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High order semi-implicit weighted compact nonlinear scheme for the all-Mach isentropic Euler system

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Abstract
The computation of compressible flows at all Mach numbers is a very challenging problem. An efficient numerical method for solving this problem needs to have shock-capturing capability in the high Mach number regime, while it can deal with stiffness and accuracy in the low Mach number regime. This paper designs a high order semi-implicit weighted compact nonlinear scheme (WCNS) for the all-Mach isentropic Euler system of compressible gas dynamics. To avoid severe Courant-Friedrichs-Levy (CFL) restrictions for low Mach flows, the nonlinear fluxes in the Euler equations are split into stiff and non-stiff components. A third-order implicit-explicit (IMEX) method is used for the time discretization of the split components and a fifth-order WCNS is used for the spatial discretization of flux derivatives. The high order IMEX method is asymptotic preserving and asymptotically accurate in the zero Mach number limit. One- and two-dimensional numerical examples in both compressible and incompressible regimes are given to demonstrate the advantages of the designed IMEX WCNS.

Keywords: High order scheme; IMEX time discretization; WCNS; Asymptotic-preserving property; Low Mach number; Isentropic Euler equations

Introduction
Low-Mach number flows in fluid dynamics that are slow compared with the speed of sound can be described by the following scaled isentropic Euler equations.

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla p &= 0,
\end{align*}
\]

where \( \rho(\mathbf{x}, t) > 0, \mathbf{u}(\mathbf{x}, t) = (u(\mathbf{x}, t), v(\mathbf{x}, t)) \) and \( \mathbf{q} = \rho \mathbf{u}(\mathbf{x}, t) \) are the density, velocity and momentum of the fluid, respectively, and \( \varepsilon > 0 \) is the scaled Mach number that is a measure of compressibility of the fluid. For example, \( \varepsilon = 1 \) corresponds to the fully compressible regime. The operators \( \nabla, \nabla \cdot \) and \( \otimes \) are the gradient, divergence and tensor product operators, respectively. The pressure \( p \) in Eqs. (1) is given by the equation of state

\[
p(\rho) = \Lambda \rho^\gamma,
\]
where $\Lambda > 0$ and $\gamma \geq 1$ are constants. Eqs. (1) are hyperbolic with wave speeds in direction $n$

$$\lambda_{1,3} = u \cdot n \pm c, \lambda_2 = u \cdot n,$$

(3)

and the sound speed $c = \sqrt{p'(\rho)/\varepsilon}$.

Klainerman and Majda [1, 2] proved that the solution of Eqs. (1) converges to the incompressible counterpart when $\varepsilon \sim 0$. The asymptotic nature of the solution to Eqs. (1) can be studied by plugging-in the ansatz

$$f(x, t) = f_0(x, t) + \varepsilon^2 f_2(x, t) + O(\varepsilon^3),$$

(4)

for variables $\rho, p$ and $u$. After performing a scale analysis, we obtain an incompressible Euler system for the velocity

$$\begin{cases}
(u_0)_{t} + \nabla \cdot (u_0 \otimes u_0) + \sum_{i} \frac{p_{2i}}{\rho_0} &= 0, \\
\nabla \cdot u_0 &= 0,
\end{cases}$$

(5)

based on well-prepared initial data, i.e.,

$$\rho(x, 0) = \text{const} + O(\varepsilon^2), \nabla \cdot u(x, 0) = O(\varepsilon),$$

(6)

and appropriate boundary conditions [3], e.g., periodic, wall or homogeneous Neumann boundary conditions. Taking the divergence of the first equation in (5) and using the incompressibility, we obtain

$$\Delta p_2 = -\nabla \cdot (\nabla \cdot (\rho_0 u_0 \otimes u_0)).$$

(7)

In low Mach number flows, the challenge faced by standard hyperbolic numerical methods for Eqs. (1) is that the system becomes stiff. The stiffness is caused by the stringent Courant-Friedrichs-Levy (CFL) restriction [3, 4, 5, 6], i.e., $\Delta t \leq \Delta x / \max |\lambda_i| = O(\varepsilon ^{-1} \Delta x)$ with the spatial and temporal grid sizes $\Delta x$ and $\Delta t$. The restriction results in an increasingly large computational time for $\varepsilon \to 0$. Moreover, the strong stability requirement for $\varepsilon \to 0$ results in the excessive numerical viscosity of the standard hyperbolic solvers that is proportional to $\max |\lambda_i| = O(\varepsilon ^{-1})$. This makes the standard hyperbolic solvers suffer from severe loss of accuracy due to the creation of spurious waves. Therefore, the stability and accuracy of the standard hyperbolic numerical methods highly depend on $\varepsilon$ [4].

In the literature, some all-speed schemes [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] have been derived for the compressible-incompressible limit problems based on the idea of the asymptotic-preserving (AP) methodology introduced by Jin [17, 18] for relaxation systems. The AP all-speed schemes are uniformly consistent with the low-Mach number limit and are uniformly stable. Although a robust AP scheme should allow both explicit and implicit time-marching techniques, the explicit strategies suffer from the CFL stability conditions and thus affect the efficiency of the method for small Mach numbers. The implicit strategies avoid the CFL restriction, but they
need to solve highly nonlinear equations and add an excessive numerical dissipation
to the slow wave, degrading the accuracy of the method [3]. To overcome the draw-
backs, the semi-implicit or implicit-explicit (IMEX) time discretization strategies
are widely applied based on the acoustic/advection splitting of the Euler equations
into the stiff/non-stiff parts (or fast and slow fluxes) [3, 4, 5, 10]. These strategies treat
with the stiff part implicitly and the non-stiff part explicitly to solve the
isentropic Euler equations effectively and efficiently. The IMEX time discretization
combined with the shock-capturing schemes, e.g., the first-order local Lax-Friedrichs
(or Rusanov) scheme [4], the second-order finite volume method [3, 19], the arbi-
trary high order accurate discontinuous Galerkin method [20] and the fifth-order
finite difference weighted essentially non-oscillatory (WENO) scheme [10], has been
shown its good performances in compressible and incompressible regimes.

Recently a class of weighted compact nonlinear schemes (WCNS’s) as high order
accurate shock-capturing schemes have been widely adopted to investigate com-
pressible flows with shock waves and discontinuities [21, 22, 23, 24, 25, 26, 27].
The WCNS’s are first proposed by Deng and Zhang [21] based on the cell-centered
compact scheme of Lele [28] and are developed in [22, 23, 24, 29]. The expression
of the classical cell-centered compact scheme is that a linear combination of the
unknown approximations of the first derivative of a function at the grid points in
a stencil is equal to another linear combination of the function itself at the grid
points in the same stencil. Compared to noncompact schemes, e.g., the classical
WENO scheme [30], the stencil can be more compact in the compact scheme for
the same order of accuracy [22]. WCNS’s that integrate WENO interpolation into
the cell-centered compact scheme for the first derivative of a function are variants
of the WENO schemes and have the good performance in terms of accuracy, ro-
bustness and efficiency for compressible flows. Compared to the classical WENO
schemes, the WCNS’s feature a better spectral resolution at the same order of ac-
curacy [21, 22]. They are more flexible in the choice of flux splitting methods, e.g.,
Roe’s flux difference splitting method, van Leer’s flux vector splitting and Liou’s
advection upstream splitting method, and flux derivatives can be computed by dif-
f erent explicit or implicit compact finite difference methods [23, 29]. In this work, in
view of the good performance of the IMEX time discretization method and the WC-
NS’s, we design a high order IMEX WCNS for the all-Mach isentropic Euler system
based on the acoustic/advection splitting strategy [10]. The stiff part governing fast
acoustic and the non-stiff part modeling slow nonlinear advection effects are solved
by the high order implicit and explicit time discretization methods, respectively.

In the zero Mach number limit, the high order IMEX scheme has the AP pro-
 perty and is asymptotically stable under a convective CFL condition independent of
the Mach number $\varepsilon$. One- and two-dimensional numerical examples are given to
demonstrate good performance of the designed IMEX WCNS in both compressible
and incompressible regimes.

**WCNS method**

In this section, a fifth-order WCNS is considered for the spatial discretization of flux
derivatives. The WCNS procedure consists of three components, i.e., midpoint-to-
ode differencing, flux evaluation at the midpoint and node-to-midpoint weighted
averaging interpolation of fluxes. Here, we briefly describe the framework of WCNS
discretization for the first derivative, \( f_x \), in the one-dimensional scalar case \( u_t + \\
\frac{\partial f}{\partial x} = 0 \). For simplicity, we take a uniform grid defined by \( x_i = i \Delta x = ih, i = \\
0, \ldots, N \), where \( h \) is the uniform grid spacing and \( N \) is the number of cells.

