DISTINCTION INSIDE NON-GENERIC L-PACKETS OF $SL(n)$

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Abstract. We extend the local results of [AP03, AP18] and the global results of [AP06, AM19] from the generic to the non-generic setting. In the local setting, using the results of [Mat14], we prove that if $E/F$ is a quadratic extension $p$-adic fields, then the $SL_n(F)$-distinguished representations inside a distinguished unitary L-packet of $SL_n(E)$ are precisely those admitting a degenerate Whittaker model with respect to a degenerate character of $N(E)/N(F)$, where $N$ is the subgroup of unipotent upper triangular matrices. For a quadratic extension $E/F$ of number fields, we prove the global analogue of this result for square-integrable L-packets of $SL_n(A_E)$, using the results of [Yam15] together with an unfolding argument. Finally when $E/F$ splits at infinity, we prove a local-global principle for $SL_{dr}(A_F)$-distinction inside square-integrable distinguished L-packets of $SL_{dr}(A_E)$ built from a cuspidal representation of $GL_r(A_E)$ with $r$ odd, similar to that proved in [AP13] for cuspidal L-packets of $SL_2(A_E)$.

1. Introduction

This work fits in the study of local distinction and periods of automorphic forms, with respect to Galois pairs of reductive groups. It is motivated by earlier works, namely [AP03, AP18] in the local context, and [AP06, AP13, AM19] in the global context, which investigated distinction in the presence of L-packets.

Let $E/F$ be a quadratic extension of $p$-adic fields with Galois involution $\theta$. In [AP03, AP18], it was proved, amongst many other results, that if $\pi$ is a generic $SL_n(F)$-distinguished representation of $SL_n(E)$, then the distinguished members of the L-packet of $\pi$ are the representations which are $\psi$-generic with respect to some non-degenerate character $\psi$ satisfying $\psi^\theta = \psi^{-1}$. Such a relationship between distinction and genericity is expected more generally [Pra15]; indeed, if $\psi$ is a non-degenerate character such that $\psi^\theta = \psi^{-1}$, according to [Pra15, Conjecture 2, (2)], for any quasi-split Galois pair, $\psi$-generic members of a distinguished L-packet are distinguished.

In this paper, we first prove a generalization of this result for unitary L-packets of $SL_n(E)$ and degenerate Whittaker models. Namely, we prove in Theorem 3.6 that if $\tilde{\pi}$ is a unitary representation of $GL_n(E)$ admitting a degenerate Whittaker model of type $(n_1, \ldots, n_d)$, where $(n_1, \ldots, n_d)$ is a partition of $n$, and if the L-packet associated to $\tilde{\pi}$ contains a representation distinguished by $SL_n(F)$, then its distinguished members are those which admit a $\psi$-degenerate Whittaker model for $\psi$ of type $(n_1, \ldots, n_d)$ satisfying $\psi^\theta = \psi^{-1}$. Our proof builds on the work of the second named author [Mat14], which classified unitary representations of $GL_n(E)$ distinguished with respect to $GL_n(F)$, making use of which we can adapt the techniques of [AP03] and

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to the unitary context. Such a result hints at the possibility of a generalization of Prasad’s prediction [Pra15], relating distinction for Galois pairs inside distinguished generic L-packets, to non-generic L-packets.

In order to state our global results, let $E/F$ be a quadratic extension of number fields with Galois involution $\theta$. It was proved in [AP06] for $\text{SL}_2(\mathbb{A}_E)$, and in [AM19] for $\text{SL}_n(\mathbb{A}_E)$, that the distinguished members in an $\text{SL}_n(\mathbb{A}_E)$-distinguished L-packet of a cuspidal automorphic representation of $\text{SL}_n(\mathbb{A}_E)$ are those on which some $\psi$-Whittaker period does not vanish, for $\psi$ a non-degenerate character such that $\psi^\theta = \psi^{-1}$. We generalize this result to the residual spectrum. Thus, if $\tilde{\pi}$ is a square-integrable automorphic representation of $\text{GL}_{dr}(\mathbb{A}_E)$ of the form $\text{Sp}(d,\sigma)$, where $\sigma$ is a cuspidal representation of $\text{GL}_r(\mathbb{A}_E)$, then we prove in Theorem 5.7 that if the L-packet determined by $\tilde{\pi}$ contains an $\text{SL}_{dr}(\mathbb{A}_F)$-distinguished representation, then the $\text{SL}_{dr}(\mathbb{A}_F)$-distinguished members of the L-packet are those which admit a degenerate Whittaker model for a degenerate character $\psi$ of type $(r_1,\ldots,r_d)$ satisfying $\psi^\theta = \psi^{-1}$. The key ingredient in the proof is the work of Yamana [Yam15], which is the global counterpart of [Mat14], which facilitates adapting the unfolding argument of [AM19] to the square-integrable context.

As an application of our global result, we establish a local global principle for square-integrable representations for $(\text{SL}_n(\mathbb{A}_E), \text{SL}_n(\mathbb{A}_F))$. For this, we assume that $E/F$ is split at the archimedean places and that $\tilde{\pi}$ as above is such that $r$ is odd. Suppose $\tilde{\pi}$ contains an $\text{SL}_{dr}(\mathbb{A}_F)$-distinguished representation in its restriction to $\text{SL}_{dr}(\mathbb{A}_E)$. We prove in Theorem 6.3 that inside the L-packet determined by $\tilde{\pi}$, the distinguished members are exactly those which are locally distinguished. This result was proved in [AP06] for cuspidal representations of $\text{SL}_2(\mathbb{A}_E)$ by quite involved arguments. In contrast, our proof is reasonably elementary, making use of the assumption that $r$ is odd.

Finally we mention that along the way we give proofs of some elementary, and probably standard, facts on archimedean and global L-packets of $\text{SL}_n$ for which we could not find accessible sources in the literature. They follow from [AGS15] in the archimedean setting, and from [JL13] in the global setting.

2. Notations

We denote by $\delta_G$ the character of a locally compact group $G$ such that $\delta_G \lambda$ is a right invariant Haar measure on $G$ if $\lambda$ is a left invariant Haar measure on $G$. We denote by $\mathcal{M}_{a,b}$ the algebraic group of $a \times b$ matrices. We denote by $G_n$ the algebraic group $\text{GL}_n$, by $T_n$ its diagonal torus and by $N_n$ the group of upper triangular matrices in $G_n$. We set

$$U_n = \{u_n(x) = \begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix}, x \in (\mathbb{A}^1)^{n-1} \} < N_n$$

where $\mathbb{A}^1$ denotes the affine line. We embed $G_{n-1}$ inside $G_n$ via $g \mapsto \text{diag}(g,1)$ and set $P_n = G_{n-1}U_n$ the mirabolic subgroup of $G_n$. We denote by $N_{n,r}$ the group of matrices

$$k(a,x,u) = \begin{pmatrix} a & x \\ 0 & u \end{pmatrix}$$
with \( a \in G_{n-r}, x \in M_{n-r,r} \) and \( u \in N_r \). We denote by \( U_{n,r} \) the unipotent radical of \( N_{n,r} \) which consists of the matrices \( k(I_{n-k}, x, u) \). Note that \( N_{n,n} = N_n \). For a subgroup \( H \) of \( G_n \), we denote by \( H^\circ \) the intersection of \( H \) with \( SL_n \).

### 3. Non-archimedean theory

Let \( E/F \) be a quadratic extension of \( p \)-adic fields with Galois involution \( \theta \). We denote by \( | | \) and \( | |_F \) the respective normalized absolute values. In this section, by abuse of notation, we set \( G = G(E) \) for any algebraic group defined over \( E \). We denote by \( v_E \) (or \( v \)), the character \( | |_E \circ \det \) of \( G_n \). We fix a non-trivial character \( \psi_0 \) of \( E \) which is trivial on \( F \).

3.1. The type of an irreducible \( GL \)-representation via derivatives. If \( \psi \) is a non-degenerate (smooth complex) character of \( N_n \), we denote by \( \psi^k \) its restriction to \( U_k \) for \( k \leq n \). We denote by \( \text{Rep}(\bullet) \) the category of smooth complex representations of \( \bullet \).

In [BZ76] and [BZ77], Bernstein and Zelevinsky have introduced the functors

\[
\Phi_{\psi^n} : \text{Rep}(P_n) \to \text{Rep}(P_{n-1})
\]

and

\[
\Psi^- : \text{Rep}(P_n) \to \text{Rep}(G_{n-1}).
\]

For \( (\tau, V) \in \text{Rep}(P_n) \), one has

\[
\Phi_{\psi^n}(V) = V/V(U_n, \psi_n)
\]

where \( V(U_n, \psi_n) \) is the space spanned by the differences \( \tau(u)v - \psi_n(u)v \) for \( u \in U_n \) and \( v \in V \), but the action of \( P_{n-1} \) on \( \Phi_{\psi^n}(V) \) is normalized by twisting by \( \delta_{P_n}^{-1/2} \).

Similarly

\[
\Psi^-(V) = V/V(U_n, 1)
\]

where the action of \( G_{n-1} \) on \( \Psi^-(V) \) is normalized by twisting by \( \delta_{P_n}^{-1/2} \) again.

