GENERALIZED COHERENT STATES FOR DYNAMICAL SUPERALGEBRAS

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ABSTRACT

Coherent states for a general Lie superalgebra are defined following the method originally proposed by Perelomov. Algebraic and geometrical properties of the systems of states thus obtained are examined, with particular attention to the possibility of defining a Kähler structure over the states supermanifold and to the connection between this supermanifold and the coadjoint orbits of the dynamical supergroup. The theory is then applied to some compact forms of contragradient Lie superalgebras.

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1. INTRODUCTION

Coherent states (CS) have been first introduced in quantum optics by Glauber\textsuperscript{1,2} as the states of minimum quantum uncertainty for the harmonic oscillator. Glauber’s CS preserve their shape during time evolution. After the work of Glauber several attempts have been made to generalize the concept of CS to systems with a larger dynamical group\textsuperscript{3,4}. The most promising among them is certainly the method proposed by Perelomov\textsuperscript{5}. Perelomov defined generalized coherent states (GCS) for an arbitrary Lie group by mimicking the group-theoretical properties of Glauber’s CS. GCS for a general dynamical system fulfill three requirements: a) they are states of minimum quantum uncertainty, b) their manifold is just the classical phase space of the system; c) the quantum evolution of the system can be described by a path on this manifold, that is the solution of Euler-Lagrange’s equations.

In the past few years several papers appeared in which the concept of coherence is carried over to systems that exhibit a dynamical symmetry with an infinite number of degrees of freedom\textsuperscript{6} and to supersymmetric systems. Supercoherent states (SCS) have been defined for series of noncompact orthosymplectic supergroups\textsuperscript{7,8,9}. A definition that already contains in itself the general idea is given in Ref. 10 in the context of a condensed matter problem, while the problem of studying SCS in their full generality is first approached by the authors\textsuperscript{11,12}, with a specific attention to the phase space properties of the SCS system, and by Nieto and co-workers\textsuperscript{13}, whose examples fulfill the requirement of minimum quantum uncertainty.

The purpose of the present paper is to define SCS associated to a general Lie superalgebra, namely to a graded algebra of even (bosonic) and odd (fermionic) operators that satisfy certain commutation and anticommutation rules. Several quantum mechanical systems are described by a second-quantized hamiltonian that lives in a dynamical algebra which is in fact a Lie superalgebra. The (super-) algebra $\mathcal{A}$ is said to be a dynamical (super-) algebra for the hamiltonian $H$ if the operators that appear in $H$ close by commutation (and/or anticommutation) on a set of generators of $\mathcal{A}$.

Dynamical superalgebras enter quantum mechanics in different ways. A
very interesting instance is the fermionic linearization of many fermion systems second quantized Hamiltonians. A product of two fermionic operators \( A \) and \( B \) describing different modes of the system can be written, under the condition \( (A - \langle A \rangle)(B - \langle B \rangle) \approx 0 \), in the form

\[
AB \approx \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle
\]  

(1.1)

that can be expected to be fulfilled in quasiclassical states, as SCS must be. Since \( A \) and \( B \) are supposed to belong to different modes, they must satisfy \( \{A, B\} = 0 \), and, for the consistency of eq. (1.1), \( \langle A \rangle \) and \( \langle B \rangle \) (which could for example be the expectation values self consistently evaluated in the ground state) must be taken as odd elements of a Grassmann algebra \( \Lambda(\alpha\text{-numbers}) \), anticommuting among themselves and with the fermionic operators. The requirement of anticommutation between fermionic operators and \( \alpha\text{-numbers} \) can be relaxed (and in fact we relax it) by writing eq. (1.1) in the form

\[
AB \approx \langle A \rangle B - \langle B \rangle A - \langle A \rangle \langle B \rangle, \quad \langle A \rangle, \langle B \rangle \in \Lambda.
\]

The paper is organized as follows. In Ch. 2 we review the theory of GCS, give the basic definitions for SCS mimicking those of GCS and study their algebraic properties as well as the geometrical structure of their supermanifold. In Ch. 3 we examine some properties of the coadjoint orbits of Lie supergroups. The main objectives are to determine whether the supermanifold is Kähler (and then can be interpreted as a classical phase space) or not and to establish the connection between SCS supermanifolds and coadjoint orbits of the dynamical supergroup. We believe that the coadjoint orbits approach should be a powerful tool to study all the SCS systems associated to a certain dynamical superalgebra, especially when the latter is nilpotent.

In Ch. 4 some examples are studied, in which the dynamical superalgebra is a compact form of a contragradient Lie superalgebra, and, finally, in Ch. 5 we summarize our results and propose a conjecture concerning the realizability of SCS systems whose manifolds are endowed with Kählerian structure.
2. GENERALIZED COHERENT STATES
FOR DYNAMICAL SUPERALGEBRAS

2.1 Definitions. General properties

First of all, we shall briefly review the fundamental definitions and properties of the theory of generalized coherent states (GCS) for ordinary (dynamical) Lie algebras\(^5,15\).

Let \( \mathcal{G} \) be the dynamical algebra of our model, \( G \) an element of the class of the corresponding Lie groups and \( T \) a unitary irreducible representation (UIR) of \( G \) in a Hilbert space \( V \).

We assume that in the representation space \( V \) there exists a fixed cyclic vector, that we shall denote by \( |\psi_0\rangle \), and call \( H \) the set of elements \( h \in G \) such that

\[
T(h)|\psi_0\rangle = e^{i\alpha(h)}|\psi_0\rangle, \quad \alpha : H \to \mathbb{R}.
\]

It is easy to verify that \( H \) is a subgroup of \( G \), which will be called isotropy subgroup of \( |\psi_0\rangle \) and that \( e^{i\alpha} \) must be a unitary character of \( H \).

Let \( \mathcal{M} = G/H \) be the left coset of \( G \) with respect to \( H \). We see that there is no arbitrariness in the choice of a particular group \( G \) among all those associated with the algebra \( \mathcal{G} \), because the manifold \( \mathcal{M} \) depends only on \( \mathcal{G} \).

Now we can define the coherent states of \( \mathcal{G} \) by means of a mapping from \( \mathcal{M} \) to \( V \), which associates to each \( x \in \mathcal{M} \) defined by the decomposition \( g = x \cdot h \), with \( g \in G \) and \( h \in H \), the state (up to a phase factor)

\[
|x\rangle = T(x)|\psi_0\rangle.
\]

Thus the coherent states are represented by the points of a manifold \( \mathcal{M} \) that will be called the coherent states manifold, on which \( G \) acts transitively by means of the left translation \( \circ : G \times \mathcal{M} \to \mathcal{M} \) defined by

\[
g \circ x = \pi \left( g \cdot \pi^{-1}(x) \right), \quad \forall g \in G, \ x \in \mathcal{M}, \quad (2.1)
\]

\( \pi \) being the natural projection of \( G \) on \( \mathcal{M} \), \( G \) being seen as a bundle over \( \mathcal{M} \). The transitivity of \( \mathcal{M} \) with respect to \( G \) reflects the first important algebraic
property of the system of states just defined: the action of the dynamical
group $G$ maps coherent states into other coherent states.

Another algebraic property is the resolution of identity, of fundamental
importance in the coherent states representation of the Feynman path
integral\textsuperscript{16}. Let $d\mu(x)$, $\forall x \in \mathcal{M}$ be a $G$-invariant measure on the coherent
states manifold (which can be deduced either from an invariant metric or,
in the simplest cases, from the invariant Haar measure on the group itself).
When the integral converges, the operator

$$B = \int d\mu(x)|x\rangle\langle x|$$  \hspace{1cm} (2.2)

commutes with all the operators of the representation $T$ (namely it is invariant with respect to $T$):

$$T(g)BT(g)^{-1} = B, \quad \forall g \in G, \quad \text{(2.3)}$$

and, by Schur’s lemma, is a multiple of the identity, $B = k\mathbb{I}$, hence

$$\frac{1}{k} \int d\mu(x)|x\rangle\langle x| = \mathbb{I}. \quad \text{(2.4)}$$

Notice that $k$ can be obtained from the relation

$$k = \langle y|B|y \rangle = \int d\mu(x)|\langle y|x \rangle|^2, \quad \text{(2.5)}$$

where $|y\rangle \in V$ is normalized.

A consequence of this property is the completeness of the system of co-
herent states: an arbitrary state $|\psi\rangle \in V$ can, in fact, be expanded in terms
of coherent states

$$|\psi\rangle = \frac{1}{k} \int d\mu(x)\psi(x)|x\rangle, \quad \psi(x) = \langle x|\psi \rangle. \quad \text{(2.6)}$$

The function $\psi(x)$ is a solution of the integral equation

$$\psi(x) = \frac{1}{k} \int d\mu(y)\langle x|y \rangle \psi(y), \quad \text{(2.7)}$$
where the kernel \( K(x, y) = \frac{1}{k} \langle x | y \rangle \) is self-reproducing:

\[
K(x, z) = \int d\mu(y) K(x, y) K(y, z).
\] (2.8)

Indeed, the system of coherent states is overcomplete. One of the features whereby this property can be derived is its cardinality, that is continuum.

Now, we consider instead of an ordinary Lie algebra \( \mathcal{G} \), a Lie superalgebra, that we shall still denote with \( \mathcal{G} \). A Lie superalgebra is a \( \mathbb{Z}_2 \)−graded space \( \mathcal{G} \), i.e. a vector space which is the direct sum of two vector subspaces \( \mathcal{G}_0 \) and \( \mathcal{G}_1 \), endowed with an algebraic structure denoted by \([ \cdot, \cdot ]\), obeying the axioms

\[
[a, b] = - (-1)^{\deg(a)\deg(b)} [b, a] \\
[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)} [b, [a, c]]
\]

where \( \deg \) is a linear mapping \( \deg : \mathcal{G} \to \mathbb{Z}_2 \) called parity obeying the property

\[
\deg : a \mapsto \alpha, \quad \forall a \in \mathcal{G}_\alpha, \quad \alpha \in \mathbb{Z}_2.
\]

We shall use in the following the definition

\[
[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba.
\]

The group associated to a Lie superalgebra by exponential mapping is a Lie supergroup\textsuperscript{17,18}, somewhat similar to a Lie group with parameters that take values in the even or odd (depending on the parity of the corresponding generator in the Lie superalgebra) subspace of a Grassmann algebra \( \Lambda \).

Once more, we consider a UIR of the supergroup \( G \) in a \( \mathbb{Z}_2 \)-graded Hilbert space \( V \), fix a cyclic vector \( |\psi_0\rangle \) and proceed as before. The left coset space \( \mathcal{M} = G/H \) will be a supermanifold\textsuperscript{17} (that is, a manifold with even and odd coordinates), and coherent states (now SCS) may be defined in the usual way. However, two points are worthy of enlightenment about the properties previously demonstrated for GCS. The integration in (2.2)-(2.8) must be intended as an ordinary integration for what concerns even variables, and as a Berezin integration\textsuperscript{19} for what concerns odd variables. We recall that if \( \eta \) is
a variable taking values in the odd subspace of a Grassmann algebra, Berezin integral is defined by

$$\int d\eta = 0 \quad \int \eta \, d\eta = 1.$$  

This is a formal definition that can be interpreted as a definite integral over the whole domain of $\eta$ but does not involve any concept of measure theory, thus we do not have notions of indefinite integral, or of integrals over subdomains. We will come back later to this problem, when talking about Berry’s phase.

