Optical signatures of shear collective modes in strongly interacting Fermi liquids

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The concept of Fermi liquid lays a solid cornerstone to the understanding of electronic correlations in quantum matter. This ordered many-body state, entailed by the Pauli exclusion principle, rigorously organizes electrons at zero temperature in progressively higher momentum states, up to the Fermi surface. As such, it displays rigidity against perturbations, with consequences like Fermi-surface resonances manifesting themselves in longitudinal and transverse collective modes, as observed in liquid helium. However, the quest for understanding and probing Fermi-liquid collective modes in charged solid-state systems remains a challenge. In this paper I analyze the transverse shear response of neutral and charged three-dimensional Fermi liquids as a function of temperature, excitation frequency and momentum, for interactions expressed in terms of the first symmetric Landau parameter. I consider the effect of momentum-conserving quasiparticle collisions and momentum-relaxing scattering in relaxation-time approximation on the coupling between photons and Fermi-surface collective modes, thus deriving the Fermi-liquid optical conductivity and dielectric function. In the high-frequency, long-wavelength excitation regime the electromagnetic response entails two coherent and frequency-degenerate polaritons, and its spatial nonlocality is encoded by a frequency- and interaction-dependent Fermi-liquid generalized shear modulus; in the opposite high-momentum low-frequency regime anomalous skin effect takes place. I identify observable signatures of propagating shear collective modes in optical spectroscopy experiments, with applications to the surface impedance and the optical transmission of thin films.

I. INTRODUCTION

The Fermi liquid represents the Rosetta Stone of electronic correlations in weakly-interacting electron systems. Such conception fostered profound insight into the phenomenology of electrical and thermal conduction in metals throughout the twentieth century, and it still reserves unexpected surprises in the application to modern-day materials.

The foundations of the Fermi liquid, introduced by Landau, interpret the complexity of thermodynamics, low-energy transport and optical properties of interacting electrons in terms of a liquid of nearly-independent quasiparticles. This nearly-free phenomenology actually stems from a remarkably ordered microscopic state: at zero temperature, long-ranged entanglement associated with the Pauli exclusion principle arranges electrons in a hierarchy of progressively higher momentum states, with the uppermost states composing the Fermi surface. The elementary excitations of such system are electron-hole quasiparticles, created by promoting an occupied state from below the Fermi surface to unoccupied levels above, and in direct correspondence with the original interacting electrons. The energy required to generate such excitations increases going deeper below the Fermi level $E_F$, so that the latter energy scale essentially dictates thermal and electrical conduction, while states at lower energies remain largely unperturbed. The Fermi liquid thus forms a stable, cohesive state of zero-temperature matter.

However, such order is not captured by the conventional language of phase transitions, whereby a generalized rigidity emerges from spontaneous symmetry breaking of a high-energy disordered phase. In fact, the Fermi liquid is rigorously defined only at temperature $T = 0$, while at energies $\hbar \omega < k_B T$ it is adiabatically connected to a classical fluid, in which thermal fluctuations ‘disorder’ the system through quasiparticle excitations across the Fermi surface but without any thermodynamic singularity. As a matter of fact, such classicalfluid is not ideal in crystalline solids: momentum conservation is already destroyed on the scale of the crystal lattice periodicity, due to momentum-relaxing scattering of quasiparticles on lattice ions, thus preventing a perfect analogy with hydrodynamics. Remarkably, at higher energies $\hbar \omega > k_B T$ the Fermi-liquid order takes over again, effectively recovering substantial remnants of the $T = 0$ physics. All this wisdom is conventionally parametrized in terms of a quasiparticle collision time $\tau_c \sim \hbar / (\hbar \omega + (k_B T))$, stemming from the phase-space restriction for collision processes entailed by the Pauli exclusion principle. Multiple experiments confirm that the low-temperature ground state of many metals indeed complies with the Fermi-liquid picture: notable examples include Sr$_2$RuO$_4$ and electron-doped BaFe$_{2-x}$Co$_x$As$_2$.

Much of the contemporary literature on transport in Fermi liquids focused on aforementioned hydrodynamic regime $\hbar \omega < k_B T$: if the collision time $\tau_c$ provides the smallest timescale in the system in a given temperature and energy window, then the establishment of local equilibrium allows for an effective hydrodynamic description of quasiparticle flow. Then, momentum and energy dissipation may be encoded in the viscosity tensor, while dissipation-
less motion may be formulated in terms of ‘generalized elasticity’ \(^{9,12-14}\) and quantified by elastic moduli. \(^{3,15}\) In this respect, quantum critical electron systems provide ideal platforms for hydrodynamic flow: their strong interactions imply an extremely low ‘Planckian’ collision time \(\tau_c \propto \frac{\hbar}{k_B T}\) in an extended range of temperatures \(T\) above the \(T = 0\) quantum critical point, thus expanding the parameter range of the hydrodynamic regime to territories where even the concept of quasiparticle breaks down.\(^{16,17}\) However, electron viscosity has eluded experimental observation in standard three-dimensional metals so far, due to multiple scattering channels, like impurities, phonons, Umklapp processes, that inevitably relax momentum and hinder the establishment of local equilibrium.\(^{18}\) On the contrary, systems with strong interactions and reduced dimensionality provided fruitful platforms for electron hydrodynamics, as testified by applications to graphene,\(^{19-23}\) delafossites,\(^{24,25}\) Weil semimetals,\(^{26,27}\) the two-dimensional electron gas in (Al,Ga)As heterostructures,\(^{28,29}\) bad metals.\(^{30}\) Experimentally, signatures of hydrodynamic behaviour have been indeed retrieved in graphene,\(^{31-35}\) PdCoO\(_2\),\(^{24,34,35}\) PtCoO\(_2\),\(^{36}\) WP\(_2\),\(^{36,37}\) PtSn\(_4\).\(^{38}\)

By contrast, the dynamical regime \(\hbar \omega > k_B T\) remains hitherto unexplored in Fermi liquids,\(^{3,14}\) thus offering untapped potential for conceptual insight and solid-state applications, as the present work aspires to highlight. In fact, it is in this dynamical regime that the \(T = 0\) order and its associated rigidity is partially recovered. Such rigidity of the Fermi-liquid order against perturbations gives rise to striking consequences like propagating collective modes of electron density and current, i.e. coherent vibrations of the Fermi surface in space and time, with a classical analogue being waves on the surface of an elastic carpet. An example of such modes is the so-called ‘zero sound’, occurring both in the longitudinal and transverse (shear) channels: its observation in liquid \(^3\)He, the archetype of electrically neutral Fermi liquid, stroke a landmark of twentieth-century low-temperature physics.\(^{39,40}\)

Nevertheless, it would be inaccurate to label Fermi-liquid collective modes as elastic phenomena, since in the proper sense elasticity refers to static reactive properties of solids, which are absent in the Fermi liquid. In fact, from the collective excitation viewpoint, the dynamical deformation of the Fermi surface behaves like a spin-1 object, akin to a ‘transverse phonon’, while shear rigidity resides in the static spin-2 channel: \(^{11}\) dynamical reactive shear in the Fermi liquid is disconnected from elasticity, as in the static limit the system rather behaves as a viscous liquid. Perhaps water offers a poignant analogy: plunging slowly into a pond produces a viscous, dissipative response typical of a fluid, while diving at high speed from above generates much more resistance at impact, whereby the liquid reacts almost like a solid.

At first sight, the analysis of charge collective modes in electrically charged Fermi liquids bears additional challenges with respect to neutral systems, due to the presence of the long-ranged Coulomb interaction\(^{41,42}\) and of momentum relaxation imposed by the breaking of translation invariance. The first conundrum is solved by the Silin theory,\(^{41,42}\) which prescribes a proper redefinition of the Landau quasiparticle interaction parameters to take into account Coulomb repulsion. Still, as we will describe later, electric charge modifies the linear dispersion of the transverse shear collective mode, introducing a ‘quadratic foot’ due to the photon and shear mode roots repelling each other at low momentum;\(^{43}\) a primary example of this phenomenon is found in the electrodynamics of the ‘isotropic Wigner crystal’.\(^{44}\) The second issue of momentum relaxation is more serious, as it prevents hydrodynamics to occur in standard 3D metals like previously mentioned. However, the reactive shear response is more robust than hydrodynamics with respect to relaxation, because the former depends mainly on quasiparticle interactions and does not rely on \(\tau_c\), being the smallest timescale in the system.

How is it possible to observe shear rigidity in solid-state Fermi liquids? The photon is an ideal candidate probe, as it exerts transverse perturbations on electrons and it couples to electric charge. Then, the fingerprint of Fermi-liquid reactive shear may be the development of a spatially nonlocal response, due to interactions that couple together all quasiparticles in the ensemble:\(^{44}\) applying a local electromagnetic field results in a perturbation spread out in an extended region of space surrounding the probe, by virtue of the nonlocal character of the optical response. Perhaps surprisingly, standard optics is formulated in terms of the spatially local Drude model, whereby the momentum dependence of the dielectric response is calculated at zero-momentum. The common justification for this approximation resides in the smallness of the momentum transferred by radiation to the solid, due to the high velocity of light with respect to the Fermi velocity. This generally suffices to interpret optical phenomena in standard metals at room temperature.\(^{1,45,46}\) However, notable known exceptions like anomalous skin effect show that, even with a perfectly local probe, the system response becomes spatially nonlocal at low temperatures due to the increase of the electronic mean free path.\(^{46,47}\) Hence, in this paper we focus on how to observe reactive shear collective modes in solid-state Fermi liquids using low-temperature optical spectroscopy.

