Multipartite steering inequalities based on entropic uncertainty relations

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We investigate quantum steering for multipartite systems by using entropic uncertainty relations. We introduce entropic steering inequalities whose violation certifies the presence of different classes of multipartite steering. These inequalities witness both steering states and genuine multipartite steerable states. Furthermore, we study their detection power for several classes of states of a three-qubit system.

Quantum steering is a type of quantum correlation, owned by some entangled states of composite systems. It enables one subsystem to influence the state of the others, with which it shares the entangled state, by applying local measurements. The concept of quantum steering, for bipartite systems, was introduced in the early days of quantum mechanics by Schrodinger [1], who recognized that this class of states allow one part “to steer” the state of the other into an eigenstate of an arbitrary observable, and hence they express the “spooky action at distance” discussed in [2]. Nowadays we are aware that three types of quantum entanglement exist: Bell nonlocality, steerability and nonseparability. Bell nonlocality correlations are the strongest ones and are owned by global states that violate some Bell inequalities [3], which are related to the non existence of local hidden variable (LHV) models. Then we have quantum steering, which was formalized in 2007 by Wiseman et al. [4] as the incompatibility of quantum mechanics predictions with a local hidden state (LHS) model, where the parties have pre-determined states. Formally, given a bipartite system owned by Alice and Bob that share a state $\rho^{AB}$, we say that the correlations demonstrate quantum steering if the joint measurement probabilities cannot be expressed as:

$$p(x_a,x_b) = \int d\lambda q(\lambda) p(x_a|\lambda) p_B(x_b), \quad (1)$$

where $x_a$ and $x_b$ are respectively the outcomes of the measurements of Alice’s observable $X_A$ and Bob’s observable $X_B$. In the above equation $p_B(x_b)$ represents the probability of $x_b$ obtained from a quantum pre-determined state $\rho^B$ that depends only on $\lambda$, which occurs with probability $q(\lambda)$ and not on $x_a$. Instead, the conditional probability of $x_b$ on an arbitrary state, which may depend on $x_a$, will be indicated as $p(x_a|x_b)$. Conversely if for any choice of measurements equation (1) holds, then the state is called nonsteerable, in the sense that it admits a LHS model.

At the bottom of the hierarchy there is entanglement [5] [6], which can be defined as the existence of states of composite systems that cannot be given as a convex combination of states of the individual subsystems, namely separable states. Interestingly, these three notions, which can be only found in nonseparable states, coincide for pure states.

All of these types of correlations have been generalized to multipartite systems. However for steerability there exist different approaches [7] [8] that go beyond the bipartite scenario. Here we consider the one discussed in [7], which also allows one to discuss the notion of post-quantum steering [10], which does not exist for bipartite systems.

In this paper we introduce a number of entropic inequalities whose violation certifies multipartite steering. Steerability is an asymmetric concept, i.e. one part steers the others. In multipartite systems there exist several different steering scenarios, depending on how many subsystems steer the others. For example in a tripartite system we can have one subsystem that tries to steer the other two, a scenario that we indicate as one-to-two steering, or two subsystem that might steer the other one, which we refer as two-to-one steering.

In the one-to-two steering scenario we say that the correlations demonstrate multipartite steering [7] if the joint measurement probabilities cannot be expressed as:

$$p(x_a,x_b,x_c) = \int d\lambda q(\lambda) p(x_a|x) p_B(x_b) p_C|x)(x_c), \quad (2)$$

where $x_a$, $x_b$, and $x_c$ are the outcomes the observables $X_A$, $X_B$ and $X_C$ of Alice, Bob and Charlie respectively. In [7] Bob and Charlie’s quantum state is predetermined, for a given $\lambda$ their state is $\rho^B \otimes \rho^C$. A state is said instead to demonstrate genuine multipartite steering [7] if the joint measurement probabilities cannot be written as:

$$p(x_a,x_b,x_c) = \int_A d\nu_A(\nu) p_A(\nu) p_B(x_b) p_C(x_c|\nu) \quad (3)$$

$$+ \int_B d\gamma q_B(\gamma) p_B(\gamma) p(x_c|x_a,\gamma) \quad (3)$$

$$+ \int_C d\omega q_C(\omega) p_C(\omega) p(x_c|x_a,\omega) p_C(x_c), \quad (3)$$

where $q_A(\nu)$, $q_B(\gamma)$ and $q_C(\omega)$ satisfy: $\int_A d\nu_A(\nu) + \int_B d\gamma q_B(\gamma) + \int_C d\omega q_C(\omega) = 1$. In [8] there are three terms: in the first there is no steering between Alice, Bob and Charlie, in the second Alice can steer Charlie but not Bob, namely only Bob’s state is pre-determined for a given $\gamma$, conversely in the third Alice can steer Bob but not Charlie, which means that only Charlie’s state is pre-determined for a given $\omega$. 

In the two-to-one steering scenario we say that the correlations demonstrate multipartite steering \(^7\) if the joint measurement probabilities cannot satisfy:

\[
p(x_a, x_b, x_c) = \int d\lambda p(\lambda) p(x_a|\lambda)p(x_b|\lambda)p(x_c|\lambda). \tag{4}
\]

Conversely if the above holds the state is nonsteerable from Alice and Bob to Charlie, indeed Charlie’s state is pre-determined by the value of \(\lambda\). In this scenario a state is said to be GMS \(^7\) if the joint measurement probabilities cannot be expressed as:

\[
p(x_a, x_b, x_c) = \int d\nu A(\nu) p(x_a|\nu)p(x_b|\nu)p(x_c|x_a, \nu)
+ \int d\gamma B(\gamma) p(x_a|\gamma)p(x_b|x_a, \gamma)
+ \int d\omega C(\omega) p(x_a, x_b|\omega)p_c(x_c), \tag{5}
\]

where \(\int_A d\nu A(\nu) + \int_B d\gamma B(\gamma) + \int_C d\omega C(\omega) = 1\). In \(^5\) the first term shows that only Bob can steer Charlie, in the second only Alice and in the third Alice and Bob cannot jointly steer Charlie. However Alice and Bob can share entanglement.

