Well-posedness for the surface quasi-geostrophic front equation

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Abstract

We consider the well-posedness of the surface quasi-geostrophic (SQG) front equation. Hunter–Shu–Zhang (2021 Pure Appl. Anal. 3 403–72) established well-posedness under a small data condition as well as a convergence condition on an expansion of the equation’s nonlinearity. In the present article, we establish unconditional large data local well-posedness of the SQG front equation, while also improving the low regularity threshold for the initial data. In addition, we establish global well-posedness theory in the rough data regime by using the testing by wave packet approach of Ifrim–Tataru.

Keywords: SQG front equation, paralinearization, modified energies, frequency envelopes, wave packet testing

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1. Introduction

In this paper we are concerned with the evolution of front solutions to the surface quasi-geostrophic (SQG) equation. This equation has drawn substantial attention over the past several years for at least two reasons. The first is that it models geophysical flows confined to a surface, like those which appear in the atmosphere or ocean. In particular, it accounts for
the formation of discontinuous temperature fronts, a phenomenon known as frontogenesis. A second motivation is that there is an interesting mathematical and physical analogy between the behavior of strongly nonlinear solutions to the SQG equation and solutions to the 3D incompressible Euler equations that potentially blow up in finite time, in the sense that level sets for the solutions of the former correspond to vortex lines for the latter [3]. To the best of our knowledge, the question of singularity formation for both the SQG equation and 3D Euler equation is still open.

The SQG equation takes the form
\begin{equation}
\theta_t + u \cdot \nabla \theta = 0, \quad u = (-\Delta)^{-\frac{1}{2}} \nabla^\perp \theta,
\end{equation}
where $\theta$ is a scalar quantity, such as a temperature, which is advected by a velocity field $u$. Here, $(-\Delta)^{-\frac{1}{2}}$ denotes a fractional Laplacian, and $\nabla^\perp = (-\partial_y, \partial_x)$. The SQG equation is one member in a family of two-dimensional active scalar equations parameterized by the transport term in (1.1), with
\begin{equation}
u = (-\Delta)^{-1+\frac{\alpha}{2}} \nabla^\perp \theta, \quad \alpha \in (0, 2].
\end{equation}
This is a natural generalization of the incompressible Euler equation, which corresponds to the case $\alpha = 2$, while the $\alpha = 1$ case gives the SQG equation (1.1) above. For additional analysis of the SQG equation, see Resnick [20].

Recall that one significant motivation for studying the SQG equation is that it admits solutions which serve as a model for frontogenesis, or the formation in finite time of a discontinuous temperature front, e.g. between masses of hot and cold air. In particular, the SQG equation admits piecewise constant solutions known as front or patch solutions. Front solutions refer to piecewise constant solutions $\theta$ of (1.1) taking the form
\begin{equation}
\theta(t, x, y) = \begin{cases} 
\theta_+ & \text{if } y > \varphi(t, x), \\
\theta_- & \text{if } y < \varphi(t, x),
\end{cases}
\end{equation}
where the front or interface between regions of constant temperature $\theta_+$ and $\theta_-$ is modeled by the graph $y = \varphi(t, x)$ with $x \in \mathbb{R}$. Front solutions are closely related with patch solutions,
\begin{equation}
\theta(t, x, y) = \begin{cases} 
\theta_+ & \text{if } (x, y) \in \Omega(t), \\
\theta_- & \text{if } (x, y) \notin \Omega(t),
\end{cases}
\end{equation}
where $\Omega$ is a bounded, simply connected domain.

In the SQG case $\alpha = 1$, the equation for the front $\varphi$ of (1.3) is derived in [11, 13] and takes the form
\begin{equation}
\partial_t \varphi(t, x) - A_\varphi \varphi_x(t, x) = 2 \log |D_x| \partial_t \varphi(t, x),
\varphi(0, x) = \varphi_0(x),
\end{equation}
where $\varphi$ is a real-valued function $\varphi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and
\begin{equation}
A_\varphi \varphi_x(t, x) = \int \left( \frac{1}{|y|} - \frac{1}{\sqrt{y^2 + (\varphi(t, x + y) - \varphi(t, x))^2}} \right)
\cdot (\varphi_x(t, x + y) - \varphi_x(t, x)) \, dy.
\end{equation}
We briefly describe one of the approaches used in [13] to derive the equation (1.4) from (1.1). One first writes the Biot–Savart law \( u = (\Delta)^{-\frac{1}{2}} \nabla \cdot \theta \) as a convolution between \( \nabla \cdot \theta \) and the corresponding Green’s function. Afterwards, one applies Green’s Theorem to the previous convolution, and then imposes the kinematic boundary condition, which asserts that the boundary moves upwards with a velocity equal to the normal component of \( u \).

We remark that this model does not apply to the case of breaking fronts. We also note that the equation (1.4) is invariant under the transformation

\[
t \to \kappa t, \quad x \to \kappa \left( x + \log |x| \right), \quad \varphi \to \kappa \varphi,
\]

which means that \( H^2(\mathbb{R}) \) is the corresponding critical Sobolev space.

When \( \alpha \in (1,2) \), contour dynamics equations for patches and fronts may be derived and analyzed in a similar way as in the SQG case \( \alpha = 1 \). Global well-posedness for small and localized data was established by Córdoba–Gómez-Serrano–Ionescu in [4]. However, when \( \alpha \in [0,1] \), the derivation of contour dynamics equations for fronts has complexities arising from the slow decay of Green’s functions. The derivation in this range was addressed by Hunter–Shu [11] via a regularization procedure, and again by Hunter–Shu–Zhang in [13].

In the case of SQG patches, the first local well-posedness results were obtained by Rodrigo [21] for initial data in \( C^\infty \). Gancedo–Nguyen–Patel proved in [6] that in a suitable parametrization, the SQG front equation is locally well-posed in \( H^s(\mathbb{T}) \), where \( s > 2 \). For the generalized SQG family, the first local well-posedness results were obtained by Gancedo [5] in the case \( \alpha \in (0,1] \), proving that the problem is locally well-posed in \( H^s(\mathbb{T}) \). Later on, Gancedo–Patel [7] studied the case \( \alpha \in (0,2) \) with \( \alpha \not= 1 \), establishing in particular local well-posedness in \( H^2 \) for \( \alpha \in (0,1) \) and in \( H^3 \) for \( \alpha \in (1,2) \).

In the case of nonperiodic SQG fronts, Hunter–Shu–Zhang studied the local well-posedness for a cubic approximation of (1.4) in [12]. In [15], they considered the local well-posedness for the full equation (1.4) with initial data in \( H^s \), \( s \geq 5 \), along with global well-posedness for small, localized, and essentially smooth (\( s \geq 1200 \)) initial data. However, these well-posedness results require a small data assumption to ensure the coercivity of the modified energies used in the energy estimates, along with a convergence condition on an expansion of the nonlinearity \( A_\varphi \varphi_x \) appearing in (1.4). These results were extended to the range \( \alpha \in (1,2] \) for the generalized SQG family in [14].

In the present article, our objective is to revisit and streamline the analysis of (1.4), while improving the established well-posedness results. Our contributions include:

- Offering substantial simplifications to the paradifferential analysis, which in a nutshell, reinterpret the nonlinear equation as an essentially linear one, by modelling it up to perturbative errors as the evolution of rough waves on a smoother background (for more details, see sections 2 and 3),
- Removing the convergence and small data assumptions in the local well-posedness result of [15],
- Establishing the local well-posedness in a significantly lower regularity setting at \( 1 + \epsilon \) derivatives above scaling, corresponding to the classical threshold of Hughes–Kato–Marsden for nonlinear hyperbolic systems [10], and
- Establishing the global well-posedness in a low regularity setting, by applying the wave packet testing method of Ifrim–Tataru (see for instance [16, 18]).
We anticipate that our streamlined analysis will also open the way to substantial simplifications and improvements in the analysis of related equations, including the generalized SQG family (1.2).

Our main local well-posedness result is as follows:

**Theorem 1.1.** Equation (1.4) is locally well-posed for initial data in \( H^s \) with \( s > \frac{5}{2} \). Precisely, for every \( R > 0 \), there exists \( T = T(R) > 0 \) such that for any \( \varphi_0 \in H^s(\mathbb{R}) \) with \( \| \varphi_0 \|_{H^s} < R \), the Cauchy problem (1.4) has a unique solution \( \varphi \in C([0,T],H^s) \). Moreover, the solution map \( \varphi_0 \mapsto \varphi \) from \( H^s \) to \( C([0,T],H^s) \) is continuous.

**Remark 1.2.** Theorem 1.1 implies via Sobolev embeddings that initial data with bounded curvature will generate solutions having the same property. Lowering the regularity threshold also shows that rougher fronts can be propagated in time, even for large initial data.

We also consider global well-posedness for small and localized data. To describe localized solutions, we define the operator

\[
L = x + 2t + 2r \log |D_x|,
\]

which commutes with the linear flow \( \partial_t - 2\log |D_x| \partial_x \), and at time \( t = 0 \) is simply multiplication by \( x \). Then we define the time-dependent weighted energy space

\[
\| \varphi \|_X := \| \varphi \|_{H^s} + \| L\partial_x \varphi \|_{L^2},
\]

where \( s > 4 \). To track the dispersive decay of solutions, we define the pointwise control norm

\[
\| \varphi \|_Y := \| |D_x|^{3/4-\delta} \varphi \|_{L^\infty} + \| |D_x|^{1+\delta} \varphi \|_{L^\infty}.
\]

**Theorem 1.3.** Consider data \( \varphi_0 \) with

\[
\| \varphi_0 \|_X \lesssim \epsilon \ll 1.
\]

Then the solution \( \varphi \) to (1.4) with initial data \( \varphi_0 \) exists globally in time, with energy bounds

\[
\| \varphi(t) \|_X \lesssim \epsilon e^{C\epsilon^2}
\]

and pointwise bounds

\[
\| \varphi(t) \|_Y \lesssim \epsilon(t)^{-\frac{1}{2}}.
\]

**Remark 1.4.** In particular, this shows that as long as the initial data is small and localized, the corresponding fronts do not break.

Further, the solution \( \varphi \) exhibits a modified scattering behavior, with an asymptotic profile \( W \), in a sense that will be made precise in section 7. There, we also observe that (1.4) has a conserved \( L^2 \) mass. We remark that Hunter–Shu–Zhang [14] makes a similar observation regarding mass conservation for the generalized SQG in the range \( \alpha \in (1,2] \).

Regarding the question of large data global well-posedness, the absence of splash singularities in the case of patches was first proved by Gancedo–Strain [8]. Recently, Kiselev–Luo [19] sharpened the criterion ruling out splash singularities. More precisely, they improved the double exponential bound of Gancedo–Strain [8] to just exponential in time.

The question of extending our results to the non-graph case is interesting and open. Our current approach does not provide an answer, as a different parametrization would be needed.
Our paper is organized as follows. In section 2, we establish notation and preliminaries used through the rest of the paper. We define a parameter-dependent paradifferential quantization which will ensure coercivity in our energy estimates, and which is key in removing the small data assumption in the local well-posedness theory. We also establish Moser estimates to be applied toward the paralinearization of $A \phi$, as well as a more general linearized counterpart $A \phi v$. Lastly, we record some elementary lemmas involving difference quotients.

In section 3, we paralinearize the operator $A \phi v$, up to a perturbative error. We then apply this result to reduce the analysis of both the equation (1.4) and its linearization

$$\partial_t v - \partial_x A \phi v = 2 \log |D_x| \partial_x v$$

(1.6)

to the analysis of a paradifferential flow with perturbative source.

In section 4, we prove energy estimates for the paradifferential flow of section 3, by defining modified energies that are comparable to the classical Sobolev energies in the spaces $H^s(\mathbb{R})$. This part crucially uses the quantization established in section 2 to remove the small data assumption made in the local well-posedness result from [15].

In section 5, we prove theorem 1.1, the local well-posedness result for (1.4). We do this by first using an iterative scheme to construct smooth solutions, and then using the method of frequency envelopes to lower the regularity exponent for the solutions. This method was introduced by Tao in [23] in order to better track the evolution of energy distribution between dyadic frequencies. A systematic presentation of the use of frequency envelopes in the study of local well-posedness theory for quasilinear problems can be found in the expository paper [17].

In section 6 we use the wave packet testing method of Ifrim–Tataru to prove the global-wellposedness part of theorem 1.3, along with the dispersive bounds for the resulting solution. This method is systematically presented in [18].

Finally, in section 7 we provide a short proof for the mass conservation for the solutions of (1.4). Then we discuss the modified scattering behavior of the global solutions constructed in section 6.

2. Notation and preliminaries

To facilitate the analysis of the operator $A \phi$, we define the smooth function

$$F(s) = 1 - \frac{1}{\sqrt{1+s^2}}$$

(2.1)

which in particular vanishes to second order at $s = 0$, satisfying $F(0) = F'(0) = 0$. Using this notation, and recalling the form of the nonlinearity $A \phi \phi x$ in (1.4), we define the operator $A \phi$ by

$$(A \phi v)(t,x) = \int F(\delta^s \phi(t,x)) \cdot |\delta|^s v(t,x) \, dy.$$  

(2.2)

This generalized operator will be useful when we study the linearized equation.

We let $0 < \delta < s - \frac{5}{2}$ refer to a small positive exponent throughout. Implicit constants may depend on our choice of $\delta$. 

5
2.1. Paradifferential operators and paraproducts

Let \( \chi \) be an even smooth function such that \( \chi = 1 \) on \( [-\frac{1}{20}, \frac{1}{20}] \) and \( \chi = 0 \) outside \( [-\frac{1}{10}, \frac{1}{10}] \), \( M \in \mathbb{R} \) with \( M > 1 \), and define

\[
\tilde{\chi}(\theta_1, \theta_2) = \chi \left( \frac{\|\theta_1\|^2}{M^2 + \|\theta_2\|^2} \right).
\]

Given a symbol \( a(x, \eta) \), we use the above cutoff symbol \( \tilde{\chi} \) to define an \( M \) dependent paradifferential quantization of \( a \) by (see also [2])

\[
\tilde{T}_a u(\xi) = (2\pi)^{-1} \int \hat{P}_{>M}(\xi) \hat{\chi}(\xi - \eta, \xi + \eta) \hat{a}(\xi - \eta, \eta) \hat{P}_{>M}(\eta) \hat{u}(\eta) \, d\eta,
\]

where the Fourier transform of the symbol \( a = a(x, \eta) \) is taken with respect to the first argument. We make the following remarks:

- The Fourier transform, and in particular \( T_a \) and Littlewood–Paley projections, will all operate with respect to the \( x \) variable throughout.
- On the Fourier side, the support conditions on \( \hat{P}_{>M} \) and \( \hat{\chi} \) imply that \( |\eta| \approx |\xi| \).
- When \( a \) is independent of \( \eta \), the above definition coincides with the Weyl quantization, which means that the operator \( T_a \) will be self-adjoint if \( a \) is real valued. This will be useful when proving energy estimates in section 4.

Implicit constants may depend on \( M > 1 \), which we view as a constant parameter except in section 5. There, we will choose \( M \) using the following lemma, which will be key for establishing local well-posedness for large data:

**Lemma 2.1.** Let \( R > 0 \), \( r > 1 \), and \( s > \frac{1}{2} \). There exists \( M \) such that \( \|T_1 - (1 - F(u))^{r} \|_{L^2 \to L^2} < 1 \) for any \( u \) such that \( \|u\|_{M^r} \leq R \).

