ASSET ALLOCATION FOR A DC PENSION PLAN WITH LEARNING ABOUT STOCK RETURN PREDICTABILITY

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Abstract. This paper investigates an optimal investment problem for a defined contribution pension plan member who receives a stochastic salary, and considers inflation risk and stock return predictability. The member aims to maximize the expected power utility from her terminal real wealth by investing her pension account wealth in a financial market consisting of a risk-free asset, an inflation-indexed bond and a stock. The expected excess return on the stock can be predicted by both an observable predictor and an unobservable predictor, and the member has to estimate the unobservable predictor by learning the history information. By using the filtering techniques and dynamic programming approach, the closed-form optimal investment strategy and the corresponding value function are derived. Finally, with the help of numerical analysis, we explore the impact of model parameters on the optimal investment strategy, and analyze the welfare benefits from leaning and using inflation-indexed bond to hedge the stock return predictors.

1. Introduction. Pension systems in most countries are facing serious challenges result from the lower fertility rate and longer lifespan. To handle the payment crisis and maintain the sustainability of pension system, more and more countries are switching their pension module from defined benefit (DB) pension plans to defined contribution (DC) pension plans. The reason lies in the fact that the contribution in a DC pension plan is predetermined while the benefit depends on the pension account’s investment return. In this sense, the effective investment management and risk control for DC pension plans are of significant importance for the members to maintain their life standard after retiring. Because the investment horizon for DC pension plans usually lasts for 20-40 years, the inflation will inevitably cut down the
actual investment return of DC pension plans during this long time period. Introducing the inflation risk into the investment management of DC pension plans will be helpful to handle the real benefit for the members after retirement. Consistent with this practical need, many studies to hedge the inflation risk for DC pension plans are conducted. For example, [21], [18], [3], [20] investigate the asset allocation problem of DC pension plans with inflation risk under different optimization objectives, respectively. These studies reveal that the inflation risk plays a decisive role in the investment strategy for DC pension plans. Investigating the investment decision for DC pension plans should not bypass the inflation risk.

In the above-mentioned literature, there exists a key assumption to assure the explicit investment strategy for DC pension plans, which is that the expected return (i.e., the appreciation rate) for the stock is a constant or a deterministic function of time. In other words, the decision-maker of DC pension plans knows the precise return of financial assets, which implies that all information of the financial market can be perfectly obtained during the investment period. This fundamental assumption results in the optimal investment strategy to be myopic. That means the investment strategy is independent of the investment horizon which is inconsistent with the investment practical in the financial market. Generally, the younger member (who has a longer investment horizon) is suggested to invest more money in the risky financial asset, while the elder member (who has a shorter investment horizon) is advised to put less money in the risky financial asset, which is called age effect. So weakening the assumption on perfect information may narrow the gap between the investment practise and the theoretical model.

Actually, due to various limitations, the decision-maker can only receive restricted information rather than complete information from the financial market. This implies that there are both observable and unobservable information in the financial market. Usually, the expected return of stock cannot be precisely figured out. It requires the decision-maker to spend time, money and effort to study the history performance of stock, which is helpful to form the decision-maker’s learning mechanism based on the history information in the financial market. Then, in the later time period, the decision-maker will predict the expected stock return by learning the history information. Meanwhile, numerous empirical studies also have confirmed stock return predictability by the realized stock return, dividend yield, earnings/price ratio, net payout yield and so on (see [1] and [7]). Meanwhile, most existing literature formulates stock return predictability by using the mean-reverting stochastic process. Accompanying with the empirical evolution on stock return predictability, [10], [14] and [16] consider the optimal portfolio selection problem when the stock return is predictable. They find that, in the optimal investment strategy, there emerges a hedging component resulting from the stock return predictability. Recently, [12] and [23] incorporate the stock return predictability into the optimal asset allocation problem of DC pension plans in continuous-time and discrete-time setting, respectively. Both [12] and [23] find that the stock return predictability leads to the result that the investment strategy depends on the investment horizon, which is consistent with the “age effect”. [6] find that the decision-maker for DC pension plans just uses partial information to determine their investment strategy rather than the whole information in the financial market. Therefore, weakening the setting of complete information to the case of partial information, and allowing to learn about stock return predictability will help to bridge the gap between the theoretical model and investment practice for DC pension plans.
Therefore, in a financial market with the stock return predictability, inflation risk and stochastic salary, we study the impact of learning on the optimal investment decision for a DC pension plan member. The financial market consists of a risk-free asset, an inflation-indexed bond and a stock. The expected excess return of the stock is predicted by an observable predictor and an unobservable predictor. The both predictors are formulated by the mean-reversion stochastic process. Different from most existing researches, for example [12], we assume that the stock, stock return predictors and inflation-indexed bond are dependent. Although the DC pension plan member cannot observe the expected stock return completely, she can learn about the unobservable predictor from the past realized stock price, inflation-indexed bond price and other observations by using the Bayesian learning mechanism. Besides, to obtain the closed-form optimal investment strategy for the DC pension plan member, the stochastic salary in our paper is assumed to be only related with the inflation risk. By applying the filtering techniques and dynamic programming approach, we solve the optimal investment decision problem of the DC pension plan member under the power utility explicitly (though, the numerical solution for simple ordinary differential equations is needed). Finally, with the help of some numerical examples, we investigate the utility loss result from ignoring learning or from not using the inflation-indexed bond to hedge the stock return predictors. The main contribution of this paper can be summarized as the following three aspects. (i) We establish a portfolio model that incorporates stock return predictability, partial information, inflation risk and stochastic salary together to study the optimal investment problem for a DC pension plan member. (ii) We indicate that stock return predictability can significantly affect the optimal inflation-indexed bond allocation. Moreover, the inflation-indexed bond’s hedging components against the observable and unobservable predictors are considerable, and can exceed the corresponding stock’s hedging components. (iii) We demonstrate that the welfare improvements either from learning or from using the inflation-indexed bond to hedge the stock return predictors can be substantial.

The rest of this paper is organized as follows. Section 2 describes the financial market with predictable stock return, and formulates the optimal investment problem for a DC pension plan member. Section 3 derives the solutions for the investment problem by using the filtering techniques and dynamic programming approach. The suboptimal investment strategies and the utility losses are analyzed in Section 4. Section 5 illustrates our theoretical results by providing some numerical examples. Section 6 concludes this paper. The proofs are presented in the appendices.

2. Problem formulation. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space armed with a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}\) satisfying the usual conditions, i.e., \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) is right-continuous and \(\mathbb{P}\)-complete, where \(\mathcal{F}_t\) denotes the information available until time \(t\). We assume that all the stochastic processes described below are well defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and adapted to \(\mathcal{F}\).

Suppose that a member participates a DC pension plan at time 0 and retires at time \(T\). Before retirement, the member receives a stochastic salary income, and contributes part of her salary to the pension account, continuously. To keep the living standard after retirement, the member invests her pension account wealth in a financial market. Without loss of generality, we assume that continuous trading is permitted in the financial market, and no transaction costs or taxes are involved in trading.
2.1. Price index. Since the investment activity for DC pension plans usually lasts for a very long time, the inflation risk is a dangerous factor that can decrease the real value of the DC pension account. Therefore, the inflation risk should be considered in the investment problem of DC pension plans. We introduce the (commodity) price index, which can measure the inflation level of the financial market (cf. [24] and [21]). We assume that the price index \( I(t) \) follows the diffusion process:

\[
\frac{dI(t)}{I(t)} = \mu_I dt + \sigma_I dZ_I(t),
\]

where the constant \( \mu_I \) denotes the expected rate of inflation, the positive constant \( \sigma_I \) represents the volatility of the price index, and \( Z_I(t) \) is a one-dimensional standard Brownian motion.

2.2. The financial market. In the financial market, there exist one risk-free asset and two risky assets: a stock and an inflation-indexed bond. Specifically, the price process of the risk-free asset follows the ordinary differential equation (ODE)

\[
\frac{dS_0(t)}{S_0(t)} = R dt,
\]

where the constant \( R \) represents a nominal interest rate.

The dynamics of the stock price process is described by the following stochastic differential equation (SDE)

\[
\frac{dS(t)}{S(t)} = \mu_S(t) dt + \sigma_S dZ_S(t),
\]

where \( \mu_S(t) \) is the appreciation rate of stock; the positive constant \( \sigma_S \) is the stock volatility; \( Z_S(t) \) is a one-dimensional standard Brownian motion, which is correlated with \( Z_I(t) \), and satisfies \( dZ_S(t) dZ_I(t) = \rho_{SI} dt \). We assume that the expected excess return on the stock, \( \mu_S(t) - R \), is an affine function of an observable predictor \( x(t) \) and an unobservable predictor \( y(t) \). In other words,

\[
\mu_S(t) - R = a + b_x x(t) + b_y y(t),
\]

where \( a, b_x \) and \( b_y \) are real constants; \( b_x \) and \( b_y \) stand for the predictive power of \( x(t) \) and \( y(t) \), respectively. Here, \( x(t) \) can be treated as one of the known predictors mentioned in Section 1, and the unobservable predictor \( y(t) \) is used to capture the variations in the expected excess return beyond those induced by the chosen predictor \( x(t) \). This implies that any predictor and even any set of predictors cannot perfectly explain the whole variations of the expected excess return. Particularly, the observable and unobservable predictors are given by

\[
\begin{align*}
 dx(t) & = \kappa_x (\mu_x - x(t)) dt + \sigma_x dZ_x(t), \\
 dy(t) & = \kappa_y (\mu_y - y(t)) dt + \sigma_y dZ_y(t),
\end{align*}
\]

where the constants \( \kappa_x \) and \( \kappa_y \) are the mean-reversion rate of \( x(t) \) and \( y(t) \); the constants \( \mu_x \) and \( \mu_y \) are the long-run mean return rates of \( x(t) \) and \( y(t) \); the positive constants \( \sigma_x \) and \( \sigma_y \) are the volatility of \( x(t) \) and \( y(t) \); \( Z_x(t) \) and \( Z_y(t) \) are two correlated one-dimensional standard Brownian motions, which are also correlated with \( Z_I(t) \) and \( Z_S(t) \), i.e., \( dZ_S(t) dZ_x(t) = \rho_{SX} dt \), \( dZ_S(t) dZ_y(t) = \rho_{SY} dt \), \( dZ_I(t) dZ_x(t) = \rho_{IX} dt \), \( dZ_I(t) dZ_y(t) = \rho_{IY} dt \).

\footnote{A similar assumption is made in [2] and [15], but none of them takes into account the inflation risk and stochastic salary. In addition, [22] use a hidden Markov chain to describe the market dynamics under the discrete-time settings, in which the stock returns depend on both the observable and unobservable market states.}
The DC pension plan member can observe \( x(t) \), and so does \( Z_x(t) \). Meanwhile, the constants \( \kappa_x, \kappa_y, \mu_x, \mu_y, \sigma_x \) and \( \sigma_y \) are assumed to be known. However, the member cannot observe the predictor \( y(t) \), meaning that \( Z_y(t) \) is unobservable.

Therefore, the dynamics of stock price (3) can be rewritten as

\[
\frac{dS(t)}{S(t)} = (R + a + b_x x(t) + b_y y(t)) dt + \sigma_S dZ_S(t). \tag{7}
\]

Note that when \( b_y \neq 0 \), the member cannot observe the expected excess return of stock. The information that can be observed by the member is the realized stock price, observable predictor and price index. Denote by \( F_t^{S,x,I} = \sigma \{ Z_S(u), Z_y(u), Z_I(u) : u \in [0,T] \} (\subset F_t) \) the \( \sigma \)-algebra generated by observing the stock price process \( S(t) \), predictor process \( x(t) \), and price index process \( I(t) \). Then the observable filtration is \( G = \{ F_t^{S,x,I} \}_{0 \leq t \leq T} (\subset F) \).

In addition, an inflation-indexed bond is introduced into our portfolio model to hedge the inflation risk. The dynamics of the inflation-indexed bond price process is determined by

\[
\frac{dP(t)}{P(t)} = r dt + \frac{dI(t)}{I(t)} = (r + \mu_I) dt + \sigma_I dZ_I(t), \tag{8}
\]

where the constant \( r \) stands for the real interest rate, and \( r + \mu_I \) is the appreciation rate of the inflation-indexed bond. From equation (8), we can see that the appreciation rate of the inflation-indexed bond relies on the expected inflation rate. The volatility of the inflation-indexed bond stems from the volatility of the inflation. Similar to [18], we further assume that the inflation-index bond has a positive risk premium \( (r + \mu_I - R > 0) \), which can be understood as a compensation to the investment risk.

2.3. The optimization problem. Before retirement, the DC pension plan member receives salary from the labor market, and continuously contributes a fixed percentage \( \xi \) of her salary to the pension account. Similar to [12] and [17], we further assume that the member’s nominal salary is stochastic and modeled as

\[
\frac{dL_N(t)}{L_N(t)} = \mu_L dt + \sigma_L dZ_I(t), \tag{9}
\]

where the constant \( \mu_L \) represents the expected growth rate of the nominal salary, and the positive constant \( \sigma_L \) is the volatility of the nominal salary.

