LIFTING CENTRAL INVARIANTS OF QUANTIZED HAMILTONIAN ACTIONS

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Abstract. Let $G$ be a connected reductive group over an algebraically closed field $K$ of characteristic $0$, $X$ an affine symplectic variety equipped with a Hamiltonian action of $G$. Further, let $*$ be a $G$-invariant Fedosov star-product on $X$ such that the Hamiltonian action is quantized. We establish an isomorphism between the center of the quantum algebra $\mathbb{K}[X][[\hbar]]^G$ and the algebra of formal power series with coefficients in the Poisson center of $\mathbb{K}[X]^G$.

1. Introduction

In this paper we establish a relation between the centers of certain Poisson algebras and their quantizations. Poisson algebras in interest are invariant algebras for Hamiltonian actions of reductive algebraic groups on affine symplectic varieties (the definition of a Hamiltonian action will be given in the beginning of Section 2). All varieties and groups are defined over an algebraically closed field $\mathbb{K}$ of characteristic $0$.

Until further notices $G$ is a connected reductive algebraic group, $X$ is an affine symplectic variety equipped with a Hamiltonian action of $G$. Construct a Fedosov star-product on $X$ and suppose that the Hamiltonian action can be quantized (all necessary definitions concerning star-products and quantized Hamiltonian actions are given in Section 3). Let $U_h^\wedge(\mathfrak{g})$ stand for the completed homogeneous universal enveloping algebra of $\mathfrak{g}$ (see the end of Section 3). We have a natural algebra homomorphism $U_h^\wedge(\mathfrak{g}) \to \mathbb{K}[X][[\hbar]]$ mapping $\xi \in \mathfrak{g}$ to $\hat{H}_\xi$.

Let $Z(\bullet), Z(\bullet)$ stand for the center of Poisson and associative algebras, respectively. In a word, the main result of this paper is that the algebras $Z(\mathbb{K}[X]^G)[[\hbar]], Z(\mathbb{K}[X][[\hbar]]^G)$ are isomorphic. In fact, a more precise statement holds. Let us state it.

We have (see Section 3) the following commutative diagram, where the vertical arrow is an isomorphism of topological $\mathbb{K}[[\hbar]]$-algebras.

\[
\begin{array}{ccc}
S(\mathfrak{g})^\theta[[\hbar]] & \cong & S(\mathfrak{g})^\theta \\
\downarrow & & \\
Z(U_h^\wedge(\mathfrak{g})) & \Rightarrow & S(\mathfrak{g})^\theta
\end{array}
\]

(1.1)

The main result of this paper is the following theorem.

Theorem 1.1. Identify $S(\mathfrak{g})^\theta[[\hbar]], Z(U_h^\wedge(\mathfrak{g}))$ by means of any topological $\mathbb{K}[[\hbar]]$-algebra isomorphism making the previous diagram commutative. Then there exists an isomorphism

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\( Z(\mathbb{K}[X]^G)[[\hbar]] \rightarrow Z(\mathbb{K}[X][[\hbar]]^G) \) of \( \mathbb{K}[[\hbar]] \)-algebras such that the following diagram is commutative.

\[
\begin{array}{ccc}
S(\mathfrak{g})[[\hbar]] & \cong & Z(\mathbb{K}[X]^G)[[\hbar]] \\
\downarrow & & \downarrow \\
Z(U_h^G(\mathfrak{g})) & \rightarrow & Z(\mathbb{K}[X][[\hbar]]^G)
\end{array}
\]

Let us describe some results related to ours.

In [Kn1] Kontsevich quantized the algebra of smooth functions on an arbitrary real Poisson manifold, say \( M \). His construction yields an isomorphism \( Z(C^\infty(M))[[\hbar]] \rightarrow Z(C^\infty(M))[[\hbar]] \).

Also let us mention the Duflo conjecture, [D]. Let \( G \) be a Lie group, \( H \) its closed subgroup. Let \( \mathfrak{g}, \mathfrak{h} \) denote the Lie algebras of \( G, H \), respectively. Further, choose \( \lambda \in (\mathfrak{h}[[\mathfrak{h}], \mathfrak{h}])^* \). Set \( \mathfrak{h}_\lambda := \{ \xi + \langle \lambda, \xi \rangle, \xi \in \mathfrak{h} \} \). The algebra \( (S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{h}_\lambda) \) has the natural Poisson structure induced from the Poisson structure on \( S(\mathfrak{g}) \). Similarly, the space \( (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda) \) has a natural structure of the associative algebra. The Duflo conjecture states that for a certain element \( \delta \in (\mathfrak{h}[[\mathfrak{h}], \mathfrak{h}])^* \) the filtered algebras \( Z((S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{h}_\lambda)) \), \( Z((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\delta}) \) are isomorphic. Of course, the Duflo conjecture makes sense for algebraic groups instead of Lie groups. Note, however, that \( (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\delta} \), in general, cannot be considered as a quantization of \( (S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{h}_\lambda) \) because the associated graded of the former does not necessarily coincide with the latter.

