We develop a variational approach in order to study qualitative properties of nonautonomous parabolic equations. Based on the method of product integrals, we discuss invariance properties and ultracontractivity of evolution families in Hilbert space. Our main results give sufficient conditions for the heat kernel of the evolution family to satisfy Gaussian-type bounds. Along the way, we study examples of nonautonomous equations on graphs, metric graphs, and domains.

**KEYWORDS**

evolution families, kernel estimates, nonautonomous parabolic problems

**MSC CLASSIFICATION**

47D06; 47D07; 47A07; 35K90

1 | INTRODUCTION

Nonautonomous evolution equations are partial differential equations in which relevant coefficients of the differential operator and/or in the boundary conditions are time dependent, thus allowing for underlying models that are variable over time.

In the autonomous case (ie, evolution equations with time-independent coefficients), well-posedness is known to be equivalent to generation of a semigroup in a suitable Banach space; in comparison, the theory for well-posedness of nonautonomous problems on general Banach spaces is more rudimentary. If the coefficients of a nonautonomous equation are piecewise constant, then one may find a solution by following the orbit of the semigroup governing a given problem as long as the coefficient stays constant; then “freeze” the system; use the final state as an initial condition for a new evolution equation with new (constant) coefficient, and so on: This boils down to consider the composition of a finite numbers of semigroups.

A theory originally developed by J.-L. Lions shows that well-posedness in Hilbert space can be proved under much weaker assumptions, most notably mere measurability of the time dependence, provided the problem has a nice variational structure: This is typically the case if the differential equation is parabolic. By adapting the setting of (time-independent) bounded elliptic forms, it is thus possible to show that the equation has a solution that is, in particular, continuous in time for a precise statement. This motivates the study of **nonautonomous forms**, a topic that has
received much attention in the last decade: We mention among others.\textsuperscript{1-5} All these articles are chiefly devoted to study properties of solutions of partial differential equations, with a focus on maximal regularity issues and hence allowing for inhomogeneous terms.

Our main aim in this paper is to develop an abstract theory with a more operator-theoretical flavor. Indeed, Lions' result paves the way to the possibility of defining an evolution family (or evolution system, or propagator), ie, a family of operators $U(\cdot, s)$ mapping each initial data

$$u(s) = x \in H$$

to the orbit of the solution of

$$\dot{u}(t) + A(t)u(t) = 0 \text{ a.e. on } [s, T].$$

Because the initial condition may well be imposed at instants $s \neq 0$, this actually defines a two-parameter family

$$U : = (U(t, s))_{(t, s) \in \Delta}$$

of bounded linear operators on $H$ by $U(t, s)x := u(t)$, where $\Delta := \{(t, s) \in (0, T)^2 : s < t\}$. Some good compendia on such evolution families are Tanabe,\textsuperscript{6} Chapt. 7 ; Pazy,\textsuperscript{7} Chapt. 5 ; Fattorini,\textsuperscript{8} Chapt. 7 ; Engel and Nagel,\textsuperscript{9} Section VI 9 or the monograph.\textsuperscript{10}

The tumultuous development of Hilbert space methods, and especially the theory of Dirichlet forms, have been fruitful also in the nonautonomous environment: A theory of nonautonomous Dirichlet forms has been recently introduced in Arnedt et al.\textsuperscript{11} If $A(t) \equiv A$, the above abstract Cauchy problem is autonomous and its solution is simply given by

$$u(t) = U(t, s)x := e^{-(t-s)A}x.$$

Hence, the findings in Arnedt et al\textsuperscript{11} can be regarded as a strict generalization of the classical theory of Markovian operators and Dirichlet forms represented, eg, in Fukushima et al.\textsuperscript{12} Our goal is to complement these results, thus setting up a nonautonomous variational program analogous to the autonomous one outlined in classical monographs like Ouhabaz\textsuperscript{13}.

Among other things, we study extrapolation to $L^p$-spaces, ultracontractivity, or Gaussian-type bounds on integral kernels of evolution families. It should be mentioned that ultracontractivity and kernel estimates have been observed already in Aronson\textsuperscript{14} and Daners\textsuperscript{15} for specific instances of parabolic nonautonomous equations; in particular, Aronson\textsuperscript{16} observed that the fundamental solution $(t, s; x, y) \mapsto \Gamma(t, s; x, y)$ of a certain class of nonautonomous diffusion equations in (domains of) $\mathbb{R}^d$ satisfies

$$\Gamma(t, s; x, y) \leq K G(t - s; x - y),$$

where $(t, x) \mapsto G(t, x)$ is the Gaussian kernel that yields the fundamental solution of the (autonomous) heat equation on $\mathbb{R}^d$. Analogous Gaussian bounds have ever since been proved for integral kernels of semigroups generated by large classes of second-order elliptic operators, possibly with complex coefficients\textsuperscript{17}; In the nonautonomous case, Aronson's original findings have been extended to operators on domains under Dirichlet or Robin boundary conditions in Aronson\textsuperscript{14} and Daners.\textsuperscript{15} In this paper, we are going to introduce a general approach, based on the so-called Davies' Trick, to prove Gaussian bounds for heat kernels of evolution families that govern nonautonomous parabolic equations. Inspired by some techniques introduced in Daners\textsuperscript{15} and Ouhabaz,\textsuperscript{17} we show the applicability of our methods by showing that a large class of elliptic operators with complex-valued, bounded measurable (both in time and space) coefficients are associated with evolution families that satisfy Gaussian bounds, thus extending the main results in Ouhabaz\textsuperscript{17} to the nonautonomous setting.

Our approach will heavily rely upon the method of product integrals, whose historical evolution is thoroughly discussed in previous works\textsuperscript{8,18, §7-10} and whose scope has been extended to nonautonomous forms with measurable dependence on time in Sani and Laasri\textsuperscript{19} and El-Menouai and Laasri.\textsuperscript{20} We adapt it to our present setting, thus deriving in Theorem 2.2 a version that we will use over and over again in different contexts throughout this paper.

The present paper is organized as follows. After describing our mathematical framework in Section 2, in Section 3, we present sufficient conditions that enforce qualitative properties based on the lattice structure of $L^2$-spaces, including stochasticity or domination. Also, we are able to discuss cases where evolution families on $L^2$-spaces extrapolate to further $L^p$-spaces. This is a key feature of the theory of Dirichlet forms in the autonomous case and shows the flexibility of the Hilbert space approach in the nonautonomous context, too. Even evolution equations on structures that change over time can be studied by means of our theory.

Gaussian-type bounds are shown to depend on ultracontractivity properties of certain operator families related to $U$. This approach requires, in turn, suitable common bounds in $L^p$-norm, uniformly on all compact subsets of $\Delta$. Inspired
by similar criteria in the autonomous setting, we show that efficient conditions based on Sobolev-type inequalities can enforce such bounds. In Section 4, we develop a theory of ultracontractive evolution families: A technical difficulty we face is related to the failure of self-adjointness of evolution families, a phenomenon that typically occurs even when all operators \( A(t) \) are self-adjoint. We take over an idea from Daners\(^{15} \) and circumvent this problem by studying some nonautonomous form associated with a tightly related backward evolution equation.

It has been known since Davies\(^{21} \) that ultracontractivity is an important ingredient to prove Gaussian bounds for semigroups. In Section 5, we are going to present different sufficient conditions under which an evolution family satisfies nonautonomous form associated with a tightly related backward equation.

Throughout this paper, \( H \) is a separable, complex Hilbert space, and \( V \) is a further complex Hilbert space that is densely and continuously embedded into \( H \). Let \( V' \) denote the antidual of \( V \) with respect to the pivot space \( H \); the duality between \( V \) and \( V' \) is denoted by \( < \cdot, \cdot > \). We also denote by \( ( \cdot | \cdot )_V \) and \( \| \cdot \|_V \) the scalar product and the norm on \( V \), respectively, and by \( ( \cdot | \cdot ) \) and \( \| \cdot \| \) the corresponding quantities in \( H \).

We fix \( T \in [0, \infty) \) and consider a time-dependent family \((a(t))_{t \in [0, T]}\) of mappings such that \( a(t; \cdot, \cdot) : V \times V \to \mathbb{C} \) is for all \( t \in [0, T] \) a sesquilinear form and

\[
[0, T] \ni t \mapsto a(t; u, v) \in \mathbb{C} \quad \text{is measurable} \quad \text{for all} \quad u, v \in V; \tag{2.1}
\]

and such that furthermore there exist constants \( M, \alpha > 0 \) and \( \omega \geq 0 \) such that the boundedness and \( H \)-ellipticity estimates

\[
|a(t; u, v)| \leq M\|u\|_V\|v\|_V \quad \text{for a.e.} \quad t \in [0, T] \quad \text{and all} \quad u, v \in V, \tag{2.2}
\]

\[
\text{Re} a(t; u, u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad \text{for a.e.} \quad t \in [0, T] \quad \text{and all} \quad u \in V, \tag{2.3}
\]

hold. In what follows, we call such a family \( a := (a(t))_{t \in [0, T]} \) bounded \( H \)-elliptic nonautonomous form: following Arendt and Dier,\(^1 \) we denote by \( \text{FORM}([0, T]; V, H) \) the class of all such forms.

By the Lax-Milgram theorem, for each \( t \in [0, T] \), there exists an operator associated with \( a(t; \cdot, \cdot) \), ie, an isomorphism \( A(t) : V \to V' \) such that

\[
\langle A(t)u, v \rangle = a(t, u, v) \quad \text{for all} \quad u, v \in V:
\]

accordingly, we refer to the family \((A(t))_{t \in [0, T]}\) as the operator family associated with \( a := (a(t))_{t \in [0, T]} \).

Regarded as an unbounded operator with domain \( V \), \(-A(t)\) generates a holomorphic semigroup on \( V' \), and in fact by Arendt\(^{25, \text{Thm. 7.1.5}} \) on \( H \) too, since \( a(t) \) is for all \( t \) a bounded, \( H \)-elliptic sesquilinear form: With an abuse of notation, we denote its generator – the part of \(-A(t)\) in \( H \) – again by \(-A(t)\), and the semigroup by

\[
T_f := \{ e^{-rA(t)} : r \geq 0 \}.
\]

Hence, for each fixed \( t, s \in [0, T] \), the Cauchy problem

\[
\dot{u}(r) + A(t)u(r) = 0, \quad r \in [s, T], \\
u(s) = x \in H,
\]

2 | EVOLUTION FAMILIES: NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, \( H \) is a separable, complex Hilbert space, and \( V \) is a further complex Hilbert space that is densely and continuously embedded into \( H \). Let \( V' \) denote the antidual of \( V \) with respect to the pivot space \( H \); the duality between \( V \) and \( V' \) is denoted by \( < \cdot, \cdot > \). We also denote by \( ( \cdot | \cdot )_V \) and \( \| \cdot \|_V \) the scalar product and the norm on \( V \), respectively, and by \( ( \cdot | \cdot ) \) and \( \| \cdot \| \) the corresponding quantities in \( H \).

We fix \( T \in [0, \infty) \) and consider a time-dependent family \((a(t))_{t \in [0, T]}\) of mappings such that \( a(t; \cdot, \cdot) : V \times V \to \mathbb{C} \) is for all \( t \in [0, T] \) a sesquilinear form and

\[
[0, T] \ni t \mapsto a(t; u, v) \in \mathbb{C} \quad \text{is measurable} \quad \text{for all} \quad u, v \in V; \tag{2.1}
\]

and such that furthermore there exist constants \( M, \alpha > 0 \) and \( \omega \geq 0 \) such that the boundedness and \( H \)-ellipticity estimates

\[
|a(t; u, v)| \leq M\|u\|_V\|v\|_V \quad \text{for a.e.} \quad t \in [0, T] \quad \text{and all} \quad u, v \in V, \tag{2.2}
\]

\[
\text{Re} a(t; u, u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad \text{for a.e.} \quad t \in [0, T] \quad \text{and all} \quad u \in V, \tag{2.3}
\]

hold. In what follows, we call such a family \( a := (a(t))_{t \in [0, T]} \) bounded \( H \)-elliptic nonautonomous form: following Arendt and Dier,\(^1 \) we denote by \( \text{FORM}([0, T]; V, H) \) the class of all such forms.

By the Lax-Milgram theorem, for each \( t \in [0, T] \), there exists an operator associated with \( a(t; \cdot, \cdot) \), ie, an isomorphism \( A(t) : V \to V' \) such that

\[
\langle A(t)u, v \rangle = a(t, u, v) \quad \text{for all} \quad u, v \in V:
\]

accordingly, we refer to the family \((A(t))_{t \in [0, T]}\) as the operator family associated with \( a := (a(t))_{t \in [0, T]} \).

Regarded as an unbounded operator with domain \( V \), \(-A(t)\) generates a holomorphic semigroup on \( V' \), and in fact by Arendt\(^{25, \text{Thm. 7.1.5}} \) on \( H \) too, since \( a(t) \) is for all \( t \) a bounded, \( H \)-elliptic sesquilinear form: With an abuse of notation, we denote its generator – the part of \(-A(t)\) in \( H \) – again by \(-A(t)\), and the semigroup by

\[
T_f := \{ e^{-rA(t)} : r \geq 0 \}.
\]

Hence, for each fixed \( t, s \in [0, T] \), the Cauchy problem

\[
\dot{u}(r) + A(t)u(r) = 0, \quad r \in [s, T], \\
u(s) = x \in H,
\]
is well-posed, its solution being given by \( u(r,x) := e^{-(r-s)A(t)}x \). However, we are rather going to focus on the nonautonomous Cauchy problem

\[
\begin{align*}
\dot{u}(t) &+ A(t)u(t) = 0, & t \in [s, T], \\
u(s) &= x \in H.
\end{align*}
\] (2.4)

In order to introduce the main objects of our investigations, let \( a \in \text{FORM}([0, T]; V, H) \). A classical well-posedness theorem by J.-L. Lions\textsuperscript{26, §XVIII.3.2–3} states that for each \( s \in [0, T] \) and each \( x \in H(2.4) \) admits a unique solution \( u \) in the maximal regularity space

\[
\text{MR}(V, V') := \text{MR}(s; T; V, V') := L^2(s; T; V) \cap H^1(s, T; V').
\]

It is well-known that \( \text{MR}(V, V') \) is continuously embedded into \( C([s, T]; H) \), see, e.g., Showalter\textsuperscript{27, Prop. III.1.2}: This allows us to introduce a family of linear operators by

\[
U(t,s) : H \ni x \mapsto U(t,s)x := u(t) \in H, \quad (t,s) \in \Delta,
\] (2.5)

where here and in the following, we adopt the notation

\[
\Delta := \{(t,s) \in (0,T)^2 : s < t\}
\]

and \( u \) is the unique solution of the Cauchy problem (2.4) in \( \text{MR}(V, V') \). Letting \( X := V' \) and \( D := V \) (wherein in particular \( T_r := H \)) in Arendt et al.\textsuperscript{28, Prop. 2.3 and Prop. 2.4} it now follows that \( U^r := (U(t,s))_{(t,s) \in \Delta} \) is a strongly continuous evolution family on \( H \), i.e., the following properties hold:

(i) \( U(s,s) = 1_d \) for all \( s \in [0,T] \),
(ii) \( U(t,s) = U(t,r)U(r,s) \) for all \( 0 \leq s \leq r \leq t \leq T \),
(iii) \( (t,s) \mapsto U(t,s)x \) is for all \( x \in H \) continuous from \( \bar{\Delta} \) into \( H \).

