Global smooth solution to the 2D Boussinesq equations with fractional dissipation

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In this paper, we consider the 2D incompressible Boussinesq system with fractional Laplacian dissipation and thermal diffusion. On the basis of the previous works and some new observations, we show that the condition $1 - \alpha < \beta < \min \left\{ 3 - 3\alpha, \frac{\alpha}{2}, \frac{3\alpha^2 + 4\alpha - 4}{8(1 - \alpha)} \right\}$ with $0.7351 \approx \frac{10 - 2\sqrt{10}}{5} < \alpha < 1$ suffices in order for the solution pair of velocity and temperature to remain smooth for all time. Copyright © 2017 John Wiley & Sons, Ltd.

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1. Introduction

In this paper, we consider the following Cauchy problem of the 2D incompressible Boussinesq equations with the fractional Laplacian dissipation:

$$\begin{aligned}
\partial_t u + (u \cdot \nabla) u + \Lambda^\alpha u + \nabla p &= \theta e_2, \quad x \in \mathbb{R}^2, \ t > 0, \\
\partial_t \theta + (u \cdot \nabla) \theta + \Lambda^\beta \theta &= 0, \quad x \in \mathbb{R}^2, \ t > 0, \\
\nabla \cdot u &= 0, \quad x \in \mathbb{R}^2, \ t > 0, \\
u(0, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \mathbb{R}^2,
\end{aligned}$$

(1.1)

where $u(x, t) = (u_1(x, t), u_2(x, t))$ is a vector field denoting the velocity, $\theta = \theta(x, t)$ is a scalar function denoting the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, $p$ is the scalar pressure, and $e_2 = (0, 1)$. Here, the numbers $\alpha \in [0, 2]$ and $\beta \in [0, 2]$ are real parameters. The fractional Laplacian operator $\Lambda^\alpha$, $\Lambda := (-\Delta)^{1/2}$ denotes the Zygmond operator that is defined through the Fourier transform, namely,

$$\hat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi), \quad \hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-\alpha |x|} f(x) \, dx.$$

The fractional Laplacian models many physical phenomena such as overdriven detonations in gasses [1]. It is also used in some mathematical models in hydrodynamics, molecular biology, and finance mathematics (e.g., [2]). We make the convention that by $\alpha = 0$, we mean that there is no dissipation in the velocity equation and similarly, $\beta = 0$ means that there is no dissipation in the temperature equation.

The classical 2D Boussinesq equations (i.e., $\alpha = \beta = 2$) model geophysical flows such as atmospheric fronts and oceanic circulation and play an important role in the study of Rayleigh–Bénard convection (e.g., [3, 4] and references therein). Moreover, there are some geophysical circumstances related to the Boussinesq equations with fractional Laplacian (see [4, 5] for details). The Boussinesq equations with fractional Laplacian also closely related equations such as the surface quasi-geostrophic equation model important geophysical phenomena (e.g., [6]).

The standard 2D Boussinesq equations and their fractional Laplacian generalizations have attracted considerable attention recently because of their physical applications and mathematical significance. On the one hand, when $\alpha = 2$ and $\beta = 2$, system (1.1) becomes the standard 2D Boussinesq equations whose global regularity has been proved previously (e.g., [7]). On the other hand, the fundamental issue of whether classical solutions to the totally inviscid Boussinesq equations (i.e., $\alpha = \beta = 0$) can develop finite time singularities
remains outstandingly open (except if \( \theta_0 \) is a constant, of course). Therefore, it is very interesting to consider the intermediate cases, that is, the fractional dissipation. We hope that the study of the intermediate cases may shed light on this extremely challenging problem. Almost at the same time, Chae [8] and Hou-Li [9] successfully established the global regularity to system (1.1) with \( \alpha = 2 \) and \( \beta = 0 \) or \( \alpha = 0 \) and \( \beta = 2 \), independently. Xu [10] proved the global unique solution of system (1.1) with \( \alpha + \beta = 2 \) and \( 1 \leq \alpha < 2 \). For the cases with weaker dissipation, that is, when \( \alpha = 0, 1 < \beta < 2 \), the corresponding system (1.1) should be globally well-posed [11]. By deeply developing the new structures of the coupling system, Hamidi, Keraani, and Rousset [12, 13] were able to establish the global well-posedness result to system (1.1) with two special critical case, namely, \( \alpha = 1 \) and \( \beta = 0 \) or \( \alpha = 0 \) and \( \beta = 1 \). The more general critical case \( \alpha + \beta = 1 \) with \( 0 < \alpha, \beta < 1 \) is extremely difficult. Very recently, the global regularity of the general critical case \( \alpha + \beta = 1 \) with \( \alpha > 2 \frac{2-\alpha}{\sqrt{177} - 24} \approx 0.9132 \) and \( 0 < \beta < 1 \) was recently examined by Ji, Miao, Wu, and Zhang [14]. This result was further improved by Stefanov and Wu [15], which requires \( \alpha + \beta = 1 \) with \( \alpha > 2 \frac{2-\alpha}{\sqrt{177} - 24} \approx 0.7981 \) and \( 0 < \beta < 1 \) (see also [16] for further improvement). Now, we want to mention some results concerning the subcritical ranges, namely, \( \alpha + \beta > 1 \) with \( 0 < \alpha, \beta < 1 \). In fact, the global regularity of (1.1) in the subcritical ranges is also definitely nontrivial and quite difficult. To the best of our knowledge, there are only a few works concerning the subcritical cases [17–23]. We note that not all subcritical cases have been resolved. As a rule of thumb, with current methods, it seems impossible to obtain the global regularity for the 2D Boussinesq equations in the supercritical ranges, while in this paper, we show that \( \alpha > 2 \frac{2-\alpha}{\sqrt{177} - 24} \approx 0.7981 \), where \( \alpha \) is an explicit function as a technical bound. Hence, this present result can be regarded as a further improvement of the results in [18, 20, 23].

To complement and improve the existing results described previously, the goal of this paper is to study the case \( 1 - \alpha < \beta < f(\alpha) \) and see how much \( \alpha > 0 \) may be reduced while preserving the global regularity result. The previous three works [18, 20, 23] required \( \alpha > \alpha_0 \approx 0.7796 \), while in this paper, we show that \( \alpha > 2 \frac{2-\alpha}{\sqrt{177} - 24} \approx 0.7351 \) suffices. More precisely, our main result reads as follows.

**Theorem 1.1**

Let \( 0.7351 \approx 2 \frac{2-\alpha}{\sqrt{177} - 24} < \alpha < 1 \) and \( 1 - \alpha < \beta < f(\alpha) \), where

\[
f(\alpha) = \min \left\{ 3 - 3\alpha, \frac{3\alpha^2 + 4\alpha - 4}{8(1 - \alpha)} \right\}.
\]

Let \((u_0, \theta_0) \in H^\vartheta (\mathbb{R}^2) \times H^\vartheta (\mathbb{R}^2)\) for \( \sigma > 2 \), then system (1.1) admits a unique global solution such that for any given \( T > 0 \),

\[
\begin{align*}
\nu &\in C([0, T]; H^\vartheta (\mathbb{R}^2)) \cap L^2([0, T]; H^{\vartheta + \frac{\vartheta}{2}} (\mathbb{R}^2)), \\
\theta &\in C([0, T]; H^\vartheta (\mathbb{R}^2)) \cap L^2([0, T]; H^{\vartheta + \frac{\vartheta}{2}} (\mathbb{R}^2)).
\end{align*}
\]

**Remark 1.2**

By combining the previous three works [18, 20, 23], the global well-posedness result of the system (1.1) was established under the condition \( 1 - \alpha < \beta < g(\alpha) \) for \( \alpha > \alpha_0 \approx 0.7796 \), where \( g(\alpha) \) is a technical bound. Hence, this present result can be regarded as a further improvement of the results in [18, 20, 23].

**Remark 1.3**

We want to point out that because of the technical reasons, the \( \beta \) is smaller than a complicated explicit function. Indeed, it is strongly believed that the diffusion term is always good term and the larger the power \( \beta \) is, the better effects it produces. Therefore, we conjecture that the aforementioned theorem should hold for all the cases \( 1 - \alpha < \beta < 1 \).

We outline the proof of this theorem. A large portion of the efforts are devoted to obtain global a priori bounds for \( u \) and \( \theta \) on the interval \([0, T]\). We begin with the following key equation (see 3.4 for details)

\[
\partial_t G + (u \cdot \nabla) G + \Lambda^\alpha G = [\mathcal{R}_\sigma, u \cdot \nabla] \theta + \Lambda^{\beta - \alpha} \partial_t \theta.
\]

Under the assumption of \( \beta > 1 - \alpha \) and \( \alpha > 2 \frac{\vartheta}{2} \), we have

\[
\sup_{0 \leq t \leq T} (\|G(t)\|^2_{L^2} + \|\Lambda^\vartheta \theta(t)\|^2_{L^2}) + \int_0^T \left( \|\Lambda^\vartheta G\|^2_{L^2} + \|\Lambda^{\vartheta + \frac{\vartheta}{2}} \theta\|^2_{L^2} \right) dt \leq C(T, u_0, \theta_0),
\]

for any \( \max \left\{ 2 - 2\alpha - \beta, 2 + \frac{2 + \vartheta - 3\alpha}{2} \right\} < \delta < \frac{\beta}{2} \). This allows us to show

\[
\|G(t)\|^m_{L^m} + \int_0^T \|G(t)\|^m_{L^m} dt \leq C(T, u_0, \theta_0),
\]

where \( m \) satisfies (3.6). Next, (1.3) and the iteration entail that

\[
\int_0^T \|\omega(t)\|_{L^2} dt \leq C(T, u_0, \theta_0), \quad 2 \leq m \leq \frac{2m}{2 - \alpha}.
\]
This further leads to the key estimate
\[ \int_0^T \| G(t) \|_{\infty, 1}^p \, dt \leq C(T, u_0, \theta_0), \]
where \( \alpha \) and \( \beta \) should satisfy \( 1 - \alpha < \beta < \min \left\{ 3 - 3\alpha, \frac{3\alpha^2 + 4\alpha - 4}{\beta(1 - \alpha)} \right\} \). Actually, if the \( \beta \) would work, then it gives rise to the restriction \( 10^{-2} < \alpha < 1 \). Thanks to the aforementioned estimate, the following key estimate holds:
\[ \int_0^T \| \omega(t) \|_{\infty, 1}^p \, dt \leq C(T, u_0, \theta_0), \]
then the global regularity of \( u \) and \( \theta \) follows by a standard approach.

