UNIFORM-IN-TIME CONTINUUM LIMIT OF THE LATTICE WINFREE MODEL AND EMERGENT DYNAMICS

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Abstract. We study a uniform-in-time continuum limit of the lattice Winfree model (LWM) and its asymptotic dynamics which depends on system functions such as natural frequency function and coupling strength function. The continuum Winfree model (CWM) is an integro-differential equation for the temporal evolution of Winfree phase field. The LWM describes synchronous behavior of weakly coupled Winfree oscillators on a lattice lying in a compact region. For bounded measurable initial phase field, we establish a global well-posedness of classical solutions to the CWM under suitable assumptions on coupling function, and we also show that a classical solution to the CWM can be obtained as a $L^1$-limit of a sequence of lattice solutions. Moreover, in the presence of frustration effect, we show that stationary states and bump states can emerge from some admissible class of initial data in a large and intermediate coupling regimes, respectively. We also provide several numerical examples and compare them with analytical results.

1. Introduction. Collective behaviors are often observed in complex systems in nature, for example, aggregation of bacteria, flocking of birds, herding of sheep and synchronous flashing of fireflies, [2, 3, 4, 5, 8, 11, 14, 12, 32, 46, 49, 54] etc. Recently, research on such collective coherent motions has received much attention from many scientific fields, such as applied mathematics, biology, computer science, statistical physics, and control theory, because of their diverse applications to the decentralized control of unmanned aerial vehicles [6, 16, 31, 46, 52]. Among them, we are mainly interested in the synchronization phenomenon arising from oscillatory systems e.g. ensembles of pacemaker cells, fireflies and neurons, etc. The rigorous mathematical study was first initiated a half century ago by two pioneers, Winfree
The Winfree model was the first mathematical model for synchronization, but compared to the vast literature on the Kuramoto model \[9, 10, 14, 15, 25, 24, 22, 28, 23, 29, 32, 33\], research on the Winfree model are relatively meager \[4, 17, 26, 27, 30, 45, 50, 51\]. This is mainly due to the lack of conservation laws that causes lots of mathematical difficulties in analyzing the dynamics of the model. However, the lack of conservation laws makes the model exhibit more diverse asymptotic patterns such as partial and complete oscillator death states, partial and complete locked states, chimera like states and incoherent states, etc.

On the other hand, coupled dynamical systems on graphs arise from diverse mathematical modeling in the realms of sciences and technology, to name a few, regulatory and neuronal networks in biology \[7, 36, 43, 48\], Josephson junctions and coupled lasers in physics \[37, 47, 53\], communication, sensor and power networks in technology \[13, 39\], etc. The continuum limit of nonlocally coupled dynamical networks is one of several analytical approaches with the potential to explain the dynamics of a broad class of networks. This limiting procedure has been used to study chimera states \[1, 34\], multistability \[18, 55\], synchronization, and the coherence-incoherence transition \[44\]. A rigorous mathematical justification for the continuum limit of the Kuramoto model on networks with increasing number of nodes has been studied by combining the theory of evolution equations and graph limit \[40, 41, 42\]. Recently, asymptotic stability analysis of the continuum Kuramoto model has been studied in \[38\] using Lojasiewicz inequality and gradient flow approach. To put the LWM in a proper framework, we begin with a suitable set of jargons.

Let \(\Omega \subset \mathbb{R}^d\) be a compact region. For some \(l > 0\) and \(b \in \mathbb{R}^d\), let \(\Gamma := \Omega \cap \left( (\mathbb{L}Z)^n + b \right)\), which is a set of all lattice points inside \(\Omega\). Since \(\Gamma\) is a finite set, it can be represented by using a finite index set \(\Lambda\), which is a collection of lattice indices. In other words, we can rewrite \(\Gamma\) as \(\{x_\alpha\}_{\alpha \in \Lambda}\). For a collection \(\{\theta_\alpha\}_{\alpha \in \Lambda}\) of lattice points in \(\Omega\), let \(\theta_\alpha(t) = \theta_\alpha(t)\) be the time-dependent phase of Winfree oscillator located at the lattice point \(x_\alpha\). Then, its dynamics is governed by the Cauchy problem to the first-order ordinary differential equations:

\[
\begin{align*}
\dot{\theta}_\alpha &= \nu_\alpha + \frac{1}{|\Lambda|} \sum_{\beta \in \Lambda} \kappa_{\alpha\beta} S(\theta_\alpha) I(\theta_\beta), \quad t > 0, \\
\theta_\alpha(0) &= \theta^{in}_\alpha, \quad \alpha \in \Lambda,
\end{align*}
\]

(1)

where \(\nu_\alpha\) and \(\kappa_{\alpha\beta}\) represent the natural frequency of the Winfree oscillator at \(x_\alpha\) and the coupling strength from oscillator located at \(x_\beta\) to oscillator located at \(x_\alpha\), respectively. Let \(S = S(\theta)\) and \(I = I(\theta)\) denote the sensitivity and influence function measuring the synchronous mechanisms, respectively (see Section 2 for their structures).

For the CWM, let \(\theta = \theta(x,t)\) be the Winfree phase field on a compact region \(\Omega \subset \mathbb{R}^d\). Then, the temporal evolution of the phase field is governed by the Cauchy problem to the following integro-differential equation:

\[
\begin{align*}
\frac{\partial \theta}{\partial t}(x,t) &= \mathcal{V}(x) + \frac{1}{\mu(\Omega)} \int_{\Omega} \kappa(x,y) S(\theta(x,t)) I(\theta(y,t)) \mu(dy), \quad (x,t) \in \Omega \times \mathbb{R}_+, \\
\theta(x,0) &= \theta^{in}(x), \quad x \in \Omega,
\end{align*}
\]

(2)
where $V = V(x)$ and $\kappa = \kappa(x, y)$ are measurable natural frequency and coupling functions, and $\mu$ is the Lebesgue measure in $\mathbb{R}^d$. Moreover, we assume that the region $\Omega$ with positive measure is compact and has measure zero boundary:

$$
\mu(\Omega) > 0 \quad \text{and} \quad \mu(\partial \Omega) = 0.
$$

In this paper, we are interested in following two questions:

- **(Q1):** Can we rigorously justify the continuum model (2) from the lattice model (1) in a suitable sense?
- **(Q2):** If so, under what condition does the continuum model (2) exhibit emergent behaviors?

Next, we briefly discuss our main results. First, we establish a global well-posedness of a classical solution to (2) under suitable structural assumptions on $S$ and $I$, for a sufficiently large coupling and bounded measurable natural frequency function. For a global existence, we construct a local classical solution using Banach’s contraction mapping principle on a suitable complete metric space and extend the local solution to a global one using continuity argument (see Theorem 3.3). The uniqueness of a solution follows from Banach’s contraction mapping principle as well.

Second, we show that a unique global classical solution $\theta$ to CWM (2) can be obtained as an $L^1$-limit of a sequence of patched lattice solutions to LWM (1). Let $(\theta^N)_{N \in \mathbb{N}}$ be the aforementioned sequence. Then, distance between $\theta$ and $\theta^N$ can be controlled by the distance between their initial states and authors use this property to show that this limiting process can be also obtained *uniformly in time* under suitable assumptions on initial datum and sufficiently large coupling strength function. We call this process as a continuum limit (see Definition 4.1 and Theorem 4.6 for details). A standard brute force argument will provide a continuum limit in any finite time interval for any initial datum. This is a kind of trade-off between finite-in-time continuum limit and uniform-in-time continuum limit.

Lastly, we provide sufficient frameworks leading to the stationary profile and bump states in sufficiently large coupling strength regime and intermediate coupling strength regime, respectively. A bump state is a mixed hybrid state consisting of phase-locked state and running state. It can be characterized by pointwise rotation number $\rho(x) := \lim_{t \to \infty} \frac{\theta(x,t)}{t}$ if it exists, i.e., for a bump state, there exists a region $U \subset \Omega$ such that

$$
\rho \equiv 0 \quad \text{a.e. in } U \quad \text{and} \quad \inf_{\Omega \setminus U} \rho(x) > 0.
$$

The detailed statements and conditions can be found in Propositions 2 and 3, respectively.

The rest of this paper is organized as follows. In Section 2, we briefly review previous results on the emergent dynamics of LWM (1) and present structural conditions on $S$ and $I$ appearing in the coupling dynamics of LWM (1). In Section 3, we study a global well-posedness of CWM (2). In Section 4, we show that a classical solution to CWM (2) can be obtained as a $L^1$-limit of a sequence of solutions to LWM (1). In Section 5, we study emergent dynamics of CWM with heterogeneous frustrations leading to the stationary asymptotic profile and bump state under some sufficient frameworks formulated in terms of the coupling strength function, the natural frequency function and initial data. In Section 6, we provide several numerical examples for the uniform stability estimate of CWM and formation of stationary
profile and bump states. Finally, Section 7 is devoted to a brief summary of our main results and some remaining issues for a future work.

**Notation:** For any measurable functions $f$ and $F$ defined on $\Omega$ and $\Omega^2 = \Omega \times \Omega$ in $L^p(\Omega)$ and $L^p(\Omega^2)$, we set

$$\|f\|_p := \|f\|_{L^p(\Omega)}, \quad \|F\|_p := \|F\|_{L^p(\Omega^2)}, \quad 1 \leq p \leq \infty.$$  

For simplicity, we set

$$\sup_{\Omega} f(x) := \sup_{x \in \Omega} f(x), \quad \inf_{\Omega} f(x) := \inf_{x \in \Omega} f(x).$$

2. **Preliminaries.** In this section, we recall several distinguished asymptotic states of LWM in terms of rotation numbers and recall previous results on the LWM which is exactly the same as the particle Winfree model. Recall the Cauchy problem to the LWM:

$$\begin{cases}
\dot{\theta}_\alpha = \nu_\alpha + \frac{1}{|\Lambda|} \sum_{\beta \in \Lambda} \kappa_{\alpha\beta} S(\theta_\alpha) I(\theta_\beta), & t > 0, \\
\theta_\alpha(0) = \theta_\alpha^{in}, & \alpha \in \Lambda,
\end{cases} \quad (3)$$

where $\kappa_{\alpha\beta}$ is a nonnegative coupling strength from oscillator located at $x_\beta$ to oscillator located at $x_\alpha$. Note that when the mutual couplings are turned off, i.e., $\kappa_{\alpha\beta} \equiv 0$, phase dynamics is given by a simple linear motion:

$$\theta_\alpha(t) = \theta_\alpha^{in} + \nu_\alpha t, \quad \alpha \in \Lambda. \quad (4)$$

As noticed in [19, 26, 27, 20, 21, 30, 45], the description of asymptotic states can be effectively described by “rotation number” $\rho_\alpha$ which corresponds to the asymptotic phase velocity of $\theta_\alpha$; for $\alpha \in \Lambda$,

$$\rho_\alpha := \lim_{t \to \infty} \frac{\theta_\alpha(t)}{t}, \quad \text{if R.H.S. exists.}$$

In the absence of mutual couplings, relation (4) yields

$$\rho_\beta - \rho_\alpha = \lim_{t \to \infty} \frac{\theta_\beta(t) - \theta_\alpha(t)}{t} = \nu_\beta - \nu_\alpha.$$  

Thus, mutual entrainment for two oscillators with different natural frequencies can not emerge. This is why we need a sufficiently large coupling to ensure the mutual entrainments between oscillators. Next, we introduce two distinguished asymptotic states, “complete oscillator death (COD) state” and “partial oscillator death (POD) state” as follows:

**Definition 2.1.** [26, 27] Let $\Theta := \{\theta_\alpha\}_{\alpha \in \Lambda}$ be the phase configuration whose dynamics is governed by (3).

