HODGE INTEGRALS AND DEGENERATE CONTRIBUTIONS

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0. Introduction

0.1. Let $X$ be a nonsingular, projective, 3 dimensional complex algebraic variety. Let $\overline{M}_{g_D,n}(X, \beta)$ be the moduli space of stable maps from genus $g_D$ curves to $X$ representing the homology class $\beta \in H_2(X, \mathbb{Z})$. The Gromov-Witten invariants of $X$ are defined via tautological integrals over these moduli spaces of maps (against their virtual fundamental classes):

$$N^g_D(\gamma_1, \ldots, \gamma_n) = \int_{[\overline{M}_{g_D,n}(X, \beta)]^{vir}} \prod_{i=1}^{n} ev_i^*(\gamma_i),$$

where $ev_i$ is the $i^{th}$ evaluation map and $\gamma_i \in H^*(X, \mathbb{Z})$. As the moduli spaces are Deligne-Mumford stacks, the Gromov-Witten invariants take values in $\mathbb{Q}$. Let $T_X$ and $K_X$ be the tangent bundle and the canonical class of $X$. For a 3-fold, the dimension formula shows the virtual dimensions do not depend upon the genus:

$$\dim_{vir}(\overline{M}_{g_D}(X, \beta)) = 3g_D - 3 + \chi(T_X) = -K_X \cdot \beta.$$ 

If we restrict attention to a fixed curve class $\beta \in H_2(X, \mathbb{Z})$, there are two basic possibilities: $-K_X \cdot \beta = 0$ or $-K_X \cdot \beta > 0$ (the negative case is of no interest here since then the Gromov-Witten invariants vanish). We will always take $\beta \neq 0$.

0.2. Case $-K_X \cdot \beta = 0$. If $X$ is Calabi-Yau, this case holds for all classes $\beta$. Let $d$ be a positive integer. Let $C \subset X$ be a nonsingular genus $g < g_D$ curve of class $\beta/d$. The moduli space $\overline{M}_{gD}(X, \beta)$ contains a substack of maps with genus $g_D$ domains which factor through a $d$-fold cover of $C$. Under suitable conditions, this substack of maps covering $C$ is a connected component of $\overline{M}_{gD}(X, \beta)$. In the latter case, the contribution of $C$ to the genus $g_D$, class $\beta$ Gromov-Witten invariant of $X$ is well-defined. It is these degenerate contributions that are studied here. Degenerate contributions play a central role in identifying the integer quantities in the Gromov-Witten theory of $X$. These

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integrality properties remain a very mysterious part of the subject. In algebraic geometry, degenerate contributions are related to Hodge integrals over the moduli space of curves \( \overline{M}_{g,n} \) [FP]. In string theory, recent progress in the study of these contributions has been made by a link to M-theory [GV1], [GV2] (see also [MM]). While the mathematical results presented here overlap with the M-theoretic results of [GV2], the precise connection between the two approaches is still not completely understood. The differences are discussed below in Section 0.3.

Let \( C \subset X \) be a nonsingular genus \( g \) curve representing the class \([C] \in H_2(X, \mathbb{Z})\). For the degenerate analysis, we assume the normal bundle to \( C \) in \( X \) is general. Consider the moduli space of maps \( \overline{M}_{g+h}(X, d[C]) \). If \( g = 0 \) or \( 1 \), this moduli space will have a connected component equal to \( \overline{M}_{g+h}(C, d[C]) \). The contribution \( C_g(h, d) \) of \( C \) to the genus \( g+h \) Gromov-Witten is thus well-defined for \( g = 0, 1 \) and all values \( h \geq 0, d > 0 \). The above component claim relies on rigidity arguments which possibly fail for multiple covers of genus \( g \geq 2 \) curves. However, in the degree 1 case, \( \overline{M}_{g+h}(X, [C]) \) has a component equal to \( \overline{M}_{g+h}(C, [C]) \) for all \( g \) and \( h \). Hence, \( C_g(h, 1) \) is always well-defined. At present, because of the possibility of deformations in \( X \) away from \( C \), the correct definition of \( C_g(h, d) \) in general is not known to the author.

The contributions in case \( g = 0 \) have recently been calculated in algebraic geometry [FP] and string theory [GV1], [MM]:

\[
\sum_{h=0}^{\infty} C_0(h, 1) t^{2h} = \left( \frac{\sin(t/2)}{t/2} \right)^{-2},
\]

\[
C_0(h, d) = d^{2h-3} C_0(h, 1),
\]

where \( C \subset X \) is a nonsingular, rigid rational curve. The contribution \( C_0(0, d) = 1/d^3 \) is the Aspinwall-Morrison formula which had been proven previously by several different methods [AM], [M], [V].

A nonsingular curve \( C \subset X \) is rigid if \( H^0(C, N) = 0 \) where \( N \) is the normal bundle of \( C \) in \( X \). For rational \( C \), rigidity is equivalent to the bundle splitting \( N = O(-1) \oplus O(-1) \). Define \( C \subset X \) to be super-rigid if, for all non-constant maps of nonsingular curves \( \mu : C' \rightarrow C \),

\[
H^0(C', \mu^*(N)) = 0.
\]

It is clear rigidity and super-rigidity are equivalent in the rational case, but differ for higher genus. Super-rigidity is a generic condition on the normal bundle for elliptic curves in \( X \). Kley has informed the author
his existence result for rigid elliptic curves on complete intersection Calabi-Yau 3-folds also shows the existence of super-rigid elliptic curves [K].

The contributions $C_1(0, d)$ are easily computed for super-rigid elliptic curves $C$. The component of the moduli space $\overline{M}_1(X, d[C])$ corresponding to maps with image $C$ is nonsingular of dimension 0 (and equal to $\overline{M}_1(C, d[C])$). The points of $\overline{M}_1(C, d[C])$ correspond naturally to the set of subgroups of $\mathbb{Z} \oplus \mathbb{Z}$ of index $d$. Hence, after accounting for automorphisms,

$$C_1(0, d) = \frac{\sigma(d)}{d} = \sum_{i|d} \frac{1}{i}$$

(see, for example, [S]). All other contributions of an elliptic curve $C$ vanish by the following result.

**Theorem 1.** Let $C \subset X$ be a super-rigid elliptic curve. Then,

$$C_1(h, d) = 0$$

for all $h > 0$, $d > 0$.

This vanishing was conjectured by Gopakumar-Vafa in [GV1] and is derived in M-theory in [GV2]. The proof given here uses basic constructions related to the virtual fundamental class.

The degree 1 contributions $C_g(h, 1)$ take a very simple form.

