Stability of trapped fermionic gases with attractive interactions

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We present a unified overview, from the mean-field to the unitarity regime, of the stability of a trapped Fermi gas with short range attractive interactions. Unlike in a system of bosons, a Fermi gas is always stable in these regimes, no matter how large the particle number. However, when the interparticle spacing becomes comparable to the range of the interatomic interactions, instability is not precluded.

I. INTRODUCTION

The freedom to tune the interparticle interactions in dilute Fermi and Bose gases using Feshbach resonances raises the question of the stability of trapped atomic gases with attractive interactions against collapse. Weakly interacting Bose gases with attractive interactions become mechanically unstable once the particle number exceeds a critical value, of order the ratio of the oscillator length to the magnitude of the s-wave scattering length. The question is whether Fermi systems are always stabilized, and by what mechanisms, or can they ever be similarly unstable? Our aim in this paper is to present an overview of the problem in the different parameter regimes, putting together and extending the various arguments that have been given for the stability (and instability) of Fermi systems within a unified framework.

At first, one might think that the Fermi energy of the atoms would be sufficient to support a trapped Fermi gas against collapse, in the way, for example, that electron degeneracy pressure supports white dwarfs against gravitational collapse. In a white dwarf the gravitational energy scales as the inverse of the system radius, $\mathcal{R}$, while the Fermi energy scales as $\mathcal{R}^{-2}$ non-relativistically; instability sets in only in the relativistic limit, when the Fermi energy begins to scale as $\sim \mathcal{R}^{-1}$. In a trapped Fermi gas with short range s-wave interactions between different components, the Fermi energy similarly scales as $\mathcal{R}^{-2}$, while the interaction energy scales, within mean-field theory, as $\mathcal{R}^{-3}$. Thus the Fermi energy is not, in itself, adequate to prevent an instability.

In the Fermi problem with low energy s-wave interactions, one must distinguish several regimes. The first is that of the weak interactions where the s-wave scattering length, $a$, is small compared with the interparticle spacing, and mean-field theory is valid. Calculating within mean-field theory one finds an instability at high density $N^{1/6} \gtrsim 0.6d/|a|$, where $k_F$ is the Fermi momentum, and $d = \sqrt{\hbar/m\omega}$ is the oscillator length, with $\omega$ the oscillator frequency (assumed isotropic); however, at such densities the interparticle spacing is comparable to the scattering length, and the mean-field calculation becomes invalid. Indeed, in the mean-field regime ($k_F|a| \ll 1$) the system is always stable. The second – “unitary” – regime is when the scattering length is large compared with the interparticle spacing, itself large compared with the range of the interatomic potential. Here the system is again always stable, since the total energy remains positive, $E > 0$. The third regime is when the interparticle spacing becomes comparable to the range, $r_0$, of the interatomic potential. In this regime the system can in principle become unstable.

![FIG. 1: Regions of stability in the $k_F r_0$, $k_F|a|$ plane of a two component Fermi gas with short range interactions. Here $k_F$ is the Fermi momentum, $a$ the s-wave scattering length, and $r_0$ the range of the interactomic potential.](image-url)
total energy of a system with an attractive square well interparticle interaction, how stability can break down in the high density regime.

II. STABILITY IN MEAN-FIELD

We consider a Fermi gas with equal numbers of two components (denoted by $\uparrow$ and $\downarrow$) in equilibrium at zero temperature in an isotropic harmonic trap, $V(r) = \frac{1}{2} m \omega^2 r^2$. The energy of the system in the local density approximation is

$$E = \frac{3\hbar^2}{10m} \left(3\pi^2\right)^{2/3} \int dr \ n^{5/3} + \frac{1}{2} m \omega^2 \int dr \ r^2 n + g \int dr \ n_{\uparrow} n_{\downarrow},$$

(1)

where $g = 4\pi\hbar^2 a/m$ is the coupling constant, and $n = 2n_{\uparrow} = 2n_{\downarrow}$ is the total density of particles. The short range s-wave interactions between particles of the same species essentially vanish due to the antisymmetry of the many-fermion wave function.

