HOMOLOGICALLY THIN, NON-QUASI-ALTERNING LINKS

JOSHUA GREENE

ABSTRACT. We exhibit the first examples of links which are homologically thin but not quasi-alternating. To show that they are not quasi-alternating, we argue that none of their branched double-covers bounds a negative definite 4-manifold with non-torsion $H_1$. Using this method, we also complete the determination of the quasi-alternating pretzel links.

1. Introduction.

Quasi-alternating links were defined by Ozsváth and Szabó [28, Definition 3.9]. They are a natural generalization of the class of alternating links.

Definition 1.1. The set $Q$ of quasi-alternating (QA) links is the smallest set of links such that

- the unknot $U$ belongs to $Q$, and
- if $L$ is a link with a projection containing a crossing for which the two resolutions $L_0$ and $L_1$ belong to $Q$, and $\det(L) = \det(L_0) + \det(L_1)$, then $L$ belongs to $Q$.

By this definition, non-split alternating links belong to $Q$. Many familiar properties of alternating links hold for a QA link $L$:

1. the branched double-cover $\Sigma(L)$ is an L-space [28 Proposition 3.3];
2. the space $\Sigma(L)$ bounds a negative definite 4-manifold $W$ with $H_1(W) = 0$ [28 Proof of Lemma 3.6];
3. the $\mathbb{Z}/2\mathbb{Z}$ knot Floer homology group $\widehat{HF}_K(L; \mathbb{Z}/2\mathbb{Z})$ is thin [18 Theorem 2];
4. the reduced ordinary Khovanov homology group $\overline{Kh}(L)$ is thin [18 Theorem 1]; and
5. the reduced odd Khovanov homology group $\overline{Kh}'(L)$ is thin [23 Remark after Proposition 5.2].

It is an interesting open problem to characterize those links that are homologically thin with respect to any of the above knot homology theories. For some time, it remained a possibility that a link was $\widehat{HF}_K$- or $\overline{Kh}$-thin if and only if it was QA. This possibility was recently refuted by Shumakovitch, who used his excellent program KhoHo [30] to show that the pretzel knots $9_{46} = P(3, -3, 3)$ and $10_{140} = P(3, -3, 4)$ have torsion in their odd Khovanov

Partially supported by an NSF Graduate Fellowship.
homology groups, although they are both $\widehat{HFK}$- and $Kh$-thin. Thus, neither of these knots is odd-thin, so neither is QA$^1$.

**Definition 1.2.** A link $L$ is homologically thin (without qualification) if it is simultaneously thin with respect to $\widehat{HFK}$, $Kh$, and $Kh'$.

It has remained a challenge to exhibit a link that is homologically thin and not QA. The purpose of this note is to describe such examples, and moreover to exploit the condition (2) as an obstruction to QA-ness.

**Theorem 1.3.** There exist homologically thin, non-QA links.

At the heart of our method is Donaldson’s celebrated “Theorem A”, which asserts that the intersection pairing of a smooth, closed, negative definite 4-manifold is diagonalizable [4]. Coupled with calculations by several researchers [1, 3, 10, 17, 30, 33], we identify $11n50$ as the only knot with up to 11 crossings which is neither QA nor odd-thick. Furthermore, combined with work of Champanerkar-Kofman [3, Theorem 3.2], we complete the determination of the QA pretzel links. For a clear, concise account of the relevant notation and facts concerning Montesinos links here and in what follows, see [21, Section 3.2].

**Theorem 1.4.** The pretzel link $P(-e; p_1, \ldots, p_n, -q_1, \ldots, -q_m) = M(-e; (p_1, 1), \ldots, (p_n, 1), (q_1, -1), \ldots, (q_m, -1))$, with $e, n, m \geq 0$, all $p_i \geq 2$, and all $q_j \geq 3$, is QA iff

1. $e > m - 1$;
2. $e = m - 1 > 0$;
3. $e = 0, n = 1$, and $p_1 > \min\{q_1, \ldots, q_m\}$ or $m \leq 1$; or
4. $e = 0, m = 1$, and $q_1 > \min\{p_1, \ldots, p_n\}$ or $n \leq 1$.

The same is true on permuting the parameters $p_i$ and $q_j$.

Any pretzel link can be put in the above form after mirroring [12, Section 2.3], which clearly preserves the QA property. Section 2 contains the proofs of Theorems 1.3 and 1.4. Section 3 contains some related examples, as well as some discussion surrounding Conjecture 3.1, which asserts that there are finitely many QA links of bounded determinant.

