Approximation Properties of the Sampling Kantorovich Operators: Regularization, Saturation, Inverse Results and Favard Classes in $L^p$-Spaces

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Abstract
In the present paper, a characterization of the Favard classes for the sampling Kantorovich operators based upon bandlimited kernels has been established. In order to achieve the above result, a wide preliminary study has been necessary. First, suitable high order asymptotic type theorems in $L^p$-setting, $1 \leq p \leq +\infty$, have been proved. Then, the regularization properties of the sampling Kantorovich operators have been investigated. Here, we show how the regularity of the kernel influences the operator itself; this has been shown for bandlimited kernels, or more in general for kernels in Sobolev spaces, satisfying a Strang-Fix type condition of order $r \in \mathbb{N}^+$. Further, for the order of approximation of the sampling Kantorovich operators, quantitative estimates based on the $L^p$ modulus of smoothness of order $r$ have been established. As a consequence, the qualitative order of approximation is also derived assuming $f$ in suitable Lipschitz and generalized Lipschitz classes. Moreover, an inverse theorem of approximation has been stated, allowing to obtain a full characterization of the Lipschitz and of the generalized Lipschitz classes in terms of convergence of the above sampling type series. These approximation results have been proved for not necessarily bandlimited kernels. From the above mentioned characterization, it remains uncovered the saturation case that, however, can be treated by a totally different approach assuming that the kernel is bandlimited. Indeed, since sampling Kantorovich (discrete) operators based upon bandlimited kernels can be viewed as double-singular integrals,

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exploiting the properties of the convolution in Fourier Analysis, we become able to get the desired result obtaining a complete overview of the approximation properties in $L^p(\mathbb{R})$, $1 \leq p \leq +\infty$, for the sampling Kantorovich operators.

**Keywords** Generalized sampling series • Sampling Kantorovich operator • Singular integral • Saturation order • Asymptotic expansion • Inverse results • Bandlimited function • Favard classes

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### 1 Introduction

In recent years, the sampling Kantorovich operators $K^\chi_w$, introduced in [4] by Butzer et al., have been deeply studied as an $L^p$-version of the well-known generalized sampling series (see, e.g., [15, 16]). In particular, they revealed to be suitable in applications to real world problems based on image reconstruction and enhancement (see [21, 22]). From the point of view of Approximation Theory, the above operators have been largely studied in connection to their basic properties, such as convergence and order of approximation, in several functions spaces, such as $L^p$-spaces, or more in general, in Orlicz spaces (see, e.g., [4, 46, 52]).

It is well-known that, the above mentioned sampling type operators, together with the so-called quasi-projection operators (see, e.g., [33, 34, 37]) are all approximate versions of the classical Wittaker-Kotel’nikov-Shannon sampling theorem (see, e.g., [32]); the study of extensions of such a classical result has been one of the most studied topic in the last forty years (see, e.g., [2, 27, 44]).

In order to completely understand what are the real approximation capabilities of a family of approximation operators, it is of fundamental importance the study of the problem of the saturation order, i.e., the best possible order of approximation that can be achieved into a certain space of functions, and to derive, if it is possible, the corresponding saturation class.

More precisely, the problem of establishing the saturation order for the family $K^\chi_w$, $w > 0$, consists into determine a class of functions $\mathcal{F}$, a certain subclass $\mathcal{E}$ of trivial functions of $\mathcal{F}$, and a positive non-increasing function $\varphi(w)$, $w > 0$, such that there exists $g \in \mathcal{F} \setminus \mathcal{E}$ with $\|K^\chi_w g - g\| = O(\varphi(w))$, as $w \to +\infty$, and with the property that, for any $f \in \mathcal{F}$ with:

$$\|K^\chi_w f - f\| = o(\varphi(w)), \quad w \to +\infty, \quad (I)$$

it turns out that $f \in \mathcal{E}$, and vice-versa. Here, $\| \cdot \|$ denotes any suitable norm on $\mathcal{F}$. In this case, $\varphi(w)$ is said the saturation order of the approximation process $K^\chi_w$, and $\mathcal{F}$ is its corresponding Favard (saturation) class.

Concerning the sampling Kantorovich operators, a partial solution of the above problem has been given in [24] in the space of uniformly continuous and bounded functions $f$, with respect to the uniform norm. In particular, we proved that if the first discrete algebraic moment of the kernel $\chi$ is constant on $\mathbb{R}$ and different from...
−1/2, the saturation order is 1/w, as w → +∞. The corresponding Favard class has been subsequently obtained in [17] assuming additional assumptions on χ. However, always in [24] we also observed that, requiring that the first discrete algebraic moment of the kernel is equal to −1/2, the saturation order of the operators $K^X_w$ can be higher than 1/w. Moreover, the problem (I) in $L^p$-spaces, which is the most suitable setting for the operators $K^X_w$, was still an open problem. Actually, for function in the Bernstein class $B^1_\sigma (\mathbb{R})$ and with respect the $L^1$-norm, the saturation problem has been partially treated in [25], where the saturation order has been established resorting to the so-called Fourier-transform method. The latter approach has been originally introduced by Butzer in the 1960 (see [11]) with the aim to obtain saturation theorems for the singular integrals in the $L^1$-setting.

All the above considerations have motivated the present work, in which a complete study of the saturation phenomenon for the sampling Kantorovich operators has been performed. Moreover, here we also establish a full characterization for the Lipschitz and the generalized Lipschitz classes of $L^p$ ($1 \leq p \leq +\infty$) in term of convergence, with respect to the $L^p$-norm, of the family of sampling Kantorovich operators; we also complete the study in the continuous case with respect to the uniform norm.

In order to reach the above results, we exploit tools and techniques of Functional Analysis, Fourier Analysis and Approximation Theory. In particular, since we cover the whole case of $L^p(\mathbb{R})$, $1 \leq p \leq +\infty$, we need to use the distributional approach and new strategies of proof.

Concerning the possible implications of the achieved results in the frame of signal (or image) reconstruction, it is clear that the possibility of characterizing functional spaces in term of the order convergence of the above operators, can help into the choice of kernels and in setting the parameter $w > 0$ in order to improve all the aspects of the reconstruction procedure, such as the quality of the approximation. All this can be particularly useful since, as shown in [22] in case of some applications involving digital images, the choice of $w$ also affect the computational time of the corresponding implementation of the considered operators, and the quality of the reconstructed images is fundamental to successfully study real world applications.

Here we give a detailed description of the contents of the paper, in which we also summarize the main results achieved section by section. We stress that most of the results established in Sects. 3, 4 and 5 played a crucial role for the proofs of the main theorems stated in the last two sections of the paper.

In Sect. 2, the main notions and definitions, together with some preliminary results have been given. In particular, some basic notions of distribution theory and Fourier Analysis, which are useful for this paper, have been recalled. Further, the definitions of the generalized and the Kantorovich sampling series have been given, together with the singular integrals. The relations of $K^X_w$ with other approximation operators revealed to be crucial in order to prove the desired results. Here, we also provide examples of both duration limited and bandlimited kernels known in the literature, such as the central B-spline of order $N$, the Bochner-Riesz and the Jackson-type kernels.

In Sect. 3, high order asymptotic type theorems have been proved for the operators $K^X_w$, with respect to the $L^p$-norm. In particular, we proved asymptotic formulas in the case of functions $f$ belonging to $C^r_c(\mathbb{R})$ and, for kernels $\chi$ with compact support, also for functions $f$ in Sobolev spaces.
Moreover, using the same procedures exploited for proving the above asymptotic expansions, we also obtain that, under suitable assumptions on $\chi$, the sampling Kantorovich operators are polynomials preserving.

Finally, also an asymptotic formula for a family of double-convolution operators has been proved. This result is an auxiliary theorem that is revealed to be crucial in order to establish the saturation order, and for the detection of the Favard classes for $K^X$ based on bandlimited kernels. The latter fact, i.e., the importance to have at disposal asymptotic type theorems in order to study the saturation phenomenon, is not surprising, and it is known since 1978 by the pioneer work of Nishishirao [45] concerning bounded linear operators.

In Sect. 4, the regularization properties of the sampling Kantorovich operators have been investigated. In particular, we proved that the regularity of the kernel function $\chi$ influences the regularity of the operator itself. More precisely, we proved that, if $\chi$ is continuous then also $K^X f$ is continuous, for every $f \in L^p(\mathbb{R})$, $1 \leq p \leq +\infty$. Similar results have been proved for kernels belonging to the Sobolev space $W^{r,1}(\mathbb{R})$ and to the Bernstein class $B^{1}_{r,\infty}(\mathbb{R})$, obtaining that $K^X f$ belong respectively to $W^{r,p}(\mathbb{R})$ and $B^{p}_{r,\infty}(\mathbb{R})$, for any $f \in L^p(\mathbb{R})$, $1 \leq p \leq +\infty$. In the bandlimited case (i.e., when $\chi$ belongs to $B^{1}_{r,\infty}(\mathbb{R})$), also a closed form for the distributional Fourier transform of the above operators has been stated.

In Sect. 5, direct and inverse theorems of approximation have been proved. In particular, we estimated the order of approximation by the order modulus of smoothness of $L^p$. Assuming $f$ in a suitable Lipschitz and generalized Lipschitz classes, also the qualitative rate of approximation has been estimated. Further, an inverse theorem of approximation has been proved, allowing to obtain a characterization of the Lipschitz and of the generalized Lipschitz classes in terms of convergence of the family $K^X$. From this characterization, the case corresponding to the saturation order is not still covered, and it has been treated in the following sections by an ad-hoc strategy.

In Sect. 6, we face the problem of the saturation order for the sampling Kantorovich operators based upon bandlimited kernels. Here, the crucial point of the proof is that the composition $K^X K^Y f$ can be written as a double-convolution allowing us to use the auxiliary asymptotic theorems given in Sect. 3. Here, we also provide examples of bandlimited kernels for which the saturation theorem (and also the results of the previous sections) holds. The main idea is to consider suitable finite linear combinations of some well-known kernels, such as those mentioned in Sect. 2.

Finally, in Sect. 7 we proved the inverse theorem of approximation corresponding to the saturation order, in fact obtaining a characterization of the Favard classes for the sampling Kantorovich operators based upon bandlimited kernels. Here, the proof is based on the well-known Helly-Bray and weak* compactness theorem.

2 Preliminaries and Auxiliary Results

Let $L^p(\mathbb{R})$, $1 \leq p < +\infty$, be the space of all Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$, for which the usual norm $\|f\|_p$ is finite, and, in the case $p = +\infty$, for the sake of simplicity, by $L^\infty(\mathbb{R})$ we refer to the space $C(\mathbb{R})$, i.e., the space of all uniformly continuous and bounded functions on $\mathbb{R}$ endowed with the usual sup-norm $\|\cdot\|_\infty$. Note
that, also the case of the usual $L^\infty(\mathbb{R})$ (when specified), i.e., the space of measurable functions with finite sup norm, it will be also considered in some result. Moreover, we denote by $C^r(\mathbb{R}), r \in \mathbb{N}^+$, and $C^\infty(\mathbb{R})$ the subspaces of $C(\mathbb{R})$ such that the derivatives $f^{(i)}$, $i = 1, \ldots, r$, and $i \in \mathbb{N}^+$, exist and belong to $C(\mathbb{R})$, respectively; $C_c^r(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ are the subsets of $C^r(\mathbb{R})$ and $C^\infty(\mathbb{R})$, respectively, of functions with compact support. Finally, we denote by $C^0(\mathbb{R})$ the space of continuous and bounded functions on $\mathbb{R}$, by $BV(\mathbb{R})$ the space of functions of bounded variation on $\mathbb{R}$, and by $AC(\mathbb{R})$ the space of absolutely continuous functions on $\mathbb{R}$.

According to the notation of the distribution theory, we will denote by $D$ the space of the test functions $C_c^\infty(\mathbb{R})$, and by $S$ the Schwartz class. Further, by $D'$ and $S'$ we define the spaces of the distributions $\Lambda : D \rightarrow \mathbb{C}$ and of the tempered distributions $T : S \rightarrow \mathbb{C}$, respectively. We recall that a function $f$ defined on $\mathbb{R}$ is said to be tempered, if there exists a positive constant $C$ and a positive integer $m$ such that:

$$|f(x)| \leq C (1 + |x|)^m, \quad x \in \mathbb{R}. $$

For any locally integrable function $f$, it is possible to define a regular distribution as follows:

$$\Lambda_f[\varphi] := f[\varphi] := \int_{\mathbb{R}} f(x) \varphi(x) \, dx, \quad \varphi \in D. \quad (1)$$

In view of the above definition, it is usual to identify the regular distribution $\Lambda_f$ by $f$ itself. If the locally integrable function $f$ in (1) is tempered, and we replace the space of test functions $D$ by $S$, one can obtain the regular tempered distribution $T_f$. As above, $T_f$ is usually identified by $f$ itself.

Suppose now that $\Lambda \in D'$ (or $T \in S'$) and $f \in C^\infty(\mathbb{R})$. We can define the multiplication of $f$ with the distribution $\Lambda$ (or $T$, see [49] p. 159) as:

$$(f \Lambda)[\varphi] := \Lambda[f\varphi], \quad \varphi \in D \quad (or \ \varphi \in S).$$

We can observe that $f \Lambda$ belongs to $D'$ (or $fT \in S'$, respectively).

A distribution $\Lambda \in D'$ (or $T \in S'$) is null on the open set $A \subset \mathbb{R}$ if for any $\varphi \in D$ (or $S$), such that $\text{supp} \, \varphi \subset A$, we have $\Lambda[\varphi] = 0$ (or $T[\varphi] = 0$). The biggest open set in which $\Lambda$ is null is called the null set of $\Lambda$ (or $T$). The complementary of the null set is called the support of the distribution $\Lambda$ (or $T$).

For any $\Lambda \in D'$ and $n \in \mathbb{N}^+$ we can define the following distribution:

$$\Lambda^{(n)}[\varphi] := (-1)^n \Lambda[\varphi^{(n)}], \quad \varphi \in D,$$

that is called the distributional derivative of order $n$ of $\Lambda$. Note that, differently from the ordinary functions, any distribution admits derivatives of any order. For more details about distribution theory, see, e.g., the monographs [49, 53]. Further, we recall the definition of Sobolev spaces:

$$W^{r,p}(\mathbb{R}) := \left\{ f \in L^p(\mathbb{R}) : f^{(n)} \in L^p(\mathbb{R}), \ 1 \leq n \leq r, \ n \in \mathbb{N} \right\}, \ 1 \leq p \leq +\infty.$$
\[ r \in \mathbb{N}^+, \text{ where the derivatives } f^{(n)} \text{ can be intended in the distributional sense. It is well-know that, the definition of the Sobolev spaces can also equivalently formulated (in term of equivalence classes) as follows:} \]

\[
W^{r,p}(\mathbb{R}) := \left\{ f \in L^p(\mathbb{R}) : f^{(r-1)} \in AC(\mathbb{R}), \text{ and } f^{(r)} \in L^p(\mathbb{R}) \right\}, \quad 1 \leq p \leq +\infty.
\]

Finally, we also introduce the space (see [26] p. 35):

\[
W^r(BV) := \left\{ f \in BV(\mathbb{R}) : f^{(r-1)} \in AC(\mathbb{R}), \text{ and } f^{(r)} \in BV(\mathbb{R}) \right\}, \quad r \in \mathbb{N}.
\]

Now, we denote by \( B^p_\sigma(\mathbb{R}) \), the so-called \textit{Bernstein class (or Bernstein space)}, for \( \sigma \geq 0 \) and \( 1 \leq p \leq +\infty \), containing the functions of \( L^p(\mathbb{R}) \) which can be extended to an entire function \( f(z) (z = x + iy \in \mathbb{C}) \) of exponential type \( \sigma \), i.e., satisfying:

\[
|f(z)| \leq e^{\sigma |y|} \|f\|_\infty, \quad z \in \mathbb{C},
\]

see e.g., [13, 14, 51]. It is well-known that:

\[
B^1_\sigma(\mathbb{R}) \subset B^p_\sigma(\mathbb{R}) \subset B^q_\sigma(\mathbb{R}) \subset B^\infty_\sigma(\mathbb{R}), \quad (2)
\]

with any \( 1 \leq p < q \). Moreover, we denote by:

\[
\widehat{f}(v) := \int_\mathbb{R} f(u) e^{-iuv} du,
\]

(3)

\( v \in \mathbb{R} \), the Fourier transform of \( f \in L^1(\mathbb{R}) \), while for \( f \in L^p(\mathbb{R}) \), \( p \geq 1 \), we denote the (distributional) Fourier transform \( \widehat{f} \) as the following regular distribution:

\[
\widehat{f}[\varphi] := \int_\mathbb{R} f(x) \widehat{\varphi}(x) dx, \quad \varphi \in S,
\]

where \( \widehat{\varphi} \) is defined as in (3). Note that, in the case \( p = 1 \) the distributional Fourier transform \( \widehat{f} \) is, in fact, a usual function and it coincides with the definition in (3), and a similar consideration can be made for the case \( 1 < p \leq 2 \). According to the Paley-Wiener-Schwartz theorem, it is possible to prove that \( f \in B^p_\sigma(\mathbb{R}) \), if and only if \( f \) is bandlimited (as a function or a distribution), with \( \text{supp } \widehat{f} \subset [-\sigma, \sigma] \). We recall that, a function \( f \) is said to be \textit{bandlimited} if the support of \( \widehat{f} \) (as a function or a distribution) is compact.

