A Characterization of Deterministic Sampling Patterns for Low-Rank Matrix Completion

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Abstract

Low-rank matrix completion (LRMC) problems arise in a wide variety of applications. Previous theory mainly provides conditions for completion under missing-at-random samplings. An incomplete \( d \times N \) matrix is \textit{finitely completable} if there are at most finitely many rank-\( r \) matrices that agree with all its observed entries. Finite completability is the tipping point in LRMC, as a few additional samples of a finitely completable matrix guarantee its unique completability. The main contribution of this paper is a full characterization of finitely completable observation sets. We use this characterization to derive sufficient deterministic sampling conditions for unique completability. We also show that under uniform random sampling schemes, these conditions are satisfied with high probability if at least \( \mathcal{O}(\max\{r, \log d\}) \) entries per column are observed.

1 Introduction

Low-rank matrix completion (LRMC) has attracted a lot of attention in recent years because of its broad range of applications, e.g., recommender systems and collaborative filtering [1] and image processing [2].

The problem entails exactly recovering all the missing entries in a \( d \times N \) rank-\( r \) matrix, given a subset of its entries. LRMC is usually studied under a missing-at-random model. Under this model, necessary and sufficient conditions for perfect recovery are known [3–8]. Other approaches have studied this problem using rigidity theory [9] as well as algebraic geometry and matroid theory [10] to derive necessary and sufficient conditions for completion of deterministic samplings, but there is a loose gap between these conditions.

We say an incomplete matrix is \textit{finitely completable} if there exist at most finitely many rank-\( r \) matrices that agree with all its observed entries. Finite completability is the tipping point in LRMC. If even a single observation of a finitely completable matrix is instead missing, then there exist \textit{infinitely} many completions. Conversely, just a few additional samples of a finitely completable matrix guarantee its unique completability.
Whether a matrix is finitely completable depends on which entries are observed. Yet characterizing the sets of observed entries that allow or prevent finite completability remained an important open question until now.

The main result of this paper is a full characterization of finitely completable observation sets, that is, sampling patterns that can be completed in at most finitely many ways. In addition, we provide deterministic sampling conditions for unique completability. Finally, we show that uniform random samplings with $\mathcal{O}(\max\{r, \log d\})$ entries per column satisfy these conditions with high probability.

Organization of the Paper

In Section 2 we formally state the problem and our main results. We present the proof of our main theorem in Section 3, and we leave the proofs of our other statements to Sections 4 and 5, where we also present an additional useful sufficient condition for finite completability.

2 Model and Main Results

Let $X_\Omega$ denote the incomplete version of a $d \times N$, rank-$r$ data matrix $X$, observed only in the nonzero positions of $\Omega$, a $d \times N$ matrix with binary entries. The LRMC problem is tantamount to identifying the $r$-dimensional subspace $S^*$ spanned by the columns in $X$, and this is how we will approach it.

First observe that since $X$ is rank-$r$, a column observed in fewer than $r + 1$ coordinates provides no information specific to $S^*$ (in general). Therefore, we will assume that $\Omega$ has exactly $r + 1$ nonzero entries per column. Under that assumption, we will see that $N \geq r(d - r)$ is necessary for finite completable, so unless otherwise stated, we will assume $N = r(d - r)$ for the rest of the paper.

The intuition for this is as follows. An $r$-dimensional subspace in $\mathbb{R}^d$ has $r(d - r)$ degrees of freedom, and the $r + 1$ nonzero entries of each column of $X_\Omega$ amount to a constraint that eliminates one of them.

Let $\text{Gr}(r, \mathbb{R}^d)$ denote the Grassmannian manifold of $r$-dimensional subspaces in $\mathbb{R}^d$. Given $S^* \in \text{Gr}(r, \mathbb{R}^d)$, let the columns of $X$ be drawn according to $\nu$, an absolutely continuous distribution with respect to the Lebesgue measure on $S^*$. Our statements hold for almost every (a.e.) $X$, that is, for almost every $S^*$, with respect to the uniform measure on $\text{Gr}(r, \mathbb{R}^d)$, and almost surely with respect to $\nu$.

The paper’s main result is the following theorem, which gives an exact deterministic necessary and sufficient condition on $\Omega$ for finite completability.

Given a matrix, let $n(\cdot)$ denote its number of columns and $m(\cdot)$ the number of its nonzero rows.
Theorem 1. For almost every $X$, there exist at most finitely many rank-$r$ completions of $X\Omega$ if and only if every matrix $\Omega'$ formed with a subset of the columns in $\Omega$ satisfies
\[ m(\Omega') \geq n(\Omega')/r + r. \] (1)

The proof of Theorem 1 is given in Section 3. The condition on $\Omega$ in Theorem 1 is that every subset of $n$ columns of $\Omega$ must have at least $n/r + r$ nonzero rows.

Example 1. The following mask, where 1 denotes a block of all 1’s and $I$ the identity matrix, satisfies the conditions of Theorem 1.

\[
\Omega = \begin{bmatrix}
1 & \cdots & 1 \\
I & \cdots & I \\
r(d-r) & & \\
\end{bmatrix}^{r \times d-r}.
\]

Unique Completability

We saw in the previous section that if $\Omega$ has the minimum required $r+1$ nonzero entries per column, then $N = r(d-r)$ columns were necessary for finite completability (hence also for unique completability). Under the same assumption, there are cases when $N = r(d-r)$ is also sufficient for unique completability, e.g., if $r = 1$, where finite completability is equivalent to unique completability (see Proposition 1).

In general, though, $N > r(d-r)$ columns are necessary for unique completability (see Example 3).

The following theorem gives deterministic sufficient conditions on $\Omega$ for unique completability that only require $N = (r+1)(d-r)$ columns. This shows that with just a little additional information, unique completability follows from finite completability.

Theorem 2. Almost every $X$ can be uniquely recovered from $X\Omega$ if $\Omega$ can be partitioned into two matrices: $\tilde{\Omega}$ with $r(d-r)$ columns and $\hat{\Omega}$ with $d-r$, such that the following two conditions are satisfied.

(i) Every matrix $\Omega'$ formed with a subset of the columns in $\tilde{\Omega}$ satisfies (1).

