MODULAR SHEAVES ON HYPERKÄHLER VARIETIES

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Abstract. A torsion free sheaf on a hyperkähler manifold $X$ is modular if its discriminant satisfies a certain condition, for example this is the case if it is a multiple of $c_2(X)$. The definition is taylor made for torsion-free sheaves on a polarized hyperkähler variety $(X, h)$ which deform to all small deformations of $(X, h)$. For hyperkähler varieties of Type $K3^2$ we prove an existence and uniqueness result for slope-stable vector bundles with certain ranks, $c_1$ and $c_2$. As a consequence we get uniqueness up to isomorphism of the tautological quotient rank 4 vector bundle on the variety of lines on a generic cubic 4-dimensional hypersurface, and on the Debarre-Voisin variety associated to a generic element of $\mathbb{A}^3\mathbb{C}^{10}$. The last result implies that the period map from the moduli space of Debarre-Voisin varieties to the relevant period space is birational.

1. Introduction

1.1. Background and motivation. The beautiful properties of vector bundles on $K3$ surfaces play a prominent rôle in algebraic geometry. Since $K3$ surfaces are the two-dimensional hyperkähler (HK) compact manifolds, one is tempted to explore the world of vector bundles on higher dimensional HK’s. In the present paper we give way to this temptation. Our proposal is to focus attention on vector bundles, or more generally (coherent) torsion free sheaves, whose Chern character satisfies a certain condition, see Definition 1.1. We call such sheaves modular. The definition is taylor made for torsion-free sheaves on a polarized HK $(X, h)$ which deform to all small deformations of $(X, h)$. With this hypothesis we may deform $(X, h)$ to a Lagrangian $(X_0, h_0, \pi)$, where $\pi: X_0 \to \mathbb{P}^n$ is a Lagrangian fibration, study stable sheaves $\mathcal{F}$ on $X_0$ by studying the restriction of $\mathcal{F}$ to a generic fiber of $\pi$ (an abelian variety of dimension $n$), and then deduce properties of the initial moduli space of sheaves on $(X, h)$. This strategy was implemented in the case of $K3$ surfaces, see [O’G97]. A successful implementation in higher dimensions requires an extension of the known results regarding the variation of $h$ slope-stability of a sheaf. More precisely one needs to know that, given a class $\chi \in H(X; \mathbb{Q})$, there exists a decomposition of the ample cone into open chambers such that, given a sheaf $\mathcal{F}$ with $\text{ch}(\mathcal{F}) = \chi$ and an open chamber $\mathcal{C}$, then $\mathcal{F}$ is either $h$ slope-stable for all $h \in \mathcal{C}$ or else for no such $h$. Modular sheaves on HK’s are exactly the sheaves for which one can prove that such a decomposition of the ample cone exists. Another remarkable consequence of our definition is the following. Let $\pi: X \to \mathbb{P}^n$ be a Lagrangian fibration, and let $\mathcal{F}$ be a modular vector bundle on $X$ whose restriction to a generic fiber of $\pi$ is slope-stable; then the restriction of $\mathcal{F}$ to a generic fiber is a semi-homogeneous vector bundle (according to Mukai’s definition, see [Muk78]), and hence it has no infinitesimal deformations fixing the determinant. This shows that, in the case of modular sheaves, the strategy outlined above in higher dimensions resembles that which has been implemented in the case of
of K3 surfaces (notice that a slope-stable vector bundle on an elliptic curve has by
default no infinitesimal deformations fixing the determinant, while this certainly
does not hold for vector bundles on abelian surfaces - it holds exactly for semi-
homogeneous ones).

1.2. Modular sheaves. Let \( \mathcal{F} \) be a rank \( r \) torsion-free sheaf on a manifold \( X \). The discriminant \( \Delta(\mathcal{F}) \in H_{2}^{2}(X) \) is defined to be
\[
\Delta(\mathcal{F}) := 2rc_{2}(\mathcal{F}) - (r - 1)c_{1}(\mathcal{F})^{2} = -2r c_{2}(\mathcal{F}) + c_{1}(\mathcal{F})^{2}.
\]

Below is our key definition.

**Definition 1.1.** Let \( X \) be a HK manifold of dimension \( 2n \), and let \( q_{X} \) be its Beauville-Bogomolov-Fuksi (BBF) bilinear symmetric form. A torsion free sheaf \( \mathcal{F} \) on \( X \) is **modular** if there exists \( d(\mathcal{F}) \in \mathbb{Q} \) such that
\[
\int_{X} \Delta(\mathcal{F}) \sim \alpha^{2n-2} = d(\mathcal{F}) \cdot (2n - 3)!! \cdot q_{X}(\alpha, \alpha)^{n-1}
\]
for all \( \alpha \in H^{2}(X) \).

**Remark 1.2.** Let \( X \) be a HK variety of dimension \( 2n \). Let \( D(\mathcal{F}) \subset H^{2}(X) \) be the image of the map \( \text{Sym} H^{2}(X) \to H(X) \) defined by cup-product. Let \( D(\mathcal{F}) := D(\mathcal{F}) \cap H^{2}(X) \). The pairing \( D(\mathcal{F}) \times D(\mathcal{F}) \to \mathbb{C} \) defined by intersection product is non degenerate, hence there is a splitting \( H(X) = D(\mathcal{F}) \oplus D(\mathcal{F})^{\perp} \), where orthogonality is with respect to the intersection pairing.

Now let \( \mathcal{F} \) be a torsion free sheaf on \( X \). Then \( \mathcal{F} \) is modular if and only if the orthogonal projection of \( \Delta(\mathcal{F}) \) onto \( D(\mathcal{F}) \) is a multiple of the class \( q_{X} \) dual to \( q_{X} \). In particular \( \mathcal{F} \) is modular if \( \Delta(\mathcal{F}) \) is a multiple of \( c_{2}(X) \).

**Remark 1.3.** Let \( X \) be a HK of Type \( K3^{[2]} \). Then \( H(X) = D(\mathcal{F}) \) (notation as in Remark 1.2). It follows that a vector bundle \( \mathcal{F} \) on \( X \) is modular if and only if \( \Delta(\mathcal{F}) \) is a multiple of \( c_{2}(X) \). It follows that if \( \mathcal{F} \) is a modular vector bundle, slope-stable for a polarization \( h \), then \( \text{End}_{h}(\mathcal{F}) \) is hyperholomorphic on \( (X, h) \), where \( \text{End}_{h}(\mathcal{F}) \) is the vector bundle of traceless endomorphisms of \( \mathcal{F} \). More generally, on an arbitrary HK polarized variety \( (X, h) \) there should be a relation between the property of being modular and that of being hyperholomorphic.

1.3. Main results. Let \( \mathcal{E} \) be a modular torsion-free sheaf on a hyperkähler manifold \( X \) of Type \( K3^{[2]} \). A simple argument, see Proposition 2.3, shows that \( r(\mathcal{E}) \) divides the square of a generator of the ideal \( \{ q_{X}(c_{1}(\mathcal{E}), \alpha) \mid \alpha \in H^{2}(X; \mathbb{Z}) \} \). We give an existence and uniqueness statement for slope-stable vector bundles \( \mathcal{E} \) such that \( r(\mathcal{E}) \) equals the square of a generator of the ideal defined above, and moreover
\[
\Delta(\mathcal{E}) = \frac{r(\mathcal{E})(r(\mathcal{E}) - 1)}{12} c_{2}(X).
\]

Before formulating our results we recall the description of the irreducible components of the moduli space of polarized HK’s of Type \( K3^{[2]} \). Let \( (X, h) \) be one such polarized HK (we emphasize that the ample class \( h \in H^{2}_{X, h} \) is primitive). Then either
\[
q(h, H^{2}(X; \mathbb{Z})) = 2\mathbb{Z}, \quad q(h) = e > 0, \quad e \equiv 0 \pmod{2}
\]
or
\[
q(h, H^{2}(X; \mathbb{Z})) = 2\mathbb{Z}, \quad q(h) = e > 0, \quad e \equiv 6 \pmod{8}.
\]
Conversely, if \( e \) is a positive natural number which is even (respectively congruent to 6 modulo 8) there exists \( (X, h) \) such that \( 1.3.2 \) (respectively \( 1.3.3 \)) holds. Let \( \mathcal{M}_{e}^{1} \) be the moduli space of polarized HK’s \( (X, h) \) of Type \( K3^{[2]} \) such that \( 1.3.2 \) holds, and let \( \mathcal{M}_{e}^{2} \) be the moduli space of polarized HK’s \( (X, h) \) of Type \( K3^{[2]} \) such that \( 1.3.3 \) holds.
Theorem 1.4. Suppose that $e, r_0$ are positive natural numbers such that
\[ e = \begin{cases} 
4r_0 - 10 \pmod{8r_0} & \text{if } r_0 = 0 \pmod{4}, \\
\frac{1}{2}(r_0 - 5) \pmod{2r_0} & \text{if } r_0 = 1 \pmod{4}, \\
4r_0 - 10 \pmod{8r_0} & \text{if } r_0 = 2 \pmod{4}, \\
-\frac{1}{2}(r_0 + 5) \pmod{2r_0} & \text{if } r_0 = 3 \pmod{4}. 
\end{cases} \quad (1.3.4) \]

Then the following hold:

1. Let $r_0$ be odd and let $[(X, h)] \in \mathcal{X}^1_{\mathbb{C}}$ be a generic point. Then up to isomorphism there exists one and only one $h$ slope-stable vector bundle $E$ on $X$ such that
\[ \text{ch}_0(E) = r_0^2, \quad \text{ch}_1(E) = r_0h, \quad \text{ch}_2(E) = \frac{1}{2}h^2 - \frac{r_0^2 - 1}{24}c_2(X). \quad (1.3.5) \]
For such $E$ we have
\[ \text{ch}_3(E) = \frac{2e - 5(r_0^2 - 1)}{12r_0e}h^3, \quad (1.3.6) \]
\[ \text{ch}_4(E) = \frac{4e^2 - 20(r_0^2 - 1)e + (r_0^2 - 1)(21r_0^2 - 25)}{32r_0^2}. \quad (1.3.7) \]

2. Let $r_0$ be even and let $[(X, h)] \in \mathcal{X}^2_{\mathbb{C}}$ be a generic point. Then up to isomorphism there exists one and only one $h$ slope-stable vector bundle $E$ on $X$ such that
\[ \text{ch}_0(E) = r_0^2, \quad \text{ch}_1(E) = \frac{r_0}{2}h, \quad \text{ch}_2(E) = \frac{1}{8}h^2 - \frac{r_0^2 - 1}{24}c_2(X). \quad (1.3.8) \]
For such $E$ we have
\[ \text{ch}_3(E) = \frac{e - 10(r_0^2 - 1)}{48r_0e}h^3, \quad (1.3.9) \]
\[ \text{ch}_4(E) = \frac{e^2 - 20(r_0^2 - 1)e + 4(r_0^2 - 1)(21r_0^2 - 25)}{128r_0^2}. \quad (1.3.10) \]

Moreover, in both items, if $[(X, h)] \in \mathcal{X}^1_{\mathbb{C}}$ is generic we have $H^p(X, \text{End}_hE) = 0$ for all $p$, where $E$ is the vector bundle above.

Remark 1.5. Let $[(X, h)] \in \mathcal{X}^2_{\mathbb{C}}$ be generic. Then $(X, h)$ is isomorphic to the variety of lines $F(Y)$ on a generic cubic hypersurface $Y \subset \mathbb{P}^5$ polarized by the Plücker embedding, and the vector bundle $E$ of Theorem [1.4] with $r_0 = 2$ is isomorphic to the restriction of the tautological quotient vector bundle on $\text{Gr}(2, \mathbb{C}^6)$. Similarly, let $[(X, h)] \in \mathcal{X}^2_{\mathbb{C}}$ be generic. Then $(X, h)$ is isomorphic to the Debarre-Voisin variety associated to a generic $\sigma \in \mathbb{A}^1V_{10}^\vee$, where $V_{10}$ is a 10 dimensional complex vector space, and
\[ X_\sigma := \{ [W] \in \text{Gr}(6, V_{10}) \mid \sigma|_W = 0 \}. \quad (1.3.11) \]
The vector bundle $E$ of Theorem [1.4] with $r_0 = 2$ is isomorphic to the restriction to $X_\sigma$ of the tautological quotient vector bundle on $\text{Gr}(6, V_{10})$. These results are proved in Section [8].

Remark 1.6. We would like the congruence relations in [1.3.4] to be forced upon us by adding to the hypotheses on $r(E), c_1(E), \Delta(E)$ the extra hypothesis that $\chi(X, \text{End}_hE) = 0$. Our computations give this for some values of $r_0$, but we do not have complete results.

The result below replaces the genericity hypothesis in Theorem [1.4] with a cohomological one.
Corollary 1.7. Assume that \( c, r_0 \) are positive natural numbers such that (1.3.4) holds. Suppose that \( [(X, h)] \in \mathcal{X}_e^i \) with \( i = r_0 \pmod{2} \), and that \( \mathcal{E} \) is an \( h \)-slope-stable vector bundle on \( X \) such that \( H^2(X, \text{End}_h \mathcal{E}) = 0 \) and such that (1.3.5) holds if \( r_0 \) is odd, and (1.3.5) holds if \( r_0 \) is even. Then any \( h \)-slope-stable vector bundle \( \mathcal{G} \) on \( X \) such that \( \text{ch}_i(\mathcal{G}) = \text{ch}_i(\mathcal{E}) \) for \( i \in \{0, 1, 2\} \) is isomorphic to \( \mathcal{E} \). Moreover Equations (1.3.0), (1.3.7) hold if \( r_0 \) is odd, and Equations (1.3.9), (1.3.10) hold if \( r_0 \) is even.

There is an interesting consequence of Theorem 1.4 involving Debarre-Voisin varieties. There is a GIT moduli space \( \mathcal{M}_{DV} := \mathbb{P}(\mathbb{P}^3 V_{10})/\text{SL}(V_{10}) \) of Debarre-Voisin varieties, see [DHOV]. In [DV10] it was proved that the moduli map \( \mathcal{M}_{DV} \to \mathcal{X}_2^2 \) has finite non zero degree.

Theorem 1.8. The moduli map \( \mathcal{M}_{DV} \to \mathcal{X}_2^2 \) is birational.

1.4. Outline of the paper. In Section 2 we give a few examples of modular sheaves, and we make the connection with semi-homogeneous vector bundles. In particular we give strong restrictions on the possible ranks of modular sheaves, under some hypotheses.

Section 3 contains the results that extend to modular sheaves the known results on the variation of slope-stability for sheaves on surfaces. In particular we show that one can extend to HK’s with a Lagrangian fibration the results that hold for sheaves on surfaces which are fibered over a curve.

In Section 4 we prove properties of slope-stable modular vector bundles on HK’s \( X \) of Type \( K3^{[2]} \) with a Lagrangian fibration \( X \to \mathbb{P}^2 \). We make certain hypotheses, in particular we assume that \( r(\mathcal{E}) \) equals the square of a generator of the ideal \( \{g_X(c_1(\mathcal{E}), \alpha) \mid \alpha \in H^2(X; \mathbb{Z})\} \). We show that the restriction of a slope-stable modular vector bundle on \( X \) to a generic Lagrangian fiber is slope-stable, and that if \( \mathcal{F} \) is another such vector bundle then the restrictions of \( \mathcal{E} \) and \( \mathcal{F} \) are isomorphic.

Section 5 discusses a construction which associates to a vector bundle \( \mathcal{F} \) on a \( K3 \) surface \( S \) two torsion free sheaves \( \mathcal{F}[n] \pm \) on \( S^{[n]} \) whose fibers over a reduced scheme \( \{x_1, \ldots, x_n\} \) are the tensor product \( \mathcal{F}(x_1) \otimes \ldots \otimes \mathcal{F}(x_n) \) of the fibers of \( \mathcal{F} \) at the points \( x_1, \ldots, x_n \). We prove that if \( \chi(S, \mathcal{F}^* \otimes \mathcal{F}) = 2 \), then \( \mathcal{F}[2] \pm \) is a modular vector bundle, and we compute its Chern character. As proved in Section 7 this construction gives (by deformation) the existence result of Theorem 1.4.

In Section 6 we let \( S \to \mathbb{P}^1 \) be an elliptic \( K3 \) surface with Picard number 2. Thus \( S^{[2]} \) has an associated Lagrangian fibration \( \pi: S^{[2]} \to (\mathbb{P}^1)^{(2)} \equiv \mathbb{P}^2 \). We prove that if \( \mathcal{F} \) is a slope-stable rigid vector bundle on \( S \) then the vector bundle \( \mathcal{F}[2] \pm \) on \( S^{[2]} \) has good properties. In particular we show that it extends to any small deformation of \( S^{[2]} \), which keeps \( c_1(\mathcal{F}[2] \pm) \) of type \((1, 1)\), and that the restriction to any fiber of the Lagrangian fibration \( \pi \) is simple.

Section 7 contains the proof of Theorem 1.4 (and of Corollary 1.4). The basic idea is as follows. Let \( \mathcal{F} \to T_e^1 \) be a complete family of polarized HK’s of Type \( K3^{[2]} \) whose moduli belong to \( \mathcal{X}_e^i \). By Gieseker and Maruyama there is a relative moduli scheme \( f: \mathcal{M}_e(r_0) \to T_e^1 \) whose fiber over \( t \in T_e^1 \) is the moduli space of slope-stable vector bundles on \( (X_t, h_t) \) with the given rank, \( c_1 \) and \( c_2 \). The map \( f: \mathcal{M}_e(r_0) \to T_e^1 \) is of finite type by a result of Maruyama. Let \( \mathcal{M}_e^+(r_0) \subset \mathcal{M}_e(r_0) \) be the (open) subset parametrizing vector bundles whose \( h_2 \) and \( h_4 \) is given by the formulae in Theorem 1.4. Because of the good properties of the vector bundles \( \mathcal{F}[2] \pm \), the image \( f(\mathcal{M}_e^+(r_0)) \) contains a dense open (in the Zariski topology) subset.
of $T^1$. On the other hand the results of Section 3 and 4 allow us to prove that, up to isomorphism, there is a unique slope-stable vector bundle with the relevant $c_1$, $c_2$, $c_3$ on a generic HK parametrized by a Lagrangian Noether-Lefschetz locus with large discriminant. By density of the union of Noether-Lefschetz divisors (with large discriminant) we conclude that $f$ has degree 1.

In Section 8 we prove Theorem 1.8. Once we have Theorem 1.4, the main point is to show that the tautological quotient vector bundle on a generic DV variety is slope-stable.

In the appendix we discuss properties of semi-homogeneous vector bundles on abelian varieties, and of Lagrangian Noether-Lefschetz divisors on moduli spaces of polarized HK’s of Type $K3^{[2]}$.

1.5. Conventions.

- Algebraic variety is sinomimous of complex quasi projective variety (not necessarily irreducible).
- Let $X$ be a smooth complex quasi projective variety and $\mathcal{F}$ a coherent sheaf on $X$. We only consider topological Chern classes $c_i(\mathcal{F}) \in H^{2i}(X(\mathbb{C}); \mathbb{Z})$.
- Let $X$ be a HK manifold of dimension $2n$. We let $q_X$, or simply $q$, be the BBF symmetric bilinear form of $X$, and we denote $q_X(\alpha, \alpha)$ by $q_X(\alpha)$. We let $c_X$ be the normalized Fuiki constant of $X$, i.e. the rational positive number such that for all $\alpha \in H^2(X)$ we have
  \[ \int_X \alpha^{2n} = c_X \cdot (2n - 1)!! \cdot q_X(\alpha)^n. \]  

A hyperkähler (HK) variety is a projective compact HK manifold.
- Let $\mathcal{F}$ be a torsion-free sheaf on a polarized projective variety $(X, h)$. A subsheaf $\mathcal{E} \subset \mathcal{F}$ is slope-stabilizing if $0 < r(\mathcal{E}) < r(\mathcal{F})$ and $\mu_h(\mathcal{E}) \geq \mu_h(\mathcal{F})$, where $r(\mathcal{E}), r(\mathcal{F})$ are the ranks of $\mathcal{E}, \mathcal{F}$, and $\mu_h(\mathcal{E}), \mu_h(\mathcal{F})$ are the $h$-slopes of $\mathcal{E}, \mathcal{F}$. If $\mu_h(\mathcal{E}) > \mu_h(\mathcal{F})$ then $\mathcal{E} \subset \mathcal{F}$ is slope-stabilizing.
- We use similar terminology for exact sequences $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$.
- A torsion-free sheaf on $(X, h)$ is strictly $h$ slope-semistable if it is $h$ slope-semistable but not $h$ slope-stable.
- Abusing notation we say that a smooth projective variety $X$ is an abelian variety if it is isomorphic to the variety underlying an abelian variety $A$. In other words $X$ is a torsor of $A$.

2. Modular sheaves

2.1. First examples. By Remark 1.2 the following are modular vector bundles:

(1) The tangent bundle $\Theta_X$.

(2) Let $V_6$ be a 6 dimensional complex vector space, and let $X \subset \text{Gr}(2, V_6)$ be the variety of lines contained in a smooth cubic hypersurface in $\mathbb{P}(V_6)$. Let $h \in H_2^{1,1}(X)$ be the Plücker polarization. Then $X$ is a HK of type $K3^{[2]}$, see [BDSS]. Let $\mathcal{D}$ be the restriction to $X$ of the tautological rank 4 quotient vector bundle on $\text{Gr}(2, V_6)$. Then
  \[ \text{ch}_0(\mathcal{D}) = 4, \quad \text{ch}_1(\mathcal{D}) = h, \quad \text{ch}_2(\mathcal{D}) = \frac{1}{8} \left(h^2 - c_2(X)\right). \]  

Thus $\Delta(\mathcal{D}) = c_2(X)$, and hence $\mathcal{D}$ is modular.

(3) Let $X \subset \text{Gr}(6, V_{10})$ be a smooth DV variety and let $h \in H_2^{1,1}(X)$ be the Plücker polarization, see [DV10]. Then $X$ is a HK of Type $K3^{[2]}$. Let $\mathcal{Z}$ be
As is well-known $q$ is generated by an ample class $\theta$. Smooth the image of the restriction map $r_2$.

