1. INTRODUCTION

Let $V$ be a six dimensional complex vector space. A smooth cubic hypersurface $X$ in $\mathbb{P}(V)$ is special if it contains an algebraic surface $S$ not homologous to a complete intersection. Denote by $h$ the hyperplane class of $X$. We say that $X$ is special of discriminant $d$ if the discriminant of the saturated lattice spanned by $h^2$ and $S$ is $d$. It is known [8] that $d > 6$ when $X$ is smooth.

The set of special cubic fourfolds of discriminant $d$ forms a divisor $C_d$ in the Hilbert scheme $\mathbb{P}(\text{Sym}^3 V^*)$. Hassett [8] gives a definition of $C_6$ via the period map (cf. §2). In this paper, we will study the degree of the these divisors in $\mathbb{P}(\text{Sym}^3 V^*)$.

Set

$$
\alpha(q) = 1 + 6 \sum_{n \geq 1} q^n \sum_{d|n} \left( \frac{d}{3} \right)
$$

$$(1.1)$$

$$
\beta(q) = \sum_{n \geq 1} q^n \sum_{d|n} (n/d)^2 \left( \frac{d}{3} \right),
$$

where $\left( \frac{d}{3} \right)$ is the Legendre symbol. Our main result is the following:
Theorem 1. Let $\Theta(q) = -2 + \sum_{d>2}^{\infty} \deg(C_d)q^{\frac{d}{3}}$ be the associated generating series. Then

$$\Theta(q) = -\alpha^{11}(q) + 162\alpha^8(q)\beta(q) + 91854\alpha^5(q)\beta^2(q)$$
$$+ 220496\alpha^2(q)\beta^3(q) - \alpha^{11}(q^\frac{1}{3}) + 66\alpha^8(q^\frac{1}{3})\beta(q^\frac{1}{3})$$
$$- 1386\alpha^5(q^\frac{1}{3})\beta^2(q^\frac{1}{3}) + 9072\alpha^2(q^\frac{1}{3})\beta^3(q^\frac{1}{3})$$
$$= -2 + 192q + 3402q^{\frac{1}{3}} + 196272q^2 + 915678q^{\frac{1}{3}} + \ldots$$

is a modular form of weight 11 and level 3.

Remark 1. The constant term of $\Theta(q)$ is determined by the degree of the Hodge bundle of a pencil of cubic fourfolds. This will be discussed in §3 and §4.

This paper is organized as follows: In section 2, we review Hodge theory on cubic fourfolds, such as the period domain of the polarized lattice $\Lambda_0 := \langle h^2 \rangle^\perp$ in the middle cohomology and the global Torelli theorem for cubic fourfolds. Moreover, we give an interpretation of the degree of $C_d$ as an intersection number on the period domain and give some examples. Section 3 is devoted to introducing Borcherd’s work on vector-valued modular forms on lattices and Heegner divisors. In particular, we review the results about the modularity of the generating series of Heegner divisors, and apply them to our lattice $-\Lambda_0$ to show the correspondence between the Heegner divisors and special cubic fourfolds. As a result, we show that the vector-valued generating function of $\deg(C_d)$ is a vector-valued modular form $\Psi$ of a certain type. In section 4, we construct basis of the space of vector-valued modular forms of that type. As a consequence, we can express $\Psi$ explicitly as some functions of vector-valued Eisenstein series. Theorem 1 is proved in section 5. In our last section, we show that the Picard group of the arithmetic quotient of the Period domain is generated by the Heegner divisors.

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2. Special Cubic Fourfolds

In this section, we review some classical results on cubic fourfolds.

2.1. Hodge theory. Let $X$ be a smooth cubic fourfold. The middle cohomology $H^4(X,\mathbb{Z}) = \Lambda$ is a rank 23 odd lattice of signature (21, 2) under the intersection form $\langle , \rangle$ (cf. [8]).
We take $\Lambda_0$ be the orthogonal complement of $h^2$ and denote by $\Lambda_0^\vee$ the dual lattice of $\Lambda_0$. It is known [8] that

$$\Lambda_0 = W \oplus U \oplus E_8^{\oplus 2},$$

where

$$W = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $E_8$ is the positive definite lattice associated to the Lie group of the same name.

The period domain

$$D = \{ \omega \in \mathbb{P}(\Lambda_0 \otimes \mathbb{Z}; \mathbb{C}) | \langle \omega, \omega \rangle = 0, -\langle \omega, \overline{\omega} \rangle > 0 \},$$

can be viewed as the analytic open subset of a quadric.

Let $X$ be the quotient $D/\Gamma$ for the arithmetic subgroup $\Gamma \subseteq \text{Aut}(\Lambda_0)$ stabilizing $D$. As in [8], $X$ is known to be a quasi-projective variety of dimension twenty.

Let $C^\circ$ be the moduli space of smooth cubic fourfolds. The period map

$$\mathcal{P}^\circ : C^\circ \to X$$

sending each smooth cubic fourfold to its period is an open immersion of analytic spaces [18, 19]. Therefore, we may regard $C^\circ$ as an open subset of the arithmetic quotient $X$.

**Definition 1.** Let $L$ be a rank-two positive definite saturated sublattice of $\Lambda$ containing $h^2$. Let $D_L$ be the hyperplane of $D$ defined by

$$D_L = \{ \omega \in D | \omega \perp L \}.$$  

Then we define $D_d$ to be the quotient by $\Gamma$ of the union of hyperplanes $D_L$ with $\det(L) = d$.

**Remark 2.** The union of $D_L$ with the given discriminant $d$ has a nontrivial stabilizer only if $d = 6$. The stabilizer acts trivially on $W^\perp$ and acts as the reflection on $W$.

The following result is shown in [8],

**Theorem 2.** If $d > 6$, $D_d \subset X$ corresponds to the special cubic fourfolds of discriminant $d$ and is a nonempty divisor if and only if $d \equiv 0, 2 \mod 6$.

**Remark 3.** Via a study of degenerations of cubic fourfolds, Hassett [8] has also shown that:

- (i) $D_2$ parametrizes the limiting mixed Hodge structures arising from determinantal cubic fourfolds.
- (ii) $D_6$ parametrizes the limiting mixed Hodge structures arising from cubic fourfolds with a single double point.
2.2. The degree of special cubic fourfolds. Let $\mathcal{U} \subset \mathbb{P}(\text{Sym}^3 V^*)$ be the Zariski open subset parametrizing cubic hypersurfaces with at worst isolated simple (A-D-E) singularities. Since a cubic fourfold with isolated simple singularities is GIT stable, it makes sense to define the moduli space $\mathcal{C}$ of such cubic fourfolds (cf. [10], [11] Theorem 2.1). Therefore, we have a canonical morphism

$$\mathcal{U} \to \mathcal{C}.$$ 

In [11], Laza has studied the image of the period map, and shown that $P^o : \mathcal{C}^o \to X$ extends to a regular morphism $P : \mathcal{C} \to X$. Let $\varphi$ be the composition

$$\mathcal{U} \to \mathcal{C} \to X.$$ 

Then the divisor $C_d$ can be considered as the Zariski closure of $\varphi^{-1}(D_d)$ in $\mathbb{P}(\text{Sym}^3 V^*)$ for $d > 2$.

