On Three Alternative Characterizations of Combined Traces

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Abstract. The combined trace (i.e., comtrace) notion was introduced by Janicki and Koutny in 1995 as a generalization of the Mazurkiewicz trace notion. Comtraces are congruence classes of step sequences, where the congruence relation is defined from two relations simultaneity and serializability on events. They also showed that comtraces correspond to some class of labeled stratified order structures, but left open the question of what class of labeled stratified orders represents comtraces. In this work, we proposed a class of labeled stratified order structures that captures exactly the comtrace notion. Our main technical contributions are representation theorems showing that comtrace quotient monoid, combined dependency graph (Kleijn and Koutny 2008) and our labeled stratified order structure characterization are three different and yet equivalent ways to represent comtraces. This paper is a revised and expanded version of Lê (in Proceedings of PETRI NETS 2010, LNCS 6128, pp. 104-124).

Key words: causality theory of concurrency, generalized trace theory, combined trace, step sequence, stratified order structure

1 Introduction

Partial orders are one of the main tools for modelling “true concurrency” semantics of concurrent systems (cf. [26]). They are utilized to develop powerful partial-order based automatic verification techniques, e.g., the partial order reduction technique for model checking of concurrent software (see, e.g., [1, Chapter 10] and [9]). Partial orders are also equipped with traces, their powerful formal language counterpart, introduced by Mazurkiewicz in his seminal paper [25]. In The Book of Traces [8], trace theory has been used to tackle problems from diverse areas including formal language theory, combinatorics, graph theory, algebra, logic, and especially concurrency theory.

However while partial orders and traces can sufficiently model the “earlier than” relationship, Janicki and Koutny argued that it is problematic to use a single partial order to specify both the “earlier than” and the “not later than” relationships [13]. This motivates them to develop the theory of relational structures, where a pair of relations is used to capture concurrent behaviors. The most well-known among the classes of relational structures proposed by Janicki and Koutny is the class of stratified order structures (so-structures) [10,16,17]. A so-structure is a triple ($X, \prec, \sqsubseteq$), where $\prec$ and $\sqsubseteq$ are binary relations on $X$. They were invented to model both the “earlier than” (the relation $\prec$) and “not later than” (the relation $\sqsubseteq$) relationships, under the assumption that system runs can be described using stratified partial orders, i.e., step sequences. So-structures have been successfully used to give semantics of inhibitor and priority systems [15,21,19,20].

The combined trace (comtrace) notion, introduced by Janicki and Koutny [14], generalizes the trace notion by utilizing step sequences instead of words. First the set of all possible steps that generates step sequences are identified by a relation $\text{sim}$, which is called simultaneity. Second a congruence relation is determined by a relation $\text{ser}$, which is called serializability and is in general not symmetric. Then a comtrace is defined to be a congruence class of step sequences. Comtraces were introduced as a formal language representation of so-structures to provide an operational semantics for Petri nets with inhibitor arcs. Unfortunately comtraces have been less often known and applied than so-structures, even though in many cases they appear to be more natural. We believe one reason is that the comtrace notion was too succinctly discussed in [14] without a full treatment dedicated to comtrace theory. Motivated by this, Janicki and the author have devoted our recent effort on the study of comtraces [23,18], yet there are too many different aspects to explore and the truth is that we could barely scratch the surface. In particular, a huge amount of results from trace theory (e.g., from [8,7]) need to be generalized to comtraces. These tasks are often challenging since we are required to develop novel techniques to deal with the complex interactions of the “earlier than” and “not later than” relations.
1.1 Motivation

In the literature of Mazurkiewicz traces, traces are defined using the following three equivalent methods. The first method is to define a trace to be a congruence class of words, where the congruence relation is induced from an independency relation on events. In the second method, a trace can be viewed as a dependence graph (cf. [8 Chapter 2]). A dependence graph is a directed acyclic graph whose vertices are labeled with events, and satisfies the condition that every two distinct vertices with dependent labels must be connected by exactly one directed edge. The third method is to define a trace as a labeled partially ordered set, whose elements are labeled with events, and we also require the partial order to be “compatible” with the independency relation (see Definition 8 for the precise formulation). Although the above three characterizations of traces can be shown to be equivalent, depending on the situation one characterization can be more convenient than the others. When studying graph-theoretic aspects of traces, the dependence graph representation is the most natural. The treatment of traces as congruence classes of words is more convenient in Ochmański’s characterization of recognizable trace languages [8 Chapter 6] and Zielonka’s theory of asynchronous automata [8 Chapter 7]. On the other hand most results on temporal logics for traces (see, e.g., [28, 29, 4, 11]) utilize the labeled poset representation of traces. The reason is that it is more natural to interpret temporal logics on the vertices (for local temporal logics) or finite downward closed subsets (for global temporal logics) of a labeled partially ordered set. Thus all of these three representations are indispensable in Mazurkiewicz trace theory.

Since our long-term goal is to generalize the results from Mazurkiewicz trace theory to comtraces, there is a strong need for all three analogous representations for comtraces. In [14], Janicki and Koutny already gave us the definition of comtraces using congruence classes of step sequences. They also showed that every comtrace can be represented by a labeled so-structure, but a direct method for defining comtraces using labeled so-structures was not given. Inspired by the dependence graph representation of traces, Kleijn and Koutny [22] recently introduced the combined dependency graph (cd-graph) notion, but a theorem showing that cd-graphs can be represented by comtraces was not given. Thus the goal of this paper is to complete the picture by giving a new characterization of comtraces using labeled so-structures and developed a unified framework to show the equivalence of these three representations of comtraces.

1.2 Organization

This paper is the revised and expanded version of the conference paper [24]. Although no new results are added, several sections and proofs are rewritten and more examples are included to improve readability of this paper. We also fix a few serious typos and mistakes found in the previous version. The paper is organized as follows.

In Section 2 we recall some preliminary definitions and notations. In Section 3 we gives a concise exposition of the theory of so-structures and comtraces [14, 17] to make the paper self-contained.

In Section 4 we introduce the concept of quotient so-structure and use it to construct our definition of comtraces using congruence classes of step sequences. They also showed that every comtrace can be represented by a labeled so-structure, but a direct method for defining comtraces using labeled so-structures was not given. Inspired by the dependence graph representation of traces, Kleijn and Koutny [22] recently introduced the combined dependency graph (cd-graph) notion, but a theorem showing that cd-graphs can be represented by comtraces was not given. Thus the goal of this paper is to complete the picture by giving a new characterization of comtraces using labeled so-structures and developed a unified framework to show the equivalence of these three representations of comtraces.

In Section 5 contains the main technical contributions. We prove the first representation theorem which establishes bijections between the set of all comtraces and the set of all lsos-comtraces over the same comtrace alphabet. Then using this theorem, we prove the second representation theorem which provides bijections between the set of all lsos-comtraces and the set of all cd-graphs [22] over the same alphabet.

In Section 6 we define composition operators for lsos-comtraces and for cd-graphs, analogous to the comtrace concatenation operator, and show that the set of all lsos-comtraces (or cd-graphs) over a fixed comtrace alphabet together with its composition operator forms a monoid. We also strengthen the representation theorems from Section 5 by showing that the bijections from these theorems are indeed monoid isomorphisms.

Finally, Section 7 contains some final remarks and future work.
2 Notation

2.1 Relations, orders and equivalences

The powerset of a set $X$ will be denoted by $\mathcal{P}(X)$. We let $id_X$ denote the identity relation on a set $X$. We write $R \circ S$ to denote the composition of relations $R$ and $S$. We write $R^*$ to denote the reflexive transitive closure of $R$ respectively.

Let $f : A \to B$ be a function, then for every set $C \subseteq A$, we write $f[C]$ to denote the image of the set $C$ under $f$, i.e., $f[C] := \{f(x) \mid x \in C\}$.

A binary relation $R \subseteq X \times X$ is an equivalence relation relation on $X$ if and only if it is reflexive, symmetric and transitive. If $R$ is an equivalence relation, we write $[x]_R$ to denote the equivalence class of $x$ with respect to $R$, and the set of all equivalence classes in $X$ is denoted as $X/R$ and called the quotient set of $X$ by $R$. We drop the subscript and write $[x]$ to denote the equivalence class of $x$ when $R$ is clear from the context.

A binary relation $\prec \subseteq X \times X$ is a partial order if and only if it is irreflexive and transitive. The pair $(X, \prec)$ in this case is called a partially ordered set (poset). The pair $(X, \prec)$ is called a finite poset if $X$ is finite. For convenience, we define:

$$\approx := \{(a,b) \in X \times X \mid a \neq b \land b \neq a\}$$  \hspace{1cm} (incomparable)

$$\sim := \{(a,b) \in X \times X \mid a \approx b \land b \neq a\}$$  \hspace{1cm} (distinctly incomparable)

$$<^\sim := \{(a,b) \in X \times X \mid a < b \lor a \sim b\}$$  \hspace{1cm} (not greater)

A poset $(X, \prec)$ is total if and only if $\sim$ is empty; and stratified if and only if $\approx$ is an equivalence relation. Evidently every total order is stratified.

2.2 Step sequences

For every finite set $E$, a set $S \subseteq \mathcal{P}(E) \setminus \{\emptyset\}$ can be seen as an alphabet. The elements of $S$ are called steps and the elements of $S^*$ are called step sequences. For example, if the set of possible steps is $S = \{a, b, c, \{a, b\}, \{a\}, \{c\}\}$, then $\{a, b\}|c| \{a, b, c\} \in S^*$ is a step sequence. The triple $(S^*, \cdot, \epsilon)$, where $\cdot$ denotes the step sequence concatenation operator (usually omitted) and $\epsilon$ denotes the empty step sequence, is a monoid.

Let $t = A_1 \ldots A_k$ be a step sequence. We define $|t|_a$, the number of occurrences of an event $a$ in $t$, as $|t|_a := |\{A_i \mid 1 \leq i \leq k \land a \in A_i\}|$. Then we can construct its unique enumerated step sequence $\overline{t}$ as

$$\overline{t} := \overline{A_1} \ldots \overline{A_k}, \text{ where } \overline{A_i} := \{e^{(|A_1| \ldots |A_i| + 1)} \mid e \in A_i\}.$$  

We will call such $a = e^{(j)} \in \overline{A_i}$ an event occurrence of $e$. We let $\Sigma_t = \bigcup_{i=1}^k A_i$ denote the set of all event occurrences in all steps of $t$. We also define $\ell : \Sigma_t \to E$ to be the function that returns the label of $a$ for each $a \in \Sigma_t$. For example, if $a = e^{(j)}$, then $\ell(a) = \ell(e^{(j)}) = e$.

**Example 1.** Given the step sequence $t = \{a, b\}|b|c|a|\{c\}$, then the enumerated step sequence is

$$\overline{t} = \{(a^{(1)}, b^{(1)}) \{b^{(2)}, c^{(1)}\} \{a^{(2)}, c^{(2)}\}\} \{a^{(3)}\}.$$  

The set of event occurrences is $\Sigma_t = \{a^{(1)}, a^{(2)}, a^{(3)}, b^{(1)}, b^{(2)}, c^{(1)}, c^{(2)}\}$.

Given a step sequence $s = B_1 \ldots B_m$ and any function $f$ defined on $\bigcup_{i=1}^m B_i$, we define

$$\text{imap}(f, s) := f[B_1] \ldots f[B_m],$$

which is the step sequence derived from $s$ by computing the image of each $A_i$ under the function $f$ successively. Using the function imap, from an enumerated step sequence $\overline{t} = \overline{A_1} \ldots \overline{A_k}$, we can recover its step sequence $t = \text{imap}(\ell, t) = \ell[\overline{A_1}] \ldots \ell[\overline{A_k}]$.

For each $a \in \Sigma_t$, we let $\text{pos}_t(a)$ denote the consecutive number of a step where $a$ belongs, i.e., if $a \in \overline{A_i}$ then $\text{pos}_t(a) = i$. For our example, $\text{pos}_t(a^{(2)}) = 3$ and $\text{pos}_t(b^{(2)}) = \text{pos}_t(c^{(1)}) = 2$. 