(ii) Every matrix $\Omega'$ formed with a subset of the columns in $\hat{\Omega}$ satisfies
\[ m(\Omega') \geq n(\Omega') + r. \] (2)
The proof of Theorem 2 is given in Section 4. Condition (ii) in Theorem 2 is that every subset of \( n \) columns of \( \hat{\Omega} \) must have at least \( n + r \) nonzero rows. Notice that (1) is a weaker condition than (2), but (1) is required to hold for all the subsets of \( r(d - r) \) columns, while (2) is required to hold only for all the subsets of \( d - r \) columns.

**Example 2.** The mask with the same pattern as in Example 1, but with \((r + 1)(d - r)\) columns, satisfies the conditions of Theorem 2.

**Random Sampling Patterns**

In general, verifying the conditions on \( \Omega \) in Theorems 1 and 2 may be computationally prohibitive, especially for large \( d \). However, as the next theorem states, sampling patterns \( \Omega \) satisfying these conditions appear with high probability under uniform random sampling schemes with only \( \mathcal{O}(\max\{r, \log d\}) \) samples per column.

**Theorem 3.** Let \( 0 < \epsilon \leq 1 \) be given. Suppose \( r \leq \frac{d}{\epsilon} \) and that each column of \( \Omega \) contains at least \( \ell \) nonzero entries, distributed uniformly at random and independently across columns, with

\[
\ell \geq \max \left\{ 12 \left( \log \left( \frac{d}{\epsilon} \right) + 1 \right), \ 2r \right\}.
\]

(3)

Then with probability at least \( 1 - \epsilon \), \( \Omega \) will satisfy the conditions of Theorem 1 (if \( N \geq r(d - r) \)) and the conditions of Theorem 2 (if \( N \geq (r + 1)(d - r) \)).

Theorem 3 is proved in Section 5.

**Remark 1.** Recall that there exist \( d \times d \) matrices that cannot be completed unless \( \mathcal{O}(r \log d) \) random entries per column are observed [4]. On the other hand, Theorem 3 implies that \( \mathcal{O}(\max\{r, \log d\}) \) observed entries per column are sufficient to uniquely complete a \( d \times (r + 1)(d - r) \) matrix. This exposes an interesting tradeoff between the required number of columns and observed entries per column for unique completion.

### 3 Proof of Theorem 1

For any subspace, matrix or vector that is compatible with a binary vector \( \omega \), we will use the subscript \( \omega \) to denote its restriction to the nonzero coordinates/rows in \( \omega \). For example, letting \( \omega_i \) denote the \( i \)th column of \( \Omega \), \( x_{\omega_i} \in \mathbb{R}^{r+1} \) and \( S_{\omega_i} \subset \mathbb{R}^{r+1} \) denote the restrictions of the \( i \)th column in \( X \), and \( S^* \), to the nonzero coordinates in \( \omega_i \). We say that a subspace \( S \in \text{Gr}(r, \mathbb{R}^d) \) fits \( X_{\Omega} \) if \( x_{\omega_i} \in S_{\omega_i} \) for every \( i \).

Let us start by studying the variety of all \( r \)-dimensional subspaces that fit \( X_{\Omega} \).
The Variety $S$

In Section 1 we mentioned that a vector observed in fewer than $r + 1$ coordinates will provide no information specific to $S^*$. This is because in general, the restriction of an $r$-dimensional subspace to $\ell \leq r$ coordinates is $\mathbb{R}^\ell$. We formalize this in the following definition, which essentially states that a subspace is non-degenerate if its restrictions to $\ell \leq r$ coordinates are $\mathbb{R}^\ell$.

**Definition 1** (Degenerate subspace). *We say $S \in \text{Gr}(r, \mathbb{R}^d)$ is degenerate if and only if there exists a binary vector $\omega$ with $\|\omega\|_1 \leq r$, such that $\dim S_{\omega} < \|\omega\|_1$.*

Since a.e. subspace is non-degenerate, let us consider only the subspaces in $\text{Gr}_r(r, \mathbb{R}^d) \subset \text{Gr}(r, \mathbb{R}^d)$, the set of all non-degenerate $r$-dimensional subspaces of $\mathbb{R}^d$.

Define $S(X_\Omega) \subset \text{Gr}_r(r, \mathbb{R}^d)$ such that every $S \in S(X_\Omega)$ fits $X_\Omega$, i.e.,

$$S(X_\Omega) := \{ S \in \text{Gr}_r(r, \mathbb{R}^d) : \{ x_{\omega_i} : x_{\omega_i} \in S_{\omega_i} \}_{i=1}^{r(d-r)} \}. $$

Let $U \in \mathbb{R}^{d \times r}$ be a basis of $S \in S(X_\Omega)$. The condition $x_{\omega_i} \in S_{\omega_i}$ is equivalent to saying that there exists a vector $\theta_i \in \mathbb{R}^r$ such that

$$x_{\omega_i} = U_{\omega_i} \theta_i. \quad (4)$$

Let $\Delta_i$ denote a binary vector with nonzero entries in the positions $r$ nonzero entries of $\omega_i$, and $\nabla_i$ the binary vector with a single nonzero entry in the position of the remaining nonzero entry of $\omega_i$.

This way, we can rewrite (4) as

$$\begin{bmatrix} x_{\Delta_i} \\ x_{\omega_i} \end{bmatrix} = \begin{bmatrix} U_{\omega_i} \\ U_{\nabla_i} \end{bmatrix} \theta_i. \quad (5)$$