**Remark 4.** Actually, Looijenga [12] and Laza [11] have both shown that the complement of $\mathcal{C}^o$ in $X$ is the union of $D_2$ and $D_6$.

A natural approach to compute the degree of $C_d$ is via intersections with test curves. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a Lefschetz pencil of cubic hyperplane sections of $\mathbb{P}(V)$. It yields a natural morphism

$$(2.5) \quad \iota_\pi : \mathbb{P}^1 \to \mathbb{P}(\text{Sym}^3 V^*)$$

factoring through $\mathcal{U}$. Then $\deg(C_d) = N_d := \int_{\mathbb{P}^1} \iota_\pi^*[C_d]$. For $d > 2$, the intersection number $N_d$ can be computed by

$$(2.6) \quad \int_{\mathbb{P}^1} \kappa_\pi^*[D_d],$$

where $\kappa_\pi : \mathbb{P}^1 \to \mathcal{X}$ is the composition of $\iota_\pi$ and $\varphi$.

If $d = 2$, since there are no determinantal cubic fourfolds in a Lefschetz pencil of cubic fourfolds, we set $N_2 = 0$.

2.3. Some examples. In this subsection, we will give some examples of $\deg(C_d)$ by enumerative geometry methods.

2.3.1. $d=6$, cubic fourfolds with double points.

The first jet bundle $J^1(\mathcal{O}_{\mathbb{P}(V)}(3))$ [16] of $\mathcal{O}_{\mathbb{P}(V)}(3)$ is defined by the following exact sequence:

$$(2.7) \quad 0 \to \Omega_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(3) \to J^1(\mathcal{O}_{\mathbb{P}(V)}(3)) \to \mathcal{O}_{\mathbb{P}(V)}(3) \to 0.$$ 

In addition, we have a natural surjection $\text{Sym}^3(V^*) \to J^1(\mathcal{O}_{\mathbb{P}(V)}(3))$ and denote by $K$ the kernel of this map. Then $K$ is a rank 50 vector bundle, corresponding to the universal subbundle parameterizing all the cubic fourfolds containing a double point. It fits into the following exact sequence:

$$(2.8) \quad 0 \to K \to \text{Sym}^3(V^*) \to J^1(\mathcal{O}_{\mathbb{P}(V)}(3)) \to 0.$$
Let $\mathbb{P}(K)$ be the projectivization of $K$ and set $\xi = c_1(\mathcal{O}_{\mathbb{P}(K)}(1))$ with 
$$\xi^{50} + c_1(K)\xi^{49} + \cdots + c_5(K)\xi^{45} = 0.$$ 
By direct computation, we have: 
$$\text{deg}(C_6) = \xi^{54} = 192.$$ 
(cf. Appendix 7.1)

2.3.2. $d=8$, cubic fourfolds containing a plane.

Let $Gr(3, V)$ be the Grassmannian parametrizing all planes in $\mathbb{P}(V)$ and $S$ be the universal subbundle of $Gr(3, V)$. Take the dual of the canonical exact sequence

$$0 \to S \to V \otimes \mathcal{O}_{Gr(3,V)} \to \underbrace{Q\text{ quotient}} \to 0,$$

we get

$$0 \to Q^* \to V^* \otimes \mathcal{O}_{Gr(3,V)} \xrightarrow{p} S^* \to 0.$$ 

Then the morphism $p$ induces a surjection $\text{Sym}^3(V^* \otimes \mathcal{O}_{Gr(3,V)}) \to \text{Sym}^3 S^*$, which gives

$$0 \to K' \to \text{Sym}^3(V^* \otimes \mathcal{O}_{Gr(3,V)}) \to \text{Sym}^3 S^* \to 0.$$ 

The kernel $K'$ corresponds to the universal subbundle parametrizing all the cubic fourfolds containing a plane.

Similar as the case of $C_6$, we obtain

$$\text{deg}(C_8) = \xi^{54} = 3402$$ 

via standard Schubert calculus, where $\xi = c_1(\mathcal{O}_{\mathbb{P}(K')}(1))$ (cf. Appendix 7.2)

3. Modular forms

We stick to the notation introduced in $\S 2$

3.1. Vector-valued modular forms. The metaplectic double cover of $SL_2(\mathbb{Z})$ consists of the elements $(A, \phi(\tau))$, where 

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad \phi(\tau) = \pm \sqrt{c\tau + d}.$$ 

It is well-known that $Mp_2(\mathbb{Z})$ is generated by 

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sqrt{\tau}.$$ 

Let $\mathcal{H}$ be the complex upper half-plane. Suppose we have a representation $\rho$ of $Mp_2(\mathbb{Z})$ on a finite dimensional complex vector space $V$, such that $\rho$
factors through a finite quotient. For any $k \in \frac{1}{2}\mathbb{Z}$, a vector-valued modular form $f(\tau)$ of weight $k$ and type $\rho$ on $\mathbb{V}$ is a function
\[ f : \mathcal{H} \to \mathbb{V} \]
holomorphic on $\mathcal{H}$ and at $\infty$, such that for all $g = (A, \phi(\tau)) \in Mp_2(\mathbb{Z})$, we have
\[ f(A\tau) = \phi(\tau)^{2k} \cdot \rho(g)(f(\tau)). \]
When $\dim \mathbb{V} = 1$, this recovers the definition of the scalar-valued modular forms with a character.