Remark 2.1. Let $X$ be a HK variety and let $\mathcal{E}, \mathcal{F}$ be modular sheaves on $X$. Then $\mathcal{E} \otimes \mathcal{F}$ is not modular in general. On the other hand $\mathcal{E} \otimes \mathcal{F}$ is modular, at least if $\mathcal{E}$ and $\mathcal{F}$ are locally free.

Remark 2.2. Let $X$ be a HK manifold of dimension $2n$, and let $\mathcal{F}$ be a torsion free modular sheaf on $X$. Then

$$\int_X \Delta(\mathcal{F}) \sim_1 \cdots \sim_{\alpha_2n-2} = d(\mathcal{F}) \cdot \sum \gamma_X(\alpha_1, \alpha_2) \cdots \cdot \gamma_X(\alpha_{2n-3}, \alpha_{12n-2}),$$

for all $\alpha_1, \ldots, \alpha_{2n} \in H^2(X)$, where $\sum$ means that in the summation we avoid repeating addends which are formally equal (i.e. are equal modulo reordering of the factors $\gamma_X(\cdot, \cdot)$'s and switching the entries in $\gamma_X(\cdot, \cdot)$).

2.2. Restrictions on the rank. Below is the result that was mentioned in Sub-section 1.3.

Proposition 2.3. Let $X$ be a HK fourfold of Type $K3^{[2]}$ or $\text{Kum}_2$. Let $\mathcal{F}$ be a modular torsion-free sheaf on $X$. Let $m$ be a generator of the ideal

$$\{ \gamma_X(c_1(\mathcal{F}), \alpha) \mid \alpha \in H^2(X; \mathbb{Z}) \}.$$

Then $r(\mathcal{F})$ divides $m^2$ if $X$ is of Type $K3^{[2]}$, and it divides $3m^2$ if $X$ is of Type $\text{Kum}_2$.

Proof. As is easily checked, there exists $\alpha \in H^2(X; \mathbb{Z})$ such that $\gamma_X(c_1(\mathcal{F}), \alpha) = m$ and $\gamma_X(\alpha) = 0$. Let $r := r(\mathcal{F})$. Since $\gamma_X(\alpha) = 0$, Equation (2.2.1) gives that

$$2r \int_X c_2(\mathcal{F}) \sim_1 \alpha^2 = (r-1) \int_X c_1(\mathcal{F})^2 \sim_1 \alpha^2 = 2(r-1)c_X \cdot \gamma_X(c_1(\mathcal{F}), \alpha) = 2(r-1)c_X \cdot m^2.$$

The result follows because $c_X = 1$ if $X$ is of Type $K3^{[2]}$, and $c_X = 3$ if $X$ is of Type $\text{Kum}_2$. □

2.3. Modular sheaves on Lagrangian fibrations. We recall that a Lagrangian fibration $\pi : X \to \mathbb{P}^n$ on a HK manifold $X$ of dimension $2n$ is a surjective map with connected fibers whose smooth fibers are abelian varieties.

Remark 2.4. For $t \in \mathbb{P}^n$ we let $X_t := \pi^{-1}(t)$ be the schematic fiber over $t$. If $X_t$ is smooth the image of the restriction map $H^2(X_t; \mathbb{Z}) \to H^2(X_t; \mathbb{Z})$ has rank one, and is generated by an ample class $\theta_t \in H^1_{\mathbb{Z}}(X_t)$, see [Wie10]. If $\mathcal{F}$ is a sheaf on $X_t$, slope-(semi)stability of $\mathcal{F}$ will always mean $\theta_t$ slope-(semi)stability.

If $\pi : X \to \mathbb{P}^n$ is a Lagrangian fibration we let

$$f := c_1(\pi^* \mathcal{O}_{\mathbb{P}^n}(1)) \in H^{1,1}_{\mathbb{Z}}(X).$$

As is well-known $q_X(\mathcal{F}) = 0$. 

Lemma 2.5. Let \( \pi: X \to \mathbb{P}^n \) be a Lagrangian fibration of a HK manifold of dimension \( 2n \). Suppose that \( \mathcal{F} \) is a modular torsion free sheaf on \( X \). Let \( t \in \mathbb{P}^n \) be a general point, and let \( \mathcal{F}_t := \mathcal{F}|_{X_t} \) be the restriction of \( \mathcal{F} \) to \( X_t \). Then
\[
\int_{X_t} \Delta(\mathcal{F}_t) \sim \theta_t^{n-2} = 0. \tag{2.3.2}
\]

Proof. There exists \( \rho > 0 \) such that \( \theta_t = \rho(\omega_{|X_t}) \). Since \( t \in \mathbb{P}^n \) is a generic point, we have \( \Delta(\mathcal{F}_t) = \Delta(\mathcal{F})|_{X_t} \). Moreover \( f^n \) is the Poincaré dual of \( X_t \). Hence
\[
\int_{X_t} \Delta(\mathcal{F}_t) \sim \theta_t^{n-2} = \rho^{n-2} \int_X \Delta(\mathcal{F}) \sim \omega^{n-2} \sim f^n. \tag{2.3.3}
\]
The integral on the right vanishes by Remark 2.2 and the equality \( q_X(f) = 0 \). \( \square \)

Example 2.6. Let \( S \) be a K3 surface, and let \( V \) be a vector bundle on \( S \). Let \( \mathcal{L} \subset S \times S^{[n]} \) be the tautological subscheme, and let \( p: \mathcal{L} \to S \), \( q: \mathcal{L} \to S^{[n]} \) be the projection maps. The locally free sheaf \( q_*(p^*V) \) is known as a tautological sheaf on \( S^{[n]} \). In general such a sheaf is not modular. In fact suppose that \( S \) is elliptic, with elliptic fibration \( S \to \mathbb{P}^1 \). The composition \( S^{[n]} \to S \to \mathbb{P}^{1\cdot n} \cong \mathbb{P}^n \) is a Lagrangian fibration with generic fiber \( X_t = C_1 \times \ldots \times C_n \), where \( C_1, \ldots, C_n \) are generic distinct fibers of the elliptic fibration \( S \to \mathbb{P}^1 \). If the restriction of \( V \) to the fibers of \( S \to \mathbb{P}^1 \) has non zero degree then Equality (2.3.2) does not hold for \( \mathcal{F} := q_*(p^*V) \), and hence \( q_*(p^*V) \) is not modular.

Proposition 2.7. Let \( \pi: X \to \mathbb{P}^n \) be a Lagrangian fibration of a HK manifold of dimension \( 2n \). Let \( \mathcal{F} \) be a modular torsion free sheaf on \( X \). Suppose that \( t \in \mathbb{P}^n \) is a regular value of \( \pi \), that \( \mathcal{F} \) is locally-free in a neighborhood of \( X_t \), and that \( \mathcal{F}_t \) is slope-stable. Then \( \mathcal{F}_t \) is a semi homogeneous vector bundle.

Proof. Follows from Lemma 2.5 and Proposition A.2. \( \square \)

The result below shows that, under suitable hypotheses, a much stronger version of Proposition 2.5 holds.

Corollary 2.8. Let \( X \) be a HK of Type K3\(^{[n]} \), Kum\(_n\) or OG6. Let \( \mathcal{F} \) be a modular torsion free sheaf on \( X \). Suppose that \( t \in \mathbb{P}^n \) is a regular value of \( \pi \), that \( \mathcal{F} \) is locally-free in a neighborhood of \( X_t \), and that \( \mathcal{F}_t \) is slope-stable. Then there exist positive integers \( r_0, d \), with \( d \) dividing \( c_X \), such that \( r(\mathcal{F}) = \frac{c_F}{2} \).

Proof. If \( X \) is of Type K3\(^{[n]} \) then \( c_X = 1 \) and \( \theta_t \) is a principal polarization, see [Wie15]. If \( X \) is of Type Kum\(_n\) or OG6 then \( c_X = n + 1 \) and \( \theta_t \) is a polarization with elementary divisors \((1, \ldots, 1, d_1, d_2)\) where \( d_1 \cdot d_2 \) divides \( n + 1 \) see [Wie15] for Kum\(_n\) and [MR] for OG6. Hence the result follows from Proposition 2.7 and Proposition A.3. \( \square \)

3. Variation of stability for modular sheaves

3.1. Main results. Let \( X \) be an irreducible smooth projective variety. If the ample cone \( \text{Amp}(X) \) has rank greater than 1 (and hence \( \dim X \geq 2 \)), slope-stability of a sheaf \( \mathcal{F} \) depends on the choice of an ample ray. If \( X \) is a surface there is a locally finite decomposition \( \text{Amp}(X) \) into chambers defined by rational walls such that slope-stability is the same for ample classes belonging to the same open chamber. One important feature is that the walls depend only on the Chern character of \( \mathcal{F} \).

If \( \dim X \geq 3 \) the picture is more intricate in general, see for example [GR19].

In the present section we show that if \( X \) is a HK variety and \( \mathcal{F} \) is a modular sheaf, then there is a decomposition of \( \text{Amp}(X) \) as if \( X \) were a surface.
Definition 3.1. Let \( a \) be a positive real number. An \( a \)-wall of \( \text{Amp}(X)_\mathbb{R} \) is the intersection \( \lambda^a \cap \text{Amp}(X)_\mathbb{R} \), where \( \lambda \in H^{1,1}_X \), \( -a \leq q_X(\lambda) < 0 \), and orthogonality is with respect to the BBF quadratic form \( q_X \).

As is well-known, the set of \( a \)-walls is locally finite, in particular the union of all the \( a \)-walls is closed in \( \text{Amp}(X)_\mathbb{R} \).

Definition 3.2. An open \( a \)-chamber is a connected component of the complement (in \( \text{Amp}(X)_\mathbb{R} \)) of the union of all the \( a \)-walls.

Definition 3.3. Let \( X \) be a HK manifold, and let \( \mathcal{F} \) be a modular torsion free sheaf on \( X \). Then

\[
a(\mathcal{F}) := \frac{r(\mathcal{F})^2 \cdot d(\mathcal{F})}{4c_X}, \tag{3.1.1}
\]

where \( d(\mathcal{F}) \) is as in Definition 3.1.

Below is the first main result.

Proposition 3.4. Let \( X \) be a HK variety of dimension \( 2n \), and let \( \mathcal{F} \) be a torsion free modular sheaf on \( X \). Then the following hold:

1. Suppose that \( h \) is an ample divisor class on \( X \) which belongs to an open \( a(\mathcal{F}) \)-chamber. If \( \mathcal{F} \) is strictly \( h \)-slope-semistable there exists an exact sequence of torsion free non zero sheaves

\[
0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0 \tag{3.1.2}
\]

such that \( r(\mathcal{F})c_1(\mathcal{E}) - r(\mathcal{G})c_1(\mathcal{G}) = 0 \).

2. Suppose that \( h_0, h_1 \) are ample divisor classes on \( X \) belonging to the same open \( a(\mathcal{F}) \)-chamber. Then \( \mathcal{F} \) is \( h_0 \)-slope-stable if and only if it is \( h_1 \)-slope-stable.

Proposition 3.4 is proved in Subsection 3.3.

The next result is about slope-stable sheaves on HK varieties which carry a Lagrangian fibration.

Definition 3.5. Let \( X \) be a HK variety equipped with a Lagrangian fibration \( \pi: X \rightarrow \mathbb{P}^n \), and let \( f := \pi^*c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \). Let \( a \) be positive integer. An ample divisor class \( h \) on \( X \) is \( a \)-suitable if the following holds. Let \( \xi \in H^{1,1}_X \) be a class such that \( -a \leq q_X(\xi) < 0 \): then either \( q_X(\xi, h) \) and \( q_X(\xi, f) \) have the same sign, or they are both zero.

Notice that the notion of \( a \)-suitable depends on the chosen Lagrangian fibration.

Proposition 3.6. Let \( \pi: X \rightarrow \mathbb{P}^n \) be a Lagrangian fibration of a HK variety of dimension \( 2n \). Let \( \mathcal{F} \) be a torsion free modular sheaf on \( X \) such that \( \text{sing} \mathcal{F} \) does not dominate \( \mathbb{P}^n \). Let \( h \) be an ample divisor class on \( X \) which is \( a(\mathcal{F}) \)-suitable. Then the following hold:

(i) If the restriction of \( \mathcal{F} \) to a generic fiber of \( \pi \) is slope-stable, then \( \mathcal{F} \) is \( h \)-slope-stable.

(ii) If \( \mathcal{F} \) is \( h \)-slope-stable then the restriction of \( \mathcal{F} \) to the generic fiber of \( \pi \) is slope-semistable.

Proposition 3.6 is proved in Subsection 3.5.

3.2. Change of slope-stability and strictly semistable sheaves. Suppose that \( \mathcal{E}, \mathcal{F} \) are sheaves on an irreducible smooth variety \( X \). We let

\[
\lambda_{\mathcal{E}, \mathcal{F}} := (r(\mathcal{F})c_1(\mathcal{E}) - r(\mathcal{E})c_1(\mathcal{F})) \in H^2(X; \mathbb{Z}). \tag{3.2.1}
\]

Lemma 3.7. Let \( (X, h) \) be a polarized HK variety, and let \( \mathcal{E}, \mathcal{F} \) be non zero torsion free sheaves on \( X \). Then
(a) \( \mu_h(\mathcal{E}) > \mu_h(\mathcal{F}) \) if and only if \( q_X(\lambda_{\mathcal{E},\mathcal{F}}, h) > 0 \).
(b) \( \mu_h(\mathcal{E}) = \mu_h(\mathcal{F}) \) if and only if \( q_X(\lambda_{\mathcal{E},\mathcal{F}}, h) = 0 \).

**Proof.** Let \( 2n \) be the dimension of \( X \). We have \( \mu_h(\mathcal{E}) > \mu_h(\mathcal{F}) \) if and only if \( \int_X \lambda_{\mathcal{E},\mathcal{F}} \sim h^{2n-1} > 0 \), and by Fujiki’s formula this holds if and only if

\[
c_X \cdot (2n-1)! q \cdot q(x(h)^{n-1} > 0.
\]

Item (a) follows, because \( c_X > 0 \) and \( q_X(h) > 0 \).

We have \( \mu_h(\mathcal{E}) = \mu_h(\mathcal{F}) \) if and only if \( \int_X \lambda_{\mathcal{E},\mathcal{F}} \sim h^{2n-1} = 0 \), and hence Item (b) follows again by Fujiki’s formula. \( \square \)

**Proposition 3.8.** Let \( X \) be a HK variety, and let \( h_0, h_1 \) be ample divisor classes on \( X \). Suppose that \( \mathcal{F} \) is a torsion free sheaf on \( X \) which is \( h_0 \) slope-stable and not \( h_1 \) slope-stable. Then there exists \( h \in (\mathbb{Q}_+ h_0 + \mathbb{Q}_+ h_1) \) such that \( \mathcal{F} \) is strictly \( h \) slope-semistable, i.e., \( \mathcal{F} \) is \( h \) slope-semistable but not \( h \) slope-stable.

**Proof.** Lemma 3.7 allows to reproduce the proof of the analogous statement valid for surfaces (see [HL10]). Let \( S \subset ([0,1] \cap \mathbb{Q}) \) be the set of \( s \) for which there exists a subsheaf \( \mathcal{E} \subset \mathcal{F} \) with \( 0 < r(\mathcal{E}) < r(\mathcal{F}) \) such that

\[
x(\lambda_{\mathcal{E},\mathcal{F}}, (1-s) h_0 + s h_1) = 0. \quad (3.2.2)
\]

Then \( S \) is non empty and finite. In fact by hypothesis there exists an \( h_1 \) destabilizing subsheaf \( \mathcal{E} \subset \mathcal{F} \). Thus \( 0 < r(\mathcal{E}) < r(\mathcal{F}) \), and \( x(\lambda_{\mathcal{E},\mathcal{F}}, \mathcal{O}_X(h_1)) > 0 \) by Lemma 3.7. On the other hand, by the same lemma, \( x(\lambda_{\mathcal{E},\mathcal{F}}, \mathcal{O}_X(h_0)) < 0 \) because \( \mathcal{F} \) is \( h_0 \) slope-stable. It follows that there exists \( s \in [0,1] \cap \mathbb{Q} \) such that (3.2.2) holds, i.e., \( S \) is not empty.

In order to prove that \( S \) is finite, assume that (3.2.2) holds. Since \( \mathcal{F} \) is \( h_0 \) slope-stable, \( x(\lambda_{\mathcal{E},\mathcal{F}}, \mathcal{O}_X(h_0)) < 0 \). By linearity of \( x(\lambda_{\mathcal{E},\mathcal{F}}, \cdot) \), it follows that \( x(\lambda_{\mathcal{E},\mathcal{F}}, h_1) > 0 \). By Lemma 3.7 it follows that

\[
\mu_{h_1}(\mathcal{E}) \geq \mu_{h_1}(\mathcal{F}). \quad (3.2.3)
\]

The set of subsheaves \( \mathcal{E} \subset \mathcal{F} \) such that (3.2.3) holds is bounded (see [HL10]), i.e., up to isomorphism each such sheaf belongs to a finite set of families, each parametrized by an irreducible quasi projective variety. It follows that \( S \) is finite because the values of \( x(\lambda_{\mathcal{E},\mathcal{F}}, \mathcal{O}_X(h_i)) \) for \( i \in \{0,1\} \) are constant for sheaves \( \mathcal{E} \) parametrized by an irreducible variety.

Since \( S \) is finite, there is a minimum \( s \), call it \( s_{\text{min}} \), such that (3.2.2) holds for some subsheaf \( \mathcal{E} \subset \mathcal{F} \) with \( 0 < r(\mathcal{E}) < r(\mathcal{F}) \). Clearly \( \mathcal{F} \) is strictly \( (h_0 + s_{\text{min}} h_1) \) slope-semistable. \( \square \)

### 3.3. Strictly semistable modular sheaves.

**Lemma 3.9.** Let

\[
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0
\]

be an exact sequence of sheaves on a smooth variety. Then

\[
r(\mathcal{F}) \cdot r(\mathcal{G}) \Delta(\mathcal{E}) + r(\mathcal{F}) \cdot r(\mathcal{E}) \Delta(\mathcal{G}) = r(\mathcal{E}) \cdot r(\mathcal{G}) \Delta(\mathcal{F}) + \lambda_{\mathcal{E},\mathcal{F}}^2.
\]

**Proof.** Follows from additivity of the Chern character, and the second equality in (1.2.1). \( \square \)

**Proposition 3.10.** Let \( (X, h) \) be a polarized HK variety of dimension \( 2n \). Let \( \mathcal{F} \) be a torsion free modular strictly \( h \) slope-semistable sheaf on \( X \), and let

\[
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0
\]

be an exact sequence of non zero torsion free sheaves which is \( h \) slope destabilizing, i.e., \( \mu_h(\mathcal{E}) = \mu_h(\mathcal{F}) \). Then

\[
a(\mathcal{F}) \leq x(\lambda_{\mathcal{E},\mathcal{F}}) \leq 0.
\]

\[ \blacksquare \]
Moreover \( q_X(\lambda_{\mathcal{E}, \mathcal{F}}) = 0 \) only if \( \lambda_{\mathcal{E}, \mathcal{F}} = 0 \).

**Proof.** Since the exact sequence in (3.3.3) is destabilizing, \( q_X(\lambda_{\mathcal{E}, \mathcal{F}}, h) = 0 \) by Lemma 3.7. Since the BBF form on \( \text{NS}(X) \) has signature \((1, p(X) - 1)\), it follows that \( q_X(\lambda_{\mathcal{E}, \mathcal{F}}) \leq 0 \) with equality only if \( \lambda_{\mathcal{E}, \mathcal{F}} = 0 \). (Recall that \( q_X(h) > 0 \), because \( h \) is ample.)

