Spontaneous excitation of a static multilevel atom coupled with electromagnetic vacuum fluctuations in Schwarzschild spacetime

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Abstract
We study the spontaneous excitation of a radially polarized static multilevel atom outside a spherically symmetric black hole in multipolar interaction with quantum electromagnetic fluctuations in the Boulware, Unruh and Hartle–Hawking vacuum states. We find that spontaneous excitation does not occur in the Boulware vacuum, and, in contrast to the scalar field case, the spontaneous emission rate is not well behaved at the event horizon as a result of the blow-up of the proper acceleration of the static atom. However, spontaneous excitation can take place both in the Unruh and the Hartle–Hawking vacua as if there were thermal radiation from the black hole. Distinctive features in contrast to the scalar field case are the existence of a term proportional to the proper acceleration squared in the rate of change of the mean atomic energy in the Unruh and the Hartle–Hawking vacua and the structural similarity in the spontaneous excitation rate between the static atoms outside a black hole and uniformly accelerated ones in a flat space with a reflecting boundary, which is particularly dramatic at the event horizon where a complete equivalence exists.

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1. Introduction

It is well known that spontaneous emission (and excitation), as one of the most important features of atoms, can be attributed to vacuum fluctuations (vf) \[1, 2\], or radiation reaction (rr) \[3\], or a combination of them \[4\]. The ambiguity in theoretical interpretation, which roots in the
freedom in the choice of ordering of commuting operators of the atom and field in a Heisenberg picture approach to the problem, was resolved by Dalibard, Dupont-Roc and Cohen-Tannoudji (DDC) [5, 6], who proposed a formalism that demands a symmetric operator ordering so that the contributions of \( \nu_f \) and \( \nu_r \) can be distinctively separated. The DDC proposal successfully resolves the problem of the stability of an inertial atom in its ground state in vacuum as a result of the delicate balance between the contributions of \( \nu_f \) and \( \nu_r \) to the rate of change of the mean atomic energy. Recent investigations—using the DDC formalism on the excitation of uniformly accelerated two-level atoms in interaction with fluctuating quantized massless scalar fields in vacuum in a flat spacetime with [7] and without boundaries [8] and that of static atoms outside a Schwarzschild black hole [9]—show that the delicate balance no longer exists in the cases under consideration, thus making the transition of the atom from ground state to excited states possible, i.e. excitation of atoms spontaneously occurs. The spontaneous excitation of uniformly accelerated atoms in the flat spacetime can be regarded as providing a physically appealing interpretation of the Unruh effect [10], since the spontaneous excitation of accelerated atoms gives a physically transparent illustration for why an accelerated detector clicks, while the spontaneous excitation of the static atoms outside a Schwarzschild black hole can be considered as providing another approach to the derivation of the Hawking radiation and it shows pleasing consistence of two different physical phenomena, the Hawking radiation and the spontaneous excitation of atoms, which are quite prominent in their own right.

However, a two-level atom interacting with a scalar field is more or less a toy model, and a more realistic system would be a multilevel atom, a hydrogen atom, for instance, in interaction with a quantized electromagnetic field. Let us note that such a system in the multipolar coupling scheme was recently examined in terms of the spontaneous excitation of an accelerated atom in both a free space [11] and cavities [12]. It has been found that both the effects of \( \nu_f \) and \( \nu_r \) on the atom are altered by the acceleration. This differs from the scalar field case where the contribution of \( \nu_r \) is not changed by the acceleration. A dramatic feature is that the contribution of electromagnetic \( \nu_f \) to the spontaneous excitation rate contains an extra term proportional to \( a^2 \), the proper acceleration squared, in addition to the usual Planckian thermal term of the Unruh temperature \( T = a/2\pi \). This is in contrast to the scalar field case where the effect of acceleration is purely thermal. As a further step along the line, in this paper, we would like to study, using the DDC formalism, the spontaneous excitation of a static multilevel atom in multipolar interaction with quantized electromagnetic fields in vacuum outside a four-dimensional Schwarzschild black hole. Our discussion will be based upon the Gupta–Bleuler quantization of free electromagnetic fields in a static spherically symmetric spacetime of arbitrary dimension in a modified Feynman gauge given by Crispino et al [13]. At this point, let us note that the DDC formalism has also been applied to calculate the radiative energy shifts of accelerated atoms both in a free space [14–16] and in cavities [17–19].

The paper is organized as follows. We introduce, in section 2, the general formalism developed in [8] and generalized in [11] to the case of a multilevel atom interacting with a quantized electromagnetic field in the multipolar coupling scheme. In section 3, we first review the Gupta–Bleuler quantization of free electromagnetic fields in the background Schwarzschild black hole [13], and then define, in analogy with the scalar field case, the Boulware, Unruh and Hartle–Hawking vacuum states, calculate the two-point functions for the electromagnetic fields in these vacuum states and analyze their properties in asymptotic regions. The calculation of the spontaneous excitation of the multilevel atom interacting with a quantized electromagnetic field in the multipolar coupling scheme in all vacuum states will be performed in section 4 and a summary will be given in section 5.
2. General formalism

The DDC formalism is carried out in the Heisenberg picture that provides a very convenient theoretical framework as it leads, for the relevant dynamical variables, to equations of motion very similar to the corresponding classical ones.

Consider a multilevel atom in interaction with vacuum electromagnetic fluctuations outside a four-dimensional spherically symmetric black hole. The line element of the spacetime is

\[ d^2s = (1 - 2M/r)^{-1} d^2t - (1 - 2M/r) d^2r - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]

with \( M \) being the mass of the black hole. Let \( x(\tau) \) represent the stationary trajectory of the atom and \( \tau \) denote its proper time. The stationary trajectory condition guarantees the existence of stationary states. By assuming the multipolar coupling, the total Hamiltonian that governs the evolution of the atom–field system with regard to the proper time \( \tau \) can be written as

\[ H(\tau) = H_A(\tau) + H_F(\tau) + H_I(\tau). \]

Here, \( H_A(\tau) \) is the Hamiltonian that determines the evolution of the atom, and it is given by

\[ H_A(\tau) = \sum_n \omega_n \sigma_{nn}(\tau), \]

where \( |n\rangle \) represents a complete set of stationary states of the atom with energies \( \omega_n \), and \( \sigma_{nn} = |n\rangle \langle n| \). \( H_F(\tau) \) is the Hamiltonian that decides the evolution of the free quantum electromagnetic field with respect to the proper time \( \tau \),

\[ H_F(\tau) = \sum_{\vec{k}} \omega_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} \frac{d\tau}{d\tau}, \]

where \( \vec{k} \) stands for the wave vector and polarization of the field modes, \( a_{\vec{k}}^{\dagger} \) and \( a_{\vec{k}} \) are the annihilation and creation operators with momentum \( \vec{k} \). \( H_I(\tau) \) describes the coupling between the multilevel atom and the electromagnetic field. In the multipolar coupling scheme [20],

\[ H_I(\tau) = -e \vec{r}(\tau) \cdot \vec{E}(x(\tau)) = -e \sum_{mn} \vec{r}_{mn} \cdot \vec{E}(x(\tau)) \sigma_{mn}(\tau), \]

where \( e \) is the electron electric charge, \( e \vec{r} \) the electric dipole moment of the atom, \( x(\tau) \) the atomic spacetime coordinates and \( \vec{E}(x(\tau)) \) the electric field operator of vacuum electromagnetic fields.

