FAST LINEAR RESPONSE ALGORITHM FOR DIFFERENTIATING CHAOS

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ABSTRACT. We devise the fast linear response algorithm for differentiating fractal invariant measures of chaos with respect to perturbations of governing equations. We first derive the first computable expansion formula for the unstable divergence, then a ‘fast’ formula by renormalized second-order tangent equations. The algorithm’s cost is solving only one orbit and \( u \), the unstable dimension, many first-order and second-order tangent equations; the main error is the sampling error of the orbit. We demonstrate it on an numerical example which is difficult for previous methods.

Keywords. SRB measure, Linear response, fast algorithm, chaos, non-intrusive shadowing algorithm.

1. Introduction

1.1. Literature review and problem definitions.

Chaos appears in many disciplines, such as fluid mechanics, geophysics, and machine learning. The derivative, or linear response, of the long-time average of some observable functions, is fundamental to many numerical tools widely used in those disciplines, such as gradient-based optimization, error analysis, and uncertainty quantification. However, because of butterfly effects and non-smooth attractors, computing the linear response is challenging.

For a map on a manifold, the SRB measure gives the long-time averaged statistics of chaotic systems, and its perturbation due to the perturbation of the map is given by the linear response formula. There are many versions of the linear response formula, and we choose to start from the ensemble version formula, which is formally an average of the orbit-wise perturbations over an ensemble of many orbits (see section 2.2, also [39, 40, 41, 24, 13]). Numerically, the ensemble formula can be directly implemented by ensemble or stochastic algorithms [26, 14, 28, 19]. Theoretically, this could give accurate linear response; however, in reality, because the orbit-wise perturbations grows exponentially fast, it is typically unaffordable for convergence to actually happen.

To get rid of the large integrand in the ensemble formula, we decompose it into the shadowing contribution and the unstable contribution. The shadowing contribution has a bounded integrand (see section 2.3), and the exponential growth of integrand is only in
the unstable contribution. It turns out that unstable contribution is small when the ratio of unstable dimension is small \cite{31}.

Shadowing algorithms accurately compute the shadowing direction and the shadowing contribution. \cite{2, 8, 36, 47, 44, 20, 6, 43, 25, 7}. The computational efficiency of shadowing algorithms is boosted by a ‘non-intrusive’ formulation \cite{34, 35}, which constrains the computation to only the unstable subspace. This makes approximate numerical differentiation affordable for the first time for several high dimensional chaos, such as computational fluid systems with $4 \times 10^6$ degrees of freedom \cite{30, 23}. In this paper, we find that non-intrusive shadowing is also useful in computing the other part of the linear response, the unstable contribution.

The large integrand of the unstable contribution is resolved by integration-by-parts on the unstable manifold (section 3.1). This gives the integrated-by-parts formula for the unstable contribution and the blended formula for the linear response, whose integrand typically grows much slower than the ensemble formula. The central term here is the divergence on the unstable manifold under the conditional SRB measure.

The unstable divergence is very difficult to compute. We can not use integration-by-parts again, which gives back the ensemble formula. Moreover, it can not be computed from summing individual directional derivatives, whose values are typically infinite. There were previous algorithms based on computing directional derivatives, such as the pioneering blended response algorithm by Majda and Abramov, and the S3 algorithm by Q.Wang and Chandramoorthy \cite{1, 10}. These algorithms use various approximations to the directional derivatives, incurring large error or high cost. Moreover, both algorithms need to further compute the oblique projection operators.

There are also linear response algorithms based on transfer operators \cite{16, 4, 12, 3, 42}. It turns out that the operator version of the linear formula is exactly integrating-by-parts the entire ensemble formula \cite{32}. For SRB measures, this yields a divergence in all directions, which has infinite sup norm, and can not be directly computed; the current practice is to compute it on an approximate measure with smooth densities, but using isotropic finite elements to approximate singular measures incur large error, and the cost increases exponentially fast at least as the dimension of the attractor \cite{32}. As a result, such operator algorithms were efficient only for low-dimensional cases, and were not applied to systems with attractor dimension larger than 2. It also turns out that the blended formula can be interpreted as using the operator formula for only the unstable contribution. Hence, our work is helpful for improving operator algorithms \cite{32}.

1.2. Main results of the paper.

A precise and efficient algorithm for the unstable divergence, and hence the linear response, was missing in previous works. Roughly speaking, this is because the lack of a computable linear response formula with a small integrand, which is neither a distribution, nor grows exponentially fast to step numbers. This paper answers this open problem for deterministic and discrete-time chaotic systems.

Our first result is the first computable formula of the unstable divergence. The main issue is to express the unstable divergence without using directional derivatives, which are true distributions with infinite values. Section 3 solves this difficulty by three steps
of transformations, following similar ideas in the proof of the regularity of the unstable divergence. The expansion formula is given in theorem 2.

The second result is the ‘fast’ formula in section 4.3, which allows the unstable contribution be computed even more efficiently and robustly than directly using the expansion formula. Using the idea for fast algorithms, we combine many small terms into a big term, then apply a uniform rule of propagation, given by second-order tangent equations, only once on the big term. A side benefit is that we get rid of oblique projections, which are summarized into a modified shadowing direction. Computing the fast formula requires solving only \( u \) many second-order tangent equations on one sample orbit.

More specifically, the fast formula says that for any \( r_0 \),

\[
U.C. = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left\langle \tilde{\beta} r_n, \tilde{e}_{n+1} \right\rangle, \quad \text{where} \quad r_{n+1} = P^\perp \tilde{\beta} r_n.
\]

Here \( \langle \cdot, \cdot \rangle \) is the Riemannian metric defined in equation (6). Roughly speaking, \( \tilde{e} \) is the unit \( u \)-dimensional cube spanned by unstable vectors, \( r_n \in D_e \) are derivatives of cubes (see Appendix B); \( \tilde{\beta} \) is the renormalized second-order tangent equation governing the propagation of derivatives of cubes (definition 1 in section 4.1); \( P^\perp \) is the orthogonal projection operator on \( D_e \) (Appendix C). The integrand in the fast formula has the same size as the blended formula and the expansion formula.

The third result is a precise, efficient, robust, and easy-to-implement algorithm for the linear response of SRB measures, the fast linear response algorithm. We first show that the renormalization in \( \tilde{\beta} \) only needs to be done after a number of steps. We further write everything in matrix notation, which is more suitable for programming. Readers mainly interested in applications can jump to the detailed procedure list in section 5.2. The algorithm is precise; it is robust, since it does not involve oblique projections; it requires little additional coding to first-order and second-order tangent solvers, where second-order tangent solvers are just first-order solvers with a second-order source term. More importantly, it is highly efficient.

Section 6 shows a numerical application on a modified solenoid map, which is a 21-dimensional system with a 20-dimensional unstable subspace, whose direction is unknown beforehand. For the same accuracy, fast linear response is about \( 10^6 \) times faster than ensemble algorithms, and approximate operator algorithms would require at least \( 10^{20} \) GB of storage. In fact, fast linear response is even faster than finite difference: no previous algorithms reached the same level of efficiency.

The last result of this paper is several basic geometry and algebra tools for second-order tangent equations, which governs the propagation of derivatives of unstable \( u \)-vectors. These tools are useful when considering \( u \)-dimensional invariant submanifolds for any \( u \geq 1 \). In particular, appendix A defines the derivative of the pushforward operator, \( \nabla f^* \).

Appendix B defines the linear space, \( D_e \), of derivatives of unstable \( u \)-vectors. Appendix C extends the projection operators on single-vectors to \( D_e \).

This paper is organized as follows. Section 2 reviews hyperbolic systems, the linear response formula, and the non-intrusive shadowing algorithm. Section 3 derives the expansion formula of the unstable divergence. Section 4 derives the fast formula. Section 5 gives more details of the fast linear response algorithm, with a procedure list. Section 6 shows a numerical application.
2. Preparations

2.1. Hyperbolicity and notations.

Let $f$ be a smooth on a smooth Riemannian manifold $M$, whose dimension is $M$. Assume that $K$ is a hyperbolic compact invariant set, that is, $T_K M$ (the tangent bundle restricted to $K$) has a continuous $f$-invariant splitting $T_K M = V^s \oplus V^u$, such that there are constants $C > 0$, $\lambda < 1$, and

$$\max_{x \in K} \| f^{-n}_s | V^u(x) \|, \| f^n_s | V^s(x) \| \leq C \lambda^n \quad \text{for} \quad n \geq 0,$$

Here $f_s$ is the pushforward operator, which is the Jacobian matrix of $f$ when $M = \mathbb{R}^M$ (see definition 2 in appendix A). We call $V^u$ and $V^s$ the stable and unstable subbundles, or subspaces; also let $u, s$ denote the dimension of the unstable and stable manifolds, hence, $u + s = M$. A stable manifold, $V^s(x)$, is as smooth as $f$, tangent to $V^s(x)$ at $x$, and there are $C > 0$ and $\lambda < 1$ such that if $y, z \in V^s(x)$,

$$d(f^n y, f^n z) \leq C \lambda^n d(y, z) \quad \text{for} \quad n \geq 0,$$

where $d$ is the distance function. Unstable manifolds are defined similarly.

More technically, we assume $f \in C^3$, and $M \in C^\infty$; we assume that $K$ is an Axiom A attractor, which is the closure of periodic orbit, and there is an open neighborhood $U$, called the basin of the attractor, such that $\cap_{n \geq 0} f^n U = K$. The SRB measure on an attractor is the weak limit of the empirical measure of almost all orbits starting from the basin. For all $x \in K$, the local unstable manifolds $V^u(x)$ lie in $K$, whereas the local stable manifolds $V^s(x)$ fill a neighborhood of $K$. For more details on hyperbolicity see [38, 45]. Due to the spectral decomposition theorem, by taking out a basic set and raise $f$ to some power, we may further assume that $f$ is mixing on $K$ [9, 46].

Under our assumptions, the SRB measure $\rho$ of $f$ on $K$ is the unique $f$-invariant measure with either of the following characterization [49]:

- $\rho$ has absolutely continuous conditional measures on unstable manifolds;
- $\rho$ is the physical measure, that is, there is a set $V \subset M$ having full Lebesgue measure such that for every continuous observable $\varphi : M \to \mathbb{R}$, almost all $x \in V$

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \to \rho(\varphi).$$

Physical measures are the limits of evolving Lebesgue measures; they are the measures most compatible with volume when volume is not preserved. Notice that typically $K$ has zero Lebesgue measure.

We explain some notation conventions used in this paper. Let $x \in M$ be the point of interest, and $x_k := f^k x$. Subscripts $n, k, m$ are only for labeling steps, and subscripts for step zero are omitted. For a tensor field $X$ on $M$, let $X_k$ be the pullback of $X$ by $f^k$,

$$X_k(x) := X(x_k).$$

The subscript $k$ always specify that the value of the tensor field is taken at $x_k$. However, when $X_k$ is differentiated, we should further specify its domain, or when the pullback happens: we leave that to be determined by the differentiating vector. For example, let $\nabla$
be the Riemannian connection, \( Y \) a vector field, then
\[
(\nabla_{Y_k} X_k)(x) := (\nabla_Y X)(x_k).
\]
Since \( Y_k(x) \in T_{x_k}M \), it must differentiate a tensor field at \( x_k \), so \( X_k \) must be a function around \( x_k \). Hence, \( X_k \) is \( X \) when being differentiated, and then the entire result is pulled-back to \( x \). The other way does not work: if \( X_k \) is \( X \circ f^k \) when being differentiated, this is a function of \( x \), and can not be differentiated by \( Y_k \). For another example, take a differentiable observable function \( \varphi \) on \( M \),
\[
f^k_\ast Y(\varphi_k)(x) := f^k_\ast Y(\varphi)(x_k) = Y(\varphi \circ f^k)(x) =: Y(\varphi_k)(x),
\]
where \( Y(\cdot) \) means to differentiate in the direction of \( Y \). Since \( f^k_\ast Y(x) \in T_{x_k}M \), the first \( \varphi_k \) is \( \varphi \) when the differentiation happens at \( x_k \), and then the result is pulled-back to \( x \); since \( Y(x) \in T_xM \), the second \( \varphi_k \) is \( \varphi \circ f^k \) when the differentiation happens right at \( x \). Finally, we omit the step subscript of the pushforward tensor \( f_\ast \), since its location is well-specified by the vector it applies to.

Given a trajectory \( \{x_n\}_{n \in \mathbb{Z}} \), we can define a sequence, say \( \{w_n(x_0, w_0) \in T_{x_n}M\}_{n \in \mathbb{Z}} \). We say such a sequence is equivariant if its definition does not depend on \( w_0 \) and further
\[
w_n(x_0) = w_0(x_n).
\]
Sequences obtained by taking values of a (rough) tensor field, such as \( X_n, X_n^u, \Phi_n, p_n \), are equivariant. We can define non-equivariant sequences by the inductive relation of some equivariant sequences. The main theorem in our paper is that, due to the stability in the induction, the non-equivariant sequence, \( r \), computed by the induction, well approximates the equivariant counterpart, \( p \), which is originally defined by complicated formulas. For other cases where the non-equivariant approximations have been previously well understood, we use the same notation for both the equivariant and non-equivariant versions, such as the shadowing direction \( v \) and the unstable hyper-cube \( e \).

2.2. Review of linear response formula.

The linear response formula is an expression of \( \delta \rho \), the derivative of the SRB measure, using \( \delta f \). The ensemble version of the linear response formula can be formally derived by averaging perturbations on each trajectories over the SRB measure [32]. Ruelle first proved that the formula indeed gives the derivative [39]. The formula can be proved for more general cases, for example, Dolgopyat proved it for the partially hyperbolic systems [13]. It is plausible that our work can also be generalized, perhaps even more so, because the fast formula does not involve oblique projections. It should also be noted that the linear response formula fails for some cases [5, 48].

**Theorem 1** (ensemble linear response formula). Denote the SRB measure of \( f \) on \( K \) by \( \rho \), assume that \( f \) is parameterized by some scalar \( \gamma \), and define
\[
\delta(\cdot) := \frac{\partial(\cdot)}{\partial \gamma}.
\]
Then the derivative of the SRB measure, for a fixed objective function \( \Phi \), is given by:
\[
\delta \rho(\Phi) = \sum_{n=0}^{\infty} \rho (\Phi_n)(x) = \lim_{W \to \infty} \rho \left( \sum_{n=0}^{W} X(\Phi_n) \right),
\]
where $X := \delta f \circ f^{-1}$, $X(\cdot)$ is to differentiate in the direction of $X$, and $\Phi_n = \Phi \circ f^n$.

Remark. The formula is more important to us than the more technical assumptions, which we choose to be the same as [39]: this is just one way to model chaotic systems. More technically, we assume that $K$ is a mixing Axiom A attractor for the $C^3$ diffeomorphism $f$ of $\mathcal{M}$. Let $\mathcal{A}$ be the space of $C^3$ diffeomorphisms sufficiently close to the $f$ in a fixed neighborhood $V$ of $K$; $\mathcal{A}$ has $C^3$ topology. Assume that $\gamma \mapsto f$ is $C^1$ from $\mathbb{R}$ to $\mathcal{A}$; $A$ has $C^3$ topology. Assume that $\gamma \mapsto f$ is $C^1$ from $\mathbb{R}$ to $A$, then $\gamma \mapsto \rho$ is $C^1$ from $\mathbb{R}$ to $\mathcal{C}^2(M)^*$. Assume $\Phi \in C^2$, then $\gamma \mapsto \rho(\Phi)$ is $C^1$ from $\mathbb{R}$ to $\mathbb{R}$.

Due to unstable components, the integrand of the ensemble formula grows exponentially to $W$:

$$\sum_{n=0}^{W} X(\Phi_n) \sim O(\lambda_{max}^W),$$

where $\lambda_{max} > 1$ is the largest Lyapunov exponent. This phenomenon is also known as the ‘exploding gradients’. Computationally, the number of samples requested to evaluate the integration to $\rho$ also grows exponentially to $W$, incurring large computational cost. Hence, algorithms based on the ensemble linear response formula suffer from very high computational cost.

