Remarks on Axion-like models

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For a recently proposed alternative to the traditional axion model, we study its long distance behavior, in particular the confinement versus screening issue, and show that a compactified version of this theory can be further mapped into the massive Schwinger model. Our calculation is based on the gauge-invariant but path-dependent variables formalism. This result agrees qualitatively with the usual axion model.

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I. INTRODUCTION

Axion-like models [1, 2, 3, 4], or simply axions models, have become the focus of intense research activity after recent results of the PVLAS collaboration [5]. This collaboration has reported measurements of the rotation of the polarization of photons passing through a vacuum cavity in an external magnetic field. As is well known, this effect can be qualitatively understood by the existence of light pseudoscalars bosons φ (the "axion"), with a coupling to two photons. In particular, the Lagrangian density of the photon-pseudoscalar system is given by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{g}{8} \phi \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m_A^2}{2} \phi^2, \]

where \( m_A \) is the mass for the axion field \( \phi \). It is worth recalling at this stage that this theory experiences mass generation when the gauge field \( F_{\mu\nu} \) takes a magnetic type expectation value [6]. If \( F_{\mu\nu} \) takes an electric type expectation value, tachyonic mass generation takes place [6]. Moreover, this theory leads to confining potentials in the presence of nontrivial constant expectation values for the gauge field strength \( F_{\mu\nu} \) [7]. In particular, in the case of a constant electric field strength expectation value the static potential remains Coulombic, while in the case of a constant magnetic field strength expectation value the potential energy is the sum of a Yukawa and a linear potential, leading to the confinement of static charges. Notice that the magnetic character of the field strength expectation value needed to obtain confinement is in agreement with the current chromo-magnetic picture of the QCD vacuum [8]. Another feature of this model is that it restores the rotational symmetry (in the potential), despite of the fact that the external fields break this symmetry. More interestingly, similar results have been obtained in the context of the dual Ginzburg-Landau theory [9], as well as for a theory of antisymmetric tensor fields that results from the condensation of topological defects as a consequence of the Julia-Toulouse mechanism [10]. Accordingly, the above interrelations are interesting from the point of view of providing unifications among diverse models as well as exploiting the equivalence in explicit calculations. We also point out that confinement as a consequence of the interaction between a non-Abelian constant chromo-magnetic background and the axion field has been recently investigated in [11].

On the other hand, we further observe that recently a novel way to describe axions has been proposed [2]. The motivation for this is mainly to reconcile the results of the PVLAS experiment with both astrophysical bounds and the results of the CAST collaboration. The crucial ingredient of this development is the existence of a new light vector field (rather than an axion field), which interacts with the photon via Chern-Simons-like terms. In such a case the Lagrangian density reads

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2(A) - \frac{1}{4} F_{\mu\nu}^2(B) + \frac{m_A^2}{2} A_{\mu}^2 + \frac{m_B^2}{2} B_{\mu}^2 - \frac{\kappa}{4} \varepsilon^{\mu\nu\lambda\rho} A_{\mu} B_{\nu} F_{\lambda\rho}(A), \]

where \( m_A \) is the mass of the photon, and \( m_B \) represents the mass for the gauge boson \( B \). In particular, this alternative theory exhibits an effective mass for the component of the photon along the direction of the external magnetic field, exactly as it happens with the theory [11].

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Given its possible relevance in order to explain the discrepancy coming from the CAST collaboration, in this work we wish to further elaborate on the physical content of the model \([12, 13]\). Our calculation is accomplished by making use of the gauge-invariant but path-dependent variables formalism along the lines of Ref. \([7]\). This approach provides a physically-based alternative to the usual Wilson loop approach, where the usual qualitative picture of confinement in terms of an electric flux tube linking quarks emerges naturally. As we shall see, our analysis reveals that although both theories \([11, 12]\) and \([13, 14]\) lead to an effective mass for the photon, the physical content is quite different. In other words, the confining nature of the potential is lost. On the other hand, if the same calculation is performed in the presence of two compact spacelike dimensions, we again find a confining potential. In this way we establish a new and peculiar connection with the Schwinger model, in the hope that this will be helpful to understand better axion-like models.