**Proof.** For \( |\xi - \eta| \leq \frac{M}{\sqrt{20}} \), we have \( \hat{\chi}(\xi - \eta, \xi + \eta) = 1 \). Thus we may write

\[
2\pi \cdot \tilde{T}_a f(\xi) = \int_{|\xi - \eta| \leq \frac{M}{\sqrt{20}}} \hat{P}_{>M}(\xi) \hat{a}(\xi - \eta) \hat{P}_{>M}(\eta) \hat{f}(\eta) \, d\eta
+ \int_{|\xi - \eta| > \frac{M}{\sqrt{20}}} \hat{P}_{>M}(\xi) \hat{\chi}(\xi - \eta, \xi + \eta) \hat{a}(\xi - \eta) \hat{P}_{>M}(\eta) \hat{f}(\eta) \, d\eta
\]

and conclude

\[
\|T_a f\|_{L^2} \leq \left( \|P_{\frac{M}{\sqrt{20}}} a\|_{L^\infty} + C \|P_{\frac{M}{\sqrt{20}}} a\|_{L^\infty} \right) \|f\|_{L^2}.
\]

We have by Sobolev embedding

\[
\|P_{\frac{M}{\sqrt{20}}} a\|_{L^\infty} \lesssim M^{-\frac{5}{4}} \|a\|_{M^r}
\]

so that for \( a = 1 - (1 - F(u))^r \) with \( \|u\|_{M^r} \leq R \), we have

\[
\|P_{\frac{M}{\sqrt{20}}} a\|_{L^\infty} \lesssim_R M^{-\frac{5}{4}}.
\]
Further, using the form of $F$ combined with Sobolev embedding, we have $\|a\|_{L^\infty} < 1 - \delta$ uniformly over $\|u\|_{H} \leq R$, for some $\delta > 0$ depending on $R$. Choosing $M$ sufficiently large depending on $R$, we obtain
\[
\|T_{1-(1-F(a))}^\gamma\|_{L^2 \rightarrow L^2} \leq \|1-(1-F(a))^\gamma\|_{L^\infty} + \frac{\delta}{2} < 1 - \frac{\delta}{2}.
\]

2.2. Classical estimates

We recall the following Moser–Schauder estimates. Note that in the present article, we typically take $F$ to be given by (2.1).

**Theorem 2.2 (Moser–Schauder).** Let $s \geq 0$ and $F : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function satisfying $F(0) = 0$.

(a) If $f \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

\[
\|F(f)\|_{H^s} \lesssim_{F,\|f\|_{L^\infty}} \|f\|_{H^s}. \tag{2.3}
\]

(b) If $f$ is integrable and $\partial_x^{-1} f \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$, where

\[
\partial_x^{-1} f(x) = \int_{-\infty}^{x} f(y) \, dy,
\]

and $F'(0) = 0$, we have

\[
\|F(f) - T_{F'(f)} f\|_{H^s} \lesssim_{F,\|f\|_{W^{1,\infty}}} \|f\|_{W^{1,\infty}} \|\partial_x^{-1} f\|_{H^s},
\]

\[
\|F(f) - T_{F'(f)} f\|_{H^s} \lesssim_{F,\|f\|_{W^{1,\infty}}} \|f\|_{W^{1,\infty}} \|\partial_x^{-1} f\|_{H^s} \tag{2.4}
\]

(c) Under the hypotheses of b), in the case $s = 0$, we have

\[
\|F(f) - T_{F'(f)} f\|_{L^2} \lesssim_{F} \|f\|_{W^{1,\infty}} \|\partial_x^{-1} f\|_{L^2}. \tag{2.5}
\]

**Proof.** For a proof of a), see [24, lemma A.9]. Here, we prove parts b) and c). In the following, denote

\[
f_{\leq k} := f_{\leq k-1} + hf_k.
\]

Then write

\[
F(f) = \sum_{k = -\infty}^{\infty} \int_{0}^{1} F'(f_{\leq k}) \cdot f_k \, dh
\]

\[
= \int_{0}^{1} \sum_{k = -\infty}^{\infty} \left( F'(f_{\leq k})_{<k-4} + F'(f_{\leq k})_{\geq k-4} \right) \cdot f_k \, dh
\]

\[
:= \int_{0}^{1} B_1 + B_2 \, dh.
\]
We consider $B_2$. Write

$$
\|B_2\|_{H^p}^2 \lesssim \sum_{k=-\infty}^{\infty} \left\| \sum_{j=-4}^{\infty} F' (f_{\leq 4k-j}) \cdot \hat{f}_{k-j} \right\|_{H^p}^2.
$$

We estimate the summand by

$$
\left\| \sum_{j=-4}^{\infty} F' (f_{\leq 4k-j}) \cdot \hat{f}_{k-j} \right\|_{H^p} \lesssim \sum_{j=-4}^{\infty} \left\| D_4^{j} F' (f_{\leq 4k-j}) \right\|_{L^\infty} \| \hat{f}_{k-j} \|_{L^2} 
$$

$$
\lesssim \sum_{j=-4}^{\infty} 2^{4j} \left\| (F'' (f_{\leq 4k-j}) \cdot \partial f_{\leq 4k-j}) \right\|_{L^\infty} \left\| \partial_{x}^{-1} \hat{f}_{k-j} \right\|_{L^2}.
$$

By the chain rule, for each $j \geq -4$,

$$
\left\| (F'' (f_{\leq 4k-j}) \cdot \partial_x f_{\leq 4k-j}) \right\|_{L^\infty} \lesssim 2^{-Nj} K(||f||_{L^\infty}, ||f_x||_{L^\infty}).
$$

Thus,

$$
\left\| \sum_{j=-4}^{\infty} F' (f_{\leq 4k-j}) \cdot \hat{f}_{k-j} \right\|_{H^p} \lesssim K(||f||_{L^\infty}, ||f_x||_{L^\infty}) \sum_{j=-4}^{\infty} 2^{(s-Nj)} \left\| \partial_{x}^{-1} \hat{f}_{k-j} \right\|_{H^p}.
$$

In this case of the frequencies $k \geq 0$ in $B_2$,

$$
\sum_{k=0}^{\infty} \left\| \sum_{j=-4}^{\infty} F' (f_{\leq 4k-j}) \cdot \hat{f}_{k-j} \right\|_{H^p}^2 \lesssim K(||f||_{L^\infty}, ||f_x||_{L^\infty})^2 \sum_{k=0}^{\infty} \left( \sum_{j=-4}^{\infty} 2^{(s-Nj)} \left\| \partial_{x}^{-1} \hat{f}_{k-j} \right\|_{H^p} \right)^2
$$

$$
\lesssim K(||f||_{L^\infty}, ||f_x||_{L^\infty})^2 \sum_{k=0}^{\infty} \left\| \partial_{x}^{-1} \hat{f}_k \right\|_{H^p}^2 
$$

$$
\lesssim K(||f||_{L^\infty}, ||f_x||_{L^\infty})^2 ||\partial_{x}^{-1} f||_{H^p}^2
$$

where $N$ has been chosen such that $N + 1 > s$. In particular, when $s = 0$, we may choose $N = 0$ to obtain

$$
\sum_{k=0}^{\infty} \left\| \sum_{j=-4}^{\infty} F' (f_{\leq 4k-j}) \cdot \hat{f}_{k-j} \right\|_{L^2}^2 \lesssim ||f_x||_{L^\infty}^2 ||\partial_{x}^{-1} f||_{L^2}^2.
$$
For the low frequency terms \( k \leq 0 \) in \( B_2 \), using the fact that \( F'(0) = 0 \),

\[
\sum_{k=-\infty}^{-1} \left\| \sum_{j=-\infty}^{\infty} F' (f_{\leq k-j})_k e^{-j} \right\|_{L^2}^2 \lesssim \sum_{k=-\infty}^{-1} \left\| \sum_{j=-4}^{\infty} F' (f_{\leq k-j})_k e^{-j} \right\|_{L^2}^2 \lesssim \sum_{k=-\infty}^{-1} \left( \sum_{j=-4}^{\infty} \| F' (f_{\leq k-j})_k \|_{L^\infty} \| e^{-j} \|_{L^2} \right)^2 \lesssim \sum_{k=-\infty}^{-1} \left( \sum_{j=-4}^{\infty} 2^{k-j} \| f \|_{L^\infty} \| \partial_x^{-1} f \|_{L^2} \right)^2 \lesssim \| f \|_{L^2}^2 \| \partial_x^{-1} f \|_{L^2}^2,
\]

We conclude in the case \( s > 0 \) that

\[
\| B_2 \|_{L^2} \lesssim K (\| f \|_{L^\infty} \| f_x \|_{L^\infty}) \| \partial_x^{-1} f \|_{L^2} + \| f \|_{L^\infty} \| \partial_x^{-1} f \|_{L^2} \lesssim K (\| f \|_{L^\infty} \| f_x \|_{L^\infty}) \| \partial_x^{-1} f \|_{L^2},
\]

and

\[
\| B_2 \|_{L^2} \lesssim \| f \|_{W^1_\infty} \| \partial_x^{-1} f \|_{L^2}
\]

when \( s = 0 \).

We now exchange \( B_1 \) for

\[
\sum_{k=-\infty}^{\infty} F' (f_{<k-4})_k f_k.
\]

The summand of the difference

\[
\sum_{k=-\infty}^{\infty} (F' (f) - F' (f_{<k}))_k f_k
\]

can be estimated by

\[
\left\| 2^k (F' (f) - F' (f_{<k}))_k \cdot 2^{-k} f_k \right\|_{L^2} \lesssim \left\| 2^k \int_0^1 \left( F'' (f_{<k} + \theta ((1-h)f_k + f_{>k})) \cdot ((1-h)f_k + f_{>k}) \right) d\theta \right\|_{L^\infty} \| \partial_x^{-1} f_k \|_{L^2} \lesssim \left( \int_0^1 \| f_k \|_{L^\infty} d\theta \right) \| \partial_x^{-1} f_k \|_{L^2} \lesssim \| f_k \|_{L^\infty} \| \partial_x^{-1} f_k \|_{L^2}.
\]

So that summing orthogonally, we have the desired bound.

It remains to exchange \( 2.6 \) with \( T_{F' (f)} f \). Write

\[
F' (f) \cdot f_k = T_{F' (f)} \cdot f_k + T_f F' (f) \cdot f_k + \Pi (F' (f) \cdot f_k).
\]

The second term contributes

\[
\| T_f F' (f) \cdot f_k \|_{L^2} \lesssim \| f \|_{L^\infty} \| \partial_x^{-1} f_k \|_{L^2}
\]
which sums orthogonally over $k$. The third term is estimated similarly. For the first, using cancellation with $T^{-1}_k\hat{f}_k'$, it remains to estimate
\[
\sum_{k=-\infty}^{\infty} T_{F'}(f)_{2k-k} \hat{f}_k,
\]
which can be done by noting that
\[
\left\| T_{F'}(f)_{2k-k} \hat{f}_k \right\|_{L^p} \lesssim \left\| (F'' f)_{2k-k} \hat{f}_k \right\|_{L^\infty} \left\| \partial_x^{-1} f_k \right\|_{H^s} \lesssim \left\| \partial_x f \right\|_{L^\infty} \left\| \partial_x^{-1} f_k \right\|_{H^s}
\]
which is estimated in a similar way as in the frequency-balanced case.

We will use the following logarithmic commutator estimate:

**Lemma 2.3.** We have for $s \geq 0$,
\[
\left\| \partial_x \left[ T_f, \log |\partial_x| \right] g \right\|_{H^s} \lesssim \left\| f \right\|_{W^{1,\infty}} \left\| g \right\|_{H^s}.
\]

**Proof.** We have
\[
\mathcal{F} \left( \partial_x \left[ T_f, \log |\partial_x| \right] g \right)(\xi) = \int_{\mathbb{R}} \hat{P}_{>M} \left( \xi - \eta, \xi + \eta \right) \hat{f}(\xi - \eta) \hat{g}(\eta) i \eta \log |\eta| - \log |\xi| \, d\eta
\]
\[
= - \int_{\mathbb{R}} \hat{P}_{>M} \left( \xi - \eta, \xi + \eta \right) \hat{f}(\xi - \eta) \hat{g}(\eta) \frac{\log \left| 1 + \frac{\eta - \xi}{\xi} \right|}{\frac{\eta - \xi}{\xi}} \, d\eta.
\]

Observe that $\log \left| 1 + \frac{\eta - \xi}{\xi} \right|$ is bounded away from $x = -1$. Thus, setting
\[
\tilde{\chi}_1(\xi - \eta, \xi + \eta) = \hat{\chi}(\xi - \eta, \xi + \eta) \hat{P}_{>M}(\eta) \frac{\log \left| 1 + \frac{\eta - \xi}{\xi} \right|}{\frac{\eta - \xi}{\xi}},
\]
and observing that $(\eta - \xi)/\xi$ remains away from $-1$ on the support of $\tilde{\chi}$ and $\hat{P}_{>M}$, where $|\eta| \geq M$ and $|\xi| \lesssim |\eta|$, we see that $\tilde{\chi}_1$ is bounded. Further, since $\tilde{\chi}_1$ satisfies the same kind of bounds as $\tilde{\chi}$, we can apply paraproduct Coifman–Meyer estimates to deduce the desired result.

We also recall the following variant of [1, Lemma 2.5].

**Lemma 2.4 (Para-products).** Assume that $\gamma_1, \gamma_2 < 1$, $\gamma_1 + \gamma_2 \geq 0$. Then
\[
\left\| T_f T_g - T_{fg} \right\|_{L^p} \lesssim \left\| D^{\gamma_1} f \right\|_{BMO} \left\| D^{\gamma_2} g \right\|_{BMO}. \tag{2.7}
\]

When $\gamma_2 = 1$,
\[
\left\| T_f T_g - T_{fg} \right\|_{L^p} \lesssim \left\| D^{\gamma_1} f \right\|_{BMO} \left\| g \right\|_{L^\infty}. \tag{2.8}
\]
2.3. Difference quotients

We denote difference quotients by
\[ \delta^y h(x) = \frac{h(x + y) - h(x)}{y}, \quad |\delta^y h(x)| = \frac{|h(x + y) - h(x)|}{|y|} \]
and the distribution \( \text{p.v.}|y|^{-1} \) by
\[ \langle \text{p.v.}|y|^{-1}, f \rangle = \int_{|y|>1} \frac{f(y)}{|y|} \, dy + \int_{|y|<1} \frac{f(y) - f(0)}{|y|} \, dy. \]

Equivalently,
\[ \langle \text{p.v.}|y|^{-1}, f \rangle = \lim_{\epsilon \to 0} \left( \int_{|y|>\epsilon} \frac{f(y)}{|y|} \, dy + 2 \log(\epsilon) f(0) \right). \]

We also use the following estimates which we will apply to averages over difference quotients:

**Lemma 2.5.** We have
\[ \left\| \int \frac{1}{|y|} f(x, y) \, dy \right\|_{H^s} \lesssim \sup_{|y|>1} \|y|^s f\|_{H^s} + \sup_{|y|<1} \|y|^{-s} f\|_{H^s} \]
and
\[ \left\| \int \frac{1}{|y|} f(x, y) \, dy \right\|_{L^\infty_x} \lesssim \sup_{|y|>1} \|y|^s f\|_{L^\infty} + \sup_{|y|<1} \|y|^{-s} f\|_{L^\infty}. \]

**Proof.** We decompose the integral
\[ \int \frac{1}{|y|} f(x, y) \, dy = \int_{|y|>1} + \int_{|y|<1} \]
and estimate
\[ \left\| \int_{|y|>1} \frac{1}{|y|} f(x, y) \, dy \right\|_{H^s} \lesssim \sup_{|y|>1} \|y|^s f\|_{H^s}, \quad \left\| \int_{|y|<1} \frac{1}{|y|} f(x, y) \, dy \right\|_{H^s} \lesssim \sup_{|y|<1} \|y|^{-s} f\|_{H^s} \]
with similar estimates for \( L^\infty \).

**Lemma 2.6.** Let \( i = 1, n \) and \( p_i, r \in [1, \infty] \) and \( \alpha_i, \beta_i \in [0, 1] \) satisfying
\[ \sum_i \frac{1}{p_i} = \frac{1}{r}, \quad n - 1 < \sum_i \alpha_i \leq n, \quad 0 \leq \sum_i \beta_i < n - 1. \]

Then
\[ \left\| \int \text{sgn}(y) \prod \delta^{\alpha_i} f_i \, dy \right\|_{L^r_y} \lesssim \prod \|D^{\alpha_i} f_i\|_{L^p}\prod \|D^{\beta_i} f_i\|_{L^q}. \]
Proof. We write
\[ \int \text{sgn}(y) \prod \delta^i f_i \, dy = \int_{|y| \leq 1} + \int_{|y| > 1}. \]
For the former integral, we have by Hölder
\[ \left\| \int_{|y| \leq 1} \text{sgn}(y) \prod \delta^i f_i \, dy \right\|_{L^2} \lesssim \int_{|y| \leq 1} \frac{1}{|y|^{\alpha - \sum} \sum \|D|^{\alpha} f_i \|_{L^p} \, dy \lesssim \prod \|D|^{\alpha} f_i \|_{L^p}. \]
The latter integral is treated similarly. \[ \square \]

3. Paralinearizations and the linearized equation

The objective of this section is to paralinearize the operator \( A_{\varphi, v} \) defined in (2.2). With this paralinearization, we reduce the analysis of (1.4) and its linearization to the analysis of a paradifferential flow with perturbative source.