To get rid of large integrands, we decompose the linear response into two parts. The first part, the shadowing contribution, accounts for the change of the location of the attractor via a conjugacy map. The second part, the unstable contribution, accounts for the fact that the pushforward of the old SRB onto the new attractor is no longer SRB. Integrate-by-part the unstable contribution on the unstable manifold gives (section 3.1)

$$\delta \rho(\Phi) = S.C. - U.C. = \rho(v(\Phi)) - \rho \left( \left( \sum_{n \in \mathbb{Z}} \Phi_n \right) \text{div}_\sigma X^u \right),$$

(3)

where $S.C. := \sum_{n \geq 0} \rho(X^s(\Phi_n)) - \sum_{n \leq -1} \rho(X^u(\Phi_n)) = \rho(v(\Phi))$,

$$U.C. := \sum_{n \in \mathbb{Z}} \rho \left( X^u(\Phi_n) \right) = \lim_{W \to \infty} \rho \left( \sum_{n=-W}^{W} \Phi_n \text{div}_\sigma X^u \right).$$

Here $S.C., U.C.$ are shadowing and unstable contributions, $v$ is the shadowing direction (section 2.3), $X^u$ and $X^s$ are the unstable and stable oblique projections of $X$ (appendix C figure 7). The unstable divergence, $\text{div}_\sigma$, is the divergence on the unstable manifold under the conditional SRB measure.

We refer to equation (3) as the blended linear response formula. For the shadowing contribution, the integrand $v$ is bounded. For the unstable contribution, notice that subtracting $\Phi$ by constant $\rho(\Phi)$ does not change the linear response, so

$$\psi \sim O \left( \sqrt{W} \right), \quad \text{where} \quad \psi := \sum_{m=-W}^{W} (\Phi_m - \rho(\Phi)).$$

Here $\sim$ means with large probability. Hence, the integrands in the blended formula are much smaller than the ensemble formula, and algorithms based on the blended formula should have much faster convergence. However, efficient computation of the unstable divergence has been an lasting open problem: this is solved in our paper.
A perhaps more familiar decomposition of the linear response is into stable and unstable contributions. However, computing the stable contribution requires computing oblique projection operators, which is twice the cost of computing the shadowing contribution by the non-intrusive shadowing algorithm; it is also not very robust. Moreover, for both decompositions, our algorithm for the unstable contribution is the same, which requires computing a modified shadowing direction anyway: there is no point using a stable-unstable decomposition.

Another remark is that our results may hold beyond the strong assumptions on hyperbolicity. Our work is essentially two new forms of the linear response formula, an expansion and a fast formula, and their equivalence to other forms can be proved more easily than the equivalence between the formula and the true derivative. Hence, there is a good chance that our results are correct whenever the linear response formula gives the correct sensitivity. It is mainly for the simplicity of discussions that we use strong assumptions on hyperbolicity.

2.3. Non-intrusive shadowing algorithm.

On an orbit \( \{x_n\}_{n \in \mathbb{Z}} \), we define the shadowing direction, \( \{v_n := v(x_n) \in T_{x_n}M\}_{n \in \mathbb{Z}} \), as the only bounded solution of the inhomogeneous tangent equation,

\[
v'_{n+1} = f_* v'_n + X_{n+1}.
\]

This equation governs the propagation of perturbations on an orbit, hence, we know that \( v \) is the first order difference between two shadowing orbits. Notice that \( v \) is a vector field and \( \{v_n\}_{n \in \mathbb{Z}} \) is an equivariant sequence. By the law of linear superposition and the exponential growth of homogeneous tangent solutions, we can write out the formula of \( v \),

\[
v = \sum_{k \geq 0} f^k_* X^s_{-k} - \sum_{k \geq 1} f^{-k}_* X^u_k.
\]

By comparing with the expression of the shadowing contribution, we can show [31]

\[
S.C. = \rho(v(\Phi)),
\]

Here \( v(\cdot) \) denotes taking derivative in the direction of \( v \).

A ‘non-intrusive’ algorithm, in our definition, is one which requires only \( u \) many solutions of the most basic inductive relations and do not compute oblique projections. Also, it does not require additional information of the governing equation, such as the Jacobian matrices. For example, the non-intrusive shadowing algorithm requires only \( u \) many tangent solutions as data. The ‘basic solutions’ of non-intrusive algorithms may vary in different context. For example, is the adjoint shadowing lemma and algorithm, non-intrusiveness is defined by \( u \) many adjoint solutions instead of tangent ones [29, 33].

The non-intrusive shadowing algorithm computes the shadowing direction, as if a non-equivariant sequence, by the constrained minimization [34],

\[
\min_{a \in \mathbb{R}^u} \frac{1}{2N} \sum_{n=0}^{N-1} \|v_n\|^2, \quad \text{s.t.} \quad v_n = v'_n + \xi_n a,
\]

where \( v' \) is an arbitrarily chosen inhomogeneous tangent solution, such as one solved from the zero initial condition. \( \xi \) is a matrix with \( u \) columns, which are homogeneous tangent solutions. In other words, the boundedness property is approximated by a minimization,
while the feasible space of all inhomogeneous tangent solutions, which has $M$ dimensions, is reduced to an affine subspace of only $u$ dimensions, almost parallel to $V^u$. The reduced feasible space significantly reduces the computational cost, yet is still capable of finding the shadowing direction [31]. The non-intrusive shadowing algorithm is an ingredient of fast linear response algorithm, and its procedure list is included in section 5.2.

If the unstable contribution is small, we may choose to neglect it or approximate it crudely. For example, when $u \ll M$, the unstable contribution is typically small, if the system further has two properties which seem to be common in practice, those are, fast decay of correlations, and $X$ and $\Phi$ not particularly aligned with the unstable direction. Then, we may well approximate the entire linear response by non-intrusive shadowing. Furthermore, it is possible to partially compute the unstable contribution using the ensemble formula, but the asymptotic cost for large $W$ is much larger than fast linear response [31]. Since non-intrusive shadowing is part of the fast linear response algorithm, it does not hurt at all to first try non-intrusive shadowing, which is faster and maybe accurate enough. If more accuracy is demanded, we may further compute the unstable contribution. Somewhat surprisingly, an efficient computation of the unstable contribution also requires non-intrusive shadowing.

3. Expanding the unstable divergence

We derive the computable expansion formula for the unstable divergence. On fractal attractors, directional derivatives of $X^u$ are true distributions with infinite values. Hence, the unstable divergence, defined as the sum of directional derivatives, is \textit{a priori} only a distribution; however, we can prove that it is in fact a Holder continuous function. Hence, the unstable divergence should not be computed by summing directional derivatives. This difficulty is resolved via three steps of transformations, following the (sketchy) roadmap given by Ruelle for proving its regularity [40]:

- Transform the unstable divergence under the conditional SRB measure to the divergence under the Lebesgue measure. The difference is the derivative of the conditional SRB measure, which can be expanded since SRB is the infinite pushforward of Lebesgue.
- The Lebesgue unstable divergence equals the derivative of the volume ratio between two projections, one along stable manifolds, the other along $X$.
- Since the projection along stable manifolds collapses after infinite pushforward, we can expand this non-differentiable projection, then expand the volume ratio.

Compared to the previous theoretical work, we make the definitions coordinate-free, and derive the detailed formulas, which are necessary for numerics, but were previously missing.

3.1. Integration by parts and measure change.

In order to obtain a smaller integrand for the unstable contribution, we integrate-by-parts on the unstable manifold under the conditional SRB measure. This yields unstable divergence under SRB measure, which relates to the Lebesgue unstable divergence by a measure change.

Throughout this paper, we make the same assumptions as theorem 1. Recall that the conditional SRB measure on an unstable manifold $V^u$ has a density, or the Radon-Nikodym derivative with respect to the $u$-dimensional Lebesgue measure, which is denoted by $\sigma$. 
Let \( \omega \) be the volume form on \( V^u \), \( \rho \) be the SRB measure. Integrations to \( \rho \) can be done by first integrating to \( \sigma \omega \) on the unstable manifold, then in the transversal direction.

We first explain the integration by parts. For any \( \varphi \in C^1 \),

\[
X^u(\varphi)\sigma + \varphi X^u(\sigma) = X^u(\varphi\sigma) = \text{div}^u(\varphi\sigma X^u) - \varphi\sigma \text{div}^u X^u.
\]

Here \( X^u(\cdot) \) means to differentiate a function in the direction of \( X^u \), \( \text{div}^u \) is the divergence on the unstable manifold under Riemannian metric. For now, we only know that \( X^u \) is Holder continuous, and its derivatives are distributions; later, we will prove the regularity of the unstable divergence. Integrate on a piece of unstable manifold \( V^u \), we have the integration-by-parts formula

\[
\int_{V^u} X^u(\varphi)\sigma\omega = \int_{V^u} \text{div}^u(\varphi\sigma X^u)\omega - \int_{V^u} \left( \frac{\varphi}{\sigma} X^u(\sigma) \right) \sigma\omega - \int_{V^u} (\varphi \text{div}^u X^u) \sigma\omega.
\]

We will deal with the first two terms on the right hand side in this subsection, and leave the last term, which involves the unstable divergence, to the next two subsections.

When further integrating over the entire attractor with SRB measure \( \rho \), the first term on the right of equation (4) becomes zero. To see this, first notice that the divergence theorem reduces it to boundary integrals. Intuitively, since unstable manifolds always lie within the attractor, and they do not have boundaries, the boundary integral would never appear when integrating over \( \rho \), and hence this term becomes zero. More rigorously, first choose a Markov partition with a small diameter, then notice that the boundary integrals cancel between two adjacent rectangles. To conclude, we have now

\[
\rho(X^u(\varphi)) = -\rho(\varphi \text{div}^u X^u), \text{ where } \text{div}^u X^u := \frac{1}{\sigma} X^u(\sigma) + \text{div}^u X^u.
\]

In the rest of this paper, let

\[
e := e_1 \wedge \cdots \wedge e_u \in \wedge^u V^u
\]

be a rough \( u \)-vector field, which is differentiable on each unstable manifold, but not necessarily continuous in all directions. Also assume that \( e \) and \( \nabla(\cdot)e \) are bounded on the attractor \( K \) under Riemannian metric, where \( \nabla(\cdot)e \) is the Riemannian connection operating on vectors in \( V^u \). In fact, we may regard \( e \) as \( C^1 \) \( u \)-vector fields on individual unstable manifolds. We can make it continuous across the foliation, for example,

\[
\tilde{e} := \frac{e}{\|e\|}
\]

is unique and continuous modulo an orientation, and it satisfies our boundedness assumption, because the unstable manifold theorem states that, in our case, unstable manifolds are continuous in \( C^3 \) topology [21]. Here \( \| \cdot \| \) is the \( u \)-dimensional volume of the hyper-cube spanned by \( \{e_i\}_{i=1}^u \). We use \( e \) instead of \( \tilde{e} \) if the statement holds more generally.

The volume of hyper-cubes, \( \| \cdot \| \), is in fact a tensor norm induced by the Riemannian metric; that is, \( \|e\|^2 = \langle e, e \rangle \). In this paper, for simple \( u \)-vectors,

\[
\langle e, r \rangle := \det \langle e_i, r_j \rangle, \text{ where } e = e_1 \wedge \cdots \wedge e_u, \ r = r_1 \wedge \cdots \wedge r_u, \ e_i, r_j \in T_M.
\]

When the operands are summations of simple \( u \)-vectors, the inner-product is the corresponding sum. Our definition is equivalent to the definition in some textbooks [22, equation 5.22, chapter 2], though may differ from some textbooks by a constant coefficient of \( u! \) [27, proposition 2.40].
SRB measure is the weak limit of pushing-forward the Lebesgue measure. Roughly speaking, pushing-forward is like a matrix multiplication. Hence, by the Leibniz rule, the measure change term, \( X^u(\sigma)/\sigma \), can be expanded into an infinite summation, with the detailed formula given below.

**Lemma 1** (expression for measure change). The following formula converges uniformly on \( K \)

\[
\frac{1}{\sigma} X^u(\sigma) = \sum_{k=1}^{\infty} \frac{-\langle \nabla_{f^{-k+1}_*X_*f_*e_{-k}}, f_*e_{-k} \rangle}{\langle f_*e_{-k}, f_*e_{-k} \rangle} + \frac{\langle \nabla_{f^{-k}_*X_*e_{-k}}, e_{-k} \rangle}{\langle e_{-k}, e_{-k} \rangle}.
\]

Here \( e_{-k} \) can be any differentiable \( u \)-vector field on \( f^{-k}(V^u) \). For example, we can take it as the one such that \( e \) and \( \nabla e \) are bounded.

**Remark.** (1) If we evaluate both side of the equation at \( x \), then \( e_{-k} = e(f^{-k}x) \), and when being differentiated, \( e_{-k} \) is a restriction of \( e \) to a neighborhood of \( x_{-k} = f^{-k}x \). (2) Due to uniform convergence, \( X^u(\sigma)/\sigma \) is uniform continuous over \( K \). (3) Moreover, an algorithm computing this expansion would converge to the true solution. (4) A more careful analysis would show that this term is in fact Holder, as claimed in [40]; we will not pursue Holder continuity here, for it does not directly affect the algorithm.

**Proof.** The conditional SRB measure \( \sigma \) on \( V^u \) is the result of evolving the Lebesgue measure starting from the infinite past. More specifically,

\[
\sigma = \prod_{k=1}^{\infty} C_k \left( J^u_{-k} \right)^{-1},
\]

where \( C_k \) is constant over \( V^u_k \) to keep the total conditional measure at 1, and \( J^u \) is the unstable Jacobian computed with respect to the Riemannian metric on \( V^u \), more specifically, for any \( e \),

\[
J^u := \frac{\|f_\cdot e\|}{\|e\|} = \left( \frac{\langle f_*e, f_*e \rangle}{\langle e, e \rangle} \right)^{0.5}.
\]

Notice \( J^u \) does not depend on the particular choice of \( e \). By the Leibniz rule of differentiation, and the notation convention explained in equation (2),

\[
\frac{1}{\sigma} X^u(\sigma) = -\sum_{k=1}^{\infty} \frac{X^u(J^u_{-k})}{J^u_{-k}} = -\sum_{k=1}^{\infty} \frac{(f_*^{-k}X^u)J^u_{-k}}{J^u_{-k}}.
\]

To get the equation in the lemma, substitute (7) into (8). For any vector \( Y \in V^u \),

\[
Y(J^u) = Y \left( \frac{\langle f_*e, f_*e \rangle}{\langle e, e \rangle} \right)^{0.5} = \frac{1}{2} (J^u)^{-1} Y \left( \frac{\langle f_*e, f_*e \rangle}{\langle e, e \rangle} \right) = \frac{1}{2} (J^u)^{-1} \left[ \frac{f_*Y \langle f_*e, f_*e \rangle}{\langle f_*e, f_*e \rangle} - \frac{Y \langle e, e \rangle}{\langle e, e \rangle} \right] = \frac{1}{2} (J^u)^{-1} \left[ \frac{\langle \nabla_{f_*Y} f_*e, f_*e \rangle}{\langle f_*e, f_*e \rangle} - \frac{\langle \nabla Y e, e \rangle}{\langle e, e \rangle} \right] .
\]

In the last equality we used the rule of differentiating Riemannian metric.

To see uniform convergence, take \( e \) be the one such that both \( e \) and \( \nabla e \) are bounded on \( K \), then the series is controlled by the exponentially shrinking term, \( \|f_*^{-k}X^u\| \leq C\lambda^{-k}\|X^u\| \).
3.2. Transforming to the derivative of volume ratio.

We have expanded the difference between two unstable divergence under two measures, which is relatively easy, since the conditional measures are smooth. Computing $\text{div}^u X^u$ is more difficult, since $X^u$ is typically not differentiable even within an unstable manifold. However, $\text{div}^u X^u$ can be more regular than typical distributions: it is Holder continuous over $K$ [40]. The main difficulty, in both proving the regularity and computing the unstable divergence, is to express it without derivatives of $X^u$ or other non-differentiable quantities. The key step is to show that the unstable divergence equals the derivative of a ratio between two volumes given by two projections, $\eta_*$ and $\xi_*$. Here $\eta_*$ is the projection along $X$; $\xi_*$ is the projection along stable manifolds. This subsection develops the coordinate-free treatment of this step.

