Charge Orbits of Extremal Black Holes in Five Dimensional Supergravity

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ABSTRACT

We derive the $U$-duality charge orbits, as well as the related moduli spaces, of “large” and “small” extremal black holes in non-maximal ungauged Maxwell-Einstein supergravities with symmetric scalar manifolds in $d = 5$ space-time dimensions.

The stabilizer groups of the various classes of orbits are obtained by determining and solving suitable $U$-invariant sets of constraints, both in “bare” and “dressed” charges bases, with various methods.

After a general treatment of attractors in real special geometry (also considering non-symmetric cases), the $\mathcal{N} = 2$ “magic” theories, as well as the $\mathcal{N} = 2$ Jordan symmetric sequence, are analyzed in detail. Finally, the half-maximal ($\mathcal{N} = 4$) matter-coupled supergravity is also studied in this context.
1 Introduction

Five-dimensional supergravity theories with non-maximal supersymmetry \(2 \leq N < 8\), emerging from Calabi-Yau compactifications of M-theory, admit extremal black \(p\)-brane solutions in their spectrum \([1]\). In particular, ungauged theories admit extremal black holes \((p = 0)\) and black strings \((p = 1)\) which are asymptotically flat, and reciprocally related through \(U\)-duality. These objects have been intensely studied along the years, due to the wide range of classical and quantum aspects they exhibit.

For asymptotically flat, spherically symmetric and stationary solutions, the Attractor Mechanism \([3, 4, 5, 6]\) proved to be a crucial phenomenon, determining, in a universal fashion, the stabilization of scalar fields in the near-horizon geometry in terms of the fluxes of the two-form field strengths of the Abelian vector fields coupled to the system. Moreover, the Attractor Mechanism turned out to be important also to unravel dynamical properties such as split attractor flows \([7]\) and wall crossing \([8]\), also in relation to string topological partition functions \([9]\) (see also \([10]\) for a recent account and list of Refs.). In \(d = 5\) space-time dimensions, progress has been achieved also with the discovery of new attractor solutions (see \([12]\)) as well as with the formulation of a first-order formalism governing the evolution dynamics of non-supersymmetric scalar flows \([13]\).

For supergravity theories with scalar manifolds which are symmetric cosets, the extremal solutions of the ungauged theory can be classified through the orbits of the relevant representation space of the \(U\)-duality group, in which the corresponding supporting charges sit. The relation between \(U\)-invariant BPS conditions and charge orbits in \(d = 5\) supergravities has been the subject of various studies along the years \([14, 15, 16, 17, 18, 19, 20]\).

The present paper extends to \(d = 5\) space-time dimensions the 4-dimensional investigation of \([21]\).

We derive the \(U\)-duality charge orbits, as well as the related moduli spaces, of “large” and “small” extremal black holes and black strings in ungauged Maxwell-Einstein supergravities with symmetric scalar manifolds. The stabilizer groups of the various classes of orbits are obtained by determining and solving suitable \(U\)-invariant sets of constraints, both in “bare” and “dressed” charges bases, as well by exploiting \(\ddot{\text{I}}\ddot{n\ddot{\text{o}}\ddot{n}\ddot{\text{ũ}}-\text{Wigner}}\) contractions and \(SO(1, 1)\)-gradings.

It is here worth pointing out that in this paper we will not deal with maximal \(N = 8, d = 5\)

\(^{1}\) Here \(U\)-duality is referred to as the “continuous” version, valid for large values of the charges, of the \(U\)-duality groups introduced by Hull and Townsend \([2]\).
supergravity, because a complete analysis of extremal black hole attractors and their “large” and “small” charge orbits is already present in literature, see e.g. [17, 14, 15, 16, 18, 22, 23, 20, 24, 25, 26]. We will just mention such a theory shortly below Eq. (4.3).

The plan of the paper is as follows.

We start and give a résumé of real special geometry in Sect. 2, setting up notation and presenting all formulæ needed for the subsequent treatment of charge orbits and attractors.

In Sect. 3 extremal black hole (black string) attractors are studied in full generality within real special geometry. Starting from the treatment of [19], various refinements and generalizations are performed, in particular addressing the issue of generic, non-symmetric vector multiplets’ scalar manifolds. In Subsect. 3.1 we analyze the various classes of critical points of the effective potential $V$, also within the so-called “new attractor” approach (see Subsubsect. 3.1.4). Then, in Subsect. 3.2 we compute the higher order covariant derivatives of the previously introduced rank-3 invariant tensor $T_{xyz}$, which will play a key role in the subsequent developments and results, exposed in Subsects. 3.3 and 3.4 respectively dealing with generic and homogeneous symmetric real special manifolds. A general analysis of the Hessian matrix of $V$, crucial in order to establish the stability of considered attractor points, is then performed in Subsect. 3.5.

In Sect. 4 all “small” charge orbits of symmetric “magic” real special geometries are explicitly determined and classified, by exploiting the properties of the functional $\hat{I}_3$ introduced in Subsubsect. 3.3.3. Note that “small” charge orbits support non-attractor solutions, which have vanishing Bekenstein-Hawking [27] entropy in the Einsteinian approximation. Nevertheless, they can be treated by exploiting their symmetry properties under $U$-duality.

Sect. 5 analyzes the “duality” relating the $\mathcal{N} = 2$ “magic” theory coupled to 14 Abelian vector multiplets and the $\mathcal{N} = 6$ “pure” supergravity, both based on the rank-3 Euclidean Jordan algebra $J^3_H$ and thus sharing the very same bosonic sector.

Then, Sect. 6 is devoted to the analysis of the “large” (Subsect. 6.1) and “small” (Subsect. 6.2) charge orbits of $\mathcal{N} = 2$ Jordan symmetric sequence. Similarly, Sect. 7 provides a detailed treatment of the “large” (Subsect. 7.1) and “small” (Subsect. 7.2) charge orbits of the half-maximal ($\mathcal{N} = 4$) matter coupled supergravity. The analysis of both Sects. 6 and 7 is made in the “bare” charges basis, and various subtleties, related to the reducible nature of the $d = 5$ $U$-duality group and disconnectedness of orbits in these two theories, are elucidated.

Some Appendices conclude the paper, containing various details concerning the determination of the “small” orbits in symmetric “magic” real special geometries.

The resolution of $U$-invariant defining (differential) constraints, both in “bare” and “dressed” charges bases, is performed in App. A.

Then, in App. B we give an equivalent derivation of all “small” charge orbits of symmetric “magic” real special geometries, relying on group theoretical procedures, namely İnönü-Wigner contractions (Sub-App. B.1) and $SO(1, 1)$-three-grading (Sub-App. B.2).

Finally, we point out that all results on charge orbits can actually be obtained in various other ways, including the analysis of cubic norm forms of the relevant Jordan systems in $d = 5$; this will be investigated elsewhere.

2 Résumé of Real Special Geometry (RSG)

Real special geometry (RSG) ([28, 29, 30, 31, 32, 33] and Refs. therein) is the geometry underlying the scalar manifold $M_5$ (with Euclidean signature) of Abelian vector multiplets coupled to the
In the present Section, we recall some basic facts about RSG, setting up notation and presenting all formulæ needed for the subsequent treatment of charge orbits and attractors. Apart from a slight changes in notation, we will adopt the conventions of [19], which are slightly different from the ones used in [34] (see the observations in [34] itself).

We start by specifying the kind and range of indices being used. \( i = 0, 1, \ldots, n_V \) is the index in the “ambient space” (in which \( M_5 \) is defined through a cubic constraint; see Eq. (2.5) below). “0” is the index pertaining to the (“bare”) \( d = 5 \) graviphoton, and \( n_V \) stands for the number of Abelian vector multiplets coupled to the supergravity multiplet. On the other hand, \( x = 1, \ldots, n_V \), and \( a = 1, \ldots, n_V \) respectively denote “curved” and (local) “flat” coordinates in \( M_5 \).

The metric \( a_{ij} \) in the “ambient space” (named \( g_{ij} \) in [34]) can be defined as follows:

\[
a_{ij} = -\frac{1}{3} \frac{\partial^2 \log V(\lambda)}{\partial \lambda^i \partial \lambda^j},
\]

where

\[
V(\lambda) \equiv d_{ijk} \lambda^i \lambda^j \lambda^k > 0
\]

is the volume of \( M_5 \) itself, and \( d_{ijk} = d_{ijk} \) is a rank-3 completely symmetric invariant tensor (see further below). In turn, the \( \lambda^i \)’s are some real functions (with suitable features of smoothness and regularity) of the set of scalars \( \phi^x \) of the theory, coordinatizing \( M_5 \):

\[
\lambda^i = \lambda^i(\phi^x).
\]

They do satisfy the inequality (2.2). As elucidated e.g. in [34], the \( \lambda^i \)’s are nothing but the (opposite of the) imaginary (“dilatonic”) part of the complex scalar fields of the special Kähler geometry (SKG) based on a cubic holomorphic prepotential (usually named \( d \)-SKG; see e.g. [32, 36]), endowing the Abelian vector multiplets’ scalar manifold of \( \mathcal{N} = 2 \) Maxwell-Einstein supergravity in 4 space-time dimensions. In this respect, the “ambient space” in 5 dimensions is nothing but the “dilatonic sector” of the \( d \)-SKG in 4 dimensions.

It is now convenient to introduce rescaled variables as follows:

\[
\hat{\lambda}^i \equiv \lambda^i V^{-1/3} \Leftrightarrow d_{ijk} \hat{\lambda}^i \hat{\lambda}^j \hat{\lambda}^k = V(\hat{\lambda}) = 1.
\]

Thus, the metric of \( M_5 \) is the pull-back of \( a_{ij} \) on the hypersurface

\[
V(\lambda) \equiv 1
\]

in the “ambient space”, namely:

\[
g_{xy} \equiv \hat{\lambda}^i_x \hat{\lambda}^j_y a_{ij} \bigg|_{V(\lambda) = 1} = -\frac{1}{3} \frac{\hat{\lambda}^i_x \hat{\lambda}^j_y \partial^2 \log V(\lambda)}{\partial \lambda^i \partial \lambda^j} \bigg|_{V(\lambda) = 1} = g_{xy}(\hat{\lambda}(\phi)) = \left(\sim\right) g_{xy}(\phi),
\]

where (the semicolon denotes Riemann-covariant differentiation throughout)

\[
\hat{\lambda}^i_x \equiv -\sqrt{\frac{3}{2}} \frac{\partial \hat{\lambda}^i}{\partial \phi^x} \equiv -\sqrt{\frac{3}{2}} \hat{\lambda}^i_x = -\sqrt{\frac{3}{2}} \hat{\lambda}^i_{,x}.
\]

Notice that the constraint (2.4) implies

\[
\frac{\partial V(\hat{\lambda})}{\partial \phi^x} = 3d_{ijk} \hat{\lambda}^i_x \hat{\lambda}^j \hat{\lambda}^k - \sqrt{6}d_{ijk} \hat{\lambda}^i_x \hat{\lambda}^j \hat{\lambda}^k = 0.
\]
Let us now introduce $T_{xyz}$, a rank-3 completely symmetric invariant tensor, related to $d_{ijk}$ through the definition

$$T_{xyz} \equiv \hat{\lambda}_x \hat{\lambda}_y \hat{\lambda}_z d_{ijk} = \left( \frac{3}{2} \right)^{3/2} \hat{\lambda}_x \hat{\lambda}_y \hat{\lambda}_z d_{ijk} = T(xyz),$$

(2.9)

whose inversion reads

$$d_{ijk} = \frac{5}{2} \hat{\lambda}_x \hat{\lambda}_y \hat{\lambda}_z - \frac{3}{2} \phi_l (ij) \hat{\lambda}_k + T_{xyz} \hat{\lambda}_x \hat{\lambda}_y \hat{\lambda}_z d_{ijk} = T_{xyz}^{(ij)},$$

(2.10)

where

$$\phi_l (\hat{\lambda} \mid \nu (\lambda) \equiv 1).$$

(2.11)

In other words, $T_{xyz}$ is the $\phi$-dependent “dressing” (through $\hat{\lambda}_x (\phi)$’s) of the constant ($\phi$-independent) tensor $d_{ijk}$. It is here worth anticipating that Eqs. (2.9) and (2.10) play the key role to relate the formalism based on “bare” charges with the formalism based on the “dressed” charges (see further below).

$T_{xyz}$ enters the so-called “RSG constraints”, relating in $M_5$ the Riemann tensor $R_{xyzw}$ to the metric tensor $g_{xy}$, as follows:

$$R_{xyzw} = \frac{4}{3} \left( g_{x[u} g_{z]y} + T_{x[w} g_{z]y} \right) = \frac{4}{3} \left( g_{x[u} g_{z]y} + T_{x[w'} g_{z]y} g^{w'} \right).$$

(2.12)

It is worth noticing a direct consequence of such “RSG constraints”: the sectional curvature (see e.g. [37] and [38]) of matter charges in RSG globally vanishes:

$$\mathcal{R} (Z) \equiv R_{xyzw} g^{[xx'} g^{[yy'} g^{[zz'} g^{[ww']} Z_{x'} Z_{y'} Z_{z'} Z_{w'} = 0.$$

(2.13)

This is trivially due to the symmetry properties of the Riemann tensor $R_{xyzw}$ (which are the ones for a generic Riemann geometry), and it is a feature discriminating RSG from SKG (in which $\mathcal{R} (Z)$ generally does not vanish; see e.g. [39] [40]).

As a consequence of the constraints (2.12) (within the metric postulate), the definition of $M_5$ to be an homogeneous symmetric manifold

$$R_{xyzw,u} = 0$$

(2.14)

yields

$$(T_{x[w'} u T_{z]y} + T_{x[w} u T_{z]y}) g^{w'} = 0 \iff T_{x[w'} u T_{z]y} g^{w'} = T_{x[w} u T_{z]y} g^{w'} = 0,$$

(2.15)

solved by

$$T_{xyzw,u} = 0.$$

(2.16)

Through Eqs. (2.9) and (2.10), and exploiting the constraints imposed by local $\mathcal{N} = 2$ supersymmetry, it can be shown that Eq. 2.16 implies the following relation between the tensors $d_{ijk}$:

$$d_{ijk} \phi_{mn} d_{pq} k = \delta_{ij}^{m} d_{pq} \phi \phi_{pk} \phi_{kq} \phi_{nl} \phi_{lr} \phi_{ri} = \delta_{ij}^{m} d_{pq} \phi \phi_{pk} \phi_{kq} \phi_{nl} \phi_{lr} \phi_{ri} = \delta_{ij}^{m} d_{pq},$$

(2.17)

where the index-raising through the contravariant metric $\phi_{ij} \phi_{il}$ has been explicited.
3 Attractors in RSG

The present Section is dedicated to the study of attractors in RSG. This has been firstly treated in [19] (and then reconsidered in [20], in connection to $d = 6$).

Starting from the treatment of [19], we will generalize and elaborate further various results obtained therein.

It is worth recalling that no asymptotically-flat dyonic solutions of Einstein Eqs. exist in $d = 5$. Thus, the $d = 5$ asymptotically flat black holes (BHs) can only carry electric charges $q_i$. Their magnetic duals are the $d = 5$ asymptotically flat black strings, which can only carry magnetic charges $p_i$.

We will perform all our treatment within the electric charge configuration. Due to the mentioned BH/black string duality, this does not imply any loss of generality. Furthermore, we will study attractors within the Ansätze of asymptotical (Minkowski) flatness, staticity, spherical symmetry and extremality of the BH space-time metric (if no scalars are coupled, this is nothing but the so-called Tangherlini extremal $d = 5$ BH). The near-horizon geometry of extremal electric BHs and extremal magnetic black strings respectively is $AdS_2 \times S^3$ and $AdS_3 \times S^2$.

3.1 Classes of Critical Points of $V$

From the general theory of Attractor Mechanism [3, 4, 5, 6], the stabilization of scalar fields in proximity of the (unique) event horizon of a static, spherically symmetric and asymptotically flat extremal BH in $\mathcal{N} = 2$, $d = 5$ Maxwell-Einstein supergravity is described by the critical points of the positive-definite effective potential function

\[ V \equiv \sum q_i q_j \left( \hat{\lambda}^i q_i \right)^2 + \frac{3}{2} g^{xy} \hat{\lambda}^y q_i \hat{\lambda}^y q_j = Z^2 + \frac{3}{2} g^{xy} Z_x Z_y, \]  

(3.1)

where the $\mathcal{N} = 2$, $d = 5$ central charge function $Z$ and its Riemann-covariant derivatives ("matter charges") have been defined as follows:

\[ Z \equiv \hat{\lambda}^i q_i; \]  
\[ Z_x \equiv \hat{\lambda}^i x q_i = Z_x = Z_{xx}. \]  

(3.2)  
(3.3)

The definitions (3.2) and (3.3) can be inverted, obtaining the fundamental identities of RSG (in electric formulation) [19]:

\[ q_i = \hat{\lambda}^i Z - \frac{3}{2} g^{xy} \hat{\lambda}^x q_y. \]  

(3.4)

The identities (3.4) relate the basis of "bare" ($\phi$-independent) electric charges $q_i$ to the basis of "dressed" (central and matter) charges $\{Z, Z_x\}$, which do depend on the scalars $\phi^x$, as yielded by definitions (3.2) and (3.3).

By recalling definitions (3.2) and (3.3), one obtains that

\[ Z_{xy} \equiv Z_{x,y} = Z_{x,y} = Z_{x,y} = \frac{2}{3} g^{xy} q_i = \frac{2}{3} g^{xy} Z - T_{xy} g^{xw} Z_w. \]  

(3.5)

Therefore, by using Eq. (3.5) the criticality conditions (alias Attractor Eqs.) for the effective potential $V$ can be easily computed to be [19]:

\[ V_x \equiv V_x = V_{xx} = 2 \left( 2 Z_x Z_x - \frac{3}{2} T_{xy} g^{xw} g^{wz} Z_x Z_z \right) = 0. \]  

(3.6)
A priori, the classes of critical points of $V$ which are non-degenerate (i.e. with $V|_{Vx=0} \neq 0$) are three:

### 3.1.1 $(\frac{1}{2})$BPS

This class is defined by the sufficient (but not necessary) criticality constraint

$$Z_x = 0, \forall x,$$

implying

$$V = Z^2.$$  \hspace{1cm} (3.8)

### 3.1.2 Non-BPS

This class is defined by the constraints

$$\begin{cases} Z \neq 0; \\ Z_x \neq 0 \text{ for at least some } x, \end{cases}$$

which are critical provided the following algebraic constraint among $Z$ and $Z_x$’s hold:

$$Z_x = \frac{1}{2Z} \sqrt{\frac{3}{2}} T_{xyz} g^{yn} g^{zt} Z_y Z_t.$$  \hspace{1cm} (3.10)

At least in symmetric RSG, this implies 19

$$V = 9Z^2.$$  \hspace{1cm} (3.11)

### 3.1.3 Remark

It is here worth recalling the Bekenstein-Hawking entropy-area formula 27, implemented for critical points of $V$:

$$\frac{S_{BH,d=5}}{\pi} = \frac{A_H}{4\pi} = \frac{R_H^2}{4} = (V|_{\partial V=0})^{3/4}. \hspace{1cm} (3.12)$$

The Attractor Mechanism 3 4 5 6 is known to hold only for the so-called “large” BHs, which, through Eq. (3.12), have a non-vanishing Bekenstein-Hawking entropy.

Therefore, attractors in strict sense are given by non-degenerate critical points of $V$. On the other hand, degenerate critical points of $V$, namely critical points such that $V|_{\partial V=0} = 0$ are trivial. Indeed, by virtue of the positive definiteness of $V$ (inherited from the strictly positive definiteness of $a^{ij}$ throughout all its domain of definition), it holds that

$$V = 0 \iff q_i = 0 \forall i,$$

which is the trivial limit of the theory with all (electric) charges switched off.

The same reasoning can be repeated in the magnetic case.

Thus, only “large” BHs do exhibit a (classical) Attractor Mechanism, implemented through non-trivial (alias non-degenerate) critical points of the effective potential itself 6.
3.1.4 “New Attractor” Approach

Through the so-called “new attractor” approach [43], an equivalent form of the \( n_V \) real criticality conditions (i.e. of the so-called Attractor Eqs.) for the various classes of critical points of \( V \) can be obtained by plugging the criticality conditions themselves into the \( n_V + 1 \) real RSG identities \( ^{2} \). By so doing, one respectively obtains:

- **BPS Attractor Eqs.**

  \[
  q_i = \hat{\lambda}_i Z. \tag{3.14}
  \]

  While Eqs. (3.7) are \( n_V \) real differential ones, the \( n_V + 1 \) real Eqs. (3.14) are purely algebraic.

- **Non-BPS Attractor Eqs.**

  \[
  q_i = \hat{\lambda}_i Z - \frac{1}{2} \left( \frac{3}{2} \right)^{3/2} \frac{1}{Z} T^{xyz} Z_y Z_z \hat{\lambda}_{i,x}. \tag{3.15}
  \]

3.2 Derivatives of \( T_{xyz} \)

Now, in order to proceed further, it is convenient to compute the Riemann-covariant derivative of the invariant tensor \( T_{xyz} \), namely \( T_{(xyz)};w \), a quantity which will be relevant in the subsequent treatment. By using the definition (2.9), one obtains

\[
T_{(xyz)};w = -\sqrt{6} \left[ \frac{1}{2} g(yz) g_{xw} + T_{r(yz)T_{xw}s} g^{rs} \right] = T_{(xyz)};w. \tag{3.16}
\]

Consequently, the condition (2.16) for the real special manifold \( M_5 \) to be a symmetric coset can be equivalently recast as follows (see e.g. page 14 of [19], and Eq. (3.2.1.9) of [20]):

\[
T_{r(yz)T_{xw}s} g^{rs} = \frac{1}{2} g(yz) g_{xw}. \tag{3.17}
\]

One can then proceed further, and compute \( T_{xyzw;q} \). Starting from Eq. (3.16) one obtains (within the metric postulate)

\[
T_{xyzw;q} = T_{(xyzw);q} = -2\sqrt{6} T_{r(yz)T_{xw}s} g^{rs} = -2\sqrt{6} T_{r(yz)T_{xw}s} g^{rs} = T_{(xyzw);q}. \tag{3.18}
\]

Through Eq. (3.16), this result can be further elaborated to give:

\[
T_{xyzw;q} = 12 \left[ \frac{1}{2} g(yz) T_{xwq} + T_{vry} T_{p|yz} T_{xw}s g^{ps} g^{rs} \right]. \tag{3.19}
\]

One can now introduce the following rank-5 completely symmetric tensor \( \tilde{E}_{xyzwq} \), which is the “RSG analogue” of the so-called E-tensor \( ^{3} \) of SKG:

\[
\tilde{E}_{xyzwq} = \frac{1}{12} T_{xyzw;q} = \frac{1}{12} T_{(xyzw);q} = \tilde{E}_{(xyzwq)}, \tag{3.20}
\]

\(^2\)The extra real degree of freedom is only apparent, and removed by the homogeneity of degree one of the RSG identities (3.4) under a real overall shift of charges

\[ q_i \rightarrow \eta q_i, \ \eta \in \mathbb{R}. \]

\(^3\)The E-tensor of SKG was firstly introduced in [32], and it has been recently considered in the theory of extremal \( d = 4 \) BH attractors in [14] [45] [39] [21] [40].
satisfying by definition the relation
\[ T_{(q)vr} T_{(p)yz} g^{pr} g^{rs} = \frac{1}{2} g_{yz} T_{xwq}, \] (3.21)
holding globally in RSG.