Let \( \pi_S(t) \) and \( \pi_P(t) \) represent the proportion of wealth invested in the stock and the inflation-indexed bond at time \( t \), respectively. So the proportion of wealth invested in the risk-free asset at time \( t \) is \( 1 - \pi_S(t) - \pi_P(t) \). The stochastic process \( \pi := \{ \pi_S(t), \pi_P(t) \} \) is called an investment strategy. Note that the whole information available to the DC pension plan member up to time \( t \) is \( F_t^{S,x,I} \), so the

\[The assumption that the parameters in equation (6) are known is strong, because a member will not generally know for certain either the predictive variable process or its parameters. A more realistic assumption is that the parameters in equation (6) are also unobservable. Although these are potentially important considerations, we simplify by modeling the unobservable predictive variable as a mean-reverting process with known parameters. Even this small source of imperfect information has major implications for the optimal portfolio choice and the member’s welfare. A more detailed explanation can be found in [19].

\[Although the expected return on the stock is unobservable due to the unobservable predictor \( y(t) \), the stock price can be observed by the member directly.
member must make the investment strategy based on \( F_t^{S,x,l} \). Denote by \( W_N^\pi(t) \) the nominal wealth of the member’s pension account at time \( t \) under the investment strategy \( \pi \). Then the dynamics of \( W_N^\pi(t) \) is governed by the following SDE

\[
dW_N^\pi(t) = \left[ W_N^\pi(t) (1 - \pi_S(t) - \pi_P(t)) \frac{dS_0(t)}{S_0(t)} + W_N^\pi(t) \pi_S(t) \frac{dS(t)}{S(t)} \right. \\
+ \left. W_N^\pi(t) \pi_P(t) \frac{dP(t)}{P(t)} + \xi L_N(t) dt \right]
\]

\[
= \left[ W_N^\pi(t) R + W_N^\pi(t) \pi_S(t) (a + b_x x(t) + b_y y(t)) + W_N^\pi(t) \pi_P(t) (r + \mu_I - R) \right. \\
+ \left. \xi L_N(t) \right] dt + W_N^\pi(t) \pi_S(t) \sigma_S dZ_S(t) + W_N^\pi(t) \pi_P(t) \sigma_I dZ_I(t).
\]

During the long-time investment horizon, the purchasing ability of the nominal wealth for the pension account will inevitably be weakened by inflation. So, it is of great importance to consider the real wealth of the pension account, which is defined as the ratio between the nominal wealth and price index. Denote by \( W^\pi(t) = \frac{W_N^\pi(t)}{\Pi(t)} \) the real wealth and \( L(t) = \frac{L_N(t)}{\Pi(t)} \) the real salary with stripping out the inflation, respectively. Applying Itô’s formula, we have the dynamics of the real wealth \( W^\pi(t) \) and real salary \( L(t) \) as below:

\[
dW^\pi(t) = \left[ W^\pi(t) (R - \mu_I + \sigma_I^2) + W^\pi(t) \pi_S(t) (a + b_x x(t) + b_y y(t)) \right. \\
- \sigma_S \sigma_I \rho_{SI} + W^\pi(t) \pi_P(t) (r + \mu_I - R - \sigma_I^2) + \xi L(t) \right] dt \\
+ W^\pi(t) \pi_S(t) \sigma_S dZ_S(t) + W^\pi(t) \pi_P(t) (\pi_P(t) - 1) \sigma_I dZ_I(t),
\]

\[
\frac{dL(t)}{L(t)} = (\mu_L - \mu_I + \sigma_I^2 - \sigma_L \sigma_I) dt + (\sigma_L - \sigma_I) dZ_I(t).
\]

**Definition 2.1.** (Admissible strategy) An investment strategy \( \pi = \{(\pi_S(t), \pi_P(t))\), \( t \in [0, T] \) is said to be admissible, if

1. \( \{(\pi_S(t), \pi_P(t)), t \in [0, T] \} \) is \( F_t^{S,x,l} \)-progressively measurable;
2. \( \mathbb{E} \left[ \int_0^T (W^\pi(t))^2 (\pi_S(t))^2 \sigma_S^2 + (\pi_P(t) - 1)^2 \sigma_I^2) dt \right] < \infty; \)
3. The SDE (10) has a unique solution.

In addition, we use \( \Pi \) to denote the set of all admissible strategies.

At the time point of retiring, the investment activity of the pension account correspondingly ends, and the member receives the benefit whose amount depends on the pension account’s investment return. Therefore, the member aims to find the optimal investment strategy based on the observable filtration \( G \) to maximize the expected power utility of her terminal real wealth. In other words, the optimization problem for the member can be formulated as follows:

\[
\max_{\pi \in \Pi} \mathbb{E} \left[ \frac{(W^\pi(T))^{1-\gamma}}{1-\gamma} \right],
\]

s.t. state equations (5), (6), (10) and (11) hold,
where the member has a constant relative risk aversion preference, and $\gamma > 1$ is the measure of relative risk aversion.\footnote{The assumption of $\gamma > 1$ is to avoid problems with infinite expected utility that may be caused by $0 < \gamma < 1$, cf. \cite{10} and \cite{11}.}

Notice that the member forms her assessment of the unobservable predictor $y(t)$ from the realized stock price $S(t)$, observable predictor $x(t)$ and price index $I(t)$. As in \cite{4}, \cite{5} and \cite{8}, the optimization problem (12) can be decomposed into two separate problems: an inference problem in which the member updates her estimate of the current value of the unobservable predictor $y(t)$; and an optimization problem in which the member uses her current estimate of $y(t)$ to determine the optimal investment strategy, taking future learning into consideration. Although the value of the unobservable predictor $y(t)$ is unknown, the separation theorem means that the member’s optimization problem (12) can be solved with the estimate of $y(t)$.

3. Solution of the optimization problem. To solve the optimization problem (12) with the unobservable expected excess stock return, we first learn about the unobservable predictor $y(t)$, and then the member’s optimization problem (12) can be solved with the estimate of $y(t)$.

3.1. The filtered models. Let \( \hat{y}(t) = E[y(t) | \mathcal{F}_t^{S,x,I}] \) and \( m(t) = E[(y(t) - \hat{y}(t))^2] \) be the conditional mean and conditional variance of the unobservable predictor $y(t)$, then \( \hat{y}(t) \) is the member’s estimate for the unobservable predictor $y(t)$ and $m(t)$ is the estimation error. This estimate process introduces learning and the estimation error into the member’s optimization problem.

By using the filtering techniques given in \cite{13}, we have an $\mathcal{G}$-adapted innovation process $B = \{(B_S(t), B_x(t), B_I(t)), 0 \leq t \leq T\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is given as follows:\footnote{We show how to derive these expressions in details in Appendix A.}:

\[
\begin{pmatrix}
\frac{dS(t)}{S(t)} \\
\frac{dx(t)}{x(t)} \\
\frac{dI(t)}{I(t)}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\rho_x} & 0 & 0 \\
-\frac{1}{\sigma_x \rho_x} & 0 & 0 \\
\left(\frac{\rho_x \rho_{SI} - \rho_x \rho_I}{\sigma_x \rho_x \rho_I}\right) & -\frac{\rho_x \rho_{SI}}{\sigma_x \rho_x \rho_I} & \frac{1}{\sigma_x \rho_x \rho_I}
\end{pmatrix}
\begin{pmatrix}
dS(t) \\
dx(t) \\
dI(t)
\end{pmatrix}
+ \begin{pmatrix}
b_S(t) \\
b_x(t) \\
b_I(t)
\end{pmatrix}
\begin{pmatrix}
R(t) + a + b_x x(t) \\
\kappa_x (\mu_x - x(t)) \\
\mu_I
\end{pmatrix}
\begin{pmatrix}
dt
\end{pmatrix}
\begin{pmatrix}
\hat{y}(t)
\end{pmatrix}.
\]

Moreover, the $\mathbb{P}$-augmentation of the natural filtration generated by $B$ coincides with the filtration $\mathcal{G}$. The innovation process $B = \{(B_S(t), B_x(t), B_I(t)), 0 \leq t \leq T\}$ is a three-dimensional standard Brownian motion with respect to (w.r.t.) the observable filtration $\mathcal{G}$, whose components are mutually independent. Correspondingly, the filtered models w.r.t. $B$ are derived in the following proposition.

**Proposition 1.** With respect to the $\mathcal{G}$-adapted innovation process $B = \{(B_S(t), B_x(t), B_I(t)), 0 \leq t \leq T\}$, the filtered models for the price index, stock price, two predictors, inflation-indexed bond price and real salary process can be rewritten as

\[
\frac{dI(t)}{I(t)} = \mu_I dt + \sigma_I \rho_{SI} dB_S(t) + \sigma_I \rho_{xI} dB_x(t) + \sigma_I(t) \rho_I dB_I(t), \tag{13}
\]
\[ \frac{dS(t)}{S(t)} = (R + a + b_x x(t) + b_y \hat{y}(t)) dt + \sigma_S dB_S(t), \] (14)
\[ dx(t) = \kappa_x (\mu_x - x(t)) dt + \sigma_x \rho_S x dB_S(t) + \sigma_x \hat{\rho}_x dB_x(t), \] (15)
\[ dy(t) = \kappa_y (\mu_y - \hat{y}(t)) dt + K_1(t) dB_S(t) + K_2(t) dB_x(t) + K_3(t) dB_l(t), \] (16)
\[ \frac{dP(t)}{P(t)} = (r + \mu_I) dt + \sigma_I \rho_S I dB_S(t) + \sigma_I \hat{\rho}_x dB_x(t) + \sigma_I \hat{\rho}_I dB_I(t), \] (17)
\[ \frac{dL(t)}{L(t)} = (\mu_L - \mu_I + \sigma_L^2 - \sigma_L \sigma_I) dt + (\sigma_L - \sigma_I) (\rho_S I dB_S(t) + \hat{\rho}_x dB_x(t) + \hat{\rho}_I dB_I(t)), \] (18)

where \( \hat{\rho}_x = \sqrt{1 - \rho_x^2}, \) \( \hat{\rho}_x \hat{\rho}_I = \frac{\rho_x - \rho_{SI} \rho_{SI} - \rho_x \hat{\rho}_x}{\rho_{SI}}, \) \( \hat{\rho}_I = \sqrt{1 - \rho_I^2}, \) \( \hat{\rho}_{xy} = \frac{\rho_{xy} - \rho_{SI} \rho_{SI} \rho_{SI} \rho_{SI} - \rho_{SI} \rho_x \rho_x}{\rho_{SI}}, \)
\( \hat{\rho}_{yI} = \frac{\rho_{yI} - \rho_{SI} \rho_{SI} \rho_{SI} \rho_{SI} - \rho_{SI} \rho_x \rho_x}{\rho_{SI}}, \) \( K_1(t) = \sigma_y \rho_S y(t) + m(t) \frac{b_y}{\sigma_S}, \) \( K_2(t) = \sigma_y \hat{\rho}_{xy} - m(t) \frac{b_y \rho_S}{\sigma_S \hat{\rho}_x}, \)
\( K_3(t) = \sigma_y \hat{\rho}_{yI} + m(t) \frac{b_y (\rho_x \rho_{SI} - \rho_{SI} \rho_{SI} \rho_{SI} \rho_{SI} - \rho_{SI} \rho_x \rho_x) \rho_{SI}}{\sigma_S \rho_x \hat{\rho}_I}, \) and \( \frac{dm(t)}{dt} = -2 \kappa_y m(t) + \sigma_y^2 - K_1^2(t) - K_2^2(t) - K_3^2(t). \)

Proof. See Appendix A. \( \square \)

Therefore, by replacing the unobserved predictor \( y(t) \) in equation (10) with its estimate \( \hat{y}(t) \), the dynamics of real wealth \( W^\pi(t) \) becomes

\[ dW^\pi(t) = \left[ W^\pi(t)(R - \mu_I + \sigma_I^2) + W^\pi(t) \pi_S(t) (a + b_x x(t) + b_y \hat{y}(t) \right] \]
\[ - \sigma_S \sigma_I \rho_S I) + W^\pi(t) \pi_P(t)(r + \mu_I - R - \sigma_I^2) + \xi L(t) \right] dt \] (19)
\[ + W^\pi(t) \pi_S(t) \sigma_S + (\pi_P(t) - 1) \sigma_I \rho_S I dB_S(t) \]
\[ + W^\pi(t) (\pi_P(t) - 1) \sigma_I \hat{\rho}_x I dB_x(t) + W^\pi(t) (\pi_P(t) - 1) \sigma_I \hat{\rho}_I dB_I(t), \]

where \( W^\pi(0) = w_0, x(0) = x_0, \hat{y}(0) = \hat{y}_0 \) and \( L(0) = l_0 \) are the initial values for the real wealth, two predictors and real salary.

Note that the filtered models given in Proposition 1 is \( \mathcal{G} \)-adapted, which means that all the state variables involved in the optimization problem (12) are accessible to the member. Therefore, the optimization problem (12) is reduced to the following equivalent optimization problem with complete observations:

\[ \max_{\pi \in \Pi} \mathbb{E} \left[ \frac{(W^\pi(T))^{1-\gamma}}{1-\gamma} \right], \] (20)

s.t. state equations (15), (16), (18) and (19) hold.

For the optimization problem (20), the expected utility achieved by an admissible strategy \( \pi = \{ (\pi_S(t), \pi_P(t)) \} \) is given as

\[ V^\pi(t, W, x, \hat{y}, \hat{l}) = \mathbb{E}_{t, W, x, \hat{y}, \hat{l}} \left[ \frac{(W^\pi(T))^{1-\gamma}}{1-\gamma} \right], \] (21)

where \( \mathbb{E}_{t, W, x, \hat{y}, \hat{l}}[\cdot] = \mathbb{E}[:W^\pi(t) = W, x(t) = x, \hat{y}(t) = \hat{y}, L(t) = l, t]. \) Thus, the value function of the optimization problem (20) is defined as follows:

\[ V(t, W, x, \hat{y}, \hat{l}) = \sup_{\pi \in \Pi} V^\pi(t, W, x, \hat{y}, \hat{l}) \] (22)
with the boundary condition \( V(T, W, x, y, l) = \frac{W^{1-\gamma}}{1-\gamma} \). For the DC pension plan member, she wants to find an optimal investment strategy \( \pi^* \in \Pi \) such that \( V^\pi^*(t, W, x, y, l) = V(t, W, x, y, l) \).

### 3.2. The optimal investment strategy

Now, we will solve the optimization problem (12) or (20) by using the dynamic programming approach. For convenience, we denote

\[
C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+),
\]

\( \psi(t, W, x, y, l) \) is once continuously differentiable on \([0, T]\), \( \psi(. \cdot, y, x, l) \) is twice continuously differentiable for \( W \) on \( \mathbb{R} \), \( x \) on \( \mathbb{R} \), \( y \) on \( \mathbb{R} \), and \( l \) on \( \mathbb{R}^+ \), respectively.