Now, in order to recall the definition of the families of sampling type operators that will be studied in this paper, we introduce the following notation.

From now on, we will say that a function \( \chi : \mathbb{R} \rightarrow \mathbb{R} \) is a \textit{kernel}, if it satisfies the following assumptions:

\[ (\chi 1) \quad \chi \in L^1(\mathbb{R}) \text{ is bounded on } \mathbb{R}; \]
(χ2) the discrete algebraic moment of order 0:

\[ m_0(\chi, u) := \sum_{k \in \mathbb{Z}} \chi(u - k) = 1, \quad u \in \mathbb{R}; \quad (4) \]

(χ3) the discrete algebraic moment of order 1:

\[ m_1(\chi, u) := \sum_{k \in \mathbb{Z}} (k - u) \chi(u - k) =: A_1^\chi \in \mathbb{R}, \quad u \in \mathbb{R}; \]

(χ4) there exists \( \beta > 2 \) such that:

\[ \chi(u) = O(|u|^{-\beta}), \quad \text{as} \quad |u| \to +\infty. \]

Now, we also introduce the following notations that will be useful later. We define for \( \chi \) the so-called continuous algebraic and absolute moments of order \( \nu \in \mathbb{N} \) and \( \nu \geq 0 \), respectively, by the following integrals:

\[ \tilde{m}_\nu(\chi) := \int_{\mathbb{R}} u^\nu \chi(u) \, du, \quad \tilde{M}_\nu(\chi) := \int_{\mathbb{R}} |u|^\nu |\chi(u)| \, du. \]

Note that, by condition (χ4) it turns out that \( \tilde{m}_\nu(\chi) \leq \tilde{M}_\nu(\chi) < +\infty \), for every \( 0 \leq \nu < \beta - 1 \) for which \( \tilde{m}_\nu(\chi) \) takes sense.

**Remark 2.1** Note that, assumption (χ4) implies that (see Lemma 2.1 of [24]):

\[ M_\nu(\chi) := \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |u - k|^\nu |\chi(u - k)| < +\infty, \quad (5) \]

for every \( 0 \leq \nu < \beta - 1 \). Moreover, (χ4) also implies that, for every \( \gamma > 0 \):

\[ \lim_{w \to +\infty} \sum_{|wx - k| > w^\gamma} |wx - k|^\nu \cdot |\chi(wx - k)| = 0, \quad (6) \]

uniformly with respect to \( x \in \mathbb{R} \), where \( 0 \leq \nu < \beta - 1 \). Finally, we also have that the 1-periodic series \( \sum_{k \in \mathbb{Z}} |\chi(u - k)| |u - k|^\nu \) are uniformly convergent on \( \mathbb{R} \). Obviously, if \( \chi \) has compact support, it turns out that assumption (χ4) is satisfied for every \( \beta > 0 \) and then (5), (6) hold for every \( \nu \geq 0 \).

Now, we recall the following lemma, from which we can deduce some equivalent conditions for (4) of (χ2) and (χ3), and that can be proved as a consequence of the well-known Poisson summation formula (see [12], p. 202).

**Lemma 2.2** Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a given continuous function satisfying (χ1). Assuming in addition that the function \( g(u) := -(iu)^j \chi(u) \), \( j \in \mathbb{N} \) \( (u \in \mathbb{R} \) and \( i \) denotes the complex unit) belongs to \( L^1(\mathbb{R}) \), we have that:

\[ \sum_{k \in \mathbb{Z}} (k - x)^j \chi(x - k) = A_j^X \in \mathbb{R}, \quad x \in \mathbb{R}, \]
if and only if
\[
\hat{\chi}^{(j)}(2\pi k) = \begin{cases} A_j, & k = 0, \\ 0, & k \in \mathbb{Z} \setminus \{0\}, \end{cases}
\]  
(7)

where \( \hat{\chi}^{(j)} \) denotes the \( j \)-th derivative of \( \hat{\chi} \). It turns out that \( A_j^0 = (-i)^j A_j \).

Clearly, in case of functions with sufficiently rapid decay, it turns out that (4) of \((\chi 2)\) and \((\chi 3)\) are equivalent to the condition of Lemma 2.2 for \( j = 0 \), \( A_0 = 1 \) and \( j = 1 \), \( A_1^1 = -i A_1 \), respectively. Condition (7) is known as the Strang-Fix type condition, see [28].

Examples of kernels satisfying the above assumptions (for the details of the proof see [4, 24]) are, e.g., the central B-splines of order \( N \) (see, e.g., [10, 16]), defined by:
\[
B_N(x) := \frac{1}{(N-1)!} \sum_{i=0}^{N} (-1)^i \binom{N}{i} \left( \frac{N}{2} + x - i \right)^{N-1}, \quad x \in \mathbb{R}, \quad N \geq 1,
\]
where \((\cdot)_+\) denotes the positive part. It is well-known that \( B_N \in C^N_c(\mathbb{R}) \), \( N \geq 2 \), i.e., they are examples of duration limited kernels with \( \text{supp} \ B_N \subset [-N/2, N/2] \), hence condition \((\chi 4)\) is satisfied for every \( \beta > 0 \). Further, we recall the Jackson-type kernels (\([18, 41]\)), defined by:
\[
J_N(x) := c_N \text{sinc}^{2N} \left( \frac{x}{2N\pi \alpha} \right), \quad x \in \mathbb{R},
\]
with \( N \in \mathbb{N} \), \( \alpha \geq 1 \), and \( c_N \) is a non-zero normalization coefficient, given by:
\[
c_N := \left[ \int_{\mathbb{R}} \text{sinc}^{2N} \left( \frac{u}{2N\pi \alpha} \right) \, du \right]^{-1}.
\]
Here, the sinc-function \( \text{sinc}(x) \) is defined as \( \sin(\pi x)/\pi x \), if \( x \neq 0 \), and 1 if \( x = 0 \). Finally, we also recall the Bochner-Riesz kernels (\([19, 50]\)), defined by:
\[
b_\eta(x) := \frac{2^n}{\sqrt{2\pi}} \Gamma(\eta + 1) |x|^{-\eta+1/2} J_{\eta+1/2}(|x|), \quad x \in \mathbb{R},
\]
with \( \eta > 0 \), and where \( J_\lambda \) is the Bessel function of order \( \lambda \) (\([20]\)), and \( \Gamma \) is the usual Euler gamma function. The functions \( J_N \) and \( b_\eta \) are examples of bandlimited (hence belonging to \( C^\infty(\mathbb{R}) \)) kernels. Moreover, by simple computations involving the inverse Fourier transform (see [24], Sect. 7) we have \( J_N^{(i)}(x) = O(|x|^{-2N}) \), \( b_\eta^{(i)}(x) = O(|x|^{-\eta-1}) \), as \( |x| \to +\infty \), \( i = 0, 1, \ldots \). Other examples of kernels can be found, e.g., in [12].

Now, we are able to recall the definition of the sampling Kantorovich operators (\([4]\)). We denote by:
\( (K^X_w f)(x) := \sum_{k \in \mathbb{Z}} w \int_{k/w}^{(k+1)/w} f(u) \, du \chi(wx - k), \quad x \in \mathbb{R}, \quad w > 0, \)

the **sampling Kantorovich operators** of \( f \), where \( f : \mathbb{R} \to \mathbb{R} \) is a locally integrable function, such that the above series is convergent for every \( x \in \mathbb{R} \).

It is well-known that, under the assumptions \((\chi 1), (\chi 2)\) and \((\chi 4)\) on \( \chi \), for every \( f \in L^p(\mathbb{R}) \), \( 1 \leq p \leq +\infty \), it turns out that (see Theorem 4.1 and Corollary 5.2 of [4]):

\[
\lim_{w \to +\infty} \|K^X_w f - f\|_p = 0, \quad \tag{8}
\]

and there holds (see Corollary 5.1 and Remark 3.2 (a) of [4]):

\[
\|K^X_w f\|_p \leq M_0(\chi)^{p-1/p} \|\chi\|_1^{1/p} \|f\|_p, \quad \text{and} \quad \|K^X_w f\|_\infty \leq M_0(\chi) \|f\|_\infty, \quad \tag{9}
\]

with \( 1 \leq p < +\infty \).

Note that, the theory of the above operators \( K^X_w \) (as usually happens for the main families of linear operators) is studied in the \( L^p \)-setting, only for \( 1 \leq p \leq +\infty \). The case \( 0 < p < 1 \) is not considered since, one of the main tools used in the proofs of the classical approximation results for \( 1 \leq p < +\infty \) (above recalled) is the Jensen inequality, that can not be applied if \( 0 < p < 1 \).

In order to study approximation properties for the above sampling Kantorovich series, we also need to recall the definition of the following operators.

We define the **generalized sampling operators** as:

\( (G^X_w f)(x) := \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \chi(wx - k), \quad x \in \mathbb{R}, \quad w > 0, \)

where \( f : \mathbb{R} \to \mathbb{R} \) is any bounded function and \( \chi \) is a kernel, see e.g., [3, 6, 14, 16, 47].

In conclusion, we also recall the notion of **singular integral** (see e.g., [12, 40, 48]):

\[
(I_w^\Phi f)(x) := (f \ast \Phi_w)(x) = w \int_{\mathbb{R}} f(x-u) \Phi(wu) \, du = w \int_{\mathbb{R}} f(y) \Phi(w[x-y]) \, dy,
\]

\( x \in \mathbb{R}, \ w > 0, \) where \( f \in L^p(\mathbb{R}), \ 1 \leq p \leq +\infty, \ \Phi_w(u) := w \Phi(wu), \ u \in \mathbb{R}, \) is a kernel of Fejér type with \( \Phi : \mathbb{R} \to \mathbb{R} \) belonging to \( L^1(\mathbb{R}) \); here the symbol “\( \ast \)” refers to the usual convolution product.

Approximation results for the singular integrals \( I_w^\Phi f \) can be proved, requiring that \( \Phi \) satisfies:

\[
\tilde{m}_0(\Phi) = \int_{\mathbb{R}} \Phi(u) \, du = 1. \quad \tag{10}
\]
Note that, if $\chi$ is a continuous kernel (according to the previous definition), then condition (10) applied to $\chi$ is satisfied, by Lemma 2.2 with $j = 0$ and $A_0 = 1$.

Obviously, it is well-known that, under the assumption (10), the family $(\Phi_{w})_{w > 0}$ turns out to be an approximate identity, see [12].

Now, we are able to recall the following lemma that is a direct consequence of the fundamental result of Fourier Analysis, which shows the connections between the (continuous) convolution integrals and the (discrete) convolution sums.

**Lemma 2.3** (Lemma 3.2 of [14]) Let $f \in B^p_{\pi w}(\mathbb{R})$, and let $\chi$ be a kernel such that $\chi \in B^q_{\pi w}(\mathbb{R})$, for some $w > 0$, $1 \leq p \leq +\infty$, $1/p + 1/q = 1$. Then: $(I_{w}^{x}f)(x) = (G_{w}^{x}f)(x), x \in \mathbb{R}$.

### 3 Asymptotic Expansions in $L^p$-Setting

We prove the following asymptotic formula for the sampling Kantorovich operators.

**Theorem 3.1** Let $\chi$ be a kernel satisfying (\chi^4) for $\beta > r + 1$, with $r \in \mathbb{N}^+$. In addition, if $r > 1$ we also require that:

$$
\sum_{k \in \mathbb{Z}} (k - x)^j \chi(x - k) =: A_j^x \in \mathbb{R}, \quad x \in \mathbb{R}, \quad j = 1, 2, \ldots, r, \quad (11)
$$

and

$$
\sum_{\ell=0}^{j} \binom{j}{\ell} A_{j-\ell}^x = 0, \quad j = 1, 2, \ldots, r - 1, \quad \sum_{\ell=0}^{r} \binom{r}{\ell} A_{r-\ell}^x \neq 0. \quad (12)
$$

Then, for any $f \in C^{r}(\mathbb{R})$:

$$
\lim_{w \to +\infty} \|w^r \left[K_w^x f - f\right] - \frac{f^{(r)}(x)}{r!} \left[\sum_{\ell=0}^{r} \binom{r}{\ell} A_{r-\ell}^x\right]\|_\infty = 0.
$$

Further, if $\chi$ satisfies (\chi^4) with $\beta > rp + 1$, $1 \leq p < +\infty$, for any $f \in C^{r}_c(\mathbb{R})$ we also have:

$$
\lim_{w \to +\infty} \|w^r \left[K_w^x f - f\right] - \frac{f^{(r)}(x)}{r!} \left[\sum_{\ell=0}^{r} \binom{r}{\ell} A_{r-\ell}^x\right]\|_p = 0.
$$

**Proof** By the Taylor formula with Lagrange remainder until the order $r$, we have:

$$
f(u) = \sum_{j=0}^{r-1} \frac{f^{(j)}(x)}{j!} (u - x)^j + \frac{1}{r!} f^{(r)}(\theta_{x,u}) (u - x)^r =: T_{r-1}(u; x) + R_r(u; x), \quad (13)
$$
\( u, x \in \mathbb{R} \), where \( \theta_{u,x} \), is a suitable value between \( u \) and \( x \). Hence, replacing (13) in the definition of the operators, and using condition (\( \chi 2 \)), the binomial Newton formula, and (12) one can obtain:

\[
(K_w^\chi f)(x) = \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} (T_{r-1}(u; x) + R_r(u; x)) \, du \right] \chi(wx - k)
\]

\[
= \sum_{k \in \mathbb{Z}} \sum_{j=0}^{r-1} \frac{f^{(j)}(x)}{j!} \left[ \sum_{\ell=0}^{j} \binom{j}{\ell} \frac{k}{w-x} \right] w \int_{k/w}^{(k+1)/w} \left( u - \frac{k}{w} \right) \ell \, du \chi(wx - k)
\]

\[
= f(x) + \sum_{j=1}^{r-1} \frac{f^{(j)}(x)}{j! w^j} \left\{ \sum_{\ell=0}^{j} \binom{j}{\ell} \frac{A^\chi_{j-\ell}}{\ell + 1} \right\} + \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} R_r(u; x) \, du \right] \chi(wx - k)
\]

\[
= f(x) + \frac{1}{r!} \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} R^{(r)}(\theta_{x,u})(u - x)^r \, du \right] \chi(wx - k).
\]

