Spin Calogero models and dynamical $r$-matrices$^1$

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Abstract

The main point of the construction of spin Calogero type classical integrable systems based on dynamical $r$-matrices, developed by L.-C. Li and P. Xu, is reviewed. It is shown that non-Abelian dynamical $r$-matrices with variables in a reductive Lie algebra $\mathcal{F}$ and their Abelian counterparts with variables in a Cartan subalgebra of $\mathcal{F}$ lead essentially to the same models.

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$^1$Based on talk by L.F. at Symposium QTS-4, Varna (Bulgaria), August 2005; to appear in the proceedings.
1 Introduction

Integrable systems of Calogero (Sutherland, Moser, Olshanetsky-Perelomov, Gibbons-Hermsen, Ruijsenaars-Schneider . . . ) type are related to many important areas of physics and mathematics. The integrability of dynamical systems is in general due to the existence of conserved quantities that reflect some (hidden) symmetries. These are usually exhibited by constructing a Lax representation for the equation of motion, and often also by deriving the system of interest as a projection of a ‘free’ system which is integrable obviously. (See for a review.) For Hamiltonian systems, Liouville integrability is linked with the St Petersburg form of the Poisson brackets (PBs) of the Lax matrix, $L$, according to the formula

$$\{L_1, L_2\} = [\rho, L_1] - [\rho_{21}, L_2], \tag{1.1}$$

where $\rho$ is a $G \otimes G$ valued function on the phase space in general if $L$ is $G$-valued. Here, $G$ can be any Lie algebra and for simplicity we restrict ourselves to spectral parameter independent cases. (Note that $\rho_{21} = Y^a \otimes X_a$ if $\rho = X_a \otimes Y^a \in G \otimes G$, and $L_1 = L \otimes 1$, $L_2 = 1 \otimes L$.) Equation (1.1) guarantees that the $G$-invariant functions (eigenvalues) of $L$ Poisson commute. In the simplest cases, like for Toda systems, $\rho$ is a constant. The $r$-matrices entering (1.1) for the $A_n$ type Calogero models were found to be coordinate dependent. They were re-derived in an inspiring way by Avan, Babelon and Billey who also related them to the classical dynamical Yang-Baxter equation (CDYBE) that arose from conformal field theory. Relying on the advance in the theory of the CDYBE thanks to Etingof and Varchenko and motivated also by the calculations in Li and Xu proposed a method to associate a spin Calogero model to any dynamical $r$-matrix as defined in Li and Xu. This method was further developed by Li in a rather abstract framework using Lie algebroids and groupoids.

In this report we wish to contribute to the ‘dynamical chapter’ of the Yang-Baxter story on integrability by presenting certain clarifications and applications of the method invented by Li and Xu. We focus on the spectral parameter independent version of the method introduced in Li and Xu. We explain its essential point in a direct manner, without any reference to Lie algebroids that feature in Li and Xu. In principle, this method can be applied to any dynamical $r$-matrix defined on the dual space of any Abelian or non-Abelian subalgebra of a Lie algebra. However, we shall demonstrate that the non-Abelian dynamical $r$-matrices with variables belonging to a reductive Lie algebra, say $F$, and their Abelian counterparts (Dirac reductions in the sense of) with variables belonging to a Cartan subalgebra of $F$ lead essentially to the same models. This ‘no go’ result was mentioned in our recent paper, where we studied spin Calogero type models built on Abelian dynamical $r$-matrices. (It provided the reason for considering there only Abelian $r$-matrices.) We proved that the models based on $r$-matrices with a certain non-degeneracy property are projections of the natural geodesic system on a corresponding Lie group. We shall briefly characterize these models in Section 4 at the end of the present report, referring to for details.

