Nearest-Neighbor Geostatistical Models for Non-Gaussian Data

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March 4, 2022

Abstract

We develop a class of nearest-neighbor geostatistical models for non-Gaussian data that provides flexibility and scalability. The model is defined on a directed acyclic graph through a weighted combination of first-order spatially varying conditional densities, for each one of a given number of neighbors. It is then extended to a proper spatial process, referred to as the nearest-neighbor mixture transition distribution process (NNMP). We provide conditions to construct general NNMP models with pre-specified stationary marginal distributions. We also establish lower bounds for the resulting strength of the tail dependence, demonstrating the flexibility of NNMPs to quantify multivariate dependence using a bivariate distribution specification. We formulate a Bayesian hierarchical model, focusing on spatially dependent weights, that is locally adaptive, and provides accurate predictions. NNMPs lay out a new computational approach to handling spatial data sets, leveraging a mixture model structure to avoid computational issues that arise from large matrix operations. We illustrate the benefits of the NNMP framework using synthetic data examples as well as analyzing sea surface temperature observations from the Mediterranean Sea.

Keywords: Bayesian hierarchical models; Copulas; Mixture transition distribution; Stationary marginal distributions; Tail dependence
1 Introduction

Gaussian processes have been widely used as an underlying structure in the model-based analysis of irregularly located spatial data in order to capture short range variability. The fruitfulness of these spatial models owes to the simple characterization of the Gaussian process by a mean and a covariance function, and the optimal prediction it provides that justifies kriging. However, the assumption of Gaussianity is restrictive in many fields where the data exhibits non-Gaussian features such as heavy tails or skewness. Moreover, it is not clear that approaches for scalable models, such as low-rank models and sparsity-inducing models, are extendable when it comes to non-Gaussian data. Accordingly, this article aims at developing a class of geostatistical models that is scalable and customizable to general non-Gaussian distributions, with particular focus on continuous data.

Gaussian process-based approaches for modeling continuous, non-Gaussian geostatistical datasets proceed by either representing skewed or long-tailed distributions as location-scale mixtures of Gaussian distributions, or by applying a transformation to a Gaussian process. In the former approach, it is possible to mix over location parameters to capture skewness (Kim and Mallick, 2004; Zhang and El-Shaarawi, 2010; Mahmoudian, 2017) or over scale parameters to capture heavy tails (Palacios and Steel, 2006; Sun et al., 2015). Morris et al. (2017) and Bevilacqua et al. (2021) provide examples of using both types of mixing. Although these models can accommodate skewness and non-standard tail behavior, modeling through mixtures of Gaussian distributions leads to the same computational issues faced by Gaussian process models.

The approach based on transforming a Gaussian process is usually applied to positive continuous data. A popular family of the so-called trans-Gaussian processes is obtained from the Box-Cox family of non-linear transformations (De Oliveira et al., 1997; Allcroft and Glasbey, 2003). Alternatively, Xu and Genton (2017) consider a different family of transformation, the Tukey g-and-h transformation. After transformation, standard Gaussian process statistical analysis can be carried out on the transformed data. However, given a transformation, it is possible that properties of the Gaussian process are not preserved for the transformed process. For example, the square-root transformation induces a non-
stationary covariance function, even though the covariance function associated with the original Gaussian process is stationary (Wallin and Bolin 2015).

An alternative to Gaussian process-based approaches is to apply a copula for the joint distribution of the underlying spatial process. A copula (Joe 2014) is used to characterize the dependence structure between random variables, separately from the specification of marginal distributions, and thus it has been used to describe non-Gaussian spatial variability with general non-Gaussian marginals; see, e.g., Bárdossy (2006), Ghosh and Mallick (2011), Beck et al. (2020). However, copulas need to be used with careful consideration of their properties in a spatial setting. For example, it is common to assume that spatial processes exhibit stronger dependence at smaller distances. Thus, copulas such as the multivariate Archimedean copula that induce an exchangeable dependence structure are inappropriate. Moreover, it is difficult to strike a good balance between the flexibility provided by a copula and the computational demand required to fit it; see, e.g., Gräler (2014).

Bayesian nonparametric methods have also been explored for geostatistical modeling, starting with the approach in Gelfand et al. (2005) which extends the Dirichlet process (Ferguson 1973) to a prior model for random spatial surfaces. We refer to Müller et al. (2018) for a review. Bayesian nonparametric methods are appealing for their large prior support on the finite-dimensional distributions of the spatial process. However, such models typically require replication at the observed spatial locations for effective inference, and they are computationally prohibitive for large data sets.

The modeling framework proposed in this article is distinctly different from the aforementioned approaches, as it builds on the class of nearest-neighbor processes. Nearest-neighbor processes are spatial processes obtained by extending a joint density over a reference set to the entire domain, based on a sparse directed acyclic graph (DAG). The joint density is derived from a stochastic process, referred to as the parent process. A Gaussian parent process results in the nearest-neighbor Gaussian process (NNGP; Datta et al. 2016a) that has received substantial attention in the recent literature; see, e.g., Datta et al. (2016b), Finley et al. (2019), and Peruzzi et al. (2020), Katzfuss and Guinness (2021).
By construction, the NNGP introduces sparsity in the precision matrix, and thus delivers computational scalability for large spatial data sets.

In this article, we propose a novel family of nearest-neighbor processes, using a mixture transition distribution (MTD) model [Le et al., 1996; Zheng et al., 2022] for the parent process. The parent MTD process is based on a weighted combination of first-order spatially varying conditional densities, each of which depends on a specific neighbor. Such local dependence, together with location-dependent mixture weights, provide flexible descriptions of the spatial variability. We refer to the resulting process as the nearest-neighbor mixture transition distribution process (NNMP). The mixture structure of the parent MTD process gives rise to mixtures for the NNMP finite-dimensional distributions, as well as the convenience of building the multivariate dependence specification through a set of bivariate distributions that define the first-order conditional densities of the parent MTD process. Utilizing this model property, we study the tail dependence and provide results that can guide modeling choices. In addition, extending the temporal MTD framework in [Zheng et al., (2022)], we develop a sufficient condition to construct NNMPs with general stationary marginal distributions. This condition provides a flexible modeling tool to describe the distributions of spatial data that are skewed, heavy-tailed, or have bounded support, as illustrated through several examples. In essence, the NNMP framework offers a flexible class of models that is able to describe complex spatial dependence, coupled with a scalable computational approach leveraged from the mixture structure of the model.

The rest of the article is organized as follows. In Section 2, we formulate the NNMP using the parent MTD process, present the constructive framework for stationary NNMPs, and study the model’s tail dependence properties. Specific examples of NNMP models illustrate different components of the methodology. Section 3 develops the general approach to Bayesian estimation and prediction under NNMP models. In Sections 4 and 5, we demonstrate different NNMP-based spatial models with synthetic data examples and with the analysis of Mediterranean Sea surface temperature data, respectively. Finally, Section 6 concludes with a summary and discussion of future work.
2 Nearest neighbor MTD processes for spatial data

2.1 Modeling framework

Consider a univariate spatial process \( \{Z(v) : v \in \mathcal{D}\} \), where \( \mathcal{D} \subset \mathbb{R}^p \), for \( p \geq 1 \). Let \( \mathcal{S} = \{s_1, \ldots, s_n\} \) be a finite collection of locations in \( \mathcal{D} \), referred to as the reference set. We write the joint density of the random vector \( \mathbf{z}_\mathcal{S} = (Z(s_1), \ldots, Z(s_n))^\top \) as

\[
p(\mathbf{z}_\mathcal{S}) = p(z(s_1)) \prod_{i=2}^n p(z(s_i) | z(s_{i-1}), \ldots, z(s_1)).
\]

(1)

If we regard the conditioning set of \( z(s_i) \) as the set of parents of \( z(s_i) \), the joint density \( p(\mathbf{z}_\mathcal{S}) \) in (1) is a factorization according to a DAG whose vertices are \( z(s_i) \). We obtain a sparse DAG by reducing the conditioning set of \( z(s_i) \) to a smaller subset, denoted as \( z_{\text{Ne}(s_i)} \), with \( \text{Ne}(s_i) \subset \mathcal{S}_i = \{s_1, \ldots, s_{i-1}\} \). We refer to \( \text{Ne}(s_i) \) as the neighbor set for \( s_i \), having at most \( L \) elements with \( L \ll n \). The resulting density for the sparse DAG is

\[
\tilde{p}(\mathbf{z}_\mathcal{S}) = p(z(s_1)) \prod_{i=2}^n p(z(s_i) | z_{\text{Ne}(s_i)}),
\]

(2)

which has been verified as a proper density [Lauritzen 1996].

Choosing the neighbor sets \( \text{Ne}(s_i) \) creates different sparse DAGs. There are different ways to select members from \( \mathcal{S}_i \) for \( \text{Ne}(s_i) \); see, for example, Vecchia (1988), Stein et al. (2004), and Gramacy and Apley (2015). Our selection is based on the geostatistical distance between \( s_i \) and \( s_j \in \mathcal{S}_i \). The selected locations \( s_j \) are placed in ascending order according to the distance, denoted as \( s_{(i_1)}, \ldots, s_{(i_{i_L})} \), where \( i_L := (i - 1) \land L \). We note that the development of the proposed framework holds true for any choice of the neighbor sets.

The crucial step of constructing the process model is the specification of a stochastic process over the reference set \( \mathcal{S} \). This process characterizes \( \tilde{p}(\mathbf{z}_\mathcal{S}) \), so it is appealing to consider processes that have a Markov property, as it naturally connects to the conditional densities in (2). To this end, we consider a MTD process such that the conditional density
for location \( s_i \in \mathcal{S} \) in (2) is

\[
p(z(s_i) \mid z_{\text{Ne}(s_i)}) = \sum_{l=1}^{i_L} w_l(s_i) f_{s_i,l}(z(s_i) \mid z(s_{il})),
\]

(3)

where \( f_{s_i,l} \) is the \( l \)th component conditional density of the mixture density \( p \) for \( s_i \in \mathcal{S} \), and the weights are subject to \( w_l(s_i) \geq 0 \), for every \( s_i \in \mathcal{S} \) and for all \( l \), and \( \sum_{l=1}^{i_L} w_l(s_i) = 1 \).

Spatial dependence characterized by (3) is twofold. First, each component \( f_{s_i,l} \) of the density \( p(z(s_i) \mid z_{\text{Ne}(s_i)}) \) is associated with spatially varying parameters indexed at \( s_i \in \mathcal{S} \), defined by a probability model or a link function. Secondly, the weights \( w_l(s_i) \) are spatially varying. As each component density \( f_{s_i,l} \) depends on a specific neighbor, the weights indicate the contribution of each neighbor of \( s_i \). Besides, the weights adapt to the change of locations. For two different \( s_i, s_j \) in \( \mathcal{S} \), the relative locations of the nearest neighbors \( \text{Ne}(s_i) \) to \( s_i \) are different from that of \( \text{Ne}(s_j) \) to \( s_j \). If all elements of \( \text{Ne}(s_i) \) are very close to \( s_i \), then values of \( (w_1(s_i), \ldots, w_{i_L}(s_i))^\top \) should be quite even. On the other hand, if, for \( s_j \), only a subset of its neighbors in \( \text{Ne}(s_j) \) are close to \( s_j \), then the weights corresponding to this subset should receive larger values. We remark that in general, probability models or link functions for the spatially varying parameters should be considered case by case, given different specifications on the components \( f_{s_i,l} \). Details of the construction for the component densities and the weights are deferred to later sections.

To obtain a properly defined spatial process in \( \mathcal{D} \), we extend (3) to an arbitrary set of non-reference locations \( \mathcal{U} = \{u_1, \ldots, u_r\} \) where \( \mathcal{U} \subset \mathcal{D} \setminus \mathcal{S} \). In particular, we define the conditional density of \( z_{\mathcal{U}} \) given \( z_{\mathcal{S}} \) as

\[
\tilde{p}(z_{\mathcal{U}} \mid z_{\mathcal{S}}) = \prod_{i=1}^{r} p(z(u_i) \mid z_{\text{Ne}(u_i)}) = \prod_{i=1}^{r} \sum_{l=1}^{L} w_l(u_i) f_{u_i,l}(z(u_i) \mid z(u_{il})),
\]

(4)

where the specification on \( w_l(u_i) \) and \( f_{u_i,l} \) for all \( i \) and all \( l \) is analogous to that for (3), except that \( \text{Ne}(u_i) = \{u_{(i1)}, \ldots, u_{(iL)}\} \) are the first \( L \) locations in \( \mathcal{S} \) that are closest to \( u_i \) in terms of geostatistical distance. Building the construction of the neighbor sets \( \text{Ne}(u_i) \) on the reference set ensures that \( \tilde{p}(z_{\mathcal{U}} \mid z_{\mathcal{S}}) \) is a proper density.

Given (3) and (4), we can obtain the joint density \( \tilde{p}(z_{\mathcal{V}}) \) of a realization \( z_{\mathcal{V}} \) over any
finite set of locations $V \subset D$. When $V \subset S$, the joint density $\tilde{p}(z_V)$ is directly available as the appropriate marginal of $\tilde{p}(z_S)$. Otherwise, we have that

$$\tilde{p}(z_V) = \int \tilde{p}(z_U \mid z_S) \tilde{p}(z_S) \prod_{s_i \in S \setminus V} dz(s_i), \quad (5)$$

where $U = V \setminus S$. If $S \setminus V$ is empty, $\tilde{p}(z_V)$ is simply $\tilde{p}(z_U \mid z_S) \tilde{p}(z_S)$.

The resulting spatial process defined using (3) and (4) is a legitimate process over the entire domain $D$. This result is a direct application of the nearest-neighbor process (Datta et al., 2016a), which is constructed from specifying a parent process that defines $\tilde{p}(z_S)$ in (2) over $S$ based on a DAG, and then extending the parent process to arbitrary finite set $U \subset D \setminus S$. We call the proposed model the nearest-neighbor MTD process (NNMP), derived from a parent MTD process. In the subsequent development of the model properties, we will use the associated conditional density

$$p(z(v) \mid z_{Ne(v)}) = \sum_{l=1}^{L} w_l(v) f_{v,l}(z(v) \mid z(v_{(l)})), \quad v \in D, \quad (6)$$

to characterize an NNMP, where $Ne(v)$ contains the first $L$ locations that are closest to $v$, selected from locations in $S$. These locations in $Ne(v)$ are placed in ascending order according to distance, denoted as $v_{(1)}, \ldots, v_{(L)}$.

In general, the joint density $\tilde{p}(z_V)$ of an NNMP is intractable. However, from (5) where both $\tilde{p}(z_U \mid z_S)$ and $\tilde{p}(z_S)$ are products of mixtures, we can recognize that $\tilde{p}(z_V)$ is a finite mixture, which suggests flexibility of the model to capture complex non-Gaussian dependence over the domain $D$. Moreover, we show in Section 2.3 that for some NNMP models, the joint density $\tilde{p}(z_V)$ has a closed-form expression.

Before closing this section, we note that spatial locations are not naturally ordered. Given a distance function, a different topological ordering on the locations $s_i$ results in different neighbor sets $Ne(s_i)$. Therefore, a different sparse DAG with density $\tilde{p}(z_S)$ is created accordingly for model inference. For the NNMP models illustrated in the data examples, we found through simulation experiments that there were no discernible differences between the inferences based on $\tilde{p}(z_S)$, given two different orderings. This observation is
coherent with that from the NNGP models and other literature that considers nearest-neighbor likelihood approximations. Since the approximation of $\tilde{p}(z_S)$ to $p(z_S)$ depends on the information borrowed from the neighbors, as outlined in Datta et al. (2016a), the effectiveness is determined by the size of $\text{Ne}(s_i)$ rather than the ordering.

2.2 NNMPs with stationary marginal distributions

In this section, we develop a sufficient condition to construct NNMPs with general stationary marginal distributions. The result is given in the following proposition, the proof of which can be found in the supplementary material. The key feature of this result is that the condition relies on the distributional assumption for the bivariate distributions that define the MTD component conditional densities in (3) and (4), without the need to impose restrictions on the parameter space.

**Proposition 1.** Consider an NNMP for which the component density $f_{v,l}$ is specified by the conditional density of $U_{v,l}$ given $V_{v,l}$, where the random vector $(U_{v,l}, V_{v,l})$ follows a bivariate distribution with marginal densities $f_{U_{v,l}}$ and $f_{V_{v,l}}$, for $l = 1, \ldots, L$. The NNMP has stationary marginal density $f_Z$ if it satisfies the invariant condition: $Z(s_1) \sim f_Z$, $s_1 \in S$, and for every $v \in D$, $f_Z(z) = f_{U_{v,l}}(z) = f_{V_{v,l}}(z)$, for all $z$ and for all $l$.

This result builds from the one in Zheng et al. (2022) where temporal MTD processes with stationary marginal distributions were constructed. It applies regardless of $Z(v)$ being a continuous, discrete or mixed random variable, thus allowing for a wide range of non-Gaussian marginal distributions and a general functional form, either linear or non-linear, for the expectation with respect to the conditional density $p$ in (3) and (4).

As previously discussed, the mixture formulation of the parent MTD process induces a finite mixture for the finite-dimensional distribution of the NNMP. On the other hand, due to the mixture form, an explicit expression for the covariance function is difficult to derive. A recursive equation can be obtained for a class of NNMP models for which the conditional expectation with respect to $(U_{v,l}, V_{v,l})$ is linear, i.e., $E(U_{v,l} \mid V_{v,l} = z) = a_l(v) + b_l(v) z$ for some $a_l(v), b_l(v) \in \mathbb{R}$, $l = 1, \ldots, L$, and for all $v \in D$. Suppose the NNMP has a stationary
marginal distribution with finite first and second moments. Without loss of generality, we assume the first moment is zero. Then the covariance over any two locations \( v_1, v_2 \in D \) is

\[
\text{Cov}(Z(v_1), Z(v_2)) = \begin{cases} 
\sum_{l=1}^{L} w_l(s_i) b_l(s_i) E(Z(s_j)Z(s_{il})), & v_1 \equiv s_i \in S, v_2 \equiv s_j \in S, \\
\sum_{l=1}^{L} w_l(v_1) b_l(v_1) E(Z(s_j)Z(v_{il})), & v_1 \notin S, v_2 \equiv s_j \in S, \\
\sum_{l=1}^{L} \sum_{l'}^{L} w_{ll'} \{a_{ll'} + b_{ll'} E(Z(v_{il})Z(v_{l'l'})}\}, & v_1, v_2 \notin S,
\end{cases}
\tag{7}
\]

where \( w_{ll'} \equiv w_l(v_1)w_{l'}(v_2) \), \( a_{ll'} \equiv a_l(v_1)a_{l'}(v_2) \), \( b_{ll'} \equiv b_l(v_1)b_{l'}(v_2) \), and without loss of generality, we assume \( i > j \). The covariance in (7) implies that, even though the process has a stationary marginal distribution, the NNMP is second-order non-stationary.

2.3 Construction of NNMP models

The spatially varying conditional densities \( f_{v,l} \) in (6) correspond to a sequence of bivariate distributions indexed at \( v \), namely, the distributions of \((U_{v,l}, V_{v,l})\), for \( l = 1, \ldots, L \). To balance model flexibility and scalability, we build spatially varying distributions by considering the distribution of random vector \((U_l, V_l)\), for \( l = 1, \ldots, L \), and extending some of its parameters to be spatially varying, i.e., indexed in \( v \). To this end, we use a probability model or a link function. We refer to the random vectors \((U_l, V_l)\) as the set of base random vectors. With a careful choice of the model for the spatially varying parameter(s), this construction method reduces significantly the dimension of the parameter space, while preserving the capability of the NNMP model structure to capture spatial dependence. The following examples illustrate the method.

