Freudenthal Duality in Gravity: 
from Groups of Type $E_7$
to Pre-Homogeneous Spaces

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ABSTRACT

Freudenthal duality can be defined as an anti-involution, non-linear map acting on symplectic spaces. It was introduced in four-dimensional Maxwell-Einstein theories coupled to a non-linear sigma model of scalar fields.

In this short review, I will consider its relation to the $U$-duality Lie groups of type $E_7$ in extended supergravity theories, and comment on the relation between the Hessian of the black hole entropy and the pseudo-Euclidean, rigid special (pseudo)Kähler metric of the pre-homogeneous spaces associated to the $U$-orbits.

Keywords : Extended Supergravity, Duality, Freudenthal Triple Systems, Special Kähler Geometry, Pre-Homogeneous Vector Spaces.

Talk presented at the Conference
Group Theory, Probability, and the Structure of Spacetime in honor of V.S.Varadarajan, UCLA Mathematics Department, Los Angeles, CA USA, November 7–9, 2014.
To appear in a special issue of “p-Adic Numbers, Ultrametric Analysis and Applications”.

1 Freudenthal Duality

We start and consider the following Lagrangian density in four dimensions \( cfr. e.g. [1] \):

\[
\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij} (\varphi) \partial_{\mu} \varphi^i \partial^{\mu} \varphi^j + \frac{1}{4} I_{\Lambda \Sigma} (\varphi) F^{\Lambda}_{\mu \nu} F^{\Sigma}_{\mu \nu} + \frac{1}{8 \sqrt{-G}} R_{\Lambda \Sigma} (\varphi) \epsilon^{\mu \nu \rho \sigma} F^{\Lambda}_{\mu \nu} F^{\Sigma}_{\rho \sigma},
\]  

(1)

describing Einstein gravity coupled to Maxwell (Abelian) vector fields and to a non-linear sigma model of scalar fields (with no potential); note that \( \mathcal{L} \) may - but does not necessarily need to - be conceived as the bosonic sector of \( D = 4 \) (ungauged) supergravity theory. Out of the Abelian two-form field strengths \( F^\Lambda \)'s, one can define their duals \( G^\Lambda \), and construct a symplectic vector:

\[
H := (F^\Lambda, G^\Lambda)^T, \quad \star G^\Lambda|_{\mu \nu} := 2 \delta L \delta F^\Lambda|_{\mu \nu}.
\]

(2)

We then consider the simplest solution of the equations of motion deriving from \( \mathcal{L} \), namely a static, spherically symmetric, asymptotically flat, dyonic extremal black hole with metric \([2]\):

\[
ds^2 = -e^{2U(\tau)} d\tau^2 + e^{-2U(\tau)} \left[ \frac{d\tau^2}{\tau^2} + \frac{1}{\tau^2} (d\theta^2 + \sin \theta d\phi^2) \right],
\]

(3)

where \( \tau := -1/r \). Thus, the two-form field strengths and their duals can be fluxed on the two-sphere at infinity \( S^2_\infty \) in such a background, respectively yielding the electric and magnetic charges of the black hole itself, which can be arranged in a symplectic vector \( Q \):

\[
p^\Lambda := \frac{1}{4\pi} \int_{S^2_\infty} F^\Lambda, \quad q_\Lambda := \frac{1}{4\pi} \int_{S^2_\infty} G_\Lambda,
\]

(4)

\[
Q := (p^\Lambda, q_\Lambda)^T.
\]

(5)

Then, by exploiting the symmetries of the background \([3]\), the Lagrangian \([1]\) can be dimensionally reduced from \( D = 4 \) to \( D = 1 \), obtaining a 1-dimensional effective Lagrangian \( (' := d/d\tau) \) \([3]\):

\[
\mathcal{L}_{D=1} = (U')^2 + g_{ij} (\varphi) \varphi^i \varphi^j + e^{2U} V_{BH} (\varphi, Q)
\]

(6)

along with the Hamiltonian constraint \([3]\)

\[
(U')^2 + g_{ij} (\varphi) \varphi^i \varphi^j - e^{2U} V_{BH} (\varphi, Q) = 0.
\]

(7)

