Decay orbital period of the binary system on gravitational waves’ detection

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Abstract. Gravitational Waves had been fifth detected which is on PSR 1913+16 at 1974 until 1981 by Arecibo Telescope, on GW150914, GW151226, and GW170104 by LIGO and the last is on GW170814 by LIGO and VIRGO. The research made a binary system dynamic modeling by using linearization theorem and post-Newtonian formalism. This modeling is a binary system with two mass identically orbiting each other on the center of circular orbit. By this modelling will get decay orbital period of system for the fifth data of gravitational waves’ detection is -1,073 x 10^-12 for PSR 1913+16, -81,956 x 10^-12 for GW150914, -2,56 x 10^-12 for GW151226, -73,73 x 10^-12 for GW170104, and -44,94 x 10^-12 for GW170814. The result can inform us about time which using system to combine be a more massive system (a black hole or a pulsar).

1. Introduction
On 2017 had been detected twice gravitational waves is at January 4th and August 14th. This detection is a proof about general relativity which was published by Einstein in 1916. The major concept of general relativity is a mass can make curvatures n curvatures can move a mass. More massive the mass is, curvatures which are made will bigger. A mass which moves on space-time can make spreading curvatures. This spreading curvature can be analogized as ripples when a stone is throwing to a pool. When the stone reaches water surface, water on that pool will be disturbed by the stone and energy will be spread out to all direction. The same phenomenon will be happened to the curvature of space-time, too. Which the ripples of space-time will be spread by the speed of light and then it will be said as gravitational waves. In this universe, there is a kind of gravitational waves’ source which is a binary pulsar system, the collision of black holes, the explosion of a supernova, and big bang [1]. Gravitational waves detection is a proof about general relativity which was published by Einstein and in this research will be studied about decay orbital period system of gravitational waves from 1974 until 2017 by using analytical modeling of linearization theorem.

2. Einsteins’ Linearization Theorem
The general characteristic of gravitational waves are transversal waves which have two perpendiculars of polarization, spreading by speed of light, it can be spread without medium, and can be detected by tidal force principal using Michelson’s interferometer of gravitational waves which can measure compression and stretching of space-time causing by the spreading of gravitational waves or it can be
said that the spreading of gravitational waves can make a length in between of two arms of interferometer’s mirror which isolated by times.

Gravitational waves can make weak ripples on space-time and can be said as a perturbation on Minkowski matrix,

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

where \( \eta_{\mu\nu} \) is a Minkowski matrix and \( h_{\mu\nu} \) is a perturbation matrix.

3. Lorentz-Gauge Transformation

Lorentz Transformation is a transformation which is connected inertia frame for two observers who moves each other in Minkowski space-time [2]. When gravitational waves reach the test particle, it will make little ripples and move test particle’s position on space-time which can be described by Lorentz transformation,

\[ x^\mu \rightarrow x'^\mu = L^\mu_\nu x^\nu \] (2)

where,

\[ L^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \] (3)

and space-time matrix is,

\[ g_{\mu\nu} \rightarrow g'_{\alpha\beta} = L^\mu_\alpha L^\nu_\beta g_{\mu\nu} = \eta_{\mu\nu} + L^\mu_\alpha L^\nu_\beta h_{\mu\nu} \] (4)

Because Minkowski matrix is invariant to Lorentz transformation, \( \eta_{\mu\nu} = \eta'_{\alpha\beta} \).

Meanwhile, Gauge transformation is an infinitisimalcoordinat transformation which had done because of space-time curvature [1]. If it is done to the matrix, the matrix will be like,

\[ g'^{(1)}_{\alpha\beta} = g^{(1)}_{\alpha\beta} - \partial_\alpha x_\beta - \partial_\beta x_\alpha \] (5)

because \( |h_{\mu\nu}| \ll 1 \) so if it have variaation to delta, \( \delta \), will have smaill value and can be neglected. Then Gauge transformation can be like,

\[ h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha x_\beta - \partial_\beta x_\alpha \] (6)

where \( |\chi_{\alpha\beta}| \ll 1 \) is a Gauge vector which have definition as an infinitesimal movement. Gauge transformation take a condition as,

\[ \partial_\mu \bar{h}_{\mu\nu} = 0 \] (7)

with

\[ \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2} \eta_{\mu\nu} \] (8)