The rest part of this paper is organized as follows. In Section 2, we collect some preliminaries materials, including the Littlewood–Paley decomposition, the definition of Besov spaces, and some useful lemmas. In Section 3, we give the proof of Theorem 1.1. In the Appendix, we give the details about the fact that a choice of \( p \) is possible.

2. Preliminaries

In this section, we are going to recall some basic facts on the Littlewood–Paley theory, the definition of Besov space, and some useful lemmas. Now, we recall the so-called Littlewood–Paley operators and their elementary properties, which allow us to define the Besov spaces (e.g., [41, 42]). Let \((\chi, \varphi)\) be a couple of smooth functions with values in \([0, 1]\) such that \( \chi \in C_0^\infty(\mathbb{R}^n) \) is supported in the ball \( B := \{ \xi \in \mathbb{R}^n, |\xi| \leq \frac{3}{2} \} \), \( \varphi \in C_0^\infty(\mathbb{R}^n) \) is supported in the annulus \( C := \{ \xi \in \mathbb{R}^n, \frac{1}{4} \leq |\xi| \leq \frac{3}{4} \} \) and satisfy
\[ \chi(\xi) + \sum_{j \in \mathbb{N}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n; \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \]

For every \( u \in S' \) (tempered distributions), we define the non-homogeneous Littlewood–Paley operators as follows:
\[ \Delta_j u = 0, \quad j \leq -2; \quad \Delta_{-1} u = \chi(D)u = F^{-1}(\chi(\xi)\hat{u}(\xi)); \]
\[ \Delta_j u = \varphi(2^{-j}D)u = F^{-1}(\varphi(2^{-j}\xi)\hat{u}(\xi)), \quad \forall j \in \mathbb{N}. \]

Meanwhile, we define the homogeneous dyadic blocks as
\[ \hat{\Delta}_j u = \varphi(2^{-j}D)u = F^{-1}(\varphi(2^{-j}\xi)\hat{u}(\xi)), \quad \forall j \in \mathbb{Z}. \]

Let us recall the definition of homogeneous and inhomogeneous Besov spaces through the dyadic decomposition.

**Definition 2.1**

Let \( s \in \mathbb{R}, (p, r) \in [1, +\infty]^2 \). The homogeneous Besov space \( \dot{B}^{s}_{p, r} \) and inhomogeneous Besov space \( B^{s}_{p, r} \) are defined as a space of \( f \in S'(\mathbb{R}^n) \) such that
\[ \dot{B}^{s}_{p, r} = \{ f \in S'(\mathbb{R}^n), \| f \|_{\dot{B}^{s}_{p, r}} < \infty \}, \quad B^{s}_{p, r} = \{ f \in S'(\mathbb{R}^n), \| f \|_{B^{s}_{p, r}} < \infty \}, \]

where
\[ \| f \|_{\dot{B}^{s}_{p, r}} = \left\{ \left( \sum_{j \in \mathbb{Z}} 2^{js} \| \hat{\Delta}_j f \|_{L^p} \right)^{\frac{1}{s}} \right\}^{\frac{1}{r}} \quad \forall r < \infty, \]
\[ \| f \|_{\dot{B}^{s}_{p, r}} = \sup_{j \in \mathbb{Z}} 2^{js} \| \hat{\Delta}_j f \|_{L^p} \quad \forall r = \infty \]

and
\[ \| f \|_{B^{s}_{p, r}} = \left\{ \left( \sum_{j \geq -1} 2^{js} \| \Delta_j f \|_{L^p} \right)^{\frac{1}{s}} \right\}^{\frac{1}{r}} \quad \forall r < \infty, \]
\[ \| f \|_{B^{s}_{p, r}} = \sup_{j \geq -1} 2^{js} \| \Delta_j f \|_{L^p} \quad \forall r = \infty. \]

For \( s > 0 \), \((p, r) \in [1, +\infty]^2\), we have the following fact:
\[ \| f \|_{\dot{B}^{s}_{p, r}} = \| f \|_{L^p} + \| f \|_{\dot{B}^{s}_{p, r}}. \]

When \( 0 < s < 1 \), the following equivalence is true [43, 44]:
\[ \| f \|_{\dot{B}^{s}_{p, r}} = \left( \int_{\mathbb{R}^n} \| f(x + .) - f(\cdot) \|_{L^p}^{p} \, dx \right)^{\frac{1}{s}}; \]
\[ \| f \|_{\dot{B}^{s}_{p, r}} = \| f \|_{L^p} + \left( \int_{\mathbb{R}^n} \| f(x + .) - f(\cdot) \|_{L^p}^{p} \, dx \right)^{\frac{1}{s}}. \]
We shall also need the mixed space–time spaces

\[ \|f\|_{L^p_t B^s_v} := \left\| (2^j \|\Delta f\|_{L^p_t})_j \right\|_{L^p_v} \]

and

\[ \|f\|_{\tilde{L}^p_t B^s_v} := \left\| (2^j \|\Delta f\|_{L^p_t})_j \right\|_{L^p_v}. \]

The following links are direct consequence of the Minkowski inequality:

\[ L^p_t B^s_v \hookrightarrow \tilde{L}^p_t B^s_v, \quad \text{if } r \geq p, \quad \text{and} \quad \tilde{L}^p_t B^s_v \hookrightarrow L^p_t B^s_v, \quad \text{if } p \geq r. \]

In particular,

\[ \tilde{L}^p_t B^s_v \approx L^p_t B^s_v. \]

Bernstein inequalities are fundamental in the analysis involving Besov spaces, and these inequalities trade integrability for derivatives.

**Lemma 2.2 ([41])**

Let \( k \geq 0, 1 \leq a \leq b \leq \infty \). Assume that

\[ \text{supp } \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^j \}, \]

for some integer \( j \), then there exists a constant \( C_1 \) such that

\[ \|\Lambda^k f\|_{L^b} \leq C_1 2^{jk(n \frac{1}{2} - \frac{1}{b})} \|f\|_{L^a}. \]

If \( f \) satisfies

\[ \text{supp } \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \approx 2^j \} \]

for some integer \( j \), then

\[ C_1 2^k \|f\|_{L^b} \leq \|\Lambda^k f\|_{L^b} \leq C_2 2^k \|f\|_{L^a}, \]

where \( C_1 \) and \( C_2 \) are constants depending on \( \alpha, a \) and \( b \) only.

To prove the theorem, we need the following commutator estimate involving \( \mathcal{R}_\alpha := \partial_x \Lambda^{-\alpha} \), which was established by Stefanov and Wu [15].

**Lemma 2.3**

Assume that \( \frac{1}{2} < \alpha < 1 \) and \( 1 < p_2 < \infty, 1 < p_3 \leq \infty \) with \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \). Then, for \( 0 \leq s_1 < 1 - \alpha \) and \( s_1 + s_2 > 1 - \alpha \), the following holds true:

\[ \left| \int_{\mathbb{R}^2} F[\mathcal{R}_\alpha, u G \cdot \nabla] \theta \, dx \right| \leq C \|\Lambda^{s_1} \theta\|_{L^{p_1}} \|F\|_{W^{s_2, p_2}} \|G\|_{L^{p_3}}. \tag{2.1} \]

Similarly, for \( 0 \leq s_1 < 1 - \alpha \) and \( s_1 + s_2 > 2 - 2\alpha \), the following holds true:

\[ \left| \int_{\mathbb{R}^2} F[\mathcal{R}_\alpha, u \theta \cdot \nabla] H \, dx \right| \leq C \|\Lambda^{s_1} \theta\|_{L^{p_1}} \|F\|_{W^{s_2, p_2}} \|H\|_{L^{p_3}}. \tag{2.2} \]

Here and in what follows, \( W^{s, p} \) denotes the standard Sobolev space.

The following lemma contains bilinear estimate.

