Limit varieties generated by finite non-J-trivial
aperiodic monoids

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Abstract

Jackson and Lee proved that certain six-element monoid generates a hered-
itarily finitely based variety \( \mathbb{E}_1 \) whose lattice of subvarieties contains an infinite
ascending chain. We identify syntactic monoids which generate finitely gen-
erated subvarieties of \( \mathbb{E}_1 \) and show that one of these finite monoids together
with certain seven-element monoid generates a new limit variety.

1 Introduction

A variety of algebras is called \textit{finitely based} (abbreviated to FB) if it has a finite basis
of its identities, otherwise, the variety is said to be \textit{non-finitely based} (abbreviated
to NFB). A variety is called \textit{limit} if it is NFB but all its proper subvarieties are FB.

The first two explicit examples of limit monoid varieties \( \mathbb{L} \) and \( \mathbb{M} \) were discovered
by Jackson [11] in 2005. In [14], Lee proved the uniqueness of the limit varieties \( \mathbb{L} \) and \( \mathbb{M} \) in the class of varieties of finitely generated aperiodic monoids with central
idempotents. In [15], Lee generalized the result of [14] and established that \( \mathbb{L} \) and \( \mathbb{M} \) are the only limit varieties within the class of varieties of aperiodic monoids with
central idempotents. In 2013, Zhang found a NBF variety of monoids that contains
neither \( \mathbb{L} \) nor \( \mathbb{M} \) [25] and, therefore, she proved that there exists a limit variety of
monoids that differs from \( \mathbb{L} \) and \( \mathbb{M} \). In [26], Zhang and Luo pointed out an explicit
example of such variety.

The following semigroup was introduced by its multiplication table and shown
to be FB in [17, Section 19]. Its presentation was recently suggested by Edmond
W. H. Lee:

\[
A = \langle a, b, c \mid a^2 = a, b^2 = b, ab = ca = 0, ac = cb = c \rangle = \{a, b, c, ba, bc, 0\}.
\]

If \( S \) is a semigroup, then the monoid obtained by adjoining a new identity element
to \( S \) is denoted by \( S^1 \) and the variety of monoids generated by \( S^1 \) is denoted by
\( S^1 \). If \( \mathbb{V} \) is a monoid variety, then \( \overline{\mathbb{V}} \) denotes the variety \textit{dual to} \( \mathbb{V} \), i.e., the variety
consisting of monoids anti-isomorphic to monoids from \( \mathbb{V} \). The variety \( \mathbb{A}^1 \vee \overline{\mathbb{A}}^1 \) is the
third example of limit variety of monoids [26] mentioned in the previous paragraph.
Figure 1 below contains the lattice of subvarieties of \( \mathbb{A}^1 \) duplicated from Figure 1 in
[26].
The next pair of limit varieties $\mathcal{J}$ and $\mathcal{J}$ was discovered by Gusev [3]. In [4], he proves that $\mathcal{L}$, $\mathcal{M}$, $\mathcal{J}$ and $\mathcal{J}$ are the only limit varieties of aperiodic monoids with commuting idempotents. In [5], Gusev and the author present the last pair $\mathcal{K}$ and $\mathcal{K}$ of limit varieties generated by finite $J$-trivial monoids and show that there are exactly seven limit varieties of $J$-trivial monoids.

Let $E$ be the semigroup given by presentation:

$$E = \langle a, b, c \mid a^2 = ab = 0, ba = ca = a, b^2 = bc = b, c^2 = cb = c \rangle = \{a, b, c, ac, 0\}.$$  

Monoid $E^1$ was first investigated by Lee and Li [16, Section 14], where it was shown to be finitely based by $\{xtx \approx xtx^2 \approx x^2tx, xy^2x \approx x^2y^2\}$.

Elements of a countably infinite alphabet $\mathcal{A}$ are called letters and elements of the free monoid $\mathcal{A}^*$ are called words. We use $1$ to denote the empty word, which is the identity element of $\mathcal{A}^*$. Words unlike letters are written in bold.

Denote:

$$u_0 = a^2b^2, v_0 = b^2a^2;$$

$$u_{k+1} = at_{k+1}u_k, v_{k+1} = at_{k+1}v_k, k = 0, 2, 4, \ldots; \tag{1}$$

$$u_{k+1} = bt_{k+1}u_k, v_{k+1} = bt_{k+1}v_k, k = 1, 3, 5, \ldots.$$  

For example:

$$u_1 = at_1a^2b^2, v_1 = at_1b^2a^2;$$

$$u_2 = bt_2at_1a^2b^2, v_2 = bt_2at_1b^2a^2;$$

$$u_3 = at_3bt_2at_1a^2b^2, v_3 = at_3bt_2at_1b^2a^2;$$

$$\ldots \ldots$$

For each $n \geq 1$, the identity $\sigma_n$ introduced in [12] is equivalent modulo $xtx \approx xtx^2 \approx x^2tx$ to $u_n \approx v_n$. We use var $\Sigma$ to denote the variety defined by a set of identities $\Sigma$. According to Proposition 5.6 in [12], the lattice of subvarieties of $E^1$ contains an infinite ascending chain $E^1\{\sigma_1\} \subset E^1\{\sigma_2\} \subset \ldots$, where for each $n \geq 1$,

$$E^1\{\sigma_n\} = \text{var}\{xtx \approx xtx^2 \approx x^2tx, xy^2x \approx x^2y^2, u_n \approx v_n\}.$$  

A copy of this lattice from Figure 4 in [12] is shown on Figure 1 below. The variety $E^1$ is the first example of a finitely generated monoid variety whose lattice of subvarieties is countably infinite. The second example of a variety with this property is contained in [7].

In Sect. 3, we present a Sufficient Condition under which the variety $A^1 \vee E^1\{\sigma_2\}$ is NFB. Using Sufficient Condition in [5] under which a monoid is FB, we show that every proper subvariety of $A^1 \vee E^1\{\sigma_2\}$ and of $\mathcal{A}^1 \vee E^1\{\sigma_2\}$ is FB. Hence $A^1 \vee E^1\{\sigma_2\}$ and $\mathcal{A}^1 \vee E^1\{\sigma_2\}$ are new limit varieties of monoids.

Let $A_0$ be the semigroup given by presentation:

$$A_0 = \langle a, b \mid a^2 = a, b^2 = b, ab = 0 \rangle = \{a, b, ba, 0\}.$$
The monoid $A_1^0$ was shown to be FB in [2].

It turns out [6], that $E_1 \{ \sigma_2 \} \lor E_1 \{ \sigma_2 \} \lor A_1^0$ is also a limit variety.

The finite monoids which generate limit varieties $L$ and $M$ were introduced by Jackson in 2005 by using the so-called Dilworth-Perkins construction, which assigns a monoid $M(W)$ to a set of words $W$. More precisely, $L = M(\{ abtbsa, atbsba \})$ and $M = M(\{ atbasb \})$ (see [11]). In [20], we generalized Dilworth-Perkins construction into $M_\tau(W)$ construction for monoids and $S_\tau(W)$ construction for semigroups, where $\tau$ is a congruence on $A^*$ (resp. $A^+$). Surprisingly, each of the ten limit varieties mentioned above can be generated by monoids of the form $M_\tau(W)$ where $W$ is an equivalence class of a word $u \in \{ abtbsa, atbsba, atbasb, atb2a, ab2ta, ab \}$ and $\tau$ is some easy-to-define congruence on the free monoid (see Sect. 6).

Another well known construction assigns a syntactic monoid $M_{synt}(W)$ or syntactic semigroup $S_{synt}(W)$ to a set of words $W$. Proposition 2.1 in [10] about syntactic algebras implies that every monoid (semigroup with zero) is equationally equivalent to a syntactic monoid (syntactic semigroup).

The usefulness and simplicity of Dilworth-Perkins construction and the universality of syntactic algebras motivated us to find a connection between monoids of the form $M_\gamma(W)$ (resp. semigroups of the form $S_\gamma(W)$) and the syntactic monoids $M_{synt}(W)$ (resp. syntactic semigroups $S_{synt}(W)$). Theorem 5.3 gives us such a connection and is used in Sect. 5 and Sect. 6 to identify syntactic algebras which generate some 0-simple semigroups, finitely generated subvarieties of $E_1$, subvarieties of $A_1^1$, and the ten limit varieties of monoids mentioned above. By computing syntactic algebras using the generalized Dilworth-Perkins construction we avoid any computations of the syntactic congruence, which tend to be very cumbersome.

