Abstract. In this paper, we show that the infinitesimal Torelli theorem implies the existence of deformations of automorphisms. In the first part, we use Hodge theory and deformation theory to study the deformations of automorphisms of complex projective manifolds. In the second part, we use crystalline cohomology to explore the $p$-adic analogues of the first part, which generalizes a result of Berthelot and Ogus. The study of the deformations of automorphisms also provides criterions characterizing when the action of the automorphism group of a variety on its cohomology is faithful.

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1. Introduction

The Hodge conjecture characterizes when a Betti cohomology class of a complex projective manifold can be represented by an algebraic cycle. Let $X_0$ be a complex projective manifold with a Betti cohomology class $Z_0$ of even degree, say $Z_0 \in H^{2d}(X_0, \mathbb{Q})$. The Betti cohomology class $Z_0$ is called a class of Hodge type if $(Z_0) \otimes \mathbb{C}$ is in $H^{2d}(X_0, \mathbb{C}) \cap H^{d}(X_0, \Omega^{d}_{X_0/\mathbb{C}})$. One expects that $Z_0$ is representable by an algebraic cycle of dimension $d$ if $Z_0$ is of Hodge type.

Let $\pi : X \to S$ be a smooth projective morphism over a complex quasi-projective variety $S$, and let $0$ be a point of $S$. Deligne [8] shows that, for a family of horizontal Betti cohomology classes $\{Z_t\}_{t \in S}$ of degree $2d$ on $X/S$, all the classes are of Hodge type if $Z_0$ is. Therefore, the Hodge conjecture predicts that $\{Z_t\}_{t \in S}$ are representable by algebraic cycles if $Z_0$ is. This prediction is what we call the variational Hodge conjecture, see [19 Conjecture 9.6].

Suppose that $S$ is smooth and connected. Assume that the Betti cohomology class $Z_0$ is representable by a local complete intersection $Z$ in $X_0$ of codimension $d$. Bloch [4] defines a semi-regularity map

$$s : H^1(Z, N_{Z/X_0}) \to H^{d+1}(X_0, \Omega^{d-1}_{X_0/\mathbb{C}})$$

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and shows that if $s$ is injective, then the Betti cohomology classes $\{Z_t\}_{t \in S}$ are representable by algebraic cycles.

In this paper, we show a similar result for automorphisms. Let
\[
\{h_t : H^m(X_t, \mathbb{Q}) \to H^m(X_t, \mathbb{Q})\}_{t \in S}
\]
be a continuous family of maps. Suppose that $h_0$ is $H^m(g_0, \mathbb{Q})$ for some automorphism $g_0$ of $X_0$ and $X_0$ satisfies the infinitesimal Torelli theorem of degree $m$, i.e., the cup product
\[
(1.1) \quad H^1(X_0, T_{X_0}) \to \bigoplus_{p+q=m} \text{Hom}(H^p(X_0, \Omega^q_{X_0/\mathbb{C}}), H^{p+1}(X_0, \Omega^{q-1}_{X_0/\mathbb{C}}))
\]
is injective. We show that there is a family of automorphisms $\{g_t : X_t \to X_t\}_{t \in U}$ over an open neighborhood $U$ of $0 \in S$ such that $h_t = H^m(g_t, \mathbb{Q})$. This motivates the following definition.

**Definition 1.1.** Let $\pi : X \to S$ be a smooth projective morphism over a complex quasi-projective variety $S$, and let $0$ be a point of $S$. For a non-negative integer $m$, an automorphism $g_0$ of $X_0$ is of $m$-Hodge type on $X/S$ if $H^m(g_0, \mathbb{Q})$ is the stalk
\[
h_0 : (R^m\pi_*\mathbb{Q})_0 \to (R^m\pi_*\mathbb{Q})_0
\]
of an endomorphism $h$ of the local system $R^m\pi_*\mathbb{Q}$ at the point 0 where $R^m\pi_*$ is the $m$-th derived functor of $\pi_*$. 

**Remark 1.2.** Let $\Omega^\bullet_{X/S}$ be the de Rham complex of $X/S$, and let $\Omega^\bullet_{X/S}^{\geq p}$ be the complex obtained from $\Omega^\bullet_{X/S}$ by replacing terms in degrees less than $p$ by 0. The Hodge filtration $F^p(R^m\pi_*\Omega^\bullet_{X/S})$ of $R^m\pi_*\Omega^\bullet_{X/S}$ is given by
\[
\text{Im} \left( R^m\pi_*\Omega^\bullet_{X/S} \to R^m\pi_*\Omega^\bullet_{X/S} \right),
\]
cf. [19, Definition 4.7]. By the results of Deligne [8], the map $h$ has the following remarkable property. Under the natural comparison
\[
R^m\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_S \cong R^m\pi_*\Omega^\bullet_{X/S},
\]
the map $h \otimes \mathcal{O}_S$ preserves the Hodge filtration of $R^m\pi_*(\Omega^\bullet_{X/S})$, namely,
\[
(1.2) \quad h \otimes \mathcal{O}_S \left( F^p R^m\pi_*(\Omega^\bullet_{X/S}) \right) \subseteq F^p R^m\pi_*(\Omega^\bullet_{X/S}) \quad \text{for all } p.
\]

**Example 1.3.** For an integer $m$, if $g_0$ is an automorphism of $X_0$ with $H^m(g_0, \mathbb{Q}) = \text{Id}_{H^m(\mathbb{Q})}$, then $g_0$ is of $m$-Hodge type on $X/S$ by taking
\[
h = \text{Id} : R^m\pi_*(\mathbb{Q}) \to R^m\pi_*(\mathbb{Q}).
\]

It is obvious that $g_0$ is of $m$-Hodge type on $X_U/U$ for all $m$ if $g_0$ has a deformation $g : X|_U \to X|_U$ over an open neighborhood $U$ of 0 in $S$ by taking $h = R^m\pi_*(g)$. It is easy to see that $h \otimes \mathcal{O}_S$ preserves the Hodge filtration of $R^m\pi_*(\Omega^\bullet_{X/U})$. Conversely, we have one of the main theorems of this paper.

**Theorem 1.4.** Let $\pi : X \to S$ be a smooth projective morphism from a complex variety $X$ to a smooth curve $S$, and let $g_0$ be an automorphism of the fiber $X_0$ of $\pi$ over a point $0$ of $S$. Suppose that $X_0$ satisfies the infinitesimal Torelli theorem of degree $m$ for some integer $m$. If the automorphism $g_0$ is of $m$-Hodge type on $X/S$, then $g_0$ has a deformation
\[
g : X|_U \to X|_U
\]
over an open neighborhood $U$ of $0 \in S$.

One interesting application of Theorem 1.4 is Corollary 3.2 characterizing when the action of the automorphism group of a variety on its cohomology is faithful. Historically speaking, the faithfulness question of the action is first explored for algebraic curves of genus at least 2 by Klein. The case of K3 surfaces is confirmed by Burns, Rapoport, Shafarevich, Ogus, among others. However, few higher dimensional varieties are known to have a positive answer to this question. Using Corollary 3.2 one can show that the automorphism groups of higher dimensional varieties (e.g. complete intersections, see [6] for details) act on their cohomology faithfully as long as they satisfy the infinitesimal Torelli theorem and the general members of these varieties have no non-trivial automorphisms.

The second part of this paper is to show the $p$-adic analogue of Theorem 1.4. Recall from [19] Definition 4.7 that the $q$-th de Rham cohomology $H_{DR}^q(X/B)$, for a smooth proper morphism $g : X \to B$ of noetherian schemes is defined as the coherent $\mathcal{O}_B$-module $\mathbb{R}^q g_* \Omega_{X/B}^\bullet$. The Hodge filtration of $H_{DR}^q(X/B)$ is given by

$$F^p(\mathbb{R}^q g_* \Omega_{X/B}^\bullet) := \text{Im} \left( \mathbb{R}^q g_* \Omega_{X/B}^{\geq p} \to \mathbb{R}^q g_* \Omega_{X/B}^\bullet \right).$$

With these notions, the $p$-adic analogue of the variational Hodge conjecture can be formulated as follows. Let $k$ be an algebraically closed field of positive characteristic, and let $W(k)$ be the Witt ring of $k$ with the fraction field $K$. Suppose that $\pi' : X \to W(k)$ is a smooth proper morphism with the closed fiber $X_0$. Let $H_{\text{cris}}^\bullet(X_0/W(k))$ be the crystalline cohomology of $X_0$. Based on the connection between crystalline cohomology and Hodge theory, Bloch, Esnault, and Kerz [5] Conjecture 1.2], Maulik and Poonen [19] Conjecture 9.2], Fontaine andMessing propose the following conjecture.

**Conjecture.** The rational crystalline cycle class of an algebraic cycle $Z_0 \in H_{\text{cris}}^2(X_0/W(k))_K$, expressed as a de Rham class on $X_K/K$ under the natural comparison $H_{\text{cris}}^2(X_0/W(k))_K \cong H_{DR}^2(X_K/K)$, is the cycle class of an algebraic cycle on $X/K$, if and only if $Z_0$ is in the right level of the Hodge filtration $F^p H_{DR}^q(X_K/K)$.

Inspired by the conjecture above, it is natural to ask for the $p$-adic analogue of Theorem 1.4. The natural way to formulate the $p$-adic analogue is to replace the morphism $\pi : X \to S$ in Theorem 1.4 by the morphism $\pi' : X \to W(k)$ over $W(k)$. In this case, $X_0$ should be the closed fiber of $\pi'$. Furthermore, by Hodge theory, a notable feature of $X/S$ is that its Hodge-de Rham spectral sequence degenerates at $E_1$ with locally free terms. Therefore, we introduce the following definition.

**Definition 1.5.** Let $k$ be an algebraically closed field. A $W(k)$-scheme $X$ over the Witt ring $W(k)$ of $k$ is of Hodge type if the structure morphism $\pi' : X \to W(k)$ is smooth and projective such that the Hodge-de Rham spectral sequence of $\pi'$ degenerates at $E_1$ and the terms are locally free.

Motivated by 1.2 in Remark 1.2 we make the following analogous definition.

**Definition 1.6.** Let $k$ be an algebraically closed field, and let $\pi' : X \to \text{Spec } W(k)$ be a smooth proper morphism over the Witt ring $W(k)$. For a non-negative integer $m$, an automorphism $g_0$ of the closed fiber $X_0$ of $\pi'$ is of $m$-Hodge type on $X/W(k)$.
if the map
\[ H_{\text{cris}}^m(g_0) : H_{\text{cris}}^m(X_0/W) \to H_{\text{cris}}^m(X_0/W) \]
preserves the Hodge filtrations under the natural identification \( H_{\text{cris}}^m(X_0/W) \cong H_{\text{DR}}^m(X/W) \).

Now, we are able to state the \( p \)-adic analogue of Theorem 1.4.

**Theorem 1.7.** Let \( k \) be an algebraically closed field, and let \( X \) be a \( W(k) \)-scheme of Hodge type with the closed fiber \( X_0 \). Suppose that \( X_0 \) satisfies the infinitesimal Torelli theorem of degree \( m \) for some integer \( m \), i.e., the cup product
\[
(1.3) \quad H^1(X_0, T_{X_0}) \to \bigoplus_{p+q=m} \text{Hom}(H^p(X_0, \Omega^p_{X_0/k}), H^{q+1}(X_0, \Omega^{q-1}_{X_0/k}))
\]
is injective. If \( g_0 \) is an automorphism of \( X_0 \) of \( m \)-Hodge type on \( X/W(k) \), then one can lift \( g_0 \) to an automorphism \( g : X \to X \) over \( W(k) \).

