Sequentially loop-generated quark and lepton mass hierarchies in an extended Inert Higgs Doublet model

A. E. Cárcamo Hernández, Sergey Kovalenko, Roman Pasechnik, and Ivan Schmidt

Universidad Técnica Federico Santa María and Centro Científico-Tecnológico de Valparaíso
Casilla 110-V, Valparaíso, Chile

Department of Astronomy and Theoretical Physics, Lund University, Sölvegatan 14A, SE-223 62 Lund, Sweden

Nuclear Physics Institute ASCR, 25068 Rež, Czech Republic

Departamento de Física, CFCM, Universidade Federal de Santa Catarina, C.P. 476, CEP 88.040-900, Florianópolis, SC, Brazil

Extended scalar and fermion sectors offer new opportunities for generating the observed strong hierarchies in the fermion mass and mixing patterns of the Standard Model (SM). In this work, we elaborate on the prospects of a particular extension of the Inert Higgs doublet model where the SM hierarchies are generated sequentially by radiative virtual corrections in a fully renormalisable way, i.e. without adding any non-renormalisable Yukawa terms or soft-breaking operators to the scalar potential. Our model has a potential to explain the recently observed $R_K$ and $R_{K^*}$ anomalies, thanks to the non universal $U_1 \times X$ assignments of the fermionic fields that yield non universal $Z'$ couplings to fermions. We explicitly demonstrate the power of this model for generating the realistic quark, lepton and neutrino mass spectra. In particular, we show that due to the presence of both continuous and discrete family symmetries in the considered framework, the top quark acquires a tree-level mass, lighter quarks and leptons get their masses at one- and two-loop order, while neutrino masses are generated at three-loop level. The minimal field content, particle spectra and scalar potential of this model are discussed in detail.

I. INTRODUCTION

The origin of various strong hierarchies in the fermion spectra of the Standard Model (SM) still remains a major unsolved problem of contemporary Particle Physics. A symmetry-based understanding of such hierarchies, in the framework of a single universal mechanism, consistent with the current phenomenological bounds on New Physics models, poses a challenging problem for the model-building community. In fact, a number of different potentially realizable mechanisms have been proposed so far, typically with certain limitations and deficiencies, and usually treating the quark, lepton and neutrino hierarchies on a separate footing. The most promising scenarios rely on the existence of horizontal (family) symmetries acting in the space of fermion generations and offering most times a more universal approach to the “fermion hierarchy problem” than other methods.

It seems natural and attractive to consider the viable prospect of a high-scale spontaneous discrete symmetry breaking, triggering the radiative generation of new mass operators in the corresponding low-energy effective field theory (EFT) [1–21]. This way, one arrives at the possible sequential generation (in general, due to a few sequential symmetry breakings at the high-energy scales) of the relevant mass (or SM Higgs Yukawa) terms in the SM, whose values are matched to zero or to a universal non-zero value in the high-scale limit of a more symmetric, and hence more fundamental theory. The search for such ultraviolet (UV) completions, possessing a low-energy SM-like EFT, and explorations of their vast potential for explaining the origin of the SM structure have only began recently [22, 23].

A model of radiatively generated fermion masses in the SM via sequential loop suppression has been proposed in Ref. [22]. In this model, the top quark mass is generated at tree level, while the bottom, tau and muon lepton masses arise at one-loop level. Meanwhile, the smaller up, down and strange quark and electron masses are generated at two-loop level, while the light active neutrinos acquire their masses at four-loop level. However, such a natural

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*Electronic address: antonio.carcamo@usm.cl
†Electronic address: sergey.kovalenko@usm.cl
‡Electronic address: Roman.Pasechnik@thep.lu.se
§Electronic address: ivan.schmidt@usm.cl
sequential generation of fermion masses at various orders of the Perturbation Theory comes with a price. Namely, this model, based on the SM gauge symmetry, supplemented with the $S_3 \times Z_2$ discrete group, has a quite low cut-off, since it incorporates non-renormalizable Yukawa terms. In addition, it has another drawback in that the $S_3 \times Z_2$ discrete group is softly broken, yielding an unknown UV completion of the theory. This situation may indicate the need for horizontal continuous symmetries for a consistent description of hierarchies in the matter sectors in a fully renormalizable way.

As a follow up to this study, in the later work of Ref. [23], we proposed such a renormalizable model based on the $SU_{3C} \times SU_{3L} \times U_{1X} \times U_{1L} \times Z_2 \times Z_4$ symmetry, which generates the SM fermion mass hierarchies with an emergent sequential loop suppression mechanism. However, that model still retains some drawbacks of the previous formulation. Namely, the $Z_2 \times Z_4$ symmetry is softly broken, and the scalar and fermion sectors are excessively large, making it very difficult to perform a reliable phenomenological analysis. In addition, that model does not explain the $R_K$ and $R_{K^*}$ anomalies, recently observed by the LHCb experiment [28–30], since it treats the first and second lepton families in the same footing. Furthermore, in the model of Ref. [23], the masses for the light active neutrinos appear at two-loop level, as well as the masses for the electron and for the light up, down and strange quarks. Thus the smallness of the light active neutrino masses with respect to the electron mass does not receive a natural explanation in the model. As a natural step forward, it would be instructive to find a new analogous formulation that enables us to generate the SM charged fermion mass hierarchies via a sequential loop suppression mechanism and to generate three-loop level light active neutrino masses without the inclusion of soft breaking mass terms. It would also be relevant to further explore the potential of such formulations for explaining the LHCb anomalies.

In the current work, we propose a first renormalizable model, an extended variant of the Inert Higgs Doublet model (IDM) [24], that enables to generate strong fermion hierarchies via another sequential loop suppression pattern, not yet discussed in the literature, without introducing any soft family symmetry breaking mass terms. Similarly to the previous formulations, in the current model the top quark and exotic fermions do acquire tree level masses, whereas the masses of the remaining SM fermions are radiatively generated. Namely, the masses for the bottom, strange and charm quarks, tau and muon leptons are generated at one-loop level, whereas the masses for the up and down quarks, as well as the electron mass, arise at two-loop level. In variance to the previous version in Ref. [22], the light active neutrinos acquire masses via radiative seesaw mechanisms at three-loop level, whereas in the former the light active neutrino masses were induced at two-loop level. Finally, the minimal field content of the model is not as complicated as in the previous formulations, enabling us to explore several key phenomenological implications of this model, which is the main subject of a follow up study.

The current article is organized as follows. In section II we discuss generic conditions for a sequential loop suppression mechanism, providing a motivation for the proposed model. In section III we set up the formalism for the extended IDM, containing the basic details about the symmetries, particle content, Yukawa interactions and scalar potential crucial for the implementation of the sequential loop suppression mechanism. The scalar mass spectrum is discussed in detail in section IV. In section V we discuss the implications of our model for the radiative generation of the quark mass and mixing hierarchies. The charged lepton and neutrino mass spectra and the corresponding mixing patterns are discussed in section VI. Finally, our conclusions are briefly stated in section VII.

II. SEQUENTIAL LOOP SUPPRESSION MECHANISM

Before describing our model in detail, let us first explain the reasoning behind introducing the additional scalar and fermion degrees of freedom and the symmetries that are required for a consistent implementation of the sequential loop suppression mechanism for generating the SM fermion hierarchies.

A. Quark sector

First of all, it is worth noticing that the top quark mass can be generated at tree level by means of a renormalizable Yukawa operator, with the corresponding coupling of order one, i.e.

$$\bar{q}_{3L} \phi_1 u_{jR}, \quad j = 1, 2, 3,$$

(1)

where $\phi_1$ is a $SU_{2L}$ scalar doublet. To generate the charm quark mass at one loop level, it is necessary to forbid the operator:

$$\bar{q}_{nL} \phi_1 u_{jR}, \quad n = 1, 2, \quad j = 1, 2, 3,$$

(2)
at tree level and to allow other operators instead, which are described in what follows. Obviously, in this case the charm quark mass is not generated in the same way as the top quark mass, i.e., from a renormalizable Yukawa term, since this would imply setting the corresponding tree-level Yukawa coupling unnaturally small.