Since $S$ is non-degenerate, $U_{\Delta_i}$ is full-rank, so we may solve for $\theta_i$ using the top block to obtain $\theta_i = U_{\Delta_i}^{-1} x_{\Delta_i}$. Plugging this on the bottom block, we have that (4) is equivalent to:

$$x_{\nabla_i} = U_{\nabla_i} U_{\Delta_i}^{-1} x_{\Delta_i}. \quad (5)$$

On the other hand, $x_{\omega_i}$ lies in $S_{\omega_i}^*$ by assumption. This implies that there exists a unique $\theta^*_i \in \mathbb{R}^r$ such that

$$x_{\omega_i} = U_{\omega_i}^* \theta^*_i, \quad (6)$$

where $U^*$ is a basis of $S^*$. Substituting (6) in (5) we obtain

$$U_{\nabla_i}^* \theta^*_i = U_{\nabla_i} U_{\Delta_i}^{-1} U_{\Delta_i}^* \theta^*_i. \quad (7)$$
Recall that $U_{\Delta_i}^{-1} = U_{\Delta_i}^\dagger / |U_{\Delta_i}|$, where $U_{\Delta_i}^\dagger$ and $|U_{\Delta_i}|$ denote the adjugate and the determinant of $U_{\Delta_i}$. Therefore, we may rewrite (7) as the following polynomial equation:

$$\left(|U_{\Delta_i}| U_{\neg i}^* - U_{\neg i} U_{\Delta_i}^\dagger U_{\Delta_i}^* \right) \theta_i^* = 0. \tag{8}$$

We conclude that a subspace $S$ with basis $U$ fits $X_\Omega$ if and only if $U$ satisfies (8) for every $i$.

Since every nontrivial subspace has infinitely many bases, even if there is only one $r$-dimensional subspace in $S(X_\Omega)$, the variety

$$\left\{ U \in \mathbb{R}^{d \times r} : \left\{ \left(|U_{\Delta_i}| U_{\neg i}^* - U_{\neg i} U_{\Delta_i}^\dagger U_{\Delta_i}^* \right) \theta_i^* = 0 \right\}_{i=1}^{r(d-r)} \right\}$$

has infinitely many solutions. Therefore, we will associate a unique $U$ with each subspace as follows. Observe that for every every $S \in \text{Gr}_r(r, \mathbb{R}^d)$, we can write $S = \text{span}\{U\}$ for a unique $U$ in the following column-echelon form:

$$U = \begin{bmatrix} I \\ V \end{bmatrix} \begin{cases} r \\ d-r. \end{cases} \tag{9}$$

On the other hand, every $V \in \mathbb{R}^{(d-r) \times r}$ defines a unique $r$-dimensional subspace of $\mathbb{R}^d$, via $\text{span}\{U\}$. Moreover, $\text{span}\{U\}$ will be non-degenerate for almost every $V$. Let $\mathbb{R}_{*}^{(d-r) \times r} \subset \mathbb{R}^{(d-r) \times r}$ denote the set of all $(d-r) \times r$ matrices $V$ whose $\text{span}\{U\}$ is non-degenerate, or equivalently, whose $U \times r$ minors are full-rank. Then we have a bijection between $\text{Gr}_r(r, \mathbb{R}^d)$ and $\mathbb{R}_*^{(d-r) \times r}$ via $S^* = \text{span}\{U\}$.

With this in mind, define

$$f_i(V|V^*, \theta_i^*) := \left(|U_{\Delta_i}| U_{\neg i}^* - U_{\neg i} U_{\Delta_i}^\dagger U_{\Delta_i}^* \right) \theta_i^*,$$

with $U$ and $U^*$ in the column-echelon form in (9). We will use $f_i$ as shorthand, with the understanding that $f_i$ is a polynomial in the elements of $V$, and that the elements of $V^*$ and $\theta_i^*$ play the role of coefficients.

Also define $\Theta^* := \{\theta_i^*\}_{i=1}^{r(d-r)}$, and let

$$\mathcal{F}(V|V^*, \Theta^*) := \{f_i\}_{i=1}^{r(d-r)}.$$

Similarly, we will use $\mathcal{F}(V)$, or simply $\mathcal{F}$ as shorthand, with the understanding that $\mathcal{F}$ is a set of polynomials in the elements of $V$, and that the elements of $V^*$ and $\Theta^*$ play the role of coefficients. We will also use $\mathcal{F} = \emptyset$ as shorthand for $\{f_i = 0\}_{i=1}^{r(d-r)}$. 

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This way, we may rewrite:

$$S(\mathbf{X}_\Omega) = \left\{ \text{span} \begin{bmatrix} 1 \\ \mathbf{V} \end{bmatrix} \in \text{Gr}_*(r, \mathbb{R}^d) : \mathcal{F}(\mathbf{V}) = 0 \right\}.$$ 

In general, the affine variety

$$\mathcal{V}(\mathcal{F}) := \left\{ \mathbf{V} \in \mathbb{R}^{(d-r) \times r} : \mathcal{F}(\mathbf{V}) = 0 \right\}$$

could contain an infinite number of elements. We are interested in conditions that guarantee there is only one or (slightly less demanding) only a finite number. The following lemma states that this will be the case if and only if the $r(d-r)$ polynomials in $\mathcal{F}$ are algebraically independent.

**Lemma 1.** For a.e. $\mathbf{X}$, $S(\mathbf{X}_\Omega)$ contains at most finitely many subspaces if and only if the $r(d-r)$ polynomials in $\mathcal{F}$ are algebraically independent.

**Proof.** By our previous discussion, for a.e. $\mathbf{X}$ there are at most finitely many subspaces in $S(\mathbf{X}_\Omega)$ if and only if there are at most finitely many points in $\mathcal{V}(\mathcal{F})$. We know from algebraic geometry that this will be the case if and only if $\dim \mathcal{V}(\mathcal{F}) = 0$ (see, for example, Proposition 6 in Chapter 9, Section 4 of [11]).

Since $\mathcal{V}(\mathcal{F}) \subset \mathbb{R}^{(d-r) \times r}$, we know that if $\dim \mathcal{V}(\mathcal{F}) = 0$, then $\mathcal{F}$ must contain $r(d-r)$ algebraically independent polynomials (see, for example, Exercise 16 in Chapter 9, Section 6 of [11]).

On the other hand, we know that $\dim \mathcal{V}(\mathcal{F}) = 0$ if the $r(d-r)$ polynomials in $\mathcal{F}$ are a regular sequence (see, for example, Exercise 8 in Chapter 9, Section 4 of [11]).

Finally, since being a regular sequence is an open condition, it follows that for a.e. $\mathbf{V}^*$ and $\Theta^*$, the polynomials in $\mathcal{F}$ are algebraically independent if and only if they are a regular sequence (see, for example, Remark 3.4 in [12]).

