Non-BPS Supersymmetric 3pt Amplitude for One Massless, Two Equally Massive Particles

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Abstract

In this paper, the non-BPS amplitudes ($Z < 2m$) are considered. Utilizing on-shell methods, the three point amplitudes of two equal-mass particles and one massless particle were constructed, where the two massive particles are non-BPS states. We verify the result by matching the $Z = 0$ and BPS limit with $\mathcal{N} = 2$ supersymmetry. As an application we derive the non-BPS coupling for $\mathcal{N} = 4$ super-Maxwell and supergravity.
1. Introduction

Spinor helicity formalism enables us to derive S-matrix by on-shell formulation. Unlike Feynman rules, the formulation does not introduce gauge redundancy into the computation. Rather, the amplitude can be fully determined by the momentum and the spin polarization of the external particles. Recently, spinor helicity formalism is adapted to describe four-dimensional scattering amplitudes for particles of any mass and spin [1]. A 3-pt amplitude of two equal massive particles with mass $m$ and a massless particle with helicity $h$ can be written in spinor helicity basis, see [1],

\[ M^{(I_1 \ldots I_2 s_1)(J_1 \ldots J_2 s_2)h}(\lambda_1^{I_1 a_1} \cdots (\lambda_1^{I_1 a_1}, (\lambda_2^{I_2 b_2} \cdots (\lambda_2^{I_2 b_2} M^{(a_1 \ldots a_2 s_1)(b_1 \ldots b_2 s_2)h}. \quad (1) \]

Supersymmetry (SUSY) requires on-shell fermionic variables. Spinor helicity formalism then must be formulated in on-shell superspace. The formulation in massless [2] and massive [3, 4] on-shell superspace were subsequently constructed. Here we consider extended SUSY with non-vanishing central charge, using superamplitudes in $\mathcal{N} = 2$ SUSY as building blocks. For extended $\mathcal{N} = 2$ SUSY, the algebra takes the form

\[
\begin{align*}
\{Q^A, \tilde{Q}_{AB}\} &= p_{ab} \delta^A_B \\
\{Q^A, Q^B\} &= \frac{1}{2} Z_{AB} \epsilon_{ab} \\
\{\tilde{Q}_{aA}, \tilde{Q}_{bB}\} &= -\frac{1}{2} Z_{AB} \epsilon_{a\dot{b}}
\end{align*}
\]

and the central charge of a massive particle has a bound, $Z \leq 2m$. BPS states are the states that saturate this bound. The solution to the amplitude of BPS states are studied in [5, 6].

\[ M_Z(1^{(I_1 \ldots I_2 s_1)}, 2^{(J_1 \ldots J_2 s_2), P^h}) = \]

Figure 1: A 3-pt superamplitude of two massive multiplets with equal mass $m$ and a massless multiplet with helicity $h$. Note that the central charges of the two massive particles are opposite to each other ($Z$ and $-Z$), due to central charge conservation.

In this paper, we will consider the non-BPS three point superamplitude shown in Figure 1. The amplitude $M_Z$ is made up of two massive multiplets, with spin-$2s_1$ and spin-$2s_2$, coupling a spin-$h$
massless multiplet, where the two massive multiplets have equal masses \( m \) and opposite central charges (\( Z \) and \(-Z\), respectively). We require \( \mathcal{M}_Z \) satisfy supersymmetry, i.e.,

\[
Q^A_a \mathcal{M}_Z = \tilde{Q}^A_a \mathcal{M}_Z = 0, \tag{4}
\]

where \( Q^A_a \) and \( \tilde{Q}^A_a \) are the supersymmetric generators, and \( A = 1, 2 \) for \( N = 2 \) SUSY. By examining the form of generators, we can show that \( \mathcal{M}_Z \) is a function of \( \delta^A, \mathcal{H}^A, \) and \( \tilde{\mathcal{H}}^A \), which are defined in (9). In addition, we also show that we can always factorize the superamplitude with spinning multiplets in \( N = 2 \) SUSY into two parts,

\[
\mathcal{M}_Z(1^{(l_1 \cdots l_{2N})}, 2^{(J_1 \cdots J_{2N})}, P^h) = M(1^{(l_1 \cdots l_{2N})}, 2^{(J_1 \cdots J_{2N})}, P^h) \cdot \mathcal{A}(Z, m) \tag{5}
\]

\[
Q^A_a \mathcal{A} = 0; \quad \tilde{Q}^A_a \mathcal{A} = 0.
\]

One of them is a bosonic factor \( M(1^{(l_1 \cdots l_{2N})}, 2^{(J_1 \cdots J_{2N})}, P^h) \), which carries the little group (LG) indices, including both the \( SU(2) \) indices for the massive and \( U(1) \) for the massless multiplets. The other is a LG neutral and SUSY invariant quantity \( \mathcal{A}(Z, m) \). Therefore, to solve \( \mathcal{M}_Z \), we need to solve \( \mathcal{A} \). There are 2 solutions for \( \mathcal{A} \), and \( \mathcal{M}_Z \) is a combination of the two solutions, see (57).

We verify the solutions of \( \mathcal{A} \) by matching to known results, i.e., the \( Z = 0 \) limit and the BPS limit (6).

As application, we use our results in \( N = 4 \) SUSY. The presence central charge breaks R-symmetry from \( SU(4) \) to \( SU(2) \otimes SU(2) \), and the central charge is

\[
Z_{AB} = \begin{bmatrix}
0 & -Z_{12} & 0 & 0 \\
Z_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & -Z_{34} \\
0 & 0 & Z_{34} & 0
\end{bmatrix}, \tag{6}
\]

where \( A, B = 1, 2, 3, 4 \). We can then treat \( \mathcal{A} \) as a ”seed” for the superamplitude in \( N = 4 \) SUSY. As an example,

\[
\mathcal{M}_{Z_{12} Z_{34}}(\Phi, \Phi, \phi) = \mathcal{A}_{12} \cdot \mathcal{A}_{34}, \tag{7}
\]

where the subscripts 12 or 34 indicate which projected \( SU(2) \) group they are describing. As in \( N = 2 \) SUSY, the superamplitude for spinning multiplets in \( N = 4 \) SUSY can be factorized as well, similar to (5) (see (75) for more details). We then proceed to consider super Maxwell theories and super-gravity (SUGRA) theories by choosing suitable bosonic factors. In the former case, the massless multiplet carries helicity +1, while in the latter case, there are 2 conjugated superamplitudes, and the massless multiplets in the superamplitudes carry helicity 2 and helicity 0.

In section 2, we show that \( \mathcal{M}_Z \) is a function of \( \delta^A, \mathcal{H}^A, \) and \( \tilde{\mathcal{H}}^A \) by examining the generators. In section 3, we solve for \( \mathcal{A} \) and \( \mathcal{M}_Z \) by requiring them being SUSY invariant. In section 4, we compare our results in the \( Z = 0 \) limit and the BPS limit with previous works. In section 5, we apply the results to \( N = 4 \) non-BPS SUSY, and explore super-Maxwell and super-gravity theories.

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1To distinguish supersymmetric amplitudes from non-supersymmetric amplitudes, we denote supersymmetric amplitudes by curly alphabet \( \mathcal{M} \).

2Note that SUSY only constraints \( \mathcal{A} \), but not the bosonic factor \( M(1^{(l_1 \cdots l_{2N})}, 2^{(J_1 \cdots J_{2N})}, P^h) \).
2. The building blocks for the 3pt amplitude

As mentioned in the introduction, we are dealing with the 3pt amplitude \( M(1, 2, P) \) consisting of two equal-mass \( m \), non-BPS particles (leg 1 and 2) and a massless particle (leg \( P \)), see Figure 1. In this chapter, we will introduce \( \mathcal{N} = 2 \), non-BPS generators. Given the generators, we find the general form of \( M(1, 2, P) \) is

\[
M(1, 2, P) = \left[ \prod_{A=1}^{2} \delta^A \right] \cdot M_{\text{res}}(\mathcal{H}, \bar{\mathcal{H}})
\]

where

\[
\delta^A \equiv \sum_{i=1}^{2} (P^i)\eta^A_{iI}, \quad \mathcal{H}^A \equiv \frac{1}{\langle \xi P \rangle} \sum_{i=1}^{2} (\zeta_i^j)\eta^A_{iI}, \quad \bar{\mathcal{H}}^A \equiv \frac{1}{\langle \bar{\xi} P \rangle} \sum_{i=1}^{2} [\bar{\xi}_i^j] \sigma_i^A \eta^A_{iI},
\]

are the building blocks for the 3pt amplitude (\( \langle \zeta \rangle \) and \( \langle \xi \rangle \) are reference spinors, and the dependence on the reference spinors will drop out if \( \mathcal{H} \) and \( \bar{\mathcal{H}} \) is evaluated on the support of \( \prod_A \delta^A \)). In other words, the superamplitude is proportional to a delta function \( \prod_A \delta^A \), and the rest of the amplitude is a function of \( \mathcal{H} \) and \( \bar{\mathcal{H}} \).

2.1. The generators

The generators for massless particles satisfy the anti-commutation rules

\[
\begin{align*}
\{Q^A_a, \bar{Q}_{ab}\} &= p_{aa} \delta^A_B \\
\{Q^A_a, \bar{Q}^B_{ab}\} &= 0 \\
\{Q_{ab}, \bar{Q}_{ab}\} &= 0
\end{align*}
\]

while for massive particles, the generators satisfy

\[
\begin{align*}
\{Q^A_a, \bar{Q}_{ab}\} &= p_{aa} \delta^A_B \\
\{Q^A_a, \bar{Q}^B_{ab}\} &= \frac{1}{2} Z_{AB} \epsilon_{ab} \\
\{Q_{ab}, \bar{Q}_{ab}\} &= -\frac{1}{2} Z_{AB} \epsilon_{ab},
\end{align*}
\]

where \( Z_{AB} = Z \cdot \epsilon_{AB} \) is the central charge. Note that \( Z \) has a bound, \( 0 \leq |Z| \leq 2m \), and BPS states are states that satisfy \( |Z| = 2m \), where calculations will be lot simpler. However, we are interested in the amplitude for general \( Z \).

In the 3pt amplitude we are considering, the central charges of the two massive particles must carry opposite signs, \( Z_1 + Z_2 = 0 \), due to central charge conservation. We then define

\[
\sin(2\theta_Z) \equiv \frac{Z_1}{2m}.
\]

The bound \( -\frac{\pi}{4} \leq \theta_Z \leq \frac{\pi}{4} \) follows directly from the bound of the central charge \( 0 \leq |Z| \leq 2m \), and BPS limit happens at \( \theta_Z = \frac{\pi}{4} \). For convenience, we also define \( c_z = \cos(\theta_Z) \) and \( s_z = \sin(\theta_Z) \).
To be a superamplitude, \( M(1, 2, \mathcal{P}) \) should satisfy

\[
Q^A_1 M(1, 2, \mathcal{P}) = 0 ; \quad \tilde{Q}_{aa} M(1, 2, \mathcal{P}) = 0 ,
\]

where \( Q^A_1 \) and \( \tilde{Q}_{aa} \) are the sum of the supersymmetric generators of the particles

\[
Q^A_1 \equiv Q^A_{1a} + Q^A_{2a} + Q^A_{pa} ; \quad \tilde{Q}_{aa} \equiv \tilde{Q}_{1aa} + \tilde{Q}_{2aa} + \tilde{Q}_{pa} .
\]

By introducing a set of Grassmann variables \( \{ \eta^A_i, \tilde{\eta}^A_i \} \) \( (i = 1, 2) \), we can express the generators as

\[
\begin{align*}
Q^A_1 &= \sum_{i=1}^2 \lambda^i_{ia} \left( c_i \eta^A_i + s_i \sigma_i \frac{\partial}{\partial \eta^A_i} \right) + \lambda_{pa} \eta^A_p , \\
\tilde{Q}_{aa} &= \sum_{i=1}^2 \tilde{\lambda}^i_{ia} \left( c_i \frac{\partial}{\partial \tilde{\eta}^A_i} - s_i \sigma_i \tilde{\eta}^A_{ia} \right) + \tilde{\lambda}_{pa} \frac{\partial}{\partial \tilde{\eta}^A_p} .
\end{align*}
\]

2.2. The fermionic delta function

From (17), we see that the generators can be separated into a multiplicative part and a differential part. By suitably combining the generators, we are able to subtract the differential part, and the superamplitude will be found to be proportional to the remaining multiplicative part. More explicitly, we can show that \( M(1, 2, \mathcal{P}) \) is proportional to the product of two fermionic delta functions

\[
M(1, 2, \mathcal{P}) \propto \prod_{A=1}^{2} \delta^A ,
\]

where \( \delta^A \equiv \sum_{i=1}^{2} \langle Pi^i \rangle \eta^A_i \).