In the same way, the invariant measure $d\mu(x)$ must be intended as invariant with respect to the integration just defined, and it will not therefore be a measure function, at least for its odd part. Given the $G$-invariant metrics $g$ that we will construct in all our examples, the invariant measure can be obtained from the relation

$$d\mu(x) = |\text{sdet}[g(x)]|^{1/2}, \quad (2.9)$$  

where sdet denotes the superdeterminant.\(^{(a)}\) As for the resolution of identity, eq. (2.3) is still valid, but it is not sufficient to ensure the validity of (2.4), because Schur’s lemma generalizes to the graded case with two alternatives. $B$ can be, in fact, either a multiple of the identity or an operator that permutes the homogeneous subspaces of $V$, when they have the same dimension. In present case, however, $B$ is even, and the second possibility has to be excluded.

### 2.2 The case of contragradient Lie superalgebras

In this section contragradient Lie superalgebras are to be intended in the sense of Ref. 20. They are Lie superalgebras for which a Cartan-Weyl basis

\(^{(a)}\) We recall that given a $(p+q) \times (p+q)$ supermatrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $A$ and $D$ respectively $p \times p$ and $q \times q$ matrices with even entries, and $B$ and $C$ $p \times q$ and $q \times p$ matrices with odd entries, one defines the superdeterminant $\text{sdet}M = \det(A - BD^{-1}C)\det D^{-1}$ and the supertrace $\text{str}M = \text{tr}A - \text{tr}D$. 
can be defined. They are particularly interesting in present context, and we restrict in this section our attention to them, because of the features of their irreducible representations theory, that are very similar to those of semisimple Lie algebras.

Actually, our examples will deal with compact forms of:

a) contragradient Lie superalgebras modulo their centers,

b) Cartan extensions of contragradient Lie superalgebras.

The most important result established by Kac in this sense is that the irreducible representations of these superalgebras admit a highest weight vector in the representation space. In order to clarify why this is important for our purposes we must go back to the theory of GCS for ordinary Lie algebras. GCS are actually "coherent", that is, they are states of minimum quantum uncertainty: in fact, it was shown in Ref. 21 that GCS for compact semisimple Lie algebras minimize the variance of the quadratic Casimir operator, which is the simplest invariant estimate of quantum uncertainty, when they are constructed choosing as base the highest weight vector $|\psi_0\rangle$ of the representation $T$. In the book by Perelomov\textsuperscript{22} this result is extended to other algebras (Weyl-Heisenberg, $su(1,1)$) by requiring that $|\psi_0\rangle$ is chosen in such a way that its isotropy subalgebra is maximal. The isotropy subalgebra $\mathcal{B}$ of a vector $|\psi_0\rangle$ is the set of elements $t$ of the complex extension $\mathcal{G}^c = \mathcal{G} \oplus i\mathcal{G}$ which satisfy

\[ t|\psi_0\rangle = \lambda_t |\psi_0\rangle, \quad \lambda_t \in \mathcal{C}, \]

and is said to be maximal when $\mathcal{B} \oplus \bar{\mathcal{B}} = \mathcal{G}^c$, where $\bar{\mathcal{B}}$ is the algebra of the hermitian conjugates of the elements belonging to $\mathcal{B}$. If $\mathcal{G}$ is semisimple and $|\psi_0\rangle$ is the highest weight vector in representation space, then $\mathcal{B}$ is the subalgebra of $\mathcal{G}^c$ generated by the Cartan and the raising operators

\[ \mathcal{B} = \mathcal{H} \oplus \sum_{\alpha \in \Delta^+} \mathcal{G}_\alpha, \quad (2.10) \]

for which, of course, $\mathcal{B} \oplus \bar{\mathcal{B}} = \mathcal{G}^c$.

The relation (2.10) still holds for contragradient Lie superalgebras (now raising operators can be either bosonic or fermionic; slight modifications are
to be made to include the case of degenerate representations, see Sec. 2.4) thus we propose to define SCS for contragradient Lie superalgebras using the highest weight vector as base vector $|\psi_0\rangle$.

Besides, we must observe that this no longer implies ”coherence”, in the sense of Delbourgo $^{21}$, because his proof relies on definite positivity of the quadratic Casimir operator, which no longer holds in general in the case of Lie superalgebras.

For a real semisimple Lie algebra the algebra of the isotropy subgroup $H$ (that is not the isotropy subalgebra $B$) is just the Cartan subalgebra $\mathcal{H}$ if the representation is non-degenerate, but can be larger if the representation is degenerate. The same happens for dynamical superalgebras. There are non-degenerate representations in which the highest weight vector is annihilated only by the raising operators: in this case the algebra of $H$ is again the Cartan subalgebra, and then $H$ is an ordinary Lie group, the Cartan subgroup. This allows us to decompose the SCS supermanifold into the (in general semidirect) product $G/H = G/G_0 \otimes_s G_0/H$, where $G_0$ is the ordinary Lie group associated to the even subalgebra $G_0$. Notice that the factor $G/G_0$ is purely fermionic (it has only odd coordinates), while the factor $G_0/H$ is purely bosonic (it is just the GCS manifold of the Lie group $G_0$). A consequence of this property is that the invariant measure (2.9) can be written as a product of two functions, one depending only on the odd coordinates, and the other on the even ones. There are also degenerate representations (see the example of $su(2|1)$) in which the highest weight vector is annihilated also by some (bosonic and/or fermionic) lowering operators: then these operators, and their conjugates, belong to the algebra of $H$ and thus appear as generators of $H$, which may then become a true Lie supergroup.

2.3 Stability

The concept of stability of coherent states is intimately related to that of dynamical (super-) algebra associated to an hamiltonian. By virtue of the definition of dynamical (super-) algebra given in the Introduction, the time-evolution operator, that can be obtained for example by Magnus’ formula$^{23}$, is
an element of the exponential group \( G = \exp \mathcal{G} \) associated with the dynamical algebra (the dynamical group), that is, \( U(t, t_0) = \exp \gamma \), for some \( \gamma \in \mathcal{G} \) and, by definition of coherent states, it will be (from now on, we shall write group and algebra elements instead of their representatives)

\[
U(t, t_0)|z_0\rangle = \exp \gamma \cdot \exp Z_0|\psi_0\rangle = \exp Z|\psi_0\rangle = e^{i\phi}|z\rangle,
\]
with \( |z_0\rangle \) and \( |z\rangle \) coherent states and \( Z_0, Z \in \mathcal{G} \).

As a consequence, a system that has been prepared in a coherent state \( |z_0\rangle \) at the time \( t_0 \) will be found in any other time in a coherent state \( |z\rangle \) still.

Actually, if \( \mathcal{G} \) is not semisimple, the ”coherence-preserving” hamiltonians may live in a larger algebra \( \mathcal{S} \) in the universal enveloping algebra of \( \mathcal{G} \), that contains \( \mathcal{G} \) as an ideal\(^\text{24}\). If we are dealing with ordinary Lie algebras (but we assume this result to be valid also for Lie superalgebras) \( \mathcal{S} \) is the algebra of the automorphism group of \( \mathcal{G} \), and when \( \mathcal{G} \) is solvable \( \mathcal{S} \) is the algebra with Levi decomposition \( \mathcal{S} = \mathcal{A} \oplus \mathcal{G} \), with \( \mathcal{G} \) maximal solvable ideal of \( \mathcal{S} \).

### 2.4 Coherent states supermanifold

Ordinary GCS are parametrized by the points of a complex manifold \( \mathcal{M} \) (e.g. the complex plane for Glauber coherent states, the 2-dimensional sphere for SU(2)-coherent states). It is very interesting to ask whether this manifold is a \( G \)-homogeneous Kähler manifold\(^\text{25}\) or not (the answer for both Glauber and SU(2) coherent states is positive). This problem has been completely solved for compact semisimple Lie algebras in Ref. \(^\text{26}\).

If it is, there exists on \( \mathcal{M} \) a \( G \)-invariant metric \( g \) and a metric-preserving complex structure \( \mathcal{J} \), namely there exists \( \mathcal{J} : T_m \mathcal{M} \to T_m \mathcal{M} \), where \( T_m \) is the space tangent to \( \mathcal{M} \) in the point \( m \), such that

\[
\mathcal{J}^2 = -\mathbb{I}, \quad g(\mathcal{J} x, \mathcal{J} y) = g(x, y) \quad \forall m \in \mathcal{M}, \forall x, y \in T_m \mathcal{M}.
\]

The metric and the complex structure can be used to define a two-form \( \omega \) by

\[
\omega(x, y) = g(\mathcal{J} x, y).
\]

The manifold \( \mathcal{M} \) is said to be Kähler if such two-form is closed (\( d\omega = 0 \)). Then \( \mathcal{M} \) is also symplectic and can be viewed as a classical phase space.
for our dynamical system. This is easy exemplified in the case of Glauber coherent states by the identification

\[ a = \frac{q + ip}{\sqrt{2\hbar}}, \quad a^\dag = \frac{q - ip}{\sqrt{2\hbar}}. \]

Let us see how coherent states allow us to map a quantum-mechanical problem into a classical dynamical one on more general grounds. First of all we recall the notion of Kähler potential. Let us introduce in \( \mathcal{M} \) an atlas of local coordinate system \( \{ z^i, \bar{z}^i \} \) such that \( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i} \) are eigenvectors of \( J \):

\[
J \frac{\partial}{\partial z^i} = i \frac{\partial}{\partial z^i}, \quad J \frac{\partial}{\partial \bar{z}^i} = -i \frac{\partial}{\partial \bar{z}^i}. \tag{2.12}
\]

Then we define the components of the metric

\[ g_{ji} = g \left( \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^i} \right), \]

while

\[ g \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right) = g \left( \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j} \right) = 0 \]

because of (2.11) and (2.12). When the two-form \( \omega \) is closed (i.e. when the manifold is Kähler) there exists a function \( K(z^i, \bar{z}^i) \) from which the metric can be obtained by derivation

\[ g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}. \]

The function \( K \) is just the Kähler potential.

The Kähler potential plays a crucial role in the dynamics on the coherent states manifold. We have already showed that coherent states are defined in such a way that the time-evolution operator maps a coherent state into another one: it follows that (up to a phase factor) all the dynamical problem is reduced to the determination of a path in the coherent states manifold, and the system is described by a state

\[ |\psi(t)\rangle = e^{i\alpha(t)} |z(t)\rangle \]
at any time \( t \).

