A central limit theorem for the rescaled Lévy area of two-dimensional fractional Brownian motion with Hurst index $H < 1/4$

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Let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional fractional Brownian motion with Hurst index $\alpha \in (0, 1/4)$. Using an analytic approximation $B(\eta)$ of $B$ introduced in [22], we prove that the rescaled Lévy area process $(s, t) \mapsto \eta^{\frac{1}{2} (1 - 4 \alpha)} \int_s^t dB^{(1)}_t (\eta) \int_s^t dB^{(2)}_t (\eta)$ converges in law to $W_t - W_s$ where $W$ is a Brownian motion independent from $B$. The method relies on a very general scheme of analysis of singularities of analytic functions, applied to the moments of finite-dimensional distributions of the Lévy area.

Keywords: fractional Brownian motion, stochastic integrals

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0 Introduction

The (two-sided) fractional Brownian motion \( t \to B_t, t \in \mathbb{R} \) (fBm for short) with Hurst exponent \( \alpha, \alpha \in (0,1) \), defined as the centered Gaussian process with covariance

\[
\mathbb{E}[B_s B_t] = \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t - s|^{2\alpha}),
\]

is a natural generalization in the class of Gaussian processes of the usual Brownian motion, in the sense that it exhibits two fundamental properties shared with Brownian motion, namely, it has stationary increments, viz.

\[
\mathbb{E}[(B_t - B_s)(B_u - B_v)] = \mathbb{E}[(B_{t+\alpha} - B_{s+\alpha})(B_{u+\alpha} - B_{v+\alpha})] \text{ for every } a, s, t, u, v \in \mathbb{R},
\]

and it is self-similar, viz.

\[
\forall \lambda > 0, \quad (B_{\lambda t}, t \in \mathbb{R}) \overset{(law)}{=} (\lambda^\alpha B_t, t \in \mathbb{R}).
\]

One may also define a \( d \)-dimensional vector Gaussian process (called: \( d \)-dimensional fractional Brownian motion) by setting \( B_t = (B_t^{(1)}, \ldots, B_t^{(d)}) \) where \( (B_t^{(i)}, t \in \mathbb{R})_{i=1,\ldots,d} \) are \( d \) independent (scalar) fractional Brownian motions.

Its theoretical interest lies in particular in the fact that it is (up to normalization) the only Gaussian process satisfying these two properties.

A standard application of Kolmogorov’s theorem shows that fBm has a version with \( (\alpha - \varepsilon) \)-Hölder paths for every \( \varepsilon > 0 \). In particular, all its paths possess finite \( q \)-variation for every \( q > \frac{1}{\alpha} \), in the sense that

\[
\sup_{n \geq 1} \sup_{s = t_0 < \ldots < t_n = t} \left( \sum_{l=0}^{n} |B_{t_l} - B_{t_{l-1}}|^q \right) < \infty \quad \text{a.s.}
\]

where the sum ranges over all partitions \( (s = t_0 < t_1 < \ldots < t_n = t) \) of any order \( n \) of the interval \([s, t]\).

There has been a widespread interest during the past ten years in constructing a stochastic integration theory with respect to fBm and solving stochastic differential equations driven by fBm, see for instance [10, 6, 4, 20, 21]. The multi-dimensional case is very different from the one-dimensional case. When one tries to integrate for instance a stochastic differential equation driven by a two-dimensional fBm \( B = (B^{(1)}, B^{(2)}) \) by using any kind of Picard iteration scheme, one encounters very soon the problem of defining the Lévy area of \( B \) which is the antisymmetric part of \( A_{s,t} := \int_s^t dB^{(1)}_t \int_s^t dB^{(2)}_t \). This is the simplest occurrence of iterated
Assume $\Gamma_{t}$ path theory due to T. Lyons, see [11, 12]. Let us describe this briefly. B\textsuperscript{d}imensional fBm – a matrix of continuous paths – is a substitute for the iterated integrals of any order and to solve differential equations driven by $\Gamma$. These data iterated integrals of any order and to solve differential equations driven by $\Gamma$.

The multiplicativity property implies in particular the following identity for the Lévy area:

$$\mathcal{A}_{s,t} = \mathcal{A}_{s,u} + \mathcal{A}_{u,t} + (B_{u}^{(1)} - B_{s}^{(1)})(B_{t}^{(2)} - B_{u}^{(2)}).$$ (0.5)

Letting $\Gamma = 1 \oplus \Gamma^{1} \oplus \Gamma^{2} \oplus \ldots \oplus \Gamma^{[q]}$ live in the truncated tensor algebra $\mathbb{R} \oplus \mathbb{R}^{d} \oplus (\mathbb{R}^{d})^{\otimes 2} \oplus \ldots \oplus (\mathbb{R}^{d})^{\otimes [q]}$, the latter property reads simply $\Gamma_{s,t} = \Gamma_{s,u} \otimes \Gamma_{u,t}$. Then there is a standard procedure which allows to define out of these data iterated integrals of any order and to solve differential equations driven by $\Gamma$.

The multiplicativity property is satisfied by smooth paths, as can be checked by direct computation. So the most natural way to construct such a multiplicative functional is to start from some smooth approximation $\Gamma(\eta)$,

integrals $B_{s,t}(i_{1}, \ldots, i_{k}) := \int_{s}^{t} dB_{t_{1}}^{(i_{1})} \ldots \int_{s}^{t_{k-1}} dB_{t_{k}}^{(i_{k})}$, $i_{1}, \ldots, i_{k} \leq d$ for $d$-dimensional fBm $B = (B^{(1)}, \ldots, B^{(d)})$ which lie at the heart of the rough path theory due to T. Lyons, see [11, 12]. Let us describe this briefly. Assume $\Gamma_{t} = (\Gamma_{t}^{(1)}, \ldots, \Gamma_{t}^{(d)})$ is some non-smooth $d$-dimensional path with bounded $q$-variation for some $q > 1$ (take for instance an $\alpha$-Hölder path with $\alpha = 1/q < 1$). Integrals such as $\int f_{1}(\Gamma_{t})d\Gamma_{t}^{(1)} + \ldots + f_{d}(\Gamma_{t})d\Gamma_{t}^{(d)}$ do not make sense a priori because $\Gamma$ is not differentiable (Young’s integral [9] works for $\alpha > 1/2$ but not beyond). In order to define the integration of a differential form along $\Gamma$, it is enough to define a truncated multiplicative functional (consisting of increments) $(\Gamma^{1}, \ldots, \Gamma^{[q]})$, $[q] = \text{entire part of } q$, where $\Gamma_{s,t}^{1} = \Gamma_{t} - \Gamma_{s}$ and each $\Gamma_{k} = (\Gamma_{k}(i_{1}, \ldots, i_{k}))_{1 \leq i_{1}, \ldots, i_{k} \leq d}$, $k \geq 2$ – a matrix of continuous paths – is a substitute for the iterated integrals $\int_{s}^{t} d\Gamma_{t_{1}}^{(i_{1})}\int_{s}^{t_{1}} d\Gamma_{t_{2}}^{(i_{2})} \ldots \int_{s}^{t_{k-1}} d\Gamma_{t_{k}}^{(i_{k})}$ with the following two properties:

(i) each component of $\Gamma_{k}$ has bounded $q/k$ variation semi-norm

$$||\Gamma_{s,t}^{k}||_{q/k}^{q/k} := \sup_{n \geq 1, s = t_{0} < t_{1} < \ldots < t_{n} = t} \frac{\sum_{l=1}^{n} |\Gamma_{t_{l-1}, t_{l}}^{k}|^{q/k}}{};$$

(ii) (multiplicativity) letting $\Gamma_{s,t}^{k} := \Gamma_{t}^{k} - \Gamma_{s}^{k}$, one requires

$$\Gamma_{s,t}^{k}(i_{1}, \ldots, i_{k}) = \Gamma_{s,u}^{k}(i_{1}, \ldots, i_{k}) + \Gamma_{u,v}^{k}(i_{1}, \ldots, i_{k}) + \sum_{k_{1} + k_{2} = k} \Gamma_{s,u}^{k_{1}}(i_{1}, \ldots, i_{k_{1}})\Gamma_{v,v}^{k_{2}}(i_{k_{1}+1}, \ldots, i_{k}).$$

(0.4)

The multiplicativity property implies in particular the following identity for the Lévy area:

The multiplicativity property is satisfied by smooth paths, as can be checked by direct computation. So the most natural way to construct such a multiplicative functional is to start from some smooth approximation $\Gamma(\eta)$,
η ≥ 0 of Γ such that each iterated integral \( \Gamma^k_{s,t}(\eta)(i_1, \ldots, i_k), k \leq [q] \) converges in the \( \frac{q}{k} \)-variation semi-norm.

This general scheme has been applied to fBm in a paper by L. Coutin and Z. Qian [5] and later in a paper by the author [22]. L. Coutin and Z. Qian used the standard \( n \)-dyadic piecewise linear approximation \( B^{CQ}(2^{-n}) \) of \( B \). Our approximation consists in seeing \( B \) as the real part of the boundary value of an analytic process \( \Gamma \) living on the upper half-plane \( \Pi^+ = \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \} \). The time-derivative of this centered Gaussian process has the following hermitian positive-definite covariance kernel:

\[
E[\Gamma'(z)\overline{\Gamma'(w)}] =: K'(z, w) = \frac{\alpha(1-2\alpha)}{2\cos \pi \alpha}(-i(z-w))^{2\alpha-2}, \quad z, w \in \Pi^+, \tag{0.6}
\]

where \( z^{2\alpha-2} := e^{(2\alpha-2)\ln z} \) (with the usual determination of the logarithm) is defined and analytic on the cut plane \( \mathbb{C} \setminus \mathbb{R}_- \). Also, by construction,

\[
E[\Gamma'(z)\overline{\Gamma'(w)}] \equiv 0 \tag{0.7}
\]

identically. It is essential to understand that \( K' \) is a singular multivalued function on \( \mathbb{C} \times \mathbb{C} \); for \( z, w \in \Pi^+ \), \( \text{Re} (-i(z-w)) > 0 \) so the kernel \( K' \) is well-defined. Then

\[
B_t(\eta) := \Gamma_{t+i_1^2} + \overline{\Gamma_{t+i_1^2}} = 2\text{Re} \, \Gamma_{t+i_1^2} \tag{0.8}
\]

is a good approximation of fBm, namely, \( B(\eta) \) converges a.s. in the \( q \)-variation distance for every \( q > \frac{1}{\alpha} \) to a process \( B_t(0) = 2\text{Re} \, \Gamma_t \) with the same law as fBm.

Both approximation schemes lead to the same semi-quantitative result, namely:

– when \( \alpha > 1/4 \), the Lévy area and volume (in other words, the multiplicative functional truncated to order 3) converge a.s. in the correct variation norm. The heart of the proof lies in the study of the piecewise linear approximation \( A_{s,t}^{CQ}(2^{-n}) \), resp. analytic approximation \( A_{s,t}(\eta) \) of the Lévy area; one may prove in particular that \( E[(A_{s,t}^{CQ}(2^{-n}))^2] \) and \( E[(A_{s,t}(\eta))^2] \) converge to the same limit when \( 2^{-n} \) and \( \eta \) go to 0;

– when \( \alpha < 1/4 \), \( E[(A_{s,t}^{CQ}(2^{-n}))^2] \) and \( E[(A_{s,t}(\eta))^2] \) diverge resp. like \( n^{(1-4\alpha)} \) and \( \eta^{-(1-4\alpha)} \). Hence the above method fails.

The latter result is of course unsatisfactory, and constitutes by no means a proof that no coherent stochastic integration theory with respect to fBm may exist when \( \alpha < 1/4 \).
We are interested in this paper in the singular case \( \alpha < 1/4 \). We give no construction of a rough path but provide results which, hopefully, may be a first step in that direction.

First of all (see Section 1), we give a precise analysis of the singular terms appearing in the moments of the Lévy area constructed out of the analytic approximation of a two-dimensional fractional Brownian motion

\[ B(\eta) = (B^{(1)}(\eta), B^{(2)}(\eta)) \]

when \( \eta \to 0 \). The main results are Theorem 1.4 and Corollary 1.19 which give in particular an equivalent of

\[ E[(A_{s,t}(\eta))^{2N}] \]

when \( \eta \to 0 \). They are in fact much more precise in that they provide a general method to find the exponents

\[ (4\alpha - 1)N = \beta_0 < \beta_1 < \beta_2 < \ldots \]

of the asymptotic expansion of the moments, namely,

\[
E[(A_{s,t}(\eta))^{2N}] = \sum_{j=0}^{J} c_j |t - s|^{4\alpha N - \beta_j} \eta^{\beta_j} + o(\eta^{\beta_j}) \tag{0.9}
\]

where the \( c_j \) are coefficients which depend only on \( \alpha \), and \( c_0 \) may be evaluated explicitly. But an easy generalization yields the same kind of results for

\[ E[A_{s_1,t_1}(\eta) \ldots A_{s_2N,t_2N}(\eta)] \]

where \( s_1 < t_1, \ldots, s_{2N} < t_{2N} \) are arbitrary arguments (see Lemma 2.3 in Section 2). The method we use is sufficiently general to be applied with some extra efforts to any kind of iterated integral of any order.

Section 1 may be seen as a long exercise in complex analysis. Let us give a simple example coming from [22]. By definition (recall \( E[\Gamma'(z)\Gamma'(w)] \equiv 0 \) identically, see (0.7))

\[
E[(A_{s,t}^\alpha)^2] = 2E \left( \int_s^t \frac{d\Gamma(1)}{x_1 + i \frac{\eta}{2}} \int_s^{x_1} \frac{d\Gamma(2)}{x_2 + i \frac{\eta}{2}} \right) \left( \int_s^t \frac{d\Gamma(1)}{y_1 + i \frac{\eta}{2}} \int_s^{y_1} \frac{d\Gamma(2)}{y_2 + i \frac{\eta}{2}} \right) + 2\text{Re} \left( \int_s^t \frac{d\Gamma(1)}{x_1 + i \frac{\eta}{2}} \int_s^{x_1} \frac{d\Gamma(2)}{x_2 + i \frac{\eta}{2}} \right) \left( \int_s^t \frac{d\Gamma(1)}{y_1 + i \frac{\eta}{2}} \int_s^{y_1} \frac{d\Gamma(2)}{y_2 + i \frac{\eta}{2}} \right)
\]

\[ = \mathcal{V}_1(\eta) + \mathcal{V}_2(\eta). \tag{0.10} \]

The first term in the right-hand side writes (using the stationarity of the increments)

\[
\mathcal{V}_1(\eta) = C \int_0^{t-s} dx_1 \int_0^{x_1} dx_2 \int_0^{t-s} dy_1 \int_0^{y_1} dy_2 (-i(x_1 - y_1) + \eta)^{2\alpha - 2} (-i(x_2 - y_2) + \eta)^{2\alpha - 2}
\]

\[ = C' \int_0^{t-s} dx_1 \int_0^{t-s} dy_1 (-i(x_1 - y_1) + \eta)^{2\alpha - 2}
\]

\[ = \left[ (-i(x_1 - y_1) + \eta)^{2\alpha} - (-ix_1 + \eta)^{2\alpha} - (iy_1 + \eta)^{2\alpha} \right]. \tag{0.11} \]
while the second term writes

\[
\mathcal{V}_2(\eta) = C' \int_0^{t-s} dx_1 \int_0^{t-s} dy_1 (-i(x_1 - y_1) + \eta)^{2\alpha - 2} [(i(x_1 - y_1) + \eta)^{2\alpha} - (ix_1 + \eta)^{2\alpha} - (-iy_1 + \eta)^{2\alpha}]
\]

Both integrals look the same except that \(\mathcal{V}_2\) (contrary to \(\mathcal{V}_1\)) involves both \(-ix_1\) and \(ix_1\), and similarly for \(y_1\). This seemingly insignificant difference is essential, since \(\mathcal{V}_1\) can be shown to have a bounded limit when \(\eta \to 0\) by using a contour deformation in \(\Pi^+ \times \Pi^-\) (where \(\Pi^-\) denotes the lower half-plane) which avoids the real axis where singularities live, while this is impossible for \(\mathcal{V}_2\) (namely, \((-i(x_1 - y_1) + \eta)^{2\alpha - 2}\) is well-defined if \((x_1, y_1)\) are in the closure of \(\Pi^+ \times \Pi^-\), while \((i(x_1 - y_1) + 2\eta)^{2\alpha}\) for instance is well-defined on the closure of \(\Pi^- \times \Pi^+\)). In fact, explicit computations using Gauss’ hypergeometric function prove that \(\mathcal{V}_2\) diverges in the limit \(\eta \to 0\) when \(\alpha < 1/4\). More general results are given in Lemmas 1.10 and 1.12. Let us state a simple consequence of them. Let

\[
I_-(\beta_1, \beta_2; 0, t)(a, b) := \int_0^t (-i(u - a) + \eta)^{\beta_1} (-i(u - b) + \eta)^{\beta_2} \, du \tag{0.13}
\]

and

\[
I_+(\beta_1, \beta_2; 0, t)(a, b) := \int_0^t (+i(u - a) + \eta)^{\beta_1} (-i(u - b) + \eta)^{\beta_2} \, du \tag{0.14}
\]

for \(\beta_1, \beta_2 \in \mathbb{R}\) such that \(\beta_2 > -1, \beta_1 + \beta_2 + 1 < 0\) and \(a, b \in (0, t)\). (Notice one retrieves terms contained in \(\mathcal{V}_1(\eta)\) or \(\mathcal{V}_2(\eta)\) when one sets \(a = b\). Take \(\eta \to 0\) and \(a - b \to 0\). Then the integral \(I_-\) converges (which follows again from a deformation of contour), while \(I_+\) diverges like \(C(i(b-a)+2\eta)^{\beta_1+\beta_2+1}\).

When evaluating the \(2N\)-th moment of \(A_{s,t}(\eta)\) for instance, iterated integrals of the same type as \(I_+\) produce multivalued power functions \((\pm i(x-y) + k\eta)^\beta\) with \(k \in \mathbb{N}\) and various exponents \(\beta = 4\alpha - 1, 6\alpha, 8\alpha - 1, 10\alpha, \ldots\). The singularities of the moments come exactly from those non-analytic terms with negative exponent when evaluated on the diagonal \(x = y\). Aside from these terms, iterated integrals also produce analytic terms of the form

\[
F(z) := \int_a^b (-i(z - u) + \eta)^\gamma (u - a)^\beta f(u) \, du
\]

\((\gamma = 2\alpha \text{ or } 2\alpha - 2, \eta > 0, \beta > -1)\) where \(f\) is analytic on some appropriate complex domain containing \([a, b]\). The question is to give a maximal domain
where $F$ is analytic and to understand its singularities around $a$. Although one is mainly interested in the behaviour of $F$ on the real axis, computations show clearly that complex analytic methods (including contour shifts e.g.) are the most appropriate in this setting, and circumvent the heavy arguments one would inevitably get using a pathwise linear approximation. A detailed analysis shows that all these terms are regular in the limit $\eta \to 0$. A shortcut (which does not give the values of the exponents though) is provided in Lemma 1.20.

The reader who is interested more in applications than in the details should essentially have in mind the two following results:

**Lemma 1.7**

The generating function $\phi_{s,t}(\eta; \lambda) := \mathbb{E}[e^{i\lambda A_{s,t}(\eta)}]$ of the Lévy area $A_{s,t}(\eta)$ is the exponential of the generating function of connected diagrams, i.e.

$$
\phi_{s,t}(\eta; \lambda) = \exp \phi_{s,t}^{(c)}(\eta; \lambda).
$$

(see subsection 1.1 for the definition of connected moments which are Gaussian cumulants of a certain type);

**Theorem 1.4**

The $2N$-th connected moment of $A_{s,t}(\eta)$ is given by the sum of two terms: the first one is regular in the limit $\eta \to 0$ and equal to $C_{reg,N} 4^{N\alpha} + O(\eta^{2\alpha})$ for some constant $C_{reg,N}$; the second one is equal to $C_{irr,N} \eta^{4\alpha N - 1}$ with

$$
C_{irr,N} = \left( \frac{\pi/2}{\cos \pi \alpha \Gamma(-2\alpha)} \right)^{2^{(N-1)}} \sin \pi \alpha \frac{\Gamma(2\alpha + 1)}{\Gamma(2 - 2\alpha)} \frac{\Gamma(1 - 4\alpha N)(2N)^{4\alpha N - 1}}{\Gamma(1 - 4\alpha N)(2N)^{4\alpha N - 1}}.
$$

(0.16)

In Section 2, we apply this analysis of singularities to convergence results concerning the Lévy area. Since the second moment of the Lévy area $A_{s,t}(\eta)$ diverges like $\eta^{-2(1-4\alpha)}$, it is natural to introduce the rescaled Lévy area

$$
\tilde{A}_{s,t}(\eta) = \eta^{\frac{1}{2}(1-4\alpha)} A_{s,t}(\eta).
$$

(0.17)

The main result of this paper is a kind of central limit theorem which we state here:
Theorem A.

The three-dimensional process \((B_1(\eta), B_2(\eta), \tilde{A}(\eta))\) converges in law to \((B_1, B_2, \sqrt{C_{\text{irr},1}} \delta W)\) where \(\delta W_{s,t} := W_t - W_s\) are the increments of a standard one-dimensional Brownian motion independent from \(B_1\) and \(B_2\).

The value of the constant \(C_{\text{irr},1} = \lim_{\eta \to 0} \mathbb{E}[(\tilde{A}_{0,1}(\eta))^2]\) is given in Theorem 1.4. We feel the exact value is not important though (different schemes of approximation lead to different values, whereas the value of \(\lim_{\eta \to 0} \mathbb{E}[(\tilde{A}_{0,1}(\eta))^2]\) when \(\alpha > 1/4\) seems to be more universal, see above).

Other central limit theorems have been obtained for sums of integer powers of increments of one-dimensional fBm (see [2, 17, 14] for instance). The most closely related result is maybe that of I. Nourdin [13] which shows an Itô-type formula for a two-dimensional fBm with Hurst index \(H = 1/4\) with a 'bracket-term' involving an independent Brownian motion, but the results are of a very different nature than ours (also, they hold precisely for \(H = 1/4\), whereas our results concern the case \(H < 1/4\)). Note also the paper by Y. Hu and D. Nualart [7] which gives a Brownian scaling limit for the self-intersection local time for \(\alpha\) large enough.

One would then like to say that the Lévy area \(A_{s,t}(\eta)\) writes (up to some finite coefficient) as a counterterm \(\eta^{-\frac{1}{2}(1-4\alpha)}(W_t - W_s)\) where \(W\) is a standard Brownian motion, plus some term which is finite in the limit \(\eta \to 0\). Unfortunately this statement is not true with the Brownian motion \(W\) constructed in Theorem A because it is independent from the Lévy area \(A_{s,t}^\eta\). But one may conjecture that some related counterterm yields a corrected Lévy area which is finite.

Let us also mention the last result of this paper, which yields a uniform exponential bound for the rescaled Lévy area when \(\alpha \in (\frac{1}{8}, \frac{1}{4})\) (see Corollary 2.5). We give no application, but it allows for instance to use the Berry-Esséen Lemma to get precise estimates of the rate of convergence of \(A_{s,t}^\eta\) to \(\sqrt{C_{\text{irr},1}} \delta W_{s,t}\). Also, Stein’s method combined with the Malliavin calculus [14], [15] may easily be applied to our setting to give convergence rates since Section 1 gives estimates of all cumulants of the Lévy area. We hope to come back to this in the future.

We shall be using a number of times the following integral representation of Gauss’ hypergeometric function \(2F_1\) (see [1], 15.3.1):

\[
2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \quad (0.18)
\]
valid if \( \text{Re } c > \text{Re } b > 0 \) and \( z \in \mathbb{C} \setminus [1; +\infty) \). Recall that \( _2F_1(a, b, c; z) \) \((c \neq 0, -1, \ldots)\) is defined around \( z = 0 \) by an infinite series with radius of convergence 1 and has an analytic extension to the cut plane \( \mathbb{C} \setminus [1; +\infty) \).