1. The phase configuration $\Theta$ tends to “complete oscillator death” state, if the rotation numbers of all oscillators are zero:

$$|\{\alpha \in \Lambda : \rho_\alpha = 0\}| = |\Lambda|,$$

where $|A|$ is the cardinality of a set $A$.

2. The phase configuration $\Theta$ tends to “partial oscillator death” state, if at least two rotation numbers are zero:

$$2 \leq |\{\alpha \in \Lambda : \rho_\alpha = 0\}| < |\Lambda|. $$
In what follows, we present structural assumptions on \( S \) and \( I \) leading to POD and COD states in Definition 2.1.

- (A1): The sensitivity function \( S \) and the influence function \( I \) satisfy
\[
S \in C^2(\mathbb{R}), \quad S(\theta + 2\pi) = S(\theta), \quad S(-\theta) = -S(\theta), \\
I \in C^2(\mathbb{R}), \quad I(\theta + 2\pi) = I(\theta), \quad I(-\theta) = I(\theta), \quad \theta \in \mathbb{R}.
\] (5)

- (A2): There exist positive constants \( \theta_* \) and \( \theta^* \), satisfying
\[
0 < \theta_* < \theta^* < 2\pi
\]
and
\[
S \leq 0 \quad \text{on } [0, \theta_*], \quad S' \leq 0, \quad S'' \geq 0 \quad \text{on } [0, \theta_*], \\
I \geq 0, \quad I' \leq 0 \quad \text{on } [0, \theta_*], \quad I'' \leq 0 \quad \text{on } [0, \theta_*], \\
(SI)' < 0 \quad \text{on } (0, \theta_*), \quad (SI)' > 0 \quad \text{on } (\theta_*, \theta^*).
\] (6)

Since \( S \) and \( I \) are \( C^2 \) and \( 2\pi \)-periodic, \( S^{(k)} \) and \( I^{(k)} \) are bounded for \( k = 0, 1, 2 \). Hence, we can define
\[
\|S^{(k)}\|_{\infty} := \|S^{(k)}\|_{L^\infty(\mathbb{R})} \quad \text{and} \quad \|I^{(k)}\|_{\infty} := \|I^{(k)}\|_{L^\infty(\mathbb{R})}.
\]

Note that the following special pair \((S, I)\) in [4]:
\[
S(\theta) = -\sin \theta, \quad I(\theta) = 1 + \cos \theta,
\] (7)

satisfies the structural conditions (5) and (6) with \( \theta^* = \pi \) and \( \theta_* = \frac{\pi}{3} \), (see Figure 1). In fact, the structural conditions (5) and (6) were introduced in [26, 27] motivated by the special pair (7) and under these conditions, emergent behaviors of (3) with \( \kappa_{\alpha \beta} = \kappa \) has been studied in aforementioned literature.

For \( \bar{\theta} \in (0, \theta^*) \), we consider the following equation on the interval \([0, \theta_*]\):
\[
(SI)(x) = (SI)(\bar{\theta}), \quad x \in [0, \theta_*].
\] (8)
Note that conditions (5) and (6) yield the following geometric shape of the coupling function $SI$ (see Figure 2):

\begin{align*}
(SI)(0) &= 0, \quad \theta^* := \arg\min_{0 \leq \theta \leq \theta^*} (SI)(\theta), \\
(SI)(\theta) &< 0 \quad \text{on} \quad \theta \in (0, \theta^*), \quad (SI)(\theta^*) \leq 0.
\end{align*}

Thus, equation (8) has a unique solution $\theta^\infty$ that is guaranteed by the relation (9).

One can see the example for the solution of (8) with special pair (7) in Figure 2. Below, we quote the result on the emergence of the POD and COD states in [27, 30] without proofs. For this, we let $\Lambda_s$ be a proper subset of $\Lambda$ and $V_s := \{ \nu_\alpha : \alpha \in \Lambda_s \}$.

**Theorem 2.2.** [27, 30] Suppose that a phase bound $\bar{\theta}$ and initial phase configuration $\Theta^{in}$ satisfy

\begin{align*}
\bar{\theta}^{in} &\in (-\bar{\theta}, \bar{\theta}), \quad \forall \alpha \in \Lambda_s,
\end{align*}

for some $\bar{\theta} \in (0, \theta^*)$. Then, there exists a coupling strength $\kappa_c = \kappa_c(V_s, \bar{\theta}^{\infty}, |\Lambda_s|) > 0$ such that if $\kappa > \kappa_c$, we have

\begin{align*}
\rho_\alpha &= 0, \quad \forall \alpha \in \Lambda_s.
\end{align*}

**Remark 1.** (1) In [26, 27], the emergence of POD states has been studied on a locally coupled networks. In [20, 21], the robustness of the COD state has been dealt in a general network under the time-delayed interactions, and the interplay of adaptive coupling and random effect, respectively. Recently, the authors have studied the emergence of POD and COD states under the effect of heterogeneous frustrations in [19].

(2) For the special pair of $(S, I) = (-\sin \theta, 1 + \cos \theta)$, since $\theta^* = \pi$, one can set $\bar{\theta}$ to be any constants in $(0, \pi)$. Hence, for generic initial data on $(-\pi, \pi)$, one can find the threshold of coupling strength $\kappa_c$ for COD states.

Lastly, we discuss the emergence of a “bump state”, investigated in [19], for the homogeneous Winfree model under heterogeneous frustrations effect (10). We first recall a concept of bump states as follows.
Definition 2.3. [19] Let \( \Theta := \{ \theta_\alpha \}_{\alpha \in \Lambda} \) be the phase configuration whose dynamics is governed by
\[
\begin{cases}
\dot{\theta}_\alpha = \nu + \frac{1}{|\Lambda|} \sum_{\beta \in \Lambda} \kappa_{\alpha\beta} S(\theta_\alpha) I(\theta_\beta + h_{\alpha\beta}), & t > 0, \\
\theta_\alpha(0) = \theta_\alpha^i, & \alpha \in \Lambda,
\end{cases}
\]
(10)
where \( h_{\alpha\beta} \) is the heterogeneous frustration from oscillator located at \( x_\beta \) to oscillator located at \( x_\alpha \). Then the phase configuration \( \Theta \) tends to a “bump state”, if there exists \( \Lambda_s \) and \( \delta > 0 \) such that
\[
\rho_\alpha = 0, \quad \forall \, \alpha \in \Lambda_s \quad \text{and} \quad \liminf_{t \to \infty} \dot{\theta}_\alpha(t) \geq \delta, \quad \forall \, \alpha \in \Lambda \setminus \Lambda_s.
\]

As can be seen from the definition, bump states are special cases of POD states. We remind the result on bump states in [19] as follows.

Theorem 2.4. [19] Suppose that a phase bound \( \overline{\theta}, \overline{\theta} \) and initial configuration \( \Theta^i \) satisfy
\[
\overline{\theta} \in (0, \theta^*), \quad \theta_\alpha^i \in (-\overline{\theta}, \overline{\theta}), \quad \forall \, \alpha \in \Lambda_s.
\]
Then, there exist a constant \( \delta > 0 \) and coupling strengths \( \kappa_c^l, \kappa_u^l > 0 \) depending on \( \nu, \overline{\theta} \) and \( |\Lambda_s| \) such that if \( \kappa_c^l < \kappa < \kappa_u^l \), we have
\[
\rho_\alpha = 0, \quad \forall \, \alpha \in \Lambda_s \quad \text{and} \quad \liminf_{t \to \infty} \dot{\theta}_\alpha(t) \geq \delta, \quad \forall \, \alpha \in \Lambda \setminus \Lambda_s.
\]

3. A global well-posedness of the CWM. In this section, we study a global existence of classical solutions to the CWM (2). First, we introduce a concept of classical solution to the integro-differential equation (2):

Definition 3.1. For \( T \in (0, \infty) \), let \( \theta = \theta(x,t) \) be a classical solution to the Cauchy problem (2) in the time-interval \( [0, T) \), if the following conditions hold.

1. The phase function \( \theta \) satisfies the following regularity condition:
\[
\theta \in C^1([0, T); L^\infty(\Omega)).
\]

2. For all \( t \in [0, T) \) and a.e. \( x \in \Omega \), the phase function \( \theta \) satisfies the following integro-differential equation pointwisely:
\[
\partial_t \theta(x,t) = \nu(x) + \frac{1}{\mu(\Omega)} \int_{\Omega} \kappa(x, y) S(\theta(x, t)) I(\theta(y, t)) \mu(dy).
\]

3. For a.e. \( x \in \Omega \),
\[
\theta(x, 0) = \theta^i(x).
\]

Next, we show that the Cauchy problem (2) admits a unique global classical solution in the sense of Definition 3.1 using the mild formulation of (2) and a contraction mapping principle on a suitable complete metric space. Similar arguments for the continuum Kuramoto model can be found in [38, 41]. First, we note that equation (2) can be rewritten as a mild form:
\[
\theta(x, t) = \theta^i(x) + t \nu(x) + \frac{1}{\mu(\Omega)} \int_0^t \int_{\Omega} \kappa(x, y) S(\theta(x, s)) I(\theta(y, s)) \mu(dy) ds.
\] (11)
Then, it is easy to see that the integral relation (11) satisfies the initial datum. Also, RHS of (11) is \( C^1 \)-regular in time \( t \) if \( \theta \) is just continuous. Hence, it suffices to show that the integral equation (11) has a solution using the contraction mapping principle on a suitably defined complete metric space. For our discussion in a proper
setting, we define a closed subspace \( \mathcal{M} = \mathcal{M}_{\tau, \theta_1} \) of \( C^1([0, \tau); L^\infty(\Omega)) \) parametrized by \( \tau \) and initial datum:

\[
\mathcal{M}_{\tau, \theta_1} := \{ \theta \in C^1([0, \tau); L^\infty(\Omega)) : \theta(x, 0) = \theta_1(x), \ a.e. \ x \in \Omega \},
\]

where \( \tau \) is chosen to satisfy

\[
0 < \tau < \frac{\mu(\Omega)}{(\sup_{\Omega} \| \kappa(x, \cdot) \|_1) \cdot (\| S \|_\infty \| I \|_\infty + \| S' \|_\infty \| I \|_\infty)},
\]

under the assumption that

\[
\sup_{\Omega} \| \kappa(x, \cdot) \|_1 < \infty,
\]

and the nonlinear map \( \mathcal{L} \) on \( \mathcal{M} \) motivated by the mild form of (11) and its norm:

\[
\begin{align*}
\mathcal{L}[\theta](x, t) & := \theta_1(x) + t \mathcal{V}(x) + \frac{1}{\mu(\Omega)} \int_{0}^{t} \int_{\Omega} \kappa(x, y) \mathcal{S}(\theta(x, s)) \mathcal{I}(\theta(y, s)) \mu(dy) ds, \\
\| \theta \|_{\mathcal{M}} & := \sup_{0 \leq t < \tau} \| \theta(\cdot, t) \|_\infty.
\end{align*}
\]

Then, it is easy to see that \( \mathcal{L}[\theta] \in \mathcal{M} \), if both \( \theta_1 \) and \( \mathcal{V} \) are contained in \( L^\infty(\Omega) \) and that the solution \( \theta \) as a mapping from \( \mathbb{R} \) to \( L^\infty(\Omega) \) corresponds to the fixed point of the nonlinear map \( \mathcal{L} \):

\[
\theta = \mathcal{L}[\theta].
\]

Next, we show that \( \mathcal{L} \) is a contraction mapping from \( \mathcal{M} \) to \( \mathcal{M} \).