**Theorem 2.** Let $g \geq 0$.

$$\sum_{h=0}^{\infty} C_g(h, 1)t^{2h} = \left( \frac{\sin(t/2)}{t/2} \right)^{2g-2}.$$  

Theorem 2 is derived in Section 2 by expressing the contributions $C_g(h, 1)$ as Hodge integrals over the moduli space of curves. The required integrals are then computed via geometric constructions, relations, and series manipulations. Theorem 2 is the main result of this paper.

The right side of Theorem 2 was encountered before in the following related result of [FP]:

$$1 + \sum_{h \geq 1} \sum_{i=0}^{h} t^{2h-1} k_i \int_{\overline{M}_{h,1}} \psi_1^{2h-2+i} \lambda_{h-i} = \left( \frac{\sin(t/2)}{t/2} \right)^{-k-1}. \quad (3)$$

Theorem 2 gives an interpretation of these Hodge integrals in the Gromov-Witten theory of Calabi-Yau 3-folds.
0.3. **M-theory predictions.** The method of [GV1], [GV2] is to con-
sider limits of type IIA string theory which may be conjecturally an-
yalyzed in M-theory. A remarkable proposal is made in [GV2] for the
form of the Gromov-Witten potential $\tilde{F}$ of a Calabi-Yau 3-fold $X$. Let

$$
\tilde{F}(t, q) = \sum_{g \geq 0} t^{2g-2} \tilde{F}_g(t, q),
$$

$$
\tilde{F}_g(t, q) = \sum_{0 \neq \beta \in H_2(X, \mathbb{Z})} N_{\beta}^g q^\beta,
$$

where $N_{\beta}^g$ is the genus $g$ Gromov-Witten invariant of $X$ in curve class
$\beta$. The potential $\tilde{F}$ differs from the full potential by the constant map
($\beta = 0$) contribution – the constant contributions have been calculated
in [FP], [GV1], [MM]. For each curve class $\beta \in H_2(X, \mathbb{Z})$ and genus $g$,
there is an integer $n_{\beta}^g$ counting BPS states in the associated M-theory.
The formula of [GV2] is:

$$
\tilde{F}(t, q) = \sum_{g, \beta} n_{\beta}^g t^{2g-2} \sum_{d > 0} \frac{1}{d^2} \left( \frac{\sin(dt/2)}{t/2} \right)^{2g-2} q^{4d\beta}.
$$

If $C^M_g(h, d)$ denotes the contribution of a single BPS state in genus
$g$ and class $\beta$ to the Gromov-Witten invariant in genus $g + h$ and class
d$\beta$, then formula (4) is equivalent to the equations:

$$
\sum_{h=0}^{\infty} C^M_g(h, 1)t^{2h} = \left( \frac{\sin(t/2)}{t/2} \right)^{2g-2},
$$

$$
C^M_g(h, d) = d^{2g+2h-3} C^M_g(h, 1).
$$

The first of these agrees with Theorem 2, so $C^M_g(h, 1) = C_g(h, 1)$. The
second is a generalization of (3) to $g \geq 0$. It is therefore reasonable to
interpret the states $n_{\beta}^0$ as counting embedded (virtual) curves of genus
0 and degree $\beta$ (even for the Calabi-Yau quintic these numbers $n_{\beta}^0$ are
at best virtual because of the existence of Vainsencher’s nodal rational
curves). However, when specialized to genus 1, the second equation
yields $C^M_1(0, d) = 1/d$ in contrast to $C_1(0, d) = \sigma(d)/d$. The (virtual)
count of embedded genus 1 curves should be derived from $\tilde{F}_1$ via the
multiple cover corrections $C^M_0(1, d)$ and $C_1(0, d)$ (as previously pursued
in [BCOV]). Gromov-Witten theory would predict the resulting number
to be virtually enumerative, and thus integral (this heuristic argument
for integrality is not a proof). The M-theoretic perspective predicts a
different correction of $\tilde{F}_1$ to yield integers via formula (3). Klemm has
checked the two genus 1 integrality predictions both hold in low degrees
for several Calabi-Yau 3 folds [Kl]. These integrality constraints are not
trivially dependent. No proofs of any of these integrality constraints are known to the author.

To find higher genus evidence for the formula (4), a direct computation of the potential \( \tilde{F} \) in the local Calabi-Yau case (\( \mathbb{P}^2 \) with canonical bundle) for low genera and degrees has been pursued by Klemm and Zaslow [KZ]. The Gromov-Witten invariants (in all genera) may be computed in this case by the virtual localization formula of [GP] and the holomorphic anomaly equation [BCOV]. The integrality predicted by (4) is a nontrivial constraint which is verified in all calculations.

At this point, it is not clear how to define or compute the general contributions \( C_g(h, d) \). One may hope a complete understanding of \( C_g(h, d) \) will lead to an integrality property of the Gromov-Witten potential of \( X \) distinct from (4).

0.4. Case \(-K_X \cdot \beta > 0\). In this case, the moduli spaces \( \overline{M}_{g_D}(X, \beta) \) have positive virtual dimensions. The Gromov-Witten invariants \( N^{g_D}_\beta(\gamma) \) of \( X \) then depend upon a vector of cohomology classes

\[
\gamma = (\gamma_1, \ldots, \gamma_k), \quad \gamma_i \in H^*(X, \mathbb{Z}).
\]

Let \( Y_i \subset X \) be general topological cycles dual to the classes \( \gamma_i \). If we wish to identify integers in this Gromov-Witten theory, degenerate contributions again play a role.

Let us assume we are in an ideal situation with respect to the moduli spaces of maps to \( X \). Let \( M^\text{Bir}_g(X, \beta) \) denote the moduli space of birational maps from smooth genus \( g \) domain curves. We assume first:

(i) The spaces \( M^\text{Bir}_g(X, \beta) \) are generically reduced and of the expected dimension for all \( g \leq g_D \).

There is then an enumerative integer \( E^{g_D}_\beta(\gamma) \) defined to equal the number of genus \( g_D \) maps of class \( \beta \) with smooth domains meeting all the cycles \( Y_i \). However, \( E^{g_D}_\beta(\gamma) \) will not equal \( N^{g_D}_\beta(\gamma) \). The difference arises from the following observation. For each \( g < g_D \), there are \( E^g_\beta(\gamma) \) maps with smooth genus \( g \) domains of class \( \beta \) satisfying the required incidence conditions. The Gromov-Witten invariant \( N^{g_D}_\beta(\gamma) \) receives a degenerate contribution from each of these lower genus solutions (via reducible genus \( g_D \) maps factoring through covers of the lower genus curves). As the genus \( g \) solution represents the class \( \beta \), the covers must be of degree 1. These degenerate contributions are therefore analogous to \( C_g(g_D - g, 1) \).