One of the most significant partial results related to the Duflo conjecture is Knop’s construction of the Harish-Chandra isomorphism for reductive group actions, [Kn2]. Namely, let \( X_0 \) be a smooth (not necessarily affine) variety and \( G \) be a connected reductive group (everything over \( \mathbb{K} \)). Set \( X := T^*X_0 \). Let \( \mathcal{D}(X_0) \) denote the algebra of linear differential operators on \( X_0 \). Knop constructed a certain monomorphism \( \mathbb{K}[X] \cap Z(\mathbb{K}[X]^G) \rightarrow Z(\mathcal{D}(X_0)^G) \), which becomes the usual Harish-Chandra homomorphism when \( X_0 = G \). We remark that it is not clear whether \( \mathbb{K}[X] \cap Z(\mathbb{K}[X]^G) = Z(\mathbb{K}[X]^G) \) and whether the image of the Harish-Chandra homomorphism coincides with whole \( Z(\mathcal{D}(X_0)^G) \). However, there is a situation when both claims hold: \( X_0 \) is affine. We derive the existence of a weaker version of the Harish-Chandra homomorphism in Section 5. This gives an alternative proof of the Duflo conjecture in the case when both \( G, H \) are reductive.

There are some other results related to the Duflo conjecture, see, for example, [R], [I].

Now let us describe the content of the paper. Sections 2–3 are preliminary. In the former we review the definition of a Hamiltonian action in the algebraic setting. Besides, we state there two results on Hamiltonian actions obtained in [L1], [L3] to be used in the proof of Theorem 1.1. In Section 3 we review Fedosov’s quantization in the context of algebraic varieties and quantization of Hamiltonian actions.

Section 4 is the central part of this paper, where Theorem 1.1 is proved. At first, we prove Theorem 4.1 which may be thought as a weaker version of Theorem 1.1. In that theorem we establish an isomorphism with desired properties between the algebras \( A[[\hbar]] \), \( A_h \), where \( A \) (resp., \( A_h \)) is the integral closure of the image of \( S(\mathfrak{g}) \) in \( Z(\mathbb{K}[X]^G) \) (resp., of the image of \( U_h(\mathfrak{g}) \) in \( Z(\mathbb{K}[X][[\hbar]]^G) \)). In its turn, Theorem 4.1 is derived from Proposition 4.3 which examines the integral closure of the image of \( U_h(\mathfrak{g}) \) in \( \mathbb{K}[X^0][[\hbar]]^G \) for certain open subvarieties \( X^0 \subset X \).
Finally, in Section 5 we discuss some applications of Theorems 1.1, 4.4. In particular, we give an alternative proof of (a weaker form of) Knop’s result cited above.

2. Hamiltonian actions

In this section G is a reductive algebraic group and X is a variety equipped with a regular symplectic form ω and an action of G by symplectomorphisms. Let \{\cdot,\cdot\} denote the Poisson bracket on X.

To any element ξ ∈ g one assigns the velocity vector field ξ. Suppose there is a linear map g → K[X], ξ → H_ξ, satisfying the following two conditions:

(H1) The map ξ → H_ξ is G-equivariant.
(H2) \{H_ξ, f\} = L_ξ f for any rational function f on X (here L_ξ denotes the Lie derivative).

Definition 2.1. The action G : X equipped with a linear map ξ → H_ξ satisfying (H1),(H2) is said to be Hamiltonian and X is called a Hamiltonian G-variety. The functions H_ξ are said to be the hamiltonians of X.

For a Hamiltonian action G : X we define the morphism µ : X → g* (called the moment map) by the formula

\langle µ(x), ξ \rangle = H_ξ(x), ξ ∈ g, x ∈ X.

Till the end of the section X is affine. Now we are going to describe the algebra Z(K[X]^G). Let ψ be the composition of µ : X → g* and the quotient morphism g* → g*/G. Denote by A the integral closure of ψ*(K[g*]^G) in K[X]^G. Set C_G,X := Spec(A). We have the decomposition ψ = τ ° \tilde{ψ}, where \tilde{ψ} is the dominant G-invariant morphism X → C_G,X induced by the inclusion A ↪ K[X] and τ is a finite morphism.

Proposition 2.2. The morphism \tilde{ψ} is open. The equality Z(K[X]^G) = \tilde{ψ}*(K[im \tilde{ψ}]) holds.

Proof. This follows from [L1], assertion 1 of Theorem 1.2.9 and assertion 2 of Proposition 5.9.1.

Now we are going to establish a certain decomposition result. Namely, we locally decompose X into the product of a vector space and of a coisotropic Hamiltonian variety. Recall that a Hamiltonian G-variety X_0 is said to be coisotropic if the Poisson field Z(K(X)^G) is commutative or equivalently is algebraic over A. Let π denote the quotient morphism X → X/G.

Proposition 2.3. There is an open subset Z^0 ⊂ X/G such that codim_{X/G}(X/G \setminus Z^0) ≥ 2 and for any point x ∈ X with π(x) ∈ Z^0 and closed G-orbit the following condition holds:

(♣) There is a vector space V, a coisotropic Hamiltonian G-variety X_0 and a point x' ∈ X_0 × {0} ⊂ X' := X_0 × V such that X_0 is a homogeneous vector bundle over Gx' and there is a Hamiltonian isomorphism ρ : X_{Gx}^G → X_{Gx'}^G ("Hamiltonian” means that ρ is a G-equivariant symplectomorphism respecting the hamiltonians).

Here X_{Gx}^G, X_{Gx'}^G denote the completions of X, X' w.r.t. Gx, Gx'.

Proof. This follows from [L3], Corollary 3.10, and the symplectic slice theorem, see [L2] and [Kn3], Theorem 5.1.
3. Fedosov Quantization

Let $B$ be a commutative associative algebra with unit equipped with a Poisson bracket.