In order to generate a small charm quark mass at one-loop level, we will use the following operators:

\[
\begin{align*}
\bar{q}_{nL}\bar{\phi}_2 T_R, & \quad T_L\sigma_3 u_{jR}, & n = 1, 2, & \quad j = 1, 2, 3, \\
T_L \sigma_1 T_R, & \quad \sigma_2^2 (\sigma_2^i)^2, & & \quad \left(\phi_1^4 \cdot \phi_2\right) \sigma_2. \\
\end{align*}
\]

For this to happen, we need to extend the SM gauge symmetry by adding the \(U_{1X} \times Z_4\) symmetry, where the \(U_{1X}\) is a spontaneously broken gauge symmetry, and \(Z_4\) is a preserved (exact) discrete symmetry (we will explain below why one should use \(Z_4\) instead of \(Z_2\)) under which the extra \(\phi_2\) scalar doublet, the \(\sigma_2\) electrically neutral scalar singlet, and the \(q_{jL}, u_{jR}\) \((j = 1, 2, 3)\) quark fields are nontrivially charged. Furthermore, the \(SU_{2L}\) singlet heavy quarks \(T_L, T_R\) with electric charges equal to \(2/3\) have to be added to the fermion spectrum in order to implement the one-loop radiative seesaw mechanism that gives rise to the charm quark mass. Besides, an electrically neutral weak-singlet scalar \(\sigma_1\) is needed in the scalar spectrum to provide a tree-level mass for the exotic \(T\) quark and to close the one-loop Feynman diagram that generates the charm quark mass. Let us note that \(\sigma_1\) is the only scalar neutral under the unbroken \(Z_4\) symmetry, and thus it conveniently acquires a vacuum expectation value (VEV) that breaks the \(U_{1X}\) gauge symmetry.

In addition, we assume that the \(q_{nL}\) \((n = 1, 2)\) fields have \(U_{1X}\) charges that are different from the charge of \(q_{3L}\), while the \(SU_{2L}\) scalar doublets \(\phi_1\) and \(\phi_2\) have different \(U_{1X}\) charges as well. Thus, the third row of the up-type quark mass matrix is generated at tree level, whereas the first and second row emerge at one-loop level. Note that, since there is only one heavy exotic \(T\) quark mediating the one-loop radiative seesaw mechanism that generates the first and second row of the up-type quark mass matrix, the determinant of this matrix is equal to zero. Therefore, the up quark is massless at one-loop level, and in order to generate an up quark mass at two-loop level, the following operators are required:

\[
\begin{align*}
\bar{q}_{nL}\bar{\phi}_2 T_R, & \quad \bar{T}_{1L}\rho_1 u_{jR}, & m_{\tilde{\tau}_1} \bar{T}_{1L} \bar{T}_{1R}, & n = 1, 2, & \quad j = 1, 2, 3, \\
T_L \sigma_1 T_R, & \quad \bar{T}_{LP_1} \bar{T}_{1R}, & (\rho_1^i)^2 \sigma_1 \sigma_2^i, & & \quad \left(\phi_1^4 \cdot \phi_2\right) \sigma_2, \\
\end{align*}
\]

where the extra \(SU_{2L}\) singlet heavy quarks \(\bar{T}_{1L}, \bar{T}_{1R}\) have electric charges equal to \(2/3\), and \(\rho_1\) is another electrically neutral weak-singlet scalar, charged under the unbroken \(Z_4\) symmetry.

Let us explain the reasons for choosing the exact \(Z_4\) discrete symmetry instead of the \(Z_2\) symmetry. Despite the fact that the \(Z_2\) symmetry is sufficient for implementing the one-loop radiative seesaw mechanism that generates the charm quark mass, this symmetry is not enough for generating the up quark mass at two-loop level. Since the \(\sigma_2\) and \(\rho_1\) scalar fields are charged under an unbroken discrete symmetry, the invariance of the \(T_L \rho_1 T_{1R}\) and \(\rho_1^2 \sigma_1 \sigma_2^i\) operators requires the use of the \(Z_4\) symmetry instead of \(Z_2\). The operators given above enable to generate the two-loop contributions to the first and second rows of the up-type quark mass matrix, and these contributions yield a nonvanishing determinant for the up-type quark mass matrix, giving rise to a suppressed two-loop up quark mass.

Turning now to a possible bottom quark mass generation at one-loop level, the following operators should be forbidden

\[
\bar{q}_{3L}\phi_1 d_{jR}, \quad j = 1, 2, 3,
\]

by means of, for example, the \(U_{1X}\) gauge symmetry. The fermion spectrum has to be extended by including the additional \(SU_{2L}\)-singlet heavy quarks \(B_{nL}, B_{nR}\) \((n = 1, 2)\) with electric charges equal to \(-1/3\), so that the mass for the bottom quark is generated with the help of the following operators:

\[
\begin{align*}
\bar{q}_{3L}\bar{\phi}_2 B_{nR}, & \quad \bar{B}_{nL}\sigma_2^i d_{jR}, & n = 1, 2, \\
B_{nL} \sigma_1 B_{nR}, & \quad \sigma_2^i (\sigma_1^i)^2, & & \quad \left(\phi_1^4 \cdot \phi_2\right) \sigma_2.
\end{align*}
\]

Note that we have added two \(SU_{2L}\)-singlet heavy quarks \(B_n\) \((n = 1, 2)\) instead of one, in order to fulfil the anomaly cancellation conditions discussed below.

In addition, in order to generate the first and second rows of the down-type quark mass matrix at one-loop level via a radiative seesaw mechanism, we need to incorporate, besides the \(SU_{2L}\)-singlet heavy quarks \(T_L, T_R\) previously introduced in the up-type quark sector, also an electrically charged weak-singlet scalar, namely, \(\varphi^+\). Thus, the first and second rows of the down-type quark mass matrix are generated by means of another set of operators given by

\[
\begin{align*}
\bar{q}_{nL}\bar{\phi}_2 T_R, & \quad \bar{T}_L \varphi^+_2 d_{jR}, & n = 1, 2, & \quad j = 1, 2, 3, \\
T_L \sigma_1 T_R, & \quad \varepsilon_{ab} (\phi_1)^a (\phi_2)^b \varphi^-_2, & a, b = 1, 2.
\end{align*}
\]
Furthermore, in order to avoid tree-level masses for the down and strange quarks one has to forbid the terms
\[ \bar{q}_{nL} \phi_j d_{jR}, \quad n = 1, 2, \quad j = 1, 2, 3. \] (8)

The latter can be excluded by assigning \( q_{nL} \) \((n = 1, 2)\) to be odd, while \( d_{jR} \) \((j = 1, 2, 3)\) to be even under the aforementioned exact \( Z_4 \) symmetry. Furthermore, since \( \phi_2 \) and \( \sigma_2 \) have the same \( Z_4 \) charge equal to \(-1\), and in order for the operators \( \overline{B}_{nL} \sigma_1^\ast B_{nR} \) to be invariant under the \( Z_4 \) symmetry, the \( SU_{2L} \)-singlet heavy quarks \( \overline{B}_{nL}, B_{nR} \) \((n = 1, 2)\) should also have \( Z_4 \) charge equal to \(-1\) (in non-additive notation). In fact, this condition implies that the \( q_{3L} \) and \( u_{3R} \) quarks should be neutral under \( Z_4 \).

Since there is only one fermionic seesaw mediator, i.e., the \( SU_{2L} \) singlet heavy quark \( T \) needed to generate the first and second rows of the down-type quark mass matrix at one-loop level, a nonvanishing one-loop strange quark mass emerges, whereas the down quark remains massless at this point. Consequently, similarly to the up-type quark sector, the two-loop contributions to the first and second rows of the down-type quark mass matrix need to be generated in order to give rise to a down-type quark mass at two-loop level. For that purpose, the following operators are required:
\[ \begin{align*}
\bar{q}_{nL} \varphi_2 T_R, & \quad \overline{T}_{2L} \phi_1^+ d_{jR}, & m_{T_2} \overline{T}_{2L} T_{2R}, \\
\overline{T}_L \sigma_1 T_R, & \quad \overline{T}_L \rho_2 T_{2R}, & \rho_2 \varphi_1^+ \varphi_2^+, & \quad \epsilon_{ab} (\phi_1)^a (\phi_2)^b \varphi_2^-, & a, b = 1, 2, \\
\end{align*} \] (9)

where an extra electrically charged weak-singlet scalar, \( \varphi_1^+ \), as well as an electrically neutral weak-singlet scalar, \( \rho_2 \), have been added to the scalar spectrum. Furthermore, the fermion spectrum has been extended by means of extra \( SU_{2L} \)-singlet heavy quarks \( \overline{T}_{2L}, T_{2R} \) with electric charges equal to \( 2/3 \). The two-loop contributions to the first and second rows of the down-type quark mass matrix provide its nonvanishing determinant, giving rise to a two-loop down quark mass.