**Remark 2.** From our previous discussion, we may assume the following equivalent generative model for $\mathbf{X}$. Draw $\mathbf{V}^*$ uniformly with respect to the Lebesgue measure on $\mathbb{R}^{(d-r) \times r}$ and construct $\mathbf{U}^*$ with the column-echelon form in (9), such that $S^* := \text{span} \{ \mathbf{U}^* \}$. This is equivalent to drawing a subspace uniformly from a dense open subset of $\text{Gr}(r, \mathbb{R}^d)$.

Next, for each $i = 1, \ldots, r(d-r)$, draw $\theta_i^* \in \mathbb{R}^r$ according to an absolutely continuous distribution with respect to the Lebesgue measure on $\mathbb{R}^r$ (analogous to $\nu$). Letting $\Theta^* \in \mathbb{R}^{r \times r(d-r)}$ be the matrix formed with $\{ \theta^*_i \}_{i=1}^{r(d-r)}$ as columns, we have that $\mathbf{X} = \mathbf{U}^* \Theta^*$.

This way, our results hold for a.e. $\mathbf{X}$, that is, for a.e. $\mathbf{V}^*$ with respect to the uniform measure on $\mathbb{R}^{(d-r) \times r}$, and for almost every $\Theta^*$ with respect to the absolutely continuous distribution on $\mathbb{R}^{r \times r(d-r)}$ corresponding to $\nu$. 

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Algebraic independence

By the previous discussion, there are at most finitely many $r$-dimensional subspaces that fit $X\Omega$ if and only if the $r(d-r)$ polynomials in $F$ are algebraically independent.

Whether this is the case depends on the supports of the polynomials in $F$, i.e., on $\Omega$. Lemma 2 shows that the polynomials in $F$ will be algebraically independent if and only if $\Omega$ satisfies the conditions in Theorem 1.

Lemma 2. For a.e. $X$, the polynomials in $F$ are algebraically dependent if and only if $m(\Omega') < n(\Omega')/r + r$ for some matrix $\Omega'$ formed with a subset of the columns in $\Omega$.

In order to show this statement, we will require Lemmas 3 and 4 below.

Let $\Omega'$ be a subset of the columns in $\Omega$, and let $F'$ be the subset of the $n = n(\Omega')$ polynomials in $F$ corresponding to such columns. Notice that $F'$ only involves the variables corresponding to the $m = m(\Omega')$ nonzero rows of $U$.

Let $\aleph(\Omega')$ be the number of algebraically independent polynomials in $F'$.

Lemma 3. For a.e. $X$, $m(\Omega') \geq \aleph(\Omega')/r + r$.

Proof. Observe that the column-echelon form in (9) was chosen arbitrarily. As a matter of fact, for every permutation of rows $\Pi$ and every $S \in \text{Gr}_r(r, \mathbb{R}^d)$, we may write $S = \text{span}(U)$, for a unique $U$ in the following permuted column-echelon form:

$$U = \Pi \begin{bmatrix} I \\ V \end{bmatrix}.$$ 

For example, we could take $\Pi$ to swap the top and bottom blocks in (9), and take $U$ in the following form:

$$U = \Pi \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} V \\ I \end{bmatrix}.$$ 

Observe that in general, $U$, $V$ and $F$ will be different for each choice of $\Pi$. Nevertheless, the condition $x_{\omega_i} \in S_{\omega_i}$ is equivalent to $f_i = 0$, regardless of the choice of basis of $S$. This implies that while different choices of $\Pi$ produce different $F$'s, the variety

$$S(X\Omega) = \left\{ \text{span} \begin{bmatrix} I \\ V \end{bmatrix} \in \text{Gr}_r(r, \mathbb{R}^d) : F(V) = 0 \right\}$$

is the same for every $\Pi$.

This implies that the number of algebraically independent polynomials in $F'$ is invariant to the choice of $\Pi$. Therefore, showing that Lemma 3 holds for one particular $\Pi$ suffices to show that it holds for every $\Pi$.

With this in mind, take $\Pi$ such that $U$ is written with the identity block in the position of $r$ nonzero rows of $\Omega'$.
Since the polynomials in $\mathcal{F}'$ only involve the elements of the $m$ rows of $\mathbf{U}$ corresponding to the nonzero rows of $\Omega'$, and $\mathbf{U}$ has the identity block in the position of $r$ nonzero rows of $\Omega'$, it follows that the polynomials in $\mathcal{F}'$ only involve the $r(m-r)$ variables in the $m-r$ corresponding rows of $\mathbf{V}$. Furthermore, $\mathcal{F}' = 0$ has at least one solution. This implies $\Re(\Omega') \leq r(m-r)$, as desired. \hfill \Box

We say $f_i$ is \textit{minimally algebraically dependent} on $\mathcal{F}'$ if $f_i$ is algebraically dependent on the polynomials in $\mathcal{F}'$, but algebraically independent of any proper subset of polynomials in $\mathcal{F}'$.

\textbf{Lemma 4.} Let $f_i$ be minimally algebraically dependent on $\mathcal{F}'$. Then $m(\Omega') = n(\Omega')/r + r$ for a.e. $\mathbf{X}$.

In order to prove Lemma 4 we will need the next lemma.

Let $\mathcal{F}''$ be a subset of the polynomials in $\mathcal{F}$, and define $\{\mathbf{V}_t, \mathbf{V}^*_t\}$ as the partition of the variables involved in $\mathcal{F}'_t \subset \mathcal{F}''$, such that all the variables in $\mathbf{V}_t$ are uniquely determined by $\mathcal{F}'' = 0$.

\textbf{Lemma 5.} Suppose $\mathbf{V}_t \neq \emptyset$ and that every $f_i \in \mathcal{F}'_t$ is a polynomial in at least one of the variables in $\mathbf{V}_t$. Then for a.e. $\mathbf{X}$, all the variables involved in $\mathcal{F}'_t$ are uniquely determined by $\mathcal{F}'' = 0$.

\textit{Proof.} Let $v^c$ be one of the variables in $\mathbf{V}^*_t$ and let $f_i$ be a polynomial in $\mathcal{F}'_t$ involving $v^c$. By assumption on $\mathcal{F}'_t$, $f_i$ also involves at least one of the variables in $\mathbf{V}_t$, say $v$.