Example 1. Gaussian and continuous mixture of Gaussian NNMP models.

For \( l = 1, \ldots, L \), take \((U_l, V_l)\) to be a bivariate Gaussian random vector with mean \( \mu_l \mathbf{1}_2 \) and covariance matrix \( \Sigma_l = \sigma_l^2 \left( \begin{array}{cc} 1 & \rho_l \\
\rho_l & 1 \end{array} \right) \), where \( \mathbf{1}_2 \) is the two-dimensional column vector of ones, resulting in a Gaussian conditional density \( f_{U_l|V_l}(u_l | v_l) = N(u_l | (1-\rho_l)\mu_l + \rho_l v_l, \sigma_l^2(1-\rho_l^2)) \). If we extend the correlation parameter to be spatially varying, \( \rho_l(v) = k_l(v, v(l)) \), for
a correlation function \( k_l \), we obtain the spatially varying conditional density,

\[
p(z(v) \mid z_{\text{Ne}(v)}) = \sum_{l=1}^{L} w_l(v) N(z(v) \mid (1 - \rho_l(v))\mu_l + \rho_l(v)z(v_{(l)}), \sigma_l^2(1 - (\rho_l(v))^2)).
\]

This NNMP is referred to as the Gaussian NNMP (GNNMP). If we take \( Z(s_i) \sim N(z \mid \mu, \sigma^2) \), and set \( \mu_l = \mu \) and \( \sigma_l^2 = \sigma^2 \), for all \( l \), the resulting GNNMP satisfies the invariant condition of Proposition 1 with stationary marginal given by the \( N(\mu, \sigma^2) \) distribution. Moreover, when \( L = 1 \), the model is a Gaussian process. The finite-dimensional distribution of the GNNMP model is characterized by the following proposition, the proof of which is included in the supplementary material.

**Proposition 2.** Consider the GNNMP in (8) with \( \mu_l = \mu \) and \( \sigma_l^2 = \sigma^2 \), for all \( l \). If \( Z(s_i) \sim N(z \mid \mu, \sigma^2) \), the GNNMP has the \( N(z \mid \mu, \sigma^2) \) stationary marginal distribution, and its finite-dimensional distributions are mixtures of multivariate Gaussian distributions.

We refer to the model in Proposition 2 as the stationary GNNMP. Based on the GNNMP, various NNMP models with different families for \((U_l, V_l)\) can be constructed by exploiting location-scale mixtures of Gaussian distributions. We illustrate the approach with the skew-GNNMP model. Denote by \( TN(\mu, \sigma^2) \) the Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \), truncated at \([0, \infty)\). Building from Example 1, we start with a conditional bivariate Gaussian distribution for \((U_l, V_l)\), given \( z_0 \sim TN(0, 1) \), where \( \mu_l \) is replaced with \( \mu_l + \lambda_l z_0 \). Marginalizing out \( z_0 \), we obtain the bivariate skew-Gaussian distribution for \((U_l, V_l)\) (Azzalini, 2013). Extending again \( \rho_l \) to \( \rho_l(v) \), for all \( l \), the spatially varying conditional density for the skew-GNNMP model can be expressed as

\[
p(z(v) \mid z_{\text{Ne}(v)}) = \sum_{l=1}^{L} w_l(v) \int_{0}^{\infty} N(z(v) \mid \mu_l(v), \sigma_l^2(v)) TN(z_0(v) \mid \mu_{0l}(v_{(l)}), \sigma_{0l}^2) dz_0(v),
\]

where: \( \mu_l(v) = \{1 - \rho_l(v)\}\{\mu_l + \lambda_l z_0(v)\} + \rho_l(v)z(v_{(l)}) \); \( \sigma_l^2(v) = \sigma_l^2\{1 - (\rho_l(v))^2\} \); \( \mu_{0l}(v_{(l)}) = \{z(v_{(l)}) - \mu_l\} \lambda_l / (\sigma_l^2 + \lambda_l^2) \); and \( \sigma_{0l}^2 = \sigma_l^2 / (\sigma_l^2 + \lambda_l^2) \). Setting \( \lambda_l = \lambda, \mu_l = \mu, \) and \( \sigma_l^2 = \sigma^2 \), for all \( l \), we obtain the stationary skew-GNNMP model, with stationary skew-Gaussian marginal \( f_Z(z) = 2 N(z \mid \mu, \lambda^2 + \sigma^2) \Phi((z - \mu)\lambda / (\sigma \sqrt{\lambda^2 + \sigma^2})) \), denoted as \( SN(\mu, \lambda^2 + \sigma^2, \lambda/\sigma) \).
The skew-GNNMP model is an example of a location mixture of Gaussian distributions. Scale mixtures can also be considered to obtain, for example, the Student-t model. In that case, we replace the covariance matrix $\Sigma_l$ with $c\Sigma_l$, taking $c$ as a random variable with an appropriate inverse-gamma distribution. Important families that admit a location and/or scale mixture of Gaussians representation include the skew-t, Laplace, and asymmetric Laplace distributions. Using the approach for the skew-GNNMP example, we can construct the corresponding NNMP models.

### 2.4 Models based on copulas

As demonstrated in Section 2.3, a practical choice to construct an NNMP model with a stationary marginal distribution is to consider the same family of bivariate distributions for all base random vectors $(U_l, V_l)$. In order to satisfy the invariant condition of Proposition 1, we require that the corresponding conditional density preserves some spatially varying parameter(s) not shared with the stationary marginal distribution. In this regard, a useful strategy to develop the bivariate distribution of $(U_l, V_l)$ is to use a copula.

A copula function $C : [0, 1]^p \to [0, 1]$ is a function such that, for any multivariate distribution $F(z_1, \ldots, z_p)$, there exists a copula $C$ for which $F(z_1, \ldots, z_p) = C(F_1(z_1), \ldots, F_p(z_p))$, where $F_j$ is the marginal distribution function of $Z_j$, $j = 1, \ldots, p$ [Sklar 1959]. If $F_j$ is continuous for all $j$, $C$ is unique. A copula enables us to separate the modeling of the marginal distributions from the bivariate dependence. Thus, the invariant condition can be attained by specifying the stationary distribution $F_Z$ as the marginal distribution of $(U_l, V_l)$ for all $l$. The copula parameter that determines the dependence of $(U_l, V_l)$ can be modeled as spatially varying to create the sequence of spatially varying bivariate vectors $(U_{v,l}, V_{v,l})$. Here, we focus on continuous stationary distributions, although this strategy can be applied for any family of distributions for $F_Z$.

For the bivariate distribution of each $(U_l, V_l)$ with marginals $f_{U_l}$ and $f_{V_l}$, we consider a copula $C_l$ with parameter $\eta_l$, for $l = 1, \ldots, L$. We obtain the spatially varying copula $C_{v,l}$ for $(U_{v,l}, V_{v,l})$ by extending $\eta_l$ to $\eta_l(v)$. The joint density of $(U_{v,l}, V_{v,l})$ is given by $c_{v,l}(z(v), z(v(l)))f_{U_{v,l}}(z(v))f_{V_{v,l}}(z(v(l)))$, where $c_{v,l}$ is the copula density of $C_{v,l}$, and $f_{U_{v,l}} = \ldots$
f_{U_l}$ and $f_{V_{v,l}} = f_{V_l}$ are the marginal densities of $U_{v,l}$ and $V_{v,l}$, respectively. Given a pre-specified stationary marginal $f_Z$ from a continuous family of distributions, we replace both $f_{U_{v,l}}$ and $f_{V_{v,l}}$ with $f_Z$, for every $v$ and for all $l$. We then obtain the conditional density

$$p(z(v) \mid \text{z}_{\text{Ne}(v)}) = \sum_{l=1}^{L} w_l(v) c_{w,l}(z(v), z(v(l))) f_Z(z(v))$$

that characterizes the stationary copula NNMP. Some of the NNMPs of Section 2.3 can be regarded as special cases of copula NNMPs; e.g., the bivariate Gaussian distribution of $(U_l, V_l)$ in the GNNMP corresponds to a Gaussian copula with Gaussian marginals.

Under the copula framework, one strategy to specify the spatially varying parameters is through the Kendall's $\tau$ coefficient. The Kendall's $\tau$, taking values in $[-1, 1]$, is a bivariate concordance measure with properties useful for non-Gaussian modeling. In particular, its existence does not require finite second moment and it is invariant under strictly increasing transformations. If $(U_l, V_l)$ is continuous with a copula $C_l$, its Kendall’s $\tau$ is $\rho_{\tau,l} = 4\int_{[0,1]^2} C_l dC_l - 1$. Taking $A_l \subset [-1, 1]$ as the range of $\rho_{\tau,l}$, we can construct a composition function $h_l := g_l \circ k_l$ for some link function $g_l : A_l \to H_l$ and kernel function $k_l : \mathcal{D} \times \mathcal{D} \to A_l$, where $H_l$ is the parameter space associated with $C_l$. The kernel $k_l$ should be specified with caution; $k_l$ must satisfy axioms in the definition of a bivariate concordance measure (Joe 2014, Section 2.12). We illustrate the strategy with the following example.

**Example 2.** The bivariate Gumbel copula is an asymmetric copula useful for modeling dependence when the marginals are positive and heavy-tailed. The spatial Gumbel copula can be defined as $C_{w,l} = \exp \left( - \left\{ - \log F_{U_{w,l}}(z(v)) \right\}^{\eta_l(v)} + \left\{ - \log F_{V_{w,l}}(z(v(l))) \right\}^{\eta_l(v)} \right)^{1/\eta_l(v)}$, where $\eta_l(v) \in [1, \infty)$ and perfect dependence is obtained if $\eta_l(v) \to \infty$. The Kendall’s $\tau$ is $\rho_{\tau,l}(v) = 1 - \eta_l^{-1}(v)$, taking values in $[0, 1]$. We define $\rho_{\tau,l}(v) := k_l(||v - v(l)||)$, an isotropic correlation function. Let $g_l(x) = (1 - x)^{-1}$. Then, the function $h_l(||v - v(l)||) = g_l \circ k_l(||v - v(l)||) = (1 - k_l(||v - v(l)||))^{-1}$. Thus, the parameter $\eta_l(v) \equiv \eta(||v - v(l)||)$ is given by $h_l(||v - v(l)||)$, and $\eta_l(v) \to \infty$ as $||v - v(l)|| \to 0$.

In addition to a convenient strategy to achieve stationarity, copula NNMP models offer avenues to capture complex dependence using general bivariate copulas. Many multivari-
ate copulas are not suitable to model finite dimensional distributions of spatial processes. Though spatial vine copula models (Gräler, 2014) were proposed to resolve this restriction, their model structure and computation are substantially more complicated than copula NNMP models.

2.5 Mixture component specification and tail dependence

A benefit of building NNMPs from a set of base random vectors is that specification of the multivariate dependence of $Z(v)$ given its neighbors is determined mainly by that of the base random vectors. In this section, we illustrate this attractive property of the model with the establishment of lower bounds for two measures used to assess the strength of the tail dependence of NNMPs.

To establish our results we rely on the assumption that the base random vector $(U_l, V_l)$ has stochastically increasing positive dependence. $U_l$ is said to be stochastically increasing in $V_l$, if $P(U_l > u_l | V_l = v_l)$ increases as $v_l$ increases. The definition is extended to a multivariate random vector $(Z_1, \ldots, Z_p)$. $Z_1$ is said to be stochastically increasing in $(Z_2, \ldots, Z_p)$ if $P(Z_1 > z_1 | Z_2 = z_2, \ldots, Z_p = z_p) \leq P(Z_1 > z_1 | Z_2 = z'_2, \ldots, Z_p = z'_p)$, for all $(z_2, \ldots, z_p)$ and $(z'_2, \ldots, z'_p)$ in the support of $(Z_2, \ldots, Z_p)$, where $z_j \leq z'_j$, for $j = 2, \ldots, p$.

The conditional density in (6) implies that

$$P(Z(v) > z | Z_{Ne(v)} = z_{Ne(v)}) = \sum_{l=1}^{L} w_l(v) P(Z(v) > z | Z(v(l)) = z(v(l))).$$

Therefore, $Z(v)$ is stochastically increasing in $Z_{Ne(v)}$ if $Z(v)$ is stochastically increasing in $Z(v(l))$ with respect to $(U_{v,l}, V_{v,l})$ for all $l$. If the sequence $(U_{v,l}, V_{v,l})$ is built from the vector $(U_l, V_l)$, then the set of base random vectors determines the stochastically increasing positive dependence of $Z(v)$ given its neighbors.

For a bivariate random vector $(U_l, V_l)$, the upper and lower tail dependence coefficients, denoted as $\lambda_{H,l}$ and $\lambda_{L,l}$, respectively, are $\lambda_{H,l} = \lim_{q \to 1^-} P(U_l > F_{U_l}^{-1}(q) | V_l > F_{V_l}^{-1}(q))$ and $\lambda_{L,l} = \lim_{q \to 0^+} P(U_l \leq F_{U_l}^{-1}(q) | V_l \leq F_{V_l}^{-1}(q))$. When $\lambda_{H,l} > 0$, we say $U_l$ and $V_l$ have upper tail dependence. When $\lambda_{H,l} = 0$, $U_l$ and $V_l$ are said to be asymptotically independent in
the upper tail. Lower tail dependence and asymptotically independence in the lower tail are similarly defined using $\lambda_{L,l}$. Let $F_{Z(v)}$ be the marginal distribution function of $Z(v)$. Analogously, we can define the upper and lower tail dependence coefficients for $Z(v)$ given its nearest neighbors, namely,

$$
\lambda_{H}(v) = \lim_{q \to 1^{-}} P(Z(v) > F_{Z(v)}^{-1}(q) \mid Z(v_{(1)}) > F_{Z(v_{(1)})}^{-1}(q), \ldots, Z(v_{(L)}) > F_{Z(v_{(L)})}^{-1}(q)),
$$

$$
\lambda_{L}(v) = \lim_{q \to 0^{+}} P(Z(v) \leq F_{Z(v)}^{-1}(q) \mid Z(v_{(1)}) \leq F_{Z(v_{(1)})}^{-1}(q), \ldots, Z(v_{(L)}) \leq F_{Z(v_{(L)})}^{-1}(q)).
$$

The following proposition provides lower bounds for the tail dependence coefficients.

**Proposition 3.** Consider an NNMP for which the component density $f_{v,l}$ is specified by the conditional density of $U_{v,l}$ given $V_{v,l}$, where the random vector $(U_{v,l}, V_{v,l})$ follows a bivariate distribution with marginal distribution functions $F_{U_{v,l}}$ and $F_{V_{v,l}}$, for $l = 1, \ldots, L$. The spatial dependence of random vector $(U_{v,l}, V_{v,l})$ is built from the base vector $(U_l, V_l)$, which has a bivariate distribution such that $U_l$ is stochastically increasing in $V_l$, for $l = 1, \ldots, L$. Then, for every $v$, the lower bound for the upper tail dependence coefficient $\lambda_{H}(v)$ is $\sum_{l=1}^{L} w_l(v) \lim_{q \to 1^{-}} P(Z(v) > F_{U_{v,l}}^{-1}(q) \mid Z(v_{(l)}) = F_{V_{v,l}}^{-1}(q))$, and the lower bound for the lower tail dependence coefficient $\lambda_{L}(v)$ is $\sum_{l=1}^{L} w_l(v) \lim_{q \to 0^{+}} P(Z(v) \leq F_{U_{v,l}}^{-1}(q) \mid Z(v_{(l)}) = F_{V_{v,l}}^{-1}(q))$.

The proof of the proposition is provided in the supplementary material. In a nutshell, Proposition 3 shows that the lower and upper tail dependence coefficients are bounded below by a convex combination of, respectively, the limits of the conditional cumulative distribution functions (c.d.f.s), and the limits of the conditional survival functions. These are fully determined by the dependence structure of the bivariate distribution for $(U_l, V_l)$. This result is best illustrated with an example.

**Example 3.** Consider a Lomax NNMP for which the bivariate distributions of the base random vectors correspond to a bivariate Lomax distribution (Arnold et al., 1999), resulting in an associated conditional density of the model as follows

$$
p(z(v) \mid z_{Ne(v)}) = \sum_{l=1}^{L} w_l(v) \text{Lo}(z(v) \mid z(v_{(l)}) + \phi_l, \alpha_l(v)),
$$
where \( L_0(x | \phi, \alpha) = \alpha \phi^{-1} (1 + x \phi^{-1})^{-(\alpha + 1)} \) denotes the Lomax density, a shifted version of the Pareto Type I density. A small value of \( \alpha \) indicates a heavy tail. The component conditional survival function of the Lomax NNMP, expressed in terms of the quantile \( q \), is 
\[
\left\{ 1 + F_{U_{w,l}}^{-1}(q)/\left( F_{V_{w,l}}^{-1}(q) + \phi_l \right) \right\}^{-\alpha_l(v)}
\]
which converges to \( 2^{-\alpha_l(v)} \) as \( q \to 1^- \). Therefore, the lower bound for \( \lambda_H(v) \) is \( \sum_{l=1}^L w_l(v) 2^{-\alpha_l(v)} \). As \( \alpha_l(v) \to 0 \) for all \( l \), the lower bound for \( \lambda_H(v) \) tends to one, and hence \( \lambda_H(v) \) tends to one, since \( \lambda_H(v) \leq 1 \). As \( \alpha_l(v) \to \infty \) for all \( l \), the lower bound tends to zero.

Proposition 3 holds for the general NNMP framework. If the distribution of \((U_l, V_l)\) with \( F_{U_l} = F_{V_l} \) has first order partial derivatives and exchangeable dependence, namely \((U_l, V_l)\) and \((V_l, U_l)\) have the same joint distribution, the lower bounds of the tail dependence coefficients depend on the component tail dependence coefficients. The result is summarized in the following corollary, the proof of which is given in the supplementary material.

**Corollary 1.** Suppose that the base random vector \((U_l, V_l)\) in Proposition 3 is exchangeable, and its bivariate distribution with marginals \( F_{U_l} = F_{V_l} \) has first order partial derivatives, for all \( l \). Then the upper and lower tail dependence coefficients \( \lambda_H(v) \) and \( \lambda_C(v) \) are bounded below by \( \sum_{l=1}^L w_l(v) \lambda_{H,l}(v)/2 \) and \( \sum_{l=1}^L w_l(v) \lambda_{C,l}(v)/2 \), where \( \lambda_{H,l}(v) \) and \( \lambda_{C,l}(v) \) are the tail dependence coefficients with respect to \((U_{w,l}, V_{w,l})\).

Under Corollary 1 if the bivariate distribution of \((U_l, V_l)\) is symmetric, e.g., an elliptically symmetric distribution, the upper and lower tail dependence coefficients coincide, and can simply be denoted as \( \lambda(v) \). Then, we have that \( \lambda(v) \geq \sum_{l=1}^L w_l(v) \lambda_l(v)/2 \), where \( \lambda_l(v) \) is the tail dependence coefficient with respect to \((U_{w,l}, V_{w,l})\).

Tail dependence can also be quantified using the boundary of the conditional c.d.f., as proposed in Hua and Joe (2014) for a bivariate random vector. In particular, the upper tail dependence of \((U_l, V_l)\) is said to have some strength if the conditional c.d.f. \( F_{U_l|V_l}(F_{U_l}^{-1}(q) | F_{V_l}^{-1}(1)) \) is positive at \( q = 1 \). Likewise, a non-zero \( F_{U_l|V_l}(F_{U_l}^{-1}(q) | F_{V_l}^{-1}(0)) \) at \( q = 0 \) indicates some strength of dependence in the lower tails. The functions \( F_{U_l|V_l}(\cdot | F_{V_l}^{-1}(0)) \) and \( F_{U_l|V_l}(\cdot | F_{V_l}^{-1}(1)) \) are referred to as the boundary conditional c.d.f.s.