The most important new result of this paper is Proposition 2 in Section 3. The content of Section 2 is not new, but it may be useful for readers who want to learn about the essence of the method due to Li and Xu keeping the technicalities to a minimum.
2 From dynamical \( r \)-matrices to integrable system

Consider a subalgebra \( K \) of a Lie algebra \( G \) and corresponding connected Lie groups \( K \) and \( G \). Let \( \check{K}^* \subset K^* \) be an open subset invariant under the coadjoint action of \( K \). A dynamical \( r \)-matrix for \( K \subset G \) is by definition \([16]\) a \( K \)-equivariant (smooth or holomorphic) map \( r : \check{K}^* \rightarrow G \otimes G \) satisfying the CDYBE

\[
[r_{12}, r_{13}] + T_i^1 \frac{\partial r_{23}}{\partial q^i} + \text{cycl. perm.} = 0,
\]

and the additional condition that the symmetric part of \( r \),

\[
r^s = \frac{1}{2}(r + r_{21}),
\]

is a \( G \)-invariant constant. As usual \( r_{23} = 1 \otimes r \), \( T_i^j = T^i \otimes 1 \otimes 1 \) etc, and \( q^i \equiv \langle q, T_i \rangle \) are the components of \( q \in K^* \) with respect to a basis \( T_i \) of \( K \). Infinitesimally, the \( K \)-equivariance property of \( r \) reads

\[
[T^i \otimes 1 + 1 \otimes T^i, r(q)] = f^{ij}_{k} q^k \frac{\partial r(q)}{\partial q^j} \quad \text{with} \quad [T^i, T^j] = f^{ij}_k T^k.
\]

Important special cases are the quasi-triangular \( r \)-matrices with symmetric part \( r^s = \frac{1}{2} T_\alpha \otimes T^\alpha \), where \( G \) is self-dual with invariant scalar product \( B_G, B_G(T_\alpha, T^\beta) = \delta^\alpha_\beta \) for dual bases of \( G \), and the triangular \( r \)-matrices characterized by \( r^s = 0 \). (The spectral parameter can be introduced in (2.1) in the standard fashion.)

To construct integrable systems from dynamical \( r \)-matrices, one starts with the phase space

\[
\mathcal{M} := T^* \check{K}^* \times G^* \simeq \check{K}^* \times K \times G^* = \{(q, p, \xi)\},
\]

and defines the ‘quasi-Lax operator’

\[
L : \mathcal{M} \rightarrow G, \quad L(q, p, \xi) = p - R(q)\xi,
\]

where \( R(q) \in \text{End}(G^*, G) \) corresponds to \( r(q) \in G \otimes G \) so that \( X \otimes Y : \zeta \mapsto \langle \zeta, Y \rangle X \) for any \( X, Y \in G, \zeta \in G^* \). Note that the map \( L \) is equivariant with respect to the natural actions of the group \( K \subset G \) on \( \mathcal{M} \) and on \( G \). Introduce also the function

\[
\chi : \mathcal{M} \rightarrow K^*, \quad \chi(q, p, \xi) := (\text{ad}^K_p)^*(q) + \xi_{K^*},
\]

where \( \xi_{K^*} \in K^* \) is the restriction of \( \xi \in G^* \) to \( K \subset G \) and \( \langle (\text{ad}^K_p)^*(q), X \rangle = \langle q, [p, X] \rangle \forall X \in K \). (We denote the pairing between any vector space and its dual by \( \langle \ , \ \rangle \).) By setting

\[
(\nabla_\chi r)(q, p, \xi) := \frac{d}{dt} r(q + t\chi(q, p, \xi))|_{t=0},
\]

the fundamental result can be formulated as follows.
Proposition 1. The quasi-Lax operator $L$ associated with any dynamical $r$-matrix as defined above satisfies
\begin{equation}
\{L_1, L_2\} = [r, L_1 + L_2] - \nabla_\chi r. \tag{2.8}
\end{equation}