The classical cell-centered compact scheme for the first derivatives [28] has no
dissipation, but the dispersive errors dominate the scheme, which may destroy the
numerical solutions. A class of hybrid cell-node and cell-edge WCNS’s proposed in
[29] overcome the disadvantages and are written as

\[
\frac{f'_i + \sum_{k=-1}^{0} q_k (f'_{i+k} + f'_{i-k})}{h} = \frac{1}{\eta} \sum_{m=1}^{\rho} \zeta_m (f_{i+(m-1)/2} + f_{i-(m-1)/2}) + \frac{1}{\eta} \sum_{l=1}^{r} \eta_l (f_{i+l+1} + f_{i-l})
\]

(8)

which introduce dissipation by approximating the cell-edge fluxes \( f_{i \pm (m-1)/2} \) based
on nonlinear weighted averaging interpolation of fluxes. The coefficients \( q_k, \eta \) and \( \zeta_m \)
in Eq. (8) can be acquired by matching the Taylor series coefficients of various
orders. If the coefficients \( q_k = 0 \), a class of explicit WCNS’s can be derived. The
explicit WCNS’s are preferred because of its low cost and simplicity for paralleliza-
tion and vectorization, compared to the implicit WCNS’s with a certain coefficient
\( q_k \neq 0 \). Here, we consider one of explicit WCNS’s and use a sixth-order explicit
central-type scheme obtained from Eq. (8) to approximate \( f_x \), i.e.,

\[
f'_i = \frac{64}{45h} (\hat{f}_{i+1/2} - \hat{f}_{i-1/2}) - \frac{2}{9h} (f_{i+1} - f_{i-1}) + \frac{1}{180h} (\hat{f}_{i+2} - \hat{f}_{i-2})
\]

(9)

By the Taylor series expansion of (9), we obtain

\[
f'_i = (f_x)_i - \frac{1}{5040} h^6 \left( \frac{\partial f}{\partial x^2} \right)_i + O(h^8) + O(h^r),
\]

(10)

where the truncation term \( O(h^r) \) comes from the numerical approximations of
the cell-edge fluxes \( f_{i \pm 1/2} \), denoted by \( \hat{f}_{i \pm 1/2} \). The accuracy of the scheme (9) is
\( \min(6, r) \). Here, the numerical fluxes \( \hat{f}_{i \pm 1/2} \) approximate \( f_{i \pm 1/2} \) to the fifth order
and thus a fifth-order WCNS is designed for the spatial discretization.

Various flux splitting methods [23] can be used in WCNS to introduce correct
upwinding. For simplicity, we use the Lax-Friedrichs flux splitting, i.e., the flux is
split into positive and negative parts:

\[
f = f^+ + f^-, \quad f^\pm = \frac{1}{2} (f \pm \alpha u),
\]

(11)

where \( \alpha = \max_u |f'_u| \) is the maximum characteristic wave speed. The numerical fluxes
\( \hat{f}_{i \pm 1/2} \) at the cell centers in (9) are calculated as

\[
\hat{f}_{i \pm 1/2} = \hat{f}^+_{i \pm 1/2} \pm \hat{f}^-_{i \pm 1/2}.
\]

(12)

A fifth-order WENO interpolation technique is considered to obtain the positive
numerical flux \( \hat{f}^+_{i+1/2} \). The interpolation process of the negative numerical flux \( \hat{f}^-_{i+1/2} \)
is symmetric with respect to $x_{i+1/2}$ and will not be shown here. For simplicity, the “+” sign in the superscript is removed.

The numerical flux $\tilde{f}_{i+1/2}$ on a cell center can be interpolated locally from the values on a five-point upwind-biased stencil $S = \{x_{i\pm 2}, x_{i\pm 1}, x_i\}$. The stencil $S$ can be divided into three sub-stencils $S_k = \{x_{i+k-2}, x_{i+k-1}, x_{i+k}\}$, $k = 0, 1, 2$. On each sub-stencil a third-order linear interpolation of $f$ at cell center $x_{i+1/2}$ can be derived, i.e.,

$$\begin{align*}
\tilde{f}_{i+1/2}^0 &= \frac{1}{8}(3f_i - 10f_{i-1} + 15f_{i+1}), \\
\tilde{f}_{i+1/2}^1 &= \frac{1}{8}(-f_{i-1} + 6f_i + 3f_{i+1}), \\
\tilde{f}_{i+1/2}^2 &= \frac{1}{8}(3f_i + 6f_{i+1} - f_{i+2}),
\end{align*}$$

(13)

(14)

(15)

where $\tilde{f}_{i+1/2}^k$, $k = 0, 1, 2$, are the approximated values at cell center $x_{i+1/2}$ on $S_k$, $k = 0, 1, 2$, respectively. On the stencil $S$ a fifth-order upwind-biased linear interpolation of $f$ at cell center $x_{i+1/2}$ can be derived, i.e.,

$$\begin{align*}
\tilde{f}_{i+1/2}^{\text{upwind}} &= \frac{1}{128}(3f_{i-2} - 20f_{i-1} + 90f_i + 60f_{i+1} - 5f_{i+2}),
\end{align*}$$

(16)

which can also be constructed linearly from the three third-order interpolations $\tilde{f}_{i+1/2}^k$:

$$\begin{align*}
\tilde{f}_{i+1/2}^{\text{upwind}} &= \sum_{k=0}^{2} d_k \tilde{f}_{i+1/2}^k,
\end{align*}$$

(17)

with the linear weights

$$\begin{align*}
d_0 &= \frac{1}{16}, & d_1 &= \frac{10}{16}, & d_2 &= \frac{5}{16}.
\end{align*}$$

(18)

The central-type scheme (9) with the upwind-biased linear interpolation (16) or (17) will generate oscillations around shock waves and discontinuities. To capture shock waves and discontinuities, the numerical flux $\tilde{f}_{i+1/2}$ should be computed by using the nonlinear combination of upwind-biased interpolations, i.e.,

$$\begin{align*}
\tilde{f}_{i+1/2} = \sum_{k=0}^{2} \omega_k \tilde{f}_{i+1/2}^k,
\end{align*}$$

(19)

where $\omega_k$, $k = 0, 1, 2$, are the upwind-biased nonlinear weights that are associated with the relative smoothness of $f$ on each sub-stencil. In smooth areas the nonlinear weights $\omega_k$ should be close to the linear weights $d_k$ to achieve the designed order and be close to zero near discontinuities to suppress spurious numerical oscillations near discontinuities.

There are several types of nonlinear weights $\omega_k$ in the literature, e.g., the WENO-JS [30], WENO-M [31] and WENO-Z [32, 33, 34] weights. The WENO-Z weights place a larger weight to the less smooth sub-stencil, and thus they are less dissipative
than the WENO-JS weights and more efficient than the WENO-M weights. Here, the WENO-Z weights $\omega_k$ proposed by Borges et al. [32] are chosen.

$$\omega_k = \frac{\alpha_k}{\sum_{k=0}^{2} \alpha_k}, \quad \alpha_k = d_k \left(1 + \frac{\tau}{\beta_k + \delta}\right),$$

(20)

where the second-order smoothness indicators $\beta_k, k = 0, 1, 2$, derived by Hu [35] are adopted to measure the smoothness of $f$ on the sub-stencils $S_k$.

$$\beta_k = \frac{1}{4}(a_k)^2 + \frac{13}{12}(b_k)^2 - |a_kb_k|,$$

(21)

with

$$a_0 = f_{i-2} - 4f_{i-1} + 3f_i, \quad a_1 = f_{i-1} - f_{i+1}, \quad a_2 = 3f_i - 4f_{i+1} + f_{i+2},$$

$$b_0 = f_{i-2} - 2f_{i-1} + f_i, \quad b_1 = f_{i-1} - 2f_i + f_{i+1}, \quad b_2 = f_i - 2f_{i+1} + f_{i+2}.$$  

In (20), the global higher order smoothness indicator $\tau = |\beta_0 - \beta_2|$ and the parameter $\delta$ is a small constant to avoid denominators becoming zero.