By an abuse of notation, we use \( F_{1|2}(\cdot | F_{Z_{Ne(v)}}^{-1}(q)) \) for \( F(\cdot | Z(v_1)) = F_{Z(v_1)}^{-1}(q), \ldots, Z(v_L) = F_{Z(v_L)}^{-1}(q) \), the conditional c.d.f. of \( Z(v) \). Then \( F_{1|2}(\cdot | F_{Z_{Ne(v)}}^{-1}(0)) \) and \( F_{1|2}(\cdot | F_{Z_{Ne(v)}}^{-1}(1)) \).
are the boundary conditional c.d.f.s for the NNMP model. The upper tail dependence is
said to be i) strongest if $F_{1|2}(F_{Z(v)}^{-1}(q) | F_{Z_{Ne(v)}}^{-1}(1))$ equals 0 for $0 \leq q < 1$ and has a mass of
1 at $q = 1$; ii) intermediate if $F_{1|2}(F_{Z(v)}^{-1}(q) | F_{Z_{Ne(v)}}^{-1}(1))$ has positive but not unit mass at
$q = 1$; iii) weakest if $F_{1|2}(F_{Z(v)}^{-1}(q) | F_{Z_{Ne(v)}}^{-1}(1))$ has no mass at $q = 1$. The strength of lower
tail dependence is defined likewise using $F_{1|2}(F_{Z(v)}^{-1}(q) | F_{Z_{Ne(v)}}^{-1}(0))$. The proposition below
provides lower bounds for the boundary conditional c.d.f.s. The proof can be found in the
supplementary material.

**Proposition 4.** Consider an NNMP for which the component density $f_{v,l}$ is specified by the
conditional density of $U_{v,l}$ given $V_{v,l}$. The spatial dependence of random vector $(U_{v,l}, V_{v,l})$ is
built from the base vector $(U_l, V_l)$, which has a bivariate distribution such that $U_l$ is stochas-
tically increasing in $V_l$, for $l = 1, \ldots, L$. Let $\lambda_{L,l}(v)$ and $\lambda_{H,l}(v)$ be the lower and upper tail
dependence coefficients corresponding to $(U_{v,l}, V_{v,l})$, respectively. If for a given $v$, there ex-
ists $\lambda_{L,l}(v) > 0$ for some $l$, then the conditional c.d.f. $F_{1|2}(F_{Z(v)}^{-1}(q) | F_{Z_{Ne(v)}}^{-1}(0))$ has strictly
positive mass $p_0(v)$ at $q = 0$ with $p_0(v) \geq \sum_{l=1}^{L} w_l(v) \lambda_{L,l}(v)$. Similarly, if for a given $v,$
there exists $\lambda_{H,l}(v) > 0$ for some $l$, then the conditional c.d.f. $F_{1|2}(F_{Z(v)}^{-1}(q) | F_{Z_{Ne(v)}}^{-1}(1))$
has strictly positive mass $p_1(v)$ at $q = 1$ with $p_1(v) \geq \sum_{l=1}^{L} w_l(v) \lambda_{H,l}(v)$.

Proposition 4 complements Proposition 3 to assess the strength of the tail dependence.
It readily applies for bivariate distributions, especially for copulas which yield explicit
expressions for the tail dependence coefficients. For example, the spatially varying Gumbel
copula $C_{v,l}$ in Example 2 has upper tail dependence coefficient $2 - 2^{1/\eta_l(v)} > 0$ for $\eta_l(v) > 1$,
so the tail dependence of a Gumbel copula NNMP model has some strength if $\eta_l(v) > 1$
for some $l$. In fact, applying the result in Hua and Joe (2014), with a Gumbel copula,
$F_{1|2}(F_{Z(v)}^{-1}(q) | F_{Z_{Ne(v)}}^{-1}(1))$ degenerates at $q = 1$, implying strongest tail dependence.

3 Bayesian hierarchical model and inference

3.1 Hierarchical model formulation

We introduce the general approach for NNMP Bayesian implementation, treating the ob-
erved spatial responses as an NNMP realization. The inferential framework can be easily
extended to incorporate model components that may be needed in practical settings, such as covariates and additional error terms. We illustrate the extensions with the real data analysis in Section 5 and provide further discussion in Section 6.

Our approach for inference is based on a likelihood conditional on the first \( L \) elements of the realization \( z_\mathcal{S} = (z(s_1), \ldots, z(s_n))^T \) over the reference set \( \mathcal{S} \subset \mathcal{D} \). Following a commonly used approach for mixture models fitting, we use data augmentation to facilitate inference. For each \( z(s_i), i = L + 1, \ldots, n \), we introduce a configuration variable \( \ell_i \), taking values in \( \{1, \ldots, L\} \), such that 

\[
P(\ell_i | w(s_i)) = \sum_{l=1}^{L} w_l(s_i) \delta_l(\ell_i),
\]

where \( w(s_i) = (w_1(s_i), \ldots, w_L(s_i))^T \) and \( \delta_l(\ell_i) = 1 \) if \( \ell_i = l \) and 0 otherwise. Conditional on the configuration variables and the vector \((z(s_1), \ldots, z(s_L))^T\), the augmented model on \( z(s_i) \)

\[
z(s_i) | z(s_{i,\ell_i}), \ell_i, \theta \overset{\text{ind.}}{\sim} f_{s_i,\ell_i}(z(s_i) | z(s_{i,\ell_i}), \theta), \quad \ell_i | w(s_i) \overset{\text{ind.}}{\sim} \sum_{l=1}^{L} w_l(s_i) \delta_l(\ell_i),
\]

(11)

where \( \theta \) collects the parameters of the densities \( f_{s_i,l} \).

A key component of the proposed model formulation is the prior model for the weights. Weights are allowed to vary in space, adjusting to the neighbor structure of different reference locations. For illustration we consider the construction for weights corresponding to a point in the reference set. For non-reference points, weights are defined analogously. Consider a collection of spatially dependent c.d.f.s \( \{G_{s_i} : s_i \in \mathcal{S}\} \) supported on \((0, 1)\). For each \( s_i \), the weights are defined as the increments of \( G_{s_i} \) with cutoff points \( r_{s_i,0}, \ldots, r_{s_i,L} \).

More specifically,

\[
w_l(s_i) = \int \mathbb{1}_{(r_{s_i,l-1}, r_{s_i,l})}(t) dG_{s_i}(t), \quad l = 1, \ldots, L,
\]

(12)

where \( \mathbb{1}_A \) denotes the indicator function for set \( A \). The cutoff points \( 0 = r_{s_i,0} < r_{s_i,1} < \cdots < r_{s_i,L} = 1 \) are such that, for \( l = 1, \ldots, L \), \( r_{s_i,l} - r_{s_i,l-1} = k'(s_i, s_{i(l)} | \zeta) / \sum_{l=1}^{L} k'(s_i, s_{i(l)} | \zeta) \),

where \( k' : \mathcal{D} \times \mathcal{D} \to [0, 1] \) is a bounded kernel function with parameters \( \zeta \). The kernel and its associated parameters affect the smoothness of the resulting random field. By default we take \( G_{s_i} \) as a logit Gaussian distribution, denoted as \( G_{s_i}(\cdot | \mu(s_i), \kappa^2) \), such that
the corresponding Gaussian distribution has mean $\mu(s_i)$ and variance $\kappa^2$. The spatial dependence across the weights is introduced through the mean $\mu(s_i) = \gamma_0 + \gamma_1 s_{i1} + \gamma_2 s_{i2}$, where $s_i = (s_{i1}, s_{i2})$. Given the cutoff points and $\kappa^2$, a small value of $\mu(s_i)$ favors large weights for the near neighbors of $s_i$. A simpler version of the model in (12) is obtained by letting $G_{s_i}$ be the uniform distribution on $(0,1)$. Then the weights become

$$k'(s_i, s_{(i)})/\sum_{l=1}^L k'(s_i, s_{(i)l}) \mid \zeta).$$

We notice that Cadonna et al. (2019) use a set of fixed, uniform cutoff points on $[0,1]$, i.e., $r_{s_i,l} - r_{s_i,l-1} = 1/L$, for spectral density estimation, with a collection of logit Gaussian distributions indexed by frequency.

The full Bayesian model is completed with prior specification for parameters $\theta, \zeta, \gamma = (\gamma_0, \gamma_1, \gamma_2)^T$, and $\kappa^2$. The priors for $\theta$ and $\zeta$ depend on the choices of the densities $f_{s_{i,l}}$ and the cutoff point kernel $k'$, respectively. For parameters $\gamma$ and $\kappa^2$, we specify $N(\gamma \mid \mu_\gamma, V_\gamma)$ and IG($\kappa^2 \mid u_\kappa^2, v_\kappa^2$) priors, respectively, where IG denotes the inverse gamma distribution.

Finally, we note that an NNMP model requires selection of the neighborhood size $L$. This can be done using standard model comparison metrics, scoring rules, or information criteria; for example, Datta et al. (2016a) used root mean square predictive error (RM-SPE) and Guinness (2018) used Kullback–Leibler (KL) divergence. In general, a larger $L$ increases computational cost. Datta et al. (2016a) conclude that a moderate value $L \leq 20$ typically suffices for the NNGP models. Peruzzi et al. (2020) point out that a smaller $L$ corresponds to a larger KL divergence of $\tilde{p}(z_S)$ from $p(z_S)$, regardless of the distributional assumption of the density. Moreover, it is possible that information from the farthest neighbors is also important (Stein et al., 2004). Therefore, for large non-Gaussian data sets with complex dependence, one may seek a larger $L$ to obtain a better approximation to the full model. The properties of our prior model for the weights enable us to take a relatively large neighbor set with less computational demand. We assign small probabilities a priori to distant neighbors. The contribution of each neighbor will be induced by the mixing, with important neighbors being assigned large weights a posteriori.
3.2 Estimation and prediction

We implement a Markov chain Monte Carlo (MCMC) sampler to simulate from the posterior distribution of the model parameters. To allow for efficient simulation of parameters $\gamma$ and $\kappa^2$, we associate each $y(s_i)$ with a latent Gaussian variable $t_i$ with mean $\mu(s_i)$ and variance $\kappa^2$, for $i = L + 1, \ldots, n$. There is a one-to-one correspondence between the configuration variables $\ell_i$ and latent variables $t_i$: $\ell_i = l$ if and only if $t_i \in (r_{s_i,l-1}^*, r_{s_i,l}^*)$ where $r_{s_i,l}^* = \log(r_{s_i,l}/(1 - r_{s_i,l}))$, for $l = 1, \ldots, L$. The posterior distribution of the model parameters, based on the new augmented model, is

$$p(\theta, \zeta, \gamma, \kappa^2, \{t_i\}_{i=L+1}^n | z_S) \propto \pi_\theta(\theta) \times \pi_\zeta(\zeta) \times N(\gamma | \mu_\gamma, V_\gamma) \times IG(\kappa^2 | u_{\kappa^2}, v_{\kappa^2})$$

$$\times N(\mathbf{t} | D\gamma, \kappa^2 \mathbf{I}_{n-L}) \times \prod_{i=L+1}^n \sum_{l=1}^L f_{s_i,l}(z(s_i) \mid z(s_{(i)}), \theta) \mathbf{1}_{(r_{s_i,l-1}^*, r_{s_i,l}^*)}(t_i),$$

where $\pi_\theta$ and $\pi_\zeta$ are the priors for $\theta$ and $\zeta$, respectively, $\mathbf{I}_{n-L}$ is an $(n - L) \times (n - L)$ identity matrix, the vector $\mathbf{t} = (t_{L+1}, \ldots, t_n)^T$, and the matrix $D$ is $(n - L) \times 3$ such that the $i$th row is $(1, s_{L+i,1}, s_{L+i,2})$.

The posterior full conditional distribution of $\theta$ depends on the choice of $f_{s_i,l}$. Details for the models implemented in Sections 4 and 5 are given in the supplementary material. To update $\zeta$, we first marginalize out the latent variables $t_i$ from the joint posterior distribution. We then update $\zeta$ using a random walk Metropolis step with target density $\pi_\zeta(\zeta) \prod_{i=L+1}^n \{G_{s_i}(r_{s_i,\ell_i} | \mu(s_i), \kappa^2) - G_{s_i}(r_{s_i,\ell_i-1} | \mu(s_i), \kappa^2)\}$. The posterior full conditional distribution of the latent variable $t_i$ is a piecewise Gaussian distribution distributed on $(r_{s_i,l-1}, r_{s_i,l})$ with probabilities proportional to $w_l(s_i)f_{s_i,l}(z(s_i) \mid z(s_{(i)}), \theta)$, where $w_l(s_i) = G_{s_i}(r_{s_i,l} | \mu(s_i), \kappa^2) - G_{s_i}(r_{s_i,l-1} | \mu(s_i), \kappa^2)$, for $l = 1, \ldots, L$. The posterior full conditional distribution of $\gamma$ is $N(\gamma \mid \mu^*_\gamma, V^*_\gamma)$ where $V^*_\gamma = (V^{-1}_\gamma + \kappa^{-2}D^T D)^{-1}$ and $\mu^*_\gamma = V^*_\gamma (V^{-1}_\gamma \mu_\gamma + \kappa^{-2}D^T \mathbf{t})$. The posterior full conditional distribution of $\kappa^2$ is $IG(\kappa^2 | u_{\kappa^2} + (n - L)/2, v_{\kappa^2} + \sum_{i=L+1}^n (t_i - \mu(s_i))^2/2)$.

Turning to the prediction, let $\mathbf{v}_0 \in \mathcal{D}$. We obtain posterior predictive samples of $z(\mathbf{v}_0)$ in the following way. If $\mathbf{v}_0 \notin \mathcal{S}$, for each posterior sample of the parameters, we first compute the cutoff points $r_{v_0,l}$ for which $r_{v_0,l} - r_{v_0,l-1} = k'(\mathbf{v}_0, \mathbf{v}(0l) \mid \zeta)/\sum_{l=1}^L k'(\mathbf{v}_0, \mathbf{v}(0l) \mid \zeta)$, and
obtain the weights \( w_l(v_0) = G_{v_0}(r_{v_0,l} | \mu(v_0), \kappa^2) - G_{v_0}(r_{v_0,l-1} | \mu(v_0), \kappa^2) \) for \( l = 1, \ldots, L \). We then predict \( z(v_0) \) using (4). If \( v_0 \equiv s_i \in S \), we generate \( z(v_0) \) similar to the earlier case but using posterior samples of the weights collected from the MCMC, and applying (3) instead of (4) to generate \( z(v_0) \).

4 Simulation study

We conduct three simulation experiments to demonstrate the benefits of the proposed modeling framework. First, we illustrate the ability of the NNMP model to handle skewed data using a skew-GNNMP model. Next, we study inference for tail dependence using copula NNMP models. Finally, we demonstrate the effectiveness of the NNMP model for bounded spatial data.

In each of the experiments, we created a regular grid of 200 × 200 resolution on a unit square domain, and generated data on each grid location. We then randomly selected 2000 locations as the reference set with a random ordering for model fitting. For the purpose of illustration, we chose neighbor size \( L = 10 \) for all cases.

Results are based on posterior samples collected every 10 iterations from a Markov chain of 30000 iterations, with the first 10000 samples being discarded. Implementation details for all models are provided in the supplementary material. The MCMC algorithms were implemented in the R programming language on a computer with a 2-GHz Intel Core i5 processor and 32-GB RAM. We integrated C++ code for the update of latent variables without particular emphasis on optimizing the code. The computing time for the models in the three experiments was around 9, 18, and 18 minutes, respectively.

4.1 First experiment

We generated data from the following skew-Gaussian process (Zhang and El-Shaarawi, 2010),

\[
y(v) = \sigma_1 |\omega_1(v)| + \sigma_2 \omega_2(v), \quad v \in \mathcal{D}
\]  

(13)
where $\omega_1(v)$ and $\omega_2(v)$ are both standard Gaussian processes with correlation matrix specified by an exponential correlation function with range parameter $1/12$. The parameter $\sigma_1 \in \mathbb{R}$ controls the skewness, whereas $\sigma_2 > 0$ is a scale parameter. The model has a stationary skew-Gaussian marginal density $\text{SN}(0, \sigma_1^2 + \sigma_2^2, \sigma_1 / \sigma_2)$. We took $\sigma_2 = 1$, and generated data with $\sigma_1 = -5$, 1 and 10, resulting in three different random fields that are, respectively, moderately negative-skewed, slightly positive-skewed, and strongly positive-skewed, as shown in Figure 1(a)-1(c).

We applied the stationary skew-GNNMP model. The model is obtained as a special case of the skew-GNNMP model in \eqref{eq:skew-GNNMP}, taking $\lambda_l = \lambda$, $\mu_l = 0$, and $\sigma_l^2 = \sigma^2$, for all $l$. Here, $\lambda \in \mathbb{R}$ controls the skewness, such that a large positive (negative) value of $\lambda$ indicates strong positive (negative) skewness. If $\lambda = 0$, the skew-GNNMP model reduces to the GNNMP model. After marginalizing out $z_0$, we obtain a stationary skew-Gaussian marginal density $\text{SN}(0, \lambda^2 + \sigma^2, \lambda / \sigma)$. We completed the full Bayesian specification for the model, by assigning priors $N(\lambda | 0, 5)$, $\text{IG}(\sigma^2 | 2, 1)$, $\text{IG}(\phi | 3, 1/3)$, $\text{IG}(\zeta | 3, 0.2)$, $N(\gamma | (-1.5, 0, 0)^T, 2I_3)$,
and $\text{IG}(\kappa^2 | 3, 1)$, where $\zeta$ is the range parameter of the exponential correlation function specified for the cutoff point kernel.

We focus on the model performance on capturing skewness. The posterior mean and 95% credible interval of $\lambda$ for the three scenarios were $-3.65 (-4.10, -3.27)$, $1.09 (0.91, 1.28)$ and $7.69 (6.88, 8.68)$, respectively, indicating the model’s ability to estimate different levels of skewness. The bottom row of Figure 1 shows that the posterior median estimates of the surfaces capture well features of the true surfaces, even when the level of skewness is small, thus demonstrating that the model is also able to recover near-Gaussian features. Figure 2 plots the posterior mean and pointwise 95% credible interval for the marginal density, overlaid on the histogram of the simulated data for each of the three cases. These estimates demonstrate the adaptability of the skew-GNNMP model in capturing skewed random fields with different levels of skewness.

### 4.2 Second experiment

The goal of the second experiment is to demonstrate the use of copulas to construct NNMPs for tail dependence modeling. We note that the focus here is to illustrate the flexibility of the NNMPs with copulas for modeling complex dependence structures, but not for extreme value modeling. To this end, we generated data from the random field

$$y(v) = F^{-1}(T_v(\omega(v))), \quad v \in \mathcal{D},$$

(14)
where \( \omega(v) \) is a standard Student-t process with tail parameter \( \nu \) and scale matrix specified by an exponential correlation function with range parameter \( \phi_w \). The distribution functions \( F \) and \( T_\nu \) correspond to a gamma distribution \( \text{Ga}(2, 2) \) and a standard Student-t distribution with tail parameter \( \nu \), respectively. For a given pair of locations in \( D \) with correlation \( \rho_0 = \exp(-d_0/\phi_w) \), the corresponding tail dependence coefficient of the random field is \( \chi_\nu = 2T_{\nu+1}( - \sqrt{(1+\nu)(1-\rho_0)/(1+\rho_0)} ) \). We took \( \phi_w = 1/12 \), and chose \( \nu = 10 \) so that the synthetic data exhibits moderate tail dependence at close distance, and the dependence decreases rapidly as the distance \( d_0 \) becomes larger. When \( \rho_0 = 0.05, 0.5, 0.95 \), \( \chi_{10} = 0.01, 0.08, 0.61 \), respectively.