The phase factor \( e^{i\alpha(t)} \) is indeed very important; the phase \( \alpha(t) \) is but the effective action that appears in a functional integral approach\(^{27}\) and its form is as follows\(^{28}\)

\[
\alpha(t) = -\int_0^t h(z(\tau), \bar{z}(\tau))d\tau + \text{Im} \int_0^t \frac{\partial K}{\partial \bar{z}} dz,
\]

where \( h(z, \bar{z}) = \langle z|\mathcal{H}|z \rangle \), \( \mathcal{H} \) denoting the hamiltonian operator, and \( K \equiv K(z, \bar{z}) \) is the Kähler potential of \( \mathcal{M} \). The second term is a geometrical one and can be viewed as a Berry’s phase\(^{29}\). It has been conjectured\(^{30}\) that in the context of the models for High-\( T_c \) superconductivity, this term could represent the element bridging Hubbard-like hamiltonians\(^{31}\) with Chern-Simons field theories\(^{32}\), and therefore possibly with anyons.

We have now an additional problem with respect to the customary Lie algebra case. In fact the second integral is not at all well-defined. This is why it is conjectured that Berezin integration is a too formal one for our purposes and the field of a-numbers might be different from pure Grassmann\(^{33}\).

Let us now return to the problem of dynamical superalgebras and study the SCS supermanifold that arises in this case.

If we restrict our attention to contragradient Lie superalgebras we can still say that the SCS supermanifold is complex, because SCS are of the form

\[
|x\rangle = \exp \left\{ \sum_{\alpha \in \Delta_0^+} (z^\alpha E_{\alpha} - z^{\alpha*} E_{-\alpha}) + \sum_{\alpha \in \Delta_1^+ \text{mod} h_1} (\zeta^\alpha E_{\alpha} - \zeta^{\alpha*} E_{-\alpha}) \right\} |\psi_0\rangle,
\]

(2.13)

where \( \Delta_0^+ \) is the set of positive roots of grade \( \alpha \), \( \alpha \in \mathbb{Z}_2 \); \( h_1 \) is the set of positive fermionic roots that enter the isotropy subalgebra of \( |\psi_0\rangle \) when the representation is degenerate (see Sec. 2.2); \( E_{-\alpha} = E_{\alpha}^\dagger \), with \( E_{\alpha} \) ladder operator corresponding to the root \( \alpha \); \( z^\alpha \in \Lambda_0 \) and \( \zeta^\alpha \in \Lambda_1 \) (\( \Lambda_0 \) and \( \Lambda_1 \) denote respectively even and odd subspaces of a Grassmann algebra \( \Lambda \) over the complex field), and \( * \) is the ordinary conjugation in \( \Lambda \) that satisfies:

\[
\text{deg}(\lambda^*) = \text{deg}(\lambda), \quad (\lambda^*)^* = \lambda, \quad (\lambda \mu)^* = \mu^* \lambda^*.
\]
In the following, we shall denote by \( \mathbb{R}_c (\mathbb{R}_a) \) the set of elements \( \lambda \in \Lambda_0 (\Lambda_1) \) for which \( \lambda^* = \lambda \). There will be a sign and notation change (see Sec. 4.3) when the representation is a grade star\(^{34}\) one.

The relevant question is again whether \( \mathcal{M} \) is a Kähler supermanifold or not.

In Ch. 4 we shall examine four examples, and derive grounds for a conjecture, presented in the Conclusions that applies to a general theory. In the examples we shall explicitly realize the following general scheme \(^{35}\). Let \( \mathcal{G} \) denote the dynamical superalgebra, \( \mathcal{H} \) the algebra of the isotropy subgroup of \( |\psi_0\rangle \) and \( \mathcal{M} \) the complementary subspace of \( \mathcal{H} \) in \( \mathcal{G} \) (this is not an algebra)

\[
\mathcal{G} = \mathcal{H} \oplus \mathcal{M}.
\]

Introducing in \( \mathcal{G} \) a basis of homogeneous generators \( \{X_\alpha\} \) such that \( \mathcal{M} \) is generated by a subset \( \{X_\mu\} \), the coset representatives will be written as

\[
x = \exp \left\{ \sum_{\mu \in \mathcal{M}} x^\mu X_\mu \right\},
\]

where the coordinates \( x^\mu \) of \( x \) will be c-numbers or a-numbers (that is, even or odd elements of the Grassmann algebra) depending on the parity of \( X_\mu \).

In order to construct a \( G \)-invariant metric on \( \mathcal{M} \) we determine the action of the group \( G \) on the manifold \( \mathcal{M} \) by means of the equation

\[
\delta g \cdot x = x' \cdot \delta h, \quad (2.14)
\]

where

\[
\delta g = \exp \left\{ \sum_{X_\alpha \in \mathcal{G}} g^\alpha X_\alpha dt \right\}
\]

is a group element close to the identity, \( x \) (given by the (2.13)) and

\[
x' = \exp \left\{ \sum_{X_\mu \in \mathcal{M}} (x^\mu + dx^\mu) X_\mu \right\}
\]
are coset representatives, and
\[
\delta h = \exp \left\{ \sum_{X_\alpha \in \mathcal{H}} dh_\alpha X_\alpha \right\}
\]
is an element of the isotropy subgroup close to the identity.

Notice that equation (2.14) is but the infinitesimal version of (2.1). It is the matrix form of a linear system in the \(\{dx^\mu, X_\mu \in \mathcal{M}\}\)'s whose solutions have the form
\[
dx^\mu = \sum_{X_\alpha \in \mathcal{G}} g^\alpha \sigma^\mu_\alpha(x) dt, \quad X_\mu \in \mathcal{M},
\]
or, in vector notation,
\[
dx = g^\alpha \sigma_\alpha dt.
\]
A \(G\)-invariant metric must have the \(\sigma_\alpha\)'s as Killing vector fields (generators of isometries):
\[
\ell_{\sigma_\alpha} g = 0 \quad \forall X_\alpha \in \mathcal{G}
\]
\((g\) denotes the metric and \(\ell\) Lie derivation) and these equations can be solved if the superalgebra \(\mathcal{G}\) admits a nondegenerate, invariant and supersymmetric bilinear form \(\gamma\). The properties of invariance and supersymmetry are fulfilled by the Cartan-Killing form \(\gamma_{CK}(X,Y) = \text{str}(\text{ad}X\text{ad}Y)\), with \((\text{ad}X)(Y) = [X,Y]\) where the symbol \([,]\) is the graded commutator, that can be identified with \(\gamma\) when it is nondegenerate. In other cases one must construct alternative forms.

The metric \(g\) must then be taken equal to \(\gamma\) at some fixed point (e.g. \(x^\mu = 0, \forall X_\mu \in \mathcal{M}\)) and ”moved” to any other point by means of the \(\sigma_\alpha\)'s.

The inverse of the metric so obtained is given by
\[
g^{\mu\nu}(x) = (-)^{\alpha(\mu+1)} \sigma^{\mu}_\alpha(x) \sigma^{\nu}_\beta(x) \gamma^{\alpha\beta}, \quad (2.15)
\]
where \((-)^{\alpha(\mu+1)}\) is a shorthand for \((-1)^{\deg(X_\alpha)[\deg(X_\mu)+1]}\).

So far we have obtained a \(G\)-invariant metric on the complex supermanifold \(\mathcal{M}\), but we still cannot say whether the supermanifold is Kähler or not. The answer to this question requires the determination of the complex structures \(\mathcal{J}\) and their eigenvector basis in \(T_m\mathcal{M}\). This (difficult) passage gives
rise to very cumbersome forms, and so (with the exception of the example of 
u(1|1), in which the metric given by (2.15) is already Kähler) we shall not give it in explicit detail in the examples with a larger dynamical superalgebra (su(2|1) and uosp(1|2)).

3. COADJOINT ORBITS APPROACH TO COHERENT STATES PROBLEM.

3.1 A review of the ordinary formulation.

In this section we shall describe the main tool of the coadjoint orbits theory, following the original formulation of Kirillov\textsuperscript{36,37} and demonstrate the connection with coherent states problem for nilpotent Lie groups, according to the approach proposed by Moscovici\textsuperscript{38}.

Let $\mathcal{G}$ be the ordinary finite dimensional Lie algebra. Since $\mathcal{G}$ is a vector space, we can construct on $\mathcal{G}$ the space $\mathcal{G}^\ast$ of exterior 1-forms, which is isomorphic to $\mathcal{G}$.

Let $G$ be the corresponding Lie group. We call the coadjoint representation of $G$, the representation of the group that we can construct on $\mathcal{G}^\ast$, by means of the relation

$$a \in \mathcal{G}, \quad K(g)a = gag^{-1} \quad \forall g \in G. \quad (3.1)$$

We can now introduce the concept of coadjoint orbit of $G$ (K-orbit). For a fixed element $x$ in $\mathcal{G}^\ast$, K-orbit of $x$ is the set

$$\mathcal{O}_x = \{ y \in \mathcal{G}^\ast \mid y = gxg^{-1} \}. \quad (3.2)$$

Let us suppose that there exists a subset $H$ of $G$, such that

$$hxh^{-1} = x \quad \forall h \in H. \quad (3.3)$$

One can easily prove that this set is a subgroup of $G$, that we call stability subgroup of the point $x$. So the orbit $\mathcal{O}_x$ of the point $x$ is isomorphic to the coset space $\mathcal{M} = G/H$. 

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The properties of this construction, that are relevant for our investigations, are listed below.

a) All the K-orbits of $G^*$, i.e. the orbits of all elements of $G^*$, are homogeneous G-spaces. Namely, the action of the group $G$ by means of $K(g)$ maps a point of a fixed orbit $O_x$ into another point of the same orbit.

b) Two different K-orbits are disjoint, i.e. we cannot connect a point of an orbit with another point of another distinct orbit by means of a coadjoint representation.

c) All the K-orbits are symplectic manifold of even dimension. One can, in fact, define on every orbit an exterior differential 2-form $\omega_f$ which is closed, skewsymmetric, nondegenerate. To this purpose, let us consider the mapping

$$\partial_x \leftrightarrow x$$

that induces a correspondence between the tangent space to $O_x$ and some subspace of $G^*$. Then for every point $f$ on $O_x$ we can define a differential 2-form through the relation

$$\omega_f(x,y) = f([x, y]), \forall x, y \in G \ \forall f \in G^*.$$  (3.5)

This form is obviously skewsymmetric, closed because of Jacobi identity, non-degenerate on the factor space $G/\mathcal{H}$, where $\mathcal{H}$ is the Lie algebra of the stabilizer of $f$, isomorphic to the tangent space to $O_x$ at point $f$.