As any type of quantum correlations, one of the problems connected with quantum steering is its detection. Several methods have been introduced in the last years for detecting steering in bipartite systems, for example \(^11\)-\(^15\). Here we are interested in entropic steering criteria such as the one defined in \(^14\)-\(^17\). In \(^14\) it was derived that a nonsteerable state satisfies:

\[
H(X_B|X_A) \geq \int d\lambda (\lambda) H_\lambda(X_B), \tag{6}
\]

where \(H(X_B|X_A)\) is the conditional entropy of \(X_B\) given \(X_A\) and \(H_\lambda(X_B)\) is the conditional entropy of \(X_B\) computed on \(p_\lambda^A\), that does not depend on Alice’s measurements. Thus any violation of \(^6\) demonstrates steering from Alice to Bob. In \(^17\) the inequality \(^6\) was generalized to state-independent entropic uncertainty relations (EUR). As an example, given any two of Alice’s observables \(X_A\) and \(Z_A\) and two of Bob’s observables \(X_B\) and \(Z_B\), for any nonsteerable state the following holds:

\[
H(X_B|X_A) + H(Z_B|Z_A) \geq -\log_2 \alpha_B, \tag{7}
\]

where \(\alpha_B = \max_{j,k} |\langle x^j_B|z^k_B \rangle|^2\), with \(\{|x^j_B\}_j\) and \(\{|z^k_B\}_k\) the eigenstates of \(X_B\) and \(Z_B\) respectively. Eq. \(^7\) is a generalization of Maaseen and Uffink’s EUR \(^18\) to nonsteerable states, which can be violated only by steerable states from Alice to Bob. Starting from \(^6\) other inequalities of the form \(^7\) can be derived simply by considering different EUR from the ones of \(^18\), for example the ones derived in \(^19\)-\(^29\).

In this paper we derive the following results:

(i) we first show that \(^6\) and \(^7\) can be generalized to tripartite steering, obtaining different sufficient conditions for both steerable and GMS states. In the case of one-to-two steering scenario where Alice, whose measurements are uncharacterized, might steer Bob and Charlie’s state, we show that any nonsteerable state satisfies the following set of entropic uncertainty relations:

\[
\sum_{O=X,Z} H(O_m|Q_A) \geq -\log_2 \alpha_m; \tag{8}
\]

where \(\alpha_m = \max_{j,k} |\langle x^m_j|z^m_k \rangle|^2\) with \(m = B, C, BC\) labeling the subsystem considered and \(\{|x^m_j\}_j\) and \(\{|z^m_k\}_k\) being the eigenstates of \(X_m\) and \(Z_m\) respectively. Here \(O_{BC}\) is given by \(O_B \otimes O_C\) for any observables.

(ii) We also show that

\[
\sum_{O=X,Z} H(O_m|Q_AO_m) \geq -\log_2 \alpha_m, \tag{9}
\]

holds for all nonsteerable states, where \(m = B, C\) and \(\hat{m}\) indicates the opposite of \(m\), i.e. \(\hat{m} = B\) if \(m = C\) and \(\hat{m} = C\) when \(m = B\).

(iii) The last inequality for nonsteerable states involves the following quantity:

\[
A(O_A, O_B, O_C) = H(O_{BC}|O_A) + \sum_{m=B,C} H(O_m|O_A, O_{\hat{m}}). \tag{10}
\]

We prove that for a nonsteerable state the following holds:

\[
\sum_{O=X,Z} A(O_A, O_B, O_C) \geq -4 \log_2 \alpha_{min}, \tag{11}
\]

where \(\alpha_{min} = \min \{\alpha_B, \alpha_C\}\).

(iv) For any non-GMS states we prove that the following inequality is satisfied:

\[
\sum_{O=X,Z} A(O_A, O_B, O_C) \geq -2 \log_2 \alpha_{min}. \tag{12}
\]

(v) Finally, we give the following state-dependent entropic uncertainty relation valid for all non-GMS states:

\[
\sum_{O=X,Z} H(O_{BC}|O_A) \geq -\log_2 \alpha_{CB}
+ \int_B d\gamma q_B(\gamma) S_{\gamma}(C|A) + \int_C d\omega q_C(\omega) S_{\omega}(B|A), \tag{13}
\]

where \(S_{\gamma}(C|A)\) represents the Von Neumann conditional entropy between the bipartition \(C|A\) when the variable \(\gamma\) occurs, while \(S_{\omega}(B|A)\) is the Von Neumann conditional between Bob and Alice when \(\omega\) occurs. Note that these quantities can be negative \(^30\) for entangled states, moreover their lowest values \(-\log_2 d_C\) and \(-\log_2 d_B\) are reached by maximally entangled states. We note that the inequality \(^13\) is not useful in the context of multipartite steering detection, if one wants to understand the
steering property of an unknown state, since it requires the knowledge of the LHS model. Conversely, the multipartite steering criteria can be exploited in the task of discovering the steering property of unknown quantum states. In order to compare the power in detecting steerability of the criteria in Section V we consider the steerability robustness of the standard GHZ and W states under white noise and we show that criterion detects more multipartite steerable states than the others.

The results are also extended, with the same techniques, to the two-to-one steering scenario.

The paper is organized as follows: in Section I we review bipartite quantum steering by following the approach of [3]. Here we also report the derivations of (6) and (7). In Section II we review the definition of multipartite steering, which was introduced in [7]. In Section III we focus on the one-to-two steering scenario and we derive the steering inequalities. In Section IV the results for the two-to-one steering are discussed. Finally in Section V some steering states are considered in order to study the detection power of these inequalities.