Generally, atomic observables evolve with time as a result of interaction between the atom and the field. The rate of change of an arbitrary observable, \( O(\tau) \), of the atom is governed by the Heisenberg equation

\[ \frac{dO(\tau)}{d\tau} = i[H_A(\tau), O(\tau)] - ie[\vec{r}(\tau) \cdot \vec{E}(x(\tau)), O(\tau)]. \]

Here, we are interested in the second part on the right-hand side of the above equation that is due to the interaction between the atom and the field:

\[ \left( \frac{dO(\tau)}{d\tau} \right)_{\text{coupling}} = -ie[\vec{r}(\tau) \cdot \vec{E}(x(\tau)), O(\tau)]. \]

The field operator \( \vec{E}(x(\tau)) \) can be divided into two parts as \( \vec{E} = \vec{E}^f + \vec{E}^s \). Here and after, the operators with superscript \( f \) represent the free parts that exist even when there is no coupling between the atom and the field and those with superscript \( s \) represent the source parts that are induced by the interaction between them. However, a tricky issue arises when we try to perform
this decomposition in the above equation, since $E'$ and $E^f$ do not separately commute with the atomic observable. As a result, we can write

$$\left(\frac{dO(t)}{dt}\right)_{\text{coupling}} = -ie(\lambda E^f(x(t)) \cdot [r(t), O(t)] + (1 - \lambda)[r(t), O(t)] \cdot E^f(x(t)))$$

- $ie(\lambda E^r(x(t)) \cdot [r(t), O(t)] + (1 - \lambda)[r(t), O(t)] \cdot E^r(x(t)))$.

So, there exists an ambiguity of operator ordering. DDC [5, 6] proposed to use a symmetric ordering ($\lambda = \frac{1}{2}$) so that the two terms on the right-hand side can be separately Hermitian and possess an independent physical meaning that can be unambiguously identified as the contributions of $vf$ and $rr$. Setting $O(t)$ to be the atomic energy $H_A(t)$, we have

$$\frac{dH_A(t)}{dt} = \left(\frac{dH_A(t)}{dt}\right)_{vf} + \left(\frac{dH_A(t)}{dt}\right)_{rr}$$

with

$$\left(\frac{dH_A(t)}{dt}\right)_{vf} = -\frac{ie}{2}(E^f(x(t)) \cdot \left[\sum_n \omega_n \sigma_{nm}(t)\right] + [r(t), \sum_n \omega_n \sigma_{nm}(t)] \cdot E^f(x(t))),$$

$$\left(\frac{dH_A(t)}{dt}\right)_{rr} = -\frac{ie}{2}(E^r(x(t)) \cdot \left[\sum_n \omega_n \sigma_{nm}(t)\right] + [r(t), \sum_n \omega_n \sigma_{nm}(t)] \cdot E^r(x(t))).$$

Accordingly, we can also divide the dynamical variables of the atom, $\sigma_{nm}$ or $r_i(t)$, into free and source parts,

$$\sigma_{nm}(t) = \sigma^f_{nm}(t) + \sigma^s_{nm}(t),$$

$$r_i(t) = r^f_i(t) + r^s_i(t).$$

Then, solving perturbatively the following Heisenberg equations of motion they satisfy

$$\frac{d}{dt} \sigma_{nm}(t) = i(\omega_m - \omega_n)\sigma_{nm}(t) - ieE^f(x(t)) \cdot [r(t), \sigma_{nm}(t)],$$

$$\frac{d}{dt} r_i(t) = i \sum_{nm} (\omega_m - \omega_n)(r_i)_{nm}\sigma_{nm}(t) - ie[r(t), r_i(t)] \cdot E^f(x(t)),$$

to order $e$, we find

$$\sigma^f_{nm}(t) = \sigma^f_{nm}(t_0) e^{i(\omega_m - \omega_n)(t - t_0)},$$

$$\sigma^s_{nm}(t) = -ie \int_{t_0}^{t} dr' E^f(x(t')) \cdot [r'(t'), \sigma^f_{nm}(t')],$$

$$\{r^f_i(t)\}_{nm} = \{r^f_i(t_0)\}_{nm} e^{i(\omega_m - \omega_n)(t - t_0)},$$

$$\{r^s_i(t)\} = -ie \int_{t_0}^{t} dr' [r_i(t'), r^f_i(t')] E^f(x(t')).$$

Here, the repeated subscript denotes the summation over all spatial components. Similarly, using the Heisenberg equation for the annihilation operator of the field,

$$\frac{d}{dt} a_i^f(t) = -i\omega_i a_i^f(t) - i e r_i(t) \cdot [E^f(x(t)), a_i(t)],$$

One can show to the same order of perturbation that

$$\{a^f_k(t)\} = \{a^f_k(t_0)\} e^{-i\omega_k(t) - t(t_0)},$$

$$\{a_k^s(t)\} = -ie \int_{t_0}^{t} dr' r^f_i(t') \cdot [E^f(x(t')), a^f_k(t)].$$
and therefore the source field $E'$ that is generated by the interaction between the atom and the free field can be expressed as

$$E'_i(x(\tau)) = -ie\int_{t_0}^t d\tau' r'_j(\tau') [E'_j(x(\tau')), E'_i(x(\tau))].$$ (20)

Now we suppose that the field is a vacuum state and the atom is in state $|b\rangle$. For simplicity, we also assume that the atom is polarized along the radial direction defined by the position of the atom relative to the black hole spacetime rotational killing fields. This assumption significantly simplifies, while keeping the physical essence of the problem, the computations we are going to perform, since we do not need to calculate the contributions associated with the polarizations in the $\theta -$ and $\phi -$ directions. Taking the expectation values of equations (10) and (11) over the state of the system $|0, b\rangle$, we can, using equations (16), (17) and (20), show that the contributions of $v_f$ and $r_r$ to the rate of change of the mean atomic energy are given, to order $e^2$, by

$$\left\langle \frac{dH_A(\tau)}{d\tau} \right\rangle_{v_f} = 2ie^2\int_{t_0}^t d\tau' C^F(x(\tau), x(\tau')) \frac{d}{d\tau} \chi^f_b(\tau, \tau'),$$

$$\left\langle \frac{dH_A(\tau)}{d\tau} \right\rangle_{r_r} = 2ie^2\int_{t_0}^t d\tau' \chi^F(x(\tau), x(\tau')) \frac{d}{d\tau} C^t_b(\tau, \tau').$$ (21) (22)

In the above equations, $C^F$ and $\chi^F$ are the two statistical functions of the electromagnetic field, i.e. the symmetric correlation function and the linear susceptibility function. The radial components of these function, which are relevant in our future calculations, are defined as

$$C^F(x(\tau), x(\tau')) = \frac{1}{2} \langle 0 | \{E'_j(x(\tau)), E'_j(x(\tau'))\} | 0 \rangle,$$ (23)

$$\chi^F(x(\tau), x(\tau')) = \frac{1}{2} \langle 0 | [E'_j(x(\tau)), E'_j(x(\tau'))] | 0 \rangle,$$ (24)

where $\langle ..., \rangle$ denotes the anti-commutator and $|0\rangle$ represents the vacuum state of the field which will be defined in the next section. Two statistical functions of the field are dependent on the trajectory of the atom. Analogously, $C^t_b(\tau, \tau')$ and $\chi^t_b(\tau, \tau')$ are two atomic statistical functions which are defined as

$$C^t_b(\tau, \tau') = \frac{1}{2} \langle b | [r'_j(\tau), r'_j(\tau')] | b \rangle,$$ (25)

$$\chi^t_b(\tau, \tau') = \frac{1}{2} \langle b | [r'_j(\tau), r'_j(\tau')] | b \rangle.$$ (26)

They do not depend on the atomic trajectory and are determined only by the internal structure of the atom itself. Their explicit forms are given, with respect to $\tau$, by

$$C^t_b(\tau, \tau') = \frac{1}{2} \sum_d |(b|r(0)|d\rangle)^2 [e^{i\omega_d(\tau-\tau')} + e^{-i\omega_d(\tau-\tau')},$$ (27)

$$\chi^t_b(\tau, \tau') = \frac{1}{2} \sum_d |(b|r(0)|d\rangle)^2 [e^{i\omega_d(\tau-\tau')} - e^{-i\omega_d(\tau-\tau')}],$$ (28)

here $\omega_{bd} = \omega_b - \omega_d$ and the sum extends over the complete set of atomic states.