Let’s contract (13) with \( \lambda^i_p \) and \( \tilde{\lambda}^i_p \) to get rid of \( \eta^i_p \)’s, and multiply them with some proper coefficients, we get

\[
\begin{align*}
c_z \langle PQ^A \rangle M(1, 2, \mathcal{P}) &= 0 \Rightarrow \sum_{i=1}^{2} \langle Pi^i \rangle \left( c_i^2 \eta^A_i + s_i \sigma_i \frac{\partial}{\partial \eta^A_i} \right) M(1, 2, \mathcal{P}) = 0 , \\
\frac{1}{x} e^{AB} s_z \langle P \tilde{Q}^B \rangle M(1, 2, \mathcal{P}) &= 0 \Rightarrow \sum_{i=1}^{2} \langle Pi^i \rangle \left( s_i \sigma_i \frac{\partial}{\partial \tilde{\eta}^A_i} + s_z^2 \tilde{\eta}^A_i \right) M(1, 2, \mathcal{P}) = 0 .
\end{align*}
\]

---

3In this paper, the Grassmann variables’ indices are defined as:

\[
\begin{align*}
i &: \text{the label of the massive particles, } i = 1, 2 \\
I &: \text{the LG indices for the massive particles} \\
A &: \text{the R-charge indices}.
\end{align*}
\]
By subtracting the two equations, we can see that if $c_2^2 - s_2^2 \neq 0$, then
\[
\sum_{i=1}^{2} \langle P^i \rangle n_i^A M(1, 2, \mathcal{P}) = 0.
\] (20)

This result implies that $M(1, 2, \mathcal{P})$ is proportional to the delta function
\[
\delta^2(\eta) = \prod_{A=1}^{2} \sum_{i=1}^{2} \langle P^i \rangle n_i^A.
\] (21)

We then write $M(1, 2, \mathcal{P})$ as
\[
M(1, 2, \mathcal{P}) = \left[ \prod_{A=1}^{2} \sum_{i=1}^{2} \langle P^i \rangle n_i^A \right] M_{res}.
\] (22)

We now want to see what constraints SUSY places on $M_{res}$. Since the calculations of $M_{res}$ later in this section will be on the support of the delta function $\delta^2(\eta)$, we introduce a notation "$\delta = \cdot \delta^2(\eta)$", and denote $A$ is equal to $B$ on the support of the delta function as $A = \delta B$. In other words,
\[
A = B \iff \delta(2) A = \delta(2) B.
\] (23)

To see what constraints SUSY places on $M_{res}$, first observe that when one imposes momentum conservation $p_1 + p_2 + P = 0$, the following two commutators vanishes
\[
\left[ Q_i^A, \prod_{A=1}^{2} \sum_{i=1}^{2} \langle P^i \rangle n_i^A \right] = 0, \quad \left[ \tilde{Q}_{iaA}, \prod_{A=1}^{2} \sum_{i=1}^{2} \langle P^i \rangle n_i^A \right] = 0.
\] (24)

Therefore, according to (13), $M_{res}$ should satisfy
\[
\left[ \prod_{A=1}^{2} \sum_{i=1}^{2} \langle P^i \rangle n_i^A, Q_i^A M_{res} \right] = 0; \quad \left[ \prod_{A=1}^{2} \sum_{i=1}^{2} \langle P^i \rangle n_i^A, \tilde{Q}_{iaA} M_{res} \right] = 0.
\] (25)

Let’s introduce two reference spinors $\langle \zeta \rangle$ and $[\xi]$, which are not parallel to $\langle P \rangle$ and $[P]$
\[
\langle \zeta P \rangle \neq 0; \quad [\xi P] \neq 0.
\] (26)

To simplify notations, let’s further define
\[
\begin{align*}
\langle \zeta D^A \rangle_+ &\equiv \sum_{i=1}^{2} \langle \zeta^i \rangle c_i \eta_d^A & \Rightarrow \langle \zeta D^A \rangle = \langle \zeta D^A \rangle_+ + \langle \zeta D^A \rangle_-

\langle \xi \tilde{D}_A \rangle_+ &\equiv \sum_{i=1}^{2} \langle \xi^i \rangle (-s_i) \sigma_i \eta_d A & \Rightarrow [\xi D_A] = [\xi \tilde{D}_A]_+ + [\xi \tilde{D}_A]_-
\end{align*}
\] (27)

\(^4\)In the rest of this paper, $\langle \zeta \rangle$ and $[\xi]$ stand for reference spinors, and they are not parallel to $\langle P \rangle$ and $[P]$. 

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and we have

\[
\langle \zeta \mathcal{D}^\Lambda \rangle \mathcal{M}_{\text{res}} + \langle \zeta P \rangle \eta^A_{\text{P}} \mathcal{M}_{\text{res}} = 0
\]
\[
[\zeta \mathcal{D}_A] \mathcal{M}_{\text{res}} + [\zeta P] \frac{\partial}{\partial \eta^A_{\text{P}}} \mathcal{M}_{\text{res}} = 0.
\]

(28)

The plus and minus signs in the subscripts are indicating that they raise/lower the \(\eta\)-order.

2.3. \(\mathcal{H}\) and \(\bar{\mathcal{H}}\)

On the delta function, we have (simply because they are proportional to \(\delta^A\))

\[
\langle P | \sum_{i=1}^{2} |\bar{\lambda}^i_I \eta^A_{iI}\rangle_{iI} \delta^A = 0 ; \quad [P | \sum_{i=1}^{2} |\bar{\lambda}^i_I \eta^A_{iI}\rangle_{iI} \delta^A = 0 ,
\]

(29)

therefore,

\[
\sum_{i=1}^{2} |\bar{\lambda}^i_I \eta^A_{iI}\rangle_{iI} \propto |P \rangle ; \quad \sum_{i=1}^{2} |\bar{\lambda}^i_I \eta^A_{iI}\rangle_{iI} \sigma \eta^A_{iI} \propto |P \rangle.
\]

(30)

Let’s define the proportionality coefficient to be \(\mathcal{H}\) and \(\bar{\mathcal{H}}\)

\[
\mathcal{H}^A \equiv \frac{1}{\langle \zeta P \rangle} \sum_{i=1}^{2} \langle \zeta \bar{\lambda}^i_I \eta^A_{iI}\rangle_{iI} ,
\]
\[
\bar{\mathcal{H}}^A \equiv \frac{1}{[\zeta P]} \sum_{i=1}^{2} [\zeta \bar{\lambda}^i_I \eta^A_{iI}\rangle_{iI} .
\]

(31)

\(\mathcal{H}\) and \(\bar{\mathcal{H}}\) are independent to those reference spinors when evaluated on the support of the delta function. For more details about \(\mathcal{H}\) and \(\bar{\mathcal{H}}\), see [Appendix C].

In [Appendix D] we show that \(\mathcal{M}_{\text{res}}\) is a function of \(\mathcal{H}\) and \(\bar{\mathcal{H}}\) only. More precisely, all \(\eta_i\) (\(\eta_i = \eta_1\) or \(\eta_2\)) in \(\mathcal{M}_{\text{res}}\) must be of the form

\[
\sum_{i=1}^{2} \bar{\lambda}^i_I \eta^A_{iI} = \lambda_P \mathcal{H}^A \quad \text{or} \quad \sum_{i=1}^{2} \bar{\lambda}^i_I \eta^A_{iI} = \lambda_P \bar{\mathcal{H}}^A .
\]

(32)

\(\delta^A, \mathcal{H}^A, \bar{\mathcal{H}}^A\) play central roles in obtaining the superamplitude. In fact, if a superamplitude satisfies (13), then all its \(\eta_i\)’s (massive Grassmann variables) will appear in the form of \(\delta^A, \mathcal{H}^A, \bar{\mathcal{H}}^A\) (proven in [Appendix D]). Therefore, \(\delta^A, \mathcal{H}^A, \bar{\mathcal{H}}^A\) serve as “building blocks” of the superamplitude. To simplify notations, we also define

\[
(\mathcal{H} \cdot \mathcal{H}) \equiv \frac{1}{2} \mathcal{H}^A \mathcal{H}_A ; \quad (\bar{\mathcal{H}} \cdot \bar{\mathcal{H}}) \equiv \frac{1}{2} \bar{\mathcal{H}}^A \bar{\mathcal{H}}_A ; \quad (\mathcal{H} \cdot \bar{\mathcal{H}}) \equiv \frac{1}{2} \mathcal{H}^A \bar{\mathcal{H}}_A .
\]

(33)

\(^{\text{In this paper, the order of } \eta_i \text{ means the order of the sum of } \eta_1 \text{ and } \eta_2. \text{ For instance, } \eta^A_1 \eta^A_2 \text{ has order } 2, \eta^A_1 \eta^A_2 \eta^A_{12} \eta_{12} \text{ has order } 3. \text{ Note that } \eta^A_1 \eta^A_2 \eta^A_3 \text{ has order } 2, \text{ since } \eta^A_1 \text{ doesn’t increase the order of } \eta_i.\)}

7
Raising and lowering $\eta$ order.

For an arbitrary quantity $X$, we can raise its $\eta_i$-order by acting $\mathcal{D}_+^A$ and $\tilde{\mathcal{D}}_{+A}$ (see (27) for their definitions)

$$
\langle \zeta \mathcal{D}^A \rangle_X = \langle \zeta P \rangle \cdot c_H X
$$
$$
[\zeta \tilde{\mathcal{D}}_A]_X = [\zeta P] \cdot (s_c) \tilde{H}_A X.
$$

On the other hand, if a quantity meets $\mathcal{D}_{-A}$ or $\tilde{\mathcal{D}}_{-A}$, the $\eta_i$-order is lowered,

$$
\left\{ \begin{align*}
\langle \zeta \mathcal{D}^A \rangle \cdot \mathcal{H}^B &= 0 \\
[\zeta \tilde{\mathcal{D}}_A] \cdot \mathcal{H}^B &= [\zeta P] \cdot (s_c) \delta_A^B \\
\langle \zeta \mathcal{D}^A \rangle \cdot \tilde{\mathcal{H}}_B &= \langle \zeta P \rangle \cdot s_c \delta^A_B \\
[\zeta \tilde{\mathcal{D}}_A] \cdot \tilde{\mathcal{H}}_B &= 0
\end{align*} \right.
$$

3. Solutions to $\mathcal{N} = 2$

In this section, we will consider $\mathcal{N} = 2$ super symmetry, and look for the explicit solution of $\mathcal{A}$ introduced in (5) (which is LG neutral and SUSY invariant), using the building blocks introduced in section 2. We will also consider spinning multiplets, and show that they can always be written in the form of (5).

The massless super field with vacuum of helicity $h$ in $\mathcal{N} = 2$ SUSY is

$$
\Psi^{+h} = |h\rangle + \eta^A |h - \frac{1}{2}\rangle^A + \eta^A \eta_A |h - 1\rangle.
$$

On the other hand, the scalar massive multiplet in $\mathcal{N} = 2$ SUSY is [5]

$$
\Phi = \phi + \eta^A \psi^A + \frac{1}{2} \eta^A \eta_J \phi^{(IJ)} + \frac{1}{2} \eta^A \eta^B \phi_{(AB)} + \frac{1}{3} \eta^A \eta_J \eta^B \psi^{I}_B + \frac{1}{12} \eta^A \eta^B \eta_J \eta^I \phi.
$$

Note that in the above equations, $I, J$ stand for LG indices, while $A, B$ stand for R-charge indices. From the above expansion, we can see that there are 5 spin-0 components ($1 \phi$, $1 \tilde{\phi}$, and $3 \phi_{(AB)}$), 4 spin-$\frac{1}{2}$ components ($2 \psi^A$, and $2 \tilde{\psi}^A$), and 1 spin-1 component ($\phi^{(IJ)}$). Since a spin-$s$ particle carries $(2s + 1)$ degree of freedom, our massive multiplet has 8 bosonic d.o.f and 8 fermionic d.o.f.