I. BIPARTITE QUANTUM STEERING

A. Definition and LHS model

Bipartite quantum steering [4] can be seen as the ability to nonlocally influence the set of possible quantum states of a given system through the measurements of another system sufficiently entangled with the first one. In the steering scenario Alice and Bob share a quantum state and Alice performs a measurement whose outcome occurs with probability . As a consequence of Alice’s measurements, Bob’s state is transformed into the state with probability . Here we do not require any characterization of Alice’s measurements, namely we only say that she performs an arbitrary measurement, and we suppose that Bob has full access to the conditional state and on his measurements. Namely, the information available to Bob is the collection of the post-measured states and their respective probabilities , which can be described with the following ensemble of unnormalized states:

\[ \{ \sigma_{x_a}^B = p(x_a) \rho_{x_a}^B \} . \quad (14) \]

Each member of (14) is given by:

\[ \sigma_{x_a}^B = \text{Tr}_A \left[ (\Pi_{x_a}^A \otimes I^B) \rho^{AB} \right] , \quad (15) \]

where \( \sum_{x_a} \Pi_{x_a}^A = I^A \) and \( \Pi_{x_a}^A \geq 0 \) are Alice’s POVM elements. The ensemble represents the set of possible quantum states that can be nonlocally influenced when steering correlations are owned by \( \rho^{AB} \). Therefore the LHS model formally represents the minimal requirement on (14) in order to avoid this nonlocal influence, then steering is defined as the possibility of remotely generating ensembles that could not be produced by a LHS model. This model can be thought in the following way: a source sends, according to a probability distribution \( q(\lambda) \), a classical message \( \lambda \) to Alice, her probability of obtaining \( x_a \) depends now on \( \lambda \): \( p(x_a|\lambda) \). To each \( \lambda \) there corresponds a pre-determined state of Bob \( \rho_{x_a}^B \), which is sent to Bob with the same probability \( q(\lambda) \). Bob’s ensemble [14], that now does not depend on Alice’s measurements, is given by:

\[ \sigma_{x_a}^B = \int d\lambda q(\lambda) p(x_a|\lambda) \rho_{x_a}^B . \quad (16) \]

The definition of steering is as follows: an ensemble is said to demonstrate bipartite steering if it does not admit a decomposition of the form (16). Moreover a quantum state \( \rho^{AB} \) is said to be steerable from Alice to Bob if there exists a measurement in Alice’s part that produces an ensemble that demonstrates steering. This is an asymmetric concept that also implies entanglement.

Indeed suppose that \( \rho^{AB} \) is separable, namely we have \( \rho^{AB}_S = \int d\lambda q(\lambda) \rho_{x_a}^A \otimes \rho_{x_b}^B \). After Alice has performed a measurement, Bob’s ensemble becomes:

\[ \sigma_{x_a}^B = \text{Tr}_{AB} \left[ (\Pi_{x_a}^A \otimes I^B) \rho^{AB}_S \right] , \quad (17) \]

which is of the form (16). Since it implies nonseparability, steering detection can be seen as an entanglement detection task where one part, the one that steers, performs arbitrary measurements and its system remains completely uncharacterized, namely we do not assume anything on it, not even its dimension. The existence of a LHS model can be written also in terms of joint probabilities of measurements, namely by the condition [1]. Indeed we have:

\[ p(x_a,x_b) = p(x_b|x_a) p(x_a) = \text{Tr}_B \left[ \Pi_{x_b}^B \sigma_{x_a}^B \right] \]

\[ = \int d\lambda q(\lambda) p(x_a|\lambda) p_{x_b} \]. \quad (18)

B. Entropic uncertainty steering inequalities

Here we review the techniques used in [14] and [15] to derive (6) and (7). Suppose that a state \( \rho^{AB} \) admits a LHS model, then (1) holds. Note first that:

\[ p(x_b|x_a) = \int d\lambda p(x_b,\lambda|x_a) , \quad (19) \]

with

\[ p(x_b,\lambda|x_a) = p(\lambda|x_a) p(x_b|x_a,\lambda) = p(\lambda|x_a) p_{\lambda}(x_b) , \quad (20) \]

where the last equality holds since the state admits a LHS model. Given \( x_a \), we consider the relative entropy
between \( p(x_b, \lambda|x_a) \) and \( p(\lambda|x_a) p(x_b|x_a) \), which is always nonnegative. Namely we have:

\[
\sum_b \int d\lambda p(x_b, \lambda|x_a) \log_2 \left( \frac{p(x_b, \lambda|x_a)}{p(\lambda|x_a) p(x_b|x_a)} \right) \geq 0.
\]

(21)

The above can be written as a sum of two terms. The first is given by:

\[
- \sum_b \int d\lambda p(x_b, \lambda|x_a) \log_2 (p(x_b|x_a)) = -\sum_b p(x_b|x_a) \log_2 (p(x_b|x_a)) = \text{H}(X_B|X_A = x_a).
\]

(22)

The second, by using (20), can be expressed as:

\[
\int d\lambda p(\lambda|x_a) \sum_b p(\lambda|x_a) \log_2 (p(\lambda|x_a)) = -\int d\lambda p(\lambda|x_a) \text{H}_\lambda(X_B).
\]

Therefore (21) implies:

\[
\text{H}(X_B|X_A = x_a) \geq \int d\lambda p(\lambda|x_a) \text{H}_\lambda(X_B),
\]

(23)

which leads to (7) by averaging over \( x_a \), that provides a sufficient condition to detect steering states, indeed any violation of it implies the presence of bipartite quantum steering. If we now consider a sum as \( \sum_{O=\{X, Z\}} \text{H}(O_B|O_A) \), we find:

\[
\sum_{O=\{X, Z\}} \text{H}(O_B|O_A) \geq \int d\lambda q(\lambda) \sum_{O=\{X, Z\}} \text{H}_\lambda(O_B).
\]

(24)

In the right-hand side of (25) \( \sum_{O=\{X, Z\}} \text{H}_\lambda(O_B) \) depends on \( \lambda \), namely the two entropies are computed over the state \( \rho_B^B \). However for any state Maassen and Uffink’s EUR [18] holds, namely we have:

\[
\sum_{O=\{X, Z\}} \text{H}_\lambda(O_B) \geq \log_2 \frac{1}{\alpha_B},
\]

which together with \( \int d\lambda q(\lambda) = 1 \), implies (7):

\[
\text{H}(X_B|X_A) + \text{H}(Z_B|Z_A) \geq -\log_2 \alpha_B.
\]

(26)

Since the above must be valid for any nonsteerable state, any violation of it indicates the presence of a steerable state.