We can see that if the value function \( V(t, W, x, y, l) \in C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+) \), then \( V(t, W, x, y, l) \) satisfies the following HJB equation:

\[
\sup_{\pi \in \Pi} \{ A^\pi V(t, W, x, y, l) \} = 0, \tag{23}
\]

for \( 0 \leq t \leq T \) with the boundary condition \( V(T, W, x, y, l) = U(W) \). Here

\[
A^\pi V = V_t + V_W[W(R - \mu_t + \sigma_t^2) + W\pi_S(a + b_x x + b_y y - \sigma_S \sigma_I \rho_{SI}) + W\pi_P(r + \mu_I - R - \sigma_I^2) + \xi l] + \frac{1}{2} V_{WW} W^2 [(\pi_S \sigma_S + (\pi_P - 1) \sigma_I \rho_{SI})^2 + (\pi_P - 1)^2 \sigma_I^2 \rho_{II}^2 + (\pi_P - 1)^2 \sigma_I^2 \rho_{II}^2 + V_{\bar{y}} \kappa_{x} (\mu_x - x) + \frac{1}{2} V_{xx} \sigma_x^2 + V_{\bar{y}} \kappa_{y} (\mu_y - \bar{y}) + \frac{1}{2} V_{\bar{y}\bar{y}} (K_1^2 + K_2^2 + K_3^2) + V_{\bar{y}} (\mu_L - \mu_I + \sigma_I^2 - \sigma_{L} \sigma_I) + \frac{1}{2} V_{il} l^2 (\sigma_L - \sigma_I)^2 + V_{WW} W [(\pi_S \sigma_S + (\pi_P - 1) \sigma_I \rho_{SI}) \sigma_S \rho_{sx} + (\pi_P - 1) \sigma_I \sigma_s \rho_{si} \rho_{sx}] + V_{\bar{y}\bar{y}} W [(\pi_S \sigma_S + (\pi_P - 1) \sigma_I \rho_{SI}) K_1 + (\pi_P - 1) \sigma_I \rho_{si} K_2 + (\pi_P - 1) \sigma_I \rho_{si} K_3 + V_{\bar{y}l} W [(\pi_S \sigma_S + (\pi_P - 1) \sigma_I \rho_{SI}) (\sigma_L - \sigma_I)] + (\pi_P - 1) \sigma_I \rho_{si} (\sigma_L - \sigma_I) + (\pi_P - 1) \sigma_I \rho_{si} (\sigma_L - \sigma_I)] + V_{x\bar{y}} (K_1 \sigma_s \rho_{sx} + K_2 \sigma_s \rho_{sx}) + V_{x\bar{y}} [\sigma_x (\sigma_L - \sigma_I) \rho_s \rho_{SI} + \sigma_s (\sigma_L - \sigma_I) \rho_s \rho_{sI}] + V_{il} [K_1 (\sigma_L - \sigma_I) \rho_{SI} + K_2 (\sigma_L - \sigma_I) \rho_{sI} + K_3 (\sigma_L - \sigma_I) \rho_{l}].
\]

Meanwhile, \( V \) is short for the value function \( V(t, W, x, y, l) \), then \( V_t, V_W, V_x, \bar{y}, V_l, V_{WW}, V_{xx}, V_{\bar{y}\bar{y}}, V_{\bar{y}l}, V_{\bar{y}l}, V_{\bar{y}l}, V_{\bar{y}l} \) and \( V_{\bar{y}l} \) represent the partial derivatives of \( V \) w.r.t. the corresponding variables.

By solving the HJB equation (23), the optimal investment strategy and the corresponding value function for the optimization problem (12) or (20) can be derived analytically, which are summarized in the following theorem.

**Theorem 3.1.** For the optimization problem (12) or (20), the optimal investment strategy \( \pi^* \) is given by

\[
\begin{align*}
\pi^*_S(t) &= A(t) \left[ 1 + h_1(t) \frac{L(t)}{W^{\pi^*_S}(t)} \right], \\
\pi^*_P(t) &= B(t) \left[ 1 + h_1(t) \frac{L(t)}{W^{\pi^*_S}(t)} \right] - \frac{\sigma_L}{\sigma_I} h_1(t) \frac{L(t)}{W^{\pi^*_S}(t)},
\end{align*}
\tag{24}
\]
and the corresponding value function is

\[
V(t,W,x,y,I) = \frac{(W + h_1(t)I)^{1-\gamma}}{1-\gamma} g(t,x,y),
\]

where

\[
A(t) = \frac{1}{\gamma \sigma_s (1 - \rho_{SI}^2)} \left[ \frac{a + b_x x + b_y y \dot{y}}{\sigma_s} - \frac{(r + \mu_I - R) \rho_{SI}}{\sigma_I} + (1 - \gamma) \sigma_x (\rho_{sx} - \rho_{SI} \rho_{sy}) \right]
\]

\[
- \rho_{SI} \rho_{xx} (H_1(\tau) + D_1(\tau)x + D_3(\tau)y) + (1 - \gamma) \left( \sigma_y (\rho_{sy} - \rho_{SI} \rho_{yI}) \right) + m(t) \frac{b_y}{\sigma_s} (H_2(\tau) + D_2(\tau)y + D_3(\tau)x) \right],
\]

\[
B(t) = \frac{1}{\gamma \sigma_I (1 - \rho_{SI}^2)} \left[ \frac{r + \mu_I - R}{\sigma_I} - \frac{a + b_x x + b_y y \dot{y}}{\sigma_s} (1 - \gamma) \sigma_I \left( 1 - \rho_{SI}^2 \right) \right] \rho_{SI} \rho_{xx} (H_1(\tau) + D_1(\tau)x + D_3(\tau)y) + (1 - \gamma) \left( \sigma_y (\rho_{sy} - \rho_{SI} \rho_{yI}) - m(t) \frac{b_y}{\sigma_s} \rho_{SI} \right) (H_2(\tau) + D_2(\tau)y + D_3(\tau)x),
\]

\[
g(t,x,y) = \exp \left\{ (1 - \gamma) \left[ H_0(\tau) + H_1(\tau)x + H_2(\tau)y \right.ight.
\]

\[
+ \frac{1}{2} D_1(\tau)x^2 + \frac{1}{2} D_2(\tau)y^2 + D_3(\tau)x \dot{y} \left. \right\},
\]

\[
h_1(t) = \frac{\xi}{\alpha_0} (e^{\alpha_{10}(\tau)} - 1),
\]

and the deterministic functions \(H_0(\tau), H_1(\tau), H_2(\tau), D_1(\tau), D_2(\tau)\) and \(D_3(\tau)\) solve the following system of ODEs:\(^6\)

\[
H_0'(\tau) = (a_1 + a_\alpha \gamma) H_1(\tau) + (a_3 + a_\alpha \alpha_8) H_2(\tau) + \frac{\sigma^2}{2} D_1(\tau)
\]

\[
+ \frac{1}{2} (K_1^2(t) + K_2^2(t) + K_3^2(t)) D_2(\tau) + \sigma_x \sigma_y \rho_{xy} D_3(\tau)
\]

\[
+ \alpha_2 H_1^2(\tau) + \alpha_4 H_2^2(\tau) + \alpha_5 H_1(\tau) H_3(\tau) + \alpha_9 + r,
\]

\[
H_1'(\tau) = (b_2 \alpha \gamma - \kappa_x) H_1(\tau) + b_2 \alpha_8 H_2(\tau) + (a_1 + a_\alpha \gamma) D_1(\tau)
\]

\[
+ (a_3 + a_\alpha \alpha_8) D_3(\tau) + 2 a_2 H_1(\tau) D_1(\tau) + 2 a_4 H_2(\tau) D_3(\tau)
\]

\[
+ \alpha_5 (H_1(\tau) D_3(\tau) + H_2(\tau) D_1(\tau)) + b_2 \alpha_6,
\]

\[
H_2'(\tau) = b_2 \alpha \gamma H_1(\tau) + (b_2 \alpha_8 - \kappa_y) H_2(\tau) + (a_3 + a_\alpha \alpha_8) D_2(\tau)
\]

\[
+ (a_1 + a_\alpha \gamma) D_3(\tau) + 2 a_2 H_1(\tau) D_3(\tau) + 2 a_4 H_2(\tau) D_2(\tau)
\]

\[
+ \alpha_5 (H_1(\tau) D_2(\tau) + H_2(\tau) D_3(\tau)) + b_2 \alpha_6,
\]

\[
D_1'(\tau) = 2 (b_2 \alpha \gamma - \kappa_x) D_1(\tau) + 2 b_2 \alpha_8 D_3(\tau) + 2 a_2 D_1^2(\tau)
\]

\[
+ 2 a_4 D_2^2(\tau) + 2 a_5 D_1(\tau) D_3(\tau) + \frac{b_2^2}{\gamma \sigma_s^2 (1 - \rho_{SI}^2)},
\]

\[
D_2'(\tau) = 2 (b_2 \alpha_8 - \kappa_y) D_2(\tau) + 2 b_2 \alpha_7 D_3(\tau) + 2 a_4 D_2^2(\tau)
\]

\[
+ 2 a_2 D_3^2(\tau) + 2 a_5 D_2(\tau) D_3(\tau) + \frac{b_2^2}{\gamma \sigma_s^2 (1 - \rho_{SI}^2)},
\]

\(^6\)In the numerical example, we obtain the numerical solution by using the function ODE45 in MATLAB.
Remark 1. See Appendix B.

\[
D'(\tau) = b_y \alpha_T D_1(\tau) + b_x \alpha_8 D_2(\tau) + (b_x \alpha_T - \kappa_x + b_y \alpha_8 - \kappa_y) D_3(\tau) \\
+ 2\alpha_2 D_1(\tau) D_3(\tau) + 2\alpha_4 D_2(\tau) D_3(\tau) \\
+ \alpha_5 (D_1(\tau) D_2(\tau) + D_3^2(\tau)) + \frac{b_x b_y}{\gamma \sigma_S^2 (1 - \rho_S^2)},
\]

here \(\tau = T - t\), and

\[
\alpha_1 = \kappa_x \mu_x + \frac{(1 - \gamma)}{\gamma} \left( \sigma_x (r + \mu_I - R)(\rho_{z_I} - \rho_S z I) - \sigma_I \rho_{z_I} \right),
\]

\[
\alpha_2 = \frac{(1 - \gamma)}{2} \sigma_x^2 + \frac{(1 - \gamma)^2 \sigma_S^2 (\rho_S^2 - 2 \rho_{z_S} \rho_{z_I} + \rho_{z_I}^2)}{2\gamma (1 - \rho_S^2)},
\]

\[
\alpha_3 = \kappa_y \mu_y + \frac{(1 - \gamma)}{\gamma} \left( \frac{r + \mu_I - R}{\sigma_I (1 - \rho_S^2)} [\sigma_y (r_y I - \rho_{z_I} \rho_{z_I}) - m(t) \frac{\rho_y I}{\sigma_S}] - \sigma_I \rho_{y I} \right),
\]

\[
\alpha_4 = \frac{(1 - \gamma)}{2} (K^2(t) + K_2^2(t) + K_3^2(t)) + \frac{(1 - \gamma)^2}{2\gamma (1 - \rho_S^2)} [\sigma_y \rho_{y I} + m(t) \frac{\rho_y I}{\sigma_S}]^2
\]

\[
- 2(\sigma_y \rho_{y I} + m(t) \frac{\rho_y I}{\sigma_S}) \sigma_y \rho_{y I} \rho_{z_I} + \sigma_y^2 \rho_{y I}^2],
\]

\[
\alpha_5 = (1 - \gamma) \sigma_x \sigma_y \rho_{y I} + \frac{(1 - \gamma)^2}{\gamma (1 - \rho_S^2)} \left[ \sigma_x \sigma_S (\sigma_y (\rho_S - \rho_{z_I} \rho_{z_I}) + m(t) \frac{\rho_y I}{\sigma_S})
\]

\[
+ \sigma_x \rho_{z_I} (\sigma_y (\rho_{z_I} \rho_{z_I}) - m(t) \frac{\rho_y I}{\sigma_S}) \right],
\]

\[
\alpha_6 = \frac{a \sigma_I - (r + \mu_I - R) \sigma_S \rho_{z_I}}{\gamma \sigma_S^2 \sigma_I (1 - \rho_S^2)},
\]

\[
\alpha_7 = \frac{(1 - \gamma) \sigma_x (\rho_S - \rho_{z_I} \rho_{z_I})}{\gamma \sigma_S (1 - \rho_S^2)},
\]

\[
\alpha_8 = \frac{(1 - \gamma) \sigma_y (\rho_S - \rho_{z_I} \rho_{z_I}) + m(t) \frac{\rho_y I}{\sigma_S}}{\gamma \sigma_S (1 - \rho_S^2)},
\]

\[
\alpha_9 = \frac{(a \sigma_I - (r + \mu_I - R) \sigma_S \rho_{z_I})^2 + (r + \mu_I - R - \sigma_I^2) \sigma_S^2 (1 - \rho_S^2)}{2\gamma \sigma_S^2 \sigma_I (1 - \rho_S^2)},
\]

\[
\alpha_{10} = \left[ \mu_L - \mu_I + \sigma_l^2 - \sigma_L \sigma_I - \frac{\sigma_L - \sigma_I}{\sigma_I (r + \mu_I - R - \sigma_I^2)} - r \right].
\]

Proof. See Appendix B. \(\square\)

Remark 1. From (24) and (26), we find that the optimal stock allocation \(\pi^*_S(t)\) has three components. The first component \(\pi^*_{S_{spec}}(t) = \frac{1}{\gamma \sigma_S (1 - \rho_S^2)} (a + b_x x + b_y y) - \frac{r + \mu_I - R}{\sigma_S (1 - \rho_S^2)} \rho_S \) reflects the speculative investment demand on the stock, which depends on both the estimated stock risk premium, \(a + b_x x + b_y y\), and the inflation-indexed bond risk premium, \(r + \mu_I - R\). As stated in Section 2, we assume that \((r + \mu_I - R) > 0\), then the speculative investment increases w.r.t. the estimated stock risk premium, and increases (decreases) w.r.t. the inflation-indexed bond risk premium if \(\rho_S < 0\) (\(\rho_S > 0\)). The second component \(\pi^*_{S_{obs}}(t) = \frac{(1 - \gamma) \sigma_x (\rho_S - \rho_{z_I} \rho_{z_I})}{\gamma \sigma_S (1 - \rho_S^2)} (H_1(\tau) + D_1(\tau) x + D_3(\tau) y)\) is used to hedge variations induced by the observable predictor \(x\). This term will vanish if there is no scope for hedging \((\rho_S = \rho_{z_I} \rho_{z_I})\), no need for hedging \((\sigma_x = 0)\), or no preference for hedging \((\gamma = 1, \log \text{utility})\). The last component
\[ \pi_{\text{unobs}}(t) = \frac{(1-\gamma)(\sigma_r (\rho_{r,t} - \rho_{y,t}) + m(t))}{\sigma_\gamma(1-\rho_{y,t}^2)} (H_2(\tau) + D_2(\tau)y + D_3(\tau)x) \] hedges against variations in the unobservable predictor \( y \). This term will disappear if there is no preference for hedging (\( \gamma = 1 \), log utility). However, owing to the unobservability of predictor \( y \) (\( m \neq 0 \)), this term does not necessarily disappear even if predictor \( y \) is locally deterministic (\( \sigma_y = 0 \)) and the correlation coefficients are related in a special way (\( \rho_{S_y} = \rho_{S_I} \rho_{y,t} \)).