By recalling the symmetricity condition (2.16), Eqs. (3.18)-(3.21) yield
\[ T_{xyz} = 0 \Rightarrow T_{xyz;w} = 0 \Leftrightarrow \tilde{E}_{xyzwq} = 0 \Leftrightarrow T_{(q)vr} T_{(p)yz} g^{pr} g^{rs} = \frac{1}{2} g_{yz} T_{xwq}. \] (3.22)

3.3 Generic RSG

Let us now consider the value of the effective potential \( V \) at the various classes of its critical points. By recalling its very definition (3.1), Eqs. (3.7) and (3.10) yield the following results:

3.3.1 BPS

Recall Eq. (3.8):
\[ V = Z^2. \] (3.23)
Through Eq. (3.12), this yields to
\[ S_{BH,d} = \frac{A_H}{4\pi} = R_H^2 = V^{3/4} = |Z|^{3/2}. \] (3.24)

3.3.2 Non-BPS and the “Dressed” Charges’ Sum Rule

\[ V = Z^2 + 3 \frac{1}{2} g^{xy} Z_x Z_y = Z^2 + 3 \frac{1}{8} Z^2 g^{xy} T_{xzt} T_{wyz} Z^x Z^t Z^w Z^z. \] (3.25)

By recalling Eq. (3.16), the second term in the r.h.s. of Eq. (7.17) can be further elaborated as follows:
\[ Z_x Z^x = -\frac{1}{8} \sqrt{\frac{3}{2} Z^2} T_{(ztw;s)} Z^z Z^t Z^w Z^s + 3 \frac{1}{16} Z^2 (Z_x Z^x)^2, \] (3.26)
yielding \( Z_x Z^x \neq 0 \):
\[ \frac{3}{2} Z_x Z^x = 8Z^2 + \sqrt{\frac{3}{2} T_{(xyz:w)} Z^x Z^y Z^z Z^w Z^u Z^v}. \] (3.27)
Consequently at non-BPS \( Z \neq 0 \) critical points of \( V \) it generally holds that:
\[ V = 9Z^2 + \tilde{\Delta}, \] (3.28)
where the real quantity
\[ \tilde{\Delta} = \frac{\sqrt{\frac{3}{2} T_{(xyz:w)} Z^x Z^y Z^z Z^w Z^u Z^v}}{Z^u Z^v} \] (3.29)
has been introduced. This latter is the “RSG analogue” of the complex quantity \( \Delta \) introduced in SKG [44] (see also [45, 39, 21, 40]). As \( \Delta \) enters the “dressed” charges’ sum rule at non-BPS \( Z \neq 0 \) critical points of \( V_{BH} \) in SKG (see e.g. Eqs. (282)-(284) of [44]), so \( \tilde{\Delta} \) enters the “dressed” charges’ sum rule (3.28) at non-BPS critical points of \( V \) in RSG, which further simplifies to (3.11) at least in symmetric RSG (having \( \tilde{\Delta} = 0 \) globally). Notice that, through Eq. (3.27) and definition
the (assumed) strictly positive definiteness of $g_{xy}$ (throughout all $M_5$, and in particular at the considered class of critical points of $V$ itself) yields

\[ Z^2 + \frac{\Delta}{8} > 0. \]  

(3.30)

Through Eq. (3.12), Eq. (3.28) yields

\[ \frac{S_{BH, d=5}}{\pi} = \frac{A_H}{4\pi} = R_H^2 = V^{3/4} = \left(9Z^2 + \Delta\right)^{3/4}. \]  

(3.31)

### 3.3.3 The Functional $\tilde{I}_3$

Within a generic RSG, let us now consider the function

\[ \tilde{I}_3 \equiv \frac{1}{6}Z^3 - \frac{3}{8}ZZ_xZ_x - \frac{1}{4}Z_x^2Z_yZ_z. \]  

(3.32)

In general, $\tilde{I}_3$ is a diffeomorphism- and symplectic- invariant function of the scalars $\phi^x$ in $M_5$, or equivalently a functional of the “dressed” charges $\{Z, Z_x\}$ in $M_5$. Its derivative reads (recalling Eq. (3.16))

\[ \tilde{I}_{3, w} = \tilde{I}_{3, w} = -\sqrt{\frac{3}{2}}T_{xyz}Z_x Z_y Z_z \]

\[ = -\frac{1}{2}Z_x Z_x Z_w + \frac{1}{3}T_{xyz}T_{xws} + T_{yzw}T_{xws} + T_{rzx}T_{yws} + T_{rzw}T_{xys} Z_x Z_y Z_z. \]  

(3.33)

From the definition (3.29), it thus follows that

\[ \tilde{\Delta} = \frac{\tilde{I}_{3, x} Z_x}{Z_y Z_z}. \]  

(3.34)

The computation of $\tilde{I}_3$ and $\tilde{I}_{3, x}$ (respectively given by Eqs. (3.32) and (3.33)) at the various classes of critical points of $V$ (specified by Eqs. (3.7)-(3.10)) respectively yield to the following results.

**BPS**

\[ \tilde{I}_3 = \frac{1}{6} Z^3; \]  

(3.35)

\[ \tilde{I}_{3, x} = 0. \]  

(3.36)

Thus, by recalling Eqs. (3.23) and (3.24), it follows that

\[ \frac{S_{BH, d=5}}{\pi} = \frac{A_H}{4\pi} = R_H^2 = |Z|^{3/2} = V^{3/4} = \sqrt{6} \left| \tilde{I}_3 \right|^{1/2}. \]  

(3.37)
Non-BPS  Eq. (3.27) and definition (3.29) yield
\[
Z_x Z^x = \frac{16}{3} Z^2 + \frac{2}{3} \tilde{\Delta}.
\] (3.38)

On the other hand, by recalling Eqs. (3.16) and (3.10), the term \( T_{xyz} Z^x Z^y Z^z \) can be further elaborated at non-BPS \( Z \neq 0 \) critical points of \( V \) as follows:
\[
T_{xyz} Z^x Z^y Z^z = -\frac{1}{2\sqrt{6}} \left( \frac{Z_x Z^x}{Z} \right) \left( \tilde{\Delta} - \frac{3}{2} Z_y Z^y \right).
\] (3.39)

Thus, definition (3.32) yields the following expression of \( \hat{I}_3 \) at non-BPS \( Z \neq 0 \) critical points of \( V \):
\[
\hat{I}_3 = -\frac{9}{2} Z^3 \left( 1 + \frac{7}{6} \tilde{\Delta} \right) \Leftrightarrow \frac{\tilde{\Delta}}{3^2 Z^2} = -\frac{6}{7} \left( \frac{2}{9} \hat{I}_3 + 1 \right).
\] (3.40)

Thus, by recalling Eqs. (3.28) and (3.31), it follows that
\[
\frac{S_{BH,d=5}}{\pi} = \frac{A_H}{4\pi} \equiv R^2_H = \left( 9Z^2 + \tilde{\Delta} \right)^{3/4} = V^{3/4} = \frac{3^{3/2}}{7^{3/4}} \left| Z \right|^{3/2} \left( 1 - \frac{4}{3} \frac{\tilde{\Delta}}{Z^3} \right)^{3/4},
\] (3.41)

thus necessarily yielding
\[
\frac{3}{4} > \frac{\hat{I}_3}{Z^3}.
\] (3.42)

3.4 Symmetric RSG and “Large” Charge Orbits

Let us now consider the case in which \(^4\) \( M_5 = \frac{G_5}{H_5} = \frac{G_5}{mcs \:(H_5)} \) (3.43)
is a symmetric coset.

(At least) in this case, \( d_{ijk} \) is the unique \( G_5 \) -invariant rank-3 completely symmetric tensor, whereas \( T_{xyz} \) is the unique \( H_5 \) -invariant rank-3 completely symmetric tensor.

“Magic” symmetric \( M_5 \)’s are reported in Table 1 (see e.g. \( [32] \), and Refs. therein; see also \( [46] \) for a brief review and list of Refs.).

Besides these four isolated cases, there are two infinite sequences of other symmetric real special manifolds, namely the so-called \( Jordan \) symmetric sequence
\[
M_{J,5,n} \equiv SO \left( 1,1 \right) \times \frac{SO \left( 1 \right)}{SO \left( n \right)}, \; n = n_V - 1 \in \mathbb{N},
\] (3.44)
and the non-Jordan symmetric sequence \( \[41\]
\[
M_{n,J,5,n} \equiv SO \left( 1 \right) \times \frac{SO \left( 1,1 \right)}{SO \left( n \right)}, \; n = n_V \in \mathbb{N},
\] (3.45)

\(^4\) “mcs” is acronym for \textit{maximal compact subgroup} (with symmetric embedding). Unless otherwise noted, all considered embeddings are symmetric. Moreover, the subscript “max” denotes the maximality of the embedding throughout.
Table 1: Homogeneous symmetric real special vector multiplets’ scalar manifolds \( M_5 \) of \( N=2, d=5 \) “magic” supergravity. \( M_5 \)’s also are: 1) the non-BPS \( Z \neq 0 \) moduli spaces of \( N=2, d=4 \) special Kähler symmetric vector multiplets’ scalar manifolds [23]; and 2) the “large” \( 1/2 \)-BPS charge orbits \( \mathcal{O}_{BPS,large} \)’s of \( N=2, d=5 \) Maxwell-Einstein supergravity itself [19]. The “large” non-BPS \( Z \neq 0 \) charge orbits \( \mathcal{O}_{nBPS,large} = M_5^* \) (see e.g. Table 5 of [48] and Refs. therein) and the related non-BPS \( Z \neq 0 \) moduli spaces \( \mathcal{M}_{nBPS,large} \) are reported, as well. The rank \( r \) of the orbit is defined as the minimal number of charges defining a representative solution. As observed in [23], for “magic” supergravities \( n_V = \dim_{\mathbb{R}} M_5 = 3q+2 \), whereas \( \dim_{\mathbb{R}} \mathcal{M}_{nBPS,large} = 2q \), and \( \text{Spin} (1 + q) \subset \tilde{h}_5 \).

\( n_V \) being the number of Abelian vector supermultiplets coupled to the \( N=2, d=5 \) supergravity one.

The sequence (3.45) is the only (sequence of) symmetric RSG which is not related to Jordan algebras of degree three. It is usually denoted by \( L (-1, n - 1) \) in the classification of homogeneous Riemannian \( d \)-spaces (see e.g. [32], and Refs. therein). It will not be further considered here, because it does not correspond to symmetric spaces in four dimensions.

\( G_5 \) and \( H_5 \) can respectively be interpreted as the reduced structure group \( \text{Str}_{0} \) and the automorphism group \( \text{Aut} \) of the corresponding Euclidean Jordan algebra of degree three (see e.g. [47] for a recent review, and Refs. therein):

\[
M_5 = \frac{G_5}{H_5} = \frac{\text{Str}_{0} (J_3)}{\text{Aut} (J_3)}.
\]

(3.46)
Furthermore, (at least\(^5\)) in symmetric RSG, due to Eqs. (2.16) and (3.33), it holds that
\[ \hat{I}_{3,x} = \hat{I}_{3,y} = 0. \]
(3.47)
In other words, \( \hat{I}_3 \) is independent on all scalars \( \phi^x \). Furthermore:
\[ \hat{I}_3 = I_3, \]
(3.48)
where \( I_3 \) is the unique cubic invariant of the relevant electric (ir)repr. \( R_Q \) of \( d = 5 \) \( U \)-duality \( G_5 \), defined by (7.2). As mentioned above, \( d_{ijk} \) is \( G_5 \)-invariant in all RSG, whereas \( d^{ijk} \) is \( G_5 \)-invariant at least in symmetric RSG.

In this framework, by virtue of the relations (7.27) or (7.31), the Bekenstein-Hawking entropy-area formula (3.12) can be completed as follows (recall Eq. (3.48)):
\[ S_{BH,d=5} = \frac{A_H}{4\pi} \equiv R^2_H = (V|_{\partial V = 0})^{3/4} = \sqrt{6}|I_3|^{1/2} = \sqrt{6} \left| \hat{I}_3 \right|^{1/2}. \]
(3.49)

Furthermore, in RSG based on symmetric cosets \( G_5/H_5 \), the representation space of the irrepr. of \( G_5 \) in which the (electric or magnetic) charges sit admit a stratification in disjoint orbits [15, 19]. Such orbits are homogeneous, in some case symmetric, manifolds.

The charge orbits supporting non-degenerate (in the sense specified above; see the end of Subsect. 3.1) critical points of \( V \) are called “large” orbits, because they correspond to the previously introduced class of “large” BHs with non-vanishing Bekenstein-Hawking entropy-area (see Eq. (3.12)). On the other hand, charge orbits corresponding to “small” BHs (having vanishing Bekenstein-Hawking entropy-area) are correspondingly dubbed “small” orbits.

In the treatment of symmetric RSG performed in present Subsection, only “large” orbits, firstly found in [19], are considered.

In Sect. 4 through the properties of the function \( \hat{I}_3 \) defined by Eq. (3.32), the stabilizers of all “small” charge orbits of symmetric RSG will be derived, by suitably solving \( G_5 \)-invariant (sets of) defining differential constraints, as well as by performing suitable group theoretical procedures.

We can now specialize the results obtained in Subsect. 3.3 and in Subsubsection 3.3.3 to “magic” symmetric RSG. The detailed treatment of \( N = 2 \) Jordan symmetric sequence (3.44) will be given in Sect. 6. Actually, the “large” charge orbits of (3.44) have been already considered in [19] (see also [23] for the study of corresponding moduli spaces), but in Sect. 6 the treatment is further refined.

3.4.1 BPS

Eqs. (3.35) and (3.48) yield to
\[ \hat{I}_3 = \frac{1}{6} Z^3 = I_3, \]
(3.50)
and thus:
\[ \frac{S_{BH,d=5}}{\pi} = \frac{A_H}{4\pi} \equiv R^2_H = (V|_{\partial V = 0})^{3/4} = \sqrt{6}|I_3|^{1/2} = \sqrt{6} \left| \hat{I}_3 \right|^{1/2} = |Z|^{3/2}. \]
(3.51)
\(^5\)Notice that, from Eq. (3.33), it follows that
\[ \hat{I}_{3,w} = 0 \iff T_{xyz} \hat{z}^x \hat{z}^y \hat{z}^z = 0, \]
whose (2.16) is a solution.
Such a “large” BH is supported by (electric) charges belonging to the “large” charge orbit (homogeneous symmetric manifold) [19]

$$\mathcal{O}_{\text{BPS,large}} = \frac{G_5}{H_5} = M_5.$$  (3.52)

The compactness of $H_5$ yields the absence of a moduli space related to $\frac{1}{2}$-BPS “large” attractor solutions, a fact that can be seen also from the expression of the Hessian matrix of $V$ evaluated along the BPS criticality constraints (3.7) (see Eq. (3.72) below).

It is worth remarking that $M_5$’s also are the non-BPS $Z \neq 0$ moduli spaces of $\mathcal{N} = 2$, $d = 4$ special Kähler symmetric vector multiplets’ scalar manifolds [23].

Notice that in general

$$\text{dim}_R M_5 = n_V.$$  (3.53)

As observed in [23], for “magic” supergravities (based on Euclidean Jordan algebras of degree three $J^3_A$ over the division algebras $\mathbb{A}$) it holds:

$$\text{dim}_R M_5 = 3q + 2,$$  (3.54)

$$q \equiv \text{dim}_R (\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}) = (8, 4, 2, 1).$$

### 3.4.2 Non-BPS

Eqs. (3.40) and (3.48) yield to

$$\tilde{I}_3 = -\frac{9}{2} Z^3 = I_3.$$  (3.55)

Indeed, from its very definition, in this framework it globally holds that

$$\tilde{\Delta} = 0,$$  (3.56)

and thus (recall Eq. (3.11)):

$$\frac{3}{2} Z_x Z^x = 8 Z^2 \iff V = 9 Z^2.$$  (3.57)

Through Eq. (3.49), it thus follows that

$$\frac{S_{BH,d=5}}{\pi} = \frac{A_H}{4\pi} = R_H^2 = (V|_{\partial V = 0})^{3/4} = \sqrt{6} |\tilde{I}_3|^{1/2} = \sqrt{6} \left| \tilde{I}_3 \right|^{1/2} = 3^{3/2} |Z|^{3/2}. $$  (3.58)

Such a “large” BH is supported by (electric) charges belonging to the “large” charge orbit (homogeneous symmetric manifold)

$$\mathcal{O}_{n\text{BPS,large}} = \frac{G_5}{\tilde{H}_5} = M_5^*;$$  (3.59)

where $\tilde{H}_5$ is the unique non-compact, real form of $H_5 = \text{mcs}(G_5)$ which admits a maximal symmetric embedding into $G_5$:

$$G_5 \supseteq_{\text{max}} \tilde{H}_5.$$  (3.60)

The homogeneous symmetric pseudo-Riemannian manifold $M_5^*$ is the “*-version” of $M_5$, obtained through timelike $d = 6 \rightarrow 5$ reduction from the corresponding anomaly-free uplifted $\mathcal{N} = (1, 0)$, $d = 6$ chiral theory (see e.g. Table 5 of [18], and Refs. therein). Notice that Eq. (3.59) yields to

$$\mathcal{O}_{n\text{BPS,large}} = \mathcal{O}_{\text{BPS,large}}^*,$$  (3.61)
in the sense we have just specified.

The non-compactness of $\tilde{H}_5$ implies the existence of a non-BPS moduli space \[ \mathcal{M}_{nBPS,large} \equiv \frac{\tilde{H}_5}{mc(H_5)} = \frac{\tilde{H}_5}{h_5}. \] (3.62)

As observed in [23], for “magic” supergravities it holds (see e.g. also Table 8 of [46], and Refs. therein):

\[
dim_{\mathbb{R}} \mathcal{M}_{nBPS,large} = 2q;
\]

\[
\text{Spin} (1 + q) \subset \tilde{h}_5,
\]

(3.63)

where $\text{Spin} (1 + q)$ is the spin group in $1 + q$ dimensions. Notice that $2q$ is the number of $d = 6$ (scalarless) vector multiplets needed for an anomaly-free uplift of the considered $N = 2, d = 5$ “magic” Maxwell-Einstein supergravity to the corresponding $N = (1, 0)$ chiral quarter-minimal “magic” supergravity in $d = 6$ (see e.g. Sect. 5 of [20], and Refs. therein).

Thus, by recalling (3.54), the number $\sharp$ of “non-flat” scalar degrees of freedom along $O_{nBPS,large}$ is

\[
\sharp_{nBPS,large} \equiv \dim_{\mathbb{R}} M_5 - \dim_{\mathbb{R}} \mathcal{M}_{nBPS,large} = q + 2. \tag{3.64}
\]

The “large” non-BPS $Z \neq 0$ charge orbits $O_{nBPS,large} = M^*_5$, and the related non-BPS $Z \neq 0$ moduli spaces $\mathcal{M}_{nBPS,large}$ for “magic” models are reported in Table 1. Furthermore, it should be recalled that the Jordan symmetric sequence (3.44) is related to the reducible rank-3 Euclidean Jordan algebra $\mathbb{R} \oplus \Gamma_{1,n}$, where $\Gamma_{1,n}$ is the rank-2 Jordan algebra with a quadratic form of Lorentzian signature $(1, n)$, i.e. the Clifford algebra of $O (n, 1)$ [49].

### 3.5 Hessian Matrix of $V$

From its very definition (3.1), the first derivative of $V$ reads (recall Eq. (3.6))

\[
V_x \equiv V_{,x} = V_x = 2 \left( 2Z_{xx} - \sqrt{\frac{3}{2}} T_{xyz} g^{ys} g^{zt} Z_s Z_t \right). \tag{3.65}
\]

By further differentiating, the global expression of the real Hessian $n_V \times n_V$ matrix of $V$ in a generic RSG can be computed as follows:

\[
V_{xy} = V_{;xy} = 8 \frac{3}{3} g_{xy} \left( Z^2 - \frac{3}{8} Z_w Z^w \right) + 2Z_x Z_y - 8 \sqrt{\frac{2}{3}} Z_{xyz} Z^z + 2 \left( T_{xys} T_{rwz} + 4T_{xsr} T_{yws} \right) g^{rs} Z^z Z^w = V_{(xy)}, \tag{3.66}
\]

where Eqs. (3.5) and (3.16) have been used.

On the other hand, by recalling definition (3.20) and Eq. (3.33), it can be computed that

\[
\tilde{I}_{3;x,y} = -3 \sqrt{\frac{3}{2}} \left( 4\tilde{E}_{xyzw} Z^w + \frac{2}{3} ZT_{xyz} w - \sqrt{\frac{2}{3}} Z_{zwst} T_{ysr} g^{ss'} Z^r \right) Z^z Z^w. \tag{3.67}
\]
Then, further elaboration of Eq. (3.66) is possible for \( Z \neq 0 \). Indeed, in such a case Eq. (3.67) implies that (recall Eq. (3.16))

\[
\begin{align*}
T_{xzw:y}Z^z Z^w &= - \frac{1}{\sqrt{6}} \frac{1}{Z} \tilde{\mathcal{I}}_{3;x:y} - \frac{6}{Z} \tilde{E}_{xyz wr} Z^z Z^w Z^r \\
&+ \frac{1}{2Z} (Z^w Z^w) T_{xyz}Z^z + \frac{1}{Z} Z_y T_{xzw}Z^z Z^w \\
&- \frac{1}{Z} (T_{xzw} T_{yr's} T_{zw} + 2 T_{xzw} T_{yr's} T_{zr}) g^{rz} g^{rv} Z^z Z^w.
\end{align*}
\]

(3.68)

Notice that the symmetry properties \( \tilde{\mathcal{I}}_{3;x:y} = \tilde{\mathcal{I}}_{3;(x:y)} \) and \( T_{xzw:y}Z^z Z^w = T_{(xzw:y)}Z^z Z^w \) are not manifest respectively from Eqs. (3.67) and (3.68), due to the presence of \( \tilde{E}_{xyz wr}, T_{xyz;w}, \) and \( \tilde{\mathcal{I}}_{3;x:y} \) itself. By plugging Eq. (3.68) back into Eq. (3.66), the following result is achieved:

\[
V_{x:y} = V_{x:y} = 4 Z_x Z_y + 8 \frac{2}{3} Z^2 g_{xy} - 8 \sqrt{\frac{2}{3}} Z T_{xyz}Z^z \\
+ \frac{1}{Z} \tilde{\mathcal{I}}_{3;x:y} + 6 \sqrt{\frac{3}{2}} \tilde{E}_{xyz wr} Z^z Z^w Z^r \\
- \sqrt{\frac{3}{2}} \frac{1}{Z} (Z^w Z^w) T_{xyz}Z^z - \sqrt{\frac{6}{Z}} Z_y T_{xzw}Z^z Z^w \\
+ \sqrt{\frac{6}{Z}} (T_{xzw} T_{yr's} T_{zw} + 2 T_{xzw} T_{yr's} T_{zr}) g^{rz} g^{rv} Z^z Z^w \\
+ 4 T_{xzw} T_{ysz} g^{uz'} Z^z Z^s,
\]

(3.69)

holding true for \( Z \neq 0 \). Once again, notice that the symmetry property \( V_{x:y} = V_{x:y} \) is not manifest from Eq. (3.69), due to the presence of \( \tilde{E}_{xyz wr} \) and \( \tilde{\mathcal{I}}_{3;x:y} \).

By inserting the global condition (2.16) into Eq. (3.66), one obtains that

\[
V_{x:y} = V_{x:y} = 4 Z_x Z_y + 8 \frac{2}{3} Z^2 g_{xy} - 8 \sqrt{\frac{2}{3}} Z T_{xyz}Z^z + 4 T_{xzw} T_{ysz} g^{uz'} Z^z Z^s \equiv V_{x:y;symm}. 
\]

(3.70)

This is the global expression of the real Hessian \( n_V \times n_V \) matrix of \( V \) (at least) in symmetric RSG, and indeed it matches the result given by Eq. (5-1) of [19] (see also [20]). Thus, Eqs. (3.66) and (3.70) yield to the following result:

\[
V_{x:y} = V_{x:y;symm} - g_{xy} (Z_u Z^u) - 2 Z_x Z_y + 2 (2 T_{xuw} T_{ysz} + T_{xyz} T_{swz}) g^{uz'} Z^u Z^s. 
\]

(3.71)

### 3.5.1 Evaluation at Critical Points of \( V \)

We will now proceed to evaluate the Hessian matrix of \( V \) given by Eq. (3.66) at the various classes of critical points of \( V \) itself, as given by Eqs. (3.7)-(3.10).

**BPS** The necessary and sufficient BPS criticality constraints (3.7) plugged into Eq. (3.66) yield

\[
V_{x:y} = \frac{8}{3} g_{xy} Z^2.
\]

(3.72)
Eq. (3.72) holds for a generic RSG, and it matches the result given by Eq. (5.2) of [19]. For a strictly positive definite $g_{xy}$ (as it is usually assumed), it implies that the Hessian matrix of $V$ at its BPS critical points has all strictly positive eigenvalues.

As mentioned above, the lack of Hessian massless modes at $\frac{1}{2}$-BPS critical points of $V$ determines the absence of a moduli space in BPS attractor solutions, which thus have all scalar fields $\phi^x$ stabilized at the (unique) event horizon of the considered (electric) $d = 5$ extremal BH.

