Algebraic equations for the exceptional eigenspectrum of the generalized Rabi model

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Received 10 August 2015, revised 15 September 2015
Accepted for publication 18 September 2015
Published 20 October 2015

Abstract
We obtain the exceptional part of the eigenspectrum of the generalized Rabi model, also known as the driven Rabi model, in terms of the roots of a set of algebraic equations. This approach provides a product form for the wavefunction components and allows an explicit connection with recent results obtained for the wavefunction in terms of truncated confluent Heun functions. Other approaches are also compared. For particular parameter values the exceptional part of the eigenspectrum consists of doubly degenerate crossing points. We give a proof for the number of roots of the constraint polynomials and discuss the number of crossing points.

Keywords: Rabi model, driven Rabi model, level crossings

(Some figures may appear in colour only in the online journal)

1. Introduction

Despite it’s simplicity, the generalized Rabi model has been solved only recently [1–5]. The Rabi model [6] describes the simplest matter-field interaction, namely between a two-level atom and a single-mode bosonic field. It is thus a fundamental textbook model in quantum optics [7]. The generalized Rabi model has Hamiltonian
where $\sigma_x$ and $\sigma_z$ are Pauli matrices for a two-level system with level splitting $\Delta$. The single-mode bosonic field is described by the creation and destruction operators $a^\dagger$ and $a$ with $[a, a^\dagger] = 1$ and frequency $\omega$. The interaction between the two systems is via the coupling $g$. The Rabi model has $Z_2$ symmetry (parity) which is broken by the addition of the term $\epsilon \sigma_z$ in the generalized version of the model. This additional term allows tunnelling between the two atomic states. The generalized Rabi model (1) is also referred to as the driven Rabi model [8] and is relevant to the description of various hybrid mechanical systems [4, 9]. Although having an analytic solution, both the Rabi and generalized Rabi models do not appear to be integrable in general in the Yang–Baxter sense [10]. However, the existence of monodromy matrices in terms of Painlevé V has now been reported [11].

The generalized Rabi model (1) has been solved in two ways: (i) by mapping the problem to the Bargmann space of analytic functions [1, 3], and (ii) by using the Bogoliubov operator method [2]. Using the former approach explicit expressions have been obtained [4, 5] for the wavefunction in terms of confluent Heun functions [12] \(^4\). Of particular relevance here is the fact that the energy spectrum of the generalized Rabi model, although possessing no parity symmetry, still includes both regular and exceptional parts. The eigenspectrum can be determined from the analytical solution. The exceptional parts, known as Juddian isolated exact solutions [16], can be systematically found from the conditions under which the confluent Heun functions are terminated as finite polynomials [4]. Our interest here is with this exceptional part of the eigenspectrum, which we obtain using a different approach.

Indeed the exceptional part of the Rabi model eigenspectrum has been obtained using a number of different (though ultimately related) approaches\(^5\). Each approach results in the simple eigenvalue expression of a shifted oscillator, however with the system parameters satisfying a constraint which becomes increasingly complicated for higher energy levels. Of most relevance here is an approach which derives a set of Bethe-like algebraic equations whose solutions define the constraint among the system parameters [21, 23]. We apply this approach in section 2 to obtain the exceptional part of the eigenspectrum of the generalized Rabi model (1), allowing an explicit connection with the results obtained for the wavefunction in terms of truncated confluent Heun functions [4]. The approach used here provides a simple product form for the wavefunction components in terms of the algebraic roots. We conclude by discussing the relation between the various approaches and the degenerate crossing points in section 3.

2. Results

In the Bargmann realization [25]

$$a^\dagger \rightarrow z, \quad a \rightarrow \frac{d}{dz}$$

(2)

the Hamiltonian (1) reads

$$H = \omega z \frac{d}{dz} + g \sigma_x \left( z + \frac{d}{dz} \right) + \Delta \sigma_z + \epsilon \sigma_x.$$  

(3)

\(^4\) The connection with confluent Heun functions was made earlier for the Rabi model [13–15].

