Contact interactions and Kronig–Penney models in Hermitian and $\mathcal{PT}$ symmetric quantum mechanics

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Abstract
The delta function potential is a simple model of zero-range contact interaction in non-relativistic quantum mechanics in one dimension. The Kronig–Penney model is a one-dimensional periodic array of delta functions and provides a simple illustration of energy bands in a crystal. Here we investigate contact interactions that generalize the delta function potential and corresponding generalizations of the Kronig–Penney model within conventional and $\mathcal{PT}$ symmetric quantum mechanics. In conventional Hermitian quantum mechanics we determine the most general contact interaction compatible with self-adjointness and in $\mathcal{PT}$ quantum mechanics we consider interactions that respect symmetry under the transformation $\mathcal{PT}$ where $P$ denotes parity and $T$ denotes time reversal. In both cases we find that the most general interaction has four independent real parameters and depending on the values of those parameters the contact interaction can support zero, one or two bound states. By contrast the conventional delta function can only support zero or one bound state. In the $\mathcal{PT}$ symmetric case moreover the two bound state energies can be both real or a complex conjugate pair. The transition from real to complex bound state energies corresponds to the spontaneous breaking of $\mathcal{PT}$ symmetry. The scattering states for the $\mathcal{PT}$ symmetric case are also found to exhibit spontaneous breaking of $\mathcal{PT}$ symmetry wherein the eigenvalues of the non-unitary $S$-matrix depart the unit circle in the complex plane. We also investigate the energy bands when the generalized contact interactions are repeated periodically in space in one dimension. In the Hermitian case we find that the two bound states result in two narrow bands generically separated by a gap. These bands intersect at a
single point in the Brillouin zone as the interaction parameters are varied. Near the intersection the bands form a massless Dirac cone. In the $\mathcal{PT}$ symmetric case we find that as the parameters of the contact interaction are varied the two bound state bands undergo a $\mathcal{PT}$ symmetry breaking transition wherein the two band energies go from being real to being a complex conjugate pair. The $\mathcal{PT}$ symmetric Kronig–Penney model provides a simple soluble example of the transition which has the same form as in other models of $\mathcal{PT}$ symmetric crystals.

Keywords: PT quantum mechanics, Kronig–Penney model, Dirac equation

(Some figures may appear in colour only in the online journal)

1. Introduction

A fundamental principle of quantum mechanics is that operators which correspond to observable quantities, most notably, the Hamiltonian, must be Hermitian. Recently there has been a surge of interest in operators that are non-Hermitian but respect the combined symmetry $\mathcal{PT}$ where $\mathcal{P}$ denotes parity and $\mathcal{T}$ is time reversal. In classical optics it has proved possible to fabricate materials with alternating regions of gain and loss that demonstrate many novel optical properties (for recent reviews see [3, 4]). In such systems the equation that governs the propagation of electromagnetic waves can be engineered to have the form of a Schrödinger equation with $\mathcal{PT}$ symmetry. In order to build intuition for wave propagation in these materials it is therefore relevant to consider simple models of $\mathcal{PT}$ quantum mechanics. In this paper we construct the $\mathcal{PT}$-symmetric generalizations of two models well-known from conventional Hermitian quantum mechanics: the delta function potential and the simplest model of a periodic crystal, the Kronig–Penney model.

The delta function is a widely used model of a zero-range contact interaction in quantum mechanics. Rigorously it is a viable model of contact interaction only in one dimension. In higher dimensions the ideal delta function potential is invisible and it is better to treat contact interactions as modified boundary conditions. Here we show that even in one dimension it is helpful to model a contact interaction as a boundary condition; adopting this point of view we find that even in Hermitian quantum mechanics in one dimension the delta function is merely a special case of the most general allowed contact interaction. This generalization of the contact interaction in one dimension has previously been discussed in [6–9].

Quite different forms of contact interaction emerge when we relax the conditions of Hermitian self-adjointness but instead impose the requirement of $\mathcal{PT}$ symmetry. We find that for both the Hermitian and $\mathcal{PT}$ symmetric generalized contact interactions there can be zero, one or two bound states depending on the parameters that characterize the interaction. In contrast the conventional delta function can only have zero or one bound states depending on whether the the potential is attractive or repulsive. In the $\mathcal{PT}$ symmetric case when there are two bound states the eigenvalues can be either both real or a complex conjugate pair depending on the parameters of the model. As the parameters pass through a critical value, the real eigenvalues degenerate and enter the complex plane, behavior that is called the $\mathcal{PT}$ transition. The $\mathcal{PT}$ transition is accompanied by spontaneous breaking of $\mathcal{PT}$ symmetry: although the interaction remains $\mathcal{PT}$ symmetric, the eigenstates are no longer invariant under $\mathcal{PT}$. We also identify a $\mathcal{PT}$ transition in the scattering states. In this case the energy is necessarily real; the transition occurs when the eigenvalues of the non-unitary $S$-matrix cease to be

\footnote{In this context we use ‘Hermitian’ synonymously with the more precise term ‘self-adjoint’; see section 2.1.}
unimodular and depart from the unit circle in the complex plane [10]. In Hermitian quantum mechanics the scattering amplitude has poles along the positive imaginary axis corresponding to bound states and it has poles in the lower half of the complex \( k \)-plane corresponding to scattering resonances. In \( \mathcal{PT} \) quantum mechanics this clean distinction is blurred. If \( \mathcal{PT} \) symmetry is broken the bound state eigenvalues become complex and although the bound state poles remain in the upper half plane they can correspond to long lived scattering resonances. Our model provides a simple example of this physics which has recently been discussed in [11, 12] in context of lattice models of coupled waveguides in optics. For long lived resonances associated with bound states the real part of the bound state energies are positive; hence these states are dubbed bound states in the continuum by Longhi [11].