Theorem 1.7 is predicted by the \( p \)-adic variational Hodge conjecture, see [11, Remark 6.5]. Ogus [22, Corollary 2.5] proves a special case of this theorem for automorphisms of K3 surfaces. Later on, Berthelot and Ogus [2, Theorem 3.15] prove a special case of the theorem for abelian varieties.

To use Theorem 1.7, we have to verify that \( X_0 \) satisfies the infinitesimal Torelli theorem. In fact, it is verified for complete intersections and cyclic coverings in [6] and [20]. Therefore, it follows from Theorem 1.7 that, for many smooth projective varieties over an arbitrary field, their automorphism groups act faithfully on their étale cohomology, see Corollaries 3.2 and 5.4. For the applications of the faithfulness of the automorphism group action on the cohomology to arithmetic and moduli problems, we refer to [23], [6], [20], [15], and [16].

To end this introduction, we outline the main ingredients of the proofs of Theorem 1.4 and Theorem 1.7. Theorem 1.4 involves some techniques of homological algebra, deformation theory and the theory of de Rham cohomology. The key step of the proof is to use the Riemann-Hilbert correspondence to translate a morphism between local systems into a morphism between vector bundles with integrable connections, see Section 3. To prove Theorem 1.7, this kind of translation is replaced by the rigidity of crystalline cohomology, see Sections 4 and 5.

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2. Homological Algebra and de Rham Cohomology

In this section, we prove some results of homological algebra. We also cite some results of the theory of de Rham cohomology. They will be used in Section 3 to prove Theorem 1.4.
Lemma 2.1. Suppose that we have the following exact sequences and diagram

\[
\begin{array}{c}
0 \rightarrow M \rightarrow C \rightarrow C_0 \rightarrow 0 \\
0 \rightarrow M \rightarrow B \rightarrow B_0 \rightarrow 0 \\
0 \rightarrow M \rightarrow A \rightarrow A_0 \rightarrow 0
\end{array}
\]

in an abelian category. There is a morphism \( C \rightarrow B \) filling in the diagram such that the whole diagram commutes.

Proof. Note that the exact sequence \( 0 \rightarrow M \rightarrow B \rightarrow B_0 \rightarrow 0 \) is the pull-back of \( 0 \rightarrow M \rightarrow A \rightarrow A_0 \rightarrow 0 \) via the morphism \( B_0 \rightarrow A_0 \) and the exact sequence \( 0 \rightarrow M \rightarrow C \rightarrow C_0 \rightarrow 0 \) is the pull-back of \( 0 \rightarrow M \rightarrow A \rightarrow A_0 \rightarrow 0 \) via the morphism \( C_0 \rightarrow A_0 \). The lemma follows from this remark. \( \Box \)

Lemma 2.2. Suppose that we have exact sequences and an diagram (in an abelian category) as follows.

\[
\begin{array}{c}
0 \rightarrow M_0 \rightarrow B_0 \rightarrow A_0 \rightarrow 0 \\
0 \rightarrow M'_0 \rightarrow B'_0 \rightarrow A_0 \rightarrow 0 \\
0 \rightarrow M_1 \rightarrow B_1 \rightarrow A_1 \rightarrow 0 \\
0 \rightarrow M'_1 \rightarrow B'_1 \rightarrow A_1 \rightarrow 0
\end{array}
\]

Assume that the morphisms \( M_0 \rightarrow M'_0 \) and \( M_1 \rightarrow M'_1 \) are monic. There is a morphism \( B_0 \rightarrow B_1 \) filling in the diagram such that the whole diagram commutes.

Proof. From the commutative diagram (2.1), we have a commutative diagram as follows
It induces a morphism \( B_0 \rightarrow B_1 \) filling into the diagram (2.1) such that the whole diagram commutes.

Let \( A \) be a \( \mathbb{C} \)-algebra with an ideal \( I \), and let \( \pi : X \rightarrow S \) be a smooth projective morphism where \( S = \text{Spec}(A) \). Denote by \( S_0 \) the affine scheme \( \text{Spec}(A_0) \) where \( A_0 = A/I \). Suppose that \( X_0 \) is the pull back of \( X/S \) along \( S_0 \hookrightarrow S \). Denote the natural immersion \( X_0 \hookrightarrow X \) by \( i \).

\[
\begin{array}{ccc}
X_0 & \hookrightarrow & X \\
\pi_0 & \downarrow & \pi \\
S_0 & \hookrightarrow & S
\end{array}
\]

Let \( K_{X/S/C} \in \text{Ext}^1_X(\Omega^1_{X/S}, \Omega^1_{S/C} \otimes \mathcal{O}_X) \) be the Kodaira-Spencer class of \( X/S \), i.e., the class of the extension

\[
0 \rightarrow \Omega^1_{S/C} \otimes \mathcal{O}_X \rightarrow \Omega^1_{X/C} \rightarrow \Omega^1_{X/S} \rightarrow 0.
\]

Let \( \beta \in \text{Ext}^1_{X_0}(\Omega^1_{X_0/S_0}, \Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) \) be the class of the extension \( K_{X/S/C} \otimes \mathcal{O}_{S_0} \)

\[
(2.2) \quad 0 \rightarrow \Omega^1_{S/C} \otimes \mathcal{O}_{X_0} \rightarrow \Omega^1_{X/C}|_{X_0} \rightarrow \Omega^1_{X_0/S_0} \rightarrow 0.
\]

Recall that \( H^q_{\text{DR}}(X/S) \) is the \( q \)-th de Rham cohomology given by \( R^q\pi_* (\Omega^\bullet_{X/S}) \) where \( \Omega^\bullet_{X/S} \) is the de Rham complex of \( X/S \) and \( R^q\pi_* \) is the \( q \)-th derived functor of \( \pi_* \).

We recall the following propositions.

**Proposition 2.3.** ([4, Theorem 3.2] and [7]) Let \( S \) be a scheme over \( \text{Spec}(\mathbb{Q}) \), and let \( \pi : X \rightarrow S \) be a proper and smooth morphism. Then

1. The sheaves \( R^q\pi_* (\Omega^p_{X/S}) \) are locally free of finite type and commute with base change.
2. The spectral sequence

\[
E^{p,q}_1 = R^q\pi_* (\Omega^p_{X/S}) \Rightarrow H^{p+q}_{\text{DR}}(X/S)
\]

degenerates at \( E_1 \).
3. The sheaves \( H^q_{\text{DR}} \) are locally free of finite type and commute with base change.

Let \( S \) be a smooth \( \mathbb{C} \)-scheme. There is a canonical integrable connection, namely, the Gauss-Manin connection

\[
\nabla : H^q_{\text{DR}}(X/S) \rightarrow H^q_{\text{DR}}(X/S) \otimes \mathcal{O}_S \Omega^1_{S/C}.
\]

The spectral sequence in Proposition 2.3 (2) induces a (Hodge) filtration

\[
0 \subseteq F^q \subseteq F^{q-1} \cdots \subseteq F^1 \subseteq F^0 = H^q_{\text{DR}}(X/S)
\]

such that

- \( F^0, F^1, \ldots, F^q \) are locally free \( \mathcal{O}_S \)-module,
- and the Griffiths’s Transversality \( \nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_{S/C} \).

**Proposition 2.4.** ([4, Proposition 3.6] and [12]) With the same notations as above, the Gauss-Manin connection \( \nabla \) is related to the Kodaira-Spencer class \( K_{X/S/C} \) by
the following commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
F^p / F^{p+1} \quad \xrightarrow{\nabla} \quad F^{p-1} / F^p \otimes \Omega^1_{S/C} \\
R^q - p \pi_* (\Omega^p_{X/S}) \quad \xrightarrow{\cup K_{X/S/C}} \quad R^q - p + 1 \pi_* (\Omega^p_{X/S} - 1) \otimes \Omega^1_{S/C}
\end{array}
\end{array}
\]

Note that we have the following proposition about integrable connections and stratifications.

**Proposition 2.5.** ([4, Proposition 3.7, Proposition 3.8, Remark 3.9])

1. Let \( k \) be a field of characteristic 0, \( M \) a finite \( k[[t_1, \ldots, t_r]] \)-module with integrable connection \( \nabla \). Let \( M^\nabla = \ker(\nabla) \). Then \( M = M^\nabla \otimes k[[t_1, \ldots, t_r]] \).
2. Let \( A \) be a complete, local, augmented \( \mathbb{C} \)-algebra (e.g. \( A \) artinian), \( S = \text{Spec}(A) \), and \( X_0 \subseteq X \) be the closed fiber. Then

\[
H^\ast_{\text{DR}}(X/S) \cong H^\ast(X_0, \mathbb{C}) \otimes \mathbb{C} A.
\]

It gives a stratification on \( H^\ast_{\text{DR}}(X/S) \). Cohomology classes of the form \( c \otimes 1, c \in H^\ast(X_0, \mathbb{C}) \)

are said to be horizontal.

3. **Deformations of Automorphisms for Algebraic Manifolds**

In this section, we use the notations as in Section 2. Let \( g_0 \) be an automorphism of \( X_0 \) over \( S_0 = \text{Spec}(A_0) \).

**Lemma 3.1.** Let \( A \) be a \( \mathbb{C} \)-algebra with square zero ideal \( \mathcal{I} \), and let \( A_0 = A/\mathcal{I} \). Denote by \( S \) (resp. \( S_0 = \text{Spec}(A_0) \)) for the definition of \( \beta \). Then

\[
g_0^* \beta = \beta,
\]

the automorphism \( g_0 \) is unobstructed, i.e., one can extend \( g_0 \) to an automorphism \( g : X/A \to X/A \) over \( \text{Spec}(A) \).

**Proof.** To extend \( g_0 \) to a morphism \( g \) over \( \text{Spec}(A) \), it suffices to find a morphism \( h : \mathcal{O}_X \to (g_0^{-1})_*(\mathcal{O}_X) \) of sheaves of rings such that it fills into the following diagram

\[
\begin{array}{cccccc}
0 & \xrightarrow{\mathcal{I}\mathcal{O}_X} & \mathcal{O}_X & \xrightarrow{i_*\mathcal{O}_X} & 0 \\
0 & \xrightarrow{\mathcal{I}(g_0^{-1})_*\mathcal{O}_X} & (g_0^{-1})_*\mathcal{O}_X & \xrightarrow{(g_0^{-1})_*i_*\mathcal{O}_X} & 0
\end{array}
\]

where we abuse \( (g_0^{-1})_* \) to represent \( i_* \circ (g_0^{-1})_* \) and \( i \) is the natural inclusion \( X_0 \hookrightarrow X \).