The analysis in this section is time independent. We will use the notation \( B^0 \) to denote the zeroth order coefficient of the paralinearization,
\[ B^0(\varphi) = \left\langle p.v., \frac{1}{y}, F(\delta^i \varphi) \right\rangle - F(\varphi_1) \left\langle p.v., \frac{1}{y}, e^{-iy} \right\rangle. \]

Proposition 3.1. We have the paralinearization
\[ A_{\varphi, v} = T_{B^0(\varphi)} v + 2T_{F(\varphi_1)} \log |D_\varphi| v + \mathcal{R}(\varphi, v) \]
where
\[ \| \partial_\varphi \mathcal{R}(\varphi, v) \|_{L^2} \lesssim \| \varphi \|_{L^2} \| |D_\varphi|^{-\delta} \varphi \|_{L^2} \| v \|_{L^2}. \]

Proof. We decompose
\[ A_{\varphi, v} = \int T_{\text{sgn}}(\varphi) \delta^i F(\delta^i \varphi) \, dy + \int \Pi(F(\delta^i \varphi), |\delta^i v|) \, dy + \int T_{F(\delta^i \varphi)} |\delta^i v| \, dy \]
\[ := A_1 + A_2 + A_3. \]

We estimate \( A_1 \) using lemma 2.5,
\[ \| \partial_\varphi A_1 \|_{L^2} \lesssim \sup_y (y^\delta \partial_{\text{sgn}}(\varphi) F(\delta^i \varphi) \|_{L^2} + y^{-\delta} \partial_{\text{sgn}}(\varphi) F(\delta^i \varphi) \|_{L^2}). \]

For the first term, we have
\[ y^\delta \partial_{\text{sgn}}(\varphi) F(\delta^i \varphi) \|_{L^2} \lesssim \| \partial_{\text{sgn}}(\varphi) \|_{L^2} \| F(\delta^i \varphi) \|_{L^2} \lesssim \| \partial_{\text{sgn}}(\varphi) \|_{L^2} \| F(\delta^i \varphi) \|_{L^2}. \]

For the second term, we have
\[ y^{\delta - \delta} \partial_{\text{sgn}}(\varphi) F(\delta^i \varphi) \|_{L^2} \lesssim |D|^{-\delta} \| F(\delta^i \varphi) \|_{L^2} \lesssim \| v \|_{L^2} \| \partial_{\text{sgn}}(\varphi) \|_{L^2}. \]

So that the contribution from \( A_1 \) may be absorbed into \( \mathcal{R}(\varphi, v) \). A similar analysis applies to absorb \( A_2 \) into \( \mathcal{R}(\varphi, v) \).
We proceed with $A_3$, which we may write as

$$A_3 = - \int T_{F(\delta^y \varphi) \frac{1-ey}{|y|}} v \, dy = - T_{F(\delta^y \varphi) \frac{1-ey}{|y|}} v.$$  

Since $F(\delta^y \varphi)(1 - e^{iy})$ vanishes at $y = 0$ and $F(\delta^y \varphi)$ satisfies the decay

$$|F(\delta^y \varphi)| \lesssim \| \varphi \|_{L^\infty} |y|^{-2}$$

in $y$, we may express the symbol of the paradifferential operator in terms of $p.v.|y|^{-1}$:

$$\int F(\delta^y \varphi) \frac{1-e^{iy}}{|y|} \, dy = \langle p.v.|y|^{-1}, F(\delta^y \varphi) (1 - e^{-iy}) \rangle.$$  

The inverse Fourier transform of $p.v.|y|^{-1}$ takes the form

$$\langle p.v.|y|^{-1}, e^{-iy} \rangle = \lim_{\epsilon \to 0} \left( \int_{|y| > \epsilon} \frac{e^{iy}}{|y|} \, dy + 2\log(\epsilon) \right)$$

$$= \lim_{\epsilon \to 0} \left( \int_{|z| > |\epsilon|} \frac{e^{iz}}{|z|} \, dz + 2\log(|\epsilon|) \right) - 2\log|\epsilon|$$

$$= \langle p.v.|y|^{-1}, e^{-iy} \rangle - 2\log|\epsilon|.$$  

Using this, we write

$$\langle p.v.|y|^{-1}, F(\delta^y \varphi) (1 - e^{-iy}) \rangle = \langle p.v.|y|^{-1}, F(\delta^y \varphi) - F(\varphi_y) e^{-iy} \rangle$$

$$+ \langle p.v.|y|^{-1}, (F(\varphi_y) - F(\delta^y \varphi)) e^{-iy} \rangle$$

$$= B^y(\varphi) + 2F(\varphi_y) \log|\eta|$$

$$+ \langle p.v.|y|^{-1}, (F(\varphi_y) - F(\delta^y \varphi)) e^{-iy} \rangle.$$  

(3.1)

Since the first two terms on the right hand side of (3.1) form the main terms of the parilinearization of $A_3$, it remains to absorb the contribution of the last term in (3.1) into $\mathcal{R}$. From lemma 6.2 in [22], we have

$$\partial_x T_{p.v.|y|^{-1}.(F(\varphi_y) - F(\delta^y \varphi))e^{-iy}} v = T'_{f(\varphi_y) - f(\delta^y \varphi) e^{iy}} v,$$

(3.2)

where $T'$ is another low-high paraproduct.

Integrating the paradifferential symbol by parts,

$$\int \frac{F(\varphi_x) - F(\delta^y \varphi)}{|y|} e^{iy} i\eta \, dy$$

$$= \int_0^\infty \frac{F(\varphi_x) - F(\delta^y \varphi)}{y} \partial_y (e^{iy}) \, dy - \int_{-\infty}^0 \frac{F(\varphi_x) - F(\delta^y \varphi)}{y} \partial_y (e^{iy}) \, dy$$

$$= F'(\varphi_y) \varphi_{xx} - \int \text{sgn}(y) \cdot \partial_y \left( \frac{F(\varphi_x) - F(\delta^y \varphi)}{y} \right) e^{iy} \, dy$$

(3.3)

$$=: F'(\varphi_y) \varphi_{xx} - \int G(x,y) e^{iy} \, dy.$$
It is immediate to see that the contribution from the first term on the right in (3.3) may be absorbed into $\mathcal{R}$. It remains to consider the contribution from the second term, which we write

$$-T'_1 G(x, y) \omega_{\phi} \psi = - \int T'_1 G(x, y) \psi(y) \, dy$$

so that

$$\left\| \int T'_1 G(x, y) \psi(y) \, dy \right\| \leq \int \|G(\cdot, y)\|_{L^\infty} \, dy \left\| \psi \right\|_{L^2}$$

$$\leq \left( \int_{|y| \leq 1} \|G(\cdot, y)\|_{L^\infty} \, dy + \int_{|y| > 1} \|G(\cdot, y)\|_{L^\infty} \, dy \right) \|\psi\|_{L^2}.$$  

Since

$$G(x, y) = \text{sgn}(y) \frac{-y \cdot \partial_y F(\frac{\delta_y \varphi}{y^2}) + F(\frac{\delta_y \varphi}{y}) - F(\varphi)}{y^2}$$

and

$$\left| \partial_y F(\frac{\delta_y \varphi}{y}) \right| \leq \|\varphi\|_{L^\infty} \frac{1}{y^2},$$

we have

$$\|G(\cdot, y)\|_{L^\infty} \leq \|\varphi\|_{L^\infty} \frac{1}{y^2}.$$  

So that the integral over $\{|y| > 1\}$ converges with the appropriate bound. On the other hand, for the integral over $\{|y| \leq 1\}$, we use instead

$$|y|^{1-\delta} \|G(\cdot, y)\|_{L^\infty} \leq \left\| \frac{1}{|y|^{\delta}} \left( \partial_y F(\frac{\delta_y \varphi}{y}) - \frac{F(\delta_y \varphi) - F(\varphi)}{y} \right) \right\|_{L^\infty}$$

$$\leq \|F(\delta_y \varphi)\|_{L^\infty} C_{\delta, t} \leq \|\partial_y \varphi\|_{C^1_t} C_{\delta, t},$$

which also suffices.  

Next, we paralinearize the nonlinear term $A_\varphi \varphi$ in (1.4), evaluating the $H^s$ higher regularity of the errors:

**Proposition 3.2.** We have the paralinearization

$$A_\varphi \varphi_x = \partial_y T_{F(\varphi)} \varphi - 2\partial_y T_{F(\varphi_x)} \log |D_x| \varphi + \mathcal{R}(\varphi),$$

where for any $s \geq 0$,

$$\|\mathcal{R}(\varphi)\|_{H^s} \lesssim \|\varphi_x\|_{C^1_t} \|D_x\|^{-\delta} \|\varphi\|_{C^{1,2,3} \delta} \|\varphi\|_{H^s}. $$

Moreover, if $\varphi$ and $\psi$ are two solutions and $\psi = \varphi - \psi$,

$$\|\mathcal{R}(\varphi) - \mathcal{R}(\psi)\|_{H^s} \lesssim K \left( \|\varphi, \psi\|_{W^{2+s, \infty}} \right) \|\psi\|_{L^2} + K \left( \|\varphi, \psi\|_{W^{2+s, \infty}} \right) \|\psi\|_{H^s},$$

where $K : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function.

We remark that the $\mathcal{R}$ difference estimate is not optimal with respect to the pointwise control coefficients. However, this estimate is only used in the construction of smooth solutions, and will play no role in the refined low regularity analysis.
Proof. We write
\[ A_\varphi \varphi_x = \int T_{[\delta^x \varphi]} F(\delta^x \varphi) \, dy + \int \Pi \left( F(\delta^x \varphi), |\delta^y \varphi_x| \right) \, dy + \int T_{F(\delta^y \varphi)} |\delta^y \varphi_x| \, dy \]
\[ = \int T_{[\delta^x \varphi]} F(\delta^x \varphi) - T_{\partial_x(F(\delta^y \varphi))} \delta^y \varphi \, dy + \int \Pi \left( F(\delta^x \varphi), |\delta^y \varphi_x| \right) \, dy \]
\[ + \partial_x \int T_{F(\delta^y \varphi)} |\delta^y \varphi| \, dy \]
\[ := A_1 + A_2 + \partial_x A_3. \]

The analysis of \( A_3 \) closely follows the analysis of the counterpart \( A_1 \) in the proof of proposition 3.1, with \( \varphi \) in the place of \( \nu \) and \( H^j \) in the place of \( L^2 \). This contributes both terms in the para-linearization. The balanced frequency term \( A_2 \) is also similar to its counterpart and may be absorbed into \( R(\varphi) \).

It remains to absorb \( A_1 \) into \( R(\varphi) \). We further decompose
\[ A_1 = \int T_{[\delta^x \varphi]} F(\delta^x \varphi) - T_{F(\delta^y \varphi)} \delta^y \varphi \, dy + \int T_{\delta^x \varphi} T_{F(\delta^y \varphi)} |\delta^y \varphi - T_{\partial_y(F(\delta^y \varphi))} |\delta^y \varphi| \, dy. \]

The first difference is estimated using the Moser estimates of theorem 2.2, while the second may be estimated using the para-product estimates of lemma 2.4.

We now prove the difference estimate. We rewrite
\[ A_\varphi \varphi_x = \int T_{[\delta^x \varphi]} F(\delta^x \varphi) \, dy + \int \Pi \left( F(\delta^x \varphi), |\delta^y \varphi_x| \right) \, dy + \int T_{F(\delta^y \varphi)} |\delta^y \varphi_x| \, dy \]
\[ := B_1 + B_2 + B_3. \]

We consider the components of the remainders arising from the terms of the form \( B_3 \). One such contribution is
\[ \int T_{\hat{\eta}_k} (F(\delta^x \varphi) - F(\psi_x)) e^{\eta_0} \varphi_x \, dy - \int T_{\hat{\eta}_k} (F(\delta^x \psi) - F(\psi_x)) e^{\eta_0} \psi_x \, dy \]
\[ = \int T_{\hat{\eta}_k} (F(\delta^x \varphi) - F(\psi_x)) e^{\eta_0} \varphi_x \, dy + \int T_{\hat{\eta}_k} ((F(\delta^x \varphi) - F(\psi_x)) - (F(\delta^x \psi) - F(\psi_x))) e^{\eta_0} \psi_x \, dy. \]

We analyze the last term, by splitting the integral over the regions \( \{|y| \leq 1\} \) and \( \{|y| > 1\} \), respectively. We first consider the former. We claim that
\[ \left\| \int_{|y| \leq 1} \partial_y T_{(F(\delta^x \varphi) - F(\psi_x)) - (F(\delta^x \psi) - F(\psi_x))} |\delta^y \psi| \, dy \right\|_{L^2} \lesssim \| \nu \|_{L^2} \| \psi_x \|_{W^1} \| \psi_x \|_{W^2}, \]

where \( \nu = \varphi - \psi \).
For this purpose, we write the low frequency component as

\[ D := \frac{1}{|y|} (F(\delta^j \varphi) - F(\varphi_1)) - (F(\delta^j \psi) - F(\psi_1)) \]

\[ = \frac{1}{|y|} \int_0^1 \varphi_x(x + \mu y) - \varphi_x(x) \, d\mu \cdot \int_0^1 F' \left( \lambda \int_0^1 \varphi_x(x + \mu y) \, d\mu + (1 - \lambda) \varphi_x(x) \right) \, d\lambda 
- \frac{1}{|y|} \int_0^1 \psi_x(x + \mu y) - \psi_x(x) \, d\mu \cdot \int_0^1 F' \left( \lambda \int_0^1 \psi_x(x + \mu y) \, d\mu + (1 - \lambda) \psi_x(x) \right) \, d\lambda. \]

Let \[ \int_0^1 f(x + \mu y) - f(x) \, d\mu := k_f(x, y), \]
and from the proof of proposition \[ \phi(j) \]
the estimate for this term now follows using ideas similar to the ones for the region \[ \{ |y| > 1 \} \] and from the proof of proposition \[ \phi(j) \].

We can now move the derivatives in the first three terms from the low frequencies to the high frequencies and obtain the claimed estimates.

For the \[ \{ |y| > 1 \} \] region, we write

\[ T_{j_{|y|>1}} (F(\delta^j \varphi) - F(\varphi_1)) - (F(\delta^j \psi) - F(\psi_1)) e^{\omega |y|} \]

integrate the low frequency component by parts (and use the notation of (3.3))

\[ \int_{|y|>1} \frac{1}{|y|} \left( (F(\delta^j \varphi) - F(\varphi_1)) - (F(\delta^j \psi) - F(\psi_1)) \partial \phi e^{\omega |y|} \right) \]

\[ = (2F(\varphi_1) - F(\delta^j \varphi) - F(\delta^{-j} \varphi)) - (2F(\psi_1) - F(\delta^j \psi) - F(\delta^{-j} \psi)) \]

\[ - T_{j_{|y|>1}} (G_{\varphi}(x, y) - G_{\psi}(x, y)) e^{\omega |y|} \Psi, \]

where \( G_{\varphi} \) is defined in the proof of proposition 3.1.

The estimate for this term now follows using ideas similar to the ones for the region \[ \{ |y| \ll 1 \} \] and from the proof of proposition 3.1.
implies that

\[ F_\text{As} \]

Using the paralinearization proposition (3.4), we will in turn be used to derive energy estimates for solutions of (1.4).

The goal of this section is to prove energy estimates for the paradifferential flow (4.1.6).

Proposition 3.3. The linearized equation (1.6) corresponding to (1.4) may be written

\[ \partial_t v - \partial_x A_v v = 2 \log |D_x| \partial_x v. \]

Using the paralinearization proposition 3.1, we obtain

\[ \partial_t v - \partial_x T_{F_\text{As}}(v) v + \mathcal{R}(v, v) = 2 \partial_x T_{1 - F_\text{As}} \log |D_x| v, \]

where

\[ \| \mathcal{R}(v, v) \|_{L^2} \lesssim \| v \|_{L^2} \| \varphi \|_{C^2} \| \| D_x^{1-\delta} \varphi_x \|_{C^{1,2}}^2 \| v \|_{L^2}. \]

4. Energy estimates

The goal of this section is to prove energy estimates for the paradifferential flow (3.4), which will in turn be used to derive energy estimates for solutions of (1.4), along with its linearization (1.6).