**Lemma 2.4**

Let \( 2 < m < \infty, 0 < s < 1 \), and \( p, q, r \in (1, \infty)^3 \) such that \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \), then it holds

\[ \|f\|_{w^{s, p}} \leq C \|f\|_{w^{s, q}} \|f\|_{t^{s, r}} \tag{2.3} \]

**Proof of Lemma 2.4**

For \( 0 < s < 1 \), we make use of the following characterization of the Sobolev space \( W^{s, p} \) [43–45]:

\[ \|f\|_{w^{s, p}} = \left\| \|f\|_{L^p} \right\|_{L^q} \approx \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(x + \cdot) - f(x + \cdot)|^p \, dx \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \]

\[ := I + J. \]

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First, by the Hölder inequality, we easily obtain

\[ l \leq \|f\|_{L^p}^p \|f^{(m-2)}\|_{L^{(m-2)p}} = \|f\|_{L^p}^p \|f^{(m-2)}\|_{L^{(m-2)p}}. \]

Note that the following simple inequality:

\[ |a|^{m-2}a - |b|^{m-2}b \leq C(m)|a - b||a|^{m-2} + |b|^{m-2}, \]

and Hölder inequality, it results in

\[ \|f\|^{(m-2)}(x, \cdot) - \|f\|^{(m-2)}(\cdot, \cdot) \|_{L^p} \leq C\|x + \cdot - f(\cdot)\|_{L^{(m-2)p}} \|f\|^{(m-2)}\|_{L^{(m-2)p}}. \]

Thus, it follows from the characterization of Besov space that [44]

\[ J \leq \int_{\mathbb{R}^2} \|f(x, \cdot) - f(\cdot)\|_{L^{(m-2)p}}^{(m-2)p} \|f\|_{L^{(m-2)p}} \|f\|_{L^{(m-2)p}} \|f\|_{L^{(m-2)p}} dx \]

\[ \leq C\|f\|_{L^{(m-2)p}} \int_{\mathbb{R}^2} \|f(x, \cdot) - f(\cdot)\|_{L^{(m-2)p}} dx \]

\[ \approx C\|f\|_{L^{(m-2)p}} \|f\|_{L^{(m-2)p}}. \]

We obtain by combining the aforementioned two estimates

\[ \|f\|_{L^{(m-2)p}} \leq C\|f\|_{L^{(m-2)p}} + C\|f\|_{L^{(m-2)p}} \|f\|_{L^{(m-2)p}} \]

\[ \approx C\|f\|_{L^{(m-2)p}} \|f\|_{L^{(m-2)p}}. \]

Thus, this completes the proof of inequality (2.3). □

The next lemma is the commutator estimate that will be used frequently.

**Lemma 2.5** ([18])

Assume that \( u \) is a smooth divergence-free vector field of \( \mathbb{R}^2 \) and \( \theta \) is a smooth function. Let \( \alpha \in (0, 1), s \in (-1, \alpha), \rho \in [2, \infty] \) and \( r \in [1, \infty] \), then

\[ \|[R_{u\theta}, u \cdot \nabla] \theta\|_{L^p} \leq C\|\nabla u\|_{L^\rho} (\|\theta\|_{L^\rho}^{1-\rho} + \|\theta\|_{L^r}). \]  

\[ \sup_{k \geq -1} 2^{(\alpha-1)k} \|[D_k u \cdot \nabla] \theta\|_{L^p} \leq C(\|\nabla u\|_{L^\rho} + \|u\|_{L^\rho}) \|\theta\|_{L^\rho}. \]  

Finally, we end the section by the following lemma gathering the smoothing effect of the transport–diffusion equation.

**Lemma 2.6**

Consider the following transport–diffusion equation with \( 0 < \beta \leq 1 \):

\[ \begin{aligned}
  \partial_t \theta + (u \cdot \nabla) \theta + \Lambda^\beta \theta &= 0, \\
  \nabla \cdot u &= 0, \\
  \theta(x, 0) &= \theta_0(x),
\end{aligned} \]

then for any \( (p, \rho) \in (1, \infty) \times [1, \infty) \), the following estimate holds:

\[ \sup_{k \in \mathbb{N}} 2^{k\beta} \|D_k \theta\|_{L^p} \leq C(\|\theta_0\|_{L^\rho} + \|\theta_0\|_{L^\rho} + \|\omega\|_{L^r}), \]

where \( \omega \) is the vorticity of the velocity \( u \), namely, \( \omega = \nabla \times u \).

**Proof of Lemma 2.6**

The case \( \rho = 1 \) and \( \beta = 1 \) has been established in [13]. The general case, namely, (2.6) can be proved by the same argument used in [13]. Here, we sketch the proof of (2.6) for the sake of completeness. Applying inhomogeneous blocks \( D_k \) operator to transport–diffusion equation, we have

\[ \partial_t \Delta_k \theta + (u \cdot \nabla) \Delta_k \theta + \Delta_k \Lambda^\beta \theta = -[\Delta_k u \cdot \nabla] \theta. \]

Multiply the equation (2.7) by \( |\Delta_k \theta|^p - 2 \Delta_k \theta \theta \) and use the Hölder inequality as well as divergence-free condition to obtain

\[ \frac{1}{p} \frac{d}{dt} \|\Delta_k \theta\|_{L^p}^p + \int_{\mathbb{R}^2} (\Lambda^\beta \Delta_k \theta) |\Delta_k \theta|^p - 2 \Delta_k \theta dx \leq \|\Delta_k (u \cdot \nabla) \theta\|_{L^p} \|\Delta_k \theta\|_{L^p}^{p-1}. \]
For $k \geq 0$, the Fourier transform of $\Delta_k \theta$ is supported away from the origin, and the dissipative part admits a lower bound \[ \int_{\mathbb{R}^d} (\Lambda^\beta \Delta_k \theta) |\Delta_k \theta|^{p-2} \Delta_k \theta \, dx \geq c2^{2k} \| \Delta_k \theta \|_{L^p}^p. \]

Therefore, one may easily deduce that
\[
\frac{d}{dt} \| \Delta_k \theta \|_{L^p} + c2^{2k} \| \Delta_k \theta \|_{L^p} \leq C \| \Delta_k (u \cdot \nabla) \theta \|_{L^p}.
\]

Note that the following fact:
\[
\| \Delta_k (u \cdot \nabla) \theta \|_{L^p} \leq C \| \nabla u \|_{L^p} \| \theta \|_{L^\infty} \leq C \| \theta_0 \|_{L^\infty} \| \omega \|_{L^p}, \quad 1 < p < \infty,
\]

it further gives
\[
\frac{d}{dt} \| e^{2t \Lambda^\beta} \Delta_k \theta \|_{L^p} \leq C e^{2t \Lambda^\beta} \| \theta_0 \|_{L^\infty} \| \omega \|_{L^p}.
\]

Integrating in time yields
\[
\| \Delta_k \theta \|_{L^p} \leq e^{-c2^2k} \| \Delta_k \theta_0 \|_{L^p} + C \| \theta_0 \|_{L^\infty} \int_0^t \| \omega(\tau) \|_{L^p} e^{-c2^2k(\tau-t)} \, d\tau.
\]

Taking the $L^p$ norm over $[0, t]$ and using convolution inequality, we have
\[
\| \Delta_k \theta \|_{L^p} \leq C 2^{-\frac{1}{p}} \| \Delta_k \theta_0 \|_{L^p} + C 2^{-\frac{1}{p}} \| \theta_0 \|_{L^\infty} \| \omega \|_{L^p} \leq C 2^{-\frac{1}{p}} \| \theta_0 \|_{L^p} + C 2^{-\frac{1}{p}} \| \theta_0 \|_{L^\infty} \| \omega \|_{L^p},
\]

which is nothing but the desired estimate. This completes the proof of Lemma 2.6. \( \square \)

3. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. First, the local well posedness of system (1.1) for smooth initial data is well known (e.g., [3]), and therefore, it suffices to prove the global in time \textit{a priori} estimate on $[0, T]$ for any given $T > 0$. Throughout this paper, we denote by $C$ an universal positive constant whose value may change from line to line. The symbol $C(x, y, z, \ldots)$ means that $C$ depends on variables $x, y, z,$ and so on.

Let us begin with the natural energy estimates of $(u, \theta)$. The proof is standard; thus, we omit it.

\textbf{Lemma 3.1}

Assume that $u_0 \in L^2$ and $\theta_0 \in L^2 \cap L^\infty$. Let $(u, \theta)$ be the corresponding solution of system (1.1). Then, for any $t > 0$, there holds
\[
\| \theta(t) \|_{L^2}^2 + 2 \int_0^t \| \Lambda^\beta \theta(\tau) \|_{L^2}^2 \, d\tau \leq \| \theta_0 \|_{L^2}^2, \quad \| \theta(t) \|_{L^p} \leq \| \theta_0 \|_{L^p}, \quad \forall p \in [2, \infty],
\]

\[
\| u(t) \|_{L^2}^2 + 2 \int_0^t \| \Lambda^\beta u(\tau) \|_{L^2}^2 \, d\tau \leq (\| u_0 \|_{L^2} + t \| \theta_0 \|_{L^2})^2.
\]

In order to obtain the $H^1$-bound for $(u, \theta)$, we apply operator curl to equation (1.1) to obtain the following vorticity equation $(\omega = \partial_1 u_2 - \partial_2 u_1)$
\[
\partial_t \omega + (u \cdot \nabla) \omega + \Lambda^\beta \omega = \partial_\alpha \theta.
\]

However, the 'vortex stretching' term $\partial_\alpha \theta$ appears to prevent us from proving any global bound for $\omega$. To overcome this difficulty, we apply the idea introduced by Hmidi, Keraani, and Rousset [12, 13] to eliminate the term $\partial_\alpha \theta$ from the vorticity equation. Now, we set $\mathcal{R}_\alpha$ as the singular integral operator
\[
\mathcal{R}_\alpha := \partial_\alpha \Lambda^{-\alpha}.
\]

Then, we can show that the new quantity $G = \omega - \mathcal{R}_\alpha \theta$ satisfies
\[
\partial_t G + (u \cdot \nabla) G + \Lambda^\beta G = [\mathcal{R}_\alpha, u \cdot \nabla] \theta + \Lambda^\beta \partial_\alpha \theta,
\]

where here and in the sequel, the following standard commutator notation is used:
\[
[\mathcal{R}_\alpha, u \cdot \nabla] \theta := \mathcal{R}_\alpha (u \cdot \nabla \theta) - u \cdot \nabla \mathcal{R}_\alpha \theta.
\]
Moreover, the velocity field $u$ can be decomposed into the following two parts:

$$ u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp \Delta^{-1} G + \nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta := u_G + u_\theta. $$

The following lemma is concerned with the $L^2$ estimate of $G$ and $\Lambda^\theta \theta$, which was already established in [20].