Recently, Gusev et al. [7] found two more pairs of limit varieties of monoids. The subvariety lattices of these varieties are much more complex than of the varieties mentioned above. In particular, it is shown in [7] that one of these varieties contains infinitely many infinite ascending chains of subvarieties. It is the first example of a finitely generated variety with this property. It can be deduced from [7] that, in contrast to the ten limit varieties of monoids discussed above, the $M_\tau(W)$ formulas for the varieties in [7] are too bulky to be useful.

2 Congruences $\tau_1, \gamma$ and $\beta$ on the free monoid $A^*$

We say that a set of words $W \subseteq A^*$ is stable with respect to a semigroup variety $V$ if $v \in W$ whenever $u \in W$ and $V$ satisfies $u \approx v$. Recall that a word $u \in A^*$ is an isoterm [18] for $V$ if the set $\{ u \}$ is stable with respect to $V$.

If $\tau$ is an equivalence relation on the free monoid $A^*$ and $V$ is a semigroup variety, then a word $u \in A^*$ is said to be a $\tau$-term for $V$ if $u \tau v$ whenever $V$ satisfies $u \approx v$. Notice that if $W \subseteq A^*$ forms a single $\tau$-class, then $W$ is stable with respect to $V$ if and only if every word in $u \in W$ is a $\tau$-term for $V$.

Let $\tau$ denote the congruence on the free monoid $A^*$ induced by the relations $a = a^2$ for each $a \in A$. 

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A letter is called simple (multiple) in a word $u$ if it occurs in $u$ once (at least twice). The set of all letters in $u$ is denoted by $\text{con}(u)$. Notice that $\text{con}(u) = \text{sim}(u) \cup \text{mul}(u)$ where $\text{sim}(u)$ is the set of all simple letters in $u$ and $\text{mul}(u)$ is the set of all multiple letters in $u$.

Let $\gamma$ be the fully invariant congruence of $\text{var}\{xy \approx yx, x^2 \approx x^3\}$. It is well-known and can be easily verified that this variety is generated by the 3-element monoid $\langle a, 1 \mid a^2 = 0 \rangle$ and that for every $u, v \in \mathfrak{A}^*$, we have $u \gamma v$ if and only if $\text{sim}(u) = \text{sim}(v)$ and $\text{mul}(u) = \text{mul}(v)$.

Given $W \subseteq \mathfrak{A}^*$ we use $W^\leq$ to denote the set of all subwords of words in $W$.

Lemma 2.1. [21] Corollary 3.5] Suppose that $W \subseteq \mathfrak{A}^*$ forms a single $(\tau_1 \land \gamma)$-class (resp. $\tau_1$-class) of $\mathfrak{A}^*$. If $W$ is stable with respect to a monoid variety $V$ then every word in $W^\leq$ is a $(\tau_1 \land \gamma)$-term (resp. $\tau_1$-term) for $V$.

A block of a word $u$ is a maximal subword of $u$ that does not contain any letters simple in $u$.

We use $u_i x$ to refer to the $i$th from the left occurrence of $x$ in a word $u$. We use $\alpha x$ to refer to the last occurrence of $x$ in $u$. If $x$ is simple in $u$ then we use $u x$ to denote the only occurrence of $x$ in $u$. If the $i$th occurrence of $x$ precedes the $j$th occurrence of $y$ in $u$, we write $(u_i x) <_u (u_j y)$.

Let $\beta$ be the fully invariant congruence of $E^1 = \text{var}\{xtx \approx xt^2 \approx x^2tx, xy^2x \approx x^2y^2\}$. It follows from [16 Section 14.3] that for every $u, v \in \mathfrak{A}^*$, we have $u \beta v$ if and only if

- (i) the identity $u \approx v$ is of the form:

$$a_0 \prod_{i=1}^m (t_i a_i) \approx b_0 \prod_{i=1}^m (t_i b_i),$$

where $\text{sim}(u) = \text{sim}(v) = \{t_1, \ldots, t_m\}$ for some $m \geq 0$ and $\text{mul}(u) = \text{con}(a_0 \ldots a_m) = \text{con}(b_0 \ldots b_m) = \text{mul}(v)$;

- (ii) for each $i = 0, 1, \ldots, m$ we have $\text{con}(a_i) = \text{con}(b_i)$;

- (iii) for each $i = 0, 1, \ldots, m$ and for each $x \neq y \in \text{con}(a_i) = \text{con}(b_i)$, we have $(1a_i, x) <_{a_i} (1a_i, y) \iff (1b_i, x) <_{b_i} (1b_i, y)$.

For example, $(x^2y^3xtyx^2) \beta (xtyy^3xy)$. Let $\beta$ denote the congruence dual to $\beta$. We use $\approx_{Q^1}$ to denote the equivalence relation on $\mathfrak{A}^*$ given by the conditions (i) and (ii) only. The proof of Proposition 4.3 in [16] implies that $\approx_{Q^1}$ is the fully invariant congruence of $Q^1$, where $Q^1$ is the monoid obtained by adjoining an identity element to the following semigroup:

$$Q = \langle e, b, c \mid e^2 = e, eb = b, ce = c, ec = be = cb = 0 \rangle.$$

The semigroup $Q$ was introduced and shown to be FB in [1, Section 6.5]. Proposition 4.3 in [16] establishes that $Q^1 = \text{var}\{xtx \approx xt^2 \approx x^2tx, x^2y^2 \approx y^2x^2\}$. 

A word \( u \) is called block-simple if every block of \( u \) involves at most one letter. For example, the word \( x^2t_1x^3t_2y^5t_3z^2t_4t_5y \) is block-simple. It is easy to see that if \( u \in \mathfrak{A}^* \) is block-simple then every word in \([u]_{\tau_1 \land \gamma} = [u]_\beta = [u]_{\beta} \) is block-simple.

We use regular expressions to describe sets of words, in particular the contents of congruence classes. For example, \([atb^2a]_\beta \) consists of all words of the form \( a^n ts is, where \( n \geq 1 \) and \( s \in \{a, b\}^+ \) starts with \( b \), contains \( b \) at least twice and contains \( a \). Using regular expression, we write \([atb^2a]_\beta = a^+tba+b\{a, b\}^+ \lor a^+tbb+a\{a, b\}^* \).

**Fact 2.2.** For a monoid variety \( \mathbb{V} \), the following are equivalent:

(i) the set \([xtx]_{\tau_1 \land \gamma} = [xtx]_\beta = x^+tx^+ \) is stable with respect to \( \mathbb{V} \);
(ii) \( \mathbb{V} \) contains \( Q^1 \);
(iii) every word is \((\sim_{Q^1})\)-term for \( \mathbb{V} \);
(iv) every block-simple word is \((\tau_1 \land \gamma)\)-term for \( \mathbb{V} \);
(v) every block-simple word is \( \beta \)-term for \( \mathbb{V} \).

**Proof.** The equivalence of (i) and (ii) follows immediately from Proposition 2.3 and Theorem 4.3(i) in [21]. Parts (ii) and (iii) are also equivalent because \( \sim_{Q^1} \) is the fully invariant congruence of \( Q^1 \). The equivalence of (iii) and (iv) are easily verified. Parts (iv) and (v) are equivalent because a block-simple word is a \((\tau_1 \land \gamma)\)-term for \( \mathbb{V} \) if and only if it is a \( \beta \)-term for \( \mathbb{V} \).

A word \( w \) is called almost-block-simple if for each \( x \neq y \in \text{mul}(w) \) at most one block in \( w \) involves both \( x \) and \( y \). For example, the word \( xt_1yt_2x^2yzt_3zt_4yp^3t_5xp \) is almost-block-simple. Notice that every block-simple word is almost-block-simple and every word in \([1] \) is almost-block-simple.

**Observation 2.3.** Let \( w \) be an almost block-simple word where \( b \) is the only block which involves letters \( a \neq b \). If \( b \in a^+b\{a, b\}^* \) then \([w]_{\sim_{Q^1}} = [w]_\beta \cup [w']_\beta \) where \( w' \) is obtained from \( w \) by replacing \( b \) by \( b^2a^2 \).