$^1$In fact, in all known examples, an odd-thin link is $Kh$-thin, and a link is $Kh$-thin iff it is $\widehat{HFK}$-thin. A conjecture of Rasmussen would imply that a $Kh$-thin link is necessarily $\widehat{HFK}$-thin [29, Section 5].
Acknowledgments.

Thanks to John Baldwin, Abhijit Champanerkar, Michael Eisermann, Slavik Jablan, and Liam Watson for helpful correspondence.

2. Proofs of the main results.

To prove the theorems, we rely on the following lemma. We use (co)homology groups with integer coefficients throughout.

**Lemma 2.1.** Suppose that $X$ and $W$ are a pair of 4-manifolds, $\partial X = -\partial W = Y$ is a rational homology sphere, and $H_1(W)$ is torsion-free. Express the map $H_2(X)/\text{Tors} \to H_2(X \cup W)/\text{Tors}$ with respect to a pair of bases by the matrix $A$. This map is an inclusion, and if some $k$ rows of $A$ contain all the non-zero entries of some $k$ of its columns, then the induced $k \times k$ minor has determinant $\pm 1$.

**Proof.** The stated assumption ensures that the restriction map $H^2(X \cup W) \to H^2(X)$ surjects. Just to be sure, consider the long exact sequences in cohomology for the pairs $(X \cup W, X)$ and $(W, Y)$, and the natural map between them. The relevant portion reads

\[
\begin{array}{ccc}
H^2(X \cup W) & \longrightarrow & H^2(X) \\
\downarrow & & \downarrow \\
H^2(Y) & \longrightarrow & H^3(W, Y).
\end{array}
\]

The second vertical arrow is an isomorphism by excision, and Poincaré-Lefschetz duality identifies this group with $H_1(W)$, which is torsion-free. Since $H^2(Y)$ is torsion, the bottom horizontal map is 0. It follows that the map $H^2(X) \to H^3(X \cup W, X)$ is 0, so the map $H^2(X \cup W) \to H^2(X)$ surjects as claimed.

Consequently, the map $H^2(X \cup W)/\text{Tors} \to H^2(X)/\text{Tors}$ surjects as well. On the other hand, this map of groups is dual to the map $H_2(X)/\text{Tors} \to H_2(X \cup W)/\text{Tors}$, so is given with respect to the pair of dual bases by the matrix $A^T$. Suppose that some $k$ rows of $A$ contain all the non-zero entries of some $k$ of its columns, and let $B$ denote the corresponding $k \times k$ minor. By permuting the basis elements if necessary, we may assume that $B$ is the top-left $k \times k$ minor, possibly changing its determinant by a sign:

\[
A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.
\]

Since the dual map $A^T$ surjects, the map $B^T$ must as well, hence $\det(B) = \pm 1$, as claimed. The fact that the map $A$ injects follows, for example, from the Mayer-Vietoris sequence for the natural decomposition of $X \cup W$, noting that $H_2(Y)$ vanishes.

\[\square\]

**Proof of Theorem 1.3.** We establish the result by showing that $K = 11n50$ is homologically thin but not QA. Additional examples appear in Subsection 3.2. The knot Floer homology
group \( \hat{HF}_K(K; \mathbb{Z}/2\mathbb{Z}) \) was calculated by Baldwin-Gillam \[2\], and the Khovanov homology groups \( \hat{Kh}(K) \) and \( \hat{Kh}_2(K) \) by Shumakovitch \[30\]; their results show that \( K \) is homologically thin.

Figure \[1\] exhibits the knot \( 11n50 \) as the Montesinos knot \( M(0; (5, 2), (3, 1), (5, -2)) \), which is equivalent to \( M(1; (5, 2), (3, 1), (5, 3)) \). As such, its branched double-cover \( Y \) is the boundary of the plumbing \( X \) on the graph \( \Gamma \) shown in Figure \[2\]. We label the vertices of \( \Gamma \) from left to right and top to bottom by \( v_1, \ldots, v_7 \). The spheres corresponding to these vertices give rise to a basis for \( H_2(X) \), with respect to which the intersection pairing on \( X \) is given by the weighted adjacency matrix \( A_\Gamma \), whose \((i, j)\)-entry records the weight of \( v_i \), and whose \((i, j)\)-entry for \( i \neq j \) is 1 or 0 according as \( v_i \) and \( v_j \) are adjacent or not. The space \( X \) is negative definite and \( H_1(X) = 0 \).

