Computer Algorithms for Pseudo-Formal Linearization Method Based on Discrete Fourier Series Expansion and Nonlinear Observer

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Abstract In this paper, we propose a computational algorithm of a pseudo-formal linearization method for nonlinear dynamic systems using the discrete Fourier series expansion in order to reduce computational burden. A nonlinear dynamic system is transformed into some augmented linear systems piecewisely with respect to a linearization function that consists of trigonometric functions by a pseudo-formal linearization method using the discrete Fourier series expansion. Then all of the linearized systems are smoothly united into a single linear system. As an application of this method, a computational algorithm for a nonlinear observer is also proposed. Numerical experiments are demonstrated to indicate the effectiveness of the proposed algorithms.

Keywords: nonlinear system, pseudo-formal linearization, discrete Fourier series expansion, nonlinear observer, linearization function, trigonometric function

1. Introduction

To deal with nonlinear dynamic systems, linearization techniques are utilized to apply linear system theories. Many studies have been presented, such as the linearizations based on Taylor expansion truncated at the first order [1],[2] and those of geometric approaches [3]-[6]. On the other hand, we have been studying a formal linearization method [7],[8] and a pseudo-formal linearization method [9]-[11], which is an advanced method of the formal linearization, in order to gain a high accuracy of the linearization.

In this paper, we propose a numerical computation algorithm for a pseudo-formal linearization method in which the discrete Fourier series expansion is used [12]. In the previous research [11], a pseudo-formal linearization method based on the Fourier series expansion was presented, and we developed the method exploiting the discrete Fourier series expansion with the aim of reducing the computational burden of the linearization and facilitating transformation using the presented algorithm. By introducing a formal linearization function that consists of the state variables and their trigonometric functions up to a higher order and dividing the given domain into some subdomains, a given nonlinear dynamic system is transformed into some linear systems piecewisely by the presented algorithm. Then linearized systems on the subdomains are smoothly united into a single linear system on a whole domain by using an automatic choosing function. As an application of this method, a computational algorithm for a nonlinear observer is presented. A measurement nonlinear system is transformed into an augmented linear one by the presented algorithm. For both dynamic and measurement linearized systems, a linear system theory is applied to synthesize a nonlinear observer.

These algorithms have the advantage that the coefficients of the linearized systems are determined by simple summations because of the orthogonality for a finite sum [12]. In other words, the computational burden for calculating the coefficients of the linearized systems can be markedly improved compared with the case of evaluating definite integrals for linearization in the previous work [11], because the presented algorithms do not need integration to determine the coefficients of the linearized systems.

Numerical experiments indicate that the performance of the proposed method is superior to that of the previous method [11].
2. Pseudo-Formal Linearization by Discrete Fourier Series Expansion

A nonlinear dynamic equation is described by
\[ \Sigma_1 : \dot{x}(t) = \mathbf{f}(x(t)) \]
\[ = A_L x(t) + f_{NL}(x(t)) \] (1)
where \( t \) denotes time, \( \dot{x} = \frac{dx}{dt}, x = [x_1, \ldots, x_n]^T \) is an \( n \)-dimensional state vector, \( A_L \) is an \( n \times n \) matrix, and \( f_{NL} = [f_{NL1}, \ldots, f_{NLn}]^T \) is a continuous nonlinear vector-valued function. \( D \) is a domain denoted by the Cartesian product
\[ D = \prod_{i=1}^{n} [m_i - p_i, m_i + p_i] \quad (m_i \in \mathbb{R}, p_i > 0) \]