We applied two copula NNMP models. The models are of the form in (10) with stationary gamma marginal \( \text{Ga}(a, b) \) with mean \( a/b \). In the first model, the component copula density \( c_{v,l} \) corresponds to a bivariate Gaussian copula that is known to be unsuitable...
for tail dependence modeling. The correlation parameter of the copula was specified by an exponential correlation function with range parameter $\phi_1$. In the second model, we consider a spatially varying Gumbel copula as in Example 2. The spatially varying parameter of the copula density is defined with the link function $\eta_l(v) \equiv \eta_l(||v - v_l||) = \min\{(1 - \exp(-||v - v(l)||/\phi_2)^{-1}, 50\}$, where the upper bound 50 ensures numerical stability. When $\eta_l(d_0) = 50$, $\exp(-d_0/\phi_2) = 0.98$. With this link function, we assume that given $\phi_2$, the strength of the tail dependence with respect to the $l$th component of the Gumbel model stays the same for any distance smaller than $d_0$ between two locations.

For the cutoff point kernels, we specified an exponential correlation function with range parameters $\zeta_1$ and $\zeta_2$, respectively, for each model. The Bayesian model is fully specified with a $\text{IG}(3, 1/3)$ prior for $\phi_1$ and $\phi_2$, a $\text{Ga}(1, 1)$ prior for $a$ and $b$, a $\text{IG}(3, 0.2)$ prior for $\zeta_1$ and $\zeta_2$, $N(\gamma | (-1.5, 0, 0)^{\top}, 2I_3)$, and $\text{IG}(\kappa^2 | 3, 1)$ priors.

We focus on the performance of the two models with respect to tail dependence inference. Table 1 presents the log-scores (Gneiting and Raftery, 2007) for subsets of the held-out data that exceed the $c$-th percentile of the held-out data. The out-of-sample log-score is the predictive log-likelihood averaging over the model parameters. It reflects the ability of a model to capture dependence structure in the data. We can see that for held-out data that exceed high sample percentiles, the Gumbel copula model gives a higher log-score.

Figure 3 shows the random fields, marginals and conditional survival probabilities estimated by the two models. From Figure 3(a)-3(c), we see that, comparing with the true field, the posterior median estimate by the Gumbel copula model seems to recover the large values better than the Gaussian copula model. Besides, as shown in Figure 3(d), the Gumbel copula NNMP model provides a more accurate estimate of the marginal distribution, especially in the tails. We computed the conditional survival probabilities at two

| $c$ | 0    | 10   | 30   | 50   | 70   | 90   | 95   |
|-----|------|------|------|------|------|------|------|
| Gaussian copula model | 140.009 | 80.713 | 30.224 | -15.479 | -28.806 | -20.377 | -13.750 |
| Gumbel copula model    | 118.684 | 63.962 | 17.242 | -24.300 | -28.232 | -16.952 | -11.422 |
different unobserved sites marked in Figure 3(a). In particular, Site 1 is surrounded with reference observations with moderate values, while Site 2 is surrounded with large reference observations. We see that the Gumbel copula model provides much closer estimates to the probabilities, indicating that the model captures better the tail dependence structure in the data. Overall, this example demonstrates that the Gumbel copula NNMP model is a useful option for modeling spatial processes with tail dependence.

4.3 Third experiment

Many spatial processes are measured over a compact interval. As an example, data on proportions are common in ecological applications. In this experiment, we demonstrate the effectiveness of the NNMP model for directly modeling bounded spatial data. In particular, we generated data using the following model

\[ y(v) = F^{-1}(\Phi(\omega(v))) , \]

where the distribution function \( F \) corresponds to a beta distribution, denoted as \( \text{Beta}(a_0, b_0) \), and \( \omega(v) \) is a standard Gaussian process with exponential correlation function with range parameter 0.1. We set \( a_0 = 3, b_0 = 6 \).

We applied a Gaussian copula NNMP model with stationary marginal \( \text{Beta}(a, b) \), with the same spatial Gaussian copula and prior specification used in the second experiment. Figure 4(b) shows the estimated random field which captures well the main features of the true field in Figure 4(a). The posterior mean and pointwise 95\% credible interval of the estimated marginal density in Figure 4(c) overlay on the data histogram. These show that the beta NNMP estimation and prediction provide good approximation to the true field.

Finally, it is worth mentioning that implementing the beta NNMP model is simpler than fitting existing models for data corresponding to proportions. For example, a spatial Gaussian copula model, that corresponds to the data generating process of this experiment, involves computations for large matrices. Alternatively, if a multivariate non-Gaussian copula is used, the resulting likelihood can be intractable and require certain approximations. Another model that is commonly used in this setting is defined analogously to a spatial
Figure 4: Synthetic data analysis - third experiment. Panels (a) and (b) are interpolated surfaces of the true field and posterior median estimate from the beta NNMP model, respectively. In Panel (c), the green dotted line corresponds to the true marginal. The red dash line and shaded region are the posterior mean and pointwise 95% credible interval for the estimated marginal.

generalized linear mixed model. The spatial element in the model is introduced through the transformed mean of the observations. A sample-based approach to fit such model requires sampling a large number of highly correlated latent variables. We conducted an additional simulation experiment to demonstrate the effectiveness of the beta NNMP to approximate the random field simulated by the link function approach. Details are provided in the supplementary material.

5 Application: Mediterranean Sea surface temperature data analysis

The study of Ocean’s dynamics is crucial for understanding climate variability. One of the most valuable sources of information regarding the evolution of the state of the ocean is provided by the centuries-long record of temperature observations recorded from the surface of the oceans. The record of sea surface temperatures (SST) consists of data collected over time at irregularly scattered locations. In this section, we examine the SST from the Mediterranean Sea area during December 2003.

It is well known that the Mediterranean Sea area produces very heterogeneous temperature fields. A goal of the spatial analysis of SST in the area is to generate a spatially continuous field that accounts for the complexity of the surrounding coastlines as well as
the non-linear dynamics of the circulation system. An additional source of complexity comes from the data collection process. Historically, SST observations are collected from different types of devices: buckets launched from navigating vessels, readings from the water intake of ships’ engine rooms, moored buoys, and drifting buoys (Kirsner and Sansó, 2020). The source of some observations is known, but not all the data are labelled. A thorough case study will be needed to include all this information in order to account for possible heterogeneities due to the different measuring devices. That is beyond the scope of this paper. We will focus on demonstrating the ability of the proposed framework to model non-Gaussian spatial processes that, hopefully, capture the complexities of the physical process and the data collection protocol better than Gaussian processes. We notice that in the original record several sites had multiple observations. In those cases we took the median of the observations, resulting in a total of 1966 observations. The data are shown in Figure 5.

In Section 5.1, we focus on a limited region that allows us to explore in detail the behavior of the GNNMP. The GNNMP has the same Gaussian marginals as the NNGP, but its finite-dimensional distribution is a mixture of Gaussian distributions. We compare the GNNMP with the NNGP in a spatially varying regression model, demonstrating the
benefit of using a non-Gaussian process to explain the SST variability. In Section 5.2 we illustrate the ability of the NNMPs to model non-Gaussian marginals by using a skew-GNNMP to analyze the whole data set.

5.1 Regional analysis

We first focus on SST over an area near the Gulf of Lion, along the islands near the shores of Spain, France, Monaco and Italy, between 0 - 9 E. longitude and 33.5 - 44.5 N. latitude. The SST observations in the region, as shown in Figure 6(a), are very heterogeneous, implying that the short range variability is likely to be non-Gaussian. To capture the variability, we consider the following spatially varying regression model,

\[ y(v) = x(v)^\top \beta + z(v) + \epsilon(v), \quad v \in D, \]

where \( y(v) \) is the SST observation, \( x(v) = (1, v_1, v_2)^\top \) includes longitude \( v_1 \) and latitude \( v_2 \) to account for the long range variability in SST with regression parameters \( \beta = (\beta_0, \beta_1, \beta_2)^\top \), \( z(v) \) is a spatial process, and \( \epsilon(v) \) \( \text{i.i.d.} \sim N(0, \tau^2) \) represents the micro-scale variability and/or the measurement error.

We model \( z(v) \) with the GNNMP defined in (8) with \( \mu_l = 0 \) and \( \sigma^2_l = \sigma^2 \), for all \( l \). For comparison, we also applied an NNGP model for \( z(v) \) with variance \( \sigma^2_0 \) and exponential correlation function with range parameter \( \phi_0 \). For the GNNMP, we used exponential correlation functions with range parameter \( \phi \) and \( \zeta \), respectively, for the correlation with respect to the component density, and the cutoff point kernel. For both models, the regression coefficients \( \beta \) were assigned flat priors. The variances \( \sigma^2_0 \) and \( \sigma^2 \) received the same inverse gamma prior \( \text{IG}(2, 1) \), and \( \tau^2 \) was assigned \( \text{IG}(2, 0.1) \). The range parameter \( \phi_0 \) of the NNGP received a uniform prior \( \text{Unif}(1/30, 1/3) \), while the range parameters \( \phi \) and \( \zeta \) of the GNNMP received inverse gamma priors \( \text{IG}(3, 1/3) \) and \( \text{IG}(3, 0.2) \), respectively. Regarding the logit Gaussian distribution parameters, \( \gamma \) and \( \kappa^2 \), we used \( N((-1.5, 0, 0)^\top, 2I_3) \) and \( \text{IG}(3, 1) \) priors, respectively.

We took around 10% of the data in the region, as the held-out data for model comparison, and used the remaining 580 observations to train models. We compare models
Based on RMSPE, 95% posterior credible interval coverage rate (95% CI coverage), deviance information criterion (DIC; Spiegelhalter et al. 2002), PPLC (Gelfand and Ghosh 1998), and continuous ranked probability score (CRPS; Gneiting and Raftery 2007). To effectively compare the GNNMP and NNGP models, we first trained the NNGP model with neighbor sizes $L$ from 5 to 20, and selected the optimal neighbor size, $L = 11$ that corresponds to the smallest RMSPE. We then trained the GNNMP model with the same $L$.

In all cases, we ran the MCMC with 120000 iterations, discarding the first 20000 samples, and collected samples every 20 iterations. Details of the MCMC algorithm to implement the GNNMP model are provided in the supplementary material. For the NNGP model, we implemented the latent NNGP algorithm from the spNNGP package in R (Finley et al. 2020). The computing time was around 12 and 9 minutes for the GNNMP and NNGP models, respectively.

We report the results for both models. The posterior mean and 95% credible interval estimates of the regression intercept from the GNNMP was higher than the NNGP. They were 28.55 (22.81, 35.34) and 27.18 (23.53, 30.94), respectively. The corresponding posterior estimates of the coefficients for longitude and latitude given by the GNNMP and the NNGP

Table 2: Performance metrics of different models

|       | RMSPE | 95% CI coverage | 95% CI width | CRPS | PPLC     | DIC    |
|-------|-------|-----------------|--------------|------|----------|--------|
| GNNMP | 0.92  | 0.97            | 4.09         | 0.50 | 135.58   | 539.53 |
| NNGP  | 1.00  | 0.97            | 4.30         | 0.56 | 526.81   | 1083.61|
were 0.09 (−0.03, 0.24); −0.32 (−0.50, −0.17) and 0.09 (0.01, 0.17); −0.29 (−0.39, −0.19), respectively. Both models indicated that there was a trend of SST decreasing in the latitude at the selected region. For the error variance \( \tau^2 \), the GNNMP provided a smaller estimate 0.12 (0.02, 0.38), compared to 0.45 (0.02, 0.92) from the NNGP.

Regarding the model performance metrics in Table 2, both the PPLC and DIC suggest that the GNNMP had a better goodness-of-fit than the NNGP. For out-of-sample prediction, the GNNMP produced smaller RMSPE and CRPS than the NNGP. Both models gave the same 95% CI coverage, while the one from GNNMP had a narrower width. Figure 6(b)-6(c) show the posterior median estimates of the temperature field from both models. We can see that both models yield estimates that resemble the pattern in the observations. The predictive surface produced by the NNGP depicts some very localized, unrealistic features. These are not present in the results from the GNNMP.

5.2 Full analysis with a skew-GNNMP model

The regional analysis results in Section 5.1 suggest that a non-Gaussian process is a better assumption for the Mediterranean SST spatial variability. In light of the evidence (Pisano et al., 2020) that the spatial patterns of SST are different over Mediterranean sub-basins, as shown in Figure 7(a), which are characterized with different dynamics and high variability of surface currents (Bouziene et al., 2020), we further investigate the SST over those sub-basins. We fitted a non-spatial linear model to all SST data, including longitude and latitude as covariates, and obtained residuals from the linear model. Figure 7(b) shows that the histograms of the residuals are asymmetric over the sub-basins, indicating skewness in the marginal SST distribution, with levels of skewness that vary across sub-basins.

The exploratory data analysis suggests the need for a spatial model that can capture skewness. The symmetric distribution assumption of \( \epsilon(v) \) in the additive model (15) may be inappropriate for modeling skewness. Moreover, the weak identifiability of its variance \( \tau^2 \) may further undermine estimation of the skewness especially when it is mild. Thus, we analyze the full SST data with an extension of the skew-GNNMP model in (9). The new model has two features that extend the skew-GNNMP: (i) it incorpor-
rates fixed effect through the location parameter of the mixture component; (ii) it allows the skewness parameter $\lambda$ to vary in space. More specifically, the spatially varying conditional density $f_{e,l}$ of the model builds from a Gaussian random vector with mean $(x(v)^\top \beta + \lambda(v)z_0(v), x(v(l))^\top \beta + \lambda(v(l))z_0(v))^\top$ and covariance matrix $\sigma^2 \left( \begin{array}{c} \rho_l(v) \\ 0 \end{array} \right)$, where $x(v) = (1, v_1, v_2)^\top$ and $z_0(v) \sim \text{TN}(0, I)$, for all $v$ and for all $l$. The associated conditional density of the extended model is

$$
p(y(v) | y_{Ne(v)}) = \sum_{l=1}^L w_l(v) \int_0^\infty N(y(v) | \mu_l(v), \sigma_l^2(v)) \text{TN}(z_0(v) | \mu_0(l(v(l)), \sigma_0(l(v(l)))) dz_0(v),

(16)$$

where $\mu_l(v) = x(v)^\top \beta + \lambda(v)z_0(v) + \rho_l(v)\{y(v(l)) - x(v(l))^\top \beta - \lambda(v(l))z_0(v)\}$, $\sigma_l^2(v) = \sigma^2\{1 - (\rho_l(v))^2\}$, $\mu_0(l(v(l)) = \{y(v(l)) - x(v(l))^\top \beta\}/\{\sigma^2 + (\lambda(v(l))^2\}$, and $\sigma_0^2(l(v(l)) = \sigma^2/\{\sigma^2 + (\lambda(v(l))^2\}$. After marginalizing out $z_0(v)$, we obtain that the marginal distribution of $Y(v)$ is $\text{SN}(x(v)^\top \beta, (\lambda(v))^2 + \sigma^2, \lambda(v)/\sigma)$, based on the result of Proposition 1. We model the spatially varying $\lambda(v)$ via a partitioning approach. In particular, we partition the Mediterranean Sea $\mathcal{D}$ according to the sub-basins, i.e., $\mathcal{D} = \cup_{k=1}^K P_k$, $P_i \cap P_j = \emptyset$ for $i \neq j$, where $K = 5$. For all $v \in P_k$, we take $\lambda(v) = \lambda_k$, for $k = 1, \ldots, K$. The partitions $P_1, \ldots, P_K$ correspond to the sub-basins: Westernmost Mediterranean Sea, Tyrrhenian Sea, Adriatic Sea, Ionian Sea, and Levantine-Aegean Sea, respectively.

We applied the extended skew-GNNMP model (16) using the whole data set as reference set with $L$ chosen to be 10, 15 or 20. The regression parameters $\beta = (\beta_0, \beta_1, \beta_2)^\top$ were assigned mean-zero, dispersed normal priors. For the skewness parameters $\lambda = (\lambda_1, \ldots, \lambda_5)$, each element received a $N(0, 5)$ prior. We used the same prior specification for other parameters as in the first simulation experiment. Posterior inference was based on thinned samples retaining every 4th iteration, from a total of 30000 samples with a burn-in of first 10000 samples. The computing time was around 14, 16, and 20 minutes, respectively, for each of the $L$ values.

We focus on the estimation of regression and skewness parameters $\beta$ and $\lambda$. We report the estimates for $L = 15$; they were similar for $L = 10$ or 20. The posterior mean and 95% credible interval estimates of $\beta_0$, $\beta_1$, and $\beta_2$ were 30.51 (28.88, 32.16), 0.12 (0.09, 0.15), 0.09 (0.07, 0.11), respectively, for the Mediterranean Sea according to the sub-basins, i.e., $\mathcal{D} = \cup_{k=1}^K P_k$, $P_i \cap P_j = \emptyset$ for $i \neq j$, where $K = 5$. For all $v \in P_k$, we take $\lambda(v) = \lambda_k$, for $k = 1, \ldots, K$. The partitions $P_1, \ldots, P_K$ correspond to the sub-basins: Westernmost Mediterranean Sea, Tyrrhenian Sea, Adriatic Sea, Ionian Sea, and Levantine-Aegean Sea, respectively.

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Figure 7: SST data analysis. Panels (a) and (b) are partitions according to Mediterranean sub-basins and histograms of the residuals obtained from a non-spatial linear model. Panels (c) and (d) are posterior median and 95% credible interval estimates of the SST from the extended skew-Gaussian NNMP model.

and $-0.37 (-0.42, -0.33)$, indicating that there was an increasing trend in SST as longitude increased and latitude decreased. The corresponding posterior estimates of $\lambda$ were $-0.38 (-0.94, 0.14)$, $-1.37 (-2.10, -0.71)$, $-2.44 (-4.03, -1.14)$, $-1.60 (-2.54, -0.86)$, and $-2.69 (-3.95, -1.82)$. These estimates suggest different levels of left skewness over the sub-basins except for the Westernmost Mediterranean Sea.

Figure 7(c) provides the posterior median estimate of the SST over a dense grid of locations on the Mediterranean Sea. Compared to Figure 5, the estimate overall resembles the observed pattern. The prediction was quite smooth even for areas with few observations. The 95% credible interval width of the SST over the gridded locations, as shown in Figure 7(d), demonstrates that the model describes the uncertainty in accordance with the observed data structure; the uncertainty is higher in areas where there are less observations or the observations are volatile.
6 Summary and discussion

We have introduced a class of geostatistical models for large, non-Gaussian data sets, based on nearest-neighbor processes. Using an MTD model as the parent process, we have demonstrated the NNMP’s flexibility for modeling complex dependence by specification of a collection of bivariate distributions indexed at space. The scope of the methodology has been illustrated through data examples with skewness, heavy tails or compact support. An additional simulation experiment, available in the supplementary material, demonstrates that the GNNMP provides a good approximation to Gaussian random fields.

To incorporate covariates, the NNMP can be embedded in an additive or multiplicative regression model. The former is illustrated in Section 5.1. Under an additive model, the MCMC algorithm requires extra care as it involves sequential updating of the elements in $z_S$. This may induce slow convergence behavior. An alternative strategy for covariate inclusion is to model the weights or some parameter(s) of the spatially varying conditional density as a function of covariates. For example, in Section 5.2 we modeled the location parameter of the skew-Gaussian marginal as a linear function of the covariates. Posterior simulation under this approach is easily developed by modifying the update of the relevant parameters discussed in Section 3.2 to that of the regression coefficients.