### B. Charged lepton sector

Now, consider the sequential loop suppression mechanism capable of explaining the observed hierarchy between the SM charged lepton masses. In what follows, let us discuss a possible way of generating the one-loop tau and muon masses as well as a two-loop electron mass. First of all, the following operators have to be forbidden by using the \( U_{1X} \) gauge symmetry, so that the SM charged lepton mass matrix is generated at one-loop level by means of the terms
\[ \begin{align*}
\mathcal{L}_{iL} \phi_1 l_{jR}, & \quad i, j = 1, 2, 3. \\
\end{align*} \]

where weak-singlet charged leptons \( E_{iL}, E_{jR} \) \((j = 2, 3)\) have been included in the fermion spectrum. Let us note that these fields mediate the one-loop radiative seesaw mechanism that generates the \((1, 1), (2, 2), (3, 3), (1, 3)\) and \((3, 1)\) entries of the charged lepton mass matrix. Consequently, at this point the determinant of the charged lepton mass matrix is equal to zero and thus only the tau and muon leptons appear to be massive at one-loop level, whereas the electron remains massless. Two-loop corrections to the \((1, 1), (3, 3), (1, 3)\) and \((3, 1)\) entries are needed in order to induce a non-zero electron mass at two-loop level, and an extra weak-singlet charged lepton \( E_1 \) would be required for this purpose. To obtain such two-loop corrections, the following operators should be considered
\[ \begin{align*}
\mathcal{L}_{kL} \phi_2 E_{3R}, & \quad \mathcal{E}_{3L} \rho_3 l_{nR}, & \left( \phi_1 \cdot \phi_2^\dagger \right) \sigma_2^+, \quad \rho_2 \sigma_2 \sigma_1^+, & \quad k, n = 1, 3, \\
\mathcal{L}_{2L} \phi_2 E_{3R}, & \quad \mathcal{E}_{2L} \rho_3 l_{nR}, & \overline{E}_{iL} \sigma_1^+ E_{jR}, & \quad i = 2, 3, \\
\end{align*} \] (10)


C. Light active neutrino sector

Turning to the neutrino sector, in order to generate the SM light active neutrino masses at three-loop level as well as a realistic lepton mixing, a few operators have to be forbidden, namely,

$$\mathcal{L}_{i\ell l} \phi_{1j} \nu_{iR} \nu_{jR}^c, \quad (m_N)_{ij} \nu_{iR} \nu_{jR}^c, \quad \nu_{iR} \sigma_{1j} \nu_{jR}^c, \quad m_\Omega \Omega_R \Omega_R^c, \quad i, j = 1, 2, 3, \quad (12)$$

while the following operators are required

$$\mathcal{L}_{k\ell l} \tilde{\phi}_2 \nu_{nR}, \quad \Omega_{1R} \Omega_{2R} \eta \Psi_R, \quad \Omega_{1R} \eta \Psi_R, \quad \Omega_{1R} \Omega_{2R} \eta \Psi_R, \quad m_\Psi \Psi_R \Psi_R^c, \quad (13)$$

where $\nu_{iR} (i = 1, 2, 3)$, $\Omega_R$ and $\Psi_R$ are the SM-singlet right-handed Majorana neutrinos, and $\eta$, $\sigma_3$ and $\rho_3$ are the extra SM-singlet scalars. The latter fields have to be added in order to ensure three-loop level generation of the SM light active neutrino masses, as well as the lepton mixing parameters $\sin^2 \theta_{23}$ and $\sin^2 \theta_{12}$.

As was mentioned above, the charged lepton mass matrix has a mixing only in the (1,3)-plane such that the lepton mixing parameters $\sin^2 \theta_{23}$ and $\sin^2 \theta_{12}$ emerge from the neutrino sector. Let us note that the $U_{1X}$ gauge symmetry prevents the light active neutrino masses to be generated at tree level, whereas the $Z_2$ symmetry, together with the $U_{1X}$ gauge symmetry, forbid the mixing terms between the right-handed Majorana neutrinos $\nu_{kR} (k = 1, 3)$ and $\nu_{2R}$. This enables us to avoid the appearance of light neutrino masses at one- and two-loop levels. Note also that the $Z_2$ symmetry, as well as the $U_{1X}$ gauge symmetry, are crucial to forbid the terms $\nu_{2R} \sigma_{1j} \nu_{jR}^c$ and $\nu_{2R} \rho_{2j} \nu_{jR}^c$ $(k = 1, 3)$ that could result in the appearance of SM neutrino masses at one loop.

III. THE EXTENDED IDM MODEL

In this section, we will summarize the main features of the first renormalizable model, an extended variant of the Inert Higgs Doublet model (IDM), that includes a sequential loop suppression mechanism for the generation of the SM fermion mass hierarchies, without the inclusion of soft breaking mass terms and, at the same time, allowing for an explanation of the $R_K$ and $R_{K^*}$ anomalies, thanks to the non-universal $U_{1X}$ assignments of the fermionic fields that yield non-universal $Z'$ couplings to fermions. In a forthcoming work, we will show in detail how our model can fit the $R_K$ and $R_{K^*}$ anomalies. As previously stated in the Introduction section, we emphasize that our model, apart from having all the means for explaining the $R_K$ and $R_{K^*}$ anomalies not previously addressed in the model of Ref. [22], has a more natural explanation for the smallness of the light active neutrino masses than the one provided in Ref. [22], since in the former the masses for the light active neutrinos are generated at three-loop level, whereas in the latter they appear at two loops. Furthermore, unlike the model of Ref. [22], our current model does not include soft breaking mass terms.

Let us summarize the structure of a minimal model where the sequential loop suppression mechanism capable of radiative generation of the mass and mixing hierarchies in the SM fermion sectors is realized. The reasons for choosing a particular field content and symmetries have been outlined in the previous section, and will be further substantiated below.

A. Particle content and charges

With the necessary ingredients introduced above, in fact, we arrive at an extension of the inert 2HDM where the SM gauge symmetry is supplemented by the exact unbroken $Z_4$ discrete group and spontaneously broken $Z_2$ discrete and $U_{1X}$ gauge symmetry groups. The unbroken $Z_4$ and continuous local $U_{1X}$ (horizontal) family symmetries are crucial for the implementation of radiative seesaw mechanism of sequential loop suppression.

Besides the SM-like Higgs doublet $\phi_1$, the implementation of this mechanism requires an additional inert scalar $SU_{2L}$-doublet, $\phi_2$, two electrically neutral, $\sigma_k (k = 1, 2)$ and one electrically charged $\varphi_2^5$ $SU_{2L}$-singlet scalars. On top of that, the scalar sector of this model contains nine electrically neutral, i.e., $\sigma_j (j = 1, 2, 3)$, $\rho_j (k = 1, 2, 3, 4, 5)$, $\eta$, and two electrically charged $\varphi_k^\pm (k = 1, 2)$, $SU_{2L}$-singlet scalars. The scalar sector of the considering model has the
following $SU_{3c} \times SU_{2L} \times U_{1Y} \times U_{1X}$ charge assignments

$$
\phi_1 \sim (1, 1, \frac{1}{2}, 1), \quad \phi_2 \sim (1, 2, \frac{1}{2}, 2), \quad \sigma_1 \sim (1, 1, 0, -1), \quad \sigma_2 \sim (1, 1, 0, -1), \\
\sigma_3 \sim (1, 1, 0, -2), \quad \rho_1 \sim (1, 1, 0, 0), \quad \rho_2 \sim (1, 1, 0, 0), \quad \rho_3 \sim (1, 1, 0, 1), \\
\eta \sim (1, 1, 0, 1), \quad \varphi_1^+ \sim (1, 1, 1, 3), \quad \varphi_2^+ \sim (1, 1, 1, 3), \\
\rho_4 \sim (1, 1, 0, 0), \quad \rho_5 \sim (1, 1, 0, 0).
$$

(14)

The corresponding $Z_2 \times Z_4$ charges are given by

$$
\phi_1 \sim (1, 1), \quad \phi_2 \sim (1, -1), \quad \sigma_1 \sim (1, 1), \quad \sigma_2 \sim (1, -1), \quad \sigma_3 \sim (-1, -1), \\
\eta \sim (-1, -1), \quad \rho_1 \sim (1, i), \quad \rho_2 \sim (1, -1), \quad \rho_3 \sim (1, i), \quad \varphi_1^+ \sim (1, -1), \\
\varphi_2^+ \sim (1, 1), \quad \rho_4 \sim (-1, -1), \quad \rho_5 \sim (-1, 1),
$$

(15)

where $\omega = e^{\frac{2\pi i}{3}}$. It is worth noting that the model does not contain a weak-singlet scalar field with the following simultaneous three features: charged under the $Z_2$ discrete symmetry, neutral under the unbroken $Z_4$ symmetry with $U_{1X}$ charge equal to $\pm 1$.