We are left with proving the second inequality in (3.3.4). Hence we assume that \( q_X(\lambda_{\mathcal{E}, \mathcal{F}}) < 0 \). Cupping both sides of the equality in (3.3.2) by \( h^{2n-2} \), and integrating, we get (here we use the hypothesis that \( \mathcal{E} \) is modular)

\[
\int_X r(\mathcal{F}) \cdot r(\mathcal{G}) \Delta(\mathcal{E}) \sim h^{2n-2} + \int_X r(\mathcal{F}) \cdot r(\mathcal{G}) \Delta(\mathcal{E}) \sim h^{2n-2} =
\]

\[
= r(\mathcal{F}) \cdot r(\mathcal{G}) \cdot d(\mathcal{F}) \cdot (2n - 3)!!q_X(h)^{n-1} + c_X \cdot q_X(\lambda_{\mathcal{E}, \mathcal{F}}) \cdot (2n - 3)!!q_X(h)^{n-1}.
\]

By hypothesis \( \mu_h(\mathcal{E}) = \mu_h(\mathcal{G}) \). Since \( \mathcal{F} \) is \( h \) slope-stable it follows that \( \mathcal{E} \) and \( \mathcal{G} \) are \( h \) slope-semistable torsion free sheaves. Thus

\[
\int_X \Delta(\mathcal{E}) \sim h^{2n-2} \geq 0, \quad \int_X \Delta(\mathcal{G}) \sim h^{2n-2} \geq 0
\]

by Bogomolov’s inequality, and hence (3.3.5) gives

\[
r(\mathcal{F}) \cdot r(\mathcal{G}) \cdot d(\mathcal{F}) \leq c_X \cdot q_X(\lambda_{\mathcal{E}, \mathcal{F}}).
\]

Dividing by \( c_X \) (which is strictly positive), we see that the second inequality in (3.3.4) follows from (3.3.6) and the inequality \( r(\mathcal{E}) \cdot r(\mathcal{G}) \leq r(\mathcal{F})^2 / 4 \). \( \square \)

3.4. **Proof of Proposition 3.4.** Item (1) follows from Proposition 3.10. We prove Item (2). By symmetry, it suffices to show that if \( \mathcal{F} \) is \( h_0 \) slope-stable, then it is \( h_1 \) slope-stable. Suppose that \( \mathcal{F} \) is not \( h_1 \) slope-stable. By Proposition 3.3 there exists \( h \in (\mathbb{Q}_+h_0 + \mathbb{Q}_+h_1) \) such that \( \mathcal{F} \) is strictly \( h \) slope-semistable. Hence there exists an \( h \) destabilizing

\[
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0
\]

exact sequence of non zero torsion free sheaves. Since \( h_0, h_1 \) belong to the same open \( a(\mathcal{F}) \) chamber, also \( h \) belongs to the same open \( a(\mathcal{F}) \)-chamber. Thus, by Proposition 3.10, we get that \( \lambda_{\mathcal{E}, \mathcal{F}} = 0 \). It follows that \( \mathcal{F} \) is not \( h_0 \) slope-stable, and that is a contradiction. \( \square \)

3.5. **Stability of modular sheaves on a lagrangian HK.**

**Lemma 3.11.** Let \( X \) be a HK variety of dimension \( 2n \) equipped with a Lagrangian fibration \( \pi: X \rightarrow \mathbb{P}^n \), and let \( f = c_1(\pi^*\mathcal{O}(1)) \). Let \( \mathcal{F} \) be a torsion free sheaf on \( X \), and let \( \mathcal{E} \subset \mathcal{F} \) be a subsheaf with \( 0 < r(\mathcal{E}) < r(\mathcal{F}) \). Then the following hold:

(a) If, for generic \( t \in \mathbb{P}^n \), the restriction \( \mathcal{F}_t := \mathcal{F}|_{X_t} \) is slope-stable, then

\[
q_X(\lambda_{\mathcal{E}, \mathcal{F}}, f) < 0.
\]

(b) If, for generic \( t \in \mathbb{P}^n \), the subsheaf \( \mathcal{E}_t := \mathcal{E}|_{X_t} \subset \mathcal{F}_t \) is \( h_1 \) slope destabilizing, then

\[
q_X(\lambda_{\mathcal{E}, \mathcal{F}}, f) > 0.
\]

**Proof.** We have

\[
\int_X \lambda_{\mathcal{E}_t, \mathcal{F}_t} \sim c_1(\mathcal{O}_X(h_t))^{n-1} = \int_X \lambda_{\mathcal{E}, \mathcal{F}} \sim c_1(\mathcal{O}_X(h))^{n-1} \sim f^n =
\]

\[
= n!c_X \cdot q_X(\mathcal{O}_X(h), f)^{n-1} \cdot q_X((\lambda_{\mathcal{E}, \mathcal{F}}, f)).
\]

In fact, the first equality holds because \( f^n \) is the Poincarè dual of any fiber of the Lagrangian fibration, and the second equality holds by Fujiki’s formula and the fact
that $q_X(f) = 0$. Items (a) and (b) follow, because $c_X$ and $q_X(\partial_X(h), f)$ are strictly positive. 

\section*{Proof of Proposition 3.16} We prove Item (i). Suppose that $\mathcal{F}$ is not $h$ slope-stable. Let $S \subset ([0, 1] \cap \mathbb{Q})$ be set of of $s$ for which there exists a subsheaf $\mathcal{E} \subset \mathcal{F}$, with $0 < r(\mathcal{E}) < r(\mathcal{F})$, such that

$$q_X(\lambda_{\mathcal{E}, \mathcal{F}}, (1-s)h + sf) = 0. \quad (3.5.4)$$

Let us show that $S$ is non empty and finite. Since $\mathcal{F}$ is not slope-stable, by Lemma 3.7 there exists a subsheaf $\mathcal{E} \subset \mathcal{F}$, with $0 < r(\mathcal{E}) < r(\mathcal{F})$, such that $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, h) \geq 0$. On the other hand, by Lemma 3.11 the inequality in (3.5.4) holds. It follows that $S$ is not empty. The argument, in the proof of Proposition 3.8 showing that the analogous $S$ is finite, applies also in the present case, and hence $S$ is finite.

Let $s_{\max}$ be the maximum element of $S$. Clearly $\mathcal{F}$ is strictly $h + s_{\max} f$ slope-semistable. Let $\mathcal{E} \subset \mathcal{F}$ be a subsheaf, with $0 < r(\mathcal{E}) < r(\mathcal{F})$ which is $h + s_{\max} f$ destabilizing, i.e. $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, h + s_{\max} f) = 0$. Then $-a(\mathcal{F}) \leq q_X(\lambda_{\mathcal{E}, \mathcal{F}}) \leq 0$ by Proposition 3.10. On the other hand, $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, f) < 0$ by Lemma 3.11 and hence $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, h) < 0$ by our hypothesis on $h$. This contradicts the equality $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, h + s_{\max} f) = 0$.

Next, we prove Item (ii). Suppose that the restriction $\mathcal{F}|_{X_1}$ is $h_1$ slope-unstable for generic $t \in \mathbb{P}^n$. As before, let $S \subset ([0, 1] \cap \mathbb{Q})$ be the subset of $s$ such that there exists a subsheaf $\mathcal{E} \subset \mathcal{F}$, with rank $0 < r(\mathcal{E}) < r(\mathcal{F})$, for which (3.5.4) holds. We claim that $S$ is not empty, and that it has a minimum (N.B.: it does not have a maximum).

In fact, since $\mathcal{F}|_{X_1}$ is $h_1$ slope-unstable for generic $t \in \mathbb{P}^n$, there exists a subsheaf $\mathcal{E} \subset \mathcal{F}$, with $0 < r(\mathcal{E}) < r(\mathcal{F})$, such that $\partial_1 \subset \mathcal{F}_1$ is $h_1$ slope descentabilizing for generic $t \in \mathbb{P}^n$. By Lemma 3.11 we have $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, f) > 0$. On the other hand, $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, h) < 0$, because $\mathcal{F}$ is slope-stable. It follows that $S$ is not empty.

It remains to show that $S$ has a minimum. Suppose that (3.5.4) holds. Since $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, h) < 0$ (because $\mathcal{F}$ is slope-stable), we get that $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, f) > 0$. Hence the sheaves $\mathcal{E} \subset \mathcal{F}$, with $0 < r(\mathcal{E}) < r(\mathcal{F})$, such that (3.5.4) holds for some $s \in [0, 1] \cap \mathbb{Q}$ are exactly those such that $\partial_1 \subset \mathcal{F}|_{X_1}$ is an $h_1$ slope destabilizing sheaf of $\mathcal{F}|_{X_1}$, for the generic $t \in \mathbb{P}^n$.

Let $\tilde{X} := X \times_{\mathbb{P}^n} \mathbb{C}(\mathbb{P}^n)$ be the abelian variety over $\mathbb{C}(\mathbb{P}^n)$ obtained from $X$ by base change. We let $\tilde{h}$ be the ample divisor on $\tilde{X}$ determined by $h$. A subsheaf $\mathcal{E} \subset \mathcal{F}$ on $X$ determines a subsheaf $\tilde{\mathcal{E}} \subset \tilde{\mathcal{F}}$ on $\tilde{X}$.

Then $\mu_{\tilde{h}}(\tilde{\mathcal{E}}) > \mu_{\tilde{h}}(\tilde{\mathcal{F}})$, i.e. $\tilde{h}$ is destabilizing for $\tilde{\mathcal{F}}$ if and only if $\partial_1 \subset \mathcal{F}_1$ is an $h_1$ slope destabilizing sheaf of $\mathcal{F}_1$, for the generic $t \in \mathbb{P}^n$. The set of $h$ destabilizing subsheaves $\partial \subset \tilde{\mathcal{F}}$ is bounded. Given such a subsheaf, there exists a unique maximal subsheaf $\mathcal{E} \subset \mathcal{F}$ such that $\partial = \partial_1$. The set $S^0$ of $s \in ([0, 1] \cap \mathbb{Q})$ such that (3.5.4) holds for such a maximal subsheaf is finite (and non empty), by boundedness. Hence there is a minimum $s_{\min}^0$ element of $S^0$. All other subsheaves $\mathcal{E} \subset \mathcal{F}$ (with $0 < r(\mathcal{E}) < r(\mathcal{F})$) such that $\tilde{\mathcal{E}} \subset \tilde{\mathcal{F}}$ is a $h$ destabilizing subsheaf, are contained in a maximal subsheaf $\mathcal{E}$, and the quotient $\mathcal{E}/\partial$ is supported on vertical divisors (i.e. divisors whose image under $\pi$ is a proper subset of $\mathbb{P}^n$). It follows that $s_{\min}^0$ is also the minimum element of $S$. The sheaf $\mathcal{F}$ is strictly $((1-s_{\min})h + s_{\min} f)$ slope-semistable by minimality of $s_{\min}$. Let $\mathcal{E} \subset \mathcal{F}$ be a subsheaf, with $0 < r(\mathcal{E}) < r(\mathcal{F})$ such that $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, (1-s_{\min})h + s_{\min} f) = 0$. By Proposition 3.10 either $-a(\mathcal{F}) \leq q_X(\lambda_{\mathcal{E}, \mathcal{F}}) < 0$ or $\lambda_{\mathcal{E}, \mathcal{F}} = 0$. The latter does not hold because $q_X(\lambda_{\mathcal{E}, \mathcal{F}}, h) < 0$ ($\mathcal{F}$ is $h$ slope-stable).
Hence \(-a(F) \leq q_X(\lambda_{e,x}) < 0\) and thus \(q_X(\lambda_{e,x}, f) < 0\) by our hypothesis on \(h\). This contradicts the equality \(q_X(\lambda_{e,x}, (1 - s_{\text{min}})h + s_{\text{min}}f) = 0\). \(\square\)

4. Stable vector bundles on Lagrangian hyperkählers

4.1. Main result. Before stating the main result we recall that \(\mathcal{M}_q^i\) is the moduli space of polarized HK's \((X, h)\) of Type \(K3[2]\) with \(q(h) = e\) and \(h\) has divisibility \(i\) (see [15, 16] and [18, 19]), which is 1 if \(e \not\equiv 6 \pmod{8}\), and is either 1 or 2 if \(e \equiv 6 \pmod{8}\).

The Noether-Lefschetz divisor \(\mathcal{M}_q^i(\ell) \subset \mathcal{M}_q^i\) parametrizes \((X, h)\) such that there exists a saturated rank 2 sublattice \(\langle h, f \rangle \subset H_{\mathbb{Z}}^1(X)\), where \(f\) is isotropic and \(q(h, f) = d\), see Definition [13, 14]. Assume that \(d > 10(e + 1)\), that \(e \not\divides 2d\) and that \(d\) is even if \(i = 2\). By Proposition [15, 16] \(\mathcal{M}_q^i(\ell)\) is of pure codimension 1 (in particular non empty), and there exists an open dense subset \(\mathcal{M}_q^i(\ell)^0 \subset \mathcal{M}_q^i(\ell)\) such that the following holds for \([X, h] \in \mathcal{M}_q^i(\ell)^0\): there exists one and only one Lagrangian fibration \(\pi : X \to \mathbb{P}^2\) (modulo automorphisms of \(\mathbb{P}^2\)) such that, letting \(f := \pi^*c_1(\mathcal{O}_{\mathbb{P}^2}(1))\), the lattice \(\langle h, f \rangle\) is as above. Below is the main result of the present section.

**Proposition 4.1.** Let \(a_0, d\) be positive integers and \(i \in \{1, 2\}\). Suppose that \(e \not\divides 2d\), that \(d\) is even if \(i = 2\), and that

\[
d > \max\left\{\frac{1}{2}a_0(e + 1), 10(e + 1)\right\}.
\] (4.1.1)

If \([X, h] \in \mathcal{M}_q^i(\ell)^0\) is generic the following hold:

1. Let \(\mathcal{E}\) be an \(h\) slope-stable vector bundle on \(X\) such that
   (a) \(a(\mathcal{E}) \leq a_0\), where \(a(\mathcal{E})\) is as in Definition [12, 13].
   (b) there exists an integer \(m\) such that \(r(\mathcal{E}) = (mi)^2\), \(c_1(\mathcal{E}) = mh\), and \(\gcd\{mi, \frac{4}{i}\} = 1\).

Then the restriction of \(\mathcal{E}\) to a generic fiber of the associated Lagrangian fibration \(\pi : X \to \mathbb{P}^2\) is slope-stable.

2. If \(\mathcal{F}, \mathcal{G}\) are \(h\) slope-stable vector bundles on \(X\) such that Items (a) and (b) hold for \(\mathcal{E} = \mathcal{F}\) and \(\mathcal{E} = \mathcal{G}\), then for generic \(z \in \mathbb{P}^2\) the restrictions of \(\mathcal{F}\) and \(\mathcal{G}\) to \(\pi^{-1}(z)\) are isomorphic.

**Remark 4.2.** Regarding Item (b) of Proposition 4.1 according to Proposition [2, 3] we always have \(r(\mathcal{E}) \mid (mi)^2\), hence the equality is an extremal case.

4.2. Preliminary results.

**Lemma 4.3.** Let \((\Lambda, q)\) be a non degenerate rank 2 lattice which represents 0, and hence \(\text{disc}(\Lambda) = -d^2\) where \(d\) is a strictly positive integer. Let \(\alpha \in \Lambda\) be primitive isotropic, and complete it to a basis \(\{\alpha, \beta\}\) such that \(q(\beta) \geq 0\). If \(\gamma \in \Lambda\) has strictly negative square (i.e. \(q(\gamma) < 0\)) then

\[
q(\gamma) \leq -\frac{2d}{1 + q(\beta)}.
\] (4.2.1)

**Proof.** There exist integers \(x, y\) such that \(\gamma = x\alpha + y\beta\). Since \(\text{disc}(\Lambda) = -q(\alpha, \beta)^2\) we have \(q(\alpha, \beta) = d\). Thus

\[
q(\gamma) = y(2dx + q(\beta)y).
\]

Since \(q(\gamma) < 0\) and since \(x, y\) are integers, we have

\[
0 < |x|, \quad 0 < |y| \leq |q(\gamma)|, \quad 0 < |2dx + q(\beta)y| \leq |q(\gamma)|.
\]

It follows that

\[
2d|x| - q(\beta)|y| \leq |2dx + q(\beta)y| \leq |q(\gamma)|
\]
because \( d \) and \( q(\beta) \) are non negative. Hence

\[
2d \leq 2|d|x| \leq q(\beta)|y| + |2dx + q(\beta)y| \leq q(\beta)|q(\gamma)| + |q(\gamma)| = (1 + q(\beta))|q(\gamma)|.
\]

Since \( q(\gamma) < 0 \) the above inequality is equivalent to \( 1 \). \( \square \)

**Proposition 4.4.** Let \((A, \theta)\) be a principally polarized abelian surface. Let \( \mathcal{F} \) be a \( \theta \) slope-semistable vector bundle on \( A \) such that \( \Delta(\mathcal{F}) = 0 \). Then we can write

\[
r(\mathcal{F}) = r_0^2 x, \quad c_1(\mathcal{F}) = r_0 b_0 x \theta,
\]

where \( r_0, x, b_0 \) are integers, the first two are positive, and \( \gcd\{r_0, b_0\} = 1 \). If \( \mathcal{F} \) is strictly \( \theta \) slope-semistable, i.e. not slope-stable, there exists such a decomposition with \( x > 1 \).

**Proof.** If \( \mathcal{F} \) is slope-stable, then it is simple semi-homogeneous by Proposition \( \text{A.2} \) and hence we may write \( \text{(4.2.2)} \) with \( x = 1 \) by Proposition \( \text{A.3} \).

Suppose that \( \mathcal{F} \) is strictly \( \theta \) slope-semistable. Hence there exists a destabilizing exact sequence of torsion free sheaves

\[
0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{4.2.3}
\]

with \( \mathcal{G} \) slope-stable. Notice that \( \mathcal{H} \) is locally free because \( \mathcal{H} \) is torsion free.

Let us prove that \( \mathcal{G} \) is simple semi-homogeneous. Since \( \text{(4.2.3)} \) is slope destabilizing, \( 0 < r(\mathcal{G}) < r(\mathcal{F}) \) and \( \sum x \lambda_{\mathcal{G}, \mathcal{F}} \subset \theta = 0 \), where \( \lambda_{\mathcal{G}, \mathcal{F}} \in H^2(A; \mathbb{Z}) \) is defined in \( \text{(2.31)} \). Since \( \mathcal{F} \) is slope-semistable, \( \mathcal{H} \) is slope-semistable. Thus \( \Delta(\mathcal{G}) \geq 0 \) and \( \Delta(\mathcal{H}) \geq 0 \) by Bogomolov. Now look at Equation \( \text{(3.2.1)} \): since \( \sum x \lambda_{\mathcal{G}, \mathcal{F}} \leq 0 \) by Hodge index, we get that \( \Delta(\mathcal{G}) = \Delta(\mathcal{H}) = 0 \) and \( \sum x \lambda_{\mathcal{G}, \mathcal{F}} = 0 \). In particular \( \mathcal{G} \) is simple semi-homogeneous by Proposition \( \text{A.2} \). By Proposition \( \text{A.3} \) we may write

\[
r(\mathcal{G}) = r_0^2, \quad c_1(\mathcal{G}) = r_0 b_0 x \theta,
\]

where \( r_0, b_0 \) are integers, \( r_0 > 0 \) and \( \gcd\{r_0, b_0\} = 1 \).

The equality \( \sum x \lambda_{\mathcal{G}, \mathcal{F}} = 0 \) gives (by Hodge index) that \( \lambda_{\mathcal{G}, \mathcal{F}} = 0 \), i.e.

\[
\frac{c_1(\mathcal{G})}{r(\mathcal{G})} = \frac{c_1(\mathcal{F})}{r(\mathcal{F})},
\]

and the lemma follows. \( \square \)

**Corollary 4.5.** Let \((A, \theta)\) be a principally polarized abelian surface. Let \( \mathcal{F} \) be a \( \theta \) slope-semistable vector bundle on \( A \) such that \( \Delta(\mathcal{F}) = 0 \). If \( r(\mathcal{F}) = r_0^2, \quad c_1(\mathcal{F}) = r_0 b_0 \theta \) where \( r_0, b_0 \) are coprime integers, then \( \mathcal{F} \) is \( \theta \) slope-stable.

**Proof.** By contradiction. Suppose that \( \mathcal{F} \) is not \( \theta \) slope-stable. By Proposition \( \text{A.4} \) we may write \( r(\mathcal{F}) = s_0^2, \quad c_1(\mathcal{F}) = s_0 c_0 x \theta \) where \( s_0, x, c_0 \) are integers (with \( s_0 > 0 \)), \( s_0, c_0 \) are coprime and \( x > 1 \). It follows that \( s_0 b_0 = c_0 r_0 \). Since \( \gcd\{r_0, b_0\} = 1 \) and \( \gcd\{s_0, c_0\} = 1 \), we get that \( r_0 = s_0 \) and hence \( x = 1 \). This is a contradiction. \( \square \)

### 4.3. Proof of Item (1) of Proposition 4.1

First we prove that \( h \) is \( a_0 \)-suitable (see Definition \( \text{3.5} \)). Suppose first that \( \rho(X) = 2 \), i.e. \( H^{1,1}_X(X) = \langle h, f \rangle \), where \( f := \pi^* c_1(\mathcal{O}_P(1)) \). Apply Lemma \( \text{1.3} \) to \( \Lambda := H^{1,1}_X(X) \), \( \alpha = f \) and \( \beta = h \); by \( \text{1.1.1} \) we get that there are no \( \xi \in H^{1,1}_X(X) \) such that \( -a_0 \leq q(\xi) < 0 \). Hence any ample divisor on \( X \) is \( a_0 \)-suitable.

Once we know that \( h \) on \( X \) is \( a_0 \)-suitable if \( \rho(X) = 2 \), it follows that the set of \( [(X, h)] \in \mathcal{M}_0^*(d) \) such that \( h \) is not \( a_0 \)-suitable belongs to the intersection of \( \mathcal{M}_0^*(d) \) with a finite union of Noether-Lefschetz divisors in \( \mathcal{M}_0^* \). In fact suppose that \( h \) is not \( a_0 \)-suitable on \( X \). Then there exists \( \gamma \in H^{1,1}_X(X) \) such that

\[
- a_0 \leq q(\gamma) < 0, \quad q(\gamma, h) > 0, \quad q(\gamma, f) < 0.
\]
Let $B$ be the (finite) index of $\langle h, f \rangle \otimes \langle h, f \rangle^\perp \cap H^{1,1}_2(X)$ in $H^{1,1}_2(X)$. Then

$$\gamma = \frac{\gamma_1}{B} + \frac{\gamma_2}{B}, \quad \gamma_1 \in \langle h, f \rangle, \quad \gamma_2 \in \langle h, f \rangle^\perp. \quad (4.3.2)$$

By the last two inequalities in (4.3.1) we have $q(\gamma_1) < 0$. Hence by the first inequality in (4.3.1) it follows that there exists a positive $M$ independent of $(X, h)$ such that $-M < q(\gamma_2) < 0$. Hence the moduli point of $(X, h)$ belongs to the intersection of $\mathcal{M}'^l_\epsilon(d)\mathcal{A}$ with a finite union of Noether-Lefschetz divisors in $\mathcal{X}'_\epsilon$, as claimed.

We have proved that if $(X, h)$ represents a generic point of $\mathcal{M}'_\epsilon(d)^0$, then $h$ is $a_0$-suitable, and hence $a(\theta)$-suitable because $a(\theta) \leq a_0$. Let $A$ be a generic (smooth) fiber of $\pi$. By Proposition 4.4 the restriction of $\theta$ to $A$ is slope-semistable with respect to the restriction of $h$.

We claim that the hypotheses of Corollary 4.5 are satisfied by $\mathcal{F} := \theta|_A$. In fact $\Delta(\mathcal{F}) = 0$ because $\theta$ is modular, see Lemma 2.4. Moreover the restriction of $h$ to $A$ is a multiple of a principal polarization $\theta$ by Theorem 1.1 in [Wie10]. From the formula $\int_A h^2 = \int_X h^2 - f^2 = 2q(h, f)^2 = 2d^2$ it follows that $h|_A = d\theta$. Hence $r(\mathcal{F}) = |(mi)|^2$ and $c_1(\mathcal{F}) = m \cdot d\theta = |(mi)|^2$. It follows that the hypotheses of Corollary 4.5 are satisfied and hence $\mathcal{F}$ is slope-stable.

