Unconventional Quantum Critical Points

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In this paper we review the theory of unconventional quantum critical points that are beyond the Landau’s paradigm. Three types of unconventional quantum critical points will be discussed: (1). The transition between topological order and semiclassical spin ordered phase; (2). The transition between topological order and valence bond solid phase; (3). The direct second order transition between different competing orders. We focus on the field theory and universality class of these unconventional quantum critical points. Relation of these quantum critical points with recent numerical simulations and experiments on quantum frustrated magnets are also discussed.

Keywords: quantum critical point, topological order, topological defect

1. Introduction

By definition, a critical point is associated with a continuous phase transition between two different phases. In classical systems, i.e. systems at finite temperature where thermal fluctuation dominates quantum fluctuation, a critical point is almost always sandwiched between a high temperature thermal disordered phase and a low temperature ordered phase where certain global symmetry of the system is spontaneously broken, thus the symmetry $H$ of the system at low temperature is a subgroup of the symmetry $G$ at high temperature. This phase transition is described by an order parameter $\Phi$ that carries a nontrivial representation of $G$. The low temperature phase is characterized by a nonzero expectation value $\langle \Phi \rangle$, which is invariant under $H$. The low temperature phase is “degenerate”, in the sense that the inequivalent states with the same free energy form a manifold $M$:

$$M = G/H.$$ (1)

The critical point can be described by either a Ginzburg-Landau (GL) theory formulated in terms of order parameter $\Phi$, or by a “Nonlinear sigma model” (NLSM) defined in manifold $M$.

As a simple example, let us consider a classical critical point of a three dimensional magnet with a full spin SU(2) symmetry. In this system, the full symmetry $G$ is SU(2). If we describe the system using a GL theory, then the order parameter should be an O(3) vector $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, and the GL theory reads

$$F = \int d^3x \sum_{i=x}^z |\nabla_i \vec{\phi}|^2 + r|\vec{\phi}|^2 + u(|\vec{\phi}|^2)^2.$$ (2)
In the disordered phase with \( r > 0 \), the thermal expectation value \( \langle \vec{\phi} \rangle = 0 \), while in the ordered phase with \( r < 0 \), the expectation value \( \langle \vec{\phi} \rangle \sim \vec{n} \neq 0 \). Here \( \vec{n} \) is a unit vector: \( |\vec{n}|^2 = 1 \). In this case \( H = U(1) \), which corresponds to the spin rotation around \( \vec{n} \). The manifold of the ordered phase is \( \mathcal{M} = G/H = SU(2)/U(1) \), which corresponds to all the configuration of \( \vec{n} \), and it is equivalent to the two dimensional sphere \( S^2 \).

We can also describe this transition using a nonlinear sigma model defined on manifold \( S^2 \). The manifold \( S^2 \) is parametrized by the unit \( O(3) \) vector \( \vec{n} \), with constraint \( |\vec{n}|^2 = 1 \). The NLSM reads

\[
F = \int d^3x \sum_{i=x}^z \frac{1}{g} |\nabla_i \vec{n}|^2, \quad |\vec{n}|^2 = 1.
\]

In Eq. 2, the phase transition is tuned by \( r \), while in Eq. 3 the phase transition is tuned by \( g \), i.e., when \( g < g_c \) \((g > g_c)\) the system is in an ordered (disordered) phase. Both theories Eq. 2 and Eq. 3 are supposed to describe a phase transition that belongs to the 3D-O(3) Wilson-Fisher universality class. In order to quantitatively show the equivalence between Eq. 2 and Eq. 3 at the critical point, one should compute the critical exponents using renormalization group (RG) for both models. Suppose we can compute the RG exactly, then it will tell us the exact scaling dimension of \( r \) in Eq. 2 and the exact scaling dimension of \( \Delta g = g - g_c \) in Eq. 3 close to the critical point. These two scaling dimensions should be identical.

A quantum critical point (QCP) is a continuous quantum phase transition between two quantum ground states at zero temperature. The formalism of classical critical point can be applied to many QCPs, and these QCPs are called conventional QCPs. A conventional QCP is usually sandwiched between an ordered phase with symmetry breaking, and a disordered phase which is gapped and nondegenerate. This disordered phase must be “featureless”, namely by locally tuning the Hamiltonian, this phase can be adiabatically connected to a fully gapped direct-product state without any nontrivial correlation or entanglement, while the system energy gap remains finite during this process.

The description of a conventional QCP is semiclassical, i.e. it is equivalent to a classical critical point. One can simply view the time coordinate as one extra spatial coordinate, and write down the GL theory or NLSM according to the symmetry. In this formalism the trivial quantum disordered state is identified as the thermal disordered high temperature phase. The only complication here is the dimension of time and energy. In the simplest case, the scaling dimension of time and energy is \(-1\) and \(1\) respectively, namely they have the same dimension as the spatial coordinate and momentum, then in this case the effectively GL theory and NLSM both have a Lorentz invariance. Thus a \( d \)-dimensional conventional QCP is equivalent to a \( D = d + 1 \) dimensional classical critical point, and all the computation techniques that were applicable to classical critical points can be straightforwardly generalized to the conventional QCPs.
This semiclassical formalism strongly relies on the nature of the quantum disordered phase, \textit{i.e.} the semiclassical formalism is only applicable when the disordered phase is completely trivial (adiabatically connected to a direct product state). However, it has been unambiguously shown that the ground states of many quantum many-body systems have certain special nontrivial structure called “topological order”\cite{1}, even though the spectrum of the system is gapped. With topological order, the quantum disordered phase can no longer be adiabatically connected to a direct product state, thus it is \textit{inequivalent} to a thermal disordered phase. Since the topological order cannot be characterized using a semiclassical formalism, significant modification should be made in our description.

The ordered phase of a quantum many-body system can also be different from a classical system, although the most important difference is usually encoded in its excitations instead of ground state. To destroy an ordered phase, one usually has to proliferate or condense the topological defects of the ordered phase. For instance, to destroy a two dimensional superfluid phase, at finite temperature the thermal fluctuation will proliferate the vortex excitation, which leads to a Kosterlitz-Thouless transition\cite{2}, at zero temperature, it is the vortex condensation that destroys the superfluid phase. By definition a topological defect will carry certain quantized topological number. For example, the ground state manifold (GSM) $\mathcal{M}$ of a superfluid phase is $S^1$. Thus in superfluid a vortex defect carries a quantized vorticity $2\pi \times \text{Integer}$ due to the homotopy group $\pi_1[S^1] = \mathbb{Z}$. However, in quantum systems, sometimes a topological defect would carry some extra physical quantum number, which is also quantized due to the quantization of its topological number. Since the topological defect carries physical quantum number, the condensation of the topological defect will lead to another ordered phase with a different symmetry breaking. For instance, it was shown by Haldane\cite{3} and Sachdev\cite{4} that the Skyrmion defect of the two dimensional Néel order parameter of a spin-$1/2$ system always carries lattice momentum, thus when the Skyrmion of the Néel order condenses, the translation symmetry of the lattice must be spontaneously broken. This type of quantum phase transitions or QCPs are also unconventional.

In order to avoid confusions, in this paper we will consistently distinguish two different concepts: topological defect condensation \textit{v.s.} proliferation. Topological defect condensation refers to the situation where topological defect are defined in space only. This type of topological defects are usually referred to as “solitons”, and they can be viewed as particles with their own dynamics, and they can condense once their kinetic energy becomes dominant. Proliferation refers to the situation where the defects are defined in space-time, and these defects are usually called “instanton”. Since these defects already live in space-time, they can no longer be viewed as quantized particles, but they will make nonzero contribution to the imaginary time path integral, and this contribution can be either relevant or irrelevant to the long wavelength continuum limit physics. When this contribution becomes relevant, these topological defects (instantons) “proliferate”.

Throughout this paper we will focus on continuous quantum phase transitions...
only, because the usual wisdom is, when two states are separated by a generic unfinetuned continuous quantum phase transition, then these two states indeed belong to two “different” phases. However, if two states are separated by a first order quantum phase transition, namely certain physical quantity jumps discontinuously, then these two phases can still belong to the same phase. For example, let us consider the following GL theory for an Ising field $\Phi$:

$$S = \int d^d x d\tau \left( (\partial\mu \Phi)^2 + r\Phi^2 + g\Phi^3 + u\Phi^4 + \cdots \right) \quad (4)$$

Notice that this GL theory has a cubic term $g\Phi^3$. With nonzero $g$, by simply minimizing this GL theory, one can see that when $r$ is tuned to certain value, the expectation value of $\Phi$ will jump discontinuously, thus there is a first order transition. However, there is no qualitative difference between these two states around this transition, because they both have no symmetry left at all. Thus a first order transition does not necessarily imply a qualitative change of the state.

This paper is organized as follows: In section 2, we will discuss one example of conventional QCP, which is the QCP between the Mott insulator phase and superfluid phase of the Bose-Hubbard model. In section 3, we will discuss the unconventional QCP between an ordered phase and a topological phase, and we will take the best-understood $\mathbb{Z}_2$ topological phase as an example. In section 4, we will discuss the unconventional QCP between two different ordered phases, for instance the QCP between Néel and Valence Bond Solid (VBS) phase. In section 5, a unified field theory that contains all the unconventional QCPs discussed in the previous sections will be discussed.