To translate these nonlinear systems into pseudo-formal linear systems using the discrete Fourier series expansion, we introduce a vector-valued separable function as
\[ C : D \rightarrow \mathbb{R}^L \] (2)
Let \( C = [I : 0] \cdot x \) (1 : \( L \times L \) unit matrix) for simplicity, where \( L \) indicates the number of state variables. Considering the nonlinearity of the given nonlinear dynamic system and letting \( D \) be a domain of \( C^{-1} \), the domain \( D \) is divided into \((M + 1)\) subdomains
\[ D = \bigcup_{k=0}^{M} D_k \] (3)
where
\[ D_M = D - \bigcup_{k=0}^{M-1} D_k \]
and \( C^{-1}(D_0) \ni 0 \). \( D_k (0 \leq k \leq M - 1) \) endowed with a lexicographic order is the Cartesian product
\[ D_k = \prod_{j=1}^{L} [a_{kj}, b_{kj}] \quad (a_{kj} < b_{kj}) \]

We here introduce an automatic choosing function of the sigmoid type [9],
\[ I_k(\zeta) = \prod_{j=1}^{L} \left(1 - \frac{1}{1 + \exp(2\mu(\zeta_j - a_{kj}))} \right) \]
\[ - \frac{1}{1 + \exp(-2\mu(\zeta_j - b_{kj}))}, \quad (0 \leq k \leq M - 1) \] (4)
\[ I_M(\zeta) = 1 - \sum_{k=0}^{M-1} I_k(\zeta) \]
so that
\[ \sum_{k=0}^{M} I_k(\zeta) = 1 \] (5)
where
\[ \zeta = [\zeta_1, \ldots, \zeta_L]^T = C(x) \] (6)
and \( \mu \) is a positive real value. \( I_k(\zeta) \) is analytic and almost unity on \( D_k \); otherwise, it is almost zero (see Fig. 1).

![Figure 1 Pseudo-formal linearization](image)

Next, the state vector \( x \) is changed into \( y \) so that we can use the discrete Fourier series expansion at each \( k \) \((0 \leq k \leq M)\). Let \( y \) be
\[ y = \mathcal{P}(k)^{-1} (x - \mathcal{M}(k)) \in D_0 \] (7)
where
\[ y = [y_1, \ldots, y_L, y_{L+1}, \ldots, y_n]^T \]
\[ \mathcal{M}(k) = [m_{1(k)}, \ldots, m_{L(k)}, m_{L+1}, \ldots, m_n]^T \]
\[ \mathcal{P}(k) = \text{diag}(p_{1(k)}, \ldots, p_{L(k)}, p_{L+1}, \ldots, p_n) \]
\[ m_{j(k)} = \frac{1}{2}(a_{kj} + b_{kj}), \quad p_{j(k)} = \frac{1}{2\pi}(b_{kj} - a_{kj}) \] (8)
\[ D_0 = \prod_{i=1}^{n} [-\pi, \pi] \] (9)
Because of Eq. (1), Eq. (7) becomes
\[ \dot{y}(t) = \mathcal{P}(k)^{-1} f(\mathcal{P}(k)y(t) + \mathcal{M}(k)) \] (10)

In this paper, we define an \( N \)th order formal linearization function that consists of the state variables and the trigonometric functions defined by
\[ \phi(x) \triangleq [\phi_{L_{10-\cdots-0}}(x), \ldots, \phi_{L_{00-\cdots-0}}(x), \phi_{SC_{10-\cdots-0}}(y(x)), \ldots, \phi_{SC_{(1N-\cdots-1)}}(y(x))] \] (11)
\[ = [\phi_{L_{10-\cdots-0}}(x), \ldots, \phi_{L_{00-\cdots-0}}(x), \phi_{SC_{10-\cdots-0}}(y(x)), \ldots, \phi_{SC_{(1N-\cdots-1)}}(y(x))] \] (12)
\[ = [x_1, \ldots, x_n, \cos y_1(x), \ldots, \cos y_n(x), \sin y_1(x), \ldots, \sin y_n(x)] \]
where \( \phi_{SC}(y) \) is a trigonometric function vector of \( y \), which can make the orthogonal systems up to the \( N \)th order, and

\[
\phi_{L,(0\ldots 0)\rightarrow (i)} (x) = x_i \\
\phi_{SC,(0\ldots 0)\rightarrow z} (y) = T_{2S-1}(y_i) \\
\phi_{SC,(0\ldots 0\rightarrow z)} (y) = T_{2S}(y_i) \\
\phi_{SC,(r_1\ldots r_n)} (y) = \prod_{i=1}^{n} T_{r_i}(y_i)
\]

where \( T_0(y_i) = 1, \ T_{2S-1}(y_i) = \cos(Sy_i) \)
\( T_{2S}(y_i) = \sin(Sy_i) \quad (S = 1, \ldots, N) \)