We generalize the conventional Kronig–Penney model by considering a periodic array of generalized contact interactions in one dimension. In the Hermitian case the two bound state bands have a simple cosine dispersion when they are well separated. However when the parameters of the contact interaction are tuned suitably the bands intersect at an isolated point in the Brillouin zone. Near the intersection the band structure is a massless Dirac cone. This behavior is reminiscent of topological insulators where gap closure is a phase boundary that separates an ordinary insulator from a topological insulator [13]. Whether that is the case here is a question we leave open for future work. For the \( \mathcal{PT} \) symmetric case the two bound state bands undergo a \( \mathcal{PT} \) symmetry breaking transition as the parameters are varied (see figure 2). Before the onset of the transition the two bands are entirely real. After the transition is complete the bands are a complex conjugate pair. For intermediate values of the parameters the bands are real over part of the Brillouin zone and a conjugate pair over the remainder. The \( \mathcal{PT} \) symmetric Kronig–Penney model thus constitutes a particularly simple and soluble model that exhibits these generic features of \( \mathcal{PT} \) symmetric crystals. The generalized Hermitian Kronig–Penney model may be useful as a description of semiconductor superlattices [14] and the \( \mathcal{PT} \) symmetric generalization may be relevant to experiments in \( \mathcal{PT} \) optics [3, 4].

Boundary conditions that respect \( \mathcal{PT} \) symmetry were first introduced by Krejcirik et al in context of a particle in a box and generalizations thereof in a series of papers [15–19]; see also [20]. The study of periodic \( \mathcal{PT} \) symmetric potentials was initiated by [21–23]. Subsequently [24] spurred experimental activity in the field by identifying practical realizations in optics and by discovering novel wave propagation effects in crystals with \( \mathcal{PT} \) symmetry. The work of [22, 23] is particularly closely related to the present work. These authors introduced and analyzed a version of the Kronig–Penney model wherein the periodic potential is piecewise constant. Here by contrast we consider a different Kronig–Penney model that consists of repetitions of zero range contact interactions that cannot be obtained from the models of [22, 23] by any limiting procedure. Motivated by very different considerations of topology change, quantum gravity and many-worlds quantum mechanics the authors of [25] have also considered Hermitian generalizations of the contact interaction. We discuss the relationship of our results to [25] in section 2.1. Finally we note that [26] has considered a contact interaction in the form of a delta function with a complex coefficient. This model is neither Hermitian nor \( \mathcal{PT} \) symmetric. Subsequently [27] studied a \( \mathcal{PT} \) symmetric model with two delta functions with complex conjugate coefficients. Here by contrast the goal is to construct a contact interaction that is intrinsically \( \mathcal{PT} \) symmetric.

2. Contact interaction

Consider the textbook problem of a non-relativistic particle of mass \( m \) in one dimension interacting with a delta function potential \( \lambda \delta(x) \) located at the origin. Rather than treating the delta function as a potential we may regard it as a boundary condition that the wave function must
satisfy, namely, continuity at the origin, \( \psi(0^+) = \psi(0^-) \), and discontinuity in the derivative given by
\[
\psi'(0^+) = \frac{2m\lambda}{\hbar^2} \psi(0^-) + \psi'(0^-).
\] (1)

Viewing the delta function as a boundary condition suggests a more general model of a contact interaction wherein the wavefunction satisfies the boundary condition
\[
\psi(0^+) = a\psi(0^-) + b\psi'(0^-)
\]
\[
\psi'(0^+) = c\psi(0^-) + d\psi'(0^-)
\]
where \( a, b, c \) and \( d \) are complex constants. This is the most general boundary condition compatible with linearity and the order of the Schrödinger equation. The conventional delta function is the special case \( a = 1, b = 0, c = 2m\lambda/\hbar^2 \) and \( d = 1 \). Below we show that imposing the requirements of self-adjointness or \( PT \) symmetry powerfully constrain the form of the boundary condition (2). However in both cases boundary conditions more general than the conventional delta function are permissible and represent new kinds of zero range contact interaction; this is a key finding of the present work.

In the remainder of this paper we will work in units wherein \( \hbar = 1 \) and the mass of the particle \( m = 1 \).

2.1. Hermitian quantum mechanics

2.1.1. The model. Consider a non-relativistic particle in one dimension, free except for a zero range contact potential at the origin. The inner product of two states \( \phi \) and \( \psi \) is given by
\[
(\phi, \psi) = \int_{-\infty}^{0^-} dx \, \phi^*(x) \psi(x) + \int_{0^+}^{\infty} dx \, \phi^*(x) \psi(x).
\] (3)

Straightforward integration by parts reveals that the free particle Hamiltonian satisfies
\[
(\phi, H\psi) = (H\phi, \psi) + \text{surface terms};
\] (4)
hence \( H \) is formally self adjoint with respect to the inner product (3). The surface term at the origin is proportional to
\[
\left[ \phi^* \psi' - \phi' \psi^* \right]_{0^+} - \left[ \phi^* \psi' - \phi' \psi^* \right]_{0^-}.
\] (5)

To determine what boundary conditions are compatible with the self adjointness of \( H \) we proceed as follows [28]. We impose the boundary condition given in equation (2) on \( \psi \) and ask what boundary condition must be imposed on \( \phi \) in order to make the surface term vanish. Let us write the boundary condition on \( \phi \) as
\[
\phi(0^+) = A\phi(0^-) + B\phi'(0^-)
\]
\[
\phi'(0^+) = C\phi(0^-) + D\phi'(0^-).
\] (6)

It is then easy to verify that the surface terms in equation (5) will vanish provided
\[
A^* = a/(ad - bc),
\]
\[
B^* = b/(ad - bc),
\]
\[
C^* = c/(ad - bc),
\]
\[
D^* = d/(ad - bc).
\] (7)
The operator \( H \) is self-adjoint when the boundary condition imposed on \( \psi \) inexorably requires the same boundary condition be imposed on \( \phi \) [28]. Hence the boundary conditions compatible with the self-adjointness of \( H \) are that

\[
\begin{align*}
    a &= \alpha \e^\theta \\
    b &= \beta \e^\theta \\
    c &= \gamma \e^\theta \\
    d &= \delta \e^\theta.
\end{align*}
\]

Here \( \alpha, \beta, \gamma \) and \( \delta \) are real and satisfy \( \alpha \delta - \beta \gamma = 1 \).

In summary the most general form of contact interaction compatible with self-adjointness is given by equation (2) with the additional constraint that the matrix of coefficients

\[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\]

is an \( SL(2, R) \) matrix (i.e. it has real entries and unit determinant) multiplied by a phase. The general contact interaction described above is time reversal symmetric for \( \theta = 0 \) or \( \pi \). This is because if a wavefunction \( \psi \) satisfies the boundary condition (2) with real coefficients, then so does its time reversed counterpart \( \psi^\ast \). Parity is respected only if we impose \( a = d \). In that case one can verify that if \( \psi(x) \) satisfies the boundary condition (2) then so does \( P \psi(x) = \psi(-x) \).