Given an even nondegenerate integral lattice $M$ of signature $(b^+, b^-)$ with a bilinear form $\langle , \rangle$, we denote by $M^\vee$ its dual lattice. From now on, we always regard $M$ as an even nondegenerate integral lattice. There is a Weil representation $\rho_M$ of $Mp_2(\mathbb{Z})$ on the group ring $\mathbb{C}[M^\vee/M]$. Let $v_\gamma$ be the standard basis of $\mathbb{C}[M^\vee/M]$ for $\gamma \in M^\vee/M$. Since $Mp_2(\mathbb{Z})$ is generated by $S$ and $T$, $\rho_M$ can be defined by the actions of the generators as follows:
\begin{align*}
\rho_M(T)v_\gamma &= e^{2\pi i (\gamma, \gamma)}v_\gamma \\
\rho_M(S)v_\gamma &= \frac{\sqrt{b^+ - b^-}}{\sqrt{|M^\vee/M|}} \sum_{\delta \in M^\vee/M} e^{-2\pi i (\gamma, \delta)}v_\delta.
\end{align*}
If $N > 0$ is the smallest integer for which $N\langle \gamma, \gamma \rangle \in \mathbb{Z}$ for all $\gamma \in M^\vee$, the representation factors through the finite index subgroup $\tilde{\Gamma}(N) \subset Mp_2(\mathbb{Z})$, where
\[ \tilde{\Gamma}(N) = \left\{ (A, \phi)|A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \]
Let $\rho_M^*$ be the dual representation of $\rho_M$ and denote by $\text{Mod}(Mp_2(\mathbb{Z}), k, \rho_M^*)$ the space of modular forms of weight $k$ and type $\rho_M^*$.}

3.2. Heegner divisors. Let $M$ have signature $(2, m)$. We consider the domain
\[ D_M = \{ \omega \in \mathbb{P}(M \otimes \mathbb{C})| \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0 \}, \]
and
\[ \Gamma_M = \{ g \in Aut(M)| g \text{ acts trivially on } M^\vee/M \}. \]
Denote by $X_M$ the quotient $D_M/\Gamma_M$ (cf. [5]). Given $v \in M^\vee$, there is an associated hyperplane
\[ v^\perp = \{ w \in D_M| \langle w, v \rangle = 0 \} \subseteq D_M. \]
It is easy to see that the value $(v, v)$ and the residue class $v \mod M$ are both invariant under the action of $\Gamma_M$. Therefore, for each pair of $n \in \mathbb{Q}^{<0}$ and $\gamma \in M^\vee/M$, one can define the Heegner divisor [3] $y_{n, \gamma}$ of $X_M$ by
\[ y_{n, \gamma} = \left( \sum_{\gamma \equiv v \mod M} v^\perp \right) / \Gamma_M. \]
In the degenerate case where \( n = 0 \), we take \( y_{0,0} \) to be the \( \mathbb{Q} \)-Cartier divisor coming from \( O(1) \) on \( D_M \subset \mathbb{P}(M \otimes \mathbb{C}) \).

Set \( q = e^{2\pi i \tau} \) and consider the vector-valued generating series

\[
\vec{\Phi}(q) = \sum_{n \in \mathbb{Z} \geq 0} \sum_{\gamma \in M^\vee / M} y_{-n,\gamma} q^n v_{\gamma} \in Pic(X_M)[[q^{1/N}]] \otimes_\mathbb{Z} \mathbb{C}[M^\vee / M];
\]

we have the following results:

**Theorem 3.** \( \text{[2][1][3]} \) Let \( M \) be an even lattice of signature \( (2,m) \). The generating series \( \vec{\Phi}(q) \) is an element of \( \text{Pic}(X_M) \otimes_\mathbb{Z} \text{Mod}(M_{p_2}(\mathbb{Z}),1 + \frac{m}{2}, \rho_M^*) \).

**Remark 5.** As a result, for any linear function

\[
(3.2) \quad \lambda : \text{Pic}(X_M) \otimes \mathbb{C} \to \mathbb{C},
\]

the corresponding linear contraction \( \lambda(\vec{\Phi}) \) is a vector-valued modular form in \( \text{Mod}(M_{p_2}(\mathbb{Z}),1 + \frac{m}{2}, \rho_M^*) \).

3.3. **Application to special cubic fourfolds.** Now, we take \( M \) to be the lattice \( \Lambda'_0 := -\Lambda_0 \) given in \([2.1]\), i.e. the bilinear form on \( \Lambda'_0 \) is the negative intersection form of \( \Lambda_0 \). By abuse of notation, we continue to use \( \langle , \rangle \) for this bilinear form on \( \Lambda'_0 \).

One can certainly identify the quotient \( X \) and \( X_{\Lambda'_0} \), and regard \( D_d \) as the natural divisor in \( X_{\Lambda'_0} \).

Since \( |\Lambda'_0 / \Lambda_0| = |\det(\Lambda'_0)| = 3 \), we can take \( \gamma_i \) to be the three elements of \( \Lambda'_0 / \Lambda_0 \) with \( \frac{1}{2} \langle \gamma_i, \gamma_i \rangle = -\frac{i^2}{3} \) mod \( \mathbb{Z} \) for \( i = 0,1,2 \). Denote by \( v_i \) the corresponding basis of \( \mathbb{C}[\Lambda'_0 / \Lambda_0] \). The following result is straightforward:

**Lemma 1.** The Heegner divisors \( y_{n,\gamma} = y_{m,-\gamma} = D_d \), where

\[
n = \frac{-d}{6} \text{ and } \gamma \equiv \frac{d}{2} \gamma_1 \mod \Lambda'_0,
\]

for \( (n,\gamma) \neq (0,0) \).

**Proof.** The redundancy \( y_{n,\gamma} = y_{m,-\gamma} \) is because of the symmetry \( \langle v \rangle^\perp = \langle -v \rangle^\perp \). Let \( L \subset -\Lambda \) be a rank 2 negative sublattice containing \( h^2 \) and of discriminant \( d \). Assume that \( L \) is generated by \( h^2 \) and \( \zeta \). Then there is a bijection between the two sets of hyperplanes as follows:

\[
D_L = \{ \omega \in D_{\Lambda'_0} | \omega \perp L \} \leftrightarrow v^\perp
\]

\[
(3.3) \quad \zeta \leftrightarrow v = \zeta + \frac{\langle \zeta, h^2 \rangle}{3} h^2,
\]
since one can verify that

\[
\frac{1}{2} \left( \zeta + \frac{\langle \zeta, h^2 \rangle}{3} h^2, \zeta + \frac{\langle \zeta, h^2 \rangle}{3} h^2 \right) = n
\]

\[
\zeta + \frac{\langle \zeta, h^2 \rangle}{3} h^2 \equiv \pm \frac{d}{2} \gamma_1 \mod \Lambda_0'.
\]

In this situation, the intersection of \( y_{0,0} \) with the test curve \( \kappa_\pi : \mathbb{P}^1 \to \mathcal{X} \) in \( \mathbb{P}^3 \) equals to the degree of the Hodge bundle \( R^3 \pi_*(\Omega^1_{\mathcal{X}/\mathbb{P}^1}) \), where \( \Omega^1_{\mathcal{X}/\mathbb{P}^1} \) is the relative sheaf of holomorphic 1-forms. After applying the linear contraction \( (2.6) \) to \( -\rightarrow \Phi \), we obtain that:

**Corollary 4.** The vector-valued meromorphic function

\[
\Psi(q) = \deg(R^3 \pi_*(\Omega^1_{\mathcal{X}/\mathbb{P}^1})) v_0 + \sum_{i=0}^2 \sum_{d \equiv i \mod 3} N_d q^d v_i
\]

is a vector-valued modular form of weight 11 and type \( \rho_{\Lambda_0}' \).