3.1. The SUSY invariant quantity $\mathcal{A}$

$\mathcal{A}$ is the quantity that satisfies

1. invariant under all LG transformations of the external particles
2. invariant under SUSY

Since $\mathcal{A}$ is annihilated by the super charges, all the discussions about $\mathcal{M}(1, 2, P)$ in section 2 applies to $\mathcal{A}$ as well. According to (22),

$$
\mathcal{A} = \delta^1 \delta^2 \mathcal{A}_{res}.
$$
where $\mathcal{A}_{res}$ is a function of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ (just like $\mathcal{M}_{res}$ is a function of $\mathcal{H}$ and $\tilde{\mathcal{H}}$). Let’s expand $\mathcal{A}_{res}$ in $\eta_P$

$$\mathcal{A}_{res} \equiv f + \eta^A B + \frac{1}{2} \eta^A C \eta_{PA} h,$$  \hspace{1cm} (39)

where $f$, $g$, $h$ are all functions of $\mathcal{H}$ and $\tilde{\mathcal{H}}$. According to (28), $f$, $g$, and $h$ should satisfy

$$
\begin{align*}
\mathcal{D}_{BA} g_b &\equiv 0 \\
\mathcal{D}_{BA} h &\equiv 0 \\
\mathcal{D}_{BA} h &\equiv 0
\end{align*}
$$

To get the full superamplitude, let’s first solve $h$, which should satisfy $\tilde{\mathcal{D}}_{BA} h = 0$. We can do this by first writing down all possible combinations of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ at each $\eta_i$ order, and fix their coefficients by demanding that $h$ satisfies $\tilde{\mathcal{D}}_{BA} h = 0$. After solving $h$, we can obtain $g_B$ and $f$ by resorting to (40).

Since the R-charge indices should be fully contracted, the $\eta_i$ orders of the terms in $h$ should be even numbers. Otherwise, there will be at least one $\eta_i$ that cannot find a partner to contract with. At each even orders of $\eta_i$, the possible terms are shown, respectively,

$$
\begin{align*}
\eta^{(0)} & : 1 \\
\eta^{(2)} & : (\mathcal{H} \cdot \mathcal{H}), (\tilde{\mathcal{H}} \cdot \tilde{\mathcal{H}}), (\mathcal{H} \cdot \tilde{\mathcal{H}}) \\
\eta^{(4)} & : (\mathcal{H} \cdot \mathcal{H})(\tilde{\mathcal{H}} \cdot \tilde{\mathcal{H}})
\end{align*}
$$

and the most general form of $h$ is

$$
\begin{align*}
\mathcal{h} = c_0 + c_1 (\mathcal{H} \cdot \mathcal{H}) + c_2 (\mathcal{H} \cdot \tilde{\mathcal{H}}) + c_3 (\tilde{\mathcal{H}} \cdot \tilde{\mathcal{H}}) + c_4 (\mathcal{H} \cdot \mathcal{H})(\tilde{\mathcal{H}} \cdot \tilde{\mathcal{H}}) .
\end{align*}
$$

Demanding $\tilde{D}_{BA} h = 0$, we get

$$
\begin{align*}
[\xi \tilde{\mathcal{D}}_A] \cdot h^{(0)} &= [\xi \tilde{\mathcal{D}}_A] \cdot h^{(2)} ; \\
[\xi \tilde{\mathcal{D}}_A] \cdot h^{(2)} &= [\xi \tilde{\mathcal{D}}_A] \cdot h^{(4)}
\end{align*}
$$

and resorting to (34) and (35), we arrive at the relations between $c_i$’s:

$$
\begin{align*}
c_2 &= -\frac{2 s_z c_0}{c_z^2}, \\
c_4 &= \frac{s_z^2}{c_z^2}, \\
c_1 &= 0
\end{align*}
$$

This implies there are two solutions, since according to (44), the general form (42) is decoupled into two linearly independent terms (the result should not be surprising, since there are also two solutions in the massless case),

$$
\begin{align*}
h^{(1)} &= 1 - \frac{2 s_z}{c_z} (\mathcal{H} \cdot \tilde{\mathcal{H}}) + \frac{s_z^2}{c_z^2} (\mathcal{H} \cdot \mathcal{H})(\tilde{\mathcal{H}} \cdot \tilde{\mathcal{H}}) \\
h^{(2)} &= (\tilde{\mathcal{H}} \cdot \tilde{\mathcal{H}})
\end{align*}
$$

(45)
Appendix C). point amplitudes in a Lorentz covariant and R-charge symmetric form. (For more details, see D

Our final solutions are

\[
\begin{align*}
\delta_{A}^{(1)} &= \frac{1}{c_{z}}\mathcal{H}_{A} + \frac{s_{z}}{c_{z}^{2}}(\mathcal{H} \cdot \mathcal{H})\tilde{\mathcal{H}}_{A} \\
\delta_{A}^{(2)} &= c_{z}\mathcal{H}_{A}(\mathcal{H} \cdot \tilde{\mathcal{H}}) + s_{z}\tilde{\mathcal{H}}_{A}.
\end{align*}
\]

(46)

Last, substitute (46) into \(\mathcal{D}_{A}g_{c} - \delta C_{A}P_{a}f = 0\), we obtain the \(f\) part of each solution

\[
\begin{align*}
f^{(1)} &= \frac{1}{c_{z}^{2}}(\mathcal{H} \cdot \mathcal{H}) \\
f^{(2)} &= s_{z}^{2} + 2s_{z}c_{z}(\mathcal{H} \cdot \tilde{\mathcal{H}}) + c_{z}^{2}(\mathcal{H} \cdot \mathcal{H})(\tilde{\mathcal{H}} \cdot \tilde{\mathcal{H}}).
\end{align*}
\]

(47)

Our final solutions are

\[
S^{(a)}(\theta_{z}) = C^{(a)} \cdot \delta^{A} \delta^{2}\{f^{(a)} + \eta_{P}^{A}g_{c}^{(a)} + \frac{1}{2}\eta_{P}^{A}\eta_{PA}h^{(a)}\},
\]

(49)

where \(a = 1, 2\), labeling the two solutions, and \(C^{(1)} = c_{z}^{2}, C^{(2)} = -s_{z}\), so as to make both solutions LG neutral \(\mathrm{F}\) (and also make \(S^{(1)}\) and \(S^{(2)}\) look more "symmetrical"). The solutions depend on \(c_{z}\) and \(s_{z}\), which are functions of \(Z\).

Up to this point, we have two solutions to \(\mathcal{A}\). However, they are written in terms of \(\mathcal{H}_{A}\) and \(\tilde{\mathcal{H}}_{A}\), which are not manifestly Lorentz covariant quantities themselves, see (31). Only when they are combined with the delta functions \(\delta^{A}\) can they be represented in a Lorentz covariant form. Let’s first define \(Q_{A}^{I}, \tilde{Q}_{A}^{I}, P_{a}\) and \(\tilde{P}_{a}\):

\[
\begin{align*}
Q_{A}^{I} &\equiv \lambda_{1a}\eta_{I1}^{A} + \lambda_{2a}\eta_{I2}^{A} ; \quad \tilde{Q}_{A}^{I} \equiv \tilde{\lambda}_{1a}\eta_{I1}^{A} - \tilde{\lambda}_{2a}\eta_{I2}^{A} \\
P_{a} &\equiv \lambda_{PA}\eta_{P}^{A} ; \quad \tilde{P}_{a} \equiv \tilde{\lambda}_{PA}\eta_{P}^{A} \\
\eta_{I} \cdot \eta_{P} &\equiv \frac{1}{2}e^{IJ}\eta_{IJ}^{A} \quad \eta_{P} \cdot \eta_{PA} \equiv \frac{1}{2}\eta_{PA}^{A}.
\end{align*}
\]

(49)

By combining \(f^{(a)}, g_{c}^{(a)},\) and \(h^{(a)}\) with the delta functions, we can finally write down the three point amplitudes in a Lorentz covariant and R-charge symmetric form. (For more details, see Appendix C).

---

\(^{6}\)One may wonder why adding x-factors is the only way to change the helicity of vacuum state of the massless multiplet. There are after all several quantities that carries helicity, including \(\lambda_{PA}\) and \(\tilde{\lambda}_{PA}\), and of course the x-factor. However, both \(\lambda_{PA}\) and \(\tilde{\lambda}_{PA}\) carry an extra \(SL(2, C)\) index, i.e., \(a\) or \(\tilde{a}\), and should be contracted with \(\lambda_{1a}\) or \(\tilde{\lambda}_{2a}\) \((i = 1, 2)\). But all \(\lambda_{1a}\) or \(\lambda_{2a}\) dependence are encoded in \(S^{(1)}\) and \(S^{(2)}\), and their \(SL(2, C)\) indices are already contracted. As a result, x-factors is the only choice remains.

\(^{7}\)Note that \(Q_{A}^{a}\) and \(\tilde{Q}_{aA}\) (normal character Q) stand for generators, while \(Q_{A}^{A}\) and \(\tilde{Q}_{aA}\) (curly character \(Q\)) stand for the little group covariant supersymmetric components of the solutions.
The first solution is

\[
S^{(1)}(\theta_2) = c^2 \cdot \delta^2 \left[ \frac{1}{2} \eta^A \eta^B \mathbf{h}^{(1)} + \eta^A g^A + f^{(1)} \right]
\]

\[
= \frac{c^2}{4} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle + \frac{c_s z}{6x} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle (\eta_P \cdot \eta_P)
\]

\[
+ \frac{s_c^2}{12 x^2} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle \left( (\eta_1^A \cdot \eta_1^B) - (\eta_2^A \cdot \eta_2^B) \right) ((\eta_1C \cdot \eta_1D) - (\eta_2C \cdot \eta_2D))
\]

\[
+ \frac{c^2}{3} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle - \frac{2s_c}{9x} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle (\eta_1B \cdot \eta_1C) - (\eta_2B \cdot \eta_2C))
\]

\[
+ \frac{1}{12} \langle Q^A Q^B \rangle \langle Q_a Q_B \rangle ,
\]

and the second solution is

\[
S^{(2)}(\theta_2) = -x \cdot \delta^2 \left[ \frac{1}{2} \eta^A \eta^B \mathbf{h}^{(2)} + \eta^A g^A + f^{(2)} \right]
\]

\[
= \frac{1}{12x} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle (\eta_P \cdot \eta_P) - \frac{c_s}{9x} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle ((\eta_1B \cdot \eta_1C) - (\eta_2B \cdot \eta_2C))
\]

\[
+ \frac{s_c^2}{3} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle + \frac{s_c^2}{6} \langle Q^A Q^B \rangle \langle Q_a Q_{P_B} \rangle (\eta_1^A \cdot \eta_1^B) - (\eta_2^A \cdot \eta_2^B)
\]

\[
+ \langle Q^A P \rangle \langle Q_a P \rangle (\eta_1^A \cdot \eta_1^B) - (\eta_2^A \cdot \eta_2^B))
\]

(50)

Note that both of the solutions are inhomogeneous in Grassmann degree. The first term in \(S^{(1)}(\theta_2)\) has Grassmann degree 4, followed by terms with Grassmann degree 6,8,4,6,4. Similar thing happens to \(S^{(2)}(\theta_2)\).

The two solutions are related. If we do Grassmann Fourier transform (defined in [4]) to either of the solutions, and change all angle brackets to square brackets (and vice versa), we get the other solution (up to an overall constant). This is in fact a direct consequence of the form of the generators [17]. If we change all \(\eta\)’s into \(\frac{\delta}{\delta \eta}\)’s (and vice versa) in \(Q_a\), and change all \(\lambda\)’s into \(\bar{\lambda}\)’s, we get exactly \(\bar{Q}_a\). We can do the same transformation to \(\bar{Q}_a\) to get \(Q_a\). Since the \(\mathcal{A}\) is invariant under transformation generated by both \(Q_a\) and \(\bar{Q}_a\), if the generators are symmetric under certain transformation, the solutions would inherit this property, and come in pairs consequently. The symmetry of the generators is the origin of why the solutions must come in pairs.

Observe that the solution (50) can be factorized into a product of two components

\[
S^{(1)}(\theta_2) = \left( c \eta_P^A \delta^1 - \delta^A \mathcal{H}_1 - c_s \eta_P^A \delta^1 \bar{\mathcal{H}}_2 \right) \cdot \left( c \eta_P^A \delta^2 - \delta^A \mathcal{H}_2 - c_s \eta_P^A \delta^1 \bar{\mathcal{H}}_1 \right).
\]

(52)

The solution (51) can be factorized in a similar expression

\[
S^{(2)}(\theta_2) = -x \cdot \left( c \delta^A \mathcal{H}_2 + \delta^A \mathcal{H}_2 \right) \cdot \left( c \delta^1 \mathcal{H}_1 + \delta^1 \mathcal{H}_1 \right)
\]

(53)
This observation will later play a crucial role in the discussion of BPS limit.