Non-BPS It is here worth noticing that Eq. (3.10) yields to

$$Z_x Z^x = \sqrt{\frac{3}{2}} \frac{1}{2Z} T_{xyz} Z^x Z^y Z^z. \quad (3.73)$$

By recalling the “dressed” charges’ sum rule given by Eq. (3.27) and definition (3.29), Eq. (3.73) implies

$$\frac{32}{3} Z^2 + \Delta = \sqrt{\frac{3}{2}} \frac{1}{Z} T_{xyz} Z^x Z^y Z^z. \quad (3.74)$$

On the other hand, by using Eq. (3.16), one can compute also that

$$Z_x Z^x = -\frac{1}{8} \sqrt{\frac{3}{2}} \frac{1}{Z^2} T_{xyzw} Z^x Z^y Z^z Z^w + \frac{3}{16} \frac{1}{Z^2} (Z_x Z^x)^2. \quad (3.75)$$

By dividing by $Z_x Z^x \neq 0$, one then obtains the “dressed” charges’ sum rule given by Eq. (3.27). However, one can also interpret Eq. (3.75) as a quadratic Eq. in the unknown $Z_x Z^x$, obtaining the result

$$0 < Z_x Z^x = \frac{8}{3} Z^2 \pm \sqrt{\frac{64}{9} Z^4 - \frac{2}{3} \hat{I}_{3x} Z^x}. \quad (3.76)$$

When $\hat{I}_{3x} = 0$ (i.e. - at least - for symmetric RSG) Eq. (3.76) consistently yields [19]:

$$\frac{3}{2} Z_x Z^x = 8Z^2. \quad (3.77)$$

4 “Small” Charge Orbits and Moduli Spaces in Symmetric “Magic” RSG

In the treatment of symmetric RSG performed in Subsect. 3.4 only “large” charge orbits, supporting solutions to the corresponding Attractor Eqs. (and firstly found in [19]; see also [20]), have been considered.

In the present Section, by exploiting the properties of the functional $\hat{I}_3$ introduced in Subsect. 3.3.3, all “small” charge orbits of “magic” symmetric RSG will be explicitly determined through the resolution of $G_5$-invariant defining (differential) constraints both in “bare” and “dressed” charges bases, as well as through group theoretical techniques.

By definition, $\hat{I}_3 = I_3$ in symmetric RSG, as discussed in Subsect. 3.4, see Eq. (3.48), vanishes for all “small” charge orbits. Consequently, such orbits do not support solutions to the Attractor Eqs. (alias criticality conditions of the effective potential $V$; see Eqs. (3.7)-(3.10), or Eqs. (3.14)-(3.15) in the so-called “new attractor” approach). In other words, the (classical) Attractor Mechanism does not hold for “small” charge orbits, which indeed do support BH states which are intrinsically quantum, in the sense that the effective description through Einstein supergravity fail for them.
Besides the condition of vanishing $\hat{I}_3$, further conditions, formulated in terms of derivatives of $\hat{I}_3$ in some charge basis, may be needed to fully characterize the class of “small” orbits under consideration. It is here worth pointing out that the (sets of) $G_5$-invariant constraints which define “small” charge orbits in homogeneous symmetric real special manifolds $G_5/S_{\text{max}}$ are characterizing equations for charges (in both “bare” and “dressed” bases), but they actually are identities in all scalar fields $\phi^x$, and thus they hold globally in $G_5/S_{\text{max}}$. This is to be contrasted with “large” charge orbits, which are defined through the Attractor Eqs. themselves, which are at the same time characterizing equations for charges (in both “bare” and “dressed” bases) and stabilization equations for the scalars $\phi^x$ at the event horizon of the extremal BH.

As it is well known [23], at non-BPS $Z \neq 0$ critical points of $V$ some scalars are actually unstabilized at the (unique) event horizon of the corresponding “large” extremal BH solutions. Such unstabilized $\phi^x$’s span the moduli space $\mathcal{M}_{\text{nBPS,large}}$ (given by Eq. (3.62)), associated with an hidden compact symmetry of the non-BPS $Z \neq 0$ Attractor Eqs. themselves, which can be traced back to the non-compactness of the stabilizer of the non-BPS $Z \neq 0$ “large” charge orbit $\mathcal{O}_{\text{nBPS,large}}$ (see Eq. (3.59), to be contrasted with Eq. (3.52)).

The “small” charge orbits are homogeneous manifolds of the form:

$$\mathcal{O}_{\text{small}} = \frac{G_5}{S_{\text{max}} \ltimes T},$$

where $\ltimes$ denotes semi-direct group product throughout, and $T$ is the non-semi-simple part of the stabilizer of $\mathcal{O}_{\text{small}}$, which in all symmetric RSG (with some extra features characterizing the symmetric Jordan sequence, see Sect. 6) can be identified with an Abelian translational subgroup of $G_5$ itself.

One can associate a moduli space also to “small” charge orbits, by observing that the non-compactness of $S_{\text{max}} \ltimes T$ yields the existence of a corresponding moduli space defined as:

$$\mathcal{M}_{\text{small}} \equiv \frac{S_{\text{max}}}{\text{mcs}(S_{\text{max}})} \ltimes T.$$

Note that, differently from “large” orbits, for “small” orbits there exists a moduli space $\mathcal{M}_{\text{small}} = T$ also when $S_{\text{max}}$ is compact. As found in [50, 51] for “large” charge orbits of $\mathcal{N} = 2$, $d = 4$ $stu$ model, and recently proved in a model-independent way in [62], the moduli spaces of charge orbits are defined all along the scalar flows, and thus they can be interpreted as moduli spaces of unstabilized scalars at the event horizon (if any) of the extremal BH, as well as moduli spaces of the ADM mass of the extremal BH at spatial infinity. In the “small” case, the interpretation at the event horizon breaks down, simply because such an horizon does not exist at all (at least in Einsteinian supergravity approximation).

In general, the number $\sharp$ of “non-flat” scalar degrees of freedom supported by a (“large” or “small”) charge orbit $\mathcal{O}$ with associated moduli space $\mathcal{M}$ is defined as follows:

$$\sharp \equiv \dim_{\mathbb{R}} M_{d=5} - \dim_{\mathbb{R}} \mathcal{M}.$$  

As an example, let us briefly consider the maximal $\mathcal{N} = 8$, $d = 5$ supergravity, whose “large” and “small” charge orbits have been classified in [15]. The scalar manifold of the theory is

$$M_{\mathcal{N}=8,d=5} \equiv \frac{E_{6(6)}}{USp(8)}, \quad \dim_{\mathbb{R}} = 42.$$  

---

We thank M. Trigiante for a discussion on the “flat” directions of “small” charge orbits.
1. The unique “large” charge orbit is $\frac{1}{8}$-BPS:

$$\mathcal{O}_{\frac{1}{8} \text{-BPS}} = \frac{E_{6(6)}}{F_{4(4)}}, \quad \text{dim}_R = 26,$$

with corresponding moduli space [23]

$$\mathcal{M}_{\frac{1}{8} \text{-BPS}} = \frac{F_{4(4)}}{USp(6) \times USp(2)}, \quad \text{dim}_R = 28.$$  

Thus, the number of “non-flat” directions along $\mathcal{O}_{\frac{1}{8} \text{-BPS}}$ reads

$$\#_{\frac{1}{8} \text{-BPS}} \equiv \dim_R \mathcal{M}_{N=5,d=5} - \dim_R \mathcal{M}_{\frac{1}{8} \text{-BPS}} = 14.$$  

Since the charge orbit is “large”, $\#_{\frac{1}{8} \text{-BPS}}$ also expresses the actual number of scalar degrees of freedom which are stabilized in terms of the electric (magnetic) charges in the near-horizon geometry of the extremal black hole (black string) under consideration.

2. The “small” $\frac{1}{4}$-BPS orbit is

$$\mathcal{O}_{\frac{1}{4} \text{-BPS}} = \frac{E_{6(6)}}{SO(5,4) \times \mathbb{R}^{16}}, \quad \text{dim}_R = 26,$$

with corresponding moduli space

$$\mathcal{M}_{\frac{1}{4} \text{-BPS}} = \frac{SO(5,4)}{SO(5) \times SO(4)} \times \mathbb{R}^{16}, \quad \text{dim}_R = 36.$$  

Thus, the number of “non-flat” directions along $\mathcal{O}_{\frac{1}{4} \text{-BPS}}$ reads

$$\#_{\frac{1}{4} \text{-BPS}} \equiv \dim_R \mathcal{M}_{N=5,d=5} - \mathcal{M}_{\frac{1}{4} \text{-BPS}} = 6.$$  

3. The “small” $\frac{1}{2}$-BPS orbit is

$$\mathcal{O}_{\frac{1}{2} \text{-BPS}} = \frac{E_{6(6)}}{SO(5,5) \times \mathbb{R}^{16}}, \quad \text{dim}_R = 17,$$

with corresponding moduli space

$$\mathcal{M}_{\frac{1}{2} \text{-BPS}} = \frac{SO(5,5)}{SO(5) \times SO(5)} \times \mathbb{R}^{16} = M_{(2,2),d=6} \times \mathbb{R}^{16}, \quad \text{dim}_R = 41,$$

where $M_{(2,2),d=6}$ is the scalar manifold of maximal (non-chiral) supergravity in $d = 6$. Thus, the number of “non-flat” directions along $\mathcal{O}_{\frac{1}{2} \text{-BPS}}$ reads

$$\#_{\frac{1}{2} \text{-BPS}} \equiv \dim_R \mathcal{M}_{N=5,d=5} - \mathcal{M}_{\frac{1}{2} \text{-BPS}} = 1.$$  

As we will point out more than once in the treatment below, result (4.13) expresses the pretty general fact that the unique “non-flat” direction along maximally supersymmetric (namely, $\frac{1}{2}$-BPS) charge orbits is the Kaluza-Klein radius in the dimensional reduction $d = 5 \rightarrow d = 4$.  

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In the treatment of Subsect. 4.1, the $G_5$-invariant constraints defining all classes of “small” charge orbits in all symmetric RSG will be derived. Then they will be solved both in “bare” and “dressed” charge bases in Subsect. A. Furthermore, in App. B the origin of “small” charge orbits (and in particular of $T$) will be elucidated through group theoretical procedures (namely, İnönü-Wigner contractions [57, 58] and $SO(1,1)$-three grading).

While the treatment of Subsect. 4.1 holds for all symmetric RSG, the treatments given in Apps. A and B strictly fit only the isolated cases of symmetric RSG provided by the so-called “magic” symmetric RSG’s [28, 29, 30, 31]. The main results of Apps. A and B are reported in Tables 3 and 4 (the symmetric Jordan sequence (3.44) is considered in Sect. 6). In the “magic” octonionic case $J^O_3$ ($q = 8$), the results of [15] are matched.

Below we summarize the main results of Apps. A and B.

- The “small” lightlike BPS charge orbit ($\dim_{\mathbb{R}} = 3q + 2$)

$$O_{\text{lightlike}, \text{BPS}} = \frac{G_5}{(SO(q + 1) \times A_q) \rtimes \mathbb{R}^{(\text{spin}(q + 1), \text{spin}(Q_q))}},$$

thus with

$$S_{\text{max, lightlike}, \text{BPS}} = SO(q + 1) \times A_q;$$

$$T_{\text{lightlike}, \text{BPS}} = \mathbb{R}^{(\text{spin}(q + 1), \text{spin}(Q_q))}.$$  

$Q_q$ and $A_q$, a further factor group in $S_{\text{max}}$, are given by Table 2. Furthermore, we define

$$\text{spin} \ (q + 1) \equiv \dim_{\mathbb{R}} (\text{Spin} \ (q + 1));$$

$$\text{spin} \ (Q_q) \equiv \dim_{\mathbb{R}} (\text{Spin} \ (Q_q)),$$

with $\text{Spin} \ (q + 1)$ and $\text{Spin} \ (Q_q)$ respectively denoting the spinor irreps. in $q + 1$ and $Q_q$ dimensions. It is worth remarking that $A_q$ is independent on the space-time dimension ($d = 3, 4, 5, 6$) in which the quarter-minimal symmetric “magic” (Maxwell-Einstein) supergravity (classified by $q = 8, 4, 2, 1$) is considered. It also holds that

$$d = 5, 6 : \hat{G}_{\text{cent}} = SO(1, 1) \times SO(q - 1) \times A_q;$$

$$d = 3, 4 : \hat{G}_{\text{cent}} = G_{\text{paint}} = SO(q) \times A_q,$$

where the groups $\hat{G}_{\text{cent}}$ and $G_{\text{paint}}$ are usually introduced in the treatment of supergravity billiards and timelike reductions (for recent treatment and set of related Refs., see e.g. [18].

| $q$ | $Q_q$ | $A_q$ |
|-----|-------|-------|
| 8   | -     | -     |
| 4   | 2     | $SO(3)$ |
| 2   | 2     | $SO(2)$ |
| 1   | -     | -     |

Table 2: $Q_q$ and $A_q$ for the various $N = 2, d = 5$ “magic” supergravities (based on $J^A_3$, $A = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$), classified by $q = \dim_{\mathbb{R}} A = 8, 4, 2, 1$.
see also Table 5 therein, also for subtleties concerning the case \( q = 8 \) in \( d = 5,6 \). The moduli space corresponding to (4.14) is purely translational:

\[
\mathcal{M}_{\text{lightlike,BPS}} = \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))},
\]

(4.21)

with real dimension

\[
\text{spin} \,(q+1) \cdot \text{spin} \,(Q_q) = 2q.
\]

(4.22)

Thus, by recalling (3.54), the number \( \# \) of scalar degrees of freedom on which the ADM mass depends along \( \mathcal{O}_{\text{lightlike,BPS}} \) is (recall Eq. (3.64))

\[
\#_{\text{light,BPS}} \equiv \dim_{\mathbb{R}} \mathcal{M}_5 - \dim_{\mathbb{R}} \mathcal{M}_{\text{lightlike,BPS}} = 3q + 2 - (\text{spin} \,(q+1) \cdot \text{spin} \,(Q_q)) = q + 2.
\]

(4.23)

By recalling Eq. (3.63), it is worth noting that \( \mathcal{M}_{n\text{BPS},\text{large}} \) and \( \mathcal{M}_{\text{lightlike,BPS}} \) have the same real dimension, but they are completely different, as yielded by Eqs. (3.62) and (4.21).

- The “small” lightlike non-BPS charge orbit (\( \dim_{\mathbb{R}} = 3q + 2 \))

\[
\mathcal{O}_{\text{lightlike,nBPS}} = \frac{G_5}{(SO(q,1) \times A_q) \rtimes \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}},
\]

(4.24)

thus with

\[
\mathcal{S}_{\text{max,lightlike,nBPS}} = SO(q,1) \times A_q,
\]

(4.25)

\[
\mathcal{T}_{\text{lightlike,nBPS}} = \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))} = \mathcal{T}_{\text{lightlike,BPS}}.
\]

(4.26)

The related moduli space reads (\( \dim_{\mathbb{R}} = 3q \))

\[
\mathcal{M}_{\text{lightlike,nBPS}} = \frac{SO(q,1)}{SO(q)} \rtimes \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}
\]

\[
= M_{n,5,q} \times \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))},
\]

(4.27)

where \( M_{n,5,q} \) is the \( q \)-th element of the generic non-Jordan symmetric sequence (3.45). Thus, by recalling (3.54), the number \( \# \) of scalar degrees of freedom on which the ADM mass depends along \( \mathcal{O}_{\text{lightlike,nBPS}} \) is

\[
\#_{\text{light,nBPS}} \equiv \dim_{\mathbb{R}} \mathcal{M}_5 - \dim_{\mathbb{R}} \mathcal{M}_{\text{lightlike,nBPS}} = 2q + 2 - (\text{spin} \,(q + 1) \cdot \text{spin} \,(Q_q)) = 2.
\]

(4.28)

- The “small” critical BPS charge orbit (\( \dim_{\mathbb{R}} = 2q + 1 \))

\[
\mathcal{O}_{\text{critical,BPS}} = \frac{G_5}{(G_6 \times A_q) \rtimes \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}},
\]

(4.29)

where

\[
G_6 = SO(1,q + 1)
\]

(4.30)

is the \( U \)-duality group of the corresponding (1,0), \( d = 6 \) chiral supergravity theory. Thus:

\[
\mathcal{S}_{\text{max,critical,BPS}} = G_6 \times A_q;
\]

\[
\mathcal{T}_{\text{critical,BPS}} = \mathcal{T}_{\text{lightlike,nBPS}} = \mathcal{T}_{\text{lightlike,BPS}}.
\]

(4.31)
The related moduli space reads \((\text{dim}_\mathbb{R} = 3q + 1)\)

\[
\mathcal{M}_{\text{critical,BPS}} = \frac{SO(q+1,1)}{SO(q+1)} \times \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}
\]

\[
= M_{nJ_5,q+1} \times \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}.
\]  

Thus, by recalling (3.54), the number \(\sharp\) of scalar degrees of freedom on which the ADM mass depends along \(O_{\text{critical,BPS}}\) is

\[
\sharp_{\text{crit,BPS}} \equiv \text{dim}_\mathbb{R} M_5 - \text{dim}_\mathbb{R} \mathcal{M}_{\text{critical,BPS}}
\]

\[
= 2q + 1 - (\text{spin} (q+1) \cdot \text{spin} (Q_q)) = 1.
\]  

The unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the \(d = 5 \rightarrow d = 4\) reduction. Furthermore, it is worth observing that:

\[
\mathcal{M}_{\text{critical,BPS}} = M_{(1,0),d=6,J_5^R} \times \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))},
\]  

where \(M_{(1,0),d=6,J_5^R}\) is the manifold of tensor multiplets’ scalars in the corresponding \((1,0),d = 6\) theory (see e.g. Sect. 5 of [20] for a recent treatment).

It should also be noticed that \(O_{n\text{BPS,large}}\) (given by Eq. (3.59)) and \(O_{\text{critical,BPS}}\) (given by Eq. (4.29)) share the same compact symmetry, or equivalently that \(M_{n\text{BPS,large}}\) (given by Eq. (3.62)) and \(M_{\text{critical,BPS}}\) (given by Eq. (4.32)) share the same stabilizer group (apart from an \(A_q\) commuting factor), but they do not coincide. This is due to the fact that \(\tilde{H}_5\) and \(G_6 \times A_q\) share the same \(\text{mcs}\), namely

\[
\tilde{h}_5 \equiv \text{mcs} (\tilde{H}_5) = \text{mcs} (G_6 \times A_q) = SO(q+1) \times A_q.
\]  

In the case \(h = \mathbb{R}\) \((q = 1)\), the following further results holds (see also Tables 3 and 4):

\[
\mathcal{M}_{n\text{BPS,large},J_5^R} \times \left\{ \begin{array}{l}
\mathbb{R}^2 \\
\mathbb{R}^{(2,2)}
\end{array} \right\} = M_{n=1,(1,0),d=6,J_5^R} \times \left\{ \begin{array}{l}
\mathbb{R}^2 \\
\mathbb{R}^{(2,2)}
\end{array} \right\} = \left\{ \begin{array}{l}
\mathcal{M}_{\text{critical,BPS},J_5^R}; \\
\mathcal{M}_{\text{lightlike,nBPS},J_5^R}.
\end{array} \right\}
\]  

Notice that \(J_5^R\) is the unique case, among \(J_5^A\) in \(d = 5\), in which \(M_{n\text{BPS,large}}\) and \(M_{\text{critical,BPS}}\) not only share the same stabilizer, but they actually do coincide (up to \(\times \mathbb{R}^{(2,2)}\)). Moreover, \(M_{n\text{BPS,large},J_5^R}\) also coincides with \(M_{\text{lightlike,nBPS},J_5^R}\) (up to \(\times \mathbb{R}^{(2,2)}\)), because the respective charge orbits \(O_{n\text{BPS,large},J_5^R}\) and \(O_{\text{lightlike,nBPS},J_5^R}\) share the same semi-simple, namely non-translational, part of the stabilizer (apart from a commuting \(A_q = SO(2)\) factor), i.e. \(SO(2,1)\).

The Jordan symmetric infinite sequence \([28, 29, 30, 31, 32, 35, 36]\) given by Eq. (3.44) needs some extra care (also at the level of “large” charge orbits), because of the factorization of \(G_5\). The “large” and “small” charge orbits for such a sequence will be treated in Sect. 6. This treatment refines and complete the ones given e.g. in [19, 23, 20].

### 4.1 \(G_5\)-invariant Defining Constraints

As mentioned above, “small” charge orbits in all symmetric RSG are all characterized by the constraint (recall Eq. (3.48)):

\[
\hat{I}_3 = I_3 = 0,
\]  

\[
(4.37)
\]
| $J_3^A$ (+ rel. data) | $\mathcal{O}_{\text{lightlike,BPS}}, \ r = 2$ | $\mathcal{M}_{\text{lightlike,BPS}}$ | $\mathcal{O}_{\text{lightlike,nBPS}}, \ r = 2$ | $\mathcal{M}_{\text{lightlike,nBPS}}$ |
|----------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $A = \mathbb{O}, q = 8$ | $\frac{E_{6(-26)}}{SO(9) \times \mathbb{R}^{16}}$ | $\mathbb{R}^{16}$ | $\frac{E_{6(-26)}}{SO(8,1) \times \mathbb{R}^{16}}$ | $SO(8,1) \times \mathbb{R}^{16}$ |
| $\text{Spin} \ (9) = 16$ | $\text{Spin} \ (Q_4) = 2$ | $\text{Spin} \ (Q_4) = 2$ | | |
| $\sharp_{\text{light,BPS}} = 10$ | $\sharp_{\text{light,nBPS}} = 2$ | $\sharp_{\text{light,BPS}} = 6$ | | |
| $\sharp_{\text{light,nBPS}} = 2$ | $\sharp_{\text{light,nBPS}} = 2$ | $\sharp_{\text{light,nBPS}} = 2$ | | |
| $A = \mathbb{H}, q = 4$ | | | | |
| $A_4 = SO \ (3)$ | $SU^* (6)$ | $\mathbb{R}^{(4,2)}$ | $SU^* (6)$ | $SO(4,1) \times \mathbb{R}^{(4,2)}$ |
| $Q_4 = 2$ | $(SO(5) \times SO(3)) \times \mathbb{R}^{(4,2)}$ | | $(SO(4,1) \times SO(3)) \times \mathbb{R}^{(4,2)}$ | |
| $\text{Spin} \ (5) = 4$ | | | | |
| $\text{Spin} \ (Q_4) = 2$ | | | | |
| $\sharp_{\text{light,BPS}} = 6$ | | | | |
| $\sharp_{\text{light,nBPS}} = 2$ | | | | |
| $A = \mathbb{C}, q = 2$ | | | | |
| $A_2 = SO \ (2)$ | $SL(3, \mathbb{C})$ | $\mathbb{R}^{(2,2)}$ | $SL(3, \mathbb{C})$ | $SO(2,1) \times \mathbb{R}^{(2,2)}$ |
| $Q_2 = 2$ | $(SO(3) \times SO(2)) \times \mathbb{R}^{(2,2)}$ | | $(SO(2,1) \times SO(2)) \times \mathbb{R}^{(2,2)}$ | |
| $\text{Spin} \ (3) = 2$ | | | | |
| $\text{Spin} \ (Q_2) = 2$ | | | | |
| $\sharp_{\text{light,BPS}} = 4$ | | | | |
| $\sharp_{\text{light,nBPS}} = 2$ | | | | |
| $A = \mathbb{R}, q = 1$ | | | | |
| $A_2 = SO \ (2)$ | $SL(3, \mathbb{R})$ | $\mathbb{R}^{2}$ | $SL(3, \mathbb{R})$ | $SO(1,1) \times \mathbb{R}^{2}$ |
| $Q_2 = 2$ | $SO(3) \times \mathbb{R}^{2}$ | | | |
| $\text{Spin} \ (2) = 2$ | | | | |
| $\sharp_{\text{light,BPS}} = 3$ | | | | |
| $\sharp_{\text{light,nBPS}} = 2$ | | | | |

Table 3: “Small” lightlike charge orbits $\mathcal{O}_{\text{lightlike,BPS}}$ and $\mathcal{O}_{\text{lightlike,nBPS}}$ (with associated moduli spaces) in symmetric “magic” RSG
| $J^A_4$  | $\mathcal{O}_{\text{critical,BPS}}$, $r = 1$ | $\mathcal{M}_{\text{critical,BPS}}$ |
|-----------------|-----------------|-----------------|
| $\mathbb{A} = \mathbb{O}, q = 8$ | $\frac{\mathcal{E}_{6(-26)}}{\text{SO}(9,1) \times \mathbb{R}^{16}}$ | $\text{SO}(9,1) \times \mathbb{R}^{16}$ |
| $\text{Spin} (9) = 16$, $\#_{\text{crit,BPS}} = 1$ | $\text{SO}(9,1) \times \mathbb{R}^{16}$ | $\text{SO}(9,1) \times \mathbb{R}^{16}$ |
| $\mathbb{A} = \mathbb{H}, q = 4$ | $\frac{\text{SU}^*(6)}{(\text{SO}(5,1) \times \text{SO}(3)) \times \mathbb{R}^{4,2}}$ | $\text{SO}(5,1) \times \mathbb{R}^{4,2}$ |
| $\mathbb{A}_1 = \text{SO} (3), Q_1 = 2$, $\text{Spin} (5) = 4$, $\text{Spin} (Q_1) = 2$, $\#_{\text{crit,BPS}} = 1$ | $\frac{\text{SU}^*(6)}{(\text{SO}(5,1) \times \text{SO}(3)) \times \mathbb{R}^{4,2}}$ | $\text{SO}(5,1) \times \mathbb{R}^{4,2}$ |
| $\mathbb{A} = \mathbb{C}, q = 2$ | $\frac{\text{SL}(3,\mathbb{C})}{(\text{SO}(3,1) \times \text{SO}(2)) \times \mathbb{R}^{2,2}}$ | $\text{SO}(3,1) \times \mathbb{R}^{2,2}$ |
| $\mathbb{A}_2 = \text{SO} (2), Q_2 = 2$, $\text{Spin} (3) = 2$, $\text{Spin} (Q_2) = 2$, $\#_{\text{crit,BPS}} = 1$ | $\frac{\text{SL}(3,\mathbb{C})}{(\text{SO}(3,1) \times \text{SO}(2)) \times \mathbb{R}^{2,2}}$ | $\text{SO}(3,1) \times \mathbb{R}^{2,2}$ |
| $\mathbb{A} = \mathbb{R}, q = 1$ | $\frac{\text{SL}(3,\mathbb{R})}{\text{SO}(2,1) \times \mathbb{R}^{2}}$ | $\text{SO}(2,1) \times \mathbb{R}^{2}$ |
| $\text{Spin} (2) = 2$, $\#_{\text{crit,BPS}} = 1$ | $\frac{\text{SL}(3,\mathbb{R})}{\text{SO}(2,1) \times \mathbb{R}^{2}}$ | $\text{SO}(2,1) \times \mathbb{R}^{2}$ |

Table 4: “Small” critical charge orbit $\mathcal{O}_{\text{critical,BPS}}$ (with associated moduli space $\mathcal{M}_{\text{critical,BPS}}$) in symmetric “magic” RSG
where $\hat{I}_3 = I_3$ is the unique cubic scalar invariant of the relevant electric representation $R_q$ of the $d = 5$ $U$-duality group $G_5$ (in which the electric charges $q_i$'s sit). By recalling definitions (3.32) and (7.2), the “smallness” condition (4.37) can be recast as follows:

$$\hat{I}_3 = 0 \iff I_3 = \frac{3}{2} \left( \frac{3}{2} \right)^{2} ZZ_x Z_x - \left( \frac{3}{2} \right)^{2} T_{xyz} Z^y Z^z = 0;$$

and

$$I_3 = 0 \iff d^{ijk} q_i q_j q_k = 0,$$

in the “dressed” and “bare” charges basis, respectively.