**Remark 1.** The value of $\delta$ in (20) will influence the convergence, accuracy and stability of the schemes. Here, a parameter-free algorithm proposed by Zheng et al. [36] is used to calculate the value of $\delta$ as follows.

$$\delta = \frac{\beta_{ave}\beta_{min}}{\beta_{std}},$$

(22)

where $\beta_{ave} = \frac{1}{3} \sum_{k=0}^{2} \beta_k$, $\beta_{min} = \min\{\beta_0, \beta_1, \beta_2\}$, $\beta_{std} = \sqrt{\frac{1}{3} \sum_{k=0}^{2} (\beta_{ave} - \beta_k)^2}$. To avoid denominators becoming zero in (20) and (22), $\beta_k + \delta$ in (20) and $\beta_{std}$ in (22) are replaced with $\max(\beta_k + \delta, \text{err})$ and $\max(\beta_{std}, \text{err})$, respectively. Here, $\text{err}$ is the square root of the smallest positive number allowed for a machine.

**IMEX method**

In this section, we briefly describe the framework of IMEX time discretization designed for the numerical integration of the stiff ordinary differential equation (ODE) as follows.

$$y_t = \bar{f}(y) + \bar{g}(y),$$

(23)

where $\bar{f}$ and $\bar{g}$ are the non-stiff and stiff parts of the ODE, respectively. The IMEX time discretization treats the non-stiff part explicitly and the stiff part implicitly in order to overcome the stringent time-step restriction for stability. There are many IMEX methods designed for various problems in the literature [37, 38, 39, 40, 41]. Among them, the IMEX Runge-Kutta (IMEX-RK) schemes as combinations of diagonally implicit A- or L-stable RK (DIRK) methods and explicit RK methods offer a precise and robust approach to define high order AP schemes. The computational cost per time step is proportional to the number of stages in the IMEX-RK schemes and the accuracy is proportional to the time-step size and the order of accuracy.
The IMEX-RKp(s, σ, p) scheme is characterized by the $s \times s$ matrices $A = (a_{ij}), \tilde{A} = (\tilde{a}_{ij})$, the vectors $b, \tilde{b} \in \mathbb{R}^s$, and the vectors $c, \tilde{c} \in \mathbb{R}^s$. This scheme with $s$ stages of the implicit scheme for the stiff term and $\sigma = (s + 1)$ stages of the explicit scheme for the non-stiff term has $p^{th}$-order temporal accuracy. Applying this scheme to solve Eq. (23) from $t^n$ to $t^{n+1}$ yields

\begin{align}
    y^{(k)} &= y^n + \Delta t \sum_{j=1}^{k-1} \tilde{a}_{kj} f(y^{(j)}) + \Delta t \sum_{j=1}^{k} a_{kj} g(y^{(j)}), k = 1, ..., s, \quad (24) \\
    y^{n+1} &= y^n + \Delta t \sum_{k=1}^{s} \tilde{b}_k f(y^{(k)}) + \Delta t \sum_{k=1}^{s} b_k g(y^{(k)}). \quad (25)
\end{align}

**Definition 1.** The IMEX-RKp(s, σ, p) scheme is of type ARS, i.e., the matrices $A, \tilde{A}$ for the implicit and explicit schemes are described respectively as

\begin{align}
    A &= \begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \quad (26)
\end{align}

with the submatrices $(\tilde{A}_{21})_{(s-1) \times 1}, (\tilde{A}_{22})_{(s-1) \times (s-1)}$ and the invertible submatrix $(A_{22})_{(s-1) \times (s-1)}$.

**Definition 2.** The IMEX-ARSp scheme is globally stiffly accurate (GSA) (see [10, 40]), i.e., the matrices $A, \tilde{A}$ and the vectors $b^T, \tilde{b}^T$ satisfy

\begin{align}
    e_s^T A = b^T, \quad e_s^T \tilde{A} = \tilde{b}^T, \quad c_s = \tilde{c}_s = 1, \quad (27)
\end{align}

where $e_s = (0, ..., 0, 1)^T$. The IMEX-ARSp scheme that satisfies the condition $e_s^T A = b^T$ is called stiffly accurate (SA).

**Remark 2.** For the GSA IMEX-ARSp scheme, $y^{n+1} = y^{(s)}$ in Eq. (25).

**IMEX-ARSp WCNS**

In this section, we design the IMEX-ARSp WCNS scheme for the isentropic Euler system. Here, the fifth-order WCNS described in the previous section is used for the spatial discretization and the $p^{th}$-order IMEX-ARSp method is used for the time discretization of flux derivatives. We first split the Euler system into two parts based on the acoustic/advection splitting strategy [10] and obtain

\begin{align}
    U_t = F + G, \quad (28)
\end{align}

where $U = (\rho, q)$, $F = -(0, \nabla \cdot \frac{q \otimes q}{\rho})$ is the non-stiff nonlinear term and $G = -\left(\nabla \cdot q, \frac{1}{\varepsilon} \nabla p\right)$ is the stiff linear term.

The $p^{th}$-order IMEX-ARSp scheme that consists of (24) and (25) is applied to each component of (28). The intermediate stages are described as

\begin{align}
    \rho^{(k)} &= \rho^n - \Delta t \sum_{j=1}^{k} a_{kj} \nabla \cdot q^{(j)}, \quad (29) \\
    q^{(k)} &= q^n - \Delta t \sum_{j=1}^{k-1} \tilde{a}_{kj} \left(\nabla \cdot \frac{q \otimes q}{\rho}\right)^{(j)} - \Delta t \sum_{j=1}^{k} \tilde{a}_{kj} \frac{1}{\varepsilon} \nabla p^{(j)} \quad (30)
\end{align}
where \( k = 1, 2, \ldots, s \), and the numerical solutions at time \( t^{n+1} \) are described as

\[
\rho^{n+1} = \rho^n - \Delta t \sum_{k=1}^{s} b_k \nabla \cdot q^{(k)},
\]

\[
q^{n+1} = q^n - \Delta t \sum_{k=1}^{s} b_k \left( \nabla \cdot \frac{q \otimes q}{\rho} \right)^{(k)} - \Delta t \sum_{k=1}^{s} b_k \frac{1}{\varepsilon^2} \nabla p^{(k)}.
\]

The intermediate stages (29) and (30) consist of two implicit steps and can be simplified by eliminating \( q^{(k)} \) between (29) and (30) \[10\], and obtain a nonlinear elliptic equation for \( \rho^{(k)} \):

\[
- \frac{(\Delta t a_{kk})^2}{\varepsilon^2} \Delta p^{(k)} + \rho^{(k)} = \hat{\rho}^{(k)} - \Delta t a_{kk} \nabla \cdot q^{(k)}.
\]

where

\[
\hat{\rho}^{(k)} = \rho^n - \Delta t \sum_{j=1}^{k-1} a_{kj} \nabla \cdot q^{(j)}
\]

\[
\hat{q}^{(k)} = q^n - \Delta t \sum_{j=1}^{k-1} a_{kj} \left( \nabla \cdot \frac{q \otimes q}{\rho} \right)^{(j)} - \Delta t \sum_{j=1}^{k-1} a_{kj} \frac{1}{\varepsilon^2} \nabla p^{(j)}.
\]

Based on (30) and (35), \( q^{(k)} \) can be calculated as

\[
q^{(k)} = \hat{q}^{(k)} - \Delta t a_{kk} \frac{1}{\varepsilon^2} \nabla p^{(k)}.
\]

**Remark 3.** To update the solutions from \( t^n \) to \( t^{n+1} \) by the \( p \)-order IMEX-ARS\(_p\) WCNS, we first solve the elliptic equation (33) to obtain \( \rho^{(k)} \) and then calculate explicitly \( q^{(k)} \) using (36). Here, (33) is a nonlinear equation for \( \rho^{(k)} \), which is solved by inexact Newton-generalized minimum residual (Newton-GMRES) algorithms. Refer to \[42, 43\] for the detailed description of the algorithms. Finally, the updates \( \rho^{n+1} \) and \( q^{n+1} \) are calculated explicitly using (31) and (32) with the intermediate values \( \rho^{(k)} \) and \( q^{(k)} \). For the GSA IMEX-ARS\(_p\) scheme, \( \rho^{n+1} = \rho^{(s)} \) and \( q^{n+1} = q^{(s)} \).

**Remark 4.** The spatial discretization of both convective terms in (33)-(35) and the gradient terms in (34) and (35) are solved by the fifth-order WCNS in the component-by-component and dimension-by-dimension forms. Here, in the interpolation process of the cell-edge pressures, the Lax-Friedrichs flux with zero numerical viscosity is considered, i.e., \( p = \frac{1}{2} p + \frac{1}{2} p \), otherwise the numerical diffusion will become large when \( \varepsilon \to 0 \) \[10\].