If \( 11n50 \) were QA, then its mirror \( \overline{11n50} \) would be as well, whose branched double-cover is \(-Y\). We proceed by way of fact (2) to derive a contradiction. According to it, there must exist a negative definite 4-manifold \( W \) with \( \partial W = -Y \) and \( H_1(W) = 0 \). Consider the 4-manifold \( X \cup W \). It is a closed, smooth, negative definite 4-manifold, so by Donaldson’s theorem, its intersection pairing is diagonalizable. That is, there exists an isomorphism \( (H_2(X \cup W)/\text{Tors}, Q_{X \cup W}) \cong -\mathbb{Z}^n \), where \( n = b_2(X \cup W) \) and \(-\mathbb{Z}^n = \langle E_1, \ldots, E_n \rangle \) denotes the space \( \mathbb{Z}^n \) equipped with minus its standard positive definite inner product.

Therefore, \( (H_2(X)/\text{Tors}, Q_X) \) embeds into \(-\mathbb{Z}^n\), for some \( n \). Let \( x_i \) denote the image of the class in \( H_2(X)/\text{Tors} \) under the inclusion into \(-\mathbb{Z}^n\) which corresponds to the vertex \( v_i \). No pair of vertices of weight -2 has the same set of neighbors, so the corresponding vectors in \(-\mathbb{Z}^n\) have distinct reductions (mod 2). This observation helps us to deduce that, by applying an automorphism to \(-\mathbb{Z}^n\), we have \( x_7 = E_1 - E_2, x_6 = E_2 - E_3, x_3 = E_3 - E_4, x_4 = E_4 - E_5 \), and \( x_1 = E_6 - E_7 \). Now, swapping \( E_6 \) and \(-E_7\) if need be, we obtain \( x_2 = E_4 + E_5 - E_6 \), and then \( x_5 = E_5 + E_6 + E_7 \). Thus, with respect to the chosen bases for \( H_2(X)/\text{Tors} \) and \( H_2(X \cup W)/\text{Tors} \), the inclusion map is given by a matrix \( A \) whose seven columns are supported in its first seven rows. Let \( B \) denote the induced \( 7 \times 7 \) minor. Then \(-B^T B = -A^T = A_\Gamma\), and this is a presentation matrix for \( H^2(Y) \cong \mathbb{Z}/25\mathbb{Z} \). Hence \( |\text{det}(B)| = 5 \neq 1 \), in contradiction to Lemma \[2.1\].

It follows that \(-Y\) does not bound a negative definite 4-manifold with torsion-free, let alone vanishing, \( H_1 \), and so the knot \( 11n50 \) is not QA.
We remark that there does exist a negative definite 4-manifold $W$ with boundary $-Y$ for which $H_1(W)$ contains torsion. The knot $11n50$ is a slice knot\footnote{Michael Eisermann points out that this is direct from the presentation of $11n50$ as a symmetric union in Figure\cite{11n50} (cf. \cite[Theorem 5]{11n50}).} so that the double-cover of $D^4$, branched along a slice disk for the mirror $\overline{11n50}$, is a rational homology ball with boundary $-Y$, which we may blow up to make undeniably negative definite. However, its first homology group contains a subgroup isomorphic to a quotient of $H_1(Y)$ of square-root order, which is $\mathbb{Z}/5\mathbb{Z}$ in this case.

**Proposition 2.2.** For $n \geq 2$, and $p_1, \ldots, p_n \geq 2$, and $q \geq 1$, the pretzel link $P(p_1, \ldots, p_n, -q)$ is QA iff $q > \min\{p_1, \ldots, p_n\}$.

**Proof.** Let $L$ denote the pretzel link $P(p_1, \ldots, p_n, -q) = M(1; (p_1, 1), \ldots, (p_n, 1), (q, q-1))$. The space $\Sigma(L)$ is the boundary of the plumbing $X$ on a star-shaped planar graph $\Gamma$. The graph $\Gamma$ has $n+1$ rays of lengths $p_1 - 1, \ldots, p_n - 1$, and 1 emanating from the star vertex in cyclic order; by length we mean the number of edges. The star vertex receives weight $-n$, the vertex on the distinguished ray of length 1 receives weight $-q$, and every other vertex receives weight $-2$. As before, the intersection pairing on $X$ is given in the natural spherical basis by the weighted adjacency matrix $A_r$.