4. **Modular forms on \( O(2, 20) \) lattice**

4.1. **Dimension of \( \text{Mod}(M_{p_2}(\mathbb{Z}), k, \rho_{\Lambda_0}' \)).** In this subsection, we will compute the dimension of the space \( \text{Mod}(M_{p_2}(\mathbb{Z}), k, \rho_{\Lambda_0}' \)) via Bruinier’s formula [4].

**Lemma 2.** \( \dim \text{Mod}(M_{p_2}(\mathbb{Z}), k, \rho_{\Lambda_0}') = \begin{cases} 1 & \text{if } k = 3, 5; \\ 2 & \text{if } k = 11. \end{cases} \)

**Proof.** According to Lemma 2 in [4], the formula yields the following evaluation:

\[
\dim \text{Mod}(M_{p_2}(\mathbb{Z}), k, \rho_{\Lambda_0}') = (2 - 1/2 - 2/3) + \frac{k}{6} \left( \frac{k}{4 \sqrt{3}} \Re[e^{\frac{(k-1)\pi i}{2}} G(2, \Lambda_0')] - \frac{1}{9} \Re[e^{\frac{(k+2)\pi i}{3}} (G(1, \Lambda_0') + G(-3, \Lambda_0'))] - \frac{1}{3} \right)
\]

\[
= \begin{cases} 1 & \text{if } k = 3, 5; \\ 2 & \text{if } k = 11. \end{cases}
\]

where \( \Re(.) \) denotes the real part of \( . \) and \( G(a, \Lambda_0') \) denotes the quadratic Gauss sum

\[
G(a, \Lambda_0') = \sum_{n=0}^2 e^{-\frac{2\pi n^2}{3} \pi i}, a \in \mathbb{Z}.
\]
Remark 6. Since $\Lambda'_0$ and the lattice $-W$ in \[\text{(2.2)}\] differ by an even unimodular lattice, we have \[\text{Mod}(M_2(Z), k, \rho^*_{\Lambda'_0}) = \text{Mod}(M_2(Z), k, \rho^*_{-W}).\] Therefore, we will study the modular forms on lattice $W' = -W$ instead.

4.2. Eisenstein series. In this subsection, we summarize some results on various Eisenstein series (cf. \[\text{(7)}\] § 4) which will be used later.

4.2.1. Scalar-valued case. Assume that $k > 2$ is even. The classical Eisenstein series
\[
E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sum_{d | n} d^{k-1} q^n
\]
is a modular form of weight $k$ for $SL_2(Z)$, where $B_k$ is the Bernoulli number (see later in \[\text{(4.7)}\]).

Let $\chi$ denote the nontrivial Dirichlet character modulo 3 on $SL_2(Z)$, i.e.
\[
\chi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \chi_{-3}(d), \forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(Z)
\]
where $\chi_{-3}: Z \to \{0, \pm 1\}$ is the nontrivial Dirichlet character modulo 3. We define the modular group $\Gamma_0(3)$ (resp. $\Gamma_0(3)$) to be the subgroup
\[
\left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(Z) \mid c \equiv 0 \mod 3 \right\} \text{(resp. } b \equiv 0 \mod 3),
\]
Then we have following results:

Proposition 4.2.2. Assume that $k > 0$ is an odd integer. Then
\[
E_k(\tau, \chi) := \begin{cases} 
1 + 6 \sum_{n \geq 1} \sum_{d | n} \chi_{-3}(\frac{a}{d}) q^n & k = 1, \\
\sum_{n \geq 1} \sum_{d | n} d^{k-1} \chi_{-3}(\frac{a}{d}) q^n & k \geq 3.
\end{cases}
\]
is a modular form of weight $k$ with character $\chi$ for $\Gamma_0(3)$.

Respectively, $E'_k(\tau, \chi) = E_k(\tau/3, \chi)$ is a modular form of weight $k$ with character $\chi$ for $\Gamma_0(3)$.

Proof. See \[\text{(3)}\] Lemma 10.2 and Lemma 10.3 for the modularity of \[\text{(4.2)}\].

Next, the modularity of $E'_k(\tau, \chi)$ comes from \[\text{(7)}\] Theorem 4.2.3 and §4.8. ♣

4.2.3. Vector-valued case. Now we introduce the case of vector-valued Eisenstein series on $C[M^\vee/M]$ constructed by Bruinier and Kuss in \[\text{(6)}\]. Let $k \in \frac{1}{2}Z$ and $f$ be a vector-valued function on $C[M^\vee/M]$. The Petersson slash operator is defined by
\[
f|_k^*(g)(\tau) = \phi(\tau)^{-2k} \rho^*_M(g)^{-1} f(A\tau)
\]
for $g = (A, \phi) \in M_2(Z)$. Then we have:
Lemma 3. Let $v_0 \in \mathbb{C}[M^\vee / M]$ be the vector corresponding to the trivial class in $M^\vee / M$ and consider $v_0$ as a constant function from $\mathcal{H}$ to $\mathbb{C}[M^\vee / M]$. The vector-valued Eisenstein series attached to $M$ with respect to $v_0$

\begin{equation}
\vec{E}_k(\tau) = \frac{1}{2} \sum_{g \in \tilde{\Gamma}_\infty \backslash Mp_2(\mathbb{Z})} v_0|_k^*(g)(\tau)
\end{equation}

is a vector-valued modular form of weight $k$ and type $\rho^*_M$ on $\mathbb{C}[M^\vee / M]$, where $\tilde{\Gamma}_\infty$ denotes the subgroup of $Mp_2(\mathbb{Z})$ generated by the elements

\begin{equation}
\left( \begin{array}{cc}
1 & n \\
0 & 1
\end{array} \right), \quad n \in \mathbb{Z}
\end{equation}

For $\gamma, \gamma' \in M^\vee / M$, we denote by $\gamma \gamma'$ the function $\langle \gamma, \gamma' \rangle \mod \mathbb{Z}$. In the case of $M = W'$ and $k > 2$, the weight $k$ vector-valued Eisenstein series attached to $W'$ is given by [6] as follows:

\begin{equation}
\vec{E}_k(q) = 2v_0 + \sum_{\gamma \in W^\vee / W} \sum_{n \in \mathbb{Z} + \frac{1}{2} \gamma^2} \frac{2^{k+1} \pi k n^{k-1} (-1)^{(k-1)/2}}{\sqrt{3} \Gamma(k) L(k, \chi_{-3})} \prod_{p|18n} \frac{L_{\gamma,n}(k,p)}{1 - \chi_{-3}(p)p^{-k}q^n v_\gamma}
\end{equation}