On the other hand:

\[
\frac{f^{(r)}(x)}{r!} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{A^\chi_{r-\ell}}{\ell + 1} \right]
\]

\[
= \frac{f^{(r)}(x)}{r!} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{\ell + 1} \sum_{k \in \mathbb{Z}} (k-wx)^{r-\ell} \chi(wx - k) \right]
\]

\[
= \frac{f^{(r)}(x)}{r!} w^r \sum_{k \in \mathbb{Z}} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{w^{-\ell}}{\ell + 1} \right] \chi(wx - k)
\]

\[
= \frac{f^{(r)}(x)}{r!} w^r \sum_{k \in \mathbb{Z}} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \left( \frac{k}{w} - x \right)^{r-\ell} w \int_{k/w}^{(k+1)/w} \left( u - \frac{k}{w} \right) \ell \, du \right] \chi(wx - k)
\]

Then, we have:

\[
J_w(x) := w^r \left[ (K_w^\chi f)(x) - f(x) \right] - \frac{f^{(r)}(x)}{r!} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{A^\chi_{r-\ell}}{\ell + 1} \right]
\]

\[
= \frac{w^r}{r!} \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} \left[ f^{(r)}(\theta_{x,u}) - f^{(r)}(x) \right] (u - x)^r \, du \right]
\]
\( \chi(wx - k). \)

We first consider the case \( p = +\infty \). By the uniform continuity of \( f^{(r)} \), for every fixed \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |f^{(r)}(y_1) - f^{(r)}(y_2)| < \varepsilon \), when \( |y_1 - y_2| \leq \delta \), \( y_1, y_2 \in \mathbb{R} \). Thus we can write what follows:

\[
|J_w(x)| \leq \frac{w^r}{r!} \left\{ \sum_{|wx-k| \leq w\delta/2} + \sum_{|wx-k| > w\delta/2} \right\} \times
\left[ w \int_{k/w}^{(k+1)/w} |f^{(r)}(\theta_{x,u}) - f^{(r)}(x)||u - x|^r \, du \right] |\chi(wx - k)| =: S_1 + S_2.
\]

We begin estimating \( S_1 \). First of all, we can observe that, if \( u \in [k/w, (k + 1)/w] \) and \( |wx - k| \leq w\delta/2 \), it results:

\[
|\theta_{x,u} - x| \leq |u - x| \leq \frac{1}{w} + \frac{1}{w}|k - wx| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

for \( w > 0 \) sufficiently large, then, using the convexity of \(|\cdot|^r\), \( r \geq 1 \):

\[
S_1 \leq \frac{\varepsilon}{r!} \sum_{|wx-k| \leq w\delta/2} \left[ w^{r+1} \int_{k/w}^{(k+1)/w} |u - x|^r \, du \right] |\chi(wx - k)|
\leq \frac{2^{r-1} \varepsilon}{r!} \sum_{|wx-k| \leq w\delta/2} \left[ w^{r+1} \int_{k/w}^{(k+1)/w} \left( \frac{u - k}{w} \right)^r \, du + |k - wx|^r \right] |\chi(wx - k)|
\leq \frac{2^{r-1} \varepsilon}{r!} \sum_{|wx-k| \leq w\delta/2} \left[ \frac{1}{r + 1} + |k - wx|^r \right] |\chi(wx - k)|
\leq \frac{2^{r-1} \varepsilon}{r!} \left[ \frac{M_0(\chi)}{r + 1} + M_r(\chi) \right] =: C \varepsilon,
\]

where \( M_0(\chi) \), \( M_r(\chi) \) are both finite by Remark 2.1, as a consequence of assumption (\( \chi 4 \)). While for \( S_2 \) we have:

\[
S_2 \leq \frac{2 \|f^{(r)}\|_{\infty}}{r!} \sum_{|wx-k| > w\delta/2} \left[ w^{r+1} \int_{k/w}^{(k+1)/w} |u - x|^r \, du \right] |\chi(wx - k)|
\leq \frac{2^{r} \|f^{(r)}\|_{\infty}}{r!} \sum_{|wx-k| > w\delta/2} \left[ \frac{1}{r + 1} + |k - wx|^r \right] |\chi(wx - k)| < \varepsilon,
\]

uniformly with respect to \( x \in \mathbb{R} \), for \( w > 0 \) sufficiently large, by (6) of Remark 2.1. This completes the first part of the proof.
Let now $1 \leq p < +\infty$ be fixed. We want to prove that:

$$\lim_{w \to +\infty} \int_{\mathbb{R}} |J_w(x)|^p \, dx = 0,$$

exploiting the Vitali convergence theorem. By the first part of this theorem, immediately follows that:

$$\lim_{w \to +\infty} \|J_w\|_{L_p}^p = 0.$$

Let now $\varepsilon > 0$ be fixed, and $\gamma > 0$ such that $\text{supp } f \subset [-\gamma, +\gamma]$. For any $M > \gamma + 1$, and using Jensen inequality twice, we can write what follows:

$$J_M := \int_{|x| > M} |J_w(x)|^p \, dx = \int_{|x| > M} |w^r \left(K_w^\chi f\right)(x)|^p \, dx$$

$$\leq M_0(\chi)^{p-1} \int_{|x| > M} w^{rp} \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} |f(u)|^p \, du \right] |\chi(wx - k)| \, dx.$$

Now, using the change of variable $t = wx - k$:

$$J_M \leq M_0(\chi)^{p-1} \int_{|x| > M} w^{rp} \sum_{|k| \leq w(\gamma + 1)} \left[ w \int_{k/w}^{(k+1)/w} |f(u)|^p \, du \right] |\chi(wx - k)| \, dx$$

$$\leq M_0(\chi)^{p-1} \|f\|_{L_p}^p \sum_{|k| \leq w(\gamma + 1)} w^{rp} |\chi(wx - k)| \, dx$$

$$\leq M_0(\chi)^{p-1} \|f\|_{L_p}^p 2 \left[w(\gamma + 1) + 1\right] w^{p-1} \int_{|t| > w(M-\gamma-1)} |\chi(t)| \, dt < \varepsilon, \quad (16)$$

for sufficiently large $w > 1$, since

$$\int_{|y| > w(M-\gamma-1)} |\chi(y)| \, dy = o(w^{-rp}), \quad \text{as } w \to +\infty,$$

by condition $(\chi 4)$ with $\beta > rp + 1$. Therefore, we just proved that in correspondence of $\varepsilon$ there exists the interval $E_\varepsilon := [-M, M]$ such that, for any measurable set $F$, with $F \cap E_\varepsilon = \emptyset$, inequality (16) holds.

Now, again by the first part of this theorem, it follows that:

$$w^r \|K_w^\chi f - f\|_\infty \leq \frac{1}{r!} \left| \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{A_r^{\chi}}{\ell + 1} \|f^{(\ell)}\|_\infty \right| + 1 =: K,$$

for sufficiently large $w > 0$; thus, for any fixed measurable set $B \subset \mathbb{R}$, with $|B| < \varepsilon 2^{-p} K^{-p}$, we finally get:
\[
\int_B |J_w(x)|^p \, dx \leq 2^{p-1} \left\{ \int_B w^r \left| (K_w^x f)(x) - f(x) \right|^p \, dx \right. \\
\left. + \frac{1}{[r!]^p} \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) A_r^x \int_B |f^{(r)}(x)|^p \, dx \right\} \leq 2^{p-1} |B| \times \left\{ \|w^r \{ K_w^x f - f \} \|_{\infty}^p + \frac{1}{[r!]^p} \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) A_r^x \int_B |f^{(r)}(x)|^p \, dx \right\} \right. \\
\left. \leq 2^p |B| K^p < \varepsilon, \right.
\]

for every \( w > 0 \) sufficiently large. This shows that the integrals \( \int_B |J_w(x)|^p \, dx \) are equi-absolutely continuous. This completes the proof. \( \square \)

Asymptotic expansions can also be proved for functions in Sobolev spaces, assuming that \( \chi \) has compact support.

**Theorem 3.2** Let \( \chi \) be a kernel with compact support. Then, for every \( f \in W^{1,p}(\mathbb{R}) \), \( 1 \leq p \leq +\infty \), we have:

\[
\lim_{w \to +\infty} \|w \{ K_w^x f - f \} - (A^x_1 + 1/2) f' \|_p = 0.
\]

Further, if \( \chi \) also satisfies (11) and (12) for \( r \in \mathbb{N}^+ \), \( r > 1 \), then for every \( f \in W^{r,p}(\mathbb{R}) \), \( 1 \leq p \leq +\infty \), it turns out that:

\[
\lim_{w \to +\infty} \|w^r \{ K_w^x f - f \} - f^{(r)}(x) \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) A_r^x \|_p = 0.
\]

**Proof** Let \( f \in W^{r,p}(\mathbb{R}) \), \( r \in \mathbb{N}^+ \), \( r > 1 \), \( 1 \leq p \leq +\infty \), be fixed. It is well-known that \( f \) can be written by the Taylor formula (13) with integral remainder \( R_r(u; x) = \int_x^u \frac{f^{(r)}(t)}{(r-1)!}(u - t)^{r-1} \, dt \) (see [26] p. 37). Hence, proceeding as in the proof of Theorem 3.1, i.e., replacing the above Taylor formula in the definition of the operators \( K_w^x f \), and noting that:

\[
\frac{f^{(r)}(x)}{r!} w^r \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} (u - x)^r \, du \right] \chi(wx - k) = \frac{f^{(r)}(x)}{(r-1)!} w^r \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} \left\{ \int_x^u (u - t)^{r-1} \, dt \right\} \, du \right] \chi(wx - k),
\]

we immediately obtain:

\[
J_w(x) := w^r \left[ (K_w^x f)(x) - f(x) \right] - \frac{f^{(r)}(x)}{r!} \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) A_r^x \int_B |f^{(r)}(x)|^p \, dx
\]

\[
= \frac{w^r}{(r-1)!} \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} \left\{ \int_x^u \left[ f^{(r)}(t) - f^{(r)}(x) \right] (u - t)^{r-1} \, dt \right\} \, du \right]
\]

\( \varepsilon \) Birkaheuser.
where the above series can be considered only for $k \in \mathbb{Z}$ such that $|wx - k| \leq T$, since $\text{supp} \, \chi \subset [-T,T]$, $T > 0$. Now, for $|wx - k| \leq T$, the integrals can be estimated in terms of:

$$\chi(wx - k),$$

(17)

$$w^{r+1} \int_{k/w}^{(k+1)/w} \left[ \int_x^u |f^{(r)}(t) - f^{(r)}(x)| |u - t|^{r-1} \, dt \right] \, du$$

$$\leq w^{r+1} \int_{k/w}^{(k+1)/w} |u - x|^r \left[ \int_x^u |f^{(r)}(t) - f^{(r)}(x)| \, dt \right] \, du$$

$$= w^{r+1} \int_{k/w}^{(k+1)/w} |u - x|^r \int_0^u |f^{(r)}(z + x) - f^{(r)}(x)| \, dz \, du$$

$$\leq w^{r+1} \int_{k/w}^{(k+1)/w} |u - x|^{r-1} \int_{|z| \leq \frac{T+1}{w}} |f^{(r)}(z + x) - f^{(r)}(x)| \, dz \, du$$

Thus, for any $w > 0$ we have:

$$|J_w(x)| \leq \frac{1}{(r-1)!} \left[ w \int_{|z| \leq \frac{T+1}{w}} |f^{(r)}(z + x) - f^{(r)}(x)| \, dz \right]$$

$$\times \sum_{|wx - k| \leq T} \left[ w^r \int_{k/w}^{(k+1)/w} |u - x|^{r-1} \, du \right] |\chi(wx - k)|$$

$$\leq \frac{2^{r-2}}{(r-1)!} \left\{ \frac{M_0(\chi)}{r} + M_{r-1}(\chi) \right\} \left[ w \int_{|z| \leq \frac{T+1}{w}} |f^{(r)}(z + x) - f^{(r)}(x)| \, dz \right]$$

$$< +\infty,$$

by Remark 2.1. Now, recalling the generalized Minkowsky type inequality (see [30], p. 148), we have:

$$|J_w|_p \leq \frac{2^{r-2}}{(r-1)!} \left\{ \frac{M_0(\chi)}{r} + M_{r-1}(\chi) \right\} \left[ w \int_{|z| \leq \frac{T+1}{w}} \|f^{(r)}(z + \cdot) - f^{(r)}(\cdot)\|_p \, dz \right]$$

$$\leq \frac{2^{r-1}}{(r-1)!} \left\{ \frac{M_0(\chi)}{r} + M_{r-1}(\chi) \right\} (T+1) \sup_{|z| \leq (T+1)/w} \|f^{(r)}(z + \cdot) - f^{(r)}(\cdot)\|_p.$$

Now, since $f^{(r)} \in L^p(\mathbb{R})$ we know that, for every $\varepsilon > 0$ there exists $\gamma > 0$ such that, for every $|z| \leq \gamma$ there holds $\|f^{(r)}(z + \cdot) - f^{(r)}(\cdot)\|_p < \varepsilon$, hence for $w$ sufficiently large we finally get: $|J_w|_p < \varepsilon.$ In the case $r = 1$, the proof can be deduced analogously. \(\square\)
Remark 3.3 Note that, in case of $p = +\infty$, Theorem 3.2 holds also if we define $W^{r,\infty}(\mathbb{R})$ as a subspace of the usual $L^\infty(\mathbb{R})$, namely, as the space of all the measurable functions with finite essential supremum.

Proceeding as in the proof of the above theorems, the following result can be easily deduced.

**Theorem 3.4** Let $\chi$ be a kernel, assumed as in Theorem 3.1 with $r \in \mathbb{N}^+$. Then, for any polynomial $p_{r-1}$ of degree at most $r - 1$, it turns out that:

$$(K_w^\chi p_{r-1})(x) = p_{r-1}(x), \quad x \in \mathbb{R}, \quad w > 0.$$  

**Proof** Let $p_{r-1}(x) = a_{r-1} x^{r-1} + \cdots + a_1 x + a_0$, be any polynomial of degree at most $r - 1$. We first observe that, for every fixed $x \in \mathbb{R}$:

$$|(K_w^\chi p_{r-1})(x)| \leq \sum_{j=0}^{r-1} |a_j| 2^j \frac{2 M_{j+1}(\chi)}{w^j} + 3 w |x|^{j+1} M_0(\chi) + \frac{M_0(\chi)}{w^j} < +\infty,$$

by condition (14), hence the series is absolutely convergent and the operators $K_w^\chi p_{r-1}$ are well-defined, for every fixed $w > 0$. Finally, proceeding as in (14), we immediately get the thesis. \hfill \Box

Theorem 3.4 shows that the sampling Kantorovich operators are polynomials-preserving, and improves Theorem 3.5 of [23].

Now, the following asymptotic formula for double-convolution type operators can be proved; this will be useful in the next sections.