**Remark 2.** Similar to the optimal stock allocation, the optimal inflation-indexed bond allocation \( \pi_P^*(t) \) also has the corresponding three components (see (24) and (27)). The first component \( \pi_P^{\text{spec}}(t) = \frac{1}{\sigma_\gamma(1-\rho_{y,t}^2)} \left( \frac{\tau + \mu - R}{\sigma_1} - a + b_x x + b_y y \right) \rho_{S_I} \) represents the member’s speculative investment demand on the inflation-indexed bond. It increases w.r.t. the inflation-indexed bond risk premium \( (\tau + \mu - R > 0) \), and increases (decreases) w.r.t. the estimated stock risk premium if \( \rho_{S_I} > 0 \) (\( \rho_{S_I} > 0 \)). In contrast to [12], the inflation-indexed bond in this paper is correlated with the stock price and two predictors, so it can also be used to hedge the risks caused by the stock price and two predictors. The second component \( \pi_P^{\text{obs}}(t) = \frac{(1-\gamma)\sigma_x (\rho_{x,t} - \rho_{y,t} \rho_{y,t})}{\sigma_\gamma(1-\rho_{y,t}^2)} (H_1(\tau) + D_1(\tau)x + D_3(\tau)y) \) reflects the hedging demand against variations in the observable predictor \( x \), which vanishes if there is no scope for hedging \( (\rho_{x,t} = \rho_{x,t} \rho_{y,t}) \), no need for hedging \( (\sigma_x = 0) \), or no preference for hedging \( (\gamma = 1, \log \text{utility}) \). The third component \( \pi_P^{\text{unobs}}(t) = \frac{(1-\gamma)\sigma_y (\rho_{y,t} - \rho_{y,t} \rho_{y,t}) - m(t)}{\sigma_\gamma(1-\rho_{y,t}^2)} (H_2(\tau) + D_2(\tau)y + D_3(\tau)x) \) hedges against variations in the unobservable predictor \( y \), and it vanishes if there is no need for hedging \( (\gamma = 1, \log \text{utility}) \). Different from the optimal stock allocation, this term also does not disappear if the unobservable predictor \( y \) is locally deterministic (\( \sigma_y = 0 \)) or the correlation coefficients satisfy \( \rho_{y,t} = \rho_{y,t} \rho_{y,t} \), and the stock price is uncorrelated with the inflation-indexed bond price (\( \rho_{S_I} = 0 \)). Furthermore, since the the volatility of both the inflation-indexed bond and the salary is attributed to the volatility of the inflation, the optimal inflation-indexed bond allocation has two additional hedging components. The one component \( \pi_P^{\text{hedge}}(t) = -\frac{(1-\gamma)\sigma_x (\rho_{x,t} - \rho_{y,t} \rho_{y,t})}{\sigma_\gamma(1-\rho_{y,t}^2)} (H_1(\tau) + D_1(\tau)x + D_3(\tau)y) \) is used to hedge the inflation risk. The other component \( \pi_P^{\text{spec}}(t) = -\frac{\sigma_y h_1(t)}{\sigma_\gamma} \frac{L(t)}{W(t)} \) hedges the changes in the stochastic salary.

In section 5, we provide some numerical examples for the optimal investment strategy and its components to further investigate the effect of the correlation coefficients and predictive powers.

4. **The utility loss from suboptimal investment strategies.** In practice, due to various limitations, the member may not be able to learn about the unobservable predictor \( y(t) \). So she cannot adopt the optimal investment strategy \( \pi^* \) given in Theorem 3.3, but choose the investment strategy \( \pi^1 \) which is optimal for the case of ignoring learning in optimization problem (12). In addition, the member may not use the inflation-indexed bond to hedge the stock return predictors, and hence may follow an investment strategy \( \pi^2 \), which is optimal for the case of not using the inflation-indexed bond to hedge the stock return predictors in optimization problem (12). Obviously, the both cases are degeneration of optimization problem (12). So \( \pi^1 \) and \( \pi^2 \) can be degenerated from \( \pi^* \) (see below for details). However, both \( \pi^1 \) and \( \pi^2 \) are not optimal for optimization problem (12), which are called the
suboptimal investment strategies of optimization problem (12). Of course, \(\pi^1\) and \(\pi^2\) also belong to the set of admissible strategies II.

By definition of the value function \(V(t, W, x, y, l)\), for an admissible strategy \(\pi\), \(V(t, W, x, y, l) \geq V^\pi(t, W, x, y, l)\), and the equality holds if and only if the investment strategy \(\pi\) is optimal. Therefore, an admissible strategy will generate a lower level of expected utility than the optimal investment strategy. To quantify the utility loss arising from an admissible strategy \(\pi\), we have to solve the partial differential equation (PDE) (23) without the supremum over \(\pi\) to derive the expected utility \(V^\pi(t, W, x, y, l)\), because in the case the investment strategy \(\pi\) is given in advance.

Clearly, both the observable predictor \(x\) and the estimate \(\hat{y}\) of the unobservable predictor are the critical factors in an admissible strategy \(\pi\). Meanwhile, after multiplying the current real wealth, we find that the investment amount in the stock is affine w.r.t \(x\) and \(\hat{y}\), and so does the inflation-indexed bond, demonstrated in the following proposition.

**Proposition 2.** For an admissible strategy \(\pi\) satisfying the affine structure

\[
W \times \pi_S(t) = (F_0(\tau) + F_1(\tau)x + F_2(\tau)\hat{y})(W + h^\pi(t)l),
\]

\[
W \times (\pi_P(t) - 1) = (\bar{F}_0(\tau) + \bar{F}_1(\tau)x + \bar{F}_2(\tau)\hat{y})(W + h^\pi(t)l) - \frac{\sigma_L - \sigma_L}{\sigma_I}h^\pi(t),
\]

where \(\tau = T - t\) and \(F_0, F_1, F_2, \bar{F}_0, \bar{F}_1,\) and \(\bar{F}_2\) are deterministic functions\(^7\), the expected utility is specified by

\[
V^\pi(t, W, x, y, l) = \frac{(W + h^\pi(t)l)^{1-\gamma}}{1-\gamma} g^\pi(t, x, \hat{y}),
\]

with

\[
g^\pi(t, x, \hat{y}) = \exp \left\{ (1 - \gamma) \left( H_0^\pi(\tau) + H_1^\pi(\tau)x + H_2^\pi(\tau)\hat{y} \right. \right.
\]

\[
\left. + \frac{1}{2} D_1^\pi(\tau)x^2 + \frac{1}{2} D_2^\pi(\tau)\hat{y}^2 + D_3^\pi(\tau)x\hat{y} \right) \right\},
\]

where \(h^\pi(t) = h_1(t)\) is given by equation (29) and the deterministic functions \(H_0^\pi(\tau), H_1^\pi(\tau), H_2^\pi(\tau), D_1^\pi(\tau), D_2^\pi(\tau)\) and \(D_3^\pi(\tau)\) solve the following system of ODEs\(^8\)

\[
(H_0^\pi)'(\tau) = [k_x \mu_x + (1 - \gamma) \sigma_S \sigma_x \rho_{Sx} F_0(\tau) + (1 - \gamma) \sigma_x \sigma_1 \rho_{x1} F_0(\tau)] H_1^\pi(\tau)
\]

\[
+ [k_y \mu_y + (1 - \gamma) \sigma_y \rho_{Sy} + m(\tau) \frac{b_y}{\sigma_S}] \sigma_S F_0(\tau) + (1 - \gamma) \sigma_1 \sigma_y \rho_{yt}
\]

\[
\times \bar{F}_0(\tau)] H_2^\pi(\tau) + \frac{\sigma_2^2}{2} D_1^\pi(\tau) + \frac{1}{2} (K_1^2(t) + K_2^2(t) + K_3^2(t)) D_2^\pi(\tau)
\]

\[
+ \frac{(1 - \gamma) \sigma_2^2}{2} H_1^\pi(\tau)^2 + \frac{(1 - \gamma) (K_1^2(t) + K_2^2(t) + K_3^2(t))}{2} H_2^\pi(\tau)^2
\]

\[
+ \sigma_x \sigma_y \rho_{xy} D_3^\pi(\tau) + (1 - \gamma) \sigma_x \sigma_y \rho_{xy} H_1^\pi(\tau) H_2^\pi(\tau) + r
\]

\[
+ (a - \sigma_1 \sigma_1 \rho_{S1}) F_0(\tau) - \gamma \sigma_2^2 F_0(\tau)^2 - \gamma \sigma_1 \sigma_1 \rho_{S1} F_0(\tau) F_0(\tau)
\]

\[
- \frac{\gamma}{2} \sigma_1^2 \bar{F}_0(\tau)^2 + (r + \mu_I - R - \sigma_1^2) \bar{F}_0(\tau),
\]

\(^7\)The specific expressions of these functions will be provided later.

\(^8\)We also obtain the numerical solution for the system of ODEs by using the function ODE45 in the numerical example.
\[(H_1^\pi)'(\tau) = [(1 - \gamma)\sigma_S\sigma_x\rho_{sz}F_1(\tau) + (1 - \gamma)\sigma_x\sigma_1\rho_{xz}\bar{F}_1(\tau) - \kappa_x]H_1^\pi(\tau)
+ (1 - \gamma)[(\sigma_y\rho_{sy} + m(t)\frac{b_y}{\sigma_S})\sigma_SF_1(\tau) + \sigma_1\sigma_9\rho_{yl}\bar{F}_1(\tau)]H_2^\pi(\tau)
+ [\kappa_x\mu_x + (1 - \gamma)\sigma_S\sigma_x\rho_{sz}F_0(\tau) + (1 - \gamma)\sigma_x\sigma_1\rho_{xz}\bar{F}_0(\tau)]D_1^\pi(\tau)
+ [\kappa_\gamma\mu_\gamma + (1 - \gamma)(\sigma_y\rho_{sy} + m(t)\frac{b_y}{\sigma_S})\sigma_SF_0(\tau) + (1 - \gamma)\sigma_1\sigma_9\rho_{yl}]
\times \bar{F}_0[D_2^\pi(\tau) + (1 - \gamma)\sigma_x^2H_1^\pi(\tau)D_3^\pi(\tau) + (1 - \gamma)(K^2_1(t) + K^2_2(t))
+ K^2_2(t)H_2^\pi(\tau)D_3^\pi(\tau) + (1 - \gamma)\sigma_x\sigma_y\rho_{xy}(H_1^\pi(\tau)D_3^\pi(\tau)
+ H_2^\pi(\tau)D_1^\pi(\tau)) + b_yF_0(\tau) + (a - \sigma_1\sigma_{1\rho_{SL}})F_1(\tau)
+ (r + \mu_1 - R - \sigma^2_\gamma)\bar{F}_1(\tau) - \gamma^2_\sigma^2\rho_\sigma(\tau)F_1(\tau)
- \gamma_\sigma\sigma_1\rho_{SL}(F_0(\tau)F_2(\tau) + F_0(\tau)F_2(\tau)) - \gamma_\sigma^2\rho^2_\sigma(\tau)F_1(\tau),
\]

\[(H_2^\pi)'(\tau) = [(1 - \gamma)\sigma_S\sigma_x\rho_{sz}F_2(\tau) + \sigma_x\sigma_1\rho_{xz}\bar{F}_2(\tau)]H_1^\pi(\tau) + [(1 - \gamma)(\sigma_y\rho_{sy})
+ m(t)\frac{b_y}{\sigma_S})\sigma_SF_2(\tau) + (1 - \gamma)\sigma_1\sigma_9\rho_{yl}\bar{F}_2(\tau) - \kappa_y]H_2^\pi(\tau)
+ [\kappa_y\mu_y + (1 - \gamma)(\sigma_y\rho_{sy} + m(t)\frac{b_y}{\sigma_S})\sigma_SF_0(\tau) + (1 - \gamma)\sigma_1\sigma_9\rho_{yl}]
\times \bar{F}_0[D_2^\pi(\tau) + [\kappa_x\mu_x + (1 - \gamma)\sigma_S\sigma_x\rho_{sz}F_0(\tau) + (1 - \gamma)
\times \sigma_1\sigma_1\rho_{xz}\bar{F}_0(\tau)]D_3^\pi(\tau) + (1 - \gamma)\sigma_x^2H_1^\pi(\tau)D_3^\pi(\tau) + (1 - \gamma)(K^2_1(t)
+ K^2_2(t))H_2^\pi(\tau)D_3^\pi(\tau) + (1 - \gamma)\sigma_x\sigma_y\rho_{xy}(H_1^\pi(\tau)D_3^\pi(\tau)
+ H_2^\pi(\tau)D_1^\pi(\tau)) + b_yF_0(\tau) + (a - \sigma_1\sigma_{1\rho_{SL}})F_2(\tau)
+ (r + \mu_1 - R - \sigma^2_\gamma)\bar{F}_2(\tau) - \gamma^2_\sigma^2\rho_\sigma(\tau)F_2(\tau)
- \gamma_\sigma\sigma_1\rho_{SL}(F_0(\tau)\bar{F}_2(\tau) + F_0(\tau)F_2(\tau)) - \gamma_\sigma^2\rho^2_\sigma(\tau)\bar{F}_2(\tau),
\]