Here, $L(k, \chi_{-3})$ denotes the Dirichlet L-series with character $\chi_{-3}$ and $L_{\gamma,n}(k,p)$ is the local Euler product defined as following:

\begin{equation}
d_\gamma = \min\{b \in \mathbb{N}, \, b\gamma \in W'\}; \\
\omega_p = 1 + 2v_p(2d_\gamma n), \, v_p \text{ is the } p\text{-evaluation}; \\
N_{\gamma,n}(a) = \sharp\{r \in (\mathbb{Z}/a\mathbb{Z})^2 | \frac{1}{2}(r - \gamma)^2 + n \equiv 0 \mod a\}; \\
L_{\gamma,n}(k,p) = (1 - p^{1-k}) \sum_{\omega_p^{-1}} N_{\gamma,n}(p^\nu) p^{-k\nu} + N_{\gamma,n}(p^{\omega_p}) p^{-k\omega_p}.
\end{equation}

When $k = 5$,

\begin{equation}
L(5, \chi_{-3}) = \frac{2^4 \pi^5}{5! \sqrt{3}} \sum_{n=1}^{3} \chi_{-3}(n) B_5(1 - n/3) = \frac{2^5 \pi^5}{3^3 \Gamma(5) \sqrt{3}}.
\end{equation}

where $B_k(x)$ is the Bernoulli polynomial [20] given by the generating function

\begin{equation}
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_k(x) \frac{t^k}{k!},
\end{equation}
and $B_k(0) = B_k$. Thus one obtains that
\begin{equation}
\vec{E}_5(q) = 2v_0 + \sum_{i=0}^{2} \sum_{n \in \mathbb{Z} + \frac{1}{4} i} 486n^4 \prod_{p \mid 18n} \frac{L_{n, i}(5, p)}{-\chi_{-3}(p) p - q^n v_i}
\end{equation}
(4.8)
\[= (2 + 492 q + 7200 q^2 + 39372 q^3 + \ldots) v_0 + (6 q^{1/3} + 1446 q^{4/3} + 14412 q^{7/3} + \ldots) v_1 + (6 q^{1/3} + 1446 q^{4/3} + 14412 q^{7/3} + \ldots) v_2.
\]

**Remark 7.** In this situation, the vector-valued Eisenstein series $\vec{E}_5(q)$ equals two times the Siegel theta series $\Theta_5(q)$ of the lattice $M = W \oplus E_8$:
\begin{equation}
\vec{\Theta}_{5,M}(q) = \sum_{\gamma \in M^\vee / M} \sum_{r \in M + \gamma} q^{2r^2} v_{\gamma}.
\end{equation}

### 4.3. Construction of modular form.

Now we are ready to find the basis of $\text{Mod}(M p_2(\mathbb{Z}), 11, \rho_{11}^\star)$. Given any two level $N$ scalar-valued modular forms $f(q), g(q)$ on the upper half plane $\mathcal{H}$ of weight $k_1$ and $k_2$. The n-th Rankin-Cohen bracket is defined as follows:
\begin{equation}
[f(q), g(q)]_n = \sum_{r=0}^{n} (-1)^r \binom{n + k_1 - 1}{n - r} \binom{n + k_2 - 1}{r} f^{(r)}(q) \cdot g^{(n-r)}(q)
\end{equation}
where $f^{(r)}$ denotes the r-th differential of $f$ with respect to $\tau$.

For a vector-valued modular form
\begin{equation}
\vec{F}(q) = \sum_{\gamma \in M^\vee / M} F_\gamma v_{\gamma} \in \text{Mod}(M p_2(\mathbb{Z}), k_1, \rho_M^\star),
\end{equation}

one can extend the Rankin-Cohen bracket to $\vec{F}(q)$ and $g(q)$ as follows,
\begin{equation}
[\vec{F}(q), g(q)]_n = \sum_{\gamma \in M^\vee / M} [F_\gamma(q), g(q)]_n v_{\gamma}
\end{equation}

The following result can be found in [13],

**Lemma 4.** If $\vec{F}(q) \in \text{Mod}(M p_2(\mathbb{Z}), k_1, \rho_M^\star)$ and $g(q)$ a scalar-valued modular form of weight $k_2$ and level 1, then
\begin{equation}
[\vec{F}, g(q)]_n \in \text{Mod}(M p_2(\mathbb{Z}), k_1 + k_2 + 2n, \rho_M^\star).
\end{equation}

Our main result is:

**Theorem 5.** The vector-valued functions
\begin{equation}
\vec{E}_n(q) = [\vec{E}_6(q), E_{6-2n}(q)]_n, n = 0, 1,
\end{equation}
form a basis of $\text{Mod}(M p_2(\mathbb{Z}), 11, \rho_{11}^\star)$. Furthermore,
\begin{equation}
\vec{\Psi}(q) = -\vec{F}_0(q) - \frac{3}{4} \vec{F}_1(q)
\end{equation}
\begin{equation}
= (-2 + 192 q + 196272 q^2 + \ldots) v_0 + (0 + 3402 q^{4/3} + 917568 q^{7/3} + \ldots) v_1 + (3402 q^{4/3} + 917568 q^{7/3} + \ldots) v_2.
\end{equation}
Proof. The modularity of $\tilde{F}_n(q)$ comes from Lemma 4. Moreover, a direct computation shows that $\tilde{F}_0$ and $\tilde{F}_1$ are linearly independent. Thus they form a basis of $\text{Mod}(Mp_2(Z), 11, \rho_\ast^{\ast V})$ by Lemma 2.

To obtain the expression (4.10), it suffices to use the following two constraint conditions:

1. The degree of the Hodge bundle $R^3\pi_*\Omega^1_{X/P}$ is $-2$, which gives the coefficient of $q^0v_0$.
   By Grothendieck-Riemann-Roch, we have the following Chern character computation:
   $$\text{ch}(\pi!\Omega^1_{X/P}) = \text{ch}(-R^1\pi_*\Omega^1_{X/P} - R^3\pi_*\Omega^1_{X/P})$$
   $$= \pi_*(\text{ch}(\Omega^1_{X/P})\text{td}(T_{X/P})$$
   $$= -2 + 2c_1(O_{P}(1)),$$
   where $T_{X/P}$ is the relative tangent bundle. Moreover, as the line bundle $R^1\pi_*\Omega^1_{X/P}$ is trivial by the Lefschetz hyperplane theorem, we obtain $\deg(\Omega^1_{X/P}) = -2$.

2. There is no discriminant 2 special cubic fourfolds in a general pencil of cubic fourfolds. It follows that the coefficient of $q^{1/3}v_1$ is 0.

\*\*\*

Remark 8. The coefficients in (4.10) coincide with our computation in \S 2.3.