II. MULTIPARTITE QUANTUM STEERING

In this section we start reviewing the concept of quantum steering for multipartite systems. We focus on the tripartite case, where there are two possible scenarios, following the approach given in [17, 20]. In the first case, which can be named one-to-two steering scenario, Alice measures her system and wants to nonlocally influence the state of the other two. The available information is encoded in the following ensemble of unnormalized states:

\[
\sigma_{x_a}^{BC} = \text{Tr}_A \left( [\Pi_A^{x_a} \otimes I_B \otimes I^C] \rho^{ABC} \right),
\]

(27)

where \( \{\Pi_A^{x_a}\}_{x_a} \) is a POVM of Alice’s measurements. The second possibility, the two-to-one steering scenario, consists in two parties, say Alice and Bob that, by measuring their systems, want to influence the states of the third party. In this case, the post-measured ensemble of states is given by:

\[
\sigma_{x_a,x_b}^{C} = \text{Tr}_{AB} \left( [\Pi_A^{x_a} \otimes \Pi_B^{x_b} \otimes I^C] \rho^{ABC} \right),
\]

(28)

where \( \{\Pi_A^{x_a}\}_{x_a}, \{\Pi_B^{x_b}\}_{x_b} \) are POVMs of Alice and Bob’s respectively. Multipartite steering scenario therefore consists of all the asymmetric scenarios, where some subset of the parties have full control on their subsystems, and they want to steer the state of the remaining subsets. Just like entanglement, which has a much richer structure in the multipartite case in than the bipartite one since different notions of separability can be introduced, also steerability have different levels for multipartite systems. In the case of a tripartite system we have two notions: multipartite steering and the genuine multipartite steering, which refers to the impossibility to explain the correlations between measurement outcomes in terms of different LHS models.

III. ONE-TO-TWO STEERING SCENARIO

A. LHS models

Let us first focus on the one-to-two steering scenario. If Alice cannot nonlocally influence Bob and Charlie the ensemble [27] becomes:

\[
\sigma_{x_a}^{BC} = \int d\lambda q(\lambda) p(x_a|\lambda) \rho_B^B \otimes \rho_A^C.
\]

(29)

In the above there is no steering from Alice to Bob and Charlie and each member of the ensemble is prepared in a separable state of Bob and Charlie. Note that the above can be thought as a multipartite LHS model where, with probabilities \( q(\lambda) \) Alice receives \( \lambda \) and outputs \( x_a \) with probability \( p(x_a|\lambda) \), while Bob and Charlie’s states are pre-determined by the value of \( \lambda \). Any tripartite state that can produce an ensemble that cannot be written as (29) is said to be multipartite steering. An example is provided by \( |\phi^+\rangle \langle \phi^+|^{AB} \otimes \rho^C \), where \( |\phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \). Indeed Alice can prepare an ensemble that cannot be written as (29). This LHS model can be expressed in terms of joint probabilities as (2), indeed:

\[
p(x_a, x_b, x_c) = \text{Tr}_{BC} \left( [\Pi_B^{x_b} \otimes \Pi_C^{x_c}] \sigma_{x_a}^{BC} \right),
\]

(30)

which implies (2) by using (29). Note that a slightly different definition of this form of multipartite steering
exists [27]. Indeed, one could require that entanglement between Bob and Charlie is present. As a consequence, Bob and Charlie’s pre-determined state would be \(\rho_{\Lambda}^{BC}\), instead of \(\rho_{\Lambda}^{B}\) \(\otimes\rho_{\Lambda}^{C}\). However, here we consider only the case where there is no entanglement between Bob and Charlie, since we are interested in detecting the possible simplest form of these quantum correlations.

If the state is non-GMS then the ensemble [27] can be expressed as:

\[
\sigma_{x,a}^{BC} = \int_A dq_{A}(\nu)\rho_{\nu,a}^{BC} + \int_B dq_{B}(\gamma)\rho_{\gamma,a}^{B} \otimes \sigma_{x,a,\gamma}^{C} + \int_C d\omega q_{C}(\omega)\rho_{\omega,a}^{C} \otimes \sigma_{x,a,\omega}^{B},
\]

where \(\sigma_{x,a,\gamma}^{C} = \text{Tr}_{A}\left[\left(\Pi_{x,a}^{A} \otimes I_{C}\right) \rho_{\gamma,a}^{AC}\right]\) and \(\sigma_{x,a,\omega}^{B} = \text{Tr}_{A}\left[\left(I_{x,a}^{A} \otimes \Pi_{\omega}^{B}\right) \rho_{\omega,a}^{AB}\right]\). Each member of this ensemble can be expressed as a sum of three terms. In the first one there is no steering between Alice and Bob-Charlie. In the other two, which are made of separable states only of Bob and Charlie, Alice can steer one of the two subsystems but not the other. This can be thought in terms of a hybrid-LHS model in the following way. The hidden variable \(\lambda\) discriminates between different situations: in the first the global state of Bob and Charlie is pre-determined, that is \(\rho_{\Lambda}^{BC}\), and this state can be entangled; in the other two \(\lambda\) determines the state of just one subsystem, the other is not pre-determined. The previous example \(|\phi^{+}\rangle = \phi^{+}_{AB} \otimes \rho_{\omega}^{C}\) can now lead to an ensemble of the form (31). Any tripartite states that cannot produce an ensemble such as (31) is said to be genuine multipartite steering. Conversely if (31) can be produced, the state is non-GMS. By using (31) and (34) we can express this hybrid-LHS model in terms of joint probabilities:

\[
p(x_a, x_b, x_c) = \int_A dq_{A}(\nu)\rho_{\nu,a}^{BC} + \int_B dq_{B}(\gamma)\rho_{\gamma,b}^{B} p(x_a|x_b,\gamma) + \int_C d\omega q_{C}(\omega)\rho_{\omega,a}^{C} p(x_b|x_a,\omega),
\]

which is exactly the requirement [3].