From Eqs. (10) and (12), the derivative of \( \phi \) is given as follows.

For \( r_1 + \ldots + r_n = 1, \ x \) is

\[
\dot{x} = [\phi_{(0\ldots 0)}(x), \ldots, \phi_{(0\ldots 1)}(x)]^T = A_L x + f_{NL}(x) \\
\quad = A_L x + f_{NL}(P^{(k)} y + \mathcal{M}^{(k)}) \\
\quad = A_L x + [G^{(k)}_{(r_1\ldots r_n)}(y)]_1
\]

For \( r_1 + \ldots + r_n \geq 2 \), each element of \( \phi_{SC} \) is

\[
\dot{\phi} = \frac{dx}{dt} = \frac{\partial y}{\partial x} \cdot \frac{\partial y}{\partial \phi_{(r_1\ldots r_n)}(y)} = f^T (P^{(k)} y + \mathcal{M}^{(k)}) \cdot P^{(k)-1} \cdot \frac{\partial y}{\partial \phi_{(r_1\ldots r_n)}(y)} \\
\quad \equiv G^{(k)}_{(r_1\ldots r_n)}(y)
\]

or

\[
\phi_{SC} = [G^{(k)}_{(r_1\ldots r_n)}(y)]_2
\]

These \( \{G^{(k)}_{(r_1\ldots r_n)}(y)\}_1, \ [G^{(k)}_{(r_1\ldots r_n)}(y)]_2 \) are approximated by the discrete Fourier series expansion [12] as

\[
G^{(k)}_{(r_1\ldots r_n)}(y) \approx \sum_{q_1=0}^{2^{N-1}-1} \cdots \sum_{q_n=0}^{2^{N-1}-1} C^{(k)}_{(q_1\ldots q_n)} \prod_{i=1}^{n} T_{q_i}(y_i)
\]

\[
= [C^{(k)}_{(q_1\ldots q_n)}] \cdot \phi_{SC} + C^{(k)}_{(0\ldots 0)}
\]

where \( C^{(k)}_{(r_1\ldots r_n)} \) is a coefficient of the discrete Fourier series expansion and is calculated by simple summation as

\[
C^{(k)}_{(q_1\ldots q_n)} = \frac{1}{2^n N^n} \sum_{j_1=-N+1}^{N-1} \cdots \sum_{j_n=-N+1}^{N-1} y_{j_1} \cdots y_{j_n}
\]

\[
\sum_{j_n=-N+1}^{N-1} G^{(k)}_{(r_1\ldots r_n)}(y_{j_1}, y_{j_2}, \ldots, y_{j_n}) \prod_{i=1}^{n} T_{q_i}(y_{j_i})
\]

\( \gamma = \{ \text{the number of } q_i = 0, 2N-1 : 1 \leq i \leq n \} \)

The interpolating points \( \{y_{j_i}\} \) are set to be

\[
y_{j_i} = \frac{j_i}{N} \quad (i = 1, \ldots, n, \ j_i = -N+1, \ldots, N)
\]

Thus,

\[
\begin{bmatrix}
[\phi_{(r_1\ldots r_n)}(y)]_1 \\
[\phi_{(r_1\ldots r_n)}(y)]_2
\end{bmatrix}
\approx
\begin{bmatrix}
[C^{(k)}_{(r_1\ldots r_n)}(y)]_1 \\
[C^{(k)}_{(r_1\ldots r_n)}(y)]_2
\end{bmatrix}
\phi_{SC}(y) + \begin{bmatrix}
A^{(k)}_{11} & b^{(k)}_1 \\
A^{(k)}_{21} & b^{(k)}_2
\end{bmatrix}
\]