\[(D_1^\pi)'(\tau) = 2[(1 - \gamma)\sigma_S\sigma_x\rho_{sz}F_1(\tau) + (1 - \gamma)\sigma_x\sigma_1\rho_{xz}\bar{F}_1(\tau) - \kappa_x]D_1^\pi(\tau)
+ 2(1 - \gamma)[(\sigma_y\rho_{sy} + m(t)\frac{b_y}{\sigma_S})\sigma_SF_1(\tau) + \sigma_x\sigma_9\rho_{yl}\bar{F}_1(\tau)]D_3^\pi(\tau)
+ (1 - \gamma)\sigma_x^2D_3^\pi(\tau)^2 + (1 - \gamma)(K^2_1(t) + K^2_2(t))D_3^\pi(\tau)^2 + 2(1 - \gamma)\sigma_x\sigma_y\rho_{xy}D_3^\pi(\tau)D_3^\pi(\tau) + 2b_yF_1(\tau)
- \kappa_x^2F_1(\tau)^2 - 2\gamma_\sigma\sigma_1\rho_{SL}F_1(\tau)\bar{F}_1(\tau) - \gamma_\sigma^2\bar{F}_1(\tau)^2,
\]

\[(D_2^\pi)'(\tau) = 2[(1 - \gamma)(\sigma_y\rho_{sy} + m(t)\frac{b_y}{\sigma_S})\sigma_SF_2(\tau) + (1 - \gamma)\sigma_x\sigma_9\rho_{yl}\bar{F}_2(\tau)
- \kappa_y]D_2^\pi(\tau) + 2(1 - \gamma)[(\sigma_y\rho_{sy} + m(t)\frac{b_y}{\sigma_S})\sigma_SF_2(\tau) + \sigma_x\sigma_1\rho_{xz}\bar{F}_2(\tau)]D_3^\pi(\tau)
+ (1 - \gamma)(K^2_1(t) + K^2_2(t))D_3^\pi(\tau)^2 + 2(1 - \gamma)\sigma_x\sigma_y\rho_{xy}D_3^\pi(\tau)^2 + 2b_yF_2(\tau)
- \gamma_\sigma^2\bar{F}_2(\tau)^2,
\]

\[(D_3^\pi)'(\tau) = [(1 - \gamma)[\sigma_S\sigma_x\rho_{sz}F_2(\tau) + \sigma_x\sigma_1\rho_{xz}\bar{F}_2(\tau)]D_2^\pi(\tau) + (1 - \gamma)((\sigma_y\rho_{sy})
+ m(t)\frac{b_y}{\sigma_S})\sigma_SF_1(\tau) + \sigma_x\sigma_9\rho_{yl}\bar{F}_1(\tau)]D_3^\pi(\tau) + [(1 - \gamma)\sigma_S\sigma_x\rho_{sx}}
× \tilde{F}_1(\tau) + (1 - \gamma)\sigma_x \sigma_1 \rho_{x_l} \tilde{F}_1(\tau) + (1 - \gamma)(\sigma_y \rho_{S_y} + m(t) \frac{b_y}{\sigma_S})\sigma_S \\
× F_2(\tau) + (1 - \gamma)\sigma_1 \sigma_y \rho_{y_i} \tilde{F}_2(\tau) - (\kappa_x + \kappa_y)D^*_3(\tau) + (1 - \gamma)\sigma^2 \\
× D^*_1(\tau)D^*_2(\tau) + (1 - \gamma)(K^2_1(t) + K^2_2(t) + K^2_3(t))D^*_2(\tau)D^*_3(\tau) \\
+ (1 - \gamma)\sigma_x \sigma_y \rho_{y_i} \sigma_1 \rho_{S1}(\tilde{F}_1(\tau)\tilde{F}_2(\tau) \\
+ \tilde{F}_1(\tau)F_2(\tau)) - \gamma \sigma^2 \sigma_1 \rho_{S1}(\tilde{F}_1(\tau)\tilde{F}_2(\tau) \\
+ \tilde{F}_1(\tau)F_2(\tau)) - \gamma \sigma^2 \sigma_1 \rho_{S1}(\tilde{F}_1(\tau)\tilde{F}_2(\tau). \\
with the terminal conditions H^*_0(0) = H^*_1(0) = H^*_2(0) = D^*_1(0) = D^*_2(0) = D^*_3(0) = 0.

Proof. See Appendix C. □

The utility loss $L^\pi := L^\pi(t,W,x,\hat{y},l)$ resulting from an admissible strategy $\pi$ is defined as

$$L^\pi = 1 - \frac{V(t,W,x,\hat{y},l)}{V^\pi(t,W,x,\hat{y},l)}.$$  
(55)

Here, since $\gamma > 1$, both $V(t,W,x,\hat{y},l)$ and $V^\pi(t,W,x,\hat{y},l)$ are negative, and then $L^\pi \geq 0$. To quantify the importance of learning about the unobservable predictor and using the inflation-index bond to hedge the stock return predictors, we will measure the utility loss suffered from using the suboptimal strategy $\pi^1$ or $\pi^2$. In turn, these utility losses can be understood as the welfare improvements obtained by considering learning and using the inflation-indexed bond to hedge the stock return predictors.

4.1. The utility loss from ignoring learning. We consider the member who neglects learning about the expected excess return on the stock from the history information. Instead, she replaces the stochastic, unobservable predictor $y(t)$ by its long-run mean $\mu_y$. In this case, the expected excess return on the stock is given by

$$\mu_S(t) = \bar{a} + b_x x(t),$$

where $\bar{a} = a + b_y \mu_y$ is constant. Then the suboptimal strategy $\pi^1$ is a special case of Theorem 3.1 with $H_2(\tau) = D_2(\tau) = D_3(\tau) = 0$ for all $\tau$, i.e.

$$W \times \pi^1_S(t) = \frac{1}{\gamma \sigma_S (1 - \rho^2_{S1})} \left[ \frac{\bar{a} + b_x x}{\sigma_S} - \frac{(r + \mu_l - R)\rho_{S1}}{\sigma_1} + (1 - \gamma) \times \sigma_x (\rho_{Sx} - \rho_{S1}\rho_{xl})(H_1(\tau) + D_1(\tau)x) \right] (W + h_1(t)l),$$

$$W \times (\pi^1_p(t) - 1) = \frac{1}{\gamma \sigma_1 (1 - \rho^2_{S1})} \left[ \frac{r + \mu_l - R}{\sigma_1} - \frac{\bar{a} + b_x x}{\sigma_S} \rho_{S1} - \sigma_l (1 - \rho^2_{S1}) \right] (W + h_1(t)l)$$

$$+ (1 - \gamma) \sigma_x (\rho_{xl} - \rho_{Sx}\rho_{S1})(H_1(\tau) + D_1(\tau)x)$$

$$\times (W + h_1(t)l) - \frac{\sigma_l - \sigma_1}{\sigma_1} h_1(t)l.$$  
(56)

The suboptimal strategy $\pi^1$ can be written by the form given by (46) with

$$F_0(\tau) = \frac{1}{\gamma \sigma_S (1 - \rho^2_{S1})} \left[ \frac{\bar{a}}{\sigma_S} - \frac{(r + \mu_l - R)\rho_{S1}}{\sigma_1} + (1 - \gamma) \sigma_x (\rho_{Sx} - \rho_{S1}\rho_{xl})H_1(\tau) \right],$$

$$F_1(\tau) = \frac{1}{\gamma \sigma_S (1 - \rho^2_{S1})} \left[ \frac{b_x}{\sigma_S} + (1 - \gamma) \sigma_x (\rho_{Sx} - \rho_{S1}\rho_{xl})D_1(\tau) \right].$$
$F_2(\tau) = 0,$

$$F_0(\tau) = \frac{1}{\gamma \sigma_I (1 - \rho_{SI}^2)} \left[ \frac{r + \mu_I - R}{\sigma_I} - \frac{\bar{a}}{\sigma_S} \rho_{SI} - \sigma_I (1 - \rho_{SI}^2) 
+ (1 - \gamma) \sigma_x (\rho_{xI} - \rho_{xS} \rho_{SI}) H_1(\tau) \right],$$

$$F_1(\tau) = \frac{1}{\gamma \sigma_I (1 - \rho_{SI}^2)} \left[ - \frac{b_x}{\sigma_S} \rho_{SI} + (1 - \gamma) \sigma_x (\rho_{xI} - \rho_{xS} \rho_{SI}) D_1(\tau) \right],$$

$$F_2(\tau) = 0.$$

Then the utility loss due to the suboptimal investment strategy $\pi^1$ can be determined by (55).

4.2. The utility loss from not using the inflation-indexed bond to hedge the stock return predictors. If the member does not use the inflation-indexed bond to hedge the stock return predictors, there is a reasonable assumption that the correlations between the inflation-indexed bond and the stock and between the inflation-indexed bond and the stock return predictors equal zero, i.e. $\rho_{SI} = \rho_{xI} = \rho_{yI} = 0$. Then the suboptimal strategy $\pi^2$ satisfies

$$W \times \pi^2_S(t) = \frac{1}{\gamma \sigma_S} \left[ \frac{a + b_x x + b_y \bar{y}}{\sigma_S} + (1 - \gamma) \sigma_x \rho_{xS} (H_1(\tau) + D_1(\tau) x
+ D_3(\tau) \bar{y}) + (1 - \gamma) \left( \sigma_y \rho_{Sy} + \bar{m} \frac{b_y}{\sigma_S} \right) (H_2(\tau) + D_2(\tau) \bar{y} + D_3(\tau) x) \right] (W + h_1(t) l),$$

$$W \times (\pi^2(t) - 1) = \frac{1}{\gamma \sigma_I} \left[ \frac{r + \mu_I - R}{\sigma_I} - \sigma_I \right] (W + h_1(t) l) - \frac{\sigma_L - \sigma_I}{\sigma_I} h_1(t) l.$$  

So we can write $\pi^2$ in the form given by (46) with

$$F_0(\tau) = \frac{1}{\gamma \sigma_S} \left[ \frac{a}{\sigma_S} + (1 - \gamma) \sigma_x \rho_{xS} x H_1(\tau) + (1 - \gamma) \left( \sigma_y \rho_{Sy} + \bar{m} \frac{b_y}{\sigma_S} \right) H_2(\tau) \right],$$

$$F_1(\tau) = \frac{1}{\gamma \sigma_S} \left[ \frac{b_x}{\sigma_S} + (1 - \gamma) \sigma_x \rho_{xS} D_1(\tau) + (1 - \gamma) \left( \sigma_y \rho_{Sy} + \bar{m} \frac{b_y}{\sigma_S} \right) D_2(\tau) \right],$$

$$F_2(\tau) = \frac{1}{\gamma \sigma_S} \left[ \frac{b_y}{\sigma_S} + (1 - \gamma) \sigma_x \rho_{xS} D_3(\tau) + (1 - \gamma) \left( \sigma_y \rho_{Sy} + \bar{m} \frac{b_y}{\sigma_S} \right) D_2(\tau) \right],$$

$$\bar{F}_0(\tau) = \frac{r + \mu_I - R - \sigma_I^2}{\gamma \sigma_I^2},$$

$$\bar{F}_1(\tau) = 0,$$

$$\bar{F}_2(\tau) = 0.$$

The utility loss form not using the inflation-indexed bond to hedge the stock return predictors can also be determined by (55).

5. Numerical example. In this section, we will numerically demonstrate the effect of model parameters on the optimal investment strategy and the utility losses arising from the suboptimal investment strategies. Since [1] show that the net payout yield (dividends plus repurchases less equity issuances) can predict the expected
stock returns, we use the net payout yield as the observable predictor in this paper. Hence, we borrow the basic values of parameters mainly from [2] and [9]. We assume that the expected excess return on the stock is 5.5%, and the values of $x$ and $y$ equal to their long-term values $\mu_x$ and $\mu_y$, respectively. Throughout the following numerical illustrations, unless otherwise specified, the basic values for model parameters are summarized in Table 1. Finally, we just focus on the analysis at time $t = 0$, and assume that the conditional variance $m(t)$ converges to its long-run level $m^9$, for simplicity. For a representative DC pension plan member, we assume that the initial real wealth of her pension account is $w_0 = 4$, the initial real salary $l_0 = 1$ and the contribution rate is $\xi = 8\%$. The investment horizon is $T = 20$ and the relative risk aversion coefficient is set to be $\gamma = 4$.

### Table 1. Value of parameters.

| $\mu_I$ | $\sigma_I$ | $\sigma_S$ | $\bar{R}$ | $b_x$ | $b_y$ | $\kappa_x$ | $\kappa_y$ |
|---------|------------|------------|-----------|-------|-------|------------|------------|
| 0.03    | 0.1        | 0.2        | 0.050     | 0.3263| 2.6807| 0.2935     | 4.0942     |

| $\mu_x$ | $\mu_y$ | $\sigma_x$ | $\sigma_y$ | $\rho_{Sx}$ | $\rho_{Sy}$ | $\rho_{xy}$ | $\rho$ |
|---------|---------|------------|------------|--------------|--------------|------------|-------|
| -2.1493| 0.0000  | 0.1467     | 0.2620     | -0.2186      | -0.2164      | -0.4913    | 0.0366 |

| $\mu_L$ | $\sigma_L$ | $\rho_{SI}$ | $\rho_{xI}$ | $\rho_{yI}$ |
|---------|------------|--------------|--------------|-------------|
| 0.04    | 0.08       | 0.0000       | 0.3000       |             |

### Figure 1. Effect of the investment horizon $T$ on the optimal investment strategy.

#### 5.1. The optimal investment strategy. Figure 1 shows the effect of the investment horizon $T$ on the optimal investment strategy. With the increasing of the investment horizon $T$, it is easy to conclude from Figure 1 that the optimal stock allocation improves. This means that the member will put more money on the stock if she has a longer investment horizon. In practice, the younger member is usually advised to allocate more money on the stock and the elder member puts less on the

\[ m^9 \text{ is determined by } -2\kappa_y m + \sigma_y^2 - K_1^2 - K_2^2 - K_3^2 = 0, \text{ where } K_1 = \sigma_y \rho_{Sy} + m \frac{b_y \eta_{SS}}{\sigma_S}, K_2 = \sigma_y \rho_{xy} - m \frac{b_y \eta_{Sx}}{\sigma_S}, K_3 = \sigma_y \rho_{yI} + m \frac{b_y (\rho_{xI} \rho_{Sx} - \rho_{xS} \rho_{SI})}{\sigma_S \rho_{SI}}. \]  

A similar assumption is also made in [2].
stock. This finding is consistent with the result of [10] and [16], who indicate that this property attributes to the assumption of stock return predictability. However, the optimal inflation-indexed bond allocation declines with the investment horizon $T$ as a result of the substitution effect.