To sum up, $\mathcal{A}$ is the linear combination of the two solutions

$$\mathcal{A}(Z, m) = \alpha_1 \cdot S^{(1)}(\theta_Z) + \alpha_2 \cdot S^{(2)}(\theta_Z).$$ (54)

Note that all $\eta$ dependencies are encoded in $S^{(1)}$ and $S^{(2)}$. There are two unfixed coefficients $\alpha_1$ and $\alpha_2$, which can be fixed by requiring parity symmetry. We will go back to this later.

### 3.2. Spinning multiplets

A 3-pt amplitude of two equal massive particles with mass $m$ and a massless particle with helicity $h$ can be written in spinor helicity basis, see \[1\],

$$M^{(I_1 \cdots I_{2s_1}) (J_1 \cdots J_{2s_2}) h} = (\lambda_1)_\alpha^{I_1} \cdots (\lambda_1)_\alpha^{I_{2s_1}} (\lambda_2)_\beta^{J_1} \cdots (\lambda_2)_\beta^{J_{2s_2}} M^{(\alpha_1 \cdots -\alpha_{2s_1}) (\beta_1 \cdots -\beta_{2s_2}) h},$$ (55)

where

$$M^{(\alpha_1 \cdots -\alpha_{2s_1}) (\beta_1 \cdots -\beta_{2s_2}) h} = \sum_{i=|s_1-s_2|}^{s_1+s_2} g_i \lambda_i^h \left[ \lambda_i^h \left( \frac{p_i \cdot \bar{\lambda}_i}{m} \right)^i \right]^{(\alpha_1 \cdots -\alpha_{2s_1}) (\beta_1 \cdots -\beta_{2s_2})}. $$ (56)

The amplitude for spinning multiplets can always be factorized into a product of a bosonic part $M(1^{(I_1 \cdots I_{2s_1})}, 2^{(J_1 \cdots J_{2s_2})}, P^h)$ and $\mathcal{A}$,

$$M_{Z}(1^{(I_1 \cdots I_{2s_1})}, 2^{(J_1 \cdots J_{2s_2})}, P^h) = M(1^{(I_1 \cdots I_{2s_1})}, 2^{(J_1 \cdots J_{2s_2})}, P^h) \cdot \left[ \alpha_1 \cdot S^{(1)}(\theta_Z) + \alpha_2 \cdot S^{(2)}(\theta_Z) \right].$$ (57)

In other words, if a spinor helicity variable that carries LG indices of the massive multiplets (e.g., $I_1 \cdots I_{2s_1}$) have their $SL(2, C)$ indices (e.g., $\alpha$ in $\lambda_1^\alpha$) contract with a spinor helicity variable whose LG indices contract with a Grassmann variable $\eta_i^\alpha$ (we say the amplitude is "polluted"), then we can always use Schouten identity to reorder the $SL(2, C)$ indices to make the amplitude "unpolluted". More explicitly, we can always write the amplitude without terms such as $(1^I 1^I) \eta_i^\alpha$. The reason is if such terms do exist, then according to the discussions in [Appendix D], they must come in combinations $\lambda_i^I \eta_i^A + \lambda_{2\alpha}^I \eta_{2\alpha}^A$ or $\tilde{\lambda}_i^I \eta_i^A - \tilde{\lambda}_{2\alpha}^I \eta_{2\alpha}^A$, and can be re-written in terms of $\mathcal{H}$ or $\tilde{\mathcal{H}}$. For example, if $(1^I 1^I) \eta_i^A$ do exist, then it must appear in the form of $\sum_i (1^I 1^I) \eta_i^A$; but according to (31), $\sum_i (1^I 1^I) \eta_i^A = (1^I P \gamma ) \mathcal{H}^A$, and thus make the term "unpolluted".

**Parity.**

Parity symmetry relates each amplitude and its conjugate amplitude, demanding they have the same couplings, and that constraints the coefficients $\alpha_1$ and $\alpha_2$ in (57). In $N = 2$ supersymmetry, the massless multiplets with helicity $+h$ and $-h+1$ are

$$\mathcal{P}^{+h} = |h\rangle + \eta^A |h - \frac{1}{2}\rangle_A + \eta^A |h - 1\rangle$$

$$\mathcal{P}^{-h+1} = |-h+1\rangle + \eta^A | -h + \frac{1}{2}\rangle_A + \eta^A | -h\rangle.$$ (58)
We can see the components are related by parity transformation, and parity invariance requires the coefficients of the superamplitudes with (+h)-helicity and (−h + 1)-helicity multiplet should be related. Let’s first write down the form of the superamplitudes

\[
M_{2}(1^{(11-12)}, 2^{(11-12)}, \mathcal{P}^{h}) = M(1^{(11-12)}, 2^{(11-12)}, \mathcal{P}) \left[ \alpha_1 \cdot S^{(1)} + \alpha_2 \cdot S^{(2)} \right]
\]

\[
M_{2}(1^{(11-12)}, 2^{(11-12)}, \mathcal{P}^{−h+1}) = M(1^{(11-12)}, 2^{(11-12)}, \mathcal{P}^{−h+1}) \left[ \beta_1 \cdot S^{(1)} + \beta_2 \cdot S^{(2)} \right],
\]

where

\[
M(1^{(11-12)}, 2^{(11-12)}, \mathcal{P}) = g_0 x^2 \prod_{i,j=1}^{2s} \langle 1^{(i)} 2^{(j)} \rangle + g_1 x M \prod_{i,j=1}^{2s} \langle 1^{(i)} 2^{(j)} \rangle \langle 1^{2s} \rangle \langle 2^{2s} \rangle P
\]

\[
+ g_2 \frac{1}{m^2} \prod_{i,j=1}^{2s-2} \langle 1^{(i)} 2^{(j)} \rangle \prod_{i,j=2s-1}^{2s} \langle 1^{(i)} \rangle \langle 2^{(j)} \rangle + \ldots
\]

\[
\bar{M}(1^{(11-12)}, 2^{(11-12)}, \mathcal{P}) = \bar{g}_0 \prod_{i,j=1}^{2s} [1^{(i)} 2^{(j)}] + \bar{g}_1 \frac{1}{x m} \prod_{i,j=1}^{2s} [1^{(i)} 2^{(j)}] [1^{2s} \rangle \langle 2^{2s} \rangle P
\]

\[
+ \bar{g}_2 \frac{1}{x^2 m^2} \prod_{i,j=1}^{2s-2} [1^{(i)} 2^{(j)}] \prod_{i,j=2s-1}^{2s} [1^{(i)} \rangle \langle 2^{(j)} \rangle P + \ldots.
\]

Parity invariance requires \((\beta_1, \beta_2) = (\alpha_2, \alpha_1)\) and \(g_i = \bar{g}_i\).

4. Limits of the amplitude

In this section, the \(Z = 0\) limit and the BPS limit of \(A\) are examined. The former is a product of two \(\mathcal{N} = 1\) SUSY invariant delta functions, and the later is a product of two components that are related to each other by Grassmann Fourier transformation.

4.1. The \(Z = 0\) limit

When \(Z_1 = Z_2 = 0, \theta_2 \rightarrow 0\), and the solution takes the form (see (52))

\[
S^{(1)} = \prod_{A=1}^{2} \left[ \langle 1^{i} 2^{j} \rangle \eta_{ij}^{A} \eta_{ij}^{A} + 2 \sum_{i=1}^{2} \langle 1^{i} \rangle \eta_{ij}^{A} \eta_{ij}^{A} + 2 m(\eta_{ij}^{A} \cdot \eta_{ij}^{A}) \right],
\]

where we defined

\[
(\eta_{ij}^{A} \cdot \eta_{ij}^{A}) = -\frac{1}{2} \epsilon^{ij} \eta_{ij}^{A} \eta_{ij}^{A} \quad \text{(no sum over} \ i \ \text{and} \ A) .
\]

This is exactly the square of \(\mathcal{N} = 1\) SUSY invariant delta function defined in [4], which is not surprising, since we can see from (17) that \(Q^{A}_{ij}\) is purely multiplicative in \(\eta\)’s, while \(Q_{ij}^{A}\) is purely differential in \(\eta\)’s, and there is no mixing terms between two R-charge indices. Therefore, the solution being multiplicative of two solutions corresponding to two R-charges is an expected result.
The other solution is

\[
S^{(2)} = \prod_{A=1}^{2} \left[ [1^I 2^J]_{\eta^A_1, \eta^A_2, \eta^A_6} + [1^I P]_{\eta^A_1} (\eta^A_2 \cdot \eta^A_6) + [2^I P]_{\eta^A_2} (\eta^A_1 \cdot \eta^A_6) + m \eta^A_P (\eta^A_1 \cdot \eta^A_2) \right],
\]

(63)

which is related to (61) by Grassmann Fourier transformation.

In the high energy limit, we can take \( \eta^A_1 \to \eta^A_i, \eta^A_2 \to \tilde{\eta}^A_i \), and the above amplitudes have the limits\(^8\) (see \([1,4]\))

\[
S^{(1)} \to \prod_{A=1}^{2} \left[ \langle 12 \rangle \eta^A_1 \eta^A_2 + \sum_{i=1}^{2} \langle iP \rangle \eta^A_P \right]
\]

\[
S^{(2)} \to \prod_{A=1}^{2} \left[ (12) \eta^A_P + [P1] \eta^A_2 + [2P] \eta^A_1 \right] \tilde{\eta}^A_1 \tilde{\eta}^A_2.
\]

(64)

We can see they are equal to the MHV and anti-MHV amplitudes in \([2]\).

4.2. The BPS limit

When the massive particles saturate the BPS limit, i.e., \( Z_1 = Z_2 = 2m, \theta_Z \to \pi/4 \), and (17) becomes

\[
\mathcal{Q}^A = \sum_{i=1}^{2} \lambda^I_{\alpha A} \left( \frac{1}{\sqrt{2}} \sigma^I_{\alpha} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial \eta^A_I} \right) + \lambda^A_{\alpha A} \eta^A_P
\]

\[
\tilde{\mathcal{Q}}_{\alpha A} = \sum_{i=1}^{2} \tilde{\lambda}^I_{\alpha A} \left( \frac{1}{\sqrt{2}} \frac{\partial}{\partial \eta^A_I} - \frac{1}{\sqrt{2}} \sigma^I_{\alpha} \eta^A_I \right) + \tilde{\lambda}^A_{\alpha A} \frac{\partial}{\partial \eta^A_P}.
\]

(65)

Although the anti-commutation relations (11) still hold, (54) is no longer valid. To see this, note that \( \mathcal{A} \) is proportional to the delta function (21) if and only if \( c_5^2 - s_5^2 = 0 \) (See subsection 2.2), and the BPS limit doesn’t meet this criteria. As a result, our solutions (54) do not apply to the BPS limit. However, in the BPS limit, the \( \eta^A_i \) degree of freedom is half the number of that in the non-BPS limit. Therefore, to obtain the amplitude of the BPS limit, we shouldn’t have started with (17), but rather one of the following set (either of them is sufficed to serve as a set of generators in the BPS limit),

\[
\begin{align*}
Q_a &= \sum_{i=1}^{2} \lambda^I_{\alpha A} \eta^A_1 + \lambda^A_{\alpha A} \eta^A_P \\
Q_a &= \sum_{i=1}^{2} \sigma^I_{\alpha} \eta^A_1 \frac{\partial}{\partial \eta^A_1} + \lambda^A_{\alpha A} \eta^A_P \\
\tilde{Q}_{\alpha A} &= \sum_{i=1}^{2} \tilde{\lambda}^I_{\alpha A} \eta^A_1 + \tilde{\lambda}^A_{\alpha A} \eta^A_P \\
\tilde{Q}_{\alpha A} &= \sum_{i=1}^{2} \sigma^I_{\alpha} \tilde{\lambda}^A_{\alpha A} \eta^A_1 + \tilde{\lambda}^A_{\alpha A} \eta^A_P
\end{align*}
\]

(66)

\(^8\)In the massless limit, the number of supercharges is reduced in half, resulting residual Grassmann variables \( \tilde{\eta}^A_i \). Those residual Grassmann variables divide the massive superfield into different massless representations \([4]\).
Although (50) and (51) are not the solutions for BPS limit, interesting things happen if we take \( \theta_2 = \pi/4 \). In this limit, the two factorized components in (52) are related. If we do Grassmann Fourier transformation to one of the components, then complex conjugate spinor helicity variables \((\lambda \leftrightarrow \bar{\lambda})\), and transform \( \eta_P \) properly, it will become the other component,

\[
\left( \frac{1}{\sqrt{2}} \eta_p \delta^2 - \delta^2 \mathcal{H}^2 + \frac{1}{2} \delta^2 \mathcal{H}_1 \eta_P \bar{\eta}_P + \frac{1}{\sqrt{2}} \delta^2 \mathcal{H} \bar{\mathcal{H}}_1 \eta_P \bar{\eta}_P \right)_{\lambda \rightarrow \bar{\lambda}, \eta \rightarrow \bar{\eta}} \nonumber \
= \text{FT} \left[ \left( \frac{1}{\sqrt{2}} \eta_p \delta^1 - \delta^1 \mathcal{H}^1 + \frac{1}{2} \delta^1 \mathcal{H}_2 \eta_P \bar{\eta}_P + \frac{1}{\sqrt{2}} \delta^1 \mathcal{H} \bar{\mathcal{H}}_2 \eta_P \bar{\eta}_P \right)_{\eta_P \rightarrow \sqrt{2} \eta_P} \right],
\]

where

\[
\text{FT} \left[ f(\eta) \right] \equiv \frac{1}{8} \left[ \prod_{i=1}^{2} \int d\eta^1_i d\eta^1_i \exp \delta^1_i \eta^1_i \right] \left[ \int d\eta^A_i d\eta^A_i \exp \bar{\eta}^A_i \eta^A_i \right] f(\eta).
\]

By comparing with (13,5), we see that \( \eta_{il}^2 \) act as the barred Grassmann number of \( \eta_{il}^1 \), and serve as another \( \eta \)-basis. In other words, (50) has unnecessary Grassmann variables in the BPS limit. We can use either of the factorized components in (52) to describe a BPS amplitude.