Proof. Let $R^a \in \text{End}(\mathcal{G}^*, \mathcal{G})$ (resp. $R^s$) correspond to the antisymmetric (resp. symmetric) part of $r$. Upon contraction with $1 \otimes X \otimes Y$, let us rewrite the CDYBE (2.1) in the equivalent form
\begin{equation}
E(R^a, X, Y) = -[R^s X, R^s Y], \quad \forall X, Y \in \mathcal{G}^*, \tag{2.9}
\end{equation}
with the $\mathcal{G}$-valued function $E(R^a, X, Y)$ on $\hat{K}^*$ given by
\begin{equation}
E(R^a, X, Y) = [R^a X, R^a Y] - R^a (\text{ad}^2 R^a Y - \text{ad} R^a Y X) + \nabla_{Y_{\mathcal{K}^*}} R^a X - \nabla_{X_{\mathcal{K}^*}} R^a Y + \langle X, (\nabla R^a) Y \rangle. \tag{2.10}
\end{equation}
Here $\text{ad}^2$ is the coadjoint representation of $\mathcal{G}$, $\text{ad}^2 Y = -(\text{ad}_T)^* (\forall T \in \mathcal{G})$, and $\langle X, (\nabla R^a) Y \rangle = T^i \frac{\partial (X^i R^a Y)}{\partial q^i}$. To see that (2.1) and (2.9) are equivalent, one must also use that $r^s$ is a $\mathcal{G}$-invariant constant. Now, for any $K$-equivariant $r^a$ for which $r^s$ is a $\mathcal{G}$-invariant constant, one obtains from (2.3) by an easy calculation
\begin{equation}
\{L_1, L_2\} - ([r, L_1 + L_2] - \nabla_\chi r) = (E(R^a, T_\alpha, T_\beta) + [R^s T_\alpha, R^s T_\beta]) T^\alpha \otimes T^\beta \tag{2.11}
\end{equation}
with dual bases $T^\alpha \in \mathcal{G}$ and $T_\alpha \in \mathcal{G}^*$. Q.E.D.

It follows from Proposition 1 that the $G$-invariant functions of $L$ yield a Poisson commuting family after introducing the constraint $\chi = 0$. This is the basic idea for constructing integrable systems out of dynamical $r$-matrices. In coordinates, the PBs on $\mathcal{M} = T^* \hat{K}^* \times \mathcal{G}^*$ are
\begin{equation}
\{q^i, p_j\} = \delta^i_j \quad \text{and} \quad \{\xi^\alpha, \xi^\beta\} = f^{\alpha\beta}_\gamma \xi^\gamma, \tag{2.12}
\end{equation}
where $f^{\alpha\beta}_\gamma$ denote the structure constants of $\mathcal{G}$ (in a basis $T^\alpha$ extending the basis $T^i$ of $\mathcal{K}$, defining $\xi^\alpha = \langle \xi, T^\alpha \rangle$). The constraints $\chi^i = 0$ are first class, since
\begin{equation}
\{\chi^i, \chi^j\} = f^{ij}_k \chi^k. \tag{2.13}
\end{equation}
In fact, $\chi$ is nothing but the momentum map generating the natural action of the group $K$ on $\mathcal{M}$. (The action of $K$ is induced by its coadjoint action on $\hat{K}^*$ and by composing the coadjoint action of $G$ on $\mathcal{G}^*$ with the inclusion $K \subset G$.) We perform Hamiltonian reduction by setting $\chi = 0$. Thus we are interested only in the gauge invariant ($K$-invariant) functions on $\mathcal{M}^{\chi=0}$, i.e., in the reduced phase space $\mathcal{M}^{\chi=0}/K$. In particular, any $G$-invariant function $h$ on $\mathcal{G}$ yields a $K$-invariant function on $\mathcal{M}$ by $h \circ L$ as $L$ (2.5) is a $K$-equivariant map.

For later purpose, note that one may also perform the Hamiltonian reduction after restriction to a symplectic leaf of $\mathcal{M}$, which has the form
\begin{equation}
T^* \hat{K}^* \times \mathcal{O}, \tag{2.14}
\end{equation}
where $\mathcal{O} \subset \mathcal{G}^*$ is a coadjoint orbit. The reduction of the subspace (2.14) of $\mathcal{M}$ leads to a union of symplectic leaves in the full reduced phase space resulting from $\mathcal{M}$. 

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The constraint $\chi = 0$ is universally applicable to remove the derivative term of \text{(2.8)}, but in some examples non-zero constants $\chi_0 \in \mathcal{K}^*$ exist, too, for which $(\nabla_{\chi_0} r)(q) = 0$ for all $q \in \mathcal{K}^*$. Then the constraint $\chi = \chi_0$ can also be used to obtain integrable systems. This occurs in particular for the standard dynamical $r$-matrices on the Cartan subalgebra $\mathcal{K}$ of $\mathcal{G} = u(n)$, for which $\chi_0$ can be taken as a multiple of the unit matrix, after the usual identification $\mathcal{K}^* \simeq \mathcal{K}$.