Therefore, we have the following on a subdomain \( D_k \):

\[
\phi(x) \approx A^{(k)} \phi(x) + b^{(k)}
\]

where

\[
A^{(k)} = \begin{bmatrix} A_L & A^{(k)}_{21} \\ 0 & A^{(k)}_{22} \end{bmatrix}, \ b^{(k)} = \begin{bmatrix} b^{(k)}_1 \\ b^{(k)}_2 \end{bmatrix}
\]

To obtain a single linear system, we unite these \( (M+1) \) linearized systems (Eq. 23) on the subdomains by using Eq. (5) as

\[
\dot{\phi}(x) = \sum_{k=0}^{M} \phi(x) I_k(\zeta) \approx \sum_{k=0}^{M} (A^{(k)} \phi(x) + b^{(k)}) I_k(\zeta)
\]

\[
= \tilde{A}(\zeta) \phi(x) + \tilde{b}(\zeta)
\]

where

\[
\tilde{A}(\zeta) = \sum_{k=0}^{M} A^{(k)} I_k(\zeta), \ \tilde{b}(\zeta) = \sum_{k=0}^{M} b^{(k)} I_k(\zeta)
\]

Finally, a pseudo-formal linearization system is defined as

\[
\Sigma_2: \dot{z}(t) = \tilde{A}(\zeta) z(t) + \tilde{b}(\zeta), \ z(0) = \phi(x(0))
\]

The resulting system (Eq. 26) is a generalization of the standard formal linearization [7] because \( \tilde{A}(\zeta) \) and \( \tilde{b}(\zeta) \) are functions of \( \zeta \) in Eq. (4). From Eq. (12), its inversion is easily carried out by evaluating

\[
\dot{x}(t) = [I, 0, \ldots, 0] z(t)
\]

as the approximated value of \( x(t) \), where \( I \) is the \( n \times n \) unit matrix.

As a result, a pseudo-formal linearization algorithm using the discrete Fourier series expansion is obtained as follows.
3. Nonlinear Observer

We synthesize a nonlinear observer as an application of the above pseudo-formal linearization. We assume that a nonlinear dynamic system is the same as Eq. (1), and a measurement equation is

$$\eta(t) = h(x(t)) = H_L x(t) + h_{NL}(x(t))$$

where $\eta \in \mathbb{R}^m$ is a measurement vector, $H_L$ is an $m \times n$ matrix, and $h_{NL} = [h_1, \ldots, h_r, \ldots, h_m]'$ is a continuous smooth nonlinear vector-valued function.

The nonlinear dynamic system in Eq. (1) is transformed into the linear system in Eq. (26) by the pseudo-formal linearization in Sect. 2. To linearize the measurement equation with respect to the linearization function $\phi$, we apply the discrete Fourier series expansion up to the $N$th order to each element of $h_{NL}$ on the same subdomain $D_k$ as in the state space. Then we have

$$h_{NL}(x) = h_{NL}(\mathbb{P}(k) y + \mathcal{M}(k)) \triangleq \mathbb{G}^{(k)}(y)$$

$$\approx \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C^{(k)}(q_1 \ldots q_n) \prod_{i=1}^{n} T_{q_i}(y_{j_i})$$

where $C^{(k)}(q_1 \ldots q_n)$ is a coefficient of the discrete Fourier series expansion and is calculated by simple summation as

$$C^{(k)}(q_1 \ldots q_n) \triangleq \frac{1}{2^n N^n} \sum_{j_1=-N+1}^{N} \sum_{j_2=-N+1}^{N} \cdots \sum_{j_n=-N+1}^{N} C^{(k)}(y_{j_1}, y_{j_2}, \ldots, y_{j_n}) \prod_{i=1}^{n} T_{q_i}(y_{j_i})$$

and a linear measurement equation with respect to $\phi$ on subdomain $D_k$ is obtained as

$$\eta \approx [H_L, H_N^{(k)}] \phi(x) + \epsilon_{NL}^{(k)} \triangleq H^{(k)} \phi(x) + d^{(k)}$$