4.4. Proof of Item (2) of Proposition 4.1. For $z \in \mathbb{P}^2$ we let $A_z := \pi^{-1}(z)$ and $\mathcal{F}_z := \mathcal{F}_|_{A_z}$, $\mathcal{G}_z := \mathcal{G}_|_{A_z}$. By Item (2) of Proposition 4.1 there exists an open dense $U \subset \mathbb{P}^2$ such that for $z \in U$ the vector bundles $\mathcal{F}_z$ and $\mathcal{G}_z$ are both slope-stable. We claim that if $z \in U$ then $\mathcal{F}_z$ and $\mathcal{G}_z$ are simple semi-homogeneous vector bundles. In fact they are simple because they are slope-stable, and they are semi-homogeneous by Lemma 2.3 and Proposition 4.2. Let $z \in U$. By Theorem 7.11 in [Muk78] the set

$$V_z := \{ [\xi] \in A' \mid \mathcal{F}_z \cong \mathcal{G}_z \otimes \xi \}$$

is non-empty, and hence it has cardinality $r(\mathcal{G})^2$ by Proposition 7.1 op. cit. Clearly $V_z$ is invariant under the monodromy action of $\pi_1(U, z)$. Now notice that $V_z \subset A_z[(mi)|^2]$, because $\mathcal{F}_z$ and $\mathcal{G}_z$ have rank $|mi|^2$ and isomorphic determinants. Hence by Corollary 4.5 we have $V_z = A[mi]$. Thus $\mathcal{F}_z \cong \mathcal{G}_z$.

5. Basic modular sheaves on Hilbert squares of K3’s

5.1. Main result. Let $S$ be a smooth projective surface. Let $X_n(S) \to S^n$ be the blow up of the big diagonal, i.e. the $n$-th isospectral Hilbert scheme of $S$, see Definition 3.2.4 and Proposition 3.4.2 in [Hai01]. The complement of the big diagonal in $S^n$ is identified with a dense open subset $U_n(S)$ of $X_n(S)$, and the natural map $U_n(S) \to S^{[n]}$ extends to a regular map $p: X_n(S) \to S^{[n]}$ (this follows from Proposition 3.4.2 in [Hai01]). Let $\tau: X_n(S) \to S^n$ be the blow up map. We let $q_i: X_n(S) \to S$ be the composition of $\tau$ and the $i$-th projection $S^n \to S$. Given a locally free sheaf $\mathcal{F}$ on $S$, let

$$X_n(\mathcal{F}) := q_1^*(\mathcal{F}) \otimes \ldots \otimes q_n^*(\mathcal{F}).$$

The action of the symmetric group $\mathcal{S}_n$ on $S^n$ by permutation of the factors maps the big diagonal to itself, and hence lifts to an action $\rho_n: \mathcal{S}_n \to \text{Aut}(X_n(S))$. The latter action lifts to a natural action $\rho_n^+\chi$ on $X_n(\mathcal{F})$. There is also a twisted action $\rho_n^+\chi$ on $X_n(\mathcal{F})$. There is also a twisted action $\rho_n^+\chi$ on $X_n(\mathcal{F})$.

Definition 5.1. Let $\mathcal{F}^{[n]} \subset p_*X_n(\mathcal{F})$ be the sheaf of $\mathcal{S}_n$-invariants for $\mathcal{F}^{[n]}$. Below is the main result of the present section.
Proposition 5.2. Let $S$ be a projective $K3$ surface, and let $\mathcal{F}$ be a locally free sheaf on $S$ such that $\chi(\text{End} \mathcal{F}) = 2$. Then $\mathcal{F}[2]^\pm$ is a locally free modular sheaf of rank $r(\mathcal{F})^2$, with

\[
\Delta(\mathcal{F}[2]^\pm) = \frac{r(\mathcal{F}[2]^\pm)(r(\mathcal{F}[2]^\pm) - 1)}{12}c_2(\text{End} \mathcal{F}[2]^\pm),
\]

(5.1.1)

\[
d(\mathcal{F}[2]^\pm) = 5 \cdot \left( \frac{r(\mathcal{F}[2]^\pm)}{2} \right),
\]

(5.1.2)

\[
a(\mathcal{F}[2]^\pm) = \frac{5}{8}r(\mathcal{F})^2 - 1.
\]

(5.1.3)

(Recall that $d(\mathcal{E})$ is defined by the equality in (1.2.2).) Moreover, if $q(c_1(\mathcal{F}[2]^\pm)) = 0$ then the Chern classes of $\mathcal{F}[2]^\pm$ remain of Hodge type for all deformations of $S[2]$ which keep $c_1(\mathcal{F}[2]^\pm)$ of Hodge type.

The proof of Proposition 5.2 is given at the end of the present section.

Remark 5.3. If $S$ is a $K3$ surface, and $\mathcal{F}$ is a locally free sheaf on $S$ such that $\chi(S, \text{End} \mathcal{F}) = 2$, then $\mathcal{F}[2]^\pm$ is not modular, and its Chern classes do not remain of Hodge type for all deformations of $S[2]$ that keep $c_1(\mathcal{F}[2]^\pm)$ of Hodge type.

5.2. Another description of $\mathcal{F}[2]^\pm$. A different definition of the sheaf $\mathcal{F}[2]^-$ was given in [DHOV]. Here we recall that construction and we give the analogous construction of $\mathcal{F}[2]^+$. The natural map $f^\pm: p^*(\mathcal{F}[2]^\pm) \to q^*_x \mathcal{F} \otimes q^*_y \mathcal{F}$ is an isomorphism away from $E$, in particular it is injective because $p^*(\mathcal{F}[2]^\pm)$ is torsion free. In order to write out the cokernel we notice that there are surjective morphisms

\[
qu^*_x \mathcal{F} \otimes q^*_y \mathcal{F} \xrightarrow{\text{ev}} \tau_E^\pm \text{Sym}^2 \mathcal{F}, \quad q^*_x \mathcal{F} \otimes q^*_y \mathcal{F} \xrightarrow{\text{ev}} \tau_E^\pm \Lambda^2 \mathcal{F}
\]

(5.2.1)

obtained by evaluating along $E$ and then projecting onto the symmetric/antisymmetric part of $(q^*_x \mathcal{F} \otimes q^*_y \mathcal{F})|_E = \tau_E^\pm (\mathcal{F} \otimes \mathcal{F})$. Let $\iota: E \hookrightarrow \check{S} \times \check{S}$ be the inclusion.

Proposition 5.4 (See Lemma 4.2 in [DHOV]). The sheaves $\mathcal{F}[2]^\pm$ are locally free of rank $r(\mathcal{F})^2$ and the following are exact sequences:

\[
0 \to p^*(\mathcal{F}[2]^+) \xrightarrow{f^+} q^*_x \mathcal{F} \otimes q^*_y \mathcal{F} \xrightarrow{\text{ev}} \tau_E^\pm (\Lambda^2 \mathcal{F}) \to 0,
\]

(5.2.2)

\[
0 \to p^*(\mathcal{F}[2]^-) \xrightarrow{f^-} q^*_x \mathcal{F} \otimes q^*_y \mathcal{F} \xrightarrow{\text{ev}} \tau_E^\pm \text{Sym}^2 \mathcal{F} \to 0.
\]

(5.2.3)

Proof. Away from $E$ the sheaves $\tau_E^\pm (\Lambda^2 \mathcal{F})$ and $\tau_E^\pm \text{Sym}^2 \mathcal{F}$ are zero and the maps $f^\pm$ are isomorphisms. Hence (5.2.2) and (5.2.3) are exact away from $E$. In particular $\mathcal{F}[2]^\pm$ is locally free of rank $r(\mathcal{F})^2$ away from $E$.

Let $x \in E$ and let $h \in \mathcal{O}_{\check{S} \times \check{S}}$ be a local generator of the ideal of $E$. Let $y = p(x)$ and let $\mathcal{F}(y)$ be the fiber of $\mathcal{F}$ at $y$. The $\pm$ eigenspaces for the action of $\rho_2^+$ on $p^*(q^*_x \mathcal{F} \otimes q^*_y \mathcal{F})_y$ are respectively

\[
\left( \text{Sym}^2 \mathcal{F}(y) \otimes \mathcal{O}_{S[2],y} \right) \oplus \left( \Lambda^2 \mathcal{F}(y) \otimes \mathcal{O}_{S[2],y} \right)
\]

and

\[
\left( \Lambda^2 \mathcal{F}(y) \otimes \mathcal{O}_{S[2],y} \right) \oplus \left( \text{Sym}^2 \mathcal{F}(y) \otimes \mathcal{O}_{S[2],y} \right).
\]

Thus $\mathcal{F}[2]^+_y$ is free of rank $r(\mathcal{F})^2$, and we get that (5.2.2) is exact at $x$. Since the $\pm$ eigenvalues of the action of $\rho_2^+$ are the $\mp$ eigenvalues of the action of $\rho_2^-$ we get also that $\mathcal{F}[2]^-_y$ is free of rank $r(\mathcal{F})^2$ and that (5.2.3) is exact at $x$.}

\footnote{The statement should be true also if $q(c_1(\mathcal{F}[2]^\pm)) = 0$.}
5.3. Preliminaries on $K3^{[n]}$. Let $\mu_\eta : H^2(S) \to H^2(S^{[n]})$ be the composition of the natural symmetrization map $H^2(S) \to H^2(S^{[n]})$ and the pull-back $H^2(S^{[n]}) \to H^2(S^{[n]})$ defined by the Hilbert-Chow map $S^{[n]} \to S^{[n]}$. Let $\Delta_n \subset S^{[n]}$ be the prime divisor parametrizing non reduced subschemes. The class $c(\Delta_n)$ is divisible by 2 in the integral cohomology of $S^{[n]}$; let $\delta_n \in H^1_{\text{et}}(S^{[n]})$ be the unique class such that $2\delta_n = c(\Delta_n)$. We have an orthogonal decomposition for the BBF quadratic form

$$H^2(S^{[n]}, \mathbb{Z}) = \mu_n(H^2(S; \mathbb{Z})) \oplus Z \delta_n.$$ 

Let $q$ be the BBF form of $S^{[n]}$. Then for $\alpha \in H^2(S)$ we have

$$q(\mu_\eta(\alpha)) = \int S \alpha^2, \quad q(\mu_\eta(\alpha), \delta_n) = 0, \quad q(\delta_n) = -2(n-1). \quad (5.3.1)$$

We will deal with $S^{[2]}$. In order to simplify notation we will drop the subscripts of $\delta_2$ and $\mu_2$. The isospectral Hilbert scheme $X_2(S)$ is the blow up of the diagonal in $S \times S$; we will denote it by $\tilde{S} \times S$. Let $E$ be the exceptional divisor of the blow up map $\tau : \tilde{S} \times S \to S^2$, and let $e \in H^2(\tilde{S} \times S)$ its Poincaré dual. We let $\tau_E : E \to S$ be the restriction of $\tau$ to $E$. Let $i \in \text{Aut}(\tilde{S} \times S)$ be the involution lifting the involution of $S^2$ exchanging the factors. Then $S^{[2]}$ is the quotient of $\tilde{S} \times S$ by the group $\langle i \rangle$, and $p : \tilde{S} \times S \to S^{[2]}$ is the quotient map. We recall that the map $q_i : \tilde{S} \times S \to S$ for $i \in \{1, 2\}$ is the composition of $\tau$ and the $i$-th projection $S \times S \to S$. We go through a few formulas that will be needed in the proof that $\mathcal{F}[2]^\pm$ is a modular sheaf. First we notice that

$$p^* (x \mu(c_1(\mathcal{F})) - y \delta) = x(q_1^* c_1(\mathcal{F}) + q_2^* c_1(\mathcal{F})) - ye, \quad (5.3.2)$$

$$p^* c_2(S^{[2]}) = 24(q_1^* \eta + q_2^* \eta) - 3e^2. \quad (5.3.3)$$

In fact $(5.3.2)$ follows directly from the definitions, and $(5.3.3)$ is the last equation on p. 84 of [DV10]. Equation $(5.3.2)$ gives

$$\int_{S^{[2]}} c_2(S^{[2]}) = 828, \quad \int_{S^{[2]}} c_2(S^{[2]}) \sim \alpha^2 = 30q(\alpha), \quad \alpha \in H^2(S^{[2]}). \quad (5.3.4)$$

**Lemma 5.5.** Let $S$ be a K3 surface. Let $\alpha \in H^2(S)$, and let $\alpha^2 = 2m_\alpha \eta$ where $\eta \in H^4(S; \mathbb{Z})$ is the orientation class. Then

$$2(q_1^* \eta \sim q_2^* \alpha + q_1^* \alpha \sim q_2^* \eta) + (q_1^* \alpha + q_2^* \alpha) \sim e^2 = 0, \quad (5.3.5)$$

$$12q_1^* \alpha \sim \alpha \sim e + m_\alpha e^3 = 0. \quad (5.3.6)$$

**Proof.** Since the cohomology of $\tilde{S} \times S$ has no torsion it suffices to check that the cup product of the left hand side of $(5.3.5)$ and $(5.3.6)$ with any class in $H^2(\tilde{S} \times S)$ vanishes. Thus we must take the cup product with $q_i^* \beta$ where $i \in \{1, 2\}$ and $\beta \in H^2(S)$, and with $e$. The easy computations are left to the reader. \hfill $\square$

5.4. Chern classes of $\mathcal{F}[2]^\pm$.

**Proposition 5.6.** Let $S$ be a K3 surface, and let $\mathcal{F}$ be a locally free sheaf of rank $r_0$ on $S$ such that $\chi(S, \text{End} \mathcal{F}) = 2$. Let $h^\pm := \mu(c_1(\mathcal{F})) - r_0 \mp \frac{1}{2} \delta$. \hfill (5.4.1)

Then

$$\text{ch}_0(\mathcal{F}[2]^\pm) = \frac{r_0^2}{2}, \quad \text{ch}_1(\mathcal{F}[2]^\pm) = r_0 h^\pm, \quad \text{ch}_2(\mathcal{F}[2]^\pm) = \frac{1}{2} (h^\pm)^2 - \frac{r_0^2 - 1}{24} c_2(S^{[2]}). \quad (5.4.4)$$
If in addition we assume that \( q(h^\pm) \neq 0 \), then
\[
\begin{align*}
\text{ch}_1(\mathcal{F}[2]^\pm) & = \frac{2q(h^\pm) - 5(r_0^2 - 1)}{12r_0 q(h^\pm)} (h^\pm)^3, \\
\text{ch}_4(\mathcal{F}[2]^\pm) & = \frac{(4q(h^\pm)^2 - 20(r_0^2 - 1)q(h^\pm) + (r_0^2 - 1)(21r_0^2 - 25)}{32r_0^6}.
\end{align*}
\]
(In the last equation we identify \( H^8(S[2]; \mathbb{Z}) \) with \( \mathbb{Z} \) via the orientation class.)

**Proof.** Of course [5.4.2] needs no proof. Let \( \text{ch}_1(\mathcal{F}) = 2m_0 \eta \) where \( \eta \in H^4(S; \mathbb{Z}) \) is the orientation class. Since \( \chi(\text{End} \mathcal{F}) = 2 \), HRR gives that
\[
2r_0 \text{ch}_2(\mathcal{F}) = \text{ch}_1(\mathcal{F})^2 - 2(r_0^2 - 1)\eta = (2m_0 - 2(r_0^2 - 1))\eta.
\]
A straightforward computation shows that
\[
2m_0 = q(h^\pm) + \frac{(r_0^2 + 1)^2}{2}.
\]
Since the pull-back \( p^* : H(S[2]; \mathbb{Z}) \to H^2(S \times S; \mathbb{Z}) \) is injective, we work on \( S \times S \). By \([5.2.2] \) and GRR, we have
\[
p^* \text{ch}(\mathcal{F}[2]^+) = -q^* \text{ch}(\mathcal{F}) q^* \text{ch}(\mathcal{F}) - q^* \left( \frac{1}{2} (r_0^2 - 1) \right) \text{ch}_1(\mathcal{F}) + \frac{1}{2} (q^* \text{ch}_1(\mathcal{F})^2 + q^* \text{ch}_2(\mathcal{F})) -
\]
\[
\frac{1}{2} \sum_{r_0^2 - 1} q^* \text{ch}_1(\mathcal{F}) + \frac{1}{2} \sum_{r_0^2 - 1} q^* \left( (2r_0^2 - 4) \text{ch}_2(\mathcal{F}) + \text{ch}_1(\mathcal{F})^2 \right) \left( e - \frac{1}{2} c^2 \right)^2.
\]
(Cup product is denoted by \( \cdot \) in order to save space.) Equation [5.4.3] for \( \mathcal{F}[2]^+ \) follows at once. Using [5.4.7], we get that
\[
p^* \text{ch}_2(\mathcal{F}[2]^+) = -(r_0^2 - 1)(q^* \text{ch}_1(\mathcal{F}) + \frac{1}{2} (q^* \text{ch}_1(\mathcal{F})^2 + q^* \text{ch}_2(\mathcal{F})) + (5.4.10)
\]
\[
+ q^* \text{ch}_1(\mathcal{F}) + q^* \text{ch}_1(\mathcal{F}) - \frac{r_0 - 1}{2} c \cdot (q^* \text{ch}_1(\mathcal{F}) + q^* \text{ch}_1(\mathcal{F})) + \frac{1}{2} \left( \frac{r_0}{2} \right) c^2.(5.4.11)
\]
By [5.3.2] and [5.3.3], we get that [5.4.4] holds for \( \mathcal{F}[2]^+ \). Using [5.4.10], [5.4.11], and the relations [5.3.3] and [5.3.5], we get that
\[
p^* \text{ch}_3(\mathcal{F}[2]^+) = \frac{2m_0 - (r_0^2 - 1)(3m_2 + 2)}{2m_0} q^* \text{ch}_1(\mathcal{F}) q^* \text{ch}_1(\mathcal{F}) + \left( \frac{r_0 - 1}{2} \right) q^* \text{ch}_1(\mathcal{F}) q^* \text{ch}_1(\mathcal{F})),
\]
\[
p^* \left( m_0 \text{ch}_1(\mathcal{F}) \right) = \frac{3r_0^2 (4m_0 - (r_0^2 - 1)(3m_2 + 2))}{2m_0} q^* \text{ch}_1(\mathcal{F}) q^* \text{ch}_1(\mathcal{F})).
\]
(Notice that \( m_0 \neq 0 \) because of [5.4.7].) Equation [5.4.3] for \( \mathcal{F}[2]^+ \) follows from the two equations above and [5.4.8].

Lastly, [5.4.10] for \( \mathcal{F}[2]^+ \) follows from [5.4.10] and [5.4.8].

The computations for \( \mathcal{F}[2]^- \) are similar. □

**Corollary 5.7.** Keep hypotheses and notation as in Proposition 5.7. Then
\[
\chi(\text{End} \mathcal{F}[2]^\pm) = 0,
\]
where \( \text{End} \mathcal{F}[2]^\pm \subset \text{End} \mathcal{F}[2]^\pm \) is the subsheaf of traceless endomorphisms.

**Proof.** A straightforward consequence of HRR, the formulae of Proposition 5.6 and [5.3.3]. □

5.5. **Proof of Proposition 5.2.** The sheaf \( \mathcal{F}[2]^\pm \) is locally free of rank \( r(\mathcal{F})^2 \) by Proposition 5.1. Equation [5.1.1] holds by Proposition 5.6. It follows that \( \mathcal{F}[2]^\pm \) is modular (see Remark 1.2). Equations [5.1.2] and [5.1.3] follow from [5.1.1] and the second equality in [5.3.3].
6. Basic modular sheaves on the Hilbert square of an elliptic $K3$

6.1. Contents of the section. We will study the vector bundle $\mathcal{F}[2]^+$ for $\mathcal{F}$ a rigid vector bundle on an elliptic $K3$ surface $S$ with Picard number 2 (analogous results hold for $\mathcal{F}[2]^-$). By varying $S$ and $\mathcal{F}$ one gets vector bundles on $S^{[2]}$ with all Chern characters appearing in Equations 1.3.5, 1.3.6, and 1.3.7 or in 1.3.8 and 1.3.9 - see Subsection 6.3. The main result of the section is Theorem 6.6, which gives key properties of $\mathcal{F}[2]^+$. In particular it gives that $\mathcal{F}[2]^+$ extends to all deformations of $S^{[2]}$ that keep $c_1(\mathcal{F}[2]^+)$ of type $(1,1)$.

6.2. Elliptic $K3$ surfaces and stable rigid vector bundles. We recall the notions of Mukai vector and Mukai pairing for a $K3$ surface $S$. If $\mathcal{F}$ is a sheaf over $S$, the Mukai vector of $\mathcal{F}$ is $v(\mathcal{F}) := \text{ch}(\mathcal{F}) \text{Td}(S)^{1/2}$. Moreover the bilinear Mukai pairing $\langle \cdot, \cdot \rangle$ on $H(S)$ has the following property: if $\mathcal{F}, \mathcal{E}$ are sheaves on $S$ then

$$\langle v(\mathcal{F}), v(\mathcal{E}) \rangle = -\chi(\mathcal{F}, \mathcal{E}) := -\sum_{i=0}^{2} (-1)^i \dim \text{Ext}^i(\mathcal{F}, \mathcal{E}).$$

(6.2.1)

Let $S$ be a $K3$ surface with an elliptic fibration $S \to \mathbb{P}^1$; we let $C$ be a fiber of the elliptic fibration. The claim below follows from surjectivity of the period map for $K3$ surfaces.

Claim 6.1. Let $m_0, d_0$ be positive natural numbers. There exist $K3$ surfaces $S$ with an elliptic fibration $S \to \mathbb{P}^1$ such that

$$H^2_{\text{ch}}(S) = \mathbb{Z}[D] \oplus \mathbb{Z}[C], \quad D \cdot D = 2m_0, \quad D \cdot C = d_0.$$ 

(6.2.2)

The result below provides us with stable vector bundles $\mathcal{F}$ on elliptic $K3$ surfaces such that $\mathcal{F}[2]^\pm$ has good properties - see Proposition 6.6.