**Lemma 3.2.** Suppose that structural condition (A1) holds, and the coupling strength function \( \kappa = \kappa(\cdot, \cdot) \) satisfies

\[
\sup_{\Omega} \| \kappa(x, \cdot) \|_1 < \infty,
\]

and let \( \theta_1 \) and \( \theta_2 \) be two functions in \( \mathcal{M} \). Then the nonlinear operator \( \mathcal{L} \) is contractive: there exists a constant \( C \in (0, 1) \) independent of \( t \in [0, \tau] \) such that

\[
\| \mathcal{L}[\theta_1] - \mathcal{L}[\theta_2] \|_{\mathcal{M}} \leq C \| \theta_1 - \theta_2 \|_{\mathcal{M}}.
\]

**Proof.** By definition of \( \mathcal{L} \), for \( t \in [0, \tau] \), one has

\[
\begin{align*}
\mu(\Omega) \| \mathcal{L}[\theta_1](\cdot, t) - \mathcal{L}[\theta_2](\cdot, t) \|_{\infty} & \leq \sup_{\Omega} \int_{0}^{t} \int_{\Omega} \kappa(x, y) \mathcal{S}(\theta_1(x, s)) \mathcal{I}(\theta_1(y, s)) - \mathcal{S}(\theta_2(x, s)) \mathcal{I}(\theta_2(y, s)) \mu(dy) ds \\
& \leq \sup_{\Omega} \int_{0}^{t} \int_{\Omega} \kappa(x, y) \mathcal{S}(\theta_1(x, s)) \cdot \left| \mathcal{I}(\theta_1(y, s)) - \mathcal{I}(\theta_2(y, s)) \right| \mu(dy) ds \\
& \quad + \sup_{\Omega} \int_{0}^{t} \int_{\Omega} \kappa(x, y) \mathcal{S}(\theta_1(x, s)) - \mathcal{S}(\theta_2(x, s)) \cdot \left| \mathcal{I}(\theta_2(y, s)) \right| \mu(dy) ds \\
& \leq \left( \sup_{\Omega} \| \kappa(x, \cdot) \|_1 \right) \cdot \left( \| S \|_\infty \| I \|_\infty + \| S' \|_\infty \| I \|_\infty \right) \int_{0}^{t} \| \theta_1(\cdot, s) - \theta_2(\cdot, s) \|_{\infty} ds \\
& \leq \tau \left( \sup_{\Omega} \| \kappa(x, \cdot) \|_1 \right) \cdot \left( \| S \|_\infty \| I \|_\infty + \| S' \|_\infty \| I \|_\infty \right) \cdot \left( \sup_{0 \leq t < \tau} \| \theta_1(\cdot, t) - \theta_2(\cdot, t) \|_{\infty} \right) \\
& = C \mu(\Omega) \| \theta_1 - \theta_2 \|_{\mathcal{M}},
\end{align*}
\]

(13)
where we used mean-value theorem, (12), and the constant $C$ defined as follows:

$$
C := \tau \left( \sup_{\Omega} \| \kappa(x, \cdot) \|_1 \right) \cdot \left( \| S \|_\infty \| I' \|_\infty + \| S' \|_\infty \| I \|_\infty \right) < 1.
$$

Hence, relation (13) yields

$$
\| L[\theta_1] - L[\theta_2] \|_M \leq C \| \theta_1 - \theta_2 \|_M.
$$

Now, we are ready to provide a global existence of classical solution to the CWM (2).

**Theorem 3.3.** Suppose that structural condition (A1) holds, and system functions $\kappa$ and $\nu$ satisfy

$$
\sup_{\Omega} \| \kappa(x, \cdot) \|_1 < \infty \quad \text{and} \quad \| \nu \|_\infty < \infty.
$$

Then, there exists a unique global classical solution $\theta(x,t) \in C^1([0, \infty); L^\infty(\Omega))$ to CWM (2).

**Proof.** We use Lemma 3.2 to find a unique fixed point $\theta \in \mathcal{M}$ of $L$, which serves a local classical solution in the time interval $[0, \tau)$. Moreover, we can continue our local solution $\theta$ in $[0, \tau)$ to the time strip $[\tau, 2\tau)$ with a new initial datum $\theta(\cdot, \tau)$. In this way, we can extend a local solution to any time interval. Detailed arguments can be found in [38, 41].

**Remark 2.** Note that

$$
|\theta(x,t+h) - \theta(x,t)| \leq \| \nu \|_\infty h + \frac{1}{\mu(\Omega)} \int_t^{t+h} \int_{\Omega} \kappa(x,y)S(\theta(x,s))I(\theta(y,s))\mu(dy)ds
$$

$$
\leq h \left[ \| \nu \|_\infty + \frac{1}{\mu(\Omega)} \left( \sup_{\Omega} \| \kappa(x, \cdot) \|_1 \right) \cdot \| S \|_\infty \cdot \| I \|_\infty \right].
$$

This implies that for fixed $x \in \Omega$, $\theta(x,t)$ is continuous in time variable $t$.

4. **From lattice model to continuum model.** In this section, we present a convergence from the lattice model to the continuum model below.

Recall the LWM and CWM:

$$
\begin{aligned}
\dot{\theta}_\alpha &= \nu_\alpha + \frac{1}{|\Lambda|} \sum_{\beta \in \Lambda} \kappa_{\alpha\beta} S(\theta_\alpha)I(\theta_\beta), \quad t > 0, \quad \alpha \in \Lambda, \\
\frac{\partial \theta}{\partial t}(x,t) &= \nu(x) + \frac{1}{\mu(\Omega)} \int_{\Omega} \kappa(x,y)S(\theta(x,t))I(\theta(y,t))\mu(dy), \quad (x,t) \in \Omega \times \mathbb{R}_+.
\end{aligned}
$$

(14)

First, we define a concept of the continuum limit from the lattice model (14) toward the continuum one (14) as follows.

**Definition 4.1.** For any $T \in (0, \infty)$, the continuum model (14) is “derivable” from the lattice model (14) in $[0, T]$ if a solution to (14) can be obtained as a suitable limit of a sequence of lattice solutions, as the spacing of between lattice points tends to zero.
Remark 3. Throughout the rest of paper, we call the above procedure as a continuum limit from the lattice model (14)$_1$ to the continuum model (14)$_2$ in $[0,T)$. Moreover, when this continuum limit is valid for whole time interval $[0,\infty)$, we call it as a uniform-in-time continuum limit.

Throughout the rest of this section, we assume the following structural conditions.

• (A3): For $\mathcal{H} \in (0,\theta^\ast)$, system functions $\kappa$ and $\mathcal{V}$ satisfy

$$\sup_{\Omega} \|\kappa(\cdot, y)\|_1 + \sup_{\Omega} \|\kappa(x, \cdot)\|_1 + \|\mathcal{V}\|_\infty < \infty, \quad \inf_{\Omega} \|\kappa(x, \cdot)\|_1 > \frac{\mu(\Omega)}{(SI)(\theta^\ast)}.$$ (15)

4.1. Preparatory lemmas. In this subsection, we study several basic estimates for the uniform-in-time continuum limit. Since $\Omega \subset \mathbb{R}^d$ is compact, there exists $L > 0$ such that $\Omega \subset [-\frac{L}{2}, \frac{L}{2}]^d$. Then, under suitable conditions, we can identify the lattice point in $\Omega$ with the lattice point of regular lattice which obtained by dividing each side of $[-\frac{L}{2}, \frac{L}{2}]^d$ into $N$ equal parts. More precisely, we define lattice indices in $\Omega$ as follows:

$$I^N_\alpha := \left[\frac{-N + 2(\alpha_1 - 1)}{2N}, \frac{-N + 2\alpha_1}{2N}\right] \times \cdots \times \left[\frac{-N + 2(\alpha_d - 1)}{2N}, \frac{-N + 2\alpha_d}{2N}\right],$$

for $\alpha = (\alpha_1, \cdots, \alpha_d)$. If $\alpha_k = N$, we use a closed interval, which means that we will use

$$\cdots \times \left[\frac{-N + 2(\alpha_k - 1)}{2N}, \frac{-N + 2\alpha_k}{2N}\right] \times \cdots,$$

so that it can contain $L/2$. Then, we define

$$\Omega^N_\alpha := \Omega \cap I^N_\alpha, \quad \alpha \in \Lambda_N := \{\beta \in \mathbb{N}^d : \text{int}(I^N_\beta) \subset \text{int}(\Omega)\}, \quad \overline{N} := \{1, \cdots, N\}.$$

Next, we study preservation of piecewise constancy of the phase field $\theta$.

Lemma 4.2. For a fixed $N \in \mathbb{N}$, suppose the system functions $\kappa, \mathcal{V}$ and the initial datum $\theta^{in}$ are piecewise constant in the sense that

$$\kappa(x, y) = \kappa^N_{\alpha, \beta}, \quad \mathcal{V}(x) = \nu^N_{\alpha}, \quad \theta^{in}(x) = \theta^{N,in}_{\alpha},$$ (16)

for $(x, y) \in \Omega^N_\alpha \times \Omega^N_\beta$, $\alpha, \beta \in \Lambda_N$, and let $\theta = \theta(x, t)$ be the solution of (2) with the initial datum $\theta^{in}$. Then the phase function $\theta$ is piecewise constant in the sense that

$$\theta(x, t) = \theta(z, t), \quad x, z \in \Omega^N_\alpha, \quad t \geq 0, \quad \alpha \in \Lambda_N.$$

Proof. For fixed $x, z \in \Omega^N_\alpha$, we define

$$\zeta(x, z, t) := \theta(x, t) - \theta(z, t).$$

Since (14)$_1$ and (14)$_2$ take the same initial datum (see (16)), one has

$$\zeta(x, z, 0) = 0.$$ (17)
Then, $\zeta$ satisfies
\[
\frac{\partial}{\partial t}(x,z,t) = \frac{1}{\mu(\Omega)} \int_{\Omega} \left[ \kappa(x,y)S(\theta(x,t)) - \kappa(z,y)S(\theta(z,t)) \right] I(\theta(y,t)) \mu(dy)
\]
\[
= \sum_{\beta \in \Lambda_N} \frac{\kappa_{\alpha\beta}}{\mu(\Omega)} \int_{\Omega^\beta} \left[ S(\theta(x,t)) - S(\theta(z,t)) \right] I(\theta(y,t)) \mu(dy)
\]
\[
= \zeta(x,z,t) \times \sum_{\beta \in \Lambda_N} \frac{\kappa_{\alpha\beta}}{\mu(\Omega)} \int_{\Omega^\beta} \tilde{S}(\theta(x,t),\theta(z,t)) I(\theta(y,t)) \mu(dy),
\]
where $\tilde{S}$ is defined by the following relation:
\[
\tilde{S}(\eta,\xi) := \begin{cases} 
S(\eta) - S(\xi) & \eta \neq \xi, \\
S'(\eta) & \eta = \xi.
\end{cases}
\]
Thus, it follows from (17) and (18) that
\[
\zeta(x,z,t) \equiv 0, \quad \forall \ t > 0,
\]
which yields the desired estimate.