I use the Landau kinetic equation approach\(^2\) to analyze the dynamical coupling of Fermi-surface oscillations to electromagnetic radiation, demonstrating that the leading-order long-wavelength correction to local Drude theory reproduces the macroscopic phenomenology of viscous charged fluids, interpreted in terms of a generalized frequency-dependent shear modulus \(\nu(\omega)\).

This leaves distinctive and potentially observable traces on state-of-the-art optical spectroscopy experiments. In par-
ticular, I show that the standard theory of anomalous skin effect in the optical response is intimately related to Fermi-liquid propagating shear: both phenomena represent optical manifestations of spatial nonlocality in the transverse response of the Fermi surface, acting in opposite momentum regimes. Likewise, such nonlocality generates the amplification of the optical transmission coefficient in Fermi-liquid films, accompanied by characteristic oscillations in frequency due to the interference of two mutually coherent polariton modes. The results of this analysis can be compared with similar kinetic equation approaches, with the general formulation in terms of the stress and viscosity tensors, and with optical conductivities derived from the AdS/CFT correspondence. Moreover, spatial nonlocality and dissipation are necessary conditions for the occurrence of negative refraction, usually realized in artificial metamaterials, but also possible in charged Fermi liquids due to the dissipative component of their shear response. 

The paper is organized as follows. Section II hosts a collection of some fundamental milestones in the analysis of the transverse shear response in electrically neutral Fermi liquids, with application to transverse sound in liquid $^3$He. Section III extends aforementioned analysis to charged Fermi liquids and presents the calculation of the transverse susceptibility in the kinetic equation approach. The latter result is used in section IV to analyze the transverse dielectric function and the optical conductivity. Section VII deals with the zero-frequency limit of the optical conductivity and its relation to hydrodynamic effects in transport experiments. Section VIII expands on the microscopic origin of electron collisions and momentum relaxation in three-dimensional Fermi liquids: using the phenomenological Matthiessen’s rule, I consider acoustic phonons, impurities and Umklapp processes as examples of independent scattering channels. In section IX I propose exemplary implications of Fermi-surface rigidity for optical spectroscopy, specifically for the surface impedance and the thin-film transmission, and I compare the results with the predictions of the Drude model. The main messages of the paper are summarized and discussed in section X.

II. TRANSVERSE RESPONSE OF NEUTRAL FERMI LIQUIDS

The prediction of Fermi-surface shear collective modes and their detection in liquid $^3$He represent milestones of Fermi-liquid theory, which lay the foundations for the analysis of solid-state charged systems performed in later sections. For these reasons, I begin by surveying the description of transverse shear rigidity in the kinetic approach of Abrikhosov and Khalatnikov, stemming from the lowest transverse harmonic in the angular expansion of quasiparticle interactions. As nomenclature often differs in the liquid-helium and solid-state communities dealing with this subject, we describe the different terminologies used while setting univocal definitions to be consistently recalled throughout this paper.

A. Transverse collective mode with collisions

In this section, we review the analysis of the shear collective mode in a neutral Fermi liquid. We start with the kinetic equation for Landau quasiparticles in the presence of collisions, the derivation of which is reported in appendix A. In essence, this kinetic equation describes the first-order deviation (or ‘displacement’) $\varepsilon_k(q, \omega)$ of the quasiparticle distribution function at the Fermi surface with respect to global thermodynamic equilibrium. Such deviation is generated by quasiparticle interactions, collisions and external driving forces. Interactions are expanded in spherical harmonics in terms of Landau parameters $F^{SA}_l$, where the label $S$ and $A$ refer to the symmetric and antisymmetric channels, which generate density and spin excitations, respectively. Momentum-conserving scattering processes are encoded by the collision integral $I_{\text{coll}}(q, \omega)$. We have

$$
\left( q v_{k,\sigma} \cos \theta - \omega \right) \varepsilon_k(q, \omega) + q v_{k,\sigma} \cos \theta \int \frac{d\Omega}{4\pi} \sum_{l=0}^{\infty} F^{SA}_l (\cos \theta') \varepsilon_{k'}(q, \omega) = I_{\text{coll}}(q, \omega)
$$

where $\theta = \arccos(q v_{k,\sigma})/|q v_{k,\sigma}|$ is the angle between the wave vector $q$ and the quasiparticle velocity $v_{k,\sigma}$. The interaction term $\sum_{l=0}^{\infty} F^{SA}_l (\cos \theta') \varepsilon_{k'}(q, \omega)$ is already expanded in terms of Legendre polynomials $p_l(\cos \theta')$ and Landau parameters $F^{SA}_l$, in accordance with their standard definitions recalled in appendix A.

We define the normalized velocity $s = \frac{\omega}{q v_{k,\sigma}}$, $v_{k,\sigma}$ being the quasiparticle velocity for the state at electronic wave vector $k$ and spin $\sigma$ on the Fermi surface: this change of variables anticipates that sound-like collective modes of linear dispersion $\omega \propto q$ appear in some momentum and frequency regime, where $s$ becomes a constant.

In terms of $s$, equation (1) is

$$
(\cos \theta - s) \varepsilon_k(q, \omega) + \cos \theta \int \frac{d\Omega}{4\pi} \sum_{l=0}^{\infty} F^{SA}_l (\cos \theta') \varepsilon_{k'}(q, \omega) = \frac{1}{q v_{k,\sigma}} I_{\text{coll}}(q, \omega)
$$

We now expand the Fermi surface displacement function $\varepsilon_k(q, \omega)$ similarly to what is done for the interactions. In three dimensions, the displacement is a function of the 3D solid angle and is expanded in spherical harmonics,

$$
\varepsilon_k(q, \omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \varepsilon_{k,l,m}(\theta, \phi)
$$

where $\hat{u}_k$ is the unit vector along the direction of $k$, and the definition of spherical harmonics $\mathcal{Y}_l^m(\theta, \phi)$ is recalled in appendix A. The label $\{S, A\}$ on the harmonic component $\varepsilon_{l,m}$...
of the Fermi-surface displacement refers to spin-symmetric and spin-antisymmetric channels, respectively.

We consider the first transverse mode with \( m = 1 \), which corresponds to transverse currents originating in the first spin-symmetric interaction channel. We truncate the sum over \( l \) in equations (2) and (3) to \( l = 1 \), so that the interaction becomes \( \sum_{l=0}^{\infty} \sum_{\alpha} F_l^S \varphi_l^{(\alpha)}(\cos \alpha) \equiv F_0^S + F_1^S \cos \alpha \), and the Fermi surface displacement can be written as \( e_{\xi}(q, \omega) = \sum_{l=0}^{\infty} e_{l} \varphi_{l}(\theta, \phi) \equiv e^{\xi} e^{\phi} \), where \( e^{\xi} \) collects the \( \theta \)-dependent portion of the displacement. The kinetic equation becomes now

\[
(\cos \theta - s) e^{\xi}(\theta) e^{\phi} + \cos \theta \int_0^{2\pi} \frac{d \phi' \sin \theta' d \theta'}{4\pi} \left( F_0^S + F_1^S \cos \alpha \right) e^{\xi}(\theta') e^{\phi'} = \mathcal{J}_{coll}(q, \omega) \tag{4}
\]

where we have defined \( \mathcal{J}_{coll}(e^{\xi}(\theta)) e^{\phi} = \frac{1}{q v_F^2} \mathcal{J}_{coll}(q, \omega) \) as the expansion of the collision integral in spherical harmonics with \( m = 1 \), similarly to what we have done for the Fermi surface displacement. By construction, the angle \( \alpha \) is such that \( \cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \). Inserting this into equation (4), we achieve

\[
(\cos \theta - s) e^{\xi}(\theta) e^{\phi} + \cos \theta \int_0^{2\pi} \frac{d \phi' \sin \theta' d \theta'}{4\pi} \left( F_0^S + F_1^S \cos \alpha \right) e^{\xi}(\theta') e^{\phi'} = \mathcal{J}_{coll}(q, \omega) \tag{5}
\]

The terms \([A] \) and \([B] \) give zero upon integration over the angles \( \theta' \) and \( \phi' \), so that we have

\[
(\cos \theta - s) e^{\xi}(\theta) e^{\phi} + \cos \theta \int_0^{\pi} \frac{\sin \theta' d \theta'}{4\pi} F_1^S \sin \theta \sin \theta' \pi e^{\phi} = \frac{1}{q v_F^2} \mathcal{J}_{coll}(q, \omega) \tag{6}
\]