**B. Entropic uncertainty multipartite steering inequalities**

In this section we derive entropic steering inequalities for a multipartite system, with the aim to discriminate also the different notions of multipartite steering. We start by considering a nonsteerable state and show that it must imply some inequalities, then we use them to formulate sufficient conditions for multipartite steering detection. If the state is nonsteerable it satisfies:

\[
p(x_b, x_c|x_a) = \int d\lambda p(x_b, x_c, \lambda|x_a),
\]

with

\[
p(x_b, x_c, \lambda|x_a) = p(\lambda|x_a) p_{\lambda}(x_b, x_c),
\]

where the last equality holds since (2) holds. As in the bipartite case, we now consider the relative entropy between \(p(x_b, x_c, \lambda|x_a)\) and \(p(\lambda|x_a) p(x_b, x_c|x_a)\), which has to verify:

\[
\sum_{b,c} d\lambda p(x_b, x_c, \lambda|x_a) \log_{2}\left(\frac{p(x_b, x_c, \lambda|x_a)}{p(\lambda|x_a) p(x_b, x_c|x_a)}\right) \geq 0.
\]

The above, with (33) and (34), implies:

\[
H(X_B, X_C|X_A = x_a) \geq \sum_{b,c} d\lambda q(\lambda) H_{\lambda}(X_B, X_C);
\]

and by averaging over \(x_a\) we arrive at:

\[
H(X_B, X_C|X_A) \geq \int d\lambda q(\lambda) H_{\lambda}(X_B, X_C).
\]

Since Bob and Charlie share a separable state, for \(m = B, C\) we also have:

\[
p(x_m, \lambda|x_a) = p(\lambda|x_a) p_{\lambda}(x_m).
\]

Now by considering the relative entropy between \(p(x_m, \lambda|x_a)\) and \(p(\lambda|x_a) p(x_m|x_a)\) for \(m = B, C\), we can derive in the same way:

\[
H(X_m|X_A) \geq \int d\lambda q(\lambda) H_{\lambda}(X_m).
\]

The entropic uncertainty relations [8], namely

\[
\sum_{O=Z,Z} H(O_m|O_A O_m) \geq -\log_2 \alpha_m,
\]

can be derived simply by noting that the following holds for any state:

\[
\sum_{O=Z,Z} H_{\lambda}(O_m) \geq -\log_2 \alpha_m,
\]

with \(m = B, C, BC\), since \(\int d\lambda q(\lambda) = 1\). From the above we can see that if we considered EUR different from the ones of [18], we could find other EUR for nonsteerable states. These relations can be used to define sufficient conditions for multipartite steering. Indeed any violation indicates its presence.

Let us consider complementary observables, namely observables whose eigenbasis are mutually unbiased [27]. In this case we have:

\[
\sum_{O=Z,Z} H_{\lambda}(O_{BC}) \geq 2 \log_2 d_{BC};
\]

\[
\sum_{O=Z,Z} H_{\lambda}(O_m) \geq \log_2 d_m,
\]

for \(m = B, C\); being \(d_m\) the dimension of the system \(m\) and \(d_{BC} = d_B d_C\).
We want now to extend these results to the case of GMS states. Suppose now that the state of the system is non-GMS, then we have:

\[
p(x_b, x_c|x_a) = \int_A d\nu p(x_b, x_c, \nu|x_a) + \int_B d\gamma p(x_b, x_c, \gamma|x_a) + \int_C d\omega q(x_b, x_c, \omega|x_a).
\]

Since (32) holds, each term can be written as follows:

\[
p(x_b, x_c, \nu|x_a) = p(\nu|x_a) p(x_b, x_c); \quad (44)
\]

\[
p(x_b, x_c, \gamma|x_a) = p(\gamma|x_a) p_\gamma(x_b) p(x_c|x_a, \gamma); \quad (45)
\]

\[
p(x_b, x_c, \omega|x_a) = p(\omega|x_a) p_\omega(x_c) p(x_b|x_a, \omega). \quad (46)
\]

Equation (43) can be also expressed as \( p(x_b, x_c|x_a) = \int d\lambda p(x_b, x_c, \lambda|x_a) \) where \( \lambda \) is a classical variable such that \( \int d\lambda p(\lambda) = \int_A d\nu p_A(\nu) + \int_B d\gamma p_B(\gamma) + \int_C d\omega q(\omega) = 1 \). We consider now the relative entropy between \( p(x_b, x_c, \lambda|x_a) \) and \( p(\lambda|x_a) p(x_b, x_c|x_a) \), which must be nonnegative:

\[
\sum_{b,c} \int d\lambda p(x_b, x_c, \lambda|x_a) \log_2 \left( \frac{p(x_b, x_c, \lambda|x_a)}{p(\lambda|x_a) p(x_b, x_c|x_a)} \right) \geq 0.
\]

The above quantity is a sum of two terms. The first one is:

\[
- \sum_{b,c} \int d\lambda p(x_b, x_c, \lambda|x_a) \log_2 p(x_b, x_c|x_a)
\]

that is \( H(X_B, X_C|X_A = X_a) \), since (43) holds. The second one is:

\[
\sum_{b,c} \int d\lambda p(x_b, x_c, \lambda|x_a) \log_2 \left( \frac{p(x_b, x_c, \lambda|x_a)}{p(\lambda|x_a)} \right),
\]

which, by using the decomposition of \( p(x_b, x_c, \lambda|x_a) \) given by (44), (45) and (46), can be written as a sum of three terms:

\[
- \int_A d\nu p(\nu|x_a) H_\nu(X_b, X_c); \quad (50)
\]

\[
- \int_B d\gamma p(\gamma|x_a) [H_\gamma(X_b) + H(X_c|X_A = x_a, \gamma)]; \quad (51)
\]

\[
- \int_C d\omega q(\omega|x_a) [H_\omega(X_c) + H(X_b|X_A = x_a, \omega)]. \quad (52)
\]

After reordering the terms and averaging over \( x_a \), that for example implies \( \sum_a p(x_a) p(\gamma|x_a) = q_B(\gamma) \) and similar relations, we arrive at:

\[
H(X_{BC}|X_A) \geq \int_A d\nu q_A(\nu) H_\nu(X_{BC}) \quad (53)
\]

\[
+ \int_B d\gamma q_B(\gamma) H_\gamma(X_B) + \int_C d\omega q_C(\omega) H_\omega(X_C) \quad (54)
\]

where \( X_{BC} = X_B \otimes X_C \). Since \( H(X_C|X_A, \gamma) \geq 0 \) and \( H(X_B|X_A, \omega) \geq 0 \) for any \( \gamma \) and \( \omega \), we finally arrive at:

\[
H(X_{BC}|X_A) \geq \int_A d\nu q_A(\nu) H_\nu(X_{BC}) \quad (55)
\]

\[
- \int_B d\gamma q_B(\gamma) H_\gamma(X_B)
\]

\[
- \int_C d\omega q_C(\omega) H_\omega(X_C).
\]

Then by using (49) we can derive the following state-dependent entropic uncertainty relations:

\[
\sum_{O=X,Z} H(O_{BC}|O_A) \geq - \int_A d\nu q_A(\nu) \log_2 \alpha_{BC} \quad (56)
\]

\[
- \int_B d\gamma q_B(\gamma) \log_2 \alpha_B
\]

\[
- \int_C d\omega q_C(\omega) \log_2 \alpha_C.
\]

Note that in general \( \alpha_{BC} \geq \min \{ \alpha_B, \alpha_C \} = \alpha_{min} \), hence the above implies:

\[
\sum_{O=X,Z} H(O_{BC}|O_A) \geq - \log_2 d_{min}, \quad (57)
\]

which for complementary observables becomes:

\[
\sum_{O=X,Z} H(O_{BC}|O_A) \geq \log_2 d_{min},
\]

where \( d_{min} = \min \{ d_B, d_C \} \).