Dimension counts show maps multiple onto their image and maps with reducible images are not expected to contribute to \( N^{g_D}_\beta(\gamma) \). This is the second ideal assumption:
(ii) Maps in $\overline{M}_{gD}(X, \beta)$ multiple onto their image or with reducible image do not satisfy incidence conditions to the cycles $Y_i$. Let $C \subset X$ be a nonsingular, genus $g$ curve of class $\beta$ satisfying incidence conditions to the cycles $Y_i$. Assume further $C$ is infinitesimally rigid with respect to these incidence conditions. The contribution $C_g(h, X, \beta)$ of $C$ to the Gromov-Witten invariant $N^g_{\beta+h}(\gamma)$ is then well-defined: it is found in Section 3 to be an integral over the moduli space $\overline{M}_{g+h}(C, [C])$. This contribution is easily seen to be independent of $\gamma$.

The final ideal assumption is:

(iii) For all $g < g_D$, the solution maps counted by $E^g_{\beta}(\gamma)$ are nonsingular embeddings infinitesimally rigid with respect to the incidence conditions.

The ideal relation between Gromov-Witten theory and the enumerative invariants is:

$$N^g_{\beta}(\gamma) = \sum_{g=0}^{g_D} C_g(g_D - g, X, \beta) E^g_{\beta}(\gamma).$$

The validity of the relation (5) for $N^g_{\beta}(\gamma)$ requires assumptions (i), (ii), and (iii) for $g_D$, $\beta$, and $\gamma$.

The easiest 3-fold to consider is $X = \mathbb{P}^3$. As the divisor $-K_{\mathbb{P}^3}$ is ample, $-K_{\mathbb{P}^3} \cdot \beta > 0$ for all nonzero curve classes. The moduli spaces of maps to $\mathbb{P}^3$ are easily seen to be ideal in the above sense for the genera $g_D = 0, 1, 2$, all degrees $d > 0$, and all $\gamma$. The rigidity statements follow as usual from Bertini arguments (see [FuP]). Therefore, the ideal relation (5) holds in these genera. The equation

$$N^0_d = E^0_d$$

is well known for $\mathbb{P}^3$ (we drop $\gamma$ in these equations). In joint work with Getzler and Graber, we had computed

$$C_0(1, \mathbb{P}^3, d) = \frac{1 - 2d}{12},$$

$$N^1_d = \frac{1 - 2d}{12} E^0_d + E^1_d.$$

Equation (7) was used in Getzler’s study [Ge] of the genus 1 enumerative geometry of $\mathbb{P}^3$. Using Xiong’s calculations of low degree genus 2 Gromov-Witten invariants of $\mathbb{P}^3$ as data, Jinzenji and Xiong conjectured the contribution equation:

$$N^2_d = \frac{3 - 11d + 10d^2}{720} E^0_d - \frac{4d}{24} E^1_d + E^2_d.$$
These equations led Jinzenji and Xiong to recently conjecture a general formula \([J]\) analogous to Theorem 2:

\[
\sum_{h=0}^{\infty} C_g(h, X, \beta)t^{2h} = \left(\frac{\sin(t/2)}{t/2}\right)^{2g-2-K_X \cdot \beta}.
\]  

(9)

The contribution \(C_g(h, X, \beta)\) is calculated here by the method used in the proof of Theorem 2.

**Theorem 3.** The degenerate contributions \(C_g(h, X, \beta)\) are determined by formula (9).

Theorem 3 and relation (3) prove formulas (3), (4), (5) for \(g = 0, 1, 2\) and all degrees \(d > 0\) in \(P^3\). For higher genera, it is known the space of curves in \(P^3\) may be of excess dimension. For example, the moduli space \(\overline{M}_3(P^3, 4)\) has a 17 dimensional component, but is expected to be 16 dimensional. The definition of enumerative invariants is therefore not clear from a space curve point of view. However, the invariants \(E_{\beta}^g(\gamma)\) may still be defined by Theorem 3 from the Gromov-Witten invariants and equation (3). Perhaps an integrality property holds for \(E_{\beta}^g(\gamma)\) in some general context.

Algebraic 3-folds are special in Gromov-Witten theory since the (virtual) dimensions of the moduli spaces of stable maps do not depend upon the genus. A similar uniform treatment of degenerate contributions in higher dimensions will require new ideas. Graber has carried out a related degenerate analysis in the genus 0 Gromov-Witten theory of the Hilbert scheme of 2 points of \(P^2\) [Gr].

**0.5. Moduli of curves.** The Hodge integral approach taken here has an application to the geometry of the moduli space of nonsingular curves \(M_g\), \((g \geq 2)\). The tautological ring \(\mathcal{R}^*(M_g)\) is the subring of \(\mathbb{A}^*(M_g)\) generated by the \(\kappa\) classes (see [Mu]). The intersection calculus of \(\mathcal{R}(M_g)\) has a very rich structure. A detailed study by Faber of \(\mathcal{R}(M_g)\) for low genera has led to very precise conjectures of this ring structure [F1]. In particular, Faber has conjectured \(\mathcal{R}^*(M_g)\) is a Gorenstein ring with socle in degree \(g - 2\). In [GeP], the (conjectural) intersection pairing of \(\mathcal{R}(M_g)\) is directly linked to Gromov-Witten theory via (conjectural) Virasoro constraints on the descendent potential of \(P^2\). The computation here of the degenerate contributions \(C_g(h, 1)\) leads to a formula in \(\mathcal{R}^*(M_g)\) conjectured previously by Faber from evidence for \(g \leq 15\).
Theorem 4. For $g \geq 2$, the relation
\[ \sum_{i=0}^{g-2} (-1)^i \lambda_i \kappa_{g-2-i} = \frac{2g-1}{g!} \kappa_{g-2} \]
holds in $\mathcal{R}^*(M_g)$.

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1. Theorem 1

1.1. Super-rigidity. Let $C \subset X$ be a nonsingular elliptic curve in a Calabi-Yau 3-fold. The normal bundle $N$ is of rank 2 with trivial determinant. If $C$ is rigid, a straightforward argument shows $N$ contains a non-trivial degree 0 line sub-bundle $L$:
\[ 0 \to L \to N \to L^{-1} \to 0. \]

Conversely, such a filtration implies the rigidity of $C$. The curve $C$ is super-rigid if and only if $L$ is not a torsion element of the Picard group of $C$. While super-rigidity is a stronger condition on $N$ than rigidity, it is an open condition. Super-rigidity is required for the equality of moduli spaces proven in Proposition 1. Note super-rigidity implies $H^0(C', \mu^*(N)) = 0$ for every non-constant stable map $\mu : C' \to C$.

