Induced cosmological constant and other features of asymmetric brane embedding

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Abstract: We investigate the cosmological properties of an ‘induced gravity’ brane scenario in the absence of mirror symmetry with respect to the brane. We find that brane evolution can proceed along one of four distinct branches. By contrast, when mirror symmetry is imposed, only two branches exist, one of which represents the self-accelerating brane, while the other is the so-called normal branch. This model incorporates many of the well-known possibilities of brane cosmology including phantom acceleration ($w < -1$), self-acceleration, transient acceleration, quiescent singularities, and cosmic mimicry. Significantly, the absence of mirror symmetry also provides an interesting way of inducing a sufficiently small cosmological constant on the brane. A small (positive) $\Lambda$-term in this case is induced by a small asymmetry in the values of bulk fundamental constants on the two sides of the brane.

Keywords: extra dimensions, cosmology with extra dimensions.
1. Introduction

Extra-dimensional ‘braneworld’ models have attracted considerable attention in recent years. This is partly due to the fact that superstring/M-theory can only be consistently formulated in a universe which has more than four space-time dimensions. A distinctive feature of this theory is that it allows some of the extra dimensions to be large and even infinite thereby accommodating the braneworld scenario [1]. It is now well known that braneworld cosmologies can display quite distinctive behaviour which departs from that in general relativity either during early or late times [2]. The former can modify standard inflationary predictions for primordial fluctuations while the latter can cause late-time acceleration. This latter class of models will form the focus of our present paper. An abundance of recent cosmological observations points to a universe which is currently accelerating [3]. Although the source of cosmic acceleration remains unknown, most observations are well described by a small cosmological constant $\Lambda/8\pi G \simeq 10^{-47}$ GeV$^4$. Such a small value for the $\Lambda$-term is difficult to explain within the context of standard field theory, which typically predicts a value for $\Lambda$ which is several orders of magnitude larger than what is indicated by observations [4].

In this setting it is natural to ask whether cosmic acceleration could arise via an infrared modification of gravity at large distances. A famous example of such a scenario is the Dvali–Gabadadze–Porrati (DGP) model [5], which has been extensively discussed in many papers [2]. The present work shall examine a family of braneworld models which contain the DGP scenario as a subclass. As we shall show, this family has many interesting
features including that of phantom acceleration \((w_{\text{eff}} < -1)\) and the possibility of inducing a small cosmological constant on the brane through bulk effects.

Our braneworld scenario will contain a single large extra dimension. Two possibilities of principle exist in this case: either the bulk space is constrained to be symmetric with respect to the \(Z_2\) group of reflections relative to the brane, or such a symmetry is not imposed. The case where the bulk is symmetric is equivalent to the geometrical setting in which the brane is just a boundary of the bulk space; this is one way of justifying this mirror symmetry. An embedded brane without the \(Z_2\) symmetry is, however, a more general case with rich possibilities for cosmology. In spite of a considerable number of papers on this class of braneworld models (see, e.g., [6, 7, 8, 9]), some of these possibilities were either not noted or insufficiently studied. They form the subject of the present work.

We consider a braneworld model described by the following simple yet generic action, which includes gravitational and cosmological constants in the bulk \((B)\) and on the brane:

\[
S = \sum_{i=1,2} M_i^3 \left[ \int_{B_i} (R_i - 2\Lambda_i) - 2 \int_{\text{brane}} K_i \right] + \int_{\text{brane}} \left( m^2 R - 2\sigma \right) + \int_{\text{brane}} L(h_{ab}, \phi). \tag{1.1}
\]

Here, \(R_i\) is the scalar curvature of the five-dimensional metric \(g_{ab}^{(i)}\) on \(B_i, i = 1,2\), the two bulk spaces on either side of the brane, and \(R\) is the scalar curvature of the induced metric \(h_{ab}\) on the brane. The quantity \(K_i = K_{ab}^{(i)} h^{ab}\) is the trace of the symmetric tensor of extrinsic curvature \(K_{ab}^{(i)}\) of the brane in the space \(B_i\). The symbol \(L(h_{ab}, \phi)\) denotes the Lagrangian density of the four-dimensional matter fields \(\phi\) the dynamics of which is restricted to the brane so that they interact only with the induced metric \(h_{ab}\). All integrations over \(B_i\) and over the brane are taken with the corresponding natural volume elements. The symbols \(M_i, i = 1,2\), and \(m\) denote the Planck masses of the corresponding spaces, \(\Lambda_i, i = 1,2\), are the five-dimensional cosmological constants on the two side of the brane, and \(\sigma\) is the brane tension. We shall focus on the asymmetric case with \(\Lambda_1 \neq \Lambda_2\) and \(M_1 \neq M_2\) which appears to be preferable from a string-theory perspective. For instance, the dilaton stabilised in different vacuum states on adjacent sides of the brane would lead to an effective five-dimensional theory with \(M_1 \neq M_2\). The string landscape is likely to favour \(\Lambda_1 \neq \Lambda_2\), which also occurs in domain wall scenarios. (The Randall–Sundrum (RS) [10] and Dvali–Gabadadze–Porrati (DGP) [5] models can be derived from (1.1) when mirror symmetry is respected.)

