Photonic systems with two-dimensional landscapes of complex refractive index via time-dependent supersymmetry

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Abstract

We present a framework for the construction of solvable models of optical settings with genuinely two-dimensional landscapes of refractive index. The solutions of the associated non-separable Maxwell equations in paraxial approximation are found with the use of the time-dependent supersymmetry. We discuss peculiar theoretical aspects of the construction. Sufficient conditions for existence of localized states are discussed. Localized solutions vanishing for large $|\vec{x}|$, that we call light dots, as well as the guided modes that vanish exponentially outside the wave guides are constructed. We consider different definitions of the parity operator and analyze general properties of the $\mathcal{PT}$-symmetric systems, e.g. presence of localized states or existence of symmetry operators. Despite the models with parity-time symmetry are of the main concern, the proposed framework can serve for construction of non-$\mathcal{PT}$-symmetric systems as well. We explicitly illustrate the general results on a number of physically interesting examples, e.g. wave guides with periodic fluctuation of refractive index or with a localized defect, curved wave guides, two coupled wave guides or a uniform refractive index system with a localized defect.

1 Introduction

In specific situations, propagation of light is governed by the same equations as matter waves in quantum mechanics. The coincidence of Maxwell equations in paraxial approximation with the Schrödinger equation makes it possible to use methods of quantum mechanics in the analysis of the optical settings.

This link proved to be particularly fruitful for investigation of optical systems where a complex refractive index representing balanced gain and loss prevents uncontrolled dimming or brightening of light [1–5]. The Hamiltonian of the associated Schrödinger equation ceases to be hermitian but possesses an antilinear symmetry. It was demonstrated two decades ago that such operators, having typically a $\mathcal{PT}$-symmetry with $\mathcal{P}$ and $\mathcal{T}$ being parity and time-reversal, can have purely real spectra [6]. It was showed later on that such models can provide consistent quantum mechanical predictions despite the non-hermicity of the Hamiltonian as long as the scalar product of the associated Hilbert space is redefined [7–11]. As much as this task proved to be difficult to accomplish in explicit quantum systems, see e.g. [12,13], it is non-existent in the realm of classical optics which, therefore, becomes an exciting field for the investigation of the systems described by $\mathcal{PT}$-symmetric (pseudo-Hermitian) Hamiltonians.

Supersymmetric quantum mechanics represents a highly efficient framework for construction of new exactly solvable models [14–16]. It is based on the Crum-Darboux transformation which is known in the analysis of Sturm-Liouville equations for a long time, see [17] and references therein. It allows to modify the potential term of the equation while preserving its solvability; the solutions of the new equation can be found by direct application of the Darboux transformation on the solutions of the original one. In the framework of the supersymmetric quantum mechanics, the transformation forms the supercharge of the supersymmetric Hamiltonian.

Supersymmetry was utilized in the analysis of $\mathcal{PT}$-symmetric quantum models [18–21]. It has been used in construction of $\mathcal{PT}$-symmetric optical systems of required properties [22–31]. There were constructed systems with invisible defects in crystal [29,31], with transparent interfaces [24], unidirectional invisibility [28] or in the...
context of coupled mode systems \[32,33\]. There were performed experiments with the supersymmetric photonic lattices \[25,26\].

It is worth mentioning the increasing interest in the non-\(\mathcal{PT}\)-symmetric systems with complex refractive index where, however, the gain and loss can still support guided, non-decaying modes associated with real spectra of the associated Hamiltonians. Explicitly solvable models of these systems were constructed via supersymmetry in \[23\]. They were also studied numerically \[34,36\] and experimental setups were proposed in \[37\].

Vast majority of the settings considered in the literature are described by effectively one-dimensional Hamiltonian. In most cases, they possess translational symmetry, typically along the axis of propagation of the light beam. The two-dimensional, exactly (analytically) solvable models possessing separability in radial coordinates were considered in \[38,39\]. \(\mathcal{PT}\)-symmetry breaking in two and three dimensions were considered in \[40\], scattering properties were studied in \[41\]. Two-dimensional periodic arrays of localized gain and loss regions, called photonic crystals, were analyzed numerically in \[42\].

In this work, we provide exactly solvable models of optical settings with non-separable complex refractive index. We focus primarily on \(\mathcal{PT}\)-symmetric optical wave guides as well as on localized defects. Our motivation is clear: as the current experimental techniques are rather ready for realization of such setting, existence of exactly solvable models with genuinely two-dimensional complex inhomogeneities of the refractive index is desirable.

The work is organized as follows. In the next section, we present the framework of the time-dependent supersymmetry \[43,44\] and discuss two definitions of the parity operator that will be distinctive for the models presented in the following sections. In the section 3, we construct exactly solvable models of \(\mathcal{PT}\)-symmetric wave guides with a defect in the form of a localized gain and loss. We show that the associated Schrödinger equation can support localized solutions, light dots. In the section 4, we construct \(\mathcal{PT}\)-symmetric wave guides where fluctuations of refractive index are vanishing for \(x \to \infty\) whereas they can be periodic along \(z\)-axis. We provide a general discussion on the existence of guided modes and illustrate the general results on explicit examples of periodically modulated \(\mathcal{PT}\)-symmetric wave guides supporting guided modes. The subsection 1.3 is devoted to a model of a non-\(\mathcal{PT}\)-symmetric wave guide with a guided mode. Section 5 is dedicated to systems generated with higher order supersymmetry representing point defects and coupled wave guides.

2 Mathematical framework

In this section, we review briefly the main mathematical tools that will be used extensively in the forthcoming text. In particular, we present construction of the time-dependent Darboux transformation for the Schrödinger equation and discuss peculiarities of the construction for the \(\mathcal{PT}\)-symmetric systems. For the sake of completeness, let us start with a short review of the relation between the Schrödinger and the Maxwell equations.

2.1 Paraxial approximation

Consider a monochromatic light beam with wavelength in vacuum \(\lambda\). Let \(X\), \(Y\), and \(Z\) be spacial coordinates. The Maxwell equation for electric and magnetic fields \(\vec{E} = \vec{E}(X,Y,Z)\) and \(\vec{H} = \vec{H}(X,Y,Z)\) of this monochromatic wave varying in time as \(\exp(-i\omega t)\) are

\[
\nabla \times \vec{E} = i\omega\mu_0 \vec{H}, \quad \nabla \times \vec{H} = -i\omega\varepsilon\vec{E},
\]

(1)

In this article we will focus on waves propagating in the \(Z\) direction in a medium with refracting index \(n(X,Y,Z) = \sqrt{\varepsilon}\). Under some circumstances that will be discussed in this subsection, equations (1) can be written as Schrödinger equations. This process is known as the paraxial approximation \[45,50\].

Let us write the electric field \(\vec{E}\) as

\[
\vec{E} = \exp(ikn_0Z) \left( \vec{\psi}_T + \hat{a}_Z \psi_Z \right),
\]

(2)

where \(k = 2\pi/\lambda\) is the wave number, \(n_0\) is a reference value of the index of refraction, \(\hat{a}_Z\) is a unit vector in the \(Z\) direction, \(T\) stands for the transverse part of the field, \(\vec{\psi}_T = \vec{\psi}_T(X,Y,Z)\) and \(\psi_Z = \psi_Z(X,Y,Z)\).

Taking the curl of the first equation in (1) the electric field must satisfy

\[
\nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = 2k^2n^2\vec{E}.
\]

(3)

Then, the substitution of the ansatz (2) in (3) and the use of the notation \(\nabla = \nabla_T + \hat{a}_Z \partial_Z\) leads to the transverse equation

\[
\nabla_T \left( \nabla_T \cdot \vec{\psi}_T + ikn_0 \psi_Z + \partial_Z \psi_Z \right) - \nabla^2 \vec{\psi}_T - \partial_Z^2 \vec{\psi}_T + k^2n_0^2\vec{\psi}_T - 2ikn_0\partial_Z \vec{\psi}_T = k^2n^2\vec{\psi}_T,
\]

(4)
and the longitudinal equation
\[ ikn_0 \nabla_T \cdot \vec{\psi}_T + \partial_Z \left( \nabla_T \cdot \vec{\psi}_T \right) - \nabla^2 \psi_Z = k^2 n^2 \psi_Z. \] (5)

Equations (4) and (5) can be approximated and simplified when introducing a small parameter. In this problem we have three different scales, first the wavelength \( \lambda \), second the characteristic size of the beam in the transverse direction \( x_0 \) and finally a longitudinal distance \( \ell \) define as \( \ell = \epsilon/|\nabla \epsilon| = n^2/|\nabla n^2| \) as scale of the inhomogeneities (another common longitudinal scale is \( \ell = kx_0^2 \) known as diffraction length). Then, if \( \lambda << x_0 < \ell \), the functions \( \vec{\psi}_T \) and \( \psi_Z \) can be expanded in the parameter \( \nu = x_0/\ell \) and introducing the scaled variables
\[ x = X/x_0, \quad y = Y/x_0, \quad z = Z/2\ell, \] (6)
the equations satisfied by \( \vec{\psi}_T = \hat{a}_X \psi_{TX} + \hat{a}_Y \psi_{TY} \) and \( \psi_Z \) in the lowest order in the expansion are
\[ i\partial_x \vec{\psi}_T + \hat{a}_X \partial^2_x \psi_{TX} + \hat{a}_Y \partial^2_y \psi_{TY} - k^2 x_0^2 (n_0^2 - n^2) \vec{\psi}_T = 0, \] (7)
\[ \psi_Z = i(\partial_x \psi_{TX} + \partial_y \psi_{TY}). \] (8)

If the index of refraction is written as \( n = n_0 + \delta n \) and if \( \delta n/n << 1 \) then (7) can be further simplified to
\[ i\partial_x \vec{\psi}_T + \hat{a}_X \partial^2_x \psi_{TX} + \hat{a}_Y \partial^2_y \psi_{TY} + 2k^2 x_0^2 n_0 \delta n \vec{\psi}_T = 0. \] (9)

Each vector component in (7) or (9) satisfies a time dependent Schrödinger equation:
\[ i\partial_t \psi + \partial^2_t \psi - V \psi = 0, \] (10)
where the \( z \) variable plays the role of time parameter and the potential \( V = k^2 x_0^2 (n_0^2 - n^2) \). Typical numbers in step index optical fibers are \([51]\): index of refraction of core and cladding of \( n_1 = 1.4 \) and \( n_0 = 1.39298 \), respectively, core diameter 50 \( \mu \)m, wavelength \( \lambda = 0.85 \) \( \mu \)m and characteristic size of beam \( x_0 = 25 \) \( \mu \)m, then using as longitudinal scale the diffraction length \( \ell = 4619 \) \( \mu \)m and the potential \( V \) is a potential well being zero in the cladding and \( V = -669.355 \) in the core region.

### 2.2 Time-dependent Darboux transformation and \( \mathcal{PT} \)-symmetry

To our best knowledge, the Darboux transformation in the context of optical systems was employed in the analysis of effectively one-dimensional models. In the current article, we shall focus on settings where the Schrödinger equation cannot be reduced to an effectively one-dimensional equation as the fluctuations of the refractive index prevent this reduction of the \( \mathcal{PT} \)-symmetry. In the current article, we shall focus on settings where the Schrödinger equation cannot be reduced to an effectively one-dimensional equation as the fluctuations of the refractive index of effectively one-dimensional models. In the current article, we shall focus on settings where the Schrödinger equation cannot be reduced to an effectively one-dimensional equation as the fluctuations of the refractive index of effectively one-dimensional models.