II. INTERACTION ENERGY

We now examine the interaction energy between static point-like sources for the model \([12]\). This can be done by computing the expectation value of the energy operator \(H\) in the physical state \(|\Phi\rangle\) describing the sources, which we will denote by \(\langle H \rangle_{\Phi}\).

Before proceeding with the determination of the interaction energy, it is worthwhile noticing that the coupling for both theories \([11, 12]\) is identical. In fact, as was first mentioned in Ref. \([2]\), substituting \(B_\mu\) by \(\partial_\mu \phi\) in \([2]\), the theory under consideration assumes the form

\[
\mathcal{L} = -\frac{1}{4} F_{\mu
u}^2 + m_\phi^2 \frac{\phi}{2} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\kappa}{4 m_B^2} \varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}.
\]  

Thus we shall begin considering the following generating functional:

\[
Z = \int \mathcal{D}\phi \mathcal{D}A \exp \left\{ i \int d^4x \mathcal{L} \right\},
\]

where the Lagrangian density is given by \([3]\). We restrict ourselves to static scalar fields, a consequence of this is that \(\Delta \phi = -\nabla^2 \phi\), with \(\Delta \equiv \partial_\mu \partial^\mu\). It also implies that, after performing the integration over \(\phi\) in \(Z\), the effective Lagrangian density reads

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + m_\phi^2 \frac{\phi}{2} A_\mu - \frac{\kappa^2}{2 m_B^2} \varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \frac{1}{\nabla^2} \varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} - A_\mu J^\mu,
\]

where \(J^\mu\) is the external current of the test charges. Furthermore, as was explained in \([7]\), this expression can be rewritten as

\[
\mathcal{L} = -\frac{1}{4} f_{\mu\nu}^2 + \frac{m_\phi^2}{2} A_\mu - \frac{\kappa^2}{8 m_B^2} \varepsilon^{\mu\nu\lambda\rho} \langle F_{\mu\nu} \rangle \varepsilon^{\lambda\rho\gamma\delta} \langle F_{\lambda\rho} \rangle \frac{1}{\nabla^2} \varepsilon^{\alpha\beta\gamma\delta} f_{\alpha\beta} - A_\mu J^\mu,
\]

where \(\langle F_{\mu\nu} \rangle\) represents the constant classical background. Here \(f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) describes fluctuations around the background. The above Lagrangian arose after using \(\varepsilon^{\mu\nu\lambda\rho} \langle F_{\mu\nu} \rangle \langle F_{\lambda\rho} \rangle = 0\) (which holds for a pure electric or a pure magnetic background). By introducing the notation \(\varepsilon^{\mu\nu\alpha\beta} \langle F_{\mu\nu} \rangle \equiv v^{\alpha\beta}\) and \(\varepsilon^{\sigma\tau\gamma\delta} \langle F_{\sigma\tau} \rangle \equiv v^{\gamma\delta}\), expression \((6)\) then becomes

\[
\mathcal{L} = -\frac{1}{4} f_{\mu\nu}^2 + \frac{m_\phi^2}{2} A_\mu - \frac{\kappa^2}{8 m_B^2} v^{\alpha\beta} f_{\alpha\beta} \frac{1}{\nabla^2} v^{\gamma\delta} f_{\gamma\delta} - A_\mu J^\mu,
\]

where the tensor \(v^{\alpha\beta}\) is not arbitrary, but satisfies \(\varepsilon^{\mu\nu\alpha\beta} v_{\mu\nu} v^{\alpha\beta} = 0\).