The connection formulas give \( _2F_1(a, b, c; z) \) in terms of a linear combination of hypergeometric functions in the transformed argument \( \phi(z) \), where \( \phi \) is any projective transformation of the Riemann sphere preserving the set of singularities of the hypergeometric differential equation, namely \( \{0, 1, \infty\} \). They relate the behaviour of the hypergeometric functions around 0 with their behaviour around 1 and \( \infty \). We reproduce here for the convenience of the reader three connection formulas, relating the behaviour around \( \infty \) with the behaviour around 0 for the first two ones, and the behaviour around 1 with the behaviour around 0 for the third one (see [1], section 15.3):

\[
_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} _2F_1(a, 1-c+a; 1-b+a; \frac{1}{z}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} _2F_1(b, 1-c+b; 1-a+b; \frac{1}{z}), \quad z \notin \mathbb{R}_+
\]

\[
(0.19)
\]

\[
_2F_1(a, b, c; z) = (1-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} _2F_1(a, c-b; a-b+1; \frac{1}{1-z}) + (1-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} _2F_1(b, c-a; b-a+1; \frac{1}{1-z}), \quad z \notin [1; +\infty)
\]

\[
(0.20)
\]

\[
_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} _2F_1(a, b; a+b-c+1; 1-z) + (1-z)^{c-a-b} - \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} _2F_1(c-a, c-b; c-a-b+1; 1-z), \quad z \notin [1; +\infty)
\]

\[
(0.21)
\]

Let us also recall that \( _2F_1(a, b, c; z) \) is symmetric in the arguments \( a, b \), constant (equal to 1) if \( a = 0 \) or \( b = 0 \), and that

\[
_2F_1(a, b; a; z) = _2F_1(b, a; a; z) = (1-z)^{-b}
\]

which is a consequence of [1], 15.3.4.
1 Moments of the Lévy area

Section 1 (by far the longest one of the article) is organized as follows.

Subsection 1.1 contains the main definitions, followed by a diagrammatic expansion of the moments of the Lévy area (see in particular Definition 1.6 for the definition of the generating function of the connected diagrams).

Subsection 1.2 gives the general scheme for subsequent computations and contains explicit closed formulas (see Lemmas 1.10 and 1.12) for the functions \(I_\pm\) defined in the Introduction, see equations (0.13), (0.14).

Subsection 1.3 (dedicated to the computations of the singularity exponents) is the heart of the section, and (unfortunately) the most technical one. The reader who is interested more in applications than in the details of the proofs may skip it, since the terms evaluated in this paragraph (called admissible functions) turn out in the end to be regular in the limit \(\eta \to 0\).

Finally, subsection 1.4 gives the asymptotic behaviour of the connected moments of the Lévy area when \(\eta \to 0\), from which one deduces easily the asymptotic behaviour of \(\mathbb{E}[(A_{s,t}(\eta))^{2N}]\).

1.1 Definitions and combinatorial arguments

Let us start by introducing three kernels which will be the fundamental objects of study in this article.

**Definition 1.1** Let, for \(\eta > 0\),

1. \[K'^{\pm}(\eta; x, y) = \frac{\alpha(1 - 2\alpha)}{2\cos \pi \alpha} (\pm i(x - y) + \eta)^{2\alpha - 2}; \quad (1.1)\]

2. \[K^{\pm}(\eta; x, y) = \int_0^x du \int_0^y dv \, K'^{\pm}(\eta; u, v)\]

\[= \frac{1}{4 \cos \pi \alpha} ((\pm ix + \eta)^{2\alpha} + (\mp iy + \eta)^{2\alpha} - (\pm i(x - y) + \eta)^{2\alpha}); \quad (1.2)\]

3. \[K^{*,\pm}(\eta; x, y) = -\frac{1}{4 \cos \pi \alpha} (\pm i(x - y) + \eta)^{2\alpha}. \quad (1.3)\]

Similarly, let \(K'(\eta; x, y) := 2\Re K'^{\pm}(\eta; x, y), K(\eta; x, y) := 2\Re K^{\pm}(\eta; x, y)\) and \(K^*(\eta; x, y) := 2\Re K^{*,\pm}(\eta; x, y)\) be the real parts (up to a coefficient 2) of the previous kernels.
As showed in [22], the kernel $K'(\eta)$ is positive and represents (for every fixed $\eta > 0$) the covariance of of a real-analytic centered Gaussian process with real time-parameter $t$. The easiest way to see it is to make use of the following explicit series expansion: letting (for $k \geq 0$)

$$f_k(z) = 2^{\alpha - 1} \frac{\sqrt{\Gamma(1 - 2\alpha)}}{2 \cos \pi \alpha} \frac{\sqrt{\Gamma(2 - 2\alpha + k)/\Gamma(2 - 2\alpha)}}{k!} \left( \frac{z + i}{2i} \right)^{2\alpha - 2} \left( \frac{z - i}{z + i} \right)^k, \quad z \in \Pi^+$$

one has

$$\sum_{k \geq 0} f_k(x_1 + i\eta_1/2) f_k(y + i\eta_2/2) = K'(\eta_1 + \eta_2; x, y). \quad (1.5)$$

Define more generally a Gaussian process with time parameter $z \in \Pi^+$ as follows:

$$\Gamma'(z) = \sum_{k \geq 0} f_k(z) \xi_k \quad (1.6)$$

where $(\xi_k)_{k \geq 0}$ are independent standard complex Gaussian variables, i.e. $E[\xi_j \xi_k] = 0$, $E[\xi_j \bar{\xi}_k] = \delta_{j,k}$. The Cayley transform $\Pi^+ \to D, z \to \frac{z+i}{2}$ ($D$=unit disk of the complex plane) makes the series defining $\Gamma'$ into a random entire series which may be shown to be analytic on the unit disk by standard arguments. Hence the process $\Gamma'$ is analytic on $\Pi^+$. Note that (restricting to the horizontal line $\mathbb{R} + i\eta/2$) $Re E[\Gamma'(x + i\eta/2)\Gamma'(y + i\eta/2)] = \Gamma'(\eta; x, y)$.

One may now integrate the process $\Gamma'$ over any path $\gamma : (0, 1) \to \Pi^+$ with endpoints $\gamma(0) = 0$ and $\gamma(1) = z \in \Pi^+ \cup \mathbb{R}$ (the result does not depend on the particular path but only on the endpoint $z$). The result is a process $\Gamma$ which is still analytic on $\Pi^+$. As mentioned in the Introduction, one may retrieve the fractional Brownian motion by considering the real part of the boundary value of $\Gamma$ on $\mathbb{R}$. Another way to look at it is to define $\Gamma_t(\eta) := \Gamma(t + i\eta)$ as a regular process living on $\mathbb{R}$, and to remark that the real part of $\Gamma(\eta)$ converges when $\eta \to 0$ to fBm. In the following Proposition, we give precise statements which summarize what has been said up to now:

**Proposition 1.2 (see [22])**

1. Let $\gamma : (0, 1) \to \Pi^+$ be a continuous path with endpoints $\gamma(0) = 0$ and $\gamma(1) = z$, and set $\Gamma_z = \int_{\gamma} \Gamma' \, du$. Then $\Gamma$ is an analytic process on $\Pi^+$. Furthermore, as $z$ runs along any path in $\Pi^+$ going to $t \in \mathbb{R}$, the random variables $\Gamma_z$ converge almost surely to a random variable called again $\Gamma_t$. 

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2. The family \( \{ \Gamma_t; t \in \mathbb{R} \} \) defines a centered Gaussian complex-valued process whose paths are almost surely \( \kappa \)-Hölder for any \( \kappa < \alpha \). Its real part \( B_t := 2 \text{Re} \Gamma_t \) has the same law as fBm.

3. The family of centered Gaussian real-valued processes \( B(\eta)_t := \text{Re} \Gamma_{t+i\eta} \) converges a.s. to \( B_t \) in the \( q \)-variation norm for very \( q > \frac{1}{\alpha} \). Its covariance kernel is \( K(\eta) \).

Let us introduce the Lévy area for a two-dimensional fBm \( B = (B^{(1)}, B^{(2)}) \).

**Definition 1.3** Let

\[
A_{s,t}(\eta) := \int_s^t dB^{(1)}_x(\eta) \int_s^t dB^{(2)}_y(\eta). \tag{1.7}
\]

In order to evaluate the moments of the Lévy area, we first need some combinatorial arguments. Recall to begin with the classical

**Proposition 1.4** Let \( (X_1, \ldots, X_{2N}) \) be a Gaussian vector with zero means. Then

\[
\mathbb{E}[X_1 \ldots X_{2N}] = \sum_{(i_1, i_2), \ldots, (i_{2N-1}, i_{2N})} \prod_{j=1}^N \mathbb{E}[X_{i_{2j-1}} X_{i_{2j+1}}], \tag{1.8}
\]

where the sum ranges over the \( (2N - 1)!! = 1.3.5 \cdots (2N - 1) \) couplings of the indices \( 1, \ldots, 2N \).

**Lemma 1.5**

\[
\mathbb{E}[A_{s,t}(\eta)^{2N}] = \int_0^{t-s} dx_1 \cdots \int_0^{t-s} dx_{2N} \sum_{(i_1, i_2), \ldots, (i_{2N-1}, i_{2N})} \sum_{(j_1, j_2), \ldots, (j_{2N-1}, j_{2N})} \prod_{k=1}^N K'(\eta; x_{i_{2k-1}}, x_{i_{2k}}) \cdot \prod_{k=1}^N K(\eta; x_{j_{2k-1}}, x_{j_{2k}}). \tag{1.9}
\]

**Proof.**
By stationarity of the increments, one may assume that \( s = 0 \). By Definition 1.3,

\[
\mathbb{E}[A_{0,t}(\eta)^{2N}] = \left( \int_0^t dx_1 \int_0^{x_1} dy_1 \right) \cdots \left( \int_0^t dx_{2N} \int_0^{x_{2N}} dy_{2N} \right) \mathbb{E} \left[ B^{(1)}_{x_1}(\eta) B^{(2)}_{y_1}(\eta) \cdots B^{(1)}_{x_{2N}}(\eta) B^{(2)}_{y_{2N}}(\eta) \right]
\]

(1.10)

Now apply Proposition 1.4 and Definition 1.1.

Every product of \( K, K' \) in the above sum may be represented by a diagram. Draw a simple line \( x \rightarrow y \) for the infinitesimal kernel \( K'(\eta; x, y) \), a dashed line \( x \rightarrow \ldots \rightarrow y \) for the integrated kernel \( K(\eta; x, y) \), and a double line \( x = y \) for the kernel \( K^\ast(\eta; x, y) \).

Each point \( x_1, \ldots, x_{2N} \) is connected to two points, with a simple line for one of them and a dashed line for the other (the two points coincide in the case of the trivial diagram with only two points). Hence each diagram falls into a number of connected components, each one consisting of a simple bipartite closed polygonal line whose \( 2n \) edges, \( n \geq 1 \) are alternatively simple and dashed lines (see figure below).

![Diagram](figure1.png)

Figure 1: Connected diagram.

There exist \((2N - 1)!\) bipartite closed polygonal lines with fixed vertices \((x_1, \ldots, x_{2N})\), i.e. \((2N - 1)!\) connected diagrams (namely, take as first point \( x_1 \), then choose any vertex among the \((2N - 1)\) remaining and connect it to \( x_1 \) by a simple line, and so on).

**Definition 1.6 (generating functions)** Let

\[
\phi_{s,t}(\eta; \lambda) := \sum_{N \geq 0} (-1)^N \mathbb{E}[A_{s,t}(\eta)^{2N}] \frac{\lambda^{2N}}{(2N)!} = \mathbb{E}[e^{i\lambda A_{s,t}(\eta)}] \quad (1.11)
\]
be the generating function of $A_{s,t}(\eta)$, and $\phi^{(c)}(\eta; \lambda)$ be the generating function of all connected diagrams (i.e. bipartite closed polygonal lines). In other words,

$$\phi^{(c)}_{s,t}(\eta; \lambda) = \sum_{N \geq 1} (-1)^N \frac{\phi_{2N}^{(c)}(\eta; s, t)}{2^N} \lambda^{2N}, \quad (1.12)$$

where

$$\phi_{2N}^{(c)}(\eta; s, t) = \int_0^{t-s} dx_1 \cdots \int_0^{t-s} dx_{2N} \left[ K(\eta; x_1, x_2) K'(\eta; x_2, x_3) \cdots K(\eta; x_{2N-1}, x_{2N}) \right] K'(\eta; x_{2N}, x_1). \quad (1.13)$$

Note one has $2N$ in the denominator instead of $(2N)!$ because of the $(2N - 1)!$ equivalent connected diagrams.

**Lemma 1.7** The generating function of the Lévy area is the exponential of the generating functional of connected diagrams, i.e.

$$\phi_{s,t}(\eta; \lambda) = \exp \phi^{(c)}_{s,t}(\eta; \lambda). \quad (1.14)$$

**Proof.**

A general bipartite diagram with $2N$ vertices $x_1, \ldots, x_{2N}$ may be decomposed into its connected components, which define a partition of the set $(x_1, \ldots, x_{2N})$ into $N_1$ subsets of two vertices, $N_2$ subsets of four vertices, $N_3$ subsets of six vertices and so on. The number of such partitions is

$$\frac{1}{(2N)!} N_1! N_2! N_3! \cdots N_1! N_2! N_3! \cdots. \quad (2N)!$$

Now

$$\phi_{s,t}(\eta; \lambda) = \sum_{N_1, N_2, N_3 \geq 0} \frac{(i\lambda)^{2(N_1+2N_2+3N_3+\ldots)}}{(2!)^{N_1}(4!)^{N_2}(6!)^{N_3} \cdots N_1! N_2! N_3! \cdots} \frac{1}{\phi_2^{(c)} \phi_4^{(c)} \phi_6^{(c)} \cdots} \frac{1}{N_1! N_2! N_3! \cdots}$$

$$= \exp \left( -\frac{\lambda^2}{2} \phi_2^{(c)} + \frac{\lambda^4}{4} \phi_4^{(c)} - \frac{\lambda^6}{6} \phi_6^{(c)} + \cdots \right)$$

$$= \exp \phi^{(c)}(\eta; \lambda). \quad (1.15)$$

□
These Feynmann diagram techniques are standard in quantum field theory, see \cite{8} section 5.3.2 for instance.

Turning now to the connected diagram of order \(2N\) – which is the main object of this section –, it may be split into the sum of a number of terms by decomposing \(K(\eta; x, y)\) into \(K(\eta; x, y) = K^*(\eta; x, y) + \frac{1}{2 \cos \pi \alpha} \text{Re} (-ix + \eta)^{2\alpha} + \frac{1}{2 \cos \pi \alpha} \text{Re} (-iy + \eta)^{2\alpha}\). Replace the \(N\) dashed lines of the closed bipartite polynomial line by a double line whenever \(K^*\) replaces \(K\), and draw a bullet (\(\bullet\)) at each point \(x\) where the function \(\text{Re} (-ix + \eta)^{2\alpha}\) has been inserted instead. Then one has:

- one closed connected bipartite diagram with alternating simple and double lines;
- and a number of open diagrams with \(n\) components, \(n \leq N\), each component consisting of alternating simple and double lines of one of the following three types:

\[
(\emptyset \emptyset) \quad -- = -- ... = --
\]

(1.16)

\[
(\emptyset \bullet) \quad -- = -- ... = -- \bullet
\]

(1.17)

\[
(\bullet \bullet) \quad \bullet -- = -- ... = -- \bullet
\]

(1.18)

1.2 Preliminary computations and general scheme

As mentioned in the Introduction and in the discussion preceding Proposition 1.2, the kernel \(K^\prime(\eta)(x, y)\) \((x, y \in \mathbb{R})\) is the trace on the horizontal line \(\mathbb{R} + i\frac{\eta}{2}\) of a positive-definite kernel \(K^\prime \) defined on \(\Pi^\pm \times \Pi^\pm\):

**Definition 1.8** For \(z \in \Pi^\pm\) and \(\bar{w} \in \Pi^\pm\), we let

\[
K^\prime(\pm)(z, \bar{w}) = \frac{\alpha(1 - 2\alpha)}{2 \cos \pi \alpha}(\pm i(z - \bar{w}))^{2\alpha - 2},
\]

(1.19)

\[
K^\pm(z, \bar{w}) = \frac{1}{4 \cos \pi \alpha}((\pm iz)^{2\alpha} + (\mp i\bar{w})^{2\alpha} - (\pm i(z - \bar{w}))^{2\alpha})
\]

(1.20)

and

\[
K^*\pm(z, \bar{w}) = -\frac{1}{4 \cos \pi \alpha}(\pm i(z - \bar{w}))^{2\alpha}
\]

(1.21)

so that

\[
K^\prime(\pm)(\eta)(x, y) = K^\prime(\pm)(x + i\frac{\eta}{2}, y + i\frac{\eta}{2})
\]

(1.22)

and similarly for \(K^\pm(\eta)\) and \(K^*\pm(\eta)\).
Definition 1.9 For \( f \in L^1([a, b], \mathbb{C}) \), \( a, b \in \mathbb{R} \), define

\[
(K'_\pm f)(z) := \int_a^b f(u)K'_\pm(z, u) \, du
\]

\[
= \frac{\alpha(1-2\alpha)}{2 \cos \pi \alpha} \int_a^b f(u)(\pm i(z-u))^{2\alpha-2} \, du, \quad z \in \Pi^\pm
\]

(1.23)

and

\[
(K^*_\pm f)(z) := \int_a^b f(u)K^*_\pm(z, u) \, du
\]

\[
= -\frac{1}{4 \cos \pi \alpha} \int_a^b f(u)(\pm i(z-u))^{2\alpha} \, du, \quad z \in \Pi^\pm.
\]

(1.24)

Both \( K'_\pm f \) and \( K^*_\pm f \) are analytic functions on \( \Pi^\pm \). Similarly, the operators \( K'_\pm(\eta) \), resp. \( K^*_\pm(\eta) \) are obtained by integrating against the \( \eta \) approximations of the kernels \( K'_\pm \), resp. \( K^*_\pm \), so that they may be extended analytically to a complex neighbourhood of the real axis,

\[
(K'_\pm(\eta)f)(z) = \frac{\alpha(1-2\alpha)}{2 \cos \pi \alpha} \int_a^b f(u)(\pm i(z-u)+\eta)^{2\alpha-2} \, du, \quad z \in \bar{\Pi}^\pm
\]

(1.25)

with \( \bar{\Pi}^+ := \{ \text{Im } z \geq 0 \} \), \( \bar{\Pi}^- := \{ \text{Im } z \leq 0 \} \), and

\[
(K^*_\pm(\eta)f)(z) = -\frac{1}{4 \cos \pi \alpha} \int_a^b f(u)(\pm i(z-u)+\eta)^{2\alpha} \, du, \quad z \in \bar{\Pi}^\pm.
\]

(1.26)

Finally, set \( K'_{[a,b]} = K'_+[a,b] + K'_-[a,b] = 2\text{Re } K'_+[a,b] \) and similarly for the five other kernels.

It is clear from the definition of \( K'_\pm \) and \( K^*_\pm \) that \( K'_\pm f \) and \( K^*_\pm f \) are well-defined and analytic on the domain \( \mathbb{C} \setminus \{ s \pm iy \mid a \leq s \leq b, y \geq 0 \} \), but one needs larger domains of convergence (including if possible the closed interval \([a, b]\) or at least the open interval \((a, b)\)) since eventually one is interested in real variables. Appropriate maximal domains of analytic extension for \( K'_\pm f \) and \( K^*_\pm f \) are given in the Appendix under some conditions on the function \( f \). The reader interested in the details of the proofs should first look at the results of the Appendix, since we shall constantly be refering to
them. Let us give the general idea for the convenience of the reader. Though
the results in the Appendix apply to somewhat more general functions som-
times, we shall only need to consider functions \( f \) which are holomorphic on
a complex neighbourhood of \( (a, b) \), with a possible power behaviour at one
of the ends \( (a, \text{say}) \) of the interval, namely \( f(z) = (z - a)\beta F(z) \) (\( \beta = 0 \) or
\( \beta \in \mathbb{R} \setminus \mathbb{Z} \)) for some function \( F \) which is holomorphic in a neighbourhood
of \( a \). The results in the Appendix show that \( K_{[a,b]}^{\epsilon,\pm}f \), \( K_{[a,b]}^{\epsilon,\pm}f \) are then hol-
omorphic on a complex neighbourhood of \( (a, b) \) and multivalued at \( a, b \) with
prescribed exponents (see the first lines of the Appendix and next subsec-
tion for more precise statements, in particular when the interval \([a, b]\) is not fixed). The computation of these exponents is fundamental for our study.

Here is the general scheme of this section. Formula (1.13) may be rewritten
in the following way. Let \( b = x_{2N} \) and \( u = x_{1} \),

\[
\phi_{2N}^{(c)}(\eta; s, t) = -\frac{1}{2\cos \pi \alpha} \Re \int_{0}^{t-s} du K'(\eta; b, u) \cdot (K_{[0,t-s]}^{\epsilon}(\eta)K_{[0,t-s]}^{\epsilon}(\eta))^{N-1} (u \mapsto (\pm i(u - b) + \eta)^{2\alpha} - (\pm iu + \eta)^{2\alpha} - (\mp ib + \eta)^{2\alpha}) (u).
\]

Replacing each occurrence of \( K(\eta; x, y) \) by \( K^{*}(\eta; x, y) + \frac{1}{2\cos \pi \alpha} \Re (-ix +
\eta)^{2\alpha} + \frac{1}{2\cos \pi \alpha} \Re (-iy + \eta)^{2\alpha} \), see end of subsection 1.1, leads (up to some
coefficient) either to the single closed diagram

\[
\int_{0}^{t-s} du K'(\eta; b, u) \cdot (K_{[0,t-s]}^{\epsilon}(\eta)K_{[0,t-s]}^{\epsilon}(\eta))^{N-1} (u \mapsto (\pm i(u - b) + \eta)^{2\alpha}) (u)
\]

or to products of terms of the type

\[
\int_{0}^{t-s} du (i\sigma u + \eta)^{\gamma}(K_{[0,t-s]}^{\epsilon}(\eta)K_{[0,t-s]}^{\epsilon}(\eta))^{n} (u \mapsto (\pm i(u - b) + \eta)^{2\alpha-2}) (u)(i\sigma'u + \eta)^{\gamma'}
\]

where \( \gamma, \gamma' = 0 \) or \( 2\alpha \), and \( \sigma, \sigma' \in \{\pm 1\} \).

The main task will be to estimate the iterated integrals

\[
(K_{[0,t-s]}^{\epsilon}(\eta)K_{[0,t-s]}^{\epsilon}(\eta))^{n} (u \mapsto (\pm i(u - b) + \eta)^{2\alpha}) (u)
\]

and

\[
(K_{[0,t-s]}^{\epsilon}(\eta)K_{[0,t-s]}^{\epsilon}(\eta))^{n} (u \mapsto (\pm i(u - b) + \eta)^{2\alpha-2}) (u).
\]

We shall content ourselves in this introductory subsection with evaluating the expression \( K_{[0,t]}^{\epsilon,\sigma_1}(\eta)(u \mapsto (i\sigma_2(u - b) + \eta)^{2\alpha})(a) \) \( (\sigma_1, \sigma_2 \in \{\pm 1\}) \), which
is (up to a coefficient) equal to the integral $I_\pm$ defined in the Introduction. We shall actually need more general formulas involving arbitrary powers. It turns out that the case $\sigma_1 = -\sigma_2$ — leading to $I_-$ (see Lemma 1.10) — is very different from the case $\sigma_1 = \sigma_2$ — leading to $I_+$ (see Lemma 1.12), the latter case involving a multivalued term of the form $(\pm i(a-b) + 2\eta)^\gamma$ for some $\gamma \in \mathbb{R}$.

Note in the following formulas that we have in mind $\beta_1 = 2\alpha$ or $2\alpha - 2$, while $\beta_2$ may be much more general (see comments just before subsection 1.3).

**Lemma 1.10** Let, for $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_2 > -1$,

$$I_-(\beta_1, \beta_2; 0, t)(a, b) := \int_0^t (-i(u-a))^{\beta_1} (-i(u-b))^{\beta_2} \, du,$$  \hspace{1cm} (1.29)

defined a priori, for every fixed complex number $b$ with $\text{Im } b \leq 0$, as an analytic function of $a$ on $\Pi^-$. We restrict to $0 < \text{Re } a < t$ and $0 < \text{Re } b < t$, $\text{Im } b \leq 0$. Let $\Omega^-_t := \{ a \in \mathbb{C} | 0 < \text{Re } a < t, \text{Im } a < \text{Im } b \}$. Then the following results hold:

(i) On the domain $\Omega^-_t$, one has

$$I_-(\beta_1, \beta_2; 0, t)(a, b) = \frac{i}{\beta_1 + \beta_2 + 1} [\Phi(\beta_1, \beta_2; t)(a, b) - \Phi(\beta_1, \beta_2; 0)(a, b)]$$  \hspace{1cm} (1.30)

with, for $s \in [0, t]$,

$$\Phi(\beta_1, \beta_2; s)(a, b) = (-i(s-b))^{\beta_1+\beta_2+1} 2F_1(-\beta_1, -\beta_1-\beta_2-1; -\beta_1-\beta_2; \frac{a-b}{s-b}).$$  \hspace{1cm} (1.31)

The function $a \mapsto \Phi(\beta_1, \beta_2; 0)(a, b)$, resp. $a \mapsto \Phi(\beta_1, \beta_2; t)(a, b)$ given by the above expression has an analytic extension to the domain $\{ 0 < c < |a/b| < C \} \cap \{ 0 < \text{Re } a < t \}$, resp. $\{ 0 < c < |\frac{t-b}{t-a}| < C \} \cap \{ 0 < \text{Re } a < t \}$ with arbitrary constants $c < 1$, $C > 1$. Both functions extend analytically to the whole domain $\{ 0 < \text{Re } a < t \}$, with different expressions given below.