Let us suppose that the following Schrödinger equation
\[ S_0 \psi = i\partial_x \psi + \partial^2_x \psi - V_0(x,z) \psi = 0, \quad x \in \mathbb{R}, \quad z \in \mathbb{R}, \] (11)
is exactly solvable and its solutions are known. We suppose that \( V_0(x,z) \) has no singularities in \( \mathbb{R}^2 \) and it is sufficiently smooth. We will use the time-dependent Darboux transformation discussed in \([43,44,52,53]\) to generate another exactly solvable equation with a different potential term. Let us present here the main points of the construction. It is based on the intertwining relation
\[ S_1 \mathcal{L} = \mathcal{L} S_0 \] (12)
that guarantees that we can get solutions of the new equation \( S_1 \phi = 0 \) defined as \( \phi = \mathcal{L} \psi \) provided that \( S_0 \psi = 0 \) and \( \mathcal{L} \) maps the domain of \( S_0 \) into the domain of \( S_1 \). The ansatz for the intertwining operator \( \mathcal{L} \) is in the form of a first order differential operator, \( S_1 \) is a Schrödinger operator with an altered potential term,
\[ \mathcal{L} = L_1(z) \left[ \partial_x - \frac{\partial_x u(x,z)}{u(x,z)} \right], \quad S_1 = i\partial_x + \partial^2_x - V_1(x,z). \] (13)

Here, \( V_1(x,z), u(x,z) \) and \( L_1(z) \) are to be fixed such that the intertwining relation (12) is satisfied. Substituting (11) and (13) into (12), one can find that the later relation can be satisfied as long as
\[ V_1(x,z) = V_0(x,z) + i\partial_z \ln L_1(z) - 2\partial_x^2 \ln u(x,z) \] (14)
and
\[ S_0 u(x, z) = c(z) u(x, z), \]
see [13] for details. As the function \( c(z) \) affects just the phase of the solution\(^1\) but not the potential \( V_1 \), it can be set to zero, \( c(z) = 0 \). In what follows, we will denote by \( u \) the solution of \( S_0 u = 0 \) used in definition of the new potential \( [14] \) and of the intertwining operator \( [13] \). It will be called transformation function.

The relations \([14]\) and \([15]\) are sufficient to make the intertwining relation formally satisfied. In addition, the function \( u(x, z) \) as well as \( L_1(z) \) are also required to be nodeless. Otherwise, the transformation would be singular and it would fail to provide the mapping between the domains of \( S_0 \) and \( S_1 \). When \( L \) as well as \( V_1 \) are regular, the relation \([12]\) guarantees that we can generate solutions of \( S_1 \phi(x, z) = 0 \) from the solutions \( \psi(x, z) \) of \([11]\) by \( L \),
\[ \phi(x, z) = L \psi(x, z). \]

We can try to find the transformation \( L^\sharp \) such that it satisfies
\[ S_0 L^\sharp = L^\sharp S_1, \]
i.e. it is the “inverse” of \( L \). Using the general formulas \([13]\), we take \( S_1 \) as the initial system and we define \( L^\sharp = \frac{1}{L_2(x)} \frac{\partial_x}{\sqrt{\phi(x,z)}} \) where \( \phi(x,z) \) solves \( S_1 v(x, z) = 0 \). Then it is granted that there holds \( L^\sharp S_1 = S_2 L^\sharp \) for
\[ S_2 = i \partial_z + \frac{\partial^2}{2} - V_0(x, z) - i \partial_t \ln L_1 L_2 + 2 \partial_x^2 \ln uv. \]

In order to identify \( S_2 = S_0 \), we have to eliminate the last two terms by setting \( 2 \partial_x^2 \ln uv = i \partial_z \ln L_1 L_2 \). As the right-hand side of the later equation is \( x \)-independent, we have to fix \( v \) such that
\[ \partial_x^2 \ln uv = 0 \]
and fix \( L_2 = L_1^{-1}(z) \exp(-2i \int z (\ln uv))'' \). Then the last two terms in \([18]\) vanish and \( L^\sharp \) represents the inverse intertwining operator.

The operator \( L \) can map any solution of \( S_0 f = 0 \) to a nontrivial solution of \( S_1 g = 0 \) as \( g = L f \), except the case where \( f \equiv u \) as it gets annihilated by \( L \), \( Lu = 0 \). Hence, the image of this state is missing in the new system. We can try to find another solution of \( S_1 g = 0 \) given in terms of the function \( u \). In the one-dimensional supersymmetric quantum mechanics, this missing state is defined as \( \frac{1}{u} \). The formula hints on the importance of the “missing state”; when \( u \) is exponentially growing, the missing state is square integrable and represents a bound state of the new system. In [13], similar formula was used for the time-dependent and hermitian systems. Inspired by these results, let us make an ansatz for the missing state in the following form
\[ u_m = \frac{1}{f(z)S u}. \]

We would like it to satisfy \( S_1 u_m = 0 \). The operator \( S \) is an operator whose properties are to be fixed such that the later equation is satisfied. We shall compute \( S_1 u_m \). We have
\[ S_1 \frac{1}{f S u} = \frac{1}{(S u)^2} f \left( -i (S u) + (S u)'' - V S u + 2 \left( \ln \frac{u}{S u} \right)'' - i (\ln (\hat{L}_1 f)) S u \right) \]

If there holds \( \partial_x^2 \ln \frac{u}{S u} = 0 \), then we can fix \( f(z) = L_1^{-1}(z) \exp(-2i \int z (\ln \frac{u}{S u}))'' \) and the last two terms in \([21]\) vanish. Hence, \( u_m \) solves \( S_1 u_m = 0 \) provided that there holds
\[ [-\partial^2_x + V_0, S] = 0, \quad \{i \partial_x, S\} = 0, \quad \partial^2_x \ln \frac{u}{S u} = 0, \]
and \( f(z) \) is fixed as
\[ f(z) = L_1^{-1}(z) \exp \left( -2i \int \left( \ln \frac{u}{S u} \right)'' \right). \]
We can see that the third relation in \([22]\) coincides with \([19]\) for \( v \equiv u_m \). Therefore, when \([22]\) are satisfied, there also exist the inverse operator,\(^2\)
\[ L^\sharp = f(z) u_m \partial_x \frac{1}{u_m}, \]
\(^1\)If there holds \( S_0 \psi = 0 \), then we can find solution of \((S_0 - c(t)) \bar{\psi} = 0 \) that reads \( \bar{\psi} = \exp(-i \int c(t)) \psi \).
\(^2\)If there holds \( S_0 \omega = 0 \), then we can find solution of \((S_0 - c(t)) \bar{\psi} = 0 \) that reads \( \bar{\psi} = \exp(-i \int c(t)) \psi \).
where $u_m$ and $f(z)$ are defined in (20), (22) and (23).

Existence of the inverse operator (24) implies another interesting fact; both $S_0$ and $S_1$ have symmetry operators

$$[S_0, L^0 L] = 0, \quad [S_1, L L^0] = 0,$$

where

$$L^0 L = \exp \left( -2i \int \frac{u}{S_u} u'' \right) \left[ \partial_x^2 - (\ln u_m u)' \partial_x + (\ln u)'(\ln u_m u)' - \frac{u''}{u} \right],$$

$$L L^0 = \exp \left( -2i \int \frac{u}{S_u} u'' \right) \left[ \partial_x^2 - (\ln u_m u)' \partial_x + (\ln u_m u)' - \frac{u''}{u_m} \right].$$

Up to now, we did not make any assumption on the hermiticity or PT-symmetry of the new potential $V_1$. In [43], both $V_0$ and $V_1$ were required to be real in order to preserve hermiticity of $S_0$ and $S_1$. In the hermitian case the operator $S$ can be identified with $S f(x, z) = f(x, z)$. Then $V_1$ is real whenever $u$ satisfies $\partial_x^2 \ln \frac{u}{u_m} = 0$ and $L_1$ is fixed as $L_1 = \exp \left( -i \int (\ln \frac{u}{u_m})' dx \right)$. This hermitian definition of $S$ complies with (22).

We are interested in the settings where $S_1$ ceases to be hermitian but possesses an antilinear symmetry that we shall identify with the simultaneous action of the operators of time-reversal $T$ and space inversion $P$. Most of the PT-symmetric systems discussed in the literature are effectively one-dimensional so that the space inversion $P$ is defined unambiguously as $P f(x) = f(-x)$. In two dimensions, we can define $P$ as the reflection with respect to a fixed point or with respect to an axis,

$$P_x f(x, z) = f(-x, z) \quad \text{or} \quad P_z f(x, z) = f(-x, -z).$$

The antilinear operator $T$ is given as

$$T f(x, z) = f(x, z).$$

We suppose that $V_0$ is PT-symmetric and we require $V_1$ to be $\mathcal{PT}$-symmetric as well,

$$\mathcal{PT} V_1(x, z) \mathcal{PT} = V_1(x, z).$$

It will restrict the possible choice of $u(x, z)$ in dependence on the actual definition of $\mathcal{P}$.

First, let us consider $\mathcal{P} \equiv \mathcal{P}_x$. The relation (30) then leads to

$$2 \partial_x^2 \ln \frac{u}{u(-x, z)} = i \partial_z \ln |L_1(z)|^2.$$

The operator $\mathcal{P}_x T$ operator anticommutes with $i \partial_x$ and it commutes with $V_0$. Hence, it fulfills the first two conditions in (22) and it is quite similar to the third one of (22). In order to explain the relation between (30) and (22), let us make the following comment: for a fixed $u$, we have a family of the new systems with the potentials (14) whose members differ by different choice of $L_1$. Now, let us identify $\mathcal{S} \equiv \mathcal{P}_x T$ and let us suppose that $\partial_x^2 \frac{u(x, z)}{u(-x, z)} = 0$. Then (22) is fulfilled and we can find the missing state (20) and the symmetry operator (26) for each member of the family. The missing state $u_m$ and $L^1$ are defined in close analogy with the hermitian case. It follows from (21) that when $u$ is exponentially expanding for large $|x|$, then $u_m$ is square integrable for fixed $z$. Despite the identification of $\mathcal{P}_x T$ with $\mathcal{S}$, the new potentials $V_1$ are not $\mathcal{P}_x T$-symmetric in general as the function $u$ can fail to solve (30).

Nevertheless, we have at least one $\mathcal{P}_x T$-symmetric potential $V_1$ in the family as the condition (30) is compatible with (22) for a constant $L_1$. We will see an explicit example of non- $\mathcal{P}_x T$-symmetric system that possesses a missing state (20) and symmetry operator (26) in the next section.

Now, we set $\mathcal{P} = \mathcal{P}_2$. The requirement (29) reduces to

$$2 \partial_x^2 \ln \frac{u(x, z)}{u(-x, z)} = i \partial_z \ln \frac{L_1(z)}{L_1(-z)}.$$

The operator $\mathcal{P}_2 T$ commutes with $i \partial_x$, so that it does not satisfy (22). In particular, it implies that we can no longer use the definition (20) for the missing state $u_m$ and we have to seek it in a different form. As $\mathcal{S} \equiv \mathcal{P}_2 T$ is not compatible with (20) or (22), we will suppose in the following text that $\mathcal{S} \equiv \mathcal{P}_2 T$ when referring to (20) or (22).
3 Light dots in wave guides

In this section, exactly solvable models will be presented where the Schrödinger equation can support solutions that are vanishing for large $|\vec{x}|$. We will call these solutions light dots. In what follows, we will take the free particle system as the initial one, i.e.

$$S_0 = i\partial_x + \partial_x^2.$$  \hspace{1cm} (32)

The function $u$ will be identified with a localized wave packet. This choice will give rise to either $P_xT$- or $P_2T$-symmetric systems.

3.1 Light dots in the wave guide with translational symmetry

We start with a simple choice of $u$ that is identified as a Gaussian wave packet,

$$u(x, z) = \frac{(2\pi)^{1/4}}{\sqrt{1-i z}} \exp \left( \frac{x^2}{4(1 - i z)} \right).$$  \hspace{1cm} (33)

The wave packet satisfies the relation (22) and it is nodeless. Therefore, the missing state (20) and the symmetry operators (26) are well defined. Taking $L_1(z) = \sqrt{1-i z}$ in the definition of the intertwining operator, the explicit form of $V_1$ and $L$ given in (13) are then

$$V_1 = -\frac{1}{2(1-i z)}, \quad L = \sqrt{1-i z} \partial_x - \frac{x}{2\sqrt{1-i z}}.$$  \hspace{1cm} (34)

We can see that the potential is $P_2T$-symmetric but it fails to be $P_xT$-symmetric as $u$ does not comply with (30). This is due to the non-trivial choice of $L_1$. If we have fixed $L_1$ as a constant, $P_2T$-symmetry of $V_1$ would be recovered.