In the absence of \(Z_2\) symmetry, cosmological evolution of the brane is described by

\[
H^2 + \frac{\kappa}{a^2} = \frac{\rho + \sigma}{3m^2} + \frac{1}{m^2} \sum_{i=1,2} \zeta_i M_i^3 \sqrt{H^2 + \frac{\kappa}{a^2} - \frac{\Lambda_i}{6} - \frac{C_i}{a^4}}, \tag{1.2}
\]

where \(\rho\) is the total energy density of matter on the brane, \(\zeta_i = \pm 1, i = 1,2\), correspond to the two possible ways of bounding each of the bulk spaces \(B_i, i = 1,2\), by the brane. We classify the resulting four branches according to the signs of \(\zeta_1\) and \(\zeta_2\) as \((+ +)\), \((+-)\), \((-+),\) or \((- -)\).\(^1\)

\(^1\)Note that, in the case of \(Z_2\) symmetry, there are only two ways of bounding the bulk by the brane, and these were called brane 1 and brane 2 in [11]. Of these, brane 2 contains the self-accelerating DGP brane as a subclass, while brane 1 can lead to phantom acceleration.
In the limit of $Z_2$ symmetry, the branches (---) and (++) become the normal branch (brane 1) and self-accelerating branch (brane 2), respectively. The other two so-called mixed branches are characterised by $\zeta_1 \zeta_2 = -1$. As we shall show in this paper, cosmology on these branches can be quite novel. For instance, a small asymmetry in the values of the bulk constants on adjacent sides of the brane can induce a cosmological constant on the brane itself. On the other hand, in the limit when the bulk constants become exactly equal, one obtains stealth branes [7] that obey the general-relativistic internal equations but do not affect the metric of the bulk space.

In this paper, we study the implications of (1.2) for a spatially flat universe ($\kappa = 0$) without dark radiation ($C_i = 0$, $i = 1, 2$). Equation (1.2) then simplifies to

$$H^2 = \frac{\rho + \sigma}{3m^2} + \frac{1}{m^2} \sum_{i=1,2} \zeta_i M_i^3 \sqrt{H^2 - \frac{\Lambda_i}{6}} = \frac{\rho + \sigma}{3m^2} + \sum_{i=1,2} \frac{\zeta_i}{\ell_i^3} \sqrt{H^2 + \lambda_i^{-2}},$$

(1.3)

where we have introduced the fundamental lengths

$$\ell_i = \frac{m^2}{M_i^3}, \quad \lambda_i = \sqrt{-\frac{6}{\Lambda_i}}, \quad i = 1, 2,$$

(1.4)

assuming negative values of the bulk cosmological constants.

Note that (1.3) can be rewritten in terms of an effective cosmological constant, $\Lambda_{\text{eff}}$, as

$$H^2 = \frac{\rho + \sigma}{3m^2} + \frac{\Lambda_{\text{eff}}}{3},$$

(1.5)

where

$$\frac{\Lambda_{\text{eff}}}{3} = \frac{\sigma}{3m^2} + \sum_{i=1,2} \frac{\zeta_i}{\ell_i} \sqrt{H^2 + \lambda_i^{-2}},$$

(1.6)

which will be useful to us when we study the cosmological properties of this braneworld later in this paper. A pictorial representation of the branches described by (1.3) is given in the appendix.

In this paper, we consider the late-time evolution of the universe, in which the energy density $\rho$ is dominated by matter with the equation of state $p = 0$. Then, introducing the cosmological parameters

$$\Omega_m = \frac{\rho_0}{3m^2 H_0^2}, \quad \Omega_\sigma = \frac{\sigma}{3m^2 H_0^2}, \quad \Omega_{\ell_i} = \ell_i^{-2} H_0^{-2}, \quad \Omega_{\lambda_i} = \lambda_i^{-2} H_0^{-2},$$

(1.7)

where $\rho_0$ and $H_0$ are the current values of the matter density and Hubble parameter, respectively, we rewrite (1.3) in terms of the cosmological redshift $z$:

$$h^2(z) \equiv \frac{H^2(z)}{H_0^2} = \Omega_m (1 + z)^3 + \Omega_\sigma + \sum_{i=1,2} \zeta_i \sqrt{\Omega_{\ell_i}} \sqrt{h^2(z) + \Omega_{\lambda_i}}.$$

(1.8)

This equation implicitly determines the function $h(z)$, and explicitly the inverse function $z(h)$. Note that the dimensionless cosmological parameters are related through the constraint equation

$$\Omega_m + \Omega_\sigma + \sum_{i=1,2} \zeta_i \sqrt{\Omega_{\ell_i}} \sqrt{1 + \Omega_{\lambda_i}} = 1.$$