We define the modified energies

\[ E^{(i)}(v) = \int g \cdot T_{1 - F_\text{As}} g \, dx, \quad g = T_{1 - F_\text{As}} \| D_x \|_k v, \]

and

\[ E^k(v) = E^{(i)}(v) + E^{(0)}(v). \]

Due to our construction of the paraproduct, these modified energies are comparable to the classical energies in the Sobolev spaces \( H^k(\mathbb{R}) \).

We prove the following:
**Proposition 4.1.** Let $v_0 \in H^r$ and $R \in C([0,T], H^r_x)$, $\varphi \in C([0,T], W^{2,\infty}_x)$. Then the initial value problem

\[\partial_t v - \partial_x T_{B^r(\varphi)} v + R = 2\partial_t T_{1-F(\varphi)} \log |D_x| v,\]

\[v(0,x) = v_0(x) \quad (4.1)\]

has a unique solution $v \in C([0,T]; H^r_x)$, satisfying the estimate

\[\frac{d}{dt} E^s(t) \lesssim (\|\varphi\|_{C^1,\delta} + \|\varphi_{10}\|_{L^\infty}) \|\varphi\|_{C^1,\delta} E^s(t) + \|R\|_{H^r} \|v\|_{H^r}. \quad (4.2)\]

**Proof.** Using that $T_{1-F(\varphi)}$ is self-adjoint,

\[\frac{d}{dt} E^s(t) = \int 2g \cdot T_{1-F(\varphi)} g - g \cdot T_{F(\varphi)} v \, dx.\]

Applying $T_{(1-F(\varphi))' |D_x|^r}$, we obtain an equation for $g$, \[\partial_t g - \partial_x T_{B^r(\varphi)} g + T_{(1-F(\varphi))' |D_x|^r} R + R_1 = 2\partial_t \log |D_x| T_{1-F(\varphi)} g,\]

where $R_1$ consists of commutators which we will record and estimate momentarily. Using this, we have

\[\frac{d}{dt} E^s(t) = \int 2 \left( \partial_x T_{B^r(\varphi)} g + 2\partial_t \log |D_x| T_{1-F(\varphi)} g - R - R_1 \right) \cdot T_{1-F(\varphi)} g \, dx\]

\[= - \int g \cdot T_{F(\varphi)} v \, dx. \quad (4.3)\]

From the first integral on the right hand side, we rewrite the contribution

\[2 \int \partial_x T_{B^r(\varphi)} g \cdot T_{1-F(\varphi)} g \, dx = 2 \int \partial_t \left( T_{B^r(\varphi)} g \cdot T_{1-F(\varphi)} g \right) \, dx + \int R_2 \cdot g \, dx,\]

where as before, $R_2$ consists of commutators which we record and estimate momentarily. Observe that the first term on the right is a divergence which thus vanishes. Similarly, since $\partial_t \log |D_x|$ is skew-adjoint, the corresponding contribution to (4.3) vanishes and we conclude

\[\frac{d}{dt} E^s(t) = \int 2 \left( R + R_1 \right) \cdot T_{1-F(\varphi)} g + g \cdot T_{F(\varphi)} v \, dx + \int R_2 \cdot g \, dx,\]

where

\[R_1 = \left[ T_{(1-F(\varphi))' |D_x|^r}, \partial_t - \partial_x T_{B^r(\varphi)} \right] v \]

\[+ 2\partial_t \left[ T_{F(\varphi)} \log |D_x| \right] g - 2 \left[ T_{(1-F(\varphi))' |D_x|^r}, \partial_t \log |D_x| T_{1-F(\varphi)} \right] v, \]

\[R_2 = \left[ T_{F(\varphi)} \partial_t B^r(\varphi) + T_{1-F(\varphi)} T_{B^r(\varphi)} \right] \partial_t + T_{(1-F(\varphi))' |D_x|^r} g.\]

The second commutator of $R_1$ may be estimated using the log $|D_x|$ commutator lemma 2.3. For the third commutator of $R_1$, applying again lemma 2.3, we may commute $\log |D_x|$ to the front, so this reduces to

\[\log |D_x| \left[ T_{(1-F(\varphi))' |D_x|^r}, \partial_t T_{1-F(\varphi)} \right] v = \log |D_x| \left[ T_{(1-F(\varphi))'}, \partial_x \right] \left[ T_{1-F(\varphi)} v \right] \]

\[+ \log |D_x| \left[ T_{1-F(\varphi)} \partial_t \left[ T_{(1-F(\varphi))'}, \partial_x \right] \left[ T_{1-F(\varphi)} v \right] + \log |D_x| \left[ T_{(1-F(\varphi))'}, \partial_t \left[ T_{1-F(\varphi)} v \right] \right.\]

\[+ \log |D_x| \partial_t \left[ T_{(1-F(\varphi))'}, T_{1-F(\varphi)} \right] |D_x|^r v.\]
The third commutator on the right may be estimated using lemma 2.4. The first may be written
\[ \log |D_x| T_{s(1-F(\varphi))^{-1}} F'(\varphi) |D_x| T_{1-F(\varphi)}^v, \]
while the principal term of the second is
\[ -\log |D_x| \partial_x T_{s(1-F(\varphi))^{-1}} [D_x]^v |D_x| \partial_x T_{F'(\varphi) \varphi_n} |D_x|^v, \]
which cancel up to commutators and paraproducts that can be estimated via lemma 2.4.

Returning to \( R_1 \), it remains to consider the first commutator. For the commutator with respect to \( \partial_x \), we have
\[ T_{s(1-F(\varphi))^{-1}} F'(\varphi) \varphi_n |D_x|^v, \]
which contributes to the right hand side of (4.2). For the commutator with respect to \( \partial_x T_{B'k(\varphi)} \), in particular to estimate \( B^q(\varphi) \), we apply lemma 2.5 to estimate
\[ \| B^q(\varphi) \|_{L_{Inc}} \lesssim \sup_{|b|>1} \sup_{|s|<1} \| y^\delta (F(\delta' \varphi) - F(\varphi) e^{-\delta}) \|_{L_{Inc}} + \sup_{|b|<1} \| y^{-\delta} (F(\delta' \varphi) - F(\varphi) e^{-\delta}) \|_{L_{Inc}} \lesssim \| \varphi \|_{C^{0,\alpha}}, \]
using the decay of \( F \) for the first term. The same estimate for \( B^q(\varphi) \) then suffices to estimate each term of \( R_2 \).

We conclude, using lemma 2.1 to pass between \( \| v \|_{H^s} \) and \( \| g \|_{L^2} \),
\[ \frac{d}{dt} E^{(1)}(t) \lesssim (\| \varphi_x \|_{C^1} + \| \varphi_{tx} \|_{L^\infty}) \| \varphi_x \|_{C^{1,\alpha}} \| g \|_{L_2} + \| R \|_{H^s} \| g \|_{L^2}, \]
which along with the similar case \( s = 0 \) gives the estimate of the proposition. By using the adjoint method (for the general theory, see theorem 23.1.2 in Hörmander [9]), it follows that the equation has a unique solution. 

**Corollary 4.2.** Let \( R > 0 \). If \( \varphi \) is a solution of (1.4) on an interval \([0,T]\) on which \( \| \varphi \|_{C^1 H^s} \lesssim R \), where \( s > \frac{3}{2} \), then we have the energy estimate
\[ \| \varphi(t,x) \|_{H^s} \lesssim C(R) e^{C(R) \int_0^t \| D_x \|_{|\delta x|, \| \varphi_x \|_{L^\infty}} d\tau} \| \varphi_0 \|_{H^s}. \]
Moreover, if \( v \) is a solution of the linearized equation (1.6) on an interval \([0,T]\) where the solution \( \varphi \) satisfies the previous conditions, then
\[ \| v(t,x) \|_{L^2} \lesssim C(R) e^{C(R) \int_0^t \| D_x \|_{|\delta x|, \| \varphi_x \|_{L^\infty}} d\tau} \| v_0 \|_{L^2}. \]
**Proof.** From proposition 4.1, we have the energy estimate
\[ \frac{d}{dt} E^{(1)}(t) \lesssim (\| \varphi_x \|_{C^1} + \| \varphi_{tx} \|_{L^\infty}) \| \varphi_x \|_{C^{1,\alpha}} E^{(1)}(t) + \| R \|_{H^s} \| v \|_{H^s}. \]
In both situations, \( \varphi \) is a solution of (1.4), so in order to control \( \| \varphi_{tx} \|_{L^\infty} \), we write
\[ \varphi_{tx} = 2 \log |D_x| \varphi_{xx} - \partial_x \int F(\delta' \varphi) |\delta'| \varphi_x \, dy. \]
From lemma 2.6 and the product rule, we have
\[ \left\| \partial_t \int F(\delta^r \varphi) |\delta^r \varphi|^2 \, dy \right\|_{L^\infty_x} \lesssim \| \varphi \| \| \varphi \|_{H^k_x} \| \| D \|_{L^{1-\delta}_x} \| \| v \|_{H^{k+2}\infty_x}. \]
We also have
\[ \| \log |D_x| \varphi_{xx} \|_{L^\infty_x} \lesssim \| |D_x|^\delta \varphi_{xx} \|_{L^\infty_x} + \| |D_x|^{1-\delta} \varphi_x \|_{L^\infty_x}. \]
In the first case, \( v = \varphi \) solves the (1.4) equation, so by proposition 3.2, we know that
\[ \| R(\varphi) \|_{W^2} \lesssim \| \varphi \|_{L^1_t H^{k+1}_x} \| \varphi_x \|_{L^2_t H^k_x} \| \varphi \|_{W^1_t H^k_x}, \]
and an application of Grönwall’s lemma, along with the coercivity bounds implies the claimed estimate. When \( v \) solves the linearized equation (1.6), proposition 3.1 similarly implies that
\[ \| R(\varphi, v) \|_{L^2_x} \lesssim \| \varphi \|_{L^2_t H^k_x} \| \varphi_x \|_{L^2_t H^k_x} \| v \|_{L^2_x}. \]

5. Local well-posedness

In this section we establish theorem 1.1, our main local well-posedness result in Sobolev spaces for the SQG equation (1.4). We do this by first constructing smooth solutions using an iterative scheme, and then we employ frequency envelopes in order to construct rough solutions as limits of smooth ones and to prove continuous dependence on the initial data.

We first consider smoother data \( \varphi_0 \in H^s \) with \( s > 4 \). Fix \( R > 0 \) and choose \( M \) as in lemma 2.1. We construct the sequence
\[ \varphi^{(0)}(x) = \varphi_0(x), \quad \varphi^{(n)} = G(\varphi^{(n-1)}), \quad n \geq 1, \]
where we define the operator \( G(\varphi) = v \) using proposition 4.1 with \( R = R(\varphi) \), the paralinearization error from proposition 3.2. We have
\[ \partial_t \varphi^{(n)} - \partial_t T^{\text{pe}} (\varphi^{(n-1)}) \varphi^{(n)} + R(\varphi^{(n-1)}) = 2\partial_t T^{1-\text{pe}} (\varphi^{(n-1)}) \log |D_x| \varphi^{(n)}. \]
An application of Grönwall’s inequality with proposition 4.1 shows that
\[ E^{(i)}(\varphi^{(n+1)}) \leq e^C R \int_0^t \left( \| \varphi^{(n)} \|_{L^2_x} + \| \varphi^{(n)} \|_{L^\infty_x} \right) \| \varphi^{(n)} \|_{H^{k+1}_x} \, dt \]
\[ \cdot \left( E^{(i)}(\varphi^{(n+1)}) (0) + \int_0^t \| \varphi^{(n)} \|_{H^{k+1}_x} \| \varphi^{(n)} \|_{H^{k+1}_x} \, dt \right), \]
along with an easy induction on \( n \) and the terms of the sequence \( (\| \varphi^{(n)} \|_{L^\infty_x})_{n \geq 0} \) show that for sufficiently small \( T > 0 \) depending on \( R \), \( \varphi^{(n)} \) has uniform \( H^k_x \) bounds.

Moreover, we show that for sufficiently small \( T > 0 \), \( \varphi^{(n)} \) is Cauchy. Let \( \varphi^{(m)} \) and \( \varphi^{(n)} \) be two terms of the sequence. Denoting \( v = \varphi^{(m)} - \varphi^{(n)} \), we have
\[\partial_t v - \partial_t T_{\mathrm{BF}}(\varphi^{(n-1)}) v + \partial_t T_{\mathrm{BF}}(\varphi^{(n-1)}) - B(\varphi^{(n-1)}) \varphi^{(n)} + R \left( \varphi^{(m-1)} \right) - R \left( \varphi^{(n-1)} \right) = 2 \partial_t T_{1-F}(\varphi^{(n-1)}) \log |D_x| v + 2 \partial_t T_{F}(\varphi^{(n-1)}) - F(\varphi^{(n-1)}) \log |D_x| \varphi^{(n)}.\]

We now apply the energy estimate of proposition 4.1 with
\[R = \partial_t T_{\mathrm{BF}}(\varphi^{(n-1)}) - B(\varphi^{(n-1)}) \varphi^{(n)} - 2 \partial_t T_{F}(\varphi^{(n-1)}) - F(\varphi^{(n-1)}) \log |D_x| \varphi^{(n)} + R \left( \varphi^{(m-1)} \right) - R \left( \varphi^{(n-1)} \right),\]
estimating
\[\|\partial_t T_{F}(\varphi^{(n-1)}) - F(\varphi^{(n-1)}) \log |D_x| \varphi^{(n)}\|_{L^2} \lesssim \|\varphi^{(m-1)} - \varphi^{(n-1)}\|_{L^2} \|\log |D_x| \varphi^{(n)}\|_{L^\infty},\]
\[\|\partial_t T_{B}(\varphi^{(n-1)}) - B(\varphi^{(n-1)}) \varphi^{(n)}\|_{L^2} \lesssim \|\varphi^{(m-1)} - \varphi^{(n-1)}\|_{L^2} \|\varphi^{(n)}\|_{L^\infty}.\]

An application of Grönwall’s inequality shows that the sequence is Cauchy in \(L^2_t\), for \(T > 0\) small enough. This settles the existence. Uniqueness follows from the energy estimate for the difference.

To establish the local well-posedness result at low regularity, we follow the approach outlined in [17]. We consider \(\varphi_0 \in H^s\) with \(s > \frac{5}{4}\). Let \(\varphi_0^h = (\varphi_0)_h\), where \(h \in \mathbb{Z}\). Since \(\varphi_0^h \to \varphi_0\) in \(H^s\), we may assume that \(|\varphi_0^h|_{H^s} < R\) for all \(h\).