3
It is important to note that step sequences and stratified orders are interchangeable concepts. Given a step sequence \( t \), define the binary relation \( \prec_t \) on \( \Sigma \) as

\[
\alpha \prec_t \beta \iff \text{pos}_t(\alpha) \prec \text{pos}_t(\beta).
\]

Intuitively, \( \alpha \prec_t \beta \) simply means \( \alpha \) occurs before \( \beta \) on the step sequence \( u \). Thus, \( \alpha \prec_t \beta \) if and only if (\( \alpha \neq \beta \land \text{pos}_t(\alpha) \leq \text{pos}_t(\beta) \)); and \( \alpha \equiv_t \beta \) if and only if \( \text{pos}_t(\alpha) = \text{pos}_t(\beta) \). Obviously, the order \( \prec_t \) is stratified and we will call it the stratified order generated by the step sequence \( t \). For instance, from the step sequence \( t \) and its enumerated step sequence \( \overline{t} \) as given in Example\( [1] \) the stratified order \( \prec_t \) is shown in Figure\( [1] \) (the edges that can be inferred by transitivity are omitted).

Fig. 1: The stratified order \( \prec_t \) defined from the step sequence \( t \) given in Example\( [1] \)

Conversely, let \( \prec \) be a stratified order on a set \( \Sigma \). The set \( \Sigma \) can be partitioned into a sequence of equivalence classes \( \Omega_\prec = B_1 \ldots B_k \ (k \geq 0) \) such that

\[
\prec = \bigcup_{i<j} B_i \times B_j \quad \text{and} \quad \equiv_\prec = \bigcup_i B_i \times B_i.
\]

The sequence \( \Omega_\prec \) is called the step sequence representing \( \prec \). For example, the let \( \prec \) be the stratified order in Figure\( [1] \) then

\[
\Omega_\prec = \{a^{(1)}, b^{(1)}\} \{b^{(2)}, c^{(1)}\} \{a^{(2)}, c^{(2)}\} \{a^{(3)}\},
\]

which is exactly the enumerated step sequence \( \overline{t} \) given in Example\( [1] \). To get back the step sequence \( t \), we need to get rid of all the superscripts as follows:

\[
\text{imap}(\ell, \overline{t}) = \ell \{a^{(1)}, b^{(1)}\} \ell \{b^{(2)}, c^{(1)}\} \ell \{a^{(2)}, c^{(2)}\} \ell \{a^{(3)}\} = (a, b) | (b, c) | (c, a) | a = t.
\]

### 3 Stratified order structures and combined traces

In this section, we review the Janicki-Koutny theory of stratified order structures and comtraces from [14][17]. This introduction might be too concise for readers who are not familiar with the subject, so we also refer to [22] for an excellent introductory tutorial on traces, comtraces, partial orders and so-structures with many motivating examples.

#### 3.1 Stratified order structures

A relational structure is a triple \( T = (X, R_1, R_2) \), where \( X \) is a set and \( R_1, R_2 \) are binary relations on \( X \). A relational structure \( T' = (X', R'_1, R'_2) \) is an extension of \( T \), denoted as \( T \subseteq T' \), if and only if \( X = X' \), \( R_1 \subseteq R'_1 \) and \( R_2 \subseteq R'_2 \).

**Definition 1** (stratified order structure [17]). A stratified order structure (so-structure) is a relational structure \( S = (X, \prec, \sqsubseteq) \), such that for all \( \alpha, \beta, \gamma \in X \), the following hold:

- S1: \( \alpha \not\prec \alpha \)
- S2: \( \alpha \prec \beta \implies \alpha \sqsubseteq \beta \)
- S3: \( \alpha \sqsubseteq \beta \sqsubseteq \gamma \land \alpha \neq \gamma \implies \alpha \sqsubseteq \gamma \)
- S4: \( \alpha \sqsubseteq \beta \sqsubseteq \gamma \land \alpha \neq \beta \sqsubseteq \gamma \implies \alpha \prec \gamma \)

When \( X \) is finite, \( S \) is called a finite so-structure.
The axioms S1–S4 imply that $<$ is a partial order and $\alpha < \beta \Rightarrow \beta \nleq \alpha$. The axioms S1 and S3 imply $\sqsubseteq$ is a strict preorder. The relation $<$ is called causality and represents the “earlier than” relationship, while the relation $\sqsubseteq$ is called weak causality and represents the “not later than” relationship. The axioms S1–S4 model the mutual relationship between “earlier than” and “not later than” relations, provided that system runs are modeled by stratified orders. Historically, the name “stratified order structure” came from the fact that stratified orders can be seen as a special kind of so-structures.

**Proposition 1 (cf. [13]).** For every stratified poset $(X, \prec)$, the triple $S_{\prec} = (X, \prec, \nprec)$ is an so-structure.

We next recall the notion of stratified order extension. This concept is important for our purpose since the relationship between stratified orders and so-structures is analogous to the one between total orders and partial orders.

**Definition 2 (stratified extension [17]).** Let $S = (X, \prec, \sqsubseteq)$ be an so-structure. A stratified order $\prec_\leq$ on $X$ is a stratified extension of $S$ if and only if $(X, \prec, \sqsubseteq) \subseteq (X, \prec_\leq, \nprec_\leq)$. The set of all stratified extensions of $S$ is denoted as $\text{ext}(S)$.

Szpilrajn’s Theorem [27] states that every poset can be reconstructed by taking the intersection of all of its total order extensions. Janicki and Koutny showed that a similar result holds for so-structures and stratified extensions.

**Theorem 1 ([17]).** Let $S = (X, \prec, \sqsubseteq)$ be an so-structure. Then

$$S = \left\{ x, \bigcup_{\alpha \in \text{ext}(\alpha)} \prec, \bigcup_{\alpha \in \text{ext}(\alpha)} \nprec \right\}.$$

This theorem holds even when $X$ is infinite, and its proof requires some version of the axiom of choice. But we are only concerned with finite so-structures in this paper. Using this theorem, we can show the following properties relating so-structures with their stratified extensions.

**Corollary 1.** For every so-structure $S = (X, \prec, \sqsubseteq)$,

1. $\exists \alpha \in \text{ext}(\alpha), \alpha \prec \beta \land \exists \beta \in \text{ext}(\beta), \beta \prec a \Rightarrow \exists \beta \in \text{ext}(\beta), \beta \nprec a$.
2. $\forall \alpha \in \text{ext}(\alpha), a \prec \beta \lor \beta \prec a \iff a \prec \beta \lor \beta \prec a$.

**Proof.** 1. See [17, Theorem 3.6].
2. Follows from 1. and Theorem [1] \hfill \square

### 3.2 Combined traces

*Comtraces (Combined traces)* were introduced in [14] as a generalization of traces to represent so-structures. The *comtrace congruence* is defined via two binary relations simultaneity and serializability on a finite set of events.

**Definition 3 (comtrace alphabet [14]).** Let $E$ be a finite set of events, and $\text{ser} \subseteq \text{sim} \subseteq E \times E$ two relations called *serializability* and *simultaneity* respectively, where $\text{sim}$ is reflexive and symmetric. The triple $\theta = (E, \text{sim}, \text{ser})$ is called a comtrace alphabet.

Note that since $\text{sim}$ is reflexive and $\text{ser} \subseteq \text{sim}$, it follows that the relation $\text{ser}$ is also reflexive. Intuitively, if $(a, b) \in \text{sim}$ then $a$ and $b$ may occur simultaneously with each other in a step $\{a, b\}$. If $(a, b) \in \text{ser}$, then $a$ and $b$ may occur together in a step $\{a, b\}$ and, moreover, such step can be split into the sequence $\{a\} \{b\}$. A step can involve more than two events as long as all events within the same step are pairwise related by the simultaneity relation. More formally, we define the set of all possible steps $S_{\theta}$ induced from the simultaneity relation $\text{sim}$ to be the set of all cliques of the graph $(E, \text{sim})$, i.e.,

$$S_{\theta} := \left\{ A \mid A \neq \emptyset \land \forall a, b \in A, \{a = b \lor (a, b) \in \text{sim}\} \right\}.$$
Definition 4 (comtrace congruence [14]). For a comtrace alphabet \( \theta = (E, sim, ser) \), we define \( \approx_{\theta} \subseteq S_{\theta}^* \times S_{\theta}^* \) to be the relation comprising all pairs \((t, u)\) of step sequences such that

\[
\begin{align*}
t &= uAz \\
u &= wBCz
\end{align*}
\]

where \(w, z \in S_{\theta}^*\) and \(A, B, C\) are steps in \(S_{\theta}\) satisfying \(B \cup C = A\) and \(B \times C \subseteq ser\).

We define comtrace congruence \(\equiv_{\theta} := (\approx_{\theta} \cup \approx_{\theta}^{-1})^\ast\), and the equivalence classes in \(S_{\theta}^*/\equiv_{\theta}\) are called comtraces.

We define the comtrace concatenation operator \(\_\_\_\_\_\) as \([r] \_\_\_\_\_\_ : = [r \ast t]\). The quotient monoid \((S_{\theta}^*/\equiv_{\theta}, \_\_\_\_\_\_, [\cdot])\) is called the comtrace monoid over the comtrace alphabet \(\theta\).

Note that since \(ser\) is irreflexive, \(B \times C \subseteq ser\) implies that \(B \cap C = \emptyset\). The fact that the comtrace concatenation operator \(\_\_\_\_\_\_\) is well-defined was shown in [14, Proposition 4.14]. We will omit the subscript \(\theta\) from \(\approx_{\theta}\) and \(\equiv_{\theta}\), and write \(\equiv\) and \(\approx\) when it causes no ambiguity. To shorten our notations, we often write \([s]\theta\) or \([s]\) instead of \([s]_{\equiv_{\theta}}\) to denote the comtrace generated by the step sequence \(s\) over the comtrace alphabet \(\theta\).

Example 2. Consider three atomic operations \(a, b\) and \(c\) as follows

\[
\begin{align*}
a &: \ y \leftarrow x + y \\
b &: \ x \leftarrow y + 2 \\
c &: \ y \leftarrow y + 1
\end{align*}
\]

Assume simultaneous reading is allowed, but simultaneous writing is not allowed. Then the events \(b\) and \(c\) can be performed simultaneously, and the execution of the step \((b, c)\) gives the same outcome as executing \(b\) followed by \(c\). The events \(a\) and \(b\) can also be performed simultaneously, but the outcome of executing the step \((a, b)\) is not the same as executing \(a\) followed by \(b\), or \(b\) followed by \(a\). Note that although executing the steps \((a, b)\) and \((b, c)\) is allowed, we cannot execute the step \((a, c)\) since that would require writing on the same variable \(y\). Thus, the simultaneity relation \(sim\) is not transitive.

Let \(E = \{a, b, c\}\) be the set of events. Then we can define the comtrace alphabet \(\theta = (E, sim, ser)\), where \(sim = \{(a, b), (b, a), (b, c), (c, b)\}\) and \(ser = \{(b, c)\}\). Thus the set of all possible steps is

\[
S_{\theta} = \{|a|, |b|, |c|, |a, b|, |b, c|\}.
\]

We observe that the set \(t = |a|\{a\}|b|\{b, c\}| = \{|a|\{a\}|b, c\}, |a|\{a\}|b, c\} \) is a comtrace. But the step sequence \(|a|\{a\}|b, c|\) is not an element of \(t\) because \((c, b) \notin ser\).

Even though traces correspond to quotient monoids over sequences and comtraces correspond to quotient monoids over step sequences, traces can be regarded as a special kind of comtraces when the relation \(ser = sim\).

For a more detailed discussion on this connection between traces and comtraces, the reader is referred to [13].

Definition 5 ([14]). Let \(u \in S_{\theta}^\ast\). We define the relations \(\prec_u, \sqsubseteq_u \subseteq \Sigma_u \times \Sigma_u\) as

1. \(a \prec_u \beta \iff a \prec_{\Sigma} \beta \cap (\ell(\alpha), \ell(\beta)) \notin ser\),
2. \(a \sqsubseteq_u \beta \iff a \sqsubseteq_{\Sigma} \beta \cap (\ell(\alpha), \ell(\beta)) \notin ser\).