**Remark 5.** For the explicit time discretization, the CFL stability condition is described as

\[
\Delta t \leq CFL \cdot \frac{h}{\max_{u \cdot n, \rho}} \left[ |u \cdot n| \sqrt{p/\rho} / \varepsilon \right] = O(\varepsilon) \xrightarrow{\varepsilon \to 0} 0,
\]

which leads to unaffordable computational costs in the low Mach number regime. For the IMEX-ARS\(_p\) time discretization, the convective CFL stability condition is
obtained as
\[ \Delta t \leq CFL \cdot \frac{h}{\max_{u,n} \{2|u \cdot n|\}} \] (38)
due to semi-implicit nature of the acoustic/advection splitting strategy. In order to avoid an excessively large time-step sizes when the velocity of the fluid is too small, Boscarino et al. [3] recommended the following the stability restriction
\[ \Delta t \leq CFL \cdot \frac{h}{\max_{u,n} \{|u \cdot n| + \min(1, 1/\varepsilon) \sqrt{\rho'(\rho)}\}} \] (39)

The stability conditions (38) and (39) of the IMEX-ARS\( \rho \) method are independent of the Mach number \( \varepsilon \) and thus are less restrictive contrary to a standard explicit method. Therefore, the IMEX-ARS\( \rho \) method is more efficient than the explicit method for low-Mach number flows.

**Remark 6.** For one-dimensional case, the term \( \Delta p = p_{xx} \) in (33) is approximated by a fourth-order central-type difference scheme:
\[ p''_i = -p_{i-2} + 16p_{i-1} - 30p_i + 16p_{i+1} - p_{i+2} \quad \frac{12h^2}{40} \] (40)

For two-dimensional case, this central-type difference scheme in a dimension-by-dimension form is applied to \( \Delta p = p_{xx} + p_{yy} \).

**AP property**

**Definition 3.** A scheme for the low Mach number limit of the Euler equations is asymptotic preserving, if it serves as a consistent and stable discretisation, independent of the Mach number \( \varepsilon \).

**Proposition 1.** Consider the system (1) subject to well-prepared initial data (i.e., \( \rho_0 = \text{const}, \nabla \cdot u_0(x) = 0 \)) and periodic, wall or homogeneous Neumann boundary conditions. The spatial derivatives are assumed to be continuous. The GSA IMEX-ARS\( \rho \) scheme ((29)-(32)) (see Definitions 1 and 2) is used for the time discretization of the system (1) from \( t = t^0 \) to \( t^1 \) with the time-step size \( \Delta t = t^1 - t^0 \) given by the stability conditions (38) or (39) independent of \( \varepsilon \), and we then have
\[ \rho^1(x, t^1) \xrightarrow{\varepsilon \to 0} \rho_0, \quad \nabla \cdot u^1(x, t^1) \xrightarrow{\varepsilon \to 0} 0, \] (41)
and
\[ u^1(x, t^1) \xrightarrow{\varepsilon \to 0} u_e(x, t^1) + O(\Delta t^{p+1}). \] (42)

Here, \( u_e \) is the exact solution of the incompressible Euler equations (5) with initial data \( u_e(x, 0) = u_0^0(x) \).

Let \( f = \nabla \cdot \frac{q_{\rho} \cdot q_{\rho}}{\rho} \tilde{\nu} = (r^{(1)}, ..., r^{(s)})^T \) for \( r = \rho \) or \( p \), and \( \tilde{\nu} = (\nu^{(1)}, ..., \nu^{(s)})^T \) for \( \nu = f \) or \( q \) generated at the IMEX-ARS\( \rho \) intermediate stages \( t_k = t^0 + c_k \Delta t \). Following [10], we give the proof process of Proposition 1.
Based on Definition 1, (53) can be simplified as

\[
\begin{align*}
\tilde{\rho} &= \rho^0 e - \Delta t A \nabla \cdot \tilde{q}, \\
\tilde{q} &= q^0 e - \Delta t \tilde{\mathbf{A}} \nabla \cdot \tilde{f} - \frac{\Delta t}{\varepsilon^2} A \nabla \tilde{\rho},
\end{align*}
\]

(43)

and

\[
\begin{align*}
\rho^1 &= \rho^0 - \Delta t b^T \nabla \cdot \tilde{q}, \\
q^1 &= q^0 - \Delta t b^T \nabla \cdot \tilde{f} - \frac{\Delta t}{\varepsilon^2} b^T \nabla \tilde{\rho},
\end{align*}
\]

(45)

where \( e = (1, \ldots, 1)^T \in \mathbb{R}^s \).

Taking the divergence of both sides of (44) and substituting \( \nabla \cdot \tilde{q} \) into (43) yields a pressure Poisson equation

\[
\frac{\Delta^2}{\varepsilon^2} A^2 \Delta \tilde{\rho} = (\tilde{\rho} - \rho^0 e) + \Delta t A \nabla \cdot q^0 e - \Delta t^2 A \tilde{\mathbf{A}} \nabla \cdot (\nabla \cdot \tilde{f}).
\]

(47)

In order to study the asymptotic behavior of the solutions, we plug the \( \varepsilon \)-asymptotic expansions \( f(x) = f_0(x) + \varepsilon^2 f_2(x) + O(\varepsilon^3) \) for variables \( \tilde{\rho}, \rho^1, \tilde{q}, q^1 \) and \( g(x) = g_0(x) + O(\varepsilon^2) \) for variables \( \tilde{q}, q^1 \) into (43)-(47). Note that the term \( O(\varepsilon) \) does not appear in Eqns. (1). At the leading order \( O(\varepsilon^{-2}) \) terms in (44), we obtain \( \nabla \tilde{p}_0 = \nabla \tilde{\rho}(\tilde{p}_0) = 0 \). Therefore, \( \tilde{p}_0 \) and \( \tilde{p}_0 \) are independent of \( x \). Collecting the order \( O(1) \) terms in \( \varepsilon \) generates

\[
\begin{align*}
\tilde{\rho}_0 - \rho^0_0 e &= -\Delta t A \nabla \cdot \tilde{q}_0, \\
\tilde{q}_0 - q^0_0 e &= -\Delta t \tilde{\mathbf{A}} \nabla \cdot \tilde{f}_0 - \Delta t A \nabla \tilde{p}_2, \\
\rho^1_0 - \rho^0_0 &= -\Delta t b^T \nabla \cdot \tilde{q}_0, \\
q^1_0 - q^0_0 &= -\Delta t b^T \nabla \cdot \tilde{f}_0 - \Delta t b^T \nabla \tilde{p}_2, \\
\Delta^2 A^2 \Delta \tilde{p}_2 &= \tilde{p}_0 - \rho^0_0 e + \Delta t A \nabla \cdot q^0_0 e - \Delta t^2 A \tilde{\mathbf{A}} \nabla \cdot (\nabla \cdot \tilde{f}_0),
\end{align*}
\]

(48)

(49)

(50)

(51)

(52)

where \( q^0_0 = \rho^0_0 u^0_0 \) and \( \tilde{f}_0 = \tilde{q}_0 \otimes q^0_0/\tilde{p}_0 \).

Integrating (48) over the whole computational domain and applying periodic, wall or homogeneous Neumann boundary conditions yields \( \tilde{p}_0 = \rho^0_0 e \), i.e., \( \tilde{p}_0 \) is independent of \( t \). Since \( \tilde{p}_0 = \rho^0_0 e \) and \( \nabla \cdot \tilde{q}_0 = \rho^0_0 \nabla \cdot u_0^0 = 0 \), (52) becomes

\[
A^2 \Delta \tilde{p}_2 = -A \tilde{\mathbf{A}} \nabla \cdot (\nabla \cdot \tilde{f}_0).
\]

(53)

Based on Definition 1, (53) can be simplified as

\[
\Delta \tilde{p}_2 = -A_{22} \left( \tilde{\mathbf{A}}_{21} \nabla \cdot (\nabla \cdot \tilde{f}_0^{(1)}) + \tilde{\mathbf{A}}_{22} \nabla \cdot (\nabla \cdot \tilde{f}_0) \right),
\]

(54)

where \( \tilde{\rho} = (\rho^{(2)}, \ldots, \rho^{(s)})^T \) and \( \tilde{f}_0 = (f^{(2)}, \ldots, f^{(s)})^T \).