Proposition 6.2. Let $m_0, r_0, s_0$ be positive integers such that $m_0 + 1 = r_0s_0$. Suppose that $d_0$ is an integer coprime to $r_0$, and that

$$d_0 > \frac{(2m_0 + 1)r_0^2 - 1}{4}.$$ 

(6.2.3)

Let $S$ be an elliptic $K3$ surface as in Claim 6.1. Then there exists a vector bundle $\mathcal{F}$ on $S$ such that the following hold:

1. $v(\mathcal{F}) = (r_0, D, s_0)$,
2. $\chi(\text{End}(\mathcal{F})) = 2$,
3. $\mathcal{F}$ is $L$ slope-stable for any polarization $L$ of $S$,
4. and the restriction of $\mathcal{F}$ to every fiber of the elliptic fibration $S \to \mathbb{P}^1$ is slope-stable.

(Notice that every fiber is irreducible by our assumptions on $\text{NS}(S)$, hence slope-stability of a sheaf on a fiber is well defined, i.e. independent of the choice of a polarization.)

Proof. (1): The Mukai vector $v = (r_0, D, s_0) \in H(S)$ has square $-2$. Let $L_0$ be a polarization of $S$. By Theorem 2.1 in [Kul90] there exists an $L_0$ slope-semistable vector bundle $\mathcal{F}$ on $S$ with $v(\mathcal{F}) = (r_0, D, s_0)$.

(2): $\chi(\text{End}(\mathcal{F})) = 2$ because $v(\mathcal{F})^2 = -2$.

(3): We claim that $\mathcal{F}$ is actually $L_0$ slope-stable, and that it is $L$ slope-stable for any polarization $L$ of $S$. This follows from the well known results on the stability chamber decomposition of $\text{Amp}(S)$ which have been extended to arbitrary HK varieties in Section 6. In fact $v(\mathcal{F})^2 = -2$ and (1.2.1) give that $\Delta(\mathcal{F}) = 2(r_0^2 - 1)$. It follows that $a(\mathcal{F}) = \frac{(2m_0 + 1)r_0^2 - 1}{2}$, and hence by Lemma 6.3 and 6.2.8 there is no $a(\mathcal{F})$ wall. Thus there is a single $a(\mathcal{F})$-chamber.
(4): By Proposition \(6.3\) the restriction of \(\mathcal{F}\) to a generic fiber \(C\) of the elliptic fibration is slope-semistable (because there is a single \(a(\mathcal{F})\)-chamber). Since \(d_0\), which is the degree of \(\mathcal{F}|_C\), is coprime to \(r_0\), which is the rank of \(\mathcal{F}|_C\), it follows that the restriction of \(\mathcal{F}\) to a generic fiber \(C\) is slope-stable. Suppose that there exists a fiber \(C_0\) such that \(\mathcal{F}|_{C_0}\) is not slope-stable. Then \(\mathcal{F}|_{C_0}\) is slope-unstable because \(d_0\) is coprime to \(r_0\). Let \(\mathcal{F}|_{C_0} \rightarrow \mathcal{B}\) be a desemistabilizing quotient, i.e. \(0 < r(\mathcal{B}) < r\), and \(\mu(\mathcal{B}) - \mu(\mathcal{F}|_{C_0}) < 0\). Let \(\mathcal{E}\) be the elementary modification of \(\mathcal{F}\) associated to the quotient \(\mathcal{B}\), i.e. the (torsion free) sheaf on \(S\) fitting into the exact sequence

\[
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow i_{0,*}\mathcal{B} \longrightarrow 0,
\]

where \(i_0: C_0 \hookrightarrow S\) is the inclusion map. Then

\[
v(\mathcal{E})^2 = v(\mathcal{F})^2 + 2r_0 \cdot r(\mathcal{B}) \cdot (\mu(\mathcal{B}) - \mu(\mathcal{F}|_{C_0})) < v(\mathcal{F})^2 = -2.
\]

By (6.2.1) it follows that \(\chi(\mathcal{E}, \mathcal{E}) > 2\). On the other hand, since \(\mathcal{E}\) is isomorphic to \(\mathcal{F}\) outside \(C_0\), the restriction of \(\mathcal{E}\) to a generic elliptic fiber is slope-stable, and this implies that \(\text{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C} \text{Id}_E\). By Serre duality it follows that \(\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) = 1\). The last two facts contradict the inequality \(\chi(\mathcal{E}, \mathcal{E}) > 2\).

\[\square\]

6.3. Dictionary. We show that, given \(e, r_0\) as in Theorem 1.4 we get a vector bundle \(\mathcal{E}\) with Chern character as in Theorem 1.4 by letting \(\mathcal{E} := \mathcal{F}[2]^+\) for suitable \(S\) and \(\mathcal{F}\) as in Proposition 6.2. First we need the result below. The elementary proof is left to the reader.

**Lemma 6.3.** Let \(e, r_0\) be positive natural numbers such that (1.3.4) holds. Let

\[
m_0 := \begin{cases} \frac{e^2}{2} + \frac{(r_0 - 1)^2}{4} & \text{if } r_0 \text{ is odd,} \\ \frac{e^2}{2} + \frac{(r_0 - 1)^2}{4} & \text{if } r_0 \text{ is even.} \end{cases}
\]  

(\(m_0\) is an integer by (1.3.4).) There exists an integer \(s_0\) such that \(m_0 + 1 = r_0s_0\).

Let \(e, r_0\) be positive natural numbers such that (1.3.4) holds, and let \(m_0\) be defined by (6.3.1). Suppose that \(d_0\) is an integer coprime to \(r_0\) such that (6.2.3) holds. Let \(S\) be an elliptic \(K3\) surface as in Claim 6.1 and let \(\mathcal{F}\) be a vector bundle \(\mathcal{F}\) on \(S\) as in Proposition 6.2 (it exists by Lemma 6.3). Let \(\mathcal{E} := \mathcal{F}[2]^+\) and let \(h^+\) be as in (1.4.1). Lastly let

\[
h := \begin{cases} h^+ & \text{if } r_0 \text{ is odd,} \\ 2h^+ & \text{if } r_0 \text{ is even.} \end{cases}
\]

**Proposition 6.4.** Keep hypotheses and notation as above. Then \(h\) is a primitive cohomology class, \(q(h) = e\) and

\[
q(h, H^2(X; \mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{if } r_0 \text{ is odd,} \\ 2\mathbb{Z} & \text{if } r_0 \text{ is odd.} \end{cases}
\]

Moreover the Chern character of \(\mathcal{E}\) is given by Equations (1.3.5), (1.3.6) and (1.3.7) if \(r_0\) is odd, and by Equations (1.3.8), (1.3.9) and (1.3.10) if \(r_0\) is even.

**Proof.** \(h\) is a primitive cohomology class because \(\text{cl}(D)\) is primitive, and \(q(h) = e\) because of Equation (5.4.3). The rest is straightforward. \[\square\]

**Remark 6.5.** In Proposition 6.4 we have set \(\mathcal{E} := \mathcal{F}[2]^+\). One gets an analogous result letting \(\mathcal{E} := \mathcal{F}[2]^-,\) provided \(\frac{(e-1)^2}{4}\) is replaced by \(\frac{(e+1)^2}{4}\) in (6.3.1).
6.4. Deformations of \((S^2, \mathcal{F}[2]^\perp)\). In the present subsection we prove the following result.

**Proposition 6.6.** Let \(S\) be a K3 surface with an elliptic fibration \(S \to \mathbb{P}^1\) as in Claim 6.1, and let \(\mathcal{F}\) be a vector bundle on \(S\) as in Proposition 6.2. Then the following hold:

1. \(H^p(S^2, \text{End}^0\mathcal{F}[2]^\perp) = 0\) for all \(p\).
2. The restriction of \(\mathcal{F}[2]^\perp\) to every fiber of the associated Lagrangian fibration \(\pi: S^2 \to \mathbb{P}^2\) is simple.

The proof is given at the end of the present subsection. The following result is a remarkable consequence of Proposition 6.6.

**Corollary 6.7.** Keep hypotheses as in Proposition 6.6. Then the natural map between deformation spaces \(\text{Def}(S^2, \mathcal{F}[2]^\perp) \to \text{Def}(S^2, \det \mathcal{F}[2]^\perp)\) is smooth.

**Proof.** This follows from Proposition 6.6 and the main result of [IM14]. \(\square\)

**Definition 6.8.** If \(S \to \mathbb{P}^1\) is an elliptically fibered K3 surface, the associated Lagrangian fibration is the composition

\[
S^2 \to S^{(2)} \to (\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2. \tag{6.4.1}
\]

Let \(S\) be as in Proposition 6.6, and let \(\pi: S^{(2)} \to \mathbb{P}^2\) be the associated Lagrangian fibration. Proposition 6.6 will be proved by showing that the restrictions of \(\mathcal{F}[2]^\perp\) to the (scheme theoretic) fibers of \(\pi\) are simple. First we describe the fibers of \(\pi\).

For \(x \in \mathbb{P}^1\) we let \(C_x\) be the fiber over \(x\) of the elliptic fibration \(S \to \mathbb{P}^1\). The set theoretic fibers of \(\pi\) are as follows:

\[
\pi^{-1}(x_1 + x_2) = \begin{cases} 
C_{x_1} \times C_{x_2} & \text{if } x_1 + x_2, \\
C_{x_1}^{(2)} \cup \mathbb{P}(\Theta_S|_{C_{x_1}}) & \text{if } x_1 = x_2 = x.
\end{cases} \tag{6.4.2}
\]

As is easily checked the fiber is reduced if \(x_1 \neq x_2\). On the other hand, the fiber over \(2x\) is not reduced. In order to prove it, we introduce some notation. Let \(V^{[2]} \subset S^{[2]}\) be the prime divisor parametrizing vertical subschemes \(Z\) (i.e. such that the scheme theoretic image \(f(Z)\) is a reduced point), let \(\Delta^{[2]} \subset S^{[2]}\) be the prime divisor parametrizing non reduced subschemes, and let \(D^{(2)} \subset (\mathbb{P}^1)^{(2)}\) be the prime divisor parametrizing multiple 0-cycles \(2x\).

**Proposition 6.9.** Let \(S\) be a K3 surface with an elliptic fibration \(S \to \mathbb{P}^1\) as in Claim 6.1. With notation as above, we have the equality of (Cartier) divisors \(\pi^*(D^{(2)}) = 2V^{[2]} + \Delta^{[2]}\).

**Proof.** By the set theoretic equality \(\pi^{-1}(D^{(2)}) = V^{[2]} \cup \Delta^{[2]}\), there exists positive integers \(a, b\) such that \(\pi^*(D^{(2)}) = aV^{[2]} + b\Delta^{[2]}\). We have a commutative square

\[
\begin{array}{ccc}
S \times S & \overset{\pi}{\longrightarrow} & S^{[2]} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times \mathbb{P}^1 & \overset{\pi}{\longrightarrow} & \mathbb{P}^2.
\end{array} \tag{6.4.3}
\]

Let \(\tilde{V}^{[2]} := p^{-1}(V^{[2]}),\) and let \(\text{Diag}_{\mathbb{P}^1}\) be the diagonal of \(\mathbb{P}^1 \times \mathbb{P}^1\). The proposition follows from the equalities

\[
a\tilde{V}^{[2]} + 2bE = \tilde{p}^* (\pi^*(D^{(2)})) = \tilde{\pi}^* (p^*(D^{(2)})) = \tilde{\pi}^* (2\text{Diag}_{\mathbb{P}^1}) = 2\tilde{V}^{[2]} + 2E.
\]

\(\square\)
Corollary 6.10. Keep hypotheses and notation as in Proposition 6.7. Let $\mathcal{L}$ be the line bundle on $S[2]$ such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_{S[2]}(\Delta[2])$. Let $x \in \mathbb{P}^1$, and let $Z_x = \pi^{-1}(2x)$ be the schematic fiber of $\pi$ over $2x$. Then

$$\mathcal{I}_{Z_x \cap Y \mid \mathbb{P}^2} \cong \mathcal{O}_{C_x}(-\mathcal{I}_x) \otimes \mathcal{L}|_{C_x},$$

(6.4.4)

where $\mathcal{I}_x := \{2p \mid p \in C_x \subset C_x^{(2)} \}$. In particular the restriction homomorphism $\theta_{Z_x} \rightarrow \theta_{Z_x^{red}}$ fits into the exact sequence

$$0 \rightarrow \theta_{C_x}(-\mathcal{I}_x) \otimes \mathcal{L}|_{C_x} \rightarrow \theta_{Z_x} \rightarrow \theta_{Z_x^{red}} \rightarrow 0.$$

(6.4.5)

Proof. It suffices to prove that (6.4.4) holds, because the kernel of the surjection $\theta_{Z_x} \rightarrow \theta_{Z_x^{red}}$ is equal to the left hand side in (6.4.4). By Proposition 6.7 we have

$$\mathcal{I}_{Z_x \cap Y \mid \mathbb{P}^2} \cong \mathcal{O}_{S[2]}(-V[2] - \Delta[2]|_{C_x}^{(2)}).$$

(6.4.6)

We have

$$\theta_{S[2]}(-\Delta[2]|_{C_x}^{(2)}) \cong \mathcal{O}_{C_x}(-\mathcal{I}_x).$$

(6.4.7)

On the other hand Proposition 6.9 gives that

$$\theta_{S[2]}(2V[2]) \cong \theta_{S[2]}(\pi^* D[2]) \cong \mu(2c_1(\mathcal{O}_{\mathbb{P}^2}(1))) \otimes \mathcal{L}^{-2}.$$  

Since $H^1(S[2]; \mathcal{O}) = 0$, it follows that

$$\theta_{S[2]}(V[2]) \cong \mu(c_1(\mathcal{O}_{\mathbb{P}^2}(1))) \otimes \mathcal{L}^{-1}.$$  

Since the restriction of $\mu(c_1(\mathcal{O}_{\mathbb{P}^2}(1)))$ to $C_x^{(2)}$ is trivial, we get that

$$\theta_{S[2]}(-V[2]|_{C_x}^{(2)}) \cong \mathcal{L}|_{C_x}.$$  

(6.4.8)

Hence (6.4.4) follows from (6.4.6), (6.4.7), and (6.4.8).

For $x \in \mathbb{P}^1$ let $\mathcal{F}_x := \mathcal{F}|_{C_x}$. If $x_1 + x_2 \in \mathbb{P}^1$, then the restriction of $\mathcal{F}$ to $\pi^{-1}(x_1 + x_2)$ is equal to $\mathcal{F}_{x_1} \boxtimes \mathcal{F}_{x_2}$. Thus we need the following result.

Proposition 6.11. Let $C_i$ be polarized irreducible curves, and let $D$ be an ample divisor on $Y := C_1 \times C_2$ such that

$$c_1(\mathcal{O}_Y(D)) = m_1 \rho_D^* \mathcal{O}_{Y_1}(p_1) + m_2 \rho_D^* \mathcal{O}_{Y_2}(p_2),$$

(6.4.9)

where $\rho_i : Y \rightarrow C_i$ is the projection, and $p_i \in C_i$. Let $\mathcal{V}_i$ be a slope-stable vector bundle on $C_i$, for $i \in \{1, 2\}$. Then $\mathcal{V}_1 \boxtimes \mathcal{V}_2$ is $D$ slope-stable.

Proof. By contradiction. Suppose the contrary. Hence exists an injection $\alpha : \mathcal{E} \rightarrow \mathcal{V}_1 \boxtimes \mathcal{V}_2$ with torsion free cokernel such that $0 < r(\mathcal{E}) < r(\mathcal{V}_1 \boxtimes \mathcal{V}_2)$ and

$$\mu_D(\mathcal{E}) < \mu_D(\mathcal{V}_1 \boxtimes \mathcal{V}_2).$$

(6.4.10)

Let $p_i \in C_i$ be generic; the restrictions of $\alpha$ to $\{p_1\} \times C_2$ and $C_1 \times \{p_2\}$ are injective maps of vector bundles (that is, injective on fibers). We have

$$\mu_D(\mathcal{E}) = m_2 \mu(\mathcal{E}|_{\{p_1\} \times C_2}) + m_1 \mu(\mathcal{E}|_{C_1 \times \{p_2\}}).$$

(6.4.11)

On the other hand

$$\mu_D(\mathcal{V}_1 \boxtimes \mathcal{V}_2) = m_2 \mu(\mathcal{V}_1) + m_1 \mu(\mathcal{V}_2).$$

(6.4.12)

Since the restrictions of $\mathcal{V}_1 \boxtimes \mathcal{V}_2$ to $C_1 \times \{p_2\}$ and $\{p_1\} \times C_2$ are isomorphic to the polystable vector bundles $\mathcal{V}_1 \boxtimes \mathcal{C}_{\mathbb{P}^2}$ and $\mathcal{V}_2$ respectively, it follows from $\mu(\mathcal{C}_{\mathbb{P}^2})$. It follows from $\mu(\mathcal{C}_{\mathbb{P}^2})$. It follows from $\mu(\mathcal{C}_{\mathbb{P}^2})$ and $\mu(\mathcal{C}_{\mathbb{P}^2})$ that $\mu(\mathcal{E}|_{\{p_1\} \times C_2}) = \mu(\mathcal{V}_1)$ and $\mu(\mathcal{E}|_{C_1 \times \{p_2\}}) = \mu(\mathcal{V}_2)$. In turn, these equalities give that there exist vector subspaces $0 + A \subset \mathcal{C}(\mathcal{V}_1)$ and $0 + B \subset \mathcal{C}(\mathcal{V}_2)$ such that $\mathcal{E}|_{\{p_1\} \times C_2} = \mathcal{V}_1 \boxtimes \mathcal{C}(\mathcal{V}_1)$ and $\mathcal{E}|_{C_1 \times \{p_2\}} = B \boxtimes \mathcal{V}_2$. Therefore, it follows that $\mathcal{E} \boxtimes \mathcal{V}_2$ is a contradiction.
Proposition 6.12. Let $S$ be a K3 surface with an elliptic fibration $S \to \mathbb{P}^1$ as in Claim 6.21 and let $\mathcal{F}$ be a vector bundle on $S$ as in Proposition 6.2. If $x_1 + x_2$ the restriction of $\mathcal{F}[2]_{\pi^{-1}(x_1 + x_2)}$ to $\pi^{-1}(x_1 + x_2) = C_{x_1} \times C_{x_2}$ is slope-stable for any product polarization. In particular it is a simple sheaf.

Proof. We have

$$\mathcal{F}[2]_{\pi^{-1}(x_1 + x_2)} \cong \mathcal{F}_{x_1} \boxtimes \mathcal{F}_{x_2}.$$ By Proposition 6.2 both $\mathcal{F}_{x_1}$ and $\mathcal{F}_{x_2}$ are slope-stable. Hence the proposition follows from Proposition 6.11. □

For $x \in \mathbb{P}^1$ let $\Delta^2_{[x]} := \mathbb{P}(\Theta_{S|[C_x]}) \subset \Delta$, and let $\tau_x : \Delta^2_{[x]} \to C_x$ be the restriction of the Hilbert-Chow map.

Lemma 6.13. Let $S$ be a K3 surface with an elliptic fibration $S \to \mathbb{P}^1$, and let $\mathcal{F}$ be a vector bundle on $S$ as in Proposition 6.2. Let $x$ be a regular value of the elliptic fibration. Then

$$\mathcal{F}[2]_{\Delta^2_{[x]}^+} \cong \left(\Theta_{\Delta^2_{[x]/C_x}} \otimes \tau_x^* \text{Sym}^2 \mathcal{F}_x\right) \oplus \tau_x^*(\mathfrak{H}^2_{\mathcal{F}}), \quad (6.4.13)$$

and the space of traceless endomorphisms of $\mathcal{F}[2]_{\Delta^2_{[x]}^+}$ has dimension 1.

Proof. The proofs for $\mathcal{F}[2]_{\Delta^2_{[x]}^\pm}$ are similar. We provide the proof for $\mathcal{F}[2]_{\Delta^2_{[x]}^+}$. Restricting the defining exact sequence (6.2.23) to $E \cong \Delta^2_{[x]}$, we get the exact sequence

$$0 \to \tau_x^*(\text{Sym}^2 \mathcal{F}_x) \otimes \Theta_{\Delta^2_{[x]/C_x}} \to \mathcal{F}[2]_{\Delta^2_{[x]}^+} \to \tau_x^*(\mathfrak{H}^2_{\mathcal{F}}) \to 0,$$

where $\tau_{\Delta^2_{[x]}} \to S$ maps a non reduced subscheme to its support (thus $\tau_{\Delta^2_{[x]}}$ may be identified with the restriction of $\pi$ to $\Delta^2_{[x]}$). Restricting to $\Delta^2_{[x]}$ the exact sequence in (6.4.14), we get the exact sequence

$$0 \to \tau_x^*(\text{Sym}^2 \mathcal{F}_x) \otimes \theta_{\Delta^2_{[x]/C_x}} \to \mathcal{F}[2]_{\Delta^2_{[x]}^+} \to \tau_x^*(\mathfrak{H}^2_{\mathcal{F}}) \to 0. \quad (6.4.16)$$

We claim that the above exact sequence splits. The extension class belongs to

$$H^1(\Delta^2_{[x]}, \tau_x^*(\text{Sym}^2 \mathcal{F}_x \otimes \mathfrak{H}^2_{\mathcal{F}}) \otimes \Theta_{\Delta^2_{[x]/C_x}}). \quad (6.4.17)$$

We compute the above cohomology group via the Leray spectral sequence of $\tau_x$. Since $(\text{Sym}^2 \mathcal{F}_x \otimes \mathfrak{H}^2_{\mathcal{F}}) \otimes R^1\tau_{\Delta^2_{[x]/C_x}}(\Theta_{\Delta^2_{[x]/C_x}}) = 0$, it suffices to show that

$$H^1(C_x, (\text{Sym}^2 \mathcal{F}_x \otimes \mathfrak{H}^2_{\mathcal{F}}) \otimes \tau_x, \Theta_{\Delta^2_{[x]/C_x}}) = H^1(C_x, (\text{Sym}^2 \mathcal{F}_x \otimes \mathfrak{H}^2_{\mathcal{F}}) \otimes \Theta_{\Delta^2_{[x]/C_x}}) = 0. \quad (6.4.18)$$

Since $C_x$ is smooth, the sheaf $\text{End}^a(\Theta_{S|[C_x]})$ has a filtration with associated graded the direct sum of three copies of $\Theta_{S|[C_x]}$. Hence the vector bundle appearing in (6.4.18) has a filtration whose associated graded is the direct sum of three copies of $\text{Sym}^3 \mathcal{F}_x \otimes \mathfrak{H}^2_{\mathcal{F}}$. The latter vector bundle has no nonzero section because it is slope-stable.
of slope 0. By HRR it follows that

\[ H^0(C_x, \tau^\bullet_x(Sym^2 \mathcal{F}_x \otimes \bigwedge^2 \mathcal{F}_x^\vee) \otimes \text{End}^0(\Theta_{S(C_x)})) = \]

\[ = H^1(C_x, \tau^\bullet_x(Sym^2 \mathcal{F}_x \otimes \bigwedge^2 \mathcal{F}_x^\vee) \otimes \text{End}^0(\Theta_{S(C_x)})) = 0. \]

Hence the extension group in (6.4.17) vanishes.