It is worth noting that the structure \((\Sigma_u, \prec_u, \sqsubseteq_u, \ell)\) is exactly the cd-graph (cf. Definition[12]) that represents the comtrace \([u]\). This gives us some intuition on how Kleijn and Koutny constructed their cd-graph definition in [22]. The structure \((\Sigma_u, \prec_u, \sqsubseteq_u, \ell)\) is usually not an so-structure since \(\prec_u\) and \(\sqsubseteq_u\) describe only basic “local” causality and weak causality invariants of the event occurrences of \(u\) by considering pairwise serializable relationships of event occurrences, and thus \(\prec_u\) and \(\sqsubseteq_u\) might not capture “global” invariants that can be inferred from S2-S4 of Definition[1]. To ensure all invariants are included, we need the following \(\diamond\)-closure operator.

Definition 6 ([14]). For every relational structure \(S = (X, R_1, R_2)\), we define \(S^\diamond\) as

\[
S^\diamond := (X, (R_1 \cup R_2)^\ast \circ R_1 \circ (R_1 \cup R_2)^\ast, (R_1 \cup R_2)^\ast \setminus id_X).
\]
Intuitively the \( \diamond \)-closure generalizes the transitive closure for relations to relational structures. The motivation is that for appropriate relations \( R_1 \) and \( R_2 \) (see assertion (3) of Proposition 2 below), the relational structure \( (X, R_1, R_2)^\diamond \) is an so-structure. The \( \diamond \)-closure satisfies the following properties.

**Proposition 2** ([14]). Let \( S = (X, R_1, R_2) \) be a relational structure.

1. If \( R_2 \) is irreflexive then \( S \subseteq S^\diamond \).
2. \( (S^\diamond)^\diamond = S^\diamond \).
3. \( S^\diamond \) is an so-structure if and only if \( (R_1 \cup R_2)^* \circ R_1 \circ (R_1 \cup R_2)^* \) is irreflexive.
4. If \( S \) is an so-structure then \( S = S^\diamond \).
5. If \( S \) is an so-structure and \( S_0 \subseteq S \), then \( S_0^\diamond \subseteq S \) and \( S_0^\diamond \) is an so-structure.

**Definition 7.** Given a step sequence \( u \in \mathbb{S}_\theta^* \) and its respective comtrace \( t = [u] \in \mathbb{S}_\theta^*/\equiv \), we define the relational structure \( S_t \) as:

\[
S_t = (\Sigma_t, <_t, \sqsubseteq_t) := (\Sigma_u, <_u, \sqsubseteq_u)^\diamond.
\]

The relational structure \( S_t \) is called the so-structure defined by the comtrace \( t = [u] \in \mathbb{S}_\theta^*/\equiv \), where \( \Sigma_t \), \( <_t \) and \( \sqsubseteq_t \) are used to denote the event occurrence set, causality relation and weak causality relation induced by the comtrace \( t \) respectively. The following nontrivial theorem and its corollary justifies the name by showing that step sequences in a comtrace \( t \) are exactly the stratified extensions of the so-structure \( S_t \), and that \( S_t \) is uniquely defined for the comtrace \( t \) regardless of the choice of the step sequence \( u \in t \).

**Theorem 2** ([14]). For each comtrace \( t \in \mathbb{S}_\theta^*/\equiv \), the structure \( S_t \) is an so-structure and the set \( \text{ext}(S_t) \) of stratified extensions of \( S_t \) is exactly the same as \( \{<_u \mid u \in t\} \), the set of stratified orders induced by the step sequences in the comtrace \( t \).

**Corollary 2.** For all comtraces \( t, q \in \mathbb{S}_\theta^*/\equiv \),

1. \( t = q \) if and only if \( S_t = S_q \)
2. \( S_t = (\Sigma_t, <_t, \sqsubseteq_t) = (\Sigma_u, \bigcap_{w \in t} <_w, \bigcap_{w \in t} \sqsubseteq_w) \)

The first part of the corollary states that two comtraces are the same if and only if they define the same so-structure. The second part of corollary gives us two equivalent methods for constructing the so-structure from the comtrace \( t \). The first method is to use the construction from Definition 7. The second method is to consider all the step sequences in \( t \), and for every two event occurrences \( \alpha \) and \( \beta \) of \( t \), define \( \alpha <_t \beta \) if \( \alpha \) always occurs strictly before \( \beta \) in all of these step sequences, and define \( \alpha \sqsubseteq_t \beta \) if \( \alpha \) always occurs before or simultaneously with \( \beta \) in all of these step sequences.

**4 Comtraces as labeled stratified order structures**

Even though Theorem 2 shows that each comtrace can be represented uniquely by a labeled so-structure, it does not give us any explicit definition describing how these labeled so-structures look like. The goal of this section is to define exactly the class of labeled so-structures representing comtraces. To provide us with more intuition, we will first recall how Mazurkiewicz traces can be characterized using labeled posets.

A trace alphabet is a pair \((E, \text{ind})\), where \( \text{ind} \) is a symmetric irreflexive binary relation on the finite set \( E \). A trace congruence \( \equiv_{\text{ind}} \) can then be defined as the smallest equivalence relation such that for all sequences \( uabv, ubav \in E^* \), if \((a, b) \in \text{ind}\), then \( uabv \equiv_{\text{ind}} ubav \). The elements of \( E^* / \equiv_{\text{ind}} \) are called traces.

Traces can also be defined alternatively as posets whose elements are labeled with symbols of a concurrent alphabet \((E, \text{ind})\) satisfying certain conditions.

Given a binary relation \( R \subseteq X \times X \), the covering relation of \( R \) is defined as

\[
R^{\text{cov}} := \{ (x, y) \mid x R y \land \exists z, x R z R y \}.
\]
In other words, \( x R^{\text{cov}} y \) if and only if \( x \) is \textit{immediately related} to \( y \) by the relation \( R \). When \( R \) is a partial order, the graph representing \( R^{\text{cov}} \) is exactly the familiar Hasse diagram of \( R \), and the notion of covering relation is easier to visualize in this case.

Using the covering relation notion, an alternative definition of Mazurkiewicz trace can be given as in the following definition.

**Definition 8 (cf. [28]).** A trace over a trace alphabet \((E, ind)\) is a finite labeled poset \( P = (X, <, \lambda) \), where \( \lambda : X \to E \) is a labeling function, such that for all elements \( a \neq b \) of \( X \),

1. \( a \prec^{\text{cov}} b \iff (\lambda(a), \lambda(b)) \notin ind \) (immediately causally related event occurrences must be labeled with dependent events), and
2. \( (\lambda(a), \lambda(b)) \notin ind \Rightarrow a < b \lor b < a \) (any two event occurrences with dependent labels must be causally related).

A trace in this definition is only identified unique up to \textit{label-preserving isomorphism}. The first condition of the above definition is particularly important since two immediately causally related event occurrences will occur next to each other in at least one of its linear extensions, and thus they cannot be labeled by independent events without violating the causality relation \(<\). This observation is the key to relate Definition 8 with quotient monoid definition of traces.

We would like to establish a analogous definition for comtraces. An immediate technical difficulty is that weak causality might be cyclic, so the notion of “immediate weak causality” does not make sense. However, we can still deal with cycles of an so-structure by taking advantage of the following simple fact: the \textit{weak causality relation is a strict preorder}.

### 4.1 Quotient so-structure

Let \( S = (X, \preceq, \sqcap) \) be an so-structure. We define the relation \( \equiv \subseteq X \times X \) as

\[
\alpha \equiv \beta \overset{df}{\iff} \alpha = \beta \lor (\alpha \sqcap \beta \land \beta \sqcap \alpha)
\]

Since \( \sqcap \) is a strict preorder, it follows that \( \equiv \) is an equivalence relation. The relation \( \equiv \) will be called the \textit{\( \sqcap \)-cycle equivalence relation} and an element of the quotient set \( X/\equiv \) will be called a \( \sqcap \)-cycle equivalence class. Define the following binary relations \( \preceq \) and \( \equiv \) on the quotient set \( X/\equiv \) as

\[
[\alpha] \preceq [\beta] \overset{df}{\iff} \alpha \neq \beta \land ([\alpha] \times [\beta]) \cap < \neq \emptyset \\
[\alpha] \equiv [\beta] \overset{df}{\iff} \alpha \neq \beta \land ([\alpha] \times [\beta]) \cap \sqcap \neq \emptyset
\]

(4.1) (4.2)

We call the relational structure \( (X/\equiv, \preceq, \equiv) \) the \textit{quotient so-structure} induced by the so-structure \( S \).

Using this quotient construction, we will show that every so-structure, whose weak causality relation might be cyclic, can be uniquely represented by an \textit{acyclic} quotient so-structure.

**Proposition 3.** The relational structure \( S/\equiv := (X/\equiv, \preceq, \equiv) \) is an so-structure, the relations \( \preceq \) and \( \equiv \) are partial orders, and for all \( \alpha, \beta \in X \),

1. \( \alpha \preceq \beta \iff [\alpha] \preceq [\beta] \)
2. \( \alpha \equiv \beta \iff [\alpha] \equiv [\beta] \lor (\alpha \neq \beta \land [\alpha] = [\beta]) \)

**Proof.** Follows from Definition [1] and how \( \preceq \) and \( \equiv \) are defined in (4.1) and (4.2). \( \square \)

Each \( \sqcap \)-cycle equivalence class is what Juhás, Lorenz and Mauser called a \textit{synchronous step} [19]. In their papers, they also used equivalence classes to capture synchronous steps but only for the special class of \textit{synchronous closed} so-structures, where \( (\sqcap \setminus <) \cup id_X \) is an equivalence relation. Note that when an so-structure is synchronous closed, the \( \sqcap \)-cycle equivalence classes of an so-structure are exactly the equivalence classes of the equivalence relation \( (\sqcap \setminus <) \cup id_X \).

We extend their ideas by using \( \sqcap \)-cycle equivalence classes to capture “synchronous steps” in arbitrary so-structures. The name is justified in the following simple yet useful proposition.
Proposition 4. Let $S = (X, \prec, \sqsubseteq)$ be an so-structure. We use $u$ and $v$ to denote some step sequences over $\varphi(X) \setminus \emptyset$. Then for all $\alpha, \beta \in X$,

1. $\alpha \in [\beta] \iff \forall \prec \in \text{ext}(S), \alpha \equiv \prec \beta$
2. There is a stratified extension $\prec$ of $S$ such that $\Omega_\prec = [\gamma_1] \ldots [\gamma_k]$.
3. If $[\alpha] \equiv \text{cov} [\beta]$, then there is a stratified extension $\prec$ of $S$ such that $\Omega_\prec = [\delta_1] \ldots [\delta_m]$ and $\alpha \in [\delta_1]$ and $\beta \in [\delta_{i+1}]$ for some $1 \leq i < m$.

Assertion (1) states that two elements belong to the same synchronous step of an so-structure if and only if they must be executed simultaneously in every stratified extension of $S$. In other words, when reinterpreting each stratified extensions of $S$ as a step sequence, all elements of a $\sqsubseteq$-cycle equivalence class must always occur together within the same step. Assertion (2) says that all elements of a synchronous step must occur together as a single step in at least one stratified extension of $S$. Assertion (3) gives a sufficient condition for two synchronous steps to occur as consecutive steps in at least one stratified extension of $S$.

Proof. 1. ($\Rightarrow$): Since $\alpha \in [\beta]$, we know that $\alpha = \beta$ or $(\alpha \sqsubseteq \beta \land \beta \sqsubseteq \alpha)$. The former case is trivial. For the latter case, by Theorem 1, we have $\forall \prec \in \text{ext}(S), \alpha \prec \beta$ and $\forall \prec \in \text{ext}(S), \beta \prec \alpha$. (Recall that $\preceq \prec \equiv \prec \cup \sim \prec$.) Thus, it follows that $\forall \prec \in \text{ext}(S), \alpha \preceq \beta$.