**Definition 3.1.** The map $*: B[[\hbar]] \otimes_{\mathbb{K}[[\hbar]]} B[[\hbar]] \to B[[\hbar]]$ is called a star-product if it satisfies the following conditions:

1. $(*)_1$ $*$ is $\mathbb{K}[[\hbar]]$-bilinear and continuous in the $\hbar$-adic topology.
2. $(*)_2$ $*$ is associative, equivalently, $(f \ast g) \ast h = f \ast (g \ast h)$ for all $f, g, h \in B$, and $1 \in B$ is a unit for $*$.
3. $(*)_3$ $f \ast g - fg \in hB[[\hbar]], f \ast g - g \ast f - h\{f, g\} \in h^2B[[\hbar]]$ for all $f, g \in B$.

By $(*)_1$, a star-product is uniquely determined by its restriction to $B$. One may write $f \ast g = \sum_{i=0}^{\infty} D_i(f, g)\hbar^i, f, g \in B, D_i : B \otimes B \to B$. Condition $(*)_3$ is equivalent to $D_0(f, g) = fg, D_1(f, g) = D_1(g, f) = \{f, g\}$. If all $D_i$ are bidifferential operators, then the star-product $*$ is called differential. When we consider $B[[\hbar]]$ as an algebra w.r.t. the star-product, we call it a quantum algebra.

**Example 3.2.** Let $X = V$ be a finite-dimensional vector space equipped with a constant nondegenerate Poisson bivector $P$. The Moyal-Weyl star-product on $\mathbb{K}[V][[\hbar]]$ (or $\mathbb{K}[[V, \hbar]]$) is defined by

$$f \ast g = \exp\left(\frac{\hbar}{2}P \right)f(x) \otimes g(y)|_{x=y}.$$ 

Here $P$ is considered as an element of $V \otimes V$. This space acts naturally on $\mathbb{K}[V] \otimes \mathbb{K}[V]$.

Let $G$ be an algebraic group acting on $B$ by automorphisms. It makes sense to speak about $G$-invariant star-products ($\hbar$ is supposed to be $G$-invariant). Now let $\mathbb{K}^x$ act on $B, (t, a) \mapsto t \cdot a$ by automorphisms. Consider the action $\mathbb{K}^x : B[[\hbar]]$ given by $t, \sum_{i=0}^{\infty} a_j \hbar^j = \sum_{j=0}^{\infty} t^j(k) a_j h^j$. If $\mathbb{K}^x$ acts by automorphisms of $*$, then we say that $*$ is homogeneous. Clearly, $*$ is homogeneous iff the map $D_t : B \otimes B \to B$ is homogeneous of degree $-kl$.

For instance, the Moyal-Weyl star-product $*$ is invariant with respect to $\text{Sp}(V)$. The action $\mathbb{K}^x : V$ given by $(t, v) \mapsto t^{-1}v$ makes $*$ homogeneous (for $k = 2$).

Now we review the Fedosov approach ([F1], [F2]) to deformation quantization of smooth affine symplectic varieties. Although Fedosov studied smooth real manifolds, his approach works as well for smooth symplectic varieties and smooth formal schemes. Let $X$ be a smooth variety with symplectic form $\omega$.

According to Fedosov, to construct a star-product one needs to fix a symplectic connection on a variety of interest.

**Definition 3.3.** By a symplectic connection we mean a torsion-free covariant derivative $\nabla$ such that $\nabla \omega = 0$.

One can also define a symplectic connection on a smooth formal scheme. In particular, if $\nabla$ is a symplectic connection on $X$ and $Y$ is a smooth subvariety of $X$, then $\nabla$ restricts to a symplectic connection on the completion $X_Y$.

It turns out that a symplectic connection on $X$ exists provided $X$ is affine. In fact, we need a stronger version of this claim.

**Proposition 3.4** ([L4], Proposition 2.22). Let $G$ be a reductive group acting on $X$ by symplectomorphisms and $\mathbb{K}^x$ act on $X$ by $G$-equivariant automorphisms such that $t \cdot \omega = t^k \omega$ for some $k \in \mathbb{Z}$. Then there is a $G \times \mathbb{K}^x$-invariant symplectic connection $\nabla$ on $X$. 


Fedosov constructed a differential star-product on $\mathbb{K}[X]$ starting with a symplectic connection $\nabla$ and $\lambda \in H^*_DR(X)[[h]]$, see [F2], Section 5.3 or [GR]. The element $\lambda$ is referred to as the characteristic cycle of $\ast$. We remark that all intermediate objects used in Fedosov’s construction are obtained from some regular objects (such as $\omega, \nabla$ or the curvature tensor of $\nabla$) by a recursive procedure and so are regular too. If a reductive group $G$ acts on $X$ by symplectomorphisms (resp., $\mathbb{K}^*$ acts on $X$ such that $t \omega = t^k \omega$), then $\ast$ is $G$-invariant, resp., homogeneous.

In the sequel we will need the following result on equivalence of two different Fedosov star-products on a formal scheme.

