Primordial non-Gaussianities in general modified gravitational models of inflation

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We compute the three-point correlation function of primordial scalar density perturbations in a general single-field inflationary scenario, where a scalar field $\phi$ has a direct coupling with the Ricci scalar $R$ and the Gauss-Bonnet term $G$. Our analysis also covers the models in which the Lagrangian includes a function non-linear in the field kinetic energy $X = -(\partial \phi)^2/2$, and a Galileon-type field self-interaction $G(\phi, X)\Box \phi$, where $G$ is a function of $\phi$ and $X$. We provide a general analytic formula for the equilateral non-Gaussianity parameter $f_{\text{NL}}^{\text{equi}}$ associated with the bispectrum of curvature perturbations. A quasi de Sitter approximation in terms of slow-variation parameters allows us to derive a simplified form of $f_{\text{NL}}^{\text{equi}}$ convenient to constrain various inflation models observationally.

If the propagation speed of the scalar perturbations is much smaller than the speed of light, the Gauss-Bonnet term as well as the Galileon-type field self-interaction can give rise to large non-Gaussianities testable in future observations. We also show that, in Brans-Dicke theory with a field potential (including $f(R)$ gravity), $f_{\text{NL}}^{\text{equi}}$ is of the order of slow-roll parameters as in standard inflation driven by a minimally coupled scalar field.

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I. INTRODUCTION

The idea of cosmic acceleration in the early Universe [1, 2] was originally introduced as a way of addressing the flatness and horizon problems plagued in the big bang cosmology. The inflationary paradigm can provide a causal mechanism for generating density perturbations responsible for the Cosmic Microwave Background (CMB) temperature anisotropies [3]. The standard, single-field slow-roll models of inflation predict adiabatic and Gaussian primordial perturbations with a nearly scale-invariant spectrum. This property is consistent with the observed CMB anisotropies [4, 5] as well as the large-scale structure data [6].

Over the past 30 years, many models of inflation have been proposed in the framework of particle physics or extended theories of gravity (see e.g., [7] for reviews). From the information of the spectral index $n_R$ of the scalar perturbations as well as the tensor-to-scalar ratio $r$, the CMB observations by the WMAP satellite have been able to rule out some of those models [3–7]. While the observables $n_R$ and $r$ are derived in linear cosmological perturbation theory, it is also possible to distinguish between a host of inflationary models further by comparing the non-Gaussianity of primordial perturbations with the WMAP data [8]. In particular it is expected that the PLANCK satellite [9] will bring us more precise data of non-Gaussianities within a few years.

The amount of non-Gaussianity can be quantified by evaluating the bispectrum of curvature perturbations $\mathcal{R}$, as

$$\langle \mathcal{R}(k_1)\mathcal{R}(k_2)\mathcal{R}(k_3) \rangle = (2\pi)^3\delta^{(3)}(k_1 + k_2 + k_3)B_R(k_1, k_2, k_3),$$

where $\mathcal{R}(k)$ is a Fourier component of $\mathcal{R}$ with a wave number $k$. Conventionally the bispectrum $B_R$ translates into a non-linear parameter $f_{\text{NL}}$ in order to confront theoretically predicted non-Gaussianities with observations [6, 10]. There are two different shapes of the primordial bispectrum: (1) “local” type [10, 12, 13], and (2) “equilateral” type [14]. The first one arises from a local, point-like non-Gaussianity given by $\mathcal{R}(x) = \mathcal{R}_L(x) + (3/5)f_{\text{NL}}^{\text{local}}\mathcal{R}_L^2(x)$, where $\mathcal{R}_L$ is a linear Gaussian perturbation. In single-field inflation models driven by a slowly varying potential, the predicted value of $f_{\text{NL}}^{\text{local}}$ is of the order of slow-roll parameters, i.e., $|f_{\text{NL}}^{\text{local}}| \ll 1$ [11, 12, 13, 21]. This small amount of non-Gaussianity is consistent with the WMAP 7-year bound: $-10 < f_{\text{NL}}^{\text{local}} < 74$ (95% CL) [6].

In the equilateral shape of non-Gaussianities, the momentum dependence of the function $B_R(k_1, k_2, k_3)$ arising from non-canonical kinetic term models (“k-inflation” [22]) can be approximated in a suitable form [23]. In k-inflation models, including Dirac-Born-Infeld (DBI) inflation [24] and (dilatonic) ghost condensate [25], one can realize the non-linear parameter $f_{\text{NL}}^{\text{equi}}$ larger than the order of unity provided that the propagation speed $c_s$ of scalar perturbations is much smaller than 1 (in unit of speed of light) [25, 30]. The WMAP 7-year bound is $-214 < f_{\text{NL}}^{\text{equi}} < 266$ (95% CL) [6], but it is expected that the PLANCK satellite can reduce this limit by one order of magnitude. This will provide us an opportunity to distinguish k-inflation from standard slow-roll inflation. We also note that it is possible to give rise to large non-Gaussianities in multiple scalar-field models [31–40], curvaton models [41–42], modulated reheating models [43, 44], models having a preheating stage after inflation [48], and models with a temporal non-slow roll stage [49] (see also Refs. [50]).
In this paper we shall compute the bispectrum $B_R$ and the equilateral-type non-linear parameter $f_{\text{NL}}^{\text{equil}}$ for general single-field inflation models described by the action (2) below. In low-energy effective string theory there is a scalar field $\phi$ called dilaton coupled to the Ricci scalar $R$ with a form $F(\phi)R$ \cite{51}. A field coupling of the form $\xi(\phi)G$, where $G$ is the Gauss-Bonnet term, also arises as a higher-order string correction to the low-energy effective string action \cite{52}. Furthermore the higher-order string correction contains a non-canonical kinetic term like $(\partial \phi)^4$ as well as a field self-interaction of the form $g(\phi)(\partial \phi)^2 c_{\Box} \phi$ in the action. For constant $g(\phi)$ the latter is linked to the Lagrangian of a covariant Galileon field that respects the Galilean symmetry $\partial_\mu \phi \to \partial_\mu \phi + b_\mu$ in the Minkowski space-time \cite{53,54}. The cosmological dynamics in the presence of the Galileon-type interaction $g(\phi)(\partial \phi)^2 c_{\Box} \phi$ have been extensively studied recently in the context of inflation \cite{55,56} and dark energy \cite{57}. We accommodate non-linear field derivative terms as the Lagrangian of the form $P(\phi, X) - G(\phi, X) \Box \phi$, where $P$ and $G$ are functions in terms of $\phi$ and $X = -(\partial \phi)^2/2$. Note that non-Gaussianities in the models described by the Lagrangian $F(\phi)R + P(\phi, X)$ without the terms $\xi(\phi)G$ and $G(\phi, X) \Box \phi$ were recently studied in Ref. \cite{58}.

Our action \cite{2} can be also viewed as describing a kind of modified gravitational theories \cite{59}. In fact, this covers the so-called scalar-tensor theories \cite{60} such as Brans-Dicke (BD) theory \cite{61}. Since metric $f(R)$ gravity is equivalent to BD theory with the BD parameter $\omega_{\text{BD}} = 0$ \cite{62}, our results can be applied also to the Starobinsky’s inflation model $f(R) = R + \alpha R^2$ \cite{63}. Since the field propagation speed $c_s$ is unity in BD theory, we will show that $|f_{\text{NL}}^{\text{equil}}| \ll 1$ as in conventional slow-roll inflation. On the other hand, the terms $\xi(\phi)G$ and $G(\phi, X) \Box \phi$ as well as $P(\phi, X)$ can give rise to $c_s$ much smaller than 1. In such theories it is possible to realize large non-Gaussianities detectable in future high-precision observations.

This paper is organized as follows. In Sec. II the field equations of motion are derived for the action (2) on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background. In Sec. III we obtain the second-order action for the curvature perturbation $\mathcal{R}$ and present the solution of its mode function at linear level on the de Sitter background. In Sec. IV the third-order perturbed action is derived in order to compute the three-point correlation function of the curvature perturbation $\mathcal{R}$ in the interacting Hamiltonian picture. We also present a general analytic formula of $f_{\text{NL}}^{\text{equil}}$ valid in the quasi de Sitter background. In Sec. V we provide a simpler expression of $f_{\text{NL}}^{\text{equil}}$ under the expansion of “slow-variation” parameters. This approximate formula is enough to estimate the amount of non-Gaussianities in practical purpose. In Sec. VI we apply our general results to a number of concrete models of inflation: (1) k-inflation, (2) generalized Galileon model, and (3) Brans-Dicke theory. Sec. VII is devoted to conclusions. In Appendix we present the detailed procedure to derive and manipulate the third-order perturbed action.

II. THE MODEL AND BACKGROUND EQUATIONS

We start with the following action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} F(\phi)R + P(\phi, X) - \xi(\phi)G - G(\phi, X) \Box \phi \right],$$

where $g$ is the determinant of the space-time metric $g_{\mu\nu}$, $M_{\text{pl}} = (8\pi G)^{-1/2}$ is the reduced Planck mass ($G$ is gravitational constant), and $\phi$ is a scalar field with a kinetic term $X = -(1/2)g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi$. The functions $F(\phi)$ and $\xi(\phi)$ depend on $\phi$ only, whereas $P(\phi, X)$ and $G(\phi, X)$ are functions of both $\phi$ and $X$. The field $\phi$ couples to the Ricci scalar $R$ as well as the Gauss-Bonnet term $G$ defined by

$$G \equiv R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta},$$

where $R_{\alpha\beta}$ is the Ricci scalar and $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor for the metric $g_{\mu\nu}$. As we mentioned in the Introduction, the action (2) covers a wide variety of single-field inflationary models.

We consider the flat FLRW space-time with a scale factor $a(t)$, where $t$ is the cosmic time. Then the background equations are given by

$$E_1 \equiv 3M_{\text{pl}}^2 F H^2 + 3M_{\text{pl}}^2 H \dot{F} + P - 2XP_X - 24H^3 \dot{\xi} - 6H \phi X G_X + 2XG_{,\phi} = 0,$$

$$E_2 \equiv 3M_{\text{pl}}^2 F H^2 + 2M_{\text{pl}}^2 H \ddot{F} + 2M_{\text{pl}}^2 \ddot{H} + M_{\text{pl}}^2 \dddot{F} + P - 16H^3 \ddot{\xi} - 16H \dddot{H} \dot{\xi} - 8H^2 \dddot{\xi} - G_X \dot{\phi} \ddot{\phi} - G_{,\phi} \dddot{\phi}^2 = 0,$$

$$E_3 \equiv (P_X + 2XP_{XX} + 6H \phi G_X + 6H \dot{\phi} X G_{XX} - 2XG_{,\phi \phi} - 2G_{,\phi}) \dddot{\phi} + (3HP_X + \phi P_{\dot{\phi} X} + 9H^2 \dot{\phi} G_X + 3\ddot{H} \phi G_X + 3H \dot{\phi}^2 G_{\phi X} - 6HG_{,\phi} - G_{,\phi \phi}) \dot{\phi} - P_{\phi} - 6M_{\text{pl}}^2 H^2 F_{,\phi} - 3M_{\text{pl}}^2 H \dot{F}_{,\phi} + 24H^4 \dot{\xi}_{,\phi} + 24H^2 \dddot{H}_{,\phi} = 0,$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, and a dot represents a derivative with respect to $t$. These equations are not independent because of the Bianchi identities, i.e. $\dot{\phi} E_3 + \dddot{\phi} E_1 + 3H(E_1 - E_2) = 0$. In Eq. (5) the term $G_{,\phi} \dot{\phi} \ddot{\phi}$ vanishes.
for the theories with \( G_X = 0 \), in which case the Lagrangian \( G(\phi)\Box \phi \) can be regarded as a part of the Lagrangian \( P(\phi, X) \). If \( G_X \neq 0 \), however, the term \( G(\phi, X)\Box \phi \) should be treated separately from the term \( P(\phi, X) \).

In order to derive the slowly varying parameter \(-\dot{H}/H^2\) we consider the combination \((E - E_1)/(M_{pl}^2 H^2 F) = 0\), which gives

\[
\epsilon \equiv -\frac{\dot{H}}{2H^2 F} = -\frac{F}{2M_{pl}^2 H^2 F} + \frac{F}{2H^2 F} + \frac{XP_X}{M_{pl}^2 H^2 F} + 4\frac{H\xi}{M_{pl}^2 F} - \frac{8\dot{H}\xi}{M_{pl}^2 H^2 F} - \frac{4\dot{\phi}XG_X}{M_{pl}^2 H^2 F} - \frac{\phi XG_X}{M_{pl}^2 H^2 F} - \frac{2XG_{,\phi}}{M_{pl}^2 H^2 F}.
\]  

(7)

During inflation the Hubble parameter changes slowly, so that the condition \( \epsilon \ll 1 \) is satisfied. Hence, in general, we require that each term on the r.h.s. of Eq. (7) is much smaller than unity. In Sec. V we shall use this property to obtain a simple expression for the equilateral non-linear parameter \( f_{NL}^{\text{equil}} \). For later convenience we introduce the following “slow-variation” parameters

\[
\delta_F = \frac{\dot{F}}{H^2 F}, \quad \delta_\xi = \frac{H\dot{\xi}}{M_{pl}^2 F}, \quad \delta_{GX} = \frac{\dot{\phi}XG_X}{M_{pl}^2 H^2 F}.
\]  

(8)

### III. SECOND-ORDER ACTION AND LINEAR PERTURBATIONS

In order to compute primordial scalar non-Gaussianities we need to expand the action (2) up to third order in the perturbations, taking into account, up to a gauge choice, both the perturbations in the scalar field, \( \delta \phi \), and in the scalar modes of the metric. The interacting Hamiltonian of perturbations follows from the third-order Lagrangian, by which the three-point correlation function of curvature perturbations \( \mathcal{R} \) can be evaluated in the framework of quantum field theory [12]. In order to calculate the vacuum expectation value of the correlation function, the mode function of \( \mathcal{R} \) should be known in the quasi de Sitter background. Once the mode function is obtained at linear level, one can derive the power spectrum of \( \mathcal{R} \) generated during inflation. The linear perturbation equation for \( \mathcal{R} \) is known by the second-order perturbed action.

For the derivation of the action expanded up to third order in the perturbations, it is convenient to work in the ADM formalism [63] with the line element

\[
ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j),
\]

(9)

where \( N \) and \( N^i \) are the lapse and shift functions, respectively. Here we consider only scalar metric perturbations about the flat FLRW background. In doing so we expand the lapse \( N \) and the shift vector \( N^i \), as \( N = 1 + \alpha \) and \( N^i = \partial_i \psi \), respectively \((i = 1, 2, 3)\). In fact, in the ADM formalism, the lapse \( N \) and the shift \( N^i \) are Lagrange multipliers, so that it is sufficient to know \( N \) and \( N^i \) up to first order. This is because the third-order and the second-order terms in \( N \) and \( N^i \) multiply the constraint equations at zero-th order and at first order, respectively, so that their contributions vanish [12, 64]. We choose the uniform-field gauge with \( \delta \phi = 0 \), which fixes the time-component of a gauge-transformation vector \( \xi^\mu \). We gauge away a field \( E \) that appears as a form \( E_{ij} \) inside \( h_{ij} \), by fixing the spatial part of \( \xi^\mu \). Then the three-dimensional metric can be written as \( h_{ij} = a^2(t)e^{2\mathcal{R}}\delta_{ij} \). This results in the following metric

\[
ds^2 = -[(1 + \alpha)^2 - a^{-2}(t)e^{-2\mathcal{R}}(\partial \psi)^2] dt^2 + 2\partial_i \psi dt dx^i + a^2(t)e^{2\mathcal{R}}(dx^2 + dy^2 + dz^2),
\]

(10)

where \((\partial \psi)^2 = (\partial \psi_1)(\partial \psi_1) = (\partial_x \psi)^2 + (\partial_y \psi)^2 + (\partial_z \psi)^2 \). Here and in the following, same lower latin indices are summed, unless otherwise specified. At level the metric (10) reduces to the standard one used in linear perturbation theory, that is [64]

\[
ds^2 = -(1 + 2\alpha) dt^2 + 2\partial_i \psi dt dx^i + a^2(t)(1 + 2\mathcal{R}) (dx^2 + dy^2 + dz^2).
\]

(11)

Expanding the action (2) up to second order, one finds

\[
S_2 = \int dt \, d^3x \, a^3 \left[ -3w_1 \frac{\dot{\mathcal{R}}}{a^2} \Box \partial^2 \psi - \frac{w_2}{a^2} \alpha \partial^2 \psi - \frac{2w_1}{a^2} \alpha \partial^2 \mathcal{R} + 3w_2 \alpha \dot{\mathcal{R}} + \frac{1}{2} w_3 \alpha^2 + \frac{w_4}{a^2} \partial_i \mathcal{R} \partial_i \mathcal{R} \right],
\]

