COXETER SYSTEM OF PLANES ARE SETS OF INJECTIVITY FOR THE TWISTED SPHERICAL MEANS ON $\mathbb{C}^n$

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Abstract. In this article, we prove that any pair of perpendicular planes is a set of injectivity for the twisted spherical means (TSM) for $L^p(\mathbb{C}^n)$ $(n \geq 2)$ with $1 \leq p \leq 2$. Then, we imitate that any Coxeter system of even number of planes is a set of injectivity for the TSM for $L^p(\mathbb{C}^n)$. We further observe that a set $S^{2n-1}_R \times \mathbb{C}$ is a set of injectivity for the TSM for a certain class of functions on $\mathbb{C}^{n+1}$.

1. Introduction

Let $\mu_r$ be the normalized surface measure on sphere $S_r(x)$. Let $\mathcal{F} \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$. We say that $S \subseteq \mathbb{R}^n$ is a set of injectivity for the spherical means for $\mathcal{F}$ if for $f \in \mathcal{F}$ with $f * \mu_r(x) = 0, \forall r > 0$ and $\forall x \in S$, implies $f = 0$.

In the work by Agranovsky and Quinto [2], it has been shown that sets of non-injectivity for the spherical means for $C_c(\mathbb{R}^n)$ ($n \geq 2$) is contained in the zero set of a certain harmonic polynomial. For non-zero function $f \in C_c(\mathbb{R}^n)$, write $S(f) = \{x \in \mathbb{R}^n : f * \mu_r(x) = 0, \forall r > 0\}$. Then they have proved that $S(f) = \bigcap_{k=0}^{\infty} Q_k^{-1}(0)$, where

$$Q_k(x) = \int_{\mathbb{R}^n} f(y)|x - y|^{2k}dy.$$  

Since all of $Q_k$ can not be identically zero, it follows that there exists the least positive integer $k_o$ such that $Q_{k_o} \neq 0$. Hence $\Delta Q_{k_o} = 2k_o(2k_o + n - 1)Q_{k_o-1} = 0$. That is, $S(f) \subseteq Q_{k_o}^{-1}(0)$. Since $Q_{k_o}$ is harmonic and a harmonic polynomial can vanish to a Coxeter system of hyperplane intersecting along a line, it follows that for $n > 2$, any Coxeter system of hyperplanes intersecting along a line may fail to be a set of injectivity for the spherical means on $\mathbb{R}^n$. In general, any real cone $K \subset \mathbb{R}^n$ ($n > 2$) is a set of injectivity for the spherical means for $C(\mathbb{R}^n)$ if and only if $K$ is not contained in the zero set of any homogeneous harmonic polynomial, (see [5]).

However, these results do not continue to hold for injectivity of the twisted spherical means on $\mathbb{C}^n$ ($n \geq 2$), because of non-commutative nature of underlying geometry of the Heisenberg group, (see [6, 7, 8, 10]). The question, any odd Coxeter system of hyperplanes can be a set of injectivity for the twisted spherical mean for $L^p(\mathbb{C}^n)$, is still unanswered.

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Using a recent result of the author ([19], p.10, Theorem 3.6) that any pair of perpendicular lines in $\mathbb{C}$ is a set of injectivity for the TSM, we prove that any pair of perpendicular planes in $\mathbb{C}^n$ ($n \geq 2$) is a set of injectivity for the TSM for function in $L^p(\mathbb{C}^n)$, $1 \leq p \leq 2$. Suppose $f \times \mu_r(z) = 0$, $\forall r > 0$ and $\forall z \in \mathbb{C}^{n-1} \times \mathbb{R} \cup \mathbb{C}^{n-1} \times \mathbb{R}$. Then $f = 0$. Using this, we conclude that any Coxeter system of even number of planes intersecting along a line is a set of injectivity for the TSM for $L^p(\mathbb{C}^n)$.

Let $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$ and $S_R^{2n-1} = \{z \in \mathbb{C}^n : |z| = R\}$. In a result of Narayanan and Thangavelu [13], it has been proved that the spheres centered at the origin are set of injectivity for the TSM on $\mathbb{C}^n$. The author has generalized their result for certain weighted twisted spherical means, (see [13]). In general, the question of set of injectivity for the twisted spherical means (TSM) with real analytic weight is still open. In view of result in [13] that $S_R^{2n-1}$ is a set of injectivity for the TSM on $\mathbb{C}^n$, we prove that the set $S_R^{2n-1} \times \mathbb{C}$ is a set of injectivity for the twisted spherical means for the functions satisfying $e^{|z|^2} f(z) \in L^p(\mathbb{C}^{n+1})$ with $1 \leq p \leq \infty$.

On account of these results, we observe an embedding property of the sets of injectivity for the TSM in higher dimensions. In the Euclidean set up, the sets of injectivity for the spherical means on the unit sphere $S^{n-1}$ can be embedded into the sets of injectivity for the spherical means on $\mathbb{R}^n$, (see [5]).