Let us note that the SM-type Higgs doublet, i.e., $\phi_1$, as well as the electroweak-singlet scalars $\sigma_1$ and $\rho_5$ are the only scalar fields neutral under the exact $Z_4$ discrete symmetry. Since the $Z_4$ symmetry is preserved, the Higgs doublet $\phi_1$ and the singlets $\sigma_1$ and $\rho_5$ are the only scalar fields that acquire nonvanishing VEVs. The VEV in $\sigma_1$ is required to spontaneously break the $U_{1X}$ local symmetry, whereas the $\rho_5$ VEV spontaneously breaks the $Z_2$ discrete symmetry, due to its nontrivial $Z_2$ charge.

Note that the exact $Z_4$ discrete symmetry guarantees the presence of several stable scalar dark matter candidates in our model. These are represented by the neutral components of the inert $SU_{2L}$ scalar doublet $\phi_2$, as well as by the real and imaginary parts of the SM-singlet scalars $\sigma_2$, $\sigma_3$, $\eta$ and $\rho_j$ ($j = 1, 2, 3, 4$).

The set of $SU_{2L}$-singlet heavy quarks $T_L, T_R, B_{kL}, B_{kR}$ ($k = 1, 2$) represents the minimal amount of exotic quark degrees of freedom needed to implement the one-loop radiative seesaw mechanism that gives rise to the charm, bottom and strange quark masses. Furthermore, in order to ensure the radiative seesaw mechanism responsible for the generation of the up and down quark masses at two-loop level, the $SU_{2L}$ singlet heavy quarks $\tilde{T}_{kL}, \tilde{T}_{kR}$ ($k = 1, 2$), as well as the electrically neutral, $\rho_1, \rho_2$, and electrically charged, $\varphi_1^+$ scalar $SU_{2L}$-singlets should also be present in the particle spectrum.

To summarize, the SM fermion sector of the considered model includes a total of six electrically charged weak-singlet leptons $E_{1L}$ and $E_{1R}$ ($j = 1, 2, 3$), four right-handed neutrinos $\nu_{jR}$ ($j = 1, 2, 3$), $\Omega_R$, and ten $SU_{2L}$-singlet heavy quarks $T_L, T_R, \tilde{T}_{1L}, \tilde{T}_{1R}, B_{kL}, B_{kR}$ ($k = 1, 2$). It is assumed that the heavy exotic $T, \tilde{T}_k$ and $B_k$ quarks have electric charges equal to $2/3$ and $-1/3$, respectively.

More specifically, the quark sector of the extended IDM under consideration has the following $SU_{3c} \times SU_{2L} \times U_{1Y} \times U_{1X}$ charges

$$
q_{nL} \sim \begin{pmatrix} 3, 2, 1, 0 \end{pmatrix}, \quad q_{3L} \sim \begin{pmatrix} 3, 2, 1, 1 \end{pmatrix}, \quad n = 1, 2,
$$

$$
u_{jL} \sim \begin{pmatrix} 3, 1, 2 \end{pmatrix}, \quad d_{jR} \sim \begin{pmatrix} 3, 1, -1, 1 \end{pmatrix}, \quad j = 1, 2, 3,
$$

$$
T_L \sim \begin{pmatrix} 3, 1, 2, 1 \end{pmatrix}, \quad T_R \sim \begin{pmatrix} 3, 1, 2, 2 \end{pmatrix}, \quad \tilde{T}_{nL} \sim \begin{pmatrix} 3, 1, 2, 2 \end{pmatrix}, \quad \tilde{T}_{nR} \sim \begin{pmatrix} 3, 1, 2, 2 \end{pmatrix},
$$

$$
B_{nL} \sim \begin{pmatrix} 3, 1, -1, 0 \end{pmatrix}, \quad B_{nR} \sim \begin{pmatrix} 3, 1, -1, 1 \end{pmatrix}, \quad n = 1, 2,
$$

(16)

while their $Z_2 \times Z_4$ charge assignments read

$$
q_{nL} \sim (1, -1), \quad q_{3L} \sim (1, 1), \quad u_{nL} \sim (1, -1), \quad u_{3L} \sim (1, 1), \quad d_{jR} \sim (1, 1), \quad j = 1, 2, 3,
$$

$$
T_L \sim (1, 1), \quad T_R \sim (1, 1), \quad \tilde{T}_{nL} \sim (1, -1), \quad \tilde{T}_{1R} \sim (1, -1), \quad \tilde{T}_{2L} \sim (1, -1), \quad \tilde{T}_{2R} \sim (1, -1), \quad B_{nL} \sim (1, -1), \quad B_{nR} \sim (1, -1).
$$

(17)

The radiative seesaw mechanism that generates the charged lepton mass hierarchy is similar to the one that produces the SM down-type quark mass hierarchy. The generation of one-loop tau and muon masses as well as two-loop
electron mass is mediated by the electrically charged weak-singlet leptons $E_{jL}$ and $E_{jR}$ ($j = 1, 2, 3$), by the inert scalar $SU_{2L}$-doublet, $\phi_2$, and by the $SU_{2L}$-singlets $\sigma_2$, $\rho_j$ ($j = 1, 2, 3$).

Moreover, the three-loop radiative seesaw mechanism responsible for the generation of the light active neutrino masses is mediated by the right-handed neutrinos $\nu_{jR}$ ($j = 1, 2, 3$), $\Omega_R$, as well as by the inert scalar $SU_{2L}$ doublet $\phi_2$ and the $SU_{2L}$-singlet $\sigma_2$. To avoid tree-level mixing between the right-handed Majorana neutrinos $\nu_{kR}$ ($k = 1, 3$) and $\nu_{2R}$ triggered by Yukawa interactions with $\sigma_1$, we need to impose a nontrivial $Z_2$ charge of $\nu_{2R}$ while keeping $\nu_{kR}$ ($k = 1, 3$) $Z_2$-neutral.

In particular, the $SU_{3c} \times SU_{2L} \times U_{1Y} \times U_{1X}$ charges of the leptonic and neutrino fields of the model are defined as follows

$$
l_{1L} \sim (1, 1, -1, -3), \quad l_{2L} \sim (1, 1, -1, 0), \quad l_{kR} \sim (1, 1, -1, -3), \quad l_{2R} \sim (1, 1, -1, 0), \quad k = 1, 3, \nonumber$$

$$
E_{1L} \sim (1, 1, -1, -3), \quad E_{1R} \sim (1, 1, -1, -6), \quad E_{2L} \sim (1, 1, -1, -6), \quad E_{2R} \sim (1, 1, -1, -5), \quad k = 1, 3, \nonumber$$

$$
\nu_{1R} \sim (1, 1, 0, 2), \quad \nu_{2R} \sim (1, 1, 0, -1), \quad \Omega_{1R} \sim (1, 1, 0, 1), \quad \Omega_{2R} \sim (1, 1, 0, 1), \quad \Psi_R \sim (1, 1, 0, 0), \quad (18)$$

whereas the corresponding $Z_2 \times Z_4$ charges are given by

$$
l_{1L} \sim (1, -1), \quad l_{2L} \sim (-1, 1), \quad l_{kR} \sim (1, -1), \quad l_{2R} \sim (-1, 1), \quad j = 1, 2, 3, \quad k = 1, 3, \nonumber$$

$$
E_{1L} \sim (1, -i), \quad E_{1R} \sim (1, -i), \quad E_{2L} \sim (-1, 1), \quad E_{2R} \sim (-1, 1), \quad E_{3L} \sim (1, 1), \quad E_{3R} \sim (1, 1), \quad k = 1, 3, \nonumber$$

$$
\nu_{1R} \sim (1, 1), \quad \nu_{2R} \sim (-1, 1), \quad \Omega_{1R} \sim (-1, 1), \quad \Omega_{2R} \sim (-1, 1), \quad \Psi_R \sim (1, 1), \quad (19)$$

The $U_{1X} \times Z_2 \times Z_4$ symmetry and the particular assignments listed above are crucial for avoiding the appearance of SM light active neutrino masses at one- and two-loop levels.