In our discussion we focus on the constraint $\chi = 0$ for definiteness. The (spectral parameter dependent variant of the) basic formula \text{(2.8)} first appeared in \cite{12} for concrete examples of quasi-Lax operators that were defined without referring to \text{(2.5)}. The statement of Proposition 1 can be found in \cite{19}, and its spectral parameter dependent version for an Abelian $\mathcal{K}$ can be found in \cite{17}. Given \text{(2.8)}, the idea to construct integrable systems by killing the derivative term arises immediately and it occurs in all the references mentioned. We thought it worthwhile to report the above elementary proof of Proposition 1, because the results are presented in \cite{17, 19} in such an abstract framework that may make it difficult to realize how simple the main point is. Incidentally, our proof clearly shows also that, in the presence of the equivariance and invariance properties of $r^a$ and $r^s$, the PB relation \text{(2.8)} for the quasi-Lax operator \text{(2.5)} does not only follow from the CDYBE, but is equivalent to it. Formulae \text{(2.5), (2.8)} and the direct verification as above work essentially in the same way for spectral parameter dependent $r$-matrices as well.

3 Abelian versus non-Abelian dynamical $r$-matrices

In the first examples \cite{14, 15} the space of variables in the CDYBE was a Cartan subalgebra of a simple Lie algebra. Later the concept was extended \cite{16} to include $r$-matrices defined on the duals of non-Abelian Lie algebras. Such ‘non-Abelian $r$-matrices’ came to light naturally in some applications (see \cite{21, 25, 26}), and Proposition 1 is valid in this general case. At first sight, it appears a natural project to construct integrable systems from non-Abelian $r$-matrices, and actually this had been one of our aims originally. However, we found that such $r$-matrices do not give rise to new integrable systems in addition to those that may be constructed using Abelian $r$-matrices, at least if one considers $r$-matrices on reductive Lie algebras of variables. In the following we describe the derivation of this ‘no go’ result.

Let $\mathcal{G}$ be a self-dual (also called quadratic) Lie algebra, equipped with a non-degenerate, symmetric, invariant bilinear form, $B_\mathcal{G}$. Identify $\mathcal{G} \otimes \mathcal{G}$ with $\text{End}(\mathcal{G})$ in such a way that $X \otimes Y : Z \mapsto B_\mathcal{G}(Y, Z)X$ for any $X, Y, Z \in \mathcal{G}$. Consider a chain of subalgebras

$$\mathcal{K} \subset \mathcal{F} \subset \mathcal{G},$$

where $\mathcal{F}$ is a reductive Lie algebra, $\mathcal{K}$ is a Cartan subalgebra of $\mathcal{F}$ and the restriction of $B_\mathcal{G}$ remains non-degenerate both on $\mathcal{F}$ and on $\mathcal{K}$. Consider also a connected Lie group $G$ with Lie algebra $\mathcal{G}$ and connected subgroups