Now it is clear that the calculation of the rate of change of the mean atomic energy requires detailed knowledge on the quantization of electromagnetic fields in the exterior region of the black hole and the specification of vacuum states. This is the main topic for the next section.
3. Quantization of electromagnetic fields in the exterior region of a Schwarzschild black hole

The quantization of the electromagnetic field in a static spherically symmetric Schwarzschild-like spacetime has been carried out by Crispino et al [13] using the Gupta–Bleuler condition in a modified Feynmann gauge. Here, we first give a brief review of their basic results, and we then define the vacuum states and calculate the statistical functions of the field. The Lagrangian density of the electromagnetic field in a modified Feynman gauge is

$$\mathcal{L}_F = \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} G^2 \right],$$

with $G = \nabla^\mu A_\mu + K^\mu A_\mu$ and $K^\mu$ being a vector independent of $A_\mu$. From the Lagrangian density, we can write down the field equations. If we choose $K^\mu$ to be $K^\mu = (0, 2M/r^2, 0, 0)$, the equation for $A_\mu$ decouples from other ones [13]. A complete set of solutions of the field equations can then be denoted by $A^{(3n; aclm)}_\mu$. Here, the label ‘$n$’ distinguishes between modes incoming from the past null infinity $J^-$ (denoted with $n = \leftarrow$) and those going out from the past horizon $H^-$ (denoted with $n = \rightarrow$). The modes with $\lambda = 0$ are

$$A^{(0; aclm)}_\mu = (R^{(0)}(\omega) Y_{lm} e^{-i\omega t}, 0, 0, 0),$$  

where $Y_{lm}$ is the spherical harmonics and $R^{(0)}(\omega)(r)$ satisfies the radial equation

$$\left[ \frac{\omega^2}{(1 - 2M/r)} + \frac{(1 - 2M/r)}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) - \frac{l(l + 1)}{r^2} \right] R^{(0)}_l(\omega)(r) = 0. \tag{31}$$

These modes are nonphysical as they do not satisfy the gauge condition $G = 0$ that are satisfied by all other modes with $\lambda \neq 0$. Modes with $\lambda = 3$ are pure-gauge modes, and they are given by

$$A^{(3n; aclm)}_\mu = \nabla_\mu \Lambda^{(aclm)},$$

with

$$\Lambda^{(aclm)} = \frac{i}{\omega} R^{(0)}(\omega)(r) Y_{lm} e^{-i\omega t}. \tag{33}$$

Those with $\lambda = 1, 2$ correspond to two classes of physical modes. For the first class of physical modes ($\lambda = 1$), $A_t = 0$, and

$$A^{(1n; aclm)}_r = R^{(1)}_l(\omega)(r) Y_{lm} e^{-i\omega t}, \tag{34}$$

with $l \geq 1$, where the radial function $R^{(1)}_l(\omega)(r)$ satisfies

$$\frac{1}{r^2} \frac{d}{dr} \left[ (1 - 2M/r) \frac{d}{dr} \left( r^2 R^{(1)}_l(\omega)(r) \right) \right] + \left[ \frac{\omega^2}{(1 - 2M/r)} - \frac{l(l + 1)}{r^2} \right] R^{(1)}_l(\omega)(r) = 0. \tag{35}$$

The angular components can be expressed as

$$A^{(1n; aclm)}_t = \frac{1 - 2M/r}{l(l + 1)} \frac{d}{dr} \left( r^2 R^{(1)}_l(\omega)(r) \right) \partial Y_{lm} e^{-i\omega t}. \tag{36}$$

For the second class of physical modes ($\lambda = 2$), $A_t = A_r = 0$,

$$A^{(2n; aclm)}_l = R^{(2)}_l(\omega)(r) Y^{(1m)}_l e^{-i\omega t}, \tag{37}$$

with $R^{(2)}_l(\omega)(r)$ obeying the following radial equation:

$$\left[ \frac{\omega^2}{(1 - 2M/r)} + \frac{d}{dr} \left( (1 - 2M/r) \frac{d}{dr} \right) - \frac{l(l + 1)}{r^2} \right] R^{(2)}_l(\omega)(r) = 0. \tag{38}$$
in which $Y_{l,m}^{(lm)}$ are the divergence-free vector spherical harmonics on the unit 2-sphere satisfying
\[ \hat{\nabla}^2 \hat{\nabla} Y_{l,m}^{(lm)} = -l(l+1) Y_{l,m}^{(lm)}, \]
\[ \int d\Omega \hat{\phi}^{(lm)} Y_{l'}^{(l'm')} = \delta_{l,l'} \delta_{m,m'}. \]

Here, the overline denotes the complex conjugation, $i$ angular variables on the unit 2-sphere $S^2$ with metric $\eta_{ij}$ and inverse metric $\eta^{ij}$ with signature $(+, -)$, $\tilde{\nabla}$ the associated covariant derivative on $S^2$ and $\tilde{\nabla}^2 \equiv \eta^{ij} \tilde{\nabla}_i \tilde{\nabla}_j$. The above four classes of modes form a complete set of basis for the quantum electromagnetic field. The normalization of them are determined from the canonical commutation relations of the fields by requiring suitable commutation relations for the annihilation and creation operators.

To quantize the electromagnetic field, let us define the general inner product as
\[ (A^{(\xi)}, A^{(\xi')}) = \int_{\Sigma} d\Sigma_{\mu} W^\mu \left[ A^{(\xi)}, A^{(\xi')} \right] \]

in which
\[ W^\mu \left[ A^{(\xi)}, A^{(\xi')} \right] = i \left[ \hat{A}^{(\xi)} \Pi^{(\xi') \mu \nu} - \Pi^{(\xi') \mu \nu} \hat{A}^{(\xi')} \right], \]
\[ \Pi^{\mu_{\nu}} = \frac{1}{\sqrt{-g}} \frac{\partial L_F}{\partial \left[ \tilde{\nabla}_\mu A^\nu \right]} = -\left[ F^{\mu_{\nu}} + g^{\mu_{\nu}} \tilde{G} \right], \]
and $d\Sigma_{\mu} = d\sigma \eta_{\mu}, \hat{A}^{(\xi)} = \hat{A}^{(\xi) (\text{phys})}$. The equal-time commutation relations for the fields and their momentum operators are
\[ [\hat{A}_\mu (t, \mathbf{x}), \hat{A}_\nu (t, \mathbf{x}')] = [\hat{\Pi}^{\mu \nu} (t, \mathbf{x}), \hat{\Pi}^{(\text{phys}) \mu \nu} (t, \mathbf{x}')] = 0, \]
\[ [\hat{A}_\mu (t, \mathbf{x}), \hat{\Pi}^{\nu} (t, \mathbf{x}')] = \frac{i \hbar^{\nu}}{\sqrt{-g}} \delta^3 (\mathbf{x} - \mathbf{x}'), \]

where $\mathbf{x}$ and $\mathbf{x}'$ represent all spatial coordinates. Expand the field operator in terms of the complete set of basic modes as
\[ \hat{A}_\mu (t, \mathbf{x}) = \sum_{\lambda, l m} \int_0^\infty \frac{d\omega}{4\pi \omega} \left[ A^{(\lambda \text{,} \text{phys}) \mu}_{\lambda m} (t, \mathbf{x}) \hat{a}^{(\lambda m)}_{\nu l m} + A^{(\lambda \text{,} \text{phys}) \mu}_{\lambda m} (t, \mathbf{x}) \hat{a}^{(\lambda m)}_{\nu l m} \right]. \]