The so-called “effective black hole potential” \( V_{BH} \) appearing in \( (6) \) and \( (7) \) is defined as \([3]\)

\[
V_{BH} (\varphi, Q) := -\frac{1}{2} Q^T \mathcal{M} (\varphi) Q,
\]

(8)

in terms of the symplectic and symmetric matrix \([1]\)

\[
\mathcal{M} := \begin{pmatrix} 1 & -R \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -R & 1 \end{pmatrix} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix},
\]

(9)

\[
\mathcal{M}^T = \mathcal{M}; \quad \mathcal{M} \Omega \mathcal{M} = \Omega,
\]

(10)

where \( I \) denotes the identity, and \( R (\varphi) \) and \( I (\varphi) \) are the scalar-dependent matrices occurring in \([1]\); moreover, \( \Omega \) stands for the symplectic metric \( (\Omega^2 = -\mathbb{I}) \). Note that, regardless of the invertibility of \( R (\varphi) \) and as a consequence of the physical consistence of the kinetic vector matrix \( I (\varphi) \), \( \mathcal{M} \) is negative-definite; thus, the effective black hole potential \([3]\) is positive-definite.
By virtue of the matrix $\mathcal{M}$, one can introduce a (scalar-dependent) anti-involution $\mathcal{S}$ in any Maxwell-Einstein-scalar theory described by (11) with a symplectic structure $\Omega$, as follows:

$$\mathcal{S}(\varphi) := \Omega \mathcal{M}(\varphi);$$

in turn, this allows to define an anti-involution on the dyonic charge vector $Q$, which has been called (scalar-dependent) Freudenthal duality [4, 5, 6]:

$$\mathfrak{F}(Q; \varphi) := -\mathcal{S}(\varphi) Q;$$

$$\mathfrak{F}^2 = -I, \quad (\forall \{\varphi\}).$$

By recalling (8) and (11), the action of $\mathfrak{F}$ on $Q$, defining the so-called ($\varphi$-dependent) Freudenthal dual of $Q$ itself, can be related to the symplectic gradient of the effective black hole potential $V_{BH}$:

$$\mathfrak{F}(Q; \varphi) = \Omega \frac{\partial V_{BH}(\varphi, Q)}{\partial Q}. \quad (15)$$

Through the attractor mechanism [7], all this enjoys an interesting physical interpretation when evaluated at the (unique) event horizon of the extremal black hole $\text{H}$ (denoted below by the subscript “H”); indeed

$$\partial_{\varphi} V_{BH} = 0 \Leftrightarrow \lim_{\tau \to -\infty} \varphi^i(\tau) = \varphi^i_{\text{H}}(Q);$$

$$S_{BH}(Q) = \frac{A_H}{4} = \pi V_{BH}|_{\partial_{\varphi} V_{BH} = 0} = -\frac{\pi}{2} Q^T \mathcal{M}_H(Q) Q,$$  \quad (17)

where $S_{BH}$ and $A_H$ respectively denote the Bekenstein-Hawking entropy [8] and the area of the horizon of the extremal black hole, and the matrix horizon value $\mathcal{M}_H$ is defined as

$$\mathcal{M}_H(Q) := \lim_{\tau \to -\infty} \mathcal{M}(\varphi(\tau)).$$

Correspondingly, one can define the (scalar-independent) horizon Freudenthal duality $\mathfrak{F}_H$ as the horizon limit of (13):

$$\tilde{Q} \equiv \mathfrak{F}_H(Q) := \lim_{\tau \to -\infty} \mathfrak{F}(Q; \varphi(\tau)) = -\Omega \mathcal{M}_H(Q) Q = \frac{1}{\pi} \Omega \frac{\partial S_{BH}(Q)}{\partial Q}. \quad (19)$$

Remarkably, the (horizon) Freudenthal dual of $Q$ is nothing but (1/π times) the symplectic gradient of the Bekenstein-Hawking black hole entropy $S_{BH}$; this latter, from dimensional considerations, is only constrained to be an homogeneous function of degree two in $Q$. As a result, $\tilde{Q} = \mathfrak{F}_H(Q)$ is generally a complicated (non-linear) function, homogeneous of degree one in $Q$.