A. Magnetic case

We now proceed to calculate the interaction energy in the \(v^{0i} \neq 0\) and \(v^{ij} = 0\) case (referred to as the magnetic case in what follows). Using this in \((7)\) we then obtain

\[
\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{m_\phi^2}{2} A_\mu - \frac{\kappa^2}{8 m_B^2} v^{0i} f_{0i} \frac{1}{\nabla^2} v^{0k} f_{0k} - A_0 J^0.
\]


with \((\mu, \nu = 0, 1, 2, 3)\) and \((i, k = 1, 2, 3)\).

Now, we move on to compute the canonical Hamiltonian. For this end we perform a Hamiltonian constraint analysis. The canonically conjugate momenta are \(\Pi^\mu = 0\) and \(\Pi_i = D_{ij}E_j\), where \(E_i \equiv f_{0i}\) and \(D_{ij} \equiv \left(\delta_{ij} - \frac{\kappa^2}{4m_B^2}v_{0i}v_{0j}\right)\).

Since \(D\) is a nonsingular matrix \((\det D = 1 - \frac{\kappa^2}{4m_B^2}v^2 \neq 0)\) with \(v^2 \equiv v^{0i}v_{0i}\), there exists the inverse of \(D\). Solving for \(E_i\) as a function of \(\Pi_i\), we get

\[
E_i = \frac{1}{\det D} \left\{ \delta_{ij} \det D + \frac{\kappa^2}{4m_B^2}v_{0i}v_{0j} \right\} \Pi_j.
\]

This leads us to the canonical Hamiltonian,

\[
H_C = \int d^3x \left\{ -A_0 \left( \nabla \cdot \Pi + \frac{m^2}{2}A^0 - J^0 \right) + \frac{1}{2} \Pi^2 + \frac{1}{2}B^2 + \frac{m^2}{2}A^2 + \frac{\kappa^2}{8m_B^2} \frac{(v \cdot \Pi)^2}{(v^2 - \frac{\kappa^2}{4m_B^2}v^2)} \right\},
\]

where \(B\) is the magnetic field. Requiring the primary constraint \(\Pi_0 = 0\) to be preserved in time yields the following secondary constraint

\[
\Gamma (x) \equiv \partial_t \Pi^i + m^2 \theta_0 - J^0 = 0.
\]

The above result reveals that there are two constraints, which are second class. It explicitly reflects the breaking of the gauge invariance of the theory under consideration. As a consequence, special care has to be exercised since it is the gauge invariance property that generally establishes unitarity and renormalizability in most quantum field theoretical models.

In order to convert the second class system into first class we will adopt the procedure described in Refs.\[14, 15\]. An important feature of this development is that the new system still has the basic features of the original one and has reestablished the gauge symmetry. As was explained in Refs.\[14, 15\], we enlarge the original phase space by introducing a canonical pair of fields \(\theta\) and \(\Pi_\theta\). Accordingly, a new set of constraints can be defined in this extended space:

\[
A_1 = \Pi_0 + m^2 \theta = 0,
\]

and

\[
A_2 = \Gamma + \Pi_\theta = 0.
\]

It can be easily checked that the new constraints are first class and in this way restore the gauge symmetry of the theory under consideration. It is worthwhile remarking at this point that the \(\theta\) fields only enlarge the unphysical sector of the total Hilbert space, not affecting the structure of the physical subspace. Therefore the new effective Lagrangian, after integrating out the \(\theta\) fields reads

\[
\mathcal{L} = -\frac{1}{4} f_{\mu\nu} \left( 1 + \frac{m^2}{\Delta} \right) f^{\mu\nu} - \frac{\kappa^2}{8m_B^2} v^{0i} f_{0i} \left( \frac{v^0}{v^2} \right) f_{0k} f_{0k} - A_0 J^0.
\]