The moduli spaces $\overline{M}_{1+h}(C, d[C])$ and $\overline{M}_{1+h}(X, d[C])$ are Deligne-Mumford stacks with possibly nonreduced structures.

Proposition 1. Let $C \subset X$ be a nonsingular, super-rigid elliptic curve. The space of maps $\overline{M}_{1+h}(C, d[C])$ is a union of connected components of $\overline{M}_{1+h}(X, d[C])$ for all $h \geq 0$, $d > 0$.

Proof. There is a natural map:
\[ \iota : \overline{M}_{1+h}(C, d[C]) \to \overline{M}_{1+h}(X, d[C]). \]

By the super-rigidity of $C$, the locus of $\overline{M}_{1+h}(X, d[C])$ corresponding to maps with support in $C$ is a union of connected components of $\overline{M}_{1+h}(X, d[C])$. We will prove $\iota$ is an isomorphism onto these connected components.
It suffices to prove a lifting property for families of stable maps over Artinian local rings $A$. Let $\xi \in \text{Spec}(A)$ be the geometric point corresponding to the maximal ideal $m \subset A$. Let

$$\pi : \mathcal{F} \to \text{Spec}(A), \quad \mu : \mathcal{F} \to X$$

be a family of stable maps satisfying

$$\mu_\xi : \mathcal{F}_\xi \to C \subset X. \quad (10)$$

We will prove $\mu$ factors through $C$. This lifting implies the desired isomorphism property of $\iota$.

Let $I$ be the ideal sheaf of $C$ in $X$. We must prove the natural map

$$\phi : \mu^* (I) \to O_{\mathcal{F}}$$

is zero. Certainly $\phi$ has image in $mO_{\mathcal{F}}$ by the assumption (10) on the geometric fiber $\xi$. Hence, $\phi$ induces a natural map on $\mathcal{F}$:

$$\mu^* (I/I^2) \to mO_{\mathcal{F}}/m^2O_{\mathcal{F}} = (m/m^2) \otimes_C O_{\mathcal{F}_\xi}. \quad (11)$$

The restriction of $\mu^* (I/I^2)$ to $\mathcal{F}_\xi$ is simply $\mu^*_\xi (N^*)$. By the super-rigidity of $C$, the map (11) is zero. We conclude $\phi$ factors through $m^2O_{\mathcal{F}}$.

The above argument may be used to prove the following implication: if $\phi$ factors through $m^kO_{\mathcal{F}}$, then $\phi$ factors through $m^{k+1}O_{\mathcal{F}}$. Since $A$ is Artinian, $m$ is nilpotent. Hence, $\phi$ vanishes.

There are two perfect obstruction theories on $\overline{M}_{1+h}(C, d[C])$ obtained from considering the moduli problem of maps to $C$ and $X$ respectively (see [B], [BF], [LT]). Let

$$\pi : \mathcal{F} \to \overline{M}_{1+h}(C, d[C]),$$

$$\mu : \mathcal{F} \to C$$

be the universal family and universal map respectively. By super-rigidity $\pi_* \mu^* (N) = 0$ and $R^1 \pi_* \mu^* (N)$ is a rank $2h$ bundle. The two obstruction theories differ exactly by the bundle $R^1 \pi_* \mu^* (N)$. From the definition of the virtual class, we conclude:

$$C_1(h, d) = \int_{[\overline{M}_{1+h}(C, d[C])]^{\text{vir}}} c_{2h}(R^1 \pi_* \mu^* (N)). \quad (12)$$

1.2. Vanishing results. Let $E$ be any bundle on $C$. Consider the complex $R\pi_* \mu^*(E)$ in the derived category of coherent sheaves on $\overline{M}_{1+h}(C, d[C])$. Let $\mathcal{L}$ be a $\pi$-relatively ample polarization on $\mathcal{F}$. We may find an exact sequence of bundles on $\mathcal{F}$:

$$0 \to K \to \oplus \mathcal{L}^{-k} \to \mu^* (E) \to 0$$
for some positive integer $k$ [H]. As both $\pi_* K$ and $\pi_* L^{-k}$ vanish, we find a two term bundle resolution of $R\pi_* \mu^*(E)$:

$$[R^1\pi_* K \to R^1\pi_* \otimes L^{-k}] \cong R\pi_* \mu^*(E).$$

The Chern classes of $R\pi_* \mu^*(E)$ are defined by $c(R^1\pi_* K)/c(R^1\pi_* \otimes L^{-k})$. This definition is independent of two term resolutions in the derived category.

As $\pi_* \mu^*(N) = 0$ and $R^1\pi_* \mu^*(N)$ is a rank 2 bundle, we see (12) may now be rewritten as:

$$C_1(h, d) = \int_{[\overline{M}_{1+h}(C, d[C])]^{vir}} [c^{-1}(R\pi_* \mu^*(N))]_{2h}.$$ 

It is easy to find flat families of bundles on $C$ connecting $N$ and the trivial rank 2 bundle $I = \mathcal{O}_C \oplus \mathcal{O}_C$. For example, if $P$ is a sufficiently ample line bundle, both $N \otimes P$ and $I \otimes P$ will have nowhere vanishing sections:

$$0 \to \mathcal{O}_C \to N \otimes P \to P^2 \to 0,$$
$$0 \to \mathcal{O}_C \to I \otimes P \to P^2 \to 0.$$ 

Hence $N$ and $I$ are connected in the family of extensions of $P$ by $P^{-1}$. The integral

$$\int_{[\overline{M}_{1+h}(C, d[C])]^{vir}} [c^{-1}(R\pi_* \mu^*(I))]_{2h}$$

is clearly constant as $E$ varies in this family (for example, the two term resolutions of $R\pi_* \mu^*(E)$ may be chosen compatibly over the family). We conclude,

$$C_1(h, d) = \int_{[\overline{M}_{1+h}(C, d[C])]^{vir}} [c^{-1}(R\pi_* \mu^*(I))]_{2h}.$$ 

Now assume $h > 0$. Let $\gamma : \overline{M}_{1+h}(C, d[C]) \to \overline{M}_{1+h}$ be the natural map to the moduli space of curves. Let $\mathcal{E}$ denote the Hodge bundle on $\overline{M}_{1+h}$: the fiber of $\mathcal{E}$ over the moduli point $[F] \in \overline{M}_{1+h}$ is $H^0(F, \omega_F)$ (see [Mu]). Since

$$\pi_* \mu^*(I) = \mathcal{O}_{\mathcal{E}} \oplus \mathcal{O}_{\mathcal{E}},$$
$$R^1\pi_* \mu^*(I) = \gamma^*(\mathcal{E}^* \oplus \mathcal{E}^*),$$
we see $[c^{-1}(R\pi_* \mu^*(I))]_{2h}$ is a cohomology class pulled-back via $\gamma$ from $\overline{M}_{1+h}$. Hence, to complete the proof of Theorem 1, it suffices to show the following vanishing.

