An analytical and numerical approach to structural stability of truss 3D elements

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Abstract
With the increasing use of high strength steels, steel structures are become more resilient and also slender than before. Therefore, the phenomena of structural instability must be considered and prevented. The goal of this paper is to help structural engineer to understand, in detail, analytically and numerically, the nonlinearities of slender structural systems. Moreover, it explains the detection and classification of critical points present in the primary equilibrium path of slender structural systems. This work uses the Total Lagrangian Formulation to describe the kinematics of a truss 3D bar element. Through this formulation the authors obtain the internal forces vector and tangent stiffness matrix that take into account the effects of geometric non-linearity. An elastic linear constitutive model for the uniaxial stress-strain state is assumed. The Green-Lagrange deformation and the axial stress of the second Piola-Kirchhoff tensor, energetically conjugated, are adopted. As a case study, a simple physical system with three degrees of freedom is presented. Such system is composed of two truss 3D bars and a linear spring. Finally, the geometric and physical conditions for the coalescence between the limit and bifurcation points are determined.

Keywords: Total Lagrangian description, geometrical nonlinearity, critical points

1 Introduction
In structures with high slenderness, it is essential to understand and to simulate nonlinear phenomena. For example, in the construction of high-rise buildings, in aeronautics, aerospace, and petroleum industries, non-linear analysis is essential in the design of different structural typologies. On the other hand, engineers must have knowledge of physics, applied mathematics and computer science to model these phenomena in a consistent manner. In the last decades, many authors have published textbooks addressing different topics of non-linear analysis in the field of numerical methods applied to engineering.

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For example, the following references are strongly recommended: Crisfield (1991), Crisfield (1997), Wriggers (2008), Neto et al. (2008), Simon and Hughes (1998), Belytschko et al (2000), Doyle (2001), Wriggers (2002), Kojic & Bathe 2005), Krenk (2009), Voyiadjis & Woelke (2010) and Hashiguchi & Yamakawa (2013). It follows, therefore, the great need to explain among engineers, the basic and fundamental concepts of non-linear analysis through the use of finite element method. Towards this aim, the goal of this work is to present the theoretical and numerical basics of structural instability for the detection and classification of singular points in the primary equilibrium path. Local buckling of truss element is not considered here. The detection of the singular points is here obtained using scalar parameter, such as determinant and pivot sign change of the tangent stiffness matrix. Other approaches can be found in Wriggers and Simo (1990), Cardona and Huespe (1999), Planinc and Saje (1999), Ibrahimbegovic and Mikad (2000), and Battini et al. (2003), among others. To easily describe the kinematics of the element movement and, analytically, obtain the vector of internal forces and the matrix of tangent stiffness, truss bar elements are employed. Such elements are employed due to their theoretical simplicity. In this way, the indispensable elements for non-linear analyses are explained. Additionally, in order to simulate finite elastic deformations, a hyperplastic constitutive model is assumed for the uniaxial tensile-strain state. This constitutive model uses the axial stress of the second Piola-Kirchhoff tensor and the Green-Lagrange deformation. Such tensor and deformation, which form a couple energetically conjugated $(\sigma, \varepsilon)$, fulfill the principle of the virtual works. In Sections 2, 3 and 4 of this paper, the kinematics of the 3D truss element is described. This description uses the Total Lagrangian Formulation (TLF) and shows the vector of the internal forces and the tangent stiffness matrix deduction, respectively. In section 5 a detailed analytical approach detects and classifies the singular points present in the primary equilibrium path. For that, a fairly simple physical system is used. This system is made of two truss 3D bars and a linear spring. In section 6 a numerical approach of the structural instability of the physical system described previously is used, for this an in-house finite element program that performs geometric non-linear analyzes is developed. The in-house program was written in Fortran90. Finally, this article presents conclusions and rich bibliographic references used in this work.
2 Kinematic description

Figure-1 shows a global coordinate system with an orthonormal base. To express the kinematic variables in the undeformed configuration, the material coordinates \((X, Y, Z)\), are used, while for the deformed configuration the spatial coordinates \((x, y, z)\) are then applied.

![Figure 1 – (a) 3D truss element in initial and displaced configurations. (b) Relative nodal displacements.](image)

In the undeformed configuration, the nodal coordinates of the end points \(A\) and \(B\) of the 3D truss element are, respectively, given by \(X_A = (X_A, Y_A, Z_A)\) and \(X_B = (X_B, Y_B, Z_B)\). Its initial position and length are, respectively, given by \(\vec{AB} = X_0 = X_B - X_A\) and \(l_0^2 = X_0^T X_0\). In the deformed configuration the nodal coordinates of the truss 3D element are given by \(x_A = (x_A, y_A, z_A)\) and \(x_B = (x_B, y_B, z_B)\), respectively. As shown in Figure 1a, the current coordinate of node \(A\) is given by \(x_A = X_A + u_A\), while the current coordinate of node \(B\) is expressed as \(x_B = X_B + u_B\), where \(u_A = (u_A, v_A, w_A)\) is the displacement of node \(A\) and \(u_B = (u_B, v_B, w_B)\) is the displacement of node \(B\). Therefore, the current position of element is given by:

\[
\overline{ab} = x = x_B - x_A = X_0 + u_{BA}
\]  

