Reduced mathematical model of the flow in a deep extended natural channel

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Abstract. A slightly curved open stationary channel flow is considered. It is assumed that the flow bed is given by a sufficiently smooth function, and the flow itself is long and deep, i.e. the length of the section under consideration is much longer than its width and depth, which, in turn, have similar dimensions. Using the previously developed technique for obtaining reduced mathematical models for channel flows, it is possible, at constant viscosity, to reduce the solution of a complex three-dimensional problem to solving a sequence of standard two-dimensional problems on cross-sections.

1. Introduction

The development of mathematical models of the hydrodynamics of natural channel flows is important for solving a huge number of practical problems: designing navigable routes, conducting hydro-construction works, predicting and assessing the consequences of natural and man-made disasters on rivers, etc. Mathematical modeling helps to optimize the planning of full-scale hydrological experiments and reduces the cost of their implementation. Modeling hydrodynamic processes in natural streams is a very difficult task in continuum mechanics and presents a real challenge for researchers [1].

Mathematical modeling of natural watercourses on the basis of complete hydrodynamic equations is very difficult and requires huge computational resources. Therefore, it is a good idea to use mathematical models that are simplified but adequately simulate the process. Such models should take into account the features of the problem, and their derivation can be based, for example, on the method of a small parameter, as described in [2].

The model equations obtained in [2] describe the channel flow as a spatially three-dimensional process, but they are much simpler than the complete three-dimensional equations of hydrodynamics, and in some cases, even two-dimensional ones. The simplification is related to the fact that we consider extended and slightly curved channel flows, which are not essentially three-dimensional objects (although, of course, they cannot be considered as one-dimensional flows).

In this paper one of the reduced models proposed in [2] is considered. We consider a model of a deep, slightly curved extended flow. It is assumed that the free flow surface is flat, and the viscosity is constant. In this case, it is possible to reduce the solution of a complex initial problem to solving a sequence of standard two-dimensional problems on cross-sections. This reduction can be used to efficiently parallelize computations. The resulting algorithm is supposed to be used in the future to develop computational schemes for numerical simulation of turbulent flows.
2. Equations of the reduced mathematical model

The equations of the reduced mathematical model of a steady flow in a deep extended natural channel with a flat free boundary have the form [2]

\[
\begin{align*}
\frac{\partial}{\partial y} \left( \nu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \nu \frac{\partial u}{\partial z} \right) &= -GI, \\
\frac{\partial}{\partial z} \left|_{z=0} \right. &= -F_x, \quad u \left|_{z=h(x, y)} = 0. \right.
\end{align*}
\]

\[
\begin{align*}
\left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2} \right) \left( \nu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right) + 2 \frac{\partial^2}{\partial y \partial z} \left( \nu \left( \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right) \right) &= 0, \\
\left. \frac{\partial v}{\partial z} \right|_{z=0} &= -F_y, \quad w \left|_{z=0} = 0, \quad v = w \left|_{z=h(x, y)} = 0. \right.
\end{align*}
\]

Here \( u, v, w \) are the components of the velocity vector in the longitudinal, transverse and vertical directions, respectively; \( h(x, y) \) is a function that sets the channel shape; \( \nu \) is a turbulent viscosity parameter (in the case of constant viscosity it is equal to 1); \( GI \) is a parameter characterizing the channel slope; \( F_x \) and \( F_y \) are parameters characterizing external influence to the surface of the stream (associated, for example, with the presence of wind).

Rectangular Cartesian coordinates \( x, y \) and \( z \) are introduced as follows, that the \( xy \) plane lies on the flow surface, and the \( z \) axis is directed towards the bottom. The \( x \) axis is directed along the slope of the stream, and the \( y \) axis is from the left bank to the right perpendicular to the \( x \)-axis. The origin is at in the inlet section of the considered section of the channel at the same distance from the banks (figure 1). The boundary value problem (1), (2) describes the hydrodynamics of a deep extended flow in a slightly curved non-eroding channel.

![Figure 1. Stream cross-section](image)

The system (1), (2) consists of two subsystems: an independent boundary value problem (1) for determining the longitudinal velocity \( u(x, y, z) \) and a boundary value problem (2) to determine the transverse \( v(x, y, z) \) and vertical \( w(x, y, z) \) velocities, where the function \( u(x, y, z) \) is already assumed to be known.