In addition, the speculative demands ($\pi_{S}^{\text{spec}}$ and $\pi_{P}^{\text{spec}}$) in both the optimal stock allocation and the inflation-indexed bond allocation are constant over time. Apart from the hedging component for the inflation risk ($\pi_{I}^{P}$), other hedging components vanish when the investment horizon $T$ moves towards zero, which implies that the myopic member does not concern about the future stock expected return and future salary. Furthermore, we find another characteristic in the optimal investment strategy that the optimal stock allocation is dominated by the speculative component ($\pi_{S}^{\text{spec}}$), whereas the optimal inflation-indexed bond allocation is mainly determined by the inflation hedging component ($\pi_{I}^{P}$). Figure 1 also indicates that although the observable and unobservable predictors only affect the stock expected return, the inflation-indexed bond’s hedging components about the two predictors are more significant than the corresponding stock’s hedging components. For instance, the inflation-indexed bond’s hedging component for the observable predictor $x$ (see Figure 1(b)) is almost four times bigger in absolute value than the corresponding stock’s hedging component (see Figure 1(a)).

![Figure 2. Effect of $\kappa_x$ and $\sigma_x$ on the optimal investment strategy.](image)

Figure 2 demonstrates the effect of the mean-reversion rate $\kappa_x$ and volatility $\sigma_x$ of the observable predictor $x$ on the optimal investment strategy. As $\kappa_x$ increases, the uncertainty in the observable predictor $x$ becomes less, and hence the hedging demand for $x$ and the optimal stock allocation go down. While a larger $\sigma_x$ usually implies a higher volatility of the observable predictor $x$, so a higher allocation to the stock is needed to hedge the risk caused by the observable predictor $x$. In contrast, the optimal inflation-indexed bond allocation increases w.r.t. $\kappa_x$ and decreases w.r.t. $\sigma_x$. This phenomenon is due to the substitution effect.

Figure 3 displays the effect of the mean-reversion rate $\kappa_y$ and volatility $\sigma_y$ of the unobservable predictor $y$ on the optimal investment strategy. The larger $\kappa_y$ is, the lower the estimation error $m$ of $y$ is, and the higher the optimal stock allocation is. When $\sigma_y$ becomes larger, the optimal stock allocation decreases, because the estimation error $m$ of $y$ grows. However, the optimal inflation-indexed bond allocation
Figure 3. Effect of $\kappa_y$ and $\sigma_y$ on the optimal investment strategy. Decreases along with $\kappa_y$ and increases along with $\sigma_y$, as a result of the substitution effect. By comparing Figure 2 with Figure 3, we find that although both $x$ and $y$ are the stock return predictors, they have different influence mechanisms on the optimal stock allocation, depending on whether they are observable.

Figure 4. Effect of $\mu_I$ and $\sigma_I$ on the optimal investment strategy. 

Figure 4 discloses the effect of the expected rate $\mu_I$ and the volatility $\sigma_I$ of the inflation on the optimal investment strategy. When $\mu_I$ becomes larger, the Sharpe ratio of the inflation-indexed bond increases. Therefore, the member prefers to improve the optimal inflation-indexed bond allocation, and the optimal stock allocation slightly decreases because of the substitution effect. However, different from $\mu_I$, with the growth of $\sigma_I$, the Sharpe ratio of the inflation-indexed bond falls, and hence the optimal inflation-indexed bond allocation drops, whereas the optimal stock allocation slightly rises. In addition, the inflation risk has a more significant effect on the optimal inflation-indexed bond allocation.

Figure 5 demonstrates the effect of the appreciation rate $\mu_L$ and volatility $\sigma_L$ of the salary on the optimal investment strategy. As $\mu_L$ increases, the optimal allocation of the risky assets improves. The growth of $\mu_L$ brings more income to the
member, so she will increase the allocation of risky assets to obtain higher returns. However, when $\sigma_L$ is growing, the stochastic salary will have more uncertainty, and hence the member will reduce the investment proportion in risky assets to avoid risk. Furthermore, since the salary risk arises from the same source of uncertainty as inflation, the optimal inflation-indexed bond allocation is more sensitive to the parameters related with the salary.

Figure 5. Effect of $\mu_L$ and $\sigma_L$ on the optimal investment strategy.

Figure 6 illustrates the effect of the correlations between the stock return predictors and the inflation-indexed bond, i.e. $\rho_{yI}$ and $\rho_{xI}$, on the inflation-indexed bond’s hedging components against the stock return predictors\textsuperscript{10}. It is obvious from Figure 6(a) that the inflation-indexed bond’s hedging component against the observable predictor $x$ is more sensitive to $\rho_{xI}$, and decreases along with $\rho_{xI}$. However, Figure 6(b) shows that the inflation-indexed bond’s hedging component against the unobservable predictor $y$ is more sensitive to $\rho_{yI}$, and declines along with $\rho_{yI}$. We

\textsuperscript{10}The behavior of the corresponding stock’s hedging components is very similar to that of the inflation-indexed bond’s hedging components, so we do not repeat it here.
also find that the hedging components can change by about 50% when the correlations vary between $-0.4$ to $0.4$. Therefore, we can conclude that the correlations between the stock return predictors and the inflation-indexed bond have important impacts on the magnitude of the inflation-indexed bond’s hedging components against the stock return predictors.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{Sample paths for $W^*(t)$, $x(t)$, $y(t)$ and $L(t)$ and the optimal investment strategy.}
\end{figure}

5.2. Monte Carlo simulations. We simulate the sample path of the real wealth process, two predictors and real salary process by adopting the Monte Carlo methods, and then obtain the simulated sample path of the corresponding optimal investment strategy ($\pi^*_S(t), \pi^*_P(t)$) shown in Figure 7. According to Figure 7(b), the optimal investment proportion in the inflation-indexed bond is generally larger than that invested in the stock. Meanwhile, the fluctuation of the optimal stock allocation is more serious than the fluctuation of the optimal inflation-indexed bond allocation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Effect of risk aversion parameter $\gamma$ on the optimal investment strategy.}
\end{figure}
Figure 8 describes the evolutions of the optimal investment strategy under three different levels of risk aversion. Here we simulate 10000 tracks of the optimal investment strategy and calculate the average portfolio weights in the two risky assets of the 10000 tracks. We find that the larger the risk aversion parameter $\gamma$ is, the smaller the proportion of wealth invested in the two risky assets is. Usually, a larger $\gamma$ means that the member prefers lower risk for the investment. So the member will put more money into the risk-free asset. In addition, the optimal investment proportion in the stock keeps a decreasing trend with the shrinking of the investment horizon, while the optimal investment proportion in the inflation-indexed bond is gradually increasing. This trend is consistent with the investment practice in DC pension plans. Usually, the members of the DC pension plan are suggested to reduce the proportion of risky stocks when they are nearing retirement.

![Figure 9](image)

**Figure 9.** Effect of $T$, $\rho_{xy}$ and $\rho_{SI}$ on the utility losses.

5.3. **Utility loss.** Figure 9 displays the effect of the investment horizon $T$, the correlation $\rho_{xy}$ between the observable and the unobservable predictors, and the correlation $\rho_{SI}$ between the inflation-indexed bond and the stock on the utility losses. Figure 9(a) shows that the utility loss from ignoring learning increases with the investment horizon $T$ and decreases as $\rho_{xy}$ changes from $-0.5$ to $0$. From Figure 9(b), we find the utility loss from not using the inflation-indexed bond to hedge the stock return predictors increases with the investment horizon $T$ and $\rho_{SI}$. During a longer investment horizon, the member will suffer a serious loss from ignoring learning or from not using the inflation-indexed bond to hedge the stock return predictors. Thus, for a DC pension plan with a long investment period, it is very important to consider learning and using the inflation-indexed bond to hedge the stock return predictors. In addition, there is an intuition behind Figure 9(a) that the member’s ability to learn reduces when $\rho_{xy}$ tends to zero. The lower the learning ability of the member, the smaller the loss she suffers from ignoring learning. Furthermore, as $\rho_{SI}$ becomes smaller, the ability of the inflation-indexed bond to hedge the stock return predictors diminishes, and hence the loss from not using the inflation-indexed bond to hedge the stock return predictors decreases.

Figure 10 shows that the effect of the correlations between the stock return predictors and the inflation-indexed bond, $\rho_{yI}$ and $\rho_{xI}$, on the utility losses. From Figure 10(a), we find that the utility loss from ignoring learning becomes larger.
when the non-zero correlations $\rho_{yI}$ and $\rho_{xI}$ have the same sign. The correlations $\rho_{yI}$ and $\rho_{xI}$ with the same sign increase the estimation error $m$ of $y$. Intuitively, along with the increases of the estimation error $m$, learning becomes less effective, and hence the utility loss from ignoring learning would be lower. Figure 10(b) shows that the utility loss from not using the inflation-indexed bond to hedge the stock return predictors also becomes higher when $\rho_{yI}$ and $\rho_{xI}$ have the same sign, but this utility loss is more sensitive to $\rho_{yI}$ and $\rho_{xI}$ than the utility loss caused by ignoring learning. Furthermore, the utility loss either from ignoring learning or from not using the inflation-indexed bond to hedge the stock return predictors can easily reach 90%, which implies that the welfare improvements from learning and from using the inflation-indexed bond to hedge the stock return predictors can be significant.

Figure 11 demonstrates the effect of the predictive power $b_y$ of the unobservable predictor and the predictive power $b_x$ of the observable predictor on the utility losses. Figure 11(a) shows that the utility loss from ignoring learning increases with the predictive powers $b_y$ and $b_x$. As the predictive powers $b_y$ and $b_x$ increase, the stock
return predictors have greater influence on the expected stock return, and hence learning about the unobservable predictor becomes more important. Figure 11(b) displays that the utility loss from not using the inflation-indexed bond to hedge the stock return predictor also grows with the predictive powers $b_y$ and $b_x$. When $b_y$ and $b_x$ become larger, the stock return predictors play a more important role in the expected stock return. Hence, using the inflation-indexed bond to hedge the stock return predictor becomes more effective, which means that the utility loss from not using the inflation-indexed bond to hedge the stock return predictors should be higher. In addition, we find that when the predictive power $b_x$ becomes bigger, the utility loss from ignoring learning or from not using the inflation-indexed bond to hedge the stock return predictors is almost insensitive to the predictive powers $b_y$.

6. Conclusion. This paper investigates an optimal investment problem for a DC pension plan member who faces inflation risk, salary risk and stock return predictability. Specifically, the expected excess stock return is predicted by both an observable predictor and an unobservable predictor, and the member has to estimate the unobservable predictor based on the history information through the learning mechanism. Therefore, by using the filtering techniques and dynamic programming method, we obtain the closed-form solutions for the optimal investment strategy and the corresponding value function. Moreover, we analyze the utility losses from ignoring learning and from not using the inflation-indexed bond to hedge the stock return predictors.

The theoretical results indicate that stock return predictability can distinctly affect the level and structure of the optimal investment strategy. In particular, because of the correlations between the stock return predictors and the inflation-indexed bond, the optimal inflation-indexed bond allocation has two hedging components against the observable and unobservable predictors. The numerical results further show that the inflation-indexed bond’s hedging components against the stock return predictors are larger than the corresponding stock’s hedging components. This means that the inflation-indexed bond plays an important role in hedging the stock return predictors, which should not be ignored. Furthermore, we find that suboptimal investment strategies, resulting either from not learning or from not using the inflation-indexed bond to hedge the stock return predictors, can lead to significant utility losses, especially for a long investment horizon. In addition, the utility loss from ignoring learning or from not using the inflation-indexed bond to hedge the stock return predictors becomes higher, when the correlations between the inflation-indexed bond and the stock return predictors have the same sign.

Appendix A.