(1.9)

We now proceed to describe some features of braneworld cosmology without $Z_2$ symmetry.
2. Induced cosmological constant on the brane

One way of accounting for cosmic acceleration within the framework of braneworld theory with mirror symmetry was suggested by Dvali, Gabadadze and Porrati [5]. An extension of this model to the case when mirror symmetry is absent is obtained by setting to zero the cosmological constants on the brane and in the bulk, so that $\sigma = 0$, $\Lambda_i = 0$, $i = 1, 2$. The expansion law (1.3) then simplifies to

$$H^2 - H \sum_{i=1,2} \frac{\zeta_i}{\ell_i} = \frac{\rho}{3m^2},$$

which evolves to a De Sitter limit at late times

$$\lim_{z \to -1} H(z) = H_{DS} = \sum_{i=1,2} \frac{\zeta_i}{\ell_i},$$

provided $\sum_{i=1,2} \zeta_i/\ell_i$ is positive, which is true for branches $(++)$ and $(+-)$, provided $M_1 > M_2$ in the latter case. If $m$ is of the order of the planck mass $M_P \simeq 10^{19}$ GeV, then the values of $M_i \sim 100$ MeV can explain the observed cosmic acceleration. [The self-accelerating DGP model corresponds to the $(++)$ branch with $\ell_1 = \ell_2$.]

The absence of mirror symmetry provides a new avenue for this mechanism. Specifically, the observed cosmic acceleration can be produced on one of the mixed branches with arbitrarily high values of the bulk Planck masses $M_1$ and $M_2$, provided these values are sufficiently close to each other. If $0 < \Delta M \equiv M_1 - M_2 \ll M_1$, then we have

$$H_{DS} = \frac{M_1^3 - M_2^3}{m^2} \approx \frac{3M_1^2}{m^2} \Delta M$$

on the $(+-)$ branch, and, by adjusting the value of $\Delta M$, one can always achieve an observationally suitable value of $H_{DS}$. For example, if $M_1, M_2 \sim m$, then one needs $\Delta M \sim H_0$.

The previous model gave one example of late-time acceleration in the absence of the (brane) cosmological constant. We now derive another model with the same property but with a more flexible assumption $\Lambda_i \neq 0$. Setting $\sigma = 0$ and $\Lambda_i \neq 0$ in (1.3) leads to

$$H^2 - \sum_{i=1,2} \frac{\zeta_i}{\ell_i} \sqrt{H^2 - \frac{\Lambda_i}{6}} = \frac{\rho}{3m^2},$$

which evolves to a different De Sitter limit, expressed by the equation

$$\lim_{z \to -1} H^2(z) = H^2_{DS} = \sum_{i=1,2} \frac{\zeta_i}{\ell_i} \sqrt{H^2_{DS} + \lambda_i^{-2}},$$

where the length scales $\ell_i$ and $\lambda_i$ are defined in (1.4).

It is interesting that a tiny asymmetry between the two bulk spaces can lead to a small cosmological constant being induced on the brane. Provided the bulk parameters $M_1$ and $M_2$ as well as $\Lambda_1$ and $\Lambda_2$ are close to each other, a neat cancelation on the right-hand
side of (2.5), which occurs for \( \zeta_1 \zeta_2 = -1 \), leads to a small value of \( H_{DS} \). Remarkably, this can happen even for very large values of the bulk constants. In particular, assuming that \( \lambda_i \ll H_{DS}^{-1} \), we have

\[
H_{DS}^2 \approx \left| \frac{1}{\ell_1 \lambda_1} - \frac{1}{\ell_2 \lambda_2} \right|
\]

for one of the mixed branches. Thus, for bulk parameters of the order of a TeV, \( M_i \sim 1 \text{ TeV} \), \( \lambda_i \sim 1 \text{ TeV}^{-1} \), we recover the current value of the Hubble parameter \( (H_{DS} \sim H_0) \) provided

\[
|\ell_1 \lambda_1 - \ell_2 \lambda_2|^{1/2} \sim 10^{-13} \text{ TeV}^{-1} \sim 10^{-30} \text{ cm}.
\]

Equations such as (2.3) or (2.6), (2.7) certainly represent fine tuning, with a tiny difference between bulk parameters only slightly breaking the smoothness of the metric across the brane. In the limit of exact equality of the bulk constants on the two sides of the brane, the branches with \( \zeta_1 \zeta_2 = -1 \) describe a smooth bulk space, and the brane approaches the limit of a stealth brane [7], evolving according to the usual Einstein equations without affecting the bulk space.

As we shall see in the next section, the cosmological scenario with an induced cosmological constant is distinguished by a property called *cosmic mimicry* which has interesting observational signatures.