We construct a uniform \(H^s\) frequency envelope \(\{c_k\}_{k \in \mathbb{Z}}\) for \(\varphi_0\) having the following properties:

a) Uniform bounds:
\[\|P_k (\varphi_0^h)\|_{H^s} \lesssim c_k,\]
b) High frequency bounds:
\[\|\varphi_0^h\|_{H^s} \lesssim 2^{h(N-s)} c_h, \quad N - s \geq 4,\]
c) Difference bounds:
\[\|\varphi_0^{h+1} - \varphi_0^h\|_{H^s} \lesssim 2^{-sh} c_h,\]
d) Limit as \(h \to \infty:\)
\[\varphi_0^h \to \varphi_0 \in H^s.\]

Let \(\varphi^h\) be the solutions with initial data \(\varphi_0^h\). Using the energy estimate for the solution \(\varphi\) of (1.4) from corollary 4.2, we deduce that there exists \(T = T(\|\varphi_0\|_{H^s}) > 0\) on which all of these solutions are defined, with high frequency bounds
\[\|\varphi^h\|_{C^1_t H^s} \lesssim \|\varphi_0^h\|_{H^s} \lesssim 2^{h(N-s)} c_h.\]

Further, by using the energy estimates for the solution of the linearized equation (1.6) from corollary 4.2, we have
\[\|\varphi^{h+1} - \varphi^h\|_{C^1_t L^2} \lesssim 2^{-sh} c_h.\]
By interpolation, we infer that
\[ \| \varphi^{h+1} - \varphi^h \|_{C^j H^s_x} \lesssim c_h. \]

As in [17], we get
\[ \| P_k \varphi^h \|_{C^j H^s_x} \lesssim c_k \]
and that
\[ \| \varphi^{h+k} - \varphi^h \|_{C^j H^s_x} \lesssim c_{h \leq h+k} = \left( \sum_{n=h}^{h+k-1} c_n^2 \right)^{\frac{1}{2}} \]
for every \( k \geq 1 \). Thus, \( \varphi^h \) converges to an element \( \varphi \) belonging to \( C^j H^s_x([0, T] \times \mathbb{R}) \). Moreover, we also obtain
\[ \| \varphi^h - \varphi \|_{C^j H^s_x} \lesssim c_{\geq h} = \left( \sum_{n=h}^{\infty} c_n^2 \right)^{\frac{1}{2}}. \] (5.1)

We now prove continuity with respect to the initial data. We consider a sequence
\[ \varphi_0^j \to \varphi_0 \in H^s_x \]
and an associated sequence of \( H^s_x \)-frequency envelopes \( \{ c_k^j \}_{k \in \mathbb{Z}} \), each satisfying the analogous properties enumerated above for \( c_k \), and further such that \( c_k^j \to c_k \) in \( \hat{L}^2(\mathbb{Z}) \). In particular,
\[ \| \varphi_j^h - \varphi_j \|_{C^j H^s_x} \lesssim c_{\geq h}^j = \left( \sum_{n=h}^{\infty} (c_n^j)^2 \right)^{\frac{1}{2}}. \] (5.2)

Using the triangle inequality with (5.1) and (5.2), we write
\[ \| \varphi_j - \varphi \|_{C^j H^s_x} \lesssim \| \varphi^h - \varphi \|_{C^j H^s_x} + \| \varphi_j^h - \varphi_j \|_{C^j H^s_x} + \| \varphi_j^h - \varphi^h \|_{C^j H^s_x} \lesssim c_{\geq h} + c_{\geq h}^j + \| \varphi_j^h - \varphi^h \|_{C^j H^s_x}. \]

To address the third term, we observe that for every fixed \( h \), \( \varphi_j^h \to \varphi^h \) in \( H^s_x \). We conclude \( \varphi_j \to \varphi \) in \( C^j H^s_x([0, T] \times \mathbb{R}) \) and therefore \( \varphi_j \to \varphi \) in \( C^j \mathcal{X}([0, T] \times \mathbb{R}) \).

6. Global well-posedness

In this section we prove global well-posedness for the SQG equation (1.4) with small and localized initial data. We do this by using the wave packet method of Iftim-Tataru, which is systematically described in [18].

6.1. Notation

Consider the linear flow

\[ i \partial_t \phi - A(D) \phi = 0 \]

and the linear operator

\[ L = x - tA' (D). \]

In our setting, we have the symbol

\[ a (\xi) = -2 \xi \log |\xi| \]

and thus

\[ A(D) = -2 D \log |D|, \quad L = x + 2t + 2t \log |D|. \]

Recall that we define the weighted energy space

\[ k \phi_k X = k \phi_k H_s + k \partial_x L \phi_k L^2, \]

which when the frequency is localized may be written

\[ k \phi_k \lambda X = k \phi_k \lambda H_s + \lambda k L \phi_k L^2. \]

We partition the frequency space into dyadic intervals \( I_\lambda \) localized at dyadic frequencies \( \lambda \in 2^\mathbb{Z} \), and consider the associated partition of velocities

\[ J_\lambda = a' (I_\lambda) \]

which form a covering of the real line, and have equal lengths. To these intervals \( J_\lambda \) we select reference points \( v_\lambda \in J_\lambda \), and consider an associated spatial partition of unity

\[ 1 = \sum_\lambda \chi_\lambda (x), \quad \text{supp} \chi_\lambda \subseteq J_\lambda, \quad \chi_\lambda = 1 \text{ on } J_\lambda, \]

where \( J_\lambda \) is a slight enlargement of \( J_\lambda \), of comparable length, uniformly in \( \lambda \).

Lastly, we consider the related spatial intervals, \( tJ_\lambda \), with reference points \( x_\lambda = tv_\lambda \in tJ_\lambda \).

6.2. Overview of the proof

We provide a brief overview of the proof.

1. We make the bootstrap assumption for the pointwise bound

\[ \| \varphi (t) \|_X \lesssim C \epsilon (t)^{- \frac{1}{2}} \]  \hspace{1cm} (6.1)

where \( C \) is a large constant, in a time interval \( t \in [0, T] \) where \( T > 1 \).

2. The energy estimates for (1.4) and the linearized equation will imply

\[ \| \varphi (t) \|_X \lesssim \langle t \rangle^{C_1} \| \varphi (0) \|_X. \]  \hspace{1cm} (6.2)
3. We aim to improve the bootstrap estimate (6.1) to
\[ \| \varphi \|_Y \lesssim \epsilon \langle t \rangle^{-\frac{1}{2}}. \]  

(6.3)

We use vector field inequalities to derive bounds of the form
\[ \| \varphi \|_Y \lesssim \epsilon \langle t \rangle^{-\frac{1}{2}} + C \epsilon^2, \]

(6.4)

which is the desired bound but with an extra \( C \epsilon^2 \) loss.

4. In order to rectify the extra loss, we use the wave packet testing method define a suitable asymptotic profile \( \gamma \), which is then shown to be an approximate solution for an ordinary differential equation. This enables us to obtain suitable bounds for the asymptotic profile without the aforementioned loss, which can then be transferred back to the solution \( \varphi \).

6.3. Energy estimates

From corollary 4.2, and by using the fact that \( \epsilon \ll 1 \),
\[ \| \varphi (t,x) \|_{Y} \lesssim \epsilon C \int_{0}^{t} ||\varphi||_{L^{1}} \, dt. \]

Let \( u = L\partial_t \varphi + \varphi \), which satisfies the linearized equation. From corollary 4.2, along with Grönwall’s lemma and the fact that \( \epsilon \ll 1 \), we have
\[ \| u(t,x) \|_{L^2} \lesssim \epsilon C \int_{0}^{t} ||\varphi||_{L^{1}} \, dt. \]

Along with the bootstrap assumptions, these readily imply that
\[ \| \varphi \|_X \lesssim \| \varphi (t) \|_{N} + ||u(t)\|_{L^2} \lesssim \epsilon C \epsilon^2 \int \, dt \lesssim \epsilon \langle t \rangle C \epsilon^2. \]  

(6.5)

6.4. Vector field bounds

Proposition 2.1 from [18] implies that
\[ \| \varphi \|_{L^2} \lesssim \frac{1}{t} \left( \| \varphi \|_{L^2} + \| L\partial_t \varphi \|_{L^2} + \| L \varphi \|_{L^2} \right). \]

When \( \lambda \leq 1 \),
\[ \| \varphi \|_{L^2} \lesssim \frac{1}{\sqrt{t}} \lambda^{-\frac{1}{2}+2\delta} \left( \| \lambda^\frac{1}{2} - \lambda^\delta \varphi \|_{L^2} \| L\partial_t \varphi \|_{L^2} + \| \lambda^\lambda - \lambda^\delta \varphi \|_{L^2} \right) \lesssim \frac{1}{\sqrt{t}} \lambda^{-\left(\frac{1}{2}-2\delta\right)} \| \varphi \|_{X} \]

and when \( \lambda > 1 \),
\[ \| \varphi \|_{L^2} \lesssim \frac{1}{\sqrt{t}} \lambda^{-2-2\delta} \left( \| \lambda^{4+4\delta} \varphi \|_{L^2} \| L\partial_t \varphi \|_{L^2} + \| \lambda^{\lambda^2} \varphi \|_{L^2} \right) \lesssim \frac{1}{\sqrt{t}} \lambda^{-\left(2+2\delta\right)} \| \varphi \|_{X}. \]

By dyadic summation and Bernstein’s inequality, we deduce the bound
\[ \| \varphi \|_{Y} = \| D^\frac{3}{4} \varphi \|_{L^2} + \| D^\frac{1}{4} \varphi \|_{L^2} \lesssim \frac{\| \varphi \|_{X}}{\sqrt{t}}. \]

(6.6)
By the localized dispersive estimate [18, proposition 5.1],
\[ |\varphi_\lambda(x)|^2 \lesssim \frac{1}{|x-x_\lambda|^\delta} \left( \|L\varphi_\lambda\|_{L^2_x} + \lambda^{-1}\|\varphi_\lambda\|_{L^2_x} \right)^2, \]
which implies that
\[ \| (1 - \chi_\lambda) \varphi_\lambda \|_{L^\infty_x} \lesssim \frac{\lambda^{-1/2}}{t} \left( \|L\partial_\lambda \varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x} \right) \lesssim \frac{\lambda^{-1/2}}{t} \|\varphi\|_x. \quad (6.7) \]

To end this section we record the following elliptic bounds:

**Lemma 6.1.** We have
\[ \|D_{\lambda}^{1+\delta} \partial_\lambda \left((1 - \chi_\lambda) \varphi_\lambda \right) \|_{L^\infty_x} \lesssim \frac{\lambda^{3/2+\delta}}{t} \|\varphi\|_x \quad (6.8) \]
\[ \|D_{\lambda}^{3/4-\delta} \left((1 - \chi_\lambda) \varphi_\lambda \right) \|_{L^\infty_x} \lesssim \frac{\lambda^{1/4-\delta}}{t} \|\varphi\|_x, \quad (6.9) \]

and
\[ \| (1 - \chi_\lambda) \varphi_\lambda \|_{L^2_x} \lesssim \frac{\lambda^{-1}}{t} \|\varphi\|_x. \quad (6.10) \]

Moreover, the difference quotient satisfies the bounds
\[ \| (1 - \chi_\lambda) \delta^\gamma \varphi_\lambda \|_{L^\infty_x} \lesssim \frac{\lambda^{1/2}}{t} \|\varphi\|_x, \]

and
\[ \| (1 - \chi_\lambda) \delta^\gamma \varphi_\lambda \|_{L^2_x} \lesssim \frac{\|\varphi\|_x}{t}. \]

**Proof.** We use the bounds
\[ |\partial_\lambda(\chi_\lambda(x/t))| \lesssim t^{-1}. \]

From (6.7) applied for \( \partial_\lambda \varphi \),
\[ \| \partial_\lambda \left((1 - \chi_\lambda) \varphi_\lambda \right) \|_{L^\infty_x} \lesssim \frac{1}{t} \|\chi_\lambda \varphi_\lambda\|_{L^\infty_x} + \| (1 - \chi_\lambda) \partial_\lambda \varphi_\lambda\|_{L^\infty_x} \lesssim \frac{\lambda^{1/2}}{t} \|\varphi\|_x. \]

The first two bounds immediately follow from (6.7), and the \( L^2 \) elliptic estimate similarly follows from [18, proposition 5.1].

For the bounds involving the difference quotient, from (6.7) applied for \( \delta^\gamma \varphi \), we have
\[ \| (1 - \chi_\lambda) \delta^\gamma \varphi_\lambda \|_{L^\infty_x} \lesssim \frac{\lambda^{1/2}}{t} \left( \|L\delta^\gamma \varphi_\lambda\|_{L^2_x} + \lambda^{-1}\|\delta^\gamma \varphi_\lambda\|_{L^2_x} \right) \lesssim \frac{\lambda^{1/2}}{t} \left( \|\delta^\gamma (L\varphi_\lambda)\|_{L^2_x} + \|\varphi_\lambda(x+y)\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x} \right) \lesssim \frac{\lambda^{1/2}}{t} \left( \|L\delta_\lambda \varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x} \right) \lesssim \frac{\lambda^{1/2}}{t} \|\varphi\|_x. \]

The other bound is proved similarly. \( \square \)
6.5. Wave packets

We construct wave packets as follows. Given the dispersion relation \( a(\xi) \), the group velocity \( v \) satisfies
\[
  v = a'(\xi) = -2 - 2 \log|\xi|,
\]
so we denote
\[
  \xi_v = -e^{-1-\frac{i}{2}}.
\]
Then we define the linear wave packet \( u^v \) associated with velocity \( v \) by
\[
  u^v = a''(\xi_v)^{-\frac{1}{2}} \chi(y) e^{i\phi(x/t)}, \quad y = \frac{x - vt}{t^2 a''(\xi_v)^{\frac{1}{2}}},
\]
where the phase \( \phi \) is given by
\[
  \phi(v) = v\xi_v - a(\xi_v),
\]
and \( \chi \) is a unit bump function, such that \( \int \chi(y) dy = 1 \).

We remark that we will typically apply frequency localization
\[
  u^v_\lambda = P_\lambda u^v
\]
with
\[
  v^2 \approx F_\lambda.
\]
We observe that since
\[
  \partial_v \left( \xi_v \right) = -\frac{1}{4} \xi_v \frac{1}{2}, \quad \partial_v \left( a''(\xi_v)^{-\frac{1}{2}} \right) = -\frac{1}{4} a''(\xi_v)^{-\frac{1}{2}},
\]
we may write
\[
  \partial_v u^v = -\tilde{L} u^v + u^{v_{\text{II}}} = t^{\frac{1}{2}} a''(\xi_v)^{-\frac{1}{2}} u^v + u^{v_{\text{II}}}
\]
where
\[
  \tilde{L} = t(\partial_v - i\phi'(x/t))
\]
and \( u^{v_{\text{II}}} \) has a similar wave packet form. We also recall from [18, lemmas 4.4 and 5.10] the sense in which \( u^v \) is a good approximate solution:

**Lemma 6.2.** The wave packet \( u^v \) solves an equation of the form
\[
  (i\partial_t - A(D)) u^v = t^{-\frac{1}{2}} (L u^{v_{\text{I}}} + r^v)
\]
where \( u^{v_{\text{I}}}, r^v \) have wave packet form,
\[
  u^{v_{\text{I}}} \approx a''(\xi_v)^{-\frac{1}{2}} u^v, \quad r^v \approx \xi_v^{-1} a''(\xi_v)^{-\frac{1}{2}} u^v.
\]

The asymptotic profile at frequency \( \lambda \) is meaningful when the associated spatial region \( tJ_\lambda \) dominates the wave packet scale at frequency \( \lambda \):
\[
  \delta x \approx t^2 a''(\lambda)^{\frac{1}{2}} \lesssim |J_\lambda| \approx t\lambda a''(\lambda).
\]
This corresponds to
\[
  t \gtrsim \lambda^{-2} a''(\lambda)^{-1} \approx \lambda^{-1}.
\]
Accordingly we define
\[
  \mathcal{D} = \{(t,v) \in \mathbb{R}^+ \times \mathbb{R} : v \in J_\lambda, \ t \gtrsim \lambda^{-1}\}.
\]
6.6. Wave packet testing

In this section we establish estimates on the asymptotic profile function

\[ \gamma^\lambda (t, v) := \langle \varphi, u^\lambda_\gamma(x) \rangle = \langle \varphi, u^\gamma \rangle_{L^2}. \]

We will see that \( \gamma^\lambda \) essentially has support \( v \in J_\lambda \).

We will also use the following crude bounds involving the higher regularity of \( \gamma^\lambda \):

**Lemma 6.3.** We have

\[
\| \chi_\lambda \partial_t^n \gamma^\lambda \|_{L^\infty} \lesssim t^{i^2} \left( 1 + t^2 \lambda^2 \right)^{n} \| \varphi_\lambda \|_{L^\infty}, \\
\| \chi_\lambda \partial_t^n \gamma^\lambda \|_{L^2} \lesssim (t \lambda)^{i^2} \left( 1 + t^2 \lambda^2 \right)^{n} \| \varphi_\lambda \|_{L^2},
\]

and

\[
\| \chi_\lambda \partial_t \gamma^\lambda \|_{L^\infty} \lesssim t^{i^2} \lambda^{-\frac{1}{2}} \| \varphi \|_{X} + t^{i^2} \| \varphi_\lambda \|_{L^\infty}.
\]

**Proof.** Using the second form of \( \partial_t u^\gamma \) in (6.11), we have

\[
\| \chi_\lambda \partial_t \gamma^\lambda \|_{L^\infty} \lesssim t^{i^2} \lambda^{-\frac{1}{2}} \| \varphi \|_{X} + t^{i^2} \| \varphi_\lambda \|_{L^\infty},
\]

where the \( i^2 \lambda \) loss in front arises from the \( L^1 \) norm of the wave packet. Higher derivatives are obtained similarly, along with the \( L^2 \) estimates.