Moreover, in Eq.(1), \(u_{BA} = u_B - u_A\) is the vector of relative nodal displacements as shown in Figure 1b. The current length is expressed as \(l_0^2 = x^T x = (X_0 + u_{BA})^T (X_0 + u_{BA})\). In this work, the Total Lagrangian
formulation is used to describe the displacement of the 3D truss element, so the material coordinates \((X, Y, Z)\) and the undeformed configuration will be used to define the deformation measure of the element. In the technical literature, among some deformation families described in material co-ordinates, the Green-Lagrange deformation measure is used to compare the squares of the current \(l\) and initial \(l_0\) lengths of the element in the following way

\[
\varepsilon_G = \frac{l^2 - l_0^2}{2l_0^2} = \frac{1}{l_0^2} \left( X_0^T u_{BA} + \frac{1}{2} u_{BA}^T u_{BA} \right)
\]  

(2)

It is noted that this measure of the deformation has quadratic terms with respect to the relative nodal displacements. In order to obtain the vector of internal forces, the Principle of Virtual Works (PVW) is utilized. Therefore, it is necessary to apply a virtual variation in the displacement field in the current equilibrium configuration, as shown in Figure 1a. This virtual variation of displacement field implies a virtual variation of the Green-Lagrange deformation that is written as

\[
\delta \varepsilon_G = \frac{1}{l_0^2} \left( X_0^T + u_{BA} \right)^T \delta u_{BA} = \frac{1}{l_0^2} x^T \delta u_{BA}
\]

(3)

where \(\delta u_{BA} = \delta u_B - \delta u_A\) is the virtual variation vector of relative displacements. Note that the virtual variation of the Green-Lagrange deformation consists of the projection of the virtual variation of displacements vector with respect to the current position of the element. Such position is defined using the \(x\) vector scaled by \(l_0^2\).

3 Nodal force vector

As shown in Figure 1a, let \(f_A = (f_{A_x}, f_{A_y}, f_{A_z})\) be the vector of forces at node \(A\), and \(f_B = (f_{B_x}, f_{B_y}, f_{B_z})\) the vector of forces at node \(B\), respectively. To obtain these force vectors the (PVW) is applied in the undeformed configuration as the Green-Lagrange deformation measure is used. Therefore, the PVW is expressed as

\[
\delta V = \int_0^{l_0} N \delta \varepsilon_G ds - f_A^T \delta u_A - f_B^T \delta u_B = 0
\]

(4)
where $N$ is the axial loading acting on the element given by $N = \sigma_G A$. Recall that the axial stress $\sigma_G$ is energetically coupled with the Green-Lagrange strain measurement. In addition, $\sigma_G$ is one of the normal stresses of the second Piola-Kirchoff stress tensor. Replacing Eq. (3) in Eq. (4), the resulting expression is

$$
\delta V = \delta u_A^T \left( -\int_0^{l_0} \frac{N}{l_0^2} x \, ds - f_A \right) + \delta u_B^T \left( \int_0^{l_0} \frac{N}{l_0^2} x \, ds - f_B \right) = 0
$$

(5a)

$$
f_A = -\frac{N}{l_0} x \quad ; \quad f_B = \frac{N}{l_0} x
$$

(5b)

In this work, it is assumed that $\sigma_G = E\varepsilon_G$, where $E$ is the longitudinal modulus of elasticity of the material. Therefore, the axial loading can be defined as $N = EA\varepsilon_G$ and Eq. (5b) may be rewritten as follows:

$$
f_A = \frac{EA}{l_0} \varepsilon_G x \quad ; \quad f_B = \frac{EA}{l_0} \varepsilon_G x
$$

(6)

### 4 Tangent stiffness matrix

By applying an infinitesimal increase in the vectors of nodal displacements $u_A$ e $u_B$ in the deformed configuration, one can obtain an infinitesimal increase of the internal forces vectors $f_A$ and $f_B$, i.e. $df_A$ and $df_B$, respectively. This increment is achieved through the tangent stiffness matrix. Therefore, the relation between the infinitesimal increments of the internal force vectors and nodal displacement vectors is

$$
\begin{bmatrix}
df_A \\
df_B
\end{bmatrix} = K_T \begin{bmatrix}
du_A \\
du_B
\end{bmatrix}
$$

(7)

where $K_T$ is the tangent stiffness matrix of order 6×6. Therefore, taking into account Eq. (1) and differentiating Eq. (5a) with respect to the vector of relative displacements, one can obtain that

$$
dq_A = -x \frac{dN}{l_0} - \frac{N}{l_0} \, dx
$$

(8a)
\[ dq_A = \left( x \frac{dN}{l_0} + \frac{N}{l_0} I \right) d (u_B - u_A) \] (8b)

\[ dq_B = -dq_A \] (8c)

where \( I \) is the identity matrix of order 3\( \times \)3. Differentiating the axial loading \( N \), with respect to the relative displacement vector, and taking into account Eqs. (1), (2) and (3), the resulting expression becomes:

\[ \frac{dN}{du_{BA}} = \frac{EA}{l_0^2} \frac{d \varepsilon}{du_{BA}} = \frac{EA}{l_0^2} \left( X_0^T + u_{BA}^T \right) = \frac{EA}{l_0^2} x^T \] (9)