Then reverse perturbation matrix on Lorentz-Gauge transformation is,

\[ \bar{h}_{\alpha\beta} = \bar{h}_{\alpha\beta} - \partial_\alpha x_\beta - \partial_\beta x_\alpha + \eta_{\alpha\beta} (\partial_\mu x_\mu) \] (9)

4. Spreading of Gravitational Waves on Vacuum Space-time

After it has the equation of space-time matrix on Lorentz-Gauge transformation, then it can get Christoffel symbol, Riemann Tensor, Ricci Tensor, and Ricci scalar which can make linearization equation of Einstein field as,

\[ \Box \bar{h}_{\mu\nu} \approx \frac{16\pi G_N}{c^4} T^{(0)}_{\mu\nu} \] (10)

which just have contemplation on perturbation matrix caused by gravitational waves on space-time. On vacuum space-time, equation (10) will be,

\[ \Box \bar{h}_{\mu\nu} = 0 \] (11)

with ansatz solution can be analogized as a plane wave,

\[ \bar{h}_{\mu\nu} = \epsilon_{\mu\nu} e^{ikx_\mu} \] (12)
where $\varepsilon_{\mu\nu}$ is a gravitational waves tensor which have two order polarisation and characteristic of symmetry with $k_a$ is number vector of wives, $\left(\frac{\omega}{c}, \vec{k}\right)$.

If equation (12) have been substituted to vacuum field equation at equation (11), then characteristic of gravitational waves will be,

a. Gravitational waves move with speed of light, $c = \frac{\omega}{|\vec{k}|}$.  

b. $h_{\mu\nu} = 0$ condition will be validated on coordinate which is invariant to Lorentz-Gauge transformation. 

c. Result of $k^\mu \varepsilon_{\mu\nu} = 0$ showed that polarization tensor of gravitational waves, $\varepsilon_{\mu\nu}$, perpendicular to it’s spreading direction, $\vec{k}$. 

Afterward, take a constrain for Gauge vector as,

$$\partial^\mu \partial_\mu \chi_\alpha = 0$$  

Then it will get ansatz solution as,

$$\chi_\alpha = B_\alpha e^{i k_{\alpha} x^\mu}$$  

with the same step as before, it can get ansatz solution for vacuum field as,

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} - i B_a k_b + i \eta_{ab} B_\mu k^\mu.$$  

Meanwhile, to establish polarization direction of waves, it can take Transverse-Traceless Gauge’s condition as,

$$\varepsilon_{\mu\mu} = 0 \text{ and } \varepsilon_{\mu\nu} U^\nu = 0$$  

where it can be analogized as a wave which spreading to the z-axis, $\vec{k} = (\omega, 0, 0)$, with velocity, $\vec{U} = (1,0,0,0)$ so it will be like,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_+ & \varepsilon_0 & 0 \\ 0 & \varepsilon_x & -\varepsilon_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$  

with $\varepsilon_{11} = -\varepsilon_{22} \equiv \varepsilon_+$ and $\varepsilon_{12} = \varepsilon_{21} \equiv \varepsilon_x$.

5. Quadrupole Moment of Gravitational Waves on TT Gauge Conditions

At the last subbed had gotten gravitational waves on vacuum space-time, then it will be studied about energy-momentum tensor which has been taken by gravitational waves, $t_{\mu\nu}$, which can make space-time’s curvature with using Einstein field equation.

$$t_{\mu\nu} = \frac{c^4}{8\pi G_N} \left( < R_{\mu\nu}^{(2)} > - \frac{1}{2} \eta_{\mu\nu} < R^{(2)} > \right).$$  

For knowing perturbation matrix of gravitational waves effect to energy-momentum tensor, it can take space-time matrix as,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + \tilde{h} & 0 \\ 0 & 0 & 1 - \tilde{h} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$  

then it can get that $\langle R^{(1)} \rangle = 0$, so will be get energy-momentum tensor as,

$$t_{\mu\nu} = \frac{c^4}{8\pi G_N} < R_{\mu\nu}^{(2)} >.$$
at initial state,
\[ t_{00} = \frac{e^4}{32\pi G_N} < \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} > \]  
so, energy flux will be like,
\[ f = c t_{00} = \frac{e^5}{32\pi G_N} < \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} > . \]  