Now we focus on the conditional entropies \( H(X_B|X_A) \) and \( H(X_C|X_A) \). Since \( p(x_b|x_a) = \sum_{x_c} p(x_b, x_c|x_a) \) the three terms (44), (45) and (46) become:

\[
p(x_b, \nu|x_a) = p(\nu|x_a) p_\nu(x_b); \quad (58)
\]

\[
p(x_b, \gamma|x_a) = p(\gamma|x_a) p_\gamma(x_b); \quad (59)
\]

\[
p(x_b, \omega|x_a) = p(\omega|x_a) p(x_b|x_a, \omega). \quad (60)
\]
From the above relations we derive, with the same arguments that we have used in the previous case,

\[ H (X_B|X_A) \geq \int_A dvq_A (\nu) H_\nu (X_B) + \int_B d\gamma q_B (\gamma) H_\gamma (X_B) + \int_C d\omega q_C (\omega) H (X_B|X_A, \omega). \]  

(61)

The same holds for Bob:

\[ H (X_C|X_A) \geq \int_A dvq_A (\nu) H_\nu (X_C) + \int_C d\omega q_C (\omega) H (X_C|X_A, \omega). \]  

(62)

As an example we consider the criteria we can also look at conditional entropies of the form \( H (X_B|X_A, X_C) \) and \( H (X_C|X_A, X_B) \) where measurements on parts different from Alice are performed. As an example we consider \( H (X_C|X_A, X_B) \), therefore we are interested in the probability:

\[ p (x_c|x_a, x_b) = \int_A d\nu p (x_c, \nu|x_a, x_b) + \int_B d\gamma p (x_c, \gamma|x_a, x_b) + \int_C d\omega p (x_c, \omega|x_a, x_b). \]  

(63)

Since the state is non-GMS, the terms in the above equation can be written as follows:

\[ p (x_c, \nu|x_a, x_b) = p (\nu|x_a, x_b) p (x_c|x_a, \nu); \]  

(64)

\[ p (x_c, \gamma|x_a, x_b) = p (\gamma|x_a, x_b) p (x_c|x_a, \gamma); \]  

(65)

\[ p (x_c, \lambda|x_a, x_b) = p (\omega|x_a, x_b) p (x_c|x_a). \]  

(66)

From the above we can derive in the usual way that:

\[ H (X_C|X_A, X_B) \geq \int_C d\omega q_C (\omega) H_\omega (X_C). \]  

(67)

The same holds for Bob:

\[ H (X_B|X_A, X_C) \geq \int_B d\gamma q_B (\gamma) H_\gamma (X_B). \]  

(68)

In terms of entropic uncertainty relations we have:

\[ \sum_{O=X,Z} H (O_B|O_A, O_C) \geq - \int_B d\gamma q_B (\gamma) \log_2 \alpha_B. \]  

(69)

\[ \sum_{O=X,Z} H (O_C|O_A, O_B) \geq - \int_C d\omega q_C (\omega) \log_2 \alpha_C. \]  

(70)

these two equations are equivalent to eqs. (9). Moreover, when combined with (59) they imply eq. (12), namely:

\[ \sum_{O=X,Z} A (O_A, O_B, O_C) \geq -2 \log_2 \alpha_{min}, \]  

(71)

where \( A (O_A, O_B, O_C) = H (O_B|O_A) + \sum_{m=B,C} H (O_m|O_A, O_m) \) and \( \alpha_{min} = \min \{ \alpha_B, \alpha_C \} \).

A non-steerable state satisfies equation (11) instead, namely:

\[ \sum_{O=X,Z} A (O_A, O_B, O_C) \geq -2 \log_2 \alpha_B \geq -4 \log_2 \alpha_{min}. \]  

(72)

Indeed for a non-steerable state we have shown that \( \sum_{O=X,Z} H (O_B|O_A) \geq - \log_2 \alpha_B \). Then in this case we also have:

\[ p (x_c|x_a, x_b) = \int d\lambda p (x_c, \lambda|x_a, x_b), \]  

(73)

with \( p (x_c, \lambda|x_a, x_b) = p (\lambda|x_a, x_b) p (x_c) \), which by using the usual procedure leads to:

\[ \sum_{O=X,Z} H (O_C|O_A, O_B) \geq - \log_2 \alpha_C. \]  

(74)

The same holds for Bob:

\[ \sum_{O=X,Z} H (O_C|O_A, O_B) \geq - \log_2 \alpha_B. \]  

(75)

The above equations, together \( \sum_{O=X,Z} H (O_B|O_A) \geq - \log_2 \alpha_B \), lead to (72), which is expressed in terms of \( \alpha_{min} \).