Let’s see how the components in (52) are related to BPS amplitudes. If we rename the \( \eta_{il} \) in the LHS of (66) as \( \eta_{il}' \), and the \( \eta_{il} \) in the RHS as \( \eta_{il}'' \), then the SUSY invariant amplitudes are (the first of them is the solution of the LHS set, while the second is that of the RHS set) [4],

\[
\mathcal{A}_{\text{BPS}}^1 = \eta_{il}^1 \delta^1 - \delta^1 \mathcal{H}^1 - \delta^1 \mathcal{H}_2 \eta_P \bar{\eta}_P + \delta^1 \mathcal{H} \bar{\mathcal{H}}_2 \eta_P \bar{\eta}_P
\]

\[
\mathcal{A}_{\text{BPS}}^2 = \eta_{il}'' \delta^2 - \delta^2 \mathcal{H}^2 + \delta^2 \mathcal{H}_2 \eta_P \bar{\eta}_P + \delta^2 \mathcal{H} \bar{\mathcal{H}}_2 \eta_P \bar{\eta}_P,
\]

where the definition of \( \delta^A, \mathcal{H}^A, \bar{\mathcal{H}}_A \) are same as the previous ones[3]. Note that the index \( A \) in the previous non-BPS calculations stands for R-charge, while in (69), \( A \) is a label, labeling the two solutions. More concretely, \( \eta_{il}' \) and \( \eta_{il}'' \) are more like a conjugate pair in (69). Despite the meaning of \( A \) in BPS limit departs radically from non-BPS case, the solutions in (69) can be related to non-BPS solution. The product of the two solutions in (69) is

\[
\mathcal{A}_{\text{BPS}}^1 \times \mathcal{A}_{\text{BPS}}^2 = \eta_{il}^1 \delta^1 \eta_{il}'' \delta^2 + 2 \delta^1 \eta_{il}^1 \eta_{il}'' \mathcal{H} \cdot \mathcal{H} - \delta^1 \delta^2 \eta_{il}^1 \eta_{il}'' \mathcal{H} \cdot \mathcal{H} + \delta^1 \delta^2 \mathcal{H} \cdot \mathcal{H} + \delta^1 \delta^2 \mathcal{H} \cdot \mathcal{H}.
\]

\[
\text{Note that in [4], the amplitude reads}
\]

\[
\mathcal{A}_{\text{BPS}} = (\delta^i \mathcal{H}^1 - \eta_P \delta^i) \cdot \exp \left( \bar{\mathcal{H}}_{2} \eta_P^2 \right) = \eta_P \delta^1 - \delta^1 \mathcal{H}^1 + \delta^1 \mathcal{H}_2 \eta_P \bar{\eta}_P + \delta^1 \mathcal{H} \bar{\mathcal{H}}_2 \eta_P \bar{\eta}_P.
\]
If we replace all $\eta^A_P$ replaced by $\frac{1}{\sqrt{2}} \eta^A_P$, it will be equal to $S^{(1)}(\theta_Z = \frac{\pi}{4})$,

$$\mathcal{A}^1_{\text{BPS}} \times \mathcal{A}^2_{\text{BPS}}\big|_{\eta^A_P \rightarrow \frac{1}{\sqrt{2}} \eta^A_P} = S^{(1)}(\theta_Z).$$  \hspace{1cm} (73)

One may wonder why there should be a $\frac{1}{\sqrt{2}} \eta_P$ in (73), and we can trace its origin from the generators (17). In the limit $\theta_Z = \frac{\pi}{4}$, if we sum the two sets of generators $Q^A_\alpha$ in (66), and modify $\eta_P$ by $\frac{1}{\sqrt{2}} \eta_P$, it will be proportional to the generators $Q^A_\alpha$ in (17). This explains why there should be a $\frac{1}{\sqrt{2}}$ factor. One may argue that once we make $\eta_P \rightarrow \frac{1}{\sqrt{2}} \eta_P$, the differential part in $\tilde{Q}_\alpha$ will acquire a $\sqrt{2}$ factor, so that the sums of $Q^A_\alpha$ in (66) are not proportional to $Q^A_\alpha$ in (17). But since non-BPS solution is the product of BPS solutions, when applying a differential operator, we would need to impose the product law of differentiation, which induces an extra factor of 2, and $\frac{1}{\sqrt{2}} \times 2 = \sqrt{2}$ explains the $\sqrt{2}$ factor.

5. $\mathcal{N} = 4$ Super-Maxwell and Supergravity

We have been working on $\mathcal{N} = 2$ SUSY amplitudes in the previous sections, and now we consider $\mathcal{N} = 4$ SUSY, where the central charge matrix can be put in the standard block-diagonal form,

$$Z_{AB} = \begin{bmatrix} 0 & -Z_{12} & 0 & 0 \\ Z_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & -Z_{34} \\ 0 & 0 & Z_{34} & 0 \end{bmatrix}.$$  \hspace{1cm} (74)

In central charge free $\mathcal{N} = 4$ SUSY, the R-symmetry group is $SU(4)$. However, central charge extension breaks the full $SU(4)$ group into $SU(2) \otimes SU(2)$ subgroup.

Since the $\mathcal{N} = 1,2$ part and the $\mathcal{N} = 3,4$ part do not mix with each other, the supersymmetric part of the amplitude is the square of $\mathcal{A}$, with $\theta_Z$’s properly modified (c.f. (57)),

$$M_{Z_{12},Z_{34}}(1^{(j_1-\ldots-j_{10})},2^{(j_1-\ldots-j_{10})},p^\parallel) = M(1^{(j_1-\ldots-j_{10})},2^{(j_1-\ldots-j_{10})},p^\parallel) \cdot \left[ \alpha_1 S^{(1)}_{12}(\theta_{Z_{12}}) + \alpha_2 S^{(2)}_{12}(\theta_{Z_{12}}) \right] \left[ \alpha_3 S^{(1)}_{34}(\theta_{Z_{34}}) + \alpha_4 S^{(2)}_{34}(\theta_{Z_{34}}) \right],$$  \hspace{1cm} (75)

where

$$2 \cos(\theta_{Z_{12}}) \sin(\theta_{Z_{12}}) = \frac{Z_{12}}{2m}; \quad 2 \cos(\theta_{Z_{34}}) \sin(\theta_{Z_{34}}) = \frac{Z_{34}}{2m}.$$  \hspace{1cm} (76)

The subscripts that $S^{(1)}_{AB}$ and $S^{(2)}_{AB}$ carries, i.e., 12 or 34, indicate which projected $SU(2)$ group they are describing.

Parity.
As we discussed in subsection 3.2, the coefficients $\alpha_n$ of different superamplitudes are related in a parity invariant theory. In $\mathcal{N} = 4$ SUSY, similar things happen. First note that the massless

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multiplet with helicity-\((+h)\) and helicity-\((-h+2)\) are related by parity

\[
\mathcal{P}^{+h} = |+h\rangle + \eta^{+} \left| h - \frac{1}{2}\right\rangle_{A'} + \eta^{+} \left| h + \frac{1}{2}\right\rangle_{A''} + \eta^{+} \eta^{+} \left| h - 1\right\rangle_{12} + \eta^{+} \eta^{+} \left| h - 1\right\rangle_{34} + \eta^{+} \eta^{+} \eta^{+} \left| h - 2\right\rangle_{A'A''}
\]

\[
\mathcal{P}^{-h+2} = |-h+2\rangle + \eta^{-} \left| h + \frac{1}{2}\right\rangle_{A'} + \eta^{-} \left| h - \frac{1}{2}\right\rangle_{A''} + \eta^{-} \eta^{-} \left| h - 1\right\rangle_{12} + \eta^{-} \eta^{-} \eta^{-} \left| h - 1\right\rangle_{34} + \eta^{-} \eta^{-} \eta^{-} \eta^{-} \left| h - 2\right\rangle_{A'A''}
\]

where \(A' = 1, 2, A'' = 3, 4\). Therefore, the amplitude with \((+h)\)-helicity multiplet and \((-h+2)\)-helicity would have their coefficients \(\alpha_s\) related. More concretely, if

\[
M_{Z_{12},Z_{34}}(1^{(j_1-j_{z1})}, 2^{(j_2-j_{z2})}, \mathcal{P}^{+h}) = M(1^{(h-j_{z1})}, 2^{(j_1-j_{z2})}, \mathcal{P}^{h}) \left[ \alpha_1 \cdot S^{(1)}_{12} + \alpha_2 \cdot S^{(1)}_{12} \right] \left[ \alpha_3 \cdot S^{(1)}_{34} + \alpha_4 \cdot S^{(1)}_{34} \right]
\]

\[
M_{Z_{12},Z_{34}}(1^{(j_1-j_{z1})}, 2^{(j_2-j_{z2})}, \mathcal{P}^{-h+2}) = M(1^{(h-j_{z1})}, 2^{(j_1-j_{z2})}, \mathcal{P}^{-h+2}) \left[ \beta_1 \cdot S^{(1)}_{12} + \beta_2 \cdot S^{(1)}_{12} \right] \left[ \beta_3 \cdot S^{(1)}_{34} + \beta_4 \cdot S^{(1)}_{34} \right],
\]

then parity invariance requires \((\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_2, \alpha_1, \alpha_4, \alpha_3)\).

If the superamplitude is self-conjugate, i.e., the massless multiplet is a spin-1 multiplet, parity invariance will require \(\alpha_1 = \alpha_2, \alpha_3 = \alpha_4\), and therefore

\[
M_{Z_{12},Z_{34}}(1^{(j_1-j_{z1})}, 2^{(j_2-j_{z2})}, \mathcal{P}^{+1}) = M(1^{(j_1-j_{z1})}, 2^{(j_2-j_{z2})}, \mathcal{P}^{1}) \cdot \left[ S^{(1)}_{12}(\theta_{Z_{12}}) + S^{(1)}_{12}(\theta_{Z_{12}}) \right] \left[ S^{(1)}_{34}(\theta_{Z_{34}}) + S^{(1)}_{34}(\theta_{Z_{34}}) \right].
\]

5.1. \(N = 4\) Super-Maxwell

If the massless multiplet carries helicity \(+1\), then the helicities of the component fields span from \(+1\) to \(-1\), which describes Super-Maxwell theory. On the other hand, if the vacuum states of the massive particles are scalars, then the component amplitudes will have massive particles’ spins up to spin 2. Demanding that parity symmetry should be preserved, see (79), we have

\[
M(\Phi, \Phi, \mathcal{P}^{+1}) = \frac{e}{m_e} \cdot \left[ S^{(1)}_{12} + S^{(2)}_{12} \right] \left[ S^{(1)}_{34} + S^{(2)}_{34} \right],
\]

where \(e\) is the electrical charge and \(S^{(1)}_{AB} \equiv S^{(1)}(\theta_{Z_{AB}})\). Note that \(\theta_{Z_{AB}}\), which are functions of \(Z_{AB}\), determines all the coefficients of the component amplitudes, including minimal and non-minimal couplings. Let’s explicitly write down the form of the component amplitude \(M(1^{v=2}, 2^{v=2}, \mathcal{P}^{+1})\) as an example, where both massive external states are spin 2, and the massless external state has
massless particle has helicity $h$, 

\[ M(1^{±2}, 2^{±2}, P^{+1}) = \frac{e}{m^2} \left\{ \left[ \left( 1 - \frac{Z_{12}}{2m} \right) \left( 1 - \frac{Z_{34}}{2m} \right) \right] x(12)^4 \right. \]
\[ \left. + \left[ \frac{Z_{12}Z_{34}}{2m^2} - \frac{Z_{12}}{2m} - \frac{Z_{34}}{2m} \right] x(12)^3 \left( \frac{1}{|P|} \right) \right. \]
\[ \left. \left. + \left[ \frac{Z_{12}Z_{34}}{4m^2} \right] x(12)^2 \left( \frac{1}{|P|} \right) \right) \right\} \]

\[ \Rightarrow (g_0, g_1, g_2) = \left( \left( 1 - \frac{Z_{12}}{2m} \right) \left( 1 - \frac{Z_{34}}{2m} \right), \frac{Z_{12}Z_{34}}{2m^2} - \frac{Z_{12}}{2m} - \frac{Z_{34}}{2m}, \frac{Z_{12}Z_{34}}{4m^2} \right) , \]

with all the LG indices symmetrized. We can see that the component amplitude consists of non-minimal couplings. In addition, the non-minimal couplings of \( M(1^{±2}, 2^{±2}, P^{+1}) \) vanish if and only if \( Z_{AB} = 0 \).