**Lemma 3.2**

Under the assumptions stated in Theorem 1.1, let $(u, \theta)$ be the corresponding solution of system (1.1). If $\beta > 1 - \alpha$ and $\alpha > \frac{1}{4}$, then the combined quantity $G$ the temperature $\theta$ admit the following bound:

$$ \sup_{0 \leq t \leq T} \left( \|G(t)\|_{L^2}^2 + \|\Lambda^\theta \theta(t)\|_{L^2}^2 \right) + \int_0^T \left( \|\Lambda^\theta \theta G\|_{L^2}^2 + \|\Lambda^\theta \theta \theta\|_{L^2}^2 \right) \, dt \leq C(T, u_0, \theta_0), $$

for any max $\left\{ \frac{2-2\alpha-\beta}{2}, \frac{2+\beta-3\alpha}{2} \right\} < \frac{\beta}{T}$. 

**Remark 3.3**

Although the Lemma 3.2 holds for max $\left\{ \frac{2-2\alpha-\beta}{2}, \frac{2+\beta-3\alpha}{2} \right\} < \frac{\beta}{T}$, yet by energy estimate (3.1) and the classical interpolation, it is actually true for any $0 \leq \delta < \frac{\beta}{T}$. We also remark that $\delta$ can be arbitrarily close to the number $\frac{\beta}{T}$, but at present, we do not know whether Lemma 3.2 is true for the case $\delta = \frac{\beta}{T}$. Meanwhile, we are not able to prove Lemma 3.2 in the critical case, namely, $\beta = 1 - \alpha$.

With the help of Lemma 3.2, we will show the following global a priori bound of $L^m$ ($2 < m < 4$) norm for the quantity $G$.

**Lemma 3.4**

Let $\frac{1}{2} < \alpha < 1$ and $1 - \alpha < \beta < \frac{\alpha}{2}$. Assume that $(u_0, \theta_0)$ satisfies the assumptions stated in Theorem 1.1, then the combined equation (3.4) admits the following bound for any $0 \leq t \leq T$:

$$ \|G(t)\|_{L^m}^m + \int_0^T \|G(t)\|_{L^{2m/p}}^{2m/p} \, dt \leq C(T, u_0, \theta_0), $$

where $m$ satisfies the following restriction:

$$ \left\{ \begin{array}{l}
\frac{4}{2-\alpha-\beta} < m < \min \left\{ \frac{1}{4}, \frac{1}{1-\alpha} \right\} , \\
(2(2-\alpha)\beta - 3\alpha + 2)m < 4(2-\alpha)\beta , \\
(4+8\beta - 4\alpha - 3\alpha^2)m < 16\beta.
\end{array} \right. $$

**Remark 3.5**

It is worthy to emphasize that under the conditions $\frac{1}{2} < \alpha < 1$ and $1 - \alpha < \beta < \frac{\alpha}{2}$, we indeed can choose some $m \in (2, 4)$ to guarantee the restriction (3.6), for example, $m = \frac{2}{\pi(2-\alpha)} + \epsilon$ for $0 < \epsilon \ll 1$.

**Proof of Lemma 3.4**

Multiplying equation (3.4) by $|G|^{m-2} G$, we have after integrating by part and using the divergence-free condition

$$ \frac{1}{m} \frac{d}{dt} \|G(t)\|_{L^m}^m + \int_{\mathbb{R}^2} (\Lambda^\theta \theta G) |G|^{m-2} G \, dx $$

$$ = \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u \cdot \nabla] \theta \ |G|^{m-2} G \, dx + \int_{\mathbb{R}^2} \Lambda^{\beta-\alpha} \delta_x \theta \ |G|^{m-2} G \, dx $$

$$ = \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_G \cdot \nabla] \theta \ |G|^{m-2} G \, dx + \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_\theta \cdot \nabla] \theta \ |G|^{m-2} G \, dx $$

$$ + \int_{\mathbb{R}^2} \Lambda^{\beta-\alpha} \delta_x \theta \ |G|^{m-2} G \, dx $$

$$ := N_1 + N_2 + N_3, $$

where in the third line, we have applied the fact $u = u_G + u_\theta$. Thanks to the maximum principle and Sobolev embedding, we deduce that there exists an absolute constant $\tilde{C} > 0$ such that

$$ \int_{\mathbb{R}^2} (\Lambda^\theta \theta G) |G|^{m-2} G \, dx \geq C \|\Lambda^{\theta} \theta G\|_{L^2}^2 \geq \tilde{C} |G|_{L^{2m/p}}^m. $$

To begin with, we handle the first term $N_1$ at the R-H-S of (3.7). As a matter of fact, $N_1$ admits a suitable estimate [20, 23]. Now, we sketch it here for reader’s convenience. Inequality (2.1) with $s_1 = 0$ as well as inequality (2.3) allows us to show...
\[ N_1 \leq \left| \int_{\mathbb{R}^2} [\mathcal{R}_{\alpha}, u_\varphi \cdot \nabla] \vartheta \ |G|^{m-2} G \, dx \right| \]
\[ \leq C |G|_{L^\infty} \|\vartheta\|_{L^\infty} \|G\|^{m-2} G \|_{W^{2-\frac{4}{n}, \frac{4}{n}}}, \quad (\delta > 0 \text{ is small enough}) \]
\[ \leq C \|\vartheta_0\|_{L^\infty} \|G\|^{m-1} G \|_{\dot{B}^2_{\frac{2}{n}, \frac{4}{n}}}, \quad (3.9) \]

where the exponents should satisfy
\[ s_2 - \frac{\delta}{2} > 1 - \alpha, \quad \frac{1}{\rho} + \frac{1}{q} = 1, \quad m - 1 < q < 2(m - 1). \]

Moreover, the embedding \( H^{\alpha - \frac{2(m-1)}{n}} \hookrightarrow \dot{B}^{2-\frac{4}{n}}_{\frac{2}{n}, \frac{4}{n}} \) has been used. Noticing the requirement \( s_2 - \frac{\delta}{2} > 1 - \alpha \), one may select a sufficiently small \( \delta > 0 \) (in fact, we can take \( \delta \leq \frac{3\alpha - 2}{\alpha} \), for example, to satisfy all the conditions) such that
\[ s_2 = 1 - \alpha + \delta. \]

By means of the following interpolation inequality:
\[ \|G\|_{H^{\alpha - \frac{2(m-1)}{n}}} \leq C \|G\|^{1-\mu} \|G\|^{\mu}_{H^{\frac{2}{n}}}, \quad \mu = \frac{-2\alpha + 2\delta + \frac{4(m-1)}{q}}{\alpha} \]
for
\[ \frac{4(m-1)}{3\alpha - 2\delta} < q < \frac{2(m-1)}{\alpha - \delta} \Rightarrow \mu \in (0, 1), \]

we infer that
\[ N_1 \leq C \|\vartheta_0\|_{L^\infty} \|G\|^{m-1} \|G\|^{1-\mu} \|G\|^{\mu}_{H^{\frac{2}{n}}} \]
\[ \leq C \|G\|^{m-1} \|G\|^{1-\mu} \|G\|^{\mu}_{H^{\frac{2}{n}}} \]
\[ \leq C \|G\|^{m-1} \|G\|^{1-\mu} \|G\|^{\mu}_{L^{2m}} \leq \frac{\tilde{C}}{4} \|G\|^{m} \|G\|^{1-\mu} \|G\|^{\mu}_{L^{2m}}, \quad (3.10) \]

where we have used the following simple interpolation:
\[ \|G\|_{L^2} \leq C \|G\|^{\frac{\mu}{2}} \|G\|^{\frac{1-\mu}{2}} \|G\|_{L^{2m}}^{\frac{2m}{\mu}}, \quad \xi = \frac{2 - \frac{2m}{\mu}}{\alpha}, \quad (3.11) \]

for
\[ m < q < \frac{2m}{2 - \alpha} \Rightarrow 0 < \xi < 1. \quad (3.12) \]

Noting the following facts:
\[ \frac{m(m-1)(1-\xi)}{m - (m-1)\xi} \leq m, \quad m \leq \frac{2}{2 - 2\alpha + \delta} \Rightarrow \frac{m\mu}{m - (m-1)\xi} \leq 2, \]

it follows that
\[ N_1 \leq \frac{\tilde{C}}{4} \|G\|^{m} \|G\|^{1-\mu} \|G\|^{\mu}_{L^{2m}} + C \|G\|^{m} \|G\|^{1-\mu} \|G\|^{\mu}_{L^{2m}}, \quad (3.13) \]

