Abstract

In this paper we show the convergence of the long-term return $t^{-\mu} \int_0^t X(s) ds$ for some $\mu \geq 1$, where $X$ is the short-term interest rate which follows an extension of Cox-Ingersoll-Ross type model with jumps and memory, and, as an application, we also investigate the corresponding behavior of two-factor Cox-Ingersoll-Ross model with jumps and memory.

AMS subject Classification: 60H10, 60H30
Key words: Cox-Ingersoll-Ross model; long-term return; two-factor model.

1 Introduction

Cox, Ingersoll and Ross [5] propose the short-term rate dynamics as

$$dS(t) = \kappa(\gamma - S(t))dt + \sigma \sqrt{S(t)}dW(t),$$

where $\kappa$, $\gamma$ and $\sigma$ are positive constants. This model is also named mean-reverting square root process or Cox-Ingersoll-Ross (CIR) model. In order to better capture the properties of the empirical data, there are many extensions of the CIR model, e.g., Chan, Karolyi, Longstaff and Sander [6] generalize the CIR model as

$$dS(t) = \kappa(\gamma - S(t))dt + \sigma S(t)^{\theta}dW(t),$$

where $\theta \geq 1/2$. Another generalization of the CIR model is to use the regime-switching such as in Ang and Bekaert [1] and Gary [12], to name a few. On the other hand, taking into consideration the influence of past events, many scholars introduce delay to the financial models. For example, in his paper [4], Benhabib considers a linear, flexible price model, where nominal interest rates are measured by a flexible distributed delay. In their paper [2], Arriojas, Hu and Mohammed take the delay into the consideration for the price process of underlying assets and develop the Black-Scholes formula. Moreover, jump processes are also used in the financial models, e.g., [3, 8, 14, 15], and the references therein.
There are extensive literature on quantitative and qualitative properties of the generalized CIR-type models. Different convergence results and corresponding applications of the long-term return can be found in [9, 10, 12]; Strong convergence of the Monte Carlo simulations are studied in [11, 13, 19, 17], and the representations of solutions are presented in [2, 7]. We here would like to point out that Deelstra and Delbaen [9, 10] investigate the long-term returns of the CIR model. Zhao [19] extends the results of [9, 10] to the jump model. In the present paper we will consider the effect of the past and jump in the determination of the interest model and study the long-term return of the stochastic interest rate model with jumps and memory.

In the following section, we will introduce the mathematical model and notation, the long-term return will be studied in section 3, and an application of the main result, Theorem 4.2, is discussed in the last section.

2 Preliminaries

Throughout this paper, let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e. it is right continuous and \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets). Let \(W(t)\) be a scalar Brownian motion. Let \(\mathcal{B}(\mathbb{R}_+)\) be the Borel \(\sigma\)-algebra on \(\mathbb{R}_+\), and \(\lambda(dx)\) a \(\sigma\)-finite measure defined on \(\mathcal{B}(\mathbb{R}_+)\). Let \(p = (p(t)), t \in D_p\), be a stationary \(\mathcal{F}_t\)-Poisson point process on \(\mathbb{R}_+\) with characteristic measure \(\lambda(\cdot)\). Denote by \(N(dt,du)\) the Poisson counting measure associated with \(p\), i.e., \(N(dt,du) = \sum_{s \in D_p, s \leq t} I_{U(p(s))}\) for \(U \in \mathcal{B}(\mathbb{R}_+)\). We assume \(\lambda(U) < \infty\) and let \(\tilde{N}(dt,du) := N(dt,du) - dt\lambda(du)\) be the compensated Poisson measure associated with \(N(dt,du)\). For the sake of convenience, we will denote \(C > 0\) a generic constant whose values may change from lines to lines.

Consider stochastic interest rate model with jumps and memory,

\[
\begin{aligned}
dX(t) &= \{2\beta X(t) + \delta(t)\}dt + \sigma X^\gamma(t-\tau)\sqrt{|X(t)|}dW(t) \\
&\quad + \int_U g(X(t-),u)\tilde{N}(dt,du), \\
X_0 &= \xi \in \mathcal{C},
\end{aligned}
\]

where \(X(t-) = \lim_{s \downarrow t} X(s)\). We make the following assumptions:

(A1) \(\beta < 0, \sigma > 0\) and \(\gamma \in [0, \frac{1}{2})\).

(A2) \(\delta: \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+,\) and there exist constants \(\mu \geq 1\) and \(\nu \geq 0\) such that

\[
\lim_{t \to \infty} \frac{1}{t^\mu} \int_0^t \delta(s)ds := \nu \quad \text{a.s.}
\]

(A3) \(g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) with \(g(0,u) = 0\) and there exists \(K > 0\) such that

\[
\int_U |g(x,u) - g(y,u)|^2 \lambda(du) \leq K|x - y|^2
\]

for arbitrary \(x,y \in \mathbb{R}\).
(A4) For any $\theta \in [0, 1]$, $x + \theta g(x, u) \geq 0$ whenever $x > 0$.