We unite $(M + 1)$ linearized systems (Eq. (32)) on subdomains into a single linear system on the whole domain, using Eq. (5), as

$$\eta \approx \sum_{k=0}^{M} (H^{(k)} \phi(x) + d^{(k)}) I_k(\zeta)$$

$$= \sum_{k=0}^{M} H^{(k)} I_k(\zeta) \phi(x) + \sum_{k=0}^{M} d^{(k)} I_k(\zeta)$$
Thus, a pseudo-formal linearization system for the measurement equation is approximately derived as

$$\eta(t) = \tilde{H}(\zeta)z(t) + \tilde{d}(\zeta)$$  \hspace{1cm} (34)

where

$$\tilde{H}(\zeta) = \sum_{k=0}^{M} H^{(k)} I_k(\zeta), \hspace{0.5cm} \tilde{d}(\zeta) = \sum_{k=0}^{M} d^{(k)} I_k(\zeta)$$

The linear observer theory [13] can be applied to the linearized systems in Eqs. (26) and (34), and an identity observer is obtained as

$$\hat{z}(t) = \tilde{A}(\zeta)\hat{z}(t) + \tilde{b}(\zeta) + K(t)(\eta(t) - \tilde{H}(\zeta)\hat{z}(t) - \tilde{d}(\zeta))$$

$$= \sum_{k=0}^{M} \left\{ (A^{(k)}\hat{z}(t) + b^{(k)}) 
+ K^{(k)}(t)(\eta(t) - H^{(k)}\hat{z}(t) - d^{(k)}) \right\} I_k(\zeta)$$  \hspace{1cm} (35)

where $\zeta = C(\hat{z})$.  $K^{(k)}(t)$ is the observer gain on sub-domain $D_k$ given by

$$K^{(k)}(t) = \frac{1}{2} P^{(k)}(t) H^{(k)T} S^{(k)}(t)$$

$P^{(k)}(t)$ satisfies the matrix Riccati differential equation,

$$\dot{P}^{(k)}(t) = A^{(k)} P^{(k)}(t) + P^{(k)}(t) A^{(k)T} + Q^{(k)}(t)$$

$$- P^{(k)}(t) H^{(k)T} S^{(k)}(t) H^{(k)} P^{(k)}(t)$$  \hspace{1cm} (36)

where $Q^{(k)}(t)$, $S^{(k)}(t)$, and $P^{(k)}(0)$ are arbitrary real symmetric positive definite matrices.

From Eq. (27), the estimate $\hat{x}(t)$ of a nonlinear observer becomes

$$\hat{x}(t) = [I, 0, \cdots, 0] \hat{z}(t)$$  \hspace{1cm} (37)

As a result, a nonlinear observer algorithm is obtained as follows.

**Nonlinear Observer Algorithm by Discrete Fourier Series Expansion**

\textbf{(O-1)} Given

$$\left\{ \begin{array}{l} \hat{x}(t) = f(x(t)), \hspace{0.5cm} x(0) = x_0 \in \mathbb{D} \\ \eta(t) = h(x(t)) \end{array} \right.$$

\textbf{(O-2)} Set

$$L, C, M, \mu, N, M^{(k)}, \mathcal{P}^{(k)}, Q^{(k)}(t), S^{(k)}(t), P^{(k)}(0)$$

\hspace{1cm} $(k = 0, \cdots, M)$

\textbf{(O-3)} \hspace{1cm} (O-3.1)

$$I_k(\zeta) = \prod_{j=1}^{L} \left( 1 - \frac{1}{1 + \exp(-2\mu(\zeta_j - a_k))} \right)$$