For the last bound, we use the first form of \( \partial_t u^\gamma \) in (6.11). The contribution from the wave packet \( u^{\gamma,h} \) is easily estimated as above. For the remaining bound, lemma 2.3 from [18] implies that

\[
|\langle \varphi_\lambda, \tilde{L} u^\gamma \rangle| \lesssim (t \lambda)^{i^2} \| \tilde{L} \varphi_\lambda \|_{L^2} \lesssim t^{i^2} \lambda^{-\frac{1}{2}} \| \varphi_\lambda \|_{X},
\]

which finishes the proof. \( \square \)

6.6.1. Approximate profile. We recall from [18] that \( \gamma^\lambda \) provides a good approximation for the profile of \( \varphi \). In our setting, we will also need to compare the profile with the differentiated flow \( \partial_\lambda \varphi \). Define

\[
r^\lambda (t, x) = \chi_\lambda (x/t) \varphi_\lambda (t, x) - t^{-\frac{1}{2}} \chi_\lambda (x/t) \gamma^\lambda (t, x/t) e^{-i \phi(x/t)}.
\]

**Lemma 6.4.** Let \( t \geq 1 \). Then we have

\[
\| \chi_\lambda (x/t) r^\lambda \|_{L^\infty} \lesssim t^{-i^2} \lambda^{-\frac{1}{2}} \| \tilde{L} \varphi_\lambda \|_{L^2}, \\
\| \chi_\lambda (x/t) \partial_t r^\lambda \|_{L^\infty} \lesssim t^{i^2} \lambda^{-\frac{1}{2}} \| \tilde{L} \partial_t \varphi_\lambda \|_{L^2} + \left( 1 + t^2 \lambda^2 \right) \| \varphi_\lambda \|_{L^\infty}.
\]

**Proof.** The first estimate may be obtained from the proof of [18, proposition 4.7]. For the latter, we use the first representation in (6.11) to write

\[
e^{i \phi(x/t)} \partial_t \gamma (t, v) e^{-i \phi(x/t)} = i (\partial_\lambda \varphi_\lambda, u^\gamma) + \langle \varphi_\lambda, i t (\phi'(x/t) - \phi'(y)) u^\gamma \rangle + \langle \varphi_\lambda, u^{\gamma,h} \rangle. \tag{6.12}
\]
To address the first term, we see that we may apply the undifferentiated estimate with \( \partial_t \varphi_\lambda \) in place of \( \varphi_\lambda \). Precisely, we may apply the first estimate on
\[
\partial_t \varphi_\lambda (t,x) - t^{-\frac{1}{2}} (\partial_t \varphi_\lambda, u^{\mu_1}) e^{-t \phi(x/t)}.
\]

We estimate the third term of (6.12) via
\[
t^{-\frac{1}{2}} |\langle \varphi_\lambda, u^{\mu_1} \rangle| \lesssim \| \varphi_\lambda \|_{L^\infty}.
\]

It remains to estimate the middle term,
\[
t^{-\frac{1}{2}} |\chi_\lambda (v) \langle \varphi_\lambda, it (\phi'' (\cdot /t) - \phi' (v)) u^\beta \rangle| \lesssim |\phi''(\lambda)| \cdot t^{\frac{1}{2}} \lambda^{\frac{1}{2}} \cdot \| \varphi_\lambda \|_{L^\infty} \lesssim t^{\frac{1}{2}} \lambda^{\frac{1}{2}} \| \varphi_\lambda \|_{L^\infty}.
\]

We also observe that on the wave packet scale, we may replace \( \gamma(t,v) \) with \( \gamma(t,x/t) \) up to acceptable errors. Denote
\[
\beta^\lambda (t,x) = t^{-1/2} \chi_\lambda (x/t) (\gamma (t,v) - \gamma (t,x/t)) e^{i t \phi(x/t)}.
\]

**Lemma 6.5.** Let \( v \in I_\lambda \), and \( (t,v) \in D \). Then, for every \( y \neq 0 \) and \( x \) such that \( |x - vt| \lesssim \delta = t^{1/2} \lambda^{-1/2} \), we have the bound
\[
|\delta^y \beta^\lambda | \lesssim t^{-3/4} \lambda^{-1/4} \| \varphi \|_x.
\]

**Proof.** We have
\[
\delta^y \beta^\lambda = -t^{-1/2} \delta^y (\gamma (t,./t)) \chi_\lambda ((x+y)/t) e^{i t \phi((x+y)/t)}
+ t^{-1/2} (\gamma (t,v) - \gamma (t,x/t)) \delta^y \left( \chi_\lambda (./t) e^{i t \phi(./t)} \right).
\]

The Mean Value Theorem ensures that
\[
|\delta^y (\gamma (t,./t)) | \lesssim t^{-1} \| \partial_t \gamma \|_{L^\infty},
\]

and that
\[
|\delta^y \beta^\lambda | \lesssim t^{-1/2} \left( t^{-1} \| \partial_t \gamma \|_{L^\infty} + t^{-1/2} \lambda^{-1/2} \| \partial_t \gamma \|_{L^\infty} (t^{-1} + \lambda) \right)
\lesssim t^{-1} \| \partial_t \gamma \|_{L^\infty} \lesssim t^{-1} \lambda^{1/2} \left( \lambda^{-3/4} \| \varphi \|_x + \| \varphi_\lambda \|_{L^\infty} t^{1/2} \right)
\lesssim t^{-3/4} \lambda^{-1/4} \| \varphi \|_x + t^{-1/2} \lambda^{1/2} \| \varphi_\lambda \|_{L^\infty} \lesssim t^{-3/4} \lambda^{-1/4} \| \varphi \|_x + t^{-1} \lambda^{-1/4} \| \varphi \|_x
\lesssim t^{-3/4} \lambda^{-1/4} \| \varphi \|_x.
\]

\( \square \)
6.7 Bounds for $Q$

Write, slightly abusing notation,

$$Q(\varphi) = Q(\varphi, \varphi, \varphi) := \frac{1}{2} \int \text{sgn}(y) \cdot |\delta^y \varphi|^2 \cdot \varphi \, dy.$$  

**Lemma 6.6.** For $0 < \delta \ll 1$, we have the difference estimates

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^\infty_{x,y}, L^1_{\lambda, \ell}} \lesssim \left( \|\partial_\lambda (\varphi_1, \varphi_2)\|_{L^\infty_{x,y}} + \|\partial_\lambda (\varphi_1, \varphi_2)\|_{L^1_{x,y}} \right) \|\partial_\lambda (\varphi_1 - \varphi_2)\|_{L^\infty_{x,y}},$$

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^1_{x,y}} \lesssim \|\partial_\lambda (\varphi_1, \varphi_2)\|_{L^2_{x,y}} \|\partial_\lambda (\varphi_1 - \varphi_2)\|_{L^\infty_{x,y}} + \|D_{x,y}^{1-\delta} (\varphi_1, \varphi_2)\|_{L^1_{x,y}} \|\varphi_1 - \varphi_2\|_{L^\infty_{x,y}} \|\partial_\lambda (\varphi_1 - \varphi_2)\|_{L^\infty_{x,y}}.$$  

**Proof.** Write

$$Q(\varphi_1) - Q(\varphi_2) = \int_{|y| < 1} + \int_{|y| > 1}$$  

where the integrand may be written

$$\text{sgn}(y) \left( |\delta^y \varphi_1|^2 |\delta^y \varphi_2 - |\delta^y \varphi_2|^2| \right) \phi,$$

$$= \text{sgn}(y) \left( \delta^y (\varphi_1 - \varphi_2) \left( |\delta^y \varphi_1|^2 + |\delta^y \varphi_2|^2 \right) + \delta^y (\varphi_1 - \varphi_2) \delta^y \varphi_1 \delta^y \varphi_2 \right).$$  

The first integral contributes to the two estimates respectively,

$$\left| \int_{|y| < 1} \right| \lesssim \|\partial_\lambda (\varphi_1, \varphi_2)\|_{L^2_{x,y}} \|\partial_\lambda (\varphi_1 - \varphi_2)\|_{L^\infty_{x,y}}$$

and

$$\left| \int_{|y| < 1} \right| \lesssim \|\partial_\lambda (\varphi_1, \varphi_2)\|_{L^2_{x,y}} \|\partial_\lambda (\varphi_1, \varphi_2)\|_{L^\infty_{x,y}} \|\partial_\lambda (\varphi_1 - \varphi_2)\|_{L^\infty_{x,y}}.$$  

For the second, using Sobolev embedding,

$$\left| \int_{|y| > 1} \right| \lesssim \|D_{x,y}^{1-\delta} (\varphi_1, \varphi_2)\|_{L^{1/\delta}_{x,y}} \|\varphi_1 - \varphi_2\|_{L^\infty_{x,y}} \|\partial_\lambda (\varphi_1 - \varphi_2)\|_{L^\infty_{x,y}}$$

and

$$\left| \int_{|y| > 1} \right| \lesssim \|D_{x,y}^{1-\delta} (\varphi_1, \varphi_2)\|_{L^1_{x,y}} \|\varphi_1 - \varphi_2\|_{L^\infty_{x,y}} \|\partial_\lambda (\varphi_1 - \varphi_2)\|_{L^\infty_{x,y}}.$$  

We will be considering separately the balanced and unbalanced components of $Q$. Precisely, we denote the diagonal set of frequencies by $\mathcal{D}$ and write

$$Q(\varphi, \varphi, \varphi) = \sum_{(\lambda_1, \lambda_2, \lambda_1, \lambda_1) \in \mathcal{D}} Q(\varphi_{\lambda_1}, \varphi_{\lambda_2}, \varphi_{\lambda_3}) + \sum_{(\lambda_1, \lambda_2, \lambda_1, \lambda_1) \notin \mathcal{D}} Q(\varphi_{\lambda_1}, \varphi_{\lambda_2}, \varphi_{\lambda_3})$$

$$= Q^{\text{bal}} (\varphi, \varphi, \varphi) + Q^{\text{unbal}} (\varphi, \varphi, \varphi) = Q^{\text{bal}} (\varphi) + Q^{\text{unbal}} (\varphi).$$

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The unbalanced portion of $Q$ satisfies the better bound as follows:

**Lemma 6.7.** $Q^{\text{unbal}}$ satisfies the bounds

$$
\| \chi_\lambda^1 \partial_x P \chi_\lambda^1 \|_{L^\infty_{t \xi}} \lesssim \lambda^{-1/4} \| \varphi \|_{X \lambda}^3
$$

and

$$
\| \chi_\lambda^1 \partial_x P \chi_\lambda^1 \|_{L^2_{t \xi}} \lesssim \lambda^{-1/4} \| \varphi \|_{X \lambda}^3
$$

where $\chi_\lambda^1$ is a cut-off widening $\chi_\lambda$.

**Proof.** We shall denote

$$
I_{\lambda_1, \lambda_2, \lambda_3} = \int_{\mathbb{R}} \text{sgn} (y) \delta^\lambda \varphi_\lambda \delta^\lambda \varphi_\lambda \delta^\lambda \varphi_\lambda \, dy
$$

and consider two cases in the frequency sum for $\partial_x P \chi_\lambda^1$.

First we consider the case in which we have two low separated frequencies. We assume without loss of generality that $\lambda_3 = \lambda$ and $\lambda_1 < \lambda_2 \ll \lambda$. In this case, the elliptic estimates will be applied for the factor $\varphi_\lambda$. Precisely, from lemma 2.6 and estimates (6.6), (6.7) and (6.10), we get that

$$
\| \chi_\lambda^1 I_{\lambda_1, \lambda_2, \lambda_3} \|_{L^\infty_{t \xi}} \lesssim \lambda \lambda_1^{-1/2} \| \varphi \|_{L^\infty_{t \xi}} \lambda_2 \lambda_3 \lambda_1 \| \varphi_\lambda \|_{X \lambda} \lambda_2 \lambda_3 \lambda_1 \| \varphi_\lambda \|_{L^\infty_{t \xi}}
$$

$$
\lesssim \lambda \lambda_1^{1/2} \lambda_2 \lambda_3 \lambda_1 \| \varphi \|_{L^\infty_{t \xi}} \lambda_2 \lambda_3 \lambda_1 \| \varphi_\lambda \|_{L^\infty_{t \xi}} \lambda_2 \lambda_3 \lambda_1 \| \varphi_\lambda \|_{L^\infty_{t \xi}}
$$

$$
\lesssim \lambda \lambda_1^{1/4} \lambda_2 \lambda_3 \lambda_1 \| \varphi \|_{L^\infty_{t \xi}} \lambda_2 \lambda_3 \lambda_1 \| \varphi_\lambda \|_{L^\infty_{t \xi}} \lambda_2 \lambda_3 \lambda_1 \| \varphi_\lambda \|_{L^\infty_{t \xi}}
$$

By using dyadic summation in $\lambda_1$ and $\lambda_2$, we deduce that

$$
\left\| \chi_\lambda^1 \partial_x \sum_{\lambda_1 < \lambda_2 \ll \lambda} I_{\lambda_1, \lambda_2, \lambda_3} \right\|_{L^\infty_{t \xi}} \lesssim \lambda^{-1/4} \| \varphi \|_{X \lambda}^3
$$

Similarly, we deduce that

$$
\left\| \chi_\lambda^1 \partial_x \sum_{\lambda_1 < \lambda_2 \ll \lambda} I_{\lambda_1, \lambda_2, \lambda_3} \right\|_{L^2_{t \xi}} \lesssim \lambda^{-1/4} \| \varphi \|_{X \lambda}^3
$$

We now analyze the situation in which $\lambda_1, \lambda_2 \gtrsim \lambda$, and $\lambda_1$ and $\lambda_2$ are comparable and both separated from $\lambda$. Thus, we will be able to use $\lambda_1$ and $\lambda_2$ interchangeably. We replace $\chi_\lambda^1$ by $\tilde{\chi}_\lambda$, which has double support, and equals 1 on a comparably-sized neighbourhood of the support of $\chi_\lambda^1$. We write

$$
\chi_\lambda^1 \partial_x P \lambda = \chi_\lambda^1 \partial_x P \lambda \tilde{\chi}_\lambda + \chi_\lambda^1 \partial_x P \lambda (1 - \tilde{\chi}_\lambda)
$$

For the first term, using lemma 2.6, along with estimates (6.6), (6.7) and (6.10), we get the bounds
We have

\[ \| \chi_\lambda P_\lambda \tilde{x} \mathcal{I}_{\lambda_1, \cdots, \lambda_N} \|_{L^\infty_x} \lesssim \lambda^{\frac{1}{2} + \delta} \frac{\lambda_1^{-2/3} + \lambda_2}{t} \| \varphi \| \| \varphi \lambda \|_{L^\infty_x} \| \varphi \lambda \|_{L^\infty_x} \]

\[ \lesssim \lambda_2^{-5/4 - \delta/2} \lambda_3^{5/2} \frac{\| \varphi \|}{\| \varphi \|} \left( \lambda_1^{1 - 5/2} + \lambda_3^{1 - \delta/2} \right) \| \varphi \lambda \|_{L^\infty_x} \lambda_2^{1/4 + 3 \delta/2} \| \varphi \lambda \|_{L^\infty_x} \]

and

\[ \| \chi_\lambda P_\lambda \tilde{x} \mathcal{I}_{\lambda_1, \cdots, \lambda_N} \|_{L^2_x} \lesssim \lambda_2^{-5/4 - \delta/2} \lambda_3^{5/2} \frac{\| \varphi \|}{\| \varphi \|} \left( \lambda_1^{1 - 5/2} + \lambda_3^{1 - \delta/2} \right) \| \varphi \lambda \|_{L^\infty_x} \lambda_2^{-1/4 + \delta/2} \| \varphi \lambda \|_{L^\infty_x} \]

By using dyadic summation in \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) (and by using the fact that \( \lambda_1 \) and \( \lambda_2 \) are close), we deduce the bound

\[ \left\| \chi_\lambda \partial_x P_\lambda \sum_{\lambda_1 \leq \lambda_2, \lambda_1 + \lambda_2 \geq \lambda} \mathcal{I}_{\lambda_1, \cdots, \lambda_N} \right\|_{L^2_x \cap L^2_t} \lesssim \lambda^{-1/4} \frac{1}{\lambda^{5/4}} \| \varphi \lambda \|_{L^\infty_x}. \]

We look at the second term. For every \( N, \) we know that

\[ \| \chi_\lambda \partial_x P_\lambda (1 - \tilde{x} \lambda) \|_{L^2_x} \lesssim \| \chi_\lambda \partial_x P_\lambda (1 - \tilde{x} \lambda) \|_{L^\infty_x} \lesssim \lambda^{1 - N}. \]

We take \( N = \frac{5}{2}. \) By carrying out a similar analysis as above, along with lemma 2.6 and dyadic summation, we deduce that the contributions corresponding to these terms are also acceptable.