Compared with the existing literature, our key contributions of this paper are as follows:

- We investigate the almost sure convergence of the long-term return $t^{-\mu} \int_0^t X(s)ds$ for some $\mu \geq 1$, and extend the results of Deelstra and Delbaen [9, 10] and Zhao [19]. Since the jumps and memory are involved, we will see the generalization is not trivial.
- As an application, we also study the long-term behaviors for a class of two-factor CIR models with jumps and memory, where we extend the result of [19, Theorem 2].

3 Almost Sure Convergence of Long-Term Returns

For our purposes we first prepare or recall several auxiliary lemmas.

**Lemma 3.1.** Under (A1)-(A4), Eq. (2.1) admits a unique nonnegative solution $(X(t))_{t \geq 0}$ for any $\xi \in \mathcal{C}$.

**Proof.** Application of [20, Theorem 2.1 & 2.2] gives that Eq. (2.1) has a unique strong solution $X(t)$ on $[0, \tau]$. Repeating this procedure we see that Eq. (2.1) also admits a unique strong solution $X(t)$ on $[\tau, 2\tau]$. Hence Eq. (2.1) has a unique strong solution $X(t)$ on the horizon $t \geq 0$. Moreover, carrying out a similar argument to that of [17, Theorem 2.1], we can deduce that there exists $C > 0$ such that for any $q > 0$

\[(3.1) \quad \mathbb{E}|X(t)|^q \leq C, \quad t \in [0, T].\]

To end the proof, it is sufficient to show the nonnegative property of the solution $(X(t))_{t \in [0,T]}$ for any $T > 0$. We adopt the method of Yamada and Watanabe [18]. Let $a_0 = 1$ and $a_k = \exp(-k(k + 1)/2), k = 1, 2 \cdots$. Then it is easy to see that $\int_{a_{k-1}}^{a_k} \frac{1}{kx}dx = 1$ and consequently there is a continuous nonnegative function $\psi_k(x), x \in \mathbb{R}_+$, which possesses the support $(a_k, a_{k-1})$, has integral 1 and satisfies $\psi_k(x) \leq \frac{2}{kx}$. Define an auxiliary function $\phi_k(x) = 0$ for $x \geq 0$ and

\[\phi_k(x) := \int_{0}^{-x} dy \int_{0}^{y} \psi_k(u)du, \quad x < 0.\]

By a straightforward computation, $\phi_k \in C^2(\mathbb{R}; \mathbb{R}_+)$ has the following properties:

(i) $-1 \leq \phi_k'(x) \leq 0$ for $-a_{k-1} < x < -a_k$, or otherwise $\phi_k'(x) = 0$;
(ii) $|\phi_k''(x)| \leq \frac{2}{k|x|}$ for $-a_{k-1} < x < -a_k$, or otherwise $\phi_k''(x) = 0$;
(iii) $x^- - a_{k-1} \leq \phi_k(x) \leq x^-, \quad x \in \mathbb{R}.$

3
An application of the Itô formula yields that for any $t \in (0, T]$

$$
\mathbb{E}\phi_k(X(t)) = \mathbb{E}\phi_k(X(0)) + \mathbb{E} \int_0^t \phi_k'(X(s)) \{2\beta X(s) + \delta(s)\} ds \\
+ \frac{\sigma^2}{2} \mathbb{E} \int_0^t \phi_k''(X(s)) X^{2\gamma}(s - \tau) X(s) ds \\
+ \mathbb{E} \int_0^t \left\{ \phi_k(X(s) + g(X(s), u)) - \phi_k(X(s)) - \phi_k'(X(s)) g(X(s), u) \right\} \lambda(du) ds.
$$

By the properties (i)-(iii), Taylor’s expansion and (A4), it then follows from (3.1) that

$$
\mathbb{E}\phi_k(X(t)) \leq \frac{C}{k} + \mathbb{E} \int_0^t \int_0^1 \int_0^1 \left\{ \phi_k'(\theta g(X(s), u) + X(s)) - \phi_k'(X(s)) \right\} d\theta \lambda(du) ds \\
= \frac{C}{k} + \mathbb{E} \int_0^t \int_0^1 \int_0^1 \left\{ \phi_k'(\theta g(X(s), u) + X(s)) - \phi_k'(X(s)) \right\} 1_{\{X(s) > 0\}} d\theta \lambda(du) ds \\
+ \mathbb{E} \int_0^t \int_0^1 \int_0^1 \left\{ \phi_k'(\theta g(X(s), u) + X(s)) - \phi_k'(X(s)) \right\} 1_{\{X(s) \leq 0\}} d\theta \lambda(du) ds \\
\leq \frac{C}{k} + 2K^{1/2} \lambda^1(U) \mathbb{E} \int_0^t X^{-}(s) 1_{\{X(s) \leq 0\}} ds \\
\leq \frac{C}{k} + 2K^{1/2} \lambda^1(U) \mathbb{E} \int_0^t \{a_{k-1} + \phi_k(X(s))\} ds \\
= \frac{C}{k} + 2K^{1/2} \lambda^1(U) T a_{k-1} + 2K^{1/2} \lambda^1(U) \int_0^t \mathbb{E}\phi_k(X(s)) ds, \quad t \in (0, T].
$$

This, together with the Gronwall inequality, gives that

$$
\mathbb{E} X^{-}(t) - a_{k-1} \leq \mathbb{E}\phi_k(X(t)) \leq C \left( \frac{1}{k} + a_{k-1} \right), \quad t \in (0, T].
$$

Thus, $\mathbb{E} X^{-}(t) = 0$ as $k \to \infty$ and therefore $X(t) \geq 0$ a.s. for any $t \in (0, T]$. Hence the nonnegative property of the solution $(X(t))_{t \geq 0}$ follows from the arbitrariness of $T > 0$. □

**Remark 3.1.** There are some examples such that (A4) holds, e.g., for $x \in \mathbb{R}$ and $u \in U$, $g(x, u) \geq 0$ or $-g(x, u) \leq x$ whenever $g(x, u) \leq 0$.