**Proposition 3.5.** Let $G$ be a reductive group acting on affine symplectic varieties $X, X’$ by symplectomorphisms, $\nabla, \nabla’$ be $G$-invariant symplectic connections on $X, X’$, and $\lambda \in H^*_DR(X)[[h]], \lambda’ \in H^*_DR(X’)[[h]]$. Further, let $x, x’$ be points of $X, X’$ with closed $G$-orbits. Suppose that there is a $G$-equivariant symplectomorphism $\psi : X^\wedge_{Gx} \rightarrow X'^\wedge_{Gx’}$ such that $\psi^*(\lambda'|_{Gx'}) = \lambda|_{Gx}$. Then there are differential operators $T_i, i = 1, 2, \ldots, \infty$, on $X^\wedge_{Gx}$ such that $(id + \sum_{i=1}^{\infty} T_i h^i) \circ \psi^*$ is an isomorphism of the quantum algebras $\mathbb{K}[X]|^\wedge_{Gx}[[h]], \mathbb{K}[X]|^\wedge_{Gx’}[[h]]$.

Again, to prove these we note that $\psi^*(\lambda'|_{X'_{Gx'}}) = \lambda|_{X_{Gx}}$ and repeat Fedosov’s argument in [F2], Theorem 5.5.3.

Now let us discuss quantization of Hamiltonian actions. Let $X$ be a smooth affine symplectic variety and $\ast$ a star-product on $\mathbb{K}[X][[h]]$. Let $G$ be a reductive group acting on $X$ by automorphisms of $\ast$. We say that this action is $\ast$-Hamiltonian if there is a $G$-equivariant linear map $\mathfrak{g} \rightarrow \mathbb{K}[X][[h]], \xi \mapsto \tilde{H}_\xi$, satisfying the equality

\[
[\tilde{H}_\xi, f] = h \xi * f, \forall \xi \in \mathfrak{g}, f \in \mathbb{K}[X].
\]

The functions $\tilde{H}_\xi$ are said to be quantum hamiltonians of the action.

Let $H_\xi$ be the classical part of $\tilde{H}_\xi$, that is, $H_\xi \in \mathbb{K}[X], \tilde{H}_\xi \equiv H_\xi (\mod h)$. Then the map $\xi \mapsto H_\xi$ turns $X$ into a Hamiltonian $G$-variety. Conversely, the following result takes place.

**Theorem 3.6** ([GR], Theorem 6.2). Let $X$ be an affine symplectic Hamiltonian $G$-variety and $\ast$ be the star-product on $\mathbb{K}[X][[h]]$ obtained by the Fedosov construction with a $G$-invariant connection $\nabla$ and $\lambda \in H^*_DR(X)[[h]]$. Then the following conditions are equivalent:

1. The $G$-variety $X$ has a $\ast$-Hamiltonian structure.
2. The 1-form $i_\xi \tilde{\lambda}$ is exact for each $\xi \in \mathfrak{g}$, where $\tilde{\lambda}$ is a representative of $\lambda$ and $i_\xi \tilde{\lambda}$ denotes the contraction of $\tilde{\lambda}$ and $\xi$.

Again, we remark that, although Gutt and Rawnsley dealt with smooth manifolds, their results remain valid in the algebraic category as well.

**Definition 3.7.** Let $\mathfrak{g}$ be a Lie algebra. By the homogeneous enveloping algebra $U_h(\mathfrak{g})$ we mean the quotient of $T(\mathfrak{g})[[h]]$ by the ideal generated by $\xi \eta - \eta \xi - h[\xi, \eta], \xi, \eta \in \mathfrak{g}$. The completed homogeneous enveloping algebra $\hat{U}_h(\mathfrak{g})$ is, by definition, $\lim_{k \rightarrow \infty} U_h(\mathfrak{g})/h^k U_h(\mathfrak{g})$.

Below $\mathfrak{g}$ is a reductive Lie algebra. Elements of $\hat{U}_h(\mathfrak{g})$ are identified with formal power series $\sum_{i=0}^{\infty} \xi_i h^i, \xi_i \in \mathfrak{g}$. It is clear that the map $\xi \mapsto \tilde{H}_\xi$ is extended to a unique continuous $\mathbb{K}[[h]]$-algebra homomorphism $\hat{U}_h(\mathfrak{g}) \rightarrow \mathbb{K}[X][[h]]$. Note that the map $\sum_{i=0}^{\infty} \xi_i h^i \mapsto \xi_0$ is a $G$-equivariant algebra epimorphism $\hat{U}_h(\mathfrak{g}) \rightarrow S(\mathfrak{g})$.

Now let us describe the center of $\hat{U}_h(\mathfrak{g})$. Clearly, $Z(\hat{U}_h(\mathfrak{g})) = U_h^0(\mathfrak{g})$. Therefore there is the natural epimorphism $Z(\hat{U}_h(\mathfrak{g})) \rightarrow S(\mathfrak{g})^0$. Since $S(\mathfrak{g})^0$ is a polynomial algebra, there is
a section $S(g)^\theta \hookrightarrow \mathcal{Z}(U_\hbar^\wedge(g))$ of the above epimorphism. Any such section is extended to a topological $\mathbb{K}[[\hbar]]$-algebra isomorphism $S(g)^\theta[[\hbar]] \to \mathcal{Z}(U_\hbar^\wedge(g))$. Note that there are unique continuous actions $\mathbb{K}^\times : S(g)^\theta[[\hbar]], \mathcal{Z}(U_\hbar^\wedge(g))$ such that $t.\xi = t\xi, t.\hbar = \hbar t, \xi \in g, t \in \mathbb{K}^\times$. An isomorphism $S(g)^\theta[[\hbar]] \to \mathcal{Z}(U_\hbar^\wedge(g))$ can be chosen $\mathbb{K}^\times$-equivariant.