Remark. (1) The $u = 1$ case is significantly easier. Because we know a posteriori that $\text{div}^u X^u$ is Holder, hence the only one directional derivative must also be Holder, and we may directly compute the directional derivative. However, such method does not help the main difficulty for $u \geq 2$. (2) Without loss of generality, we assume that the angle between the vector field $X$ and $\mathcal{V}^u$ is uniformly away from zero. If not, then we can find a vector field in the stable subspace, mollify it, then multiply by a large constant: denote the resulting vector field by $X'$, which is always non-parallel to $\mathcal{V}^u$. We can compute the sensitivity caused by $X$ as the sum of those caused by $X + CX'$ and $-CX'$.

As illustrated in figure 1, fix an unstable manifold $\mathcal{V}^u$, let $q \in \mathbb{R}$ be a small parameter, for any $y \in \mathcal{V}^u$, define $\eta(y) : \mathcal{V}^u \rightarrow \mathcal{M}$ as the unique curve such that $\partial \eta/\partial q = X$ and $\eta(0) = y$. For fixed $q$, $\mathcal{V}^{uq} := \{\eta(y) : y \in \mathcal{V}^u\}$ is a $u$-dimensional $C^3$ manifold; for a small interval of $q$, $\hat{\mathcal{V}}^u := \cup q \mathcal{V}^{uq}$ is a $u + 1$ dimensional manifold. For any $y \in \mathcal{V}^u$, denote the stable manifold that goes through it by $\mathcal{V}^s(y)$, which does not vary differentiably with $y$ in $C^3$. Define $\xi^q(y)$ as the unique intersection point of $\mathcal{V}^s(y)$ and $\mathcal{V}^{uq}$.

Define $\pi_\eta, \pi_\xi : \hat{\mathcal{V}}^u \rightarrow \mathcal{V}^u$ such that $\pi_\eta(x) = (\eta^0)^{-1} x$, $\pi_\xi(x) = (\xi^q)^{-1} x$, for any $x \in \mathcal{V}^{uq}$ and small $q \in \mathbb{R}$. Denote $\eta^q, \xi^q$ as the pushforward operator of $\eta^0, \xi^q$. For any small $q \in \mathbb{R}$, define

$$\eta_* e(x) := \eta^q e(\pi_\eta(x)), \quad \xi_* e := \xi^q e(\pi_\xi(x)), \quad \text{for } x \in \mathcal{V}^{uq}.$$ 

Then $\eta_* e$ and $\xi_* e$ are two parallel $u$-vector fields on $\hat{\mathcal{V}}^u$. (We may equivalently define $\eta_* e := \cup q \eta^q e$, $\xi_* e := \cup q \xi^q e$.) Define a function on $\hat{\mathcal{V}}^u$, $\varpi$, as the volume ratio,

$$\frac{\eta_* e}{\|e \circ \pi_\eta\|} = \varpi \frac{\xi_* e}{\|e \circ \pi_\xi\|},$$

By transversal absolute continuity, $\varpi$ is a well-defined measurable function.

Lemma 2 (expression of $\nabla_{X_*} \eta_* e$). Denote $\nabla e X := \sum e_1 \wedge \cdots \wedge \nabla e_i X \wedge \cdots \wedge e_u$. On $\mathcal{V}^u$, $\nabla_{X_*} (\eta_* e_i) = \nabla_{e_i} X - \nabla_{X^u e_i}$, $\nabla_{X_*} (\eta_* e) = \nabla e X - \nabla_{X^u e}$.

Remark. $\nabla e X$ is a function, not a distribution, because $X$ is differentiable on $\mathcal{M}$. $\nabla_{X_*} e$ also is a function, since it only requires $e$ be differentiable along the direction of $X^u$. Differentiability of $X^u$ is not and should not be required, since $X^u$ is not differentiable.
Proof. With our notation, the proof below works for both $e$ and $e_i$. First decompose,
$$\nabla_{X^s}(\eta_*e) = \nabla_X(\eta_*e) - \nabla_{X^u}(\eta_*e)$$
Since $\eta$ is the flow of $X$, the Lie derivative $L_X(\eta_*e) = 0$, hence, on $\mathcal{V}^u$,
$$\nabla_X(\eta_*e) = \nabla_{\eta_*e}X = \nabla eX,$$
where $\eta_*e = e$ on $\mathcal{V}^u$. Since $X^u$ is a vector field on $\mathcal{V}^u$, $\nabla_{X^u}(\eta_*e) = \nabla_{X^u}e$.

Lemma 3. Further assume that $\mathcal{V}^s(y)$ varies smoothly with $y$, then on $\mathcal{V}^u$,
$$\nabla_{X^s}(\xi_*e_i) = \nabla_{e_i}X^s, \quad \nabla_{X^s}(\xi_*e) = \nabla eX^s.$$

Remark. (1) The extra smoothness assumption is for $\xi_*e$ to be a differentiable $u$-vector field on $\mathcal{V}^{u\!\!q}$. Without this assumption, we can still make sense of this equation as distributional derivatives. (2) The roughness of $\xi_*e$ hints us to further replace it in section 3.3. (3) More technically, it is enough to assume that the stable foliation is $C^1$ with $C^3$ leaves, then $\partial\xi^q/\partial q$ is a $C^1$ vector field on $\hat{\mathcal{V}}^u$.

Proof. On $\hat{\mathcal{V}}^u$, $\partial\xi^q/\partial q$ is a smooth vector field. We claim that $\partial\xi^q/\partial q = X^s$ on $\mathcal{V}^u$. To see this, define $y^q : \mathcal{V}^u \to \mathcal{V}^u$ as
$$y^q(x) = \pi_\eta(\xi^q(x)), \quad \text{or equivalently, } \xi^q(x) = \eta^q(y^q(x)).$$
Fix $x$, differentiate the second equation to $q$, evaluate at $q = 0$, where $\partial\eta^q/\partial y = Id$,
$$\frac{\partial\xi^q}{\partial q} = \frac{\partial\eta^q}{\partial y} \frac{\partial y^q}{\partial q} + \frac{\partial\eta^q}{\partial q} = \frac{\partial y^q}{\partial q} + X.$$ 
Since $\partial y^q/\partial q \in V^u$, $\partial\xi^q/\partial q \in V^s$, and that $X$ is uniquely decomposed in $V^u \oplus V^s$ into $X = X^u \oplus X^s$, we see that
$$\frac{\partial\xi^q}{\partial q} = X^s, \quad \frac{\partial y^q}{\partial q} = -X^u \text{ at } q = 0.$$ 
Hence, on $\mathcal{V}^u$,
$$\nabla_{X^s}(\xi_*e) = \nabla_{\partial\xi^q/\partial q}(\xi_*e) = \nabla_{\xi_*e}\partial\xi^q/\partial q = \nabla eX^s.$$
Here second equality is because the Lie derivative $L_{\partial\xi^q/\partial q}(\xi_*e) = 0$ by definitions.

Lemma 4. Under the same assumption as lemma 2, $X^s(\varpi) = \text{div}^u X^u$ on $\mathcal{V}^u$. 

Remark. In [40], Ruelle went on to show that this equivalence persists into general cases, where the stable foliation is not smooth. The main technique is to approximate the stable foliation by evolving a smooth foliation backward in time; we will not reproduce that proof here, for it does not directly help the computation.

Proof. By definitions,

\[ g^q(x) = \pi_\eta(\xi^q(x)), \quad x = \pi_\xi(\xi^q(x)). \]

Differentiate to \( q \), and use equation (10), we get

\[ -X^u = \frac{\partial g^q}{\partial q} = \pi_\eta \frac{\partial \xi^q}{\partial q} = \pi_\eta \xi^q, \quad 0 = \pi_\xi \frac{\partial \xi^q}{\partial q} = \pi_\xi \xi^q \quad \text{on} \quad \mathcal{V}^u. \]

Hence,

\[ X^s||e \circ \pi_\eta||^2 = (\pi_\eta \xi^s)||e||^2 = \nabla X^u ||e||^2 = 2 \langle \nabla X^u e, e \rangle, \quad X^s||e \circ \pi_\xi||^2 = 0. \]

Take inner product of each side of equation (9) with itself,

\[ \|\eta^* e\|^2 \|e \circ \pi_\xi\|^2 = \varpi^2 \|\xi^* e\|^2 \|e \circ \pi_\eta\|^2. \]

Differentiate in the direction of \( X^s \), notice that \( \varpi = 1, \eta_\ast = \xi_\ast = Id \) on \( \mathcal{V}^u \), hence

\[ \langle \nabla X^s(\eta^* e), e \rangle = X^s(\varpi) \langle e, e \rangle + \langle \nabla X^s(\xi^* e), e \rangle + \langle \nabla X^s e, e \rangle. \]

By lemma 2 and 3, we have

\[ X^s(\varpi) = \frac{1}{\langle e, e \rangle} \langle \nabla e X - \nabla X^s e - \nabla e X^s + \nabla X^s e, e \rangle = \frac{1}{\langle e, e \rangle} \langle \nabla e X^u, e \rangle. \]

Since the Riemannian connection within a submanifold \( \mathcal{V}^u \) is the orthogonal projection of that on the background manifold \( M \),

\[ X^s(\varpi) = \frac{1}{\langle e, e \rangle} \langle \nabla^u e X^u, e \rangle = \frac{1}{\langle e, e \rangle} \sum_{i=1}^u \left( e_1 \wedge \cdots \wedge \sum_j e_j^i (\nabla^u e_i X^u) e_j \wedge \cdots \wedge e_u, e \right). \]

Here \( \nabla^u, \langle \cdot, \cdot \rangle_u, \) and \( \{e^i_u\}_{j=1}^u \) are the Riemannian connection, metric, and dual basis of \( \{e_j^i\}_{j=1}^u \) within \( \mathcal{V}^u \). Terms with \( j \neq i \) vanish due to that the same direction appears twice in the exterior product, hence

\[ X^s(\varpi) = \sum_{i=1}^u e^i_u (\nabla^u e_i X^u). \]

This is a contraction of \( \nabla^u X^u \) within \( \mathcal{V}^u \), which is the definition of \( \text{div}^u X^u \). \( \square \)

3.3. Expanding the volume ratio.

We show that \( X^s(\varpi) \) is continuous by expanding the volume ratio into a converging summation. The main problem now is that \( \xi^* e \) is not yet differentiable. Hence, we further write \( \xi^* e \) as an infinite pushforward, whose derivative is an infinite summation. This finally expands the derivative of the volume ratio, and hence Lebesgue unstable divergence.
Lemma 5 (expansion of Lebesgue unstable divergence). The following formula converges uniformly on $K$:

$$\text{div}^u X^u = \langle \nabla e X, e \rangle \sum_{k=0}^{\infty} \frac{\langle \nabla f^k_X X^s f^{k+1} e, f^{k+1} e \rangle}{\| f^{k+1} e \|^2} - \frac{\langle \nabla f^k_X X^s f^{k} \eta e, f^{k} e \rangle}{\| f^{k} e \|^2}.$$  

Remark. Due to uniform convergence, $\text{div}^u X^u$ is uniform continuous over $K$. A more careful analysis would show that it is Holder continuous over $K$, as claimed in [40].

Proof. By definition of the stable manifold, for $x \in V^u$, $\lim_{k \to \infty} f^k(\pi_\xi x) = \lim_{k \to \infty} f^k(x)$. Hence,

$$\lim_{k \to \infty} \| f^k \eta e \|^2/\| f^k (e \circ \pi_\xi) \|^2 = 1.$$  

Because $\xi e$ is parallel to $\eta e$,

$$\| \eta e \|^2 = \lim_{k \to \infty} \| f^k \eta e \|^2 = \lim_{k \to \infty} \| f^k (e \circ \pi_\xi) \|^2.$$  

We can use this to replace $\xi e$ in the definition of $\varpi$ in equation (9), which yields an infinite pushforward:

$$\varpi = \lim_{k \to \infty} \| f^k (e \circ \pi_\xi) \|^2 = \lim_{k \to \infty} \| f^k e \circ \pi_\xi \|^2.$$  

Further differentiating in direction $X^s$ would yield an infinite summation. More specifically, notice that at $V^u$, $\eta e = e \circ \pi_\xi = e$, $\varpi = 1$, and apply equation (2),

$$X^s(\varpi) = \langle \nabla X^s \eta e, e \rangle - \langle \nabla X^s e, e \rangle + \sum_{k=0}^{\infty} \frac{\langle \nabla f^k_X X^s f^{k+1} e, f^{k+1} e \rangle}{\| f^{k+1} e \|^2} - \frac{\langle \nabla f^k_X X^s f^{k} \eta e, f^{k} e \rangle}{\| f^{k} e \|^2}.$$  

The equality in the lemma is obtained by lemma 2 and 4.

For the uniform convergence, use the Leibniz rule in appendix A lemma 15, and the projection operators defined in appendix C, we can decompose the $k$-th term in the summation of the lemma, $S_k$, into $S_k = S_{k1} + S_{k2} + S_{k3}$, where

$$S_{k1} := \frac{1}{\| f^{k+1} e \|^2} \langle \nabla f^k_X X^s f^{k} e, f^{k+1} e \rangle \leq C u \lambda^k \| X^s \|.$$  

Here $\nabla f_s$ is the Riemannian connection of $f_s$ (appendix A definition 3). This inequality is straightforward when $u = 1$, because $X^s$ decays exponentially via pushforwards, and that $\nabla f_s$ is bounded. For general $u$, recall that we can freely select $e_i$’s so long as their wedge product is $e$; now let $\{ f^k e_i \}_{i=1}^u$ be orthogonal at $f^k x$, hence

$$\| \langle \nabla f^k_X X^s f^{k} e_1 \wedge \cdots \wedge f^{k+1} e_u \rangle \|_{\| f^{k+1} e \|} \leq \langle \nabla f^k_X X^s f^{k} e_1 \| f^{k+1} e_2 \wedge \cdots \wedge f^{k+1} e_u \| \| f^{k+1} e_1 \| \| f^{k+1} e_2 \wedge \cdots \wedge f^{k+1} e_u \|,$n

which reduces to case $u = 1$.

The term $S_{k2}$ is defined as

$$S_{k2} := \frac{1}{\| f^{k+1} e \|^2} \langle f_s^u P^u \nabla f^{k+1} X^s f^{k} \eta e, f^{k+1} e \rangle - \frac{1}{\| f^{k} e \|^2} \langle P^u \nabla f^{k+1} X^s f^{k} e, f^{k} e \rangle = 0,$$  

This is because $(V^n)^u$ is 1-dimensional, so $P^u \nabla e_{\pi_\xi} f^{k} e(y)$ and $f^{k} e$ increase by same amounts via the pushforward.
Finally, $S_{k3}$ is defined as
\[ S_{k3} := \frac{1}{\|f^{k+1}_* e\|^2} \langle f_* P^s \nabla f^{k+1}_* X, f^k_* e \rangle - \frac{1}{\|f^k_* e\|^2} \langle P^s \nabla f^k_* X, f^k_* e \rangle. \]

The sum $\sum_{k \geq 0} S_{k3}$ also converges uniformly on $K$, since
\[ \frac{\|P^s \nabla f^k_* X, f^k_* e\|}{\|f^k_* e\|} \leq \frac{\|P^s (\nabla e f^k_* X - \nabla X f^k_* e)\|}{\|f^k_* e\|} + \sum_{n=0}^{k-1} \frac{\|P^s (\nabla f^{k-n}_* X f^k_* e)\|}{\|f^k_* e\|} \leq C u \lambda^k \|X\| C. \]

Here the second inequality is again straightforward when $u = 1$; for general $u$, let $\{f^k_* e_i\}_{i=1}^u$ be orthogonal at $f^k_* X$, and use the same trick for $S_{k1}$.

Hence, $\sum_{k \geq 0} S_k$ uniformly converges.