**Lemma 2.4.** Let \( u_1, u_2, u_3, \ldots; v_1, v_2, v_3, \ldots \) be the words defined recursively in \([1] \). If for some \( n \geq 1 \) either \([u_n]_\beta \) or \([v_n]_\beta \) is stable with respect to a monoid variety \( \mathbb{V} \) then every almost-block-simple word with at most \( n \) simple letters is \( \beta \)-term for \( \mathbb{V} \).

**Proof.** First, we establish the following.

**Claim.** Every block-simple word is \( \beta \)-term for \( \mathbb{V} \).

**Proof.** We verify this claim only for \( v_1 = atb^2a^2 \), because the argument for \( u_n \) and \( v_n \) with \( n \geq 1 \) is similar but bulkier. So, we are given that the set \([atb^2a^2]_\beta = [atb^2a]_\beta \) is stable with respect to \( \mathbb{V} \). First, notice that \( t \) must be an isoterm for \( \mathbb{V} \), because otherwise, \( \mathbb{V} \) satisfies \( xty^2x \approx x^kt^y^2x \) for some \( k \geq 2 \), which contradicts the fact that \( xty^2x \) is a \( \beta \)-term for \( \mathbb{V} \). Since an assumption that \( \mathbb{V} \) is commutative also easily leads to a contradiction, the word \( xy \) must be an isoterm for \( \mathbb{V} \).

If the set \( x^+tx^+ \) is not stable with respect to \( \mathbb{V} \), then \( \mathbb{V} \models x^ntxm \approx x^kt \) or \( \mathbb{V} \models x^ntxm \approx tx^k \) for some \( n, m \geq 1, k \geq 2 \). The first identity implies \( x^nty^2xm \approx x^kt^y^2x \) and the second implies \( x^nty^2xm \approx ty^2x^k \). Since the left side of each of these
identities is in \([xty^2x]_\beta\) but the right side is not, the set \(x^+tx^+\) must be stable with respect to \(V\).

If \(x^+tsx^+\) is not stable with respect to \(V\), then \(V \models x^ntsx^n \approx x^ktx^qtx^p\) for some \(n, m, k, q, p \geq 1\). This identity implies \(x^ny^2tx^m \approx x^ktx^qy^2tx^p\), which contradicts the fact that \(x^nty^2x^m\) is a \(\beta\)-term for \(V\). We conclude that the set \(x^+tsx^+\) is stable with respect to \(V\). Consequently, every block-simple word is a \(\beta\)-term for \(V\) by Fact 2.2.

Next we verify the following.

**Claim.** For each \(i = 0, \ldots, n\) both \([u_i]_\beta\) and \([v_i]_\beta\) are stable with respect to \(V\).

**Proof.** Since every block-simple word is \(\beta\)-term for \(V\), Fact 2.2(iii) implies that for each \(i = 0, \ldots, n\), the set

\[ [u_i]_{\sim Q^1} = [v_i]_{\sim Q^1} \]

Observation 2.3 \([u_i]_\beta \cup [v_i]_\beta\)

is stable with respect to \(V\). Since \([u_n]_\beta\) or \([v_n]_\beta\) is stable with respect to \(V\), both \([u_n]_\beta\) and \([v_n]_\beta\) are stable with respect to \(V\). If either \([u_{n-1}]_\beta\) or \([v_{n-1}]_\beta\) is not stable with respect to \(V\), then \(V \models u \approx v\) such that \(u \in [u_{n-1}]_\beta\) but \(v \notin [v_{n-1}]_\beta\). This identity implies \(a_{n-1}u \approx a_{n-1}v\), where \(a_{n-1}u \in [u_{n-1}]_\beta\) but \(a_{n-1}v \in [v_{n-1}]_\beta\). To avoid the contradiction, both \([u_{n-1}]_\beta\) and \([v_{n-1}]_\beta\) must be stable with respect to \(V\). And so on, until we show that both \([u_0]_\beta\) and \([v_0]_\beta\) are stable with respect to \(V\).

In view of Fact 2.2 and the first claim every identity of \(V\) holds on \(Q^1\). Now let \(w\) be an almost-block-simple word with at most \(n\) simple letters. To obtain a contradiction, assume that \(w\) is not a \(\beta\)-term for \(V\). Then \(V\) satisfies an identity \(w \approx w'\) such that \(w \sim_{Q^1} w'\), and for some block \(a\) in \(w\) and \(x \neq y \in \text{mul}(a)\) we have \((1ax) <_a (1ay)\) but \((1by) <_b (1bx)\), where \(b\) is the block in \(w'\) which corresponds to the block \(a\) in \(w\). Let \(\{t_1, \ldots, t_k\} \subseteq \text{sim}(w)\) be the (possibly empty) set of simple letters which appear in \(w\) on the left of block \(a\). Since \(w\) is almost-block-simple, for some \(\Xi \subseteq \{t_1, \ldots, t_k\}\), the variety \(V\) satisfies \(w(x, y, \Xi) \approx w'(x, y, \Xi)\) such that modulo renaming letters \(w(x, y, \Xi) \in [u_m]_\beta\) but \(w'(x, y, \Xi) \in [v_m]_\beta\), where \(m\) is the number of letters in \(\Xi\). Since \(0 \leq m \leq k \leq n\), this contradicts the second claim. To avoid the contraction we conclude that every almost-block-simple word must be \(\beta\)-term for \(V\).

**3 Sufficient condition under which a monoid is NFB**

**Fact 3.1.** [19, Fact 2.1] Suppose that for infinitely many \(n\), a semigroup variety \(V\) satisfies an identity \(U_n \approx V_n\) in at least \(n\) letters such that \(U_n\) has some property \(P_n\) but \(V_n\) does not. Suppose that for every word \(U\) such that \(V \models U \approx U_n\) and \(U\) has property \(P_{n}\), the word \(\Theta(U)\) also has property \(P_n\) for every substitution \(\Theta : \text{A} \rightarrow \text{A}^+\).
and every identity \( u \approx v \) in less than, say, \( n/2 \) letters such that \( \Theta(u) = U \). Then \( V \) is NFB.

If \( U = \Theta(u) \) for some endomorphism \( \Theta \) of \( \mathbb{A}^+ \) and \( u \cdot v \) is an occurrence of a letter \( x \) in \( U \) then \( \Theta^{-1}(u \cdot v) \) denotes an occurrence \( j \cdot z \) of a letter \( z \) in \( u \) such that \( \Theta(j \cdot z) \) regarded as a subword of \( U \) contains \( u \cdot v \).

**Sufficient Condition.** Let \( V \) be a monoid variety that satisfies the identity

\[
U_n = xy_1^2y_2^2 \cdots y_{n-1}^2y_n^2x \approx xy_1^2xy_2^2 \cdots y_{n-1}^2y_n^2x = V_n
\]

for any \( n \geq 1 \). If the sets \([ab^2ta]_{\gamma \cdot \gamma} = a^+bb^+ta^+ \) and \([atb^2a]_\beta \) are stable with respect to \( V \) then \( V \) is NFB.

**Proof.** Consider the following property of a word \( U \) with \( \text{con}(U) = \{x, y_1, \ldots, y_n\} \):

(P): There is no \( x \) in \( U \) between the first occurrence of \( y_1 \) and the first occurrence of \( y_n \).

Notice that \( U_n \) satisfies property (P) but \( V_n \) does not.

Let \( U \) be such that \( V \models U_n \approx U \). Since \([a^2b^2]_{\gamma \cdot \gamma} = a^+bb^+ \) is stable with respect to \( V \) by Lemma [2.1], we have:

\[
(1u^n) < _u (1uy_1) < _u (1uy_2) < _u (1uy_2) < _u (1uy_2) < _u (1uy_{n-1}) < _u (1uy_n) < _u (1ux).
\]

(3)

Let \( u \approx v \) be an identity of \( V \) in less than \( n/2 \) letters and let \( \Theta : \mathbb{A} \to \mathbb{A}^+ \) be a substitution such that \( \Theta(u) = U \). In view of (3), the following holds:

(*) If \( \Theta(t)v \) contains both \( y_i \) and \( y_j \) for some \( 1 \leq i < j \leq n \) then letter \( t \) is simple in \( u \).

Suppose that \( U \) has Property (P). Then in view of (3), every subword \( A \) of \( U \) has the following property:

(**) if \( A \) contains \( xy_i \) then \( i = 1 \); if \( A \) contains \( y_jx \) then \( j = n \).