4. The proof of the main theorem

Set $A := \mathbb{K}[C_{G,X}]$. Let $A_h$ denote the integral closure of the image of $\mathcal{Z}(U_\hbar^\wedge(g))$ in $\mathcal{Z}(\mathbb{K}[X][[\hbar]]^G)$. We have the natural algebra homomorphism $A_h \to A, \sum_{i=0}^\infty f_i\hbar^i \mapsto f_0$.

Here is the main result of this section.

**Theorem 4.1.** Suppose that $\mathbb{K}^\times$ acts on $X$ such that $*$ is homogeneous with $t.\hbar = t^k\hbar, k \in \mathbb{Z}, t.\hat{H}_\xi = t^k\hat{H}_\xi$. There is a $\mathbb{K}^\times$-equivariant isomorphism $A[[\hbar]] \to A_h$ of $\mathbb{K}[[\hbar]]$-algebras such that the following diagram is commutative.

\[
\begin{array}{ccc}
S(g)^\theta[[\hbar]] & \xrightarrow{\cong} & A[[\hbar]] \\
\downarrow & & \downarrow \\
\mathcal{Z}(U_\hbar^\wedge(g)) & \xrightarrow{\sim} & A_h
\end{array}
\]

Let $B$ be a commutative algebra and $V$ a symplectic vector space. We denote by $\hat{W}_V(B)$ the algebra $B[[V^*, \hbar]]$ equipped with the Moyal-Weyl star-product.

**Lemma 4.2.** Let $B$ be an integral domain and $V$ a symplectic vector space. If $f \in \hat{W}_V(B)$ is integral over $B[[\hbar]]$, then $f \in B[[\hbar]]$.

**Proof.** Let $F$ denote the algebraic closure of the fraction field $\text{Quot}(B)$ of $B$. There is the natural embedding $\hat{W}_V(B) \hookrightarrow \hat{W}_V(F)$ and we may assume that $B = F$. Let $P \in F[t]$ be a monic polynomial such that $P(f) = 0$. The algebraic closure of $\text{Quot}(F[[\hbar]])$ coincide with the field $F\{\hbar\}$ consisting of all expressions of the form $g(\hbar^\alpha)$, where $\alpha \in \mathbb{Q}, g \in F[[\hbar]]$, see, for example, [2]. So there are $g_1, \ldots, g_k \in F\{\hbar\}$ such that $P(t) = \prod_{i=1}^k(t - g_i)$. Let $\nu = \hbar^{1/r}$ be such that $g_1, \ldots, g_k$ are Laurent power series in $\nu$. Consider the algebra $\hat{W}$ that coincides with $F[[V^*, \nu]]$ as the vector space and is equipped with the Moyal-Weyl star-product with $\hbar = \nu^r$. The algebra $\hat{W}_V(F)$ is naturally embedded into $\hat{W}$. For sufficiently large $n$ we have $\nu^n P(f) = \prod_{i=1}^k P_i(f)$, where $P_i$ is a linear polynomial. To prove the claim of the lemma it remains to verify that $\hat{W}$ has no zero divisors. Indeed, let $a = \sum_{i=0}^\infty a_i\nu^i, b = \sum_{i=0}^\infty b_i\nu^i, a_i, b_i \in F[[V^*]]$ be such that $a \ast b = 0$. Then $a_j b_l = 0$, where $j, l$ be the minimal integers such that $a_j \neq 0, b_l \neq 0$, which is nonsense. \qed

**Proposition 4.3.** Denote by $\pi$ the quotient morphism $X \to X//G$. Let $g$ lie in $\psi^\ast(S(g)^\theta)$. Set $X^0 := \{x \in X | g(x) \neq 0\}$. If an element $\hat{f} \in \mathbb{K}[X^0][[\hbar]]^G$ is integral over the image of $\mathcal{Z}(U_\hbar^\wedge(g))$, then $\hat{f} \in \mathcal{Z}(\mathbb{K}[X][[\hbar]]^G)$.

**Proof.** It is enough to check that $\hat{f} \in \mathcal{Z}(\mathbb{K}[X^0][[\hbar]]^G)$ and that $\hat{f}$ is defined in all points $y \in (X//G)^{reg}$ such that

1. $y$ satisfies condition (⋆) of Proposition [2.3];
2. either $g(y) \neq 0$ or $y$ is a smooth point of the zero locus of $g$. 


Let $X_0, V, X', x', \rho$ be such as in (1). We may assume that $\rho(x) = x'$ and identify $Gx \cong Gx'$.