The conservation of particle number, energy and momentum in collisions imposes constraints on the form of the collision integral \( \mathcal{J}_{coll}(q, \omega) \): in fact, the moments obtained by phase-space integration of equation (6) have to yield the continuity equation for particle density, as well as the conservation of energy and of momentum. Aforementioned prescriptions lead to the following form of the collision integral: \(^{39}\)

\[
\frac{1}{q v_F^2} \mathcal{J}_{coll}(q, \omega) = \mathcal{J}_{coll} \left[ e^{\xi}(\theta) e^{\phi} \right] = -\frac{e^{\xi}(\theta) - \left[ e^{\xi}(\theta) \right]_{av} - 3 \left[ e^{\xi}(\theta) \sin \theta \right]_{av} \sin \theta}{i \omega \tau_c} \quad \tag{7}
\]

where the notation \( [\cdot]_{av} \) denotes the angular average with respect to \( \theta \) performed as for the integral over \( \theta' \). In this approach, the effective collision time \( \tau_c \) parametrizes the integral \( \mathcal{J}_{coll} \left[ e^{\xi}(\theta) \right] \) phenomenologically: this allows one to model collisions independently from the microscopic origin of scattering, as done in the application to liquid \(^{3}^{3}He. \) In later sections, we will use the Fermi-liquid expression for \( \tau_c \) stemming from the Pauli exclusion principle – see also appendix F. We notice that, contrarily to a longitudinal mode with \( m = 0 \), the transverse mode with \( m = 1 \) does not generate a net density flow, as it couples to transverse currents but not directly to density fluctuations. As a result, in our case \( [e^{\xi}(\theta)]_{av} = 0.\)\(^{39}\) From here on, the solution for the displacement \( e^{\xi}(\theta) \) follows from equation (4) with straightforward mathematical steps, which are detailed in appendix B. We quote here the final result for the dispersion relation of transverse sound: \(^{39,40}\)

\[
\left( \xi^2 - 1 \right) \left[ \frac{\xi}{2 \ln \left( \frac{\xi + 1}{\xi - 1} \right)} - 1 \right] = \frac{F_1^S - 6 - 9 \beta}{3F_1^S - 9 \beta} \tag{8}
\]

where

\[
\xi = s \left( 1 + \frac{i}{\omega \tau_c} \right) \tag{9a}
\]

\[
s = \frac{\omega}{qv_F^2} \tag{9b}
\]

\[
\beta = \frac{1}{i \omega \tau_c - 1} \tag{9c}
\]

Equation (8) determines the dispersion (9b) of the transverse collective mode generated by the interaction parameter \( F_1^S \), with collisions encoded by \( \tau_c \). This mode has been labeled ‘transverse sound’ in the liquid-helium literature, although physically it amounts to a shear oscillation of the Fermi surface. Apart from collisions, Landau damping strongly attenuates transverse sound in some regions of the \( q \)-\( \omega \) plane, due to the mode exchanging energy and momentum with incoherent electron-hole excitations: \(^{5,58}\) at small momentum, this happens when \( \omega < v_F^e q \), i.e. Re \( (s) < 1 \). Therefore, observing a substantially undamped propagation of the shear mode requires Re \( (s) > 1 \).

Figure 1(a) displays the dispersion relation of the shear collective mode (8), showing the normalized wave vector \( q v_F^e \tau_c \) as a function of \( \omega \tau_c \) for \( F_1^S = 20 \), with blue and red...
this establishes local thermodynamic equilibrium, so that the system behaves like a viscous fluid,\textsuperscript{5,54} with the collective mode essentially being governed by quasiparticle collisions. The relaxational mode\textsuperscript{56} velocity in the hydrodynamic case at leading order in $\omega \tau_c \to 0$ is

$$s = \sqrt{\frac{i}{5 \omega \tau_c} \left( 1 + \frac{F_s^*}{3} \right) \frac{i \omega \tau_c}{i + \omega \tau_c}} = \frac{1}{v_F} \sqrt{i \omega \nu(0)} \frac{1}{i + \omega \tau_c},$$

(10)

where

$$\nu(0) = \frac{1}{5} (v_F^2 \tau_c \frac{1}{1 + \frac{v_F^2}{c^2}})$$

(11)

is the static viscosity coefficient of the isotropic Fermi liquid in three dimensions.\textsuperscript{1,2} Equation (10) implies

$$\text{Re} \{s\} = \text{Im} \{s\} = \frac{1}{v_F} \frac{\sqrt{\omega \nu(0)}}{\sqrt{2(1 + \omega \tau_c)}},$$

(12)

that is, the mode is critically damped in hydrodynamic regime, having equal real and imaginary parts. Figure 1(b) shows the real and imaginary parts of equation (10) as a function of $\omega \tau_c$, as purple and red dashed lines, respectively. Notice that equation (10) implies the dispersion relation

$$q = \frac{\omega}{v_F^*} \frac{\sqrt{\nu(0)}}{2 \nu(0)} (1 + \omega \tau_c)(1 - i),$$

(13)

which means $\omega \propto q^2$ in hydrodynamic regime for uncharged transverse sound, with the proportionality coefficient governed by the static shear viscosity (11). Physically, the viscosity-driven dissipative character (13) of transverse sound is due to the latter being submerged in the Lindhard continuum for $\omega \tau_c \to 0$ – see figure 1(a) – which dissipates the collective mode into incoherent electron-hole quasiparticle excitations. The quadratic dispersion (13) in hydrodynamic regime will be further modified by the inclusion of electric charge in section III.

At higher values of $\omega \tau_c$, the evolution of the shear response depends on the first Landau parameter $F_s^*$. If the interaction is sufficiently strong, such that the transverse mode persists in the absence of collisions, then the mode propagates undamped for $\omega \tau_c \to +\infty$: this is the limit of transverse zero sound\textsuperscript{50}, in which the system responds out of equilibrium without dissipation, in a reactive way reminiscent of solid elasticity but here occurring only at finite frequency.\textsuperscript{5,54} Then, the transverse zero sound velocity reaches the real constant determined by the numerical solution $s = s_{\text{inf}}$ of the equation\textsuperscript{56}

$$\left( s^2 - 1 \right) \left[ \frac{s}{2} \ln \left( \frac{s+1}{s-1} \right) - 1 \right] = \frac{F_s^* - 6}{3 F_1^s}. \quad (14)$$

Strictly speaking, only in this limit the labeling ‘sound’ for the transverse mode is fully justified, since $\omega \propto q$. For

FIG. 1. Dispersion relation of the transverse shear collective mode from equation (8). (a) Dimensionless wave vector $q v_F^* \tau_c$ as a function of dimensionless frequency $\omega \tau_c$ for first symmetric Landau parameter $F_1^s = 20$. The blue and red curves show the real and imaginary part of the mode wave vector, respectively. The shaded purple area indicates the Lindhard electron-hole continuum of quasi-particle excitations, from which transverse sound emerges. The black horizontal line indicates the elastic-like limit $\omega \tau_c = 1$, according to $\text{Re} \{q\} < \omega / v_F^*$. Figure 1(b) shows the corresponding real and imaginary parts of the mode velocity (9b) as a function of $\omega \tau_c$, as blue and red solid lines respectively.

In the hydrodynamic/collisional regime we have $\omega \tau_c \ll 1$: the Lindhard electron-hole continuum of quasiparticle excitations.

\begin{align*}
\text{Re} \{q v_F^* \tau_c\} &\quad \text{and} \quad \text{Im} \{q v_F^* \tau_c\}, \quad \text{respectively. The former represents the dissipationless, i.e. reactive, component of the collective mode responding to an excitation of real frequency $\omega$, while the latter encodes dissipative processes and is connected to the mode damping. The purple shadow}\end{align*}
strong interactions $F_1^S \gg 1$, equation (14) approaches the simpler analytical result $s = s_{el}$ satisfying

$$s_{el} = \sqrt{\frac{1}{5} \left(1 + \frac{F_1^S}{3}\right)} = \sqrt{\frac{v_F^2}{\nu_{el}^2}} : F_1^S \gg 1$$

with

$$\nu_{el} = \frac{1}{5} \frac{(v_F)^2}{1 + F_1^S/3}$$

reactive shear modulus of the Fermi liquid. We see that, for strong interactions, the Fermi surface resonates with transverse zero sound in a reactive way, characterized by the shear modulus $\mu_s$ – see figure 1(c). Notice the relation $\nu(0)/\tau_c$ between the Fermi-liquid static viscosity and reactive shear modulus. Reducing the interaction to $F_1^S \rightarrow 6^+$, the exact mode velocity (14) approaches the renormalized Fermi velocity, i.e. $s \rightarrow 1^+$, and it significantly differs from the elastic-like estimation (15). This is shown in figure 1(c) as a function of $F_1^S$, and by the dashed blue and gold horizontal lines in figure 1(b) for $s = s_{el}$ and $s = s_{el}$, respectively. Such discrepancy is a consequence of the mode velocity being only slightly higher than $v_F^s$, so that the mode significantly suffers from Landau damping, the mode velocity being only slightly higher than $v_F^s$, so that the mode significantly suffers from Landau damping. Mathematically, this is expressed by equation (8) not having any solution, damped or undamped, for $\omega \tau_c = (\omega \tau_c)_{crit}$. Such a disappearance of solutions above a critical value $\omega \tau_c = (\omega \tau_c)_{crit}$ was pointed out by Brooker for longitudinal sound in a Fermi liquid, and reported by Lea et al. for transverse sound: physically, the Fermi surface stops resonating with collective modes in these conditions, and only incoherent electron-hole quasiparticles can be excited.