$$K \subset F \subset G$$

corresponding to the subalgebras \text{(3.1)}. Let us assume that

$$R_\mathcal{F} : \mathcal{F} \rightarrow \text{End}(\mathcal{G})$$

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is an $F$-equivariant map on a domain for which
\[ \text{ad}_q|_{\mathcal{K} \cap \mathcal{F}^\perp} \text{ is invertible} \quad \forall q \in \mathcal{K} := \mathcal{K} \cap \mathcal{F}, \tag{3.4} \]
and $\mathcal{F}$ consists of orbits of $F$ through $\mathcal{K}$, i.e.,
\[ \mathcal{F} = \{ \text{Ad}_f q \mid f \in F, q \in \mathcal{K} \}. \tag{3.5} \]
The properties expressed by the last two equations can always be arranged by a restriction of the domain of any $F$-equivariant map $R_F$. We then define $R_K : \mathcal{K} \to \text{End}(G)$ as follows:
\[ R_K(q)(X) = \left\{ \begin{array}{ll} R_F(q)(X) & \text{if } X \in (\mathcal{K} + \mathcal{F}^\perp) \\ R_F(q)(X) + (\text{ad}_q|_{\mathcal{K} \cap \mathcal{F}^\perp})^{-1}(X) & \text{if } X \in \mathcal{K}^\perp \cap \mathcal{F}. \end{array} \right. \tag{3.6} \]
It is known that $R_F$ is a solution of the CDYBE associated with $F \subseteq G$ if and only if $R_K$ is a solution of the CDYBE associated with $K \subset G$. Of course, to view $R_F$ and $R_K$ as dynamical $r$-matrices, one takes into account the identifications $F^* \simeq F$ and $K^* \simeq K$ based on $B_G$. For reasons explained in [22], we call $R_K$ the Dirac reduction of $R_F$. In the main examples [25], $G$ is semi-simple with Killing form $B_G$, and $F$ is the fixed point set of a (possibly trivial) automorphism of $G$. For a semi-simple Lie algebra $G$, all non-Abelian $r$-matrices that are known to us are related to corresponding Abelian $r$-matrices in the manner in (3.6).

Now we show that the integrable systems that result by applying the construction outlined in Section 2 to the non-Abelian $r$-matrix $R_F$ and to its Abelian counterpart $R_K$ are essentially (up to factoring by a discrete symmetry) the same. For the proof, it is convenient to proceed by first restricting the ‘spin’ variable $\xi \in G^*$ to a coadjoint orbit $O \subset G^* \simeq G$, so that the construction based on $R_F$ starts with the phase space
\[ \mathcal{M}_F = T^*\mathcal{F} \times \mathcal{O} = \mathcal{F} \times F \times \mathcal{O} = \{(Q, P, \xi)\}. \tag{3.7} \]
$\mathcal{M}_F$ carries the symplectic form $\Omega_F$,
\[ \Omega_F(Q, P, \xi) = B_G(dP \wedge dQ) + \omega_\mathcal{O}(\xi), \tag{3.8} \]
where $\omega_\mathcal{O}$ is the symplectic form of the orbit $\mathcal{O}$, and the quasi-Lax operator $L_F$,
\[ L_F(Q, P, \xi) = P - R_F(Q)\xi. \tag{3.9} \]
The construction based on $R_K$ works by reducing $\mathcal{M}_K = \mathcal{K} \times \mathcal{K} \times \mathcal{O} = \{(q, p, \xi)\}$, which is equipped with its symplectic form $\Omega_K$,
\[ \Omega_K(q, p, \xi) = B_G(dp \wedge dq) + \omega_\mathcal{O}(\xi), \tag{3.10} \]
and the quasi-Lax operator $L_K$ defined using $R_K$.

Continuing with the Abelian case of $R_K$, we decompose $\xi$ as $\xi_K + \xi_K^\perp$ and introduce the first class constrained manifold
\[ \mathcal{M}_K^0 = T^*\mathcal{K} \times \mathcal{O}^0 = \{(q, p, \xi_K^\perp) \mid q \in \mathcal{K}, p \in \mathcal{K}, \xi_K^\perp \in \mathcal{O} \cap \mathcal{K}^\perp \}, \tag{3.11} \]
where \( \chi_K(q, p, \xi) = \xi_K = 0 \). The corresponding reduced phase space

\[
\mathcal{M}^{\text{red}}_K = \mathcal{M}^0_K / K
\]

(3.12)
is a (in general singular, stratified\(^2\)) symplectic manifold, whose symplectic structure is induced by the restriction (pull-back) of \( \Omega_K \) to \( \mathcal{M}^0_K \subset \mathcal{M}_K \). A commuting family of Hamiltonians on \( \mathcal{M}^{\text{red}}_K \) is obtained by the application of the \( G \)-invariant functions on \( G, I(G) \subset C^\infty(G) \), to the quasi-Lax operator \( L_K \), since these Hamiltonians survive the reduction. In fact, \( h \circ L_K \) yields \( \forall h \in I(G) \) a \( K \)-invariant function on \( \mathcal{M}^0_K \), on account of the \( K \)-equivariance of \( L_K \).