In fact, we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \xrightarrow{\mathcal{I}\mathcal{O}_X} & \mathcal{O}_X & \xrightarrow{i_*\mathcal{O}_X} & 0 \\
0 & \xrightarrow{\mathcal{I}(g_0^{-1})_*\mathcal{O}_X} & (g_0^{-1})_*\mathcal{O}_X & \xrightarrow{(g_0^{-1})_*i_*\mathcal{O}_X} & 0
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \xrightarrow{\mathcal{I}\mathcal{O}_X(= \mathcal{I} \otimes \mathcal{O}_{X_0})} & \mathcal{O}_X & \xrightarrow{i_*\mathcal{O}_X} & 0 \\
0 & \xrightarrow{\mathcal{I}(g_0^{-1})_*\mathcal{O}_X(= \mathcal{I} \otimes \mathcal{O}_{X_0})} & (g_0^{-1})_*\mathcal{O}_X & \xrightarrow{(g_0^{-1})_*i_*\mathcal{O}_X} & 0
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \xrightarrow{\mathcal{I}\mathcal{O}_X} & \mathcal{O}_X & \xrightarrow{i_*\mathcal{O}_X} & 0 \\
0 & \xrightarrow{\mathcal{I}(g_0^{-1})_*\mathcal{O}_X} & (g_0^{-1})_*\mathcal{O}_X & \xrightarrow{(g_0^{-1})_*i_*\mathcal{O}_X} & 0
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \xrightarrow{i_*\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}} & i_*\Omega^1_{X_0} & \xrightarrow{i_*d} & i_*\Omega^1_{X_0/S_0} & \xrightarrow{0}
\end{array}
\]
where $\pi$ is the quotient $\Omega^1_{X/S} \to \Omega^1_{X/C}|_{X_0}$. We denote the first exact sequence in the previous diagram by $\alpha$. Recall that $\beta$ (see Section 2) is the class of the extension

$$0 \to \Omega^1_{S/C} \otimes \mathcal{O}_{X_0} \to \Omega^1_{X/C}|_{X_0} \to \Omega^1_{X_0/S_0} \to 0.$$ 

We have that

$$(i_*(d))^* (i_*(\beta)) = (d \otimes 1)_*(\alpha).$$

We hope it will cause no confusion if we also denote $\beta$ by $i_*(\beta)$. It gives rise to the following diagram

(3.2)

$$0 \to \mathcal{I} \mathcal{O}_X \to \mathcal{O}_X \to i_* \mathcal{O}_{X_0} \to 0$$

$$(0 \to i_* (\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) \to \mathcal{B} \to i_* \mathcal{O}_{X_0} \to 0)$$

$$(0 \to i_* (\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) \to \iota_* (\Omega^1_{X/C}|_{X_0}) \to \iota_* \Omega^1_{X_0/S_0} \to 0)$$

where the middle short exact sequence is represented by $(i_*(d))^*(\beta)$. Note that $g_0^* = (g_0^{-1})^*$, $g_0^* = (g_0^{-1})_*$, and the bottom of (3.2) consists of $\mathcal{O}_{X_0}$-modules. We can pull back the diagram above via $g_0^*$. It gives rise to a commutative diagramm as (3.2).

(3.3)

$$0 \to \mathcal{I} (g_0^{-1})_* \mathcal{O}_X \to (g_0^{-1})_* \mathcal{O}_X \to (g_0^{-1})_* \mathcal{O}_{X_0} \to 0$$

$$0 \to g_0^* (\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) \to D \to (g_0^{-1})_* \mathcal{O}_{X_0} \to 0$$

$$0 \to g_0^* (\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) \to (i \circ g_0)_* (\Omega^1_{X/C}|_{X_0}) \to g_0^* \Omega^1_{X_0/S_0} \to 0$$

$$0 \to i_* (\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) \to C \to i_* (\Omega^1_{X_0/S_0}) \to 0$$

where $X_0$ is $X_0 \xrightarrow{\text{res}} X_0 \xrightarrow{i} X$ and we abuse $g_0^*$ to represent $i_* \circ g_0^*$. Note that the pull-back $g_0^*$ is given by

$$\text{Ext}^1 (\Omega^1_{X_0/S_0}, \Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) \to \text{Ext}^1 (g_0^* \Omega^1_{X_0/S_0}, g_0^* (\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}))$$

$$g_0^* \to \text{Ext}^1 (\Omega^1_{X_0/S_0}, \Omega^1_{S/C} \otimes \mathcal{O}_{X_0})$$
where the vertical identification follows from the differential map
\[ dg_0 : g_0^*Ω^1_{X_0/S_0} \xrightarrow{\sim} \Omega^1_{X_0/S_0}. \]

The assumption \( g_0^*\beta = \beta \) follows that the exact sequence at the bottom of (3.3) is the exact sequence at the bottom of (3.2). It follows from Lemma 2.1 that there is a map \( u : B \to D \) filling into the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{i_*} (\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) & B \\
\downarrow & & \downarrow u \\
0 & \xrightarrow{g_0^*} (\Omega^1_{S/C} \otimes \mathcal{O}_{X_0}) & D \\
\end{array}
\]

such that the whole diagram commutes.

Recall that \( d : \mathcal{I} \to \Omega^1_{S/C} \otimes \mathcal{O}_{S_0} \) is injective and \( X \to S \) is smooth. It implies that the maps \( d \otimes 1 \) in (3.2) and (3.3) are injective. It follows from Lemma 2.2 and (3.4) that there is a map \( h : \mathcal{O}_X \to (g_0^{-1})_*\mathcal{O}_X \) filling into the diagram (3.1) above such that the whole diagram commutes. We have proved the lemma. \( \square \)

Now we are able to prove Theorem 1.4.

**Proof.** We translate the assumption of the theorem in terms of the Gauss-Manin connection as follows, cf. Proposition 2.3, Proposition 2.4 and Proposition 2.5. By the Riemann-Hilbert correspondence, the horizontal map \( h \) in Definition 1.1 (cf. 1.2) induces the following commutative diagram

\[
\begin{array}{ccc}
R^{m-p}π_* (\Omega^p_{X/S}) & \xrightarrow{\nabla} & R^{m-p+1}π_* (\Omega^p_{X/S} \otimes \Omega^1_{S/C}) \\
\downarrow h_C & & \downarrow h_C \otimes \text{id}_{Ω^1_{S/C}} \\
R^{m-p}π_* (\Omega^p_{X/S}) & \xrightarrow{\nabla} & R^{m-p+1}π_* (\Omega^p_{X/S} \otimes \Omega^1_{S/C})
\end{array}
\]

where \( \nabla \) is given by the cup product of \( K_{X/S/C} \). We first show the theorem when \( S = \text{Spec}(\mathbb{C}[[t]]) \).

Let \( S_N \) be \( \text{Spec}(\mathbb{C}[[t]]/(t)^{N+1}) \). Note that the maps
\[
d : (t)^N/(t)^{N+1} \to Ω^1_{S_N/C} \otimes \mathcal{O}_{S_N-1}
\]
are injective, cf. [4] Theorem 7.1. Denote by \( X_N \) the pull-back \( X \times_S S_N \). Assume that \( g_0 \) can extend to an automorphism \( g_N \) over \( S_N \). We base change to \( S_N \) via \( S_N \subseteq S \). We have the Kodaira-Spencer class \( K_{X_{N+1}/S_{N+1}/C} \) and

\[
\beta = K_{X_{N+1}/S_{N+1}/C} \otimes \mathcal{O}_{S_N} \text{ with } \nabla(-) = (-) \cup \beta,
\]

cf. [4] (4.1), Proposition 2.4 and Proposition 2.5. Note that
\[
Ω^1_{S/C} = \mathbb{C}[t]/(t)^{N+1}dt = S_Ndt = Ω^1_{S/C} \otimes \mathcal{O}_S S_N
\]
and \( h_C \otimes S_N \) is given by \( g_N : H^{m-p}(X_N, Ω^p_{X_N/S_N}) \to H^{m-p}(X_N, Ω^p_{X_N/S_N}) \). Therefore, the commutativity of the diagram (3.5) \( (h_C \nabla = \nabla h_C) \) gives that
\[
g_N^*(-) \cup \beta = g_N^*(- \cup \beta) \text{ in } H^{m-p+1}(X_N, Ω^p_{X_N/S_N}) \otimes Ω^1_{S_{N+1}/C},
\]
cf. [4, Proposition 4.2]. It follows that \( g_N^*(-) \cup \beta = g_N^*(-) \cup g_N^*(\beta) \).

We claim that the element 
\[
g_N^*(\beta) - \beta \in \text{Ext}^1(\Omega^{1}_{X_N/S_N}, \Omega^{1}_{S_{N+1}/C} \otimes \mathcal{O}_{X_N})
\]
is zero. If the claim holds, then it follows from Lemma 3.1 that \( g_N \) is liftable to an automorphism \( g_{N+1} : X_{N+1} \to X_{N+1} \) over \( S_{N+1} \). Therefore, by the Grothendieck existence theorem, we have an automorphism \( \hat{g} : X \to X \) over \( S \) such that \( \hat{g}|_{X_0} = g_0 \).

Note that \( \Omega^1_{S_{N+1}/C} = \mathbb{C}[t]/(t)^{N+1}dt = S_Ndt \). Therefore, we have 
\[
\Omega^1_{S_{N+1}/C} \otimes \mathcal{O}_{X_N} = \mathcal{O}_{X_N}.
\]

To show the claim, it suffices to show the cup product
\[
\text{(3.6)} \quad \text{Ext}^1(\Omega^{1}_{X_N/S_N}, \Omega^{1}_{S_{N+1}/C} \otimes \mathcal{O}_{X_N}) \to \bigoplus_{p+q=m} \text{Hom}(H^p(X_N, \Omega^q_{X_N/S_N}), H^{p+1}(X_N, \Omega^{q-1}_{X_N/S_N} \otimes \Omega^{1}_{S_{N+1}/C} \otimes \mathcal{O}_{X_N}))
\]
is injective. We show the injectivity of the cup product by the induction on the length of \( S_N \).

Firstly, we define two functors as follows:
\[
T(M) = \text{Ext}^1_{\mathcal{O}_{X_N}}(\Omega^{1}_{X_N/S_N}, M)
\]
and
\[
S(M) = \bigoplus_{p+q=m} \text{Hom}(H^p(X_N, \Omega^q_{X_N/S_N}), H^{p+1}(X_N, \Omega^{q-1}_{X_N/S_N} \otimes M))
\]
from the category of \( \mathcal{O}_{X_N} \)-modules to the category of \( S_N \)-modules. The cup product is a natural transformation between these two functors
\[
\cup_M : T(M) \to S(M).
\]
The injectivity of the vertical arrow \((3.6)\) is the same as the injectivity of
\[
\cup_{\mathcal{O}_{X_N}} : T(\mathcal{O}_{X_N}) \to S(\mathcal{O}_{X_N}).
\]
We consider the following exact sequence
\[
0 \to (t)^N \otimes \mathcal{O}_{X_N} (= (t)^N \otimes_{\mathbb{C}} \mathcal{O}_{X_0}) \to \mathcal{O}_{X_N} \to \mathcal{O}_{X_N-1} \to 0.
\]
By the base change theorem [17, Chapter III, Section 12], we have the following commutative diagram of exact sequences
\[
\begin{array}{c}
(t^N) \otimes_{\mathbb{C}} T(\mathcal{O}_{X_0}) \longrightarrow T(\mathcal{O}_{X_N}) \longrightarrow T(\mathcal{O}_{X_N-1}) \\
\downarrow (t^N) \otimes_{\mathbb{C}} \bigcirc \cup \downarrow \bigcirc \cup \\
0 \longrightarrow (t^N) \otimes_{\mathbb{C}} S(\mathcal{O}_{X_0}) \longrightarrow S(\mathcal{O}_{X_N}) \longrightarrow S(\mathcal{O}_{X_N-1}) \longrightarrow 0.
\end{array}
\]
The first vertical arrow \( (t^N) \otimes_{\mathcal{O}} \) is injective by the assumption of the theorem that \( X_0 \) satisfies the infinitesimal Torelli theorem of degree \( m \). So the cup product \( \cup_{X_0} \) is injective by the induction. It implies the theorem in the case \( S = \text{Spec}(\mathbb{C}[[t]]) \).

The general case follows from the case when \( S = \text{Spec}(\mathbb{C}[[t]]) \) by base change along \( \text{Spec}(\mathcal{O}_{S,0}) = \text{Spec}(\mathbb{C}[[t]]) \). □

Apply Theorem 1.3 with \( h = \text{Id}_{R^m \pi_* Q} \) in Definition 1.1. We obtain the following corollary.

**Corollary 3.2.** Suppose that a smooth projective \( X_0 \) is a fiber of a smooth family of projective variety \( \pi : X \rightarrow S \). Assume that the cup product

\[
H^1(X_0, T_{X_0}) \rightarrow \bigoplus_{p+q=m} \text{Hom}(H^p(X_0, \Omega^q_{X_0}), H^{p+1}(X_0, \Omega^{q-1}_{X_0}))
\]

is injective for some \( m \). Then the kernel of \( \text{Aut}(X_s) \rightarrow \text{Aut}(H^m(X_s, \mathbb{Q})) \) is trivial for all \( s \in S \) if the kernels are trivial over an open dense subset of \( S \).

