Recovery of coefficients for a weighted $p$-Laplacian perturbed by a linear second order term

Cătălin I Cărstea$^{1,*}$ and Manas Kar$^2$

$^1$ School of Mathematics, Sichuan University, Chengdu, Sichuan, 610064, People’s Republic of China

$^2$ Indian Institute of Science Education and Research Bhopal, Indore By-pass Road, Bhauri Bhopal, 462066, Madhya Pradesh, India

E-mail: catalin.carstea@gmail.com

Received 28 July 2020, revised 24 November 2020
Accepted for publication 27 November 2020
Published 17 December 2020

Abstract
This paper considers the inverse boundary value problem for the equation

$$\nabla \cdot \left( \sigma \nabla u + a |\nabla u|^{p-2} \nabla u \right) = 0.$$

We give a procedure for the recovery of the coefficients $\sigma$ and $a$ from the corresponding Dirichlet-to-Neumann map, under suitable regularity and ellipticity assumptions.

Keywords: inverse boundary value problems, $p$-Laplacian, quasilinear elliptic equations

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with smooth boundary $\partial \Omega$. In this domain, we may consider the following quasilinear boundary value problem for the function $u$

$$\begin{cases}
\nabla \cdot (J(x, u, \nabla u)) = 0, \\
u|_{\partial \Omega} = f.
\end{cases} \quad (1)$$

Here we assume $J$ is a vector valued function.

In the linear case $J(x, u, \nabla u) = \sigma(x) \nabla u$, with a function $\sigma$ that has both upper and lower bounds that are positive, (1) may be interpreted as the equation describing conduction in an object whose spatial extension coincides with $\Omega$, and whose (possibly inhomogeneous) conductivity matrix is $\sigma$. In this case $u$ would be the electric potential (with boundary values $f$) and $J(x, u(x)) = \sigma(x) \nabla u(x)$ would be the current density, as given by Ohm’s law. The general case could then be seen (for example, and not exclusively) as describing nonlinear conduction phenomena.

*Author to whom any correspondence should be addressed.
Assuming the equation (1) can be solved for some class of Dirichlet boundary data, with solutions having suitable regularity, we can define the Dirichlet-to-Neumann map (which may also be called the ‘potential-to-current map’ in the conduction interpretation of the equation) to be

\[ \Lambda f = \nu \cdot (x, u, \nabla u)_{\partial \Omega}, \]  

where \( u \) is the solution to equation (1) and \( \nu \) is the unit outer pointing normal vector to the boundary \( \partial \Omega \). From an experimental point of view, \( \Lambda \) represents all the information that may be obtained from boundary measurements of pairs of Dirichlet and Neumann data (potential and current, if the equation is describing conduction).

The inverse problem proposed by Calderón in [7] is to invert the mapping \( J \rightarrow \Lambda J \). A nongeometric subproblem is the question of uniqueness, i.e. the question of the injectivity of the mapping \( J \rightarrow \Lambda J \). In the linear case, much work has already been done towards solving these problems, especially in the isotropic case (i.e. when \( \sigma \) is a multiple of the identity matrix). For \( n \geq 3 \), uniqueness was proven in [32] and a method for reconstructing \( \sigma \) from the Dirichlet-to-Neumann map was given in [26]. The anisotropic case is more difficult, as uniqueness does not hold, and the problem is still open when \( n \geq 3 \).

In the case of elliptic semilinear or quasilinear equations, there are a number of uniqueness results that are known. For the semilinear case, see [12, 18, 19, 22–24, 30]. For the quasilinear case, see [1, 8] for quasilinear time-harmonic Maxwell systems.

In this paper we will consider the following boundary value problem for the complex valued function \( u \)

\[
\begin{aligned}
\nabla \cdot (\sigma(x) \nabla u + a(x)|\nabla u|^{p-2} \nabla u) &= 0, \\
u|_{\partial \Omega} &= f,
\end{aligned}
\]

where \( p \in (1, \infty) \setminus \{2\} \),

\[ |\nabla u|^2 = \nabla u \cdot \nabla u, \]  

\[ \sigma \in C^\infty(\Omega), \quad a \in L^\infty(\Omega), \]  

and for some positive constants \( \lambda, m > 0 \) we have

\[ 0 < \lambda < \sigma < \lambda^{-1}, \quad 0 < m < a < m^{-1}. \]  

This can be interpreted as a nonlinear conductivity equation, with nonlinear conductivity

\[ \gamma(x, \nabla u) = \sigma(x) + a(x)|\nabla u|^{p-2}. \]  

The associated Dirichlet-to-Neumann map is

\[ \Lambda_{\sigma, a, p} f = (\sigma(x) + a(x)|\nabla u|^{p-2}) \partial_n u|_{\partial \Omega}. \]
where \( u \) is the solution to (3). From here on, \( q \) will be the Hölder conjugate of \( p \), i.e. \( q^{-1} + p^{-1} = 1 \).

The nonlinear term in equation (3) is known as the weighted \( p \)-Laplacian. Inverse problems for the \( \sigma = 0 \) case have been investigated in [3–6, 13, 21]. Except for the two-dimensional case, a uniqueness result has not been obtained yet. Part of the interest of this paper lies in showing that the addition of a linear term to the equation renders the problem much more tractable.

We also need to mention the paper [14], where the authors consider the inverse problem for the equation

\[
\nabla (\sigma \tau^2 + |\nabla u|^2)^{\frac{2}{p-2}} \nabla u = 0, \tag{9}
\]

from a numerical point of view. We can isolate the linear part of their equation by writing it as

\[
\nabla (\sigma \nabla u) + \nabla \left[ \sigma \left( (1 + \tau^2 |\nabla u|^2)^{\frac{2}{p-2}} - 1 \right) \nabla u \right] = 0, \tag{10}
\]

where the second term is clearly purely nonlinear. Though not identical to the equation we are considering here, this model can be treated in a similar way to what we will present in this paper.

1.1. Main results and outline

In order to discuss the inverse problem, we must first give an account of when and in what sense the equation (3) can be solved. Using energy minimization methods it is relatively straightforward to prove the existence of weak solutions. To this end we introduce the spaces of Dirichlet data

\[
X_p = \begin{cases} 
W^{1-\frac{1}{p}}(\partial \Omega) & \text{if } p > 2, \\
W^{1,p}(\partial \Omega) & \text{if } p \in (1,2). 
\end{cases} \tag{11}
\]

We will denote by \( X'_p \) the dual of the space \( X_p \). We will show in section 2 that the Dirichlet-to-Neumann map \( \Lambda_{\sigma,a,p} \) maps \( X_p \) into \( X'_p \).

For \( p > 2 \) and Dirichlet boundary data \( f \in W^{1-\frac{1}{p}}(\partial \Omega) \) with sufficiently small norm, we will show the existence of strong \( W^{2,p}(\Omega) \) solutions of (3). This will be done via a contraction principle argument. In the \( p \in (1,2) \) case, we do not need to use strong solutions in our treatment of the inverse problem, so we forgo investigating this topic here.

Our main result will be

**Theorem 1.** If the assumptions (5), (6) hold, then the parameters \( \sigma, a, \) and \( p \) can be reconstructed from the Dirichlet-to-Neumann map \( \Lambda_{\sigma,a,p} \).

As in many other works on inverse boundary value problems for semilinear and quasilinear elliptic equations, we will use a version of a so called ‘second linearization’ trick originally used in [16]. In our case, when \( p > 2 \), this amounts to considering solutions \( u_\epsilon \) of (3) with Dirichlet data \( \epsilon f \), where \( \epsilon \) is a small parameter. One can show that these solutions have an asymptotic expansion of the form

\[
u_\epsilon = \epsilon u_0 + \epsilon^{p-1} u_1 + \mathcal{O}(\epsilon^{2p-3}), \tag{12}\]

as \( \epsilon \to 0 \). Then the Dirichlet-to-Neumann map will also have a similar expansion. For \( w \) a solution to \( \nabla \cdot (\sigma \nabla w) = 0 \) we have

\[
\langle \Lambda_{\sigma,a,p}(\epsilon f), w \rangle_{|\partial \Omega} = \epsilon \langle \Lambda_{\sigma}(f), w \rangle + \epsilon^{p-1} I(u_0, w) + \mathcal{O}(\epsilon^{2p-3}), \tag{13}\]

3
where $\Lambda_\sigma$ is the Dirichlet-to-Neumann map for the linear problem, $u_0$ is the solution of the linear equation with Dirichlet data $f$, and $I(u_0, w)$ is a map that is homogeneous of order $p - 1$ in $u_0$. Both $\Lambda_\sigma$ and $I$ are determined by the full Dirichlet-to-Neumann map $\Lambda_{\sigma,a,p}$. The coefficient $\sigma$ can then be reconstructed from $\Lambda_{\sigma,a,p}$ using the method of [26].

It is clear that $p$ is also determined by $\Lambda_{\sigma,a,p}$, since it appears in the exponent of $\epsilon$ in the second term of the expansion.