**Remark 3.2.** Wu, Mao and Chen [16] study the strong convergence of Monte Carlo simulations of the mean-reverting square root process with jump

$$(3.2) \quad dS(t) = \alpha [\mu - S(t)] dt + \sigma \sqrt{|S(t)|} dW(t) + \delta S(t-) d\tilde{N}(t),$$

where $\alpha, \mu, \sigma > 0$, and, in particular, investigate the nonnegative property of $S(t)$. It is easy to see that our model is a generalization of model (3.2). Zhao [19] also showed the nonnegative property of (2.1) with $\gamma = 0$, $g(x, u) = 0$ for $x < 0$ and $\int_U g^2(x, u) \lambda(du) \leq K|x|$ for some constant $K > 0$. Moreover, we would like to point that our goal is to study the long-term return, which is different from those of [16] [17].
**Lemma 3.2.** Let (A1)-(A4) hold and assume further that $4\beta + K < 0$. Then there exist $\kappa > 0$ and $C > 0$ such that

$$
(3.3) \quad \mathbb{E}(e^{-\kappa \rho} X^2(\rho)) \leq C + C \mathbb{E} \int_{0}^{\rho} e^{-\kappa \beta s}(\delta^2(s) + 1)ds,
$$

where $\rho > 0$ is a bounded stopping time.

**Proof.** We first recall the Young inequality: for any $a, b > 0$ and $\alpha \in (0, 1)$

$$
(3.4) \quad a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b.
$$

Let $\kappa > 0$ and $\epsilon > 0$ be arbitrary. By the Itô formula, (A3) and the Young inequality (3.4), we obtain that

$$
d(e^{-\kappa \beta t} X^2(t)) = -\kappa \beta e^{-\kappa \beta t} X^2(t)dt + e^{-\kappa \beta t} dX^2(t)
$$

$$
= e^{-\kappa \beta t} \left\{(4 - \kappa) \beta X^2(t) + \sigma^2 X(t) X^{2\gamma}(t - \tau) + 2\delta(t)X(t)
$$

$$
+ \int_{U} g^2(X(t), u)\lambda(du) \right\}dt + M_1(t) + M_2(t)
$$

$$
\leq e^{-\kappa \beta t} \left\{((4 - \kappa) \beta + \epsilon + K) X^2(t) + C_1(\epsilon) X^{4\gamma}(t - \tau) + C_1(\epsilon) \delta^2(t) \right\}dt
$$

$$
+ M_1(t) + M_2(t)
$$

$$
\leq e^{-\kappa \beta t} \left\{((4 - \kappa) \beta + \epsilon + K) X^2(t) + \epsilon e^{\kappa \beta \tau} X^2(t - \tau) + C_1(\epsilon) \delta^2(t) + C_2(\epsilon) \right\}dt
$$

$$
+ M_1(t) + M_2(t)
$$

for some constants $C_1(\epsilon) > 0$ and $C_2(\epsilon) > 0$, dependent on $\epsilon$, where $M_1(t) := 2\sigma e^{-\kappa \beta t} X^2(t) X^{2\gamma}(t - \tau) dW(t)$ and $M_2(t) := e^{-\kappa \beta t} \int_{U} \{g^2(X(t), u) + 2X(t)g(X(t), u)\} \tilde{N}(dt, du)$. Integrating from 0 to $\rho$ and taking expectations on both sides, we arrive at

$$
\mathbb{E}(e^{-\kappa \beta \rho} X^2(\rho)) \leq C\|\xi\|^2 + ((4 - \kappa) \beta + 2\epsilon + K) \mathbb{E} \int_{0}^{\rho} e^{-\kappa \beta s} X^2(s)ds
$$

$$
+ (C_1(\epsilon) \vee C_2(\epsilon)) \mathbb{E} \int_{0}^{\rho} (\delta^2(s) + 1) ds.
$$

Due to $4\beta + K < 0$, we can choose $\kappa > 0$ and $\epsilon > 0$ such that $(4 - \kappa) \beta + 2\epsilon + K = 0$, and therefore (3.3) follows immediately. \hfill \Box

For the future use, we cite the following as a lemma.

**Lemma 3.3.** (Kronecker’s lemma, p164]) Assume that $Y(t)$ is a càdlàg semimartingale and that $f(t)$ is a strictly positive increasing function with $f(t) \to \infty$ as $t \to \infty$. If $\int_{0}^{\infty} \frac{dY(t)}{f(t)}$ exists a.s., then $\frac{Y(t)}{f(t)} \to 0$ a.s.