The potential term is $x$-independent. It has the form of a straight wave guide divided symmetrically by gain and loss regions, see Fig. 1 for illustration. The missing state (20) reads

$$\psi_m(x, z) = \frac{1}{(2\pi)^{1/4}(1+z^2)^{1/2}} \exp \left( -\frac{x^2}{4(1 + iz)} \right),$$  \hspace{1cm} (35)

where $\psi_m$ fulfills $S_1\psi_m = 0$. The solution is strongly localized in the wave guide; it vanishes exponentially along $x$-axis while it has $\sim z^{-1}$ decay along the $z$-axis.

Symmetry operators can, as well, be constructed

$$L^2 = (1 + z^2)\partial_x^2 - iz\partial_x - \frac{1}{4}(x^2 + 2iz + 2), \quad [S_0, L^2] = 0,$$

$$L^3 = (1 + z^2)\partial_x^2 + iz\partial_x - \frac{1}{4}(x^2 + 2iz - 2), \quad [S_1, L^3] = 0.$$  \hspace{1cm} (36)

By construction, $L^2\psi_m = 0$. In order to illustrate the action of the symmetry operator $L^2$, let us consider the wave packet $\psi = (2\pi)^{-1/4}(1 + z^2)^{-1/2} \exp(- (x + 1/2)^2/4(1 + iz))$ solving $S_0\psi = 0$. We transform it into the solution of $S_1\phi = 0$ by the application of the intertwining operator,

$$\phi = L\psi = -\frac{4x - iz + 1}{4(2\pi)^{1/4}(1 + iz)\sqrt{1 + z^2}} \exp \left( -\frac{(x + \frac{1}{2})^2}{4(1 + iz)} \right), \quad S_1\phi = 0.$$  \hspace{1cm} (37)

Through the successive applications of the symmetry operator $L^3$ we can obtain a whole family of solutions. After the first iteration, we get

$$\chi = L^3\phi = -\frac{16x^2 + 56x + 72ixz + 2iz + 33z^2 + 31}{64(2\pi)^{1/4}(1 + z^2)^{1/2}(1 + iz)^2} \exp \left( -\frac{(x + \frac{1}{2})^2}{4(1 + iz)} \right), \quad S_1\chi = 0.$$  \hspace{1cm} (38)

The solutions $\psi_m, \phi$ and $\chi$ are illustrated in the Fig. 1.

As we can see, identification of $u$ with the Gaussian wave packet (33) resulted in the separable potential (34). It is not quite satisfactory result as we seek for a non-separable, two-dimensional system. In the next part of this section, we will show how to get wave packets that will serve well for definition of $u$ that will provide the new potentials of desired properties.
Figure 1: A straight wave guide divided symmetrically by gain and loss regions. The potential term is $V_1 = -\frac{1}{2}(1-iz^2)$, plots show its real (top left) and imaginary (top center) parts. Three different solutions of the time dependent Schrödinger equation for this potential, see (35), (37) and (38), $|u_m|^2$ (top right), $|\phi|^2$ (bottom left) and $|\chi|^2$ (bottom right) are also graphed.

3.2 Free particle solutions via harmonic oscillator

The harmonic oscillator and free particle systems are related through a specific point transformation. It allows to map solutions of one system into solutions of the other system, see [54–60]. Here, we discuss the main results of this transformation as we will use it extensively in this section. We will follow the notation introduced in [60].

Let us consider a one dimension time independent Schrödinger equation with the spatial variable $y$ as

$$\partial_y^2 \tilde{\psi}(y) + [E - \tilde{V}_0(y)]\tilde{\psi}(y) = 0,$$  \hspace{1cm} (39)

where $E$ is a real energy parameter and $\tilde{V}_0(y)$ is a known potential. Now, let $y$ be defined in terms of the new variables $z$ and $x$ as

$$y(x,z) = x \exp \left( 4 \int A \, dz \right) + 2 \int B \exp \left( 4 \int A \, dz \right) \, dz,$$ \hspace{1cm} (40)

where $A = A(z)$ and $B = B(z)$ are some (at the moment unspecified) functions. As a result, the function

$$\tilde{\psi}(x,z) = \tilde{\psi}(y(x,z)) \exp \left\{ -i \left[ A \, x^2 + B \, x + E \int \exp \left( 8 \int A \, dz \right) \, dz + \int (2iA + B^2) \, dz \right] \right\},$$ \hspace{1cm} (41)

is solution of the time dependent Schrödinger equation $(i\partial_z + \partial_z^2 - V_0)\psi = 0$, where the potential $V_0$ is given by

$$V_0(x,z) = \tilde{V}_0(y(x,z)) \exp \left( 8 \int A \, dz \right) + \left( \frac{d}{dz}A - 4A^2 \right) x^2 + \left( \frac{d}{dz}B - 4A \, B \right) x.$$ \hspace{1cm} (42)

In order to map the harmonic oscillator potential $\tilde{V}_0(y) = y^2/4$ into the free particle system $V_0(x,z) = 0$, we set $A = -z/(1 + z^2)$ and $B = 0$. Then the relation (40) results into $y = x(z^2 + 1)^{-1/2}$. A general solution $\tilde{u}$ of (39) for $\tilde{V}_0(y) = y^2/4$ and $E_n = n + 1/2$, where $n$ is an arbitrary real number, can be written as a superposition of two linearly independent solutions $\tilde{u}_{I,n}$ and $\tilde{u}_{II,n}$, see [61], where

$$\tilde{u}_{I,n}(y) = y \, \text{F}_1 \left( \frac{1-n}{2}, \frac{1}{2}; \frac{1}{2}; y^2 \right) \exp \left( -\frac{1}{4} y^2 \right),$$

$$\tilde{u}_{II,n}(y) = y \, \text{F}_1 \left( \frac{1-n}{2}, \frac{3}{2}; \frac{1}{2}; y^2 \right) \exp \left( -\frac{1}{4} y^2 \right).$$ \hspace{1cm} (43)
Here, \( \text{$_1F_1$(a;b;z)} \) is a confluent hypergeometric function \cite{62,63}. The functions satisfy \( \tilde{u}_{I,n}(y) = \tilde{u}_{I,n}(-y) \) and \( \tilde{u}_{II,n}(-y) = -\tilde{u}_{II,n}(y) \), i.e. they are even and odd functions, respectively. It implies that all the functions \( \tilde{u}_{II,n} \) share at least one zero at \( y = 0 \), \( \tilde{u}_{II,n}(0) = 0 \). The Wronskian of the two solutions is constant, \( W(\tilde{u}_{I,n},\tilde{u}_{II,n}) = 1 \).

The point transformation \eqref{41} maps the solutions \eqref{43} to
\[
\begin{align*}
    u_{I,n}(x,z) &= \frac{1}{(1+z^2)^{1/4}} \exp \left\{ \frac{i}{4} \left[ \frac{x^2}{z-i} - 4E_n \arctan(z) \right] \right\} \, _1F_1 \left( \frac{n}{2} ; \frac{1}{2} ; \frac{x^2}{2(z^2+1)} \right), \\
    u_{II,n}(x,z) &= \frac{x}{(1+z^2)^{1/4}} \exp \left\{ \frac{i}{4} \left[ \frac{x^2}{z-i} - 4E_n \arctan(z) \right] \right\} \, _1F_1 \left( \frac{1-n}{2} ; \frac{3}{2} \frac{x^2}{2(z^2+1)} \right).
\end{align*}
\]

They satisfy
\[
S_0 u_{I,n} = S_0 u_{II,n} = 0.
\]

In the special case of \( n \) being a non-negative integer, either \eqref{44} or \eqref{45} reduces to a square integrable function as the confluent hypergeometric function is truncated to a Hermite polynomial.

Let us analyze its zeros. When \( \alpha \) can be simplified to
\[
\text{\( u \approx \delta \text{nodeless in this case. When } u = 0 \) we used the abbreviation \( \delta = z^2 + 1 \).}
\]

\[
\begin{align*}
    u(x, z) &= \sum_{j=1}^{N} \left( \alpha_{I,n} u_{I,n} + i \alpha_{II,n} u_{II,n} \right), \\
    \alpha_{I(n),n,j} \in \mathbb{R}, \quad n_j \in \mathbb{R}.
\end{align*}
\]

We can see that it satisfies \eqref{2} \( \mathcal{P}_2 \mathcal{T} u = \epsilon u \), where \( \epsilon \in \{-1, 1\} \). Considering the other definition of the \( \mathcal{P} \) operator, there holds
\[
\mathcal{P}_2 \mathcal{T} u_{(I),n} = u_{(I),n} \exp \left( 2iE_n \arctan(z) + i \frac{z}{2(z^2+1)} x^2 \right).
\]

One can see that the function \( u \) complies with \eqref{22} provided that it is a linear combination of the solutions associated with the same energy. It can be written as
\[
\begin{align*}
    u(x, z) &= \alpha_{I,n} u_{I,n} + i \alpha_{I,n} u_{II,n}, \\
    \alpha_{I(n),n} \in \mathbb{R}.
\end{align*}
\]

In the rest of the section, we will consider special cases of \eqref{47} (and of \eqref{49} in particular) for construction of exactly solvable settings. We will also discuss regularity of new potentials for the explicit choices of \( u \).

### 3.3 Optical wave guide with a localized defect

In this subsection, we utilize the transformation function \eqref{49}. It can be written as
\[
\begin{align*}
    u(x, z) &= \frac{1}{(1+z^2)^{1/4}} \exp \left\{ \frac{i}{4} \left[ \frac{x^2}{z-i} - 4 \left( \frac{n+1}{2} \right) \arctan(z) \right] \right\} \times
    \left[ \alpha_{I,n} \, _1F_1 \left( \frac{n}{2} ; \frac{1}{2} ; \frac{x^2}{2(z^2+1)} \right) + i \alpha_{II,n} \frac{x}{(1+z^2)^{1/2}} \, _1F_1 \left( \frac{1-n}{2} ; \frac{3}{2} \frac{x^2}{2(z^2+1)} \right) \right].
\end{align*}
\]

Let us analyze its zeros. When \( \alpha_{I,n} \alpha_{I,n} \neq 0 \), then \( u_{I,n} \) and \( u_{II,n} \) cannot vanish in the same points as we have \( W(u_{I,n},u_{II,n}) = 1 \). Hence, the function \( u \) is nodeless in this case. When \( \alpha_{I,n} = 0 \) and \( n \leq 0 \), \( u \equiv u_{II,n} \) is nodeless by the oscillation theorem. As we discussed above, \( u \) satisfies \eqref{2} so that the missing state \( u_m \) can be constructed as in \eqref{22} (in the definition of \( u_m \), the function \( f(z) \) in \eqref{25} reads \( f(z) = L_1^1/(z)(z^2+1) \)).

In order to present an explicit model, let us take \( n = -2 \), \( \alpha_{I,-2} = 1 \), \( \alpha_{I,-2} = \alpha \) and \( L_1 = \sqrt{z^2+1} \). Then \eqref{50} can be simplified to
\[
\begin{align*}
    u(x, z) &= \frac{1}{\delta^{1/4}} \exp \left\{ \frac{i}{4} \left[ \frac{x^2}{z-i} + 6 \arctan(z) \right] \right\} \left[ 1 + \sqrt{\frac{2\delta}{x^{2}}} \text{erf} \left( \frac{x}{\sqrt{2\delta}} \right) \exp \left( \frac{x^2}{2\delta} \right) + i \alpha \frac{x}{\sqrt{\delta}} \exp \left( \frac{x^2}{2\delta} \right) \right],
\end{align*}
\]

where we used the abbreviation \( \delta = z^2 + 1 \), and \( \text{erf}(\cdot) \) is the error function \cite{62}. The new potential \eqref{14} reads
\[
\begin{align*}
    V_1(x,z) &= \frac{2 \left\{ e^{\frac{\pi}{\delta}} \left( x^2 - 2\delta \right) \left[ \sqrt{2}\alpha^2 - i\sqrt{\pi} \text{erf} \left( \frac{x}{\sqrt{2\delta}} \right) \right]^2 - 4\sqrt{2}\pi \delta x e^{\frac{x^2}{2\delta}} \text{erf} \left( \frac{x}{\sqrt{2\delta}} \right) - 8i\alpha x \sqrt{\delta e^{\frac{x^2}{2\delta}}} - 6\delta \right\}}{\delta \left( 2\sqrt{\delta} + xe^{\frac{x^2}{2\delta}} \left( \sqrt{2\pi} \text{erf} \left( \frac{x}{\sqrt{2\delta}} \right) + 2i\alpha \right) \right)^2}.
\end{align*}
\]

\footnote{There holds \( \mathcal{P}_2 \mathcal{T} u_{I,n} = u_{I,n} \) and \( \mathcal{P}_2 \mathcal{T} u_{II,n} = -u_{II,n} \).}
Figure 2: Optical wave guide with a localized defect. Plots of the real (top left) and imaginary (top center) parts of $V_1$, see (52), when $\alpha = -(2\pi)^{-1/2}$. Three different solutions for the corresponding time dependent Schrödinger equation are shown: first $|u_m|^2$ (top right), see (53), then $|\phi_0|^2 = |L\psi_0|^2$ (bottom left) and finally $|\phi_1|^2 = |L\psi_1|^2$ (bottom right).