(ii) Suppose $|a/b| < c < 1$. Then

$$\Phi(\beta_1, \beta_2; 0)(a, b) = (ib)^{\beta_1+\beta_2+1} \left\{ \frac{\Gamma(-\beta_1-\beta_2)\Gamma(1+\beta_1)}{\Gamma(-\beta_2)} (1 - \frac{a}{b})^{\beta_1+\beta_2+1} \right. \right.$$

$$+ \frac{\beta_1 + \beta_2 + 1}{\beta_1 + 1} \left( \frac{a}{b} \right)^{1+\beta_1} 2F_1(-\beta_2, 1; \beta_1 + 2; a/b) \right\}$$  \hspace{1cm} (1.32)
extends analytically to the domain \( \{|a/b| < c < 1\} \cap \{0 < \Re a < t\} \).

(iii) Suppose \(|a/b| > C > 1\). Then

\[
\Phi(\beta_1, \beta_2; 0)(a, b) = (ib)^{\beta_1+\beta_2+1} \cdot \left\{ \frac{\Gamma(-\beta_1 - \beta_2)\Gamma(1 + \beta_2)}{\Gamma(-\beta_1)} \left( \frac{b}{a} \right)^{-\beta_1-\beta_2-1}
\right.
\]

\[
(1 - \frac{b}{a})^{\beta_1+\beta_2+1} + \frac{\beta_1 + \beta_2 + 1}{\beta_2 + 1} \left( \frac{b}{a} \right)^{-\beta_1} 2F_1(-\beta_1, 1; \beta_2 + 2; b/a)
\]

extends analytically to the domain \( \{|a/b| > C > 1\} \cap \{0 < \Re a < t\} \).

(iv) Similarly, suppose \(|\frac{t-a}{t-b}| < c < 1\). Then

\[
\Phi(\beta_1, \beta_2; t)(a, b) = (-i(t - b))^{\beta_1+\beta_2+1} \cdot \left\{ \frac{\Gamma(-\beta_1 - \beta_2)\Gamma(1 + \beta_1)}{\Gamma(-\beta_2)} \left( \frac{a - b}{t - b} \right)^{\beta_1+\beta_2+1}
\right.
\]

\[
+ \frac{\beta_1 + \beta_2 + 1}{\beta_1 + 1} \left( \frac{t - a}{t - b} \right)^{1+\beta_1} 2F_1(-\beta_2, 1; \beta_1 + 2; \frac{t - a}{t - b})
\]

extends analytically to the domain \( \{|\frac{t-a}{t-b}| < c < 1\} \cap \{0 < \Re a < t\} \).

(v) Suppose \(|\frac{t-a}{t-b}| > C > 1\). Then

\[
\Phi(\beta_1, \beta_2; t)(a, b) = (-i(t - b))^{\beta_1+\beta_2+1} \cdot \left\{ \frac{\Gamma(-\beta_1 - \beta_2)\Gamma(1 + \beta_2)}{\Gamma(-\beta_1)} \left( \frac{t - b}{t - a} \right)^{-\beta_1-\beta_2-1}
\right.
\]

\[
\left( \frac{b - a}{t - a} \right)^{\beta_1+\beta_2+1} + \frac{\beta_1 + \beta_2 + 1}{\beta_2 + 1} \left( \frac{t - b}{t - a} \right)^{-\beta_1} 2F_1(-\beta_1, 1; \beta_2 + 2; \frac{t - b}{t - a})
\]

extends analytically to the domain \( \{|\frac{t-a}{t-b}| > C > 1\} \cap \{0 < \Re a < t\} \).

Remark 1.11 Note that \(I_-\) behaves as a sum of power functions when \(a\) and/or \(b\) are in the neighbourhood of either of the interval ends. All together, one gets the following expressions (assuming to simplify notations that \(a, b \in (0, t)\) are real):

\[
\Phi(\beta_1, \beta_2; 0)(a, b) \sim C_1(\max(a, b))^{\beta_1+\beta_2+1} + C_2a^{\beta_1+1}b^{\beta_2}1_{a < b} + C_3b^{\beta_2+1}a^{\beta_1}1_{a > b}
\]

(1.36)
if \( a \) or \( b \) is close to 0, and similarly
\[
\Phi(\beta_1, \beta_2; t)(a,b) \sim C_1(\max(t-a, t-b))^{\beta_1+\beta_2+1} + C_2(t-a)^{\beta_1+1}(t-b)^{\beta_2}1_{t-a < t-b} + C_3(t-b)^{\beta_2+1}(t-a)^{\beta_1}1_{t-a > t-b}
\]  
(1.37)
if \( a \) or \( b \) is close to \( t \). Now sum up the contributions of these two terms to get the exponents of \( I_{-} \).

**Proof.**
(i) is proved in [22], Lemma 4.1 with slightly different hypotheses. Let us give a self-contained proof with easier arguments. Decompose the integral
\[
\int_0^t du = \int_0^b du + \int_b^t du.
\]
The first integral writes
\[
\int_0^b (-i(u-a))^{\beta_1}(-i(u-b))^{\beta_2} du = b \int_0^1 [i(vb + (a-b))]^{\beta_1} (ivb)^{\beta_2} dv.
\]  
(1.38)
By hypothesis, \( \text{Im} \ a < \text{Im} \ b \leq 0 \) and \( 0 < \text{Re} \ a, \text{Re} \ b < t \). Suppose for the moment that \( \text{Re} \ a < \text{Re} \ b \). Then one has \(-\pi/2 < \text{Arg}(i(a-b)) < 0 \) and \( 0 < \text{Arg}(1 - \frac{vb}{b-a}) < \pi \), hence
\[
[i(vb + (a-b))]^{\beta_1} = (i(a-b))^{\beta_1}(1 - \frac{vb}{b-a})^{\beta_1}
\]  
(1.39)
and one obtains
\[
\int_0^b (-i(u-a))^{\beta_1}(-i(u-b))^{\beta_2} du = -\frac{(ib)^{\beta_2+1}}{\beta_2+1} (-i(b-a))^{\beta_1} 2F1(-\beta_1, \beta_2+1; \beta_2+2; \frac{b}{b-a}).
\]  
(1.40)
Similarly,
\[
\int_b^t (-i(u-a))^{\beta_1}(-i(u-b))^{\beta_2} du = (t-b) \int_0^1 [-i(w(t-b) + (b-a))]^{\beta_1} (-i(t-b)w)^{\beta_2} dw.
\]  
(1.41)
Now (still assuming \( \text{Re} \ a < \text{Re} \ b \)) \(-\pi/2 < \text{Arg}(-i(b-a)) < 0 \) and \(-\pi/2 < \text{Arg}(1 - \frac{w(t-b)}{t-b-a}) < \pi/2 \), hence
\[
[-i(w(t-b) + (b-a))]^{\beta_1} = (-i(b-a))^{\beta_1}(1 - \frac{w(t-b)}{t-b-a})^{\beta_1}
\]  
(1.42)
and one obtains
\[
\int_{b}^{t} (-i(u-a))^{\beta_1}(-i(u-b))^{\beta_2} \, du = i^{(\beta_1 - \beta_2 + 1)}/(\beta_2 + 1) \left( (-i(b-a))^{\beta_1} \right)_{\beta_2+1}(t-b+a; \beta_2+2; \frac{b-t}{b-a}) \tag{1.43}
\]

Observe one has obtained
\[
I_{-}(\beta_1, \beta_2; t)(a, b) = \frac{i}{\beta_2 + 1}(-i(b-a))^{\beta_1} [F(\beta_1, \beta_2; t)(a, b) - F(\beta_1, \beta_2; 0)(a, b)] \tag{1.44}
\]

with
\[
F(\beta_1, \beta_2; s)(a, b) = (-i(s-b))^{\beta_2 + 1} \left( 2F_1(-\beta_1, \beta_2 + 1; \beta_2 + 2; \frac{s-b}{a-b}) \right), \quad s \in (0, t). \tag{1.45}
\]

One may lift the restriction \( \text{Re} \ a < \text{Re} \ b \) since the last expression makes sense for all \( a \in \mathbb{C} \) such that \( \text{Im} \ a < \text{Im} \ b \leq 0 \) (namely, \( s - b, a-b \in \Pi^+ \) so that \( \frac{s-b}{a-b} \notin \mathbb{R}_+ \)).

Now, by the connection formula (0.19) (reproducing an argument in [22])
\[
2F_1(-\beta_1, \beta_2; 1; \beta_2 + 2; \frac{s-b}{a-b}) = \frac{\beta_2 + 1}{\beta_1 + \beta_2 + 1} \left( \frac{s-b}{a-b} \right)^{\beta_1} \tag{1.46}
\]

Now \( \left( \frac{s-b}{a-b} \right)^{1+\beta_2} = (s-b)^{1+\beta_2}, \) so the second term, multiplied by the prefactor \( (-i(s-b))^{\beta_2 + 1} \), is independent of \( s \in (0, t) \) and hence makes no contribution to \( I_- \).

Now (ii) and (iv), resp. (iii) and (v), are consequences of the connection formula (0.21), resp. (0.20).

Note (as mentioned in the Introduction) that the fact that \( I_- \) is analytic in the parameters \( a, b \) away from 0, \( t \) may easily be proved by using a deformation of contour in \( \Pi^+ \).

Let us now turn to the non-analytic case:

**Lemma 1.12** Let, for \( \beta_1, \beta_2 \in \mathbb{R} \) with \( \beta_2 > -1 \),
\[
I_{+}(\beta_1, \beta_2; 0, t)(a, b) := \int_{0}^{t} (i(u-a))^{\beta_1}(-i(u-b))^{\beta_2} \, du, \tag{1.47}
\]
defined a priori, for every fixed complex number $b$ such that $\text{Im } b \leq 0$, as an analytic function of $a$ on $\Pi^+$. We restrict to $0 < \text{Re } a < t$ and $0 < \text{Re } b < t$. Let $\Omega^+_t := \{a \in \Pi^+ | 0 < \text{Re } a < t\}$. Then the following formula holds on $\Omega^+_t$:

$$I_+(\beta_1, \beta_2; 0, t)(a, b) = \frac{1}{\beta_1 + \beta_2 + 1} \left[ e^{i\pi \beta_1} \Phi(\beta_1, \beta_2; t)(a, b) - e^{-i\pi \beta_1} \Phi(\beta_1, \beta_2; 0)(a, b) \right]$$

$$- \frac{\Gamma(\beta_2 + 1)\Gamma(-\beta_1 - \beta_2 - 1)}{\Gamma(-\beta_1)} \cdot 2 \sin \pi \beta_2 \cdot (i(b - a))^{\beta_1 + \beta_2 + 1}$$

(1.48)

where $\Phi$ is the same analytic function as in Lemma 1.10.

Contrary to what happens for the $I_-$ integral, the last term does not admit an analytic extension in $a$ to any neighbourhood of $b$ (one cannot 'circle' around $b$).

**Proof.**

A variant of this Lemma is also proved in Lemma 4.1 of [22], but let us give an independent proof. The beginning is as in Lemma 1.10. Namely,

$$\int_0^b du (i(u - a))^{\beta_1} (-i(u - b))^{\beta_2} = b \int_0^1 dv \left[ -i(vb + (a - b)) \right]^{\beta_1} (ivb)^{\beta_2} \quad (1.49)$$

and

$$\int_b^t du (i(u - a))^{\beta_1} (-i(u - b))^{\beta_2} + (t - b) \int_0^1 dw \left[ i(w(t - b) + (b - a)) \right]^{\beta_1} (-i(t - b)w)^{\beta_2}. \quad (1.50)$$

Mind $a \in \Pi^+$ this time. Suppose provisorily that $\text{Re } a < \text{Re } b$: then

$$0 < \text{Arg}(-i(a - b)) < \pi/2, \quad -\pi < \text{Arg}(1 - \frac{vb}{b - a}) < 0 \quad (1.51)$$

(since $\frac{b}{a - b} = (\frac{b}{a})^{-1} \in \Pi^-$) and

$$0 < \text{Arg}(-i(a - b)) + \text{Arg}(1 - \frac{b - t}{b - a}) < \pi \quad (1.52)$$

since $0 < \text{Arg}(1 - \frac{b - t}{b - a}) < \text{Arg}(\frac{a - b}{b - a}) < \pi$ and $-i(a - b), \frac{t - b}{b - a} = i(t - b) \in \Pi^+$.

Hence

$$I_+(\beta_1, \beta_2; t)(a, b) = (i(b - a))^{\beta_1} \frac{i}{\beta_2 + 1} \left\{ (-i(t - b))^{\beta_2 + 1} \begin{array}{c} 2F_1(-\beta_1, \beta_2 + 1; \beta_2 + 2; \frac{t - b}{a - b}) \\ - (ib)^{\beta_2 + 1} \begin{array}{c} 2F_1(-\beta_1, \beta_2 + 1; \beta_2 + 2; \frac{b}{b - a}) \end{array} \end{array} \right\} \quad (1.53)$$
\((a \in \Pi^+)\) which is the same formula as in the proof of Lemma 1.10 except \(a \in \Pi^+\) and the prefactor is \((i(b-a))^{\beta_1}\) instead of \((-i(b-a))^{\beta_1}\). Apply the connection formula (0.19). Unfortunately the second terms in equation (1.46) do not cancel each other this time. Namely, they contribute \((\text{up to a constant prefactor})\) the following expression:

\[
J := (i(b-a))^{\beta_1} \left[ (-i(t-b))^{\beta_2+1} \left( \frac{t-b}{b-a} \right)^{-\beta_2-1} - (ib)^{\beta_2+1} \left( \frac{b}{a-b} \right)^{-\beta_2-1} \right]
\]

\[= (i(b-a))^{\beta_1} \left[ e^{-i\frac{\pi}{2}(\beta_2+1)}(b-a)^{\beta_2+1} - e^{i\frac{\pi}{2}(\beta_2+1)}(a-b)^{\beta_2+1} \right].\]

(1.54)

Since \(a \in \Pi^+\), one has \((b-a)^{\beta_2+1} = e^{-i\frac{\pi}{2}(\beta_2+1)}(i(b-a))^{\beta_2+1}\) and \((a-b)^{\beta_2+1} = e^{i\frac{\pi}{2}(\beta_2+1)}(-i(a-b))^{\beta_2+1}\), whence

\[J = (i(b-a))^{\beta_1+\beta_2+1} 2i \sin \pi \beta_2.\]

(1.55)

Now the first terms in equation (1.46) come up with a supplementary prefactor \(e^{i\pi\beta_1}\) with respect to the formulas in Lemma 1.10 because

\[(i(b-a))^{\beta_1} \cdot \left( \frac{t-b}{b-a} \right)^{\beta_1} = (i(b-a))^{\beta_1} \left( \frac{i(t-b)}{i(b-a)} \right)^{\beta_1} = (i(t-b))^{\beta_1} = (-i(t-b))^{\beta_1} e^{i\pi\beta_1}\]

(1.56)

and similarly

\[(i(b-a))^{\beta_1} \cdot \left( \frac{-b}{b-a} \right)^{\beta_1} = (-ib)^{\beta_1} = (ib)^{\beta_1} e^{-i\pi\beta_1}.
\]

(1.57)

\[\square\]

**Remark 1.13** Both integrals \(I_{\pm}(\beta_1, \beta_2; 0; t; a, b)\) extend analytically to the product of the cut planes \((\mathbb{C} \setminus \{\mathbb{R}_- \cup (t + \mathbb{R}_+)\})^2\), and behave as in Lemma 1.10 \((\text{i.e. with the same power functions, see Remark 1.11 and without the non-analytic term in } (i(b-a))^{\beta_1+\beta_2+1})\) if \(\text{Re } a \in \mathbb{R} \setminus [0, t]\) or \(\text{Re } b \in \mathbb{R} \setminus [0, t]\). This result is a consequence of Lemmas 1.11 and 1.12 and also Lemmas 3.2, 3.3 in the Appendix. If \(\text{Re } a, \text{Re } b < 0 \text{ (or similarly if } \text{Re } a, \text{Re } b > t)\) then \(I_{\pm}(\beta_1, \beta_2; 0; t; a, b) = I_{\pm}(\beta_1, \beta_2; -s; t; a, b) - I_{\pm}(\beta_1, \beta_2; -s, 0; a, b)\) for \(-s < \text{Re } a, \text{Re } b, \text{so the non-analytic term (in the case of } I_{+}) \text{ disappears. If } \text{Re } a < 0 \text{ and } \text{Re } b > t \text{ for instance (so } a \text{ and } b \text{ are far away) then one may cut the interval of integration into } [0, t/2] \cup [t/2, t] \text{ and apply Lemmas 3.2, 3.3. Finally, if } \text{Re } b \in (0, t) \text{ and, say, } \text{Re } a < 0, \text{ then } I_{+} \text{ reduces to } I_{-} \text{ since } (i(u-a))^{\beta_1} = e^{i\pi\beta_1}(-i(u-a))^{\beta_1} \text{ for every } u \in (0, t). \text{ Then the formulas appearing in Lemma 1.10 may be extended analytically to } \text{Re } a < 0 \text{ with a little care.}
\]
The extra non-analytic power term proportional to $a \mapsto (-i(a-b))^{\beta_1 + \beta_2 + 1}$ in equation (1.48) may in turn be integrated against $K'_{\eta,b}$ or $K_{\eta,b}$. Generally speaking, alternate chains of the form $(K_{\eta,b}^{\pm} (\eta) K_{\eta,b}^{-} (\eta))^n (a \mapsto (-i(a-b) + \eta)^{2\alpha})$ or $K_{\eta,b}^{\pm} (\eta) K_{\eta,b}^{-} (\eta))^n (a \mapsto (-i(a-b) + \eta)^{2\alpha})$ (or conjugate) contain an extra power term with increasing exponent (hence the need for a general exponent $\beta_2$). Note that integrating against $K_{\eta,b}^{\pm}(\eta)$ or $K_{\eta,b}^{-}(\eta)$ at some point kills the power term (by Lemma 1.10) and produces a function (depending on $b$) which is analytic on a complex neighbourhood of $(0,t)$. This remark is fundamental to understand the divergence of the Lévy area for $\alpha < 1/4$, which is due to the non-analytic power terms only, as we shall show in the next paragraph.

1.3 Convergence of the analytic part of the moments

Leaving aside the non-analytic extra power terms coming from Lemma 1.12, one is led (see preceding subsection, equation (1.28)) to evaluate alternating integral chains of the form $\cdots K_{\eta,b}^{\pm} (\eta) K_{\eta,b}^{-} (\eta) \cdots f$ for some function $f$, depending on $b$, which is analytic on a complex neighbourhood of $(0,t)$, and multivalued (of power type) at $0$ or $t$. Leaving aside the dependence on the variable $b$ and on the parameter $\eta$, the results in the Appendix show that such integrals are also analytic on a complex neighbourhood of $(0,t)$, multivalued of power type (with some exponents) at $0$ and $t$, and allow to compute the exponents. Let us make precise statements. We first need to define an appropriate class of analytic functions $f(\eta,b,t;u)$ (called admissible in the sequel), of the form $\eta H bB u^r F(\eta,b,u,t)$ or $\eta H bB (t-u)^r F(\eta,b,u,t)$ (where $u \mapsto F(\eta,b,t;u)$ is supposed to be analytic on a neighbourhood of the closed interval $[0,t]$) for which such alternating integral chains make sense. Such functions $F$ may only be defined locally because the values of the exponents $H,B,U$ depend on the relative positions of $\eta,b,u$. This makes the exact definition look a little complicated at first:

Definition 1.14 (admissible analytic functions) Let $\eta > 0$ and $\sigma_f \in \mathbb{Z}, \bar{\sigma} \in \{\pm 1\}$. Assume $f(\eta,b,t;u) := f_b(\eta,b,t;u) + f_\eta(\eta,b,t;u)$ where both $f_b$ and $f_\eta$ are analytic in $\eta, b$ and $u$ for $(\eta,b)$ on a complex neighbourhood $\Omega'$ of $\{\eta \geq 0, b \geq 0\}$ and $u$ on the cut domain

$$\Omega := \left\{ \frac{u + i \sigma_f \eta}{t} \right\} < 2 \right\} \left\{ \frac{u}{\eta} \in -i \sigma + \mathbb{R}_- \right\} \cup \left\{ \frac{t - u}{\eta} \in -i \sigma + \mathbb{R}_+ \right\}$$

for some $\sigma \in \{\pm 1\}$. Let $\Omega'_1 = \{ \frac{\eta}{b + i \sigma_f \eta}; (\eta,b) \in \Omega' \}$ and $\Omega'_2 = \{ \frac{b + i \sigma_f \eta}{t}; (\eta,b) \in \Omega' \}$. Assume also that $f_b$ may be written as:
\((i)_b \sum_{j=1}^{J} (b + i\sigma \eta)B_j^{(b)}(u+i\sigma \eta)U_j^{(b)} F_j^{(u+i\sigma \eta, \frac{\eta}{b+i\sigma \eta}, \frac{\eta}{b+i\sigma \eta})} \text{ on the subdomain } (i)_b \text{ of } \Omega \text{ such that } 0 < |u+i\sigma \eta| < 2|b+i\sigma \eta|/3;\)

\((ii)_b \sum_{j=1}^{J'} (b + i\sigma \eta)B_j^{(b)'}F_j^{(u+i\sigma \eta, \frac{\eta}{b+i\sigma \eta}, \frac{\eta}{b+i\sigma \eta})} \text{ on the subdomain } (ii)_b \text{ of } \Omega \text{ such that } |b+i\sigma \eta|/3 < |u+i\sigma \eta| < 3|b+i\sigma \eta|;\)

\((iii)_b \sum_{j=1}^{J''} (b + i\sigma \eta)B_j^{(b)''}U_j^{(b)''} F_j^{(u+i\sigma \eta, \frac{\eta}{b+i\sigma \eta}, \frac{\eta}{b+i\sigma \eta})} \text{ on the subdomain } (iii)_b \text{ of } \Omega \text{ such that } 2|b+i\sigma \eta| < |u+i\sigma \eta|\)

where \(F_j, F_j'\) are holomorphic on \(B(0, 1) \times \Omega'_1, \text{ and } F_j'(w, \zeta)\) are holomorphic on \(\{1/3 < |w| < 3\} \times \Omega'_1\)

with \(U_j^{(b)} > -1\) for all \(j, \text{ and } \{(B_j^{(b)} + U_j^{(b)}), j = 1 \ldots J\} = \{(B_j^{(b)'}), j' = 1 \ldots J'\} = \{(B_j^{(b)''} + U_j^{(b)''}), j'' = 1 \ldots J''\}, \text{ while } f_{\eta} \text{ may be written as}\)

\((ii)_{\eta} \sum_{j=1}^{J'} \eta^H_j^{(\eta)'} (b + i\sigma \eta)B_j^{(\eta)'}(u + i\sigma \eta)U_j^{(\eta)'} F_j^{(u+i\sigma \eta, \frac{\eta}{b+i\sigma \eta}, \frac{\eta}{b+i\sigma \eta})} \text{ on the subdomain } (ii)_{\eta} \text{ of } \Omega \text{ such that } 0 < |u+i\sigma \eta| < 3\eta;\)

\((iii)_{\eta} \sum_{j=1}^{J''} \eta^H_j^{(\eta)''} (b + i\sigma \eta)B_j^{(\eta)''}(u + i\sigma \eta)U_j^{(\eta)''} F_j^{(u+i\sigma \eta, \frac{\eta}{b+i\sigma \eta}, \frac{\eta}{b+i\sigma \eta})} \text{ on the subdomain } (iii)_{\eta} \text{ of } \Omega \text{ such that } 2\eta < |u+i\sigma \eta| < 2t/3;\)

\((\tilde{iii})_{\eta} \sum_{j=1}^{\tilde{J}''} \eta^{\tilde{H}_j^{(\eta)''}} (b + i\sigma \eta)\tilde{B}_j^{(\eta)''}(1 - \frac{u + i\sigma \eta}{t})\tilde{U}_j^{(\eta)''} F_j^{(u+i\sigma \eta, \frac{\eta}{b+i\sigma \eta}, \frac{\eta}{b+i\sigma \eta})} G_j^{(1 - \frac{u + i\sigma \eta}{t}, \frac{\eta}{b+i\sigma \eta}, \frac{\eta}{b+i\sigma \eta})} \text{ on the subdomain } (\tilde{iii})_{\eta} \text{ of } \Omega \text{ such that } 2\eta/t < \left|1 - (u+i\sigma \eta)/t\right| < 2/3;\)
\[ \sum_{j=1}^{J} \eta \hat{F}_j^{(n)'} (b + i\sigma \eta)\hat{b}_j^{(n)'} (1 - (u + i\sigma \eta)/t)\hat{\eta}_j^{(n)'} \]

\[ \hat{F}_j'(1 - (u + i\sigma \eta)/t) \eta/n \frac{b + i\sigma \eta}{t} \]

on the subdomain \((\hat{ii})_\eta\) of \(\Omega\) such that \(0 < |1 - (u + i\sigma \eta)/t| < 3\eta/t\)

where \(F_j'(w, \zeta, \xi)\), resp. \(\hat{F}_j'(w, \zeta, \xi)\) are holomorphic on \(B \times \Omega_1 \times \Omega_2\), resp. \(\tilde{B} \times \Omega_1' \times \Omega_2'\), with \(B := \{|w| < 3, w - i\sigma \not\in -i\sigma + \mathbb{R}_-\}\), \(\tilde{B} := \{|w| < 3, w + i\sigma \not\in i\sigma + \mathbb{R}_-\}\) and \(F_j''(w, \zeta, \xi)\), \(\tilde{F}_j''(w, \zeta, \xi)\), \(G_j''(w, \zeta, \xi)\), \(\tilde{G}_j''(w, \zeta, \xi)\) are holomorphic on \(B(0,1) \times \Omega_1 \times \Omega_2\),

with \(U_j^{(n)}, \tilde{U}_j^{(n)} > -1\) for all \(j\) and \(\{(B_j^{(n)'}, H_j^{(n)'}) + U_j^{(n)'}, j' = 1 \ldots J'\} = \{(B_j^{(n)'}, H_j^{(n)'}) + U_j^{(n)'}, j' = 1 \ldots J''\}\) (and similarly for the exponents with a tilde), \(\{(H_j^{(n)'}, B_j^{(n)'}) + U_j^{(n)'}, j = 1 \ldots J''\}\)

Then one says that \(f\) is an admissible analytic function with \(b\)-exponents \(\{(B_j^{(b)'}, U_j^{(b)'})\}\) on the domain \((\hat{i})_b\) and \(\{(B_j^{(b)'}, U_j^{(b)'})\}\) on the domain \((\hat{ii})_b\), and \(\eta\)-exponents \(\{(H_j^{(n)'}, B_j^{(n)'}, U_j^{(n)'})\}\) on the domain \((\hat{ii})_\eta\), \(\{(H_j^{(n)'}, B_j^{(n)'}, U_j^{(n)'})\}\) on the domain \((\hat{iii})_\eta\) (and similarly for the domains with a tilde).