(12)

where

\[
w_1 \equiv M_{pl}^2 F - 8H \dot{\xi}, \quad (13)
\]

\[
w_2 \equiv M_{pl}^4 (2HF + \dot{F}) - 2\dot{\phi}XG_{,X} - 24H^2 \dot{\xi}, \quad (14)
\]

\[
w_3 \equiv -9M_{pl}^4 FH - 9M_{pl}^2 HF \dot{F} + 2XG_{,X} + 3XP_X + 2X^2 P_{,XX} + 144H^3 \dot{\xi}
\]

\[
+ 18H^2 \dot{\phi}(2XG_{,X} + X^2 G_{,XX}) - 6(\dot{XG}_{,\phi} + X^2 G_{,\phi X}), \quad (15)
\]

\[
w_4 \equiv M_{pl}^2 F - 8\dot{\xi}.
\]

(16)
In the action \( [12] \), both the coefficients of the terms \( \alpha \mathcal{R} \) and \( \mathcal{R}^2 \) vanish by using the background equations of motion. This is related to the fact that the field \( \mathcal{R} \) does not have an explicit mass term. Furthermore, in \( [12] \), the term quadratic in \( \psi \) vanishes after integrations by parts. The equations of motion for \( \psi \) and \( \alpha \), derived from \( [12] \), lead to the following two constraints

\[
\alpha = L_1 \dot{\mathcal{R}}, \quad \frac{1}{a^2} \partial^2 \psi = \frac{2w_3}{3w_2} \alpha + 3 \dot{\mathcal{R}} - \frac{2w_1}{w_2} \frac{1}{a^2} \partial^2 \mathcal{R},
\]

where

\[
L_1 \equiv \frac{2w_1}{w_2} = \frac{2(M_{pl}^2 F - 8H^2 \dot{\xi})}{M_{pl}^2 (2HF + F) - 2\phi X G_{,X} - 24H^2 \dot{\xi}}.
\]

In k-inflation with \( F = 1, \xi = 0 \), and \( G = 0 \) one has \( L_1 = 1/H \). In general cases, expansion in terms of the slow-variation parameters defined in Eq. \( [8] \) gives

\[
L_1 = \frac{1}{H} \left[ 1 - \frac{1}{2} \frac{\delta F}{H} + 4 \delta \dot{\xi} + \delta G_{,X} + \mathcal{O}(\epsilon^2) \right].
\]

Plugging the relation \( \alpha = (2w_1/w_2) \dot{\mathcal{R}} \) into Eq. \( [12] \) and integrating the term \( \ddot{\mathcal{R}} \partial^2 \mathcal{R} \) by parts as

\[
\epsilon(t) \ddot{\mathcal{R}} \partial^2 \mathcal{R} = \dot{\epsilon} (\partial \mathcal{R})^2/2 + \text{total derivatives},
\]

one finds

\[
S_2 = \int dt d^3 x a^3 Q \left[ \ddot{\mathcal{R}}^2 - \frac{\epsilon^2}{a^2} (\partial \mathcal{R})^2 \right],
\]

where

\[
Q = \frac{w_1(4w_1w_3 + 9w_2^2)}{3w_2^2},
\]

\[
\epsilon_s^2 = \frac{3(2w_1^2 w_2 H - w_2^3 w_4 + 4w_1 \dot{w}_1 w_2 w_2 - 2w_1^2 \dot{w}_2)}{w_1(4w_1 w_3 + 9w_2^2)}. \quad (24)
\]

More explicit expressions for \( Q \) and \( \epsilon_s^2 \) are given in Appendix \( A \). In order to avoid the appearance of ghosts and Laplacian instabilities we require that \( Q > 0 \) and \( \epsilon_s^2 > 0 \), respectively. Using Eqs. \( [17] \) and \( [23] \), we can rewrite Eq. \( [18] \) as

\[
\psi = -L_1 \dot{\mathcal{R}} + \chi, \quad \text{where} \quad \partial^2 \chi = a^2 Q \frac{w_1}{w_1} \dot{\mathcal{R}}.
\]

We introduce a parameter \( \epsilon_s \) defined by

\[
\epsilon_s \equiv \frac{Q \epsilon_s^2}{M_{pl}^2 F} = \frac{2w_1^2 w_2 H - w_2^3 w_4 + 4w_1 \dot{w}_1 w_2 w_2 - 2w_1^2 \dot{w}_2}{M_{pl}^2 F w_2^2}.
\]

In k-inflation with \( F = 1, \xi = 0 \), and \( G = 0 \) this reduces to the slow-roll parameter \( \epsilon = -\dot{H}/H^2 \). In general, one can expand \( \epsilon_s \) in terms of slow-variation parameters, as

\[
\epsilon_s = \epsilon + \frac{1}{2} \delta F + \delta G_{,X} - 4 \delta \dot{\xi} + \mathcal{O}(\epsilon^2).
\]

The equation of motion for \( \mathcal{R} \) follows by varying the Lagrangian \( \mathcal{L}_2 = a^3 Q [\ddot{\mathcal{R}}^2 - (\epsilon_s^2/a^2) (\partial \mathcal{R})^2] \) in terms of \( \mathcal{R} \). We define

\[
\frac{\delta \mathcal{L}_2}{\delta \mathcal{R}} \bigg|_{\mathcal{R}} \equiv -2 \left[ \frac{d}{dt} (a^3 Q \dot{\mathcal{R}}) - aQ \epsilon_s^2 \partial^2 \mathcal{R} \right].
\]
Then the curvature perturbation obeys the equation \( \delta L_2 / \delta \mathcal{R} |_{\mathcal{R}} = 0 \) at linear level, i.e.

\[
\frac{d}{dt}(a^3Q\dot{\mathcal{R}}) - aQc_s^2\dot{\mathcal{R}} = 0.
\]

(29)

We write \( \mathcal{R} \) in Fourier space, as

\[
\mathcal{R}(\tau, x) = \frac{1}{(2\pi)^3} \int d^3k \mathcal{R}(\tau, k)e^{ikx}, \quad \mathcal{R}(\tau, k) = u(\tau, k)a(k) + u^*(\tau, -k)a^\dagger(-k),
\]

(30)

where \( a(k) \) and \( a^\dagger(k) \) are the annihilation and creation operators, respectively, satisfying the commutation relations

\[
[a(k_1), a^\dagger(k_2)] = (2\pi)^3\delta^{(3)}(k_1 - k_2), \quad [a(k_1), a(k_2)] = [a^\dagger(k_1), a^\dagger(k_2)] = 0.
\]

(31)

Note that \( \tau \equiv \int a^{-1}dt \) is the conformal time, which is given by \( \tau = -1/(aH) \) in the de Sitter background. The asymptotic past and future correspond to \( \tau \to -\infty \) and \( \tau \to 0 \), respectively.

Let us derive the solution of \( \mathcal{R} \) during inflation at linear order. The equation for the Fourier mode \( u \) is given by

\[
\ddot{u} + \frac{(a^3Q)}{a^3Q} \dot{u} + \epsilon_s^2 k^2 u = 0.
\]

(32)

In the large-scale limit \( (k \to 0) \) the solution to this equation is \( u = c_1 + c_2 \int (a^3Q)^{-1} dt \), where \( c_1 \) and \( c_2 \) are integration constants. Provided that the variable \( Q \) changes slowly in time, \( u \) approaches a constant after the perturbations exit the Hubble radius \( (c_s k \lesssim aH) \). Introducing a field \( v = zu \) with \( z = a\sqrt{2Q} \) the kinetic term in the second-order action \[22\] can be rewritten as \( \int d\tau d^3x \nu''/2 \), where a prime represents a derivative with respect to \( \tau \). In other words, \( v \) is the canonical field that should be quantized. Equation \[32\] can be written as

\[
v'' + \left(\epsilon_s^2 k^2 - \frac{z''}{z}\right) v = 0.
\]

(33)

In the de Sitter background with a slow variation of the quantity \( Q \), we can approximate \( z''/z \approx 2/\tau^2 \). In the asymptotic past \( (k\tau \to -\infty) \) we choose the Bunch-Davis vacuum characterized by the mode function \( v = e^{-ic_s k\tau}/\sqrt{2c_s k} \). Then the solution of Eq. \[33\] is given by

\[
u(\tau, k) = \frac{i H e^{-ic_s k\tau}}{2(c_s k)^{3/2} \sqrt{Q}} (1 + ic_s k\tau).
\]

(34)

The deviation from the exact de Sitter background gives rise to a small modification to the solution \[34\], but this difference appears as a next-order correction to the power spectrum and to the non-Gaussianity parameter \[28\].

The two-point correlation function, some time after the Hubble radius crossing, is given by the vacuum expectation value \( \langle 0|\mathcal{R}(\tau, \mathbf{k}_1)\mathcal{R}(\tau, \mathbf{k}_2)|0 \rangle \) at \( \tau \approx 0 \). We define the power spectrum \( P_R(k_1) \), as \( \langle 0|\mathcal{R}(0, \mathbf{k}_1)\mathcal{R}(0, \mathbf{k}_2)|0 \rangle = (2\pi^2/k_1^3)P_R(k_1) (2\pi^3\delta^{(3)})(k_1 + k_2) \). Using the solution \[34\], we obtain

\[
P_R = \frac{H^2}{8\pi^2Qc_s^3} = \frac{H^2}{8\pi^2M_{pl}^2 c_s^3 c_s},
\]

(35)

where we have used \( c_s \) defined in Eq. \[20\]. Since the curvature perturbation soon approaches a constant for \( c_s k < aH \), we just need to evaluate the power spectrum \[35\] at \( c_s k = aH \) during inflation \[65\]. The spectral index of \( \mathcal{R} \) is given by

\[
n_\mathcal{R} - 1 \equiv \frac{d \ln P_R}{d \ln k} \bigg|_{c_s k = aH} = -2\epsilon - \delta F - \eta_s - s = -2\epsilon_s - \eta_s - s - 8\delta\xi + 2\delta G_X,
\]

(36)

where

\[
\eta_s = \frac{\dot{c}_s}{Hc_s}, \quad s = \frac{\ddot{c}_s}{Hc_s}.
\]

(37)

Here we assumed that both \( H \) and \( c_s \) slowly vary, such that \( d\ln k \) at \( c_s k = aH \) is approximated as \( d\ln k = d\ln a = H dt \). In the second equality of Eq. \[36\] we used Eq. \[27\] to convert \( \epsilon \) to \( c_s \) at linear order.
Let us also derive the spectrum of tensor perturbations generated during inflation. In addition to the three-dimensional metric $a^2(t)e^{2\Phi}\delta_{ij}$ coming from the scalar part, we consider the intrinsic tensor perturbation $h_{ij}^{(T)}$. In terms of the two polarization tensors $e_{ij}^+$ and $e_{ij}^\times$ we can write $h_{ij}^{(T)} = h_+ e_{ij}^+ + h_\times e_{ij}^\times$, where both the symmetric tensors $e_{ij}$ are transverse and traceless. We also impose the normalization condition, $e_{ij}(k) e_{ij}(-k)^* = 2$, for each polarization, whereas $e_{ij}^+(k) e_{ij}^\times(-k)^* = 0$. The second-order action for the tensor modes is given by

$$S_T = \sum_\lambda \int dt \, d^3x \, a^3 Q_T \left[ \dot{h}_\lambda^2 - \frac{c_s^2}{a^2} (\partial h_\lambda)^2 \right],$$  

(38)

where $\lambda = +, \times$, and

$$Q_T = \frac{1}{4} w_1 = \frac{1}{4} M_{pl}^2 F (1 - 8\delta\xi),$$  

(39)

$$e^2_T = \frac{w_4}{w_1} = \frac{M_{pl}^2 F - 8\xi}{M_{pl}^2 F - 8H\xi} = 1 + 8\delta\xi + {\cal O}(\epsilon^2).$$  

(40)

A canonical field associated with $h_\lambda$ corresponds to $v_T = \zeta_T h_\lambda$ and $\zeta_T = a\sqrt{2Q_T}$. Following the same procedure as before, the solution to $h_\lambda$ recovering the Bunch-Davis vacuum in the asymptotic past is

$$h_\lambda = \frac{i H e^{-ic_T k\tau}}{2(\zeta_T k)^{3/2}\sqrt{Q_T}} \left( 1 + ic_T k\tau \right),$$  

(41)

which approaches $h_\lambda \to i H/[2(\zeta_T k)^{3/2}\sqrt{Q_T}]$ after the Hubble radius crossing. According to the chosen normalization for the tensors $e_{ij}$, the two-point correlation function leads to a tensor power spectrum, whose expression is given by

$$P_T = 4 \cdot k^3 |h_{ij}|^2/(2\pi^2),$$  

(42)

where in the last approximate equality we have used Eqs. (39) and (40) at leading order. The spectral index of $P_T$ is

$$n_T \equiv \frac{d}{d\ln k} \left[ \frac{P_T}{k^3} \right]_{c_T k = aH} = -2\epsilon - \delta_F = -2\epsilon_s - 8\delta\xi + 2\delta_{GX},$$  

(43)

which is valid up to first order of slow-variation parameters.

For those times before the end of inflation ($\epsilon \ll 1$) when both $P_R$ and $P_T$ remain approximately constants, we can estimate the tensor-to-scalar ratio as

$$r \equiv \frac{P_T}{P_R} = 4 \frac{Q e^3_T}{Q_T c_T^3} \approx 16c_s,\epsilon_s,$$  

(44)

where in the last approximate equality the terms at second order of slow-variation parameters are not taken into account. Using Eqs. (43) and (44), we obtain the consistency relation

$$r = 8c_s (-n_T - 8\delta\xi + 2\delta_{GX}).$$  

(45)

The Gauss-Bonnet term, as well as the Galileon term, modifies the consistency relation $r = -8c_s n_T$ valid in k-inflation [65].

IV. THIRD-ORDER ACTION AND PRIMORDIAL NON-GAUSSIANITIES

In order to evaluate the three-point correlation function of curvature perturbations, we need to expand the action up to third order in the perturbation fields. From the computational point of view, the fact that we just need to expand the lapse and shift only up to first order (as already stated in Sec. III) is very useful for deriving the action at cubic order.
After many integrations by parts, the third-order action following from (2) can be written as

\[ S_3 = \int dt \, dx \, a^3 \{ a_1 \, a^3 + a^2 (a_2 \, R + a_3 \, \dot{R} + a_4 \, \dot{\phi}^2 R/a^2 + a_5 \, \dot{\phi}^2 \psi/a^2) \\
+ a_6 [a_6 \, \partial_i R \partial_j \psi/a^2 + a_7 \, \dot{R} \, R + a_8 \, \dot{R} \, \dot{\phi}^2 R/a^2 + a_9 \, (\partial_i \partial_j \psi \partial_i \partial_j \psi - \dot{\phi}^2 \dot{\psi}^2 \psi)/a^4] \\
+ a_{10} (\partial_i \partial_j \psi \partial_i \partial_j \psi - \dot{\phi}^2 \dot{\psi}^2 R)/a^4 + a_{11} \, \dot{R} \, \dot{\phi}^2 \psi/a^2 + a_{12} \, \dot{R} \, \dot{\phi}^2 \psi/a^2 + a_{13} \, \dot{R} \, \dot{\phi}^2 \psi/a^2 + a_{14} (\partial \phi^2 R)/a^2 + a_{15} \, \dot{\phi}^2 R|^2 \}
\]

where

\[
\begin{align*}
    a_1 &= 3M^2_{pl}F^2 + 3M^2_{pl}\dot{F}H - XP_X - 4X^2P_{XX} - 4X^3P_{XXX}/3 - 8H^3 \dot{\xi} \\
    &\quad - 2H \dot{\phi}(10XG_{,XX} + 11X^2G_{,XXX}) + 2XG_{,\phi} + 14X^2G_{,\phi X}/3 + 4X^3G_{,\phi XX}/3, \\
    a_2 &= w_3 = -9M^2_{pl}F^2 - 9M^2_{pl}\dot{F}H + 3(XP_X + 2X^2P_{XX}) + 144H^3 \dot{\xi} \\
    &\quad + 18H \dot{\phi}(2XG_{,XX} + X^2G_{,XXX}) - 6(XG_{,\phi} + X^2G_{,\phi X}), \\
    a_3 &= -3a_5 = -3(2M^2_{pl}F^2 + M^2_{pl}\dot{F}H - 48H^2 \dot{\xi} - 2\dot{\phi}(2XG_{,XX} + X^2G_{,XXX})) \\
    a_4 &= -16H \dot{\xi}, \\
    a_6 &= -a_7/9 = a_{11} = -w_2 = -[M^2_{pl}(2HF + \dot{F}) - 2\dot{\phi}XG_{,XX} - 24H^2 \dot{\xi}], \\
    a_8 &= 2a_{10} = 2b_1 = -2c_2 = -4d_1 = 16 \dot{\xi}, \\
    a_9 &= a_{12}/4 = -a_{15}/6 = -(M^2_{pl}F - 24H^2 \dot{\xi})/2, \\
    a_{13} &= 2a_{14} = 2b_1/9 = -c_1 = -c_3 = -4d_2/3 = d_3 = -2w_1 = -2(M^2_{pl}F - 8H^2 \dot{\xi}), \\
    b_2 &= w_4 = M^2_{pl}F - 8H^2 \dot{\xi}.
\end{align*}
\]