Starting from the 2-form (3.5) we can then easily construct a Poisson bracket on $O_x$ through the relation

$$\{\varphi_1, \varphi_2\} = (\omega_f)^{ik}\partial_i\varphi_1\partial_k\varphi_2.$$  (3.6)

For a vast class of Lie algebras, all the coadjoint orbits are Kähler manifold. We can, in that case, introduce on $O_x$ a complex structure $\mathbb{J}$

$$\mathbb{J} : O_x \to O_x, \quad \mathbb{J}^2 = -\mathbb{I}$$  (3.7)

and construct a metric $g$ from $\omega_f$

$$g(x, \mathbb{J}y) = \omega_f(x, y), \quad g(\mathbb{J}x, \mathbb{J}y) = g(x, y).$$  (3.8)
By means of the relations (3.5) and (3.8) we can define on the orbit a hermitian scalar product \((x, y)\):

\[(x, y) = g(x, y) + i\omega_f(x, y).\]  (3.9)

(3.9) allows us to obtain the dual \(x^\star\) of an orbit point \(x\) with respect to the scalar product \((,\) and to give a resolution of the identity (when this is possible) through the relation (2.2). Furthermore, in correspondence with every orbit we can construct an irreducible unitary representation of the dynamical group. In fact if we consider the maximal admissible subalgebra subordinate \(^{(a)}\) to any point of the orbit we can obviously construct a 1-dimensional representation of the corresponding subgroup of \(G\). By means of Mackey induction\(^{40,41}\), from this representation we can recover a representation of \(G\). It has been shown by Kirillov\(^{36,37}\), Kostant and Auslander\(^{42,43}\), that for a vast class of Lie groups this is the way to obtain all irreducible unitary representations. In particular, for the class of the nilpotent Lie algebras, one can show, in a fairly straightforward way, that all coadjoint orbits that are linear manifolds, are coherent states manifolds for the dynamical Lie algebra as well\(^{38}\). It is expected that it should be possible to classify all coherent states manifolds for a wider class of dynamical algebras by means of the coadjoint orbits method.

### 3.2 A possible extension to the superalgebraic case.

We look now for a way of generalizing the orbits method to the super case, trying to obtain an algorithmic procedure which gives us all possible supersymplectic (Kähler) mechanics and all possible UIR related to our dynamical superalgebra.

The first step is to introduce a correct generalization of the concept of space of exterior 1-forms on the algebra.

\(^{(a)}\) We recall that a subalgebra \(N \subseteq \mathcal{G}\) is subordinate to the 1-form \(f\) if the form \(\omega_f\) vanishes identically on \(N\). A subordinate subalgebra is called admissible if \(\text{codim} N = \frac{1}{2} \dim \mathcal{O}_f\) and \(N^\perp + f \subset \mathcal{O}_f\), where \(N^\perp\) is the set of elements of \(\mathcal{G}^\star\), whose extensions over \(\mathcal{G}_c\) vanish identically on \(N\).
Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a finite dimensional Lie superalgebra (typically, the dynamical algebra for some physical model) with $\mathcal{G}_0$ ($\mathcal{G}_1$) even (odd) part. Let us consider in $\mathcal{G}$ a standard homogeneous base $(q_i, r_j)$ with $q_i$ ($r_j$) even (odd) generators. We can then define the space $\mathcal{G}^*$ of 1-forms on $\mathcal{G}$, which is isomorphic to $\mathcal{G}$. \(\mathcal{G}^*\) is a $\mathbb{Z}_2$-graded space, with the standard homogeneous base $(e_i, f_j)$ defined through the relations

\[
e_i(q_l) = \delta_{il}, \quad e_i(r_k) = 0
\]
\[
f_j(q_l) = 0, \quad f_j(r_k) = \delta_{jk}\]  \(3.10\)

Let $\Lambda = \Lambda_0 \oplus \Lambda_1$ be a finite or infinite dimensional Grassmann algebra, $\Lambda_0$ ($\Lambda_1$) being the even (odd) sector of $\Lambda$. We introduce the left $\Lambda_0$-module on $\mathcal{G}$, as the set of objects of the form

\[a_i q_i + \alpha_j r_j, \quad a_i \in \Lambda_0, \quad \alpha_j \in \Lambda_1.\]  \(3.11\)

We call space of exterior even 1-forms on $\mathcal{G}$, denoted $\mathcal{G}_\Lambda^*$, the set of elements

\[b_i e_i + \beta_j f_j, \quad b_i \in \Lambda_0, \quad \beta_j \in \Lambda_1.\]  \(3.12\)

We can identify $\mathcal{G}_\Lambda^*$ as the super analogue of the dual of the superspace $\mathcal{G}_\Lambda$, defining in $\mathcal{G}_\Lambda^*$ a standard homogeneous base $(E_i, F_j)$ by means of

\[
E_l(a_i q_i + \alpha_j r_j) = a_i e_l(q_i) + \alpha_j e_l(r_j) = a_l
\]
\[
F_k(a_i q_i + \alpha_j r_j) = a_i f_k(q_i) + \alpha_j f_k(r_j) = \alpha_k
\]  \(3.13\)

and a generic $\mathcal{G}_\Lambda^*$ element by

\[b_i E_i + \beta_j F_j.\]  \(3.15\)

There are good reasons to believe that $\mathcal{G}_\Lambda$ and $\mathcal{G}_\Lambda^*$ are isomorphic (see conjecture in section 3.3 for an intuitive proof). Then we can construct on the space $\mathcal{G}_\Lambda^* \simeq \mathcal{G}_\Lambda$ a representation $K_0(g)$ of supergroup $G$ corresponding to superalgebra $\mathcal{G}$, obtained by exponentiating $\mathcal{G}_\Lambda$, by means of

\[K_0(g)x = gxg^{-1},\]  \(3.16\)
and introduce the concept of $K_0(g)$-orbit

$$\mathcal{O}_x = \{ y \in \mathcal{G}^* \mid y = gxg^{-1} \},$$

(3.17)

in a way strictly analogous to the ordinary case.

Obviously, orbits constructed in this way are homogeneous $G$–spaces, transitive with respect to the representation $K_0(g)$ and pairwise disjoint. Implementing a symplectic construction in the super case is a non trivial task. We can, of course, define different generalization of the 2-form $\omega_f$. The method we adopt is to consider in every point $f$ of $\mathcal{O}_x$ the 2-form

$$\varpi_f = f([x,y]), \quad \forall x, y \in \mathcal{G}^*.$$

(3.18)

The form $\varpi_f$ is super skewsymmetric. Since $\mathcal{G}^*_\Lambda$ is a (formal) ordinary Lie algebra, the closure of the form $\varpi_f$ is guaranteed. In all examples $\varpi_f$ turns out to be nondegenerate as well. Then $\varpi_f$ is supersymplectic. From $\varpi_f$ we can recover superpoisson brackets by means of

$$\{ \varphi_1, \varphi_2 \} = \varphi_1 \overset{\to}{\partial}_i (\varpi_f)^{ik} \overset{\to}{\partial}_k \varphi_2$$

(3.19)

and, when the supermanifold is Kähler, a metric $g$ in the customary way.

We are finally able to identify, among all $K_0(g)$-orbits of a generic contragradient Lie superalgebra, the class of orbits which are isomorphic to the coherent states supermanifolds corresponding to the non degenerate representations of the same Lie superalgebra. This class of orbits is simply obtained acting by the $K_0(g)$-representation on elements of the Cartan subalgebra of the Lie superalgebra considered. Such orbits, isomorphic to coherent states supermanifolds are of the form

$$\mathcal{O}_h = \{ y \in \mathcal{G}_\Lambda \mid y = ghg^{-1} \} \quad h \in \mathcal{H}.$$  

(3.20)

This shows how, for non degenerate representations of contragradient Lie superalgebras, results obtained by means of generalized Rasetti-Perelomov method and coadjoint orbits method are in agreement.
3.3 Matrix formulation.

A (faithful) linear representation of a finite dimensional Lie superalgebra $G$ is an isomorphism, which maps $G$ into a finite dimensional matrix Lie superalgebra. Then every element $a \in G$ is a matrix. Let the matrix Lie superalgebra have dimension $(m, n)$ over the field $\mathbb{K}$. Every element $a$ is a block matrix with elements in $\mathbb{K}$, of the form

$$ a = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, $$

with $\dim A_{11} = m \times m$, $\dim A_{12} = m \times n$, $\dim A_{21} = n \times m$, $\dim A_{22} = n \times n$. We call $\text{Mat}(m|n)$ the set of matrices of this kind.

A generic element $\alpha$ of the $\Lambda_0$-module on $G$, denoted $G_\Lambda$, has the same form, but matrices $A_{11}$ and $A_{22}$ have elements in $\Lambda_0$, while $A_{11}$ and $A_{22}$ have elements in $\Lambda_1$. We denote this set of matrices by $\text{Mat}(m|n)_\Lambda$.

We can construct a matrix representation of the space of even 1-forms on $G$, by means of the mapping $\langle , \rangle$ defined through the relation

$$ \langle , \rangle : \text{Mat}(m|n)_\Lambda \times G \longrightarrow \Lambda_0 \\
\langle M_f, a \rangle = \text{str}(M_f \cdot a). $$(3.22)

The matrix $M_f$ corresponds to the even 1-form $f$, $f(a) = \langle M_f, a \rangle$. We conjecture that, at least for $m \neq n$, such correspondence is one-to-one in general.

**Conjecture:**

For all finite dimensional matrix Lie superalgebras, there exists a one to one correspondence between matrices of even 1-forms and elements of the superalgebra. This correspondence is defined by the mapping $\langle , \rangle$.

A plausibility argument for this conjecture is the following. A finite dimensional matrix Lie superalgebra is constructed by means of some commutation invariant relations on the matrix. We can impose the same relations on the set of matrices corresponding to all even 1-forms, which we construct through the mapping $\langle , \rangle$. Thus we can attribute this set, which is in one to one correspondence with even 1-forms space, the same superalgebra structure. Also for every superalgebra element there exists an even 1-form.
Now, we can construct $K_0(g)$-orbits for each superalgebra in this set, acting with $K_0(g)$ on the (faithful) matrix representation of the Lie superalgebra with the corresponding (faithful) matrix representation of the Lie supergroup, which we obtain from superalgebra by means of the exponential mapping. Classifying $K_0(g)$-orbits is thus the same as classifying classes of conjugate elements of the Lie superalgebra.

In order to classify $K_0(g)$-orbits, we classify first all subsuperalgebras of the superalgebra, and then we consider only those whose corresponding subsupergroup stabilizes a point of the space of the even 1-forms, i.e. of the superalgebra.

We finally construct the generalized Kirillov-Kostant 2-form in the same manner as in (3.18), where superalgebra elements are now matrices.

In the following section we describe a simple application as exemplification of the method just exposed.