In the non-Abelian case of \( \mathcal{R}_F \), we start by introducing the first class constraints

\[
\chi_F(Q, P, \xi) = [Q, P] + \xi_F = 0,
\]

(3.13)

using the decomposition \( \xi = \xi_F + \xi_{F^\perp} \). This defines the constrained manifold \( \mathcal{M}^0_F \subset \mathcal{M}_F \), and we wish to compare \( \mathcal{M}^{\text{red}}_F \) to \( \mathcal{M}^{\text{red}}_F = \mathcal{M}^0_F / F \). The reduced (stratified) symplectic structure and the commuting Hamiltonians on \( \mathcal{M}^{\text{red}}_F \) are induced by the restrictions of \( \Omega_F \) and \( h \circ L_F, h \in I(G) \), to \( \mathcal{M}^0_F \). By the assumption (3.5), every \( F \)-orbit in \( \mathcal{M}^0_F \) intersects the submanifold \( \mathcal{M}^0_{F, K} \subset \mathcal{M}^0_F \) defined by

\[
\mathcal{M}^0_{F, K} = \{(q, P, \xi) \in \mathcal{M}^0_F | q \in \hat{K}\},
\]

(3.14)
i.e., any \( Q \in F \) can be conjugated into \( \hat{K} \). Since \( \hat{K} \) consists of regular elements (3.4), the ‘residual gauge transformations’ that preserve the ‘partial gauge fixing’ defined by \( \mathcal{M}^0_{F, K} \) are given by the normalizer subgroup

\[
N_F(K) := \{ f \in F | Ad_f k \in K, \forall k \in K \}
\]

(3.15)
of \( K \) inside \( F \). In other words, an arbitrarily fixed element of \( \mathcal{M}^0_{F, K} \) is mapped to \( \mathcal{M}^0_{F, K} \) precisely by those \( f \in F \) that lie in \( N_F(K) \). Note that \( K \subset N_F(K) \) is a normal subgroup, and

\[
W := N_F(K) / K
\]

(3.16)
is a discrete group since the Lie algebra of \( N_F(K) \) equals \( K \). These observations imply the second and third equalities in

\[
\mathcal{M}^{\text{red}}_F = \mathcal{M}^0_F / F = \mathcal{M}^0_{F, K} / N_F(K) = (\mathcal{M}^0_{F, K} / K) / W.
\]

(3.17)

In order to compare with the Abelian case, notice that on \( \mathcal{M}^0_{F, K} \) the constraint (3.13) is uniquely solved as

\[
\xi_K = 0, \quad P = P(q, p, \xi) = p - (\text{ad}_{q}|_{K^\perp \cap F})^{-1} \xi_{K^\perp \cap F} \quad \text{with} \quad p \in K,
\]

(3.18)

where we use the decomposition \( \xi = \xi_K + \xi_{K^\perp \cap F} + \xi_{F^\perp} \). We see from this that the map

\[
m: \mathcal{M}^0_K \to \mathcal{M}^0_{F, K}, \quad m: (q, p, \xi) \mapsto (q, P(q, p, \xi), \xi)
\]

(3.19)

\(^2\)To understand the fine structure of the reduced phase spaces \( \mathcal{M}^{\text{red}}_K \) and \( \mathcal{M}^{\text{red}}_F \), one may wish to apply the theory of singular symplectic reduction [27]. This is directly applicable if \( K \) and \( F \) are compact Lie groups, but actually our construction works without this assumption, too.
is a $K$-equivariant diffeomorphisms, and it is also easy to check that this map converts the restrictions of the relevant symplectic forms and quasi-Lax operators into each other:

$$m^* (\Omega|_{\mathcal{M}^0_{F,K}}) = \Omega_K|_{\mathcal{M}^0_K} \quad \text{and} \quad m^* (L|_{\mathcal{M}^0_{F,K}}) = L_K|_{\mathcal{M}^0_K}. \tag{3.20}$$

Here, $\Omega_K|_{\mathcal{M}^0_K}$ denotes $i^*\Omega_K$ with the natural map $i: \mathcal{M}^0_K \to \mathcal{M}_K$ and, similarly, $\Omega_F|_{\mathcal{M}^0_{F,K}}$ is the pull-back of $\Omega_F$. Since $m$ is $K$-equivariant, it induces a one-to-one map, $\tilde{m}: \mathcal{M}^0_{F,K} \to \mathcal{M}^0_{F,K}/K$, whereby we can identify these spaces of $K$-orbits. Because of (3.20), this identification converts the Poisson structures and commuting Hamiltonians carried by these spaces into each other. By combining the map $\tilde{m}$ with the last equality in (3.17), we arrive at the following conclusion.