3. Braneworld expansion can mimic \( \Lambda \)CDM

For large values of the bulk parameters, we encounter the phenomenon of *cosmic mimicry* which, in the context of \( Z_2 \) symmetry, was described in [12]. Note that, during the radiation and matter-dominated epochs, the expansion of the universe follows the general-relativistic prescription

\[
H^2 \approx \frac{\rho + \sigma}{3 m^2},
\]

where \( \sigma/m^2 \) plays the role of the cosmological constant on the brane. However, at very late times, cosmic expansion gets modified due to extra-dimensional effects. Indeed, if \( \lambda_i \ll H_0^{-1} \), then the square root in the last term of (1.3) can be expanded in the small parameter \( \lambda_i^2 H^2 \) at late times, and the braneworld expands according to \( \Lambda \)CDM, namely

\[
H^2 = \frac{8 \pi G \rho}{3} + \frac{\Lambda}{3}
\]

with

\[
8 \pi G = m^{-2} \left( 1 - \sum_{i=1,2} \frac{\zeta_i \lambda_i}{2 \ell_i} \right)^{-1},
\]

\[
\Lambda = \left( \frac{\sigma}{m^2} + \sum_{i=1,2} \frac{3 \zeta_i}{\ell_i \lambda_i} \right) \left( 1 - \sum_{i=1,2} \frac{\zeta_i \lambda_i}{2 \ell_i} \right)^{-1}.
\]

\(^2\)Perhaps, the small asymmetry in the fundamental constants characterising the bulk can be explained by the presence of the brane itself. For instance, the presence of the brane could lead to a small difference in the quantum contribution to the effective action of the bulk on its two sides, inducing slightly different bulk constants.
Note that both $G$ and $\Lambda$ are independent of time. Equations (3.2)–(3.4) have important ramifications. They inform us that the ‘bare’ value of the cosmological constant on the brane, $\sigma$, is ‘screened’ at late times by extra-dimensional effects resulting in its effective value $\Lambda$. Thus, the early-time and late-time values of the cosmological constant are likely to be different, and this makes our model open to verification.

Also note that one can have $\Lambda \neq 0$ even if $\sigma = 0$. Then, a small $\Lambda$-term can be induced during late-time evolution on the brane solely by extra-dimensional effects, as pointed out in the previous section. The mechanism by which the induced $\Lambda$-term becomes relatively small consists in a compensation of two potentially large terms with opposite signs in equation (3.4). Specifically, for small values of $\lambda_i$ (which correspond to large values of $\Lambda_i$) such that $\lambda_i/\ell_i \ll 1$, in the case $\sigma = 0$, we have, approximately,

$$\Lambda \approx \sum_{i=1,2} \frac{3\zeta_i}{\ell_i \lambda_i},$$

which is another form of the result (2.6) for one of the mixed branches. What is remarkable here is that a positive cosmological constant on the brane can be sourced by bulk cosmological constants which are negative.

From (3.2), (3.3) we also find that the effective gravitational constants during the early and late epochs are related by a multiplicative factor

$$1 - \sum_{i=1,2} \frac{\zeta_i \lambda_i}{2\ell_i},$$

which can be larger as well as smaller than unity, depending on the braneworld branch. This factor will be closer to unity for the mixed branches ($\zeta_1 \zeta_2 = -1$) than it is for the usual branches ($\zeta_1 \zeta_2 = 1$) which survive in the case of $Z_2$ symmetry.

Focussing on the important case where $\sigma = 0$ and the effective four-dimensional cosmological constant is induced entirely by five-dimensional effects, we find that, at redshifts significantly below the mimicry redshift

$$z_m \simeq \left(\frac{\Omega_{\lambda_1}}{\Omega_m}\right)^{1/3} - 1, \quad \Omega_{\lambda_1} \simeq \Omega_{\lambda_2},$$

the brane expansion mimics $\Lambda$CDM

$$h^2(z) = \Omega_m (1 + z)^3 + \Omega_\Lambda, \quad z \ll z_m,$$

with ‘screened’ values of the cosmological parameters:

$$\Omega_m = \Omega_m \left(1 - \sum_{i=1,2} \frac{\zeta_i}{2 \sqrt{\Omega_{\ell_i} / \Omega_{\lambda_i}}} \right)^{-1},$$

$$\Omega_\Lambda = \sum_{i=1,2} \zeta_i \sqrt{\Omega_{\ell_i} / \Omega_{\lambda_i}} \left(1 - \sum_{i=1,2} \frac{\zeta_i}{2 \sqrt{\Omega_{\ell_i} / \Omega_{\lambda_i}}} \right)^{-1}.$$
On the other hand, from (3.2) it follows that, at high redshifts, the universe expands as SCDM

\[ h^2(z) = \Omega_m(1 + z)^3, \quad z \gg z_m. \]  

An important distinguishing feature of this model is that the (screened) matter density, \( \tilde{\Omega}_m \), inferred via geometrical tests based on standard candles and rulers, may not match its (bare) dynamical value \( \Omega_m \). This allows cosmic mimicry to be distinguished from other cosmological scenarios by means of the Om diagnostic suggested in [13]. The fact that brane expansion also follows different laws at low and high redshift provides another important observational test of this model.