We now state our main result.
Theorem 3.4. Let \((A1)-(A4)\) hold and \(4\beta + K < 0\). Assume further that there exist \(\lambda > 0\) and \(\theta \in [1, 2\mu]\) such that

\[
\limsup_{t \to \infty} \frac{1}{t^\mu} \int_0^t \delta^2(s) \, ds \leq \lambda \quad \text{a.s.}
\]

Then

\[
\lim_{t \to \infty} \frac{1}{t^\mu} \int_0^t \left\{ X(s) + \frac{\delta(s)}{2\beta} \right\} \, ds = 0 \quad \text{a.s.}
\]

Proof. It is easy to see from Eq. (2.1) that

\[
\int_0^t \left\{ X(s) + \frac{\delta(s)}{2\beta} \right\} \, ds = \frac{X(t) - \xi(0)}{2\beta} - \frac{\sigma}{2\beta} \int_0^t X^\gamma(s - \tau) \sqrt{|X(s)|} \, dW(s)
\]

\[
- \frac{1}{2\beta} \int_0^t \int_U g(X(s-), u) \tilde{N}(ds, du).
\]

On the other hand, application of Itô’s formula to \(e^{-2\beta t} X(t)\) yields that

\[
X(t) = e^{2\beta t} \left\{ \xi(0) + \int_0^t e^{-2\beta s} \delta(s) \, ds + \sigma \int_0^t e^{-2\beta s} X^\gamma(s - \tau) \sqrt{|X(s)|} \, dW(s)
\]

\[
+ \int_0^t \int_U e^{-2\beta s} g(X(s-), u) \tilde{N}(ds, du) \right\}.
\]

Thus, substituting this into (3.7) one has

\[
\frac{1}{t^\mu} \int_0^t \left\{ X(s) + \frac{\delta(s)}{2\beta} \right\} \, ds = \frac{(e^{2\beta t} - 1)\xi(0)}{2\beta t^\mu}
\]

\[
+ \frac{1}{2\beta t^\mu} \int_0^t e^{2\beta(t-s)} \delta(s) \, ds
\]

\[
+ \frac{\sigma}{2\beta} \left(1 + \frac{1}{t}\right)^\mu \frac{1}{(1 + t)^\mu} \int_0^t e^{-2\beta s} \delta(s) X^\gamma(s - \tau) \sqrt{|X(s)|} \, dW(s)
\]

\[
- \frac{\sigma}{2\beta} \left(1 + \frac{1}{t}\right)^\mu \frac{1}{(1 + t)^\mu} \int_0^t X^\gamma(s - \tau) \sqrt{|X(s)|} \, dW(s)
\]

\[
- \frac{1}{2\beta} \left(1 + \frac{1}{t}\right)^\mu \frac{1}{(1 + t)^\mu} \int_0^t \int_U g(X(s-), u) \tilde{N}(ds, du)
\]

\[
+ \frac{1}{2\beta} \left(1 + \frac{1}{t}\right)^\mu \frac{1}{(1 + t)^\mu} \int_0^t \int_U e^{-2\beta s} g(X(s-), u) \tilde{N}(ds, du)
\]

\[
:= I_1(t) + I_2(t) + \frac{1}{2\beta} \left(1 + \frac{1}{t}\right)^\mu I_3(t) - \frac{\sigma}{2\beta} \left(1 + \frac{1}{t}\right)^\mu I_4(t)
\]

\[
- \frac{1}{2\beta} \left(1 + \frac{1}{t}\right)^\mu I_5(t) + \frac{1}{2\beta} \left(1 + \frac{1}{t}\right)^\mu I_6(t).
\]
To derive the desired assertion (3.6), it is sufficient to verify that \( I_i(t) \to 0 \) a.s., \( i = 1, \cdots, 6 \), as \( t \to \infty \) respectively. Due to \( \beta < 0 \) and \( \mu \geq 1 \), it is trivial that \( I_i(t) \to 0 \) as \( t \to \infty \). Following a similar argument to that of [9, p.168] and noting that \( \lim_{t \to \infty} [(1 + t)^{\mu} - (1 + t - \sqrt{t})^\mu] / (1 + t)^\mu = 0 \), by (A2) we can also deduce that \( I_2(t) \to 0 \) a.s. for \( t \to \infty \). Next, in order to show \( I_3(t) \to 0 \) a.s. and \( I_4(t) \to 0 \) a.s. whenever \( t \to \infty \), respectively, by Lemma 3.3 it suffices to check that

\[
\int_0^\infty \frac{X^\gamma(t - \tau) \sqrt{|X(t)|}}{(1 + t)^\mu} dW(t) \quad \text{exists a.s.} \tag{3.8}
\]

For each \( n > \| \xi \| \) define a stopping time

\[
\tau_n := \inf \left\{ t \geq 0 \ \bigg| \int_0^t \frac{\delta^2(s)}{(1 + s)^{2\mu}} ds \geq n \right\}.
\]

In the light of (3.5) there exists an \( L > 0 \) such that

\[
\int_0^t \delta^2(s) ds \leq L(1 + t)^{\theta} \quad \text{a.s.} \tag{3.9}
\]