In past work for semilinear or quasilinear equations with polynomial type nonlinearities (e.g. [1, 8, 9, 12, 20, 23, 24]) a polarization trick has been used on the equivalent of our term $I$, in order to obtain an integral identity involving multiple independently chosen solutions of the linear equation. Since here $p$ is not necessarily an integer, we use a trick inspired by polarization to obtain our final integral identity. We then use a method developed in [9] to obtain a reconstruction of the coefficient $a$. All this is covered in section 3.

When $p \in (1, 2)$, the above method needs to be modified since in the limit $\epsilon \to 0$ the nonlinear term (which is now sublinear) would be the more significant one, so we would not be able to derive asymptotic expansions in the same way. Instead, we take Dirichlet data of the form $\epsilon^{-1} f$ and eventually obtain the expansion

$$
\langle \Lambda_{\sigma,a,p} (\epsilon^{-1} f), w|_{\partial \Omega} \rangle = \epsilon^{-1} \langle \Lambda_\sigma, w|_{\partial \Omega} \rangle + \epsilon^{1-p} I(u_0, w) + O(\epsilon^{3-2p}).
$$

(14)

From here the method proceeds identically. We cover this case in section 4.

Some vector identities that we need to use throughout the paper are collected in appendix A.

2. The forward problem

In this section we will derive existence results and estimates for solutions of equation (3) with Dirichlet data in $X_p$. Since we are principally concerned with the inverse problem, we do not attempt to obtain a comprehensive existence and regularity theory for this equation. Instead, we only derive results that are sufficient for our subsequent arguments. In particular, we will prove the existence of weak solutions for all $p \in (1, \infty)$. However, we will only derive the existence of strong solutions in the $p > 2$ case, and only for sufficiently small source and boundary data.

2.1. Weak solutions

**Proposition 2.1.** If $p \in (1, \infty)$, $f \in X_p$, and $F \in L^q(\Omega; \mathbb{R}^n)$, then the boundary value problem

$$
\begin{align*}
\nabla \cdot (\sigma(x) \nabla u + a(x)|\nabla u|^{p-2} \nabla u) &= \nabla \cdot F, \\
u|_{\partial \Omega} &= f,
\end{align*}
$$

(15)

has a unique solution $u \in W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$. Furthermore, there exists $C > 0$, independent of $f, F$ such that

$$
\|u\|_{W^{1,2}(\Omega)} \leq C \left( \|f\|_{X_p} + \|F\|_{L^q(\Omega)} \right), \quad \text{if} \ p > 2,
$$

(16)

$$
\|u\|_{W^{1,2}(\Omega)} \leq C \left( \|f\|_{X_p} + \|F\|_{L^q(\Omega)} \right), \quad \text{if} \ p \in (1, 2).
$$

(17)

**Proof.** Let $S = W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$ be equipped with the norm

$$
\|u\|_S = \max \left( \|\nabla u\|_{W^{1,2}(\Omega)}, \|u\|_{W^{1,p}(\Omega)} \right).
$$

(18)
With this norm, \( S \) is a Banach space. Its dual is \( S' = W^{1,p}(\Omega)' + W^{1,2}(\Omega)' \), with the norm
\[
\|l\|_{S'} = \inf \left\{ \|l_1\|_{W^{1,2}(\Omega)'} + \|l_2\|_{W^{1,p}(\Omega)'} : l = l_1 + l_2, \right. \\
\left. \quad l_1 \in W^{1,2}(\Omega)', l_2 \in W^{1,p}(\Omega)' \right\}.
\] (19)

The space \( S \) is reflexive. (For example, see [2].)

Define the energy
\[
E[v] = \int_\Omega \frac{1}{2} \sigma(x)|\nabla v|^2 + \frac{1}{p} a(x)|\nabla v|^p + \Re \nabla \phi \cdot F, \quad v \in S.
\] (20)

Note that
\[
E[v] \geq \int_\Omega \left( \frac{\lambda}{2} |\nabla v|^2 + \frac{m}{2p} |\nabla v|^p \right) - C \|F\|_{L^p(\Omega)}^p \\
\quad \geq C_1 \left[ \max \left( \|\nabla v\|_{L^2(\Omega)}^2, \|\nabla v\|_{L^p(\Omega)}^p \right) \right] - \|F\|_{L^p(\Omega)}^p.
\] (21)

and, similarly
\[
E[v] \leq C_2 \left[ \max \left( \|\nabla v\|_{L^2(\Omega)}^2, \|\nabla v\|_{L^p(\Omega)}^p \right) \right] + \|F\|_{L^p(\Omega)}^p.
\] (22)

Here \( C_1, C_2 > 0 \) are constants that depend on \( \lambda, m, p \), and \( |\Omega| \).

Let \( u_k \in S \) be a minimizing sequence such that \( u_k|_{\partial \Omega} = f \) and
\[
\lim_{k \to \infty} E[u_k] = \min_{v \in S, v|_{\partial \Omega} = f} E[v].
\] (23)

Since \( u_k \) is bounded in \( S \), up to extracting a subsequence, it converges weakly to a limit \( u \in S \). By the convexity in \( \nabla v \) of the integrand in the definition of \( E[v] \), we have that \( E \) is weakly lower semicontinuous (see [11, section 8.2]). It is then clear that \( E[u] = \min_{v \in S, v|_{\partial \Omega} = f} E[v] \) and (by the convexity of \( E \)) it is the unique minimum.

Let \( \phi \in C_0^\infty(\Omega) \) and, for \( z \in \mathbb{C} \) we should have
\[
\partial_z E[u + z\phi]_{z=0} = \partial_z E[u + z\phi]_{z=0} = 0.
\] (24)

In particular, since
\[
\partial_z |\nabla(u + z\phi)|_{z=0}^2 = \nabla \phi \cdot \nabla u,
\] (25)
\[
\partial_z |\nabla(u + z\phi)|_{z=0}^p = \partial_z (|\nabla(u + z\phi)|_{z=0}^2)^{p/2} = \frac{p}{2} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi,
\] (26)

it follows that
\[
\int_\Omega \nabla \phi \cdot \left( \sigma(x) \nabla u + a(x) |\nabla v|^{p-2} \nabla u + F \right) = 0.
\] (27)

Therefore, \( u \) is a weak solution to equation (15).

Conversely, if \( u \) is a weak solution to equation (15), then for any \( w \in W^{1,2}_0(\Omega) \cap W^{1,p}_0(\Omega) \) we have that
\[
\int_\Omega \nabla \phi \cdot \left( \sigma(x) \nabla u + a(x) |\nabla v|^{p-2} \nabla u + F \right) = 0.
\] (28)
This, and its complex conjugate, give that
\[ \partial z E[u + zw]|_{z=0} = \partial \bar{z} E[u + zw]|_{z=0} = 0, \quad (29) \]
so, by the uniqueness of the energy minimum, we conclude the uniqueness of the solutions to (3).

Suppose \( p > 2 \). We define the functionals
\[ E_2[v] = \int_{\Omega} \frac{1}{2} \sigma(x)|\nabla v|^2, \quad \text{and} \quad E_p[v] = \int_{\Omega} \frac{1}{p} \alpha(x)|\nabla v|^p + \Re \nabla \bar{v} \cdot F. \quad (30) \]
Consider now the function \( v_2 \in W^{1,p}(\Omega) \) which is the solution of the boundary value problem
\[ \begin{aligned}
\nabla (\sigma(x) \nabla v_2) &= 0, \\
v_2|_{\partial \Omega} &= f.
\end{aligned} \quad (31) \]
We know that
\[ \|v_2\|_{W^{1,p}(\Omega)} \leq C\|f\|_p \quad (32) \]
and that it minimizes the energy \( E_2 \). Indeed, regarding the last claim, suppose \( w \in W^{1,p}_0(\Omega) \).
Then
\[ E_2[v_2 + w] = E[v_2] + \Re \int_{\Omega} \nabla \bar{w} \cdot (\sigma \nabla v_2) + \int_{\Omega} \frac{1}{2} \sigma |\nabla w|^2 \geq E[v_2], \quad (33) \]
as the middle term is zero and the last term is positive.
Since \( E[v] = E_2[v] + E_p[v] \), it is clear that
\[ E[u] \leq E_2[v_2] + E_p[v_2]. \quad (34) \]
Then
\[ E_p[v_2] \geq E[u] - E_2[v_2] = E_2[u] - E_2[v_2] + E_p[u] \geq E_p[u], \quad (35) \]
so
\[ E_p[u] \leq E_p[v_2]. \quad (36) \]
From this it follows that
\[ \|\nabla u\|_{L^p(\Omega)} \leq C \left( \|\nabla v_2\|_{L^p(\Omega)}^p + \left| \int_{\Omega} \nabla \bar{v} \cdot F \right| + \left| \int_{\Omega} \nabla \bar{v}_2 \cdot F \right| \right) \leq C \left( \|\nabla v_2\|_{L^p(\Omega)}^p + \|F\|_{L^q(\Omega)}^q \right) \]
\[ + \frac{1}{2} \|\nabla u\|_{L^p(\Omega)}^p, \quad (37) \]
so
\[ \|\nabla u\|_{L^p(\Omega)} \leq C \left( \|f\|_p + \|F\|_{L^q(\Omega)} \right). \quad (38) \]
Suppose now that \( p \in (1, 2) \). Let
\[ \tilde{E}_2[v] = \int_{\Omega} \frac{1}{2} \sigma(x)|\nabla v|^2 + \Re \nabla \bar{v} \cdot F, \quad \text{and} \quad \tilde{E}_p[v] = \int_{\Omega} \frac{1}{p} \alpha(x)|\nabla v|^p. \quad (39) \]
Suppose
\[
\begin{align*}
\nabla (\sigma(x) \nabla \tilde{v}_2) &= \nabla \cdot F, \\
\tilde{v}_2|_{\partial \Omega} &= f.
\end{align*}
\]
(40)

Then
\[
\|\tilde{v}_2\|_{W^{1,2}(\Omega)} \leq C(\|f\|_{X_p} + \|F\|_{L^q(\Omega)}),
\]
(41)
and \(\tilde{v}_2\) minimizes the energy \(\tilde{E}_2\).