In the sequel, for each $N \in \mathbb{N}$, we define a sequence of the Cauchy problem for the LWM:
\[
\begin{cases}
\dot{\theta}^N_\alpha = \nu^N_\alpha + \frac{1}{|\Lambda_N|} \sum_{\beta \in \Lambda_N} \kappa_{\alpha\beta}^N S(\theta^N_\alpha) I(\theta^N_\beta), \quad t > 0, \\
\theta^N_\alpha|_{t=0^+} = \theta^{N,\text{in}}_\alpha, \quad \alpha \in \Lambda_N,
\end{cases}
\]
where the sequences $\{\nu^N_\alpha\}$, $\{\theta^{N,\text{in}}_\alpha\}$ and $\{\kappa_{\alpha\beta}^N\}$ are defined as follows: for $\alpha \in \Lambda_N$,
\[
\nu^N_\alpha := \left( \frac{N}{L} \right)^d \int_{\Omega^\alpha} \nu(x) \mu(dx), \quad \theta^{N,\text{in}}_\alpha := \left( \frac{N}{L} \right)^d \int_{\Omega^\alpha} \theta^{\text{in}}(x) \mu(dx),
\]
\[
\kappa_{\alpha\beta}^N := \left( \frac{N}{L} \right)^{2d} \int_{\Omega^\alpha \times \Omega^\beta} \kappa(x,y) \mu(dx) \mu(dy), \quad \beta \in \Lambda_N.
\]
Now, we set
\[
\nu^N := \sum_{\alpha \in \Lambda_N} \nu^N_\alpha 1_{\Omega^\alpha}, \quad \theta^{N,\text{in}} := \sum_{\alpha \in \Lambda_N} \theta^{N,\text{in}}_\alpha 1_{\Omega^\alpha},
\]
\[
\kappa^N := \sum_{\alpha, \beta \in \Lambda_N} \frac{\mu(\Omega) N^d}{|\Lambda_N| L T} \kappa_{\alpha\beta}^N 1_{\Omega^\alpha} \otimes 1_{\Omega^\beta},
\]
where $1_E$ is the characteristic function of a set $E$. Then, Lemma 4.2 and a simple calculation guarantee that
\[
\theta^N := \sum_{\alpha \in \Lambda_N} \theta^N_\alpha 1_{\Omega^\alpha}
\]
is a solution of
\[
\begin{cases}
\frac{\partial}{\partial t} \theta(x,t) = \nu^N(x) + \frac{1}{\mu(\Omega)} \int_{\Omega} \kappa^N(x,y) S(\theta(x,t)) I(\theta(y,t)) \mu(dy), \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
\theta(x,0) = \theta^{N,\text{in}}(x), \quad x \in \Omega.
\end{cases}
\]
This allows us to compare the solutions of the lattice model (14) and the continuum model (14)\textsuperscript{2}.

In the following lemma, we show that sequences \( \{ V_N \} \), \( \{ \kappa_N \} \) and \( \{ \theta^{N, in} \} \) defined in (21) with (20) converge to \( \kappa \), \( V \), and \( \theta^{in} \), respectively as \( N \to \infty \).

Lemma 4.3. Suppose \( \kappa \) and \( V \) satisfy (15) and let \( V_N \), \( \kappa_N \) and \( \theta^{N, in} \) be sequences of functions given in (21). Then, one has
\[
\lim_{N \to \infty} \| V_N - V \|_1 = \lim_{N \to \infty} \| \kappa_N - \kappa \|_1 = \lim_{N \to \infty} \| \theta^{N, in} - \theta^{in} \|_1 = 0.
\]

Proof. Below, we treat \( L^1 \)-convergence of \( V_N \), \( \theta^{N, in} \) and \( \kappa_N \), separately.

• (\( L^1 \)-convergence of \( V_N \) and \( \theta^{N, in} \)): We first show the \( L^1 \)-convergence of \( V_N \). For \( x \in \text{int}(\Omega) \), we can apply Lebesque differentiation theorem to obtain the convergence of \( V_N(x) \) to \( V(x) \). Moreover, since \( \mu(\partial \Omega) = 0 \), we know that \( V_N \) converges to \( V \) in almost everywhere sense. Furthermore, since
\[
|V_N - V| \leq 2\| V \|_{\infty}, \quad \text{a.e.,}
\]
and \( \Omega \) is compact, one can apply the Lebesgue dominated convergence theorem to obtain the \( L^1 \)-convergence. Similarly, we can show \( L^1 \)-convergence of \( \theta^{N, in} \).

• (\( L^1 \)-convergence of \( \kappa_N \)): By the same argument as in the case of \( V_N \) and \( \theta^{N, in} \), it suffices to show
\[
\lim_{N \to \infty} \frac{\mu(\Omega)}{|\Lambda_N|^d} = 1,
\]
to verify the \( L^1 \)-convergence of \( \kappa_N \). For this, we use \( \mu(\partial \Omega) = 0 \) to obtain
\[
\lim_{N \to \infty} |\Lambda_N| \left( \frac{L}{N} \right)^d = \lim_{N \to \infty} |\Lambda_N| \mu(\Omega_N) = \lim_{N \to \infty} \mu \left( \bigcup_{\alpha \in \Lambda_N} \Omega_N^{\alpha} \right) = \mu(\text{int}(\Omega)) = \mu(\Omega).
\]
Hence, we can get the desired result. \( \square \)

Next, we show that a solution to the CWM (2) is uniformly bounded.

Lemma 4.4. Suppose that structural conditions (A1) – (A3) hold, and let \( \theta \) be a solution of (2) with the initial datum \( \theta^{in} \):
\[
\| \theta^{in} \|_{\infty} < \tilde{\theta}^\infty < \theta^*.
\]
Then, \( \theta \) is bounded by the same constant \( \tilde{\theta}^\infty \):
\[
\sup_{0 \leq t < \infty} \| \theta(\cdot, t) \|_{\infty} \leq \tilde{\theta}^\infty.
\]

Proof. Suppose there exists \( T > 0 \) such that
\[
\| \theta(\cdot, T) \|_{\infty} > \tilde{\theta}^\infty.
\]
Then, if follows from the definition of essential supremum that
\[
\mu(\{ x \in \Omega : |\theta(x, T)| > \tilde{\theta}^\infty \}) \neq 0.
\]
Since \( \theta^{in} \) satisfies
\[
\mu(\{ x \in \Omega : |\theta^{in}(x)| > \tilde{\theta}^\infty \}) = 0,
\]
there exists \( x_0 \in \Omega \) such that
\[
|\theta(x_0, T)| > \tilde{\theta}^\infty \quad \text{but} \quad |\theta^{in}(x_0)| \leq \tilde{\theta}^\infty.
\]
By Remark 2, there exists \( t_0 \in [0, T) \) such that 
\[
|\theta(x_0, t_0)| = \bar{\theta}^\infty \quad \text{and} \quad \left. \frac{\partial}{\partial t} \right|_{t=t_0} |\theta(x_0, t)| \geq 0.
\]

We set
\[
t_1 := \min \left\{ t_0 \in [0, T) : |\theta(x_0, t_0)| = \bar{\theta}^\infty, \left. \frac{\partial}{\partial t} \right|_{t=t_0} |\theta(x_0, t)| \geq 0, x_0 \in \Omega \right\}. \tag{23}
\]

Then, since \( S \) is odd and decrease on \([0, \bar{\theta})\), we have
\[
\left. \frac{\partial}{\partial t} \right|_{t=t_1} |\theta(x_0, t)| = \text{sgn}(\theta(x_0, t)) \left[ \mathcal{V}(x_0) + \frac{S(\theta(x_0, t))}{\mu(\Omega)} \int_\Omega \kappa(x_0, y) I(\theta(y, t_1)) \mu(dy) \right]
\leq \|\mathcal{V}\| + \frac{S(\bar{\theta}^\infty)}{\mu(\Omega)} \int_\Omega \kappa(x_0, y) I(\theta(y, t_1)) \mu(dy)
\leq \|\mathcal{V}\| + \frac{(SI)(\bar{\theta}^\infty)}{\mu(\Omega)} \cdot \inf_\Omega \|\kappa(x, \cdot)\|_1 < 0.
\]

This leads to contradiction to the definition (23), and we conclude the desired result. \( \square \)

**Lemma 4.5.** Suppose \( S \) and \( I \) satisfy the structural conditions (A1) and (A2) and let \( \theta \) and \( \bar{\theta} \) be two global classical solutions to (2) with \((\mathcal{V}, \kappa, \bar{\theta}^\infty)\) and \((\bar{\mathcal{V}}, \bar{\kappa}, \hat{\bar{\theta}}^\infty)\) satisfying conditions (A3) and (22), respectively. Then, for \( t \geq 0 \),
\[
\frac{d}{dt} \|\theta(t) - \bar{\theta}(t)\|_1 \leq \frac{\mu(\Omega)}{\|\mathcal{V}\|^2} \|\mathcal{V} - \bar{\mathcal{V}}\|_1 + \frac{\|S\|^2}{\|\mathcal{V}\|} \|\kappa - \bar{\kappa}\|_1
+ \frac{(SI)(\bar{\theta}^\infty)}{\mu(\Omega)} \inf_\Omega \|\bar{\kappa}(\cdot, \cdot)\|_1 \cdot \frac{(SI)(\hat{\bar{\theta}}^\infty)}{\mu(\Omega)} \sup_\Omega \|\bar{\kappa}(\cdot, \cdot)\|_1 \|\theta(t) - \bar{\theta}(t)\|_1.
\]

**Proof.** Let \( \theta \) and \( \bar{\theta} \) be two solutions to (2) with \((\theta^\infty, \mathcal{V}, \kappa)\) and \((\hat{\theta}^\infty, \hat{\mathcal{V}}, \hat{\kappa})\), respectively:
\[
\begin{cases}
\partial_t \theta = \mathcal{V} + \frac{1}{\mu(\Omega)} \int_\Omega \kappa(x, y) S(\theta(x, t)) I(\theta(y, t)) \mu(dy), \quad (x, t) \in \Omega \times \mathbb{R}_+,
\theta(x, 0) = \theta^\infty(x), \quad x \in \Omega,
\end{cases} \tag{24}
\]
and
\[
\begin{cases}
\partial_t \bar{\theta} = \hat{\mathcal{V}} + \frac{1}{\mu(\Omega)} \int_\Omega \bar{\kappa}(x, y) S(\bar{\theta}(x, t)) I(\bar{\theta}(y, t)) \mu(dy), \quad (x, t) \in \Omega \times \mathbb{R}_+,
\bar{\theta}(x, 0) = \hat{\theta}^\infty(x), \quad x \in \Omega.
\end{cases} \tag{25}
\]