**Theorem 2** (expansion of unstable divergence). Define
\[ \psi := \sum_{m=-W}^W (\Phi_m - \rho(\Phi)), \quad \Psi := \psi X. \]

By lemma 1 and lemma 5, the unstable contribution is $U.C. = \lim_{W \to \infty} U.C.\,^W$, where
\[ U.C.\,^W := \rho(\psi \text{div}_\sigma X^u) = \rho \left( \psi \text{div}_\sigma X^u + \frac{\psi}{\sigma} X^u(\sigma) \right) \]
\[ = \rho \left[ \left( \psi \nabla e X, e \right) \|e\|^2 + \sum_{k=0}^{\infty} \left( \frac{\langle \nabla f^{k+1}_* \Psi, f^k_\ast e \rangle}{\|f^k_\ast e\|^2} - \frac{\langle \nabla f^k_\ast \Psi, f^k_\ast e \rangle}{\|f^k_\ast e\|^2} \right) \right] \]
\[ - \sum_{k=1}^{\infty} \left( \frac{\langle \nabla f^{k-1}_* \Psi, f_\ast e_{-k} \rangle}{\|f_\ast e_{-k}\|^2} - \frac{\langle \nabla f^{k-1}_* \Psi, e_{-k} \rangle}{\|e_{-k}\|^2} \right). \]

Here $\sigma$ is the density of the conditional SRB measure, $e_{-k}$ is a $u$-vector field on $f^{-k}Y^u$ as defined in equation (5), and $\langle \cdot, \cdot \rangle^u, \langle \cdot, \cdot \rangle^s$ are the unstable and stable projections of a vector.

**Remark.** (1) The uniform convergence in lemma 1 and 5 shows that this formula also converges uniformly. In fact, we did not use the full strength of uniform hyperbolicity to prove uniform convergences; moreover, weaker forms of convergence could also suffice our computational purpose. (2) Note that adding a constant to $\Phi$ does not change the linear response, but it helps to reduce numerical errors.

Since the integrand is much smaller than the ensemble linear response formula, directly computing the expansion formula in theorem 2 would have much faster convergence than ensemble or stochastic algorithms based on the ensemble linear response formula. However, that would require solving at least $u^2$ second-order tangent equations, which shall be defined later. Directly computing the expansion formula would also require the oblique projections, which can be computed via a ‘little-intrusive’ algorithm which uses not only the tangent solvers, but also the adjoint solvers [29, 31]. In the next section, we further transform the expansion formula to a ‘fast’ formula, which requires only $u$ many second-order tangent solutions and do not involve oblique projections.
4. Fast formula of unstable divergence

Fast algorithms, such as the fast Fourier transformation \[11\] and the fast multipole method \[17, 18\], found a non-obvious ‘fast’ structure, which allows us to combine many small terms into a few big terms. Because the rules of propagation on these small terms are similar, we can then apply some averaged rule of propagation on big terms only a few times, instead of many times on each small term. In our case, we find an ‘embarrassingly fast’ structure for the expansion formula of the unstable divergence, where we only need one uniform rule of propagation for one big term. Here the most expensive operation is the renormalized second-order tangent equation, which governs the propagation of derivatives of vectors. We use the linearity of \( \nabla e \) to define a summation, \( p \); hence, the second-order tangent equation only needs to be applied to \( p \) once, instead many times on each \( \nabla e \).

In this section, we show that both the unstable contribution and the inductive relation of \( p \) can be expressed via this ‘first combine then propagate’ strategy. Then we show that \( p \) is uniquely determined by the inductive relation it satisfies. It also turns out that, in the fast formula, the oblique projection disappears; hence, the fast formula is non-intrusive, requiring only \( u \) many first and second order tangent solutions.

4.1. Definitions of \( p, \beta, \) and \( U \).

The expression in theorem 2 computes too many times \( \nabla f_z \), which is the Riemannian connection of the pushforward tensor defined in appendix A. To save computational efforts, we seek to combine terms with \( \nabla f_z \) at the same step. To achieve this, first let \( e_{-k} \) be \( \tilde{e}_{-k} \) in the second summation, where \( \tilde{e} = e / \| e \| \); then, by the invariance of the SRB measure,

\[
U.C. W = \rho \left[ \sum_{k=0}^{\infty} \left( \frac{\langle \nabla f_{k+1}^{\ast} \Psi_{-k} f_k^{k+1} \eta e_{-k}, f_k^{k+1} e_{-k} \rangle}{\| f_k^{k+1} e_{-k} \|^2} - \frac{\langle \nabla f_k^{\ast} \Psi_{-k} f_k^{k} \eta e_{-k}, f_k^{k} e_{-k} \rangle}{\| f_k^{k} e_{-k} \|^2} \right) \right]
\]

\[+ \frac{\langle \psi_1 \nabla e_1 X_1, e_1 \rangle}{\| e_1 \|^2} - \sum_{k=1}^{\infty} \left( \frac{\langle \nabla f_1^{k} \Psi_{-k} f_1^{k} e \tilde{e}, f_1^{k} e \tilde{e} \rangle}{\| f_1^{k} e \tilde{e} \|^2} - \frac{\langle \nabla f_1^{\ast} \Psi_{-k} f_1 \tilde{e}, \tilde{e} \rangle}{\| f_1^{\ast} \tilde{e} \|^2} \right) \right],
\]

where the subscript (\( \cdot \)\( _1 \)) labels steps, \( \Psi_k := \psi_k X_k, e \) is the \( u \)-dimensional hyper-cube defined in equation (5), \( \tilde{e} \) is the normalized hyper-cube, \( \eta \) is the projection map along \( X = \delta f \circ f^{-1} \) (see section 3.2), and the definition of \( \nabla (\cdot)_{k} \) is in equation (1).

Because \( \wedge V^u \) is one-dimensional, \( f_k e_{-k} / \| f_k e_{-k} \| = \pm \tilde{e} \) for all \( k \). Here the sign is determined by the orientations of \( e_k \) and \( \tilde{e} \). In our algorithm, the SRB measure is given by the empirical measure of an aperiodic trajectory, and \( \{ e_k \}_{k \in \mathbb{Z}} \) is obtained by pushing forward on this trajectory. For this case, the orientations of \( e_k \)’s are the same; hence, in our paper, we assume

\[
\frac{f_k e_n}{\| f_k e_n \|} = \tilde{e}_{n+k}, \quad \text{for any } n, k.
\]

Should we choose to consider the more general case, we only need to keep track of the sign generated by the different orientations of \( e_k \)’s. Because this sign is canceled within the inner products in equation (6), our work below would not change.
Hence, we may pull out \( \tilde{e} \) from the second term of each summand in equation (12), and define

\[
p := \sum_{k=0}^{\infty} \frac{1}{\|f_k \psi_{-k}\|} P^\perp \nabla f_k \psi_{-k} f_k^* \eta_{-k} - \sum_{k=1}^{\infty} P^\perp - \sum_{k=1}^{\infty} p_k \rho_k,
\]

where \( p_k := \frac{\nabla f_k \psi_{-k} f_k^* \eta_{-k}}{\|f_k \psi_{-k}\|} \), \( p_k := P^\perp p_k \), \( p_k := \nabla f_k \psi_{-k} \).

Here \( P^\perp \) is the orthogonal projection operator (appendix C definition 5). and notice that \( p_{k+1} \neq p_k \). We will see later that \( P^\perp \) keeps only a convergent part of the first summation, while the normalized \( \tilde{e} \) makes the second summation in \( D_e^\perp := P^\perp D_e \), which is the orthogonal projection of the space of derivative-like \( u \)-vectors (appendix B). Also notice that \( p \) is the same for any \( e_{-k} \) with the same orientation, since for any \( C^\infty \) function \( g \geq 0 \), \( P^\perp \nabla g e = g P^\perp \nabla e \).

**Lemma 6.** \( p \) is a convergent summation in \( D_e^\perp \).

**Proof.** Since \( \tilde{e} \) has constant volume, we have \( \langle \nabla \tilde{e}, \tilde{e} \rangle = 0 \). By the linearity of the projection operator, \( p \in D_e^\perp \) should it converge. To see the convergence of the first summation, use the estimation in equation (11) and that \( P^\perp = P^\perp P^u \) from lemma 18. The convergence of the second summation is because \( f_{-k}^* \Psi_k^u \) decays exponentially. \( \square \)

To simplify our writing, we define two maps and show some of their properties. The map \( \beta \)'s are renormalized second-order tangent equations, which governs the propagation of the derivatives of \( u \)-vectors, and \( U \)'s are the integrand in the unstable contribution.

**Definition 1.** For any \( r \in D_e, Y \in T_x M \), define

\[
\beta_Y (r) := (f_r + \langle \nabla_Y f, \tilde{e} \rangle) / \| f_r \|,
\]

\[
\bar{\beta} (r) := \beta_Y (r) + \psi_1 \nabla \tilde{e}, X_1,
\]

\[
U_Y (r) := \langle \bar{\beta} (r), \tilde{e}_1 \rangle - \langle r, \tilde{e} \rangle,
\]

\[
\bar{U} (r) := \langle \bar{\beta} (r), \tilde{e}_1 \rangle - \langle r, \tilde{e} \rangle = U_Y (r) + \langle \psi_1 \nabla \tilde{e}, X_1, \tilde{e}_1 \rangle.
\]

Here \( \nabla f_r \) is the Riemannian connection of \( f_r \) (appendix A definition 3), \( \tilde{v} \) is the shadowing direction of \( \Psi \),

\[
\tilde{v} := \sum_{k=0}^{\infty} f^k \Psi_{-k}^u = \sum_{k=1}^{\infty} f_{-k}^k \Psi_k^u.
\]

**Remark.** (1) Similar to pushforward operators, for \( r \in D_e(x) \), we have \( \beta_Y (r), \bar{\beta} (r) \in D_e(f x) \).

(2) The oblique projections are summarized into the modified shadowing direction \( \tilde{v} \). By section 2.3, \( \tilde{v} \) can be efficiently computed by non-intrusive shadowing algorithms, which does not require computing oblique projections. (3) The second-order tangent equation, \( \bar{\beta} \), is in fact also an inhomogeneous first-order tangent equation with a first-order inhomogeneous term, \( \nabla \tilde{v} X \), and a second-order inhomogeneous term, \( \nabla \tilde{v} f_r \).

**Lemma 7** (properties of \( \beta \) and \( U \)). For any \( r, r' \in D_e \), and \( X, Y \in T_x M \),

(1) \( \beta_X r' \pm \beta_Y r = \beta_{X \pm Y} (r' \pm r) \), \( U_X r' \pm U_Y r = U_{X \pm Y} (r' \pm r) \);

(2) for both \( \beta \) and \( \bar{\beta} \), we have \( \beta \in D_e, P^\perp \beta r = P^\perp \beta r \), where \( r^\perp := P^\perp r \);

(3) for both \( U \) and \( \bar{U} \), \( U (r) = U (r^\perp) \).
Proof. (1) By definition. (2) By definition, then lemma 19 in appendix C. (3) Since $\wedge^u V^u$ is one-dimensional, all of its $u$-vectors are increased by same amounts by the pushforward. Hence, $\langle f_x r^\parallel, \tilde{e}_1 \rangle / \| f_x \tilde{e} \parallel = \langle r^\parallel, \tilde{e} \rangle$, and

$$\begin{align*}
U(r) - U(r^\perp) &= \langle \beta(r) - \beta(r^\perp), \tilde{e}_1 \rangle - \langle r - r^\perp, \tilde{e} \rangle \\
&= \langle \beta_0 (r^\parallel) \tilde{e}_1 \rangle - \langle r^\parallel, \tilde{e} \rangle = \langle \frac{f_x r^\parallel}{\| f_x \tilde{e} \parallel}, \tilde{e}_1 \rangle - \langle r^\parallel, \tilde{e} \rangle = 0.
\end{align*}$$

Here $\beta_0$ means $Y = 0$ in definition 1.

4.2. Unstable contribution expressed by $p$.

The most expensive operation in the expression of the unstable contribution is to propagate the derivatives of $u$-vectors by second-order tangent equations, $\beta$. With the definition of the summation, $p$, now we only need to apply $\beta$ once on $p$, instead of many times on each summand in the definition of $p$. Hence the computation of $U$ from $p$ is much more efficient. Moreover, we no longer need to compute oblique projections, since $\tilde{v}$ can be computed by non-intrusive shadowing. In the next subsection, we show that similar reductions happen for the inductive relation of $p$.

Lemma 8. $U.C.W = \rho(\tilde{U}(p))$.

Proof. By the Leibniz rule in appendix A lemma 15, $\eta_s = Id$ at $q = 0$, and the definition of $p_{(k)}$ and $p'_{(k)}$ in equation (13), we have

$$\begin{align*}
\frac{\nabla f_{k+1}^s \eta_s f_{k+1}^s \eta_s e_{-k}}{\| f_{k+1}^s e_{-k} \|^2} &= \frac{1}{\| f_x \tilde{e} \parallel} \left( f_x p_{(k)} + \langle \nabla f_{k}^s \eta_s e_{-k}, f_x \tilde{e} \rangle \right) = \beta f_{k}^q \eta_s e_{-k} (p_{(k)}), \\
\frac{\nabla f_{k+1}^s \eta_s f_{k+1}^s \eta_s e_{-k}}{\| f_x \tilde{e} \parallel} &= \frac{1}{\| f_x \tilde{e} \parallel} \left( f_x p_{(k)} + \langle \nabla f_{k}^s \eta_s e_{-k}, f_x \tilde{e} \rangle \right) = \beta f_{k}^q \eta_s e_{-k} (p_{(k)}).
\end{align*}$$

The summand in the first summation of equation (12) becomes

$$\begin{align*}
\langle \nabla f_{k+1}^s \eta_s f_{k+1}^s \eta_s e_{-k}, f_{k+1}^s \eta_s e_{-k} \rangle - \langle \nabla f_{k}^s \eta_s e_{-k}, f_{k}^s e_{-k} \rangle \\
= \langle \beta f_{k}^q \eta_s e_{-k} (p_{(k)}), \tilde{e}_1 \rangle - \langle p_{(k)}, \tilde{e} \rangle = U f_{k}^q \eta_s (p_{(k)}) = U f_{k}^q \eta_s (p_{(k)}).
\end{align*}$$

The summand in the second summation becomes

$$\begin{align*}
\frac{\langle \nabla f_{k+1}^s \eta_s f_{k+1}^s \eta_s e_{-k}, f_{k+1}^s \eta_s e_{-k} \rangle - \langle \nabla f_{k}^s \eta_s e_{-k}, f_{k}^s e_{-k} \rangle}{\| f_{k+1}^s e_{-k} \|^2} - \langle \nabla f_{k+1}^s \eta_s f_{k+1}^s \eta_s e_{-k}, f_{k+1}^s \eta_s e_{-k} \rangle \\
= \langle \beta f_{k}^q \eta_s e_{-k} (p_{(k)}), \tilde{e}_1 \rangle - \langle p_{(k)}, \tilde{e} \rangle = U f_{k}^q \eta_s (p_{(k)}).
\end{align*}$$

The lemma is proved by summing over $k$ and lemma 7 (1). □

4.3. Characterizing $p$ by induction.

We further transform the expansion formula to a ‘fast’ formula, which allows even faster computation than directly using the expansion. Roughly speaking, we will show that $p$ satisfies an inductive relation given by a renormalized second-order tangent equation, whose stability indicates that any non-equivariant sequences satisfying this equation will eventually converge to $p$. This is the non-intrusive formulation for the unstable contribution,
because it requires only $u$ many first-order and second-order tangent solutions; here we also allow second-order tangent solutions because they are inevitable in this topic.

**Lemma 9** (inductive relation of $p$). $p_1 := p \circ f = P^\perp \tilde{\beta} p$.

**Remark.** (1) Should we know the correct $p(x_0)$, we can solve all $p_n$ inductively by $p_{n+1} = P^\perp \tilde{\beta}(p_n)$; this is more efficient than computing from the definition of $p$, because the most expensive operation, $\tilde{\beta}$, now only needs to operate once on $p$, instead of many times on each summand in the definition of $p$. (2) Compared to oblique projections, computing the orthogonal projection is easier and faster; in particular, it can be done with only $u$ many first order tangent solutions. (3) The subscript convention is explained in equation (1).