Let us note that the left-handed quark $SU_{2L}$ doublets of the first and second generations are distinguished from the third generation by means of the $U_{1X}$ charge assignments. In addition, the local $U_{1X}$ family symmetry distinguishes the second generation left-handed lepton $SU_{2L}$ doublet from the first and third generation ones. Such nonuniversal $U_{1X}$ charge assignments in the fermion sector are crucial for implementing the sequential loop suppression mechanism and, hence, for the induced strong hierarchies in the SM fermion mass spectrum. Besides, as will be explicitly demonstrated in a forthcoming paper, such assignments are also relevant for explaining the $R_K$ and $K^*_L$ anomalies.

With the above assignments, we have numerically checked that the gauge anomaly cancellation conditions

$$
A_{[SU_{3c}]^2 U_{1X}} = \sum Q X_{QL} - \sum Q X_{QR}, \quad A_{[SU_{2L}]^2 U_{1X}} = \sum L X_{LL} + 3 \sum Q X_{QL}, \nonumber$$

$$
A_{[U_{1Y}]^2 U_{1X}} = \sum_{L,Q} (Y^2_{LL} X_{LL} + 3 Y^2_{QL} X_{QL}) - \sum_{L,Q} (Y^2_{LR} X_{LR} + 3 Y^2_{QR} X_{QR}), \nonumber$$

$$
A_{[U_{1X}]^2 U_{1Y}} = \sum_{L,Q} (Y_{LL} X^2_{LL} + 3 Y_{QL} X^2_{QL}) - \sum_{L,Q} (Y_{LR} X^2_{LR} + 3 Y_{QR} X^2_{QR}), \nonumber$$

$$
A_{[U_{1X}]^3} = \sum_{L,Q} (X^3_{LL} + 3 X^3_{QL}) - \sum_{L,Q} (X^3_{LR} + X^3_{NR} + 3 X^3_{QR}), \nonumber$$

$$
A_{[Gravity]^2 U_{1X}} = \sum_{L,Q} (X_{LL} + 3 X_{QL}) - \sum_{L,Q,N} (X_{LR} + X_{NR} + 3 X_{QR}) \quad (20)$$

are satisfied in our model.

**B. Yukawa interactions**

With the above specified particle content and charge assignments, the most general renormalizable Lagrangian of Yukawa interactions and the exotic fermion mass terms, invariant under the $SU_{3c} \times SU_{2L} \times U_{1Y} \times U_{1X} \times Z_2 \times Z_4$
symmetry, takes the following form

\[
\mathcal{L}_F = y_{3j}^{(u)} \bar{q}_{3L} \phi_1 u_{3R} + \sum_{n=1}^{2} x_n^{(u)} \phi_2 \bar{T}_R + \sum_{n=1}^{2} w_j^{(u)} \bar{T}_L \sigma_2 u_{nR} + \sum_{n=1}^{2} z_n^{(u)} \bar{T}_1 L \rho_1 u_{nR} + y_l \bar{T}_L \sigma_1 T_R
\]

\[
+ \sum_{n=1}^{2} m_{T_n} \bar{T}_n L \bar{T}_n R + \sum_{n=1}^{2} x_n^{(T)} \bar{T}_L \rho_n \bar{T}_n R + \sum_{n=1}^{2} x_n^{(d)} \bar{q}_{3L} \phi_2 B_{nR} + \sum_{n=1}^{2} y_{nj}^{(d)} \bar{B}_{nL} \sigma_2 d_{jR}
\]

\[
+ \sum_{n=1}^{2} \frac{2}{n} y_{nm}^{(B)} \bar{B}_{nL} \sigma_1^* B_{mR} + \sum_{j=1}^{3} w_j^{(d)} \bar{T}_2 L \phi_1^* d_{jR} + \sum_{j=1}^{3} z_j^{(d)} \bar{T}_L \phi_2^* d_{jR}
\]

\[
+ \sum_{k=1,3} x^{(l)}_{k} \bar{k}_L \phi_2 \bar{E}_3 R + \sum_{k=1,3} y^{(l)}_{3k} \bar{E}_{3L} \rho_2 \bar{k}_R + x^{(l)}_{22} \bar{E}_{2L} \phi_2 \bar{E}_{2R} + y^{(l)}_{22} \bar{E}_{2L} \rho_2 \bar{d}_2 R
\]

\[
+ z_{31}^{(E)} \bar{E}_{3L} \rho_3 \bar{E}_{1R} + \sum_{k=1,3} y_{1k}^{(E)} \bar{E}_{1L} \rho_1 \bar{k}_R + \sum_{i=1}^{3} y_{i}^{(E)} \bar{E}_{iL} \sigma_1^* E_{iR} + x^{(v)}_{22} \bar{E}_{2L} \phi_2 \nu_{2R}
\]

\[
+ \sum_{k=1,3} \sum_{n=1,3} x^{(v)}_{kn} \bar{k}_L \phi_2 \nu_{nR} + \sum_{k=1,3} y_{k}^{(v)} \bar{E}_1 R \eta_1 \nu_{kR} + y^{(l)} \bar{\Omega}_{1R}^\sigma_3 \nu_{2R}
\]

\[
+ x_{1}^{(\Psi)} \bar{\Omega}_{1R}^\eta \Psi_R + x_{2}^{(\Psi)} \bar{\Omega}_{2R}^\eta \Psi_R + z_{1} \bar{\Omega}_{1R} \sigma_2^* \bar{\Omega}_{2R} + m_\Psi \overline{\Psi} R + h.c.,
\]

(21)

where the Yukawa couplings are $O(1)$ parameters. From the quark Yukawa terms it follows that the top quark mass emerges due to an interaction involving the SM-like Higgs doublet $\phi_1$ only. After spontaneous breaking of the electroweak symmetry, the observed hierarchies of SM fermion masses arise by means of a sequential loop suppression, according to the following pattern: tree-level top quark mass; one-loop bottom, strange, charm, tau and muon masses; two-loop masses for the up, down quarks as well as for the electron. Furthermore, the SM light active neutrinos get their masses by means of a three-loop radiative seesaw mechanism.
C. Scalar potential

The most general renormalizable scalar potential invariant under the gauge and discrete symmetries of our model is given by