It can be proved that the entropy $S_{BH}$ itself is invariant along the flow in the charge space $Q$ defined by the symplectic gradient (or, equivalently, by the horizon Freudenthal dual) of $Q$ itself:

$$S_{BH}(Q) = S_{BH}(\mathfrak{F}_H(Q)) = S_{BH} \left( \frac{1}{\pi} \Omega \frac{\partial S_{BH}(Q)}{\partial Q} \right) = S_{BH}(\tilde{Q}). \quad (20)$$

It is here worth pointing out that this invariance is pretty remarkable: the (semi-classical) Bekenstein-Hawking entropy of an extremal black hole turns out to be invariant under a generally non-linear map acting on the black hole charges themselves, and corresponding to a symplectic gradient flow in their corresponding vector space.

For other applications and instances of Freudenthal duality, see [9, 10, 11].
2 Groups of Type $E_7$

The concept of Lie groups of type $E_7$ as introduced in the 60s by Brown [12], and then later developed e.g. by [13, 14, 15, 16, 17].

Starting from a pair $(G, R)$ made of a Lie group $G$ and its faithful representation $R$, the three axioms defining $(G, R)$ as a group of type $E_7$ read as follows:

1. Existence of a (unique) symplectic invariant structure $\Omega$ in $R$:
   \[ \exists ! \Omega \equiv 1 \in R \times_a R, \] (21)

   which then allows to define a symplectic product $\langle \cdot, \cdot \rangle$ among two vectors in the representation space $R$ itself:
   \[ \langle Q_1, Q_2 \rangle := Q_1^M Q_2^N \Omega_{MN} = - \langle Q_2, Q_1 \rangle. \] (22)

2. Existence of (unique) rank-4 completely symmetric invariant tensor ($K$-tensor) in $R$:
   \[ \exists ! K \equiv 1 \in (R \times R \times R \times R)_s, \] (23)

   which then allows to define a degree-4 invariant polynomial $I_4$ in $R$ itself:
   \[ I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q. \] (24)

3. Defining a triple map $T$ in $R$ as
   \[ T : R \times R \times R \to R; \] (25)
   \[ \langle T (Q_1, Q_2, Q_3) , Q_4 \rangle := K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q, \] (26)

   it holds that
   \[ \langle T (Q_1, Q_1, Q_2) , T (Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q. \] (27)

This property makes a group of type $E_7$ amenable to a description as an automorphism group of a Freudenthal triple system (or, equivalently, as the conformal groups of the underlying Jordan triple system - whose a Jordan algebra is a particular case -).

All electric-magnetic duality ($U$-duality\(^1\)) groups of $\mathcal{N} \geq 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are of type $E_7$. Among these, degenerate groups of type $E_7$ are those in which the $K$-tensor is actually reducible, and thus $I_4$ is the square of a quadratic invariant polynomial $I_2$. In fact, in general, in theories with electric-magnetic duality groups of type $E_7$ holds that

\[ S_{BH} = \pi \sqrt{|I_4(Q)|} = \pi \sqrt{|K_{MNPQ} Q^M Q^N Q^P Q^Q|}, \] (28)

whereas in the case of degenerate groups of type $E_7$ it holds that $I_4(Q) = (I_2(Q))^2$, and therefore the latter formula simplifies to

\[ S_{BH} = \pi \sqrt{|I_4(Q)|} = \pi |I_2(Q)|. \] (29)

\(^1\)Here $U$-duality is referred to as the “continuous” symmetries of [13]. Their discrete versions are the $U$-duality non-perturbative string theory symmetries introduced by Hull and Townsend [19].
Table 1: Simple, non-degenerate groups $G$ related to Freudenthal triple systems $\mathfrak{M}(J_3)$ on simple rank-3 Jordan algebras $J_3$. In general, $G \cong Conf(J_3) \cong Aut(\mathfrak{M}(J_3))$ (see e.g. [20, 21, 22] for a recent introduction, and a list of Refs.). $\mathbb{O}$, $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{R}$ respectively denote the four division algebras of octonions, quaternions, complex and real numbers, and $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ are the corresponding split forms. Note that the $G$ related to split forms $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ is the maximally non-compact (split) real form of the corresponding compact Lie group. $M_{1,2}(\mathbb{O})$ is the Jordan triple system generated by $2 \times 1$ vectors over $\mathbb{O}$ [23]. Note that the $STU$ model, based on $J_3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, has a semi-simple $G_4$, but its triality symmetry [24] renders it “effectively simple”. The $D = 5$ uplift of the $T^3$ model based on $J_3 = \mathbb{R}$ is the pure $\mathcal{N} = 2$, $D = 5$ supergravity. $J_3^\mathbb{R}$ is related to both 8 and 24 supersymmetries, because the corresponding supergravity theories are “twin”, namely they share the very same bosonic sector [23, 25, 26, 27].