**Theorem 3.5** Let $\chi, \Phi \in L^1(\mathbb{R})$ both satisfying conditions (10) and (14) for $\beta > r + 1$, with $r \in \mathbb{N}^+$. If $r > 1$, we assume in addition that:

$$\sum_{\ell=0}^{j} \binom{j}{\ell} \tilde{m}_\ell(\Phi) \tilde{m}_{j-\ell}(\chi) = 0, \quad j = 1, 2, \ldots, r - 1,$$

$$\sum_{\ell=0}^{r} \binom{r}{\ell} \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \neq 0. \quad (18)$$

Then, for any $f \in C_c^r(\mathbb{R})$, and $1 \leq p \leq +\infty$, we have:

$$\lim_{w \to +\infty} \|w^r [I_w^\chi (I_w^\Phi f) - f] - \frac{f(r)}{r!} \left[ (-1)^r \sum_{\ell=0}^{r} \binom{r}{\ell} \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \right] \|_p = 0.$$  

**Proof** By the Taylor formula (13), and using (10) for both $\chi$ and $\Phi$, we can write what follows:

$$(I_w^\chi [I_w^\Phi f])(x)$$
On the other hand, we have:

\[
\int_{\mathbb{R}} \chi(w[x - y]) \left\{ \int_{\mathbb{R}} w \Phi(w[y - u]) \left\{ \sum_{j=0}^{r-1} \frac{f^{(j)}(x)}{j!} (u - x)^j \right\} du \right\} dy
\]

\[
+ \frac{1}{r!} f^{(r)}(\theta_{x,u}) (u - x)^r \right\} du \right\} dy
\]

\[
= \int_{\mathbb{R}} \chi(w[x - y]) \left\{ \int_{\mathbb{R}} w \Phi(w[y - u]) \left\{ \sum_{j=0}^{r-1} \frac{f^{(j)}(x)}{j!} (u - x)^j \right\} du \right\} dy
\]

\[
+ \frac{1}{r!} w \int_{\mathbb{R}} \chi(w[x - y]) \left\{ \int_{\mathbb{R}} w \Phi(w[y - u]) f^{(r)}(\theta_{x,u}) (u - x)^r \right\} du \right\} dy
\]

\[
= f(x) + \sum_{j=1}^{r-1} \frac{f^{(j)}(x)}{j!} w \int_{\mathbb{R}} \chi(w[x - y]) \left\{ \int_{\mathbb{R}} w \Phi(w[y - u]) (u - x)^j \right\} du \right\} dy
\]

\[
+ \frac{1}{r!} w \int_{\mathbb{R}} \chi(w[x - y]) \left\{ \int_{\mathbb{R}} w \Phi(w[y - u]) f^{(r)}(\theta_{x,u}) (u - x)^r \right\} du \right\} dy
\]

Now, by the change of variable \( z = w[y - u] \), the Fubini theorem, the change of variable \( t = w[x - y] \), and using (18), respectively, we obtain:

\[
(I_w^x [I_w f]) (x) = f(x) + \sum_{j=1}^{r-1} \frac{f^{(j)}(x)}{j!} w \int_{\mathbb{R}} \chi(t) t^j (-1)^j \frac{\hat{m}_{j-\ell}(\chi)}{\ell} \hat{m}_{\ell}(\Phi) \]

\[
+ \frac{w^{-r}}{r!} \int_{\mathbb{R}} \chi(t) t^{r-\ell} \left\{ \int_{\mathbb{R}} \Phi(z) z^\ell f^{(r)}(\theta_{x,z-(t/w) - (z/w)}) dz \right\} dt
\]

\[
= f(x) + \frac{w^{-r}}{r!} \int_{\mathbb{R}} \chi(t) t^{r-\ell} \left\{ \int_{\mathbb{R}} \Phi(z) z^\ell f^{(r)}(\theta_{x,z-(t/w) - (z/w)}) dz \right\} dt,
\]

\( w > 0 \). On the other hand, we have:

\[
\frac{f^{(r)}(x)}{r!} (-1)^r \sum_{\ell=0}^{r} \left( \frac{r}{\ell} \right) \hat{m}_{\ell}(\Phi) \hat{m}_{r-\ell}(\chi)
\]

\[
= \frac{(-1)^r}{r!} \sum_{\ell=0}^{r} \left( \frac{r}{\ell} \right) \int_{\mathbb{R}} \chi(t) t^{r-\ell} dt \left\{ \int_{\mathbb{R}} \Phi(z) z^\ell f^{(r)}(x) dz \right\},
\]

thus, by Fubini theorem we easily obtain:

\[
w^r \left\{ (I_w^x [I_w f]) (x) - f(x) \right\} - \frac{f^{(r)}(x)}{r!} (-1)^r \sum_{\ell=0}^{r} \left( \frac{r}{\ell} \right) \hat{m}_{\ell}(\Phi) \hat{m}_{r-\ell}(\chi)
\]
\[
\begin{aligned}
= \frac{(-1)^r}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \int_{\mathbb{R}} \chi(t) t^{r-\ell} \left\{ \int_{\mathbb{R}} \Phi(z) z^\ell \left[ f^{(r)}(\theta_{x_+,x-(t/w)}-(z/w)) - f^{(r)}(x) \right] dz \right\} dt,
\end{aligned}
\]

(19)

\( w > 0 \) and \( x \in \mathbb{R} \). Now, for \( 1 \leq p \leq +\infty \), using (19) and the generalized Minkowski type inequality, we obtain:

\[
I_w := \| w^{\ell} \{ T_w^{\Phi} f - f \} - \frac{f^{(r)}}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \overline{m}_\ell(\Phi) \overline{m}_{r-\ell}(\chi) \|_p
\leq \frac{1}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \int_{\mathbb{R}} |\chi(t)| |t|^{r-\ell} \left\{ \int_{|t| \leq w\delta/2} |\Phi(z)| |z|^{\ell} \| f^{(r)}(\theta_{(\cdot),-(t/w)}-(z/w)) - f^{(r)}(\cdot) \|_p dz dt + \int_{|z| > w\delta/2} |\Phi(z)| |z|^{\ell} \| f^{(r)}(\theta_{(\cdot),-(t/w)}-(z/w)) - f^{(r)}(\cdot) \|_p dz dt \right\}
\leq \frac{1}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \int_{|t| > w\delta/2} |\chi(t)| |t|^{r-\ell} \int_{\mathbb{R}} |\Phi(z)| |z|^{\ell} \| f^{(r)}(\theta_{(\cdot),-(t/w)}-(z/w)) - f^{(r)}(\cdot) \|_p dz dt
\leq \begin{align*}
&=: I_{w,1} + I_{w,2} + I_{w,3}.
\end{align*}
\]

We begin estimating \( I_{w,1} \). First of all, we can note that, in the domain of integration of \( I_{w,1} \) it results:

\[
|\theta_{x_+,x-(t/w)}-(z/w) - x| \leq |t/w| + |z/w| \leq \delta/2 + \delta/2 = \delta,
\]

then:

\[
I_{w,1} \leq \frac{\varepsilon}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \overline{M}_\ell(\Phi) \overline{M}_{r-\ell}(\chi) < +\infty,
\]

in view of (\( \chi^4 \)). While for \( I_{w,2} \) we have:

\[
I_{w,2} \leq \frac{2 \| f^{(r)} \|_p}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \overline{M}_{r-\ell}(\chi) \int_{|z| > w\delta/2} |\Phi(z)| |z|^{\ell} dz < \varepsilon,
\]

\( \varepsilon \)

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for $w > 0$ sufficiently large, since by (χ4) the integrals $\int_{|z|>w\delta/2} |\Phi(z)| \, |z|^\ell \, dz$ goes to zero, as $w \to +\infty$. Similarly to above,

$$I_{w,3} \leq \frac{2 \|f^{(r)}\|_p}{r!} \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) \tilde{M}_\ell(\Phi) \int_{|t|>w\delta/2} |\chi(t)| \, |t|^{r-\ell} \, dt < \varepsilon,$$

for $w > 0$ sufficiently large. This completes the proof. □

For references about double-convolution type operators, see, e.g., [42, 43]. From Theorem 3.5 the following useful corollary which involves the above double-convolution with the sampling Kantorovich operators can be proved.

**Corollary 3.6** Let $\chi$ be a kernel, and let $\Phi \in L^1(\mathbb{R})$ both assumed as in Theorem 3.5. Then, for any $f \in C^n_c(\mathbb{R})$, $r \in \mathbb{N}^+$, and $g \in L^p(\mathbb{R})$, $1 \leq p \leq +\infty$, we have:

$$\lim_{w \to +\infty} \|w^r \left[I_w^\chi (I_w^\Phi f) - f \right] K_w^\chi g - \frac{f^{(r)}(\cdot)}{r!} g \left[ (-1)^r \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \right] \|_p = 0.$$

Further, we also have for $g \in L^p(\mathbb{R})$:

$$\lim_{w \to +\infty} \|w^r \left[I_w^\chi (I_w^\Phi f) - f \right] K_w^\chi g - \frac{f^{(r)}(\cdot)}{r!} g \left[ (-1)^r \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \right] \|_1 = 0.$$

**Proof** Let $1 \leq p \leq +\infty$ be fixed. Using the Minkowski and the Hölder inequalities:

$$\begin{align*}
\|w^r \left[I_w^\chi [I_w^\Phi f] - f \right] K_w^\chi g &- \frac{f^{(r)}(\cdot)}{r!} g \left[ (-1)^r \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \right] \|_p^p \\
&\leq \|w^r \left[I_w^\chi [I_w^\Phi f] - f \right] \left[ K_w^\chi g - g \right] \|_p^p \\
&\quad + \left\| \left[ I_w^\chi [I_w^\Phi f] - f \right] - \frac{f^{(r)}(\cdot)}{r!} \left[ (-1)^r \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \right] g \right\|_p^p \\
&\leq \|w^r \left[I_w^\chi [I_w^\Phi f] - f \right] \|_{L^p} \|K_w^\chi g - g\|_p^p \\
&\quad + \|w^r \left[I_w^\chi [I_w^\Phi f] - f \right] - \frac{f^{(r)}(\cdot)}{r!} \left[ (-1)^r \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \right] \|_{L^\infty} \|g\|_p^p \\
&\leq C_p \|K_w^\chi g - g\|_p^p \\
&\quad + \|w \left[I_w^\chi [I_w^\Phi f] - f \right] - \frac{f^{(r)}(\cdot)}{r!} \left[ (-1)^r \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \right] \|_{L^\infty} \|g\|_p^p,
\end{align*}$$

where $C > 0$ is a suitable positive constant depending only on $\chi$, $\Phi$ and $f^{(r)}$, that can be determined since $(w^r \left[I_w^\chi [I_w^\Phi f] - f \right])_{w>0}$ converges uniformly to $\frac{f^{(r)}(\cdot)}{r!} \left[ (-1)^r \sum_{\ell=0}^r \left( \frac{r}{\ell} \right) \tilde{m}_\ell(\Phi) \tilde{m}_{r-\ell}(\chi) \right]$, as $w \to +\infty$. Now, recalling the convergence result of (8) and using Theorem 3.5 in the case $p = +\infty$ the proof of the first part.
of the corollary follows immediately. For what concerns the second part of the thesis, using H"{o}lder inequality, we have:

\[
\| w^r \{ I_w^x \{ I_w^\Phi f \} - f \} \| K_w^x g - \frac{f^{(r)}}{r!} \left[ (-1)^r \sum_{\ell=0}^{r} \left( \begin{array}{c} r \\ \ell \end{array} \right) \tilde{m}_{\ell}(\Phi) \tilde{m}_{r-\ell}(\chi) \right] \|_1 \\
\leq \| w^r \{ I_w^x \{ I_w^\Phi f \} - f \} \|_q \| K_w^x g - g \|_p \\
+ \| w^r \{ I_w^x \{ I_w^\Phi f \} - f \} - \frac{f^{(r)}}{r!} \left[ (-1)^r \sum_{\ell=0}^{r} \left( \begin{array}{c} r \\ \ell \end{array} \right) \tilde{m}_{\ell}(\Phi) \tilde{m}_{r-\ell}(\chi) \right] \|_q \| g \|_p,
\]

where \( 1/p + 1/q = 1 \). Hence the proof follows as above by Theorem 3.5 (applied with \( 1 \leq q \leq +\infty \)) and (8).

Note that, the second part of Corollary 3.6 (i.e., the case with the \( L^1 \)-norm) provides a theorem which allows to pass the limit under the integral, for the sequence \( (w^r \{ I_w^x \{ I_w^\Phi f \} - f \} )_{w>0} \) with \( f \in C_c^r(\mathbb{R}) \) and \( g \in L^p(\mathbb{R}) \), \( 1 \leq p \leq +\infty \). This will be useful in the proofs of the last two sections of the present paper.

4 On Regularization Properties of Sampling Kantorovich Operators

We begin this section with the following theorems about the regularity of the sampling Kantorovich operators.

**Theorem 4.1** Let \( \chi \) be a continuous kernel, and \( f \in L^p(\mathbb{R}) \), \( 1 \leq p \leq +\infty \), be fixed. Then \( K_w^x f \in C^0(\mathbb{R}) \), for every fixed \( w > 0 \). In particular, if \( \chi \in C(\mathbb{R}) \), hence \( K_w^x f \in C(\mathbb{R}) \).

**Proof** Let \( 1 \leq p < +\infty \) and \( w > 0 \) be fixed. For every \( m \in \mathbb{N}^+ \) we consider the sequence:

\[
h_w^m(x) := \sum_{|k| \leq m} \left[ w \int_{k/w}^{(k+1)/w} f(u) \, du \right] \chi(wx-k), \quad x \in \mathbb{R}.
\]

Then, we can write the following inequality:

\[
|(K_w^x f)(x) - h_w^m(x)| \leq \sum_{|k| > m} \left[ w \int_{k/w}^{(k+1)/w} |f(u)| \, du \right] |\chi(wx-k)|, \quad x \in \mathbb{R}.
\]

Let now \( x \in \mathbb{R} \) be fixed. Using Jensen inequality twice, we obtain:

\[
|(K_w^x f)(x) - h_w^m(x)| \\
\leq \left\{ \sum_{|k| > m} \left[ w \int_{k/w}^{(k+1)/w} |f(u)| \, du \right] |\chi(wx-k)| \right\}^{1/p}
\]

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\[
\begin{align*}
\leq M_0(\chi) \frac{p^{-1}}{p} \left\{ \sum_{|k| > m} \left[ \left( \sum_{k=1}^{(k+1)/w} |u|^{p} \right)^{p} \chi(wx-k) \right] \right\}^{1/p} \\
\leq M_0(\chi) \frac{p^{-1}}{p} \|\chi\|_{\infty}^{1/p} \left\{ \sum_{|k| > m} \left[ \left( \sum_{k=1}^{(k+1)/w} |u|^p \right)^{p} \right] \right\}^{1/p} \\
\leq M_0(\chi) \frac{p^{-1}}{p} \|\chi\|_{\infty}^{1/p} \left\{ \sum_{|u| \geq m/w} |f(u)|^p du \right\}^{1/p}.
\end{align*}
\]

Now, passing to the limit for \( m \to +\infty \) in the above inequality we provide the uniform convergence of the sequence \((h_m^w)_{m \in \mathbb{N}^+}\) to \(K_w^X f\) on the whole \(\mathbb{R}\). Now, observing that any \(h_m^w\) is continuous on \(\mathbb{R}\) as a finite sum of continuous functions, we finally get that also \(K_w^X f\) is continuous on \(\mathbb{R}\). Further, proceeding as above, one can also prove that:

\[
|(K_w^X f)(x)| \leq M_0(\chi) \frac{p^{-1}}{p} \|\chi\|_{\infty}^{1/p} w^{1/p} \|f\|_p < +\infty, \quad x \in \mathbb{R},
\]

i.e., the operator is bounded on \(\mathbb{R}\) for every fixed \(w > 0\).