\textbf{(O-5)} Obtain $P^{(k)}$ by solving the Riccati equation:

$$\dot{P}^{(k)}(t) = A^{(k)} P^{(k)}(t) + P^{(k)}(t) A^{(k)T} + Q^{(k)}(t)$$

\hspace{1cm} $- \frac{1}{1 + \exp(-2\mu(\zeta_j - b_{kj}))} \} \hspace{1cm} (0 \leq k \leq M - 1)$

\textbf{(O-3.2)}

$$T_0(y_i) = 1, \hspace{0.5cm} T_{2S-1}(y_i) = \cos(Sy_i)$$

$$T_{2S}(y_i) = \sin(Sy_i), \hspace{0.5cm} y_{ji} = \frac{j_i}{N} \pi$$

\hspace{1cm} $(i = 1, \cdots, n, \hspace{0.5cm} j_i = -N + 1, \cdots, N, \hspace{0.5cm} S = 1, \cdots, N)$

\textbf{(O-3.3)}

$$G^{(k)}(y) = f^T(p^{(k)} y + M^{(k)}) \cdot p^{(k)-1} \frac{\partial}{\partial y} \phi_{(r_1 \cdots r_n)}(y)$$

\textbf{(O-3.4)}

$$\gamma = \{ \text{the number of } q_i = 0, 2N - 1 : \hspace{0.5cm} 1 \leq i \leq n \}$$

\textbf{(O-3.5)}

$$H^{(k)}_{NL} = [C^{(k)}_{(q_1 \cdots q_n)}], \hspace{0.5cm} d^{(k)}_{NL} = [C^{(k)}_{(0 \cdots 0)}]$$

\textbf{(O-4)}

$$A^{(k)} = \begin{bmatrix} A_{12}^{(k)} \\ 0 \end{bmatrix}, \hspace{0.5cm} b^{(k)} = \begin{bmatrix} b_1^{(k)} \\ b_2^{(k)} \end{bmatrix}$$

$$H^{(k)} = [H_L, H^{(k)}_{NL}], \hspace{0.5cm} d^{(k)} = d^{(k)}_{NL}$$

$$\hat{A}(\zeta) = \sum_{k=0}^{M} A^{(k)} I_k(\zeta), \hspace{0.5cm} \hat{b}(\zeta) = \sum_{k=0}^{M} b^{(k)} I_k(\zeta),$$

$$\tilde{H}(\zeta) = \sum_{k=0}^{M} H^{(k)} I_k(\zeta), \hspace{0.5cm} \tilde{d}(\zeta) = \sum_{k=0}^{M} d^{(k)} I_k(\zeta)$$

\textbf{(O-5)} Obtain $P^{(k)}$ by solving the Riccati equation:

$$\dot{P}^{(k)}(t) = A^{(k)} P^{(k)}(t) + P^{(k)}(t) A^{(k)T} + Q^{(k)}(t)$$
Pseudo-formal linearization

4.1 Pseudo-formal linearization

Consider a simple nonlinear dynamic system

\[ \dot{x} = x - x^2, \quad D = [0, 1.2] \subset R \]  

(38)

The parameters are set as \( M = 5, \mu = 80, \) and \( \zeta = x \) in Eqs. (3) and (4). \( D \) is divided into \( D = \bigcup_{k=0}^{5} D_k \)

where

\[
D_0 = [0, 0.2), D_1 = [0.2, 0.4), D_2 = [0.4, 0.6) \\
D_3 = [0.6, 0.8), D_4 = [0.8, 1), D_5 = [1, 1.2)
\]

The system parameters are set as

\[ A^{(k)} = 0.1 + 0.2k, \quad P^{(k)} = \frac{1}{10\pi} \quad (k = 0, \ldots, 5) \]

\[ \mathcal{M}^{(k)} = 0.1 + 0.2k, \quad \mathcal{P}^{(k)} = \frac{1}{10\pi} \quad (k = 0, \ldots, 5) \]