The functor \( \Phi_{\psi^n} \) does not in fact depend on \( \psi \) in the sense that for \( \tau \in \text{Rep}(P_n) \) one has \( \Phi_{\psi^n}(\tau) \simeq \Phi_{\psi'}(\tau) \) whenever \( \psi \) and \( \psi' \) are non-degenerate characters of \( N_n \). Hence we simply write \( \Phi^-(\tau) \) for it. For \( \tau \in \text{Rep}(P_n) \), we set

\[
\tau(k) = (\Phi^-)^{k-1}(\tau) \in \text{Rep}(P_{n+1-k}),
\]

and

\[
\tau(k) = \Psi^-(\Phi^-)^{k-1}(\tau) \in \text{Rep}(G_{n-k}),
\]

which is called the \( k \)-th twisted derivative of \( \tau \). The \( k \)-th shifted derivative of \( \tau \) is given by

\[
\tau[k] = v^{1/2} \tau(k).
\]

Note that these definitions apply when \( \tau \) is a representation of \( G_n \) which we consider as a representation of \( P_n \) by restriction.

Let \( \tilde{\pi} \) be an irreducible smooth representation of \( G_n \). We denote by \( \tilde{\pi}^{[n_1]} \) its highest (non-zero) twisted derivative, by \( \tilde{\pi}^{[n_1,n_2]} := (\tilde{\pi}^{[n_1]})^{[n_2]} \) the highest twisted derivative of \( \tilde{\pi}^{[n_1]} \), and so on. All the representations \( \tilde{\pi}^{[n_1,n_2,\ldots,n_d]} \) are irreducible thanks to [Zel80] Theorem 8.1. This defines a finite sequence of positive integers \( (n_1, \ldots, n_d) \) such that \( n_1 + \cdots + n_d = n \). In fact, [Zel80] Theorem 8.1 implies that this sequence is a partition of \( n \), i.e., \( n_1 \geq n_2 \geq \cdots \geq n_d \). We call \( (n_1, \ldots, n_d) \) the partition associated to \( \tilde{\pi} \). We will
also say that \( \widetilde{\pi} \) is of type \((n_1, \ldots, n_d)\). Note that by [Ber84, Section 7.4], if \( \widetilde{\pi} \) is unitary, then all the representations \( \widetilde{\pi}^{[n_1, n_2, \ldots, n_d]} \) are unitary as well.

**Example 1.** Using the product notation for normalized parabolic induction, if \( \delta \) is an essentially square-integrable representation of \( G_r \) we set

\[
\text{Sp}(d, \delta) = \text{LQ}(|\delta|_{E}^{(d-1)/2} \times \cdots \times |\delta|_{E}^{(1-d)/2})
\]

to be the Langlands quotient of the parabolically induced representation

\[
|\delta|_{E}^{(d-1)/2} \times \cdots \times |\delta|_{E}^{(1-d)/2}.
\]

More generally, if \( \tau = \delta_1 \times \cdots \times \delta_t \) is a generic unitary representation of \( G_r \) written as a commutative product of essentially square-integrable representations ([Zel80, Theorem 9.7]), we set

\[
\text{Sp}(d, \tau) = \text{Sp}(d, \delta_1) \times \cdots \times \text{Sp}(d, \delta_t)
\]

which is a commutative product by the results of Tadic ([Iad86, Theorem D]). In this situation [BZ77, 4.5, Lemma] together with the computation of the highest derivative of Speh representations ([OS08, §3.5 (3.3)] and [Iad87, §6.1]) imply that the partition of \( n = rd \) associated to \( \text{Sp}(d, \tau) \) is \( r^d := (r, \ldots, r) \). Conversely one can check using the same results, that an irreducible unitary representation of \( G_n \) of type \( r^d \) is of the form \( \text{Sp}(d, \tau) \) for a unitary generic representation \( \tau \) of \( G_r \). We refer to Section 4.2 for the details in the archimedean setting, which are the same as in the non-archimedean setting.

### 3.2. Degenerate Whittaker models and L-packets

Let \( \psi_{n_i} \) be a non-degenerate character of the group \( N_{n_i} \). By [Zel80, Section 8], if the representation \( \widetilde{\pi} \) is of type \((n_1, \ldots, n_d)\), then it has a unique degenerate Whittaker model with respect to

\[
(\psi_{n_1} \otimes \cdots \otimes \psi_{n_d})(u_d \ldots 1) \cdots (\psi_{n_1} \otimes \cdots \otimes \psi_{n_d})(u_d)
\]

for \( u_i \in N_{n_i} \). We often use the notation

\[
\psi_{n_1, \ldots, n_d} := \psi_{n_1} \otimes \cdots \otimes \psi_{n_d},
\]

it has the advantage of being short but could mislead the reader so we insist on the fact that \( \psi_{n_1, \ldots, n_d} \) depends on the characters \( \psi_{n_i} \) and not only on the positive integers \( n_i \). We will say that \( \psi_{n_1, \ldots, n_d} \) is of type \((n_1, \ldots, n_d)\). If all the \( n_i \) are equal then we set

\[
\psi_{1, \ldots, d} := \psi_{n_1, \ldots, n_d}.
\]

The L-package associated to \( \widetilde{\pi} \) is the finite set of irreducible representations of \( G_n = \text{SL}_n(E) \) appearing in the restriction of \( \widetilde{\pi} \), and is denoted by \( L(\pi) \). We refer to [HS12, Section 2] for its basic properties that we now state (see also [GK82] or [Iad92]). Any irreducible representation \( \pi \) of \( G_n \) arises in the restriction of an irreducible representation of \( G_n \) and two irreducible representations of \( G_n \) containing \( \pi \) are twists of each other by a character. Hence it makes sense to set \( L(\pi) = L(\widetilde{\pi}) \). We define the type of \( \pi \) (or the type...
of $L(\pi))$ to be that of $\pi$. Of course two irreducible representations of $G_n$ determining the same L-packet have the same type.

Clearly the group $\text{diag}(E^\times, I_{n-1})$ acts transitively on $L(\pi)$ and the existence of a degenerate Whittaker model for irreducible representations of $G_n$ then has the following immediate consequence.

**Lemma 3.1.** Let $\pi$ be an irreducible representation of $G_n$ of type $(n_1, \ldots, n_d)$, then the group $\text{diag}(E^\times, I_{n-1})$ acts transitively on $L(\pi)$ and every member of $L(\pi)$ has a (necessarily unique) degenerate $\psi$-Whittaker model for some $\psi$ of type $(n_1, \ldots, n_d)$.

Uniqueness of degenerate Whittaker models for $\pi$ together with Lemma 3.1 then has the following well-known consequence.

**Proposition 3.2.** If $\pi$ is an irreducible representation of $G_n$ then the representations in $L(\pi)$ appear with multiplicity one in the restriction of $L(\pi)$ to $G_n^\circ$.

In fact we can be more precise. The following lemma follows from the fact that if $\pi$ is of type $(n_1, \ldots, n_d)$ then $\pi^{[n_1, \ldots, n_k-1]}$ is of type $(n_k, \ldots, n_d)$ (cf. 3.1).

**Lemma 3.3.** If $\pi$ is an irreducible representation of $G_n$ of type $(n_1, \ldots, n_d)$, then $L(\pi^{[n_1, \ldots, n_k-1]})$ contains a unique irreducible representation of $G_{n_k+\cdots+n_d}$ with a degenerate Whittaker model with respect to $\psi_{n_k, \ldots, n_d}$.

Again $L(\pi^{[n_1, \ldots, n_k-1]})$ only depends on $L(\pi) = L(\pi)$ (because derivatives commute with character twists), and we set

$$L(\pi)^{[n_1, \ldots, n_k-1]} := L(\pi^{[n_1, \ldots, n_k-1]})$$

for any irreducible representation $\pi$ of $G_n$ such that $\pi \in L(\pi)$.

**Definition 1.** Let $\pi$ be an irreducible representation of $G_n^\circ$. Let $\pi^{[n_1, \ldots, n_k-1]}(\psi_{n_k, \ldots, n_d})$ denote the irreducible representation of $G_{n_k+\cdots+n_d}$ isolated in Lemma 3.3, i.e., the unique representation in $L(\pi)^{[n_1, \ldots, n_k-1]}$ with a degenerate Whittaker model with respect to $\psi_{n_k, \ldots, n_d}$. In particular, $\pi^{[n_1, \ldots, n_k-1]}(\psi_{n_k, \ldots, n_d})$ denotes the unique irreducible representation of $G_n^\circ$ in $L(\pi)$ with a degenerate Whittaker model with respect to $\psi_{n_k, \ldots, n_d}$.

**Remark 1.** We do not claim that if $\pi(\psi) = \pi(\psi')$, then $\psi$ and $\psi'$ are in the same $T_n^\circ$-conjugacy class.

### 3.3. Distinguished representations inside a distinguished L-packet

Let $\pi$ be an irreducible representation of $G_n$. We start by making explicit the relation between the degenerate Whittaker models $\mathcal{W}(\pi, \psi_{n_1, \ldots, n_d})$ and $\mathcal{W}(\pi^{[1]}, \psi_{n_2, \ldots, n_d})$.

**Lemma 3.4.** The map

$$W \mapsto W_{|G_{n-n_1}}$$

is surjective from $\mathcal{W}(\pi, \psi_{n_1, \ldots, n_d})$ to $\mathcal{W}(\pi^{(n_1-1)/2}E_{1}, \psi_{n_2, \ldots, n_d})$.