\[
V = \sum_{i=1}^{2} \left( \mu_{pi}^2 |\phi_i|^2 + \lambda_{pi} |\phi_i|^4 \right) + \sum_{j=1}^{3} \left( \mu_{sj}^2 |\sigma_j|^2 + \lambda_{sj} |\sigma_j|^4 \right) + \sum_{j=1}^{5} \left( \mu_{rj}^2 |\rho_j|^2 + \lambda_{rj} |\rho_j|^4 \right) + \mu_c^2 |\eta|^2 + \lambda_c |\eta|^4 \\
+ \sum_{i=1}^{2} \left( \mu_{fi}^2 \phi_i^+ \phi_i + \lambda_{fi} (\phi_i^+ \phi_i)^2 \right) + \sum_{i=1}^{5} \left( \mu_{fi}^2 \phi_i^+ \phi_i + \lambda_{fi} (\phi_i^+ \phi_i)^2 \right) + \sum_{i=1}^{5} \left( \mu_{fi}^2 \phi_i^+ \phi_i + \lambda_{fi} (\phi_i^+ \phi_i)^2 \right) \\
+ \frac{\kappa_1}{2} \left[ \varepsilon_{ab} \varepsilon_{cd} (\phi_1)^a (\phi_2)^b (\phi_1)^c (\phi_2)^d \right] + h.c.] \\
+ \sum_{i=1}^{3} \sum_{j=1}^{3} \beta_{ij} |\phi_i|^2 |\phi_j|^2 + \sum_{i=1}^{5} \kappa_{pi} |\phi_i|^2 |\phi_i|^2 + \sum_{i=1}^{5} \kappa_{pi} |\phi_i|^2 |\phi_i|^2 + \sum_{i=1}^{5} \kappa_{pi} |\phi_i|^2 |\phi_i|^2 \\
+ \sum_{i=1}^{3} \sum_{j=1}^{3} \gamma_{ij} |\sigma_i|^2 |\sigma_j|^2 + \sum_{i=1}^{5} \alpha_{ci} |\eta|^2 |\sigma_i|^2 + \sum_{i=1}^{5} \beta_{ci} |\eta|^2 |\sigma_i|^2 + \sum_{i=1}^{5} \gamma_{ci} |\phi_i|^2 |\phi_i|^2 + \sum_{i=1}^{5} \beta_{ci} |\eta|^2 |\phi_i|^2 + \sum_{i=1}^{5} \gamma_{ci} |\phi_i|^2 |\phi_i|^2 \\
+ \sum_{i=1}^{2} \sum_{j=1}^{2} \lambda_{ij} (\phi_i^+ \phi_i)|\phi_j|^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \gamma_{ij} (\phi_i^+ \phi_i)|\sigma_j|^2 + \sum_{i=1}^{5} \sum_{j=1}^{5} \beta_{ij} (\phi_i^+ \phi_i)|\rho_j|^2 + \sum_{i=1}^{5} \gamma_{ij} (\phi_i^+ \phi_i)|\rho_j|^2 \\
+ A_1 \left[ (\phi_1^+ \phi_2) \sigma_2 + h.c. \right] + A_2 \left[ \varepsilon_{ab} (\phi_1)^a (\phi_2)^b \varphi_2 + h.c. \right] + A_3 \left( \rho_2 \varphi_1 \varphi_2 + h.c. \right) \\
+ A_4 \left( \rho_1 \rho_2 \sigma_1 + h.c. \right) + A_5 \left( \rho_1 \rho_2 \sigma_1^* + h.c. \right) + A_6 \left[ \sigma_3 \sigma_2 \rho_5 + h.c. \right] + A_7 \left[ \sigma_2 \sigma_1 \rho_5 + h.c. \right] \\
+ A_8 \left( \rho_2 \rho_4 \rho_5 + h.c. \right) + A_9 \left( \rho_4 \sigma_1 + h.c. \right) + A_{10} \left( \sigma_3 \sigma_1^* \eta + h.c. \right) \\
+ \zeta_1 \left[ \varepsilon_{ab} (\phi_1)^a (\phi_2)^b \varphi_1 \rho_2 + h.c. \right] + \zeta_2 \left[ \sigma_2^* (\sigma_2)^2 + h.c. \right] + \zeta_3 \left[ (\rho_1)^2 \sigma_1 \sigma_2^* + h.c. \right] \\
+ \zeta_4 \left[ (\rho_1)^2 \sigma_2 \sigma_1^* + h.c. \right] + \zeta_5 \left[ \varphi_1^* \varphi_2 \sigma_2 (\sigma_1)^* + h.c. \right] + \zeta_6 \left[ \sigma_1 \sigma_2 (\eta)^* \rho_5 + h.c. \right] \\
+ \zeta_7 \left[ \rho_1 \rho_2 \sigma_3 \sigma_1 + h.c. \right] + \zeta_9 \left[ \sigma_3 \rho_4 (\sigma_1)^2 + h.c. \right] + \zeta_{10} \left[ \sigma_1 \sigma_2 \rho_5 + h.c. \right] \\
+ \zeta_{11} \left[ \sigma_2 \rho_4 \rho_5 + h.c. \right] + \zeta_{12} \left[ \rho_2 \eta \rho_5 \sigma_1 + h.c. \right] + \zeta_{13} \left[ \rho_2 \left( \phi_1^+ \phi_2^+ \right) \sigma_1^* + h.c. \right] \\
+ \zeta_{14} \left[ \rho_2^2 \left( \phi_1^+ \phi_2^+ \right) \sigma_1^* + h.c. \right]. \tag{22}
\]

From the minimization conditions for this potential, we find the following simple relations

\[
\mu_{p1}^2 = \frac{1}{2} \left( -2v^2 \lambda_{p1} - \beta_{15} v_\rho^2 - \alpha_{11} v_\sigma^2 \right) , \\
\mu_{s1}^2 = \frac{1}{2} \left( -\gamma_{15} v_\rho^2 - 2v_\rho^2 \lambda_{s1} - \alpha_{11} v^2 \right) , \\
\mu_{r5}^2 = \frac{1}{2} \left[ -2 (2\kappa_{r5} + \lambda_{r5}) v_\rho^2 - \gamma_{15} v_\sigma^2 - \beta_{15} v^2 \right]. \tag{23}
\]

that will be used below in a discussion of the scalar mass spectrum of the model.

IV. SCALAR MASS SPECTRUM

Considering the scalar potential given above, we find that the squared mass matrices for the CP-even neutral scalar sector are have the following form

\[
M_{\text{CP even}} = \begin{pmatrix}
M_{\text{CP even}}^{(1)} & 0_{3 \times 8} \\
0_{8 \times 3} & M_{\text{CP even}}^{(2)}
\end{pmatrix}, \tag{24}
\]
where $M^{(1)}_{\text{CPEven}}$ and $M^{(2)}_{\text{CPEven}}$ are the squared mass matrices for the $Z_4$-neutral and $Z_4$-charged scalars, respectively. The matrix $M^{(1)}_{\text{CPEven}}$ in the basis $(\text{Re}(\phi_1^0), \text{Re}(\sigma_1), \rho_5)$ (remind, $\rho_5$ is a real SM-singlet scalar), takes the form:

$$M_{\text{CPEven}}^{(1)} = \begin{pmatrix}
\nu^2 \lambda_{p1} & \frac{1}{2} \nu \nu \alpha_{11} & \frac{1}{2} \nu \nu \beta_{15} \\
\frac{1}{2} \nu \nu \alpha_{11} & \nu^2 \lambda_{s1} & \frac{1}{2} \nu \nu \beta_{15} \\
\frac{1}{2} \nu \nu \beta_{15} & \frac{1}{2} \nu \nu \beta_{15} & v_\sigma^2 (2 \kappa_{5} + \lambda_{s5})
\end{pmatrix}.$$  

The second mass form $M_{\text{CPEven}}^{(2)}$ in the basis $(\text{Re}(\sigma_2), \text{Re}(\sigma_3), \text{Re}(\rho_2), \text{Re}(\rho_4), \text{Re}(\eta), \text{Re}(\phi_2^0), \text{Re}(\rho_1), \text{Re}(\rho_3))$ reads

$$M_{\text{CPEven}}^{(2)} = \begin{pmatrix}
M_{\text{CPEven}}^{(2a)} & 0_{6 \times 2} \\
0_{2 \times 6} & M_{\text{CPEven}}^{(2b)}
\end{pmatrix},$$

$$M_{\text{CPEven}}^{(2a)} = \begin{pmatrix}
\frac{1}{4} \left( \beta_{11} \nu^2 + 2 \mu_{r1}^2 + v_\sigma^2 \gamma_{11} \right) & \frac{A \mu_{r1}}{2 \nu^2} & 0 \\
\frac{A \mu_{r1}}{2 \nu^2} & \frac{1}{4} \left( \beta_{13} \nu^2 + 2 \mu_{r3}^2 + v_\sigma^2 \gamma_{13} \right) & 0 \\
0 & 0 & \frac{1}{2} (4 - \sqrt{5}) \nu \nu \sigma^2
\end{pmatrix},$$

$$M_{\text{CPEven}}^{(2b)} = \begin{pmatrix}
\frac{A \nu \sigma}{2 \nu^2} & 0 & 0 \\
0 & \frac{1}{2} (4 + \sqrt{5}) \nu \nu \sigma^2 & 0 \\
0 & 0 & \frac{1}{2} (4 - \sqrt{5}) \nu \nu \sigma^2
\end{pmatrix}.$$  

Since the 126 GeV SM-like Higgs boson is found in the squared mass matrix $M_{\text{CPEven}}^{(1)}$, and considering the fact that the scalar potential has a very large number of parameters, in this first study it is sufficient to diagonalize only $M_{\text{CPEven}}^{(1)}$ in the scalar sector. In addition, since this matrix cannot be diagonalized in analytically closed form, and for the sake of simplicity, here we focus on a particular scenario with $v_\sigma = v_\rho$. In this scenario, the matrix $M_{\text{CPEven}}^{(1)}$ can be diagonalized as follows

$$R_{\text{CPEven}}^{(1)} \approx \begin{pmatrix}
\frac{1}{2} (4 - \sqrt{5}) \nu \nu \sigma^2 & 0 & 0 \\
0 & \frac{1}{2} (4 + \sqrt{5}) \nu \nu \sigma^2 & 0 \\
0 & 0 & \frac{1}{2} (4 - \sqrt{5}) \nu \nu \sigma^2
\end{pmatrix},$$

$$R_{\text{CPEven}}^{(1)} \approx \begin{pmatrix}
-1 + \frac{13}{12} \nu^2 & -\frac{1}{12} \nu \nu \sigma & \frac{1}{12} \nu \nu \sigma \\
\frac{13}{12} \nu \nu \sigma & \nu \nu \sigma & \frac{1}{12} \nu \nu \sigma \\
\frac{1}{12} \nu \nu \sigma & \frac{1}{12} \nu \nu \sigma & \nu \nu \sigma
\end{pmatrix}, \quad x = \frac{\nu}{\nu \sigma}.$$  