The transverse response of a Fermi liquid for $F_1^S < 6$ is devoid of transverse collective modes above an $F_1^S$-dependent product $(\omega \tau_c)_{crit}$ between frequency $\omega$ and collision time $\tau_c$: equation (8) has no solution, either real or complex, for $\omega \tau_c > (\omega \tau_c)_{crit}$. In this section we detail how to find $(\omega \tau_c)_{crit}$ for transverse sound, following the analogous treat-
ment by Brooker for longitudinal sound. First, we notice that the real and imaginary parts of equation (8) are discontinuous when \( \xi \) is a real number between \(-1\) and \(1\), because of the discontinuity inherent in the logarithmic term \( \frac{s}{\xi} \ln \left( \frac{s}{\xi} \right) \) in passing from the lower half of the complex plane to the upper half or vice versa. Such discontinuity occurs for \( \xi \in (-1, 1) \): if this happens at a real value of \( (\omega \tau_c)_{\text{crit}} \), that means that the transverse wave solution existing for \( \omega \tau_c < (\omega \tau_c)_{\text{crit}} \) is interrupted by the discontinuity. Therefore, in order to find the critical value \( (\omega \tau_c)_{\text{crit}} \) at a given interaction \( F^S \), we first impose that \( \xi \) is real, which means \( \text{Im} \left\{ s \left( 1 + \frac{i}{\omega \tau_c} \right) \right\} = 0 \) from equation (9). For \( s \in \mathbb{C} \), i.e. damped solutions, we have \( \text{Im} \left\{ s \left( 1 + \frac{i}{\omega \tau_c} \right) \right\} = \text{Im} \left\{ s \right\} + \text{Re} \left\{ s \right\} \frac{1}{\omega \tau_c} = 0 \), so \( \text{Im} \left\{ s \right\} = -\frac{\text{Re} \left\{ s \right\}}{\omega \tau_c} \). We impose the latter condition in equation (8), which now depends on \( \text{Re} \left\{ s \right\} \) and \( \omega \tau_c \). We then solve the imaginary part of equation (8) numerically for \( \text{Re} \left\{ s \right\} \), and we insert the latter found value in the real part of equation (8), solving numerically for real-valued \( \omega \tau_c = (\omega \tau_c)_{\text{crit}} \). The result is shown in figure 3. The red-colored region in figure 3 identifies the viscous regime, defined as the regime where a transverse damped sound, or relaxational mode, exists for an interaction comprising only the first Landau parameter \( F^S \). The blue-colored area marks the Lindhard regime, where no collective modes are allowed in the neutral Fermi liquid. The red solid line separating the viscous and Lindhard regions is the critical value \( (\omega \tau_c)_{\text{crit}} \) which determines the disappearance of transverse damped sound, as found from equation (8) for \( F^S = 6 \). Such line approaches zero at \( F^S = -3 \): there, the Fermi liquid ground state becomes unstable against collective modes, which means a negative effective interaction \( A^S = \frac{F^S}{F^S - 3} \) between quasiparticles. The corresponding criterion for longitudinal sound is \( F^S = -1 \). Notice that any solution for the sound velocity \( v_s = \frac{\omega}{k} \) will always be complex-valued, i.e. damped, at any allowed \( \omega \tau_c < (\omega \tau_c)_{\text{crit}} \), for \( F^S = 6 \). The blue arrow identifies the Fermi gas limit, meaning truly noninteracting quasiparticles, that is \( F^S = 0 \): here we see that the Fermi gas sustains a relaxational mode until \( \omega \tau_c < (\omega \tau_c)_{\text{crit}} \approx 0.87 \), a mechanism analogous to the disappearance of longitudinal sound in a Fermi gas.\(^{62}\) For \( F^S > 6 \) a smooth crossover takes place between the hydrodynamic/collisional character of the viscous regime and the propagating shear regime, denoted by the gold-colored area: in the latter regime, the collective mode lies at frequencies higher than the electron-hole continuum, thus becoming propagating transverse zero sound. The dashed orange curve marks the product \( \omega \tau_c \) above which the collective mode emerges from the electron-hole continuum and starts propagating, as shown in equation (17).

\[
\omega = \frac{\omega_c}{\sqrt{q}} = 1.
\]

III. TRANSVERSE RESPONSE OF CHARGED FERMI LIQUIDS

Electrons in standard metals compose an electrically charged Fermi liquid. In the kinetic equation approach of appendix A, the inclusion of electric entails that quasiparticles perceive finite electromagnetic potentials \( \phi_e (r, t) \) and \( A(r, t) \) – see the quasiclassical force (A5) – even in the absence of external fields.\(^{64}\) Therefore, the interactions among Landau quasiparticles are modified by the long-ranged Coulomb interaction \( V_{\text{Coul}} = \frac{e^2}{r^2} \), which poses a challenge for the Fermi-liquid picture since \( V_{\text{Coul}} (q) \) is divergent for \( q \rightarrow 0 \), while in neutral systems Landau theory assumes short-ranged quasiparticle interactions with a well-defined long-wavelength limit. The solution was first found by Silin\(^{41,42}\) effects due to Coulomb interactions can be separated into a long-ranged part, representing the classical polarization field that provides dielectric screening, and a short-ranged quantum component, driven by the virtual creation of electron hole pairs around a charged particle. The polarization field cuts the range of Coulomb interactions to a finite range by dielectric screening, while electron-hole quantum fluctuations modify the short-ranged interactions with respect to the neutral case. The consequence is that the long-ranged and spherically symmetric polarization screens quasiparticle interactions in the isotropic \( l = 0 \) channel, thereby modifying the value of \( F^S \) only. This alters Landau parameters according to \( F^S = \frac{e^2 N_e (0)}{e_0 q^2} \) for \( F^S > 0 \). In particular, for transverse excitations in the \( l = 1 \) channel, electric charge does not introduce any momentum dependence of \( F^S \). The only consequence of Coulomb interaction is that \( F^S \) differs from the corresponding value in the neutral system,
due to the electron-hole short-ranged quantum component. Furthermore, one can expect that the dispersion relations of collective modes are modified by the presence of electric charge. The transverse $f = 1$ channel is particularly intriguing: such transverse excitations can couple to photons, which allows one to probe Fermi liquid collective modes in solids using electromagnetic fields. With this in mind, we approach the analysis of the transverse susceptibility, which models the current response.

A. Interacting transverse susceptibility in the absence of collisions

In order to study the response of the Fermi liquid to transverse perturbations at any momentum and frequency, we have to calculate the full transverse susceptibility, which is the proportionality coefficient between the transverse current density $J_x(q, \omega)$ and the respective Fourier component of the applied vector potential $A_x(q, \omega)$. We first consider the case with no short-range interactions between quasiparticles and no collisions.\(^5\) The derivation is reported in appendix C for the interested reader, while here we only quote the final result. It is

$$X_T^0(q, \omega) \equiv X_T(q, \omega) = \frac{3nm^*}{m^2} \mathcal{J}(s)$$

$$= \frac{3nm^*}{4m^2} \left[ -\frac{4}{3} + 2s^2 + s(1-s) \ln \left( \frac{s+1}{s-1} \right) \right]$$

(18)

with the integral

$$\mathcal{J}(s) = \frac{1}{4} \int_0^{\pi} (\sin \theta)^3 \cos \theta \frac{d\theta}{s - \cos \theta}$$

$$= -\frac{1}{3} + \frac{1}{2} s^2 + \frac{s}{4} (1-s^2) \ln \left( \frac{s+1}{s-1} \right).$$

(19)

We see that the noninteracting transverse susceptibility (18) only depends on the ratio $\frac{\omega}{q} \propto v_F$, and not on momentum and frequency separately, in the absence of collisions.

In the static limit $\omega = 0$, equation (18) gives $X_T^0(0) = -\frac{n}{m} \left( 1 + \frac{F_1}{3} \right)$. In the quasi-static nonlocal limit $\omega \ll q$, an expansion of the integral $\mathcal{J}(s)$ around $s = 0$ gives $\mathcal{J}(s) = -\frac{1}{2} + i \pi s^2 + o(s^2)$; this gives $X_T^0(0) = \frac{2}{3} \left[ 1 - \left( 1 + \frac{F_1}{3} \right) i \pi s^2 \right]$. In the local limit $q = 0$, we have $s \to +\infty$ and $\lim_{s \to +\infty} \mathcal{J}(s) = 0$, so that $\lim_{s \to +\infty} X_T^0(s) = 0$. Quasiparticle collisions modify the result (18) as analyzed in the following section.