**Proof.** By the same proof as in [CPS17 Proposition 1.2], the map

$$W \mapsto W_{|n-n_1+1}$$

is surjective from $\mathcal{W}(\pi, \psi_{n_1, \ldots, n_d})$ to $\mathcal{W}(\pi^{[n_1]}, \psi_{n_2, \ldots, n_d})$. 


is a surjection from $\mathcal{W}(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$ to $\mathcal{W}(v_E^{(n-n_1+1)/2} \tilde{\pi}(n_{-1}), \psi_{n_2, \ldots, n_d})$. But then, because $W(gu) = W(g)$ for $W \in \mathcal{W}(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$, $g \in G_{n-n_1}$ and $u \in U_{n-n_1+1}$, we deduce that $W_{|G_n-1} \in \mathcal{W}(v_E^{n_1/2} \tilde{\pi}(n_1), \psi_{n_2, \ldots, n_d})$ and that

\[ W \mapsto W_{|G_n-1} \]

is surjective from $\mathcal{W}(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$ to $\mathcal{W}(v_E^{n_1/2} \tilde{\pi}(n_1), \psi_{n_2, \ldots, n_d})$. The result follows. \qed

We denote by $\mathcal{K}(\tilde{\pi}, \tilde{\pi}[n_1], \psi_{n_1})$, the generalized Kirillov model of $\tilde{\pi}$ (see [Zel80, Section 5]) with respect to $\tilde{\pi}[n_1]$ and $\psi_{n_1}$. It is, by definition, the image of the unique embedding of $\tilde{\pi}|_{\mathcal{P}_n}$ into the space of functions $K : \mathcal{P}_n \to \tilde{\pi}[n_1]$ which satisfy

\[ K(k(a, x, u_1)p) = v(a)^{(n_1-1)/2} \psi_{n_1}(u_1) \tilde{\pi}(n_1)(a)K(p) \]

for $k(a, x, u_1) \in N_{n,n_1}$. Let $\tilde{\pi}$ be an irreducible representation of $G_n$ with degenerate Whittaker model $\mathcal{W}(\tilde{\pi}, \phi_{n_1, \ldots, n_d})$. Then, by Lemma 3.4 for any $W \in \mathcal{W}(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$ and $g \in G_n$, the map

\[ g \mapsto v_E^{(n_1-1)/2}W(g_1) \]

belongs to $\mathcal{W}(\tilde{\pi}[n_1], \psi_{n_2, \ldots, n_d})$. We set

\[ I(W) : G_n \to \mathcal{W}(v_E^{(n_1-1)/2} \tilde{\pi}[n_1], \psi_{n_2, \ldots, n_d}) \]

to be the map defined by

\[ I(W)(g) : g_1 \in G_{n-n_1} \mapsto W(g_1) \]

Hence $I$ realizes $\mathcal{W}(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$ inside the induced representation

\[ \text{Ind}_{\mathcal{N}_{n,n_1}}^{G_n}(\mathcal{W}(v_E^{(n_1-1)/2} \tilde{\pi}[n_1], \psi_{n_2, \ldots, n_d}) \otimes \psi_{n_1}). \]

Then the map $W \mapsto I(W)|_{\mathcal{P}_n}$ is a bijection from

\[ \mathcal{W}(\tilde{\pi}, \psi_{n_1, \ldots, n_d}) \to \mathcal{K}(\tilde{\pi}, \mathcal{W}(\tilde{\pi}[n_1], \psi_{n_2, \ldots, n_d}), \psi_{n_1}). \]

The following is now a consequence of the results of [Mat14].

**Proposition 3.5.** Let $\tilde{\pi}$ be an irreducible unitary representation of $G_n$ which is $G_n^0$-distinguished, with degenerate Whittaker model $\mathcal{W}(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$, and suppose that $\psi_{n_1, \ldots, n_d}$ is trivial on $N_n(F)$. Then the invariant linear form on $\tilde{\pi}$ is expressed as a local period on $\mathcal{W}(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$ by

\[ \lambda(W) = \int_{\mathcal{N}_n \setminus \mathcal{N}_n} \int_{\mathcal{N}_n \setminus \mathcal{N}_n} \cdots \int_{\mathcal{N}_n \setminus \mathcal{N}_n} W(p_1 \cdots p_2 p_1)dp_1 \cdots dp_2 dp_1. \]

**Proof.** The proof is by induction on $d$. For $d = 1$, the representation is unitary generic, and the fact that $W \mapsto \int_{\mathcal{N}_n \setminus \mathcal{N}_n} W(p)dp$ is well-defined is due to Flicker [Fli88, Section 4], and that it is $G_n^0$-invariant is a result due to Youngbin Ok (see [Mat14, Proposition 2.5] for a more general statement in the unitary context). Then for a general $d$, by [Mat14, Proposition 2.4], if $\tilde{\pi}$ is
Theorem 3.6. Propositions 2.2 and 2.5], the linear form
\[
\lambda_K : K \mapsto \int_{N_{n-1}^\theta} L(K(p_1))dp_1
\]
is, up to scaling, the unique $G_n^\theta$-invariant linear form on $K(\tilde{\pi}, \tilde{\pi}^{[n_1]}, \psi_{n_1})$. We realize $\tilde{\pi}$ as $W(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$ and $\tilde{\pi}^{[n_1]}$ as $W(\tilde{\pi}^{[n_1]}, \psi_{n_2, \ldots, n_d})$. Then by induction
\[
L(W') = \int_{N_{n-1}^\theta \setminus \mathbb{R}^n} \cdots \int_{N_{n-1}^\theta \setminus \mathbb{R}^{n_d}} W'(p_r \ldots p_2)dp_r \ldots dp_2.
\]
Applying it to $W' = K(p_1) = I(W_{K})(p_1)$ for the unique $W_K \in W(\tilde{\pi}, \psi_{n_1, \ldots, n_d})$ such that the previous equality holds, the result follows in view of the discussion preceding the proposition.

**Theorem 3.6.** Let $\pi$ be an irreducible unitary representation of $\text{SL}_n(E)$ of type $(n_1, \ldots, n_d)$ which is $\text{SL}_n(F)$-distinguished. Then the $\text{SL}_n(F)$-distinguished representations in $L(\pi)$ are precisely the representations $\pi(\psi)$ for a character $\psi$ of $N_n$ of type $(n_1, \ldots, n_d)$ such that $\psi|_{N_n^\theta} \equiv 1$.

**Proof.** The proof follows exactly along the same lines of the generic case as in [AP03, Section 3] and [AP18, Section 4], making use of Proposition 3.5 in lieu of Flicker’s invariant linear form mentioned above.

**Theorem 3.6** has the following consequences.

**Proposition 3.7.** Let $\pi$ be an irreducible unitary representation of $G_n^\theta$ of type $(n_1, \ldots, n_d)$, and fix $\psi_{n_1, \ldots, n_d}$ a character of $N_n$ of this type trivial on $N_n^\theta$. If $\pi(\psi_{n_1, \ldots, n_d})$ is $\text{SL}_n(F)$-distinguished, then the representation $\pi^{[n_1, \ldots, n_{k-1}]}(\psi_{n_{k}, \ldots, n_d})$ is $\text{SL}_n(\mathbb{A})$-distinguished for all $i = 1, \ldots, d$.

**Proof.** According to [AP18, Lemma 3.2], up to twisting $\tilde{\pi}$ by an appropriate character we can suppose that it is $\text{GL}_n(F)$-distinguished. Then $\tilde{\pi}^{[n_1, \ldots, n_{k-1}]}(\psi_{n_k, \ldots, n_d})$ belongs to $L(\tilde{\pi}^{[n_1, \ldots, n_{k-1}]}(\psi_{n_k, \ldots, n_d}))$ and it has a degenerate Whittaker model with respect to the distinguished character $\psi_{n_k, \ldots, n_d}$, hence the result follows from Theorem 3.6.

**Proposition 3.7** can be strengthened for Speh representations.

**Theorem 3.8.** Let $\tau$ be a generic representation of $G_r$ and let $\psi_i$ be a non-degenerate character of $N_r$ trivial on $N_i^\theta$ for $i = 1, \ldots, d$. Fix $1 \leq k \leq d$, then $\pi(\psi_{1, \ldots, d}) \in L(\text{Sp}(d, \tau))$ is $\text{SL}_n(F)$-distinguished if and only if $\pi^{[d-k]}(\psi_{d-k+1, \ldots, d}) \in L(\text{Sp}(k, \tau))$ is $\text{SL}_kr(F)$-distinguished.