5. Proof of Theorem 1

In this section, we will study the modularity of certain functions coming from a vector-valued modular function on $\mathbb{C}[M'/M]$.

5.1. Modularity of $\Psi_i$. Denote by $\mathcal{M}$ the group $M'/M$ and let $m_c : \mathcal{M} \to \mathcal{M}$ be the multiplication map by $c \in \mathbb{Z}$. Let $\mathcal{M}^c$ be the image of $m_c$ and $\mathcal{M}_c$ be the kernel. The set $\mathcal{M}^{c*}$ defined by

$$(5.1)\quad \{\nu \in \mathcal{M} \mid c\gamma^2/2 + \nu\gamma = 0 \mod \mathbb{Z}, \forall \gamma \in \mathcal{M}_c\},$$

is a coset of $\mathcal{M}^c$ with representative $x_c$ (cf. [17]). Then there is a well-defined $\mathbb{Q}/\mathbb{Z}$ valued function $\mathcal{M}^{c*} \to \mathbb{Q}/\mathbb{Z}$ given by

$$\nu = x_c + c\gamma \mapsto \nu^2 = c\gamma^2/2 + \gamma x_c \mod \mathbb{Z}.$$ 

Write $e(x) = e^{2\pi i x}$; the Weil representation can be expressed as follows:

Theorem 6. [17] Let $g = (A, \phi) \in Mp_2(\mathbb{Z})$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$(5.2)\quad \rho_\ast^\ast(g)v_\gamma = \xi^{\sqrt{\left|\mathcal{M}_c\right|}}\sqrt{\left|\mathcal{M}\right|} \sum_{\nu \in \mathcal{M}^{c*}} e(-ar^2_c/2 - bv\gamma - bd\gamma^2)v_{d\gamma + \nu},$$

where $\xi = e(\text{sign}(\mathcal{M})/4) \prod_p \xi_p$, and the local factor $\xi_p$ is determined by the Jordan components of $\mathcal{M}$.
In particular, if $N$ is an integer such that $\frac{N\gamma^2}{2} \in \mathbb{Z}$ for all $\gamma \in M$, then
\begin{equation}
(5.3)
\rho_M(g)v_\gamma = \left( \frac{a}{|M|} \right)e((a-1)\mathrm{oddy}(M)/8)e(-bd\gamma^2/2)v_d\gamma, \text{ if } g \in \tilde{\Gamma}_0(N)
\end{equation}
where $\tilde{\Gamma}_0(N) \subset M_{p2}(\mathbb{Z})$ is the preimage of $\Gamma_0(N)$ and $(\frac{a}{m})$ is the Jacobi symbol.

The definitions of $\xi_p$ and the oddity of $M$ are given in [17]. Now we consider the case where $M = W'$ and denote $W = W' \vee W'/W'$. The following result is straightforward by (5.3):

**Corollary 7.** Let $\vec{F} = \sum_{i=0}^{2} F_i v_i$ be a vector-valued modular form of weight $k$ and type $\rho_{W'}^*$. Then

(i). $F_0$ is a scalar-valued modular form for $\Gamma_0(3)$ of weight $k$ with character $\chi$.

(ii). $F_1 = F_2$ is a scalar-valued modular form for $\Gamma_1(3)$ of weight $k$ with character $\chi'(A) = e^{2\pi i A}$, where

$$\Gamma_1(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a \equiv d \equiv 1, \ b \equiv 0 \mod 3 \right\}.$$ 

The space $\text{Mod}(\Gamma_0(3), \chi)$ of modular forms with character $\chi$ for $\Gamma_0(3)$ is a polynomial ring generated by $E_1(\tau, \chi)$ and $E_3(\tau, \chi)$ in (4.2) (cf. [3]). Therefore, the holomorphic function $F_0$ can be expressed as a polynomial of $E_1(\tau, \chi)$ and $E_3(\tau, \chi)$.

Since the Legendre symbol of 3 and the Dirichlet character $\chi_{-3}$ coincide, we can see that
\begin{align*}
(5.4) \quad & \alpha(q) = E_1(q, \chi) = 1 + 6q + 6q^3 + 12q^7 + \ldots \\
& \beta(q) = E_3(q, \chi) = q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + \ldots
\end{align*}

Then we have

**Corollary 8.** Set $\vec{\Psi} = \sum_{i=0}^{2} \Psi_i v_i$. Then
\begin{equation}
(5.5) \quad \Psi_0(q) = -2\alpha^{11} + 324\alpha^8\beta + 183708\alpha^5\beta^2 + 4408992\alpha^2\beta^3.
\end{equation}

*Proof.* The computation is done via checking the first four terms of $\Psi_0$. ☐

**5.2. Modularity of $\Theta(q)$.** With the notation above, we consider the scalar-valued holomorphic function
\begin{equation}
(5.6) \quad \Theta'(q) = \Psi_0 + \Psi_1 + \Psi_2.
\end{equation}

Recalling that $\Theta(q) = \Psi_0 + \frac{1}{2}(\Psi_1 + \Psi_2)$, we have
\begin{equation}
(5.7) \quad \Theta(q) = \frac{1}{2}(\Theta'(q) + \Psi_0(q)).
\end{equation}
Since \( \ker \rho^*_W = \Gamma(3) \), \( \Theta(q) \) is a modular form of level 3. Moreover, one can check that \( \Theta'(q) \) is a scalar-valued modular form with character \( \chi \) for the congruence subgroup \( \Gamma^0(3) \) with generators
\[
\begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}, \begin{pmatrix}
2 & 3 \\
-1 & -1
\end{pmatrix}
\]
(5.8)

Remark 9. This can be verified by either Theorem 6 or using the generators (5.8) to check the modularity.

Similar as \( \text{Mod}(\Gamma_0(3), \chi) \), the space \( \text{Mod}(\Gamma_0(3), \chi) \) of modular forms for \( \Gamma^0(3) \) with character \( \chi \) is also a polynomial ring generated by the Eisenstein series \( E_1'(\tau, \chi) \) and \( E_3'(\tau, \chi) \).