\(^5\) See, e.g., [1, 17–24]. A different generalized Rabi model is considered in [23] with both rotating and counter-rotating terms, i.e., interpolating between the Jaynes–Cummings and Rabi models.
Following, e.g., [4], in terms of the two-component wavefunction
\[
\psi'(z) = \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix}
\] (4)
the Schrödinger equation \( H\psi = E\psi \) gives rise to a pair of coupled equations for \( \psi_+(z) \) and \( \psi_-(z) \), namely
\[
(\omega z + g) \frac{d\psi_+}{dz} + (gz + \epsilon - E) \psi_+ + \Delta \psi_+ = 0,
\]
\[
(\omega z - g) \frac{d\psi_-}{dz} - (gz + \epsilon + E) \psi_- + \Delta \psi_- = 0.
\]
(5)
(6)
Two sets of solutions for \( \psi_+(z) \) and \( \psi_-(z) \) can be obtained. For the first set, the substitution \( \psi_+(z) = e^{-\xi / \omega} \phi^1_{+}(z) \) leads to the coupled equations
\[
\left[ (\omega z + g) \frac{d}{dz} - \left( \frac{g^2}{\omega} + E - \epsilon \right) \right] \phi^1_{+}(z) = -\Delta \phi^1_{+}(z),
\]
\[
\left[ (\omega z - g) \frac{d}{dz} - \left( 2gz - \frac{g^2}{\omega} + E + \epsilon \right) \right] \phi^1_{-}(z) = -\Delta \phi^1_{+}(z).
\]
(7)
(8)
Eliminating \( \phi^1_{+}(z) \) gives the second order differential equation
\[
(\omega z - g)(\omega z + g) \frac{d^2\phi^1_{+}(z)}{dz^2}
+ \left[ -2g\omega z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2 - 2E\omega) \right] \frac{d\phi^1_{+}(z)}{dz}
+ \left[ 2g \left( \frac{g^2}{\omega} + E - \epsilon \right)z + E^2 - \Delta^2 - \epsilon^2 + \frac{2g^2}{\omega^2} - \frac{g^4}{\omega^2} \right] \phi^1_{+}(z) = 0.
\]
(9)
This is a case of the general second order differential equation considered by Zhang [26]. Applying the result of theorem 1.1 therein [26] gives the wavefunction component in the factorized form
\[
\psi_+(z) = e^{-\xi / \omega} \prod_{i=1}^{N} (z - z_i),
\]
(10)
where the roots \( z_i \) satisfy the set of algebraic equations (details are given in appendix A)
\[
\sum_{j \neq i}^{N} \frac{2\omega}{z_i - z_j} = \frac{2\omega^2 g^2}{2} + \left( 2N\omega - \omega^2 + 2E \right)\omega z_i^2 + \omega^2 g + 2\omega g - 2g^3
= \frac{N\omega^2 + 2E\omega}{\omega z_i^2 - g} + \frac{N\omega^2 - \omega^2}{\omega z_i + g} + 2g
\]
(11)
for \( i = 1, \ldots, N \). The system parameters obey the constraint
\[
\Delta^2 + 2Ng^2 + 2\omega g \sum_{i=1}^{N} z_i = 0.
\]
(12)
The energy of these states is given by

$$E = N \omega - \frac{g^2}{\omega} + \epsilon.$$  \hfill (13)

The corresponding wavefunction component $\psi^2(z)$ can be determined using the result (10) and equation (7). For $\epsilon = 0$ the algebraic equations (11) reduce to those obtained by the same approach [21]. The energy expression (13) has been given in [4], where it follows as a condition for the general solution given in terms of the confluent Heun functions to truncate to a polynomial with $N$ terms.

Another set of solutions follow from the substitution $\psi^2(z) = e^{i\epsilon}/\omega \phi^2(z)$, leading to the coupled equations

$$\left[ (\omega z + g) \frac{d}{dz} + \left( 2gz + \frac{g^2}{\omega} - E + \epsilon \right) \right] \phi^2_+(z) = -\Delta \phi^2_+(z),$$  \hfill (14)

$$\left[ (\omega z - g) \frac{d}{dz} - \left( \frac{g^2}{\omega} + E + \epsilon \right) \right] \phi^2_-(z) = -\Delta \phi^2_-(z).$$  \hfill (15)

Proceeding as above, these equations can be solved for the wavefunction components in the form

$$\psi^2(z) = e^{i\epsilon/\omega} \prod_{i=1}^{N} -\left(z - z_i\right)$$  \hfill (16)

where the roots $z_i$ satisfy the algebraic equations

$$\sum_{j \neq i}^{N} \frac{2\omega}{z_i - z_j} = -\frac{2\omega^2g^2}{\omega z_i - g} + \left( 2N\omega - \omega^2 \right) g^2 - 2g^3$$

$$= \frac{N\omega^2 - \omega^2}{\omega z_i - g} + \frac{N\omega - 2\omega g}{\omega z_i + g} - 2g$$  \hfill (17)

for $i = 1, \ldots, N$. The system parameters now obey the constraint

$$\Delta^2 + 2Ng^2 - 2\omega g \sum_{i=1}^{N} z_i = 0,$$  \hfill (18)

with energy

$$E = N \omega - \frac{g^2}{\omega} - \epsilon.$$  \hfill (19)