For the case \(p = +\infty\), i.e., for \(f \in C(\mathbb{R})\), the proof follows similarly recalling (9) and observing that:

\[
|(K_w^X f)(x) - h_m^w(x)| \leq \|f\|_{\infty} \sup_{x \in \mathbb{R}} \sum_{|k| > m} |\chi(wx-k)|,
\]

and that \(\sum_{|k| > m} |\chi(wx-k)|\) is the remainder of a uniformly convergent series on \(\mathbb{R}\) (see Remark 2.1).

For what concerns the second part of the thesis, if \(\chi \in C(\mathbb{R})\), for any fixed \(w > 0\), \(\varepsilon > 0\) and \(x, y \in \mathbb{R}\), we can write:

\[
|(K_w^X f)(x) - (K_w^X f)(y)| \leq |(K_w^X f)(x) - h_m^w(x)| + |h_m^w(x) - h_m^w(y)| + |h_m^w(y) - (K_w^X f)(y)| \leq \frac{2}{3} \varepsilon + |h_m^w(x) - h_m^w(y)|,
\]

for a fixed sufficiently large \(m \in \mathbb{N}^+\). Moreover, \(h_m^w\) is uniformly continuous on \(\mathbb{R}\) since it is defined as a finite sum of uniformly continuous functions, thus, if we choose the parameter \(\delta > 0\) of the uniform continuity of \(h_m^w\) corresponding to \(\varepsilon/3\), and \(|x - y| < \delta\), we finally get: \(|(K_w^X f)(x) - (K_w^X f)(y)| \leq \varepsilon\), i.e., \(K_w^X f \in C(\mathbb{R})\). \(\Box\)

The above theorem can be generalized as follows.

**Theorem 4.2** Let \(\chi \in W^{r,1}(\mathbb{R})\), \(r \in \mathbb{N}^+\), be a given kernel, such that \(M_0(\chi^{(i)}) < +\infty\), for every \(i = 1, \ldots, r\). Then, for every \(f \in L^p(\mathbb{R})\), \(1 \leq p \leq +\infty\), it turns out that \(K_w^X f \in W^{r,p}(\mathbb{R})\), \(w > 0\), and the distributional derivatives of \(K_w^X f\) can be expressed as follows:

\[
(K_w^X f)^{(i)}(x) = w^i \sum_{k \in \mathbb{Z}} \left[ \sum_{k=1}^{(k+1)/w} f(u) du \right] \chi^{(i)}(wx-k), \quad x \in \mathbb{R},
\]

\(\Box\)

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where \( \chi^{(i)}, i = 1, \ldots, r, \) are the distributional derivatives of \( \chi. \)

**Proof** Proceeding as in the proof of Theorem 4.1, it turns out that, \( K_w^X f \) is bounded a.e. on \( \mathbb{R} \), then the operator \( K_w^X f \in \mathcal{D}' \), i.e., it defines a regular distribution\(^1\). Let now \( 1 \leq p < +\infty \) be fixed. We can immediately observe that the series \( K_w^X f \) is absolutely convergent on \( \mathbb{R} \) by the estimate given in (20). Then, for every \( \varphi \in \mathcal{D} \), passing the integral under the series, and using the Fubini theorem, we can compute the distributional derivatives of \( K_w^X f \) as follows:

\[
(K_w^X f)^{(i)}[\varphi] = (-1)^i \int_{\mathbb{R}} (K_w^X f)(x) \varphi^{(i)}(x) \, dx
\]

\[
= \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} f(u) \, du \right] [\chi(wx - k)]^{(i)}[\varphi]
\]

\[
= \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} f(u) \, du \right] w^{i} \chi^{(i)}(w \cdot -k)[\varphi],
\]

\( i = 1, \ldots, r. \) Now, since \( \chi^{(i)} \in L^1(\mathbb{R}) \), it turns out that \( (K_w^X f)^{(i)} \) is an ordinary function, and moreover, using Jensen inequality twice, Fubini-Tonelli theorem and the change of variable \( y = wx - k \):

\[
\int_{\mathbb{R}} |(K_w^X f)^{(i)}(x)|^p \, dx \leq M_0(\chi^{(i)})^{p-1}w^i \sum_{k \in \mathbb{Z}} \left[ w \int_{k/w}^{(k+1)/w} |f(u)|^p \, du \right]
\]

\[
\int_{\mathbb{R}} |\chi^{(i)}(wx - k)| \, dx = M_0(\chi^{(i)})^{p-1}w^i \sum_{k \in \mathbb{Z}} \left[ \int_{k/w}^{(k+1)/w} |f(u)|^p \, du \right] \|\chi^{(i)}\|_1
\]

\[
= M_0(\chi^{(i)})^{p-1}w^i \|f\|_p^p \|\chi^{(i)}\|_1 < +\infty,
\]

\( i = 1, \ldots, r. \) In the case \( p = +\infty \), the proof follows similarly. \( \square \)

For \( p = +\infty \), Theorem 4.2 provide a generalization of the result established in Proposition 4 of [17] in the case \( r = 1. \) Clearly, in this case all the derivatives involved must be considered in the usual sense. Further, we can also observe that, for \( p = +\infty \), Theorem 4.2 holds also if we consider the classical definition of \( L^\infty(\mathbb{R}) \) and \( W^{r,\infty}(\mathbb{R}) \) (that we recalled in Remark 3.3).

**Theorem 4.3** Let \( \chi \in B^1_{\pi w}(\mathbb{R}), w > 0, \) be a given kernel. Then, for every \( f \in L^p(\mathbb{R}) \) it turns out that \( K_w^X f \in B^p_{\pi w}(\mathbb{R}), 1 \leq p \leq +\infty. \)

**Proof** As in Theorem 4.2, \( K_w^X f \in C^0(\mathbb{R}), \) then \( K_w^X f \in \mathcal{S}'. \) Let now \( 1 \leq p < +\infty \) be fixed. First we can note that the series \( K_w^X f \) is absolutely convergent on \( \mathbb{R} \) by the

---

\(^1\) In fact, \( K_w^X f \) is a regular tempered distribution.
estimate given in (20). Then, for every \( \varphi \in \mathcal{S} \), passing the integral under the series, and by the Fubini theorem, we can write what follows:

\[
(\widehat{K_w f})(\varphi) = \int_{\mathbb{R}} (K_w f)(x) \hat{\varphi}(x) \, dx \\
= \sum_{k \in \mathbb{Z}} \left[ \int_{k/w}^{(k+1)/w} f(u) \, du \right] \int_{\mathbb{R}} \chi(wx - k) \hat{\varphi}(x) \, dx \\
= \sum_{k \in \mathbb{Z}} \left[ \int_{k/w}^{(k+1)/w} f(u) \, du \right] \int_{\mathbb{R}} \chi(wx - k) \left( \int_{\mathbb{R}} \varphi(y) e^{-ixy} \, dy \right) \, dx \\
= \sum_{k \in \mathbb{Z}} \left[ \int_{k/w}^{(k+1)/w} f(u) \, du \right] \int_{\mathbb{R}} \varphi(y) \left( \int_{\mathbb{R}} \chi(wx - k) e^{-ixy} \, dx \right) \, dy \\
= \sum_{k \in \mathbb{Z}} \left[ \int_{k/w}^{(k+1)/w} f(u) \, du \right] \int_{\mathbb{R}} \varphi(y) e^{-y(k/w)} \hat{\chi}(y/w) \, dy.
\]

Recalling that \( \text{supp} \hat{\chi} \subset [-\pi w, +\pi w] \) we finally obtain:

\[
(\widehat{K_w f})(\varphi) = \sum_{k \in \mathbb{Z}} \left[ \int_{k/w}^{(k+1)/w} f(u) \, du \right] \int_{-\pi}^{\pi} \varphi(y) e^{-y(k/w)} \hat{\chi}(y/w) \, dy. \quad (22)
\]

Let now \( A := (-\infty, -\pi) \cup (+\pi, +\infty) \) be a given open set of \( \mathbb{R} \). For every \( \varphi \in \mathcal{S} \) with \( \text{supp} \varphi \subset A \), by (22) it turns out that:

\[
(\widehat{K_w f})(\varphi) = 0.
\]

This shows that \( A \) is the null set of \( (\widehat{K_w f}) \) and then \( \text{supp} (\widehat{K_w f}) \subset [-\pi, +\pi] \). This completes the proof. In the case \( p = +\infty \), the proof can be done similarly. \( \Box \)

At the end of this section, in view of the result proved in Theorem 4.3, the following closed form for the Fourier transform of the sampling Kantorovich operators can be established.

**Theorem 4.4** Let \( \chi \in B_{1,\pi w}(\mathbb{R}) \), \( w > 0 \) be a given kernel, and \( f \in B_{p,\pi w}(\mathbb{R}) \), \( 1 \leq p \leq +\infty \). Then the distributional Fourier transform of the sampling Kantorovich operator \( K_w^\chi f \) is:

\[
(\widehat{K_w^\chi f}) = e^{ix/2w} \text{sinc} \left( \frac{x}{2\pi w} \right) \hat{\chi} \left( \frac{x}{w} \right) \hat{f}.
\]

where the above distribution is given by the multiplication of the smooth function \( e^{ix/2w} \text{sinc} \left( \frac{x}{2\pi w} \right) \hat{\chi} \left( \frac{x}{w} \right) \) and the distribution \( \hat{f} \).
Proof First of all, we can note that, denoting by \( \Theta : \mathbb{R} \rightarrow \mathbb{R} \) the characteristic function of the interval \([-1, 0]\), we can write what follows:

\[
W \int_{k/w}^{(k+1)/w} f(u) \, du = (I^\Theta_W f)(k/w), \quad k \in \mathbb{Z}, \quad w > 0,
\]

hence the following relation can be deduced:

\[
(K^X_W f)(x) = (G^X_W[I^\Theta_W f])(x), \quad x \in \mathbb{R}.
\]  \hspace{1cm} (23)

Now, we claim that \( I^\Theta_W f \in B^p_{\pi w}(\mathbb{R}) \), when \( f \in B^p_{\pi w}(\mathbb{R}) \). Indeed, for every \( \varphi \in \mathcal{S} \) and by the Fubini theorem:

\[
(I^\Theta_W f)[\varphi] = \int_{\mathbb{R}} (I^\Theta_W f)(x) \varphi(x) \, dx = \int_{\mathbb{R}} f(y) \left[ \int_{\mathbb{R}} \Theta(w[x-y]) \varphi(x) \, dx \right] \, dy = \int_{\mathbb{R}} f(y) \left( I^\varphi_W \varphi \right)(y) \, dy,
\]

where \( \Psi : \mathbb{R} \rightarrow \mathbb{R} \) denotes the characteristic function of the interval \([0, 1]\). Hence:

\[
(I^\Theta_W f)[\varphi] = \int_{\mathbb{R}} f(y) \left( \int_{0}^{1/w} \varphi(y-u) \, du \right) \, dy = \int_{0}^{1/w} du \int_{\mathbb{R}} f(y) \varphi(y-u) \, dy = \int_{0}^{1/w} du \int_{\mathbb{R}} f(y) \left( \int_{\mathbb{R}} \eta_u(z) e^{-izy} \, dz \right) \, dy = \int_{0}^{1/w} du \int_{\mathbb{R}} f(y) \eta_u(y) \, dy = \int_{0}^{1/w} \tilde{f}[\eta_u] \, du.
\]

Now, since \( f \in B^p_{\pi w}(\mathbb{R}) \) we know that \( A = (-\infty, -\pi w) \cup (\pi w, +\infty) \) is the null set of \( \tilde{f} \), and then for every \( \varphi \in \mathcal{S} \) such that \( \text{supp} \varphi \subset A \) it turns out that \( \tilde{f}[\varphi] = 0 \). Clearly, in the latter case also \( \text{supp} \eta_u \subset A \), then we must have \( \tilde{f}[\eta_u] = 0 \) which also implies that \( (I^\Theta_W f)[\varphi] = 0 \). Now, recalling the inclusions (2) it is clear that \( \chi \in B^q_{\pi w}(\mathbb{R}) \), with \( q \) such that \( 1/p + 1/q = 1 \), and using Lemma 2.3 we can finally obtain:

\[
(K^X_W f)(x) = (G^X_W[I^\Theta_W f])(x) = (I^X_W[I^\Theta_W f])(x), \quad x \in \mathbb{R}.
\]  \hspace{1cm} (24)

Exploiting (24), for every \( \varphi \in \mathcal{S} \) we have:

\[
(K^X_W f)[\varphi] = \int_{\mathbb{R}} (I^X_W[I^\Theta_W f])(x) \varphi(x) \, dx = \int_{\mathbb{R}} w \chi(wu) \left\{ \int_{\mathbb{R}} w \Theta(wt) \left[ \int_{\mathbb{R}} f(x-u-t) \varphi(x) \, dx \right] dt \right\} \, du,
\]

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\[
\begin{align*}
&= \int_{\mathbb{R}} w \chi(\omega u) \left\{ \int_{\mathbb{R}} w \Theta(wt) e^{-i(u+t)x} \hat{f}[\varphi] \, dt \right\} \, du \\
&= \int_{\mathbb{R}} w \chi(\omega u) e^{-ixu} \Theta\left(\frac{x}{w}\right) \hat{f}[\varphi] \, du = \hat{\chi}\left(\frac{x}{w}\right) \Theta\left(\frac{x}{w}\right) \hat{f}[\varphi],
\end{align*}
\]

where \(\hat{\chi}\) and \(\hat{\Theta}\) denote the usual \(L^1\)-Fourier transform. Now, recalling that:
\[
\hat{\Theta}\left(\frac{x}{w}\right) = e^{ix/2w} \text{sinc}\left(\frac{x}{2\pi w}\right), \quad x \in \mathbb{R},
\]
we get the thesis. \(\square\)

Note that, Theorem 4.4 extend to all \(1 \leq p \leq +\infty\) the result originally proved only for \(p = 1\), in Lemma 3.1 of [25].

5 Direct and Inverse Results of Approximation in \(L^p\)-Setting

Now, we recall the definition of Lipschitz classes:

\[
\text{Lip}(\alpha, L^p) := \left\{ f \in L^p(\mathbb{R}) : \omega_1(f, \delta)_p = O(\delta^{\alpha}), \text{ as } \delta \to 0^+ \right\}, \quad 0 < \alpha \leq 1,
\]

where \(\omega_1(f, \delta)_p\) denotes the usual first-order modulus of smoothness of \(f \in L^p(\mathbb{R})\), \(1 \leq p \leq +\infty\). In general, the modulus of smoothness of order \(r \in \mathbb{N}^+\) of any given \(f \in L^p(\mathbb{R})\), \(1 \leq p \leq +\infty\), is defined by ( [39]):

\[
\omega_r(f, \delta)_p := \sup \left\{ \| \Delta^r_t(f, \cdot) \|_p, \ |t| \leq \delta \right\}, \quad \delta > 0,
\]

with:

\[
\Delta^r_t(f, x) := \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} f(x+jt), \quad x \in \mathbb{R}.
\]

Moreover, the generalized Lipschitz classes are given by:

\[
\text{Lip}^*(\alpha, L^p) := \left\{ f \in L^p(\mathbb{R}) : \omega_r(f, \delta)_p = O(\delta^{\alpha}), \text{ as } \delta \to 0^+ \right\}, \quad \alpha > 0,
\]

where \(r\) is the smallest integer such that \(r > \alpha\), i.e., \(r = \lfloor \alpha \rfloor + 1\) (where \(\lfloor \cdot \rfloor\) denotes the integer part of a given number). It is well-known that \(\text{Lip}^*(\alpha, L^p) = \text{Lip}(\alpha, L^p)\), for \(1 \leq p \leq +\infty\) when \(\alpha > 0\) is not an integer. Note that, the definition of \(\text{Lip}(\alpha, L^p)\) can be extended for \(\alpha > 1\) setting \(\text{Lip}(\alpha, L^p) := W^{\alpha_0}(\text{Lip}(\beta, L^p))\), where \(\alpha = \alpha_0 + \beta\), with \(\alpha_0 \in \mathbb{N}, 0 < \beta \leq 1\), and:

\[
W^{\alpha_0}(\text{Lip}(\beta, L^p)) := \left\{ f \in L^p(\mathbb{R}) : f, f', \ldots, f^{(\alpha_0-1)} \in AC_{loc}(\mathbb{R}), f^{(\alpha_0)} \in \text{Lip}(\beta, L^p) \right\}.
\]
It is known that \( \text{Lip} (r, L^p) = W^{r,p}(\mathbb{R}) \subset \text{Lip}^*(r, L^p) \), if \( 1 < p \leq +\infty \), and \( \text{Lip} (r, L^1) = W^{r,-1}(BV) \cap L^1(\mathbb{R}) \subset \text{Lip}^* (r, L^1) \), \( r \in \mathbb{N}^+ \), and for any function \( f \) in these spaces, there holds:

\[
\omega_r(f, \delta)_p \leq M \delta^r, \quad \text{as } \delta \to 0^+,
\]

for \( 1 < p \leq +\infty \) and \( p = 1 \), respectively, where \( M > 0 \) is a suitable positive constant.