**Proof.** Write down the definition of $p_1$, relabel subscripts, we get

$$p_1 = \sum_{k=0}^\infty \frac{P^\perp \nabla f^k \eta_{s_k} e_{1-k}}{\|f^k e_{1-k}\|} - \sum_{k=1}^\infty \nabla f^k \eta_{s_k} e_1$$

$$= \sum_{k=-1}^\infty \frac{P^\perp \nabla f^{k+1} \eta_{s_k} e_{-k}}{\|f^{k+1} e_{-k}\|} - \sum_{k=2}^\infty \nabla f^k \eta_{s_k} e_1$$

The induction of the first summation is achieved by substituting equation (14). For the second summation, denote $Y := f^k \eta_k'$, we have

$$\nabla f_{*Y} e_1 = \nabla f_{*Y} \frac{f_* e}{\|f_* e\|} = \beta_Y p_{(k)} + f_* Y \left( \frac{1}{\|f_* e\|} \right) f_* e.$$ 

Since $\nabla e \in D^\perp$, $\nabla e = P^\perp \nabla^e e$; since $f_* e \in \perp uV^u_1$, $P^\perp f_* e = 0$. Hence,

$$\nabla f_{*Y} e_1 = P^\perp \nabla_{f_* Y} e_1 = P^\perp \beta_Y p_{(k)}.$$ 

Substitute into the expression for $p_1$, we have

$$p_1 = \frac{\psi_1}{\|e_1\|} P^\perp \nabla X^e \eta e_1 + \sum_{k=0}^\infty P^\perp \beta f^k \eta_{s_k} p_{(k)} + \psi_1 \nabla X^e \nabla \nabla^e e_1 - \sum_{k=1}^\infty P^\perp \beta f_{s_k} \eta_{s_k} p_{(k)}$$

$$= \frac{\psi_1}{\|e_1\|} P^\perp \nabla X^e \eta e_1 + \psi_1 \nabla X^e \nabla \nabla^e e_1 + P^\perp \beta_5 p$$

To add the first two terms, use lemma 2 on the first term, and that $P^\perp e = 0$,

$$\frac{\psi}{\|e\|} P^\perp \nabla X^e \eta e = \frac{\psi}{\|e\|} P^\perp \left( \nabla e X - \nabla X^e e \right) = \psi P^\perp \left( \nabla e X - \frac{1}{\|e\|} \nabla X^e e - e X^u \left( \frac{1}{\|e\|} \right) \right)$$

$$= \psi P^\perp \left( \nabla e X - \nabla X^e e \right) = \psi P^\perp \nabla e X - \psi \nabla X^e e.$$ 

Hence $p_1 = \psi_1 P^\perp \nabla \nabla^e e X^e + P^\perp \beta_5 p$, as claimed. 

Hence, we have found that the sequence $\{p_n\}_{n \geq 0}$ satisfies an inductive relation. This is an equivariant sequence, that is, $p_n(x_0) = p(x_n)$. We may as well define a non-equivariant sequence, $r$, only from the inductive relation of $p$. It turns out that $r$, the non-equivariant version of $p$, convergences exponentially fast to the true $p$ on any trajectory.
Lemma 10 (stability of renormalized second-order tangent equation). For any $x_0 \in K$, $r_0 \in D_e(x_0)$, define a non-equivariant sequence, $\{r_n\}_{n \geq 0}$, by the inductive relation of $p$,

$$r_n \in D_e(x_n), \quad r_{n+1} := P^\perp \tilde{\beta} r_n.$$ 

Then $\{r_n\}_{n \geq 0}$ approximates its equivariant counterpart, that is,

$$\lim_{n \to \infty} r_n - p_n(x_0) = \lim_{n \to \infty} P^\perp \tilde{\beta}^n r_0 - P^\perp \tilde{\beta}^n p(x_0) = 0.$$ 

Remark. (1) An easy choice of elements in $D_e$ is zero. (2) The notation $\beta^n$ means applying the inhomogeneous propagation operator $n$ times. This is done similarly to the pushforward operator, with $\tilde{v}$ and $\tilde{e}$ evaluated at suitable steps.

Proof. $P^\perp \tilde{\beta}^n r_0 - P^\perp \tilde{\beta}^n p = P^\perp f^n (r_0 - p)/\|f^n \tilde{e}\| = P^\perp f^n P^s (r_0 - p)/\|f^n \tilde{e}\| \leq C\lambda^{2n}$. □

Finally, we can prove the main theorem of this paper.

Theorem 3 (‘fast’ formula for the unstable divergence). Make same assumptions as theorem 1. Let $\{x_n := f^n x_0\}_{n \geq 0}$ be an orbit on the attractor, for any $r_0 \in D_e(x_0)$, define the sequence $\{r_n\}_{n \geq 0}$,

$$r_n \in D_e(x_n), \quad r_{n+1} := P^\perp \tilde{\beta} r_n.$$ 

Then for almost all $x_0$ according to the SRB measure, the unstable contribution,

$$U.C.^W = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \tilde{\beta} r_n, \tilde{e}_{n+1} \rangle.$$ 

Remark. (1) In practice, a $\rho$-typical point $x_0$ can be found by running almost all trajectories starting from the attractor basin for some time. (2) In practice, the unstable subspace, $V^u$, can be obtained by pushing-forward $u$ many randomly initiated vectors, because unstable vectors grow faster than stable ones. (3) A more careful analysis should prove this theorem for almost all $x_0$ in the attractor basin, and almost all initial guess of $V^u(x_0)$. (4) This is a new form of the linear response formula: it has a small integrand without involving oblique projections.

Proof. By the same tail bound in lemma 1 and 5, we can show that $\langle p, \tilde{e} \rangle$ is continuous on the attractor $K$. Hence, $\tilde{U}(p)$ is continuous on $K$; by the ergodic theorem and lemma 8, we see that almost surely according to the SRB measure,

$$U.C.^W = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{U}(p_n).$$ 

By lemma 9, the definition of $\tilde{U}$, the orthogonal condition, we have

$$U.C.^W = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{U}((P^\perp \tilde{\beta})^n p)$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \tilde{\beta}(P^\perp \tilde{\beta})^n p, \tilde{e}_{n+1} \rangle - \langle (P^\perp \tilde{\beta})^n p, \tilde{e}_n \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \tilde{\beta}(P^\perp \tilde{\beta})^n p, \tilde{e}_{n+1} \rangle$$

Finally, apply lemma 10 to replace $p$ by its non-equivariant approximation, $r$. □
5. Fast linear response algorithm

This section concerns the practical aspect of the algorithm. First, to further reduce computational cost and round-off error, we show that the renormalization only needs to be done intermittently. We will also write major equations in matrix notations, which are more suitable for computer programming. Then we give a procedure list of the algorithm. Finally, we give several remarks on the implementation of the algorithm.

5.1. Intermittent renormalization.

This subsection explains how renormalization only needs to be done once after a segment of several steps. Here renormalization refers to

- The orthogonal projection, $P_{\perp}$, used in the fast formula, defined in appendix C.
- The rescaling in $\beta$, that is, dividing by the $\|f\epsilon\|$ factor in definition 1.
- Othonormalizing the basis vectors of $V^u$.

The third operation does not explicitly appear in the fast formula, but a good basis improves numerical performance. We first show that both orthogonal projection and rescaling can be done intermittently. Then, we show that all three renormalizing operations can be done together by a short formula using matrix notation.

The subscript convention for multiple segments, shown in figure 2, is similar to that of the non-intrusive shadowing algorithm [34, 33]. We divide a trajectory into small segments, each containing $N$ steps. The $\alpha$-th segment consists of step $\alpha N$ to $\alpha N + N$, where $\alpha$ runs from 0 to $A - 1$; notice that the last step of segment $\alpha$ is also the first step of segment $\alpha + 1$.

We use double subscript, such as $x_{\alpha,n}$, to indicate the $n$-th step in the $\alpha$-th segment, which is the $(\alpha N + n)$-th step in total. Note that for some quantities defined on each step, for example, $e_{\alpha,N} \neq e_{\alpha+1,0}$, since renormalization is performed at the interface across segments. Continuity across interfaces is true only for some quantities, such as shadowing directions $v, \tilde{v}$, and unit unstable cube $\tilde{e}$. Later, we will define some quantities on the $\alpha$-th segment, such as $C_{\alpha}, d_{\alpha}$ in equation (15), their subscripts are the same as the segment they are defined on. For quantities to be defined at interfaces, such as $Q_{\alpha}, R_{\alpha}, b_{\alpha}$ in equation (16), their subscripts are the same as the total step number of the interface, divided by $N$.

![Figure 2. Subscript convention on multiple segments.](image)

**Lemma 11** (intermittent orthogonal projection). For any $\tilde{r}_{0,0} \in D_\epsilon$, let

$$\tilde{r}_{\alpha,n} := \tilde{\beta} \tilde{r}_{\alpha,n-1}, \quad \tilde{r}_{\alpha+1,0} := P_{\perp} \tilde{r}_{\alpha,N},$$
then almost surely

\[
U.C.W = \lim_{A \to \infty} \frac{1}{NA} \sum_{\alpha=0}^{A-1} \left\langle \bar{r}_{\alpha,N}, \bar{e}_{\alpha,N} \right\rangle.
\]

Proof. Denote \((P^\perp \tilde{\beta})^{\alpha N} \bar{r}_{0,0}\) by \(r'\), by lemma 7, lemma 8, and the definition of the SRB measure, the unstable contribution from step \(\alpha N\) to \(\alpha N + N - 1\) is,

\[
\sum_{n=0}^{N-1} \bar{U}((P^\perp \tilde{\beta})^{\alpha N+n} \bar{r}_{0,0}) = \sum_{n=0}^{N-1} \bar{U}((P^\perp \tilde{\beta})^{n} r') = \sum_{n=0}^{N-1} \bar{U}(\tilde{\beta}^n r') = \sum_{n=0}^{N-1} \left\langle \tilde{\beta}^{n+1} r', \bar{e}_{\alpha N+n+1} \right\rangle - \left\langle \tilde{\beta}^n r', \bar{e}_{\alpha N+n} \right\rangle = \left\langle \tilde{\beta}^{N} r', \bar{e}_{\alpha N+N} \right\rangle.
\]

Average over all steps and adopt the subscript convention to prove the lemma.

Lemma 12 (intermittent rescaling). Let \(e\) be first-order tangent solutions,

\[
e_{0,0} = \tilde{e}_{0,0}, \quad e_{\alpha,n} := f_* e_{\alpha,n-1}, \quad e_{\alpha+1,0} := e_{\alpha,N}/\|e_{\alpha,N}\| = e_{\alpha+1,0};
\]

For any \(r_{0,0} \in \mathcal{D}_e\), let \(r\) be governed by the second-order tangent equation,

\[
r_{\alpha,n} := f_* r_{\alpha,n-1} + (\nabla \bar{e}_{\alpha,n-1} f_*) e_{\alpha,n-1} + \psi_{\alpha,n} \nabla e_{\alpha,n} X_{\alpha,n}, \quad r_{\alpha+1,0} := P^\perp r_{\alpha,N}/\|e_{\alpha,N}\|.
\]

Then almost surely

\[
U.C.W = \lim_{A \to \infty} \frac{1}{NA} \sum_{\alpha=0}^{A-1} \left\langle r_{\alpha,N}, e_{\alpha,N} \right\rangle.
\]

Proof. We prove by induction that if we choose \(r_{0,0} = \tilde{r}_{0,0}\), then \(r_{\alpha,n} = \|e_{\alpha,n}\| \bar{r}_{\alpha,n}\). Within segment \(\alpha\), assuming we have this relation hold for \(n - 1\), then

\[
r_{\alpha,n} = \|e_{\alpha,n-1}\| f_* \bar{r}_{\alpha,n-1} + \|e_{\alpha,n-1}\| (\nabla \bar{e}_{\alpha,n-1} f_*) \bar{e}_{\alpha,n-1} + \|e_{\alpha,n}\| \psi_{\alpha,n} \nabla \bar{e}_{\alpha,n} X_{\alpha,n} = \|e_{\alpha,n}\| \bar{r}_{\alpha,n}.
\]

Hence the relation also holds for \(n\); it also holds across interfaces, since

\[
r_{\alpha+1,0} = P^\perp r_{\alpha,N}/\|e_{\alpha,N}\| = P^\perp \bar{r}_{\alpha,N} = \bar{r}_{\alpha+1,0} = \bar{r}_{\alpha+1,0}\|e_{\alpha+1,0}\|,
\]

where \(\|e_{\alpha+1,0}\| = 1\) by construction. Finally, substitute into lemma 11.

Proposition 13 (intermittent renormalization). Neglecting the first two subscripts, \(\alpha\) and \(n\), let \(e := \wedge_{i=1}^u e_i, \quad r := \sum e_1 \wedge \cdots \wedge r_i \wedge \cdots \wedge e_u\). Denote matrices \(e := [e_1, \cdots, e_u]\), \(r := [r_1, \cdots, r_u]\). Then between segments, the renormalization in lemma 12 is realized by:

\[
\mathcal{L}_{\alpha,N} = Q_{\alpha+1} R_{\alpha+1}, \quad \mathcal{L}_{\alpha+1,0} = Q_{\alpha+1}, \quad \mathcal{L}_{\alpha,N} = \mathcal{L}_{\alpha,N} - Q_{\alpha+1} Q_{\alpha+1} \mathcal{L}_{\alpha,N}, \quad \mathcal{L}_{\alpha+1,0} = \mathcal{L}_{\alpha,N} R_{\alpha+1}^{-1}.
\]

Here the first equation means to perform QR factorization, and \(Q^T \mathcal{L} := (\left\langle Q_i, r_j \right\rangle)\) is a matrix. Use \(\text{Tr}(\cdot)\) to denote trace of a matrix, the unstable contribution is,

\[
U.C.W = \lim_{A \to \infty} \frac{1}{NA} \sum_{\alpha=0}^{A-1} \text{Tr} \left( R_{\alpha+1}^{-1} Q_{\alpha+1}^T \mathcal{L}_{\alpha,N} \right).
\]
Remark. (1) Rewriting \( r \) on the new basis does not change \( r \) as a \( u \)-vector, but it makes the numerical properties better. (2) Computing \( e \) via pushforwards and renormalization is also part of the non-intrusive showing algorithm. (3) The pushforward relation inside a segment is the same as that in lemma 12.

Proof. The renormalization on \( e \) is due to the definition of QR factorization. For \( r \), first substitute the QR factorization into the expression for projection,

\[
L_{\alpha,N}^{\perp} = L_{\alpha,N} - L_{\alpha,N}(L_{\alpha,N}^T L_{\alpha,N})^{-1}(L_{\alpha,N}^T L_{\alpha,N}) = L_{\alpha,N} - Q_{\alpha+1}Q_{\alpha+1}^T L_{\alpha,N}.
\]

Use appendix B lemma 16 to rewrite \( r_{\alpha,N}^{\perp} \) on to the new basis, \( Q_{\alpha+1} \), then rescale,

\[
E_{\alpha+1,0} = \det(R_{\alpha+1}) E_{\alpha,N}^{\perp} R_{\alpha+1}^{-1}/\|e_{\alpha,N}\| = E_{\alpha,N}^{\perp} R_{\alpha+1}^{-1},
\]

since \( \det(R_{\alpha+1}) = \|e_{\alpha,N}\| \) by definition of QR factorization.