4. Phantom branes

In the presence of \( Z_2 \) symmetry, the brane1 branch of the generic model (1.2) exhibits phantom-like behaviour [11] which is in excellent agreement with observations [14] (see also [15]). Let us see whether this behaviour persists when mirror symmetry is absent. Note first that the condition for phantom acceleration, \( w(z) < -1 \), where

\[ w(z) = \frac{2q(z) - 1}{3[1 - \Omega_m(z)]}, \quad q(z) = \frac{d \log H(z)}{d \log(1 + z)} - 1, \quad \Omega_m(z) = \frac{\Omega_m(1 + z)^3}{h^2(z)}, \]  

has two equivalent formulations:

\[ \Omega_m(z) > \frac{2}{3} \frac{d \log H(z)}{d \log(1 + z)} \quad \text{and} \quad \dot{\Lambda}_{\text{eff}} > 0, \]  

where \( \Lambda_{\text{eff}} \) is the effective cosmological constant in (1.6), and differentiation is carried out with respect to the physical time variable. In the case of the \( (-\cdot- \) brane \( \zeta_1 = \zeta_2 = -1 \), one has

\[ \Lambda_{\text{eff}} = \frac{\sigma}{3m^2} - \sum_{i=1,2} \sqrt{\frac{H^2 + \lambda_i^{-2}}{\ell_i}}, \]  

and we find immediately that \( \Lambda_{\text{eff}} \) increases with time when the expansion rate, \( H \), decreases. It is also quite clear that one (and only one) of the mixed branches will necessarily have a negative value of the sum term in (1.6), again exhibiting phantom behaviour.

It is straightforward to verify that \( \dot{\Lambda}_{\text{eff}} > 0 \) and \( \dot{H} < 0 \) on the two branches exhibiting phantom behaviour. Differentiating (1.3) and (1.6), we find

\[ \dot{H} = -\frac{\rho}{m^2} \left[ 2 - \sum_{i=1,2} \frac{\zeta_i}{\ell_i \sqrt{H^2 + \lambda_i^{-2}}} \right]^{-1} < 0, \]  

\[ \dot{\Lambda}_{\text{eff}} = 3H\dot{H} \sum_{i=1,2} \frac{\zeta_i}{\ell_i \sqrt{H^2 + \lambda_i^{-2}}} > 0 \]  

for

\[ \sum_{i=1,2} \frac{\zeta_i}{\ell_i \sqrt{H^2 + \lambda_i^{-2}}} < 0. \]
Note that phantom models [16] with constant equation of state, \( w < -1 \), are marked by \( \dot{\Lambda} \,_{\text{eff}} > 0 \) and super-acceleration: \( \dot{H} > 0 \) at late times.\(^3\) This is related to the fact that the dark-energy (phantom) density in such models \textit{increases}, as the universe expands, according to

\[
\rho_{\text{phantom}} \propto a^{3|1+w|}, \quad w < -1, \tag{4.7}
\]

which causes the Hubble parameter to grow at late times, eventually leading to a Big-Rip singularity at which \( H \) diverges. By contrast, although the behaviour of our braneworld is phantom-like (\( w_{\text{eff}} < -1 \)), the universe never super-acCELERATES since \( \dot{H} < 0 \) always holds. Furthermore, since \( H \) decreases during expansion, a Big-Rip-type future singularity which plagues phantom cosmology is absent in the braneworld. From the definition

\[
q = -
\left(1 + \frac{\dot{H}}{H^2}\right) \tag{4.8}
\]

and property \( \dot{H} < 0 \), we find \( q > -1 \). In fact, the deceleration parameter in our model always remains larger than the De Sitter value of \( q = -1 \), approaching it only in the limit of \( t \to \infty \).

5. Transient acceleration

An important property of this class of braneworld models is that the current acceleration of the universe need not be eternal. In other words, for a specific relationship between the fundamental parameters in (1.3), the acceleration of the universe is a \textit{transient} phenomenon, and the universe reverts back to matter-dominated expansion in the future. Within the context of mirror symmetry, this scenario was called \textit{disappearing dark energy} and discussed in [11]. In the absence of mirror symmetry, it was studied in [8] under the name \textit{stealth-acceleration} (which should not be confused with the ‘stealth brane’ of [7]).