Then, by Lemma 4.4, we know that
\[
\sup_{0 \leq t < \infty} \|\theta(t)\|_\infty \leq \bar{\theta}^\infty \quad \text{and} \quad \sup_{0 \leq t < \infty} \|\bar{\theta}(t)\|_\infty \leq \hat{\bar{\theta}}^\infty.
\]
Now, we subtract (25) from (24) to get
\[
\partial_t (\theta - \tilde{\theta}) = \mathcal{V} - \tilde{\mathcal{V}} + \frac{S(\theta)}{\mu(\Omega)} \int_{\Omega} [\kappa(x, y) - \tilde{\kappa}(x, y)] I(\theta(y, t)) \mu(dy)
+ \frac{S(\theta) - S(\tilde{\theta})}{\mu(\Omega)} \int_{\Omega} \tilde{\kappa}(x, y) I(\theta(y, t)) \mu(dy)
+ \frac{S(\tilde{\theta})}{\mu(\Omega)} \int_{\Omega} \tilde{\kappa}(x, y) [I(\theta(y, t)) - I(\tilde{\theta}(y, t))] \mu(dy).
\]

We multiply both sides of (26) by \( \text{sgn}(\theta - \tilde{\theta}) \) and integrate the resulting relation over \( \Omega \) to obtain, for \( t \geq 0 \),
\[
\frac{d}{dt} \| \theta(\cdot, t) - \tilde{\theta}(\cdot, t) \|_1
= \int_{\Omega} \text{sgn}(\theta(x, t) - \tilde{\theta}(x, t)) [\mathcal{V}(x) - \tilde{\mathcal{V}}(x)] \mu(dx)
+ \frac{1}{\mu(\Omega)} \int_{\Omega} \int_{\Omega} \text{sgn}(\theta(x, t) - \tilde{\theta}(x, t)) [\kappa(x, y) - \tilde{\kappa}(x, y)]
\times S(\theta(x, t)) I(\theta(y, t)) \mu(dy) \mu(dx)
+ \frac{1}{\mu(\Omega)} \int_{\Omega} \int_{\Omega} \text{sgn}(\theta(x, t) - \tilde{\theta}(x, t)) \tilde{\kappa}(x, y)
\times [S(\theta(x, t)) - S(\tilde{\theta}(x, t))] I(\theta(y, t)) \mu(dy) \mu(dx)
+ \frac{1}{\mu(\Omega)} \int_{\Omega} \int_{\Omega} \text{sgn}(\theta(x, t) - \tilde{\theta}(x, t)) \tilde{\kappa}(x, y)
\times S(\tilde{\theta}(x, t)) [I(\theta(y, t)) - I(\tilde{\theta}(y, t))] \mu(dy) \mu(dx)
= I_{11} + I_{12} + I_{13} + I_{14}.
\]

Below, we provide estimates for \( I_{11} \) separately.

- (Estimates of \( I_{11} \) and \( I_{12} \)): Direct calculation provides us
\[
|I_{11}| \leq \| \mathcal{V} - \tilde{\mathcal{V}} \|_1 \quad \text{and} \quad |I_{12}| \leq \frac{\| S \|_\infty \| I \|_\infty}{\mu(\Omega)} \| \kappa - \tilde{\kappa} \|_1.
\]

- (Estimates of \( I_{13} \) and \( I_{14} \)): We use mean-value theorem to obtain
\[
I_{13} \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \int_{\Omega} \tilde{\kappa}(x, y) \left( \max_{|\xi| \leq \frac{S'(\tilde{\theta})}{\mu(\Omega)}} S'(%(\xi)) \right) |\theta(y, t)| - \tilde{\theta}(x, t)| \mu(dy) \mu(dx)
\leq \frac{S'(\tilde{\theta})}{\mu(\Omega)} \int_{\Omega} \int_{\Omega} \tilde{\kappa}(x, y) \left( \min_{|\xi| \leq \frac{S'(\tilde{\theta})}{\mu(\Omega)}} I(\xi) \right) |\theta(y, t)| - \tilde{\theta}(x, t)| \mu(dy) \mu(dx)
\leq \left( \frac{S'(\tilde{\theta})}{\mu(\Omega)} \right) \inf_{\Omega} \| \tilde{\kappa}(x, .) \|_{1} \cdot |\theta(t) - \tilde{\theta}(t)|,
\]
where we used the negativity of \( S'(%(\tilde{\theta})). \) We use a similar argument to have
\[ |I_{14}| \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \int_{\Omega} \tilde{k}(x, y) \left( \max_{|\xi| \leq \tilde{\theta}_\infty} |S(\xi)| \right) \times \left( \max_{|\xi| \leq \tilde{\theta}_\infty} |I'(\xi)| \right) (\theta(y, t) - \tilde{\theta}(y, t)) \mu(dy) \mu(dx) \]

\[ \leq \frac{(SI'(\tilde{\theta}_\infty)}{\mu(\Omega)} \int_{\Omega} (\theta(y, t) - \tilde{\theta}(y, t)) \int_{\Omega} \tilde{k}(x, y) \mu(dx) \mu(dy) \]

\[ \leq \frac{(SI'(\tilde{\theta}_\infty)}{\mu(\Omega)} \cdot \left( \sup_{\Omega} \|\tilde{k}(\cdot, y)\|_1 \right) \cdot \|\theta(t) - \tilde{\theta}(t)\|_1. \]

In (27), we combine (28)–(29) to obtain the desired result. \hfill \Box

4.2. Uniform-in-time continuum limit. In this section, we use Lemma 4.5 to show that the global classical solution to (2) can be constructed by a sequence of lattice solutions to (1) and it converges to the unique stationary state.

**Theorem 4.6.** Suppose \( S, I, \kappa \) and \( V \) satisfy assumptions \( (A1) - (A3) \) and the following additional conditions: for \( \tilde{\theta}_\infty \in (0, \theta_*) \),

\[ \|\theta^{in}\|_\infty \leq \tilde{\theta}_\infty, \quad \|\theta^{N, in}\|_\infty \leq \tilde{\theta}_\infty, \quad \inf_{\Omega} \|\kappa(x, \cdot)\|_1 \leq \sup_{\Omega} \|\kappa(\cdot, y)\|_1 > \frac{(SI'(\tilde{\theta}_\infty)}{(SI'(\tilde{\theta}_\infty)}, \quad (30) \]

and let \( \theta \) and \( \theta^N \) be solutions of (2) and (19) corresponding to data-parameter sets \((\theta^{in}, V, \kappa)\) and \((\theta^{N, in}, V^N, \kappa^N)\), respectively. Then, we have

\[ \lim_{N \to \infty} \sup_{0 < t < \infty} \|\theta(t) - \theta^N(t)\|_1 = 0. \]

**Proof.** We apply Lemma 4.5 with \((\theta, V, \kappa)\) and \((\theta^N, V^N, \kappa^N)\) to obtain

\[ \frac{d}{dt} \|\theta(\cdot, t) - \theta^N(\cdot, t)\|_1 \leq D^N - \lambda \|\theta(\cdot, t) - \theta^N(\cdot, t)\|_1, \quad t \geq 0, \]

where the coefficients \( D^N \) and \( \lambda \) are given by

\[ D^N := \frac{\mu(\Omega)\|V - V^N\|_1 + \|S\|\|I\|_\infty \|\kappa - \kappa^N\|_1}{\mu(\Omega)}, \]

\[ \lambda := -\frac{(SI'(\tilde{\theta}_\infty)}{\mu(\Omega)} \inf_{\Omega} \|\kappa(x, \cdot)\|_1 + (SI'(\tilde{\theta}_\infty)}{\sup_{\Omega} \|\kappa(\cdot, y)\|_1}. \]

Then, it follows from the last relation in (30) and Lemma 4.3 that

\[ \lambda > 0, \quad \lim_{N \to \infty} D^N = 0. \]

Gronwall’s lemma implies

\[ \|\theta(\cdot, t) - \theta^N(\cdot, t)\|_1 \leq e^{-\lambda t} \|\theta^{in} - \theta^{N, in}\|_1 + \frac{D^N}{\lambda} (1 - e^{-\lambda t}), \quad t \geq 0. \]

This yields

\[ \sup_{0 < t < \infty} \|\theta(t) - \theta^N(t)\|_1 \leq \max \left\{ \|\theta^{in} - \theta^{N, in}\|_1, \frac{D^N}{\lambda} \right\}. \]

Finally, we take \( N \to \infty \) and use Lemma 4.3 and (32) to obtain the desired estimate:

\[ \lim_{N \to \infty} \sup_{0 < t < \infty} \|\theta(t) - \theta^N(t)\|_1 = 0. \]

\hfill \Box
As a direct application of Lemma 4.5, we can also get the existence of equilibria to (2).

**Proposition 1.** Suppose $S, I, \kappa$ and $V$ satisfy assumptions $(A1) - (A3)$ and the following additional conditions: for $\varrho^\infty \in (0, \theta_\ast)$,

$$
\|\theta^n\|_\infty \leq \varrho^\infty, \quad \inf_{\Omega} \|\kappa(x, \cdot)\|_1 > \frac{(SI')(\varrho^\infty)}{(S'I)(\varrho^\infty)},
$$

and let $\theta$ be the solution of (2) with the initial datum $\theta^0$. Then, there exists a stationary state $\theta^\infty = \theta^\infty(\cdot) \in L^\infty(\Omega)$ such that

$$
\lim_{t \to \infty} \theta(x, t) = \theta^\infty(x), \quad a.e.
$$

**Proof.** For $h > 0$, we define

$$
\theta^h(x, t) := \theta(x, t + h)
$$

and apply (31) with $(\theta, V, \kappa)$ and $(\theta^h, V, \kappa)$ to obtain,

$$
\frac{d}{dt}\|\theta(\cdot, t) - \theta^h(\cdot, t)\|_1 \leq -\lambda\|\theta(\cdot, t) - \theta^h(\cdot, t)\|_1, \quad t \geq 0.
$$

This implies

$$
\|\theta(\cdot, t) - \theta^h(\cdot, t)\|_1 \leq e^{-\lambda t}\|\theta^0 - \theta(\cdot, h)\|_1, \quad t \geq 0.
$$

Hence, $\{\theta(\cdot, t)\}_{t \geq 0}$ is Cauchy sequence in $L^1(\Omega)$ and we can get $\theta^\infty \in L^1(\Omega)$ such that

$$
\lim_{t \to \infty} \int_{\Omega} |\theta(x, t) - \theta^\infty(x)|\mu(dx) = 0.
$$

(33)

Note that $\|\theta(\cdot, t)\|_\infty$ is uniformly bounded by Lemma 4.4. So, it is enough to show that $\theta^\infty \in L^\infty(\Omega)$ so that one can use the Lebesgue dominated convergence theorem with (33) to see

$$
\lim_{t \to \infty} |\theta(x, t) - \theta^\infty(x)| = 0, \quad a.e.
$$

- **(Proof of claim):** for all $t > 0$, one has $\|\theta(\cdot, t)\|_\infty \leq \varrho^\infty$ so that

$$
\int_{\Omega} |\theta(x, t) - \theta^\infty(x)|\mu(dx) \geq \int_{\{x \in \Omega : |\theta^\infty(x)| \geq \varrho^\infty + 1/n\}} |\theta(x, t) - \theta^\infty(x)|\mu(dx)
$$

$$
\geq \frac{1}{n}\mu\{x \in \Omega : |\theta^\infty(x)| \geq \varrho^\infty + 1/n\} \geq 0, \quad n \geq 1.
$$

Letting $t \to \infty$, we get

$$
\mu\{x \in \Omega : |\theta^\infty(x)| \geq \varrho^\infty + 1/n\} = 0, \quad n \geq 1.
$$

We take a countable union over $n \in \mathbb{N}$ to have

$$
\mu\{x \in \Omega : |\theta^\infty(x)| \geq \varrho^\infty\} = 0,
$$

which verifies $\theta^\infty \in L^\infty(\Omega)$. 