It still holds that \(\tilde{E}_p[u] \leq \tilde{E}_p[\tilde{v}_2]\), so
\[
\|\nabla u\|_{L^p(\Omega)} \leq C\|\nabla \tilde{v}_2\|_{L^p(\Omega)} \leq C\|\tilde{v}_2\|_{L^2(\Omega)},
\]
(42)
and the same conclusion follows. \(\square\)

**Corollary 2.1.** The DN map \(\Lambda_{\sigma,a,p} : X_p^r \rightarrow X_p^r\) and
\[
\|\Lambda_{\sigma,a,p} f\|_{X_p^r} \leq C\left(\|f\|_{X_p^r} + \|f\|_{X_p^{r-1}}\right).
\]
(43)

**Proof.** Case (i) \(p > 2\). Let \(w \in W^{1,p}(\Omega)\). Then
\[
|\langle \Lambda_{\sigma,a,p} f, w |_{\partial \Omega} \rangle| = \left| \int_{\Omega} (\sigma(x) + a(x)|\nabla u|^{p-2})\nabla u \cdot \nabla w \right|
\leq \lambda^{-1}\|\nabla u\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}0 + m^{-1}\|\nabla u\|_{L^p(\Omega)}\|\nabla w\|_{L^p(\Omega)}
\leq C(\|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)}^{p-1})\|w\|_{W^{1,p}(\Omega)}.
\]
(44)
case (ii) \(p \in (1,2)\). Let \(w \in W^{1,2}(\Omega)\). Then, similarly to the above estimate,
\[
|\langle \Lambda_{\sigma,a,p} f, w |_{\partial \Omega} \rangle| \leq C\left(\|u\|_{W^{1,2}(\Omega)} + \|u\|_{W^{1,2}(\Omega)}^{p-1}\right)\|w\|_{W^{1,2}(\Omega)}.
\]
(45)
\(\square\)

2.2. Strong solutions

Next we will consider strong solutions to our equation, in the \(p > 2\) case.

**Theorem 2.** Suppose \(r > n, p > 2\). Let \(F \in L^r(\Omega)\) and \(f \in W^{2, \frac{n}{r}}(\partial \Omega)\). There exists \(\delta > 0\) so that if
\[
\|F\|_{L^r(\Omega)}, \|f\|_{W^{1, \frac{n}{r}}(\partial \Omega)} < \delta,
\]
(46)
then equation
\[
\begin{align*}
\nabla \cdot (\sigma(x) \nabla u + a(x)|\nabla u|^{p-2}\nabla u) &= F, \\
u|_{\partial \Omega} &= f,
\end{align*}
\]
(47)
has a unique solution \(u \in W^{2,p}(\Omega)\). Furthermore, there exists \(C > 0\) such that
\[
\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|f\|_{W^{2, \frac{n}{r}}(\partial \Omega)} + \|F\|_{L^r(\Omega)}\right).
\]
(48)
**Proof.** Let \( G_r \) be the inverse of the operator \( \nabla \cdot (\sigma \nabla \cdot) \), with zero Dirichlet boundary conditions. That is, if

\[
\begin{aligned}
\nabla \cdot (\sigma \nabla v) &= S, \quad \text{in } \Omega, \\
v|_{\partial \Omega} &= 0,
\end{aligned}
\]  

(49)

then \( G_r(S) = v \). \( G_r \) is a bounded operator form \( L^r(\Omega) \) into \( W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \).

Suppose \( f \in W^{2,1, r}(\partial \Omega) \) and let \( v_f \in W^{2,r}(\Omega) \) be the solution of

\[
\begin{aligned}
\nabla \cdot (\sigma \nabla v_f) &= F, \quad \text{in } \Omega, \\
v_f|_{\partial \Omega} &= f.
\end{aligned}
\]  

(50)

There exists a constant \( C > 0 \) such that

\[
\|v_f\|_{W^{2,r}(\Omega)} \leq C \left( \|f\|_{W^{2,1,r}(\partial \Omega)} + \|F\|_{L^r(\Omega)} \right).
\]  

(51)

For \( v \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \) we define the operator

\[
T(v) = -G_r \left( \nabla \cdot (a|\nabla(v_f + v)|^{p-2} \nabla(v_f + v)) \right).
\]  

(52)

Let \( A(v), B(v) \), and \( H(v) \) be the matrix valued functions with coefficients

\[
A_{jk}(v) = |\nabla(v_f + v)|^{p-4} \partial_j(v_f + v) \partial_k(v_f + v),
\]  

(53)

\[
B_{jk}(v) = |\nabla(v_f + v)|^{p-4} \partial_j(v_f + v) \partial_k(v_f + v),
\]  

(54)

\[
H_{jk}(v) = \partial_{jk}(v_f + v).
\]  

(55)

Then

\[
\nabla \cdot (a|\nabla(v_f + v)|^{p-2} \nabla(v_f + v)) = |\nabla(v_f + v)|^{p-2} (\nabla a) \cdot \nabla(v_f + v)
\]

\[
+ a|\nabla(v_f + v)|^{p-2} \Delta (v_f + v)
\]

\[
+ \frac{p-2}{2} a \Sigma (AH + B^T),
\]  

(56)

which belongs to \( L^r(\Omega) \), since, by Sobolev embedding, \( W^{1, r}(\Omega) \subset L^\infty(\Omega) \). We then see that

\( T: W^{2, r}(\Omega) \cap W_0^{1, r}(\Omega) \to W^{2, r}(\Omega) \cap W_0^{1, r}(\Omega) \).

Let \( v_1, v_2 \in W^{2, r}(\Omega) \cap W_0^{1, r}(\Omega) \). Then, applying lemma A.1, we have

\[
\begin{aligned}
\| |\nabla(v_f + v_1)|^{p-2} \nabla(v_f + v_1) - |\nabla(v_f + v_2)|^{p-2} \nabla(v_f + v_2)\|_{L^r(\Omega)}
\leq C \left( \|\nabla v_f\|_{L^\infty(\Omega)} + \|\nabla v_1\|_{L^\infty(\Omega)} + \|\nabla v_2\|_{L^\infty(\Omega)} \right)^{p-2}
\times \|\nabla(v_1 - v_2)\|_{L^r(\Omega)}
\leq C \left( \|f\|_{W^{2,1,r}(\partial \Omega)} + \|F\|_{L^r(\Omega)} + \|v_1\|_{W^{2,r}(\Omega)} + \|v_2\|_{W^{2,r}(\Omega)} \right)^{p-2}
\times \|v_1 - v_2\|_{W^{2,r}(\Omega)}.
\end{aligned}
\]  