Transient acceleration implies the property \( H \to 0 \) in the asymptotic future, which requires the following condition to be satisfied:

\[
\frac{\sigma}{3m^2} + \sum_{i=1,2} \frac{\zeta_i}{\ell_i \lambda_i} = 0 \quad \Rightarrow \quad \Omega_{\sigma} + \sum_{i=1,2} \zeta_i \sqrt{\Omega_{\ell_i} \Omega_{\lambda_i}} = 0. \tag{5.1}
\]

On the (--) and (++) branches, this condition is realised with the following respective values of the brane tension:

\[
\frac{\sigma}{3m^2} = \pm \left( \frac{1}{\ell_1 \lambda_1} + \frac{1}{\ell_2 \lambda_2} \right) \quad \Rightarrow \quad \Omega_{\sigma} = \pm \left( \sqrt{\Omega_{\ell_1} \Omega_{\lambda_1}} + \sqrt{\Omega_{\ell_2} \Omega_{\lambda_2}} \right). \tag{5.2}
\]

\(^3\)It is easy to show that, in phantom models, the turning point \( \dot{H} = 0 \) occurs at

\[
1 + z_* \equiv \frac{a_0}{a(t_*)} = \left( \frac{1 - \Omega_m}{\Omega_m} \right)^{1/3|w|}, \quad w < -1. \tag{4.6}
\]
On the new mixed branches (−+) and ((−)), the required brane tension is smaller by absolute value:

\[
\frac{\sigma}{3m^2} = \pm \left| \frac{1}{\ell_1 \lambda_1} - \frac{1}{\ell_2 \lambda_2} \right| \quad \Rightarrow \quad \Omega_\sigma = \pm \left| \sqrt{\Omega_{\ell_1} \Omega_{\lambda_1}} - \sqrt{\Omega_{\ell_2} \Omega_{\lambda_2}} \right|.
\]  

(5.3)

Under constraint (5.1), the cosmological evolution equation (1.8) becomes

\[
h^2(z) = \Omega_m (1 + z)^3 + \sum_{i=1,2} \zeta_i \sqrt{\Omega_{\ell_i}} \left( \sqrt{h^2(z) + \Omega_{\lambda_i}} - \sqrt{\Omega_{\lambda_i}} \right).
\]  

(5.4)

Condition (5.1) is necessary but not sufficient to speak about transient acceleration on a particular branch. A distinguishing property of a transiently accelerating brane is that \(q(z) \to 0.5\) in the remote past \((10^5 \gg z \gg 1)\) as well as in the remote future \((z \to -1)\), reflecting the fact that the universe is matter dominated in the past and in the future, while, during the current phase, the deceleration parameter is negative, \(q_0 < 0\). This last condition is realised only if the cosmological expansion law \(H^2(\rho)\) is convex upwards to a sufficiently high degree. Specifically, in view of the second expression in (4.1), the condition \(q_0 < 0\) can be presented in the form

\[
\frac{dH^2(\rho_0)}{d\rho} < \frac{2H^2_0}{3\rho_0}.
\]  

(5.5)

Looking at figures 3 and 4 in the appendix, one can see that this property can be realised only on two of the four branches: on the (++) branch and on one of the mixed branches.

The expression for the current value of the deceleration parameter can be calculated by using the formula

\[
q_0 = \frac{3\Omega_m}{2 - \sum_i \zeta_i \sqrt{\frac{\Omega_{\ell_i}}{1 + \Omega_{\lambda_i}}} - 1}.
\]  

(5.6)

One should note that only four out of five \(\Omega\) parameters are independent in this expression because of the normalization condition \(h^2(0) = 1\) applied to the evolution equation (5.4). In the \(Z_2\)-symmetric case, there remain only two independent \(\Omega\) parameters. It is clear then that transient acceleration can be realised more easily in the \(Z_2\)-asymmetric case. This is illustrated in figures 1 and 2, which show the corresponding behaviour of the deceleration parameter \(q(z)\).

In a transiently accelerating universe, cosmic acceleration is sandwiched between two matter-dominated regimes. A transiently accelerating braneworld clearly does not possess the Big Rip of phantom cosmology, nor even the event horizon of De Sitter space! An in-depth study of this class of models [8] has revealed the existence of regions in parameter space which are stable (ghost-free).

We have demonstrated that it is possible to construct braneworld models with transient acceleration. What is less clear is whether such transiently accelerating branches will pass key cosmological tests based on observations of high-redshift type Ia supernovae, baryon acoustic oscillations, etc. This important issue is open for further study.
Figure 1: The deceleration parameter versus redshift is plotted for the $(++)$ branch in the case of $Z_2$ symmetry. The model has the parameters $\Omega_{\lambda_1} = \Omega_{\lambda_2} = 2$. We present plots for two different values of the matter density parameter: $\Omega_m = 0.3$ and $\Omega_m = 0.2$. The sets of other parameters are calculated to be $\Omega_{\ell_1} = \Omega_{\ell_2} = 1.21$, $\Omega_\sigma = -3.11$ and $\Omega_{\ell_1} = \Omega_{\ell_2} = 1.58$, $\Omega_\sigma = -3.56$, respectively. In this case, an accelerated regime is not realised, although deceleration is significantly slowed down at the present cosmological epoch.