By using the inner products of the field and the commutation relations between the field and momentum operators, the commutation relations between the annihilation and creation operators are found to be
\[ \left[ \hat{a}^{(\lambda m)}_{\nu l m}, \hat{a}^{(\lambda' m')}_{\nu' l' m'} \right] = -\left[ \hat{a}^{(\lambda m)}_{\nu l m}, \hat{a}^{(\lambda m')}_{\nu' l' m'} \right] = \delta_{\lambda \lambda'} \delta_{l l'} \delta_{m m'} \delta (\omega - \omega'), \]
\[ \left[ \hat{a}^{(\lambda m)}_{\nu l m}, \hat{a}^{(\lambda' m')}_{\nu' l' m'} \right] = \left[ \hat{a}^{(\lambda m)}_{\nu l m}, \hat{a}^{(\lambda m')}_{\nu' l' m'} \right] = \delta_{\omega \omega'} \delta_{l l'} \delta_{m m'} \delta (\omega - \omega'), \]
and all other commutators vanish. The Gupta–Bleuler condition \cite{21} requires that
\[ \hat{G}^+ \mid \text{phys} \rangle = 0, \]
where $\hat{G}$ is the positive-frequency part of $\hat{G} = \nabla^\mu \hat{A}_\mu + K^\mu \hat{A}_\mu$ and $| \text{phys} \rangle$ represents an arbitrary physical state. This is equivalent to the following statements: the states obtained by applying the creation operator $\hat{a}^{(\lambda m)}_{\nu l m}$ are all nonphysical; the physical states of the form $a^{(\lambda m)}_{\nu l m} \mid \text{phys} \rangle$ have zero norm and are orthogonal to any physical state.

Now we can define vacuum states and calculate the correlation functions of the field. We start with the Boulware vacuum state, $| 0_B \rangle$, which is defined by
\[ a^{(\lambda m)}_{\nu l m} | 0_B \rangle = 0, \quad \text{for all } (\lambda, \nu, l, m) \text{ with } (\omega > 0). \]
In the computation of the correlation function $B(0 | \hat{E}_r(t), \hat{E}_r(x') | 0)_B$ with \( \hat{E}_r = \hat{A}_{r,t} - \hat{A}_{r,t} \), the contributions of nonphysical modes and pure-gauge modes are found to be canceled out, and only the contribution of the first class of physical modes is left:

$$
B(0 | \hat{E}_r(x), \hat{E}_r(x') | 0)_B = \frac{1}{4\pi} \sum_{l,m} \int_0^\infty d\omega \omega e^{-i\omega(x-t)} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')
$$

\[ \times \left[ \vec{R}_l^{(1)}(\omega|\omega') \vec{R}_l^{(1)}(\omega|\omega') + \vec{R}_l(\omega|\omega) \vec{R}_l(\omega|\omega') \right]. \tag{51} \]

Here and after, for the sake of brevity, we omit the label \( \lambda = 1 \) as others will not appear in the correlation functions. For the Unruh vacuum state, \( |0\>_U \), similar to that in the scalar field case \([10, 22]\), we define the positive frequency for modes stemming from \( H^- \), the past horizon of the black hole, with respect to the Killing vector \( \xi = \hat{\partial}_t \) and those originating at infinity with respect to the Killing vector \( \eta = \hat{\partial}_r \). For the Hartle–Hawking vacuum state, \( |0\>_H \), we define the incoming modes to be positive frequency with respect to \( \tilde{r} \), the canonical affine parameter on the future horizon, and the outgoing modes to be positive frequency with respect to \( \tilde{r} \), the canonical affine parameter on the past horizon. Repeat the same steps as that in the Boulware vacuum case, the correlation function satisfying the corresponding boundary conditions in both the Unruh and Hartle–Hawking vacuums are also found to be only associated with the first class of physical modes. They are given respectively by

$$
u_U(0 | \hat{E}_r(x), \hat{E}_r(x') | 0)_U = \frac{1}{4\pi} \sum_{l,m} \int_{-\infty}^\infty d\omega \omega e^{-i\omega(x-t')} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')
$$

\[ \times \left[ \frac{\vec{R}_l^{(1)}(\omega|\omega) \vec{R}_l^{(1)}(\omega|\omega') + \theta(\omega) \vec{R}_l(\omega|\omega) \vec{R}_l(\omega|\omega')}{1 - e^{-2\pi\omega/\kappa}} \right]. \tag{52} \]

and

$$
u_H(0 | \hat{E}_r(x), \hat{E}_r(x') | 0)_H = \frac{1}{4\pi} \sum_{l,m} \int_{-\infty}^\infty d\omega \omega \left[ e^{-i\omega(x-t')} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \frac{\vec{R}_l(\omega|\omega) \vec{R}_l(\omega|\omega')}{1 - e^{-2\pi\omega/\kappa}} + e^{i\omega(x-t')} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \frac{\vec{R}_l^{(1)}(\omega|\omega) \vec{R}_l^{(1)}(\omega|\omega')}{e^{2\pi\omega/\kappa} - 1} \right], \tag{53} \]

where \( \kappa = 1/4M \) is the surface gravity of the black hole. Details about the features of the correlation functions calls for the specific properties of the radial functions, so now we turn our attention to the analysis of the radial functions. To do so, let us further write the radial function as

$$
\frac{R_l^{(n)}(\omega|\omega)}{\omega} = \frac{\sqrt{l(l+1)} \varphi_{al}^{(n)}(r)}{r^2}, \tag{54} \]

then equation (35) becomes

$$
\left[ \frac{d^2}{dr^2} + \omega^2 - \left( 1 - \frac{2M}{r} \right) \frac{l(l+1)}{r^2} \right] \varphi_{al}^{(n)}(r) = 0, \tag{55} \]

where \( r_* = r + 2M \ln(r/2M - 1) \) is the Regge–Wheeler tortoise coordinate. It is difficult to exactly solve this equation in terms of elementary functions, but fortunately, two classes of physical solutions in the two asymptotic regions single out

$$
\varphi_{al}(r) \sim \begin{cases} e^{\text{hor}}, & r \to 2M, \\ \frac{\sqrt{l}(\omega)}{\omega} e^{i\omega r}, & r \to \infty, \end{cases} \tag{56} \]

$$
$$
\varphi_{al}(r) \sim \begin{cases} \frac{\sqrt{l}(\omega)}{\omega} e^{i\omega r}, & r \to 2M, \\ e^{-i\omega r} + \frac{\sqrt{l}(\omega)}{\omega} e^{i\omega r}, & r \to \infty. \end{cases} \tag{57} \]

8
Here $\mathcal{R}$ and $\mathcal{T}$ are, respectively, the reflection and transmission coefficients and the following relationships exist among them and their complex conjugates,

\[
\begin{align*}
\mathcal{T}_l(\omega) &= \mathcal{T}_l(\omega), \\
|\mathcal{R}_l(\omega)| &= |\mathcal{R}_l(\omega)|, \\
1 - |\mathcal{R}_l(\omega)|^2 &= 1 - |\mathcal{T}_l(\omega)|^2, \\
\mathcal{R}_l^*(\omega)\mathcal{T}_l(\omega) &= -\mathcal{T}_l^*(\omega)\mathcal{R}_l(\omega).
\end{align*}
\]

Although the exact forms for these coefficients demand an exact solution of equation (55) which is a formidable task, further studies (see the appendix) show that the summation concerning the radial functions in the two asymptotic regions, i.e. $r \to 2M$ and $r \to \infty$, behaves as

\[
\sum_l (2l + 1) \left| \mathcal{R}_l(\omega) \right|^2 \sim \frac{\sum l(l+1)(2l+1)|\mathcal{T}_l(\omega)|^2}{(2M)^3 \omega^2}, \quad r \to 2M,
\]

and

\[
\sum_l (2l + 1) \left| \mathcal{R}_l(\omega) \right|^2 \sim \frac{\sum l(l+1)(2l+1)|\mathcal{T}_l(\omega)|^2}{\omega r^2}, \quad r \to \infty.
\]

### 4. Spontaneous excitation of an atom interacting with the vacuum electromagnetic field in Schwarzschild spacetime

Suppose that a static multilevel atom is held at the radial distance $r$ and it interacts with fluctuating quantum electromagnetic fields in vacuum. Using the DDC formalism introduced in section 2 and the results in the preceding section, we now calculate the rate of change of the mean atomic energy.