As a final result we want to derive a state-dependent entropic uncertainty relation starting from (59), which we remind that it is valid for any non-GMS states. To derive (74) from (59) we have used the fact that the conditional entropies must be always greater than zero, however we can take advantage of these terms by considering the entropic uncertainty relations in presence of quantum memories (31-33), which, for the cases considered, can be expressed in terms of conditional Shannon entropies (33) as:

\[ \sum_{O=O} H (O_C|O_A, O_C) \geq - \log_2 \alpha_C + S_{\gamma} (C|A), \]  

(76)

\[ \sum_{O=O} H (O_B|O_A, O_B) \geq - \log_2 \alpha_B + S_{\omega} (B|A), \]  

(77)

where \( S_{\gamma} (C|A) \) represents the Von Neumann conditional entropy between Charlie and Alice over the state \( p^{AC} \) and \( S_{\omega} (B|A) \) the Von Neumann conditional entropy between Bob and Alice over \( p^{AB} \). Note that these quantities can be negative (32) for entangled states, moreover their lowest values \( - \log_2 d_C \) and \( - \log_2 d_B \) are reached by maximally entangled states.
Therefore, by using (40), (76) and (77), the quantity \( \sum_{O=\{X, Z\}} H(OBC|OA) \) can be lower bounded as follows:

\[
\begin{align*}
\sum_{O=\{X, Z\}} H(OBC|OA) & \geq -\log_2 \alpha_{CB} \\
+ \int_B d\gamma q_B(\gamma) S_\gamma(C|A) + \int_C d\omega q_C(\omega) S_\omega(B|A),
\end{align*}
\]

where we have used also the relation \( \log_2 \alpha_{BC} = \log_2 \alpha_B + \log_2 \alpha_C \). Then we can conclude that for a non-GMS state the following inequality holds (13):

\[
\begin{align*}
\sum_{O=\{X, Z\}} H(OBC|OA) & \geq -\log_2 \alpha_{CB} \\
+ \int_B d\gamma q_B(\gamma) S_\gamma(C|A) + \int_C d\omega q_C(\omega) S_\omega(B|A).
\end{align*}
\]

IV. TWO-TO-ONE STEERING SCENARIO

A. LHS models

We now consider the two-to-one steering scenario, where Alice and Bob want to nonlocally influence Charlie’s state. In the case of a nonsteerable state the ensemble (28) becomes:

\[
\sigma^C_{x_a, x_b} = \int d\lambda q(\lambda) p(x_a|\lambda) p(x_b|\lambda) \rho^C_{x_a, x_b}, (80)
\]

which can be written in term of probabilities as:

\[
p(x_a, x_b, x_c) = \int d\lambda q(\lambda) p(x_a|\lambda) p(x_b|\lambda) p(x_c|\lambda). (81)
\]

In the above Alice and Bob can share local correlations but are jointly unable to steer Charlie. This case corresponds to a LHS model where, conversely from the previous scenario Bob, receives \( \lambda \) instead of \( \rho^C_{x_a, x_b} \) and obtains \( x_b \) with probability \( p(x_b|\lambda) \). Charlie, as in the previous, is represented by a pre-determined state \( \rho^C_{x_a, x_b} \) which depends solely on \( \lambda \). If the ensemble (28) cannot be written as (80) we say that the state is steerable from Alice and Bob to Charlie. For a non-GMS state the ensemble (28) becomes:

\[
\sigma^C_{x_a, x_b} = \int_A d\nu q_A(\nu) p(x_a|\nu) \sigma^C_{x_a, x_b|\nu} \\
+ \int_B d\gamma q_B(\gamma) p(x_b|\gamma) \sigma^C_{x_a, x_b|\gamma} \\
+ \int_C d\omega q_{AB}(\omega) p(x_a, x_b|\omega) \rho^C_{x_a, x_b|\omega}, (82)
\]

which is as a sum of three terms; in the first only Bob can steer Charlie, in the second only Alice can steer the state of Charlie, whereas in the third Alice and Bob cannot jointly steer Charlie, but they can share quantum correlations. In terms of probabilities we have:

\[
p(x_a, x_b, x_c) = \int_A d\nu q_A(\nu) p(x_a|\nu) p(x_b|\nu) p(x_c|x_b\nu) \\
+ \int_B d\gamma q_B(\gamma) p(x_a|\gamma) p(x_b|\gamma) p(x_c|x_a, \gamma) \\
+ \int_C d\omega q_{AB}(\omega) p(x_a, x_b|\omega) p(x_c|\omega). (83)
\]

B. Entropic steering inequalities

Let us start with a nonsteerable state: it is characterized by:

\[
p(x_c|x_a, x_b) = \int d\lambda p(x_c, \lambda|x_a, x_b), (84)
\]

with

\[
p(x_c, \lambda|x_a, x_b) = p(\lambda|x_a, x_b) p(x_c|\lambda). (85)
\]

The above relation allows us to consider, given \( x_a \) and \( x_b \), the relative entropy between \( p(x_c, \lambda|x_a, x_b) \) and \( p(x_c|x_a, x_b) p(\lambda|x_a, x_b) \) that by definition is always nonnegative, namely we have:

\[
\sum_c d\lambda p(x_c, \lambda|x_a, x_b) \log_2 \left( \frac{p(x_c, \lambda|x_a, x_b)}{p(\lambda|x_a, x_b) p(x_c|x_a, x_b)} \right) \geq 0. (86)
\]

By proceeding as in the previous section we can arrive at:

\[
H(X_c|X_A, X_B) \geq \int d\lambda H(\lambda|X_c), (87)
\]

where also in this case \( \lambda \) indicates that the Shannon entropy \( H(\lambda|X_c) \) is calculated over the state \( \rho^C_{x_a, x_b} \). By using the EUR \( H(\lambda|X_c) + H(\lambda|Z_c) \geq -\log_2 \alpha_C \) we arrive at:

\[
\sum_{O=\{X, Z\}} H(OC|OA, OB) \geq -\log_2 \alpha_C. (88)
\]

For a nonsteerable state also the following holds:

\[
p(x_c, \lambda|x_m) = p(\lambda|x_m) p(x_c), (89)
\]

with \( m = A, B \). With the same derivation for \( m = A, B \) we can therefore arrive at:

\[
H(X_c|X_m) \geq \int d\lambda H(\lambda|X_c), (90)
\]

which then implies:

\[
\sum_{O=\{X, Z\}} H(OC|O_m) \geq -\log_2 \alpha_C. (91)
\]

The entropic uncertainty relations (88), (91) represent sufficient criteria to steerability from Alice and Bob to
Charlie. Note that if we sum the above inequality over \( m \) we derive:

\[
\sum_{m=A,B} \left( \sum_{O=X,Z} H(O_C|O_m) \right) \geq -2 \log_2 \alpha_C. \tag{92}
\]

Let us now focus on non-GMS states. The probability \( p(x_c|x_a, x_b) \) can be now written as:

\[
p(x_c|x_a, x_b) = \int_A dv p(x_c, \nu|x_a, x_b) + \int_B d\gamma p(x_c, \gamma|x_a, x_b) + \int_C d\omega p(x_c, \omega|x_a, x_b). \tag{93}
\]