The computation of the NNMP not only bypasses all the potential issues from large matrix operations, but also enhances modeling power. Kernel functions, such as wave co-variance functions, that are impractical for Gaussian process-based models due to numerical instability from matrix inversion, can be used as link functions for the spatially varying parameter of the NNMP. One limitation of the NNMP’s computation, similar to mixture models, is that the MCMC algorithm may experience slow convergence issues. Further development is needed on efficient algorithms for fast computation, especially when dealing with massive, complex data sets.

In this article, we have focused on developing the framework for continuous data. The proposed approach can be naturally extended to modeling discrete data. Modeling options for geostatistical count data in the existing literature involve either spatial generalized linear mixed models (Diggle et al., 1998) or spatial copula models (Madsen, 2009). However,
owing to their structures, both models have limitations with respect to the distributional assumption for the spatial random effects, as well as in computational efficiency. The extension to discrete NNMP models has the potential to provide both inferential and computational benefits to modeling large discrete data sets.

It is also interesting to explore the opportunities for the analysis of spatial extremes using the NNMP framework. We developed guidelines in Section 2.3 to choose NNMP mixture components based on strength of tail dependence. The results highlight the ability of the NNMP model structure to capture local tail dependence in different levels, controlled by the mixture component bivariate distributions, e.g., with a class of bivariate extreme-value copulas. Moreover, using NNMPs for spatial extreme modeling allows for efficient computation for implementation of inference which is typically a challenge with existing approaches (Davison et al., 2012).

Other research directions include extensions to multivariate and spatio-temporal settings. The former extension requires families of high-dimensional multivariate distributions to construct an NNMP. Effective strategies will be needed to define the spatially varying multivariate distributions that balance flexibility and scalability. When it comes to a joint model over time and space, there is large scope for exploring the integration of the time component into the model, including extending the NNMP weights or the NNMP mixture components.

Supplementary Material

The supplementary material includes proofs of the propositions, additional simulation experiments, and MCMC implementation details.

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A Proofs

Proof of Proposition 1. We consider a univariate spatial process \( \{Z(v), v \in \mathcal{D}\} \), where \( Z(v) \) takes values in \( \mathcal{X} \subseteq \mathbb{R} \), and \( \mathcal{D} \subseteq \mathbb{R}^p, p \geq 1 \). Let \( \mathcal{S} \subseteq \mathcal{D} \) be a reference set. Without loss of generality, we consider the continuous case, i.e., \( Z(v) \) has a continuous distribution for which its density exists, for all \( v \in \mathcal{D} \). To verify the proposition, we partition the domain \( \mathcal{D} \) into the reference set \( \mathcal{S} \) and the nonreference set \( \mathcal{U} \).

Given any \( v \in \mathcal{D} \), consider a bivariate random vector indexed at \( v \), denoted as \( (U_{v,l}, V_{v,l}) \) taking values in \( \mathcal{X} \times \mathcal{X} \). We denote \( f_{v,l} \) as the conditional density of \( U_{v,l} \) given \( V_{v,l} \), and \( f_{U_{v,l}}, f_{V_{v,l}} \) as the marginal densities of \( U_{v,l}, V_{v,l} \), respectively. Using the proposition assumption that \( f_Z = f_{U_{v,l}} = f_{V_{v,l}} \), we have that

\[
\int_{\mathcal{X}} f_{v,l}(u | v) f_Z(v) dv = \int_{\mathcal{X}} f_{v,l}(u | v) f_{V_{v,l}}(v) dv = f_{U_{v,l}}(u) = f_Z(u),
\]

for every \( v \in \mathcal{D} \) and for all \( l \).

We first prove the result for the reference set \( \mathcal{S} \). By the model assumption, locations in \( \mathcal{S} \) are ordered. In this regard, using the proposition assumptions, we can show that \( Z(s) \sim f_Z \) for all \( s \in \mathcal{S} \) by applying Proposition 1 in Zheng et al. (2022).

Turning to the nonreference set \( \mathcal{U} \). Let \( g_u(z(u)) \) be the marginal density of \( Z(u) \) for every \( u \in \mathcal{U} \). Denote by \( \tilde{p}(z_{Ne(u)}) \) the joint density for the random vector \( z_{Ne(u)} \) where \( Ne(u) = \{u_{(1)}, \ldots, u_{(L)}\} \subset \mathcal{S} \), so every element of \( Z_{Ne} \) has marginal density \( f_Z \). Then, the
marginal density for $Z(u)$ is given by:

$$g_u(z(u)) = \int_{\mathcal{X}^L} p(z(u) \mid z_{Ne(u)}) \tilde{p}(z_{Ne(u)}) \prod_{\{s_i \in Ne(u)\}} d(z(s_i))$$

$$= \sum_{i=1}^{L} w_i(v) \int_{\mathcal{X}^L} f_{v,i}(z(u) \mid z(u(l))) \tilde{p}(z_{Ne(u)}) \prod_{\{s_i \in Ne(u), s_i \neq u(l)\}} d(z(s_i))$$

$$= \sum_{i=1}^{L} w_i(v) \int_{\mathcal{X}^L} f_{v,i}(z(u) \mid z(u(l))) f_Z(z(u(l))) d(z(u(l)))$$

$$= f_Z(z(u)),$$

where the second-to-last equality holds by the result that $Z(s) \sim f_Z$ for all $s \in \mathcal{S}$ and $Ne(u) \subset \mathcal{S}$ for every $u \in \mathcal{U}$. The last equality follows from (1).

\[\square\]

**Proof of Proposition 2.** We verify the proposition by partitioning the domain $\mathcal{D}$ into the reference set $\mathcal{S}$ and the nonreference set $\mathcal{U}$. We first prove by induction the result for the joint distribution $\tilde{p}(z_s)$ over $\mathcal{S}$. Then to complete the proof, it suffices to show that for every location $u \in \mathcal{U}$, the joint density $\tilde{p}(z_{U_1})$ is a mixture of multivariate Gaussian distributions, where $U_1 = S \cup \{u\}$.

Without loss of generality, we assume $\mu = 0$ for the stationary GNNMP, i.e., the GNNMP has invariant marginal $f_Z(z) = N(z \mid 0, \sigma^2)$. The conditional density for the reference set is

$$p(z(s_i) \mid z_{Ne(s_i)}) = \sum_{i=1}^{i_L} w_i(s_i) N(z(s_i) \mid \rho_i(s_i) z(s_{i(L)}), \sigma^2 (1 - (\rho_i(s_i))^2)),$$

where for, $i = 2, \ldots, L$, $i_L = i - 1$, and for $i > L$, $i_L = L$. For each $i$, we denote as $\{w_{i,L}\}_{i=1}^{i_L}$ the vector of mixture weights, as $\{\rho_{i,L}\}_{i=1}^{i_L}$ the vector of the correlation coefficients, and as $\{z_{i,L}\}_{i=1}^{i_L}$ the vector of the nearest neighbors of $z_i \equiv z(s_i)$, for $i \geq 2$, where $w_{i,L} \equiv w_{i}(s_i)$, $\rho_{i,L} \equiv \rho_{i}(s_i)$, $z_{i,L} \equiv z(s_{i(L)})$. Let $z_1 \equiv z(s_1)$. We denote by $z_{1:k} = (z(s_1), \ldots, z(s_k))$ the realization of $Z(s)$ over locations $(s_1, \ldots, s_k)^\top$ for $k \geq 2$, and use $z_{1:k}^{-z_j}$ to denote the random vector $z_{1:k}$ with element $z_j$ removed, $1 \leq j \leq k$. In the following, for a vector $a = (a_1, \ldots, a_m)^\top$, we have that $ac = (a_1c, \ldots, a_mc)^\top$, where $c$ is a scalar.
Take $Z_1 \sim N(z_1 | 0, \sigma^2)$. When $i = 2$, $i_L = 1$ and $w_{2,1} = 1$. The joint density of $z_{1:2}$ is

$$
\tilde{p}(z_{1:2}) = N(z_2 | \rho_{2,1} z_1, \sigma^2 (1 - \rho_{2,1}^2)) N(z_1 | 0, \sigma^2) = N(z_{1:2} | 0, \sigma^2 \Omega_{2,1})
$$

where $\Omega_{2,1} = \begin{pmatrix} 1 & \rho_{2,1} \\ \rho_{2,1} & 1 \end{pmatrix}$.

The joint density of $z_{1:3}$ is

$$
\tilde{p}(z_{1:3}) = \tilde{p}(z_{1:2}) p_3(z_3 | z_{1:2})
\begin{align*}
= \sum_{l_3 = 1}^{2} w_{3,l_3} N(z_3 | \rho_{3,l_3} z_{3,l_3}, \sigma^2 (1 - \rho_{3,l_3}^2)) N(z_{1:2} | 0, \sigma^2 \Omega_{2,1}) \\
= \sum_{l_3 = 1}^{2} w_{3,l_3} N(z_3 | \rho_{3,l_3} z_{3,l_3}, \sigma^2 (1 - \rho_{3,l_3}^2)) N(z_{1:2} | \rho_{2,l_2} z_{3,l_3}, \sigma^2 (1 - \rho_{2,1}^2)) N(z_{3,l_3} | 0, \sigma^2) \\
= \sum_{l_3 = 1}^{2} w_{3,l_3} N((z_3, z_{1:2}^{\top}) | m_{3,l_3} z_{3,l_3}, V_{3,l_3}) N(z_{3,l_3} | 0, \sigma^2)
\end{align*}
$$

where $m_{3,l_3} = (\rho_{3,l_3}, \rho_{2,1})^\top$, and $V_{3,l_3} = \begin{pmatrix} \sigma^2 (1 - \rho_{3,l_3}^2) & 0 \\ 0 & \sigma^2 (1 - \rho_{2,1}^2) \end{pmatrix}$. The last equality follows from the fact that a product of conditionally independent Gaussian densities is a Gaussian density. By the properties of the Gaussian distribution and the property of the model that has a stationary marginal $N(0, \sigma^2)$, for each $l_3$, we have that

$$
N(\tilde{z}_{1:3,l_3} | 0, \sigma^2 R_{3,l_3}) = N((z_3, z_{1:2}^{\top}) | m_{3,l_3} z_{3,l_3}, V_{3,l_3}) N(z_{3,l_3} | 0, \sigma^2),
$$

where $\tilde{z}_{1:3,l_3} = (z_3, z_{1:2}^{\top}, z_{3,l_3})^\top$, with the following partition relevant to the vector $\tilde{z}_{1:3,l_3}$,

$$
\tilde{z}_{1:3,l_3} = \begin{pmatrix} (z_3, z_{1:2}^{\top}) \\ z_{3,l_3} \end{pmatrix}, \quad E(\tilde{z}_{1:3,l_3}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad R_{3,l_3} = \begin{pmatrix} R_{3,l_3}^{(11)} & R_{3,l_3}^{(12)} \\ R_{3,l_3}^{(21)} & R_{3,l_3}^{(22)} \end{pmatrix},
$$

where $R_{3,l_3}^{(22)} = 1$ corresponds to $z_{3,l_3}$. It follows that

$$
\begin{align*}
& m_{3,l_3} z_{3,l_3} = E((Z_3, \tilde{Z}_{1:2}^{\top}) | Z_{3,l_3} = z_{3,l_3}) = R_{3,l_3}^{(2)} z_{3,l_3}, \\
& V_{3,l_3} = \sigma^2 (R_{3,l_3}^{(11)} - R_{3,l_3}^{(12)} R_{3,l_3}^{(21)}).
\end{align*}
\tag{2}
$$

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From [2], we obtain \( m_{3,l_3} = R_{3,l_3}^{(12)} \) and \( R_{3,l_3} = \begin{pmatrix} 1 & \rho_{3,l_3} \rho_{3,l_3} \\ \rho_{2,1} \rho_{3,l_3} & 1 & \rho_{2,1} \end{pmatrix} \) for \( l_3 = 1, 2 \). Then we reorder \( \tilde{z}_{1:3,l_3} \) with a matrix \( B_{3,l_3} \) such that \( \tilde{z}_{1:3} = B_{3,l_3} \tilde{z}_{1:3,l_3} \). It follows that \( \Omega_{3,l_3} = B_{3,l_3} R_{3,l_3} B_{3,l_3}^T \), and the joint density is

\[
p(\tilde{z}_{1:3}) = \sum_{l_3=1}^{2} w_{3,l_3} N(\tilde{z}_{1:3} | 0, \sigma^2 \Omega_{3,l_3})
\]

Similarly, the joint density of \( z_{1:4} \) is given by

\[
\bar{p}(z_{1:4}) = p_{4}(z_{1:3}|z_{1:3}) \bar{p}(z_{1:3})
\]

\[
= \sum_{l_4=1}^{3} w_{4,l_4} N(z_{1:3} | 0, \sigma^2(1 - \rho^2_{4,l_4})) \sum_{l_3=1}^{2} w_{3,l_3} N(\tilde{z}_{1:3} | 0, \sigma^2 \Omega_{3,l_3})
\]

\[
= \sum_{l_4=1}^{3} \sum_{l_3=1}^{2} w_{4,l_4} w_{3,l_3} N(z_{1:3} | 0, \sigma^2(1 - \rho^2_{4,l_4}))
\]

\[
N((z_{1:3} | \Omega_{3,l_3})^{(12)} z_{1:3} | \Omega_{3,l_3})^T, \sigma^2(\Omega_{3,l_3} - \Omega_{3,l_3}^{(12)} \Omega_{3,l_3}^{(21)})) N(z_{1:3} | 0, \sigma^2)
\]

\[
= \sum_{l_4=1}^{3} \sum_{l_3=1}^{2} w_{4,l_4} w_{3,l_3} N((z_{1:3}, z_{1:3} | \Omega_{3,l_3})^{(12)} z_{1:3} | \Omega_{3,l_3})^T | m_{4,l_4} z_{1:3}, V_{4,l_4}^T) N(z_{1:3} | 0, \sigma^2)
\]

where \( \tilde{\Omega}_{3,l_3} = B_{4,l_4} \Omega_{3,l_1} B_{4,l_4}^T \), and \( B_{4,l_4} \) is a rotation matrix such that \( (z_{1:3} | \tilde{\Omega}_{3,l_3})^{(12)} = B_{4,l_4} z_{1:3} \). We partition the matrix \( \tilde{\Omega}_{3,l_3} \) such that \( \tilde{\Omega}_{3,l_1}^{(11)} \) and \( \tilde{\Omega}_{3,l_1}^{(22)} \) correspond to \( z_{1:3} \) and \( z_{4,l_4} \), respectively. We have that for \( l_3 = 1, 2, l_4 = 1, 2, 3 \),

\[
N(\tilde{z}_{1:4,l_4} | 0, \sigma^2 R_{4,l_4}) = N((z_{1:3}, z_{1:3} | \tilde{\Omega}_{3,l_3}^{(12)} z_{1:3} | \tilde{\Omega}_{3,l_3})^T | m_{4,l_4} z_{1:3}, V_{4,l_4}) N(z_{1:3} | 0, \sigma^2),
\]

and

\[
\tilde{z}_{1:4,l_4} = (z_{1:3}, z_{1:3} | \tilde{\Omega}_{3,l_3}^{(12)} z_{1:3} | \tilde{\Omega}_{3,l_3})^T, \quad m_{4,l_4} z_{1:3} = (\rho_{4,l_4}, (\tilde{\Omega}_{3,l_3}^{(12)})^T)^T,
\]

\[
V_{4,l_4} z_{1:3} = \begin{pmatrix} \sigma^2(1 - \rho^2_{4,l_4}) & 0 \\ 0 & \sigma^2(\tilde{\Omega}_{3,l_3}^{(11)} - \tilde{\Omega}_{3,l_3}^{(12)} \tilde{\Omega}_{3,l_3}^{(21)}) \end{pmatrix},
\]

\[
R_{4,l_4}^{(12)} = (R_{4,l_4} z_{1:3})^T = m_{4,l_4} z_{1:3}, \quad R_{4,l_4}^{(11)} = V_{4,l_4} z_{1:3} / \sigma^2 + m_{4,l_4} m_{4,l_4}^T,
\]

We reorder \( \tilde{z}_{1:4,l_4} \) with a matrix \( B_{4,l_4} \) such that \( \tilde{z}_{1:4} = B_{4,l_4} \tilde{z}_{1:4,l_4} \) and let \( \Omega_{4,l_4} = \)
\[ B_{4,1} R_{4,4} B_{4,4}^T. \] Then we obtain the joint density of \( z_{1:4} \) as

\[
\tilde{p}(z_{1:4}) = \sum_{i_4=1}^{3} \sum_{i_3=1}^{2} w_{4,i_4} w_{3,i_3} N(z_{1:4} | 0, \sigma^2 \Omega_{4,i_4,i_3}).
\]

Applying the above technique iteratively for \( \tilde{p}(z_{1:j}) \) for \( 5 \leq j \leq k \), we obtain the joint density \( \tilde{p}(z_{1:k}) \equiv \tilde{p}(z_S) \), for \( k \geq 2 \), namely,

\[
\tilde{p}(z_{1:k}) = \sum_{l_k=1}^{k_L} \cdots \sum_{l_2=1}^{2_{l_2}} w_{k,l_k} \cdots w_{3,l_3} w_{2,l_2} N(z_{1:k} | 0, \sigma^2 \Omega_{k,l_k \ldots l_3 l_2}).
\]

where \( k_L := (k - 1) \wedge L \), \( w_{2,1} = 1 \), and for \( k \geq 3 \),

\[
\tilde{\Omega}_{k-1,l_{k-1} \ldots l_3 l_1} = \tilde{B}_{k,l_{k-1}} \Omega_{k-1,l_{k-1} \ldots l_3 l_1} \tilde{B}_{k,l_k}, \quad m_{k,l_k \ldots l_3} = (\rho_{k,l_k}, (\tilde{\Omega}_{k-1,l_{k-1} \ldots l_3 l_1}^{(12)})^\top),
\]

\[
V_{k,l_k \ldots l_3} = \begin{pmatrix}
\sigma^2 (1 - \rho_{k,l_k}^2) & 0 \\
0^T & \sigma^2 (\tilde{\Omega}_{k-1,l_{k-1} \ldots l_3 l_1}^{(11)} - \tilde{\Omega}_{k-1,l_{k-1} \ldots l_3 l_1}^{(12)} \tilde{\Omega}_{k-1,l_{k-1} \ldots l_3 l_1}^{(21)})
\end{pmatrix},
\]

\[
R_{k,l_k \ldots l_3}^{(12)} = (R_{k,l_k \ldots l_3}^{(21)})^\top = m_{k,l_k \ldots l_3}, \quad R_{k,l_k \ldots l_3}^{(11)} = V_{k,l_k \ldots l_3} / \sigma^2 + m_{k,l_k \ldots l_3} m_{k,l_k \ldots l_3}^\top,
\]

\[
\Omega_{k,l_k \ldots l_3} = B_{k,l_k} R_{k,l_k \ldots l_3} B_{k,l_k}^T,
\]

where \( \tilde{B}_{k,l_k} \) is the rotation matrix such that \( (z_{1:k-1}^{-z_{k,l_k}}, z_{k,l_k})^\top = \tilde{B}_{k,l_k} z_{1:(k-1)} \) and \( B_{k,l_k} \) is the rotation matrix such that the vector \( z_{1:k} = B_{k,l_k} \tilde{z}_{1:k,l_k} \), where \( \tilde{z}_{1:k,l_k} = (z_k, z_{1:(k-1)}, z_{k,l_k})^\top \).