Finally, by replacing Eq. (9) in Eq. (8b) and taking Eq. (8c) into account, one can obtain that

\[ \begin{bmatrix} dq_A \\ dq_B \end{bmatrix} = \begin{bmatrix} x \otimes x & -x \otimes x \\ -x \otimes x & x \otimes x \end{bmatrix} + \frac{N}{l_0} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \begin{bmatrix} du_A \\ du_B \end{bmatrix} \] (10)

where:

\[ K_M = \frac{EA}{l_0^3} \begin{bmatrix} x \otimes x & -x \otimes x \\ -x \otimes x & x \otimes x \end{bmatrix} \] (11a)

\[ K_\sigma = \frac{N}{l_0} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \] (11b)

\[ x \otimes x = \begin{bmatrix} x_{BA}^2 & x_{BA}y_{BA} & x_{BA}z_{BA} \\ x_{BA}y_{BA} & y_{BA}^2 & y_{BA}z_{BA} \\ x_{BA}z_{BA} & y_{BA}z_{BA} & z_{BA}^2 \end{bmatrix} \] (11c)

where \( K_M \) is the material stiffness matrix of order 6\( \times \)6, which depends on the current position vector of the element \( x \) whose components are: \( x_{BA} = x_B + x_A \), \( y_{BA} = y_B + y_A \) and \( z_{BA} = z_B + z_A \). \( K_\sigma \) is the geometric stiffness matrix which depends on the axial loading \( N \). \( K_\sigma \) is a matrix of order 6\( \times \)6. The symbol \( \otimes \) is the tensor (or open) product. Therefore, the tangent stiffness matrix is expressed as \( K_T = K_M + K_\sigma \).
5 Analytical formulation

In order to analytically detect the critical points in the primary equilibrium path, a structural system composed of two truss 3D elements are considered. Figure 2 shows the displacement boundary conditions, loading conditions, and the mechanical and geometrical properties of this structural system.

It should be noted that in the upper vertex node, along the \( z \)-axis direction, the boundary condition there is represented as a linear spring of constant \( k \). In this case, and in that node, the loads \( f_2 \) and \( f_3 \) are applied in the \( y \) and \( z \) axes directions, respectively. It is also assumed that there will be no displacement in the direction of the \( x \)-axis due to the lack of load applied in this direction. Also, the length \( c \) in the direction of the \( z \)-axis is assumed to be an imperfection of the structural system.

According to Fig. 2 the initial length of each truss bar is given by \( l_0 = \sqrt{a^2 + b^2 + c^2} \).

After the vertex displacement given by \( (0, u_2, u_3) \), the current length of each bar will be given by \( l = \sqrt{(a - u_2)^2 + b^2 + (c + u_3)^2} \). First, the stability analysis is done considering the structural system as perfect, which means \( c = 0 \) and \( f_3 = 0 \). Note that this perfect condition does not necessarily imply that \( u_3 \) would be equal to zero - as will be shown hereinafter. Therefore, for the perfect system the primary equilibrium path (the \( f_2 \) vs. \( u_2 \) curve), would take place in the \((x, y)\) plane. To make algebraic development easier, the following non-dimensional parameters are adopted:
\[
\alpha = \frac{a}{l_0}; \beta = \frac{b}{l_0}; \gamma = \frac{c}{l_0}; \kappa = \frac{k_0}{EA}; \mu_2 = \frac{u_2}{l_0}; \mu_3 = \frac{u_3}{l_0}; \lambda_2 = \frac{f_2}{EA}; \lambda_3 = \frac{f_3}{EA}.
\] (12)

Taking into account the definitions of \(l_0\) and \(l\) and the parameters given by equation (12), the Green-Lagrange deformation of each truss bar is given by

\[
\varepsilon_G = \frac{l^2 - l_0^2}{2l_0^2} = -\mu_2 \left(\alpha - \frac{1}{2} \mu_2\right) + \mu_3 \left(\gamma + \frac{1}{2} \mu_3\right)
\] (13)

The virtual variation of the Green-Lagrange deformation, with respect to the virtual variation of the displacements \((0, \delta u_2, \delta u_3)\), can be expressed as:

\[
\delta \varepsilon_G = -\delta\mu_2 \left(\alpha - \mu_2\right) + \delta\mu_3 \left(\gamma + \mu_3\right)
\] (14)

In addition, taking into account the parameters defined in Eq. (12), the total potential energy of the structural system is given by the following expression

\[
\pi = EAl_0 \left(\varepsilon_G^2 + \frac{1}{2} \kappa \mu_3^2 - \lambda_2 \mu_2 - \lambda_3 \mu_3\right)
\] (15)