**Proof.** Let \( f_0 \) be an automorphism of \( X_0 \) in

\[
\text{Ker}_0 : (\text{Aut}(X_0) \rightarrow \text{Aut}(H^m(X_0, \mathbb{Q}))).
\]

Definition 1.1 can be verified by taking \( h = \text{Id}_{R^m \pi_* (\mathbb{Q})} \). Therefore, by Theorem 1.3, \( f_0 \) can be deformed to an automorphism \( f_s \in \text{Ker}_s \) of \( X_s \) where \( s \in S \) is a general point. By the assumption of the corollary, we conclude that \( f_0 \) is a specialization of identities, therefore, \( f_0 = \text{Id}_{X_0} \). □

**Remark 3.3.** One can use Corollary 3.2 to show that the automorphism groups of higher dimensional varieties (e.g. complete intersections) act on their cohomology faithfully as long as they satisfy the infinitesimal Torelli theorem and the general members of these varieties have no non-trivial automorphisms.

4. **Crystalline Cohomology and Obstructions**

In this section, we use the notations following 1 and 2. We will carry out a proof of Theorem 1.7 through the rest of the paper. Suppose that \( S \rightarrow T \) is a closed immersion of affine schemes with the square zero ideal sheaf \( \mathcal{I} \) and the prime \( p \) is nilpotent on \( T \). The pair \( (T, \mathcal{I}) \) has a natural P.D structure such that \( \mathcal{I}^a = 0 \) if \( a \geq 2 \). Suppose that \( X \) and \( Y \) are smooth projective schemes over \( T \) with reductions \( X_0 \) and \( Y_0 \) over \( S \). We have two inclusions

\[
i : Y_0 \hookrightarrow Y \quad \text{and} \quad j : X_0 \hookrightarrow X.
\]

Let \( f_0 : X_0 \rightarrow Y_0 \) be an isomorphism between \( X_0 \) and \( Y_0 \). It gives rise to a map

\[
H^k_{\text{cris}}(f_0/T) : H^k_{\text{cris}}(Y_0/T) \rightarrow H^k_{\text{cris}}(X_0/T).
\]

By the comparison theorem of crystalline cohomology and de Rham cohomology, we can view this map as \( H^k_{\text{cris}}(f_0) : H^k_{\text{DR}}(Y/T) \rightarrow H^k_{\text{DR}}(X/T) \).

If we restrict \( S \) in Theorem 1.4 to a small disk \( \Delta \) with center 0, then the local system \( R^m \pi_* Q|_{\Delta} \) is trivial and the the map \( H^m(g_0) \) induces the horizontal map

\[
h_\Delta : R^m \pi_* Q|_{\Delta} \rightarrow R^m \pi_* Q|_{\Delta}
\]

(cf. Definition 1.4) with the stalk \( h_0 = H^m(g_0) \). In the case of mixed characteristic, the map \( H^k_{\text{cris}}(f_0) \) replaces the role of the horizontal morphism \( h_\Delta \).
Proposition 4.1. With the notations and assumptions in 4.1, let

\[ \text{an isomorphism between } X \rho \text{.} \]

The following proposition relates extending \( f \)

\[ \text{Since } H^k \text{ is zero. The map } H^k \]

\[ \text{(4.2)} \]

\[ \text{is nilpotent on } T. \text{ Let } S \text{ be the closed subscheme of } T \text{ defined by } I. \text{ Let } X \text{ (resp. } Y) \text{ be a smooth projective scheme over } T \text{ with reductions } X_0 \text{ (resp. } Y_0) \text{ over } S. \]

Suppose that the Hodge-de Rham spectral sequences of \( X/T \) and \( Y/T \) degenerate at \( E_1 \) and the terms are locally free, so that the Hodge and de Rham cohomology sheaves commute with base change.

We denote \( F_X \) (resp. \( F_Y \)) the Hodge filtration \( (F_{Hdg}) \) for \( X/T \) (resp. \( Y/T \)). Since \( H^k_{\text{cris}}(f_0) \otimes \text{Id}_S = H^k_{\text{DR}}(f_0) \) preserves the Hodge filtrations, the map

\[ F^p_Y H^k_{\text{DR}}(Y/T) \rightarrow g r^p_{F_X} H^k_{\text{DR}}(X/T) \otimes O_S \]

is zero. The map \( H^k_{\text{cris}}(f_0) \) induces a map:

\[ F^p_Y H^k_{\text{DR}}(Y/T) \rightarrow g r^p_{F_X} H^k_{\text{DR}}(X/T) \otimes I = H^{k-p+1}(X, \Omega^{p-1}_{X/T}) \otimes I \]

which factors through

\[ \rho(f_0)_p : F^p_Y H^k_{\text{DR}}(Y_0/S) \rightarrow H^{k-p+1}(X_0, \Omega^{p-1}_{X_0/S}) \otimes I. \]

We denote \( \rho(f_0)_p \) by \( \rho(f_0) \) for simplicity. On the other hand, the obstruction \( ob(f_0) \) of extending \( f_0 \) to a \( T \)-morphism \( X \rightarrow Y \) is an element of \( \text{Ext}^1_{X_0}(f_0, \Omega^1_{Y_0/S}, \Omega_{X_0/S}) \). The following proposition relates \( \rho(f_0) \) and \( ob(f_0) \).

**Proposition 4.1.** With the notations and assumptions in 4.1, let \( f_0 : X_0 \rightarrow Y_0 \) be an isomorphism between \( X_0 \) and \( Y_0 \). We have the following commutative diagram

\[ \begin{array}{ccc}
F^p_Y H^k_{\text{DR}}(Y_0/S) & \xrightarrow{\rho(f_0)} & g r^p_{F_X} H^k_{\text{DR}}(X_0/S) \otimes I \\
\downarrow \text{proj} & & \downarrow \text{proj} \\
H^{k-p}(Y_0, \Omega^p_{Y_0/S}) & \xrightarrow{\pm ob(f_0) \cup} & H^{k-p+1}(X_0, \Omega^{p-1}_{X_0/S}) \otimes I.
\end{array} \]

For \( p = 1 \), Proposition [4.1] is [3 Proposition 3.20]. The idea of the proof of Proposition [4.1] is similar to the proof of [3 Proposition 3.20], but the argument for the case of higher degrees \( p \geq 2 \) is much more complicated. The proof of Proposition [4.1] depends on the concrete descriptions of the maps of \( \rho(f_0) \) and \( ob(f_0) \). We give these concrete descriptions in the rest of this section and present a proof of Proposition [4.1] in Section 5.

Let us cite some results from [2]. Recall that there are short exact sequences as follows ([2 Chapter 5, 5.2 (3)]):

\[ 0 \rightarrow J_{X_0/T} \rightarrow O_{X_0/T} \rightarrow i_{X_0/T*}(O_{X_0}) \rightarrow 0 \]

and

\[ 0 \rightarrow J_{X/T} \rightarrow O_{X/T} \rightarrow i_{X/T*}(O_X) \rightarrow 0. \]
There is a natural morphism of topoi:

\[ u_{X/T} : (X/T)_{\text{cris}} \to X_{\text{zar}}, \]

given by

\[(u_{X/T,*}(\mathcal{F}))(U) = \Gamma((U/T)_{\text{cris}}, j_{\text{cris}}^* \mathcal{F})\]

where \( \mathcal{F} \in (X/T)_{\text{cris}} \) and \( j : U \to X \) is an open immersion, cf. [2, Proposition 5.18].

**Lemma 4.2.** Let \( \mathcal{J} \) be the quotient of \( j_{\text{cris}}(J_{X_0/T}') \) by \( J_{X/T}' \). We have a short exact sequence

\[ 0 \to J_{X/T}' \to j_{\text{cris}}(J_{X_0/T}') \xrightarrow{Q} \mathcal{J} \to 0. \]

Then (in the derived category)

\[ Ru_{X/T,*}(\mathcal{J}) = I \otimes \Omega^{p-1}_{X/T}[−p+1]. \]

**Proof.** By [21, Theorem 2.1], [1, Proposition 3.4.1], \((X_0)_{\text{zar}} = X_{\text{zar}}\) and \( I^2 = 0\), we have that (in the derived category)

\[(4.4) \quad Ru_{X/T,*}(J_{X/T}') \xrightarrow{\cong} Ru_{X/T,*}(J_{X_0/T}') \xrightarrow{\cong} Ru_{X/T,*}\mathcal{J} \xrightarrow{+1} \]

\[ 0 \to F_{X_0}^p \Omega^\bullet_{X/T} \xrightarrow{\cong} F_{X_0}^p \Omega^\bullet_{X/T} \xrightarrow{h} I \otimes \Omega^{p-1}_{X/T}[−p+1] \to 0 \]

where

\[ F_{X_0}^p \Omega^\bullet_{X/T} = \left( \ldots \to 0 \to \Omega^p_{X/T} \to \Omega^{p+1}_{X/T} \to \ldots \right) \]

and

\[(4.5) \quad F_{X_0}^p \Omega^\bullet_{X/T} = \sum_{a+b=p} \mathcal{I}^a \Omega^b_{X/T} = \left( \ldots \to 0 \to \mathcal{I} \Omega^{p-1}_{X/T} \to \Omega^p_{X/T} \to \Omega^{p+1}_{X/T} \to \ldots \right), \]

see [21, Theorem 2.1] and [2, Theorem 7.2]. Since a derived category is a triangulated category, the induced isomorphism \( h \) between the distinguished triangles is an isomorphism. We have proved the lemma.

**Lemma 4.3.** With the notations as in Lemma 4.2, we have

\[ H^k_{\text{cris}}(X, J_{X/T}') \cong H^k(X, \Omega^\bullet_{X/T}) \cong F_{X}^p H^k_{\text{DR}}(X/T) \]

**Proof.** The lemma follows from [2, 7.2.1] and our assumption [1,1].

**Description of \( \rho(f_0) \).** In this subsection, we define a map

\[ \theta : Ru_{Y/T,*}(J_{Y/T}') \to Rf_{0,*}(I \otimes \Omega^{p-1}_{X/T}[−p+1]) \]

which induces the map \( \rho(f_0) \).
First of all, we make a map $\rho$ as follows. Recall that we have a map $j_{0,cris}^{-1}J_{Y_0/T}^{[p]} \to J_{X_0/T}$ (see [1] Chapter III). The map $\rho$ is given as follows:

$$
\begin{array}{c}
Ru_{Y/T,*}J_{Y/T}^{[p]} \xrightarrow{\rho} Ru_{Y/T,*}i_{cris,*}J_{Y_0/T}^{[p]} \\
\downarrow \qquad \downarrow \\
izar RF_{0,*}RuX_{0,T,*}J_{X_0/T}^{[p]} \xleftarrow{\psi} izar RF_{0,*}J_{Y_0/T}^{[p]}
\end{array}
$$

where the map $\psi$ is induced by

$$
Ru_{Y_0,T,*}J_{Y_0/T}^{[p]} \to Ru_{Y_0,T,*}RF_{0,cris,*}(J_{X_0/T}^{[p]}) \cong RF_{0,zar,*}RuX_{0,T,*}(J_{X_0/T}^{[p]}).
$$

The quotient $Q$ in Lemma [4.2] gives rise to a map (4.6)

$$
\hat{\rho} = i_* RF_{0,*}(Ru_{X/T,*}(Q)) : i_* RF_{0,*}[Ru_{X/T,*}(j_{*}J_{X_0/T}^{[p]})] \to i_* RF_{0,*}(Ru_{X/T,*}(J)).
$$