Now, in order to prove quantitative estimates in \( L^p \)-setting by means of the modulus of smoothness of order \( r \), we recall the definition of the well-known K-functionals:

\[
K(f, t; L^p, W^{r,p}) := \inf_{g \in W^{r,p}(\mathbb{R})} \left\{ \| f - g \|_p + t \| g^{(r)} \|_p \right\}, \quad t > 0,
\]

\( f \in L^p(\mathbb{R}) \), \( 1 \leq p \leq +\infty \). From a classical result of Johnen [31], we know that:

\[
C_1 \omega_r(f, t)_p \leq K(f, t^r; L^p, W^{r,p}) \leq C_2 \omega_r(f, t)_p, \quad t > 0,
\]

for \( f \in L^p(\mathbb{R}) \), \( 1 \leq p \leq +\infty \), for suitable positive constants \( C_1 \) and \( C_2 \). Now we are able to prove the following.

**Theorem 5.1** Let \( \chi \) be a given kernel. In what follows, if the parameter \( r \in \mathbb{N} \) is greater than \( 1 \), we also assume that \((\chi^4)\) is satisfied for \( \beta > r + 1 \), and conditions (11), (12) hold. Then:

(i) for every \( f \in L^\infty(\mathbb{R}) \), there exists \( C > 0 \) such that:

\[
\| K^\chi_w f - f \|_\infty \leq C \omega_r(f, 1/w)_\infty, \quad w > 0;
\]

(ii) if in addition \( \chi \) satisfies \((\chi^4)\) with \( \beta > rp+1, 1 \leq p < +\infty \), for any \( f \in L^p(\mathbb{R}) \), and a suitable \( C > 0 \):

\[
\| K^\chi_w f - f \|_p \leq C \omega_r(f, 1/w)_p, \quad w > 0;
\]

(iii) if \( f \in \text{Lip}^*(\alpha, L^\infty) \), with \( 0 < \alpha < r \), it turns out that:

\[
\| K^\chi_w f - f \|_\infty = O(w^{-\alpha}), \quad \text{as } w \to +\infty;
\]

while for \( f \in \text{Lip}(r, L^\infty) = W^{r,\infty}(\mathbb{R}) \), we have:

\[
\| K^\chi_w f - f \|_\infty = O(w^{-r}), \quad \text{as } w \to +\infty;
\]

(iv) under the assumptions required in item (iii) for \( 1 \leq p < +\infty \), if \( f \in \text{Lip}^*(\alpha, L^p) \), where \( 0 < \alpha < r \), it turns out that:

\[
\| K^\chi_w f - f \|_p = O(w^{-\alpha}), \quad \text{as } w \to +\infty;
\]

\[\text{Note that, } W^{r,1}(\mathbb{R}) \subset W^{r,-1}(BV) \cap L^1(\mathbb{R}), r \in \mathbb{N}^+, r > 1.\]
while if \( f \in \text{Lip}(r, L^p) = W^{r,p}(\mathbb{R}) \), we have:

\[
\|K_w^x f - f\|_p = O(w^{-r}), \quad \text{as } w \to +\infty.
\]

In particular, for \( p = 1 \) and \( f \in W^{r-1}(BV) \cap L^1(\mathbb{R}) \), we also have:

\[
\|K_w^x f - f\|_1 = O(w^{-r}), \quad \text{as } w \to +\infty.
\]

**Proof** The proof of (i) with \( r = 1 \) can be found in [5] for the case \( p = +\infty \), while for the case \( 1 \leq p < +\infty \) the proof is analogous of that of item (ii) given below.

Now, we suppose that \( \chi \) satisfies (\( \chi \) \ref{4}) for \( \beta > r + 1 \), with \( r \in \mathbb{N}, r > 1, (11) \) and (12). Then, let \( f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty \) be fixed and \( g \in W^{r,p}(\mathbb{R}) \). We can write what follows:

\[
\|K_w^x f - f\|_p \\
\leq \|K_w^x f - K_w^x g\|_p + \|K_w^x g - g\|_p + \|g - f\|_p \\
\leq (M + 1) \|f - g\|_p + \|K_w^x g - g\|_p,
\]

\( w > 0 \), where \( M := M_0(\chi)^{p-1/p}\|\chi\|_1^{1/p} \) if \( p \neq +\infty \), or \( M := M_0(\chi) \) if \( p = +\infty \) (see (9)). Now, since \( g \in W^{r,p}(\mathbb{R}) \), the Taylor expansion (13) with integral remainder \( R_r(u; x) = \int_x^u g^{(r)}(t) (u - t)^{r-1} \, dt \) holds. Hence, as in the proof of Theorem 3.2, using (11) and (12), we obtain:

\[
(K_w^x g)(x) = g(x) + \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left[ \int_x^u \frac{g^{(r)}(t)}{(r-1)!} (u - t)^{r-1} \, dt \right] du \right\} \chi(wx - k),
\]

\( x \in \mathbb{R} \).

(i) Now, in the case \( p = +\infty \), we immediately have:

\[
|(K_w^x g)(x) - g(x)| \\
\leq \frac{1}{(r-1)!} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left[ \int_x^u |g^{(r)}(t)| |u - t|^{r-1} \, dt \right] du \right\} |\chi(wx - k)| \\
\leq \|g^{(r)}\|_\infty \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} |u - t|^{r-1} \, dt \right\} |\chi(wx - k)| \\
\leq \|g^{(r)}\|_\infty \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} |u - x|^{r-1} \, du \right\} |\chi(wx - k)| \\
\leq \|g^{(r)}\|_\infty \sum_{i=0}^{r} \binom{r}{i} \sum_{k \in \mathbb{Z}} \left\{ \frac{w}{k} |x|^{r-i} w \int_{k/w}^{(k+1)/w} |u - \frac{k}{w}|^{r-i} \, du \right\} |\chi(wx - k)| \\
\leq w^{-r} \|g^{(r)}\|_\infty \sum_{i=0}^{r} \binom{r}{i} \frac{M_{r-i}(\chi)}{i+1}. \]
Hence, using \((\chi^4)\) we obtain:

\[
\| K_w^X g - g \|_{\infty} \leq w^{-r} \| g^{(r)} \|_{\infty} \sum_{i=0}^{r} \binom{r}{i} \frac{M_{r-i}(\chi)}{i+1} =: w^{-r} \| g^{(r)} \|_{\infty} H < +\infty,
\]

(27)

from which, together with (26) we finally get:

\[
\| K_w^X f - f \|_{\infty} \leq (M + 1) K \left( f, w^{-1} \frac{H}{M + 1}, L^{\infty}, W^{r, \infty} \right) \\
\leq C_2 (M + 1) \omega_r \left( f, w^{-1} H^{1/r} (M + 1)^{-1/r} \right)_{\infty} \\
\leq C_2 (M + 1) [H^{1/r} (M + 1)^{-1/r} + 1] \omega_r \left( f, w^{-1} \right)_{\infty} \\
=: C \omega_r \left( f, w^{-1} \right)_{\infty}.
\]

(ii) Let now \(1 \leq p < +\infty\) be fixed. Proceeding as above, and using Jansen inequality three times, we have:

\[
\| K_w^X g - g \|_{p}^p \leq \frac{1}{(r - 1)!} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} u \int_{x}^{u} |g^{(r)}(t)| |u - t|^{-1} dt | du \right\} |\chi(wx - k)|^p dx \\
\leq \frac{M_0(\chi)^{p-1}}{(r - 1)!} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} u \int_{x}^{u} |g^{(r)}(t)| |u - t|^{-1} dt |^p du \right\} |\chi(wx - k)| dx \\
\leq \frac{M_0(\chi)^{p-1}}{(r - 1)!} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} u - x |^{p(r-1)} \int_{x}^{u} |g^{(r)}(t)| dt |^p du \right\} |\chi(wx - k)| dx \\
\leq \frac{M_0(\chi)^{p-1}}{(r - 1)!} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} u - x |^{pr-1} \int_{x}^{u} |g^{(r)}(t)|^p dt | du \right\} |\chi(wx - k)| dx.
\]

Now, we can also write:

\[
\| K_w^X g - g \|_{p}^p \leq \frac{M_0(\chi)^{p-1}}{(r - 1)!} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} u - x |^{pr-1} \int_{x}^{u} |g^{(r)}(t)|^p dt - \int_{u+x-k/w}^{u} |g^{(r)}(t)|^p dt | du \right\} |\chi(wx - k)| dx.
\]
Further, we can estimate

\[ \chi(wx - k)dx \]

\[ + \frac{M_0(\chi)^{p-1}}{(r - 1)!^p} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} |u - x|^{pr - 1} \left| \int_x^{u+k/w} |g^{(r)}(t)|^p dt \right| du \right\} |\chi(wx - k)|dx \]

For what concerns \( T_1 \), using the change of variable \( y = t - x \) and Fubini-Tonelli theorem, we have:

\[ T_1 = \frac{M_0(\chi)^{p-1}}{(r - 1)!^p} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} |u - x|^{pr - 1} \left[ \int_0^{u-k/w} |g^{(r)}(y + x)|^p dy \right] du \right\} |\chi(wx - k)|dx \]

\[ \leq w^{-pr + 1} \frac{M_0(\chi)^{p-1}}{(r - 1)!^p} \int_0^{1/w} \left[ \int_{\mathbb{R}} \left| g^{(r)}(y + x) \right|^p \sum_{i=0}^{pr-1} \binom{pr-1}{i} \frac{M_{r-1-i}(\chi)}{i + 1} \right] dy \]

\[ \leq w^{-pr} \frac{M_0(\chi)^{p-1}}{(r - 1)!^p} \sum_{i=0}^{pr-1} \binom{pr-1}{i} \frac{M_{r-1-i}(\chi)}{i + 1} \|g^{(r)}\|_p^p =: w^{-pr} \|g^{(r)}\|_p^p N_1 < +\infty. \]

Further, we can estimate \( T_2 \). By the change of variable \( y = t - u \) and Fubini-Tonelli theorem, we have:

\[ T_2 = \frac{M_0(\chi)^{p-1}}{(r - 1)!^p} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} |u - x|^{pr - 1} \left[ \int_0^0 \left| g^{(r)}(y + u) \right|^p dy \right] du \right\} |\chi(wx - k)|dx \]

\[ \leq 2^{pr - 1} \frac{M_0(\chi)^{p-1}}{(r - 1)!^p} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left[ \left( \frac{1}{w} \right)^{pr - 1} \right] \left[ \left( \frac{1}{w} \right)^{pr - 1} \right] \right\} |\chi(wx - k)|dx \]
\[
\left| \int_{x-k/w}^{0} |g^{(r)}(y+u)|^p dy \right| du \right| \chi(wx-k)dx.
\]

Now, setting \( z = wx - k \), applying Fubini-Tonelli theorem again, and using (\( \chi 4 \)):

\[
T_2 \leq 2^{pr-1} \frac{M_0(\chi)^{-1}}{[(r-1)!]^p} \sum_{k \in \mathbb{Z}} \left\{ \int_{k/w}^{(k+1)/w} \left[ \left( \frac{1}{w} \right)^{pr-1} + \left| \frac{z}{w} \right|^{pr-1} \right] \times \right.
\]

\[
\left. \left| \int_{z/w}^{0} |g^{(r)}(y+u)|^p dy \right| du \right| \chi(z)dz \\]

\[
\leq 2^{pr-1} \frac{M_0(\chi)^{-1}}{[(r-1)!]^p} \int_{\mathbb{R}} |\chi(z)| \sum_{k \in \mathbb{Z}} \left\{ \int_{k/w}^{(k+1)/w} \left[ \left( \frac{1}{w} \right)^{pr-1} + \left| \frac{z}{w} \right|^{pr-1} \right] \times \right.
\]

\[
\left. \left( \int_{|y|\leq|z|/w} |g^{(r)}(y+u)|^p dy \right) du \right\} dz \\
= w^{-pr+1} \frac{2^{pr-1} M_0(\chi)^{-1}}{[(r-1)!]^p} \int_{\mathbb{R}} \left[ 1 + |z|^{pr-1} \right] |\chi(z)| \times \right.
\]

\[
\left. \int_{|y|\leq|z|/w} \left( \sum_{k \in \mathbb{Z}} \left\{ \int_{k/w}^{(k+1)/w} |g^{(r)}(y+u)|^p du \right\} dy \right) dz \\
= w^{-pr+1} \frac{2^{pr} M_0(\chi)^{-1}}{[(r-1)!]^p} \left\{ \frac{\|g^{(r)}\|_p}{\|\|g^{(r)}\|_p} \left( \int_{\mathbb{R}} \left[ |z| + |z|^{pr} \right] |\chi(z)|dz \right) + \tilde{M}_1(\chi) + \tilde{M}_{pr}(\chi) \right\} + \infty.
\]

Rearranging all the above estimates, we finally get:

\[
\| K_w g - g \|_p \leq w^{-r} \left\{ N_1 + \frac{2^{pr} M_0(\chi)^{-1}}{[(r-1)!]^p} \left[ \tilde{M}_1(\chi) + \tilde{M}_{pr}(\chi) \right] \right\}^{1/p}
\]

\[
= w^{-r} \| g^{(r)} \|_p N_2,
\]

from which, together with (\( 26 \)), we easily deduce that:

\[
\| K_w f - f \|_p \leq (M + 1) K \left( f, w^{-r} \frac{N_2}{M+1}, L^p, W^{r,p} \right)
\]

\[
\leq (M + 1) C_2 \omega_r \left( f, w^{-1} N_2^{1/r} (M + 1)^{-1/r} \right)_p
\]

\[
\leq (M + 1) C_2 \left( 1 + N_2^{1/r} (M + 1)^{-1/r} \right)^r \omega_r \left( f, w^{-1} \right)_p
\]

\[
= : C \omega_r \left( f, w^{-1} \right)_p.
\]
Finally, we can observe that the items (iii) and (iv) of the statement follows immediately from (i), (ii), (25), and the well-known inequality:

\[
\omega_r(f, \delta)_p \leq 2^{r-m} \omega_m(f, \delta)_p, \quad \delta > 0 \quad 1 \leq m < r, \quad 1 \leq p \leq +\infty.
\]

This completes the proof. □

**Remark 5.2** Note that some results concerning quantitative estimates by suitable modulus of smoothness for an abstract class of operators (including those considered in the present paper) are given in [35, 36, 38]. The main assumptions assumed in the above quoted results are similar (but given in a more abstract form) to those assumed in Theorem 5.1. One of the main differences between the results established in [35, 36, 38] and Theorem 5.1 is that, there certain local condition on the Fourier transform of the involved kernels and their derivatives are assumed, while here we simply require a sufficiently rapid decay of \(\chi\) at \(\pm\infty\). Even if in some special cases the estimates established in Theorem 5.1 can be deduced from the above quoted results, here in the proposed approach the required assumptions are immediate to check, and the proposed proofs are completely different. Further, the families of kernels for which all the above results can be applied are not exactly the same. Indeed, as will be showed in Sect. 6 and can also be deduced, e.g., from Example 4 of [35], in order to provide examples of kernels satisfying all the required assumptions of both theories, suitable finite linear combinations of classical kernels (such as those mentioned in Sect. 2) must be considered. Obviously, such combinations are not, in general, the same.