To simplify the expression for \( U.C.W \), first use lemma 20 in appendix C,

\[
\langle r_{\alpha,N}, e_{\alpha,N} \rangle = \langle E_{\alpha,N}^{\perp}, e_{\alpha,N} \rangle.
\]

Because \( r_i^{\parallel} \in V^u \), we can write it as

\[
r_i^{\parallel} = (e_u^i r_i) e_j,
\]

where \( e_u^i \) is the covector of \( e_j \) in \( V^u \), and \( i, j \) are summed from 1 to \( u \). Hence,

\[
r^{\parallel} = \sum_{i=1}^u e_1 \wedge \cdots \wedge (e_u^i r_i^{\parallel}) e_j \wedge \cdots \wedge e_u = \sum_{i=1}^u e_1 \wedge \cdots \wedge (e_u^i r_i^{\parallel}) e_i \wedge \cdots \wedge e_u,
\]

because the wedge product between parallel vectors is zero. Moreover, by lemma 12,

\[
U.C.W = \lim_{A \to \infty} \frac{1}{NA} \sum_{\alpha=0}^{A-1} \langle r_{\alpha,N}, e_{\alpha,N} \rangle = \lim_{A \to \infty} \frac{1}{NA} \sum_{\alpha=0}^{A-1} \sum_{i=1}^u e_u^{i,\alpha,N} r_{\alpha,N,i}^{\parallel}.
\]

To write this expression in matrix notation, first notice that \( (e_u^i r_i^{\parallel}) e_i = r_i^{\parallel} \), so

\[
(e_u^i r_i^{\parallel}) \langle e_i, e_j \rangle = \langle r_i^{\parallel}, e_j \rangle = \langle r_i, e_j \rangle.
\]

This is a linear equation system, with solution

\[
e_u^{i,\parallel} = i - \text{th entry of the vector } (e^T e)^{-1} (e^T r_i).
\]

Hence we can further write the expression of \( U.C.W \) in matrix notation,

\[
U.C.W = \lim_{A \to \infty} \frac{1}{NA} \sum_{\alpha=0}^{A-1} \text{Tr} \left( (L_{\alpha,N}^T L_{\alpha,N})^{-1}(L_{\alpha,N}^T L_{\alpha,N}) \right).
\]

Substituting the QR factorization of \( \varepsilon \), we have

\[
(e^T \varepsilon)^{-1} (e^T \varepsilon) = (R^T Q^T Q R)^{-1} R^T Q^T R = R^{-1} Q^T R.
\]  
\[\square\]
5.2. Procedure list.

This subsection gives a detailed procedure list of the fast linear response algorithm. It does not require differential geometry knowledge to understand this list when $\mathcal{M} = \mathbb{R}^M$, and corresponding simplifications are explained. All sequences in this subsection are non-equivariant, which means that they are defined by some inductive relations. In particular, here the unstable vectors $e$ and shadowing directions $v, \tilde{v}$ are non-equivariant, but they are very good approximations of their equivariant counterparts. We still use $r$ as the non-equivariant version of $p$, since the approximation has just been established in this paper. The subscript explanation is in figure 2.

i. Evolve the dynamical system for a sufficient number of steps before $n = 0$, so that $x_0$ is on the attractor at the beginning of our algorithm. Then, evolve the system from segment $\alpha = 0$ to $\alpha = A - 1$, each containing $N$ steps, to obtain the trajectory,

$$x_{\alpha,n+1} = f(x_{\alpha,n}), \quad x_{\alpha+1,0} = x_{\alpha,N}.$$  

ii. Start with initial condition $v' = 0$ and $\tilde{v}' = 0$, and random initial conditions for each column in $e := [e_1, \ldots, e_u]$. Then, repeat the following procedures for all $\alpha$.

(a) From initial conditions, solve first-order tangent equations, $\alpha$ neglected,

$$\xi_{n+1} = f_\alpha \xi_n, \quad v'_{n+1} = f_\alpha v'_n + X_{n+1}, \quad \tilde{v}'_{n+1} = f_\alpha \tilde{v}'_n + \Psi_{n+1}.$$  

Here $X := \delta f \circ f^{-1} = (\partial f/\partial \gamma) \circ f^{-1}$, where $\gamma$ is the parameter of the dynamical system; $\Psi_n := \psi_n X_n, \psi := \sum_{m=-W}^W \Phi \circ f^m$ for a large $W$, $f_\ast$ is the pushforward operator. When $\mathcal{M} = \mathbb{R}^M$, $f_\ast$ is the Jacobian matrix,

$$f_\ast = [\partial f^i/\partial z^j]_{ij}$$

where $\cdot_{ij}$ is the matrix with $(i, j)$-th entry given inside the bracket, $f^i$ is the $i$-th component of $f$, $z^j$ is the $j$-th coordinate of $\mathbb{R}^M$.

(b) Compute and store the covariance matrix and the inner product,

$$C_\alpha := \sum_{n=0}^{N-1} e_{\alpha,n}^T \xi_{\alpha,n} := \frac{1}{2} e_{\alpha,0}^T \xi_{\alpha,0} + \sum_{n=1}^{N-1} e_{\alpha,n}^T \xi_{\alpha,n} + \frac{1}{2} e_{\alpha,N}^T \xi_{\alpha,N},$$

$$d_\alpha := \sum_{n=0}^{N-1} e_{\alpha,n}^T v'_{\alpha,n}, \quad \tilde{d}_\alpha := \sum_{n=0}^{N-1} e_{\alpha,n}^T \tilde{v}'_{\alpha,n} dt,$$

where $\sum'$ is the summation with $1/2$ weight at the two end points. Here $\xi^T \xi := [(e_i, e_j)]$ is a matrix, same for $\xi^T \xi$ and $Q^T \xi$ later.

(c) At step $N$ of segment $\alpha$, orthonormalize $\xi$ with a QR factorization, and compute

$$\xi_{\alpha,N} = Q_{\alpha+1} R_{\alpha+1}, \quad b_{\alpha+1} = Q_{\alpha+1} v'_{\alpha,N}, \quad \tilde{b}_{\alpha+1} = Q_{\alpha+1} \tilde{v}'_{\alpha,N}.$$  

(d) Set initial conditions of the next segment,

$$\xi_{\alpha+1,0} = Q_{\alpha+1}, \quad v'_{\alpha+1,0} = v'_{\alpha,N} - Q_{\alpha+1} b_{\alpha+1}, \quad \tilde{v}'_{\alpha+1,0} = \tilde{v}'_{\alpha,N} - Q_{\alpha+1} \tilde{b}_{\alpha+1}.$$  

iii. Solve the non-intrusive shadowing problem,

$$\min_{\{a_\alpha\}} \sum_{\alpha=0}^{A-1} 2d_{\alpha}^T a_\alpha + a_\alpha^T C_\alpha a_\alpha$$

s.t. $a_\alpha = R_\alpha a_{\alpha-1} + b_\alpha, \quad \alpha = 1, \ldots, A - 1.$
We solve this via the Schur complement, which is a tridiagonal block matrix, and it can be solved with \( O(A) \) cost (section 3.4 of [35], also [6]). Solve the same problem again, with \( b \) replaced by \( \tilde{b} \), for \( \tilde{\alpha} \). Then compute \( v_\alpha \) and \( \tilde{v}_\alpha \),

\[
v_\alpha = v'_\alpha + \mathcal{L}_\alpha a_\alpha, \quad \tilde{v}_\alpha = \tilde{v}'_\alpha + \mathcal{L}_\alpha \tilde{a}_\alpha.
\]

iv. Compute the shadowing contribution,

\[
S.C. = \lim_{A \to \infty} \frac{1}{AN} \sum_{\alpha=0}^{A-1} \sum_{n=0}^{N} v_{\alpha,n}(\Phi_{\alpha,n}),
\]

where \( v(\cdot) \) means to differentiate a function in the direction of \( v \).

v. Denote \( \underline{r} := [r_1, \ldots, r_u] \). Set initial condition \( r_{0,i} = 0 \) for all \( 1 \leq i \leq u \). Then repeat the following procedures for all \( \alpha \).

(a) From initial conditions, solve second-order tangent equations, \( \alpha \) neglected,

\[
0 = \frac{\partial f}{\partial \delta f} \bigg|_{\Phi_{\alpha,N}}.
\]

Here \( \nabla_{\cdot} f_* \) is the Riemannian connection of the pushforward operator defined in appendix A. In \( \mathbb{R}^M \), \( \nabla f_* \) is the derivative of Jacobian, or the Hessian tensor, and

\[
\nabla f_* = [\nabla \partial f/\partial z_i]_{i,j}.
\]

To compute \( \nabla_{\alpha,n+1} X_{n+1} \), denote the coordinate at \( x_n \) and \( x_{n+1} \) by \( \zeta \) and \( z \), then

\[
\nabla_{\alpha,n+1} X_{n+1} = \nabla f_{\alpha,n} \circ f = \nabla f_{\alpha,n} \delta f_n = \nabla f_{\alpha,n} \left( \delta f_n \frac{\partial}{\partial z_j} \right)
\]

\[
= f_n \delta f_n \left( \delta f_n \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_j} \nabla f_{\alpha,n} \delta f_n \right) = f_n \delta f_n \left( \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_j} \nabla f_{\alpha,n} \delta f_n \right).
\]

Here \( \frac{\partial}{\partial z_j} \) is the \( j \)-th coordinate vector. In \( \mathbb{R}^M \), both coordinates \( \zeta \) and \( z \) are the canonical coordinate, both denoted by \( z \), and \( \nabla \frac{\partial}{\partial z_j} \) is zero, hence

\[
\nabla_{\alpha,n+1} X_{n+1} = \delta f_n \left( \frac{\partial}{\partial z_j} \nabla f_{\alpha,n} \delta f_n \right) = \delta f_n \left( \left[ \frac{\partial}{\partial z_j} \delta f_n \right]_{i,j} \right).
\]

This is a vector at \( x_{n+1} \), where \([\cdot]_{i,j} \) is a vector in \( \mathbb{R}^M \) whose \( j \)-th entry is given in the bracket. The differentiation \( \delta f_n \left( \delta f_n \right) \) happens at \( x_n \), when \( \delta f_n \left( \delta f_n \right) \) is a function in a neighborhood of \( x_n \).

(b) To set initial conditions of the next segment, first orthogonally project,

\[
\mathcal{L}_{\alpha,N}^+ = \mathcal{L}_{\alpha,N} - Q_{\alpha+1} Q_{\alpha+1}^T \mathcal{L}_{\alpha,N}.
\]

Then change basis and rescale,

\[
\mathcal{L}_{\alpha+1,0} = \mathcal{L}_{\alpha,N}^+ R_{\alpha+1}.
\]

vi. Let \( \text{Tr}(\cdot) \) be the trace of a matrix, compute the unstable contribution,

\[
U.C. = \lim_{A \to \infty} \frac{1}{NA} \sum_{\alpha=0}^{A-1} \text{Tr} \left( R_{\alpha+1}^{-1} Q_{\alpha+1}^T \mathcal{L}_{\alpha,N} \right).
\]

vii. The linear response is

\[
\delta \rho(\Phi) = \lim_{W \to \infty} S.C. - U.C. + W.
\]
5.3. Remarks on implementation.

When the number of homogeneous tangent solutions computed, $u'$, is strictly larger than $u$, the unstable contribution part of our algorithm may or may not work, depending on whether the renormalized second order tangent solutions has a meaningful average. This is different from the non-intrusive shadowing algorithm, which works for any $u' \geq u$. It remains to be investigated whether and how much error is incurred for using a large $u'$, especially how the error relates to the spectrum of the Lyapunov exponents.

Fast linear response could converge even when the system slightly fails the uniform hyperbolicity assumption we used in the proof, because it does not compute oblique projections. This extra robustness is an upshot of non-intrusiveness, and was intentionally pursued during our algorithm design. The convergence of the fast linear response actually depends on the integrability of shadowing directions and renormalized second-order tangent solutions, which might be more abundant in applications than uniform hyperbolicity. Failure of fast linear response can be caused by a large region where the stable and unstable directions are close to each other [37], such as the Henon map [15]. This situation is difficult for many algorithms and theoretical analysis for linear response, even for SRB measures.

We discuss how to choose the number of steps in each segment, $N$. Large $N$ gives fewer segments for the same total number of steps, thus saving some computational cost. However, the major computational cost in the algorithm comes from computing first and second order tangent equations. Hence, the benefit for choosing a very large $N$ is limited. Still, if we really want a large $N$, the upper bound is from the numerical stability of first and second order tangent equations. Notice that second order tangent equations are essentially first order tangent equations with a second-order inhomogeneous term. Hence, the limiting factors for large $N$ is that, over one segment, the $u$ many first-order homogeneous tangent solutions can not grow too large or too parallel to each other.

Depending on the sparsity of the problem, the main computational complexity of the fast linear response may come from different places. For the worst case, the Hessian tensor $\nabla f_*$ is dense, and contracting it with $\bar{v}$ takes $O(M^3)$ operations. If further assuming that each entry in $\nabla f_*$ takes $O(1)$ operations, then it also takes $O(M^3)$ operations to get $\nabla f_*$; in $f_*$, each entry depends on $O(M)$ variables, and it typically also takes $O(M^3)$ operations to get $f_*$. Hence, for the dense case, the cost for each step of fast linear response is dominated by obtaining $\nabla \bar{v} f_*$ and $f_*$, which only need to be done once, and cost is typically $O(M^3)$. On the other hand, for many problems from engineering, for example fluid mechanics and image processing, $\nabla f_*$ and $f_*$ typically has only $O(M)$ entries, with each entry taking $O(1)$ operations to compute. Then the cost mainly comes from computing terms such as $f_\epsilon$, whose complexity is $O(uM)$ for each step, and can be computed faster via vectorized programming, as explained below. The sparse case is perhaps more common in real-life applications.

There are several places in the algorithm where we contract a high-order tensor with several one-dimensional vectors, which can be done more efficiently via the so-called ‘vectorized’ programming. More specifically, when contracting one tensor with several vectors, we should load the tensor into computer once, and inner-product with all vectors, instead of loading the tensor once for each vector: this saves computer time [33, 30]. Such vectorization can be used in the non-intrusive shadowing algorithm, where several
first-order tangent solutions, \( v' \) and \( \{e_i\}_{i=1}^u \), are multiplying with the same Jacobian matrix \( f_* \), which is a two-dimensional tensor. Vectorization can also be used when solving second-order tangent equations, for the contraction between \( f_* \) and \( \{r_i\}_{i=1}^u, \nabla X \) and \( \{e_i\}_{i=1}^u \).

Finally, we give a very crude and formal estimation on the error and total cost of fast linear response, using the decorrelation step number \( W \), the total number of steps \( T := AN \), unstable dimension \( u \), and system dimension \( M \). First, the error for using finite \( W \), \( U.C. - U.C. \), is \( O(\theta^W) \) for some \( 0 < \theta < 1 \), which is the rate of decay of correlation. We make the simplifying assumption that, on any trajectory, for any function \( \varphi \) we care about, and for large enough \( N \),

\[
\sum_{n=1}^N \varphi(x_n) \sim O(\sqrt{N}),
\]

where \( \sim \) means approximately equal with large probability. This assumption is verified later in our numerical example, and it can be achieved by more basic assumptions, such as all summands are i.i.d.

Passing \( \Phi \) to \( \Phi - \rho(\Phi) \) makes \( \rho(\Phi) = 0 \), so \( \psi \sim O(\sqrt{W}) \); further notice that \( \psi X \) is the inhomogeneous term for \( \tilde{v} \), and that \( \tilde{v} \) and \( \psi \) are the inhomogeneous terms in \( \tilde{b} \); hence, \( \tilde{U}(p) \sim O(\sqrt{W}) \). Hence the error caused by averaging \( \tilde{U}(p) \) on a finite trajectory of length \( T \) is \( \sim O(\sqrt{W/T}) \). Hence the total error for the unstable contribution, which is also the major error for the fast linear response, denoted by \( h \), is

\[
h \sim O(\theta^W) + O(\sqrt{W/T})
\]

In practice we want the two errors to be roughly equal to each other, hence \( T \sim \theta^{-2W}W \).