To conclude this subsection we discuss the connection of these results to the findings of [25]. We can rewrite equation (2) as

\[
\begin{align*}
    \psi(0^+) &= \frac{\alpha}{\gamma} \psi'(0^+) - \frac{\e^{i\theta}}{\gamma} \psi'(0^-) \\
    \psi(0^-) &= \frac{\e^{-i\theta}}{\gamma} \psi'(0^+) - \frac{\delta}{\gamma} \psi'(0^-).
\end{align*}
\]

From equation (2) we see that \( \alpha = \delta = 1 \) and \( \beta = \theta = 0 \) and \( \gamma = 0 \) corresponds to zero interaction. In this case the wave function and its derivative are continuous and the wave function is smooth across the origin. From equation (10) we see that for \( \gamma \to \infty \) (with \( \alpha, \delta \) finite) the positive and negative half lines become disconnected with Dirichlet boundary conditions applied at the origin on either side. It follows that if we fix \( \alpha = \delta = 1 \) and \( \beta = \theta = 0 \) then as \( \gamma \) goes from zero to \( \infty \) we interpolate continuously from the smooth case to the disconnected case. This interpolation is the topological transition discussed by Shapere et al [25].

Another continuous trajectory through the space of Hermitian boundary conditions is to choose \( \alpha = \delta = \cosh s \) and \( \beta = \xi \sinh s \) and \( \gamma = \sinh s / \xi \) where \( \xi \) is a fixed constant and \( s \) varies from \( s = 0 \) to \( s = \infty \). This trajectory starts from zero contact interaction and terminates in the disconnection of the two half lines but with the boundary conditions \( \psi(0^+) = \xi \psi'(0+) \) and \( \psi(0^-) = -\xi \psi'(0^-) \) on either side of the origin in place of Dirichlet boundary conditions. These boundary conditions allow for the possibility of bound states that are confined close to the origin on both sides if \( \xi < 0 \).

Naively one might suppose that it is possible to eliminate the phase \( \theta \) by making the following gauge transformation: \( \psi \to \e^{-i\theta} \phi \) for \( x > 0 \) and \( \psi \to \phi \) for \( x < 0 \). However even for an open one dimensional system physics is invariant only under small gauge transformations \( \psi \to \psi \xi^\ast \) where \( \gamma \to 0 \) for \( x \to \pm \infty \). Indeed we will see below that although the bound states are independent of \( \theta \) the scattering matrix does depend on \( \theta \) in a non-trivial way. This is a particularly simple example of the subtle distinction between large and small gauge transformations in quantum mechanics.
2.1.2. **Bound states.** We seek a solution of the form
\[
\psi = A \exp(-\kappa x) \quad \text{for} \quad x > 0,
\]
\[
= B \exp(\kappa x) \quad \text{for} \quad x < 0.
\]
This solution satisfies the free particle Schrödinger equation and has an energy \(-\kappa^2/2\).

Application of the boundary condition (2) reveals that \(\kappa\) must satisfy
\[
\beta \kappa^2 + (\alpha + \delta) \kappa + \gamma = 0.
\]
(12)

For the case \(\beta \neq 0\) this equation has two roots which can be written in the form
\[
\kappa_\pm = -\frac{\alpha + \delta}{2\beta} \pm \frac{\sqrt{4 + (\alpha - \delta)^2}}{2\beta}.
\]
(13)

(Here we have made use of the condition \(\alpha \delta - \beta \gamma = 1\).) Thus both roots are necessarily real. For the root to correspond to a viable bound state it must also be positive (negative roots are said to correspond to anti-bound states). Depending on the choice of \(\alpha, \beta\) and \(\delta\) it is possible that zero, one or both of the roots are positive. Thus in contrast to the conventional delta function which can only have zero or one bound states, our generalized zero range potential is capable of having two bound states. The existence of up to two bound states for the generalized contact interaction has been discussed previously in the literature; see for example the monograph by Albeverio et al [8].

The case \(\beta = 0\) includes the conventional delta function as a special case. For this case equation (12) is linear and has just one root
\[
\kappa = -\frac{\alpha \gamma}{\alpha^2 + 1}.
\]
(14)

Here we have made use of \(\alpha \delta = 1\) to write the root in a particularly transparent form. Evidently the root corresponds to a bound state if \(\alpha \gamma < 0\). For the conventional delta function \((\alpha = 1, \gamma = 2\lambda)\) equation (14) translates to the familiar condition that \(\lambda < 0\) for the existence of a bound state.

Note that the bound states are independent of the phase \(\theta\). This is because the bound states decay exponentially as \(x \to \pm \infty\); hence in this case it is permissible to gauge away the phase \(\theta\) by a large gauge transformation. That the bound states are independent of the phase \(\theta\) is implicit in the work of [7]; however we will see below that the \(S\)-matrix does depend on \(\theta\) in a non-trivial way.

Finally we note for later use that equation (12) suggests an alternative way to parametrize a Hermitian contact interaction using \(\beta\) and the two real roots of the quadratic form equation (12) as independent parameters. Denoting the roots \(\kappa_1\) and \(\kappa_2\) with \(\kappa_1 > \kappa_2\) we can easily reconstruct \(\alpha + \delta = -\beta(\kappa_1 + \kappa_2)\) and \(\gamma = \beta \kappa_1 \kappa_2\) from the fact that \(\kappa_1\) and \(\kappa_2\) are roots of (12). In order to reconstruct \(\alpha - \delta\) we make use of \(\alpha \delta - \beta \gamma = 1\) to show that
\[
\alpha - \delta = \pm \sqrt{\beta^2 (\kappa_1 - \kappa_2)^2 - 4}. \tag{15}
\]
Hence we see that we may use \(\beta, \kappa_1\) and \(\kappa_2\) as an alternative set of parameters provided we also specify the sign of \(\alpha - \delta\) and respect the constraint that \((\kappa_1 - \kappa_2)|\beta| \geq 2\).