2. An example of Conventional QCP

The most well-known quantum critical point, is the QCP between the Mott insulator (MI) and superfluid (SF) phase in the Bose-Hubbard model. This model was first studied as a toy model in Ref. 5, and later it was shown that this is actually a perfect model to describe the spinless bosonic atoms trapped in an optical lattice.\[6\]

The Bose Hubbard model reads

$$H = \sum_{<i,j>} -tb_i^\dagger b_j + H.c. + \frac{U}{2}(n_i - \bar{n})^2. \quad (5)$$

This model has a global U(1) symmetry $b_i \to \exp(i\theta)b_i$, which corresponds to the conservation of the total boson number. The phase diagram of this model is tuned by two parameters $\bar{n}$ and $t/U$. In the SF phase, the expectation value $\langle b_i \rangle \neq 0$, and the global U(1) symmetry of the model is spontaneously broken. When $\bar{n}$ is an integer, the MI phase of this simple model is a trivial quantum disordered phase, namely it is adiabatically connected to a direct product state: $\prod_i (b_i^\dagger)^{n_i}|0\rangle$. Thus in this model the SF-MI transition is a conventional QCP, and it can be described semiclassically.
Fig. 1. (a). The phase diagram of Eq. 6, tuned by both \( \tilde{n} \) and \( t/U \). There is an emergent Lorentz invariance when \( \tilde{n} \) is an integer (dashed lines in this phase diagram). (b). Interpretation of the MI-SF transition in terms of bosons, and in terms of vortices.

In order to describe the MI-SF transition, we should first introduce the continuum limit order parameter \( \psi(x) \sim b_i \), which carries the same representation of the global U(1) symmetry as \( b_i \). The MI-SF transition can be described by the following field theory:

\[
S = \int d\tau d^d x \left[ \mu \psi^* \partial_\tau \psi + |\partial_x \psi|^2 + \sum_i c_i^2 |\partial_i \psi|^2 + r |\psi|^4 + g |\psi|^4 + \cdots \right] \tag{6}
\]

The ellipses include other irrelevant terms allowed by symmetry. When \( \mu = 0 \), the field theory Eq. 6 has a particle-hole symmetry \( \psi \rightarrow \psi^* \). However, there is no precise particle-hole (PH) symmetry for boson systems, thus in the ellipses of Eq. 6, there are PH-symmetry breaking terms like \( \psi^* (\partial_x)^3 \psi + H.c. \). This term breaks the PH symmetry of the field theory Eq. 6, but it is irrelevant at the QCP \( r = 0 \). Thus when \( \mu = 0 \), the PH symmetry becomes exact in the continuum limit, where all the irrelevant terms flow to zero.

The mean field phase diagram of Eq. 6 is apparently PH symmetric at the lines \( \tilde{n} = k \), where \( k \in \text{Integers} \). Thus close to the lines \( \tilde{n} = k \), we can identify \( \mu \sim \tilde{n} - k \), i.e. the system has an emergent Lorentz invariance when \( \tilde{n} = k \). Let us focus on the spatial dimension \( d = 2 \), then when \( d = 2 \) and \( \mu = 0 \) this QCP is precisely described by a classical three dimensional GL theory with U(1) symmetry, thus this QCP belongs to the 3D O(2) (or 3D XY) Wilson-Fisher universality class.

We can also describe this MI-SF transition in a different way. In this phase diagram, SF is the ordered phase with global symmetry breaking. Thus this transition can also be viewed as the condensation of the topological defects of the SF phase, i.e. the vortices of the SF phase. In order to describe this transition in terms of vortices, we need to go to the dual picture. Inside the superfluid phase, the low
energy physics can be described by the following rotor model:

$$H = \int d^2x \frac{\hat{n}}{2}(\hat{\delta n})^2 + \rho_s(\hat{\theta})^2.$$  \hspace{1cm} (7)

where $\hat{\delta n} = \hat{n} - \bar{n}$ is the density fluctuation above the average filling of the bosons; $\theta$ is defined as $\psi \sim \sqrt{\rho_s} \exp(i\theta)$, and $\rho_s$ is the superfluid stiffness, $\hat{\delta n}$ and $\theta$ are a pair of conjugate variables, namely $[\hat{\delta n}_x, \theta_{x'}] = i\delta_{x,x'}$.

The duality transformation is formulated as follows: define dual vector field $\vec{E}$ and $\vec{A}$ as

$$\partial_i \theta = 2\pi \epsilon_{ij} E_j, \quad \hat{\delta n} = \frac{1}{2\pi}(\vec{\partial} \times \vec{A}),$$ \hspace{1cm} (8)

the commutation relation between $\hat{\delta n}$ and $\theta$ guarantees that $\vec{A}$ and $\vec{E}$ satisfy the algebra of a pair of vector canonical variables: $[E^a_x, A^b_{x'}] = i\delta_{ab}\delta_{x,x'}$. Also, since only the curl of $\vec{A}$ is related to a physical quantity, the dual description in terms of $\vec{A}$ and $\vec{E}$ must be invariant under the following gauge transformation: $\vec{A} \to \vec{A} + \vec{\partial} f$, which is the familiar gauge transformation for U(1) gauge field. With the new variables, the rotor model Hamiltonian is mapped to the Hamiltonian of a U(1) gauge field:

$$H = \int d^2x \frac{\hat{n}}{8\pi^2}(\vec{\partial} \times \vec{A})^2 + 4\pi^2 \rho_s(\vec{E})^2.$$ \hspace{1cm} (9)

The Goldstone mode of the SF phase is dual to the photon of the gauge field. In 2+1 dimension, an ordinary vector field has two polarizations at each momentum $\vec{k}$. However, for a vector gauge boson, one of the two polarizations is an unphysical gauge degree of freedom, thus a gauge boson at 2+1d only has one transverse physical mode at each momentum $\vec{k}$, this is why it can be dual to a real scalar Goldstone mode.

This duality implies the following identity:

$$\frac{1}{2\pi} \epsilon_{ij} \partial_i \partial_j \theta = \partial_i E_i.$$ \hspace{1cm} (10)

The left side of this equation vanishes when $\theta$ is smooth in the entire space, while it does not vanish when $\theta$ has a singular vortex defect, i.e. the vortex of $\theta$ is precisely the gauge charge of the dual gauge field: $\partial_i E_i = n_v$, here $n_v$ is the density of vortices. Inside the SF phase, an isolated vortex has logarithmic divergent energy; in the dual picture, an isolated gauge charge also has logarithmic divergent energy due to its coupling to the dual U(1) gauge field. Thus the dual theory of the SF phase is the following bosonic QED:

$$S = \int d\tau d^2x |(\partial_\mu - iA_\mu)\Phi|^2 + r_v|\Phi|^2 + u|\Phi|^4 + \frac{1}{\epsilon^2}(F_{\mu\nu})^2.$$ \hspace{1cm} (11)

The complex field $\Phi$ is the vortex field, i.e. $\Phi(\vec{r})$ annihilate a vortex at position $\vec{r}$. The SF-MI phase transition is driven by the condensation of the vortices. In this theory, the phase with $r_v > 0$ is an “uncondensed” phase of vortex, and in this phase there is one gapless photon, i.e. the dual of the SF Goldstone mode. On the other hand, in the phase with $r_v < 0$, the vortex condenses, and the system is completely
gapped due to the Higgs mechanism, which is equivalent to the MI phase. Thus we can claim that the 3D $O(2)$ Wilson-Fisher critical point is dual to the critical point of the bosonic QED.

Conventional QCPs also exist in some quantum spin models. However, it was proved that for a spin-$1/2$ system with a local Hamiltonian, the ground state has to be either gapless or gapped but degenerate, thus a fully gapped nondegenerate direct product ground state does not exist for an SU(2) invariant spin-$1/2$ model on a lattice with one site per unit cell [213]. Thus a conventional QCP cannot exist in ordinary spin-$1/2$ systems on square or triangular lattices, unless the spin Hamiltonian explicitly breaks the translation symmetry i.e. the unit cell is enlarged. For instance, let us investigate the following $J - \lambda$ model (Fig. 2) [213]:

$$H = \sum_{<i,j>} \lambda J \vec{S}_i \cdot \vec{S}_j + \sum_{<i,j>} J \vec{S}_i \cdot \vec{S}_j. \quad (12)$$

In this equation $<i,j>$ denotes solid links of the square lattice with Heisenberg coupling $\lambda J$; $<i,j>$ denotes dashed links with Heisenberg coupling $J$. When $\lambda \sim 1$, the system has an ordinary Néel order; when $\lambda \gg 1$, the Néel order disappears, and the system is disordered with a nondegenerate ground state that is adiabatically connected to the following direct product state:

$$\prod_{<i,j>} | \uparrow_{i} \downarrow_{j} \rangle - | \downarrow_{i} \uparrow_{j} \rangle. \quad (13)$$

This is the exact ground state wave function when $\lambda = \infty$. Notice that this state is a direct product between different unit cells, while it is a maximally entangled state within one unit cell.