The space $X$ is negative definite if and only if $p_1^{-1} + \cdots + p_n^{-1} - q^{-1} > 0$ \cite[Theorem 5.2]{Eisermann}. If it is not – implying that $q < \min\{p_1, \ldots, p_n\}$ – then we claim that $L$ is not QA. Consider the space $-Y = \Sigma(L)$. It is the boundary of plumbing on a star-shaped graph $\Gamma'$ with $n$ rays of length 1 and one of length $q - 1$ emanating from the star vertex. The vertices on rays of length 1 receive weights $-p_1, \ldots, -p_n$ in cyclic order, the vertices on the ray of length $q - 1$ receive weight $-2$, and the star vertex receives weight $-1$. Under the assumption that $\Gamma$ is not negative definite, the graph $\Gamma'$ is by another application of \cite[Theorem 5.2]{Eisermann}.

Now we appeal to some facts for which a detailed account would extend too far beyond the scope of this note. Since the star vertex has degree $\geq 3$ and weight $-1$, an application of Laufer’s algorithm terminates at the $0^{th}$ iteration, and shows at once that the space $-Y = Y_{\Gamma'}$ is not the link of a rational surface singularity \cite[Section 4]{Laufer}. The invariant $HF^+(-Y_{\Gamma'})$ is identified with a particular $\mathbb{Z}[U]$-module $\mathbb{H}^\Gamma(\Gamma')$, as detailed in \cite[Section 2]{Nemethi}. On the other hand, Némethi has proven that if $HF^+(-Y_{\Gamma'}) \cong \mathbb{H}^\Gamma(\Gamma')$ and $Y_{\Gamma'}$ is an L-space, then $Y_{\Gamma'}$ is the link of a rational surface singularity \cite[Proposition 4.1.2]{Nemethi}. It follows that $Y$ is not an L-space, so $L$ is not QA in this case.

Therefore, we may assume henceforth that $X$ is negative definite. Now suppose that $L$ were QA, so that $-\Sigma(L) = \partial W$, where $W$ is a negative definite 4-manifold with $H_1(W) = 0$. Since $\det(L) > 0$, it follows that $-\Sigma(L)$ is a rational homology sphere. We proceed as in the proof of Theorem\cite{Eisermann} and analyze how $(H_2(X), Q_X)$ can embed into the lattice $\mathbb{Z}^n = \langle E_1, \ldots, E_n \rangle$. To every vertex of $\Gamma$ corresponds a vector in $-\mathbb{Z}^n$. If two distinct vertices of weight $-2$ gave rise to vectors with the same reduction (mod 2), then a change of basis of $-\mathbb{Z}^n$ puts these vectors in the form $E_1 + E_2$ and $E_1 - E_2$. These in turn correspond to a pair of columns of the matrix $A$ representing the map $H_2(X)_{\text{Tors}} \to H_2(X \cup \partial W)_{\text{Tors}}$ supported in the first two columns. The induced $2 \times 2$ minor has determinant $\pm 2$, in contradiction to Lemma\cite{Eisermann}.
It follows that, moving away from the star vertex, the vectors corresponding to the vertices along a ray of length \( p_i - 1 \) can be put in the form \( E_1^i - E_2^i, \ldots, E_{p_i-1}^i - E_{p_i}^i \), where all the basis vectors \( E_j^i \), \( 1 \leq i \leq n, 1 \leq j \leq p_i \), are distinct. It then follows that the star vertex corresponds to the vector \(-E_1^i - \cdots - E_p^i\). Consider the vector \( x \) corresponding to the vertex of weight \(-q\). Its inner product with \(-E_1^i - \cdots - E_p^i\) is non-zero, so in its expansion with respect to the chosen basis of \(-Z^n\), it has some term of the form \( a \cdot E_1^i \) with \( a \neq 0 \). Since \( x \) is orthogonal to all those vectors corresponding to the \( i \)-th ray of \( \Gamma \), its expansion takes the form \( a(E_1^i + \cdots + E_{p_i}^i) + \) (additional terms). It follows that \( q \geq |a| \cdot p_i \geq \min\{p_1, \ldots, p_n\} \). Now suppose by way of contradiction that equality held throughout. Then in fact \( x = E_1^i + \cdots + E_{p_i}^i \). Consider the rows of the matrix \( A \) corresponding to the vectors \( E_1^i - E_2^i, \ldots, E_{p_i-1}^i - E_{p_i}^i \), and \( E_1^i + \cdots + E_{p_i}^i \). These are \( p_i \) rows, whose induced \( p_i \times p_i \) minor has determinant \( \pm p_i \neq \pm 1 \), in contradiction to Lemma \[2.3\] Consequently, \( q > \min\{p_1, \ldots, p_n\} \), as desired.

The converse statement in \[3\] Theorem 3.2(1) \] completes the proof of the Proposition.

