ADDING MANY RANDOM REALS MAY ADD MANY COHEN REALS

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abstract. Let $\kappa$ be an infinite cardinal. Then, forcing with $\mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$ adds a generic filter for $\mathbb{C}(\kappa)$; where $\mathbb{R}(\kappa)$ and $\mathbb{C}(\kappa)$ are the forcing notions for adding $\kappa$-many random reals and adding $\kappa$-many Cohen reals respectively.

1. introduction

For a cardinal $\kappa > 0$ let $\mathbb{R}(\kappa)$ be the forcing notion for adding $\kappa$-many random reals and let $\mathbb{C}(\kappa)$ be the Cohen forcing for adding $\kappa$-many Cohen reals\(^1\).

It is a well-know fact that forcing with $\mathbb{R}(1) \times \mathbb{R}(1)$ adds a Cohen real; in fact, if $r_1, r_2$ are the added random reals, then $c = r_1 + r_2$ is Cohen [1]. This in turn implies all reals $c + a$, where $a \in \mathbb{R}^V$, are Cohen, and so, we have continuum many Cohen reals over $V$. However, the sequence $\langle c + a : a \in \mathbb{R}^V \rangle$ fails to be $\mathbb{C}(\mathbb{Q}(\kappa)^V)$-generic over $V$. In fact, there is no sequence $\langle c_i : i < \omega_1 \rangle \in V[r_1, r_2]$ of Cohen reals which is $\mathbb{C}(\omega_1)$-generic over $V$.

In this paper, we extend the above mentioned result by showing that if we force with $\mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$, then in the resulting extension, we can find a sequence $\langle c_i : i < \kappa \rangle$ of reals which is $\mathbb{C}(\kappa)$-generic over the ground model:

**Theorem 1.1.** Let $\kappa$ be an infinite cardinal. Then, forcing with $\mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$ adds a generic filter for $\mathbb{C}(\kappa)$.

In Section 2, we briefly review the forcing notions $\mathbb{C}(\kappa)$ and $\mathbb{R}(\kappa)$. Then in Section 3, we state some results from analysis which are needed for the proof of above theorem and in Section 4, we give a proof of Theorem 1.1.

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1See Section 2 for the definition of the forcing notions $\mathbb{R}(\kappa)$ and $\mathbb{C}(\kappa)$.
2. **Cohen and Random forcings**

In this section we briefly review the forcing notions $\mathbb{C}(\kappa)$ and $\mathbb{R}(\kappa)$, and present some of their properties.

2.1. **Cohen forcing.** Let $I$ be a non-empty set. The forcing notion $\mathbb{C}(I)$, the Cohen forcing for adding $|I|$-many Cohen reals is defined by

$$\mathbb{C}(I) = \{ p : I \times \omega \to 2 : |p| < \aleph_0 \},$$

which is ordered by reverse inclusion.

**Lemma 2.1.** $\mathbb{C}(I)$ is c.c.c.

Assume $G$ is $\mathbb{C}(I)$-generic over $V$, and set $F = \bigcup G : I \times \omega \to 2$. For each $i \in I$ set $c_i : \omega \to 2$ be defined by $c_i(n) = F(i, n)$. Then:

**Lemma 2.2.** For each $i \in I, c_i \in 2^\omega$ is a new real and for $i \neq j$ in $I, c_i \neq c_j$. Further, $V[G] = V[\langle c_i : i \in I \rangle]$.

The reals $c_i$ are called Cohen reals. By $\kappa$-Cohen reals over $V$, we mean a sequence $\langle c_i : i < \kappa \rangle$ which is $\mathbb{C}(\kappa)$-generic over $V$.