Let us verify that \( V = \Theta(v) \) also has Property (P). To obtain a contradiction, assume that there is an occurrence of \( x \) in \( V \) such that

\[
(1vy_1) < _v (1vy_n) < _v (1vy_n).
\]

Consider two cases.

**Case 1:** There is an occurrence of \( x \) in \( V \) such that

\[
(1vy_{n/2}) < _v (1vy_n) < _v (1vy_n).
\]

Since \( u \) has less than \( n/2 \) letters, for some \( t \in \text{con}(u) \) the word \( \Theta(t)v \) contains both \( y_i \) and \( y_j \) for some \( 1 \leq i < j \leq n/2 \). In view of (*), the letter \( t \) is simple in \( u \). Since \( t \) is an isoterm for \( V \) by Lemma [2.1], the letter \( t \) is simple in \( v \) as well. \( \Theta^{-1}(1vy_n) = pvz \) is an occurrence of some letter \( z \) in \( v \) such that \( \Theta(z) \) contains \( x \). Since the empty word \( 1 \) is an isoterm for \( V \) by Lemma [2.1], the letter \( z \) occurs in \( u \) as well.
Every monoid variety \( y, y \) satisfy Property (P). Therefore, the variety \( V \) contains either \( y \) contains an occurrence of \( \Theta(y) \) are subwords of \( U \), this is impossible by Property (**). Therefore, \( t \neq z \) and \( y \neq z \).

Since \( U \) has Property (P), no \( z \) occurs between \( t \) and \( 1u'y \). Hence \( u(z, y, t) \in z^\ast t^\ast \) if \( t = y \) and \( u(z, y, t) \in z^\ast y \{ y, z \}^\ast \) if \( t \neq y \). On the other hand, in view of (4), we have \( (vt) < \text{v} (pvz) < \text{v} (1vy) \). If \( t = y \) this is impossible, because \( t \) is simple in \( v \). If \( t \neq y \) then \( v(z, y, t) \in z^\ast t^\ast \{ y, z \}^\ast \). This is impossible, because \( u(z, y, t) \) is \( \beta \)-term for \( V \) by Lemma 2.4.

Case 2: There is an occurrence of \( x \) in \( V \) such that

\[
(1vyt) < \text{v} (1vx) < \text{v} (1vn/2).
\] (5)

Since \( u \) has less than \( n/2 \) letters, for some \( t \in \text{con}(u) \) the word \( \Theta(t) \) contains both \( y_i \) and \( y_j \) for some \( n/2 \leq i < j \leq n \). In view of (**), the letter \( t \) is simple in \( u \). Since \( t \) is an isoterms for \( V \), \( t \) is simple in \( v \) as well. \( \Theta^{-1}(tvx) = pvz \) is an occurrence of some letter \( z \) in \( u \) such that \( \Theta(z) \) contains \( x \). Since the empty word \( 1 \) is an isoterms for \( V \) by Lemma 2.4 the letter \( z \) occurs in \( u \) as well.

In view of Fact 2.6 in [19], \( \Theta^{-1}(1uyt) = 1uyt \) and \( \Theta^{-1}(1vyu) = 1vyt \) for some \( y, y' \in \text{con}(u) = \text{con}(v) \). If \( y \neq y' \) then \( (1uyt) < \text{u} (1uyt) \) but \( (1vyt) < \text{v} (1vyt) \). This is impossible, because \( u(y, y') \) is \( \beta \)-term for \( V \) by Lemma 2.4. Thus \( y = y' \).

If \( t = z \) (resp. \( y = z \)) then in view of (5), \( \Theta(t) = \Theta(z) \) (resp. \( \Theta(y) = \Theta(z) \)) contains either \( xy_i \) for some \( 1 < i \leq n \) or \( y_jx \) for some \( 1 \leq j < n \). Since both \( \Theta(t) \) and \( \Theta(y) \) are subwords of \( U \), this is impossible by Property (**). Therefore, \( t \neq z \) and \( y \neq z \).

Subcase 2.1: \((1uyt) < \text{u} (1uz)\).

In this case, \((1vyt) < \text{v} (1vz)\) because \( u(z, y) \) is \( \beta \)-term for \( V \) by Lemma 2.4. Since \( U \) has Property (P), \( \text{v} (1vt) < \text{u} (1uz) \). But in view of (5), we have \( (1vz) < \text{v} (1vt) \). This is impossible, because \( u(z, t) \) is \( \beta \)-term for \( V \) by Lemma 2.4.

Subcase 2.2: \((1uz) < \text{u} (1vyt)\).

In this case, \((1vyt) < \text{v} (1vz)\) because \( u(z, y) \) is \( \beta \)-term for \( V \) by Lemma 2.4. Since \( U \) has Property (P), we have \( u(z, y, t) \in z^\ast t^\ast \) if \( t = y \) and \( u(z, y, t) \in z^\ast y^\ast t^\ast \) if \( t \neq y \). But in view of (5), the word \( v \) contains an occurrence of \( z \) between \( 1vyt \) and \( t \). If \( t = y \) this is impossible, because \( t \) is simple in \( v \). If \( t \neq y \), this contradicts the fact that \([zy^2t^2]_{y^2t} \land \gamma \) and \([z^2y^2t]_{y^2t} \land \gamma \) are stable with respect to \( V \).

Since we obtain a contradiction in every case we conclude that \( V \) must also satisfy Property (P). Therefore, the variety \( V \) is NFB. By Fact 3.1. □

**Corollary 3.2.** Every monoid variety \( V \) that contains \( A^1 \lor E^1 \{ \sigma_2 \} \) and is contained in \( A^1 \lor E^1 \) is NFB.
Proof. The fact that $A^1$ satisfies (2) is verified in [20]. The variety $E^1$ satisfies (2) because $U_n \beta V_n$ where $\beta$ is the fully invariant congruence of $E^1$. Therefore, $A^1 \lor E^1$ satisfies (2).

Theorem 4.3(iii) in [21] implies that $[ab^2ta]_{\tau_n \wedge \gamma} = a^+bb^+ta^+$ is stable with respect to $A^1$ and can be used to recheck that $A^1$ satisfies (2). The fact that $[atb^2a]_{\beta}$ is stable with respect to $E\{\sigma_2\}$ is, in essence, verified in the proof of Lemma 5.7 in [12]. Hence $V$ is NFB by the Sufficient Condition.

4 New pair of limit varieties of aperiodic monoids

Given a congruence $\tau$ on the free monoid $A^*$, we use $\circ$ to denote the binary operation on the quotient monoid $A^*/\tau$. We refer to the elements of $A^*/\tau$ as $\tau$-classes. The subword relation $\leq$ on $A^*$ can be naturally extended to $\tau$-classes as follows. Given two $\tau$-classes $u, v \in A^*/\tau$ we write $v \leq_\tau u$ if $u = p o_\tau v o_\tau s$ for some $p, s \in A^*/\tau$.

Let $\tau$ be a congruence on the free monoid $A^*$ and $W \subseteq A^*$ be a union of $\tau$-classes. If $A^* = W \leq$ then we define $M_\tau(W) = M_\tau(\bar{w}) = A^*/\tau$, where $\bar{w}$ is the set of all $\tau$-classes formed by words in $W$. If $A^* \setminus W \leq$ is not empty then it is a union of $\tau$-classes containing all words which are not subwords of any word in $W$. In this case we define $M_\tau(W) = M_\tau(\bar{w})$ as the Rees quotient of $A^*/\tau$ over the ideal $(A^*/\tau) \setminus W^{\leq_\tau}$, where $W^{\leq_\tau}$ is the closure of $W$ in quasi-order $\leq_\tau$.

Here is the connection between monoids of the form $M_\tau(W)$ and $\tau$-terms for monoid varieties.

Proposition 4.1. [21, Proposition 2.3] Let $\tau$ be a congruence on the free monoid $A^*$ such that the empty word $1$ forms a singleton $\tau$-class. Let $W \subseteq A^*$ be a set of words which is a union of $\tau$-classes. Let $W \subseteq A^*/\tau$ denote the set of all $\tau$-classes contained in $W$. Then for every monoid variety $\mathbb{V}$ the following are equivalent:

(i) $\mathbb{V}$ contains $M_\tau(W) = M_\tau(\bar{w})$;
(ii) every word in $W \leq$ is $\tau$-term for $\mathbb{V}$;
(iii) every $\tau$-class in $W^{\leq_\tau}$ is stable with respect to $\mathbb{V}$.