The corresponding wavefunction component $\psi^2_+(z) = e^{i\epsilon/\omega} \phi^2_+(z)$ follows from the result (16) and equation (15). The energy expression (19) has also been given in [4], again following from the condition for truncation of the general solution given in terms of the confluent Heun functions. This other set of solutions was not considered for $\epsilon = 0$ [21]. The resemblance of algebraic equations of this type with Richardson BCS equations of Gaudin type has been noted [23].

There is clearly a symmetry between the two sets of solutions. Namely the algebraic equations (11) and (17) are equivalent under the transformation $z_i \leftrightarrow -z_i$, $\epsilon \leftrightarrow -\epsilon$. This corresponds to the related symmetry $\psi^2_+(z, \epsilon) = \psi^2_-(z, -\epsilon)$, $\psi^2_-(z, \epsilon) = \psi^2_+(z, -\epsilon)$ in the wavefunction components. This symmetry is further discussed in [4] and is well known in
the \( \epsilon = 0 \) case (see, e.g., [19]). The \( - \) sign has been inserted into equation (16) to ensure this symmetry.

### 2.1. Examples

We now turn to some specific examples. First consider \( N = 1 \). The energy is

\[
E = \omega - \frac{g^2}{\omega} + \epsilon
\]

and the algebraic equations (11) reduce to

\[
2\omega^2 g z_1^2 + (\omega + 2\epsilon)\omega^2 z_1 + \omega^2 g + 2\epsilon \omega g - 2g^3 = 0.
\]

The two solutions are

\[
z_1 = -\frac{g}{\omega}, \quad \frac{2g^2 - \omega^2 - 2\epsilon \omega}{2\omega g}.\]

Substitution into the constraint (12) gives \( \Delta^2 = 0 \) and

\[
\Delta^2 + 4g^2 = \omega^2 + 2\epsilon \omega,
\]

respectively. The value \( \Delta^2 = 0 \) obtained from the first solution corresponds to the degenerate atomic limit, which we discuss further below. The second solution in (22) gives the wavefunction components

\[
\psi_+^1(z) = e^{-gz/\omega} \left( \frac{2\omega gz + \omega^2 + 2\epsilon \omega - 2g^2}{2\omega g} \right),
\]

\[
\psi_-^1(z) = e^{-gz/\omega} \frac{\Delta}{2g},
\]

where in the last equation, we made use of the simplifying constraint (23).

On the other hand, equation (17) becomes

\[
-2\omega^2 g z_1^2 + (\omega - 2\epsilon)\omega^2 z_1 - \omega^2 g + 2\epsilon \omega g + 2g^3 = 0.
\]

The two solutions are

\[
z_1 = \frac{g}{\omega}, \quad \frac{-2g^2 + \omega^2 - 2\epsilon \omega}{2\omega g}.\]

Substitution into the constraint (18) gives \( \Delta^2 = 0 \) and \( \Delta^2 + 4g^2 = \omega^2 - 2\epsilon \omega \), respectively. The relevant energy and wavefunction components are

\[
E = \omega - \frac{g^2}{\omega} - \epsilon,
\]

\[
\psi_+^2(z) = e^{gz/\omega} \frac{\Delta}{2g},
\]

\[
\psi_-^2(z) = e^{gz/\omega} \left( \frac{-2\omega gz + \omega^2 - 2\epsilon \omega - 2g^2}{2\omega g} \right).
\]