Figure 2: The deceleration parameter versus redshift is plotted for the $(+-)$ branch in the case of absence of $Z_2$ symmetry. The model has the parameters $\Omega_{\lambda_1} = 2$ and $\Omega_{\lambda_2} = 2.1$. We present plots for two different sets of values of the remaining two independent parameters: $(\Omega_m, \Omega_{\ell_1}) = (0.3, 10000)$, which results in $(\Omega_{\ell_2}, \Omega_\sigma) = (9954.68, 3.16)$, and $(\Omega_m, \Omega_{\ell_1}) = (0.2, 5000)$, which results in $(\Omega_{\ell_2}, \Omega_\sigma) = (4840.15, 0.82)$. Both plots show acceleration at the present cosmological epoch, which generically becomes more prominent for lower values of $\Omega_m$.

6. Quiescent singularities

A new feature of brane cosmology is a possible presence of quiescent singularities at which
the density, pressure and expansion rate remain finite, while the deceleration parameter and the Kretchman invariant, $R_{ikln}R^{ikln}$, diverge [17]. The universe encounters such a singularity in the future if a point is reached during expansion where the derivative of $H^2$ with respect to $\rho$ goes to infinity or, equivalently, where the derivative of $\rho$ with respect to $H^2$ vanishes. Using (1.3), we can express this condition as the existence of a positive root $H^2_s$ of the equation

$$\sum_{i=1,2} \frac{\zeta_i}{\ell_i \sqrt{H^2_s + \lambda_i^{-2}}} = 2,$$

and a quiescent singularity is approached as $H \to H_s$. At this moment, expansion formally ceases, and one cannot extend the classical evolution of the brane beyond this point. Such a singular point obviously exists on the $(++)$ branch if and only if

$$\frac{\lambda_1}{\ell_1} + \frac{\lambda_2}{\ell_2} > 2 \quad \Rightarrow \quad \sqrt{\frac{\Omega_{\ell_1}}{\Omega_{\lambda_1}}} + \sqrt{\frac{\Omega_{\ell_2}}{\Omega_{\lambda_2}}} > 2,$$

and it is reachable on this branch if the brane tension $\sigma$ is sufficiently negative:

$$\frac{\sigma}{3m^2} < H^2_s - \sum_{i=1,2} \frac{1}{\ell_i} \sqrt{H^2_s + \lambda_i^{-2}} < 0,$$

or, equivalently,

$$\Omega_\sigma < \frac{H^2_s}{H^2_0} - \sum_{i=1,2} \sqrt{\Omega_{\ell_i}} \sqrt{\frac{H^2_s}{H^2_0} + \Omega_{\lambda_i}} < 0.$$

Condition (6.1) may or may not be realised on the mixed branches. For example, in the simplifying case $\ell_1 = \ell_2 = \ell$, condition (6.1) is realised on the mixed branch $(+-)$ provided $\lambda_1 > \lambda_2$. One can show that the values of the parameters $\ell_i$, $\lambda_i$, $i = 1, 2$, in principle can be chosen so that equation (6.1) has positive roots for three branches $(++)$, $(+-)$ and $(-+)$. To achieve this, one only needs to satisfy the conditions $\ell_1 > \ell_2$ and $\lambda_1 \ell_2 > \lambda_2 \ell_1$ and choose sufficiently small values of $\ell_1$, $\ell_2$.

For a graphical presentation of the reasons for the existence of quiescent singularities, the reader can look into the appendix. As in the case of mirror symmetry, quantum effects may play an important role in the vicinity of a quiescent singularity [18]; see also [19].

We also note that, in the case of mirror symmetry, realisation of quiescent singularity requires either negative brane tension or positive bulk cosmological constant (both conditions are suspicious from the viewpoint of possible instabilities). However the quiescent singularity can easily be realised without these assumptions in the asymmetric case on a mixed branch.

The presence of a quiescent singularity in the future of the cosmological evolution does not threaten the past cosmological scenario. Therefore, this issue, just like the issue of Big Rip of phantom cosmology, is mainly of academic interest. Here, we only wish to point out that the possibility of quiescent singularity can be realised rather easily in braneworld theory in certain domain of its parameters without any additional ingredients (such as phantom fields, which lead to Big Rip singularities).
7. Discussion

In this paper, we derive the expansion laws for an induced-gravity brane model in the general case where mirror ($Z_2$) symmetry of the bulk space with respect to the brane may or may not be present. We find that, depending upon the choice of brane embedding, cosmological expansion on the brane can proceed along four independent branches, two of which survive in the case of $Z_2$ symmetry. An important property of this class of models is that the four-dimensional gravitational and cosmological constants are effective quantities derivable from five-dimensional physics. In this case, brane expansion mimics $\Lambda$CDM at low redshifts, but the ‘screened’ matter density parameter $\tilde{\Omega}_m$ does not equal its bare (dynamical) value $\Omega_m$. This opens a new avenue for testing such models against observations (see [12, 13] in this respect). Another important property of these models would be the growth of density perturbations which is likely to differ from $\Lambda$CDM (see [20], [21] and references therein). This issue lies beyond the scope of the present paper but we may return to it in the future. Braneworld models can be phantom-like and also exhibit transient acceleration. Thus, brane phenomenology, with its basis in geometry, provides an interesting alternative to ‘physical’ dark energy scenario’s such as quintessence.