**Boulware vacuum.** In the Boulware vacuum state, the two statistical functions of the electromagnetic field, i.e. the symmetric correlation function and the linear susceptibility function, can easily be found from equation (51) to be

\[
C^F(x(\tau), x(\tau')) = \frac{1}{32\pi^2} \int_0^\infty d\omega \omega \left[ e^{-\frac{i\omega}{\Delta t}} + e^{\frac{i\omega}{\Delta t}} \right] \sum_{l=1}^\infty (2l + 1)\left[ |\mathcal{R}_l(\omega)|^2 + |\mathcal{T}_l(\omega)|^2 \right],
\]

and

\[
\chi^F(x(\tau), x(\tau')) = \frac{1}{32\pi^2} \int_0^\infty d\omega \omega \left[ e^{-\frac{i\omega}{\Delta t}} - e^{\frac{i\omega}{\Delta t}} \right] \sum_{l=1}^\infty (2l + 1)\left[ |\mathcal{R}_l(\omega)|^2 + |\mathcal{T}_l(\omega)|^2 \right].
\]

In obtaining the above results, we have used the relation $\Delta \tau = \sqrt{\Delta t} g_{00}$ with $g_{00} = (1 - 2M/r)$ and the following property of the spherical harmonics:

\[
\sum_{m=-l}^l |Y_{lm}(\theta, \phi)|^2 = \frac{2l + 1}{4\pi}.
\]
Inserting equations (61) and (28) into equation (21), we obtain the contribution of vacuum electromagnetic fluctuations to the rate of change of the mean atomic energy,

\[
\left\langle \frac{dH_A(\tau)}{d\tau} \right\rangle_{vf} = -\frac{e^2 g_{00}}{16\pi} \left[ \sum_{\omega_b > \omega_d} |\langle b|\tau(0)|d\rangle|^2 \omega_d^2 P(\omega_d, r) - \sum_{\omega_b < \omega_d} |\langle b|\tau(0)|d\rangle|^2 \omega_d^2 P(-\omega_d, r) \right],
\]

(64)

where we have defined

\[
P(\omega, r) = \vec{P}(\omega, r) + \vec{P}(\omega, r)
\]

(65)

with

\[
\vec{P}(\omega, r) = \sum_{l} (2l + 1) \left| \vec{R}(\omega \sqrt{g_{00}} | r) \right|^2.
\]

(66)

\[
\vec{P}(\omega, r) = \sum_{l} (2l + 1) \left| \vec{R}(\omega \sqrt{g_{00}} | r) \right|^2.
\]

(67)

Hereafter, the summation over \(l\) is implied to range from 1 to \(\infty\). Similarly, by inserting equations (62) and (27) into equation (22), the contribution of \(rr\) to the rate of change of the mean atomic energy is calculated out to be

\[
\left\langle \frac{dH_A(\tau)}{d\tau} \right\rangle_{rr} = -\frac{e^2 g_{00}}{16\pi} \left[ \sum_{\omega_b > \omega_d} |\langle b|\tau(0)|d\rangle|^2 \omega_d^2 P(\omega_d, r) + \sum_{\omega_b < \omega_d} |\langle b|\tau(0)|d\rangle|^2 \omega_d^2 P(-\omega_d, r) \right].
\]

(68)

Adding up equations (64) and (68) yields the total rate of change of the mean atomic energy

\[
\left\langle \frac{dH_A(\tau)}{d\tau} \right\rangle_{tot} = -\frac{e^2 g_{00}}{16\pi} \sum_{\omega_b > \omega_d} |\langle b|\tau(0)|d\rangle|^2 \omega_d^2 P(\omega_d, r).
\]

(69)

Here, only the term with \(\omega_b > \omega_d\) survives after the addition and it is negative. This means that for an atom originally in the ground state, the contributions of \(vf\) and \(rr\) to the total rate of change of the mean atomic energy cancel, and, as a result, the ground state is stable. However, an atom originally in an excited state can transition to lower level states, since the total rate of change of the mean atomic energy is negative. Although, qualitatively, these features are the same as those for the atom in the Minkowski vacuum state in a flat spacetime, the total rate of change also displays quantitative difference which is embodied in the factor \(P(\omega_d, r)\).

A comparison of equation (69) with equation (23) in [7], which gives the rate of change of the mean atomic energy for an inertial atom in a flat space with a reflecting boundary, shows that the two rates are quite similar, and the appearance of \(P(\omega_d, r)\) in equation (69) can be understood as a result of backscattering of the vacuum electromagnetic field modes off the spacetime curvature in much the same way as the reflection of the field modes at the reflecting boundary in a flat spacetime. To further understand the effect of backscattering caused by the spacetime curvature, let us now analyze what occurs in two asymptotic regions, i.e. at the spatial infinity and at the event horizon, which are regions of particular physical interest to us.

At spatial infinity, i.e. \(r \rightarrow \infty\), a further simplification of the total rate of change by use of equations (59) and (60) yields

\[
\left\langle \frac{dH_A(\tau)}{d\tau} \right\rangle_{tot} \approx -\frac{e^2}{3\pi} \sum_{\omega_b > \omega_d} |\langle b|\tau(0)|d\rangle|^2 \omega_d^4 \left[ 1 + f(\omega_d, r) \right],
\]

(70)
in which
\[ f(\omega_{bd}, r) = \frac{3 \sum_l l(l + 1)(2l + 1) |T_l(\omega_{bd} \sqrt{g_{00}})|^2}{8r^4 \omega_{bd}^3} \]

is a gray-body factor that characterizes the backscattering of the electromagnetic field modes off the spacetime curvature. Note that at infinity, \( f(\omega_{bd}, r) \sim 0 \), so the rate of change reduces to that of an inertial atom in the Minkowski vacuum in flat spacetimes with no boundaries, suggesting that the Boulware vacuum at large radii is equivalent to the Minkowski vacuum.

When the atom is fixed near the event horizon, i.e. when \( r \to 2M \), further simplification gives
\[ \left\langle d\mathcal{H}_A(\tau) \right\rangle_{\text{tot}} \approx -\frac{e^2}{32\pi} \int_0^\infty d\omega \omega^2 \left( \sum_l l(l + 1)(2l + 1) |b^l(r(0)\omega)|^2 \omega_{bd}^4 \left[ \left( 1 + \frac{a^2}{\omega_{bd}^2} \right) + f(\omega_{bd}, r) \right] \right), \]

with
\[ a = \frac{M}{r^2 \sqrt{g_{00}}} = \frac{M}{r^2 \sqrt{1 - 2M/r}}. \]

where \( a \) is the proper acceleration of the static atom. Note the appearance of an extra term proportional to \( a^2 \) here in contrast to the scalar field case [9]. Let us note however that the presence of the terms proportional to the proper acceleration squared also appear in cases when the Unruh–DeWitt monopole detector is replaced by a dipole detector which couples to the derivatives of a scalar field [23, 24]. The proper acceleration \( a \) diverges as the event horizon is approached, so does the rate of change of the mean atomic energy. However, at infinity, where the spacetime is asymptotically flat, \( a \sim 0 \), so its contribution to the rate is negligible. This result is in sharp contrast to that of the scalar field case, where the rate of change is always finite [9]. Except for the gray-body factor, \( f(\omega_{bd}, r) \), here the result agrees with that of a co-accelerated atom with a proper acceleration \( a \) in interaction with fluctuating electromagnetic fields in the Rindler vacuum [25] (the case with the temperature of the thermal bath set to zero). The above discussions reveal that the Boulware vacuum is the vacuum state of static observers outside a black hole, and it resembles the Rindler vacuum in the flat spacetime since the static atom accelerates with the proper acceleration \( a \) with respect to locally free-falling observers.