Since Laguerre function $\varphi_k^{n-1}$ is an eigenfunction of the special Hermite operator $A = -\Delta_z + \frac{1}{4} |z|^2$, with eigenvalue $2k + n$, the projection $f \times \varphi_k^{n-1}$ is also an eigenfunction of $A$ with eigenvalue $2k + n$. As $A$ is an elliptic operator and eigenfunction of an elliptic operator is real analytic [11], the projection $f \times \varphi_k^{n-1}$ must be a real analytic function on $\mathbb{C}^n$. By polar decomposition, the conditions $f \times \mu_r(z) = 0$, $\forall r > 0$ is equivalent to $f \times \varphi_k^{n-1}(z) = 0$, $\forall k \in \mathbb{Z}_+$, set of non-negative integers. Therefore, any determining set for the real analytic functions is a set of injectivity for the TSM on $L^p(\mathbb{C}^n)$ with $1 \leq p \leq \infty$. For example, let $\gamma(t) = r(t)e^{i\theta}$, where $r(t)$ be a non-periodic real analytic function on $[0, \infty)$ with $\lim_{t \to \infty} r(t) = 0$. Then $\mathbb{C}^{n-1} \times \{\gamma(t) : t \in [0, \infty)\}$ is a set of injectivity for the TSM for $L^p(\mathbb{C}^n)$ with $1 \leq p \leq \infty$. For details on determining sets for real analytic functions, see [14], [15].

In an interesting result, Courant and Hilbert (9, p. 699) had proved that if the circular averages of a function $f$ which is even with respect to a line $L$, vanishes over all circles centered at points of $L$, then $f \equiv 0$. As a consequence of this result, the circular averages of a function $f$ vanish over all circles centered at points of $L$ if and only if $f$ is odd with respect to $L$, (see [2], Lemma 6.3). Hence, any line $L$ in $\mathbb{R}^2$ is not a set of injectivity for the spherical means for the odd functions about $L$.

In 1996, Agranovsky and Quinto have completely characterized the sets of injectivity for the Euclidean spherical means for compactly supported functions on $\mathbb{R}^2$. Their result says that the exceptional set for the sets of injectivity is a very thin set which consists of a Coxeter system of lines union finitely many points. Following theorem is their main result.
Theorem 1.1. [2] A set $S \subset \mathbb{R}^2$ is a set of injectivity for the spherical means for $C^r(\mathbb{R}^2)$ if and only if $S \not\subseteq \omega(\Sigma_N) \cup F$, where $\omega$ is a rigid motion of $\mathbb{R}^2$, $\Sigma_N = \bigcup_{n=1}^{N-1}\{te^{\frac{i\pi}{n}}: t \in \mathbb{R}\}$ is a Coxeter system of $N$ lines and $F$ is a finite set in $\mathbb{R}^2$.

In particular, any closed curve is a set of injectivity for $C^r(\mathbb{R}^2)$. In fact, Agranovsky et al. [1] further prove that the boundary of any bounded domain in $\mathbb{R}^n$ ($n \geq 2$) is set of injectivity for the spherical means on $L^p(\mathbb{R}^n)$, with $1 \leq p \leq \frac{2n}{n+1}$. For $p > \frac{2n}{n+1}$, unit sphere $S^{n-1}$ is an example of non-injectivity set in $\mathbb{R}^n$. This result has been generalized for certain weighted spherical means, (see [12]). In general, the question of set of injectivity for the spherical means with real analytic weight is still open. In [12], it has been shown that $S^{n-1}$ is a set of injectivity for the spherical means with real analytic weight for the class of radial functions on $\mathbb{R}^n$.

An analogue of Theorem 1.1 in the higher dimensions is still open and appeared as a conjecture in their work [2]. It says that the sets of non-injectivity for the Euclidean spherical means are contained in a certain algebraic variety. Following is their question.

Conjecture [2]. A set $S \subset \mathbb{R}^n$ is a set of injectivity for the spherical means for $C^r(\mathbb{R}^2)$ if and only if $S \not\subseteq \omega(\Sigma) \cup F$, where $\omega$ is a rigid motion of $\mathbb{R}^n$, $\Sigma$ is the zero set of a homogeneous harmonic polynomial and $F$ is an algebraic variety in $\mathbb{R}^n$ of co-dimension at most 2.

This conjecture remains unsolved, however a partial result related to this conjecture has been proved by Kuchment et al. [4]. They also present a survey on the recent developments towards the above conjecture. However, in this article, we observe that this conjecture does not continue to hold for the spherical means on the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$. In fact result on $\mathbb{H}^1$ (see [19]) is an adverse to the Euclidean result, Theorem 1.1 on $\mathbb{R}^2$.

In more general, let $f$ be a non-zero function in $L^2(\mathbb{C}^n)$ and write $S(f) = \{z \in \mathbb{C}^n: f \times \mu_r(z) = 0, \forall r > 0\}$. Our main problem is to describe completely the geometrical structure of $S(f)$ that would characterize which “sets” are set of injectivity for the TSM. For a type function $f(z) = \tilde{a}(|z|)P(z) \in L^2(\mathbb{C}^n) \cap C(\mathbb{C}^n)$, where $P \in H_{p,q}$ is the space of bigraded homogeneous harmonic polynomials on $\mathbb{C}^n$. In the article [19], it has been shown that $S(f) = P^{-1}(0) \cup F$, where $F$ is the union of finitely many spheres centered at the origin. Since $P$ is harmonic, by maximal principle $P^{-1}(0)$ can not contain the boundary of any bounded domain in $\mathbb{C}^n$. Hence the boundary of any bounded domain would be a possible candidate for set of injectivity for the TSM, (see [3] [13] [17]).