The results for \( N = 1 \) agree with those obtained from the truncation of the confluent Heun functions [4], within a harmless renormalization of the wavefunction components.
As a further check, consider $N = 2$ for which equation (11) are seen to give six sets of solutions. For the first set, $z_1 = z_2 = -g/\omega$, the constraint relation (12) gives $\Delta^2 = 0$. The solution $z_1 = z_2 = g/\omega$ gives the unphysical constraint $\Delta^2 + 8g^2 = 0$. As in dealing with Bethe Ansatz equations, the equations need to be solved numerically for finite sizes. For the simplest case $\epsilon = 0$, the solutions with distinct roots have

\[
z_1 + z_2 = \frac{-5 + 4\tilde{g}^2 \pm \sqrt{9 + 8\tilde{g}^2 + 16\tilde{g}^4}}{4\tilde{g}},
\]

with $\tilde{g} = g/\omega$. Substitution into the constraint relation (12) and squaring gives the known result, namely $\Delta^4 + 12\Delta^2g^2 - 5\Delta^2\omega^2 + 32g^4 - 32\omega^2g^2 + 4\omega^4 = 0$. Equations (17) and (18) give the same constraint. The explicit wavefunction components for $N = 2$ and $\epsilon = 0$ are

\[
\psi_1^a(z) = e^{-g\Delta/\omega}(z^2 + a_1z + a_2^\pm),
\]

\[
\psi_1^b(z) = e^{-g\Delta/\omega}(b_1z + b_2^\pm),
\]

\[
\psi_2^a(z) = e^{g\Delta/\omega}(-b_1z + b_2^{-\epsilon}),
\]

\[
\psi_2^b(z) = e^{g\Delta/\omega}(z^2 - a_1z + a_2^{-\epsilon}),
\]

where

\[
a_1 = \frac{\Delta^2 + 4g^2}{2g\omega}, \quad a_2^\pm = \frac{\Delta^4 + 8\Delta^2g^2 + 8g^4 - \Delta^2\omega^2 - 2\Delta^2\epsilon \omega}{8g^2\omega^2},
\]

\[
b_1 = \frac{\Delta}{2g}, \quad b_2^\pm = \frac{\Delta^4 + 6\Delta^2g^2 - \Delta^2\omega^2 - 2\Delta^2\epsilon \omega}{4\Delta g^2\omega^2}.
\]

2.2. Degenerate atomic limit

Some comments can be made about the degenerate atomic limit $\Delta = 0$ for general $N$. The degenerate solutions $z_i = -g/\omega$, for $i = 1, \ldots, N$ satisfy the algebraic equations (11), with $\Delta^2 = 0$ following from the constraint relation (12). The energy is given by (13) with (10) giving the wavefunction component

\[
\psi_1^a(z) = e^{-g\Delta/\omega}\left(z + \frac{g}{\omega}\right)^N.
\]

This is precisely the solution obtained for the equivalent displaced harmonic oscillator in the Bargmann space [25]. The related solution [18] similarly follows from equations (16)–(19).

3. Discussion

It is interesting to compare the various approaches for deriving the exceptional part of the eigenspectrum. We have derived a set of algebraic equations (11) for the exceptional part of the eigenspectrum of the generalized Rabi model (1) using a method [21, 26] akin to the functional or analytic Bethe Ansatz. Although the energies have a simple form (13), the constraint relations (12) and wavefunction components (10) are given in terms of the Bethe-like roots $z_i$. The constraint relations can be generated by a number of methods. It is known, for example, that the coefficients of the wavefunction components satisfy a system of $2N + 1$ linear equations, with the constraint emerging as a condition for the determinant to vanish.
One can also determine a recurrence relation leading to the constraint relations (see, e.g., [20]).

For the generalized Rabi model considered here, in terms of the series expansion coefficients \( h_n \) for the confluent Heun function \( \sum h_n x^n \), where \( x = \frac{z - z_+}{2g} \), the recurrence relation is

\[
A_n h_n = B_n h_{n-1} + C_n h_{n-2},
\]

with initial conditions \( h_{-1} = 0 \), \( h_0 = 1 \). The coefficients are given by

\[
A_n = n(-1 + n - N - 2\epsilon/\omega),
\]

\[
B_n = (1 - n + N)^2 - 4(n - 1)g^2/\omega^2 - \Delta^2/\omega^2 + 2(1 - n + N)\epsilon/\omega,
\]

\[
C_n = 4(-2 + n - N)g^2/\omega^2.
\]

This result follows from [4] specified to the exceptional points\(^6\). Indeed, the general three-term recurrence relation is central to the analytic solution of the generalized Rabi model [1–5]. The approach taken here effectively gives the factorization of the truncated confluent Heun functions at the exceptional points.