### 3.4 Weyl-Heisenberg superalgebra.

Let us consider the five dimensional Lie superalgebra $\mathcal{W}$, with the set of generators

$$\{\mathbb{1}, a, a^\dagger, b, b^\dagger\}$$

and basic commutation relations

$$[a, a^\dagger] = \{b, b^\dagger\} = \mathbb{1}$$

$$[a, \mathbb{1}] = [b, \mathbb{1}] = [a, b] = [a^\dagger, b] = 0.$$  

We can introduce a new set of generators

$$e_1 = \frac{1}{\sqrt{2}}(a^\dagger + a), \quad e_2 = \frac{i}{\sqrt{2}}(a^\dagger - a), \quad e_3 = i\mathbb{1}, \quad e_4 = b, \quad e_5 = ib^\dagger.$$  

In this basis, a faithful five-dimensional matrix representation for a generic element in $\mathcal{W}_\Lambda$ is

$$x = \begin{pmatrix} 0 & x_1 & x_3 & \alpha_1 & \alpha_2 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & 0 \end{pmatrix} \quad x_i \in \mathbb{R}_c, \quad \alpha_j \in \mathbb{R}_a.$$
The matrix \( M_f \) of generic exterior even 1-form \( f \) on \( W \) takes the form

\[
M_f = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
y_1 & 0 & 0 & 0 & 0 \\
y_3 & y_2 & 0 & 0 & 0 \\
\delta_1 & 0 & 0 & 0 & 0 \\
\delta_2 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad y_i \in \mathbb{R}_c, \quad \delta_j \in \mathbb{R}_a.
\]

Exponentiating the generic element \( w \) of \( W_\Lambda \), we can obtain a generic element \( g \) of the corresponding supergroup \( W_\Lambda \). By utilizing Berezin’s decomposition \( g = g_e \circ g_o \), we recover \( g_e \) (\( g_o \)) by exponentiating the even (odd) part of \( w \).

\[
g = g_e \circ g_o = \begin{pmatrix}
1 & x_1 & x_3 + \frac{1}{2} x_1 x_3 & 0 & 0 \\
0 & 1 & x_2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 & \alpha_1 & \alpha_2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & \alpha_2 & 1 & 0 & 0 \\
0 & \alpha_1 & 0 & 1 & 0 \\
\end{pmatrix}
\]

whose inverse is

\[
g^{-1} = \begin{pmatrix}
1 & -x_1 & -x_3 + \frac{1}{2} x_1 x_2 & -\alpha_1 & -\alpha_2 \\
0 & 1 & -x_2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\alpha_2 & 1 & 0 \\
0 & 0 & -\alpha_1 & 0 & 1 \\
\end{pmatrix}.
\]

The \( K_0(g) \)-orbit in generic position has thus matrix form

\[
O_f = P\{g_o^{-1}g_e^{-1}fg_ego\} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
y_1 + x_2 y_3 & 0 & 0 & 0 & 0 \\
y_3 & y_2 - x_1 y_3 & 0 & 0 & 0 \\
\delta_1 + \alpha_2 y_3 & 0 & 0 & 0 & 0 \\
\delta_2 + \alpha_1 y_3 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

where \( P \) is the projection from Mat\((3|2)_\Lambda\) onto \( W_\Lambda^* \). \( K_0(g) \)-representation maps coordinates of the point \( f \) into points

\[
y_1 \rightarrow y_1 + x_2 y_3 \\
y_2 \rightarrow y_2 - x_1 y_3 \\
y_3 \rightarrow y_3 \\
\delta_1 \rightarrow \delta_1 + \alpha_2 y_3 \\
\delta_2 \rightarrow \delta_2 + \alpha_1 y_3.
\]
There are obviously only two classes of orbits, in analogy with the ordinary
Weyl-Heisenberg algebra case:

a) Orbits of the points \( f_0 = \{y_1, y_2, 0 \mid \delta_1, \delta_2\} \) that are points in \( \mathcal{W}^* \)
labelled by their coordinate set. The corresponding supermanifold is trivial. The representations \( T^{y_1, y_2, \delta_1, \delta_2} \) constructed from orbits belonging to this
class are 1-dimensional UIR. In fact, the maximal admissible subsuperalgebra
subordinate to the 1-form \( f_0 \) is the whole superalgebra \( \mathcal{W} \). We obtain

\[ T^{y_1, y_2, \delta_1, \delta_2} = e^{i(y_1 x_1 + y_2 x_2 + \delta_1 \alpha_1 + \delta_2 \alpha_2)}. \]

b) Orbits of the points \( f_1 \), which have \( y_3 \neq 0 \) are four dimensional super
planes labelled by the value of this coordinate. On this class of orbits, the
matrix of the generalized supersymplectic 2-form \( \varpi_f \) in the \( \{x_1, x_2, \alpha_1, \alpha_2\} \)
basis is

\[ \varpi_f = y_3 \cdot \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}. \]

\( \varpi_f \) is closed and nondegenerate. We can define on the orbit a complex
structure \( \mathbb{J} \) of type (3.7) by means of the relations

\[ \mathbb{J} : x_1 \mapsto -x_2, \quad \mathbb{J} : x_2 \mapsto x_1 \\
\mathbb{J} : \alpha_1 \mapsto -\alpha_2, \quad \mathbb{J} : \alpha_2 \mapsto \alpha_1. \]

From the latter we can construct on the orbit a metric tensor as in (3.8), by
writing first the Kähler potential

\[ K = y_3(x_1 x_2 + \alpha_1 \alpha_2) \]

whereby the metric is recovered by the relation

\[ g_{ij} = \frac{\partial K}{\partial z_i \partial \bar{z}_j}, \]

with \( z_i, z_j \) in \( \{x_1, x_2, \alpha_1, \alpha_2\} \). The superpoisson brackets, constructed resorting
to (3.19), are

\[ \{\varphi_1, \varphi_2\}_s = \partial_a \varphi_1 \partial_b \varphi_2 - \partial_b \varphi_1 \partial_a \varphi_2 + \varphi_1 \partial_a \partial_\beta \varphi_2 + \varphi_1 \partial_\beta \partial_a \varphi_2. \]
The UIR’s of supergroup $W$ corresponding to this class of orbits, can be induced from the 1-dimensional representations

$$U^{y_3}(x_3) = e^{iy_3x_3}$$

of the 1-dimensional subsupergroup of $W$, generated by the center of Lie superalgebra $\mathcal{W}$, which is the maximal admissible subalgebra subordinate to the 1-form $f_1$.

These results are in concordance with those obtained in the ordinary way by Nieto$^{13}$.

4. EXAMPLES

4.1 The simplest case: $u(1|1)$

$l(1|1)$ is the superalgebra of $(1 + 1) \times (1 + 1)$ matrices, with

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

as even generators,

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

as odd generators, and commutation-anticommutation relations

$$[H, E_\pm] = \pm E_\pm \quad \{E_+, E_-\} = C;$$

all other commutators-anticommutators being equal to zero.

$l(1|1)$ can be viewed as the contragradient Lie superalgebra of rank 1 and null Cartan matrix, extended by the Cartan element $H$, and can be realized in terms of fermionic creation-annihilation operators, by means of the correspondence

$$C \mapsto \mathbb{1}, \quad H \mapsto a^\dagger a, \quad E_+ \mapsto a^\dagger, \quad E_- \mapsto a.$$
Coherent states supermanifold

The defining representation is equivalent to that in one-fermion Fock space, and has as its highest weight vector

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

$u(1|1)$ is the compact form of $l(1|1)$ defined by

$$u(1|1) = \{ M = i\phi_0 C + i\phi H + \eta E_+ - \eta^* E_- | \phi_0, \phi \in \mathbb{R}_c, \eta \in \Lambda_1 \},$$

where $\mathbb{R}_c = \{ \lambda \in \Lambda_0 | \lambda^* = \lambda \}$. The exponential group of matrices associated with $u(1|1)$, denoted by $U(1|1)$, is the group of matrices

$$g = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}; \quad a, d \in \Lambda_0, \beta, \gamma \in \Lambda_1$$

such that $g^+ g = g g^+ = I$, where the adjoint is defined by

$$g^+ = \begin{pmatrix} a^* & \gamma^* \\ \beta^* & d^* \end{pmatrix}.$$ 

Group elements can be written in the form $g = \xi \cdot u$, with $\xi = \exp(\eta E_+ - \eta^* E_-)$ and $u = \exp(i\phi_0 C + i\phi H)$. The isotropy subgroup is just $U(1) \otimes U(1)$, generated by the $u$'s, and the SCS supermanifold is the coset space $M = \frac{U(1|1)}{U(1) \otimes U(1)}$, parametrized by the coset representatives $x \equiv x(\eta, \eta^*) = \xi$.

We are thus able to apply the method described at the end of previous chapter: acting on the point $x \in M$ by the infinitesimal group element

$$\delta g = \exp[dt(i\gamma^0 C + i\gamma^1 H + \zeta E_+ - \zeta^* E_-)],$$

we obtain the variation of the coordinates on the supermanifold:

$$\frac{d\eta}{dt} = i\gamma^1 \eta + \zeta, \quad \frac{d\eta^*}{dt} = -i\gamma^1 \eta^* + \zeta^*.$$ 

The Cartan-Killing form on $l(1|1)$ is degenerate, so we define the nondegenerate form $\gamma_d$ by

$$\gamma_d(X, Y) = \text{str}(XY) \quad \forall X, Y \in l(1|1).$$
Finally, (2.15) gives $g^{\eta\eta^*} = -g^{\eta^*\eta} = 1$, from which we obtain the metric

$$g_{\eta^*\eta} = -g_{\eta\eta^*} = 1, \quad g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

This admits the natural complex structure defined by

$$J \left( \frac{\partial}{\partial \eta} \right) = i \frac{\partial}{\partial \eta}, \quad J \left( \frac{\partial}{\partial \eta^*} \right) = -i \frac{\partial}{\partial \eta^*}$$

that one can use to define the 2-form $\omega = -2i d\eta \wedge d\eta^*$, which is closed. Then $g$ is Kähler, with Kähler potential $K(\eta, \eta^*) = \eta \eta^*$, and induces the invariant measure $d\mu(x) \equiv 1$, as we could expect from the invariant measure $d\mu(g) \equiv 1$ on the $U(1|1)$ group manifold.

### Coadjoint orbits approach

Let us consider the superalgebra $u(1|1)$. A generic element of the left $\Lambda_0$-module over $u(1|1)$, that we denote $u(1|1)_{\Lambda}$, takes the matrix form

$$e = \begin{pmatrix} ia & \delta \\ -\delta^* & i d \end{pmatrix} \quad a, d \in \mathbb{R}_c, \quad \delta \in \Lambda_1.$$ 

Even though $m = n$, we can resort here to the conjecture exposed in section 3.3 giving to the space $u(1|1)_{\Lambda}^*$ of exterior even 1-forms on $u(1|1)_{\Lambda}$ the form

$$M_f = \begin{pmatrix} ix & \alpha \\ -\alpha^* & iy \end{pmatrix} \quad x, y \in \mathbb{R}_c, \quad \alpha \in \Lambda_1.$$ 

The orbit $\mathcal{O}_f$ of $f$, obtained from (3.17) takes then the matrix form

$$\mathcal{O}_f = \begin{pmatrix} i(1 - \delta\delta^*)x - \alpha\delta^* \exp(-iu) - i\delta(x - y) + \alpha \exp(-iu) \\ -\alpha^*\delta \exp(iu) + i \delta\delta^* y + i \delta^*(x - y) - \alpha^* \exp(iu) \end{pmatrix}$$

where $u = a - d$.