Let $\lambda$ be the characteristic class used in the construction of $*$. Choose a representative $\lambda$ of $\lambda$ in $\Omega^2(X)[h]$. Set $\lambda = \pi^*(\iota^*(\lambda))$, where $\iota$ denotes the inclusion $Gx \hookrightarrow X$ and $\pi$ the projection $X' \to Gx$. Finally, let $\lambda'$ denote the class of $\lambda'$. It is easy to see that $i_\xi \lambda'$ is an exact form. Construct the star-product on $\mathbb{K}[X'][[h]]$ w.r.t. some $G$-invariant connection and the characteristic class $\lambda'$. By Theorem 3.6, $X', X_0$ have $*$-Hamiltonian structures. Then $\rho$ induces an isomorphisms $\iota : \mathbb{K}[X]_G \to \mathbb{K}[X]_{G', G}$, $\mathbb{K}[X]_{G}[[g^{-1}]] \to \mathbb{K}[X]_{G'}[[g^{-1}]]$. The characteristic classes of the star-products on $\mathbb{K}[X]_G[[h]], \mathbb{K}[X]_{G'}[[h]]$ coincide, so there is an isomorphism $\iota_h := (id + \sum_{i=1}^\infty T_i h^i) \circ \iota : \mathbb{K}[X]_{G}[[h]] \to \mathbb{K}[X]_{G'}[[h]]$ of quantum algebras, where all $T_i$ are differential operators, see Proposition 3.6. So $\iota_h$ is extended to the isomorphism $\iota_h : \mathbb{K}[X]_G[[g^{-1}]][[h]] \to \mathbb{K}[X]_{G'}[[g^{-1}]][[h]]$ of quantum algebras. Taking $G$-invariants, we get the isomorphism $\mathbb{K}[X/G]_G[[g^{-1}]][[h]] \cong \mathbb{K}[X/G]_{G'}[[g^{-1}]][[h]]$. From construction it is clear that $\iota(g) \in \mathbb{K}[X_0]^G \hookrightarrow \mathbb{K}[X/G]_{G'}[[g^{-1}]] = \mathbb{K}[X_0/G]_{G'}[[V^H]][[g^{-1}]].$

Set $B := \mathbb{K}[X_0/G]_{y}, \tilde{B} := B[\iota(g)^{-1}]$. Let us check that the structures of the classical and quantum algebras on $\tilde{B}[[h]]$ are isomorphic. The algebra $B$ is isomorphic to the formal power series algebra $\mathbb{K}[x_1, \ldots, x_n]$. By the assumptions on $y$, we may assume that either $g(y) \neq 0$, and then $\tilde{B} = B$, or $\tilde{B} = \mathbb{K}[x_1, \ldots, x_n][x_1^{-1}]$. Now let us show that the quantum algebra $B[[h]]$ is commutative. Indeed, $\mathbb{K}[X_0]^G$ is algebraic over the image of $S(g)^G$, for $X_0$ is coisotropic. From that and the observation that the star-product on $\mathbb{K}[X_0]^G[[h]]$ is differential we see that the quantum algebra $\mathbb{K}[X_0/G]_{G'}[[h]]$ is commutative. On the other hand, $\mathbb{K}[X_0/G]$ is dense in $B$ and we are done. Since the star-product on $B[[h]]$ is differential, we see that the quantum algebra $\tilde{B}[[h]]$ is also commutative. So both classical and quantum algebras $B[[h]]$ are commutative complete local algebras. Therefore there are $q_1, \ldots, q_n \in B[[h]]$ such that the map $\varphi : x_i \to x_i + hq_i$ defines the homomorphism from the classical algebra to the quantum one. Since $\tilde{B}[[h]]$ is naturally identified with $B[\varphi(x_1)^{-1}][h]]$, we see that $\varphi$ is extended to the isomorphism $\tilde{B}[[h]] \to \tilde{B}[[h]]$.

So we have the isomorphism of quantum algebras

$$\mathbb{K}[X'/G]_y[\iota(g)^{-1}][h]] \cong \tilde{B}[[h]] \otimes_{\mathbb{K}[h]} W_{V^H}(\mathbb{K}) \cong \hat{W}_{V^H}(\tilde{B}).$$

By Lemma 1.2, $\iota_h(f) \in \tilde{B}[[h]]$. Since $\iota_h(f)$ is integral even over the image of $\mathbb{Z}(U_n^H(g))$ in $B[[h]]$, we easily get $\iota_h(f) \in B[[h]]$. It follows that $\hat{f}$ is defined in $y$ and commutes with the whole algebra $\mathbb{K}[X/G]_y$. This completes the proof.

Proof of Theorem 4.4. By the graded version of the Noether normalization theorem, there are algebraically independent homogeneous elements $g_1, \ldots, g_k \in \mu^*(S(g)^G)$ that are algebraically independent such that $\mu^*(S(g)^G)$ is finite over $B_0 := \mathbb{K}[g_1, \ldots, g_k]$.

Let $B$ be a subalgebra of $A$ containing $B_0$. Suppose there is a continuous homomorphism $\iota : B[[h]] \to A_h$ of $\mathbb{K}[h]$-algebras such that the diagram analogous to that of the theorem (with $B$ instead of $A$) is commutative. Automatically, the image of $\iota$ is closed and $\iota$ is a topological isomorphism onto its image.

Clearly, $B = B_0$ satisfies the above conditions. By definition, $A$ is a finite $B_0$-module, so any ascending chain of subalgebras between $B_0$ and $A$ is finite. So we assume that $B$ is a maximal subalgebra of $A$ satisfying the conditions of the previous paragraph. Theorem 4.4 is equivalent to the equality $A = B$. Assume that $A \neq B$.