We define $\theta := \hat{\rho} \circ \rho$. In other words, the map

$$
\theta : Ru_{Y/T,*}J_{Y/T}^{[p]} \to RF_{0,*}(\mathcal{I} \otimes \Omega_{X/T}^{p-1}[−p + 1])
$$

is defined as

$$
\begin{array}{c}
Ru_{Y/T,*}J_{Y/T}^{[p]} \xrightarrow{\rho} i_* RF_{0,*}RuX_{0,T,*}(J_{X_0/T}^{[p]}) \xrightarrow{i_* RF_{0,*}j_* RuX_{0,T,*}(J_{X_0/T}^{[p]})} \\
i_* RF_{0,*}[Ru_{X/T,*}(j_{*}J_{X_0/T}^{[p]})] \xrightarrow{\hat{\rho}} i_* RF_{0,*}(Ru_{X/T,*}(J)) \xrightarrow{n} RF_{0,*}(\mathcal{I} \otimes \Omega_{X/T}^{p-1}[−p + 1])
\end{array}
$$

where the identity in the first row follows from $(X_0)_{zar} = X_{zar}$ and the last identity in the second row follows from Lemma [4.2] and $(Y_0)_{zar} = Y_{zar}$. Furthermore, the following diagram is commutative, cf. [3] the proof of Proposition 3.20, page 184:

$$
\begin{array}{c}
H^k(Y/T, J_{Y/T}^{[p]}) \xrightarrow{F} H^k_{DR}(Y/T) \xrightarrow{\rho(f_0)} H^k_{DR}(X_0/S) \otimes \mathcal{I} \\
\downarrow \quad \downarrow \quad \downarrow \\
H^k(Y/T, i_* J_{Y_0/T}^{[p]}) \xrightarrow{G_0} H^k(X/T, j_* J_{X_0/T}^{[p]}) \xrightarrow{G_1} H^k(X, \Omega_{X/T}^{p-1} \otimes \mathcal{I}) \\
\downarrow \quad \downarrow \quad \downarrow \\
H^k(Y/T, \mathcal{O}_{Y/T}) \xrightarrow{h_{cris}(f_0)} H^k(X/T, \mathcal{O}_{X/T}) \xrightarrow{G_1} H^{k-1}(X, \Omega_{X/T}^{p-1})
\end{array}
$$

where the identity in the first row is due to Lemma [4.3], the map $G_0$ is induced by the map $Q$ in Lemma [4.2] and the vertical map $G_1$ is induced by

$$
j_{cris,*}(J_{X_0/T}^{[p]}) \mapsto j_{cris,*}\mathcal{O}_{X_0/T} \cong \mathcal{O}_{X/T}.
$$

The diagram (4.7) makes it clear that $\rho(f_0)$ is induced by the local map $\theta$.

On the other hand, we have the following diagram:
With the notations as above, we have

Lemma 4.4.

The second row of the diagram gives rise to a map

(4.9) \( R_{Y/T_\ast} \left( \mathcal{J}_{Y/T}^{[p]} / \mathcal{J}_{Y/T}^{[p+1]} \right) \rightarrow R_f_{0\ast} \left( \mathcal{I} / \mathcal{I}^{[2]} \otimes \Omega_{X/T}^{p-1}[-p+1] \right). \)

Let us describe the map \( \Psi \) more precisely. The map \( \Psi \) is given by

\[
Ru_{Y/T_\ast} \left( \mathcal{J}_{Y/T}^{[p]} / \mathcal{J}_{Y/T}^{[p+1]} \right) \xrightarrow{\psi} Ru_{X_0/T_\ast} \left( \mathcal{J}_{X_0/T}^{[p]} / \mathcal{J}_{X_0/T}^{[p+1]} \right) \rightarrow R_f_{0\ast} \left( \mathcal{I} / \mathcal{I}^{[2]} \otimes \Omega_{X/T}^{p-1}[-p+1] \right).
\]

where the existence of the first arrow in the first row follows from the fact that \( i_{criss} \) is exact (\textit{\cite{2} Proposition 6.2}), the identities in the first and the second rows follow from \textit{\cite{1} Proposition 3.4.1}, the arrow \( \phi \) is induced by the morphism

\( \mathcal{J}_{Y_0/T} \rightarrow f_{0criss} \mathcal{J}_{X_0/T} \).

Lemma 4.4. With the notations as above, we have

\( Ru_{Y/T_\ast} \left( \mathcal{J}_{Y/T}^{[p]} / \mathcal{J}_{Y/T}^{[p+1]} \right) \cong \Omega_{Y/T}^{p}[-p] \).

Proof. By the proof of the filtered Poincaré lemma \textit{\cite{2} 6.13 and 7.2}, we have (in the derived category)

\[
\begin{array}{ccc}
0 & \rightarrow & F_Y^{p+1} \Omega_{Y/T}^{\bullet} \\
\simeq & \rightarrow & \rightarrow \\
0 & \rightarrow & F_Y^{p} \Omega_{Y/T}^{\bullet}
\end{array}
\]

where the map \( h \) is an isomorphism. We have proved the lemma. \( \square \)

By Lemma 4.4, the bottom arrow \( \text{(4.9)} \) of the diagram \( \text{(4.8)} \) induces a morphism

(4.10) \( \widetilde{\rho(f_0)} : \Omega_{Y/T}^{p}[-p] \cong Ru_{Y/T_\ast} \left( \mathcal{J}_{Y/T}^{[p]} / \mathcal{J}_{Y/T}^{[p+1]} \right) \rightarrow R_f_{0\ast} \left( \mathcal{I} / \mathcal{I}^{[2]} \otimes \Omega_{X/T}^{p-1}[-p+1] \right). \)

The morphism \( \text{(4.10)} \) gives rise to an “adjoint” morphism (in the derived category) and we denote it by \( \rho(f_0) \) as well:

(4.11) \( \rho(f_0) : f_0^\ast \Omega_{Y_0/S}^{p}[-p] \rightarrow \Omega_{X_0/S}^{p-1}[-p+1] \otimes \mathcal{I} / \mathcal{I}^{[2]} = \Omega_{X_0/S}^{p}[-p] \otimes \mathcal{I}. \)

This "is" an element of

\[ H^0 \mathcal{R}Hom \left( f_0^\ast \Omega_{Y_0/S}^{p}[-p], \Omega_{X_0/S}^{p-1}[-p+1] \otimes \mathcal{I} / \mathcal{I}^{[2]} \right) = \mathcal{E}xt_{\mathcal{O}_{X_0}} \left( f_0^\ast \Omega_{Y_0/S}^{p}, \Omega_{X_0/S}^{p-1} \otimes \mathcal{I} / \mathcal{I}^{[2]} \right). \]
4.2. De Rham description of $\rho(f_0)$. In this subsection, we describe the map $\rho(f_0)$ in (4.11) in terms of de Rham theory. Using the graph of $f_0$, we have an immersion

$$X_0 \xrightarrow{(j,i \circ f_0)} X \times_T Y.$$ 

Let $D$ be the PD envelope of $X_0$ in $X \times_T Y$ and denote by $\pi_D : D \to X \times_T Y$. Let $\mathcal{J}$ be the ideal of $X_0$ in $D$. It follows from [2, Theorem 7.2] that

$$Ru_{X_0/T} \mathcal{J}_X^k \cong F^*_X \Omega^1_D \cong \left( \mathcal{J}^{[k]} \to \mathcal{J}^{[k-1]} \Omega^1_{D/T} \to \mathcal{J}^{[k-2]} \Omega^2_{D/T} \to \ldots \right)$$

where $\Omega^*_D$ is the de Rham complex of $D/T$ induced by the natural connection

$$\nabla : \mathcal{O}_D \to \mathcal{O}_D \otimes \Omega^1_{X \times_T Y} = \Omega^1_{D/T}$$

of $\mathcal{O}_D$, cf. [2, Exercise 6.4 and Theorem 7.1]. The $s$-th term of $\Omega^*_D$ is given by $\mathcal{O}_D \otimes \Omega^s_{X \times_T Y}$.

**Lemma 4.5.** With the same notations as above, we have that

$$gr_{F_{X_0}} (\Omega^*_D) = (\mathcal{J}^{[p]} / \mathcal{J}^{[p+1]} \xrightarrow{d} \mathcal{J}^{[p-1]} / \mathcal{J}^{[p]} \xrightarrow{d} \ldots \mathcal{O}_{X_0} \otimes \Omega^p_{X \times_T Y})$$

$$gr_{F_{X_0}} (\Omega^s_{X \times T Y}) = (\mathcal{I}^{[p]} / \mathcal{I}^{[p+1]} \xrightarrow{d} \mathcal{I}^{[p-1]} / \mathcal{I}^{[p]} \xrightarrow{d} \ldots \mathcal{O}_{X_0} \otimes \Omega^p_{X_0/S})$$

where the first terms of both complexes are of degree zero and the differentials follow the rule [1, Page 238 (1.3.6)] (2, Exercise 6.4)).

**Proof.** It follows from the proof of the filtered Poincaré Lemma [2, (6.13) and (7.2)] or [1, Chapter V, 2]. See (4.3). □

**Remark 4.6.** Let $J$ be the ideal sheaf of $X_0$ in $X \times_T Y$. It follows from [2, Remark 3.20 3)] that $J$ is mapped into $\mathcal{J}$. Therefore, we have $J \mathcal{J} \subseteq \mathcal{J}^{s+1}$. It follows that all the terms of $gr_{F_{X_0}} (\Omega^*_D)$ in Lemma 4.5 are $\mathcal{O}_{X_0}$-module in a natural way, i.e., the $s$-th component is

$$gr_{F_{X_0}} (\Omega^*_D) = (\mathcal{J}^{[p-s]} / \mathcal{J}^{[p-s+1]} \Omega^s_{X \times_T Y} = \mathcal{J}^{[p-s]} / \mathcal{J}^{[p-s+1]} \Omega^s_{X_0 \times Y_0/S | X_0}.$$

Furthermore, we have a diagram as follows

$$
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & D \\
& \xrightarrow{\pi_Y} & \xrightarrow{\pi_X} X \\
& \xrightarrow{\pi_D} & \xrightarrow{\Pr_X} Y \times_T Y
\end{array}
$$

where $\Pr_X$ and $\Pr_Y$ are the natural projections. It follows from the construction of the natural connection $\nabla$ (2, Corollary 6.3 and Exercise 6.4)) that there is a natural commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{X \times_T Y} & \xrightarrow{d} & \mathcal{O}^1_{X \times_T Y/T} \\
\pi_D & & \\
\mathcal{O}_D & \xrightarrow{\nabla} & \mathcal{O}_D \otimes \Omega^1_{X \times_T Y/T}
\end{array}$$
which induces a map \( \pi_Y^* : \Omega^\bullet_{X \times Y/T} \to \Omega^\bullet_{D/T} \). Therefore, we have a commutative diagram

\[
\begin{array}{ccc}
Pr_Y^* \Omega^1_{Y/T} & \xrightarrow{\pi_Y^*} & \Omega^1_{X \times Y/T} \\
\downarrow{\pi_D^*} & & \downarrow{\pi_X^*} \\
\Omega^1_{D/T} & \xrightarrow{\pi_D^*} & \Omega_X^1/T
\end{array}
\]

The map (4.11)

\[ H^k_{\text{cris}}(f_0/T) : H^k_{\text{cris}}(Y_0/T) \to H^k_{\text{cris}}(X_0/T) \]

can be considered as “topological extension” of \( H^k_{\text{cris}}(f_0/S) \). The map \( f_0 \) does not necessarily have a “real” extension \( f : X \to Y \). However, we can use \( \pi_Y \) to calculate \( H^k_{\text{cris}}(f_0/T) \) by [2] Remark 7.5 [and [1] Chapter V, Lemma 2.3.3, Corollary 2.3.4]. Namely, the map \( H^k_{\text{cris}}(f_0/T) \) is given by

\[ H^k_{\text{cris}}(Y_0/T) = H^k(Y, \Omega^\bullet_{Y/T}) \xrightarrow{\pi_Y^*} H^k(X \times_T Y, \Omega^\bullet_{D/T}) = H^k_{\text{cris}}(X_0/T) \]

Moreover, the map \( \pi_X^* \) induces a filtered quasi-isomorphism

\[ (\Omega^\bullet_{X/T}, F_{X_0}) \xrightarrow{\cong} (\Omega^\bullet_{D/T}, F_{X_0}) \]

which is also denoted by \( \pi_X^* \), see [2] Theorem 7.2.2 and Remark 7.5 [and [1] Chapter V, Corollary 2.3.5]. Recall from the last subsection that the map \( \rho(f_0) \) is induced by the map \( \theta \). Therefore, by Remark 4.7 and the diagrams (4.7), (4.8) and (4.4), the morphism \( \rho(f_0) \) (4.11) is given by (in the derived category):

(4.13)

\[
\begin{array}{ccc}
f_0^\bullet \Omega^p_{Y_0/S}[-p] & \xrightarrow{\pi_Y^*} & gr^p_{X_0} \Omega^\bullet_{D/T} \\
\rho(f_0) & \downarrow & \pi_X^* \xrightarrow{\cong} \rho(f_0) \\
\Omega^p_{X_0/S} \otimes T/T[2][-p + 1] & \xrightarrow{\text{proj}} & \Omega^\bullet_{X/T}
\end{array}
\]

where \( \text{proj} \) is the natural map following from Lemma 4.3.