Therefore, the \( q \)-expansion of \( \Phi'(\tau) \) can be expressed as a linear combination of
\[
\alpha_{11}(q^{\frac{1}{3}}), \alpha_8(q^{\frac{1}{3}})\beta(q^{\frac{1}{3}}), \alpha_5(q^{\frac{1}{3}})\beta^2(q^{\frac{1}{3}}), \alpha_2(q^{\frac{1}{3}})\beta^3(q^{\frac{1}{3}}).
\]
The following formula is then obtained by checking coefficients:

Corollary 9.

\[
\Theta'(q) = -2\alpha_{11}(q^{\frac{1}{3}}) + 132\alpha_8(q^{\frac{1}{3}})\beta(q^{\frac{1}{3}}) - 2\alpha_5(q^{\frac{1}{3}})\beta^2(q^{\frac{1}{3}}) + 1814\alpha_2(q^{\frac{1}{3}})\beta^3(q^{\frac{1}{3}}),
\]
(5.10)

Hence,
\[
\Theta(q) = -\alpha_{11}(q) + 162\alpha_8(q)\beta(q) + 91854\alpha_5(q)\beta^2(q) + 2204496\alpha_2(q)\beta^3(q) - 1386\alpha_5(q^{\frac{1}{3}})\beta^2(q^{\frac{1}{3}}) + 9072\alpha_2(q^{\frac{1}{3}})\beta^3(q^{\frac{1}{3}}).
\]
(5.11)

6. Picard group of the arithmetic quotient

Let \( M \) be an even integral lattice of signature \((2, m)\) and \( X_M \) the arithmetic quotient defined in §3.2. A natural question is whether the Picard group with rational coefficients \( \text{Pic}_Q(X_M) \) is generated by the Heegner divisors.

We denote by \( \text{Pic}_Q(X_M)^{\text{Heegner}} \) the subgroup of \( \text{Pic}_Q(X_M) \) generated by the Heegner divisors and \( \text{Cusp}(Mp_2(\mathbb{Z}), 1 + \frac{m}{2}, \rho^*_M) \) the space of vector-valued cusp forms (i.e. vector-valued modular forms vanish at 0) of weight \( 1 + \frac{m}{2} \) and type \( \rho^*_M \).

Lemma 5. [1][4] If \( M \) contains \( U^{\otimes 2} \) as a direct summand, then
\[
\dim \text{Pic}_Q(X_M)^{\text{Heegner}} = 1 + \dim \text{Cusp}(Mp_2(\mathbb{Z}), 1 + \frac{m}{2}, \rho^*_M).
\]

Remark 10. If \( M = \mathbb{Z}\omega \oplus U^{\otimes 2} \oplus (-E_8)^{\otimes 2} \) is an even lattice of signature \((2, 19)\), where \( M \) corresponds to the primitive cohomology of a quasi-polarized \( K3 \) surface of degree \( l \) and \( \langle \omega, \omega \rangle = -l \). The question above is equivalent to asking whether \( \text{Pic}_Q(M_l) \) is spanned by the Noether-Lefschetz
divisors on $M_l$, where $M_l$ is the moduli space of quasi-polarized $K3$ surfaces of degree $l$. This question remains open when $l > 4$ (cf. [13] §7).

In this section, we are interested in the case $M = -\Lambda_0$, where $X_M = X$. As we computed in §4, $\text{Cusp}(M\wp_2(\mathbb{Z}), 1 + \mathbb{Z}, \rho_M^*)$ is a one-dimensional space generated by $F_1$. Therefore, we have

$$\dim \text{Pic}_Q(X)^{\text{Heegner}} = 1 + 1 = 2.$$  

On the other hand, let $C^* = U^*/\text{SL}_6(\mathbb{C})$ denote the moduli space of stable cubic hypersurfaces in $\mathbb{P}(V)$ in the sense of Geometric Invariant Theory, where $U^*$ is the open subset of $\mathbb{P}(\text{Sym}^3 V^*)$ consisting of all points which are stable under the action of $\text{SL}_6(\mathbb{C})$.

In [9], Kirwan has shown that $H^i(C^*, \mathbb{Q}) \cong H^i(\text{BGL}_6(\mathbb{C}), \mathbb{Q})$ for $i < 12$, where $\text{BGL}_6(\mathbb{C})$ is the classifying space of the complex general linear group $\text{GL}_6(\mathbb{C})$. The cohomology ring of $\text{BGL}_6(\mathbb{C})$ is a polynomial ring generated by 6 generators in degree 2, 4, 6, ..., 12 (cf. [9]). In particular, $\dim \mathbb{Q}H^2(C^*, \mathbb{Q}) = 1$ and hence $\text{Pic}_Q(C^*)$ has rank one.

As indicated in §2, the moduli space of cubic fourfold with at worst isolated simple singularities $\mathcal{C}$ is an open subset of $C^*$. Furthermore, the boundary component $C^* \setminus C$ has codimension bigger than one (cf. [8] [10]).

By the definition of stability, the Lie group $\text{SL}_6(\mathbb{C})$ acts on $U^*$ and $U$ smoothly with only finite stabilizers. It is easy to see that there are natural orbifold structures on $C^*$ and $C$ (cf. [15] §8.4). Then $C^*$ and $C$ are complex orbifolds and thus $\mathbb{Q}$-factorial.

Let $\text{Div}(C^*)$ be the group of Weil divisor classes on $C^*$ and $\text{Div}(C)$ the group of Weil divisors classes on $C$. Then $\text{Div}(C^*) \cong \text{Div}(C)$ as $\text{codim}(C, C^*) \geq 2$. It follows that $\text{Pic}_Q(C^*) \cong \text{Div}(C^*) \otimes \mathbb{Q}$ and $\text{Pic}_Q(C) \cong \text{Div}(C) \otimes \mathbb{Q}$ by the $\mathbb{Q}$-factority. We thus conclude that

**Theorem 10.** The Picard group with rational coefficients $\text{Pic}_Q(C)$ has rank one.

Recalling that $C$ is an open subset of $X$ via the open immersion $\mathcal{P} : C \to X$ and the boundary $X \setminus C$ is $D_2$ (see Remark [4]), we have the following result as a corollary:

**Corollary 11.** $\text{Pic}_Q(X)$ has rank two and is spanned by $y_{0,0}$ and $D_2 = y_{1/3,\gamma_1}$.

7. Appendix

7.1. **Computation of the degree of $C_6$.** Recall the two exact sequences (2.8) and (2.7). Let $H$ denote the hyperplane class on $\mathbb{P}(V)$ and $c_i = c_i(K)$, the chern classes of $K$. By (2.7), Chern polynomial of $J^1(O_{\mathbb{P}(V)}(3))$ can be
written as:

\[
c_t(J^1(\mathcal{O}_{\mathbb{P}(V)}(3))) = (1 + 3Ht) \sum_{i=0}^{5} (1 + 3Ht)^{5-i} \binom{6}{i} (-tH)^i.
\]

Then (2.8) implies the equation

\[
1 = c_t(K) \times c_t(J^1(\mathcal{O}_{\mathbb{P}^5}(3)))
= c_t(K) \left( (1 + 3Ht) \sum_{i=0}^{5} (1 + 3Ht)^{5-i} \binom{6}{i} (-tH)^i \right),
\]

which yields

\[
c_1 = -12H, c_2 = 84H^2, c_3 = -448H^3, c_4 = 2016H^4, c_5 = -8064H^5.
\]