(57)
Since
\[
|\nabla (v_f + v_1)|^{p-2} \Delta (v_f + v_1) - |\nabla (v_f + v_2)|^{p-2} \Delta (v_f + v_2)
\]
\[
= \left(|\nabla (v_f + v_1)|^{p-2} - |\nabla (v_f + v_2)|^{p-2}\right) \Delta (v_f + v_1)
\]
\[
+ |\nabla (v_f + v_2)|^{p-2} \Delta (v_1 - v_2),
\]
(58)
and, by lemma A.2,
\[
\|\nabla (v_f + v_1)|^{p-2} - |\nabla (v_f + v_2)|^{p-2}\|_{L^\infty(\Omega)}
\]
\[
\leq C \left( \|f\|_{W^{2,\frac{1}{2},\gamma}(\partial \Omega)} + \|F\|_{L^2(\Omega)} + \|v_1\|_{W^{2,\gamma}(\Omega)} + \|v_2\|_{W^{2,\gamma}(\Omega)} \right)^{\mu_1(p-2)}
\times \|v_1 - v_2\|_{W^{1,\gamma}(\Omega)}^{\mu_2(p-2)},
\]
(59)
we also have
\[
\|\nabla (v_f + v_1)|^{p-2} \Delta (v_f + v_1) - |\nabla (v_f + v_1)|^{p-2} \Delta (v_f + v_1)\|_{L^2(\Omega)}
\]
\[
\leq C \left( \|f\|_{W^{2,\frac{1}{2},\gamma}(\partial \Omega)} + \|F\|_{L^2(\Omega)} + \|v_1\|_{W^{2,\gamma}(\Omega)} + \|v_2\|_{W^{2,\gamma}(\Omega)} \right)^{p-2}
\times \|v_1 - v_2\|_{W^{2,\gamma}(\Omega)} + C \left( \|f\|_{W^{2,\frac{1}{2},\gamma}(\partial \Omega)} + \|F\|_{L^2(\Omega)} + \|v_1\|_{W^{2,\gamma}(\Omega)} + \|v_2\|_{W^{2,\gamma}(\Omega)} \right)^{\mu_1(p-2)}
\times \|v_1 - v_2\|_{W^{1,\gamma}(\Omega)}^{\mu_2(p-2)},
\]
(60)
Finally, applying lemma A.3, we have
\[
\|\nabla \left(A(v_1)H(v_1) + B(v_1)\nabla H(v_1) - A(v_2)H(v_2) - B(v_2)\nabla H(v_2)\right)\|_{L^2(\Omega)}
\]
\[
\leq C \left( \|f\|_{W^{2,\frac{1}{2},\gamma}(\partial \Omega)} + \|F\|_{L^2(\Omega)} + \|v_1\|_{W^{2,\gamma}(\Omega)} + \|v_2\|_{W^{2,\gamma}(\Omega)} \right)^{p-2}
\times \|v_1 - v_2\|_{W^{2,\gamma}(\Omega)}
\]
\[
+ C \left( \|f\|_{W^{2,\frac{1}{2},\gamma}(\partial \Omega)} + \|F\|_{L^2(\Omega)} + \|v_1\|_{W^{2,\gamma}(\Omega)} + \|v_2\|_{W^{2,\gamma}(\Omega)} \right)^{p-2}
\times \left( \|f\|_{W^{2,\frac{1}{2},\gamma}(\partial \Omega)} + \|F\|_{L^2(\Omega)} + \|v_1\|_{W^{2,\gamma}(\Omega)} + \|v_2\|_{W^{2,\gamma}(\Omega)} \right)^{\mu_1(p-2)}
\times \|v_1 - v_2\|_{W^{1,\gamma}(\Omega)}^{\mu_2(p-2)},
\]
(61)
where \(s = \frac{1}{\alpha} \min(p - 2, 1)\).

We see now that there is a \(\delta > 0\) such that if
\[
\|f\|_{L^2(\Omega)}, \|f\|_{W^{2,\frac{1}{2},\gamma}(\partial \Omega)} \leq \delta,
\]
(62)
then \(\mathcal{T}\) is a contraction (taking the distance to be \(d(v_1, v_2) = \|v_1 - v_2\|_{W^{2,\gamma}(\Omega)}\) on the ball of radius \(\delta\) in \(W^{2,\gamma}(\Omega) \cap W_0^{1,\gamma}(\Omega)\). That is, there exists \(\kappa \in (0, 1)\) such that if \(\|v_1\|_{W^{2,\gamma}(\Omega)}, \|v_2\|_{W^{2,\gamma}(\Omega)} < \delta\) then
\[
\|\mathcal{T}(v_1) - \mathcal{T}(v_2)\|_{W^{2,\gamma}(\Omega)} \leq \kappa \|v_1 - v_2\|_{W^{2,\gamma}(\Omega)},
\]
(63)
Let \(v \in W^{2,\gamma}(\Omega) \cap W_0^{1,\gamma}(\Omega)\) be the fixed point of this contraction. Then \(u = v_f + v\) is the solution whose existence we needed to prove.
Furthermore,
\[ \|v\|_{W^2, r(\Omega)} \leq \|T(v) - T(0)\|_{W^2, r(\Omega)} + \|T(0)\|_{W^2, r(\Omega)} \]
\[ \leq \kappa_{\frac{1}{2}} \|v\|_{W^2, r(\Omega)} + C \left( \|f\|_{W^{-1}, r(\partial\Omega)} + \|F\|_{L^r(\Omega)} \right)^{p-1}, \tag{64} \]
which implies that
\[ \|v\|_{W^2, r(\Omega)} \leq C \left( \|f\|_{W^{-1}, r(\partial\Omega)} + \|F\|_{L^r(\Omega)} \right)^{p-1}. \tag{65} \]
This gives
\[ \|u\|_{W^2, r(\Omega)} \leq C \left( \|f\|_{W^{-1}, r(\partial\Omega)} + \|F\|_{L^r(\Omega)} \right) + \left( \|f\|_{W^{-1}, r(\partial\Omega)} + \|F\|_{L^r(\Omega)} \right)^{p-1} \right). \tag{66} \]

3. Reconstruction when \( p > 2 \)

In this section we will give a reconstruction procedure for the parameters \( \sigma, a, \) and \( p, \) in the case when \( p > 2. \) For now we will assume that it is \textit{a priori} known that \( p > 2. \) In section 4 we will show how it can be determined if \( p \) is in the range or in the interval \((1, 2). \) Here we will impose boundary data multiplied by a parameter \( \epsilon \) and then derive an asymptotic expansion of the Dirichlet-to-Neumann map in \( \epsilon, \) as \( \epsilon \to 0. \) Using he order \( \epsilon^{2p-3} \) part of the DN map we obtain an integral identity involving \( a \) and several solutions of the linear part of equation (3).

3.1. Asymptotics of solutions

For \( 0 < \epsilon < 1, \) and \( f \in W^{2, \frac{1}{2}, r}(\partial\Omega) \) with \( \|f\|_{W^{2, \frac{1}{2}, r}(\partial\Omega)} \), let \( u_\epsilon \) be the solution to the equation
\[ \begin{aligned}
\nabla \cdot (\sigma \nabla u_\epsilon + a |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon) &= 0, \\
u_\epsilon|_{\partial\Omega} &= \epsilon f.
\end{aligned} \tag{67} \]
By theorem 2, we have that
\[ \|\epsilon^{-1} u_\epsilon\|_{W^{2, r}(\Omega)} \leq C \|f\|_{W^{2, \frac{1}{2}, r}(\partial\Omega)}. \tag{68} \]

We make the following Ansatz
\[ u_\epsilon = \epsilon u_0 + \epsilon^{p-1} v_\epsilon, \tag{69} \]
where \( u_0 \in W^{2, r}(\Omega) \) satisfies the equation
\[ \begin{aligned}
\nabla \cdot (\sigma \nabla u_0) &= 0, \\
u_0|_{\partial\Omega} &= f.
\end{aligned} \tag{70} \]
We have that
\[ \|u_0\|_{W^{2,r}(\Omega)} \leq C \|f\|_{W^{2,1}(\partial\Omega)}. \] (71)

Note that \( v_\epsilon \) satisfies
\[ \begin{cases} \nabla \cdot (\sigma \nabla v_\epsilon) + \epsilon^{1-p} \nabla \cdot (a|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon) = 0, \\ v_\epsilon|_{\partial\Omega} = 0. \end{cases} \] (72)

By elliptic regularity
\[ \|v_\epsilon\|_{W^{2,r}(\Omega)} \leq C \|\epsilon^{-1} u_\epsilon\|_{W^{p-1,2} \cap W^{0,0}(\partial\Omega)} \leq C \|f\|_{p-1} \|\nabla v_\epsilon\|_{\partial\Omega}. \] (73)

This shows that
\[ \text{Lemma 3.1.} \quad v_\epsilon \text{ is bounded in } W^{2,r}(\Omega) \cap W^{1,2}_0(\Omega) \text{ uniformly as } \epsilon \to 0. \]

### 3.2. Asymptotics of the DN map

The expansion of the solution \( u_\epsilon \) gives us an expansion of the DN map in powers of \( \epsilon \). Let \( w \in W^{2,r}(\Omega) \). Using lemma A.1 we obtain that
\[ \left| \int_\Omega a \left( |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon - \epsilon^{p-1} |\nabla u_0|^{p-2} \nabla u_0 \right) \cdot \nabla w \right| \leq C \epsilon^{2p-3} \int_\Omega \left( |\epsilon^{-1} \nabla u_\epsilon|^{p-2} + |\nabla u_0|^{p-2} \right) |\nabla v_\epsilon| |\nabla w| = \mathcal{O}(\epsilon^{2p-3}). \] (74)

Then
\[ \langle \Lambda_{\sigma,a,p}(\epsilon f), w|_{\partial\Omega} \rangle = \int_\Omega (\sigma + a|\nabla u_\epsilon|^{p-2}) \nabla u_\epsilon \cdot \nabla w \]
\[ = \epsilon \int_\Omega \sigma \nabla u_0 \cdot \nabla w + \epsilon^{p-1} \int_\Omega \sigma \nabla v_\epsilon \cdot \nabla w + \epsilon^{p-1} \int_\Omega a|\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla w + \mathcal{O}(\epsilon^{2p-3}). \] (75)

Here we can observe that, since \( p > 2 \), we have
\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \langle \Lambda_{\sigma,a,p}(\epsilon f), w|_{\partial\Omega} \rangle = \int_\Omega \sigma \nabla u_0 \cdot \nabla w = \langle \Lambda_\sigma(f), w \rangle. \] (76)

The left-hand side is exactly the DN map associated to the equation
\[ \nabla \cdot (\sigma \nabla u_0) = 0. \] (77)

We can observe here that, by the theorem of [26], \( \sigma \) can be recovered from \( \Lambda_\sigma \), and hence from \( \Lambda_{\sigma,a,p} \).