The result about traceless endomorphisms of \( \mathcal{F}[2]^\pm \) follows from the direct sum decomposition in (6.4.13) and the vanishing in (6.4.18).

\[ \square \]

**Lemma 6.14.** Let \( S \) be a K3 surface with an elliptic fibration \( S \to \mathbb{P}^1 \) as in Claim 6.1, and let \( \mathcal{F} \) be a vector bundle on \( S \) as in Proposition 6.2. Let \( x \) be a regular value of the elliptic fibration. Let \( \pi_x : C^2_x \to S^{[2]} \) be composition of the quotient map \( C^2_x \to C^{(2)}_x \) and the inclusion \( C^{(2)}_x \to S^{[2]} \). Then \( \pi_x^* \mathcal{F}[2]^\pm \) is simple, and hence the restriction of \( \mathcal{F}[2]^\pm \) to \( C^{(2)}_x \) is simple.

**Proof.** The proofs for \( \mathcal{F}[2]^\pm \) are similar. We give the proof for \( \mathcal{F}[2]^+ \).

Let \( q_x, r_x : C^2_x \to C_x \) be the two projections, and let \( \text{Diag}_x \subset C^2_x \) be the diagonal. By the exact sequence in (6.2.23), we have the following exact sequence

\[ 0 \to \pi_x^* \mathcal{F}[2]^+ \to q_x^* \mathcal{F}_x \otimes r_x^* \mathcal{F}_x \overset{\text{ev}^+}{\to} \text{Sym}^2 \mathcal{F}_x \mid \text{Diag}_x \to 0. \]

It follows that we have the following exact sequence

\[ 0 \to q_x^* \mathcal{F}_x \otimes r_x^* \mathcal{F}_x \otimes \Theta_{C^2_x}(-\text{Diag}_x) \overset{\lambda}{\to} \pi_x^* \mathcal{F}[2]^+ \overset{\text{ev}^-}{\to} \bigwedge^2 \mathcal{F}_x \mid \text{Diag}_x \to 0. \]

Notice that the restriction of \( \lambda \) to \( \text{Diag}_x \) is identified with the natural map \( \text{ev}^+ : \mathcal{F}_x \otimes \mathcal{F}_x \to \text{Sym}^2 \mathcal{F}_x \), in particular its image is \( \text{Sym}^2 \mathcal{F}_x \) (we identify \( \text{Diag}_x \) and \( C_x \)). Equivalently, the restriction to \( \text{Diag}_x \) of the exact sequence in (6.4.20) gives rise to the exact sequence

\[ 0 \to \text{Sym}^2 \mathcal{F}_x \to \pi_x^* \mathcal{F}[2]^+ \overset{\text{ev}^-}{\to} \bigwedge^2 \mathcal{F}_x \to 0, \]

which is split by Lemma 6.13. Now let \( \varphi \) be an endomorphism of \( \pi_x^* \mathcal{F}[2]^+ \). The restriction of \( \varphi \) to \( \text{Diag}_x \) preserves the exact sequence in (6.4.21) because the vector bundles \( \text{Sym}^2 \mathcal{F}_x, \bigwedge^2 \mathcal{F}_x \) are slope-stable of equal slopes. It follows that \( \varphi \) induces an endomorphism of the kernel of \( \text{ev}^- \), i.e. of \( q_x^* \mathcal{F}_x \otimes r_x^* \mathcal{F}_x \otimes \Theta_{C^2_x}(-\text{Diag}_x) \). Since \( \mathcal{F}_x \) is simple, the latter is a simple sheaf. It follows that \( \varphi \) is a scalar. \( \square \)

**Proposition 6.15.** Let \( S \) be a K3 surface with an elliptic fibration \( S \to \mathbb{P}^1 \) as in Claim 6.1, and let \( \mathcal{F} \) be a vector bundle on \( S \) as in Proposition 6.2. Let \( x \) be a regular value of the elliptic fibration. Then the restriction of \( \mathcal{F}[2]^\pm \) to the scheme theoretic fiber \( \pi^{-1}(2x) \) is a simple sheaf.

**Proof.** Let \( Z_x := \pi^{-1}(2x) \) be the scheme theoretic fiber. By Corollary 6.10 we have an exact sequence

\[ 0 \to \text{End}^0(\mathcal{F})|_{C^{(2)}_x} \otimes \Theta_{C^{(2)}_x}(-\Xi_x) \otimes \left( \mathcal{L}_{|C^{(2)}_x} \right) \to \text{End}^0(\mathcal{F})|_{Z_x} \to \text{End}^0(\mathcal{F})|_{Z_x^{red}} \to 0. \]

Taking global sections, we get an isomorphism

\[ H^0(C^{(2)}_x, \text{End}^0(\mathcal{F})|_{C^{(2)}_x} \otimes \Theta_{C^{(2)}_x}(-\Xi_x) \otimes \left( \mathcal{L}_{|C^{(2)}_x} \right)) \cong H^0(C^{(2)}_x, \text{End}^0(\mathcal{F})|_{Z_x}). \]

because of Lemmas 6.13 and 6.14. On the other hand, since

\[ p_x^* \left( \Theta_{C^{(2)}_x}(-\Xi_x) \otimes \left( \mathcal{L}_{|C^{(2)}_x} \right) \right) \cong \Theta_{C^2_x}(-\text{Diag}_x), \]
we have an embedding
\[ H^0(C^2, \text{End}^0(\mathcal{F}|_{C^2}) \otimes \mathcal{O}_{C^2}(\Xi_x)) \hookrightarrow H^0(C^2, \text{End}^0(p_x^*\mathcal{F}))(\text{Diag}_x), \]
and the latter vanishes by Lemma 6.14.

**Proof of Item (1) of Proposition 6.6.** By Serre duality
\[ h^p(\text{End}^0(\mathcal{F}|[2]^\pm)) = h^{4-p}(\text{End}^0(\mathcal{F}[2]^\pm)), \]
and by Corollary 5.7 we have \( \chi(\text{End}^0(\mathcal{F}|[2]^\pm)) = 0 \). Hence it suffices to prove that \( h^p(\text{End}^0(\mathcal{F}[2]^\pm)) = 0 \) for \( p \in \{0, 1\} \). By the Leray spectral sequence for \( \pi: S^2 \rightarrow \mathbb{P}^2 \), it suffices to show that
\[ R^p\pi_*\text{End}^0(\mathcal{F}[2]^\pm) = 0, \quad p \in \{0, 1\}. \]
(6.4.25)

Let \( B \subset \mathbb{P}^2 \) be the set parametrizing \( 2x \in (\mathbb{P}^2)^2 \), where \( x \) is a critical value of the elliptic fibration \( S \rightarrow \mathbb{P}^1 \). Let \( (x_1 + x_2) \in (\mathbb{P}^2 \setminus B) \). Let us prove that
\[ h^p(\pi^{-1}(x_1 + x_2), \text{End}^0(\mathcal{F}[2]^\pm)_{|\pi^{-1}(x_1 + x_2)}) = 0 \quad \forall p. \]
(6.4.26)

To see this, first notice that since \( \pi^{-1}(x_1 + x_2) \) is a local complete intersection with trivial dualizing sheaf, and \( \chi(\pi^{-1}(x_1 + x_2), \text{End}^0(\mathcal{F}[2]^\pm)_{|\pi^{-1}(x_1 + x_2)}) = 0 \), it suffices to check that (6.4.26) holds for \( p = 0 \).

If \( x_1 + x_2 \), then (6.4.26) holds for \( p = 0 \) by Proposition 6.12. If \( x_1 = x_2 = x \), and \( x \) is a regular value of the elliptic fibration, then (6.4.26) holds for \( p = 0 \) by Proposition 6.13. This proves that (6.4.26) holds for all \( (x_1 + x_2) \in (\mathbb{P}^2 \setminus B) \). Then (6.4.26) follows by Proposition 2.26 in Muk87, because \( B \) is a finite set.

**Proof of Item (2) of Proposition 6.6.** Suppose that there exists \( x_1 + x_2 \in \mathbb{P}^2 \) such that the restriction of \( \mathcal{F}[2]^\pm \) to \( \pi^{-1}(x_1 + x_2) \) is not simple. Then \( x_1 + x_2 = 2x \), where \( x \) is a critical value of the elliptic fibration. It follows (since the fibers of \( x \) are surfaces) that \( R^2\pi_*\text{End}^0(\mathcal{F}[2]^\pm) \) is a non zero Artinian sheaf. By the Leray spectral sequence for \( \pi \) it follows that \( H^2(S^2, \mathcal{F}[2]^\pm) \neq 0 \). This contradicts Proposition 6.6.

## 7. Proof Theorem 1.4 and Corollary 1.7

### 7.1. Summary.

In Subsection 7.2 we prove that if \( d > 0 \) and \( [(X,h)] \in \mathcal{A}_i^c(d) \) is generic, there exists an \( h \) slope-stable vector bundle \( \mathcal{E} \) on \( X \) with the relevant \( c_0, c_1, c_2 \) which has good properties, namely its restriction to Lagrangian fibers is slope-stable (actually slope-stable on the normalization), with the possible exception of a finite set of fibers. We also pin down one component of the relative moduli space of slope-stable vector bundles on polarized HK’s parametrized by \( \mathcal{X}_e^i \) with given \( c_0, c_1, c_2 \) which dominates the moduli space \( \mathcal{X}_e^i \).

In Subsection 7.3 we prove that if \( [(X,h)] \in \mathcal{A}_i^c(d) \) is as above, there is a single \( h \) slope-stable vector bundle with the relevant Chern character.

Theorem 1.4 follows from the above results and density of the union of the Noether-Lefschetz divisors in \( \mathcal{X}_e^i \), while Corollary 1.7 follows from Theorem 1.4 and known results on deformations of vector bundles. The proofs are in Subsection 7.4.

### 7.2. Good vector bundles over Lagrangian HK’s.

Below is the first main result of the present subsection.

**Proposition 7.1.** Let \( r_0 \geq 2 \). Suppose that \( i \equiv r_0 \pmod{2} \), that (1.3.4) holds, that \( e / 2d \) and that
\[ d > \frac{5}{16} r_0^2(r_0^2 - 1)(e + 1). \]
(7.2.1)
(Notice that with these assumptions, the hypotheses of Proposition 4.2 hold. We let \( \pi : X \to \mathbb{P}^2 \) be the associated Lagrangian fibration.) If \( [(X, H)] \in \mathcal{N}^i(d) \) is generic there exists an h slope-stable vector bundle \( \mathcal{E} \) on \( X \) such that

1. the Chern character of \( \mathcal{E} \) is given by Equations (1.3.5), (1.3.6) and (1.3.7) if \( r_0 \) is odd, and by Equations (1.3.8), (1.3.9) and (1.3.10) if \( r_0 \) is even,

2. and, except possibly for a finite set of \( z \in \mathbb{P}^2 \), the restriction of \( \mathcal{E} \) to \( \pi^{-1}(z) \) is slope-stable (for the restricted polarization), and also its pull-back to the normalization of \( \pi^{-1}(z) \) (for the pull-back of the restricted polarization).

The proof of Proposition 7.1 is given at the end of the subsection.

Let \( S \) be an elliptic \( K3 \) surface as in Subsection 6.3 and let us adopt the notation of that subsection. Let \( X_0 = S^{[2]} \), and let \( \mathcal{E}_0 := \mathcal{F}[2]^* \) be the vector bundle on \( X_0 \) of loc. cit. Let \( h_0 := h \), where \( h \) is given by (6.3.2). Let \( C \subset S \) be a fiber of the elliptic fibration and let \( f_0 := \mu(\text{cl}(C)) \). Lastly let

\[
d := \begin{cases} d_0 & \text{if } r_0 = 1 \pmod{2}, \\ 2d_0 & \text{if } r_0 = 0 \pmod{2}. \end{cases}
\]

(7.2.2)

Then the sublattice \( \langle f_0, h_0 \rangle \subset H^{1,1}_\mathbb{Z}(X_0) \) is saturated and

\[
q(f_0) = 0, \quad q(h_0, f_0) = d, \quad q(h_0) = e.
\]

(7.2.3)

Let \( \tau_0 : X_0 \to \mathbb{P}^2 \) be the Lagrangian fibration associated to the elliptic fibration of \( S \), see Definition 6.8. Notice that \( f_0 = c_1(\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1)) \).

Let \( \varphi : \mathcal{X} \to B \) be an analytic representative of the deformations space of \((X,\langle h_0, f_0 \rangle)\) i.e. deformations of \( X_0 \) that keep \( h_0 \) and \( f_0 \) of Hodge type. We assume that \( B \) is contractible. Let \( 0 \in B \) the base point, in particular \( X_0 \) is isomorphic to \( \varphi^{-1}(0) \). For \( b \in B \) we let \( X_b := \varphi^{-1}(b) \). If \( B \) is small enough, then by Proposition 6.6 and Corollary 6.7 the vector bundle \( \mathcal{E}_0 \) on \( X_0 \) deforms to a vector bundle \( \mathcal{E}_b \) on \( X_b \) (unique up to isomorphism because \( H^1(X_0, \text{End}^0 \mathcal{E}_0) = 0 \)). Notice that \( \langle h_0, f_0 \rangle \) deforms by Gauss-Manin parallel transport to a saturated sublattice

\[
\Lambda_b := \langle h_b, f_b \rangle \subset H^{1,1}_\mathbb{Z}(X_b).
\]

(7.2.4)

Possibly after shrinking \( B \) around \( 0 \) there exists a map \( \pi : \mathcal{X} \to \mathbb{P}^2 \) which restricts to a Lagrangian fibration on every \( X_b \), and is equal to \( \tau_0 \) on \( X_0 \). We let \( \pi_b \) be the restriction of \( \pi \) to \( X_b \). Notice that the fiber of \( \varphi \times \pi : \mathcal{X} \to B \times \mathbb{P}^2 \) over \((b, z)\) is the Lagrangian fiber of \( X_b \to \mathbb{P}^2 \) over \( z \); we denote it by \( X_{b,z} \). Of course \( f_b = c_1(\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1)) \).

**Proposition 7.2.** With the hypotheses of Proposition 7.1 the following holds.

For \( b \in B \) outside a proper analytic subset \( h_0 \) is ample and hence \([(X_b, h_0)] \in \mathcal{N}^i(d) \) where \( i \equiv r_0 \pmod{2} \). Moreover \( \mathcal{E}_0 \) is \( h_b \) slope-stable and Items (1), (2) of Proposition 7.1 hold for \( \mathcal{E} = \mathcal{E}_b \).

**Proof.** Since \( r_0 > 2 \), Inequality (7.2.1) implies that \( d > 10(e + 1) \). Hence the hypotheses of Proposition 6.2 hold. In the proof of that proposition we showed that \( h_0 \) is ample for \( b \) outside a proper analytic subset of \( B \).

Next we prove that \( h_b \) is \( a(\mathcal{E}_b) \)-suitable, see Definition 5.5. We claim that the hypotheses of Proposition 6.4 hold with \( a_0 = a(\mathcal{E}_b) \). This is clear for all the hypotheses, except perhaps for the inequality in (4.1.1). By Proposition 6.4 we have \( a(\mathcal{E}_b) = \frac{3}{2} \rho_0 (\rho_0^2 - 1) \), and hence the inequality in (4.1.1) follows from (7.2.1).

Since \( h_b \) is \( a(\mathcal{E}_b) \)-suitable, in order to prove that \( \mathcal{E}_b \) is \( h_b \) slope-stable it suffices to show that the restriction to a generic fiber of the Lagrangian fibration is slope-stable, see Proposition 5.6. This is true for \( \mathcal{E}_0 \) by Proposition 6.12. By openness of slope-stability, it follows that it is true also for \( b \in B \) outside a proper analytic subset.
Next we prove that Items (1) and (2) of Proposition 7.1 hold for $\mathcal{E} = \mathcal{E}_b$.

Item (1) holds by Proposition 6.4.

Let us prove that Item (2) holds. First we notice that for $b \in B$ outside a proper closed analytic subset the restriction of $\mathcal{E}_b$ to every Lagrangian fiber is simple. In fact this holds for $b = 0$ by Proposition 6.6 and hence the assertion we made holds by openness of "simplicity". Let us prove that for $b \in B$ outside a proper closed analytic subset the restriction of $\mathcal{E}_b$ to a smooth Lagrangian fiber is slope-stable.

By Proposition 6.12 the restriction of $\mathcal{E}_b$ to a generic Lagrangian fiber is slope-stable, and hence the restriction of $\mathcal{E}_b$ (for $b \in B$ outside...) to a generic smooth Lagrangian fiber is slope-stable by openness of slope-stability. By Proposition A.2 we get that the restriction of $\mathcal{E}_b$ to any smooth Lagrangian fiber is simple semi-homogeneous (note: the fact that the restriction is simple is crucial). By Proposition 6.16 in [Muk78] the restriction of $\mathcal{E}_b$ is simple semi-homogeneous. It follows that the restriction to any smooth Lagrangian fiber is simple semi-homogeneous (note: the fact that the restriction is simple is crucial).

By Proposition 6.12 in [Muk78] the restriction of $\mathcal{E}_b$ is simple semi-homogeneous. It follows that the restriction to any smooth Lagrangian fiber is simple semi-homogeneous (note: the fact that the restriction is simple is crucial). By Corollary 4.5 it follows that it is actually slope-stable.

Next we claim that the restriction of $\mathcal{E}_b$ (for $b \in B$ outside...) to a generic singular Lagrangian fiber is slope-stable, except possibly for a finite set of singular fibers. The singular Lagrangian fibers of $X_b$ are parametrized by the discriminant curve of $\pi_b$, which, for $b \in B$ outside a proper closed analytic subset, is the dual curve of a generic sextic plane curve, see Proposition 5.4. On the other hand the discriminant curve of $\pi_0$ is the union of 24 lines (each corresponding to a critical value of the elliptic fibration) and a conic (the "diagonal"). The restriction of $\mathcal{E}_b$ to the Lagrangian surface parametrized by a generic point of one of the lines is slope-stable by Proposition 6.12. By openness of slope-stability, it follows that the locus of singular Lagrangian fibers on which $\mathcal{E}_b$ restricts to a slope-stable vector bundle is non empty (for $b \in B$ outside...). Since (for $b \in B$ outside...) the discriminant curve is irreducible, this proves that, with the possible exception of a finite set of singular fibers, the restriction of $\mathcal{E}_b$ (for $b \in B$ outside...) to a generic singular Lagrangian fiber is slope-stable.

In order to finish proving that Item (2) holds, we must show that, with the possible exception of a finite set of singular fibers, the pull-back of $\mathcal{E}_b$ to the normalization of $\pi^{-1}(z)$ is slope-stable. By the arguments given above it suffices to look at the vector bundle $\mathcal{E}_b$ on $X_b$. More precisely it suffices to check that the stated property holds for $z$ a generic point of the union of 24 lines mentioned above, i.e. such that $\pi^{-1}(z) = C_x \times C_y$, where $C_x, C_y$ are fibers of the elliptic fibration (see Claim 6.1), with $C_x$ smooth and $C_y$ singular, i.e. with a single singular point, in fact a node. Hence the normalization is $\nu: C_x \times \mathbb{P}^1 \to C_x \times C_y$. Let us prove that $\nu^*(\mathcal{E}_{0|C_x \times C_y})$ is slope-stable for any polarization. First we notice that the restriction of $\nu^*(\mathcal{E}_{0|C_x \times C_y})$ to any fiber of the fibration $C_x \times \mathbb{P}^1 \to \mathbb{P}^1$ is slope-stable because $\mathcal{E}_0 = \mathcal{F}[2]^1$ where $\mathcal{F}$ is as in Proposition 6.2. It follows that $\nu^*(\mathcal{E}_{0|C_x \times C_y})$ is slope-stable for polarizations of the form $NC_x \times \{q\} + \{p\} \times \mathbb{P}^1$ for $N \gg 0$. Hence if there exists a polarization for which $\nu^*(\mathcal{E}_{0|C_x \times C_y})$ is not slope-stable, then there exists one for which it is strictly slope-semistable. But this leads to a contradiction because $\Delta(\nu^*(\mathcal{E}_{0|C_x \times C_y})) = 0$, see Equation (4.3.2).

\[\square\]

Proof of Proposition 7.1. Let $X \to T^1_e$ and $X \to T^2_e$ be complete families of polarized HK's of Type $K3^{[2]}$ such that $1.3.2$, respectively $1.3.3$, holds - e.g. the families parametrized by the relevant open subsets of suitable Hilbert schemes. We may, and will, assume that $T^2_e$ is irreducible. For $t \in T^2_e$ we let $(X_t, h_t)$ be the
corresponding polarized HK of Type $K3^{[2]}$. Let $m: T^i_\epsilon \rightarrow \mathcal{X}^i_{\epsilon}$ be the moduli map, which sends $t$ to $[(X_t, H_t)]$.

