ON THE MEROMORPHIC CONTINUATION OF THE RESOLVENT FOR THE WAVE EQUATION WITH TIME-PERIODIC PERTURBATION AND APPLICATIONS

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Abstract. Consider the wave equation \( \partial_t^2 u - \Delta_x u + V(t, x)u = 0 \), where \( x \in \mathbb{R}^n \) with \( n \geq 3 \) and \( V(t, x) \) is \( T \)-periodic in time and decays exponentially in space. Let \( U(t, 0) \) be the associated propagator and let \( R(\theta) = e^{-D(x)U(T, 0)} - e^{-\theta \theta} - 1 \) be the resolvent of the Floquet operator \( U(T, 0) \) defined for \( \Im(\theta) > BT \) with \( B > 0 \) sufficiently large. We establish a meromorphic continuation of \( R(\theta) \) from which we deduce the asymptotic expansion of \( e^{-(D + \epsilon)(x)}U(t, 0)e^{-D(x)}f \), where \( f \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), as \( t \to +\infty \) with a remainder term whose energy decays exponentially when \( n \) is odd and a remainder term whose energy is bounded with respect to \( t^l \log(t)^m \), with \( l, m \in \mathbb{Z} \), when \( n \) is even. Then, assuming that \( R(\theta) \) has no poles lying in \( \{ \theta \in \mathbb{C} : \Im(\theta) \geq 0 \} \) and is bounded for \( \theta \to 0 \), we obtain local energy decay as well as global Strichartz estimates for the solutions of \( \partial_t^2 u - \Delta_x u + V(t, x)u = F(t, x) \).

Introduction

Consider the Cauchy problem

\[
\begin{cases}
\partial_t^2 u - \Delta_x u + V(t, x)u = F(t, x), & t > \tau, \ x \in \mathbb{R}^n, \\
(u, u_t)(\tau, x) = (f_1(x), f_2(x)) = f(x), & x \in \mathbb{R}^n,
\end{cases}
\]

(0.1)

where \( n \geq 3 \) and the potential \( V(t, x) \in C^\infty(\mathbb{R}^{1+n}, \mathbb{R}) \) satisfies the conditions

\begin{enumerate}
\item[(H1)] there exist \( D, \varepsilon > 0 \) such that for all \( \alpha \in \mathbb{N}^n, \beta \in \{0, 1, 2\} \) we have

\[
\left| \partial_\alpha \partial_\beta u V(t, x) \right| \leq C_{\alpha, \beta} e^{-\varepsilon(2D+3\varepsilon)x},
\]

\item[(H2)] there exists \( T > 0 \) such that \( V(t + T, x) = V(t, x) \).
\end{enumerate}

Let \( \dot{H}^\gamma(\mathbb{R}^n) = \Lambda^{-\gamma}(L^2(\mathbb{R}^n)) \) be the homogeneous Sobolev spaces, where \( \Lambda = \sqrt{-\Delta_x} \) is determined by the Laplacian in \( \mathbb{R}^n \). Set \( \dot{\mathcal{H}}_\gamma(\mathbb{R}^n) = \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n) \) and notice that for \( \gamma < \frac{n}{2} \) and \( r > 0 \) the multiplication with \( e^{-r(x)} \) is continuous from \( \dot{H}^\gamma(\mathbb{R}^n) \) to \( \dot{H}^\gamma(\mathbb{R}^n) \) (this follows from the fact that \( e^{-r(x)} \in S(\mathbb{R}^n) \)). The solution of (0.1) with \( F = 0 \) is given by the propagator

\[
U(t, \tau) : \dot{\mathcal{H}}_\gamma(\mathbb{R}^n) \ni (f_1, f_2) = f \mapsto U(t, \tau)f = (u, u_t)(t, x) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)
\]

and we refer to [22], Chapter V, for the properties of \( U(t, \tau) \). Let \( U_0(t) \) be the unitary propagator in \( \dot{\mathcal{H}}_\gamma(\mathbb{R}^n) \) related to the Cauchy problem (0.1) with \( V = 0, s = 0 \) and let \( U(T) = U(T, 0) \). We have the representation

\[
U(t, \tau) = U_0(t - \tau) - \int_\tau^t U(t, s)Q(s)U_0(s)ds, \quad t \geq \tau
\]

where

\[
Q(t) = \begin{pmatrix}
0 & 0 \\
V(t, x) & 0
\end{pmatrix}.
\]

By interpolation it is easy to see that

\[
\|U(t, s)\|_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} \leq C_\gamma e^{k_\gamma|t-s|}, \quad t \geq \tau
\]

(0.2)

where \( k_\gamma \) is bounded if \( \gamma \) runs in a compact interval.
It is well known that for stationaries potentials \((V(t,x))\) independent of \(t\), in order to establish large time estimates of solutions of \((0.1)\), the main points (see [18], [27] and [29]) are the meromorphic continuation and estimates of the resolvent \((P - \lambda^2)^{-1}\) associate to the hamiltonian \(P = -\Delta_x + V\). For time dependent potential, the hamiltonian \(P\) is also time-dependent and we can not deduce the large time behavior of solutions of \((0.1)\) from the properties of the resolvent \((P - \lambda^2)^{-1}\). Moreover, the time dependence of the potential \(V(t,x)\) leads to many difficulties (see [4] and [22]). The analysis of the Floquet operator \(U(T)\) make it possible to overcome some of these difficulties. Following an idea of [5], in [1] and [22] the authors used the Lax-Phillips theory and proved many results including local energy decay and existence of scattering operator. In [23], Petkov treats the case of even dimensions by considering the meromorphic continuation of the cut-off resolvent of the Floquet operator \(U(T)\) defined by

\[ R_{\psi_1, \psi_2}(\theta) = \psi_1(U(T) - e^{-i\theta})^{-1}\psi_2, \quad \psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^n). \]

Note that all these arguments hold only for potentials \(V(t,x)\) which are compactly supported with respect to \(x\). Let us introduce, for \(B > 0\) sufficiently large, the resolvent

\[ R(\theta) = e^{-D(x)}(U(T) - e^{-i\theta})^{-1}e^{-D(x)} : \mathcal{H}_1(\mathbb{R}^n) \to \mathcal{H}_1(\mathbb{R}^n), \quad \Im(\theta) \geq BT \]

with \(D\) the constant of (H1). The purpose of this paper is to extend the works of [1], [2], [4], [22] and [23] to perturbations of the Laplacian which are time-periodic and not-compact supported in \(x\) by showing the meromorphic continuation of \(R(\theta)\).

We recall that the properties of \(R(\theta)\) are closely related to the asymptotic expansion of \(e^{-(D+\varepsilon)(x)}U(t,0)e^{-D(x)f}\), with \(f \in \mathcal{H}_1(\mathbb{R}^n)\), as \(t \to +\infty\). Indeed, we can establish (see [12]) the inversion formula

\[ e^{-D(x)}U(kT,0)e^{-D(x)} = -\frac{1}{2\pi i} \int_{[-\pi+BT, \pi+BT]} e^{-i(k+1)\theta} R(\theta) d\theta, \quad k \in \mathbb{N}. \quad (0.3) \]

We use the following definition of meromorphic family of bounded operators.

**Definition 1.** Let \(H_1\) and \(H_2\) be two Hilbert spaces. A family of bounded operators \(Q(\theta) : H_1 \to H_2\) is said to be meromorphic in a domain \(D \subset \mathbb{C}\), if \(Q(\theta)\) is meromorphically dependent on \(\theta\) for \(\theta \in D\) and for any pole \(\theta = \theta_0\) the coefficients of the negative powers of \(\theta - \theta_0\) in the appropriate Laurent extension are finite-rank operators.

Now assume that there exists \(B_1 \in \mathbb{R}\) such that \(R(\theta)\) admits a meromorphic continuation with respect to \(\theta\) for \(\theta \in \{\theta \in \mathbb{C} : \Im(\theta) > B_1T\}\). Then, applying (0.3) and integrating \(e^{-i(k+1)\theta} R(\theta)\), for \(k \in \mathbb{N}\), on a well chosen contour (see Lemma 3 of [12] and the proof of the Main Theorem in [28]), we can establish the asymptotic expansion of \(e^{-(D+\varepsilon)(x)}U(t,0)e^{-D(x)f}\) as \(t \to +\infty\) with a remainder term whose energy is bounded with respect to \(e^{(B_1+\varepsilon)t}\) for all \(\varepsilon_1 > 0\).

The goal of this paper is to establish a meromorphic continuation of \(R(\theta)\) that allows us to study the asymptotic expansion of \(e^{-(D+\varepsilon)(x)}U(t,0)e^{-D(x)f}\) as \(t \to +\infty\) with a remainder term whose energy decays exponentially when \(n\) is odd and a remainder term whose energy is bounded with respect to \(t^l \log(t)^m\), with \(l,m \in \mathbb{Z}\), when \(n\) is even. For this purpose, it suffices to show that there exists \(B_1 > 0\) such that \(R(\theta)\) admits a meromorphic continuation to \(\theta \in \{\theta \in \mathbb{C} : \Im(\theta) > -B_1T\}\) for \(n\) odd and to \(\theta \in \{\theta \in \mathbb{C} : \Im(\theta) > -B_1T, \ \theta \notin 2\pi \mathbb{Z} + i\mathbb{R}^-\}\) for \(n\) even, and, for \(n\) even, to establish the asymptotic expansion as \(\theta \to 0\) of \(R(\theta)\) (see Lemma 3, Lemma 4 in [12] and the proof of the Main Theorem in [28]). The main result of this paper is the following.

**Theorem 1.** Assume (H1), (H2) fulfilled and let \(n \geq 3\). Then, for \(D > 0\) the constant of (H1), \(R(\theta)\) admits a meromorphic continuation to \(\{\theta \in \mathbb{C} : \Im(\theta) > -\frac{DT}{4}, \ \theta \notin 2\pi \mathbb{Z} + i\mathbb{R}^-\}\) for \(n\) odd and to

\[ \{\theta \in \mathbb{C} : \Im(\theta) > -\frac{DT}{4}, \ \theta \notin 2\pi \mathbb{Z} + i\mathbb{R}^-\} \]
for \( n \) even. Moreover, for \( n \) even, there exists \( \epsilon_0 > 0 \) such that for \( \theta \in \{ \theta \in \mathbb{C} : |\theta| \leq \epsilon_0, \ \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^-\} \) we have the representation

\[
R(\theta) = \sum_{i,j} \sum_{m,n} R_{i,j} \theta^i (\log(\theta))^{-j}, \tag{0.4}
\]

where \( \log \) is the logarithm defined on \( \mathbb{C} \setminus i\mathbb{R}^- \) and, for \( i < 0 \) or \( j \neq 0 \), \( R_{i,j} \) are finite-rank operators.

Let us recall that in [23], the author established the meromorphic continuation of the cut-off resolvent \( R_{\psi_1,\psi_2}(\theta) \) by applying some arguments of [28]. Since the analysis of [28] is mainly based on the fact that the perturbations are compactly supported in \( x \), we can not apply the arguments used in [23]. Nevertheless, we follow the analysis of [28] and we use the transformation \( F' \) defined by

\[
F'(\varphi(t,x))(t,\theta) = e^{i\theta t} \left( \sum_{k=-\infty}^{+\infty} \varphi(t+kt,x)e^{ik\theta} \right), \quad \varphi \in C_0^\infty(\mathbb{R}^{1+n})
\]

which plays the role of the Fourier transform with respect to \( t \) for non-stationary and time-periodic problems. Let us remark (see Lemma 2 in [28] and Section 1) that an application of \( F' \) transforms the solutions of (0.1), whose energy can grow exponentially in time (see [4]), into time-periodic functions. As in [28], in our analysis, this property will play a crucial role.

We prove Theorem 1 in three steps. First, in Section 1, we reduce the problem using integrals equations and we show that the meromorphic continuation of \( R(\theta) \) follows from an analytic continuation of the transformation \( F' \) with respect to \( \theta \) of the solutions of the free wave equation extended by 0 for \( t < 0 \). Then, in Section 2, we establish this analytic continuation. Finally, in Section 3, we conclude by applying the analytic Fredholm theorem and some results of [28].

Let us observe that the resolvent \( R(\theta) \) is defined in homogeneous Sobolev space. In our approach, we need to consider the resolvent \((U(T) - e^{-it\theta})^{-1}\) acting in weighted inhomogeneous Sobolev space. For this purpose, we exploit the fact that, for \( r > 0 \), the multiplication with \( e^{-r(x)} \) maps continuously \( H^1(\mathbb{R}^n) \) into \( H^1(\mathbb{R}^n) \) for \( n \geq 3 \). Notice that for \( n \leq 2 \) this property does not hold. Therefore, our analysis does not work for \( n = 2 \). To treat the case \( n = 2 \), one can consider the resolvent

\[
e^{-D(x)}(U(T) - e^{-it\theta})^{-1}e^{-D(x)} : \mathcal{H}_\gamma(\mathbb{R}^n) \to \mathcal{H}_\gamma(\mathbb{R}^n)
\]

with \( \gamma < 1 \), and repeat our arguments.

According to [12] and the proof of the Main Theorem in [28], from Theorem 1 we can establish the following results:

1. Assume \( n \geq 3 \) odd and let \( \theta_1, \ldots, \theta_N \) be the poles of \( R(\theta) \) lying in

\[
\{ \theta \in \mathbb{C} : \Im(\theta) \geq -\frac{DT}{4} + \epsilon_1 T, \ -\pi \leq \Re(\theta) < \pi \}
\]

with \( 0 < \epsilon_1 < \frac{\pi}{2} \). Then we have the representation

\[
e^{-D(x)}U(kT,0)e^{-D(x)} = -i \left[ \sum_{j=1}^{N} \text{res}_{\theta = \theta_j} \left( e^{-i(k+1)\theta} R(\theta) \right) \right] + G_k, \quad k \in \mathbb{N}, \tag{0.5}
\]

where the remainder term \( G_k \) satisfies

\[
\|G_k\|_{L(\mathcal{H}_\gamma(\mathbb{R}^n))} \lesssim e^{(-\frac{\pi}{2}+\epsilon_1)kT}.
\]

Moreover, from (0.5), we deduce the asymptotic expansion of \( e^{-(D+(\epsilon_1/2))\theta}U(t,0)e^{-D(x)} f \) as \( t \to +\infty \) with a remainder term whose energy is bounded with respect to \( e^{(-\frac{\pi}{2}+\epsilon_1)h} \).