By our discussion of the cost at each step, for typical problems with sparsity, the numerical complexity of fast linear response is

\[
\text{Complexity of FLR} \sim O(uM\theta^{-2W}W).
\]

In comparison, for the ensemble linear response formula, the size of the integrand is \( \sim \lambda_{\text{max}}^W \), where \( \lambda_{\text{max}} > 1 \) is the largest Lyapunov exponent. By similar arguments,

\[
h \sim O(\theta^W) + O(\lambda_{\text{max}}^W/\sqrt{N'}), \quad T = N'W \sim \theta^{-2W}\lambda_{\text{max}}^W W.
\]

Here \( N' \) is the number of sample trajectories, each with \( W \) steps, and \( T \) is the total number of steps in all sample trajectories. Recall that only one first-order tangent equation and one \( f \) is computed at each step, hence, for the sparse case, the cost per step is \( O(M) \), and in total,

\[
\text{Complexity of ensemble} \sim O(M\theta^{-2W}\lambda_{\text{max}}^W W),
\]

Hence, for typical problems with sparsity, the numerical complexity of the ensemble linear response formula can be about \( O(\lambda_{\text{max}}^W/u) \) times higher than fast linear response. Should we be interested in estimating the dense case, note that for a fair comparison with fast linear response, we should assume \( \nabla f_* \) is dense, in which case computing \( f_* \) is \( O(M^3) \), as we discussed. Moreover, as we discussed, several techniques can help further reducing the actual computing time of fast linear response.
6. A numerical example

This section illustrates the fast linear response algorithm on a 21-dimensional solenoid map with 20 unstable dimensions. Here $\mathcal{M} = \mathbb{R} \times \mathbb{T}^{20}$, and the governing equation is

$$x_{n+1}^1 = 0.05x_n^1 + \gamma + 0.1 \sum_{i=2}^{21} \cos(5x_n^i)$$

$$x_{n+1}^i = 2x_n^i + \gamma(1 + x_n^1) \sin(2x_n^1) \mod 2\pi, \text{ for } 2 \leq i \leq 21$$

where the superscript labels the coordinates. The perturbation is caused by changing $\gamma$, and the instantaneous objective function is

$$\Phi(x) := (x^1)^3 + 0.005 \sum_{i=2}^{21} (x^i - \pi)^2.$$  

The default setting, $N = 20$ steps in each segment, $A = 200$ segments, $\gamma = 0.1$, and $W = 10$, is used unless otherwise noted. The code is at https://github.com/niangxiu/flr.git. Figure 3 shows a typical trajectory. Figure 4 shows that the variance of the computed derivative is proportional to $A^{−0.5}$. Figure 5 shows that the bias in the averaged derivative decreases as $W$ increases, but the variance increases like $W^{0.5}$, indicating that we should increase $A$ together with $W$. This square-root trend verifies the assumption we made for the error estimation in section 5.3. Finally, figure 6 shows that the derivative computed by fast linear response correctly reflects the trend of the objective as $\gamma$ changes.

![Figure 3](image1.png)  

**Figure 3.** The empirical measure of a trajectory with default setting.

On a single-core 3.0GHz CPU, for $10^4$ segments, which is a total of $2 \times 10^5$ steps, the time for computing the orbit is 7.4 seconds; whereas the fast linear response algorithm on the same orbit takes another 47 seconds. So the time cost is about 6 times of simulating the orbit. In fact, our implementation of the fast linear response algorithm is not optimal, for example, we still use dense matrix representations for $f_*$ and $\nabla f_*$. Our algorithm can run even faster if we use the sparsity of the system, either via sparse matrices or graph tracing.
Our example is difficult for ensemble or stochastic algorithms. With $W = 10$, the magnitude of the integrand is

$$\| f^W X(\Phi) \| \sim \lambda^{W}_{\text{max}} \| X \| \| d\Phi \| \sim 2^{10} \times \sqrt{21} \times 1 \sim 4700,$$

In order to get the standard deviation within $7 \times 10^{-3}$, the number of sample trajectories required is about $4.5 \times 10^{11}$. Hence, the total number of steps computed is about $4.5 \times 10^{12}$, where each step contains one application of $f$ and one step of first order tangent equation. In comparison, as shown in figure 4, the same setting requires running the fast linear response for $2 \times 10^5$ steps, where each step contains one application of $f$, about 30 first and second order tangent equations. Say the cost per step is 6 times of ensemble algorithms, then, overall, fast linear response is about $10^6$ times faster.

Our example is difficult for operator algorithms where SRB measures are approximated by isotropic finite-elements. By the estimation in [32], the error for computing only the
first term in the linear response via such operator algorithm is

\[ \rho'(X(\Phi)) - \rho(X(\Phi)) \sim \rho'(\frac{1}{2} \frac{\partial^2 X(\Phi)}{\partial (x^1)^2} (x^1)^2) \sim \frac{1}{\varepsilon} \int_{-0.5\varepsilon}^{0.5\varepsilon} 3y^2 dy \sim \frac{1}{4} \varepsilon^2. \]

Here \( \varepsilon \) is the side length of the finite-element, and \( \rho' \) is the approximated SRB measure. To reduce this error to \( 7 \times 10^{-3} \), we need \( \varepsilon \approx 0.17 \), hence each side of \( \mathbb{T} \) needs \( 2\pi/0.17 \approx 37 \) elements, and the entire attractor needs about \( 37^{20} \approx 10^{30} \) finite elements to cover. Assuming each element only needs one float number to store, it would cost \( 10^{20} \) GB storage only for the elements.

Our example is also difficult for previous blended algorithms. For the default \( \gamma \), the shadowing contribution is about 0.4, and the unstable contribution is about 0.2. Algorithms using approximations on the unstable contribution, such as non-intrusive shadowing, blended response, and S3, have large error. Also note that if we did not use a small coefficient in the unstable part of \( \Phi \), then the unstable contribution takes the major portion: this is predicted in [31] for cases where \( u/M \) is large. Moreover, since \( u \geq 2, s \geq 1 \), and the unstable and stable directions are unknown beforehand, algorithms intended for \( u = 1 \) systems or expanding maps does not apply.

Finally, we compare with finite difference, or some functional regression method. In figure 6, because \( \rho(\Phi) \) computed from finite trajectories has oscillations, revealing the correct trend between \( \rho(\Phi) \) and \( \gamma \) takes about 20 data points with different \( \gamma \). Although it is hard to quantify the error in such regression method, which would require a probability distribution on regression models; we can still roughly say that, overall, the fast linear response is a few times faster than finite difference. This is very encouraging, because in the most simple non-chaotic situations, computing the derivative function is typically a few times faster than finite difference: fast linear response recovers such cost in chaotic systems. This hints that the fast linear response could perhaps be close to the best possible efficiency.
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Appendix A. Pushforward Operators as Tensors

In $\mathbb{R}^M$, the pushforward operator $f_*$ is a matrix. In particular, when differentiating a composition of several pushforwards, the Leibniz rule applies. In this section, we establish the Leibniz rule for general pushforward operators on Riemannian manifolds. To achieve this, we will define the pushforward operator on vectors as a $(1,1)$-tensor, and define its Riemannian connection. Finally, we extend this Leibniz rule to $u$-vectors.

Definition 2 ($f_*$). Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a $C^\infty$ diffeomorphism. Let $e \in \mathcal{X}(\mathcal{M}_1)$ be a $C^\infty$ vector field over $\mathcal{M}_1$; $\alpha \in \mathcal{X}(\mathcal{M}_2)^*$ be a $C^\infty$ 1-form over $\mathcal{M}_2$. Define

$$f_*(\alpha, e) := \alpha(f_* e) = e(f^* \alpha).$$

Let $z_1, z_2$ be coordinates on $\mathcal{M}_1, \mathcal{M}_2$ respectively. Written in coordinates, we have

$$f_* = \frac{\partial}{\partial z_2^i} f^i_1 dz_1^1, \quad f_*(\alpha, e) = (\alpha \frac{\partial}{\partial z_2^i}) f^i_1 (edz_1^1),$$

where $f^i_1$ is the Jacobian matrix under $z_1$ and $z_2$. Under our definition, $f_*$ is a tensor field, in the sense that it is a $C^\infty$-multilinear function $f_* : \mathcal{X}(\mathcal{M}_1) \times \mathcal{X}^*(\mathcal{M}_2) \to C^\infty(\mathcal{M}_2)$. We then define the Riemannian connection of this pushforward operator.

For $\frac{\partial}{\partial \eta} \in T \mathcal{M}_1$, the text-book way of defining connections of tensors is

$$(\nabla_{\frac{\partial}{\partial \eta}} f_*)(\alpha, e) := (f_* \frac{\partial}{\partial \eta} ) (\alpha f_* e) - (\nabla f_* \frac{\partial}{\partial \eta}) f_* e - \alpha f_* \nabla_{\frac{\partial}{\partial \eta}} e = \alpha \nabla_{\nabla_{\frac{\partial}{\partial \eta}} f_*} f_* e - \alpha f_* \nabla_{\frac{\partial}{\partial \eta}} e.$$

$(\nabla_{\cdot}) f_*(\cdot, \cdot)$ is a tensor field in the sense that it is a $C^\infty$-multilinear function : $\mathcal{X}(\mathcal{M}_1) \times \mathcal{X}(\mathcal{M}_1) \times \mathcal{X}(\mathcal{M}_2) \to C^\infty(\mathcal{M}_2)$. Notice that, similar to typical Riemannian connections, $\nabla f_*$ only requires the value of $\frac{\partial}{\partial \eta}$ at a point. Since $\alpha$ is a common factor, we may neglect it on both sides of the equation. Hence, we define the Riemannian connection of the pushforward operator, $(\nabla_{\cdot}) f_* : \mathcal{X}(\mathcal{M}_1) \times \mathcal{X}(\mathcal{M}_1) \to \mathcal{X}(\mathcal{M}_2)$, as

$$(\nabla_{\frac{\partial}{\partial \eta}} f_*) e := \nabla_{\nabla_{\frac{\partial}{\partial \eta}} f_*} f_* e - f_* \nabla_{\frac{\partial}{\partial \eta}} e.$$

To write $\nabla f_*$ in coordinates, by the coordinate form of $f_*$, we have

$$(\nabla_{\frac{\partial}{\partial \eta}} f_*) e = (\nabla f_* \frac{\partial}{\partial \eta}) f^i_1 (dz_1^1 e) + (\frac{\partial}{\partial \eta}) ( df^i_1 \frac{\partial}{\partial \eta} ) (dz_1^1 e) + (\frac{\partial}{\partial \eta}) f^i_1 (e \nabla_{\frac{\partial}{\partial \eta}} dz_1^1 ).$$

Lemma 14 (Leibniz rule for composition of pushforward operators). Let $g : \mathcal{M}_2 \to \mathcal{M}_3$ be a diffeomorphism. Then

$$\nabla_{g_* f_* \frac{\partial}{\partial \eta}} (g_* f_* e) = (\nabla f_* \frac{\partial}{\partial \eta} g_*) f_* e + g_* (\nabla_{\frac{\partial}{\partial \eta}} f_*) e + g_* f_* \nabla_{\frac{\partial}{\partial \eta}} e.$$
Remark. Besides the proof below, readers may find it consolidating to prove by writing everything in coordinates, which also helps us to check that the coordinate form is correct. When doing that, use the following relations to cancel or combine terms:

$$\frac{\partial}{\partial z^2} \frac{\partial}{\partial t} d \frac{\partial}{\partial z_2} + d \frac{\partial}{\partial t} \frac{\partial}{\partial z_2} = \frac{\partial}{\partial q} \delta^j_i = 0, \quad e \frac{\partial}{\partial t} d z^1 + d \frac{\partial}{\partial t} e = \frac{\partial}{\partial q} (edz^1).$$

Proof. By definition,

$$\nabla_{g,f,\frac{\partial}{\partial t}} (g(f,e)) = (\nabla_{f,\frac{\partial}{\partial t}} g)(f,e) + g \nabla_{f,\frac{\partial}{\partial t}} (f,e) = (\nabla_{f,\frac{\partial}{\partial t}} g)(f,e) + g \nabla_{f,\frac{\partial}{\partial t}} (f,e) + e \nabla_{f,\frac{\partial}{\partial t}} (f,e).$$

Finally, we extend the Leibniz rule to the case where \(e = e_1 \wedge \cdots \wedge e_u\) is a \(u\)-vector. Now \(f,e = f, e_1 \wedge \cdots \wedge f, e_u\), and

$$\nabla e = \sum_{i=1}^u e_i \wedge \cdots \nabla e_i \wedge \cdots \wedge e_u, \quad \nabla (f,e) = \sum_{i=1}^u f, e_1 \wedge \cdots \nabla (f, e_i) \wedge \cdots \wedge f, e_u.$$

Definition 3 (\(\nabla f, e\)). The Riemannian connection of pushforward operators on \(u\)-vectors,

$$\nabla f, \frac{\partial}{\partial t} e := \nabla_{f,\frac{\partial}{\partial t}} (f,e) - f, e \nabla \frac{\partial}{\partial t} e = \sum_{i=1}^u f, e_i \wedge \cdots \nabla f, e_i \wedge \cdots \wedge f, e_u.$$

In the last expression, \(\nabla f, e\) acts on single vectors, as defined earlier. Notice that by our definition, \(\nabla f, e\) does not depend on the choice of \(e_i\)'s, as long as their wedge product is \(e\).

Lemma 15 (Leibniz rule for differentiating \(u\)-vectors). Let \(e\) be a \(C^\infty\) \(u\)-vector, then

$$\nabla_{g,f,\frac{\partial}{\partial t}} (g,f,e) = (\nabla_{f,\frac{\partial}{\partial t}} g)(f,e) + g \nabla_{f,\frac{\partial}{\partial t}} (f,e) + g \nabla_{f,\frac{\partial}{\partial t}} (f,e) + f \nabla_{f,\frac{\partial}{\partial t}} (f,e),$$

$$\nabla_{k,\frac{\partial}{\partial t}} f^n = \sum_{n=0}^{k-1} f^{k-n}(\nabla f, e) f^n e + f^k \nabla f, e.$$

Remark. (1) There is no need to add a subscript to \(\nabla f, e\) to indicate steps, since the two vectors it applies to, \(f, e\) and \(f, \frac{\partial}{\partial t}\), already well-locate this tensor. (2) We use \(C^\infty\) in this section only because ‘\(C^\infty\)-multilinear’ is a conventional terminology in differential geometry text. In fact, we only require the differentiation to be meaningful at the particular point and the particular direction. For example, the second equation of lemma applies to the rough \(u\)-vector field, \(e\), defined in equation (5), differentiated in an unstable direction.

Proof. Inductively apply that \(\nabla (f,e) = (\nabla f, e) + f, \nabla e\). □

Appendix B. Derivative-like \(u\)-vectors

Let there be a \(u\)-vector field \(e := e_1 \wedge \cdots \wedge e_u\), smoothly defined along a direction \(\frac{\partial}{\partial q}\). The derivative is \(\nabla \frac{\partial}{\partial q} e = \sum e_1 \wedge \cdots \nabla \frac{\partial}{\partial q} e_i \wedge \cdots \wedge e_u\), that is, one entry in each summand is typically not in the span of \(\{e_i\}_{i=1}^u\). However, it is not trivial that the summation of derivatives still has this form, especially when directions \(\frac{\partial}{\partial q}\) and the basis of \(e\) in each summand are different. Motivated by this, we define a subspace of \(u\)-vectors which looks like these derivatives, and show some related properties.
Remark. which does not rely on the fact that $p$ is a derivative or a summation of derivatives.

**Definition 4.** At point $x$, the collection of derivative-like $u$-vectors of $e = \wedge_{i=1}^{u} e_i$, written on basis $\{e_i\}_{i=1}^{u}$, is defined as

$$D_e := \{p \in \wedge^u T_x M : p = \sum_{i} e_1 \wedge \cdots \wedge p_i \wedge \cdots \wedge e_u, \ p_i \in T_x M\}.$$  

In our paper, the basis $\{e_i\}_{i=1}^{u}$ is typically the basis of the unstable subspace. Letting $\{e_i\}_{i=1}^{M}$ be a full basis of $T_x M$ whose first $u$ vectors spans $V^u$, then by decomposing $p$ onto the basis of $\wedge^u T_x M$, we can see that $D_e$ is the direct sum

$$D_e = \text{span}\{e\} \oplus \sum_{i=1}^{u} \sum_{j>u} \text{span}\{e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_u \wedge e_j\}.$$  

Hence, under a given basis, $p_i$ is unique modulo $\text{span}\{e_i\}_{i=1}^{u}$.

So long as $\text{span}\{e_1, \ldots, e_u\}$ is the same, $D_e$ as a subspace is independent of the selection of basis. To see this, let $\{e'_i\}_{i=1}^{M}$ be another full basis such that $\text{span}\{e_1, \ldots, e_u\} = \text{span}\{e'_1, \ldots, e'_u\} = V^u$, and $D_{e'}$ be the corresponding subspace. Write each term in equation (17) by the other basis. we can see that

$$\text{span}\{e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_u \wedge e_j\}$$

$$\subset \text{span}\{e'\} \oplus \sum_{i=1}^{u} \sum_{j>u} \text{span}\{e'_1 \wedge \cdots \wedge e'_{i-1} \wedge e'_{i+1} \wedge \cdots \wedge e'_u \wedge e'_j\}.$$  

Hence, $D_e \subset D_{e'}$. By symmetry, $D_e = D_{e'}$.