To complete the proof, what remains to be shown is that the density \( \tilde{p}(z_{u_{l_1}}) \) is a mixture of multivariate Gaussian distributions, where \( U_1 = S \cup \{u\} \). We have that

\[
\tilde{p}(z_{u_{l_1}}) = \sum_{l=1}^{L} w_l(u) N(z(u) \mid \rho_l(u) z(u_{l_1}), \sigma^2 (1 - (\rho_l(u))^2)) \tilde{p}(z_{1:k}),
\]

where \( z(u_{l_1}) \) is an element of \( z_{1:k} \), for \( l = 1, \ldots, L \). We can express each component density \( N(z_{1:k} \mid 0, \sigma^2 \Omega_{k,l_k \ldots l_3 l_1}) \) of the joint density \( \tilde{p}(z_{1:k}) \) as the product of a Gaussian density of \( Z_1^{-Z(u_{l_1})} \) conditional on \( Z(u_{l_1}) = z(u_{l_1}) \) and a Gaussian density of \( Z(u_{l_1}) \). Using the approach in deriving the joint density over \( S \), we can show that \( \tilde{p}(z_{u_{l_1}}) \) is a mixture of multivariate Gaussian distributions.
Proof of Proposition 3. For an NNMP $Z(\mathbf{v})$, The conditional probability that $Z(\mathbf{v})$ is greater than $z$ given its neighbors $Z_{\text{Ne}(\mathbf{v})} = z_{\text{Ne}(\mathbf{v})}$, where $z_{\text{Ne}(\mathbf{v})} = (z_{v(1)}, \ldots, z_{v(l)})$, is

$$P(Z(\mathbf{v}) > z \mid Z_{\text{Ne}(\mathbf{v})} = z_{\text{Ne}(\mathbf{v})}) = \sum_{l=1}^{L} w_l(\mathbf{v}) P(Z(\mathbf{v}) > z \mid Z(\mathbf{v}(l)) = z(\mathbf{v}(l))),$$

where the conditional probability $P(Z(\mathbf{v}) > z \mid Z(\mathbf{v}(l)) = z(\mathbf{v}(l)))$ corresponds to the bivariate random vector $(U_{v,l}, V_{v,l})$. If $U_l$ is stochastically increasing in $V_l$ for all $l$, by the assumption that the sequence $(U_{v,l}, V_{v,l})$ is built from the base random vectors $(U_l, V_l)$ for all $l$, we have that $Z(\mathbf{v})$ is stochastically increasing in $Z_{\text{Ne}(\mathbf{v})}$ for every $\mathbf{v} \in D$, i.e.

$$P(Z(\mathbf{v}) > z \mid Z_{\text{Ne}(\mathbf{v})} = z_{\text{Ne}(\mathbf{v})}) \leq P(Z(\mathbf{v}) > z \mid Z_{\text{Ne}(\mathbf{v})} = z'_{\text{Ne}(\mathbf{v})})$$

for all $z_{\text{Ne}(\mathbf{v})}$ and $z'_{\text{Ne}(\mathbf{v})}$ in the support of $Z_{\text{Ne}(\mathbf{v})}$, such that $z_{v(l)} \leq z'_{v(l)}$ for all $l$.

Let $F_{Z(\mathbf{v})}$ and $F_{Z(\mathbf{v}(1)), \ldots, Z(\mathbf{v}(L))}$ be the distribution functions of $Z(\mathbf{v})$ and $Z_{\text{Ne}(\mathbf{v})}$, respectively. Denote by $S_{Z(\mathbf{v}(1)), \ldots, Z(\mathbf{v}(L))}(z_1, \ldots, z_L) = P(Z(\mathbf{v}(1)) > z_1, \ldots, Z(\mathbf{v}(L)) > z_L)$ the joint survival probability. Then for every $\mathbf{v} \in D$ and $q \in (0, 1),$

$$P(Z(\mathbf{v}) > F_{Z(\mathbf{v})}^{-1}(q) \mid Z(\mathbf{v}(1)) > F_{Z(\mathbf{v}(1))}^{-1}(q), \ldots, Z(\mathbf{v}(L)) > F_{Z(\mathbf{v}(L))}^{-1}(q)) = \int_{F_{Z(\mathbf{v}(1))}^{-1}(q)}^{\infty} \ldots \int_{F_{Z(\mathbf{v}(L))}^{-1}(q)}^{\infty} P(Z(\mathbf{v}) > F_{Z(\mathbf{v})}^{-1}(q) \mid Z(\mathbf{v}(1)) = z_1, \ldots, Z(\mathbf{v}(L)) = z_L)$$

$$dF_{Z(\mathbf{v}(1)), \ldots, Z(\mathbf{v}(L))}(z_1, \ldots, z_L) \bigg\} \bigg/ S_{Z(\mathbf{v}(1)), \ldots, Z(\mathbf{v}(L))} F_{Z(\mathbf{v}(1))}^{-1}(q), \ldots, F_{Z(\mathbf{v}(L))}^{-1}(q)$$

$$\geq \int_{F_{Z(\mathbf{v}(1))}^{-1}(q)}^{\infty} \ldots \int_{F_{Z(\mathbf{v}(L))}^{-1}(q)}^{\infty} P(Z(\mathbf{v}) > F_{Z(\mathbf{v})}^{-1}(q) \mid Z(\mathbf{v}(1)) = F_{Z(\mathbf{v}(1))}^{-1}(q), \ldots, Z(\mathbf{v}(L)) = F_{Z(\mathbf{v}(L))}^{-1}(q))$$

$$dF_{Z(\mathbf{v}(1)), \ldots, Z(\mathbf{v}(L))}(z_1, \ldots, z_L) \bigg\} \bigg/ S_{Z(\mathbf{v}(1)), \ldots, Z(\mathbf{v}(L))} F_{Z(\mathbf{v}(1))}^{-1}(q), \ldots, F_{Z(\mathbf{v}(L))}^{-1}(q)$$

$$= P(Z(\mathbf{v}) > F_{Z(\mathbf{v})}^{-1}(q) \mid Z(\mathbf{v}(l)) = F_{Z(\mathbf{v}(l))}^{-1}(q), \ldots, Z(\mathbf{v}(l)) = F_{Z(\mathbf{v}(l))}^{-1}(q))$$

$$= \sum_{l=1}^{L} w_l(\mathbf{v}) P(Z(\mathbf{v}) > F_{U_{v,l}}^{-1}(q) \mid Z(\mathbf{v}(l)) = F_{V_{v,l}}^{-1}(q)),$$

(3)

where the first inequality follows from the stochastically increasing positive dependence of $Z(\mathbf{v})$ given $Z_{\text{Ne}(\mathbf{v})}$. 45
Taking $q \to 1^-$ on both sides of (3), we obtain
\[
\lambda_H(v) \geq \sum_{l=1}^{L} w_l(v) \lim_{q \to 1^-} P(Z(v) > F_{V_{v,l}}^{-1}(q) \mid Z(v) = F_{V_{v,l}}^{-1}(q)),
\]
where $F_{U_v,l}$ and $F_{V_{v,l}}$ are the marginal distribution functions of $(U_{v,l}, V_{v,l})$.

Similarly, we can obtain the lower bound for $\lambda_L(v)$.

\[\square\]

**Proof of Corollary 1.** We prove the result for $\lambda_L(v)$. The result for $\lambda_H(v)$ is obtained in a similar way.

Consider a bivariate distribution $F_{U_i,V_i}$ for random vector $(U_i, V_i)$, with marginal distributions $F_{U_i} = F_{V_i} = F_i$ and marginal densities $f_{U_i} = f_{V_i} = f_i$, for all $l$. The lower tail dependence coefficient is expressed as $\lambda_{L,l} = \lim_{q \to 0^+} \frac{\partial F_{U_i,V_i}(F_i^{-1}(q), F_i^{-1}(q)) + \partial F_{U_i,V_i}(F_i^{-1}(q), F_i^{-1}(q))}{f_i(F_i^{-1}(q))}$ with $q \in [0,1]$. If $F_{U_i,V_i}$ has first order partial derivatives, applying the L’Hopital’s rule, we obtain

\[
\lambda_{L,l} = \lim_{q \to 0^+} P(U_i \leq F_i^{-1}(q) \mid V_i = F_i^{-1}(q)) + \lim_{q \to 0^+} P(V_i \leq F_i^{-1}(q) \mid U_i = F_i^{-1}(q)).
\]

The above is a reproduced result from Theorem 8.57 of Joe (2014). If $(U_i, V_i)$ is exchangeable, we have

\[
\lambda_{L,l} = 2 \lim_{q \to 0^+} P(U_i \leq F_i^{-1}(q) \mid V_i = F_i^{-1}(q)).
\]

If the sequences $(U_{v,l}, V_{v,l})$ of an NNMP model are built from the base random vectors $(U_i, V_i)$. By our assumption that $F_{U_i} = F_{V_i}$ for all $l$, the marginal distributions of $(U_{v,l}, V_{v,l})$ extended from $(U_i, V_i)$ are $F_{v,l} = F_{U_{v,l}} = F_{V_{v,l}}$ for all $v$ and all $l$. Then we have

\[
\lambda_{L,l}(v) = 2 \lim_{q \to 0^+} P(U_{v,l} \leq F_{v,l}^{-1}(q) \mid V_{v,l} = F_{v,l}^{-1}(q)).
\]

Using the result of Proposition 3, we obtain

\[
\lambda_L(v) \geq \sum_{l=1}^{L} w_l(v) \lim_{q \to 0^+} P(U_{v,l} \leq F_{v,l}^{-1}(q) \mid V_{v,l} = F_{v,l}^{-1}(q)) = \sum_{l=1}^{L} w_l(v) \lambda_{L,l}(v)/2.
\]

\[\square\]
**Proof of Proposition 4.** By the assumption that \( U_l \) is stochastically increasing in \( V_l \) and that \((U_{v,l}, V_{v,l})\) is constructed based on \((U_l, V_l)\), \( U_{v,l} \) is stochastically increasing in \( V_{v,l} \) for all \( v \in \mathcal{D} \) and for all \( l \). Then for \((Z(v), Z(v(l)))\) with respect to the bivariate distribution of \((U_{v,l}, V_{v,l})\) with marginal distributions \( F_{U_{v,l}} \) and \( F_{V_{v,l}} \), we have that

\[
P(Z(v) \leq F_{V_{v,l}}^{-1}(q) \mid Z(v(l)) \leq F_{V_{v,l}}^{-1}(q))
\]

\[
= \int_{F_{V_{v,l}}^{-1}(0)}^{F_{V_{v,l}}^{-1}(q)} P(Z(v) \leq F_{V_{v,l}}^{-1}(q) \mid Z(v(l)) = z_l) dF_{V_{v,l}}(z_l) / \int_{F_{V_{v,l}}^{-1}(0)}^{F_{V_{v,l}}^{-1}(q)} dF_{V_{v,l}}
\]

\[
\leq \int_{F_{V_{v,l}}^{-1}(0)}^{F_{V_{v,l}}^{-1}(q)} P(Z(v) \leq F_{V_{v,l}}^{-1}(q) \mid Z(v(l)) = F_{V_{v,l}}^{-1}(0)) dF_{V_{v,l}}(z_l) / \int_{F_{V_{v,l}}^{-1}(0)}^{F_{V_{v,l}}^{-1}(q)} dF_{V_{v,l}}
\]

\[
= P(Z(v) \leq F_{V_{v,l}}^{-1}(q) \mid Z(v(l)) = F_{V_{v,l}}^{-1}(0)).
\]

It follows that the boundary cdf of the NNMP model

\[
F_{1|2}(F_{Z(v)}^{-1}(q) \mid F_{Z_{N0}(v)}^{-1}(0))
\]

\[
= P(Z(v) \leq F_{Z(v)}^{-1}(q) \mid Z(v(1)) = F_{Z(v(1))}^{-1}(0), \ldots, Z(v(L)) = F_{Z(v(L))}^{-1}(0))
\]

\[
= \sum_{l=1}^{L} w_l(v) P(Z(v) \leq F_{U_{v,l}}^{-1}(q) \mid Z(v(l)) = F_{V_{v,l}}^{-1}(0))
\]

\[
\geq \sum_{l=1}^{L} w_l(v) P(Z(v) \leq F_{U_{v,l}}^{-1}(q) \mid Z(v(0)) \leq F_{V_{v,l}}^{-1}(q)),
\]

Taking \( q \to 0^+ \) on both sides of (4), we obtain

\[
F_{1|2}(F_{Z(v)}^{-1}(0) \mid F_{Z_{N0}(v)}^{-1}(0)) \geq \sum_{l=1}^{L} w_l(v) \lambda_{L,l}(v).
\]

Hence, if there exists some \( l \) such that \( \lambda_{L,l}(v) > 0 \), the conditional cdf \( F_{1|2}(F_{Z(v)}^{-1}(q) \mid F_{Z_{N0}(v)}^{-1}(0)) \) has strictly positive mass at \( q = 0 \). We can prove the result for \( F_{1|2}(F_{Z(v)}^{-1}(q) \mid F_{Z_{N0}(v)}^{-1}(1)) \) in a similar way.  

\[ \square \]
B Additional Simulation Experiments

We provide two additional simulation experiments to demonstrate the benefits of the proposed NNMP framework. In each of the two experiments, we created a regular grid of 200 × 200 resolution on a unit square domain, and generated data on each grid location. We then randomly selected a subset of the data as the reference set with a random ordering for model fitting. For the purpose of illustration, we chose neighbor size $L = 10$ for both cases. Results are based on posterior samples collected every 10th iteration from a Markov chain of 30000 iterations, with the first 10000 samples as a burn-in.

B.1 Additional simulation experiment 1

We generated data from a spatially varying regression,

$$y(v) = x(v)^\top \beta + z(v) + \epsilon(v), \quad v \in \mathcal{D},$$

where $z(v)$ is a standard Gaussian process with an exponential correlation function with range parameter 1/12. We included an intercept and a covariate drawn from $N(0, 1)$ in the model, and chose $\beta = (\beta_0, \beta_1)^\top = (1, 5)^\top$, and $\tau^2 = 0.1$. The setting followed the simulation experiment in Datta et al. (2016a).

We applied two models. The first one assumes that $z(v)$ follows an NNGP model with variance $\sigma_0^2$ and exponential correlation function with range parameter $\phi_0$. The second one assumes that $z(v)$ follows a stationary GNNMP model, i.e., $\mu_l = 0$ and $\sigma_l^2 = \sigma^2$ for all $l$, such that $z(v)$ has a stationary marginal $N(0, \sigma^2)$. For the GNNMP, we used exponential correlation functions with range parameter $\phi$ and $\zeta$, respectively, for the correlation with respect to the component density and the cutoff points kernel function. For the NNGP model, we implement the latent NNGP algorithm from the spNNGP package in R (Finley et al., 2020). We trained both models using 2000 of the 2500 observations, and used the remaining 500 observations for model comparison.

For both models, the regression coefficients $\beta$ were assigned flat priors. The variances $\sigma_0^2$ and $\sigma^2$ received the same inverse gamma prior IG(2, 1), and $\tau^2$ was assigned IG(2, 0.1). The range parameter $\phi_0$ of the NNGP received a uniform prior Unif(1/30, 1/3), while the range parameters $\phi$ and $\zeta$ of the GNNMP received inverse gamma priors IG(3, 1/3) and...
Figure 1: Additional simulation experiment 1 data analysis: interpolated surface of the true Gaussian random field and posterior median estimates from the NNGP and GNNMP models.

Table 1: Performance metrics of different models

|        | RMSPE | 95% CI | 95% CI width | CRPS | PPLC  | DIC    |
|--------|-------|--------|--------------|------|-------|--------|
| GNNMP  | 0.57  | 0.96   | 2.51         | 0.32 | 461.40| 2656.67|
| NNGP   | 0.54  | 0.95   | 2.09         | 0.30 | 382.15| 2268.79|

IG(3, 0.2), respectively. Regarding the logit Gaussian distribution parameters, $\gamma$ and $\kappa^2$, we used $N(\gamma | (-1.5, 0, 0)^\top, 2I_3)$ and IG(3, 1) priors, respectively.

The posterior estimates from the two models for the common parameters, $\beta$ and $\tau^2$, were quite close. The posterior mean and 95% credible interval estimates of $\beta_0$ and $\beta_1$ were $1.32 (1.11, 1.54)$ and $5.01 (4.99, 5.04)$ from the GNNMP model, and $1.25 (0.83, 1.62)$ and $5.01 (4.99, 5.04)$ from the NNGP model. The corresponding estimates of $\tau^2$ were $0.12 (0.09, 0.15)$ and $0.10 (0.07, 0.12)$ from the GNNMP and NNMP models, respectively.

Table 1 shows the performance metrics of the two models. The performance metrics of the GNNMP model are comparable to those of the NNGP model, the model assumptions of which are more well suited to the particular synthetic data example. The posterior median estimate of the spatial random effects from both models are shown in Figure 1. We can see that the predictive field given by the GNNMP looks close to the true field and that predicted by the NNGP. On the whole, the GNNMP model provides a reasonably good approximation to the Gaussian random field.
B.2 Additional simulation experiment 2

We generated bounded data using the following model,

\[ y(v) | \mu(v), \psi \sim \text{Beta}(\mu(v)\psi, (1 - \mu(v))\psi), \]
\[ \text{logit}(\mu(v)) = \mu_0 + \sigma_0 \omega(v). \]

The above model is analogous to a spatial generalized linear mixed model where the mean \( \mu(v) \) of the beta distribution is modeled via a logit link function, and \( \omega(v) \) is a standard Gaussian process with exponential correlation function with range parameter 0.1. We set \( \psi = 20, \mu_0 = -0.5 \) and \( \sigma_0 = 0.8 \).

Since our purpose is primarily demonstrative, we applied a Gaussian copula NNMP model with a stationary beta marginal \( \text{Beta}(a, b) \), referred to as the beta NNMP model. The correlation parameter of the Gaussian copula was specified by an exponential correlation function with range parameter \( \phi \). We specified an exponential correlation function for the random cutoff points kernel function with range parameter \( \zeta \). The Bayesian model is fully specified with a IG(3, 1/3) prior for \( \phi \), a Ga(1, 1) prior for \( a \) and \( b \), a IG(3, 0.2) prior for \( \zeta \), \( N(\gamma | (-1.5, 0, 0)^T, 2I_3) \) and IG(\( \kappa^2 | 3, 1 \)).

We trained the model using 2000 observations. Figure 2(a)-(b) shows the interpolated surface of the true field and the predictive field given by the beta NNMP model. Although the beta NNMP’s stationary marginal distribution assumption does not align with the true model, we can see that the predictive field was able to capture the main feature of the true field.
Moreover, it is worth mentioning that the MCMC algorithm for the beta NNMP to fit the data set took around 18 minutes with 30000 iterations. This is substantially faster than the MCMC algorithm for fitting the true model which involves sampling a large number of highly correlated latent variables.

C NNMP models and MCMC Implementation

In this section, we provide details of the models implemented in the data examples of the main paper. In particular, Section 3.1 corresponds to the real data example of the main paper and additional simulation experiment 1 of this supplementary material. Section 3.2 discusses the stationary skew-GNNMP model for the first simulation experiment of the main paper and the extended skew-GNNMP model for the real data example of the main paper. Section 3.3 introduces the Gaussian and Gumbel copula NNMP models implemented in the second and third simulation experiments of the main paper and the additional simulation experiment 2 of this supplementary material.

C.1 GNNMP models

We consider the spatially varying regression model,

\[ y(v) = x(v)^\top \beta + z(v) + \epsilon(v), \quad v \in D \]

where \( \epsilon(v) \overset{i.i.d.}{\sim} N(0, \tau^2) \), and the spatial random effect \( z(v) \) follows a stationary GNNMP model. The associated conditional density of the model is

\[ p(z(v) \mid z_{Ne(v)}) = \sum_{l=1}^{L} w_l(v) N(z(v) \mid \rho_l(v) z(v(l)), \sigma^2(1 - (\rho_l(v))^2)), \quad (5) \]

where \( \rho_l(v) \equiv \rho_l(||v - v(l)||) = \exp(-||v - v(l)||/\phi) \). For the weights, we consider an exponential correlation function with range parameter \( \zeta \) for the kernel function that defines the random cutoff points. The Bayesian model is completed with priors \( N(\beta \mid \mu_\beta, V_\beta) \), \( \text{IG}(\sigma^2 \mid u_{\sigma^2}, v_{\sigma^2}) \), \( \text{IG}(\tau^2 \mid u_{\tau^2}, v_{\tau^2}) \), \( \text{IG}(\phi \mid u_\phi, v_\phi) \), \( \text{IG}(\zeta \mid u_\zeta, v_\zeta) \), \( N(\gamma \mid \mu_\gamma, V_\gamma) \), \( \text{IG}(\kappa^2 \mid u_{\kappa^2}, v_{\kappa^2}) \).