2. Notation and Preliminaries

We define the twisted spherical means which arise in the study of spherical means on Heisenberg group. The group $\mathbb{H}^n$, as a manifold, is $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t)(w, s) = (z + w, t + s + \frac{1}{2} \Im(z, \bar{w})), \quad z, w \in \mathbb{C}^n \text{ and } t, s \in \mathbb{R}.$$
Let $\mu_s$ be the normalized surface measure on the sphere $\{(z,0) : |z| = s\} \subset \mathbb{H}^n$. The spherical means of a function $f$ in $L^1(\mathbb{H}^n)$ are defined by

$$f * \mu_s(z,t) = \int_{|w|=s} f((z,t)(-w,0)) \, d\mu_s(w).$$

Thus the spherical means can be thought of as convolution operators. An important technique in many problems on $\mathbb{H}^n$ is to take partial Fourier transform in the $t$-variable to reduce matters to $\mathbb{C}^n$. Let

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t) e^{i\lambda t} \, dt$$

be the inverse Fourier transform of $f$ in the $t$-variable. Then a simple calculation shows that

$$(f * \mu_s)^\lambda(z) = \int_{-\infty}^{\infty} f * \mu_s(z,t) e^{i\lambda t} \, dt$$

$$= \int_{|w|=s} f^\lambda(z-w) e^{i\hat{\mu}(z,w)} \, d\mu_s(w)$$

$$= f^\lambda \times_{\lambda} \mu_s(z),$$

where $\mu_s$ is now being thought of as normalized surface measure on the sphere $S_0(t) = \{z \in \mathbb{C}^n : |z| = s\}$ in $\mathbb{C}^n$. Thus the spherical mean $f * \mu_s$ on the Heisenberg group can be studied using the $\lambda$-twisted spherical mean $f^\lambda \times_{\lambda} \mu_s$ on $\mathbb{C}^n$. For $\lambda \neq 0$, a further scaling argument shows that it is enough to study these means for the case of $\lambda = 1$.

Let $\mathcal{F} \subseteq L^1_{\text{loc}}(\mathbb{C}^n)$. We say $S \subseteq \mathbb{C}^n$ is a set of injectivity for twisted spherical means for $\mathcal{F}$ if for $f \in \mathcal{F}$ with $f \times \mu_r(z) = 0, \forall r > 0$ and $\forall z \in S$, implies $f = 0$ a.e. The results on set of injectivity differ in the choice of sets and the class of functions considered. We would like to refer to [3] [13, 18], for some results on the sets of injectivity for the TSM.

We need the following basic facts from the theory of bigraded spherical harmonics (see [20], p.62 for details). We shall use the notation $K = U(n)$ and $M = U(n-1)$. Then, $S^{2n-1} \cong K/M$ under the map $kM \to k.e_n, k \in U(n)$ and $e_n = (0,\ldots,1) \in \mathbb{C}^n$. Let $\hat{K}_M$ denote the set of all equivalence classes of irreducible unitary representations of $K$ which have a nonzero $M$-fixed vector. It is known that each representation in $\hat{K}_M$ has a unique nonzero $M$-fixed vector, up to a scalar multiple.

For a $\delta \in \hat{K}_M$, which is realized on $V_\delta$, let $\{e_1,\ldots,e_{d(\delta)}\}$ be an orthonormal basis of $V_\delta$ as the $M$-fixed vector. Let $t_j^\delta(k) = \langle e_i, \delta(k)e_j \rangle$, $k \in K$ and $\langle,\rangle$ stand for the inner product on $V_\delta$. By Peter-Weyl theorem, it follows that $\{\sqrt{d(\delta)} t_j^\delta : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M\}$ is an orthonormal basis of $L^2(K/M)$ (see [20], p.14 for details). Define $Y_j^\delta(\omega) = \sqrt{d(\delta)} t_j^\delta(k)$, where $\omega = k.e_n \in S^{2n-1}$, $k \in K$. It then follows that $\{Y_j^\delta : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M,\}$ forms an orthonormal basis for $L^2(S^{2n-1})$. 


For our purpose, we need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way. See [10], p.253, for details. For $p, q \in \mathbb{Z}_+$, let $P_{p,q}$ denote the space of all bigraded polynomials $P$ in $z$ and $\bar{z}$ of the form
\[
P(z) = \sum_{|\alpha| = p} \sum_{|\beta| = q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.
\]
Let $H_{p,q} = \{P \in P_{p,q} : \Delta P = 0\}$. The elements of $H_{p,q}$ are called the bigraded solid harmonics on $\mathbb{C}^n$. The group $K$ acts on $H_{p,q}$ in a natural way. It is easy to see that the space $H_{p,q}$ is $K$-invariant. Let $\pi_{p,q}$ denote the corresponding representation of $K$ on $H_{p,q}$. Then representations in $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_{p,q} : p, q \in \mathbb{Z}_+\}$.