2.2. **Random forcing.** In this subsection we briefly review random forcing. Suppose $I$ is a non-empty set and consider the product measure space $2^{I \times \omega}$ with the standard product measure $\mu_I$ on it. Let $\mathcal{B}(I)$ denote the class of Borel subsets of $2^{I \times \omega}$. Recall that $\mathcal{B}(I)$ is the $\sigma$-algebra generated by the basic open sets

$$[s_p] = \{ x \in 2^{I \times \omega} : x \supseteq p \},$$

where $p \in \mathbb{C}(I)$. Also $\mu_I([s_p]) = 2^{-|p|}$.

For Borel sets $S, T \in \mathcal{B}(I)$ set

$$S \sim T \iff S \triangle T \text{ is null},$$

where $S \triangle T$ denotes the symmetric difference of $S$ and $T$. The relation $\sim$ is easily seen to be an equivalence relation on $\mathcal{B}(I)$. Then $\mathbb{R}(I)$, the forcing for adding $|I|$-many random reals, is defined as
\( R(I) = \mathbb{B}(I)/\sim \).

Thus elements of \( R(I) \) are equivalent classes \([S]\) of Borel sets modulo null sets. The order relation is defined by

\[ [S] \leq [T] \iff \mu(S \setminus T) = 0. \]

The following fact is standard.

**Lemma 2.3.** \( R(I) \) is c.c.c.

Using the above lemma, we can easily show that \( R(I) \) is in fact a complete Boolean algebra. Let \( \widetilde{F} \) be an \( R(I) \)-name for a function from \( I \times \omega \) to \( 2 \) such that for each \( i \in I, n \in \omega \) and \( k < 2, \| \widetilde{F}(i, n) = k \|_{R(I)} = p^{i,n}_k \), where

\[ p^{i,n}_k = [x \in 2^{I \times \omega} : x(i, n) = k]. \]

This defines \( R(I) \)-names \( \mathcal{L}_i \in 2^\omega, i \in I \), such that

\[ \| \forall n < \omega, \mathcal{L}_i(n) = \mathcal{F}(i, n) \|_{R(I)} = 1_{R(I)} = [2^{I \times \omega}]. \]

**Lemma 2.4.** Assume \( G \) is \( R(I) \)-generic over \( V \), and for each \( i \in I \) set \( r_i = \mathcal{L}_i[G] \). Then each \( r_i \in 2^\omega \) is a new real and for \( i \neq j \) in \( I \), \( r_i \neq r_j \). Further, \( V[G] = V[\langle r_i : i \in I \rangle] \).

The reals \( r_i \) are called random reals. By \( \kappa \)-random reals over \( V \), we mean a sequence \( \langle r_i : i < \kappa \rangle \) which is \( R(\kappa) \)-generic over \( V \).

3. Some results from analysis

A famous theorem of Steinhaus [2] from 1920 asserts that if \( A, B \subseteq \mathbb{R}^n \) are measurable sets with positive Lebesgue measure, then \( A + B \) has an interior point; see also [3]. Here, we need a version of Steinhaus theorem for the space \( 2^{\kappa \times \omega} \).

For \( S, T \subseteq 2^{\kappa \times \omega} \), set \( S + T = \{ x + y : x \in S \text{ and } y \in T \} \), where \( x + y : \kappa \times \omega \to 2 \) is defined by

\[ (x + y)(\alpha, n) = x(\alpha, n) + y(\alpha, n) \pmod{2}. \]

Note that the above addition is continuous.
Lemma 3.1. Suppose $S \subseteq 2^{\kappa \times \omega}$ is Borel and non-null. Then $S - S$ contains an open set around the zero function 0.

Proof. We follow [3]. Set $\mu = \mu_\kappa$ be the product measure on $2^{\kappa \times \omega}$. As $S$ is Borel and non-null, there is a compact subset of $S$ of positive $\mu$-measure, so may suppose that $S$ itself is compact. Let $U \supseteq S$ be an open set with $\mu(U) < 2 \cdot \mu(S)$. By continuity of addition, we can find an open set $V$ containing the zero function 0 such that $V + S \subseteq U$.