Since $\text{erf}(-y) = -\text{erf}(y)$ and $V_1$ is actually function of $x$ and $z^2$, we can verify explicitly that the potential is both $P_xT$-symmetric and $P_x2T$-symmetric. It is not surprising; $u$ satisfies both (30) and (31). The expression (52) represents a one-parameter family of potentials where $\alpha$ can acquire any real value, see Fig. 2. For $\alpha = 0$, the potential is real function and $S_1$ is Hermitian.

The missing state $u_m$ can be written as

$$u_m(x, z) = \frac{1}{\delta^{1/4}} \exp \left\{ \frac{1}{4} \left[ -\frac{x^2}{1+z} + 6i \arctan(z) \right] \right\} \left[ 1 + \sqrt{\frac{\pi}{2\delta}} x \text{ erf} \left( \frac{x}{\sqrt{2\delta}} \right) \exp \left( \frac{x^2}{2\delta} \right) + i \frac{x}{\sqrt{16\delta}} \exp \left( \frac{x^2}{2\delta} \right) \right]^{-1}$$

It is exponentially vanishing for large $|x|$. In order to obtain other exponentially vanishing solutions, it is convenient to introduce the following notation

$$\psi_n(x, z) = \begin{cases} \frac{1}{\sqrt{\sqrt{2\pi}x^n n!}} u_{I,n}(x, z), & n \text{ is even}, \\ \frac{1}{\sqrt{\sqrt{2\pi}x^n n!}} u_{II,n}(x, z), & n \text{ is odd}, \end{cases}$$

where $\psi_n$ are square integrable functions for fixed $z$ that are obtained from the bound states of the harmonic oscillator by the point transformation. Then $\phi_n \equiv L\psi_n$ represent light dots in the current system as they vanish both for large $x$ and $z$, see Fig. 2 for illustration.

The symmetry operators (26) can be found explicitly as

$$L^2L = (1+z^2)\partial_x^2 - i z x \partial_x - \frac{1}{4}(x^2 + 2iz + 6),$$

$$LCL = (1+z^2)\partial_x^2 - i z x \partial_x - \frac{1}{4}(x^2 + 2iz + 6) - (z^2 + 1)V_1,$$

where $[S_0, L^2L] = [S_1, LCL] = 0$. Notice that there holds $LCL - L^4L = -L_1(z)^2V_1$. The action of $LCL^2$ on $\phi_n$ can produce new solutions of $S_1 LCL^2 \phi_n = 0$. 

9
3.4 Light dots in epic landscapes of refractive index

Let us depart from (47) for definition of $u$, fixing it as a linear combination of $\psi_n$ defined in (53). There are some properties inherited from the eigenstates of the harmonic oscillator to consider. First, $\psi_0$ is the only solution without nodes. Second, for all odd $n$, $\psi_n(0, z) = 0$, and third, two functions $\psi_m$ and $\psi_n$, $m \neq n$, can vanish simultaneously only at $x = 0$. This information helps us to find a set of coefficients such that $u(x, z) \neq 0$.

Let us take $u = \psi_j + i \alpha \psi_{j+1}$, where $\alpha$ is a real constant and $j$ is a positive integer number, (note that up to a global constant phase these are particular cases of (47))

\[
u(x, z) = \frac{1}{\sqrt{2\pi j!}} (1 + z^2)^{1/4} \exp \left\{ \frac{i}{4} \left[ \frac{x^2}{z-i} - 4 \left( j + \frac{1}{2} \right) \arctan(z) \right] \right\} \\ \times \left[ H_j \left( \frac{x}{\sqrt{2(z^2+1)}} \right) + i \frac{\alpha}{\sqrt{2(j+1)}} \left( \frac{1-iz}{1+z^2} \right) \right] H_{j+1} \left( \frac{x}{\sqrt{2(z^2+1)}} \right). \]

This function is never zero. It follows from the fact that $H_j(x/\sqrt{2(z^2+1)})$ and $H_{j+1}(x/\sqrt{2(z^2+1)})$ are real functions and their zeros do not coincide. Indeed, the imaginary part of the linear combination of the Hermite polynomials in brackets is a real multiple of $H_{j+1}'(\cdot)$ whereas the real part is a combination of $H_j(\cdot)$ and $H_{j+1}(\cdot)$, so that their zeros are mismatched.

Fixing $L_1(z) = 1$, the operator $\mathcal{L}$ in (13)

\[\mathcal{L} = \partial_x + \frac{x}{2(1+iz)} - \partial_x \ln \left[ H_j \left( \frac{x}{\sqrt{2(z^2+1)}} \right) + i \frac{\alpha}{\sqrt{2(j+1)}} \left( \frac{1-iz}{1+z^2} \right) H_{j+1} \left( \frac{x}{\sqrt{2(z^2+1)}} \right) \right] \]

intertwines $S_0$ with the new Schrödinger operator whose potential $V_1$ defined in (14) can then be written as

\[V_1 = \frac{1}{1+iz} - 2\partial^2_x \ln \left[ H_j \left( \frac{x}{\sqrt{2(z^2+1)}} \right) + i \frac{\alpha}{\sqrt{2(j+1)}} \left( \frac{1-iz}{1+z^2} \right) H_{j+1} \left( \frac{x}{\sqrt{2(z^2+1)}} \right) \right]. \]

The potential is $\mathcal{P}, \mathcal{T}$-symmetric and it is a rational function in the $x$ variable for any (positive integer) $j$. It is remarkable that $\{V_1\}$ follows a star-like pattern where the number of rays in the “star-burst” correlates with the value of $j$. As we will show below, the asymptotic, star-like behavior of the potential can be understood explicitly with the use of the fact that $V_1$ is rational function in $x$ variable.

The solutions of the corresponding Schrödinger equation can be written as $\phi_n = \mathcal{L} \psi_n$. Using the property $H_n'(y) = 2nH_{n-1}(y)$ of Hermite polynomials, we can write them as

\[\phi_n = \sqrt{n \pi} \frac{1}{1+iz} \psi_{n-1} \left[ \partial_x \ln \left[ H_j \left( \frac{x}{\sqrt{2(z^2+1)}} \right) + i \frac{\alpha}{\sqrt{2(j+1)}} \left( \frac{1-iz}{1+z^2} \right) H_{j+1} \left( \frac{x}{\sqrt{2(z^2+1)}} \right) \right] \right] \psi_n. \]

It follows from the definition of $\mathcal{L}$ that for $n \equiv j$, there holds $\phi_j = i \alpha \phi_{j+1}$. Indeed, on one side we have $\phi_j = \mathcal{L} \psi_j = \psi_j - (\ln(\psi_j + i \alpha \psi_{j+1}))/\psi_j = i \alpha (\psi_j \psi_{j+1} - \psi_j \psi_{j+1})/\psi_j + i \alpha \psi_{j+1}$. On the other side, we get $\phi_{j+1} = \mathcal{L} \psi_{j+1} = \psi_{j+1} - (\ln(\psi_j + i \alpha \psi_{j+1}))/\psi_j = (\psi_j \psi_{j+1} - \psi_j \psi_{j+1})/\psi_j + i \alpha \psi_{j+1}$. Let us present two explicit examples. First, we fix $j = 0$. Then (57) reduces to

\[u(x, z) = \psi_0 + i \alpha \psi_1 = \frac{1 + iz + i \alpha x}{(2\pi)^{1/4}(1 + z^2)^{3/4}} \exp \left\{ -\frac{1}{4} \left( \frac{x^2}{1+iz} + 6i \arctan z \right) \right\}. \]

The new potential $V_1$ defined in (59) and the intertwining operator can then be written as

\[V_1(x, z) = \frac{1}{1+iz} + \frac{2\alpha^2}{(ax + z - i)^2}, \quad \mathcal{L} = \partial_x + \frac{x}{2(1+iz)} - \frac{\alpha}{ax + z - i}. \]

The $\alpha$-dependent part of the potential is translationally invariant along $z + \alpha x = \text{const.}$. Hence, the first term in $V_1$ represents a barrier along $z$-axis that is crossed by a wave guide in an angle given by $\alpha$, represented by the second term of $V_1$, see Fig. 9 for illustration. It is worth mentioning that for $\alpha = 0$, the potential is $x$-independent and proportional to the conjugate of the potential (53).

As the function $u$ does not comply with (23), it is not clear how to define the missing state. However, we can still find localized solutions for this new system. They can be constructed from the square integrable solutions of the harmonic oscillator as in (60):

\[\phi_n = \sqrt{n \pi} \frac{1}{1+iz} \psi_{n-1} - \frac{\alpha}{ax + z - i} \psi_n. \]
Figure 3: Plots of the real (top left) and imaginary (top center) parts of $V_1$ when $\alpha = 1$, see (62). Moreover, three different solutions, or light dots, $|\phi_0|^2$ (top right), $|\phi_2|^2$ (bottom left) and $|\phi_3|^2$ (bottom right), see (63), are shown.

The first three functions (notice that $\phi_0 = i\alpha\phi_1$) are illustrated in Fig. 3.

As the second example, let us set $u = \psi_1 + i\alpha\psi_2$ and $L_1 = 1$. The corresponding potential term $V_1$ and the intertwining operator are:

$$V_1(x, z) = \frac{1}{1 + iz} - \frac{4\alpha}{\alpha x^2 + \sqrt{2}(z - i)x - (1 + z^2)\alpha} + \frac{2}{\alpha x^2 + \sqrt{2}(z - i)x - (1 + z^2)\alpha} \left( \frac{2\alpha x + \sqrt{2}(z - i)}{\alpha x^2 + \sqrt{2}(z - i)x - (1 + z^2)\alpha} \right)^2,$$

$$L = \partial_x + \frac{x}{2(1 + iz)} - \frac{2\alpha x + \sqrt{2}(z - i)}{\alpha x^2 + \sqrt{2}(z - i)x - (1 + z^2)\alpha}. \quad (64)$$

As the explicit formulas suggest, $V_1$ is regular as the denominators cannot vanish. The potential represents two asymptotically straight wave guides that come together at a specific angle, but they avoid intersection, see the first two graphs (top left and top center) in Fig. 4. It is possible to understand the asymptotic behavior in the following manner: rewriting the potential as a single fraction, we substitute $x = az + b$. This way, we get a polynomial of order four in the numerator, while there is a polynomial of order five in $z$ in the denominator. We require the polynomials to be of the same order, so that the coefficient of the leading term in the denominator has to vanish. This condition fixes the values of $a$ as $a = \frac{1}{\sqrt{2}} \left( -1 + (1 + (1 + 2\alpha^2)^2) \right)$, $\epsilon = 1, 2$. The asymptotic behavior of the potential can be then calculated as

$$\lim_{|z| \to \infty} V_1|_{x \to a_1z + b} = \frac{4\alpha^2(1 + 2\alpha^2)}{(b\alpha \sqrt{2} + 2\alpha^2 - i(1 + \sqrt{1 + 2\alpha^2}))^2}, \quad (65)$$

$$\lim_{|z| \to \infty} V_1|_{x \to a_2z + b} = \frac{2\alpha^2(1 + 2\alpha^2)}{-1 + \sqrt{1 + 2\alpha^2} + \alpha(-\alpha + b^2\alpha(1 + 2\alpha^2) + i\sqrt{2}(1 - 2\alpha^2 + \sqrt{1 + 2\alpha^2})). \quad (66)$$

The asymptotic values are invariant with respect to conjugation joined by substitutions $z \to -z$ and $b \to -b$ which is just the manifestation of the $P_2\mathcal{T}$-symmetry of the potential.