5.2. \( \mathcal{N} = 4 \) Supergravity

For \( \mathcal{N} = 4 \) supergravity, since we have only 4 \( \eta_{\mu} \)'s, we can’t obtain whole graviton spectrum (from +2 to -2) in one superamplitude. Rather, the spectrum is composed of two superamplitudes, one of which, \( \mathcal{M}(\Phi, \Phi, \mathcal{P}^{+2}) \), has its massless spectrum range from +2 to 0, and the other, \( \mathcal{M}(\Phi, \Phi, \mathcal{P}^{0}) \), has it range from 0 to -2. In general, \( \mathcal{M}(\Phi, \Phi, \mathcal{P}^{+2}) \) and \( \mathcal{M}(\Phi, \Phi, \mathcal{P}^{0}) \) can be of the form

\[ \mathcal{M}(\Phi, \Phi, \mathcal{P}^{+2}) = \frac{k}{2m^2} \left[ \alpha_1 S_{12}^{(1)} + \alpha_2 S_{12}^{(2)} \right] \left[ \alpha_3 S_{34}^{(1)} + \alpha_4 x S_{34}^{(2)} \right] \]
\[ \mathcal{M}(\Phi, \Phi, \mathcal{P}^{0}) = \frac{k}{2m^2} \left[ \beta_1 S_{12}^{(1)} + \beta_2 S_{12}^{(2)} \right] \left[ \beta_3 S_{34}^{(1)} + \beta_4 x S_{34}^{(2)} \right] . \]

We require the following in order to preserve parity symmetry, see (82).

\[ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\beta_2, \beta_1, \beta_4, \beta_3) . \]

The \( g_1 \) problem of gravitational interaction.

The most general form of the 3-pt amplitude of two massive spinning particles with equal masses \( m \) interacting with a massless particle, where the two massive particles are spin \( S_1 \) and \( S_2 \), and the massless particle has helicity \( h \), is of the form (see (11))

\[ M(1^{(l_1 \cdots l_3)}, 2^{(j_1 \cdots j_3)}, P^h) = g_0 x^{h} \prod_{i,j=1}^{2} \langle 1^{(l_i)} 2^{(j_i)} \rangle + g_1 x^{h-1} \prod_{i,j=1}^{2s-1} \langle 1^{(l_i)} 2^{(j_i)} \rangle \langle 1^{(2i)} P \rangle \langle 2^{(2i)} P \rangle \]
\[ \left. + g_2 x^{h-2} \prod_{i,j=1}^{2s-2} \langle 1^{(l_i)} 2^{(j_i)} \rangle \prod_{i,j=2s-1}^{2s} \langle 1^{(l_i)} P \rangle \langle 2^{(l_i)} P \rangle + \cdots . \right\} \]

It is shown in (7) that, if \( h = 2 \) in the above equation, i.e., 3-pt amplitude includes a graviton, then \( g_1 \)-term must vanish. Otherwise, we can’t write down a Lagrangian with local operators that give raise to the amplitude.
Let’s consider the component amplitude $M(1^{s=2}, 2^{s=2}, P^{s=2})$ (one of the component amplitudes of $M(\Phi, \Phi, P^{s=2})$), where both massive external states are spin 2, and the massless external state has helicity $+2$ (i.e. the graviton),

$$M(1^{s=2}, 2^{s=2}, P^{s=2}) = \frac{k}{2m^2}\left\{\alpha_1\alpha_3 \left(1 - \frac{\alpha_2 Z_{12}}{\alpha_1 2m}\right)\left(1 - \frac{\alpha_4 Z_{34}}{\alpha_3 2m}\right) \right\} x^2 \langle 12 \rangle^4$$

$$+ \left[\alpha_2\alpha_4 \frac{Z_{12}Z_{34}}{2m^2} - \alpha_2\alpha_3 \frac{Z_{12}Z_{34}}{2m} - \alpha_1\alpha_4 \frac{Z_{34}}{2m}\right] x^2 \langle 12 \rangle^3 \frac{\langle 1|P|2 \rangle}{m}$$

$$+ \left[\alpha_2\alpha_4 \frac{Z_{12}Z_{34}}{4m^2}\right] x^2 \langle 12 \rangle^2 \left(\frac{\langle 1|P|2 \rangle}{m}\right)^2$$

and the coupling constants are

$$\Rightarrow (g_0, g_1, g_2) = \left(\alpha_1\alpha_3 \left(1 - \frac{\alpha_2 Z_{12}}{\alpha_1 2m}\right)\left(1 - \frac{\alpha_4 Z_{34}}{\alpha_3 2m}\right), \alpha_2\alpha_4 \frac{Z_{12}Z_{34}}{2m^2} - \alpha_2\alpha_3 \frac{Z_{12}Z_{34}}{2m} - \alpha_1\alpha_4 \frac{Z_{34}}{2m}, \alpha_2\alpha_4 \frac{Z_{12}Z_{34}}{4m^2}\right) \quad (86)$$

We can see the component amplitudes contains $g_1$ term, which is forbidden. In order to get rid of the $g_1$-term, we are forced to choose $\alpha_2 = \alpha_4 = 0$\footnote{Neither can we choose $\alpha_1 = 0$ nor $\alpha_3 = 0$, because we will get $g_0 = 0$, and this will violate the equivalence principle, which is not what Einstein would like to see.}. This choice not only excludes the $g_1$ term in $M(1^{s=2}, 2^{s=2}, P^{s=2})$, but $g_1$ term in all component amplitudes that includes graviton, e.g., $M(1^{s=1}, 2^{s=1}, P^{s=2})$. Substitute $\alpha_2 = \alpha_4 = 0$ into (82), we get

$$M(\Phi, \Phi, P^{s=2}) = \frac{k}{2m^2} x^2 S^{(1)}_{12} S^{(1)}_{34} \quad (87)$$

$$M(\Phi, \Phi, P^{s=1}) = \frac{k}{2m^2} S^{(2)}_{12} S^{(2)}_{34} \quad (87)$$

In fact, not only $g_1$-terms are excluded in all component amplitudes of this superamplitude, but also $g_2$ terms, leaving with only minimal coupling terms. This indicates that supersymmetry implies all components in massive scalar multiplets interact with graviton only through minimal couplings.

6. Summary and outlook

In this paper, we calculated the supersymmetric part in (5), i.e., $A(Z, m)$. Starting from $N = 2$, the delta function is first extracted, then building blocks of the superamplitude, i.e., $\mathcal{H}$ and $\mathcal{H}$ are examined. There are two solutions in $N = 2$, which are related to each other by Grassmann Fourier transformation, see (50) and (51). The most general solution of $A(Z, m)$ in $N = 2$ is the linear combination of the two.

Two special cases are discussed, the $Z = 0$ case and the BPS case ($Z = 2m$). In the $Z = 0$ case, the solutions can be factorized into products of $N = 1$ amplitudes, see (61) and (63). In the BPS limit, we know that half of the Grassmannian degrees drop out. It is shown in (73) that in BPS
limit, the amplitude can be factorized into two components that are related by Grassmann Fourier transform, each of which is a BPS amplitude. In other words, non-BPS amplitude is the product of two BPS amplitudes, explaining the drop out of half of the Grassmannian degrees in the BPS limit.

The $\mathcal{N} = 4$ supersymmetry is also considered, and especially super-Maxwell amplitude and SUGRA amplitude. The super-Maxwell amplitude indicates that there is non-minimal coupling of photons, see (80). The SUGRA amplitude showed that, if we require $g_1 = 0$, then all non-minimal couplings in graviton exchange vanish as well, see (87).

Any massive spinning body can be described as a spinning particle at large distances, and black holes are no exception. For instance, a Kerr black hole can be described as a massive particle with large spin, interacting with graviton fields through minimal couplings [8]. It would be interesting to see what spinning particles are able to describe a supersymmetric black hole at large distance. Recently, relative entanglement entropy of binary Kerr black holes is found to be nearly zero for minimal coupling in the Eikonal limit, and increases when spin multipole moments are turned on [9]. We are interested in the relative entanglement entropy of supersymmetric black holes, and examine whether BPS limit results the lowest entropy, relative to non-BPS amplitudes [10].

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**Appendix A. Spinor helicity formalism**

For a more detailed introduction of massive spinor helicity formalism, see [1].

**Contractions and the Levi-Civita Tensor.**

We choose the convention of contracting the dotted and undotted spinors into square and angle brackets as:

$$\langle \lambda \mu \rangle \equiv \lambda^\alpha \mu_\alpha = \epsilon_{\alpha \beta} \lambda^\alpha \mu^\beta, \quad [\lambda \mu] \equiv \tilde{\lambda}_\alpha \tilde{\mu}^\alpha = \epsilon^{\dot{\alpha} \dot{\beta}} \tilde{\lambda}_{\dot{\alpha}} \tilde{\mu}_{\dot{\beta}}. \quad (A.1)$$

Same for massive spinors that carry SU(2) indices. Here the Levi-Civita tensor in matrix form is given by:

$$\epsilon_{\alpha \beta} = \epsilon^{\dot{\alpha} \dot{\beta}} = -\epsilon_{\dot{\alpha} \dot{\beta}} = -\epsilon^{\dot{\alpha} \dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (A.2)$$

such that

$$\epsilon^{\alpha \beta} \epsilon_{\beta \gamma} = \delta^\alpha_{\gamma}. \quad (A.3)$$
The Massless and Massive Momenta.

The momentum of the massless particle $P$ can be written as a product of two two-component spinors:

$$P_{\alpha\tilde{\alpha}} = \lambda_{\alpha} \tilde{\lambda}_{\tilde{\alpha}} \equiv |\lambda\rangle\langle\bar{\lambda}|.$$  \hfill (A.4)

and a massive momentum $p_i$ can be written as a product of two 2-by-2 matrices:

$$p_{i\alpha\tilde{\alpha}} = \lambda_{i\alpha} \tilde{\lambda}_{i\tilde{\alpha}} \equiv |i\rangle\langle i|.$$  \hfill (A.5)

where the $I$ is the SU(2) index.

Explicit kinematics.

For massless particles with momentum $p^\mu = (E, \ E \sin(\theta) \cos(\phi), \ E \sin(\theta) \sin(\phi), \ E \cos(\theta)),$

we have

$$\lambda_{\alpha} = \sqrt{2E} \left(\begin{array}{c}s \\ -c \end{array}\right), \ \tilde{\lambda}_{\tilde{\alpha}} = \sqrt{2E} \left(\begin{array}{c}s^* \\ c \end{array}\right).$$  \hfill (A.7)

For massive particles with momentum $p^\mu = (E, \ p \sin(\theta) \cos(\phi), \ p \sin(\theta) \sin(\phi), \ p \cos(\theta)),$

we have

$$\lambda_{i\alpha} = \sqrt{E + ps} \left(\begin{array}{cc}\sqrt{E - pc} & \sqrt{E - ps} \\ \sqrt{E + pc} & \sqrt{E + ps} \end{array}\right), \ \tilde{\lambda}_{i\tilde{\alpha}} = \left(\begin{array}{cc}\sqrt{E - pc} & \sqrt{E - ps} \\ -\sqrt{E + pc} & -\sqrt{E + ps} \end{array}\right),$$  \hfill (A.9)

where $c \equiv \cos(\frac{\theta}{2}) e^{i\phi}, \ s \equiv \sin(\frac{\theta}{2}).$

Contractions of Massive Spinors.