As the structural system is considered only with 2 degrees of freedom, its equilibrium condition may be represented by a nonlinear system of 2 equations with 2 unknowns. The equation system can be found applying the principle of stationarity of the functional expressing the total potential energy which may be written as

\[
\delta \pi = \frac{\delta \pi}{\delta u_2} \delta u_2 + \frac{\delta \pi}{\delta u_3} \delta u_3 = 0.
\]

Taking into account the parameters defined in Eq. (12) and applying the condition of stationarity in the Eq. (15), the following system of equations can be found:

\[
\begin{align*}
-2\varepsilon_G \left(\alpha - \mu_2\right) &= \lambda_2 \\
2\varepsilon_G \left(\gamma + \mu_3\right) + \kappa \mu_3 &= \lambda_3
\end{align*}
\] (16)

Eq. (16) represents the conditions of equilibrium in the deformed configuration of the space truss in the directions of the axes \(y\) and \(z\), respectively. It should be pointed out that both equations depend on cubic terms with respect to the parameters \(\mu_2\) and \(\mu_3\). In fact, from Eq. (13), deformation \(\varepsilon_G\), depends upon the quadratic terms in \(\mu_2\)
and \( \mu_3 \). To obtain the tangent stiffness matrix, the load parameters are differentiated in relation to the displacement parameters in both equations of equilibrium such that:

\[
\begin{bmatrix}
\frac{d\lambda_2}{d\lambda_3} \\
\frac{d\lambda_3}{d\lambda_3}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial \lambda_2}{\partial \mu_2} & \frac{\partial \lambda_2}{\partial \mu_3} \\
\frac{\partial \lambda_3}{\partial \mu_2} & \frac{\partial \lambda_3}{\partial \mu_3}
\end{bmatrix}
\begin{bmatrix}
\frac{d\mu_2}{d\lambda_3} \\
\frac{d\mu_3}{d\lambda_3}
\end{bmatrix}
\]  

(17)

The coefficients of the tangent stiffness matrix are determined by differentiating the Eqs. (13) and (16) according to Eq. (17). After some basic algebraic developments, it follows that

\[
\begin{bmatrix}
\frac{d\lambda_2}{d\lambda_3} \\
\frac{d\lambda_3}{d\lambda_3}
\end{bmatrix} = 
\begin{bmatrix}
2(\alpha - \mu_2)^2 + 2\varepsilon_G & -2(\alpha - \mu_2)(\gamma - \mu_3) \\
-2(\alpha - \mu_2)(\gamma + \mu_3) & 2(\gamma + \mu_3)^2 + 2\varepsilon_G + \kappa
\end{bmatrix}
\begin{bmatrix}
\frac{d\mu_2}{d\lambda_3} \\
\frac{d\mu_3}{d\lambda_3}
\end{bmatrix}
\]  

(18)

The tangent stiffness matrix can be decomposed into the material stiffness matrix \( K_M \) and the geometric stiffness matrix \( K_\sigma \). Such matrices can be then expressed as:

\[
K_M = 
\begin{bmatrix}
2(\alpha - \mu_2)^2 & -2(\alpha - \mu_2)(\gamma - \mu_3) \\
-2(\alpha - \mu_2)(\gamma + \mu_3) & 2(\gamma + \mu_3)^2 + \kappa
\end{bmatrix}; 
K_\sigma = 
\begin{bmatrix}
2\varepsilon_G & 0 \\
0 & 2\varepsilon_G
\end{bmatrix}
\]  

(19)

Now, the equilibrium path of the perfect system can be analyzed, i.e., making \( c = 0 \Rightarrow \gamma = 0 \) and with \( f_3 = 0 \Rightarrow \lambda_3 = 0 \). These conditions means that the equilibrium equation in the \( z \)-axis must comply that \( (2\varepsilon_G + \kappa)\mu_3 = 0 \). This equation can be satisfied in two conditions: \( \mu_3 = 0 \) or \( (2\varepsilon_G + \kappa) \). For the first condition, the primary equilibrium path takes place in the \( (x, y) \) plane. This means that the path is symmetric, since there is only the vertical displacement of the node at the vertex of space truss. In this case, according to Eq. (13), the deformation is \( \varepsilon_G = -\mu_2 \left( \alpha - \frac{1}{2} \mu_2 \right) \). Substituting the expression of \( \varepsilon_G \) into Eq. (16a) yields the primary equilibrium path which is expressed as

\[
\lambda_2 = 2\alpha^2 \mu_2 - 3\alpha \mu_2^2 + \mu_2^3
\]  

(20)
Eq. (20) is a third degree polynomial written in function of $\mu_2$ which graph is shown in Fig. 3. The curve in that figure was obtained for the following values: $a = 1$, $b = 2$ and $c = 0$. As can be seen in that figure, the equilibrium path shows two extremes: points $A$ and $B$. Those points are determined with condition $d\lambda_2 = d\mu_2 = 0$. In this way, the maximum and minimum points, $A$ and $B$, respectively are obtained as

$$
\mu_2^a = \alpha \left( 1 - \frac{1}{\sqrt{3}} \right) \Rightarrow \lambda_2^a = \frac{2}{3\sqrt{3}} \alpha^3
$$
$$
\mu_2^b = \alpha \left( 1 + \frac{1}{\sqrt{3}} \right) \Rightarrow \lambda_2^b = -\frac{2}{3\sqrt{3}} \alpha^3
$$

