ON THE GRAVITATIONAL POTENTIAL OF AN INHOMOGENEOUS ELLIPSOID OF REVOLUTION (SPHEROID)

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Abstract

It is shown that the gravitational potential outside an inhomogeneous ellipsoid of revolution (spheroid) whose isodensity surfaces are confocal spheroids is identical to the gravitational potential of a homogeneous spheroid of the same mass.

1 Introduction

The result presented in this note was published in our paper entitled “Rotation of gas above the Galactic disk” (Astrophysics, 1988, 28, 57-64; see also Proceedings of the Joint Varenna-Abastumani International School and Workshop on Plasma Astrophysics, 1986, ESA SP-251, 551-556). (Below we give Sect. 2 of this paper, as it appeared in Astrophysics.) Recently B.P. Kondratyev (2003, The Potential Theory and Equilibrium Figures1, Moskow-Izhevsk: The Institute of Computer Science, 624 p.) showed that the same result is valid in the more general case of confocal triaxial ellipsoids.

2 The Gravitational Potential of the Disk

In this section, we obtain the gravitational potential of an ellipsoid of revolution having as isodensity surfaces confocal ellipsoids. The gravitational potential of a body of arbitrary shape is

$$\Phi(\vec{r}) = -G \int \frac{dM}{R} = G \int \frac{\rho(\vec{r})d^3r'}{|\vec{r} - \vec{r}'|},$$

(1)

where the integration is over the complete volume of the body. If the body possesses some symmetry (exact or approximate), then the most effective method

1We thank O.G. Chkhetiani for bringing this monography to our attention.
for finding the potential is an expansion with respect to orthogonal functions. The particular choice of the orthogonal system of functions depends on the symmetry.

We shall assume that the stellar disk of the Galaxy has the form of an oblate ellipsoid of revolution (spheroid). The section of the disk perpendicular to the plane of rotation has the form of an ellipse. It is then natural to make all the calculations in a system of oblate spheroidal coordinates:

\[
\begin{align*}
  x &= c[(\xi^2 + 1)(1 - \eta^2)]^{1/2} \cos \phi, \\
y &= c[(\xi^2 + 1)(1 - \eta^2)]^{1/2} \sin \phi, \\
z &= c\xi \eta, \quad 0 \leq \xi < \infty, \quad -1 \leq \eta < 1, \quad 0 \leq \phi \leq 2\pi.
\end{align*}
\] (2)

The Lamé parameters in this system of coordinates have the form

\[
\begin{align*}
h_{\xi} &= c\left[\frac{\xi^2 + \eta^2}{\xi^2 + 1}\right]^{1/2}, \\
h_{\eta} &= c\left[\frac{\xi^2 + \eta^2}{1 - \eta^2}\right]^{1/2}, \\
h_{\phi} &= c[(\xi^2 + 1)(1 - \eta^2)]^{1/2},
\end{align*}
\]

and the element of volume can be expressed in terms of the introduced coordinates as follows:

\[
d^3r = c^3(\xi^2 + \eta^2)d\xi d\eta d\phi,
\] (3)

where \(c = \sqrt{a^2 - b^2}\) is the half-distance between the focuses of the spheroid, and \(a\) and \(b\) are, respectively, the semimajor and semiminor axes of the spheroid.

The boundary of the spheroid is determined by the relation \(\xi = \xi_0 = b/c\).

In the oblate spheroidal coordinates we can expand \(R^{-1}\) in a series in associated Legendre functions [7,8]:

\[
\begin{align*}
  \frac{1}{R} &= \frac{1}{c} \sum_{n=0}^{\infty} (2n + 1) \sum_{m=0}^{\infty} \epsilon_m i^{m+1} \left[\frac{(n-m)!(n+m)!}{(2n+1)!}\right]^2 \cos[m(\phi - \phi')] \times \\
  &\quad \times P^m_n(\eta') P^m_n(\eta), \\
  &\quad \text{for } m = 0, \\
  &\quad \epsilon_m = 1, \quad \text{for } m > 0,
\end{align*}
\] (4)

where \(P^m_n\) and \(Q^m_n\) are associated Legendre functions of the first and second kinds, respectively.

Substituting the expansion (4) for the case \(\xi > \xi_0 \geq \xi'\), i.e., outside the disk, in the general formula (1) and taking into account (3), we obtain a representation of the gravitational potential of the ellipsoid in the oblate spheroidal coordinates:

\[
\Phi(\xi, \eta, \phi) = -Gc^2 \int \rho(\xi', \eta', \phi') (\xi'^2 + \eta'^2) \sum_{n=0}^{\infty} (2n + 1) \sum_{m=0}^{\infty} \epsilon_m i^{m+1} \times \\
  \left[\frac{(n-m)!(n+m)!}{(n+m)!}\right]^2 \cos[m(\phi - \phi')] P^m_n(\eta') P^m_n(\eta) Q^m_n(i\xi) d\xi' d\eta' d\phi'.
\]
The integration over $\xi'$ is from 0 to $\xi_0$, over $\eta'$ from $-1$ to $+1$, and over $\phi$ from 0 to $2\pi$.