With this in hand, the canonical momenta are \(\Pi^\mu = -\left(1 + \frac{m^2}{\Delta}\right) f^{0\mu} - \frac{\kappa^2}{4m_B^2} v^{0\mu} \frac{1}{v^2} v^{0i} f_{0i}\), and one immediately identifies the usual primary constraint \(\Pi_0^0 = 0\), and \(\Pi_i = -\left(1 + \frac{m^2}{\Delta}\right) f_{0i} - \frac{\kappa^2}{4m_B^2} v_{0i} \frac{1}{v^2} v_{0j} f_{0j}\). The canonical Hamiltonian can be worked out as usual and is given by the expression

\[
H_C = \int d^3x \left\{ -A_0 \left( \partial_t \Pi^i - J^0 \right) + \frac{1}{2} \Pi^i \left( 1 + \frac{m^2}{\Delta} \right)^{-1} \right\} + B^i \left( 1 + \frac{m^2}{\Delta} \right) B^i \right\} + \frac{\kappa^2}{8m_B^2} \frac{(v \cdot \Pi)^2}{(v^2 - \frac{\kappa^2}{4m_B^2}v^2)} \left( \nabla \cdot \Pi \right).
\]

Temporal conservation of the primary constraint \(\Pi_0\) leads to the secondary constraint \(\Gamma_1 (x) \equiv \partial_t \Pi^i - J^0 = 0\). The preservation in time of \(\Gamma_1\) does not give rise to any further constraints. The extended Hamiltonian that generates translations in time then reads \(H = H_C + \int d^3x \left( c_0 (x) \Pi_0 (x) + c_1 (x) \Gamma_1 (x) \right)\), where \(c_0 (x)\) and \(c_1 (x)\) are the Lagrange multiplier fields. Moreover, it is straightforward to see that \(A_0 (x) = [A_0 (x), H] = c_0 (x)\), which is an arbitrary
function. Since $\Pi^0 = 0$ always, neither $A^0$ nor $\Pi^0$ are of interest in describing the system and may be discarded from the theory. Thus the Hamiltonian takes the form

$$H_C = \int d^3x \left\{ \frac{1}{2} \Pi^i \left( 1 + \frac{m_x^2}{2} \right)^{-1} \Pi^i + \frac{i}{2} B^i \left( 1 + \frac{m_x^2}{2} \right) B^i + c(x) \left( \partial_\xi \Pi^i - J^0 \right) \right\} +$$
$$+ \int d^3x \frac{x^2}{8m_B^4} \left( v \cdot \Pi \right) \frac{\Delta^2}{\left( \Delta + m_z^2 \right)^2 - \frac{v^2}{m_B^4}} \left( v \cdot \Pi \right),$$

(16)

where $c(x) = c_1(x) - A_0(x)$.

According to the usual procedure we introduce a supplementary condition on the vector potential such that the full set of constraints becomes second class. A convenient choice is found to be

$$\Gamma_2 (x) \equiv \int_C dz^\nu A_\nu (z) \equiv \frac{1}{0} d\lambda x^i A_i (\lambda x) = 0,$$

(17)

where $\lambda (0 \leq \lambda \leq 1)$ is the parameter describing the spacelike straight path $x^i = \xi^i + \lambda (x - \xi)^i$, and $\xi$ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\xi^i = 0$. This allows us to write the only nonvanishing equal-time Dirac bracket

$$\{ A_i (x), \Pi^j (y) \}^s = \delta^s_i \delta^{(3)} (x - y) - \partial_i \frac{1}{0} d\lambda x^j \delta^{(3)} (\lambda x - y).$$

(18)

Finally we are ready to tackle the question of the interaction energy between pointlike sources, where a fermion is localized at $y^\nu$ and an antifermion at $y$. Since the fermions are taken to be infinitely massive (static) we can substitute $\Delta$ by $-\nabla^2$ in Eq. (16). In such a case we write $\langle H \rangle_\Phi$ as

$$\langle H \rangle_\Phi = \langle \Phi \rangle \int d^3x \left\{ \frac{1}{2} \Pi^i \nabla^2 \Pi^i - \frac{1}{2} B^i \left( 1 - \frac{m_x^2}{\nabla^2} \right) B^i \right\} \langle \Phi \rangle,$$

(19)

with $M^2 = m_x^2 + \frac{\kappa^2}{4m_B^4} v^2 = m_x^2 + \frac{B^2}{v^2}$, where we have employed $v^2 = 4B^2$ and $\kappa = \frac{m_B}{v_B}$.