Define the bigraded spherical harmonic by $Y^{p,q}_j(\omega) = \sqrt{d(p,q)} f^{p,q}_j(k)$. Then $\{Y^{p,q}_j : 1 \leq j \leq d(p,q), p, q \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{2n-1})$. Therefore, for a continuous function $f$ on $\mathbb{C}^n$, writing $z = \rho \omega$, where $\rho > 0$ and $\omega \in S^{2n-1}$, we can expand the function $f$ in terms of spherical harmonics as
\[
f(\rho \omega) = \sum_{p,q \geq 0} \sum_{j=1}^{\text{dim}(p,q)} a^{p,q}_j(\rho) Y^{p,q}_j(\omega),
\]
where the series on the right-hand side converges uniformly on every compact set $K \subseteq \mathbb{C}^n$. The functions $a^{p,q}_j$ are called the spherical harmonic coefficients of $f$ and function $a^{p,q}(\rho)Y^{p,q}(\omega)$ is known as the type function.

We also need an expansion of functions on $\mathbb{C}^n$ in terms of Laguerre functions $\varphi_k^{n-1}$s, which is know as special Hermite expansion. The special Hermite expansion is a useful tool in the study of convolution operators and is related to the spectral theory of sub-Laplacian on the Heisenberg group $H^n$. However, more details can be found in [20].

For $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, let $\pi_\lambda$ be the unitary representation of $H^n$ on $L^2(\mathbb{R}^n)$ given by
\[
\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda |x|} \varphi(x + y), \varphi \in L^2(\mathbb{R}^n).
\]
A celebrated theorem of Stone and von Neumann says that $\pi_\lambda$ is irreducible and up to unitary equivalence $\{\pi_\lambda : \lambda \in \mathbb{R}\}$ are all the infinite dimensional unitary irreducible representations of $H^n$. Let
\[
T = \frac{\partial}{\partial t}, X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, j = 1, 2, \ldots, n.
\]
Then $\{T, X_j, Y_j : j = 1, \ldots, n\}$ is a basis for the Lie Algebra $\mathfrak{h}^n$ of all left invariant vector fields on $H^n$. Define $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$, the second order differential operator which is known as the sub-Laplacian of $H^n$. The representation $\pi_\lambda$ induces a representation $\pi_\lambda^* \mathfrak{h}^n$, on the space of $C^\infty$ vectors in $L^2(\mathbb{R}^n)$ is defined by
\[
\pi_\lambda^*(X)f = \left. \frac{d}{dt} \right|_{t=0} \pi_\lambda(\exp tX)f.
\]
An easy calculation shows that $\pi^*(X_j) = i\lambda x_j$, $\pi^*(Y_j) = \frac{\partial}{\partial x_j}$, $j = 1, 2, \ldots, n$. Therefore, $\pi_\lambda^*(\mathcal{L}) = -\Delta_x + \lambda^2|\lambda|^2 =: H(\lambda)$, the scaled Hermite operator. The eigenfunction of $H(\lambda)$ are given by $\phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{2}}\phi_\alpha(\sqrt{|\lambda|}x)$, $\alpha \in \mathbb{Z}_n^+$, where $\phi_\alpha$ are the Hermite functions on $\mathbb{R}^n$. Since $H(\lambda)\phi_\alpha^\lambda = (2|\lambda| + n)|\lambda|\phi_\alpha^\lambda$. Therefore,

$$\mathcal{L} (\pi_\lambda(z,t)\phi_\alpha^\lambda, \phi_\beta^\lambda) = (2|\lambda| + n)|\lambda| \left(\pi_\lambda(z,t)\phi_\alpha^\lambda, \phi_\beta^\lambda\right).$$

Thus the entry functions $(\pi_\lambda(z,t)\phi_\alpha^\lambda, \phi_\beta^\lambda)$, $\alpha, \beta \in \mathbb{Z}_n^+$ are eigenfunctions for $\mathcal{L}$. As $(\pi_\lambda(z,t)\phi_\alpha^\lambda, \phi_\beta^\lambda) = e^{i\lambda t}(\pi_\lambda(z)\phi_\alpha^\lambda, \phi_\beta^\lambda)$, these eigenfunctions are not in $L^2(H^n)$. However for a fix $t$, they are in $L^2(\mathbb{C}^n)$. Define $\mathcal{L}_\lambda$ by $\mathcal{L} (e^{i\lambda t}f(z)) = e^{i\lambda t}L_\lambda f(z)$. Then the functions

$$\phi_{\alpha\beta}^\lambda(z) = (2\pi)^{-\frac{n}{2}}(\pi_\lambda(z)\phi_\alpha^\lambda, \phi_\beta^\lambda),$$

are eigenfunction of the operator $L_\lambda$ with eigenvalue $2|\lambda| + n$. The functions $\phi_{\alpha\beta}^\lambda$’s are called the special Hermite functions and they form an orthonormal basis for $L^2(\mathbb{C}^n)$ (see [21], Theorem 2.3.1, p.54). Thus, for $g \in L^2(\mathbb{C}^n)$, we have the expansion

$$g = \sum_{\alpha,\beta} \langle g, \phi_{\alpha\beta}^\lambda \rangle \phi_{\alpha\beta}^\lambda.$$

To further simplify this expansion, let $\varphi_{k,\lambda}^{n-1}(z) = \varphi_k^{n-1}(\sqrt{|\lambda|}z)$, the Leguerre function of degree $k$ and order $n - 1$. The special Hermite functions $\phi_{\alpha\alpha}^\lambda$ satisfy the relation

$$(2.3) \quad \sum_{|\alpha| = k} \phi_{\alpha\alpha}^\lambda(z) = (2\pi)^{-\frac{n}{2}}|\lambda|^{\frac{n}{2}}\varphi_{k,\lambda}^{n-1}(z).$$

Let $g$ be a function in $L^2(\mathbb{C}^n)$. Then $g$ can be expressed as

$$g(z) = (2\pi)^{-\frac{n}{2}}|\lambda|^{\frac{n}{2}}\sum_{k=0}^{\infty} g_x \varphi_{k,\lambda}^{n-1}(z),$$

whenever $\lambda \in \mathbb{R}^+$, (see [21], p.58). In particular, for $\lambda = 1$, we have

$$(2.4) \quad g(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} g_x \varphi_{k}^{n-1}(z),$$

which is called the special Hermite expansion for $g$.