Suppose now that \( w \) is such that \( \nabla \cdot (\sigma \nabla w) = 0 \). Since \( v_\epsilon \in W^{1,2}_0(\Omega) \),
\[ \int_\Omega \sigma \nabla v_\epsilon \cdot \nabla w = 0. \] (78)
Define
\[
I(u_0, w) = \lim_{\epsilon \to 0^+} \epsilon^{1-p} \left[ (\Lambda_{\sigma, a, p}(u_0)_{|\partial \Omega}), w_{|\partial \Omega} \right] - \epsilon \int_{\Omega} \sigma \nabla u_0 \cdot \nabla w
\]
\[
= \int_{\Omega} a |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla w. \tag{79}
\]
With this notation we can write
\[
(\Lambda_{\sigma, a, p}(\epsilon f), w_{|\partial \Omega}) = \epsilon (\Lambda_{\sigma, f}, w_{|\partial \Omega}) + \epsilon^{p-1} I(u_0, w) + O(\epsilon^{2p-3}). \tag{80}
\]

The expansion (80) allows us to determine the parameter \( p \). Namely, \( p \) is the supremum of the set
\[
\left\{ r > 2 : \lim_{\epsilon \to 0^+} \frac{1}{\epsilon^{r-1}} \left( (\Lambda_{\sigma, a, p}(\epsilon f), f) - \epsilon (\Lambda_{\sigma, f}, f) \right) = 0, \quad \forall f \in C(\partial \Omega) \right\}. \tag{81}
\]

3.3. Recovery of the parameter \( a \)

Let the functions \( u_0, u_1, u_2, u_3 \in C^\infty(\Omega) \) be such that \( \nabla \cdot (\sigma \nabla u_j) = 0, j = 0, 1, 2, 3, 4 \). We will further require that \( \nabla u_1 \) does not have any zeros.

If the DN map \( \Lambda_{\sigma, a, p} \) is known, then \( I(u_0, u_3) \) will also be known. Note that
\[
\partial_z \left[ |\nabla (u_1 + z\bar{u}_2)|^{p-2} \nabla (u_1 + z\bar{u}_2) \right]_{z=0} = \frac{p-2}{2} \nabla u_1|^{p-4}(\nabla u_1 \cdot \nabla u_2)\nabla u_1, \tag{82}
\]
and
\[
\partial_z \left[ |\nabla (u_1 + z\bar{u}_2)|^{p-2} \nabla (u_1 + z\bar{u}_2) \right]_{z=0} = \frac{p-2}{2} \nabla u_1|^{p-4}(\nabla \bar{u}_1 \cdot \nabla u_2)\nabla u_1
+ |\nabla u_1|^{p-2}\nabla u_2. \tag{83}
\]

Define
\[
I(u_1, u_2, u_3) = \frac{2}{p-2} \partial_z I(u_1 + z\bar{u}_2, u_3)_{z=0} = \int_{\Omega} a |\nabla u_1|^{p-4}(\nabla u_1 \cdot \nabla u_2)(\nabla u_1 \cdot \nabla u_3), \tag{84}
\]
and
\[
J(u_1, u_2, u_3) = \frac{2}{p-2} \partial_z I(u_1 + z\bar{u}_2, u_3)_{z=0} = \int_{\Omega} a |\nabla u_1|^{p-4} \left[ (\nabla \bar{u}_1 \cdot \nabla u_2)(\nabla u_1 \cdot \nabla u_3) \right.
+ \frac{p-2}{2} |\nabla u_1|^{p-2}\nabla u_2 \cdot \nabla u_3, \tag{85}
\]

This functionals are also known from the boundary data.

We will choose the solution \( u_1 \) to be real valued and set \( \beta_1 = a |\nabla u_1|^{p-2} \). In this case we observe that
\[
K(u_1, u_2, u_3) = \frac{2}{p-2} [J(u_1, u_2, u_3) - I(u_1, u_2, u_3)] = \int_{\Omega} \beta_1 |\nabla u_2| \cdot \nabla u_3 \tag{86}
\]
is known from the boundary data.

We will recover $\beta_{u_1}$ for any $u_1$ from $K$. To do this, we will choose $u_2$ and $u_3$ to be complex geometric optics (CGO) solutions for the linear part of the equation. Such solutions were constructed in [32]. We recall their properties here.

**Proposition 3.1 (see [32, theorem 1.1]).** If $\zeta \in \mathbb{C}^n$ is such that $\zeta \cdot \zeta = 0$, then the equation

$$\nabla \cdot (\sigma \nabla u) = 0 \quad (87)$$

has complex geometric optics (CGO) solutions of the form

$$u(x) = e^{i \zeta \cdot x} \sigma^{-1/2}(x)(1 + r(x, \zeta)), \quad (88)$$

where $r$ satisfies the equation

$$\triangle r + \zeta \cdot \nabla r - Vr = V, \quad V = \frac{\triangle \sigma^{1/2}}{\sigma^{1/2}}, \quad (89)$$

and the estimates

$$\|r\|_{W^{2,2}(\Omega)} \leq C \frac{|\zeta|}{|\zeta|}, \quad s > \frac{n}{2}. \quad (90)$$

Now let $\xi \in \mathbb{R}^n$ be arbitrary and choose $\eta, \mu \in S^{n-1}$ such that

$$\xi \cdot \eta = \xi \cdot \mu = \eta \cdot \mu = 0, \quad (91)$$

Using these, we define $\zeta_2, \zeta_3 \in \mathbb{C}^n$ to be

$$\zeta_2 = \tau \mu - i \left( \frac{\xi}{2} + s \eta \right), \quad \zeta_3 = -\tau \mu - i \left( \frac{\xi}{2} - s \eta \right), \quad (92)$$

where $s \in \mathbb{R}$ and $r$ satisfies

$$\tau^2 = \frac{|\xi|^2}{4} + s^2. \quad (93)$$

With these choices, we have

$$\zeta_2 \cdot \zeta_2 = \zeta_3 \cdot \zeta_3 = 0, \quad \zeta_2 + \zeta_3 = -i \xi. \quad (94)$$

We will take $u_2 = e^{i \zeta_2 \cdot x} \sigma^{-1/2}(1 + r_2)$, $u_3 = e^{i \zeta_3 \cdot x} \sigma^{-1/2}(1 + r_3)$ to be the corresponding CGO solutions.

Note that

$$\nabla u_i = e^{i \zeta_i \cdot x} \left[ \zeta_i \sigma^{-\frac{1}{2}}(1 + r_i) + \nabla \left( \sigma^{-\frac{1}{2}} \right)(1 + r_i) + \sigma^{-\frac{1}{2}} \nabla r_i \right], \quad (95)$$
\[
\n\n\n\]

so

\[
\n\n\n\]

Here the \(\mathcal{O}(s^{-1})\) is to be taken in the sense of \(L^\infty(\Omega)\) norms.

Using (89), we compute

\[
\n\n\n\]

Similarly

\[
\n\n\n\]

Then

\[
\n\n\n\]

We can now take the limit \(s \to \infty\) in (86) to obtain

\[
\n\n\n\]

Then

\[
\n\n\n\]

on \(\mathbb{R}^n\), in the sense of distributions. In the identity above we have extended \(\sigma\) so that it is \(C^\infty\) on the whole of \(\mathbb{R}^n\) and the support of \(\sigma - 1\) is compact. Since

\[
\n\n\n\]

and

\[
\n\n\n\]
we can conclude that $\beta_{u_1}$ satisfies
\[
\triangle \beta_{u_1} - 3V\beta_{u_1} = 2\sigma^{1/2} F^{-1} \lim_{s \to \infty} K(u_1, u_2, u_3),
\]
(104)
on $\mathbb{R}^n$, in the sense of distributions. Suppose there exists $f \in \mathcal{E}'(\mathbb{R}^n)$ such that
\[
\triangle f - 3V f = 0.
\]
(105)
Then for any $\varphi \in C^\infty(\mathbb{R}^n)$ we have
\[
\langle f, \triangle \varphi - 3V \varphi \rangle = 0.
\]
(106)
Since for any $\psi \in \mathcal{S}(\mathbb{R}^n)$ we can find $\varphi$ such that
\[
\triangle \varphi - 3V \varphi = \psi,
\]
(107)
it follows that $\langle f, \psi \rangle = 0$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$, so $f = 0$. This implies (104) can have at most one compactly supported solution in distributions. We can then invert the operator $\triangle - 3V$ and we obtain the reconstruction formula
\[
a = 2|\nabla u_1|^{2-p}(\triangle - 3V)^{-1} \left[ \sigma^{1/2} F^{-1} \lim_{s \to \infty} K(u_1, u_2, u_3) \right].
\]
(108)

4. Reconstruction when $p \in (1, 2)$

In this section we give the reconstruction of the parameters $\sigma$, $a$, and $p$, in the case when $p \in (1, 2)$. In this case the nonlinear term is sublinear, so we consider Dirichlet data of the form $\epsilon^{-1}f$, where $\epsilon \to 0$. Once an asymptotic expansion for the DN map is established, the reconstruction method is the same as in the previous case. We begin with a discussion of how we may determine if $p > 2$ or $p \in (1, 2)$.