Choose some homogeneous elements $a_0, \ldots, a_{n-1} \in B, a_0 \neq 0$ and $f \in A$. Set $P(t) = \sum_{i=0}^{n} a_i t^i, a_n := 1$. Suppose $P(f) = 0$ and $Q(f) \neq 0$ for any monic polynomial $Q \in B[x]$.
with \( \deg Q < \deg P \). Set \( \widehat{a}_i = \iota(a_i), \ i = 0, n - 1 \). We are going to show that there exists a homogeneous element \( \widehat{f} = \sum_{i=0}^{\infty} f_i h^i \in \mathbb{K}[X][[h]]^G \) such that \( f_0 = f \) and

\[
(*) \quad \widehat{f}^* + \widehat{a}_{n-1} * \widehat{f}^{(n-1)} + \ldots + \widehat{a}_0 = 0.
\]

Suppose we have already constructed such \( \widehat{f} \). Let \( \widehat{B} \) denote the subalgebra in \( A \) generated by \( B \) and \( f \). By Proposition 4.3, \( \widehat{f} \) is a central element of \( \mathbb{K}[X][[h]]^G \). Let \( \overline{\iota} \) denote the continuous \( \mathbb{K}[[h]] \)-algebra homomorphism \( \overline{\iota} : \mathbb{K}[X][[h]]^G \to \mathbb{K}[X][[h]]^G \) defined by \( \overline{\iota}(b) = \iota(b), b \in B, \overline{\iota}(f) = \widehat{f} \). By construction, \( \overline{\iota} \) is well-defined and satisfies the assumptions of the first paragraph of the proof.

At first, we show that there is an open \( G \)-stable subvariety \( X^1 \) such that there is an element \( \widehat{f} \in \mathbb{K}[X]][[h]]^G \) satisfying \( (*) \). Namely, for \( X^1 \) we take the set of all points \( x \in X \) such that \( \frac{d_{P}(f)}{dt} \) is nonzero in \( x \). Since \( X^1 \) contains \( X^0 \) of the form indicated in Proposition 4.3, we will automatically get \( \widehat{f} \in \mathbb{K}[X][[h]]^G \).

We will construct such \( \widehat{f} \) recursively. Clearly, if \( \widehat{f} \) satisfies \( (*) \) iff \( \widehat{f}^{(m)} := \sum_{i=0}^{m} f_i h^i \) satisfies

\[
(*_m) \quad \widehat{f}^{(m)} + \widehat{a}_{n-1} * \widehat{f}^{(n-1)} + \ldots + \widehat{a}_0 \in h^{m+1} \mathbb{K}[X][[h]]
\]

for any \( m \in \mathbb{N} \).

Suppose we have already found \( f_1, \ldots, f_m \in \mathbb{K}[X]^G \) such that \( \widehat{f}^{(m)} \) satisfies \( (*_m) \). Let us check that there is a unique element \( \widehat{f}^{(m+1)}(x) \in \mathbb{K}[X]^G \) such that \( \widehat{f}^{(m+1)}(x) \) satisfies \( (*_{m+1}) \). This follows from the observation that the coefficient of \( h^{m+1} \) in the l.h.s. of \( (*_{m+1}) \) is equal to \( \frac{d_{P}(f)}{dt} \) \( f_{m+1} - Q \), where \( Q \) depends only on \( f_0, \ldots, f_m \). So \( \widehat{f}^{(m+1)} \) is constructed. By construction, \( \widehat{f}^{(m+1)} \) is homogeneous, the degree of \( \widehat{f}^{(m+1)} \) coincides with that of \( f \) and \( \widehat{f}^{(m+1)} \) satisfies \( (*_{m+1}) \). \( \square \)

Proof of Theorem 1.1. Let us construct an algebra homomorphism \( \mathcal{Z}(\mathbb{K}[X]^G) \to \mathcal{Z}(\mathbb{K}[X][[h]]^G) \) that is a section of \( \mathcal{Z}(\mathbb{K}[X][[h]]^G) \to \mathcal{Z}(\mathbb{K}[X]^G) \), \( \sum_{i=0}^{\infty} f_i h^i \mapsto f_0 \). Let \( \overline{\iota} \) denote an isomorphism \( A[[h]] \to A_h \) constructed in Theorem 4.1.

By Proposition 2.2, \( \mathcal{Z}(\mathbb{K}[X]^G) = \overline{\psi}^* (\mathcal{Z}(\mathbb{K}[X][[h]]^G)) \). So for any \( f \in \mathcal{Z}(\mathbb{K}[X]^G) \), there exist elements \( f_1, \ldots, f_k, g_1, \ldots, g_t \in A \) such that \( f_g = f_i \) and for any \( y \in \text{im} \overline{\psi} \) there is \( i \) with \( g_i(y) \neq 0 \). Then \( \frac{\overline{\iota}(f)}{\overline{\iota}(g_i)} = \frac{\overline{\iota}(f)}{\overline{\iota}(g_i)} \). By construction \( \overline{\iota}(g_i) - g_i \in hA_h \), so if \( g_i(y) \neq 0 \), then \( \frac{\overline{\iota}(f)}{\overline{\iota}(g_i)} \) is defined in \( y \). So the fractions \( \frac{\overline{\iota}(f)}{\overline{\iota}(g_i)} \) are glued together into an element \( \widehat{f} \in \mathbb{K}[X][[h]]^G \). Since \( \overline{\iota}(f_i), \overline{\iota}(g_i) \in \mathcal{Z}(\mathbb{K}[X][[h]]^G) \), we get \( \widehat{f} \in \mathcal{Z}(\mathbb{K}[X][[h]]^G) \). Set \( \overline{\iota}(f) := \widehat{f} \). It is clear from the construction that the map \( \overline{\iota} : \mathcal{Z}(\mathbb{K}[X]^G) \to \mathcal{Z}(\mathbb{K}[X][[h]]^G) \) has the desired properties.