($\Leftarrow$): By definition, we have $\sim \prec \equiv \sim \prec \cup \sim X$, so we consider two cases. The first case when $\alpha = \beta$ is trivial. The second case when we have $\forall \prec \in \text{ext}(S), \alpha \sim \prec \beta$ and $\alpha \neq \beta$. Thus, by Theorem 1, $\alpha \sqsubseteq \beta$ and $\beta \sqsubseteq \alpha$, which implies $\alpha$ and $\beta$ belong to the same equivalence class.

2. We will construct a step sequence $s = [\gamma_1] \ldots [\gamma_k]$ which can be converted back to a stratified extension $\prec$ of the so-structure $S$ as follows. Since $P = \{X \equiv \sqsubseteq, \sim X\}$ is a poset, we can simply choose $s$ to be an arbitrary total order extension of $P$. From this, it is not hard to check that the stratified order $(X, \prec)$ representing the step sequence $s$ is an extension of $S$.

3. Similarly to 2., we can choose the step sequence $t = [\delta_1] \ldots [\delta_m]$ to be a total order extension of the poset $P$. Moreover, since $[\alpha]$ and $[\beta]$ are immediately related by the partial order $\sim$, we can easily choose the step sequence $t$ such that $[\alpha]$ and $[\beta]$ occur consecutively on $t$. \qed

4.2 Using quotient so-structure to define comtrace

We need to define label-preserving isomorphisms for labeled so-structures more formally. A tuple $T = (X, P, Q, \lambda)$ is a labeled relational structure if and only if $(X, P, Q)$ is a relational structure and $\lambda$ is a function with domain $X$. If $(X, P, Q)$ is an so-structure, then $T$ is a labeled so-structure.

Definition 9 (label-preserving isomorphism). Let $T_1$ and $T_2$ be labeled relational structures, where we let $T_i = (X_i, P_i, Q_i, \lambda_i)$. We write $T_1 \equiv T_2$ if and only if $T_1$ and $T_2$ are label-preserving isomorphic (lp-isomorphic). In other words, there is a bijection $f : X_1 \to X_2$ such that for all $\alpha, \beta \in X_1$,

1. $(\alpha, \beta) \in P_1 \iff (f(\alpha), f(\beta)) \in P_2$
2. $(\alpha, \beta) \in Q_1 \iff (f(\alpha), f(\beta)) \in Q_2$
3. $\lambda_1(\alpha) = \lambda_2(f(\alpha))$

Such function $f$ is called a label-preserving isomorphism (lp-isomorphism).

Note that all notations, definitions and results for so-structures are applicable to labeled so-structures. We also write $|T|$ or $[X, P, Q, \lambda]$ to denote the lp-isomorphic class of a labeled relational structure $T = (X, P, Q, \lambda)$. We will not distinguish an lp-isomorphic class $|T|$ with a single labeled relational structure $T$ when it does not cause ambiguity.

We are now ready to give an alternative definition for comtraces. To avoid confusion with the comtrace notion by Janicki and Koutny in [14], we will use the term Isos-comtrace to denote a comtrace defined using our definition.

Definition 10 (Isos-comtrace). A Isos-comtrace over a comtrace alphabet $\theta = (E, sim, ser)$ is (an lp-isomorphic class of) a finite labeled so-structure $\{X, \prec, \sqsubseteq, \lambda\}$ such that $\lambda : X \to E$ and for all elements $\alpha \neq \beta$ of $X$,
A characterization of comtraces using the quotient so-structure.

Let \( T \) be a comtrace alphabet, where
\[
E = \{a, b, c\}
\]
\[
sim = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\}
\]
\[
\text{ser} = \{(a, b), (b, a), (a, c)\}
\]
The set of all possible steps is \( S = \{(a, b), (b, c), (c, b)\} \). The lp-isomorphic class of the labeled so-structure \( T = (X, \prec, \sqsubseteq, \lambda) \) shown in Figure 2 is an Isos-comtrace. The dashed edges denote the \( \sqsubseteq \) relation and the solid edges denote the \( \prec \) relation. The graph in Figure 3 represents the labeled quotient so-structure \( T/\sqsubseteq = (X/\sqsubseteq, \hat{\prec}, \hat{\sqsubseteq}, \lambda') \) of \( T \), where we define \( \lambda'(A) := \lambda[A] \). It is important to note that the quotient construction collapses

**Example 3.** Let \( \theta = (E, \sim, \text{ser}) \) be a comtrace alphabet, where

\[
\begin{align*}
E &= \{a, b, c\} \\
\sim &= \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\} \\
\text{ser} &= \{(a, b), (b, a), (a, c)\}
\end{align*}
\]

We write \( \text{LCT}(\theta) \) to denote the class of all Isos-comtraces over \( \theta \).

To understand the first three conditions LC1–LC3 of this definition, it is more intuitive to consider the Hasse diagram \( (X/\equiv, \prec, \sqsubseteq) \) of the poset \( (X/\equiv, \preceq) \). Each vertex \([a]\) of this Hasse diagram is a \( \sqsubseteq \)-cycle equivalence class, so all events in \([a]\) must happen simultaneously, and thus each equivalence class \([a]\) can be seen as a “composite event”. Hence, condition LC3 imposes that we cannot serialize \([a]\) into two separate steps. Also, given \([a]\sqsubseteq [\beta]\) on this Hasse diagram, we know that the “composite event” \([a]\) must happen not later than the “composite event” \([\beta]\), and there are two possible cases. If \([a]\) must happen earlier than \([\beta]\), then condition LC1 ensures that the events in \([a]\) and \([\beta]\) cannot be put back together into a single step, i.e., \( \lambda([a]) \times \lambda([\beta]) \not\subseteq \text{ser} \). Condition LC2 is the dual of condition LC1 and takes care of the remaining case when \([a]\) does not have to happen earlier than \([\beta]\).

The last two conditions are needed to ensure that the relationships \( \text{ser} \) and \( \sim \) are properly translated into the “earlier than” relation \( \prec \) and the “not later than” relation \( \sqsubseteq \) of the so-structure. It is important to note that the definition is still valid if we substitute LC4 and LC5 with the following two conditions:

\[
\begin{align*}
\text{LC4}': \quad \lambda([a]) \times \lambda([\beta]) \not\subseteq \text{ser} \land [a] \neq [\beta] & \implies [a] \prec [\beta] \lor [\beta] \sqsubseteq [a] \\
\text{LC5}': \quad \lambda([a]) \times \lambda([\beta]) \not\subseteq \sim \land [a] \neq [\beta] & \implies [a] \sqsubseteq [\beta] \lor [\beta] \prec [a]
\end{align*}
\]

Then all conditions will involve only the \( \sqsubseteq \)-cycle equivalence classes. In other words, this definition can be seen as a characterization of comtraces using the quotient so-structure \( (X/\equiv\sqsubseteq, \hat{\prec}, \hat{\sqsubseteq}) \).

Fig. 2: Isos-comtrace \( T \)

Fig. 3: the quotient structure \( T/\equiv\sqsubseteq \)
the two rightmost nodes of the graph in Figure 2 since they belong to the same ⊑-cycle equivalence class. The result is a much simpler and more compact representation in Figure 3.

The lsos-comtrace \([T]\) actually corresponds to the comtrace

\[
\{[a, b] | c | b, c\} = \{[a, b] | [c | b, c], [a] | [b] | [c | b, c], [b] | [a] | [c | b, c], [b] | [a, c] | [b, c]\},
\]

since the step sequences in this comtrace when reinterpreted as stratified orders are exactly the two stratified extensions of the lsos-comtrace \([T]\). We will show this relationship formally in Section 5.1.

**Remark 1.** Definition 10 can be extended to define *infinite comtrace* as follows. Instead of asking \(X\) to be finite, we require a labeled so-structure to be *initially finite* (cf. [17]), i.e., \(\{x \in X \mid x \sqsubseteq \beta\}\) is finite for all \(\beta \in X\). The initial finiteness gives us a sensible interpretation that before any event there can only be finitely many events.

### 4.3 Canonical representation of lsos-comtrace

Since each lsos-comtrace is defined as a class of \(lp\)-isomorphic labeled so-structures, it might seem tricky to work with lsos-comtraces. Fortunately, the “no autoconcurrency” property, i.e., the relations \(sim\) and \(ser\) are irreflexive, gives us a *canonical* way to enumerate the events of an lsos-comtrace similar to how the events of a comtrace are enumerated.

Given an lsos-comtrace \(T = [X, <, \sqsubseteq, \lambda]\) over a comtrace alphabet \(\theta = (E, sim, ser)\), a stratified order \(\triangleleft \in ext(T)\) can be seen as a step sequence \(\Omega_{\triangleleft} = A_1 \ldots A_k\) satisfying the following properties.

**Proposition 5.**

1. For all \(1 \leq i \leq k, |A_i| = |\lambda(A_i)|\)
2. \(imap(\lambda, \Omega_{\triangleleft}) = \lambda(A_1) \ldots \lambda(A_k)\) is a step sequence over the comtrace alphabet \(\theta\).

This proposition ensures that \(u = imap(\lambda, \Omega_{\triangleleft})\) is a valid step sequence over the comtrace alphabet \(\theta\).

**Proof.** 1. Intuitively, this follows from the fact that \(sim\) is irreflexive, which guarantees that different occurrences of the same events cannot occur simultaneously in any stratified extension of \(T\). More formally, assume \(a, \beta \in A_i\) and \(a \neq \beta\). Thus, \(a \not< \beta\). By Corollary 1(2), this implies that \(\alpha \not< \beta\). Hence, by LC5 of Definition 10 \((\lambda(a), \lambda(\beta)) \in sim\).

Since \(sim\) is irreflexive, this implies that any two distinct \(a\) and \(\beta\) in \(A_i\) have different labels. Thus, \(|A_i| = |\lambda(A_i)|\) for all \(1 \leq i \leq k\).

2. From the proof of 1., we know that for any two distinct \(a, \beta \in A_i\) we have \((\lambda(a), \lambda(\beta)) \in sim\). Thus, \(\lambda(A_i) \in \Sigma_\theta\) for all \(1 \leq i \leq k\).

Given an lsos-comtrace \(T = [X, <, \sqsubseteq, \lambda]\) as above, we define a function \(enum\) with domain \(X\) as

\[
enum(x) = \lambda(x)_{(i)},
\]

where the index \(i := |\{y \in X \mid y < x \wedge \lambda(y) = \lambda(x)\}| + 1\). In other words, \(i - 1\) is the number of elements that occur earlier than \(x\) and have the same labels as \(x\). Thus, the index \(i = 1\) when \(x\) is the first occurrence of the event \(\lambda(x)\) with respect to the relation \(<\), and in general \(enum(x) = \lambda(x)_{(i)}\) if \(x\) is the \(i\)th occurrence of the event \(\lambda(x)\) with respect to \(<\). Note that we can enumerate events with the same label in this way because it follows from LC5 of Definition 10 that events with the same label are totally ordered by \(<\).

Fix a stratified extension \(\triangleleft\) of the above lsos-comtrace \(T\), and assume that \(\Omega_{\triangleleft} = A_1 \ldots A_k\). Then by Proposition 5, we know that the step sequence \(u = imap(\lambda, \Omega_{\triangleleft})\) is a valid step sequence over the comtrace alphabet \(\theta\). Recall that \(\mathcal{U} = A_1 \ldots A_k\) denotes the enumerated step sequence of \(u\) and \(\Sigma_u\) denotes the set of event occurrences in \(u\).

Then it is not hard to see that the range of the function \(enum\) defined above is exactly the set \(\Sigma_u\), i.e., \(enum[X] = \Sigma_u\), and this holds regardless of the choice of the stratified extension \(\triangleleft \in ext(T)\).