We can classify $u(1|1)_{\Lambda}$’s $K_0(g)$-orbits, by means of the method of sub-superalgebras exposed in section 3.4. As we conjectured above (section 3.3), since $u(1|1)$ is contragradient, orbits of elements of the kind

$$f_1 = \begin{pmatrix} ix & 0 \\ 0 & iy \end{pmatrix} \quad x \neq y \wedge x \neq 0 \wedge y \neq 0$$
are isomorphic to coherent states supermanifolds corresponding to nondegenerate representations of $u(1|1)\Lambda$. The subgroup that stabilizes points belonging to this class is the (ordinary) subgroup $U(1)\otimes U(1)$. The orbits are supermanifolds of dimension $(0,2)$. The generalized Kirillov-Kostant 2-form $\varpi_f$, which is nondegenerate, has matrix form

$$\varpi_f = i(x - y) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

We can introduce a complex structure of kind (3.7) by means of the relations

$$\mathcal{J} : \text{Re}(\delta) \mapsto -\text{Im}(\delta), \quad \mathcal{J} : \text{Im}(\delta) \mapsto \text{Re}(\delta)$$

and recover a metric tensor of the form (3.8), which admits a Kähler potential

$$K = (x - y)\delta\delta^*.$$ 

The superpoisson brackets, defined by (3.19), can be finally written as

$$\{\varphi_1, \varphi_2\} = \varphi_1 \partial_\delta \partial_{\delta^*} \varphi_2 + \varphi_1 \partial_{\delta^*} \partial_\delta \varphi_2$$

There are four other classes of orbits, which, however, are not relevant for our scope.

### 4.2 Another "ordinary unitary" example: $su(2|1)$

$su(2|1)^{44}$ is the superalgebra of $(2 + 1) \times (2 + 1)$ supermatrices with zero supertrace. It has even sector generated by $\{Q, J_3, J_+, J_-\}$ and odd sector generated by $\{W_+, W_-, V_+, V_-\}$.

These generators satisfy the following commutation rules:

$$[J_3, J_\pm] = \pm J_\pm \quad [J_+, J_-] = 2J_3$$

$$[Q, J_3] = [Q, J_\pm] = 0$$

$$[Q, V_\pm] = \frac{1}{2} V_\pm \quad [Q, W_\pm] = -\frac{1}{2} W_\pm$$

$$[J_3, V_\pm] = \pm \frac{1}{2} V_\pm \quad [J_3, W_\pm] = \pm \frac{1}{2} W_\pm$$

$$[J_\pm, V_\mp] = V_\pm \quad [J_\pm, W_\mp] = W_\pm$$

$$[J_\pm, V_\pm] = [J_\pm, W_\pm] = 0$$

$$\{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = 0$$

$$\{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0$$

$$\{V_\pm, W_\pm\} = \pm J_\pm \quad \{V_\pm, W_\mp\} = -J_3 \pm Q,$$
from which one can easily realize that the \( J \)'s are the generators of a Lie algebra \( su(2) \), whereas \( Q \) generates a commuting \( u(1) \).

Before defining \( su(2|1) \) (of which \( sl(2|1) \) is the complexification) we must summarize a few properties of the irreducible representations (IRREP) of \( sl(2|1) \).

These IRREP are labelled by two quantum numbers, \( j \) and \( q \) (referred to as spin and charge, respectively), which are defined by

\[
J^2|\psi_0\rangle = j(j+1)|\psi_0\rangle, \quad J_3|\psi_0\rangle = j|\psi_0\rangle, \quad Q|\psi_0\rangle = q|\psi_0\rangle,
\]

|\( \psi_0 \rangle \) denoting the highest weight vector, \( j \in \{0, \frac{1}{2}, 1, \ldots \} \) and \( q \in \mathbb{C} \). The representation space may contain up to 4 multiplets spanned by the \((Q, J^2, J_3)\) eigenvectors

\[
|q, j, m\rangle, \quad m = -j, -j + 1, \ldots, j \\
|q \pm \frac{1}{2}, j - \frac{1}{2}, m\rangle, \quad m = -j + \frac{1}{2}, -j + \frac{3}{2}, \ldots, j - \frac{1}{2} \\
|q, j - 1, m\rangle, \quad m = -j + 1, -j + 2, \ldots, j - 1.
\]

Apart from the trivial one-dimensional one, there are three series of IRREP:

a) a nondegenerate series, in which the highest weight vector is annihilated only by the raising operators \( J_+, V_+, W_+ \);

b) a degenerate series, in which \( W_- \) also annihilates \( |\psi_0\rangle \);

c) a second degenerate series, obtained from the previous one by means of the automorphism

\[
J_3 \mapsto J_3, \quad J_\pm \mapsto J_\pm, \quad Q \mapsto -Q, \quad V_\pm \mapsto W_\pm, \quad W_\pm \mapsto V_\pm.
\]

Finally, we can introduce in \( sl(2|1) \) the ”normal adjoint” operation \( ^\dagger \) defined by

\[
Q^\dagger = Q, \quad J_3^\dagger = J_3, \quad V_\pm^\dagger = \pm W_\mp, \quad W_\pm^\dagger = \mp V_\mp.
\]

It is the possibility of defining a ”normal adjoint” (that is, an involution with all the properties of Hermitian adjoint) that justifies the different approaches of present section and of the next one.
Turning now to the superalgebra $su(2|1)$, it is but the compact form of $sl(2|1)$ defined by

$$su(2|1) = \{ i\phi_0 Q + i\phi J_3 + zJ_+ - z^* J_- + \eta_1 W_+ + \eta_1^* V_+ + \eta_2 W_- - \eta_2^* V_- |$$

$$\phi_0, \phi \in \mathbb{R}_c, z \in \Lambda_0, \eta_k \in \Lambda_1 \}.$$

The exponential group associated with $su(2|1)$, denoted by $SU(2|1)$, is the group of $(2 + 1) \times (2 + 1)$ unitary, unimodular supermatrices.

**Coherent states supermanifold**

Looking at the IRREP’s theory of $sl(2|1)$ one can understand how SCS (and their supermanifold) depend on the choice of the IRREP of $sl(2|1)$ from which the UIR of $SU(2|1)$ is obtained. Taking an IRREP of the (a) series, the isotropy subgroup of $|\psi_0\rangle$ will be $U(1) \otimes U(1)$, generated by $Q$ and $J_3$. Taking an IRREP of the (b) series, the isotropy subgroup will be a true supergroup, $U(1|1)$, generated by $Q, J_3, V_+, W_-$.  

We study in detail the first case, constructing first an $SU(2|1)$-invariant metric on the SCS supermanifold $\mathcal{M} = \frac{SU(2|1)}{U(1) \otimes U(1)}$ and trying to establish whether $\mathcal{M}$ is Kähler or not.

The group elements can again be written in the form $g = \xi \cdot u$, where now $\xi = \exp(\eta_1 W_+ + \eta_1^* V_+ + \eta_2 W_- - \eta_2^* V_-)$ and $u = \exp(i\phi_0 Q + i\phi J_3 + zJ_+ - z^* J_-)$, and then the coset representatives are of the form $x = \xi \cdot v$, with $v = \exp(zJ_+ - z^* J_-)$.

Acting on the point $x \in \mathcal{M}$ by the infinitesimal group element

$$\delta g = I + \left( \sum_{i=0}^{3} g^i A_i + \zeta^1 W_+ + \zeta^1* V_+ + \zeta^2 W_- - \zeta^2* V_- \right) dt,$$

where $A_0 = iQ, A_1 = \frac{i}{2}(J_+ + J_-), A_2 = \frac{i}{2}(J_- - J_+), A_3 = iJ_3$, after the substitutions $-i \frac{z}{|z|} \tan|z| \mapsto z$ and

$$\tanh \sqrt{-BB^t} B \mapsto \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad B = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

we obtain (introducing the simplified notation $e_i = \eta_i^* \eta_i$) the variations of
the coordinates on the supermanifold
\[
\frac{dz}{dt} = ig^3 z + \frac{1}{2} g^1 (1 + z^2) + \frac{i}{2} g^2 (1 - z^2)
\]
\[
- \frac{1}{2} \left[ (\zeta^1 \eta_1 + \zeta^1 \eta_1^*) \left( 1 - \frac{1}{2} e_2 \right) - (\zeta^2 \eta_2 + \zeta^2 \eta_2^*) \left( 1 - \frac{1}{2} e_2 \right) \right] z
\]
\[
+ \frac{i}{2} \left[ (\zeta^1 \eta_2 + z^2 \zeta^1 \eta_2) \left( 1 - \frac{1}{2} e_1 \right) + (\zeta^2 \eta_1 + z^2 \zeta^2 \eta_1^*) \left( 1 - \frac{1}{2} e_2 \right) \right],
\]
\[
\frac{d\eta_1}{dt} = -\frac{i}{2} (g^0 - g^3) \eta_1 + \frac{i}{2} (g^1 + ig^2) \eta_2 + \zeta^1 + \zeta^2^* \eta_1,
\]
\[
\frac{d\eta_2}{dt} = -\frac{i}{2} (g^0 + g^3) \eta_2 + \frac{i}{2} (g^1 - ig^2) \eta_1 + \zeta^2 + \zeta^1^* \eta_2.
\]

(4.1)

The Cartan-Killing form is nondegenerate and thus from (2.15) we obtain
the inverse of the metric tensor, which has the form 
\[
g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
(the explicit expression of each block is given in the Appendix). The metric can be obtained by taking the inverse of this block supermatrix, and is straightforwardly given by the formula
\[
g = \begin{pmatrix} \mathbb{I} & 0 \\ -D^{-1}C & \mathbb{I} \end{pmatrix} \cdot \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I} & -BD^{-1} \\ 0 & \mathbb{I} \end{pmatrix}.
\]

From the latter (we do not report it explicitly because too cumbersome) we can calculate the invariant measure
\[
d\mu(x) = (\text{sdet}(g^{-1}))^{-1/2} = (1 + z^* z)^{-2} (1 + e_1 + e_2),
\]
which allows us to derive information relevant to our main question.

The SCS supermanifold \( \mathcal{M} \) can be written (see Sec. 2.2) as the product
\[
SU(2|1) \otimes U(2) \otimes U(1) \otimes U(1),
\]
where the second (even) factor is the 2-dimensional sphere \( S_2 \equiv \mathbb{C}P^1 \) — an Einstein-Kähler symmetric space. The first (odd) factor is a symmetric space in some sense similar to \( \mathbb{C}P^1 \equiv SU(3)/U(2) \). This is again a Kähler supermanifold, with invariant metric
\[
g = D^{-1} = \begin{pmatrix}
0 & 1 - \eta_2^* \eta_2 & 0 & \eta_2^* \eta_2 \\
-(1 - \eta_2^* \eta_2) & 0 & -\eta_2^* \eta_1 & 0 \\
0 & \eta_2^* \eta_1 & 0 & \eta_1^* \eta_1 \\
-\eta_1^* \eta_2 & 0 & -(1 - \eta_1^* \eta_1) & 0
\end{pmatrix},
\]

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and Kähler potential (up to an additive constant) given by $K = \eta_1^* \eta_1 + \eta_2^* \eta_2 - \eta_1^* \eta_1 \eta_2^* \eta_2$. It is worth pointing out that this Kähler potential can be obtained by taking the logarithm of the invariant measure function

$$d\mu(x) = (1 + \eta_1^* \eta_1 + \eta_2^* \eta_2),$$

exactly as for an Einstein space.