Points $A$ and $B$ are referred as limit points in the technical literature. At these points, the tangent stiffness matrix of Eq. (18) becomes singular. Therefore, the structure reaches its maximum loading carrying capacity at point $A$. From this point on, the structure becomes unstable. As shown in Fig. 3, the structure jumps forward from point $A$ to point $A'$ - where the equilibrium configuration becomes stable. In the technical literature, this phenomenon is called snap-through. Additionally, the second condition takes place when the deformation reaches the value $\varepsilon_G = -\frac{1}{2} \kappa$ and $\mu_3 \neq 0$. In this case, initially the space truss is in the primary equilibrium path. However, when this equilibrium becomes unstable, the truss moves out of the plane $(x, y)$ seeking a stable secondary equilibrium path. At this moment the buckling phenomena takes place.

![Figure 3 –Primary equilibrium path.](image)
To determine the displacement in the direction of the $z$-axis, Eq. (13) is used. In this case, Eq. (13) can be expressed as

$$\varepsilon_G = -\mu_2 \left( \alpha - \frac{1}{2} \mu_2 \right) + \frac{1}{2} \mu_3^2.$$  

Making $\varepsilon_G$ equal to $-\frac{1}{2} \kappa$, it is obtained that:

$$\mu_3 = \pm \sqrt{-\kappa + 2 \mu_2 \left( \alpha - \frac{1}{2} \mu_2 \right)}$$  

(22)

Which is valid for the range: $\alpha - \sqrt{\alpha^2 - \kappa} \leq \mu_2 \leq \alpha + \sqrt{\alpha^2 - \kappa}$. The relationship between the displacement parameters of $\mu_2$ and $\mu_3$ is shown in Fig. 4. In order to obtain the graphic of this force, the following values for the variables $a = 1$, $b = 2$, $c = 0$ and $\kappa = 0.1$, therefore, $0.131 \leq \mu_2 \leq 0.763$. From the analytical point of view, in Fig. 4, for the truss buckling out of the $(x, y)$ plane, there are two possible directions in $z$-axis. The truss can buckle in the positive direction of the $z$-axis (when $u_3 > 0$), or in the negative direction of the $z$-axis (when $u_3 < 0$). The maximum displacement, in the $z$-axis direction, points $E$ and $F$, is given by

$$\frac{d \mu_3}{d \mu_2} = 0 \Rightarrow \mu_3 = \pm \sqrt{\alpha^2 - \kappa}.$$  

According to Fig. 4, after the bifurcation represented by point C, the displacement $u_3$ of the vertex of the truss increases until a maximum of $\pm 0.316$. After reaching the maximum, displacement decreases until it cancels out at point D. This point represents that the truss element returns to the plane $(x, y)$. From the physical point of view, it is important to emphasize that what makes possible the truss to return to the $(x, y)$ plane is the existence of the spring in the $z$-axis direction at the vertices of the truss – see Fig. 2.
Figure 4 – Graph of the secondary equilibrium path on the $\mu_2$ vs. $\mu_3$ plane.

Figure 5 shows the secondary equilibrium path is obtained by taking into account Eq. (16a) and the constraint $\varepsilon_G = -\frac{1}{2}\kappa$ knowing that the expression for $\lambda_2$ is given by

$$\lambda_2 = -2\varepsilon_G (\alpha - \mu_2) = \kappa (\alpha - \mu_2)$$

(23)

Figure 5 – Secondary equilibrium path.

In Figure 5, points $C$ and $D$ are called bifurcation points. At these points, there is the intersection between the primary and secondary equilibrium paths. At these points the tangent stiffness matrix given by Eq. (18) becomes singular. At the bifurcation point $C$, the structure changes to the secondary equilibrium path while at the bifurcation point D it returns the primary equilibrium path. Substituting the values of $\mu_2$, obtained
by imposing that \( \mu_3 = 0 \) in Eq. (23), one can obtain, from Eq. (23), the bifurcation points, in the following way:

\[
\begin{align*}
\mu_2^c &= \alpha - \sqrt{\alpha^2 - \kappa} \Rightarrow \lambda_2^c = \kappa \sqrt{\alpha^2 - \kappa} \\
\mu_2^d &= \alpha - \sqrt{\alpha^2 - \kappa} \Rightarrow \lambda_2^d = -\kappa \sqrt{\alpha^2 - \kappa}
\end{align*}
\]  

(24)