Figure 2 shows the true value \( x \) of Eq. (38) and the estimate \( \hat{x} \) obtained by the pseudo-formal linearization of Eq. (27) when the order \( N \) is 4. For comparison, \( \hat{x} \) (old) is the result obtained by the previously reported method [11] when the parameters are the same, that is, \( N = 4, \mathcal{M}^{(k)} = 0.1 + 0.2k, \) and \( \mathcal{P}^{(k)} = \frac{1}{10\pi} \quad (k = 0, \ldots, 5) \). \( \hat{x} \) (Taylor) is the result obtained by the ordinary method [1] using Taylor expansion truncated at the first order when the operating point is 0.

Figure 3 shows the logarithmic trajectories of the integral square error

\[ J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^2 d\tau \]  

(39)

to indicate small errors between \( x \) and \( \hat{x} \) when the order \( N \) is varied from 1 to 4 in this case. \( J(t) \) (old, \( N=4) \) is the error of \( \hat{x} \) (old) [11] and \( J(t) \) (Taylor) is the error of \( \hat{x} \) (Taylor) [1] for comparison. These results show that the error of the presented method improves as the order \( N \) increases. Between \( J(t)(N=4) \) and \( J(t)(old, N=4) \), the performance of the presented method is worse than that of the previous method when \( t < 4 \); otherwise, it is better. This may be affected by small interpolating errors of the presented method.

Table 1 shows running times for calculating the coefficients of the pseudo-formal linearization using the proposed algorithm and those in the previous work [11] and the difference between them. We used a DELL Optiplex 3040 computer (Intel Core i7 CPU, 3.4GHz, 16 GB RAM) and the software was PTC Mathcad.

The results in Fig. 3 and Table 1 show that the proposed method has an accuracy of linearization similar to that of the previous method [11], but the running time for the linearization is greatly shortened.
The reason is because the previous method requires numerical integration techniques, whereas the proposed method simply employs the finite sum on the interpolation points (Eq. (21)). The new method reduces the running time to 5% – 18% of that for the previous method at $N = 1 - 4$.

Therefore, when considering the performance in terms of both the approximation error $J(t)$ and the running time, this newly proposed method is much better than the previous ones.

4.2 Nonlinear observer

We simulate a nonlinear observer using the presented method Eq. (35). As a tentative system, we consider a nonlinear dynamic system and a nonlinear measurement system

$$
\begin{aligned}
\dot{\delta}(t) + \delta(t) + c \sin \delta(t) &= u(t), \quad \delta(t) \in R \\
y(t) &= \sin(\delta(t))
\end{aligned}
$$

When $x_1 = \delta$, $x_2 = \dot{\delta}$, $c = 0.4$, and $u = 0$, the system yields

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} x_2(t) \\ -0.4 \sin(x_1(t)) - x_2(t) \end{bmatrix} \\
y(t) &= \sin(x_1(t)) \triangleq h(x(t))
\end{aligned}
$$

A domain $D$ is considered to be

$$
D = [-0.05, 1.95] \times [-0.365, 0.035]
$$

To apply the pseudo-formal linearization method, we set $C(x) = x_1$ and $L = 1$ because the nonlinearity in $\sin(x_1)$ is the highest. We divide the whole domain into four subdomains as

$$
D_0 = [-0.05, 0.45], \quad D_1 = [0.45, 0.95] \\
D_2 = [0.95, 1.45], \quad D_3 = [1.45, 1.95]
$$