Likewise in the previous case, we can find solutions $\phi_n = L\psi_n$ that are square integrable for fixed $z$:

$$\phi_n = \frac{\sqrt{m}}{1 + iz}\psi_{n-1} - \frac{2\alpha x + \sqrt{2}(z - i)}{\alpha x^2 + \sqrt{2}(z - i)x - (1 + z^2)\alpha}\psi_n. \quad (67)$$

They represent light dots that are concentrated at the bending of the wave guides, see Fig. 4 for illustration.
that when the absolute value of the potential term is integrable, \( |V| \) is integrable.

Let us suppose that the initial potential \( V \) is integrable.

4.1 On the construction of guided modes

In this section, we will focus on models of optical wave guides. By a asymptotic analysis of the involved quantities, we will obtain general results for a large class of initial potentials that are just required to be integrable and to possess translational symmetry. In particular, we will show how to find solutions of the associated Schrödinger equation that are vanishing exponentially in the transverse direction to the wave guide, and hence, represent guided modes.

4 Guided modes in optical waveguides

In this section, we will focus on models of optical wave guides. By a asymptotic analysis of the involved quantities, we will obtain general results for a large class of initial potentials that are just required to be integrable and to possess translational symmetry. In particular, we will show how to find solutions of the associated Schrödinger equation that are vanishing exponentially in the transverse direction to the wave guide, and hence, represent guided modes.

4.1 On the construction of guided modes

Let us suppose that the initial potential \( V_0 \) does not depend on \( z \). It is known (see e.g. Th. 4.1, p.70 in [64]) that when the absolute value of the potential term is integrable, \( \int_{\mathbb{R}} |V_0(x)| dx < \infty \), then the stationary equation \( -\partial_x^2 + V_0(x) = \lambda^2 \psi \) has two solutions \( \psi^+(\lambda, x) \) and \( \phi^+ (\lambda, x) \) for a complex \( \lambda \neq 0 \) satisfying

\[
\psi^+(\lambda, x) = e^{i\lambda x} (1 + o(1)), \quad \phi^+ (\lambda, x) = e^{-i\lambda x} (1 + o(1)) \quad \text{as} \quad x \to +\infty. \tag{68}
\]

Here, \( o(1) \) is a function of \( x \) and \( k \) that vanishes for the corresponding values of \( x \). As the integrability of \( V_0(x) \) is invariant with respect to \( x \to -x \), there are also two solutions \( \psi^-(\lambda, x) \) and \( \phi^-(\lambda, x) \) satisfying

\[
\psi^-(\lambda, x) = e^{-i\lambda x} (1 + o(1)), \quad \phi^-(\lambda, x) = e^{i\lambda x} (1 + o(1)) \quad \text{as} \quad x \to -\infty. \tag{69}
\]

The two sets \([68]\) and \([69]\) represent two possible choices of the fundamental solutions of the stationary equation, i.e. a function from one set can be written as a linear combination of the functions from the other set.

We fix \( \lambda_j = -ik_j, \ j = 1, \ldots, N, \) where \( k_1 > k_2 > \cdots > k_N > 0 \) and \( \lambda_{\ell} = r_\ell > 0, \ \ell = 1, \ldots, M. \) We denote the corresponding functions \([68]\) or \([69]\) as \( \psi_{k_j}^\pm(x) \equiv e^{\mp k_j x}, \) \( \phi_{k_j}^\pm(x) \equiv e^{\pm k_j x}, \) \( \psi_{r_\ell}^\pm(x) \equiv e^{\mp r_\ell x}, \) and \( \phi_{r_\ell}^\pm(x) \equiv e^{\pm r_\ell x}. \) We suppose that the derivatives of \( \psi_{k_j}^\pm \) satisfy

\[
(\psi_{k_j}^\pm)' = \pm k_j e^{\mp k_j x} (1 + o(1)), \quad (\phi_{k_j}^\pm)' = \mp k_j e^{\pm k_j x} (1 + o(1)), \quad (x \to \pm\infty) \tag{70}
\]

and \( (\psi_{r_\ell}^\pm)' \) and \( (\phi_{r_\ell}^\pm)' \) are bounded functions. These requirements can be met provided that the functions do not have asymptotically small but rapidly oscillating terms.

Let us compose the following functions

\[
u = \sum_{j=0}^N F_j e^{ik_j^2 z} + \sum_{\ell=0}^M G_{\ell} e^{-ir_\ell^2 z}, \quad v = \sum_{j=0}^N \tilde{F}_j e^{ik_j^2 z} + \sum_{\ell=0}^M \tilde{G}_{\ell} e^{-ir_\ell^2 z}, \tag{71}
\]
where $F_0 = G_0 = 0$, $\tilde{F}_0 = \tilde{G}_0 = 0$ and

$$
F_j = a_j^\pm \psi_{k_j}^\pm + b_j^\pm \phi_{k_j}^\pm, \quad G_\ell = c_\ell^\pm \psi_{r_\ell}^\pm + d_\ell^\pm \phi_{r_\ell}^\pm, \\
\tilde{F}_j = \tilde{a}_j^\pm \psi_{k_j}^\pm + \tilde{b}_j^\pm \phi_{k_j}^\pm, \quad \tilde{G}_\ell = \tilde{c}_\ell^\pm \psi_{r_\ell}^\pm + \tilde{d}_\ell^\pm \phi_{r_\ell}^\pm,
$$

(72)
The functions $G_\ell$ and $\tilde{G}_\ell$ are asymptotically vanishing and bounded. The functions $F_j$ and $\tilde{F}_j$ are exponentially expanding,

$$
F_j = a_j^\pm e^{\pm kj x} + o(e^{\pm kj x}), \quad (x \to \pm \infty), \\
\tilde{F}_j = \tilde{a}_j^\pm e^{\pm kj x} + o(e^{\pm kj x}), \quad (x \to \pm \infty), \\
(F_j)' = \pm kj a_j^\pm e^{\pm kj x} + o(e^{\pm kj x}), \quad (x \to \pm \infty), \\
(\tilde{F}_j)' = \pm kj \tilde{a}_j^\pm e^{\pm kj x} + o(e^{\pm kj x}), \quad (x \to \pm \infty).
$$

(73)

We define the Darboux transformation

$$
L = L_1(z) \left( \partial_x - \frac{u'}{u} \right).
$$

We suppose that $L_1(z)$ is bounded and non-vanishing for all real $z$. The function $u$ fixes the new potential $V_1 = V_0 + \delta V_1 + i\partial_x \ln L_1(z)$, where $\delta V_1 = -\frac{2z}{u} + \frac{2z\partial_x}{u}$ which is asymptotically vanishing for large $|x|$. Indeed, we have

$$
\delta V_1 = \frac{N}{u^2} \sum_{j,s} [(V_0 + k_j^2) F_j F_s - F_j F_s^*] e^{i(k_j^2 + k_s^2)z} + \frac{N,M}{u^2} \sum_{j,\ell} [(2V_0 - r_\ell^2 + k_j^2) F_j G_\ell - 2F_j G_\ell^*] e^{i(k_j^2 - r_\ell^2)z} \\
+ \frac{M}{u^2} \sum_{\ell,s} [(V_0 - r_\ell^2) G_\ell G_s - G_\ell G_s^*] e^{-i(r_\ell^2 + r_s^2)z}.
$$

(74)

As we have $u^2 = a_j^\pm e^{\pm 2kj x}(1 + o(1))$ for $x \to \pm \infty$, the last two terms vanish for large $|x|$. The first term will vanish as well provided that $(V_0 + k_j^2) F_j^2 - (F_j^*)^2 \to 0$ for $|x| \to \infty$. But taking into account (73) and integrability of $V_0$ that implies $V_0 = o(1)$ for $|x| \to \infty$, we can see that this is indeed the case. We get

$$
\delta V_1 = O(e^{\pm (-k_1 + k_2)x}) \quad \text{for} \quad x \to \pm \infty.
$$

(75)
The term $\delta V_1$ is regular provided that $u$ has no zeros. It is rather nontrivial to guarantee in general. We will discuss this point in the explicit examples in the end of this section.

Let us inspect the asymptotic behavior of the function $L v$. We shall fix the coefficients $\tilde{a}_j^\pm$ and $\tilde{b}_j^\pm$ such that $Lv$ is asymptotically vanishing for large $|x|$ or it is at least bounded. Denoting $W(f, g) = fg' - f'g$, we have

$$
L_1^{-1} L v = \frac{W(u, v)}{u} = \frac{\sum_{j,s=0}^N W(F_j, F_s) e^{i(k_j^2 + k_s^2)z}}{u} + \frac{N,M}{u} \sum_{j,\ell=0}^M \left( W(G_\ell, \tilde{F}_j) + W(F_j, \tilde{G}_\ell) \right) e^{i(k_j^2 - r_\ell^2)z} \\
+ \frac{M}{u} \sum_{\ell,s=0}^M W(G_\ell, \tilde{G}_s) e^{-i(r_\ell^2 + r_s^2)z}.
$$

(76)

In the numerator of the first sum, there are terms of order $e^{\alpha x}$ with $\alpha \geq k_1$. We would like to fix $\tilde{a}_j^\pm$ and $\tilde{b}_j^\pm$ such that these terms vanish. Using (73), we get

$$
W(F_j, \tilde{F}_s) + W(F_j, \tilde{F}_s) = e^{\pm (k_j + k_s)x} \left( \pm kj (a_j^\pm \tilde{a}_s^\pm - a_s^\pm \tilde{a}_j^\pm) \pm k_s (a_j^\pm \tilde{a}_j^\pm - a_j^\pm \tilde{a}_j^\pm) \right)
$$

(77)

We can make the first term proportional to $e^{\pm (k_j + k_s)x}$ vanishing by fixing

$$
\tilde{a}_j^\pm = e^c a_j^\pm, \quad \tilde{a}_j^\pm = e^c a_j^\pm.
$$

(78)
where $c^\pm$ are constants. However, the condition (78) cannot guarantee that $\mathcal{L} v$ will vanish exponentially; the condition (78) does not nullify the term $o(e^{\pm(k_1^\pm_k + k_\ell)} x)$ in (77) in general, so that it only forces the first term in (76) to behave as $o(e^{\pm 2k_1^\pm_k x})$ for $x \to \pm \infty$. However, (78) can serve well as the guiding relation in the construction of guided modes in the explicit examples discussed later on.

Let us focus on the second term in (76). It vanishes asymptotically provided that the function at the term $e^{\pm k_1^\pm_k x}$ is vanishing. Using (78), we get

$$F_1 \tilde{G}_\ell - F_1' \tilde{G}_\ell + G_\ell \tilde{F}_1' - G_\ell' \tilde{F}_1 = \left( a_1^\pm (\tilde{G}_\ell - e^{\pm k_1^\pm_k x}) \pm k_1 a_1^\pm (e^{\pm k_1^\pm_k x} - \tilde{G}_\ell) \right) e^{\pm k_1^\pm_k x} + o(e^{\pm k_1^\pm_k x}), \quad x \to \pm \infty. \quad (79)$$

The leading term above can be made zero provided that

$$\tilde{G}_\ell = e^{\pm k_1^\pm_k} G_\ell, \quad (80)$$

where $c^\pm$ are the two constants introduced in (78). When $c^+ \neq c^-$, the only way how to make the second term of (76) asymptotically vanishing for both $x \to \pm \infty$ is to make it identically zero by setting $G_\ell = \tilde{G}_\ell = 0$. Fixing either $\tilde{G}_\ell = e^{+ k_1^\pm_k} G_\ell$ or $\tilde{G}_\ell = e^{- k_1^\pm_k} G_\ell$ will make the second term vanishing either at $x \to +\infty$ or $x \to -\infty$. Hence, when $M > 0$, $\mathcal{L} v$ can vanish only on one side of the wave guide represented by $V_1$.