Next, as was first established by Dirac [17], the physical state can be written as

$$| \Phi \rangle \equiv | \overline{\Psi} (y) \Psi (y) \rangle = | \overline{\Psi} (y) \rangle \exp \left( i e \int^y_{y'} dz^\nu A_i (z) \right) \psi (y') | 0 \rangle,$$

(20)

where $| 0 \rangle$ is the physical vacuum state and the line integral appearing in the above expression is along a spacelike path starting at $y'$ and ending at $y$, on a fixed time slice. It is worth noting here that the strings between fermions have been introduced in order to have a gauge-invariant function $| \Phi \rangle$. In other terms, each of these states represents a fermion-antifermion pair surrounded by a cloud of gauge fields sufficient to maintain gauge invariance.

From the foregoing Hamiltonian discussion, we first observe that

$$\Pi_i (x) | \overline{\Psi} (y) \Psi (y') \rangle = | \overline{\Psi} (y) \Psi (y') \Pi_i (x) | 0 \rangle + e \int^y_{y'} dz_\nu \delta^{(3)} (z - x) \langle \Phi \rangle.$$  

(21)

Substituting this in (19), we get

$$\langle H \rangle_\Phi = \langle H \rangle_0 - \frac{g^2 e^{-ML}}{4\pi} \frac{L}{L},$$

(22)

where $\langle H \rangle_0 = \langle 0 | H | 0 \rangle$ and $| y - y' | \equiv L$. Since the potential is given by the term of the energy which depends on the separation of the two fermions, from the expression (22) we obtain

$$V = - \frac{g^2 e^{-ML}}{4\pi} \frac{L}{L}.$$  

(23)

As already stated, $M$ is the effective mass for the component of the photon along the direction of the external magnetic field. Accordingly, the above analysis reveals that, although both theories [11] and [22] contain the same coupling, the physical content is quite different. It is important to realize that expression (23) is spherically symmetric, although the external fields break the isotropy of the problem in a manifest way. Another example where the rotational symmetry is restored was studied in the case of non commutative QED [15].
B. Electric case

We shall now consider the case $v^{0i} = 0$ and $v^{ij} \neq 0$ (referred to as the electric one in what follows). In such a case the density Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}f_{\mu\nu}^2 + \frac{m^2}{2}A_\mu^2 - \frac{\kappa^2}{8m_B^2}v^{ij}f_{ij}\frac{1}{\sqrt{2}}v^{kl}f_{kl}, \quad (24)$$

($\mu, \nu = 0, 1, 2, 3$) and $(i, j, k, l = 1, 2, 3)$. The above Lagrangian will be the starting point of the Dirac constrained analysis. As before, the above Lagrangian follows from Eq. (25) are $\Pi^\mu = -\left(1 + \frac{m^2}{\Delta}\right)f^{0\mu}$, which results in the usual primary constraint $\Pi^0 = 0$ and $\Pi^i = -\left(1 + \frac{m^2}{\Delta}\right)f^{0i}$. Defining the electric and magnetic fields by $E^i = f^{i0}$ and $B^i = \frac{1}{2}\varepsilon^{ijk}f_{jk}$, respectively, the canonical Hamiltonian assumes the form

$$H_C = \int d^3x \left\{ \frac{1}{2}E_i \left(1 + \frac{m^2}{\Delta}\right)E^i + \frac{1}{2}B_i \left(1 + \frac{m^2}{\Delta}\right)B^i + \frac{\kappa^2}{8m_B^2}\varepsilon_{ijm}\varepsilon_{kln}v^{ij}B^n\frac{1}{\sqrt{2}}v^{kl}B^m \right\} - \int d^3x A_0 \left\{ \partial_i \Pi^i - J^0 \right\}. \quad (26)$$