**Proof.** One direction follows from Proposition 3.7. Conversely suppose that $\pi^{[d-k]}(\psi_{d-k+1, \ldots, d}) \in L(\text{Sp}(k, \tau))$ is $\text{SL}_kr(F)$-distinguished. Then up to a twist $\text{Sp}(k, \tau)$, hence $\tau$, and hence $\text{Sp}(d, \tau)$ is distinguished, thanks to [Mat14, Theorem 2.13]. But then because $\pi(\psi_{1, \ldots, d}) \in L(\text{Sp}(d, \tau))$ has a $\psi_{1, \ldots, d}$-degenerate Whittaker model and that $\psi_{1, \ldots, d}$ is trivial on $N_i^\theta$, we deduce that $\pi(\psi_{1, \ldots, d})$ is $\text{SL}_n(F)$-distinguished, thanks to Theorem 3.6.
We will give the global analogue of this result in Theorem 5.1

4. Archimedean fields

In this section $E = \mathbb{C}$ or $\mathbb{R}$ and by abuse of notation we write $G = G(E)$ for any algebraic group defined over $E$. We set $|a + ib|_C = a^2 + b^2$ and denote by $|\cdot|_E$ the usual absolute value on $\mathbb{R}$. We then denote by $\nu_E$ the character of $G_n$ obtained by composing $|\cdot|_E$ with det. For $G$ a reductive subgroup of $G_n$ we write $SAF(G)$ for the category of smooth admissible Fréchet representations of $G$ of moderate growth as in [AGS15], in which we work. We use the same product notation for parabolic induction in $SAF(G_n)$ as in [AGS15].

We only consider unitary characters of $N_n$. The non-degenerate characters of $N_n$ are of the form

$$
\psi_{\lambda} = \begin{pmatrix} 1 & z_1 & \ldots & \ldots & \cdot \\
1 & z_2 & \ldots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \ldots & \ldots & 1
\end{pmatrix} \mapsto \exp(i \sum_{i=1}^{n-1} \Re(\lambda_i z_i))
$$

with $\lambda_i \in E^*$. Then for a partition $(n_1, \ldots, n_r)$ of $n$ and non-degenerate characters $\psi_{n_i}$ of $N_{n_i}$ we define the degenerate character $\psi_{n_1, \ldots, n_r}$ of $N_n$ as in Section 3.2 and we also write $\psi_{n_1, \ldots, n_d} := \psi_{n_1, \ldots, n_d}$ when all the $n_i$’s are equal. We again say $\psi_{n_1, \ldots, n_d}$ is of type $(n_1, \ldots, n_d)$ so that the set of characters of a given type forms a single $T_n$-conjugacy class. We call a member of this conjugacy class a degenerate character of type $(n_1, \ldots, n_d)$. For a degenerate character $\psi$ of $N_n$ and an irreducible representation $\bar{\pi}$ of $G_n$, by a $\psi$-Whittaker functional, we mean a non-zero continuous linear form $L$ from $\bar{\pi}$ to $\mathbb{C}$ satisfying

$$L(\bar{\pi}(n)v) = \psi(n)L(v)$$

for $n \in N_n$ and $v \in \bar{\pi}$. We will say that $\bar{\pi}$ has a unique $\psi$-Whittaker model if the space of $\psi$-Whittaker functionals on the space of $\bar{\pi}$ is one-dimensional.

4.1. The Tadić classification of the unitary dual of $G_n$. We recall that irreducible square-integrable representations of $G_n$ for $n \geq 1$ exist only when $n = 1$ if $E = \mathbb{C}$ and when $n = 1$ or 2 if $E = \mathbb{R}$. When $n = 1$ these are just the unitary characters of $E^\times$. For $d \in \mathbb{N}$ and an irreducible square-integrable representation $\delta$ of $G_n$ ($n = 1$ or $n \in \{1,2\}$ depending on $E$ is $\mathbb{C}$ or $\mathbb{R}$) we denote by

$$\text{Sp}(d,\delta) = \text{LQ}(v_E^{(d-1)/2}\delta \times \cdots \times v_E^{(1-d)/2}\delta)$$

the Langlands quotient of $v_E^{(d-1)/2}\delta \times \cdots \times v_E^{(1-d)/2}\delta$. In particular, $\text{Sp}(d,\chi) = \chi \circ \det$ when $\chi$ is a unitary character of $G_1$. By [Tad09], the representations

$$\pi(\text{Sp}(d,\delta),\alpha) := v^\alpha \text{Sp}(d,\delta) \times v^{-\alpha} \text{Sp}(d,\delta)$$

are irreducible unitary when $\alpha \in (0,1/2)$, and any irreducible representation $\pi$ of $G_n$ can be written in a unique manner as a commutative product

$$\bar{\pi} = \prod_{i=1}^{r} \text{Sp}(d_i,\delta_i) \prod_{j=r+1}^{s} \pi(\text{Sp}(d_j,\delta_j),\alpha_j).$$
When all the $d_i$ and $d_j$ are equal to one, the representation

$$\tau = \prod_{i=1}^{r} \delta_i \prod_{j=r+1}^{s} \pi(\delta_j, \alpha_j)$$

is generic unitary (it has a unique $\psi$-Whittaker model for any non-degenerate character $\psi$ of $N_\mathbb{R}$), according to [Jac09, p.4], and we set

$$\text{Sp}(d, \tau) = \prod_{i=1}^{r} \text{Sp}(d, \delta_i) \prod_{j=r+1}^{s} \pi(\text{Sp}(d, \delta_j), \alpha_j)$$

which is thus an irreducible unitary representation.

We note that according to the proof of [GS13, 4.1.1], which refers to [Vog86] and [SS90], a Speh representation $\text{Sp}(d, \delta)$ for $\delta$ an irreducible square-integrable representation of $G_2$ is the same thing as the Speh representations of Vogan’s classification as presented in [AGS15, 4.1.2 (c)]. Hence the Vogan classification as stated in [AGS15, 4.1.2] is immediately related to that of Tadić:

- the unitary characters of [AGS15, 4.1.2 (a)] are the representations of the form $\text{Sp}(d, \chi)$ for $\chi$ a unitary character of $G_1$,
- the Stein complementary series of [AGS15, 4.1.2 (b)] are the representations of the form $\pi(\text{Sp}(d, \chi), \alpha)$ for $\chi$ a unitary character of $G_1$,
- the Speh representations of [AGS15, 4.1.2 (c)] are the representations of the form $\text{Sp}(d, \delta)$ for $\delta$ an irreducible square-integrable representation of $G_2$,
- the Speh complementary series of [AGS15, 4.1.2 (d)] are the representations of the form $\pi(\text{Sp}(d, \delta), \alpha)$ for $\delta$ an irreducible square-integrable representation of $G_2$.

The third and fourth cases occur only when $E = \mathbb{R}$.

4.2. Degenerate Whittaker models of irreducible unitary representations. In this section we recall the results of Aizenbud, Gourevitch, and Sahi on degenerate Whittaker models for $\text{GL}_n(E)$ for $E = \mathbb{C}$ or $\mathbb{R}$. We believe that with the material developed by these authors, together with the real analogue of Ok’s result due to Kemarsky ([Kem13]), the results obtained in [Mat14] and Section 3 are in reach. However, being inexperienced in such matters, we leave this for experts, and simply recall immediate implications of the results in [AGS15] that we will need for our global applications.

In [Sah89], to any irreducible representation $\pi$ of $G_n$, Sahi attached an irreducible representation $A(\pi)$ of $G_{n-n_1}$ for some $0 < n_1 \leq n$, the adduced representation of $\pi$ and proved that it satisfied

$$A(\prod_{i=1}^{r} \text{Sp}(d_i, \delta_i) \prod_{j=r+1}^{s} \pi(\text{Sp}(d_j, \delta_j), \alpha_j)) = \prod_{i=1}^{r} A(\text{Sp}(d_i, \delta_i)) \prod_{j=r+1}^{s} A(\pi(\text{Sp}(d_j, \delta_j), \alpha_j))$$

with respect to the Tadić classification. The adduced representation is the archimedean highest shifted derivative, and from [Sah90], [GS13] and [AGS15] (see [AGS15, Section 4]) one has

$$A(\prod_{i=1}^{r} \text{Sp}(d_i, \delta_i) \prod_{j=r+1}^{s} \pi(\text{Sp}(d_j, \delta_j), \alpha_j))$$

(2)
Unitary representation model.

Theorem 4.1. Let \( \pi \) be an irreducible unitary representation of \( G_n \) of type \( (n_1, \ldots, n_d) \), and \( \psi \) be any character of \( N_n \) of type \( (n_1, \ldots, n_d) \), then \( \pi \) has a unique degenerate \( \psi \)-Whittaker model.

For an irreducible representation \( \pi \) of \( SL_n(E) \), the notion of a degenerate Whittaker model is defined similarly, though this time the notion does depend on the \( T_n \)-conjugacy class of the degenerate character \( \psi \) and not just its type. The L-packet of \( \pi \) is defined as in the \( p \)-adic case, and we refer to [HS12, End of Section 2]. Note that [HS12] deals with Harish-Chandra modules but their results remain valid in the context of \( S \mathcal{A} \mathcal{F}(G_n) \), thanks to the Casselman-Wallach equivalence of categories (see [Wal88, Chapter 11]). If \( \pi \) is an irreducible unitary representation of \( G_n \), it follows from [GS13, Theorem A] that the type of \( \pi \) depends only on \( L(\pi) \), and we define the type of an irreducible unitary representation \( \pi \) of \( G_n \) to be that of any irreducible representation \( \tilde{\pi} \) of \( G_n \) such that \( \pi \in L(\tilde{\pi}) \).

Remark 2. If \( \tilde{\pi} \) is an irreducible representation of \( G_n^0 \), then \( \tilde{\pi}|_{G_n} \) contains an irreducible unitary representation if and only if it is unitary up to a character twist.