If \(f = f_b\), resp. \(f = f_\eta\), then one says that \(f\) is of \(b\)-type, resp. of \(\eta\)-type.

In particular, the function \(\Phi(\beta_1, \beta_2; 0)(u, b)\) appearing in Lemma 1.10 is an admissible analytic function of \(b\)-type with \(b\)-exponents \(\{(B_j^{(b)'}, U_j^{(b)'})\}\) \(\{(B_j^{(b)'}, U_j^{(b)'})\}\)

while \(\Phi(\beta_1, \beta_2; t)(u, b)\) is of the same type up to the symmetry \(u \rightarrow t - u, b \rightarrow t - b\).

Remarks.

- The extra conditions \(w - i\sigma \not\in \frac{u}{\eta} \not\in -i\sigma + \mathbb{R}_-,\) resp. \(w + i\sigma \not\in \frac{t - u}{\eta} \not\in i\sigma + \mathbb{R}_-\) (which are true when integrating over the real axis or, more generally, over any deformed contour \(\gamma : 0 \rightarrow t\)) avoid considering unnecessary complications (the values of the exponents are different when \(u\) or \(t - u\) is in a neighbourhood of \(\pm \eta\)). They also appear in the definition of the cut domain \(\Omega\). Note that the functions \(F_j,\)
$F'_j$, $F''_j$ associated to the domains $(i)_b$, $(ii)_b$, $(iii)_b$ are bounded on their respective domains, and so are the functions $F''_j$, $G''_j$, $\tilde{F}''_j$, $\tilde{G}''_j$ on $(iii)_{\eta_j}$, $(iii)_{\tilde{\eta}}$, while $F'_j$ are possibly unbounded on $(ii)_{\eta_j}$, $(ii)_{\tilde{\eta}}$. But they are bounded on the subdomain $\Omega_{res} := \Omega \setminus \{B(-i\sigma_0, \eta/3) \cup B(t-i\sigma_0, \eta/3)\}$, see also proof of Theorem 1.5, which is already satisfactory since one is ultimately interested in the behaviour around $[0, t] \subset \Omega_{res}$.

The reason for the appearance of the $\sigma$-sign is that admissible analytic functions are in the image of the integral transformation $K^{*, \sigma}([0, t])$ or $K^{*, \sigma}([0, t])$ for some $\sigma \in \{\pm 1\}$. Note that $\int^t_0 (i\sigma(v - u) + \eta)^\beta g(v) \, dv$, $\beta = 2\alpha$ or $2\alpha - 2$ (for appropriate functions $g$) is possibly singular on the boundary of $\Omega$, but always regular on the closure of $\Omega_{res}$.

- Note that the family of $b$-exponents $\{(B_j^{(b)}, U_j^{(b)}), (B_j^{(b)'}, U_j^{(b)'})\}$ and the family of $\eta$-exponents $\{(H_j^{(\eta)'}, B_j^{(\eta)'}, U_j^{(\eta)'}, (H_j^{(\eta)'}, B_j^{(\eta)'}, U_j^{(\eta)'})\}$ (together with the relative exponents with a tilde) determine all the exponents of the function $f$. The conditions on the $U_j$’s ensure in particular that $f_b$ is integrable when $\eta = 0$.

- The number $\sigma$ does not change when one integrates $f$ against $K''([\eta])$ or $K([\eta])$. When computing the contribution to the $2N$-th moment of the $\eta$-Lévy area of the terms containing the kernel $(i\sigma_2 (x_{2N-1} - x_{2N}))^{2\alpha} = (i\sigma_2 (x_{2N-1} - x_{2N}))^{2\alpha}$, one simply has $\sigma = \sigma_2$.

On the contrary, the value of $\sigma_f$ is shifted by $\pm 1$ after each integration (see Theorem 1.4).

- Depending on whether the context is clear of not, we shall sometimes drop the upper indices ($b$) or ($\eta$) of the exponents.

- The constants $1/3$, $2/3$, $\ldots$ appearing in the definition of the domains are arbitrary and may be replaced by any other set of positive constants, as long as the subdomains intersect.

- The splitting of $f$ into $f_b + f_\eta$ is essentially a 'pedagogical' artefact. One should actually consider a single function with different expressions and exponents depending on the relative position of $0, \eta, b, t$ and $u$. That would make the above Definition even more technical, with the following advantage however: assuming $\sigma_f \neq \sigma$, so $z := u + i\sigma_f \eta$
may be arbitrary close to 0 on $\Omega_{res}$, both functions $f_b$ and $f_\eta$ have
a singularity when $z = 0$ if some exponent $U_j^{(b)}$ or $U_j^{(\eta')}$ is negative
(which does happen in our case), while $f_b + f_\eta$ is analytic at $z = 0$ (by
definition). Fortunately this is not a real problem for the convergence
proof.

The existence of the $b$-exponents follows naturally from the discussion
before Definition 1.14. They describe the power behaviour of the function
$f$ near 0 and $t$. The complications come from the fact that the parameter
$b$ itself may be close to 0 (on the contrary, if $b$ is bounded from below, say
$b > t/2$, then $b$ may be considered as a constant and the above Definition
may be drastically simplified). The presence of the $\eta$-exponents follows in
a less straightforward manner from the fact that one integrates against the
$\eta$-approximation of the power kernels $K^{*, \pm}$, $K^{*, \pm}$ (see the proof of Theorem
1.1 for a computational explanation).

Theorem 1.1 (action of the kernels $K^{*, \pm}(\eta)$ of the admissible analytic functions)

Assume $f$ is an admissible analytic function with exponents as in Definition
1.14. Then $g := K^{*, \sigma}_{b}[0,t](\eta)f$ ($\sigma = \pm 1$) is admissible, with $b$-exponents:

\begin{equation}
(i)_b \quad \{ (B_j^{(b)} + U_j^{(b)} + 2\alpha + 1, 0)_{j=1...J}, (B_j^{(b)} + U_j^{(b)} + 2\alpha + 1)_{j=1...J} \}
\end{equation}

on the domain $(i)_b$;

\begin{equation}
(iii)_b
\{ (B_j^{(b)} + U_j^{(b)} + 1, 2\alpha)_{j=1...J} \} \\
\cup \{ (B_j^{(\eta')}, U_j^{(\eta')} + 2\alpha + 1)_{j=1...J'} \}
\end{equation}

on the domain $(iii)_b$,

while its $\eta$-exponents are:

\begin{equation}
(ii)_\eta
\{ (0, B_j^{(\eta)}, U_j^{(\eta)} + 2\alpha + 1)_{j=1...J}, (U_j^{(\eta)} + 2\alpha + 1, B_j^{(\eta)}, 0)_{j=1...J} \} \\
\cup \{ (H_j^{(\eta')}, U_j^{(\eta')} + 2\alpha + 1, B_j^{(\eta')}, 0)_{j=1...J'}, (H_j^{(\eta')}, B_j^{(\eta')}, 0)_{j=1...J'} \} \\
\cup \{ (\tilde{H}_j^{(\eta')}, \tilde{B}_j^{(\eta')}, 0)_{j=1...J''}, (\tilde{H}_j^{(\eta')}, \tilde{U}_j^{(\eta')} + 2\alpha + 1, \tilde{B}_j^{(\eta')}, 0)_{j=1...J''} \}
\end{equation}
Theorem 1.2 (action of the kernels $K^{.,\pm}(\eta)$ of the admissible analytic functions) (same hypotheses). Add the following stronger assumption on the exponents: $U_j^{(b)} > -2\alpha$ ($j = 1, \ldots, J$). Then $g := K^{.,\pm}_{[0,\hat{f}]}$ is admissible, with exponents given as in Theorem 1.1 except that $\alpha$ should be replaced everywhere by $\alpha - 1$.

Proof of Theorems 1.1 and 1.2

The proof is exactly the same for both Theorems. Note that the assumption on the exponents $(U_j)$, $(V_j)$ in the case of Theorem 1.2 ensures
that the exponents \((U_j^{(b)'}),(V_j^{(b)'''})\) of \(g = K^\prime,\pm(\eta)f\) are larger than \(-1\). No such assumption is needed in the case of Theorem 1.1. We shall prove for instance Theorem 1.1 and assume that \(\bar{\sigma} = -1\) (otherwise one should take the complex conjugate of all expressions).

By Lemma 3.2 \(K^*,\sigma_{[0,t]}(\eta)f\) is analytic on \(\Omega\). The question is: what are its exponents?

Let us first look at the contribution of \(f_b\). We shall use the variable \(z := a + i(\sigma + \sigma f)\eta\) throughout the proof. By splitting the interval of integration \([0,t]\) into three pieces (corresponding to the domains \((i)b, (ii)b, (iii)b\)), there appears spurious singularities near the inner boundaries of each subdomain, which cancel when summing up all terms. Hence we shall not consider these boundary regions.

\((i)b\) Let

\[
h(a) = \int_{0}^{\frac{b-i\eta}{U}} (i\sigma(u-a) + \eta)^{2\alpha}(b-i\eta)^B(u + i\sigma f\eta)^U F\left(\frac{u + i\sigma f\eta}{b-i\eta}, \frac{\eta}{b-i\eta}\right) du
\]

\[
= (b-i\eta)^{-2\alpha+1} \int_{\frac{b+i\eta(2\sigma f-1)}{b-i\eta}}^{\frac{1}{b+i\eta}} (i\sigma(w-z) + \eta)^{2\alpha} w^U F\left(\frac{w}{b-i\eta}, \frac{\eta}{b-i\eta}\right) dw
\]

(1.66)

(with \(w = \frac{u+i\sigma f\eta}{b-i\eta}\) for some \(U > -1\). It is the sum of two terms (see Lemma 3.5). We may assume \(|b/\eta| > c > 0\) (otherwise apply simply Lemmas 3.2, 3.3), so \(\frac{1}{b+i\eta(2\sigma f-1)} > \frac{\sigma f\eta}{b-i\eta}\). Apply now Lemma 3.5 with the 'crude' version of case (iv) – see last Remark after Definition 1.14 on the non-optimality of the \((f_b,f_\eta)\)-splitting –. Then \(h(a) =: h_b(z) + h_\eta(z)\) with:

- \((b\text{-exponents})\)

One finds

\[
h_b(z) = (b-i\eta)^{-B+U+2\alpha+1} G\left(\frac{z}{b-i\eta}, \frac{\eta}{b-i\eta}\right)
\]

\[
+ (b-i\eta)^{-B+U+2\alpha+1} H\left(\frac{z}{b-i\eta}, \frac{\eta}{b-i\eta}\right)
\]

(1.67)

if \(|\frac{z}{b+i\eta(2\sigma f-1)}| < c < 1\), and

\[
h_b(z) = (b-i\eta)^{-B+U+1} z^{2\alpha} H\left(\frac{b-i\eta}{z}, \frac{\eta}{b-i\eta}\right)
\]

(1.68)
\[
|\frac{z}{b+\eta(2\sigma_f-1)}| > C > 1. \text{ In other words, one has the following sets of } \ b\text{-exponents: } \{(B+U+2\alpha+1,0),(B+2\alpha+1)\} \text{ on the domain } \ (i)_b, \text{ and } \{(B+U+1,2\alpha)\} \text{ on the domain } \ (iii)_b.
\]

• (\eta\text{-exponents})

Suppose \(\sigma_f \neq 0\) (otherwise \(h_\eta \equiv 0\)). One finds (on the cut domain \(\Omega\))

\[
h_\eta(z) = (b - i\eta)^{B+U+2\alpha+1} \left[ \left( \frac{z}{b - i\eta} \right)^{U+2\alpha+1} F(z, \eta, b - i\eta) + \left( \frac{\eta}{b - i\eta} \right)^{U+2\alpha+1} G(z, \eta, b - i\eta) \right]
\]

(1.69)

if \(0 < |\frac{z}{\eta}| < c < 1\), and

\[
h_b(z) = (b - i\eta)^{B+U+2\alpha+1} \left( \frac{i\sigma_f \eta}{b - i\eta} \right)^{U+1} \left( \frac{z}{b - i\eta} \right)^{2\alpha} F(\eta, \frac{\eta}{b - i\eta})
\]

(1.70)

if \(|\frac{z}{\eta}| > C > 1\). The function \(F\) may go to infinity when \(z \to i\sigma_f \eta\), i.e. \(a \to -i\sigma_\eta\), on the boundary of the cut domain \(\Omega\). In other words, one has the following sets of \(\eta\)-exponents: \(\{(0, B,U+2\alpha+1),(U+2\alpha+1,B,0)\}\) on the domain \((ii)_\eta\); \(\{(U+1,B,2\alpha)\}\) on the domain \((iii)_\eta\) and \(\{(U+1,B,0)\}\) on the relative tilded domains \(\tilde{(iii)}_\eta, \tilde{(ii)}_\eta\).

(ii) Let

\[
h(a) = \int_{\eta b = i\eta}^{\eta b = i\eta /2} (i\sigma(u-a) + \eta)^{2\alpha} (b - i\eta)^{B'} F\left( \frac{u + i\sigma_f \eta}{b - i\eta}, \frac{\eta}{b - i\eta} \right) \, du
\]

(1.71)

with the same change of variables. Applying Lemma 3.5 (or Lemma 3.2) to the above integral, one gets:

\[
h(a) = (b - i\eta)^{B'+2\alpha+1} G\left( \frac{z}{b - i\eta}, \frac{\eta}{b - i\eta} \right)
\]

(1.72)
\[ h(a) = (b - i\eta)^{B' + 1}z^{2\alpha}G\left( \frac{b - i\eta}{z}, \frac{\eta}{b - i\eta} \right) \quad (1.73) \]

if \( \left| \frac{z}{b + in(2\sigma_f - 1)} \right| < c < 1 \) and

\[ h(a) = (b - i\eta)^{B' + 1}z^{2\alpha}G\left( \frac{b - i\eta}{z}, \frac{\eta}{b - i\eta} \right) \quad (1.75) \]

if \( \left| \frac{z}{b + in(\sigma_f - 1/2)} \right| > C > 1 \). Hence \( h \) is admissible of \( b \)-type, with the following sets of \( b \)-exponents: \( \{(B' + 2\alpha + 1, 0)\} \) on the domain \((i)_b\), \( \{(B' + 1, 2\alpha)\} \) on the domain \((iii)_b\).

\((iii)_b\) Let

\[ h(a) = \int_{b-in/2}^{t} (i\sigma(u - a) + \eta)^{2\alpha}(b - i\eta)^{B''}(u + i\sigma\eta)^{U''}F\left( \frac{b - i\eta}{u + i\sigma\eta}, \frac{\eta}{b - i\eta} \right) \, du \]

\[ \quad = \frac{(b - i\eta)^{B''+U''+1}z^{2\alpha}}{U''} \int_{b-in}^{b-in(\sigma_f - 1/2)} (i\sigma(u) - w)^{2\alpha}w^{-2\alpha-2-U''}F\left( \frac{b - i\eta}{u}, \frac{\eta}{b - i\eta} \right) \, dw \]

\[(1.74)\]

(set \( w = \frac{b-in}{u+i\sigma_f\eta} \)). By Lemma 3.5, it is the sum of two terms, \( h = h_b + h_\eta \). One gets:

• (\( b \)-exponents)

\[ h_b(a) = (b - i\eta)^{B''+U''+2\alpha+1}F\left( \frac{z}{b - in}, \frac{\eta}{b - i\eta} \right) \quad (1.77) \]

if \( \left| \frac{z}{b + in(\sigma_f - 1/2)} \right| < c < 1 \);

\[ h_b(a) = (b - i\eta)^{B''}z^{U''+2\alpha+1}F\left( \frac{b - i\eta}{z}, \frac{\eta}{b - i\eta} \right) \]

\[ \quad + (b - i\eta)^{B''}z^{U''+1}2\alpha G\left( \frac{b - i\eta}{z}, \frac{\eta}{b - i\eta} \right) \quad (1.76) \]

if \( \left| \frac{z}{b + in(\sigma_f - 1/2)} \right| > C > 1 \).

In other words, \( h_b \) is admissible of \( b \)-type, with the following sets of \( b \)-exponents: \( \{(B''+U''+2\alpha+1, 0)\} \) on the domain \((i)_b\), and \( \{(B'', U'' + 2\alpha + 1), (B'' + U'' + 1, 2\alpha)\} \) on the domain \((iii)_b\).

• (\( \eta \)-exponents)

One finds:

\[ \text{32} \]
$$h_\eta(a) = (b - i\eta)^{B''} H(\frac{b - i\eta}{t + i\sigma f \eta}, \frac{z}{t + i\sigma f \eta}, \frac{\eta}{b - i\eta})$$ (1.77)

if \(|\frac{z}{t + i\sigma f \eta}| < c < 1; \)

$$h_\eta(a) = (b - i\eta)^{B''} \left( G(\frac{b - i\eta}{t + i\sigma f \eta}, 1 - \frac{t + i\sigma f \eta}{z}) + (1 - \frac{t + i\sigma f \eta}{z})^{2\alpha + 1} H(\frac{b - i\eta}{t + i\sigma f \eta}, 1 - \frac{t + i\sigma f \eta}{z}) \right)$$ (1.78)

if \(|1 - z/t| < c < 1; \)

Hence \(h_\eta\) is admissible of \(\eta\)-type, with the following sets of \(\eta\)-exponents:

\(\{(0, B''(0, 0)\}\) on \((ii)_\eta\) and \((iii)_\eta; \{(0, B'', 0), (0, B'', 2\alpha + 1)\}\) on \((iii)_\eta, (ii)_\eta.\)

There remains to analyze the contribution to \(K_{[0,t]}^{\sigma,\eta}(\eta)f\) of \(f_\eta.\) These terms are simpler since (as we shall presently see) they only contribute admissible functions of \(\eta\)-type.

\((ii)_\eta\) Consider

$$h(a) = \int_0^{2\eta} \left( i\sigma(u - a) + \eta \right)^{2\alpha} \eta^{H'} (b - i\eta)^{B'} (u + i\sigma f \eta)^{U'} \eta^{F}(\frac{u + i\sigma f \eta}{\eta}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t}) \, du$$

$$= (b - i\eta)^{B''} \eta^{H'' + U'' + 2\alpha + 1} \int_{i\sigma f}^{2(1+i\sigma f/2)} \left( i\sigma(w - \frac{z}{\eta}) \right)^{2\alpha} w^{U'} F(w, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t}) \, dw$$ (1.79)

where one has set \(w = \frac{u + i\sigma f \eta}{\eta}.\) By Lemma 3.3

$$h(a) = (b - i\eta)^{B'} \eta^{H' + U' + 2\alpha + 1} G(\frac{z}{\eta}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t})$$ (1.80)

if \(|\frac{z}{2(1+i\sigma f/2)\eta}| < c < 1, \) where \(G\) becomes possibly infinite when \(z \rightarrow i\sigma f \eta;\)

$$h(a) = (b - i\eta)^{B'} \eta^{H' + U' + 1} z^{2\alpha} H(\frac{\eta}{z}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t})$$ (1.81)
\[ h(a) = \int_{2\eta}^{t/2} \left( F\left( \frac{\eta}{u + i\sigma_f h}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t} \right) + G\left( \frac{u + i\sigma_f h}{t + i\sigma_f h}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t} \right) \right) du \]

\[ = \eta^{H''+U''+1}(b - i\eta)^{B''} z^{2\alpha} \eta \int_{\frac{2\eta}{t+2\sigma_f h}}^{t/2} (i\sigma(\frac{\eta}{z} - w))^{2\alpha} w^{-2\alpha - 2 - U''} F(w, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t}) dw \]

\[ + \eta^{H''}(b - i\eta)^{B''} \int_{\frac{2\eta}{t+2\sigma_f h}}^{t/2} (i\sigma(w - \frac{z}{t + i\sigma_f h}))^{2\alpha} w^{U''} G(w, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t}) dw \]

\[ =: h_1(a) + h_2(a) \] (1.82)
if $\left|\frac{z}{\sigma f}t\right| > C > 1$, $|z/t| < 2$. (Note that $\frac{\eta}{z} = \frac{\eta}{t} 1/(1-z/t)$ is an analytic function of $\frac{\eta}{t} = \frac{\eta}{b-i\eta}$ and $1 - \frac{z}{t}$, while the powers in $z$ in factor in the last equation may be skipped since $(\frac{z}{t})^\beta = (1 - (1 - \frac{z}{t}))^\beta$ is an analytic function of $\frac{z}{t}$).

Similarly,

$$h_2(a) = \eta H''(b - i\eta) B'' G(\frac{z}{t}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t}) + \eta H'' + U'' + 2\alpha + 1(b - i\eta) B'' H(\frac{z}{\eta}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t})$$

(1.86)

if $\left|\frac{z}{(2+i\sigma f)\eta}\right| < \eta$;

$$h_2(a) = \eta H''(b - i\eta) B'' z U'' + 2\alpha + 1 F(\frac{z}{t}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t})$$

$$+ \eta H''(b - i\eta) B'' G(\frac{z}{t}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t})$$

$$+ \eta H'' + U'' + 1(b - i\eta) B'' z 2\alpha H(\frac{\eta}{z}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t})$$

(1.87)

if $\left|\frac{z}{(2+i\sigma f)\eta}\right| < \eta$ and $\left|\frac{z}{(1+i\sigma f/2)\eta}\right| > C > 1$;

$$h_2(a) = \eta H''(b - i\eta) B'' \tilde{G}(1 - \frac{z}{t}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t})$$

$$+ \eta H'' + U'' + 1(b - i\eta) B'' H(\frac{\eta}{z}, \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t})$$

(1.88)

if $\left|\frac{z}{(2+i\sigma f)\eta}\right| > C > 1$, $|z/t| < 2$.