If in fact \( a \in \text{FORM}([0, \infty[; V, H) \), then arguing as above we deduce that

\[
\begin{align*}
\dot{u}(t) &+ A(t)u(t) = 0, & t \in [s, \infty[,
\end{align*}
\]

has for all \( s > 0 \) and all \( x \in H \) a unique solution \( u \in L^2_{\text{loc}}(s, \infty; V) \cap H^1_{\text{loc}}(s, \infty; V) \), hence \( u \in C([0, \infty[; H) \): This defines a strongly continuous evolution family \( (U(t,s))_{0 \leq s \leq t < \infty} \), hence \( u \in C([0, \infty[; H) \): This defines a strongly continuous evolution family

\[
(\hat{U}(t,s))_{(t,s) \in \Delta} \text{ as the evolution family associated with the nonautonomous form or with the operator family } (A(t))_{t \in [0,T]}.
\]

Remark 2.1. A nonautonomous form is called coercive if (2.3) is satisfied with \( \omega = 0 \). Now observe that \( a \) satisfies (2.3) if and only if the form \( a_\omega \) given by

\[
a_\omega(t; u, v) := a(t; u, v) + \omega(u \mid v)
\]

is coercive: because \( a_\omega \in \text{FORM}([0, T]; V, H) \) in its own right, it is associated with an evolution family. Moreover, \( u \) is a solution of class \( \text{MR}(V, V') \) of (2.4) if and only if \( v = e^{-\omega t-s}u \) is a solution of class \( \text{MR}(V, V') \) of

\[
\begin{align*}
\dot{v}(t) &+ (\omega + A(t))v(t) = 0, & t \in [s, T], \\
v(s) &= x.
\end{align*}
\]

Thus, the evolution family associated with \( a_\omega \) is simply obtained by rescaling, i.e,

\[
U_\omega(t,s) := e^{-\omega(t-s)}U(t,s), \quad (t,s) \in \bar{\Delta}.
\] (2.6)

The earliest well-posedness results for (2.4) were obtained by Kato based on an approximation method under strong regularity assumptions on the dependence \( t \mapsto a(t) \). Kato's approach has been extended to nonautonomous form of class \( \text{FORM}([0, T]; V, H) \) in previous works.\textsuperscript{20,29} We sketch the construction of evolution families proposed in Sani and Laasri\textsuperscript{19} for the sake of self-containedness, since we are going to use it repeatedly in the next sections.
Let $\Lambda = (\lambda_0, \ldots, \lambda_{n+1})$ be a partition of $[0, T]$, i.e., $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T$. Let $(\alpha_k)_{k \in \mathbb{N}}$ be a family of sesquilinear forms defined by
\[ a_k : V \times V \ni (u, v) \mapsto \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} a(r; u, v)dr \in \mathbb{C}, \quad k = 0, 1, \ldots, n. \tag{2.7} \]

All these forms are bounded and $H$-elliptic with constants $M$, $\alpha$, and $\omega$. The associated operators $A_k \in \mathcal{L}(V, V')$ are given by
\[ A_k : V \ni u \mapsto \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r)udr \in V', \quad k = 0, 1, \ldots, n. \tag{2.8} \]

The mapping $A(\cdot) : [0, T] \to \mathcal{L}(V, V')$ is strongly measurable by Pettis’ Theorem\(^{30}\) since $t \mapsto A(t)u$ is weakly measurable and $V'$ is assumed to be separable. On the other hand, $\|A(t)u\|_{V'} \leq M\|u\|_{V}$ for all $u \in V$ and a.e. $t \in [0, T]$. Thus $[0, T] \ni t \mapsto A(t)u \in V'$ is Bochner integrable for all $u \in V$. Hence the integrals in (2.7) and (2.8) are well defined.

Next, consider the bounded $H$-elliptic nonautonomous form $a_\Lambda := (a_\Lambda(t))_{t \in [0, T]}$ defined by
\[ a_\Lambda(t; \cdot, \cdot) : V \times V \ni (u, v) \mapsto \begin{cases} a_k(u, v) & \text{if } t \in [\lambda_k, \lambda_{k+1}] \\ a_n(u, v) & \text{if } t = T. \end{cases} \tag{2.9} \]

Its associated time-dependent operator family $A_\Lambda := (A_\Lambda(t))_{t \in [0, T]} \subset \mathcal{L}(V, V')$ is given by
\[ A_\Lambda(t) := \begin{cases} A_k & \text{if } t \in [\lambda_k, \lambda_{k+1}] \\ A_n & \text{if } t = T. \end{cases} \tag{2.10} \]

For each $k = 0, 1, \ldots, n$, we denote by $T_k := \{e^{-rA_k} \mid r \geq 0\}$ the $C_0$-semigroup generated by $-A_k$. For each $a, b \in [0, T]$ such that
\[ \lambda_{m-1} \leq a < \lambda_m < \ldots < \lambda_{l-1} \leq b < \lambda_l \tag{2.11} \]

we define the operator families $U_\Lambda := (U_\Lambda(t, s))_{(t, s) \in \Delta} \subset \mathcal{L}(V')$ by
\[ U_\Lambda(b, a) = e^{-(b-a)A_1}e^{-(\lambda_{l-1}-\lambda_{l-2})A_{l-2}} \cdots e^{-(\lambda_{m-1}-\lambda_{m-2})A_{m-2}}e^{-(\lambda_{m-1}-a)A_{m-1}}, \ldots \tag{2.12} \]

and for $\lambda_{l-1} \leq a \leq b < \lambda_l$ by
\[ U_\Lambda(b, a) = e^{-(b-a)A_1}. \tag{2.13} \]

Remark that $U_\Lambda$ defines an evolution family on $H$ (as well as on $V'$ and $V$), since all semigroups $T_k$ consist of bounded linear operators on $H$. Additionally, one sees that the conditions (2.1) to (2.3) are satisfied by the forms $a_\Lambda$, too. Moreover, for all $x \in H$, the function $u_\Lambda(\cdot) := U_\Lambda(\cdot, s)x$ is the unique solution of class $MR(s, T; V, V')$ of the problem
\[ u_\Lambda(0) + A_\Lambda(0)u_\Lambda(t) = 0, \quad t \in [s, T], \]
\[ u_\Lambda(s) = x. \tag{2.14} \]

A similar approximation scheme was introduced in El-Mennaoui and Laasri\(^{20}\) in the more general context of inhomogeneous nonautonomous problems; several convergence results could be deduced there, depending on conditions satisfied by the nonautonomous form. In the language of evolution families, we can paraphrase Proposition 3.1 in El-Mennaoui and Laasri\(^{20}\) and state the following.

**Theorem 2.2.** Let $a \in \text{FORM}[\{0, T\}; V, H]$ and let $U'$ and $U_\Lambda$ be the evolution families associated with $a$ and $a_\Lambda$, respectively. Then
\[ \lim_{|\Lambda| \to 0} U_\Lambda(t, s) = U(t, s) \quad \text{for all } (t, s) \in \Delta \]

in the strong operator topology of $\mathcal{L}(H)$.

The proof of Theorem 2.2 is very similar to that of El-Mennaoui and Laasri\(^{20}\), Prop. 3.1 and will be omitted.

While all strongly continuous semigroups are exponentially bounded, this is not the case for general evolution families, cf, Engel and Nagel.\(^9\), §VI.9 Evolution families associated with nonautonomous forms are rather special, though the
product integral method can be applied to deduce two known results about their long-time behavior. The assertion about quasi-contraction is Lasri,\textsuperscript{31, Prop. 2.1} whereas strong stability was proved by similar means in a special case in Arendt et al.\textsuperscript{32, Thm. 5.4}

**Proposition 2.3.** Let \( a \in \text{FORM}([0, T]; V, H) \). Then the associated evolution family \( U \) is quasi-contraction, ie,

\[
\| U(t, s) \|_{L(H)} \leq e^{\beta(t) dr} \quad \text{for all} \quad (t, s) \in \Delta,
\]

for some \( \beta \in L^1(0, T) \) such that

\[
\Re a(t; u, u) + \beta(t)\| u \|_H^2 \geq 0 \quad \text{for a.e.} \quad t \in [0, T] \quad \text{and} \quad u \in V.
\]  \( \text{(2.15)} \)

If in particular \( a \in \text{FORM}([0, \infty]; V, H) \), \( \beta \in L^1_{\text{loc}}([0, \infty]), \) and \( \limsup_{t \to \infty} \frac{1}{t-t_0} \int_{t_0}^t \beta(r) dr = \Omega < 0 \) for some \( t_0 \), then \( U \) is uniformly exponentially stable, ie,

\[
\| U(t, s) \|_{L(H)} \leq M_0 e^{(t-t_0) \Omega} \quad \text{for some} \quad M_0 \quad \text{and all} \quad t > t_0.
\]

In view of Proposition 2.3 and Remark 2.1 and up to scalar perturbations of \((A(t))_{t \in [0, T]}\), we will thus often assume without loss of generality that the associated evolution family \( U \) is contractive.

**Proof.** Let \([a, b] \subset [0, T], \Lambda \) be a partition of \([a, b] \) as in (2.11) and consider the discretized evolution family \( U_{\Lambda} \). Let \( \beta_k \in \mathbb{R} \) be defined by

\[
\beta_k := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} \beta(r) dr, \quad k = 0, 1, \ldots, n.
\]  \( \text{(2.16)} \)

Then by definition of \( a_k \) in (2.7), (2.15) implies

\[
\Re a_k(u, u) + \beta_k\| u \|_H^2 \geq 0 \quad \text{for all} \quad u \in V \quad \text{and} \quad k = 0, 1, \ldots, n;
\]

hence, the associated semigroup satisfies

\[
\| e^{-rA_k} \|_{L(H)} \leq e^{\rho_k} \quad \text{for all} \quad k = 0, 1, \ldots, n, \quad r \geq 0.
\]  \( \text{(2.17)} \)

Now we obtain from (2.12)

\[
\| U_{\Lambda}(b, a) \|_{L(H)} \leq \prod_{k=1}^n e^{\beta_k^{b_k} \rho_k^{a_k} (r) dr} = e^{\sum_{k=1}^n \beta_k^{b_k} \rho_k^{a_k} r}. \]  \( \text{(2.18)} \)

The claim now follows from Theorem 2.2 and Fatou’s Lemma.

The second assertion follows by observing that

\[
\| U(t, s)x \| \leq \| U(t_0, s)x \| \leq e^{(t-t_0) \Omega} \| U(t_0, s)x \| =: M_0 e^{(t-t_0) \Omega} \| x \|
\]

for all \( 0 \leq s \leq t < \infty \) and \( x \in H \). \( \square \)

## 3 | INVARIANCE PROPERTIES

Let us discuss invariance of a given subset \( C \) of \( H \) under \( U \), ie, whether \( u(s) \in C \) implies that the solution \( u(t) \) of (2.4) lies in \( C \) for any \( (t, s) \in \Delta \). The following criterion is known: It combines Sani and Lasri\textsuperscript{19, Thm. 4.1} with an extension to nonaccretive forms\textsuperscript{33, Thm. 2.1} of Ouhabaz’ classical invariance criterion.\textsuperscript{34, Thm. 2.1}

**Proposition 3.1.** Let \( a \in \text{FORM}([0, T]; V, H) \). Let \( C \) be a closed convex subset of \( H \) and denote by \( P \) the projector of \( H \) onto \( C \). Consider the following assertions:

(i) \( C \) is invariant under the semigroup \( T_t \) associated with \( a(t) \) for a.e. \( t \in [0, T] \);

(ii) \( C \) is invariant under the evolution family \( U \).

(iii) \( Pu \in V \) and \( \Re a(t; Pu, Pu) \geq 0 \) for all \( u \in V \) and a.e. \( t \geq 0 \).

Then (i) is equivalent to (iii) and both imply (ii).
The implication \((iii) \Rightarrow (ii)\) has been proved in Arendt et al\(^{11}\), Thm. 2.2 in the more general case of inhomogeneous equations. Special instances of the same assertion have been obtained in Thomaschewski.\(^{35,13,5} \) The implication \((i) \Rightarrow (ii)\) allows us to deduce invariance properties for \(\mathcal{U}\) even if \(P\) is not explicitly known, eg, when \(T_t\) is known to preserve convexity for a.e. \(t.\)\(^{36}\)

For our purposes, a particularly interesting instance of closed convex sets are order intervals in Hilbert lattices: We hence assume in the following \(H\) to be a Hilbert lattice. It is known that each separable Hilbert lattice is isometrically lattice isomorphic to a Lebesgue space \(L^2(X)\) for some \(\sigma\)-finite measure space \((X, \Sigma, \mu),\) see, eg, Meyer-Nieberg.\(^{37, \text{Cor. 2.7.5}}\)

Accordingly, we can consider the set \(H_\mathbb{R} := L^2(X; \mathbb{R})\) of real-valued functions. Let \(a, b \in \mathbb{R} \cup \{\pm \infty\},\) we introduce the (bounded or unbounded) order intervals

\[
[a, b]_H := \{ f \in H_\mathbb{R} : a \leq f(x) \leq b \quad \text{for a.e.} \quad x \in X \}
\]

which are closed convex subsets of \(H.\) Many qualitative properties of solutions to evolution equations can be described by means of invariance of order intervals under the flow that governs the associated Cauchy problems.