Let $\mathbf{w}$ denote the set of all variables involved in $f_i$ except $v$. Observe that $v^c \in \mathbf{w}$. This way, $f_i$ is shorthand for $f_i(v, \mathbf{w}, |\mathbf{V}^*, \theta^*_i|)$.

We will show that for a.e. $\mathbf{X}$, all the variables in $\mathbf{w}$ are also uniquely determined by $\mathcal{F}'' = 0$.

Suppose there exists a solution to $\mathcal{F}'' = 0$ with $\mathbf{w} = \eta$, and define the univariate polynomial
\[ g(v|\mathbf{V}^*, \theta^*_i) := f_i(v, \mathbf{w}, |\mathbf{V}^*, \theta^*_i)|_{\mathbf{w} = \eta}. \]

Now assume for contradiction that there exists another solution to $\mathcal{F}'' = 0$ with $\mathbf{w} \neq \eta$. Let $\mathbf{w} = \eta'$ be an other solution to $\mathcal{F}'' = 0$, and define
\[ g'(v|\mathbf{V}^*, \theta^*_i) := f_i(v, \mathbf{w}, |\mathbf{V}^*, \theta^*_i)|_{\mathbf{w} = \eta'}. \]

We will now show that $g \neq g'$. To see this, recall the definition of $f_i$, and observe that it depends on the choice of $\nabla_i$. Nevertheless, it is easy to see that $f_i = 0$ describes the same variety regardless of the choice of $\nabla_i$. Intuitively, this means that even though $f_i$ might look different for each choice of $\nabla_i$, it really is the same.

Therefore, we may select $\nabla_i$ to be the row of $\omega_i$ corresponding to the position of a variable of $\mathbf{w}$ that takes different values in $\eta$ and $\eta'$. This way, a variable with multiple solutions is located in the location of $\mathbf{U}_{\nabla_i}$. Since $f_i$ is linear in $\mathbf{U}_{\nabla_i}$, it follows that $g \neq g'$.
Now observe that since \( v \) is uniquely determined by \( F'' = 0 \), \( g \) and \( g' \) have a common root.

We know from elimination theory that two distinct polynomials \( g, g' \) have a common root if and only if their resultant \( \text{Res}(g, g') \) is zero (see, for example, Proposition 8 in Chapter 3, Section 5 of [11]).

But \( \text{Res}(g, g') \) is a polynomial in the coefficients of \( g \) and \( g' \). In other words, \( \text{Res}(g, g') = h(V^*, \theta_i^*) \) for some non constant polynomial \( h \) in \( V^* \) and \( \theta_i^* \). Therefore, \( h \neq 0 \) for almost every \( V^* \) and \( \theta_i^* \) (since the variety defined by \( h = 0 \) has measure zero). Equivalently, \( h \neq 0 \) for almost every \( S^* \) and almost surely with respect to \( \nu \), hence for a.e. \( X \).

Since \( \text{Res}(g, g') \neq 0 \), it follows that \( g \) and \( g' \) do not have a common root \( v \), which is the desired contradiction.

This shows that for a.e. \( X \), all the variables in \( w \) (including \( v^c \)) are uniquely determined by \( F'' = 0 \).

Since \( v^c \) was an arbitrary element in \( V_i^c \), we conclude that all the variables in \( V_i^c \) are also uniquely determined by \( F'' = 0 \). \( \Box \)

With this, we are now ready to present the proofs of Lemma 4, Lemma 2 and Theorem 1.

Proof. (Lemma 4) By the same arguments as in Lemma 3, whether a polynomial \( f_i \) is minimally algebraically dependent on \( F' \) is invariant to any permutation \( \Pi \) of the rows of the column-echelon form in (9). Therefore, showing that Lemma 4 holds for one particular choice of \( \Pi \) suffices to show it holds for every \( \Pi \).

With this in mind, take \( \Pi \) such that \( U \) is written with the identity block in the rows of the first \( r \) nonzero entries of \( \omega_i \), and let \( v_i \) denote the row of \( V \) corresponding to the last row of \( U_{\omega_i} \), such that

\[
U_{\omega_i} = \begin{bmatrix} I \\ v_i \\ 1 \end{bmatrix}.
\]

Suppose \( f_i \) is minimally algebraically dependent on \( F' \). We will first show that for a.e. \( X \), \( v_i \) is finitely determined by \( F' = 0 \).

To see this, suppose for contrapositive that there exist infinitely many solutions \( v_i \) to \( F' = 0 \). Each of these solutions defines a different subspace \( S_{\omega_i} = \text{span}(U_{\omega_i}) \). Let \( S'_{\omega_i} \) be the intersection of all these subspaces \( S_{\omega_i} \), and notice that \( \dim S'_{\omega_i} < r \). Then \( \nu \)-almost surely, \( x_{\omega_i} \notin S'_{\omega_i} \) (because \( \dim S'_{\omega_i} \cap S^*_{\omega_i} < r \), hence \( S'_{\omega_i} \cap S^*_{\omega_i} \) is a set of \( \nu \)-measure zero).

Since \( x_{\omega_i} \notin S'_{\omega_i} \), there are some solutions \( v_i \) for which \( f_i \neq 0 \). This implies \( f_i \) is a nonzero polynomial in \( v_i \). Since the variety that satisfies \( f_i = 0 \) has measure zero, it follows that almost none of the solutions \( v_i \) to \( F' = 0 \) satisfy \( f_i = 0 \), which implies \( f_i \) is algebraically independent of \( F' \).

Since this is true for almost every \( S^* \in \text{Gr}(r, \mathbb{R}^d) \), and since this holds \( \nu \)-almost surely, this is true for a.e. \( X \).
Let $\mathcal{F}' = \{ f_i \}$. We will now show that for a.e. $x$, $v_i$ is uniquely determined by $\mathcal{F}' = 0$, or equivalently, that every solution to $\mathcal{F}' = 0$ satisfies $v_i = v_i^*$. 