Hence one gets the following $\eta$-exponents: \{(H'', B'', 0), (H'' + U'' + 2\alpha + 1, B''), (H'' + U'' + 2\alpha + 1, B'', 2\alpha), (H'', B'', 0)\}
on (iii)$_\eta$; \{(H'' + U'' + 1, B'', 0), (H'', B'', 0)\} on (iii)$_\eta$ and (ii)$_\eta$. 35
\((ii)\), \((iii)\) The expansions may be derived very simply from the previous ones by using the symmetry \(u \leftrightarrow t - u\).

\[\Box\]

**Corollary 1.15 (integrability and b-exponents)**

1. **(integrability)** Suppose \(f\) is an admissible analytic function of \(b\)-type with exponents \((U^{(b)})_j = 1 \ldots j\) given by \(U^{(b)}_1 = 0\), \(U^{(b)}_2 = 2\alpha - 1\). Then \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\), resp. \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\) are well-defined admissible analytic functions for any \(m = 0, 1, \ldots\), with exponents \(\{(U^{(b)})_j\} \subset \{4\alpha n, 4\alpha n + 2\alpha - 1 \mid n = 0, \ldots, m\}\), resp. \(\{(U^{(b)}_j) \subset \{0\} \cup \{4\alpha n + 2\alpha + 1, 4\alpha n + 4\alpha \mid n = 0, \ldots, m\}\). In particular, the exponents \((U^{(b)}_j)_{j = 1 \ldots J}\) of \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\), resp. \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\) are all non-negative, resp. \(\geq 2\alpha - 1\).

Similarly, if \(f\) is an admissible analytic function of \(b\)-type with exponents \((U^{(b)}_j)_{j = 1 \ldots J}\) given by \(U^{(b)}_1 = 0\), \(U^{(b)}_2 = 2\alpha + 1\) then \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\), resp. \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\) are well-defined admissible analytic functions for any \(m \geq 0\), and the exponents \((U^{(b)}_j)_{j = 1 \ldots J}\) of \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\), resp. \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\) are all non-negative, resp. \(\geq 2\alpha - 1\).

2. **(on some other b-exponents)** Let \(n \geq 1\). Suppose \(f\) is an admissible analytic function of \(b\)-type with exponents \((B^{(b)}_j, U^{(b)}_j)_{j = 1 \ldots J}\) such that \(B^{(b)}_j + U^{(b)}_j = 4\alpha n - 1\), resp. \(4\alpha n + 2\alpha\) for all \(j\) and \(\{B^{(b)}_j\} = \{0, 4\alpha(n - 1) + 2\alpha + 1\}\), resp. \(\{0, 4\alpha n\}\). Then the \(B^{(b)}_j\) -exponents of \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\), resp. \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\) \((m \geq 0)\) are all non-negative, and so are all the elements of the set \(\{B^{(b)}_j + U^{(b)}_j, j = 1 \ldots J\} = \{B^{(b)}_j + U^{(b)}_j, j = 1 \ldots J\}\).

Also, the \(B^{(b)}_j\) -exponents of \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\), resp. \((K^{\prime, \pm\eta}(\eta)K^{\prime, \pm\eta}(\eta))^{m} f\) are all non-negative, and the elements of the set \(\{B^{(b)}_j + U^{(b)}_j, j = 1 \ldots J\} = \{B^{(b)}_j + U^{(b)}_j, j = 1 \ldots J\}\) are all \(\geq 2\alpha - 1\).

3. More precisely, if one of these sums of exponents \((X_j, \text{ say})\) satisfies instead a strict inequality, namely, \(X_j > 0\), resp. \(X_j > 2\alpha - 1\), then \(X_j \geq 2\alpha\), resp. \(X_j \geq 0\).
Proof.

1. Elementary induction on n using Theorems 1.1 and 1.2

2. The inequalities hold true for f. Then one may prove that if the b-exponents of any admissible function f (of b-type) satisfy the relations $B_j^{(b)''} + U_j^{(b)''} \geq 0$, then the b-exponents of $K_{[0,\xi]}(\eta)f$ satisfy the corresponding relations $B_j^{(b)''} \geq 0$, $B_j^{(b)''} + U_j^{(b)''} \geq 2\alpha - 1$, and vice versa if one considers $K_{[0,\xi]}^{\pm}(\eta)f$ instead.

3. is proved along the same lines as 2.

\[ \blacksquare \]

Corollary 1.16 (η-exponents) Let f be an admissible analytic function of b-type with exponents $(B_j^{(b)}, U_j^{(b)})_{j=1,...,j}$ such that $B_j^{(b)} + U_j^{(b)} = 4\alpha n - 1$, resp. $4\alpha n + 2\alpha$ for all $j$ and $\{(B_j^{(b)''})\} = \{0, 4\alpha(n-1) + 2\alpha + 1\}$, resp. $\{0, 4\alpha n\}$ (see Corollary 1.15, point 2). Then:

1. Let $g_0 := K_{[0,\xi]}^{\pm}(\eta)(K_{[0,\xi]}^{\pm}(\eta)K_{[0,\xi]}^{\pm}(\eta))m f$, resp. $(K_{[0,\xi]}^{\pm}(\eta)K_{[0,\xi]}^{\pm}(\eta))m f$ for some $m \geq 0$, $g_1 := (K_{[0,\xi]}^{\pm}(\eta)((g_0)_\xi))_{\eta}$ be the η-part of the integral of the b-part of $g_0$ against $K_{[0,\xi]}^{\pm}(\eta)$, and $h := (K_{[0,\xi]}^{\pm}(\eta)K_{[0,\xi]}^{\pm}(\eta))n K_{[0,\xi]}^{\pm}(\eta)g_1$, resp. $(K_{[0,\xi]}^{\pm}(\eta)K_{[0,\xi]}^{\pm}(\eta))n g_1$ for some $n \geq 0$. Then $h$ is of η-type, with η-exponents such that

\[ H_j^{(n)''}, \tilde{H}_j^{(n)''}, H_j^{(n)''} + B_j^{(n)''}, \tilde{H}_j^{(n)''} + \tilde{B}_j^{(n)''} \geq 0 \] (1.89)

and

\[ U_j^{(n)'}, \tilde{U}_j^{(n)'}, H_j^{(n)''} + B_j^{(n)''} + U_j^{(n)''}, \tilde{H}_j^{(n)''} + \tilde{B}_j^{(n)''} + \tilde{U}_j^{(n)''}, H_j^{(n)''} + U_j^{(n)''}, \tilde{H}_j^{(n)''} + \tilde{U}_j^{(n)''} \geq 0, \] (1.90)

resp. (depending on h)

\[ U_j^{(n)'}, \tilde{U}_j^{(n)'}, H_j^{(n)''} + B_j^{(n)''} + U_j^{(n)''}, \tilde{H}_j^{(n)''} + \tilde{B}_j^{(n)''} + \tilde{U}_j^{(n)''}, H_j^{(n)''} + U_j^{(n)''}, \tilde{H}_j^{(n)''} + \tilde{U}_j^{(n)''} \geq 2\alpha - 1. \] (1.91)

More precisely, if one of these sums of exponents $(X_j, \text{say})$ satisfies instead a strict inequality, namely, $X_j > 0$, resp. $X_j > 2\alpha - 1$, then $X_j \geq 2\alpha$, resp. $X_j \geq 0$.

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2. Similarly, let \( g_0 := (K'_{\pm}(\eta)K_{[0,t]}(\eta))^m f \), resp. \( K'_{\pm}(\eta)(K_{[0,t]}(\eta))^m \) for some \( m \geq 0 \), \( g_1 := (K_{[0,t]}(\eta))(g_0)_0 \) be the \( \eta \)-part of the integral of the \( b \)-part of \( g_0 \) against \( K'_{\pm}(\eta) \), and \( h := (K_{[0,t]}(\eta)K'_{\pm}(\eta))^n g_1 \), resp. \( K'_{[0,t]}(\eta)^n g_1 \) for some \( n \geq 0 \). Then \( h \) is of \( \eta \)-type, with \( \eta \)-exponents \((H_j^{(\eta')}, B_j^{(\eta')}, U_j^{(\eta')})\) and \((\tilde{H}_j^{(\eta')}, \tilde{B}_j^{(\eta')}, \tilde{U}_j^{(\eta')})\) satisfying the same relations.

**Proof.**

1. Let \((B_{g_0}^{(b)}, U_{g_0}^{(b)})\) be the \((B^{(b)}, U^{(b)})\)-exponent of any term in \((g_0)_b\). By Corollary 1.15 \( U_{g_0}^{(b)} + B_{g_0}^{(b)} \geq 0 \). Now rules (1.62), (1.65), (1.64) and (1.64) in Theorem 1.1 imply that the \( \eta \)-exponents of \( g_1 \) satisfy relations (1.89) and (1.91). Now one may check very easily that \( K_{[0,t]}(\eta)f_\eta \) satisfies relations (1.89) and (1.90) if \( f_\eta \) satisfies relations (1.89), (1.91) and vice-versa if one considers \( K'_{[0,t]}(\eta)f_\eta \) instead.

2. The proof is similar, with initial relations \( U_{g_0}^{(b)}, B_{g_0}^{(b)} + U_{g_0}^{(b)} \geq 2\alpha - 1 \) and \( B_{g_0}^{(b)} \geq 0 \) this time.

\( \square \)

**Remark.** Note that the exponents \( U_j^{(\eta')'} \) and \( \tilde{U}_j^{(\eta')'} \) are simply 0, except those which come from an admissible function of \( \tilde{b} \)-type (and are actually only due to the \((f_\tilde{b}, f_\tilde{\eta})\)-splitting, which is somewhat unfortunate in this respect).

### 1.4 Asymptotic behaviour of the moments of the Lévy area

Now the computation of the exponents is over, we may study the singularities of \( \mathbb{E}[(A_{k,t}(\eta))^{2N}] \) when \( \eta \to 0 \). It turns out eventually that only one term of the \( 2N \)-th connected moment is singular (see Theorem 1.4 below, cited in the Introduction). This term comes from the only closed bipartite diagram with alternating simple and double lines, see end of subsection 1.1, and obtained by iterating \( I^+ \)-type integrals, see comments at the end of subsection 1.2.

Elementary arguments relying on Lemma 1.7 allow then to deduce the asymptotic behaviour of \( \mathbb{E}[(A_{k,t}(\eta))^{2N}] \) from that of the connected moments.

The proof of Theorem 1.4 requires first a separate analysis of all terms coming from the splitting of the kernel \( K \) (see end of subsection 1.1).
Lemma 1.17  Let $I_n(\eta, b, t; u) = (K_{[s,t]}^*(\eta)K_{[s,t]}'(\eta))^n \left( u \mapsto (i\sigma(u-b) + \eta)^{2\alpha} \right)(u), \ \sigma \in \{\pm 1\}$. Then $I_n$ is the sum of an admissible analytic function of type $b$ with $b$-exponents such that $U_j^{(b)}, B_j^{(b)} + U_j^{(b)} \geq 0$ for all possible indices $j$, of an admissible analytic function of type $\eta$ with $\eta$-exponents such that

$$H_j^{(\eta)} + U_j^{(\eta)} + B_j^{(\eta)} + B_j^{(\eta)} + U_j^{(\eta)} + U_j^{(\eta)} \geq 0 \quad (1.92)$$

(plus the same inequalities with a tilde), and of the non-analytic function

$$R_n(b, u) = C_n \operatorname{Re} (i(b - u) + (2n + 1)\eta)^{2\alpha + 4\alpha n} \quad (1.93)$$

with

$$C_n = \frac{1}{2\pi} \left( \frac{\pi/2}{\cos \pi \alpha \Gamma(-2\alpha)} \right)^{2n} \sin \pi \alpha \Gamma(2\alpha + 1) \Gamma(-2\alpha - 4\alpha n). \quad (1.94)$$

Remark. As appears clearly in the proof below, the non-analytic function $R_N$ doesn’t show up when one considers the moments of the Lévy area of the analytic process $\Gamma$ (see Proposition 1.2), namely $\int_0^t d\Gamma_2^{(1)}(t) \int_0^\pi d\Gamma_2^{(2)}(\eta)$.

Proof.

By definition (choosing for instance $\sigma = -1$),

$$I_n = \sum_{\sigma_2(N-n), \ldots, \sigma_2N \in \{\pm 1\}} \left( u \mapsto (-i(u-(b-in)))^{2\alpha} \right)(u). \quad (1.95)$$

Iterating the non-analytic term $(\pm i(b-a))^{\beta_1+\beta_2+1}$ appearing in Lemma 1.12 one obtains (up to a certain coefficient) $(\pm i(u-b) + (2n+1)\eta)^{2\alpha + 4\alpha m}$ after $2m$ integrations, resp. $(\pm i(u-b) + (2n+2)\eta)^{4\alpha - 1 + 4\alpha m}$ after $2m + 1$ integrations. This is possible only if the $2m$, resp. $2m + 1$ iterated integrals are of $I_4$-type, i.e. if $-1 = \sigma_2N = \ldots = \sigma_2(N-m)$, resp. $-1 = \sigma_2N = \ldots = \sigma_2(N-m)-1$. Integrate once again and look instead at the analytic term this time: by Definition 1.14 it is an admissible analytic function with exponents $\{(B_j, U_j)\} = \{(4\alpha - 1 + 4\alpha m, 0), (2\alpha + 4\alpha m, 2\alpha - 1)\}$, $\{(B_j'', U_j'')\} = \{(0, 4\alpha - 1 + 4\alpha m), (2\alpha + 4\alpha m, 2\alpha - 2)\}$, resp. $\{(B_j, U_j)\} = \{(6\alpha + 4\alpha m, 0), (4\alpha - 1 + 4\alpha m, 2\alpha + 1)\}$, $\{(B_j'', U_j'')\} = \{(0, 6\alpha + 4\alpha m), (4\alpha + 4\alpha m, 2\alpha)\}$, possibly up to the symmetry $u \to t-u$ (so $b$ must be replaced with $t-b$ in that case). Now Theorem 1.1 shows that integrating such an admissible function alternatively
against the kernels $K^*,\pm(\eta)$ and $K',\pm(\eta)$ yields admissible functions. The lower bounds on the exponents come from Corollaries 1.15 and 1.16 in the 'good' case where exponents are non-negative.

Suppose now all signs $\sigma_{2N},\ldots,\sigma_{2(N-n)}$ are equal. Then one obtains in the end a non-analytic term by iterating $2n$ times Lemma 1.12. The coefficient before that term may be checked by an easy induction using the complement formula for the Gamma function, namely, $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$.

We may now proceed to estimate the connected diagrams. Let us first analyze the contribution of the open diagrams (see end of subsection 1.1). The following Theorem shows that they are all regular in the limit $\eta \to 0$.

**Theorem 1.3** The contribution of the open diagrams writes $C_N t^{4n\alpha} + O(\eta^{2n\alpha})$ for some constant $C_N$.

**Proof.**

Recall that the open diagrams are products of terms of three types, $(\emptyset\emptyset)$, $(\emptyset\bullet)$ and $(\bullet\bullet)$. Let us analyze these three cases separately.

**Case $(\bullet\bullet)$.**

Let

$$I_t^{\bullet\bullet} := \sum_{\sigma_0,\ldots,\sigma_{2n-1}} \int_0^t \cdots \int_0^t (i\sigma_0 x_1 + \eta)^{2\alpha}(i\sigma_1(x_1 - x_2) + \eta)^{2\alpha-2} \cdots \cdots (i\sigma_{2n-1}(x_{2n-1} - x_{2n}) + \eta)^{2\alpha-2}(i\sigma_{2n}x_{2n} + \eta)^{2\alpha} dx_1 \cdots dx_{2n}$$

with $\sigma_0,\ldots,\sigma_{2n} \in \{\pm 1\}$. Then $I_t^{\bullet\bullet} = I_n(\eta, 0, t; 0)$ in the notation of Lemma 1.17. The non-analytic part $R_n(0, 0)$ of $I_n$ is negligible, of order $O(\eta^{2n+4\alpha n})$, so let us consider the analytic part. Assume first $\sigma_f \neq 0$, then $\eta^H(b + i\sigma_f)^B(u + i\sigma_f\eta)^U = C\eta^{H+B+U}$ with

$$(H, B, U) \in \{(0, B^{(b)}_{j}, U^{(b)}_{j}), (0, B^{(b)}_{j}', 0), (0, B^{(b)''}_{j}, U^{(b)''}_{j})\},$$

$$(H^{(n)}_{j}, B^{(n)}_{j}, U^{(n)}_{j}), (H^{(n)'}_{j}, B^{(n)'}_{j}, U^{(n)'}_{j}), (H^{(n)''}_{j}, B^{(n)''}_{j}, U^{(n)''}_{j})\}. \quad (1.97)$$

In any case, by Lemma 1.17, $I_n(\eta, 0, t; 0)$ converges when $\eta \to 0$. More precisely, Corollaries 1.15 and 1.16 imply: $I_n(\eta, 0, t; 0) - I_n(0, 0, t; 0) = O(\eta^{2n\alpha})$. 40
If now \( \sigma_f = 0 \), then \( \eta^H (b + \sigma \eta)^B (u + \sigma_f \eta)^U = 0 \) except if \( U = 0 \), with \( U = U_j^{(b)} \) or \( U_j^{(b)'} \) (by Lemma 1.17 these exponents are non-negative). The rest of the proof is identical.

Case (\( \emptyset \bullet \)).

Let

\[
I_t^{\emptyset \bullet} := \sum_{\sigma_2, \ldots, \sigma_{2n-1}} \int_0^t \cdots \int_0^t (i \sigma_1 (x_1 - x_2) + \eta)^{2\alpha-2} \cdots (i \sigma_{2n-1} (x_{2n-1} - x_{2n}) + \eta)^{2\alpha-2} dx_1 \cdots dx_{2n}
\]

(1.98)

with \( \sigma_1, \ldots, \sigma_{2n} \in \{\pm 1\} \). Then

\[
I_t^{\emptyset \bullet} = \int_0^t (K'_{[0,t]} (\eta) I_{n-1} (\eta, 0, t; \cdot))(u) \ du.
\]

(1.99)

The non-analytic part of \( (K'_{[0,t]} (\eta) I_{n-1} (\eta, 0, t; \cdot))(u) \) writes (up to a coefficient) \( (\pm u + 2n \eta) \) \( 4\alpha-1+4\alpha(n-1) \) which is uniformly integrable in \( \eta \); the integral writes \( C + O(\eta^{4\alpha}) + O(\eta) \) \( n \geq 1 \). As for the analytic part, one must integrate \( (b+i\sigma \eta)^B (u+i \sigma_f \eta)^U \) \( C \eta^B (u+i \sigma_f \eta)^U \) (with the \( b \)-exponents), resp. \( \eta^H (b+i\sigma \eta)^B (u+i \sigma_f \eta)^U \) \( C \eta^H + B (u+i \sigma_f \eta)^U \) (with the \( \eta \)-exponents). The integral over \( (i)B : 0 < |u+i \sigma_f \eta| < \eta/3 \), resp. \( (iii)B : 2 \eta < |u+i \sigma_f \eta| \) yields an expression bounded by \( O(\eta^{B(b)'} + U_j^{(b)}) \), resp. \( O(\eta^{B(b)''} + O(\eta^{B(b)'''} + U_j^{(b)''''} + 1) \). Similar statements hold for the \( \eta \)-exponents (one must essentially replace \( B \) with \( H + B \)). Now use the relations given in Corollaries 1.15 and 1.16 in the 'bad' case where exponents may be negative.

Case (\( \emptyset \emptyset \)).

Let

\[
I_t^{\emptyset \emptyset} := \sum_{\sigma_2, \sigma_{2n-2}} \int_0^t \cdots \int_0^t (i \sigma_1 (x_1 - x_2) + \eta)^{2\alpha-2} \cdots (i \sigma_{2n-1} (x_{2n-1} - x_{2n}) + \eta)^{2\alpha-2} dx_1 \cdots dx_{2n}
\]

\[
= C \int_0^t dx_{2n-1} \left( \int_0^t J_{n-2}(\eta, x_{2n-1}, t; x_1) \ dx_1 \right)
\]

\[
[(i \sigma_{2n-1} (x_{2n-1} - t) + \eta)^{2\alpha-1} - (i \sigma_{2n-1} (x_{2n-1} + \eta)^{2\alpha-1}]
\]

(1.100)

\( (\sigma_1, \ldots, \sigma_{2n-1} \in \{\pm 1\}) \), where \( (n \geq 2) \)

\[
J_{n-2}(\eta, x_{2n-1}, t; x_1) = \left( K'_{[0,t]} (\eta) I_{n-2}(\eta, x_{2n-1}, t; \cdot) \right)(x_1).
\]

(1.101)
Set \( b = x_{2n-1} \) for convenience. The non-analytic part of \( J_{n-2} \) writes (up to a coefficient) \( (\pm i(x_1 - b) + 2(n - 1)\eta)^{4\alpha - 1 + 4\alpha(n - 2)} \), hence its contribution to \( I_{\emptyset}^{\emptyset} \) is of the form
\[
C_1 \int_0^t db \ (\pm ib + 2(n - 1)\eta)^{4\alpha(n-1)}(\pm ib + \eta)^{2\alpha-1}
+ C_2 \int_0^t db \ (\pm i(t - b) + 2(n - 1)\eta)^{4\alpha - 1}(\pm ib + \eta)^{2\alpha - 1} \quad (1.102)
\]
plus two similar terms (that reduce to the previous ones by the symmetry \( b \leftrightarrow t - b \)). By splitting the integral into
\[
\int_\eta^0 db \ (\pm ib + \eta)^{2\alpha - 1} + \int_\frac{t}{2}^{\eta} db \ (\pm ib + \eta)^{2\alpha - 1} + \int_{t-\eta}^{\frac{t}{2}} db \ (\pm ib + \eta)^{2\alpha - 1} + \int_{t}^{t-\eta} db \ (\pm ib + \eta)^{2\alpha - 1},
\]
it is easy to prove that the limit when \( \eta \to 0 \) writes \( C_1 + O(\eta^2) \).

Turning to the analytic part, one must integrate in \( u := x_1 \) separately over each of the 7 domains \((i)_b, \ldots, (\tilde{i})_\eta\). The precise dependence in \( \eta \) may be computed by rewriting the proof of Theorem 1.1 with \( \alpha = 0 \), which makes things essentially trivial. Let us write in details for instance the case \( u \in (i)_b \). One has
\[
\int_{b - i\eta}^{b + i\eta} J_{n-1}(\eta, b, t; u) \ du = (b - i\eta)^{B_{j}^{(b)} + U_{j}^{(b)} + 1} \int_{\frac{\eta}{b - i\eta}}^{\frac{\eta + (2\sigma f - 1)}{b - i\eta}} w^{U_{j}^{(b)}} F(w, \frac{\eta}{b - i\eta}) \ dw
= (b - i\eta)^{B_{j}^{(b)} + U_{j}^{(b)} + 1} \left( G\left(\frac{\eta}{b - i\eta}\right) + \left(\frac{\eta}{b - i\eta}\right)^{U_{j}^{(b)} + 1} H\left(\frac{\eta}{b - i\eta}\right)\right). \quad (1.103)
\]

Now integrate in \( b \) the product of this function by \( (\pm i(t - b) + \eta)^{2\alpha - 1} - (\mp ib + \eta)^{2\alpha - 1} \) by the same method as for the non-analytic term.

The other domains are left to the reader. \( \square \)

We now turn to the contribution of the unique closed diagram \( \int_0^t duK'(\eta, b, u)I_{N-1}(\eta, b, t; u) \). The function \( I_{N-1} \) has been analyzed in Lemma 1.17. There only remains to integrate it against the kernel \( K^{\prime, \pm}(\eta) \).

