\alpha}(x_i) a_i \psi(x_i),
\] (4.6)
where \( n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X, \) and \( a_1, a_2, \ldots, a_n \in \mathfrak{A} \). Set \( b := \sum_{i=1}^{n} k_{x_i} a_i \). From (4.5), we conclude that
\[
c^2 \langle b, b \rangle \geq \varphi_{\alpha}^*(b^*) \varphi_{\alpha}(b)
\]
for every \( \alpha \in I \). Hence \( \varphi_{\alpha} \) is a well-defined bounded \( \mathfrak{A} \)-linear map. Since \( X \in \cup(\mathcal{E}) \), the set \( D \) is dense in \( \mathcal{E} \). Hence, we can extend \( \varphi_{\alpha} \) to \( \mathcal{E} \). For simplicity, we denote it by the same \( \varphi_{\alpha} \), so that \( \varphi_{\alpha} \in \mathcal{E}' \). From (4.5), we reach
\[
c^2|g|^2 \geq \sum_{\alpha \in I} |\varphi_{\alpha}(g)|^2 \quad (g \in \mathcal{E}).
\] (4.7)
Utilizing Lemma 4.3 and (4.6), we arrive at
\[
\varphi_{\alpha}(x) = \varphi_{\alpha}(k_x) = e_{\alpha}(x) \psi(x) \quad (x \in X, \alpha \in I).
\]
Then
\[
e_{\alpha}(x) \varphi_{\beta}(x) = \varphi_{\beta}(x) e_{\alpha}(x) \quad (x \in X, \alpha, \beta \in I).
\] (4.8)
Since \( e_{\alpha} \in \mathcal{M}(\mathcal{E}) \) and \( X \in \cup(\mathcal{E}) \), from (4.8), we infer that
\[
e_{\alpha}(s) \varphi_{\beta}(s) = \varphi_{\beta}(s) e_{\alpha}(s), \quad (s \in S).
\]
Now, fix \( \alpha \in I \) and \( s \in S \). Putting \( g = k_s \) in (4.7), we arrive at
\[
c^2 K(s,s) \geq \sum_{\beta \in I} \varphi_{\beta}(s)^* \varphi_{\beta}(s).
\]
Since \( \text{ran}(e_{\alpha}) \subseteq \mathcal{Z}(\mathfrak{A}) \), we have
\[
c^2 K(s,s) e_{\alpha}(s)^* e_{\alpha}(s) \geq \sum_{\beta \in I} (e_{\alpha}(s) \varphi_{\beta}(s))^* e_{\alpha}(s) \varphi_{\beta}(s)
= \sum_{\beta \in I} e_{\beta}(s)^* \varphi_{\alpha}(s)^* e_{\beta}(s) \varphi_{\alpha}(s)
= |\varphi_{\alpha}(s)|^2 K(s,s).
\]
Now, the invertibility of \( K(s,s) \) entails that \( |\varphi_{\alpha}(s)| \leq c |e_{\alpha}(s)| \). \( \square \)

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