Taking the divergence of both sides of (49) and applying Definition 1 and (54) yields

\[
\nabla \cdot \tilde{q}_0 = 0, \quad \nabla \cdot \tilde{u}_0 = 0.
\]

(55)
Based on (50) and (55), we have $\rho_1^0 = \rho_0^0$, i.e., $\rho^1(x, t^1) \xrightarrow{\epsilon \to 0} \rho_0^0$. Taking the divergence of both sides of (51) and applying Definition 2 generates

$$\nabla \cdot q_0^1 = -\Delta t e^T(\tilde{A} \nabla \cdot (\nabla \cdot \tilde{f}_0) + A \Delta \tilde{p}_2).$$  \hspace{1cm} (56)$$

By applying Definition 1 and (54), (56) is simplified as

$$\nabla \cdot q_0^1 = -\Delta t e^T \left( \frac{1}{\rho_0^0} \nabla \cdot (\nabla \cdot \tilde{f}_0) + \frac{1}{\rho_0^0} \nabla \tilde{p} \right).$$

Both sides of (49) and (51) are divided by the constant density $\rho_0^0$ and we obtain the asymptotic limit of scheme for the incompressible Euler system (5):

$$\bar{u}_0 = u_0^0 e - \Delta t \tilde{A} \frac{1}{\rho_0^0} \nabla \cdot \tilde{f}_0 - \Delta t A \frac{1}{\rho_0^0} \nabla \tilde{p}_2,$$  \hspace{1cm} (57)

$$u_1^0 = u_0^0 - \Delta t b^T \frac{1}{\rho_0^0} \nabla \cdot \tilde{f}_0 - \Delta t b^T \frac{1}{\rho_0^0} \nabla \tilde{p}_2,$$  \hspace{1cm} (58)

where $\tilde{p}_2$ is given by (53). If the IMEX-ARS$p$ scheme that has $p^{th}$-order temporal accuracy is adopted to solve the system (5) subject to initial data $u_e(x, 0) = u_0^0$, the asymptotic accuracy can be attained, i.e., (42) is proved.

**Remark 7.** A similar procedure can be adopted when going from $t^n$ to $t^{n+1}$ to prove that the limiting IMEX-ARS$p$ scheme is a consistent discretization of the limiting equation for fixed $h$ and $\Delta t$. The IMEX-ARS$p$ scheme is also asymptotically stable with the time-step restrictions (38) or (39) independent of the Mach number $\epsilon$.

**Remark 8.** The AP property of the GSA IMEX-ARS$p$ scheme is proved based on the assumption that the Euler equations are subject to well-prepared initial data. For the SA IMEX-ARS$p$ scheme, the AP property can also be proved using a similar procedure under the same assumption. The proof of the AP property of the GSA IMEX-ARS$p$ scheme combined with the fifth-order WCNS is relatively complex and has not been performed. In the next section, we will verify numerically the AP property of the IMEX-ARS$p$ WCNS. In addition, the IMEX-ARS$p$ scheme combined some other shock capturing schemes, e.g., the MUSCL (monotone upstream-centered scheme for conservation laws) type finite volume scheme proposed in [44], can also maintain numerically the AP property.

**Numerical examples**

The third-order GSA IMEX-ARS3 WCNS is applied to solve the following numerical examples. In some convergence tests, the third-order SA IMEX-ARS3 WCNS is also tested. See Appendix for the characteristics of the GSA and SA IMEX-ARS3 scheme. In addition, the third-order explicit total variation diminishing Runge-Kutta time-stepping method [30] coupled with the fifth-order WCNS (TVD-RK3 WCNS) and the third-order GAS IMEX-ARS3 scheme coupled with the third-order MUSCL type finite volume scheme (IMEX-ARS3 MUSCL) are used to solve some examples for comparisons.

The order of temporal accuracy is numerically calculated as

$$p = \log_2(E_{\Delta t_1}/E_{\Delta t_2}),$$  \hspace{1cm} (59)$$

where $E_{\Delta t_1}$ obtained with $\Delta t_1 = O(h)$ and $E_{\Delta t_2}$ obtained with $\Delta t_2 = \Delta t_1/2$ are the global errors in the $L^\infty$ and $L^1$ norms, respectively. The space-step size $h$ decreases
with the time-step size $\Delta t$. In the convergence tests, to test the temporal order of accuracy the time-step size is taken as $\Delta t = CFL \cdot h$ ($CFL = 0.1$), while it is set as $\Delta t = CFL \cdot h^k$ ($CFL = 0.1$) to test the spatial order of accuracy. In the other tests, the time-step sizes of the IMEX-ARS3 WCNS and the IMEX-ARS3 MUSCL are computed by the stability conditions (38) or (39), while the time-step size of the TVD-RK3 WCNS is calculated by the stability condition (37).

**Example 1.** Convergence test for one-dimensional case [3]

This example is adopted to test the temporal order of accuracy of the GSA and SA IMEX-ARS3 WCNS for one-dimensional case. The computational domain is $[-2.5, 2.5]$ with periodic boundary conditions. The initial data for one-dimensional isentropic Euler equations (1) is given by

$$u(x, 0) = \sin \left( \frac{2\pi x}{L} \right), \quad \rho(x, 0) = \left(1 + \frac{(\gamma - 1)u(x, 0)}{c}\right)^2, \quad p = \rho^\gamma,$$  \hspace{1cm} (60)

where $\gamma = 2, c = 2\sqrt{\gamma}/\varepsilon, L = 5$ and the final time is fixed to $T_f = 0.1$. To measure the temporal convergence of the GSA and SA IMEX-ARS3 WCNS, the solution computed on a fine mesh with $N = 2560$ cells is used as a reference (“exact”) solution since the exact solution of this problem is unknown. Tables 1 and 2 report the numerical $L^\infty$ and $L^1$-norm errors and orders of accuracy in time for the density obtained with the GSA IMEX-ARS3 WCNS at different Mach numbers, while Tables 3 and 4 show the corresponding results obtained with the SA IMEX-ARS3 WCNS. It is observed that both methods show a third-order temporal convergence, as expected, for different values of the Mach number, from compressible to incompressible regimes. In the following examples, we only investigate the performance of the GSA IMEX-ARS scheme. Here, we also test the spatial accuracy of the GSA IMEX-ARS3 WCNS. The numerical $L^\infty$ and $L^1$-norm errors are computed by comparing the numerical solutions of the density on third consecutive mesh sizes [13]. Tables 5 and 6 report the corresponding results at $t = 0.05$, which demonstrate a fifth-order spatial convergence of the GSA IMEX-ARS3 WCNS at different Mach numbers.

**Example 2.** One-dimensional Riemann problem [10, 4, 6]

This example that consists in several interacting Riemann problems is used to validate the asymptotic stability of the IMEX-ARS3 WCNS. The computational domain is $[0, 1]$ with periodic boundary conditions. The initial data is described as

$$\begin{align*}
\rho(x, 0) &= 1, \quad q(x, 0) = 1 - 0.5\varepsilon^2, \quad x \in [0, 0.2] \cup [0.8, 1], \\
\rho(x, 0) &= 1 + \varepsilon^2, \quad q(x, 0) = 1, \quad x \in [0.2, 0.3], \\
\rho(x, 0) &= 1, \quad q(x, 0) = 1 + 0.5\varepsilon^2, \quad x \in [0.3, 0.7], \\
\rho(x, 0) &= 1 - \varepsilon^2, \quad q(x, 0) = 1, \quad x \in [0.7, 0.8],
\end{align*}$$  \hspace{1cm} (61)

and the pressure is $p(\rho) = \rho^2$. The time-step size is computed by (38) with $CFL = 0.3$ and the final time is fixed to $T_f = 0.05$. The values of the Mach number are chosen as $\varepsilon = \sqrt{0.99}, 0.3$ and 0.01 corresponding to the compressible, intermediate and incompressible regimes, respectively. The numerical solutions calculated by the GSA IMEX-ARS3 WCNS and the GSA IMEX-ARS3 MUSCL with $N = 200$ cells
are compared to a reference solution computed by the explicit TVD-RK 3 WCNS with \( N = 2000 \) cells. Figure 1 illustrates the solutions for the density and momentum at different Mach numbers. From this figure, we observe the IMEX-ARS3 WCNS and the TVD-RK 3 WCNS can capture shocks and contact discontinuities that are stronger when \( \varepsilon \) is bigger, while the IMEX-ARS3 MUSCL suffers from excessive numerical diffusion. Spurious oscillations are observed with the TVD-RK 3 WCNS when \( \varepsilon = \sqrt{0.99} \) (see Figure 1(a) and (b)). The utilization of WENO interpolation based on the characteristic variables instead of the flux functions can effectively suppress spurious oscillations near discontinuities. Compared to the TVD-RK 3 WCNS, the IMEX-ARS3 WCNS and the IMEX-ARS3 MUSCL capture the limit incompressible solution faster when \( \varepsilon \) is small. When \( \varepsilon \) approaches 0, the TVD-RK 3 WCNS cannot capture the correct solution, due to the excessive numerical viscosity.