The system parameters are set as $M = 3$, $\mu = 100$, $\zeta = x_1$,

$$
\mathcal{M}^{(0)} = \begin{bmatrix} 0.2 \\ -0.165 \end{bmatrix}, \mathcal{M}^{(1)} = \begin{bmatrix} 0.7 \\ -0.165 \end{bmatrix}
$$

$$
\mathcal{M}^{(2)} = \begin{bmatrix} 1.2 \\ -0.165 \end{bmatrix}, \mathcal{M}^{(3)} = \begin{bmatrix} 1.7 \\ -0.165 \end{bmatrix}
$$

$$
\mathcal{P}^{(k)} = \begin{bmatrix} 1 \\ \frac{1}{4\pi} \\ \frac{1}{5\pi} \end{bmatrix}, \quad (k = 0, 1, 2, 3)
$$

and the order of the formal linearization function is $N = 3$. The parameters for a nonlinear observer are set as

$$
\begin{aligned}
x(0) &= [1.5, 0]^T, \quad \hat{x}(0) = [1.85, -0.3]^T \\
Q^{(k)}(t) &= P^{(k)}(0) = I, \quad S^{(k)}(t) = 1, \quad (k = 0, 1, 2, 3)
\end{aligned}
$$

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y(t) &= \sin(x_1(t)) \triangleq h(x(t))
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D = [-0.05, 1.95] \times [-0.365, 0.035]
$$

To apply the pseudo-formal linearization method, we set $C(x) = x_1$ and $L = 1$ because the nonlinearity in $\sin(x_1)$ is the highest. We divide the whole domain into four subdomains as

$$
D_0 = [-0.05, 0.45], \quad D_1 = [0.45, 0.95] \\
D_2 = [0.95, 1.45], \quad D_3 = [1.45, 1.95]
$$

The system parameters are set as $M = 3$, $\mu = 100$, $\zeta = x_1$,

$$
\mathcal{M}^{(0)} = \begin{bmatrix} 0.2 \\ -0.165 \end{bmatrix}, \mathcal{M}^{(1)} = \begin{bmatrix} 0.7 \\ -0.165 \end{bmatrix}
$$

$$
\mathcal{M}^{(2)} = \begin{bmatrix} 1.2 \\ -0.165 \end{bmatrix}, \mathcal{M}^{(3)} = \begin{bmatrix} 1.7 \\ -0.165 \end{bmatrix}
$$

$$
\mathcal{P}^{(k)} = \begin{bmatrix} 1 \\ \frac{1}{4\pi} \\ \frac{1}{5\pi} \end{bmatrix}, \quad (k = 0, 1, 2, 3)
$$

Table 1 Running times for the linearization

| $N$ | Proposed(s) | Previous(s) | Diff.(s) |
|-----|-------------|-------------|----------|
| 1   | 0.0020      | 0.0350      | -0.0330  |
| 2   | 0.0070      | 0.0450      | -0.0380  |
| 3   | 0.0100      | 0.0560      | -0.0460  |
| 4   | 0.0210      | 0.1120      | -0.0910  |

Figure 4 shows the true value $x$ and the estimated $\hat{x}$ in Eq. (37) obtained by the presented method. $\hat{x}(t)$ is the result obtained by the ordinary method [1] using Taylor expansion truncated at the first order with the operating point of $[0.75, 0]^T$. The parameters are set as $Q(t) = P(0) = 0.5I$ and $S(t) = 0.5$. Figure 5 shows the logarithmic integral square errors of the estimation

$$
J(t) = \int_0^t \|x(t) - \hat{x}(\tau)\|^2 d\tau
$$

in these cases.

These results show that the performance of the presented pseudo-formal linearization is better than that of the previous method. The estimation accuracy will be improved if the parameters are properly chosen.
5. Conclusions

We have studied a pseudo-formal linearization method for nonlinear systems using the discrete Fourier series expansion and proposed a computational algorithm. We have also synthesized a nonlinear observer as an application of the method and presented its computational algorithm. From the results of the numerical experiments, the performance of the presented algorithms was found to be better than those of the previous methods. The following is left for future works: problems of optimal selection of parameters \( \{M, \mu, N, \mathcal{M}^k(t), \mathcal{P}^k(t), \mathcal{Q}^k(t), \mathcal{S}^k(t), \mathcal{P}^k(0)\} \) specifying this approach and application to practical systems such as electric power systems.

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