1. Introduction

The first realization of Bose–Einstein condensates (BECs) more than two decades ago has announced the entering of a new field in many-body physics with ultracold quantum gases, in which many-body physics and matter-wave coherence properties can be sufficiently described by a single macroscopic wave function, and the fundamental quantum mechanics (in such systems) can be studied in a macroscopic, accessible, and controlled way accordingly.[1–7] The evolutionary dynamics of weakly ultracold Bosonic gases has been well described by a mean-field theory called Gross–Pitaevskii equation, which, formally speaking, bears the same expression of the nonlinear Schrödinger equation used in diverse fields. It is well established that one of the most interesting and key issues of emergent nonlinear phenomena in the BEC context lies in the formation, dynamics, property, and management of coherent matter waves like bright and dark solitons, which, in terms of the basic theory, root in the BECs with attractive and repulsive nonlinearities induced by atom–atom collisions, respectively.[14–17] However, the bright solitons, immersed in 2D and 3D self-focusing (attractive) nonlinearity, are vulnerable to critical and supercritical collapses (blowup), respectively.[3–5,8–13] The same problems exist in BECs and other contexts. Various strategies to stabilize multidimensional solitons were thus conceived.[4,7,14,15]

The stabilization of multidimensional matter-wave solitons can be realized by two important, powerful, and practical methods—optical lattices,[4,7,16,17] created by counterpropagating laser beams—and Feshbach resonance (a nonlinear way).[4,18–20] In particular, by loading the BECs onto optical lattices, bright matter-wave gap solitons[21] and truncated nonlinear Bloch waves (gap waves)[22] have also been predicted; both 2D and 3D solitons of gap and vortical types can also be stabilized by low-dimensional optical lattices,[42–45]–49] and photonic Moiré lattices,[46,47] wherein both can be realized in experiments. From a theoretical viewpoint, 2D matter-wave gap solitons and vortical ones, both bright and dark types, have been predicted in the pancake/disc-shaped BECs trapped by 2D optical lattice potentials,[23–31] 3D Weyl solitons,[32] gap solitons, and vortices of a spherical (repulsive) BEC[33,34] and a dipolar one[35] confined in 3D optical lattices have also been predicted; both 2D and 3D solitons of gap and vortical types can also be stabilized by low-dimensional optical lattices[36–39]; self-trapping and stable multidimensional solitons supported by 2D and 3D periodic optical lattice potentials and described by the mean-field hydrodynamic model have entered the context of superfluid (degenerate) Fermi gases.[40,41] In particular, the novel 2D linear periodic lattices as parity-time (PT) symmetric potentials[42–45] and photonic Moiré lattices,[46,47] wherein both can find their optical lattices counterparts in BECs, were recently applied to stabilizing various solitons.

Feshbach resonance offers an alternative and nonlinear scheme for stabilizing matter-wave solitons. The value and even the sign of the nonlinear strength of the cubic (Kerr) term...
characterizing the interaction between atoms (in the Gross–Pitaevskii equation) can be tuned as periodic or nonuniform forms, controlled by Feshbach resonance management. Previous experimental work has demonstrated the realization of nonlinear lattices based on optical Feshbach resonance mediated by periodic standing waves. The combined use of optical lattices and Feshbach resonance techniques has attracted great attention for stabilizing gap solitons of different types, where the commensurate model of linear–nonlinear lattices (both have the same modulation form) has been discussed and can be realized using the optical lattice and Feshbach resonance techniques. Although gap solitons can be stabilized in media with homogeneous nonlinearity (as stated earlier), the periodic modulation can make the model more flexible for creating solitons of diverse types and for stabilizing them in different scenarios like commensurability (spatial resonance) and incommensurability between the linear and nonlinear lattices. Very few studies have focused on generating stable full spatial (3D) localized ordinary and gap modes by 3D optical lattices. The combination of the 3D optical lattices and nonlinear lattices created by Feshbach resonance, however, has not been used to explore multidimensional solitons.