2.1.3. **Scattering states.** Next we turn to positive energy scattering states. A state that is incoming from the left has the behavior
\[
\psi = e^{ikx} + re^{-ikx} \quad \text{for} \quad x < 0,
\]
\[
= te^{ikx} \quad \text{for} \quad x > 0.
\]
(15)
The scattering coefficients $t$ and $r$ are determined by imposing the boundary condition equation (2). For simplicity let us suppose initially that the phase $\theta$ is zero. For the case $\beta \neq 0$ the transmission coefficient

$$ t = \frac{2ik}{\beta(k - i\kappa_+)(k - i\kappa_-)} $$

(16)

and for $\beta = 0$

$$ t = \frac{1}{\alpha + \delta} \frac{2k}{k - i\kappa}. $$

(17)

Thus in each case the transmission coefficient has poles along the positive imaginary axis in the $k$ plane at locations determined by the bound states, consistent with the general analytic properties of the $S$-matrix in quantum mechanics. One can similarly analyze a scattering state that is incoming from the right. The scattering coefficients in this case are denoted $t'$ and $r'$.

For the sake of brevity we omit expressions for $r$ and $r'$ but note that explicit calculation confirms that

$$ S = \begin{pmatrix} t & r' \\ r & t \end{pmatrix} $$

(18)

is unitary as expected on general grounds. Furthermore $t = t'$ which is a general consequence of time reversal symmetry combined with the unitarity of the $S$-matrix. On the other hand $r \neq r'$ unless $\alpha = \delta$ ensuring that parity is also a symmetry.

Now let us consider the effect of the phase angle $\theta$ which has so far been set equal to zero in this subsection. By explicit calculation or use of a gauge argument one can show that the $S$ matrix now becomes

$$ S = \begin{pmatrix} te^{-i\theta} & r' \\ r & te^{i\theta} \end{pmatrix}. $$

(19)

Hence as claimed the $S$ matrix has a non-trivial dependence on the phase $\theta$. Moreover the transmission coefficient for incidence from the left and right is no longer the same once time reversal symmetry is broken.

2.2. $\mathcal{PT}$ quantum mechanics

2.2.1. The model. In $\mathcal{PT}$ quantum mechanics we eschew the condition of self-adjointness with respect to the inner product expressed in equation (3) but instead require the Hamiltonian to respect $\mathcal{PT}$ symmetry. In the present context we require that if $\psi$ satisfies the boundary condition equation (2) then so should $\mathcal{PT}\psi(x) = \psi^*(-x)$. Straightforward analysis shows that this condition is met provided the coefficients are given by equation (8) together with the conditions (i) $\beta$ and $\gamma$ are real (ii) $\alpha = \delta^*$ and (iii) $\alpha\delta - \beta\gamma = 1$. Thus the primary departure from the Hermitian case is that $\alpha$ and $\delta$ are no longer required to be real but are required to be a complex conjugate pair. Hence the number of independent parameters that specify the interaction is the same in both cases.

If we impose the condition that both $\mathcal{P}$ and $\mathcal{T}$ symmetry should be separately respected we find exactly the same conditions as in the Hermitian case with $\mathcal{P}$ and $\mathcal{T}$ symmetry; namely, that $a, b, c, d$ must be real, $a = d$ and $ad - bc = 1$. For the record we note that if we impose
only $T$ symmetry we obtain the condition that $a, b, c, d$ are real. If we impose only $\mathcal{P}$ symmetry we need $a = d$ and $ad - bc = 1$ but there is no restriction to real values for any of the coefficients.

2.2.2. Bound states. We now turn to the analysis of the bound states of a $\mathcal{PT}$ symmetric contact interaction. The analysis closely parallels that for the Hermitian case. We seek states of the same form as in the Hermitian case given by equation (11) but this time we no longer require $\kappa$ to be real. We do need the real part of $\kappa$ to be positive to ensure that the solution vanishes for $x \to \pm\infty$. The ansatz (11) remains a solution to the Schrödinger equation with energy $-\kappa^2/2$. Application of the boundary condition equation (2) again reveals that $\kappa$ satisfies equation (12) but the subsequent analysis departs from the Hermitian case.

First let us consider the case $\beta \neq 0$. Taking into account that $\delta = \alpha^*$ we may write the roots of equation (12) in the transparent form

$$\kappa \pm = -\frac{\alpha_R}{\beta} \pm \frac{1}{\beta} \sqrt{1 - \alpha_I^2}, \quad \kappa \pm = -\frac{\alpha_R}{\beta} \pm \frac{1}{\beta} \sqrt{1 - \alpha_I^2}$$

(20)

where $\alpha_R$ and $\alpha_I$ are respectively the real and imaginary parts of $\alpha$. We now separately consider the cases $|\alpha_I| > 1$ and $|\alpha_I| \leq 1$. (i) For $|\alpha_I| > 1$ the roots $\kappa_\pm$ are a complex conjugate pair. For $-\alpha_R/\beta > 0$ the real parts of both $\kappa_+$ and $\kappa_-$ are positive and hence there are two bound states. The energies of the two bound states are complex conjugates of each other. On the other hand for $-\alpha_R/\beta < 0$ the real parts of $\kappa_\pm$ are negative and hence there are no bound states. (ii) For $|\alpha_I| < 1$ the roots $\kappa_\pm$ are both real. The roots correspond to bound states only if their real parts are positive. It follows from equation (20) that there can be zero, one or two bound states depending on the values of $\alpha_R, \alpha_I$ and $\beta$.

Next consider the case that $\beta = 0$. In this case equation (12) is linear and there is only one root

$$\kappa = -\frac{\gamma}{2\alpha_R}. \quad \kappa = -\frac{\gamma}{2\alpha_R}$$

(21)

For $-\gamma/2\alpha_R > 0$ this root corresponds to a bound state; otherwise there are zero bound states.

In summary $|\alpha_I| > 1$ and $-\alpha_R/\beta > 0$ corresponds to the case of spontaneously broken $\mathcal{PT}$ symmetry. In this phase there are two bound states with complex conjugate energies. Otherwise $\mathcal{PT}$ symmetry is unbroken and there can be zero, one or two bound states all with real energy. The behavior of the bound state energies across the $\mathcal{PT}$ symmetry breaking transition exhibits a characteristic complementary pitchfork form illustrated in figure 1.

2.2.3. Scattering states. We now turn to the scattering of waves by a $\mathcal{PT}$ symmetric contact interaction. In $\mathcal{PT}$ quantum mechanics the $S$-matrix is no longer unitary and hence its eigenvalues are not required to be unimodular. When the eigenvalues are nonetheless unimodular $\mathcal{PT}$ symmetry is said to be intact; when they cease to unimodular $\mathcal{PT}$ is said to be spontaneously broken [10].