It is here worth noticing that Eq. (4.38) can be recast as a cubic algebraic equation:

$$Z^3 + p Z - q = 0;$$

with polynomial discriminant

$$D \equiv \frac{p^3}{9} + \frac{q^2}{4} = \frac{3^3}{2^6} \left[ 2 (T_{xyz} Z^y Z^z)^2 - (Z_x Z^x)^3 \right].$$

Thus, for $D > 0$ one gets one real and two complex conjugate (unacceptable) roots, whereas for $D < 0$ all roots are real and unequal. In the particular case

$$D = 0 \iff 2 (T_{xyz} Z^y Z^z)^2 = (Z_x Z^x)^3,$$

all roots are real, and at least two equal.

Let us proceed further, by differentiating the functional $\hat{I}_3$ with respect to the “dressed” charges $Z \equiv \{Z, Z_x\}$, as well as function $I_3$ with respect to the “bare” charges $\{q_i\}$. One respectively obtains:

$$\frac{\partial \hat{I}_3}{\partial Z} = \begin{cases} \frac{\partial^2 \hat{I}_3}{\partial Z^2} = \frac{3}{2} ZZ_x Z_x - \frac{3}{2} Z_x Z^x; \\
\frac{\partial^2 \hat{I}_3}{\partial Z_x Z_x} = -
\frac{3}{4} T_{xyz} Z^y Z^z - \frac{3}{2} \left( \frac{3}{2} \right)^{3/2} T_{xyz} Z^y Z^z; \\
\frac{\partial^2 \hat{I}_3}{\partial Z_x Z_y} = \frac{1}{2} d^{ijk} q_i q_j q_k, \\
\frac{\partial^2 \hat{I}_3}{\partial q_i \partial q_j} = \frac{\partial^2 I_3}{\partial q_i \partial q_j}.
\end{cases}$$

where it should be recalled once again that here we are considering symmetric real special manifolds $G_5^\mathbb{R}$, where Eqs. (2.17) and (2.16) all hold true.

A further differentiation with respect to $Z$ or $\{q_i\}$ respectively yields

$$
\frac{\partial^2 \hat{I}_3}{(\partial Z)^2} = \begin{cases}
\frac{\partial^2 I_3}{\partial Z^2} = Z; \\\n\frac{\partial^2 I_3}{\partial Z_x Z_x} = -\frac{3}{4} Z_x Z^x; \\\n\frac{\partial^2 I_3}{\partial Z_x Z_y} = -\frac{3}{4} Z g^{xy} - \left( \frac{3}{2} \right)^{3/2} T_{xyz} Z^z = \frac{\partial^2 \hat{I}_3}{\partial Z_x \partial Z_y}; \\
\frac{\partial^2 I_3}{\partial q_i \partial q_j} = d^{ijk} q_k = \frac{\partial^2 I_3}{\partial q_i \partial q_j}.
\end{cases}
$$

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By further differentiating, one then obtains:

$$\frac{\partial^3 \hat{I}_3}{(\partial Z)^3} = \begin{cases} 
\frac{\partial^3 \hat{I}_3}{(\partial Z)^3} = 1; \\
\frac{\partial^3 \hat{I}_3}{(\partial Z)^3} Z_x = 0; \\
\frac{\partial^3 \hat{I}_3}{\partial Z_x \partial Z_y} = \frac{3}{4} g^{xy} = \frac{\partial^3 \hat{I}_3}{\partial Z_x \partial Z_y}; \\
\frac{\partial^3 \hat{I}_3}{\partial Z_x \partial Z_y \partial Z_z} = -\left(\frac{3}{2}\right)^{3/2} T^{xyz} = \frac{\partial^3 \hat{I}_3}{\partial Z_x \partial Z_y \partial Z_z}; \\
\partial^3 \hat{I}_3 = 0 
\end{cases}$$

(4.48)

Starting from the fourth order of differentiation, all derivatives vanish. This is no surprise, because \( \hat{I}_3 \) is a functional polynomial homogeneous of degree three in “dressed” charges \( Z \), as well as (equivalently) \( I_3 \) is a polynomial homogeneous of degree three in “bare” charges \( q_i \)’s.

At this point, it is possible to classify the various “small” charge orbits through \( G_5 \)-invariant conditions involving \( \hat{I}_3 \) and its non-vanishing functional derivatives with respect to \( Z \), or equivalently through \( G_5 \)-invariant conditions involving \( I_3 \) and its non-vanishing derivatives with respect to \( q_i \)’s.

4.1.1 “Small” Lightlike Orbits

The “small” lightlike charge orbits are defined by the constraints (recall Eqs. (4.38) and (4.39)):

$$\begin{cases} 
\hat{I}_3 = 0 \iff Z^3 - \left(\frac{3}{2}\right)^2 Z Z_x Z_x^2 - \left(\frac{3}{2}\right)^2 T_{xy} Z_x Z_y Z_z = 0; \\
\frac{\partial^2 I_3}{\partial Z_x \partial Z_x} \neq 0 \iff \begin{cases} 
Z^2 - \frac{3}{4} Z_x Z_z^2 \neq 0; \\
and/or \\
Z Z_x + \sqrt{\frac{3}{2} T_{xy} Z_y Z_z} \neq 0 \text{ (at least for some } x), 
\end{cases} 
\end{cases}$$

(4.50)

or equivalently:

$$\begin{cases} 
I_3 = 0 \iff d^{ijk} q_i q_j q_k = 0; \\
\frac{\partial I_3}{\partial q_i} \neq 0 \iff d^{ijk} q_i q_j q_k \neq 0 \text{ (at least for some } i); 
\end{cases}$$

(4.51)

The sets of constraints (4.50) and (4.51) are both \( G_5 \)-invariant, but their manifest invariance is different. Indeed, the “dressed” charge basis \( Z \) is covariant with respect to \( H_5 \), and thus the set of constraints (4.50) exhibits a manifest \( H_5 \)-invariance. Instead, the “bare” charge basis \( \{q_i\} \) is \( G_5 \)-covariant, and thus the set of constraints (4.50) is manifestly \( G_5 \)-invariant.

In the “dressed” charge basis, it is immediate to realize that two classes of “small” lightlike charge orbits exist, namely:
• “small” lightlike charge orbit for which the constraints (4.50) are solved with $Z = 0$:
\[
\left\{ \begin{array}{c}
\frac{\partial \hat{T}_3}{\partial Z} \bigg|_{Z=0} = 0 \quad \Leftrightarrow \quad T_{xyz} Z_x Z_y Z_z = 0; \\
\frac{\partial \hat{T}_3}{\partial Z} \bigg|_{Z=0} \neq 0 \quad \Leftrightarrow \quad \begin{cases} 
Z_x Z^x \neq 0; \\
\text{and/or} \\
T_{xy} Z_y Z^y Z^z \neq 0 \quad \text{(at least for some } x). \end{cases}
\end{array} \right.
\]

Notice that the constraint $Z_x Z^x \neq 0$ is automatically satisfied, because: 1) $g_{xy}$ is assumed to be strictly positive definite, and 2) $Z_x \neq 0$ \textit{at least} for some $x$ (otherwise, since $Z = 0$, one would obtain the trivial limit in which all charges vanish).

• “small” lightlike charge orbit for which the constraints (4.50) are solved with $Z \neq 0$ (also recall Eqs. (4.40)-(4.42)):
\[
\left\{ \begin{array}{c}
\frac{\partial \hat{T}_3}{\partial Z} \bigg|_{Z \neq 0} = 0 \quad \Leftrightarrow \quad Z^3 - \left(\frac{3}{2}\right)^2 Z Z_x Z^x - \left(\frac{3}{2}\right)^3 T_{xyz} Z_x Z_y Z_z = 0; \\
\frac{\partial \hat{T}_3}{\partial Z} \bigg|_{Z \neq 0} \neq 0 \quad \Leftrightarrow \quad \begin{cases} 
Z^2 - \frac{3}{4} Z_x Z^x \neq 0; \\
\text{and/or} \\
Z Z^x + \sqrt{\frac{3}{2}} T_{xy} Z_y Z^z \neq 0 \quad \text{(at least for some } x). \end{cases}
\end{array} \right.
\]

4.1.2 “Small” Critical Orbit

The “small” critical charge orbit is defined by the constraints (recall Eqs. (4.38) and (4.39)):
\[
\left\{ \begin{array}{c}
\hat{T}_3 = 0 \quad \Leftrightarrow \quad Z^3 - \left(\frac{3}{2}\right)^2 Z Z_x Z^x - \left(\frac{3}{2}\right)^3 T_{xyz} Z_x Z_y Z_z = 0; \\
\frac{\partial \hat{T}_3}{\partial Z} = 0 \quad \Leftrightarrow \quad \begin{cases} 
Z^2 - \frac{3}{4} Z_x Z^x = 0; \\
Z Z^x + \sqrt{\frac{3}{2}} T_{xy} Z_y Z^z = 0, \end{cases}
\end{array} \right.
\]

or equivalently:
\[
\left\{ \begin{array}{c}
T_3 = 0 \quad \Leftrightarrow \quad d^{ijk} q_i q_j q_k = 0; \\
\frac{\partial \hat{T}_3}{\partial q_i} = 0 \quad \Leftrightarrow \quad d^{ijk} q_j q_k = 0.
\end{array} \right.
\]

As noticed above for the sets of constraints (4.50) and (4.51), the sets of constraints (4.54) and (4.55) are both $G_5$-invariant: while (4.54) is manifestly invariant only under $H_5 = mcs (G_5)$, (4.55) is actually manifestly $G_5$-invariant.

Once again, in the “dressed” charges basis it is immediate to realize that only one class of “small” critical charge orbits exists, namely:
• “small” critical charge orbit for which the constraints (4.54) are solved with \( Z \neq 0 \):

\[
\begin{align*}
\frac{\partial \hat{I}_3}{\partial Z} \bigg|_{Z \neq 0} &= 0 \iff Z^3 - \left( \frac{3}{2} \right)^2 Z Z_x Z^x - \left( \frac{3}{2} \right)^2 T_{xyz} Z^x Z^y Z^z = 0; \\
\frac{\partial \hat{I}_3}{\partial Z} \bigg|_{Z \neq 0} &= 0 \iff \\
Z^2 - \frac{3}{4} Z_x Z^x &= 0; \\
Z Z^x + \sqrt{\frac{3}{2}} T_{xyz} Z^y Z^z &= 0.
\end{align*}
\]

(4.56)

Notice that, for the same reason the constraint \( \frac{\partial \hat{I}_3}{\partial Z} \bigg|_{Z = 0} \neq 0 \) is automatically satisfied for the “small” lightlike charge orbit whose a representative in the “dressed” charges basis is given by Eq. (4.52), a “small” critical charge orbit with representative having \( Z = 0 \) cannot exist. Indeed, such an orbit should have \( Z = 0 \) and \( Z_x Z^x = 0 \). Due to the assumed strictly positive definiteness of \( g_{xy} \), this would be possible only in the trivial limit of the theory in which all charges do vanish. This can be formally stated as follows:

\[
\frac{\partial \hat{I}_3}{\partial Z} \bigg|_{Z = 0} = 0 \iff Z = 0.
\]

(4.57)

5 \( J_3^H : \mathcal{N} = 2 \) versus \( \mathcal{N} = 6 \)

The rank-3 Euclidean Jordan algebra \( J_3^H (q = 4) \) is related to two different theories, namely an \( \mathcal{N} = 2 \) theory coupled to 14 Abelian vector multiplets and the \( \mathcal{N} = 6 \) “pure” theory. These two theories share the same bosonic sector \([15, 19, 59]\), but their fermionic sectors, exploiting the supersymmetric completion of the bosonic one, are different.

Thus, it also follows that the supersymmetry-preserving features of the “large” and “small” charge orbits of the relevant irrep. 15 of \( G_5 = SU^* (6) \) are different. The \( \mathcal{N} \)-dependent supersymmetry properties of the various orbits are given in Table 5 (notice they are consistent with the results of [53]). In the “large” (attractor) cases, these match the results of [20].

6 \( \mathcal{N} = 2, d = 5 \) Jordan Symmetric Sequence

The Jordan symmetric sequence of \( \mathcal{N} = 2, d = 5 \) supergravity coupled to \( n_V = n + 1 \) vector multiplets reads (\( \text{dim}_\mathbb{R} = n + 1, \text{rank} = 2, n \in \mathbb{N} \cup \{0\} \))

\[
M_{\mathcal{N}=2,d=5,Jordan,symm} = SO (1,1) \times \frac{SO (1,n)}{SO (n)}.
\]

(6.1)

This sequence is associated to the rank-3 Euclidean reducible Jordan algebra \( \mathbb{R} \oplus \Gamma_{1,n} \). In the following treatment, we will determine the “large” and “small” orbits of the irrepr. \( (1,1 + n) \) of the \( U \)-duality group \( SO (1,1) \times SO (1,n) \).

For brevity’s sake, we will do this only through an analysis in the “bare” charges’ basis.

Without any loss in generality, one can choose to treat only \( d = 5 \) extremal (electric) BHs. Indeed, due to the symmetry of the reducible coset [6.1], the treatment of \( d = 5 \) extremal (magnetic) black strings is essentially analogous.

Two disconnected geometric structures emerge in the treatment, namely:
Table 5: $\mathcal{N}$-dependent supersymmetry-preserving features of “large” and “small” charge orbits of the irrepr. 15 of the $d = 5$ $U$-duality group $SU^*(6)$, related to $J_3^H$. This corresponds to two “twin” theories, sharing the same bosonic sector: an $\mathcal{N} = 2$ Maxwell-Einstein theory and the $\mathcal{N} = 6$ “pure” theory. The subscript “$H$” stands for “(evaluated at the) horizon”

| $J_3^H$ | $\mathcal{N} = 2$ | $\mathcal{N} = 6$ |
|---------|--------------------|--------------------|
| $SU^*(6)/USp(6)$ | $\frac{1}{2} - BPS$ | $nBPS$, $Z_{AB,H} = 0$, $X_H \neq 0$ |
| “large”, $I_3 \neq 0$ | | |
| $SU^*(6)/USp(4,2)$ | $nBPS, Z_H \neq 0$ | $\frac{1}{2} - BPS$, $Z_{AB,H} \neq 0$, $X_H \neq 0$ |
| “large”, $I_3 \neq 0$ | | |
| $SU^*(6)/(SO(5) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}$ | $\frac{1}{2} - BPS$ | $\frac{1}{6} - BPS$ |
| “small”, $I_3 = 0$ | | |
| $SU^*(6)/(SO(4,1) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}$ | $nBPS$ | $\frac{1}{3} - BPS$ |
| “small”, $I_3 = 0$ | | |
| $SU^*(6)/(SO(5,1) \times SO(3)) \rtimes \mathbb{R}^{(4,2)}$ | $\frac{1}{2} - BPS$ | $\frac{1}{2} - BPS$ |
| “small”, $\partial I_3 = 0$ | | |

- Timelike two-sheet hyperboloid $T_n$, with the two disconnected sheets $T_n^\pm$ respectively related to $q_0 \geq 0$:
  $$T_n \equiv \frac{SO(1,n)}{SO(n)} \bigg|_{q_I^2 > 0} = T_n^+ \cup T_n^-; \ T_n^+ \cap T_n^- = \emptyset.$$  
  (6.2)

- Forward/backward light-cone $\Lambda_n$ of $(n + 1)$-dimensional Minkowski space with metric $\eta_{IJ}$ defined by (6.5), with two (forward $\Lambda_n^+$ and backward $\Lambda_n^-$) cone branches, respectively related to $q_0 \geq 0$:
  $$\Lambda_n \equiv \frac{SO(1,n)}{SO(n - 1) \times \mathbb{R}^{n-1}} = \Lambda_n^+ \cup \Lambda_n^-; \ \Lambda_n^+ \cap \Lambda_n^- = 0.$$  
  (6.3)

0 here denoting the origin of $\Lambda_n$ itself.

Due to such structures, as well as to the lower ($\mathcal{N} = 2$) supersymmetry, the case study of “large” and “small” charge orbits in $\mathcal{N} = 2$, $d = 5$ Jordan symmetric sequence exhibits some subtleties absent in the $\mathcal{N} = 4, d = 5$ theory analyzed in Sect. 7.

In the “bare” charges’ basis, the electric cubic invariant of the $(1, 1 + n)$ of $SO(1,1) \times SO(1,n)$ reads as follows ($I = 0, i$, where $i = 1, ..., n$, throughout; “0” pertains to the $d = 5$ graviphoton field, which through the dimensional reduction $d = 5 \rightarrow d = 4$ becomes the Maxwell vector field of
the axio-dilatonic vector multiplet):

\[ I_{3,el} \equiv q_H q_I q_J \eta^{IJ} \equiv q_H q_I^2 = q_H \left( q_0^2 - \sum_{i=1}^{n} q_i^2 \right) \] (6.4)

where \( q_H \) is the electric charge of the dilatonic vector multiplet: it is an \( SO(1,n) \)-singlet, with \( SO(1,1) \)-weight +2. On the other hand, the \( SO(1,n) \)-vector \( q_I \) has \( SO(1,1) \)-weight −1, such that \( I_{3,el} \) defined by (6.4) is \( SO(1,1) \times SO(1,n) \)-invariant. Notice that the action of the \( U \)-duality group does not mix \( q_H \) and \( q_I \), and this originates more charge orbits with respect to the irreducible cases. Moreover, \( \eta_{IJ} = \eta^{IJ} \) is the Lorentzian metric of \( SO(1,n) \):

\[ \eta_{IJ} = \eta^{IJ} \equiv \text{diag} \left( +1, -1, \ldots, -1 \right) \] (6.5)

In \( \mathcal{N} = 2, d = 5 \) Jordan symmetric sequence, as well as in \( \mathcal{N} = 4, d = 5 \) theory, the reducibility of the associated rank-3 Jordan algebra gives rise to many subtleties and differences with respect to the theories associated to irreducible Euclidean rank-3 Jordan algebras. In the \( \mathcal{N} = 2 \) case under consideration, the major difference consists in a higher number of “large” and “small” orbits with respect to the “magic” supergravities.

6.1 “Large” Orbits

- BPS (3-charge) orbit, defined as follows:

\[
\begin{cases}
q_H > 0; \\
q_0^2 - \sum_{i=1}^{n} q_i^2 > 0; \\
q_0 > 0;
\end{cases}
\quad \text{or} \quad
\begin{cases}
q_H < 0; \\
q_0^2 - \sum_{i=1}^{n} q_i^2 > 0; \\
q_0 < 0.
\end{cases}
\] (6.6)

By recalling definition (6.2), the orbit reads \( (n \geq 0) \):

\[ \mathcal{O}_{\text{BPS,large}} = [SO(1,1)^+ \times T_n^+] \cup [SO(1,1)^- \times T_n^-], \] (6.7)

with no related moduli space. In particular, for \( n = 0 \), namely in the so-called \( \mathcal{N} = 2, d = 5 \) \( SO(1,1) \) model \( (d = 5 \) uplift of the \( d = 4 \) \( st^2 \) model), in which only the dilatonic vector multiplet is coupled to the gravity multiplet, this orbit is actually 2-charge, and it is given by

\[ \mathcal{O}_{\text{BPS,large,SO(1,1)} } = \{ (q_H, q_0) = (+, +), (-, -) \}. \] (6.8)

On the other hand, for \( n = 1 \), i.e. in the so-called \( \mathcal{N} = 2, d = 5 \) \([SO(1,1)]^2 \) model \( (d = 5 \) uplift of \( stu \) model), the cubic invariant (6.4) can be rewritten as follows:

\[ I_{3,el} \equiv q_H q_I q_J \eta^{IJ} \equiv q_H \left( q_0^2 - q_1^2 \right) = q_H q_+ q_-; \]
\[ q_{\pm} \equiv q_0 \pm q_1, \] (6.9)

and thus the hyperboloid (6.2) and light-cone (6.3) structures get respectively factorized as follows (“+”, “−” and “0” respectively denote strictly positive, strictly negative and vanishing
values):

\[ T_1 = \text{SO}(1,1)|_{q_+ q_- > 0} = T^+_1 \cup T^-_1; T^+_1 \cap T^-_1 = \emptyset, \]

\[
\begin{cases}
T^+_1 = \{(q_+, q_-) = (+, +)\}; \\
T^-_1 = \{(q_+, q_-) = (-, -)\}.
\end{cases}
\]  

(6.10)

\[ \Lambda_1 = \text{SO}(1,1) = \Lambda^+_1 \cup \Lambda^-_1; \quad \Lambda^+_1 \cap \Lambda^-_1 = \emptyset, \]

\[
\begin{cases}
\Lambda^+_1 = \{(q_+, q_-) = (+, 0), (0, +)\}; \\
\Lambda^-_1 = \{(q_+, q_-) = (-, 0), (0, -)\}.
\end{cases}
\]

(6.11)

For \( n = 1 \), orbit (6.7) reads

\[ \mathcal{O}_{BPS,3\text{-charge},[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, +, +), (-, -,-)\}. \]  

(6.12)

This is invariant under triality permutation symmetry of \( q_H, q_+ \) and \( q_- \), and it is consistent with the analysis of [33].