As in the \( p \)-adic case, Theorem 4.1 has the following consequence.

Corollary 4.2. Let \( \tilde{\pi} \) be a smooth irreducible unitary representation of \( G_n^0 \) of type \( (n_1, \ldots, n_d) \). Then the group \( \text{diag}(E^\times, I_{n-1}) \) acts transitively on \( L(\tilde{\pi}) \) and every \( \pi \in L(\tilde{\pi}) \) has a (necessarily unique) degenerate \( \psi \)-Whittaker model for some character \( \psi \) of \( N_n \) of type \( (n_1, \ldots, n_d) \). Moreover, \( \tilde{\pi}|_{G_n^0} \) is multiplicity free.

We note that the computation of the adduced representation given in Equation (2) implies:

Theorem 4.3 (Aizenbud, Gourevitch, and Sahi). Let \( \tau \) be an irreducible generic representation of \( G_r \). The Speh representation \( \text{Sp}(d, \tau) \) has type \( r^d \), and conversely an irreducible unitary representation of \( G_n \) of type \( r^d \) is of the form \( \text{Sp}(d, \tau) \) for some unitary generic representation \( \tau \) of \( G_r \).

We end by giving the archimedean analogue of Definition 1 for Speh representations.

Definition 2. Let \( \pi \) be an irreducible unitary representation of \( G_n^0 \) of type \( r^d \), and let \( \tau \) be an irreducible unitary generic representation of \( G_r \) such that \( \pi \in L(\text{Sp}(d, \tau)) \). For \( \psi_{d-k+1, \ldots, d} \) a character of \( N_n \) of type \( r^d \), we denote by \( \pi^{[r^d]}(\psi_{d-k+1, \ldots, d}) \) the unique representation in \( L(\text{Sp}(k, \tau)) \) with a \( \psi_{d-k+1, \ldots, d} \)-degenerate Whittaker model.

Remark 3. The representation \( \pi^{[r^d]}(\psi_{d-k+1, \ldots, d}) \) above depends only on \( L(\pi) \).
5. Global theory

In this section, $E/F$ is a quadratic extension of number fields with associated Galois involution $\theta$. We denote by $A_E$ and $A_f$ the rings of adeles of $E$ and $F$ respectively. We denote by $GL_n(A_E)^1$ the elements of $GL_n(A_E)$ which have determinant of adelic norm equal to 1, and for any subgroup $H$ of $GL_n(A_E)$, by $H^1$ we denote the intersection of $H$ with $GL_n(A_E)^1$. We recall that $A_F^\times = A_F^1 \times (A_F)_>$, where $(A_F)_>$ is $\mathbb{R}_{>0} \otimes \mathbb{Q} 1 \subset \mathbb{R} \otimes \mathbb{Q} F$ sitting inside $A_F$. In particular passing to the groups of unitary characters we have $A_F^\times = \widehat{A_F}^1 \times (\widehat{A_F})_>$, and for $\lambda \in \mathbb{R}$ we denote by $a_\lambda$ the unitary character of $A_F^\times$ corresponding to $(\alpha, |\mathfrak{a}|_\mathbb{A}_F^1) \in \widehat{A_F}^1 \times (\widehat{A_F})_>$. In particular $a_\lambda$ is automorphic if and only if $\lambda \in F^{\times} \backslash \widehat{A_F}^1$.

5.1. Degenerate Whittaker models and residual L-packets for $SL_n(A_E)$. Let $\sigma$ be a smooth unitary cuspidal automorphic representation of $GL_{r^d}(A_E)$. Let

$$\tilde{\pi} = \text{Sp}(d, \sigma) = \otimes_v \text{Sp}(d, \sigma_v),$$

for $v$ varying through the places of $E$, be the square-integrable automorphic representation of $GL_n(A_E)$ associated to $d$ and $\sigma$ ([Jac84], [MgW89]), where $n = dr$. The paper [MgW89] proves that any irreducible square-integrable automorphic representation of $GL_n(A_E)$ is of this form for a unique pair $(\sigma, d)$. We set $U_{rd}$ to be the unipotent radical of the parabolic subgroup of type $r^d$ of $GL(n)$, denoted by $P_{rd}$. Let

$$\psi_{1,\ldots,d}(\text{diag}(n_1, \ldots, n_d)u) = \prod_{i=1}^d \psi_i(n_i)$$

where $\psi_i$ is a non-degenerate character of $N_i(A_E)$ trivial on $N_i(E)$ and $u \in U_{rd}(A_E)$. For $\varphi \in \pi$, we set

$$p_{\psi_{1,\ldots,d}}(\varphi) = \int_{N_r(E) \backslash N_r(A_E)} \varphi(n)\psi_{1,\ldots,d}^{-1}(n)dn.$$

By [JL13, Corollary 3.4], there exists $\varphi \in \text{Sp}(d, \sigma)$ such that $p_{\psi_{1,\ldots,d}}(\varphi) \neq 0$: we will say that $\varphi$ has a non-zero Fourier coefficient of type $r^d$.

Remark 4. The result [JL13, Corollary 3.4] could also be deduced by the techniques used in Section 5.2 using the $E = F \times F$-analogue of Yamana’s formula [Yam15, Theorem 1.1] (see Theorem 5.4). Also following Section 5.2 in the case where $E$ is split, one would conclude that any square-integrable representation of $SL_n(A_E)$ in the L-packet determined by $\text{Sp}(d, \sigma)$ has a degenerate Whittaker model of type $r^d$. However for the sake of variety we offer a different proof of this fact here, using the results of [JL13] rather than those of [Yam15] (or rather its split analogue).

We recall from [HS12, Remark 4.23] the following facts. If $\pi$ is a square-integrable automorphic representation of $SL_n(A_E)$ (realized in the space of smooth $L^2$ automorphic functions), then there is a square-integrable automorphic representation $\tilde{\pi}$ of $GL_n(A_E)$ such that $\pi$ is a submodule of $\text{Res}(\tilde{\pi})$ where $\text{Res}$ is the restriction of automorphic forms from $GL_n(A_E)$ to $SL_n(A_E)$, and such a $\tilde{\pi}$ is unique up to twisting by an automorphic character of $A_E^\times$. We denote by $L(\pi)$ or $L(\tilde{\pi})$ the set of irreducible submodules of $\text{Res}(\tilde{\pi})$. We say that a square-integrable representation $\pi$ of $SL_n(A_E)$ is...
of type $r^d$ if it belongs to $L(Sp(d,\sigma))$ for $\sigma$ an irreducible (unitary) cuspidal automorphic representation of $G_r(\mathbb{A}_E)$.

We say that $\pi$ (resp. $\pi$) has a degenerate Whittaker model of type $r^d$ if there is $\varphi \in \pi$ (resp. $\varphi \in \pi$) with a non-zero Fourier coefficient of type $r^d$. In particular $Sp(d,\sigma)$ has a degenerate Whittaker model of type $r^d$.

We denote by $\psi$ a non-degenerate character of $N_r(\mathbb{A}_E)$ trivial on $N_r(E)$. We set

$$(1 \otimes \psi) \begin{pmatrix} I_{n-r} & x \\ 0 & u_1 \end{pmatrix} = \psi(u_1)$$

for

$$\begin{pmatrix} I_{n-r} & x \\ 0 & u_1 \end{pmatrix} \in U_{n,r}(\mathbb{A}_E).$$

For $\varphi \in \pi$, we set

$$\varphi_{U_{n,r}\psi}(g) = \int_{U_{n,r}(E) \setminus U_{n,r}(\mathbb{A}_E)} \varphi(ug)(1 \otimes \psi^{-1})(u)du$$

for $g \in GL_n(\mathbb{A}_E)$.

**Remark 5.** Note that the function $\varphi_{U_{n,r}\psi}$ is nothing but the integral of the constant term of $\varphi$ along the $(n-r, r)$ parabolic against $\psi^{-1}$ on $N_r(E) \setminus N_r(\mathbb{A}_E)$. By [Yam15] Lemma 6.1 there is a positive character $\delta$ of $GL(n-r)(\mathbb{A}_E)$ such that the function $\delta \otimes \varphi_{U_{n,r}\psi}$ belongs to $Sp(d-1, \sigma)$, in particular $(\varphi_{U_{n,r}\psi})|_H$ belongs to $Res_{H}(Sp(d-1, \sigma))$ (restriction of cusp forms) for any subgroup $H$ of $GL_n(\mathbb{A}_E)^1$, for example $H = SL_n(\mathbb{A}_E)$.

We now can prove the following result.

**Proposition 5.1.** A square-integrable automorphic representation $\pi$ of $SL_n(\mathbb{A}_E)$ of type $r^d$ has a degenerate Whittaker model of type $r^d$.