The recurrence relation (39), and in particular the condition \( C_{N+2} = 0 \), ensures that the infinite series expansion for the confluent Heun function terminates with \( h_n = 0 \) for \( n > N \). The value \( h_{N+1} = 0 \) determines the constraint relation for given \( N \). The first few polynomials obtained in this way are

\[
\Delta^2 = 0,
\]

\[
\Delta^2(\Delta^2 + 4g^2 - \omega^2 - 2\epsilon\omega) = 0,
\]

\[
\Delta^2(\Delta^2 + \Delta^2(12g^2 - 5\omega^2 - 6\epsilon\omega) + 32g^4 - 32\epsilon g^2\omega + 8\epsilon^2\omega^2
\]

\[ - 32\omega^2 g^2 + 12\epsilon \omega^3 + 4\omega^4) = 0,
\]

\[
\Delta^2(\Delta^2 + 2\Delta^2(12g^2 - 7\omega^2 - 6\epsilon\omega) + \Delta^2(49\omega^4 + 44\epsilon^2\omega^2 - 232\omega^2 g^2
\]

\[ + 176g^4 + 16\epsilon \omega(6\omega^2 - 11g^2)) - 12(-32g^6 + 24\omega g^4(2\epsilon + 3\omega)
\]

\[ - 12\epsilon^2 g^2(2\epsilon^2 + 5\epsilon \omega + 3\omega^2) + 4\epsilon \omega^3
\]

\[ + 12\epsilon^2 \omega^4 + 11\epsilon \omega^5 + 3\omega^6) = 0,
\]

for \( N = 0, 1, 2, 3 \), respectively. The \( N = 1 \) result (44) is as given in (23), with (45) the example given in [4]. The constraint polynomials for given \( N \) are generated readily enough via the recurrence relation. A similar recurrence relation can be written down corresponding to the solutions \( \psi_{\pm}^i(z) = e^{\epsilon/\omega} \phi_{\pm}^i(z) \). This results in the same constraint polynomials as given in the above examples, however with \( \epsilon \leftrightarrow -\epsilon \). In contrast the approach used here gives the closed form expressions (12) and (18), albeit in terms of the roots of the algebraic equations (11) and (17). These equations remain to be explored.

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\(^6\) Note that, taking \( \omega = 1 \) and \( \epsilon = 0 \) for simplicity, this recurrence relation can also be written in the form \((m + 1)(m - N)h_{m+1} + (\Delta^2 + 4g^2 m - (N - m)^2)h_m + 4g^2(N + 1 - m)h_{m-1} = 0\), which differs from the recurrence relations given elsewhere, e.g., in [18, 20]. Presumably this is because the coefficients in the recurrence relation change with the expansion variable, in this case either \( \epsilon \) defined above or \( \omega \).
3.1. Degenerate crossing points

It has been noted that when $\epsilon$ is an integer multiple of $\omega/2$ the exceptional eigenvalues considered here for the generalized Rabi model are crossing points in the eigenspectrum as a function of the coupling $g$. For this particular value of $\epsilon$ the exceptional part of the eigenspectrum consists of the doubly degenerate crossing points. The blue (thick) lines are the energy curves $E = N\omega - g^2/\omega + \epsilon$ for $N = 1, \ldots, 5$.

Figure 1. The first few energy levels $E$ in the eigenspectrum of the generalized Rabi model as a function of the coupling $g$ denoted by grey (thin) lines. The parameter values are $\epsilon = \frac{1}{4}\omega$ with $\Delta = 1.2$ and $\omega = 1$. For this particular value of $\epsilon$ the exceptional part of the eigenspectrum consists of the doubly degenerate crossing points.

Figure 2. The first few energy levels $E$ in the eigenspectrum of the generalized Rabi model as a function of the coupling $g$ denoted by grey (thin) lines. The parameter values are $\epsilon = \frac{1}{4}\omega$ with $\Delta = 1.5$ and $\omega = 1$. The blue (thick) lines are the energy curves $E = N\omega - g^2/\omega + \epsilon$ for $N = 1, \ldots, 5$.
crossings for $\Delta = 1.2$. We expect that this is indeed the case for all $N$ in the range $0 < \Delta/\omega < \sqrt{2}$. Following Kus [18] we are able to prove a theorem by induction in appendix B giving the number of roots of the constraint polynomial and thus the number of exceptional points for given $N$ and $\epsilon$. Specifically, defining the function $Q_k(x) = (1/k!)(\partial^{k})P(x)$, where $P_k(x) = 0$ is the constraint polynomial, we have established the following theorem.