Note that for the buckling phenomenon to take place, the condition \( \alpha^2 - \kappa > 0 \) must be satisfied. Considering the equilibrium paths shown in Figs. 3 and 5, it can be concluded that the primary equilibrium path of the truss element analyzed has four critical points, two limit points and two bifurcation points. As mentioned previously, the tangent stiffness matrix given by Eq. (18) for the two conditions \( \gamma = 0 \) and \( \mu_3 = 0 \) becomes singular at the critical points, i.e., its determinant is zero. It is important to note that the manifestation sequence of these two critical points depends on the relation between the dimensionless parameters \( \alpha \) and \( \kappa \). Therefore, for the bifurcation point \( C \) takes place before the limit point \( A \), it is necessary that \( \mu_2^c < \mu_2^a \). For the bifurcation point \( C \) takes place after the limit point \( A \), it is necessary that \( \mu_2^c > \mu_2^a \) and \( \kappa < \alpha^2 \). Finally, for the bifurcation point \( C \) corresponds to the limit point \( A \), it is necessary that \( \mu_2^c = \mu_2^a \). These conditions are summarized in the equations.

\[
\begin{align*}
0 < \kappa < \frac{2}{3} \alpha^2 &\Rightarrow \text{the bifurcation point, } C, \text{ takes place before the limit point, } A; \\
\frac{2}{3} \alpha^2 < \kappa < \alpha^2 &\Rightarrow \text{the bifurcation point, } C, \text{ takes place after the limit point, } A; \\
\kappa > \alpha^2 &\Rightarrow \text{there is no bifurcation;}
\end{align*}
\]

(25)

\[
\kappa = \frac{2}{3} \alpha^2 \Rightarrow \text{the bifurcation point, } C, \text{ corresponds to the limit point, } A.
\]
Fig. 6 shows the equilibrium paths also shown in Figs. 3 and 5 in a three-dimensional reference system. Note in this Fig. 6 that the projection of the secondary equilibrium path produces a circle on the plane \((u_2, u_3)\) shown in Fig. 4, and this projection over the plane \((f_2, u_2)\) produces a line connecting points \(C\) and \(D\) as shown in Figure 5. As can be seen in Fig. 6, on the bifurcation points \(C\) and \(D\) there is an intersection between the primary and secondary equilibrium paths, respectively. In addition, at points \(E\) and \(F\) the displacement \(u_3\) reaches the maximum absolute values of the same magnitude. The limit points \(A\) and \(B\) are extremes with respect to the load and belong to the primary equilibrium path. The next step is the calculation of the determinant of the tangent stiffness matrix to the primary equilibrium path; that is, considering the space truss contained in the plane \((x, y)\). Taking into account Eq. (18), the tangent stiffness matrix can be defined for the conditions \(\gamma = 0\) and \(\mu_3 = 0\), as

\[
K = \begin{bmatrix}
2(\alpha - \mu_2)^2 + 2\varepsilon_G & 0 \\
0 & 2\varepsilon_G + \kappa
\end{bmatrix}
\]

with \(\varepsilon_G = -\mu_2\left(\alpha - \frac{1}{2}\mu_2\right)\) \(\quad\) (26)

In which the determinant is given by \(\det K = (2(\alpha - \mu_2)^2 + 2\varepsilon_G)(2\varepsilon_G - \kappa)\).

Making this determinant zero, one can obtain that

\[
\det K = 3\mu_2^4 - 12\alpha\mu_2^3 + \left(14\alpha^2 + 3\kappa\right)\mu_2^2 - \left(4\alpha^3 + 6\alpha\kappa\right)\mu_2 + 2\alpha^2\kappa = 0
\]

(27)
Note that in Eq. (27) it is a fourth-degree polynomial with 4 real roots as shown be observed in Fig. 7. These four roots represent the four critical points defined in Eqs. (21) and (24), that is; two limit points \((A, B)\) and two bifurcation points \((C, D)\).

![Figure 7 – Determinant of the tangent stiffness matrix.](image)

### 6 Numerical analysis

To make the numerical analysis of the theoretical example presented in the previous section, a program written in Fortran90 language named “gnla_truss.f90” was written by the authors. Such program executes the incremental-iterative analysis using the Newton-Raphson method together with the arc length method. The Total Lagrangian description of the 3D truss element as described in sections 2, 3 and 4 is used. In the computational implementation, Eq. (5b) is used for the calculation of the internal force vector, expressed in global coordinates, and Eq. (10) is used for the calculation of the tangent stiffness matrix. The coefficients of the tangent stiffness matrix are also expressed in global coordinates. To capture the secondary equilibrium path, the space truss with a small imperfection in the direction and positive direction of \(z\)-axis was considered. The small imperfection was assumed to be \(c = 0.001\). The space truss was discretized with two 3D truss elements. The modulus of elasticity of the truss was taken as \(E = 200GPa\), the cross section of the truss bar is \(A = 5cm^2\). Other important variables assumed are: \(a = 1m\), \(b = 2m\) and \(\kappa = 0.1\). Therefore, according to Eq. (12), the spring stiffness is given by
\[ k = \frac{EA}{l_0} \Rightarrow k = 2000\sqrt{5} \quad kN/m. \] In the calculation of \( k \), the undeformed length considered did not take into consideration the imperfection \( c = 0.001. \)

Figs. 8a and 8b show the evolution of the displacements of the 3D truss vertex in the directions of the two axes \( y \) and \( z \), respectively. For the secondary equilibrium path expressed by the curve \((\mu_2 \times \lambda_2)\), Fig. 8a shows the comparison between the analytical and the numerical values. As can be seen in that figure, there is a good agreement between the analytical and the numerical values. As pointed out before, \( C \) and \( D \) are the bifurcation points and the line connecting these two points is the projection of the secondary equilibrium path projected on the plane curve \( \mu_2 \times \lambda_2 \) – see Fig.4.