We define the *enumerated so-structure* of \(T\) to be the labeled so-structure \(T_0 = (\Sigma, <_0, \sqsubseteq_0, \ell)\), where we let \(\Sigma = enum[X]\), the function \(\ell : \Sigma \to E\) is as defined as in Section 2.2. and the two relations \(<_0, \sqsubseteq_0 \subseteq \Sigma \times \Sigma\) are defined as:

\[
\begin{align*}
\alpha <_0 \beta & \iff enum^{-1}(\alpha) < enum^{-1}(\beta) \\
\alpha \sqsubseteq_0 \beta & \iff enum^{-1}(\alpha) \sqsubseteq enum^{-1}(\beta)
\end{align*}
\]
Here \( \text{enum}^{-1} \) denotes the inverse of the function \( \text{enum} \). The inverse of \( \text{enum} \) is well-defined since it is not hard to see that the function \( \text{enum} : X \to \Sigma \) is bijective.

Clearly the enumerated so-structure \( T_0 \) can be uniquely determined from \( T \) in this definition. Since \( T_0 \) is constructed from \( T \) by renaming the elements in \( X \) using the function \( \text{enum} \), we can easily show the following relationships:

**Proposition 6.**

1. \( T \) and \( T_0 \) are \( \text{lp} \)-isomorphic under the mapping \( \text{enum} \).
2. For every stratified extension \( \prec \) of \( T \), if \( u = \text{imap}(\lambda, \Omega, \prec) \), then \((X, \prec, \sqsubseteq, \lambda)\) and \((\Sigma, \prec_u, \sqsubseteq_u, \ell)\) are \( \text{lp} \)-isomorphic under the mapping \( \text{enum} \), and we also have \( \prec_u \in \text{ext}(T_0) \).

In other words, the mapping \( \text{enum} : X \to \Sigma \) plays the role of both the \( \text{lp} \)-isomorphism from \( T \) to \( T_0 \) and the \( \text{lp} \)-isomorphism from the stratified extension \((X, \prec, \lambda)\) of \( T \) to the stratified extension \((\Sigma, \prec_u, \ell)\) of \( T_0 \). These relationships can be best captured using the following commutative diagram.

\[
\begin{array}{ccc}
(X, \prec, \sqsubseteq, \lambda) & \xrightarrow{\text{enum}} & (\Sigma, \prec_0, \sqsubseteq_0, \ell) \\
\downarrow{id_X} & & \downarrow{id_{\Sigma}} \\
(X, \prec, \sqsubseteq, \lambda) & \xrightarrow{\text{enum}} & (\Sigma, \prec_u, \sqsubseteq_u, \ell)
\end{array}
\]

We can also observe that two lsos-comtraces are identical if and only if they define the same enumerated so-structure. Henceforth, we define the **canonical representation** of \( T \) to be the enumerated so-structure of \( T \).

**Example 4.** The canonical presentation \( T_0 = (\Sigma, \prec_0, \sqsubseteq_0, \ell) \) of the lsos-comtrace \( T \) from Example 3 is given in the figure on the right. The solid edges of the graph represent the relation \( \prec_0 \) and the dashed edges represent the relation \( \sqsubseteq_0 \). The nodes of this graph are labeled with the elements of \( \Sigma \). The labeling function \( \ell \) returns the label of each element of \( \Sigma \) by simply getting rid of its superscript. For example, \( \ell(a^{(1)}) = a \) and \( \ell(b^{(2)}) = b \).

**5 Representation theorems**

This section contains the main technical contribution of this paper by showing that for a given comtrace alphabet \( \theta \), the comtrace monoid \( \mathbb{S}^*_{\theta}/\equiv_{\theta} \), the set of lsos-comtraces \( \text{LCT}(\theta) \) and the set of cd-graphs \( \text{CDG}(\theta) \) are three equivalent ways of talking about the same class of objects.

**5.1 Representation theorem for comtraces and lsos-comtraces**

The goal of this section is to prove the first representation theorem which establishes the representation mappings between \( \mathbb{S}^*_{\theta}/\equiv_{\theta} \) and \( \text{LCT}(\theta) \). Before doing so, we need some preliminary results.

**Proposition 7.** Let \( S = (X, \prec, \sqsubseteq) \) and \( S' = (X', \prec', \sqsubseteq') \) be stratified order structures such that \( \text{ext}(S) \subseteq \text{ext}(S') \). Then \( S' \subseteq S \).

**Proof.** Follows from Theorem[1]

\( \square \)
For the next two lemmas, we let $T$ be an Isos-contrace over a contrace alphabet $\theta = (E, sim, ser)$. Let $T_0 = (\Sigma, <_0, \sqcap, \ell)$ be the canonical representation of $T$. Let $<_0 \in ext(T_0)$ and $u = \text{imap}(\ell, \Omega_{<0})$. Since $u$ is a valid step sequence in $S^*$ (by Proposition 5), we can construct the so-structure $S_{|u|} = (\Sigma_u, <_{|u|}, \sqsubseteq_{|u|})$ as in Definition 7 where we have seen in Section 4.3 that $\Sigma = \Sigma_u$. Our goal is to show that the so-structure $S_{|u|}$ is exactly the canonical representation $T_0$ of $T$.

**Lemma 1.** $S_{|u|} \subseteq (\Sigma_u, <_0, \sqcap_0)$.

**Proof.** By Proposition 2 to show $S_{|u|} = (\Sigma_u, <_{|u|}, \sqsubseteq_{|u|}) \subseteq (\Sigma_u, <_0, \sqcap_0)$, it suffices to show $(\Sigma_u, <_{|u|}, \sqsubseteq_{|u|}) \subseteq (\Sigma_u, <_0, \sqcap_0)$. Intuitively, the lemma holds since $u$ is the step sequence representation of the stratified extension $<_0$ of $T_0$, and so the relations $<_|u| \text{ and } \sqsubseteq_{|u|}$ defined from $u$ cannot ’violate’ (i.e. must be contained in) the relations $<_0$ and $\sqcap_0$ respectively. The formal proof goes as follows.

$(<_|u| \subseteq <_0)$: Assume $\alpha <_u \beta$. Then from Definition 5 $\alpha <_u \beta$ and $(\ell(\alpha), \ell(\beta)) \notin ser$. Since $(\ell(\alpha), \ell(\beta)) \notin ser$, it follows from Definition 10 that $\alpha <_0 \beta$ or $\beta \sqcap_0 \alpha$. Suppose for a contradiction that $\beta \sqcap_0 \alpha$, then by Theorem 10 $\forall \nu <_0 \text{ ext}(T_0)$, $\beta <_0^\nu \alpha$. Since $T_0$ is the canonical representation of $T$, it is important to observe that $<_0 \approx \num{u}$. But since we assumed that $<_0 \in \text{ext}(T_0)$, it follows that $<_u \in \text{ext}(T_0)$ and $\alpha <_u \beta$, which contradicts that $\beta \sqcap_0 \alpha$. Hence, $\alpha <_0 \beta$.

$(\sqsubseteq_{|u|} \subseteq \sqcap_0)$: Can be shown similarly. □

**Lemma 2.** $S_{|u|} \supseteq (\Sigma_u, <_0, \sqcap_0)$.

In this proof, we will include subscripts for equivalence classes to avoid confusing the elements from quotient set $\Sigma_u/\equiv_{\sqcap_0}$ with the elements from the quotient contrace monoid $S^*/\equiv_{\theta}$. In other words, we write $[\alpha]_{\equiv_{\sqcap_0}}$ to denote an element of the quotient set $\Sigma_u/\equiv_{\sqcap_0}$, and we write $[u]_{\theta}$ to denote the contrace generated by the step sequence $u$.

**Proof.** Let $S' = (\Sigma_u, <_0, \sqcap_0)$. To show $S_{|u|} \supseteq S'$, by Proposition 2 it suffices to show $\text{ext}(S_{|u|}) \subseteq \text{ext}(S')$. From Theorem 2 we know that $\text{ext}(S_{|u|}) = \{<_u | w \in [u]_{\theta}\}$. Thus we only need to show that $<_u \in \text{ext}(S')$ for all $w \in [u]_{\theta}$. The main idea of the proof is simple: since every step sequence in $[u]_{\theta}$ is generated from the sequence $u$ according to how $\text{sim}$ and $\text{ser}$ are defined and $u$ is the step sequence representation of the stratified extension $<_0$ of $S'$, it follows from how Isos-contraces are defined that every stratified order $<_s$, where $s \in [u]$, must also be a stratified extension of $S'$. Unfortunately, the actual proof is a bit technical and goes as follows.

We observe that from $u$, by Definition 3 we can generate all the step sequences in the contrace $[u]_{\theta}$ in stages using the following recursive definition:

$$D^0(u) := [u]$$

$$D^n(u) := \{w | w \in D^{n-1}(u) \lor \exists v \in D^{n-1}(u), (v \approx_\theta w \lor v \approx_\theta^{-1} w)\}$$

Since the set $[u]_{\theta}$ is finite, $[u]_{\theta} = D^n(u)$ for some stage $n \geq 0$. The proof is complete if we can show the following claim.

**Claim 1.** For all $n \in \mathbb{N}$, we have $<_u \in \text{ext}(S')$ for all $w \in D^n(u)$.

We prove Claim 1 by induction on $n$.

**Base case:** When $n = 0$, $D^0(u) = [u]$. Since $<_0 \in \text{ext}(T)$, it follows from Proposition 5 that $<_u \in \text{ext}(S')$.

**Inductive case:** When $n > 0$, let $w$ be an element of $D^n(u)$. Then either $w \in D^{n-1}(u)$ or $w \in (D^n(u) \setminus D^{n-1}(u))$. For the former case, by inductive hypothesis, $<_u \in \text{ext}(S')$. For the latter case, there must be some element $v \in D^{n-1}(u)$ such that $v \approx_\theta w$ or $v \approx_\theta^{-1} w$. By induction hypothesis, we already known $<_u \in \text{ext}(S')$. We want to show that $<_u \in \text{ext}(S')$. There are two cases to consider:

**Case (i):**
When $v \approx_\theta w$, by Definition 3 there are some $y, z \in S^*_\theta$ and steps $A, B, C \in S_\theta$ such that $v = yAz$ and $w = yBCz$ where $A, B, C$ satisfy $B \cap C = \emptyset$ and $B \cup C = A$ and $B \times C \subseteq ser$. Let $\overline{y} = yAz$ and $\overline{w} = yBCz$ be enumerated step sequences of $v$ and $w$ respectively.
Suppose for a contradiction that \(<_w \not\in ext(S')\). By Definition\[2\] there are \(\alpha \in \overline{C}\) and \(\beta \in \overline{F}\) such that \(\alpha \sqsubseteq_0 \beta\). We now consider the quotient set \(\overline{A}/\equiv_0\). By Proposition\[3\] (1), \(\overline{A}/\equiv_0 \subseteq \Sigma/\equiv_0\). Since \(\alpha \sqsubseteq_0 \beta\), it follows that \([\alpha] \equiv_0 \overline{C} \sqsubseteq_0 [\beta] \equiv_0 \overline{C}\). Thus, from the fact that \(\overline{C}\) is a partial order, there must exist a chain

\[\vdash_{0 \equiv_0} [\alpha] \equiv_0 [\gamma_1] \equiv_0 [\gamma_2] \equiv_0 \ldots \equiv_0 [\gamma_k] = [\beta] \equiv_0 \overline{C}\]  

(5.1)

We want to show the following claim.

**Claim 2.** There are two consecutive elements \([\gamma_i] \equiv_0 \overline{C}\) and \([\gamma_{i+1}] \equiv_0 \overline{C}\) on the chain such that

(a) \(\ell([\gamma_{i+1}] \equiv_0 \overline{C}) \supseteq \ell([\gamma_i] \equiv_0 \overline{C}) \subseteq ser\)

(b) \(\neg (\ell([\gamma_i] \equiv_0 \overline{C}) \prec_0 \ell([\gamma_{i+1}] \equiv_0 \overline{C}))\)

Note that Claim 2 gives us the desired contradiction for this case by violating the assumption that \(T_0\) satisfies condition LC2 of Definition\[10\]. We can show Claim 2 as follows.

Our proof of Claim 2 (a) is an instance of the following simple combinatorial observation.

Assume that the elements of a chain are colored such that each element can only be colored either black or white, and furthermore we know that the first element of the chain is colored black and the last element of the chain is colored white. Then there must be some point on the chain where the color is switched from black to white (i.e. there must be two consecutive elements on the chain that are colored black and white respectively).