**Lemma 6.8.** We have

\[ \chi_\lambda \left( (x/t)^3 \mathcal{Q} \left( e^{i \phi(x/t)} \right) \right) = (\chi_\lambda (x/t))^3 e^{i \phi(x/t)} q(\phi'(x/t) + h(\lambda, t)), \]

where for every \( a \in (0, 1) \)

\[ |h(\lambda, t)| \lesssim \lambda \frac{\lambda_1}{t - 3a} + \frac{\lambda_2}{t^a} + \frac{1}{t^{2a}}. \]

**Proof.** We write

\[ e^{-i \phi(x/t)} \delta e^{i \phi(x/t)} = \frac{e^{i \phi'(x/t)} (e^{2i \phi''(x,y)/y} - 1)}{y} + \frac{e^{i \phi'(x/t)} - 1}{y} =: a + b, \]

where \( c_{x,y} \) is between \( x \) and \( x + y. \) We now use the fact that \( x/t \) belongs to the support of \( \chi_\lambda. \) We have

\[ |\chi_\lambda (x/t)| \|b\| \lesssim \lambda. \]
Moreover, when $|y| \leq r^t$, $|c_{x,y}/t - x/t| \leq |y/t| \leq r^{-1}$. This implies that $c_{x,y}/t$ belongs to the support of the enlarged cut-off $\chi_{1/\lambda}$, hence $\phi''((c_{x,y}/t)) \simeq \lambda$. We note the bound
\[
|\chi(x/t)| |a| \leq \chi(x/t) |y/2| \phi''((c_{x,y}/t)) \left| \frac{e^{\pm i \phi''((c_{x,y}/t))y^2/(2t)}}{\phi''((c_{x,y}/t))y^2/(2t)} - 1 \right| \lesssim \lambda r^{t-1}.
\]
Thus, we have the bounds
\[
|\chi(x/t)| |a| \lesssim \lambda r^{t-1} \\
|\chi(x/t)| |b| \lesssim \lambda.
\]
(6.13)

We also note the cruder bounds
\[
|\chi(x/t)| |a| + |\chi(x/t)| |b| \lesssim \frac{1}{|y|}.
\]
(6.14)

We write
\[
(\chi(x/t))^3 Q(\phi(x/t)) = (\chi(x/t))^3 e^{i \phi(x/t)} \int |b|^2 b \, dy + (\chi(x/t))^3 e^{i \phi(x/t)} \int a^2 \bar{\alpha} + a^2 b + 2|a|^2 b + b^2 |\bar{\alpha}| \, dy := T_1 + T_2.
\]
We note that
\[
T_1 = (\chi(x/t))^3 e^{i \phi(x/t)} \int |b|^2 b \, dy = (\chi(x/t))^3 e^{i \phi(x/t)} q(\phi'(x/t)),
\]
so we only need to analyze $T_2$.

We first bound the contribution over the region $|y| \leq r^t$, which we shall denote by $T_2^1$. We denote the contribution over the region $|y| > r^t$ by $T_2^2$. We have
\[
T_2^1 = (\chi(x/t))^3 e^{i \phi(x/t)} \int_{|y| \leq r^t} a^2 \bar{\alpha} + a^2 b + 2|a|^2 b \, dy \\
+ (\chi(x/t))^3 e^{i \phi(x/t)} \int_{|y| \leq r^t} 2a|b|^2 + b^2 |\bar{\alpha}| \, dy := T_{2a} + T_{2b},
\]
(6.13) implies that
\[
|T_{2a}| \lesssim |\chi(x/t)|^3 \int_{|y| \leq r^t} |a|^3 + |a|^2 |b| \, dy \lesssim \int_{|y| \leq r^t} \frac{\lambda^2}{r^2 - 2a} \left( \frac{\lambda}{r - a} + \lambda \right) \lesssim \int_{|y| \leq r^t} \frac{\lambda^3}{r^2 - 2a} \, dy \\
\lesssim r^t \frac{\lambda^3}{r^2 - 2a} \lesssim \frac{\lambda^3}{r^2 - 3a}.
\]
(6.13) and (6.14) imply the bound
\[
|\chi(x/t)|^3 |b|^2 |a| = |\chi(x/t)|^3 |b|^2 |y/2| \phi''((c_{x,y}/t)) \left| \frac{e^{\pm i \phi''((c_{x,y}/t))y^2/(2t)}}{\phi''((c_{x,y}/t))y^2/(2t)} - 1 \right| \\
\lesssim \frac{1}{|y|} |\chi(x/t)| |y/2| \phi''((c_{x,y}/t)) \lesssim \frac{\lambda^2}{t}.
\]
It follows that $T_{2b}$ satisfies the bound
\[ |T_{2b}| \lesssim |\chi_\lambda(x/t)| \int_{|y| \leq a} |h|^2 |a| dy \lesssim \int_{|y| \leq a} \frac{\lambda^2}{t} \, dy \lesssim \frac{\lambda^2}{t^{1/2}}. \]

For $T_2^2$, (6.14) implies that
\[ |T_2^2| \lesssim \int_{|y| > a} \frac{1}{|y|^3} \, dy \lesssim \frac{1}{t^{2a}}. \]

6.8. The asymptotic equation for $\gamma$

Here we prove the following:

**Proposition 6.9.** Let $v \in J_\lambda$. Under the assumption $(t,v) \in \mathcal{D}$, we have
\[ \dot{\gamma}(t,v) = iq(\xi_i)\xi_i t^{-1} \gamma(t,v) |\gamma(t,v)|^2 + f(t,v), \]
where
\[ |f(t,v)| + \|f(t,v)\|_{L^2(J_\lambda)} \lesssim \lambda^{-1/4} t^{-6/5 + C_\lambda^2} \epsilon. \]

**Proof.** We have
\[ \dot{\gamma}(t,v) = \langle \dot{\phi}, u^\lambda \rangle + \langle \phi, u^\lambda \dot{\phi} \rangle = \langle p_A \phi, \varphi, u^\gamma \rangle + i \langle \phi, (i\partial_t - A(D)) u^\gamma \rangle := I_1 + I_2. \]

We first analyze $I_2$. We use lemma 6.2 to write
\[ (i\partial_t - A(D)) u^\gamma = t^{-2} \left( L u^\gamma + F \right) \]
\[ |\langle \phi, (i\partial_t - A(D)) u^\gamma \rangle| \lesssim t^{-2} \left( \|L\phi\|_{L^2} \cdot \lambda^{1/2} \lambda^{-1/4} t^{1/4} \right) \lesssim \lambda^{-1/4} t^{-5/4} \|\phi\|_X \lesssim \lambda^{-1/4} t^{-5/4} C \epsilon, \]

and
\[ \|\chi_\lambda \langle \phi, (i\partial_t - A(D)) u^\gamma \rangle\|_{L^2} \lesssim t^{-3/2} \left( \|L\phi\|_{L^2} \lambda^{1/2} + \|\phi\|_{L^2} \lambda^{-1/2} \right) \lesssim \lambda^{-1/4} t^{-5/4} \|\phi\|_X \lesssim \lambda^{-1/4} t^{-5/4} C \epsilon, \]

(we have used the condition $(t,v) \in \mathcal{D}$.)

In the remaining part of this section we shall analyze the term $I_1$. We first exchange $F$ for its principal quadratic term, expanding
\[ F(\delta^\gamma \varphi) - \frac{1}{2} (\delta^\gamma \varphi)^2 = \int_0^1 \frac{(1 - h)^2}{2} (\delta^\gamma \varphi)^3 F'''(h \delta^\gamma \varphi) \, dh. \]
From Moser’s estimate (the nonlinear version, as well as the one for products), lemma 2.6, and Sobolev embedding, we get that

$$\lambda^{\frac{1}{2}} \left\| P_{\lambda} \int (\delta' \varphi)^3 F''' (t \delta^2 \varphi) |\delta^3 \varphi_{x} dy \right\|_{L^\infty_{t}} \leq \int \| D_{\lambda} \|_{L^\infty_{t}} \left\| \left( \delta' \varphi \right)^{3} |\delta^3 \varphi_{x} F''' (t \delta^2 \varphi) \right\|_{L^1_{t}} dy \leq \| \varphi_{x} \|_{L^\infty_{t}} \| D_{\lambda} \|_{L^\infty_{t}} \left\| \left( \delta' \varphi \right)^{3} F''' (t \delta^2 \varphi) \right\|_{L^1_{t}} dy \leq \frac{1}{\lambda^{5/4}} \epsilon^5 \langle t \rangle C^2.$$

We have also used Sobolev embedding and the classical Moser estimate, keeping in mind $F'''(0) = 0$. Similarly,

$$\left\| \int_{\mathbb{R}} P_{\lambda} \left( F (\delta' \varphi) - \frac{1}{2} (\delta' \varphi)^2 \right) |\delta^3 \varphi_{x} dy \right\|_{L^1_{t}} \leq \lambda^{-1/4} \frac{1}{\lambda^{5/2}} \epsilon^5 \langle t \rangle C^2.$$

By Hölder’s inequality and Young’s inequality respectively,

$$\left\| \int_{\mathbb{R}} P_{\lambda} \left( F (\delta' \varphi) - \frac{1}{2} (\delta' \varphi)^2 \right) |\delta^3 \varphi_{x} dy, \varphi_{x} \right\|_{L^1_{t}} \leq \lambda^{-1/4} \frac{1}{\lambda^{5/2}} \epsilon^5 \langle t \rangle C^2,$$

$$\left\| \int_{\mathbb{R}} P_{\lambda} \left( F (\delta' \varphi) - \frac{1}{2} (\delta' \varphi)^2 \right) |\delta^3 \varphi_{x} dy, \varphi_{x} \right\|_{L^2_{t}(J_{\lambda})} \leq \lambda^{-1/4} \frac{1}{\lambda^{5/2}} \epsilon^5 \langle t \rangle C^2.$$

We are left to estimate

$$\left\langle \int_{\mathbb{R}} (\delta' \varphi)^2 |\delta^3 \varphi_{x} dy, u^x \right\rangle = \left\langle \partial_{x} P_{\lambda} Q^{\text{bal}} (\varphi), u^x \right\rangle + \left\langle \chi_{\lambda}^1 \partial_{x} P_{\lambda} Q^{\text{unbal}} (\varphi), u^x \right\rangle,$$

where $\chi_{\lambda}^1$ be a cut-off function enlarging $\chi_{\lambda}$. Due to the fact that $u^x$ is supported in the region $|x - v| \leq \lambda^{-1/2} r^{-1/2}$, the condition $(t, v) \in D$ will imply that the third term is identically zero, while lemma 6.7 implies that the second term is an acceptable error. Thus, we only have to analyze

$$\left\langle \partial_{x} P_{\lambda} Q^{\text{bal}} (\varphi), u^x \right\rangle = \left\langle \partial_{x} P_{\lambda} Q (\varphi_{x}), u^x \right\rangle.$$

Let $\chi^1$ be a cut-off function that is equal to 1 on the support of the wave packet $u_{x}$. Let $\tilde{\chi}$ be another cut-off function whose support is slightly larger than the one of $\chi^1$. We write

$$\left\langle \partial_{x} P_{\lambda} Q (\varphi_{x}), u^x \right\rangle = \left\langle \partial_{x} P_{\lambda} \tilde{Q} (\varphi_{x}), u^x \right\rangle + \left\langle \chi^1 \partial_{x} P_{\lambda} (1 - \tilde{\chi}) Q (\varphi_{x}), u^x \right\rangle.$$
As in the proof of lemma 6.7, we note that the operator norm bounds
\[ \| \chi^1 \partial_t P \lambda (1 - \tilde{x}) \|_{L^\infty \to L^\infty} + \| \chi^1 \partial_t P \lambda (1 - \tilde{x}) \|_{L^2 \to L^2} \lesssim \lambda^{1 - 2N} r^{-N} \]
for every \( N \) imply that the second term is acceptable error. This leaves us with the first.

We first replace \( \varphi_\lambda \) by \( \chi_\lambda \varphi_\lambda \). From lemma 6.6, we have
\[
|\langle \partial_t P \tilde{x} (Q(\varphi_\lambda) - Q(\chi_\lambda \varphi_\lambda)) , u^r \rangle | \\
\lesssim \lambda \| \partial_t (\chi_\lambda \varphi_\lambda) , \partial_t (1 - \chi_\lambda) \varphi_\lambda \|_{L^\infty} \left( \| u^r \|_{L^1} + \| u^r \|_{L^1}^{1 \to 2} \right) \\
+ \lambda \| \partial_t (\chi_\lambda \varphi_\lambda) , \partial_t (1 - \chi_\lambda) \varphi_\lambda \|_{L^2} \left( \| u^r \|_{L^1} + \| u^r \|_{L^1}^{1 \to 2} \right).
\]

By interpolation, along with lemma 6.1 and the condition \((t,v) \in D\), it follows that the errors are acceptable. The \( L^2 \)-bound is similar.

We now denote
\[
\psi(t,x) = t^{-1} \chi_\lambda (x/t) \gamma(t,x/t) e^{i\phi(x/t)}
\]
and replace \( \chi_\lambda \varphi_\lambda \) by \( \psi \). From lemma 6.6, we have
\[
|\langle \partial_t P \tilde{x} (Q(\varphi_\lambda) - Q(\psi)) , u^r \rangle | \\
\lesssim \lambda \| \partial_t (\chi_\lambda \varphi_\lambda) , \partial_t (1 - \chi_\lambda) \varphi_\lambda \|_{L^\infty} \left( \| u^r \|_{L^1} + \| u^r \|_{L^1}^{1 \to 2} \right) \\
+ \lambda \| \partial_t (\chi_\lambda \varphi_\lambda) , \partial_t (1 - \chi_\lambda) \varphi_\lambda \|_{L^2} \left( \| u^r \|_{L^1} + \| u^r \|_{L^1}^{1 \to 2} \right).
\]

By interpolation, along with lemmas 6.4 and 6.3, and the condition \((t,v) \in D\), it follows that the errors are acceptable. The \( L^1 \)-bound is similar.

We now denote
\[
\theta(t,x) = t^{-1} \chi_\lambda (x/t) \gamma(t,x/t) e^{i\phi(x/t)}
\]
and replace \( \psi \) by \( \theta \). We evaluate
\[
\langle \partial_t P \tilde{x} (Q(\theta) - Q(\theta)) , u^r \rangle.
\]
We have
\[
|\tilde{x} (Q(\psi) - Q(\theta))| \lesssim \tilde{x} \left( \frac{x - vt}{\sqrt{|t\partial t^r(x)|}} \right) \int \left( |\delta \psi|^2 + |\delta \theta|^2 \right) |\delta \beta_\lambda(x) | \, dy.
\]
The support condition of \( \tilde{x} \) implies that \( x \) is in the region \(|x - vt| \lesssim \delta x = t^{1/2} \lambda^{-1/2} \). From lemma 6.5 we now get that
\[
|\tilde{x} (Q(\psi) - Q(\theta))| \lesssim t^{-3/2} \lambda^{-1/2} \|\psi\| \int \left( |\delta \psi|^2 + |\delta \theta|^2 \right) \, dy.
\]
Bernstein’s inequality, and lemma 2.6, and Sobolev embedding, imply that
\[
|\chi(Q(\psi) - Q(\theta))| \lesssim t^{-3/4} \lambda^{3/4} ||\varphi||_x \left( ||\psi_x||^2_{L^\infty} + ||\theta_x||^2_{L^\infty} \right) ||u^r||_{L^2_t} \\
+ t^{-3/4} \lambda^{3/4} ||\varphi||_x \left( ||\psi||_{L^\infty} |||D_1|^{1-\delta} \psi||_{L^2_t} + ||\theta||_{L^\infty} |||D_1|^{1-\delta} \theta||_{L^2_t} \right) ||u^r||_{L^2_t} \\
\lesssim t^{-3/4} \lambda^{3/4} ||\varphi||_x \left( ||\psi_x||^2_{L^\infty} + ||\theta_x||^2_{L^\infty} \right) ||u^r||_{L^2_t} \\
+ t^{-3/4} \lambda^{3/4} ||\varphi||_x \left( ||\psi||_{L^\infty} ||\psi_x||^{1/3}_{L^\infty} + ||\theta||_{L^\infty} ||\theta_x||^{1/3}_{L^\infty} \right) ||u^r||_{L^2_t}.
\]

From lemma 6.3 along with the condition \((t, v) \in \mathcal{D}\), it follows that this error is acceptable.