![Figure 8 - Secondary equilibrium paths](image)

Fig. 8b shows the curve between the applied load at the 3D truss vertex in the \( y \)-axis direction and the displacement of this vertex in the \( z \)-axis direction when buckling takes place. In this figure, point E represents the maximum displacement that the vertex accomplished out of the \((x, y)\) plane in the direction of the imperfection \( c \). In addition, it is observed in that figure that after the displacement \( u_3 \) reaches the
maximum value at point \( E \), its value decreases to zero. At last, Fig. 8c shows the relation between the displacements of the vertex in the direction of the \( y \) and \( z \) axes, respectively. Note that, initially, the displacement \( u_3 = 0 \), but when the bifurcation in \( C \) takes place, there is a smooth transition of \( u_3 \) to values different than zero due to the imperfection \( c = 0.001 \). Again, it is shown in Fig. 8 that \( u_3 \) displacement reaches a maximum in \( E \). After this point, \( u_3 \) decreases until zero at point \( D \), which is the second bifurcation point. At the bifurcation point \( D \), the 3D truss returns to the primary equilibrium path. Also in Fig. 8c, the numerical and the analytical results are compared showing good agreement between both results. To get the secondary equilibrium paths shown in Figs. 8a, 8b and 8c, respectively, we used an incremental-iterative process with a constant arc length of 0.025 for 120 load steps. To test the residual convergence force a tolerance of \( 10^{-5} \) was adopted. The average number of iteration per load step was 2.1. To detect and classify the singular points along the primary equilibrium path, two test functions were adopted, among many other test functions described in Crisfield (1991, 1997). The first test function used counts the number of positive or negative pivots of the tangent stiffness matrix. When the equilibrium of the structure passes from stable to unstable; the tangent stiffness matrix changes from a positive-definite matrix to an indefinite matrix. In other words, the tangent stiffness matrix, before the critical points, has only positive pivots; but beyond the critical points this matrix shows also negative pivot(s). In the finite element software developed in this research, the solution of the system of linear equations follows the decomposition of the tangent stiffness matrix shown in Eq. (28).

\[
\begin{align*}
K &= LDL^T \\
\text{where } L &\text{ lower triangular matrix} \\
\text{with } L_{ii} &= 1 \text{ and } L^T = L^{-1}
\end{align*}
\]

and \( D \) is the following diagonal matrix
\[ D = \text{diag} \left[ D_{ii} \right] = \begin{bmatrix} D_{11} & 0 & \ldots & 0 \\ 0 & D_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D_{nn} \end{bmatrix} \] (29)

Theoretically, it is important to note that the pivot at matrix \( D \) reaches zero in the singular points. Numerically, in the singular points there is a signal change of some pivot. In this way, this test function is defined as

\[ \tau = \begin{cases} +1, & \text{if the number of positive (or negative) pivots of matrix } D \\ \text{is the same between load steps } (n-1) \text{ and } n \\ -1, & \text{if the number of positive (or negative) pivots of matrix } D \\ \text{is different between load steps } (n-1) \text{ and } n \end{cases} \] (30)

The second test function is specific for determining limit points. When a structure reaches a limit point it loses its load carrying capacity, therefore, besides the tangent stiffness matrix that becomes singular at the limit point, after this critical point the structure supports no more loading. This test function will be named “stiffness parameter” and it is defined as

\[ k = \frac{\Delta u_p^T p}{\Delta u_p^T \Delta u_p}, \quad \Delta u_p = K^{-1} p \] (31)

where \( \Delta u_p \) is the increment of the nodal displacement vector of the predictor phase and \( p \) is the vector of the external nodal forces. This test function is normalized using \( S_p = k^n / k^0 \) - where \( k^0 \) is the stiffness parameter estimated at the first iteration with the first loading step and \( k^n \) is the stiffness parameter evaluated in the predictive phase of the current loading step. The calculation of \( S_p \) is carried out in the predictive phase of each loading step. When a limit point is reached, the stiffness parameter \( k \) (and, consequently, \( S_p \)) tends to zero - for the bifurcation point, both \( k \) and \( S_p \) reach up values different than zero. Therefore, the rigidity parameter \( S_p \) and the test function \( \tau \) are both used to detect and classify the singular points in the following way

For limit points : \( \tau \) changes signal, and \( S_p = 0 \)

For bifurcation points : \( \tau \) changes signal, and \( S_p \neq 0 \) (32)

In addition to the test functions given in equations (30) and (31), in the software “gnla_truss.f90”, other test functions were also implemented as described in Crisfield
To detect the four singular points in the primary equilibrium path, this research also analyzed the 3D truss but without the imposed imperfection, that is, $c = 0$. The values assumed for the other variables were: $E = 200 \text{GPa}$, $A = 5 \text{cm}^2$, $a = 1 \text{m}$, $b = 2 \text{m}$, $\kappa = 0.1$ and $k = 2000 \sqrt{5} \text{ kN/m}$. The truss was discretized with 2 3D truss elements. In this case, an incremental-iterative process with a constant arc length of 0.025 for 90 load steps was used. To test the convergence of the residual force a tolerance of $10^{-5}$ was adopted. The average number of iterations was 2.0. Fig. 9a shows the graphics of the determinant of the tangent stiffness matrix obtained analytically and numerically. In this figure one can observe the good agreement between these two approaches.