In order to determine the eigenvalues of the $S$-matrix we consider scattering states of the form

$$\psi = Ae^{ikx} + \mu Be^{-ikx} \quad \text{for} \quad x < 0,$$

$$= \mu Ae^{ikx} + Be^{-ikx} \quad \text{for} \quad x > 0.$$

(22)

Here the amplitudes of the incoming waves from the left and right, denoted $A$ and $B$ respectively, are amplified by the eigenvalue $\mu$ in the corresponding outgoing waves. Making use
of the boundary condition equation (2) yields two conditions connecting the ratio \( AB \) and \( \mu \).

Imposing consistency between these expressions reveals that the eigenvalues of the \( S \)-matrix are the roots of the quadratic equation

\[
\Delta \mu^2 + 4i k \cos \theta \mu - \Delta^* = 0
\]

where

\[
\Delta = \gamma - \beta k^2 - ik(\alpha + \delta).
\]

In obtaining equation (23) we have assumed that \( \beta, \gamma \) and \( \alpha + \delta \) are real and that \( \alpha \delta - \beta \gamma = 1 \). Hence our analysis to this point applies both to the Hermitian and the \( \mathcal{PT} \) symmetric contact interaction models.

It is evident from equation (23) that if \( \mu \) is an eigenvalue of the \( S \)-matrix then so is \( 1/\mu^* \). It is also clear that the product of the magnitudes of the two eigenvalues must be \( 1 \). This also follows more generally because the \( S \) matrix satisfies \( S^\dagger S = \mathcal{I} \) and \( S^* S = \mathcal{I} \) for the Hermitian and \( \mathcal{PT} \) symmetric cases respectively. (Here \( \mathcal{I} \) denotes the \( 2 \times 2 \) identity matrix.)

Figure 1. \( \mathcal{PT} \) symmetry breaking for the \( \mathcal{PT} \) symmetric delta function model. The two bound state eigenvalues are calculated using equation (20) with \( \alpha_R = -1 \) and \( \beta = 1 \). The upper plot shows the real parts of the two eigenvalues and the lower plot shows the imaginary parts. The eigenvalues are plotted as a function of the parameter \( \alpha_I \). The eigenvalues are found to undergo a characteristic complementary pitchfork bifurcation. Below the critical value \( \alpha_I = 1 \) the eigenvalues are real and above they are a complex conjugate pair. The real parts of the eigenvalues degenerate at the critical value forming one pitchfork while the imaginary parts become non-zero forming the second complementary pitchfork.
By writing down the explicit solution to equation (23) it can be seen that if $|\Delta| > 2k |\cos \theta|$ the eigenvalues are unimodular. On the other hand if $|\Delta| < 2k |\cos \theta|$ the $S$-matrix eigenvalues no longer lie on the unit circle in the complex plane. One has a magnitude bigger than unity; the other, smaller, in order to ensure that the product of the magnitudes is still unity. Physically one eigenmode of the $S$-matrix is amplified upon scattering from the contact interaction; the other is attenuated.

Making use of equation (24) and exploiting $\alpha \delta - \beta \gamma = 1$ yields the useful formula

$$|\Delta|^2 - 4k^2 \cos^2 \theta = (\gamma + \beta k^2)^2 + (\alpha - \delta)^2 k^2 + 4k^2 \sin^2 \theta.$$  \hspace{1cm} (25)

From equation (25) it is evident that in the Hermitian case $\alpha = \delta$ and the right hand side is positive; hence $|\Delta| > 2k |\cos \theta|$ always. In other words in the Hermitian case we see by explicit calculation that the eigenvalues of the $S$ matrix must be unimodular as expected on general grounds also. However for the $\mathcal{PT}$ symmetric case $(\alpha - \delta)^2 = -4\alpha_I^2$ and hence the middle term on the right hand side of equation (25) is negative. If it is sufficiently negative the eigenvalues of the $S$ matrix no longer have to lie on the unit circle and $\mathcal{PT}$ symmetry is said to be broken. Equation (25) reveals that there will always be a range of $k$ for which $\mathcal{PT}$ symmetry is broken so long as $\alpha^2 I > \sin^2 \theta$ and $\alpha^2 R < \cos^2 \theta$.

It is also interesting to examine the scattering amplitudes for the $\mathcal{PT}$ symmetric contact interaction; for simplicity we only consider the case $b \neq 0$ and $\theta = 0$. If we consider a wave incoming from the left as in equation (15), then the scattering amplitude $t$ is still given by equation (16) but with $\kappa_{\pm}$ now given by equation (20). As for the Hermitian case we see that the scattering amplitude has poles in the upper half $k$-plane that are determined by the bound states if any. Furthermore in case the bound state energies are a complex conjugate pair the scattering amplitude has a Lorentzian resonance whose location is determined by $\text{Im} \kappa_{\pm}$ and width by $\text{Re} \kappa_{\pm}$. This resonance has no counterpart in conventional Hermitian quantum mechanics. The effect of a non-zero $\theta$ on the scattering amplitude is relatively innocuous; the amplitude is multiplied by $\exp(\pm i\theta)$ depending on whether the incident wave comes from $x \to \mp \infty$.

The existence of resonances associated with poles of the scattering amplitude in the upper half of the complex $k$-plane is a feature of $\mathcal{PT}$ quantum mechanics that has no Hermitian counterpart. It has been remarked upon previously by the authors of [11, 12] who refer to such resonances as bound states in the continuum since the real part of the complex energies of these bound states are degenerate with the positive energy continuum for sufficiently long lived resonances. Moreover if the parameters of the model are tuned so that the bound states transition to anti-bound states, then precisely at the transition, where the poles are crossing the real axis in the $k$-plane, the states are termed resonant states in the continuum by Garmon \textit{et al} [12].

3. Kronig–Penney model

The Kronig–Penney model in one dimension is the simplest model of a crystal, originally introduced to provide a simple illustration of energy bands and band gaps in the early days of solid state physics. The model describes a particle that interacts with a periodic comb of delta functions that are separated by a distance $\ell$. Here we consider two generalizations of the textbook model wherein the ordinary delta function is replaced by either the generalized Hermitian or the $\mathcal{PT}$ symmetric contact interactions introduced here. In the textbook case an isolated attractive delta function would have a single bound state. For a well separated array of delta function potentials this bound state fans out into a narrow band characterized by the
energy dispersion $E(k)$ where $E$ is the energy of the Bloch state and $k$ its crystal momentum which lies in the Brillouin zone $-\pi/\ell < k < \pi/\ell$.