**Unruh vacuum.** For a multilevel atom in interaction with quantum electromagnetic fluctuations in the Unruh vacuum, two statistical functions of the field are easily obtained from equation (52) as follows:
\[ C^F(x(\tau), x(\tau')) = \frac{1}{32\pi^2} \int_{-\infty}^{\infty} d\omega \omega \left( e^{i\omega x(\tau')} + e^{-i\omega x(\tau')} \right) \times \sum_l (2l + 1) |\tilde{R}_l(\omega)|^2 \right)^2 \right], \]

and
\[ \chi^F(x(\tau), x(\tau')) = \frac{1}{32\pi^2} \int_{-\infty}^{\infty} d\omega \omega \left( e^{i\omega x(\tau')} - e^{-i\omega x(\tau')} \right) \times \sum_l (2l + 1) |\tilde{R}_l(\omega)|^2 \right)^2 \right]. \]

Now the contributions of \( v_f \) and \( r_r \) to the rate of change of the mean atomic energy can be calculated by inserting the statistical functions into equations (21) and (22). For the contribution
of vacuum electromagnetic fluctuations, we have
\[
\frac{dH_A(t)}{dt}_{vf} = -\frac{e^2 g_{00}}{16\pi} \sum_{\omega_{bd} > \omega_b} |\langle b| r(0)|d\rangle|^2 \omega_{bd}^2 \\
	imes \left[ \left( 1 + \frac{1}{e^{2\pi\omega_{bd}/\kappa_r} - 1} \right) \widetilde{P}(\omega_{bd}, r) + \frac{\widetilde{P}(\omega_{bd}, r)}{e^{2\pi\omega_{bd}/\kappa_r} - 1} + \overline{P}(\omega_{bd}, r) \right] \\
- \sum_{\omega_{bd} < \omega_b} |\langle b| r(0)|d\rangle|^2 \omega_{bd}^2 \\
	imes \left[ \left( 1 + \frac{1}{e^{2\pi\omega_{bd}/\kappa_r} - 1} \right) \widetilde{P}(\omega_{bd}, r) + \frac{\widetilde{P}(\omega_{bd}, r)}{e^{2\pi\omega_{bd}/\kappa_r} - 1} + \overline{P}(\omega_{bd}, r) \right].
\]
(76)

where we have defined
\[
\kappa_r = \frac{\kappa}{\sqrt{1 - 2M/r}}.
\]
(77)

For the contribution of the rr,
\[
\frac{dH_A(t)}{dt}_{rr} = -\frac{e^2 g_{00}}{16\pi} \sum_{\omega_{bd} > \omega_b} |\langle b| r(0)|d\rangle|^2 \omega_{bd}^2 \\
	imes \left[ \left( 1 + \frac{1}{e^{2\pi\omega_{bd}/\kappa_r} - 1} \right) \widetilde{P}(\omega_{bd}, r) - \frac{\widetilde{P}(\omega_{bd}, r)}{e^{2\pi\omega_{bd}/\kappa_r} - 1} + \overline{P}(\omega_{bd}, r) \right] \\
+ \sum_{\omega_{bd} < \omega_b} |\langle b| r(0)|d\rangle|^2 \omega_{bd}^2 \\
	imes \left[ \left( 1 + \frac{1}{e^{2\pi\omega_{bd}/\kappa_r} - 1} \right) \widetilde{P}(\omega_{bd}, r) - \frac{\widetilde{P}(\omega_{bd}, r)}{e^{2\pi\omega_{bd}/\kappa_r} - 1} + \overline{P}(\omega_{bd}, r) \right].
\]
(78)

Compared with the case of the Boulware vacuum, both the contributions of vacuum electromagnetic fluctuations and rr are altered by the appearance of a Planckian factor. Adding them up, we obtain the total rate of change of the mean atomic energy.
\[
\frac{dH_A(t)}{dt}_{tot} = -\frac{e^2 g_{00}}{8\pi} \sum_{\omega_{bd} > \omega_b} |\langle b| r(0)|d\rangle|^2 \omega_{bd}^2 \left[ 1 + \frac{1}{e^{2\pi\omega_{bd}/\kappa_r} - 1} \right] \overline{P}(\omega_{bd}, r) \\
+ \sum_{\omega_{bd} < \omega_b} |\langle b| r(0)|d\rangle|^2 \omega_{bd}^2 \overline{P}(\omega_{bd}, r).
\]
(79)

Now the delicate balance between \(vf\) and \(rr\) that ensures the stability of the atom in its ground state in the Boulware vacuum no longer exists. The \(\omega_{bd} < \omega_b\) term which gives a positive contribution to the total rate of change of the mean atomic energy makes the transition of the atom from the ground state to an excited state possible, i.e. excitation spontaneously occurs in the Unruh vacuum outside a black hole. The structure of the rate of change also suggests that there is thermal radiation from the black hole (represented by the Planckian term) which is backscattered by spacetime curvature (represented by \(\overline{P}\)). It is this thermal radiation that renders the spontaneous excitation possible. The temperature of the thermal radiation is given by
\[
T = \frac{\kappa_r}{2\pi} = \frac{\kappa}{2\pi \sqrt{1 - 2M/r}} = (g_{00})^{-1/2} T_H.
\]
(80)
with $T_H = \kappa/2\pi$ being the usual Hawking temperature of the black hole. This is actually the Tolman relation which gives the temperature felt by a local observer.

In order to gain more understanding, let us now examine the behavior of the rate of change in two asymptotic regions. First, when the atom is fixed near the event horizon, i.e. when $r \to 2M$, we can approximate, by use of equations (59) and (60), the rate of change as follows:

$$\left(\frac{dH_A(\tau)}{d\tau}\right)_{\text{tot}} \approx -\frac{e^2}{3\pi} \sum_{\omega_b < \omega_{d'}} |\langle br(0)|d\rangle|^2 \omega_{bd}' \left[ 1 + \frac{1}{e^{\omega_{bd}'/T} - 1} \right] \left( 1 + \frac{\alpha^2_{bd} + f(\omega_{bd}, r)}{\omega_{bd}^2} \right),$$

(81)

A distinct feature in contrast to the case of the scalar fields [9] is the existence of an extra term proportional to $a^2$, the proper acceleration squared. It is worth pointing out here that the appearance of such a term has also been found when one studies the spontaneous excitation of an accelerated multilevel atom coupled with electromagnetic $v_f$ in a flat spacetime [11]. Noteworthily, one can show that, close to the event horizon, $T \approx a/2\pi$ holds. This leads to a remarkable observation, that is, the $\omega_{bd} < \omega_{d}$ term, which makes the spontaneous excitation possible, can be viewed completely as a result of the Unruh effect due to the proper acceleration that must exist to hold the atom static (refer to equation (29) in [11]). This demonstrates a complete equivalence between the effect of acceleration and that of a gravitational field at the event horizon.