**Theorem.** For $0 < \Delta/\omega < \sqrt{1 + 2\epsilon/\omega}$, $Q_k(x)$ has exactly $k$ different, positive roots $a^{(k)}_1, a^{(k)}_2, \ldots, a^{(k)}_k$, moreover

\[0 < a^{(k)}_1 < a^{(k-1)}_1 < a^{(k)}_2 < a^{(k-1)}_2 < \cdots < a^{(k-1)}_{k-1} < a^{(k)}_k,\]  

(47)

where $a^{(k-1)}_1, \ldots, a^{(k-1)}_{k-1}$ denote the roots of $Q_{k-1}(x)$.

More generally, as discussed in appendix B, one can prove that there are $N - k$ roots of the constraint polynomial for $\Delta$ in the range

\[\sqrt{k^2 + 2\epsilon/\omega} < \Delta/\omega < \sqrt{(k + 1)^2 + 2(k + 1)\epsilon/\omega}.\]  

(48)

Indeed, in figure 2 we see that for the first few values of $N$ there are $N - 1$ crossings for $\Delta = 1.5$ when $\epsilon = \frac{1}{2}\omega$. However, to prove the number of level crossings for $\epsilon = \frac{1}{2}\omega$ requires a further step. Figure 3 shows the first few exceptional points in the eigenspectrum at the typical value $\omega = 0.3\omega$, where there are no crossing points. It was pointed out how these exceptional points merge to form doubly degenerate crossing points as $\epsilon \to \frac{1}{2}\omega$ [4]. For the number of level crossings, it is necessary to prove that the two corresponding sets of roots of the constraint polynomials coincide for given $N$ when $\epsilon = \frac{1}{2}\omega$, see appendix B. This does not seem so straightforward to prove however, for general $N$. Nevertheless, we are confident that there are indeed $N - k$ doubly degenerate crossing points in the range (48). It would be fascinating, though seemingly unlikely, if such theorems could be proved via the algebraic equations obtained in this paper.

Figure 3. The first few energy levels $E$ in the eigenspectrum of the generalized Rabi model as a function of the coupling $g$ denoted by grey (thin) lines. The parameter values are $\Delta = 1.2$ and $\omega = 1$ with now $\epsilon = 0.3$. The blue (thick) lines are the energy curves $E = N\omega - g^2/\omega + \epsilon$ for $N = 1, \ldots, 5$. The red (thick) lines are the energy curves $E = N\omega - g^2/\omega - \epsilon$ for $N = 1, \ldots, 5$. For this value of $\epsilon$ there are no crossing points. The first nine exceptional points in the eigenspectrum are indicated by red ($N = 1$), blue ($N = 2$) and green ($N = 3$) circles.
Acknowledgments

It is a pleasure to thank Professor Huan-Qiang Zhou for insightful discussions. We also thank Daniel Braak for suggesting to prove the number of crossings for $\epsilon = \frac{1}{2} \omega$ and the anonymous referees for a number of useful suggestions. MTB gratefully acknowledges support from Chongqing University and the 1000 Talents Program of China. This work is also supported by the Australian Research Council through grant DP130102839.

Appendix A. Derivation of the algebraic equations

The second order differential equation (9) is of the general form

$$\left[ X(z) \frac{d^2}{dz^2} + Y(z) \frac{d}{dz} + Z(z) \right] S(z) = 0, \quad (A.1)$$

where

$$X(z) = \sum_{k=0}^{4} a_k z^k, \quad Y(z) = \sum_{k=0}^{3} b_k z^k, \quad Z(z) = \sum_{k=0}^{2} c_k z^k. \quad (A.2)$$

Comparing with equation (9), the nonzero coefficients are

$$a_0 = -g^2, \quad a_2 = \omega^2, \quad (A.3)$$

$$b_0 = \frac{2g^2}{\omega} - 2\epsilon g - \omega g, \quad b_1 = \omega^2 - 2g^2 - 2\epsilon \omega, \quad b_2 = -2\omega g, \quad (A.4)$$

$$c_0 = E^2 - \Delta^2 - \epsilon^2 + \frac{2\epsilon g^2}{\omega} - \frac{g^4}{\omega^2}, \quad c_2 = \frac{2g^3}{\omega} + 2\epsilon g - 2\epsilon g. \quad (A.5)$$