**Remark 4.** Above proposition is the continuum analogue of Theorem 2.2. It follows from the convergence of $\theta$ that

$$
\lim_{t \to \infty} \frac{\theta(x, t)}{t} = 0.
$$

This implies

$$
\rho(x) = 0 \quad \text{for all } x \in \Omega.
Again, this corresponds to the COD state in the sense of Definition 2.1.

5. The continuum Winfree model with heterogeneous frustrations. In this section, we study the emergent dynamics of the CWM with frustrations. In particular, we provide sufficient frameworks for the emergence of equilibrium state and bump states.

Consider the Cauchy problem to the CWM with frustration:

\[
\begin{align*}
\frac{\partial \theta}{\partial t} &= V + \frac{1}{\mu(\Omega)} \int_\Omega \kappa(x, y) S(\theta(x, t)) I(\theta(y, t) + H(x, y)) \mu(dy), \quad (x, t) \in \Omega \times \mathbb{R}_+, \\
\theta(x, 0) &= \theta^1(x), \quad x \in \Omega,
\end{align*}
\]

where \( H \) is a bounded measurable function with \( \|H\|_{\infty} < \pi \). ‘Frustration’, in other words, ‘phase transition’ acts as an error in communications among oscillators. For fixed \( x \) and \( y \), we set \( H(x, y) \) to be constant without any randomness. Hence, this can also be interpreted as a bias in the communication between two specific oscillators.

Using a similar argument in Section 3, we can obtain the existence of global solutions to (34) so that we omit those process in this section. For notational simplicity, we define

\[
I_U(\xi, H) := \inf_{(x, y) \in U^2} I(H(x, y) + \text{sgn}(H(x, y))\xi), \quad U \subset \Omega, \quad \xi \in \mathbb{R},
\]

In addition, throughout the rest of this section, we assume (15)\(_1\), structural conditions on coupling strength and natural frequency. First, we study a uniform boundedness of phase function \( \theta \) as follows.

**Lemma 5.1.** Suppose \( S \) and \( I \) satisfy (A1) and (A2), and let \( \theta \) be a global classical solution of (34) with the initial datum \( \theta^1 \). Then the following assertions hold.

1. If initial datum and system functions satisfy

\[
\|\theta^1\|_{\infty} \leq \bar{\theta} < \theta^*, \quad \|H\|_{\infty} < \theta^* - \bar{\theta}, \quad \inf_{\Omega} \|\kappa(x, \cdot)\|_1 > -\frac{\mu(\Omega)\|V\|_{\infty}}{S(\bar{\theta})I_{\Omega}(\bar{\theta}, H)},
\]

then the phase field \( \theta \) is uniformly bounded:

\[
\sup_{0 \leq t < \infty} \|\theta(t)\|_{\infty} \leq \bar{\theta}.
\]

2. For fixed \( \bar{\theta} \in (0, \theta_*) \) and \( \bar{\theta} \in [\theta_*, \theta^*] \), if initial datum and system functions satisfy

\[
\|\theta^1\|_{\infty} \leq \bar{\theta}, \quad \|H\|_{\infty} < \theta^* - \bar{\theta}, \quad \inf_{\Omega} \|\kappa(x, \cdot)\|_1 > -\frac{\mu(\Omega)\|V\|_{\infty}}{\max_{\theta \leq \theta \leq \bar{\theta}} S(\theta)}I_{\Omega}(\bar{\theta}, H),
\]

then there exists \( t_* \geq 0 \) such that

\[
\sup_{t \geq t_*} \|\theta(\cdot, t)\|_{\infty} \leq \bar{\theta}.
\]
Proof. (i) For fixed $x \in \Omega$, $\theta$ is continuous in $t$-variable. As in the proof of Lemma 4.4, without loss of generality we can assume that
\[ |\theta^\text{in}(x)| \leq \overline{\theta}, \quad x \in \Omega, \]
and furthermore, we can assume that there exists $x_0 \in \Omega$ such that
\[ |\theta^\text{in}(x_0)| = \overline{\theta}. \]
Then, we use $S(\overline{\theta}) < 0$ to see
\[
\frac{\partial}{\partial t} \bigg|_{t=0} |\theta(x_0, t)| \leq \mathcal{V}(x_0) + \frac{S(\overline{\theta})}{\mu(\Omega)} \int_{\Omega} \kappa(x_0, y) I(\theta^\text{in}(y) + H(x_0, y)) \mu(dy)
\leq \|\mathcal{V}\|_{\infty} + \frac{S(\overline{\theta}) I_{\Omega}(\overline{\theta}, H)}{\mu(\Omega)} \left( \inf_{\Omega} \|\kappa(x, \cdot)\|_1 \right) < 0.
\]
Hence, we have
\[ \sup_{0 \leq t < \infty} \|\theta(t)\|_{\infty} \leq \overline{\theta}. \]
(ii) As in the proof of Lemma 4.4, without loss of generality, we assume
\[ |\theta^\text{in}(x)| \leq \overline{\theta}, \quad x \in \Omega. \]
Suppose there exists $x_0 \in \Omega$ such that $|\theta^\text{in}(x_0)| \in [\theta, \overline{\theta}]$. Then, one has
\[
\frac{\partial}{\partial t} \bigg|_{t=0} |\theta(x_0, t)| \leq \|\mathcal{V}\|_{\infty} + \frac{S(|\theta^\text{in}(x_0)|)}{\mu(\Omega)} \int_{\Omega} \kappa(x_0, y) I(\theta^\text{in}(y) + H(x_0, y)) \mu(dy)
\leq \|\mathcal{V}\|_{\infty} + \frac{T_{\Omega}(\overline{\theta}, H)}{\mu(\Omega)} \left( \inf_{\Omega} \|\kappa(x, \cdot)\|_1 \right) \cdot \left( \max_{0 \leq \theta \leq \overline{\theta}} S(\theta) \right) =: -C < 0.
\]
Since $|\theta(x_0, t)|$ is continuous by Remark 2, one has
\[ |\theta(x_0, t)| \leq |\theta^\text{in}(x_0)| - tC \quad \text{if} \quad |\theta(x_0, t)| \in [\theta, \overline{\theta}]. \]
Then, since (35) guarantees that once $|\theta(x_0, t)|$ reaches $\theta$, it remains in $[0, \theta]$ after that, we can obtain
\[ |\theta(x_0, t)| \leq \theta \quad \text{for} \quad t \geq \frac{|\theta^\text{in}(x_0)| - \theta}{C}. \]
Therefore, we can get the desired result for
\[ t_* := \sup_{|\theta^\text{in}(x_0)| \in [\theta, \overline{\theta}]} \frac{|\theta^\text{in}(x_0)| - \theta}{C} < \infty. \]

In next two subsections, we study emergence of stationary states and bump states, respectively.
5.1. Emergence of stationary states. In this subsection, we introduce a sufficient framework \((\mathcal{F}_S)\) for the emergence of stationary states:

\begin{align*}
\inf_{\Omega} \|\kappa(x, \cdot)\|_1 > \frac{\mu(\Omega)\|V\|_\infty}{\max_{-\theta < \theta \leq \theta^*} S(\theta)} I\Omega(\|\partial_t \mathcal{F}\|_\infty) - S(\theta) I\Omega(\|\partial_t \mathcal{F}\|_\infty), \quad (\text{36})
\end{align*}

First, we study \(L^1\)-contraction property of (34).

**Lemma 5.2.** Suppose \(S\) and \(I\) satisfy (A1) and (A2), and initial data and system parameters satisfy \((\mathcal{F}_S)\), and let \(\theta\) and \(\tilde{\theta}\) be two global classical solutions to (34) with the initial data \(\theta^n\) and \(\tilde{\theta}^n\), respectively. Then, there exist positive constants \(\lambda\) and \(t_\ast\) such that

\[
\|\theta(\cdot, t) - \tilde{\theta}(\cdot, t)\|_1 \leq e^{-\lambda(t - t_\ast)}\|\theta(\cdot, t_\ast) - \tilde{\theta}(\cdot, t_\ast)\|_1, \quad t \geq t_\ast.
\]

**Proof.** Let \(\theta\) and \(\tilde{\theta}\) be two global classical solutions of (34) corresponding to initial data \(\theta^n\) and \(\tilde{\theta}^n\), respectively. Then, by Lemma 5.1, without loss of generality, we assume

\[
\sup_{t_\ast \leq t < \infty} \|\theta(t)\|_\infty < \theta \quad \sup_{t_\ast \leq t < \infty} \|\tilde{\theta}(t)\|_\infty < \tilde{\theta}.
\]

Note that \(\theta - \tilde{\theta}\) satisfies

\[
\frac{\partial}{\partial t} (\theta - \tilde{\theta}) = \frac{S(\theta(x, t)) - S(\tilde{\theta}(x, t))}{\mu(\Omega)} \int_\Omega \kappa(x, y)\left[I(\theta(y, t) + H(x, y))\mu(dy) - I(\tilde{\theta}(y, t) + H(x, y))\right] \mu(dy).
\]

We multiply both sides of (37) by \(\text{sgn}(\theta - \tilde{\theta})\) and integrate the resulting relation over \(\Omega\) to obtain, for \(t \geq t_\ast\),

\[
\begin{align*}
\frac{d}{dt}\|\theta(t) - \tilde{\theta}(t)\|_1 &= \frac{1}{\mu(\Omega)} \int_\Omega \kappa(x, y)\left[I(\theta(y, t) + H(x, y))\text{sgn}((\theta - \tilde{\theta})(x, t))\right] \mu(dy) dx
\end{align*}
\]

Below, we estimate the term \(I_{21}\) one by one.
Then, there exists $\theta > 5.2$. Emergence of bump states. Proposition 2. Suppose

1. (Estimate of $I_{21}$): We use the same arguments in Lemma 4.5 and the mean value theorem to see

$$I_{21} \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \int_{\mathbb{R}^2} \kappa(x, y) \left( \max_{|\xi| \leq 2} S'(\xi) \right) I(\theta(y, t) + H(x, y)) |\theta(x, t) - \tilde{\theta}(x, t)| \mu(dy) \mu(dx)$$

$$\leq S'(\theta) \frac{\mu(\Omega)}{\mu(\Omega)} \int_{\Omega} \int_{\mathbb{R}^2} \kappa(x, y) I(\theta(y, t) + H(x, y)) |\theta(x, t) - \tilde{\theta}(x, t)| \mu(dy) \mu(dx)$$

$$\leq S'(\theta) I_{21}(\theta, H) \frac{\mu(\Omega)}{\mu(\Omega)} \left( \inf_{\tilde{t}} \|\kappa(\cdot, \cdot)\|_1 \right) \|\theta(t) - \tilde{\theta}(t)\|_1.$$  