If $\tau$ is the trivial congruence on $A^*$ then we simply write $M(W)$ instead of $M_\tau(W)$. Proposition 4.1 generalizes Lemma 3.3 in [11] which gives a connection between monoids of the form $M(W)$ and $\tau$-terms for monoid varieties.

For each set of words $W$ we use $M_\tau(W)$ to denote the monoid variety generated by $M_\tau(W)$.

Example 4.2. For each $n \geq 1$,

(i) $E^1\{\sigma_{n+1}\} = M_\beta([v_n]_\beta)$, in particular, $E^1\{\sigma_2\} = M_\beta([atb^2a]_\beta)$;
(ii) a monoid variety $\mathbb{V}$ contains $E^1\{\sigma_{n+1}\}$ if and only if $\beta$-class $[v_n]_\beta$ is stable with respect to $\mathbb{V}$.

Proof. (i) The argument used in the proof of Lemma 5.7 in [12], in essence, establishes that the set $[v_n]_\beta$ is stable with respect to $E^1\{\sigma_{n+1}\}$ for each $n \geq 1$. Since
every word in (|v_n|)[β] is β-term for E^1{σ_{n+1}} by Lemma 2.4, we have E^1{σ_{n+1}} ⊇ M_β(|v_n|)[β] by Proposition 4.1.

According to Proposition 5.6 in [12], for each n ≥ 1, the variety E^1{σ_n} is a unique maximal subvariety of E^1{σ_{n+1}}. Since E^1{σ_n} satisfies u_n ≈ v_n, the word v_n is not β-term for E^1{σ_n}. In view of Proposition 4.1, the variety E^1{σ_n} does not contain M_β(|v_n|)[β]. Therefore, E^1{σ_{n+1}} = M_β(|v_n|)[β].

(ii) If [v_n] is stable with respect to V, then every word in (|v_n|)[β] is β-term for V by Lemma 2.4. Hence V contains M_β(|v_n|)[β] by Proposition 4.1. Consequently, V contains E^1{σ_{n+1}} = M_β(|v_n|)[β] by Part (i). Conversely, if V contains E^1{σ_{n+1}} = M_β(|v_n|)[β] then [v_n] is stable with respect to V by Proposition 4.1. □

**Fact 4.3.** [4] Corollary 3.6] A monoid variety is FB whenever it satisfies one of the following:

(i) \{xtx ≈ x^2tx, xy^2tx ≈ xy^2tx\};
(ii) \{xtx ≈ x^2tx, xy^2x ≈ xty^2x\}.

**Lemma 4.4.** Let V be a monoid variety that satisfies xtx ≈ x^2tx ≈ xtx^2 and contains neither E^1{σ_2} nor \(\bigwedge\). Then V is FB.

**Proof.** Consider two cases.

**Case 1:** The set \(x^+\) is not stable with respect to V.

If \(xy\) is not an isoterm for V then V is either commutative or idempotent (see Lemma 2.6 in [5], for instance) and consequently, is FB \[8, 24\]. If xy is an isoterm for V then \(V ≡ x^2tx \lor V ≡ xtx^2\). Consequently, V satisfies either \(xy^2tx ≈ xy^2tx\) or \(xy^2x ≈ xty^2x\) and is FB by Fact 4.3.

**Case 2:** The set \(x^+\) is stable with respect to V.

Since V does not contain E^1{σ_2}, the β-class [atb^2a] is not stable with respect to V by Example 4.2. This means that \(V ≡ u ≈ v\) such that \(u \in [atb^2a] \land v \notin [atb^2a]\). Since \(V ≡ (yx)^2 ≈ (yx)^3\), we may assume that

\[u \in \{xy^2x, xtyxy, xt(yx)^2, xt(yx)^2y\};\]
\[v \in \{xy^2x, xty^2x, xt(xy)^2, xt(xy)^2y\}.\]

Since V does not contain \(\bigwedge\), the dual of Lemma 4.5 in [5] implies that \(V ≡ xty^2x ≈ xt(yx)^2\). Hence we may assume that

\[u \in \{xy^2x, xtyxy\}.\]

Since \(xty^2x ≈ xt(yx)^2\) implies \(xy^2x ≈ x(yx)^2\), we may assume that

\[v \in \{xty^2x, xty^2x, xt(yx)^2\}.\]

Then \(u ≈ v\) together with \(xtx ≈ x^2tx ≈ xtx^2\) implies one of the following identities:

**Subcase 2.1:** \(xty^2x ≈ xtxy^2x \lor xty^2x ≈ xt(yx)^2x.\)
Notice that \(xty^2x ≈ xt(yx)^2x\) together with \(xy^2x ≈ x(yx)^2\) implies \(xty^2x ≈ xt(yx)^2x.\) Therefore, V is FB by Fact 4.3(ii).

**Subcase 2.2:** \(xtxyy ≈ xty^2x \lor xtyxy ≈ xt(yx)^2.\)
Each of these identities together with \(xy^2x ≈ x(yx)^2\) implies \(xt(yx)^2 ≈ xty^2x.\) Therefore, V is FB by Fact 4.3(ii). □
Theorem 4.5. \(A^1 \lor E^1\{\sigma_2\}\) and \(\overline{A^1} \lor \overline{E^1}\{\sigma_2\}\) are new limit varieties of monoids.

Proof. The variety \(A^1 \lor E^1\{\sigma_2\}\) is NFB by Corollary 3.2.

According to [21, Theorem 4.3(iii)] we have \(A^1 = M_{\tau_1 \wedge \gamma}(\{ab^2ta\}_{\tau_1 \wedge \gamma})\) and \(\overline{A^1} = M_{\tau_1 \wedge \gamma}(\{atb^2a\}_{\tau_1 \wedge \gamma})\). Therefore, the following identity holds on \(A^1\) but fails on \(\overline{A^1}\):

\[
x t y s x y \approx x t y s y x.
\]

Since \(E^1\{\sigma_2\}\) satisfies (6) by the definition, the variety \(A^1 \lor E^1\{\sigma_2\}\) satisfies (6).

Let \(V\) be a proper subvariety of \(A^1 \lor E^1\{\sigma_2\}\). If \(V\) does not contain \(E^1\{\sigma_2\}\) then \(V\) is FB by Lemma 4.4. So, we may assume that \(V\) does not contain \(A^1\). Then

\[
V \models xy^2tx \overset{[5, Lemma 4.5]}{\approx} xxytx \approx xyyxtx.
\]

Hence \(V\) is FB by Fact 4.3(i).

The variety \(A^1 \lor E^1\{\sigma_2\}\) is also limit by dual arguments. \(\square\)

5 Syntactic monoids \(M_{\text{synt}}(W)\) (semigroups \(S_{\text{synt}}(W)\)) and monoids of the form \(M_{\tau}(W)\) (semigroups of the form \(S_{\tau}(W)\))

Recall that given a set of words (language) \(W \subseteq \mathfrak{A}^+\), the syntactic congruence or Myhill congruence \(\sim_W\) on the free monoid \(\mathfrak{A}^*\) (resp. free semigroup \(\mathfrak{A}^+\)) is defined by \(u \sim_W v\) if and only if for any \(p, s \in \mathfrak{A}^*\) we have \(pus \in W \iff pvs \in W\). It is well-known and can be easily verified that the syntactic congruence \(\sim_W\) is the largest congruence on \(\mathfrak{A}^*\) (resp. \(\mathfrak{A}^+\)) for which \(W\) is a union of congruence classes.

The quotient \(\mathfrak{A}^*/ \sim_W\) (\(\mathfrak{A}^+\, / \sim_W\)) is called the syntactic monoid (resp. syntactic semigroup) of \(W\) and denoted by \(M_{\text{synt}}(W)\) (resp. \(S_{\text{synt}}(W)\)) (see [1], for instance).

Let \(\tau\) be a congruence on the free semigroup \(\mathfrak{A}^+\) and \(W \subseteq \mathfrak{A}^+\) be a union of \(\tau\)-classes. We extend \(\tau\) to the free monoid \(\mathfrak{A}^*\) by adding \(\{(1,1)\}\) to it and define \(S_{\tau}(W) = M_{\tau}(W) \setminus \{1\}\). The following proposition is similar to Proposition 4.1 above.