The on-shell condition for massive particle is given by

$$p_i^2 = \langle i\rangle^{i'} \langle i'\rangle_i = m_i^2,$$  \hfill (A.10)

where we choose

$$\langle i\rangle^{i'} = -m_i \epsilon^{ij}, \ \ [i^\dagger i] = +m_i \epsilon^{ij}.$$  \hfill (A.11)

which will be repeatedly used throughout massive amplitude calculations. With these conventions, we find that momentum contracting with the massive spinors are:

$$m_i |i\rangle_{i'} = +p_i |i\rangle_{i'}, \ m_i |i\rangle_{i'} = +p_i |i\rangle_{i'},$$

$$m_i |i\rangle_{i'} = -|i\rangle_{i'} p_i, \ m_i |i\rangle_{i'} = -|i\rangle_{i'} p_i.$$  \hfill (A.12)

For massive spinors associated with the same particle whose LG indices contracted:

$$|i\rangle_{i'} \epsilon_{iJ} |i\rangle_{i'} = -m_i \epsilon^{i\alpha}, \ |i\rangle_{i'} \epsilon_{iJ} |i\rangle_{i'} = m_i \epsilon^{i\alpha},$$

$$|i\rangle_{i'} \epsilon_{iJ} |i\rangle_{i'} = p_{i\alpha\alpha}, \ |i\rangle_{i'} \epsilon_{iJ} |i\rangle_{i'} = -p_i^{i\alpha \alpha}.$$  \hfill (A.13)
Definition and convention of the $x$-factor.
The external momenta satisfy
\[ p_1 + p_2 + P = 0 \, . \] (A.14)
The momentum conservation condition (A.14) and the on-shell condition yields:
\[ 2p_1 \cdot P = \langle P | p_1 | P \rangle = \lambda_P^\mu p_{1\alpha} \tilde{\lambda}_P^\alpha = 0 \, , \] (A.15)
so that $\lambda_{P\alpha}$ is proportional to $p_{1\alpha} \tilde{\lambda}_P^\alpha$. This allow us to define the $x$-factor:
\[ x \lambda_{P\alpha} = \frac{p_{1\alpha} \tilde{\lambda}_P^\alpha}{m} \quad , \quad \frac{\tilde{\lambda}_P^\alpha}{x} = \frac{p_{1\alpha} \lambda_{P\alpha}}{m} \] (A.16)

Appenidix B. Grassmann variables

Convention of the Grassmann variables $\eta$.
When contract with Levi-Civita tensors, the massive Grassmann variables $\eta$'s transform as
\[ \eta_{iA} \equiv \varepsilon_{AB} \eta_{iB} \Rightarrow \frac{\partial}{\partial \eta_{iA}} = -\varepsilon_{AB} \frac{\partial}{\partial \eta_{iB}} \]
\[ \eta_i^A \equiv \varepsilon_{ij} \eta_j^A \Rightarrow \frac{\partial}{\partial \eta_i^A} = -\varepsilon_{ij} \frac{\partial}{\partial \eta_j^A} \] (B.1)
To simplify equations, let's denote the contraction rules for massive Grassmann variables $\eta_i$ and for massless Grassmann variables $\eta_P$
\[ \eta_i^A \cdot \eta_i^B \equiv -\frac{1}{2} \varepsilon_{ij} \eta_j^A \eta_i^B \]
\[ \eta_P \cdot \eta_P \equiv \frac{1}{2} \eta_P^A \eta_P^A \] (B.2)
Grassmann Fourier transformation.
The Grassmann Fourier transformation of a function $f(\eta)$ in $\eta$ basis to $\tilde{\eta}$ basis is
\[ f(\tilde{\eta}) = \int d\eta \cdot e^{\tilde{\eta} \eta} f(\eta) \]
\[ f(\eta) = \int d\tilde{\eta} \cdot e^{-\tilde{\eta} \eta} f(\tilde{\eta}) \, . \] (B.3)
For Grassmann variables of massless particles, the Grassmann Fourier transformation is
\[ \tilde{f}(\tilde{\eta}) = \int d\eta^A d\eta_A \cdot e^{\tilde{\eta} \eta_A} f(\eta) \, , \] (B.4)
while for Grassmann variables of massive particles
\[ \tilde{f}(\tilde{\eta}) = \int d\eta_i^A d\eta_i^B d\eta_{iA} d\eta_{iB} \cdot e^{\tilde{\eta}_i^A \eta_{iA}} f(\eta) \, . \] (B.5)
Appendix C. More on the building blocks

$H^A$ and $\bar{H}^A$ play crucial rules in this paper, they serve as basic building blocks of the super-amplitude. This appendix is devoted to introducing several important properties of them, which will be useful if one wants to reproduce the calculations we have done in this paper. To simplify equations, let’s denote

$$ (\eta^A_i \cdot \eta^A_i) \equiv -\frac{1}{2} \epsilon^{ij} \eta^A_i \eta^A_j \quad (\text{no sum over } i \text{ and } A) \Rightarrow \eta^A_i \eta^A_j = \epsilon_{ij}(\eta^A_i \cdot \eta^A_i), \quad (C.1) $$

which is just a special case of (B.2).

Lemma Appendix C.1.

$$ \langle P^1 I \rangle \eta^A_{1I} = \frac{1}{x} H^A - \bar{H}^A. \quad (C.2) $$

Proof.

$$ [\xi P] \left( \frac{1}{x} H^A - \bar{H}^A \right) = [\xi | P_{1I} | P] H^A - [\xi P] \bar{H}^A $$

$$= \sum_{i=1}^{2} [\xi | P_{1I} | I] \eta^A_{iI} - \sum_{i=1}^{2} [\xi | I] \sigma^A \eta^A_{iI} $$

$$= [\xi | (P_1 + P_2) | 2J] \eta^A_{2J} = [\xi P] \langle P^1 I \rangle \eta^A_{1I}. \quad \square $$

Lemma Appendix C.2.

$$ \langle 1^I 2^J \rangle \eta^A_{1I} \eta^A_{2J} = -\frac{2}{x} (H \cdot \bar{H}) + 2(H \cdot \bar{H}) \quad (C.4) $$

$$ [1^I 2^J] \eta^A_{1I} \eta^A_{2J} = -2x(H \cdot \bar{H}) - 2(H \cdot \bar{H}). $$

Proof.

Let’s just proof the first equation in the lemma, since the proof of the second is similar to that of the first.

$$ \langle 1^I 2^J \rangle \eta^A_{1I} \eta^A_{2J} $$

$$= \langle 1^I 2^J \rangle \eta^A_{1I} \eta^A_{1J} + \langle 1^I 2^J \rangle \eta^A_{2I} \eta^A_{2J} $$

$$= \langle 1^I P \rangle \eta^A_{1I} \eta^A_{1J} + \langle 1^I P \rangle \eta^A_{2I} \eta^A_{2J} $$

$$= \frac{2}{x} (H \cdot H) + 2(H \cdot \bar{H}). \quad \square $$

Lemma Appendix C.3.

$$ \left\{ \begin{array}{l}
\delta^1 H^1 \doteq -\frac{1}{2} \sum_{i,j} \langle i^I j^J \rangle \eta^A_i \eta^A_j \\
\delta^2 H^2 \doteq -\frac{1}{2} \sum_{i,j} \langle i^I j^J \rangle \bar{\eta}^A_i \bar{\eta}^A_j \\
\delta^1 \bar{H}^1 \doteq \frac{1}{2} \sum_{i,j} \langle i^I j^J \rangle \eta^A_i \eta^A_j \\
\delta^2 \bar{H}^2 \doteq \frac{1}{2} \sum_{i,j} \langle i^I j^J \rangle \bar{\eta}^A_i \bar{\eta}^A_j \\
\end{array} \right. \quad (C.6) $$
Proof.

\[ \delta^{1}\mathcal{H}^{1} = \delta^{1} \cdot \frac{1}{\langle \zeta P \rangle} \sum_{j=1}^{2} \langle \xi \mathbf{j} \rangle \eta_{\mathbf{j},j} \]

\[ = \frac{1}{\langle \zeta P \rangle} \sum_{i,j} \langle Pi \mathbf{i} \rangle \langle \xi \mathbf{j} \rangle \eta_{\mathbf{i},i} \eta_{\mathbf{j},j} \]

\[ = - \frac{1}{2 \langle \zeta P \rangle} \sum_{i,j} \langle Pi \mathbf{i} \rangle \langle j \mathbf{j} \rangle \eta_{\mathbf{i},i} \eta_{\mathbf{j},j} , \]

where we used Schouten identity. The other equations can be proven in similar ways. \( \square \)

Lemma Appendix C.4.

\[ \left\{ \begin{array}{l}
\delta^{1}\mathcal{H}^{1}\mathcal{H}_{2} \equiv -\frac{1}{x} \left( \langle P1^{1} \rangle \eta_{1,1} \cdot \eta_{2,1} \right) + \langle P2^{1} \rangle \eta_{1,2} \cdot \eta_{1,1} \\
\delta^{2}\mathcal{H}^{2}\mathcal{H}_{1} \equiv \frac{1}{x} \left( \langle P1^{2} \rangle \eta_{2,1} \cdot \eta_{2,2} + \langle P2^{2} \rangle \eta_{2,2} \cdot \eta_{1,1} \right)
\end{array} \right. \] (C.8)

Proof. Let \( m = 1, \)

\[ \langle \zeta P \rangle \delta^{1}\mathcal{H}^{1}\mathcal{H}_{2} \]

\[ = - \left( \langle \zeta P \rangle \mathcal{H}^{1} \right) \left( \delta^{1}\mathcal{H}_{2} \right) \]

\[ = \frac{1}{x} \left( \langle \zeta \mathbf{1} \rangle \eta_{1,1}^{1} + \langle \zeta \mathbf{2} \rangle \eta_{2,2}^{2} \right) \left( \langle \mathbf{1}^{1} \mathbf{2}^{2} \rangle \eta_{1,1}^{2} \right) + \left( \eta_{1,1} \cdot \eta_{1,1}^{1} \right) + \left( \eta_{1,1} \cdot \eta_{1,1}^{2} \right) \]

\[ = \frac{1}{x} \left( \langle \zeta \mathbf{1} \rangle \eta_{2,1}^{1} \cdot \eta_{1,1} \right) + \left( \eta_{1,1} \cdot \eta_{1,1}^{1} \right) + \left( \eta_{1,1} \cdot \eta_{1,1}^{2} \right) \]

\[ = - \langle \zeta P \rangle \frac{1}{x} \left( \langle P1^{1} \rangle \eta_{1,1} \cdot \eta_{2,1} + \langle P2^{2} \rangle \eta_{2,2} \cdot \eta_{1,1} \right) . \]

The second equation can be proven in similar fashion. \( \square \)

When \( \mathcal{H}^{A}, \mathcal{H}_{A}, \) and their products meet the delta function, we are able to recast them into a Lorentz invariant and R-charge symmetric form. To simplify notations, let’s define

\[ Q_{a}^{A} \equiv \lambda_{1,a} \eta_{1,1}^{A} + \lambda_{2,a} \eta_{2,1}^{A} ; \quad \bar{Q}_{a}^{A} \equiv \lambda_{1,a} \eta_{1,1}^{A} - \lambda_{2,a} \eta_{2,1}^{A} \]

\[ Q_{Pa}^{A} \equiv \lambda_{Pa} \eta_{P}^{A} ; \quad \bar{Q}_{Pa}^{A} \equiv \lambda_{Pa} \eta_{P}^{A} \]

\[ \eta_{I}^{A} \cdot \eta_{I}^{B} \equiv \frac{1}{2} \varepsilon^{IJ} \eta_{I,1}^{A} \eta_{I,1}^{B} ; \quad \eta_{P} \cdot \eta_{P} \equiv \frac{1}{2} \eta_{P}^{A} \eta_{PA} . \] (C.10)
The following equations follow from \( \text{(C.6)} \)

\[
\begin{align*}
\delta^1 \delta^2 &= -\frac{1}{2} \langle Q^A P \rangle \langle Q_A P \rangle \\
\delta^1 \delta^2 (\eta_P \cdot \eta_P) &= \frac{1}{4} \langle Q^A Q^B \rangle \langle Q_A Q_{PB} \rangle \\
\delta^1 \delta^2 (H \cdot H) &= \frac{1}{12} \langle Q^A Q^B \rangle \langle Q_A Q_B \rangle \\
\delta^1 \delta^2 (H \cdot \bar{H}) &= -\frac{1}{12x} \langle Q^A Q^B \rangle [\bar{Q}_A \bar{Q}_B] \\
\delta^1 \delta^2 (H \cdot \bar{H}) &= -\frac{1}{12x^2} [\bar{Q}^A \bar{Q}^B] [\bar{Q}_A \bar{Q}_B] \\
\delta^1 \delta^2 \eta^A \eta^A = 1/3 \langle Q^A Q^B \rangle \langle Q_A Q_{PB} \rangle \\
\delta^1 \delta^2 \eta^2 \tilde{H}_A = -\frac{1}{3x} [\bar{Q}^A \bar{Q}^B] \langle Q_A Q_{PB} \rangle.
\end{align*}
\]