**Remark 4.7.** To let the maps in (4.13) be more explicit, we make the following remarks.

- The map \( \text{proj} \) is given by

\[ gr^p_{F_{X_0}} (\Omega^\bullet_{X/T}) = (\cdots \to T/T[2] \otimes \Omega^p_{X_0/S} \to \cdots) \to T/T[2] \otimes \Omega^p_{X_0/S}, \]

cf. Lemma 4.3. By the diagram in the proof of Lemma 4.2, the map \( \text{proj} \) is induced by \( Ru_{X/T}(Q) \) where \( Q \) is the quotient in Lemma 4.2.

- By Lemma 4.3, it is clear that \( \pi_Y^* : f_0^\bullet \Omega^p_{Y_0/S}[-p] \to gr^p_{F_{X_0}} \Omega^\bullet_{D/T} \) maps \( u \in f_0^\bullet \Omega^p_{Y_0/S} \) to the element \((0,0,0,u,0)\) in

\[ \Omega^p_{X_0/S} \otimes (\Omega^{p-1}_{X_0/S} \otimes f_0^\bullet \Omega^1_{Y_0/S}) \oplus (\Omega^1_{X_0/S} \otimes f_0^\bullet \Omega^{p-1}_{Y_0/S}) \oplus f_0^\bullet \Omega^p_{Y_0/S} \oplus (\cdots) \]

\[ = \Omega^p_{X_0 \times Y_0/S}(x_0) \xrightarrow{\rho_{F_{X_0}}(\Omega^\bullet_{D/T})} p. \]
4.3. Description of cup product $\text{ob}(f_0) \cup -$. Note that we have a natural injection

$$\text{incl} : \Omega^p_{Y_0/S} \hookrightarrow \Omega^1_{Y_0/S} \otimes \Omega^{p-1}_{Y_0/S},$$

associating to $dx_1 \wedge \ldots \wedge dx_p$ the element

$$\sum_{i=1}^{p} (-1)^i dx_i \otimes dx_1 \wedge \ldots \wedge \hat{dx}_i \wedge \ldots \wedge dx_p.$$

Given an element in

$$\text{Ext}^1_{\mathcal{O}_{X_0}}(f_0^* \Omega^1_{Y_0/S}, \mathcal{O}_{X_0} \otimes \mathcal{I}/\mathcal{I}^2),$$

say the obstruction element $\text{ob}(f_0)$ of the map $f_0$ with respect to $S \hookrightarrow T$, the cup product of this element induces a morphism (in the derived category) from $f_0^* \Omega^p_{Y_0/S}$ to $\Omega^{p-1}_{X_0/S} \otimes \mathcal{I}/\mathcal{I}^2[1]$ via the inclusion (4.14). Denote the cup product (in the derived category) by

$$\text{ob}(f_0) \cup : f_0^* \Omega^p_{Y_0/S} \to \Omega^{p-1}_{X_0/S} \otimes \mathcal{I}/\mathcal{I}^2[1] = \Omega^{p-1}_{X_0/S} \otimes \mathcal{I}[1].$$

In the following, we describe the map $\text{ob}(f_0) \cup$. Let $I_0$ be the ideal of $X_0$ in $X \times_T Y$, and let $I_1$ be the ideal of $X_0$ in the $X_0 \times_S Y_0$ (via the graph of $f_0$). There is an exact sequence of $\mathcal{O}_{X_0}$-modules:

$$0 \to \mathcal{I} \otimes \mathcal{O}_{X_0} \to \mathcal{I}_0/\mathcal{I}^2_0 \to \mathcal{I}_1/\mathcal{I}^2_1 \to 0,$$

see [21, (2.7)]. This extension corresponds to the obstruction element

$$\text{ob}(f_0) \in \text{Ext}^1_{\mathcal{O}_{X_0}}(I_1/I_1^2, \mathcal{I} \otimes \mathcal{O}_{X}).$$

We identify $I_1/I_1^2$ with $f_0^* \Omega^1_{Y_0/S}$, which gives the exact sequence [3, Page 186]

$$0 \to \mathcal{O}_{X_0} \otimes \mathcal{I}/\mathcal{I}^2 \to \mathcal{J}/\mathcal{J}^2[t] \to f_0^* \Omega^1_{Y_0/S} \to 0.$$

Let $A^\bullet$ be the two terms complex:

$$\mathcal{J}/\mathcal{J}^2[t] \to I_1/I_1^2 = f_0^* \Omega^1_{Y_0/S},$$

where the first term is of degree zero. In particular, there is a quasi-isomorphism as follows

$$w : \mathcal{O}_{X_0} \otimes \mathcal{I}/\mathcal{I}^2[t] \longrightarrow A^\bullet.$$

It gives rise to an element

$$\overline{\text{ob}(f_0)} \in \text{Ext}^1(f_0^* \Omega^1_{Y_0/S}, \mathcal{I}/\mathcal{I}^2[t] \otimes \mathcal{O}_{X_0})$$

as follows.

$$\mathcal{O}_{X_0} \otimes \mathcal{I}/\mathcal{I}^2[t][1] \longrightarrow A^\bullet \cong A^\bullet$$

We also have a natural map

$$\text{gr}^1_{\mathcal{O}_{X_0} \otimes \mathcal{I}/\mathcal{I}^2[t]} \to A^\bullet.$$
between the complexes,

\[
\begin{array}{c}
\text{gr}^1_{\mathcal{F}_{X_0}} \Omega^{{\bullet}\mathcal{D}/T} \\
\downarrow A^\bullet \\
\text{gr}^1_{\mathcal{F}_{X_0}} \Omega^{{\bullet}\mathcal{D}/T}
\end{array}
\Rightarrow
\begin{array}{c}
\mathcal{J}/\mathcal{J}^2 \\
 \downarrow \phi \\
\mathcal{J}/\mathcal{J}^2
\end{array}
\Rightarrow
\begin{array}{c}
\Omega^1_{\mathcal{D}/T} \\
\downarrow f_0^* \Omega^1_{Y_0/S}
\end{array}
\]

where \(\Omega^1_{\mathcal{D}/T}\) is \(\mathcal{O}_D \otimes \Omega^1_{X \times Y/T}\) by definition (see [2, Chapter 7]) and \(\phi\) is the natural projection

\[
\Omega^1_{\mathcal{D}/T} \mathcal{J}/\mathcal{J}^2 \Omega^1_{\mathcal{D}/T} \left( = \Omega^1_{X_0/S} \oplus f_0^* \Omega^1_{Y_0/S} \right) \rightarrow f_0^* \Omega^1_{Y_0/S}.
\]

Recall that there is a natural quasi-isomorphism: \(\text{gr}^1_{\mathcal{F}_{X_0}} \Omega^{{\bullet}\mathcal{D}/T} \simeq \text{gr}^1_{\mathcal{F}_{X_0}} \Omega^{{\bullet}\mathcal{X}/T}\).

It gives rise to a commutative diagram as follows.

\[
\begin{array}{c}
f_0^* \Omega^1_{Y_0/S}[-1] \\
\downarrow ob(f_0) \\
\mathcal{O}_{X_0} \otimes \mathcal{I}/\mathcal{I}^2
\end{array}
\Rightarrow
\begin{array}{c}
gr^1_{\mathcal{F}_{X_0}} \Omega^{{\bullet}\mathcal{D}/T} \\
\downarrow \simeq \\
gr^1_{\mathcal{F}_{X_0}} \Omega^{{\bullet}\mathcal{X}/T}
\end{array}
\Rightarrow
\begin{array}{c}
f_0^* \Omega^1_{Y_0/S} \\
\downarrow \simeq \\
f_0^* \Omega^1_{Y_0/S}
\end{array}
\]

It gives rise to the morphism \(ob(f_0)\cup \mathcal{I}/\mathcal{I}^2\) as the composition of the following maps:

\[
\begin{array}{c}
f_0^* \Omega^p_{Y_0/S}[-1] \\
\downarrow \text{qis} \\
\mathcal{I}/\mathcal{I}^2 \otimes f_0^* \Omega^{p-1}_{Y_0/S} \otimes \mathcal{I}/\mathcal{I}^2 \otimes \Omega^{p-1}_{X_0/S} \\
\downarrow \text{Id} \otimes f_0^* \\
f_0^* \Omega^p_{Y_0/S} \otimes f_0^* \Omega^{p-1}_{Y_0/S} \otimes f_0^* \Omega^{p-1}_{Y_0/S}
\end{array}
\]

where \(f_0^* : f_0^* \Omega^{p-1}_{Y_0/S} \rightarrow \Omega^{p-1}_{X_0/S}\) is the natural differential induced by

\[
df_0 : f_0^* \Omega^1_{Y_0/S} \rightarrow \Omega^1_{X_0/S}.
\]

The first arrow \(f_0^* \Omega^p_{Y_0/S}[-1] \rightarrow gr^1_{\mathcal{F}_{X_0}} \Omega^{{\bullet}\mathcal{D}/T} \otimes f_0^* \Omega^{p-1}_{Y_0/S}\) is given by

\[
f_0^* \Omega^p_{Y_0/S} \xrightarrow{f_0^* (\text{incl})} f_0^* \Omega^1_{Y_0/S} \otimes f_0^* \Omega^{p-1}_{Y_0/S} \xrightarrow{h} \Omega^1_{X_0/S} \otimes f_0^* \Omega^{p-1}_{Y_0/S} \oplus f_0^* \Omega^1_{Y_0/S} \otimes f_0^* \Omega^{p-1}_{Y_0/S}
\]

where the term at the bottom is the degree-one term of the complex

\[
gr^1_{\mathcal{F}_{X_0}} \mathcal{I}/\mathcal{I}^2 \otimes f_0^* \Omega^{p-1}_{Y_0/S}
\]

and

\[
h = (f_0^* \otimes \text{Id}) f_0^* \Omega^{p-1}_{Y_0/S} \otimes f_0^* \Omega^{p-1}_{Y_0/S}.
\]
The map (4.20) induces the cup product map
\[ \text{ob}(f_0) \cup - : H^{k-p}(Y_0, \Omega^p_{X_0/S}) = H^{k-p}(X_0, f_0^* \Omega^p_{Y_0/S}) \to H^{k-p+1}(X_0, \Omega^{p-1}_{X_0/S}) \otimes \mathcal{I}. \]

5. \textit{p-Adic Deformations of Automorphisms}

In this section, we provide a general criterion lifting automorphisms of smooth projective varieties from positive characteristic to characteristic zero, see Theorem [4.7]. The key point to show this criterion is Proposition 4.1. In the following, we will show Proposition 4.1 in a cautious way. We follow the notations and assumptions in Section 4. For \( p = 1 \), Proposition 4.1 is [3, Proposition 3.20]. In the following, we assume \( p \geq 2 \).