Proposition 5.1. Let \(\tau\) be a congruence on the free semigroup \(\mathfrak{A}^+\) and \(W \subseteq \mathfrak{A}^+\) be a set of words which is a union of \(\tau\)-classes. Let \(\mathfrak{W} \subseteq \mathfrak{A}^+/\tau\) denote the set of all \(\tau\)-classes contained in \(W\). Then for every semigroup variety \(\mathfrak{V}\) the following are equivalent:

(i) \(\mathfrak{V}\) contains \(S_{\tau}(W) = S_{\tau}(\mathfrak{W})\);

(ii) every word in \(W^\leq\) is a \(\tau\)-term for \(\mathfrak{V}\);

(iii) every \(\tau\)-class in \(\mathfrak{W}^\leq\) is stable with respect to \(\mathfrak{V}\).

Proof. The equivalence of (i) and (ii) follows Lemma 7.1 in [20] and its proof. The equivalence of (ii) and (iii) follows from Lemma 2.1 in [21]. \(\square\)
Observation 5.2. (i) For any congruence \( \tau \) on \( \mathfrak{A}^* \) (resp. on \( \mathfrak{A}^+ \)) and for any set of words \( W \subseteq \mathfrak{A}^+ \) such that \( W \) is a union of \( \tau \)-classes, the syntactic monoid \( M_{\text{syt}}(W) \) (resp. semigroup \( S_{\text{syt}}(W) \)) is a homomorphic image of \( M_\tau(W) \) (resp. \( S_\tau(W) \)).

(ii) \( \exists \) The monoids \( M(\{w\}) \) (resp. semigroups \( S(\{w\}) \)) and \( M_{\text{syt}}(\{w\}) \) (resp. \( S_{\text{syt}}(\{w\}) \)) are isomorphic.

Proof. (i) Since the syntactic congruence \( \sim_W \) is larger than \( \tau \), the monoid \( M_{\text{syt}}(W) = \mathfrak{A}^*/\sim_W \) (resp. semigroup \( S_{\text{syt}}(W) = \mathfrak{A}^+/\sim_W \)) is a homomorphic image of \( \mathfrak{A}^*/\tau \) (resp. \( \mathfrak{A}^+/\tau \)). The rest follows from the fact that if \( \mathfrak{A}^* \setminus W \leq \) is not empty then it forms a single \( \sim_W \)-class.

(ii) If \( W \) consists of a single word \( w \in \mathfrak{A}^+ \) then the syntactic congruence \( \sim_W \) is diagonal on \( W \leq \).

\[ \square \]

Given a set of words \( W \) we use \( M_{\text{syt}}(W) \) (resp. \( S_{\text{syt}}(W) \)) to denote the monoid (resp. semigroup) variety generated by \( M_{\text{syt}}(W) \) (resp. \( S_{\text{syt}}(W) \)). Let \( S_\tau(W) \) denote the semigroup variety generated by \( S_\tau(W) \).

Theorem 5.3. Let \( \tau \) be a congruence on the free semigroup \( \mathfrak{A}^+ \). Let \( W \subseteq \mathfrak{A}^+ \) be a set of words such that:

(i) \( W \) forms a single \( \tau \)-class, that is, \( W = [w]_\tau \) for some \( w \in \mathfrak{A}^+ \);

(ii) if \( W \) is stable with respect to a monoid (resp. semigroup) variety \( V \), then every word in \( W \leq \) is \( \tau \)-term for \( V \).

Then \( M_\tau(W) = M_{\text{syt}}(W) \) (resp. \( S_\tau(W) = S_{\text{syt}}(W) \)).

Proof. If \( M_{\text{syt}}(W) \) (resp. \( S_{\text{syt}}(W) \)) satisfies an identity \( u \approx v \) then \( u \sim_W v \). Hence \( v \in W \) whenever \( u \in W \). Since \( W \) is a \( \tau \)-class we have \( uv \sim_W v \) whenever \( u \in W \). Consequently, every word in \( W \) is a \( \tau \)-term for \( M_{\text{syt}}(W) \) (resp. \( S_{\text{syt}}(W) \)).

This implies that \( W \) is stable with respect to the monoid variety \( M_{\text{syt}}(W) \) (resp. semigroup variety \( S_{\text{syt}}(W) \)). Then every word in \( W \leq \) is a \( \tau \)-term for \( M_{\text{syt}}(W) \) (resp. \( S_{\text{syt}}(W) \)) by our assumption. Using Proposition 4.1 (resp. Proposition 5.1) we obtain \( M_\tau(W) \leq M_{\text{syt}}(W) \) (\( S_\tau(W) \leq S_{\text{syt}}(W) \)). In view of Observation 5.2 we have \( M_\tau(W) = M_{\text{syt}}(W) \) (\( S_\tau(W) = S_{\text{syt}}(W) \)).

\[ \square \]

Let \( \sim_2 \) be the 2-testable congruence on \( \mathfrak{A}^* \) defined by

- \( u \sim_2 v \) if and only if \( u \) and \( v \) begin and end with the same letter(s) and share the same set of subwords of length two.

Let \( A_2 \) and \( B_2 \) be two 0-simple semigroups given by presentations:

\[ A_2 = \langle a, b \mid a^2 = aba = a, b^2 = 0, bab = b \rangle = \{a, b, ab, ba, 0\}; \]

\[ B_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle = \{a, b, ab, ba, 0\}. \]

Let \( W_A \) be the set of all words in \( \{x, y\}^+ \) which do not contain \( y^2 \) and \( W_B \) be the set of all words in \( \{x, y\}^+ \) which contain neither \( x^2 \) nor \( y^2 \).

Example 5.4. Let \( B_2 \) and \( A_2 \) denote the varieties generated by semigroups \( B_2 \) and \( A_2 \) respectively. Then

(i) \( B_2 = S_{\sim_2}(W_B) = S_{\sim_2}([xy]_{\sim_2}) = S_{\text{syt}}((x(yx)^+)\sim_2); \]

(ii) \( A_2 = S_{\sim_2}(W_A) = S_{\text{syt}}([xyx^2]_{\sim_2}). \)
Proof. (i) It is easy to check that every word in \( W_B \) is a \( \sim_2 \)-term for \( \mathbb{B}_2 \). Hence \( \mathbb{B}_2 \) contains \( S_{\sim_2}(W_B) \) by Proposition 5.1. On the other hand, \( S_{\sim_2}(W_B) \) contains the following subsemigroup isomorphic to \( B_2 \):

\[
\{a = [xyx]_{\sim_2}, b = [yxy]_{\sim_2}, ab = [xyxy]_{\sim_2}, ba = [yxyx]_{\sim_2}, 0 \}.
\]

Therefore, \( \mathbb{B}_2 = S_{\sim_2}(W_B) \). It is easy to see that \( W_B \leq W_B \cup \{1\} = ([xyx]_{\sim_2})_\leq \). Thus the semigroups \( S_{\sim_2}(W_B) \) and \( S_{\sim_2}([xyx]_{\sim_2}) \) coincide by the definition.

To verify Condition (ii) in Theorem 5.3, suppose that the set \( [xyx]_{\sim_2} = x(xy)^+ \) is stable with respect to a semigroup variety \( \mathbb{V} \). Take some \( v \in W_B \) and suppose that \( \mathbb{V} \models v \approx w \) for some \( w \). If \( w \) contains either \( x^2 \) or \( y^2 \) then multiplying the identity \( v \approx w \) by \( x \) on the left or (and) on the right if necessary, we obtain that \( \mathbb{V} \models u \approx w' \) such that \( u \in x(xy)^+ \) but \( w' \not\in x(xy)^+ \). Since this contradicts the fact that the set \( x(xy)^+ \) is stable with respect to \( \mathbb{V} \), we must assume that \( w \) contains neither \( x^2 \) nor \( y^2 \). In a similar way, one can argue that \( v \) and \( w \) must begin and end with the same letters. Hence every word in \( W_B \) must be a \( \sim_2 \)-term for \( \mathbb{V} \). Consequently, \( S_{\sim_2}([xyx]_{\sim_2}) = S_{synt}([xyx]_{\sim_2}) \) by Theorem 5.3.