The next step is to consider how to express derivative-like $u$-vectors under a new basis of the same span. Should $p$ indeed be the derivative of a $u$-vector field, there is a formula for changing to a new basis which is a constant linear combination of the old basis. We will show that the same formula is true when $p$ is only derivative-like.

**Lemma 16** (change of basis formula). Let $\{e'_i\}_{i=1}^{u}$ be another basis of $V^u$ such that $e'_j = a^i_j e_i$; let $p_k, p'_k \in T_x M$ satisfy $p'_k := a^i_k p_i$ (notice that typically $p_k, p'_k \notin V^u$), then

$$\sum_{k=1}^{u} e'_1 \wedge \cdots \wedge p'_k \wedge \cdots \wedge e'_u = \det(a^i_k) \sum_{k=1}^{u} e_1 \wedge \cdots \wedge p_k \wedge \cdots \wedge e_u.$$  

**Remark.** (1) If $e$ and $e'$ are differentiable, $a^i_j$ are constants, $p_k = \nabla_Y e_k$ for some vector $Y$, then this lemma is just the result of applying $\nabla_Y$ on $e' = \det(a^i_j) e$. (2) We use this lemma for proving lemma 13, where $p$ is a summation of derivatives. For this particular case, we may as well first prove the lemma for each summand, then apply linearity to obtain the lemma for the summation. However, here we give a more general and algebraic proof, which does not rely on the fact that $p$ is a derivative or a summation of derivatives.

**Proof.** We can reorder $e'_i$ so that for all $i$, $S_i := \{e'_1, \ldots, e'_i, e_{i+1}, \ldots, e_u\}$ is an independent set of vectors. Changing basis from $S_0$ to $S_u$ can be achieved by sequentially changing from $S_0$ to $S_{i+1}$, where only one vector is changed in each step. Hence, by induction, it suffices to show that the lemma is true for any one step, for example, the first step. Hence, it suffices to prove for the case where $e'_1 = a^i_i e_i$, and $e'_j = e_j$ for all $j \geq 2$. Now $p'_1 = a^i_i p_i$, and
\[ p'_j = p_j \] for all \( j \geq 2 \). the left hand side is

\[
LHS = \sum_{k=1}^{n} e'_1 \wedge \cdots \wedge p'_k \wedge \cdots \wedge e'_u = p'_1 \wedge e'_2 \wedge \cdots \wedge e'_u + \sum_{k=2}^{n} e'_1 \wedge \cdots \wedge p'_k \wedge \cdots \wedge e'_u
\]

\[
= \sum_i a^i p_i \wedge e_2 \wedge \cdots \wedge e_u + \sum_{k \geq 2} \sum_i a^i e_i \wedge e_2 \wedge \cdots \wedge p_k \wedge \cdots \wedge e_u
\]

\[
= \sum_i a^i p_i \wedge e_2 \wedge \cdots \wedge e_u + a^1 \sum_{k \geq 2} e_1 \wedge \cdots \wedge p_k \wedge \cdots \wedge e_u + \sum_{k \geq 2} \sum_{i \geq 2} a^i e_i \wedge e_2 \wedge \cdots \wedge p_k \wedge \cdots \wedge e_u
\]

In the last summation, notice that an exterior product vanishes if \( e_i \) appears twice. Hence,

\[
LHS = \sum_{k \geq 2} e_1 \cdots e_u + \left( \sum_{k \geq 2} a^i p_i \wedge e_2 \cdots e_u + \sum_{k \geq 2} a^k e_k \wedge e_2 \cdots p_k \wedge e_u \right)
\]

Comparing the two summations in the parenthesis, notice that interchanging the position of \( p_k \) and \( e_k \) in an exterior product changes the sign, hence all terms cancel, except the term with \( i = 1 \), and

\[
LHS = a^1 \sum_{k \geq 2} e_1 \wedge \cdots \wedge p_k \wedge \cdots \wedge e_u + \sum_{i=1} a^i p_i \wedge e_2 \wedge \cdots \wedge e_u
\]

\[
= a^1 \sum_{k \geq 2} e_1 \wedge \cdots \wedge p_k \wedge \cdots \wedge e_u = a^1 p.
\]

Since \( a^1 \) is the determinant of our current transformation matrix, we have proved the lemma for one step. The lemma is proved by induction. \( \square \)

**Appendix C. Projection Operators of Derivative-like \( u \)-Vectors**

For derivative-like \( u \)-vectors, \( D_e \), defined in appendix B, only one entry is not in \( V^u \) in each summand, hence, we can extend the definition of projection operators from one-vectors to \( D_e \), by applying the projection on the one exceptional entry in each summand. Notice that, in general, it should be hard to further extend our definitions to \( ^u \wedge K \wedge M \) while keeping all the good properties. In this section, \( e \in ^u V^u \) by default.

**Definition 5.** The projection operator \( P \) on \( p \in D_e \) is

\[
Pp = \sum_i e_1 \wedge \cdots \wedge Pp_i \wedge \cdots \wedge e_u,
\]

where \( P \) on the right side is the projection operator on one-vectors.

The projection operators used in this paper are \( P^u, P^s, P^\|, \) and \( P^\perp \). The first two operators are oblique projections along stable or unstable subspace to the unstable or stable subspace, respectively. The last two operators are orthogonal projections onto the unstable subspace and its orthogonal complement. These operators, applied on one-vectors, are illustrated in figure 7. Notice that computing \( P^u \) and \( P^s \) both require both \( V^u \) and \( V^s \); in contrast, computing \( P^\| \) and \( P^\perp \) only require \( V^u \), thus are faster. For both single and \( u \)-vectors, denote

\[
p^u := P^u p, \ p^s := P^s p; \ p^\| := P^\| p, \ p^\perp := P^\perp p.
\]
For fixed $V^u$, $P^\parallel$ and $P^\perp$ do not depend on the choice of basis. To see this, notice that $D^\perp_e := \sum_{i=1}^{n} \sum_{j>u} \text{span}\{e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_u \wedge e_j\}$ is the same so long as $e_{u+1}, \ldots, e_M$ is a basis of $(V^u)^\perp$; also notice that $P^\parallel$ and $P^\perp$ are operators taking components in the decomposition $D_e = \text{span}\{e\} \oplus D^\perp_e$. Similarly, if both $V^u$ and $V^s$ are fixed, $P^u$ and $P^s$ do not depend on the choice of basis.

**Lemma 17** (composing projections with addition). For $p, p' \in D_e$, $P(p + p') = Pp + Pp'$.

**Lemma 18** (composing projection operators). $P^\parallel P^u = P^u P^\parallel = P^\parallel, P^\perp P^s = P^\perp, P^s P^\perp = P^s; P^\perp P^u = P^s P^\parallel = 0$.

**Remark.** Notice that typically $P^u P^\perp \neq 0, P^\parallel P^s \neq 0$.

**Lemma 19** (composing projections with pushforwards). $f_* P^u = P^u f_* = P^u f_* P^u, f_* P^s = P^s f_* P^s$; $f_* P^\parallel = P^\parallel f_* P^\parallel, P^\parallel f_* = P^\parallel f_* P^\perp$.

Here the projection $P$’s are evaluated at suitable locations.

**Proof.** The first three equalities are because of the invariance of stable and unstable subspace. The last equality is because

$$P^\perp f_* = P^\perp P^s f_* = P^\perp P^s f_* P^\perp = P^\perp P^s f_* P^\perp = P^\perp f_* P^\perp.$$ 

**Lemma 20** (expressing $P^\parallel$ by inner products). For any $p \in D_e$, $P^\parallel p = \langle p, e \rangle \frac{e}{\|e\|^2}; \quad P^\perp p = p - \langle p, e \rangle \frac{e}{\|e\|^2}$.

**Proof.** For the first equation, since both sides are in $\wedge^u V^u$, which is a one-dimensional subspace, it suffices to prove the equation taken inner product with $e$, that is,

$$\langle P^\parallel p, e \rangle = \langle p, e \rangle \frac{\langle e, e \rangle}{\|e\|^2} = \langle p, e \rangle.$$
Further adding $\langle P^\perp p, e \rangle = 0$ to the left proves this equality. The second equality is because $P^\perp + P^\parallel = Id$, hence $P^\perp p = p - P^\parallel p$. □

REFERENCES

[1] R. V. Abramov and A. J. Majda. New Approximations and Tests of Linear Fluctuation-Response for Chaotic Nonlinear Forced-Dissipative Dynamical Systems. *Journal of Nonlinear Science*, 18(3):303–341, 2008.

[2] D. V. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.*, 90:1–235, 1967.

[3] F. Antown, G. Froyland, and S. Galatolo. Optimal linear response for Markov Hilbert-Schmidt integral operators and stochastic dynamical systems. *arXiv:2101.09411*, 2021.

[4] W. Bahsoun, S. Galatolo, I. Nisoli, and X. Niu. A rigorous computational approach to linear response. *Nonlinearity*, 31(3):1073–1109, 2018.

[5] V. Baladi. Linear response, or else. *ICM Seoul 2014 Proceedings*, 3:525–545, 2014.

[6] P. J. Blonigan. Adjoint sensitivity analysis of chaotic dynamical systems with non-intrusive least squares shadowing. *Journal of Computational Physics*, 348:803–826, 2017.

[7] P. J. Blonigan and Q. Wang. Multiple shooting shadowing for sensitivity analysis of chaotic dynamical systems. *Journal of Computational Physics*, 354:447–475, 2018.

[8] R. Bowen. Markov Partitions for Axiom A Diffeomorphisms. *American Journal of Mathematics*, 92(3):725–747, 1970.

[9] R. Bowen. Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms Second revised edition. Springer, 2008.

[10] N. Chandramoorthy and Q. Wang. A computable realization of Ruelle’s formula for linear response of statistics in chaotic systems. *arXiv:2002.04117*, 2020.

[11] J. W. Cooley and J. W. Tukey. An Algorithm for the Machine Calculation of Complex Fourier Series. *Mathematics of Computation*, 19(90):297, 1965.

[12] H. Crimmins and G. Froyland. Fourier approximation of the statistical properties of Anosov maps on tori. *Nonlinearity*, 33(11):6244–6296, 2020.

[13] D. Dolgopyat. On differentiability of SRB states for partially hyperbolic systems. *Inventiones Mathematicae*, 155(2):389–449, 2004.

[14] G. L. Eyink, T. W. N. Haine, and D. J. Lea. Ruelle’s linear response formula, ensemble adjoint schemes and Lévy flights. *Nonlinearity*, 17(5):1867–1889, 2004.

[15] J. E. Fornæss and E. A. Gavosto. Existence of generic homoclinic tangencies for henon mappings. *The Journal of Geometric Analysis*, 2(5):429–444, 1992.

[16] S. Gouëzel and C. Liverani. Banach spaces adapted to Anosov systems. *Ergodic Theory and Dynamical Systems*, 26(1):189–217, 2006.

[17] L. Greengard and V. Rokhlin. A fast algorithm for particle simulations. *Journal of Computational Physics*, 135(2):280–292, 1987.

[18] L. Greengard and J. Strain. The Fast Gauss Transform. *SIAM Journal on Scientific and Statistical Computing*, 12(1):79–94, 1991.

[19] A. Gritsun and V. Lucarini. Fluctuations, response, and resonances in a simple atmospheric model. *Physica D: Nonlinear Phenomena*, 349:62–76, 2017.

[20] S. Günther, N. R. Gauger, and Q. Wang. A framework for simultaneous aerodynamic design optimization in the presence of chaos. *Journal of Computational Physics*, 328:387–398, 2017.

[21] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant Manifolds*. Lecture Notes in Mathematics. Springer, Berlin, 1977.

[22] C. W. Huan and L. X. Xiao. *Introduction to Riemannian Geometry*. Peking University Press, Beijing, 2004.

[23] F. Huhn and L. Magri. Stability, sensitivity and optimisation of chaotic acoustic oscillations. *Journal of Fluid Mechanics*, 882, 2020.

[24] M. Jiang. Differentiating potential functions of SRB measures on hyperbolic attractors. *Ergodic Theory and Dynamical Systems*, 32(4):1350–1369, 2012.
[25] D. Lasagna, A. Sharma, and J. Meyers. Periodic shadowing sensitivity analysis of chaotic systems. Journal of Computational Physics, 391:119–141, 2019.

[26] D. J. Lea, M. R. Allen, and T. W. N. Haine. Sensitivity analysis of the climate of a chaotic system. Tellus Series a-Dynamic Meteorology and Oceanography, 52(5):523–532, 2000.

[27] J. M. Lee. Introduction to Riemannian Manifolds. Springer, 2018.

[28] V. Lucarini, F. Ragone, and F. Lunkeit. Predicting Climate Change Using Response Theory: Global Averages and Spatial Patterns. Journal of Statistical Physics, 166(3-4):1036–1064, 2017.

[29] A. Ni. Adjoint shadowing directions in hyperbolic systems for sensitivity analysis. arXiv:1807.05568, pages 1–23, 2018.

[30] A. Ni. Hyperbolicity, shadowing directions and sensitivity analysis of a turbulent three-dimensional flow. Journal of Fluid Mechanics, 863:644–669, 2019.

[31] A. Ni. Approximating linear response by non-intrusive shadowing algorithms. https://arxiv.org/abs/2003.09801, to appear in SIAM Journal on Numerical Analysis, pages 1–12, 2020.

[32] A. Ni. On computing derivatives of transfer operators and linear responses in higher dimensions. arXiv:2108.13863, 2021.

[33] A. Ni and C. Talnikar. Adjoint sensitivity analysis on chaotic dynamical systems by Non-Intrusive Least Squares Adjoint Shadowing (NILSAS). Journal of Computational Physics, 395:690–709, 2019.

[34] A. Ni and Q. Wang. Sensitivity analysis on chaotic dynamical systems by Non-Intrusive Least Squares Shadowing (NILSS). Journal of Computational Physics, 347:56–77, 2017.

[35] A. Ni, Q. Wang, P. Fernandez, and C. Talnikar. Sensitivity analysis on chaotic dynamical systems by Finite Difference Non-Intrusive Least Squares Shadowing (FD-NILSS). Journal of Computational Physics, 394:615–631, 2019.

[36] S. Y. Pilyugin. Shadowing in Structurally Stable Flows. Journal of Differential Equations, 140(2):238–265, 1997.

[37] E. R. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. Annals of Mathematics, 151(3):961–1023, 2000.

[38] D. Ruelle. Elements of Differentiable Dynamics and Bifurcation Theory. Academic Press, San Diego, 1989.

[39] D. Ruelle. Differentiation of SRB States. Commun. Math. Phys, 187:227–241, 1997.

[40] D. Ruelle. Differentiation of SRB States: Correction and Complements. Communications in Mathematical Physics, pages 185–190, 2003.

[41] D. Ruelle. Differentiation of SRB states for hyperbolic flows. Ergodic Theory and Dynamical Systems, 28(02):613–631, 2008.

[42] M. Santos Gutiérrez and V. Lucarini. Response and Sensitivity Using Markov Chains. Journal of Statistical Physics, 179(5-6):1572–1593, 2020.

[43] K. Shawkä and G. Papadakis. A preconditioned multiple shooting shadowing algorithm for the sensitivity analysis of chaotic systems. Journal of Computational Physics, 398:108861, 2019.

[44] Y. S. Shimizu and K. J. Fidkowski. Output-based error estimation for chaotic flows using reduced-order modeling. AIAA Aerospace Sciences Meeting, 2018, (210059), 2018.

[45] M. Shub. Global Stability of Dynamical Systems. Springer, Berlin, 1987.

[46] S. Smale. Differentiable Dynamical Systems. Bull. Amer. Math. Soc., 73:747–817, 1967.

[47] Q. Wang. Convergence of the Least Squares Shadowing Method for Computing Derivative of Ergodic Averages. SIAM Journal on Numerical Analysis, 52(1):156–170, 2014.

[48] C. L. Wormell and G. A. Gottwald. Linear response for macroscopic observables in high-dimensional systems. Chaos, 29(11), 2019.

[49] L.-S. Young. What are SRB measures, and which dynamical systems have them? Journal of Statistical Physics, 108(5):733–754, 2002.

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