Let \( y(s_i), i = 1, \ldots, n \), be the observations over reference set \( S = (s_1, \ldots, s_n) \). We
introduce the MCMC sampler for the spatially varying regression model in which the GNNMP is used as a prior for the latent spatial random effect. The MCMC sampler involves sampling the latent variables \( z(s_i) \), but it is easily developed based on the sampler described in the main paper.

For each \( z(s_i), i = 3, \ldots, n \), we introduce a configuration variable \( \ell_i \), taking values in \( \{1, \ldots, i_L\} \) where \( i_L = (i - 1) \wedge L \), such that \( \Pr(\ell_i \mid w(s_i)) = \sum_{i=1}^{i_L} w_i(s_i) \delta_i(\ell_i) \), where \( w(s_i) = (w_1(s_i), \ldots, w_{i_L}(s_i))^\top \) and \( \delta_i(\ell_i) = 1 \) if \( \ell_i = l \) and 0 otherwise. To allow for efficient simulation of parameters \( \gamma = (\gamma_0, \gamma_1, \gamma_2) \) and \( \kappa^2 \) for the weights, we associate each \( z(s_i) \) with a latent Gaussian variable with mean \( \mu(s_i) = \gamma_0 + s_i \gamma_1 + s_i^2 \gamma_2 \) and variance \( \kappa^2 \), where \( s_i = (s_{i1}, s_{i2}) \), for \( i = 3, \ldots, n \). There is a one-to-one correspondence between the configuration variables \( \ell_i \) and latent variables \( t_i \): \( \ell_i = l \) if and only if \( t_i \in (r_{s_i,l-1}, r_{s_i,l}) \) where \( r_{s_i,l} = \log((r_{s_i,l}/(1 - r_{s_i,l}))) \), for \( l = 1, \ldots, i_L \). The posterior distribution of the model parameters, based on the latent variables \( t_i \), is given by

\[
N(\beta \mid \mu_\beta, V_\beta) \times IG(\tau^2 \mid u_{\tau_2}, v_{\tau_2}) \times IG(\sigma^2 \mid u_{\sigma^2}, v_{\sigma^2}) \times IG(\phi \mid u_\phi, v_\phi) \times IG(\zeta \mid u_\zeta, v_\zeta)
\times N(\gamma \mid \mu_\gamma, V_\gamma) \times IG(\kappa^2 \mid u_{\kappa^2}, v_{\kappa^2}) \times \prod_{i=1}^{n-1} N(y(s_i) \mid x(s_i)^\top \beta + z(s_i), \tau^2)
\times N(t \mid D_\gamma \kappa^2 \mathbb{I}_{n-2}) \times N(z(s_1) \mid 0, \sigma^2) \times N(z(s_2) \mid \rho_1(s_2)z(s_1), \sigma^2(1 - (\rho_1(s_2))^2))
\times \prod_{i=3}^{n} \sum_{l=1}^{i_L} N(z(s) \mid \rho_i(s)z(s_{(i,l)}), \sigma^2(1 - (\rho_i(s_i))^2)) \mathbb{1}_{(r_{s_i,l-1}, r_{s_i,l})(t_i)},
\]

where the vector \( t = (t_3, \ldots, t_n)^\top \), and the matrix \( D \) is \( (n - 2) \times 3 \) such that the \( i \)th row is \( (1, s_{2+i,1}, s_{2+i,2}) \).

We describe the MCMC sampler to simulate from the posterior distribution of model parameters \( (\beta, \gamma, \sigma^2, \phi, \zeta, \tau^2, \kappa^2) \) and latent variables \( \{t_i\}_{i=3}^{n}, \{z(s_i)\}_{i=1}^{n} \). Denote by \( y_S = (y(s_1), \ldots, y(s_n))^\top \), \( z_S = (z(s_1), \ldots, z(s_n))^\top \), and let \( X \) be the covariate matrix with the \( i \)th row being \( x(s_i) \). The posterior full conditional distribution for \( \beta \) is \( N(\beta \mid \mu_\beta^*, V_\beta^*) \) where \( V_\beta^* = (V_\beta^{-1} + \tau^{-2}X^\top X)^{-1} \) and \( \mu_\beta^* = V_\beta^*(V_\beta^{-1} \mu_\beta + \tau^{-2}X^\top(y_S - z_S)) \). An inverse gamma prior for \( \tau^2 \) yields an IG(\( \tau^2 \mid u_{\tau_2}, v_{\tau_2} + \sum_{i=1}^{n} e_i^2/2 \)) posterior full conditional, where \( e_i = y(s_i) - x(s_i)^\top \beta - z(s_i) \).

Note that given the latent variable \( t_i \), we have the configuration variable \( \ell_i = l \) if \( t_i \in (r_{s_i,l-1}, r_{s_i,l}) \), for \( i = 3, \ldots, n \). To update \( \zeta \), we first marginalize out the latent variables \( t_i \)
from the joint posterior distribution. The posterior full conditional distribution of \( \zeta \) is proportional to \( \text{IG}(\zeta \mid u_\zeta, v_\zeta) \prod_{i=3}^{n} \{ G_{s_i}(r_{s_i, \ell_i} \mid \mu(s_i), \kappa^2) - G_{s_i}(r_{s_i, \ell_i-1} \mid \mu(s_i), \kappa^2) \} \). We update \( \zeta \) on its log scale using a random walk Metropolis step. The posterior full conditional distribution of the latent variable \( t_i, i = 3, \ldots, n \), is a piecewise Gaussian distribution on \((r_{s_i, l-1}, r_{s_i, l})\) with probabilities proportional to \( w_l(s_i) f_{s_i,l}(z(s_i) \mid z(s_{(i)}), \theta) \), where \( w_l(s_i) = G_{s_i}(r_{s_i, l} \mid \mu(s_i), \kappa^2) - G_{s_i}(r_{s_i, l-1} \mid \mu(s_i), \kappa^2) \). The posterior full conditional distribution of \( \gamma \) is \( N(\gamma \mid \mu_\gamma, V_\gamma^\ast) \) where \( V_\gamma^\ast = (V_\gamma^{-1} + \kappa^{-2} D^\top D)^{-1} \) and \( \mu_\gamma = V_\gamma^\ast (V_\gamma^{-1} \mu_\gamma + \kappa^{-2} D^\top t) \). The posterior full conditional distribution of \( \kappa^2 \) is \( \text{IG}(\kappa^2 \mid u_\kappa, 2(n-2)/2, v_\kappa + \sum_{i=3}^{n} (t_i - \mu(s_i))^2/2) \).

Since \( N(s_2) = z(s_1) \), we take \( \ell_2 = 1 \). To make expressions more compact, we let \( \ell_1 = 0, \rho_0(s_1) = 0, \) and \( z(s_{(1,0)}) = 0 \). The posterior full conditional distribution of \( \sigma^2 \) is \( \text{IG}(\sigma^2 \mid u_{\sigma^2} + n/2, v_{\sigma^2} + \sum_{i=1}^{n} (z(s_i) - \rho_\ell(s_i) z(s_{(i, \ell_i)}))^2/\{2(1 - (\rho_\ell(s_i))^2)\}) \). The posterior full conditional distribution of \( \phi \) is proportional to \( \text{IG}(\phi \mid u_\phi, v_\phi) \prod_{i=2}^{n} N(z(s_i) \mid \rho_\ell(s_i) z(s_{(i, \ell_i)}), \sigma^2_\ell (1 - (\rho_\ell(s_i))^2)) \). We update \( \phi \) on its log scale with a random walk Metropolis step. Denote by \( A_j^{(i)} = \{ j : z(s_{(j, \ell_j)}) = z(s_i) \} \). The posterior full conditional of the latent spatial random effects \( z(s_i) \), for \( i = 1, \ldots, n \), is \( N(z(s_i) \mid \tilde{\sigma}^2 \tilde{\mu}_i, \tilde{\sigma}^2_\ell) \), where \( \tilde{\sigma}^2 = \left( \tau^2 + \sigma^2 (1 - (\rho_\ell(s_i))^2)^{-1} + \sum_{j \in A_j^{(i)}} \tilde{s}_{ij}^{-2} \right)^{-1} \), and \( \tilde{\mu}_i = \tau^2 (y(s_i) - x(s_i) \beta) + \sigma^2 (1 - (\rho_\ell(s_i))^2)^{-1} \rho_\ell(s_i) z(s_{(i, \ell_i)}) + \sum_{j \in A_j^{(i)}} z(s_j) (\rho_\ell(s_j))^{-1} \tilde{s}_{ij}^{-2} \) with \( \tilde{s}_{ij}^2 = \sigma^2 (1 - (\rho_\ell(s_j))^2)/(\rho_\ell(s_j))^2 \).

### C.2 Skew-GNNMP models

#### C.2.1 Bivariate skew-Gaussian distribution

Exploiting the location mixture representation of the skew-Gaussian distribution [Azzalini and Valle, 1996] for bivariate random vector \( Z = (U, V) \), we can write

\[
 f(z \mid z_0) \sim N \left( \begin{pmatrix} \xi_u + \lambda_u z_0 \\ \xi_v + \lambda_v z_0 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right), \quad z_0 \sim N(z_0 \mid 0, 1) I(z_0 \geq 0). \tag{6}
\]

It follows that, conditional on \( Z_0 = z_0 \), the marginal densities of \( Z \) are \( N(u \mid \xi_u + \lambda_u z_0, \sigma^2) \) and \( N(v \mid \xi_v + \lambda_v z_0, \sigma^2) \), respectively. Then the conditional density of \( Z_0 \) given \( V = v \) is

\[
p(z_0 \mid v) \propto N(v \mid \xi_v + \lambda_v z_0, \sigma^2) N(z_0 \mid 0, 1) I(z_0 \geq 0)
\]

\[
\propto N(z_0 \mid (v - \xi_v)/\lambda_v, \sigma^2/\lambda_v^2) N(z_0 \mid 0, 1) I(z_0 \geq 0).
\]
where the marginal density of $Z$ is given by

$$f_{\mathbf{Z}}(z) = 2N(z \mid \mu_0(v), \tau_0^2(v)),$$

where $\tau_0^2(v) = \sigma^2/(\sigma^2 + \lambda_v^2)$, truncated at $[0, \infty)$, denoted as $\text{TN}_0(z_0 \mid \mu_0(v), \tau_0^2(v))$.

Then the conditional distribution of $U$ given $V$ can be written as

$$f_{U \mid V}(u \mid v) = \int_0^{\infty} N(u \mid \mu_u + \rho(v - \mu_v), \sigma^2(1 - \rho^2)) \text{TN}(z_0 \mid \mu_0(v), \tau_0^2(v)) dz_0,$$

where $\mu_u = \xi_u + \lambda_u z_0$, $\mu_v = \xi_v + \lambda_v z_0$.

Let $\xi = (\xi_u, \xi_v)^T$ and $\lambda = (\lambda_u, \lambda_v)$. After marginalizing out $z_0$, the joint density of $Z$ is given by

$$f(z) = 2N(z \mid \xi, \Sigma) \Phi((1 - \lambda^T \Sigma^{-1} \lambda)^{-1/2} \lambda^T \Sigma^{-1}(y - \xi)),$$

where $\Sigma = \sigma^2 \mathbf{R} + \lambda \lambda^T$, $\mathbf{R} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, and $\Phi$ is the cdf of a standard Gaussian distribution.

The marginal density of $U$ is

$$f_U(u) = 2N(u \mid \xi_u, \omega_u^2) \Phi(\alpha_u(u - \xi_u)/\omega_u),$$

where $\omega_u^2 = \lambda_u^2 + \sigma^2$ and $\alpha_u = \lambda_u/\sigma$. We denote $f_U(u)$ as $\text{SN}(u \mid \xi_u, \omega_u^2, \alpha_u)$. Similarly, the marginal density of $V$ is $f_V(v) = \text{SN}(\xi_v, \omega_v^2, \alpha_v)$.

It follows that the conditional density of $U$ given $V = v$ is

$$f_{U \mid V}(u \mid v) = f(z)/f_V(v)$$

$$= N(u \mid \xi_u + \gamma(v - \xi_v), \omega^2) \Phi(\alpha_1(u - \xi_u) + \alpha_2(v - \xi_v))/\Phi(\alpha_v(v - \xi_v)/\omega_v),$$

where $\gamma = (\rho \sigma^2 + \lambda_u \lambda_v)/(\sigma^2 + \lambda_v^2)$, $\omega^2 = \sigma^2 + \lambda_v^2 - (\rho \sigma^2 + \lambda_u \lambda_v)^2/(\sigma^2 + \lambda_v^2)$, $\alpha_1 = (\lambda_u - \rho \lambda_v)/m$, $\alpha_2 = (\lambda_v - \rho \lambda_u)/m$, and $m = \sqrt{(1 - \rho)s^2} \sqrt{(1 - \rho)s^2 + \lambda_u^2 + \lambda_v^2 - 2\rho \lambda_u \lambda_v}$.

In the special case where $\xi_u = \xi_v = 0$ and $\lambda_u = \lambda_v = \lambda$, let $\omega^2 = \lambda^2 + \sigma^2$ and $\alpha = \lambda/\sigma$. The joint density of $\mathbf{Z}$ can be written as

$$f(z) = 2N(z \mid 0, \Sigma) \Phi(\lambda(1 - \lambda^2 1_2^T \Sigma^{-1} 1_2)^{-1/2} 1_2^T \Sigma^{-1} z),$$

where the marginal density of $\mathbf{Z}$ is $\text{SN}(x \mid 0, \omega^2, \alpha)$. The conditional density of $U$ given...
For the weights $w$ with prior specifications for the kernel functions of the random cutoff points. The Bayesian model is completed then extend (4) of the stationary skew-GNNMP is $U$ function such that $\gamma y z m$ marginalize out $b$ where $\lambda = \lambda v = \lambda$. We then extend $(U, V)$ to $(U, V, V, V)$ by extending $\rho$ to $\rho_t(\mathbf{v})$ using an exponential correlation function such that $\rho_t(\mathbf{v}) \equiv \rho(\|\mathbf{v} - \mathbf{v}(l)\|) = \exp(-\|\mathbf{v} - \mathbf{v}(l)\|/\phi)$, for $l = 1, \ldots, L$. Using the resulting bivariate distribution for $(U, V, V, V)$, we define the spatially varying density $f_{v,t} = f_{U,v,t|v,t}$ based on the formulation in (7). The resulting associated conditional density of the stationary skew-GNNMP is

$$p(y(\mathbf{v}) | y_{Ne(\mathbf{v})}) = \sum_{l=1}^{L} w_l(\mathbf{v}) \int_{0}^{\infty} N(y(\mathbf{v}) | \mu_l(\mathbf{v}), \sigma_l^2(\mathbf{v}))TN(z_0(\mathbf{v}) | \mu_0(\mathbf{v}), \sigma_0^2(\mathbf{v}))dz_0(\mathbf{v}), \quad (10)$$

where $\mu_l(\mathbf{v}) = (1 - \rho_t(\mathbf{v}))(\lambda z_0(\mathbf{v}) + \rho_t(\mathbf{v})y(\mathbf{v}(l)))$, $\sigma_l^2(\mathbf{v}) = \sigma^2(1 - (\rho_t(\mathbf{v}))^2)$, $\mu_0(\mathbf{v}) = y(\mathbf{v}(l))\lambda/(\sigma^2 + \lambda^2)$, and $\sigma_0^2 = \sigma^2/(\sigma^2 + \lambda^2)$.

The component conditional density in (10) is sampled via a latent variable $z_0(\mathbf{v})$. We marginalize out $z_0(\mathbf{v})$ to facilitate computation. Based on (9), we obtain the associated conditional density of the skew-GNNMP as

$$p(y(\mathbf{v}) | y_{Ne(\mathbf{v})}) = \sum_{l=1}^{L} w_l(\mathbf{v}) b_l(\mathbf{v}) N(y(\mathbf{v}) | \tilde{\rho}_l(\mathbf{v})y(\mathbf{v}(l)), \omega^2(1 - (\tilde{\rho}_l(\mathbf{v}))^2)), \quad (11)$$

where $b_l(\mathbf{v}) = \Phi(\alpha_l(\mathbf{v})(y(\mathbf{v}) + y(\mathbf{v}(l)))\omega_l(\mathbf{v}))/\Phi(\alpha y(\mathbf{v}(l))\omega_l(\mathbf{v}))$, $\tilde{\rho}_l(\mathbf{v}) = (\rho_t(\mathbf{v})\sigma^2 + \lambda^2)/(\sigma^2 + \lambda^2)$, $\alpha_l(\mathbf{v}) = \lambda/\sqrt{(1 + \rho_t(\mathbf{v}))\sigma^2}$, $\omega_l^2(\mathbf{v}) = (1 + \rho_t(\mathbf{v}))\sigma^2 + 2\lambda^2$, $\alpha = \lambda/\sigma$, and $\omega^2 = \sigma^2 + \lambda^2$.

For the weights $w_l(\mathbf{v})$, we use an exponential correlation function with range parameter $\zeta$ for the kernel functions of the random cutoff points. The Bayesian model is completed with prior specifications for $\lambda, \sigma, \phi, \zeta, \gamma, \kappa$. In particular, we consider priors $N(\lambda | \mu_\lambda, \sigma_\lambda^2)$, $\text{IG}(\sigma^2 | u_{\sigma^2}, v_{\sigma^2})$, $\text{IG}(\phi | u_\phi, v_\phi)$, $\text{IG}(\zeta | u_\zeta, v_\zeta)$, $N(\gamma | \mu_\gamma, V_\gamma)$ and $\text{IG}(\kappa^2 | u_\kappa^2, v_\kappa^2)$.
Given observations \( y(s_i), i = 1, \ldots, n \), over reference set \( S = (s_1, \ldots, s_n) \), we perform Bayesian inference based on likelihood conditional on the first \( L \) observations. The posterior distribution of the model parameters, given the conditional likelihood, is given by

\[
N(\lambda | \mu_\lambda, \sigma_\lambda^2) \times \text{IG}(\sigma^2 | u_\sigma^2, v_\sigma^2) \times \text{IG}(\phi | u_\phi, v_\phi) \times N(\zeta | u_\zeta, v_\zeta) \times N(\gamma | \mu_\gamma, V_\gamma) \times \text{IG}(\kappa^2 | u_\kappa^2, v_\kappa^2) \\
\times N(t | D\gamma, \kappa^2 I_{n-L}) \times \prod_{i=L+1}^n \prod_{l=1}^{L} b_l(s_i)N(y(s_i) | \tilde{\rho}_l(s_i)y(s_{(i)})^\top, \omega^2(1 - (\tilde{\rho}_l(s_i))^2))^1_{(r_{s_i,l-1}^*, r_{s_i,l}^*)}(t_i),
\]

where the vector \( t = (t_{L+1}, \ldots, t_n)^\top \), and the matrix \( D \) is \((n-L) \times 3\) such that the \( i \)th row is \((1, s_{L+i,1}, s_{L+i,2})\).