(ii)

\[ A_2 \cong S_{\sim_2}(\mathbb{A}^+) \supseteq S_{\sim_2}(W_A) \cong S_{\sim_2}([xyx^2]_{\sim_2}) \text{ Theorem 5.3 } S_{synt}([xyx^2]_{\sim_2}). \]

By the result of Trahtman [22, 23], the relation \( \sim_2 \) is the fully invariant congruence of \( A_2 \). Hence \( S_{\sim_2}(\mathbb{A}^+) = \mathbb{A}^+/\sim_2 \) is the relatively free semigroup in \( A_2 \). This establishes the equality \( \cong \) above. It is easy to see that \( W_A \leq W_A \cup \{1\} = ([xyx^2]_{\sim_2})_\leq \). Thus the semigroups \( S_{\sim_2}(W_A) \) and \( S_{\sim_2}([xyx^2]_{\sim_2}) \) coincide by the definition. This establishes the equality \( \cong \) above. Notice that the following 5-elements subset of \( S_{\sim_2}(W_A) \) forms a subsemigroup of \( S_{\sim_2}(W_A) \) isomorphic to \( A_2 \):

\[
\{a = [xyx^2]_{\sim_2}, b = [yxyx]_{\sim_2}, ab = [x^2yxy]_{\sim_2}, ba = [yxyx^2]_{\sim_2}, 0 \}.
\]

Hence we have \( S_{\sim_2}(W_A) \supseteq A_2 \) and consequently, \( A_2 = S_{\sim_2}(W_A) \). Similar arguments as in the proof of Part (i) show that the semigroups \( S_{\sim_2}([xyx^2]_{\sim_2}) \) and \( S_{synt}([xyx^2]_{\sim_2}) \) are equationally equivalent by Theorem 5.3. \( \square \)

Example 5.5. Let \( v_1, v_2, v_3, \ldots \) be the words defined recursively in (1). Then for each \( n \geq 1 \),

\[ \mathbb{B} \{\sigma_{n+1}\} \cong \mathbb{M}_{\beta}(\langle v_n \rangle_{\beta}) = \mathbb{M}_{synt}(\langle v_n \rangle_{\beta}). \]

Proof. \( \mathbb{M}_{\beta}(\langle v_n \rangle_{\beta}) = \mathbb{M}_{synt}(\langle v_n \rangle_{\beta}) \) by Lemma 2.4 and Theorem 5.3. \( \square \)

Let \( L_2 \) denote the left-zero semigroup of order two:

\[ L_2 = \langle e, f \mid e^2 = ef = e, f^2 = fe = f \rangle = \{e, f\}. \]

Let \( B_0 \) be the semigroup given by presentation:

\[ B_0 = \langle e, f, c \mid c^2 = e, f^2 = f, ef = fe = 0, ec = cf = c \rangle, \]
The five-element monoid $B^1_0$ was introduced and shown to be FB in [2].

In contrast with Example 5.5 the next example together with Figure 4 in [12] implies that for $v_0 = b^2a^2$, the three varieties $E^1\{\sigma_1\}$, $M_\beta([v_0]_\beta)$ and $M_{\text{synt}}([v_0]_\beta)$ are pairwise distinct (see Figure 1).

Example 5.6.

\begin{itemize}
  \item[(i)] $L^1_2 \lor M(x) = M_{\text{synt}}([a^2b^2]_\beta)$.
  \item[(ii)] $L^1_2 \lor B_0^1 = M_\beta([a^2b^2]_\beta)$.
\end{itemize}

Proof. Let $\alpha$ denote the fully invariant congruence of $L^1_2 = \text{var}\{xy \approx xyx\}$. It is well known and easily verified that for any $u, v \in \mathfrak{X}$ we have:

$u \alpha v$ if and only if $\text{con}(u) = \text{con}(v)$ and $(1_u x) <_u (1_u y) \Leftrightarrow (1_v y) <_v (1_v x)$ for any $x, y \in \text{con}(u)$. 

Figure 1: Subvariety lattices of $E^1$ (cf. Fig. 4 in [12]) and of $A^1$ (cf. Fig. 1 in [26]).
example, if $V$ is a join of two varieties then $E$ is a subvariety of the form $E_{xytxy}$ with $M$ to the lattice of subvarieties of $V$. Therefore, $M_a \land (a^2 b^2) = M_{synt}(a^2 b^2)$ by Theorem 5.3. Consequently, $M_a \land (a, b) = M_{synt}(a^2 b^2)$.

Since $\alpha$ is the fully invariant congruence of $L_2^1$ and $\gamma$ is the fully invariant congruence of $M(x)$, their intersection $(\alpha \land \gamma)$ is the fully invariant congruence of $L_2^1 \lor M(x)$ and $M_a \land (a, b) = M_{synt}(a^2 b^2)$.

Because $L_2^1$ is isomorphic to the submonoid $\{1, [a^2 b^2], [b^2 a^2], [a^2 b^2] \beta \}$ of $M_a \land (a, b)$ and $x$ is an isoterm for $M_a \land (a, b)$. Hence $L_2^1 \lor M(x) = M_{synt}(a^2 b^2)$. Overall we have $L_2^1 \lor M(x) = M_{synt}(a^2 b^2)$. 

(ii) According to Figure 4 in [12], the variety $L_2^1 \lor B_1$ contains $L_2^1 \lor M(x)$. Hence every word in $[a^2 b^2] \land \beta = [a^2 b^2] \beta$ is an $(\alpha \land \gamma)$-term for $L_2^1 \lor B_1$ by Part (i) and Proposition 4.1. Consequently, every word in $[a^2 b^2] \land \beta = [a^2 b^2] \beta$ is a $\beta$-term for $L_2^1 \lor B_1$.

According to Figure 4 in [12], the variety $L_2^1 \lor B_1$ has two dual covers $L_2^1 \lor M(x)$ and $B_1$. Since $L_2^1 \lor M(x) = xtx \approx x^2t$ and $B_1 = x^2y^2 \approx y^2x^2$, Proposition 4.1 implies that neither $L_2^1 \lor M(x)$ nor $B_1$ contains $M_{\beta}(a^2 b^2)$. Therefore, $L_2^1 \lor B_1 = M_{\beta}(a^2 b^2)$.

Figure 1 combines Figure 4 in [12] and Figure 1 in [21]. Figure 1 in [21] duplicates the lattice of subvarieties of $A^1$ from [20], where each variety $V$ is labeled by a monoid of the form $M_{(\sigma, \gamma)}(W)$ which generates $V$. When $W$ is a single $(\sigma, \gamma)$-class, we use Theorem 5.3 together with Lemma 2.1 to replace the $M_{(\sigma, \gamma)}(W)$ generator of $V$ by the corresponding syntactic monoid $M_{synt}(W)$. If a variety $V$ on Figure 1 is a join of two varieties then $V$ can be generated by two syntactic monoids. For example, if $V$ is a non-J-trivial subvariety of $E^1 \{\sigma_1\}$ then $V$ is the join of $L_2^1$ with a subvariety of $A^1$. It is easy to see that the three-element monoid $L_2^1$ is isomorphic to $M_{synt}(a\{a, b\})$, where $a\{a, b\}$ is the set of all words in $\{a, b\}$ which begin with $a$.

For each $n \geq 1$ we have $E^1 \{\sigma_{n+1}\} = M_{synt}(\{\nu_n\})$ by Example 5.5. Since $\beta$ is the fully invariant congruence of $E^1$, every word is $\beta$-term for $E^1$. In particular, every word in $([abtab])^\subseteq \beta$ is $\beta$-term for $E^1$. Hence $E^1$ contains $M_{\beta}(abtab)$ by Proposition 4.1. Since $E^1 \{\sigma_{\infty}\}$ is a unique maximal subvariety of $E^1$ which satisfies $xtyxy \approx xtytx$ [12], we have $E^1 = M_{\beta}(abtab)$. Now suppose that $abtab$ is stable with respect to a monoid variety $V$. Since every word in $([abtab])^\subseteq \beta$ is either almost-block-simple or belongs to $[abtab]$, every word in $([abtab])^\subseteq \beta$ is $\beta$-term for $V$ by Lemma 2.1. Consequently, $E^1 = M_{\beta}(abtab) = M_{synt}(abtab)$ by Theorem 5.3.
Overall, using Theorem 5.3, for every variety on Figure 1 other than $E^1 \{\sigma_\infty\} = E \{xytxy \approx xytxy\}$, one can readily identify one or two syntactic monoids which generate it. In view of Proposition 5.11 in [12], $E^1 \{\sigma_\infty\}$ is the only non-finitely generated variety on Figure 1. It is easy to verify that $E^1 \{\sigma_\infty\} = M_{\beta}(W_\infty)$, where $W_\infty$ is the set of all almost block-simple words.