In Appendix we present more details for the derivation of Eq. (40). Terms of the forms \( R^3 \) and \( \alpha R^2 \) vanish because of the background equations of motion, which is similar to what happens for the terms \( R^2 \) and \( \alpha R \) in the second-order Lagrangian (12). Another analogy between the two actions \( S_2 \) and \( S_3 \) consists of the cancellation of the self-coupling term for the field \( \psi \); that is why the cubic term in \( \psi \) is absent in Eq. (40). In addition to the curvature perturbation \( \mathcal{R} \) the action (40) involves the terms \( \alpha \) and \( \psi \). Using the constraint equation (17) to eliminate \( \alpha \), the action (46) reduces to

\[ S_3 = \int dt \, dx \, a^3 \{ A_1 \, \mathcal{R}^3 + A_2 \mathcal{R}^2 \dot{\mathcal{R}} \partial^2 \mathcal{R}/a^2 + A_3 \mathcal{R} \dot{\mathcal{R}}^2 \dot{\phi}^2 \psi/a^2 + A_4 \mathcal{R} \dot{\mathcal{R}}^2 + (A_5 \mathcal{R} + A_6 \mathcal{R}) (\partial_i \partial_j \psi \partial_i \partial_j \psi - \dot{\phi}^2 \dot{\psi}^2 \psi)/a^4 \\
+ A_7 \mathcal{R} (\partial_i \partial_j \psi \partial_i \partial_j \mathcal{R} - \dot{\phi}^2 \dot{\psi}^2 \partial^2 \mathcal{R})/a^4 + A_8 \mathcal{R} (\partial \mathcal{R}^2)/a^2 + A_9 \mathcal{R} \partial \mathcal{R} \partial \mathcal{R} \partial^2 \psi/a^4 \}
\]

where

\[
\begin{align*}
    A_1 &= b_1 + L_1 a_{15} + L_1^2 a_3 + L_1^3 a_1, \\
    A_2 &= L_1 (L_1 a_4 + a_8), \\
    A_3 &= c_2 + L_1 a_{12} + L_1^2 a_5, \\
    A_4 &= b_3 + L_1 a_7 + L_1^2 a_2, \\
    A_5 &= L_1 a_9 + d_1, \\
    A_6 &= d_2, \\
    A_7 &= L_1 a_{10}, \\
    A_8 &= b_2 + a_{13} L_1/2 + L_1(a_{13} + Ha_{13})/2, \\
    A_9 &= d_3.
\end{align*}
\]

For the derivation of Eq. (56) the term proportional to \( \mathcal{R} \partial \mathcal{R} \partial^2 \mathcal{R} \) has been integrated by parts, as

\[ c(t) \mathcal{R} \partial \mathcal{R} \partial^2 \mathcal{R} = \frac{c(t) \partial \mathcal{R} (\partial \mathcal{R}^2)/2}{2} - \frac{c(t) \partial \mathcal{R} (\partial \mathcal{R}^2)/2}{2} + \text{total derivatives} \]
where

\[
f_1 = \left( A_1 + A_3 \frac{Q}{w_1} - A_5 \frac{Q^2}{w_1^2} \right) \dot{R}^3 + \left( A_4 - A_6 \frac{Q^2}{w_1^2} \right) R \dot{R}^2 + A_7 \frac{Q}{w_1^3} R \dot{R} \partial_i R \partial_j X,
\]

\[
+ \frac{1}{w_1} \left( A_5 \dot{R} + A_6 \bar{R} \right) \left( \partial_i \partial_j X \right)(\partial_i \partial_j X),
\]

\[
f_2 = \left( A_2 - A_3 L_1 + A_5 \frac{2L_1 Q}{w_1} - A_7 \frac{Q}{w_1} \right) \dot{R}^2 \partial^2 R + A_6 \frac{2L_1 Q}{w_1} R \dot{R} \partial^2 R + A_8 R (\partial R)^2 - A_9 \frac{L_1 Q}{w_1} R (\partial R)^2
\]

\[
+ \frac{A_7}{w_1} - 2A_5 \frac{L_1}{w_1} \dot{R} \left( \partial_i \partial_j R \right) \left( \partial_i \partial_j X \right) - \frac{2A_6 L_1}{w_1} R \left( \partial_i \partial_j R \right) \left( \partial_i \partial_j X \right) - \frac{A_7 L_1}{w_1} \partial^2 R \partial_i R \partial_j X
\]

\[
f_3 = \left( A_5 L_1^2 - A_7 L_1 \right) \dot{R} \left( \partial_i \partial_j R \right) \left( \partial_i \partial_j X \right) - (\partial^2 R)^2 + A_6 L_1^2 R \left( \partial_i \partial_j R \right) \left( \partial_i \partial_j X \right) - (\partial^2 R)^2 + A_9 L_1^2 (\partial R)^2 \partial^2 R.
\]

In standard inflation with a canonical kinetic term (i.e. \( F = 1, P = X - V(\phi), \xi = 0, G = 0 \)) one has \( A_1 = -M_p^2 \phi / H, A_2 = A_3 = A_7 = 0, A_4 = 3M_p^2 \epsilon, A_5 = -M_p^2 / (2H), A_6 = 3M_p^2 / 2, A_8 = -M_p^2 \epsilon, A_9 = -2M_p^2, L_1 = 1/H, Q = M_p^2 \epsilon, \) and \( w_1 = M_p^2, \) where \( \epsilon = \dot{\phi}^2 / (2H^2 M_p^2). \) It then follows that

\[
f_1 = M_p^2 \epsilon \left( 1 - \epsilon / 2 \right) \dot{R}^2 (3R - \dot{R} / H) - 2 \dot{R} \partial_i \partial_j R \partial_i \partial_j X + (3R - \dot{R} / H) \left( \partial_i \partial_j X \right) \left( \partial_i \partial_j X \right) / (2M_p^2),
\]

\[
f_2 = -M_p^2 \epsilon (R - \dot{R} / H) (\partial R)^2 + (M_p^2 \epsilon / H) (3R - \dot{R} / H) \dot{R} \partial^2 R - (1/H) (3R - \dot{R} / H) \left( \partial_i \partial_j R \right) \left( \partial_i \partial_j R \right)
\]

\[
+ (2H) \partial^2 R \partial_i R \partial_j X
\]

\[
f_3 = M_p^2 / (2H^2) (3R - \dot{R} / H) \left[ \left( \partial_i \partial_j R \right) \left( \partial_i \partial_j R \right) - (\partial^2 R)^2 \right] - (2M_p^2 / H^2) (\partial R)^2 \partial^2 R,
\]

which agree with Eq. (114) in Ref. 67 in units where \( M_p^2 = 1. \)

Although the expressions for the building blocks of the third-order action are correctly given in Eqs. (60)–(62), it is more convenient to perform several integrations by parts to bring each contribution into a simpler and more usable form. In Appendix A we show that the integrand \( f_3/a \) in Eq. (60) reduces to

\[
f_3/a = (q_3/a) \left\{ (\partial R)^2 \left( \partial^2 R \right) - R \partial_i \partial_j \left[ (\partial_i R) (\partial_j R) \right] \right\} + \text{total derivatives},
\]

where

\[
q_3 = A_6 L_1^2 - \frac{a}{3} \frac{d}{dt} \left( \frac{A_5 L_1^2 - A_7 L_1}{a} \right) + \frac{2}{3} A_9 L_1^2.
\]

Along the same lines the term \( af_2 \) can be written as (see Appendix E for details)

\[
af_2 = -\frac{A_7 - 2A_5 L_1}{2w_1 a^2} \frac{d}{dt} \left( a^3 \dot{R} \right) \left\{ (\partial R)^2 - \partial^2 R \partial_i \partial_j \left[ (\partial_i R) (\partial_j R) \right] \right\} + \frac{aq_2}{2} \left\{ (\partial_i X) (\partial_i R) (\partial^2 R) - R \partial_i \partial_j \left[ (\partial_i R) (\partial_j X) \right] \right\}
\]

\[
- 2aL_1 Q R \dot{R} \partial^2 R + a \left( A_2 - A_3 L_1 \right) \dot{R} \partial^2 R + a \left[ A_8 + \frac{1}{a} \frac{d}{dt}(a L_1 Q) \right] R \partial(\partial R)^2 + \text{total derivatives},
\]

where

\[
q_2 = \frac{-4A_6 L_1}{w_1} - \frac{a^2}{3} \frac{d}{dt} \left( \frac{A_7 - 2A_5 L_1}{a^2 w_1} \right) - \frac{2A_9 L_1}{w_1}.
\]

Finally, the term \( a^3 f_1, \) as shown in Appendix E, is equivalent to

\[
a^3 f_1 = a^3 \left[ A_4 + q_1 (\dot{Q} + 3HQ) - Q \dot{q}_1 \right] R \dot{R}^2 - 2q_1 \dot{R} \dot{R} \frac{d}{dt} (a^3 \dot{R})
\]

\[
+ a^3 \left[ A_4 + A_3 \frac{Q}{w_1} - q_1 Q \right] \dot{R}^3 + \frac{a^3}{w_1} \left[ A_9 \frac{Q}{w_1} + Q A_6 \frac{w_1}{Q} - Q w_1 \frac{d}{dt} \left( \frac{A_5}{w_1^2} \right) + \frac{3QH A_5}{w_1} \right] \dot{R} \partial_i R \partial_j X
\]

\[
+ \frac{a^3}{2} \left[ \frac{A_6}{w_1^2} - \frac{d}{dt} \left( \frac{A_5}{w_1^2} \right) + \frac{3HA_5}{w_1^2} \right] \left( \partial^2 R \right) \left( \partial R \right)^2 - \frac{2A_5}{w_1^2} \frac{d}{dt} (a^3 \dot{R}) \left\{ (\partial_i R) (\partial_j X) - \partial^2 R \partial_i \partial_j \left[ (\partial_i R) (\partial_j X) \right] \right\}
\]

\[
+ \text{total derivatives}.
\]
In what follows we omit to write the terms corresponding to total derivatives. We determine the coefficient $q_1$ such that the term $-2q_1 \dot{R} \ddot{R}$ in Eq. (70) is merged with the term $-2aL_1Q\ddot{R} \ddot{R}$ in Eq. (68) to give rise to a combination proportional to the linear equation of motion for $R$ [i.e. Eq. (29)]. This can be achieved by demanding

$$q_1 = -L_1/c_s^2.$$  

(71)

Since, in general, we have that

$$c(t)\{ (\partial_i \dot{X}) (\partial_i \dot{X}) (\partial_j \dot{R}) - \dot{R} \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{X})] \} = c(t)\{ (\partial_i \dot{X}) (\partial_i \dot{X}) (\partial_j \dot{R}) - (\partial_i \partial_j \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{X})]) \}$$

(72)

the last term in Eq. (70) can be merged with the term including $q_2$ in Eq. (68) as follows:

$$\frac{2A_5}{w_1^2} \left[ \frac{d}{dt} (a^3 Q \ddot{R}) + \frac{\alpha q_2}{2} \partial^2 \dot{R} \right] \{ (\partial_i \dot{R}) (\partial_i \dot{R}) - \partial^2 \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{X})] \}$$

(73)

$$= \frac{A_5}{w_1^2} \{ (\partial_i \dot{R}) (\partial_i \dot{R}) - \partial^2 \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{X})] \} \frac{\delta L_2}{\delta \dot{R}}_{1}$$

$$+ a \left[ q_2 - \frac{2c_s^2 A_5 Q}{w_1^2} \right] \{ (\partial^2 \dot{R}) \} \{ (\partial_i \dot{R}) (\partial_i \dot{R}) - \partial^2 \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{X})] \},$$

where $\delta L_2/\delta \dot{R}_{1}$ is defined in Eq. (28). Along the same lines one can show that

$$c(t)\{ (\partial \dot{R})^2 (\partial^2 \ddot{R}) - \dot{R} \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{R})] \} = c(t)\{ (\partial \dot{R})^2 (\partial^2 \ddot{R}) - (\partial_i \partial_j \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{R})]) \}$$

(74)

We use this result to merge two different contributions, namely the term $f_s/a$ in Eq. (66) and the first term on the r.h.s. of Eq. (68). We have

$$\frac{1}{a^2} \left[ -\frac{1}{2w_1^2} (A_7 - 2A_5 L_1) \frac{d}{dt} (a^3 Q \ddot{R}) + \alpha q_3 \partial^2 \dot{R} \right] \{ (\partial \dot{R})^2 (\partial^2 \ddot{R}) - \partial^2 \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{R})] \}$$

(75)

$$= \left[ \frac{1}{4w_1^2} (A_7 - 2A_5 L_1) \{ (\partial \dot{R})^2 (\partial^2 \ddot{R}) - \partial^2 \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{R})] \} \frac{\delta L_2}{\delta \dot{R}}_{1} \right]$$

$$+ \frac{1}{a} \left[ q_3 - \frac{Q^2 c_s^2}{2w_1^2} (A_7 - 2A_5 L_1) \right] \{ (\partial \dot{R})^2 (\partial^2 \ddot{R}) - \partial^2 \partial_i \partial_j [(\partial_i \dot{R}) (\partial_j \dot{R})] \}.$$}

Finally, merging all the contributions together, the third-order action (60) can be written as

$$S_3 = \int dt \mathcal{L}_3,$$  

(66)

where

$$\mathcal{L}_3 = \int d^3x \left\{ a^3 C_1 M^2_{pl} \dot{R} \ddot{R}^2 + a C_2 M^2_{pl} \dot{R} (\partial \dot{R})^2 + a^3 C_3 M^2_{pl} \dot{R} (\partial \dot{R}) (\partial \dot{R}) + a^3 (C_5/M^2_{pl}) \partial^2 \dot{R} (\partial \dot{R})^2 + a C_6 \dot{R} \ddot{R} (\partial \dot{R})^2 + \left[ (C_7/a) \left( \partial^2 \dot{R} (\partial \dot{R})^2 - \dot{R} \partial_i \partial_j (\partial_i \dot{R}) (\partial_j \dot{R}) \right) + a (C_8/M^2_{pl}) \left[ \partial^2 \dot{R} \partial_i \partial_j \dot{R} (\partial_i \dot{R}) (\partial_j \dot{R}) \right] \right] + \mathcal{F}_1 \frac{\delta L_2}{\delta \dot{R}}_{1} \right\}.$$  

(76)
The dimensionless coefficients $C_i$ ($i = 1, \ldots, 8$) are given by

\[
C_1 = \frac{1}{M_{\text{pl}}} \left[ A_4 + q_1(\dot{Q} + 3HQ) - Qq_1 \right] = \frac{Q}{M_{\text{pl}}} \left[ 3 - \frac{L_1H}{c_s^2} \left( 3 + \frac{\dot{Q}}{HQ} \right) + \frac{d}{dt} \left( \frac{L_1}{c_s^2} \right) \right],
\]
\[
C_2 = \frac{1}{M_{\text{pl}}} \left[ A_8 + \frac{1}{a} \frac{d}{da} (aL_1Q) \right] = \frac{1}{M_{\text{pl}}} \left[ M_{\text{pl}}^2F - 8\xi + \frac{1}{a} \frac{d}{da} (aL_1(Q - w_1)) \right],
\]
\[
C_3 = \frac{1}{M_{\text{pl}}} \left( A_1 + A_3 \frac{Q}{w_1} - q_1Q \right) = \frac{1}{M_{\text{pl}}} \left\{ L_1 \left[ L_1(L_1a_1 + a_3) + a_15 + (a_{12} + L_1a_5) \frac{Q}{w_1} + \frac{Q}{c_s^2} \right] + 8\xi \left( 1 - \frac{Q}{w_1} \right) \right\},
\]
\[
C_4 = \frac{Q}{w_1} \left[ \frac{1}{w_1} (A_6 + A_9) - w_1 \frac{d}{dt} \left( \frac{A_5}{w_1} \right) + 3HA_5 \right] = -\frac{Q}{2w_1} \left\{ 1 + 2w_1 \left[ \frac{d}{dt} \left( \frac{A_5}{w_1} \right) - 3H A_5 \right] \right\},
\]
\[
C_5 = \frac{M_{\text{pl}}^2}{2} \left[ \frac{A_6}{w_1^2} - \frac{d}{dt} \left( \frac{A_5}{w_1} \right) + 3HA_5 \right] = \frac{M_{\text{pl}}^2}{2w_1^2} \left[ \frac{3}{2} M_{\text{pl}}^2 F (1 - HL_1) - 24H \xi \left( 1 - \frac{3}{2} HL_1 \right) \right] - \frac{1}{2w_1^2} \frac{d}{dt} \left( \frac{A_5}{w_1} \right) M_{\text{pl}}^2,
\]
\[
C_6 = A_2 - A_3L_1 = L_1 \left[ L_1(2M_{\text{pl}}^2 F - L_1a_5 - 64H \xi) + 24\xi \right],
\]
\[
C_7 = q_3 - \frac{Q^2}{2w_1} (A_7 - 2A_5L_1) = \frac{1}{6} L_1^2 \left[ M_{\text{pl}}^2 F (1 - HL_1) - 8H \xi (4 - 3HL_1) \right] - \frac{\xi}{2w_1} \left[ L_1(M_{\text{pl}}^2 F - 24H \xi) + 16\xi \right]
\]
\[
+ \frac{1}{4w_1^2} \left( M_{\text{pl}}^2 \left[ L_1(M_{\text{pl}}^2 F - 24H \xi) + 24\xi \right] \right),
\]
\[
C_8 = M_{\text{pl}} \left( \frac{Q}{w_1} - \frac{2\xi A_5Q}{w_1^2} \right) = \left\{ \frac{L_1}{w_1} M_{\text{pl}}^2 F (24H \xi) (HL_1 - 1) + \frac{\xi}{w_1^2} \left[ L_1(M_{\text{pl}}^2 F - 24H \xi) + 8\xi \right]
\]
\[
- \frac{d}{dt} \left[ \frac{L_1[L_1(M_{\text{pl}}^2 F - 24H \xi) + 16\xi]}{2w_1} \right] \right\} M_{\text{pl}},
\]

with $A_5 = -(L_1/2)(M_{\text{pl}}^2 F - 24H \xi) - 4\xi$. Note that we have used $A_4 = 3Q$ to simplify $C_1$. The coefficient in front of $\delta L_2/\delta R|_1$ in Eq. (77) is

\[
\mathcal{F}_1 = \frac{A_5}{w_1^2} \left\{ \partial_k R \partial_k \mathcal{X} - \partial^2 \partial_i \partial_j \left[ (\partial_i R) (\partial_j \mathcal{X}) \right] \right\} + q_1 R \mathcal{R} + \frac{A_7 - 2A_5L_1}{4w_1^2} \frac{1}{\alpha} \left\{ (\partial \mathcal{R})^2 - \partial^2 \partial_i \partial_j \left[ (\partial_i \mathcal{R}) (\partial_j \mathcal{R}) \right] \right\}.
\]