We assume that the density does not depend on $\phi$ (axial symmetry), and then all integrals with $m > 0$ are zero. The expression for the gravitational potential simplifies,

$$\Phi(\xi, \eta) = -Gc^2 \int \rho(\xi', \eta')(\xi'^2 + \eta'^2) \times$$

$$\left[ i \sum_{n=0}^{m} (2n+1) P_n(\eta') P_n(\xi') P_n(\xi) Q_n(i\xi') \right] d\xi' d\eta' d\phi' .$$

We consider the terms of this series

$$\Phi(\xi, \eta) = \Phi_0(\xi, \eta) + \Phi_1(\xi, \eta) + \Phi_2(\xi, \eta) + ... ,$$

where $\Phi_0, \Phi_1, \Phi_2$, etc., are determined by

$$\begin{cases} 
\Phi_0(\xi, \eta) = -c^2GI_0\arctg(1/\xi), \\
I_0 = 2\pi \int_0^{\xi_0} \int_{-1}^{1} \rho(\xi', \eta')(\xi'^2 + \eta'^2) P_0(\eta') P_0(i\xi') d\xi' d\eta' ; 
\end{cases}$$

$$\begin{cases} 
\Phi_1(\xi, \eta) = -3c^2GI_1\eta[\arctg(1/\xi) - 1], \\
I_1 = 2\pi i \int_0^{\xi_0} \int_{-1}^{1} \rho(\xi', \eta')(\xi'^2 + \eta'^2) P_1(\eta') P_1(i\xi') d\xi' d\eta' ; 
\end{cases}$$

$$\begin{cases} 
\Phi_2(\xi, \eta) = \frac{5c^2GI_2(3\eta^2 - 1)}{2}(1 + 3\xi^2)[\arctg(1/\xi) - 3\xi], \\
I_2 = 2\pi \int_0^{\xi_0} \int_{-1}^{1} \rho(\xi', \eta')(\xi'^2 + \eta'^2) P_2(\eta') P_2(i\xi') d\xi' d\eta' . 
\end{cases}$$

The constants $I_0, I_1, I_2$, etc., can be readily determined by specifying the particular form of the density function. For $\rho = \rho(\xi)$, i.e., when the density is distributed over confocal spheroids (and also in the special case $\rho = \text{const}$), all the constants except $I_0$ and $I_2$ are zero. This can be readily seen by representing $\xi^2 + \eta^2$ in the form

$$\xi^2 + \eta^2 \equiv P_0(\eta) \left( \xi^2 + \frac{1}{3} \right) + \frac{2}{3} P_2(\eta)$$

and by using in the integration over $\eta$ the orthogonality property of the Legendre functions:

$$\int_{-1}^{1} P_n(\eta) P_m(\eta) d\eta = \delta_{nm} \frac{2}{2n+1} .$$

In the well-known (see, for example, [9]) special case $\rho = \text{const}$, the constants $I_0$ and $I_2$ have the form

$$I_0 = \frac{4\pi}{3} \rho \xi_0 (1 + \xi_0^2) = \frac{1}{c^3} \rho V = \frac{M}{c^3} ,$$

$$I_2 = -\frac{1}{5} I_0 .$$
In (8) $V = (4\pi/3)a^2b$ is the volume of the spheroid, and $M$ is its mass.

After substitution of (6), (7), and (9) in (5) we obtain

$$\Phi(\xi, \eta) = -\frac{GM}{c} \left\{ \arctg(1/\xi) + \frac{1}{4}(3\eta^2 - 1)(1 + 3\xi^2)\arctg(1/\xi) - 3\xi \right\}. \quad (10)$$

The oblate spheroidal coordinates $\xi$ and $\eta$ are related to the cylindrical coordinates $r$ and $z$ by

$$\xi = \frac{1}{\sqrt{2}c} \sqrt{\kappa + \rho}^{1/2}, \quad \eta = \frac{1}{\sqrt{2}c} \sqrt{\kappa - \rho}^{1/2},$$
$$\kappa = \sqrt{\rho^2 + 4z^2c^2}^{1/2}, \quad \rho = z^2 + r^2 - c^2.$$ 

After some manipulations the expression (10) can be shown to be identical to the well-known expression for the gravitational potential of a homogeneous spheroid [9]. However, it is found that this expression is unchanged in the more general case $\rho = \rho(\xi)$, since the relation (9) between the constants $I_0$ and $I_2$ remains true. Thus, the gravitational potential outside an inhomogeneous ellipsoid of revolution whose isodensity surfaces are confocal ellipsoids is identical to the gravitational potential of a homogeneous ellipsoid of revolution of the same mass. Similarly, the gravitational potential outside a spherically symmetric mass distribution does not depend on the particular distribution of the density.

We note that the gravitational potential of an ellipsoid of revolution for the special case of isodensity surfaces in the form of confocal ellipsoids was obtained in [10], but the identity of the obtained potential and the potential of a homogeneous ellipsoid was not pointed out.

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