B. Interacting transverse paramagnetic susceptibility with collisions

When we include short-ranged quasiparticle interactions and collisions, we have to employ the full kinetic equation (6). From there, we perform the same steps as in section II A, but now explicitly including an external vector potential $A(q, \omega)$. The result is

$$\langle \cos \theta - s \rangle e^{S(\theta)e^{i\phi}} + \cos \theta \int_0^{2\pi} \int_0^{\pi} d\phi' \sin \theta' d\theta'$$

$$\left\{ \begin{array}{c}
F_T^0 \bigg[ \frac{\cos \theta \cos \phi' + \sin \theta \sin \phi' \cos (\phi - \phi')}{e^{S'(\theta)e^{i\phi'}}} \bigg] e^{S'(\theta)e^{i\phi'}}
\end{array} \right. + F_T^1 \left[ \cos \theta \cos \phi' + \sin \theta \sin \phi' \cos (\phi - \phi') \right]$$

$$- \frac{e}{mv_F} \cos \theta k \cdot A(q, \omega) = \frac{1}{qv_F} \mathcal{J}_{\text{coll}}(q, \omega)$$

(20)

The integration along $\phi'$ gives zero for the terms $A$ and $B$ in equation (20), and we are left with

$$\langle \cos \theta - s \rangle e^{S(\theta)e^{i\phi}}$$

$$+ \cos \theta \left[ \int_0^{\pi} \sin \theta' d\theta' \right] e^{S(\theta)e^{i\phi}}$$

$$= \frac{e}{mv_F} \left( k \cdot A(q, \omega) \right)$$

(21)

We now employ the parametrization (7) for the collision integral $\mathcal{J}_{\text{coll}}(q, \omega)$, as done for the neutral case of section II A. Hence, we assume that the inclusion of electric charge does not qualitatively modify the collision integral. In terms of the variables (9), the kinetic equation becomes

$$\langle \cos \theta - \xi \rangle e^{S(\xi)} + \left[ 3e^{S(\xi)} \sin \theta \right] \sin \theta \left[ \frac{F_3}{3} \cos \theta - \beta \xi \right]$$

$$= \cos \theta \frac{e}{mv_F} k \cdot A(q, \omega)$$

(22)

We now utilize the definition of the paramagnetic current density in a Fermi liquid, $J_\rho(q, \omega) = \frac{i}{2} \sum_{k,\sigma} \frac{\hbar^2}{m} v_F \delta(\xi_k) \epsilon_{k,\sigma}(q, \omega)$ and we define the paramagnetic susceptibility as the ratio between aforementioned current density $J_\rho(q, \omega)$ and the vector potential $A(q, \omega)$. The details of this calculation are reported in appendix D. The final result for the interacting paramagnetic transverse response function is

$$X_T^\rho(\xi) = \frac{X_T^0(\xi)}{1 - 3 \left( \frac{F_1}{3} - \beta \right) \frac{m^2}{3nm^*} X_T^0(\xi) + \beta}$$

$$= \frac{n}{m} \frac{3 \left( 1 + \frac{F_1}{3} \right) \mathcal{J}(\xi)}{1 - 3 \left( \frac{F_1}{3} - \beta \right) \mathcal{J}(\xi) + \beta}$$

(23)
where \( X^p_T(\xi) \) is the noninteracting transverse response function (18) in terms of \( \xi \).

Now we have the microscopic expression (23) for the transverse paramagnetic susceptibility of a Fermi liquid in Landau theory, for an interaction of the form \( F^p_0 + F^p_1 \cos \alpha \). Arguments similar to the ones developed so far apply to the longitudinal susceptibility \( X^c_T(q, \omega) \). The result (23) allows us to study the electromagnetic response of the charged system.

IV. FERMI-LIQUID OPTICAL CONDUCTIVITY AND DIELECTRIC FUNCTION

From the paramagnetic response function (23), we can calculate the optical properties of the charged Fermi liquid. In particular, the transverse dielectric function is obtained in two steps: first we calculate the optical conductivity by the means of the Kubo formula for a translation-invariant system:

\[
\sigma_{\tau,\mu\nu}(q, \omega) = \frac{i e^2}{\omega} \left[ \chi_{\mu\nu}^c(q, i\Omega_n) \right]_{\omega \to \omega + i\Omega_n} + \delta_{\mu\nu} \frac{n}{m},
\]

(24)

where \( \chi_{\mu\nu}^c(q, \omega) \) is the current-current correlation function. We take the diagonal component \( \mu = \nu \), so that the current-current correlation function corresponds to our transverse susceptibility \( X^c_T(q, \omega) \):

\[
\sigma_T(q, \omega) = \frac{i e^2}{\omega} \left[ X^c_T(q, \omega) + \frac{n}{m} \right].
\]

(25)

The second term in square brackets in equation (25) is the diamagnetic susceptibility, which is connected to the diamagnetic part of the current response. Then, we can write

\[
\sigma_T(q, \omega) = \frac{i e^2}{\omega} \left[ X^c_T(q, \omega) + \frac{n}{m} \right] = \frac{i e^2}{\omega} X_T(q, \omega).
\]

(26)

In equation (26) we have defined the total current susceptibility \( X_T(q, \omega) \), which considers both the paramagnetic term (23) and the diamagnetic term \( \frac{n}{m} ; \)

\[
X_T(q, \omega) = X^c_T(q, \omega) + \frac{n}{m} \equiv \frac{n}{m} \left( 1 + \beta \right) \left[ 3 \mathcal{S}(\xi) + 1 \right]
\]

(27)

In the collisionless limit \( \tau_c \to +\infty \), from equation (27) we retrieve the known result

\[
\lim_{\tau_c \to +\infty} X_T(\xi) \equiv X_T(s) = \frac{X^c_T(s)}{1 - F_1^3 \left[ \frac{n}{m} \chi^c_T(\xi) \right]} = \frac{n}{m} \left( 3 \mathcal{S}(\xi) + 1 \right)
\]

(28)

Employing the general relation

\[
\sigma_T(q, \omega) = -i e \omega \left[ \epsilon_T(q, \omega) - 1 \right],
\]

(29)

we arrive at the transverse dielectric function:

\[
e_T(q, \omega) = 1 - \frac{e^2}{\epsilon_0 \omega^2} \left[ X^c_T(q, \omega) + \frac{n}{m} \right]
\]

(30)

Using equations (30) and (26), we have

\[
e_T(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2} \left[ \frac{1 + 3 \mathcal{S}(\xi) + 1}{1 - 3 \left( \frac{\omega_p}{\omega} - \beta \right) \mathcal{S}(\xi) + \beta} \right]
\]

(31)

where \( \omega_p = \sqrt{n e^2/(m \epsilon_0)} \) is the electron plasma frequency. The pole of the dielectric function (31) corresponds exactly to the transverse sound dispersion relation (8). Indeed, in general the poles of \( e_T(q, \omega) \) yield the collective modes existing in the matter sector of the theory, i.e. the modes which exist in the material in the absence of external electromagnetic fields. In the collisionless limit \( \tau_c \to +\infty \), the transverse dielectric function (31) reduces to

\[
\lim_{\tau_c \to +\infty} e_T(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2} \left[ \frac{1 + \beta \mathcal{S}(\xi)}{1 - F_1^3 \mathcal{S}(\xi)} \right]
\]

(32)

The self-consistent solutions of Maxwell equations inside the Fermi liquid give the collective modes of the system in the charged case. The latter modes are polaritons, i.e. modes of electromagnetic radiation coupled to the Fermi-surface oscillations. Formally, these collective modes satisfy

\[
\frac{q^2 c^2}{\omega^2} = e_T(q, \omega).
\]

(33)

We can obtain analytical solutions by expanding equations (33) and (31) in various physically important limits: the leading term for \( s \to +\infty \) at finite \( \tau_c \) yields the phenomenology of a ‘dynamical viscoelastic’ charged fluid, whereby the Fermi liquid responds to radiation as a continuum characterized by a generalized shear modulus \( \nu(\omega) \) in accordance with the laws of continuum mechanics. On the other hand, in the \( s \to 0 \) regime at finite \( \tau_c \), we retrieve the theory of anomalous skin effect, long-known to the optics community to be a consequence of nonlocal correlations in the optical response. Finally, in the limit \( \omega \tau_c \to 0 \) at finite \( s \) the response yields electron hydrodynamics. The following sections are devoted to the analysis of all aforementioned regimes.

A. Propagating shear regime

We first analyze the dielectric function (31) in the high-frequency, long-wavelength regime \( \omega >> \nu_f \text{Re} \{ q \} \), i.e. outside of the electron-hole continuum. Then, we have \( s \to \)
The system satisfy equation (33). Using equation (35), we obtain two complex-valued solutions:

\[
q^2 = \frac{1}{2q^2 \nu(\omega)} \left\{ ic^2 \omega + \nu(\omega) \omega^2 \right\}
\]

\[
\pm i \sqrt{\nu(\omega) [c^2 + i \nu(\omega) \omega^2 - 4c^2 \nu(\omega) \omega^2]} \}
\]

Figure 4(a) displays the dispersion relation of the polariton branches in propagating shear regime from equation (37), for first symmetric Landau parameter \(F_s = 20\), renormalized Fermi velocity \(\nu_F = 10^{-3}c\), and plasma frequency fixed by \(\omega_\tau = 1\).