4.1. Determination of the range of $p$

It is an easy consequence of the asymptotic expansions of the previous section that if $p > 2$ then the limit
\[
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \langle \Lambda_{\sigma,a,p}(\epsilon f), f \rangle = \langle \Lambda_{\sigma}(f), f \rangle,
\]
(109)
exists for all $f \in C(\partial \Omega)$. Also note that the limit $\Lambda_{\sigma}(f)$ is not zero for all choices of $f$.

Suppose now that $p \in (1, 2)$ and that the limit $\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \Lambda_{\sigma,a,p}(\epsilon f)$ exists for all $f \in C(\partial \Omega)$. For one such $f$, let $w_\epsilon$ be the solution to the equation
\[
\begin{cases}
\nabla \cdot (\sigma \nabla w_\epsilon + a |\nabla w_\epsilon|^{p-2} \nabla w_\epsilon) = 0, \\
w_\epsilon |\partial \Omega| = \epsilon f.
\end{cases}
\]
(110)
By proposition 2.1 we have that
\[
\|w_\epsilon\|_{W^{1,2}(\Omega)} \leq C\epsilon \|f\|_{W^{1,2}(\partial \Omega)}.
\]
(111)
Since
\[
\epsilon^{-1} \langle \Lambda_{\sigma,a,p}(\epsilon f), f \rangle = \epsilon^{-2} \int \sigma |\nabla w_\epsilon|^2 + a |\nabla w_\epsilon|^p,
\]
(112)
we see that there exists $C > 0$ (which may depend on $f$) such that
\[ \| \nabla w_\epsilon \|_{L^p(\Omega)} \leq C \epsilon^\frac{p}{2}. \] (113)

The family of functions $\epsilon^{-1} w_\epsilon$ is bounded in $W^{1,2}(\Omega)$, and hence in $W^{1,p}(\Omega)$. Therefore their averages
\[ (\epsilon^{-1} w_\epsilon)_\Omega = \frac{1}{\text{vol}(\Omega)} \int_{\Omega} \epsilon^{-1} w_\epsilon \] (114)
are also bounded. Up to a subsequence, which we also denote $\epsilon^{-1} w_\epsilon$, we have that these average converge to a number $c_0 \in \mathbb{C}$. By Poincaré’s inequality and (113) we have that
\[ \| \epsilon^{-1} w_\epsilon - c_0 \|_{L^p(\Omega)} \leq \| \epsilon^{-1} w_\epsilon - (\epsilon^{-1} w_\epsilon)_\Omega \|_{L^p(\Omega)} + \| (\epsilon^{-1} w_\epsilon)_\Omega - c_0 \| \| \text{vol}(\Omega) \|^{\frac{1}{p}} \]
\[ \leq C \left( \epsilon^\frac{p}{2} + \| (\epsilon^{-1} w_\epsilon)_\Omega - c_0 \| \right) \to 0. \] (115)

Again by (113), it follows that
\[ \lim_{\epsilon \to 0^+} \| \epsilon^{-1} w_\epsilon - c_0 \|_{W^{1,p}(\Omega)} = 0. \] (116)

Therefore we have that
\[ \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \langle \Lambda_{\sigma,a,p} (\epsilon f), f \rangle = 0, \quad \forall f \in C(\partial \Omega). \] (117)

We have obtained a constructive method for determining from the DN map which range the parameter $p$ belongs to: if $\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \langle \Lambda_{\sigma,a,p} (\epsilon f), f \rangle$ is identically zero or does not exist for all $f \in C(\Omega)$, then $p \in (1/2)$; otherwise $p > 2$.

4.2. Asymptotics of solutions

For $0 < \epsilon < 1$, and $f \in \mathcal{X}_p = W^{1,2}(\partial \Omega)$, let $u_\epsilon$ be the solution to the equation
\[ \begin{cases} \nabla \cdot (\sigma \nabla u_\epsilon + a |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon) = 0, \\ u_\epsilon |_{\partial \Omega} = \epsilon^{-1} f. \end{cases} \] (118)

By proposition 2.1, we have that
\[ \| \epsilon u_\epsilon \|_{W^{1,2}(\Omega)} \leq C \| f \|_{\mathcal{X}_p}. \] (119)

We make the following Ansatz
\[ u_\epsilon = \epsilon^{-1} u_0 + \epsilon^{1-p} v_\epsilon, \] (120)
where $u_0 \in W^{1,2}(\Omega)$ satisfies the equation
\[ \begin{cases} \nabla \cdot (\sigma \nabla u_0) = 0, \\ u_0 |_{\partial \Omega} = f. \end{cases} \] (121)

We have that
\[ \| u_0 \|_{W^{1,2}(\Omega)} \leq C \| f \|_{\mathcal{X}_p}. \] (122)
Note that \( v_\epsilon \) satisfies
\[
\begin{cases}
\nabla \cdot (\sigma \nabla v_\epsilon) + \epsilon^{p-1} \nabla \cdot (a|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon) = 0 \\
v_\epsilon|_{\partial \Omega} = 0.
\end{cases}
\]
(123)

Since \( p < 2 \), we have that \( |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \in L^2(\Omega) \) and
\[
\| |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \|_{L^2(\Omega)} \leq \| \nabla u_\epsilon \|_{L^2(\Omega)}^{p-1}.
\]
(124)

Elliptic regularity then gives
\[
\| v_\epsilon \|_{W^{1,2}(\Omega)} \leq C \| \epsilon u_\epsilon \|_{W^{1,2}(\Omega)}^{p-1}.
\]
(125)

this shows that
Lemma 4.1. \( v_\epsilon \) is bounded in \( W^{1,2}(\Omega) \) uniformly as \( \epsilon \to 0 \).

4.3. Asymptotics of the DN map

We will expand the DN map in powers of \( \epsilon \). Let \( w \in W^{1,2}(\Omega) \). Using lemma A.1 we obtain that
\[
\left| \int_\Omega a \left( |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - \epsilon^{1-p}|\nabla u_0|^{p-2}\nabla u_0 \right) \cdot \nabla \bar{w} \right| 
\leq C \epsilon^{3-2p} \int_\Omega \left( |\epsilon| |\nabla u_\epsilon|^{p-2} + |\nabla u_0|^{p-2} \right) |\nabla u_\epsilon| |\nabla \bar{w}| \equiv O(\epsilon^{2p-3}).
\]
(126)

Then
\[
\langle \Lambda_{\sigma,a,p}(\epsilon^{-1}f), w \rangle = \int_\Omega \left( \sigma + a|\nabla u_\epsilon|^{p-2} \right) \nabla u_\epsilon \cdot \nabla \bar{w}
\]
\[
= \epsilon^{-1} \int_\Omega \nabla \sigma \cdot \nabla \bar{w} + \epsilon^{1-p} \int_\Omega \nabla u_\epsilon \cdot \nabla \bar{w} + \epsilon^{1-p}
\]
\[
\times \int_\Omega a|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla \bar{w} + O(\epsilon^{3-2p}).
\]
(127)

Since \( p \in (1, 2) \), we have
\[
\lim_{\epsilon \to 0} \epsilon \langle \Lambda_{\sigma,a,p}(\epsilon^{-1}f), w \rangle = \int_\Omega \sigma \nabla u_0 \cdot \nabla \bar{w} = \langle \Lambda_\sigma(f), w \rangle.
\]
(128)

The left-hand side is exactly the DN map associated to the equation
\[
\nabla \cdot (\sigma \nabla u_0) = 0.
\]
(129)

By the the result of [26], we have that \( \sigma \) can be recovered from \( \Lambda_{\sigma,a,p} \).