\( \square \)

**Proof of Theorem \[1.4\] ** Let \( L \) denote the pretzel link appearing in the statement of the Theorem. As a Montesinos link, it is notated by \( M(e; (p_1, 1), \ldots, (p_n, 1), (q_1, -1), \ldots, (q_m, -1)) \).

If \( e < m - 1 \), then \( L \) has the equivalent description \( M(0; (p_1, 1), \ldots, (p_n, 1), (q_1, q_1 - 1), \ldots, (q_m, q_m - 1)) \). If \( e + n \geq 2 \), then the diagram resulting from this description is adequate and non-alternating. It follows that \( L \) is \( KH \)-thick \[13\] Proposition 5.1 \] and hence not QA in this case. If \( e + n \leq 1 \), then either \( e = 1, n = 0 \), or \( e = 0, n = 1 \).

In the first case, \( L \) is QA if \( m \leq 1 \) (falling under Case (1) of the Theorem) and non-QA if \( m \geq 2 \) (applying Proposition \[2.2\] to \( \overline{T} \)). In the second case, Proposition \[2.2\] applies once again to \( T \) to determine when \( L \) is QA (Case (3)). This establishes the Theorem in case \( e < m - 1 \).

If \( e = m - 1 \), then \( L \) has the equivalent description \( M(e - m; (p_1, 1), \ldots, (p_n, 1), (q_1, q_1 - 1), \ldots, (q_m, q_m - 1)) \). Its associated diagram is connected and alternating, so \( L \) is QA in this case (Case (1)). Also, if \( e = m - 1 = 0 \), then the Theorem follows by a combination of Proposition \[2.2\] and \[3\] Theorem 3.2(2) \] (Case (4)).

It stands to consider the case that \( e = m - 1 > 0 \) (Case (2)). We prove that \( L \) is QA by induction on \( e + q_1 + \cdots + q_m \). Consider a crossing appearing in the tassel with \(-q_m\) half-twists.

The resolution \( L_0 \) is the link \( P(e; p_1, \ldots, p_n, -q_1, \ldots, -q_{m-1}) \), while the resolution \( L_1 \) is the link \( P(e; p_1, \ldots, p_n, -q_1, \ldots, -q_m) \). We calculate

\[
\det(L_0) = p_1 \cdots p_n q_1 \cdots q_{m-1} (e + p_1^{-1} + \cdots + p_n^{-1} - q_1^{-1} - \cdots - q_{m-1}^{-1}),
\]

\[
\det(L_1) = p_1 \cdots p_n q_1 \cdots q_{m-1} (q_m - 1) (e + p_1^{-1} + \cdots + p_n^{-1} - q_1^{-1} - \cdots - q_{m-1}^{-1} - q_m^{-1} - (q_m - 1)^{-1}),
\]

\[
\det(L) = p_1 \cdots p_n q_1 \cdots q_m (e + p_1^{-1} + \cdots + p_n^{-1} - q_1^{-1} - \cdots - q_m^{-1}).
\]

Note in particular that the expression for each determinant is positive, since \( e = m - 1 \) and there are at most \( m \) negative terms in each sum, with each term \( \geq -1/2 \). The equality \( \det(L) = \det(L_0) + \det(L_1) \) is immediate. Now, \( L_0 \) has the same value \( e \) and one fewer
negative term, so as in the case $e > m - 1$ treated above, this link has a connected, alternating diagram, hence is QA. If $q_m > 3$, then the link $L_1$ is QA by induction. Otherwise, $q_m = 3$, and so $L_1 = P(e; p_1, \ldots, p_n, -q_1, \ldots, -q_{m-1}, -2) = P(e - 1; p_1, \ldots, p_n, -q_1, \ldots, -q_{m-1}, 2) = P(e - 1; 2, p_1, \ldots, p_n, -q_1, \ldots, -q_{m-1})$. If $e - 1 = m - 2 > 0$, then $L_1$ is QA by induction, while if $e - 1 = m - 2 = 0$, then $L_1$ is QA by Proposition 2.2. Thus, $L_1$ is QA regardless, and it follows that $L$ is QA as well. This completes the induction step.

The preceding argument carries over \textit{mutatis mutandis} to the case of a pretzel link which differs from $L$ by a permutation of the parameters $p_i$ and $q_j$. This completes the proof of the Theorem.

\[\square\]

3. Discussion.

3.1. \textbf{Further obstructions.} The main principle at work in this note is the fact that for a QA link $L$, there is naturally associated to it a smooth, negative definite 4-manifold $X_L$ with vanishing $H_1$ and boundary $\Sigma(L)$. It is therefore of interest to have on hand obstructions to a 3-manifold bounding a negative definite 4-manifold with torsion-free or vanishing $H_1$, and to examine more closely the topology of $X_L$ in the hopes of developing finer obstructions to QA-ness.