Again for contrapositive, observe that any solution to $\mathcal{F}'' = 0$ is also a solution to $\mathcal{F}' = 0$, so there exist at most finitely many solutions $v_i$ to $\mathcal{F}'' = 0$. For each of these solutions, if $v_i = v_i^*$, then $\nu$-almost surely, $x_{\omega_i} \notin S_{\omega_i}$ (because almost surely $x_{\omega_i} \notin S_{\omega_i}$), whence $v_i$ does not satisfy $f_i = 0$.

Now that we know that $v_i$ is uniquely determined by $\mathcal{F}' = 0$, we will iteratively use Lemma 5 to show that all variables in $\mathcal{F}''$ (which are the same as the variables in $\mathcal{F}'$) are also uniquely determined by $\mathcal{F}'' = 0$. This will imply that all the variables in $\mathcal{F}'$ are finitely determined by $\mathcal{F}' = 0$, and that $\mathcal{F}'$ contains the same number of polynomials, $n$, as variables, $r(m-r)$, which is the desired conclusion.

First observe that since $v_i$ is finitely determined by $\mathcal{F}' = 0$, $\mathcal{F}'$ must contain at least $r$ polynomials in $v_i$. Denote these polynomials by $\mathcal{F}_1 \subset \mathcal{F}'$.

We will proceed inductively, indexed by $t \geq 1$. First, set $t = 1$ and define $V_1 = \{ v_i \}$. We showed above that the variables in $V_1$ are uniquely determined by $\mathcal{F}'' = 0$. Suppose that $\mathcal{F}_1'$ involves some variables other than those in $V_1$. Note that every polynomial in $\mathcal{F}_1'$ involves at least one of the variables in $V_1$. Define $V_2$ to be the set of all variables involved in $\mathcal{F}_1'$. By Lemma 5, all the variables in $V_2$ are uniquely determined by $\mathcal{F}'' = 0$.

We will now proceed inductively. For any $t \geq 2$, let $V_t$ be a subset of $M_t$ variables in $V$. Assume that all the variables in $V_t$ are uniquely determined by $\mathcal{F}'' = 0$. Since $\dim V(\mathcal{F}') = \dim V(\mathcal{F}'')$, it follows that all the variables in $V_t$ are finitely determined by $\mathcal{F}' = 0$. It follows that $\mathcal{F}'$ must contain at least $M_t$ algebraically independent polynomials, each involving at least one of the variables in $V_t$. Let $\mathcal{F}_t'$ be this set of polynomials. Suppose $\mathcal{F}_t'$ involves some variables other than $V_t$. Define $V_{t+1}$ to be the set of all variables involved in $\mathcal{F}_t'$. By Lemma 5, all the variables in $V_{t+1}$ are uniquely determined by $\mathcal{F}'' = 0$.

Since this is true for every $t$, and there are finitely many variables, this process must terminate at some finite step $T$, at which point $\mathcal{F}_T'$ is a set of $M_T$ algebraically independent polynomials in $M_T$ variables.

This means that all the variables in $\mathcal{F}_T'$ are finitely determined by $\mathcal{F}_T' = 0$, and uniquely determined by $\{ \mathcal{F}_T', f_i \} = 0$. This implies that $f_i$ is algebraically independent on $\mathcal{F}_T' \subset \mathcal{F}'$. Furthermore, since $f_i$ is minimally algebraically dependent on $\mathcal{F}'$ by assumption, we have that $\mathcal{F}_T' = \mathcal{F}'$.

Finally, observe that $\mathcal{F}'$ contains $r(m-r)$ variables and $n$ polynomials. Since $\mathcal{F}' = \mathcal{F}_T'$, and $\mathcal{F}_T'$ has $M_T$ polynomials in $M_T$ variables, it follows that $n = r(m-r)$, as desired.  

*Proof.* (Lemma 2)

(⇒) Let $\Omega''$ be a subset of the columns in $\Omega$, and let $\mathcal{F}''$ be the subset of the polynomials in $\mathcal{F}$ corresponding to such columns. Suppose $f_i$ is minimally algebraically dependent on the polynomials in $\mathcal{F}''$. By Lemma 4, $n(\Omega'') = r(m(\Omega'') - r)$. Let $\Omega' = \{ \Omega'' \mid \omega_i \}$. It is clear that $m(\Omega') = m(\Omega'')$ and $n(\Omega') = n(\Omega'') + 1$. Thus $m(\Omega') < n(\Omega')/r + r$, and we have the first implication.
Suppose there exists an \( \Omega' \) with \( m(\Omega') < n(\Omega')/r \). By Lemma 3, \( n(\Omega') > \mathfrak{n}(\Omega') \), which implies \( \mathcal{F}' \), and hence \( \mathcal{F} \), has an algebraically dependent polynomial.

**Proof. (Theorem 1)**

\( \Rightarrow \) Suppose for contrapositive that there exists an \( \Omega' \) such that \( m(\Omega') < n(\Omega')/r \). Lemma 2 implies that the polynomials in \( \mathcal{F}' \), and hence \( \mathcal{F} \), are not algebraically independent. It follows by Lemma 1 that there are infinitely many subspaces in \( \mathcal{S}(X_\Omega) \).

\( \Leftarrow \) Suppose every \( \Omega' \) satisfies \( m(\Omega') \geq n(\Omega')/r \), including \( \Omega \). By Lemma 2, \( \mathcal{F} \) has \( r(d-r) \) algebraically independent polynomials. It follows by Lemma 1 that there are at most finitely many subspaces in \( \mathcal{S}(X_\Omega) \), hence at most finitely many rank-\( r \) completions of \( X_\Omega \).

### 4 Unique Completability

In this section we give the proof of Theorem 2.

We will use \( \tilde{X} \) and \( \hat{X} \) to denote the \( d \times r(d-r) \) and \( d \times (d-r) \) sub matrices of \( X \) corresponding to \( \tilde{\Omega} \) and \( \hat{\Omega} \). In addition, let \( \tilde{\omega}_i \) and \( x_{\tilde{\omega}_i} \) denote the \( i^{\text{th}} \) columns of \( \tilde{\Omega} \) and \( \tilde{X}_\tilde{\Omega} \). Recall that we assume the underlying complete data matrices are drawn independently from an Lesbegue absolutely continuous probability measure \( \nu \) with support over the entire subspace \( S^* \).