For the longitudinal velocity \( u(x, y, z) \) we have the standard mixed boundary value problem for Poisson’s equation (1), which is solved independently, moreover, the coordinate \( x \) enters the boundary condition as a parameter.

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For the transverse and vertical velocities \( v(x, y, z) \) and \( w(x, y, z) \), we have an inhomogeneous boundary value problem (2), where the right-hand sides of the equations contain \( u(x, y, z) \) and
may be considered known. The $x$ coordinate is also included in the boundary condition and in the right-hand side (2) as a parameter.

Thus, the hydrodynamic subsystem (1), (2), being formally three-dimensional, actually represents two two-dimensional boundary value problems that must be solved sequentially on sections along the flow. Such a splitting can be very convenient when used for calculations multiprocessor computing systems.

Note that to calculate such an important hydraulic flow characteristic as flow rate, it is sufficient to find only the longitudinal velocity $u(x, y, z)$ in a given section.

3. The case of constant viscosity

In general, the system (1), (2), must be solved numerically, in any case, for flows with an arbitrary shape of the channel cross-section and a variable coefficient of turbulent viscosity $\nu$. However, in the case of constant viscosity (i.e., for $\nu = 1$), further simplifications and reduction to a sequence of standard boundary value problems are possible.

The longitudinal component of the velocity $u(y, z)$ is determined from the mixed boundary value problem for the Poisson equation

$$\Delta u = -GI, \quad \frac{\partial u}{\partial z}\bigg|_{z=0} = -F_x, \quad u\bigg|_{z=h(y)} = 0.$$ (3)

Hereinafter, $\Delta$ is the two-dimensional Laplace operator with respect to the variables $y$ and $z$.

The solution of the mixed problem (3) could be conveniently represented in the form

$$u(y, z) = U(y, z) + F_x(h(y) - z),$$ (4)

where $U(y, z)$ is the solution to the auxiliary standard Dirichlet problem for the Poisson equation

$$\Delta U = -g(y), \quad g(y) = GI + F_x h''(y), \quad U\big|_{\Gamma} = 0.$$ (5)

The domain for which the problem (5) is solved is obtained by symmetric reflection of the flow cross-section (see figure 1) into the half-plane $z < 0$, and its boundary $\Gamma$ is determined by the equation $h^2(y) - z^2 = 0$.

For the vertical and transverse velocity components $v(y, z)$ and $w(y, z)$ from (2) at constant viscosity, we have a boundary value problem of the form

$$\frac{\partial^3 v}{\partial y^2 \partial z} + \frac{\partial^3 v}{\partial z^3} - \frac{\partial^3 w}{\partial y \partial z^2} - \frac{\partial^3 w}{\partial y^3} = 0, \quad \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x},$$ (6)

$$\frac{\partial v}{\partial z}\bigg|_{z=0} = -F_y, \quad w\bigg|_{z=0} = 0, \quad v\bigg|_{z=h(y)} = w\bigg|_{z=h(y)} = 0.$$

The first equation (6) can be rewritten as

$$\Delta \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = 0.$$ (7)

Then, expressing the transverse and vertical components of the velocity vector in terms of the potential function $\phi(y, z)$ and the stream function $\psi(y, z)$ by the formulas [3]

$$v = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial y},$$ (8)

we arrive after substituting (8) in first equation (6) and taking into account (7), to the boundary value problem for the biharmonic equation

$$\Delta^2 \psi = 0, \quad \Delta \psi|_{z=0} = -F_y,$$
which, by replacing $\Omega = -\Delta \psi$, is reduced to the boundary value problem for the Laplace equation

$$\Delta \Omega = 0, \quad \Omega|_{z=0} = F_y. \quad (9)$$

Note that the function $\Omega$ is a vortex of the vector $(v, w)$ [3]

$$\Omega = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}. \quad (8)$$

As a solution to the problem (9), we define a function that is harmonic in the half-plane $z \geq 0$ and vanishes everywhere on the $y$ axis, except for the segment $-a \leq y \leq a$, where it has the value $F_y$. Figure 1 explains the meaning of the $a$ parameter. In this case, the Poisson integral has the form [4]

$$\Omega(y, z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Omega|_{z=0} z}{(\tau - y)^2 + z^2} d\tau = \frac{F_x}{\pi} \int_{-a}^{a} \frac{z}{(\tau - y)^2 + z^2} d\tau = F_x \left( \arctan \frac{a + y}{z} + \arctan \frac{a - y}{z} \right). \quad (10)$$