The \(L^2_t\)-bound is similar.

We are left to analyze
\[
t^{-3/2} \gamma(t, v) |\gamma(t, v)|^2 \left< \partial_t \chi, \chi \lambda e^{i\phi(s)} \right>, u^r \right>.
\]

Since by lemma 6.3,
\[
||t^{-3/2} \gamma(t, v) |\gamma(t, v)|^2 ||L^\infty_{(t, x)} \lesssim ||\varphi||^2_{L^\infty_x}, \\
||t^{-3/2} \gamma(t, v) |\gamma(t, v)|^2 ||L^2_{(t, x)} \lesssim t^{-1/2} ||\varphi||_{L^\infty_x} ||\varphi||_{L^2_t}
\]

it suffices to estimate
\[
|\left< \partial_t \chi, \chi \lambda e^{i\phi(s)} \right>, u^r | - t^{i} q(\xi \tau) \xi \left< \chi \lambda (v) \right>^3 \lesssim \lambda^{-1/4} \lambda^{3/10 + C \epsilon}.
\]

We note that
\[
\delta^y \left( \chi \lambda e^{i\phi(s)} \right) = \chi \lambda \delta^y \left( e^{i\phi(s)} \right) + \delta^y \left( \chi \lambda \right) e^{i\phi(s+\gamma)/2}.
\]

Lemma 2.6, implies that for every \(\delta > 0\) we have
\[
\left| \left< \partial_t \chi, \chi \lambda e^{i\phi(s+\gamma)/2} \right>, \delta^y \left( \chi \lambda e^{-i\phi(s)}/2 \right) \delta^y \left( \chi \lambda e^{i\phi(s)} \right) dy, u^r \right> \\
\lesssim \lambda \left( t^{-1/2} + t^{-1/2} \lambda^2 \right).
\]

The most problematic contribution is the one that arises from the first term. We have
\[
\lambda^{9/4} \lambda^{-1/2} ||\varphi||^3_{L^\infty_x} \lesssim \lambda^{-2} ||\varphi||^3_{L^\infty_x} \lesssim t^{-6/5} \lambda^3 C \epsilon^2.
\]

The other term is analogous.

The \(L^2_t\)-bound is treated similarly, and so is the case in which one chooses the term \(\delta^y \left( \chi \lambda e^{-i\phi(s+\gamma)/2} \right)\) in the expansion of \(\delta^y \left( \chi \lambda e^{i\phi(s)} \right)\). This leaves us with
\[
\left< \partial_t \chi, \left( \chi \lambda (s/t)^3 e^{i\phi(s)} \right) \right>, u^r \right>.
\]

Lemma 6.8, implies that we can replace the latter with
\[
\left< \partial_t \chi, \left( \chi \lambda (s/t)^3 e^{i\phi(s)} (\phi' (s/t)^3 q (1)) \right) \right>, u^r \right>.
\]
with error bounded by
\[ \lambda^{1/2} \left( \frac{\lambda^3}{t^{2-a}} + \frac{\lambda^2}{t^{1-a}} + \frac{1}{t^a} \right). \]

We note that one problematic contribution is the one arising from the last term. We have the bound
\[ \frac{\lambda^{5/4}}{t^{2a-1/2}} \| \varphi \|_L^3 \lesssim t^{-2a} \| \varphi \|_X^3. \]

By picking \( a = \frac{3}{7} \), we deduce that this contribution is acceptable. The only other problematic contribution is the one arising from the first term, for which we bound
\[ \frac{\lambda^{17/4}}{t^{3/2-3a}} \| \varphi \|_L^3 \lesssim t^{3a-3} \| \varphi \|_X^3 \lesssim t^{-6/5} \epsilon^3 \xi^2. \]

The contribution arising from the second term can be immediately bounded by
\[ \frac{\lambda^{13/4}}{t^{1/2-a}} \| \varphi \|_L^3 \lesssim t^{\alpha-2} \| \varphi \|_X^3 \lesssim t^{-6/5} \epsilon^3 \xi^2. \]

The \( L^2 \)-bound is similar.

This means that we have to analyze
\[
q(1) \left( \partial_t \left( \chi \left( \frac{x}{t} \right)^3 \phi'(x/t)^2 e^{itb(x/t)} \right), u^t_\lambda \right) = q(1) \left( \left( \chi \left( \frac{x}{t} \right) \phi'(x/t)^3 e^{itb(x/t)} \right), u^t_\lambda \right) \\
+ q(1) t^{-1} \left( \left( 3 \chi \left( \frac{x}{t} \right)^2 \chi'_t \left( \frac{x}{t} \right) \phi'(x/t)^2 + 2 \chi \left( \frac{x}{t} \right)^3 \phi'(x/t) \phi''(x/t) e^{itb(x/t)} \right), u^t_\lambda \right),
\]

where the last contribution can be immediately shown to be an acceptable error by using the condition \( (t, v) \in D \). Further, we may replace \( u^t_\lambda \) by \( u^t \). To see this, from the proof of lemma 5.8 in [18], we have
\[ |P_{\neq \lambda} u^t| \lesssim \lambda^{1/2} (1 + |y|)^{-1-\delta} t^{-1-\delta} \lambda^{-1-\delta}, \quad y = (x - vt) \left( a''(\xi_v) \right)^{-1}, \]

and
\[ \left( \chi \left( \frac{x}{t} \right) \phi'(x/t)^3 e^{itb(x/t)} \right) \lesssim \lambda^3. \]

Thus,
\[
\left| \left( \chi \left( \frac{x}{t} \right) \phi'(x/t)^3 e^{itb(x/t)}, P_{\neq \lambda} u^t \right) \right| \lesssim \lambda^3 t^{-1/2-\delta} \lambda^{-1/2} \lesssim t^{-1/2-\delta} \lambda^{2-\delta},
\]

which along with the condition \( (t, v) \in D \) shows that this is an acceptable error.

As \( u^t \) is supported in the region \( \left| \frac{x}{t} - v \right| \lesssim t^{-1/2} \lambda^{-1/2} \), we can replace \( x/t \) by \( v \) in \( \chi \left( \frac{x}{t} \right) \phi'(x/t) \), with acceptable errors. As \( \chi \left( v \right) = 1 \), the remaining term is now
\[ iq(1) \left( \chi \left( v \right) \xi_v \right)^3 \left( e^{ib(x/t)}, u^t \right) = t^2 iq(1) \xi_v, \]

as desired.
6.9. Closing the bootstrap argument

We recall that
\[ \| \varphi_\lambda \|_{L^\infty_t} \lesssim \frac{1}{\sqrt{t}} \lambda^{-3/4-\delta_1} \| \varphi \|_{X} \lesssim \frac{1}{\sqrt{t}} \lambda^{-(3/4-\delta_1)} t \epsilon C^2, \]
when \( \lambda \leq 1 \) and
\[ \| \varphi_\lambda \|_{L^\infty_t} \lesssim \frac{1}{\sqrt{t}} \lambda^{-3/4+3\delta_2/2} \| \varphi \|_{X} \lesssim \frac{1}{\sqrt{t}} \lambda^{-(3/4+3\delta_2/2)} t \epsilon C^2, \]
when \( \lambda > 1 \).

Thus, if \( t \lesssim \lambda^N \) when \( \lambda > 1 \), and if \( t \lesssim \lambda^{-N} \) when \( \lambda \leq 1 \), where \( N \) can be chosen arbitrarily, we get the desired bounds. We are left to analyze \( t \gtrsim \lambda^N \) when \( \lambda > 1 \), and \( t \gtrsim \lambda^{-N} \) when \( \lambda \leq 1 \).

We recall the following bounds in the elliptic region:
\[ \| D_t^{1/4-\delta} ((1 - \chi_\lambda) \varphi_\lambda (x)) \|_{L^\infty_t} \lesssim \lambda^{1/4-\delta} \| \varphi \|_{X} \lesssim \lambda^{1/4-\delta} t \epsilon C^2 \]
\[ \| D_t^{1/4+\delta} \partial_\lambda ((1 - \chi_\lambda) \varphi_\lambda (x)) \|_{L^\infty_t} \lesssim \lambda^{3/4+\delta} \| \varphi \|_{X} \lesssim \lambda^{3/4+\delta} t \epsilon C^2 \]
which gives the desired bounds when \( t \gtrsim \lambda^N \) (\( \lambda > 1 \)), and \( t \gtrsim \lambda^{-N} \) (\( \lambda \leq 1 \)). We still have to bound \( \chi_\lambda \varphi_\lambda \). We recall that, if \( x/t \in J_\lambda \), and \( r(t,x) = \chi_\lambda \varphi_\lambda (t,x) - \frac{1}{\sqrt{t}} \lambda \gamma (t, x/t) e^{i \varphi (x/t)} \),
\[ t^{1/2} \| r_\lambda \|_{L^\infty_t} \lesssim t^{-1/4} \lambda^{-5/4} \epsilon C^2. \]

We note that
\[ t^{-1/4} \lambda^{-5/4} \epsilon C^2 \lesssim \lambda^{-(3/4-\delta_1)} t \epsilon \]
when \( \lambda \leq 1 \), because this is equivalent to
\[ \lambda^{-1/2+\delta_1} \lesssim t^{1/4-\epsilon C^2} \]
(this is true when \( t \gtrsim \lambda^{-N} \)), and that
\[ t^{-1/4} \lambda^{-5/4} \epsilon C^2 \lesssim \lambda^{-(2+3\delta_2/2)} \epsilon \]
when \( \lambda > 1 \), because this is equivalent to
\[ \lambda^{3/4+3\delta_2/2} \lesssim t^{1/4-\epsilon C^2} \]
(this is true when \( t \gtrsim \lambda^N \)).

This means that we only need the bounds
\[ | \gamma (t,v) | \lesssim \epsilon \lambda^{-(3/4-\delta_1)} \]
when \( \lambda \leq 1 \), and
\[ | \gamma (t,v) | \lesssim \epsilon \lambda^{-(2+3\delta_2/2)} \]
when \( \lambda > 1 \). By initializing at time \( t = 1 \), up to which the bounds are known to be true from the energy estimates, and by using proposition 6.9, we reach the desired conclusion.
7. Modified scattering

In this section we discuss the modified scattering behaviour of the global solutions constructed in section 6. We begin by proving the conservation of mass for the solutions of (1.4):

**Proposition 7.1.** For solutions $\varphi$ of (1.4), $\|\varphi(t)\|_{L^2}^2$ is conserved in time.

**Proof.** We have

$$\frac{d}{dt} \|\varphi\|_{L^2}^2 = \int \varphi_t \cdot \varphi \, dx$$

$$= -2 \int F(\delta^s \varphi) \delta^s \varphi_t \cdot \varphi \, dx - 2 \int \varphi \cdot D_x |\varphi_x| \, dx$$

$$= -2 \int F(\delta^s \varphi) \delta^s \varphi_t \cdot \varphi \, dx := -2I.$$

We note that by the change of variables $(x, y) \mapsto (x + y, -y)$,

$$I = -\int \int F(\delta^s \varphi) |\delta^s \varphi_x| \cdot \varphi (x + y) \, dx \, dy.$$

Thus,

$$-2I = \int \int F(\delta^s \varphi) |\delta^s \varphi_x| \cdot (\varphi (x + y) - \varphi (x)) \, dx \, dy$$

$$= \int |y| \int F(\delta^s \varphi) \delta^s \varphi \delta^s \varphi_x \, dx \, dy$$

$$= \int |y| \int \partial_y (G(\delta^s \varphi)) \, dx \, dy = 0,$$

where $G(x) = \frac{x^2}{2} - \sqrt{1 + x^2}$.

Recall the asymptotic equation

$$\gamma^s(t, v) = iq(\xi_\epsilon)\xi_\epsilon^{-1} |\gamma(t, v)|^2 \gamma(t, v) + f(t, v).$$

As $t \to \infty$, $\gamma(t, v)$ converges to the solution of the equation

$$\hat{\gamma}^s(t, v) = iq(\xi_\epsilon)\xi_\epsilon^{-1} \hat{\gamma}(t, v) |\hat{\gamma}(t, v)|^2,$$

whose solution is

$$\hat{\gamma}(t, v) = W(v) \phi(q(\xi_\epsilon)\xi_\epsilon^{-1} |W(v)|^2).$$

We can immediately see that $W(v)$ is well-defined, as $|W(v)| = |\hat{\gamma}(t, v)|$, which is a constant, and

$$W(v) = \lim_{t \to \infty} \hat{\gamma}(t, v) \phi(q(\xi_\epsilon)\xi_\epsilon^{-1} |W(v)|^2).$$

**Corollary 7.2.** Let $v \in J_\lambda$. Under the assumption $(t, v) \in D$, we have the asymptotic expansions

$$\|\gamma(t, v) - W(v) e^{iq(\xi_\epsilon)\xi_\epsilon^{-1} |W(v)|^2} \|_{L^2 \cap L^\infty(J_\lambda)} \lesssim \lambda^{-1/4} t^{-1/5} e^{C_1 \epsilon}.$$ (7.1)
**Proof.** This is an immediate consequence of proposition 6.9.

**Proposition 7.3.** Under the assumption

\[\|\varphi_0\|_X \lesssim \epsilon \ll 1,\]

the asymptotic profile \(W\) defined above satisfies

\[
\|W(v)\|_{L^2} + \|e^{-\frac{\lambda}{2} \epsilon} e^{i|\xi|\epsilon^2} [D\epsilon]^{1-C\epsilon^2} W(v)\|_{L^2} \lesssim \epsilon.
\]

**Proof.** We fix \(\lambda\), and let \(t \gtrsim \max\{1, \lambda^{-1}\} := t_\lambda\). From corollary 7.2 we know that

\[
\|W(v) - e^{-i\mathbf{q}(\xi)\mathbf{x}, \log(t)}\|_{L^2(t)} \lesssim \lambda^{-1/4} t^{-1/5} + C\epsilon^2 \epsilon.
\]

From the product and chain rules with lemma 6.3, we have

\[
\|\partial_v \left(e^{-i\mathbf{q}(\xi)\mathbf{x}, \log(t)}\right)\|_{L^2(t)} \lesssim \lambda^{-1} \log(t) \epsilon t C^2 \epsilon.
\]

In this case,

\[
W(v) = O_{H^1(t)} \left(\lambda^{-1} \log(t) \epsilon t C^2 \epsilon\right) + O_{L^2(t)} \left(\lambda^{-1/4} t^{-1/5} + C\epsilon^2 \epsilon\right), \quad t \gtrsim t_\lambda.
\]

By interpolation this will imply that for \(C_1\) large enough we have

\[
\|W(v)\|_{H^\frac{1}{2} - C_1 \epsilon^2 (t)} \lesssim \lambda^{-\frac{C_1 \epsilon^2}{4}} \epsilon, \quad \lambda > 1
\]

respectively

\[
\|W(v)\|_{H^\frac{1}{2} - C_1 \epsilon^2 (t)} \lesssim \lambda^{-\frac{C_1 \epsilon^2}{4}} \epsilon, \quad \lambda < 1.
\]

By dyadic summation over \(\lambda \geq 1\) and \(\lambda \leq 1\),

\[
\|e^{-\frac{\lambda}{2} \epsilon} e^{i|\xi|\epsilon^2} [D\epsilon]^{1-C\epsilon^2} W(v)\|_{L^2} \lesssim \epsilon.
\]

**Data availability statement**

No new data were created or analysed in this study.

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