In order to prove Theorem 2, we will require Theorem 1 in [13], which we state here as the following lemma, with some minor adaptations to our context.

**Lemma 6.** Suppose \( \hat{\Omega} \) is a \( d \times (d-r) \) binary matrix for which (ii) holds and let \( S \in \text{Gr}(r, \mathbb{R}^d) \). Then for a.e. \( S^* \), \( \{ S_{\tilde{\omega}_i} = S^*_{\tilde{\omega}_i} \}_{i=1}^{d-r} \) if and only if \( S = S^* \).

With this, we are now ready to give the proof of Theorem 2.

**Proof. (Theorem 2)** Suppose there is a partition of \( \Omega \) into matrices \( \tilde{\Omega} \) and \( \hat{\Omega} \) satisfying the conditions of Theorem 2.

Since \( \tilde{\Omega} \) satisfies (i), by Theorem 1 there are at most finitely many \( r \)-dimensional subspaces that fit \( \tilde{X}_\tilde{\Omega} \). Let \( S \) be one of these subspaces, and assume \( S \) fits \( \tilde{X} \). Then \( x_{\tilde{\omega}_i} \in S_{\tilde{\omega}_i} \cap S^*_{\tilde{\omega}_i} \).

Suppose that \( S_{\tilde{\omega}_i} \neq S^*_{\tilde{\omega}_i} \). Then \( \nu \)-almost surely, \( x_{\tilde{\omega}_i} \) cannot belong to \( S_{\tilde{\omega}_i} \cap S^*_{\tilde{\omega}_i} \) (since \( x_{\tilde{\omega}_i} \) will have a non-zero component outside of \( S^*_{\tilde{\omega}_i} \)), contradicting the assumption that \( S_{\tilde{\omega}_i} \neq S^*_{\tilde{\omega}_i} \). The same reasoning holds for all of the finitely many subspaces that fit \( \tilde{X}_\tilde{\Omega} \).

Therefore, every \( S \) that fits both \( \tilde{X} \) and \( \hat{X} \) must satisfy \( \{ S_{\tilde{\omega}_i} = S^*_{\tilde{\omega}_i} \}_{i=1}^{d-r} \). Since \( \hat{\Omega} \) satisfies (ii), it follows by Lemma 6 that \( S = S^* \).
In Section 2 we mentioned that there are cases where $N = r(d-r)$ is sufficient for unique completability. The next result states that this is indeed the case if $r = 1$.

**Proposition 1.** If $r = 1$, finite completability is equivalent to unique completability.

*Proof.* Assume $r = 1$. Then $U_{\Delta_i}$ and $U_{\nabla_i}$ are scalars, so $f_i$ simplifies into:

$$f_i = (U_{\Delta_i} U_{\psi_i}^* - U_{\nabla_i} U_{\theta_i}^*) \theta_i^*.$$  

This implies that $\mathcal{F} = 0$ is a system of linear equations in $U$, hence if it has finitely many solutions, it has only one. \hfill $\Box$

In Section 2 we also mentioned that in general, $N > r(d-r)$ is necessary for unique completability. We would like to close this section with an example where this is the case.

**Example 3.** Consider $r = 2$ and $d = 4$, such that $N = r(d-r) = 4$. Let

$$\Omega = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}.$$  

It is easy to see that $\Omega$ satisfies the conditions of Theorem 1. One may also verify (for example, solving explicitly $\mathcal{F}(V) = 0$) that for a.e. $X$ there exist two subspaces that fit $X \Omega$. As a matter of fact, this will also be the case for any permutation of the rows and columns of this matrix. One may construct similar samplings with the same property for any $r \geq 2$ and $d \geq 4$. All this to say that this is not a singular pathological example; there are many samplings that cannot be uniquely recovered with only $N = r(d-r)$ columns.

### 5 Random Sampling Patterns

In this section we present the proof of Theorem 3. To do so, we will use the following lemma, which is an additional sufficient condition for finite completability. This useful result also shows the tight relation between the conditions for finite completability and the condition in (ii).

**Lemma 7.** $\Omega$ satisfies the conditions of Theorem 1 if there is a partition of $\Omega$ into matrices $\{\Omega_{\tau}\}_{\tau=1}^r$, each of size $d \times (d-r)$, such that (ii) holds for every $\Omega_{\tau}$.

*Proof.* Suppose $\Omega$ can be partitioned into matrices $\{\Omega_{\tau}\}_{\tau=1}^r$ satisfying the conditions of Lemma 7. Let $\Omega'$ be a matrix formed with a subset of the columns in $\Omega$. Then $\Omega' = [\Omega'_1 \mid \ldots \mid \Omega'_r]$ for some matrices $\{\Omega'_\tau\}_{\tau=1}^r$ formed with subsets of the columns in $\{\Omega_{\tau}\}_{\tau=1}^r$. 

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It follows that
\[ n(\Omega') = \sum_{\tau=1}^{r} n(\Omega'_\tau) \leq \sum_{\tau=1}^{r} \max n(\Omega'_\tau). \]
Assume without loss of generality that this maximum is achieved when \( \tau = 1 \). Then
\[ n(\Omega') \leq r n(\Omega'_1) \leq r (m(\Omega'_1) - r) \leq r (m(\Omega') - r), \]
where the last two inequalities follow because (2) holds for every \( \Omega'_\tau \) by assumption, and because \( m(\Omega') \geq m(\Omega'_\tau) \) for every \( \tau \).

Since \( \Omega' \) was arbitrary, we conclude that (1) holds for every matrix \( \Omega' \) formed with a subset of the columns in \( \Omega \).

**Example 4.** The partition \( \Omega = [ \hat{\Omega}_1 | \cdots | \hat{\Omega}_r ] \) in Example 1 satisfies the conditions in Lemma 7.

The following lemma shows that (ii) is satisfied with high probability under uniform random sampling schemes with only \( \mathcal{O}(\max\{r, \log d\}) \) samples per column.