Using the representation (10), we write the expression for the stream function as a solution of the Poisson equation $\Delta \psi = -\Omega$

$$\psi(y, z) = \frac{1}{2\pi} \int_{\partial D} \left( \frac{\partial \psi}{\partial n} \ln \frac{1}{r} - \psi \frac{\partial}{\partial n} \left( \ln \frac{1}{r} \right) \right) d\gamma + \frac{1}{2\pi} \int _D \Omega \ln \frac{1}{r} d\omega. \quad (11)$$

Further, after substituting the representation (8) into the continuity equation (the second equation in the problem (6)), we obtain the Neumann boundary value problem for the Poisson equation $\Delta \phi = -\Omega$

$$\frac{\partial \phi}{\partial z}|_{z=0} = \frac{\partial \psi}{\partial y}|_{z=0}, \quad \frac{\partial \phi}{\partial y}|_{z=h} = -\frac{\partial \psi}{\partial z}|_{z=h}, \quad \frac{\partial \phi}{\partial z}|_{z=h} = \frac{\partial \psi}{\partial y}|_{z=h}. \quad (13)$$

The formulas (13) define the normal derivative of the function $\phi(y, z)$ on the boundary of the section area. Indeed, if the boundary of the flat region is given by the equation $\eta(y, z) = 0$, then the normal derivative of the function $\phi(y, z)$ on this boundary is calculated by the formula

$$\frac{\partial \phi}{\partial n}|_{\eta=0} = (\nabla \phi, \mathbf{n}) = \frac{1}{\|\nabla \eta\|} (\nabla \phi, \nabla \eta). \quad (14)$$

In the case when $\eta(y, z) = h(y) - z$ (see figure 1), the formula (14) takes the form

$$\frac{\partial \phi}{\partial n}|_{z=h} = \frac{1}{\sqrt{1 + \left( \frac{\partial h}{\partial y} \right)^2}} \left( \frac{\partial \phi}{\partial y} \frac{\partial h}{\partial y} + \frac{\partial \phi}{\partial z} \frac{\partial h}{\partial z} \right)|_{z=h} = \frac{1}{\sqrt{1 + \left( \frac{\partial h}{\partial y} \right)^2}} \left( \frac{\partial \psi}{\partial y} \frac{\partial h}{\partial y} + \frac{\partial \psi}{\partial y} \frac{\partial h}{\partial y} \right)|_{z=h}$$

and on the flat free boundary $z = 0$

$$\frac{\partial \phi}{\partial n}|_{z=0} = \frac{\partial \phi}{\partial z}|_{z=0} = -\frac{\partial \psi}{\partial y}|_{z=0}. \quad (15)$$
Thus, the solution to the problem (12), (13) can be written as an integral of the form

\[ \phi(y, z) = \frac{1}{2\pi} \int_{\partial D} \left( \frac{\partial \phi}{\partial n} \ln \frac{1}{r} - \phi \frac{\partial}{\partial n} \left( \ln \frac{1}{r} \right) \right) d\gamma - \frac{1}{2\pi} \int_{\partial \Omega} \frac{\partial u}{\partial x} \ln \frac{1}{r} d\omega. \] (15)

The solvability condition for the problem (12), (13) implies an additional condition that the stream function (11) and its normal derivative on the boundary of the flow cross-section

\[ \int_{-a}^{a} \frac{1}{1 + \left( \frac{\partial h}{\partial y} \right)^2} \left( \frac{\partial \psi}{\partial z} \frac{\partial h}{\partial y} + \frac{\partial \psi}{\partial y} \right) \bigg|_{z=h} dy - \psi(a, 0) + \psi(-a, 0) = \int_{-a}^{a} \int_{0}^{h(y)} \frac{\partial u}{\partial x} dz dy. \] (16)

So, the solution of the system of differential equations (1), (2) of the reduced mathematical model of the flow in a deep extended and slightly curved natural channel, when the viscosity can be considered constant, consists of the following steps:

- find the solution to the auxiliary standard Dirichlet problem for the Poisson equation (5), and obtain the longitudinal velocity (4);
- taking into account (10), we find the stream function (11) satisfying the condition (16);
- find the potential (15) using the boundary conditions (13);
- finally we find the transverse and vertical speed by the formulas (8).

For channel flows with simple sections (rectangle, semicircle), the indicated solution algorithm can be implemented analytically.

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