Revision of the fractional exclusion statistics

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Abstract. - I discuss the concept of fractional exclusion statistics (FES) and I show that in order to preserve the thermodynamic consistency of the formalism, the exclusion statistics parameters should change if the species of particles in the system are divided into subspecies. Using a simple and intuitive model I deduce the general equations that have to be obeyed by the exclusion statistics parameters in any FES system.

Introduction. – In Ref. [1] Haldane introduced the fruitful concept of fractional exclusion statistics (FES). Although many authors analyzed the physical properties of FES systems and the microscopic reasons for the manifestation of this type of statistics (see [2–15] and references therein, just as examples), there are important properties that have been overlooked. In Ref. [16] I proved that if the mutual exclusion statistics parameters (see below the definitions) are defined in the typical way (e.g. like in [1, 2]), then the thermodynamics of the system is inconsistent. To restore the thermodynamics, I conjectured in the same paper that any of the mutual exclusion statistics parameters based on simple, general arguments and I show that the conjectures introduced in [16] are, simply, necessary conditions for the logical consistency of the formalism. This is not surprising, since the inconsistency of the thermodynamics proved in Ref. [16] could have been only a consequence of an inconsistent underlying physical model.

In this letter I analyze the basic properties of the mutual exclusion statistics parameters based on simple, general arguments and I show that the conjectures introduced in [16] are, simply, necessary conditions for the logical consistency of the formalism. This is not surprising, since the inconsistency of the thermodynamics proved in Ref. [16] could have been only a consequence of an inconsistent underlying physical model.

A simple model. – Let us assume that we have a system formed of only two species of particles, 0 and 1, like in Fig. 1. We denote the exclusion statistics parameters of this system by \(\tilde{\alpha}_{00}, \tilde{\alpha}_{01}, \tilde{\alpha}_{10}\) and \(\tilde{\alpha}_{11}\), and we start in the standard way [1, 2] by writing the total number of configurations corresponding to \(N_0\) particles of species 0 and \(N_1\) particles of species 1 as

\[
W_{(0,1)} = \prod_i \frac{\left\{ \begin{array}{l} 0,1 \\ \end{array} \right\}_{i} \left( G_i + N_i - 1 - \sum_{j}^{0,1} \tilde{\alpha}_{ij}(N_j - \delta_{ij}) \right)!}{N_i! \left( G_i - 1 - \sum_{j}^{0,1} \tilde{\alpha}_{ij}(N_j - \delta_{ij}) \right)!},
\]

where \(G_0\) and \(G_1\) are the number of single-particle states corresponding to the two species of particles. We recall here that the physical interpretation of the exclusion statistics parameters is that at the variations \(\delta N_0\) and \(\delta N_1\) of the particle numbers \(N_0\) and \(N_1\), the number of single-particle states available for the two species changes by \(\delta G_0 = -\tilde{\alpha}_{00}\delta N_0 - \tilde{\alpha}_{01}\delta N_1\) and \(\delta G_1 = -\tilde{\alpha}_{10}\delta N_0 - \tilde{\alpha}_{11}\delta N_1\).

If all the \(G_0\) states have the same energy, say \(\epsilon_0\), and all the \(G_1\) states have the energy \(\epsilon_1\), we may write the grandcanonical partition function of the system as

\[
Z_{(0,1)} = W_{(0,1)} \prod_i e^{\beta N_i (\mu_i - \epsilon_i)},
\]

where \(\beta \equiv (k_B T)^{-1}\), \(\mu_0\) and \(\mu_1\) are the chemical potentials of the two species of particles, and \(T\) is the temperature, common to both species.