This, together with \( \theta \in [1, 2\mu] \), leads to

\[
\int_0^\infty \frac{\delta^2(s)}{(1 + s)^{2\mu}} ds = \lim_{s \to \infty} \int_0^s \frac{\delta^2(u) du}{(1 + s)^{2\mu}} + 2\mu \int_0^\infty \left( \int_0^s \frac{\delta^2(u) du}{(1 + s)^{2\mu}} \right) \frac{ds}{(1 + s)^{2\mu - 1}} \leq \lim_{s \to \infty} \frac{L}{(1 + s)^{2\mu - \theta}} + 2L \int_0^\infty \frac{1}{(1 + s)^{2\mu + 1 - \theta}} ds < \infty \quad \text{a.s.}
\]

Hence \( \{ \tau_n = \infty \} \uparrow \Omega \) and consequently, it is sufficient to verify (3.8) on \( \{ \tau_n = \infty \} \). Furthermore, observing that

\[
J(t) := \int_0^t \frac{X^\gamma(s - \tau) \sqrt{|X(s)|}}{(1 + s)^\mu} 1_{\{s \leq \tau_n\}} dW(s)
\]

is a local martingale, we only need to check that \( J(t) \) is an \( L^2 \)-bounded martingale. By the Itô isometry and the Young inequality (3.4) we can obtain that

\[
\mathbb{E} |J(t)|^2 = \int_0^t \mathbb{E} \{ X^{2\gamma}(s - \tau) X(s) \} 1_{\{s \leq \tau_n\}} ds
\]

\[
\leq \int_0^t \frac{\mathbb{E} \{ X^{2\gamma}(s) 1_{\{s \leq \tau_n\}} \}}{2(1 + s)^{2\mu}} ds + \int_0^t \frac{\mathbb{E} \{ X^{4\gamma} 1_{\{s \leq \tau_n\}} \}}{2(1 + s)^{2\mu}} ds
\]

\[
\leq \int_0^t \frac{(1 - 2\gamma)}{2(1 + s)^{2\mu}} ds + \int_0^t \frac{\mathbb{E} \{ X^{2\gamma}(s) 1_{\{s \leq \tau_n\}} \}}{2(1 + s)^{2\mu}} ds + \gamma \int_0^t \frac{\mathbb{E} \{ X^{2}(s - \tau) 1_{\{s \leq \tau_n\}} \}}{(1 + s)^{2\mu}} ds
\]

\[
\leq \frac{1 - 2\gamma}{2(2\mu - 1)} + \int_0^t \frac{e^{-\kappa \beta s}}{2(1 + s)^{2\mu}} \mathbb{E} \{ e^{\kappa \beta (s \wedge \tau_n)} X^2(s \wedge \tau_n) \} ds + \gamma \int_0^t \frac{e^{-\kappa \beta (s - \tau)} \mathbb{E} \{ e^{\kappa \beta (s \wedge \tau_n - \tau)} X^2(s \wedge \tau_n - \tau) \}}{(1 + s)^{2\mu}} ds
\]

\[:= (1 - 2\gamma)/(4\mu - 2) + J_1(t) + J_2(t).
\]
By (3.3) with $\kappa > 0$ it follows that

\[
J_1(t) \leq C \int_0^t \left\{ \frac{1 + s + e^{\kappa s} \mathbb{E} \int_0^{s \wedge \tau_n} e^{-\kappa \beta r} \delta^2(r) dr}{2(1 + s)^{2\mu}} \right\} ds
\]

\[
\leq C \int_0^t \left\{ \frac{1 + s + e^{\kappa s} \mathbb{E} \int_0^s e^{-\kappa \beta r} \mathbb{E}(\delta^2(r) 1_{\{r \leq \tau_n\}}) dr}{2(1 + s)^{2\mu}} \right\} ds
\]

\[
\leq C + C \int_0^t \frac{e^{\kappa s}}{2(1 + s)^{2\mu}} \int_0^s \mathbb{E}(\delta^2(r) 1_{\{r \leq \tau_n\}}) ds dr
\]

\[
\leq C + C \mathbb{E} \int_0^{\tau_n} \frac{\delta^2(r)}{(1 + r)^{2\mu}} dr
\]

\[
\leq C \left( 1 + n \right).
\]

By (3.3) with $\kappa > 0$ it follows that

\[
J_1(t) \leq C \int_0^t \left\{ \frac{1 + s + e^{\kappa \beta s} \mathbb{E} \int_0^{s \wedge \tau_n} e^{-\kappa \beta r} \delta^2(r) dr}{2(1 + s)^{2\mu}} \right\} ds
\]

\[
\leq C \int_0^t \left\{ \frac{1 + s + e^{\kappa \beta s} \mathbb{E} \int_0^s e^{-\kappa \beta r} \mathbb{E}(\delta^2(r) 1_{\{r \leq \tau_n\}}) dr}{2(1 + s)^{2\mu}} \right\} ds
\]

\[
\leq C + C \int_0^t \frac{e^{\kappa \beta s}}{2(1 + s)^{2\mu}} \int_0^s \mathbb{E}(\delta^2(r) 1_{\{r \leq \tau_n\}}) ds dr
\]