We show that $V \subseteq S - S$. Thus suppose $x \in V$. Then $(x + S) \cap S \neq \emptyset$, as otherwise we will have $(x + S) \cup S \subseteq U$, and hence $\mu(U) \geq 2 \cdot \mu(S)$, which is in contradiction with our choice of $U$. Thus let $y_1, y_2 \in S$ be such that $x + y_1 = y_2$. Then $x = y_2 - y_1 \in S - S$ as required.

Similarly, we have the following:

Lemma 3.2. Suppose $S, T \subseteq 2^{\kappa \times \omega}$ are Borel and non-null. Then $S + T$ contains an open set.

Suppose $S, T \subseteq 2^{\kappa \times \omega}$ are Borel and non-null. It follows from Lemma 3.2 that for some $p \in \mathbb{C}(\kappa)$, $[s_p] \subseteq S + T$. Thus, by continuity of the addition, we can find $x \in S$ and $y \in T$ such that:

- $(x + y) \restriction \text{dom}(p) = p$.
- The sets $S \cap [s_x]_{\text{dom}(p)}$ and $T \cap [s_y]_{\text{dom}(p)}$ are Borel and non-null.

4. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. Thus force with $\mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$ and let $G \times H$ be generic over $V$. Let $\langle \langle r_\alpha : \alpha < \kappa \rangle, \langle s_\alpha : \alpha < \kappa \rangle \rangle$ be the sequence of random reals added by $G \times H$.

For $\alpha < \kappa$ set $c_\alpha = r_\alpha + s_\alpha$. The following completes the proof:

Lemma 4.1. The sequence $\langle c_\alpha : \alpha < \kappa \rangle$ is a sequence of $\kappa$-Cohen reals over $V$.

Proof. It suffices to prove the following:

For every $([S], [T]) \in \mathbb{R}(\kappa) \times \mathbb{R}(\kappa)$, and every open dense subset $D \in V$

\[ (\ast) \quad \text{of } \mathbb{C}(\kappa), \text{there is } ([\bar{S}], [\bar{T}]) \leq ([S], [T]) \text{ such that } ([\bar{S}], [\bar{T}]) \models \langle \mathbb{L}_\alpha : \alpha < \kappa \rangle \]
extends some element of \( D^n \).

Thus fix \((S, T) \in \mathbb{R}(\kappa) \times \mathbb{R}(\kappa)\) and \( D \in V \) as above, where \( S, T \subseteq 2^{\kappa \times \omega} \) are Borel and non-null. By Lemma 3.2 and the remarks after it, we can find \( p \in C(\kappa) \) and \((x, y) \in S \times T \) such that:

1. \([s_p] \subseteq S + T\).
2. \((x + y) \upharpoonright \text{dom}(p) = p\).
3. The sets \( S \cap [s_x]_{\text{dom}(p)} \) and \( T \cap [s_y]_{\text{dom}(p)} \) are Borel and non-null.

Now let \( q \in D \) be such that

\[
([S \cap [s_x]_{\text{dom}(p)}], [T \cap [s_y]_{\text{dom}(p)}]) \models \text{"} q \leq C(\kappa) \text{"}.
\]

Using continuity of the addition and further application of Lemma 3.2 and the remarks after it, we can find \( x', y' \) such that:

4. \( x' \in S \cap [s_x]_{\text{dom}(q)} \) and \( y' \in T \cap [s_y]_{\text{dom}(q)} \).
5. \((x' + y') \upharpoonright \text{dom}(q) = q\).
6. The sets \( S \cap [s_{x'}]_{\text{dom}(q)} \) and \( T \cap [s_{y'}]_{\text{dom}(q)} \) are Borel and non-null.

It is now clear that

\[
([S \cap [s_{x'}]_{\text{dom}(q)}], [T \cap [s_{y'}]_{\text{dom}(q)}]) \models \text{"} \langle \xi_{\alpha} : \alpha \in \kappa \rangle \text{ extends } q^\kappa \text{"}.
\]

The result follows. \( \square \)

References

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