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Fig. 2. The phase diagram of the $J - \lambda$ model Eq. (12). The solid links and dashed links have Heisenberg coupling $\lambda J$ and $J$ respectively. This QCP is a conventional QCP that belongs to the 3D $O(3)$ universality class.
The phase transition in Eq. 12 can be described by the semiclassical GL theory Eq. 2 and NLSM Eq. 3, and it belongs to the O(3) Wilson-Fisher fixed point. If we start with the Néel phase of this model, then this transition can be viewed as condensing the Skyrmion defect of the Néel order parameter. In this particular model, due to the explicit breaking of the translation and rotation symmetry of the square lattice, the Skyrmion will not carry any nontrivial physical quantum number, thus the condensate of the Skyrmion is a featureless direct product state. It is in fact a little surprising that one has to cook up a relatively complicated spin model Eq. 12 in order to realize a simple O(3) Wilson-Fisher transition in quantum spin systems.

3. Unconventional QCP between topological phase and ordered phase

3.1. \( \mathbb{Z}_2 \) topological phase

Starting with this section we will discuss unconventional QCPs, and we will first discuss the QCP between semiclassical ordered phases and phases with topological order. As we discussed in the introduction section, when a phase has topological order, it can no longer be adiabatically connected to a trivially gapped direct product state, thus the semiclassical description needs significant modifications.

In this section we will take the best understood \( \mathbb{Z}_2 \) topological phase in two spatial dimension as an example. The \( \mathbb{Z}_2 \) topological phase is described by the \( \mathbb{Z}_2 \) gauge field, which can be obtained by spontaneously breaking a compact U(1) gauge symmetry. The Hamiltonian of the compact U(1) gauge theory reads:

\[
H = \sum_{j,\mu} -t \cos(\vec{\nabla} \times \vec{A}) + \frac{U}{2} E_{j,\mu}^2.
\] (14)

Both \( A_{j,\mu} \) and \( E_{j,\mu} \) are defined on links of a square lattice. A link around site \( j \) is denoted as \((j, \mu)\), with \( \mu = \hat{x}, \hat{y} \). This Hamiltonian is always accompanied with the Gauss law constraint:

\[
\sum_{\mu} \nabla_\mu E_{\mu} = \rho_j.
\] (15)

\( \rho_j \) is the local charge density on site \( j \). The compact U(1) gauge field \( A_{j,\mu} \) is defined periodically: \( A_{j,\mu} = A_{j,\mu} + 2\pi \), and its canonical conjugate variable \( E_{j,\mu} \) must be discrete integers only.

In order to break the U(1) gauge symmetry to \( \mathbb{Z}_2 \) gauge symmetry, we can couple the compact U(1) gauge field to a U(1) rotor matter field \( \exp(i\phi) \):

\[
H = \sum_{\mu} -t \cos(\nabla_\mu \phi - 2A_\mu) + \cdots
\] (16)

The rotor field \( \exp(i\phi) \) can be viewed as a Cooper pair, which carries two unit gauge charges. When \( \phi \) is ordered, Eq. 16 is reduced to \(-t \cos(2A_\mu)\), which prefers \( A_\mu \) to
take only two values 0 and $\pi$. When $t$ is strong enough, we can effectively describe the physics of gauge field $A_\mu$ using the following Ising variables $\sigma^z$ and $\sigma^x$ defined on the links of the lattice:

$$
\sigma^z_{j,\mu} = \exp(iA_{j,\mu}), \quad \sigma^x_{j,\mu} = \exp(i\pi E_{j,\mu}) = \cos(\pi E_{j,\mu}).
$$

Here $A_{j,\mu}$ only takes two values 0 and $\pi$. Please note that $\sigma^z$ and $\sigma^x$ so defined satisfy the ordinary Pauli matrix algebra.

Introducing $\sigma^z$ and $\sigma^x$ as Eq. 17, the Hamiltonian of compact QED in Eq. 14 is reduced to the following form:

$$
H_{z^2} = \sum_i -K\sigma^z_i\sigma^z_{i+1,x}\sigma^z_{i+1,y}\sigma^z_{i+2,x,y} - h\sigma^x_{i,\mu}.
$$

This Hamiltonian Eq. 18 has a special discrete symmetry:

$$
\sigma^z_{i,\mu} \rightarrow \eta_i \sigma^z_{i,\mu} \eta_{i+\mu}.
$$

$\eta_i = \pm 1$ is an arbitrary $Z_2$ function on the lattice. Eq. 19 is precisely the $Z_2$ discrete gauge transformation. The model Eq. 18 is actually the minimal model that describes the $Z_2$ topological phase, and the quantum $Z_2$ gauge theory.

Just like the U(1) gauge theory, the $Z_2$ gauge theory is always subject to the following local gauge constraint:

$$
Q_i = \sigma^z_{i,x}\sigma^z_{i-y,x}\sigma^z_{i,y}\sigma^z_{i-y,y} = \chi_i.
$$

Here $\chi_i = \exp(i\pi \rho_i)$. This $Z_2$ gauge theory is called even or odd $Z_2$ gauge theory, when $\rho_i$ is an even or odd integer. For example, if a $Z_2$ topological phase is realized in a spin system, $\rho_i$ is usually the average density of “spinons” on every site: $\rho_i = \sum_\alpha f^\dagger_{\alpha,i} f_{\alpha,i}$, and for spin-1/2 systems there is precisely one spinon on every site, i.e. $\rho_j = 1$. Recently, it was demonstrated numerically that the $Z_2$ topological phase does exist in the $J_1 - J_2$ spin-1/2 Heisenberg model on the square lattice. Based on our analysis, this $Z_2$ topological phase must be an odd theory.

In condensed matter systems, the $Z_2$ gauge symmetry is usually obtained from spontaneously breaking a U(1) or even SU(2) gauge symmetry by condensing a matter field which is the analogue of “Cooper pair”, and this gauge symmetry breaking occurs at a rather high energy scale. At low energy we can safely ignore this “Cooper pair”, and describe everything using the effective $Z_2$ gauge theory.

When $K \gg h$ in Eq. 18 the system is in the “deconfined phase” of the $Z_2$ gauge theory. In the ground state of the deconfined phase, in addition to the constraint $Q_i = \chi_i$, the ground state (approximately) satisfies $\sigma^z_{i,x}\sigma^z_{i,y}\sigma^z_{i+1,y,x}\sigma^z_{i+1,x,y} = 1$ on every plaquette. In this deconfined phase, there are two types of local excitations above the ground state: the first type of excitation is a “electric” excitation, or the $Z_2$ charge excitation, which is a violation of the gauge constraint, i.e. $Q_i = -\chi_i$ at some site $i$; the second type of excitation is a “magnetic” excitation, which corresponds to $\sigma^z_{i,x}\sigma^z_{i,y}\sigma^z_{i+1,y,x}\sigma^z_{i+1,x,y} = -1$ on certain plaquette. The magnetic excitations are usually called the “visons”, and in terms of the original U(1) gauge theory it is simply a local $\pi$–flux through one plaquette. The unit electric and magnetic excitations satisfy the
mutual semion statistics, namely when a magnetic excitation adiabatically encircles an electric excitation through a closed loop, the system wave function will acquire a minus sign \[11][12].

The ground state of the deconfined phase of the $Z_2$ gauge theory is four fold degenerate on a torus. Starting with one of the ground states, the other three ground states can be obtained by inserting a vison ($\pi$–flux) through either hole of the torus. This degeneracy is topological, in the sense that in the thermal dynamical limit this degeneracy cannot be lifted through any weak local perturbation in the Hamiltonian, even if this perturbation breaks the $Z_2$ gauge symmetry.

Although the $Z_2$ topological phase is fully gapped, its topological nature can be described by the following mutual Chern-Simons field theory:

$$S_{mcs} = \int d^2 x d\tau \frac{i}{\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu b_\rho.$$  \hspace{1cm} (21)

$a_\mu$ and $b_\mu$ are two different U(1) gauge fields. This mutual Chern-Simons (CS) theory leads to precisely four fold degenerate ground states on a torus. Also, $a_\mu$ and $b_\mu$ are minimally coupled to the currents of electric and magnetic excitations of the $Z_2$ topological phase, and the mutual CS theory guarantees that the electric and magnetic excitations see each other as a $\pi$–flux, i.e. they automatically have the mutual semion statistics. Thus the mutual CS theory describes all the key properties of a $Z_2$ topological phase.

Although the minimal model Eq. 18 for the $Z_2$ topological phase looks quite abstract, the $Z_2$ topological order can be very reliably realized in various (quasi-)realistic models, such as the quantum dimer model on the triangular lattice \[13\], a XXZ spin-1/2 model on the Kagome lattice \[14\], and also a quantum spin Hall-Superconductor-Ferromagnet Josephson array \[15\]. For a model that is not exactly soluble, the best way to verify the $Z_2$ topological order is by computing the topological entanglement entropy, which was introduced in Ref. \[17][16\].

A series of exactly soluble models have been constructed in Ref. \[18\], and the phase diagram of these models have both the $Z_2$ topological order and ordered phases with symmetry breaking. In our paper we will focus on a more general discussion about this type of transition, that is driven by the condensation of topological excitations of this $Z_2$ topological state.