**Proof.** We will prove the stronger claim: for any $\varphi \in \pi$ such that $\varphi|_{SL_n(\mathbb{A}_E)} \neq 0$, there is $h_0 \in SL_r(\mathbb{A}_E)$ (embedded in $SL_n(\mathbb{A}_E)$ in the upper left block) such that $\rho(h_0)\varphi$ has a non-zero Fourier coefficient of type $r^d$. If $d = 1$, we are in the cuspidal (and hence generic) case and the result follows from the same inductive procedure of [AM19], Lemma 3.3 and Proposition 3.4], but applied to $E$ diagonally embedded inside $E \times E$ (instead of $F \subset E$ considered in [AM19]). If $d \geq 2$, by [LL13] Proposition 3.1 (1) applied to $\varphi$ there is a non-degenerate character $\psi$ of $N_r(\mathbb{A}_E)$ trivial on $N_r(E)$ such that $\varphi_{U_{n,r}\psi}$ is non-zero on $SL_{n-r}(\mathbb{A}_E)$ (because $N_{n,r}(E)\backslash P_n(E) = N^0_{n,r}(E)\backslash P^0_{n}(E)$ as $d \geq 2$). We conclude by induction, thanks to Remark 5.1.

We have the following corollary of Proposition 5.1.

**Corollary 5.2.** If $\pi$ is an irreducible square-integrable automorphic representation of $GL_n(\mathbb{A}_E)$ of type $r^d$, then $Res(\pi)$ is multiplicity free. Moreover, for any automorphic character $\psi$ of $N_n(\mathbb{A}_E)$ of type $r^d$, the $L$-packet $L(\pi)$ contains a unique member $\pi(\psi)$ with a $\psi$-Whittaker model, and the group $\text{diag}(E^\times, I_{n-1})$ acts transitively on $L(\pi)$.

**Proof.** Thanks to multiplicity one inside local $L$-packets (cf. Proposition 3.2 and Corollary 4.2), it follows that the representations in $L(\pi)$ appear with multiplicity one in $Res(\pi)$. Moreover, we deduce that $T_n(E)$ acts transitively on $L(\pi)$: by Proposition 5.1...
any representation in $L(\pi)$ has a degenerate Whittaker model of type $r^d$. Note that two automorphic characters of type $r^d$ of $N_n(A_E)$ are conjugate to each other by $T_n(E)$ and this implies that for each automorphic character $\psi$ of type $r^d$ of $N_n(A_E)$ there is a representation $\pi(\psi)$ in $L(\pi)$ with a $\psi$-Whittaker model. Moreover $L(\pi)$ has at most one representation with a $\psi$-Whittaker model by local multiplicity one of degenerate Whittaker models and this implies the uniqueness of $\pi(\psi)$ in the statement. Finally for $t \in T_n(E)$ and $t' = \text{diag}(\det(t), I_{n-1})$, the representations $\pi^t$ and $\pi^{t'}$ in $L(\pi)$ are isomorphic, hence equal by multiplicity one inside $L(\pi)$.

5.2. Distinction inside distinguished $L$-packets. As in Section 5.1 we consider $\sigma$ be a cuspidal automorphic representation of $GL_r(A_E)$, and $\pi = \text{Sp}(d, \sigma)$ the residual representation of $GL_n(A_E)$ associated to it for $n = dr$. We first have the following basic result.

**Proposition 5.3.** The period integral

$$\varphi \mapsto \int_{SL_n(F) \backslash SL_n(A_F)} \varphi(h) dh$$

is given by an absolutely convergent integral on $\text{Res}(\pi)$. Moreover for any $\varphi \in \pi$, then

$$\int_{SL_n(F) \backslash SL_n(A_F)} \varphi(h) dh = \sum_{\alpha} \int_{GL_n(F) \backslash GL_n(A_F)^1} \varphi(h) \alpha(\det(h)) dh$$

where the sum is over all characters $\alpha$ of the compact abelian group $F^\times \backslash A_1^\times$.

**Proof.** The proof is essentially that of [AP06, Proposition 3.2], the only additional ingredient for the case at hand being [Yam15, Lemma 3.1]. Note that, by [Yam15, Lemma 3.1], square-integrable automorphic forms on $GL_n(A_E)$ are (absolutely) integrable over $GL_n(F) \backslash GL_n(A_F)^1$.

Now, by applying the integral decomposition

$$\int_{H_1 \backslash G} f(g) dg = \int_{H_2 \backslash G} \left( \int_{H_1 \backslash H_2} f(hg) dh \right) dg$$

where $H_1 \subset H_2$ are closed subgroups of a locally compact group $G$, all three unimodular, and $f \in L^1(H_1 \backslash G)$, for

$$GL_n(F) \subset GL_n(F) SL_n(A_F) \subset GL_n(A_F)^1$$

and $f = |\varphi|$, it follows, as $\varphi$ is continuous, that

$$\int_{SL_n(F) \backslash SL_n(A_F)} \varphi(h \text{ diag}(x, I_{n-1})) dh$$

is absolutely convergent for each $x \in A_1^\times$.

The second assertion follows from the decomposition

$$\int_{GL_n(F) \backslash GL_n(A_F)^1} \varphi(h) dh = \int_{F^\times \backslash A_1^\times} \left( \int_{SL_n(F) \backslash SL_n(A_F)} \varphi(h \text{ diag}(x, I_{n-1})) dh \right) dx,$$

as in [AP06, Proposition 3.2], by Fourier inversion on the compact abelian group $F^\times \backslash A_1^\times$. $\square$
Indeed if $\bar{\pi}$ is distinguished with respect to $GL_n(A_F)$ (indeed if $\bar{\pi}$ is $(GL_n(A_F), \alpha)$-distinguished, then it is $(GL_n(A_F), \alpha')$-distinguished for a character $\alpha'$ of $A_F^\times$ extending $\alpha$ and equal to the central character $\omega_{\bar{\pi}}$ on $(A_F)_0$). Observe that this implies that $\bar{\pi}$ is Galois conjugate self-dual by strong multiplicity one for the residual spectrum ([MgW89]) and the fact that $Sp(d, \sigma)$ is distinguished and hence Galois conjugate self-dual for any finite place $v$ ([Fl91]). Now, if the $(GL(n, A_F), \alpha)$-period is also non-zero then we have $\bar{\pi} \cong \pi \otimes \alpha' \circ N_{E/F}$ for some $\alpha'$ extending $\alpha$ to $A_F^\times$. As $\bar{\pi} = Sp(d, \sigma)$, we see that $\sigma \cong \sigma \otimes \alpha' \circ N_{E/F}$. As $\sigma$ is a cuspidal representation, the finiteness of the set of such characters $\alpha'$ (hence of that of the characters $\alpha$) follows from [Ram00, Lemma 3.6.2] (which is [HS12, Lemma 4.11]).

Our aim in this section is to show that if $\pi$ is distinguished then $\pi$ has a non-vanishing Fourier coefficient with respect to a character of type $\beta$ of type $A_F$ which is trivial on $N_n(E + A_F)$ (see Theorem 5.7). The key ingredient in achieving this is Proposition 5.3 below.

The following result is [Yam15, Theorem 1.1] slightly reformulated for our purposes.

**Theorem 5.4.** Let $n = rd$ with $r \geq 2$ and $d \geq 2$, and let $\psi$ be a non-degenerate unitary character of $N_n(A_E)$ trivial on $N_n(E + A_F)$. Fix a character $\alpha$ of $F^\times \setminus A_E^\times$. Then for $\phi \in \pi = Sp(d, \sigma)$, we have

$$\int_{GL_n(F) \setminus GL_n(A_F)} \phi(h) \alpha(\det h) dh =$$

$$\int_{N^{-1}_{n-1,r-1}(A_F) \setminus SL_{n-1}(A_F)} \int_{GL_{n-r}(F) \setminus GL_{n-r}(A_F)} (\alpha \psi)_{U_{n,r}}(\psi(\text{diag}(m, I_r) \text{diag}(h, 1))) dmdh.$$

**Proof.** We denote by $\omega_{\sigma}$ the central character of $\sigma$. We extend $\alpha$ as $\alpha_0$ to $A_F^\times$. We then extend $\alpha_0$ to an automorphic character of $\beta$ of $A_E^\times$. Then we claim that the following equality holds

$$\int_{GL_n(F) \setminus GL_n(A_F)} \phi(h) \alpha_0(\det h) dh =$$

$$\int_{N^{-1}_{n-1,r-1}(A_F) \setminus GL_{n-1}(A_F)} \int_{GL_{n-r}(F) \setminus GL_{n-r}(A_F)} (\alpha_0 \psi)_{U_{n,r}}(\psi(\text{diag}(m, I_r) \text{diag}(h, 1))) dmdh.$$

Indeed if $\alpha_0' \cdot \omega_{\sigma} | A_F^\times$ is trivial this follows from the second part of Theorem [Yam15, Theorem 1.1] applied to $\beta \otimes \pi$. If $\alpha_0' \cdot \omega_{\sigma} | A_F^\times \neq 1$ then follows from the first part [Yam15, Theorem 1.1] applied to $\beta \otimes \pi$, with the extra-observation that the right hand side of the equality also vanishes thanks to Remark 5 and the first part of Theorem [Yam15, Theorem 1.1] again if $d \geq 3$, and for central character reasons when $d = 2$. We can now replace the quotient $N_{n-1,r-1}(A_F) \setminus GL_{n-1}(A_F)$ by $N_{n-1,r-1}(A_F) \setminus SL_{n-1}(A_F)$ and the statement follows. \qed

From Theorem 5.4, we deduce its $SL(n)$ version by making use of Proposition 5.3.
Proposition 5.5. With notations and assumptions \((r, d \geq 2)\) as in Theorem 5.4 for \(\varphi \in \text{Res}(\pi)\) we have

\[
p_n(\varphi) := \int_{\text{SL}_n(F) \backslash \text{SL}_n(A_F)} \varphi(h) dh = \\
\int_{N_{n-1,F}(A_F) \backslash \text{SL}_{n-1}(A_F)} \int_{\text{SL}_{n-r}(F) \backslash \text{SL}_{n-r}(A_F)} \varphi U_{n,r}(\text{diag}(m, I_r) \text{diag}(h, 1)) dmdh.
\]

Proof. We relate the \(\text{SL}(n, A_F)\)-period \(p_n\) to the \((\text{GL}(n, A_E)^1, \alpha)\)-periods via Proposition 5.3. Applying Theorem 5.4 to each summand of the sum over characters \(\alpha\) of \(F^\times \backslash A_F^\times\) just selected, we once again apply Proposition 5.3 to the right hand side sum to conclude the proof. \(\square\)

Setting

\[
(\rho(g) \varphi)_{n-r, \varphi} := m \in \text{GL}_{n-r}(A_E) \mapsto \varphi U_{n,r}(\text{diag}(m, I_r)g),
\]

Proposition 5.5 implies the following observation which we state as a lemma.