The stability issues of the class of braneworld models without $Z_2$ mirror symmetry were studied in [8, 9]. It is notable that ghost-free settings of the model with transient acceleration (and phantom-acceleration) appear to exist [8]. On the other hand, the analysis of the recent paper [9] reveals the presence of ghosts on a background with a De Sitter vacuum brane on the three branches $(++)$, $(+-)$, $(-+)$ (i.e., which have at least one ‘+’, so that the bulk at least on one side of the brane has ‘infinite volume’ in terminology of [9]). Whether this situation is critical for the cosmology under investigation remains to be seen. In this connection, it should be noted that the $(++)$ branch, surviving in the $Z_2$ symmetric case, contains a ghost and is, therefore, linearly unstable [22]. On the other hand, the $(-–)$ branch is ghost-free in the $Z_2$ symmetric case and, apparently, also in the general case without $Z_2$ symmetry. As we have demonstrated in this paper, this branch is responsible for ‘phantom acceleration’ ($w_0 < -1$); also see [11, 14, 15].

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Appendix

A. Visual representation of brane evolution

The variables $\{X, Y\}$:

$$X \equiv \frac{\rho_{\text{tot}}}{3m^2} - H^2, \quad Y \equiv H^2, \quad \rho_{\text{tot}} = \rho + \sigma,$$

(A.1)
allow us to rewrite equation (1.3) in the form

\[ X = - \sum_{i=1,2} \frac{\zeta_i}{\ell_i} \sqrt{Y + \lambda_i^{-2}}, \]  

(A.2)

which has a convenient visual interpretation. Equation (A.2) describes four branches in the physically restricted range \( Y \geq 0 \), with the symmetry of reflection with respect to the \( Y \) axis. If there exists a positive root \( Y_c \) of the right-hand side of (A.2) for \( \zeta_1 \zeta_2 = -1 \), then the two mixed branches intersect each other at the point

\[ Y_c = H_c^2 = \frac{\ell_2^2 \lambda_2^{-2} - \ell_1^2 \lambda_1^{-2}}{\ell_2^2 - \ell_1^2}. \]  

(A.3)

The condition for the existence of this intersection point is that the constant on the right-hand side of (A.3) be positive.

The four branches (with and without intersection) are shown in Figs. 3 and 4, respectively.

Figure 3: Four branches described by Eq. (A.2) in the \((X, Y)\) plane and in the \((\rho_{\text{tot}}, H^2)\) plane in the case where two of them intersect. The horizontal dotted line indicates the position of the \( H^2 = 0 \) axis in the case \( \Lambda_1 \Lambda_2 = 0 \). The region below this axis is nonphysical.

The brane evolves along one of the four branches towards decreasing values of \( \rho_{\text{tot}} \) (during expansion). Depending (in particular) on the value of the brane tension \( \sigma \), three distinct possibilities can arise:

(i) The trajectory may reach the value of \( H = 0 \), after which the universe recollapses and evolves along the same branch in the opposite direction. This happens when the value of \( \rho_{\text{tot}} \) at this point is greater than its minimum value \( \sigma \).

(ii) The trajectory may asymptotically tend to either de Sitter space or the Minkowski universe with the minimum value \( \rho_{\text{tot}} = \sigma \). The second possibility occurs when the minimum value of \( \rho_{\text{tot}} = \sigma \) is exactly the point where the corresponding graph crosses the axis \( H^2 = 0 \), which, therefore, requires some amount of fine tuning. This possibility can be realised as transient acceleration.
Figure 4: Four branches described by Eq. (A.2) in the $(X, Y)$ plane and in the $(\rho_{\text{tot}}, H^2)$ plane in the case of absence of intersection. The horizontal dotted line indicates the position of the $H^2 = 0$ axis in the case $\Lambda_1\Lambda_2 = 0$. The region below this axis is nonphysical.

(iii) The trajectory may end in a quiescent singularity at a finite value of $H$. This happens when the critical minimum point of $\rho_{\text{tot}}$ on the evolution curve is reached, and if this value of $\rho_{\text{tot}}$ is greater than its minimum value $\sigma$. The reasons for the existence of quiescent singularities can be seen from the right panels in Figs. 3 and 4. They occur at the points of infinite derivative $dH^2/d\rho_{\text{tot}}$.

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