Zhang’s theorem 1.1 [26] states that (A.1) has a degree $n$ polynomial solution

$$S(z) = \prod_{i=1}^{n} (z - z_i), \quad (A.6)$$

with distinct roots $z_1, z_2, \ldots, z_n$. The values of the coefficients $c_0, c_1, c_2$ are given by

$$c_2 = -n(n - 1)a_4 - nb_3, \quad (A.7)$$

$$c_1 = -\left[ 2(n - 1)a_4 + b_3 \right] \sum_{i=1}^{n} z_i - n(n - 1)a_3 - nb_2, \quad (A.8)$$

$$c_0 = -\left[ 2(n - 1)a_4 + b_3 \right] \sum_{i=1}^{n} z_i^2 - 2a_4 \sum_{i<j}^{n} z_i z_j$$

$$- \left[ 2(n - 1)a_3 + b_2 \right] \sum_{i=1}^{n} z_i - n(n - 1)a_2 - nb_1. \quad (A.9)$$

The roots $z_1, z_2, \ldots, z_n$ satisfy the algebraic equations

$$\sum_{j=1}^{n} \frac{2}{z_i - z_j} + \frac{b_3 z_i^3}{a_4 z_i^4} + \frac{b_2 z_i^2}{a_3 z_i^3} + \frac{b_1 z_i}{a_2 z_i^2} + \frac{b_0}{a_1 z_i} + a_0 = 0, \quad (A.10)$$

for $i = 1, 2, \ldots, n$. 
Substituting the values (A.3)–(A.5) into (A.7)–(A.10) with \( n = N \) gives \( 0 = 0 \), the energy expression (13), the constraint (12) and the algebraic equation (11), respectively. In a similar fashion we arrive at equations (17)–(19).

**Appendix B. Proof for the number of exceptional points**

To prove the number of roots of the constraint polynomial it is convenient to generalize the recurrence relation obtained by Kus [18] rather than use the recurrence relation (39). For nonzero \( \epsilon \) we thus use the recurrence relation

\[
P_0 = 1, \quad P_1 = 4g^2 + \Delta^2 - \omega^2 - 2\epsilon \omega, \]
\[
P_k = \left[ k(2g)^2 + \Delta^2 - k^2\omega^2 - 2k\epsilon \omega \right] P_{k-1} - k(k-1)(n-k+1)(2g)^2\omega^2 P_{k-2}. \tag{B.1}
\]

The equation \( P_k = 0 \) when \( k = N \) defines the constraint polynomial, which can be written here in the form

\[
\left[ N(2g)^2 + \Delta^2 - N^2\omega^2 - 2N\epsilon\omega \right] P_{N-1} - N(\omega - 1)(2g)^2\omega^2 P_{N-2} = 0. \tag{B.2}
\]

We now fix the value of \( N \) and set \( x = (2g)^2 \). Thus

\[
Q_0(x) = 1, \quad Q_1(x) = x - \alpha_1, \]
\[
Q_k(x) = (x - \alpha_k)Q_{k-1}(x) - \beta_k x Q_{k-2}(x), \tag{B.3}
\]

where \( Q_k(x) = (1/k!)P_k(x) \) and

\[
\alpha_k = \left( k^2\omega^2 + 2k\epsilon \omega - \Delta^2 \right)/k, \tag{B.4}
\]
\[
\beta_k = (n-k+1)\omega^2. \tag{B.5}
\]

Now, following Kus [18], we can prove the theorem stated in section 3.1.

**Proof.** For \( 0 < \Delta/\omega < \sqrt{1 + 2\epsilon/\omega} \) we have \( \alpha_k > 0 \) with always \( \beta_k > 0 \). From the definitions we also have \( a_1^{(1)} = \alpha_1 > 0 \) and \( Q_2(x) = (x - \alpha_2)(x - a_1^{(1)}) - \beta_2 x \). Thus \( Q_2(0) = \alpha_2 a_1^{(1)} > 0, Q_2(a_1^{(1)}) = - \beta_2 a_1^{(1)} < 0 \) and \( \text{sgn} \ Q_2(\infty) = 1 \), where

\[
\text{sgn} \ a = \begin{cases} 
-1 & a < 0 \\
0 & a = 0 \\
1 & a > 0. 
\end{cases}
\]

These results and relations (47) prove that \( Q_2(x) = (x - a_1^{(2)})(x - a_2^{(2)}) \) and \( 0 < a_1^{(2)} < a_1^{(1)} < a_2^{(2)} \).