(39)

2. (Estimate of $I_{22}$): Similarly, one has

$$|I_{22}| \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \int_{\mathbb{R}^2} \left( \max_{|\xi| \leq 2} S(\xi) \right) \left( \max_{|\xi| \leq 2} |I'(\xi)| \right) \kappa(x, y) |\theta(y, t) - \tilde{\theta}(y, t)| \mu(dy) \mu(dx)$$

$$\leq \frac{S(\theta) I'(\theta_*)}{\mu(\Omega)} \int_{\Omega} \int_{\mathbb{R}^2} \kappa(x, y) |\theta(y, t) - \tilde{\theta}(y, t)| \mu(dy) \mu(dx)$$

$$\leq \frac{S(\theta) I'(\theta_*)}{\mu(\Omega)} \left( \sup_{\tilde{t}} \|\kappa(\cdot, \cdot)\|_1 \right) \|\theta(t) - \tilde{\theta}(t)\|_1.$$  

(40)

In (38), we combine (39) with (40) to obtain

$$\frac{d}{dt} \|\theta(\cdot, t) - \tilde{\theta}(\cdot, t)\|_1 \leq -\hat{\lambda} \|\theta(\cdot, t) - \tilde{\theta}(\cdot, t)\|_1, \quad t \geq t_*,$$

where

$$\hat{\lambda} := -\frac{1}{\mu(\Omega)} \left[ \left( \inf_{\tilde{t}} \|\kappa(\cdot, \cdot)\|_1 \right) S'(\theta) I_{21}(\theta, H) + \left( \sup_{\tilde{t}} \|\kappa(\cdot, \cdot)\|_1 \right) S(\theta) I'(\theta_*) \right] > 0.$$  

This implies desired exponential contraction.

Proposition 2. Suppose $S$ and $I$ satisfy (A1) and (A2), and initial data and system parameters satisfy $(F_S)$, and let $\theta$ be a solution of (34) with the initial datum $\theta^\circ$. Then, there exists $\theta^\infty \in L^\infty(\Omega)$ such that

$$\lim_{t \to \infty} \theta(x, t) = \theta^\infty(x), \quad a.e.$$  

Proof. We set $\tilde{\theta}$ as a translation of $\theta$:

$$\tilde{\theta}(x, t) := \theta(x, t + h)$$

so that we can conclude that for $x \in \Omega$, the sequence $\{\theta(x, \cdot)\}_{t \geq 0}$ is Cauchy in $L^1(\Omega)$. Then, boundedness of domain $\Omega$ and uniform boundedness of $|\theta(x, t)|$ guaranteed by Lemma 5.1 make it possible to use the same arguments in Proposition 1 to obtain the existence of $\theta^\infty \in L^\infty(\Omega)$ and the almost everywhere convergence of $\theta(x, t)$ to $\theta^\infty \in L^\infty(\Omega)$.\]

5.2. Emergence of bump states. In this subsection, we provide a sufficient framework for a bump state for identical CWM with heterogeneous frustrations. More precisely, we assume that in (34), natural frequency function $V$ is positive constant and influence function $I$ is nonnegative:

$$V = V^\infty, \quad I(\theta) \geq 0,$$

and let $U$ be a proper subset of $\Omega$ with a positive measure:

$$0 < \mu(U) < \mu(\Omega).$$
For a particle model, a bump state is a mixed hybrid state consisting of phase-locking oscillators and running oscillators in a homogeneous Winfree ensemble with the same natural frequency. For a brief discussion on bump states, we refer to \[35, 36\]. Similarly, for the continuum model, if the pointwise phase velocity has a positive infimum in the region \( \Omega \setminus U \) so that

\[
\rho(x) > 0, \quad x \in \Omega \setminus U,
\]

if it exists and \( \rho \) vanishes a.e. in \( U \), we call this state as a bump state.

**Lemma 5.3.** Suppose \( S \) and \( I \) satisfy \((A1)\) and \((A2)\). Then, the following assertions hold.

1. If initial datum and system functions satisfy

\[
0 < \theta < \theta^*, \quad \| H \|_{L^\infty(U^2)} < \theta^* - \theta,
\]

and let \( \theta \) be global solution to \((34)\). Then, there exists \( t_* \geq 0 \) such that, for \( t \geq t_* \) and \( x \in U \),

\[
0 \leq \theta(x,t) \leq \theta, \quad \text{a.e.}
\]

2. If initial datum and system functions satisfy

\[
0 < \theta \leq \theta < \theta^*, \quad \| H \|_{L^\infty(U^2)} < \theta^* - \theta,
\]

and let \( \theta \) be global solution to \((34)\). Then, there exists \( t_* \geq 0 \) such that for \( t \geq t_* \) and \( x \in U \),

\[
0 \leq \theta(x,t) \leq \theta, \quad \text{a.e.}
\]

**Proof.** (i) Recall that for fixed \( x \in \Omega, \theta(x, \cdot) \) is continuous in time variable \( t \). Hence, as in the proof of Lemma 5.1, without loss of generality we can assume that

\[
|\theta^n(x)| \leq \theta, \quad x \in U,
\]

and furthermore, we can assume that there exists \( x_0 \in U \) such that

\[
|\theta^n(x_0)| = \theta.
\]

Then, one has

\[
\frac{\partial}{\partial t} \bigg|_{t=0} |\theta(x_0,t)| \leq \nu^{\infty} + \frac{S(\theta)}{\mu(\Omega)} \int_U \kappa(x_0,y)I(\theta^n(y) + H(x_0,y))\mu(dy)
\]

\[
\leq \nu^{\infty} + \frac{S(\theta)I_U(\theta,H)}{\mu(\Omega)} \int_U \kappa(x_0,y)\mu(dy)
\]

\[
\leq \nu^{\infty} + \frac{S(\theta)I_U(\theta,H)}{\mu(\Omega)} \left( \inf_U \| \kappa(x, \cdot) \|_{L^1(U)} \right) < 0.
\]

This implies

\[
\sup_{t \geq 0} \| \theta(\cdot,t) \|_{L^\infty(U)} \leq \theta.
\]
However, if $\theta(x,t) \in [-\overline{\theta},0]$, then $S(\theta(x,t))$ is positive so that

$$\frac{\partial}{\partial t}\theta(x,t) = V^\infty + \frac{1}{\mu(\Omega)} \int_{\Omega} \kappa(x,y) S(\theta(x,t)) I(\theta(y,t)) + H(x,y)\mu(dy) \geq V^\infty > 0.$$ 

This implies that $\theta(x,t)$ increases until it becomes positive and never becomes negative again. We combine this fact with (42) to get the desired result.

(ii) As in the proof of Lemma 5.1, without loss of generality, we can assume that

$$0 \leq \theta^\infty(x) \leq \overline{\theta}, \quad x \in U,$n

and suppose that there exists $x_0 \in U$ such that $\theta^\infty(x_0) \in [\underline{\theta},\overline{\theta}]$. Then, since $S$ has negative value on $[\underline{\theta},\overline{\theta}]$,

$$\frac{\partial}{\partial t}\theta(x_0,t) \leq V^\infty + \frac{S(\theta^\infty(x_0))}{\mu(\Omega)} \int_{U} \kappa(x_0,y) I(\theta^\infty(y)) + H(x_0,y)\mu(dy)$$

$$\leq V^\infty + \frac{\max_{0 \leq \theta < \pi} S(\theta)}{\mu(\Omega)} \int_{U} \kappa(x_0,y) I(\theta^\infty(y)) + H(x_0,y)\mu(dy) \quad (43)$$

$$\leq V^\infty + \frac{\mathcal{I}_U(\overline{\theta},H)}{\mu(\Omega)} \left( \inf_{U} \|\kappa(x,\cdot)\|_{L^1(U)} \right) \cdot \left( \max_{\underline{\theta} \leq \theta \leq \overline{\theta}} S(\theta) \right) < 0,$$

where we used the fact that

$$\int_{\Omega} \kappa(x_0,y) I(\theta^\infty(y)) + H(x_0,y)\mu(dy) \geq \int_{U} \kappa(x_0,y) I(\theta^\infty(y)) + H(x_0,y)\mu(dy)$$

in the first inequality. Since $\theta(x_0,t)$ is continuous by Remark 2, one has

$$\theta(x_0,t) \leq \theta^\infty(x_0) - tC \quad \text{if} \quad \theta(x_0,t) \in [\underline{\theta},\overline{\theta}].$$

Then, since (43) guarantees that once $\theta(x_0,t)$ reaches $\overline{\theta}$, it remains in $[0,\overline{\theta}]$ after that, we can obtain

$$\theta(x_0,t) \leq \overline{\theta} \quad \text{for} \quad t \geq \frac{\theta^\infty(x_0) - \overline{\theta}}{C}.$$ 

Therefore, we can get the desired result for

$$t_* := \sup_{\theta^\infty(x_0) \in [\underline{\theta},\overline{\theta}]} \frac{\theta^\infty(x_0) - \overline{\theta}}{C} < \infty.$$

Remark 5. Note that Lemma 5.3 yields the pointwise rotation number $\rho(x) := \lim_{t \to \infty} \frac{\theta(x,t)}{t}$ which corresponds to the pointwise phase velocity vanishes a.e. in $U$ under assumption (41).

Now, we introduce a sufficient framework ($\mathcal{F}_B$) for the emergence of a bump state:
constant $\delta > 0$ and $\kappa$

Suppose Proposition 3. provides the emergence of bump states.

First, (34)

Proof. First, (34)

\[
(F_B) : \quad \begin{cases} 
0 < \theta < \theta^*, \quad \|\theta^\alpha\|_{L^\infty(U)} \leq \bar{\theta}, \quad \|H\|_{L^\infty(U^2)} < \theta^* - \bar{\theta}, \\
\inf_U \kappa(x, \cdot) \|_{L^1(U)} > -\frac{\mu(\Omega)\nu}{\mu(\Omega)} \left( \max_{0 \leq \theta \leq \bar{\theta}} S(\bar{\theta}) \right) \left( I_U(\bar{\theta}, H) \right), \\
\left( \sup_{U \setminus \Omega} \kappa(x, \cdot) \right) \cdot \left( \sup_{0 \leq \theta \leq \bar{\theta}} I(H(x, y) + z) \right), \\
\|I\|_\infty \left( \sup_{\Omega \setminus \U} \kappa(x, \cdot) \right) \|_{L^1(\Omega, U)} < -\frac{\mu(\Omega)\nu}{\mu(\Omega)} \left( \inf_{0 \leq \theta \leq 2\pi} S(\theta) \right).
\end{cases}
\]

(44)

Note that the last two conditions on $\kappa$ say that $\kappa$ is neither that large nor that small, i.e. $\kappa$ should take a suitable intermediate value. The following proposition provides the emergence of bump states.