The equilibrium density minimizes the total energy with respect to the density, at fixed particle number, $N = \int dr \ n(r)$. Taking the constraint of fixed total number of particles into account via a chemical potential, $\mu = \mu_{\uparrow} = \mu_{\downarrow}$, we find that the first functional derivative of the energy is

$$\delta E - \mu \delta N = \int dr \left( \left(3\pi^2\right)^{2/3} \frac{\hbar^2}{2m} n^{2/3} \right) \frac{1}{2} m \omega^2 r^2 + \frac{g}{2} n - \mu \right) \delta n = 0, \quad (2)$$

which yields the density-position relation,

$$(3\pi^2)^{2/3} \frac{\hbar^2}{2m} n^{2/3} + \frac{1}{2} m \omega^2 r^2 + \frac{g}{2} n = \mu. \quad (3)$$

It is most convenient to regard this equation as determining $r^2$ in terms of $n$:

$$r^2 = \frac{1}{\left(3\pi^2 n\right)^{2/3} - 4\pi |a| n} d^2 + R^2(a, N), \quad (4)$$

where $d = \sqrt{\hbar/m \omega}$ is the oscillator length, and the radius of the cloud is $R = \sqrt{2\mu/m \omega^2}$. Equation (4) can be inverted either graphically, or by simply solving the cubic equation explicitly. The results are shown in Fig. 2. Note that when $a < 0$, $r(n)^2$ has a minimum as a function of density at

$$r_0^2 = -\frac{\pi^2 d^4}{12 |a|^2} + R^2, \quad (5)$$

where

$$n_0 = \frac{\pi}{24 |a|^3}. \quad (6)$$

In order for the density profile to be stable it must minimize, not maximize, the energy functional, i.e., the second variation of $E - \mu N$:

$$\delta^2 E - \mu \delta^2 N = \int dr \left( \left(3\pi^2\right)^{2/3} \frac{\hbar^2}{3m} n^{-1/3} \right) \frac{g}{2} \delta n^2, \quad (7)$$

must be positive for all $\delta n$. This condition requires,

$$k_F(r)|a| < \frac{\pi}{2} \approx 1.57, \quad (8)$$

at all positions, or equivalently $n \leq \pi/24 |a|^3$ (cf. Eq. (6)). As we see, Eq. (8) is the condition that the density profile corresponds to a minimum of the energy functional [11]. Note that we have derived this result in terms of the Thomas-Fermi profile including interactions. This argument is similar to that of [12]. The result (8) has also been obtained heuristically by comparing the repulsive force on an atom arising from the kinetic energy term with the attractive force due to the mean field interaction energy [5], and derived via a field-theoretical approach [15].

When the minimum of $r^2$ vs. $n$ is at negative $r^2$, the system is stable because then $n$ is less than $n_0$, so that $k_F|a|$ is always $< \pi/2$. On the other hand, when the minimum is at positive values, the curve does not intersect the horizontal axis, and therefore there is no solution for the density at the center of the trap. (An unphysical solution for the density would have a hole of radius $r_0$ about the center of the trap.) However, in this case the density equals $n_0$ at $r = r_0$, and vanishes as $r \to R$. Then, $k_F(0)|a| = \pi/2$, and the solution is marginally stable, not actually minimizing the energy functional [11]. We conclude that the system is stable when $r_0^2 \leq 0$, and the
critical point is \( r_0 = 0 \), corresponding to
\[
R^2 = \frac{\pi^2}{12} \frac{d^4}{|a|^2},
\]
and then the density of the system at the center of the trap is given by Eq. (6).

When Eq. (9) is satisfied the system reaches its maximum number of particles in equilibrium, given by the area under the \( n \) vs \( r \) curve, \( N_{\text{max}} = 4\pi \int_0^\infty n(r)r^2 dr \). Integrating with respect to the \( n \), we find
\[
N_{\text{max}} = \frac{\pi^5}{48} \frac{d^6}{|a|^6} I,
\]
where
\[
I = \int_0^1 \left( \frac{2}{3} x^3 - x^2 + \frac{1}{3} \right)^{1/2} (1-x)x^3 dx
\]
\[
= \frac{54}{5005} - \frac{40}{9000\sqrt{3}} \approx 0.00823.
\]

As long as the local density approximation is valid, the condition (10) is equivalent to the condition (8). However, the use of mean field is no longer valid for \( k_F|a| > 1 \). From (10), the condition for stable equilibrium is \( N^{1/6} \lesssim 0.612d^2 |a| \), which agrees with that found numerically in Ref. 6.

Table I lists the number of particles in representative experiments on \( ^6\text{Li} \) gases with two-spin components. Since these experiments were not testing the maximum number of particles that could be trapped, they provide only lower bounds on the maximum number of particles. For small \( k_F|a| < 1 \), the experimental particle numbers are well within the bound (10). However fermionic systems at larger \( k_F|a| \), exceeding \( \pi/2 \), do remain stable. See Fig. 3. The argument leading to Eq. (10) breaks down at large \( k_F|a| \), for then correlations beyond mean field become important. In addition, when \( n(0) \to n_0 \), the local density approximation fails. A necessary condition for the local density approximation to be valid is that the density varies slowly over an interparticle spacing, or \( |\partial \ln n/\partial r| \ll k_F \). From Eq. (11), we thus find the requirement, for \( n(0) \approx n_0 \), that \( 1 - 2k_F|a|/\pi \gg (|a|/d)^2 \) for the local density approximation to be valid, where we have taken \( r^2 \) to be bounded by the critical \( R^2 \) in Eq. (6). This condition clearly breaks down as \( n(0) \to n_0 \), where \( dn/dr \) diverges, as one sees in Fig. 2.