\[
\leq C + C \mathbb{E} \int_0^{\tau_n} \frac{\delta^2(r)}{(1 + r)^{2\mu}} dr
\]

\[
\leq C \left( 1 + n \right).
\]

and carrying out the similar argument to that of (3.10), we can conclude that there exists $C(n, \mu, \alpha) > 0$ such that $J_2(t) \leq C(n, \mu, \alpha)$. Finally, $I_5(t) \to 0$ a.s. and $I_6(t) \to 0$ a.s. whenever $t \to \infty$ by observing

\[
\mathbb{E} \left( \int_0^t \int_U \frac{g(X(s-), u)}{(1 + s)^{2\mu}} 1_{\{s \leq \tau_n\}} \tilde{N}(ds, du) \right)^2 = \mathbb{E} \int_0^t \int_U \frac{g^2(X(s-), u)}{(1 + s)^{2\mu}} 1_{\{s \leq \tau_n\}} \lambda(du) ds,
\]

and following the previous argument, and the proof is therefore complete.

\[\square\]

**Remark 3.3.** For $\delta(t) = t^{\mu-1}, t \geq 0$, and $\theta = 2\mu - 1$, it is trivial to see that both (A2) and (3.3) are true.

**Remark 3.4.** For $\beta < 0, \sigma > 0$ and $\gamma \in [0, \frac{1}{2})$, Theorem 3.4 clearly applies to the generalized mean-reverting model

\[
\begin{align*}
\frac{dX(t)}{dt} &= \{2\beta X(t) + \delta(t)\} dt + \sigma |X(t)|^{\frac{1}{2}+\gamma} dW(t), \\
X(0) &= x > 0,
\end{align*}
\]

where Deelstra and Delbaen [9] investigated the long-term returns of such model with $\gamma = 0$. Moreover, Zhao [19] discussed the long-time behavior of the stochastic interest rate model (2.1) with $\gamma = 0, g(x, u) = 0$ for $x < 0, u \in U$, and

\[
\int_U g^2(x, u) \lambda(du) \leq K|x| \quad \text{for some constant } K > 0.
\]

Clearly, the linear case $g(x, u) = C|u|x$ for some $C > 0$ does not satisfy (3.11), however, Theorem 3.4 is available for such fundamental case.
4 An Application to Two-Factor CIR Model

In this section we turn to an application of Theorem 3.4. Let $W_1(t), W_2(t)$ be Brownian motions, and $N_1(dt, du), N_2(dt, du)$ Poisson counting measures with characteristic measures $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ respectively, defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. Consider the following two-factor model with jumps and memory

\[
\begin{aligned}
\text{(4.1)} \quad & \quad \text{d}X(t) = \left\{2\beta_1 X(t) + \delta(t)\right\}dt + \sigma_1 X^{\gamma_1}(t - \tau)\sqrt{X(t)}dW_1(t) \\
& + \vartheta_1 X(t) \int_U u\tilde{N}_1(dt, du), \\
\text{d}Y(t) = \left\{2\beta_2 Y(t) + X(t)\right\}dt + \sigma_2 Y^{\gamma_2}(t - \tau)\sqrt{Y(t)}dW_2(t) \\
& + \vartheta_2 Y(t) \int_U u\tilde{N}_2(dt, du)
\end{aligned}
\]

with initial data $(X(t), Y(t)) = (\xi(t), \eta(t)), t \in [-\tau, 0]$, where $\xi, \eta \in \mathcal{C}$.

We assume that

(A5) $\beta_1 < 0, \sigma_1 > 0$ and $\gamma_1 \in [0, \frac{1}{2})$, $\vartheta_1 > 0$, $\delta(t)$ satisfies (A2);

(A6) $\beta_2 < 0, \sigma_2 > 0$, $\gamma_2 \in [0, \frac{1}{2})$, $\vartheta_2 > 0$ and $\vartheta_2^2 \int_U u^2\lambda_2(du) < -4\beta_2$;

(A7) For $\theta \in [1, 2\mu]$ (where $\mu$ is defined in (A2)), $\int_0^\infty \frac{\delta^4(t)}{(1+e)^2}dt < \infty$ a.s.

Lemma 4.1. Let (A5) and (A6) hold and assume that

\[
\text{(4.2)} \quad \vartheta_1^2 \int_U u^2(6 + 4\vartheta_1 u + \vartheta_1^2 u^2)\lambda_1(du) =: \Gamma(\vartheta_1, \lambda_1) < -8\beta_1.
\]

Then (4.1) admits a unique nonnegative solution $(X(t), Y(t))_{t \geq 0}$, and there exist $\kappa > 0$ and $C > 0$ such that

\[
\text{(4.3)} \quad \mathbb{E}(e^{-\kappa \beta_1 \rho}X^4(\rho)) \leq C + C\mathbb{E} \int_0^\rho e^{-\kappa \beta_1 s}(\delta^4(s) + 1)ds,
\]

where $\rho > 0$ is a bounded stopping time.