2. Assume \( n \geq 4 \) even and let \( \theta_1, \ldots, \theta_N \) be the poles of \( R(\theta) \) lying in

\[
\{ \theta \in \mathbb{C} : \Im(\theta) \geq 0, \ -\pi \leq \Re(\theta) < \pi, \ \theta \neq 0 \}.
\]
Now, remove from (0.4) the terms at which $i \geq 0$ and $j = 0$ and let $B \theta^\mu (\log(\theta))^\nu$ be the leading term of the remaining term with non-zero coefficients. Then, we obtain the representation

\[ e^{-D(x)} U(kT,0)e^{-D(x)} = -i \sum_{j=1}^{N} \text{res}_{\theta=\theta_j} \left( e^{-i(k+1)\theta} R(\theta) \right) + \lambda_k B(1+kT)^l (\ln(2+kT))^m + G_k, \quad k \in \mathbb{N}, \quad (0.6) \]

where $\lambda_k \in \mathbb{C}$ is bounded with respect to $k$, $(l,m) = (-\mu -1, \nu)$ for $\mu < 0$ and $(l,m) = (-\mu -1, -\nu)$ for $\mu \geq 0$, the remainder term $G_k$ satisfies

\[ \|G_k\|_{L^1(\mathfrak{H}_1(\mathbb{R}^n))} = o_{k \to +\infty} \left( (1+kT)^l (\ln(2+kT))^m \right). \]

In addition, applying (0.5), we obtain the asymptotic expansion of $e^{-(D+\varepsilon)(x)} U(t,0)e^{-(D+\varepsilon)x} f$ as $t \to +\infty$ with a remainder term whose energy is bounded with respect to $t^l \ln(t)^m$.

From the meromorphic continuation of $R(\theta)$, we deduce sufficient conditions for local energy decay and global Strichartz estimates. More precisely, we have seen the link between the meromorphic continuation of $R(\theta)$ and the asymptotic expansion of $e^{-(D+\varepsilon)(x)} U(t,0)e^{-(D+\varepsilon)x} f$ as $t \to +\infty$. Consequently, it seems natural to consider the meromorphic continuations of $R(\theta)$ that imply dispersive estimates. Introduce the following condition.

(H3) The resolvent $R(\theta)$ has no poles on $\{ \theta \in \mathbb{C} : \Im(\theta) \geq 0 \}$ for $n$ odd and on $\{ \theta \in \mathbb{C} : \Im(\theta) \geq 0, \theta \notin 2\pi \mathbb{Z} \}$ for $n$ even. Moreover, for $n$ even we have

\[ \limsup_{\lambda \to 0, \lambda \geq 0} \| R(\lambda) \| < \infty. \]

Assuming (H3) fulfilled and applying the arguments in [11], [12] and [23], in Subsection 4.1 we establish a local energy decay of the form

\[ \|e^{-(D+\varepsilon)(x)} U(t,\tau)e^{-(D+\varepsilon)x}\|_{L^1(\mathfrak{H}_1(\mathbb{R}^n))} \lesssim p(t-\tau), \quad t \geq \tau \quad (0.7) \]

with $p(t) \in L^1(\mathbb{R}^+)$. More precisely, we obtain the following.

**Theorem 2.** Assume (H1), (H2) and (H3) fulfilled and let $n \geq 3$. Then, we have (0.7) with

\[ \begin{cases} p(t) = e^{-\delta t} & \text{for } n \geq 3 \text{ odd}, \\
 p(t) = \frac{1}{(t+1) \ln^2(t+\varepsilon)} & \text{for } n \geq 4 \text{ even}. \end{cases} \quad (0.8) \]

The decay of local energy for time dependent perturbations has been investigated by many authors (see [1], [5], [23] and [26]). The main hypothesis is that the perturbations are non-trapping (see [5], [10] and [25]). In contrast to the stationary case (see [14], [15], [27] and [29]) the non-trapping condition is not sufficient for a local energy decay. In particular, the problem (0.1) is non-trapping but we may have solutions with exponentially growing local energy. Indeed, if $R(\theta)$ has a pole $\theta_0 \in \mathbb{C}$ with $\Im(\theta_0) > 0$, from the representations (0.5) and (0.6), one may show that

\[ \|e^{-(D+\varepsilon)(x)} U(t,0)e^{-(D+\varepsilon)x}\|_{L^1(\mathfrak{H}_1(\mathbb{R}^n))} \geq e^{\frac{1}{\log^2(t+\varepsilon)}}, \quad t > 0 \]

and deduce from this estimate the existence of a solution whose energy grows exponentially. In [4], the authors established an example of positive potential such that this phenomenon occurs for problem (0.1). In our case assumption (H3) excludes the existence of such solutions. Moreover, according to (0.5) and (0.6), (H3) is an optimal condition for a local energy decay (0.7) with $p(t)$ satisfying (0.8) (see also Lemma 3 and Lemma 4 of [12]).
Let us recall that the local energy decay is a crucial point (see [10], [11], [15], [16] and [23]) to obtain estimates of the form
\[
\|u\|_{L^p_t L^q_x(\mathbb{R}^+,\mathbb{R}^n)} + \|\left(\frac{\partial_t}{\tau}, u(\tau)\right)\|_{L^\infty_t L^\infty_x(\mathbb{R}^+,\mathbb{T}_n(\mathbb{R}^n))} \leq C(p,q,n,\rho,T,\gamma) \left(\|f\|_{H^\infty_\infty} + \|F\|_{L^p_t L^q_x(\mathbb{R}^+,\mathbb{R}^n)}\right).
\]
Estimates (0.9) are called global Minkowski Strichartz estimates and they are important in order to prove existence and uniqueness results for non-linear equations (see for examples [9], [13], [20] and [21]).

Following the results of [23], in Subsection 4.2 we apply estimate (0.7) and prove estimates (0.9) for solutions of (0.1). We obtain the following.

**Theorem 3.** Assume (H1), (H2) and (H3) fulfilled and let \(n \geq 3\). Let \(1 \leq \tilde{p}, \tilde{q} \leq 2 \leq p, q \leq \infty, 0 < \gamma \leq 1,\)
\(p > 2, q < \frac{2(n-1)}{n-3}\) and \(\tilde{q}' < \frac{2(n-1)}{n-3}\) satisfy
\[
\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{\tilde{p}} + \frac{n}{\tilde{q}} - 2, \quad \frac{1}{p} \leq \frac{(n-1)(q-2)}{4q}, \quad \frac{1}{\tilde{p'}} \leq \frac{(n-1)(\tilde{q}'-2)}{4\tilde{q'}}.
\]
Then, the solution \(u\) of (0.1) with \(\tau = 0\) satisfies (0.9).

It is well known (see [9] and Corollary 3.2 of [13]) that for the solution of (0.1) with \(V = 0\) and \(\tau = 0\), estimates (0.9) hold for \(0 < \gamma < 1\) if \(1 \leq \tilde{p}, \tilde{q} \leq 2 \leq p, q \leq \infty, 0 < \gamma < 1, p > 2, q < \frac{2(n-1)}{n-3}\) and \(\tilde{q}' < \frac{2(n-1)}{n-3}\) satisfy (0.10). Since we apply the Krist-Chiselev lemma in the proof of Theorem 3 (see [22]) we must exclude \(p = 2\). In fact, the result of Theorem 3 holds for any values of \(p, q, \gamma, \tilde{p}, \tilde{q}\) for which the solutions of the free wave equation \((V(t,x) = 0)\) satisfy (0.9) as long as \(p > 2\) and \(0 < \gamma \leq 1\).

For the assumption (H3), in Subsection 4.3 we give examples of potentials \(V(t,x)\) such that (H1), (H2) and (H3) are fulfilled.

We like to mention that many authors proved local energy decay as well as global Strichartz estimates for the Schrödinger equation with non-compactly supported in \(x\) and time-periodic potentials (see [6], [8], [30] and [31]). It seems that our paper is the first work where one treats local energy decay and global Strichartz estimates for the wave equations with perturbations depending on \((t,x)\) which are not compactly supported with respect to \(x\), without any assumptions on the size of the perturbations (see the work of [16] and [17] for small perturbations of the Laplacian).

1. Reduction of the problem

For all \(\gamma \in \mathbb{R}\), we denote by \(\mathcal{H}_\gamma(\mathbb{R}^n)\) the space
\[
\mathcal{H}_\gamma(\mathbb{R}^n) = H^{\gamma}(\mathbb{R}^n) \times H^{\gamma-1}(\mathbb{R}^n).
\]
Since, for \(n \geq 3\) and for \(0 < \delta < D\), the multiplication by \(e^{-\delta(x)}\) is a bounded operator from \(\mathcal{H}_1(\mathbb{R}^n)\) to \(\mathcal{H}_1(\mathbb{R}^n)\), estimate (0.2) implies that, for all \(t > 0\), the operator
\[
e^{-\delta(x)}U(t,0)e^{-\delta(x)} : \mathcal{H}_1(\mathbb{R}^n) \rightarrow \mathcal{H}_1(\mathbb{R}^n)
\]
is bounded. In this section, as well as in Sections 2 and 3, we assume that
\[
U(t,\tau) = 0 \quad \text{for} \quad t < \tau \quad \text{and} \quad U_0(t) = 0 \quad \text{for} \quad t < 0.
\]
This assumption and estimate (0.2) allow us to consider, for \(\mathcal{J}(\theta) \geq BT\), with \(B > 0\) sufficiently large, and for \(0 < \delta < D\), the families of bounded operators
\[
F'\left[e^{-\delta(x)}U(t,0)e^{-\delta(x)}\right](t,\theta) = e^{\frac{i\pi t}{2}} \sum_{k=-\infty}^{+\infty} \left(e^{-\delta(x)}U(t+kT,0)e^{-\delta(x)}e^{ik\theta}\right) : \mathcal{H}_1(\mathbb{R}^n) \rightarrow \mathcal{H}_1(\mathbb{R}^n)
\]
and
\[
F'\left[e^{-\delta(x)}U_0(t)e^{-\delta(x)}\right](t,\theta) = e^{\frac{i\pi t}{2}} \sum_{k=-\infty}^{+\infty} \left(e^{-\delta(x)}U_0(t+kT)e^{-\delta(x)}e^{ik\theta}\right) : \mathcal{H}_1(\mathbb{R}^n) \rightarrow \mathcal{H}_1(\mathbb{R}^n).
\]
The resolvent operator in the homogeneous Sobolev space \( H \). We start by proving that the meromorphic continuation of the disturbed wave equation and the propagator \( U \) admits a meromorphic continuation to \( \mathcal{H}_1(\mathbb{R}^n) \) defined as a family of bounded operators mapping \( \mathcal{H}_1(\mathbb{R}^n) \) to \( \mathcal{H}_1(\mathbb{R}^n) \). To prove the meromorphic continuation of \( R(\theta) \), we will use the following result.

**Lemma 1.** Assume that for all \( 0 < \theta < D \), the family

\[
e^{-\delta(x)}(U(T) - e^{-i\theta})^{-1}e^{-\delta(x)} : \mathcal{H}_1(\mathbb{R}^n) \to \mathcal{H}_1(\mathbb{R}^n)
\]

admits a meromorphic continuation, with respect to \( \theta \), to \( \{ \theta \in \mathbb{C} : \mathcal{I}(\theta) > -\frac{\delta T}{4} \} \) for \( n \) odd and to \( \{ \theta \in \mathbb{C} : \mathcal{I}(\theta) > -\frac{\delta T}{4}, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \) for \( n \) even, with an asymptotic expansion as \( \theta \to 0 \) which take the form \((0.4)\). Then,

\[
R(\theta) : \mathcal{H}_1(\mathbb{R}^n) \to \mathcal{H}_1(\mathbb{R}^n)
\]

admits the meromorphic continuation described in Theorem 1.

**Proof.** Let \( 0 < \theta < D \). Notice that, for \( \mathcal{I}(\theta) > BT \), we have

\[
R(\theta) = e^{-(D-\delta)(x)} \left( e^{-\delta(x)}(U(T) - e^{-i\theta})^{-1}e^{-\delta(x)} \right) e^{-(D-\delta)(x)}.
\]

Since the multiplication by \( e^{-(D-\delta)(x)} \) is a bounded operator from \( \mathcal{H}_1(\mathbb{R}^n) \) to \( \mathcal{H}_1(\mathbb{R}^n) \), for \( \theta \in \{ \theta \in \mathbb{C} : \mathcal{I}(\theta) > -\frac{\delta T}{4} \} \) when \( n \) is odd and \( \theta \in \{ \theta \in \mathbb{C} : \mathcal{I}(\theta) > -\frac{\delta T}{4}, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \) when \( n \) is even, \( R(\theta) \) admits the following meromorphic continuation

\[
R(\theta) = e^{-(D-\delta)(x)} \left( e^{-\delta(x)}(U(T) - e^{-i\theta})^{-1}e^{-\delta(x)} \right) e^{-(D-\delta)(x)} : \mathcal{H}_1(\mathbb{R}^n) \to \mathcal{H}_1(\mathbb{R}^n).
\]

Hence, for all \( 0 < \delta < D \), \( R(\theta) \) admits a meromorphic continuation to \( \{ \theta \in \mathbb{C} : \mathcal{I}(\theta) > -\frac{\delta T}{4} \} \) for \( n \) odd and to \( \{ \theta \in \mathbb{C} : \mathcal{I}(\theta) > -\frac{\delta T}{4}, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \) when \( n \) is even, and we deduce Theorem 1. \( \square \)

Following Lemma 1, we set \( 0 < \delta < D \) and we will establish the meromorphic continuation of the family

\[
e^{-\delta(x)}(U(T) - e^{-i\theta})^{-1}e^{-\delta(x)} : \mathcal{H}_1(\mathbb{R}^n) \to \mathcal{H}_1(\mathbb{R}^n)
\]

described in Lemma 1. Let us recall (see [12]) that, for \( \mathcal{I}(\theta) \geq BT \) with \( B > 0 \) sufficiently large, we have

\[
e^{-\delta(x)}(U(T) - e^{-i\theta})^{-1}e^{-\delta(x)} = -e^{i\theta} F' \left[ e^{-\delta(x)}U(t, 0)e^{-\delta(x)} \right] (T, \theta).
\] \hspace{1cm} (1.2)

Thus, for our purpose it suffices to establish the meromorphic continuation of the family

\[
F' \left[ e^{-\delta(x)}U(t, 0)e^{-\delta(x)} \right] (t, \theta) : \mathcal{H}_1(\mathbb{R}^n) \to \mathcal{H}_1(\mathbb{R}^n)
\]
with respect to $\theta$. We will show that the meromorphic continuation of
\[ F' \left[ e^{-\delta(x)}U(t, 0)e^{-\delta(x)} \right] (t, \theta) : H_1(\mathbb{R}^n) \to H_1(\mathbb{R}^n) \]
follows from an analytic continuation of
\[ F' \left[ e^{-\delta(x)}U_0(t)e^{-\delta(x)} \right] (t, \theta) : H_1(\mathbb{R}^n) \to H_1(\mathbb{R}^n). \]

Now, let us define two Hilbert spaces and let us recall some results of [28].