- non-BPS (3-charge) orbit, with \( Z \neq 0 \) at the horizon, defined as follows:

\[
\begin{cases}
q_H > 0; \\
q_0^2 - \sum_{i=1}^{n} q_i^2 > 0; \quad \text{or} \quad q_0^2 - \sum_{i=1}^{n} q_i^2 > 0; \\
q_0 < 0.
\end{cases}
\]

(6.13)

By recalling definition (6.2), the orbit reads \( (n \geq 0) \):

\[ \mathcal{O}_{nBPS,large,I} = \left[\text{SO}(1,1)^+ \times T_n^-\right] \cup \left[\text{SO}(1,1)^- \times T_n^+\right], \]  

(6.14)

with no related moduli space. In particular, for \( n = 0 \), this orbit is actually 2-charge, and it is given by

\[ \mathcal{O}_{nBPS,large,SO(1,1)} = \{(q_H, q_0) = (+, -), (-, +)\}. \]  

(6.15)

On the other hand, for \( n = 1 \), orbit (6.14) reads

\[ \mathcal{O}_{nBPS,large,I,[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, -, -), (-, +, +)\}. \]  

(6.16)

The supersymmetry properties of \( \mathcal{O}_{BPS,large} \) and \( \mathcal{O}_{nBPS,large,I} \) can be understood by noticing that the flip of the sign of \( q_H \) amounts, in the dressed charges’ basis, to the exchange \( Z \leftrightarrow \partial_s Z \), where \( s \) is the real dilaton scalar field, parametrizing \( \text{SO}(1,1) \) of (6.1).

It is worth pointing out both the \( \mathcal{N} = 2 \) orbits \( \mathcal{O}_{BPS,large} \) and \( \mathcal{O}_{nBPS,large,I} \) (respectively given by (6.7) and (6.14)) uplift to the same \( \mathcal{N} = 4 \) orbit \( \mathcal{O}_{-BPS,large,N=4;d=5} \) given by Eq. (7.4). As mentioned, this is due to the fact that in \( \mathcal{N} = 4 \), \( d = 5 \) \( q_H > 0 \leftrightarrow q_H < 0 \) amounts to exchanging the two gravitinos in the gravity multiplet, i.e. the two (opposite) skew-eigenvalues of the skew-traceless central charge matrix \( \mathcal{Z}_{AB} \) \( (A, B = 1, ..., 4) \).
Another non-BPS (3-charge) orbit, with \( Z \neq 0 \) at the horizon, is defined as follows \[19\]:

\[
\begin{align*}
q_H & \geq 0; \\
q_0^2 - \sum_{i=1}^{n} q_i^2 & < 0.
\end{align*}
\] (6.17)

Thus, the resulting orbit reads (existing only for \( n \geq 1 \))

\[
O_{nBPS,large,II} = SO(1,1) \times \frac{SO(1,n)}{SO(1,n-1)},
\] (6.18)

with related moduli space (recall (3.44) and (3.45)):

\[
M_{nBPS,large,II} = \frac{SO(1,n-1)}{SO(n-1)} = M_{J,5,n-1} = M_{(1,0),d=6|_{n-1}},
\] (6.19)

where \( M_{J,5,n-1} \) denotes the \( N = 2, d = 5 \) non-Jordan symmetric sequence with \( n - 1 \) vector multiplets \[41\], and \( M_{(1,0),d=6|_{n-1}} \) is the scalar manifold of \( (1,0), d = 6 \) supergravity with \( n_T = n - 1 \) tensor multiplets. Thus, by recalling (6.1), the number \( \sharp \) of “non-flat” scalar degrees of freedom along \( O_{nBPS,large,II} \) is independent on \( n > 1 \):

\[
\sharp_{nBPS,large,II} \equiv \dim_{\mathbb{R}} M_{N=2,d=5,Jordan,symm} - \dim_{\mathbb{R}} M_{nBPS,large,II} = 2.
\] (6.20)

For \( n = 1 \), orbit (6.18) reads

\[
O_{nBPS,large,II,[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, +, -), (+, -), (-, +, -), (-, +, +), (-, -), (+, -)\}.
\] (6.21)

with no corresponding moduli space. (6.21) is equivalent to (6.16) through triality permutation symmetry of \( q_H, q_+, \) and \( q_- \). Thus, consistent with the analysis of \[34\], the non-BPS “large” orbit of \( [SO(1,1)]^2 \) model is given, up to permutations of the triplet \( (q_H, q_+, q_-) \), by

\[
O_{nBPS,3-charge,[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, +, -), (+, -), (-, +)\}.
\] (6.22)

### 6.2 “Small” Orbits

Let us now consider the “small” orbits, and compute the criticality and double-criticality conditions on \( \mathcal{I}_{3,el} \) defined by (6.4):

\[
\frac{\partial \mathcal{I}_{3,el}}{\partial Q} = \left\{ \begin{array}{l}
\frac{\partial \mathcal{I}_{3,el}}{\partial q_H} = q_H^2; \\
\frac{\partial \mathcal{I}_{3,el}}{\partial q_I} = 2q_H q_J \eta^{IJ}; \\
\end{array} \right.
\] (6.23)

\[
\frac{\partial^2 \mathcal{I}_{3,el}}{\partial Q^2} = \left\{ \begin{array}{l}
\frac{\partial^2 \mathcal{I}_{3,el}}{\partial q_H^2} = 0; \\
\frac{\partial^2 \mathcal{I}_{3,el}}{\partial q_H \partial q_I} = \frac{\partial \mathcal{I}_{3,el}}{\partial q_H} = 2q_J \eta^{IJ}; \\
\frac{\partial \mathcal{I}_{3,el}}{\partial q_I \partial q_J} = 2q_H; \\
\end{array} \right.
\] (6.24)
where
\[ Q \equiv (q_H, q_I) \] (6.25)
is shorthand for the vector of electric charges. As expected from the fact that \( \mathcal{I}_{3, \text{el}} \) is homogeneous of degree three, (6.24) implies that the unique doubly-critical orbit is the trivial one with all charges vanishing, because
\[ \frac{\partial^2 \mathcal{I}_{3, \text{el}}}{\partial Q^2} = 0 \iff Q = 0. \] (6.26)
The “small” orbits of the \((1, 1 + n)\) of the \(U\)-duality group \(SO(1, 1) \times SO(1, n)\) list as follows:

1. BPS lightlike \( (\mathcal{I}_{3, \text{el}} = 0, \frac{\partial \mathcal{I}_{3, \text{el}}}{\partial Q} \neq 0: 2\)-charge) orbit with vanishing \(q_H\) and timelike \(q_I\):
   \[
   \begin{cases}
   q_H = 0; \\
   q_0^2 - \sum_{i=1}^{n} q_i^2 > 0.
   \end{cases}
   \] (6.27)
   By recalling definition (6.2), the orbit reads \((n \geq 0)\):
   \[ \mathcal{O}_{\text{BPS, small}, I} = SO(1, 1) \times T_n, \] (6.28)
   with no corresponding moduli space. In particular, for \(n = 0\) this orbit is actually 1-charge, and it is given by
   \[ \mathcal{O}_{\text{BPS, small}, I, SO(1,1)} = \{ (q_H, q_0) = (0, +), (0, -) \}. \] (6.29)
   On the other hand, for \(n = 1\), the orbit (6.28) reads
   \[ \mathcal{O}_{\text{BPS, small}, I, [SO(1,1)]^2} = \{ (q_H, q_+, q_-) = (0, +, +), (0, -, -) \}, \] (6.30)
   with no corresponding moduli space, and thus
   \[ \sharp_{\text{BPS, small}, I, [SO(1,1)]^2} = 2. \] (6.31)

2. Non-BPS lightlike \( (\mathcal{I}_{3, \text{el}} = 0, \frac{\partial \mathcal{I}_{3, \text{el}}}{\partial Q} \neq 0: 2\)-charge) orbit with vanishing \(q_H\) and spacelike \(q_I\):
   \[
   \begin{cases}
   q_H = 0; \\
   q_0^2 - \sum_{i=1}^{n} q_i^2 < 0.
   \end{cases}
   \] (6.32)
   It reads (existing only for \(n \geq 1\))
   \[ \mathcal{O}_{\text{nBPS, small}, I} = SO(1, 1) \times \frac{SO(1, n)}{SO(1, n - 1)}, \] (6.33)
   with corresponding moduli space (recall Eq. (6.19))
   \[ \mathcal{M}_{\text{nBPS, small}, I} = \mathcal{M}_{\text{nBPS, large}, I}. \] (6.34)
   Thus, by recalling (6.1), the number \(\sharp\) of “non-flat” scalar degrees of freedom along \(\mathcal{O}_{\text{nBPS, small}, I}\) is independent on \(n \geq 1\):
   \[ \sharp_{\text{nBPS, small}, I} \equiv \dim \mathcal{M}_{N=2,d=5,\text{Jordan,sym}} - \dim \mathcal{M}_{\text{nBPS, small}, I} = 2. \] (6.35)
   For \(n = 1\), orbit (6.33) reads
   \[ \mathcal{O}_{\text{nBPS, small}, I, [SO(1,1)]^2} = \{ (q_H, q_+, q_-) = (0, +, -), (0, -, +) \}, \] (6.36)
   with no corresponding moduli space.
3. BPS critical ($I_{3,el} = 0, \frac{\partial I_{3,el}}{\partial q_l} = 0$: 1-charge) orbit with vanishing $q_H$ and lightlike $q_I$:

$$\begin{cases} 
q_H = 0; \\
q_0^2 - \sum_{i=1}^{n} q_i^2 = 0.
\end{cases}$$  \hspace{1cm} (6.37)

By recalling definition (6.3), the orbit reads (existing only for $n \geq 1$)

$$\mathcal{O}_{BPS,small,II} = \Lambda_n,$$  \hspace{1cm} (6.38)

and the corresponding moduli space is ($n \geq 1$)

$$\mathcal{M}_{BPS,small,II} = SO(1,1) \times \mathbb{R}^{n-1}. $$  \hspace{1cm} (6.39)

Thus, by recalling (6.1), the number $\sharp$ of “non-flat” scalar degrees of freedom along $\mathcal{O}_{BPS,small,II}$ is independent on $n \geq 1$:

$$\sharp_{BPS,small,II} \equiv dim_{\mathbb{R}} M_{N=2,d=5, Jordan, symm} - dim_{\mathbb{R}} \mathcal{M}_{BPS,small,II} = 1.$$  \hspace{1cm} (6.40)

Analogously to what holds for symmetric “magic” RSG (noted below Eq. (4.33)), the unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 5 \to d = 4$ reduction. For $n = 1$, orbit (6.38) reads

$$\mathcal{O}_{BPS,small,II, [SO(1,1)]^2} = \{ (q_H, q_+, q_-) = (0, 0, +), (0, +, 0), (0, 0, -), (0, -, 0) \}. $$  \hspace{1cm} (6.41)

4. BPS lightlike ($I_{3,el} = 0, \frac{\partial I_{3,el}}{\partial q_l} \neq 0$: 2-charge) orbit, defined as follows:

$$\begin{cases} 
q_H > 0; \\
q_0^2 - \sum_{i=1}^{n} q_i^2 = 0; \hspace{0.5cm} or \hspace{0.5cm} \\
q_0 > 0.
\end{cases}$$  \hspace{1cm} (6.42)

By recalling definition (6.3), the orbit reads ($n \geq 2$)

$$\mathcal{O}_{BPS,small,III} = \left[ SO(1,1)^+ \times \Lambda_n^+ \right] \cup \left[ SO(1,1)^- \times \Lambda_n^- \right],$$  \hspace{1cm} (6.43)

and the corresponding moduli space is purely translational ($n \geq 2$):

$$\mathcal{M}_{BPS,small,III} = \mathbb{R}^{n-1} = \mathcal{M}_{BPS,small,II}.$$  \hspace{1cm} (6.44)

Thus, by recalling (6.1), the number $\sharp$ of “non-flat” scalar degrees of freedom along $\mathcal{O}_{BPS,small,III}$ is independent on $n \geq 2$:

$$\sharp_{BPS,small,III} \equiv dim_{\mathbb{R}} M_{N=2,d=5, Jordan, symm} - dim_{\mathbb{R}} \mathcal{M}_{BPS,small,III} = 2.$$  \hspace{1cm} (6.45)

This orbit exists also for $n = 1$, and it reads

$$\mathcal{O}_{BPS,small,III, [SO(1,1)]^2} = \{ (q_H, q_+, q_-) = (+, 0, +), (+, +, 0), (-, 0, -), (-, -, 0) \},$$  \hspace{1cm} (6.46)

with no corresponding moduli space. (6.46) is equivalent to (6.39) through triality permutation symmetry of $q_H, q_+$, and $q_-$. Thus, the BPS 2-charge orbit of $[SO(1,1)]^2$ model is given, up to permutations of the triplet $(q_H, q_+, q_-)$, by

$$\mathcal{O}_{BPS,2\text{-charge},[SO(1,1)]^2} = \{ (q_H, q_+, q_-) = (+, +, 0), (-, -, 0) \}. $$  \hspace{1cm} (6.47)
5. Non-BPS lightlike \((\mathcal{I}_{3,el} = 0, \frac{\partial \mathcal{I}_{3,el}}{\partial Q} \neq 0): 2\)-charge) orbit, defined as follows:

\[
\begin{cases}
q_H < 0; \\
q_0^2 - \sum_{i=1}^n q_i^2 = 0; \quad \text{or} \quad q_0^2 - \sum_{i=1}^n q_i^2 = 0; \\
q_0 > 0;
\end{cases}
\] (6.48)

By recalling definition (6.3), the orbit reads \((n \geq 2)\)

\[
\mathcal{O}_{nBPS,small,II} = \left[ SO(1,1)^+ \times \Lambda_n^- \right] \cup \left[ SO(1,1)^- \times \Lambda_n^+ \right],
\] (6.49)

with corresponding moduli space \((n \geq 2)\)

\[
\mathcal{M}_{nBPS,small,II} = \mathbb{R}^{n-1} = \mathcal{M}_{BPS,small,II} = \mathcal{M}_{BPS,small,III}
\] (6.50)

Thus, by recalling (6.1), the number \(\sharp\) of “non-flat” scalar degrees of freedom along \(\mathcal{O}_{nBPS,small,II}\) is independent on \(n \geq 2:\)

\[
\sharp_{nBPS,small,II} \equiv \dim_{\mathbb{R}} M_{N=2,d=5,\text{Jordan, symm}} - \dim_{\mathbb{R}} \mathcal{M}_{nBPS,small,II} = 2.
\] (6.51)

This orbit exists also for \(n = 1\), and it reads

\[
\mathcal{O}_{nBPS,small,II,[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, 0, -), (+, -, 0), (-, 0, +), (-, +, 0)\},
\] (6.52)

with no corresponding moduli space. (6.52) is equivalent to (6.36) through triality permutation symmetry of \(q_H, q_+\) and \(q_-\). Thus, the non-BPS 2-charge orbit of \([SO(1,1)]^2\) model is given, up to permutations of the triplet \((q_H, q_+, q_-)\), by

\[
\mathcal{O}_{nBPS,2\text{-charge},[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, -, 0)\}.
\] (6.53)

6. BPS critical \((\mathcal{I}_{3,el} = 0, \frac{\partial \mathcal{I}_{3,el}}{\partial Q} = 0): 1\)-charge) orbit with vanishing \(q_I\) and non-vanishing \(q_H\):

\[
\begin{cases}
q_H \in \mathbb{R}; \\
q_I = 0.
\end{cases}
\] (6.54)

It exists for every \(n \geq 0\), and it reads

\[
\mathcal{O}_{BPS,small,IV} = SO(1,1),
\] (6.55)

with moduli space \((n \geq 1; \text{recall (3.45)})\)

\[
\mathcal{M}_{BPS,small,IV} = \frac{SO(1,n)}{SO(n)} = M_{n,5,n}.
\] (6.56)

Thus, by recalling (6.1), the number \(\sharp\) of “non-flat” scalar degrees of freedom along \(\mathcal{O}_{BPS,small,IV}\) is independent on \(n \geq 1:\)

\[
\sharp_{BPS,small,IV} \equiv \dim_{\mathbb{R}} M_{N=2,d=5,\text{Jordan, symm}} - \mathcal{M}_{BPS,small,IV} = 1.
\] (6.57)
Analogously to what holds for symmetric “magic” RSG (noted below Eq. (4.33)), the unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 5 \rightarrow d = 4$ reduction. Furthermore, as in the corresponding $\mathcal{N} = 4$, $d = 5$ “small” orbit (given by Eq. (7.34)), the sign of $q_H$ does not matter here. Orbit (6.55) is originated by the $d = 6 \rightarrow d = 5$ reduction of (1,0) theory with all charges switched off. Indeed, $q_H$ is the electric charge of the Kaluza-Klein vector in the reduction $d = 6 \rightarrow d = 5$.

In particular, for $n = 0$, this orbit reads

$$O_{BPS,small,IV,SO(1,1)} = \{(q_H, q_0) = (+, 0), (-, 0)\},$$

(6.58)

with no corresponding moduli space. On the other hand, for $n = 1$ the orbit (6.55) reads

$$O_{BPS,small,IV,[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, 0, 0), (-, 0, 0)\},$$

(6.59)

which is equivalent to (6.41) through triality permutation symmetry of $q_H$, $q_+$ and $q_-$. Thus, the BPS 1-charge orbit of $[SO(1,1)]^2$ model is given, up to permutations of the triplet $(q_H, q_+, q_-)$, by

$$O_{BPS,1-charge,[SO(1,1)]^2} = \{(q_H, q_+, q_-) = (+, 0, 0), (-, 0, 0)\}.$$

(6.60)

Thus, the stratification structure of the $(1,1+n)$-repr. space of the $d = 5$ $U$-duality group $SO(1,1) \times SO(1,n)$ can be given through the following two chains of relations, proceeding (left to right) from 1-charge orbits to 2-charge and then 3-charge orbits:

$$O_{BPS,small,II} \rightarrow \left\{ \begin{array}{l}
O_{BPS,small,I} \rightarrow \left\{ \begin{array}{l}
O_{BPS,small,IV} \rightarrow \left\{ \begin{array}{l}
O_{BPS,small,II} \rightarrow \left\{ \begin{array}{l}
O_{BPS,large} \\
O_{nBPS,large,I}
\end{array} \right.
\end{array} \right.
\end{array} \right.
\end{array} \right.
\end{array}$$

(6.61)

For the $SO(1,1)$ model ($n = 0$), such a stratification structure simplifies as follows:

$$SO(1,1) : \left\{ \begin{array}{l}
1\text{-charge} \quad O_{BPS,small,I} \\
2\text{-charge} \quad O_{BPS,large} \right. \end{array} \right.
\end{array}$$

(6.63)
On the other hand, for the $[SO(1,1)]^2$ model ($n = 1$), stratification structure (6.61)-(6.62) reads:

$$[SO(1,1)]^2 : \mathcal{O}_{\text{BPS,1-charge}} \rightarrow \begin{cases} \mathcal{O}_{\text{BPS,2-charge}} \rightarrow \mathcal{O}_{\text{BPS,3-charge}} \\ \mathcal{O}_{\text{nBPS,2-charge}} \rightarrow \mathcal{O}_{\text{nBPS,3-charge}} \end{cases} \ (6.64)$$

Thus, summarizing, $\mathcal{N} = 2$, $d = 5$ Jordan symmetric sequence admits six “small” charge orbits describing the flux configurations supporting static, spherically symmetric, asymptotically flat “small” BHs: four $\frac{1}{2}$-BPS and two non-BPS. Furthermore, the “large” orbits are three, namely one $\frac{1}{2}$-BPS and two non-BPS (with $Z \neq 0$ at the horizon).

7 $\mathcal{N} = 4$, $d = 5$ Supergravity

The scalar manifold of $\mathcal{N} = 4$, $d = 5$ supergravity coupled to $n_V = n \in \mathbb{N} \cup \{0\}$ matter (vector) multiplets reads ($\dim_{\mathbb{R}} = 1 + 5n$, rank$ = 1 + \min(5, n)$)

$$M_{\mathcal{N}=4,d=5} = SO(1,1) \times \frac{SO(5,n)}{SO(5) \times SO(n)}. \quad (7.1)$$

This theory is associated to the rank-3 Euclidean reducible Jordan algebra $\mathbb{R} \oplus \Gamma_{5,n}$. In the following treatment, we will determine the “large” and “small” orbits of the irrepr. $(1, 5 + n)$ of the $U$-duality group $SO(1,1) \times SO(5,n)$.

For brevity’s sake, we will do this only through an analysis in the “bare” charges’ basis. Indeed, due to the symmetricity of the reducible coset (7.1), the treatment of $d = 5$ extremal (magnetic) black strings is essentially analogous.

In the “bare” charges’ basis, the electric cubic invariant of the $(1, 5 + n)$ of $SO(1,1) \times SO(5,n)$ reads as follows ($I = 1, \ldots, 5 + n$ throughout; the indices $1, \ldots, 5$, with positive signature, pertain to the five $\mathcal{N} = 4$, $d = 5$ graviphotons):

$$\mathcal{I}_{3,\text{el}} = q_H q_I q_J \eta_{IJ} \equiv q_H q_I^2, \quad (7.2)$$

where $q_H$ is the electric charge of the 3-form field strength of the 2-form $B_{\mu\nu}$ ($\mu, \nu = 0, 1, \ldots, 4$) in the gravity multiplet (see e.g. [60, 61]). $q_H$ is an $SO(5,n)$-singlet, with $SO(1,1)$-weight $+2$. On the other hand, the $SO(5,n)$-vector $q_I$ has $SO(1,1)$-weight $-1$, such that $\mathcal{I}_{3,\text{el}}$ defined by (7.2) is $SO(1,1) \times SO(5,n)$-invariant. Notice that the action of the $U$-duality group does not mix $q_H$ and $q_I$, and this originates more charge orbits with respect to the irreducible cases. Moreover, $\eta_{IJ} = \eta^{IJ}$ is the pseudo-Euclidean metric of $SO(5,n)$, with signature $(\begin{smallmatrix} 5 \\ +, \ldots, +, -, \ldots, - \end{smallmatrix})$.