Proof. By Lemma 3.1 under (A5) and (A6), (4.1) admits a unique nonnegative solution $(X(t), Y(t))_{t \geq 0}$. By the Itô formula and the Young inequality (3.4), compute

\[
\begin{aligned}
\text{d}(e^{-\kappa \beta_1 t}X^4(t)) &= -\kappa \beta_1 e^{-\kappa \beta_1 t}X^4(t)dt + e^{-\kappa \beta_1 t}dX^4(t) \\
&= e^{-\kappa \beta_1 t}\left\{(8 - \kappa)\beta_1 X^4(t) + 4\delta(t)X^3(t) + 6\sigma_1^2 X^2(t)X^{2\gamma_1}(t - \tau) \\
&+ \int_U ((1 + \vartheta_1 u)^4 - 1 - 4\vartheta_1 u)\lambda_1(du)X^4(t)\right\} + \tilde{M}_1(t) + \tilde{M}_2(t)
\end{aligned}
\]

\[
\text{(4.4)} \quad \leq e^{-\kappa \beta_1 t}\left\{(8 - \kappa)\beta_1 + \epsilon + \Gamma(\vartheta_1, \lambda_1)\right\}X^4(t) + \epsilon e^{\kappa \beta_1 \tau}X^4(t - \tau) \\
&+ C(e)(\delta^4(t) + 1)dt + \tilde{M}_1(t) + \tilde{M}_2(t)
\]

for any $\kappa > 0$ and sufficiently small $\epsilon > 0$, where $\tilde{M}_1(t)$ and $\tilde{M}_2(t)$ are two local martingales. Then (4.3) can be obtained by integrating from $0$ to $\rho$, taking expectations on both sides of (4.4) and, in particular, choosing $\kappa > 0$ and $\epsilon > 0$ such that $(8 - \kappa)\beta_1 + 2\epsilon + \Gamma(\vartheta_1, \lambda_1) = 0$ due to (4.2). \qed
Remark 4.1. In fact, (4.1) admits a unique nonnegative solution \((X(t), Y(t))_{t \geq 0}\) under the weaker condition
\[ m(\vartheta_1, \lambda_1) := \vartheta_1^2 \int_{U} u^2 \lambda_1(du) < -4\beta_1, \]
rather than (4.2), which is imposed just to guarantee (4.3).

For the two-factor model, we have the following result.

**Theorem 4.2.** Under \((A_5) - (A_7)\) and (4.2),
\[
\lim_{t \to \infty} \frac{1}{t^\mu} \int_0^t Y(s)ds = \frac{\nu}{4\beta_1 \beta_2}, \quad \text{a.s.}
\]

**Proof.** By \((A_5)\) and Theorem 3.4 we can deduce that
\[
\lim_{t \to \infty} \frac{1}{t^\mu} \int_0^t X(s)ds = -\frac{\nu}{2\beta_1}, \quad \text{a.s.}
\]

On the other hand, for \(\theta \in [1, 2\mu]\) such that (3.5), if there exists \(C > 0\) such that
\[
\limsup_{t \to \infty} \frac{1}{t^\theta} \int_0^t X^2(s)ds \leq C, \quad \text{a.s.,}
\]
which, together with (4.5) and Theorem 3.4, leads to
\[
\lim_{t \to \infty} \frac{1}{t^\mu} \int_0^t Y(s)ds = \frac{\nu}{4\beta_1 \beta_2}, \quad \text{a.s.}
\]