### 3.2. QCP between $Z_2$ topological phase and superfluid

Now let us assume that the $Z_2$ gauge field introduced in the previous section is coupled to a bosonic matter field, and this matter field carries a U(1) global quantum number in addition to the $Z_2$ gauge charge. The simplest lattice model that describes this physics reads

$$H = \sum_{<j,\mu>} -t\sigma_j^\mu \cos \left( \frac{\phi_j}{2} - \frac{\phi_{j+\mu}}{2} \right) + \frac{U}{2} (n_j - \bar{n})^2 + H_{\text{ss}}$$  \hspace{1cm} (22)
Here $\phi_j$ is the phase angle of the boson matter field: $\Psi_j \sim \exp(i\phi_j)$, and $\psi_j \sim \exp(i\phi_j/2)$ creates one half of the boson. When $\phi$ is disordered, i.e. $t$ is weak compared with $U$, $\phi$ can be safely integrated out, and the system is in the $Z_2$ topological phase; when $t$ is strong, or $\phi$ is condensed, the system is in a SF phase.

Why do we couple the $Z_2$ gauge field to a half boson operator instead of a single boson? The reason is that we want to make sure that the condensate of $\phi$ is an ordinary superfluid whose smallest vortex excitation has a $2\pi$ vorticity. In the condensate of $\phi$, the smallest vortex is a $2\pi$-vortex of $\phi$ ($\pi$-vortex of $\phi/2$), and it is bound with a vison excitation of $Z_2$ gauge field $\sigma^z$. If the $Z_2$ gauge field is coupled with a single boson $\exp(i\phi_j)$, then the superfluid phase becomes a paired boson condensate, whose smallest vortex has a $\pi$-vorticity.

As we discussed in the previous section, the SF-MI transition can be interpreted as the condensation of the vortex of the SF phase. Since the $2\pi$-vortex of the SF phase is bound with a vison, and in the $Z_2$ deconfined phase the vison is a well-defined excitation, thus the $Z_2$ topological phase is not a condensate of the $2\pi$-vortex. Instead, the $Z_2$ topological phase is a condensate of the $4\pi$-vortex, or the double-vortex of the SF phase, which is not bound with any vison. This transition driven by double-vortex condensation is usually referred to as “3D XY∗” transition.

Since the $Z_2$ gauge theory is fully gapped, it will not generate any singular correlation for $\psi \sim \exp(i\phi/2)$ in the infrared limit. Thus, this phase transition can be effectively described by Eq. 6, although $\psi$ is not really a gauge invariant operator. Let us focus on the case with $\mu = 0$, where there is an emergent Lorentz invariance. In this case, this transition belongs to the 3D XY universality class if we take $\psi$ as an “order parameter”, namely the correlation length of $\psi$ diverges as $\xi \sim r^{-\nu}$, and $\nu \sim 0.67$. However, the scaling behavior of the physical order parameter $\Psi \sim \psi^2$ is very different from the 3D XY universality class. For example, let us consider the anomalous dimension $\eta_\Psi$ of the physical order parameter $\Psi$, which is defined as

$$\Delta[\Psi] = (D - 2 + \eta_\Psi)/2,$$

where $\Delta[\Psi]$ is the scaling dimension of $\Psi$ at the critical point. $\Psi$ corresponds to a bilinear composite field at the 3D XY transition, thus $\Delta[\Psi] = \Delta[\psi^2]$. The scaling dimensions of composite fields at a Wilson-Fisher critical point have been calculated numerically with high precision and quoting these results, we can conclude that $\eta_\Psi = 1.49$ at this 3D XY∗ transition.

This anomalous dimension is enormous compared with the ordinary 3D Wilson-Fisher transition. For instance, the ordinary 3D XY transition has anomalous dimension $\eta \sim 0.03$, which is orders of magnitude smaller than the 3D XY∗ transition. This anomalous dimension can be verified numerically by computing the scaling of the order parameter in the ordered phase close to the critical point:

$$\langle \Psi \rangle \sim r^{\beta}, \quad \beta = \nu(D - 2 + \eta_\Psi)/2.$$
by quantum Monte Carlo simulation on a Hard-core boson model on the Kagome lattice [20].

3.3. QCP between $Z_2$ topological phase and spin order

Recently a lot of efforts have been devoted to searching for spin liquid phases in frustrated quantum spin models using various numerical methods. So far it has been proposed that a fully gapped spin liquid phase exists in the Kagome lattice spin-1/2 antiferromagnetic Heisenberg model [21,22], the honeycomb lattice Hubbard model [23], and the $J_1 - J_2$ spin-1/2 Heisenberg model on the square lattice [9]. In all these models, the numerical simulations have found a phase without any symmetry breaking, and there is a finite gap for both spinful and spin singlet excitations.

In this section we will only consider spin systems with a full SU(2) symmetry (if the SU(2) spin symmetry is broken down to the inplane U(1) symmetry, the situation reduces to the case discussed in the previous section). Based on our understanding of spin liquid state, when we see a fully gapped spin liquid state in either experiments or numerical simulations, the first idea that we have in mind is the $Z_2$ spin liquid, i.e. the $Z_2$ topological phase. Then presumably in the $Z_2$ spin liquid phase the electric excitation carries certain representation of the spin SU(2) symmetry group, and the transition between the liquid phase and the spin ordered phase is driven by the condensation of the spin-carrying excitation (usually called spinon). Then the nature of the spin order and the universality class of this transition depend on the particular representation of this spin excitation.

The smallest representation of SU(2) is spin-1/2 representation, and there is no consistent “fractional” representation of SU(2) group that is smaller than spin-1/2. Thus let us first assume the spinon is a spin-1/2 boson, which is described by a two component complex boson field $z_\alpha = (z_1, z_2)^t$, and $z_\alpha$ is subject to the constraint $|z_1|^2 + |z_2|^2 = 1$. Just like the previous section, $z_\alpha$ is coupled to a $Z_2$ gauge field in the following way:

$$ H = \sum_{i,\mu} \sum_\alpha - t \sigma_{i,\mu} \bar{z}_{\alpha,i} \bar{z}_{\alpha,i+\mu} + H.c. + \cdots $$

(25)

The condensed phase of $z_\alpha$ is the spin ordered phase, while the disordered phase of $z_\alpha$ is the deconfined $Z_2$ topological phase.

Since $z_\alpha$ has in total two complex bosonic fields, i.e. four real fields, then with the constraint $|z_1|^2 + |z_2|^2 = 1$, the entire configuration of $z_\alpha$ is equivalent to a three dimensional sphere $S^3$. Since the spinon field $z_\alpha$ is coupled to a $Z_2$ gauge field, then the physical configuration of the condensate of $z_\alpha$ is $S^3/Z_2$, which is mathematically equivalent to group manifold SO(3). The universality class of this transition is the 3D O(4)$^*$ transition, which is an analogue of the 3D XY$^*$ transition discussed in the previous section. Since $z_\alpha$ itself is not a physical observable, inside the condensate of $z_\alpha$ the physical observables are the following three vectors:

$$ \tilde{N}_1 = \text{Re}[z^t i \sigma^y \bar{z}], \quad \tilde{N}_2 = \text{Im}[z^t i \sigma^y \bar{z}], \quad \tilde{N}_3 = z^t \tilde{\sigma} z. $$

(26)
A simple application of the Fierz identity
\[ \sum_{\alpha} \sigma^a_{\alpha \beta} \sigma^a_{\gamma \rho} = 2 \delta_{\alpha \rho} \delta_{\beta \gamma} - \delta_{\alpha \beta} \delta_{\gamma \rho} \]
proves that these three vectors are orthogonal with each other. At the 3D O(4)* quantum critical point, the anomalous dimension of \( \vec{N}_i \) is also very large:
\[ \eta_{\vec{N}_i} \sim 1.37 \] [24].

One type of spin orders that has ground state manifold (GSM) SO(3), is the noncollinear spin density wave (SDW), for instance the standard 120 degree SDW on the triangular lattice, with order wave vector \( \vec{Q} = (4\pi/3, 0) \). In this case, the vector \( \vec{N}_i \) are defined as
\[ \vec{S}(\vec{r}) \sim \vec{N}_1 \cos(\vec{Q} \cdot \vec{r}) + \vec{N}_2 \sin(\vec{Q} \cdot \vec{r}), \]
\[ \vec{N}_3 = \vec{N}_1 \times \vec{N}_2. \] (27)

It is straightforward to check that when vector fields \( \vec{N}_1 \) and \( \vec{N}_2 \) are ordered uniformly on the lattice, \( \vec{S} \) has the standard 120 degree state on the triangular lattice. The 3D O(4)* QCP between noncollinear SDW and \( Z_2 \) topological phase has been used to explain the spin liquid phenomena observed in the organic frustrated magnet \( \kappa-(ET)_2Cu_2(CN)_3 \). [24]

Since the first homotopy group of SO(3) is \( \pi_1[SO(3)] = Z_2 \), inside this spin ordered phase there are half-vortex excitations, which are bound with the visons of the \( Z_2 \) gauge field. Two of these half-vortices can annihilate each other.