(a) Dimensionless wave vector \(q\), of the polariton branches as a function of dimensionless frequency \(\omega_\tau\). The blue and red curves show the real and imaginary part of the mode wave vector, respectively. The shaded purple area indicates the Lindhard continuum.

(b) Dimensionless wave vector \(q\), of the plasmon-polariton as a function of dimensionless frequency \(\omega_\tau\). The blue and red curves show the real and imaginary part of the mode wave vector, respectively. The inset zooms the same data at low frequencies, showing the quartic dispersion (38) valid for \(\omega \to 0\).
the plasma frequency $\omega_p$: for this reason, here I label this solution as ‘plasmon-polariton’. The other root lies much closer to the Lindhard continuum and emerges from the latter at a $F_{1}^{\nu}$-dependent frequency, analogously to the uncharged chaise of transverse sound: in the following, I refer to this solution as ‘shear-polariton’. In the hydrodynamic/collisional regime $\omega \tau_c \ll 1$, the real part of the generalized shear modulus $\nu(\omega) \approx \nu(0)$ prevails over the imaginary part, giving rise to viscous dissipation. Then, equation (37) admits the leading-order expansions for $q \to 0$

$$q = \sqrt{\frac{\omega_p}{c} \left[ \frac{\omega}{\nu(0)} \right]} \frac{\sin \left( \frac{\pi}{8} \right) - i \cos \left( \frac{\pi}{8} \right)}{\sqrt{2} \mu} + o \left( \sqrt{\omega} \right),$$

(38a)

$$q = \frac{\omega_p}{c} \frac{\sin \left( \frac{\pi}{8} \right) - i \cos \left( \frac{\pi}{8} \right)}{\sqrt{2} \mu} + o \left( \sqrt{\omega} \right),$$

(38b)

for the plasmon-polariton and the shear-polariton, respectively. Equations (38) show that the dispersion is quartic, i.e. $\omega \propto q^4$, in the collisional/hydrodynamic limit, as shown by panel (a) and by the inset of panel (b) in figure 4. For the shear-polariton (38b), this is in contrast with the uncharged case (13), where the dispersion was quadratic. Such difference is due to the combined effects of electric charge and of Landau damping. Firstly, radiation-matter interaction modifies the dispersion of the shear-polariton at momenta $q \sim \omega_p/c$ with respect to uncharged transverse sound, due to the latter being ‘repelled’ by the nearby photon root $\omega = cq$ as a result, the charged shear-polariton resulting from the photon-matter mixing acquires quadratic dispersion and is pushed down inside the electron-hole continuum – see also section IV C. Such ‘quadratic bending’ of the dispersion due to electromagnetic forces is also retrieved for reactive shear stresses in the isotropic version of the Wigner crystal formed by charged bosonic constituents, brought about by the propagation of electromagnetic fields. A similar mechanism is also at play in particular viscoelastic-like holographic models of strange metals. Secondly, at vanishing frequencies deep inside the continuum, Landau damping further modifies the shear-polariton dispersion to the quartic result (38b).

In the opposite, collisionless regime $\omega \tau_c \gg 1$, the generalized shear modulus $\nu(\omega) \approx \nu(0)$ is predominantly imaginary, which leads to a dissipationless reactive dispersion. Then, equation (37) is expanded for $\omega \to +\infty$ as

$$q = \frac{\omega}{c} + o(\omega^{-2}),$$

(39a)

$$q = \frac{\omega}{\sqrt{\mu}} + o(\omega^{-2}),$$

(39b)

for the plasmon-polariton and shear-polariton, respectively; with $\mu$, Fermi-liquid shear modulus (16). Physically, equation (39a) is the standard phenomenon whereby the plasmon dispersion asymptotically reaches the uncoupled photon root $\omega = cq$ at very large frequencies $\omega \gg \omega_p$, where radiation does not feel the coupling to the fermionic quasiparticles. Furthermore, equation (39b) is equivalent to the uncharged case (15): at high momenta $q \gg \omega_p/c$, the Fermi-surface oscillation is essentially uncoupled from radiation, and we recover the dispersion of uncharged transverse zero sound, which is analogous to a dynamical version of elasticity in the limit $\omega \gg v_{F}^{\nu} \Re \{q\}$, that is $F_{1}^{\nu} \gg 1$. In conclusion, equations (39) tell us that, in the high-frequency limit, radiation decouples from the Fermi-surface oscillation.

B. Momentum dependence of the generalized shear modulus

As mentioned in section IV, starting from the limit $\omega/(v_{F}^{\nu} \Re \{q\}) \to +\infty$, an expansion in $q \to 0$ of equation (31) at order $q^{2n}$, $n \in \mathbb{N}$ gives $2n$ frequency-degenerate charge collective modes. Such procedure corresponds to a gradient expansion for the coordinate-dependent current density in real space, of which the first term defines the generalized shear modulus. Therefore, $\epsilon_{\parallel}(q, \omega)$ entails an increasing number of polaritons for progressively higher momenta, up to the limit $\omega/(v_{F}^{\nu} \Re \{q\}) \to 1$ where the expansion breaks down. Technically, terms of order higher than $q^{2}$ may be reformulated in terms of a momentum dependence of the generalized shear modulus $\nu(\omega)$. Thus, the latter becomes a scale-dependent quantity in real space, as it occurs in 2D systems. Such an analysis for Fermi liquids in three dimensions is left for future work, while this paper focuses on the leading-order $q^{2}$ correction to the dielectric properties.

C. Emergence of the shear collective mode from the Lindhard continuum

As mentioned in section IV A, the results there presented hold whenever $\omega \gg v_{F}^{\nu} \Re \{q\}$ and the system sustains two polariton modes. The latter condition breaks down when the slower-dispersing mode, i.e. the shear-polariton, submerges into the Lindhard electron-hole continuum at frequencies $\omega \leq v_{F}^{\nu} q$. Such constraint on the shear-polariton can be deduced directly from its dispersion (37), by equating the latter to (17). Figure 5 shows the frequency $\omega_{CH,VE}$ obtained from the numerical solution of the system (37), (17), as a function of the first Landau parameter $F_{1}^{\nu}$. We see that, within the assumptions of section IV A, the shear-polariton never exits from the continuum if $F_{1}^{\nu} < 12$.

More generally, we can obtain a precise statement on the emergence of the shear-polariton from the continuum directly from the uncharged dispersion relation (8) for transverse sound. In fact, since the upper bound on the renormalized Fermi velocity is $v_{F}^{\nu}/c \approx 0.001$ in standard metals, the condition $\omega = v_{F}^{\nu} q$ also implies $\omega \ll c q$. Therefore, in this regime we can let $c q / \omega \to +\infty$ at the left-hand side of equation (33), which means that the polariton solution reduces to the pole of the dielectric function (31), i.e. to un-
charged transverse sound: the collective mode is the same as in the uncharged system.\footnote{1} Furthermore, a microscopic analysis of the Fermi-liquid collision time $\tau_c(\omega, T)$ – see section VIII – implies that, for first Landau parameter $F_1 > 6$, the condition (17) occurs for $\omega \tau_c(\omega, T) \gg 1$. The latter implies that $\xi \approx s$ in equation (8). Moreover, one verifies that, in such regime, the shear collective mode is underdamped, i.e. Re $\{q\} \gg \text{Im } \{q\}$. Therefore, to a high degree of accuracy we can assume $s = \omega/(\nu_p q) \in \mathbb{R}$ in equations (8) and (17). Then, the latter two equations can be solved analytically for $\omega$, which gives

$$\omega_{ch} = \frac{\omega_p}{c} \frac{3\nu_p^2}{\sqrt{(F_1^2 - 6)\left[1 - (\nu_p q)^2\right]^2}}.$$  \hspace{1cm} (40)

Equation (40) produces the dashed black curve in figure 5. It confirms the early order-of-magnitude estimate by Nozières and Pines,\footnote{1} that is the shear-polariton plunges into the continuum at momenta lower than Re $\{q\} = \omega / \nu_p \approx \omega_p / \epsilon$. Consistently with section II B, the shear-polaron never exits from the continuum if $F_1^2 < 6$, since equation (40) implies $\lim_{\nu_p q \to 6} \omega_{ch} = +\infty$. The discrepancy between $\omega_{ch,V_F}$ and $\omega_{ch}$ underlines the limits of the continuum-mechanics picture for shear modes: for $\omega_{ch} < \omega < \omega_{ch,V_F}$ the shear-polariton still propagates in the system, however this propagation cannot be described with continuum mechanics applied to the Fermi liquid as a whole. This reflects the arguments of section IV B: when the shear-polariton starts approaching the continuum from above, that is $\omega/(\nu_p q) \to 1^+$, higher-order momentum corrections with respect to the $q^2$-expansion (35) become non-negligible, which limits the validity of the model (35) for the dielectric response.