We have thus proved that our supermanifold is the (semidirect) product of two Einstein-Kähler symmetric spaces. We conjecture that it is Kähler as well; but not Einstein. In fact, if it were, its Kähler potential would be the sum of the Kähler potentials of the two factor manifolds, and the metric would have a block matrix form, thus exhibiting a complete separation (orthogonality) between even and odd coordinates. However, this is not the case, in that it appears evident from (4.1) that the variation of even coordinates under the group action depends on the odd ones.

**Coadjoint orbits approach**

Let us consider the contragradient Lie superalgebra $su(2|1)$. The (fundamental) matrix representation of a generic $su(2|1)$ element is

$$e = \begin{pmatrix} i(a + b) & c & \alpha \\ -c^* & i(a - b) & \beta \\ -\alpha^* & -\beta^* & 2ia \end{pmatrix} \quad a, b, c \in \mathbb{R}, \quad \alpha, \beta \in \Lambda_1.$$ 

Because of the conjecture formulated in section 3.3, we are able to equip the space $su(2|1)_\Lambda^*$ of even 1-forms on $su(2|1)_\Lambda$ with the same superalgebra structure as for $su(2|1)_\Lambda$. Then, the matrix $M_f$ corresponding to a generic 1-form $f$ takes the form

$$M_f = \begin{pmatrix} i(x + y) & z & \xi \\ -z^* & i(x - y) & \chi \\ -\xi^* & -\chi^* & 2ix \end{pmatrix} \quad x, y \in \mathbb{R}, \quad z \in \Lambda_0, \quad \chi, \xi \in \Lambda_1.$$ 

As shown before, we can recover representation $K_0(g)$ by exponentiating the fundamental representation. Thus we are able to study a generic $K_0(g)$-orbit. We restrict here our attention to some cases interesting for the physical applications we have in mind. Applying the conjecture exposed in section
3.4, the orbits isomorphic to the coherent states supermanifolds are those obtained acting trough $K_0(g)$ on elements of the form

$$f_0 = \begin{pmatrix} i(x + y) & 0 & 0 \\ 0 & i(x - y) & 0 \\ 0 & 0 & 2ix \end{pmatrix} \quad x \neq y \wedge x \neq 0 \wedge y \neq 0.$$

The elements belonging to this class are stabilized by the (ordinary) Lie subgroup $U(1) \otimes U(1)$. Also, orbits of this form are isomorphic to the coset space $\frac{SU(2|1)}{U(1) \otimes U(1)}$. We can obtain another interesting class of orbits starting from elements of $su(2|1)_{\Lambda}^*$ of type

$$f_1 = ix \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad x \neq 0.$$

These are elements stabilized by the (ordinary) Lie subgroup $U(2) = SU(2) \otimes U(1)$. Orbits that belong to this class are isomorphic to the coset space $\frac{SU(2|1)}{U(2)}$, which is a supermanifold of dimension $(0,4)$. In the Appendix we report the generalized Kirillov-Kostant 2-form. Some of its properties are not yet known.

4.3 The "graded unitary" example: $uo\text{sp}(1|2)$

So far we have used expressions like "normal adjoint", "ordinary conjugation in a Grassmann algebra" without completely explaining what these expressions mean. Before studying the SCS supermanifold for $uo\text{sp}(1|2)$ we must briefly discuss how hermitian conjugation in a Lie algebra and complex conjugation in the field of complex numbers generalize to the graded case. Let us start with hermitian conjugation.

Hermitian conjugation can be generalized in two different ways to the case of Lie superalgebras. The first one, called "normal adjoint" and denoted by $^\dagger$, is defined by the ordinary axioms

$$\deg(X) = \deg(X^\dagger)$$

$$(aX + bY)^\dagger = aX^\dagger + bY^\dagger$$
\[[X, Y]^\dagger = [Y^\dagger, X^\dagger]\]

\((X^\dagger)^\dagger = X,\)

where \(X, Y \in \mathcal{G}, a, b \in \mathfrak{C},\) the bar means complex conjugation and \([,]\) is the graded commutator. In \(l(1|1)\) and \(sl(2|1)\) we have already defined operations of this kind. The second one, called "grade adjoint" and denoted by \(^\dagger\), is defined by the following axioms

\[
\text{deg}(X) = \text{deg}(X^\dagger) \\
(aX + bY)^\dagger = \bar{a}X^\dagger + \bar{b}Y^\dagger \\
[X, Y]^\dagger = (-)^{XY}[Y^\dagger, X^\dagger] \\
(X^\dagger)^\dagger = (-)^X X,
\]

where \(X\) and \(Y\) must be homogeneous elements and an element and its degree are represented by the same symbol. Given a matrix representation of a superalgebra over the complex field we have

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad M^\dagger = \begin{pmatrix} A^+ & C^+ \\ B^+ & D^+ \end{pmatrix}, \quad M^\dagger = \begin{pmatrix} A^+ & -C^+ \\ B^+ & D^+ \end{pmatrix},
\]

where \(^\dagger\) denotes hermitian conjugate. Finally, the generalization of hermitian representations will be called "star" or "grade star", depending on the kind of adjoint that has been defined in the superalgebra.

A very similar situation occurs when one defines conjugation in a Grassmann algebra\(^{45}\). The ordinary conjugation, denoted by \(^*\), satisfies

\[
\text{deg}(\lambda) = \text{deg}(\lambda^*) \\
(c\lambda + d\mu)^* = \bar{c}\lambda^* + \bar{d}\mu^* \\
(\lambda\mu)^* = \mu^*\lambda^* \\
(\lambda^*)^* = \lambda
\]

where \(\lambda, \mu \in \Lambda\) and \(c, d \in \mathfrak{C}\), while the graded conjugation, denoted by \(^\diamond\), obeys

\[
\text{deg}(\lambda) = \text{deg}(\lambda^\diamond) \\
(c\lambda + d\mu)^\diamond = \bar{c}\lambda^\diamond + \bar{d}\mu^\diamond \\
(\lambda\mu)^* = \lambda^\diamond\mu^\diamond \\
(\lambda^\diamond)^\diamond = (-)^\lambda \lambda,
\]

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with \( \lambda \) and \( \mu \) homogeneous elements.

In our examples we deal with compact forms that have the general structure \((\Lambda \otimes G)_0\), because even (odd) parameters are associated to even (odd) generators; in these algebras the hermitian conjugation \( ^+ \) can be defined either by

\[
(\lambda X + \mu Y)^+ = \lambda^* X^\dagger + \mu^* Y^\dagger
\]

or by

\[
(\lambda X + \mu Y)^+ = \lambda \hat{\circ} X^\dagger + \mu \hat{\circ} Y^\dagger,
\]

depending on which kind of adjoint can be defined in \( G \); in both cases, \( \forall Z, W \in (\Lambda \otimes G)_0 \), we have

\[
(Z^+)^+ = Z \quad [Z, W]^+ = [W^+, Z^+].
\]

\(osp(1|2)\) is the superalgebra of \((1 + 2) \times (1 + 2)\) supermatrices with \( \{J_3, J_+, J_-\} \) as even generators and \( \{R_+, R_-\} \) as odd generators. The following commutations rules are valid:

\[
[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2 J_3
\]

\[
[J_3, R_{\pm}] = \pm \frac{1}{2} R_{\pm}, \quad [J_+, R_{\pm}] = R_\mp, \quad [J_-, R_{\pm}] = 0
\]

\[
\{R_\pm, R_{\pm}\} = \pm \frac{1}{2} J_{\pm}, \quad \{R_+, R_-\} = -\frac{1}{2} J_3,
\]

from which one sees that the even sector of \(osp(1|2)\) is just \( su(2) \). The IRREP of \(osp(1|2)\) are labelled by the \( su(2)\)-quantum number \( j \) and the representation space is the direct sum of two \( su(2)\) representation spaces, one with spin \( j \) and even (odd) degree and the other with spin \( j - \frac{1}{2} \) and odd (even) degree. Depending on the choice of the parity of the two subspaces (and then of the highest weight vector) one can extend the \( su(2)\) hermitian conjugation \((J_3^\dagger = J_3, J_{\pm}^\dagger = J_{\mp})\) to a grade adjoint operation over the whole superalgebra, by setting \( R_+^\dagger = \pm R_- \) (the upper sign corresponding to odd highest weight vector), and consequently \( R_-^\dagger = \mp R_+ \). It is easy to verify that the defining representation has \( j = \frac{1}{2} \) and odd highest weight vector.

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The algebra $uosp(1|2)$ can now be defined by

$$uosp(1|2) = \{ X = i\phi J_3 + z J_+ - z^\diamond J_- + \eta^\diamond R_+ + \eta R_- \mid \phi \in \mathbb{R}_c, z \in \Lambda_0, \eta \in \Lambda_1 \}.$$ 

The exponential group associated with $uosp(1|2)$, denoted by $UOSP(1|2)$, is the group of $(1 + 2) \times (1 + 2)$ orthosymplectic, unimodular supermatrices.

**Coherent states supermanifold**

The IRREP being nondegenerate, the isotropy subgroup of the highest weight vector is just the Cartan subgroup, generated by $J_3$, the SCS supermanifold is the coset space $\mathcal{M} = \frac{UOSP(1|2)}{U(1)}$, and the coset representatives can be written in the usual form $x = \xi \cdot v$, where $\xi = \exp(\eta^\diamond R_+ + \eta R_-)$ and $v = \exp(zJ_+ - z^\diamond J_-)$. Acting on the coset representative by the infinitesimal group element $\delta g = \mathbb{I} + \left( \sum_{i=1}^{3} g_i A_i + \zeta^\diamond R_+ + \zeta R_- \right) dt$, after the substitution $-i \frac{z}{|z|} \tan |z| \mapsto z$ we obtain the variation of the coordinates

$$\frac{dz}{dt} = \frac{1}{2} g^1(1 + z^2) + \frac{i}{2} g^2(1 - z^2) + ig^3z$$

$$+ \frac{1}{4} \zeta(i\eta + z\eta^\diamond) + \frac{1}{4} \zeta^\diamond(z\eta - iz^2\eta^\diamond),$$

$$\frac{d\eta}{dt} = \frac{i}{2} (g^1 + ig^2) \eta^\diamond + \frac{i}{2} g^3 \eta + \zeta \left( 1 + \frac{1}{4} \eta^\diamond \eta \right),$$

$$\frac{d\eta^\diamond}{dt} = \frac{i}{2} (g^1 - ig^2) \eta - \frac{i}{2} g^3 \eta^\diamond + \zeta^\diamond \left( 1 + \frac{1}{4} \eta^\diamond \eta \right).$$

The Cartan-Killing form is nondegenerate and, once more, from (2.15) we obtain the inverse of the metric tensor and from this the invariant measure

$$d\mu(x) = (1 + z^\diamond z)^{-2} \left( 1 - \frac{1}{4} \eta^\diamond \eta \right).$$

Once more we can write $\mathcal{M}$ as a product $\frac{UOSP(1|2)}{SU(2)} \otimes_s \frac{SU(2)}{U(1)}$, and again the bosonic factor is $S_2$. The invariant metric on the fermionic factor is

$$g = \begin{pmatrix} 0 & 1 + \frac{1}{4} \eta^\diamond \eta \\ - \left( 1 + \frac{1}{4} \eta^\diamond \eta \right) & 0 \end{pmatrix},$$

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that admits a natural complex structure, from which the 2-form
\[ \omega = 2i \left( 1 - \frac{1}{4} \eta \overset{\wedge}{\eta} \right) d\eta \wedge d\eta \]
can be derived. In this case the 2-form is not closed. Indeed, closure and supersymmetry imply that the 2-form should be constant, and hence manifestly non-invariant. Then the fermionic factor is not a Kähler supermanifold and we assume that the same holds for the SCS supermanifold.