In the first direction, Ozsváth-Szabó [24, Section 9.2] have developed an obstruction which makes use of the \textit{correction terms} in Heegaard Floer homology, and which was subsequently sharpened by Owens-Strle [22, Theorem 2]. For the case of $-Y = \Sigma(11n50)$, the Owens-Strle obstruction does not rule out the possibility that this space bounds a negative definite 4-manifold with torsion-free $H_1$. Indeed, using the plumbing graph $\Gamma$, we can calculate the correction terms of $-Y$ according to [25, Corollary 1.5]. The largest correction term has the value $8/25$, which passes their obstruction since it is $> 1/4$. Therefore, the argument given in Theorem 1.3 provides information where this obstruction does not.

In the second direction, Ozsváth-Szabó have shown that $X_L$ is a \textit{sharp} 4-manifold when $L$ is an alternating link ([27, Section 2.8], [28, Theorem 3.4]), and an early arXiv version of the paper [28] suggested that the same is true for an arbitrary QA link $L$ (math.GT/0309170 after Proposition 3.3). However, this is not the case. Indeed, $L = S_{20} = P(3, -2, -2)$ does not bound \textit{any} sharp 4-manifold [2 Proposition 7.3]. This negative result begs for an efficient means of calculating the correction terms of $\Sigma(L)$ for a QA link $L$ in general. Is it still possible to utilize $X_L$ in some way towards this end? What further information can we glean from $X_L$ to develop an obstruction to QA-ness?

3.2. \textbf{Further examples.} The connect-sum of $11n50$ with any QA link $L$ will result in a homologically thin link which is not QA. The fact that it is homologically thin follows from the behavior of the relevant knot homology groups under the connect-sum operation. The fact that it is not QA follows the proof of Theorem 1.3 noting that $\Sigma(11n50#L)$ is the boundary of $X_{11n50}#X_L$, whose intersection pairing decomposes as a direct sum.

In the interest of giving examples of \textit{prime} links that are homologically thin but not QA, consider the family of examples given by the prime Montesinos links $L(m, n) = M(0; (m^2 +
for positive integers $m, n \geq 2$. Thus, \(11n50\) is the knot \(L(2, 3)\). The proof of Theorem 1.3 easily generalizes to show that for all \(n > m\), the link \(L(m, n)\) is not QA. Jablan-Sazdanović [10] suggested the family \(L(2, n)\) with \(n \geq 3\) as an infinite family of non-QA, homologically thin links. Indeed, an application of the skein sequence in knot Floer homology shows that \(L(2, n)\) is $\widetilde{HF}_\mathcal{K}$-thin for all \(n \geq 6\), and a calculation in KhoHo shows that both the relevant Khovanov homology groups are thin for \(n \geq 6\).\footnote{\(Kh\) \(Kh'\) (L\(2, n\)) possesses 5-torsion, and this persists for all \(L(2, n), n \geq 6\), by an application of the long exact sequence in Khovanov homology. Indeed, small calculations suggest that \(Kh'(L(m, n))\) will possess torsion of order \(m^2 + 1\) for all \(n \geq m\). Furthermore, the pretzel link \(P(q, -q, q)\) has \(q\)-torsion for all \(q \leq 6\) [30, 31]. It would be very interesting to explain these torsion phenomena. However, it is reasonable to conjecture that the links \(L(m, m + 1)\) are homologically thin for all \(m \geq 2\), and thereby provide an infinite source of examples of prime, homologically thin, non-QA links. We discuss another potential family at the end of this note.

The family of links \(L(m, n)\), together with work of Champanerkar-Kofman [3] and Widmer [33], indicate some progress in extending Theorem 1.4 to the more general case of Montesinos links. We hope to address this question more fully in future work.

Lastly, extensive calculations by Jablan-Sazdanović [9, 10] (including corrections to some of the ones that originally appeared in [10]) indicate that amongst multi-component links with up to 11 crossings, all except \(L11n77\) and \(L11n90\) are either odd-thick or non-QA. The method described here can be used to show that \(L11n90\) is not QA, although it is odd-thin. We were unable to conclude anything further about \(L11n90\). Therefore, it may require additional ideas to prove that it is non-QA, if indeed this is the case.