Each contribution to $\mathcal{F}_1$ includes spatial and time derivatives of $\mathcal{R}$, which vanish in the large-scale limit ($k \to 0$). Moreover the term $\delta L_2/\delta R|_1$ survives only at second order in $\mathcal{R}$. When we evaluate the three-point correlation function of $\mathcal{R}$ below, we neglect the contribution of the last term in Eq. (77) relative to those coming from other terms. In Refs. [22, 27, 28] the term proportional to $\mathcal{R}^2$ is present in the definition of $\mathcal{F}_1$. After a suitable field redefinition this gives rise to a correction of the order of the slow-roll parameter $\eta = \xi/(H \epsilon)$ in standard inflation [66]. We have absorbed such a contribution to other terms in Eq. (77) [such as $C_1$] and hence the field redefinition is not required in our method.

The Hamiltonian in the interacting picture is given by $\mathcal{H}_{\text{int}} = -L_3 [12, 27, 28]$. The vacuum expectation value of $\mathcal{R}$ for the three-point operator at the conformal time $\tau = \tau_f$ can be expressed as

\[
\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle = -i \int_{\tau_i}^{\tau_f} d\tau a \sqrt{a} \langle 0 | \mathcal{R}(\tau_f, k_1) \mathcal{R}(\tau_f, k_2) \mathcal{R}(\tau_f, k_3), \mathcal{H}_{\text{int}}(\tau_f) | 0 \rangle,
\]

where $\tau_i$ is the initial time when the perturbations are deep inside the Hubble radius. Since $\tau \sim -1/(aH)$ during inflation it is a good approximation to take $\tau_i \to -\infty$ and $\tau_f \to 0$, where the latter corresponds to some time after the Hubble radius crossing.

In order to evaluate the vacuum expectation value [87] we use the curvature perturbation [80] in Fourier space with the mode function $u(\tau, k)$ given in Eq. (54). Each term in the third-order Lagrangian (77) includes the phase factor of the form \(\int d^3x e^{i(k_4+k_5+k_6)\cdot x}\), which gives rise to the delta function \((2\pi)^3 \delta(3)(k_4 + k_5 + k_6)\). Among the
condition is not satisfied, one needs to numerically solve the integrals without approximations. In the following we
relative to the scale factor $a$, so that it is a good approximation to treat them as constants for the integration. If this
condition is not satisfied, one needs to numerically solve the integrals without approximations. In the following we
present each contribution of the three-point correlation function coming from the integrals given in Eq. (77).

- (1) $H_{\text{int}}^{(1)} = - \int d^3x a^2 C_1 M_{\text{pl}}^2 \mathcal{R} \dot{\mathcal{R}}^2$

$$\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle^{(1)} = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \frac{C_1 M_{\text{pl}}^2 H^4}{16 Q^3 c_s^3} \left( \frac{k_1 k_2 k_3}{K} + \frac{k_1 k_2 k_3}{K^2} + \text{sym} \right),$$

where $K = k_1 + k_2 + k_3$. The symbol "sym" means the symmetric terms with respect to $k_1, k_2, k_3$.

- (2) $H_{\text{int}}^{(2)} = - \int d^3x a^2 C_2 M_{\text{pl}}^2 \mathcal{R} (\partial \mathcal{R})^2$

$$\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle^{(2)} = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \frac{C_2 M_{\text{pl}}^2 H^4}{16 Q^3 c_s^3} \left( \frac{1}{k_1 k_2 k_3} \left( -K + \frac{k_1 k_2 + k_2 k_3 + k_3 k_1}{K} + \frac{k_1 k_2 k_3}{K^2} \right) \right).$$

- (3) $H_{\text{int}}^{(3)} = - \int d^3x a^2 C_3 M_{\text{pl}} \mathcal{R} \dot{\mathcal{R}}^3$

$$\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle^{(3)} = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \frac{3 C_3 M_{\text{pl}} H^5}{8 Q^3 c_s^3} \left( \frac{1}{k_1 k_2 k_3} \right).$$

- (4) $H_{\text{int}}^{(4)} = - \int d^3x a^3 C_4 \dot{\mathcal{R}} (\partial \mathcal{R})(\partial \mathcal{X})$

$$\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle^{(4)} = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \frac{C_4 H^4}{32 Q^3 c_s^3} \left( \frac{1}{k_1 k_2 k_3} \right) \left[ \frac{(k_1 \cdot k_2)^2}{K} \left( 2 + \frac{k_1 + k_2}{K} \right) + \text{sym} \right].$$

- (5) $H_{\text{int}}^{(5)} = - \int d^3x a^3 (C_5 / M_{\text{pl}}) \partial^2 \mathcal{R} (\partial \mathcal{X})^2$

$$\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle^{(5)} = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \frac{C_5 H^4}{16 Q^3 M_{\text{pl}} c_s^3} \left( \frac{1}{k_1 k_2 k_3} \right) \left[ \frac{k_1^2 (k_2 \cdot k_3)}{K} \left( 1 + \frac{k_1}{K} \right) + \text{sym} \right].$$

- (6) $H_{\text{int}}^{(6)} = - \int d^3x a C_6 \dot{\mathcal{R}}^2 \partial^2 \mathcal{R}$

$$\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle^{(6)} = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \frac{3 C_6 H^6}{4 Q^3 c_s^3} \left( \frac{1}{k_1 k_2 k_3} \right).$$

- (7) $H_{\text{int}}^{(7)} = - \int d^3x (C_7 / a) \left[ \partial^2 \mathcal{R} (\partial \mathcal{R})^2 - \mathcal{R} \partial_i \partial_j (\partial_i \mathcal{R})(\partial_j \mathcal{R}) \right]$

$$\langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle^{(7)} = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \frac{C_7 H^6}{8 Q^3 c_s^3} \left( \frac{1}{k_1 k_2 k_3} \right) \left[ 1 + \frac{k_1 k_2 + k_2 k_3 + k_3 k_1}{K^2} + \frac{3 k_1 k_2 k_3}{K^3} \right] \times \left[ k_1^2 (k_2 \cdot k_3) - (k_1 \cdot k_2)(k_1 \cdot k_3) + \text{sym} \right].$$

(95)
\[ H_{\text{int}}^{(8)} = - \int d^3 x \, a(c_s/M_{\text{pl}}) \left[ \partial^2 \mathcal{R} \partial_i \mathcal{R} \partial_j \mathcal{X} - \mathcal{R} \partial_i \partial_j (\partial_i \mathcal{R} \partial_j \mathcal{X}) \right] \]

\[ \langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle^{(8)} = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) \frac{C_s H^5}{32Q^2 M_{\text{pl}}^2 c_s^8 (k_1k_2k_3)^3 K} \times \left\{ \left(2 + \frac{2k_1 + k_2 + k_3}{K} + \frac{2k_1(k_2 + k_3)}{K^2} \right) [k_1^2(k_2 \cdot k_3) - (k_1 \cdot k_2)(k_1 \cdot k_3)] + \text{sym} \right\} \cdot (96) \]

We write the three-point correlation function of the curvature perturbation, as

\[ \langle \mathcal{R}(k_1) \mathcal{R}(k_2) \mathcal{R}(k_3) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) B_{\mathcal{R}}(k_1, k_2, k_3), \] (97)

where

\[ B_{\mathcal{R}}(k_1, k_2, k_3) = \frac{(2\pi)^4 (P_{\mathcal{R}})^2}{\prod_{i=1}^3 k_i^3} A_{\mathcal{R}}(k_1, k_2, k_3). \] (98)

Recall that the power spectrum \( P_{\mathcal{R}} \) is given by Eq. (85). Collecting all the terms in Eqs. (89)-(96), it follows that

\[ A_{\mathcal{R}} = \frac{M_{\text{pl}}^2}{Q} \left\{ \frac{1}{4} \left( \frac{2}{K} \sum_{i \neq j} k_i^3 k_j^3 - \frac{1}{K^2} \sum_{i \neq j} k_i^4 k_j^2 \right) C_1 + \frac{1}{4c_s^4} \left( \frac{1}{2} \sum_i k_i^4 + \frac{2}{K} \sum_{i \neq j} k_i^2 k_j^2 - \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^2 \right) C_2 \right. \]

\[ + \frac{3}{2} \frac{H}{M_{\text{pl}}} \left( \frac{k_1k_2k_3}{K^3} \right) C_3 + \frac{1}{8} \frac{Q}{M_{\text{pl}}^2} \left( \sum_{i} k_i^3 - \frac{1}{2} \sum_{i \neq j} k_i k_j^2 - \frac{2}{K^2} \sum_{i \neq j} k_i^3 k_j^2 \right) C_4 \]

\[ + \frac{1}{4} \left( \frac{Q}{M_{\text{pl}}^2} \right)^2 \frac{1}{K} \left[ \sum_{i} k_i^3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 - \frac{3}{2} \sum_{i \neq j} k_i^2 k_j^2 - k_1k_2k_3 \sum_{i \neq j} k_i k_j \right] C_5 + \frac{3}{c_s^2} \left( \frac{H}{M_{\text{pl}}} \right)^2 \left( \frac{k_1k_2k_3}{K^3} \right) C_6 \]

\[ + \frac{1}{2c_s^4} \left( \frac{H}{M_{\text{pl}}} \right)^2 \frac{1}{K} \left[ \left( \frac{1}{K} + \frac{1}{2} \sum_{i \neq j} k_i k_j + \frac{3}{2k_1k_2k_3} \right) \left( \frac{3}{4} \sum_{i} k_i^4 - \frac{3}{2} \sum_{i \neq j} k_i^2 k_j^2 \right) C_7 \right. \]

\[ + \frac{1}{8c_s^2} \frac{H}{M_{\text{pl}}} \frac{Q}{M_{\text{pl}}^2} \frac{1}{K} \left[ \left( \frac{3}{2} k_1k_2k_3 \sum_{i} k_i^2 - \frac{5}{2} k_1k_2k_3 K^2 - 6 \sum_{i \neq j} k_i^2 k_j^2 - \sum_{i} k_i^5 + \frac{7}{2} K \sum_{i} k_i^4 \right) C_8 \right], \] (99)

which has an explicit dependence on the wave numbers. The bispectrum (99) is the central result of our work.

In general the non-linear parameter \( f_{\text{NL}} \) associated with non-Gaussianities of the curvature perturbation is defined as \( \frac{A_{\mathcal{R}}}{\sum_{i=1}^3 k_i^3} \) (100),

which matches with the notation of the WMAP group. For the equilateral configuration with \( k_1 = k_2 = k_3 \) one has

\[ f_{\text{NL}}^{\text{equil}} = \frac{40}{9} \frac{M_{\text{pl}}^2}{Q} \left[ \frac{1}{12} C_1 + \frac{17}{96c_s^4} C_2 + \frac{1}{72} \frac{H}{M_{\text{pl}}} C_3 - \frac{1}{24} \frac{Q}{M_{\text{pl}}^2} C_4 - \frac{1}{24} \left( \frac{Q}{M_{\text{pl}}^2} \right)^2 C_5 + \frac{1}{36c_s^2} \left( \frac{H}{M_{\text{pl}}} \right)^2 C_6 \right. \]

\[ - \frac{13}{96c_s^4} \left( \frac{H}{M_{\text{pl}}} \right)^2 C_7 - \frac{17}{192c_s^4} \frac{H}{M_{\text{pl}}} \frac{Q}{M_{\text{pl}}^2} C_8 \right], \] (101)

which is independent of the wave numbers. Recall that the coefficients \( C_i \) (\( i = 1, \cdots, 8 \)) can be evaluated by Eqs. (78)-(85).

V. Expansion in Terms of Slow Variation Parameters

Let us derive a simpler form of \( f_{\text{NL}}^{\text{equil}} \) by using the approximation that the slow-variation terms defined in Eq. (8) are much smaller than 1. We also use the expansion of \( L_1 \) and \( \epsilon_s \) given in Eqs. (20) and (27), respectively. First we
write the variables \( Q \) and \( c_s^2 \) in the forms
\[
Q = \frac{4M_{pl}^4 \Sigma}{w_f^2}, \quad c_s^2 = \frac{2w_f^2 w_2 H - w_f^2 w_4 + 4w_1 w_1 w_2 - 2w_f^2 w_2}{4M_{pl}^4 \Sigma},
\]
where
\[
\Sigma = \frac{w_1(4w_1 w_3 + 9w_f^2)}{12M_{pl}^4}.
\]

We also define the following quantities
\[
\lambda \equiv F^2 \left[ X^2 P_{,XX} + 2X^3 P_{,XXX} / 3 + \phi H (X G_{,X} + 5X^2 G_{,XX} + 2X^3 G_{,XXX}) - 2(2X^2 G_{,X} + X^3 G_{,XXX}) / 3 \right],
\]
and
\[
\lambda_G \equiv X G_{,X} X / G_{,X}.
\]

It is convenient to introduce \( \Sigma \) and \( \lambda \) in order to obtain a compact expression of \( f_{NL} \). Note that these parameters have been already introduced in the context of k-inflation \([27, 28]\).