Example 3. Sod shock tube problem [5]

This problem with non-well prepared initial data is used to investigate the performance of the GSA IMEX-ARS3 WCNS in the intermediate and incompressible regimes. The computational domain is \([0, 1]\) with zero Neumann boundary conditions. The pressure is \( p(\rho) = \rho \). The non-well prepared initial data including discontinuity is

\[
(r, u) = \begin{cases} (1.000, 0.000), & x \leq 0.5; \\
(0.125, 0.000), & \text{otherwise.} \end{cases}
\] (62)

The time-step size is computed by (39) with \( CFL = 0.1 \). The values of the Mach number are chosen as \( \varepsilon = 0.3 \) and 0.03 corresponding to the intermediate and incompressible regimes, respectively. For both \( \varepsilon = 0.3 \) and \( \varepsilon = 0.03 \), reference solutions are generated using the GSA IMEX-ARS3 MUSCL with \( N = 2000 \) cells. Figure 2 depicts the density and momentum calculated by the GSA IMEX-ARS3 WCNS with \( N = 200 \) cells at time \( T_f = 0.05 \) for \( \varepsilon = 0.3 \) and at time \( T_f = 0.005 \) for \( \varepsilon = 0.03 \). From this figure, the density and momentum are not close to the solutions of the incompressible Euler equations. The solutions include the rarefaction waves moving to the left and shock waves moving to the right for both \( \varepsilon = 0.3 \) and \( \varepsilon = 0.03 \). From Figure 2(a) and (b), it is observed that when \( \varepsilon = 0.3 \), the numerical solution obtained with the IMEX-ARS3 WCNS has a good agreement with the reference solution. When \( \varepsilon = 0.03 \) in the incompressible regime, nonphysical oscillations arise near discontinuities based on the IMEX-ARS3 WCNS (see Figure 2(c) and (d)). Nonphysical oscillations are a common phenomenon when shocks develop in the low Mach number regime [4, 5] and can be eliminated by decreasing \( \Delta t \) and taking a smaller \( CFL = 0.01 \).

Example 4. Convergence test for two-dimensional case [19]

This example is used to test the temporal order of accuracy of the GSA IMEX-ARS3 WCNS for two-dimensional case. The computational domain is \([0, 1] \times [0, 1]\) with periodic boundary conditions. The pressure is \( p(\rho) = \frac{1}{2} \rho^2 \). The final time is
where \( r = 4\pi \sqrt{(x - 0.6t - 0.5)^2 + (y - 0.5)^2} \), \( k(r) = 2\cos(r) + 2r\sin(r) + 0.125\cos(2r) + 0.25r^2\sin(2r) + 0.75r^2 \) and

\[
\delta(r) = \begin{cases} 
1, & r < \pi, \\
0, & \text{otherwise}. 
\end{cases}
\]  

(64)

Tables 7 and 8 show the numerical \( L^\infty \) and \( L^1 \)-norm errors and orders of accuracy in time for the momentum \( \rho u \) based on different values of the Mach number, from compressible to incompressible regimes, which validates the GSA IMEX-ARS3 WCNS for two-dimensional case can attain the designed temporal accuracy. In addition, the \( L^1 \)-norm convergence is better than the \( L^\infty \)-norm convergence under the same conditions.

Example 5. Two-dimensional Riemann problem [10, 8]

This example is used to test the performance of the GSA IMEX-ARS3 WCNS at a high Mach number for two-dimensional case. The computational domain is \([0, 1] \times [0, 1]\) with zero Neumann boundary conditions. The pressure is \( p(\rho) = \rho^{1.4} \).

This initial data includes four shock waves and is described as

\[
(\rho, u, v) = \begin{cases} 
(0.1308, 1.206, 1.206), & x < 0.5, y < 0.5, \\
(0.5323, 0.000, 1.206), & x \geq 0.5, y < 0.5, \\
(0.5323, 1.206, 0.000), & x < 0.5, y \geq 0.5, \\
(1.5000, 0.000, 0.000), & \text{otherwise}. 
\end{cases}
\]  

(65)

The time-step size is computed by (38) with \( CFL = 0.4 \). The values of the Mach number are chosen as \( \varepsilon = 1 \) and 2 to test the robust shock-capturing capability of the IMEX-ARS3 WCNS in the compressible regime. Figure 3 plots the surfaces of the density profile obtained with the IMEX-ARS3 WCNS and the IMEX-ARS3 MUSCL on a mesh 50 \( \times \) 50 at \( T_f = 0.1 \). From this figure, we see that the IMEX-ARS3 WCNS has a higher resolution, compared to the IMEX-ARS3 MUSCL.

Example 6. Cylindrical explosion problem [6, 11]

The computational domain is \([-1, 1] \times [-1, 1]\) with periodic boundary conditions and the pressure is given by \( p(\rho) = \rho^2 \). The initial density of the fluid is given by

\[
\rho(x, 0) = \begin{cases} 
1 + \varepsilon^2, & \text{if } r < 0.5, \\
1, & \text{otherwise}, 
\end{cases}
\]  

(66)

where \( r = \sqrt{x^2 + y^2} \) is the distance to the center \((0, 0)\) of the domain. The initial velocity of the fluid is given by

\[
\begin{align*}
u(x, 0) &= -\frac{x}{\rho(x, 0)}, \\
v(x, 0) &= -\frac{y}{\rho(x, 0)},
\end{align*}
\]  

(67)
where \( \beta = \max(0, 1 - r)e^{-16r^2} \) and \((u, v) = (0, 0)\), if \( r \leq 10^{-15} \). The computational domain is divided into \( 50 \times 50 \) cells. The values of the Mach number are chosen as \( \varepsilon = 1 \) and \( 0.001 \). The time-step sizes are computed by (38) with \( CFL = 0.4 \) for \( \varepsilon = 1 \) and by (39) with \( CFL = 0.03 \) for \( \varepsilon = 0.001 \), respectively. Figure 4 plots the surfaces of the density profile and the velocity fields for \( \varepsilon = 1 \) obtained with the GSA IMEX-ARS3 WCNS at different times. Figure 5 shows the density profile and the discrete divergence of the velocity \( \nabla \cdot u \) at time \( t = 0.05 \) for a small value of \( \varepsilon = 0.001 \). It is observed that when \( \varepsilon = 0.001 \), the density converges to the constant value 1 and the divergence \( \nabla \cdot u \) is close to 0.

**Example 7.** Shear flow problem [3, 10, 16]

This example is used to verify numerically the AP property of the IMEX-ARS3 WCNS in the low Mach number limit. The computational domain is \([0, 2\pi] \times [0, 2\pi]\) with periodic boundary conditions and the pressure is set to \( p(\rho) = \rho^{1.4} \). Initially, a constant density and an incompressible velocity field are given by \( \rho(x, 0) = \pi/15 \) and

\[
\begin{align*}
    u(x, 0) &= \begin{cases} 
        \tanh((y - 0.5\pi)/\rho), & y \leq \pi, \\
        \tanh((1.5\pi - y)/\rho), & \text{otherwise},
    \end{cases} \\
    v(x, 0) &= 0.05 \sin(x).
\end{align*}
\]  

The time-step size is computed by (39) with \( CFL = 0.2 \). The Mach number is \( \varepsilon = 10^{-4} \) and the final time is \( T_f = 10 \). Figure 6 demonstrates the discrete vorticity \( \omega = v_x - u_y \) on a mesh \( 50 \times 50 \) at different times. From this figure, we see the vorticity quickly develops into roll-ups with smaller and smaller scales with the time evolution. Figure 7 shows the time history of the \( L_1 \)-norm error of the discrete divergence of the velocity \( \nabla \cdot u \). It is observed that the \( L_1 \)-norm error increases with time, which is in agreement with the phenomenon in [10] and is caused by the appearance of the smaller and smaller scale structures in the shear flow. The maximum error is smaller than that obtained with a finer mesh in [10].

**Example 8.** Kelvin-Helmholtz instability problem [10]

This problem is solved in the domain \([0, 2\pi] \times [0, 4\pi]\) with periodic boundary conditions. The pressure is \( p(\rho) = \rho^2 \) and the initial data reads

\[
\begin{align*}
    \rho(x, 0) &= 1, \\
    u(x, 0) &= \cos(y), \\
    v(x, 0) &= 0.03 \sin(0.5x).
\end{align*}
\]

The time-step size is computed by (39) with \( CFL = 0.2 \). The Mach number is \( \varepsilon = 10^{-3} \) and the final time is \( T_f = 40 \). The discrete vorticity \( \omega = v_x - u_y \) on a mesh \( 50 \times 50 \) at different times is depicted in Figure 8, which shows the emergence and subsequent evolution of small scale flow features. The time history of the \( L_1 \)-norm error of the discrete divergence of the velocity \( \nabla \cdot u \) is shown in Figure 9, from which we also observe that the \( L_1 \)-norm error increases with time and the maximum error is smaller than that obtained with a finer mesh in [10].