III. NEAR THE UNITARY REGIME

When \( k_F|a| \) becomes large perturbation methods fail, and the question of stability is more involved. In the unitarity regime \( (k_F|a| \to \infty) \), the energy of the ground state is bounded and can be written in the scale-free form \([6,8,9,11]\) \( E_0 = (1 + \beta)E_{FG} \), where \( \beta \) is universal constant \( \sim -0.5 \). To approach the problem, we interpolate the energy between small \( |a| \) and \( -\infty \) by calculating in mean field theory with a density dependent effective scattering length,
\[
a_{\text{eff}} = \frac{a}{1 - \gamma k_F a},
\]
for \( k_F a < 0 \), and replacing the coupling \( g \) in Eq. (11) by the local coupling \( g_{\text{eff}} = 4\pi\hbar^2 a_{\text{eff}}/m \). The choice \( \gamma = -20/9\pi\beta \) reproduces the known energies in the weakly interacting limit and in the unitary regime [21].

Repeating the above Thomas-Fermi calculation with this density-dependent scattering length, we find, first, the position as a function of density,
\[
r^2 = -\left[ k_F^2 - 4\pi |a_{\text{eff}}| n \left( 1 + \frac{\gamma}{6} k_F |a_{\text{eff}}| \right) \right] d^4 + R^2.
\]

The density profile is similar to that of the stable configurations shown in Fig. 2. We show in that figure, as a dashed line, the density profile calculated with Eq. (12) \( a = -0.5 \), and \( \beta = -0.5 \).
The second functional derivative of the energy with respect to $n$ is,

$$\frac{\delta^2 E - \mu \delta^2 N}{(\delta n)^2} = \frac{\pi h^2}{mk_F} (\pi + 2k_F a_{\text{eff}} + \frac{10}{9} \gamma(k_F a_{\text{eff}})^2 \frac{2}{\pi} \gamma^2(k_F a_{\text{eff}})^3).$$

In the limit $\gamma k_F |a| \ll 1$, $a_{\text{eff}} \sim a$, and Eq. (13) predicts that the gas is stable for all $k_F |a|$, in distinction to the prediction of Eq. (8). In the opposite limit, $\gamma k_F |a| \gg 1$, $|a_{\text{eff}}| \rightarrow 9 \pi \beta / 20 k_F$, a constant, and $(\delta^2 E - \mu \delta^2 N) / (\delta n^2) = (\pi^2 h^2) / (mk_F) (1 + \beta / 2)$. Hence the gas is stable for $\beta > -2$, as can be seen in Fig. 4, a plot of $(mk_F / \pi h^2) (\delta^2 E - \mu \delta^2 N) / (\delta n)^2$ as a function of $-1/k_F |a|$ for various $\beta$. Experiments on $^6$Li find $\beta \sim -0.5$; for these values, the second variation is always positive, predicting that the gas should be stable for all $k_F |a|$, in agreement with experiment.

**IV. EVEN HIGHER DENSITY**

In the scale-free unitary regime the interparticle spacing is large compared with the range of the potential, and the system is stable. However, at higher densities when the interparticle spacing becomes comparable to the range of the potential, the system can in fact become unstable, as we see from the following model calculation. Let us assume that the interparticle interaction is an attractive square well potential with range $r_0$ and depth $V(<0)$. The energy calculated assuming a Hartree-Fock wave function is a rigorous upper bound to the exact total energy, $E$; thus

$$\frac{E}{N} \leq \frac{3h^2 k_F^2}{10m} + \frac{2 \pi r_0^3}{3} nV.$$  \hspace{1cm} (14)

As $n$ increases, the energy becomes unbounded below; the kinetic energy does not save the system from collapse. We note that at $V = -h^2 \pi^2 / 4mr_0^2$ the two-body-scattering amplitude has a resonance, the scattering length diverges, and the interaction energy per particle becomes $E_{\text{int}} / N = -(5\pi/54) k_F r_0 E_{FG}$. The instability in this model should not occur, however, for realistic interatomic potentials which have a repulsive core.

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than scattering lengths: $E_{\text{int}} \simeq -4\pi\hbar^2\delta n/km$, where $k \lesssim k_F$ is an averaged relative momentum in the interaction, $\sim k_F$, and $\delta = -\tan^{-1}(ka)$ is the scattering phase shift. Thus in this case, we would write $ka_{\text{eff}} \sim \tan^{-1}(ka)$, which is similar in structure to Eq. (11). This result corresponds to $\gamma = 2k/\pi k_F$. 