We first replace \( w_3 \) with respect to \( \Sigma \), which in turn can be written in terms of \( \epsilon_s \). Using Eq. (29), the parameter \( \epsilon_s \) can be expressed in terms of other slow-variation parameters. Then we replace \( F \) as \( F = Qc_s^2 / (\epsilon_s M_{pl}^2) \). Finally we expand the coefficients \( \mathcal{C}_i \) \( (i = 1, \cdots, 8) \) in terms of the slow-variation parameters, by keeping \( c_s^2 \) and \( \lambda_G \) as unknown parameters. A similar method is applied to all the other terms, except for \( \mathcal{C}_3 \). In this case the terms \( P_{,X} \) and \( G_{,XXX} \) are expressed in terms of \( \Sigma \) and \( \lambda \) respectively. We replace any left \( G_{,XX} \) with \( \lambda_G \). We also multiply and divide \( \lambda \) by \( \Sigma \), and we consider the ratio \( \lambda / \Sigma \) as a free parameter of the theories (as we do for \( c_s^2 \) and \( \lambda_G \)). Then, as done for the other terms, we replace \( \Sigma \) in terms of \( Q \), and any left \( F \) in terms of \( Q \) and \( \epsilon_s \). Finally we use the relation between \( \epsilon \) and other slow-variation parameters. By following this procedure, the final expression of each \( f_{NL}^{\text{equil}(i)} \) \( (i = 1, \cdots, 8) \) only depends on \( \epsilon_s, \eta_s, s, \delta_F, \delta_\xi, \delta_{GX} \), and the free parameters \( c_s^2, \lambda_G \), and \( \lambda / \Sigma \). This form allows an expansion with respect to the slow-variation parameters. Then the leading contributions to each \( f_{NL}^{\text{equil}(i)} \) coming from the coefficients \( \mathcal{C}_i \) are given by

\[
\begin{align*}
\mathcal{F}_{NL}^{\text{equil}(1)} & = \frac{10}{9} \left( 1 - \frac{1}{c_s^2} \right) - \frac{10}{27c_s^2} (8 \delta_\xi + \eta_s + 4 \delta_{GX} - \epsilon_s), \\
\mathcal{F}_{NL}^{\text{equil}(2)} & = \frac{85}{108} \left( \frac{1}{c_s^2} - 1 \right) + \frac{85}{108c_s^2} (\epsilon_s + 8 \delta_\xi - 2 s + \eta_s), \\
\mathcal{F}_{NL}^{\text{equil}(3)} & = \frac{5}{81} \left( \frac{1}{c_s^2} - 1 \right) - \frac{10 \lambda}{81 \Sigma} + \frac{5}{162c_s^2}(8 \delta_\xi + 2 \delta_{GX} - \delta_F) - \frac{5}{162} \left( 6 \delta_{GX} - \delta_F + 4 \lambda_G \delta_{GX} + 8 \delta_\xi \right) \\
& \quad + \frac{5}{27c_s^2} \delta_{GX} (1 + \lambda_G) \left( \delta_F - 8 \delta_\xi - 2 \delta_{GX} \right), \\
\mathcal{F}_{NL}^{\text{equil}(4)} & = \frac{10}{27} \epsilon_s / c_s^2, \\
\mathcal{F}_{NL}^{\text{equil}(5)} & = - \frac{5}{108c_s^2} (\epsilon_s - 4 \delta_{GX} + 16 \delta_\xi) \epsilon_s, \\
\mathcal{F}_{NL}^{\text{equil}(6)} & = \frac{20}{81} \left( 1 + \lambda_G \right) \delta_{GX}, \\
\mathcal{F}_{NL}^{\text{equil}(7)} & = \frac{65}{162c_s^2} \epsilon_s, \\
\mathcal{F}_{NL}^{\text{equil}(8)} & = - \frac{85}{108} \delta_{GX} / c_s^2.
\end{align*}
\]

Let us now add up all the contributions together. In doing so we take the largest contributions, that is, we discard the corrections of the order of slow-variation parameters relative to other existent terms. For example, if a term \( \delta_{GX} / (\epsilon_s c_s^2) \) already exists, we ignore the terms like \( \delta_{GX} / c_s^2 \). Then we have that
\[
\mathcal{F}_{NL} \approx \mathcal{F}_{NL}^{\text{equil}} \left( \delta_\xi \right) + \frac{5}{162} \delta_F \left( 1 - \frac{1}{c_s^2} \right) - \frac{10}{81} \delta_\xi \left( 2 - \frac{29}{c_s^2} \right) + \delta_{GX} \left[ \frac{20}{81} \left( 1 + \lambda_G \right) + \frac{65}{162c_s^2} \epsilon_s \right],
\]

where
\[
\epsilon_s = \frac{2w_f^2 w_2 H - w_f^2 w_4 + 4w_1 w_1 w_2 - 2w_f^2 w_2}{4M_{pl}^4 \Sigma},
\]
and
\[
\lambda_G \equiv X G_{,X} X / G_{,X}.
\]
where we have tacitly assumed that \( c_s^2 < \mathcal{O}(1) \). The result (114) is valid for any quasi de Sitter background.

VI. CONCRETE MODELS

In this section we apply the results derived in previous sections to concrete models of inflation. This includes (i) k-inflation, (ii) generalized Galileon model, and (iii) models based on BD theories.

A. k-inflation

Let us first consider k-inflation models with \( F = 1, \xi = 0, \) and \( G = 0 \). Since \( \epsilon_s = \epsilon = -\dot{H}/H^2 = X P_X/(M_{pl}^2 H^2) \) and \( L_1 = 1/H \) in those models, it is possible to derive an exact expression of \( f_{NL}^{\text{equil}} \) without employing the approximation in terms of slow-variation parameters. The coefficients (115) are given by

\[
C_1 = \frac{\epsilon}{c_s^2} (\epsilon - 3 + 3c_s^2 - \eta), \quad C_2 = \frac{\epsilon}{c_s^2} (1 - c_s^2 + \epsilon + \eta - 2s), \quad C_3 = -\frac{1}{H^3 M_{pl}^2} \left[ \left( 1 - \frac{1}{c_s^2} \right) \Sigma + 2\lambda \right],
\]

\[
C_4 = \frac{\epsilon}{c_s^2} (-2 + \frac{1}{2^5}), \quad C_5 = \frac{1}{4^4}, \quad C_6 = C_7 = C_8 = 0,
\]

where \( \eta \equiv \eta_s = \epsilon/(H\dot{\epsilon}) \), \( \Sigma = XP_X + 2X^2P_{XX} \) and \( \lambda = X^2 P_{XX} + 2X^3 P_{XXX}/3 \).

The field propagation speed squared is

\[
c_s^2 = \frac{P_X}{P_X + 2XP_{XX}} = \frac{XP_X}{\Sigma} = \frac{M_{pl}^2 H^2 \epsilon}{\Sigma}.
\]

Plugging the coefficients (115) into Eq. (101), it follows that

\[
f_{NL}^{\text{equil}} = 85 \left( 1 - \frac{1}{c_s^2} \right) \left[ \frac{10}{81} \frac{\lambda}{\Sigma} + \frac{55}{36} \frac{\epsilon}{c_s^2} + \frac{5}{12} \frac{\eta}{c_s^2} + \frac{85}{54} \frac{s}{c_s^2} \right],
\]

where we have ignored the second-order term \( \epsilon^2/(2c_s^2) \) in the expression of \( C_4 \). The non-linear parameter (114) derived under the slow-variation approximation also gives the same result. Note that this coincides with the result in Refs. [27, 28].

In standard inflation driven by a potential energy \( V(\phi) \) of the field \( \phi \), i.e. \( P = X - V(\phi) \), one has \( c_s^2 = 1, \lambda = 0, \) and \( s = 0 \). Then the equilateral non-linear parameter (117) reduces to

\[
f_{NL}^{\text{equil}} = \frac{55}{36} \epsilon + \frac{5}{12} \eta,
\]

which means that \( |f_{NL}^{\text{equil}}| \ll 1 \).

B. Generalized Galileon model

Let us consider the generalized Galileon model with \( F = 1 \) and \( \xi = 0 \). Since the exact formula of \( f_{NL}^{\text{equil}} \) is complicated, we employ the result (114) derived under the slow-variation approximation:

\[
f_{NL}^{\text{equil}} \simeq \frac{85}{324} \left( 1 - \frac{1}{c_s^2} \right) \left[ \frac{10}{81} \frac{\lambda}{\Sigma} + \frac{55}{36} \frac{\epsilon_s}{c_s^2} + \frac{5}{12} \frac{\eta_s}{c_s^2} + \frac{85}{54} \frac{s}{c_s^2} + \frac{\delta_{GX}/c_s}{c_s} \frac{20}{81} \left( 1 + \lambda G \right) + \frac{65}{162 c_s^2} \right].
\]

Using the background equation (7), the term \( \delta_{GX}/c_s \) can be expressed as

\[
\frac{\delta_{GX}}{c_s} \simeq \frac{\delta_{GX}}{\epsilon_s} = \frac{H \dot{\phi} G_X}{P_X + (4H \dot{\phi} - \dot{\phi})G_X - 2G_{\phi}}.
\]

In the limit that \( c_s^2 \ll 1 \) the dominant contribution to \( f_{NL}^{\text{equil}} \) is

\[
f_{NL}^{\text{equil}} \simeq -\frac{85}{324} \frac{1}{c_s^2} \left[ \frac{10}{81} \frac{\lambda}{\Sigma} + \frac{H \dot{\phi} G_X}{P_X + (4H \dot{\phi} - \dot{\phi})G_X - 2G_{\phi}} \left[ \frac{20}{81} \left( 1 + \lambda G \right) + \frac{65}{162 c_s^2} \right] \right].
\]
This matches with the result of Ref. [56] in which the authors ignored the terms $\ddot{\phi} G_X$ and $G_{,\phi}$ relative to $H\dot{\phi} G_X$. The last term in the parenthesis of Eq. [121] gives rise to an additional contribution to the terms appearing in $k$-inflation.

C. Brans-Dicke theories

The action in BD theory with a field potential $V(\phi)$ is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M_{pl} \dot{\phi} R + \frac{M_{pl}}{\phi} \omega_{BD} X - V(\phi) \right],$$

(122)

where $\omega_{BD}$ is the BD parameter. Compared to the original paper [61] we have introduced the reduced Planck mass $M_{pl}$, so that the field $\phi$ has a dimension of mass. In this theory the background equation (7) gives

$$2\epsilon_1 = -\epsilon_2 + \epsilon_3 + \omega_{BD} \epsilon_2^2,$$

(123)

where

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = \frac{\dddot{\phi}}{H\dot{\phi}}.$$

(124)

Let us derive a full expression of $f_{NL}^{\text{equil}}$ without using the slow-variation approximation. From Eqs. (23) and (24) it follows that

$$Q = \frac{(3 + 2\omega_{BD})\epsilon_2^2 \phi M_{pl}}{(2 + \epsilon_2)^2}, \quad c_s^2 = 1.$$

(125)

The time-derivatives appearing in the coefficients $C_i$ can be expressed in terms of $\epsilon_i$ ($i = 1, 2, 3$) by using the relation

$$\dot{\epsilon}_2 = H\epsilon_2(\epsilon_1 - \epsilon_2 + \epsilon_3).$$

(126)

Then the coefficients $C_i$ ($i = 1, \cdots, 8$) are given by

$$C_1 = -\frac{\phi}{M_{pl}} \frac{(3 + 2\omega_{BD})\epsilon_2^2 \left[(2\omega_{BD} - 3)\epsilon_2^2 + 4(\epsilon_3 - 3)\epsilon_2 + 8\epsilon_3 \right]}{(2 + \epsilon_2)^4},$$

(127)

$$C_2 = \frac{\phi}{M_{pl}} \frac{(3 + 2\omega_{BD})\epsilon_2^2 \left[8\epsilon_3 + (4\epsilon_3 - 12)\epsilon_2 + (3 + 6\omega_{BD})\epsilon_2^2 \right]}{(2 + \epsilon_2)^4},$$

(128)

$$C_3 = C_6 = C_7 = C_8 = 0,$$

(129)

$$C_4 = \frac{(3 + 2\omega_{BD})\epsilon_2^2 \left[-16 - 16\epsilon_2 + (2\omega_{BD} - 1)\epsilon_2^2 \right]}{2(2 + \epsilon_2)^4},$$

(130)

$$C_5 = \frac{M_{pl}}{\phi} \frac{(3 + 2\omega_{BD})\epsilon_2^2}{4(2 + \epsilon_2)^2},$$

(131)

which lead to

$$f_{NL}^{\text{equil}} = \frac{5}{18} \frac{48\epsilon_3 - 72(1 - \epsilon_3)\epsilon_2 - (6 - 68\omega_{BD} - 36\epsilon_3)\epsilon_2^2 + (48 + 68\omega_{BD} + 6\epsilon_3)\epsilon_3^2 + (12 + 11\omega_{BD} - 2\omega_{BD}^2)\epsilon_2^4}{(2 + \epsilon_2)^4}.$$  

(132)

Since $|\epsilon_2| \ll 1$ and $|\epsilon_3| \ll 1$ during inflation, we have

$$f_{NL}^{\text{equil}} \simeq -\frac{5}{4} \epsilon_2 + \frac{5}{6} \epsilon_3,$$

(133)

which gives $|f_{NL}^{\text{equil}}| \ll 1$. It is worth mentioning that the same result as Eq. (133) follows by using the approximated expression of $f_{NL}^{\text{equil}(i)}$ ($i = 1, \cdots, 8$) derived in Sec. IV. The above analysis covers the case of $f(R)$ gravity in which the BD parameter is $\omega_{BD} = 0$. Hence, the Starobinsky’s inflation model, $f(R) = R + R^2/(6M^2)$, leads to $|f_{NL}^{\text{equil}}| \ll 1$ as in standard inflation.
D. Potential driven inflation in the presence of the nonminimal coupling and the Gauss-Bonnet term

The result (114) shows that, in the limit $c_s^2 \ll 1$, the terms coming from the nonminimal coupling and the Gauss-Bonnet coupling are suppressed relative to the term proportional to $1/c_s^2$. The effect of those couplings on $f_{\text{NL}}^{\text{equil}}$ appears indirectly through the change of $c_s^2$.

Let us consider the following action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M^2_\text{pl}}{2} F(\phi) R + \omega(\phi) X - V(\phi) - \xi(\phi) G \right], \tag{134} \]

which corresponds to inflation driven by the field potential $V(\phi)$. Expanding the scalar propagation speed $\omega(\phi)$ up to second order, we obtain

\[ c_s^2 \approx 1 - \frac{2\delta_\xi(\delta_F - 8\delta_\xi)(3\delta_F - 24\delta_\xi - 4\delta_{PX})}{\delta_{PX}}, \tag{135} \]

where $\delta_{PX} = \omega X/(M^2_\text{pl} H^2 F)$ is another slow-variation parameter. Since $\{|\delta_\xi|, |\delta_F|, |\delta_{PX}| \ll 1$, the scalar propagation speed is very close to 1, i.e. $|c_s^2 - 1| = \mathcal{O}(\epsilon^2)$. From Eq. (114) we have

\[ f_{\text{NL}}^{\text{equil}} \approx \frac{55}{36} c_s + \frac{5}{12} \eta_s + \frac{10}{3} \delta_\xi, \tag{136} \]

and hence the non-Gaussianity is small.

The non-Gaussianity can be large for the kinetically driven inflation in the presence of the terms $P(\phi, X)$, $\xi(\phi) G$, and $G(\phi, X) \Box \phi$. We will study such models in a separate publication [68].

VII. CONCLUSIONS

For the general single-field models described by the action (2) we have calculated the three-point correlation function of curvature perturbations $\mathcal{R}$ generated during inflation. This covers the inflationary models motivated by low-energy effective string theory, scalar-tensor theories, and Galileon-inspired gravity. The Gauss-Bonnet term $\xi(\phi) G$ and the generalized Galileon term $G(\phi, X) \Box \phi$ give rise to the scalar propagation speed $c_s$ different from 1, as it happens in k-inflation. These models can lead to large non-Gaussianities of primordial perturbations detectable in future observations.

Using the ADM metric in uniform-field gauge ($\dot{\phi} = 0$), we have expanded the action (2) up to third order in the perturbations. The second-order action associated with linear curvature perturbations is given by Eq. (22), where both $Q$ and $c_s^2$ need to be positive to avoid the appearance of ghosts and Laplacian instabilities respectively. In the quasi de Sitter background we have solved the equation for curvature perturbations and derived the scalar power spectrum and its spectral index. The tensor spectrum as well as the tensor-to-scalar ratio is also evaluated to confront the models with observations.

After the derivation of the third-order perturbed action (159), we have expressed it in a more convenient form for the calculation of non-Gaussianities by making a lot of integrations by parts. The coefficient $F_3$ in front of $\delta \mathcal{L}_2/\delta \mathcal{R}|_1$ includes only the derivative terms of $\mathcal{R}$, which vanish in the large-scale limit. Moreover the term $\delta \mathcal{L}_2/\delta \mathcal{R}|_1$ survives only at second order of perturbations. When the vacuum expectation value of $\mathcal{R}$ for the three-point operator is evaluated, we just need to compute the contributions coming from the 8 terms in Eq. (77) by treating the coefficients $C_i$ ($i = 1, \ldots, 8$) as slowly varying parameters relative to the scale factor. Finally we have derived the three-point correlation function as in the form (99). For the equilateral configuration ($k_1 = k_2 = k_3$) the non-linear parameter $f_{\text{NL}}^{\text{equil}}$ reduces to Eq. (111), which does not have any momentum dependence.

Under the approximation in terms of slow-variation parameters we showed that $f_{\text{NL}}^{\text{equil}}$ reduces to a simple form (114). Provided that $c_s^2 \ll 1$, one can realize large non-Gaussianities with $|f_{\text{NL}}^{\text{equil}}| \gg 1$. For $c_s^2 \ll 1$ the terms involving $\delta_F$ and $\delta_\xi$ are sub-dominant contributions with respect to the first two terms in Eq. (114), but the last term in Eq. (114), coming from the Galileon-type self interaction, can be comparable to the dominant contributions. Compared to k-inflation the presence of the terms $\xi(\phi) G$ and $G(\phi, X) \Box \phi$ leads to different values of $c_s$, so that $f_{\text{NL}}^{\text{equil}}$ is subject to change.

We have applied our results of the equilateral non-linear parameter to a number of concrete models of inflation. In k-inflation and in the generalized Galileon model, our formula of $f_{\text{NL}}^{\text{equil}}$ reproduces the previous results known in literature. In the inflationary models based on Brans-Dicke theories, including $f(R)$ gravity, the non-linear parameter
is of the order of slow-roll parameters. This is associated with the fact that the scalar propagation speed is unity in those models.

It will be of interest to distinguish between various inflationary models observationally by using our formula of $f_{\text{NL}}^{\text{equil}}$ as well as $n_R$ and $r$. Especially the potential detectability of non-Gaussianities by the PLANCK satellite may open up a new opportunity to approach the origin of inflation.