**Definition 3.2.** Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space. An evolution family \(\mathcal{U}\) on the Hilbert lattice \(L^2(X)\) is called

\begin{enumerate}
\item[a.] *real* if \(U(t, s)H_\mathbb{R} \subset H_\mathbb{R}\) for all \((t, s) \in \Delta;\)
\item[b.] *positive* if it is real and \(U(t, s)[0, \infty)_H \subset [0, \infty)_H\) for all \((t, s) \in \Delta;\)
\item[c.] \(L^p\)-contractive, \(p \in [1, \infty],\) if \(U(t, s)\) maps \(\{f \in L^2(X) \cap L^p(X) : \|f\|_{L^p} \leq 1\}\) into itself for all \((t, s) \in \Delta;\)
\item[d.] *completely contractive* if it is both \(L^1\)-contractive and \(L^\infty\)-contractive;
\item[e.] *completely quasi-contractive* if there is some constant \(\bar{\omega}\) such that the rescaled evolution family \(U_{\bar{\omega}}\) defined by

\[
U_{\bar{\omega}}(t, s) := e^{-\bar{\omega}(t-s)}U(t, s), \quad (t, s) \in \Delta, \tag{3.1}
\]

is completely contractive;
\item[f.] *sub-Markovian* if it is positive and \(L^\infty\)-contractive; *Markovian* if additionally \(\|U(t, s)\|_{L(L^\infty)} = 1;\)
\item[g.] *sub-stochastic* if it is positive and \(L^1\)-contractive; *stochastic* if additionally and \(\|U(t, s)f\|_{L^1(X)} = \|f\|_{L^1(X)}\) for all \(0 \leq f \in L^2(X) \cap L^1(X)\) and all \((t, s) \in \Delta.\)
\end{enumerate}

**Remark 3.3.** Let \(\mathcal{U}\) be a completely quasi-contractive evolution family on \(L^2(X).\) Then by the Riesz-Thorin theorem, the rescaled evolution family \(U_{\bar{\omega}}\) is \(L^p\)-contractive for all \(p \in [1, \infty]\) and each \(U(t, s)\) can be extended from \(L^p(X) \cap L^2(X)\) to a quasi-contractive operator \(U_p(t, s)\) on \(L^p(X)\) for all \(p \in [1, \infty].\) The extrapolated family \(U_p := \{U_p(t, s)| (t, s) \in \Delta\}\) is consistent, ie, for all \(p \in [1, \infty]\)

\[
U_p(t, s)f = U_p(t, r)U_p(r, s)f \quad \text{for all} \quad (t, s) \in \Delta \quad \text{and all} \quad f \in L^p(X) \cap L^2(X).
\]

Clearly, \(U_p(s, s) = I_H\) and \(U_p(t, t) = U_p(t, r)U_p(r, s)\) for all \(0 \leq s \leq r \leq t \leq T\) and \(p \in [1, \infty].\) Moreover, by the interpolation inequality (Hölder inequality), we obtain that \(U_p\) is strongly continuous on \(\Delta\) for all \(p \in ]1, \infty[.\) Using a similar argument as in Voigt,\(^{38, \text{Prop. 4}}\) we conclude that \(U_p\) is a strongly continuous evolution family on \(L^p(X)\) for all \(p \in [1, \infty].\)

For future reference, let us note explicitly the following consequence of Proposition 3.1.

**Proposition 3.4.** The evolution family \(\mathcal{U}\) associated with \(a \in \text{FORM}([0, T]; V, H)\) is

\begin{enumerate}
\item[a.] *positive provided* \((\text{Re } v)^+ \in V, \text{ Re } a(t; v) \in \mathbb{R}, \text{ and Re } a(t; (\text{Re } v)^+, (\text{Re } v)^-) \leq 0\) for all \(v \in V\) and a.e. \(t \in [0, T].\)
\item[b.] \(L^\infty\)-contractive, provided \(1 \land |v| \text{sgnv} \in V \text{ and Re } a(t; (1 \land |v|) \text{sgnv}, (|v|-1)^+ \text{sgnv}) \geq 0\) for all \(v \in V\) and a.e. \(t \in [0, T].\)
\end{enumerate}

Let us state a further consequence of Proposition 3.1 concerning irreducibility of evolution families on \(L^2(X)\) on a given \(\sigma\)-finite measure space \((X, \Sigma, \mu).\) We denote by \(1_{\Xi}\) the characteristic function of any given \(\Xi \in \Sigma.\)
We can now provide sufficient conditions for the evolution family to converge towards a rank-one projector, thus extending to the nonautonomous setting one of the main results of the classical theory of positive semigroups. Observe that given $a \in \text{FORM}([0, T]; V, H)$, if

$$(\text{Re } v)^\ast \in V, \quad \text{Re } a(t; v, \text{Im } v) \in \mathbb{R} \quad \text{and}$$

$$(\text{Re } a(t; v)^\ast, (\text{Re } v)^\ast) \leq 0 \quad \text{for all } v \in V \quad \text{and a.e. } t \in [0, T]$$

and

$$\text{for all } \Xi \in \Sigma \ 1_\Xi V \subset V \quad \text{implies} \quad \mu(\Xi) = 0 \quad \text{or} \quad \mu(X \setminus \Xi) = 0,$$
and for all \( x \in H \) by Theorem 2.2

\[
    \lim_{t \to \infty} \| U(t, t_0)x - P_x \| = \lim_{t \to \infty} \| U_*(t, t_0)x - P_x \| \leq \lim_{t \to \infty} e^{-\kappa(t-t_0)}\| x \|.
\]

This concludes the proof. \( \square \)

In the following sections, we will often need to discuss complete contractivity. In order to find sufficient conditions, therefore, observe that \( U^* \) is \( L^1 \)-contractive if and only if \( U(t, s)^* \) is \( L^\infty \)-contractive for all \( (t, s) \in \Delta \). How to prove \( L^\infty \)-contractivity of all \( U(t, s)^* \)? Consider the nonautonomous adjoint form \( a^* : [0, T] \times V \times V \to \mathbb{C} \) of \( a \) defined by \( a^*(t; u, v) := a(t; v, u) \) for all \( t \in [0, T] \) and \( u, v \in V \). While \( a^* \in \text{FORM}([0, T]; V, H) \), too, and hence \( a^* \) is associated with an evolution family \( (U(t, s))_{(t, s)\in \Delta} \), one has in general \( U(t, s)^* \neq U(t, s)^* \). However, it was observed in Daners\textsuperscript{15}, Thm. 2.6 that the returned adjoint form \( \overrightarrow{a^*} : [0, T] \times V \times V \to \mathbb{C} \) of \( a^* \) defined by

\[
    \overrightarrow{a^*}(t; u, v) := a^*(T - t; v, u), \quad t \in [0, T], \quad u, v \in V,
\]

which clearly belongs to \( \text{FORM}([0, T]; V, H) \), too, is associated with an evolution family \( \overrightarrow{U^*} \) that satisfies

\[
    \left[ \overrightarrow{U^*}(t, s) \right]^* f = U(T - s, T - t)f \quad \text{for all } f \in H \quad \text{and} (t, s) \in \Delta. \tag{3.4}
\]

In particular, \( U^* \) is \( L^1 \)-contractive if and only if \( \overrightarrow{U^*} \) is \( L^\infty \)-contractive; \( U^* \) is completely contractive if so is \( \overrightarrow{U^*} \); and by Proposition 3.4, we conclude the following.

**Proposition 3.6.** The evolution family \( U \) associated with \( a \in \text{FORM}([0, T]; V, H) \) is

1. \( L^1 \)-contractive provided \( (1 \land |u|)sgnu \in V \) and \( \text{Re} a(t; (|v| - 1)^+sgnv, (1 \land |v|)sgnv) \geq 0 \) for all \( v \in V \) and a.e. \( t \in [0, T] \);
2. completely contractive provided \( (1 \land |u|)sgnu \in V \) and \( \text{Re} a(t; (|v| - 1)^+sgnv, (1 \land |v|)sgnv) \geq 0 \), \( \text{Re} a(t; (|v| - 1)^+sgnv, (1 \land |v|)sgnv) \geq 0 \) for all \( v \in V \) and a.e. \( t \in [0, T] \).

We can now give a sufficient condition for \( L^p \)-quasi-contractivity of the evolution family \( U \) that governs the Cauchy problem (2.4).

**Theorem 3.7.** Let \( a \in \text{FORM}([0, T]; V, H) \) and let \( p \in [1, \infty[ \) be given. Assume that there exists a function \( \hat{a}_p \in L^\infty(0, T) \) such that \( (T_t)_{t \in [0, T]} \) satisfies

\[
    \| e^{-rA(t)}f \|_{L^p(X)} \leq \phi^{\hat{a}_p(t)}\| f \|_{L^p(X)} \quad \text{for all } f \in L^2(X) \cap L^p(X), \quad r \geq 0, \quad \text{and a.e. } t \in [0, T]. \tag{3.5}
\]

Then the evolution family \( U \) associated with a extrapolates to a consistent evolution family on \( L^p(X) \) and

\[
    \| U(t, s)f \|_{L^p(X)} \leq \phi^{\hat{a}_p(t)dr}\| f \|_{L^p(X)} \quad \text{for all } f \in L^2(X) \cap L^p(X) \quad \text{and} (t, s) \in \Delta. \tag{3.6}
\]

**Proof.** The case where \( \hat{a}_p \equiv 0 \) follows directly from Proposition 3.1; the general case is slightly more delicate. First, applying\textsuperscript{30}, Thm. 4.1 to the nonautonomous form \( a(t, \cdot, \cdot) + \hat{a}_p(t)(|\cdot|_{L^p}) \), we see that the assumption (3.5) is equivalent to the following condition: \( P_{B^p} V \subset V \) and for a.e. \( t \in [0, T] \)

\[
    \text{Re} a(t; u, |u|^{p-2}u) + \hat{a}_p(t)|u|_{L^p}^p \geq 0 \quad \text{for all } u \in V \quad \text{s.t.} \quad |u|^{p-2}u \in V. \tag{3.7}
\]

(Here, \( B^p \) denotes the \( L^p \)-unit ball and \( P_{B^p} \) is the projector of \( L^2(X) \) onto \( B^p \).)

Let now \( [s, t] \subset [0, T] \) and let \( \Delta = (\lambda_0, \lambda_1, \ldots, \lambda_{k+1}) \) be a partition of \( [s, t] \) and let \( a_k : V \times V \to \mathbb{C} \), \( k = 0, 1, \ldots, n \), be the family of bounded \( H \)-elliptic forms given by (2.7) and \( (e^{-rA_k})_{r \geq 0} \) be the associated \( C_0 \)-semigroup. Furthermore, define the finite real sequence \( \hat{a}_{k, p}, k = 0, 1, \ldots, n \), as follows

\[
    \hat{a}_{k, p} := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} \hat{a}_p(r)dr \quad k = 0, 1, \ldots, n. \tag{3.8}
\]
Then (3.7) implies that for all \( k = 0, 1, \ldots, n \)
\[
\Re a_k(u, |u|^{p-2}u) + \hat{a}_k |u|^p \geq 0, \quad \text{for all } u \in V \quad \text{s.t.} \quad |u|^{p-2}u \in V. \tag{3.9}
\]
Again applying Nittka\textsuperscript{39, Thm. 4.1} to the form \( a_k + \hat{a}_k \rho(-\cdot)_{L^1} \), we obtain
\[
\|e^{-t\hat{a}_k}\|_{L^p(X)} \leq e^{\hat{a}_k t} \quad \text{for all } s \geq 0 \quad \text{and} \quad k = 0, 1, \ldots, n. \tag{3.10}
\]
Thus, using (2.12) to (2.13), we find
\[
\|U(t,s)\|_{L^p(X)} \leq e^{\int_s^t \hat{a}_k(r)dr} \quad \text{for all } (t,s) \in \Delta \quad \text{and each partition } \Lambda \text{ of } [s,t]. \tag{3.11}
\]
Thus, the desired estimate (3.6) follows from Theorem 2.2 and Fatou's lemma.

In a similar way, we can discuss stochasticity, another feature that cannot be easily interpreted as an invariance property.

**Proposition 3.8.** The evolution family \( U \) associated with \( a \in \text{FORM}([0, T]; V, H) \) is stochastic provided \( (\Re u)^+ \in V, \Re a(t, Re u, Im u) \in \mathbb{R}, \Re a(t; (Re u)^+, (Re u)^- ) \leq 0, I \in V, \) and \( a(t; Re u, I) = 0 \) for all \( v \in V \) and a.e. \( t \in [0, T] \).

Our last result in this section is devoted to the issue of domination of evolution families.

**Proposition 3.9.** Let \( a \in \text{FORM}([0, T]; V, H) \) and denote as usual by \( U \) the associated evolution family. Let furthermore \( W \) be a separable Hilbert space that is densely and continuously embedded in \( H \) and \( b \in \text{FORM}([0, T]; W, H) \): We denote by \( V := (V(t,s)_{(t,s) \in \Delta} \) the associated evolution family. Assume that
- \( \Re u \in V \) and \( (\Re u)^+ \in V \) for all \( u \in V \);
- \( V \) is a generalized ideal of \( W \), i.e,
  - \( u \in V \) implies \( |u| \in W \) and
  - \( u_1 \in V \) and \( u_2 \in W \) are such that \( |u_2| \leq |u_1| \), then \( u_2 \text{sgn} u_1 \in V \);
- \( \Re a(t; Re u, Im u) \in \mathbb{R} \) for all \( u \in V \);
- \( \Re a(t; (Re u)^+, (Re u)^- ) \leq 0 \) for all \( u \in V \);
- \( \Re a(t; u,v) \geq b(t; |u|, |v|) \) for all \( u, v \in V \) s.t. \( uv \geq 0 \).

Then \( U \) is dominated by \( V \), i.e,
\[
|U(t,s)f| \leq V(t,s)|f| \quad \text{for all } (t,s) \in \Delta \quad \text{and} \quad f \in H. \tag{3.12}
\]

**Proof.** Let \( \Lambda \) be a partition of \([0, T]\). Define the piecewise constant \( b_\Lambda \in \text{FORM}([0, T]; W, H) \) via formulae which are analogous to (2.7) and (2.10) and let \( V_\Lambda \) be the associated evolution family. By Manavi et al.\textsuperscript{33, Thm. 4.1} we see that the semigroups associated with the averaged forms \( b_\Lambda \) dominate \( T_\Lambda \), hence that \( V_\Lambda \) dominates \( U_\Lambda \). By Theorem 2.2, we conclude that \( U \) is dominated by \( V \).

\section{ULTRACONTRACTIVITY}

In this section and the next section, we are going to restrict to the case of \( H = L^2(X) \), where \( X \) is a \( \sigma \)-finite measurable space. Recall that a \( C_0 \)-semigroup \( S \) on \( L^2(X) \) is said to be \textit{ultraccontractive} if there exist constants \( c_0, n > 0 \), and \( \hat{\alpha} \in \mathbb{R} \) such that
\[
\|S(r)f\|_{L^\infty(X)} \leq c_0 r^{-\frac{n}{2}} e^{\frac{\alpha}{2} t} \|f\|_{L^1(X)} \quad \text{for all } r \geq 0 \quad \text{and all} \quad f \in L^2(X) \cap L^1(X). \tag{4.1}
\]
In this section, we are going to develop a theory of ultraccontractive evolution families.
Definition 4.1. We call an evolution family \( U \) on \( L^2(X) \) ultracontractive if there exist constants \( c_0, n > 0 \) and \( \tilde{\omega} \in \mathbb{R} \) such that
\[
\|U(t, s)f\|_{L^\infty(X)} \leq c_0(t - s)^{-\frac{n}{2}} e^{\tilde{\omega}(t-s)} \|f\|_{L^1(X)} \quad \text{for all} \ (t, s) \in \Delta \quad \text{and} \quad f \in L^2(X) \cap L^1(X). \tag{4.2}
\]

By a direct consequence of the Kantorovitch–Vulikh Theorem (see, eg, Arendt and Bukhvalov\(^\text{"40, Theorem 1.3"}\)) any ultracontractive evolution family \( U \) is given by an integral kernel: More precisely, there exists a family \( (\Gamma(t, s))_{(t, s) \in \Delta} \subset L^\infty(\Omega \times \Omega) \) such that
\[
U(t, s)f(x) = \int_\Omega \Gamma(t, s; x, y)f(y)dy \quad \text{for all} \ (t, s) \in \Delta, \quad f \in L^2(X) \cap L^1(X) \quad \text{and a.e.} \ x \in \Omega,
\]
with
\[
\|U(t, s)\|_{L(L^1(X), L^\infty(X))} = \|\Gamma(t, s)\|_{L^\infty(\Omega \times \Omega)}, \quad \text{for all} \ (t, s) \in \Delta.
\]

It is well known that ultracontractivity of semigroups can be deduced from the Nash or Gagliardo-Nirenberg inequalities for the domain of the associated form; see Ouhabaz.\(^\text{13, Chapt. 6"}\) We are going to extend this result to the nonautonomous setting. For that, we need the following definitions.