We apply this observation to show Claim 2 (a) as follows. By Theorem\[1\] and the fact that \(<_{w} \in ext(S')\), we know that \(\gamma_i \in \overline{A}\) for all \(i\). This follows from the fact that the chain \(\overline{S}\) implies that every \(\gamma_i\) must always occur between \(\alpha\) and \(\beta\) in all stratified extensions of \(S'\) and \(\alpha\), \(\beta\) \(\in \overline{A}\). Hence, by Proposition\[3\] (1), we have \([\gamma_i] \equiv_0 \overline{C}\) for all \(i\), \(1 \leq i \leq k\). Also from LC3 of Definition\[10\] and that \(B \times C \subseteq ser\), we know that either \([\gamma_i] \equiv_0 \overline{C}\) or \([\gamma_i] \equiv_0 \overline{C}\) for all \(i\), \(1 \leq i \leq k\). Now we note that the first element on the chain \([\gamma_1] \equiv_0 \overline{C}\) is \([\alpha] \equiv_0 \overline{C}\) and the last element on the chain \([\gamma_k] \equiv_0 \overline{C}\) is \([\beta] \equiv_0 \overline{C}\). Thus, there exist two consecutive elements \([\gamma_i] \equiv_0 \overline{C}\) and \([\gamma_{i+1}] \equiv_0 \overline{C}\) on the chain such that \([\gamma_i] \equiv_0 \overline{C}\) and \([\gamma_{i+1}] \equiv_0 \overline{C}\). Since \(B \times C \subseteq ser\), we have just shown Claim 2 (a).

Since \(<_{w} \in ext(S')\) and \(\gamma_i \prec_{w} \gamma_{i+1}\), by Theorem\[1\] Claim 2 (b) also follows.

**Case (ii):**

When \(v \approx_{\theta}^{-1} w\), by Definition\[4\] there are some \(y, z \in \Sigma_{\theta}\) and steps \(A, B, C \in \Sigma_{\theta}\) such that \(v = yBCz\) and \(w = yAz\). Since \(A, B, C\) satisfy \(B \cap C = \emptyset\) and \(B \cup C = A\) and \(B \times C \subseteq ser\), we can show that this leads to a contradiction with LC1 of Definition\[10\].

We also need to show that the labeled so-structure defined from each comtrace is indeed an lsos-comtrace. In other words, we need to show the following lemma.

**Lemma 3.** Let \(\theta = (E, sim, ser)\) be a comtrace alphabet. Given a step sequence \(u \in \Sigma_{\theta}^\ast\), the \(lp\)-isomorphic class \([\Sigma_{\theta}] u, \prec_{\theta} u, \sqsubseteq_{\theta} u, \ell\) is an lsos-comtrace over \(\theta\).

**Proof.** Let \(T = ([\Sigma_{\theta}] u, \prec_{\theta} u, \sqsubseteq_{\theta} u, \ell)\). From Theorem\[2\] \(T\) is a labeled so-structure. It only remains to show that \(T\) satisfies conditions LC1-LC5 of Definition\[10\].

**LC1:** Assume \([\alpha] \sqsubseteq_{\theta} \preceq_{\theta} [\beta]\) and suppose for a contradiction that \(\ell([\alpha]) \times \ell([\beta]) \subseteq ser\). Then from Proposition\[4\] (3), there exists \(\epsilon \in ext(T)\) such that \(\Omega_{\epsilon} = v([\alpha]) [\beta] w\). By Theorem\[2\] we have \(\epsilon \in \{\epsilon_i\mid s \in [u]\} = ext(\Sigma_{\theta}([u]))\). Thus, the step sequence \(\text{imap}(\ell, v([\alpha]) [\beta] w)\) is in the comtrace \([u]\). Since \(\ell([\alpha]) \times \ell([\beta]) \subseteq ser\), the step sequence \(\text{imap}(\ell, u([\alpha] \cup [\beta]) v)\) is also in \([u]\). Hence, the stratified order \(\prec_{\epsilon}\) satisfying \(\Omega_{\epsilon} = u([\alpha] \cup [\beta]) v\) is also an extension of \(T\), where \(\alpha \prec_{\epsilon} \beta\). But this contradicts that \([\alpha] \approx_{\theta} [\beta]\). We can also show LC2 and LC3 similarly.

**LC4:** Follows from Definitions\[5\] and\[7\] and the \(\Diamond\)-closure definition.

**LC5:** Assume that \(\alpha \not\prec \beta\) and \(\beta \not\prec \alpha\), then \(\alpha \prec_{\theta} \beta\). Thus, it follows from Corollary\[11\] that there exists \(\epsilon \in ext(T)\) where \(\alpha \prec_{\theta} \beta\). Since \(\{\epsilon_i\mid s \in [u]\} = ext(\Sigma_{\theta}([u]))\), there exists a step sequence \(s \in [u]\) such that \(s = \text{imap}(\ell, \Omega_{\epsilon})\). This implies \(\alpha\) and \(\beta\) belong to the same step in the step sequence \(T\). Thus, \((\ell(\alpha), \ell(\beta)) \in sim\).
**Definition 11** (representation mappings ct2lct and lct2ct). Let $\theta$ be a comtrace alphabet.

1. The mapping $\text{ct2lct} : S_\theta^*/\equiv_\theta \rightarrow \text{LCT}(\theta)$ is defined as

$$\text{ct2lct}(t) := [\Sigma_t, \prec_t, \sqsubseteq_t, \ell],$$

where the function $\ell : \Sigma_t \rightarrow E$ is defined in Section 2.2 and $S_t = (\Sigma_t, \prec_t, \sqsubseteq_t)$ is the so-structure defined by the comtrace $t$ from Definition 7.

2. The mapping $\text{lct2ct} : \text{LCT}(\theta) \rightarrow S_\theta^*/\equiv_\theta$ is defined as

$$\text{lct2ct}(\{X, \prec, \sqsubseteq, \lambda\}) := \{\text{map}(\lambda, \Omega_\searrow) \mid \diamond \in \text{ext}((X, \prec, \sqsubseteq))\}.$$

Intuitively, the mapping $\text{ct2lct}$ is used to convert a comtrace to an Isos-comtrace while the mapping $\text{lct2ct}$ is used to transform an Isos-comtrace into a comtrace. The next theorem will show that $\text{ct2lct}$ and $\text{lct2ct}$ are valid representation mappings for $S_\theta^*/\equiv_\theta$ and $\text{LCT}(\theta)$.

**Theorem 3 (The 1st Representation Theorem).** Let $\theta$ be a comtrace alphabet.

1. For every $t \in S_\theta^*/\equiv_\theta$, $\text{lct2ct} \circ \text{ct2lct}(t) = t$.
2. For every $T \in \text{LCT}(\theta)$, $\text{ct2lct} \circ \text{lct2ct}(T) = T$.

In other words, the following diagram commutes.

```
\begin{center}
\begin{tikzpicture}
  \node (ct2lct) at (0,0) {$S_\theta^*/\equiv_\theta$};
  \node (lct2ct) at (4,0) {$\text{LCT}(\theta)$};
  \node (id) at (2,0) {$\Sigma_\theta^*/\equiv_\theta$};
  \node (idLCT) at (6,0) {\text{id}_{\text{LCT}(\theta)}};

  \draw[->] (ct2lct) -- (lct2ct) node[midway, above] {$\text{ct2lct}$};
  \draw[->] (lct2ct) -- (idLCT) node[midway, above] {$\text{id}_{\text{LCT}(\theta)}$};
  \draw[->] (ct2lct) -- (id) node[midway, left] {$\text{lct2ct}$};
  \draw[->] (id) -- (lct2ct) node[midway, right] {$\text{id}$};
\end{tikzpicture}
\end{center}
```

**Proof.**

1. The fact that $\text{ran}(\text{ct2lct}) \subseteq \text{LCT}(\theta)$ follows from Lemma 3. Now for a given $t \in S_\theta^*/\equiv_\theta$, we have $\text{ct2lct}(t) = (\Sigma_t, \prec_t, \sqsubseteq_t, \ell)$. Thus, it follows that

$$\text{lct2ct}(\text{ct2lct}(t)) = \{\text{imap}(\ell, \Omega_\searrow) \mid \diamond \in \text{ext}(S_t)\}$$

$$= \{\text{imap}(\ell, \Omega_\searrow) \mid \diamond \in \{s \mid s \in t\}\}$$

(by Theorem 2)

$$= \{\text{imap}(\ell, \Omega_\searrow) \mid s \in t\}$$

$$= t$$

2. Let $T_0 = (\Sigma_t, \prec_t, \sqsubseteq_t, \ell)$ be the canonical representation of $T$. Note that since $T_0 \equiv T$, we have

$$\{\text{imap}(\ell, \Omega_\searrow) \mid \diamond \in \text{ext}(T_0)\} = \{\text{imap}(\lambda, \Omega_\searrow) \mid \diamond \in \text{ext}(T)\}.$$

Let $\Delta = \{\text{imap}(\ell, \Omega_\searrow) \mid \diamond \in \text{ext}(T_0)\}$. We will show that $\Delta \in S_\theta^*/\equiv_\theta$ and $\text{ct2lct}(\Delta) = [T_0]$. Fix an arbitrary $u \in \Delta$, then by Lemmas 10 and 2 it follows that $S_{[u]} = (\Sigma_t, \prec_t, \sqsubseteq_t)$. Thus, from Theorem 2 we have

$$\Delta = \{\text{imap}(\ell, \Omega_\searrow) \mid \diamond \in \text{ext}(S_{[u]})\} = [u].$$

And the rest follows. $\square$

The theorem says that the mappings ct2lct and lct2ct are inverses of each other and hence are both bijective.
5.2 Representation theorem for lsos-comtraces and combined dependency graphs

Recently, inspired by the dependency graph notion for Mazurkiewicz traces (cf. [8, Chapter 2]), Kleijn and Koutny claimed without a proof that their combined dependency graph notion is another alternative way to define comtraces [22]. In this section, we will give a detailed proof of their claim. We will now recall the combined dependency graph definition.

**Definition 12 (combined dependency graph [22]).** A combined dependency graph (cd-graph) over a comtrace alphabet $\mathcal{A} = (E, \text{ser}, \text{sim})$ is an lp-isomorphic class of a finite labeled relational structure $D = [X, \rightarrow, \rightarrow, \lambda]$ such that
\[
\begin{align*}
\text{CD1:} & \quad (\lambda(\alpha), \lambda(\beta)) \not\in \text{sim} \implies \alpha \rightarrow \beta \lor \beta \rightarrow \alpha \\
\text{CD2:} & \quad (\lambda(\alpha), \lambda(\beta)) \not\in \text{ser} \implies \alpha \rightarrow \beta \lor \beta \rightarrow \alpha \\
\text{CD3:} & \quad \alpha \rightarrow \beta \implies (\lambda(\alpha), \lambda(\beta)) \not\in \text{ser} \\
\text{CD4:} & \quad \alpha \rightarrow \beta \implies (\lambda(\beta), \lambda(\alpha)) \not\in \text{ser}
\end{align*}
\]

We will write $\text{CDG}(\mathcal{A})$ to denote the class of all cd-graphs over $\mathcal{A}$.

Cd-graphs can be seen as a reduced graph-theoretic representation for lsos-comtraces, where some arcs that can be recovered using $\diamond$-closure are omitted. It is interesting to observe that the non-serializable sets of a cd-graph are exactly the strongly connected components of the directed graph $(X, \rightarrow)$ and can easily be found in time $O(|X| + |\rightarrow|)$ using any standard algorithm (cf. [2, Section 22.5]).

**Remark 2.** Cd-graphs were called dependence comdags in [22]. But this name could be misleading since the directed graph $(X, \rightarrow)$ is not necessarily acyclic. For example, the graph on the right is the cd-graph that corresponds to the lsos-comtrace from Figure 2, but it is not acyclic (here, we use the dashed edges to denote the relation $\rightarrow$ and the solid edges to denote only the relation $\rightarrow$). Thus, we use the name “combined dependency graph” instead.