Let us lift \( \overline{\iota} : \mathcal{Z}(\mathbb{K}[X]^G) \to \mathcal{Z}(\mathbb{K}[X][[h]]^G) \) to the continuous \( \mathbb{K}[[h]] \)-algebra homomorphism \( \iota_h : \mathcal{Z}(\mathbb{K}[X]^G)[[h]] \to \mathcal{Z}(\mathbb{K}[X][[h]]^G) \). By the construction of \( \iota \), the homomorphism \( \iota_h \) is injective and makes the right triangle of the theorem diagram commutative. From the commutativity of the triangle one easily deduces that this homomorphism is also surjective. It follows from the properties of the isomorphism \( A[[h]] \to A_h \) that the left square is also commutative. \( \square \)

5. SOME SPECIAL CASES

At first, let \( X = V \) be a symplectic vector space with constant symplectic form and Moyal-Weyl star-product \( * \). Suppose \( G \) is a connected reductive group acting on \( V \) by linear symplectomorphisms. This action is Hamiltonian with quadratic hamiltonians. Equip \( V \) with the action of \( \mathbb{K}^* \) given by \( (t, v) \mapsto t^{-1} v \). Setting \( t_0 = t^2 h \) we make \( * \) a homogeneous
star-product. The quantum hamiltonians are homogeneous of degree 2. By \[L1\], Theorem 1.2.7, \(\tilde{\psi}_{G,V}\) is surjective whence \(\mathbb{Z}(\mathbb{K}[V][\hbar]) = \mathbb{K}[C_{G,V}]\). Further, (\[L3\], Corollary 3.12. or \[Kn4\], Section 1) \(C_{G,V}\) is an affine space. By Theorem \[L1\] there is a \(\mathbb{K}^\times\)-equivariant isomorphism \(\mathbb{K}[C_{G,V}][[\hbar]] \to \mathcal{Z}(\mathbb{K}[V][\hbar])^{G}\). The \(\mathbb{K}^\times\)-finite parts of these two algebras coincide with \(\mathbb{K}[C_{G,V}][\hbar], \mathcal{Z}(\mathbb{K}[V][\hbar])^{G}\). Taking quotients of these algebras by the ideal generated by \(\hbar - 1\), we get an isomorphism \(\mathbb{K}[C_{G,V}] \to \mathcal{Z}(W(V)^G)\), where \(W(V)\) is the Weyl algebra of \(V\). In particular, \(\mathcal{Z}(W(V)^G)\) is a polynomial algebra.

Now we consider the case when \(X = T^*X_0\) for some smooth affine \(G\)-variety \(X_0\). We consider the action \(\mathbb{K}^\times : X\) given by \(t.(x_0, \alpha) = (x_0, t^{-1}\alpha), t \in \mathbb{K}^\times, x_0 \in X_0, \alpha \in T^*_x X_0\). Choose a \(G \times \mathbb{K}^\times\)-invariant symplectic connection \(\nabla\) on \(X\). Construct the star-product on \(X\) by means of \(\nabla\) and the zero characteristic class. Setting \(t.h = th\), we make \(*\) a homogeneous star-product. Again, the morphism \(\tilde{\psi}\) is surjective and \(C_{G,X}\) is a polynomial algebra. The latter follows from results of \[Kn1\]. The quantum algebra \(\mathbb{K}[X][[\hbar]]\) is \(G \times \mathbb{K}^\times\)-equivariantly isomorphic to the completed homogeneous algebra \(D^h_\hbar(X_0)\) of differential operators on \(X_0\), which is defined as follows. Let \(\text{Der}(X_0)\) denotes the space of all vector fields on \(X_0\). By the homogeneous algebra of differential operators on \(X_0\) we mean the quotient \(D^h_\hbar(X_0)\) of \(T(\mathbb{K}[X_0] \oplus \text{Der}(X_0))[[\hbar]]\) by the relations \(f \otimes g = fg, \xi \otimes f - f \otimes \xi = h\xi, f, \xi \otimes \eta - \eta \otimes \xi = \hbar\xi, f, g \in \mathbb{K}[X_0], \xi, \eta \in \text{Der}(X_0)\). We have a natural action \(G \times \mathbb{K}^\times : D^h_\hbar(X_0)\), the group \(\mathbb{K}^\times\) acts as follows: \(t.f = f, t.\xi = t\xi, t.h = h, f \in \mathbb{K}[X_0], \xi \in \text{Der}(X_0)\). By definition, \(D^h_\hbar(X_0)\) is the completion of \(\text{Der}(X_0)\) in the \(\hbar\)-adic topology.

For a particular choice of \(\nabla\) an isomorphism \(\mathbb{K}[X][[\hbar]] \to D^h_\hbar(X_0)\) was constructed in \[BNW\] and the algebra \(\mathbb{K}[X][[\hbar]]\) does not depend (up to a \(G \times \mathbb{K}^\times\)-equivariant isomorphism) on the choice of \(\nabla\). So we get an isomorphism \(\mathbb{K}[C_{G,X}] \cong \mathcal{Z}(D(X_0)^G)\). This is a weak version of the main theorem of \[Kn2\].

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\(^1\)ADDED IN PROOF. In fact, the quantization \(D^h_\hbar(X_0)\) of \(\mathbb{K}[X]\) corresponds not to the zero characteristic class but to the half of the Chern class of the canonical bundle. This is implicitly contained in [BNW]. However, the precise value of the correspondent characteristic class is not important for our argument. Indeed, any quantization of \(\mathbb{K}[X]\) is isomorphic to a Fedosov one with some characteristic class. Theorem 1.1 works for an arbitrary characteristic class as long as the comoment map can be quantized. The last condition definitely holds for \(D^h_\hbar(X_0)\).
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