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Appendix A: Explicit expression of $Q$ and $c_s$

Using the expressions for $w_1$, $w_2$, $w_3$ and $w_4$, it is possible to write down explicitly the form of $Q$ defined in Eq. (23). Namely we have

$$Q \equiv \frac{w_1}{w_2} \left[ 3 M_{\text{Pl}}^4 \ddot{F}^2 - 2 \dot{\phi}^4 M_{\text{Pl}}^2 F G,\phi + 4 M_{\text{Pl}}^2 F G,\phi^2 + 2 P_X \dot{\phi}^2 M_{\text{Pl}}^2 F - 48 \dot{\phi}^5 H^2 G,XX \dot{\xi} 
+ 2 \dot{\phi}^4 P_X X M_{\text{Pl}}^2 F + 12 \dot{\phi}^3 G,XX M_{\text{Pl}}^2 FH + 6 \dot{\phi}^5 H G,XX M_{\text{Pl}}^2 F - 48 M_{\text{Pl}}^2 \ddot{F} H^2 \dot{\xi} 
- 6 M_{\text{Pl}}^2 \ddot{F} \dot{\phi}^3 G,XX - 48 H^2 \dot{\xi} \dot{\phi}^2 G,XX + 16 \dot{\phi}^4 H^2 G,XX - 16 P_X \dot{\phi}^2 H^2 \dot{\xi} 
- 16 P_X \dot{\phi}^2 H \dot{\xi} + 192 H^4 \dot{\xi}^2 + 32 \dot{\phi}^2 H^2 G,\phi + 3 \dot{\phi}^3 G,XX - 16 \dot{\phi}^4 P_X X H \dot{\xi} \right].$$

Along the same lines the speed of propagation $c_s^2$ given in Eq. (23) can be written as

$$c_s^2 = \frac{L_3 \ddot{\phi} + L_4}{L_2},$$

where

$$L_2 = (M_{\text{Pl}}^2 F - 8 H \xi,\phi \ddot{\phi}) \left[ 3 M_{\text{Pl}}^4 \dot{F}^2 - 2 \dot{\phi}^4 M_{\text{Pl}}^2 F G,\phi + 4 M_{\text{Pl}}^2 F G,\phi + 2 M_{\text{Pl}}^2 F P_X 
+ 2 M_{\text{Pl}}^2 \dot{\phi}^2 P_X X + 12 M_{\text{Pl}}^2 \dot{F} \dot{H} G,XX + 6 M_{\text{Pl}}^2 \dot{F} \dot{\phi}^3 H G,XX - 48 M_{\text{Pl}}^2 \ddot{F} \dot{\phi} H^2 \xi,\phi 
- 6 M_{\text{Pl}}^2 \ddot{F} \dot{\phi}^3 G,XX - 48 H^2 \dot{\xi} \dot{\phi}^2 G,XX + 16 \dot{\phi}^3 H \dot{\xi} G,\phi X - 16 P_X \dot{\phi} H \dot{\xi,\phi} + 192 H^4 \dot{\xi}^2 
+ 32 \ddot{\phi} H \dot{\xi} G,\phi + 3 \dot{\phi}^3 G,XX - 16 \dot{\phi}^4 P_X X \ddot{H} \dot{\xi,\phi} - 48 \dot{\phi}^3 H^2 G,XX \ddot{\xi,\phi} \right].$$

$$L_3 = 1536 H^4 \dot{\xi}^3 \dot{\phi} + 640 H^2 \ddot{\xi}^2 \dot{\phi} G,XX + 128 H^2 \ddot{\xi} \dot{\phi}^2 H G,XX + 2 M_{\text{Pl}}^4 \dot{F} \ddot{\phi}^2 G,XX 
- 384 M_{\text{Pl}}^2 \ddot{\xi} \ddot{\phi}^2 F,\phi G,X + 24 \dot{\phi}^3 G,XX \ddot{\xi,\phi} + 4 M_{\text{Pl}}^4 \dot{F}^2 G,XX 
+ 24 M_{\text{Pl}}^4 \ddot{F} \dot{\phi}^2 \xi,\phi - 32 M_{\text{Pl}}^2 \ddot{F} H \ddot{\phi}^3 G,XX - 64 M_{\text{Pl}}^2 \ddot{F} \ddot{H} \dot{\xi,\phi} \ddot{\phi} G,XX,$$}

$$L_4 = 8 G,XX \ddot{\xi,\phi} \dot{\phi} - 48 G,XX \ddot{H} \ddot{\xi,\phi} \ddot{\phi} + [32 \ddot{\xi} G,XX G,\phi - M_{\text{Pl}}^2 \dot{F} G,XX^2 + 256 \ddot{\xi} G,XX H^2 \ddot{\xi,\phi} 
- 16 M_{\text{Pl}}^2 F,\phi G,XX \ddot{\xi,\phi} - 16 \ddot{\xi} G,XX P_X + 128 H^2 \ddot{H} \ddot{\xi} G,XX - 16 M_{\text{Pl}}^2 \ddot{F} G,XX \ddot{\xi,\phi} 
- 32 \ddot{F} G,XX \dot{H} + 2 M_{\text{Pl}}^2 \ddot{F} \ddot{H} G,XX - 3 M_{\text{Pl}}^2 \ddot{F} \ddot{\phi} G,XX \right] \dot{\phi}^4 
+ [-112 M_{\text{Pl}}^2 \ddot{F}^2 \ddot{H} \ddot{\xi,\phi} G,XX + 16 M_{\text{Pl}}^2 \ddot{F} \ddot{\phi} P_X + 8 M_{\text{Pl}}^4 \ddot{F} \ddot{\phi}^2 \ddot{\xi,\phi} - 2 M_{\text{Pl}}^4 \ddot{F} \ddot{\phi} G,XX 
- 256 M_{\text{Pl}}^2 \ddot{\xi} F,\phi H^2 \ddot{\xi,\phi} + 16 M_{\text{Pl}}^2 \ddot{\xi} F,\phi \ddot{F,\phi} - 128 M_{\text{Pl}}^2 H^2 \ddot{\xi} \ddot{\xi} F,\phi + 2 M_{\text{Pl}}^4 \ddot{F} \ddot{G} \ddot{\phi,\phi} 
- 32 M_{\text{Pl}}^2 \ddot{F} \ddot{G} \ddot{F,\phi} + 1536 H^4 \ddot{\xi} \ddot{\xi} G,XX \dot{\phi}^2 
+ [8 H (\dddot{\xi} \dddot{\phi} - 96 M_{\text{Pl}}^2 \ddot{\phi} \ddot{H} \ddot{\phi} + 8 M_{\text{Pl}}^2 \ddot{F} \ddot{\phi} G,\phi - 6 M_{\text{Pl}}^4 \ddot{F} \ddot{\phi}^2 \ddot{\phi} + M_{\text{Pl}}^4 \ddot{F} G,XX - 4 M_{\text{Pl}}^2 \ddot{F} \ddot{P_X} \ddot{\phi} G,\phi) \right] \dot{\phi}^4 
+ M_{\text{Pl}}^2 \ddot{F} (2 M_{\text{Pl}}^2 \ddot{F} P_X - 4 M_{\text{Pl}}^2 \ddot{F} \ddot{G,\phi} + 192 H^4 \dddot{\xi} \dddot{\phi} + 3 M_{\text{Pl}}^4 \ddot{F} \ddot{\phi}^2 - 48 M_{\text{Pl}}^2 \ddot{F} \ddot{\phi} H^2 \ddot{\xi,\phi}).$$

Appendix B: Third-order action

In the Appendices[3] and [12] we shall use the symbol $\dddot{\phi}$ for the quantities which are valid up to total derivatives.
We first perturb the action \( S = \int d^4x \sqrt{-g} \mathcal{L} \) given in Eq. 2 up to third order. Since we choose the gauge in which the scalar field \( \phi \) is unperturbed (\( \delta \phi = 0 \)), the functions \( P \) and \( G \) are expanded due to their \( X \) dependence. After the Taylor-expansion of the action up to third order, we can perform the following steps in order to simplify the result.

1. We start by removing the cubic term in \( \psi \). One can employ the following relations

\[
\begin{align*}
    c(t, x) \partial_i^2 \psi & \doteq - (\partial_t c)(\partial_i^2 \psi) & \text{for} & \quad i = 1, 2, 3, \\
    c(t, x)(\partial_{ij} \psi)(\partial_j \partial_i^2 \psi) & \doteq - (\partial_t c)(\partial_{ij} \psi)(\partial_i \partial_j \psi)/2 & \text{for} & \quad i \neq j,
\end{align*}
\]

where \( c \) depends on other perturbation variables, and a hatted index is not summed. The \( \psi \)-cubic term automatically cancels out after the integration by parts.

2. Now we simplify the term cubic in \( \alpha \). This can be done by employing the following relations

\[
    c(t) \alpha^2 \alpha \doteq - \dot{c} \alpha^3 / 3, \quad c(t, x) \partial_t^2 \alpha \doteq - (\partial_t \alpha)(\partial_t c).
\]

3. Next, we simplify the cubic term in \( \mathcal{R} \). In this case we can make use of the following relations

\[
    c(t) \mathcal{R}^2 \doteq - \dot{c} \mathcal{R}^3 / 3, \quad c(t) \mathcal{R}^2 \mathcal{R} \doteq - \dot{c} \mathcal{R}^3 / 3 - 2c \mathcal{R} \mathcal{R}^2, \quad c(t, x) \mathcal{R} \mathcal{R} \doteq - \dot{c} \mathcal{R}^2 / 2, \quad c(t) \mathcal{R}^2 \mathcal{R} \doteq - \dot{c} \mathcal{R}^3 / 3. \quad (B4)
\]

Other useful relations are

\[
\begin{align*}
    c(t) \mathcal{R}(\partial_t \mathcal{R})^2 & \doteq - \dot{c} \mathcal{R}(\partial_t \mathcal{R})^2 + 2c \partial_t \mathcal{R} \partial_t \mathcal{R}, \\
    c(t) \mathcal{R} \mathcal{R}(\partial_t \mathcal{R})^2 & \doteq c \mathcal{R}(\partial_t \mathcal{R})^2 + \mathcal{R}(\partial_t \mathcal{R})(\partial_t \mathcal{R}) + 3c \mathcal{R}(\partial_t \mathcal{R})(\partial_t \mathcal{R}).
\end{align*}
\]

After performing these integrations by parts, we find that the \( \mathcal{R} \)-cubic term can be written in the form \( c_1(t) \mathcal{R}^3 + c_2(t) \mathcal{R} \mathcal{R}^2 + c_3(t) \mathcal{R}^3 + c_4(t) \mathcal{R}(\partial_t \mathcal{R})^2 \).

4. Now let us simplify the term quadratic in \( \psi \) and linear in \( \mathcal{R} \). First we integrate by parts any derivative for the field \( \mathcal{R} \), so that these terms can be written as \( \mathcal{R} \times (\text{quadratic term in } \psi) \). Afterwards, we use the following relations (where \( i \neq j \))

\[
\begin{align*}
    c(t, x)(\partial_t^2 \psi)(\partial_j \partial_i \psi) & \doteq - c(\partial_t^2 \psi)(\partial_j \partial_i \psi) - \dot{c}(\partial_t \psi)(\partial_j \partial_i \psi) / 2, \\
    c(t, x)(\partial_t \psi)(\partial_j \partial_i \psi) & \doteq - c(\partial_t \psi)(\partial_j \partial_i \psi) / 2, \\
    c(t, x) \partial_t \partial_j \partial_i \psi & \doteq - (\partial_t \alpha)(\partial_j \partial_i \psi), \\
    c(t, x)(\partial_t \partial_j \partial_i \psi) & \doteq - (\partial_t \partial_j \partial_i \psi) / 2.
\end{align*}
\]

5. Along the same lines we can simplify the term quadratic in \( \psi \) and linear in \( \alpha \) after integrating by parts any derivative of \( \alpha \).

6. Let us next consider the term quadratic in \( \alpha \) and linear in \( \mathcal{R} \). In this case one can first integrate by parts all the second derivatives for the field \( \alpha \). Then one can integrate the following first derivative terms

\[
    c(t, x) \alpha \partial_t \alpha \doteq - \alpha^2 \partial_t c / 2.
\]

7. Now we consider the term quadratic in \( \mathcal{R} \). First of all we can remove any derivative from the subset of terms which possess the field \( \alpha \). Then we eliminate the second derivatives for the field \( \mathcal{R} \) by using

\[
\begin{align*}
    c(t, x) \mathcal{R} \partial_t \mathcal{R} & \doteq - \dot{c} \mathcal{R} \partial_t \mathcal{R} / 2, \\
    c(t, x) \mathcal{R} \mathcal{R} & \doteq - \dot{c} \mathcal{R} \mathcal{R} / 2, \\
    c(t, x) \mathcal{R} \mathcal{R} \partial_t \mathcal{R} & \doteq - c(\mathcal{R} \mathcal{R} \partial_t \mathcal{R})^2 / 2, \\
    c(t)(\partial_t \mathcal{R})(\partial_j \partial_i \mathcal{R}) & \doteq - c(\partial_t \mathcal{R})(\partial_j \partial_i \mathcal{R})^2 / 2, \\
    c(t)(\partial_t \mathcal{R})(\partial_j \partial_i \mathcal{R}) & \doteq - c(\partial_t \mathcal{R})(\partial_j \partial_i \mathcal{R}) + c \mathcal{R}^2(\partial_t \mathcal{R})^2 / 2, \\
    c(t)(\partial_t \mathcal{R})(\partial_j \partial_i \mathcal{R}) & \doteq - c(\partial_t \mathcal{R})(\partial_j \partial_i \mathcal{R}) + c \mathcal{R}^2(\partial_t \mathcal{R})^2 / 2, \\
    c(t)(\partial_t \mathcal{R})(\partial_j \partial_i \mathcal{R}) & \doteq - c(\partial_t \mathcal{R})(\partial_j \partial_i \mathcal{R}) + c \mathcal{R}^2(\partial_t \mathcal{R})^2 / 2. \quad (B8)
\end{align*}
\]

where \( i \neq j \). The remaining terms can be simplified by using the following expressions

\[
\begin{align*}
    c(t, x) \mathcal{R}(\partial_t \mathcal{R}) & \doteq - \dot{c} \mathcal{R}(\partial_t c) / 2, \\
    c(t, x) \partial_t \mathcal{R}(\partial_t \mathcal{R}) & \doteq - \dot{c} \mathcal{R}(\partial_t \mathcal{R}) / 2, \\
    c(t) \mathcal{R}(\partial_t \mathcal{R})(\partial_t \mathcal{R}) & \doteq - \dot{c} \mathcal{R}^2 \partial_t \mathcal{R} / 2 + c \mathcal{R}^2(\partial_t \mathcal{R})^2 / 2 - c(\partial_t \mathcal{R})(\partial_t \mathcal{R}), \\
    c(t, x)(\partial_t \mathcal{R}) & \doteq - \dot{c} \mathcal{R}(\partial_t c). \quad (B9)
\end{align*}
\]

8. For the terms which consist of a combination of three different fields, we can integrate by parts any derivative of \( \alpha \).
Appendix C: Integration of the term $f_3/a$

We rewrite the integrand $f_3/a$ in Eq. (59) in a more convenient form. We define

$$\dot{X} = c(t) \mathcal{R} \left[ (\partial^2 \mathcal{R})^2 - (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) \right],$$  \hspace{1cm} (C1)$$
$$\dot{X}_k = c(t) \mathcal{R} \left[ \partial_k \mathcal{R} (\partial^2 \mathcal{R}) - \partial_i \mathcal{R} (\partial_i \partial_k \mathcal{R}) - c(t) \mathcal{R} \left[ (\partial_k \mathcal{R}) (\partial^2 \mathcal{R}) - (\partial_i \mathcal{R}) (\partial_i \partial_k \mathcal{R}) \right] \right].$$  \hspace{1cm} (C2)$$

Then, for any function $c(t)$ dependent on $t$, we find

$$\dot{X} + 2 \partial_k \dot{X}_k = \dot{c} \mathcal{R} \left[ (\partial^2 \mathcal{R})^2 - (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) \right] + 3 c \dot{\mathcal{R}} \left[ (\partial^2 \mathcal{R})^2 - (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) \right],$$  \hspace{1cm} (C3)$$

which implies that

$$c(t) \dot{\mathcal{R}} \left[ (\partial^2 \mathcal{R})^2 - (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) \right] = -\dot{c} \mathcal{R} \left[ (\partial^2 \mathcal{R})^2 - (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) \right]/3.$$  \hspace{1cm} (C4)$$

Let us also define

$$Y = c(t) \mathcal{R} \left[ (\partial \mathcal{R})^2 \right],$$  \hspace{1cm} (C5)$$
$$Y_k = c(t) \left[ 2 \mathcal{R} \partial_j \mathcal{R} \partial_k \partial_j \mathcal{R} - 2 \mathcal{R} \partial_k \mathcal{R} \partial^2 \mathcal{R} - \partial_k \mathcal{R} (\partial \mathcal{R})^2 \right].$$  \hspace{1cm} (C6)$$

It then follows that

$$3Y + \partial_k Y_k = -2 c \mathcal{R} \left[ (\partial^2 \mathcal{R})^2 - (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) \right],$$  \hspace{1cm} (C7)$$

so that

$$c(t) \mathcal{R} \left[ (\partial \mathcal{R})^2 \right] = -2 c \mathcal{R} \left[ (\partial^2 \mathcal{R})^2 - (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) \right]/3.$$  \hspace{1cm} (C8)$$

By using Eqs. (C4) and (C8), one finds

$$\frac{f_3}{a} \leq \frac{1}{a} \left[ (A_6 L_1^2 - A_7 L_1) \mathcal{R} + A_6 L_1^2 \mathcal{R} \right] \left[ (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) - (\partial^2 \mathcal{R})^2 \right] + \frac{A_6 L_1^2}{a} (\partial \mathcal{R})^2 \partial^2 \mathcal{R}$$
$$\leq \frac{1}{a} \left[ A_6 L_1^2 - \frac{1}{3} \frac{d}{dt} (A_6 L_1^2 - A_7 L_1) + \frac{H}{3} (A_5 L_1^2 - A_7 L_1) + \frac{2}{3} A_6 L_1^2 \right] \left[ (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) - (\partial^2 \mathcal{R})^2 \right].$$  \hspace{1cm} (C9)$$

Employing the following relation

$$c(t) \mathcal{R} \left[ (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) - (\partial^2 \mathcal{R})^2 \right] \leq -c (\partial \mathcal{R}) (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{R}) - c (\partial \mathcal{R}) (\partial \mathcal{R}) (\partial \mathcal{R}) - c \mathcal{R} (\partial^2 \mathcal{R})^2$$
$$\leq -c \mathcal{R} \partial_i \partial_j [I (\partial_i \mathcal{R}) (\partial_j \mathcal{R})] + c (\partial \mathcal{R})^2 (\partial^2 \mathcal{R}),$$  \hspace{1cm} (C10)$$

we obtain the form of $f_3/a$ given in Eq. (59).