Suppose now that \( w \) is such that \( \nabla (\sigma \nabla w) = 0 \). Since \( v_\epsilon \in W^{1,2}_0(\Omega) \),
\[
\int_\Omega \sigma \nabla v_\epsilon \cdot \nabla \bar{w} = 0.
\]
(130)

This leaves us with the expansion
\[
\langle \Lambda_{\sigma,a,p}(\epsilon f), w \rangle = \epsilon^{-1} \int_\Omega \sigma \nabla u_0 \cdot \nabla \bar{w} + \epsilon^{1-p} \int_\Omega a|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla \bar{w} + O(\epsilon^{3-2p}).
\]
(131)
Define
\[ I(u_0, w) = \lim_{\epsilon \to 0^+} \epsilon^{p-1} \left[ \langle \Lambda_{\sigma, u_0}(\epsilon^{-1} u_0 |_{\partial \Omega}), w |_{\partial \Omega} \rangle - \epsilon^{-1} \int_{\Omega} \sigma \nabla u_0 \cdot \nabla w \right] = \int_{\Omega} a |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla w. \] (132)

With this notation we can write
\[ \langle \Lambda_{\sigma, u_0}(\epsilon^{-1} f), w |_{\partial \Omega} \rangle = \epsilon^{-1} \langle \Lambda_{\sigma} f, w |_{\partial \Omega} \rangle + \epsilon^{1-p} I(u_0, w) + O(\epsilon^{3-2p}). \] (133)

We can recover the parameter \( p \) as the minimum of the set
\[ \left\{ r \in (1,2) : \lim_{\epsilon \to 0^+} \frac{1}{\epsilon^{r-1}} \left( \langle \Lambda_{\sigma, u_0}(\epsilon f), f \rangle - \epsilon \langle \Lambda_{\sigma} f, f \rangle \right) = 0, \quad \forall f \in C(\partial \Omega) \right\}. \] (134)

The coefficient \( a \) can then be recovered from the functional \( I \) using the same method as in the \( p > 2 \) case.

**Acknowledgments**

Cătălin I Cărstea was supported by NSF of China under Grant 11931011.

**Appendix A. Some vector estimates**

Here we gather a few useful inequalities for vectors in \( \mathbb{C}^n \). Some of these are well known, and we give them without a proof. We do give a proof to the estimate in lemma A.3, which we need in order to prove the existence of strong solutions.

**Lemma A.1.** Suppose \( \xi, \zeta \in \mathbb{C}^n, p \in (1, \infty), \) then
\[ \| |\xi|^{p-2} - |\zeta|^{p-2} \| \leq C(\|\xi\| + \|\zeta\|)^{p-2} |\xi - \zeta|, \] (135)
\[ 2 \text{Re} \left[ (|\xi|^{p-2} - |\zeta|^{p-2}) \cdot (\overline{\xi} - \overline{\zeta}) \right] \sim (\|\xi\| + \|\zeta\|)^{p-2} |\xi - \zeta|^2, \] (136)
\[ \| |\xi|^{p} - |\zeta|^{p} \| \leq p (\|\xi|^{p-1} + \|\zeta|^{p-1}) |\xi - \zeta|. \] (137)

These are standard, well known inequalities. A list containing them as well as others is collected in [27, appendix A].

**Lemma A.2.** Let \( p > 2, \xi, \zeta \in \mathbb{C}^n, R > 0 \) such that \( |\xi|, |\zeta| < R, \) then
\[ \| |\xi|^{p-2} - |\zeta|^{p-2} \| \leq CR^{\mu} |\xi - \zeta|^{\min(p-2,1)}, \] (138)
where \( C > 0, \mu \geq 0 \) are constants that only depend on \( p \).

**Proof.** If \( p \geq 3 \), we can apply lemma A.1 to obtain
\[ \| |\xi|^{p-2} - |\zeta|^{p-2} \| \leq 2(p-2)R^{p-3} |\xi - \zeta|. \] (139)
Suppose now that \( p \in (2, 3) \), and also, without loss of generality, that \( |\xi| \geq |\zeta| \). Then
\[
| |\xi|^{p-2} - |\zeta|^{p-2}| = |\xi + \zeta - |\zeta|^{p-2} - |\xi|^{p-2} \\
\leq |\zeta|^{p-2} + |\xi - |\zeta|^{p-2} - |\xi|^{p-2} \\
= |\xi - |\zeta|^{p-2} |.
\] (140)

Finally, the following technical lemma will also be of use.

**Lemma A.3.** Let \( p > 2 \), \( \xi, \zeta \in \mathbb{C}^n \), \( R > 0 \) such that \( |\xi|, |\zeta| < R \), and let \( A \) and \( B \) be the matrices with coefficients
\[
A_{jk} = |\xi|^{p-4} \xi_j \overline{\zeta_k} - |\zeta|^{p-4} \zeta_j \overline{\xi_k},
\] (141)
\[
B_{jk} = |\xi|^{p-4} \xi_j \overline{\zeta_k} - |\zeta|^{p-4} \zeta_j \overline{\xi_k}.
\] (142)

For any \( n \times n \) matrix \( H \) such that \( H^T = H \), we have that
\[
|\mathfrak{I} (AH + B\overline{H})| \leq C R^{\min(p-2,1)} ||H|| |\xi - |\zeta|^{p-2} |,
\] (143)

where \( ||H|| = \max_{|j|} |H_{j.|} |\) and \( C, \mu_2 > 0 \) are constants that depend only on \( p \).

**Proof.** The case when \( p \geq 3 \) is relatively easier, so we will not write it out in full. For now assume that \( 2 < p < 3 \).

First we define a new matrices \( \tilde{A} = A - \frac{2}{n} \mathfrak{I} (A) I_{n \times n} \), \( \tilde{B} = B - \frac{2}{n} \mathfrak{I} (B) I_{n \times n} \), with coefficients
\[
\tilde{A}_{jk} = |\xi|^{p-4} \left( \xi_j \overline{\zeta_k - \frac{2}{n} |\xi|^2 \delta_j} \right) - |\zeta|^{p-4} \left( \zeta_j \overline{\xi_k - \frac{2}{n} |\zeta|^2 \delta_j} \right),
\] (144)
\[
\tilde{B}_{jk} = |\xi|^{p-4} \left( \xi_j \zeta_k - \frac{2}{n} |\xi|^2 \delta_j \right) - |\zeta|^{p-4} \left( \zeta_j \xi_k - \frac{2}{n} |\zeta|^2 \delta_j \right).
\] (145)

Note that
\[
\frac{1}{n} |\mathfrak{I} (\mathfrak{I} (A) I_{n \times n})| = | |\xi|^{p-2} - |\zeta|^{p-2} | |\mathfrak{I} (H)| \leq |\mathfrak{I} (H)| |\xi - |\zeta|^{p-2} |,
\] (146)

and also
\[
\frac{1}{n} |\mathfrak{I} (\mathfrak{I} (B) I_{n \times n})| = | |\xi|^{p-4} \xi^2 - |\zeta|^{p-4} \zeta^2 | |\mathfrak{I} (H)|.
\] (147)

We observe that
\[
| |\xi|^{p-4} \xi^2 - |\zeta|^{p-4} \zeta^2 | = \left| \left( |\xi|^{\frac{p-1}{2}} |\xi|^{\frac{p+1}{2}} \xi + |\zeta|^{\frac{p-1}{2}} |\zeta|^{\frac{p+1}{2}} \zeta \right) \cdot \left( |\xi|^{\frac{p-1}{2}} |\xi|^{\frac{p+1}{2}} \xi - |\zeta|^{\frac{p-1}{2}} |\zeta|^{\frac{p+1}{2}} \zeta \right) \right| \\
\leq \left( |\xi|^{\frac{p-1}{2}} + |\zeta|^{\frac{p-1}{2}} \right) \left( |\xi|^{\frac{p-1}{2}} |\xi|^{\frac{p+1}{2}} - |\zeta|^{\frac{p-1}{2}} |\zeta|^{\frac{p+1}{2}} \right) \\
\cdot \left( |\xi|^{\frac{p-1}{2}} |\xi|^{\frac{p+1}{2}} - |\zeta|^{\frac{p-1}{2}} |\zeta|^{\frac{p+1}{2}} \right)^{\frac{1}{2}}
\]
\[
\leq 2R^{\frac{p-2}{2}} \left| |\xi|^{p-2} + |\xi|^{p-2} - 2|\xi|^{\frac{p-2}{2}} |\xi|^{\frac{p-2}{2}} \right| \\
+ |\xi|^{\frac{p-4}{2}} |\xi|^{\frac{p-4}{2}} (2|\xi| |\xi - \zeta - \bar{\zeta}|) \right|^\frac{1}{2} \\
= 2R^{\frac{p-2}{2}} \left| \left( |\xi|^{\frac{p-2}{2}} - |\xi|^{\frac{p-2}{2}} \right)^2 + |\xi|^{\frac{p-4}{2}} |\xi|^{\frac{p-4}{2}} \right| \\
\times (2|\xi| |\xi - \zeta - \bar{\zeta}|)^\frac{1}{2}. \tag{148}
\]

In order to complete the estimate, suppose $|\xi| \leq |\zeta|$ and look at the following separately

\[
| |\xi| | - \xi \cdot \bar{\zeta}| = | |\xi| | - |\xi|^2 + \xi \cdot (\bar{\xi} - \bar{\zeta})| \leq 2|\xi| |\xi - \zeta| \\
\leq 2|\xi|(|\xi| + |\xi|)^{1-\theta}| \xi - \zeta|^\theta \leq 2^{2-\theta} |\xi| |\zeta|^{1-\theta}| \xi - \zeta|^\theta, \tag{149}
\]

where $\theta \in [0,1]$ can be chosen arbitrarily. We choose $\theta = \frac{p-2}{p}$. In general, after repeating this argument in all the cases that arise, we have

\[
|2|\xi| |\zeta| - \xi \cdot \bar{\zeta} - \zeta \cdot \bar{\xi}| \leq 2^{3-\frac{p-2}{p}} \min(|\xi|, |\zeta|) \max(|\xi|, |\zeta|)^{\frac{p-2}{p}} |\xi - \zeta|^{\frac{p-2}{p}}. \tag{150}
\]