**Lemma 1.18** The double integral
\[
F(\eta; t) := \int_0^t db \int_0^t du \ (\pm i(u - b) + \eta)^{2\alpha - 2} I_{N-1}(\eta, b, t; u) \quad (1.104)
\]
tends to a finite limit \( F(t) \) when \( \eta \to 0 \). More precisely, \( F(\eta; t) = F_0(t) + O(\eta^{2\alpha}) \) where \( F_0 \) is independent of \( \eta \).
Proof.
Integrating with respect to $u$ gives (applying Theorem 1.2 with $u = b$)

$$F(\eta; t) = \int_0^t db \ g_b(\eta, b, t) + \int_0^t db \ g_\eta(\eta, b, t)$$  \hspace{1cm} (1.105)

where

$$g_b(\eta, b, t) = \sum_{j=1}^{J'} (b - i\eta)^{B_j(b)'} \ G_j \left( \frac{\eta}{b - i\eta} \right),$$  \hspace{1cm} (1.106)

and

$$g_\eta(\eta, b, t) = \sum_{j=1}^{J'} \eta^{H_j(\eta)'} (b - i\eta)^{B_j(\eta)'} + U_j(\eta) \ H_j \left( \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t} \right)$$  \hspace{1cm} (1.107)

if $|b/\eta| < C$,

$$g_\eta(\eta, b, t) = \sum_{j=1}^{J''} \eta^{H_j(\eta)''} (b - i\eta)^{B_j(\eta)''} + U_j(\eta) \ F_j \left( \frac{\eta}{b - i\eta}, \frac{b - i\eta}{t} \right)$$  \hspace{1cm} (1.108)

if $|b/\eta| > C > 0$, $|1 - b/t| > C' > 0$, and similarly for the domains with a tilde. Adequate lower bounds for the exponents are given in Corollaries 1.15 and 1.16.

There remains to integrate over $b$. Consider for instance $g_b$. One has

$$\int_0^t db \ (b - i\eta)^B G \left( \frac{\eta}{b - i\eta} \right) = \int_0^{\eta} db \ (b - i\eta)^B G \left( \frac{\eta}{b - i\eta} \right) + \int_{\eta}^t db \ (b - i\eta)^B G \left( \frac{\eta}{b - i\eta} \right)$$

$$= C \eta^{B+1} (1 + O(\eta)) + C' (1 + O(\eta))$$  \hspace{1cm} (1.109)

(use for instance a series expansion for $G$ in the second integral). In the present case, $B = B_j(b)' = B_j(b) + U_j(b) \geq 2\alpha - 1$. The other cases are similar.

\square

We may now prove the main result of this section.
Theorem 1.4 The $2N$-th connected moment of the $\eta$-approximation of the Lévy area $\phi^{(c)}_{2N}(\eta; t)$ is given by the sum of two terms: the first one is regular in the limit $\eta \to 0$ and equal to $C_{\text{reg},N} t^{4N\alpha} + O(\eta^{2\alpha})$ for some constant $C_{\text{reg},N}$; the second one is equal to $C_{\text{irr},N} t^{4N\alpha-1}$ with

$$C_{\text{irr},N} = \left( \frac{\pi/2}{\cos \pi \alpha \Gamma(-2\alpha)} \right)^{2(N-1)} \sin \pi \alpha \frac{\Gamma(2\alpha + 1)}{\Gamma(2 - 2\alpha)} \Gamma(1 - 4\alpha N)(2N)^{4\alpha N-1}. \quad (1.110)$$

Proof.

The first term is obtained by summing the contribution of all admissible functions, see Theorem [1.3] and Lemma [1.18]. The second one is obtained from the single irregular term (see Lemma [1.17])

$$\frac{\alpha(1 - 2\alpha)}{2 \cos \pi \alpha} \int_0^t du \int_0^t db ( \pm i(u - b) + \eta )^{2\alpha-2} R_{N-1}(b, u). \quad (1.111)$$

Up to a coefficient (essentially $C_N$, see Lemma [1.17]), and forgetting the regular $I_-$ integral, this is

$$2 \text{Re} \int_0^t db \ I_+(2\alpha - 2, 2\alpha + 4\alpha(N - 1); 0, t)(b + i\eta, b - i(2N - 1)\eta), \quad (1.112)$$

whose irregular part is (see Lemma [1.12])

$$-2 \frac{\Gamma(2\alpha + 1 + 4\alpha(N - 1)) \Gamma(1 - 4\alpha N)}{\Gamma(2 - 2\alpha)} \cdot 2 \sin \pi [2\alpha + 4\alpha(N - 1)] \cdot (2N\eta)^{4\alpha N-1}, \quad (1.113)$$

hence the result.

Remark. Stirling’s formula implies: $|C_{\text{irr},N}| \leq C'^N$ for some constant $C'$.

Corollary 1.19 The $2N$-th moment of the $\eta$-approximation of the Lévy area $\phi_{2N}(\eta; t)$ writes

$$\mathbb{E}[(A_{0,t}(\eta))^{2N}] = (2N - 1)!! C_{\text{irr},1} t^N \eta^{(4\alpha - 1)N} (1 + O(\eta^{1 - 4\alpha})). \quad (1.114)$$
Proof.

Decompose \( \mathbb{E}[(A_{0,t}(\eta))^{2N}] \) into a sum of diagrams as in subsection 1.1. The \((2N-1)!!\) totally disconnected diagrams (with \(N\) trivial components) contribute the main term in the expansion, namely

\[
(2N-1)!! \left( C_{\text{irr},1} t^4 \eta^{4\alpha-1} + O(1) \right)^N = (2N-1)!! C_{\text{irr},1}^N t^N \eta^{(4\alpha-1)N} (1+O(\eta^{1-4\alpha})).
\]

(1.115)

Then 'almost' totally disconnected diagrams (i.e. with \(N-2\) trivial components and only one bipartite polynomial line with 4 legs) contribute

\[
c_N \left( C_{\text{irr},1} t^4 \eta^{4\alpha-1} + O(1) \right)^{N-2} \left( C_{\text{irr},2} t^8 \eta^{8\alpha-1} + O(1) \right) \sim_{\eta \to 0} c'_N t^{N-1} \eta^{(4\alpha-1)N+1}
\]

(1.116)

for some constants \(c_N, c'_N\). It is clear that the other diagrams are even less irregular. \(\square\)

Remark. The coefficient \((2N-1)!!\) in front of \((t^4 \eta^{4\alpha-1})^N\) is equal to \(\mathbb{E}[X^{2N}]\) if \(X\) is a standard Gaussian variable. This simple remark leads after some more computations to Theorem A, our main result (see Introduction) which states that the rescaled Lévy area converges to Brownian motion.

We shall also need the following refinement of Theorem 1.4 in section 2 to establish the existence of a uniform exponential moment for the rescaled Lévy area.

**Theorem 1.5** Let \(t \in [0,T]\), \(T\) fixed. Then there exists a constant \(C\) (depending only on \(\alpha\)) such that the regular part \(C_{\text{reg},N} t^{4N\alpha} + O(\eta^{2\alpha})\) of the \(2N\)-th connected moment \(\phi_{2N}^{(c)}(\eta; t)\) satisfies: \(|\phi_{2N}^{(c)}(\eta; t)| \leq C^N\) for all \(N \geq 1\).

The proof relies mainly on the following Lemma.

**Lemma 1.20** Assume \(t \in [0,T]\) for some fixed \(T > 0\), and \(\eta > 0\) small enough. Let also

\[
\Omega' := \{ u \in \mathbb{C} \mid |u/t| < 2 \} \setminus \left( \left\{ \frac{u}{\eta} \in -i\sigma + \mathbb{R}_- \right\} \cup \left\{ \frac{u-t}{t} \in -i\sigma + \mathbb{R}_+ \right\} \cup B(-i\sigma \eta, \frac{\eta}{3}) \cup B(t - i\sigma \eta, \frac{\eta}{3}) \right).
\]

(1.117)
Let \( f \in C([0, t]) \) be a continuous function such that \( f(z) = z^{\beta_1} \phi_1(z) \) if \( z \in [0, t/2] \), \( z^{\beta_2} \phi_2(z) \) if \( z \in [t/2, t] \), where \( \beta_1, \beta_2 \geq 0 \), and \( \phi_1, \phi_2 \) are analytic and bounded on \( \Omega' \). Finally, let \( \sigma_1, \sigma_2 \in \{ \pm 1 \} \). Then \( \left( K_{[0,t]}^{\sigma_1} \right)^* (\eta) K_{[0,t]}^{\sigma_2} (\eta) \) is analytic in \( \Omega' \) and furthermore

\[
\sup_{z \in \Omega'} \left| \left( K_{[0,t]}^{\sigma_1} \right)^* (\eta) K_{[0,t]}^{\sigma_2} (\eta) \right| f \leq C \sup_{z \in \Omega'} |f(z)|
\]  

(1.118)

where \( C \) depends only on \( \alpha \).

Note that \( \Omega' \) is essentially the domain \( \Omega_{res} \) defined in the first Remark after Definition 1.14.

**Proof.**

Consider first the easier case \( \sigma_1 = -\sigma_2 \): let for instance

\[
g(z) := \left( K_{[0,t]}^{\sigma_1} \right)^* (\eta) K_{[0,t]}^{\sigma_2} (\eta) f(z) = \int_0^t \left( -i(z - u) + \eta \right)^{2\alpha} (K_{[0,t]}^{\sigma_2} (\eta) f(u)) du
\]

\[
= C' \int_0^t \left( -i(z - u) + \eta \right)^{2\alpha} du \int_0^t \left( i(u - v) + \eta \right)^{2\alpha - 2} f(v) dv
\]

\[
= C' \int_0^t f(v) dv \cdot \int_0^t \left( i(u - z) + \eta \right)^{2\alpha} (i(u - v))^{2\alpha - 2} dw
\]

\[
= \int_0^t f(v) dv \cdot I_-(2\alpha - 2, 2\alpha; 0, t)(v + i\eta, z + i\eta) \quad (z \in \Pi^+) \quad (1.119)
\]

The \( I_- \)-function extends analytically to \( \Omega' \); it is bounded when its arguments \( v + i\eta, z + i\eta \) are bounded away from 0 and \( t \), and has an integrable singularity (with negative exponents \( 2\alpha - 1, 4\alpha - 1 \), see Lemma 1.10 and the Remark following it) near 0 and \( t \). Hence \( |g(z)| \leq C \sup_{z \in [0, t]} |f(z)| \).

Consider now, for \( z \in \Pi^+ \),

\[
g(z) := \left( K_{[0,t]}^{\sigma_1} \right)^* (\eta) K_{[0,t]}^{\sigma_2} (\eta) f(z) = \int_0^t \left( -i(z - w) + \eta \right)^{2\alpha} (K_{[0,t]}^{\sigma_2} (\eta) f(u)) du
\]

\[
= C' \int_{\Gamma_1} \left( -i(z - w) + \eta \right)^{2\alpha} dw \int_0^t \left( -i(w - v) + \eta \right)^{2\alpha - 2} f(v) dv
\]

\[
= C' \int_0^t f(v) dv \cdot \int_{\Gamma_1} \left( i(w - z) + \eta \right)^{2\alpha} (i(w - v) + \eta)^{2\alpha - 2} dw
\]

\[
= C' \int_0^t f(v) dv \cdot I_+(2\alpha - 2, 2\alpha; 0, t)(v - i\eta, z + i\eta)
\]

(1.120)

where \( \Gamma_1 : 0 \to t, \Gamma((0, t)) \subset \Pi^+ \) and \( \Gamma \) passes below \( z \).
The same conclusions hold as in the previous case except (see Lemma 1.12) for the supplementary non-analytic term of the form $C(i(v - z))^{4\alpha - 1}$. This term may be integrated against $f$ as in Lemma 3.2, yielding for $z \in \Pi^-$ an integral

$$\int_{\bar{\Gamma}_2} f(\bar{\omega}) (i(\bar{\omega} - z))^{4\alpha - 1} d\bar{\omega},$$

where $\bar{\Gamma}_2$ passes below $z$ while staying in $\Omega$ when it leaves the real axis. Now the singularity of the kernel $(i(\bar{\omega} - z))^{4\alpha - 1}$ is integrable, hence the result. □

We may now prove briefly Theorem 1.5. Consider, as in the proof of Lemma 1.17, a $(2m)$- or $(2m + 1)$-iterated non-analytic term, and then the analytic term obtained by integrating once more this non-analytic term. Integrate against $K^*$ once more if the last integration was against $K'$, so the result, $f$, satisfies the hypotheses of Lemma 1.20. Namely, it has positive exponents $U_j^{(b)}, U_j^{(\eta)}, U_j^{(\eta')}$ by Lemma 1.17, hence it is bounded on $\Omega'$ (see first remark after Definition 1.14), with an overall constant $|C_{\text{irr}, m}| \leq C^m$ by the remark following Theorem 1.4. One may now iterate Lemma 1.20 till the last two iterated integrals.

□

2 Convergence in law of the rescaled Lévy area

Using the analysis of singularities developed in Section 1, we mainly aim to prove in this section the following Theorem.

Definition 2.1 Let $\tilde{A}_{s,t}(\eta) := \eta^{\frac{1}{2} (1 - 4\alpha)} A_{s,t}(\eta)$ be the rescaled Lévy area.

Theorem A

The three-dimensional process $(B^{(1)}(\eta), B^{(2)}(\eta), \tilde{A}(\eta))$ converges in law in the Skohorod topology to $(B^{(1)}, B^{(2)}, \sqrt{C_{\text{irr}, 1}} dW)$ where $\delta W_{s,t} := W_t - W_s$ are the increments of a standard one-dimensional Brownian motion independent from $B^{(1)}$ and $B^{(2)}$.

Theorem A is a consequence of the following Theorem which generalizes the asymptotics obtained in section 1 for the moments of $A_{s,t}$ to the case of the moments of finite-dimensional distributions.

Theorem 2.1 Let $(W_t)_{t \in \mathbb{R}}$ be a (two-sided) standard one-dimensional Brownian motion. Then, for every $s_1 < t_1, \ldots, s_n < t_n$, and
\[ u(1,1), \ldots, u(1,k_1), u(2,1), \ldots, u(2,k_2) \in \mathbb{R}, \]
\[
\mathbb{E} \left[ \left( B_{u(1),1}^{(1)}(\eta) \ldots B_{u(1),k_1}^{(1)}(\eta) \right) \left( B_{u(2),1}^{(2)}(\eta) \ldots B_{u(2),k_2}^{(2)}(\eta) \right) \tilde{A}_{s_1,t_1}(\eta) \ldots \tilde{A}_{s_n,t_0}(\eta) \right] \to_{\eta \to 0} 0
\]
\[
C_{irr,1}^{n/2} \mathbb{E}[(W_{t_1} - W_{s_1}) \ldots (W_{t_n} - W_{s_n})]. \quad \mathbb{E}[B_{u(1),1}^{(1)} \ldots B_{u(2),k_1}^{(1)}] \mathbb{E}[B_{u(2),1}^{(2)} \ldots B_{u(2),k_2}^{(2)}].
\]

(2.1)

The fact that the moments of \( \tilde{A}(\eta) \) and \( B(\eta) \) 'factorize' in the limit \( \eta \to 0 \) is of course an indication of the asymptotic independence of \( \delta W \) and \( B \).

For the proof of Theorem 2.1 we shall need the following two Lemmas.

**Lemma 2.2** Let \( s_1 < t_1, s_2 < t_2 \), then
\[
\mathbb{E}[A_{s_1,t_1}(\eta)A_{s_2,t_2}(\eta)] = C_{irr,1} \lambda([s_1,t_1] \cap [s_2,t_2]) \eta^{4\alpha-1} + O(1) \quad (\alpha = \text{Lebesgue measure}) \text{ when } \eta \to 0.
\]

(2.2)

**Proof.**

Consider first the case when the intervals \([s_1,t_1]\) and \([s_2,t_2]\) are disjoint, say, \( s_1 < t_1 \leq s_2 < t_2 \). Then
\[
|\mathbb{E}[A_{s_1,t_1}(\eta)A_{s_2,t_2}(\eta)]|
\leq C \int \int \int_{s_1 \leq x_1, y_1 \leq t_1} dx_1 dy_1 \int_{s_2 \leq x_2, y_2 \leq t_2} dx_2 dy_2 (x_2 - x_1)^{2\alpha-2} (y_2 - y_1)^{2\alpha-2}
\]
\[
= C \left[ \int \int_{s_1 \leq x_1 \leq t_1, s_2 \leq x_2 \leq t_2} (x_2 - x_1)^{2\alpha-2} dx_1 dx_2 \right]^2
\]
\[
= C' \left[ (t_2 - t_1)^{2\alpha} + (s_2 - s_1)^{2\alpha} - (t_2 - s_1)^{2\alpha} - (s_2 - t_1)^{2\alpha} \right] < \infty.
\]

(2.3)

For the general case, set \([s_1,t_1] \cap [s_2,t_2] := [s, t] \) \( (s < t) \) so that \([s_i,t_i] = [s_i,s] \cup [s,t] \cup [t,t_i] \) \( (i = 1, 2) \) and \( \lambda([s_1,t_1] \cap [s_2,t_2]) = t - s \). Write (using the multiplicative property \((0.5)\))
\[
A_{s_1,t_1}(\eta) = A_{s_1,s}(\eta) + A_{s,t}(\eta) + A_{t,t}(\eta)
\]
\[
+ (B_{s_1}^{(1)}(\eta) - B_{s_1}^{(1)}(\eta))(B_{t_1}^{(2)}(\eta) - B_{t_1}^{(2)}(\eta)) + (B_{t_1}^{(1)}(\eta) - B_{t_1}^{(1)}(\eta))(B_{s_1}^{(2)}(\eta) - B_{s_1}^{(2)}(\eta)).
\]

(2.4)

Forgetting about the products of increments appearing on the last line, the only singular term comes from \( \mathbb{E}[(A_{s,t}(\eta))^2] = C_{irr,1}|t - s|^4 \eta^{4\alpha-1} \). Now
the covariance between the area terms and the products of increments is regular in the limit \( \eta \to 0 \), as follows from the general arguments in Lemma 2.3 below (which does not use the result of this Lemma).

**Lemma 2.3** Let \( s_1 < t_1, \ldots, s_{2N} < t_{2N}, \) and

\[
u(1), \ldots, u(1), k_1, u(2), 1, \ldots, u(2), k_2 \in \mathbb{R} \] as in Theorem 2.1. Consider the closed connected diagrams coming from the evaluation of

\[
\mathbb{E} \left[ \left( B_{u(1),1}^{(1)}(\eta) \cdots B_{u(1),k_1}^{(1)}(\eta) \right) \left( B_{u(2),1}^{(2)}(\eta) \cdots B_{u(2),k_2}^{(2)}(\eta) \right) \mathcal{A}_{s_1,t_1}(\eta) \cdots \mathcal{A}_{s_{2N},t_{2N}}(\eta) \right].
\]

(2.5)

Then their sum is regular in the limit \( \eta \to 0 \) unless \( k_1 = k_2 = 0 \). In the latter case, their sum writes \((2N-1)! C_{1,\alpha} \lambda^{N-1} \) plus a regular term in the limit \( \eta \to 0 \), where \( t = \lambda([s_1, t_1] \cap \ldots \cap [s_{2N}, t_{2N}]) \).

The same expression with an odd number of \( \mathcal{A} \)'s is always regular in the limit \( \eta \to 0 \).

**Proof.**

Set \( s_{2N+1}, \ldots, s_{2N+k_1+k_2} = 0 \) and

\[
(t_{2N+1}, \ldots, t_{2N+k_1}) = (u(1), 1, \ldots, u(1), k_1), \quad (t_{2N+k_1+1}, \ldots, t_{2N+k_1+k_2}) = (u(2), 1, \ldots, u(2), k_2).
\]

Decompose \([s_1, t_1] \cup \ldots \cup [s_{2N+k_1+k_2}, t_{2N+k_1+k_2}]\) into a finite union of intervals with disjoint interiors \( I_1, \ldots, I_m \), with \( m \) minimal, so that for each \( j \) and \( k \), one has either \( I_k \subset [s_j, t_j] \) or \( I_k \cap [s_j, t_j] = \emptyset \). Let \( B_I := B_t - B_s \) and \( \mathcal{A}_I := \mathcal{A}_{k,t} \) if \( I = [s, t] \). Then the multiplicativity property for the area, see equation (0.5), implies:

\[
\mathbb{E} \left[ \left( B_{u(1),1}^{(1)}(\eta) \cdots B_{u(1),k_1}^{(1)}(\eta) \right) \left( B_{u(2),1}^{(2)}(\eta) \cdots B_{u(2),k_2}^{(2)}(\eta) \right) \mathcal{A}_{s_1,t_1}(\eta) \cdots \mathcal{A}_{s_{2N},t_{2N}}(\eta) \right]
\]

\[
= \mathbb{E} \left[ \prod_{j=1}^{2N} \left( \mathcal{A}_{I_{j,1}}(\eta) + \mathcal{A}_{I_{j,2}}(\eta) + \ldots + P_j(s_1, \ldots, s_{2N+k_1+k_2}, t_1, \ldots, t_{2N+k_1+k_2}) \right) \right]
\]

\[
\prod_{k=1}^{k_1} \left( B_{I_{k,1}}^{(1)}(\eta) + B_{I_{k,1}}^{(1)}(\eta) + \ldots \right) \prod_{k=1}^{k_2} \left( B_{I_{k,2}}^{(2)}(\eta) + B_{I_{k,2}}^{(2)}(\eta) + \ldots \right)
\]

(2.6)

for some polynomials \( P_j \) in the variables

\( B_{s_1}(\eta), \ldots, B_{s_{2N+k_1+k_2}}(\eta), B_{t_1}(\eta), \ldots, B_{t_{2N+k_1+k_2}}(\eta) \). Expanding the product, one gets terms of the type \( \mathbb{E}[\mathcal{A}_{I_{1}}(\eta) \cdots \mathcal{A}_{I_{k}}(\eta)Q] \) for some polynomial \( Q \) as above.
Assume to begin with that $Q = 1$. Then the singularities of $\mathbb{E}[A_{I_{s_1}}(\eta) \ldots A_{I_{s_k}}(\eta)]$ may be investigated as in section 1, with the only difference that the non-analytic terms come from the iterated integrals
\[
K_{s_1}^\gamma(\eta)K_{s_2}^\gamma(\eta) \ldots K_{s_{2k+1}}^\gamma(\eta)(u \mapsto (-i(u - (b - i\eta)))^{2\alpha})(u)
\]  
(2.7)

or
\[
K_{s_1}^\gamma(\eta)K_{s_2}^\gamma(\eta) \ldots K_{s_{2k+1}}^\gamma(\eta)(u \mapsto (-i(u - (b - i\eta)))^{2\alpha})(u)
\]  
(2.8)

(and their conjugates) for all possible choices of subsets of intervals \(\{I_{s_1}, \ldots, I_{s_k}\} \subset \{I_1, \ldots, I_m\}\). Now Remark 1.13 after the proof of Lemma 1.12 proves that the corresponding non-analytic term is 0 except if \(I_{s_1} = \ldots = I_{s_k}\). This gives the singular term \((2N - 1)! C_{irr,N} t \eta^{4N\alpha - 1}\) by Theorem 1.4.

If now $Q$ is of degree $\geq 1$, then $Q$ may be written as
\[
\left( \int_{I_{s_1}} dB_{s_1}(\eta) \int_{I_{s_2}} dB_{s_2}(\eta) \right) \ldots \left( \int_{I_{s_k}} dB_{s_k}(\eta) \right), \quad k \geq 1
\]
for some intervals $I_{s_1}, I_{s_2}$ (indeed, the expectation is simply zero if there isn’t the same number of $B_{s_1}'s$ and $B_{s_2}'s$). Consider any connected diagram and evaluate it by the same method as usual (see subsection 1.1). The integration $\int_{I_{s_1}} dt_1$ yields a result which does not depend on $s_1$, hence the next integration $\int_{I_{s_2}} ds_1$ contains no non-analytic term. The results of subsection 1.4 show then that $\mathbb{E}[A_{I_{t_1}}(\eta) \ldots A_{I_{t_h}}(\eta)Q]$ is regular in the limit $\eta \to 0$.