**Conclusions**

In this work, we have designed a high order IMEX WCNS for the compressible-incompressible limit problems described as scaled isentropic Euler equations. Based
on the acoustic/advection splitting strategy, the semi-implicit method combines
the third-order IMEX-ARS3 scheme for the time integration with the fifth-order
WCNS for the spatial discretization. The GSA IMEX-ARS3 scheme has been proven
to be asymptotic preserving, i.e., it converges to a consistent scheme in the zero
Mach number limit, and asymptotically stable under a convective CFL condition.
The IMEX-ARS3 WCNS has been verified numerically that it can achieve a third-
order temporal accuracy for different values of the Mach number, from compressible
to incompressible regimes. Numerical results show that the IMEX-ARS3 WCNS
with the AP property can capture shocks and contact discontinuities with high
resolution in the compressible regime, while it can capture incompressible features
in the incompressible regime. The future work will focus on the implementation
of the proof of the AP property of the fully discrete scheme and the extension to
compressible Navier-Stokes flows at any Mach number.

Appendix

This appendix is used to characterize the GSA and SA IMEX-ARS<sub>p</sub> schemes adopt-
ed in this paper.

The third-order GSA IMEX-RK3 scheme is characterized by

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 \\
0 & 0.1666666666 & 0.5 & 0 & 0 \\
0 & -0.5 & 0.5 & 0.5 & 0 \\
0 & 1.5 & -1.5 & 0.5 & 0.5 \\
\end{bmatrix},
\]

\[
\tilde{A} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 \\
0.6111111111 & 0.0555555555 & 0 & 0 & 0 \\
0.8333333333 & -0.8333333333 & 0.5 & 0 & 0 \\
0.25 & 1.75 & 0.75 & -1.75 & 0 \\
\end{bmatrix},
\]

and \(b^T = (0, 1.5, -1.5, 0.5, 0.5), \tilde{b}^T = (0.25, 1.75, 0.75, -1.75, 0).\) The third-order
SA IMEX-RK3 scheme is characterized by

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0.4358665215 & 0 & 0 & 0 \\
0 & 0.2820667392 & 0.4358665215 & 0 & 0 \\
0 & 1.2084966491 & -0.644363171 & 0.4358665215 & 0 \\
\end{bmatrix},
\]

\[
\tilde{A} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0.4358665215 & 0 & 0 & 0 & 0 \\
0.3212788860 & 0.3966543747 & 0 & 0 & 0 \\
-0.1058582961 & 0.5529291480 & 0.5529291480 & 0 & 0 \\
\end{bmatrix},
\]

and \(b^T = b^T = (0, 1.208496649, -0.644363171, 0.4358665215).\)
Abbreviations

WCNS: Weighted compact nonlinear scheme
IMEX: Implicit-explicit
CFL: Courant-Friedrichs-Levy
AP: Asymptotic-preserving
WENO: Weighted essentially non-oscillatory
ODE: Ordinary differential equation
RK: Runge-Kutta
DIRK: Diagonally implicit Runge-Kutta
GSA: Globally stiffly accurate
SA: Stiffly accurate
Newton-GMRES: Newton-generalized minimum residual
MUSCL: Monotone upstream-centered scheme for conservation laws
TVD-RK3: Third-order total variation diminishing Runge-Kutta

Declarations

Availability of data and materials
All data generated or analysed during this study are included in this published article.

Competing interests
The authors declare that they have no competing interests.

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Figure 1. Example 2. Density (left) and momentum (right) calculated by the GSA IMEX-ARS3 WCNS and the GSA IMEX-ARS3 MUSCL with $N = 200$ cells and the TVD-RK3 WCNS with $N = 2000$ cells at $t = 0.05$. The values of the Mach number are $\varepsilon = \sqrt{0.99}$ (top), 0.3 (middle) and 0.01 (bottom).
Table 1 Example 1. $L^\infty$ errors and numerical orders of temporal accuracy obtained with the GSA IMEX-ARS WCNS at $t = 0.1$.

| $N$  | $\varepsilon = 0.01$ | $\varepsilon = 0.1$ | $\varepsilon = 1$ |
|------|---------------------|---------------------|------------------|
|      | $L^\infty$ error | Order | $L^\infty$ error | Order | $L^\infty$ error | Order |
| 40   | $7.59E - 03$       | 3.12  | $7.16E - 05$       | 3.03  | $1.23E - 03$       | 3.63  |
| 80   | $3.03E - 03$       | 2.65  | $8.78E - 06$       | 2.99  | $9.95E - 05$       | 2.92  |
| 320  | $9.85E - 05$       | 2.69  | $1.48E - 07$       | 2.99  | $6.58E - 07$       | 4.32  |
| 640  | $1.28E - 05$       | 2.94  | $1.95E - 08$       | 2.93  | $8.75E - 09$       | 6.23  |

Table 2 Example 1. $L^1$ errors and numerical orders of temporal accuracy obtained with the GSA IMEX-ARS WCNS at $t = 0.1$.

| $N$  | $\varepsilon = 0.01$ | $\varepsilon = 0.1$ | $\varepsilon = 1$ |
|------|---------------------|---------------------|------------------|
|      | $L^1$ error | Order | $L^1$ error | Order | $L^1$ error | Order |
| 40   | $4.68E - 03$       | 1.52  | $2.59E - 05$       | 2.99  | $2.18E - 04$       | 4.22  |
| 80   | $1.63E - 03$       | 1.52  | $3.19E - 06$       | 2.99  | $1.18E - 05$       | 4.22  |
| 160  | $2.79E - 04$       | 2.55  | $4.00E - 07$       | 2.99  | $5.48E - 07$       | 4.42  |
| 320  | $3.88E - 05$       | 2.85  | $5.02E - 08$       | 3.00  | $1.80E - 08$       | 4.93  |
| 640  | $4.94E - 06$       | 2.97  | $6.34E - 09$       | 2.98  | $1.40E - 09$       | 3.69  |

Table 3 Example 1. $L^\infty$ errors and numerical orders of spatial accuracy obtained with the SA IMEX-ARS WCNS at $t = 0.1$.

| $N$  | $\varepsilon = 0.01$ | $\varepsilon = 0.1$ | $\varepsilon = 1$ |
|------|---------------------|---------------------|------------------|
|      | $L^\infty$ error | Order | $L^\infty$ error | Order | $L^\infty$ error | Order |
| 40   | $7.90E - 03$       | 2.50  | $9.18E - 05$       | 3.07  | $6.50E - 04$       | 2.78  |
| 80   | $3.34E - 03$       | 1.24  | $1.09E - 05$       | 2.97  | $9.50E - 05$       | 2.47  |
| 160  | $7.45E - 04$       | 2.16  | $1.40E - 06$       | 2.97  | $1.71E - 05$       | 2.47  |
| 320  | $1.19E - 04$       | 2.65  | $1.75E - 07$       | 3.00  | $1.97E - 07$       | 6.44  |
| 640  | $1.57E - 05$       | 2.92  | $2.19E - 08$       | 3.00  | $6.74E - 10$       | 8.19  |

Table 4 Example 1. $L^1$ errors and numerical orders of spatial accuracy obtained with the SA IMEX-ARS WCNS at $t = 0.1$.

| $N$  | $\varepsilon = 0.01$ | $\varepsilon = 0.1$ | $\varepsilon = 1$ |
|------|---------------------|---------------------|------------------|
|      | $L^1$ error | Order | $L^1$ error | Order | $L^1$ error | Order |
| 40   | $4.87E - 03$       | 1.45  | $3.10E - 05$       | 3.03  | $1.05E - 04$       | 3.65  |
| 80   | $1.79E - 03$       | 1.45  | $3.94E - 06$       | 2.98  | $8.39E - 06$       | 3.65  |
| 160  | $3.29E - 04$       | 2.44  | $4.86E - 07$       | 3.02  | $4.77E - 07$       | 4.13  |
| 320  | $4.73E - 05$       | 2.80  | $6.08E - 08$       | 3.00  | $5.39E - 09$       | 6.47  |
| 640  | $6.10E - 06$       | 2.95  | $7.53E - 09$       | 3.01  | $1.27E - 10$       | 5.41  |