7.1 “Large” Orbits

- $\frac{1}{3}$-BPS (3-charge) orbit, defined by a timelike $q_I$ vector, with $q_H$ of any sign:

$$q_H \in \mathbb{R}_0, \ q_I q_J \eta^{IJ} > 0. \quad (7.3)$$
The resulting form of the orbit reads \( n \geq 0 \)
\[
\mathcal{O}_{\frac{1}{2}-\text{BPS,large}} = SO(1,1) \times \frac{SO(5,n)}{SO(4,n)}, \tag{7.4}
\]
with related moduli space:
\[
\mathcal{M}_{\frac{1}{2}-\text{BPS,large}} = \frac{SO(4,n)}{SO(4) \times SO(n)} = \frac{M_{(1,1),d=6}}{SO(1,1)}, \tag{7.5}
\]
where \( M_{(1,1),d=6} \) is the scalar manifold of non-chiral half-maximal supergravity in \( d = 6 \) with \( n \) matter (vector) multiplets. The exchange between \( q_H > 0 \) and \( q_H < 0 \) amounts to exchanging the two gravitinos in the gravity multiplet, \( i.e. \) the two (opposite) skew-eigenvalues of the skew-traceless central charge matrix \( \check{Z}_{AB} \) (\( A, B = 1, \ldots, 4 \)). Thus, the number \( \sharp \) of “non-flat” scalar degrees of freedom along \( \mathcal{O}_{\frac{1}{2}-\text{BPS,large}} \) is (for \( n \geq 1 \))
\[
\sharp_{\frac{1}{2}-\text{BPS,large}} \equiv \dim_{\mathbb{R}} M_{N=4,d=5} - \dim_{\mathbb{R}} \mathcal{M}_{\frac{1}{2}-\text{BPS,large}} = n + 1. \tag{7.6}
\]
In \( \mathcal{N} > 2 \)-extended supergravity theories, in general \( \frac{1}{N} \)-BPS attractors have a related moduli space \( [23] \). It corresponds to the hypermultiplets’ scalar manifold in the supersymmetry reduction \( \mathcal{N} > 2 \rightarrow \mathcal{N} = 2 \) of the theory under consideration. In this case, it is amusing to observe that \( \mathcal{M}_{\frac{1}{4}-\text{BPS,large}} \) given by (7.5) is the c-map of the vector multiplets’ scalar manifold of the \( \mathcal{N} = 2, d = 4 \) Jordan symmetric sequence:
\[
\mathcal{M}_{\frac{1}{4}-\text{BPS,large}} = c \left( \frac{SU(1,1)}{U(1)} \times \frac{SO(2,n-2)}{SO(2) \times SO(n-2)} \right). \tag{7.7}
\]
Thus, \( \mathcal{M}_{\frac{1}{4}-\text{BPS,large}} \) admits an interpretation either as 1) scalar manifold of \( \mathcal{N} = 4, d = 3 \) Jordan symmetric sequence in \( d = 3 \), or as 2) the hypermultiplets’ scalar manifold of Jordan symmetric sequence in \( d = 4, 5 (\mathcal{N} = 2) \) and \( 6 ((1,0)) \). In particular, \( \mathcal{M}_{\frac{1}{4}-\text{BPS,large}} \) parametrizes the \( \mathcal{N} = 2 \) hyperscalar degrees of freedom in the supersymmetry/Jordan algebra reduction:
\[
d = 5 : \quad \mathcal{N} = 4 \quad \mathbb{R} \oplus \Gamma_{5,n} \quad \rightarrow \quad \mathcal{N} = 2 \quad \mathbb{R} \oplus \Gamma_{1,n-3}. \quad \tag{7.8}
\]
The pure theory (\( i.e. \) \( n = 0 \)) limit of orbit (7.4) is actually 2-charge (indeed, \( SO(5) \) symmetry can be used to make only one component of the Euclidean vector \( q_I \) non-vanishing), and it reads
\[
\mathcal{O}_{\frac{1}{2}-\text{BPS,large},n=0} = SO(1,1) \times \frac{SO(5)}{SO(4)} \equiv SO(1,1) \times S^4, \tag{7.9}
\]
with no corresponding moduli space, and thus trivially
\[
\sharp_{\frac{1}{2}-\text{BPS,large},n=0} = 1. \tag{7.10}
\]
• non-BPS (3-charge) orbit with \( \check{Z}_{AB} = 0 \) (at the horizon), defined by a spacelike \( q_I \) vector, and \( q_H \) of any sign:
\[
q_H \in \mathbb{R}_0, \quad q_I q_J \eta^{IJ} < 0. \tag{7.11}
\]
Notice that both signs of $q_H$ are allowed, due to the fact that the non-BPS $Z_{AB} = 0$ Attractor Eqs. are quadratic in $q_H$ (see e.g. [20]). The resulting orbit reads ($n \geq 1$, not existing in pure theory) [20]

$$O_{nBPS,large} = SO(1,1) \times \frac{SO(5,n)}{SO(5,n-1)},$$

(7.12)

with related moduli space:

$$M_{nBPS,large} = \frac{SO(5,n-1)}{SO(5) \times SO(n-1)} = M_{(2,0),d=6|_{n-1}},$$

(7.13)

where $M_{(2,0),d=6|_{n-1}}$ is the scalar manifold of $(2,0)$, $d = 6$ supergravity with $n_T = n - 1$ tensor multiplets. Note that $N = 4$, $d = 5$ and $(2,0)$, $d = 6$ supergravities share the same $\mathcal{R}$-symmetry $SO(5) \sim USp(4)$. Thus, the number $\sharp$ of “non-flat” scalar degrees of freedom along $O_{nBPS,large}$ is independent on $n \geq 2$:

$$\sharp_{nBPS,large} \equiv \dim_{\mathbb{R}} M_{N=4,d=5} - \dim_{\mathbb{R}} M_{nBPS,large} = 6.$$  

(7.14)

### 7.2 “Small” Orbits

The conditions on $I_{3,el}$ defined by (7.2) are formally the same as the ones holding in $N = 2$, $d = 5$ Jordan symmetric sequence, and given by Eqs. (6.23) and (6.24). Thus, analogously to the case of $N = 2$, $d = 5$ Jordan symmetric sequence, and as expected from the fact that $I_{3,el}$ is homogeneous of degree three, (6.24) implies that the unique doubly-critical orbit is the trivial one with all charges vanishing (namely, 0-charge orbit; recall Eq. (6.26)).

The “small” orbits of the $(1,5+n)$ of the $U$-duality group $SO(1,1) \times SO(5,n)$ list as follows:

1. Lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial q} \neq 0$: 2-charge) orbit with vanishing $q_H$ and timelike $q_I$:

$$\begin{aligned}
q_H & = 0; \\
q_I^2 & > 0.
\end{aligned}$$

(7.15)

This orbit is $\frac{1}{2}$-BPS [14]. It reads ($n \geq 0$)

$$O_{\frac{1}{2}-BPS,small,I} = SO(1,1) \times \frac{SO(5,n)}{SO(4,n)},$$

(7.16)

with corresponding moduli space (recall Eq. (7.5))

$$M_{\frac{1}{2}-BPS,small,I} = M_{\frac{1}{2}-BPS,large}.$$  

(7.17)

Thus, the number $\sharp$ of “non-flat” scalar degrees of freedom along $O_{\frac{1}{2}-BPS,small,I}$ is (for $n \geq 1$):

$$\sharp_{\frac{1}{2}-BPS,small,I} \equiv \dim_{\mathbb{R}} M_{N=4,d=5} - \dim_{\mathbb{R}} M_{\frac{1}{2}-BPS,small,I} = n + 1.$$  

(7.18)

The pure theory (i.e. $n = 0$) limit of orbit (7.16) is actually 1-charge, and it reads

$$O_{\frac{1}{2}-BPS,small,I,n=0} = SO(1,1) \times S^4,$$  

(7.19)
with no related moduli space, and thus
\[ \#_{1/2-BPS, small, I, n=0} = 1. \quad (7.20) \]

2. Lightlike ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} \neq 0$: 2-charge) orbit with vanishing $q_H$ and spacelike $q_I$:
\[ \begin{cases} 
q_H = 0; \\
q_I^2 < 0. 
\end{cases} \quad (7.21) \]

This orbit is non-BPS. It reads ($n \geq 1$, not existing in pure theory)
\[ O_{nBPS, small} = SO(1, 1) \times \frac{SO(5, n)}{SO(5, n-1)}, \quad (7.22) \]
with corresponding moduli space (recall Eq. (7.13))
\[ M_{nBPS, small} = M_{nBPS, large}. \quad (7.23) \]
Thus, the number $\#$ of “non-flat” scalar degrees of freedom along $O_{nBPS, small}$ is independent on $n \geq 1$:
\[ \#_{nBPS, small} \equiv \dim_{\mathbb{R}} M_{N=4,d=5} - \dim_{\mathbb{R}} M_{nBPS, small} = 6. \quad (7.24) \]

3. Critical ($I_{3,el} = 0$, $\frac{\partial I_{3,el}}{\partial Q} = 0$: 1-charge) orbit with vanishing $q_H$ and lightlike $q_I$:
\[ \begin{cases} 
q_H = 0; \\
q_I^2 = 0. 
\end{cases} \quad (7.25) \]

This orbit is $1/2$-BPS [14]. It reads ($n \geq 1$, not existing in pure theory)
\[ O_{1/2-BPS, small, II} = \frac{SO(5, n)}{SO(4, n-1) \times \mathbb{R}^{4,n-1}}, \quad (7.26) \]
with corresponding moduli space (recall Eq. (7.17))
\[ M_{1/2-BPS, small, II} = \left. \frac{SO(1, 1) \times M_{1/2-BPS, small, I}}{SO(4, n-1) \times \mathbb{R}^{4,n-1}} \right|_{n \rightarrow n-1} \times \mathbb{R}^{4,n-1}. \quad (7.27) \]
Thus, the number $\#$ of “non-flat” scalar degrees of freedom along $O_{1/2-BPS, small, II}$ is independent on $n \geq 1$:
\[ \#_{1/2-BPS, small, II} \equiv \dim_{\mathbb{R}} M_{N=4,d=5} - \dim_{\mathbb{R}} M_{1/2-BPS, small, II} = 1. \quad (7.28) \]

Analogously to what holds for symmetric “magic” RSG (noted below Eq. (4.33)) and for $N = 2$, $d = 5$ Jordan symmetric sequence treated in Sect. [9] the unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 5 \rightarrow d = 4$ reduction.
4. Lightlike ($I_{3,el} = 0$, $\partial I_{3,el}/\partial Q \neq 0$: 2-charge) orbit with non-vanishing $q_H$ and lightlike $q_I$:

$$\begin{cases} q_H \in \mathbb{R}_0; \\ q_I^2 = 0. \end{cases}$$ (7.29)

This orbit is $\frac{1}{4}$-BPS. It reads ($n \geq 1$)

$$O_{\frac{1}{4}-BPS,small} = SO(1,1) \times \frac{SO(5,n)}{SO(4,n-1) \times \mathbb{R}^{4,n-1}},$$ (7.30)

with corresponding moduli space (recall Eq. (7.27))

$$M_{\frac{1}{4}-BPS,small} = M_{\frac{1}{2}-BPS,small,I} \bigg|_{n \rightarrow n-1} \times \mathbb{R}^{4,n-1} \quad = \quad M_{\frac{1}{2}-BPS,large} \bigg|_{n \rightarrow n-1} \times \mathbb{R}^{4,n-1}.$$ (7.31)

Thus, the number $\sharp$ of “non-flat” scalar degrees of freedom along $O_{\frac{1}{4}-BPS,small,II}$ is independent on $n \geq 1$:

$$\sharp_{\frac{1}{4}-BPS,small} \equiv \dim_{\mathbb{R}}M_{N=4,d=5} - \dim_{\mathbb{R}}M_{\frac{1}{4}-BPS,small} = 2.$$ (7.32)

5. Critical ($I_{3,el} = 0$, $\partial I_{3,el}/\partial Q = 0$: 1-charge) orbit with vanishing $q_I$ and non-vanishing $q_H$:

$$\begin{cases} q_H \in \mathbb{R}_0; \\ q_I = 0. \end{cases}$$ (7.33)

This orbit is $\frac{1}{2}$-BPS [13]. It reads (independent on $n \geq 0$)

$$O_{\frac{1}{2}-BPS,small,III} = SO(1,1),$$ (7.34)

with moduli space

$$M_{\frac{1}{2}-BPS,small,III} = \frac{SO(5,n)}{SO(5) \times SO(n)}.$$ (7.35)

Thus, the number $\sharp$ of “non-flat” scalar degrees of freedom along $O_{\frac{1}{2}-BPS,small,III}$ is independent on $n \geq 0$:

$$\sharp_{\frac{1}{2}-BPS,small,III} \equiv \dim_{\mathbb{R}}M_{N=4,d=5} - M_{\frac{1}{2}-BPS,small,III} = 1.$$ (7.36)

Notice that $O_{\frac{1}{2}-BPS,small,III}$ can also be seen as the “$n = 0$ formal limit” of $O_{\frac{1}{4}-BPS,small}$ given by Eq. (7.30). Indeed, the $n = 0$ limit of (7.29) is given by (7.33) itself. Furthermore, analogously to what holds for symmetric “magic” RSG (noted below Eq. (4.33)) and for $N = 2$, $d = 5$ Jordan symmetric sequence treated in Sect. 6, the unique scalar degree of freedom on which the ADM mass depends can be interpreted as the Kaluza-Klein radius in the $d = 5 \rightarrow d = 4$ reduction. Orbit (7.34) is originated by the $d = 6 \rightarrow d = 5$ reduction of (2,0) theory with all charges switched off. Indeed, $q_H$ is the electric charge of the Kaluza-Klein vector in the reduction $d = 6 \rightarrow d = 5$. Notice that in the pure theory (i.e. $n = 0$) $M_{\frac{1}{2}-BPS,small,III}$ vanishes, and thus:

$$\sharp_{\frac{1}{2}-BPS,small,III,n=0} = 1.$$ (7.37)
Thus, the stratification structure of the \((1, 5 + n)\)-repr. space of the \(d = 5\) \(U\)-duality group \(SO(1, 1) \times SO(5, n)\) can be given through the two chains of relations, proceeding (left to right) from 1-charge orbits to 2-charge and then 3-charge orbits:

\[
\begin{align*}
O_{\frac{1}{4}}^{1-BPS, small, I} &\rightarrow O_{\frac{1}{4}}^{1-BPS, large} \\
O_{nBPS, small} &\rightarrow O_{nBPS, large} \\
O_{\frac{1}{4}}^{1-BPS, small} &\rightarrow \begin{cases} 
O_{\frac{1}{4}}^{1-BPS, large} \\
O_{nBPS, large} \end{cases}
\end{align*}
\]

(7.38)

\[
\begin{align*}
O_{\frac{1}{4}}^{1-BPS, small, III} &\rightarrow O_{\frac{1}{4}}^{1-BPS, small} \rightarrow \begin{cases} 
O_{\frac{1}{4}}^{1-BPS, large} \\
O_{nBPS, large} \end{cases}
\end{align*}
\]

(7.39)

For pure \(\mathcal{N} = 4, d = 5\) supergravity, such a stratification structure simplifies as follows:

\[
\begin{align*}
\left\{ O_{\frac{1}{4}}^{1-BPS, small, I,n=0} \right\} &\rightarrow O_{\frac{1}{4}}^{1-BPS, large,n=0} \\
O_{\frac{1}{4}}^{1-BPS, small, III} &\rightarrow O_{\frac{1}{4}}^{1-BPS, small} \rightarrow \begin{cases} 
O_{\frac{1}{4}}^{1-BPS, large} \\
O_{nBPS, large} \end{cases}
\end{align*}
\]

(7.40)

Thus, summarizing, \(\mathcal{N} = 4, d = 5\) supergravity theory admits five “small” charge orbits describing the flux configurations supporting static, spherically symmetric, asymptotically flat “small” BHs: one \(\frac{1}{4}\)-BPS, three \(\frac{1}{2}\)-BPS and one non-BPS. The “large” orbits are two, namely one \(\frac{1}{4}\)-BPS and one non-BPS (with \(Z_{AB} = 0\) at the horizon).

The relations among the charge orbits of \(\mathcal{N} = 4, d = 5\) supergravity and the charge orbits of \(\mathcal{N} = 2, d = 5\) Jordan symmetric sequence can be determined through the supersymmetry reduction

\[
d = 5 : \quad \mathcal{N} = 4 \quad R \oplus \Gamma_{5,n} \quad \rightarrow \quad \mathcal{N} = 2 \quad R \oplus \Gamma_{1,n},
\]

(7.41)

yielding to the results summarized in Table 6.

Finally, it is worth summarizing the results obtained about the number \(\sharp\) of “non-flat” scalar degrees of freedom, within the symmetric RSG studied in previous Sections. For the “magic” supergravities, it holds

\[
J^k_3 : \begin{cases} 
"large" \ (rank = 3) : \\
"small" : \\
rank = 3 : \\
rank = 3 : \quad BPS : \sharp = q + 2; \\
BPS : \sharp = 2; \\
BPS : \sharp = 1,
\end{cases}
\]

(7.42)

whereas for \(\mathcal{N} = 4\) supergravity and \(\mathcal{N} = 2\) Jordan symmetric sequence the results are reported in Table 6. As pointed out above, in the symmetric RSG’s under consideration the unique scalar degree of freedom on which the ADM mass depends along the 1-charge \(\frac{1}{4}\)-BPS (maximally symmetric) charge orbits can be interpreted as the Kaluza-Klein radius in the \(d = 5 \rightarrow d = 4\) reduction.
Table 6: “Large” (rank = 3) and “small” (rank = 1 and 2) charge orbits of the repr. \((1, 5 + n)\) and \((1, 1 + n)\) of the \(d = 5\) U-duality groups \(SO(1, 1) \times SO(5, n)\) and \(SO(1, 1) \times SO(1, n)\) of \(\mathcal{N} = 4\) supergravity (based on \(\mathbb{R} \oplus \Gamma_{5,n}\)) and \(\mathcal{N} = 2\) Jordan symmetric sequence (based on \(\mathbb{R} \oplus \Gamma_{1,n}\)), respectively. The rank \(r\) of the orbit is defined as the minimal number of charges defining a representative solution. “\(\uparrow \ast\)” denotes the fact the orbits are related through a flip of the sign of \(q_H\). The disconnected timelike hyperboloid \(T_n\) and lightcone \(\Lambda_n\) structures are defined by (6.2) and (6.3), respectively. \(\sharp\), defined in (4.3), denotes the number of “non-flat” scalar degrees of freedom supported by the charge orbit.
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A Resolution of $G_5$-invariant Constraints

In this Appendix, we explicitly solve the $G_5$-invariant defining constraints of “small” charge orbits in “magic” symmetric RSG, both in the “bare” (Sub-App. A.1) and “dressed” (Sub-App. A.2) charges bases.

A.1 “Bare” Charges Basis

Let us start by noticing that for each of the four “magic” symmetric RSG’s a unique maximal symmetric embedding into $G_5$ exists containing a factor $SO (1,1)$. It reads (recall Eq. (4.30) [54]

\[ G_5 \supseteq_{\text{max}} G_6 \times A_q \times SO (1,1) , \]  

(A.1)

where the group $A_q$ has been defined in Table 2. Notice that, in the cases $q = 4$ and 2, $G_6 \times SO (1,1)$ is not embedded maximally (also considering non-symmetric embeddings [55]) into $G_5$ itself.

When removing $A_q$ in the cases $q = 4$ and 2 (and thus losing the maximality), the embedding (A.1) has a nice interpretation in terms of truncation of the “magic” supergravity to theories belonging to the Jordan symmetric sequence $\{3.44\} [19]:$

\[
\begin{align*}
J_3^D \supseteq_{\text{max}} \mathbb{R} \oplus J_2^D : & \quad E_6(-26) \supseteq_{\text{max}} SO (1,1) \times SO (1,9) ; \\
J_3^R \supseteq \mathbb{R} \oplus J_2^R : & \quad SU^* (6) \supseteq SO (1,1) \times SO (1,5) ; \\
J_3^C \supseteq \mathbb{R} \oplus J_2^C : & \quad SL (3, \mathbb{C}) \supseteq SO (1,1) \times SO (1,3) ; \\
J_3^R \supseteq_{\text{max}} \mathbb{R} \oplus J_2^R : & \quad SL (3, \mathbb{R}) \supseteq_{\text{max}} SO (1,1) \times SO (1,2) ,
\end{align*}
\]  

(A.2)

where it should be recalled that $(q = 8, 4, 2, 1; \text{see e.g. } [47])$

\[ J_2^R \sim \Gamma_{1,q+1} . \]  

(A.3)
A.1.1 $O_{\text{lightlike,BPS}}$

In order to solve the “small” lightlike $G_5$-invariant defining constraints [4.51] in “bare” charges in a way consistent with an orbit representative having $Z \neq 0$, let us further embed the $mcs$ of the group in the right-hand side of Eq. (A.1), thus obtaining

$$G_5 \supseteq_{\text{max}} G_6 \times A_q \times SO(1,1) \supseteq_{\text{mcs}} SO(q+1) \times A_q.$$  \hspace{1cm} \text{(A.4)}

Thus, under the “branching” (A.4) the irrepr. $R_Q$ of $G_5$ in which the electric charges $q_i$’s sit decomposes as follows:

$$R_Q \rightarrow (1,1)_{+4} + (q + 2,1)_{-2} + (\text{Spin}(q+2),\text{Spin}(Q_q))_{+1} \hspace{1cm} \text{(A.5)}$$

$$\rightarrow (1,1)_I + (q+1,1) + (1,1)_{II} + (\text{Spin}(q+1),\text{Spin}(Q_q)).$$

This in turn entails the “branching”

$$q_i \rightarrow \left( q(1,1)_I, q(1,1)_{II}, q(q+1,1), q(\text{Spin}(q+1),\text{Spin}(Q_q)) \right).$$  \hspace{1cm} \text{(A.6)}

In the first line of (A.5) subscripts denote the weight with respect to $SO(1,1)$, whereas in the second line they just discriminate between the two singlets of $SO(q+1) \times A_q$. Also recall that, as given in Table 2, $A_q$ and $Q_q$ are absent for $q = 8$ and $q = 1$.

Therefore, with respect to $SO(q+1) \times A_q$, one obtains:

- two singlets (note that $(1,1)_I$ is a singlet of $SO(q+1,1) \times A_q$, as well);
- one vector $(q + 1, 1)$;
- a (double-)spinor $(\text{Spin}(q+1), \text{Spin}(Q_q))$.

The representation decomposition (A.5) yields that $d^{ijk}$, the rank-3 completely symmetric $G_5$-invariant tensor (namely, the unique singlet in the tensor product $(R_Q)^3$) decomposes in a such way that $(1,1)_I$ and $(q+1,1)$ have the same couplings inside $(R_Q)^3$.

Details concerning the various “magic” symmetric RSG’s are given further below.

The position which solves (with maximal - compact - symmetry $SO(q+1) \times A_q$) the “small” lightlike $G_5$-invariant defining constraints [4.51] in “bare” charges (and in a way consistent with an orbit representative having $Z \neq 0$) reads as follows:

$$\begin{cases} 
q(1,1)_I = 0; \\
q(q+1,1) = 0; \\
q(\text{Spin}(q+1),\text{Spin}(Q_q)) = 0; \\
q(1,1)_{II} \neq 0.
\end{cases}$$ \hspace{1cm} \text{(A.7)}

Since $SO(q+1) \times A_q$ is the unique group maximally (and symmetrically) embedded into $G_6 \times A_q \times SO(1,1)$ which has $SO(q+1) \times A_q$ as (in this case improper) $mcs$, it follows that $SO(q+1) \times A_q$ is also the maximal semi-simple symmetry of $O_{\text{lightlike,BPS}}$, which is thus given by Eq. (4.14).

The origin of the non-semi-simple Abelian (namely, translational) factor $\mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}$ in the stabilizer of $O_{\text{lightlike,BPS}}$ will be explained through the procedure of suitable İnönü-Wigner contraction performed in Sub-App. B.1.

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A.1.2 $\mathcal{O}_{\text{critical,BPS}}$

Eq. (A.4) and subsequent ones are also relevant for the resolution of the “small” critical $G_5$-invariant defining constraints (4.55) in “bare” charges in a way consistent with an orbit representative having $Z \neq 0$ (which is the unique possible case; see treatment above). In this case, the position which solves (with maximal - non-compact - symmetry $G_6 \times A_q$) the constraints (4.55) in “bare” charges reads as follows:

$$\begin{align*}
q(1,1)_{II} = 0; \\
q(q+1,1) = 0; \\
q(q+1,1) = 0; \quad (A.8)
\end{align*}$$

At least for the relevant values $q = 8, 4, 2, 1$ it holds that $spin (q + 2) = spin (q + 1)$ (recall definition (4.17)). Therefore, since

$$\begin{align*}
q(1,1)_{II} = 0; \\
q(q+1,1) = 0; \quad (A.9)
\end{align*}$$

it follows that the position (A.8) exhibits maximal - non-compact - symmetry $G_5$, which then is the maximal semi-simple symmetry of $\mathcal{O}_{\text{critical,BPS}}$, which is thus given by Eq. (4.29).

The origin of $\mathbb{R}^{(spin(q+2),spin(Q_q))}$ in the stabilizer of $\mathcal{O}_{\text{critical,BPS}}$ will be explained through the procedure of suitable $SO(1,1)$-(three-)grading performed in Sub-App. B.2.

A.1.3 $\mathcal{O}_{\text{lightlike,nBPS}}$

In order to solve the “small” lightlike $G_5$-invariant defining constraints (4.51) in “bare” charges in a way consistent with an orbit representative having $Z = 0$, the embedding (A.1) has to be further elaborated as follows:

$$G_5 \supseteq_{\text{max}} G_6 \times A_q \times SO(1,1) \supseteq_{\text{max}} SO(q,1) \times A_q \times SO(1,1) mcs \supseteq \ SO(q) \times A_q. \quad (A.10)$$

Thus, under the “branching” (A.10) the irrepr. $R_Q$ decomposes as follows:

$$\begin{align*}
R_Q & \rightarrow (1,1)_{+4} + (q+2,1)_{-2} + (\text{Spin} (q+2), \text{Spin} (Q_q))_{+1} \\
& \rightarrow (1,1)_{+4} + (q+1,1)_{-2} + (1,1)_{-2} + (\text{Spin} (q+1), \text{Spin} (Q_q))_{+1} \\
& \rightarrow (1,1)_{II} + (q,1) + (1,1)_{III} + (1,1)_{II} + (\text{Spin} (q), \text{Spin} (Q_q)) + (\text{Spin}'' (q), \text{Spin} (Q_q)), \quad (A.11)
\end{align*}$$

where, besides the obvious irrepr. decompositions determining the last line of (A.11), one should recall that

$$\begin{align*}
(\text{Spin} (q+1), \text{Spin} (Q_q)) & \rightarrow (\text{Spin}' (q), \text{Spin} (Q_q)) + (\text{Spin}'' (q), \text{Spin} (Q_q)), \quad (A.12)
\end{align*}$$

where the primes discriminate between the two spinor irreps. of $SO(q) \times A_q$. The “branching” of electric charges corresponding to (A.11) reads

$$q_i \rightarrow \left( q_{(1,1)I} q_{(1,1)II}, q_{(1,1)III}, q_{(q,1)}, q_{(\text{Spin} (q), \text{Spin} (Q_q))}, q_{(\text{Spin}'' (q), \text{Spin} (Q_q))} \right). \quad (A.13)$$

In the first and second line of (A.11) subscripts denote the weight with respect to $SO(1,1)$, whereas in the third line they just discriminate between the three singlets of $SO(q) \times A_q$.