Both the new potential term $V_1$ and $\mathcal{L} v$ are periodic in $z$ provided that $k_1, \ldots, k_N$ and $r_1, \ldots, r_M$ are commensurable. For $M = 0$, $V_1$ offers a strong confinement of the guided mode as $\mathcal{L} v$ vanishes outside exponentially. When $M \neq 0$, the potential $V_1$ offers rather weak confinement as the guided mode “leaks” out of the wave guide, performing non-vanishing oscillations in $|x| \to \infty$.

Up to now, we did not make any assumption on the $PT$-symmetry of $V_1$. As we do not see how the function $u$ in (71) could satisfy (30), we look for $P_2 T$-symmetry of the new potential. It is sufficient to fix the function $u$ such that it has definite $P_2 T$-parity. It can be granted by taking the coefficients in (72) such that

$$P_2 T F_j = \epsilon F_j, \quad P_2 T G_\ell = \epsilon G_\ell, \quad \epsilon \in \{-1, 1\}. \quad (81)$$

These relations imply $a_j^- = \epsilon a_j^+$. The following examples will differ by the choice of the transformation function $u$; when it consists of exponentially expanding components only, the resulting systems will represent strongly confining wave guides as the guided mode will disappear exponentially out of the wave guide. For $u$ containing the bounded oscillating components, weakly localizing wave guides will be obtained. We will consider both $P_2 T$-symmetric models as well as non-$P_2 T$-symmetric ones. In all the cases, we will set $L_1 = 1$.

4.2 Examples: $P_2 T$-symmetric deformations of the Pöschl-Teller potential

Our initial system will be the free particle again, so that we fix $S_0$ as in (32). We can identify the fundamental solutions of the stationary Schrödinger equation with prescribed asymptotic behavior (68) and (69) as

$$\psi^+(\lambda, x) = \phi^-(\lambda, x) = e^{i\lambda x}, \quad \phi^+(\lambda, x) = \psi^-(\lambda, x) = e^{-i\lambda x}. \quad (82)$$

We shall construct $P_2 T$-symmetric model. Taking into account (75) and (81), the functions $F_j$ and $G_\ell$ with definite $P_2 T$-parity have to be fixed in the following form

$$F_{j, \epsilon} = a_j^+ e^{k_j^+ x} + \epsilon a_j^- e^{-k_j^- x} = \begin{cases} |a_j^+| \cosh(k_j^+ x + i\delta_j), & \epsilon = 1, \\ |a_j^-| \sinh(k_j^- x + i\delta_j), & \epsilon = -1, \end{cases}, \quad a_j^+ = |a_j^+| e^{i\delta_j}, \quad \delta_j \in \mathbb{R}, \quad (83)$$

$$G_{\ell, \epsilon} = e^{i\epsilon r_\ell x} e^{-\epsilon r_\ell x} = \begin{cases} |c_{\ell}^+| \cos(r_\ell x + i\mu_\ell), & \epsilon = 1, \\ |c_{\ell}^-| \sin(r_\ell x + i\mu_\ell), & \epsilon = -1, \end{cases}, \quad c_{\ell}^+ = |c_{\ell}^+| e^{i\mu_\ell}, \quad \mu_\ell \in \mathbb{R}, \quad (84)$$

that guarantees that $u_\epsilon = \sum_{j=0}^N F_{j, \epsilon} e^{ik_j^+ x} + \sum_{\ell=0}^M G_{\ell, \epsilon} e^{-i\epsilon r_\ell x}$ will satisfy

$$PT u_\epsilon = \epsilon u_\epsilon. \quad (85)$$

In what follows, we will discuss the systems related with the following fixed form of the function $u_\epsilon$,

$$u_\epsilon = F_{1, \epsilon} e^{i k_1^+ x} + F_{2, \epsilon} e^{i k_2^+ x} + G_{1, \epsilon} e^{-i r_1^+ x}, \quad (86)$$

As we argued below (80), the character of the guided modes given by $v$ depends on the presence of $G_{1, \epsilon}$ in $u$. When it is absent in (86), i.e. $G_{1, \epsilon} \equiv 0$, then we can construct wave functions that are exponentially vanishing for large $x$. Otherwise, $v$ is asymptotically non-vanishing, bounded and oscillating.
For \( x > 0 \), we have \( 0 < \cosh x < 1 \). Indeed, the equation \( \cosh x = 0 \) can be written as \( \tanh x = 0 \), which leads to \( x = 0 \). We can show that the function \( u \) has no zeros. Indeed, the equation \( u = 0 \) can be written as \( \frac{\sinh k_1 x e^{ik_2^2 z} + \alpha \sinh k_2 x e^{ik_2^2 z}}{\cosh k_1 x (1 + e^{ik_2^2 z})} = 0 \). Comparing with (87), we can see that \( e^+ = 1 \) while \( e^- = -1 \). We can show that the function \( u \) has no zeros. Indeed, the equation \( u = 0 \) can be written as \( \frac{\sinh k_1 x e^{ik_2^2 z} + \alpha \cosh k_2 x e^{ik_2^2 z}}{\cosh k_1 x (1 + e^{ik_2^2 z})} = 0 \). Comparing with (88), we can see that \( e^+ = 1 \) while \( e^- = -1 \). We can show that the function \( u \) has no zeros.

Figure 5: A strongly confining waveguide. Plots of the real (top left) and imaginary (top right) parts of \( V_1 \), for the parameters \( k_1 = 0.4, k_2 = 0.1, \alpha = 0.5 \) are presented. Furthermore, the corresponding power density of the guided mode \( \mathcal{L}v \), see (90), is shown (bottom left). The power \( P(\mathcal{L}v) = \int_{-\infty}^{\infty} |\mathcal{L}v|^2 dx \) can be seen in the last frame (bottom right), the power of the guided mode is oscillating.

4.2.1 \( G_1 = 0 \): Strongly confining wave guides

Let us discuss two choices of the function \( u \). First, we fix

\[
\begin{align*}
u &= \cosh k_1 x e^{ik_2^2 z} + \alpha \cosh k_2 x e^{ik_2^2 z}, \quad (87) \\
v &= \sinh k_1 x e^{ik_2^2 z} + \alpha \sinh k_2 x e^{ik_2^2 z}, \quad \text{where } |\alpha| < 1, \quad \alpha \in \mathbb{R}. \quad (88)
\end{align*}
\]

Comparing with (78), we can see that \( e^+ = 1 \) while \( e^- = -1 \). We can show that the function \( u \) has no zeros. Indeed, the equation \( u = 0 \) can be written as \( \frac{\cosh k_1 x e^{ik_2^2 z} + \alpha \cosh k_2 x e^{ik_2^2 z}}{\cosh k_1 x (1 + e^{ik_2^2 z})} = 0 \). Comparing with (87), we can see that \( e^+ = 1 \) while \( e^- = -1 \). We can show that the function \( u \) has no zeros. Indeed, the equation \( u = 0 \) can be written as \( \frac{\cosh k_1 x e^{ik_2^2 z} + \alpha \cosh k_2 x e^{ik_2^2 z}}{\cosh k_1 x (1 + e^{ik_2^2 z})} = 0 \). Comparing with (88), we can see that \( e^+ = 1 \) while \( e^- = -1 \). We can show that the function \( u \) has no zeros.

The potential \( V_1 \) and the guided mode \( \mathcal{L}v \) acquire the following forms

\[
V_1 = -2 k_1^2 + \frac{\alpha^2 k_2^2 e^{2i(k_2^2-k_1^2)z}}{k_1 x} = \frac{2 e^{2i(k_2^2-k_1^2)z} \alpha \cosh k_2 x ((k_2^2 + k_1^2) - 2k_1 k_2 \tanh k_1 x \tanh k_2 x)}{k_1 x (1 + e^{2i(k_2^2-k_1^2)z} \alpha \cosh k_1 x \cosh k_2 x)^2} \quad (89)
\]

and

\[
\mathcal{L}v = \frac{e^{2ik_1^2 z} k_1 + e^{2ik_2^2 z} k_2 \alpha^2 + e^{i(k_1^2+k_2^2)z} (k_1 + k_2) \alpha \cosh(k_1 - k_2) x}{e^{ik_2^2 z} \cosh k_1 x \left(1 + e^{i(k_2^2-k_1^2)z} \alpha \cosh k_1 x \right)} \quad (90)
\]

In Fig. 5 we present the real (top left) and imaginary (top right) parts of the potential as well as the power density \( |\mathcal{L}v|^2 \) (bottom left) of the guided mode, for the parameters \( k_1 = 0.4, k_2 = 0.1, \alpha = 0.5 \). The power \( P(\mathcal{L}v) = \int_{-\infty}^{\infty} |\mathcal{L}v|^2 dx \) can be seen in the last frame (bottom right), the power of the guided mode is oscillating.

Now, let us consider a different choice of the function \( u \) and of the preimage \( v \) of the guided mode,

\[
\begin{align*}
u &= \cosh k_1 x e^{ik_2^2 z} + \iota \sinh k_2 x e^{ik_2^2 z}, \quad (91) \\
v &= \sinh k_1 x e^{ik_2^2 z} + \iota \cosh k_2 x e^{ik_2^2 z}, \quad \text{where } |\alpha| < 1. \quad (92)
\end{align*}
\]

We can prove that \( u \) has no zeros. Indeed, the equation \( u = 0 \) leads to \( \frac{\sinh k_2 x}{\cosh k_1 x} = \iota^{-1} e^{i(k_1^2-k_2^2)z} \). We can show that the absolute value of the left-hand side is smaller than one: for \( x > 0 \), we have \( 0 < \frac{\sinh k_2 x}{\cosh k_1 x} < \frac{\sinh |k_2|}{\cosh |k_1|} = \tanh k_1 x < 1 \). For \( x \leq 0 \), we have \( 0 > \frac{\sinh k_2 x}{\cosh k_1 x} = \tanh k_1 x + \frac{\sinh k_2 x - \sinh k_1 x}{\cosh k_1 x} > \tanh k_1 x > -1 \). Hence, the potential \( V_1 \) is regular.
We fix $G_{2.2}$ $V$ the same manner, as the function (95) can be obtained from (91) by the substitution side is always greater or equal to one, whereas the absolute value of the right-hand side is smaller or equal to $|L_v|$. The function has no zeros. Writing the equation $\cosh kV$ the potential $\mathcal{L}_v$, see (94), is shown (bottom left). The power $P(\mathcal{L}_v) = \int_{-\infty}^{\infty} |\mathcal{L}_v|^2 dx$ can be seen in the last frame (bottom right), the power of the guided mode is oscillating.

and it can be written as

$$V_1 = -2 \frac{k_1^2 + \alpha^2 k_2^2 e^{2i(k_2^2 - k_1^2)z}}{\cosh^2 k_1 x \left( 1 + i e^{i(k_2^2 - k_1^2)z} \frac{\sinh k_2 x}{\cosh k_1 x} \right)} - 2 \frac{e^{i(k_2^2 - k_1^2)z} \alpha \cosh k_2 x \left( k_1^2 + k_2^2 \tanh k_2 x - 2 k_1 k_2 \tanh k_1 x \right)}{\cosh k_1 x \left( 1 + i e^{i(k_2^2 - k_1^2)z} \frac{\sinh k_2 x}{\cosh k_1 x} \right)^2}$$

(93)

whereas the guided mode $\mathcal{L}_v$ acquires the following form

$$\mathcal{L}_v = \frac{e^{i k_2^2 z} k_1 + e^{2i k_2^2 z} \alpha^2 k_2 - i e^{i(k_1^2 + k_2^2)z} (k_1 + k_2) \alpha \sinh k_1 - k_2 x}{e^{k_1^2 k_2^2} \sinh k_1 x \left( 1 + i e^{i(k_2^2 - k_1^2)z} \frac{\sinh k_2 x}{\cosh k_1 x} \right)}.$$  

(94)

The potential $V_1$ as well as the guided mode are illustrated in Fig. 6. The power of the guided mode $P(\mathcal{L}_v) = \int_{-\infty}^{\infty} |\mathcal{L}_v|^2 dx$ (bottom right) is also oscillating. Let us remark that both the potentials (89) and (93) reduce to the $z$-independent Pöschl-Teller potential for $\alpha = 0$.