**Definition 2.** Let $\gamma \in \mathbb{R}$, $A \in \mathbb{R}$. We denote by $\mathcal{H}_A^\gamma(\mathbb{R}^{1+n})$ the closure of $C_0^\infty(0, +\infty[\mathbb{R}^n) \times C_0^\infty(0, +\infty[\mathbb{R}^n)$ with respect to the norm $\| \cdot \|_{\mathcal{H}_A^\gamma}$. Here, $\| \cdot \|_{\mathcal{H}_A^\gamma}$ is defined by
\[ \| (\varphi_1, \varphi_2) \|_{\mathcal{H}_A^\gamma}^2 = \| e^{-At}\varphi_1 \|_{H^\gamma(\mathbb{R}^{1+n})}^2 + \| e^{-At}\varphi_2 \|_{H^{\gamma-1}(\mathbb{R}^{1+n})}^2. \]

**Remark 1.** To understand the use of the spaces $\mathcal{H}_A^\gamma(\mathbb{R}^{1+n})$, let $h \in C^\infty(\mathbb{R})$ be such that $\text{supp}(h) \subset [0, +\infty[$. Then, estimate (0.2) implies that, for $A > k_1$, $h(t)e^{-\delta(x)}U(t, 0)e^{-\delta(x)}$ is a bounded operator from $H_1(\mathbb{R}^n)$ to $H_1^\gamma(\mathbb{R}^{1+n})$.

**Definition 3.** Let $\gamma \in \mathbb{R}$. We denote by $H_{\gamma, \text{per}}(\mathbb{R}^{1+n})$ the set of functions $T$-periodic with respect to $t$ lying in the closure of the set
\[ \{ \varphi \in C_0^\infty(\mathbb{R}, C_0^\infty(\mathbb{R}^n)) : \varphi(t, x) \text{ is } T \text{-periodic with respect to } t \} \]
with respect to the norm
\[ \| \cdot \|_{H_{\gamma, \text{per}}(\mathbb{R}^{1+n})}. \]
We set $H_{\gamma, \text{per}}(\mathbb{R}^{1+n}) = H_{\text{per}}^\gamma(\mathbb{R}^{1+n}) \times H_{\text{per}}^{\gamma-1}(\mathbb{R}^{1+n})$.

**Remark 2.** Let $\theta \in \mathbb{C}$ and $A > 0$ be such that $\Im(\theta) > AT$. Then, Vainberg proved in Lemma 2 of [28] that the operator
\[ F_0' : H_{\gamma, \text{per}}^\gamma(\mathbb{R}^{1+n}) \to H_{\gamma, \text{per}}(\mathbb{R}^{1+n}), \tag{1.3} \]
\[ \varphi(t, \cdot) \mapsto F'(\varphi)(t, \theta) \]
is bounded. This result shows the remarkable property of $F'$. Namely, $F'$ transforms functions, which are exponentially growing in time, into time-periodic functions.

Applying (0.2), we deduce easily the following (see also Lemma 1 and Lemma 2 in [28]).

**Proposition 1.** Let $B > k_1$ with $k_1$ the constant of (0.2) when $\gamma = 1$. Then, for $\Im(\theta) > BT$, the operator
\[ F' \left[ e^{-\delta(x)}U(t, 0)e^{-\delta(x)} \right] (t, \theta) : H_1(\mathbb{R}^n) \to H_{1, \text{per}}(\mathbb{R}^{1+n}), \]
\[ f \mapsto F' \left[ e^{-\delta(x)}U(t, 0)e^{-\delta(x)} \right] (t, \theta)f \]
is bounded.

Let us recall a result of [28].

**Proposition 2.** (Lemma 6, [28]) For all $r > 0$, the operator
\[ W_0 : H_1^\gamma(\mathbb{R}^{1+n}) \to H_1^\gamma(\mathbb{R}^{1+n}), \tag{1.4} \]
\[ \varphi(t, \cdot) \mapsto \int_{-\infty}^{t} U_0(t - s)\varphi(s)ds. \]
is bounded, and for $\Im(\theta) > rT$, we have
\[ F' [W_0 \varphi](t, \theta) = J(t, \theta)F' (\varphi)(t, \theta), \]
where

\[ J(t, \theta) : \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n}) \to \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n}), \]  

\[ \psi(t, \cdot) \mapsto \int_0^T F^\ast[U_0](t-s, \theta) \psi(s) \, ds \]  

is a family of bounded operators defined for \( \theta \in \mathbb{C}, \mathcal{J}(\theta) > T \).

In fact in [28], Lemma 6, Vainberg shows this result for \( F^\ast[W_0^{\alpha_1}](t, \theta) \) where \( W_0^{\alpha_1} \) is defined by

\[ W_0^{\alpha_1}(\varphi)(t) = \int_{-\infty}^t \alpha_1(t-s)U_0(t-s)\varphi(s) \, ds, \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n}) \]

with \( \alpha_1 \in \mathcal{C}_0^\infty(\mathbb{R}) \) is such that \( \alpha_1(t) = 0 \) for \( t \leq t_0 \) and \( \alpha_1(t) = 1 \) for \( t \geq t_0 + 1 \). Following the same arguments, we can easily obtain the same result for \( W_0^{(1-\alpha_1)} \) and deduce (1.5).

From the Duhamel’s principle, we deduce the representation

\[ U(t, 0) = U_0(t) - \int_0^t U_0(t-s)Q(s)U(s, 0) \, ds. \]

and since \( U(t, 0) = 0 \) for \( t < 0 \) we obtain

\[ U(t, 0) = U_0(t) - W_0[Q(t)U(t, 0)](t). \]

**Lemma 2.** Let \( B > k_1 \) with \( k_1 \) the constant of (0.2) when \( \gamma = 1 \). Then, for \( \mathcal{J}(\theta) > BT \), we have

\[ F^\ast \left[ W_0 \left[ Q(t)U(t, 0)e^{-\delta(x)} \right] \right](t, \theta) = J(t, \theta)F^\ast \left[ \alpha(t)Q(t)U(t, 0)e^{-\delta(x)} \right](t, \theta). \]

**Proof.** Choose \( \alpha \in \mathcal{C}_0^\infty(\mathbb{R}) \) such that \( 0 \leq \alpha(t) \leq 1, \alpha(t) = 0 \) for \( t \leq \frac{T}{3} \) and \( \alpha(t) = 1 \) for \( t \geq \frac{T}{4} \). Then, assumption (H1) and (0.2) imply

\[ \alpha(t)Q(t)U(t, 0)e^{-\delta(x)} \in \mathcal{L} \left( \mathcal{H}_1(\mathbb{R}^n), \mathcal{H}_1^B(\mathbb{R}^{1+n}) \right) \]

and Proposition 2 yields

\[ F^\ast \left[ W_0 \left[ \alpha(t)Q(t)U(t, 0)e^{-\delta(x)} \right] \right](t, \theta) = J(t, \theta)F^\ast \left[ \alpha(t)Q(t)U(t, 0)e^{-\delta(x)} \right](t, \theta). \]

Moreover, since \((1-\alpha(t))Q(t)U(t, 0)e^{-\delta(x)} = 0\) for \( t \notin [0,T] \), we have

\[ F^\ast \left[ (1-\alpha(t))Q(t)U(t, 0)e^{-\delta(x)} \right](t, \theta) = e^{\frac{\alpha(t)}{BT}}(1-\alpha(t))Q(t)U(t, 0)e^{-\delta(x)}, \quad t \in [0,T] \]

and, since \( U_0(t) = 0 \) for \( t < 0 \), we get

\[ W_0 \left[ (1-\alpha(t))Q(t)U(t, 0)e^{-\delta(x)} \right](t) = \int_{\mathbb{R}} U_0(t-s)(1-\alpha(s))Q(s)U(s, 0)e^{-\delta(x)} \, ds, \]

\[ = \int_0^T U_0(t-s)(1-\alpha(s))Q(s)U(s, 0)e^{-\delta(x)} \, ds. \]

Applying \( F^\ast \) to this representation, for \( \mathcal{J}(\theta) > BT \), we obtain

\[ F^\ast \left[ \int_0^T U_0(t-s)(1-\alpha(s))Q(s)U(s, 0)e^{-\delta(x)} \, ds \right](t, \theta) \]

\[ = \int_0^T F^\ast[U_0](t-s, \theta)e^{\frac{\alpha(s)}{BT}}(1-\alpha(s))Q(s)U(s, 0)e^{-\delta(x)} \, ds, \]

\[ = J(t, \theta)F^\ast \left[ (1-\alpha(t))Q(t)U(t, 0)e^{-\delta(x)} \right](t, \theta). \]

Combining this equality with (1.8), we obtain (1.7). 

\( \square \)
Following (1.7), applying $F'$ to both sides of (1.6) (see also [11] and [28] for the properties of the transformation $F'$) and multiplying by $e^{-\delta(x)}$ on the right and on the left of (1.6) for $\mathcal{H}(\theta) \geq BT$ with $B > 0$ sufficiently large, we obtain

$$
F' \left[ e^{-\delta(x)} U(t, 0) e^{-\delta(x)} \right] (t, \theta) = F' \left[ e^{-\delta(x)} U_0(t) e^{-\delta(x)} \right] (t, \theta) - e^{-\delta(x)} J(t, \theta) F' \left[ Q(t) U(t, 0)e^{-\delta(x)} \right] (t, \theta).
$$

(1.9)

Notice that, for $\mathcal{H}(\theta) > BT$, we have

$$
F' \left[ Q(t) U(t, 0)e^{-\delta(x)} \right] (t, \theta) = e^{\frac{\delta}{T}} \left( \sum_{k=-\infty}^{+\infty} Q(t + kT) U(t + kT, 0)e^{-\delta(x)} e^{ik\theta} \right).
$$

Since $Q(t)$ is $T$-periodic and, from (H1), $Q(t)e^{\delta(x)} \in \mathcal{L}(\mathcal{H}_1(\mathbb{R}))$, we obtain

$$
F' \left[ Q(t) U(t, 0) e^{-\delta(x)} \right] (t, \theta) = Q(t) e^{\delta(x)} e^{\frac{\delta}{T}} \left( \sum_{k=-\infty}^{+\infty} e^{-\delta(x)} U(t + kT, 0)e^{-\delta(x)} e^{ik\theta} \right)
$$

$$
= Q(t) e^{\delta(x)} F' \left[ e^{-\delta(x)} U(t, 0) e^{-\delta(x)} \right] (t, \theta).
$$

Applying this formula to (1.9), we get

$$
F' \left[ e^{-\delta(x)} U(t, 0) e^{-\delta(x)} \right] (t, \theta) = F' \left[ e^{-\delta(x)} U_0(t) e^{-\delta(x)} \right] (t, \theta) - e^{-\delta(x)} J(t, \theta) Q(t) e^{\delta(x)} F' \left[ e^{-\delta(x)} U(t, 0) e^{-\delta(x)} \right] (t, \theta)
$$

and it follows

$$
\left( \text{Id} + e^{-\delta(x)} J(t, \theta) Q(t) e^{\delta(x)} \right) F' \left[ e^{-\delta(x)} U(t, 0) e^{-\delta(x)} \right] (t, \theta) = F' \left[ e^{-\delta(x)} U_0(t) e^{-\delta(x)} \right] (t, \theta).
$$

Thus, to prove the meromorphic continuation of $R(\theta)$, we only need to show that the families

$$
\left( \text{Id} + e^{-\delta(x)} J(t, \theta) Q(t) e^{\delta(x)} \right)^{-1}, \quad F' \left[ e^{-\delta(x)} U_0(t) e^{-\delta(x)} \right] (t, \theta)
$$

admit a meromorphic continuation with respect to $\theta$.

**Proposition 3.** Let $G$ be the operator defined by

$$
G : \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n}) \rightarrow \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n}), \quad \varphi(t,.) \mapsto e^{\delta(x)} Q(t) e^{\delta(x)} \varphi(t,.).
$$

Then $G$ is a compact operator.

**Proof.** Let $h = (h_1, h_2) \in \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n})$. We have

$$
e^{\delta(x)} Q(t) e^{\delta(x)} h = (0, e^{2\delta(x)} Vh_1).
$$

Applying (H1), we show that $e^{2\delta(x)} Vh_1$ takes value in $e^{-\varepsilon(x)} \mathcal{H}^1_{\text{per}}(\mathbb{R}^{1+n})$ and the Rellich-Kondrachov theorem for unbounded domains implies that $G$ is a compact operator on $\mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n})$. \hfill \square

Let us remark that for $\mathcal{H}(\theta) \geq BT$, we have

$$
e^{-\delta(x)} J(t, \theta) Q(t) e^{\delta(x)} = e^{-\delta(x)} J(t, \theta)e^{-\delta(x)} e^{\delta(x)} Q(t) e^{\delta(x)}
$$

with

$$
e^{-\delta(x)} J(t, \theta)e^{-\delta(x)} \varphi = \int_0^T F' \left[ e^{-\delta(x)} U_0(t) e^{-\delta(x)} \right] (t-s, \theta) \varphi(s) ds.
$$
Theorem 4. The following.

is well defined and analytic with respect to $\theta$ is invertible.

The main point of Theorem 4 is to consider, for

and showing that for all $\theta \in \{ \theta \in C : \Im(\theta) > 0 \}$ to prove Theorem 1 it remains to show that $\mathcal{H}_1(\mathbb{R}^n)$ to $H_1(\mathbb{R}^n)$, we can easily prove (see Lemma 2 in [28]) that

is well defined and analytic with respect to $\theta$ in $\{ \theta \in C : \Im(\theta) > 0 \}$. The goal of this section is to establish the following.

Theorem 4. The family of operators

admits an analytic continuation with respect to $\theta$, which is continuous with respect to $t$ in $\mathbb{R}$, from $\{ \theta \in \mathbb{C} : \Im(\theta) > 0 \}$ to $\{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{4T}{3} \}$ for $n$ odd and to $\{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{4T}{3}, \theta \notin 2\pi \mathbb{Z} + i\mathbb{R}^{-} \}$ for $n$ even. Moreover, for $n$ even, there exists $\varepsilon_0 > 0$ such that for $\theta \in \{ \theta \in \mathbb{C} : |\theta| \leq \varepsilon_0, \theta \notin 2\pi \mathbb{Z} + i\mathbb{R}^{-} \}$ we have the representation

where log is the logarithm defined on $\mathbb{C} \setminus i\mathbb{R}^{-}, C(t,\theta)$ is $C^\infty$ and $T$-periodic with respect to $t$ and analytic with respect to $\theta$, $B(\theta)$ is analytic with respect to $\theta$ for $|\theta| \leq \varepsilon_0$, and for all $j \in \mathbb{N}$, $(\partial_\theta^jB(\theta))_{|\theta=0}$ are finite rank operators.