Therefore, with respect to $SO(q) \times A_q$, one obtains:
- three singlets (notice that $(1, 1)_I$ is also singlet of $SO(q, 1) \times A_q$ and of $G_6 \times A_q$, and that $(1, 1)_{II}$ is singlet of $SO(q, 1) \times A_q$, as well);

- a vector $(q, 1)$;

- two (double-)spinors $(\text{Spin}'(q), \text{Spin}(Q_q))$ and $(\text{Spin}''(q), \text{Spin}(Q_q))$.

As a feature peculiar to (A.11), the vector $(q, 1)$ and the two (double-)spinors $(\text{Spin}'(q), \text{Spin}(Q_q))$ and $(\text{Spin}''(q), \text{Spin}(Q_q))$ do exhibit a “triality symmetry”, realized differently depending on $q = 8, 4, 2, 1$, as given in Sub-App. A.4.

The representation decomposition (A.11) yields that $d_{ijk}$ decomposes in such a way that the manifest “triality” exhibited by the “branching” of $R_Q$ is removed, and the two (double-)spinors are put on a different footing with respect to the vector. As a consequence:

- $(1, 1)_{II}$, $(1, 1)_{III}$ and $(q, 1)$;

- $(\text{Spin}'(q), \text{Spin}(Q_q))$ and $(\text{Spin}''(q), \text{Spin}(Q_q))$

separately have the same couplings inside $(R_Q)^3$.

The position which solves (with maximal - compact - symmetry $SO(q) \times A_q$) the “small” lightlike $G_5$-invariant defining constraints (4.51) in “bare” charges (and in a way consistent with an orbit representative having $Z = 0$) reads as follows:

\[
\begin{align*}
q_{(q, 1)} &= 0; \\
q_{(\text{Spin}'(q), \text{Spin}(Q_q))} &= 0; \\
q_{(\text{Spin}''(q), \text{Spin}(Q_q))} &= 0; \\
\end{align*}
\]  

(A.14)

with the three singlets $q_{(1, 1)_I}$, $q_{(1, 1)_{II}}$ and $q_{(1, 1)_{III}}$ constrained by

\[
q_{(1, 1)_I} \begin{bmatrix}
\begin{array}{c}
d_{(1,1)_I,(1,1)_I,(1,1)_I}, q_{(1,1)_I}^2 \\
+ 2d_{(1,1)_I,(1,1)_I,(1,1)_{II}}, q_{(1,1)_I}, q_{(1,1)_{II}} \\
+ d_{(1,1)_I,(1,1)_{II),(1,1)_{II}}, q_{(1,1)_{II}}^2
\end{array}
\end{bmatrix} = 0.
\]  

(A.15)

Notice that in (A.14) the charges related to the vector and to the two (double-)spinors are on equal footing, thus exhibiting a “triality symmetry”, as already mentioned above.

Notice that $SO(q, 1) \times A_q$ is the unique group which is maximally (if one consider also the factor $SO(1, 1)$) and symmetrically embedded into $G_6 \times A_q \times SO(1, 1)$, and also which has $SO(q) \times A_q$ as mcs. Therefore, it follows that $SO(q, 1) \times A_q$ is also the maximal semi-simple symmetry of $O_{\text{lightlike},nBPS}$, which is thus given by Eq. (4.24).

As mentioned above, the origin of $R_{\text{spin}(q+1),\text{spin}(Q_q)}$ in the stabilizer of $O_{\text{lightlike},nBPS}$ will be explained through the procedure of suitable İnönü-Wigner contraction performed in Sub-App. B.1.

**A.1.4 Details**

We now explicit some details of the treatment of symmetric “magic” RSG.

We start by giving the explicit form of Eqs. (A.4) and (A.5) for all $q = 8, 4, 2, 1$ classifying symmetric “magic” RSG.
\( q = 8 \) \((j_3^C)\)

\[
E_{6(-26)} \supseteq_{\text{max}} SO(9,1) \times SO(1,1) \supseteq_{\text{mcs}} SO(9);
\]
\[
27 \rightarrow 1_{+4} + 10_{-2} + 16_{+1} \rightarrow 1_I + 9 + 1_{II} + 16. \tag{A.16}
\]

\( q = 4 \) \((j_3^H)\) \((SO(5,1) \sim SU^*(4), SO(5) \sim USp(4))\)

\[
SU^*(6) \supseteq_{\text{max}} SO(5,1) \times SO(3) \times SO(1,1) \supseteq_{\text{mcs}} SO(5) \times SO(3); \tag{A.17}
\]
\[
15 \rightarrow (1,1)_{+4} + (6,1)_{-2} + (4,2)_{+1} \rightarrow (1,1)_I + (5,1) + (1,1)_{II} + (4,2).
\]

\( q = 2 \) \((j_3^C)\) \((SL(2, \mathbb{C}) \sim SO(3,1), GL(1, \mathbb{C}) \sim SO(2) \times SO(1,1))\)

\[
SL(3, \mathbb{C}) \supseteq_{\text{max}} SL(2, \mathbb{C}) \times SL(1, \mathbb{C}) \times GL(1, \mathbb{C}) \supseteq_{\text{mcs}} SO(3) \times SO(2); \tag{A.18}
\]
\[
9 \rightarrow (1_0)_{+4} + (3_0 + 1_0)_{-2} + (2_3 + \overline{2}_-3)_{+1} \rightarrow (1_0)_I + 3_0 + (1_0)_{II} + 2_3 + 2_-3,
\]

where the first subscript in the second step and the subscript in the last step denote charges w.r.t. \( SO(2) \sim U(1) \), as well as the second subscript in the second step denotes weights w.r.t. \( SO(1,1) \). In order to derive (A.18), the decompositions of the irreps. of \( SL(3, \mathbb{C}) \) under \( SL(2, \mathbb{C}) \times SL(1, \mathbb{C}) \times GL(1, \mathbb{C}) \sim SL(2, \mathbb{C}) \times SO(2) \times SO(1,1) \) have been recalled (the charges and weights w.r.t. \( SO(2) \) and \( SO(1,1) \) are given):

\[
\begin{align*}
3 & \rightarrow (2,1,-1) + (1,-2,2); \tag{A.19} \\
\overline{3} & \rightarrow (\overline{2},-1,-1) + (1,2,2); \tag{A.20} \\
3' & \rightarrow (2,-1,1) + (1,2,-2); \tag{A.21} \\
\overline{3}' & \rightarrow (\overline{2},1,1) + (1,-2,-2). \tag{A.22}
\end{align*}
\]

Thus, through (A.19) and (A.20), the irrepr.

\[
R_{q=2} = 9 \equiv 3 \times \overline{3} \tag{A.23}
\]

branches as given by (A.18).

\( q = 1 \) \((j_3^H)\) \((SL(2, \mathbb{R}) \sim SO(2,1))\)

\[
SL(3, \mathbb{R}) \supseteq_{\text{max}} SO(2,1) \times SO(1,1) \supseteq_{\text{mcs}} SO(2); \tag{A.24}
\]
\[
6' \rightarrow 1_{+4} + 3_{-2} + 2_{+1} \rightarrow 1_I + 2 + 1_{II} + 2,
\]

where the the normalizations and conventions of Table 58 of [55] have been adopted.

Next, we explicit Eqs. (A.10) and (A.11) for all \( q = 8, 4, 2, 1 \) classifying symmetric “magic” RSG.
\[ q = 8 \left( j_3^C \right) \]

\[ E_{6(-26)} \supseteq_{\text{max}} SO(9, 1) \times SO(1, 1) \supseteq_{\text{max}} SO(8, 1) \times SO(1, 1) \supseteq_{\text{max}} SO(8); \]

\[ 27 \to 1_{+4} + 10_{-2} + 16_{+1} \to 1_{+4} + 9_{-2} + 1_{-2} + 16_{+1} \to 1_I + 8_v + 1_{II} + 1_{II} + 8_s + 8_c. \]

The “triality” in irreps. of \( SO(q) \) is here implemented through the triality of \( (8_v, 8_s, 8_c) \) of \( SO(8). \)

\[ q = 4 \left( j_3^{\text{mcs}} \right) \]

\[ SU^* (6) \supseteq_{\text{max}} SO(5, 1) \times SO(3) \times SO(1, 1) \supseteq_{\text{max}} SO(4) \times SO(3) \sim SU(2) \times SU(2) \times SU(2); \]

\[ 15 \to (1, 1)_{+4} + (6, 1)_{-2} + (4, 2)_{+1} \to (1, 1)_{+4} + (5, 1)_{-2} + (1, 1)_{-2} + (4, 2)_{+1} \to (1, 1, 1)_I + (2, 2, 1) + (1, 1, 1)_{III} + (1, 1, 1)_{II} + (1, 2, 2) + (2, 1, 2). \]

Thus, the “triality” in irreps. of \( SO(q) \times A_q \) is implemented for \( q = 4 \) through the triality of \((2, 2, 1), (2, 1, 2), (1, 2, 2)) \) of \( SU(2) \times SU(2) \times SU(2). \)

\[ q = 2 \left( j_3^C \right) \]

\[ SL(3, \mathbb{C}) \supseteq_{\text{max}} SL(2, \mathbb{C}) \times SL(1, \mathbb{C}) \times GL(1, \mathbb{C}) \supseteq_{\text{max}} SO(2, 1) \times SO(2) \times SO(1, 1) \supseteq_{\text{max}} SO(2) \times SO(2); \]

\[ 9 \to (1_0)_{+4} + (3_0 + 1_0)_{-2} + (2_3 + 2_{-3})_{+1} \to (1_0)_{+4} + (3_0)_{-2} + (1_0)_{-2} + (2_3)_{+1} + (2_{-3})_{+1} \to (1_0)_I + 2_0 + (1_0)_{III} + (1_0)_{II} \]

Thus, the triality in irreps. of \( SO(q) \times A_q \) is implemented for \( q = 2 \) through the triality of \((2_0, 2_3, 2_{-3}) \) of \( SO(2) \times SO(2) \) (notice the different charges w.r.t. \( A_{q=2} = SO(2) \sim U(1) \)).

\[ q = 1 \left( j_3^{\text{mcs}} \right) \]

\[ SL(3, \mathbb{R}) \supseteq_{\text{max}} SO(2, 1) \times SO(1, 1) \supseteq_{\text{max}} SO(1, 1) \supseteq_{\text{max}} 1; \]

\[ 6' \to 1_{+4} + 3_{-2} + 2_{+1} \to 1_{+4} + 2_{-2} + 1_{-2} + 2_{+1} \to 1_I + 1_{II} + 1_{III} + 1_{IV} + 1_{V} + 1_{VI}, \]

where in the first line 1 denotes the identity element. Notice that there is no compact symmetry in \( \mathcal{O}_{\text{lightlike}, nBPS, J_3^C, d=5} \), as also given by the fact that \( \mathcal{M}_{\text{lightlike}, nBPS, J_3^C, d=5} = SO(1, 1) \times \mathbb{R}^2 \) (see Table 3). Thus, the “triality” of irreps. of \( SO(q) \) in this case trivially degenerates into a “sexuality” (six singlets in the r.h.s. of the second line of (A.30)).
A.2 “Dressed” Charges Basis

Concerning the resolution of the \( G_5 \)-invariant (sets of) constraints in the basis of “dressed” charges, one should notice that for each of the four “magic” symmetric RSG’s a unique non-compact, real form \( \tilde{H}_5 \) of the compact group \( H_5 \equiv mcs \left( G_5 \right) \) exists with maximal symmetric embedding into \( G_5 \) (see e.g. \[54\]; also recall Subsect. 3.4 and Table 1):

\[
G_5 \supseteq_{\text{max}} \tilde{H}_5. \tag{A.31}
\]

A.2.1 \( \mathcal{O}_{\text{lightlike,BPS}} \)

In order to solve the “small” lightlike \( G_5 \)-invariant defining constraints in “dressed” charges in a way consistent with an orbit representative with \( Z \neq 0 \), let us further embed

\[
\tilde{h}_5 \equiv mcs \left( \tilde{H}_5 \right) = SO \left( q + 1 \right) \times \mathcal{A}_q, \tag{A.32}
\]

thus obtaining

\[
G_5 \left( \supseteq_{\text{max}} \tilde{H}_5 \right) \overset{\text{mcs} \left( \tilde{H}_5 \right)}{\supseteq} SO \left( q + 1 \right) \times \mathcal{A}_q, \tag{A.33}
\]

where the brackets denote the auxiliary nature of the embedding. Thus, under the “branching” \( (A.33) \) \( R_Q \) decomposes as follows:

\[
R_Q \left( \rightarrow 1 + \hat{R} \right) \rightarrow (1,1)_I + (q+1,1) + (\text{Spin} \left( q + 1 \right), \text{Spin} \left( Q_q \right)) + (1,1)_{II}, \tag{A.34}
\]

where \( \hat{R} \) is an irrepr. of \( \tilde{H}_5 \) used as an intermediate step. Eq. \( (A.34) \) corresponds to the “branching”

\[
Z \equiv (Z,Z_x) \rightarrow \left( Z, Z_{(1,1)_I}, Z_{(q+1,1)}, Z_{\left( \text{Spin} \left( q + 1 \right), \text{Spin} \left( Q_q \right) \right)} \right), \tag{A.35}
\]

where

\[
Z_{(1,1)_I} \equiv Z \tag{A.36}
\]

throughout. Therefore, with respect to \( SO \left( q + 1 \right) \times \mathcal{A}_q \), one obtains:

- two singlets;
- one vector \( (q + 1,1) \);
- one (double-)spinor \( (\text{Spin} \left( q + 1 \right), \text{Spin} \left( Q_q \right)) \).

The position which solves (with maximal - compact - symmetry \( SO \left( q + 1 \right) \times \mathcal{A}_q \)) the “small” lightlike \( G_5 \)-invariant defining constraints in “dressed” charges (and in a way consistent with an orbit representative having \( Z \neq 0 \)) reads as follows:

\[
\left\{ \begin{array}{l}
Z_{(q+1,1)} = 0; \\
Z_{\left( \text{Spin} \left( q + 1 \right), \text{Spin} \left( Q_q \right) \right)} = 0,
\end{array} \right. \tag{A.37}
\]

with \( Z \) and \( Z_{(1,1)_{II}} \) constrained by:

\[
Z^3 - \left( \frac{3}{2} \right)^2 ZZ^2_{(1,1)_{II}} - \left( \frac{3}{2} \right)^3 T_{(1,1)_{II}(1,1)_{II}(1,1)_{II}} Z^3_{(1,1)_{II}} = 0. \tag{A.38}
\]
Notice that $SO(q + 1) \times A_q$ is the unique group which is maximally (and symmetrically) embedded into $\tilde{H}_5$ and which has $SO(q + 1) \times A_q$ as (in this case improper) mcs (actually, $SO(q + 1) \times A_q = mcs(\tilde{H}_5)$). Therefore, it follows that $SO(q + 1) \times A_q$ is also the maximal semi-simple symmetry of $O_{\text{lightlike,BPS}}$, which is thus given by Eq. (4.14).

The explicit form of Eqs. (A.33) and (A.34) for all $q = 8, 4, 2, 1$ classifying symmetric “magic” RSG is given below.

- $q = 8$ \(J^0_3\)
  \[
  E_6(-26) \left( \supseteq_{\text{max}} F_{4(-20)} \right) \supseteq \text{SO}(9);
  \]
  \[
  27 \, (\rightarrow 1 + 26) \rightarrow 1_I + 9 + 16 + 1_{II}.
  \]

- $q = 4$ \(J^R_3\)
  \[
  SU^*(6) \left( \supseteq_{\text{max}} USp(4,2) \right) \supseteq USp(4) \times USp(2) \sim SO(5) \times SO(3);
  \]
  \[
  15 \, (\rightarrow 1 + 14) \rightarrow (1,1)_I + (5,1) + (4,2) + (1,1)_{II}.
  \]

- $q = 2$ \(J^C_3\)
  \[
  SL(3,C) \left( \supseteq_{\text{max}} SU(2,1) \right) \supseteq SU(2) \times U(1) \sim SO(3) \times SO(2);
  \]
  \[
  9 \, (\rightarrow 1 + 8) \rightarrow (1_0)_I + 2_{-3} + 2_3 + 3_0 + (1_0)_{II}.
  \]

- $q = 1$ \(J^R_3\)
  \[
  SL(3,R) \left( \supseteq_{\text{max}} SO(2,1) \right) \supseteq SO(2);
  \]
  \[
  6' \, (\rightarrow 1 + 5) \rightarrow 1_I + 2 + 2 + 1_{II}.
  \]

As mentioned in the resolution in the basis of “bare” (electric) charges $q_i$’s, the origin of $R(\text{spin}(q+1),\text{spin}(q))$ in the stabilizer of $O_{\text{lightlike,BPS}}$ will be explained through the procedure of suitable Inönü-Wigner contraction performed in Sub-App. B.1.

**A.2.2 $O_{\text{lightlike,nBPS}}$**

In order to solve the “small” lightlike $G_5$-invariant defining constraints [4.51] in “dressed” charges in a way consistent with an orbit representative having $Z = 0$, the embedding (A.31) has to be further elaborated as follows:

\[
G_5 \left( \supseteq_{\text{max}} \tilde{H}_5 \right) \supseteq_{\text{max}} \tilde{h}_5 \supseteq \text{SO}(q) \times A_q,
\]

where

\[
\tilde{h}_5 = SO(q,1) \times A_q
\]

is the unique non-compact form of $\tilde{h}_5$ (defined by (A.32)) to be embedded maximally and symmetrically into $\tilde{H}_5$ (see e.g. [5]).
Thus, under the “branching” \( \text{Eqs. (A.43)-(A.45)} \) \( R_Q \) decomposes as follows:

\[
R_Q \rightarrow 1 + \tilde{R}
\]

\[
\rightarrow (1, 1)_I + (q + 1, 1) + (\text{Spin} (q + 1), \text{Spin} (Q_q)) + (1, 1)_{II}
\]

\[
\rightarrow (1, 1)_I + (q, 1) + (1, 1)_{III} + (\text{Spin}' (q), \text{Spin} (Q_q)) + (\text{Spin}'' (q), \text{Spin} (Q_q)) + (1, 1)_{III}.
\]

(Eq. (A.45))

Eq. (A.45) corresponds to the “branching” (recall Eq. (A.36))

\[
Z \equiv (Z, Z_x) \rightarrow \left( Z, Z{}_{(1,1)_{I}}, Z{}_{(1,1)_{II}}, Z{}_{(q,1)}, Z{}_{(\text{Spin}' (q), \text{Spin} (Q_q))}, Z{}_{(\text{Spin}'' (q), \text{Spin} (Q_q))} \right).
\]

Therefore, with respect to \( SO (q) \times A_q \), besides \( Z \), one obtains:

- two singlets (note that \((1,1)_{II}\) is a singlet of \( SO (q,1) \times A_q \), as well):
- one vector \((q,1)\):
- two (double-)spinors \((\text{Spin}' (q), \text{Spin} (Q_q)) \) and \((\text{Spin}'' (q), \text{Spin} (Q_q)) \).

The position which solves (with maximal - compact - symmetry \( SO (q) \times A_q \)) the “small” lightlike \( G_5 \)-invariant defining constraints (4.50) in “dressed” charges (and in a way consistent with an orbit representative having \( Z = 0 \)) reads as follows:

\[
\begin{align*}
Z &= Z{}_{(1,1)_{I}} = 0; \\
Z{}_{(q,1)} &= 0; \\
q_{(\text{Spin}' (q), \text{Spin} (Q_q))} &= 0; \\
q_{(\text{Spin}'' (q), \text{Spin} (Q_q))} &= 0,
\end{align*}
\]

(Eq. (A.47))

with the two singlets \( Z_{(1,1)_{II}} \) and \( Z_{(1,1)_{III}} \) constrained by

\[
T_{(1,1)_{II}(1,1)_{II}(1,1)_{II} Z_{(1,1)_{II}}}^{2} + 3T_{(1,1)_{II}(1,1)_{II}(1,1)_{II} Z_{(1,1)_{III}}}^{2} = 0.
\]

(Eq. (A.48))

Besides \( SO (q + 1) \times A_q \), the only other group which is maximally (and symmetrically) embedded into \( \tilde{H}_5 \) and which has \( SO (q) \times A_q \) as \((m)\) cs, is \( SO (q, 1) \times A_q \). Therefore, \( SO (q, 1) \times A_q \) is also the maximal semi-simple symmetry of \( \mathcal{O}_{\text{lightlike,BPS}} \), which is thus given by Eq. (4.24).

The explicit form of Eqs. (A.43)-(A.45) for all \( q = 8, 4, 2, 1 \) classifying symmetric “magic” RSG is given below.

- \( q = 8 \) \( \left( J_{3}^{0} \right) \)

\[
E_{6(-26)} \left( \mathcal{F}_{\text{max}} \left( F_{4(-20)} \right) \right) \mathcal{O}_{\text{max}} \left( SO (8, 1) \right) \gtrless m_{cs} \mathcal{O}_{\text{max}} \left( SO (8) \right);
\]

(Eq. (A.49))

\[
27 \rightarrow 1 + 26 \rightarrow 1_{I} + 9 + 16 + 1_{II} \rightarrow 1_{I} + 8_{v} + 1_{III} + 1_{II} + 8_{s} + 8_{c}.
\]

- \( q = 4 \) \( \left( J_{3}^{3} \right) \) \( USp (2, 2) \sim SO (5, 1), USp (2) \sim SU (2) \)

\[
SU^{*} (6) \left( \mathcal{F}_{\text{max}} \left( USp (4, 2) \right) \right) \mathcal{O}_{\text{max}} \left( USp (2, 2) \times USp (2) \right) \gtrless m_{cs} \mathcal{O}_{\text{max}} \left( USp (2) \times USp (2) \times USp (2) \right);
\]

(Eq. (A.50))

\[
15 \rightarrow 1 + 14 \rightarrow (1, 1)_{I} + (5, 1) + (4, 2) + (1, 1)_{II} \rightarrow (1, 1, 1)_{I} + (1, 1, 1)_{II} + (2, 2, 1) + (2, 1, 2) + (1, 2, 2) + (1, 1, 1)_{III}.
\]

(Eq. (A.51))
• $q = 2 \left( J_3^C \right)$

$$SL(3, \mathbb{C}) \supseteq_{\text{max}} SU(2,1) \supseteq_{\text{max}} SU(1,1) \times U(1) \supseteq_{\text{mcs}} U(1) \times U(1);$$

$$9 \left( \rightarrow 1 + 8 \right) \rightarrow (1_0)_I + 2_3 + 2_{-3} + 3_0 + (1_0)_{II} \rightarrow (1_0)_I + 2_0 + 2_3 + 2_{-3} + (1_0)_{III} + (1_0)_{II}. \quad (A.52)$$

• $q = 1 \left( J_3^R \right)$

$$SL(3, \mathbb{R}) \supseteq_{\text{max}} SO(2,1) \supseteq_{\text{max}} SO(1,1) \supseteq_{\text{mcs}} 1;$$

$$6' \left( \rightarrow 1 + 5 \right) \rightarrow 1_I + 2 + 1_{II} + 2 \rightarrow 1_I + 1_{II} + 1_{III} + 1_{IV} + 1_{V} + 1_{VI}, \quad (A.53)$$

where 1 denotes the identity element.