**Proposition 2.** Under the foregoing assumptions (in particular, choosing the domains of $\mathcal{R}_F$ and $\mathcal{R}_K$ according to (3.4)-(3.5)), the Hamiltonian systems associated (using $\forall h \in I(\mathcal{G})$) with the non-Abelian $r$-matrix $\mathcal{R}_F$ by the construction outlined in Section 2 are identical to the systems associated with the Abelian $r$-matrix $\mathcal{R}_K$ (3.6) up to the discrete symmetry given by the group $W$ (3.16). That is the corresponding phase spaces are related as

$$\mathcal{M}^\text{red}_F = \mathcal{M}^\text{red}_K/W. \tag{3.21}$$

**Remark 1.** Suppose that the semi-simple factor of $F$ is compact, and notice that $W$ is then the Weyl group of $K \subset F$. Thus the space of $W$-orbits $\mathcal{M}^\text{red}_K/W$ can be realized simply by restricting the variable $q$ to a fundamental domain of $W$ in $\mathcal{K}$, i.e., to an open Weyl chamber. This means that the system on $\mathcal{M}^\text{red}_F$ associated with $\mathcal{R}_F$ also arises by performing the construction of Section 2 using $\mathcal{R}_K$ restricted to a Weyl chamber. This strengthens our claim that the systems associated with the non-Abelian and Abelian $r$-matrices are ‘essentially’ the same.

**Remark 2.** A non-compact reductive Lie algebra $F$ possesses non-conjugate Cartan subalgebras, say $\mathcal{K}_a$ for $a = 1, \ldots, N > 1$, in general. If $\mathcal{R}_F$ is defined on a dense open subset of $F$ (which can be achieved for any $r$-matrix on $F \subseteq \mathcal{G}$), then the associated reduced phase space $\mathcal{M}^\text{red}_F$ contains the reduced spaces $\mathcal{M}^\text{red}_{\mathcal{K}_a}$ associated with the non-conjugate Cartan subalgebras as disjoint open subsets.

### 4 On the resulting family of spin Calogero models

We explain below that the dynamical $r$-matrix method presented in Section 2 leads to a large family of integrable systems of spin Calogero type. The Hamiltonians of these systems are induced by the quadratic form of an invariant scalar product using Abelian dynamical $r$-matrices.

Let us take an Abelian, self-dual subalgebra $\mathcal{K}$ of a self-dual Lie algebra $\mathcal{G}$ and suppose that

$$\mathcal{R}: \mathcal{K} \to \text{End}(\mathcal{G}) \tag{4.1}$$
is a dynamical $r$-matrix for $K \subset G$, where we made the identifications $G \cong G^*$ and $K \cong K^*$.

Suppose also that the operator $R(q)$ ($q \in \mathcal{K}$) is compatible with the decomposition $G = K + K^\perp$.

This compatibility condition holds for all examples we are aware of. The simplest Hamiltonian of interest on the phase space $M = \mathcal{K} \times \mathcal{K} \times G$ is

$$H(q, p, \xi) = \frac{1}{2} B_G(L(q, p, \xi), L(q, p, \xi)), \quad (4.3)$$

which corresponds to the quadratic Casimir associated with the invariant scalar product $B_G$.

Upon imposing the first class constraint $\xi_K = 0$ on $\xi = \xi_K + \xi_K^\perp$ and recalling (2.5), the Hamiltonian takes the following form:

$$H(q, p, \xi) = \frac{1}{2} B(p, p) + \frac{1}{2} B_G(R(q) \xi_K, R(q) \xi_K), \quad (4.4)$$

If $R(q)$ depends on $q$ through rational or trigonometric (hyperbolic) functions of its components, which holds in all known examples, then $H(q, p, \xi)$ is a Hamiltonian of spin Calogero type. The first term of (4.4) represents kinetic energy and the second term is a rational or trigonometric (or hyperbolic) potential containing the 'spin' degrees of freedom as well. The restriction of $B_G$ to $K$ must be positive or negative definite for this interpretation to be valid in the strict sense.