If the atom is placed far away from the black hole, i.e. $r \to \infty$, the total rate of change of the mean atomic energy becomes

$$\left(\frac{dH_A(\tau)}{d\tau}\right)_{\text{tot}} \approx -\frac{e^2}{3\pi} \sum_{\omega_b < \omega_{d'}} |\langle br(0)|d\rangle|^2 \omega_{bd}' \left[ 1 + \frac{1}{e^{\omega_{bd}'/T} - 1} \right] \left( 1 + \frac{\alpha^2_{bd} \frac{f(\omega_{bd}, r)}{2\pi(\omega_{bd}/\kappa) - 1}}{1/\kappa - 1} \right).$$

(82)

Note that thermal terms are all multiplied by a gray-body factor, $f(\omega_{bd}, r)$, which vanishes at spatial infinity. The above results are in accordance with the common belief that thermal flux emanates from the black hole horizon and is backscattered and partly depleted by the curved spacetime geometry on its way to infinity. So, as the atom is placed further and further away, the flux it feels becomes weaker and weaker.

**Hartle–Hawking vacuum.** Now let us look at the case in which the electromagnetic fields are in the Hartle–Hawking vacuum state. Two statistical functions of the field can be easily found from equation (53) to be

$$C^F(x(\tau), x'(\tau')) = \frac{1}{32\pi^2} \int_{-\infty}^{\infty} d\omega \alpha(\omega) \left( e^{-\frac{\omega d}{\kappa}} + e^{\frac{\omega d}{\kappa}} \right) \times \left[ \sum_{l}(2l+1) \left| \mathcal{R}_l(\omega|\alpha) \right|^2 \right. \left. \frac{1}{1 - e^{-2\pi\omega/\kappa}} + \sum_{l}(2l+1) \left| \mathcal{R}_l(\omega|\alpha) \right|^2 \right. \left. \frac{1}{e^{2\pi\omega/\kappa} - 1} \right],$$

(83)

and

$$\chi^F(x(\tau), x'(\tau')) = \frac{1}{32\pi^2} \int_{-\infty}^{\infty} d\omega \alpha(\omega) \left( e^{-\frac{\omega d}{\kappa}} - e^{\frac{\omega d}{\kappa}} \right) \times \left[ \sum_{l}(2l+1) \left| \mathcal{R}_l(\omega|\alpha) \right|^2 \right. \left. \frac{1}{1 - e^{-2\pi\omega/\kappa}} - \sum_{l}(2l+1) \left| \mathcal{R}_l(\omega|\alpha) \right|^2 \right. \left. \frac{1}{e^{2\pi\omega/\kappa} - 1} \right].$$

(84)
Inserting them into equations (21) and (22) yields the contributions of $v_f$ and $r_r$ to the rate of change of the mean atomic energy

$$
\frac{dH_A(\tau)}{d\tau}_{v_f} = \frac{e^2 g_{00}}{16\pi} \sum_{|b| > |a_d|} |\langle b | r(0) | d \rangle|^2 \omega_{ld}^2 \left[ \frac{P(-\omega_{bd}, r) + \bar{P}(\omega_{bd}, r)}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right]
$$

\begin{align*}
&+ \left( 1 + \frac{1}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right) \left( P(\omega_{bd}, r) + \bar{P}(-\omega_{bd}, r) \right) \\
&- \sum_{|b| < |a_d|} |\langle b | r(0) | d \rangle|^2 \omega_{ld}^2 \left[ \frac{P(-\omega_{bd}, r) + \bar{P}(\omega_{bd}, r)}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right] \\
&+ \left( 1 + \frac{1}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right) \left( P(-\omega_{bd}, r) + \bar{P}(\omega_{bd}, r) \right) \right],
\end{align*}

\begin{align*}
\text{and}
\frac{dH_A(\tau)}{d\tau}_{rr} &= \frac{e^2 g_{00}}{16\pi} \sum_{|b| > |a_d|} |\langle b | r(0) | d \rangle|^2 \omega_{ld}^2 \left[ \frac{P(-\omega_{bd}, r) + \bar{P}(\omega_{bd}, r)}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right] \\
&- \left( 1 + \frac{1}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right) \left( P(\omega_{bd}, r) + \bar{P}(-\omega_{bd}, r) \right) \\
&+ \sum_{|b| < |a_d|} |\langle b | r(0) | d \rangle|^2 \omega_{ld}^2 \left[ \frac{P(-\omega_{bd}, r) + \bar{P}(\omega_{bd}, r)}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right] \\
&- \left( 1 + \frac{1}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right) \left( P(-\omega_{bd}, r) + \bar{P}(\omega_{bd}, r) \right) \right].
\end{align*}

Then, the total rate of change readily follows

\begin{align*}
\frac{dH_A(\tau)}{d\tau}_{\text{tot}} &= \frac{e^2 g_{00}}{8\pi} \sum_{|b| > |a_d|} |\langle b | r(0) | d \rangle|^2 \omega_{ld}^2 \left( 1 + \frac{1}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right) \\
&\times \left( P(\omega_{bd}, r) + \bar{P}(-\omega_{bd}, r) \right) - \sum_{|b| < |a_d|} |\langle b | r(0) | d \rangle|^2 \omega_{ld}^2 \\
&\times \frac{1}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \left( P(\omega_{bd}, r) + \bar{P}(-\omega_{bd}, r) \right) \right].
\end{align*}

As in the Unruh vacuum case, the positive $\omega_{bd} < \omega_{ld}$ term also appears, and this term leads to spontaneous excitation of static atoms in the Hartle–Hawking vacuum in the exterior region of the black hole. Besides, the Planckian factor in the total rate of change is a revelation of the thermal nature of the Hartle–Hawking vacuum. To learn more, let us further study what happens in the asymptotic regions.

When the atom is fixed at infinity, i.e., $r \to \infty$, the rate of change of the mean atomic energy is

\begin{align*}
\frac{dH_A(\tau)}{d\tau}_{\text{tot}} \approx \frac{e^2}{3\pi} \sum_{|b| > |a_d|} |\langle b | r(0) | d \rangle|^2 \omega_{ld}^4 \left( 1 + \frac{1}{e^{2\pi \omega_{ld}/\kappa_s} - 1} \right) [1 + f(\omega_{bd}, r)] \\
- \sum_{|b| < |a_d|} |\langle b | r(0) | d \rangle|^2 \omega_{ld}^4 \frac{1}{e^{2\pi \omega_{ld}/\kappa_s} - 1} [1 + f(\omega_{bd}, r)]
\end{align*}

Furthermore, at spatial infinity where $f(\omega_{bd}, r)$ can be taken as zero, the total rate of change is what one would obtain when the atom is immersed in a thermal bath at the Hawking
temperature $T = T_H$ in a flat spacetime, while at the event horizon, i.e. $r \rightarrow 2M$,

$$\frac{dH_\lambda(\tau)}{d\tau}_{\text{tot}} \approx \frac{e^2}{3\pi} \left\{ \sum_{\omega_b > \omega_d} |\langle b|\tau(0)|d\rangle|^2 \omega_d^4 \left( 1 + \frac{1}{e^{2\pi\omega_d/\kappa} - 1} \right) \left[ (1 + \frac{a^2}{\omega_d^2}) + f(\omega_d, r) \right] \right. 
- \left. \sum_{\omega_b < \omega_d} |\langle b|\tau(0)|d\rangle|^2 \omega_d^4 \frac{1}{e^{2\pi|\omega_d|/\kappa} - 1} \left[ (1 + \frac{a^2}{\omega_d^2}) + f(\omega_d, r) \right] \right\},$$

it is divergent as $a \rightarrow \infty$. In addition to the contribution of the outgoing thermal radiation from the event horizon that exists in the Unruh vacuum (refer to equation (81)), there is another contribution to the total rate of change, the thermal term multiplied by $f(\omega_d, r)$, which can be regarded as resulting from the incoming thermal radiation from infinity. Both incoming and outgoing thermal radiation are backscattered off the spacetime curvature on their way. This result is consistent with our usual understanding that the Hartle–Hawking vacuum is not empty at infinity but corresponds instead to a thermal distribution of quanta at the Hawking temperature; thus, it describes a black hole in equilibrium with an infinite sea of black-body radiation.