Definition 4.2. Let \( V \) be a subspace of \( L^2(X) \). The space \( V \) is said to satisfy
\[(i) \text{ a Nash inequality} \text{ if there exist constants } C_N, \mu > 0 \text{ such that}
\|u\|_{L^2(X)}^{2+\frac{4}{q}} \leq C_N \|u\|_{W^1_v}^{\frac{2}{q}} \|u\|_{L^1(X)}^{1-\frac{1}{q}} \quad \text{for all} \ u \in L^1(X) \cap V; \tag{4.3}
\]
\[(ii) \text{ a Gagliardo-Nirenberg inequality} \text{ if there exist constants } C_G, N > 0 \text{ such that}
\|u\|_{L^2(X)} \leq C_G \|u\|_{L^1(X)}^{1-N\frac{\epsilon}{2q}} \|u\|_{v}^{N\frac{\epsilon}{2q}} \quad \text{for all} \ u \in V \tag{4.4}
\]
holds for all \( q \in ]2, \infty[ \) such that \( N\frac{\epsilon}{2q} \leq 1 \).

Sobolev spaces \( H^1(I) \) on intervals \( I \subset \mathbb{R} \) satisfy e.g. the Nash inequality, see, eg, Maz'ya.\(^\text{41, § 1.4.}\) More generally, the same is true for each closed subspace \( V \) of \( H^1(\Omega) \), which has the \( L^1 - H^1 \)-extension property,\(^\text{42, Lemma 2.7"}\) where \( \Omega \) is an arbitrary open set of \( \mathbb{R}^d \). Several geometric conditions on \( \Omega \subset \mathbb{R}^d \) under which a Sobolev space \( V = H^k(\Omega) \) satisfies a Gagliardo-Nirenberg inequality are known; see, eg, Adams and Fournier.\(^\text{43, Chapter 5"}\)

Here and in the following, we are adopting the usual notations introduced in (2.1) to (2.3).

Theorem 4.3. Let \( a \in \text{FORM}([0, T]; V, H) \) such that the associated evolution family \( U \) is completely quasi-contractive with constant \( \tilde{\omega} \in \mathbb{R} \). If \( V \) satisfies a Nash inequality (4.3) for some constants \( \mu, C_N > 0 \), then \( U \) is ultracontractive and
\[
\|U(t, s)\|_{L(L^1(X), L^\infty(X))} \leq \left( \frac{\mu C_N}{4\tilde{\omega}} \right)^{\frac{1}{2}} (t - s)^{-\frac{n}{2}} e^{\max\{\tilde{\omega}, \tilde{\omega}\}(t-s)} \quad \text{for all} \ (t, s) \in \Delta. \tag{4.5}
\]

Proof. The first part of the proof is similar to that of Arendt and Elst.\(^\text{42, Prop. 3.8"}\) Upon rescaling \( U(t, s) \) by \( e^{-\max\{\tilde{\omega}, \tilde{\omega}\}(t-s)} \), we can without loss of generality assume both \( a \) to be coercive and the evolution family \( U \) to be completely contractive. Let \( f \in L^1(X) \cap V \) and let \( s \in [0, T] \) be fixed. If \( y \in MR(V, V') \) then \( \|y\|_{H}^2 \in W^{1,1}(s, T; V') \) and
\[
\frac{d}{dt} \|y\|_{H}^2 = 2\text{Re} \langle \dot{y}(\cdot), y(\cdot) \rangle \tag{4.6}
\]
by Showalter\(^\text{27, Prop. III.1.2"}\); accordingly, using (2.3) and since \( t \mapsto U(t, s)f \in MR(s, T; V, V') \), we obtain that for all \( f \in V \cap L^1(X) \) and a.e. \( (t, s) \in \Delta \)
\[
\frac{d}{dt} \|U(t, s)f\|_{L^2(X)}^2 = 2\text{Re} \left( \frac{d}{dt} U(t, s)f, U(t, s)f \right) 
= -2\text{Re} \langle A(t)U(t, s)f, U(t, s)f \rangle
\]
where $\alpha$ is the constant in (2.3). It follows that

$$\frac{\partial}{\partial t} \left( \| U(t,s)f \|_{L^2(X)}^2 \right)^{-\frac{1}{2}} = - \frac{2}{\mu} \| U(t,s)f \|_{L^2(X)}^{-2} - \frac{\partial}{\partial t} \| U(t,s)f \|_{L^2(X)}^2 \geq \frac{4\alpha}{\mu C_N} \| U(t,s)f \|_{L^2(X)}^{-\frac{1}{2}}$$

since $U^*$ is completely contractive. Integrating this inequality between $s$ and $t$, we find

$$\| U(t,s)f \|_{L^2(X)} \leq \left( \frac{\mu C_N}{4\alpha} \right)^{\frac{1}{4}} (t-s)^{-\frac{1}{4}} \quad \text{for all} \quad (t,s) \in \Delta. \quad (4.7)$$

In order to obtain the $L^2 - L^\infty$-bound and thus prove the claimed ultracontractivity we will use the returned adjoint form $\overrightarrow{a^*}$ introduced in Section 3. In fact, arguing as in the first part of the proof, we find that the evolution family $\overrightarrow{U^*}$ associated with $\overrightarrow{a^*}$ satisfies (4.7) with the same bound. Then using the identity (3.4), we conclude that

$$\| U(t,s)f \|_{L^2(X)} \leq \left( \frac{\mu C_N}{4\alpha} \right)^{\frac{1}{4}} (t-s)^{-\frac{1}{4}} \quad \text{for all} \quad (t,s) \in \Delta. \quad (4.8)$$

Finally, the evolution law satisfied by $U^*$ completes the proof. $\square$

**Definition 4.4.** An evolution family $U^*$ on $L^2(X)$ is called *linearly quasi-contractive* if for some constants $\alpha_1, \alpha_2$ independent of $p$

$$\| U(t,s)f \|_{L^p(X)} \leq e^{(t-s)(\alpha_1 + \alpha_2 p)} \| f \|_{L^p(X)} \quad (4.9)$$

for all $(t,s) \in \Delta, f \in L^2(X) \cap L^p(X)$, and $p \in [2, \infty]$. Linear $L^p$-quasi-contractivity turns out to be a key notion when it comes to checking ultracontractivity when the domain of the form satisfies a Gagliardo-Nirenberg inequality. Indeed, in this case, we have the following result.

**Theorem 4.5.** Let $a \in \text{form}([0, T]; V, H)$. Assume that $U^*$ and $\overrightarrow{U^*}$ are both linearly quasi-contractive with constants $\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*$. If $V$ satisfies a Gagliardo-Nirenberg inequality for some $C_G, N > 0$, then $U^*$ is ultracontractive and we have

$$\| U(t,s)f \|_{L^2(X), L^\infty(X)} \leq \frac{C_N^N}{4^{N/2}} e^{\alpha(t-s)} (t-s)^{-\frac{N}{2}} \quad \text{for all} \quad (t,s) \in \Delta. \quad (4.10)$$

where

$$\alpha := \alpha_1 + \alpha_1^* + \frac{2(N+2)}{N} [a_2 + a_2^*].$$

In the proof of Theorem 4.5, we will need the following lemma.

**Lemma 4.6.** Let $a \in \text{form}([0, T]; V, H)$ and let $U^*$ and $\overrightarrow{U^*}$ be the evolution families associated with $a$ and $\overrightarrow{a^*}$, respectively. Assume that $U^*$ and $\overrightarrow{U^*}$ are both linearly quasi-contractive (with constants $\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*$). In addition, we assume that there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$\| U(t,s)f \|_{L^2(X), L^2(X)} \leq \kappa (t-s)^{-\frac{2}{N}} e^{(t-s)[\alpha_1 + \alpha_2]} \quad \text{for all} \quad (t,s) \in \Delta \quad (4.11)$$

and

$$\| \overrightarrow{U^*}(t,s)f \|_{L^2(X), L^2(X)} \leq \kappa (t-s)^{-\frac{2}{N}} e^{(t-s)[\alpha_1^* + \alpha_2^*]} \quad \text{for all} \quad (t,s) \in \Delta \quad (4.12)$$
where \( \tilde{\omega} := [\alpha_1 + \alpha_1^2] + \mu[\alpha_2 + \alpha_2^2] \).

**Proof.** For the proof, we follow similar argument as in Ouhabaz\(^{17,}\)Thm. 5.2 and Coulhon\(^{44}\) where ultracontractivity for semigroups are treated.

**Step 1.** We will first prove that

\[
\|U(t,s)\|_{L(L^1(X), L^\infty(X))} \leq \kappa^t C(t-s)^{-\frac{\alpha_t N}{r}} e^{\tilde{\omega} t(s-t)} e^{\alpha_2 t(s-t)},
\]

for some positive constants \( c, \mu > 0 \) that depend only on \( N, \kappa, \) where

\[
\tilde{\omega} := [\alpha_1 + \alpha_1^2] + \mu[\alpha_2 + \alpha_2^2].
\]

where the positive constants \( C \) and \( \mu \) depend only on \( N \) and \( \kappa_1 \). For some \( r > 2 \) that will be fixed later, we can combine (4.11) with the linear \( L^r \)-quasi-contractivity of \( U \) and obtain by a version of Riesz-Thorin interpolation theorem\(^{45,}\)Thm. 2.2.14 that for any \( \theta \in [0, 1] \)

\[
\|U(t,s)\|_{L(L^1(X), L^\infty(X))} \leq \kappa^t (t-s)^{-\frac{\alpha_t N}{r}} e^{(1-\theta)(t-s)[\alpha_t + \alpha_1][\alpha_t + 2\alpha_1]},
\]

where \( \frac{1}{p_k} := \frac{1-\theta}{r} + \frac{\theta}{q_1} \). Let now \( p \in [2, \infty[ \). Choosing \( \theta := \frac{1}{p} \) and \( r = 2(p - 1) \) in the above equation, we obtain that

\[
\|U(t,s)\|_{L(L^1(X), L^\infty(X))} \leq \kappa^t (t-s)^{-\frac{\alpha_t N}{r}} e^{(1-\theta)(t-s)[\alpha_t + 2\alpha_1][\alpha_t + 2\alpha_1]},
\]

holds for all \( p \in [2, \infty[ \) where \( N_p := \frac{N}{N-1} \).

Next, set \( R = \frac{N}{N-1}, p_k = 2R^k \) and \( t_k = \frac{N+1}{2N} 2^{-k} \) for all \( k \in \mathbb{N} \). Moreover, let \( s_0 = s \) and \( s_{k+1} = s_k + t_k \) for each integer \( k > 0 \). Then we have \( \sum_k t_k = 1, \sum_k \frac{1}{p_k} = \frac{N}{2} \). Furthermore, \( s_{k+1} < s_k \) for all \( k \in \mathbb{N} \) and \( t = \lim_{k \to \infty} s_k \). Thus, applying (4.15) for \( p = p_k \), using (4.9) and the evolution law satisfied by \( U \), we deduce that

\[
\|U(t,s)\|_{L(L^1(X), L^\infty(X))} \leq \prod_{k \geq 0} \|U(s_{k+1}, s_k)\|_{L(L^1(X), L^\infty(X))}
\]

where the positive constants \( C \) and \( \mu \) depend only on \( N \) and \( \kappa_1 \).

**Step 2.** It remains to estimate \( U(t,s) \) in \( L(L^1(X), L^2(X)) \). To this end, we will follow an idea in Daners\(^{15,}\)Corollary 5.3 and use the returned adjoint form \( \tilde{a}^* \). Indeed, by assumption \( \tilde{a}^* \) is linearly contractive. Thus, one can just repeat the argument in Step 1 and obtain

\[
\| \tilde{U}^*(t,s)\|_{L(L^1(X), L^\infty(X))} \leq \tilde{C}(t-s)^{-\frac{\alpha_t N}{r}} e^{\tilde{\omega}^* t(s-t)} e^{\tilde{\omega}^*_2 t(s-t)},
\]

for some positive constants \( \kappa, \mu > 0 \) that depend only on \( N, \kappa, \) where

\[
\tilde{\omega}^* := [\alpha_1 + \alpha_1^2] + \mu[\alpha_2 + \alpha_2^2].
\]
for each \((t,s) \in \Delta\) and some constants \(\hat{\mathcal{C}}, \hat{\mu}\) that depend only on \(N, \kappa_1\). This yields, in turn, an estimate of \(U(t, s)\) from \(L^2(X)\) to \(L^1(X)\), thanks to (4.16). Finally, using again the evolution law satisfied by \(U\), we conclude that \(U\) is ultracontractive and (4.13) holds.