Consequently, the physical scalar states contained in the matrix $M_{\text{CPEven}}^{(1)}$ are given by:

$$\begin{pmatrix} h \\ \chi_1 \\ \chi_2 \end{pmatrix} \approx \begin{pmatrix}
-1 + \frac{13}{12} \nu^2 & -\left(\frac{\nu}{\nu \sigma}\right) & \frac{1}{12} \nu \nu \sigma \\
\frac{1}{12} \nu \nu \sigma & \nu \nu \sigma & \frac{1}{12} \nu \nu \sigma \\
\frac{1}{12} \nu \nu \sigma & \frac{1}{12} \nu \nu \sigma & \nu \nu \sigma
\end{pmatrix} \begin{pmatrix}
\phi_{1R}^0 \\ \sigma_{1R} \\ \rho_5
\end{pmatrix},$$

where $h$ is the 126 GeV SM-like Higgs boson, whereas $\chi_1$ and $\chi_2$ are the physical heavy scalar fields, which acquire masses at the scale of $U_{1\chi}$ breaking. The squared masses of these fields are given by

$$m_h^2 \approx \frac{8}{12} \nu \nu \sigma^2, \quad m_{\chi_1}^2 \approx \frac{1}{2} (4 - \sqrt{5}) \nu \nu \sigma^2, \quad m_{\chi_2}^2 \approx \frac{1}{2} (4 + \sqrt{5}) \nu \nu \sigma^2.$$  

Furthermore, we find that the SM-like Higgs boson $h$ has the couplings that are very close to the SM expectation, with small deviations of the order of $\sim v^2/v_\sigma^2$.

Considering the CP-odd neutral scalar sector, we find that the squared mass matrices for the electrically neutral CP-odd scalars in the basis, are $(\text{Im}(\phi_1^0), \text{Im}(\sigma_1), \text{Im}(\sigma_2), \text{Im}(\sigma_3), \text{Im}(\rho_2), \text{Im}(\rho_4), \text{Im}(\eta), \text{Im}(\phi_2^0), \text{Im}(\rho_1), \text{Im}(\rho_3))$
are given by

\[ M^\text{CPodd} = \begin{pmatrix} M^{(1)}_{\text{CPodd}} & 0_{2 \times 8} \\ 0_{8 \times 2} & M^{(2)}_{\text{CPodd}} \end{pmatrix}, \quad M^{(1)}_{\text{CPodd}} = 0_{2 \times 2}, \quad M^{(2)}_{\text{CPodd}} = \begin{pmatrix} M^{(2a)}_{\text{CPodd}} & 0_{5 \times 2} \\ 0_{2 \times 5} & M^{(2b)}_{\text{CPodd}} \end{pmatrix}, \quad (30) \]

where \( M^{(1)}_{\text{CPodd}} \) and \( M^{(2)}_{\text{CPodd}} \) are the squared mass matrices for the CP-odd scalars, neutral and charged under \( Z_4 \), respectively. Note that the squared mass matrix \( M^{(1)}_{\text{CPodd}} \) (which is written in the basis \( \text{Im}(\phi^0_1) \), \( \text{Im}(\sigma_1) \)) is exactly zero, since \( \text{Im}(\phi^0_1) \) and \( \text{Im}(\sigma_1) \) are the Goldstone bosons associated with the longitudinal components of the \( Z \) and \( Z' \) gauge bosons, respectively.

Finally, the squared mass matrix for the charged scalar fields in the basis \( (\phi^+_1, \phi^+_2, \varphi^+_1, \varphi^+_2) \) reads

\[ M_{\text{C}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} \left( \kappa_1 v^2 + 2 \mu^2_1 + v_\sigma^2 \gamma_{11} \right) & 0 & 0 & 0 \\ \frac{v_{A_2}}{\sqrt{2}} & 0 & \frac{1}{2} \left( \lambda_{11} v^2 + 2 \mu^2_2 + v_\sigma^2 \gamma_{12} + v_\rho^2 \gamma_{15} + v_\sigma^2 \gamma_{11} \right) & 0 \\ \frac{v_{A_2}}{\sqrt{2}} & 0 & 0 & \frac{1}{2} \left( \lambda_{21} v^2 + 2 \mu^2_2 + v_\sigma^2 \gamma_{25} + v_\rho^2 \gamma_{21} \right) \end{pmatrix}, \quad (31) \]

such that \( \phi^\pm_i \) are the electrically charged massless scalar states corresponding to the Goldstone bosons associated with the longitudinal components of the \( W^\pm \) gauge bosons.

V. RADIATIVELY GENERATED QUARK MASS AND MIXING HIERARCHIES

The quark Yukawa interactions determined by Eq. (21) give rise to the following up and down mass matrices for the SM quarks, respectively,

\[ M_U = \begin{pmatrix} \varepsilon^{(u)}_{11} + \varepsilon^{(d)}_{11} & \varepsilon^{(u)}_{12} + \varepsilon^{(d)}_{12} & 0 \\ \varepsilon^{(u)}_{21} + \varepsilon^{(d)}_{21} & \varepsilon^{(u)}_{22} + \varepsilon^{(d)}_{22} & 0 \\ 0 & 0 & y_{\lambda(3)}^{(u)} \end{pmatrix} \frac{v}{\sqrt{2}}, \quad M_D = \begin{pmatrix} \varepsilon^{(d)}_{11} + \varepsilon^{(d)}_{11} & \varepsilon^{(d)}_{12} + \varepsilon^{(d)}_{12} & \varepsilon^{(d)}_{13} + \varepsilon^{(d)}_{13} \\ \varepsilon^{(d)}_{21} + \varepsilon^{(d)}_{21} & \varepsilon^{(d)}_{22} + \varepsilon^{(d)}_{22} & \varepsilon^{(d)}_{23} + \varepsilon^{(d)}_{23} \\ 0 & 0 & y_{\rho(3)}^{(d)} \end{pmatrix} \frac{v}{\sqrt{2}}, \quad (32) \]

where the dimensionless parameters \( \varepsilon_{nm}^{(u)}, \varepsilon_{ij}^{(d)} \) \((n, m = 1, 2)\) and \( \varepsilon_{ij}^{(d)} \) \((i, j = 1, 2, 3)\) are generated at one-loop level, whereas \( \varepsilon_{nm}^{(d)} \) and \( \varepsilon_{ij}^{(d)} \) arise at two-loop level. The characteristic Feynman loop diagrams contributing to the entries of the SM quark mass matrices are shown in Fig. [4].

In what follows, we demonstrate that the mass matrices for SM quarks given above incorporate the observed hierarchies in the SM quark mass spectrum and the Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix. To this end, we proceed with a parametrization of the SM quark mass matrices in the following form

\[ M_U = \begin{pmatrix} a^{(u)}_{11} & 0 & 0 \\ a^{(u)}_{12} & l & 0 \\ a^{(u)}_{21} & l & 0 \\ 0 & 0 & y_{\lambda(3)}^{(u)} \end{pmatrix} \frac{v}{\sqrt{2}}, \quad M_D = \begin{pmatrix} a^{(d)}_{11} & 0 & 0 \\ a^{(d)}_{12} & l & 0 \\ a^{(d)}_{21} & l & 0 \\ a^{(d)}_{31} & l & 0 \\ 0 & 0 & y_{\rho(3)}^{(d)} \end{pmatrix} \frac{v}{\sqrt{2}}, \quad (33) \]

where \( l \approx (1/4\pi)^2 \approx 2.0 \times \lambda^4 \) is the loop suppression factor, and \( \lambda = 0.225 \) is the Wolfenstein parameter. As a consequence, we expect that \( a_{nm}, a_{ij}^{(d)} \) \((n, m = 1, 2)\) and \( i, j = 1, 2, 3 \) be \( \mathcal{O}(1) \) parameters.
Figure 1: One- and two-loop Feynman diagrams contributing to the entries of the SM quark mass matrices. Here, $n, m = 1, 2$ and $j = 1, 2, 3$. 
We remark that the Feynman diagrams contributing to the entries of the SM fermion mass matrices contain a large number of uncorrelated parameters that belong to the fermion and scalar sectors of our model. Nevertheless, these parameters can be absorbed into a limited number of effective parameters $\varepsilon_{nm}$, $\varepsilon_{ij}$, $\xi_{ij}$, and $\zeta_{ij}$ ($n, m = 1, 2$ and $i, j = 1, 2, 3$), which can be used to reproduce the experimental values of the physical observables in the quark sector.$^{[33]}$

\[
\begin{align*}
 m_u(MeV) &= 1.45^{+0.56}_{-0.45}, & m_d(MeV) &= 2.9^{+0.5}_{-0.4}, & m_s(MeV) &= 57.7^{+16.8}_{-15.7}, \\
m_t(MeV) &= 635 \pm 86, & m_t(GeV) &= 172.1 \pm 0.6 \pm 0.9, & m_b(GeV) &= 2.82^{+0.09}_{-0.04}, \\
\sin \theta_{12} &= 0.2254, & \sin \theta_{23} &= 0.0414, & \sin \theta_{13} &= 0.00355, & J &= 2.96^{+0.20}_{-0.16} \times 10^{-5}.
\end{align*}
\]

Here, $m_{t,u,c,d,s,b}$ are the SM quark masses, $\theta_{12}, \theta_{23}, \theta_{13}$ are the mixing angles, and $J$ is the Jarlskog parameter.