**Proposition 2.** Let $h > 0$. Then, $\gamma_* ([\overline{M}_{1+h}(C, d[C])]^{vir}) = 0.$
Fix a base point $p \in C$ for the course of the proof. We will consider the moduli space of 1-pointed maps $\overline{M}_{1+h,1}(C, d[C])$. Let 
$$\text{ev}^{-1}_1(p) = \overline{M}_{1+h,p}(C, d[C]) \subset \overline{M}_{1+h,1}(C, d[C]),$$

denote the subspace of maps for which the marking has image $p$. There is a canonical isomorphism obtained by the group law on $C$: 
$$\overline{M}_{1+h,1}(C, d[C]) \cong C \times \overline{M}_{1+h,p}(C, d[C]).$$

Let $\rho : \overline{M}_{1+h,1}(C, d[C]) \to \overline{M}_{1+h,p}(C, d[C])$ denote the canonical projection.

The perfect obstruction theory on $\overline{M}_{1+h,1}(C, d[C])$ may be obtained from a canonical distinguished triangle involving the cotangent complex of the Artin stack of prestable curves and the perfect obstruction theory relative to this Artin stack (see [B], [BF], [GrP]). These objects are naturally equivariant with respect to the natural group law on $C$ (see the constructions of [B], [BF]). Hence, the virtual class of $\overline{M}_{1+h,1}(C, d[C])$ is a pull-back of an algebraic cycle class on $\overline{M}_{1+h,p}(C, d[C])$. As the map $\gamma_1 : \overline{M}_{1+h,1}(C, d[C]) \to \overline{M}_{1+h,1}$ factors through $\overline{M}_{1+h,p}(C, d[C])$, we obtain

$$\gamma_1^*([\overline{M}_{1+h,1}(C, d[C])]_{\text{vir}}) = 0. \quad (13)$$

Consider now the commutative diagram obtained from the 1-pointed moduli spaces:

$$\begin{array}{ccc}
\overline{M}_{1+h,1}(C, d[C]) & \longrightarrow & \overline{M}_{1+h,1} \\
\downarrow & & \downarrow \\
\overline{M}_{1+h}(C, d[C]) & \longrightarrow & \overline{M}_{1+h}.
\end{array} \quad (14)$$

While (14) is not a fiber square, it is easy to see the following equality holds

$$\gamma_1^* \pi^* = \pi^* \gamma_s. \quad (15)$$

From the Axiom of contracting a point [BM], we see 
$$\pi^*([\overline{M}_{1+h}(C, d[C])]_{\text{vir}}) = [\overline{M}_{1+h,1}(C, d[C])]_{\text{vir}}.$$

Then, equations (13) and (15) imply:

$$\pi^* \gamma_s([\overline{M}_{1+h}(C, d[C])]_{\text{vir}}) = 0. \quad (16)$$

For any class $\xi \in A_*(\overline{M}_{1+h})$, 
$$\pi_* (\psi_1 \cdot \pi^* (\xi)) = 2h \cdot \xi,$$
where $\psi_1$ is the Chern class of the cotangent line on $\overline{M}_{1+h,1}$. Hence, the pull-back $\pi^* : A_*(\overline{M}_{1+h}) \to A_*(\overline{M}_{1+h,1})$ is injective. The Proposition now follows from (16).

2. Theorem 2

2.1. Rigidity. Let $C \subset X$ be a rigid, nonsingular genus $g$ curve with normal bundle $N$. The contribution $C_g(0, 1)$ is certainly 1, so we may assume $h$ is a positive integer. The proof of Proposition 1 also establishes the following facts. First, the moduli space $\overline{M}_{g+h}(C, [C])$ is a component (easily seen to be connected) of $\overline{M}_{g+h}(X, [C])$. Second, the contribution $C_g(h, 1)$ is determined by:

$$C_g(h, 1) = \int_{[\overline{M}_{g+h}(C, [C])]} c_{2h}(R_{g, h}).\tag{17}$$

Here, $R_{g, h}$ denotes the rank $2h$ bundle $R^1\pi_*\mu^*(N)$. Note the virtual dimension of $\overline{M}_{g+h}(C, [C])$ is also $2h$. The arguments of Section 1 are valid because a rigid curve is super-rigid in degree 1.

2.2. Irreducible components of $\overline{M}_{g+h}(C, [C])$. Let $C$ be a nonsingular genus $g$ curve. Let $h$ be a positive integer. We first analyze the moduli space of degree 1 maps $\overline{M}_{g+h}(C, [C])$. Let $P(h)$ denote the set of partitions $h$. There is a natural set-theoretic function:

$$\nu : \overline{M}_{g+h}(C, [C]) \to P(h)$$

defined by the following method. Let $\mu : F \to C$ correspond to a point $[\mu] \in \overline{M}_{g+h}(C, [C])$. The domain $F$ must contain a unique irreducible component $F_C$ mapped isomorphically to $C$ by $\mu$. The arithmetic genera of the connected components $\{F_i\}$ of $F \setminus F_C$ form a partition of $h$. Let $\nu([\mu])$ equal this partition. The irreducible components of $\overline{M}_{g+h}(C, [C])$ are in bijective correspondence with $P(h)$ by the value of $\nu$ on a general element.

Let $\tau = (h_1 \geq \ldots \geq h_l)$ be a partition of $h$ of length $l$. Consider the Fulton-MacPherson configuration space $C[l]$ of $l$ marked points in $C$: $C[l]$ is a natural compactification of the space of $l$ distinct points of $C$ [FuM]. If $C$ has no automorphisms, $C[l]$ is simply the fiber of $\overline{M}_{g+l} \to \overline{M}_g$ over the moduli point $[C]$. Define the nonsingular Deligne-Mumford stack $I_\tau$ by:

$$I_\tau = C[l] \times \prod_{i=1}^{l} \overline{M}_{h_i,1} \tag{18}$$
There is a natural family,
\[ \pi : \mathcal{F} \to I_\tau, \]
of prestable curves over \( I_\tau \) obtained by attaching a 1-pointed genus \( h_i \) curve to the \( i \)th marking of the universal \( l \)-pointed curve over \( C[l] \). Moreover, there is canonical projection \( \mu : \mathcal{F} \to C \). The induced morphism:

\[ \gamma_\tau : I_\tau \to \overline{M}_h \subset \overline{M}_{g+h}(C, [C]) \]
is finite and surjective onto the irreducible component \( \overline{M}_\tau \) corresponding to the partition \( \tau \).