Simple, non-degenerate groups of type $E_7$ relevant to $\mathcal{N} \geq 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are reported in Table 1.

Semi-simple, non-degenerate groups of type $E_7$ of the same kind are given by $G = SL(2, \mathbb{R}) \times...
SO(2, n) and \( G = SL(2, \mathbb{R}) \times SO(6, n) \), with \( R = (2, 2 + n) \) and \( R = (2, 6 + n) \), respectively relevant for \( N = 2 \) and \( N = 4 \) supergravity.

Moreover, degenerate (simple) groups of type \( E_7 \) relevant to the same class of theories are \( G = U(1, n) \) and \( G = U(3, n) \), with complex fundamental representations \( R = n + 1 \) and \( R = 3 + n \), respectively relevant for \( N = 2 \) and \( N = 3 \) supergravity [16].

The classification of groups of type \( E_7 \) is still an open problem, even if some progress have been recently made e.g. in [28] (in particular, cfr. Table D therein).

In all the aforementioned cases, the scalar manifold is a symmetric cosets \( \mathbb{G}/H \), where \( H \) is the maximal compact subgroup (with symmetric embedding) of \( G \). Moreover, the \( K \)-tensor can generally be expressed as [17]

\[
K_{MNPQ} = -\frac{n(2n+1)}{6d} \left[ t^a_{MN} t^b_{PQ} - \frac{d}{n(2n+1)} \Omega_{M(P} \Omega_{Q)N} \right],
\]

where \( \dim R = 2n \) and \( \dim G = d \), and \( t^a_{MN} \) denotes the symplectic representation of the generators of \( G \) itself. Thus, the horizon Freudenthal duality can be expressed in terms of the \( K \)-tensor as follows [1]:

\[
\mathcal{H}_H(Q)_M \equiv \bar{\varpi}_M = \frac{\partial \sqrt{|I_4(Q)|}}{\partial Q^M} = \epsilon \frac{2}{\sqrt{|I_4(Q)|}} K_{MNPQ} Q^N P^P Q^Q,
\]

where \( \epsilon := I_4/|I_4| \); note that the horizon Freudenthal dual of a given symplectic dyonic charge vector \( Q \) is well defined only when \( Q \) is such that \( I_4(Q) \neq 0 \). Consequently, the invariance (20) of the black hole entropy under the the horizon Freudenthal duality can be recast as the invariance of \( I_4 \) itself:

\[
I_4(Q) = I_4(\bar{Q}) = I_4 \left( \Omega \frac{\partial \sqrt{|I_4(Q)|}}{\partial Q} \right).
\]

In absence of “flat directions” at the attractor points (namely, of unstabilized scalar fields at the horizon of the black hole), and for \( I_4 > 0 \), the expression of the matrix \( \mathcal{M}_H(Q) \) at the horizon can be computed to read

\[
\mathcal{M}_{H|MN}(Q) = \frac{1}{\sqrt{I_4}} \left( 2 \bar{\varpi}_M \bar{Q}_N - 6 K_{MNPQ} Q^P Q^Q + Q_M Q_N \right),
\]

and it is invariant under horizon Freudenthal duality:

\[
\mathcal{H}_H(\mathcal{M}_H)_{MN} := \mathcal{M}_{H|MN}(\bar{Q}) = \mathcal{M}_{H|MN}(Q).
\]