Let $(\varphi_m)_{m\in\mathbb{N}} \subset C^\infty(\mathbb{R}^n)$ be a partition of unity such that

$$\sum_{m=0}^{+\infty} \varphi_m(x) = 1,$$

$$\|\partial_x^\alpha \varphi_m\|_{L^\infty(\mathbb{R}^n)} \leq C_{\alpha}, \quad 0 \leq \varphi_m \leq 1, \quad m \in \mathbb{N}, \quad \alpha \in \mathbb{N}^n,$$

supp$\varphi_m \subset \{ x \in \mathbb{R}^n : 2^{m-1}T \leq |x| \leq 2^{m+1}T \}, \quad m \geq 1,$$

supp$\varphi_0 \subset \{ x \in \mathbb{R}^n : |x| \leq 2T \}.$$

The main point of Theorem 4 is to consider, for $\Im(\theta) > 0$, the following representation

$$F^T \left[ e^{-\delta(x)}U_0(t)e^{-\delta(x)} \right] (t,\theta) = \sum_{m,l=0}^{+\infty} e^{-\delta(x)} F^T \left[ \varphi_m U_0(t) \varphi_l \right] (t,\theta)e^{-\delta(x)}. \quad (2.1)$$

Using this representation we will obtain Theorem 4, by proving the analytic continuation of

$$e^{-\delta(x)} F^T \left[ \varphi_m U_0(t) \varphi_l \right] (t,\theta)e^{-\delta(x)} : \mathcal{H}_1(\mathbb{R}^n) \rightarrow \mathcal{H}_1(\mathbb{R}^n)$$

and showing that for all $\theta \in \{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{4T}{3} \}$ for $n$ odd (respectively $\theta \in \{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{4T}{3} \}, \theta \notin 2\pi \mathbb{Z} + i\mathbb{R}^{-} \}$ for $n$ even)

$$\sum_{m,l\in\mathbb{N}} e^{-\delta(x)} F^T \left[ \varphi_m U_0(t) \varphi_l \right] (t,\theta)e^{-\delta(x)}$$
converge to $\mathcal{G}(t, \theta)$ which satisfies the properties described in Theorem 4. We treat separately, the case of odd and even dimensions.

2.1. Analytic continuation of $F'[e^{-\delta(x)}U_0(t)e^{-\delta(x)}](t, \theta)$ for $n$ odd. In this subsection we treat the case of odd dimensions. For $n$ odd, the Huygens principle implies that, for all $m, l \in \mathbb{N}$, the infinite sum

$$\sum_{k=-\infty}^{+\infty} e^{-\delta(x)} \varphi_m U_0(t + kT) e^{ik\theta} \varphi_l e^{-\delta(x)}$$

is in fact a finite sum. Thus, $e^{-\delta(x)} F'[\varphi_m U_0(t)\varphi_l](t, \theta) e^{-\delta(x)}$ admits an analytic continuation on $\mathbb{C}$ and we can estimate it to obtain the following.

**Lemma 3.** Let $n \geq 3$ be odd. Then, the family of operators

$$F'[e^{-\delta(x)}U_0(t)e^{-\delta(x)}](t, \theta) : \mathcal{H}_1(\mathbb{R}^n) \rightarrow \mathcal{H}_1(\mathbb{R}^n)$$

admits an analytic continuation with respect to $\theta$, which is continuous with respect to $t \in \mathbb{R}$, from $\{ \theta \in \mathbb{C} : \Im(\theta) > 0 \}$ to $\{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{\delta(T)}{2} \}$.

**Proof.** Since $e^{-\delta(x)} F'[\varphi_m U_0(t)\varphi_l](t, \theta) e^{-\delta(x)}$ is $T$-periodic (see [11] and [28]) we will only need to estimate it for $0 \leq t \leq T$. Let $0 < t \leq T$. Since, for all $m, l \in \mathbb{N}$, we have

$$|x| + |x_0| \leq 2^{m+1}T + 2^{l+1}T, \quad x \in \text{supp} \varphi_m, \quad x_0 \in \text{supp} \varphi_l,$$

an application of the Huygens’s principle yields

$$\varphi_m U_0(t + kT)\varphi_l = 0, \quad \text{for} \quad k \geq 2^{m+1} + 2^{l+1} + 1.$$ 

Thus, for all $m, l \in \mathbb{N}$, $F'[\varphi_m U_0(t)\varphi_l](t, \theta)$ admits an analytic continuation to $\mathbb{C}$ and we have

$$F'[\varphi_m U_0(t)\varphi_l](t, \theta) = e^{\frac{i\theta}{2}} \left( \sum_{k=0}^{2^{m+1} + 2^{l+1}} \varphi_m U_0(t + kT) \varphi_l e^{ik\theta} \right).$$

Since for $m \geq 1 \text{ supp} \varphi_m \subset \{ x : |x| \geq 2^{m-1}T \}$, one may show that, for all $0 < r < \delta$, for $m, l \geq 1$, we have

$$\left\| e^{-\delta(x)} F'[\varphi_m U_0(t)\varphi_l](t, \theta) e^{-\delta(x)} \right\| \leq e^{-(\delta-r)2^{-m-1}T} e^{-(\delta-r)2^{-l-1}T} \left\| F'[e^{-r(x)} \varphi_m U_0(t) \varphi_l e^{-r(x)}](t, \theta) \right\|, \quad \text{(2.2)}$$

and get

$$\left\| e^{-\delta(x)} F'[\varphi_m U_0(t)\varphi_l](t, \theta) e^{-\delta(x)} \right\| \leq e^{-(\delta-r)2^{-m-1}T} \left\| F'[e^{-r(x)} \varphi_m U_0(t) \varphi_l e^{-r(x)}](t, \theta) \right\|, \quad m \geq 1, \quad \text{(2.3)}$$

$$\left\| e^{-\delta(x)} F'[\varphi U_0(t)\varphi_l](t, \theta) e^{-\delta(x)} \right\| \leq e^{-(\delta-r)2^{-l-1}T} \left\| F'[e^{-r(x)} \varphi U_0(t) \varphi_l e^{-r(x)}](t, \theta) \right\|, \quad l \geq 1. \quad \text{(2.4)}$$

For this purpose, we can choose $\psi_m \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\psi_m = 1$ on $\text{supp} \varphi_m$ supported on $\{ x : 2^{m-1}T - \frac{T}{4} \leq |x| \leq 2^{m+1}T + \frac{T}{4} \}$ with derivatives bounded independently of $m$ and notice that

$$\varphi_m e^{-\delta(x)} = e^{-\delta(x)} \varphi_m = \left( e^{-\delta-r(x)} \psi_m \right) e^{-r(x)} \varphi_m = \varphi_m e^{-r(x)} \left( e^{-\delta-r(x)} \psi_m \right)$$

with

$$\left\| e^{-\delta-r(x)} \psi_m h \right\|_{\mathcal{H}_1(\mathbb{R}^n)} \leq e^{-(\delta-r)2^{-m-1}T} \left\| h \right\|_{\mathcal{H}_1(\mathbb{R}^n)}, \quad h \in \mathcal{H}_1(\mathbb{R}^n).$$
Let $K \subset \{ \theta \in \mathbb{C} : \mathcal{I}(\theta) > -\frac{\delta T}{4} \}$ be a compact set. According to representation (2.1) and estimates (2.2), (2.3) and (2.4), it remains to show that there exists $0 < r < \delta$ such that we have

$$
\sum_{m,l=1}^{+\infty} \sup_{m,l=1} \left| e^{-(\delta-r)2m^{-1}T} e^{-(\delta-r)2^{l-1}T} \right| \left\| F' \left[ e^{-r(x)} \phi_m U_0(t) \phi_l e^{-r(x)} \right] (t, \theta) \right\|_{L(H_1(\mathbb{R}^n))} < +\infty
$$

(2.5)

Since $K$ is a compact set, there exists $\delta_1 > 0$, such that, for all $\theta \in K$, $\mathcal{I}(\theta) \geq \delta_1 - \frac{\delta T}{4}$. Then, for all $\theta \in K$, $t \in \mathbb{R}$, $0 < r < \delta$ and $m,l \in \mathbb{N}$, we obtain

$$
\left\| F' \left[ e^{-r(x)} \phi_m U_0(t) \phi_l e^{-r(x)} \right] (t, \theta) \right\|_{L(H_1(\mathbb{R}^n))} \leq \sum_{k=0}^{2m+1+2^l+1} \left\| e^{-r(x)} \phi_m U_0(t+kT) \phi_l e^{-r(x)} \right\| e^{(\frac{\delta T}{4} - \delta_1)k}.
$$

Since the multiplication by $e^{-r(x)}$ is continuous from $H_1(\mathbb{R}^n)$ to $H_1(\mathbb{R}^n)$, for all $k \in \mathbb{N}$, we obtain

$$
\left\| e^{-r(x)} \phi_m U_0(t+kT) \phi_l e^{-r(x)} \right\|_{L(H_1(\mathbb{R}^n))} = \left\| \phi_m e^{-r(x)} U_0(t+kT) e^{-r(x)} \phi_l \right\|_{L(H_1(\mathbb{R}^n))} \leq \left\| e^{-r(x)} U_0(t+kT) e^{-r(x)} \right\|_{L(H_1(\mathbb{R}^n))} \leq \left\| U_0(t+kT) \right\|_{L(H_1(\mathbb{R}^n))} \leq 1.
$$

Thus, we get

$$
\left\| F' \left[ e^{-r(x)} \phi_m U_0(t) \phi_l e^{-r(x)} \right] (t, \theta) \right\|_{L(H_1(\mathbb{R}^n))} \leq \sum_{k=0}^{2m+1+2^l+1} e^{(\frac{\delta T}{4} - \delta_1)k}
$$

and it follows

$$
\left\| F' \left[ e^{-r(x)} \phi_m U_0(t) \phi_l e^{-r(x)} \right] (t, \theta) \right\|_{L(H_1(\mathbb{R}^n))} \leq \max \left( e^{(\frac{\delta T}{4} - \delta_1)(2m+1+2^l+1)}, 2m+1+2^l+1 \right).
$$

Combining this inequality with estimates (2.2), (2.3) and (2.4) for $r < \min (\delta, \delta_1)$, we obtain (2.5). This completes the proof.

**2.2. Analytic continuation of $F' [e^{-\delta(x)U_0(t)} e^{-\delta(x)}] (t, \theta)$ for $n$ even.** In this subsection we treat the case of even dimensions. Since for $n$ even the Huygens principle does not hold, we must use other arguments to prove Theorem 4. Let $h_{m,l} \in C^\infty(\mathbb{R})$ be such that

$$
0 \leq h_{m,l} \leq 1, \sup_{m,l} \left\| \partial^j_x h_{m,l} \right\|_{L^\infty(\mathbb{R})} \leq C_j < \infty,
$$

$$
h_{m,l}(t) = 1 \quad \text{for} \quad t \geq 2^{m+1}T + 2^{l+1}T + \frac{T}{2},
$$

$$
h_{m,l}(t) = 0 \quad \text{for} \quad t \leq 2^{m+1}T + 2^{l+1}T + \frac{T}{3}.
$$

Then for $\mathcal{I}(\theta) > 0$ and $m,l \in \mathbb{N}$, we obtain

$$
F' [\phi_m U_0(t) \phi_l] (t, \theta) = F' [(1 - h_{m,l}(t)) \phi_m U_0(t) \phi_l] (t, \theta) + F' [h_{m,l}(t) \phi_m U_0(t) \phi_l] (t, \theta).
$$

Since $1 - h_{m,l}(t) = 0$ for $t \geq 2^{m+1}T + 2^{l+1}T + \frac{T}{2}$, repeating the arguments used in the proof of Lemma 3, we obtain the following.
Lemma 4. Let \( n \geq 4 \) be even. Then, for \( \theta \in \{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{\pi}{4} \} \),
\[
\sum_{m,l \in \mathbb{N}} F' \left[ e^{-\delta(x)} (1 - h_{m,l}(t)) \varphi_m U_0(t)\varphi_l e^{-\delta(x)} \right] (t, \theta) : \mathcal{H}_1(\mathbb{R}^n) \to \mathcal{H}_1(\mathbb{R}^n)
\]
converge to a family of operator analytic with respect to \( \theta \), continuous with respect to \( t \).

Following Lemma 4, if we show that
\[
\sum_{m,l \in \mathbb{N}} F' \left[ e^{-\delta(x)} h_{m,l}(t) \varphi_m U_0(t)\varphi_l e^{-\delta(x)} \right] (t, \theta) : \mathcal{H}_1(\mathbb{R}^n) \to \mathcal{H}_1(\mathbb{R}^n)
\]
(2.6)
converge to a family of operators analytic with respect to \( \theta \) satisfying the properties described in Theorem 4, we will deduce the analytic continuation of \( F' \left[ e^{-\delta(x)} U_0(t) e^{-\delta(x)} \right] (t, \theta) \) for \( n \) even.

The main point in the proof of Theorem 4 for \( n \) even is the convergence of (2.6). In order to prove this result we will apply the Herglotz-Petrovskii formula (see [7], Chapter X of [27] and the proof of Lemma 6 in [28]) that gives an explicit formula of the Green’s function associate to some solutions of the free wave equation and replaces the Huygens principle which is lacking when the dimension is even.

Let us recall the Herglotz-Petrovskii formula for the free wave equation. Consider \( E^0(t, x, x_0) \) the Green’s function of (0.1) with \( n \geq 2 \) even, \( V = 0 \), \( \tau = 0 \), \( F = 0 \) and \( f_1 = 0 \). More precisely, \( E^0(t, x, x_0) \) is the solution of
\[
\begin{cases}
(\partial_t^2 - \Delta_x) E^0 = 0, & t > 0, \ x \in \mathbb{R}^n, \\
E^0(0, x, x_0) = 0, \\
\partial_t E^0(0, x, x_0) = \delta(x - x_0),
\end{cases}
\]
(2.7)
where \( \delta(x - x_0) \) is the delta-function. For all \( a, b, c > 0 \) and all domains
\[
Q_{a,b,c} = \{ (t, x, x_0) : t \geq a + b + c, \ |x| \leq a, \ |x_0| \leq b \},
\]
by the Herglotz-Petrovskii formula, we have
\[
E^0(t, x, x_0) = \int_{S_{n-1}} \frac{d\omega(\sigma)}{(|x - x_0, \sigma| + t)^{n-1}}, \ (t, x, x_0) \in Q_{a,b,c}.
\]
(2.8)

Now, consider the operator
\[
\frac{\sin(t\Lambda)}{\Lambda} : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)
\]
with \( \Lambda = \sqrt{-\Delta} \). The solution \( u \) of (0.1) with \( V = 0 \), \( \tau = 0 \), \( F = 0 \) and \( f_1 = 0 \) is given by
\[
u(t) = \frac{\sin(t\Lambda)}{\Lambda} f_2.
\]

Applying (2.8), for all \( m, l \in \mathbb{N} \), we obtain an explicit formula of the kernel of \( h_{m,l}(t) \varphi_m \frac{\sin(t\Lambda)}{\Lambda} \varphi_l \) (see the proof of Lemma 5). Using this formula, we deduce the analytic continuation with respect to \( \theta \) of the family
\[
F' \left[ e^{-\delta(x)} h_{m,l}(t) \varphi_m \frac{\sin(t\Lambda)}{\Lambda} \varphi_l e^{-\delta(x)} \right] (t, \theta).
\]

Then, in Lemma 5, we establish the analytic continuation with respect to \( \theta \) of the family
\[
\sum_{m,l = 0}^{+\infty} F' \left[ e^{-\delta(x)} h_{m,l}(t) \varphi_m \frac{\sin(t\Lambda)}{\Lambda} \varphi_l e^{-\delta(x)} \right] (t, \theta).
\]

Combining the result of Lemma 5 with the Duhamel’s principle we show the convergence of (2.6) and we deduce Theorem 4 for \( n \) even.
Lemma 5. Let \( n \geq 4 \) be even. Then
\[
\sum_{m,l \in \mathbb{N}} F' \left[ e^{-\delta(x)} h_{m,l}(t) \varphi_m \frac{\sin(t\Lambda)}{\Lambda} \varphi_1 e^{-\delta(x)} \right] \left( t, \theta \right) : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)
\]
converge to a family of operators analytic with respect to \( \theta \), continuous with respect to \( t \in \mathbb{R} \) and satisfying the properties described in Theorem 4. Moreover,
\[
\sum_{m,l \in \mathbb{N}} F' \left[ e^{-\delta(x)} h_{m,l}(t) \varphi_m \frac{\sin(t\Lambda)}{\Lambda} \varphi_1 e^{-\delta(x)} \right] \left( t, \theta \right) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)
\]
converge to a family of operators analytic with respect to \( \theta \), \( C^1 \) with respect to \( t \in \mathbb{R} \) and satisfying the properties described in Theorem 4.