While our model does not predict the exact values of the physical observables, it offers a natural explanation of the observed (strong) hierarchies. As was previously mentioned, it only pretends to reproduce the existing pattern of quark masses and mixing caused by a sequential loop suppression predicted by the model. To this end, for the SM quark mass matrices given above, we look for the eigenvalue problem solutions reproducing the experimental values of the quark masses and the CKM parameters given by Eq. (34), requiring that $a^{(u,d)}, b^{(u,d)}$ are all of the same order of one. The standard procedure renders the following solution

\[
\begin{align*}
 a^{(u)}_{11} &\simeq 0.708, & a^{(u)}_{12} = a^{(u)}_{21} \simeq 0.567, & a^{(u)}_{22} \simeq 0.456, & y^{(u)}_{33} = 0.989, \\
a^{(d)}_{11} &\simeq 0.191, & a^{(d)}_{12} = a^{(d)}_{21} \simeq 0.182, & a^{(d)}_{13} \simeq 0.325 + 0.009i, \\
a^{(d)}_{23} &= a^{(d)}_{32} \simeq 0.269 - 0.016i, & a^{(d)}_{22} \simeq 0.190, & a^{(d)}_{33} \simeq 1.771.
\end{align*}
\]

The above $O(1)$ values exactly reproduce the measured central values of the SM quark masses and CKM parameters given in Eq. (34). Hence, our model is consistent with and successfully reproduces the existing pattern of SM quark masses caused by the sequential loop suppression mechanism, with different quark flavors getting mass at different orders in Perturbation Theory as discussed above.

VI. RADIATIVELY GENERATED LEPTON MASSES AND MIXINGS

The lepton and neutrino Yukawa interactions and mass terms given in Eq. (21) give rise to the characteristic Feynman loop diagrams illustrated in Figs. 2 and 3 that necessarily generate the following SM charged lepton and light active neutrino mass forms:

\[
M_l = \begin{pmatrix} e^{(l)}_{11} + \zeta^{(l)}_{11} & 0 & e^{(l)}_{12} + \zeta^{(l)}_{12} \\
0 & e^{(l)}_{22} + \zeta^{(l)}_{22} & 0 \\
0 & 0 & e^{(l)}_{33} + \zeta^{(l)}_{33} \end{pmatrix} \frac{v}{\sqrt{2}}, \quad M_\nu = \begin{pmatrix} a^{(l)}_{11} & a^{(l)}_{12} & a^{(l)}_{13} \\
a^{(l)}_{21} & a^{(l)}_{22} & a^{(l)}_{23} \\
a^{(l)}_{31} & a^{(l)}_{32} & a^{(l)}_{33} \end{pmatrix}
\]

where $e^{(l)}_{ii}, e^{(l)}_{ij}, \zeta^{(l)}_{ij}$ are the dimensionless parameters generated at one-loop level, whereas the parameters $\zeta^{(d)}_{ii}, \zeta^{(d)}_{ij}$ and $\zeta^{(d)}_{ij}$ appear at two-loop level. In what follows, we show that the lepton mass matrices given above can accommodate the experimental data on the SM lepton masses and mixing. For this purpose, following the same strategy as for the quark mass forms discussed in the previous section, we parametrize the SM charged lepton mass matrix as follows:

\[
M_l = \begin{pmatrix} a^{(l)}_{11} & 0 & a^{(l)}_{13} \\
0 & a^{(l)}_{22} & 0 \\
a^{(l)}_{31} & 0 & a^{(l)}_{33} \end{pmatrix} \frac{v}{\sqrt{2}}.
\]

In order to fit the measured values of the charged lepton masses, as well as the neutrino mass squared differences and lepton mixing parameters $^{[32]}$, we proceed by solving the eigenvalue problem for the SM lepton and light neutrino
The neutrino mass matrix satisfying $O(1)$ values for the $a_{nm}^{(l)}$ ($n, m = 1, 3$), $a_{22}^{(l)}$ parameters and the above specified entries of the neutrino mass matrix satisfying $O(10^{-3}) \lesssim O(10^{-2})$ ($i, j = 1, 2, 3$), the experimental values for the physical observables of the lepton sector, i.e., the three charged lepton masses, the two neutrino mass squared splittings, the three leptonic mixing parameters and the leptonic Dirac CP violating phase can be successfully reproduced.
Consequently, our model is consistent with and successfully reproduces the existing pattern of SM charged lepton masses generated by the sequential loop suppression mechanism.

VII. DISCUSSIONS AND CONCLUSIONS

We have constructed a first renormalizable extension of the Inert Doublet Model that enables an implementation of a sequential loop suppression mechanism, capable of explaining the observed SM fermion mass hierarchy without invoking soft breaking mass terms. In our model, the SM gauge symmetry is supplemented by the $U_{1X} \times Z_2 \times Z_4$ family symmetry, where the gauge $U_{1X}$ and discrete $Z_2$ symmetries are spontaneously broken, whereas the $Z_4$ symmetry is preserved. Our model is consistent with the observed SM fermion mass spectrum and fermionic mixing parameters and allows for an explanation of the recently observed $R_K$ and $R_K^*$ anomalies, thanks to the non-universal $Z'$ couplings to fermions.
We focused on an extended of the Inert Higgs Doublet model (IDM) that allows the implementation of the sequential loop suppression mechanism for the generation of SM fermion masses instead of an extension of the inert 3-3-1 model (model based on the $SU_{AC} \times SU_{3L} \times U_{1X}$ gauge symmetry). As previously mentioned, the extension of the inert 3-3-1 model of Ref. [24] does not explain the $R_K$ and $R_{K^*}$ anomalies and the light active neutrino masses appear at two-loop level like the masses of the light SM charged fermions. Addressing the $R_K$ and $R_{K^*}$ anomalies in the framework of a 3-3-1 model would require to consider five families of $SU(3)_L$ leptonic triplets as done in Ref. [31], in order to have different $U(1)_{X}$ charge assignments for the first and second lepton families, without spoiling the anomaly cancellation conditions. Thus, modifying the inert 3-3-1 model of Ref [24] to account for the $R_K$ and $R_{K^*}$ anomalies, and to generate the hierarchy of SM fermion masses by sequential loop suppression mechanism, with the light active neutrino masses appearing at three-loop level, will require a much larger particle content than the one adopted in the framework of an extended IDM.

In our model only the top quark and exotic fermions acquire tree-level masses, whereas the masses of the remaining SM fermions emerge from a radiative seesaw-like mechanism: the masses for the bottom, strange and charm quarks, tau and muon leptons are generated at one-loop level, whereas the masses for the up and down quarks as well as the electron mass appear at two-loop level. Furthermore, light active neutrino acquire masses by means of a radiative seesaw mechanism at three-loop level.

Due to an unbroken $Z_4$ discrete symmetry, our model has several stable scalar dark matter candidates, which can be the neutral components of the inert $SU_{2L}$ scalar doublet $\phi_2$ as well as the real and imaginary parts of the SM scalar singlets $\sigma_2, \sigma_3, \eta$ and $\rho_j$ ($j = 1, 2, 3, 4$). Furthermore, the model can have a fermionic dark matter candidate which is the only SM-singlet Majorana neutrino $\Omega_{IR}$ with a non-trivial $Z_4$ charge. A study of the phenomenological implications of our model goes beyond the scope of the present paper and will be performed in a forthcoming work.

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