3.3. A conjecture. We close with a conjecture.

**Conjecture 3.1.** There exist only finitely many QA links with a given determinant.

In support of Conjecture 3.1, note that there are finitely many alternating links with a given determinant. Moreover, we have the following result for very small determinants.

**Proposition 3.2.** If \(L\) is a QA link with determinant \(\leq 3\), then \(L\) is alternating.

**Proof.** Suppose that \(L\) is a QA link. If \(\det(L) = 1\), then the assertion is trivial.

Next, suppose that \(\det(L) = 2\). Let \(c\) denote a QA crossing in a diagram of \(L\), and \(L_0\) and \(L_1\) the two resolutions of \(L\) at \(c\). Of course, both \(L_0\) and \(L_1\) are unknots. Let \(\gamma\) denote a small unknotted arc connecting the two strands nearby the resolution in \(L_0\), and let \(K\) denote its preimage in \(\Sigma(L_0) = S^3\). Then \(\Sigma(L_1) = S^3\) is a non-trivial surgery on \(K\); by the Dehn surgery characterization of the unknot ([14, Theorem 1.1], or [6, Theorem 2] in this special case), it follows that \(K\) is the unknot. The space \(\Sigma(L)\) is an integer surgery on \(K\) as well, and since \(\det(L) = 2\) it follows that \(\Sigma(L) \cong \mathbb{R}P^3\). A result of Hodgson-Rubinstein [8, Corollary 4.12] characterizes 2-bridge links as those links whose branched double-covers are lens spaces. It follows that \(L\) is the Hopf link.

Lastly, suppose that \(L\) is QA and \(\det(L) = 3\), and proceed as above. In this case, \(L_0\) is the unknot, while \(L_1\) is the Hopf link (or vice versa). Now \(\Sigma(L_1) \cong \mathbb{R}P^3\) is a surgery on \(K \subset S^3\),
and the Dehn surgery characterization of the unknot once again shows that \( K \) is the unknot. Hence \( \Sigma(L) \) is the lens space \( \pm L(3,1) \), and citing [8 Corollary 4.12] again shows that \( L \) is a trefoil knot.

John Baldwin gives an alternative argument for the case of a determinant 3 QA link. Such a link is \( \widehat{HF}(\cdot) \)-thin. Now using the facts that \( \widehat{HF}(\cdot) \) is the \( E_1 \) term in a spectral sequence converging to \( \widehat{HF}(S^3) \cong \mathbb{Z}(0) \), and that \( \widehat{HF}(d,L,i) \cong \widehat{HF}(d-2i,L,-i) \) [26], it follows that \( L \) has the knot Floer homology of a trefoil knot. Since a trefoil is uniquely determined by its knot Floer homology [5], the result follows.

We remark that if Conjecture 3.1 were false, and there were infinitely many QA links of some fixed determinant, then amongst their branched double covers we would obtain an infinite family of L-spaces with the same graded Heegaard Floer homology groups. No such family is known as of this writing. The details of this argument will appear in a separate paper.

In contrast to Conjecture 3.1, Liam Watson points out that there exist infinitely many homologically thin knots with the same homological invariants as the knot 11\( n \)50 (in particular, determinant 25), which we now describe. The knot 11\( n \)50 occurs as \( K(0,3) \) in Kanenobu’s two-parameter family of knots \( K(p,q) \) whose HOMFLY polynomial depends only on \( p+q \) [11]. Watson showed that the unreduced ordinary Khovanov homology of \( K(p,q) \) depends only on the value \( p+q \) as well [32 esp. Lemma 3.1 and Section 7.4], and the same is true in the context of unreduced odd Khovanov homology by a similar argument. Since the unreduced groups are thin for \( K(0,3) \), they are thin and equal for all \( K(n,3-n) \), hence the reduced versions are thin and equal as well. Furthermore, an application of the long exact sequence in knot Floer homology implies that \( \widehat{HF}(K(n,3-n)) \) is independent of \( n \). Finally, Kanenobu’s work implies that the knots \( K(n,3-n) \) are distinguished by their Alexander modules. Therefore, the knots \( K(n,3-n) \) provide the desired source of examples. Amongst them, the knot \( K(1,2) = K(2,1) = 11n132 \) is QA. However, we speculate that this is the only knot in this family that is QA, and this is the subject of work in progress.

**References**

[1] J. A. Baldwin. Heegaard Floer homology and genus one, one-boundary component open books. *J. Topol.*, 1(4):963–992, 2008.