Lemma 5.6. With notations and assumptions \((r, d \geq 2)\) as in Theorem 5.4, suppose that \(\varphi \in \text{Res}(\pi)\) is such that \(p_n(\varphi) \neq 0\), then there is \(h \in \text{SL}_{n-1}(A_F)\) such that

\[
p_{n-r}((\rho(\text{diag}(h, 1)) \varphi)_{n-r, \varphi}) \neq 0.
\]

We now state the main theorem of this section which follows from Lemma 5.6 by an inductive argument (see also the proof of [AM19, Proposition 3.4]).

Theorem 5.7. Let \(\pi\) be an irreducible automorphic representation of \(\text{SL}_n(A_E)\) of type \(r^d\) which is distinguished with respect to \(\text{SL}_n(A_F)\); thus there exists \(\varphi \in \pi\) such that \(p_n(\varphi) \neq 0\). Then there exist \(d\) non-degenerate characters \(\psi_i\) of \(N_r(A_E)\) trivial on \(N_r(E + A_F)\) and \(\varphi' \in \pi\) such that

\[
p_{\psi_1, \ldots, \psi_d}(\varphi') = \int_{N_r(E) \backslash N_r(A_E)} \varphi'(n) \psi_1, \ldots, \psi_d^{-1}(n) dn \neq 0.
\]

Moreover, \(\varphi'\) can be chosen to be a right \(\text{SL}_{n-1}(A_F)\)-translate of \(\varphi\).

Proof. The theorem is immediate from Lemma 5.6 by an inductive argument, however we have to treat the case \(r = 1\) separately. If \(r = 1\) then \(\pi\) is the trivial character of \(\text{SL}_n(A_E)\) and the claim is obvious. So we suppose that \(r \geq 2\). If \(d = 1\) the result was proved in [AM19, Theorem 3.1], so we assume \(d \geq 2\). Since \(\varphi \in \pi\) is such that \(p_n(\varphi) \neq 0\), by Lemma 5.6 we get \(h \in \text{SL}_{n-1}(A_F)\) such that \(p_{n-r}((\rho(\varphi)_{n-r, \varphi}) \neq 0\).

Therefore, by induction thanks to Remark 5 we get \(d - 1\) non-degenerate characters \(\psi_i, i = 2, \ldots, d\) of \(N_r(A_E)\) trivial on \(N_r(E + A_F)\) such that

\[
p_{\psi_2, \ldots, \psi_d}(\rho(x)(\rho(h) \varphi)_{n-r, \varphi}) = \int_{N_{n-r}(E) \backslash N_{n-r}(A_E)} (\rho(h) \varphi)_{n-r, \varphi}(nx) \psi_2, \ldots, \psi_d^{-1}(n) dn \neq 0,
\]

for some \(x = \text{diag}(y, 1)\) for \(y \in \text{SL}_{n-r-1}(A_F)\). But setting \(\psi_1 := \varphi, \)

\[
\int_{N_{n-r}(E) \backslash N_{n-r}(A_E)} (\rho(h) \varphi)_{n-r, \varphi}(nx) \psi_2, \ldots, \psi_d{-1}(n) dn
\]
Then \([AM19, Proposition 2.3]\) tells us that \(\sigma\) and Remark 5 allow an induction to conclude that the L-packet \(L\) selfdual hence an automorphic twist of \(\tilde{\pi}\) results of Flicker recalled in \([AM19, Theorem 2.2]\). Hence by \([Yam15, Theorem 1.2]\) an automorphic representation of \(GL_n(\mathbb{A}_E)\) extends thanks to the results of Yamana and Proposition 5.5, so we end this section by proving this generalization.

\[\pi\] is distinguished if it contains a distinguished member.

**Theorem 5.9.** Let \(\tilde{\pi} = Sp(d, \sigma)\) an irreducible square-integrable automorphic representation of \(GL_n(\mathbb{A}_E)\), with \(\sigma\) a unitary cuspidal automorphic representation of \(GL_r(\mathbb{A}_E)\) (hence \(n = dr\)). If \(\tilde{\pi} \sim \mu \otimes \tilde{\pi}^\theta\) for a character \(\mu\) of \(\mathbb{A}_E^\times\) of the form \(\alpha \circ N_{E/F}\) where \(\alpha \in F^\times \backslash \mathbb{A}_F\), then \(L(\tilde{\pi})\) is distinguished. Conversely, assume that \(F\) and \(r\) are such that the Grunwald-Wang theorem is applicable, then if an L-packet \(L(\tilde{\pi})\) is distinguished, there is a Hecke \(\mu\) character of \(\mathbb{A}_E^\times\) of the form described above such that \(\tilde{\pi}^\vee \sim \mu \oplus \tilde{\pi}^\theta\).

**Proof.** If \(\tilde{\pi}^\vee \sim \mu \otimes \tilde{\pi}^\theta\) then \(\sigma^\vee \sim \mu \otimes \sigma^\theta\) for \(\mu = \alpha \circ N_{E/F}\), then \(\alpha \otimes \sigma\) is conjugate selfdual hence an automorphic twist of \(\sigma\) distinguished by \(GL_r(\mathbb{A}_F)\) thanks to the results of Flicker recalled in \([AM19, Theorem 2.2]\). Hence by \([Yam15, Theorem 1.2]\) an automorphic twist of \(\tilde{\pi}\) is distinguished by \(GL_n(\mathbb{A}_F)\). Then \(L(\tilde{\pi})\) is distinguished by a straightforward generalization of \([AP06, Proposition 3.2]\). Conversely, suppose that \(F\) and \(r\) are such that the Grunwald-Wang theorem is applicable. Then Proposition 5.5 and Remark 5 allow an induction to conclude that the L-packet \(L(\sigma)\) is distinguished. Then \([AM19, Proposition 2.3]\) tells us that \(\sigma^\vee \sim \mu \otimes \sigma^\theta\) with \(\mu = \alpha \circ N_{E/F}\) for some \(\alpha \in F^\times \backslash \mathbb{A}_F^\times\), hence \(\tilde{\pi}^\vee \sim \mu \otimes \tilde{\pi}^\theta\).
5.4. Automorphy and distinction of the highest derivative for $\text{SL}_n(\mathbb{A}_E)$. We end this section with an analogue of [Yam15, Theorem 1.2] in the context of $\text{SL}_n(\mathbb{A}_E)$.

**Lemma 5.10.** Let $\pi$ be an irreducible square-integrable representation of $\text{SL}_n(\mathbb{A}_E)$ of type $r^d$, and write

$$\pi = \otimes_v \pi_v.$$

Then for any $k \in [1, d]$, the representation

$$\pi_{r^d-k}({\psi_{d-k+1,...,d}}) := \otimes_v \pi_{r^d-k}({\psi_{d-k+1,...,d,v}})$$

(see Definitions 1 and 2) is automorphic. If $\sigma$ is a cuspidal automorphic representation of $\text{GL}_r(\mathbb{A}_F)$ such that $L(\pi) = L(\text{Sp}(d, \sigma))$, then $\pi_{r^d-k}({\psi_{d-k+1,...,d}})$ is in fact the unique element of $L(\text{Sp}(k, \sigma))$ with a $\psi_{d-k+1,...,d}$-Whittaker model.

**Proof.** Let $\mu$ be the member of $L(\text{Sp}(k, \sigma))$ with a $\psi_{d-k+1,...,d}$-Whittaker model. Then for all places $v$ the representation $\mu_v$ is the member of $L(\text{Sp}(k, \sigma_v))$ with a $\psi_v$-Whittaker model, hence it must be $\pi_{r^d-k}({\psi_{d-k+1,...,d,v}})$ and the result follows.  

Here is our SL-analogue of [Yam15, Theorem 1.2].

**Theorem 5.11.** Suppose that $\psi_{1,...,d}$ is a character of $N_n(\mathbb{A}_E)$ of type $r^d$ trivial on $N_n(E + \mathbb{A}_F)$. Let $\pi$ be an irreducible square-integrable representation of $\text{SL}_n(\mathbb{A}_E)$ of type $r^d$ and fix $k \in [1, d]$, then $\pi(\psi_{1,...,d})$ is $\text{SL}_n(\mathbb{A}_F)$-distinguished if and only if $\pi_{r^d-k}({\psi_{d-k+1,...,d}})$ is $\text{SL}_{kr}(\mathbb{A}_F)$-distinguished.