Proof. Let \( E_{m,l}(t, x, x_0) \) be the kernel of \( e^{-\delta(x)} h_{m,l}(t) \varphi_m \frac{\sin(t\Lambda)}{\Lambda} \varphi_1 e^{-\delta(x)} \). Since
\[
|x| + |x_0| \leq 2^{m+1} T + 2^{l+1} T, \quad x \in \text{supp} \varphi_m, \quad x_0 \in \text{supp} \varphi_l
\]
and \( h_{m,l}(t) = 0 \) for \( t \leq 2^{m+1} T + 2^{l+1} T + \frac{T}{\delta} \), the formula (2.8) implies
\[
E_{m,l}(t, x, x_0) = \int_{\mathbb{R}^{n-1}} e^{-\delta(x)} h_{m,l}(t) \varphi_m(x) \varphi_l(x_0) e^{-\delta(x_0)} \frac{\omega}{\langle x-x_0, \sigma \rangle + t} e^{-\delta(x)} \varphi_m(x_0) \varphi_l(x_0) e^{-\delta(x_0)}. \tag{2.9}
\]

Now notice that for \( \mathcal{J}(\theta) > 0 \), we have the representation
\[
F'[E_{m,l}(t, x, x_0)](t, \theta) = \int_{\mathbb{R}^{n-1}} e^{-i\theta(x-x_0, \sigma)} v_{m,l}(t, x, x_0, \sigma, \theta) d\omega(\sigma)
\]
with
\[
v_{m,l}(t, x, x_0, \sigma, \theta) = \sum_{k=0}^{+\infty} \left( e^{-\delta(x)} h_{m,l}(t + kT) \varphi_m(x) \varphi_l(x_0) e^{-\delta(x_0)} \right) \frac{e^{i\theta(x-x_0, \sigma) + kT + 1}}{\left( \langle x-x_0, \sigma \rangle + t + kT \right)^{n-1}}.
\]

Then, for \( 0 \leq t < T \), we get
\[
(-iT \partial_\theta)^{n-1} v_{m,l}(t, x, x_0, \sigma, \theta) = \sum_{k=0}^{+\infty} h_{m,l}(t + kT) e^{i\theta(x-x_0, \sigma) + kT + 1} e^{-\delta(x)} \varphi_m(x_0) \varphi_l(x_0) e^{-\delta(x_0)},
\]
\[
(-iT \partial_\theta)^{n-1} v_{m,l}(t, x, x_0, \sigma, \theta) = \left( h_{m,l}(t + 2^{m+1} T + 2^{l+1} T) e^{i(2^{m+1} + 2^{l+1}) \theta} + \frac{1}{(1-e^{\theta})} \right) e^{i\theta(x-x_0, \sigma) + kT + 1} e^{-\delta(x)} \varphi_m(x_0) \varphi_l(x_0) e^{-\delta(x_0)}. \tag{2.9}
\]
Let \( E(t, x, x_0) \) be the kernel of
\[
\sum_{m,l=0}^{+\infty} h_{m,l}(t) \varphi_m \frac{\sin(t\Lambda)}{\Lambda} \varphi_1.
\]
For \( \mathcal{J}(\theta) > 0 \), we have
\[
F'[E(t, x, x_0)](t, \theta) = \sum_{m,l=0}^{+\infty} F'[E_{m,l}(t, x, x_0)](t, \theta) = \int_{\mathbb{R}^{n-1}} e^{-i\theta(x-x_0, \sigma)} v(t, x, x_0, \sigma, \theta) d\omega(\sigma), \tag{2.10}
\]
where
\[
v(t, x, x_0, \sigma, \theta) = \sum_{m,l=0}^{+\infty} v_{m,l}(t, x, x_0, \sigma, \theta).
\]
Repeating the arguments used in the proof Lemma 3, we deduce that
\[
\sum_{m,l=0}^{+\infty} (1 - e^{i\theta}) (-iT \partial_\theta)^{n-1} v_{m,l}(t, x, x_0, \sigma, \theta)
\]
admits an analytic continuation to \( \{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{\delta T}{4} \} \). Notice that, for \( \Im(\theta) > 0 \), we have
\[
\sum_{m,l=0}^{+\infty} (1 - e^{i\theta}) (-iT\partial_\theta)^{n-1} v_{m,l}(t,x,\sigma,\theta) = (1 - e^{i\theta}) (-iT\partial_\theta)^{n-1} v(t,x,\sigma,\theta).
\]
Hence, \((-iT\partial_\theta)^{n-1} v\) admits a meromorphic continuation in \( \{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{\delta T}{4} \} \) with first order poles only at the points \( 2k\pi, k \in \mathbb{Z} \). Therefore, upon integration of \((-iT\partial_\theta)^{n-1} v\) with respect to \(\theta\) we find that in the region \( t \in [0,T], \sigma \in \mathbb{S}^{n-1}, x, x_0 \in \mathbb{R}^n, \theta \in \{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{\delta T}{4}, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \) the function \( v \) is analytic with respect to \(\theta\) and infinitely differentiable with respect to the remaining variables. In the same way we can establish this result for \( t \in [-T,T] \). Here, for \( \theta \in \{ \theta \in \mathbb{C} : |\theta| \leq \varepsilon_0, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \) with \( \varepsilon_0 \) sufficiently small, we have
\[
v = Ce^{-\delta(x)} e^{-\delta(x_0)} \theta^{n-2} \log(\theta) + \tilde{v},
\]
where \( \tilde{v} \) is analytic at \( \theta = 0 \) and \( C \) is constant. Hence from (2.9), (2.10) and (2.11), we deduce that \( F'[E(t,x,0)](t,\theta) \) admits an analytic continuation, with respect to \( \theta \), to
\[
\{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{\delta T}{4}, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \}.
\]
Moreover, for all \( \theta \in \{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{\delta T}{4}, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \), we have
\[
F'[E(t,x,0)](t,\theta) = e^{-(\frac{\delta T}{4}+\frac{\delta T}{4})} H(t,x,0,\theta) e^{-(\frac{\delta T}{4}+\frac{\delta T}{4})} (x_0),
\]
where \( H(t,x,0,\theta) \) is \( C^\infty \) and bounded with respect to \( t \in \mathbb{R}, x \in \mathbb{R}^n \) and \( x_0 \in \mathbb{R}^n \). Also, for \( \theta \in \{ \theta \in \mathbb{C} : |\theta| \leq \varepsilon_0, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \) with \( \varepsilon_0 \) sufficiently small, we have the representation
\[
F'[E(t,x,0)](t,\theta) = C \left( \int_{\mathbb{S}^{n-1}} e^{-\theta(x - x_0,x_0)} \sin(t\Lambda) \frac{\Lambda}{\varphi \theta^\delta(x)} \right) e^{-\theta(x)} e^{-\delta(x_0)} \theta^{n-2} \log(\theta) + \tilde{E},
\]
where \( \tilde{E} \) is analytic at \( \theta = 0 \). Thus, the family of operators
\[
\sum_{m,l=0}^{+\infty} F' \left[ e^{-\theta(x)} h_{m,l}(t) \frac{\sin(t\Lambda)}{\Lambda} \varphi \theta^\delta(x) \right] (t,\theta)
\]
admits an analytic continuation to \( \{ \theta \in \mathbb{C} : \Im(\theta) > -\frac{\delta T}{4}, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \) which is defined by
\[
\left( \sum_{m,l=0}^{+\infty} F' \left[ e^{-\theta(x)} h_{m,l}(t) \frac{\sin(t\Lambda)}{\Lambda} \varphi \theta^\delta(x) \right] (t,\theta) g \right) (x) = \int_{\mathbb{R}^n} F'[E(t,x,0)](t,\theta) g(x) dx_0
\]
and, for \( \varepsilon_0 \) sufficiently small and \( \theta \in \{ \theta \in \mathbb{C} : |\theta| \leq \varepsilon_0, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \), we have the following representation
\[
\sum_{m,l=0}^{+\infty} F' \left[ e^{-\theta(x)} h_{m,l}(t) \varphi \frac{\sin(t\Lambda)}{\Lambda} \theta^\delta(x) \right] (t,\theta) = B_1(\theta) \theta^{n-2} \log(\theta) + C_1(t,\theta).
\]
Here \( C_1(t,\theta) \) is \( C^\infty \) and \( T \)-periodic with respect to \( t \) and analytic with respect to \( \theta \) and \( B_1(\theta) \) is analytic with respect to \( \theta \) for \( |\theta| \leq \varepsilon_0 \). From (2.13), the operators \( \left( \partial^j_\theta B_1(\theta) \right) |_{\theta=0} \) take the following form
\[
\left( \partial^j_\theta B_1(\theta) \right) |_{\theta=0} = e^{-\delta(x)} \left( P_j(x) \right) + \left( e^{-\delta(x)} g \right), \quad j \in \mathbb{N}, \ g \in L^2(\mathbb{R}^n)
\]
with \( P_j \) an homogeneous polynomial of order \( j \). This completes the proof. \( \square \)

Consider the operator
\[
\cos(t\Lambda) : \dot{H}^1(\mathbb{R}^n) \rightarrow \dot{H}^1(\mathbb{R}^n)
\]
with \( \Lambda = \sqrt{-\Delta} \). The solution \( u \) of (0.1) with \( V = 0, \tau = 0, F = 0 \) and \( f_2 = 0 \) is given by
\[
u(t) = \cos(t\Lambda) f_1.
\]
To prove the meromorphic continuation of the transformation $F'$ of all solutions of the free wave equation we must show the following.

**Lemma 6.** Let $n \geq 4$ be even. Then, the family of operators

$$
\sum_{m,l \in \mathbb{N}} F' \left[ e^{-\delta(x)} h_{m,l}(t) \varphi_m \cos(t\Lambda) e^{-\delta(x)} \right] (t, \theta) : H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n),
$$

converges to a family of operators analytic with respect to $\theta$, continuous with respect to $t \in \mathbb{R}$ and satisfying the properties described in Theorem 4 and

$$
\sum_{m,l \in \mathbb{N}} F' \left[ e^{-\delta(x)} h_{m,l}(t) \varphi_m \cos(t\Lambda) e^{-\delta(x)} \right] (t, \theta) : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)
$$

converges to a family of operators analytic with respect to $\theta$, $C^1$ with respect to $t \in \mathbb{R}$ and satisfying the properties described in Theorem 4.

**Proof.** Choose $\alpha \in C^\infty(\mathbb{R})$ such that $0 \leq \alpha(t) \leq 1$, $\alpha(t) = 0$ for $t \leq \frac{T}{n}$ and $\alpha(t) = 1$ for $t \geq \frac{T}{n}$. An application of the Duhamel’s principle (see [11]) yields

$$
\cos(t\Lambda) = \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda} \left[ \partial_t^2, \alpha \right] (s) \cos(s\Lambda) ds, \quad t \geq T.
$$

Let $(\chi_m)_m$ be a family of functions lying in $C^\infty_0(\mathbb{R}^n)$ such that for $m \geq 2$, $\chi_m = 1$ on

$$
\{ x : 2^{m-1}T - T \leq |x| \leq 2^{m+1}T + T \},
$$

and

$$
\supp \chi_m \subset \{ x : 2^{m-1}T - \frac{T}{4} \leq |x| \leq 2^{m+1}T + \frac{T}{4} \},
$$

$$
\supp \chi_1 \subset \{ x : |x| \leq 4T + \frac{T}{4} \},
$$

$$
\supp \chi_0 \subset \{ x : |x| \leq 2T + \frac{T}{4} \},
$$

and

$$
\sum_{m=0}^{+\infty} \chi_m(x) \leq 2.
$$

An application of the finite speed of propagation yields

$$
\varphi_m \cos(t\Lambda) \varphi_l = \int_0^T \varphi_m \frac{\sin((t-s)\Lambda)}{\Lambda} \left[ \partial_t^2, \alpha \right] (s) \cos(s\Lambda) \varphi_l ds, \quad t \geq T \quad (2.14)
$$

Let $g_{m,l} \in C^\infty(\mathbb{R})$ be such that

$$
0 \leq g_{m,l}(t) \leq 1, \quad \sup_{m,l \in \mathbb{N}} \left\| \partial_t g_{m,l} \right\|_{L^\infty(\mathbb{R})} \leq C_j < \infty,
$$

$$
g_{m,l}(t) = 1 \quad \text{for} \quad t \geq 2^{m+1}T + 2^{l+1}T + 2T + \frac{T}{2},
$$

$$
g_{m,l}(t) = 0 \quad \text{for} \quad t \leq 2^{m+1}T + 2^{l+1}T + 2T + \frac{T}{3},
$$