Now let us assume the spin excitation of the \( Z_2 \) topological phase carries a spin-1 representation. A spin-1 representation is a vector representation of SU(2), i.e. it can be parametrized as a unit real vector \( \vec{n}, |\vec{n}|^2 = 1 \). Now the coupling between the spin excitation and \( Z_2 \) gauge theory reads
\[ H = \sum_{i, \mu} \sum_a -t \sigma^a_{i,\mu} \eta^a_i \eta^{a+\mu} + H.c. + \cdots \] (28)

Again, since \( \vec{n} \) couples to a \( Z_2 \) gauge field, it is not a physical observable: \( \vec{n} \) and \( -\vec{n} \) are physically equivalent. If vector \( \vec{n} \) condenses, the condensate is in fact a spin nematic, or quadrupole order, with quadrupolar order parameter
\[ Q^{ab} = n^a n^b - \frac{1}{3} \delta_{ab}. \] (29)

This spin order has manifold \( S^2/Z_2 \), which also supports half-vortex excitations since \( \pi_1[S^2/Z_2] = Z_2 \). The condensation transition of the vector \( \vec{n} \) belongs to the 3D O(3)* universality class.

In this section we have discussed two types of unconventional QCPs between \( Z_2 \) liquid phase and spin orders. In either case, the spin ordered phase is different from the ordinary collinear Néel order, because a Néel order should have GSM \( S^2 \). In particular, in both cases we have considered, the spin ordered phase must have a nontrivial homotopy group \( \pi_1 \), which corresponds to the vison excitation of the \( Z_2 \) topological phase. In Ref. [20] and Ref. [23], a continuous quantum phase transition between a fully gapped spin liquid phase and a Néel order was reported. If the fully gapped spin liquid discovered in these numerical works is indeed a \( Z_2 \) spin liquid as
we expected, then such continuous quantum phase transition is beyond our current understanding of unconventional QCP.

3.4. QCP between $Z_2$ topological order and VBS

As we have mentioned, it has been proved that the ground state of a spin-1/2 quantum magnet cannot be trivially gapped without any degeneracy\[17\]. Thus if the ground state of a spin-1/2 system has a short range spin-spin correlation, then besides topological order, another possible scenario is the valence bond solid (VBS) phase. The most naive picture of VBS order is that, each spin forms a spin-singlet with one of its neighboring spins, and these spin singlets form a crystal pattern that breaks lattice symmetry, thus the ground state also has degeneracy, although this degeneracy is due to spontaneous symmetry breaking.

If there is a continuous quantum phase transition between the $Z_2$ topological phase and the VBS phase, then this transition can only be interpreted as the condensation of spinless excitations of the $Z_2$ liquid phase, and this spinless excitation must carry lattice momentum, in order to break the lattice symmetry in its condensate. As we discussed in the previous section, in the $Z_2$ liquid phase, the electric excitations carry spin, then the only excitation that can drive the transition into VBS is the magnetic excitation, or the vison.

In spin-1/2 $Z_2$ liquid phase, the $Z_2$ gauge theory is usually odd. This is because the $Z_2$ gauge theory is subject to the gauge constraint

$$\prod_{\text{links around site } i} \sigma_{ij}^z = (-1)^{\rho_i}, \quad (30)$$

where $\rho_i$ corresponds to the density of spinons on every site, and in spin-1/2 systems, no matter we use bosonic or fermionic spinons, $\rho_i$ is always 1. Let us consider the $Z_2$ gauge theory on the honeycomb lattice first. With this $Z_2$ gauge constraint, we can write down the simplest $Z_2$ gauge theory on the honeycomb lattice as follows:

$$H = \sum_{\text{o}} -K \prod_{\text{links in o}} \sigma_{ij}^z - \sum_{i,j} h\sigma_{ij}^x + \cdots \quad (31)$$

The first term is a sum of the ring product of the $Z_2$ gauge field $\sigma_{ij}^z$ in every hexagon, and the second term is a $Z_2$ “string tension”. The ellipses include other interaction terms between $Z_2$ electric field.

When the $K$ term dominates everything else in Eq. (31) the system is in the deconfined phase of the $Z_2$ gauge theory, with topological degeneracy. When $h$ or other interaction terms between $\sigma^x$ dominate $K$, the system enters the confined phase. In order to analyze the confined phase, it is convenient to go to the dual picture of the $Z_2$ gauge theory. Dual variables $\tau^z$ and $\tau^x$ are defined on the dual lattice sites $\bar{m}$, which are located at the center of the hexagons (Fig. 3c):}

$$\sigma_{ij}^z = -\tau^+_{\bar{p}} \tau^+_{\bar{q}}, \quad \bar{p} \text{ and } \bar{q} \text{ share link } ij,$$
\[
\prod_{\text{links around } \vec{p}}^{6} \sigma_{ij}^{z} = \tau_{\vec{p}}^{x}.
\] (32)

Introduction of \( \tau_{\vec{p}}^{z} \) automatically solves the odd \( Z_2 \) gauge constraint Eq. 30. Now the Hamiltonian becomes an antiferromagnetic transverse field Ising model on the dual triangular lattice:

\[
H = \sum_{\vec{p}} -K\tau_{\vec{p}}^{x} + \sum_{\vec{p},\vec{q}} J_{\vec{p},\vec{q}}\tau_{\vec{p}}^{z}\tau_{\vec{q}}^{z}
\] (33)

For nearest neighbor sites \( \vec{p}, \vec{q} \), \( J_{\vec{p},\vec{q}} = h \). When \( J_{\vec{p},\vec{q}} \) dominates \( K \), \( \tau_{\vec{p}}^{z} \) takes on a non-zero expectation value forming some pattern which optimizes the \( J_{\vec{p},\vec{q}} \) term. The non-zero “condensate” of \( \tau_{\vec{p}}^{z} \) signals that the \( Z_2 \) gauge theory has entered the confined phase.

![Diagram](image)

Fig. 3. (a), c—VBS order. \( \vec{p} \) and \( \vec{q} \) are the dual triangular lattice sites. We consider the nearest and 2nd neighbor hopping for vison (vortex). (b), the \( s-\)VBS pattern, realized when \( h/8 < J < h \) in the dual Ising Hamiltonian Eq. 33 (c), the four sublattice plaquette order, realized when \( w > 0 \) in Eq. 42 (d), the vison (vortex) Brillouin zone. For weak 2nd neighbor vison (vortex) hopping, the minima of band structure are located at the corner of the BZ (circles); with intermediate 2nd neighbor hopping, there are three inequivalent minima located at the center of the edges of BZ (square); There are six inequivalent incommensurate minima with strong 2nd neighbor hopping (hexagon).

The pattern of order in \( \tau_{\vec{p}}^{z} \) depends upon the detailed form of \( J_{\vec{p},\vec{q}} \). This can be analyzed by treating \( \tau_{\vec{p}}^{z} \) as a “soft” scalar field taking all possible real values, rather than the integers \( \pm 1 \); this approximation describes well the critical region in which fluctuations on short time scales render the average of \( \tau_{\vec{p}}^{z} \) non-integral. Then, the quadratic form defined by \( J_{\vec{p},\vec{q}} \) can be diagonalized in wavevector space and
generically has multiple minima in its Brillouin zone. Physically the eigenvalues of this quadratic form define the dispersion relation of visons in the $Z_2$ phase. On entering the confined phase, the location of these minima determines the VBS pattern. Notice that the physical VBS order parameter should always be a bilinear of $\langle \tau_z \rangle$, since under transformation $\tau_z \rightarrow -\tau_z$ the physical quantity $\sigma^x$ is unchanged.

In the following we will discuss four types of VBS patterns on the honeycomb lattice.

3.4.1. $c-$VBS order on the honeycomb lattice

Now let us take the simplest case, with nonzero $J_{\bar{p},\bar{q}}$ only between nearest neighbor dual sites $\bar{p}, \bar{q}$. Taking $h > 0$, the model becomes the nearest neighbor frustrated quantum Ising model with transverse field. This model was studied in Ref. 31. Solving the band structure of $\tau_z$, we find two inequivalent minima at the corners of the vison BZ: $Q = (\pm \frac{4\pi}{3}, 0)$. Expanding $\tau_z$ at these two minima, we obtain a complex local order parameter $\psi$:

$$\tau_z \sim \psi e^{i\frac{4\pi}{3}x} + \psi^* e^{-i\frac{4\pi}{3}x}. \quad (34)$$

The low energy physics of visons should be fully characterized by $\psi$.

Under discrete lattice symmetry, $\psi$ transforms as

$T_1 : x \rightarrow x + 1, \quad \psi \rightarrow e^{i\frac{2\pi}{3}}\psi,$

$T_2 : x \rightarrow x + \frac{1}{2}, \quad y \rightarrow y + \frac{\sqrt{3}}{2}, \quad \psi \rightarrow e^{i\frac{2\pi}{3}}\psi,$

$P_y : x \rightarrow -x, \quad \psi \rightarrow \psi^*,$

$P_x : y \rightarrow -y, \quad \psi \rightarrow \psi,$

$T : t \rightarrow -t, \psi \rightarrow \psi^*,$

$R_{\frac{2\pi}{3}} : \psi \rightarrow \psi. \quad (35)$

$R_{\frac{2\pi}{3}}$ is the rotation by $2\pi/3$ around the center of hexagon.