3. SETS OF INJECTIVITY FOR THE TWISTED SPHERICAL MEANS

In this section, we prove that any Coxeter system of even number of hyperplanes intersecting along a line is a set of injectivity for the TSM for $L^p(\mathbb{C}^n)$ ($n \geq 2$), for $1 \leq p \leq 2$. Then, we prove that the set $S^{n-1}_{R^2} \times \mathbb{C}$ is a set of injectivity for the TSM for a certain class of functions on $\mathbb{C}^{n+1}$. In the first case, we deduce a density result for $L^p(\mathbb{C}^n)$ with $2 \leq p < \infty$.

In order to prove our main results, we need the following results from [13, 19].
Theorem 3.1. [13] Let \( f \) be a function on \( \mathbb{C}^n \) such that \( e^{\frac{|z|^2}{2}} f(z) \in L^p(\mathbb{C}^n) \), for \( 1 \leq p \leq \infty \). Suppose \( f \times \mu_r(z) = 0, \forall \, r > 0 \) and \( \forall \, z \in S^{2n-1}_R \). Then \( f = 0 \) a.e. on \( \mathbb{C}^n \).

Remark 3.2. For \( \eta \in \mathbb{C}^n \), define the left twisted translate by

\[ \tau_{\eta} f(\xi) = f(\xi - \eta)e^{i\text{Im}(\xi, \eta)}. \]

Then \( \tau_{\eta}(f \times \mu_r) = \tau_{\eta} f \times \mu_r \). Since the function space considered as in the above Theorem 3.1 is not twisted translation invariant, it follows that a sphere centered off the origin is not set of injectivity for the TSM on \( \mathbb{C}^n \). The author has generalized Theorem 3.1 for certain weighted twisted spherical means, (see [15]).

As the space \( L^p(\mathbb{C}^n) \) is twisted translations invariant, to prove any Coxeter system of even number of hyperplanes intersecting along a line is a set of injectivity for the TSM for \( L^p(\mathbb{C}^n) \), it is enough to prove that the set \( \mathbb{C}^n \times \Sigma_{2N} \), where \( \bigcup_{l=0}^{2N-1} \{ t e^{i\alpha} : t \in \mathbb{R} \} \), is a set of injectivity for the TSM for \( L^p(\mathbb{C}^n) \).

Theorem 3.3. [19] Let \( f \in L^p(\mathbb{C}) \), for \( 1 \leq p \leq 2 \). Suppose \( f \times \mu_r(z) = 0, \forall \, r > 0 \) and \( \forall \, z \in \Sigma_{2N} \). Then \( f = 0 \) a.e. on \( \mathbb{C} \).

Using Theorem 3.3 we prove the following result. Let \( S_n = \mathbb{C}^n \times \Sigma_{2N} \).

Theorem 3.4. Let \( f \in L^p(\mathbb{C}^n) \), for \( 1 \leq p \leq 2 \). Suppose \( f \times \mu_r(z) = 0, \forall \, r > 0 \) and \( \forall \, z \in S_n \). Then \( f = 0 \) a.e. on \( \mathbb{C}^n \).

Proof. Since \( f \times \mu_r(z) = 0, \forall \, r > 0 \), by polar decomposition, it follows that \( f \times \varphi_k^{-1}(z) = 0, \forall \, k \in \mathbb{Z}_+ \). Given that \( f \in L^p(\mathbb{C}^n) \). By convolving \( f \) with a right and radial compactly supported smooth approximate identity, we can assume \( f \in L^2(\mathbb{C}^n) \). In order to prove the result on \( \mathbb{C}^n \), We first prove the result on \( \mathbb{C}^2 \) and then by induction hypothesis on \( n \), we deduce it for \( \mathbb{C}^n \). Since \( \varphi_k^1(z_1, z_2) = \sum_{\beta_1 + \beta_2 = k} \varphi_{\beta_1}^0(z_1) \varphi_{\beta_2}^0(z_2) \).