### 3 Duality Orbits, Rigid Special Kähler Geometry and Pre-Homogeneous Vector Spaces

For \( I_4 > 0 \), \( \mathcal{M}_H(Q) \) given by (33) is one of the two possible solutions to the set of equations [29]

\[
\begin{cases}
M^T (Q) \Omega M (Q) = \epsilon \Omega; \\
M^T (Q) = M (Q); \\
Q^T M (Q) Q = -2 \sqrt{|I_4(Q)|},
\end{cases}
\]

which describes symmetric, purely \( Q \)-dependent structures at the horizon; they are symplectic or anti-symplectic, depending on whether \( I_4 > 0 \) or \( I_4 < 0 \), respectively. Since in the class
of (super)gravity $D = 4$ theories discussed the sign of $I_4$ actually determines a stratification of the representation space $R$ of charges into distinct orbits of the action of $G$ into $R$ itself (usually named duality orbits), the symplectic or anti-symplectic nature of the solutions to the system (35) is $G$-invariant, and supported by the various duality orbits of $G$ (in particular, by the so-called “large” orbits, for which $I_4$ is non-vanishing).

One of the two possible solutions to the system (35) reads

$$M_+(Q) = - \frac{1}{\sqrt{|I_4|}} \left(2 \bar{Q}_M \bar{Q}_N - 6 \epsilon K_{MNPQ} Q^P Q^Q + \epsilon Q_M Q_N \right);$$

$$\mathfrak{g}_H(M_+)_{MN} = M_{+\,|MN}(\bar{Q}) = \epsilon M_{+\,|MN}(Q).$$

For $\epsilon = +1 \iff I_4 > 0$, it thus follows that

$$M_+(Q) = M_H(Q),$$

as anticipated.

On the other hand, the other solution to system (35) reads

$$M_-(Q) = \frac{1}{\sqrt{|I_4|}} \left(\bar{Q}_M \bar{Q}_N - 6 \epsilon K_{MNPQ} Q^P Q^Q \right);$$

$$\mathfrak{g}_H(M_-)_{MN} = M_{-\,|MN}(\bar{Q}) = \epsilon M_{-\,|MN}(Q).$$

By recalling the definition of $I_4$ (23), it is then immediate to realize that $M_-(Q)$ is the (opposite of the) Hessian matrix of $(1/\pi)$ times the black hole entropy $S_{BH}$:

$$M_{-\,|MN}(Q) = - \partial_M \partial_N \sqrt{|I_4|} = - \frac{1}{\pi} \partial_M \partial_N S_{BH}. \quad (39)$$

The matrix $M_-(Q)$ is the (opposite of the) pseudo-Euclidean metric of a non-compact, non-Riemannian rigid special Kähler manifold related to the duality orbit of the black hole electromagnetic charges (to which $Q$ belongs), which is an example of pre-homogeneous vector space (PVS) [30]. In turn, the nature of the rigid special manifold may be Kähler or pseudo-Kähler, depending on the existence of a $U(1)$ or $SO(1,1)$ connection.3

In order to clarify this statement, let us make two examples within maximal $N = 8$, $D = 4$ supergravity. In this theory, the electric-magnetic duality group is $G = E_7(7)$, and the representation in which the e.m. charges sit is its fundamental $R = 56$. The scalar manifold has rank-7 and it is the real symmetric coset $G/H = E_7(7)/SU(8)$, with dimension 70.

1. The unique duality orbit determined by the $G$-invariant constraint $I_4 > 0$ is the 55-dimensional non-symmetric coset

$$O_{I_4>0} = \frac{E_7(7)}{E_6(2)}. \quad (40)$$

By customarily assigning positive (negative) signature to non-compact (compact) generators, the pseudo-Euclidean signature of $O_{I_4>0}$ is $(n_+, n_-) = (30, 25)$. In this case, $M_-(Q)$ given by (39) is the 56-dimensional metric of the non-compact, non-Riemannian rigid special Kähler non-symmetric manifold

$$O_{I_4>0} = \frac{E_7(7)}{E_6(2)} \times \mathbb{R}^+, \quad (41)$$

with signature $(n_+, n_-) = (30, 26)$, thus with character $\chi := n_+ - n_- = 4$. Through a conifiction procedure (amounting to modding out $\mathbb{C} \cong SO(2) \times SO(1,1) \cong U(1) \times \mathbb{R}^+$, with signature along the $\mathbb{R}^+$-direction is negative [29].