The transformations in Eq. (35) determine that the low energy Lagrangian for $\psi$ reads

$$\mathcal{L} = |\partial_\mu \psi|^2 + r|\psi|^2 + u|\psi|^4 + w(\psi^6 + \psi^{*6}), \quad (36)$$

*i.e.* The condensation of $\psi$ is described by a 3D XY transition with $Z_6$ anisotropy, which is an irrelevant perturbation at the 3D XY universality class. The physical VBS order parameter $V$ should be a bilinear of $\psi$, i.e. $V \sim \psi^2$. It is straightforward to check that $V$ transforms in the same way as the columnar VBS ($c-$VBS) order parameter on the honeycomb lattice. Thus more precisely, this transition belongs to the 3D $XY^*$ universality class, where the anomalous dimension VBS order parameter $V$ is $\eta_V \sim 1.49$. Notice that on the honeycomb lattice the $c-$VBS and the $\sqrt{3} \times \sqrt{3}$
plaqette order have the same symmetry, hence the condensate of $\psi$ can be either the $c-$VBS or the plaquette order depending on the sign of $w$.

If we approach this transition from the $c-$VBS side of the phase diagram, this transition can be interpreted as a proliferation of the vortex of $\psi$ i.e. double vortex of VBS order parameter $V$, while the single vortex of $V$ is still gapped. In fact, the single vortex core of the $c-$VBS is attached with a spinon (analogous to the square lattice case discussed in Ref. [26]), condensation of single vortex will lead to a spinon condensate, which corresponds to certain spin order. However, if the spinon gap is finite, the finite temperature thermal fluctuation can proliferate the single vortex. Therefore although the quantum phase transition is driven by double vortices, the finite temperature phase transition is still driven by single vortex, hence at finite temperature the $Z_6$ anisotropy of Eq. 3.4 becomes the $Z_3$ anisotropy, and there is no algebraic Kosterlitz-Thouless phase at finite temperature. This is a key difference between our current case and a physical transverse field frustrated quantum Ising model, where a finite temperature algebraic phase is expected [31].

3.4.2. $s-$VBS order and four-fold plaquette order on the honeycomb lattice

Now we modify the $Z_2$ gauge theory in Eq. 31 by turning on the interaction between $Z_2$ electric field $\sigma^\tau$ on second nearest neighbor links:

$$H_J = \sum_{\text{2nd neighbor links}} J \sigma^\tau_{i,j} \sigma^\tau_{k,l}. \tag{37}$$

In the dual theory this electric field interaction becomes a next nearest neighbor hopping of $\tau^z$, and the full dual Hamiltonian reads

$$H = \sum_p -K \tau^z_p + \sum_{\langle p,q \rangle} h \tau^z_p \tau^z_q + \sum_{\langle\langle p,q \rangle\rangle} J \tau^z_p \tau^z_q. \tag{38}$$

The vison minima ($\pm \frac{4\pi}{3},0$) are stable with $J/h < 1/8$. When $1/8 < J/h < 1$, the minima of the vison band structure are shifted to three inequivalent points on the edges of BZ (Fig. [44]):

$$\vec{Q}_1 = (0, \frac{2\sqrt{3}\pi}{3}), \quad \vec{Q}_2 = (-\pi, -\frac{\sqrt{3}\pi}{3}), \quad \vec{Q}_3 = (\pi, -\frac{\sqrt{3}\pi}{3}). \tag{39}$$

Notice that $-\vec{Q}_a$ are equivalent to $\vec{Q}_a$ in the BZ.

Now three low energy modes can be defined by expanding $\tau^z$ at momenta $\vec{Q}_a$:

$$\tau^z \sim \sum_a \varphi_a e^{i\vec{Q}_a \cdot \vec{r}}. \tag{40}$$

Since $\vec{Q}_a$ and $-\vec{Q}_a$ are equivalent, all three fields $\varphi_a$ are real. Under lattice symmetry, $\varphi_a$ transform as

$$T_1 : \varphi_1 \rightarrow \varphi_1, \quad \varphi_2, \varphi_3 \rightarrow -\varphi_2, -\varphi_3,$$

$$T_2 : \varphi_1, \varphi_2 \rightarrow -\varphi_1, -\varphi_2, \quad \varphi_3 \rightarrow \varphi_3,$$
Now the symmetry allowed Lagrangian for $\varphi_a$ up to the quartic order reads
$$L = \sum_a (\partial_\mu \varphi_a)^2 + r \varphi_a^2 + u (\sum_a \varphi_a^2)^2 + w (\sum_a \varphi_a^4). \tag{42}$$

This is an O(3) model with cubic anisotropy. There are two possible types of condensates of $\varphi_a$:

(i) When $w > 0$, the condensate $\langle \vec{\varphi} \rangle$ are along the diagonal directions, and there are in total four independent states with $\langle \vec{\varphi} \rangle \sim (1,1,1), (-1,-1,1), (-1,1,-1)$, $(1,-1,-1)$. According to the transformation of $\vec{\varphi}$, these four states correspond to the four-sublattice plaquette phase (Fig. 3c).

(ii) When $w < 0$, the condensate $\langle \vec{\varphi} \rangle$ has three fold degeneracy: $\langle \vec{\varphi} \rangle \sim (1,0,0), (0,1,0)$ and $(0,0,1)$. These three condensates break the rotation symmetry of the lattice, but they do not break the translation symmetry. This is again because physical order parameters are bilinears of $\varphi_a$, hence they are insensitive to the sign change of $\varphi_a$ under translation. These three states correspond precisely to the three staggered VBS ($s$-VBS) pattern (Fig. 3b). Unlike the $c$-VBS, the $s$-VBS is no longer described by an XY order parameter, and the phase transition is not driven by vortex-like VBS defect.

The universality class of the QCPs described by Eq. (42) was discussed carefully in Ref. [28].

### 3.4.3. $Z_2$ topological phase and VBS on the square lattices

The vison dynamics on the square lattice is technically more complicated than the honeycomb lattice, because in order to solve the odd $Z_2$ gauge constraint, now the dual quantum Ising model has to apparently break the lattice symmetry in any specific gauge choice. The correct lattice symmetry transformation for the dual vison field $\tau^z$ must be combined with a nontrivial $Z_2$ gauge transformation, i.e. $\tau^z$ carries a projective representation of the symmetry group. The dual quantum Ising model has to be invariant under the projective symmetry group (PSG).

One of the dual quantum Ising model that is consistent with all the PSG is
$$H = \sum_p -K \tau_p^z + \sum_{<p,q>} J_{p,q} \tau_p^z \tau_q^z + \sum_{<p,q>} J'_{p,q} \tau_p^z \tau_q^z. \tag{43}$$

$J$ and $J'$ denote the nearest and fourth nearest neighbor Ising couplings. $J$ and
Fig. 4. (a), the dual square lattice. The vison (vortex) hopping on the dashed bonds are negative. (b), (c), the $c$–VBS and $s$–VBS patterns. (d), the vison (vortex) Brillouin zone. When the nearest neighbor vison (vortex) hopping is dominant, there are two inequivalent minima located at $(0, \pm \frac{\pi}{2})$ (circles); when the 4th neighbor hopping is dominant, there are four inequivalent minima described by Eq. 39.

$J'$ are chosen to be positive on all the solid bonds, but negative on all the dashed bonds in Fig. 4.

This quantum Ising model can be analyzed in the same way as the previous subsection. And with different choices of $J'$ and $J$ we will find that the $\tau^z$ band structure has multiple minima in the BZ. The condensate on these minima corresponds to different VBS pattern. If $J'/J < 0.0858$, there are two inequivalent minima in the vison band structure, located at $\vec{Q} = (0, \pm \frac{\pi}{2})$. Again we can expand $\tau^z$ at these two minima as

$$\tau^z \sim \varphi e^{i\frac{\pi}{2} y} + \varphi^* e^{-i\frac{\pi}{2} y}. \quad (44)$$

The PSG for $\varphi$ reads

$$T_x : x \to x + 1, \quad \varphi \to e^{i\frac{\pi}{2} x} \varphi^*,$$

$$T_y : y \to y + 1, \quad \varphi \to e^{-i\frac{\pi}{2} x} \varphi^*,$$

$$P_y : x \to -x, \quad \varphi \to \varphi,$$
Notice that the reflection $P_x$ and $P_y$ are site-centered reflection of the dual lattice (bond-centered reflection of the original lattice). The PSG allowed field theory for $\varphi$ reads

$$L = |\partial_\mu \varphi|^2 + r|\varphi|^2 + g|\varphi|^4 + w(\varphi^8 + \varphi^{*8}).$$

(46)

The gauge invariant physical order parameters are the columnar VBS orders (Fig. 4b):

$$c - \text{VBS}_x: e^{i\frac{\pi}{4}} \varphi^2 + e^{-i\frac{\pi}{4}} \varphi^{*2},$$

$$c - \text{VBS}_y: e^{-i\frac{\pi}{4}} \varphi^2 + e^{i\frac{\pi}{4}} \varphi^{*2}.$$

(47)

The quantum phase transition between the $Z_2$ liquid and the $c$-VBS is a 3D XY* transition, since the $Z_8$ anisotropy in Eq. 46 is highly irrelevant at the 3D XY* fixed point. This result is consistent with previous studies on fully frustrated Ising model on the cubic lattice 29, 30.