Therefore, we can write

\[
\begin{align*}
f \times \varphi_k^1(z_1, z_2) &= \sum_{\beta_1 + \beta_2 = k} \int_{\mathbb{C}^2} f(z_1 - w_1, z_2 - w_2) \varphi_{\beta_1}^0(w_1) \varphi_{\beta_2}^0(w_2) \\
& \quad \times e^{i\text{Im}(z_1, \bar{w_1} + z_2, \bar{w_2})} dw_1 dw_2,
\end{align*}
\]

\[
= \sum_{\beta_1 + \beta_2 = k} \int_{\mathbb{C}} f \times_2 \varphi_{\beta_2}(z_1 - w_1, z_2) \varphi_{\beta_1}(w_1) e^{i\text{Im}(z_1, \bar{w_1})} dw_1
\]

\[
= \sum_{\beta_1 + \beta_2 = k} \int_{\mathbb{C}} F_{z_2, \beta_2}(z_1 - w_1) \varphi_{\beta_1}^0(w_1) e^{i\text{Im}(z_1, \bar{w_1})} dw_1
\]

\[
= \sum_{\beta_1 + \beta_2 = k} F_{z_2, \beta_2} \times_1 \varphi_{\beta_1}^0(z_1),
\]

(3.1)
where the function \( F_{z_2,\beta_2} \) is defined by
\[
F_{z_2,\beta_2}(z_1) = \int_{\mathbb{C}} f(z_1, z_2 - w_2) \varphi_{\beta_2}^0(w_2) e^{\frac{i}{2} \Im(z_2 \bar{w}_2)} dw_2.
\]

For fixed \( z_2 \), using the Minkowski integral inequality, it can show that the function \( F_{z_2,\beta_2} \in L^2(\mathbb{C}) \). By the given condition, \( f \times \varphi_1^\epsilon(z_1, z_2) = 0, \forall (z_1, z_2) \in S_2 \) and \( \forall k \in \mathbb{Z}_+ \).

Therefore, from equation (3.1), we can write
\[
\sum_{\beta_1 + \beta_2 = k} F_{z_2,\beta_2} \times_1 \varphi_{\beta_1}^0(z_1) = 0, \forall (z_1, z_2) \in S_2 \text{ and } \forall k \in \mathbb{Z}_+.
\]

As the above equation is valid for each \( k \in \mathbb{Z}_+ \), it follows that the sum over each of the diagonal \( \beta_1 + \beta_2 = k \) is zero. Using the facts that set \( \{ \varphi_{\beta_1}^0 : \beta_1 \in \mathbb{Z}_+ \} \) form an orthogonal basis for \( L^2(\mathbb{C}) \) and \( S_2 = \mathbb{C} \times \Sigma_{2N} \), it follows that
\[
F_{z_2,\beta_2} \times_1 \varphi_{\beta_1}^0(z_1) = 0, \forall \beta_1, \beta_2 \in \mathbb{Z}_+.
\]

By equations (3.1), we have
\[
f \times (\varphi_{\beta_1}^0 \varphi_{\beta_2}^0)(z_1, z_2) = 0, \forall (z_1, z_2) \in S \text{ and } \forall \beta_1, \beta_2 \in \mathbb{Z}_+.
\]

Now, we can write,
\[
f \times (\varphi_{\beta_1}^0 \varphi_{\beta_2}^0)(z_1, z_2) = \int_{\mathbb{C}} G_{z_1,\beta_1}(z_2 - w_2) \varphi_{\beta_1}^0(w_2) e^{\frac{i}{2} \Im(z_2 \bar{w}_2)} dw_1 = G_{z_1,\beta_1} \times_2 \varphi_{\beta_2}^0(z_2).
\]

By the given condition, we have \( G_{z_1,\beta_1} \times_2 \varphi_{\beta_2}^0(z_2) = 0, \forall \beta_1, \beta_2 \in \mathbb{Z}_+ \). For each fixed \( \beta_1 \), in view of Theorem 3.3, we can conclude that \( G_{z_1,\beta_1}(z_2) = 0, \forall (z_1, z_2) \in \mathbb{C}^2 \).

That is, \( f \times_1 \varphi_{\beta_1}(z_1, z_2) = 0, \forall \beta_1 \in \mathbb{Z}_+ \). Therefore,
\[
f \times \varphi_1^0(z_1, z_2) = \sum_{\beta_1 + \beta_2 = k} \int_{\mathbb{C}} f \times_1 \varphi_{\beta_1}(z_1, z_2 - w_2) \varphi_{\beta_1}^0(w_2) e^{\frac{i}{2} \Im(z_2 \bar{w}_2)} dw_2 = 0,
\]

for all \( k \in \mathbb{Z}_+ \). Hence, we conclude that \( f = 0 \), a.e. \( \mathbb{C}^2 \). In order to prove the result for \( n > 2 \), we use the induction hypothesis on \( n \). Suppose the result is true for \( n - 1 \) with \( n > 2 \). That is, the set \( S_{n-1} = \mathbb{C}^{n-2} \times \Sigma_{2N} \) is a set of injectivity for the TSM for \( L^2(\mathbb{C}^{n-1}) \). Let \( k = \beta_1 + \beta_2 + \cdots + \beta_n = \beta_1 + |\gamma| \) and \( z = (z_1, z_2, \ldots, z_n) = (z_1, z') \). Then, as similar to \( \mathbb{C}^2 \) case, we can write
\[
f \times \varphi_k^{n-1}(z_1, z') = \sum_{\beta_1 + |\gamma| = k} F_{z'_\gamma,\gamma} \times_1 \varphi_{\beta_1}^0(z_1).
\]

Given that
\[
f \times \varphi_k^{n-1}(z_1, z') = \sum_{\beta_1 + |\gamma| = k} F_{z'_\gamma,\gamma} \times_1 \varphi_{\beta_1}^0(z_1) = 0, \forall (z_1, z') \in S_n \text{ and } \forall k \in \mathbb{Z}_+.
\]