D. Anomalous skin effect regime

In this section we analyze the Fermi-liquid optical conductivity in the low-frequency, high-momentum regime $\text{Re} \{q\} \nu_p \gg \omega$ (quasi-static limit\footnote{1}) at finite $\omega \tau_c$. An expansion of equation (26) at leading order in $s \to 0$ at finite $\tau_c$ gives

$$\sigma_T(q, \omega) = \epsilon_0(\omega_p) \frac{3\pi}{4} \frac{1}{\nu_p q} :$$ \hspace{1cm} (41)

this represents dissipative processes in which radiation transfers energy to electrons moving perpendicularly to $q$ – see equation (3.122) in reference 1. Using equation (29), we obtain the transverse dielectric function:

$$\varepsilon_T(q, \omega) = 1 + i \frac{(\omega_p)^2}{\omega} \frac{3\pi}{4} \frac{1}{\nu_p q}.$$  \hspace{1cm} (42)

The collective modes of the charged system satisfy equation (33). Using the dielectric function (42), we arrive at

$$\frac{q^2 c^2}{\omega^2} = 1 + i \frac{(\omega_p)^2}{\omega} \frac{3\pi}{4} \frac{\omega}{\nu_p q}.$$  \hspace{1cm} (43)

In general, equation (43) has three polariton roots for $q$ in the complex plane. However, for $\omega_p q \to 0$ we can approximately write

$$\left[1 + i \frac{(\omega_p)^2}{\omega} \frac{3\pi}{4} \frac{\omega}{\nu_p q}\right]^{-1} = \frac{\omega^2}{q^2(\nu_p q)^2} \approx 0.$$  \hspace{1cm} (44)

Employing the series expansion $(1 + x)^{-1} = 1 - x + o(x^2)$ in equation (44) with $x = \frac{(\omega_p)^2}{\omega} \frac{3\pi}{4} \frac{\omega}{\nu_p q} \to 0$, we obtain

$$q = \frac{3\pi}{4} \frac{\omega_p q}{\nu_p},$$  \hspace{1cm} (45)

which is an entirely imaginary dispersion, again signaling a dissipative coupling between radiation and Fermi-surface quasiparticles. Physically, this happens because photons excite only incoherent electron-hole pairs in the Lindhard continuum for $\omega \ll \nu_p q$, contrarily to the regime $\omega \gg \nu_p q$ of section IV A where the shear mode (8) in the matter sector of the theory strongly couples to radiation and produces the polaritons (37).

Notice that the optical conductivity (41) is entirely real and is independent of interactions and of frequency, but only depends on momentum in the quasi-static limit:\footnote{1,46} it is exactly the momentum-dependent optical conductivity used in the theory of anomalous skin effect.\footnote{46,47} Hence, we can directly use the latter formalism to analyze the optical properties of charged Fermi liquids in the low-temperature quasi-static limit, as we will appreciate in section IX B 2.
V. HYDRODYNAMIC REGIME

A qualitatively distinct regime arises when $\tau_c$ is the smallest timescale in the system: in such condition, the resulting local thermodynamic equilibrium among quasiparticles allows for hydrodynamic flow. For the dielectric function (31), this happens in the low-frequency and low-momentum regime, so that we can expand along radial lines in the $(\omega, \text{Re} \{q\})$ plane at fixed finite $s$ and $\omega \tau_c \to 0$. For a clean Fermi liquid characterized by the collision time $\tau_c \propto [((\hbar \omega)^2 + (\pi k_B T)^2)^{-1}$ – see section VII and appendix F – we have $\omega \tau_c \to +\infty$ at any temperature and $\omega \to 0$ at $T > 0$, while at $T = 0$ $\omega \tau_c \propto 1/\omega \to +\infty$ if $\omega \to 0$. This qualitative distinction between the zero- and finite-temperature cases produces ‘viscous’ hydrodynamic behaviour only in the latter case at vanishing frequencies. The leading-order expansion of the dielectric function (31) in $\omega \tau_c \to 0$ at fixed $s$ produces

$$\epsilon_\tau(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2} \left[ 1 - \frac{1}{5} \left( 1 + \frac{F_5}{3} \right) \frac{\omega \tau_c}{s^2} \right]. \quad (46)$$

As done in section IV A, we can resum equation (46) with $\frac{1}{1 + \tau_c} = 1 - x + o(x^2)$ for $x = \frac{\omega \tau_c}{s^2} \to 0$, so that

$$\epsilon_\tau(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2 + i \omega v(0) q^2}. \quad (47)$$

The dielectric function (47) describes the electromagnetic response of viscous charged fluids, written in terms of the static Fermi-liquid shear viscosity $\nu(0)$ – see equation (11). It corresponds to the limit of equation (35) for $\omega \tau_c \to 0$, consistently. Hence, there are still two polariton branches in hydrodynamic regime, which formally follow the dispersion (37) upon substituting $\nu(\omega)$ with $\nu(0)$. In particular, the ‘shear-polariton’ root is critically damped in hydrodynamic regime, i.e. its dispersion has equal real and imaginary parts, see equation (38b) in section IV A.

For later discussions on the DC conductivity, it is convenient to emphasize the dichotomy between the low-frequency, low-momentum Fermi-liquid electrodynamics in the $T = 0$ and $T > 0$ cases. At zero temperature $\omega \tau_c$ diverges, so that the limit $\omega \to 0$ is equivalent to $s \to 0$, hence we retrieve the results of section IV D: anomalous skin effect. On the other hand, at finite temperature $\omega \tau_c$ vanishes at vanishing frequency, so that the latter condition gives equation (47), i.e. hydrodynamics.

Figure 6 summarizes the different regimes for the dielectric response analyzed so far. The situation will be complicated by the presence of momentum relaxation, as analyzed in the next sections.

VI. OPTICAL CONDUCTIVITY WITH MOMENTUM RELAXATION

In crystalline solids, the presence of the ionic lattice, of defects and impurities invariably breaks Galilean invariance, so that quasiparticle momentum is not a conserved quantity. This induces momentum relaxation in the electron ensemble, which is essential in preventing the divergence of conductivities in metals and implies a finite mean free path

$$l_{\text{MFP}} = v_F \tau_k, \quad (48)$$

renormalized by interactions as $l_{\text{MFP}}^* = v_F^* \tau_k$. Therefore, we expect that the electrodynamic functions obtained in section IV A, valid for Galilean-invariant Fermi liquids, will be altered in the presence of momentum-relaxing processes. The interplay of momentum-conserving collisions and momentum-relaxing scattering depends on the specific lattice symmetry and quasiparticle dispersion. Its complexity has been carefully analyzed in two-dimensional strongly-interacting systems. While such detailed analysis is beyond the scope of the present paper, a qualitative discussion is in order here, to assess whether shear stress propagation can be observed in solid-state 3D Fermi liquids. Essentially, broken translational symmetry implies that quasiparticle momentum is not a conserved quantity.

Figure 6. Sketch of the different regimes for the dielectric response of charged Fermi liquids at $T > 0$ in the $(\text{Re} \{q\}, \omega)$ plane: the condition $\omega \gg v_F^* \text{Re} \{q\}$ corresponds to the propagating shear regime (section IV A); at both vanishing momentum and frequency and for $\omega \tau_c \to 0$ the response becomes hydrodynamic (section V); in the high-momentum, low-frequency regime $\omega \ll v_F^* \text{Re} \{q\}$ the Lindhard continuum dominates the response and produces anomalous skin effect (section IV D). At $T \neq 0$ the hydrodynamic regime is absent – see discussion in section V.

FIG. 6. Sketch of the different regimes for the dielectric response of charged Fermi liquids at $T > 0$ in the $(\text{Re} \{q\}, \omega)$ plane: the condition $\omega \gg v_F^* \text{Re} \{q\}$ corresponds to the propagating shear regime (section IV A); at both vanishing momentum and frequency and for $\omega \tau_c \to 0$ the response becomes hydrodynamic (section V); in the high-momentum, low-frequency regime $\omega \ll v_F^* \text{Re} \{q\}$ the Lindhard continuum dominates the response and produces anomalous skin effect (section IV D). At $T \neq 0$ the hydrodynamic regime is absent – see discussion in section V.
To include damping of transverse currents due to momentum relaxation, we modify the kinetic equation (21) into
\[
(\cos \theta - s)e^S(\theta)e^{i\phi} + \cos \theta \left[ \int_0^\tau \sin \theta'd\theta' \frac{F^S_i}{4\pi} \sin \theta \sin \theta' \pi e^{i\phi} e^S(\theta') \right] - \frac{e_i}{mv_F^S} \mathbf{k} \cdot \mathbf{A}(q, \omega) = \frac{1}{qv_F^S} \mathcal{S}(q, \omega) + \frac{1}{qv_F^S} \mathcal{S}_{\text{coll}}(q, \omega),
\]
where \( \mathcal{S}(q, \omega) \) describes momentum-relaxing scattering. In order to relate the solution of equation (49) to the one without momentum relaxation, reference 9 outlines a convenient approach: using a single-relaxation time approximation, we can write
\[
\frac{1}{qv_F^S} \mathcal{S}(q, \omega) = -\frac{e^S(\theta) - e^S(\theta)}{i\omega\tau_k} e^{i\phi},
\]
so that momentum damping attempts to restore a 'locally relaxed' equilibrium distribution function, characterized by the displacement \( e^S(\theta) \), which would be in equilibrium in the absence of collisions and in the presence of a vector potential \( \mathbf{A}(q, \omega) \). The scattering processes described by the relaxation time \( \tau_k \) conserve particle number but not current. In this case, the transverse susceptibility in the presence of relaxation and collisions may be written in terms of the one without relaxation as outlined in appendix E.