This work devotes to examining the existence, property, and stability of 3D localized matter-wave gap modes in BECs loaded on optical lattices and space-periodic nonlinear potentials induced by Feshbach resonance, constituting combined linear–nonlinear lattices or 3D nonlinear photonic crystals that were recently fabricated in the optics context. An intriguing diversity of 3D localized gap modes, represented as fundamental gap solitons, on-axis or off-axis higher-order gap-soliton clusters of planar and vertical (solid) modes, as well as vortex gap solitons with vorticity (topological charge) \( s = 1 \), is demonstrated numerically and theoretically. Stability and instability diagrams of all the 3D localized gap modes are obtained by means of direct simulations of the underlying perturbed modes. The setting can be directly realized in BECs by combining the optical lattices and Feshbach resonance techniques, providing a new path to generating and stabilizing 3D localized gap modes of different types in the real 3D physical space, the unique soliton physics which cannot be provided by low-dimensional systems.

2. Theoretical Model

The mean-field theory of a BEC trapped by a 3D optical lattice and space-periodic nonlinear potential, the combined linear–nonlinear lattice model, is based on the Gross–Pitaevskii equation, whose scaled form yields

\[ \frac{\partial \psi}{\partial t} = -\frac{1}{2} \nabla^2 \psi + V_{OL}(r)(1 + g |\psi|^2) \psi \] (1)

Here \( \psi \) is the macroscopic matter-wave wave function, \( r = (x, y, z) \) and Laplacian \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \). As we discuss on the localized gap modes of BECs with repulsive (self-defocusing) nonlinearity, the nonlinear coefficient \( g \) is thus set as \( g > 0 \). The 3D simple cubic optical lattice takes the form

\[ V_{OL}(r) = V_0 (\sin^2 x + \sin^2 y + \sin^2 z) \] (2)

with \( V_0 \) being the modulation depth (amplitude) of the optical lattice with periodicity \( d = \pi \), which is scaled by the lattice recoil energy \( E_{rec} = \hbar^2 \pi^2 / (2md^2) \). The profile of such a 3D cubic optical lattice is displayed in Figure 1a.

![Figure 1](image-url)

**Figure 1.** Profile, the first Brillouin zone, and matter-wave bandgap spectrum of a 3D simple cubic optical lattice. a) Isosurface of the cubic lattice, the corresponding first Brillouin zone, b) and bandgap spectra in reciprocal lattice space with lattice strength c) \( V_0 = 3 \) and d) \( V_0 = 6 \).
3. Numerical Results

3.1. 3D Fundamental Gap Solitons

In this work, we are first interested in fundamental modes of the 3D localized gap modes called fundamental gap solitons that are isotropic along each spatial dimension. Our direct perturbed numerical simulations demonstrate that, for the fundamental gap solitons localized within the first finite bandgap, they are stable modes in the left edge of the first gap and are unstable close to the right edge of the gap (see Figure 2a). Here, we judge the stability of the solitons in direct perturbed dynamics over evolution time \( t = 1000 \) corresponding to approximate 1s in the experiment for \(^{87}\text{Rb}\).[34] It is also seen from Figure 2a that the curve \( N(\mu) \) for the gap solitons coincides with the empirical stability criterion, the “anti-Vakhitov–Kolokolov” (anti-VK) criterion, \( dN/d\mu > 0 \), which, essentially, is a necessary but not a sufficient condition for the stability of gap solitons under repulsive nonlinearity.[54,56,64] The norm \( N \) of the solitons is defined by \( N = \int_{-L/2}^{L/2} |\psi(r)|^2 dr \), with \( L \) the total integration length. The characteristic profile of a fundamental matter-wave gap soliton looks like a “quantum ball,” isotropic spatially; it exhibits a cusp-like modulation when it is prepared initially close to the gap edge (which resembles its counterpart supported by 3D optical lattices and constant nonlinearity[14]), according to their profiles in Figure 2b,c and their bird’s eye views at the plane \( z = 0 \) in Figure 2d,e.