In the models considered here there can be two bound states in the isolated limit that fan into a pair of narrow bands when the contact potentials are well separated. We find that in the Hermitian case the gap between the bands can close when the parameters are tuned suitably. The energy dispersion near the intersection of the two bands is approximately that of a massless Dirac particle. In the $\mathcal{PT}$ symmetric case we find that as the parameters are tuned the bands undergo a $\mathcal{PT}$ symmetry breaking transition. More precisely recall that for an isolated contact interaction the $\mathcal{PT}$ transition takes place for $|\alpha_I| = 1$. For the corresponding Kronig–Penney model we find that for $|\alpha_I| < \alpha_{c1} < 1$ the band energies $E_{\pm}(k)$ are entirely real; for $1 < \alpha_{c2} < |\alpha_I|$ the band energies become complex conjugate; and for an intermediate range $\alpha_{c1} < |\alpha_I| < \alpha_2$ the band energies are real for a small range of $k$ and complex conjugate elsewhere in the Brillouin zone.

3.1. Bloch analysis

For a periodic potential in one dimension with a period $\ell$ the eigenfunctions must have the Bloch form

$$\psi_{nk}(x) = \Pi_{nk}(x) \exp(ikx).$$  \hspace{1cm} (26)

Here $n$ is an index that labels the bands and $k$ is the crystal momentum which lies in the Brillouin zone $-\pi/\ell < k < \pi/\ell$. The form factor $\Pi_{nk}$ is a periodic function of $x$ with period $\ell$. Hence the Bloch wave-function obeys the quasi-periodic condition

$$\psi_{nk}(x + \ell) = \psi_{nk}(x) \exp(ik\ell).$$  \hspace{1cm} (27)

In the Kronig–Penney model considered here we assume that the particle experiences a contact interaction at the points $x = \nu\ell$ where $\nu$ is an integer. This includes the origin which corresponds to $\nu = 0$. Hence $\psi_{nk}$ must obey the boundary condition equation (2) at the origin. Since we are interested in bound state bands with negative energy we take the Bloch wave-function to have the form

$$\psi_{nk} = A \exp(\kappa x) + B \exp(-\kappa x) \text{ for } -\ell/2 < x < 0,$$

$$= C \exp(\kappa x) + D \exp(-\kappa x) \text{ for } 0 < x < \ell/2.$$  \hspace{1cm} (28)

The energy of this state is $E = -\hbar^2\kappa^2/2$.

Imposing the quasi-periodicity condition (27) and the boundary condition (2) leads to the quantization condition

$$\kappa^2\beta + \kappa(\alpha + \delta) + \gamma = 4\kappa \cos(\kappa \ell - \theta) e^{-\kappa \ell}$$

$$+ |\kappa^2\beta - \kappa(\alpha + \delta) + \gamma| e^{-2\kappa \ell}.  \hspace{1cm} (29)$$

Our task now is to solve the transcendental equation (29) for $\kappa$. By determining the dependence of $\kappa$ on $k$ we can determine the energy dispersion $E(k)$. In the limit $\ell \to \infty$ the right hand side of equation (29) vanishes and the allowed $\kappa$ values are the same as for an isolated contact interaction as expected. The analysis for finite $\ell$ is undertaken separately below for the Hermitian and $\mathcal{PT}$ symmetric cases. In both cases for simplicity we will take $\theta = 0$ since the only effect of non-zero $\theta$ is to shift the bands in $k$-space.
3.2. $\mathcal{PT}$ symmetric bands

Recall that for the isolated $\mathcal{PT}$ symmetric contact interaction the allowed $\kappa$ values are given by equation (20). Since we are interested in the $\mathcal{PT}$ symmetry breaking transition we consider values of $|\alpha_I|$ near to the threshold value of unity. For brevity we write $\kappa_\pm = \pi \pm \epsilon$ for $|\alpha_I| > 1$ and $\kappa_\pm = \pi \pm \epsilon$ for $|\alpha_I| < 1$ respectively. We assume that $\ell$ is sufficiently large that the bands will be narrow and hence posit that $\kappa = \pi + \Delta$ where $\Delta$ is small. To enforce that the bands are narrow we assume $\exp(-\kappa \ell) \ll 1$ and we also assume that $\Delta \ell \ll 1$. The condition $\exp(-\kappa \ell) \ll 1$ allows us to neglect the second term on the right hand side of the quantization condition and the condition that $\Delta \ell \ll 1$ allows us to approximate $\exp(-\kappa \ell) \approx \exp(-\pi \ell)$. Making these approximations yields

$$\Delta_\pm(k) = \pm W \sqrt{\epsilon^2 + \cos k \ell}$$

(30)

for $|\alpha_I| < 1$ and

$$\Delta_\pm(k) = \pm W \sqrt{\cos k \ell - \epsilon^2}$$

(31)

for $|\alpha_I| > 1$. Here for simplicity we have defined

$$W = \left[ \frac{4\pi}{\beta} \exp(-\pi \ell) \right]^{1/2}$$

(32)

and $\epsilon$ is a rescaled version of $\epsilon$ given by $\epsilon = W \epsilon$.

In summary the energy bands near the $\mathcal{PT}$ transition in the narrow band limit are given by the simple expression

$$E_\pm(k) = -\frac{1}{2} \pi^2 - \pi \Delta_\pm(k)$$

(33)

where $\Delta_\pm(k)$ is given by equations (30) and (31) for the cases $|\alpha_I| < 1$ and $|\alpha_I| > 1$ respectively. $W$ is a measure of the bandwidth and $\epsilon$ measures the distance of $|\alpha_I|$ from the transition value of unity. It is evident from these expressions that there are four regimes. (i) For the $\mathcal{PT}$ symmetric regime $|\alpha_I| < 1$ and $\epsilon > 1$ the bands are pure real. (ii) For the broken $\mathcal{PT}$ symmetry regime $|\alpha_I| > 1$ and $\epsilon > 1$ the two bands are a complex conjugate pair. (iii) The range $|\alpha_I| < 1$ and $\epsilon < 1$ corresponds to the onset of the $\mathcal{PT}$ transition. In this regime the bands are real for small $k$ and complex conjugate elsewhere in the Brillouin zone. (iv) The range $|\alpha_I| > 1$ and $\epsilon < 1$ corresponds to the range over which the $\mathcal{PT}$ transition is completed. Over this range too the bands are partially real at small $k$ and complex conjugate elsewhere in the Brillouin zone. These behaviors are shown in figure 2.