By fundamental results of Gieseker and Maruyama there exists a map of schemes $f: \mathcal{M}^i_\epsilon (r_0) \rightarrow T^i_\epsilon$ such that for every $t \in T^i_\epsilon$ the (scheme theoretic) fiber $f^{-1}(t)$ is isomorphic to the (coarse) moduli space of $h_t$ slope-stable vector bundles $\mathcal{E}$ on $X_t$ with Chern character given by Equations $(1.3.5)$, $(1.3.6)$ and $(1.3.7)$ if $r_0$ is odd, and by Equations $(1.3.8)$, $(1.3.9)$ and $(1.3.10)$ if $r_0$ is even. By Proposition $7.2$, the image of $f: \mathcal{M}^i_\epsilon (r_0) \rightarrow T^i_\epsilon$ contains a non empty subset of $m^{-1}(\mathcal{X}^i_{\epsilon}(d))$ which is open in the classical topology. Since the image of $f$ is a Zariski-constructible set, it follows that it contains a Zariski open dense subset of $m^{-1}(\mathcal{X}^i_{\epsilon}(d))$. By openness of slope-stability, the result follows from Proposition $7.2$. □

The second main result of the present subsection is the following.

**Proposition 7.3.** With notation as above, the map $f: \mathcal{M}^i_\epsilon (r_0) \rightarrow T^i_\epsilon$ is dominant.

**Proof.** Follows at once from Corollary $6.7$. □

7.3. Unicity of stable vector bundles on lagrangian HK’s. We will prove the result below.

**Proposition 7.4.** Let $r_0 \geq 2$. Suppose that $i \equiv r_0 \mod 2$, that $(1.3.1)$ holds, that $e \neq 2d$ and that

$$d > \frac{5}{16} \frac{\delta}{\epsilon} (r_0^2 - 1)(e + 1).$$

(7.3.1)

Let $[(X, H)] \in \mathcal{X}^i_{\epsilon}(d)$ be generic. Then, up to isomorphism, there exists one and only one $h$ slope-stable vector bundle $\mathcal{E}$ on $X$ such that $(1.3.3)$ holds if $r_0$ is odd, and $(1.3.5)$ holds if $r_0$ is even. For such $\mathcal{E}$ Equations $(1.3.6)$, $(1.3.7)$ hold if $r_0$ is odd, and Equations $(1.3.9)$, $(1.3.10)$ hold if $r_0$ is even. In other words $[\mathcal{E}] \in \mathcal{M}^i_\epsilon (r_0)$.

(Note as in the Proof of Proposition $7.1$)

We first prove the following auxiliary result.

**Lemma 7.5.** Let $S$ be an irreducible smooth projective surface, and $p: S \rightarrow T$ be a dominant map to a smooth curve with integral fibers. Let $\mathcal{F}$ and $\mathcal{G}$ be vector bundles on $S$ such that the following hold:

1. $\text{ch}(\mathcal{F}) = \text{ch}(\mathcal{G})$.
2. The restriction of $\mathcal{F}$ to every fiber of $p$ is slope-stable (there is only one notion of stability because the fibers are integral).
3. The restrictions of $\mathcal{F}$ and $\mathcal{G}$ to a generic fiber of $p$ are isomorphic.

Then the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to all fibers of $p$ are isomorphic.

**Proof.** Assume first that the restrictions of $\mathcal{G}$ to all fibers of $p$ are slope-semistable. For $t \in T$ let $C_t := p^{-1}(t)$. By Item $(2)$ we have $h^0(\mathcal{F} \otimes \mathcal{G}_{C_t}) > 0$ for $t$ a generic point of $T$. Let $t \in T$ be any point. By upper semicontinuity we get that $h^0(\mathcal{F} \otimes \mathcal{G}_{C_t}) > 0$, and since the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to $C_t$ are respectively slope-stable and slope-semistable, it follows that they are isomorphic.

Now suppose that there exists $t \in T$ such that the restriction of $\mathcal{G}$ to $C_t$ is not slope-semistable. Let $\mathcal{G}_{C_t} \rightarrow \mathcal{B}$ be a demestabilizing quotient, i.e.

$$0 < r(\mathcal{B}) < r(\mathcal{G}), \quad \mu(\mathcal{B}) < \mu(\mathcal{G}_{C_t}) = \frac{c_1(\mathcal{G}_{C_t}) \cdot C_t}{r(\mathcal{G})} 

(7.3.2)$$

Let $\mathcal{H}$ be the elementary modification of $\mathcal{G}$ associated to the quotient $\mathcal{B}$, i.e. the (torsion free) sheaf on $S$ fitting into the exact sequence

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{G} \rightarrow i_* \mathcal{B} \rightarrow 0.

(7.3.3)$$
where \( i : C_t \hookrightarrow S \) is the inclusion map. A computation gives that

\[
\Delta(H) = \Delta(F) - 2r(F) \cdot r(H)(\mu(F) - \mu(H)) < \Delta(F) .
\] (7.3.4)

Now replace \( H \) by \( \mathcal{G} := H \wedge \nu \), which is locally free, and notice that \( \Delta(H \wedge \nu) \leq \Delta(H) \). We claim that by iterating the procedure described above we get, after a finite number of steps, a vector bundle \( \mathcal{G}_n \) on \( S \) with the following properties. The restrictions of \( \mathcal{G}_n \) to all fibers of \( \rho \) are slope-semistable, and the restrictions of \( \mathcal{G} \) and \( \mathcal{G}_n \) to a generic fiber of \( \rho \) are isomorphic. Moreover \( \Delta(\mathcal{G}_n) < \Delta(\mathcal{G}) \). All we need to prove is that the procedure stops after a finite number of steps. This follows either from Langton’s algorithm for semistable reduction, or from Bogomolov’s inequality because by (7.3.4) there exists a suitable polarization of \( S \) for which all the \( \mathcal{G}_n \)'s are slope-stable, hence their discriminants (which are strictly decreasing integers) are non-negative.

Arguing as we did under the hypothesis that the restrictions of \( \mathcal{G} \) to all fibers of \( \rho \) are slope-semistable, we get that the sheaf \( \mathcal{G}_n \) is slope-stable, hence their discriminants (which are strictly decreasing integers) are non-negative.

The above inequality contradicts Item (1).

**Proof of Proposition 7.2** Existence has been proved in Proposition 7.1. Let \( \mathcal{E} \) be the vector bundle of that proposition. Then \( [\mathcal{E}] \in \mathcal{M}^\alpha_s(r_0) \).

Now let \( \mathcal{A} \) be any h slope-stable vector such that

\[
ch_i(\mathcal{A}) = ch_i(\mathcal{E}), \quad \forall i \in \{0, 1, 2\} .
\] (7.3.6)

We must prove that \( \mathcal{A} \) is isomorphic to \( \mathcal{E} \).

Let \( \pi : X \to \mathbb{P}^2 \) be the associated Lagrangian fibration of \([X, h]\). Since \([X, h]\) is generic point of \( \mathcal{M}_\alpha^\nu(d) \) the discriminant divisor of \( \pi \) is the dual of a smooth plane sextic curve (see Proposition 7.1.4), and hence it is smooth away from a finite set \( B_0 \subset \mathbb{P}^2 \). By Item (2) of Proposition 7.1 there is a finite (possibly empty) \( B_1 \subset \mathbb{P}^2 \) of z such that the pull-back of \( \mathcal{E} \) to the normalization of \( \pi^{-1}(z) \) is not slope-stable.

Let \( z \in (\mathbb{P}^2 \setminus (B_0 \cup B_1)) \). The key step is to prove that \( \mathcal{A}_{\pi^{-1}(z)} \) is slope-stable.

In order to achieve this, let \( T \subset \mathbb{P}^2 \) be a smooth curve containing \( z \) and intersecting transversely the discriminant divisor of \( \pi \). Thus \( \pi^{-1}(T) \) is a smooth threefold. Now let \( D \) be a generic effective divisor on \( X \) such that \( cl(D) = Nh + mf \), for \( N \) and \( m \) large, and let \( S := \pi^{-1}(T) \cap D \). Then \( S \) is a smooth (projective) surface, and the restriction of \( \pi \) to \( S \) defines a dominant map \( \rho : S \to T \). We apply Lemma 7.5 to \( \mathcal{F} := \mathcal{E}_S \) and \( \mathcal{G} := \mathcal{A}_S \). Let us check that the hypotheses of that lemma are satisfied. Item (1) holds by (7.3.6). Item (2) holds by Bogomolov’s effective version of the restriction Theorem for slope-stable bundles (see Theorem 7.1 in [Bog94] and Theorem 7.3.5 in [HL10]) and Item (2) of Proposition 7.1. Notice that Bogomolov’s Theorem holds for smooth surfaces, and that the normalization of a generic singular Lagrangian fiber is smooth. Lastly Item (3) of Lemma 7.5 holds by Proposition 7.1. In fact by that proposition the set of \( z \in \mathbb{P}^2 \) such that \( \mathcal{A}_{\pi^{-1}(z)} \) is not isomorphic to \( \mathcal{A}_{\pi^{-1}(z)} \) is contained in a proper closed subset \( Z \subset \mathbb{P}^2 \), and hence it suffices to choose \( T \) so that it is not contained in \( Z \). Thus by Lemma 7.5 the restriction of \( \mathcal{E} \) to \( \mathcal{F} \) to the fibers of \( S \to T \) are isomorphic. In particular, since the restriction of \( \mathcal{E} \) to such a fiber is slope-stable, so is the restriction of \( \mathcal{A} \). This implies that \( \mathcal{A} \) restricted to \( \pi^{-1}(z) \) is slope-stable because the fiber of \( S \to T \) over \( z \) is a section of a multiple of the polarization.

We have proved that \( \mathcal{A}_{\pi^{-1}(z)} \) is slope-stable for \( z \in (\mathbb{P}^2 \setminus (B_0 \cup B_1)) \). Since also \( \mathcal{E}_{\pi^{-1}(z)} \) is slope-stable for \( z \in (\mathbb{P}^2 \setminus (B_0 \cup B_1)) \), and the restrictions of \( \mathcal{E} \) and \( \mathcal{A} \) to
\( \pi^{-1}(z) \) are isomorphic for a generic \( z \), it follows that the restrictions of \( \mathcal{E} \) and \( \mathcal{E}' \) to \( \pi^{-1}(B_0 \cup B_1) \) are isomorphic, see the proof of Lemma 7.5. By Hartogs it follows that \( \mathcal{E} \) is isomorphic to \( \mathcal{E}' \).

\[ \square \]

7.4. Proofs of Theorem 1.4 and Corollary 1.7

**Proof of Theorem 1.4**. If \( r_0 = 1 \) the result is trivially true, hence we may assume that \( r_0 \geq 2 \). Let \( \mathcal{E} \rightarrow T^1_\varepsilon \) and \( \mathcal{E} \rightarrow T^2_\varepsilon \) be complete families of polarized HK’s of Type \( K^3[2] \) such that (1.3.2), respectively (1.3.3), holds - e.g. the families parametrized by the relevant open subsets of suitable Hilbert schemes. Since \( \mathcal{X}_{\varepsilon}' \) is irreducible we may, and will, assume that \( T^1_\varepsilon \) is irreducible. By normalization we may also assume that \( T^1_\varepsilon \) is normal. For \( t \in T^1_\varepsilon \) we let \( (X_t, h_t) \) be the corresponding polarized HK of Type \( K^3[2] \). We let \( m : T^1_\varepsilon \rightarrow \mathcal{X}_{\varepsilon}' \) be the moduli map, sending \( t \) to \([X_t, h_t]\).

By Gieseker and Maruyama there exists a map of schemes \( f : \mathcal{M}_e(r_0) \rightarrow T^1_\varepsilon \) such that for every \( t \in T^1_\varepsilon \) the (scheme theoretic) fiber \( f^{-1}(t) \) is isomorphic to the (coarse) moduli space of \( h_t \) slope-stable vector bundles \( \mathcal{E} \) on \( X_t \) such that (1.3.3) holds if \( r_0 \) is odd, and (1.3.3) holds if \( r_0 \) is even.

Notice that if \( f : \mathcal{M}_e(r_0) \rightarrow T^1_\varepsilon \) is of finite type by Maruyama [Mar81], and hence \( f(\mathcal{M}_e(r_0)) \) is a constructible subset of \( T^1_\varepsilon \).

By Proposition 7.3 for \( t \in T^1_\varepsilon \) the preimage \( f^{-1}(t) \) consists of a single point, and this point belongs to \( \mathcal{M}^{\text{GM}}_e(r_0) \). Since \( \bigcup_{d \geq 0} \mathcal{M}^{\text{GM}}_e(r_0) \) is Zariski dense in \( T^1_\varepsilon \), and \( f(\mathcal{M}_e(r_0)) \) is a constructible subset of \( T^1_\varepsilon \), it follows that for generic \( t \in T^1_\varepsilon \) the fiber \( f^{-1}(t) \) consists of a single point, and by Proposition 7.3 this point belongs to \( \mathcal{M}^{\text{GM}}_e(r_0) \).

By Proposition 5.4, Equations (1.3.6), (1.3.7) hold if \( r_0 \) is odd, and Equations (1.3.9), (1.3.10) hold if \( r_0 \) is even. The last sentence of Theorem 1.4 holds by Proposition 6.6.

\[ \square \]

**Proof of Corollary 1.7**. By Gieseker and Maruyama there exists a *projective* map of schemes \( g : \mathcal{M}_e(r_0)^{\text{GM}} \rightarrow T^1_\varepsilon \) such that for every \( t \in T^1_\varepsilon \) the (scheme theoretic) fiber \( g^{-1}(t) \) is isomorphic to the (coarse) moduli space of Gieseker-Maruyama \( h_t \) semistable torsion-free sheaves \( \mathcal{E} \) on \( X_t \) with Chern character given by (1.3.5), (1.3.6) and (1.3.7) if \( r_0 \) is odd, and by (1.3.8), (1.3.9) and (1.3.10) if \( r_0 \) is even. Moreover \( \mathcal{M}_e(r_0) \) is an open subset of \( \mathcal{M}_e(r_0)^{\text{GM}} \), and \( f \) is the restriction of \( g \) to \( \mathcal{M}_e(r_0) \).

Let \( \mathcal{M}_e(r_0) \) be the closure of \( \mathcal{M}_e(r_0) \) in \( \mathcal{M}_e(r_0)^{\text{GM}} \). The restriction of \( g \) is a projective map \( f : \mathcal{M}_e(r_0) \rightarrow T^1_\varepsilon \). The fiber of \( f \) over a generic point is the closure of \( f^{-1}(t) \), and hence is irreducible by Theorem 1.4. Since \( T^1_\varepsilon \) is normal, it follows by Zariski’s main theorem that every fiber of \( f \) is connected.

Now let \( t \in T^1_\varepsilon \) such that \((X_t, h_t)\) is isomorphic to \((X, h)\); we identify \( X_t \) with \( X \). Then \( \mathcal{E} \) is represented by a point \( x \in f^{-1}(t) \), and \( x \in \mathcal{M}_e(r_0) \) because \( h^2(X, \text{End}_0 \mathcal{E}) = 0 \). Moreover \( x \) is a component of \( f^{-1}(t) \) because \( h^1(X, \text{End}_0 \mathcal{E}) = 0 \) (this holds because \( \chi(X, \text{End}_0 \mathcal{E}) = 0 \), \( h^0(X, \text{End}_0 \mathcal{E}) = 0 \) by stability, hence \( h^3(X, \text{End}_0 \mathcal{E}) = 0 \) by Serre duality, and \( h^2(X, \text{End}_0 \mathcal{E}) = 0 \) by hypothesis). Since \( f^{-1}(t) \) is connected, it equals \( x \). This proves Corollary 1.7.

\[ \square \]

8. Moduli of DV varieties

8.1. Debarre-Voisin vector bundles. Let \( X \subset \text{Gr}(6, V_{10}) \) be a DV variety, and let

\[
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X \otimes V_{10} \longrightarrow \mathcal{D} \longrightarrow 0
\]

(8.1.1)
be the restriction to $X$ of the tautological exact sequence of vector bundles on $\text{Gr}(6, V_{10})$. Thus $r(\mathcal{S}) = 6$ and $r(\mathcal{D}) = 4$.

**Lemma 8.1.** Let $X$ be a DV variety, and let $h \in H^{1,1}_2(X)$ be the Plücker polarization. Then

$$
\begin{align*}
\text{ch}_0(\mathcal{D}) &= 4, \\
\text{ch}_1(\mathcal{D}) &= h, \\
\text{ch}_2(\mathcal{D}) &= \frac{1}{8} (h^2 - c_2(X)), \\
\text{ch}_3(\mathcal{D}) &= -\frac{1}{264} h^3, \\
\text{ch}_4(\mathcal{D}) &= -\frac{1}{4} \eta_X,
\end{align*}
$$

where $\eta_X \in H^8(X; \mathbb{Z})$ is the fundamental class.

**Proof.** The first two equations are obvious. Equation (8.1.4) follows from the third-to-last equation on p. 83 of [DV10] (beware that $c = c_i(\mathcal{S}^\vee)$). Equation (8.1.5) follows from Equation (11) in loc. cit. and the fact that $\text{ch}_2(\mathcal{D})$ is a multiple of $h^3$. The last equation is a straightforward of Equation (11) in loc. cit. \( \square \)

**Remark 8.2.** Let $(X, h)$ be a smooth DV variety. Then $q(h) = 22$ and the divisibility of $h$ is 2, i.e. $[(X, h)] \in \mathcal{X}^2_2$. If one sets $r_0 = 2$ and $e = 22$ in Equations (1.3.8), (1.3.9) and (1.3.10) one gets the Chern character of the quotient vector bundle $\mathcal{D}$ described above.

**Remark 8.3.** Let $X$ be the variety of lines in a smooth cubic fourfold in $\mathbb{P}(V_6)$. Let

$$
0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}_X \otimes V_6 \rightarrow \mathcal{D} \rightarrow 0
$$

be the restriction to $X$ of the tautological exact sequence of vector bundles on $\text{Gr}(2, V_6)$. Hence the rank of $\mathcal{D}$ is 4. Let $h \in H^{1,1}_2(X)$ be the Plücker polarization. Then $c_1(\mathcal{S}) = -h$ and $c_2(\mathcal{S}) = \frac{1}{4}(5h^2 - c_2(X))$. It follows that

$$
\text{ch}_1(\mathcal{D}) = h, \quad \text{ch}_2(\mathcal{D}) = \frac{1}{8} (h^2 - c_2(X)), \quad \text{ch}_3(\mathcal{D}) = -\frac{1}{24} h^3, \quad \text{ch}_4(\mathcal{D}) = \frac{3}{4} \eta_X.
$$

Next notice that $q(h) = 6$ and the divisibility of $h$ is 2, i.e. $[(X, h)] \in \mathcal{X}^2_2$. The equations above show that the Chern character of $\mathcal{D}$ is identified with the Chern character appearing in Theorem 1.4 for $r_0 = 2$ and $e = 6$.

**Proposition 8.4.** If $X$ is a smooth DV variety with cyclic Picard group both $\mathcal{D}$ and $\mathcal{S}$ (see 8.1.1) are slope-stable vector bundles.

**Proof.** Let $h \in H^{1,1}_2(X)$ be the Plücker polarization. Let us prove that $\mathcal{D}$ is $h$ slope-stable. Suppose that

$$
0 \rightarrow \mathcal{S} \rightarrow \mathcal{D} \rightarrow \mathcal{B} \rightarrow 0
$$

is a desemistabilizing sequence. Thus $0 < r(\mathcal{S}) < 4$,

$$
\frac{c_1(\mathcal{S}) \cdot h^3}{r(\mathcal{S})} = \mu(\mathcal{S}) \geq \mu(\mathcal{D}) := \frac{h^4}{4} = 363,
$$

and we may assume that $\mathcal{B}$ is torsion-free. By hypothesis $c_1(\mathcal{S}) = x h$ for some $x \in \mathbb{Z}$, and hence $x \geq 1$. It follows that $c_1(\mathcal{B}) = (1 - x) h$. Since $\mathcal{D}$ is globally generated, so is $\mathcal{B}$. Thus $x = 1$, i.e. $c_1(\mathcal{B}) = 0$, and $\mathcal{B}$ is trivial because it is globally generated. Hence $c_4(\mathcal{D}) = 0$. This is a contradiction. In fact, following the notation on p. 83 of [DV10], we let $c_1 = c_i(\mathcal{S}^\vee)$. Then (using the formulae in Equation (11) of loc. cit.)

$$
c_4(\mathcal{D}) = c_4^1 - 3c_1^2c_2 + c_2^3 + 2c_1c_3 - c_4 = 9\eta_X.
$$
An analogous proof gives slope-stability of $\mathcal{F}$.

By openness of slope-stability we also get the following result.

**Corollary 8.5.** If $X$ is a generic DV variety both $\mathcal{Q}$ and $\mathcal{F}$ are slope-stable vector bundles (for the Plücker polarization).

**Remark 8.6.** Let $X$ be the variety of lines in a smooth cubic fourfold in $\mathbb{P}(V_0)$, and let $\mathcal{Q}$ be quotient vector bundle appearing in (8.1.7). If Pic($X$) is cyclic one can prove that $\mathcal{Q}$ is slope-stable by proceeding as in the proof of Proposition 8.4 except that in the end one does not conclude by a Chern class computation (we have $c_4(\mathcal{Q}) = 0$). Rather one shows directly that there is no trivial quotient $\mathcal{Q} \to \mathcal{O}_X$. In fact, assume there is such a quotient; then there is a non zero section of $\mathcal{Q}^\vee$, and one gets that the latter is false by considering the dual of the exact sequence in (8.1.7).

**Proposition 8.7.** If $X$ is a generic DV variety, then the map $V_{10} \to H^0(X, \mathcal{Q})$ induced by (8.1.1) is an isomorphism.