The proof is the same for an odd number of $A$’s.

\[\square\]

**Proof of Theorem 2.1**

Assume $n = 2N$ (the proof is the same for $n$ odd). Decompose $[s_1, t_1] \cup \ldots \cup [s_{2N+k_1+k_2}, t_{2N+k_1+k_2}]$ into a finite union of disjoint intervals $I_1, \ldots, I_m$ as in the proof of Lemma 2.3. Then the multiplicativity property for the area implies once again:
\[
\mathbb{E} \left[ \left( B_{u_{(1),1}}^{(1)}(\eta) \ldots B_{u_{(1),k_1}}^{(1)}(\eta) \right) \left( B_{u_{(2),1}}^{(2)}(\eta) \ldots B_{u_{(2),k_2}}^{(2)}(\eta) \right) \tilde{A}_{s_1,t_1}(\eta) \ldots \tilde{A}_{s_{2N},t_{2N}}(\eta) \right]
\]
\[
= \mathbb{E} \left[ \prod_{j=1}^{2N} \left( \tilde{A}_{l_{j,1}}(\eta) + \tilde{A}_{l_{j,2}}(\eta) + \ldots + \eta^{2(1-4\alpha)}P_j \right) \right]
\]
\[
= \prod_{k=1}^{k_1} \left( B_{l_{k,1}}^{(1)}(\eta) + B_{l_{k,2}}^{(1)}(\eta) + \ldots \right) \prod_{k=1}^{k_2} \left( B_{l_{k,1}}^{(2)}(\eta) + B_{l_{k,2}}^{(2)}(\eta) + \ldots \right)
\]  
(2.9)

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Expanding the product, one gets terms of the type
\[ E[A_{i_1}(\eta) \ldots A_{i_k}(\eta)R] \] (2.10)
for some polynomial \( R \) as above. Use a diagrammatic expansion now. Lemma 2.3 shows that connected diagrams involving both Lévy areas and a non-trivial product of increments are regular, hence go to 0 in the limit \( \eta \to 0 \) because of the rescaling. This implies the 'factorization' of the moments of \( \bar{A}(\eta) \) and \( B(\eta) \) in the limit \( \eta \to 0 \). Now the same Lemma implies that (if \( R = 1 \)) only the totally disconnected diagrams survive in the limit \( \eta \to 0 \), still because of the rescaling (see proof of Corollary 1.19 for the power-counting). Then Lemma 2.2 allows one to evaluate such diagrams, and the result follows now from a simple combinatorial argument by 're-gluing' together the intervals \( I_1, \ldots, I_m \). \( \Box \)

We may now prove Theorem A.

**Proof of Theorem A.**

The moments of the finite-dimensional distributions of \((B^{(1)}(\eta), B^{(2)}(\eta), \bar{A}(\eta))\) converge to those of \((B^{(1)}, B^{(2)}, \delta W)\) as Theorem 2.1 shows. Furthermore,
\[ E[|B_t(\eta) - B_s(\eta)|^2] \leq C|t - s|^{2\alpha} \] (2.11)
for all \( \eta \) (which results from a simple computation using the explicit formula for the covariance, or from [22], Lemma 1.5) and (assuming for instance \( s < t \leq u < v \), but similar estimates hold in all cases), using once again the multiplicative property (0.5)
\[ E[(\bar{A}_{s,t}(\eta) - \bar{A}_{u,v}(\eta))^2] \]
\[ = E[(\bar{A}_{s,u}(\eta) - \bar{A}_{u,v}(\eta)) + \eta \frac{1}{2}(1-4\alpha)(\delta B_{s,u}^{(1)}\delta B_{u,t}^{(2)} - \delta B_{s,u}^{(1)}\delta B_{u,v}^{(2)})]^2] \]
\[ \leq C \left( |u - s| + |v - t| + \eta^{1-4\alpha}|u - s|^{2\alpha}|v - t|^{2\alpha} \right) \] (2.12)
(where \( \delta B_{x,y} := B_y - B_x \)), which implies that the sequence of processes is tight (by standard arguments for processes in a Gaussian chaos of finite order, see [3]). Since Gaussian laws are uniquely characterized by their moments, this implies Theorem A. \( \Box \)

**Remark.** By using the central limit theorem due to D.Nualart and G. Peccati [18], it would have been enough to compute the second and fourth moments of the joint process to prove the convergence in law. But we feel
that these particular cases are not much easier than the general case, and that the general and powerful asymptotic analysis given in Section 1 may be used for a wide range of applications.

Let us end this paragraph by giving an estimate of the characteristic function of the rescaled Lévy area $\tilde{A}_{s,t}(\eta)$ for $\alpha \in (\frac{1}{8}, \frac{1}{4})$, implying a uniform exponential bound.

**Lemma 2.4** Let $\alpha \in (\frac{1}{8}, \frac{1}{4})$, $0 < s, t < T$ (T fixed) and $\tilde{\phi}_{s,t}(\eta; \lambda) = \mathbb{E}[e^{i\lambda \tilde{A}_{s,t}(\eta)}]$ be the characteristic function of the Lévy area. Then there exist two constants $\lambda_0, C_0 > 0$ such that, for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| \leq \lambda_0 \eta^{-\frac{1}{8}(1-4\alpha)}$, and for all $\eta$ small enough,

$$|\tilde{\phi}_{s,t}(\eta; \lambda) - e^{-\frac{1}{2}C_{irr,1}|t-s|\lambda^2}| \leq C_0 \lambda^2 \eta^{1-4\alpha} e^{-\frac{1}{2}C_{irr,1}|t-s|\lambda^2}. \quad (2.13)$$

**Proof.**

Theorems 1.4 and 1.5 yield

$$\tilde{\phi}_{s,t}^{(c)}(\eta; \lambda) = \phi_{s,t}^{(c)}(\eta; \eta^{\frac{1}{2}(1-4\alpha)} \lambda)$$

$$= -\frac{1}{2} \lambda^2 (C_{irr,1}|t-s| + C_{reg,1}(\eta|t-s|)\eta^{1-4\alpha}) + \ldots +$$

$$+ \frac{\lambda^{2N}}{2N}(-1)^N \left( C_{irr,N}\eta^{N-1}|t-s| + C_{reg,N}(\eta|t-s|)\eta^{(1-4\alpha)N} \right) + \ldots \quad (2.14)$$

where $|C_{reg,N}(\eta|t-s|)| = |C_{reg,N}|t-s|^{4N\alpha} + O(\eta^{2\alpha})| \leq C_N$ is the regular part of the $2N$-th connected moment. Recall also $C_{irr,N} \leq C'N$ for some constant $C'$. Note that the condition $\alpha > \frac{1}{8}$ implies: $\eta^{N-1} < \eta^{(1-4\alpha)N}$ for all $N \geq 2$.

Let $\lambda_0 = \frac{1}{2\sqrt{\max(C,C')}}$, then the series converges for $|\lambda| \leq \lambda_0 \eta^{-\frac{1}{8}(1-4\alpha)}$ and $|\tilde{\phi}_{s,t}^{(c)}(\eta; \lambda) + \frac{1}{2}C_{irr,1}|t-s|\lambda^2|$ is bounded by a constant times $\lambda^2 \eta^{1-4\alpha}$ when $\eta \to 0$. Now $\tilde{\phi}_{s,t}(\eta; \lambda) = \exp \tilde{\phi}_{s,t}^{(c)}(\eta; \lambda)$ (see Lemma 1.7) which yields the result. \qed

**Corollary 2.5 (uniform exponential moment)** Fix $T > 0$ and $\alpha \in (\frac{1}{8}, \frac{1}{4})$.

Then there exist constants $\lambda_0, C_0$ such that, for every $0 < s, t < T$ and $0 \leq \lambda \leq \lambda_0 \eta^{-\frac{1}{8}(1-4\alpha)}$,

$$\mathbb{E} \left[ \exp \lambda \tilde{A}_{s,t}(\eta) \right] \leq C_0 e^{\frac{1}{2}C_{irr,1}|t-s|\lambda^2}. \quad (2.15)$$

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Proof. Straightforward.

For instance, this implies in particular by Markov’s inequality:

\[ P[A_{s,t}(\eta) \geq A] \leq C_0 e^{-\frac{A^2}{C_{irr,1}|t-s|}} \]  

for every \( A \leq \lambda_0 C_{irr,1}|t-s|\eta^{-\frac{1}{2}(1-4\alpha)} \).

3 Appendix

Assume \( \eta = 0 \) to begin with. Recall from Definition 1.9 that \( K'_{[a,b]} \) and \( K^*_{[a,b]} \) are integral operators from \( L^1((a,b)) \) to \( \text{Hol}(\Pi^\mp) \) (the space of holomorphic functions on one of the half-planes) defined by

\[
(K'_{[a,b]} f)(z) = \frac{(1-2\alpha)}{2 \cos \pi \alpha} \int_a^b f(u)(\pm i(z-u))^{2\alpha-2} du, \quad z \in \Pi^\mp
\]

and

\[
(K^*_{[a,b]} f)(z) = -\frac{1}{4 \cos \pi \alpha} \int_a^b f(u)(\pm i(z-u))^{2\alpha} du, \quad z \in \Pi^\mp.
\]

The function \( u \mapsto (\pm i(z-u))^{2\alpha-2} \), \( u \in [a,b] \) depends analytically on \( z \) if \( z \) belongs to the domain \( \mathbb{C} \setminus \{s \pm iy \mid s \in [a,b], y \geq 0\} \). Outside this domain, the functions \( (\pm i(z-u))^{2\alpha-2} \) or \( (\pm i(z-u))^{2\alpha} \) are multivalued and admit singularities.

One of the first (and easiest) results established in this Appendix (see Lemma 3.2) is that \( K'_{[a,b]} f \) and \( K^*_{[a,b]} f \) may be extended analytically to a neighbourhood of any point \( u \in (a,b) \) in a neighbourhood of which \( f \) is analytic. Hence (supposing \( f \) is analytic on a neighbourhood of \( (a,b) \)) problems of multivaluedness and singularities are concentrated at the ends \( a,b \) of the interval of integration.

Lemmas 3.2, 3.3 give the local behaviour of \( K'_{[a,b]} f \) and \( K^*_{[a,b]} f \) around \( a \) and \( b \) under some hypotheses on \( f \). Then Lemmas 3.4, 3.5, 3.6 generalize the previous results to the case where \( a \) or \( b \) is a varying parameter going to 0, with \( f \) possibly possessing a non-integrable singularity at 0.

The generalization to \( \eta > 0 \) is straightforward in principle, but leads (see subsection 1.3) to some complications in practice (to be specific, they necessitate the introduction of the \( \eta \)-exponents, see Definition 1.14).
The proofs of the Lemmas in this Appendix depend crucially on the properties of Gauss’ hypergeometric function $\,_{2}F_{1}$ recalled in the Introduction.

Let us start with a technical Lemma.

**Lemma 3.1** Let, for $2\alpha \in \mathbb{R} \setminus \mathbb{Z}$, $\beta > -1$ and $n = 0, 1, \ldots$,

$$F_n(\alpha, \beta; t; z) := \int_{0}^{t} \frac{(t-u)^n}{n!} u^{\beta} (-i(z-u))^{2\alpha-2} \, du, \quad z \in \Pi^+ \quad (3.3)$$

be the $n$-times iterated integral of the function $u \to u^{\beta} (-i(z-u))^{2\alpha-2}$.

Then $F_n(\alpha, \beta; t; z)$ has an analytic extension to $\mathbb{C} \setminus (\mathbb{R} \cup \{ t - iy \mid y \geq 0 \})$ given by

$$F_n(\alpha, \beta; t; z) = -i^{\beta+n+1} \left[ \frac{\Gamma(1+\beta)\Gamma(n-1+2\alpha)}{\Gamma(2\alpha+n+\beta)} e^{-i\pi \alpha} z^{2\alpha-1} \, \, _{2}F_{1}(2-2\alpha, 1+\beta; -n+2-2\alpha; 1-t/z) \right.ight.$$

$$\left. + \frac{\Gamma(-n+1-2\alpha)}{\Gamma(2-2\alpha)} (-i(z-t))^{2\alpha-1} (1-t/z)^n \, \, _{2}F_{1}(n+2\alpha+\beta, n+1; n+2\alpha; 1-t/z) \right] . \quad (3.4)$$

with restriction to $B(0, t) \setminus (-t, 0) = \{ z \in \mathbb{C} \mid |z| < t \} \setminus (-t, 0)$ given by

$$F_n(\alpha, \beta; t; z) = -i^{\beta+n+1} \left[ e^{i\pi \alpha} \frac{\Gamma(2\alpha+\beta+1)}{\Gamma(2\alpha+\beta+n)} t^{2\alpha-2} \, \, _{2}F_{1}(2-2\alpha, 1-2\alpha-\beta-n; 2-2\alpha-\beta; z/t) \right.$$

$$\left. + \frac{\Gamma(1+\beta)\Gamma(1-2\alpha-\beta)}{\Gamma(2-2\alpha)n!} e^{-i\pi (\alpha+\beta+1)} t^{-1-\beta} z^{2\alpha+\beta-1} \, \, _{2}F_{1}(1+\beta, -n; 2\alpha+\beta; z/t) \right] . \quad (3.5)$$

**Proof.**

Suppose $v \in (0, 1)$. If $0 < \text{Arg}(z) < \pi/2$ then $-\pi/2 < \text{Arg}(-iz) < 0$ and $0 < \text{Arg}(1 - \frac{t}{z}v) < \pi$.

If $\pi/2 < \text{Arg}(z) < \pi$ then $0 < \text{Arg}(-iz) < \pi/2$ and $0 < \text{Arg}(1 - \frac{t}{z}v) < \pi/2$.

In both cases (hence for any $z \in \Pi^+$) one has $|\text{Arg}(-iz) + \text{Arg}(1 - \frac{t}{z}v)| < \pi$. Hence
\[ F_n(\alpha, \beta; t; z) = \frac{t^{\beta+n+1}}{n!} \int_0^1 v^\beta (1 - v)^n (-iz)^{2\alpha-2} (1 - \frac{t}{z}v)^{2\alpha-2} \, dv \]
\[ = t^{\beta+n+1} (-iz)^{2\alpha-2} \frac{\Gamma(1 + \beta)}{\Gamma(n + 2 + \beta)} \, _2F_1(2 - 2\alpha, 1 + \beta; n + 2 + \beta; \frac{t}{z}). \]  
(3.6)

By the connection formula (0.21),
\[ _2F_1(2 - 2\alpha, 1 + \beta; n + 2 + \beta; \frac{t}{z}) = \frac{\Gamma(n + 2 + \beta) \Gamma(-n + 1 - 2\alpha)}{\Gamma(1 + \beta) \Gamma(2 - 2\alpha)} \, _2F_1(n + 2\alpha + \beta, n + 1; n + 2\alpha; 1 - \frac{t}{z}). \]
(3.7)

If \( 0 < \text{Arg}(z) < \frac{\pi}{2} \) then \(-\frac{\pi}{2} < \text{Arg}(-i(z - t)) < \frac{\pi}{2} \) and \( 0 < \text{Arg}(\frac{z}{t}) < \frac{\pi}{2} \).

If \( \frac{\pi}{2} < \text{Arg}z < \pi \) then \( 0 < \text{Arg}(-i(z - t)) < \frac{\pi}{2} \) and \(-\frac{\pi}{2} < \text{Arg}(\frac{z}{t}) < 0 \).

In both cases \( |\text{Arg}(-i(z - t)) + \text{Arg}(\frac{z}{t})| < \pi \). Hence, if \( z \in \Pi^+ \),
\[ (1 - \frac{t}{z})^{2\alpha-1} = (\frac{i}{z})^{2\alpha-1} (-i(z - t))^{2\alpha-1}. \]

Whence the first result.

Alternatively, if \( z \in B(0, t) \setminus (-t, 0] \), then the functions \(-e^{-i\alpha z} z^{2\alpha-2} \), resp. \( e^{-i\gamma z} (z/t)^\gamma \), extend the function \((-iz)^{2\alpha-2} \), resp. \((-z/t)^\gamma \) defined on \( \Pi^+ \), whence the second result by applying the connection formula (0.19) to (3.6).

We now look at the case where \([a, b] = [0, t]\) is a fixed interval and present two Lemmas. It is important to understand that we omit the dependence in \( t \) of the results since \( t \) is assumed to be a non-zero constant (one might just as well have assumed that \( t = 1 \), but this would make the use of the lemmas somewhat awkward). This remark is valid for the whole Appendix (and for the whole article).

**Lemma 3.2** Let \( f \in L^1([0, t], \mathbb{C}) \) and \( \phi : z \mapsto \int_0^t (-i(z - u))^{2\alpha-2} f(u) \, du \) with \( 2\alpha \in \mathbb{R} \setminus \mathbb{Z} \). (The result applies in particular to \( K^{\alpha, \pm}_{[0, t]}(\eta) f \) and \( K^{\alpha, \pm}_{[0, t]}(\eta) f \)).
1. Assume \( f \) is analytic in a (complex) neighbourhood \( \Omega \) of \( s \in (0, t) \). Then \( \phi \) has an analytic extension to a complex neighbourhood of \( s \).

2. Assume \( f \) is analytic in a complex neighbourhood \( \Omega \) of \( 0 \). Then \( \phi \) may be written on a small enough neighbourhood of \( 0 \) as

\[
\phi(z) = (-iz)^{2\alpha - 1}F(z) + G(z)
\]

where both \( F \) and \( G \) are analytic. The function \( F \) has the following expression near \( 0 \):

\[
F(z) = \frac{i}{2\alpha - 1} \sum_{n \geq 0} a_n \frac{n!}{(2\alpha)_n} z^n.
\]

where \((2\alpha)_n = \frac{\Gamma(2\alpha+n)}{\Gamma(2\alpha)}\) is the Pochhammer symbol.

3. (no analyticity assumption is required here) Assume \( |z/t| > C > 1 \). Then \( \phi(z) = z^{2\alpha - 2}F(t/z) \) for some analytic function \( F \).

Proof.

1. Assume \( \Omega \supset B(s, 2r) \) for some \( r > 0 \). Then the contour of integration may be deformed into \( \Gamma = [0, s-r] \cup \{s+re^{i\phi} : \phi : \pi \to 0 \} \cup [s+r, t] \). If \( z \in B(s, r/2) \) then \(-i(z - \bar{w}) \not\in (-\infty, 0] \) for any \( \bar{w} \in \Gamma \), so \( \phi(z) = \int_{\Gamma} f(\bar{w})(-i(z - \bar{w}))^{2\alpha - 2} d\bar{w} \) is well-defined and analytic on \( B(s, r/2) \).

2. Suppose \( f(\bar{w}) \) is given by the convergent series \( \sum_{n \geq 0} a_n \bar{w}^n \) on \( B(0, 4r) \). A first integration by parts

\[
\int_0^r f(u)(-i(z - u))^{2\alpha - 2} du
\]

\[
= -\frac{i}{2\alpha - 1} [f(r)(-i(z - r))^{2\alpha - 1} - f(0)(-i z)^{2\alpha - 1}]
\]

\[
+ \frac{i}{2\alpha - 1} \int_0^r f'(u)(-i(z - u))^{2\alpha - 1} du
\]

(3.10)

yields a function \( \psi_r(z) = \int_0^r f'(u)(-i(z - u))^{2\alpha - 1} du \) which has a continuous extension to the real axis. Then successive integrations by parts yield
\[
\int_0^r u^n (-i(z-u))^{2\alpha-1} \, du \\
= -\frac{i}{2\alpha} r^n (-i(z-r))^{2\alpha} + \frac{i}{2\alpha} \int_0^r nu^{n-1} (-i(z-u))^{2\alpha} \, du \\
= \ldots = -\sum_{m=1}^{n} \frac{i^m}{(2\alpha)_m} n(n-1) \ldots (n-m+1)r^{n-m} (-i(z-r))^{2\alpha-1+m} \\
+ \frac{i^n}{(2\alpha)_n} \int_0^r (-i(z-u))^{2\alpha-1+n} \, du \\
= -\sum_{m=0}^{n} \frac{i^{m+1}}{(2\alpha)_{m+1}} n(n-1) \ldots (n-m+1)r^{n-m} (-i(z-r))^{2\alpha+m} \\
+ \frac{i^{n+1}}{(2\alpha)_{n+1}} n! (-iz)^{2\alpha+n} \\
(3.11)
\]

so

\[
\psi_r(z) = (-iz)^{2\alpha} \left( \sum_{n \geq 0} (n+1)a_{n+1} \cdot \frac{n!}{(2\alpha)_n} (-iz)^n \right) \\
- (-i(z-r))^{2\alpha} \sum_{n \geq 0} (n+1)a_{n+1} \cdot \sum_{m=0}^{n} \frac{i^{m+1}}{(2\alpha)_{m+1}} n(n-1) \ldots (n-m+1)r^{n-m} (-i(z-r))^{m} \\
(3.12)
\]

The first series is easily seen to be convergent for \(|z| < 4r\) since \(\frac{\Gamma(n+2\alpha)}{\Gamma(2\alpha)} \sim n^{2\alpha} \). The multivalued function \(z \mapsto (-i(z-r))^{2\alpha}\) is well-defined on \(B(0,r)\). The double series converges in the supremum norm \(||\cdot||_{\infty}\) on \(B(0,r)\) since \(|z-r| < 2r\) (hence \(||r^{n-m}(-i(z-r))^{m}||_{\infty} \leq (2r)^n\)) and

\[
\sum_{m=0}^{n} \frac{n(n-1) \ldots (n-m+1)}{(2\alpha)_{m+1}} \leq \Gamma(2\alpha) \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) = \Gamma(2\alpha) \cdot 2^n.
\]

Finally, \(\int_r^t f(u)(-i(z-u))^{2\alpha-2} \, du\) is analytic on \(B(0,r)\).
3. Expand \((-i(z-u))^{2\alpha-2}\) \((u \leq t, |z/t| > C > 1)\) into 
\(-e^{-i\pi \alpha} z^{2\alpha-2} \sum_{k \geq 0} \frac{(2-2\alpha)^k}{k!} \left(\frac{u}{z}\right)^k.\)

This Lemma has the following generalization:

**Lemma 3.3** Let \(f \in L^1([0,t], \mathbb{C})\) be analytic in a neighbourhood \(\Omega\) of 0, \(2\alpha \in (0,4) \setminus \{1, 2, 3\}\), \(\beta > -1\) and

\[
\phi : z \mapsto \int_0^t u^\beta f(u)(-i(z-u))^{2\alpha-2} \, du \quad (z \in \Pi^+). 
\]

(The result applies in particular to \(K'_{[0,t]}(\eta)(u \mapsto u^\beta f(u))\) and \(K^*_{[0,t]}(\eta)(u \mapsto u^\beta f(u))).\)

Then \(\phi\) may be written on a small neighbourhood of 0 as \(z^{2\alpha+\beta-1}F(z) + G(z),\) where both \(F\) and \(G\) are analytic.

**Proof.**
Suppose \(f(\bar{w})\) is given by the convergent series \(\sum_{n \geq 0} a_n \bar{w}^n\) on \(B(s,4r)\).
Here again, \(\int_0^t u^\beta f(u)(-i(z-u))^{2\alpha-2} \, du\) is readily shown to be analytic on \(B(0,r)\). A first integration by parts

\[
\int_0^r f(u)u^\beta(-i(z-u))^{2\alpha-2} \, du = f(r)F_0(\alpha, \beta; r; z) - \int_0^r f'(u)F_0(\alpha, \beta; u; z) \, du
\]

(see Lemma 3.1 for the definition of the functions \(F_n\)) yields a function \(\psi_r(z) = \int_0^r f'(u)F_0(\alpha, \beta; u; z) \, du\) which has a continuous extension to \(\mathbb{R}^*_+.\)

Then successive integrations by parts yield

\[
\int_0^r u^n F_0(\alpha, \beta; u; z) \, du = r^n F_1(\alpha, \beta; r; z) - \int_0^r nu^{n-1} F_1(\alpha, \beta; u; z) \, du
\]

\[
= \ldots = \sum_{m=0}^{n-1} (-1)^m r^{n-m} n(n-1) \ldots (n-m+1) F_{1+m}(\alpha, \beta; r; z)
\]

\[
+ (-1)^n n! \int_0^r F_n(\alpha, \beta; u; z) \, du
\]

\[
= \sum_{m=0}^n (-1)^m r^{n-m} n(n-1) \ldots (n-m+1) F_{m+1}(\alpha, \beta; r; z)
\]
\( \psi_r(z) = \sum_{n \geq 0} (n+1)a_{n+1} \sum_{m=0}^{n} (-1)^m r^{n-m} n(n-1) \ldots (n-m+1) F_{m+1}(\alpha, \beta; r; z) \)

is the sum of two terms (see last formula in Lemma 3.1) for \(|z| < r, z \not\in \mathbb{R}_-\).

- The first one is

\[
\psi_r^{(1)}(z) := -e^{i\pi \alpha} \Gamma(2\alpha + \beta + 1) r^{2\alpha+\beta} \sum_{n \geq 0} (n+1)a_{n+1} r^n \sum_{m=0}^{n} \frac{(-1)^m}{\Gamma(2\alpha + \beta + m + 1)} n(n-1) \ldots (n-m+1) \frac{1}{2} F_{m+1}(2-2\alpha, -2\alpha - \beta - m; 2-2\alpha - \beta; z/r) \]

Let us assume \(2\alpha \in (0, 2)\) (otherwise apply once or twice the formula \(2 F_1(a, b, c; z) = \frac{ab}{c} \int_0^z 2 F_1(a+1, b+1, c+1; u) \, du\) [1] to reduce to the following computations).