Let \( \partial M_{h_i,1} \) denote the boundary of the moduli space: the locus of curves with at least one node. Similarly, let \( \partial C[l] \subset C[l] \) denote the locus of nodal curves (\( \partial C[l] \) may also be viewed as the locus lying over the diagonals of the product \( C^l \)). Let \( \partial I_\tau \) denote the union of the pull-backs of the boundaries of the factors \( (18) \) via the \( l+1 \) projections. Let

\[ \partial \gamma_\tau : \partial I_\tau \to \overline{M}_{g+h}(C, [C]) \]
denote the natural map. The main geometric result used in the proof of Theorem 2 is the following vanishing.

**Proposition 3.** For all partitions \( \tau \) of \( h \),

\[ c_{2h}(\partial \gamma_\tau^*(R_{g,h})) = 0. \]

**Proof.** By definition, \( \partial I_\tau \) is union of the pull-backs of the boundary divisors of the \( l+1 \) product factors of \( (18) \). We show \( c_{2h}(\partial \gamma_\tau^*(R_{g,h})) \) restricts to 0 on each of these pull-backs.

Let \( \text{pr}_j \) denote the projection of \( I_\tau \) onto the \((j+1)\)th factor of \( (18) \) for \( 0 \leq j \leq l \). There are \( l \) natural evaluation maps \( \text{ev}_i : C[l] \to C \) obtained from the \( l \) markings. Define \( \text{ev}_i : I_\tau \to C \) by the composition

\[ I_\tau \xrightarrow{\text{pr}_0} C[l] \xrightarrow{\text{ev}_i} C \]

for \( 1 \leq i \leq l \).

The bundle \( \gamma_\tau^*(R_{g,h}) \) is easily analysed via the natural normalization sequence of the family \( \mathcal{F} \). We find a decomposition:

\[ \gamma_\tau^*(R_{g,n}) = \bigoplus_{i=1}^l E_i^* \otimes \text{ev}_i^*(N) \tag{19} \]

where \( E_i \) is the Hodge bundle on \( \overline{M}_{h_i,1} \). We denote the pull-back of these Hodge bundles to \( I_\tau \) by the same symbols. An important relation among the Chern classes of the Hodge bundle has been established by Mumford in [Mu]. Mumford’s relation is: \( c(E_i) \cdot c(E_i^*) = 1 \) in \( A^*(\overline{M}_{h_i,1}) \).
From (19), we deduce:

\[ c_{2h}(\gamma^*_\tau(R_{g,h})) = \prod_{i=1}^l c_{2h_i}(E^*_i \otimes \text{ev}_i^*(N)). \]

Algebra and Mumford’s relation then yield:

\[ c_{2h}(\gamma^*_\tau(R_{g,h})) = \prod_{i=1}^l \lambda_{h_i} \lambda_{h_i-1} c_1(\text{ev}_i^*(N^*)). \]

(20)

Here, \( \lambda_k \) denotes the \( k \)th Chern class of the Hodge bundle.

First, consider a boundary divisor \( \Delta \subset \overline{M}_{g,1} \). The pull-back of \( \Delta \) to \( I_\tau \) is simply:

\[ \text{pr}^{-1}_j(\Delta) = C[l] \times \Delta \times \prod_{i \neq j} \overline{M}_{h_i,1}. \]

The restriction of the factor \( \lambda_{h_j} \lambda_{h_j-1} \) of (20) to \( \Delta \) has been proven by Faber to vanish [F1] (the reducible divisors of \( \overline{M}_{1,1} \) have non-trivial genus splittings). Hence, the restriction of \( c_{2h}(\gamma^*_\tau(R_{g,h})) \) to \( \text{pr}^{-1}_j(\Delta) \) vanishes.

Second, consider a boundary divisor \( \Delta \) of \( C[l] \). The divisor \( \Delta \) corresponds to a locus in which a subset \( S \subset [l] \) (of at least 2 elements) of the marked points coincide over \( C \). The evaluation maps \( \{\text{ev}_i\}_{i \in S} \) coincide when restricted to \( \text{pr}^{-1}_0(\Delta) \). Therefore, since \( c_1(N^*)^2 = 0 \), the restriction of \( c_{2h}(\gamma^*_\tau(R_{g,h})) \) to \( \text{pr}^{-1}_0(\Delta) \) vanishes. \( \square \)

2.3. Hodge integrals. Let \( \partial \overline{M}_r = \gamma_r(\partial I_r) \), and let \( M_r = \overline{M}_r \setminus \partial \overline{M}_r \). \( M_r \) is open in \( \overline{M}_{g,h}(C, [C]) \) and corresponds to the moduli space of degree 1 maps which consist of nonsingular curves of genus \( h_i \) attached to distinct point of \( C \).

A deformation theory argument shows \( M_r \) is a nonsingular moduli stack of dimension \( \sum_{i=1}^l (3h_i - 1) \). More precisely, for \( [\mu : F \rightarrow C] \in M_r \), there is a canonical exact sequence:

\[ 0 \rightarrow \text{Aut}_{[F]} \rightarrow H^0(F, \mu^*(T_C)) \rightarrow \text{Def}_{[\mu]} \rightarrow \text{Def}_{[F]} \rightarrow H^1(F, \mu^*(T_C)) \rightarrow \text{Obs}_{[\mu]} \rightarrow 0 \]

where \( \text{Aut}_{[F]} \) is the infinitesimal automorphisms of \( F \) and \( \text{Def}_{[F]}, \text{Def}_{[\mu]} \) are the infinitesimal deformation spaces of \( F, \mu \) respectively. It is easy to prove the cokernel of \( \iota \) is equal to a vector space \( V \) with filtration

\[ 0 \rightarrow \text{Def}_{[C]} \rightarrow V \rightarrow \bigoplus_{i=1}^l (T_{p_i} \otimes T_{p_i'}) \rightarrow 0. \]
Here, the component $F_i \subset F$ of genus $h_i$ is attached to $C$ at the points $p_i \in F_i$ and $p'_i \in C$. The cokernel computation amounts to showing the map $\mu$ has no infinitesimal deformations which smooth any of the $l$ nodes of $F$. We then see $\text{Def}_[\mu]$ is of constant dimension $\sum_{i=1}^{l} (3h_i - 1)$. Moreover, the obstruction space is a bundle over $M_\tau$ with fiber

$$
\frac{H^1(F, \mu^*(T_C))}{\text{Im}(V)} = \bigoplus_{i=1}^{l} \frac{H^1(F_i, \mathcal{O}_{F_i} \otimes T_{p_i})}{T_{p_i} \otimes T_{p'_i}}
$$

over $[\mu]$. The essential point here is the deformation theory of maps in $M_\tau$ is very simple.