3.2. 3D Gap-Soliton Clusters: Planar and Vertical Soliton Composites

Next, we search the 3D higher-order matter-wave gap solitons of the combined linear–nonlinear lattice model. The planar gap-soliton clusters, with all the elements (gap solitons) arranged in a flat plane (e.g., \( z = 0 \)), are the most simple higher-order gap modes. Their stability diagram represented by the dependency \( N(\mu) \) and generic examples of such planar gap-soliton clusters are displayed in Figure 3, which showcases the off-axis modes constituted by five gap solitons (with one centered at the origin of coordinates, the point with coordinates \((0,0,0)\)) and quadruple-mode solitons spaced with a distance \( 2\pi \) (twice to the period of the optical lattice) in Figure 3b,c respectively, and their bird’s eye contours at the plane \( z = 0 \) in Figure 3d,e. Our direct simulations of the evolution of the underlying perturbed soliton clusters demonstrate...
that the stability region of such planar soliton composites narrows with an increase in the number of gap solitons (see Figure 3a). The property can be understood by the fact that the destructive interaction between each individuals becomes more intense as the number of participants (gap solitons) increases; such a common effect exists also for 2D gap-soliton clusters in other physical periodic systems with low-dimensional optical lattice.[15] Physically, the coherence of nonlinear matter waves with more atoms could induce intense site-to-site tunneling dynamics,[22] which will destroy the stability of soliton clusters.

We further verify that, besides the off-axis modes centered not on the axes \((x, y) = 0\), the model also supports the on-axis modes with all the constituted gap solitons centered at axes \((x, y) = 0\); typical examples of them and their stability regions depending on the relation \(N(\mu)\) are depicted in Figure 4. It is observed from Figure 4a that the stability regions for on-axis five-soliton clusters and four-soliton ones, with their isosurface plots and the corresponding densities \(|\psi|\) at the plane \(z = 0\) displayed respectively in the second (see Figure 4b,c) and third rows (see Figure 4d,e), shrink greatly compared with their off-axis modes (see Figure 3a).

The planar soliton composites are the simplest gap-soliton clusters in the underlying 3D physical geometry. A fascinating possibility is to construct the vertical soliton composites (alias...
solid modes) in the 3D Euclidean space; it is a significant difference from their low-dimensional systems where only planar modes can be built in 2D space. Families of 3D higher-order matter-wave gap-soliton clusters (on-axis solid modes) centered at 3D space and consisting of seven and six gap solitons are accumulated in Figure 5, which displays the relationship between atom number of wave function $N$ and chemical potential $\mu$ (see Figure 5a) and isosurface plots of a stable seven-soliton cluster and an unstable six-soliton cluster in Figure 5b,c, respectively. Through the comparison of the results of their on-axis planar modes (see Figure 4a), although the stability region of the on-axis seven-soliton solid cluster (with a gap soliton centered at the origin of coordinates $(0,0,0)$) makes not a big difference with its planar counterpart (on-axis five-soliton planar cluster), the stability region expands greatly for the on-axis six-soliton solid cluster compared with its planar gap-soliton cluster mode (quadruple mode), highlighting that the 3D optical lattice is robust to the formation of stable 3D gap solitons of solid modes.

As mentioned earlier, the relationships $N(\mu)$ for all the fundamental gap solitons and their diverse higher-order modes depicted in the panels of Figure 2a,3a,4a,5a, with stable and unstable modes marked by solid and dashed lines, respectively, have been verified by our direct perturbed numerical simulations. In the first two rows of Figure 6, perturbed evolution over time of a stable fundamental gap soliton and an unstable one corresponding to the marked points A1 and A2 in Figure 2a are depicted, from which one can see that the unstable soliton first expands and then becomes a highly localized mode with multiple side peaks. Instead of expanding or decaying, the stable 3D higher-order matter-wave gap-soliton cluster, corresponding to the off-axis mode B2 in Figure 3a, maintains its coherence during evolution, according to the third row in Figure 6. For the unstable on-axis higher-order gap-soliton modes, represented as the planar soliton composite at plane $z = 0$ and vertical one (solid mode) and marked as points C2 and D2 in Figure 4a,5a respectively, their evolutions are displayed in the fourth and fifth rows in Figure 6, showing that the unstable on-axis gap-soliton composites undergo expansion and then evolve into a smaller localized mode with highly modulated peaks, which resemble the evolutional process of their 3D fundamental counterparts—unstable fundamental gap solitons.

Our model (Equation (1)) is closely associated with ultracold atom experiment, with the condition that the total density $N = \int_{-\infty}^{+\infty} |U|^2 dr = 1/(2\pi a_0 L |a_0|)$ in atoms $m^{-3}$, with $a_L = p/\pi$ and $a_s$ the scattering length of atoms. For $^{87}\text{Rb}$ atoms loaded onto a 3D optical lattice having period ($p$) of half-laser wavelength ($\lambda = 800 \text{ nm}$), $p = 400 \text{ nm}$, the generated 3D fundamental gap solitons populating inside a single lattice have about 600 atoms, corresponding to the normalized norm $N = 20$ in Figure 2. For soliton clusters shown in Figure 3–5, the total atoms range from 2400 to 4200.