Now, in order to obtain a complete characterization of the spaces Lip*(\(\alpha, L^p\)), \(1 \leq p \leq +\infty,\) also for \(0 < \alpha < r,\) we need of the following lemma.

**Lemma 5.3** Let \(\chi \in W^{r,1}(\mathbb{R}), \ r \in \mathbb{N}^+,\) be a kernel satisfying (\(\chi 4\)) with \(\beta > r + 1,\) and (11). Then:

\[
m_j(\chi^{(i)}_r, x) = \begin{cases} 0, & i \neq j \\ i!, & i = j, \end{cases}
\]

where \(i = 1, \ldots, r,\) and \(j = 0, \ldots, i.\)

**Proof** Let \(i = 1\) and \(j = 0, 1,\) be fixed. We have that the series \(m_j(\chi, x)\) is absolutely convergent on \(\mathbb{R}\) since it is 1-periodic and (\(\chi 4\)) holds. Now, using the Fubini theorem, noting that \(\chi(\cdot - k), \ k \in \mathbb{Z},\) defines a regular (tempered) distribution, and that (by (11)) \(m_j(\chi, x)\) is constant on \(\mathbb{R},\) we have, for every fixed \(\varphi \in \mathcal{D}:\)

\[
0 = (m_0(\chi, x))'[\varphi] = -\int_{\mathbb{R}} m_0(\chi, x) \varphi'(x) \, dx = -\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \chi(x - k) \varphi'(x) \, dx = \sum_{k \in \mathbb{Z}} (\chi(x - k))'[\varphi],
\]

\[
0 = (m_1(\chi, x))'[\varphi] = -\int_{\mathbb{R}} m_1(\chi, x) \varphi'(x) \, dx
\]
\[- \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} (k - x) \varphi(x) \, dx = \sum_{k \in \mathbb{Z}} ((k - x) \varphi(x))' [\varphi].\]

Now, in view of the regularity of the monomial \((k - \cdot)\), and recalling the Leibnitz formula (see e.g., [49], p. 160) we immediately get:

\[
\sum_{k \in \mathbb{Z}} \chi'(x - k) = 0, \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \left[ -\chi(x - k) + (k - x) \chi'(x - k) \right] = 0,
\]

from which we obtain:

\[
m_0(\chi', x) = 0, \quad \text{and} \quad m_1(\chi', x) = m_0(\chi, x) = 1.
\]

Proceeding as above, and noting that the following general relation holds:

\[
m_j(\chi^{(i)}, x) = jm_{j-1}(\chi^{(i-1)}, x), \quad j = 0, 1, \ldots, i,
\]

the proof follows. \(\square\)

Note that, Lemma 5.3 represents a generalization of Lemma 7 of [1] in case of non-smooth kernels. Now, we are able to prove the following inverse theorem of approximation.

**Theorem 5.4** Let \(\chi \in W^{r, 1}(\mathbb{R}), r \in \mathbb{N}^+, \) be a kernel satisfying (11). Moreover, we assume in addition that \(\chi^{(i)}\) satisfies (\(\chi 4\)) with \(\beta > r + 1\), for \(i = 0, 1, \ldots, r\). Now, let \(f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty, \) such that:

\[
\|K^w_\chi f - f\|_p = O(w^{-\alpha}), \quad \text{as} \quad w \to +\infty, \quad 0 < \alpha < r. \tag{28}
\]

Then, if \(p = +\infty\), it turns out that \(f \in \text{Lip}^\ast(\alpha, L^\infty)\), while if \(1 \leq p < +\infty\) and any \(\chi^{(i)}\) satisfies (\(\chi 4\)) with \(\beta > rp + 1, i = 1, \ldots, r\), it turns out that \(f \in \text{Lip}^\ast(\alpha, L^p)\).

**Proof** Let \(1 \leq p \leq +\infty\). By (28), there exist \(C, \bar{w} > 0\) such that:

\[
\|K^w_f - f\|_p \leq C/w^\alpha, \quad w \geq \bar{w}.
\]

Now, we set \(\bar{\delta} := 1/\bar{w}\), and we consider fixed \(0 < \delta \leq \bar{\delta}, w \geq \bar{w}\).

Since \(0 < \alpha < r\), we can also fix \(m \in \mathbb{N}^+, 1 \leq m \leq r\) such that, \(m = 1\) if \(0 < \alpha < 1, \) or \(2 \leq m \leq r\) if \(m - 1 \leq \alpha < m\).

Observing that \(K^w_f \in W^{m, p}(\mathbb{R})\) in view of Theorem 4.2, and using (26), for any \(g \in W^{m, p}(\mathbb{R})\) we can write what follows:

\[
\omega_m(f, \delta)_p \leq \omega_m \left(f - K^w_f, \delta\right)_p + \omega_m \left(K^w_f, \delta\right)_p \\
\leq 2^m \|K^w_f - f\|_p + \frac{1}{C_1} K(K^w_f, \delta^m, L^p, W^{m, p}) \\
\leq \frac{2^m C}{w^\alpha} + \frac{2^m}{C_1} \|K^w_f|^{(m)}\|_p.
\]

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Recalling again Lemma 5.3 we finally have:

\[
\leq \frac{2^m C}{w^\alpha} + \frac{\delta^m}{C_1} \left\{ \| (K_w^X f)^{(m)} - (K_w^X g)^{(m)} \|_p + \| (K_w^X g)^{(m)} \|_p \right\} 
\]

\[
\leq \frac{2^m C}{w^\alpha} + \frac{\delta^m}{C_1} \left\{ w^m \| \chi^{(m)} \|_1 \| f - g \|_p + \| (K_w^X g)^{(m)} \|_p \right\}.
\]

Now, by Lemma 5.3 we know that \( m_0(\chi^{(m)}, x) = 0, x \in \mathbb{R} \), hence:

\[
(K_w^X g)^{(m)}(x) = (K_w^X g)^{(m)}(x) - g(x) w^m \sum_{k \in \mathbb{Z}} \chi^{(m)}(wx - k)
\]

\[
= w^m \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} [g(u) - g(x)] \, du \right\} \chi^{(m)}(wx - k),
\]

and using the Taylor expansion with integral remainder of \( g \) as in the proof of Theorem 5.1 we get:

\[
(K_w^X g)^{(m)}(x) = w^m \sum_{i=0}^{m-1} \frac{g^{(i)}(x)}{i!} \sum_{v=0}^{i} \left( \begin{array}{c} i \vspace{1mm} \end{array} \right) \left( \begin{array}{c} j \vspace{1mm} \end{array} \right) \sum_{k \in \mathbb{Z}} \left( \frac{k}{w} - x \right)^{i-v} \times \left\{ w \int_{k/w}^{(k+1)/w} \left( u - \frac{k}{w} \right)^{i-v} \, du \right\} \chi^{(m)}(wx - k)
\]

\[
+ w^m \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left( \int_{x}^{u} \frac{g^{(m)}(t)}{(m-1)!} (u-t)^{m-1} \, dt \right) \, du \right\} \chi^{(m)}(wx - k)
\]

\[
+ w^m \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left( \int_{x}^{u} \frac{g^{(m)}(t)}{(m-1)!} (u-t)^{m-1} \, dt \right) \, du \right\} \chi^{(m)}(wx - k).
\]

Recalling again Lemma 5.3 we finally have:

\[
(K_w^X g)^{(m)}(x) = w^m \sum_{k \in \mathbb{Z}} \left\{ w \int_{k/w}^{(k+1)/w} \left( \int_{x}^{u} \frac{g^{(m)}(t)}{(m-1)!} (u-t)^{m-1} \, dt \right) \, du \right\} \chi^{(m)}(wx - k).
\]

Now, estimating \( \| (K_w^X g)^{(m)} \|_p \) by the same procedure used in the proof of Theorem 5.1 (here we need to exploit the assumption (\( \chi 4 \)) for \( \chi^{(m)} \)), we obtain:

\[
\omega_m(f, \delta)_p \leq \frac{2^m C}{w^\alpha} + \frac{\delta^m}{C_1} w^m \left\{ \| \chi^{(m)} \|_1 \| f - g \|_p + T w^{-m} \| g^{(m)} \|_p \right\},
\]
for a suitable positive constant $T$, and where we suppose that $\|\chi^{(m)}\|_1 \neq 0$ (otherwise the proof become trivial). Passing to the infimum with respect to $g \in W^{m,p}(\mathbb{R})$, and using (26) again, we have:

$$\omega_m(f, \delta)_p \leq \frac{2^m C}{w^\alpha} + \frac{\delta^m}{C_1} \|\chi^{(m)}\|_1 K \left( f, \frac{T}{\|\chi^{(m)}\|_1} w^{-m}, L^p, W^{m,p} \right) + \frac{K_1}{w^\alpha} + K_2 \delta^m w^m \omega_m(f, 1/w)_p,$$

(29)

for suitable positive constants $K_1$ and $K_2$.

Now, we fix a sufficiently large $A > 0$, such that $A > \bar{w}$ and $K_2 < \frac{1}{2} A^{m-\alpha}$, with $m-\alpha > 0$. Setting $\delta_n := A^{-n}, n \in \mathbb{N}^+$, we claim:

$$\omega_m(f, \delta_n)_p \leq \tilde{M} \delta_n^\alpha,$$

(30)

for every $n \in \mathbb{N}^+$, for a suitable positive constant $\tilde{M}$. In order to prove (30) we proceed by induction on $n$. First, we choose the constant $\tilde{M}$ as follows:

$$\tilde{M} := \max \{ \omega_m(f, \delta_1)_p A^{-\alpha}, 2 K_1 A^{\alpha} \}.$$

Now, for $n = 1$ it follows immediately:

$$\omega_m(f, \delta_1)_p = \omega_m(f, \delta_1)_p A^{\alpha} A^{-\alpha} \leq \tilde{M} \delta_1^\alpha.$$

Suppose now that (30) holds for $n-1 \geq 1$. Then, for $n$, exploiting (29) with $w = A^{n-1} > \bar{w}$, and noting that $A^{-\alpha} \delta_n^{\alpha} = \tilde{\delta}_n$, we have:

$$\omega_m(f, \delta_n)_p \leq K_1 \delta_n^{\alpha} + K_2 A^{m(n-1)} \delta_n^m \omega_m(f, \delta_n-1)_p \leq K_1 A^{\alpha} \delta_n^{\alpha} + K_2 A^{m-\tilde{M}} \tilde{\delta}_n \delta_n^{\alpha} \leq \frac{1}{2} \tilde{M} \delta_n^{\alpha} + \frac{1}{2} \tilde{M} \delta_n^{\alpha} = \tilde{M} \delta_n^{\alpha}.$$

In order to conclude the proof, we finally need to show that $\omega_m(f, \delta)_p \leq L \delta^\alpha$, for every $0 < \delta < \delta_1$, with $\delta_1 \leq \tilde{\delta}$, and for a suitable constant $L > 0$. Let now $0 < \delta < \delta_1$ be fixed. Then, there exists $n \in \mathbb{N}^+, n \geq 2$, such that $\delta_n \leq \delta \leq \delta_{n-1}$, and we have:

$$\omega_m(f, \delta)_p \leq \omega_m(f, \delta_{n-1})_p \leq \tilde{M} \delta_{n-1}^{\alpha} = \tilde{M} A^{\alpha} \delta_n^{\alpha} =: L \delta_n^{\alpha} \leq L \delta^\alpha.$$

This completes the proof. \qed

Note that, an analogous of Theorem 5.4 for $p = +\infty$ and $r = 1$ has been proved in Theorem 6.4 of [24]. The proof of Theorem 5.4 has been inspired by those given in [7–9] for studying inverse results for Bernstein polynomials. However, the present approach differs from those just quoted in two main aspects: (i) here the usual
differentiability of the operators has been replaced by the (weaker) distributional differentiability, and (ii) we are able to achieve inverse results of approximation also for \( r > 2 \).

Now, by Theorem 5.1 and Theorem 5.4 it is immediate to deduce a characterization of the generalized Lipschitz classes \( \text{Lip}^*(\alpha, L^p) \), \( 0 < \alpha < r, r \geq 1 \), in term of convergence of sampling Kantorovich operators.

**Remark 5.5** Note that, all the results established in Sect. 5 for the case \( p = +\infty \), hold also if we consider the usual definition of \( L^\infty(\mathbb{R}) \) and \( W^{r,\infty}(\mathbb{R}) \) in place of those recalled in Sect. 2.

### 6 Saturation Theorems for the Sampling Kantorovich Operators with Bandlimited Kernels in \( L^p \)-Setting

We can prove what follows.

**Theorem 6.1** Let \( \chi \in B_{\pi w}^1(\mathbb{R}), w > 0 \), be a given kernel satisfying (\( \chi 4 \)) with \( \beta > r + 1 \), and such that, \( \tilde{m}_1(\chi) \neq 1/2 \) if \( r = 1 \), or

\[
\sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^{j-\ell} \tilde{m}_{j-\ell}(\chi) = 0, \quad j = 1, 2, \ldots, r - 1, \quad \sum_{\ell=0}^{r} \binom{r}{\ell} (-1)^{r-\ell} \tilde{m}_{r-\ell}(\chi) \neq 0.
\]

(31)

Let now \( f \in L^p(\mathbb{R}), 1 \leq p \leq +\infty \), such that:

\[
\|K_\chi^X f - f\|_p = o(w^{-r}), \quad \text{as} \quad w \to +\infty.
\]

Then, if \( 1 \leq p < +\infty \) it turns out that \( f \equiv 0 \) a.e. on \( \mathbb{R} \), while if \( p = +\infty \) it turns out that \( f \) is constant over \( \mathbb{R} \).