Time conservation of the primary constraint leads to the secondary constraint $\Gamma_1(x) \equiv \partial_i \Pi^i - J^0 = 0$, and the time stability of the secondary constraint does not induce more constraints, which are first class. Following our earlier procedure, we will compute the expectation value of the Hamiltonian in the physical state $|\Phi\rangle$, that is,

$$\langle H \rangle_\Phi = \langle \Phi | \int d^3x \left\{ \frac{1}{2}E_i \left(1 + \frac{m^2}{\Delta}\right)E^i \right\} |\Phi\rangle. \quad (27)$$

Hence we see that the potential is given by

$$V = -\frac{q^2}{4\pi}e^{-m_\gamma L} \frac{L}{4\pi}, \quad (28)$$

where $L \equiv |y - y'|$. It must be observed that in this case $M$ reduces to $m_\gamma$, and in the limit $m_\gamma \to 0$ the potential reduces to the Coulomb one [7].

III. FINAL REMARKS

Our discussion, so far, has concentrated on the confinement versus screening issue for the recently proposed axion model [2]. As we have seen, the coupling for this theory is identical to the traditional axion model [1]. We have also seen that this alternative theory exhibits an effective mass for the component of the photon along the direction of the external magnetic field, as it happens with the theory [11]. From this point of view, it is meaningful to ask whether the confining nature of the potential can be recovered in some approximations. We now address this question.

To that end, we will discuss the mapping of the theory [7] into the massive Schwinger model. As was explained in [19], we illustrate this by making a dimensional compactification (à la Kaluza-Klein) on Eq. (7). Then we see that the new theory takes the form:

$$\mathcal{L}^{(1+1)} = -\frac{1}{4}f_{\mu\nu}^2 \sum_n \left(1 + \frac{\zeta^2}{\Delta^{(1+1)} + a^2}\right)f^{\mu\nu} - A_\mu J^\mu, \quad (29)$$

where $\zeta^2 = m_\gamma^2 + \frac{\kappa^2}{8m_B^2} \left[g^\alpha\mu g^\beta\nu v_{\alpha\beta}g_\gamma\mu g_\delta\nu v^{\gamma\delta}\right]$ and $a^2 \equiv \hbar^2/R^2$. We immediately recognize the above to be the massive Schwinger model with mass $m^2 \equiv a^2$. Notwithstanding, in order to put our discussion into context it is useful to
summarize the relevant aspects of the analysis described previously [19]. We shall begin by recalling the interaction energy for the massive Schwinger model taking a contribution of a single mode in Eq. (29). We obtain [13]:

$$V = \frac{q^2}{2} \left( 1 + \frac{a^2}{\lambda^2} \right) \left( 1 - e^{-\lambda L} \right) + \frac{q^2}{2} \left( 1 - \frac{4\zeta^2}{\lambda^2} \right) L,$$

(30)

where $\lambda^2 \equiv 4\zeta^2 + a^2$ and $|y| \equiv L$. Effectively, therefore, our initial theory [7] is mapped into the massive Schwinger model, which displays both the screening and the confining part of this interaction. Of course, if we consider the zero mode case, i.e. $a = 0$, the static potential above shows that confinement disappears. As in [19], we will concentrate on the second term of Eq. (30), which represents confinement. The expression for the coefficient of the linear potential between two static point sources is:

$$T = \frac{q^2}{2} \sum_{n_1, n_2} \frac{\left( n_1^2 + n_2^2 \right)}{\zeta^2 + n_1^2 + n_2^2}.$$

(31)