Since, the set \( \{ \varphi_{\beta_1}^0 : \beta_1 \in \mathbb{Z}_+ \} \) form an orthogonal basis for \( L^2(\mathbb{C}) \) and \( S_n = \mathbb{C}^{n-1} \times \Sigma_{2N} \), it follows that \( F_{z'_\gamma,\gamma} \times_1 \varphi_{\beta_1}^0(z_1) = 0, \forall \beta_1 \in \mathbb{Z}_+ \) and \( \forall |\gamma| \in \mathbb{Z}_+ \). This in turn implies that for each fixed \( \gamma \in \mathbb{Z}_+^{n-1} \), we get \( F_{z'_\gamma,\gamma} \times_1 \varphi_{\beta_1}^0(z_1) = 0, \forall \beta_1 \in \mathbb{Z}_+ \) and \( \forall (z_1, z') \in \mathbb{C} \times S_{n-1} \). Once again using the orthogonality of the set \( \{ \varphi_{\beta_1}^0 : \beta_1 \in \mathbb{Z}_+ \} \).
injectivity for the TSM for \( L \) for a certain class of functions on the TSM for \( L \).

Remark 3.5. (a) By the proof of Theorem 3.4, it reveals that if \( S \) is a set of injectivity for the TSM for \( L^p(\mathbb{C}) \), then the set \( \mathbb{C}^{n-1} \times S \) will be a set of injectivity for the TSM for \( L^p(\mathbb{C}^n) \). That is, the sets of injectivity for the TSM on \( \mathbb{C}^n \) can be embedded into the sets injectivity for the TSM on \( \mathbb{C}^{n+1} \).

(b) We would like to mention that the question of Coxeter system of odd number of hyperplanes intersecting along a line is a set of injectivity for the TSM, and it follows that for each fixed \( z \), \( z \in \mathbb{C} \) and \( |z| = 1 \).

Next, we prove that the set \( S_{n+1} = S_{R}^{n+1} \times \mathbb{C} \) is a set of injectivity for the TSM for a certain class of functions on \( \mathbb{C}^{n+1} \). Let \( z = (z', z_{n+1}) \in \mathbb{C}^{n+1} \).

Theorem 3.6. Let \( f \) be a function on \( \mathbb{C}^{n+1} \) such that \( e^{\frac{1}{2}|z'|^2} f(z) \in L^p(\mathbb{C}^{n+1}) \) for \( 1 \leq p \leq \infty \). Suppose \( f \times \mu_r(z) = 0, \forall r > 0 \) and \( \forall z \in S_{n+1} \). Then \( f = 0 \) a.e. on \( \mathbb{C}^{n+1} \).

Proof. Let \( k = \beta_1 + \beta_2 + \cdots + \beta_{n+1} = |\gamma| + \beta_{n+1} \). Then, we can write

\[
f \times \varphi_k^n(z', z_{n+1}) = \sum_{|\gamma| + \beta_{n+1} = k} F_{z', \gamma} \times (n+1) \varphi_{\beta_{n+1}}^0(z_{n+1}).
\]

By the given conditions, we have

\[
f \times \varphi_k^n(z', z_{n+1}) = \sum_{|\gamma| + \beta_{n+1} = k} F_{z', \gamma} \times (n+1) \varphi_{\beta_{n+1}}^0(z_{n+1}) = 0,
\]

for all \( (z', z_{n+1}) \in S_{n+1} \) and \( \forall k \in \mathbb{Z}_+ \). Since, the set \( \{ \varphi_{\beta_1}^0 : \beta_{n+1} \in \mathbb{Z}_+ \} \) form an orthogonal basis for \( L^2(\mathbb{C}) \), it follows that \( F_{z', \gamma} \times (n+1) \varphi_{\beta_1}^0(z_{n+1}) = 0, \forall \beta_{n+1} \in \mathbb{Z}_+ \) and \( \forall |\gamma| \in \mathbb{Z}_+ \). This in turn implies that for each fixed \( \gamma \in \mathbb{Z}_+^n, F_{z', \gamma} \times (n+1) \varphi_{\beta_1}^0(z_{n+1}) = 0, \forall \beta_{n+1} \in \mathbb{Z}_+ \) and \( \forall (z', z_{n+1}) \in S_{n+1} \). Once again using the orthogonality of the set \( \{ \varphi_{\beta_{n+1}}^0 : \beta_{n+1} \in \mathbb{Z}_+ \} \) in \( L^2(\mathbb{C}) \), we conclude that \( F_{z', \gamma}(z_{n+1}) = 0, \forall \gamma \in \mathbb{Z}_+^n \) and \( \forall \gamma \in \mathbb{Z}_+^n \). Hence, we can write

\[
\sum_{|\gamma| = k} F_{z', \gamma}(z_{n+1}) = \sum_{|\gamma| = k} \int_{\mathbb{C}^n} f(z' - w', z_{n+1}) \Pi_{j=1}^n \varphi_{\beta_j}^0(w_j) e^{\frac{i}{2} \text{Im}(z', \bar{w}')} dw' = 0,
\]

\( \forall k \in \mathbb{Z}_+ \) and \( \forall z' \in S_{R}^{2n-1} \). Therefore, in view of Theorem 3.1, we infer that \( f = 0 \) a.e. \( \mathbb{C}^n \).
Since the set $S_n = \mathbb{C}^{n-1} \times \Sigma_{2N}$ is a set of injectivity for the TSM for $L^p(\mathbb{C}^n)$, with $1 \leq p \leq 2$. As a dual problem, it is natural to ask that $S_n$ is a set of density for $L^q(\mathbb{C}^n)$, for $2 \leq q < \infty$. Let $C^\circ_q(\mathbb{C})$ denote the space of radial compactly supported continuous functions on $\mathbb{C}$. Let $\tau_z f(w) = f(z-w)e^{\frac{\pi}{2} \text{Im}(z, w)}$.