**Lemma 8.** Let the assumptions of Theorem 3 hold, and let \( \hat{\Omega} \) be a matrix formed with \( d - r \) columns of \( \Omega \). With probability at least \( 1 - \frac{\epsilon}{d} \), \( \hat{\Omega} \) will satisfy (ii).

**Proof.** Let \( \mathcal{E} \) be the event that \( m(\Omega') < n(\Omega') + r \) for some matrix \( \Omega' \) formed with a subset of the columns in \( \hat{\Omega} \). It is easy to see that this will only occur if there is a matrix \( \Omega' \) formed with \( n \) columns of \( \hat{\Omega} \) that has all its nonzero entries in the same \( n + r - 1 \) rows. Let \( \mathcal{E}_n \) denote the event that the matrix formed with the first \( n \) columns from \( \hat{\Omega} \) has all its nonzero entries in the first \( n + r - 1 \) rows. Then
\[ \mathbb{P}(\mathcal{E}) \leq \sum_{n=1}^{d-r} \binom{d-r}{n} \binom{n}{d} \mathbb{P}(\mathcal{E}_n) \tag{10} \]
If each column of \( \hat{\Omega} \) contains at least \( \ell \) nonzero entries, distributed uniformly and independently at random with \( \ell \) as in (3), it is easy to see that \( \mathbb{P}(\mathcal{E}_n) = 0 \) for \( n \leq \ell - r \), and for \( \ell - r < n \leq d - r \),
\[ \mathbb{P}(\mathcal{E}_n) \leq \left( \frac{n+r-1}{\ell} \right)^n \left( \frac{n}{d} \right)^\ell n. \]
Since \( \left( \frac{d-r}{d} \right) < \left( \frac{d}{n+r-1} \right) \), continuing with (10) we obtain:
\[ \mathbb{P}(\mathcal{E}) < \sum_{n=\ell}^{d-r} \left( \frac{d}{n+r-1} \right)^2 \left( \frac{n+r-1}{d} \right)^\ell n \]
\[ < \sum_{n=\ell}^{d} \left( \frac{d}{n} \right)^2 \left( \frac{n}{d} \right)^\ell (n+r-1) + \sum_{n=1}^{d-r} \left( \frac{d}{d-n} \right)^2 \left( \frac{d-n}{d} \right)^\ell (d-n+r-1). \tag{11} \]
For the terms in the first sum of (11), write
\[
\binom{d}{n}^2 \binom{n}{\frac{n}{d}}^\ell(n-r+1) \leq \left(\frac{de}{n}\right)^{2n} \binom{n}{\frac{n}{d}}^\ell(n-r+1).
\] (12)

Since \( n \geq \ell \geq 2r \),
\[
(12) < \left(\frac{de}{n}\right)^{2n} \binom{n}{\frac{n}{d}}^\ell = e^{2n} \left(\frac{n}{d}\right)^\ell (\frac{\ell - 2}{2})^n,
\] (13)
and since \( n \leq \frac{d}{2} \),
\[
(13) \leq e^{2n} \left(\frac{1}{2}\right)^{(\frac{\ell - 2}{2})^n} = \left(2e^2 \cdot \frac{\ell - 2}{2}\right)^n < \frac{e}{d^2},
\] (14)
where the last step follows because \( \ell > 2 \log_2 \left(\frac{(de)^2}{e}\right) + 4 \).

For the terms in the second sum of (11), write
\[
\binom{d}{d-n}^2 \binom{d-n}{\frac{d-n}{d}}^\ell(d-r+1) \leq \left(\frac{de}{n}\right)^{2n} \binom{d-n}{\frac{d-n}{d}}^\ell(d-r+1).
\] (15)

In this case, since \( 1 \leq n \leq \frac{d}{2} \) and \( r \leq \frac{d}{6} \), we have
\[
(15) < (de)^{2n} \left(\frac{d-n}{d}\right)^\ell \left(\frac{d-n}{d}\right)^\ell (1 - \frac{n}{d})^{\frac{\ell}{d}} \leq (de)^{2n} \left(\frac{e-n}{n}\right)^{\frac{\ell}{d}},
\]
which we may rewrite as
\[
\left(2e^2 \log d\right)^n \left(\frac{1}{3}\right)^n \left(\frac{e^{-\frac{\ell}{d}}}{3}\right)^n = \left(2e^2 \log d + 2 - \frac{\ell}{d}\right)^n < \frac{e}{d^2},
\] (16)
where the last step follows because \( \ell > 3 \log \left(\frac{d^2}{e}\right) + 6 \log d + 6 \).

Substituting (14) and (16) in (11), we have that \( P(\mathcal{E}) < \frac{e}{d} \).

We are now ready to give the proof of Theorem 3.

Proof. (Theorem 3) If \( N = r(d-r) \), randomly partition \( \Omega \) into matrices \( \{\Omega_{\tau}\}_{\tau=1}^r \), each of size \( d \times (d-r) \).

Union bounding over \( \tau \), we may upper bound the event that \( \Omega \) fails to satisfy the conditions of Lemma 7 by
\[
\sum_{\tau=1}^r P(\mathcal{E}) < \sum_{\tau=1}^r \frac{\epsilon}{d} \leq \sum_{\tau=1}^r \frac{\epsilon}{r} < \epsilon.
\]

The first part of the statement follows because the conditions on \( \Omega \) in Lemma 7 imply the conditions on \( \Omega \) in Theorem 1.
If $N = (r + 1)(d - r)$, randomly partition $\Omega$ into matrices $\{\tilde{\Omega}_\tau\}_{\tau=1}^{r+1}$. By the same arguments, the probability that $\Omega$ fails to satisfy the conditions of Theorem 2 is upper bounded by:

$$\sum_{\tau=1}^{r+1} P(\mathcal{E}) < \sum_{\tau=1}^{r+1} \frac{\epsilon}{d} \leq \sum_{\tau=1}^{r+1} \frac{\epsilon}{r+1} < \epsilon.$$

6 Conclusions

In this paper we give a full characterization of finitely completable observation sets, that is, sampling patterns that can be completed in at most finitely many ways. We also provide deterministic sampling conditions for unique completability. In addition, we show that uniform random samplings with only $O(\max\{r, \log d\})$ observed entries per column satisfy these conditions with high probability.

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