Table 5 Example 1. $L^\infty$ errors and numerical orders of spatial accuracy obtained with the GSA IMEX-ARS WCNS at $t = 0.05$.

| $N$  | $\varepsilon = 0.01$ | $\varepsilon = 0.1$ | $\varepsilon = 1$ |
|------|---------------------|---------------------|------------------|
|      | $L^\infty$ error | Order | $L^\infty$ error | Order | $L^\infty$ error | Order |
| 20   | $8.70E - 03$       | 3.22  | $3.22E - 04$       | 4.49  | $4.49E - 03$       | 3.63  |
| 40   | $1.80E - 04$       | 5.59  | $1.30E - 05$       | 4.63  | $5.49E - 04$       | 3.03  |
| 80   | $2.83E - 06$       | 5.99  | $2.61E - 07$       | 5.12  | $6.68E - 05$       | 3.04  |
| 160  | $4.36E - 08$       | 6.05  | $3.34E - 10$       | 8.13  | $1.96E - 06$       | 5.09  |
| 320  | $6.59E - 10$       | 6.05  | $1.08E - 10$       | 3.11  | $2.44E - 09$       | 9.65  |
Table 6. Example 1. \( L^1 \) errors and numerical orders of spatial accuracy obtained with the GSA IMEX-ARS WCNS at \( t = 0.05 \).

| \( N \) | \( \epsilon = 0.01 \) | \( \epsilon = 0.1 \) | \( \epsilon = 1 \) |
|-------|----------------|----------------|----------------|
|       | \( \varepsilon \) | \( \varepsilon \) | \( \varepsilon \) |
|       | \( L^1 \) Error | Order | \( L^1 \) Error | Order | \( L^1 \) Error | Order |
| 20    | 5.58E−04       |       | 4.21E−05       |       | 1.10E−03       |       |
| 40    | 3.13E−05       | 4.16  | 6.63E−07       | 5.99  | 1.29E−05       | 6.42  |
| 80    | 4.57E−06       | 2.78  | 5.99E−09       | 6.79  | 1.13E−07       | 6.83  |
| 160   | 9.96E−08       | 5.52  | 9.91E−12       | 9.24  | 8.92E−10       | 6.98  |
| 320   | 5.53E−11       | 10.81 | 9.86E−13       | 3.33  | 5.42E−12       | 7.36  |

Table 7. Example 4. \( L^\infty \) errors and numerical orders of temporal accuracy obtained with the GSA IMEX-ARS WCNS at \( t = 0.02 \).

| \( M \times N \) | \( \epsilon = 0.01 \) | \( \epsilon = 0.1 \) | \( \epsilon = 1 \) |
|-----------------|----------------|----------------|----------------|
|                 | \( \varepsilon \) | \( \varepsilon \) | \( \varepsilon \) |
|                 | \( L^\infty \) Error | Order | \( L^\infty \) Error | Order | \( L^\infty \) Error | Order |
| 40 \times 40    | 2.08E−03       |       | 2.10E−03       |       | 2.24E−03       |       |
| 80 \times 80    | 8.02E−04       | 1.37  | 7.98E−04       | 1.40  | 7.76E−04       | 1.53  |
| 160 \times 160  | 3.01E−04       | 1.42  | 3.02E−04       | 1.40  | 3.02E−04       | 1.36  |
| 320 \times 320  | 5.64E−05       | 2.41  | 5.76E−05       | 2.39  | 6.53E−05       | 2.21  |
| 640 \times 640  | 8.36E−06       | 2.76  | 8.35E−06       | 2.79  | 8.36E−06       | 2.96  |

Table 8. Example 4. \( L^1 \) errors and numerical orders of temporal accuracy obtained with the GSA IMEX-ARS WCNS at \( t = 0.02 \).

| \( M \times N \) | \( \epsilon = 0.01 \) | \( \epsilon = 0.1 \) | \( \epsilon = 1 \) |
|-----------------|----------------|----------------|----------------|
|                 | \( \varepsilon \) | \( \varepsilon \) | \( \varepsilon \) |
|                 | \( L^1 \) Error | Order | \( L^1 \) Error | Order | \( L^1 \) Error | Order |
| 40 \times 40    | 1.08E−04       |       | 1.15E−04       |       | 9.39E−05       |       |
| 80 \times 80    | 2.31E−05       | 2.23  | 2.51E−05       | 2.20  | 2.13E−05       | 2.14  |
| 160 \times 160  | 4.04E−06       | 2.51  | 4.17E−06       | 2.59  | 4.10E−06       | 2.38  |
| 320 \times 320  | 3.87E−07       | 3.38  | 3.93E−07       | 3.41  | 4.04E−07       | 3.34  |
| 640 \times 640  | 4.10E−08       | 3.24  | 4.12E−08       | 3.25  | 6.37E−08       | 2.66  |
Figure 2 Example 3. Density (left) and momentum (right) calculated by the GSA IMEX-ARS3 WCNS with $N = 200$ cells and the GSA IMEX-ARS3 MUSCL with $N = 2000$ cells at $t = 0.05$ for $\varepsilon = 0.3$ and at $t = 0.005$ for $\varepsilon = 0.03$. The values of the Mach number are $\varepsilon = 0.3$ (top) and $0.03$ (bottom).
Figure 3 Example 5. Density obtained with the GSA IMEX-ARS WCNS (left) and the GSA IMEX-ARS MUSCL (right) on a mesh $50 \times 50$ at $t = 0.1$. The values of the Mach number are $\varepsilon = 1$ (top) and 2 (bottom).
Figure 4 Example 6. Density (left) and velocity field (right) calculated by the GSA IMEX-ARS3 WCNS on a mesh $50 \times 50$ at the Mach number $\varepsilon = 1$. 
Figure 5 Example 6. Density (left) and divergence (right) calculated by the GSA IMEX-ARS\textsuperscript{3} WCNS on a mesh $50 \times 50$ at $t = 0.05$.

Figure 6 Example 7. Vorticity $\omega = v_x - u_y$ obtained with the GSA IMEX-ARS\textsuperscript{3} WCNS on a mesh $50 \times 50$ at different times.
Figure 7 Example 7. Time history of $L_1$ error of the divergence of the velocity $\nabla \cdot u$.

Figure 8 Example 8. Vorticity $\omega = v_x - u_y$ obtained with the GSA IMEX-ARS WCNS on a mesh $50 \times 50$ at different times.
**Figure 9** Example 8. Time history of $L_1$ error of the divergence of the velocity $\nabla \cdot \mathbf{u}$. 
Figure 1

See figure for legend.

Figure 1. Example 2: Density (left) and momentum (right) calculated by the GSA IMEX-ARS3 WCONS and the GSA IMEX-ARS/MUSCL with $N = 200$ cells and the TVD-ARS WCONS with $N = 2000$ cells at $x = 0.05$. The values of the Mach number are $x = \sqrt{0.59}$ (top), 0.3 (middle) and 0.01 (bottom).
Figure 2: Example 3. Density (left) and momentum (right) calculated by the GSA IMEX-ARS3 WCNS with $N = 200$ cells and the GSA IMEX-ARS3 MUSCL with $N = 2000$ cells at $t = 0.05$ for $\varepsilon = 0.3$ and at $t = 0.005$ for $\varepsilon = 0.03$. The values of the Mach number are $\varepsilon = 0.3$ (top) and $0.03$ (bottom).

Figure 2

See figure for legend.
Figure 3 Example 5. Density obtained with the GSA IMEX-ARS3 WCNS (left) and the GSA IMEX-ARS3 MUSCL (right) on a mesh 50 × 50 at t = 0.1. The values of the Mach number are ε = 1 (top) and 2 (bottom).

Figure 3

See figure for legend.
Figure 4

See figure for legend.
Figure 5

See figure for legend.

Figure 5

Example 6. Density (left) and divergence (right) calculated by the GSA IMEX-ARS3 WCNS on a mesh $50 \times 50$ at $t = 0.05$. 

Figure 6 Example 7. Vorticity $\omega = v_x - u_y$ obtained with the GSA IMEX-ARS3 WCNS on a mesh $50 \times 50$ at different times.

See figure for legend.
Figure 7

Example 7. Time history of $L_1$ error of the divergence of the velocity $\nabla \cdot \mathbf{u}$. 

See figure for legend.
Figure 8 Example 8. Vorticity $\omega = u_x - u_y$ obtained with the GSA IMEX-ARS3 WCNS on a mesh $50 \times 50$ at different times.

Figure 8

See figure for legend.
Figure 9 Example 8. Time history of $L_1$ error of the divergence of the velocity $\nabla \cdot \mathbf{u}$. 

Figure 9

See figure for legend.