**Proof.** The proof is the same as that of Theorem 3.8 using [Yam15, Theorem 1.2] in lieu of [Mat14, Theorem 2.13].

6. Local global principle for distinguished $L$-packets when $r$ is odd

This section establishes a local global principle for distinction inside a square-integrable $L$-packet of type $r^d$ of $\text{SL}_n(\mathbb{A}_E)$, when $r$ is odd.

Our proof makes use of the set up of [AP13, Section 7] where such a result is proved for a cuspidal $L$-packet of $\text{SL}_2(\mathbb{A}_F)$. The proof over there is somewhat intricate and relied crucially on an analysis of the fibers of the Asai lift (see [AP13, Remark in Section 7]). Here our arguments are more elementary due to the fact that $r$ is odd. This is consistent with the earlier works [Ana05, AP18, AM19].

For the moment however $r$ is general. Let $\pi$ be an irreducible square-integrable automorphic representation of $\text{SL}_n(\mathbb{A}_E)$ and denote by $\tilde{\pi}$ a square-integrable automorphic representation of $\text{GL}_n(\mathbb{A}_E)$ such that $\pi$ is realized in $\text{Res}(\tilde{\pi})$.

We borrow the notations of [AP13, Section 7]. We consider $\mathbb{A}_E^\times$ as a subgroup of $\text{GL}_n(\mathbb{A}_E)$ via $x \mapsto \text{diag}(x, I_{n-1})$. This group acts by conjugation on isomorphism classes of an irreducible representation $\pi$ of $\text{SL}_n(\mathbb{A}_E)$. The orbit of $\pi$ under this action is the representation theoretic $L$-packet of $\pi$, say $L'(\pi)$. Let $G(\pi) < \mathbb{A}_E^\times$ be the stabilizer of $\pi$. Then, see [HS12, p. 23],

$$G_\pi = \bigcap_{\chi \in \widetilde{X(\pi)}} \text{Ker} \chi$$
Proposition 5.1. However \( L \) (member of \( L' \)) clearly \( L \) Whittaker model according to Corollary 5.2. We conclude that
\[
\supseteq \theta \quad \text{under the action of } \theta.
\]

Remark 7. Note that \( L(\pi) \) consists of the automorphic members of \( L'(\pi) \). Indeed clearly \( L(\pi) \) is included in this set. On the other hand, if \( \pi' \) is an automorphic member of \( L'(\pi) \), then it has a degenerate \( \psi \)-Whittaker model of type \( r^d \) thanks to Proposition 5.1. However \( L(\pi) \) also contains a member \( \pi'' \) with a degenerate \( \psi \)-Whittaker model according to Corollary 5.2. We conclude that \( \pi' \simeq \pi'' \) by local uniqueness of degenerate Whittaker models.

We start with an elementary observation.

Proposition 6.1. Suppose \( \tilde{\pi} \) is a square-integrable automorphic representation of \( \text{GL}_n(\mathbb{A}_E) \) which is Galois conjugate self-dual; i.e., \( \tilde{\pi}^\vee \simeq \tilde{\pi}^\theta \), and that \( \pi \in L(\tilde{\pi}) \). Then \( G_\pi \) is stable under the action of \( \theta \).

Proof. As \( \tilde{\pi} \) is Galois conjugate self-dual, it follows that the finite abelian group \( X(\tilde{\pi}) \) is stable under the Galois action, and thus \( G_\pi \) is Galois stable. Alternatively, note that if \( \pi_1 \) and \( \pi_2 \) are in the same \( L \)-packet then \( G_{\pi_1} = G_{\pi_2} \). Indeed, \( \pi_2 = \tilde{\pi}_1^y \), for some \( y \in \mathbb{A}_E^\times \), and by definition, \( G_{\pi_1} = y^{-1}G_{\pi_1}y = G_{\pi_1} \) as the groups are abelian. In particular, \( G_{\pi_1} = G_{\pi_1}^\vee \) as \( \tilde{\pi}^\vee \simeq \tilde{\pi}^\theta \). Observe also that \( G_{\pi}^\vee = G_\pi \). Thus, if \( x \in G_\pi \) then \( x^\theta \in G_{\pi^\theta} = G_{\pi^\vee} = G_\pi \). \qed

From now on, we assume that \( E \) is split at the archimedean places, so that the archimedean analogue of Theorem 3.6 obviously holds.

As in \[\text{AP13 Section 7}\], we define the following groups,
\[
H_0 = \mathbb{A}_E^\times,
\]
\[
H_1 = \mathbb{A}_E^\times G_\pi,
\]
\[
H_2 = E^\times G_\pi,
\]
\[
H_3 = F^\times G_\pi,
\]
and we observe that

(1) The set \( H_0 \cdot \pi \) is the \( L \)-packet of representations of \( \text{SL}_n(\mathbb{A}_E) \) determined by \( \pi \) (see, for instance, \[\text{HST12 Corollary 2.8}\]).
(2) The set \( H_1 \cdot \pi \) is the set of locally distinguished representations in the \( L \)-packet of \( \text{SL}_n(\mathbb{A}_E) \) determined by \( \pi \) (by Theorem 3.6 and its archimedean analogue).
(3) The set \( H_2 \cdot \pi \) is the set of automorphic representations in the \( L \)-packet of \( \text{SL}_n(\mathbb{A}_E) \) determined by \( \pi \) (by Proposition 5.2).
(4) The set \( H_3 \cdot \pi \) is the set of globally distinguished representations in the \( L \)-packet of \( \text{SL}_n(\mathbb{A}_E) \) determined by \( \pi \) (by Proposition 5.3).

We also record the following observation as a lemma.

Lemma 6.2. Let \( \pi \) as above be of type \( r^d \), then for an \( x \in \mathbb{A}_E^\times \), we have \( x^\pi \in G_\pi \).

Proof. We observe that if \( \pi \) has a \( \psi_1, \ldots, d \)-Whittaker model with respect to the automorphic character \( \psi_1, \ldots, d \), then
\[
\pi^{\text{diag}(x_1, \ldots, x_n)} \in L'(\tilde{\pi}).
\]
In particular, for finite places $v$, the local representation $\pi_{\operatorname{diag}(x_d, I_n)}$ has a $\psi_{1, \ldots, d, v}$-Whittaker model because $\pi_{\operatorname{diag}(x_d, I_n)}$ fixes $\psi_{1, \ldots, d, v}$ by conjugation, hence both $\pi_{\psi_{1, \ldots, d, v}}$ and $\pi_{\operatorname{diag}(x_d, I_n)}$ have a $\psi_{1, \ldots, d, v}$-Whittaker model inside $L(\pi_v)$, so they are equal, and the lemma follows. \hfill $\square$

Next we state the local global principle for $(\SL_n(\mathbb{A}_E), \SL_n(\mathbb{A}_F))$ for square-integrable automorphic representations (for $r$ odd).

**Theorem 6.3.** Let $\pi$ be an irreducible square-integrable automorphic representation of $\SL_n(\mathbb{A}_E)$ such that $L(\pi)$ is distinguished (see Section 5.3). Assume that $r$ is odd and write $\pi = \otimes_v \pi_v$ but this time for $v$ varying through the places of $F$ (hence here $\pi_v$ is $\pi_w$ for $w$ the place in $E$ lying over $v$ if $v$ does not split in $E$, and $\pi_v = \pi_w \otimes \pi_{w_2}$ if $v$ splits into $(w_1, w_2)$). Then, $\pi$ is distinguished with respect to $\SL_n(\mathbb{A}_F)$ if and only if each $\pi_v$ is $\SL_n(F_v)$-distinguished.

**Proof.** One direction is obvious, so we suppose that $\pi$ is locally distinguished. We can always suppose that $\pi$ is Galois conjugate self-dual (cf. Remark 6).

The group $G_\pi$ is Galois stable by Proposition 6.1. As in [AP13, Theorem 7.1], we need to prove that the group

$$(H_1 \cap H_2) / H_3$$

is trivial. In order to show that $H_1 \cap H_2 \subseteq H_3$, we claim that $H_2 \cap \mathbb{A}_F^\times \subseteq H_3$.

So let $x \in E^\times G_\pi \cap \mathbb{A}_F^\times$. Note that $x^2 = xx \theta$, as $x \in \mathbb{A}_F^\times$. Since $G_\pi$ is Galois stable, we get $x^2 \in F^\times G_\pi = H_3$. Indeed, writing $x = hk$, $h \in E^\times$, $k \in G_\pi$, we get

$$x^2 = xx \theta = hkh \theta k \theta = hh \theta kk \theta \in F^\times G_\pi.$$ 

Also $x^r \in G_\pi$ by Lemma 6.2. We have thus shown that both $x^2$ and $x^r$ are in $H_3$. It follows that $x \in H_3$, as $r$ is odd. \hfill $\square$

**Remark 8.** The simplifying role played by the fact that $r$ is odd in the proof of Theorem 6.3 is quite analogous to its role in the proof of local multiplicity one, when $n$ is odd, for the pair $(\SL_n(E), \SL_n(F))$ (see [Ana05, p. 183] or [AP18, p. 1703]).

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