6 Syntactic monoids generating limit varieties

Let $\zeta$ be the fully invariant congruence of $\text{var}\{xtxs \approx xttsx\}$. Given $w \in A^*$ we use $\text{ini}_2(w)$ to denote the word obtained by retaining the first two occurrences of each letter in $w$. It is easy to see that for every $u, v \in A^*$ we have:

$u \zeta v \iff \text{ini}_2(u) = \text{ini}_2(v)$.

Let $\overline{\zeta}$ denote the congruence dual to $\zeta$.

The goal of this section is to show that the ten limit varieties mentioned in the introduction can be assembled from a single ‘lego set’, which contains ten types of pieces: six words $\{abtbsa, atbsba, atbasb, atb^2a, ab^2ta, ab\}$ and four congruences $\{\tau_1, \gamma, \beta, \zeta\}$. Notice that each of the four congruences $\{\tau_1, \gamma, \beta, \zeta\}$ is defined by a simple formula. We collect the formulas for the syntactic monoids which generate these varieties in Table 1.

The first two rows of Table 1 contain syntactic monoids which generate limit varieties $L$ and $M$ from [11]. As we mentioned in the introduction, $L$ is generated by $M(\{abtbsa, atbsba\})$ and $M$ is generated by $M(\{atbasb\})$. Observation 5.2 implies that $M(\{atbasb\})$ is isomorphic to $M_{\text{synt}}(\{atbasb\})$. In view of Lemma 5.1 in [13], we have $L = M_{\text{synt}}(\{abtbsa\}) \lor M_{\text{synt}}(\{atbsba\})$.

**Fact 6.1.** [21, Lemma 6.3] Suppose that every letter occurs at most twice in $u$. If $[u]_{\tau_1 \wedge \zeta}$ is stable with respect to a monoid variety $V$ then every word in $([u]_{\tau_1 \wedge \zeta})^\leq$ is $(\tau_1 \wedge \zeta)$-term for $V$.

The third row contains syntactic monoid which generates the limit variety $J$ from [3]. According to Theorem 7.2 in [21], we have $J = M_{\tau_1 \wedge \zeta}(\{atbasb\}_{\tau_1 \wedge \zeta})$. Using Theorem 5.3 together with Fact 6.1, we obtain

$J = M_{\text{synt}}(\{atbasb\}_{\tau_1 \wedge \zeta}) = M_{\text{synt}}(atba^+sb^+)$. 

Dually, we have

$\overline{J} = M_{\tau_1 \wedge \zeta}(\{atbasb\}_{\tau_1 \wedge \zeta}) = M_{\text{synt}}(a^+tb^+asb)$.

The limit variety $K$ in the fourth row of Table 1 was introduced in [5] as the variety generated by $M_{\tau_1 \wedge \zeta}(\{atb^2a\}_{\tau_1 \wedge \zeta})$. Using Theorem 5.3 together with Fact 6.1, we obtain

$K = M_{\text{synt}}(\{atb^2a\}_{\tau_1 \wedge \zeta}) = M_{\text{synt}}(atbb^+a^+)$. 

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Lemma 2.1, we obtain

\[ \mathbb{K} = M_{\tau_1 \wedge \zeta}(ab^2ta) = M_{\text{synt}}(a^+b^+bta). \]

Using Fact 6.1 one can verify that the variety \( \mathbb{K} \) can be also generated by

\[ M_{\tau_1 \wedge \zeta}(atbsba) = M_{\tau_1 \wedge \zeta}(atbsb^+a^+). \]

Hence, we also have

\[ \mathbb{K} = M_{\text{synt}}(atbsb^+a^+), \quad \overline{\mathbb{K}} = M_{\text{synt}}(a^+b^+tbsa). \]

Dually, we have

\[ \mathbb{K} = M_{\text{synt}}(atbsb^+a^+), \quad \overline{\mathbb{K}} = M_{\text{synt}}(a^+b^+tbsa). \]

The fifth row contains two syntactic monoids which generate the limit variety \( \overline{A^1} \lor A^1 \) from [26]. Indeed, according to Theorem 4.3 in [21], the variety \( A^1 \) is generated by \( M_{\tau_1 \wedge \gamma}([ab^2ta]_{\tau_1 \wedge \gamma}) = M_{\tau_1 \wedge \gamma}(a^+b^+ta^+) \) and \( \overline{A^1} \) is generated by \( M_{\tau_1 \wedge \gamma}([atb^2a]_{\tau_1 \wedge \gamma}) = M_{\tau_1 \wedge \gamma}(a^+tb^+a^+) \). Using Theorem 5.3 together with Lemma 2.1 we obtain

\[ \overline{A^1} \lor A^1 = M_{\text{synt}}(a^+tb^+b^+a^+) \lor M_{\text{synt}}(a^+bb^+ta^+). \]

The sixth row in Tables 1 contains two syntactic monoids which generate the limit variety \( \overline{A^1} \lor E^1\{\sigma_2\} \) from Theorem 4.3 above. In view of the previous paragraph and the dual of Example 5.5 we have

\[ \overline{A^1} \lor E^1\{\sigma_2\} = M_{\text{synt}}(a^+tb^+b^+a^+) \lor M_{\text{synt}}([ab^2ta]_{\overline{\sigma}}). \]

Dually, we have

\[ A^1 \lor E^1\{\sigma_2\} = M_{\text{synt}}(a^+bb^+ta^+) \lor M_{\text{synt}}([atb^2a]_{\beta}). \]

Finally, the seventh row contains three syntactic monoids which generate the limit variety \( E^1(\sigma_2) \lor \overline{E^1(\sigma_2)} \lor A^1_0 \) from [6]. According to Sect. 7 in [20], the monoid \( A^1_0 \) is isomorphic to \( M_{\tau_1}([ab]_{\tau_1}) = M_{\tau_1}(a^+b^+) \). Using Theorem 5.3 together with Lemma 2.1 we obtain \( A^1_0 = M_{\text{synt}}(a^+b^+) \). Example 5.5 and its dual imply that

\[ E^1(\sigma_2) \lor \overline{E^1(\sigma_2)} \lor A^1_0 = M_{\text{synt}}([atb^2a]_{\beta}) \lor M_{\text{synt}}([ab^2ta]_{\overline{\beta}}) \lor M_{\text{synt}}(a^+b^+). \]

In contrast with the ten limit varieties whose generating syntactic monoids up to duality are listed in the first seven rows of Table 1, it seems that monoids of the form \( M_\tau(W) \) are not useful for describing the (finitely generated) limit varieties \( J_1, J_2, J_3 \) and \( J_4 \) found in [11]. For instance, Lemma 5.3 in [11] implies that \( J_1 = M_\delta([asbtabzsz]_\delta) \), for some congruence \( \delta \) such that \( [asbtabzsz]_\delta = [asbtabzsz]_\zeta \cup [asbtabzsz]_\zeta \). The formula for the congruence \( \delta \) can be extracted from the proof of Proposition 5.4 in [11] and is very technical.

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limit variety of monoids generated by found by
\[ M_{\text{synt}}(\{abthbsa\}) \times M_{\text{synt}}(\{athbsba\}) \] Jackson in [11]
\[ M_{\text{synt}}(\{atbsb\}) \] Jackson in [11]
\[ M_{\text{synt}}(\{atbsb\}) \times M_{\text{synt}}(\{atbsba\}) \] Gusev in [8]
\[ M_{\text{synt}}(\{atb^2a_1\}) \] or by \[ M_{\text{synt}}(\{athbsba\}) \] Gusev-Sapir in [5]
\[ M_{\text{synt}}(\{atb^2a_1\}) \times M_{\text{synt}}(\{ab^2ta_1\}) \] Zhang-Luo in [26]
\[ M_{\text{synt}}(\{atb^2a_1\}) \times M_{\text{synt}}(\{ab^2ta_2\}) \] Theorem [4.5]
\[ M_{\text{synt}}(\{atb^2a_1\}) \times M_{\text{synt}}(\{ab^2ta_2\}) \times M_{\text{synt}}(\{ab\}) \] Gusev-Sapir in [6]
Contains \[ M_{\text{synt}}(\{atbsba\}) \] Gusev-Li-Zhang in [7]
Contains \[ M_{\text{synt}}(\{atb^2a_1\}) \] Gusev-Li-Zhang in [7]

Table 1: 14 limit varieties of aperiodic monoids

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