To calculate the thermodynamics of the system, we assume that all the numbers involved in our problem are very big, i.e. \(N_0, N_1, G_0 - 1 + \tilde{\alpha}_{00} - \tilde{\alpha}_{01}N_0 - \tilde{\alpha}_{10}N_1\), and \(G_1 - 1 + \tilde{\alpha}_{10} - \tilde{\alpha}_{11}N_0 - \tilde{\alpha}_{11}N_1\) are much bigger than 1. Maximizing \(Z\)–by calculating its logarithm and using the Stirling approximation–we obtain the maximum probability populations, which are given by the system of equa-
by setting $\delta N_1 = \delta N_{10} = \delta N_1 = 0$, $\delta N_0 = \delta N_{10} = 0$ or $\delta N_0 = \delta N_{11} = 0$. Then, writing the variation of $G_1$ as $\delta G_1 = \delta G_{10} + \delta G_{11}$, and using the general expressions $\delta G_1 = -\tilde{\alpha}_{10}\delta N_0 - \tilde{\alpha}_{11}\delta (\delta N_0 + \delta N_{11})$, $\delta G_{10} = -\tilde{\alpha}_{10}\delta N_0 - \tilde{\alpha}_{10}\delta N_{10} - \tilde{\alpha}_{10}\delta N_{11}$, and $\delta G_{11} = -\tilde{\alpha}_{11}\delta N_0 - \tilde{\alpha}_{11}\delta N_{10} - \tilde{\alpha}_{11}\delta N_{11}$, we obtain

$$\tilde{\alpha}_{10} = \tilde{\alpha}_{10} + \tilde{\alpha}_{11},$$
$$\tilde{\alpha}_{11} = \tilde{\alpha}_{10} + \tilde{\alpha}_{11},$$
$$\tilde{\alpha}_{11} = \tilde{\alpha}_{10} + \tilde{\alpha}_{11},$$

by setting the independent fluctuations $\delta N_0$, $\delta N_{10}$, and $\delta N_{11}$ to zero in proper order.

Now we write the total number of configurations in the system, considering species 1 and 1 as distinct,

$$W_{(0,1,1)} = \prod_i \frac{(1 + \alpha)_{N_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})}}{N_i! \left(1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})\right)!}$$

and we compare $\log W_{(0,1)}$ and $\log W_{(0,1,1)}$, within the approximation of large numbers.

After some obvious simplifications, we obtain

$$\log W_{0,1} = (F_0 + N_0) \log F_0 - F_0 \log F_0 - F_1 \log F_1 - N_0 \log N_0 - N_1 \log N_1,$$
$$\log W_{0,1,1} = (F_0' + N_0) \log F_0' - F_0' \log F_0' - F_1 \log F_1 - N_0 \log N_0 - N_1 \log N_1,$$

with

$$F_0 = G_0 + N_0 - 1 + \tilde{\alpha}_{00} - \tilde{\alpha}_{00}N_0 - \tilde{\alpha}_{01}N_1,$$  
$$F_1 = G_1 + N_1 - 1 + \tilde{\alpha}_{10} - \tilde{\alpha}_{10}N_0 - \tilde{\alpha}_{11}N_1,$$  
$$F_0' = G_0 + N_0 - 1 + \tilde{\alpha}_{00} - \tilde{\alpha}_{00}N_0 - \tilde{\alpha}_{01}N_1,$$  
$$F_1' = G_1 + N_1 - 1 + \tilde{\alpha}_{10} - \tilde{\alpha}_{10}N_0 - \tilde{\alpha}_{11}N_1,$$  

But using Eqs. (4) and (5), one can easily show that

$$F_0 = F_0'$$

Now notice that if $M$ is a big number and $c$ is a number between 0 and 1, then

$$M \log M - c M \log (1 - c) M - (1 - c) M \log (1 - c) M = -c \log c \log (1 - c) \log M = O(\log M) \ll 1.$$
Therefore from Eqs. (9) and (10) we obtain that

$$\frac{\log W_{(0,1)} - \log W_{(0,1,0)}}{\log W_{(0,1)}} = O(\log^{-1} N) \ll 1,$$

(11)

where $N$ is a number comparable to $N_0$ and $N_1$. So indeed, as mentioned in the beginning of this subsection, in the limit of large numbers the splitting of the systems species into sub-species does not change the thermodynamics of the system, provided that the consistency conditions (5) are imposed on the $\alpha$.