Suppose \(|w| < 1\) and \(\beta < 0\), then

\[
2 F_1(2-2\alpha, -2\alpha - \beta - m; 2-2\alpha - \beta; w) = \frac{\Gamma(2-2\alpha - \beta)}{\Gamma(2-2\alpha) \Gamma(-\beta)} \int_0^1 t^{1-2\alpha} (1-t)^{-1-\beta} (1-tw)^{2\alpha+\beta+m} \, dt.
\]

(3.16)

If \(2\alpha + \beta + m \geq 0\) (in particular as soon as \(m \geq 1\)) then \(|(1-tw)^{2\alpha+\beta+m}| \leq 2^{2\alpha+\beta+m}\), so

\[
|2 F_1(2-2\alpha, -2\alpha - \beta - m; 2-2\alpha - \beta; w)| \leq C 2^m
\]

for all \(|w| < 1\). On the other hand, if \(2\alpha + \beta < 0\), then \(|(1-tw)^{2\alpha+\beta}| \leq (1-t)^{2\alpha+\beta}\), so \(|2 F_1(2-2\alpha, -2\alpha - \beta - m; 2-2\alpha - \beta; w)| \leq C\).

Now, if \(\beta \geq 0\), the preceding arguments must be slightly adapted. Let \(p := \lceil \beta \rceil + 1\), then (by [1], formula (15.2.4))

\[
2 F_1(2-2\alpha, -2\alpha - \beta - m; 2-2\alpha - \beta; w) = \frac{w^{\alpha+\beta-1}}{(2-2\alpha - \beta)_p} \left( \frac{d}{dw} \right)^p \left[ w^{2-2\alpha-\beta+\lceil \beta \rceil} \cdot 2 F_1(2-2\alpha, -2\alpha - \beta - m; 2-2\alpha - \beta + \lceil \beta \rceil + 1; w) \right]
\]

(3.17)
with
\[ _2F_1(2-2\alpha, -2\alpha - \beta - m; 2 - 2\alpha - \beta + [\beta] + 1; w) = \int_0^1 t^{1-2\alpha} (1-t)^{[\beta]-\beta} (1-tw)^{2\alpha + \beta + m} dt. \]

The same kind of estimates also apply to the latter hypergeometric function, together with its derivatives up to order \( p \).

All together one has proved in all cases:
\[ | _2F_1(2-2\alpha, -2\alpha - \beta - m; 2 - 2\alpha - \beta; w) | \leq C.2^m \tag{3.18} \]
for all \(| w | < 1 \).

Now
\[ \sum_{m=0}^n \frac{n(n-1)\ldots(n-m+1)}{\Gamma(2\alpha + \beta + m + 1)} 2^m \leq \sum_{m=0}^n \binom{n}{m} C_1 2^m \leq C_2 n C_3 n \tag{3.19} \]
for some constants \( C_1, C_2, C_3 \). Hence the series giving \( \psi_r^{(1)} \) converges in the supremum norm \( \| \cdot \|_\infty \) on \(| z | < r \) to an analytic function \( F(z) \);

- The second one is
\[ \psi_r^{(2)}(z) := -\frac{\Gamma(1+\beta)\Gamma(1-2\alpha-\beta)}{\Gamma(2-2\alpha)} e^{-i\pi(\alpha+\beta+1)} \left[ \sum_{n\geq0} (n+1)a_{n+1}r^{n+1} \right] \sum_{m=0}^n \frac{(-1)^m n(n-1)\ldots(n-m+1)}{(m+1)!} _2F_1(1+\beta, -m-1; 2\alpha + \beta; z/r) z^{2\alpha + \beta - 1} \tag{3.20} \]
which (using the same type of estimates) converges in the supremum norm to a multivalued function \( z^{2\alpha + \beta - 1}G(z) \) (\( G \) analytic) on \(| z | < r \).

We now generalize the previous results to the case when \([a,b] = [0, \varepsilon] \) or \([\varepsilon, t] \) where \( \varepsilon \) and \( t \) are assumed to be complex. (By definition, \([z,w] := \{(1-s)z+sw \mid s \in [0,1]\} \) if \( z, w \in \mathbb{C} \). We assume \( t \) is bounded and bounded away from 0, i.e. \( 0 < c < |t| < C \), and \(|\varepsilon| < |t| \) may be arbitrarily close to 0 (actually, the following Lemmas are meaningful only when \(|\varepsilon| \) is small, otherwise they are redundant with Lemmas 3.2. and 3.3).

Recall \( \Omega \subset \mathbb{C} \) is star-shaped with respect to 0 if \( (z \in \Omega \Rightarrow \lambda z \in \Omega \forall \lambda \in [0,1]) \).
Lemma 3.4 Let $f$ be analytic on a fixed complex star-shaped neighbourhood $\Omega_f$ of $0$, $\beta > -1$, $2\alpha \in (0, 4) \setminus \{1, 2, 3\}$, and, for $\varepsilon \in \Omega_f \setminus \mathbb{R}_{-}$ such that $0 < |\varepsilon| < C$,

$$g(\varepsilon; z) := \int_0^\varepsilon (-i(z - u))^{2\alpha - 2} u^\beta f(u) \, du.$$  \hfill (3.21)

The function $z \mapsto g(\varepsilon; z)$, initially defined as an analytic function on $\text{Im} \, z > \max(0, \text{Im} \, \varepsilon)$, may be extended into an analytic function on the cut domain

$$\Omega := \Omega_f \setminus (\varepsilon(1 - i\mathbb{R}_{+}) \cup \mathbb{R}_{-}).$$

The behaviour of the function $g$ on $\Omega$ is given as follows.

(i) Suppose $z \in \Omega$, $|z/\varepsilon| > C > 1$: then

$$g(\varepsilon, z) = \varepsilon^{\beta + 1} z^{2\alpha - 2} F(\varepsilon, \varepsilon/z)$$  \hfill (3.22)

where $F(\varepsilon, \zeta)$ is holomorphic on the domain \{$(\varepsilon, \zeta) \in \mathbb{C}^2 \mid \varepsilon \in \Omega_f, |1/\zeta| > C$\};

(ii) Suppose $z \in \Omega$, $|1 - z/\varepsilon| < c < 1$: then

$$g(\varepsilon, z) = \varepsilon^{2\alpha - 1 + \beta} \left[ F(\varepsilon, 1 - \frac{z}{\varepsilon}) + (-i(\frac{z}{\varepsilon} - 1))^{2\alpha - 1} G(\varepsilon, 1 - \frac{z}{\varepsilon}) \right]$$  \hfill (3.23)

where $F(\varepsilon, \zeta)$ is holomorphic on the domain \{$(\varepsilon, \zeta) \in \mathbb{C}^2 \mid \varepsilon, \varepsilon(1 - \zeta) \in \Omega_f, |\zeta| < c$\};

(iii) Suppose $z \in \Omega$, $|z/\varepsilon| < c < 1$, then

$$g(\varepsilon, z) = z^{2\alpha - 1 + \beta} F(\varepsilon, \frac{z}{\varepsilon}) + \varepsilon^{2\alpha - 1 + \beta} G(\varepsilon, \frac{z}{\varepsilon})$$  \hfill (3.24)

where $F(\varepsilon, \zeta), G(\varepsilon, \zeta)$ are holomorphic on the domain \{$(\varepsilon, \zeta) \in \mathbb{C}^2 \mid \varepsilon, \varepsilon\zeta \in \Omega_f, |\zeta| < c$\};

(iv) Suppose $z \in \Omega$ is in the cut $\varepsilon$-ring $\Omega_{\varepsilon,c,c'} := \{0 < c' < |\frac{z}{\varepsilon}| < C', \mid 1 - \frac{z}{\varepsilon} \mid > c > 0\}$ \{$0 < c' < 1 < C'$\}, then

$$g(\varepsilon, z) = \varepsilon^{2\alpha - 1 + \beta} F(\varepsilon, z/\varepsilon)$$  \hfill (3.25)

where $F(\varepsilon, \zeta)$ is holomorphic on the domain \{$(\varepsilon, \zeta) \in \mathbb{C}^2 \mid \varepsilon, \varepsilon\zeta \in \Omega_f, \varepsilon' < |\zeta| < C', |1 - \zeta| > c$\};
Remark.
We shall need in the sequel to define the above function \( g \) on the complex plane cut along non-intersecting half-lines in a general position. The exponents remain of course the same, but the determination of the power functions should be chosen in a different way.

Proof.
Rewrite \( g(\varepsilon; z) \) in the following form:
\[
g(\varepsilon; z) = \varepsilon^{2\alpha-1+\beta} \int_0^1 (-i(\varepsilon^{-1} - v))^{2\alpha-2} v^\beta f(\varepsilon v) \, dv,
\]
(3.26)
a priori valid for \(-iz \in [A; +\infty), A \) large enough, which defines it as an analytic function on \( \{ z \in \mathbb{C} \mid \frac{z}{\varepsilon} \notin [0, 1] - i\mathbb{R}_+ \} \).

The previous results show that \( g \) may be extended analytically to a cut domain \( \Omega \subset \Omega_f \) excluding two non-intersecting half-lines ending at \( 0 \) and \( \varepsilon \), for instance, \( \Omega = \Omega_f \setminus (\varepsilon(1 - i\mathbb{R}_+) \cup \mathbb{R}_-). \)

Now apply Lemmas 3.2, 3.3 (more precisely, the obvious extension of these Lemmas to the case of a function \( f(\varepsilon; v) := f(\varepsilon v) \) depending analytically on a parameter \( \varepsilon \)).

Lemma 3.5 (integration against the infinitesimal kernel on a general interval)
Let, for \( \alpha \in (0, \frac{1}{2}), -1 < \beta < 0 \) and \( p = 0, 1, \ldots \), and for some fixed function \( f \) analytic on a neighbourhood \( \Omega_f \) of the triangle \( T \) with vertices \( \{ 0, \varepsilon, t \} \),
\[
h(\varepsilon, t; z) = \int_\varepsilon^t (i(z - u))^{2\alpha-2} u^\beta f(u) \, du
\]
(3.27)
where \( \varepsilon, t \in \mathbb{C}, 0 < |\varepsilon| < |t|, c < |t| < C \). Choose three non-intersecting half-lines \( D_1, D_2, D_3 \) in the exterior of \( T \), with endings at the vertices \( 0, \varepsilon, t \). The function \( z \to h(\varepsilon, t; z) \), initially defined as an analytic function for \( \text{Im } z > \text{max(Im } \varepsilon, \text{Im } t) \), may be extended into an analytic function on \( \Omega = \mathbb{C} \setminus (D_1 \cup D_2 \cup D_3) \). The behaviour of \( h \) on \( \Omega \) is given as follows, where the functions \( F, G, \tilde{G}, H, \tilde{H} \) are assumed to be holomorphic:

(i) Suppose \( z \in \Omega, \ |z/t| > C > 1 \). Then
\[
h(\varepsilon, t; z) = z^{2\alpha-2} \left( t^\beta-p+1\tilde{G}(t, t/z) + \varepsilon^{\beta-p+1} H(\varepsilon, \varepsilon/z) \right).
\]
(3.28)

(ii) Suppose \( z \in \Omega, \ |z/\varepsilon| > C > 1, \ |z/t| < c < 1 \): then
\[
h(\varepsilon, t; z) = z^{2\alpha-1+(\beta-p)} F(z) + G(z) + \varepsilon^{(\beta-p)+1} z^{2\alpha-2} H(\varepsilon, \varepsilon/z).
\]
(3.29)
(iii) Suppose \( z \in \Omega, \ |1 - z/\varepsilon| < c < 1, \ |z/t| < c' < 1: \) then
\[
h(\varepsilon, t; z) = G(z) + \varepsilon^{2\alpha-1+(\beta-p)} \left[ \tilde{H}(\varepsilon, 1 - \frac{z}{\varepsilon}) + (-i(\frac{z}{\varepsilon} - 1))^{2\alpha-1}H(\varepsilon, 1 - \frac{z}{\varepsilon}) \right].
\] (3.30)

(iv) Suppose \( z \in \Omega, \ |z/\varepsilon| < c < 1: \) then
\[
h(\varepsilon, t; z) = G(z) + \varepsilon^{2\alpha-1+(\beta-p)}H(\varepsilon, z/\varepsilon). \quad (3.31)
\]

(v) Assume \( z \in \Omega \) is in the cut \( \varepsilon \)-ring, i.e. \( 0 < c' < |z/\varepsilon| < C' < |t/\varepsilon| \) \((0 < c' < 1 < C'), \ |1 - z/\varepsilon| > c > 0. \) Then
\[
h(\varepsilon, t; z) = \left( z^{2\alpha-1+(\beta-p)}F(z) + G(z) \right) + \varepsilon^{2\alpha-1+(\beta-p)}H(\varepsilon, z/\varepsilon). \quad (3.32)
\]

Remarks.

1. Domains of holomorphy for the functions \( F, G, \ldots \) follow from Lemma 3.4.

2. We do not give the behaviour of \( h \) in the \( t \)-ring \( |\varepsilon/t| < c < |z/t| < C \) because we shall not need it.

3. The results for cases (i), (ii), (iii) and (v) follow essentially from splitting the integral into \( \int_0^t du - \int_0^\varepsilon du \) (with some extra care when \( p \geq 1 \) since the singularity is not integrable at 0). The method also applies in case (iv) but yields a spurious extra term of the form \( z^{2\alpha-1+\beta-p}\tilde{H}(\varepsilon, z/\varepsilon) \) which is singular when \( z = 0 \). Yet we use this so-called 'crude' versions of case (iv) at some places in the course of the proof of Theorem 1.1 because this splitting yields the required \((f_b, f_n)\)-splitting. Eq. (3.31) – called the 'refined' version of case (iv) – follows from a different splitting which avoids integrating around 0.

Proof.

Let us first prove briefly the 'refined' version of case (iv) as stated in the Lemma, see eq. (3.31). Write
\[
h(\varepsilon, t; z) = C \int_\varepsilon^t u^{2\alpha-2+\beta-p}(1 - z/u)^{2\alpha-2} f(u) \, du \quad (3.33)
\]
and expand \((1 - z/u)^{2\alpha-2} = \sum_{k \geq 0} \frac{(2-2\alpha)k}{k!} (\frac{z}{u})^k, f(u) = \sum_{n \geq 0} a_n u^n \ (|u| \leq c). \) Splitting \( \int_\varepsilon^t du \) into \( \int_\varepsilon^{c/2} du + \int_{c/2}^t du \) and exchanging the order of summation for the first integral leads to the expression \( G(z) + \varepsilon^{2\alpha-1+\beta-p}H(\varepsilon, z/\varepsilon), \)
while the second integral is trivially analytic in \( z \). (Easy details are left to the reader).

Let us now prove the 'crude' version of the Lemma (which does not differ from the 'refined' version, except for case (iv)).

Suppose first \( p = 0 \). Then

\[
h(\varepsilon, t; z) = \int_0^t (-i(z - u))^{2\alpha - 2} u^\beta f(u) \, du - \int_0^\varepsilon (-i(z - u))^{2\alpha - 2} u^\beta f(u) \, du.
\]

The first integral is estimated in Lemma 3.3 and the second one in Lemma 3.4. One gets:

\[
h(\varepsilon, t; z) = \left( z^{2\alpha - 1 + \beta} F(z) + G(z) \right) + \varepsilon^{\beta + 1} z^{2\alpha - 2} H(\varepsilon, \varepsilon/z) \tag{3.35}
\]

if \( |z/\varepsilon| > C > 1, |z/t| < c < 1; \)

\[
h(\varepsilon, t; z) = \left( z^{2\alpha - 1 + \beta} F(z) + G(z) \right) + \varepsilon^{2\alpha - 1 + \beta} \left( \tilde{H}(\varepsilon, 1 - z/\varepsilon) + (-i(1 - \varepsilon/1))^{2\alpha - 1} H(\varepsilon, 1 - \varepsilon) \right) \tag{3.36}
\]

if \( |1 - z/\varepsilon| < c < 1, |z/t| < c' < 1 \) (note that the function \( F \) may be discarded since \( z^{2\alpha - 1 + \beta} = \varepsilon^{2\alpha - 1 + \beta} (1 - \varepsilon/1)^{2\alpha - 1} = \varepsilon^{2\alpha - 1 + \beta} H(1 - \varepsilon); \)

\[
h(\varepsilon, t; z) = \left( z^{2\alpha - 1 + \beta} F(z) + G(z) \right) + \left( z^{2\alpha - 1 + \beta} \tilde{H}(\varepsilon, z/\varepsilon) + \varepsilon^{2\alpha - 1 + \beta} H(\varepsilon, z/\varepsilon) \right) \tag{3.37}
\]

if \( |z/\varepsilon| < c < 1 \) (once again, one may discard the function \( F \)); and

\[
h(\varepsilon, t; z) = \left( z^{2\alpha - 1 + \beta} F(z) + G(z) \right) + \varepsilon^{2\alpha - 1 + \beta} H(\varepsilon, z/\varepsilon) \tag{3.38}
\]

in the cut \( \varepsilon \)-ring. On the other hand, if \( |z/t| > C > 1 \), then

\[
h(\varepsilon, t; z) = z^{2\alpha - 2} \left( \varepsilon^{\beta + 1} F(t, t/z) + \varepsilon^{\beta + 1} G(\varepsilon, \varepsilon/z) \right). \tag{3.39}
\]

Suppose now \( p > 0 \). Let \( \tilde{B}(0, 2\rho|t|) \subset \Omega_f, \rho \in (0, 1] \) maximal, and set \( t' = \rho t, \varepsilon' = \max\{s \in [0, 1] | s \in \tilde{B}(0, 2\rho|t|) \} \cdot \varepsilon \), so that \( \varepsilon', t' \) are proportional to \( \varepsilon, t \) and contained in \( \tilde{B}(0, 2\rho|t|) \). Then the integral \( \int_{\varepsilon}^t \) splits into \( I_1 + I_2 \), where \( I_1 = \int_\varepsilon^{\varepsilon'} + \int_t^{t'}, \text{ and } I_2 = \int_{\varepsilon'}^{t'} \). The domain of integration of \( I_1 \) is bounded away from the origin, hence \( I_1 \) does not 'feel' the singularity at \( 0 \) and its behaviour may be deduced from Lemmas 3.2 and 3.3. The multivalued terms in \( I_2 \) at \( \varepsilon', t' \) are compensated by those of \( I_1 \). So one may just as well assume that \( \varepsilon = \varepsilon' \) and \( t = t' \), which we do in the sequel.
Write \( f(u) = \sum_{k=0}^{p-1} a_k u^k + u^p \tilde{f}(u) \) with \( \tilde{f} \) holomorphic in a neighborhood of 0, so that
\[
h(\varepsilon, t; z) = \int_\varepsilon^t (-i(z-u))^{2\alpha-2} u^{\beta-p+k} du + \sum_{k=0}^{p-1} a_k \int_\varepsilon^t (-i(z-u))^{2\alpha-2} u^{\beta-p+k} du.
\]
(3.40)

The first integral in the right hand side is estimated as in the case \( p = 0 \). As for the sum,
\[
\int_\varepsilon^t (-i(z-u))^{2\alpha-2} u^{\beta-p+k} du = \int_\varepsilon^t (-i(z-u))^{2\alpha-2} u^{\beta-p+k} du - \int_t^\infty (-i(z-u))^{2\alpha-2} u^{\beta-p+k} du
\]
= \( g_{p-k}(\varepsilon) - g_{p-k}(t) \).
(3.41)

(Note that the integrals converge since \( 2\alpha - 2 + \beta < 0 \)).

Now
\[
g_{p-k}(s) = -e^{i\pi \alpha} s^{2\alpha-1+\beta-p+k} \int_0^1 (1 - \frac{z}{s} v)^{2\alpha-2} u^{-2\alpha-\beta+p-k} dv
\]
\[
= \frac{e^{i\pi \alpha}}{2\alpha - 1 + \beta - p + k} s^{2\alpha-1+\beta-p+k} \binom{2-2\alpha}{2-2\alpha-\beta+p+k}(-z/s)^{2\alpha-1} 2F_1(-\beta+p-k,1;2\alpha;1-z/s).
\]
(3.42)

If \( |1-z/s| < c < 1 \) then the connection formula (0.21) yields
\[
g_{p-k}(s) = \frac{e^{i\pi \alpha}}{2\alpha - 1 + \beta - p + k} \frac{\Gamma(2-2\alpha-\beta+p-k)\Gamma(2\alpha-1)}{\Gamma(-\beta+p-k)} z^{2\alpha-1+\beta-p+k}
\]
\[
-\frac{1}{1-2\alpha} e^{i\pi \alpha} s^{2\alpha-1+\beta-p+k} (1-z/s)^{2\alpha-1} 2F_1(-\beta+p-k,1;2\alpha;1-z/s).
\]
(3.43)

If \( |z/s| > C > 1 \), then the connection formula (0.19) yields
\[
g_{p-k}(s) = \frac{e^{i\pi \alpha}}{2\alpha - 1 + \beta - p + k} s^{2\alpha-1+\beta-p+k}.
\]
\[
\left\{ \frac{2\alpha - 1 + \beta - p + k}{1 + \beta - p + k} (-z/s)^{2\alpha-2} 2F_1(2-2\alpha,1+\beta-p+k;2+\beta-p+k;s/z)
\]
\[
+ \frac{\Gamma(2-2\alpha-\beta+p-k)\Gamma(1+\beta-p+k)}{\Gamma(2-2\alpha)} (-z/s)^{2\alpha-1+\beta-p+k} \right\}.
\]
(3.44)
Using the formulas (3.43), (3.44) gives the result (the most singular terms are obtained for \( k = 0 \)).

Exactly the same results hold when one integrates against the kernel \( K^{*; \pm}(\eta) \), but the proof is different.

**Lemma 3.6 (integration against the integrated kernel on a general interval)**

Let, for \( \alpha \in (0, \frac{1}{2}) \), \(-1 < \beta < 0 \) and \( p = 0, 1, \ldots \), and for some function \( f \) analytic on an \( \varepsilon \)-independent neighbourhood of 0,

\[
h(\varepsilon, t; z) = \int_{\varepsilon}^{t} (-i(z - u))^{2\alpha} u^{\beta-p} f(u) \, du.
\]  
(3.45)

Then the results of Lemma 3.5 hold if one replaces \( \alpha \) with \( \alpha + 1 \).

**Proof.**

The proof is the same as for Lemma 3.5 except for the computation of

\[
\int_{\varepsilon}^{t} (-i(z-u))^{2\alpha} u^{\beta-p+k} \, du = (\int_{z}^{t} - \int_{z}^{\varepsilon}) (-i(z-u))^{2\alpha} u^{\beta-p+k} \, du =: g_{p-k}(t) - g_{p-k}(\varepsilon).
\]  
(3.46)

Setting \( v := \frac{u-z}{s-z} \), one obtains:

\[
g_{p-k}(s) = e^{i\pi \alpha (s-z)^{2\alpha+1}} \frac{1}{2\alpha + 1} z^{\beta-p+k} 2F_{1}(\beta-p-k; 2\alpha + 1; 2\alpha + 2; 1-s/z).
\]  
(3.47)

If \( |s/z| < c < 1 \) then the connection formula (0.21) entails

\[
2F_{1}(\beta-p-k, 2\alpha + 1, 2\alpha + 2; 1-s/z) = \frac{\Gamma(2\alpha + 2)\Gamma(\beta + 1 - p + k)}{\Gamma(2\alpha + 2 + \beta - p + k)} (1-s/z)^{-2\alpha-1}
\]

\[
+ \left( \frac{s}{z} \right)^{\beta+1-p+k} \frac{2\alpha + 1}{-\beta - 1 + p - k} 2F_{1}(2\alpha + 2 + \beta - p + k, 1; 2 + \beta - p + k; s/z)
\]  
(3.48)
On the other hand, if \(|s/z| > C > 1\) then the connection formula (0.20) entails

\[
2F_1(-\beta + p - k, 2\alpha + 1, 2\alpha + 2; 1 - s/z) = \\
\frac{\Gamma(2\alpha + 2)\Gamma(-2\alpha - 1 - \beta + p - k)}{\Gamma(-\beta + p - k)} (s/z - 1)^{-2\alpha - 1} \\
+ \frac{2\alpha + 1}{2\alpha + 1 + \beta - p + k} (z/s)^{-\beta + p - k} 2F_1(-\beta + p - k, 1; -2\alpha - \beta + p - k; z/s) 
\]

(3.49)

One may check that these expansions lead to the same leading exponents as in Lemma 3.5. \(\square\)
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