**Proof.** The vector bundle $\mathcal{F}$ has no global sections because it is slope-stable (by Corollary 8.5), with negative slope. Hence it suffices to prove that $h^0(X, \mathcal{Q}) = 10$. We do this by considering a $K3$ surface $S$ as in Claim 6.4 with $h_0 = 3$ and large odd $d_0$. The vector bundle $\mathcal{F}$ on $S$ of Proposition 8.2 has Mukai vector $v(\mathcal{F}) = (2, D, 2)$. We claim that

$$h^0(S, \mathcal{F}) = 4, \quad h^1(S, \mathcal{F}) = 0, \quad h^2(S, \mathcal{F}) = 0. \quad (8.1.9)$$

at least for $d \gg 0$ and “most” $S$. In fact let $(S', D')$ be a generic polarized $K3$ surface with $D'$ of square 6. Then there exists a unique $D'$ slope-stable vector bundle $\mathcal{F}'$ on $S'$ with Mukai vector $(2, D', 2)$. As is easily checked $h^0(S', \mathcal{F}') = 4$. Since moduli of elliptic $K3$’s that we are considering are dense in the moduli space of polarized $K3$’s of degree 6, we get that $h^0(S, \mathcal{F}) = 4$ for “most” $S$. By stability of $\mathcal{F}$ we have $h^2(S, \mathcal{F}) = 0$. Hence also the middle equality in (8.1.9) holds because

$$\chi(S, \mathcal{F}) = 4.$$  

Let $\mathcal{E}_0 := \mathcal{F}[2]^+$. By definition there is a canonical isomorphism $H^0(S[2], \mathcal{E}_0) \cong \text{Sym}^2 H^0(S, \mathcal{F})$ and hence $h^0(S[2], \mathcal{E}_0) = 10$. From the second equality in (8.1.9) we also get that $h^1(S[2], \mathcal{E}_0) = 0$. Now let $X_0 := S[2]$ and let $h_0 := h$ where $h$ is given by (6.3.2). Notice that $q(h_0) = 22$ and the divisibility of $h_0$ is 2.

Let $\mathcal{F} \to T$ be an analytic representative of the deformations space of $(X_0, h_0)$. Let $t \in T$; by Proposition 6.6 and Corollary 6.7 there is one and only one vector bundle $\mathcal{E}_t$ on $X_t$ which is a deformation of $\mathcal{E}_0$. By Proposition 7.2 $h_t$ is a polarization on $X_t$ for $t$ generic in $T$, and $\mathcal{E}_t$ is $h$-slope-stable. But for $t \in T$ generic $(X_t, h_t)$ is isomorphic to a DV variety parametrized by an analytic open subset of $\mathbb{P}(\Lambda^3 V_{10})$. Hence $\mathcal{E}_t$ is isomorphic to the corresponding quotient DV vector bundle $\mathcal{Q}_t$ on $X_t$ by Theorem 1.4 and Corollary 8.5. Hence $h^0(X_t, \mathcal{E}_t) = h^0(X_0, \mathcal{E}_0) = 10$ because $h^1(X_0, \mathcal{E}_0) = 0$.

---

**8.2. Proof of Theorem 1.8**

**Proof.** Let $d$ be the degree of the moduli map

$$\mathcal{M}_{DV} \to \mathcal{K}_{22}. \quad (8.2.1)$$

We have $d \geq 1$ because the moduli map is dominant. We need to prove that $d = 1$. Let $[(X, h)] \in \mathcal{K}_{22}$ be a generic point. Then there exist $[\sigma_1], \ldots, [\sigma_d] \in \mathbb{P}(\Lambda^1 V_{10})$ such that the corresponding polarized DV varieties $(X_1, h_1), \ldots, (X_d, h_d)$ are smooth and all isomorphic to $(X, h)$, but the $\text{PGL}(V_{10})$-orbits of $[\sigma_1], \ldots, [\sigma_d]$ are pairwise distinct. Let $\mathcal{Q}_i$ be the DV quotient vector bundle on $X$ determined by $\sigma_i$. By
Corollary 8.8. Each $\mathcal{E}_i$ is $h$ slope-stable, and hence by Theorem 1.3 all the $\mathcal{E}_i$ are isomorphic to a single vector bundle $\mathcal{E}$. By Proposition 5.1 the surjection $\mathcal{O}_X \otimes V_0 \to \mathcal{E}_i$ is identified with the canonical map $\mathcal{O}_X \otimes H^0(X, \mathcal{E}) \to \mathcal{E}$. It follows that $d = 1$.

\textbf{Remark 8.8.} Let $|\mathcal{O}_P(3)|/\text{PGL}(6) \to \mathcal{H}_0^2$ be the moduli map one gets by associating to a smooth cubic 4-fold the variety of its lines. This map is birational by Voisin’s Global Torelli Theorem for cubics. Charles [Cha12] inverted the argument: he proved independently that the moduli map $|\mathcal{O}_P(3)|/\text{PGL}(6) \to \mathcal{H}_0^2$ is isomorphic to a single vector bundle $\mathcal{E}$ is not globally generated, or it is globally generated but the corresponding map $X \to \text{Gr}(6, H^0(\mathcal{E}))$ is not an embedding, or $\mathcal{E}$ is not locally free (one should also take into account the possibility of getting a degenerate $\sigma$ because $h$ is not ample). An example: in the proof of Proposition 8.7 we discussed a case in which $H^0(X, \mathcal{E}) = 10$ but $\mathcal{E}$ is not globally generated but the corresponding map $X \to \text{Gr}(6, H^0(\mathcal{E}))$ is not an embedding. The “inverse” approach should allow to complete the discussion of the family appearing in Section 8 of [DHOV].

Appendix A. (Semi)homogeneous vector bundles on abelian varieties

A.1. Basics. Let $A$ be an abelian variety, and let $A^\vee := \text{Pic}^0(A)$ be its dual abelian variety. For $a \in A$, let $T_a : A \to A$ be the translation by $a$. For an invertible sheaf $\xi$ on $A$, we let $[\xi] \in A^\vee$ be its isomorphism class.

\textbf{Definition A.1.} A vector bundle $\mathcal{F}$ on $A$ is homogeneous if $T_a^* \mathcal{F} \cong \mathcal{F}$ for every $a \in A$, it is semi-homogeneous if, for every $a \in A$, there exists $[\xi] \in \text{Pic}^0(A)$ such that $T_a^* \mathcal{F} \cong \mathcal{F} \otimes \xi$.

\textbf{Proposition A.2.} Let $(A, \theta)$ be a polarized abelian variety of dimension $n$, and let $\mathcal{F}$ be a $\theta$ slope-stable vector bundle on $A$. If

$$\int_A \Delta(\mathcal{F}) \sim \theta^{n-2} = 0 \quad \text{(A.1.1)}$$

(the condition is to be understood to be empty if $n = 1$) then $\mathcal{F}$ is simple semi-homogeneous. Moreover $\Delta(\mathcal{F}) = 0$.

\textbf{Proof.} Of course $\mathcal{F}$ is simple because it is slope-stable.

If $n = 1$ then $\mathcal{F}$ is semi-homogeneous by Atiyah’s classification of simple vector bundles on elliptic curves.

Suppose that $n \geq 2$. By the Kobayashi-Hitchin correspondence, $\mathcal{F}$ has a $\theta$ Hermitian-Einstein metric, and hence so does the vector bundle $\text{End}\mathcal{F}$. Equation (A.1.1) is equivalent to $\int_A c_2(\text{End}\mathcal{F}) \sim \theta^{n-2} = 0$. Since $c_2(\text{End}\mathcal{F}) = 0$, the Hermite-Einstein connection on $\text{End}\mathcal{F}$ is flat, see §4 in [Kob82]. Hence $\text{End}\mathcal{F}$ is homogeneous, and thus $\mathcal{F}$ is semi-homogeneous by Theorem 5.8 in [Muk78].
Since \( \text{End}\mathcal{F} \) is homogeneous, it is a direct sum of vector bundles which have filtrations whose associated graded bundles are direct sums of a topologically trivial line bundles - see Theorem 4.17 in \[\text{Muk78}\]. Thus \( \Delta(\mathcal{F}) = c_2(\text{End}\mathcal{F}) = 0. \)

A.2. Rank of semi-homogeneous vector bundles. The rank of a simple semi-homogeneous vector bundle with assigned first Chern class is not arbitrary. Below we extend a result of Mukai, see Theorem 7.11 and Remark 7.13 in \[\text{Muk78}\], so that we cover all the canonical polarizations that occur on Lagrangian fibrations in the known deformation classes of HK's with the exception of OG10.

**Proposition A.3** (Mukai \[\text{Muk78}\]). Let \((A, \theta)\) be a polarized abelian variety of dimension \(n\). Suppose that the elementary divisors of \(\theta\) are \((1, \ldots, 1, d_1, d_2)\) where \(d_1|d_2\). Let \(\mathcal{F}\) be a semi-homogeneous vector bundle on \(A\) such that \(c_1(\mathcal{F}) = a\theta\). Then there exists a positive integer \(r_0\) such that, letting \(g_i := \gcd\{r_0, d_i\}\) we have

\[
\gcd(r(\mathcal{F}), a) = \frac{r_0^{n-1}}{g_1 \cdot g_2}, \quad r(\mathcal{F}) = \frac{r_0^n}{g_1 \cdot g_2}. \tag{A.2.1}
\]

**Proof.** Let \(r := r(\mathcal{F})\) and let \(c := \gcd\{r, a\}\). We let \(r_0 := \frac{c}{a}\) and \(a_0 := \frac{a}{c}\). Let us prove that (A.2.1) holds. Let \(\mathcal{L}\) be a line bundle on \(A\) such that \(c_1(\mathcal{L}) = a_0\theta\), and let \(\varphi_{\mathcal{L}} : A \to A^\vee\) be the homomorphism defined by \(\varphi_{\mathcal{L}}(a) := T^*_a(\mathcal{L}) \otimes \mathcal{L}^{-1}\).

Let \(K(\mathcal{L})\) be the kernel of \(\varphi_{\mathcal{L}}\). Lastly, following Mukai, we let

\[
\Sigma(\mathcal{F}) := \{[\xi] \in A^\vee \mid \exists \mathcal{F} \hookrightarrow \mathcal{F} \otimes \xi\}. \tag{A.2.2}
\]

(Since we are in characteristic 0, the above set-theoretic definition coincides with the schematic one, see Proposition 5.9 in \[\text{Muk78}\].) By Theorem 7.11 in \[\text{Muk78}\], we have an exact sequence of groups:

\[
0 \longrightarrow A[r_0] \cap K(\mathcal{L}) \longrightarrow A[r_0] \xrightarrow{\varphi_{\mathcal{L}}} \Sigma(\mathcal{F}) \longrightarrow 0. \tag{A.2.3}
\]

Because of our hypothesis on the elementary divisors of \(\theta\) we have

\[
K(\mathcal{L}) \cong (\mathbb{Z}/(a_0)^2) \oplus \ldots \oplus (\mathbb{Z}/(a_0d_1)^2) \oplus (\mathbb{Z}/(a_0d_2)^2). \tag{A.2.4}
\]

Since \(a_0, r_0\) are coprime, it follows that \(A[r_0] \cap K(\mathcal{L}) \cong (\mathbb{Z}/(g_1))^2 \oplus (\mathbb{Z}/(g_2))^2\). Thus \(|\Sigma(\mathcal{F})| = \frac{r_0^n}{g_1 \cdot g_2}\). On the other hand the cardinality of \(\Sigma(\mathcal{F})\) is equal to \(r^2\) by Proposition 7.1 in \[\text{Muk78}\]. Thus we get that \(r = \frac{r_0^n}{g_1 \cdot g_2}\). Since \(c := \frac{r_0}{r}\) it follows that \(\gcd\{r, a\} = c = \frac{r_0^{n-1}}{g_1 \cdot g_2}\).

\[\square\]

**Appendix B. Polarized Lagrangian HK’s of Type K3\(^{[2]}\)**

B.1. **Lagrangian Noether-Lefschetz loci.** We recall that \(\mathcal{X}_i^p\) is the moduli space of polarized HK's of Type K3\(^{[2]}\) with polarization of BBF square \(e\) and divisibility given by \(i\) (which is either 1 or 2) - see Subsection 1.3 If \((X, H)\) represents a point of \(\mathcal{X}_i^p\) we let \(h := \text{cl}(H)\).

**Definition B.1.** For \(d\) a strictly positive integer let \(\mathcal{N}_e^p(d) \subset \mathcal{X}_i^p\) be the closure of the locus parametrizing polarized HK's \((X, h)\) such that \(H_{2i}^1(X)\) contains a saturated rank 2 sublattice generated by \(h, f\), where

\[
q(f) = 0, \quad q(h, f) = d. \tag{B.1.1}
\]

**Proposition B.2.** Keeping notation as above, suppose in addition that \(d\) is even if \(i = 2\), and that

\[
d > 10(e + 1), \quad e \not\equiv 2d. \tag{B.1.2}
\]

Then \(\mathcal{N}_e^p(d)\) is closed of pure codimension 1 (in particular non empty), and if \([[(X, h)]] \in \mathcal{N}_e^p(d)\) is generic there is one and only one Lagrangian fibration \(\pi : X \to \mathbb{P}_2\).
$\mathbb{P}^2$ (modulo automorphisms of $\mathbb{P}^2$) such that, letting $f := c_1(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))$, the equalities in (B.1.1) hold and the sublattice $(h, f) \subset H^1_{\mathbb{Z}}(X)$ is saturated.

Proof. Before starting the proof we emphasize that $\mathcal{M}_e^\ell(d)$ might have several irreducible components, and that “generic point” of $\mathcal{M}_e^\ell(d)$ means belonging to an open dense subset of $\mathcal{M}_e^\ell(d)$. By surjectivity of the period map there exists a HK $X$ of Type $K3^{[2]}$ such that $H^2_{\mathbb{Z}}(X) = \langle h, f \rangle$ where

$$q(h) = e, \quad \{q(h, \alpha) \mid \alpha \in H^2(X; \mathbb{Z})\} = (i),$$

and the equalities in (B.1.1) hold. There are no $\xi \in H^2_{\mathbb{Z}}(X)$ with $-10 \leq q(\xi) < 0$ by Lemma 4.3 and the inequality in (B.1.2). It follows that the ample cone of $X$ is equal to the intersection of $H^2_{\mathbb{Z}}(X)$ and the positive cone. Hence either $h$ or $-h$ is ample. If the former holds then $\left\{ [X, h] \right\} \in \mathcal{M}_e^\ell(d)$, if the latter holds, we may replace $h, f$ by $-h, -f$ respectively, and again we get that $\left\{ [X, h] \right\} \in \mathcal{M}_e^\ell(d)$. Moreover $\mathcal{M}_e^\ell(d)$ is closed of pure codimension 1 because it is a Noether-Lefschetz divisor.

A straightforward computation shows that there are exactly two primitive nef isotropic classes, namely $f$ and $\alpha := \frac{d}{\gcd(d, e)}(2dh - ef)$. By our “non divisibility” hypothesis in (B.1.2), we get that $q(\alpha, h) = -\frac{d}{\gcd(d, e)}$ is not equal to $d$. Hence $f$ is the unique primitive nef isotropic class such that $q(h, f) = d$.

By Theorem 1.3 in [Mar14] (see also Remark 1.8) there exist a Lagrangian fibration $\pi: X \to \mathbb{P}^2$ such that $f := c_1(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))$. By Theorem 1.2 in [Mar17] it follows that there exists a Zariski open neighborhood $\mathcal{W}$ of $\left\{ [X, h] \right\}$ in $\mathcal{M}_e^\ell(d)$ such that each representative $(X', h')$ of a points in $\mathcal{W}$ has a Lagrangian fibration as required. Since the set of points of $\mathcal{M}_e^\ell(d)$ representing $(X, h)$ such that $\rho(X) = 2$ is dense, this proves the result about existence of the required Lagrangian fibration.

It remains to prove that if $\left\{ [X, h] \right\} \in \mathcal{M}_e^\ell(d)$ is generic then there is a unique isotropic class $f$ such that $q(h, f) = d$. We checked that this is the case if $\rho(X) = 2$. It follows that the statement holds for the generic point of $\mathcal{M}_e^\ell(d)$; the argument is similar to that given to show that $h$ is a ($\mathcal{E}$)-suitable in the proof of Proposition 1.11.

Definition B.3. Suppose that (B.1.2) holds. We let $\mathcal{M}_e^\ell(d)^0 \subset \mathcal{M}_e^\ell(d)$ be an open dense subset such that the thesis of Proposition B.2 holds for any $\left\{ [X, h] \right\} \in \mathcal{M}_e^\ell(d)^0$. For $\left\{ [X, h] \right\} \in \mathcal{M}_e^\ell(d)^0$ the associated Lagrangian fibration $\pi: X \to \mathbb{P}^2$ is the unique fibration (modulo automorphisms of $\mathbb{P}^2$) of Proposition B.2.

B.2. Tate-Shafarevich twists. A basic example of Lagrangian fibration is constructed as follows. Let $S \to \mathbb{P}^2$ be the double cover ramified over a smooth sextic curve $B$, i.e. a polarized $K3$ surface of degree 2. Let $\mathcal{F}(S)$ be the moduli space of rank 0 pure $\mathcal{O}_S(1)$ semistable sheaves $\xi$ with $\chi(\xi) = -1$. The generic point of $\mathcal{F}(S)$ is represented by $i_*\mathcal{L}$, where $i: C \to S$ is the inclusion of a smooth $C \in \mathcal{O}_S(1)$, and $\mathcal{L}$ is a line bundle of degree 0. Then for generic $B$ every semistable sheaf is stable (the precise condition is that $B$ have no tritangents), and hence $\mathcal{F}(S)$ is smooth. In fact it is a HK of Type $K3^{[2]}$, and the support map $\mathcal{F}(S) \to (\mathbb{P}^2)^{\vee}$ is a Lagrangian fibration.

A Lagrangian fibration parametrized by a generic point of $\mathcal{M}_e^\ell(d)$ is related to a generic $\mathcal{F}(S)$ via a Tate-Shafarevich twist. In order to be more precise, we recall a result of Markman. Let $(X, h)$ be a representative of a generic point of $\mathcal{M}_e^\ell(d)$. Then there is an associated polarized $K3$ surface $(S, D)$ of degree 2, and moreover $(S, D)$ is generic - see Subsection 4.1 in [Mar14].

Proposition B.4. Keep the hypotheses of Proposition B.2. Let $\left\{ [X, h] \right\}$ be a generic point of $\mathcal{M}_e^\ell(d)$. Let $\pi: X \to \mathbb{P}^2$ and $S$ be the associated Lagrangian fibration.
and K3 surface of degree 2 respectively. Then X is isomorphic to a Tate-Shafarevich twist of \( \mathcal{S}(S) \to (\mathbb{P}^2) \) via an identification \( \mathbb{P}^2 \to (\mathbb{P}^2) \).

**Proof.** Suppose first that \( \rho(X) = 2 \). Then, as shown in the proof of Proposition B.2, the ample cone of X is equal to the positive cone (because of the inequality in (B.1.2)), and hence every bimeromorphic map \( X \to X' \), where \( X' \) is a HK, is actually an isomorphism. It follows that X is isomorphic to a Tate-Shafarevich twist of \( \mathcal{S}(S) \to (\mathbb{P}^2) \) by Theorem 7.13 in [Mar14]. The result follows from this because the locus in \( \mathcal{M}_d^4(d) \) parametrizing \((X,h)\) such that \( \rho(X) = 2 \) is dense. □

Let \( X \to \mathbb{P}^2 \) be as in Proposition B.3 and let \( \text{Pic}^0(X/\mathbb{P}^2) \) be the relative Picard scheme (notice that all fibers of \( X \to \mathbb{P}^2 \) are irreducible by Proposition B.3). If \( U \subset \mathbb{P}^2 \) is the open dense set of regular values of \( X \to \mathbb{P}^2 \) and \( z \in U \), the fiber of \( \text{Pic}^0(X/\mathbb{P}^2) \to \mathbb{P}^2 \) over \( z \) is an abelian surface \( A_z \) and the fundamental group \( \pi_1(U,z) \) acts by monodromy on the subgroup \( A_z \) of torsion points.

**Corollary B.5.** Keep hypotheses as above, and suppose that \( V \subset A_z[r^3_0] \) is a coset (of a subgroup) of cardinality \( r^3_0 \) invariant under the action of monodromy. Then \( V = A_z[r^3_0] \).

**Proof.** Let \( S \) be the polarized K3 surface of degree 2 associated to X following Markman, and let \( S \to \mathbb{P}^2 \) be the double cover ramified over a sextic curve \( B \). Let \( \mathcal{S}(S) \subset \mathcal{S}(S) \) be the open dense subset of smooth points (i.e. smooth points of \( \mathcal{S}(S) \) with surjective differential) of the map \( \mathcal{S}(S) \to (\mathbb{P}^2) \). By Proposition B.3 \( \text{Pic}^0(X/\mathbb{P}^2) \to \mathbb{P}^2 \) is isomorphic to \( \mathcal{S}(S) \to (\mathbb{P}^2) \), for a certain identification \( \mathbb{P}^2 \to (\mathbb{P}^2) \). Under this identification \( z \in \mathbb{P}^2 \) corresponds to a line \( R \subset (\mathbb{P}^2) \) transverse to \( B \), and the corresponding Lagrangian fiber \( A_z \) is the Jacobian of the double cover of \( R \) ramified over \( R \cap B \). Hence we have a natural isomorphism

\[
H_1(C; \mathbb{Q})/H_1(C; \mathbb{Z}) \cong A_{z,\text{tors}}, \quad \text{(B.2.1)}
\]

and the identification is compatible with the monodromy actions.

First we prove the result under the assumption that \( V \) is a subgroup \( G \). By the structure theorem for finite abelian groups \( G \cong \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_i) \), where \( r \leq 4 \) (because \( A_z[r^3_0] \cong \mathbb{Z}/(r^3_0)^{d_1} \) and \( d_i[r^3_0] \) for all \( i \). Since the monodromy action on \( H_1(C; \mathbb{Z}) \) is transitive on non zero elements, it follows from the isomorphism in (B.2.1) that \( r = 4 \) and \( d_1 = \cdots = d_i \). Thus \( d_i = r_0 \) for all \( i \in \{1, \ldots, 4\} \) because \( G = r^3_0 \). This proves the result under the assumption that \( V \) is a subgroup.

Now let \( V \) be a translate of a group \( G \). Then \( G = \{a - b \mid a, b \in V\} \), and hence \( G \) is also invariant for the monodromy action. Thus \( G = A_z[r_0] \), and hence the coset \( V \) gives a point of the quotient \( A_z[r^3_0]/A_z[r_0] \) which is invariant for the monodromy action. By the isomorphism in (B.2.1) it follows that 0 is the unique invariant element, and hence \( V = A_z[r_0] \). □

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