**Comparison of the maps \( \rho(f_0) \) and \( \text{ob}(f_0) \cup - \)**

To show Proposition 4.1, it suffices to prove
\[ \rho(f_0) = - [\text{ob}(f_0) \cup -] \]
for \( p \geq 2 \). Recall that \( D \) is the PD envelope of \( X_0 \) in \( X \times Y \) and \( A^\bullet \) is the two term complex (4.16). Note the concrete descriptions of \( \rho \).

In the following, we assume \( p = 1 \). Proposition 4.1 is [3, Proposition 3.20]. In the following, we assume \( p \geq 2 \).

In the following, we prove Proposition 4.1.

To prove the identity (5.1), it suffices to show the existence of the following diagrams:

\[
\begin{array}{ccc}
    f_0^* \Omega^p_{Y_0/S}[-p] & \xrightarrow{\pi_X} & gr^p_{X_0} \Omega^\bullet_{X/T} \\
    \downarrow & & \downarrow \\
    gr^p_{X_0} \Omega^\bullet_{X/T} & \xrightarrow{\pi_X} & gr^p_{X_0} \Omega^\bullet_{X/T}
\end{array}
\]

and

\[
\begin{array}{ccc}
    gr^p_{X_0} \Omega^\bullet_{X/T} & \xrightarrow{-\text{proj}} & \mathcal{I}/\mathcal{I}^2 \otimes \Omega^{p-1}_{X_0/S}[-p+1] \\
    \downarrow & & \downarrow \\
    \mathcal{I}/\mathcal{I}^2 \otimes f_0^* \Omega^{p-1}_{Y_0/S}[-p+1] & \xrightarrow{-H} & \mathcal{I}/\mathcal{I}^2 \otimes \Omega^{p-1}_{X_0/S}[-p+1]
\end{array}
\]

where the map \( \text{proj} \) is the natural map (cf. Lemma 4.5)

\[ gr^p_{X_0} \Omega^\bullet_{X/T} = (\ldots \to \mathcal{I}/\mathcal{I}^2 \otimes \Omega^{p-1}_{X_0/S} \to \ldots) \to \mathcal{I}/\mathcal{I}^2 \otimes \Omega^{p-1}_{X_0/S}[-p+1]. \]

In the following, we construct the maps \( F \) and \( H \) so that the diagrams commute.

The map \( H \) is given by

\[ H \cdot \text{ob}(f_0) \cup - = \text{id}_{\mathcal{I}/\mathcal{I}^2} \otimes (f_0^*)^{-1}. \]

A direct diagram chasing can verify the commutativity of diagram I. We will only show the commutativity of diagrams II and III.

\[ \text{ob}(f_0) \cup - : H^{k-p}(X_0, f_0^* \Omega^p_{Y_0/S}) \to H^{k-p+1}(X_0, \Omega^{p-1}_{X_0/S}) \otimes \mathcal{I}. \]
First of all, we define the map $F : gr^{p}_F \mathcal{X}_0 \to A^\bullet \otimes f^*_0 \Omega^1_{0/S}[-p + 1]$. Define the map $F_{p-1}$ as follows ($p \geq 2$):

$$\begin{array}{c}
\mathcal{J}/\mathcal{J}^{[2]} \otimes \Omega^{p-1}_{X_0 \times Y_0/S}|_{x_0} \xrightarrow{F_{p-1}} \mathcal{J}/\mathcal{J}^{[2]} \otimes (\Omega^{p-1}_{X_0/S} \oplus \ldots \oplus f^*_0 \Omega^{p-1}_{Y_0/S})
\\
\xrightarrow{h_1} \mathcal{J}/\mathcal{J}^{[2]} \otimes f^*_0 \Omega^{p-1}_{Y_0/S}
\end{array}$$

where $h_1 = \text{Id}_{\mathcal{J}/\mathcal{J}^{[2]}} \otimes (-f^*_0)^{-1}, 0, \text{Id}_{f^*_0 \Omega^{p-1}_{Y_0/S}}$. Define the map $F_p$ as follows:

$$\begin{array}{c}
\mathcal{J}/\mathcal{J}^{[2]} \otimes \Omega^{p}_{X_0 \times Y_0/S}|_{x_0} \xrightarrow{F_p} \Omega^{p}_{X_0/S} \oplus \Omega^{p-1}_{X_0/S} \oplus f^*_0 \Omega^1_{Y_0/S} \oplus f^*_0 \Omega^{p-1}_{Y_0/S} \oplus f^*_0 \Omega^p_{Y_0/S} \oplus (\ldots)
\\
\xrightarrow{h_2} f^*_0 \Omega^1_{Y_0/S} \oplus f^*_0 \Omega^{p-1}_{Y_0/S}
\end{array}$$

where $h_2 = (0, -(f^*_0)^{-1} \otimes \text{Id}_{f^*_0 \Omega^{p}_Y/S}, 0, f^*_0 (\text{incl}), 0)$.

The map $F$ : (cf. Lemma 4.1)

$$\begin{array}{c}
(\ldots \to \mathcal{J}/\mathcal{J}^{[2]} \otimes \Omega^{p-1}_{X_0 \times Y_0/S}|_{x_0} \to \Omega^{p}_{X_0 \times Y_0/S}|_{x_0} \to \mathcal{J}/\mathcal{J}^{[2]} \otimes f^*_0 \Omega^{p-1}_{Y_0/S} \to f^*_0 \Omega^1_{Y_0/S} \otimes f^*_0 \Omega^{p-1}_{Y_0/S})
\end{array}$$

is given by the diagram (see Lemma 5.1)

$$(5.2) \quad \mathcal{J}/\mathcal{J}^{[2]} \otimes \Omega^{p}_{X_0 \times Y_0/S}|_{x_0} \xrightarrow{F_p} f^*_0 \Omega^1_{Y_0/S} \oplus f^*_0 \Omega^{p-1}_{Y_0/S}$$

$$(d) \quad \mathcal{J}/\mathcal{J}^{[2]} \otimes \Omega^{p-1}_{X_0 \times Y_0/S}|_{x_0} \xrightarrow{F_{p-1}} \mathcal{J}/\mathcal{J}^{[2]} \otimes f^*_0 \Omega^{p-1}_{Y_0/S}.$$ 

**Lemma 5.1.** The map $F$ defined above is a morphism between complexes, i.e. the diagram (5.2) is commutative.

**Proof.** We show the commutativity of diagram (5.2). In fact, let $B$ be an element $u \otimes (a, \ldots, b)$ of

$$\mathcal{J}/\mathcal{J}^{[2]} \otimes (\Omega^{p-1}_{X_0/S} \oplus \ldots \oplus f^*_0 \Omega^{p-1}_{Y_0/S}) = \mathcal{J}/\mathcal{J}^{[2]} \otimes \Omega^{p-1}_{X_0 \times Y_0/S}|_{x_0}.$$ 

Then we have

$$d(B) = u(da + db) + (du) \cdot (a, \ldots, b)$$

$$= 0 + (du) \cdot (a, \ldots, b) = (du) \cdot (a, \ldots, b) \in \Omega^{p}_{X_0 \times Y_0/S}|_{x_0}$$

where "·" is the wedge product, $\Omega^1_{X_0 \times Y_0/S}|_{x_0} = \Omega^1_{X_0/S} \oplus f^*_0 \Omega^1_{Y_0/S}$ with projections $\text{Pr}_1$, $\text{Pr}_2$ and

$$d(u) = (\text{Pr}_1(d(u)), \text{Pr}_2(d(u))).$$

Therefore, we have

$$F_p(d(B)) = F_p((du) \cdot (a, \ldots, b))$$

$$= \text{Pr}_2(d(u)) \cdot -(f^*_0)^{-1}(a) + \text{Pr}_2(d(u)) \cdot b$$

$$= \text{Pr}_2(d(u)) \cdot (b - (f^*_0)^{-1}(a)).$$
On the other hand, we have
\[
d(F_{p-1}(B)) = d \left( \text{Id}_{\mathcal{J}/\mathcal{J}^{[2]}} \otimes (-(f_0^*)^{-1}, 0, \text{Id}_{f_0^*\Omega_{0/S}})(u \otimes (a, \ldots, b)) \right)
\]
\[
= d(u \otimes (b - (f_0^*)^{-1}(a)))
\]
\[
= \text{Pr}_2(du) \cdot (b - (f_0^*)^{-1}(a))
\]
where the last equality follows from the definition of the complex $A^*$, see (4.16), namely, we have $d_{A^*} = \text{Pr}_2 \circ d$

\[
\begin{array}{c}
\mathcal{J}/\mathcal{J}^{[2]} \xrightarrow{d_{A^*}} f_0^*\Omega_{Y_0/S} \\
\downarrow d \hspace{1cm} \downarrow \text{Pr}_2
\end{array}
\]
\[
\mathcal{J}/\mathcal{J}^{[2]} \xrightarrow{\text{Id}_{\mathcal{J}/\mathcal{J}^{[2]}} \otimes [-(f_0^*)^{-1}] \otimes \text{Id}_{f_0^*\Omega_{0/S}}} \mathcal{J}/\mathcal{J}^{[2]} \otimes \mathcal{J}/\mathcal{J}^{[2]} \otimes (\Omega_{X_0 \times Y_0/S} \otimes \ldots) \xrightarrow{F_{p-1}} \mathcal{J}/\mathcal{J}^{[2]} \otimes f_0^*\Omega_{Y_0/S}.
\]

By the equalities (5.3) and (5.4), we have proved the lemma. □

Let $(\text{Id}$, 0) be the natural inclusion
\[
\Omega_{X_0/S}^{p-1} \hookrightarrow \Omega_{X_0 \times Y_0/S}^{p-1} |x_0 = \Omega_{X_0/S}^{p-1} \oplus \ldots
\]
and similar for
\[
\Omega_{X_0/S}^{p} \hookrightarrow \Omega_{X_0 \times Y_0/S}^{p} |x_0 = \Omega_{X_0/S}^{p} \oplus \ldots
\]
Denote by “in” the natural map
\[
in: \mathcal{I}\mathcal{O}_D \rightarrow \mathcal{J}.
\]

**Lemma 5.2.** The diagram II commutes.

**Proof.** To prove the commutativity of the diagram II, it suffices to prove that the following diagrams (5.5) and (5.6) commute.