Under the constraint $\xi_K = 0$, the evolution equation generated by $H$ (4.3) implies

$$\dot{L} = [R(q)L, L]. \quad (4.5)$$

Together with $\dot{q} = p$, the Lax equation (4.5) is actually equivalent to the constrained Hamiltonian equation of motion if $R(q)$ maps $K^\perp$ to $K^\perp$ in an invertible manner. Indeed, for such non-degenerate $r$-matrices meeting the compatibility and non-degeneracy conditions, $H(q, p, \xi)$ can be recovered from the decomposition of $L$ according to (4.2).

Note that $q$, $p$, and $H$ are gauge invariant, while $\xi_K$ matters only up to conjugation by the elements of $K$ since the gauge transformations generated by $\xi_K^\perp$ act as

$$\xi_K \mapsto e^\kappa \xi_K e^{-\kappa}, \quad (4.6)$$

where $\kappa$ is an arbitrary $K^*$-valued function.

The rational and trigonometric (hyperbolic) $r$-matrices on the Cartan subalgebra of a simple Lie algebra were classified in [17], and the corresponding examples in $R(q)(K^\perp)$ were described in [16]. More recently, we studied the family of spin Calogero models in the quasi-triangular case. In all such $r$-matrices are provided up to an irrelevant freedom in $R(q)(K)$ by the formula

$$R(q)(K) = \frac{1}{2} \text{id}_K, \quad R(q)(K^\perp) = (1 - \theta^{-1} e^{-\text{ad}_q(K^\perp)})^{-1}, \quad (4.7)$$

where $\theta$ is an automorphism of $G$ that preserves also the scalar product. $K$ lies in the fixpoint set of $\theta$, and the inverse that occurs is well-defined for a non-empty open subset $\mathcal{K} \subset K$.

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the general case this formula is due to Alekseev and Meinrenken [24], its uniqueness property mentioned above was proved in [23].

It turned out that the spin Calogero models associated with the \( r \)-matrices (4.7) can be also derived by Hamiltonian reduction of the geodesic system on (an open submanifold of) \( T^*G \). The geodesics in question are simply the orbits of the one-parameter subgroups of \( G \), since the underlying metric on \( G \) is induced from the invariant bilinear form \( B_{\mathcal{G}} \) and is thus bi-invariant. The reduction relies on the Hamiltonian action of \( G \) arising from twisted conjugations. The twisted conjugation by \( k \in G \) acts on the group manifold \( G \) according to

\[
\text{Ad}_k^\Theta : g \mapsto \Theta^{-1}(k)g_k^{-1} \quad \forall g \in G,
\]

if \( \Theta \in \text{Aut}(G) \) lifts \( \theta \in \text{Aut}(\mathcal{G}) \). The Hamiltonian reduction method leads to a simple algorithm for integrating the spin Calogero equation of motion with the aid of the geodesics of \( G \). The reader is referred to [23] for a detailed presentation of the Hamiltonian reduction picture as well as for several examples. The examples include systems built on the non-trivial diagram automorphisms of the simply laced simple Lie algebras and systems built on the cyclic permutation automorphisms of semi-simple Lie algebras composed of \( N > 1 \) identical factors. The former systems seem to be new, while the latter were studied earlier in the \( A_n \) case by Blom-Langmann [28] and by Polychronakos [29] by means of different methods. It should be possible to quantize these systems with the aid of quantum Hamiltonian reduction, which is one the topics of our interest for future work.

Acknowledgements. The work of L.F. was supported in part by the Hungarian Scientific Research Fund (OTKA) under the grants T043159, T049495, M045596 and by the EU networks ‘EUCLID’ (contract number HPRN-CT-2002-00325) and ‘ENIGMA’ (contract number MRTN-CT-2004-5652). B.G.P. is grateful for support by a CRM-Concordia Postdoctoral Fellowship and he especially wishes to thank J. Harnad for hospitality in Montreal.

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