We point out that the choice of the number \( q \) is possible if we select \( \delta < \frac{3\alpha - 2}{2} \). Actually, combining all the requirement on the number \( q \), it should be
\[ \max \left\{ m - 1, \frac{4(m-1)}{3\alpha - 2\delta}, m \right\} < q < \min \left\{ 2(m-1), \frac{2m - 1}{\alpha - \delta}, \frac{2m}{2 - \alpha} \right\}. \]

The second term \( N_2 \) and the last term \( N_3 \) at the R-H-S of (3.7) will be treated differently compared with the first term. Here, the estimates for the terms \( N_2 \) and \( N_3 \) are the main difference compared with the ones in [20, 23]. By making use of the estimate (2.2) with \( s_1 = 1 + \beta - \alpha - \eta \) and \( s_2 = \eta \) (note \( \beta > 1 - \alpha \Rightarrow s_1 + s_2 > 2 - 2\alpha \)), one can show that for any \( 1 < p < 2 \).
\[
N_2 \leq \int_{\mathbb{R}^2} |\mathcal{R}_{\alpha, u_0} \cdot \nabla \theta | |G|^{m-2} G \, dx \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|\Lambda^\eta (|G|^{m-2} G)\|_{L^p} \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|G\|_{L^p} \|G|^{m-2} \|_{L^{(2m-2)/m}} \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|G|^{m-2} \|_{L^{(2m-2)/m}} 
\]

(3.14)

Here and in what follows, we select the parameter \( \eta \) such that

\[
\beta < \eta < \min \left\{ 1 + \beta - \alpha, \frac{\alpha}{2} \right\}. 
\]

(3.15)

Under assumption (3.15), we are now resorting to inequality (2.3) to find

\[
N_3 \leq \int_{\mathbb{R}^2} \Lambda^{1+\beta-\alpha-\eta} \partial_x \theta |G|^{m-2} G \, dx \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|\Lambda^\eta (|G|^{m-2} G)\|_{L^p} \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|G\|_{L^p} \|G|^{m-2} \|_{L^{(2m-2)/m}} \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|G|^{m-2} \|_{L^{(2m-2)/m}} \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|G\|_{H^s} \|G|^{m-2} \|_{L^{(2m-2)/m}} \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|G\|_{H^s} \|G|^{m-2} \|_{L^{(2m-2)/m}} \\
\leq C \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|G\|_{H^s} \|G|^{m-2} \|_{L^{(2m-2)/m}} 
\]

(3.16)

where the following facts have been applied (e.g., [43, 44]):

\[
B_{p,p}^\eta \hookrightarrow F_{p,2}, \quad 1 < p \leq 2 \quad \text{and} \quad B_{p,p}^\eta \approx W^{\alpha p}, \quad 0 < \eta \neq \mathbb{N}. 
\]

Now, let us recall the following fractional version of the Gagliardo–Nirenberg inequality:

\[
\|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \leq C \|\Lambda^\eta \theta\|_{L^p} \|\theta\|_{L^{\frac{2l}{l-1}}(\mathbb{R}^2)} \quad \text{(see Theorem 1.2 of [47])} \\
\leq C \|\theta\|_{L^p} \|\Lambda^{1+\beta-\alpha-\eta} \theta\|_{L^p} \|\theta\|_{L^{\frac{2l}{l-1}}(\mathbb{R}^2)}, 
\]

(3.17)

where we need the following restrictions:

\[
\frac{p-1}{p} = 1 + \beta - \alpha - \eta \quad \text{or} \quad l = \frac{2(p-1) - (1 + \beta - \alpha - \eta)p}{(1-s)p} \in (0, 1), \\
1 + \beta - \alpha - \eta < sl, \quad 1 + \beta - \alpha - \eta < s < \beta.
\]

One can easily check that inequality (3.17) holds as long as \( p \) satisfies

\[
\frac{2s}{2s + \alpha + \eta - \beta - 1} < p < \frac{2}{\alpha + \eta - \beta - s}. 
\]

(3.18)

Let us also recall the following simple interpolation:

\[
\|G\|_{L^{\frac{2m}{m-2}}(\mathbb{R}^2)} \leq C \|G\|_{H^s} \|G\|_{L^{\frac{2m}{m-2}}}^{\lambda}, \quad \lambda = \frac{2}{\alpha} - \frac{(2-p)m}{(m-2)ap},
\]

(3.19)

where

\[
\frac{2m}{3m-4} < p < \frac{2m}{(2-\alpha)(m-2) + m} \Rightarrow \lambda \in (0, 1). 
\]

(3.20)
The estimates (3.17), (3.19), and the Hölder inequality give directly

\[
C \| \Lambda^1 + \beta - \alpha \theta \|_{L^2(B)} \| G \|_{H^\frac{\alpha - \eta}{2}} \| G \|_{H^\frac{(m-2)(1-\lambda)}{2}} \leq C \| \theta \|_{L^2(B)} \| \Lambda^1 + \beta - \alpha \theta \|_{L^2(B)} \| G \|_{H^\frac{\alpha - \eta}{2}} \| G \|_{H^\frac{(m-2)(1-\lambda)}{2}} \leq C \Lambda^1 + \beta - \alpha \theta \|_{L^2} \| G \|_{H^\frac{\alpha - \eta}{2}} \| G \|_{H^\frac{(m-2)(1-\lambda)}{2}} \leq \hat{\mathcal{C}} \| \Lambda^1 + \beta - \alpha \theta \|_{L^2} \| G \|_{H^\frac{\alpha - \eta}{2}} \| G \|_{H^\frac{(m-2)(1-\lambda)}{2}} \leq \frac{\hat{\mathcal{C}}}{8} \| G \|_{L^2} + C(1 + \| \Lambda^1 + \beta - \alpha \theta \|_{L^2}^2 + \| G \|_{H^\frac{\alpha - \eta}{2}}^2) (1 + \| G \|_{H^\frac{\alpha - \eta}{2}}),
\]

where in the last line, we have used the following requirement:

\[
\left(\frac{2s}{2\alpha + \beta} + 1\right) \frac{m}{m - (m - 2)\lambda} \leq 2.
\]

Notice that \( \delta < \frac{\beta}{2} \) and \( \delta \) can be arbitrarily close to the number \( \frac{\beta}{2} \), we can check the following requirement instead of (3.22):

\[
\left(\frac{s}{\beta} + 1\right) \frac{m}{m - (m - 2)\lambda} < 2.
\]

Here, we mention that because of the presence of parameter \( \delta \) in (3.22), the requirement (3.23) is simpler than (3.22). Moreover, considering (3.23) will not affect our main result. Inserting \( \delta \) and \( \lambda \) into (3.23), we obtain the following restriction:

\[
p < \frac{4(1 - s)\beta + 2as}{(6 - \alpha - \frac{s}{m}) (1 - s)\beta + as(1 + \alpha + \eta - \beta)}.
\]

Putting all the restrictions (3.18), (3.20), and (3.24) on \( p \) gives

\[
\mathcal{P} < p < \mathcal{P}^*,
\]

where

\[
\mathcal{P} = \max \left\{ \frac{2m}{3m - 4}, \frac{2s}{2s + \alpha + \eta - \beta - 1} \right\}, \quad \mathcal{P}^* = \min \left\{ \frac{s}{\beta} + 1, \frac{2m}{(6 - \alpha - \frac{s}{m})(1 - s)\beta + as(1 + \alpha + \eta - \beta)} \right\}.
\]

It should be noted that under condition (3.6), the number \( p \) would work (see Appendix section for a detailed explanation). Estimate (3.21) ensures

\[
N_2 + N_3 \leq \frac{\hat{\mathcal{C}}}{4} \| G \|_{L^2} + C \left(1 + \| \Lambda^1 + \beta - \alpha \theta \|_{L^2}^2 + \| G \|_{H^\frac{\alpha - \eta}{2}}^2 \right) (1 + \| G \|_{H^\frac{\alpha - \eta}{2}}).
\]

By substituting estimates (3.8), (3.13), and (3.27) into (3.7), it leads to

\[
\frac{d}{dt} \| G(t) \|_{L^2} + \| G(t) \|_{H^\frac{\alpha - \eta}{2}} \leq C \left(1 + \| \Lambda^1 + \beta - \alpha \theta \|_{L^2}^2 + \| G \|_{H^\frac{\alpha - \eta}{2}}^2 \right) (1 + \| G \|_{H^\frac{\alpha - \eta}{2}}).
\]

Thanks to the estimates of Lemma 3.2, the combination of the inequality (3.28) with the Gronwall inequality thus leads to

\[
\| G(t) \|_{L^2} + \int_0^T \| G(t) \|_{H^\frac{\alpha - \eta}{2}} \, dt \leq C \Rightarrow \infty.
\]

Therefore, we conclude the proof of Lemma 3.4. \( \square \)

With bound (3.5) in hand, we are now in the position to derive the following lemmas (i.e., Lemmas 3.6 and 3.7), which play an important role in proving the main theorem and are also the main difference compared with the recent papers [20,23].