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2For a thorough introduction to special Kähler geometry, see e.g. [31].

3To be more precise, it is worth mentioning that the actual relevant coset manifold is $E_7(7)/[SU(8)/\mathbb{Z}_2]$, because spinors transform according to the double cover of the stabilizer of the scalar manifold (see e.g. [32, 33], and Refs. therein).

4The signature along the $\mathbb{R}^+$-direction is negative [29].
Table 2: Non-generic, nor irregular PVS with simple $G$, of type 2 (in the complex ground field). To avoid discussing the finite groups appearing, the list presents the Lie algebra of the isotropy group rather than the isotropy group itself [34]. The interpretation (of suitable real, non-compact slices) in $D = 4$ theories of Einstein gravity is added; remaining cases will be investigated in a forthcoming publication.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$G$ & $V$ & $n$ & isotropy alg. & degree & interpr. $D = 4$ \\
\hline
$SL(2, \mathbb{C})$ & $S^3 \mathbb{C}^2$ & 1 & 0 & 4 & $\mathcal{N} = 2, \mathbb{R} \ (T^3)$ \\
$SL(6, \mathbb{C})$ & $\Lambda^3 \mathbb{C}^6$ & 1 & $\mathfrak{sl}(3, \mathbb{C})^{\oplus 2}$ & 4 & $\mathcal{N} = 2, J_3^C$, $\mathcal{N} = 0, J_1^C$, $\mathcal{N} = 5, M_{1,2}(\mathbb{C})$ \\
$SL(7, \mathbb{C})$ & $\Lambda^3 \mathbb{C}^7$ & 1 & $\mathfrak{g}_2^C$ & 7 & \\
$SL(8, \mathbb{C})$ & $\Lambda^3 \mathbb{C}^8$ & 1 & $\mathfrak{sl}(3, \mathbb{C})$ & 16 & \\
$SL(3, \mathbb{C})$ & $S^2 \mathbb{C}^3$ & 2 & 0 & 6 & \\
$SL(5, \mathbb{C})$ & $\Lambda^2 \mathbb{C}^5$ & 3 & $\mathfrak{sl}(2, \mathbb{C})$ & 5 & \\
$SL(6, \mathbb{C})$ & $\Lambda^2 \mathbb{C}^6$ & 2 & $\mathfrak{sl}(2, \mathbb{C})^{\oplus 3}$ & 6 & \\
$SL(3, \mathbb{C})^{\oplus 2}$ & $\mathbb{C}^3 \otimes \mathbb{C}^3$ & 2 & $\mathfrak{gl}(1, \mathbb{C})^{\oplus 2}$ & 6 & \\
$Sp(6, \mathbb{C})$ & $\Lambda^3_0 \mathbb{C}^6$ & 1 & $\mathfrak{sl}(3, \mathbb{C})$ & 4 & $\mathcal{N} = 2, J_3^R$ \\
$Spin(7, \mathbb{C})$ & $\mathbb{C}^8$ & 2 & $\mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$ & 2 & \\
$Spin(9, \mathbb{C})$ & $\mathbb{C}^{16}$ & 1 & $\mathfrak{spin}(7, \mathbb{C})$ & 2 & \\
$Spin(10, \mathbb{C})$ & $\mathbb{C}^{16}$ & 2 & $\mathfrak{g}_2^C \oplus \mathfrak{sl}(2, \mathbb{C})$ & 2 & \\
$Spin(11, \mathbb{C})$ & $\mathbb{C}^{32}$ & 3 & $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$ & 4 & \\
$Spin(12, \mathbb{C})$ & $\mathbb{C}^{32}$ & 1 & $\mathfrak{sl}(5, \mathbb{C})$ & 4 & \\
$Spin(14, \mathbb{C})$ & $\mathbb{C}^{64}$ & 1 & $\mathfrak{g}_2^C \oplus \mathfrak{g}_2^C$ & 8 & \\
$G_2^C$ & $\mathbb{C}^7$ & 1 & $\mathfrak{sl}(3, \mathbb{C})$ & 2 & $\mathcal{N} = 2, 6, J_3^{II}$ $\mathcal{N} = 0, J_3^H$ \\
$G_2^C$ & $\mathbb{C}^7$ & 2 & $\mathfrak{gl}(2, \mathbb{C})$ & 2 & \\
$E_6^C$ & $\mathbb{C}^{27}$ & 1 & $\mathfrak{f}_4^C$ & 3 & \\
$E_6^C$ & $\mathbb{C}^{27}$ & 2 & $\mathfrak{so}(8, \mathbb{C})$ & 6 & \\
$E_7^C$ & $\mathbb{C}^{56}$ & 1 & $\mathfrak{e}_6^C$ & 4 & $\mathcal{N} = 2, J_3^O$ $\mathcal{N} = 8, J_3^O$ \\
\hline
\end{tabular}
\end{table}