Let $\text{Aut}_\tau$ denote the stabilizer of the permutation $S_l$-action on the $l$-tuple $\tau$. The map $\gamma_\tau : I_\tau \to \overline{M}_\tau$ is $\text{Aut}_\tau$-invariant. Moreover, the quotient map induces a proper, bijective morphism

$$
\tilde{\gamma}_\tau : I_\tau / \text{Aut}_\tau \to \overline{M}_\tau.
$$

Let $I_\tau^0 = I_\tau \setminus \partial I_\tau$. Certainly, $\tilde{\gamma}_\tau$ induces an isomorphism $I_\tau^0 / \text{Aut}_\tau \cong M_\tau$.

The restriction of the virtual class $\xi_{\tau}^\text{vir} = [\overline{M}_{g+h}(C, [C])]^\text{vir}$ to the disjoint open union $\bigcup_{\tau \in P(h)} M_\tau$ is:

$$
\bigoplus_{\tau \in P(h)} \xi_{\tau}^\text{vir},
$$

where $\xi_{\tau}^\text{vir} \in A_{2h}(M_\tau)$. The pull-back of $\xi_{\tau}^\text{vir}$ to $I_\tau^0$ is identified from the obstruction theory (21) to be:

$$
\gamma_\tau^*(\xi_{\tau}^\text{vir}) = \prod_{i=1}^{l} c_{h_i - 1} \left( \frac{c(E_i^* \otimes \text{ev}_i^*(T_C))}{1 - \psi_1 + c_1(\text{ev}_i^*(T_C))} \right).
$$

Since $M_\tau$ is nonsingular, the restriction of the virtual class is the Euler class of the obstruction bundle.

The virtual class $\xi_{\tau}^\text{vir}$ may be (non-canonically) expressed at a sum:

$$
\bigoplus_{\tau \in P(h)} \tilde{\xi}_{\tau}^\text{vir},
$$

where $\tilde{\xi}_{\tau}^\text{vir} \in A_{2h}(\overline{M}_\tau)$. Using the proper bijection (22), we see:

$$
C_g(h, 1) = \sum_{\tau \in P(h)} \int_{I_\tau / \text{Aut}_\tau} \tilde{\xi}_{\tau}^\text{vir} \cap c_{2h}(\tilde{\gamma}_\tau^*(R_{g,h})).
$$

By the vanishing of Proposition 3, equation (24) remains valid if $\xi_{\tau}^\text{vir}$ is replaced with any cycle class which restricts to $\xi_{\tau}^\text{vir}$ on $M_\tau$. This
observation together with (23) yields the equality:

\[ C_g(h,1) = \sum_{\tau \in P(h)} \frac{1}{|Aut_{\tau}|} \int_{I_{\tau}} c_{2h}(\gamma_{\tau}^*(R_g,h)) \cdot \prod_{i=1}^{l} c_{h_i-1}(1 - \psi_1 + c_1(\text{ev}_i^*(T_C))) \cdot \int_{\text{vir}} C_{c_1}(N^*) \cdot \prod_{i=1}^{l} \int_{M_{h_i,1}} \lambda_h \lambda_{h_i-1}(\sum_{j=0}^{h_i-1} (-1)^j \lambda_j \psi_1^{h_i-1-j}). \]

Equation (20) together with basic algebraic manipulations then prove the main integral formula:

\[ C_g(h,1) = \sum_{\tau \in P(h)} \frac{(2 - 2g)^l}{|Aut_{\tau}|} \prod_{i=1}^{l} \int_{M_{h_i,1}} \lambda_h \lambda_{h_i-1}(\sum_{j=0}^{h_i-1} (-1)^j \lambda_j \psi_1^{h_i-1-j}). \]

The only aspect of \( N \) which affects the integral

\[ C_g(h,1) = \int_{[\overline{M}_{g+h}(C;[C])]^{vir}} c_{2h}(R_g,h). \]

is \( \int_{C} c_1(N^*) \). This Chern class enters enters (24) via equation (21) yielding the factor

\[ (\int_{C} c_1(N^*))^l = (2 - 2g)^l. \]

Theorems 2-4 will directly follow from formula (26).

For \( q \geq 1 \), define \( \alpha_q = \int_{M_{q+1}} \lambda_q \lambda_{q-1}(\sum_{j=0}^{q-1} (-1)^j \lambda_j \psi_1^{q-1-j}) \). Define the generating series:

\[ Q(t) = \sum_{q \geq 1} \alpha_q t^{2q}. \]

An immediate consequence of formula (26) is the equation:

\[ \sum_{h\geq 0} C_g(h,1)t^{2h} = \exp((2 - 2g)Q(t)) \]

\[ = \exp(2Q(t))^{1-g} \]

\[ = \left( \sum_{h\geq 0} C_0(h,1)t^{2h} \right)^{1-g} \]

\[ = \left( \frac{\sin(t/2)}{t/2} \right)^{2g-2}. \]

The last equality follows from the previous computations of \( C_0(h,1) \) in [FP]. The proof of Theorem 2 is complete.
3. Theorem 3

We follow here the notation of Section 0.4. Let $C$ be a nonsingular genus $g$ curve in a 3-fold $X$ representing the homology class $\beta$. We now assume $-K_X \cdot \beta > 0$, so the moduli space $\overline{M}_g(X, \beta)$ is of positive expected dimension. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a vector of cohomology classes defining a Gromov-Witten invariant $N^g_\beta(\gamma)$. For each $i$, let $Y_i \subset X$ be a topological cycle dual to $\gamma_i$. Let $p_i \in C \cap Y_i$. We let $(C)$ denote the identity map $\pi : C \to C \subset X$ defining a point in the moduli space of stable maps. The contribution of $C$ to $N^g_{g+h}(\gamma)$ via covers will require two general position hypotheses analogous to rigidity in the Calabi-Yau case:

(i) $(C, p_1, \ldots, p_n)$ is a nonsingular point of $\overline{M}_{g,n}(X, \beta)$ lying on a component of expected dimension $-K_X \cdot \beta + n$.

(ii) The topological intersection of the cycles $ev_i^{-1}(Y_i)$ in $\overline{M}_{g,n}(X, \beta)$ is transverse at $(C, p_1, \ldots, p_n)$.

Under these hypotheses, the degenerate contribution of $C$ may be expressed directly as an integral over $\overline{M}_{g+h}(C, [C])$.