**Proposition 3.** Suppose $S$ and $I$ satisfy (A1) and (A2), and the framework $(F_B)$ holds. Then, for any global classical solution $\theta$ to (34), there exist positive constants $t_\ast$ and $C$ such that

\[
\inf_{t \leq t_\ast} \inf_{x \in U} \frac{\partial \theta}{\partial t}(x, t) \geq C > 0.
\]

Proof. First, (34)

\[
\frac{\partial \theta}{\partial t}(x, t) = \nu + \frac{S(\theta(x, t))}{\mu(\Omega)} \int_U \kappa(x, y) I(\theta(y, t) + H(x, y)) \mu(dy)
\]

\[
+ \frac{S(\theta(x, t))}{\mu(\Omega)} \int_{\Omega \setminus U} \kappa(x, y) I(\theta(y, t) + H(x, y)) \mu(dy).
\]

We use Lemma 5.3 to see that for $t \geq t_\ast > 0$ and $x \in U$,

\[
0 \leq \theta(x, t) \leq \bar{\theta}, \quad \text{a.e.}
\]

which implies, for $t \geq t_\ast$ and $x \in \Omega \setminus U$,

\[
0 \leq \int_U \kappa(x, y) I(\theta(y, t) + H(x, y)) \mu(dy)
\]

\[
\leq \left( \sup_{\Omega \setminus U} \kappa(x, \cdot) \right) \cdot \left( \sup_{0 \leq \theta \leq \bar{\theta}} I(H(x, y) + z) \right),
\]

and

\[
0 \leq \int_{\Omega \setminus U} \kappa(x, y) I(\theta(y, t) + H(x, y)) \mu(dy) \leq \left( \sup_{\Omega \setminus U} \kappa(x, \cdot) \right) \|I\|_\infty.
\]

Now, we fix $x_0 \in \Omega \setminus U$ and $t \geq t_\ast$.

- **Case A** $(S(\theta(x_0, t)) \leq 0)$: We use the last condition in (44) to obtain positive constant $\delta > 0$ which does not depend on $x_0$ and satisfies

\[
\frac{\partial \theta}{\partial t}(x_0, t) \geq \nu + \frac{\sup_{0 \leq \theta \leq \bar{\theta}} I(H(x, y) + z)}{\mu(\Omega)} \left( \sup_{\Omega \setminus U} \kappa(x, \cdot) \right) \left( \inf_{0 \leq \theta \leq 2\pi} S(\theta) \right)
\]

\[
+ \|I\|_\infty \left( \sup_{\Omega \setminus U} \kappa(x, \cdot) \right) \left( \inf_{0 \leq \theta \leq 2\pi} S(\theta) \right) \geq \delta > 0.
\]
Case B ($S(\theta(x_0, t)) > 0$): Since $I \geq 0$, we have
\[
\frac{\partial}{\partial t} \theta(x_0, t) = \mathcal{V}^\infty + \frac{S(\theta(x, t))}{\mu(\Omega)} \int_\Omega \kappa(x, y)I(\theta(y, t) + H(x, y)) \mu(dy) \geq \mathcal{V}^\infty.
\]
We combine two results in Case A and Case B to obtain the desired positive constants $C = \min\{\delta, \mathcal{V}^\infty\}$.

Remark 6. Above proposition and Lemma 5.3 are the continuum analogues of Theorem 2.4. In particular, Lemma 5.3 guarantees the boundedness of $\theta(x, t)$ for almost every $x \in U$, and this implies that
\[
\rho(x) = \lim_{t \to \infty} \frac{\theta(x, t)}{t} = 0, \quad \text{a.e. } x \in U.
\]
Moreover, Proposition 3 guarantees the lower bound of $\partial_t \theta$ for each $x$ which belongs to outside of $U$. This results exactly correspond to Definition 2.3.

6. Numerical simulations. In this section, we provide several numerical simulations to confirm analytical results in Sections 4 and 5 on the emergence of stationary and bump states. For the numerical integration of the nonlocal term in the CWM, we used a trapezoidal rule and the following special pair of sensitivity and influence functions satisfying conditions (A1) and (A2), and compact domain $\Omega$:
\[
S(\theta) = -\sin \theta, \quad I(\theta) = 1 + \cos \theta
\]
In this case, $\theta_*$ and $\theta^*$ appearing in (A1) and (A2) are explicitly given as follows.
\[
\theta_* = \frac{\pi}{3} \quad \text{and} \quad \theta^* = \pi.
\]

6.1. Emergence of stationary states. In this subsection, we perform numerical simulations for both continuum model with and without frustration to see the emergence of unique stationary states independent of initial data.

6.1.1. The CWM without frustration. In this part, we perform numerical simulations for the CWM without frustration. In the simulation, we assume that $V(x) = |x|$, $\bar{\theta} = \frac{5\pi}{6}$ and $\kappa \equiv 22$,
where $\kappa$ was chosen to satisfy the condition in (15):
\[
22 = \inf_{\Omega} \|\kappa(x, \cdot)\|_1 > \frac{\mu(\Omega)\|\mathcal{V}\|_\infty}{(SI)(\bar{\theta}^\infty)} \approx 21.1117.
\]

For numerical simulations, we set time step $\Delta t = 0.005$ and observe the dynamics in the time interval $[0, 0.5]$. To check the uniqueness of the stationary state, we employ two homogeneous settings except initial configurations. Two different initial phase configurations $\theta_1^{in}$ and $\theta_2^{in}$ are supposed to be simple functions chosen randomly in $[-\bar{\theta}, \bar{\theta}]$.

In Figure 3, we plot $L^1$-difference between two solutions. The linear decay of $\log \|\theta_1 - \theta_2\|_1$ implies the exponential decay of the $L^1$-difference, $\|\theta_1 - \theta_2\|_1$.

In Figure 4, we plot point trajectories of $\theta_1$ and $\theta_2$ at $(0.5, 0.5)$. For each trajectory, one can observe the decrease of the absolute value of slope, which implies the convergence toward the stationary state.

In Figure 5, we compared the terminal configuration of $\theta_1$ and $\theta_2$ with configuration of natural frequency. From the last observation, we know that terminal...
configuration of $\theta_1$ and $\theta_2$ are the same. Hence, we just plot terminal configuration of $\theta_1$ in Figure 5 (a). One can easily observe the positive correlation between the terminal configuration and the natural frequency function.

6.1.2. The CWM with frustration. In this part, we perform numerical simulations for the CWM with frustration. In the simulation, we assume that $H$ is a simple function whose range chosen randomly in $[-0.2, 0.2]$ and we also assume that

$$\mathcal{V} \equiv 0.1, \quad \bar{\theta} = \frac{5\pi}{6}, \quad \theta = \frac{\pi}{6} \quad \text{and} \quad \kappa \equiv 4,$$

where the coupling strength $\kappa$ was chosen to satisfy the conditions in (36):

$$4 = \inf_{\Omega} \|\kappa(x, \cdot)\|_1 > -\frac{\mu(\Omega)\|\mathcal{V}\|_\infty}{\left(\max_{\theta \leq \theta \leq \bar{\theta}} S(\theta)\right) I_{\Omega}(\bar{\theta}, H)} \approx 3.8534$$
For a numerical simulation, we set time step $\Delta t = 0.005$ and observe the dynamics in the time interval $[0, 4]$. To check the uniqueness of the stationary profile, we employ two homogeneous settings except initial configurations. Two different initial phase configurations $\theta_1^{in}$ and $\theta_2^{in}$ are supposed to be simple function whose range chosen randomly in $[-\theta, \theta]$.

In Figure 6, we plot $L^1$-difference between two solutions. The linear decay of $\log \|\theta_1 - \theta_2\|_1$ implies the exponential decay of the $L^1$-difference.

In Figure 7, we plot point trajectories of $\theta_1$ and $\theta_2$ at $(0.5, 0.5)$. For each trajectory, one can observe the decrease of the absolute value of slope, which implies the convergence toward the stationary state.
In Figure 8, we compared the terminal configuration of $\theta_1$ and $\theta_2$ with configuration of natural frequency. We just plot terminal configuration of $\theta_1$ in Figure 8 (a), as we did in Figure 5. In Figure 8 (a), one can observe that terminal state is highly oscillated, which is induced by the effects of frustration $H$.

6.2. Emergence of a bump state. In this subsection, we perform numerical simulations for the CWM with heterogeneous frustration to observe a bump state. In the simulation, we chose an open subset $U \subset \Omega$ satisfying

$$\Omega \setminus U = [0.45, 0.55] \times [0.45, 0.55],$$

and assumed that $H$ is a simple function on $U \times \Omega$ whose range takes values in $[-0.05, 0.05]$ randomly, and takes constant value on $\Omega \setminus U \times \Omega$ with value $\frac{5\pi}{6}$. We also assume that

$$\mathcal{V}(x) \equiv 1, \quad \underline{\theta} = \frac{2\pi}{3}, \quad \theta = \frac{\pi}{3} \quad \text{and} \quad \kappa \equiv 3,$$
where $\kappa$ was chosen to satisfy the conditions in (44):

$$2.97 = \inf_{U} \| \kappa(x, \cdot) \|_{L^1(U)} > -\frac{\mu(\Omega)V^\infty}{\left(\max_{2 \leq r \leq \theta} S(\overline{\theta})\right) I_U(\overline{\theta}, H)} \approx 2.5248,$$

and

$$0.4508 \approx \left(\sup_{U \backslash U} \| \kappa(x, \cdot) \|_{L^1(U)}\right) \cdot \left(\sup_{0 \leq z \leq \theta} I(H(x, y) + z)\right)$$
$$+ \| I \|_\infty \left(\sup_{\Omega \backslash U} \| \kappa(x, \cdot) \|_{L^1(\Omega \backslash U)}\right) \leq -\frac{\mu(\Omega)V^\infty}{\left(\inf_{0 \leq \theta \leq 2\pi} S(\theta)\right)} = 1.$$

For numerical simulation, we set time step $\Delta t = 0.01$ and set initial phase configuration $\theta^0$ to a simple function whose range takes values in $[-\overline{\theta}, \overline{\theta}]$ randomly. We observe the dynamics in the time interval $[0, 20]$. We calculate $\theta(x, T)$ to observe a bump state.

![Image](image1.png)

**Figure 9.** $\theta(x, T)$ for a sufficiently large $T$, here $T = 20$ to observe a bump state.

7. **Conclusion.** In this paper, we have derived a uniform-in-time continuum limit of the lattice Winfree model toward the continuum Winfree model. Here the continuum limit from lattice model to the continuum one roughly means that the continuum solution can be constructed as a suitable limit of a sequence of lattice solutions. This kind of concept for continuum limit is consistent with mean-field limit from particle system to kinetic system and hydrodynamic limit from kinetic system to macroscopic system. In fact, Gronwall type arguments based on the brutal force analysis will provide a continuum limit which is valid only for any finite-time interval for a general initial data. In contrast, if we restrict our discussion on a set of initial data leading to the collective dynamics, we can extend the finite-in-time continuum limit to the uniform-in-time continuum limit using the uniform structural stability estimate. For the derived continuum model, we also establish a global
existence of classical solutions. Among possible asymptotic scenarios, we studied sufficient frameworks leading to the stationary profile and bump state in terms of initial data, coupling and natural frequency functions and special structures of the sensitivity and influence functions. Of course, aforementioned conditions are only sufficient, and not optimal at all. So more refined sharp conditions on the coupling function need to be done. Moreover, we only consider two possible asymptotic scenarios, say emergence of stationary profile and bump states. However, there might be other asymptotic scenarios, say emergence of complete locking-solutions and partial locking-solutions for the CWM. These will be left for a future work.

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1032 SEUNG-YEAL HA, MYEONGJU KANG AND BORA MOON

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