[2] J. A. Baldwin and W. D. Gillam. Computations of Heegaard-Floer knot homology. math.GT/0610167, 2007.

[3] A. Champanerkar and I. Kofman. Twisting quasi-alternating links. arXiv:0712.2590, 2009.

[4] S. K. Donaldson. The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Differential Geom.*, 26(3):397–428, 1987.

[5] P. Ghiggini. Knot Floer homology detects genus-one fibre d knots. *Amer. J. Math.*, 130(5):1151–1169, 2008.

[6] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2(2):371–415, 1989.

[7] J. Greene. On closed 3-braids with unknotting number one. arXiv:0902.1573, 2009.

[8] C. Hodgson and J. H. Rubinstein. Involutions and isotopies of lens spaces. In *Knot theory and manifolds (Vancouver, B.C., 1983)*, volume 1144 of *Lecture Notes in Math.*, pages 60–96. Springer, Berlin, 1985.
[9] S. Jablan. Private communication, 2009.
[10] S. Jablan and R. Sazdanović. Quasi-alternating links and odd homology: computations and conjectures. arXiv:0901.0075, 2009.
[11] T. Kanenobu. Infinitely many knots with the same polynomial invariant. Proc. Amer. Math. Soc., 97(1):158–162, 1986.
[12] A. Kawauchi. A survey of knot theory. Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author.
[13] M. Khovanov. Patterns in knot cohomology. I. Experiment. Math., 12(3):365–374, 2003.
[14] P. B. Kronheimer, T. S. Mrowka, P. Ozsváth, and Z. Szabó. Monopoles and lens space surgeries. Ann. of Math. (2), 165(2):457–546, 2007.
[15] C. Lamm. Symmetric unions and ribbon knots. Osaka J. Math., 37(3):537–550, 2000.
[16] H. B. Laufer. On rational singularities. Amer. J. Math., 94:597–608, 1972.
[17] C. Manolescu. An unoriented skein exact triangle for knot Floer homology. Math. Res. Lett., 14(5):839–852, 2007.
[18] C. Manolescu and P. Ozsváth. On the Khovanov and knot Floer homologies of quasi-alternating links. In Proceedings of the 14th Gökova Geometry-Topology Conference, pages 60–81. International Press, Berlin, 2007.
[19] A. Némethi. Lattice cohomology of normal surface singularities. Publ. Res. Inst. Math. Sci., 44(2):507–543, 2008.
[20] W. D. Neumann and F. Raymond. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. In Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977), volume 664 of Lecture Notes in Math., pages 163–196. Springer, Berlin, 1978.
[21] B. Owens and S. Strle. Rational homology spheres and the four-ball genus of knots. Adv. Math., 200(1):196–216, 2006.
[22] B. Owens and S. Strle. A characterisation of the $\mathbb{Z}^n \oplus \mathbb{Z}(\delta)$ lattice and definite nonunimodular intersection forms. arXiv:0802.1495, 2008.
[23] P. Ozsváth, J. Rasmussen, and Z. Szabó. Odd Khovanov homology. arXiv:0710.4300, 2008.
[24] P. Ozsváth and Z. Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. Adv. Math., 173(2):179–261, 2003.
[25] P. Ozsváth and Z. Szabó. On the Floer homology of plumbed three-manifolds. Geom. Topol., 7:185–224 (electronic), 2003.
[26] P. Ozsváth and Z. Szabó. Holomorphic disks and knot invariants. Adv. Math., 186(1):58–116, 2004.
[27] P. Ozsváth and Z. Szabó. Knots with unknotting number one and Heegaard Floer homology. Topology, 44(4):705–745, 2005.
[28] P. Ozsváth and Z. Szabó. On the Heegaard Floer homology of branched double-covers. Adv. Math., 194(1):1–33, 2005.
[29] J. Rasmussen. Knot polynomials and knot homologies. In Geometry and topology of manifolds, volume 47 of Fields Inst. Commun., pages 261–280. Amer. Math. Soc., Providence, RI, 2005.
[30] A. Shumakovitch. KhoHo pari package. www.geometrie.ch/KhoHo/, 2009.
[31] A. Shumakovitch. Patterns in odd Khovanov homology. In preparation, 2009.
[32] L. Watson. Knots with identical Khovanov homology. Algebr. Geom. Topol., 7:1389–1407, 2007.
[33] T. Widmer. Quasi-alternating Montesinos links. arXiv:0811.0270, 2008.

Department of Mathematics, Princeton University, Princeton, NJ 08542

E-mail address: jegreene@math.princeton.edu