4.2.2 $G_1 \neq 0$: weakly confining wave guides

We fix

$$u = \cosh k_1 x e^{i k_2^2 z} + i \alpha \sin r_1 x e^{-ir_2^2 z}, \quad \alpha \in (-1, 1).$$

(95)

The function has no zeros. Writing the equation $\cosh k_1 x = -i \alpha \sin r_1 x e^{-i(r_1^2 + r_2^2)z}$, we can see that the left hand side is always greater or equal to one, whereas the absolute value of the right-hand side is smaller or equal to $|\alpha|$. As the function (95) can be obtained from (91) by the substitution $k_2 \rightarrow i r_1$, the potential $V_1$ is related to (93) in the same manner,

$$V_1 = -2 \frac{k_1^2 + \alpha^2 r_1^2 e^{-2i(r_1^2 + r_2^2)z}}{\cosh^2 k_1 x \left( 1 + i e^{-i(r_1^2 + r_2^2)z} \frac{\sin r_1 x}{\cosh k_1 x} \right)} - 2 \frac{i e^{-i(r_1^2 + r_2^2)z} \alpha \cosh k_1 x \left( (r_1^2 + r_2^2) \sin r_1 x - 2 k_1 r_1 \cos r_1 x \tanh k_1 x \right)}{\cosh^2 k_1 x \left( 1 + i e^{-i(r_1^2 + r_2^2)z} \frac{\sin r_1 x}{\cosh k_1 x} \right)^2}.$$  

(96)

Let us consider the function $v_1$ and $v_2$,

$$v_1 = \sinh k_1 x e^{i k_2^2 z} - i \alpha \sin r_1 x e^{-ir_2^2 z}, \quad v_2 = \sinh k_1 x e^{i k_2^2 z} - \alpha \sin r_1 x e^{-ir_2^2 z}.$$  

(97)
Figure 7: Weakly confining wave guides. Plots of the real (top left) and imaginary (top right) parts of $V_1$ when $k_1 = 0.4$, $r_x = 0.5$, $\alpha = 0.2$, see (96). The intensity densities of $\mathcal{L}v_1$ and $\mathcal{L}v_2$ are shown as well (bottom left and right, respectively).

The state $v_1$ fulfills the condition (80) so that $\mathcal{L}v_1$ vanishes exponentially on one side of the potential barrier $V_1$. Yet it breaks manifestly $\mathcal{P}_2\mathcal{T}$ symmetry (it can be written as a linear combination of two $\mathcal{P}_2\mathcal{T}$-symmetric solutions). The solution $v_2$ has definite $\mathcal{P}_2\mathcal{T}$ parity but does not comply with (80). Therefore, $\mathcal{L}v_2$ has non-vanishing oscillations on both sides of the barrier, see Fig. 7 for illustration.

4.3 Non-$\mathcal{P}\mathcal{T}$-symmetric systems

The results of section 4.1 are valid for large class of potentials, including those where the parity-time symmetry is manifestly broken. These systems, where guided modes still exist despite the lack of the symmetry, can be constructed in the same vain as the $\mathcal{P}_2\mathcal{T}$-symmetric ones. Let us present briefly a simple example where the transformation function $u$ and the preimage $v$ of the guided mode are fixed as

$$u = \cosh k_1xe^{ik_2^1z} + \alpha \sinh (k_2x + \delta) e^{ik_2^2z}, \quad v = \sinh k_1xe^{ik_2^1z} + \alpha \cosh (k_2x + \delta) e^{ik_2^2z}, \quad \delta \in \mathbb{R}. \quad (98)$$

We can see that when $\alpha \not\in \mathbb{R}$ and $\delta \neq 0$, the function $u$ does not comply with (31) and, hence, the resulting potential $V_1$ ceases to be $\mathcal{P}_2\mathcal{T}$-symmetric. Analysis of the range of parameters where $u$ is nodeless can be performed similarly to preceding cases and we will not present it here explicitly. The new potential $V_1$ reads

$$V_1 = -2 \frac{k_1^2 - e^{2i(k_2^2 - k_1^2)z} k_2^2 \alpha^2}{\cosh^2 k_1 x \left(1 + e^{i(k_2^2 - k_1^2)z} \alpha \sinh (k_2x + \delta) / \cosh k_1 x \right)^2} - 2 \alpha e^{i(k_2^2 - k_1^2)z} \cosh k_1 x \cosh (k_2x + \delta) \left((k_1^2 + k_2^2) \tanh (k_2x + \delta) - 2k_1k_2 \tanh k_1 x \right) \cosh^2 k_1 x \left(1 + e^{i(k_2^2 - k_1^2)z} \alpha \sinh (k_2x + \delta) / \cosh k_1 x \right)^2. \quad (99)$$

The guided mode is obtained in the following form

$$\mathcal{L}v = \frac{e^{2ik_2^1z}k_1 - e^{ik_2^1z}k_2 \alpha^2 - e^{i(k_2^2 + k_1^2)z}(k_1 + k_2) \alpha \sinh ((k_1 - k_2)x - \delta)}{e^{ik_2^1z} \cosh k_1 x \left(1 + e^{i(k_2^2 - k_1^2)z} \alpha \sinh (k_2x + \delta) / \cosh k_1 x \right)}. \quad (100)$$

Both the potential $V_1$ and the guided mode are plotted in Fig. 8.
In this subsection we will utilize the second-order Crum-Darboux transformation and the point transformation to get the required system with a localized and bounded potential term that depends both on $x$ and $z$.

When $V_0$ is $z$-independent, we can find the solutions $S_0u = 0$ in terms of the stationary states $u(x, z) = e^{i\varphi}(x)$, $(-\partial_z^2 + V_0 - \epsilon)\psi_\epsilon = 0$. When $u_1$ and $u_2$ are the stationary states of $S_0$, then both, $V_0$ and $V_1$, are $z$-independent and the transformation $L_{12}$ can be identified with the $N = 2$ (time-independent) Crum-Darboux transformation transformation.

It can happen that $S_1$ ceases to have regular potential, however, $S_2$ has regular potential. It stems from the fact that despite $u_1$ can have zeros (that introduce singularities into $S_1$), the Wronskian of $u_1$ and $u_2$ can be nodeless, keeping $S_2$ regular. We will discuss such a case in the following text.

### 5.1 Localized defects in a homogeneous crystal

In this subsection we will utilize the second-order Crum-Darboux transformation and the point transformation to derive a exactly solvable model of localized $\mathcal{PT}$-symmetric defect of the refractive index. First, we will use the stationary version of \[14\] to get a deformed harmonic oscillator with $z$-independent potential and then we apply the point transformation to get the required system with a localized and bounded potential term that depends both on $x$ and $z$.

For the purpose of this subsection, we take $S_0$ as the Schrödinger operator of harmonic oscillator, $S_0 = i\partial_z + \partial_y^2 - y^2/4$. We also fix $L_1 = L_2 = 1$. Let us select $u_1$ as the stationary solution $S_0u_1 = 0$,

\[ u_1(y, z) = \tilde{\psi}_m(y)e^{-iE_m z}, \quad \text{where} \quad \tilde{\psi}_m(y) = H_m \left( \frac{y}{\sqrt{2}} \right) e^{-\frac{y^2}{4}} \],

\[ E_m = m + \frac{1}{2}. \]
where $H_m(y)$ is Hermite polynomial. As the second function $u_2$, we take

$$u_2(y, z) = e^{-iE_m z} \psi_m \left(\int_{y_0}^{y} \frac{1}{\psi_m} \left(\int_{r_0}^{r} \psi_m^2(r) dr + \alpha\right) + a\right).$$

where $a$ is a constant. It satisfies $S_1 L_1 u_2 = 0$ but $S_0 u_2 \neq 0$. Instead it fulfills $S_0^2 u_2 = 0$, see \[31\]. The operator $L_{12}$ is called the confluent Crum-Darboux transformation in the literature, see e.g. \[17, 31, 65–68\] and references therein. The new Schrödinger operator can be written in terms of $\tilde{\psi}_m$ as

$$S_2 = i \partial_z + \partial_y^2 - V_2(y), \quad V_2(y) = \frac{y^2}{4} - 4 \alpha \psi_m \partial_y \psi_m + \frac{1}{2} \frac{\tilde{\psi}_m^4}{(\alpha + \int_0^y \psi_m^2(s) ds)^2}.$$  \hspace{1cm} \text{(105)}

As $\tilde{\psi}_m$ is a real function, the new potential will be free of singularities provided that $\alpha$ is a complex constant with a non-vanishing imaginary part. The stationary states $f_n$ of $S_2$ for $n \neq m$ can be found by direct application of $L_{12}$,

$$f_n(y) = L_{12} \tilde{\psi}_n(y) = \left(\partial_{y} - \frac{\partial_{y} u_2}{\tilde{u}_2}\right) \left(\tilde{\psi}_n(y), \tilde{u}_2 = \partial_{y} \left(u_1 \partial_{y} u_2 / u_1\right)\right).$$ \hspace{1cm} \text{(106)}

For $n = m$, we can find the following square integrable (in $y$-coordinate) solution (see e.g. \[31\])

$$f_m(y) = \frac{\tilde{\psi}_m}{\alpha + \int_0^y \psi_m^2(s) ds}.$$ \hspace{1cm} \text{(107)}

Now, we will perform the point transformation \[39, 10, 11, 42\] of $S_2$ that will generate $z$-dependent potential term. We fix $A = -z/4(1 + z^2)$ and $B = 0$. The new potential can then be written as

$$V_2(x, z) = -2\partial_x^2 \ln \left[\alpha + \int_0^{y = \sqrt{1 + z^2}} \phi_m^2(s) ds\right],$$ \hspace{1cm} \text{(108)}

and solutions of the corresponding time dependent Schrödinger equation from \[106\] and \[107\] as

$$\phi_n(x, z) = \frac{1}{(1 + z^2)^{1/4}} \exp \left\{ i \frac{z}{4} \left[\frac{1}{1 + z^2} \right] \right\} f_n \left(\frac{x}{\sqrt{1 + z^2}}\right).$$ \hspace{1cm} \text{(109)}

These functions satisfy $P_x \mathcal{T} \phi_n(x, z) = (-1)^n \phi_n(x, z)$. We can see that for large $|x|$ and fixed $z$, the functions vanish like an exponential multiplied by a polynomial. Along the curves $x = c\sqrt{1 + z^2}$ where the argument of $f_n$ is constant, the wave functions behave as $|\phi_n(c\sqrt{1 + z^2})| \sim 0 \left(1 + (1 + z^2)^{-1/4}\right)$, see Fig. 3.

For explicit illustration, let us fix $m = 0$. It means that the ground state of the harmonic oscillator has been selected as the new potential function to perform the confluent SUSY transformation and $E_m \equiv E_0 = 1/2$. Then the potential term \[108\] acquires the following explicit form

$$V_2(x, z) = \frac{2}{z^2 + 1} \left\{ \frac{xe^{-z^2/(2(z^2 + 1))}}{\sqrt{2\pi(z^2 + 1)}} \left[\alpha + \frac{1}{2} \text{erf} \left(\frac{x}{\sqrt{2(z^2 + 1)}}\right)\right] + \frac{e^{-z^2/(2(z^2 + 1))}}{2\pi \alpha + \frac{1}{2} \text{erf} \left(\frac{x}{\sqrt{2(z^2 + 1)}}\right)^2} \right\},$$ \hspace{1cm} \text{(110)}

Choosing $\alpha$ as a pure imaginary parameter and since $\text{erf}(-x) = -\text{erf}(x)$, one can check that the new potential $V_2$ is invariant with respect to both $P_x \mathcal{T}$ and $P_x \mathcal{T}$. The potential $V_2$ represents a well localized defect of a uniform refractive index. Indeed, the potential behaves as $V_2(x, z) \sim O \left(x e^{-z^2/(2(z^2 + 1))}\right)$ for large $|x|$ and fixed $z$, whereas for fixed $x$ and large $z$, it behaves as $V_2(x, z) = O \left(e^{-z^2/(2(z^2 + 1))}\right)$.

In Fig. 3 plots of the real (top left) and imaginary (top center) parts of $V_2$ for $\alpha = i$ are shown. In the same Fig. the absolute value squared of three solutions are plotted: $|\phi_0|^2$ (top right), $|\phi_1|^2$ (bottom left) and $|\phi_2|^2$ (bottom right), see \[109\].