We are going to show that the combined dependency graph notion is another correct alternative definition for comtraces. We will define several representation mappings that are needed for our proofs.

**Definition 13 (representation mappings ct2dep, dep2lct and lct2dep).** Let $\mathcal{A}$ be a comtrace alphabet.

1. The mapping $\text{ct2dep} : \mathbb{S}_\mathcal{A}^* /_{\equiv_{\mathcal{A}}} \rightarrow \text{CDG}(\mathcal{A})$ is defined as
   \[
   \text{ct2dep}(t) := (\Sigma_t, <_u, \sqcup_u, \ell),
   \]
   where $u$ is any step sequence in $t$ and $<_u$ and $\sqcup_u$ are defined as in Definition 5.

2. The mapping $\text{dep2lct} : \text{CDG}(\mathcal{A}) \rightarrow \text{LCT}(\mathcal{A})$ is defined as $\text{dep2lct}(D) := D^\diamond$.

3. The mapping $\text{lct2dep} : \text{LCT}(\mathcal{A}) \rightarrow \text{CDG}(\mathcal{A})$ is defined as
   \[
   \text{lct2dep}(T) := \text{ct2dep} \circ \text{lct2ct}(T).
   \]

Before proceeding further, we want to make sure that:

**Lemma 4.** 1. The function $\text{dep2lct} : \text{CDG}(\mathcal{A}) \rightarrow \text{LCT}(\mathcal{A})$ is well-defined.
2. The function $\text{ct2dep} : \mathbb{S}_\mathcal{A}^* /_{\equiv_{\mathcal{A}}} \rightarrow \text{CDG}(\mathcal{A})$ is well-defined.
**Proof.** 1. Given a cd-graph $D = [X, \rightarrow, \rightarrow, \lambda] \in \text{CDG}(\theta)$, let $T = [X, \prec, \sqsubseteq, \lambda] = D^{\circ}$. We know from Definition[2] that $(X, \prec, \sqsubseteq)$ is an so-structure. It remains to show that $T$ is indeed an isos-comtrace by verifying that $T$ satisfies the conditions LC1-LC5 of Definition[10].

We first verify LC1. Suppose for contradiction that there are two distinct $\square$-cycle equivalence classes $[\alpha], [\beta] \in X$ satisfying $[\alpha](\triangleleft \cap \prec)[\beta]$ but $\lambda(\alpha) \times \lambda(\beta) \subseteq \text{ser}$. Clearly, this implies that $\alpha < \beta$, and thus by the $\circ$-closure definition, $\beta$ is reachable from $\alpha$ on the directed graph $G = (X, \rightarrow)$, where $\rightarrow = \rightarrow \cup \rightarrow$. Now we consider a shortest path $P$

$$\alpha = \delta_1 \rightarrow \delta_2 \ldots \rightarrow \delta_{k-1} \rightarrow \delta_k = \beta$$

on $G$ that connects $\alpha$ to $\beta$. Our strategy is to show that there exist two consecutive elements $\delta_i$ and $\delta_{i+1}$ on $P$ such that $\delta_i \in [\alpha]$ and $\delta_{i+1} \in [\beta]$ and $(\lambda(\delta_i), \lambda(\delta_{i+1})) \notin \text{ser}$, which contradicts with $\lambda(\alpha) \times \lambda(\beta) \subseteq \text{ser}$. By CD3 it suffices to show the following claim.

**Claim:** There are two consecutive elements $\delta_i$ and $\delta_{i+1}$ on $P$ such that $\delta_i \in [\alpha]$ and $\delta_{i+1} \in [\beta]$ and $\delta_i \rightarrow \delta_{i+1}$.

We will prove this claim by induction on the number of elements on the path $P$, where $k \geq 2$.

**Base case:** when $k = 2$, then $\alpha \rightarrow \beta$. Since $[\alpha](\triangleleft \cap \prec)[\beta]$, we have $\alpha \rightarrow \beta$.

**Inductive case:** when $k > 2$, we consider first two elements on the path $\delta_1$ and $\delta_2$. If $\delta_1 \in [\alpha]$ and $\delta_2 \in [\beta]$, then by the assumption $[\alpha](\triangleleft \cap \prec)[\beta]$, it must be the case that $\delta_1 \rightarrow \delta_2$. Otherwise, we have $\delta_2 \notin [\alpha] \cup [\beta]$ or $[\delta_1, \delta_2] \subseteq [\alpha]$. For the first case, we get $[\alpha][\delta_2] \subseteq [\beta]$, which contradicts that $[\alpha]|_{\triangleleft \cap \prec}[\beta]$. For the latter case, we can apply the induction hypothesis on the path $\delta_2 \rightarrow \delta_3 \rightarrow \ldots \rightarrow \delta_{k-1} \rightarrow \delta_k$.

LC2 and LC3 can also be shown similarly using a “shortest path” argument as above. These proofs are easier since we only need to consider paths with edges in $\rightarrow \rightarrow$. LC4 and LC5 easily follow from the fact that the cd-graph $D$ satisfies CD1 and CD2.

2. By the proof of [14] Lemma 4.7, for any two step sequences $t$ and $u$ in $S^*_\theta$, we have $u \equiv t$ if and only if $\text{ct2dep}(u) = \text{ct2dep}(t)$. This ensures that $\text{ct2dep}$ gives us the same cd-graph no matter how we choose the step sequence $u \in t$. It is also not hard to check that the range of $\text{ct2dep}$ consists only of cd-graphs over $\theta$. Thus the mapping $\text{ct2dep}$ is well-defined.

**Lemma 5.** The mapping $\text{dep2lct} : \text{CDG}(\theta) \rightarrow \text{LCT}(\theta)$ is injective.

**Proof.** Assume that $D_1, D_2 \in \text{CDG}(\theta)$, such that

$$\text{dep2lct}(D_1) = \text{dep2lct}(D_2) = T = [X, \prec, \sqsubseteq, \lambda].$$

Since $\circ$-closure does not change the labeling function, we can assume $D_i = [X, \rightarrow_i, \rightarrow_i, \lambda]$ and $(X, \rightarrow_i, \rightarrow_i)^{\circ} = (X, \prec, \sqsubseteq)$. We want to show that $(X, \rightarrow_i, \rightarrow_i) \subseteq (X, \rightarrow_i, \rightarrow_i)^{\circ}$.

$(\rightarrow_i \rightarrow_i \rightarrow_i)$: Let $\alpha, \beta \in X$ such that $\alpha \rightarrow_i \beta$. Suppose for a contradiction that $\neg(\alpha \rightarrow_i \beta)$. Since $\alpha \rightarrow_i \beta$, by CD3, $(\lambda(\alpha), \lambda(\beta)) \notin \text{ser}$. Thus, by CD2, we must have $\beta \rightarrow_i \alpha$. But since $(X, \rightarrow_i, \rightarrow_i)^{\circ} = (X, \prec, \sqsubseteq)$, it follows that $(X, \rightarrow_i, \rightarrow_i) \subseteq (X, \prec, \sqsubseteq)$ (by Proposition[2]). Thus, $\alpha < \beta$ and $\beta \sqsubseteq \alpha$, a contradiction.

$(\rightarrow_i \rightarrow_i \rightarrow_i)$: Can be proved similarly.

By reversing the role of $D_1$ and $D_2$, we have $(X, \rightarrow_i, \rightarrow_i) \supseteq (X, \rightarrow_i, \rightarrow_i)$. Thus, $D_1 = D_2$. □

We are now ready to show the following representation theorem which ensures that $lct2dep$ and $dep2lct$ are valid representation mappings for LCT$(\theta)$ and CDG$(\theta)$.

**Theorem 4 (The 2$^\text{nd}$ Representation Theorem).** Let $\theta$ be a comtrace alphabet.

1. For every $T \in \text{LCT}(\theta)$, $\text{dep2lct} \circ \text{lct2dep}(T) = T$.
2. For every $D \in \text{CDG}(\theta)$, $\text{lct2dep} \circ \text{dep2lct}(D) = D$.
In other words, the following diagram commutes.

\[
\begin{array}{c}
\text{id}_{\text{LCT}(\theta)} \\
\text{lct2dep} \\
\text{dep2lct}
\end{array}
\quad
\begin{array}{c}
\text{LCT}(\theta) \\
\text{CDG}(\theta) \\
\text{id}_{\text{CDG}(\theta)}
\end{array}
\quad
\begin{array}{c}
\text{id}_{\text{LCT}(\theta)} \\
\text{lct2dep} \\
\text{dep2lct}
\end{array}
\]

**Proof.** 1. Let \( T \in \text{LCT}(\theta) \) and let \( D = \text{lct2dep}(T) \). Suppose for a contradiction that \( Q = \text{dep2lct} \circ \text{lct2dep}(T) \) and \( Q \neq T \). Since \( \text{lct2dep} = \text{ct2dep} \circ \text{lct2ct} \), if we let \( t = \text{lct2ct}(T) \), then \( Q = \text{dep2lct} \circ \text{ct2dep}(t) \neq T \). Thus, we have shown that \( t = \text{lct2ct}(T) \) and by the way we construct so-structure from comtraces, we also have \( \text{ct2lct}(t) = \text{dep2lct} \circ \text{ct2dep}(t) = Q \). Since \( Q \neq T \), we have \( \text{ct2lct} \circ \text{lct2ct}(T) \neq T \), which contradicts Theorem 3 (2).

2. Let \( D \in \text{CDG}(\theta) \) and \( T = \text{dep2lct}(D) \). Suppose for a contradiction that \( E = \text{lct2dep} \circ \text{dep2lct}(D) \) and \( E \neq D \). From 1., we know that \( \text{dep2lct} \circ \text{lct2dep}(T) = T \), and thus it must be the case that \( \text{dep2lct}(E) = T \). Hence, we have \( \text{dep2lct}(E) = T = \text{dep2lct}(D) \), but \( E \neq D \), which contradicts the injectivity of \( \text{dep2lct} \) from Lemma 5. \( \square \)

This theorem shows that both \( \text{lct2dep} \) and \( \text{dep2lct} \) are bijective. Note that we do not need to prove another representation theorem for cd-graphs and comtraces since their representation mappings are simply the composition of the representation mappings from Theorems 3 and 4. In other words, we have shown that the following diagram commutes.

\[
\begin{array}{c}
\text{id}_{S^*/\equiv_\theta} \\
\text{ct2lct} \\
\text{lct2ct} \\
\text{dep2lct} \\
\text{lct2dep} \\
\text{CDG}(\theta) \\
\text{id}_{\text{CDG}(\theta)}
\end{array}
\quad
\begin{array}{c}
S^*/\equiv_\theta \\
\text{LCT}(\theta) \\
\text{id}_{\text{LCT}(\theta)}
\end{array}
\quad
\begin{array}{c}
\text{id}_{S^*/\equiv_\theta} \\
\text{ct2lct} \\
\text{lct2ct} \\
\text{dep2lct} \\
\text{lct2dep} \\
\text{CDG}(\theta) \\
\text{id}_{\text{CDG}(\theta)}
\end{array}
\]

In Section 6 after constructing suitable composition operators for lsos-comtraces and cd-graphs, we will show that the representation mappings in this diagram are indeed monoid isomorphisms. Thus, lsos-comtraces and cd-graphs are equivalent representations for comtraces.

### 6 Composition operators

For a comtrace monoid \((S^*/\equiv_\theta, \circ, [e])\), the comtrace operator \(_\circ_\) was defined as \([r] \circ [t] = [r \circ t]\). We will construct analogous composition operators for lsos-comtraces and cd-graphs. We will then show that lsos-comtraces (cd-graphs) over a comtrace alphabet \(\theta\) together with its composition operator form a monoid isomorphic to the comtrace monoid \((S^*/\equiv_\theta, \circ, [e])\). In other words, we need to show that the mappings from Theorems 3 and 4 are compatible with the corresponding monoid operators.