Proof of Proposition 1: We rewrite the models for the stock price, observable predictor, price index and unobservable predictor as

\[
\begin{bmatrix}
\frac{dS(t)}{S(t)} \\
\frac{dx(t)}{x(t)} \\
\frac{dI(t)}{I(t)} \\
\frac{dy(t)}{y(t)}
\end{bmatrix} = \begin{bmatrix}
R(t) + a + b_x x(t) + b_y y(t) \\
\kappa_x (\mu_x - x(t)) \\
\kappa_I (\mu_I - I(t)) \\
\kappa_y (\mu_y - y(t))
\end{bmatrix} dt + \begin{bmatrix}
\sigma_S \\
\sigma_x \\
\sigma_I \\
\sigma_y
\end{bmatrix} \begin{bmatrix}
dZ_S(t) \\
dZ_x(t) \\
dZ_I(t) \\
dZ_y(t)
\end{bmatrix},
\]

where $[dZ_S(t), dZ_x(t), dZ_I(t), dZ_y(t)]'$ can be decomposed into
We still apply the notation as in [13]. We rewrite the equations for the observable
mutually independent one-dimensional standard Brownian motions. Then (A.1) becomes
\[
\begin{align*}
\frac{dS(t)}{S(t)} & = \left[ R(t) + a + b_x x(t) + b_y y(t) \right] dt + \left[ \begin{array}{c}
\sigma_S \\
\sigma_x \\
\sigma_I \\
\sigma_y \\
0 \\
\rho_{Sx} \\
\rho_{SI} \\
\rho_{Sy} \\
\rho_x \\
\rho_I \\
\rho_y \\
\end{array} \right] \left[ \begin{array}{c}
\frac{dZ_1(t)}{\sqrt{1-\rho_{Sx}^2}} \\
\frac{dZ_2(t)}{\sqrt{1-\rho_{SI}^2}} \\
\frac{dZ_3(t)}{\sqrt{1-\rho_{Sy}^2}} \\
\frac{dZ_4(t)}{\sqrt{1-\rho_{xy}^2}} \\
\end{array} \right] \\
& = \left[ \begin{array}{c}
\kappa_x (\mu_x - x(t)) \\
\kappa_x (\mu_S - \bar{S}_x) \\
\kappa_I (\mu_I - \bar{I}_x) \\
\kappa_y (\mu_y - y(t)) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right] dt + \left[ \begin{array}{c}
1 \\
0 \\
0 \\
0 \\
0 \\
\rho_{Sx} \\
\rho_{SI} \\
\rho_{Sy} \\
\rho_x \\
\rho_I \\
\rho_y \\
\end{array} \right] \left[ \begin{array}{c}
\frac{dZ_1(t)}{\sqrt{1-\rho_{Sx}^2}} \\
\frac{dZ_2(t)}{\sqrt{1-\rho_{SI}^2}} \\
\frac{dZ_3(t)}{\sqrt{1-\rho_{Sy}^2}} \\
\frac{dZ_4(t)}{\sqrt{1-\rho_{xy}^2}} \\
\end{array} \right] .
\end{align*}
\]
\tag{A.2}
\]
Equivalently,
\[
\begin{align*}
\frac{dS(t)}{S(t)} & = \left[ R(t) + a + b_x x(t) + b_y y(t) \right] dt + \left[ \begin{array}{c}
\sigma_S \\
\sigma_x \\
\sigma_I \\
\sigma_y \\
0 \\
\rho_{Sx} \\
\rho_{SI} \\
\rho_{Sy} \\
\rho_x \\
\rho_I \\
\rho_y \\
\end{array} \right] \left[ \begin{array}{c}
\frac{dZ_1(t)}{\sqrt{1-\rho_{Sx}^2}} \\
\frac{dZ_2(t)}{\sqrt{1-\rho_{SI}^2}} \\
\frac{dZ_3(t)}{\sqrt{1-\rho_{Sy}^2}} \\
\frac{dZ_4(t)}{\sqrt{1-\rho_{xy}^2}} \\
\end{array} \right] \\
& = \left[ \begin{array}{c}
\kappa_x (\mu_x - x(t)) \\
\kappa_x (\mu_S - \bar{S}_x) \\
\kappa_I (\mu_I - \bar{I}_x) \\
\kappa_y (\mu_y - y(t)) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right] dt + \left[ \begin{array}{c}
\sigma_S \\
\sigma_x \\
\sigma_I \\
\sigma_y \\
0 \\
\rho_{Sx} \\
\rho_{SI} \\
\rho_{Sy} \\
\rho_x \\
\rho_I \\
\rho_y \\
\end{array} \right] \left[ \begin{array}{c}
\frac{dZ_1(t)}{\sqrt{1-\rho_{Sx}^2}} \\
\frac{dZ_2(t)}{\sqrt{1-\rho_{SI}^2}} \\
\frac{dZ_3(t)}{\sqrt{1-\rho_{Sy}^2}} \\
\frac{dZ_4(t)}{\sqrt{1-\rho_{xy}^2}} \\
\end{array} \right] .
\end{align*}
\tag{A.3}
\]
We still apply the notation as in [13]. We rewrite the equations for the observable
processes $S(t)$, $x(t)$ and $I(t)$ separately from the unobservable process $y(t)$. So, we
have the observable processes
\[
\begin{align*}
\frac{dS(t)}{S(t)} & = \left[ R(t) + a + b_x x(t) \right] dt + \left[ \begin{array}{c}
\kappa_x (\mu_x - x(t)) \\
\kappa_x (\mu_S - \bar{S}_x) \\
\kappa_I (\mu_I - \bar{I}_x) \\
\kappa_y (\mu_y - y(t)) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
\frac{dZ_1(t)}{\sqrt{1-\rho_{Sx}^2}} \\
\frac{dZ_2(t)}{\sqrt{1-\rho_{SI}^2}} \\
\frac{dZ_3(t)}{\sqrt{1-\rho_{Sy}^2}} \\
\frac{dZ_4(t)}{\sqrt{1-\rho_{xy}^2}} \\
\end{array} \right] \\
& = \left[ \begin{array}{c}
\kappa_x (\mu_x - x(t)) \\
\kappa_x (\mu_S - \bar{S}_x) \\
\kappa_I (\mu_I - \bar{I}_x) \\
\kappa_y (\mu_y - y(t)) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{array} \right] dt + \left[ \begin{array}{c}
1 \\
0 \\
0 \\
0 \\
0 \\
\rho_{Sx} \\
\rho_{SI} \\
\rho_{Sy} \\
\rho_x \\
\rho_I \\
\rho_y \\
\end{array} \right] \left[ \begin{array}{c}
\frac{dZ_1(t)}{\sqrt{1-\rho_{Sx}^2}} \\
\frac{dZ_2(t)}{\sqrt{1-\rho_{SI}^2}} \\
\frac{dZ_3(t)}{\sqrt{1-\rho_{Sy}^2}} \\
\frac{dZ_4(t)}{\sqrt{1-\rho_{xy}^2}} \\
\end{array} \right] .
\end{align*}
\tag{A.4}
\]
and the unobservable process
\[
\begin{align*}
dy(t) & = \left( \kappa_y \mu_y + (\kappa_y - \kappa) y(t) \right) dt + \sigma_y \hat{\rho}_y dy(t) + \left[ \sigma_y \rho_{Sy}, \sigma_y \rho_{xy}, \sigma_y \rho_{yI} \right] \left[ \begin{array}{c}
\frac{dZ_1(t)}{\sqrt{1-\rho_{Sy}^2}} \\
\frac{dZ_2(t)}{\sqrt{1-\rho_{xy}^2}} \\
\frac{dZ_3(t)}{\sqrt{1-\rho_{yI}^2}} \\
\end{array} \right] .
\end{align*}
\tag{A.5}
We evaluate the matrix $BB = B_1B_1' + B_2B_2'$. We have

$$BB = B_2B_2', \quad (BB)^{-1} = (B_2B_2)^{-1} = B_2^{-1}B_2^{-1},$$

where

$$B_2^{-1} = \begin{bmatrix}
\frac{1}{\sigma_x} & 0 & 0 \\
\frac{\rho_{Sx} \rho_{xI}}{\sigma_x \rho_{xI}} & \frac{1}{\rho_{xI}} & 0 \\
\frac{\rho_{Sx} \rho_{SI}}{\sigma_x \rho_{xI}} & \frac{1}{\rho_{xI}} & \frac{1}{\sigma_{xI}}
\end{bmatrix}. \quad (A.6)$$

Since the elements in $B_2^{-1}$ are uniformly bounded, then the elements in $(BB)^{-1}$ are also uniformly bounded.

According to Theorem 12.7 in [13], we have that the process $\hat{g}(t) = E\left[y(t) \middle| \mathcal{F}_t^{S,x,I}\right]$ with the variance $m(t) = E\left[(y(t) - \hat{g}(t))^2 \middle| \mathcal{F}_t^{S,x,I}\right]$ satisfies

$$
\begin{align*}
\frac{d\hat{g}(t)}{dt} &= (\kappa_y \mu_y + (-\kappa_y) \hat{g}(t))dt + \left[b_2B_2 + m(t) [b_y 0 0]\right] \times \\
&\times \begin{bmatrix}
\frac{1}{\sigma_S} & \frac{\rho_{SBx}}{\sigma_{xS}} & \frac{\rho_{SBxI}}{\sigma_{xS}\rho_{xI}} \\
0 & \frac{1}{\sigma_{xS}} & \frac{\rho_{SBxI}}{\sigma_{xS}\rho_{xI}} \\
0 & 0 & \frac{1}{\sigma_{xS}\rho_{xI}}
\end{bmatrix}^{-1} \begin{bmatrix}
\begin{bmatrix}
\frac{dS(t)}{S(t)} \\
\frac{dx(t)}{d(t)} \\
\frac{dT(t)}{T(t)}
\end{bmatrix} - \begin{bmatrix}
R(t) + a + b_x x(t) \\
\kappa_x (\mu_x - x(t)) \\
\mu_I
\end{bmatrix}
\end{bmatrix} \\
&\quad + \begin{bmatrix}
b_y \\
0 \\
0
\end{bmatrix} \hat{g}(t) dt.
\end{align*}
$$

On the one hand, considering that vector

$$B_2^{-1} \times \begin{bmatrix}
\frac{dS(t)}{S(t)} \\
\frac{dx(t)}{d(t)} \\
\frac{dT(t)}{T(t)}
\end{bmatrix} - \begin{bmatrix}
R(t) + a + b_x x(t) \\
\kappa_x (\mu_x - x(t)) \\
\mu_I
\end{bmatrix} \times \begin{bmatrix}
\sigma_S & 0 & 0 \\
\sigma_{xS} & \sigma_x \rho_{xI} & 0 \\
\sigma_{xSI} & \sigma_{xI} \rho_{xI} & \sigma_{xI}\rho_{xI}
\end{bmatrix} \begin{bmatrix}
\frac{dB_S(t)}{dS(t)} \\
\frac{dB_x(t)}{dx(t)} \\
\frac{dB_I(t)}{dT(t)}
\end{bmatrix}.
$$

On the other hand, we also obtain

$$
\begin{align*}
\frac{dS(t)}{S(t)} &= \begin{bmatrix}
R + a + b_x x(t) + b_y \hat{g}(t) \\
\kappa_x (\mu_x - x(t)) \\
\mu_I
\end{bmatrix} dt + \begin{bmatrix}
\sigma_S & 0 & 0 \\
\sigma_{xS} & \sigma_x \rho_{xI} & 0 \\
\sigma_{xSI} & \sigma_{xI} \rho_{xI} & \sigma_{xI}\rho_{xI}
\end{bmatrix} \begin{bmatrix}
\frac{dB_S(t)}{dS(t)} \\
\frac{dB_x(t)}{dx(t)} \\
\frac{dB_I(t)}{dT(t)}
\end{bmatrix}.
\end{align*}
$$

\begin{align*}
\text{(A.8)}
\end{align*}
Therefore, the dynamics of \( y(t) \) becomes

\[
d\hat{y}(t) = \kappa_y(\mu_y - \hat{y}(t))dt + K_1(t)dBS(t) + K_2(t)dB_x(t) + K_3(t)dB_t(t). \tag{A.11}
\]

According to Theorem 12.7 in [13], the variance \( m(t) \) is given by

\[
\frac{dm(t)}{dt} = -2\kappa_y m(t) + \frac{\sigma_y^2}{\kappa_y} - K_1^2(t) - K_2^2(t) - K_3^2(t). \tag{A.12}
\]

According to (A.9), we have the following representation

\[
dW_I(t) = \rho_{S1}dB_S(t) + \hat{\rho}_{x1}dB_x(t) + \hat{\rho}_I dB_t(t). \tag{A.13}
\]

Thus, we can obtain the filtered models (13)-(18).

\[\square\]

**Appendix B.**

**Proof of Theorem 3.1:** Assume that \( V(t, W, x, \hat{y}, l) \) is a solution of HJB equation (23). We try to conjecture \( V(t, W, x, \hat{y}, l) \) in the following form

\[
V(t, W, x, \hat{y}, l) = \frac{(W + h(t,l))^{1-\gamma}}{1-\gamma} g(t, x, \hat{y}), \tag{B.1}
\]

with the boundary conditions given by \( h(T, l) = 0, g(T, x, \hat{y}) = 1 \). Then

\[
V_t = (W + h)^{-\gamma} h_t g + \frac{(W + h)^{1-\gamma}}{1-\gamma} g_t, \quad V_W = (W + h)^{-\gamma} g, \quad V_x = (W + h)^{1-\gamma} g_x,
\]

\[
V_y = \frac{(W + h)^{1-\gamma}}{1-\gamma} g_y, \quad V_{yy} = \frac{(W + h)^{1-\gamma}}{1-\gamma} g_{yy},
\]

\[
V_{xx} = \frac{(W + h)^{1-\gamma}}{1-\gamma} g_{xx}, \quad V_{xy} = \frac{(W + h)^{1-\gamma}}{1-\gamma} g_{xy}, \quad V_{yy} = \frac{(W + h)^{1-\gamma}}{1-\gamma} g_{yy},
\]

\[
V_{xg} = \frac{(W + h)^{1-\gamma}}{1-\gamma} g_{xg}, \quad V_{yl} = (W + h)^{-\gamma-1} h_t g_y, \quad V_{yl} = (W + h)^{-\gamma-1} h_t g_y.
\tag{B.2}
\]

Differentiating (23) w.r.t. \( \sigma_S \) and \( \pi_P \) implies

\[
\pi_{S} = -\frac{1}{V_{WW}W_S(1 - \rho_{S1}^2)} \left[ V_W \left( \frac{a + b_x x + b_y \hat{y}}{\sigma_S} - \frac{(r + \mu_I - R)\rho_{SI}}{\sigma_I} \right) + V_{xS} \sigma_x (\rho_{Sx} - \rho_{x1P_S1}) + V_{yg} (\sigma_y (\rho_{Sy} - \rho_{S1P_{SI}}) + m \frac{b_y}{\sigma_S}) \right],
\]

\[
\pi_{P} - 1 = -\frac{1}{V_{WW}W_{\sigma_I}^2(1 - \rho_{S1}^2)} \left[ V_W \left( \frac{r + \mu_I - R}{\sigma_S} - \frac{a + b_x x + b_y \hat{y}}{\sigma_S} \right) - \sigma_I^2 (1 - \rho_{S1}^2) \right] + V_{xS} \sigma_x (\rho_{x1} - \rho_{Sx} - \rho_{SISS}) + V_{yg} \sigma_I (\sigma_y (\rho_{yI})
\]
Plugging (B.3) into (23) gives

$$-\rho_{Sy}\rho_{SI} + \frac{b_y}{\sigma_y}\rho_{SI} + V_{Wi}l\sigma_1(\sigma_L - \sigma_I)(1 - \rho_{SI}^2)) \right]. \tag{B.3}$$