**Lemma 3.6**

Under the assumptions stated in Lemma 3.4, the vorticity \( \omega \) admits the following key global \( a \ priori \) bound:

\[
\int_0^T \| \omega(t) \|_{L^2} \, dt \leq C(T, \omega_0, \theta_0), \quad 2 \leq \bar{m} \leq \frac{2m}{2 - \alpha},
\]

where \( m \) is the same as in Lemma 3.4.
Proof of Lemma 3.6
Notice the fact $\beta > 1 - \alpha$, then there exist some $\rho > 1$ such that $\frac{\beta}{\rho} > 1 - \alpha$. Recalling the definition of $G$ and the bound (3.5), we have
\[
\|\omega\|_{L^4}^4 \leq \|G\|_{L^4}^4 + \|R_{\alpha} \theta\|_{L^4}^4
\leq C(t) + \|R_{\alpha} \theta\|_{L^4}^4,
\]
\[
\leq C(t) + \|R_{\alpha} \theta\|_{L^4}^4.
\]
(3.31)
The Littlewood–Paley technique and the estimate (2.6) allow us to show
\[
\|R_{\alpha} \theta\|_{L^4}^4 \leq \|\Delta_{-1} R_{\alpha} \theta\|_{L^4}^4 + \|\|I - \Delta_{-1}\|_\theta\|_{L^4}^4
\leq C\|\theta\|_{L^4}^4 + \|\|I - \Delta_{-1}\|_\theta\|_{L^4}^4
\leq C(t) + \|\theta\|_{L^4}^4 + \|\theta\|_{L^4}^4
\leq C(t) + C(t) + C(t).
\]
(3.32)
Combining (3.31) and (3.32) yields
\[
\|\omega\|_{L^4}^4 \leq C(t) + \|\omega\|_{L^4}^4,
\]
where constant $C$ is independent of $t$. Denoting $T_0 := (2C)^{-\frac{\alpha}{\beta}}$, one can conclude that for any $t \leq T_0$
\[
\|\omega\|_{L^4}^4 \leq 2C(t).
\]
Adopting the same argument, we obtain that for any $t \leq T_0$
\[
\|\omega\|_{L^4(t + T_0)} \leq 2C(T + T_0).
\]
By the same iteration, it ensures for any $t \leq T$
\[
\|\omega\|_{L^4}^4 \leq C(T).
\]
This completes the proof of Lemma 3.6.

On the basis of estimate (3.30), the next lemma is concerned with the global a priori bound $\int_0^T \|G(t)\|_{B_{\infty}^0}^4 \ d\tau$.

Lemma 3.7
Assume $(u_0, \theta_0)$ satisfies the assumptions stated in Theorem 1.1. Under the assumption $1 - \alpha < \beta < \min \left\{ 3 - 3\alpha, \frac{3\alpha + 4\alpha - 4}{8(1-\alpha)} \right\}$ with $\frac{10 - \sqrt{10}}{5} < \alpha < 1$, the quantity $G$ admits the following key global a priori bound:
\[
\int_0^T \|G(t)\|_{B_{\infty}^0}^4 \ d\tau \leq C(T, u_0, \theta_0).
\]
(3.34)
Proof of Lemma 3.7
Apply inhomogeneous blocks $\Delta_k (k \in \mathbb{N})$ operator to the combined equation (3.4) to obtain
\[
\partial_t \Delta_k G + (u \cdot \nabla) \Delta_k G + \Lambda^\alpha \Delta_k G = \Delta_k [R_{\alpha}, u \cdot \nabla] \theta - [\Delta_k, u \cdot \nabla] G + \Delta_k \Lambda^{1-\alpha} \partial_t \theta.
\]
(3.35)
For notational convenience, we denote
\[
f_k := \Delta_k [R_{\alpha}, u \cdot \nabla] \theta - [\Delta_k, u \cdot \nabla] G + \Delta_k \Lambda^{1-\alpha} \partial_t \theta.
\]
Multiplying equation (3.35) by $|\Delta_k G|^{-2} \Delta_k G$ and using the divergence-free condition, we can conclude that
\[
\frac{1}{r} \frac{d}{dt} \|\Delta_k G\|_{L^r}^r + \int_{\mathbb{R}^2} (\Lambda^\alpha \Delta_k G) |\Delta_k G|^{-2} \Delta_k G dx = \int_{\mathbb{R}^2} f_k |\Delta_k G|^{-2} \Delta_k G dx,
\]
where $2 \leq \eta \leq \frac{2m}{2-\alpha}$ is to be fixed hereafter. Thanks to (3.30), we have
\[
\int_0^T \|\omega(t)\|_{L^r}^r \ d\tau < \infty, \quad 2 \leq \eta \leq \frac{2m}{2-\alpha}.
\]
(3.37)
By the following lower bound [46]:
\[
\int_{\mathbb{R}^2} (\Lambda^\alpha \Delta_k G) |\Delta_k G|^{-2} \Delta_k G dx \geq c \omega_k |\Delta_k G|, \quad k \geq 0,
\]
(3.35)
for an absolute constant $c > 0$ independent of $k$, one arrives at
\[
\frac{1}{r} \frac{d}{dt} \| \Delta_k G \|_{L^r} + c 2^{\alpha k} \| \Delta_k G \|_{L^r} \leq C \| f_k \|_{L^r} \| \Delta_k G \|_{L^r}^{-1}.
\]
Consequently, making use of the Gronwall inequality to the aforementioned inequality leads to
\[
\| \Delta_k G(t) \|_{L^r} \leq C e^{-c 2^{\alpha k} t} \| \Delta_k G_0 \|_{L^r} + C \int_0^t e^{-c 2^{\alpha k} (t-\tau)} \| f_k(\tau) \|_{L^r} \, d\tau.
\]
(3.38)

Integrating over time variable and using the convolution Young inequality yield
\[
\int_0^t \| \Delta_k G(\tau) \|_{L^r} \, d\tau \leq C 2^{-\alpha k} \| \Delta_k G_0 \|_{L^r} + C 2^{-\alpha k} \int_0^t \| f_k(\tau) \|_{L^r} \, d\tau
\]
\[
\leq C 2^{-\alpha k} \| \Delta_k G_0 \|_{L^r} + C 2^{-\alpha k} (J_1 + J_2 + J_3),
\]
where
\[
J_1 = \int_0^t \| \Delta_k [R_{\alpha', u \cdot \nabla}] \theta(\tau) \|_{L^r} \, d\tau, \quad J_2 = \int_0^t \| [\Delta_k, u \cdot \nabla] G(\tau) \|_{L^r} \, d\tau,
\]
\[
J_3 = \int_0^t \| \Delta_k \Lambda^{\beta - \alpha} \partial_\alpha \theta(\tau) \|_{L^r} \, d\tau.
\]

According to the estimate (2.4) with $s = \alpha - 1$, we immediately obtain
\[
J_1 \leq 2^{-(\alpha-1) k} \int_0^t 2^{(\alpha-1) k} \| \Delta_k [R_{\alpha', u \cdot \nabla}] \theta(\tau) \|_{L^r} \, d\tau
\]
\[
\leq C 2^{-(\alpha-1) k} \int_0^t \| \nabla u(\tau) \|_{L^r} \| \theta_0 \|_{L^r} + \| \theta_0 \|_{L^\infty} \, d\tau
\]
\[
\leq C 2^{-(\alpha-1) k} \int_0^t \| \omega(\tau) \|_{L^r} \, d\tau
\]
\[
\leq C(t) 2^{-(\alpha-1) k},
\]
where in the last line, we have used estimate (3.37). By means of the commutator estimate (2.5), we find
\[
J_2 \leq 2^{-(\alpha-1) k} \int_0^t 2^{(\alpha-1) k} \| [\Delta_k, u \cdot \nabla] G(\tau) \|_{L^r} \, d\tau
\]
\[
\leq C 2^{-(\alpha-1) k} \int_0^t \| \nabla u \|_{H^{-1}} \| \theta \|_{H^{1}} \| G \|_{H^{\frac{1}{2}}} \, d\tau
\]
\[
\leq C 2^{-(\alpha-1) k} \int_0^t \| \| \theta \|_{H^{1}} \| u \|_{L^r} + \| \theta \|_{L^\infty} \| G \|_{H^{\frac{1}{2}}} \, d\tau
\]
\[
\leq C 2^{-(\alpha-1) k} \int_0^t \| \| G \|_{H^{\frac{1}{2}}} \, d\tau,
\]
where we have used the following estimate:
\[
\| G \|_{H^{-1}} \leq C \| G \|_{L^\infty} < \infty, \quad \frac{2}{m} \leq 1 - \alpha + \frac{2}{r}.
\]
Finally, by estimate (2.6) and estimate (3.37),
\[
J_3 \leq \int_0^t \| \Delta_k \Lambda^{\beta - \alpha} \partial_\alpha \theta(\tau) \|_{L^r} \, d\tau
\]
\[
\leq C 2^{-(\alpha-1) k} \int_0^t 2^{\beta k} \| \Delta_k \theta(\tau) \|_{L^r} \, d\tau
\]
\[
\leq C 2^{-(\alpha-1) k} (\| \theta_0 \|_{L^r} + \| \theta_0 \|_{L^\infty} \| \theta \|_{L^r})
\]
\[
\leq C(t) 2^{-(\alpha-1) k}.
\]
Putting all the aforementioned estimates \( J_1, J_2, \) and \( J_3 \) together, one obtains
\[
\int_0^t \| \Delta_x G(t) \|_{L^r} \, dt \leq C 2^{-\alpha k} \| \Delta_x G_0 \|_{L^r} + C(t) 2^{-(2\alpha - 1)k} \\
+ C(t) 2^{-(2\alpha - 1)k} \int_0^t \| G \|_{\dot{B}^\frac{\alpha}{r}} \, dt.
\]  
(3.42)