one can obtain the corresponding 54-dimensional non-compact, non-Riemannian special Kähler symmetric manifold

$$O_{I_4>0}/\mathbb{C} \cong \hat{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)} \times U(1)}.$$ (42)

2. The unique duality orbit determined by the $G$-invariant constraint $I_4 < 0$ is the 55-
dimensional non-symmetric coset

\[ \mathcal{O}_{I_4<0} = \frac{E_7(7)}{E_6(6)}, \]  

(43)

with pseudo-Euclidean signature given by \((n_+, n_-) = (28, 27)\), thus with character \(\chi = 0\).

In this case, \(M_-(Q)\) given by (39) is the 56-dimensional metric of the non-compact, non-Riemannian rigid special pseudo-Kähler non-symmetric manifold

\[ \mathcal{O}_{I_4<0} = \frac{E_7(7)}{E_6(6)} \times \mathbb{R}^+, \]  

(44)

with signature \((n_+, n_-) = (28, 28)\). Through a “pseudo-conification” procedure (amounting to modding out \(\mathbb{C}_s \cong SO(1, 1) \times SO(1, 1) \cong \mathbb{R}^+ \times \mathbb{R}^+\), one can obtain the corresponding 54-dimensional non-compact, non-Riemannian special pseudo-Kähler symmetric manifold

\[ \mathcal{O}_{I_4<0}/\mathbb{C}_s \cong \hat{\mathcal{O}}_{I_4<0} = \frac{E_7(7)}{E_6(6)} \times SO(1, 1). \]  

(45)

(41) and (44) are non-compact, real forms of \(\frac{E_7}{E_6} \times GL(1)\), which is the type 29 in the classification of regular, pre-homogeneous vector spaces (PVS) worked out by Sato and Kimura in [34]. From its definition, a PVS is a finite-dimensional vector space \(V\) together with a subgroup \(G\) of \(GL(V)\), such that \(G\) has an open dense orbit in \(V\). PVS are subdivided into two types (type 1 and type 2), according to whether there exists an homogeneous polynomial on \(V\) which is invariant under the semi-simple (reductive) part of \(G\) itself. For more details, see e.g. [30, 35, 36].

In the case of \(\frac{E_7}{E_6} \times GL(1)\), \(V\) is provided by the fundamental representation space \(\mathbb{R} = 56\) of \(G = E_7\), and there exists a quartic \(E_7\)-invariant polynomial \(I_4\) in the \(56\); \(H = E_6\) is the isotropy (stabilizer) group.

Amazingly, simple, non-degenerate groups of type \(E_7\) (relevant to \(D = 4\) Einstein (super)gravities with symmetric scalar manifolds) almost saturate the list of irreducible PVS with unique \(G\)-invariant polynomial of degree 4 (cfr. Table 2); in particular, the parameter \(n\) characterizing each PVS can be interpreted as the number of centers of the regular solution in the (super)gravity theory with electric-magnetic duality (\(U\)-duality) group given by \(G\). This topic will be considered in detail in a forthcoming publication.

Acknowledgments

My heartfelt thanks to Raja, for being so kind to invite me to participate into such a beautiful and inspiring conference.

I would like to thank Leron Borsten, Mike J. Duff, Sergio Ferrara, Emanuele Orazi, Mario Trigiante, and Armen Yeranyan for collaboration on the topics covered in this review.

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