**Proof of Theorem 4.5.** Upon rescaling the evolution family by \(e^{-\omega(t-s)}\), we can without loss of generality assume \(U\) to be contractive. Let \(q > 2, f \in L^q(X) \cap L^2(X)\) and set

\[
\hat{U}(t,s) := e^{-(t-s)|a_1 + qa_2|}U(t,s) \quad \text{for each} \quad (t,s) \in \Delta.
\]

Because of (4.9), we have that \(t \mapsto \|\hat{U}(t,s)f\|_{L^q(X)}\) is decreasing on \([s, T]\) for each \(s \in [0, T]\). Let now \((t,s) \in \Delta\): the contractivity of \(U\) together with the Gagliardo-Nirenberg inequality and (4.6) imply that for all \(q > 2\) and all \((t,s) \in \Delta\)

\[
(t-s)\|\hat{U}(t,s)f\|_{L^q(X)}^{\frac{4q}{Nq-2}} \leq \int_s^t \|\hat{U}(r,s)f\|_{L^q(X)}^{\frac{4q}{Nq-2}} \|\hat{U}(r,s)f\|_2^2 dr
\]

\[
\leq C_G \frac{4q}{Nq-2} -1 \int_s^t \|\hat{U}(r,s)f\|_{L^q(X)}^{\frac{4q}{Nq-2}} -2 \|\hat{U}(r,s)f\|_2^2 dr
\]

\[
= -C_G \frac{4q}{Nq-2} -1 \int_s^t \|\hat{U}(r,s)f\|_{L^q(X)}^{\frac{4q}{Nq-2}} -2 \|\hat{U}(r,s)f\|_2^2 dr
\]

\[
= -C_G \frac{4q}{Nq-2} -1 \|f\|_{L^q(X)}^{\frac{4q}{Nq-2}} \int_s^t \frac{\partial}{\partial r} \|U(r,s)f\|^2_{L^2(X)} dr
\]

\[
\leq C_G \frac{4q}{Nq-2} -1 \|f\|_{L^q(X)}^{\frac{4q}{Nq-2}} \left( \|U(t,s)f\|_{L^2(X)}^2 - \|f\|_{L^2(X)}^2 \right)
\]

Here, we have used that \(t \mapsto \hat{U}(t,s)f\) solves (2.4) with \(A(t)\) replaced by \(A(t) + [a_1 + qa_2]\). It follows that for all \((t,s) \in \Delta\) and all \(q > 2\)

\[
\|U(t,s)\|_{L^q(X), L^3(X)} \leq C_G \alpha^{-N \frac{q+2}{4q}} (t-s)^{-N \frac{q+2}{4q}} e^{\omega(t-s)|a_1 + qa_2|} \quad \text{(4.17)}
\]

and likewise

\[
\|\hat{U}(t,s)\|_{L^q(X), L^3(X)} \leq C_G \alpha^{-N \frac{q+1}{4q}} (t-s)^{-N \frac{q+1}{4q}} e^{\omega(t-s)|a_1 + qa_2|} \quad \text{(4.18)}
\]

Choosing now \(q = \frac{2(N+2)}{N+2} = \frac{2(N+2)}{N}\) in (4.17) to (4.18), we obtain that (4.12) to (4.11) are fulfilled with \(\kappa = C_G \alpha^{-\frac{N}{2(N+2)}}\) and \(\kappa_1 = \frac{N}{N+2}\). Thus, we conclude by Lemma 4.6 that \(U\) is ultracontractive and (4.10) holds.

**Remark 4.7.** Theorem 4.5 holds in particular for \(N+2 = 1\): in this case, the Gagliardo-Nirenberg inequality becomes

\[
\|u\|_{L^q(X)} \leq C_G \|u\|_V \quad \text{for all} \quad u \in V,
\]

ie, (4.4) reduces to the elementary assumption that \(V\) is continuously embedded in some \(L^q(X)\): a classical Sobolev inequality. More precisely, if there exists \(N > 2\) such that

\[
V \subset L^q(X) \quad \text{for} \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{N},
\]

then \(U\) is ultracontractive and (4.10) holds.

### 5 | GAUSSIAN BOUNDS

The existence of integral kernels of the evolution family, established in the previous section, paves the way to the discussion of kernel estimates.
**Definition 5.1.** Let \( U \) be an evolution family on \( L^2(\mathbb{R}^d) \) with an integral kernel \( \Gamma \). Then \( U \) is said to satisfy Gaussian bounds if there exist \( b, c > 0 \), \( n > 0 \), and \( \omega \in \mathbb{R} \) such that

\[
|\Gamma(t,s;x,y)| \leq c e^{\rho(t-s)}(t-s)^{-\frac{n}{2}} \exp\left(-\frac{b|x-y|^2}{t-s}\right)
\]  

(5.1)

for all \((t,s) \in \Delta\) and a.e. \( x, y \in \mathbb{R}^d \).

We regard \( L^2(\Omega) \) as a closed subspace of \( L^2(\mathbb{R}^d) \), extending operators on \( L^2(\Omega) \) to \( L^2(\mathbb{R}^d) \) by 0. In this way, we can naturally define Gaussian bounds for operators on \( L^2(\Omega) \). Here \( \Omega \) is an arbitrary open set of \( \mathbb{R}^d \).

Gaussian bounds for evolution equations can be characterized by ultracontractivity. Well-known for autonomous closed forms, this characterization is based on the so-called Davies' trick, first appeared in Davies\(^{21}\); see also Arendt and Elst\(^{42}\), Thm. 3.3 and Arendt\(^{25}\), Thm. 13.1.4 for more general versions. Davies' trick is essentially an algorithm centered around an auxiliary result, whose nonautonomous counterpart is Theorem 5.2 below.

To begin with we introduce a suitable space

\[
W := \{ \psi \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) : \|D_j \psi\|_\infty \leq 1, \|D_i D_j \psi\|_\infty \leq 1, i, j = 1, \ldots, n \}
\]  

(5.2)

of smooth functions. By Robinson,\(^{46}\), p. 200–202 the function \( d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ \) defined by

\[
d(x, y) := \sup\{|\psi(x) - \psi(y)| \mid \psi \in W\}
\]

is a metric equivalent to the Euclidean one: there exists \( \beta > 0 \) such that

\[
\beta|x - y| \leq d(x, y) \leq \beta^{-1}|x - y| \quad \text{for all} \quad x \in \mathbb{R}^d.
\]  

(5.3)

Let \( U \) be an evolution family on \( L^2(\Omega) \) and, as usual, extend it if needed to \( L^2(\mathbb{R}^d) \). For a fixed \( \psi \in W \), we define perturbed evolution families \( U_\rho \) on \( L^2(\mathbb{R}^d) \) by

\[
U_\rho(t,s) := U_\psi^\rho(t,s) := M_\rho U(t,s) M_\rho^{-1}, \quad \rho \in \mathbb{R},
\]

where \( M_\rho \) is the isomorphism on \( L^2(\mathbb{R}^d) \) defined by

\[
(M_\rho g)(x) := (M_\rho^\psi g)(x) := e^{-\rho \psi(x)} g(x), \quad g \in L^2(\mathbb{R}^d), \quad x \in \mathbb{R}^d.
\]

Gaussian bounds for \( U \) can now be derived from uniform ultracontractivity of the perturbed evolution families \( U_\rho \) with respect to \( \rho \) and \( \psi \). The proof of this fact is very similar to that of the autonomous case studied in Arendt and Elst\(^{42}\), Prop. 3.3 and we omit it: Our result contains a previous work's theorem\(^{15}\), Thm. 6.1 as a special case.

**Theorem 5.2.** Let \( U \) be an evolution family on \( L^2(\Omega) \). Then the following are equivalent:

(i) There exist \( c > 0, n > 0, \) and \( \bar{\omega} \in \mathbb{R} \) such that

\[
\|U_\rho(t,s)\|_{L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d)} \leq c(t-s)^{-\frac{n}{2}} e^{\bar{\omega}(1+\rho^3)(t-s)}
\]  

(5.4)

for all \( \rho \in \mathbb{R}, \psi \in W, \) and \((t,s) \in \Delta\).

(ii) \( U \) satisfies Gaussian bounds.

In this case \( U \) satisfies (5.1) with \( b = \frac{\bar{\omega}^2}{4c}, \) \( c, \bar{\omega} \) as in (5.4) and \( \beta \) in (5.3).

The form domain \( V \) of \( a \in \text{FORM}([0, T]; V, H) \) is said to be \( W \)-invariant if \( M_\rho V \subset V \) for all \( \rho \in \mathbb{R} \) and \( \psi \in W \). In this case, the family of mappings \( a^\rho \) given by

\[
a^\rho(t; u, v) := a(t; M_\rho u, M_\rho^{-1} v), \quad \rho \in \mathbb{R}, \quad t \in [0, T], \quad u, v \in V,
\]  

(5.5)
is well defined. Let now \( a^\rho \in \text{FORM}([0, T]; V, H) \); for each \( \rho \in \mathbb{R}, \psi \in W, \) and \( t \in [0, T], \) we can hence consider the operator family

\[
A^\rho(t) := \mathcal{M}_\rho^{-1} A(t) \mathcal{M}_\rho,
\]

\[
D(A^\rho(t)) := \{ u \in L^2(\Omega) : \mathcal{M}_\rho u \in D(A(t)) \},
\]

and \(-A^\rho(t)\) is for all \( \rho \in \mathbb{R} \) and all \( t \in [0, T] \) the generator of the semigroup \( T^\rho_t \) given by

\[
T^\rho_t(r) := \mathcal{M}_\rho^{-1} e^{-rA(t)} \mathcal{M}_\rho, \quad r \geq 0.
\]

**Lemma 5.3.** Assume that \( V \) is \( W \)-invariant and \( a^\rho \in \text{FORM}([0, T]; V, H) \) for each \( \rho \in \mathbb{R} \) with constants \( M_\rho, a_\rho > 0 \) and \( \omega_\rho \in \mathbb{R}, \) i.e.,

\[
|a^\rho(t; u, v)| \leq M_\rho |u|_V |v|_V \quad \text{for all } t \in [0, T], u, v \in V.
\]

Then \( (A^\rho(t))_{t \in [0, T]} \) and \( U_\rho \) are the operator family and the evolution family on \( H \) associated with \( a^\rho, \) respectively.

The easy proof is left to the reader.

After all these preparatory results we are finally in the position to present our main theorems: given \( a \in \text{FORM}([0, T]; V, H) \) we introduce two sets of assumptions, which impose a Sobolev-like embedding on \( V \) and a contractivity condition on the perturbed semigroups \( T^\rho_t, \) and show that each of them implies Gaussian bounds for the evolution family associated with \( a. \)

**Theorem 5.4.** Let \( a \in \text{FORM}([0, T]; V, H) \). Assume that \( V \) is \( W \)-invariant and that (5.6) holds for a uniform choice of \( a \) and for \( \omega_\rho \) such that

\[
\omega_\rho \leq \omega(1 + \rho^2)
\]

for some constant \( \omega > 0 \) that is independent of \( \rho. \) Assume \( V \) satisfies a Nash inequality and the semigroups \( (e^{-\omega_\rho t} T^\rho_t(r))_{r \geq 0} \) are completely contractive for a.e. \( t \in [0, T] \) and all \( \rho \in \mathbb{R}. \) Then the evolution family \( U \) associated with \( a \) satisfies Gaussian bounds.

**Proof.** By Propositions 3.4 and 3.6, the evolution family \( (e^{-\omega_\rho(t-s)} U_\rho(t, s))_{(t, s) \in \Delta} \) is completely contractive. Thus, by Theorem 4.3

\[
\|e^{-\omega_\rho(t-s)} U_\rho(t, s)\|_{L^1(\Omega), L^\infty(\Omega)} \leq \left( \frac{\mu e^{\frac{\mu}{4\alpha}}} {4\alpha} \right)^{\frac{1}{2}} (t-s)^{-\frac{1}{2}} e^{\omega_\rho(t-s)} \quad \text{for all } (t, s) \in \Delta.
\]

Now, using (5.7) we obtain that \( U_\rho \) satisfies (5.4): the claim follows from Theorem 5.2.

**Theorem 5.5.** Let \( a \in \text{FORM}([0, T]; V, H) \) with associated evolution family \( U. \) Assume that \( V \) is \( W \)-invariant and that (5.6) holds for a uniform choice of \( a \) and all \( t, u, v \). Assume that \( V \) satisfies a Gagliardo–Nirenberg inequality and both \( U_\rho \) and \( U_{\rho}^{\ast} \) are linearly quasi-contractive for all \( \rho \in \mathbb{R}. \) Then \( U_\rho \) is ultracontractive for all \( \rho \in \mathbb{R} \) with

\[
\|U_\rho(t, s)\|_{L^1(\Omega), L^\infty(\Omega)} \leq \frac{C^2}{\tilde{\omega}_\rho} (t-s)^{-\frac{1}{2}} \quad \text{for all } (t, s) \in \Delta,
\]

where

\[
\tilde{\omega}_\rho := \left( \omega_\rho + a_{\rho, 1}^2 \right) + \left( \frac{2(N + 2)}{N} \right) |a_{\rho, 2} + a_{\rho, 2}^2|
\]

and \( a_{\rho, 1}, a_{\rho, 1}^2 \) are the constants that appear in the linear quasi-contractivity estimate. Thus, if additionally \( \omega_\rho, a_{\rho, i}, a_{\rho, i}^2, i = 1, 2, \) can be chosen in such a way that

\[
\tilde{\omega}_\rho \leq \omega_0 (1 + \rho^2)
\]

for some constant \( \omega_0 > 0 \) independent of \( \rho, \) then \( U \) satisfies Gaussian bounds.

**Proof.** The assertion can be proved similarly to Theorem 5.4, based in this case on Theorem 5.2 and Theorem 4.5.
6 | APPLICATIONS

6.1 | Diffusion equations on dynamic graphs

Consider a (finite or infinite) simple graph $G$ with vertex set $V$ and edge set $E$, with $V$ vertices and $E$ edges (i.e., $V = |V|$ and $E = |E|$). Fix an orientation of $G$ and introduce the $V \times E$ (signed) incidence matrix $I = (i_{ve})$ of $G$ by

$$i_{ve} := \begin{cases} -1 & \text{if } v \text{ is initial endpoint of } e, \\ +1 & \text{if } v \text{ is terminal endpoint of } e, \\ 0 & \text{otherwise.} \end{cases}$$

Let $m \in \ell^\infty(E)$ be a family of edge weights and consider the (weighted) Laplacian $\mathcal{L} := IMM^T$ on $\ell^2(V)$, where $0 \leq M := \text{diag} \,(m(e))_{e \in E}$. ($\mathcal{L}$ can be shown to be independent of the orientation.)

We assume that $G$ is uniformly locally finite, i.e., there is $M < \infty$ such that $\sum_{e \in E} |i_{ve}| \leq M$ for all $v \in V$: in this case $I$ is a bounded linear operator from $\ell^2(E)$ to $\ell^2(V)$, hence $\mathcal{L}$ is a positive semi-definite, bounded self-adjoint operator on $\ell^2(V)$; we thus take $V = H = \ell^2(V)$. It is well-known that the semigroup generated by $-\mathcal{L}$ is sub-Markovian; see, e.g., Mugnolo\cite{Mugnolo:2014}, §6.4.1: if the graph is finite, then it is Markovian and stochastic too.