Appendix D: Integration of the term $a f_2$

Regarding the integrands of the term $a f_2$ in Eq. (59), we start by considering the contribution written in the form $c(t) \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X})$. We define

$$Z = c(t) \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X}) \right],$$  \hspace{1cm} (D1)$$
$$Z_k = c(t) (\mathcal{R} \partial_k \mathcal{R} - \mathcal{R} \partial_k \mathcal{R}) (\partial_k \partial_i \mathcal{X}),$$  \hspace{1cm} (D2)$$

so that

$$\dot{Z} + \partial_k Z_k = \dot{c} \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X}) + 2c \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X}) + c \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X}) + c Q (\mathcal{R} \partial_i \mathcal{R} - \mathcal{R} \partial_i \mathcal{R}) \partial_i \mathcal{R} \right].$$  \hspace{1cm} (D3)$$

This gives

$$c(t) \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X}) \leq -[(\dot{c} + a q_2) \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X}) + c \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X}) + c Q (\mathcal{R} \partial_i \mathcal{R} - \mathcal{R} \partial_i \mathcal{R}) \partial_i \mathcal{R})/2$$
$$+ a q_2 \mathcal{R} (\partial_i \partial_j \mathcal{R}) (\partial_i \partial_j \mathcal{X})/2.$$  \hspace{1cm} (D4)$$
For later convenience we added and subtracted the quantity \(aq_2 \mathcal{R}(\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X})/2\).

Let us focus on the term containing \(q_2\) alone. Then we have

\[
aq_2(t) \mathcal{R}[(\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X})]/2 \doteq \left[-aq_2(\partial_t \mathcal{X})(\partial_t \mathcal{R})(\partial_t \partial_j \mathcal{R}) - aq_2(\partial_t \mathcal{R}) (\partial_t \partial^2 \mathcal{R})/2 \right]
\]

\[
\doteq aq_2 \{ (\partial_t \mathcal{X})(\partial_t \mathcal{R})(\partial^2 \mathcal{R}) - \mathcal{R} \partial_t \partial_j[(\partial_t \mathcal{R})(\partial_t \mathcal{X})]\}/2 + aq_2 Q \mathcal{R} \hat{\mathcal{R}}(\partial^2 \mathcal{R})/2. \tag{D5}
\]

For the term \((\partial_t \mathcal{R})(\partial_t \mathcal{X})(\partial_t \mathcal{R})\) the following equalities hold

\[
c(t)(\partial_t \mathcal{R})(\partial_t \mathcal{X})(\partial_t \mathcal{R}) \doteq -c\partial_t \mathcal{R} \partial_t \mathcal{X} \partial_t \mathcal{R} - c\partial_t \mathcal{R} \partial_t \partial_j \mathcal{X} \partial_t \mathcal{R} \doteq 2c \mathcal{R} \partial_t \partial_k \mathcal{R} \partial_t \mathcal{X} + c\mathcal{R} \partial_t \partial^2 \mathcal{R} \partial_t \mathcal{X} + c\mathcal{R} \partial_t \partial_t \partial_t \partial^2 \mathcal{X}, \tag{D6}
\]

so that

\[
c(t)(\partial_t \mathcal{R})(\partial_t \mathcal{X})(\partial_t \mathcal{R}) \doteq c\mathcal{R} \partial_t \partial_t \partial_t \partial_t \partial_t \mathcal{X} - cQ \mathcal{R} \partial_t \partial_t \mathcal{R} /2 + cQ \mathcal{R} \partial_t \partial_t \mathcal{R} \doteq 2c \mathcal{R} \partial_t \partial_k \mathcal{R} \partial_t \mathcal{X} + c\mathcal{R} \partial_t \partial_t \partial_t \partial_t \mathcal{X} + cQ \mathcal{R} \partial_t \partial_t \mathcal{R} /2, \tag{D7}
\]

where we have defined \(\partial_t \partial_j \mathcal{A} \equiv \partial_t \partial_j \mathcal{A}\). Putting all these partial results together, it follows that

\[
a \left[ \frac{2a_6 L_1}{w_1} \mathcal{R} (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X}) + \frac{1}{w_1} (A_7 - 2A_5 L_1) \mathcal{R} (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X}) - \frac{A_9 L_1}{w_1} \partial_t \mathcal{R} \partial_t \partial_j \mathcal{X} \right]
\]

\[
\doteq -a \left[ \frac{2a_6 L_1}{w_1} + \frac{d}{dt} \left( \frac{A_7 - 2A_5 L_1}{2w_1} \right) + H \left( \frac{A_7 - 2A_5 L_1}{2w_1} \right) + \frac{q_2}{2} + \frac{A_9 L_1}{w_1} \right] \mathcal{R} (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X})
\]

\[
- \frac{a}{2w_1} (A_7 - 2A_5 L_1) \mathcal{R} (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X}) + \frac{aq_2}{2} \{ (\partial_t \mathcal{X})(\partial_t \mathcal{R})(\partial_t \mathcal{R}) - \mathcal{R} \partial_t \partial_j[(\partial_t \mathcal{R})(\partial_t \mathcal{X})]\}
\]

\[
- \frac{aQ}{2w_1} (A_7 - 2A_5 L_1) (\mathcal{R} \partial_t \mathcal{R} - \mathcal{R} \partial_t \hat{\mathcal{R}}) \partial_t \hat{\mathcal{R}} + \frac{aq_2 Q}{2} \mathcal{R} \hat{\mathcal{R}} (\partial_t \mathcal{R}) - \frac{a}{w_1} A_9 L_1 Q \mathcal{R} (\partial_t \mathcal{R})(\partial_t \hat{\mathcal{R}})
\]

\[
- \frac{a}{2w_1} A_9 L_1 Q \mathcal{R} (\partial_t \mathcal{R})^2. \tag{D8}
\]

We choose \(q_2\) such that

\[
\frac{2a_6 L_1}{w_1} + \frac{d}{dt} \left( \frac{A_7 - 2A_5 L_1}{2w_1} \right) + H \left( \frac{A_7 - 2A_5 L_1}{2w_1} \right) + \frac{q_2}{2} + \frac{A_9 L_1}{w_1} = \frac{3H}{2w_1} (A_7 - 2A_5 L_1). \tag{D9}
\]

In this case we have

\[
a \left[ \frac{2a_6 L_1}{w_1} \mathcal{R} (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X}) + \frac{1}{w_1} (A_7 - 2A_5 L_1) \mathcal{R} (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X}) - \frac{A_9 L_1}{w_1} \partial_t \mathcal{R} \partial_t \partial_j \mathcal{X} \right]
\]

\[
\doteq -a \left[ \frac{2a_6 L_1}{w_1} + \frac{d}{dt} \left( \frac{A_7 - 2A_5 L_1}{2w_1} \right) + H \left( \frac{A_7 - 2A_5 L_1}{2w_1} \right) + \frac{aq_2}{2} \{ (\partial_t \mathcal{X})(\partial_t \mathcal{R})(\partial_t \mathcal{R}) - \mathcal{R} \partial_t \partial_j[(\partial_t \mathcal{R})(\partial_t \mathcal{X})]\}
\]

\[
- \frac{aQ}{2w_1} (A_7 - 2A_5 L_1) (\mathcal{R} \partial_t \mathcal{R} - \mathcal{R} \partial_t \hat{\mathcal{R}}) \partial_t \hat{\mathcal{R}} + \frac{aq_2 Q}{2} \mathcal{R} \hat{\mathcal{R}} (\partial_t \mathcal{R}) - \frac{a}{w_1} A_9 L_1 Q \mathcal{R} (\partial_t \mathcal{R})(\partial_t \hat{\mathcal{R}})
\]

\[
- \frac{a}{2w_1} A_9 L_1 Q \mathcal{R} (\partial_t \mathcal{R})^2. \tag{D10}
\]

Consider the following combination

\[
ac(t) \mathcal{R} (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X}) + 3aH c(t) \mathcal{R} (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X})
\]

\[
\doteq -ac(\partial_t \mathcal{R}) \{ (\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X}) + \mathcal{R} \partial_t \partial_t \mathcal{X} \} - 3aH c(\partial_t \mathcal{R}) (\partial_t \partial_t \mathcal{X} + \mathcal{R} \partial_t \partial_t \mathcal{X})
\]

\[
\doteq -ac(\partial_t \partial_j \mathcal{R})(\partial_t \partial_j \mathcal{X}) + ac(\partial^2 \mathcal{X}) (\partial_t \mathcal{R})^2 + \mathcal{R} \partial_t \partial_t \mathcal{X} + 3aH c(\partial_t \mathcal{X}) (\partial_t \mathcal{R})^2 + \mathcal{R} \partial_t \partial_t \mathcal{X}
\]

\[
\doteq ac(\partial_t \partial_t \mathcal{X} + 3H \mathcal{X}) - ac(\partial_t \partial_t \mathcal{X} \partial_t \partial_j [(\partial_t \mathcal{R})(\partial_j \mathcal{R})] + ac \mathcal{R} \partial_t \partial_t \mathcal{X} + 3aH Q \mathcal{R} \hat{\mathcal{R}} \partial_t \partial_t \mathcal{R}, \tag{D11}
\]

where we have used the properties

\[
\mathcal{X} = Q \partial_t \partial_t \mathcal{X}, \quad \text{and} \quad \partial_t \partial_t (\mathcal{X} + 3H \mathcal{X}) = \left( \frac{d}{dt} + 3H \right) (Q \mathcal{R}) = a^{-3} \frac{d}{dt} (a^3 Q \mathcal{R}). \tag{D12}
\]
The reason why we have introduced this combination is that it is a part of the linear equation of motion \([29]\). Then Eq. \((D10)\) reduces to
\[
a \left[ -\frac{2A_9L_1}{w_1} \mathcal{R}(\partial_i \partial_j \mathcal{R})(\partial_i \partial_j \mathcal{X}) + \frac{1}{w_1} (A_7 - 2A_5L_1) \hat{\mathcal{R}}(\partial_i \partial_j \mathcal{R})(\partial_i \partial_j \mathcal{X}) - \frac{A_9L_1}{w_1} \partial^2 \mathcal{R}(\partial_i \partial_j \mathcal{X}) \right] \\
\approx -\frac{A_7 - 2A_5L_1}{2w_1a^2} \frac{d}{dt}(a^3 \dot{\mathcal{R}}) \left\{ (\partial \mathcal{R})^2 - \partial^{-2} \partial_i \partial_j [(\partial_i \mathcal{R})(\partial_j \mathcal{R})] \right\} + \frac{aq_2}{2} \left\{ (\partial \mathcal{X})(\partial_i \mathcal{R})(\partial_j \mathcal{X}) \right\} \\
- \frac{a}{w_1} (A_7 - 2A_5L_1) \mathcal{R} \partial^2 \mathcal{R}(\partial^2 \mathcal{X}) - \frac{3aH}{w_1} Q (A_7 - 2A_5L_1) \mathcal{R} \partial \mathcal{R} \partial \mathcal{X} - \frac{aQ}{w_1} (A_7 - 2A_5L_1) \left( \mathcal{R} \partial \mathcal{R} - \mathcal{R} \partial \mathcal{X} \right) \partial \mathcal{R} \\
+ \frac{aq_2Q}{2} \mathcal{R} \mathcal{R}(\partial^2 \mathcal{X}) - \frac{a}{w_1} A_9L_1 Q \mathcal{R}(\partial_i \mathcal{R})(\partial_i \mathcal{R}) - \frac{a}{w_1} A_9L_1 Q \mathcal{R}(\partial^2 \mathcal{X}). \quad \text{(D13)}
\]
Combining these terms with the other ones in the \(af_2\) term, we find
\[
a f_2 \approx -\frac{A_7 - 2A_5L_1}{2w_1a^2} \frac{d}{dt}(a^3 \dot{\mathcal{R}}) \left\{ (\partial \mathcal{R})^2 - \partial^{-2} \partial_i \partial_j [(\partial_i \mathcal{R})(\partial_j \mathcal{R})] \right\} + \frac{aq_2}{2} \left\{ (\partial \mathcal{X})(\partial_i \mathcal{R})(\partial_j \mathcal{X}) \right\} \\
+ a \left[ \frac{2L_1Q}{w_1} - \frac{3H}{w_1} Q (A_7 - 2A_5L_1) + \frac{aq_2Q}{2} \right] \mathcal{R} \partial^2 \mathcal{R} - \frac{3a}{w_1} A_9L_1 Q \mathcal{R}(\partial^2 \mathcal{X}) + a A_9 \mathcal{R}(\partial^2 \mathcal{X}) \\
+ a \left( A_2 - A_3L_1 + \frac{2L_1Q}{w_1} - A_7 \right) \mathcal{R} \mathcal{R}(\partial \mathcal{R}) - \frac{a}{w_1} (A_7 - 2A_5L_1) \mathcal{R}(\partial \mathcal{R}) \mathcal{R}(\partial^2 \mathcal{X}) \\
- \frac{aQ}{w_1} (A_7 - 2A_5L_1) \left( \mathcal{R} \partial \mathcal{R} - \mathcal{R} \partial \mathcal{X} \right) \partial \mathcal{R} - \frac{a}{w_1} A_9L_1 Q \mathcal{R}(\partial_i \mathcal{R})(\partial_i \mathcal{R}). \quad \text{(D14)}
\]
This is similar to the formula (131) of Ref. \([67]\). Let us define
\[
A = \mathcal{R} \mathcal{R}(\partial^2 \mathcal{X}), \quad B = \mathcal{R}(\partial^2 \mathcal{X}), \quad C = \mathcal{R} \mathcal{R}(\partial^2 \mathcal{X}), \quad D = \mathcal{R}(\partial^2 \mathcal{X}), \quad E = \mathcal{R}(\partial^2 \mathcal{X}) \mathcal{R}(\partial^2 \mathcal{X}), \\
F = \mathcal{R} \mathcal{R}(\partial \mathcal{R} \mathcal{R}) \mathcal{R}(\partial \mathcal{R}), \quad H = \mathcal{R} \mathcal{R}(\partial \mathcal{R} \mathcal{R}) \mathcal{R}(\partial \mathcal{R}), \quad I = \mathcal{R} \mathcal{R}(\partial \mathcal{R} \mathcal{R}) \mathcal{R}(\partial \mathcal{R}), \quad J = \mathcal{R}(\partial^2 \mathcal{X}), \quad K = \mathcal{R}(\partial^2 \mathcal{X}). \quad \text{(D15)}
\]
Then the following relations hold
\[
\begin{align*}
B & \doteq -A - I, \quad D \doteq -2F, \quad K \doteq -2I, \quad J \doteq -2C, \quad cB \doteq -cC - 2cJ, \\
& cE \doteq c(\partial^2 \mathcal{X}) (\hat{\mathcal{R}} + Q \mathcal{R}) + cQ \mathcal{R}(\hat{\mathcal{R}} + Q \mathcal{R}) \mathcal{R}(\partial \mathcal{R} \mathcal{R} + cQ \mathcal{R}(\partial^2 \mathcal{X})) \\
& \doteq -cA + cQ \mathcal{R} + cQ \mathcal{R}(\hat{\mathcal{R}} + Q \mathcal{R}) \mathcal{R}(\partial \mathcal{R} \mathcal{R} + cQ \mathcal{R}(\partial^2 \mathcal{X})).
\end{align*} \quad \text{(D16)}
\]
Because of these identities, we can rewrite Eq. \((D14)\) in the form
\[
a f_2 \approx -\frac{A_7 - 2A_5L_1}{2w_1a^2} \frac{d}{dt}(a^3 \dot{\mathcal{R}}) \left\{ (\partial \mathcal{R})^2 - \partial^{-2} \partial_i \partial_j [(\partial_i \mathcal{R})(\partial_j \mathcal{R})] \right\} + \frac{aq_2}{2} \left\{ (\partial \mathcal{X})(\partial_i \mathcal{R})(\partial_j \mathcal{X}) \right\} \\
+ \left[ \frac{2L_1Q}{w_1} - \frac{3H}{w_1} Q (A_7 - 2A_5L_1) + \frac{aq_2Q}{2} \right] \mathcal{R} \partial^2 \mathcal{R} - \frac{3a}{w_1} A_9L_1 Q \mathcal{R}(\partial^2 \mathcal{X}) + a A_9 \mathcal{R}(\partial^2 \mathcal{X}) \\
- \frac{3aA_9 L_1 Q}{w_1} \mathcal{R}(\partial \mathcal{R})^2 + a A_9 \mathcal{R}(\partial^2 \mathcal{X}) + a \left( A_2 - A_3L_1 \right) \mathcal{R} \mathcal{R}(\partial \mathcal{R}) - \frac{aQ}{w_1} (A_7 - 2A_5L_1) \mathcal{R}(\partial \mathcal{R}) \mathcal{R}(\partial^2 \mathcal{X}) \\
- \frac{a}{w_1} A_9L_1 Q \mathcal{R}(\partial_i \mathcal{R})(\partial_i \mathcal{R}). \quad \text{(D17)}
\]
Using the relations \(A_9 = -2w_1, cI \doteq -cC + cA, cB \doteq -cC - 2cA\) and the definition of \(q_2\), we obtain the form of \(af_2\) given in Eq. \((68)\). 