Things will be more complicated when \( \delta^1 \bar{H}^1 \bar{H}_2 \) or \( \delta^2 \bar{H}^2 \bar{H}_1 \) appear, but as we can see from \( \text{(C.8)} \), we have

\[
\begin{align*}
\delta^1 \delta^2 (H \cdot H) \eta_{PA} \bar{H}_A &= -\frac{2}{9x} \langle Q^A Q^B \rangle [\bar{Q}_A \bar{Q}_C] ((\eta_{1B} \cdot \eta_{1C}) - (\eta_{2B} \cdot \eta_{2C})) \\
\delta^1 \delta^2 (H \cdot \bar{H}) H^A \eta_{PA} &= \frac{2}{9x} \langle Q^A Q^B \rangle [\bar{Q}_A \bar{Q}_C] ((\eta_{1B} \cdot \eta_{1C}) - (\eta_{2B} \cdot \eta_{2C}))
\end{align*}
\]

and finally

\[
\begin{align*}
\delta^1 \delta^2 (H \cdot \bar{H}) (\bar{H} \cdot \bar{H}) &= -\frac{1}{6x} [\bar{Q}^A P] [\bar{Q}_A P] ((\eta_{1B} \cdot \eta_{1C}) - (\eta_{2B} \cdot \eta_{2C})) \\\n\delta^1 \delta^2 (H \cdot \bar{H}) (\bar{H} \cdot \bar{H}) (\eta_P \cdot \eta_P) &= \frac{1}{12x^2} [\bar{Q}^A \bar{Q}^B] [\bar{Q}_A \bar{Q}_{PB}] ((\eta_{1C} \cdot \eta_{1D}) - (\eta_{2C} \cdot \eta_{2D}) \).
\end{align*}
\]

**Appendix D. More on \( \mathcal{M}_{res} \)**

The amplitude is proportional to \( \delta^1 \delta^2 \) follows straightforwardly given the generators, see \( \text{(22)} \). The main goal in this section is to study the rest of the amplitude, i.e., \( \mathcal{M}_{res} \), in more detail, and proof that it is a function of \( H \) and \( \bar{H} \), in other words, all \( \eta_i \) (\( \eta_i = \eta_1 \) or \( \eta_2 \)) in \( \mathcal{M}_{res} \) must be of the form

\[
\sum_{i=1}^{2} \lambda_i^i \eta_i^A \text{ or } \sum_{i=1}^{2} \lambda_i^i \sigma_i \eta_i^A.
\]

\( \mathcal{M}_{res} \) is a function of \( \eta_i \)'s, and we can expand it according to the order of \( \eta_i \)'s

\[
\mathcal{M}_{res} = \mathcal{M}_{res}^{(0)} + \mathcal{M}_{res}^{(1)} + \mathcal{M}_{res}^{(2)} + \cdots.
\]
Observe the form of the generators \((17)\), they have a multiplicative part in \(\eta\) and a differential part in \(\eta\). Therefore, since \(\mathcal{M}_{res}\) must satisfy \((28)\), we have (see \((27)\) for definitions)

\[
\langle \zeta \mathcal{D}^A \rangle \mathcal{M}_{res}^{(n-2)} + \langle \zeta \mathcal{D}^A \rangle \mathcal{M}_{res}^{(n)} + \langle \zeta \mathcal{P} \rangle \eta^I A \mathcal{M}_{res}^{(n-1)} \frac{\partial}{\partial \eta_I^A} \mathcal{M}_{res}^{(n)} \delta = 0 \tag{D.3}
\]

**Lemma Appendix D.1.**

The lowest order of \(\mathcal{M}_{res}\) must be \(\mathcal{M}_{res}^{(0)}\).

**Proof.**

Assume the lowest order of \(\mathcal{M}_{res}\) is \(\mathcal{M}_{res}^{(low)}\), and we must have

\[
\langle \zeta \mathcal{D}^A \rangle \mathcal{M}_{res}^{(low)} \delta = 0 \Rightarrow \sum_{i=1}^2 \langle \zeta I^I \rangle \frac{\partial}{\partial \eta^I A} \mathcal{M}_{res}^{(low)} \delta = 0 \tag{D.4}
\]

\[
[\zeta \mathcal{D}^A] \mathcal{M}_{res}^{(low)} \delta = 0 \Rightarrow \sum_{i=1}^2 [\zeta I^I] \frac{\partial}{\partial \eta^I A} \mathcal{M}_{res}^{(low)} \delta = 0 .
\]

Since \(\frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(n)}\) carries a LG index \(I\), we have three possibilities for \(\frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(n)}\)

\[
\frac{\partial}{\eta^I A} \mathcal{M}_{res}^{(low)} = \eta^I A \text{ or } \langle i_I f_{iA} \rangle \text{ or } [i_I f_{iA}] .
\]

However, the first one is impossible since it would imply \(\mathcal{M}_{res}^{(low)} \propto \eta^I A \eta^I A = 0\). In addition, one of the second and the third is redundant, since we can always use \(p_i |i'\rangle = |i'\rangle\) to convert \(|i'\rangle\) to \(|i'\rangle\).

Let’s keep \(\frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(low)} = \langle i_I f_{iA} \rangle\) and \((D.3)\) implies

\[
\langle \zeta f_{iA} \rangle - \langle \zeta f_{2A} \rangle \frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(low)} = 0 \tag{D.6a}
\]

\[
[\zeta] p_1 |f_{iA} \rangle + [\zeta] p_2 |f_{2A} \rangle \frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(low)} = 0 \tag{D.6b}.
\]

\((D.6a)\) implies \(|f_{iA} \rangle = |f_{2A} \rangle \equiv |f_A \rangle\). Set \(\zeta = |\zeta| p_2\), and add \((D.6a)\) to \((D.6b)\), we get

\[
[\zeta](p_1 + p_2) |f_{iA} \rangle = [\zeta] P |f_{iA} \rangle \frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(low)} = 0 \Rightarrow \langle P f_{iA} \rangle \frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(low)} = 0 \Rightarrow |f_{iA} \rangle \frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(low)} = |f_A \rangle P .
\]

Since \(\frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(low)} = \langle i_I f_{A} \rangle \cdot |f_A \rangle\) can not be a function of \(\eta^I_{1I}\) and \(\eta^I_{2I}\), and the solution to \(\frac{\partial}{\eta_I^A} \mathcal{M}_{res}^{(low)} \delta = f_{A} \langle i_I P \rangle \delta\) is

\[
\mathcal{M}_{res}^{(low)} = f_{A} \sum_i \eta^I_{i} \langle i_I P \rangle + (\eta_i\text{-free term}) \delta = (\eta_i\text{-free term}) ,
\]

where in the last equation, we used \(\sum_i \eta^I_{i} \langle i_I P \rangle \delta = 0\). Therefore, \(\mathcal{M}_{res}^{(low)}\) is \(\eta_i\)-free, in other words, \(\mathcal{M}_{res}^{(low)} = \mathcal{M}_{res}^{(0)}\). \(\Box\)

\(^{11}\)Note that \(|f_{iA}\rangle\) might carry Grassmann variables.
Lemma Appendix D.2.
If \( M_{res}^{(n-1)} \) and \( M_{res}^{(n-2)} \) is a function of \( \mathcal{H} \) and \( \bar{\mathcal{H}} \), then so is \( M_{res}^{(n)} \).

Proof.
Define \( \frac{\partial}{\partial \eta_i} M_{res}^{(n)} \equiv \langle i, f_1^{(n)} \rangle \) and (D.3) implies
\[
\begin{align*}
    s_z \sum_i \sigma_i \langle \xi f_1^{(n)A} \rangle &= \langle \xi P \rangle \left( c_z H^A M_{res}^{(n-2)} + \eta_i^A M_{res}^{(n-1)} \right) \quad \text{(D.9a)} \\
    c_z \sum_i [\xi| p_i | f_1^{(n)A}] &\delta_i \approx \langle \xi P \rangle \left( -s_z \bar{H}^A M_{res}^{(n-2)} + \frac{\partial}{\partial \eta_i^A} M_{res}^{(n-1)} \right) . \quad \text{(D.9b)}
\end{align*}
\]

If we set \( \langle \xi \rangle = [\xi| p_i \rangle \) and linear combine (D.9a) and (D.9b), we get
\[
\begin{align*}
    c_z s_z \langle P f_1^{(n)A} \rangle &\delta_i = -\left( \frac{c_z^2}{x} H^A + s_z \bar{H}^A \right) M_{res}^{(n-2)} - c_z \eta_i^A M_{res}^{(n-1)} + s_z \frac{\partial}{\partial \eta_i^A} M_{res}^{(n-1)} , \text{if } \langle \xi \rangle = [\xi| p_2 \rangle . \quad \text{(10)}
    c_z s_z \langle P f_2^{(n)A} \rangle &\delta_i = -\left( \frac{c_z^2}{x} H^A + s_z \bar{H}^A \right) M_{res}^{(n-2)} - c_z \eta_i^A M_{res}^{(n-1)} + s_z \frac{\partial}{\partial \eta_i^A} M_{res}^{(n-1)} , \text{if } \langle \xi \rangle = [\xi| p_1 \rangle .
\end{align*}
\]

This not only tells us that both \( f_1^{(n)A} \) and \( f_2^{(n)A} \) are functions of \( \mathcal{H} \) and \( \bar{\mathcal{H}} \), but also implies
\[
\langle P f_1^{(n)A} \rangle = \langle P f_2^{(n)A} \rangle . \quad \text{(11)}
\]

This implies whenever \( \lambda^t_{ia} \eta^A_i \) appears in \( M_{res}^{(n)} \), it must be either \( \langle \xi 1^t \rangle \eta^A_{1i} + \langle \xi 2^t \rangle \eta^A_{2i} \) or \( \langle P 1^t \rangle \eta^A_{1i} \). In other words, \( \langle \xi 1^t \rangle \eta^A_{1i} - \langle \xi 2^t \rangle \eta^A_{2i} \) can’t appear in \( M_{res} \). In addition, \( \langle \xi 1^t \rangle \eta^A_{1i} - \langle \xi 2^t \rangle \eta^A_{2i} \) is also a possible choice, since
\[
[\xi 1^t \rangle \eta^A_{1i} - [\xi 2^t \rangle \eta^A_{2i} = -[\xi| p_2 | 1^t \rangle \eta^A_{1i} - [\xi| p_2 | 2^t \rangle \eta^A_{2i} + \langle P 1^t \rangle \eta^A_{1i} . \quad \text{(12)}
\]

By similar arguments, \( \langle \xi 1^t \rangle \eta^A_{1i} + [\xi 2^t \rangle \eta^A_{2i} \) is prohibited. Therefore, only the following terms can exist in \( M_{res}^{(n)} \),
\[
\begin{align*}
    \langle \xi 1^t \rangle \eta^A_{1i} + \langle \xi 2^t \rangle \eta^A_{2i} = \langle \xi P \rangle \mathcal{H}^A \\
    [\xi 1^t \rangle \eta^A_{1i} - [\xi 2^t \rangle \eta^A_{2i} = [\xi P \rangle \bar{\mathcal{H}}^A \quad \text{(13)}
\end{align*}
\]

The last identity follows from Lemma Appendix C.1. As a result, \( M_{res}^{(n)} \) is a function of \( \mathcal{H} \) and \( \mathcal{H} \).

Given Lemma Appendix D.2 and given \( M_{res}^{(n-1)} = 0 \), we can conclude that \( M_{res}^{(n)} \) is function of \( \mathcal{H} \) and \( \bar{\mathcal{H}} \). By iteration, we can conclude all \( M_{res}^{(n)} \), and thus \( M_{res} \) are functions of \( \mathcal{H} \) and \( \bar{\mathcal{H}} \).
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