3.3. Hermitian bands

We now analyze the quantization condition equation (29) for the Hermitian case. We focus on the case that for an isolated contact interaction there are two bound states. In the literature periodic arrays of delta functions and of the derivative of the delta function have been studied [8]. In the former case there is at most one bound state and in the latter there are none. Thus the case of bands formed from two bound states does not appear to have been investigated earlier. It is more convenient to work with the parameters $\kappa_1, \kappa_2$ and $\beta$ that were introduced at the end of section 2.2.1 to describe the contact interaction. In terms of these parameters the exact quantization condition (29) may be rewritten
\[(\kappa - \kappa_1)(\kappa - \kappa_2) = \frac{4\kappa}{\beta} \cos k\ell e^{-\kappa\ell} + (\kappa + \kappa_1)(\kappa + \kappa_2)e^{-2\kappa\ell}. \tag{34}\]

First for simplicity we assume that the two bound states are well separated in comparison to the width of the bands that they form. Now in order to analyze the band associated with the first isolated bound state we write \(\kappa = \kappa_1 + \Delta\) where \(\Delta\) is assumed to be small. More precisely we assume that \(\Delta \ell \ll 1\) and also that \(\exp(-\kappa_1\ell) \ll 1\). The latter assumption allows us to ignore the second term on the right hand side of equation (34) and we obtain

\[\Delta = \frac{4\kappa_1}{\beta(\kappa_1 - \kappa_2)} e^{-\kappa_1\ell} \cos k\ell. \tag{35}\]

Recalling that the energy is given by \(E = -\kappa^2/2\) we find that the first band has the energy dispersion

\[E_1(k) = -\frac{\kappa_1^2}{2} - \frac{4\kappa_1^2}{\beta(\kappa_1 - \kappa_2)} e^{-\kappa_1\ell} \cos k\ell. \tag{36}\]

Similarly the second band is given by

\[E_2(k) = -\frac{\kappa_2^2}{2} + \frac{4\kappa_2^2}{\beta(\kappa_1 - \kappa_2)} e^{-\kappa_2\ell} \cos k\ell. \tag{37}\]

More interesting behavior results when \(\kappa_1 \approx \kappa_2\). In this regime as the parameters of the generalized delta function potential are tuned appropriately the bands intersect at an isolated point in \(k\)-space before moving apart again. At the intersection the bands form a Dirac cone. To demonstrate this behavior we write \(\kappa_1 = \pi + \epsilon\) and \(\kappa_2 = \pi - \epsilon\) where \(\epsilon\) is positive and assumed to be small in a sense to be made precise. As before we write \(\kappa = \pi + \Delta\) where \(\Delta\) is also assumed to be small. For simplicity we assume that the delta potentials are well separated, \(\exp(-\pi\ell) \ll 1\), but that \(\Delta\) is sufficiently small that \(\Delta \ell \ll 1\). Due to the constraint \((\kappa_1 - \kappa_2)|\beta| = 2e|\beta| \geq 2\), a small value of \(\epsilon\) implies a large value of \(\beta\); hence we can no longer neglect the second term on the right hand side of equation (34) in comparison to the first. In fact the two terms are of the same order if we take

\[\epsilon = 2\pi e^{-\pi\ell} \tag{38}\]

where \(\epsilon\) is of order unity. We also write \(1/\beta = f\epsilon\) with \(-1 < f < 1\) (in order to respect the constraint \(|\beta|\epsilon > 1\)). Making these assumptions and using equation (34) we obtain

\[\Delta = \pm 2\pi e^{-\pi\ell} \left(1 + \epsilon^2 + 2\epsilon f \cos k\ell\right)^{1/2}. \tag{39}\]

This corresponds to the energy bands

\[E_{\pm}(k) = -\frac{\pi^2}{2} \pm 2\pi^2 e^{-\pi\ell} \left[1 + \epsilon^2 + 2\epsilon f \cos k\ell\right]^{1/2}. \tag{40}\]

From equation (40) we see that for \(\epsilon = 1\) and \(f = -1\) or \(f = +1\) the bands touch at \(k = 0\) or \(k = \pi\) respectively. For \(k\) near the intersection the energy dispersion is approximately linear and the bands form a massless Dirac cone. In terms of the original parameters \(\epsilon = 1\) and \(f = \pm 1\) translates to \((\kappa_1 - \kappa_2)|\beta| = 2\) where the sign of \(\beta\) is the same as that of \(f\). Put another way there are two gapped phases corresponding to \(\beta > 2/(\kappa_1 - \kappa_2)\) and \(\beta < -2/(\kappa_1 - \kappa_2)\) respectively.
4. Conclusion

The delta function potential is a simple model of zero range contact interaction in one dimension. In this paper we have introduced generalizations of the delta function for conventional Hermitian quantum mechanics and $\mathcal{PT}$ quantum mechanics in one dimension. We find that the corresponding generalizations of the Kronig–Penney model exhibit interesting behavior in both Hermitian and $\mathcal{PT}$ quantum mechanics. In $\mathcal{PT}$ quantum mechanics we find bands that undergo $\mathcal{PT}$ symmetry breaking, providing a particularly simple example of this phenomenon. In Hermitian quantum mechanics we find that the gap between the two bound state bands closes when the parameters of the interaction are appropriately tuned yielding a conical intersection between the bands at a single point in the Brillouin zone. Near the intersection the dispersion relation is that of a massless Dirac fermion. Whether the gapped phase on either side of gap closure is a topological insulator is an intriguing question we leave open for future work.