**Proposition 3.7.** The subspace $\mathcal{F}(S_n) = \text{Span}\{\tau_z f : z \in S_n, f \in C^\circ_q(\mathbb{C}^n)\}$ is dense in $L^q(\mathbb{C})$, for $2 \leq q < \infty$.

**Proof.** Let $\frac{1}{p} + \frac{1}{q} = 1$. Then $1 \leq p \leq 2$. By Hahn-Banach theorem, it is enough to show that $\mathcal{F}(S_n)^\perp = \{0\}$. Let $g \in L^p(\mathbb{C}^n)$ be such that

$$\int_{\mathbb{C}} \tau_z f(w)g(w)dw = 0, z \in \Sigma_{2N}, \forall f \in C^\circ_q(\mathbb{C}^n).$$

That is,

$$\bar{g} \times f(z) = f \times g(z) = 0.$$

Let the support of $f$ be contained in $[0, t]$. Then by passing to the polar decomposition, we get

$$\int_{r=0}^{t} \bar{g} \times \mu_r(z) f(r)r^{2n-1}dr = 0.$$

By differentiating the above equation, it follows that $\bar{g} \times \mu_r(z) = 0, \forall t > 0$ and $\forall z \in S_n$. Thus by Theorem 3.4, we conclude that $g = 0$ a.e. on $\mathbb{C}^n$.

**Remark 3.8.** (a) In the article by Agranovskv et al. [3], it has been shown that boundary of any bounded domain in $\mathbb{C}^n$ is a set of injectivity for the TSM for a class of functions $f$ satisfying $f(z)e^{\frac{1}{4} \epsilon |z|^2} \in L^p(\mathbb{C}^n)$, for some $\epsilon > 0$ and $1 \leq p \leq \infty$. However to prove this result for $\epsilon = 0$ is an open problem. The sphere $S^{2n-1}_R$ is an example with $\epsilon = 0$, as mentioned in Theorem 3.1.

We are thinking to do away with exponential condition. Though it is not possible for sphere, because of the relations $\varphi_k^{n-1} \times \mu_r(z) = B(n, k)\varphi_k^{n-1}(r)\varphi_k^{n-1}(|z|)$. We are working for the real analytic curves $\gamma$ having non-constant curvature can be the sets of injectivity for the TSM for $L^q(\mathbb{C})$ with $1 \leq q \leq 2$. We know that the spectral projections $Q_k = f \times \varphi_k^{n-1}$ is a real analytic function on $\mathbb{C}^n$. Using the Hecke-Bochner identity for the spectral projections, we have derived a real analytic expansion for $Q_k$ in the article [19] as

$$Q_k(z) = \sum_{p=0}^{k} C_{k-p}^{p} \varphi^{p}_{k-p}(z) + \sum_{q=0}^{\infty} C^{0q}_k z^q \varphi^{q}_k(z).$$

Suppose $Q_k(z) = 0, \forall k \in \mathbb{Z}_+$, and $\forall z \in \gamma$. For a curve $\gamma(t) = r(t)e^{it}$ of non-constant curvature, radius $r(t)$ will vary in an interval. The fact that $Q_k$ is a real analytic function, $Q_k(\gamma(t))$ must be a real analytic function, which vanishes over an interval. It is interesting to know that can $Q_k(\gamma(t))$ will vanish for all $t \in \mathbb{R}$. In particular, it can be possible when $r(t)$ is a non-periodic function. For example, spiral $\gamma(t) = \{(e^t \cos t, e^t \sin t) : t \in (\infty, 0]\}$. Hence spiral is a set of injectivity for the TSM for $L^q(\mathbb{C})$ with $1 \leq q \leq \infty$. For a more general example, let $\gamma(t) = r(t)e^{it}$, where $r(t)$ be a non-periodic real analytic function on $[0, \infty)$ with $\lim_{t \to \infty} r(t) = 0$. Then $\gamma(t)$ is a set of injectivity for the TSM for $L^q(\mathbb{C})$ with
1 ≤ q ≤ ∞. Moreover, γ(t) is a determining curve for any real analytic function on C.

(b) We are working for the set S = R × R ∪ R × iR can be set of injectivity for the TSM for L^q(C^2), 1 ≤ q ≤ 2. We start with a small class of functions \( f(z_1, z_2) = z_1^p \varphi_0(z_1) h(z_2) \). Suppose \( f \times \mu_r(z_1, z_2) = 0, \forall r > 0 \) and \( \forall (z_1, z_2) \in R \times R \cup R \times iR \). Then \( f = 0 \) a.e. on C^2. If it goes to happen that S is a set of injectivity for the TSM for L^q(C^2) then it would be a surprise contrast to the sets of injectivity for the Euclidean spherical means on R^4, where the minimal dimension of a set of injectivity is three.

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