In view of the definition of the Besov space, we deduce that
\[
\| G \|_{\dot{B}^\frac{\alpha}{r}} \leq \sum_{k < N_0} 2^\frac{k}{r} \| \Delta_x G \|_{L^r} + \sum_{k \geq N_0} 2^\frac{k}{r} \| \Delta_x G \|_{L^r} \\
\leq C 2^\frac{k}{N_0} \| G \|_{L^r} + C(t) \sum_{k \geq N_0} 2^{-(\alpha - \frac{1}{r})k} \| \Delta_x G_0 \|_{L^r} + C(t) \sum_{k \geq N_0} 2^{-(2\alpha - 1 - \frac{1}{r})k} \\
+ C(t) \sum_{k \geq N_0} 2^{-(2\alpha - 1 - \frac{1}{r})k} \| G \|_{\dot{B}^\frac{\alpha}{r}} \\
\leq C 2^\frac{k}{N_0} \| G \|_{L^r} + C(t) + C(t) 2^{-(2\alpha - 1 - \frac{1}{r})N_0} \| G \|_{\dot{B}^\frac{\alpha}{r}},
\]  
(3.43)

where we have applied the following restriction:
\[
2\alpha - \frac{2}{r} > 0.
\]  
(3.44)

Choosing \( N_0 \) as
\[
\frac{1}{4} \leq C(t) 2^{-(2\alpha - 1 - \frac{1}{r})N_0} \leq \frac{1}{2},
\]
we conclude
\[
\| G \|_{\dot{B}^\frac{\alpha}{r}} \leq C(t) < \infty.
\]

By the embedding theorem, we arrive at
\[
\| G \|_{L^r} \leq C \| G \|_{\dot{B}^\frac{\alpha}{r}} \leq C(t) < \infty.
\]

Finally, let us check that the numbers \( r \) and \( m \) can be fixed. Combining all the requirement on the number \( r \), we have
\[
2\alpha - \frac{2}{r} > 0 \Rightarrow \frac{2}{r} < 2\alpha - 1, \quad \frac{2m}{2} \leq 1 - \frac{2}{r} \Rightarrow \frac{2m}{2} + \alpha - 1 \leq \frac{2}{r}, \\
2 \leq r \leq \frac{2m}{2 - \alpha} \Rightarrow \frac{2 - \alpha}{m} \leq \frac{2}{r} \leq 1.
\]

Therefore, it gives rise to
\[
\max \left\{ \frac{2}{m} + \alpha - 1, \frac{2 - \alpha}{m} \right\} \leq \frac{2}{r} < \min \left\{ 2\alpha - 1, 1 \right\} = 2\alpha - 1,
\]
which would work as long as
\[
m > \max \left\{ \frac{2}{\alpha}, \frac{2 - \alpha}{2\alpha - 1} \right\} = \frac{2}{\alpha}, \quad \left( \alpha > \sqrt{3} - 1 \approx 0.7321 \right).
\]  
(3.45)

Recall condition (3.6), namely,
\[
\left\{ \begin{array}{l}
\frac{2}{\alpha - \beta} < m < \min \left\{ 4, \frac{1}{1 - \alpha} \right\}, \\
(2(2 - \alpha) - 3\alpha + 2)m < 4(2 - \alpha)\beta, \\
(4 + 8\beta - 4\alpha - 3\alpha^2)m < 16\beta.
\end{array} \right.
\]  
(3.46)

Noticing \( \alpha > \frac{3}{4} \), we substitute the number \( m = \frac{2}{\alpha} \) into (3.46) to obtain
\[
1 - \alpha < \beta < \min \left\{ 3 - 3\alpha, \frac{\alpha}{2}, \frac{3\alpha - 2}{2(2 - \alpha)(1 - \alpha)}, \frac{3\alpha^2 - 4\alpha - 4}{8(1 - \alpha)} \right\}.
\]  
(3.47)

Thanks to the following simple fact:
\[
\frac{3\alpha - 2}{2(2 - \alpha)(1 - \alpha)} > \frac{3\alpha^2 + 4\alpha - 4}{8(1 - \alpha)}, \quad \text{for any} \quad \alpha > \frac{2}{3},
\]
the restriction (3.47) reduces to
\[
1 - \alpha < \beta < \min \left\{ 3 - 3\alpha, \frac{\alpha}{2}, \frac{3\alpha^2 + 4\alpha - 4}{8(1 - \alpha)} \right\}.
\]
It is not difficult to check that the assumption for $\beta$ will work as long as
\[
1 - \alpha < \frac{3\alpha^2 + 4\alpha - 4}{8(1 - \alpha)} \implies \alpha > \frac{10 - 2\sqrt{10}}{5} \approx 0.7351.
\]

Here, it is worth particularly mentioning that this is the only place where in the proof we need the key assumption $\alpha > \frac{10 - 2\sqrt{10}}{5}$. If inequality (3.46) holds true when $m = \frac{2}{\alpha}$, then one may take $m = \frac{2}{\alpha} + \epsilon$ for some sufficiently small $\epsilon$ ($\epsilon > 0$ may depend on $\alpha$ and $\beta$) such that both inequalities (3.46) and (3.45) are fulfilled. Such a choice of $\epsilon > 0$ is possible because both inequalities (3.46) and (3.45) are strict. Consequently, this completes the proof of Lemma 3.7.

Finally, we would like to establish the following global \textit{a priori} bound $\int_0^T \|\omega(t)\|_{\theta_{\alpha,1}}^2 \, dt$.

\textbf{Lemma 3.8}

Under the assumptions stated in Lemma 3.7, the vorticity $\omega$ admits the following key global \textit{a priori} bound:
\[
\int_0^T \|\omega(t)\|_{\theta_{\alpha,1}}^2 \, dt \leq C(T, u_0, \theta_0).
\]  

\textbf{Proof of Lemma 3.8}

Using the Bernstein inequality and choosing $r_0 \in \left( \frac{2}{\alpha + \beta - 1}, \infty \right)$, it is clear that
\[
\|\omega\|_{L^1(T)} \leq \|G\|_{L^\infty} + \|R_\alpha \theta\|_{L^\infty} \leq C(t) + \|\Delta_{\beta-1} R_\alpha \theta\|_{L^\infty} + \|\Delta_{\beta-1} \Delta_\theta\|_{L^\infty} \leq C(t) \leq C(t) + \sup_{k \in \mathbb{N}} 2^{\frac{k}{2}} \|\Delta_{\beta} \theta\|_{L^1(T)}.
\]

where we have used the following fact:
\[
\|\Delta_{\beta-1} R_\alpha \theta\|_{L^\infty} \leq C \|\Delta_{\beta-1} L^{-\alpha} \theta\|_{L^\infty} \leq C \|\theta\|_{L^\infty} \leq C(t).
\]

Again, estimate (2.6) ensures
\[
\sup_{k \in \mathbb{N}} 2^{\frac{k}{2}} \|\Delta_{\beta} \theta\|_{L^1(T)} \leq C \|\theta_0\|_{L^\infty} + \|\Delta_{\beta} \theta\|_{L^\infty} \|\omega\|_{L^1(T)}.
\]

By the estimate (3.49), we obtain
\[
\|\omega\|_{L^1(T)} \leq \|G\|_{L^\infty} + \|R_\alpha \theta\|_{L^\infty} \leq \|G\|_{L^\infty} + \|R_\alpha \theta\|_{L^\infty} \leq C(t) + \sup_{k \in \mathbb{N}} 2^{\frac{k}{2}} \|\Delta_{\beta} \theta\|_{L^1(T)} \leq C(t) + \sup_{k \in \mathbb{N}} 2^{\frac{k}{2}} \|\Delta_{\beta} \theta\|_{L^1(T)}.
\]

An argument similar to that used in the proof of (3.32) yields
\[
\|R_\alpha \theta\|_{L^1(T)} \leq \|\Delta_{\beta-1} R_\alpha \theta\|_{L^\infty} + \|\Delta_{\beta-1} \Delta_\theta\|_{L^\infty} \leq C(t) + \sup_{k \in \mathbb{N}} 2^{\frac{k}{2}} \|\Delta_{\beta} \theta\|_{L^1(T)} \leq C(t) + \sup_{k \in \mathbb{N}} 2^{\frac{k}{2}} \|\Delta_{\beta} \theta\|_{L^1(T)}.
\]

By the iterative process as used in proving Lemma 3.6, we thus obtain
\[
\|\omega\|_{L^1(T)} \leq C(t) < \infty,
\]

which along with (3.50) guarantees that
\[
\sup_{k \in \mathbb{N}} 2^{\frac{k}{2}} \|\Delta_{\beta} \theta\|_{L^1(T)} \leq C(t) < \infty.
\]

Thus, we conclude the desired bound (3.48). This ends the proof of Lemma 3.8.