**Coadjoint orbits approach**

Let us consider the contragradient Lie superalgebra \( uosp(1|2) \) and its left \( \Lambda_0 \)-module \( uosp(1|2)_\Lambda \). A generic element \( a \) of \( uosp(1|2)_\Lambda \) admits a (fundamental) matrix representation of type
\[
a = \begin{pmatrix}
0 & -\beta \overset{\wedge}{\cdot} & \beta \\
-\beta & ik & z \\
-\beta \overset{\wedge}{\cdot} & -z \overset{\wedge}{\cdot} & -ik
\end{pmatrix} \quad k \in \mathbb{R}_c, z \in \Lambda_0, \beta \in \Lambda_1.
\]

Also in this case, by means of the conjecture in section 3.3, we can give to the generic \( uosp(1|2)_\Lambda^* \) element matrix form
\[
M_f = \begin{pmatrix}
0 & -\alpha \overset{\wedge}{\cdot} & \alpha \\
-\alpha & i\lambda & w \\
-\alpha \overset{\wedge}{\cdot} & -w \overset{\wedge}{\cdot} & -i\lambda
\end{pmatrix} \quad \lambda \in \mathbb{R}_c, w \in \Lambda_0, \alpha \in \Lambda_1.
\]

Being \( uosp(1|2) \) contragradient, one can obtain the orbits corresponding to coherent states supermanifolds from elements of the form
\[
f_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & i\lambda & 0 \\
0 & 0 & -i\lambda
\end{pmatrix}
\]

stabilized by the (ordinary) Lie subgroup \( U(1) \). These orbits are isomorphic to coset space \( \frac{UOSP(1|2)}{U(1)} \), i.e are supermanifolds of dimension \( (2, 2) \). We can then write the matrix for the generalized Kirillov-Kostant 2-form \( \varpi_f \), which is nondegenerate on the factor space \( \mathcal{G}/\mathcal{H} \) (see Appendix). Other properties of \( \varpi_f \) for this class of orbits are not considered here, because irrelevant in this context.
5. CONCLUSIONS

We have given a generalized definition of coherent states for a system described by a Hamiltonian which lives in a dynamical superalgebra and have showed that for the SCS systems so obtained the properties of transitivity with respect to the action of the dynamical group, the identity resolution and overcompleteness, as well as stability during time evolution, still hold. For what concerns the problem of establishing whether or not the SCS are states of minimum quantum uncertainty we have shown that this concept has not a well-defined, unambiguous meaning and we can only refer to the cited literature for some examples.

We have studied in detail the SCS supermanifold for three dynamical superalgebras. In the simplest case of $u(1|1)$ we have found an homogeneous Kähler supermanifold, determined explicitly its Kähler structure, derived the Kähler potential, and have determined the connection between the SCS supermanifold and the coadjoint orbits. In the case of $su(2|1)$ we have decomposed the homogeneous supermanifold into the product of two Einstein-Kähler symmetric spaces, one purely fermionic and one purely bosonic, for which we have determined Kähler structures and potentials: we expect the product supermanifold to be Kähler as well. In the case of $uosp(1|2)$ we showed that one cannot construct a Kähler structure over the purely fermionic factor supermanifold, we expect then the whole supermanifold not to be Kähler. On the basis of these results, and looking at the particular properties of the conjugation operation that one has to introduce in the Grassmann algebra when working with superalgebras whose IRREP can not be turned into star representations, we conjecture that a necessary condition for our SCS supermanifolds to be Kähler is that the dynamical superalgebra admits a star representation, and that the UIR used to define the SCS system is derived from one such IRREP.

In addition, we have established a connection between a certain orbit of a generic contragradient Lie superalgebra and the coherent states supermanifold related to non-degenerate representations of that superalgebra. Finally, in the examples, we have found symplectic structures over these orbits.
Appendix

Inverse metric tensor for the $su(2|1)$ case.

The inverse of the invariant metric tensor on \( SU(2|1) \) has the usual block form \( g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where, in the basis \( \{ \partial_z, \partial_{z^*}, \partial_{\eta_1}, \partial_{\eta_2}, \partial_{\eta_2^*} \} \),

\[
A = (1 + z^* z)^2 \left( 1 - \frac{1}{4} e_1 - \frac{1}{4} e_2 + \frac{1}{2} e_1 e_2 \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
B = C = \begin{bmatrix}
-\frac{i}{2} z \eta_1 (1 - \frac{1}{2} e_2) + \frac{i}{2} z^2 \eta_2 (1 - \frac{1}{2} e_1) & \frac{1}{2} z^* \eta_1 (1 - \frac{1}{2} e_2) + \frac{i}{2} \eta_2 (1 - \frac{1}{2} e_1) \\
\frac{1}{2} z \eta_1^* (1 - \frac{1}{2} e_2) - \frac{1}{2} z^* \eta_2 (1 - \frac{1}{2} e_1) & -\frac{1}{2} z^* \eta_1^* (1 - \frac{1}{2} e_2) - \frac{i}{2} \eta_2^* (1 - \frac{1}{2} e_1) \\
-\frac{i}{2} z^2 \eta_1^* (1 - \frac{1}{2} e_2) - \frac{1}{2} z \eta_2^* (1 - \frac{1}{2} e_1) & -\frac{i}{2} \eta_1^* (1 - \frac{1}{2} e_2) + \frac{i}{2} \eta_2^* (1 - \frac{1}{2} e_1)
\end{bmatrix}
\]

and \( D = \begin{pmatrix} 0 & -(1 + e_2 - e_1 e_2) & 0 & \eta_2^* \eta_1 \\ 1 + e_2 - e_1 e_2 & 0 & -\eta_1^* \eta_2 & 0 \\ 0 & \eta_1^* \eta_2 & 0 & -(1 + e_1 - e_1 e_2) \\ -\eta_2^* \eta_1 & 0 & 1 + e_1 - e_1 e_2 & 0 \end{pmatrix} \).

\( su(2|1)'s \) orbit isomorphic to \( SU(2|1)/U(2) \).

\[
\Theta_{\mathcal{F}_1} = ix \begin{pmatrix}
1 - \alpha^* \alpha + \eta & \alpha \beta^* & -\alpha + \alpha^* \beta \\
-\alpha^* \beta & 1 - \beta^* \beta + \eta & -\beta + \alpha \beta^* \\
-\alpha^* + \alpha \beta^* \beta & -\beta^* + \alpha^* \alpha \beta^* & 2 - \alpha^* \alpha - \beta^* \beta + 2 \eta
\end{pmatrix}
\]

where \( \eta = \alpha^* \alpha \beta^* \beta \).

Generalized Kirillov-Kostant 2-form for \( su(2|1) \).

We list only a subset of nonzero elements, the others can be recovered by
means of superskewsymmetry

\[ \varpi_{f_{1,5}} = - \varpi_{f_{2,5}} = - \varpi_{f_{3,6}} = - \alpha^* \eta_\beta \]

\[ \varpi_{f_{1,7}} = \varpi_{f_{2,7}} = \varpi_{f_{3,8}} = - \beta \eta_\alpha \]

\[ \varpi_{f_{2,3}} = 2 \varpi_{f_{5,7}} = - 2 \alpha^* \beta \]

\[ \varpi_{f_{3,4}} = \beta^* \beta - \alpha^* \alpha \]

\[ \varpi_{f_{4,7}} = \varpi_{f_{1,8}} = - \varpi_{f_{2,8}} = \alpha \eta_\beta \]

\[ \varpi_{f_{5,8}} = 1 - \beta^* \beta \eta_\alpha \]

\[ \varpi_{f_{2,4}} = 2 \varpi_{f_{6,8}} = - 2 \alpha \beta^* \]

\[ \varpi_{f_{1,6}} = \varpi_{f_{2,6}} = - \varpi_{f_{4,5}} = - \beta^* \eta_\alpha \]

\[ \varpi_{f_{6,7}} = - 1 + \alpha^* \alpha \eta_\beta \]

where \( \eta_\alpha = 1 - \alpha^* \alpha, \eta_\beta = 1 - \beta^* \beta, \varpi_{f_{i,j}} = \varpi_f(e_i, e_j) \) and \( \{e_i, i = 1, 2, \ldots, 8\} \) is a suitable homogeneous basis.

**Coherent states orbit for the \( uosp(1|2) \) case.**

The coordinate set of an element in generic position on an orbit belonging to this class is

\[
\{ -\xi \sin \theta e^{i\psi}, -\xi \sin \theta e^{-i\psi}, i\xi \cos \theta | \\
\beta \sin \theta e^{i\psi} - i\beta^* \cos \theta, -\beta^* \sin \theta e^{-i\psi} + i \beta \cos \theta \}
\]

where \( \xi = 1 - \beta^* \beta \).

**Generalized Kirillov-Kostant 2-form for \( uosp(1|2) \).**

We list in the following only a subset of non zero elements, the others can be obtained by means of the skewsymmetry properties of \( \varpi_f \).

\[ \varpi_{f_{1,3}} = - \varpi_{f_{5,5}} = - 2 \xi \sin \theta e^{-i\psi} \]

\[ \varpi_{f_{2,3}} = \varpi_{f_{4,4}} = 2i \xi \sin \theta e^{i\psi} \]

\[ \varpi_{f_{1,5}} = \varpi_{f_{3,4}} = - 2 \beta^* \sin \theta e^{-i\psi} + 2i \beta \cos \theta \]

\[ \varpi_{f_{2,4}} = \varpi_{f_{3,5}} = 2 \beta \sin \theta e^{i\psi} - 2i \beta^* \cos \theta \]

\[ \varpi_{f_{4,5}} = - 2i \xi \cos \theta \]
where $\varpi_{i,j} = \varpi_j(e_i, e_j)$ and $\{e_i, i = 1, 2, \ldots, 5\}$ is a suitable homogeneous basis.

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