Let us now regard $G$ as a reference graph (one may e.g. think of a complete graph, or else of a lattice graph $\mathbb{Z}^d$) and consider a family $(G(t))_{t \in [0,T]}$ of modifications of $G$ – in other words, a graph-valued dynamical system, or dynamic graph.\cite{Fattorini:2005} We describe the dependence of $G(t)$ on $t$ by introducing a measurable function $[0, T) \ni t \mapsto m(t) \in \ell^\infty(E)$: this allows e.g. for sudden switching of edges (as in the case of adjacency driven by a Poisson process). In particular, we consider the nonautonomous form $a$ defined by

$$a(t; u, v) := \left(\text{diag} \,(m(t,e))I^T u \left| I^T v\right.)_{\ell^2(E)}, \quad t \in [0,T], \quad u, v \in \ell^2(V).$$

It is easy to see that $a \in \text{FORM}([0,T]; \ell^2(V), \ell^2(V))$ and the associated operators are the Laplacians $(\mathcal{L}_{G(t)})_{t \in [0,T]}$. (We are not assuming boundedness from below on $m$: this is made unnecessary by the boundedness of the operator $\mathcal{L}_{G(t)}$ for all $t$; in fact, even negative weights and hence signed graphs are allowed.) We deduce by Proposition 3.1 that the nonautonomous Cauchy problem

$$\dot{u}(t, v) + \mathcal{L}_{G(t)}u(t, v) = 0, \quad t \geq s, \quad v \in V, \quad u(s, v) = x_v, \quad v \in V$$

is governed by an evolution family on $\ell^2(V)$; in fact, for all $x \in \ell^2(V)$ the above equation enjoys backward well-posedness, too, and the unique solution $u$ is of class $\mathcal{H}^1(\mathbb{R}; \ell^2(V))$: the corresponding evolution family $(U(t, s))_{t, s \in \mathbb{R}}$ can be defined via product integrals. As observed in Arendt and Dier, Example 7.3 $U(t, s)$ is sub-Markovian for all $t, s$; in particular, $U$ extrapolates to a consistent family of contractive evolution families on $\ell^2(V)$ for all $p \in [1, \infty]$. The evolution family is also positivity improving, and additionally stochastic if $G$ is finite. Furthermore, Laasri\cite{Laasri:2015}, Thm. 2.6 yields that the evolution family is immediately norm-continuous if $[0, T) \ni t \mapsto m(t) \in \ell^\infty(E)$ is Hölder continuous with exponent $\alpha > 1/2$; by Fattorini,\cite{Fattorini:2005}, Thm. 7.4.1 it is even holomorphic if additionally $[0, T) \ni t \mapsto m(t) \in \ell^\infty(E)$ extends to a holomorphic function on an open convex neighborhood in $C$ of $[0, T]$. To conclude, let us study Laplacians on subgraphs $G_t$ induced by subsets $V_t$ of $V$ as in Chung\cite{Chung:2007}, Chapt. 8 in the unweighted case $(m(t, e) \in \{0, 1\})$. Even in the autonomous case, Laplacian on (nontrivial) subgraphs of $G$ generate semigroup that neither are dominated by nor dominate $(e^{-t\mathcal{L}_G})_{t \geq 0}$: This can, e.g., be seen by applying Ouhabaz.\cite{Ouhabaz:2013}, Cor. 2.22 Things change, however, if Dirichlet boundary conditions are imposed, e.g., if $\mathcal{L}$ on $G$ is restricted to

$$D_t := \{ f \in \ell^2(V) : f|_{V \setminus V_t} \equiv 0 \}, \quad t \in [0,T].$$

Because $D_t$ is for all $t$ a generalized ideal of $V = \ell^2(V)$, the associated Laplacian $\mathcal{L}|_{D_t}$ generates for all $t$ a semigroup $(e^{-t\mathcal{L}|_{D_t}})_{t \geq 0}$ which is—again by Ouhabaz,\cite{Ouhabaz:2013}, Cor. 2.22—dominated by $(e^{-t\mathcal{L}})_{t \geq 0}$. Therefore, by Proposition 3.1 and Proposition 3.9, the evolution family $(U(t, s))_{t, s \in \Delta}$ satisfies

$$|U(t, s)f| \leq e^{-\Delta(t-s)}|f| \quad \text{for all} \quad (t, s) \in \Delta \quad \text{and} \quad f \in H.$$  \hfill (6.1)
We have seen in the introduction that if $A(t) \equiv A$, then the evolution family that governs the nonautonomous problem is given by $U(t,s) = e^{(t-s)A}$, hence $U^*$ satisfies Gaussian bounds if and only if so does $(e^{rA})_{r \geq 0}$. We can now show a less trivial instance of Gaussian-type bounds.

Gaussian-type kernel estimates on $(e^{-tC})_{t \geq 0}$ have been proved in Delmotte\(^49\) for certain classes of $G$. Thus, if $(G_t)_{t \in [0,T]}$ is a family of subgraphs of a reference graph $G$ with measurable $t \mapsto m(t, \mathbf{e})$ for all $\mathbf{e} \in E$, and if $U^*$ is the evolution family associated with the corresponding Laplacians $-L|_{D_t}$, then (6.1) yields a Gaussian-type kernel estimate. If we, eg, take $G$ to be $\mathbb{Z}$, then

$$0 \leq \Gamma(t,s; n_1, n_2) \leq G(t-s; n_1, n_2), \quad (t,s) \in \Delta, \quad n_1, n_2 \in \mathbb{Z},$$

where

$$G(r; n_1, n_2) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((n_1 - n_2)q)e^{-2r(1-\cos q)}dq$$

is the heat kernel on $\mathbb{Z}$ explicitly computed, eg, in Davies.\(^45\), Exa. 12.3.3

6.2 | Time-dependent pageranks

Let us study a model similar to that of Example 6.1: It is based on an idea proposed in Chung\(^50\); cf Gleich\(^51\) for later developments, where the connectivity of $G$ describes the links within a server network—possibly the whole World Wide Web.

We thus consider an orientation of a finite complete graph (ie, a graph such that either $(v, w) \in E$ or $(w, v) \in E$ for any $v, w \in V$ with $v \neq w$). As in 6.1, we assign a weight $m$ to each edge: if eg, $m(t, \mathbf{e}) \in \{0,1\}$ for all $\mathbf{e} \in E$ and all $t \in [0,T]$, then we are effectively shutting off/switching on certain links in the considered network. We then consider the matrix

$$A := IM(I^-)^T(D^{out})^{-1} \quad (6.2)$$

where $I$ is again the incidence matrix of $G$ (see Example 6.1), $I^- := (l_{ve})$ is its negative part, $M = \text{diag} \big( m(\mathbf{e}) \big)_{\mathbf{e} \in E}$, and $D^{out} := \text{diag} \big( \deg^{\text{out}}(V) \big)_{v \in V}$, where $\deg^{\text{out}}(V) := \sum_{\mathbf{e} \in E} l_{ve} m(\mathbf{e})$.

Then, $A$ defines a so-called heat kernel pagerank $e^{-rA}x$ of $G$ with parameters $r$ and $x$: here $r$ is a positive time and $x$ a probability distribution on $V$, ie, $x \in \mathbb{R}^V$, $x_v \geq 0$ for all $v \in V$ and $\|x\|_1 = 1$. The rationale behind this definition is that $A$ is a column stochastic matrix, hence $(e^{-rA})_{r \geq 0}$ is a stochastic semigroup and $e^{-rA}f$ is thus again a probability distribution for all $r \geq 0$, which can be used to measure the relevance of a certain node within a network in a way similar to Google’s classical PageRank, cf Mugnolo.\(^47\), § 2.1.7.3

We can now consider a measurable function $t \mapsto (m(t, \mathbf{e}))_{\mathbf{e} \in E}$ and accordingly a time-dependent matrix family $(A(t))_{t \in [0,T]}$ as in (6.2) these matrices will in general not be symmetric, but in view of finiteness of $V$ they are certainly associated with a form $a \in \text{FORM}(\{0,T\}; c^2(V), c^2(V))$. Accordingly, in view of Proposition 3.8 the associated evolution family $(U(t,s))_{t,s \in \Delta}$ consists of stochastic operators and hence $U(t,s)f$ is a probability distribution on $V$ for all $(t,s) \in \Delta$ and all probability distributions $f \in \mathbb{R}^V$.

There is a correspondence between linear transport differential equations on networks and flows on their underlying graphs\(^52\): accordingly, our results also extend to the space-continuous case. The well-posedness result in Bayazit\(^53\), § 6—which relies on the assumption that the dependence of the graph on time is absolutely continuous—can thus be strengthened: We omit the details.

6.3 | Black–Scholes equation with time-dependent volatility

The Cauchy problem consisting of the backward parabolic equation

$$u_t(t,x) + \frac{1}{2} \sigma^2 u_{xx}(t,x) + rxu_x(t,x) - ru(t,x) = 0, \quad t \in [0,\tau), \quad x \in ]0,\infty[,$$

along with the final value assignment

$$u(\tau, x) = h(x) \quad x \in ]0,\infty[$$

was derived in Black and Scholes\(^54\) and is currently considered among the main mathematical tool in the pricing theory of European options: the positive constants $\sigma, r$ describe volatility and interest rate of the system, respectively, whereas $\tau$ is the maturity time of an option.
An effective variational approach to the relevant operator appearing in the Black–Scholes equation has been discussed in Einemann\textsuperscript{55}: It is based on studying the sesquilinear form

\[
a(u, v) := \frac{\sigma^2}{2} \int_0^\infty x^2 u'(x)v'(x) \, dx + (\sigma^2 - r) \int_0^\infty x u'(x)v(x) \, dx + r \int_0^\infty u(x)v(x) \, dx, \quad u, v \in V,
\]

defined on the form domain

\[
V := \{ u \in W^{1,1}_0[0, \infty] \cap L^2[0, \infty] : id \cdot u' \in L^2[0, \infty], \}
\]

which is a Hilbert space with respect to the inner product

\[
(u|v)_V := \int_0^\infty x^2 u'(x)v'(x) \, dx + \int_0^\infty u(x)v(x) \, dx.
\]

Then it was proved in Einemann\textsuperscript{55, § 7.2} that \( V \) is continuously dense in \( L^2[0, \infty] \) and that furthermore

\[
|a(u, v)| \leq \left( \frac{\sigma^2}{2} + |\sigma^2 - r| + |r| \right) \| u \|_V \| v \|_V
\]

for all \( u \in V \),

\[
\text{Re } a(u, u) + \left( \sigma^2 - \frac{3r}{2} \right) \| u \|_V^2 = \frac{\sigma^2}{2} \| u \|_V^2
\]

ie, \( a \) is bounded and elliptic; and \( \text{Re } a(u, u) \geq 0 \) if \( 3r \geq \sigma^2 \).

The original Black–Scholes-theory assumes \( \sigma \) to be time-independent but is rather unrealistic and has been questioned ever since: We mention the celebrated Heston model,\textsuperscript{23} which leads to a nonautonomous PDE similar to \( (6.3) \), based on the assumption that the volatility evolves following a certain Brownian-like motion. This justifies the study of

\[
u_t(t, x) + \frac{1}{2} x^2 \sigma^2(t) u_{xx}(t, x) + 3 x \sigma(t) u_x(t, x) - ru(t, x) = 0, \quad t \in [0, \tau], \quad x \in [0, \infty],
\]

with measurable dependence \( t \mapsto \sigma(t) \). The computations in \( (6.5) \) show that if \( 0 < \sigma_0 \leq \sigma(t) \leq \sigma_1 \) for a.e. \( t \in [0, \tau] \), then \( a \in \text{FORM}([0, \tau]; V; H) \), where \( a \) is defined by

\[
a(t; u, v) := \frac{\sigma^2(t)}{2} \int_0^\infty x^2 u'(x)v'(x) \, dx + (\sigma^2(t) - r) \int_0^\infty x u'(x)v(x) \, dx + r \int_0^\infty u(x)v(x) \, dx, \quad u, v \in V.
\]

Furthermore, the semigroup associated with \( a \) is quasi-contractive and sub-Markovian: we deduce from Proposition 3.1 that such nonautonomous Black–Scholes equation is governed by a sub-Markovian evolution family \( U^\tau \) that extrapolates to all \( L^p[0, \infty] \) spaces, \( p \in [2, \infty] \). In view of Proposition 3.1 we can also apply Einemann\textsuperscript{55, Thm. 7.2.5} and deduce that \( U^\tau \) leaves invariant the order interval \( ] - \infty, id]_H \). By Einemann,\textsuperscript{55, Rem. 7.2.4} min\{log, 0\} \( \in V \), hence \( V \nsubseteq L^\infty[0, \infty] \); however, it is unclear whether \( V \) satisfies a Nash or Gagliardo-Nirenberg inequality, which would imply ultracontractivity of the evolution family.

(A manifold of financial models exist that display such a similar mathematical structure, albeit their meaning is different: the popular Cox–Ingersoll–Ross along with several other so-called short-rate models surveyed in Chan et al\textsuperscript{56} involve time-dependent \( \sigma \) and/or \( r \) and can be discussed with only minor variations to our treatment above.)

### 6.4 Second-order elliptic operators on networks

With the purpose of introducing a differential operator on a network-like structure, we consider once again a graph \( G = (V, E) \) (like in Example 6.1) and identify each edge \( e \in E \) with an interval \([0, 1] \). In other words, we are considering
a collection of copies of $[0,1]$ and gluing them in a graph-like fashion: we thus obtain what are often called metric graphs or networks in the literature.\cite{Mugnolo2017, Lumer1964} (For the sake of simplicity we are going to assume such a metric graph to be connected.) The history of nonautonomous diffusion equations on networks goes back at least to pioneering investigations by von Below, Lumer, and Schnaubelt: well-posedness results could be proved in previous works,\cite{Arendt2007, Mugnolo2007} further results on long-time asymptotics have been deduced in Arendt et al.\cite{Arendt2012}

We are going to apply in this context the theory developed in the previous sections: when introducing operators on metric graphs we avoid to go into full details and refer the interested reader to Mugnolo.\cite[Chapter 6]{Mugnolo2017} To fix the ideas, consider a possibly infinite, but uniformly locally finite (see Example 6.1) graph $G = (V, E)$ and upon rescaling identify each edge $e$ with an interval $(0,1)$. On each interval $e$ we consider the operator family

$$A_e(t) : u_e \mapsto -\frac{d}{dx} \left( c_e(t) \frac{du_e}{dx} \right) - p_e(t) u_e, \quad t \in [0, T] :$$

we assume the coefficients

$$[0, T] \ni t \mapsto c_e(t) \in L^\infty(0,1; L^\infty(E)) \quad \text{and} \quad [0, T] \ni t \mapsto p_e(t) \in L^1(0,1, L^\infty(E))$$

to be measurable: this defines in a natural way an operator $A$ with domain $D(A) := \hat{H}^2(G) := H^2(0,1; E)$ on the Hilbert space

$$H := L^2(G) := L^2(0,1; E).$$

We will additionally assume that the operator family is uniformly elliptic, ie,

$$c_e(t,x) \geq \gamma \quad \text{for all} \quad e \in E \quad \text{and a.e.} \quad x \in (0,1), \quad t \in [0, T]. \quad (6.6)$$

for some $\gamma > 0$. In order to reflect the topology of the graph, transmission conditions in the vertices are required: the most common conditions are usually referred to as continuity/Kirchhoff and amount to asking that

- $u$ is continuous, ie, the boundary values of $u_e$ and $u_f$ agree whenever evaluated at endpoints of the intervals $e,f$ that are glued together in the network $G$ (continuity);
- $u$ satisfies a Kirchhoff-type rule, ie, at any vertex the sum over all neighboring edges of the normal derivatives evaluated at the vertex vanishes.