When $J'/J > 0.0858$, the minima of the vison band structure are shifted to four other inequivalent momenta in the BZ (Fig. 4f):

$$Q_1 = (0, 0), \quad Q_2 = (0, \pi), \quad Q_3 = (\frac{\pi}{2}, \frac{\pi}{2}), \quad Q_4 = (-\frac{\pi}{2}, \frac{\pi}{2}).$$

(48)

Notice all these four modes are real fields, because $Q_a$ are equivalent to $-Q_a$. Thus these four minima correspond to four different real fields, which correspond to four different staggered-VBS state described in Fig. 4c.

These analysis can be parallelly generalized to the triangular lattice. For odd $Z_2$ gauge theory on the triangular lattice, the dual theory is a frustrated quantum Ising model on the honeycomb lattice. For the simplest nearest neighbor frustrated quantum Ising model on the honeycomb lattice, there are four minima in the vison Brillouin zone, and the low energy field theory of the QCP between the $Z_2$ liquid and the VBS has a large emergent O(4) symmetry 31, and the liquid-VBS transition belongs to the 3D O(4)* universality class. The $Z_2$ topological phase to VBS transition on the Kagome lattice was recently studied in Ref. 27.

4. Unconventional QCP between ordered phases

In this section we discuss unconventional QCP between two different types of ordered phases, i.e. two ordered phases with different symmetry breaking. More precisely, the GSM of one of the phases around this QCP should not be the submanifold of the other phase. When one phase diagram involves two or even more ordered phases like this, these orders are usually called “competing orders”. For instance, in the phase diagram of High $T_c$ cuprates, there are both Néel order and superconductor, as well as other possible orders such as spin or charge density wave especially at
certain commensurate doping. The classical way of describing competing orders, is to start with a GL theory that involves all the relevant competing orders. However, in this approach it is impossible to get a generic unfine-tuned continuous quantum phase transition between two different competing orders. The GL theory will conclude that two competing orders are always separated by one first order transition, or two (or even more) continuous transitions.

Fig. 5. The schematic phase diagram of unconventional QCP between two different competing orders. The phase $A$ has GSM $S^2$, and it spontaneously breaks symmetry SU(2)$_A$; the phase $B$ with GSM $S^1$ spontaneously breaks symmetry U(1)$_B$. This QCP can be interpreted as the condensation of Skyrmions of phase $A$, it can also be interpreted as condensation of vortices of phase $B$.

An unfine-tuned direct second order quantum transition between two competing orders must be an unconventional QCP. The existence of this unconventional QCP implies that, suppressing one of the orders necessarily leads to the other order. This effect is guaranteed when the topological defect of one of the orders carries the quantum number of the other order. So far, almost all the unconventional QCP of this type can be roughly described with the general formalism described in the following section.

4.1. General Formalism

The system has a global symmetry SU(2)$_A \times$ U(1)$_B$. Phase $A$ of the phase diagram spontaneously breaks the SU(2)$_A$ symmetry down to U(1)$_A$ symmetry, thus phase $A$ has GSM SU(2)$_A$/U(1)$_A = S^2$; phase $B$ of the phase diagram spontaneously breaks the U(1)$_B$ symmetry, thus the phase $B$ has GSM $S^1$. The Skyrmion defect of phase $A$ carries the quantum number of U(1)$_B$, thus when this Skyrmion condenses, it not only destroys order $A$, it also induces order $B$. Meanwhile, the vortex defect of order $B$ carries a fundamental representation of SU(2)$_A$, thus the condensate of this vortex not only destroys order $B$, it also leads to phase $A$ that spontaneously breaks SU(2)$_A$ symmetry.
Now let us describe this QCP from phase A. Phase A has GSM $S^2$, thus it can be described by a unit $O(3)$ vector $\vec{n}$ and the NLSM Eq. [3]. The special property of phase A is that its Skyrmion carries a global $U(1)_B$ symmetry, thus this Skyrmion is conserved. In order to describe this Skyrmion as a local excitation instead of a topological defect, it is most convenient to use the CP(1) field representation: 

$$z_\alpha = (z_1, z_2) = \left( e^{i(\gamma/2) \cos(\theta/2) + i\phi/2} \sin(\theta/2), e^{-i\phi/2} \right).$$

(49)

Here the angle $\gamma$ is a gauge degree of freedom. Because $z_\alpha$ is not gauge invariant, if we describe phase A using $z_\alpha$, then $z_\alpha$ is automatically coupled to a $U(1)$ gauge field $a_\mu$, and the field theory that describes phase A reads

$$L = \sum_\alpha |(\partial_\mu - ia_\mu)z_\alpha|^2 + r|z_\alpha|^2 + u(\sum_\alpha |z_\alpha|^2)^2 + \frac{1}{e^2}(f_{\mu\nu})^2.$$  

(50)

In this equation we have softened the constraint $|z_1|^2 + |z_2|^2 = 1$, whose effect has been replaced by the interaction $u$ term.

This CP(1) representation has a great advantage: the Skyrmion of the vector $\vec{n}$ is precisely the flux quantum of $a_\mu$:

$$\frac{1}{8\pi} \epsilon_{abc} \epsilon_{ij} n^a \partial_i n^b \partial_j n^c = \frac{1}{2\pi} \epsilon_{ij} \partial_i a_j.$$  

(51)

Because the Skyrmion carries a global $U(1)_B$ symmetry, the Skyrmion is conserved, thus when it condenses the system will have a Goldstone mode. Because the Skyrmion is mapped to the $U(1)$ gauge flux quantum, this Skyrmion Goldstone mode is precisely dual to the photon excitation of $a_\mu$.

In terms of the CP(1) field theory, this phase diagram is interpreted as follows: phase A is the condensate of CP(1) field $z_\alpha$, and the gauge field $a_\mu$ is gapped due to the Higgs mechanism. Phase A has GSM $S^2$ characterized by vector $z_\dagger \sigma z$. Phase B is the gapped phase of $z_\alpha$, and in this phase the gauge field $a_\mu$ is in its photon phase, which is precisely the condensate of its gauge flux (the duality discussed in section 2). Because the gauge flux carries the $U(1)_B$ quantum number, this photon phase spontaneously breaks the $U(1)_B$ symmetry, and it has GSM $S^1$.

We can also understand this QCP from phase B. Phase B has GSM $S^1$, which is equivalent to a superfluid phase. In section 2, we derived the dual description of the SF phase, which is the bosonic QED Eq. [11] where the vortex of the SF phase is described by a bosonic scalar field $\Phi$ that couples to the dual $U(1)$ gauge field $a_\mu$. In the current case, since we assumed that the vortex of phase B carries a fundamental representation of the $SU(2)_A$ symmetry, then the dual theory actually becomes precisely the CP(1) field theory Eq. [50] and the CP(1) field $z_\alpha$ precisely corresponds to the vortex of phase B.
The phase transition between phase $A$ and $B$ is described by “fractionalized particles” $z_\alpha$ instead of physical order parameters, thus this type of QCP is called “deconfined QCP”. Although the CP(1) field theory was “derived” from the O(3) model, the universality class of the QCP described by Eq. 50 is very different from the O(3) Wilson-Fisher fixed point. The O(3) Wilson-Fisher quantum phase transition is sandwiched between a phase with GSM $S^2$ and a fully gapped trivial disordered phase, which is very different from phase $B$. In fact, the O(3) Wilson-Fisher fixed point is equivalent to the case where the Skyrmion (the gauge flux) is not conserved, i.e., the $U(1)_B$ symmetry is absent. In the language of the CP(1) model, an unconserved Skyrmion number corresponds to an unconserved $U(1)$ gauge flux of $a_\mu$, thus the flux condensate has no Goldstone mode, i.e., the photon excitation is fully gapped. The $U(1)$ gauge field with unconserved flux is precisely the compact $U(1)$ gauge theory. Thus the O(3) Wilson-Fisher universality class is equivalent to a compact-$CP(1)$ model, while the deconfined QCP is described by a noncompact-$CP(1)$ model.

4.2. Examples of deconfined QCPs

In this subsection we discuss two examples of deconfined QCPs.

The first example of deconfined QCP that was discussed is the Néel-VBS transition of spin-1/2 quantum magnet on the square lattice: the phase $A$ is the AF Néel order that breaks the spin rotation $SU(2)$ symmetry, while phase $B$ is the VBS phase that only breaks lattice translation and rotation symmetry. It appears that this transition is different from the general case discussed in the previous subsection, since the VBS phase breaks a discrete four fold rotation lattice symmetry, instead of a continuous $U(1)_B$ symmetry. However, there is a strong analytical and numerical evidence which suggests that the discrete four fold rotation symmetry is enlarged to a continuous $U(1)$ symmetry at the QCP. Thus the GSM of phase $B$ is enlarged to $S^1$ close to the QCP.

The essence of the deconfined QCP is the physical quantum number carried by topological defects in both phase $A$ and $B$. It was shown by Haldane and Sachdev that the Skyrmion of Néel order carries lattice momentum, thus when the Skyrmion of Néel order condenses, it spontaneously breaks the lattice symmetry, i.e., the system automatically enters the VBS order. Later on Senthil and Levin also demonstrated that the discrete $Z_4$ vortex of the VBS order carries a spin-1/2 spinon, thus as long as the $Z_4$ rotation symmetry of the lattice is enlarged to $U(1)_B$ symmetry at the QCP, this QCP is exactly equivalent to the general formalism discussed in the previous subsection.