Following our earlier procedure, in the limit $R_1, R_2 \to \infty$ we obtain

$$T = \pi q^2 R_1 R_2 \int_0^\Lambda d\rho \frac{\rho^3}{\zeta^2 + \rho^2},$$

(32)

that is,

$$T = \frac{\pi q^2}{2} R_1 R_2 \left[ \Lambda^2 - \zeta^2 \ln \left( \frac{\zeta^2 + \Lambda^2}{\zeta^2} \right) \right],$$

(33)

again if $R_1, R_2 \to \infty$, we obtain a transcendental equation for $\Lambda/\zeta^2$:

$$\frac{\Lambda^2}{\zeta^2} - \ln \left( 1 + \frac{\Lambda^2}{\zeta^2} \right) = 0.$$

(34)

From (32) here we can deduce that as $\frac{\Lambda^2}{\zeta^2} \sim \sqrt{\frac{2\pi T}{q^2 R_1 R_2}} \to 0$, which means that $\zeta$ has to grow stronger than $\Lambda$ when $\Lambda \to \infty$, in order to obtain a finite coefficient of the linear potential. It is clear from this discussion that our phenomenological result (30) agrees qualitatively with the magnetic case of the usual axion model [7], in the limit of large $L$. Thus, only after the compactification, both theories are equivalent in the low energy regime. In this way we establish an intriguing analogy with the massive Schwinger model, which simulates the features of the usual (3 + 1)-dimensional axion model.

IV. ACKNOWLEDGMENTS

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[1] E. Masso and J. Redondo, JCAP 0509, 015 (2005); Phys. Rev. Lett. 97, 151802 (2006).
[2] I. Antoniadis, A. Boyarsky and O. Ruchayskiy, hep-ph/0606306.
[3] D. Lai and J. Heyl, astro-ph/0609775.
[4] G. G. Raffelt, hep-ph/0611118.
[5] E. Zavattini et al. [PVLAS collaboration], Phys. Rev. Lett. 96, 110406 (2006).
[6] E. I. Guendelman, S. Ansoldi and E. Spallucci, JHEP 09, 044 (2003).
[7] P. Gaete and E. I. Guendelman, Mod. Phys. Lett. A20, 319 (2005).
[8] S. G. Matinian and G. K. Savvidy, Nucl. Phys. B134, 539 (1978); G. K. Savvidy, Phys. Lett. B71, 133 (1977); P. Olesen, Physica Scripta, 23, 1000 (1981); H. B. Nielsen and P. Olesen, Nucl. Phys. B160, 380 (1979); H. B. Nielsen and M. Ninomiya, Nucl. Phys. B156, 1 (1979).
[9] H. Suganuma, S. Sasaki and H. Toki, Nucl. Phys. B435, 207 (1995).
[10] P. Gaete and C. Wotzasek, Phys. Lett. B601, 108 (2004).
[11] P. Gaete and E. Spallucci, J. Phys. A39, 6021 (2006).
[12] J. Schwinger, Phys. Rev. \textbf{128}, 2435 (1962).
[13] P. Gaete and I. Schmidt, Phys. Rev. \textbf{D61}, 125002 (2000).
[14] C. Wotzasek, Int. J. Mod. Phys. \textbf{A5}, 1123 (1990).
[15] N. Banerjee and R. Banerjee, Mod. Phys. Lett. \textbf{A11}, 1919 (1996).
[16] P. Gaete, Z. Phys. \textbf{C76}, 355 (1997); Phys. Lett. \textbf{B515}, 382 (2001); Phys. Lett. \textbf{B582}, 270 (2004).
[17] P. A. M. Dirac, \emph{The Principles of Quantum Mechanics} (Oxford University Press, Oxford, 1958); Can. J. Phys. \textbf{33}, 650 (1955).
[18] P. Gaete and I. Schmidt, Int. J. Mod. Phys. \textbf{A19}, 3247 (2004).
[19] P. Gaete, E. I. Guendelman and E. Spallucci, \texttt{hep-th/0607151} Phys. Lett. \textbf{B} (in press).