However, many more boundary conditions are conceivable: indeed, a parametrization of a large class of boundary conditions that fits very well the setting of sesquilinear forms has been discussed in Mugnolo,\cite[§ 6.5.1]{Mugnolo2017} based on the finite case treated in Kuchment.\cite[Thm. 5]{Kuchment1994} Indeed, fix a closed subspace $Y$ of the Hilbert space $E \times E$, let $(\Sigma(t))_{t \in [0,T]}$ be a family of bounded linear operators on $Y$, and consider the nonautonomous form $a$ defined by

$$a(t; u, v) = \int_0^1 \left[ \left( c_e(t,x) u_e'(x) | v_e'(x) \right)_{E(x)} + (p_e(t,x) u_e(x) | v_e(x))_{E(x)} \right] dx + (\Sigma(t) u | v)_Y$$

with time-independent form domain

$$H_1^\Sigma(G) := \{ u \in \oplus_{e \in E} H^1(0,1; E) : u \in Y \}$$

where

$$u := \left( \begin{array}{c} (u_e(0))_{e \in E} \\ (u_e(1))_{e \in E} \end{array} \right).$$

Then the conditions in the vertices satisfied by functions in the domain of each operator $A(t)$ associated with $a$ can be written in a compact form as

$$u \in Y \quad \text{and} \quad u + \Sigma(t) u \in Y^\perp. \quad (6.7)$$

where

$$u := \left( \begin{array}{c} (-c_e(0) u_e'(0))_{e \in E} \\ (c_e(1) u_e'(1))_{e \in E} \end{array} \right).$$
We finally assume that for some $P, S > 0$

$$
\|p(t)\|_{L^1} \leq P \quad \text{and} \quad \|\Sigma(t)\|_{L^2(Y)} \leq S \quad \text{for all} \quad t \in [0, T].
$$

Using an obvious extension of Mugnolo47, Lemma 6.22 to nonautonomous forms we see that $\Lambda$.

**Proposition 6.1.** Under the above assumptions on the coefficients $c_e, p_e$, the space $Y$, and the operators $\Sigma$, the form $a$ is associated with a strongly continuous evolution family $U$ on $H$. If $p_e(t) \geq 0$ and $\Sigma(t)$ is accretive for a.e. $t \in [0, T]$, then $U$ is contractive.

If all these coefficients are defined on the whole interval $[0, \infty]$, then $U$ extends to an evolution family on $\{(t, s) : 0 \leq s \leq t < \infty\}$.

We denote by $P_Y$ the orthogonal projector of $\ell^2(E) \times \ell^2(E)$ onto $Y$; the latter inherits the lattice structure of $\ell^2(E) \times \ell^2(E)$. Owing to Proposition 3.4, we can formulate the following generalization of Mugnolo47, Thm. 6.85 (see also Cardanobile and abstract results in the previous sections hence yield the following.

**Corollary 6.2.** (1) If $e^{-r^2(t)}$ (for all $r \geq 0$ and a.e. $t \in [0, T]$) and $P_Y$ are positive, then $U$ is positive. If additionally $p_e \equiv 0$, $e^{-r^2(t)}$ is (sub-)stochastic and $I \in Y$, then $U$ is (sub-)stochastic.

(2) Let $p_e(t, x) \geq 0$ for all $e \in E$ and a.e. $t \in (0, T)$ and $x \in (0, 1)$. If $e^{-r^2(t)}$ (for all $r \geq 0$ and a.e. $t \in [0, T]$) and $P_Y$ are $\ell^\infty$-contractive, then $U$ is $L^\infty(0, 1; \ell^2(E))$-contractive.

(3) Under the assumptions of (2), let additionally $e^{-r^2(t)}$ be $\ell^\infty$-contractive for all $r \geq 0$ and a.e. $t \in [0, T]$. Then $U$ is completely contractive; accordingly, it extrapolates to a strongly continuous, contractive evolution family on all spaces $L^p(0, 1; \ell^2(E)), p \in [1, \infty]$.

**Example 6.3.** The continuity/Kirchhoff vertex conditions are special cases of the general conditions in (6.7). Indeed, denote by $c_V$ the vector in $\ell^2(E) \times \ell^2(E)$ that consists of vertex-wise constants, i.e., entries of $c_V$ agree whenever they correspond to endpoints of edges the same vertex $v \in V$ is incident with. Let by $Y$ the subspace of $\ell^2(E) \times \ell^2(E)$ spanned by $c_V$ and take $\Sigma = 0$: then (6.7) agrees with continuity Kirchhoff conditions: We denote by

$$
H^1(G)
$$

the Sobolev space $H^1(G)$ with respect to this distinguished space $Y$. Under stronger assumptions on $p_e, c_e$, a well-posedness result comparable to Proposition 6.1 has been obtained in Arendt et al.32, Thm. 3.1. Because $H^1(G) \hookrightarrow C(G)$, we furthermore deduce that $U(t, s)$ maps $C(G) \cap L^2(G)$ into $C(G)$ for all $s \in [0, T]$ and a.e. $t \in (s, T)$.

Due to the standing assumption that $G$ is uniformly locally finite, $P_Y$ is a block operator matrix whose blocks are of the form $\frac{1}{n}J_n$, where $J_n$ denotes the $n \times n$ all-1-matrix, $n$ the degree of the corresponding vertex). Because $P_Y$ leaves invariant the order interval $[-1, 1]_{j=0}^E \Sigma \equiv 0$, we deduce that $(e^{-rA(t)})_{r \geq 0}$ is positive and -- if $p_e \geq 0$ -- sub-Markovian for a.e. $t \in [0, T]$: hence by Proposition 3.1 so is the evolution family $U$.

As concerns the long time behavior if $T = \infty$, we can hence discuss two cases:

- If $\liminf_{t \to \infty} \frac{1}{t-H} \int_0^t e^{r(s)}dr > 0$ for all $e \in E$, then $U$ is by Proposition 2.3 uniformly exponentially stable.
- Let $p_e \equiv 0$ for all $e \in E$. The network is always assumed to be connected; if it is additionally finite (i.e., $|E| < \infty$), then by Proposition 3.5 and Mugnolo47, Prop. 6.70 0 is a simple eigenvalue of each $\lambda(t); \lambda(t)$ is self-adjoint and the its null space consists of all constant functions. Furthermore, not only has each $\lambda(t)$ a spectral gap, but there is a uniform lower bound on them: By Nicaise’ inequality62, Théo. 3.1 $S(t) \geq \frac{\pi \sqrt{a}}{|E|} > 0$ for a.e. $t \in [0, \infty]$, and we conclude by Proposition 3.5 that

$$
\left\| U(t, s) f - \frac{1}{\sqrt{|E|}} \int_G f(x) \cdot 1 \right\| \leq e^{\frac{\pi \sqrt{a}}{|E|} (s-t)} \| f \| \quad \text{for all} \quad f \in H \quad \text{and} \quad 0 \leq s \leq t < \infty.
$$

Similar convergence results have been obtained in the strong topology for general inhomogeneous diffusion equations in Arendt et al.32, § 5.1.
Proposition 6.4. Let \( p_e(t, x) \geq 0 \) for all \( e \in E \) and a.e. \( t \in [0, T], x \in (0, 1) \). Let furthermore \( P_T = (\pi^T) \) and \( \Sigma(t) = (\sigma(t)) \) satisfy

- \( \sum_j |\sigma_{ij}| \leq 1 \) for all i;
- \( \Re \sigma(t)_i \geq \sum_j |\sigma(t)_j| \) for all i and a.e. \( t \in [0, T] \);
- \( \Re \sigma(t)_i \geq \sum_j |\sigma(t)_j| \) for all i and a.e. \( t \in [0, T] \).

Then \( U \) is completely contractive. If additionally \( Y \) is a generalized ideal of \( (c_V) \), then \( U \) is ultracontractive.

Proof. It follows from Mugnolo\(^{24}\), Lemma 6.1 and Corollary 6.2 that \( U \) is completely contractive.

Under the assumption that \( Y \) is a generalized ideal of \( (c_V) \) it has been shown in Pröpper\(^{63}\), Chapt. 3 that \( H^1(G) \) satisfies a Nash inequality whenever \( G \) is a connected, locally finite metric graph with edge lengths uniformly bounded away from 0: accordingly, the nonautonomous form with domain \( H^1(G) \) is associated with an ultracontractive \( U \), owing to Theorem 4.3.

In order to complete the proof, observe that by Cardanobile and Mugnolo\(^{61}\), Thm. 6.2 the semigroup associated with \( a(t) \equiv a \) with domain \( H^1_1(G) \) is dominated by the semigroup associated with the same form with domain \( H^1(G) \), provided \( Y \) is a generalized ideal of \( (c_V) \) (in fact by Nagel\(^{64}\), Thm. C.II-5.5 the latter is the modulus semigroup of the former one). By Proposition 3.1, the same holds for the associated evolution families, hence the former heat kernel inherits ultracontractivity from the latter one.

The following result seems to be new even in the autonomous case: in Mugnolo\(^{24}\) Gaussian bounds for heat kernels on finite networks have been proved only in the special case of \( Y = (c_V) \), see also Mugnolo\(^{47}\), Chapt. 7 for an abstract approach based on the theory of Dirichlet forms.

Corollary 6.5. Under the assumptions of Proposition 6.4, let \( Y \) be \( (c_V) \)-invariant (ie, the entrywise product \( \psi c_V \) lies in \( Y \) for all \( \psi \in Y \)), where the vector \( c_V \) is defined as in Example 6.3. If furthermore \( \Sigma(t) \) is diagonal for a.e. \( t \in [0, T] \), then \( U \) satisfies Gaussian bounds.

We stress that our assumption on \( P_T \) and \( \Sigma \) are only enforcing complete contractivity, but the evolution family need not be positive. An example is given by the nonautonomous parabolic equation on a loop with boundary conditions defined by \( \Sigma \equiv 0 \) and \( Y = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \), which is \( (c_V) \)-invariant (here \( c_V = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) ); this equation is governed by a completely contractive and (in view of the Nash inequality for \( H^1(G) \)) ultracontractive evolution family \( U \), which therefore enjoys Gaussian bounds. However, \( U \) is not positive, since neither is \( P_T \).

Proof of Corollary 6.5. We apply Davies’ Trick in a slightly different version. Indeed, we adapt the usual setting to our network environment by introducing the space

\[
W_G := \{ \psi \in H^1(\mathcal{G}) \cap C^\infty(0, 1; \ell^2(E)) : \|\psi\|_\infty \leq 1, \|\psi\|_\infty \leq 1 \}.
\]

then one can check that

\[
d(x, y) := \sup \{|\psi(x) - \psi(y)| : \psi \in W_G \}, \quad x, y \in \mathcal{G},
\]
defines a metric on \( \mathcal{G} \) that is equivalent to the canonical one.\(^{47}\), § 3.2 (Recall that \( H^1(G) \) denotes the space \( H^1_1(G) \), where \( \hat{Y} = (c_V) \) is the space spanned by \( c_V \); each function in \( H^1(G) \) is by definition continuous on the metric space \( G \). Observe that \( (c_V) \) is in fact an algebra with respect to the entry-wise product.) By definition, \( H^1_1(G) \) is \( W_G \)-invariant if and only if \( u \in Y \) implies \( e^\psi u \in Y \), ie, if and only if \( Y \) is \( (c_V) \)-invariant.

If \( \Sigma(t) \equiv 0 \), then the assertion has been proved in Mugnolo\(^{24}\), Thm. 4.7 by showing that the relevant form (let us denote it by \( a_0 \) to stress the absence of boundary terms) induces perturbed forms \( a'_0 \) that are associated with completely contractive perturbed evolution families (with the form domain being unchanged and still satisfying a Nash inequality). In the general case of \( a(t; u, v) = a_0(t; u, v) + (\Sigma(t)u)v_Y \), we find that \( a'(t; u, v) = a'_0(t; u, v) + (\Sigma(t)u)v_Y \). These forms are associated with completely contractive evolution families, hence the claim follows.
6.5 Second-order elliptic operators with complex coefficients on open domains

Let $\Omega \subset \mathbb{R}^d$ be an open set. On the complex Hilbert space $L^2(\Omega)$ we consider the nonautonomous form $a_V : [0, T] \times V \times V \to \mathbb{C}$ defined by

$$a_V(t; u, v) := \sum_{k,j=1}^d \int_{\Omega} a_{kj}(t; x) D_k u D_j \overline{v} \, dx + \sum_{k=1}^d \int_{\Omega} \left[ b_k(t; x) D_k u \overline{v} + c_k(t; x) u D_k \overline{v} \right] \, dx + \int_{\Omega} a_0(t; x) u \overline{v} \, dx$$

(6.8)

for all $u, v \in V$ where $V$ is a closed subspace of $H^1(\Omega)$ that contains $H_0^1(\Omega)$. We assume that the coefficients $a_{kj}, b_k, c_k, a_0$ lie in $L^\infty([0, T] \times \Omega; \mathbb{C})$. Moreover, we assume that the principal part is uniformly elliptic, i.e., there exist a constant $\nu > 0$ such that

$$\text{Re} \sum_{k,j=1}^d a_{kj}(t; x) \xi_k \xi_j \geq \nu |\xi|^2 \quad \text{for a.e. } t \in [0, T], \ x \in \Omega \quad \text{and all } \xi \in \mathbb{C}^n.$$  

(6.9)

Then $a_V$ defined in (6.8) belongs to FORM([0, T]; $V, L^2(\Omega)$). In fact, we have

$$|a_V(t; u, v)| \leq M \|u\|_V \|v\|_V$$

and

$$\text{Re} \ a_V(t; u, u) + \omega \|u\|_{L^2(\Omega)}^2 \geq \nu \|u\|_V^2,$$

where $M > 0$ is a constant depending only on $\|a_{ij}\|_{\infty}, \|b_k\|_{\infty}, \|c_k\|_{\infty}, \text{ and } \|a_0\|_{\infty}$, and one can choose $\omega = \sum_{k=1}^d \frac{1}{2} \left( \|\text{Re}(b_k + c_k)\|_{\infty} + \|\text{Im}(b_k - c_k)\|_{\infty} \right) + \|\text{Re} a_0\|_{\infty}$. Here $(\text{Re} a_0)^- = \max\{0, -\text{Re} a_0\}$. We can then associate a family of operators $A_V(t) \in \mathcal{L}(V, V')$, $t \in [0, T]$, with the form $a_V$ which are formally given by

$$A_V(t) = -\sum_{k,j=1}^d D_j a_{kj} D_k u + \sum_{k=1}^d b_k D_k u - \sum_{k=1}^d D_k (c_k u) + c_0 u.$$

(6.10)

Let $A_V(t)$ be the operator associated with $a(t; \cdot, \cdot)$ on $L^2(\Omega)$. Thus $A_V(t)$ is the realization of $A_V(t)$ in $L^2(\Omega)$ with various boundary condition which are determined by the form domain $V$. For example $A_V(t)$ is the realization of $A_V(t)$ with

a. Dirichlet boundary condition if $V = H_0^1(\Omega)$.

b. Neumann boundary condition if $V = H^1(\Omega)$.

c. Mixed boundary condition if $V = \{ u_{\Omega} : u \in C^\infty_c(\mathbb{R}^d \setminus \Gamma) \}^H(\Omega)$

where $\Gamma$ is a closed subset of the boundary of $\Omega$.