Since the rank of \( K \) is 50, it gives

\[
(7.1) \quad \xi^{50} + c_1\xi^{49} + \cdots + c_5\xi^{45} = 0.
\]

It follows that

\[
\xi^{54} = (-c_1^5 + 4c_1^3c_2 - 3c_1^2c_3 - 3c_1c_2^2 + 2c_2c_3 + 2c_2c_3 + c_2c_4 - c_5)\xi^{49} = 192.
\]

7.2. Computation of the degree of \( C_8 \). Recall the two exact sequences (2.9) and (2.10). Still, we let \( c_i \) denote Chern classes of \( K' \) and set \( \sigma_a \) to be Schubert class of \( Gr(3,V) \), where \( a = \{a_1, a_2, a_3\} \) satisfies \( 3 \geq a_1 \geq a_2 \geq a_3 \geq 0 \). Since \( K' \) is of rank 46, we have the equation

\[
(7.2) \quad \xi^{46} + c_1\xi^{45} + \cdots + c_9\xi^{37} = 0.
\]

This implies

\[
\xi^{54} = (-c_1^9 + 8c_1^7c_2 - 7c_1^6c_3 - 21c_1^5c_2^2 + 6c_1^4c_4 + 30c_1^4c_2c_3 - 5c_1^4c_5 - 20c_1^3c_2^3 + 10c_1^3c_2c_4 + 4c_1^3c_6 - 30c_1^2c_2^2c_3 + 12c_1^2c_2c_5 + 12c_1^2c_3c_4 - 3c_1c_7
- 5c_1c_2^4 + 12c_1c_2^2c_4 + 12c_1c_2c_3^2 - 6c_1c_2c_6 - 6c_1c_3c_5 - 3c_1c_4^2 + 2c_1c_8
+ 4c_2^3c_3 - 3c_2^2c_5 - 6c_2c_3c_4 + 2c_2c_7 - c_3^3 + 2c_2c_6 + 2c_4c_5 - c_0) \cdot \xi^{45}.
\]

According to the exactness of (2.9), we get

\[
c_1(S) = -\sigma_1, \quad c_2(S) = \sigma_{1,1}, \quad c_3(S) = -\sigma_{1,1,1}.
\]
Then $c_i$ can be computed via the exact sequence (2.10) as following:

$$c_1 = -10 \cdot \sigma_1$$
$$c_2 = 60 \cdot \sigma_1^2 - 15 \cdot \sigma_{1,1}$$
$$c_3 = -282 \cdot \sigma_1^2 + 189 \cdot \sigma_1 \cdot \sigma_{1,1} - 27 \cdot \sigma_{1,1,1}$$
$$c_4 = 1149 \cdot \sigma_1^4 - 1395 \cdot \sigma_1^2 \cdot \sigma_{1,1} + 351 \cdot \sigma_1 \cdot \sigma_{1,1,1} + 162 \cdot \sigma_{1,1}^2$$
$$c_5 = -4272 \cdot \sigma_1^5 + 7911 \cdot \sigma_1^3 \cdot \sigma_{1,1} - 2673 \cdot \sigma_1^2 \cdot \sigma_{1,1,1} - 2484 \cdot \sigma_1 \cdot \sigma_{1,1}^2$$
$$+ 648 \cdot \sigma_{1,1} \cdot \sigma_{1,1,1}$$
$$c_6 = 14932 \cdot \sigma_1^6 - 38268 \cdot \sigma_1^4 \cdot \sigma_{1,1} + 15629 \cdot \sigma_1^3 \cdot \sigma_{1,1,1} + 21898 \cdot \sigma_1^2 \cdot \sigma_{1,1}^2$$
$$- 10188 \cdot \sigma_1 \cdot \sigma_{1,1} \cdot \sigma_{1,1,1} - 1570 \cdot \sigma_{1,1}^3 + 702 \cdot \sigma_{1,1,1}^2$$
$$c_7 = -49996 \cdot \sigma_1^7 + 166590 \cdot \sigma_1^5 \cdot \sigma_{1,1} - 77858 \cdot \sigma_1^4 \cdot \sigma_{1,1,1} - 146032 \cdot \sigma_1^3 \cdot \sigma_{1,1}^2$$
$$+ 92952 \cdot \sigma_1^2 \cdot \sigma_{1,1} \cdot \sigma_{1,1,1} + 28522 \cdot \sigma_1 \cdot \sigma_{1,1}^3 - 11232 \cdot \sigma_1 \cdot \sigma_{1,1}^2$$
$$- 10206 \cdot \sigma_{1,1}^3 \cdot \sigma_{1,1,1}$$
$$c_8 = 162369 \cdot \sigma_1^8 - 673530 \cdot \sigma_1^6 \cdot \sigma_{1,1} + 348538 \cdot \sigma_1^5 \cdot \sigma_{1,1,1} + 819728 \cdot \sigma_1^4 \cdot \sigma_{1,1}^2$$
$$- 628656 \cdot \sigma_1^3 \cdot \sigma_{1,1} \cdot \sigma_{1,1,1} - 293408 \cdot \sigma_1^2 \cdot \sigma_{1,1}^3 + 103302 \cdot \sigma_1^2 \cdot \sigma_{1,1,1}^2$$
$$+ 189162 \cdot \sigma_1 \cdot \sigma_{1,1}^2 \cdot \sigma_{1,1,1} + 14583 \cdot \sigma_{1,1}^4 - 23490 \cdot \sigma_{1,1} \cdot \sigma_{1,1,1}^2$$
$$c_9 = -515886 \cdot \sigma_1^9 + 2580498 \cdot \sigma_1^7 \cdot \sigma_{1,1} - 1446718 \cdot \sigma_1^6 \cdot \sigma_{1,1,1}$$
$$- 4093280 \cdot \sigma_1^5 \cdot \sigma_{1,1}^2 + 3609936 \cdot \sigma_1^4 \cdot \sigma_{1,1} \cdot \sigma_{1,1,1} + 2253992 \cdot \sigma_1^3 \cdot \sigma_{1,1}^3$$
$$- 717984 \cdot \sigma_1^3 \cdot \sigma_{1,1,1}^2 - 1983960 \cdot \sigma_1^2 \cdot \sigma_{1,1}^2 \cdot \sigma_{1,1,1} - 307242 \cdot \sigma_1 \cdot \sigma_{1,1}^4$$
$$+ 441774 \cdot \sigma_1 \cdot \sigma_{1,1} \cdot \sigma_{1,1,1}^2 + 134244 \cdot \sigma_{1,1}^3 \cdot \sigma_{1,1,1} - 18954 \cdot \sigma_{1,1}^2$$

Hence we obtain $\deg(C_8) = \xi^{54} = 3402$.

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