The three terms in the above equation can be expressed as follow:

\[
p(x_c, \nu|x_a, x_b) = p(\nu|x_a, x_b) p_c(x_c|x_b); \tag{94}
\]

\[
p(x_c, \gamma|x_a, x_b) = p(\gamma|x_a, x_b) p_\gamma(x_c|x_a); \tag{95}
\]

\[
p(x_c, \omega|x_a, x_b) = p(\omega|x_a, x_b) p_\omega(x_c). \tag{96}
\]

Now we can consider the relative entropy between \( p(x_c, \lambda|x_a, x_b) \) and \( p(x_c|x_a, x_b) p(\lambda|x_a, x_b) \), where \( \lambda \) is such that \( \int d\lambda q(\lambda) = \int_A d\nu q_A(\nu) + \int_B d\gamma q_B(\gamma) + \int_C d\omega q_C(\omega) = 1 \). The non-negativity of the relative entropy implies:

\[
\sum_{c} \int d\lambda p(x_c, \lambda|x_a, x_b) \log_2 \left( \frac{\rho(x_c, \lambda|x_a, x_b)}{p(\lambda|x_a, x_b) p_c(x_c|x_b)} \right) \geq 0. \tag{97}
\]

By proceeding as in the derivation of \((54)\) we arrive in this case at:

\[
H(X_C|X_A, X_B) \geq \int_C d\omega q_C(\omega) H_\omega(X_C). \tag{98}
\]

We can also consider the relative entropy between \( p(x_c, \lambda|x_a) \) and \( p(x_c|x_a) p(\lambda|x_a) \). The same arguments used to derive \((62)\) leads us to:

\[
H(X_C|X_A) \geq \int_A d\nu q_A(\nu) H_\nu(X_C) + \int_C d\omega q_\omega(\omega) H_\omega(X_C). \tag{99}
\]

The same holds by conditioning on Bob:

\[
H(X_C|X_A) \geq \int_B d\gamma q_\gamma(\gamma) H_\gamma(X_C) + \int_C d\omega q_\omega(\omega) H_\omega(X_C). \tag{100}
\]

The corresponding EUR of \((98)\) and \((100)\) are then given by:

\[
\sum_{O=X,Y} H(O_C|O_A, O_B) \geq -\log_2 \alpha_C \int_C d\omega q_C(\omega); \tag{101}
\]

\[
\sum_{O=X,Y} H(O_C|O_A) \geq -\log_2 \alpha_C \left( \int_A d\nu q_A(\nu) + \int_C d\omega q_C(\omega) \right); \tag{102}
\]

\[
\sum_{O=X,Y} H(O_C|O_B) \geq -\log_2 \alpha_C \left( \int_B d\gamma q_\gamma(\gamma) + \int_C d\omega q_C(\omega) \right). \tag{103}
\]

The sum of the two last inequalities above implies:

\[
\sum_{O=X,Y} (H(O_C|O_A) + H(O_C|O_B)) \geq -\log_2 \alpha_C \left( 1 + \int_C d\omega q_C(\omega) \right), \tag{104}
\]

then by using \(-\log_2 \alpha_C \int_C d\omega q_C(\omega) \geq 0\), we arrive at:

\[
\sum_{O=X,Y} (H(O_C|O_A) + H(O_C|O_B)) \geq -\log_2 \alpha_C, \tag{105}
\]

which in the case of complementary observables is:

\[
\sum_{O=X,Y} (H(O_C|O_A) + H(O_C|O_B)) \geq \log_2 d_C. \tag{106}
\]

Any violation of the above inequality indicates the presence of genuine multipartite steering from Alice and Bob to Charlie.

V. STEERING DETECTION

In this section we study the steering detection power of the relations derived in the previous sections. Here we consider only multiqubit systems and complementary observables, namely \( X \) and \( Z \) are always the Pauli matrices \( \sigma_x \) and \( \sigma_z \) respectively. We study the following quantities:

\[
S_1 = \sum_{O=X,Z} H(O_{BC}|O_A); \tag{107}
\]

\[
S_2 = \sum_{O=X,Z} H(O_B|O_A); \tag{108}
\]

\[
S_3 = \sum_{O=X,Z} H(O_C|O_A); \tag{109}
\]

\[
C = \sum_{O=X,Z} H(O_B|O_A, O_C); \tag{110}
\]

\[
A = \sum_O H(O_{BC}|O_A) + \sum_{m=B,C} H(O_m|O_A, O_m). \tag{111}
\]

As a first example we consider the GHZ class of three qubit states, which can be expressed as:

\[
|GHZ\rangle = a |000\rangle + \sqrt{1-a^2} |111\rangle, \tag{112}
\]

with \( 0 \leq a \leq 1 \). One can check that \( S_2 \) and \( S_3 \) are always greater or equal to 1, that is the threshold below which quantum steering is detected. Hence for GHZ these two relations do not see any form of steering. In
Figure 1. Plot of $S_1$ for the state $a|000\rangle + \sqrt{1-a^2}|111\rangle$ as a function of $a$. The upper dashed line represents the threshold to detect steering (which is then seen for all $a$), the lower dashed line is the threshold for GMS.

Figure 2. Plot of $A$ for the state $a|000\rangle + \sqrt{1-a^2}|111\rangle$ as a function of $a$. The upper dashed line represents the threshold to detect steering (which is then seen for all $a$), the lower dashed line is the threshold for GMS. Some GMS states are detected with $A$.

Figure 3. Plot of $A$ for the state $p|GHZ\rangle \langle GHZ| + \frac{1-p}{8}I$ as a function of $p$. The upper dashed line represents the threshold to detect steering, while the lower dashed line is the threshold for GMS.

VI. CONCLUSIONS

In conclusion, we derived and characterized a certain number of entropic uncertainty inequalities whose violation guarantees the presence of different classes of multipartite steering. Most of all these criteria enable to distinguish between multipartite steering and genuine multipartite steering and, being state-independent, they allow to study the steering property of an unknown multipartite state. Recently, for bipartite systems some new steering criteria based on generalized entropic uncertainty relations, namely defined in terms of Tsallis entropies instead of Shannon one, have been derived [39]. It will be therefore interesting to extend our results by considering these generalized entropies.

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