3.3. 3D Gap Vortices: Planar Vortices

A more interesting issue is to construct 3D gap vortices in the introduced model with the combined optical lattice and space-periodic nonlinear potentials. We concentrate on the planar vortices composed of quadruple-soliton modes centered at the plane $z = 0$. In Figure 7a,b, we produced the dependencies $N(\mu)$ for the off-axis and on-axis quadruple-soliton modes of such planar vortices carrying the topological charge $s = 1$, with their typical isosurface plots displayed, respectively, in Figure 7c,e, and the corresponding phase structures projected onto plane $z = 0$ (bird’s eye views) in Figure 7d,f.

The stabilization and destabilization dynamics of the 3D planar gap vortices imprinted with vorticity (topological charge) $s = 1$, corresponding to the off-axis and on-axis quadruple-soliton modes and as marked points E1 and E2 in Figure 7a,b, are respectively depicted in Figure 8a,b. It is seen that, for the stable gap vortex soliton, its coherence remains under small perturbations; the situation is more complex for the unstable one, evidenced by the fact that the unstable on-axis quadruple-soliton vortex mode experiences shrinkage and expansion through transforming itself into a mode with multiple tails and then undergoes a rotating configuration by exchanging phases and energies between the four elements which are phase connected by

![Figure 6](image-url)  
**Figure 6.** Dynamics for 3D fundamental gap-soliton and higher-order solitons. A1, A2) Fundamental gap modes; cluster modes of B1) off-axis and C2) on-axis types centered at the plane $z = 0$, and D2) on-axis gap-soliton cluster centered at 3D space.
vorticity. In contrast, for the 3D higher-order gap-soliton clusters with phase unconnected \( s = 0 \), they do not rotate at all during unstable evolutions (see Figure 6c,d). To distinguish the gap vortices from the soliton clusters in Figure 3–5, the former may be called vector modes (with spiraling phase around the central point), while the latter belongs to scalar modes (i.e., no phase distribution information).

4. Conclusion and Discussion

In closing, we have surveyed, theoretically and numerically, the formation and dynamics of 3D nonlinear localized gap modes in a BEC trapped by a 3D optical (linear) lattice and space-periodic nonlinear potential—which takes the same form (and period) as its linear counterpart—controlled by Feshbach resonance, thus forming combined linear–nonlinear lattices or a 3D nonlinear photonic crystal in optics. Besides the fundamental gap solitons that feature an isotropic shape, the model also supports a vast variety of 3D high-order matter-wave gap solitons in forms of planar and vertical soliton composites; both can be classified as the on-axis and off-axis modes if they are centered on and centered not on the axes \((x, y, z) = 0\). In particular, the solid localized gap modes are unique to the 3D periodic physical systems and are unreachable in low-dimensional space. The 3D gap vortices (vortex solitons) with topological charge \( s = 1 \) were also constructed, the stability of which and the other localized gap-soliton modes were identified by simulating the evolution of the perturbed soliton solutions. The setting can be accomplished in a BEC by means of the combined techniques of optical lattice induced by multiple laser beams and Feshbach resonance controlled by periodic magnetic/optical fields, offering a new way for the formation and stabilization of 3D gap solitons and soliton composites that are not possible in low-dimensional systems.

The predicted 3D localized gap modes here are restricted to the first finite bandgap of the underlying linear spectrum, making stable 3D gap modes in higher bandgaps. The composite vortex states (solid mode) of “vortex atoms” and “vortex crystals” groups\(^{[34]}\) can also be further investigated. An extension of this work is to consider the two-component BECs loaded onto 3D optical lattices and study the multidimensional soliton physics therein. It is also constructive to consider the nonperiodic spatially modulated Feshbach resonances.\(^{[65,66]}\)

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Conflict of Interest

The authors declare no conflict of interest.
Data Availability Statement

Research data are not shared.

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Bose–Einstein condensates, Feshbach resonances, localized gap modes, optical lattices

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