Substituting (B.1) and (B.2) into (B.4), we have

$$V_i + V_WWr + V_W\xi l + V_xK_x(\mu_x - x) + \frac{1}{2}V_{xx}\sigma_x^2 + V_y\kappa_y(\mu_y - \hat{y})$$

$$+ \frac{1}{2}V_{y\hat{y}}(K_1^2 + K_2^2 + K_3^2) + V_l(\mu_L - \mu_I + \sigma_I^2 - \sigma_L\sigma_I) + \frac{1}{2}V_l\sigma^2(\sigma_L - \sigma_I)^2$$

$$+ V_{xy}\sigma_x\rho_{xy} + V_{yl}(\sigma_L - \sigma_I)\rho_{xI} + V_y\sigma_s(\sigma_L - \sigma_I)\sigma_{ys}\rho_{SI}$$

$$- \frac{1}{2}V_{yy}W^2[(\pi_2^2\sigma_S + (\pi_p^2 - 1)\sigma_1\rho_{SI})^2 + (\pi_p^2 - 1)^2\sigma_I^2(1 - \rho_{SI}^2)] = 0. \tag{B.4}$$

Substituting (B.1) and (B.2) into (B.4), we have

$$h_{tg} + \frac{W + h}{1 - \gamma}g_{xk}(\mu_x - x) + \frac{W + h}{2(1 - \gamma)}g_{xx}\sigma_x^2$$

$$+ \frac{W + h}{1 - \gamma}g_{y\hat{y}}(K_1^2 + K_2^2 + K_3^2) + h_{lg}(\mu_L - \mu_I + \sigma_I^2)$$

$$- \sigma_L\sigma_I + \frac{1}{2}[-\gamma(W + h)^{-1}h_{tg}^2g + h_{lg}g^2(\sigma_L - \sigma_I)^2 + \frac{W + h}{1 - \gamma}g_{xy}\sigma_x\rho_{xy}$$

$$+ h_{lg}\sigma_s(\sigma_L - \sigma_I)\rho_{xI} + h_{ly}\sigma_s(\sigma_L - \sigma_I)\rho_{ys} + \frac{(W + h)}{2\gamma}\left\{g[(a + 2b_x + b_y\hat{y})\sigma_s + \gamma(W + h)^{-1}h_{lg}\sigma_L(1 - \rho_{SI})^2]\right.$$}

Furthermore, we guess

$$h(t, l) = h_1(t)l + h_2(t),$$

g(t, x, \hat{y}) = \exp\left\{(1 - \gamma)\left[H_0(t) + H_1(t)x + H_2(t)\hat{y}\right.ight.$$}

$$\left.+ \frac{1}{2}D_1(t)x^2 + \frac{1}{2}D_2(t)\hat{y}^2 + D_3(t)xy\right]\right\}, \tag{B.6}$$

with $h_1(T) = h_2(T) = H_0(T) = H_1(T) = H_2(T) = D_1(T) = D_2(T) = D_3(T) = 0$. Then we obtain

$$h_t = h_{1t}l + h_{2t}, \quad h_l = h_1, \quad h_{ll} = 0,$$

$$g_t = (1 - \gamma)g\left[H_{0t} + H_{1t}x + H_{2t}\hat{y} + \frac{1}{2}D_{1t}x^2 + \frac{1}{2}D_{2t}\hat{y}^2 + D_{3t}xy\right],$$

$$g_x = (1 - \gamma)g[H_1 + D_1x + D_3\hat{y}], \quad g_{xx} = (1 - \gamma)g[D_1 + (1 - \gamma)(H_1 + D_1x + D_3\hat{y})^2],$$

$$g_{\hat{y}} = (1 - \gamma)g[H_2 + D_2\hat{y} + D_3x], \quad g_{\hat{y}\hat{y}} = (1 - \gamma)g[D_2 + (1 - \gamma)(H_2 + D_2\hat{y} + D_3x)^2],$$

$$g_{x\hat{y}} = (1 - \gamma)g[D_3 + (1 - \gamma)(H_1 + D_1x + D_3\hat{y})(H_2 + D_2\hat{y} + D_3x)]. \tag{B.7}$$
Inserting (B.6) and (B.7) into (B.5), we get

\[ (W + h) \left\{ x \left[ H_{1t} + (b_x \alpha_7 - \kappa_x) H_1 + b_x \alpha_8 H_2 + (\alpha_1 + a \alpha_7) D_1 + (\alpha_3 + a \alpha_8) D_3 \\
+ 2 \alpha_2 H_1 D_1 + 2 \alpha_4 H_2 D_3 + \alpha_5 (H_1 D_3 + H_2 D_1) + b_x \alpha_6 \right] + \dot{y} \left[ H_{2t} + b_y \alpha_7 H_1 \\
+ (b_y \alpha_8 - \kappa_y) H_2 + (\alpha_3 + a \alpha_8) D_2 + (\alpha_1 + a \alpha_7) D_3 + 2 \alpha_2 H_1 D_3 + 2 \alpha_4 H_2 D_2 \\
+ \alpha_5 (H_1 D_2 + H_2 D_3) + b_y \alpha_6 \right] + x^2 \left[ \frac{1}{2} D_{1t} + (b_x \alpha_7 - \kappa_x) D_1 + b_x \alpha_8 D_3 + \alpha_2 D_1^2 \\
+ \alpha_4 D_3^2 + \alpha_5 D_1 D_3 + \frac{b_x^2}{2 \gamma \sigma_5^2 (1 - \rho_8^2)} \right] + \dot{y}^2 \left[ \frac{1}{2} D_{2t} + (b_y \alpha_8 - \kappa_y) D_2 + b_y \alpha_7 D_3 \\
+ \alpha_4 D_3^2 + \alpha_2 D_3^2 + \alpha_5 D_2 D_3 + \frac{b_y^2}{2 \gamma \sigma_5^2 (1 - \rho_8^2)} \right] + x \dot{y} \left[ D_{3t} + b_y \alpha_7 D_1 + b_x \alpha_8 D_2 \\
+ (b_x \alpha_7 - \kappa_x + b_x \alpha_8 - \kappa_y) D_3 + 2 \alpha_2 D_1 D_3 + 2 \alpha_4 D_2 D_3 + \alpha_5 (D_1 D_2 + D_3^2) \\
+ \frac{b_y b_x}{\gamma \sigma_5^2 (1 - \rho_8^2)} \right] + H_{0t} + (\alpha_1 + a \alpha_7) H_1 + (\alpha_3 + a \alpha_8) H_2 + \frac{\sigma_5^2}{2} D_1 \\
+ \frac{K_1^2 + K_2^2 + K_3^2}{2} D_2 + \sigma_x \sigma_y \rho_{xy} D_3 + \alpha_2 H_1^2 + \alpha_4 H_2^2 + \alpha_5 H_1 H_2 + \alpha_9 + r \right\} \\
+ l[h_{1t} + \alpha_1 h_1 + \xi] + h_{2t} - rh_2 = 0. \] (B.8)

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10} \) are given by (36)-(45). Our guess (B.6) does solve the PDE (B.8), if the functions \( H_0(\cdot), H_1(\cdot), H_2(\cdot), D_1(\cdot), D_2(\cdot) \) and \( D_3(\cdot) \) solve the system of ODEs (30)-(35).

**Appendix C.**

**Proof of Proposition 2:** To prove this proposition, we conjecture that

\[ V^\pi(t, W, x, \dot{y}, l) = \frac{(W + h^\pi(t))^1 - \gamma}{1 - \gamma} g^\pi(t, x, \dot{y}). \] (C.1)

We plug an admissible strategy \( \pi \) in the form (46) and the guess for \( V^\pi \) into the (23) without the supremum over \( \pi \). Through the similar derivation in Appendix B, we obtain \( h^\pi(t) = h_1(t) \) and the following PDE:

\[
\frac{1}{1 - \gamma} \dot{g}^\pi_t + \left[ r + (F_0 + F_1 x + F_2 \dot{y})(a + b_x x + b_y \dot{y} - \sigma_S \sigma_1 \rho_{SI}) + (F_0 + F_1 x + F_2 \dot{y}) \right] x (r + \mu_t - R - \sigma_7^2) - \frac{\gamma}{2} \left( (F_0 + F_1 x + F_2 \dot{y}) \sigma_S + (F_0 + F_1 x + F_2 \dot{y}) \sigma_1 \rho_{SI} \right)^2 \\
- \frac{\gamma}{2} \left( (F_0 + F_1 x + F_2 \dot{y})^2 \sigma_7^2 (1 - \rho_8^2) \right) g^\pi_t + \frac{1}{1 - \gamma} \kappa_x \sigma_x (\mu_x - x) + ((F_0 + F_1 x + F_2 \dot{y}) \sigma_S + (\dot{F}_0 + \dot{F}_1 x + \dot{F}_2 \dot{y}) \sigma_1 \rho_{SI}) g^\pi_x \\
+ \left( \dot{F}_0 + \dot{F}_1 x + \dot{F}_2 \dot{y} \right) \sigma_1 \rho_{SI} + \sigma_x \rho_{sx} + (\dot{F}_0 + \dot{F}_1 x + \dot{F}_2 \dot{y}) \sigma_1 \sigma_x (\rho_{xt} - \rho_{sx} \rho_{SI}) \right] g^\pi_x \\
+ \left[ \frac{1}{1 - \gamma} \kappa_y \sigma_y (\mu_y - \dot{y}) + ((F_0 + F_1 x + F_2 \dot{y}) \sigma_S + (\dot{F}_0 + \dot{F}_1 x + \dot{F}_2 \dot{y})) K_1 \\
+ (F_0 + F_1 x + F_2 \dot{y}) \sigma_1 (\rho_{yt} - \rho_{SI} \rho_{Sy}) - m \frac{b_y}{\sigma_S \rho_{SI}} \right] g^\pi_y
\]
We substitute the conjecture of $g^x$ from (48) into (C.2) and divide by $\hat{g}^y$. Then the left-hand side of the above equation is an affine function for $x$, $y$, $x^2$, $\hat{y}^2$ and $x\hat{y}$. The equation must hold for all values of $x$ and $\hat{y}$, which leads to the functions $H_0^x(\cdot)$, $H_1^x(\cdot)$, $H_2^x(\cdot)$, $D_1^x(\cdot)$, $D_2^x(\cdot)$, and $D_3^x(\cdot)$ satisfy the system of ODEs (49)-(54).

REFERENCES

[1] J. Boudoukh, R. Michaely, M. Richardson and M. R. Roberts, On the importance of measuring payout yield: Implications for empirical asset pricing, The Journal of Finance, 62 (2007), 877–915.
[2] N. Branger, L. S. Larsen and C. Munk, Robust portfolio choice with ambiguity and learning about return predictability, Journal of Banking and Finance, 37 (2013), 1397–1411.
[3] Z. Chen, Z. Li, Y. Zeng and J. Sun, Asset allocation under loss aversion and minimum performance constraint in a DC pension plan with inflation risk, Insurance: Mathematics and Economics, 75 (2017), 137–150.
[4] J. B. Detemple, Asset pricing in a production economy with incomplete information, The Journal of Finance, 41 (1986), 383–391.
[5] M. U. Dothan and D. Feldman, Equilibrium interest rates and multiperiod bonds in a partially observable economy, The Journal of Finance, 41 (1986), 369–382.
[6] D. Duffie, Presidential address: Asset price dynamics with slow-moving capital, The Journal of Finance, 65 (2010), 1237–1267.
[7] E. F. Fama and K. R. French, Dividend yields and expected stock returns, Journal of Financial Economics, 22 (1988), 3–25.
[8] G. Gennette, Optimal portfolio choice under incomplete information, The Journal of Finance, 41 (1986), 733–746.
[9] N.-W. Han and M.-W. Hung, Optimal asset allocation for DC pension plans under inflation, Insurance: Mathematics and Economics, 51 (2012), 172–181.
[10] T. S. Kim and E. Omberg, Dynamic nonmyopic portfolio behavior, Review of Financial Studies, 9 (1996), 141–161.
[11] R. Korn and H. Kraft, On the stability of continuous-time portfolio problems with stochastic opportunity set, Mathematical Finance, 14 (2004), 403–414.
[12] Y. W. Li, S. Y. Wang, Y. Zeng and H. Qiao, Equilibrium investment strategy for a DC plan with partial information and mean–variance criterion, IEEE Systems Journal, 11 (2017), 1492–1504.
[13] R. S. Liptser and A. N. Shiryaev, Statistics of Random Processes. Applications, vol. II., Springer-Verlag, 2001.
[14] J. Liu, Portfolio selection in stochastic environments, Review of Financial Studies, 20 (2007), 1–39.
[15] J. H. van Binsbergen and R. S. J. Koijen, Predictive regressions: A present-value approach, The Journal of Finance, 65 (2010), 1439–1471.
[16] J. A. Wachter, Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets, Journal of Financial and Quantitative Analysis, 37 (2002), 63–91.
[17] P. Wang, Z. Li and J. Sun, Robust portfolio choice for a DC pension plan with inflation risk and mean-reverting risk premium under ambiguity, Optimization, 70 (2021), 191–224.
[18] H. Wu, L. Zhang and H. Chen, Nash equilibrium strategies for a defined contribution pension management, Insurance: Mathematics and Economics, 62 (2015), 202–214.
[19] Y. Xia, Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation, The Journal of Finance, 56 (2001), 205–246.
[20] H. Yao, P. Chen, M. Zhang and X. Li, Dynamic discrete-time portfolio selection for defined contribution pension funds with inflation risk, Journal of Industrial and Management Optimization.
[21] A. Zhang and C.-O. Ewald, Optimal investment for a pension fund under inflation risk, Mathematical Methods of Operations Research, 71 (2010), 353–369.
[22] L. Zhang, Z. Li, Y. Xu and Y. Li, Multi-period mean variance portfolio selection under incomplete information, *Applied Stochastic Models in Business and Industry*, 32 (2016), 753–774.

[23] L. Zhang, H. Zhang and H. Yao, Optimal investment management for a defined contribution pension fund under imperfect information, *Insurance: Mathematics and Economics*, 79 (2018), 210–224.

[24] Q. Zhao, R. Wang and J. Wei, Time-inconsistent consumption-investment problem for a member in a defined contribution pension plan, *Journal of Industrial and Management Optimization*, 12 (2016), 1557–1585.

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