Moreover, in Fig.9, there are 4 singular points, $(A, B, C$ and $D)$, where the determinant is zero. Only with the condition of $\det K = 0$, as shown in Fig. 9a, there are not sufficient features to classify these points. In Fig. 9b, the numerical results obtained for the test functions in Eqs. (30) and (32) are shown. For the limit points $A$ and $B$, the test function, Eq.(30), changes sign while the test function, Eq.(32), becomes zero. For the bifurcation points $C$ and $D$ the test function (based on pivot signal) changes sign, while the test function based on stiffness parameter is different than zero.
7 Conclusions

Steel structures are gradually becoming slender due to high strength steel material commonly used today in construction of high-rise buildings, aerospace structures, petroleum industries, among others. Non-linear analysis is essential in the design of steel structures. Very often finite element modeling is used to help the designer. Sometimes it is necessary to consider the stability phenomena, or critical points along the equilibrium path associated to finite element modeling of slender members. When taking into account the stability of the structural system, it is important for engineers to understand the effect of the geometric nonlinearity. The main objective of this paper was to explain the nonlinear phenomena, and consequently, make steel design safer. It is important to emphasize that, for the understanding of such non-linear phenomena, knowledge in the field of applied computational mathematics is necessary. This work described objectively, and in a short way the analytical approach for the detection, classification and sequencing of critical points in the primary equilibrium path of a simple physical system. The geometry and physical conditions for the coalescence between the limit points and bifurcation points were explained. The condition of coalescence of critical points should be taken into account for the stability study of slender structural systems. The existence of critical points in the equilibrium path can be detected from the tangent stiffness matrix singularity. The approach followed in this paper was simple and easy so that one can implement it computationally. The formulation presented was able to explain the detection and classification of critical points. For slender steel structures, it is necessary to take into consideration the non-linearity in the analysis of the structural systems equilibrium, when one takes into account the stability. It is hoped that this article has contributed for the understanding of the geometrically non-linear analysis of 3D truss structures.

8 References

Battini, J. M., Pacoste, C. and Ericksson, A. Improved minimal augmentation procedure for the direct computation of critical points. Comp. Meth. Appl. Mech. Eng. vol. 192, p. 2169-2185, 2003.

Belytschko, T.; Liu, W. K.; Moran, B. Nonlinear finite elements for continua and structures. John Wiley, 2000.
Cardona, A. and Huespe, A. *Evaluation of simple bifurcation points and post-critical path in large finite rotation problems*. Comp. Meth. Appl. Mech. Eng., vol. 175, p. 137-156, 1999.

Crisfield, M. A. *Non-linear finite element analysis of solids and structures*. Volume 1: Essentials, John Wiley, 1991.

Crisfield, M. A. *Non-linear finite element analysis of solids and structures*. Volume 2: Advanced Topics, John Wiley, 1997.

Doyle, J. F. *Nonlinear analysis of thin-walled structures. Statics, Dynamics and Stability*. Springer, 2001.

Hashiguchi, K.; Yamakawa, Y. *Introduction to finite strain theory for continuum elastoplasticity*. Wiley, 2013.

Ibrahimbergovic, A. and Mikdad, M. A. *Quadratically convergent direct calculation of critical points for 3D structures undergoing finite rotations*. Comp. Meth. Appl. Mech. Eng., vol. 189, p. 107-120, 2000.

Kojic, M.; Bathe, K. J. *Inelastic analysis of solids and structures*. Springer, 2005.

Krenk, S. *Non-linear modeling and analysis of solids and structures*. Cambridge University Press, 2009.

Neto, E. A. S; Peric, D.; Owen, D. R. J. *Computational methods for plasticity. Theory and applications*. John Wiley, 2008.

Planinc, I. and Saje, M. *A quadratically convergent algorithm for the computation of stability points: the application of the determinant of the tangent stiffness matrix*. Comp. Meth. Appl. Mech. Eng., vol. 169, p. 89-105, 1999.

Simo, J. C.; Hughes, T. J. R. *Computational Inelasticity*. Springer, 1998.

Voyiadjis, G. Z.; Woelke, P. *Elasto-plastic and damage analysis of plates and shells*. Springer, 2010.

Wriggers, P.; Simo, J. C. *A general procedure for the direct computation of turning and bifurcation points*. Comp. Meth. Appl. Mech. Eng., vol. 30, p. 155-176, 1990.

Wriggers, P. *Nonlinear finite element methods*. Springer, 2008.

Wriggers, P. *Computational contact mechanics*. 2nd Edition, Springer, 2002.