The general proof now proceeds by induction. Assume that the theorem is valid for \( \ell < k \), i.e.

\[
Q_{k-1}(x) = \left( x - a_1^{(k-1)} \right) \ldots \left( x - a_1^{(1)} \right),
\]
\[
Q_{k-2}(x) = \left( x - a_1^{(k-2)} \right) \ldots \left( x - a_1^{(2)} \right), \tag{B.6}
\]
and

\[
0 < a_1^{(k-1)} < a_1^{(k-2)} < a_2^{(k-1)} < a_2^{(k-2)} < \cdots < a_k^{(k-2)} < a_k^{(k-1)}. \tag{B.7}
\]
Then from (B.3) we have
\[ Q_k(x) = (x - \alpha_k) \left( x - a_1^{(k-1)} \right) \cdots \left( x - a_{k-1}^{(k-1)} \right) \]
\[ - \beta_k x \left( x - a_1^{(k-2)} \right) \cdots \left( x - a_{k-2}^{(k-2)} \right). \] (B.8)

Thus
\[ \text{sgn } Q_k(0) = \text{sgn } (-1)^{k} \alpha_k a_1^{(k-1)} a_2^{(k-1)} \cdots a_{k-1}^{(k-1)} = (-1)^{k}, \]
\[ \text{sgn } Q_k(a_i^{(k-1)}) = - \text{sgn } \left( \beta_k a_1^{(k-1)} \left( a_i^{(k-1)} - a_1^{(k-2)} \right) \cdots \left( a_i^{(k-1)} - a_{i-1}^{(k-2)} \right) \right) \]
\[ \times \left( a_i^{(k-1)} - a_1^{(k-2)} \right) \cdots \left( a_i^{(k-1)} - a_{k-2}^{(k-2)} \right) = (-1)^{k-i}, \] (B.9)

This implies that
\[ Q_k(x) = \left( x - a_1^{(k)} \right) \cdots \left( x - a_k^{(2)} \right), \] (B.10)

and \( a_1^{(k-1)}, \ldots, a_i^{(k-1)} \) fulfill the inequalities (47). The theorem is thus proved.

Again following Kus [18] similar theorems can be proved for other values of \( \Delta \). The ranges of \( \Delta \) follow from the values of \( k \) for which \( \alpha_i \) defined in (B.4) is no longer positive. In this way one can prove that there are \( N - k \) roots of the constraint relation in the range (48). When \( \epsilon = \frac{1}{2} \omega \) there are thus expected to be \( N - k \) roots of the constraint polynomial. We can repeat the above working to arrive at similar results for the number of roots and thus number of exceptional points. Now one can prove that the function \( Q_N'(x) = (1/k!) P_N'(x) \) has \( N - k \) different positive roots for \( \Delta \) in the range
\[ \sqrt{k^2 + 2 \epsilon / \omega} < \Delta / \omega < \sqrt{(k + 1)^2 - 2(k + 1) \epsilon / \omega}. \] (B.12)

When \( k = 0 \), \( Q_N'(x) \) has \( N \) roots for \( 0 < \Delta / \omega < \sqrt{1 - 2 \epsilon / \omega} \). We thus have that for \( \sqrt{k^2 + 2 \epsilon / \omega} < \Delta / \omega < \sqrt{(k + 1)^2 - 2(k + 1) \epsilon / \omega} \) both \( Q_N'(x) \) and \( Q_N''(x) \) have \( N - k \) roots. In particular, for \( \sqrt{(k + 1)^2 - 2(k + 1) \epsilon / \omega} < \Delta / \omega \) both \( Q_N'(x) \) and \( Q_N''(x) \) have \( N - k \) roots. Precisely at \( \epsilon = \frac{1}{2} \omega \), the intervals
\[ \left( \sqrt{k^2 + 2 \epsilon / \omega}, \sqrt{(k + 1)^2 - 2(k + 1) \epsilon / \omega} \right) \] vanish. This implies that for arbitrary \( \Delta \), \( Q_N'(x) \) has \( N - k \) roots and \( Q_N''(x) \) has \( N - k - 1 \) roots. This is the situation seen, for example, in figure 3. We have been able to show numerically that the roots of the polynomials \( Q_N'(x) \) and \( Q_{N+1}'(x) \) coincide for all values of \( N \) when \( \epsilon = \frac{1}{2} m \omega \). More generally the roots of \( Q_N'(x) \) and \( Q_{N+1}'(x) \) coincide when \( \epsilon = \frac{1}{2} m \omega \).
doubly degenerate at the crossing points. However, we have not so far been able to prove this analytically.

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