Applying the arguments of Lemma 5, we can show that

$$
\sum_{m,l = 0}^{+\infty} F' \left[ e^{-\delta(x)} g_{m,l}(t) \varphi_m \frac{\sin((t-s)\Lambda)}{\Lambda} \chi_l e^{-\delta(x)} \right] (t, \theta)
$$
admits an analytic continuation described in Lemma 5 uniformly with respect to \( s \in [0, \frac{T}{4}] \). Also, since 
\[ 1 - g_{m,t}(t) = 0 \text{ for } t \geq 2^{m+1}T + 2^l + 1 + 3T, \]
repeating the arguments used for \( n \) odd we show that
\[
\sum_{m,l=0}^{+\infty} F' \left[ e^{-\delta(x)} h_{m,t}(t)(1 - g_{m,t}(t)) \varphi_m \frac{\sin((t-s)\Lambda)}{\Lambda} \chi(t)e^{-\delta(x)} \right](t, \theta)
\]
admits an analytic continuation to \( \mathcal{H}(\theta) > -\frac{T}{4} \) uniformly with respect to \( s \in [0, \frac{T}{4}] \). Moreover, since \( [\partial_t^2, \alpha](t) = 0 \) for \( t \notin \left[ \frac{T}{4}, \frac{T}{2} \right] \) we have
\[
e^{-\delta(x)} \left[ \partial_t^2, \alpha \right](s) \cos(s\Lambda)e^{-\delta(x)} \in L(H^1(\mathbb{R}^n))
\]
and
\[
\left\| e^{-\delta(x)} \left[ \partial_t^2, \alpha \right](s) \cos(s\Lambda)\varphi_t e^{-\delta(x)} \right\|_{L(H^1(\mathbb{R}^n))} \lesssim \left\| e^{-\delta(x)} \left[ \partial_t^2, \alpha \right](s) \cos(s\Lambda)\varphi_t e^{-\delta(x)} \right\|_{L(H^1(\mathbb{R}^n))} \lesssim 1.
\]

Thus, applying \( F' \) to (2.14), for \( \mathcal{H}(\theta) > 0 \), we obtain
\[
F' \left[ e^{-\delta(x)} h_{m,t}(t) \varphi_m \cos(t\Lambda)\varphi_t e^{-\delta(x)} \right](t, \theta)
\]
\[
= \int_0^{\frac{T}{2}} F' \left[ e^{-\delta(x)} g_{m,t}(t) \varphi_m \frac{\sin((t-s)\Lambda)}{\Lambda} \chi(t)e^{-\delta(x)} \right](t, \theta)e^{-\delta(x)} \left[ \partial_t^2, \alpha \right](s) \cos(s\Lambda)\varphi_t e^{-\delta(x)} ds
\]
\[
+ \int_0^{\frac{T}{2}} F' \left[ e^{-\delta(x)} h_{m,t}(t)(1-g_{m,t}(t)) \varphi_m \frac{\sin((t-s)\Lambda)}{\Lambda} \chi(t)e^{-\delta(x)} \right](t, \theta)e^{-\delta(x)} \left[ \partial_t^2, \alpha \right](s) \cos(s\Lambda)\varphi_t e^{-\delta(x)} ds.
\]

From this last representation we deduce the analytic continuation of
\[
\sum_{m,l=0}^{+\infty} F' \left[ e^{-\delta(x)} h_{m,t}(t) \varphi_m \cos(t\Lambda)\varphi_t e^{-\delta(x)} \right](t, \theta).
\]

\[ \square \]

**Proof of Theorem 4.** We obtain the analytic continuation of \( F' \left[ e^{-\delta(x)} U_0(t)e^{-\delta(x)} \right](t, \theta) \) by combining Lemma 4, Lemma 5 and Lemma 6 (see also the proof of Lemma 2 in [12]).

Applying Theorem 4 to the representation (1.5), we show easily the following (see also Lemma 6 in [28]).

**Corollary 1.** The family of operators
\[
e^{-\delta(x)} J(t, \theta)e^{-\delta(x)} : \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n}) \to \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n})
\]
admits an analytic continuation with respect to \( \theta \), from \( \{ \theta \in \mathbb{C} : \mathcal{H}(\theta) > 0 \} \) to \( \{ \theta \in \mathbb{C} : \mathcal{H}(\theta) > -\frac{T}{4} \} \) for \( n \) odd and to
\[ \{ \theta \in \mathbb{C} : \mathcal{H}(\theta) > -\frac{T}{4}, \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^- \} \]
for \( n \) even. Moreover, for \( n \) even, there exists \( \varepsilon_0 > 0 \) such that for \( \theta \in \{ \theta \in \mathbb{C} : \theta \notin 2\pi\mathbb{Z} + i\mathbb{R}^-, |\theta| \leq \varepsilon_0 \} \) we have the representation
\[
e^{-\delta(x)} J(t, \theta)e^{-\delta(x)} = B_2(\theta)\theta^{n-2}\log(\theta) + C_2(t, \theta),
\]
where \( \log \) is the logarithm defined on \( \mathbb{C} \setminus i\mathbb{R}^- \), \( C_2(t, \theta) \) is \( C^\infty \) and \( T \)-periodic with respect to \( t \) and analytic with respect to \( \theta \), \( B_2(\theta) \) is analytic with respect to \( \theta \) for \( |\theta| \leq \varepsilon_0 \) for \( |\theta| \leq \varepsilon_0 \), for all \( l \in \mathbb{N} \) \( (\partial_\theta^l (B_2(t, \theta)))_{|\theta=0} \) are finite rank operators.
3. The meromorphic continuation of the resolvent

In this section we will apply the results of Sections 1 and 2 to show the meromorphic continuation of \( R(\theta) \) described in Theorem 4. According to Corollary 1 and Proposition 3, if we show that there exists \( \theta_0 \) such that for all \( G \in \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n}) \) the equation

\[
g + e^{-\delta(x)} J(t, \theta_0) Q(t)e^{\delta(x)} g = G \tag{3.1}
\]

has a solution \( g \in \mathcal{H}_{1,\text{per}}(\mathbb{R}^{1+n}) \), since \( e^{-\delta(x)} J(t, \theta_0) Q(t)e^{\delta(x)} \) is a compact operator, the operator

\[
\text{Id} + e^{-\delta(x)} J(t, \theta_0) Q(t)e^{\delta(x)}
\]

will be invertible and we can conclude by applying Theorem 8 of [28] and the analytic Fredholm Theorem. In order to solve (3.1), we introduce the following Hilbert spaces.

**Definition 4.** Let \( I \) be an interval of \( \mathbb{R} \). Then, we denote by \( \mathcal{H}_r(I \times \mathbb{R}^n) \) the space

\[
\mathcal{H}_r(I \times \mathbb{R}^n) = H^r(I \times \mathbb{R}^n) \times H^{r-1}(I \times \mathbb{R}^n).
\]

We can easily prove that there exists \( r > 0 \) such that \( W_0 \in \mathcal{L}(\mathcal{H}_r^* (\mathbb{R}^{1+n})) \). To solve (3.1) we will use the following result that we show by applying some arguments of Vainberg (see Theorem 5 in [28]).

**Theorem 5.** There exists \( A > r \) such that for all \( h \in \mathcal{H}_r^* (\mathbb{R}^{1+n}) \) the equation

\[
\varphi(t) + e^{-\delta(x)} W_0 \left( Q(t)e^{\delta(x)} \varphi(t) \right) (t) = h(t) \tag{3.2}
\]

admits a unique solution \( \varphi \in \mathcal{H}_r^* (\mathbb{R}^{1+n}) \).

**Proof.** For all \( A > r \), let \( \varphi = (\varphi_1, \varphi_2) \in \mathcal{H}_r^* (\mathbb{R}^{1+n}) \) be such that

\[
\varphi(t) + e^{-\delta(x)} \int_0^t U_0(t-s)Q(s)e^{\delta(x)} \varphi(s)ds = 0.
\]

Since \( \int_0^t U_0(t-s)Q(s)e^{\delta(x)} \varphi(s)ds \in \mathcal{H}_r^* (\mathbb{R}^{1+n}) \), we get \( e^{\delta(x)} \varphi \in \mathcal{H}_r^* (\mathbb{R}^{1+n}) \) and

\[
e^{\delta(x)} \varphi(t) = -\int_0^t U_0(t-s)Q(s)e^{\delta(x)} \varphi(s)ds.
\]

Thus, \( Z = e^{\delta(x)} \varphi_1 \) will be the solution of

\[
\begin{cases}
\partial_t^2(Z) - \Delta_x Z + V(t, x)Z = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
Z(t, \partial_t(Z))(0, x) = (0,0), & x \in \mathbb{R}^n,
\end{cases}
\]

and \( e^{\delta(x)} \varphi_2 = \partial_t(Z) \). Hence, \( \varphi = (0,0) \) and (3.2) admits a unique solution in \( \mathcal{H}_r^* (\mathbb{R}^{1+n}) \). For the proof of Theorem 5 it now suffices to show that (3.2) has a solution for \( h \) on a dense set of functions in \( \mathcal{H}_r^* (\mathbb{R}^{1+n}) \) and that this solution satisfies

\[
\|\varphi\|_{\mathcal{H}_r^* (\mathbb{R}^{1+n})} \lesssim \|h\|_{\mathcal{H}_r^* (\mathbb{R}^{1+n})}, \tag{3.3}
\]

for some \( A > r \). We take \( C^\infty(\mathbb{R}^{1+n}) \times C^\infty(\mathbb{R}^{1+n}) \cap e^{-\delta(x)} \mathcal{H}_r^* (\mathbb{R}^{1+n}) \) as our dense set. Clearly, we have

\[
(\text{Id} + W_0Q(t)) \varphi \in C^\infty(\mathbb{R}^{1+n}) \times C^\infty(\mathbb{R}^{1+n}) \cap \mathcal{H}_r^* (\mathbb{R}^{1+n}), \quad \varphi \in C^\infty(\mathbb{R}^{1+n}) \times C^\infty(\mathbb{R}^{1+n}) \cap \mathcal{H}_r^* (\mathbb{R}^{1+n}).
\]

Let \( h \in C^\infty(\mathbb{R}^{1+n}) \times C^\infty(\mathbb{R}^{1+n}) \cap e^{-\delta(x)} \mathcal{H}_r^* (\mathbb{R}^{1+n}) \) and consider the equation

\[
\varphi_1 + W_0Q(t)\varphi_1 = e^{\delta(x)} h. \tag{3.4}
\]

Equation (3.4) is a Volterra equation and it is uniquely solvable in \( C^\infty(\mathbb{R}^{1+n}) \times C^\infty(\mathbb{R}^{1+n}) \cap \mathcal{H}_r^* (\mathbb{R}^{1+n}) \). Now choose \( \varphi = e^{-\delta(x)} \varphi_1 \). We obtain

\[
e^{\delta(x)} \varphi + W_0Q(t)e^{\delta(x)} \varphi = e^{\delta(x)} h
\]

and it follows that \( \varphi \) is the solution of (3.2). It remains to show (3.3) and conclude by density. Let

\[
\psi = e^{-rt} \varphi, \quad \tilde{U}_0(t) = e^{-rt} U_0(t).
\]
Then, (3.2) can be rewritten in the form
\[
\psi + e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi = e^{-rt} h, \quad \text{where } (\hat{W}_0 Q(t)\psi)(t) = \int_0^t \hat{U}_0(t-s)Q(s)\psi(s)ds.
\] (3.5)

Repeating the arguments used in Proposition 3, we can easily prove that for any \( g \in \mathcal{H}_1(\mathbb{R}^{1+n}) \), we get
\[
\left\| e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} g \right\|_{\mathcal{H}_2([0,T]\times\mathbb{R}^n)} \lesssim \|g\|_{\mathcal{H}_1(\mathbb{R}^{1+n})}.
\]

Now let \( T_1 > 0 \). We have
\[
\left\| e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} g \right\|_{\mathcal{H}_2([0,T_1]\times\mathbb{R}^n)} \lesssim \|g\|_{\mathcal{H}_1(\mathbb{R}^{1+n})}.
\] (3.6)

Suppose that the function \( \hat{\psi} \) is equal to \( \psi \) for \( t < T_1 \) and
\[
\left\| \hat{\psi} \right\|_{\mathcal{H}_1(\mathbb{R}^{1+n})} \leq 2 \left\| \psi \right\|_{\mathcal{H}_1([0,T_1]\times\mathbb{R}^n)}.
\] (3.7)

Clearly, such a function exists (it is constructed by the means of extension operator). We recall that \( \psi \in C^\infty(\mathbb{R}^{1+n}) \times C^\infty(\mathbb{R}^{1+n}) \cap \mathcal{H}_1(\mathbb{R}^{1+n}) \) and, consequently, the right-hand side of (3.7) is bounded. The left hand-side of (3.6) is independent of the values of \( g(t) \) for \( t > T_1 \). It therefore follows from (3.6) for \( g = \hat{\psi} \) and (3.7) that
\[
\left\| e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi \right\|_{\mathcal{H}_2([0,T_1]\times\mathbb{R}^n)} \leq 2C_1 \left\| \psi \right\|_{\mathcal{H}_1([0,T_1]\times\mathbb{R}^n)}.
\] (3.8)

For all \( h = (h_1, h_2) \in \mathbb{C}^2 \), we set \( (h_1) = h_1 \). We denote by \( E(T_1, g) \) the quantity
\[
E(T_1, g) = \left\| g_1(T_1) \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \partial_t (g_1)(T_1) \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| g_2(T_1) \right\|_{L^2(\mathbb{R}^n)}^2,
\]
where \( g = (g_1, g_2) \). Notice that \( e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi \in C^\infty(\mathbb{R}^{1+n}) \times C^\infty(\mathbb{R}^{1+n}) \cap \mathcal{H}_2([0,T_1]\times\mathbb{R}^n) \),
\[
\left[ \left( e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi \right) \right]_{t=0}^1 = 0 \quad \text{and} \quad \left[ \partial_t \left( e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi \right) \right]_{t=0}^1 = 0.
\]
Thus, we have
\[
E(0, e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi) = 0.
\]

Then, for \( \psi \) compactly supported in \( x \), we obtain
\[
E(T_1, e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi) \leq \int_0^{T_1} \left\| \partial_t \left[ E(t, e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi) \right] \right\| dt,
\]
and an application of the Hölder’s inequality yields
\[
E(T_1, e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi) \lesssim \left\| e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi \right\|_{\mathcal{H}_2([0,T_1]\times\mathbb{R}^n)}^2.
\]
By density, we can extend this estimate to all \( \psi \in C^\infty(\mathbb{R}^{1+n}) \times C^\infty(\mathbb{R}^{1+n}) \cap \mathcal{H}_1(\mathbb{R}^{1+n}) \). Since \( \psi(0) = 0 \), combining this result with (3.8), we obtain
\[
E(T_1, e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi) \leq C \left\| \psi \right\|_{\mathcal{H}_1([0,T_1]\times\mathbb{R}^n)}^2 = C \int_0^{T_1} E(t, \psi) dt
\] (3.9)
with \( C > 0 \) independent of \( \psi \) and \( T_1 \). Clearly, for any \( \varphi_1, \varphi_2 \) we have
\[
E((T_1, \varphi_1 + \varphi_2) \leq 2 \left[ E((T_1, \varphi_1) + E((T_1, \varphi_2)) \right].
\]
Consequently, it follows from (3.5) and (3.9) that
\[
E(T_1, \psi) = E(T_1, e^{-rt} h - e^{-\delta(x)} \hat{W}_0 Q(t)e^{\delta(x)} \psi) \lesssim 2C \int_0^{T_1} E(t, \psi) dt + 2E(T_1, e^{-rt} h).
\]
Hence, by the Gronwall’s lemma we get
\[
E(T_1, \psi) = 2E(T_1, e^{-rt} h)e^{2CT_1}.
\]
Moreover, applying Theorem 8 of [28], for \( \alpha(t) = 0 \) for \( t \leq \frac{T}{2} \) and \( \alpha(t) = 1 \) for \( t \geq \frac{T}{2} \). Then, applying the analytic Fredholm theorem, we obtain that

\[
\mathcal{A} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

and we have

\[
\mathcal{A}^{-1}(t) = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

If we now multiply this inequality by \( e^{-CNT} \) and integrate with respect to \( T > 0 \) we obtain (3.3) with

\[
A = \sqrt{2C} + r.
\]