It follows directly from the formulas \[109\] that the power of the light beam defined as $P(\phi_n) = \int_0^{\infty} |\phi_n(x, z)|^2 dx$ is constant, i.e. it does not depend on $z$. However, different superpositions of states $\phi_n$ would not have a constant power. Using the same example ($m = 0, \alpha = i$), consider the following four superpositions:

$$\phi_a = \frac{1}{\sqrt{2}} (N_0 \phi_0 + iN_1 \phi_1), \quad \phi_b = \phi_a, \quad \phi_c = \frac{1}{\sqrt{2}} (N_0 \phi_0 + N_1 \phi_1), \quad \phi_d = \frac{1}{\sqrt{2}} (N_0 \phi_0 - N_1 \phi_1)$$ \hspace{1cm} \text{(111)}
Figure 9: Localized defect in a homogenous crystal. Plots of the real (top left) and imaginary (top center) parts of $V_2$ when $\alpha = i$, see (110). Moreover, the absolute value squared of three solutions are plotted: $|\phi_0|^2$ (top right), $|\phi_1|^2$ (bottom left) and $|\phi_2|^2$ (bottom right), see (109).

where $N_0$, $N_1$ are (real) normalization constants. For such states, the power is a nontrivial function of $z$. In Fig. 10, the power of these states is plotted. First in blue the power of the first superposition $P(\phi_a)$ was plotted. It can be seen that power decreases near $z = 0$, the interaction of the light with the defect results in its absorption. In purple, we have $P(\phi_b)$ representing the opposite case, power increases after the interaction zone. In yellow, $P(\phi_c)$ has a minimum around zero whereas $P(\phi_d)$ in green has a maximum. In the same Fig. the functions $|\phi_0|^2$ (top center), $|\phi_1|^2$ (top right), $|\phi_2|^2$ (bottom right), $|\phi_3|^2$ (bottom left) were plotted.

5.2 Coupled wave guides

We fix

$$u_1 = \cosh k_1 xe^{ik_2^2 z} + i\alpha \sinh k_3 xe^{ik_3^2 z}, \quad u_2 = \sinh k_2 xe^{ik_2^2 z}$$

(112)

where we suppose that the constants $k_1$, $k_2$, $k_3$ and $\alpha$ are all real. The explicit form of the new potential term $V_2 = -2\partial_x^2 \ln W(u_1, u_2)$ is not quite compact, so that we refer to (103) from which it can be obtained directly when substituting (112). For $\alpha = 0$, both $u_1$ and $u_2$ correspond to the stationary states of the free particle Hamiltonian and the Darboux transformation $L_{12}$ renders $z$-independent potential

$$V_2|_{\alpha=0} = \frac{(k_1^2 - k_2^2)(k_2^2 - k_1^2 + k_1^2 \cosh 2k_2 x + k_2^2 \cosh 2k_1 x)}{(k_2 \cosh k_1 x \cosh k_2 x - k_1 \sinh k_1 x \sinh k_2 x)^2}.$$  

(113)

It corresponds to two parallel wave guides that were discussed in [69].

We can find two guided modes $\tilde{v}_1$ and $\tilde{v}_2$ of $S_0$. They can be obtained as

$$\tilde{v}_a = L_{12} v_a, \quad a = 1, 2$$

(114)

where the functions $v_1$ and $v_2$ are fixed as

$$v_1 = \sinh k_1 xe^{ik_2^2 z} + i\alpha \cosh k_3 xe^{ik_3^2 z}, \quad v_2 = \cosh k_2 xe^{ik_2^2 z}.$$  

(115)
As one can see directly from their explicit form,

\[
\bar{v}_1 = e^{ik_2 z} \frac{e^{i(k_1^2 + k_2^2)z}k_1(k_1^2 - k_2^2) + \alpha^2 e^{i(k_1^2 + k_2^2)}k_3(k_3^2 - k_2^2) \sinh k_2 x}{W(u_1, u_2)} \sinh k_2 x - i\alpha e^{i(k_1^2 + k_2^2)z}k_2(k_1^2 - k_2^2) \left(\cosh k_2 x \cosh(k_1 - k_3)x + \frac{k_1k_3 - k_2^2}{k_2(k_1 - k_3)} \sinh k_2 x \sinh(k_1 - k_3)x\right), 
\]

(116)

\[
\bar{v}_2 = e^{ik_2 z} k_2 e^{i(k_1^2 + k_2^2)z} \cos k_1 x + i\alpha e^{i(k_1^2 + k_2^2)z} \sinh k_3 x, 
\]

(117)

they are decaying exponentially for large \(|x|\).

It is worth noticing that if we took \(u_2\) as the only transformation function, the potential \(V_1 = -2\partial_x^2 \ln u_2\) would be singular as \(u_2\) vanishes at the origin. Hence, the current model is an example where the intermediate potential \(V_1\) can be singular, however, the final one \(V_2\) is regular.

Let us discuss regularity of the new system. The Wronskian \(W(u_1, u_2)\) can be written as

\[
W(u_1, u_2) = e^{i(k_1^2 + k_2^2)z} \left(k_1 \sinh k_1 x \sinh k_2 x - k_2 \cosh k_1 x \cosh k_2 x\right)
\]

(118)

\[
- i\alpha e^{i(k_1^2 + k_2^2)z} \left(k_3 \cosh k_3 x \sinh k_2 x - k_2 \cosh k_2 x \sinh k_3 x\right). 
\]

(119)

It should be free of zeros for all real \(x\) and \(z\) in order to have \(V_2\) regular. It is convenient to consider its real and imaginary part separately. For the later one, we have

\[
\text{Re} \left( \frac{W(u_1, u_2)}{e^{i(k_1^2 + k_2^2)z}} \right) = -k_2 \cosh k_1 x \cosh k_2 x \left( 1 - \frac{k_1}{k_2} \tanh k_1 x \tanh k_2 x \right) - \alpha k_2 \sin((k_3^2 - k_2^2)z) \cosh k_2 x \cosh k_3 x \left( \frac{k_3}{k_2} \tanh k_2 x - \tanh k_3 x \right), 
\]

(120)

\[
\text{Im} \left( \frac{W(u_1, u_2)}{e^{i(k_1^2 + k_2^2)z}} \right) = \alpha k_2 \cos((k_3^2 - k_2^2)z) \cosh k_2 x \cosh k_3 x \left( \frac{k_3}{k_2} \tanh k_2 x - \tanh k_3 x \right). 
\]

(121)
First, let us focus on the imaginary part \( \text{Im}(W(u_1, u_2)) \). One can show\(^3\) that the term in brackets is a monotonic function which is increasing for \( |k_2| > |k_3| \), decreasing for \( |k_3| > |k_2| \) and it has a single zero at \( x = 0 \). Therefore, \( \text{Im}(W(u_1, u_2)) = 0 \) for \( x = 0 \) and \( z = \frac{(n+1/2)\pi}{k_2^3-k_1^3} \) where \( n \) is an integer.

Considering \( \text{Re}(W(u_1, u_2)) \), we can see that it is nonvanishing for \( x = 0 \) for \( k_2 \neq 0 \). For \( z = \frac{(n+1/2)\pi}{k_2^3-k_1^3} \), the zeros of \( \text{Re}(W(u_1, u_2)) \) coincide with the zeros of

\[
\frac{\cosh k_3 x}{\cosh k_3 x} \left( 1 - \frac{k_1}{k_2} \tanh k_1 x \tanh k_2 x \right) + \alpha \left( \tanh k_3 x - \frac{k_3}{k_2} \tanh k_2 x \right).
\]

Let us suppose that \( |k_1| < |k_2| \). Then the first term is positive. We also take \( |k_1| > |k_3| \). Then \( \frac{\cosh k_3 x}{\cosh k_3 x} > 1 \) and we can see that the first term is bounded from below by \( 1 - \frac{|k_1|}{|k_2|} \) and from above by \( \alpha \left( 1 + \frac{|k_1|}{|k_2|} \right) \). So that it is granted that the term \( 123 \) is positive when \( \left( 1 - \frac{|k_1|}{|k_2|} \right) > |\alpha| \left( 1 + \frac{|k_1|}{|k_2|} \right) \). However, this estimate is very rough and the term remains nodeless (and \( V_2 \) regular) for larger range of \( \alpha \). In the Fig. 11 we present plots of \( V_2 \), its real (top left) and imaginary (top right) parts and also the intensity densities of the guided modes: \( |\hat{v}_1(x,z)|^2 = |L_{12}v_1|^2 \) (bottom left) and \( |\hat{v}_2(x,z)|^2 = |L_{12}v_2|^2 \) (bottom right), for the parameters \( k_1 = 1, k_2 = 1.09, k_3 = 0.95, \) and \( \alpha = 0.5 \).

6 Summary

The aim of the current article was to construct exactly solvable models of optical setting with complex refractive index, where propagation of light in paraxial approximation is governed by a two-dimensional, non-separable Schrödinger equation. We utilized the time-dependent Darboux transformation discussed in \([43]\). It allowed us to construct systems with localized defects of refractive index that can accommodate localized solutions, called by us light dots, or systems with periodically structured wave guides with exponentially localized guided modes.

In order to get the models with parity-time symmetry, we considered two different definitions of parity operator; reflection with respect to the axis \( x \) denoted as \( P_x \) and reflection with respect to the origin \( P_2 \), see \([27]\). Actual

\(^3\)We have \( \partial_z \left( \frac{1}{k_2^2} \tanh k_2 x - \tanh k_3 x \right) = k_3 (\text{sech}^2 k_3 x - \text{sech}^2 k_2 x) \). The monotonicity follows from \( \text{sech}^2 x_1 > \text{sech}^2 x_2 \) whenever \( |x_1| < |x_2| \).
choice of the parity operator determined the whole construction to a large extend. The transformation function $u$, the solution of the initial Schrödinger equation in terms of which both the Darboux transformation and the new potential (14) were defined, had to comply with either (30) or (31), dependently on the definition of the parity operator. We showed that existence of the missing state (20), which can represent a localized state in the new systems, can be guaranteed provided that the transformation function satisfies (22) where the operator $S$ is identified with $P_x T$.

In section 3, we focused on the construction of systems that can possess localized solutions, light dots. In order to get a transformation function $u$ of required properties, we utilized a mapping between Schrödinger equations of the harmonic oscillator and of the free particle. The wave packets (44), (45) obtained this way served as the basis for construction of solvable models. We presented a solvable model of a wave guide with a localized defect (52), or curved wave guides (64). We found the light dot solutions for these systems, see Fig. 4 for illustration.

In section 4, we proposed the construction of wave guides that are exponentially vanishing along $x$-axis and periodic along $z$-axis. Construction of the guided modes was provided. We showed that dependently on the choice of $u$, the wave guides differ by the strength of confinement. In the strong wave guides, the guided mode is vanishing exponentially in the perpendicular direction to the wave guide. In weak wave guides, the guided modes leak from the wave guide and exhibit non-vanishing oscillations in transverse direction. We illustrated the general results on explicit examples of optical wave guides with two-dimensional fluctuations of refractive index, distinguished by different choices of the transformation function. Strong wave guides were generated in (89) and (93), a weak wave guide was presented in (96). It is worth mentioning that these solvable models are two-dimensional $P_2 T$-symmetric generalizations of the reflectionless Pöschl-Teller potential. Indeed, setting $\alpha = 0$ in (89), (93), (96), the expressions reduce to $V_1 = -2k_1 \text{sech}^2 k_1 x$. The presented construction of wave guides and guided modes is applicable to a large class of initial systems with an integrable potential. It is not restricted to parity-time-symmetric operator, so that it can be utilized for construction of systems where $PT$ symmetry is manifestly broken. We exemplified construction of such setting in the end of Sec. 4, see Fig. 8.

In the Section 5, we used second order Darboux transformation to generate localized defect of a uniform refractive index (110) that can support light dots, see Fig. 9. We also constructed a system with two coupled wave guides (113) and calculated two associated guided modes.

In our constructions, the parity operator $P_2$, which corresponds to the reflection with respect to origin, proved to be rather universal as all the presented parity-time symmetric systems possessed $P_2 T$-symmetry. Only two of them, namely (52) and (110) possessed both $P_2 T$- and $P_2 T$-symmetry.

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