Therefore, we only need to verify (4.6). By the Itô formula and the Young inequality 3.4, it follows from (4.1) that
\[
dX^2(t) = \left\{ (4\beta_1 + m(\vartheta_1, \lambda_1))X^2(t) + 2\delta(t)X(t) + \sigma_1^2 X(t)X^{2\gamma_1}(t - \tau) \right\} dt + 2\sigma_1 X \hat{X}(t)X^\gamma_1(t - \tau)dW_1(t) + \vartheta_1 \int_{U} (2u + \vartheta_1 u^2)X^2(t)\tilde{N}_1(du, dt)
\]
\[
\leq \left\{ (4\beta_1 + \epsilon + m(\vartheta_1, \lambda_1))X^2(t) + \epsilon X^2(t - \tau) + C(\epsilon)(1 + \delta^2(t)) \right\} dt + 2\sigma X \hat{X}(t)X^\gamma_1(t - \tau)dW_1(t) + \vartheta_1 \int_{U} (2u + \vartheta_1 u^2)X^2(t)\tilde{N}_1(du, dt)
\]
for sufficiently small \(\epsilon > 0\) and some constant \(C(\epsilon) > 0\). Integrating from 0 to \(t\) on both sides leads to
\[
X^2(t) - \xi^2(0) \leq \epsilon\|\xi\|^2 \tau + (4\beta_1 + 2\epsilon + m(\vartheta_1, \lambda_1)) \int_0^t X^2(s)ds + C(\epsilon) \int_0^t (1 + \delta^2(s))ds
\]
\[
+ 2\sigma \int_0^t X \hat{X}(s)X^\gamma_1(s - \tau)dW_1(s) + \vartheta_1 \int_{U} (2u + \vartheta_1 u^2)X^2(s)\tilde{N}_1(du, ds).
\]
By virtue of (4.2), we can choose $\epsilon > 0$ such that $\tilde{\kappa} := 4\beta_1 + 2\epsilon + m(\vartheta_1, \lambda_1) < 0$. Thus for $\theta \in [1, 2\mu]$ such that (A5)
\[
\frac{1}{t^\theta} \int_0^t X^2(s)ds \leq \frac{C(1 + t)}{t^\theta} + \frac{C}{\kappa t^\theta} \int_0^t \delta^2(s)ds + \frac{2\sigma}{\kappa t^\theta} \int_0^t X^2(s)X^{\gamma_1}(s - \tau)dW_1(s) + \frac{\vartheta_1}{\kappa t^\theta} \int_0^t \int_U (2u + \vartheta_1 u^2)X^2(s)\tilde{N}_1(du, ds).
\]
By virtue of $\theta \in [1, 2\mu]$ and (3.5), note that the first two terms on the right hand side are finite almost surely. In order to prove (4.6), by Lemma 3.3 we only need to show that
\[
J_1(\infty) := \int_0^\infty \frac{X^2(s)X^{\gamma_1}(s - \tau)}{(1 + s)^\theta}dW_1(s) \quad \text{and} \quad J_2(\infty) := \int_0^\infty \int_U \frac{X^2(s)}{(1 + s)^\theta}\tilde{N}_1(du, ds)
\]
exist a.s. For each $n > \|\xi\|$ define a stopping time
\[
\rho_n := \inf\left\{t \geq 0 \mid \int_0^t \frac{\delta^4(s)}{(1 + s)^{2\theta}}ds \geq n\right\}.
\]
By (A5) it is easy to see that $\{\rho_n = \infty\} \uparrow \Omega$. Following the argument of Theorem 3.4 in what follows we only need to show that
\[
M(t) := \int_0^t X^2(s)X^{\gamma_1}(s - \tau)1_{\{s \leq \rho_n\}}dW_1(s)
\]
is $L_2$-bounded. By the Itô isometry and the Young inequality (3.4), compute that
\[
\mathbb{E}|M(t)|^2 = \mathbb{E} \int_0^t X^3(s)X^{2\gamma_1}(s - \tau)1_{\{s \leq \rho_n\}}ds
\leq C + \mathbb{E} \int_0^t X^4(s)(1 + s)^{2\theta}1_{\{s \leq \rho_n\}}ds + \mathbb{E} \int_0^t X^4(s - \tau)(1 + s)^{2\theta}1_{\{s \leq \rho_n\}}ds
:= C + J_1(t) + J_2(t).
\]
For $\kappa > 0$ by (4.3),
\[
J_1(t) \leq \int_0^t e^{\kappa \beta_1 s}\mathbb{E}\left\{e^{-p\beta_1(s \wedge \tau_n)}X^4(s \wedge \tau_n)\right\}ds
\leq C \int_0^t \frac{e^{\kappa \beta_1 s}\left\{1 + s + \int_0^s e^{-\kappa \beta_1 r}\mathbb{E}(\delta^4(r)1_{\{r \leq \tau_n\}})dr\right\}}{(1 + s)^{2\theta}}ds
\]
(4.7)
\[
\leq C + C \int_0^t \frac{e^{\kappa \beta_1 s}}{(1 + s)^{2\theta}} \int_0^s e^{-\kappa \beta_1 r}\mathbb{E}(\delta^4(r)1_{\{r \leq \tau_n\}})drds
\]
\[
= C + C \int_0^t e^{-\kappa \beta_1 r}\mathbb{E}(\delta^4(r)1_{\{r \leq \tau_n\}}) \int_r^t \frac{e^{\kappa \beta_1 s}}{(1 + s)^{2\theta}}dsdr
\leq C(1 + n).
\]
Similarly, we can get that $J_2(t) \leq C(1 + n)$ and $J_2(\infty)$ exists. The proof is therefore complete. \[\square\]
Remark 4.2. By checking the argument of Theorem 4.2, it is easy to see that Theorem 4.2 is still true for the two-factor CIR-type mode (4.1) with delay \( \tau = 0 \) whenever \( \gamma_i \in [0, \frac{1}{2}) \), \( i = 1, 2 \). On the other hand, the model (4.1) is not covered by [19, Theorem 2] due to the fact that the jump-diffusion coefficient is Lipschitz continuous, but not Hölder continuous with exponent \( \frac{1}{2} \).

Remark 4.3. Stochastic models under regime-switching have recently been developed to model various financial quantities, e.g., option pricing, stock returns, and portfolio optimisation. In particular, the CIR-type model under regime-switching has found its considerable use as a model for volatility and interest rate. Hence, it is also interesting to discuss the long-term behavior of CIR-type model under regime-switching

\[
\begin{cases}
    dX(t) = \{2\beta(r(t))X(t) + \delta(t)\}dt + \sigma(r(r))|X(t)|^\theta dW(t), \\
    X(0) = x \text{ and } r(0) = i_0,
\end{cases}
\]

where \( \theta \in [\frac{1}{2}, 1] \) and \( r(t) \) is a continuous-time Markov chain with a finite state space. This will be presented in a forthcoming paper.

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