The origin of $\mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}$ in the stabilizer of $O_{\text{lightlike}, \text{BPS}}$ will be explained through the procedure of suitable İnönü-Wigner contraction performed in Sub-App. B.1

### B Equivalent Derivations

In this Appendix, we determine the general form of “small” charge orbits of symmetric “magic” RSG (see Eqs. (4.14), (4.24) and (4.29)) through suitable group theoretical procedures, namely:

• İnönü-Wigner contractions, for “small” lightlike orbits, Sub-App. B.1

• $SO(1,1)$-three-grading, for “small” critical orbit, Sub-App. B.2

Such procedures will clarify the origin of the non-semi-simple Abelian (namely, translational) factor (recall Eq. (4.1), definitions (4.17)-(4.18), and see Eq. (B.41) below)

$$\mathcal{T} = \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))} \quad \text{(B.1)}$$

in all three classes (lightlike BPS, lightlike non-BPS, and critical BPS) of “small” orbits (for each relevant $q = 8, 4, 2, 1$).

#### B.1 İnönü-Wigner Contractions

##### B.1.1 $O_{\text{lightlike}, \text{BPS}}$

In order to deal with $O_{\text{lightlike}, \text{BPS}}$, we start from the group embedding [A.33]. This determines the following decompositions of irreprs. ($\text{Adj}$ and $\text{Fund}$ respectively denoting the adjoint and fundamental irrepr.):

$$\text{Adj} (G_5) \rightarrow \text{Adj} \left( \tilde{H}_5 \right) + \text{Fund} \left( \tilde{H}_5 \right), \quad \text{(B.2)}$$

and further

$$\text{Adj} \left( \tilde{H}_5 \right) \rightarrow (\text{Adj} (SO(q+1)), 1) + (1, \text{Adj} (A_q)) + (\text{Spin} (q+1), \text{Spin} (Q_q))_I; \quad \text{(B.3)}$$

$$\text{Fund} \left( \tilde{H}_5 \right) \rightarrow (1, 1) + (q+1, 1) + (\text{Spin} (q+1), \text{Spin} (Q_q))_{II}. \quad \text{(B.4)}$$
where trivially $\text{Adj}(SO(q+1)) = \frac{q(q+1)}{2}$. Eqs. (B.2)-(B.4) thus imply

$$\text{Adj}(G_5) \rightarrow (\text{Spin}(q+1), \text{Spin}(Qq))_I + (\text{Adj}(SO(q+1)), 1) + (1, \text{Adj}(A_q)) + (1, 1) + (q, 1, 1) + (\text{Spin}(q+1), \text{Spin}(Qq))_{II}.$$  \hspace{1cm} (B.5)

The decomposition of the branching (B.3) yields to

$$\text{Adj}(\tilde{\mathfrak{g}}_{H_5}) \rightarrow (\text{Adj}(SO(q+1)), 1) + (1, \text{Adj}(A_q)) \oplus_s (\text{Spin}(q+1), \text{Spin}(Qq)).$$  \hspace{1cm} (B.6)

The coset (recall Eq. (3.62))

$$\tilde{H}_5 \rightarrow \text{mcs} \left( \tilde{H}_5 \right) = \frac{\tilde{H}_5}{SO(q+1) \times A_q} = \mathcal{M}_{nBPS,\text{large}}$$  \hspace{1cm} (B.7)

is symmetric, with real dimension, Euclidean signature and character respectively (see e.g. [54, 56]; here “c” and “nc” respectively stand for “compact” and “non-compact”):

$$\text{dim}_\mathbb{R} = 2q;$$

$$(c, nc) = (0, 2q);$$

$$\chi \equiv c - nc = -2q.$$  \hspace{1cm} (B.8)

By definition, the symmetricity of $\mathcal{M}_{nBPS,\text{large}}$ implies that

$$\begin{align*}
\left[ \mathfrak{h}_{\tilde{H}_5}, \mathfrak{h}_{\tilde{H}_5} \right] &= \mathfrak{h}_{\tilde{H}_5}; \\
\left[ \mathfrak{h}_{\tilde{H}_5}, \mathfrak{k}_{\tilde{H}_5} \right] &= \mathfrak{k}_{\tilde{H}_5}; \\
\left[ \mathfrak{k}_{\tilde{H}_5}, \mathfrak{k}_{\tilde{H}_5} \right] &= \mathfrak{h}_{\tilde{H}_5}.
\end{align*}$$  \hspace{1cm} (B.9)

The “decoupling” of $\mathfrak{h}_{\tilde{H}_5}$, with subsequent transformation of the irrepr. $$(\text{Spin}(q+1), \text{Spin}(Qq))_{II}$$ of $SO(q+1) \times A_q$ into the non-semi-simple, Abelian (namely, translational) part of the stabilizer of $\mathcal{O}_{\text{lightlike},\text{BPS}}$ is achieved by performing a uniform rescaling of the generators of $\mathfrak{k}_{\tilde{H}_5}$:

$$\mathfrak{k}_{\tilde{H}_5} \rightarrow \lambda \mathfrak{k}_{\tilde{H}_5}, \quad \lambda \in \mathbb{R}_0^+,$$  \hspace{1cm} (B.10)

and then by letting $\lambda \rightarrow \infty$. This amounts to performing an İnönü-Wigner (IW) contraction [57, 58] on $\mathfrak{k}_{\tilde{H}_5}$. Thus (recall Eqs. (4.14) and (4.16)):

$$\text{IW} \left( \mathcal{O}_{nBPS,\text{large}} = \frac{G_5}{H_5} \right) \rightarrow G_5 \rightarrow \left( SO(q+1) \times A_q \right) \times \mathbb{R}(\text{spin}(q+1), \text{spin}(Qq));$$  \hspace{1cm} (B.11)

$$\mathcal{T}_{\text{lightlike},\text{BPS}} \equiv \mathbb{R}(\text{spin}(q+1), \text{spin}(Qq)).$$  \hspace{1cm} (B.12)

Thus, $\mathcal{T}_{\text{lightlike},\text{BPS}}$ given by (B.12) is the $\mathfrak{k}_{\tilde{H}_5}$-part of the decomposition (B.6) of the Lie algebra $\mathfrak{g}_{\tilde{H}_5}$ of $\tilde{H}_5$ with respect to $\text{mcs} \left( \tilde{H}_5 \right) = SO(q+1) \times A_q$, which then gets “decoupled” from $\mathfrak{g}_{\tilde{H}_5}$ and Abelianized through the IW contraction procedure (B.10)-(B.11).
B.1.2 $\mathcal{O}_{\text{lightlike,nBPS}}$

On the other hand, the treatment of $\mathcal{O}_{\text{lightlike,nBPS}}$ requires to start from the embedding (A.43) (actually, without the last step involving $SO(q) \times A_q = mcs(\hat{h}_5)$; recall Eq. (A.44):

$$G_5 \supseteq_{\text{max}} \tilde{H}_5 \supseteq_{\text{max}} \hat{h}_5 = SO(q,1) \times A_q.$$ (B.13)

The subsequent decompositions of $\text{Adj}(G_5)$, $\text{Adj}(\tilde{H}_5)$ and $\text{Fund}(\tilde{H}_5)$ are given by Eqs. (B.2), (B.3) and (B.4), respectively, thus yielding the same decomposition as in (B.5). Consequently, the decomposition of the branching (B.3) yields the same result as in (B.6).

The coset (recall Eq. (3.62))

$$\tilde{H}_5 = \tilde{H}_5 \supseteq_{\text{max}} SO(q,1) \times A_q,$$ (B.14)

is symmetric, with real dimension, Euclidean signature and character respectively:

$$\text{dim}_R = 2q; \quad (c, nc) = (q, q); \quad \chi \equiv c - nc = 0.$$ (B.15)

By definition, the symmetricity of $\tilde{H}_5$ implies the same relations as in (B.9).

Thus, the “decoupling” of $b_{\tilde{H}_5}$, with subsequent transformation of the irrepr. $(\text{Spin}(q+1), \text{Spin}(Q_q))_I$ of $SO(q,1) \times A_q$ into the non-semi-simple, Abelian (namely, translational) part of the stabilizer of $\mathcal{O}_{\text{lightlike,nBPS}}$ is achieved by performing a uniform rescaling of the generators of $t_{\tilde{H}_5}$ as given by Eq. (B.10), and then by letting $\lambda \to \infty$. This amounts to performing an IW contraction [57, 58] on $t_{\tilde{H}_5}$. Therefore, one obtains (recall Eqs. (4.24) and (4.26)):

$$\text{IW}(\mathcal{O}_{\text{nBPS,large}}) \xrightarrow{\text{A.43}} \mathcal{O}_{\text{lightlike,nBPS}} = \frac{G_5}{(SO(q,1) \times A_q) \times \mathbb{R}(\text{spin}(q+1), \text{spin}(Q_q))},$$ (B.16)

$$\mathcal{T}_{\text{lightlike,nBPS}} = \mathcal{T}_{\text{lightlike,BPS}} = \mathbb{R}(\text{spin}(q+1), \text{spin}(Q_q)).$$ (B.17)

Thus, $\mathcal{T}_{\text{lightlike,nBPS}}$ given by (B.17) is the $t_{\tilde{H}_5}$-part of the decomposition (B.6) of the Lie algebra $\mathfrak{g}_{\tilde{H}_5}$ of $\tilde{H}_5$ with respect to $\hat{h}_5 = SO(q,1) \times A_q$, which then gets “decoupled” from $\mathfrak{g}_{\tilde{H}_5}$ and Abelianized through the IW contraction procedure (see Eqs. (B.10) and (B.16)).

Note that the IW contraction does not change the dimension of the starting orbit. Indeed $\mathcal{O}_{\text{lightlike,BPS}}$, obtained through the IW contraction of $\mathcal{O}_{\text{nBPS,large}}$ along the branching (A.33), has the same real dimension of $\mathcal{O}_{\text{nBPS,large}}$ itself. Analogously, also $\mathcal{O}_{\text{lightlike,nBPS}}$, obtained through the IW contraction of $\mathcal{O}_{\text{nBPS,large}}$ along the branching (A.43), has the same real dimension of $\mathcal{O}_{\text{nBPS,large}}$ itself.

B.1.3 Details

Below, we explicit in order, besides (B.2)-(B.4), the relevant formulae of the derivations given above, namely Eqs. (B.7), (B.8), (B.11), and (B.14), (B.15), (B.16), for all $q = 8, 4, 2, 1$ classifying symmetric “magic” RSG.
• \( q = 8 \left( J^O_3 \right) \)

\[
\begin{align*}
78 & \rightarrow 26 + 52; \\
52 & \rightarrow 36 + 16_I; \\
26 & \rightarrow 1 + 9 + 16_{II};
\end{align*}
\]  \hspace{1cm} (B.18)

\[
\frac{\tilde{H}_5}{\text{mcs}(H_5)} = \frac{\bar{H}_5}{\text{SO}(q+1) \times \mathcal{A}_q} \bigg|_{q=8} = \mathcal{M}_{nBPS,large,J^O_3,d=5} = \frac{F_{4(-20)}}{\text{SO}(9)};
\]

\( \dim \mathbb{R} = 16; \; (c, nc) = (0, 16); \; \chi = -16; \)  \hspace{1cm} (B.19)

\[
\text{IW} \left( \mathcal{O}_{nBPS,large,J^O_3} \right) = \frac{F_{6(-26)}}{\mathcal{F}_4(-20)} \longrightarrow \mathcal{O}_{\text{lightlike},BPS,J^O_3} = \frac{F_{6(-26)}}{\text{SO}(9) \times \mathbb{R}^{16}};
\]

\[
\frac{\tilde{H}_5}{\bar{H}_5} = \frac{\bar{H}_5}{\text{SO}(q+1) \times \mathcal{A}_q} \bigg|_{q=8} = \frac{F_{4(-20)}}{\text{SO}(8,1)};
\]

\( \dim \mathbb{R} = 16; \; (c, nc) = (8, 8); \; \chi = 0; \)  \hspace{1cm} (B.20)

\[
\text{IW} \left( \mathcal{O}_{nBPS,large,J^O_3} \right) = \frac{F_{6(-26)}}{\mathcal{F}_4(-20)} \longrightarrow \mathcal{O}_{\text{lightlike},nBPS,J^O_3} = \frac{F_{6(-26)}}{\text{SO}(8,1) \times \mathbb{R}^{16}}.
\]

• \( q = 4 \left( J^E_3 \right) \)

\[
\begin{align*}
35 & \rightarrow 14 + 21; \\
21 & \rightarrow (4, 2)_I + (10, 1) + (1, 3); \\
14 & \rightarrow (1, 1) + (5, 1) + (4, 2)_{II};
\end{align*}
\]  \hspace{1cm} (B.21)

\[
\frac{\tilde{H}_5}{\text{mcs}(H_5)} = \frac{\bar{H}_5}{\text{SO}(q+1) \times \mathcal{A}_q} \bigg|_{q=4} = \mathcal{M}_{nBPS,large,J^E_3,d=5} = \frac{\text{USp}(4,2)}{\text{USp}(4) \times \text{USp}(2)};
\]

\( \dim \mathbb{R} = 8; \; (c, nc) = (0, 8); \; \chi = -8; \)  \hspace{1cm} (B.22)

\[
\text{IW} \left( \mathcal{O}_{nBPS,large,J^E_3} \right) = \frac{SU^*(6)}{\text{USp}(4,2)} \longrightarrow \mathcal{O}_{\text{lightlike},BPS,J^E_3} = \frac{SU^*(6)}{(\text{SO}(5) \times \text{SO}(3)) \times \mathbb{R}^{(4,2)}};
\]

\[
\frac{\tilde{H}_5}{\bar{H}_5} = \frac{\bar{H}_5}{\text{SO}(q+1) \times \mathcal{A}_q} \bigg|_{q=4} = \frac{\text{USp}(4,2)}{\text{USp}(2,2) \times \text{USp}(2)};
\]

\( \dim \mathbb{R} = 8; \; (c, nc) = (4, 4); \; \chi = 0; \)  \hspace{1cm} (B.23)

\[
\text{IW} \left( \mathcal{O}_{nBPS,large,J^E_3} \right) = \frac{SU^*(6)}{\text{USp}(4,2)} \longrightarrow \mathcal{O}_{\text{lightlike},nBPS,J^E_3} = \frac{SU^*(6)}{(\text{SO}(4,1) \times \text{SO}(3)) \times \mathbb{R}^{(4,2)}}.
\]

• \( q = 2 \left( J^C_3 \right) \). Notice that in this case Eq. (B.2) gets modified into

\[
\text{Adj} \left( G_5 \right) \rightarrow \text{Adj} \left( \tilde{H}_5 \right) + \text{Adj} \left( \bar{H}_5 \right);
\]

\[
16 \rightarrow 8 + 8;
\]

\[
8 \rightarrow 3_0 + 1_0 + 2_3 + 2_{-3}.
\]
Everything fits also because for $q = 2$ it holds that

$$(q + 1, 1) = (\text{Adj} (SO (q + 1)), 1) = 3_0;$$

$$(1, \text{Adj} (A_q)) = (1, 1) = 1_0.$$  \hfill (B.25)

$$\tilde{H}_5 = \tilde{H}_5 \bigg|_{q=2} \equiv M_{nBPS,large,J_3^{\mathbb{R}}}, d=5 = SU(2,1) / U(1);$$  \hfill (B.26)

$\text{dim}_R = 4; (c, nc) = (0, 4); \chi = -4;$$

$$IW \left( O_{nBPS,large,J_3^{\mathbb{R}}} \right) = \frac{SL(3,\mathbb{C})}{SU(2,1)} \bigg| \rightarrow O_{lightlike,BPS,J_3^{\mathbb{R}}} = \frac{SL(3,\mathbb{C})}{(SO(3) \times SO(2)) \times \mathbb{R}^{1,2}.}$$

• $q = 1 \left( J_3^{\mathbb{R}} \right)$. Notice that in this case Eq. (B.2) gets modified into

$$\text{Adj} (G_5) \rightarrow \text{Adj} \left( \tilde{H}_5 \right) + \text{Spin}_{s=2} \left( \tilde{H}_5 \right);$$

$$8 \rightarrow 3 + 5;$$

$$3 \rightarrow 1_{II} + 2_I;$$

$$5 \rightarrow 1_I + 2_{III} + 2_{II};$$

$\text{dim}_R = 4; (c, nc) = (0, 4); \chi = -4;$$

$$IW \left( O_{nBPS,large,J_3^{\mathbb{R}}} \right) = \frac{SL(3,\mathbb{C})}{SU(2,1)} \bigg| \rightarrow O_{lightlike,BPS,J_3^{\mathbb{R}}} = \frac{SL(3,\mathbb{C})}{(SO(3) \times SO(2)) \times \mathbb{R}^{1,2}.}$$

$\tilde{H}_5 = \tilde{H}_5 \bigg|_{q=1} \equiv M_{nBPS,large,J_3^{\mathbb{R}}}, d=5 = SU(2,1) / U(1);$$  \hfill (B.30)

$\text{dim}_R = 2; (c, nc) = (0, 2); \chi = -2;$$

$$IW \left( O_{nBPS,large,J_3^{\mathbb{R}}} \right) = \frac{SL(3,\mathbb{R})}{SO(2,1)} \bigg| \rightarrow O_{lightlike,BPS,J_3^{\mathbb{R}}} = \frac{SL(3,\mathbb{R})}{SO(2) \times \mathbb{R}^{2}.}$$

$\tilde{H}_5 = \tilde{H}_5 \bigg|_{q=1} \equiv M_{nBPS,large,J_3^{\mathbb{R}}}, d=5 = SU(2,1) / U(1);$$  \hfill (B.31)

$\text{dim}_R = 2; (c, nc) = (1, 1); \chi = 0;$$

$$IW \left( O_{nBPS,large,J_3^{\mathbb{R}}} \right) = \frac{SL(3,\mathbb{R})}{SO(1,1) \times \mathbb{R}^{2}.}$$
B.2 \(SO(1,1)\)-Three Grading and \(O_{\text{critical,BPS}}\)

In order to deal with \(O_{\text{critical,BPS}}\), we start from the group embedding \([A.1]\). As pointed out above, this is the unique maximal embedding (at least among the symmetric ones; see e.g. \([54]\)) into \(G_5\) to exhibit a commuting factor \(SO(1,1)\).

Therefore, the Lie algebra \(\mathfrak{g}_{G_5}\) of \(G_5\) admits a three-grading with respect to the Lie algebra \(\mathbb{R}\) of \(SO(1,1)\) as follows:

\[
\mathfrak{g}_{G_5} = \mathcal{W}^+ + \mathcal{W}^0 + \mathcal{W}^-,
\]

where as above the subscripts denote the weights with respect to \(SO(1,1)\) itself. At the level of “branching” of \(\text{Adj}(G_5)\), the \((1,1)\)-three-grading reads as follows:

\[
\text{Adj}(G_5) \rightarrow (1,1)_0 + (\text{Adj}(G_6),1)_0 + (1,\text{Adj}(A_q))_0 + \\
+ (\text{Spin}(q+2),\text{Spin}(Q_q))_{-3} + \\
+ (\text{Spin}'(q+2),\text{Spin}(Q_q))_{+3}.
\]

Thus, the decomposition \([B.33]\) yields the following identification of the graded terms in \([B.32]\):

\[
\mathcal{W}^0 \equiv (1,1)_0 + (\text{Adj}(G_6),1)_0 + (1,\text{Adj}(A_q))_0,
\]

\[
\mathcal{W}^+ \equiv (\text{Spin}'(q+2),\text{Spin}(Q_q))_{+3};
\]

\[
\mathcal{W}^- \equiv (\text{Spin}(q+2),\text{Spin}(Q_q))_{-3};
\]

with “exp” denoting the exponential mapping.

Thus, \(O_{\text{critical,BPS}}\) is obtained by cosetting \(G_5\) with the +3 (or equivalently −3)-graded extension of \(\mathcal{W}^0 - (1,1)_0\), namely:

\[
O_{\text{critical,BPS}} = \frac{G_5}{N_{+3}(-3)},
\]

where

\[
N_{+3} \equiv \exp [(\mathcal{W}^0 - (1,1)_0) + s \mathcal{W}^+]
\]

\[
= \exp [(\text{Adj}(G_6),1)_0 + (1,\text{Adj}(A_q))_0 + s (\text{Spin}'(q+2),\text{Spin}(Q_q))_{+3}]
\]

\[
= (G_6 \times A_q) \rtimes \mathbb{R}^{(\text{spin}(q+2),\text{spin}(Q_q))};
\]

\[
N_{-3} \equiv \exp [(\mathcal{W}^0 - (1,1)_0) + s \mathcal{W}^-]
\]

\[
= \exp [(\text{Adj}(G_6),1)_0 + (1,\text{Adj}(A_q))_0 + s (\text{Spin}(q+2),\text{Spin}(Q_q))_{-3}]
\]

\[
= (G_6 \times A_q) \rtimes \mathbb{R}^{(\text{spin}(q+2),\text{spin}(Q_q))}.
\]

Thus, it holds that Eqs. \([B.37]\) and \([B.38]\) (or equivalently Eqs. \([B.37]\) and \([B.39]\)) are consistent with the general form of \(O_{\text{critical,BPS}}\) given by Eq. \([4.29]\).

Therefore, in the stabilizer of \(O_{\text{critical,BPS}}\), the factor

\[
\mathcal{T}_{\text{critical,BPS}} = \mathbb{R}^{(\text{spin}(q+2),\text{spin}(Q_q))} = \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}
\]
is given by the exponential mapping of the Abelian subalgebra of $\mathfrak{g}_{G_6}$ contained into the $+3$ (or equivalently $-3$)-graded extension of $\mathcal{W}^0 - (1,1)_0$ through the $SO(1,1)$-three grading (B.32), corresponding to the irrepr. $(\text{Spin}(q+2), \text{Spin}(Q_q))_{+3}$ (or equivalently $(\text{Spin}(q+2), \text{Spin}(Q_q))_{-3}$) of $G_6 \times A_q(\times SO(1,1))$.

The results obtained in Sub-Apps. [B.1] and [B.2] (and reported in Tables 3 and 4) allows one to conclude that all “small” charge orbits of symmetric “magic” RSG (classified by $q = 8, 4, 2, 1$) share the same non-semi-simple, Abelian (namely, translational) part of the stabilizer. Namely, Eqs. (B.17) and (B.40) yield to:

$$\mathcal{T}_{\text{lightlike, BPS}} = \mathcal{T}_{\text{lightlike, nBPS}} = \mathcal{T}_{\text{critical, BPS}} = \mathbb{R}^{(\text{spin}(q+1),\text{spin}(Q_q))}.$$  \hspace{1cm} (B.41)

### B.2.1 Details

Below, we explicit Eqs. (B.33)-(B.36) for all $q = 8, 4, 2, 1$ classifying symmetric “magic” RSG.

- $q = 8$ ($J^D_3$)
  
  \[
  78 \rightarrow \underbrace{1_0 + 45_0}_{} + \underbrace{16_{-3}}_{-3} + \underbrace{16'_{+3}}_{+3}. \hspace{1cm} (B.42)
  \]

- $q = 4$ ($J^E_3$)
  
  \[
  35 \rightarrow (1,1)_0 + (15,1)_0 + (1,3)_0 + (4,2)_{-3} + (4,2)_{+3}. \hspace{1cm} (B.43)
  \]

- $q = 2$ ($J^C_3$). In this case it should be recalled that
  
  \[
  \text{Adj}(SL(3, \mathbb{C})) = 16 \equiv 3 \times 3' + 3 \times 3' - 2 \text{ singlets}. \hspace{1cm} (B.44)
  \]

Thus, by recalling Eqs. (A.22)-(A.22), one can compute that under $SL(3, \mathbb{C}) \supseteq \max SL(2, \mathbb{C}) \times SL(1, \mathbb{C}) \times GL(1, \mathbb{C})$:

\[
3 \times 3' \rightarrow (3_0)_0 + (1_0)_0 + (2_3)_{-3} + (\overline{2}_{-3})_{-3} + (1_0)_0;
\]

\[
3 \times 3' \rightarrow (3_0)_0 + (1_0)_0 + (\overline{2}_{-3})_{-3} + (2_3)_3 + (1_0)_0. \hspace{1cm} (B.45)
\]

Therefore:

\[
\text{Adj}(SL(3, \mathbb{C})) = 16 \rightarrow \underbrace{2 (3_0)_0 + 2 (1_0)_0 + (2_3)_{-3}}_{-3} + \underbrace{(\overline{2}_{-3})_{-3}}_{-3} + \underbrace{(2_3)_3 + (\overline{2}_{-3})_{+3}}_{+3}. \hspace{1cm} (B.46)
\]

- $q = 1$ ($J^D_3$)
  
  \[
  8 \rightarrow \underbrace{1_0 + 3_0 + 2_{-3}}_{-3} + \underbrace{2_{+3}}_{+3}. \hspace{1cm} (B.47)
  \]
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