Figure 2. The energy bands of the $\mathcal{PT}$ symmetric Kronig–Penney model. The plot on top left shows the energy plotted as a function of wave-vector for the case of unbroken $\mathcal{PT}$ symmetry. This is the case (i) in the main text. The energy bands in this case are entirely real and have an approximate particle-hole symmetry. The plot on bottom right by contrast corresponds to case (iv) where the $\mathcal{PT}$ symmetry breaking transition is complete. The energy bands are a complex conjugate pair. The plot only shows the dispersion of the imaginary parts of the energy because the real part of the energies is found to be constant. The symmetry between the bands in this case is required by $\mathcal{PT}$ symmetry. The plots on top right and bottom left correspond to the onset of the $\mathcal{PT}$ transition. This is case (iii) in the text. The plot on top right shows the real part of the energy and the plot on the lower right shows the imaginary part. The bands have been offset so as to be centered about zero energy and the parameter $W = 1$ and $\ell = 1$ throughout whereas $\varepsilon^2 = 1.2$ (top left and bottom right) or $\varepsilon^2 = 0.5$ (top right and bottom left).
Another interesting application of our generalized contact interaction may be to many body physics in one dimension. There are only a handful of exactly soluble non-trivial models of quantum many body systems. In a seminal paper Lieb and Liniger showed that a one dimensional gas of bosons interacting via a delta function contact interaction was soluble via Bethe ansatz [29, 30]. There has been a resurgence of interest in this class of integrable models due to their experimental realization in cold atoms [31–33]. A natural generalization of the Lieb–Liniger model suggested by this paper is to replace the delta function interaction with the generalized form of contact interaction studied here. This model also should be soluble via Bethe ansatz and may be realizable with cold atoms.

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References

[1] Bender C M and Boettcher S 1998 Real spectra in non-Hermitian Hamiltonians having \( PT \) symmetry Phys. Rev. Lett. 80 5243
[2] Bender C M 2007 Making sense of non-Hermitian Hamiltonians Rep. Prog. Phys. 70 947
[3] El-Ganainy R et al 2018 Non-Hermitian physics and \( PT \) symmetry Nat. Phys. 14 11
[4] Feng L, El-Ganainy R and Ge L 2017 Non-Hermitian photonics based on parity-time symmetry Nat. Photon. 11 752
[5] Jackiw R 1991 Delta function potentials in two and three dimensional quantum mechanics M.A.B. Bég Memorial Volume ed A Ali and P Hoodbhoy (Singapore: World Scientific)
[6] Seba P 1986 The generalized point interaction in one dimension Czech. J. Phys. B 36 667
[7] Tsutsui I 2001 Möbius structure of the spectral space of Schrödinger operators with point interaction J. Math. Phys. 42 5687
[8] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2005 Solvable Models in Quantum Mechanics 2nd edn (Providence, RI: American Mathematical Society)
[9] Cheon T and Shigehara T 1998 Realizing discontinuous wave functions with renormalized short range potentials Phys. Lett. A 243 111
[10] Chong Y D, Ge L and Stone A D 2011 \( PT \)-symmetry breaking and laser-absorber modes in optical scattering systems Phys. Rev. Lett. 106 093902
[11] Longhi S 2014 Bound states in the continuum in \( PT \)-symmetric optical lattices Opt. Lett. 39 1697
[12] Garmon S, Gianfreda M and Hatano N 2015 Bound states, scattering states, and resonant states in \( PT \)-symmetric open quantum systems Phys. Rev. A 92 022125
[13] Hasan M Z and Kane C L 2010 Topological insulators Rev. Mod. Phys. 82 3045
[14] Davies J H 1997 The Physics of Low-dimensional Semiconductors (Cambridge: Cambridge University Press)
[15] Krejcirík D, Bila H and Znojil M 2006 Closed formula for the metric in the Hilbert space of a \( PT \)-symmetric model J. Phys. A: Math. Gen. 39 10143
[16] Krejcirík D 2008 Calculation of the metric in the Hilbert space of a \( PT \)-symmetric model via the spectral theorem J. Phys. A: Math. Theor. 41 244012
[17] Krejcirík D and Siegl P 2010 \( PT \)-symmetric models in curved manifolds J. Phys. A: Math. Theor. 43 485204
[18] Hernandez-Coronado H, Krejcirik D and Siegel P 2011 Perfect transmission scattering as a $\mathcal{PT}$ symmetric spectral problem Phys. Lett. A 375 2149
[19] Krejcirik D, Siegel P and Zelezny J 2014 On the similarity of Sturm–Liouville operators with non-Hermitian boundary conditions to self-adjoint and normal operators Complex Anal. Oper. Theory 8 255
[20] Dasarthy A, Isaacson J P, Jones-Smith K, Tabachnik J and Mathur H 2013 The particle in a box in $\mathcal{PT}$ quantum mechanics and an electromagnetic analog Phys. Rev. A 87 062111
[21] Bender C M, Dunne G V and Meisinger P N 1999 Complex periodic potentials with real band spectra Phys. Lett. A 252 272
[22] Jones H F 1999 The energy spectrum of complex periodic potentials of the Kronig–Penney type Phys. Lett. A 262 242
[23] Ahmed Z 2001 Energy band structure due to a complex periodic $\mathcal{PT}$ invariant potential Phys. Lett. A 286 231
[24] Makris K G, El-Ganainy R and Christodoulides D N 2008 Beam dynamics in $\mathcal{PT}$ symmetric optical lattices Phys. Rev. Lett. 100 103904
[25] Shapere A D, Wilczek F and Xiong Z 2012 Models of topology change (arXiv:1210.3545)
[26] Mostafazadeh A 2006 Delta-function potential with a complex coupling J. Phys. A: Math. Gen. 39 13495
[27] Zezyulin D A and Konotop V V 2016 Nonlinear currents in a ring-shaped waveguide with balanced gain and dissipation Phys. Rev. A 94 043853
[28] Stone M and Goldbart P M 2009 Mathematics for Physics: a Guided Tour for Graduate Students (Cambridge: Cambridge University Press)
[29] Lieb E H and Liniger W 1963 Exact analysis of an interacting Bose gas. I. The general solution and the ground state Phys. Rev. 130 1605
[30] Mattis D 1993 The Many Body Problem. An Encyclopedia of Exactly Solved Models in One Dimension (Singapore: World Scientific)
[31] Kinoshita T, Wenger T and Weiss D S 2004 Observation of a one-dimensional Tonks–Girardeau gas Science 305 1125
[32] Paredes B et al 2004 Tonks–Girardeau gas of ultracold atoms in an optical lattice Nature 429 277
[33] Bloch I, Dalibard J and Zwerger W 2008 Many-body physics with ultracold gases Rev. Mod. Phys. 80 885