**Proof** First of all, we can observe that, if \( 1 \leq p < +\infty \), by (9) we have:

\[
\|K_\chi^X f - f\|_p = \|K_\chi^X [K_\chi^X f - f]\|_p \leq M_0(\chi) \frac{(p-1)/p}{p} \|\chi\|_1 \|K_\chi^X f - f\|_p = o(w^{-r}),
\]

(32)

as \( w \to +\infty \), while for \( p = +\infty \):

\[
\|K_\chi^X [K_\chi^X f - f]\|_\infty \leq M_0(\chi) \|K_\chi^X f - f\|_\infty = o(w^{-r}),
\]

(33)

as \( w \to +\infty \). Now, in both the above cases, for any fixed \( \varphi \in \mathcal{D} \), we have:

\[
\lim_{w \to +\infty} \int_{\mathbb{R}} w^r [ (K_\chi^X [K_\chi^X f])(x) - (K_\chi^X f)(x) ] \varphi(x) \, dx = 0,
\]

(34)
since the limit can be brought under the integral because, using the Holder inequality and (32) (or (33) respectively), we have:

\[
w^r \| \left\{ K_{w}^p [K_{w}^q f] - K_{w}^p f \right\} \varphi \|_1 \leq w^r \| K_{w}^p [K_{w}^q f] - K_{w}^p f \|_p \| \varphi \|_q \to 0, \quad \text{as} \quad w \to +\infty,
\]

where \(1/p + 1/q = 1\). Now, by Theorem 4.3 we know that \(K_{w}^p f \in B^p_1(\mathbb{R}) \subset B^p_{1, w}(\mathbb{R})\) for \(w \geq 1\), and then, in view of what has been established in (24), it turns out that:

\[
\mathcal{A}_w := \int_{\mathbb{R}} w^r \left[ (K_{w}^{\Theta}(K_{w}^p f)) (x) - (K_{w}^p f) (x) \right] \varphi(x) \, dx
\]

\[
= \int_{\mathbb{R}} w^r \left[ (I_{w}^{\Theta}(K_{w}^p f)) (x) - (K_{w}^p f) (x) \right] \varphi(x) \, dx. \quad (35)
\]

Further, we also recall that \(\chi\) satisfies (10) by (\(\chi\), 2) and Lemma 2.2. Now, using the well-known properties of the convolution, the Fubini theorem and suitable changes of variables:

\[
\int_{\mathbb{R}} \left( I_{w}^{\Theta}(K_{w}^p f) \right) (x) \varphi(x) \, dx
\]

\[
= \int_{\mathbb{R}} w \chi(wu) \left[ \int_{\mathbb{R}} \varphi(u+t) \left\{ \int_{\mathbb{R}} w \Theta(wz)(K_{w}^p f)(t-z) \, dz \right\} \, dt \right] \, du
\]

\[
= \int_{\mathbb{R}} w \chi(wu) \left[ \int_{\mathbb{R}} (K_{w}^p f)(s) \left\{ \int_{\mathbb{R}} w \Theta(w(y-u-s))(y) \, dy \right\} \, ds \right] \, du
\]

\[
= \int_{\mathbb{R}} w \chi(wu) \left[ \int_{\mathbb{R}} (K_{w}^p f)(s) \left\{ \int_{\mathbb{R}} w \Psi(w(s-u-y))(y) \, dy \right\} \, ds \right] \, du
\]

\[
= \int_{\mathbb{R}} w \chi(wu) \left[ \int_{\mathbb{R}} (K_{w}^p f)(s) \left( I_{w}^{\psi}(\chi) \right) (s+u) \, ds \right] \, du
\]

\[
= \int_{\mathbb{R}} (K_{w}^p f)(s) \left( I_{w}^{\psi}(\chi) \right) (s+u) \, ds
\]

\[
= \int_{\mathbb{R}} (K_{w}^p f)(s) \left( I_{w}^{\psi}(\chi) \right) (s+u) \, ds
\]

where \(\Psi\) denotes again the characteristic function of the interval \([0, 1]\) and \(\chi(u) := \chi(-u), u \in \mathbb{R}\). Now, by the relation in (35) we get:

\[
\mathcal{A}_w = \int_{\mathbb{R}} w^r \left[ (I_{w}^{\psi}(\chi) \right) (x) - \varphi(x) \right] (K_{w}^p f)(x) \, dx
\]
hence:

$$\lim_{w \to +\infty} A_w = \left[ \frac{(-1)^r}{r!} \sum_{\ell=0}^{r} \left( \frac{r}{\ell} \tilde{m}_{r-\ell}(\tilde{\chi}) \right) \right] \int_{\mathbb{R}} f(x) \varphi(r)(x) dx$$

$$= \frac{(-1)^r}{r!} \left[ \sum_{\ell=0}^{r} \left( \frac{r}{\ell} (-1)^{r-\ell} \tilde{m}_{r-\ell}(\tilde{\chi}) \right) \right] \int_{\mathbb{R}} f(x) \varphi(r)(x) dx,$$  \hspace{1cm} (36)

since the limit can pass under the integral, in view of the second part of Corollary 3.6 (applied with $\tilde{\chi}$ and $\Psi_1$, respectively). By (34), (36) and the uniqueness of the limit we obtain:

$$\frac{(-1)^r}{r!} \left[ \sum_{\ell=0}^{r} \left( \frac{r}{\ell} (-1)^{r-\ell} \tilde{m}_{r-\ell}(\tilde{\chi}) \right) \right] \int_{\mathbb{R}} f(x) \varphi(r)(x) dx = 0,$$  \hspace{1cm} (37)

and so, by the arbitrariness of $\varphi \in D$, in the sense of distribution (37) can be written as $f^{(r)} = 0$. It is well-known that this implies that $f$ is a polynomial of degree at most $r - 1$, and since $f \in L^p(\mathbb{R})$, $1 \leq p \leq +\infty$, it must be $f \equiv 0$ a.e. in case of $1 \leq p < +\infty$, and $f \equiv c$, $c \in \mathbb{R}$, if $p = +\infty$. This completes the proof. \hspace{1cm} $\Box$

Note that, the moment assumption of Theorem 6.1 that $\tilde{m}_1(\chi) \neq \frac{1}{2}$ if $r = 1$ or (31) if $r > 1$, are not restrictive. Indeed, it is not difficult to see that, e.g., for even kernels $\chi$ one has $\tilde{m}_1(\chi) = 0$, i.e., the moment condition for $r = 1$ holds.

Further, since any kernel $\chi \in B^1_{\pi w}(\mathbb{R})$, $w > 0$, satisfies $(\chi 4)$ and it is also continuous, by the Poisson summation formula the following relation between the first order continuous and discrete algebraic moments holds:

$$\tilde{m}_1(\chi) = -A_1^{X}.$$  \hspace{1cm} (38)

Relation (38) shows the accordance between the moment-type assumptions of Theorem 3.1 and Theorem 6.1 in case of $r = 1$. Similarly to above, by the Poisson summation formula again, also for $r > 1$ it turns out that $\tilde{m}_{j-\ell}(\chi) = (-1)^{\ell} A_1^{X}$, $\ell = 0, \ldots, j, j = 1, \ldots, r - 1$, from which we obtain:

$$(-1)^j \sum_{\ell=0}^{j} \left( \frac{j}{\ell} \right) \frac{1}{\ell + 1} \left[ (-1)^{-\ell} \tilde{m}_{j-\ell}(\chi) - A_1^{X} \right] = 0, \hspace{0.5cm} j = 1, 2, \ldots, r - 1,$$

that is, the moment-type assumptions of Theorem 3.1 and Theorem 6.1 are equivalent.

Now, in order to provide an example of bandlimited kernel satisfying the assumptions of Theorem 6.1 (and then also of Theorem 3.1), one can simply start by a continuous and even kernel $\chi : \mathbb{R} \rightarrow \mathbb{R}$, i.e., by a general kernel such that $\tilde{m}_1(\chi) = 0$, and such that condition $(\chi 4)$ holds for $\beta > r + 1, r \in \mathbb{N}^+$. In this way, one can simply define the following combination:

$$\bar{\chi}(x) := \frac{1}{2} [\chi(x) + \chi(x - 1)], \hspace{0.5cm} x \in \mathbb{R}.$$
It is not difficult to see that \( \chi \) is a kernel satisfying (\( \chi 1 \)), (\( \chi 2 \)), (\( \chi 3 \)), and \( \tilde{m}_0(\bar{\chi}) = 1 \), \( \tilde{m}_1(\bar{\chi}) = 1/2 \), \( \tilde{m}_2(\bar{\chi}) = \tilde{m}_2(\chi) + 1/2 > 1/6 \), and that (31) of Theorem 6.1 (and Theorem 3.1) is satisfied with \( r = 2 \).

While, if we consider the combination:

\[
\bar{\chi}_k(x) := \left[ \left( \frac{1}{3} - \frac{1}{2}k \right) \chi(x - 1) + \left( k + \frac{5}{6} \right) \chi(x) - \left( \frac{1}{6} + \frac{1}{2}k \right) \chi(x + 1) \right],
\]

\( k := \tilde{m}_2(\chi) > 0, x \in \mathbb{R} \), it turns out that \( \bar{\chi}_k \) is a kernel, with:

\[
\tilde{m}_0(\bar{\chi}_k) = 1, \quad \tilde{m}_1(\bar{\chi}_k) = 1/2, \quad \tilde{m}_2(\bar{\chi}_k) = 1/6, \quad \tilde{m}_3(\bar{\chi}_k) \neq 0,
\]

namely, (31) of Theorem 6.1 (and (12) of Theorem 3.1) is satisfied with \( r = 3 \). For instance, we can choose \( \chi \) in the above examples among one of those mentioned in Sect. 2, such as the Bochner-Riesz or the Jackson-type kernels. By the above procedure we can simply construct kernels which satisfy (31) also for \( r > 3 \).

### 7 Inverse Results and Favard Classes for the Sampling Kantorovich Operators with Bandlimited Kernels

Now, we can establish the following inverse result of approximation in the case of the saturation order.

**Theorem 7.1** Let \( \chi \) be a kernel assumed as in Theorem 6.1, with \( r \in \mathbb{N}^+ \). Moreover, let \( f \in L^p(\mathbb{R}) \), \( 1 \leq p \leq +\infty \), such that:

\[
\| K_\chi^w f - f \|_p = \mathcal{O}(w^{-r}), \quad \text{as} \quad w \to +\infty.
\]

Then, it turns out that:

\[
f \in W^{r-1}(BV) \cap L^1(\mathbb{R}), \quad \text{if} \quad p = 1,
\]

while,

\[
f \in W^{r-p}(\mathbb{R}), \quad \text{if} \quad 1 < p \leq +\infty.
\]

**Proof** First, we consider the case \( p = 1 \). Proceeding as in (32) and using assumption (39), it is not difficult to observe that there exist a suitable constant \( M > 0 \) such that:

\[
w^r \| K_\chi^w [K_\chi^w f] - K_\chi^w f \|_1 \leq M,
\]

for sufficiently large \( w > 0 \). Now, since the indefinite integral of a function belonging to \( L^1(\mathbb{R}) \) is absolutely continuous and so of bounded variation, one can apply the Helly-Bray theorem for BV functions (see, e.g., Proposition 0.8.15, p. 23 of [12]).
hence there exists a sub-sequence \( \{ w_{\nu}(K_{w_{\nu}}^f[K_{w_{\nu}}^f f] - K_{w_{\nu}}^f f) \} \) and a function \( G \in BV(\mathbb{R}) \), such that:

\[
\lim_{\nu \to +\infty} \int_{\mathbb{R}} w_{\nu}' \left[ (K_{w_{\nu}}^f[K_{w_{\nu}}^f f]) - (K_{w_{\nu}}^f f) \right] \phi(x) \, dx = \int_{\mathbb{R}} G'(x) \phi(x) \, dx, \quad \phi \in \mathcal{D}.
\]

On the other hand, by (36) we can obtain:

\[
\int_{\mathbb{R}} G'(x) \phi(x) \, dx = \frac{(-1)^{r}}{r!} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{(-1)^{r-\ell} \tilde{m}_{r-\ell}(\chi)}{\ell + 1} \right] \int_{\mathbb{R}} f(x) \phi^{(r)}(x) \, dx, \quad \phi \in \mathcal{D},
\]

that in the sense of distribution theory means:

\[
G' = \frac{(-1)^{r}}{r!} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{(-1)^{r-\ell} \tilde{m}_{r-\ell}(\chi)}{\ell + 1} \right] f^{(r)} =: \mathcal{L}_{r}(\chi) f^{(r)},
\]

with \( \mathcal{L}_{r}(\chi) \neq 0 \) by (31). Now, if we denote by \( G_{r-1} \) the \((r-1)\)-th antiderivative of \( G \), we obtain:

\[
\mathcal{L}_{r}(\chi) f = G_{r-1}, \quad a.e.,
\]

and so \( f^{(r-1)} \in BV(\mathbb{R}) \). Thus, by Theorem 5.4, p. 37 of [26] we get \( f \in W^{r-1}(BV) \cap L^{1}(\mathbb{R}) \). For what concerns \( 1 < p \leq +\infty \), proceeding similarly to above, and using the weak* compactness theorem for \( L^{p} \) (see, e.g., Proposition 0.8.14, p. 23 of [12]) in place of Helly-Bray theorem, there exists \( g \in L^{p}(\mathbb{R}) \), such that:

\[
\int_{\mathbb{R}} g(x) \phi(x) \, dx = \frac{(-1)^{r}}{r!} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{(-1)^{r-\ell} \tilde{m}_{r-\ell}(\chi)}{\ell + 1} \right] \int_{\mathbb{R}} f(x) \phi^{(r)}(x) \, dx, \quad \phi \in \mathcal{D},
\]

i.e., in the sense of distribution:

\[
g = \frac{(-1)^{r}}{r!} \left[ \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{(-1)^{r-\ell} \tilde{m}_{r-\ell}(\chi)}{\ell + 1} \right] f^{(r)} \in L^{p}(\mathbb{R}),
\]

i.e., \( f^{(r)} \in L^{p}(\mathbb{R}) \). Now, recalling again that \( f \in L^{p}(\mathbb{R}) \) is locally absolutely integrable and that Theorem 5.4, p. 37 of [26] holds, we have that, in term of equivalence class, \( f \) belongs to the Sobolev space \( W^{r,p}(\mathbb{R}) \).

We stress again that, due to the application of the weak* compactness theorem, in Theorem 7.1 the case \( p = +\infty \) holds in the usual sense only.

---

3 if \( r \geq 2 \), \( G_{r-1} \) is defined by: \( G_{r-1}(x) := \frac{1}{(r-2)!} \int_{0}^{x} (x-u)^{r-2} G(u) \, du, \quad x \in \mathbb{R} \).

4 For the sake of completeness, we specify that for \( p = +\infty \) we have \( f^{(r)} \in L^{\infty}(\mathbb{R}) \), where \( L^{\infty}(\mathbb{R}) \) is defined in the usual sense, i.e., \( f^{(r)} \) is measurable and \( \sup_{x \in \mathbb{R}} |f^{(r)}(x)| < +\infty \).
Now, for $r = 1$, in view of what has been recalled in Sect. 5, by Theorem 5.1, Remark 5.5, Theorem 6.1 and Theorem 7.1, and observing that the inclusion $B^1_r(\mathbb{R}) \subset W^{r,1}(\mathbb{R})$, $r \in \mathbb{N}^+$, holds, it follows that:

$$BV(\mathbb{R}) \cap L^1(\mathbb{R}) = \text{Lip}(1, L^1), \quad \text{if } p = 1,$$

and

$$\text{Lip}(1, L^p) = W^{1,p}(\mathbb{R}), \quad \text{if } 1 < p \leq +\infty,$$

are the so-called Favard (saturation) classes of the sampling Kantorovich operators based upon bandlimited kernels $\chi$, with $\tilde{m}_1(\chi) = -A^1_1 \neq 1/2$.

While, if $\chi \in B^1_r w$, $w > 0$, is a kernel assumed as in Theorem 6.1, with $r \in \mathbb{N}^+$, $r > 1$, and it satisfies $(\chi 4)$ for suitable values of $\beta$, from Theorem 5.1, Remark 5.5, Theorem 6.1 and Theorem 7.1, it follows that, the spaces $\text{Lip}(r, L^p) = W^{r,p}(\mathbb{R})$, for $1 < p \leq +\infty$, and $\text{Lip}(r, L^1) = W^{r-1}(BV) \cap L^1(\mathbb{R})$, for $p = 1$, are the Favard classes for the sampling Kantorovich operators.

**Remark 7.2** Note that, results concerning the saturation classes for the quasi-projection operators have been also established in Theorem 24 of [38] using the well-known Mikhlin’s condition (see [29], p. 367). In the present instance, other than the considerations already given in Remark 5.2, we can also stress that the results here proved hold for every $1 \leq p \leq +\infty$ and without requiring the above mentioned Mikhlin’s condition, while those proved in [38] hold for $1 < p < +\infty$ only.

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