Proof of Theorem 1. As it was mentioned in the beginning of this section, it remains to solve (3.1) for some \( \theta \in \mathbb{C} \) with \( \Re(\theta) > rT \). Choose \( \theta = i(A + 1)T \) with \( A \) the constant of Theorem 5. Let \( G \in \mathcal{H}_{1, \text{per}}(\mathbb{R}^{1+n}) \) and let \( \alpha \in C^{\infty}(\mathbb{R}) \) be such that \( 0 \leq \alpha(t) \leq 1 \), \( \alpha(t) = 0 \) for \( t \leq \frac{T}{2} \) and \( \alpha(t) = 1 \) for \( t \geq \frac{T}{2} \). Then, \( h(t) = \alpha(t)G(t) \in \mathcal{H}_{1}^{[1]}(\mathbb{R}^{1+n}) \). Let \( \varphi \) be the solution of (3.2) with \( h(t) = \alpha(t)G(t) \). We have

\[
\varphi(t) + e^{-\delta(x)}W_{0}(Q(t)e^{\delta(x)}\varphi(t)) = \alpha(t)G(t).
\]

According to Proposition 2, since \( \varphi \in \mathcal{H}_{1}^{[1]}(\mathbb{R}^{1+n}) \), applying \( F' \) to both sides of (3.10) at \( \theta = \theta_{0} \), we obtain

\[
F' [\varphi](t, \theta_{0}) + e^{-\delta(x)}J(t, \theta_{0})Q(t)e^{\delta(x)}F' [\varphi](t, \theta_{0}) = F'[\alpha(t)G(t)](t, \theta_{0}).
\]

Since \( G(t) \) is \( T \)-periodic with respect to \( t \), for \( 0 \leq t < T \), we have

\[
F'[\alpha(t)G(t)](t, \theta_{0}) = \left( e^{-(A+1)t} \sum_{k=0}^{+\infty} \alpha(t + kT)e^{-(A+1)kT} \right) G(t)
\]

\[
= e^{-(A+1)t} \left( \alpha(t) + \frac{e^{-(A+1)T}}{1 - e^{-(A+1)T}} \right) G(t).
\]

Let \( p_{1}(t) \) be the \( C^{\infty} \) and \( T \)-periodic function defined by

\[
p_{1}(t) = e^{-(A+1)t} \left( \alpha(t) + \frac{e^{-(A+1)T}}{1 - e^{-(A+1)T}} \right), \quad 0 \leq t < T.
\]

Since \( \alpha(t) \geq 0 \), we have \( p_{1}(t) > 0 \) and according to (3.11),

\[
g(t) = \frac{F'[\varphi(t)](t, \theta_{0})}{p_{1}(t)}
\]

is a solution of (3.1). Thus, from Proposition 3, we deduce that

\[
\text{Id} + e^{-\delta(x)}J(t, \theta_{0})Q(t)e^{\delta(x)}
\]

is invertible. Then, applying the analytic Fredholm theorem, we obtain that

\[
F'[e^{-\delta(x)}U(t, 0)e^{-\delta(x)}](t, \theta) : \mathcal{H}_{1}(\mathbb{R}^{n}) \to \mathcal{H}_{1}(\mathbb{R}^{n})
\]

admits the following meromorphic continuation

\[
F'[e^{-\delta(x)}U(t, 0)e^{-\delta(x)}](t, \theta) = \left( \text{Id} + e^{-\delta(x)}J(t, \theta)Q(t)e^{\delta(x)} \right)^{-1} F'[e^{-\delta(x)}U_{0}(t)e^{-\delta(x)}](t, \theta).
\]

Moreover, applying Theorem 8 of [28], for \( n \) even and \( \theta \in \{ \theta \in \mathbb{C} : \theta \notin 2\pi \mathbb{Z} + i\mathbb{R}^{-}, |\theta| \leq \varepsilon_{0} \} \), we obtain the following representation

\[
F'[e^{-\delta(x)}U(t, 0)e^{-\delta(x)}](t, \theta) = \theta^{-m} \sum_{j \geq 0} \left( \frac{\theta}{R_{j}(\log \theta)} \right)^{j} P_{j,t}(\log \theta) + C(t, \theta),
\]

where \( C(t, \theta) \) is analytic with respect to \( \theta \), \( R_{j} \) is a polynomial, the \( P_{j,t} \) are polynomials of order at most \( j \), and \( \log \) is the logarithm defined on \( \mathbb{C} \setminus i\mathbb{R}^{-} \). Also, \( C(t, \theta) \) and the coefficients of the polynomials \( R_{j} \) and \( P_{j,t} \)

are \( C^{\infty} \) and \( T \)-periodic with respect to \( t \). We conclude by applying (0.10) and Lemma 1. \( \square \)
4. Applications to local energy decay and Strichartz estimates

In this section, we apply the result of Theorem 1 to prove estimates (0.7) and (0.9). We start by showing that assumption (H3) implies the local energy decay (0.7). Then, by combining this result with some arguments of [23] and [19], we establish the global Strichartz estimates (0.9). In Subsection 4.3 we give examples of potential $V(t,x)$ such that (H3) is fulfilled.

4.1. Local energy decay. The goal of this subsection is to prove the local energy decay introduced in Theorem 2. For this purpose, we need the following.

Lemma 7. Assume (H1) and (H2) fulfilled. Then, for all $0 \leq t \leq T$,

$$e^{-(D+\varepsilon)(x)}U(t,0)\in \mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))$$

and we have

$$\left\| e^{-(D+\varepsilon)(x)}U(t,0) \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \leq C;$$

$$\left\| e^{D(x)}U(t,0)e^{-(D+\varepsilon)(x)} \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \leq C$$

with $C$ independent of $t$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$. Applying the Duhamel’s principle we obtain these representations

$$U(t,0) = U_0(t) - \int_0^t U(t,s)Q(s)U_0(s)ds, \quad (4.3)$$

$$U(t,0) = U_0(t) - \int_0^t U_0(t-s)Q(s)U(s,0)ds. \quad (4.4)$$

It is well known that $e^{\mp(D+\varepsilon)(x)}U_0(t)e^{\pm(D+\varepsilon)(x)} \in \mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))$ and

$$C_1 = \sup_{t\in[0,T]} \left\| e^{\mp(D+\varepsilon)(x)}U_0(t)e^{\pm(D+\varepsilon)(x)} \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} < +\infty.$$

Since the multiplication by $e^{-\varepsilon(x)}$ is continuous from $\mathcal{H}_1(\mathbb{R}^n)$ to $\mathcal{H}_1(\mathbb{R}^n)$ we have

$$e^{-(D+\varepsilon)(x)}U_0(t)e^{D(x)}, \quad e^{D(x)}U_0(t)e^{-(D+\varepsilon)(x)} \in \mathcal{H}_1(\mathbb{R}^n))$$

and, for all $0 \leq t \leq T$, we get

$$\left\| e^{-(D+\varepsilon)(x)}U_0(t)e^{D(x)} \right\|_{\mathcal{H}_1(\mathbb{R}^n)} \leq C_1 \left\| e^{-(D+\varepsilon)(x)}U_0(t)e^{D(x)} \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \leq C_2; \quad (4.5)$$

$$\left\| e^{D(x)}U_0(t)e^{-(D+\varepsilon)(x)} \right\|_{\mathcal{H}_1(\mathbb{R}^n)} \leq C_1 \left\| e^{(D+\varepsilon)(x)}U_0(t)e^{-(D+\varepsilon)(x)} \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \leq C_2. \quad (4.6)$$

Hence, multiplying the representation (4.3) on the right by $e^{-(D+\varepsilon)(x)}$ and on the left by $e^{D(x)}$, we obtain

$$e^{-(D+\varepsilon)(x)}U(t,0)e^{D(x)}\varphi = e^{-(D+\varepsilon)(x)}U_0(t)e^{D(x)}\varphi$$

$$- \int_0^t \left( e^{-(D+\varepsilon)(x)}U(t,s)Q(s)e^{(D+\varepsilon)(x)} \right) e^{-(D+\varepsilon)(x)}U_0(s)e^{D(x)}\varphi ds.$$

Then applying (H1), we have $e^{-(D+\varepsilon)(x)}U(t,s)Q(s)e^{(D+\varepsilon)(x)} \in \mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))$ and, from (4.5) we deduce, for all $0 \leq t \leq T$, the estimate

$$\left\| e^{-(D+\varepsilon)(x)}U(t,0)e^{D(x)} \varphi \right\|_{\mathcal{H}_1(\mathbb{R}^n)} \leq \left( C_2 + C_2 \int_0^t \left\| e^{-(D+\varepsilon)(x)}U(t,s)Q(s)e^{(D+\varepsilon)(x)} \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} ds \right) \left\| \varphi \right\|.$$

It follows

$$\left\| e^{-(D+\varepsilon)(x)}U(t,0)e^{D(x)} \varphi \right\|_{\mathcal{H}_1(\mathbb{R}^n)} \leq (C_2 + C_2T) \left\| \varphi \right\|_{\mathcal{H}_1(\mathbb{R}^n)}, \quad 0 \leq t \leq T.$$


Thus, by density $e^{-(D+\varepsilon)(x)}U(t,0)e^{D(x)} \in \mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))$ and
\[
\left\| e^{-(D+\varepsilon)(x)}U(t,0)e^{D(x)} \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \leq C.
\]
Combining (4.4) and (4.6) with the same arguments, we obtain
\[
e^{-(D+\varepsilon)(x)}U(t,0)e^{D(x)} \in \mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))
\]
and
\[
\left\| e^{-(D+\varepsilon)(x)}U(t,0)e^{D(x)} \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \leq C_2 + C_4T, \quad 0 \leq t \leq T.
\]

Proof of Theorem 2. From the results of Theorem 1 and assumption (H3), by applying the arguments of the proof of Lemma 3 and Lemma 4 of [12], we obtain
\[
\left\| e^{-D(x)}U(kT,0)e^{-D(x)} \right\|_{\mathcal{L}(\mathcal{H}_1(\mathbb{R}^n))} \lesssim p(kT), \quad k \in \mathbb{N}.
\]
Combining this result with Lemma 7 we obtain (0.7) (see the proof of Lemma 3 and Lemma 4 in [12]).

4.2. Global Strichartz estimates. The goal of this subsection is to prove Theorem 3. For this purpose we need to show the $L^2$-integrability of the local energy (see [3], [23] and [15]) which takes the following form.

Proposition 4. Assume (H1), (H2) and (H3) fulfilled and let $n \geq 3$, $0 \leq \gamma \leq 1$. Let $f \in \mathcal{H}_\gamma(\mathbb{R}^n)$ and $F \in e^{-(\frac{3D+\varepsilon}{2})(x)}L^2_{\frac{n}{2}}(\mathbb{R}^+, \mathcal{H}_\gamma^2(\mathbb{R}^n))$. Then the solution $u$ of (0.1) with $\tau = 0$ satisfies the estimate
\[
\int_0^{+\infty} \left\| \left( e^{-(D+2\varepsilon)(x)}u(t), e^{-(D+2\varepsilon)(x)}u_0(t) \right) \right\|^2_{\mathcal{H}_\gamma(\mathbb{R}^n)} dt \lesssim \left( \| f \|_{\mathcal{H}_\gamma(\mathbb{R}^n)} + \| e^{(\frac{3D+\varepsilon}{2})(x)}F \|_{L^2_{\frac{n}{2}}(\mathbb{R}^+, \mathcal{H}_\gamma^2(\mathbb{R}^n))} \right)^2 \quad (4.7)
\]
with $C$ only depending on $n$ and $\gamma$.

In fact estimate (4.7) is the main point in the proof of Theorem 3. Combining (4.7) with some arguments of [23], including applications of the Kriest-Chisliev and the Duhamel’s principle, we establish estimates (0.9). We start with the proof of Proposition 4.

Proof of Proposition 4. First notice that for the free wave equation, $f \in \mathcal{H}_\gamma(\mathbb{R}^n)$ and $r > 0$ we have
\[
\int_{\mathbb{R}} \left\| e^{-r(x)}U_0(t)f \right\|^2_{\mathcal{H}_\gamma(\mathbb{R}^n)} dt \lesssim \| f \|^2_{\mathcal{H}_\gamma(\mathbb{R}^n)}. \quad (4.8)
\]
To obtain this estimate for $0 \leq \gamma \leq 1$ we can apply a result of Smith and Sogge.

Lemma 8. ([19], Lemma 2.2) Let $\gamma \leq \frac{n-1}{2}$ and let $\varphi \in S(\mathbb{R}^n)$. Then
\[
\int_{\mathbb{R}} \left\| \varphi e^{\pm it\Lambda}f \right\|^2_{H^r(\mathbb{R}^n)} dt \leq C(\varphi, n, \gamma) \| f \|^2_{H^r(\mathbb{R}^n)}. \quad (4.9)
\]

In [19] the authors consider only odd dimensions $n \geq 3$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, but the proof of this lemma goes without any change for even dimensions. Moreover, to prove (4.9) for $\varphi \in C_0^\infty(\mathbb{R}^n)$, [19] consider the following
\[
\int_{\mathbb{R}} \left\| \varphi e^{\pm it\Lambda}f \right\|^2_{H^r(\mathbb{R}^n)} dt = \int_0^{+\infty} \int_{\mathbb{R}^n} \left( 1 + |\xi|^2 \right)^\gamma \left| \int_{\mathbb{R}^n} \hat{\varphi}(\xi - \eta)\hat{f}(\eta)\delta(\tau - |\eta|)d\eta \right|^2 d\xi d\tau
\]
and they only exploit the fact that $\hat{\varphi} \in S(\mathbb{R}^n)$. Thus, (4.9) is still true for $\varphi \in S(\mathbb{R}^n)$. Setting $(u_0(t), \partial_\tau(u_0)(t)) = U_0(t)f$ we have the representation
\[
u_0(t) = \cos(t\Lambda)f_1 + \frac{\sin(t\Lambda)}{\Lambda}f_2
\]
and, since $e^{-t(x)} \in \mathcal{S}(\mathbb{R}^n)$, (4.8) follows immediately.