**Appendix E: Integration of the term \(a^a f_i\)**

By defining
\[
W = c(t) \mathcal{R}(\partial_i \mathcal{X})(\partial_i \mathcal{X}), \tag{E1}
\]
\[
W_k = c(t) \mathcal{R}(\partial_j \mathcal{X}) - c(t) \partial_k \mathcal{R}(\partial_j \mathcal{X})(\partial_j \mathcal{X}) + c(t) \hat{\mathcal{R}}(\partial_j \mathcal{X})(\partial_j \mathcal{X}), \tag{E2}
\]
we have
\[
2\partial_k W - \hat{W} - 2c \mathcal{R}(\partial_k \mathcal{X}) - 2c \hat{\mathcal{R}}(\partial_k \mathcal{R})(\partial_k \mathcal{X}) + c \hat{\mathcal{R}}(\partial_k \mathcal{X})(\partial_k \mathcal{X}) = -\hat{c} \mathcal{R}(\partial_k \mathcal{X})(\partial_k \mathcal{X}). \tag{E3}
\]
It follows that
\[
c(t) \dot{R}(\partial_{ik}X)(\partial_{ik}X) \doteq -\dot{c}R(\partial_{ik}X)(\partial_{ik}X) + 2cR(\partial_{ik}X)(\partial_{ik}\dot{X}^2) + 2c\dot{X}\partial_{ik}(\partial_{ik}R\partial_{ik}X) \\
\doteq -\dot{c}R(\partial_{ik}X)(\partial_{ik}X) + 2c\dot{X}\partial_{ik}(\partial_{ik}R\partial_{ik}X) - 2c(\partial_{ik}R)(\partial_{ik}X)\partial_{ik}\dot{X} - 2cR(\partial_{ik}X)(\partial_{ik}\dot{X}). \tag{E4}
\]

There is also another relation
\[
d(t) R(\partial_{kl}X)(\partial_{kl}X) \doteq -d(\partial_{kl}X)(\partial_{kl}R) - d(\partial_{kl}X)R(\partial_{kl}\dot{X}) \\
\doteq -dX\partial_{kl}[(\partial_{kl}X)(\partial_{kl}R)] + d(\partial_{kl}X)(\partial_{kl}R)(\partial_{kl}\dot{X}) + dR(\partial_{kl}X)(\partial_{kl}\dot{X}). \tag{E5}
\]

We study the contribution of the last term in Eq. (E10), i.e. \((c_1\dot{R} + c_2R)(\partial_{ik}\partial_{j}X)(\partial_{ik}\partial_{j}X)\), where \(c_1 = A_5/w_1^2\) and \(c_2 = A_6/w_1^2\). Using Eqs. (E11) and (E5), we have
\[
a^3c_1\dot{R}(\partial_{ik}X)(\partial_{ik}X) + a^3(c_2 - p_1)\dot{R}(\partial_{kl}X)(\partial_{kl}X) + a^3p_1\dot{R}(\partial_{kl}X)(\partial_{kl}X) \\
\doteq -a^3(\dot{c}_1 + 3Hc_1 - p_1)R(\partial_{ik}X)(\partial_{ik}X) + a^3[2c_1\dot{X}\partial_{kl}(\partial_{kl}R\partial_{kl}X) - 2c_1\dot{X}\partial_{kl}X\partial_{kl}\dot{X} - 2c_1R(\partial_{kl}X)(\partial_{kl}\dot{X})] \\
+ a^3[(p_1 - c_2)\partial_{kl}[(\partial_{kl}X)(\partial_{kl}R)] + (c_2 - p_1)(\partial_{kl}X)(\partial_{kl}R)(\partial_{kl}\dot{X}) + (c_2 - p_1)R(\partial_{kl}X)(\partial_{kl}\dot{X})], \tag{E6}
\]
where we added and subtracted the term \(a^3p_1\dot{R}(\partial_{kl}X)(\partial_{kl}X)\). Let us define \(p_1\) such that
\[
p_1 - c_2 = 6Hc_1, \tag{E7}
\]
in which case
\[
a^3c_1\dot{R}(\partial_{ik}X)(\partial_{ik}X) + a^3(c_2 - p_1)\dot{R}(\partial_{kl}X)(\partial_{kl}X) + a^3p_1\dot{R}(\partial_{kl}X)(\partial_{kl}X) \\
\doteq -a^3(\dot{c}_1 + 3Hc_1 - p_1)R(\partial_{ik}X)(\partial_{ik}X) + a^3[2c_1\dot{X}\partial_{kl}(\partial_{kl}R\partial_{kl}X) - 2c_1\dot{X}\partial_{kl}X\partial_{kl}\dot{X} - 2c_1R(\partial_{kl}X)(\partial_{kl}\dot{X})] \\
+ 2c_1a^3\dot{X}^2\partial_{kl}(\partial_{kl}R\partial_{kl}X) - (\partial_{kl}R)(\partial_{kl}X)\partial_{kl}\dot{X}^2 - 2c_1R(\partial_{kl}X)(\partial_{kl}\dot{X}) - 2c_1\dot{X}\partial_{kl}X\partial_{kl}\dot{X} - 2c_1R(\partial_{kl}X)(\partial_{kl}\dot{X})] \\
\doteq -a^3(\dot{c}_1 + 3Hc_1 - p_1)R(\partial_{ik}X)(\partial_{ik}X) - 2c_1\dot{X}\partial_{kl}X\partial_{kl}\dot{X} - 2c_1R(\partial_{kl}X)(\partial_{kl}\dot{X}) \\
- 2c_1Q R \dot{R} \frac{d}{dt}(a^3Q \dot{R}), \tag{E8}
\]
where we used Eq. (D12). The last term of Eq. (E8) is expressed as
\[
- 2c_1Q R \dot{R} \frac{d}{dt}(a^3Q \dot{R}) \doteq -\frac{c_1}{a^3} \dot{R} \left[ (a^3Q \dot{R})^2 \right] \doteq (a^3Q \dot{R})^2 \frac{d}{dt} \left( \frac{c_1}{a^3} \right) \\
\doteq a^3c_1Q^2 \dot{R}^3 + a^3(\dot{c}_1 - 3Hc_1)Q^2 \dot{R}^2. \tag{E9}
\]
The integrand \(a^3f_1\) can be written as
\[
a^3f_1 \doteq a^3 \left( A_1 + A_2 \frac{Q}{w_1} - q_1Q \right) \dot{R}^3 + a^3 \left[ A_4 - A_6 \frac{Q}{w_1^2} - Q^2 \frac{d}{dt} \left( A_5 \frac{Q}{w_1^2} - \frac{3HA_5Q^2}{w_1^2} \right) \right] \dot{R}^2 + a^3q_1Q^2 \dot{R}^3 \\
+ a^3 \left[ A_6 \frac{Q}{w_1^2} - \frac{d}{dt} \left( A_5 \frac{Q}{w_1^2} + \frac{3HA_5}{w_1^2} \right) \right] \dot{R}(\partial_{ik}X)(\partial_{ik}X) + 2A_5 \frac{Q}{w_1^2} \left[ (a^3Q \dot{R})(\partial_{ik}R)(\partial_{ik}X) + \partial_{ik}\partial_{kl}[(\partial_{ik}R)(\partial_{ik}X)] \right] \\
+ a^3 \left( A_9 \frac{Q}{w_1} \right) \dot{R} \partial_{ik} \partial_{ik} \dot{X}, \tag{E10}
\]
where we added and subtracted the quantity \(a^3q_1Q^2 \dot{R}^3\). Finally we integrate this term by parts as follows
\[
a^3q_1Q^2 \dot{R}^3 \doteq -\dot{R} \left[ q_1a^3Q \dot{R}^2 + q_1 \frac{d}{dt} (a^3Q) \dot{R}^2 + 2q_1a^3Q \dot{R} \dot{R} \right] \\
\doteq -\dot{R} \dot{R} \left[ q_1a^3Q \dot{R} + q_1 \frac{d}{dt} (a^3Q) \dot{R} + 2q_1a^3Q \dot{R} \right] \doteq -\dot{R} \left[ 2q_1 \frac{d}{dt} (a^3Q \dot{R}) - q_1 \frac{d}{dt} (a^3Q) \dot{R} + q_1a^3Q \dot{R} \right] \\
\doteq -2q_1R \dot{R} \frac{d}{dt} (a^3Q \dot{R}) + a^3[q_1(\dot{Q} + 3HQ) - Q\dot{q}_1] \dot{R} \dot{R}^2. \tag{E11}
\]
Then Eq. (E10) reads
\[
a^3 f_1 = a^3 \left[ A_4 - A_0 \frac{Q^2}{w_1^2} + \frac{Q^2}{w_1^2} \left( \frac{A_5}{w_1^3} \right) - \frac{3H A_5 Q^2}{w_1^2} + q_1 (\dot{Q} + 3HQ) - \dot{Q} \dot{q}_1 \right] \mathcal{R} \dot{\mathcal{R}}^2
\]
\[
+ a^3 \left( A_1 + A_3 \frac{Q}{w_1} - q_1 Q \right) \mathcal{R}^2 - 2q_1 \mathcal{R} \mathcal{R} \frac{d}{dt} (a^3 Q \dot{R}) + \frac{a^3}{w_1} \left( A_3 \frac{Q}{w_1} \right) \mathcal{R} \partial_{ij} \mathcal{R} \partial_{ij} \mathcal{X}
\]
\[
+ a^3 \left[ \frac{A_5}{w_1^2} - \frac{d}{dt} \left( \frac{A_5}{w_1^3} \right) + \frac{3H A_5}{w_1^2} \right] \mathcal{R} (\partial_{ik} \mathcal{X}) (\partial_{ik} \mathcal{X}) - \frac{2A_5}{w_1^2} \frac{d}{dt} (a^3 Q \dot{R}) (\partial_{ik} \mathcal{X}) (\partial_{ik} \mathcal{X}) - \partial^{-2} \partial_i \partial_j [(\partial_i \mathcal{R}) (\partial_j \mathcal{X})] \right].
\]

The value of \( q_1 \) is chosen to match another term in \( a f_2 \), see Eq. (71). Using the following relation
\[
c(t) (\partial_j \mathcal{X}) (\partial_i \mathcal{X}) \equiv -c (\partial^2 \mathcal{R}) (\partial^2 \mathcal{X})^2 / 2,
\]
we finally obtain the expression of \( a^3 f_1 \) given in Eq. (70).

[1] A. A. Starobinsky, Phys. Lett. B 91, 99 (1980).
[2] D. Kazanas, Astrophys. J. 241 L59 (1980); K. Sato, Mon. Not. R. Astron. Soc. 195, 467 (1981); Phys. Lett. 99B, 66 (1981); A. H. Guth, Phys. Rev. D 23, 347 (1981).
[3] V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33, 532 (1981); A. H. Guth and S. Y. Pi, Phys. Rev. Lett. 49 (1982) 1110; S. W. Hawking, Phys. Lett. B 115, 295 (1982); A. A. Starobinsky, Phys. Lett. B 117 (1982) 175; J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D 28, 679 (1983).
[4] G. F. Smoot et al., Astrophys. J. 396, L1 (1992).
[5] D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003); E. Komatsu et al. [WMAP Collaboration], Astrophys. J. Suppl. 180, 330-376 (2009).
[6] E. Komatsu et al., Astrophys. J. Suppl. 123, 18 (2011).
[7] S. M. Leach and A. R. Liddle, Phys. Rev. D 68, 123508 (2003); W. H. Kinney, E. W. Kolb, A. Melchiorri and A. Riotto, Phys. Rev. D 69, 103516 (2004); L. Alabidi and D. H. Lyth, JCAP 0605, 016 (2006); JCAP 0608, 013 (2006); L. Alabidi and J. E. Lidsey, Phys. Rev. D78, 103519 (2008); L. Alabidi and I. Huston, JCAP 1008, 037 (2010); J. Martin and C. Ringeval, JCAP 0608, 009 (2006); L. Lorenz, J. Martin and C. Ringeval, JCAP 0804, 001 (2008); J. Martin, C. Ringeval and R. Trotta, arXiv:1009.4157 [astro-ph.CO].
[8] M. Tegmark et al. [SDSS Collaboration], Phys. Rev. D69, 103501 (2004); U. Seljak et al. [SDSS Collaboration], Phys. Rev. D71, 103515 (2005); J. K. Adelman-McCarthy et al. [SDSS Collaboration], Astrophys. J. Suppl. 175, 297-313 (2008); K. N. Abazajian et al. [SDSS Collaboration], Astrophys. J. Suppl. 182, 543 (2009).
[9] D. H. Lyth and A. Riotto, Phys. Rept. 314, 1 (1999); A. D. Linde, arXiv:hep-th/9905320 B. A. Bassett, S. Tsujikawa and D. Wands, Phys. Rev. D 85, 537 (2006).
[10] E. Komatsu and D. N. Spergel, Phys. Rev. D63, 063002 (2001).
[11] N. Bartolo, S. Matarrese, A. Riotto, Phys. Rev. D65, 103502 (2002).
[12] J. M. Maldacena, JHEP 0305, 013 (2003).
[13] P. Creminelli, JCAP 0310, 003 (2003).
[14] G. I. Rigopoulos and E. P. S. Shellard, Phys. Rev. D 68, 123518 (2003); JCAP 0510, 006 (2005).
[15] D. H. Lyth and Y. Rodriguez, Phys. Rev. Lett. 95, 121302 (2005); D. H. Lyth and Y. Rodriguez, Phys. Rev. D 71, 123508 (2005); D. H. Lyth, K. A. Malik and M. Sasaki, JCAP 0504, 005 (2005).
[16] C. T. Byrnes, M. Sasaki and D. Wands, Phys. Rev. D 74, 123519 (2006).
[17] [PLANCK Collaboration], arXiv:astro-ph/0604069.
[18] A. Gangui, F. Lucchin, S. Matarrese and S. Mollerach, Astrophys. J. 430, 447 (1994).
[19] L. Verde, L. M. Wang, A. Heavens and M. Kamionkowski, Mon. Not. Roy. Astron. Soc. 313, L141 (2000).
[20] P. Creminelli, A. Nicols, L. Senatore, M. Tegmark and M. Zaldarriaga, JCAP 0605, 004 (2006).
[21] D. S. Salopek and J. R. Bond, Phys. Rev. D 42, 3936 (1990).
[22] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999).
[23] E. Silverstein and D. Tong, Phys. Rev. D 70, 103505 (2004).
[24] N. Arkani-Hamed, H. C. Cheng, M. A. Luty and S. Mukohyama, JHEP 0405, 074 (2004); N. Arkani-Hamed, P. Creminelli, S. Mukohyama and M. Zaldarriaga, JCAP 0404, 001 (2004); F. Piazza and S. Tsujikawa, JCAP 0407, 004 (2004).
[25] A. Gruzinov, Phys. Rev. D 71, 027301 (2005).
[26] M. Alishahiha, E. Silverstein and D. Tong, Phys. Rev. D 70, 123505 (2004).
V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. 215, 203 (1992); K. A. Malik and D. Wands, Phys. Rept. 475, 1-51 (2009).
[65] J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, 219 (1999).
[66] K. Koyama, Class. Quant. Grav. 27, 124001 (2010).
[67] H. Collins, [arXiv:1101.1308 [astro-ph.CO]].
[68] A. De Felice, J. Elliston, R. Tavakol, and S. Tsujikawa, in preparation.