In Ref. 37, the authors proposed another deconfined QCP. In this phase diagram, phase $A$ is a quantum spin Hall insulator on the honeycomb lattice, but the quantum spin Hall (QSH) state is generated by spontaneously breaking the spin symmetry, while preserving the time-reversal symmetry, thus the QSH state discussed in Ref. 37 has GSM $S^2$, which is equivalent to phase $A$ in the general formalism. Also, it was
demonstrated in Ref. [38] that the Skyrmion of the QSH vector carries charge-2e. Thus the Skyrmion of the QSH vector is conserved, and if this Skyrmion condenses, the system enters a $s$-wave superconductor. The transition between the QSH and $s$-wave SC is precisely described by Eq. [50]. In this model, SU(2)$_A$ is the spin SU(2) symmetry, while the U(1)$_B$ is the charge U(1) symmetry.

5. Global phase diagram of unconventional QCPs

So far in our paper we have discussed three types of unconventional QCPs:

1. QCP between $Z_2$ topological order and spin ordered phase;

2. QCP between $Z_2$ topological order and VBS phase;

3. Deconfined QCP between two different ordered phases that spontaneously break two different symmetries.

In this section we will discuss a single unified theory that contains all these phenomena in one phase diagram. This theory was introduced in Ref. [39] and Ref. [40], and it was applied to different microscopic systems. Before we discuss the physical motivation of this theory, let us first write down the Lagrangian of the unified field theory:

$$\mathcal{L} = \sum_{\alpha=1}^{N_z} |(\partial_\mu - ia_\mu)z_\alpha|^2 + s_z|z_\alpha|^2 + \sum_{\alpha=1}^{N_v} |(\partial_\mu - ib_\mu)v_\alpha|^2 + s_v|v_\alpha|^2 + \frac{i}{\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu b_\rho + \cdots$$  \hspace{1cm} (52)

In this field theory, there are two types of matter fields, $z_\alpha$ and $v_\alpha$, and they are interacting with each other through a mutual Chern-Simons theory, which grants them a mutual semion statistics i.e. when $v_\alpha$ adiabatically encircles $z_\alpha$ through a closed loop, the system wave-function acquires a minus sign.

The field theory Eq. [52] has symmetry SU($N_z$)$\times$SU($N_v$). However, depending on the details of the microscopic model, the higher order interactions between matter fields can break this symmetry down to its subgroups. We will first ignore this high order symmetry breaking effects, and focus on the case with $N_z = 2$, and $N_v = 1$.

In Ref. [39], the authors used the model Eq. [52] with $N_z = 2$, $N_v = 1$ to describe the global phase diagram of spin-$1/2$ quantum magnets on a distorted triangular lattice, which is a very common structure in many materials. Here $z_\alpha$ is a bosonic spin-$1/2$ spinon, and $v$ is the low energy mode of vison, which is a complex scalar field, like the complex $\psi$ and $\varphi$ field introduced in Eq. [34] and Eq. [40]. The phase diagram of this model is tuned by two parameters: $s_z$ and $s_v$, and depending on the sign of these two parameters, there are in total four different phases (Fig. 6):

**Phase 1.** This is the phase with $s_z > 0$, $s_v > 0$. In this phase, both matter fields
$z_\alpha$ and $v$ are gapped, and they have a topological statistic interaction through the mutual CS theory. Since all the matter fields are gapped, the low energy properties of phase 1 is described by the mutual CS theory only. Thus phase 1 is the $Z_2$ topological phase, described by $Z_2$ gauge theory Eq. [18].

**Phase 2.** $s_v > 0$, $s_z < 0$. When $N_z = 2$, this phase corresponds to a condensate of CP(1) field $z_\alpha = (z_1, z_2)$ while coupling to a $Z_2$ gauge field, thus this phase has GSM SO(3). Physically this phase corresponds to the incommensurate spiral SDW.

**Phase 3.** $s_v < 0$, $s_z > 0$. This is a phase where $v$ condenses while $z_\alpha$ is gapped out. This phase is the VBS phase that breaks the reflection and translation symmetry of the lattice.

**Phase 4.** $s_v < 0$, $s_z < 0$. This is a phase where both $z_\alpha$ and $v$ condense, and a careful analysis will conclude that this is precisely the collinear Néel phase with GSM $S^3$.

According to the unified theory Eq. [52], the QCP between phase 1 and 2 ($Z_2$ topological phase and spiral SDW) is the 3D O(4) transition that was described by Eq. [25]. The QCP between phase 1 and 3 ($Z_2$ topological phase and VBS) is the 3D XY transition described by Eq. [46]. The QCP between phase 3 and 4 is the deconfined QCP that is described by the noncompact CP(1) field theory Eq. [50]. A more detailed discussion of the phases and QCPs of Eq. [52] can be found in Ref. [39].

All of these phases have been observed in real frustrated quantum mag-
nets on the (distorted) triangular lattice. For example, a noncollinear spiral SDW was observed in Cs$_2$CuCl$_4$ [46], spin liquid phases were discovered in κ-(ET)$_2$Cu$_2$(CN)$_3$ [141, 143, 144, 145], EtMe$_3$Sb[Pd(dmit)$_2$]$_2$ [47, 51, 52], Ba$_3$CuSb$_2$O$_9$ [53] and Ba$_3$NiSb$_2$O$_9$ [54], a VBS phase was observed in (C$_2$H$_5$)(CH$_3$)$_3$P[Pd(dmit)$_2$]$_2$ [49], and many materials that belong to the dmit family have collinear Néel order at low temperature. All of these phases, including the QCPs between them can be unified using one single Lagrangian Eq. 52.

In Ref. 40, Eq. 52 was used to describe the phase diagram of the Hubbard model on the honeycomb lattice. $z_\alpha$ and $v_\alpha$ are the fundamental excitations of spin SU(2) and charge SU(2) symmetry of the Hubbard model at half-filling. Since the maximal symmetry of interacting electron systems is $\text{SO}(4) \sim (\text{SU}(2)_{\text{spin}} \times \text{SU}(2)_{\text{charge}})/Z_2$, there is an extra factor of $Z_2$ in the GSM of all the phases in this phase diagram. For example, here the phase with $s_z > 0$ and $s_v > 0$ is a $Z_2 \times Z_2$ topological phase, and Ref. 40 identified this phase as the fully gapped spin liquid phase observed by quantum Monte Carlo simulation on the Hubbard model on a honeycomb lattice [23].

6. Summary and Extensions

So far we have discussed unconventional QCPs around topological phases, and the QCPs between competing orders. However, this discussion is far from being general. We have a more or less complete understanding about QCPs around the $Z_2$ topological phase, and it is straightforward to generalize this understanding to $Z_N$ topological phases. However, the QCPs around other topological phases are less understood. One major limitation of our description is that, the physical picture of the QCPs discussed so far all relies on “condensation” of certain bosonic point particles. But there is no reason to believe this picture can be applied to all the QCPs in strongly interacting many-body systems. For example, a large class of topological phases can be described using loop or string like variables, instead of point particles [55, 56]. Some of the phases described by loop variables have a dual description in terms of point particles [57], thus the formalism described in this paper may still apply, but we do not have a general formalism to describe QCPs driven by extended objects.

Another complication of topological phases is that, their low energy excitations can carry nontrivial anyonic or even nonabelian statistics. Some of these excitations can be described as bosons coupled to a Chern-Simons field, but a more general and complete formalism of dealing with particles with nontrivial statistics is still demanded. The condensation of anyons with nontrivial statistics usually drives the system into a different topological phases, and in this case the two states around the QCP has the same symmetry. Examples of quantum critical points between different topological orders have been studied in Ref. [58, 59, 60, 61].

Unconventional QCPs are very easy to detect experimentally, because of its large anomalous dimension associated with physical order parameters. For example, in 2+1 dimension, if there is a QCP between a magnetic ordered phase and a...
disordered phase, then in the quantum critical regime with finite temperature the NMR relaxation rate $1/T_1$ has the universal scaling $1/T_1 \sim T^\eta$. Thus the unusual anomalous dimension $\eta$ can be probed conveniently in experiments. In Ref. [42], it was reported that the NMR relaxation rate of material $\kappa-(ET)_2Cu_2(CN)_3$ scales as $1/T_1 \sim T^a$, where $a \sim 1.5$. This is qualitatively consistent with the 3D $XY^*$ and $O(4)^*$ QCP discussed in section 3. This observation led to the conjecture that the organic material $\kappa-(ET)_2Cu_2(CN)_3$ is close to an unconventional QCP [24].

Unconventional quantum critical point is a rapidly developing field, and it is impossible for us to review every related topic. Besides the subjects included in this paper, there are a few other types of exotic QCPs that are beyond the Landau’s paradigm. For example, our paper has focused on the unconventional QCPs in two spatial dimensions, while the idea of deconfined QCPs has been generalized to three dimensional lattices as well [62,63]. Another special type of exotic QCP in fermionic systems was reviewed in Ref. [64]. Unconventional phase transitions at finite temperature in classical systems have also been discussed in special models, for example the classical dimer models on three dimensional lattices [65,66,67].

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