A distinctive feature in contrast to the scalar field case is the existence of the term proportional to $a^2$ that is nontrivial at the event horizon and thus is physically important. Remarkably, equation (89) is in structural similarity to the total rate of change of the mean atomic energy of a uniformly accelerated atom interacting with electromagnetic field fluctuations in a flat space with a reflecting boundary (refer to equation (60)) [12], reflecting again that the scattering of the electromagnetic field modes off the spacetime curvature plays similar role as the reflection of the field modes at boundaries in a flat spacetime. This similarity is particularly striking at the event horizon, where $T \approx a/2\pi$.

5. Summary

In summary, using the Gupta–Bleuler quantization of free electromagnetic fields in a static spherically symmetric spacetime of arbitrary dimension in a modified Feynman gauge given by Crispino et al [13], we have defined, in analogy to the scalar field case, the Boulware, Unruh and Hartle–Hawking vacuum states outside a four-dimensional Schwarzschild black hole, calculated the two-point functions for the electromagnetic fields in these vacuum states and analyzed their properties in asymptotic regions. We then computed the contributions of vacuum fluctuations (vf) and radiation reaction (rr) to the total rate of change of the mean energy for a radially polarized static multilevel atom in interaction with quantum electromagnetic fluctuations in all the three vacuum states.

Our results show that the static atoms in the ground state in the Boulware vacuum are stable and this is in qualitative agreement with the case where the atom is assumed to be in interaction with quantized massless scalar fields [9]. However, the spontaneous emission rate of the excited atoms close to the horizon contains an extra term proportional to the squared proper acceleration of the atom in contrast to the scalar field case, and this rate is not well behaved at the event horizon as a result of the blow-up of the proper acceleration of the static atom with respect to the free-falling local observers there (note that this acceleration vanishes however at the spatial infinity). This is in sharp contrast to that of the scalar field case, where the rate of change of the mean atomic energy is always finite [9].

For the static atoms in both the Unruh and the Hartle–Hawking vacua, the delicate balance between the contributions of vf and rr that ensures the stability of the static atoms in ground state in the Boulware vacuum no longer exists, so spontaneous excitation occurs in the exterior region of the black hole. The spontaneous excitation rate of the static atoms is in accordance
with our usual understanding that the Unruh vacuum describes a black hole with thermal radiation emitting from its event horizon, whereas the Hartle–Hawking vacuum depicts a radiating black hole in equilibrium with an infinite sea of black-body radiation.

Distinctive features in contrast to the scalar field case are the existence of the term proportional to the proper acceleration squared in the rate of change of the mean atomic energy in the Unruh and the Hartle–Hawking vacua and the structural similarity in the spontaneous excitation rate between the static atoms outside a black hole and uniformly accelerated atoms interacting with electromagnetic field fluctuations in a flat space with a reflecting boundary, which is particularly dramatic at the event horizon where a complete equivalence exists.

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Appendix. On the asymptotic evaluation of mode sums

Here, we derive the properties of the quantity \( \sum_l (2l + 1) |\mathcal{R}_l(\omega; r)|^2 \) in two asymptotic regions. Let us start with the incoming modes \( \leftarrow \mathcal{R}_l(\omega; r) \), by examining, at a fixed radial distance \( r \), the correlation function of the field in the Boulware vacuum, equation (51)

\[
\langle 0 | \hat{E}_r(\tau) \hat{E}_r(\tau') | 0 \rangle_B = \frac{1}{16\pi^2} \int_0^\infty d\omega \sum_{l=1}^\infty (2l + 1) \left[ |\mathcal{R}_l(\omega; r)|^2 + |\mathcal{R}_l^\ast(\omega; r)|^2 \right] e^{-\frac{\omega r}{\sqrt{g_{00}}}} \\
\sim \frac{1}{\pi^2(\Delta t)^4}.
\]  

(A.1)

Here, \( \Delta t \) is the interval between the coordinate time and the approximation is taken at spatial infinity, where the spacetime is asymptotically flat. So, the two point function in the Boulware vacuum at large radii and that in the Minkowski vacuum should agree. At spatial infinity \( (r \to \infty) \), it can be deduced from equation (56) that

\[
\sum_l (2l + 1) |\mathcal{R}_l(\omega; r)|^2 \approx \sum_l l(l + 1)(2l + 1) |T_l(\omega)|^2 \omega^2 r^4.
\]  

(A.2)

This is very small as \( r \to \infty \), and thus can be neglected in equation (A.1). Using \( \int_0^\infty \omega^2 e^{-i\omega t} = \frac{\delta}{\omega} \), we obtain

\[
\sum_l (2l + 1) |\mathcal{R}_l(\omega; r)|^2 \approx \frac{8\omega^2}{3g_{00}}.
\]  

(A.3)

The summation over the outgoing modes, \( \mathcal{R}_l(\omega; r) \), in the region \( r \sim 2M \) can be obtained by solving the equation of the corresponding radial function as follows. Let \( \xi^2 = r/2M - 1 \) and \( q = 4M\omega \), the radial equation (55) can be simplified to be

\[
\xi^2 \frac{d^2 \varphi_l}{d\xi^2} + \xi \frac{d\varphi_l}{d\xi} + [q^2 - (2l\xi)^2] \varphi_l = 0.
\]  

(A.4)
we have approximated \( l(l + 1)\xi^2 \) by \((l\xi)^2\) since \( \xi \sim 0 \). The general solution of this equation can be expressed in terms of the modified Bessel functions as

\[
\varphi_l |_{r \to 2M} \sim a_l K_{l}(2l\xi) + b_l I_{l}(2l\xi) .
\]  
(A.5)

To estimate the coefficients \( a_l \) and \( b_l \), let us look at the radial equation, equation (55), in the limiting case, \( l \to \infty \). Now the effective potential in the equation is very large as compared to the other two terms for fixed \( r \) and \( \omega \); therefore, one can deduce that \( \varphi_l \sim 0 \) for large \( l \).

As a result, \( b_l \) is an exponentially small function of \( l \) when \( l \) is large, as \( I_{l}(2l\xi) \sim e^{2l\xi} / 2 l^{1/2} (l \gg 1) \). The second part in equation (A.5) therefore makes a bounded contribution to the summation of \( \sum (2l + 1) |\mathcal{R}_l(\omega)|^2 \) and it is negligible as the contribution of the first term is proportional to \( \xi^{-4} \). The coefficient \( a_l \) can then be determined by comparing the general solution, equation (A.5), with the asymptotic solution

\[
\varphi_{al}(r) \sim e^{i\omega r} + \mathcal{R}_l(\omega) e^{-i\omega r} ,
\]  
(A.6)

and the result is

\[
a_l \sim \frac{2l^{-i\omega} e^{i\omega/2}}{\Gamma(lq)} .
\]  
(A.7)

Thus, the summation to the leading order is

\[
\sum_l (2l + 1) |\mathcal{R}_l(\omega)|^2 \sim \frac{4}{\omega^2 r^4 \Gamma(lq) \Gamma(-iq)} \sum_l l(l + 1)(2l + 1) |K_{l}(2l\xi)|^2 
\approx \frac{8\alpha^2}{3g_{00}} + \frac{1}{6M^2 r^4} .
\]  
(A.8)

Here, we have appealed to the trick of approximating the infinite summation over \( l \) by an integral.

The summation over the outgoing modes, \( \mathcal{R}_l(\omega)|r \), at infinity can be easily deduced from the asymptotic solution of the outgoing mode, equation (56) as

\[
\sum_l (2l + 1) |\mathcal{R}_l(\omega)|^2 \approx \frac{\sum_l l(l + 1)(2l + 1) |J_l(\omega)|^2}{\omega^2 r^4} .
\]  
(A.9)

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