Passing to the estimate of $e^{-(D+2\varepsilon)(x)} U(t,0)f$, we obtain from (4.3) that
\[
\left\| e^{-(D+2\varepsilon)(x)} U(t,0)f \right\|_{\mathcal{H}_s(\mathbb{R}^n)} \leq \left\| e^{-(D+2\varepsilon)(x)} U_0(t)f \right\|_{\mathcal{H}_s(\mathbb{R}^n)} + \int_0^t \left\| e^{-(D+2\varepsilon)(x)} U(t,s)e^{-(D+\varepsilon)(x)} \left( e^{(D+\varepsilon)(x)} Q(s)U_0(s)f \right) \right\|_{\mathcal{H}_s(\mathbb{R}^n)} ds.
\]

Notice that for $0 \leq \gamma \leq 1$
\[
\left\| \int_0^t e^{-(D+2\varepsilon)(x)} U(t,s)e^{-(D+\varepsilon)(x)} \left( e^{(D+\varepsilon)(x)} Q(s)U_0(s)f \right) ds \right\|_{\mathcal{H}_s(\mathbb{R}^n)} \leq \left\| e^{-\varepsilon(x)} \int_0^t e^{-(D+\varepsilon)(x)} U(t,s)e^{-(D+\varepsilon)(x)} \left( e^{(D+\varepsilon)(x)} Q(s)U_0(s)f \right) ds \right\|_{\mathcal{H}_s(\mathbb{R}^n)}.
\]

Since the multiplication by $e^{-\varepsilon(x)}$ is continuous from $\mathcal{H}_1(\mathbb{R}^n)$ to $\mathcal{H}_1(\mathbb{R}^n)$, it follows
\[
\left\| \int_0^t e^{-(D+2\varepsilon)(x)} U(t,s)e^{-(D+\varepsilon)(x)} \left( e^{(D+\varepsilon)(x)} Q(s)U_0(s)f \right) ds \right\|_{\mathcal{H}_s(\mathbb{R}^n)} \leq \int_0^t \left\| e^{-(D+\varepsilon)(x)} U(t,s)e^{-(D+\varepsilon)(x)} \left( e^{(D+\varepsilon)(x)} Q(s)U_0(s)f \right) ds \right\|_{\mathcal{H}_s(\mathbb{R}^n)} ds
\]

and applying Theorem 2 it follows
\[
\left\| \int_0^t e^{-(D+2\varepsilon)(x)} U(t,s)e^{-(D+\varepsilon)(x)} \left( e^{(D+\varepsilon)(x)} Q(s)U_0(s)f \right) ds \right\|_{\mathcal{H}_s(\mathbb{R}^n)} \leq \left( p(t)1_{\mathbb{R}^+}(t) + \left( e^{(D+\varepsilon)(x)} Q(t)U_0(t)f \right) \right)(t), \quad t > 0.
\]

It is clear that (4.8) and (H1) imply
\[
\int_0^{+\infty} \left\| e^{(D+\varepsilon)(x)} Q(t)U_0(t)f \right\|_{\mathcal{H}_{s+1}(\mathbb{R}^n)}^2 dt = \int_0^{+\infty} \left\| e^{(D+\varepsilon)(x)} V(t,x)u_0(t) \right\|_{\mathcal{H}_{s+1}(\mathbb{R}^n)}^2 dt \lesssim \|f\|_{\mathcal{H}_s(\mathbb{R}^n)}^2.
\]

Since $p(t)1_{\mathbb{R}^+}(t) \in L^1(\mathbb{R})$, an application of the Young inequality for the convolution, combined with (4.8) and (4.10), yields (4.7) with $F = 0$.

In the general case ($F \neq 0$) consider the solution $v$ of (0.1) with $\tau = 0$, $f = (0,0)$ and $F \in e^{-\frac{D\varepsilon}{2}+2\varepsilon}(x) L^2_2(\mathbb{R}, \mathcal{H}_s(\mathbb{R}^n))$. Then
\[
\left( e^{-(D+2\varepsilon)(x)} v(t), e^{-(D+2\varepsilon)(x)} v(t) \right) = \int_0^t e^{-(D+2\varepsilon)(x)} U(t,s)e^{-(D+\varepsilon)(x)} \left( 0, e^{(D+\varepsilon)(x)} F(s) \right) ds.
\]

Clearly, we have
\[
\left( 0, e^{(D+\varepsilon)(x)} F(t) \right) \in L^2_2(\mathbb{R}^+, e^{-\frac{D\varepsilon}{2}(x)} \mathcal{H}_{s+1}(\mathbb{R}^n)) \hookrightarrow L^2_2(\mathbb{R}^+, \mathcal{H}_1(\mathbb{R}^n)) \hookrightarrow L^2_2(\mathbb{R}^+, \mathcal{H}_s(\mathbb{R}^n)).
\]

Exploiting the local energy decay of Theorem 2 and repeating the above arguments we get for $u = v$ estimate (4.7) with $F = (0,0)$. This completes the proof.

Now let us return to Theorem 3. In order to show (0.9) we need these two results.
Proposition 5. Let \( n \geq 3 \) and let \( 1 \leq \tilde{p}, \bar{q} \leq 2 \) and \( 0 \leq \gamma \leq 1 \) satisfy (0.9) with \( 1 < \tilde{p} \) and \( \bar{q}' < \frac{2(n-1)}{n-3} \). Let \( f \in \mathcal{H}_\gamma(\mathbb{R}^n) \), \( F \in L^p_t(\mathbb{R}, L^q_s(\mathbb{R}^n)) \) and let \( u \) be the solution of (0.1) with \( \tau = 0 \) and \( V = 0 \). Then, for all \( r > 0 \), we have

\[
\int_{\mathbb{R}} \left\| e^{-r(x)}u(t), e^{-r(x)}u(t) \right\|_{\mathcal{H}_\gamma(\mathbb{R}^n)}^2 dt \lesssim \left( \|f\|_{\mathcal{H}_\gamma(\mathbb{R}^n)} + \|F\|_{L^p_t(\mathbb{R}, L^q_s(\mathbb{R}^n))} \right)^2.
\]  

(4.11)

We prove Proposition 5 by combining Lemma 8 (for \( \varphi = e^{-r(x)} \)) with the arguments used in Proposition 2 of [23].

Proposition 6. Assume (H1), (H2) and (H3) fulfilled. Let \( n \geq 3 \) and let \( 1 \leq \tilde{p}, \bar{q} \leq 2 \) and \( 0 \leq \gamma \leq 1 \) satisfy (0.9) with \( 1 < \tilde{p} \) and \( \bar{q}' < \frac{2(n-1)}{n-3} \). Let \( f \in \mathcal{H}_\gamma(\mathbb{R}^n) \), \( F \in L^p_t(\mathbb{R}, L^q_s(\mathbb{R}^n)) \) and let \( u \) be the solution of (0.1) with \( \tau = 0 \). Then, we have

\[
\int_0^{+\infty} \left\| e^{-(D+2\epsilon)u(t)}(x) \right\|_{\mathcal{H}_\gamma(\mathbb{R}^n)}^2 dt \lesssim \left( \|f\|_{\mathcal{H}_\gamma(\mathbb{R}^n)} + \|F\|_{L^p_t(\mathbb{R}, L^q_s(\mathbb{R}^n))} \right)^2.
\]  

(4.12)

Combining (4.7) with Proposition 5 (see also the proof of Corollary 2 in [23]), we prove easily (4.12).

Proof of Theorem 3. We present the solution of (0.1) as a sum \( u = u_0 + v \), where \( u_0 \) is the solution of

\[
\begin{cases}
\partial_t^2 u_0 - \Delta_x u_0 = F(t, x), & t > 0, \ x \in \mathbb{R}^n, \\
(u_0, \partial_t u_0)(0, x) = (f_1(x), f_2(x)) = f(x), & x \in \mathbb{R}^n,
\end{cases}
\]

while \( v \) is the solution of the problem

\[
\begin{cases}
\partial_t^2 v - \Delta_x v + V(t, x)v = -V u_0, & t > 0, \ x \in \mathbb{R}^n, \\
(v, v_t)(0, x) = (0, 0), & x \in \mathbb{R}^n.
\end{cases}
\]

It follows from [9], that for \( u = u_0 \) estimates (0.9) hold. It remains to estimate \( v \). Next, we have

\[
v(t) = -\int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda} (V(s, \cdot)u_0(s) + V(s, \cdot)v(s)) ds.
\]

For \( e^{\epsilon(x)} V u_0 \) we have (4.11). Moreover, (H1), (4.7) and (4.11) imply

\[
\left\| e^{\epsilon(x)} V u_0 \right\|_{L^2_t(\mathbb{R}^+, \mathcal{H}_\gamma(\mathbb{R}^n))} \lesssim \left\| e^{(\frac{2\epsilon}{\epsilon-1})\epsilon(x)} V u_0 \right\|_{L^2_t(\mathbb{R}^+, \mathcal{H}_\gamma(\mathbb{R}^n))} \lesssim \left( \|f\|_{\mathcal{H}_\gamma(\mathbb{R}^n)} + \|F\|_{L^p_t(\mathbb{R}, L^q_s(\mathbb{R}^n))} \right).
\]  

(4.13)

Then we have

\[
\left\| e^{\epsilon(x)} (V u_0 + V v) \right\|_{L^2_t(\mathbb{R}^+, \mathcal{H}_\gamma(\mathbb{R}^n))} \lesssim \left( \|f\|_{\mathcal{H}_\gamma(\mathbb{R}^n)} + \|F\|_{L^p_t(\mathbb{R}, L^q_s(\mathbb{R}^n))} \right).
\]  

(4.14)

An application of Lemma 8 (with \( \varphi = e^{\epsilon(x)} \)) shows that the operator

\( S : \mathcal{H}_\gamma(\mathbb{R}^n) \ni g \rightarrow e^{-\epsilon(x)} e^{\pm i t \Lambda} g \in L^2(\mathbb{R}^+, \mathcal{H}_\gamma(\mathbb{R}^n)) \)

is bounded. The adjoint operator

\( (S^*(G))(x) = \int_0^{+\infty} e^{\pm i s \Lambda} e^{-\epsilon(x)} G(s, x) ds \)

is bounded as an operator from \( L^2(\mathbb{R}^+, \mathcal{H}_\gamma(\mathbb{R}^n)) \) to \( \mathcal{H}_\gamma(\mathbb{R}^n) \) and this yields

\[
\left\| e^{\epsilon(x)} h(s) \right\|_{\mathcal{H}_\gamma(\mathbb{R}^n)} \lesssim \|h\|_{L^2(\mathbb{R}^+, \mathcal{H}_\gamma(\mathbb{R}^n))}.
\]

Combining this result with the arguments exploited in the last section of [23], we obtain

\[
\left\| \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda} (V(s, \cdot)u_0(s) + V(s, \cdot)v(s)) ds \right\|_{L^2_t(\mathbb{R}^+, \mathcal{H}_\gamma(\mathbb{R}^n))} \lesssim \left\| e^{\epsilon(x)} (V u_0 + V v) \right\|_{L^2_t(\mathbb{R}^+, \mathcal{H}_\gamma(\mathbb{R}^n))}
\]  

(4.15)
where $C$.

An application of the Duhamel’s principle yields

\begin{equation}
\left\| \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda} (V(s, \cdot)u_0(s) + V(s, \cdot)v(s)) \, ds \right\|_{\dot{H}^\gamma(\mathbb{R}^n)} \leq \left\| e^{\varepsilon(x)} (V_{t_0} + Vv) \right\|_{L^2(\mathbb{R}^+, \dot{H}^\gamma(\mathbb{R}^n))}.
\end{equation}

Combining (4.14), (4.15) and (4.16) we deduce that for $u = v$ estimates (0.9) hold. This completes the proof of Theorem 3.

4.3. Examples of potential such that (H3) is fulfilled. The purpose of this subsection is to give examples of potential $V(t, x)$ satisfying (H1) and (H2) such that (H3) is fulfilled. Let $T_0 > 0$ and let $(V_{T_1})_{T_1 \geq T_0}$ be a set of functions lying in $C^\infty(\mathbb{R}^{1+n})$ and satisfying (H1) such that $V_{T_1}(t, x)$ is $T_1$-periodic with respect to $t$ and

\[ V(t, x) = V_0(x), \quad \text{for } T_0 \leq t \leq T_1. \]

Now consider the equation

\begin{equation}
\begin{cases}
\partial_t^2 v - \Delta v + V_0(x)v = 0, & t > 0, \ x \in \mathbb{R}^n, \\
(v, v_0)(0, x) = (f_1, f_2), & x \in \mathbb{R}^n
\end{cases}
\end{equation}

and let $\mathcal{V}(t)$ be the associate propagator. Notice that for $V(t, x) = V_{T_1}(t, x)$ we have

\[ U(t, s) = \mathcal{V}(t - s), \quad \text{for } T_0 \leq t, s \leq T_1. \]

Assume that $V_0(x)$ is chosen such that we have the following estimate

\[ \left\| e^{-D(x)} \mathcal{V}(t)e^{-D(x)} \right\|_{L(H_1(\mathbb{R}^n))} \lesssim p(t), \quad t > 0, \]

where $p(t)$ satisfies (0.8). We refer to [18] and [27] for sufficient conditions to obtain this estimate for stationary potentials (we can also choose $V_0(x) = 0$). Now consider the following result.

**Lemma 9.** Assume (H1), (H2) fulfilled. Then, for all $0 \leq t \leq T_0$,

\[ e^{D(x)} (U(t, 0) - \mathcal{V}(t)) e^{D(x)} \in L(H_1(\mathbb{R}^n)) \]

and we have

\[ \left\| e^{D(x)} (U(t, 0) - \mathcal{V}(t)) e^{D(x)} \right\|_{L(H_1(\mathbb{R}^n))} \lesssim C(T_0), \]

where $C(T_0)$ depend only on $T_0$.

**Proof.** An application of the Duhamel’s principle yields

\[ U(t, 0) = \mathcal{V}(t) + \int_0^t \mathcal{V}(t - s)Q_1(s)U(s, 0) \, ds \]

with

\[ Q_1(t) = \begin{pmatrix}
0 & 0 \\
V_0(x) - V(t, x) & 0
\end{pmatrix}. \]

Thus, we conclude by applying Lemma 7 and (H1).

Following the arguments exploited in the last section of [11], by applying Lemma 9, one can show that for $T = T_1$ with $T_1$ sufficiently large and for $V(t, x) = V_{T_1}(t, x)$ conditions (H1), (H2) and (H3) are fulfilled.