#### 6.1 Monoid of lsos-comtraces

Given two sets \(X_1\) and \(X_2\), we write \(X_1 \uplus X_2\) to denote the **disjoint union** of \(X_1\) and \(X_2\). Such disjoint union can be easily obtained by renaming the elements in \(X_1\) and \(X_2\) so that \(X_1 \cap X_2 = \emptyset\). We define the lsos-comtrace composition operator as follows.
Definition 14 (composition of Isos-comtraces). Let $T_1$ and $T_2$ be Isos-comtraces over an alphabet $\theta = (E, \text{sim}, \text{ser})$, where $T_i = (X_i, \vartriangleleft_i, \sqsubseteq_i, \lambda_i)$. The composition $T_1 \odot T_2$ is defined as (an lp-isomorphic class of) a labeled so-structure $(X, \vartriangleleft, \sqsubseteq, \lambda)$ such that $X = X_1 \uplus X_2$, $\lambda = \lambda_1 \uplus \lambda_2$, and $(X, \vartriangleleft, \sqsubseteq) = (X, \vartriangleleft_{(1,2)}, \sqsubseteq_{(1,2)}, \lambda_{(1,2)})$, where

\[
\vartriangleleft_{(1,2)} = \vartriangleleft_1 \cup \vartriangleleft_2 \cup \{ (\alpha, \beta) \in X_1 \times X_2 \mid (\lambda(\alpha), \lambda(\beta)) \not\in \text{ser} \}
\]

\[
\sqsubseteq_{(1,2)} = \sqsubseteq_1 \sqcup \sqsubseteq_2 \sqcup \{ (\alpha, \beta) \in X_1 \times X_2 \mid (\lambda(\beta), \lambda(\alpha)) \not\in \text{ser} \}
\]

The operator $\_ \odot \_$ is well-defined since we can easily check that:

**Proposition 8.** For every $T_1, T_2 \in \text{LCT}(\theta)$, $T_1 \odot T_2 \in \text{LCT}(\theta)$.

We will next show that this composition operator $\_ \odot \_$ properly corresponds to the operator $\_ \otimes \_$ of the comtrace monoid over $\theta$.

**Lemma 6.** Let $\theta$ be a comtrace alphabet. Then

1. For every $R, T \in \text{LCT}(\theta)$, $\text{lct2ct}(R \odot T) = \text{lct2ct}(R) \circ \text{lct2ct}(T)$.
2. For every $r, t \in S^*_{\theta} / \equiv_{\theta}$, $\text{lct2ct}(r \odot t) = \text{ct2lct}(r) \odot \text{ct2lct}(t)$.

**Proof.** 1. Without loss of generality, we can assume that $R = [X_1, \vartriangleleft_1, \sqsubseteq_1, \lambda_1], T = [X_2, \vartriangleleft_2, \sqsubseteq_2, \lambda_2]$ and $Q = R \odot T = [X_1 \uplus X_2, \vartriangleleft, \sqsubseteq, \lambda]$. We can pick any $\vartriangleleft_1 \in \text{ext}(R)$ and $\vartriangleleft_2 \in \text{ext}(T)$. Then observe that the stratified order $\vartriangleleft$ satisfying $\Omega_1 \subseteq \Omega_2 \cup \Omega_2$ is also a stratified extension of $Q$. Thus, by Theorem 5 we get

\[
\text{lct2ct}(R) \circ \text{lct2ct}(T) = [\text{imap}(\lambda_1, \vartriangleleft_1)] \circ [\text{imap}(\lambda_2, \vartriangleleft_2)] = [\text{imap}(\lambda, \vartriangleleft)] = \text{lct2ct}(Q)
\]
as desired.

2. Without loss of generality, we assume that $r = [r], t = [t]$ and $q = [q] = r \otimes t$, where $q = r \ast t$. By reindexing $\Sigma_t$ appropriately, we can also assume that $\Sigma_q = \Sigma_r \uplus \Sigma_t$. Under these assumptions, let

\[
T_1 = [\Sigma_r, \vartriangleleft_r, \sqsubseteq_r, \ell_1], T_2 = [\Sigma_t, \vartriangleleft_t, \sqsubseteq_t, \ell_2]
\]

where $\ell = \ell_1 \uplus \ell_2$ is simply the standard labeling functions. It will now suffice to show that $T_1 \odot T_2 = T$.

($\subseteq$): Let $T_1 \odot T_2 = (\Sigma_r \uplus \Sigma_t, \vartriangleleft_{(r,t)}, \sqsubseteq_{(r,t)}, \ell_{(r,t)})$. By Definitions 5 and 7 we have

\[
\vartriangleleft_{(r,t)} = \vartriangleleft_r \cup \vartriangleleft_t \cup \{ (\alpha, \beta) \in \Sigma_r \times \Sigma_t \mid (\lambda(\alpha), \lambda(\beta)) \not\in \text{ser} \}
\]

\[
\sqsubseteq_{(r,t)} = \sqsubseteq_r \uplus \sqsubseteq_t \sqcup \{ (\alpha, \beta) \in \Sigma_r \times \Sigma_t \mid (\lambda(\beta), \lambda(\alpha)) \not\in \text{ser} \}
\]

Thus, from Proposition 2(5), we have $(\Sigma_r \uplus \Sigma_t, \vartriangleleft_{(r,t)}, \sqsubseteq_{(r,t)}, \ell_{(r,t)}) \subseteq (\Sigma_q, \vartriangleleft_q, \sqsubseteq_q, \ell_q)$.

($\supseteq$): By Definitions 5 and 7, we have $\vartriangleleft_q \subseteq \vartriangleleft_{(r,t)}$ and $\vartriangleleft_q \subseteq \vartriangleleft_{(r,t)}$. From Definition 14 we already know that $\vartriangleleft_q$ is irreflexive since $\vartriangleleft_q$ and $\vartriangleleft_q$ are irreflexive. Thus, by Proposition 2(1),

\[
(\Sigma_r \uplus \Sigma_t, \vartriangleleft_{(r,t)}, \sqsubseteq_{(r,t)}, \ell_{(r,t)}) \subseteq (\Sigma_q, \vartriangleleft_q, \sqsubseteq_q, \ell_q)
\]

Hence, we have $(\Sigma_q, \vartriangleleft_q, \sqsubseteq_q, \ell_q) \subseteq (\Sigma_r \uplus \Sigma_t, \vartriangleleft_{(r,t)}, \sqsubseteq_{(r,t)}, \ell_{(r,t)})$. Thus, from Proposition 2(5),

\[
T = (\Sigma_q, \vartriangleleft_q, \sqsubseteq_q, \ell_q) = (\Sigma_q, \vartriangleleft_q, \sqsubseteq_q, \ell_q) \subseteq (\Sigma_r \uplus \Sigma_t, \vartriangleleft_{(r,t)}, \sqsubseteq_{(r,t)}, \ell_{(r,t)}) = T_1 \odot T_2.
\]

Thus, we have shown that $T_1 \odot T_2 = T$. \(\square\)

Let $I$ denote the lp-isomorphic class $[\varnothing, \varnothing, \varnothing, \varnothing]$. Then observe that $\text{ct2lct}(\{I\}) = I$ and $\text{lct2ct}(I) = \{I\}$. By Lemma 5 and Theorem 2 it follows that the structures $(\text{LCT}(\theta), \odot, I)$ and $(S^*_{\theta} / \equiv_{\theta}, \otimes, \{I\})$ are isomorphic under the isomorphisms $\text{ct2lct} : S^*_{\theta} / \equiv_{\theta} \rightarrow \text{LCT}(\theta)$ and $\text{lct2ct} : \text{LCT}(\theta) \rightarrow S^*_{\theta} / \equiv_{\theta}$. Thus, the triple $(\text{LCT}(\theta), \odot, I)$ is also a monoid. We can summarize these facts in the following theorem:

**Theorem 5.** The mappings $\text{ct2lct}$ and $\text{lct2ct}$ are monoid isomorphisms between $(S^*_{\theta} / \equiv_{\theta}, \otimes, \{I\})$ and $(\text{LCT}(\theta), \odot, I)$. 

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6.2 Monoid of cd-graphs

Similarly to the previous section, for a given a comtrace alphabet, we can also define a composition operator for cd-graphs.

Definition 15 (composition of cd-graphs). Let \( D_1 \) and \( D_2 \) be cd-graphs over an alphabet \( \theta = (E, \text{sim}, \text{ser}) \), where \( D_i = [X_i, \rightarrow_i, \cdots, \lambda_i] \). The composition \( D_1 \odot D_2 \) is defined as (an lp-isomorphic class of) a labeled so-structure \([X, \rightarrow, \cdots, \lambda]\) such that \( X = X_1 \uplus X_2 \), \( \lambda = \lambda_1 \uplus \lambda_2 \), and

\[
\rightarrow = \rightarrow_1 \uplus \rightarrow_2 \cup \{(\alpha, \beta) \in X_1 \times X_2 \mid (\lambda(\alpha), \lambda(\beta)) \notin \text{ser}\}
\]

\[
\sigma_{\rightarrow} = \sigma_{\rightarrow_1} \uplus \sigma_{\rightarrow_2} \cup \{(\alpha, \beta) \in X_1 \times X_2 \mid (\lambda(\beta), \lambda(\alpha)) \notin \text{ser}\}
\]

The operator \( \_ \odot \_ \) is well-defined since we can easily check that:

Proposition 9. For every \( D_1, D_2 \in \text{CDG}(\theta) \), \( D_1 \odot D_2 \in \text{CDG}(\theta) \).

Using techniques similar to the proofs of Lemma 6 and Theorem 4 it is not hard to show the following lemma.

Lemma 7. Let \( \theta \) be a comtrace alphabet. Then

1. For every \( R, T \in \text{LCT}(\theta) \), \( \text{lct2dep}(R \circ T) = \text{lct2dep}(R) \odot \text{lct2dep}(T) \).
2. For every \( D, E \in \text{CDG}(\theta) \), \( \text{dep2lct}(D \odot E) = \text{dep2lct}(D) \odot \text{dep2lct}(E) \).

Putting the preceding lemma and Theorem 4 together, we conclude:

Theorem 6. The mappings lct2dep and dep2lct are monoid isomorphisms between \((\text{LCT}(\theta), \odot, \text{lp})\) and \((\text{CDG}(\theta), \odot, \text{lp})\).

By composing the isomorphisms from Theorems 5 and 6 we have:

Corollary 3. The monoids \((\langle S_\theta \rangle_\theta / \equiv_\theta, \odot, [\varepsilon])\) and \((\text{CDG}(\theta), \odot, \text{lp})\) are isomorphic.

7 Conclusion

The simple yet useful construction we used extensively in this paper is to build a quotient so-structure modulo the \( \sqcup \)-cycle equivalence relation. Intuitively, each \( \sqcup \)-cycle equivalence class consists of events that must be executed simultaneously with one another and hence can be seen as a single “composite event”. The resulting quotient so-structure is technically easier to handle since both relations of the quotient so-structure are acyclic. From this construction, we were able to give a labeled so-structure definition of comtraces analogous to the labeled poset definition of traces.

We have also formally shown that the quotient monoid of comtraces, the monoid of Isos-comtraces and the monoid of cd-graphs over the same comtrace alphabet are isomorphic by constructing monoid isomorphisms between them. These three models are formal linguistic, order-theoretic, and graph-theoretic respectively, which allows us to apply a variety of tools and techniques. We believe the ability to conceptualize on three alternative representations is the main advantage of trace theory in general.

An immediate future task is to develop a framework similar to the one in this paper for generalized comtraces, proposed and developed in \([23, 18]\). Generalized comtraces extend comtraces with the ability to model events that can be executed \( \text{earlier than or later than but never simultaneously} \). We believe that the quotient so-structure technique developed in this paper can be used to simplify some proofs in \([18]\).

The labeled so-structure definition of comtraces can easily be extended to define infinite comtraces to model nonterminating concurrent processes, and thus it would be interesting to generalize the results in \([12, 3]\) for comtraces. It is also promising to use Isos-comtraces and cd-graphs to develop logics for comtraces similarly to what have been done for traces (see, e.g., \([23, 29, 54, 11, 16]\)).
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