Meanwhile, perturbation matrix of gravitational waves can be written on integral with Long-Wavelength approximation.
\[ \tilde{h}_{\mu\nu}(x, t) = \frac{4G_N}{c^4} \int \frac{d^3x'}{r} T_{\mu\nu}^{(0)} \left( x', t - \frac{r}{c} \right) = \frac{2G_N}{c^4} \frac{\partial^2}{\partial t^2} \left[ \tilde{l}_{ij} \left( t - \frac{r}{c} \right) \right] \]  

where \( \tilde{l}_{ij} \) is a quadropole moment of gravitational waves which have shown as spreading waves and if it is projecting to TT Gauge operator, it can make as
\[ \tilde{l}_{ij} = I_{ij} - \frac{1}{2} \delta_{ij} I \]

using energy flux, energy which disappeared during spreading can be gotten,
\[ \frac{dE}{dt} = \int f \cdot r^2 d\Omega = \frac{G_N}{2c^5} < \left( \frac{\partial^3}{\partial t^3} \tilde{l}_{ij}^{TT} \right) \left( \frac{\partial^3}{\partial t^3} \tilde{l}_{ij}^{TT} \right) > \]

6. Analytical Modelling of Binary System in Gravitational Waves’ Detection

Modeling which using in this research is a binary system with identical mass and circular trajectory which have \( r \) radius and initial orbital speed is \( \omega_0 \).

According to Figure 1,
\[ x_1(t) = r \cos (\omega_b t) \]
\[ x_2(t) = r \sin (\omega_b t) \]
\[ x_3(t) = 0. \]

Meanwhile, quadropole moment is,
\[ I_{ij} = \int d^3x \rho(x) x_i x_j \]  

Then quaduple moment systems are,
\[ I_{11} = 2Mr^2 \cos^2 (\omega_b t) \]
\[ I_{22} = 2Mr^2 \sin^2 (\omega_b t) \]
\[ I_{33} = 2Mr^2 \sin^2 (\omega_b t) \cos^2 (\omega_b t) \]  

and quadropole moment for TT Gauge condition is,
\[ \tilde{I}_{11}^{TT} = Mr^2 \cos 2(\omega_b t) \]
\[ \tilde{I}_{22}^{TT} = -Mr^2 \cos 2(\omega_b t) \]
\[ I_{33}^{TT} = Mr^2 \sin 2(\omega_bt). \]  

(27)

A disappeared energy during spreading gravitational waves is,

\[ \frac{dE}{dt} = \frac{64G_NM^2r^2\omega_b^6}{c^5} \]  

(28)

by using the relation between decay orbital period system and disappeared energy which is,

\[ \frac{dP_b}{P_b} \propto \frac{dE}{E} \]  

(29)

and Newton equation,

\[ E = MV^2 - \frac{G_NM^2}{2r} \]  

(30)

It will get decay orbital period binary system to time is,

\[ \dot{P}_b = -\frac{24\pi}{c^5}\left(\frac{4\pi G_NM}{P_b}\right)^{\frac{5}{3}} \]  

(31)

By using observation data of gravitational waves since September 1974 until August 2017 [3-7], getting decay orbital period system,

1. PSR 1913+16, \( \dot{P}_b = -1.073 \times 10^{-12} \)
2. GW150914, \( \dot{P}_b = -81.956 \times 10^{-12} \)
3. GW151226, \( \dot{P}_b = -2.56 \times 10^{-12} \)
4. GW170104, \( \dot{P}_b = -73.73 \times 10^{-12} \)
5. GW170814, \( \dot{P}_b = -44.94 \times 10^{-12} \)

This observation data about decay orbital period can be only observed in PSR 1913+16 at 1974 until 1981 is \( \dot{P}_b = -2.30 \times 10^{-12} \).

7. Conclusion

According to modeling result on this research getting that have been observed fifth gravitational waves phenomena which is PSR 1913+16, GW150914, GW151226, GW170104, and GW170814 by Arecibo radio telescope (PSR 1913+16), LIGO (GW150914, GW151226, GW170104), and LIGO-VIRGO (GW170814).

Then according to analytical modeling of linearization theory with circular trajectory getting that decay orbital period binary system for fifth gravitational waves' observation is \(-1.073 \times 10^{-12}, -81.956 \times 10^{-12}, -2.56 \times 10^{-12}, -73.73 \times 10^{-12}, \text{dan} -44.94 \times 10^{-12} \).

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