The MCMC sampler to obtain samples from the joint posterior distribution is described in the main paper. We present the posterior updates of \( \lambda, \sigma^2 \) and \( \phi \). Note that the configuration variables \( \ell_i \) are such that \( \ell_i = l \) if \( t_i \in (r_{s_i,l-1}^*, r_{s_i,l}^*) \) for \( i \geq L + 1 \). Denote by \( f_{s_i,l} = b_l(s_i)N(y(s_i) | \tilde{\rho}_l(s_i)y(s_{(i)})^\top, \omega^2(1 - (\tilde{\rho}_l(s_i))^2)) \). We use a random walk Metropolis step to update \( \lambda \) with target density \( N(\lambda | \mu_\lambda, \sigma_\lambda^2) \prod_{i=L+1}^n f_{s_i,\ell_i} \). The posterior full conditional distributions of \( \sigma^2 \) and \( \phi \) are proportional to \( \text{IG}(\sigma^2 | u_\sigma^2, v_\sigma^2) \prod_{i=L+1}^n f_{s_i,\ell_i} \), and \( \text{IG}(\phi | u_\phi, v_\phi) \prod_{i=L+1}^n f_{s_i,\ell_i} \), respectively. For each parameter, we update it on its log scale with a random walk Metropolis step.

### C.2.3 Extended skew-GNNMP models

Again, we take a set of base random vectors \((U_l, V_l) \equiv (U, V)\) for all \( l \), where \((U, V)\) is a bivariate skew-Gaussian vector with distribution given by [7]. We extend \((U_l, V_l)\) to \((U_{v,l}, V_{v,l})\) by extending \( \rho_l(v) \) using an exponential correlation function such that \( \rho_l(v) \equiv \rho(||v - v_{(l)}||) = \exp(-||v - v_{(l)}||/\phi) \), and extending \( \xi_u = x(v)^\top \beta, \xi_v = x(v_{(l)})^\top \beta, \lambda_u \) to \( \lambda(v) \), and \( \lambda_v \) to \( \lambda(v_{(l)}) \), for \( l = 1, \ldots, L \). Using the resulting bivariate distribution for \((U_{v,l}, V_{v,l})\), we define the spatially varying density \( f_{v,l} = f_{U_{v,l}}|V_{v,l} \) based on the formulation in [7]. The resulting associated conditional density of the extended skew-GNNMP is

\[
p(y(v) | y_{N(v)}) = \sum_{l=1}^L w_l(v) \int_0^\infty N(y(v) | \mu_t(v), \sigma_t^2(v))TN(z_0(v) | \mu_{0l}(v_{(l)}), \sigma_{0l}^2(v_{(l)}))dz_0(v),
\]

(12)

where \( \mu_t(v) = x(v)^\top \beta + \lambda(v)z_0(v) + \rho_l(v)(y(v_{(l)}) - x(v_{(l)})^\top \beta - \lambda(v_{(l)})z_0(v)) \), \( \sigma_t^2(v) = \sigma^2(1 - (\rho_l(v))^2) \), \( \mu_{0l}(v_{(l)}) = (y(v_{(l)}) - x(v_{(l)})^\top \beta)\lambda(v_{(l)})/(\sigma^2 + (\lambda(v_{(l)})^2) \), and \( \sigma_{0l}^2(v_{(l)}) = \)
\[ \sigma^2 / (\sigma^2 + (\lambda(\mathbf{v}(l)))^2). \]

After marginalizing out \( z_0(\mathbf{v}) \), the conditional density \([12]\) based on formulation \([8]\) can be written as

\[
p(y(\mathbf{v}) \mid y_{Ne(\mathbf{v})}) = \sum_{l=1}^{L} w_l(\mathbf{v}) \tilde{b}_l(\mathbf{v}) N(y(\mathbf{v}) \mid \tilde{\mu}_l(\mathbf{v}), \tilde{\omega}_l^2(\mathbf{v})),
\]

where \( \tilde{\mu}_l(\mathbf{v}) = x(\mathbf{v})^\top \beta + \hat{\gamma}_l(\mathbf{v})(y(\mathbf{v}(l)) - x(\mathbf{v}(l))^\top \beta), \hat{\gamma}_l(\mathbf{v}) = (\rho_l(\mathbf{v})\sigma^2 + \lambda(\mathbf{v})\lambda(\mathbf{v}(l))) / \omega^2(\mathbf{v}(l)), \)
\( \tilde{\omega}_l^2(\mathbf{v}) = \omega(\mathbf{v})^2 - (\rho_l(\mathbf{v})\sigma^2 + \lambda(\mathbf{v})\lambda(\mathbf{v}(l)))^2 / \omega^2(\mathbf{v}(l)), s_l^2(\mathbf{v}) = (1 + \rho_l(\mathbf{v}))\sigma^2, \alpha(\mathbf{v}) = \lambda(\mathbf{v}) / \sigma, (\omega(\mathbf{v}))^2 = \lambda(\mathbf{v})^2 + \sigma^2, \) and

\[
\tilde{b}_l(\mathbf{v}) = \frac{\Phi(\alpha_1(\mathbf{v}, \mathbf{v}(l))(y(\mathbf{v}) - x(\mathbf{v}))^\top \beta + \alpha_2(\mathbf{v}, \mathbf{v}(l))(y(\mathbf{v}(l)) - x(\mathbf{v}(l))^\top \beta))}{\Phi(\alpha(\mathbf{v}(l))(y(\mathbf{v}(l)) - x(\mathbf{v}(l))^\top \beta) / \omega(\mathbf{v}(l)))},
\]
\( \alpha_1(\mathbf{v}, \mathbf{v}(l)) = (\lambda(\mathbf{v}) - \rho_l(\mathbf{v})\lambda(\mathbf{v}(l))) / \omega(\mathbf{v}(l)), \alpha_2(\mathbf{v}, \mathbf{v}(l)) = (\lambda(\mathbf{v}(l)) - \rho_l(\mathbf{v})\lambda(\mathbf{v})) / \omega(\mathbf{v}(l)), \)
\( m(\mathbf{v}) = \sqrt{(1 - \rho_l(\mathbf{v}))s_1^2(\mathbf{v})\sqrt{(1 - \rho_l(\mathbf{v}))s_1^2(\mathbf{v}) + (\lambda(\mathbf{v}))^2 + (\lambda(\mathbf{v}(l)))^2 - 2\rho_l(\mathbf{v})\lambda(\mathbf{v})\lambda(\mathbf{v}(l))}. \)

We model the spatially varying \( \lambda(\mathbf{v}) \) via a partitioning approach. In particular, we partition the domain \( D \) such that \( D = \bigcup_{k=1}^{K} P_k, P_i \cap P_j = \emptyset \) for \( i \neq j \). For all \( \mathbf{v} \in P_k \), we take \( \lambda(\mathbf{v}) = \lambda_k \), for \( k = 1, \ldots, K \). For the weights \( w_l(\mathbf{v}) \), we use an exponential correlation function with range parameter \( \zeta \) for the kernel function of the random cutoff points. The Bayesian model is completed with prior specifications for \( \beta, \lambda = (\lambda_1, \ldots, \lambda_K), \sigma^2, \phi, \zeta, \gamma, \kappa^2 \).

We assign a \( N(\beta, \mu_\beta, V_\beta) \) to the regression parameter \( \beta \) and \( N(\lambda | \mu_{\lambda k}, \sigma^2_{\lambda k}) \) to \( \lambda_k \), for \( k = 1, \ldots, K \). For other parameters, we take \( IG(\sigma^2 | u_{\sigma^2}, v_{\sigma^2}) \), \( IG(\phi | u_\phi, v_\phi) \), \( IG(\zeta | u_\zeta, v_\zeta) \), \( N(\gamma | \mu_\gamma, V_\gamma) \) and \( IG(\kappa^2 | u_\kappa, v_\kappa) \).

Given observations \( y(s_i), i = 1, \ldots, n \), over reference set \( S = (s_1, \ldots, s_n) \), we perform Bayesian inference based a likelihood conditional on the first \( L \) observations. The posterior distribution of the model parameters, given the conditional likelihood, is given by

\[
N(\beta \mid \mu_\beta, V_\beta) \times \prod_{k=1}^{K} N(\lambda_k \mid \mu_{\lambda k}, \sigma^2_{\lambda k}) \times IG(\sigma^2 \mid u_{\sigma^2}, v_{\sigma^2}) \times IG(\phi \mid u_\phi, v_\phi) \times IG(\zeta \mid u_\zeta, v_\zeta) \times N(\gamma \mid \mu_\gamma, V_\gamma) \times IG(\kappa^2 \mid u_\kappa, v_\kappa) \times N(t \mid D_\gamma, \kappa^2 I_{n-L}) \\
\times \prod_{i=L+1}^{n} \sum_{l=1}^{L} \tilde{b}_l(s_i)N(y(s_i) \mid \tilde{\mu}_l(s_i), \tilde{\omega}_l^2(s_i))I_{(r_{s_i, l-1}^\ast, r_{s_i, l}^\ast)}(t_i),
\]

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where the vector \( \mathbf{t} = (t_{L+1}, \ldots, t_n)^\top \), and the matrix \( \mathbf{D} \) is \((n - L) \times 3\) such that the \( i \)th row is \((1, s_{L+i,1}, s_{L+i,2})\).

The MCMC sampler to obtain samples from the joint posterior distribution is described in the main paper. We present the posterior updates of \( \mathbf{\beta}, \lambda, \sigma^2 \) and \( \phi \). Note that the configuration variables \( \ell_i \) are such that \( \ell_i = l \) if \( t_i \in (r_{s_i,l-1}^*, r_{s_i,l}^*) \) for \( i \geq L + 1 \). Denote by \( f_{s_i,l} = \tilde{b}_l(s_i)N(y(s_i) | \tilde{\mu}_l(s_i), \tilde{\sigma}_l^2(s_i)) \). We use a random walk Metropolis step to update \( \mathbf{\beta} \) with target density \( N(\mathbf{\beta} | \mathbf{\mu}_B, \mathbf{V}_B) \prod_{i=1}^n f_{s_i, \ell_i} \). Let \( B_k = \{ i : s_i \in P_k \} \cup \{ i : s_i(\ell_i) \in P_k \} \). The posterior full conditional of \( \lambda_k \) is proportional to \( N(\lambda_k | \mu_{\lambda k}, \sigma_{\lambda k}^2) \prod_{i \in B_k} f_{s_i, \ell_i} \), and we use a random walk Metropolis step to sample \( \lambda_k \) for \( k = 1, \ldots, K \). The posterior full conditional distributions of \( \sigma^2 \) and \( \phi \) are proportional to \( IG(\sigma^2 | u_{\sigma^2}, v_{\sigma^2}) \prod_{i=1}^n f_{s_i, \ell_i} \), and \( IG(\phi | u_{\phi}, v_{\phi}) \prod_{i=1}^n f_{s_i, \ell_i} \), respectively. For each parameter, we update it on its log scale with a random walk Metropolis step.

### C.3 Copula NNMP models

#### C.3.1 The Gaussian and Gumbel copula

We consider a continuous bivariate vector \((X_1, X_2)\) with marginal cdfs \( F_1 \) and \( F_2 \) such that \( F_1(x_1) = t_1 \) and \( F_2(x_2) = t_2 \). We introduce basic properties of the Gaussian and Gumbel copulas. For more details we refer to [Joe] (2014).

**Gaussian copula** A Gaussian copula with correlation \( \rho \in (0, 1) \) for \((X_1, X_2)\) is

\[
C(F_1(x_1), F_2(x_2)) = C(t_1, t_2 \mid \rho) = \Phi(\Phi^{-1}(t_1) + \Phi^{-1}(t_2)).
\]

The copula is asymptotically independent in both the lower and the upper tails. The corresponding copula density is given by:

\[
\frac{1}{\sqrt{1 - \rho^2}} \exp \left( \frac{2\rho \Phi^{-1}(t_1)\Phi^{-1}(t_2) - \rho^2 \{(\Phi^{-1}(t_1))^2 + (\Phi^{-1}(t_2))^2\}}{2(1 - \rho^2)} \right).
\]

Denote by \( C_{1|2}(t_1 \mid t_2) \) the conditional cdf of \( T_1 \) given \( T_2 = t_2 \). Then we have

\[
C_{1|2}(t_1 \mid t_2) = \frac{\partial C(t_1, t_2)}{\partial t_2} = \Phi \left( \frac{\Phi^{-1}(t_1) - \rho \Phi^{-1}(t_2)}{\sqrt{1 - \rho^2}} \right).
\]

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We sample $X_1$, given $X_2 = x_2$, with the following steps. Given a realization $x_2$ of $X_2$, we compute $t_2 = F_2(x_2)$. We then generate a random number $z$ from a uniform distribution on $[0, 1]$, and compute $t_1 = C^{-1}_{12}(z | t_2)$ where $C^{-1}_{12}(z | t_2) = \Phi \left( \sqrt{(1 - \rho^2)} \Phi^{-1}(z) + \rho \Phi^{-1}(t_2) \right)$ is the inverse of $C_{12}(t_1 | t_2)$. Finally, we obtain $x_1$ from the inverse cdf $F^{-1}_1(t_1)$.

**Gumbel copula** A Gumbel copula with parameter $\eta \in [1, \infty)$ is

$$ C(F_1(x_1), F_2(x_2)) = C(t_1, t_2 | \eta) = \exp(-[(-\log(t_1))^\eta + (-\log(t_2))^\eta]^{1/\eta}). $$

It is asymptotically independent in the lower tail and asymptotically dependent in the upper tail with tail dependence coefficient $2 - 2^{1/\eta}$. The corresponding copula density is

$$ \exp(-(u_1^\eta + u_2^\eta)^{1/\eta})((u_1^\eta + u_2^\eta)^{1/\eta} + \eta - 1)(u_1^\eta + u_2^\eta)^{1/\eta-2}(u_1u_2)^{\eta-1}(t_1t_2)^{-1}. \quad (15) $$

Let $u_1 = -\log(t_1)$ and $u_2 = -\log(t_2)$. The Gumbel copula can be written as $\overline{C}(u_1, u_2 | \eta) = \exp(-(u_1^\eta + u_2^\eta)^{1/\eta})$, which is a bivariate exponential survival function, with marginals corresponding to a unit rate exponential distribution. Then the conditional cdf of $T_1$ given $T_2 = t_2$ is

$$ C_{1|2}(t_1 | t_2) = \overline{C}_{1|2}(u_1 | u_2) = u_2^{-1}\exp(-(u_1^\eta + u_2^\eta)^{1/\eta})(1 + (u_1/u_2)^\eta)^{1/\eta-1}. $$

The inverse conditional cdf $C^{-1}_{1|2}(\cdot | t_2)$ does not have a closed form. To generate $X_1$ given $X_2 = x_2$, following [Joe (2014)], we first define $y = (u_1^\eta + u_2^\eta)^{1/\eta}$. Then we have a realization of $X_1$, say $x_1 = (y_0 - u_2^\eta)^{1/\eta}$, where $y_0$ is the root of $h(y) = y + (\eta - 1)\log(y) - (u_2 + (\eta - 1)\log(u_2) - \log z) = 0$, where $y \geq u_2$, and $z$ is a random number generated from a uniform distribution on $[0, 1]$.

**C.3.2 Copula NNMP models and inference**

We take a set of base random vectors $(U_l, V_l) \equiv (U, V)$ where its bivariate distribution is specified by a Gaussian copula with correlation parameter $\rho$. We extend $(U_l, V_l)$ to $(U_{v,l}, V_{v,l})$ by extending $\rho$ to $\rho_l(v) \equiv \rho_0(||v - v(l)||) = \exp(-||v - v(l)||/\phi)$, creating a sequence of spatially varying Gaussian copula $C_{v,l}$ for $(U_{v,l}, V_{v,l})$ with marginal cdfs $F_{U_{v,l}} = F_{V_{v,l}} = F_Y$ for all $v$ and all $l$. The cdf $F_Y$ corresponds to the stationary marginal distribution
of the model. The associated conditional density of the Gaussian copula NNMP is

\[ p(y(v) \mid v_{Ne(v)}) = \sum_{l=1}^{L} w_l(v) c_{v,l}(y(v), y(v(l))) f_Y(y(v)), \]  

(16)

where \( c_{v,l}(y(v), y(v(l))) \) is the Gaussian copula density obtained by replacing \( \rho \) in (14) with \( \rho_l(v) \), and \( f_Y \) is the density of \( F_Y \).

Similarly, we can obtain the Gumbel copula NNMP model using a collection of spatially varying Gumbel copulas by extending \( \eta \) in (15) to \( \eta_l(v) = \eta_l(||v - v(l)||) = \min\{(1 - \exp(-||v - v(l)||/\phi))^{-1}, 50\} \), where the upper bound 50 ensures numerical stability.

We discuss the inferential approach for the Gaussian copula NNMP; the approach for the Gumbel copula NNMP model is similar. Assume the stationary marginal density is a gamma density, denoted as \( f_Y = \text{Ga}(a, b) \), with mean \( E(Y) = a/b \). For the weights \( w_l(v) \), we use an exponential correlation function with range parameter \( \zeta \) to define the random cutoff points. The Bayesian model is completed with prior specifications for \( a, b, \phi, \zeta, \gamma, \kappa^2 \). In particular, we consider priors \( \text{Ga}(u_a, v_a) \), \( \text{Ga}(u_b, v_b) \), \( \text{IG}(\phi \mid u_\phi, v_\phi) \), \( \text{IG}(\zeta \mid u_\zeta, v_\zeta) \), \( \text{N}(\gamma \mid \mu_\gamma, V_\gamma) \) and \( \text{IG}(\kappa^2 \mid u_{\kappa^2}, v_{\kappa^2}) \).

Given observations \( y(s_i), i = 1, \ldots, n \), over reference set \( S = (s_1, \ldots, s_n) \), we perform Bayesian inference using a likelihood conditional on \( (y(s_1), \ldots, y(s_L)) \). The posterior distribution of the model parameters, given the conditional likelihood, is given by

\[ \text{Ga}(u_a, v_a) \times \text{Ga}(u_b, v_b) \times \text{IG}(\phi \mid u_\phi, v_\phi) \times \text{IG}(\zeta \mid u_\zeta, v_\zeta) \times \text{N}(\gamma \mid \mu_\gamma, V_\gamma) \times \text{IG}(\kappa^2 \mid u_{\kappa^2}, v_{\kappa^2}) \]
\[ \times N(t \mid D\gamma, \kappa^2 I_{n-L}) \times \prod_{i=L+1}^{n} \sum_{l=1}^{L} c_{s_i,l}(y(s_i), y(s(i))) f_Y(y(s_i)) I_{(r_{s,i,l-1}, r_{s,i,l})}(t_i), \]

where the vector \( t = (t_{L+1}, \ldots, t_n)^T \), and the matrix \( D \) is \( (n - L) \times 3 \) such that the \( i \)th row is \( (1, s_{L+i,1}, s_{L+i,2}) \).

We provide the updates for parameters \( (a, b, \phi) \). Note that the configuration variables \( \ell_i \) are such that \( \ell_i = l \) if \( t_i \in (r_{s,i,l-1}, r_{s,i,l}) \) for \( i \geq L + 1 \). Denote by \( f_{s_i,l} = c_{s_i,l}(y(s_i), y(s(i))) f_Y(y(s_i)) \). The posterior full conditional distributions for parameters \( a, b \) and \( \phi \) are proportional to \( \text{IG}(a \mid u_a, v_a) \prod_{i=L+1}^{n} f_{s_i,\ell_i}, \text{IG}(b \mid u_b, v_b) \prod_{i=L+1}^{n} f_{s_i,\ell_i}, \) and \( \text{IG}(\phi \mid u_\phi, v_\phi) \prod_{i=L+1}^{n} c_{s_i,l}(y(s_i), y(s(i))), \) respectively. Each parameter is updated on its log scale with a random walk Metropolis step.
Additional References

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