We then have that

\[
||\xi|^{p-4} \xi^2 - |\xi|^{p-4} \zeta^2| \leq CR^{\frac{p-2}{2}} \left( |\xi - \xi|^{\frac{p-2}{2}} + R^{\frac{p-4}{2}} |\xi - \zeta|^{\frac{p-2}{2}} \right), \tag{151}
\]

so

\[
\frac{1}{n} |\Im (\Im (B)_{\mu \nu} \Pi)| = CR^{\frac{p-2}{p}} |\Im (H)| |\xi - \zeta|^{\frac{p-2}{p}}. \tag{152}
\]

We can now replace the matrices $A$ and $B$ by $\tilde{A}$ and $\tilde{B}$, respectively. By the Cauchy–Schwartz inequality, we have

\[
|\Im (\tilde{A}H + \tilde{B}H)| \leq |\Im (\tilde{A}H)| + |\Im (\tilde{B}H)| \\
\leq \left[ (\Im (\tilde{A} \tilde{A}^*) + (\Im (\tilde{B} \tilde{B}^*))^\frac{1}{2} \right] (\Im (HH^*))^\frac{1}{2} \\
\leq 2\left[ (\Im (\tilde{A} \tilde{A}^*) + (\Im (\tilde{B} \tilde{B}^*))^\frac{1}{2} \right] (\Im (HH^*))^\frac{1}{2}. \tag{153}
\]

We can compute that

\[
\Im (\tilde{A} \tilde{A}^*) = (|\xi|^{p-2} - |\xi|^{p-2})^2 + |\xi|^{p-4} |\xi|^{p-4} (2|\xi|^2) |\xi|^2 - 2|\xi \cdot \bar{\zeta}|^2, \tag{154}
\]

and

\[
\Im (\tilde{B} \tilde{B}^*) = (|\xi|^{p-2} - |\xi|^{p-2})^2 + |\xi|^{p-4} |\xi|^{p-4} (2|\xi|^2) |\xi|^2 - (\xi \cdot \bar{\xi})^2 - (\zeta \cdot \bar{\zeta})^2. \tag{155}
\]
Then
\[
\|\nabla (A^h) + \nabla (B^h)\| \leq 2|\xi - \zeta|^{2(p-2)} + |\xi|^{p-4}|\zeta|^{p-4} \left[ 4|\xi|^2|\zeta|^2 - (\xi \cdot \zeta)^2 \right.
\]
\[
- (\xi \cdot \zeta)^2 - 2(\xi \cdot \zeta)(\xi \cdot \zeta)
\]
\[
= 2|\xi - \zeta|^{2(p-2)} + |\xi|^{p-4}|\zeta|^{p-4}
\]
\[
\times \left[ 4|\xi|^2|\zeta|^2 - (\xi \cdot \zeta + \xi \cdot \zeta)^2 \right].
\]
(156)

Notice that
\[
4|\xi|^2|\zeta|^2 - (\xi \cdot \zeta + \xi \cdot \zeta)^2 = |2|\xi| |\zeta| + \xi \cdot \zeta + \xi \cdot \zeta| |2|\xi| |\zeta| - \xi \cdot \zeta - \xi \cdot \zeta|
\]
\[
\leq 2^{5-\frac{p}{2}} |\xi| |\zeta| \min(|\xi|, |\zeta|) \max(|\xi|, |\zeta|) \frac{2^{p-2}}{2}
\]
\[
\times |\xi - \zeta|^{\frac{p-2}{2}}.
\]
(157)

It follows that
\[
\|\nabla (A^h) + \nabla (B^h)\| \leq CR_{\frac{3p-2}{4}} |\xi - \zeta|^{\frac{p-2}{2}}.
\]
(158)

The lemma is now proven in the \( p \in (2, 3) \) case.

The main differences in the proof when \( p \geq 3 \) are that now
\[
|\xi|^{p-2} - |\zeta|^{p-2} \leq (p-2)(|\xi|^{p-3} + |\zeta|^{p-3})|\zeta - \zeta|,
\]
(159)

and that it is enough to choose \( \theta = \frac{1}{p} \).

\[\square\]

**ORCID iDs**

Cătălin I Cârstea 🌐 https://orcid.org/0000-0001-8644-2704

**References**

[1] Assylbekov Y M and Zhou T 2017 Direct and inverse problems for the nonlinear time-harmonic Maxwell equations in Kerr-type media (arXiv:1709.07767)

[2] Bennett C and Sharpley R C 1988 Interpolation of Operators vol 129 (New York: Academic)

[3] Brander T 2016 Calderón problem for the p-Laplacian: first order derivative of conductivity on the boundary Proc. Am. Math. Soc. 144 177–89

[4] Brander T, Ilmavirta J and Kar M 2018 Superconductive and insulating inclusions for linear and non-linear conductivity equations Inverse Problems Imaging 12 91–123

[5] Brander T, Kar M and Salo M 2015 Enclosure method for the p-Laplace equation Inverse Problems Imaging 31 045001

[6] Brander B, Harrach B, Kar M and Salo M 2018 Monotonicity and enclosure methods for the SpS-Laplace equation SIAM J. Appl. Math. 78 742–58

[7] Calderón A P 1980 On an inverse boundary value problem Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro: Sociedade Brasileira de Matemática) pp 65–73

[8] Cârstea C I 2018 On an inverse boundary value problem for a nonlinear time harmonic Maxwell system (arXiv:1804.09586)

[9] Cârstea C I, Nakamura G and Vashisth M 2019 Reconstruction for the coefficients of a quasilinear elliptic partial differential equation Appl. Math. Lett. 98 121–7
[10] Egger H, Pietschmann J-F and Schlottbom M 2014 Simultaneous identification of diffusion and absorption coefficients in a quasilinear elliptic problem Inverse Problems 30 035009

[11] Evans L C 2010 Partial Differential Equations 2nd edn (Providence, RI: American Mathematical Society)

[12] Feizmohammadi A and Oksanen L 2019 An inverse problem for a semi-linear elliptic equation in Riemannian geometries (arXiv:1904.00608)

[13] Guo C-Y, Kar M and Salo M 2016 Inverse problems for p-Laplace type equations under monotonicity assumptions Rend. Istit. Mat. Univ. Trieste 48 79–99

[14] Hannukainen A, Hyvonen N and Mustonen L 2018 An inverse boundary value problem for the p-Laplacian: a linearization approach Inverse Problems 35 034001

[15] Hervas D and Sun Z 2002 An inverse boundary value problem for quasilinear elliptic equations Commun. PDE 27 2449–90

[16] Isakov V 1993 On uniqueness in inverse problems for semilinear parabolic equations Arch. Ration. Mech. Anal. 124 1–12

[17] Isakov V 2001 Uniqueness of recovery of some quasilinear partial differential equations Commun. PDE 26 1947–73

[18] Isakov V and Nachman A I 1995 Global uniqueness for a two-dimensional semilinear elliptic inverse problem Trans. Amer. Math. Soc. 347 3375–90

[19] Isakov V and Sylvester J 1994 Global uniqueness for a semilinear elliptic inverse problem Comm. Pure Appl. Math. 47 1403–10

[20] Kang H and Nakamura G 2002 Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map Inverse Problems 18 1079

[21] Kar M and Wang J-N 2020 Size estimates for the weighted p-Laplace equation with one measurement Discrete Continuous Dyn. Sys. B 22

[22] Krupchyk K and Uhlmann G 2019 Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities (arXiv:1909.08122)

[23] Lassas M, Liiimatainen T, Lin Y-H and Salo M 2019 Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations (arXiv:1905.02764)

[24] Lassas M, Liiimatainen T, Lin Y-H and Salo M 2019 Inverse problems for elliptic equations with power type nonlinearities (arXiv:1903.12562)

[25] Munoz C and Uhlmann G 2018 The Calderón problem for quasilinear elliptic equations (arXiv:1806.09586)

[26] Nachman A I 1988 Reconstructions from boundary measurements Ann. Math. 128 531–76

[27] Salo M and Zhong X 2012 An inverse problem for the SpS-Laplacian: boundary determination SIAM J. Math. Anal. 44 2474–95

[28] Shankar R 2019 Recovering a quasilinear conductivity from boundary measurements (arXiv:1910.07890)

[29] Sun Z 1996 On a quasilinear inverse boundary value problem Math. Z. 221 293–305

[30] Sun Z 2010 An inverse boundary-value problem for semilinear elliptic equations Electron. J. Differ. Equ. 2010 1–5

[31] Sun Z and Uhlmann G 1997 Inverse problems in quasilinear anisotropic media Am. J. Math. 119 771–97

[32] Sylvester J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary value problem Ann. Math. 153 169