In particular, $a_V$ is associated with an evolution family $U_V$ that governs the nonautonomous problem driven by the operator family $(A_V(t))_{t \in [0, T]}$. Each of these evolution families is positive, it dominates the evolution family $U_{H_0^1}$ and is dominated by $U_{H^1}$. Following Ouhabaz, we introduce the following notations:

$$f_k(t, x) := \sum_{j=1}^d D_j (\text{Im} a_{kj}(t, x)), \quad m(t, x) := \frac{1}{4\nu} \sum_{k=1}^d \left[ f_k(t, x) + \text{Im}(c_k(t, x) - b_k(t, x)) \right],$$

$$\mathcal{R}_V(t; u, v) := \sum_{k,j=1}^d \text{Re}(a_{kj}) D_k u D_j \overline{v} \, dx + \sum_{k=1}^d \int_{\Omega} \left[ \text{Re}(b_k) D_k u \overline{v} \, dx + \text{Re}(c_k) u D_k \overline{v} \right] \, dx + \int_{\Omega} \text{Re}(a_0) u \overline{v} \, dx.$$

Lemma 6.6. Let $a_V$ be given by (6.8) and denote by $U_V$ the associated evolution family on $L^2(\Omega)$. Assume that $(|v| \wedge 1) \text{sgnv} \in V$ for all $v \in V$. Moreover, we assume that $a_{kj}(t, \cdot)$ are real-valued functions for all $k, j = 1, 2, \ldots, d$. Then the evolution family $U_V$ is $L^p$-quasi-contractive for all $p \in (1, \infty]$ and we have

$$\| U_V(t, s) f \|_{L^p(\Omega)} \leq e^{(1-\delta)\nu t} \| f \|_{L^p(\Omega)} \quad \text{for all } (t, s) \in \Delta.$$  

(6.11)
where

\[
\hat{\omega}_p := \begin{cases} 
\|(\text{Re} a_0^-)^\sim\|_\infty + \frac{1}{p} \left( \frac{1}{p} + \frac{1}{2} \right) \sum_{k=1}^d \|b_k - c_k\|_\infty^2 + \frac{d}{2} \sum_{k=1}^d \|\text{Re} c_k\|_\infty^2 & \text{if } p \in [2, \infty[, \\
\|(\text{Re} a_0^-)^\sim\|_\infty + \frac{1}{p} \left( \frac{1}{2} + \frac{p-1}{p} \right) \sum_{k=1}^d \|b_k - c_k\|_\infty^2 + \frac{d}{p(p-1)} \sum_{k=1}^d \|\text{Re} b_k\|_\infty^2 & \text{if } p \in [1, 2].
\end{cases}
\]

(6.12)

Proof. The assertion follows from Theorem 3.7 and Ouhabaz.\textsuperscript{17}, Thm. 4.3

We can also discuss the case where \(a_{kj}\) are complex-valued functions.

**Lemma 6.7.** Let \(a_V\) be given by (6.8) such that \(|v| \wedge 1\)sgnv \(\in V\) for all \(v \in V\) and denote by \(U_V\) the associated evolution family on \(L^2(\Omega)\). Assume that \(f_k \in L^\infty([0, T] \times \Omega)\), \(\text{Im } (a_{kj}(t, \cdot) + a_{j,k}(t, \cdot)) = 0\) for all \(k, j = 1, 2, \ldots, d\) and a.e. \(t \in [0, T]\).

If either of the conditions

(i) \(V = H_0^1(\Omega);\)

(ii) \(V \neq H_0^1(\Omega), (\text{Re } u)^+ \in V\) for all \(u \in V\) and there exists two constants \(c_1, c_2 > 0\) such that

\[
\int_{\Omega} m(t, x)|u|^2 \, dx \geq c_1 \int_{\Omega} |u|^2 \, dx + c_2 \text{Re } R_V(t; u, u) \quad u \in V \quad \text{and} \quad t \in [0, T];
\]

are satisfied, then \(U_V\) is \(L^p\)-quasi-contractive and (6.11) holds (up to replacing \(\text{Re } a_0(t, \cdot)\) by \(\text{Re } a_0(t, \cdot) - m\) in the expression of \(\hat{\omega}_p\)).

Proof. The assertion follows again from Theorem 3.7 and Ouhabaz.\textsuperscript{17}, Thm. 4.4

**Remark 6.8.** If \(a_V\) fulfills the assumptions of Lemma 6.6 or those of Lemma 6.7, then we see that the evolution family \(U_V\) is linearly quasi-contractive where (4.9) is satisfied with

\[
\alpha_1 = \|(\text{Re } a_0 - m\|_\infty + \frac{1}{p} \sum_{k=1}^d \|b_k - c_k\|_\infty^2
\]

(6.13)

and

\[
\alpha_2 = \frac{1}{p} \sum_{k=1}^d \|\text{Re } c_k\|_\infty^2.
\]

(6.14)

Likewise, the evolution family \(\overline{U_V}\) associated with \(\overline{a_V}\) is linearly quasi-contractive and (4.9) is satisfied with \(\alpha_1^* = \alpha_1\) and

\[
\alpha_2^* = \frac{1}{p} \sum_{k=1}^d \|\text{Re } b_k\|_\infty^2
\]

(6.15)

In view of Remark (6.8), the following corollary follows directly from Lemma 6.6 and Lemma 6.7.

**Corollary 6.9.** Let \(a \) be given by (6.8) and denote by \(U_V\) the associated evolution family on \(L^2(\Omega)\). Suppose that \(V\) satisfies a Gagliardo–Nirenberg inequality and that the assumptions of Lemma 6.6 (or those of Lemma 6.7) hold. Then \(U_V\) is ultracontractive and satisfies 4.10 with \(\mu = \nu\) and \(\alpha_1 = \alpha_1^*, \alpha_2, \alpha_2^*\) defined by (6.13)-(6.15).

Now we are going to prove that the evolution family \(U_V\) governed by the time-dependent elliptic operator (6.10) satisfies Gaussian bounds. We known from Theorem 5.2 that \(U_V\) satisfies Gaussian bounds if and only if there exist a constants \(c > 0, n > 0\) and \(\omega \in \mathbb{R}\) such that

\[
\|\mathcal{M}_p U_V(t, s) \mathcal{M}_p^{-1}\| \in L^1([0, T] \times \mathbb{R}) \leq c|t - s|^\frac{n}{2} e^{\omega^2(1 + p^2)(t - s)}
\]
for all \( \rho \in \mathbb{R}, \psi \in W \) and \( 0 \leq s < t \leq T \). Let \( a \) given by (6.8). Then the nonautonomous form \( a^\iota(t,u,v) := a(t,M_\rho u,M_\rho^{-1}v) \) is given by

\[
a_\iota(t;u,v) := \sum_{k,j=1}^d \int_\Omega a_{kj}(t,x)D_ku D_jv \, dx + \sum_{k=1}^d \int \left[ b_{k,\rho}(t,x)D_ku\bar{v} + c_{k,\rho}(t,x)uD_k\bar{v} \right] \, dx + \int a_{0,\rho}(t;x)u\bar{v} \, dx \quad (6.16)
\]

where

\[
b_{k,\rho} = b_k - \rho \sum_{j=1}^d a_{kj}D_j \psi, \quad c_{k,\rho} = c_k + \rho \sum_{j=1}^d a_{jk}D_j \psi
\]

and

\[
a_{0,\rho} = a_0 - \rho^2 \sum_{j,k=1}^d a_{jk}D_j \psi D_k v + \rho \sum_{k=1}^d b_k D_k \psi - \rho \sum_{k=1}^d c_k D_k \psi.
\]

In the following we define for each \( \rho \in \mathbb{R}, \psi \in W \) the constants \( a_{i,\rho}, a_{i,\rho}^*, i = 1, 2 \), via formulas which are analogous to (6.13), (6.14), and (6.15) where \( Re a_0 \) is replaced by \( Re a_0 - m \) if \( a_{kj} \) are complex-valued functions. Further, we set

\[
c_0 := \max\{ ||a_{k,j}||_\infty, ||b_k||_\infty, ||c_k||_\infty, ||c_0||_\infty, k, j = 1, \ldots, d \}
\]

and

\[
\omega := 4c_0d^2 + 4c_0d^3v^{-1}. \quad (6.18)
\]

**Lemma 6.10.**

a. For all \( \rho \in \mathbb{R}, \psi \in W \)

\[
Re a^\iota(t;u,u) + \omega(1 + \rho^2)||u||^2 \geq \frac{\nu}{2}||u||^2 \quad \text{for a.e. } t \in [0, T], \text{ and all } u \in V \quad (6.19)
\]

b. Assume that \( f_k \in L^\infty([0, T] \times \Omega), \quad Im [a_{k,j}(t, \cdot) + a_{j,k}(t, \cdot)] = 0 \) for all \( k, j = 1, 2, \ldots, d \) and \( t \in [0, T] \). Then for all \( \rho \in \mathbb{R}, \psi \in W \) we have

\[
a_{1,\rho} = a_{1,\rho}^* \leq 2\alpha_1 + \rho^2(1 + 2d^2c_0 + 4d^3c_0^2v^{-1}) + c_0d^2 \quad (6.20)
\]

\[
a_{2,\rho} \leq 2\alpha_2 + 2d^3\rho^2c_0^2v^{-1} \quad (6.21)
\]

\[
a_{i,\rho}^* \leq 2\alpha_i^* + 2d^3\rho^2c_0^2v^{-1}. \quad (6.22)
\]

**Proof.** (a) We first show (6.19). Let \( k = 1, \ldots, d \) and \( u \in V \)

\[
\left| Re \left[ b_{k,\rho}(t;x)D_ku\bar{u} + c_{k,\rho}(t;x)u\bar{D}_k u \right] \right|
\]

\[
\leq \left| b_k(t;x)D_ku\bar{u} + c_k(t;x)u\bar{D}_k u \right| + |\rho| \left| \sum_{j=1}^d a_{kj}(t;x)D_j\psi D_ku\bar{u} - \sum_{j=1}^d a_{jk}(t;x)D_j\psi u\bar{D}_ku \right|
\]

\[
\leq 2c_0||D_k u||u| + 2d||\rho||c_0||D_k u||u| = 2c_0(1 + d||\rho||)|D_k u||u|
\]

\[
\leq \frac{\nu}{2}||D_k u||^2 + 2c_0^2(1 + d||\rho||)^2v^{-1}||u||^2
\]

\[
\leq \frac{\nu}{2}||D_k u||^2 + 4c_0^2d^2(1 + \rho^2)v^{-1}||u||^2,
\]

Here we used that \( |D_j \psi| < 1, i = 1, 2, \ldots, d \), the Young inequality and that \( d \geq 1 \). Thus we have

\[
\left| Re \sum_{k=1}^d \left[ b_{k,\rho}(t;x)D_ku\bar{u} + c_{k,\rho}(t;x)u\bar{D}_k u \right] \right| \leq \frac{\nu}{2} \sum_{k=1}^d ||D_k u||^2 + 4c_0^2d^3(1 + \rho^2)v^{-1}||u||^2
\]

(6.23)

Likewise,

\[
| Re a_{0,\rho}(t;x)u\bar{u} | \leq 4c_0d^2(1 + \rho^2)||u||^2
\]

(6.24)
Combining (6.9), (6.23) and (6.24) yields (6.19).

(b) Using again that $|D_j \psi| < 1$, one easily prove (6.21) and (6.22). Further,

$$\frac{1}{\nu} \sum_k \|b_k - c_k\|^2_{\infty} = \frac{1}{\nu} \sum_k \left[ \|b_k - a_k - \rho \sum_{j=1}^d a_{kj} D_j \psi - \rho \sum_{i=1}^d a_{ik} D_i \psi\|_{\infty} \right]$$

$$\leq \frac{1}{\nu} \sum_k \left[ 2\|b_k - a_k\|^2_{\infty} + 4d^2 \rho^2 c_0^2 \right]$$

$$\leq \frac{2}{\nu} \sum_k \|b_k - a_k\|^2_{\infty} + 4d^2 \rho^2 c_0^2$$

(6.25)

Since $\text{Im}(a_{kj} + a_{j,k}) = 0$ for all $k, j = 1, 2, \ldots, d$, we deduce that $m_\rho = m$ for every $\rho \in \mathbb{R}$. It follows that

$$\| (\text{Re} a_{0,\rho} - m_\rho) \|_{\infty} \leq \| (\text{Re} a_0 - m) \|_{\infty} + \| \rho^2 \sum_{i=1}^d a_{ik} D_i \psi D_k \psi + \rho \sum_{k=1}^d b_k D_k \psi - \rho \sum_{k=1}^d c_k D_k \psi \|_{\infty}$$

$$\leq \| (\text{Re} a_0 - m) \|_{\infty} + \rho^2 d^2 c_0 + \rho^2 + c_0 d^2.$$ 

This equality together with (6.5) prove (6.20).

Combining Theorem 5.5 with Lemma 6.10 and Corollary 6.9 we can finally prove Gaussian bounds for evolution families associated with families of uniform elliptic operators of the form (6.10).

**Theorem 6.11.** Let $V$ be $W^-$-invariant and satisfy (4.4). If the assumptions of Lemma 6.6 or those of Lemma 6.7 are satisfied, then $U_V$ satisfies Gaussian bounds. More precisely we have

$$\langle U_V(t, s) f \rangle(x) = \int_{\mathbb{R}^d} \Gamma_V(t, s, x, y) f(y) dy$$

(6.26)

where

$$|\Gamma_V(t, s, x, y)| \leq c_0 e^{c_0 (t-s)} (t-s)^{-\frac{d}{2}} \exp \left( -\frac{\beta_0 |x-y|^2}{4(t-s)} \right)$$

(6.27)

for a.e. $x \in \mathbb{R}^d$, all $(t, s) \in \Delta$, and all $f \in L^2(\mathbb{R}^d)$, where $c, c_0, n$ and $\beta_0$ are positive constants that depend only on $C_G, N, d, \nu, c_0$ and on the constant $\beta$ defined in (5.3).

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**CONFLICT OF INTEREST**

This work does not have any conflicts of interest.

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