Now let us compare the equilibrium distributions of particles in the two descriptions of the system. If we maximize the partition function $Z_{(0,1,0)} = W_{(0,1,0)} \prod_{i=0}^{1} e^{\beta N_i (\mu_i - \epsilon_i)}$, with respect to the populations we obtain the new system of equations

$$e^{\beta (\epsilon_0 - \mu_0)} = (1 + w_0') \left( \frac{w_0'}{1 + w_0'} \right)^{\tilde{\alpha}_{00}} \left( \frac{w_{10}}{1 + w_{10}} \right)^{\tilde{\alpha}_{100}},$$

(12a)

$$e^{\beta (\epsilon_1 - \mu_1)} = (1 + w_{10}) \left( \frac{w_0'}{1 + w_0'} \right)^{\tilde{\alpha}_{01}} \left( \frac{w_{10}}{1 + w_{10}} \right)^{\tilde{\alpha}_{101}}$$

(12b)

$$e^{\beta (\epsilon_1 - \mu_1)} = (1 + w_{11}) \left( \frac{w_0'}{1 + w_0'} \right)^{\tilde{\alpha}_{11}} \left( \frac{w_{11}}{1 + w_{11}} \right)^{\tilde{\alpha}_{11}},$$

(12c)

where $G_0 = w_0' N_0 + \tilde{\alpha}_{00} N_0 + \tilde{\alpha}_{01} N_{10} + \tilde{\alpha}_{01} N_{11}$, $G_1 = w_1' N_{10} + \tilde{\alpha}_{10} N_0 + \tilde{\alpha}_{11} N_{10} + \tilde{\alpha}_{11} N_{11}$, $G_{11} = w_{11} N_{10} + \tilde{\alpha}_{11} N_0 + \tilde{\alpha}_{11} N_{10} + \tilde{\alpha}_{11} N_{11}$.

The “extensivity” of the mutual exclusion statistics parameters. Notice that the property (14b) of the mutual exclusion statistics parameters is satisfied for a given pair of species, $i$ and $j$, $i \neq j$, if $\tilde{\alpha}_{ij}$ satisfy the relation

$$\tilde{\alpha}_{ij}/G_j = \tilde{\alpha}_{ij} N_j + \tilde{\alpha}_{ij} N_j + \ldots = \alpha_{ij},$$

(15)

for any division of the space $G_j$, where $\alpha_{ij}$ is a constant for the pair $(i, j)$. In such a situation $\tilde{\alpha}_{ij}$ is proportional to the dimension of the space on which it acts $– G_j$ and $G_j$, in Eq. (15); we say $\tilde{\alpha}_{ij}$ “extensive” [16].

Let us assume that for a given system, we can find a fine enough division into species, such that the extensivity condition (15) is satisfied. Therefore we can write

$$\tilde{\alpha}_{ij} = G_i \alpha_{ij},$$

(16)

and we apply the general formalism introduced in Ref. [16]. The populations of the single-particle levels are given by the set of equations

$$\beta (\mu_i - \epsilon_i) + \ln \left[ \frac{1 + n_i}{n_i} \right] = \sum_{j(\neq i)} G_{1j} \ln [1 + n_j] \alpha_{ji},$$

(17)

where $\mu_i$ and $\epsilon_i$, are the chemical potential and the energy level of species $i$ ($i = 0, 1, \ldots$).

Some care should be taken with Eq. (17), since species $i$ of the l.h.s may be divided into sub-species and this would modify both sides of the equation. Therefore Eq. (17) is applicable without any ambiguities in the limit in which the subspecies $i$ is sufficiently small, so that further division would not modify the equation significantly. Nevertheless, in the thermodynamic (quasi-continuous) limit the summations are transformed into integrals and we obtain the integral equation

$$\beta (\mu_i - \epsilon_i) + \ln \left[ \frac{1 + n_i}{n_i} \right] = \int \sigma_j \ln [1 + n_j] \alpha_{jj} d_j.$$
Conclusions. – In this letter I deduced the general conditions necessary for the consistency of the fractional exclusion statistics (FES) formalism. In accordance with Refs. [16–18], I showed that the exclusion statistics parameters, $\alpha_{ij}$, are not constants, but they change with the species of particles in the system. The consistency conditions on $\alpha$s are given as Eqs. (14).

A particular case for which Eqs. (14) are satisfied is when the mutual exclusion statistics parameters are proportional to the dimension of the space on which they act (see Eq. 15), as conjectured in Ref. [16]. One can eventually find in a physical system a fine enough coarse-graining for which Eq. (15) is satisfied; in such a case the most probable particle occupation numbers are given by Eqs. (17) or (18).

In Ref. [17] I showed that general systems of interacting particles may be described as ideal systems with FES. The exclusion statistics parameters were calculated and it was proven that the mutual parameters obey Eq. (15) mentioned above.

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