Let $W \subset \overline{M}_g(X, \beta)$ be the open, nonsingular, expected dimensional subset of the moduli space of maps. Let $U \subset W$ be the open subset corresponding to embeddings of nonsingular genus $g$ curves in $X$. As such embeddings have no nontrivial automorphisms, $U$ is a nonsingular variety (not just a Deligne-Mumford stack). Moreover, by assumption (i), $U$ is nonempty of dimension $-K_X \cdot \beta$ and contains $(C)$. After discarding a finite number of points of $U$, we may assume $(C)$ is the only point of $U$ meeting all the cycles $Y_i$. Note the moduli space $U$ is also naturally an open set of a component of the Hilbert scheme of curves in $X$. Let

$$\eta : C \to U$$

denote the universal family of curves over $U$. Let $\overline{M}_{g+h}(\eta, \beta)$ denote the $\eta$-relative moduli space of maps representing the fundamental class of the fibers of $\eta$. There is a natural morphism of Deligne-Mumford stacks:

$$\iota : \overline{M}_{g+h}(\eta, \beta) \to \overline{M}_{g+h}(X, \beta)$$

obtained by composition. There are several tautological morphisms (over $U$):

$$\pi : \mathcal{F} \to \overline{M}_{g+h}(\eta, \beta),$$

$$\mu : \mathcal{F} \to C,$$

$$\tau : \overline{M}_{g+h}(\eta, \beta) \to U.$$
Let $N$ denote the universal normal bundle $N$ on $C$. $N$ is the family of normal bundles of the fibers of $\eta$ in $X$. As $U$ is nonsingular of expected dimension, $\eta_*(N)$ is isomorphic to the tangent bundle of $U$ and $R^1\eta_*(N) = 0$.

A deformation theoretic check over Artinian rings shows $\iota$ is an open immersion. We see the stack $\overline{M}_{g+h}(\eta, \beta)$ has two natural fundamental classes. The first is $[\overline{M}_{g+h}(\eta, \beta)]^{\text{vir}}$ obtained from the structure of a $\eta$-relative moduli space of maps. Second, the open inclusion $\iota$ endows $\overline{M}_{g+h}(\eta, \beta)$ with the perfect obstruction theory on $\overline{M}_{g+h}(X, \beta)$. A direct comparison of these two obstruction theories on $\overline{M}_{g+h}(\eta, \beta)$ shows they differ exactly by the bundle $R_{g,h} = R^1\pi_*\mu^*(N)$:

$$\iota^*([\overline{M}_{g+h}(X, \beta)]^{\text{vir}}) = [\overline{M}_{g+h}(\eta, \beta)]^{\text{vir}} \cap c_{2h}(R_{g,h}).$$

Relations (27) and (28) are valid when considered in the context of $n$-pointed stable maps (this may be deduced from the above unpointed relations together with the natural properties of these virtual structures under the morphisms forgetting the markings [BM]).

By relation (28) and the definition of the Gromov-Witten invariants, the contribution of $(C, p_1, \ldots, p_n)$ to $N_{g+h}^+(\gamma)$ is equal to the intersection product:

$$[\overline{M}_{g+h,n}(\eta, \beta)]^{\text{vir}} \cap c_{2h}(R_{g,h}) \cap \prod_{i=1}^n \operatorname{ev}^{-1}_i(Y_i),$$

with value in the zeroth homology of the compact space $\cap_{i=1}^n \operatorname{ev}^{-1}_i(Y_i)$. By assumption (ii) and the pull-back properties of the virtual class, intersection (29) is (numerically) equal to:

$$[\overline{M}_{g+h}(\eta, \beta)]^{\text{vir}} \cap c_{2h}(R_{g,h}) \cap \tau^{-1}(C).$$

The latter class (30) is an integral over the virtual class of the fiber $\tau^{-1}(C) = \overline{M}_{g+h}(C, [C])$. We find:

$$C_g(h, X, \beta) = \int_{[\overline{M}_{g+h}(C, [C])]^{\text{vir}}} c_{2h}(R_{g,h}).$$

This integral is identical to (17) except for the different normal bundles $N$ occurring in the definition of $R_{g,h}$.

The method in Section 2 to compute (17) also yields a computation of (31). As remarked after equation (26), the bundle $N$ affects the integral (31) through $\int_C c_1(N^*)$:

$$C_g(h, X, \beta) = \sum_{\tau \in \mathcal{P}(h)} \frac{\int_C c_1(N^*)}{|\operatorname{Aut}_\tau|} \prod_{i=1}^t \int_{M_{h_i,1}} \lambda_{h_i} \lambda_{h_i-1} \left( \prod_{j=0}^{h_i-1} (-1)^j \lambda_j \psi_1^{h_i-1-j} \right).$$
Since $\int_C c_1(N^*) = 2 - 2g + K_X \cdot \beta$, Theorem 3 follows via the series analysis of Section 2.

4. Theorem 4

Let $\pi : \overline{M}_{g,1} \to \overline{M}_q$ be the universal curve (for $q \geq 2$). The class $\psi_1$ is the Chern class of the cotangent line bundle on $\overline{M}_{g,1}$. The kappa classes are defined by $\kappa_j = \pi_*(\psi_1^{j+1})$. Define

$$\beta_{q-2} = \pi_*(\sum_{j=0}^{q-1} (-1)^j \lambda_j \psi_1^{q-1-j}) = \sum_{j=0}^{q-2} (-1)^j \lambda_j \kappa_{q-2-j}.$$ 

In the notation of Section 2.3, we see:

$$Q(t) = t^2/24 + \sum_{q \geq 2} t^{2q} \int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \beta_{q-2}.$$ 

The results of Section 2.3 applied in case $g = 0$ prove:

$$\exp(2Q(t)) = \left( \frac{t/2}{\sin(t/2)} \right)^2.$$ 

After taking the logarithm, we find:

$$Q(t) = \log \left( \frac{t/2}{\sin(t/2)} \right).$$

The right series in (33) may be expanded as

$$\log \left( \frac{t/2}{\sin(t/2)} \right) = \sum_{q \geq 1} \frac{|B_{2q}|}{(2q)(2q)!} t^{2q}$$

by Lemma 3 of [FP]. Faber has computed

$$\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \kappa_{q-2} = \frac{1}{2^{2q-1}(2q-1)!!} \frac{|B_{2q}|}{2q}$$

from Witten’s conjectures/ Kontsevich’s theorem [F2]. It is known $R^{g-2}(M_g)$ is exactly 1 dimensional ([F2], [L]). Since $\lambda_q \lambda_{q-1}$ vanishes when restricted to $\partial \overline{M}_g$, we find

$$\beta_{q-2} = \frac{\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \beta_{q-2}}{\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \kappa_{q-2}} \cdot \kappa_{q-2}.$$ 

Theorem 4 now follows from the computation:

$$\frac{\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \beta_{q-2}}{\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \kappa_{q-2}} = \frac{2^{q-1}}{q!}.$$
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