Now, we are ready to prove Theorem 1.1.
Proof of Theorem 1.1
Bearing in mind the bound (3.48) and the Littlewood–Paley technique, it is clear that
\[
\int_0^T \|\nabla u(t)\|_{L^\infty}^q \, dt \leq C \int_0^T (\|u(t)\|_{L^2} + \|\omega(t)\|_{H^3}) \, dt < \infty.
\] (3.53)

From the temperature \( \theta \) equation, we can immediately obtain for any \( q \in [1, \infty) \)
\[
\frac{1}{q} \frac{d}{dt} \|\nabla \theta\|_{L^q}^q \leq \|\nabla u\|_{L^\infty} \|\nabla \theta\|_{L^q} \Rightarrow \frac{d}{dt} \|\nabla \theta\|_{L^q} \leq \frac{\|\nabla \theta\|_{L^q}}{\|\nabla u\|_{L^\infty}}.
\]

By integrating the later differential inequality over \((0, t)\), we have
\[
\|\nabla \theta\|_{L^q} \leq \|\nabla \theta_0\|_{L^q} \exp \left[ \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right].
\]

Setting \( q \rightarrow \infty \), we thus show that
\[
\|\nabla \theta\|_{L^\infty} \leq C \|\nabla \theta_0\|_{L^\infty} \exp \left[ \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right] < \infty.
\] (3.54)

Through the standard energy estimates and the Kato–Ponce inequality, we can conclude that
\[
\mathcal{X}(t) := \|u(t)\|_{H^{2/3}}^2 + \|\theta(t)\|_{H^{1/3}}^2
\]
obey the following differential inequality:
\[
\frac{d}{dt} \mathcal{X}(t) + \|u\|_{H^{2/3}}^2 + \|\theta\|_{H^{1/3}}^2 \leq C (1 + \|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^\infty}) \mathcal{X}(t).
\] (3.55)

Consequently, noticing the two estimates (3.53)-(3.54) and applying the classical Gronwall inequality to (3.55) imply the desired estimates of Theorem 1.1. The continuity of \( u \) and \( \theta \) in time, namely, \( u \in C([0, T]; H^s(\mathbb{R}^2)) \) and \( \theta \in C([0, T]; H^s(\mathbb{R}^2)) \) can be obtained by a standard approach (e.g., [3, 22, 39]); thus, the details are omitted. Moreover, the uniqueness is clear because the velocity and the temperature are both in Lipschitz spaces. As a result, we finish the proof of Theorem 1.1. \( \square \)

Appendix A. Details about a choice of \( p \)

In this appendix, we will give the details that a choice of \( p \) is possible. As a matter of fact, \( p \) would work as long as all the following conditions hold:

\[
\frac{2m}{3m-4} \begin{cases} 
< \frac{2}{\alpha + \eta - \beta + s} & \Rightarrow s < \frac{(3 + \beta - \alpha - \eta)m - 4}{m}, \quad (A1) \\
\left(\frac{6 - \alpha - \frac{8}{m}}{m}(1-s)\right)^\beta + 2as \end{cases}
\]

\[
\frac{2m}{3m-4} \begin{cases} 
< \frac{2m}{2s + \alpha + \eta - \beta - 1} < \frac{2m}{2s + \alpha + \eta - \beta} & \Rightarrow s > \frac{1 + \beta - \alpha - \eta}{\alpha(\alpha + \eta - 1) + (\frac{8}{m} - 2)\beta}, \quad (A4)
\end{cases}
\]

Here, we say some words about (A5). By direct computation, we obtain
\[
\frac{2s}{2s + \alpha + \eta - \beta} \leq \left(\frac{8}{m} - 2\right)\beta + \alpha(\alpha + \eta - 1) + 2s \left(\frac{8}{m} - 2\right)\beta + \alpha(\alpha + \eta - 1) - 2\beta(\alpha + \eta - 1) < 0
\]
\[
\Leftrightarrow (s - 1) \left( s - \frac{2(1 + \beta - \alpha - \eta)\beta}{\alpha(\alpha + \eta - 1) + (\frac{8}{m} - 2)\beta} \right) < 0.
\]

\[ \tag{A5} \]
Thanks to $s < 1$, we deduce
\[ s > \frac{2(1 + \beta - \alpha - \eta)\beta}{\alpha(\alpha + \eta - 1) + \left(\frac{\eta}{m} - 2\right)\beta}. \]

Noting estimates (A1)–(A5) and $s < \beta$, if the following requirement is true, then the number $s$ would work
\[ \mathcal{S} < s < \bar{\mathcal{S}}, \tag{A6} \]
where
\[ \mathcal{S} = \max \left\{ \frac{(1 + \beta - \alpha - \eta)m}{m - (2 - \alpha)(m - 2)}, \frac{2(1 + \beta - \alpha - \eta)\beta}{\alpha(\alpha + \eta - 1) + \left(\frac{\eta}{m} - 2\right)\beta} \right\}, \]
\[ \bar{\mathcal{S}} = \min \left\{ \beta, \frac{(3 + \beta - \alpha - \eta)m - 4}{m}, \frac{\beta m}{(\alpha + \eta - 2)m + 4} \right\}. \]

We now further assume $\eta$ satisfying
\[ \eta < 3 - \alpha - \frac{s}{m}, \]
which implies
\[ \frac{(3 + \beta - \alpha - \eta)m - 4}{m} > \beta, \quad \frac{\beta m}{(\alpha + \eta - 2)m + 4} > \beta. \]

We thus obtain
\[ \bar{\mathcal{S}} = \beta \]
and
\[ \beta < \eta < \min \left\{ 1 + \beta - \alpha, \frac{\alpha}{2}, 3 - \alpha - \frac{4}{m} \right\}. \]

Direct computations yield
\[ \frac{(1 + \beta - \alpha - \eta)m}{m - (2 - \alpha)(m - 2)} < \beta \Rightarrow \eta > \frac{(1 + \beta - \alpha)m - [2(2 - \alpha) - (1 - \alpha)m]\beta}{m}, \tag{A7} \]
\[ \frac{2(1 + \beta - \alpha - \eta)\beta}{\alpha(\alpha + \eta - 1) + \left(\frac{\eta}{m} - 2\right)\beta} < \beta \Rightarrow \eta > \frac{2(1 + \beta - \alpha) + \alpha(1 - \alpha) - \left(\frac{\eta}{m} - 2\right)\beta}{2 + \alpha}. \tag{A8} \]

Combining all the restriction on $\eta$, it is obvious to find that $\eta$ should satisfy
\[ \underline{\mathcal{A}} < \eta < \bar{\mathcal{A}}, \]
where $\underline{\mathcal{A}}$ and $\bar{\mathcal{A}}$ are given by
\[ \underline{\mathcal{A}} = \max \left\{ \beta, \frac{(1 + \beta - \alpha)m - [2(2 - \alpha) - (1 - \alpha)m]\beta}{m}, \right. \]
\[ \left. \frac{2(1 + \beta - \alpha) + \alpha(1 - \alpha) - \left(\frac{\eta}{m} - 2\right)\beta}{2 + \alpha} \right\}, \]
\[ \bar{\mathcal{A}} = \min \left\{ 1 + \beta - \alpha, \frac{\alpha}{2}, 3 - \alpha - \frac{4}{m} \right\}. \]

Then, it is easy to check that $\eta$ would work provided that
\[ \beta < 3 - \alpha - \frac{s}{m} \Rightarrow m > \frac{4}{3 - \alpha - \beta}, \]
\[ \frac{(1 + \beta - \alpha)m - [2(2 - \alpha) - (1 - \alpha)m]\beta}{m} < 1 + \beta - \alpha \Rightarrow \eta < \frac{2(2 - \alpha)}{1 - \alpha}, \]
\[ \frac{2(1 + \beta - \alpha) + \alpha(1 - \alpha) - \left(\frac{\eta}{m} - 2\right)\beta}{2 + \alpha} < 1 + \beta - \alpha \Rightarrow m < \frac{8}{2 - \alpha}. \]
\[ \frac{(1 + \beta - \alpha)m - [2(2 - \alpha) - (1 - \alpha)m]\beta}{m} < \frac{\alpha}{2} \Rightarrow (2(2 - \alpha)\beta - 3\alpha + 2)m < 4(2 - \alpha)\beta, \]
\[ \frac{2(1 + \beta - \alpha) + \alpha(1 - \alpha) - \left(\frac{\eta}{m} - 2\right)\beta}{2 + \alpha} < \frac{\alpha}{2} \Rightarrow (4 + 8\beta - 4\alpha - 3\alpha^2)m < 16\beta, \]
\[ \frac{(1 + \beta - \alpha)m - [2(2 - \alpha) - (1 - \alpha)m]\beta}{m} < 3 - \alpha - \frac{s}{m} \Rightarrow m > 2, \]
\[ \frac{2(1 + \beta - \alpha) + \alpha(1 - \alpha) - \left(\frac{\eta}{m} - 2\right)\beta}{2 + \alpha} < 3 - \alpha - \frac{4}{m} \Rightarrow m > 2. \]
Recall the following restriction:

\[ m \leq \frac{2}{2 - 2\alpha + \delta}, \quad \delta > 0, \]

and notice the fact that \( \delta > 0 \) can be arbitrarily small, then the \( m \) only needs to meet the following condition:

\[ m < \frac{1}{1 - \alpha}. \]

Finally, we obtain that \( m \) should satisfy

\[
\begin{align*}
\frac{4}{4 - \alpha - \beta} &< m < \min\{4, \frac{1}{1 - \alpha}\}, \\
(2(2 - \alpha)\beta - 3\alpha + 2)m &< 4(2 - \alpha)\beta, \\
(4 + 8\beta - 4\alpha - 3\alpha^2)m &< 16\beta.
\end{align*}
\]

(A9)

Therefore, under restriction (A9) on \( m \), a choice of \( p \) is possible.

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