Recall that $w : \mathcal{O}_{X_0} \otimes \mathcal{I}/\mathcal{I}^{[2]} \rightarrow A^*$ is a quasi-isomorphism, see (4.17)). It is induced by the map “in”. We claim there is a commutative diagram as follows:

\[
\begin{array}{c}
\mathcal{I}/\mathcal{I}^{[2]} \otimes \Omega_{X_0 \times Y_0/S}^{p-1} \xrightarrow{H = \text{Id}_{\mathcal{I}/\mathcal{I}^{[2]}} \otimes [-(f_0^*)^{-1}]} \mathcal{I}/\mathcal{I}^{[2]} \otimes f_0^*\Omega_{Y_0/S}^{p-1} \\
\downarrow \text{in} \otimes (\text{Id}, 0) \hspace{1cm} \downarrow w \otimes \text{Id}_{\Omega_{Y_0/S}^{p-1}}
\end{array}
\]
\[
\mathcal{J}/\mathcal{J}^{[2]} \otimes \Omega_{X_0 \times Y_0/S}^{p-1} |x_0 \xrightarrow{\text{in}} \mathcal{J}/\mathcal{J}^{[2]} \otimes (\Omega_{X_0 \times Y_0/S}^{p-1} \oplus \ldots) \xrightarrow{F_{p-1}} \mathcal{J}/\mathcal{J}^{[2]} \otimes f_0^*\Omega_{Y_0/S}^{p-1}.
\]

In fact, we have
\[
w \otimes \text{Id}_{\Omega_{Y_0/S}^{p-1}}(H(u \otimes a)) = w \otimes \text{Id}_{\Omega_{Y_0/S}^{p-1}}(u \otimes [- (f_0^*)^{-1}(a)])
\]
\[
= \text{in}(u) \otimes [- (f_0^*)^{-1}(a)]
\]
for $u \otimes a \in \mathcal{I}/\mathcal{I}^{[2]} \otimes \Omega_{X_0 \times Y_0/S}^{p-1}$. On the other hand, we have
\[
F_{p-1}(\text{in} \otimes (\text{Id}, 0)(u \otimes a)) = F_{p-1}(\text{in}(u) \otimes (a, 0))
\]
\[
= \text{in}(u) \otimes [- (f_0^*)^{-1}(a)].
\]
We have proved that the diagram (5.5) commutes. To show the lemma, it remains to verify the commutativity of the following diagram

(5.6) \[
\begin{array}{ccc}
\Omega^p_{X_0/S} & \rightarrow & 0 \\
\downarrow & & \\
\Omega^p_{X_0 \times Y_0/S} \mid X_0 & \rightarrow & \Omega^p_{X_0/S} \oplus (\ldots) \\
\end{array}
\]

In fact, it is clear that

\[F_p((\text{Id}, 0)(a)) = F_p((a, 0, \ldots, 0)) = 0\]

for \(a \in \Omega^p_{X_0/S}\). We have proved that the diagram commutes.

In summary, we show the diagram (II) commutes. \(\square\)

Lemma 5.3. The diagram (III) commutes.

Proof. It suffices to show the following diagram commutes.

\[
\begin{array}{ccc}
f^*_0 \Omega^p_{Y_0/S} & \rightarrow & \Omega^p_{X \times Y \mid X_0} \\
\downarrow f^*_0 (\text{incl}) & & \\
f^*_0 \Omega^1_{Y_0/S} \otimes f^*_0 \Omega^{p-1}_{Y_0/S} & \rightarrow & f^*_0 \Omega^1_{Y_0/S} \otimes f^*_0 \Omega^{p-1}_{Y_0/S} \\
\downarrow h & & \\
\Omega^1_{X_0/S} \otimes f^*_0 \Omega^{p-1}_{Y_0/S} \oplus f^*_0 \Omega^1_{Y_0/S} \otimes f^*_0 \Omega^{p-1}_{Y_0/S} & \rightarrow & f^*_0 \Omega^1_{Y_0/S} \otimes f^*_0 \Omega^{p-1}_{Y_0/S} \\
\end{array}
\]

where \(h\) is the map (4.21) and the term \(f^*_0 \Omega^1_{Y_0/S} \otimes f^*_0 \Omega^{p-1}_{Y_0/S}\) at the right corner is the \(p\)-th term of the complex \(A^* \otimes f^*_0 \Omega^{p-1}_{Y_0/S}[-p + 1]\).

In fact, we have

\[\pi^*_Y(u) = (0, 0, 0, u, 0) \in \Omega^p_{X_0/S} \oplus \Omega^{p-1}_{X_0/S} \oplus \Omega^1_{X_0/S} \oplus \Omega^{p-1}_{X_0/S} \oplus \Omega^1_{X_0/S} \oplus \Omega^p_{Y_0/S} \oplus (\ldots)\]

for \(u \in f^*_0 \Omega^p_{Y_0/S}\), see Remark 4.7. Therefore, we conclude that

\[F_p(\pi^*_Y(u)) = F_p((0, 0, 0, u, 0)) = f^*_0 (\text{incl}(u)).\]

On the other hand, we have

\[\Pr_2(h(f^*_0 (\text{incl}(u)))) = \Pr_2 \left( (f^*_0 \otimes \text{Id}_{f^*_0 \Omega^{p-1}_{Y_0/S}}, \text{Id}_{f^*_0 \Omega^1_{Y_0/S}} \otimes f^*_0 \Omega^{p-1}_{Y_0/S}) (\text{incl}(u)) \right) = f^*_0 (\text{incl}(u))\]

Comparing the identities above, we have proved the commutativity of the diagram. \(\square\)

Now, we can finish the proof of Proposition 4.1.

Proof. Proposition 4.1 follows from Lemma 5.1, Lemma 5.2 and Lemma 5.3. \(\square\)

We are able to prove Theorem 1.7.
Proof. Let \( h : X \to W(k) \) be the structure map of \( X \) over the Witt ring \( W(k) \). Suppose that \( \pi \) is the uniformizer of \( W(k) \). We have smooth morphisms

\[
h_n : X_n \to W_n
\]

where \( h_n = h|_{W_n} \) is the restriction of \( h \) to \( W_n = W(k)/(\pi^{n+1}) \). Note that \((\pi^{n+1})\) is square-zero ideal of \( W_{n+1} \). Therefore, the \( W_{n+1} \)-module \( (\pi^{n+1}) \) is a \( W_n \)-module as well. For each \( h_n \), we have the natural cup product \( \Psi_n \) as follows

\[
\bigoplus_{p+q=m} \text{Hom}(R^q\pi h_n^*(\mathcal{O}_{X_n/W_n}^p), R^{q+1}h_n^*(\mathcal{O}_{X_n/W_n}^{p-1}) \otimes (\pi^{n+1})).
\]

It follows from our assumptions that the Hodge-de Rham spectral sequence of \( X_n/W_n \) degenerates at \( E_1 \) and their terms are locally free so that the Hodge and de Rham cohomology sheaves commute with base change. Therefore, the cup product \( \Psi_n \) is \( \Psi_0 \otimes_k (\pi^{n+1}) \). Since \( \Psi_0 \) is the cup product \( \llbracket 1,3 \rrbracket \) which is injective, the cup product \( \Psi_n \) is injective.

Let \( g_n : X_n \to X_n \) be a lifting of \( g_0 \) over \( W_n \). Note that \( g_n \) is an automorphism of \( X_n \) over \( W_n \). The map \( \Psi_n \) induces an injection \( \hat{\Psi}_n \)

\[
\bigoplus_{p+q=m} \text{Hom}(R^q\pi h_n^*(\mathcal{O}_{X_n/W_n}^p), R^{q+1}h_n^*(\mathcal{O}_{X_n/W_n}^{p-1}) \otimes (\pi^{n+1})).
\]

On the other hand, the map \( g_n : X_n/W_{n+1} \to X_n/W_{n+1} \) induces a map \( H^{p+q}_{\text{cris}}(g_n) \) as follows

\[
\begin{align*}
H^{p+q}_{\text{cris}}(X_n/W_{n+1}) & \to H^{p+q}_{\text{cris}}(X_n/W_{n+1}) \\
H^{p+q}_{\text{DR}}(X_{n+1}/W_{n+1}) & \to H^{p+q}_{\text{DR}}(X_{n+1}/W_{n+1}).
\end{align*}
\]

The map \( H^{p+q}_{\text{cris}}(g_n) \otimes W_n \) can be identified with \( H^{p+q}_{\text{DR}}(g_n) \) and hence it preserves the Hodge filtrations. Therefore, as \( \llbracket 1,2 \rrbracket \), the map \( H^{p+q}_{\text{cris}}(g_n) \otimes W_n \) induces a diagram

\[
\begin{CD}
F^p_{Hdg} H^{p+q}_{\text{DR}}(X_{n+1}/W_{n+1}) @>>> g^n_{\mathbb{F}_p} H^{p+q}_{\text{DR}}(X_{n+1}/W_{n+1}) \otimes_{W_{n+1}} (\pi^{n+1}) \\
@. @VVV \\
\quad H^{p+1}(X_n, \mathcal{O}_{X_n/W_n}^{p-1}) \otimes W_n (\pi^{n+1})
\end{CD}
\]
by the fact that \((\pi^{n+1})\) is a square zero ideal in \(W_{n+1}\). In particular, we have that
\[
F^p_{\text{Hdg}} H_{\text{DR}}^{p+q}(X_n/W_n) \xrightarrow{\text{proj}} \mathcal{H}^{p+q+1}(X_n, \Omega^{p+q+1}_{X_n/W_n}) \otimes_{W_n} (\pi^{n+1})
\]
see \([122]\) for the definition of \(\rho(g_n)\). We apply Proposition 4.1 to the case \(f_0 = g_n, S = \text{Spec} W_n, T = \text{Spec}(W_{n+1})\). It follows that
\[
\bigoplus_{p+q=m} \rho(g_n)q = \pm \Psi_n(ob(g_n))
\]
where \(ob(g_n)\) is the obstruction element in
\[
\mathcal{H}^1(X_n, g_n T_{X_n/W_n} \boxtimes (\pi^{n+1})) = \mathcal{H}^1(X_0, \mathcal{g}_n T_{X_0/k}) \boxtimes_k (\pi^{n+1}).
\]
Since \(H^{m}_{\text{cris}}(g)\) preserves the Hodge filtration of \(H^{m}_{\text{cris}}(X_0/W)\), we conclude that \(\rho(g_n)q\) are zero by the construction of \(\rho(g_n)q\), cf. \([122]\).

It follows from the injectivity of \(\Psi_n\) that \(ob(g_n)\) is zero. Hence, we have a formal automorphism \(\lim g_n\) on the formal scheme \(\lim X_n\). By the Grothendieck’s existence theorem, the formal automorphism comes from an automorphism \(g : X/W \rightarrow X/W\). In other words, we can lift \(g_0\) over \(k\) to \(g\) over \(W(k)\).

**Corollary 5.4.** With the notations and assumptions as in Theorem 1.7, we suppose that \(g_0\) is an automorphism of \(X_0\) over \(k\) such that the order of \(g_0\) is finite. If \(H_{\text{et}}^m(0, \mathcal{Q}_l) = \text{Id}\) where \(l \neq \text{char}(k)\), then one can lift \(g_0\) to an automorphism over \(W(k)\)
\[
g : X/W(k) \rightarrow X/W(k).
\]
In particular, if the automorphism group \(\text{Aut}(X_0)\) is finite and \(\text{Aut}(X_K)\) acts on \(H^{m}_{\text{et}}(X_K, \mathcal{Q}_l)\) faithfully where \(K\) is the fraction field of \(W(k)\), then \(\text{Aut}(X_0)\) acts on \(H^{m}_{\text{et}}(X_0, \mathcal{Q}_l)\) faithfully.

**Proof.** Note that
\[
\text{det}(\text{Id} - g_0^* t, H^{m}_{\text{cris}}(X_0/W)_K) = \text{det}(\text{Id} - g_0^* t, H^{m}_{\text{et}}(X_0, \mathcal{Q}_l)),
\]
see \([138]\) Theorem 2] and \([133]\) 3.7.3 and 3.10]. The finiteness of the order \(\text{ord}(g_0)\) implies that \(H^{m}_{\text{et}}(g_0, \mathcal{Q}_l) = \text{Id}\) if and only if \(H^{m}_{\text{cris}}(g_0)_K = \text{Id}\) since both \(H^{m}_{\text{et}}(g_0, \mathcal{Q}_l)\) and \(H^{m}_{\text{cris}}(g_0)_K\) can be diagonalizable. The corollary follows from Theorem 1.7. \(\square\)

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