Explicitly
\[
X_F^T(q, \omega) = \frac{n}{m} \frac{\omega}{\omega + i\tau_k} \left( 1 + \beta \right) \left[ 3\mathcal{S}(\xi) + 1 \right],
\]
where
\[
\xi = \xi + s + \frac{i}{\omega\tau_k} = \frac{s}{1 + \frac{i}{\omega\tau_k}} + \frac{i}{\omega\tau_k},
\]
and we have defined
\[
\tau_m = \left[ \frac{1}{\tau_c} + \frac{1}{\tau_k} \right]^{-1}.
\]

Physically, equation (52) tells us that the nonlocal part of the dielectric response is dominated by the smallest timescale in the system, the latter being related either to momentum relaxation or to collisions in accordance with the average (53). Once inserted into the Kubo formula (26), the momentum-relaxing susceptibility (51) yields the optical conductivity
\[
\sigma_T(q, \omega) = \frac{ie^2}{\omega} X_F^T(q, \omega) = \frac{ine^2}{m} \frac{(1 + \beta)\left[ 3\mathcal{S}(\xi) + 1 \right]}{1 - 3\left( \frac{\tau_c}{\tau} - \beta \right) \mathcal{S}(\xi) + \beta}.
\]

The dielectric function descending from equations (54) and (29) is
\[
e_T(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2} \frac{(1 + \beta)\left[ 3\mathcal{S}(\xi) + 1 \right]}{(\omega + \frac{i}{\omega\tau_m}) - 3\left( \frac{\tau_c}{\tau} - \beta \right) \mathcal{S}(\xi) + \beta}. \tag{55}
\]

Equations (54) and (55) are the fundamental results of this section. In the following, we will specialize them to the regimes \( \omega \gg v_F^S \text{Re} \{ q \} \) and \( \omega \ll v_F^S \text{Re} \{ q \} \), to connect the present Fermi-liquid kinetic theory with the electrodynamics of viscous charged fluids, with the Drude model of metals, and with anomalous skin effect.

A. Propagating shear regime

Momentum relaxation alters the results (35) for the dielectric function in the regime \( \omega \gg v_F^S \text{Re} \{ q \} \). To see this, we start from equation (55) and we perform an expansion of the latter in \( s \to +\infty \), analogous to the one in section IV A. The result depends on the ratio between the quantities \( \omega \tau_c, \omega \tau_k, qv_F^S \tau_c \) and \( qv_F^S \tau_k \). Taking the limit \( s \to +\infty \) at finite \( \omega \tau_k \) and \( \omega \tau_c \), gives at leading order
\[
e_T(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2 + i\omega \tau_c} \frac{1}{1 + \frac{F^S_i}{3}} \left( \frac{\tau_c}{(1 - i\omega \tau_m)} \right)^{\frac{1}{2}} \left( \frac{1 + \frac{F^S_i}{3}}{i} \right) \mathcal{S}(\xi) + \beta.
\]

In the present regime, we can resum the \( 1/s \to 0^+ \)-dependent term using the Taylor series \( \frac{1}{1 + \frac{s}{\tau^2}} = 1 - x + o(x^2) \), which gives
\[
e_T(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2 + i\omega \tau_c} \frac{1}{1 + \frac{F^S_i}{3}} \left( \frac{\tau_c}{(1 - i\omega \tau_m)} \right)^{\frac{1}{2}} \left( \frac{1 + \frac{F^S_i}{3}}{i} \right) \mathcal{S}(\xi) + \beta.
\]

The optical conductivity associated with the dielectric function (57a) is
\[
\sigma_T(q, \omega) = \frac{ie^2/m}{\omega + i\tau_k} \frac{in e^2/m}{1 + (1 + \frac{i}{\omega \tau_m})} \mathcal{S}(\xi) + \beta.
\]

Instead, taking the limit \( \xi \to +\infty \), i.e. \( s \to +\infty \) and \( qv_F^S \tau_m \to 0 \), produces
\[
\sigma_T(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2 + i\omega \tau_c} \frac{1}{1 + \frac{F^S_i}{3}} \left( \frac{\tau_c}{(1 - i\omega \tau_m)} \right)^{\frac{1}{2}} \left( \frac{1 + \frac{F^S_i}{3}}{i} \right) \mathcal{S}(\xi) + \beta.
\]
Taking the limit $s \to +\infty$ with $\tau_c \ll \tau_k$ yields

$$
\epsilon_r(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega(\omega + \frac{i}{\tau_c})} \left[ 1 + \frac{1}{3} \left( 1 + \frac{F_{\perp}^2}{\omega^2} \right) \frac{(v_{\perp})^2 q^2}{\omega(i + \omega \tau_c)} \right].
$$

(60)

Resuming the $q$-dependent term in equation (60) at finite $\omega \tau_c$, as done for equation (56), we obtain

$$
\epsilon_r(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2 + i \frac{\omega}{\tau_c} + i(\omega + \frac{1}{\tau_c}) v(\omega)q^2},
$$

(61)

where $v(\omega)$ is the Fermi-liquid generalized shear modulus (36). Equation (61) is the generalization of the dielectric function (35) to finite (weak) momentum relaxation characterized by the time $\tau_k \gg \tau_c$.

Moreover, the term $i/\tau_k v(\omega) \propto \tau_c/\tau_k \to 0$ for $\tau_c \ll \tau_k$, so that

$$
\epsilon_r(q, \omega) \approx 1 - \frac{(\omega_p)^2}{\omega^2 + i \omega + i v(\omega)q^2},
$$

(62)

which is exactly the dielectric function stemming from the macroscopic phenomenology of viscous charged fluids in the presence of the momentum-relaxing term $\omega/\tau_c$. We emphasize that this holds in the regime $\omega \gg v_\parallel^2 \text{Re} \{q\}$, $\tau_k \gg \tau_c$ and $\omega \tau_k \gg 1$: the Fermi liquid responds akin to a viscous charged fluid for excitations of energy far above the electron-hole continuum, and for momentum relaxation occurring at a rate lower than both collision rate and radiation frequency. The optical conductivity corresponding to the dielectric function (62) is

$$
\sigma(q, \omega) = i e_0 \frac{(\omega_p)^2}{\omega + i(\tau_k)^{-1} + v(\omega)q^2}.
$$

(63)

Conversely, if $s \to +\infty$ and $\tau_k \ll \tau_c$ equation (56) becomes

$$
\epsilon_r(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega(\omega + \frac{i}{\tau_c})} \left[ 1 + \frac{1}{3} \left( 1 + \frac{F_{\perp}^2}{\omega^2} \right) \frac{(v_{\perp})^2 q^2}{\omega(i + \omega \tau_c)} \frac{\tau_k \tau_c}{i + \omega \tau_c} \right]
$$

(64)

$$
\approx 1 - \frac{(\omega_p)^2}{\omega(\omega + \frac{i}{\tau_c})},
$$

(65)

which demonstrates that the $q$-dependent nonlocal term is negligible when relaxing processes dominate over collisions, and we retrieve the Drude/Ohmic dielectric function.

$$
\epsilon_r(q, \omega) = 1 - \frac{(\omega_p)^2}{\omega^2 + i \omega / \tau_k}
$$

(66)

for $i\omega v(\omega)q^2 \ll \frac{\omega}{\tau_k}$ — see also the discussion on the DC conductivity in section VII. How are the collision time $\tau_c$ and the relaxation time $\tau_k$ related in a realistic solid-state system? Recent magnetotransport experiments on the strongly-interacting electrons in WP_2 suggest that $\tau_c \approx \tau_k$ at high temperatures, while $\tau_c \gg \tau_k$ in the low-temperature regime. In particular, for WP_2 the momentum-relaxing time saturates to a nearly-constant value below $T \approx 10$ K, possibly due to residual impurity scattering, while $\tau_c \propto T^{-1}$ as expected for a quantum critical fluid. On the other hand, reference assumed $\tau_c = \alpha \tau_k$ at all temperatures: this amounts to assuming that the long fraction of quasiparticle momentum, that is dumped into the lattice at each collision, is constant and given by the ‘Umklapp efficiency’ $\alpha U \in (0, 1)$. We defer a more detailed discussion on the physical nature of quasiparticle scattering in Fermi liquids to section VIII, while here we just comment on the qualitative impact of weak momentum relaxation on polaritons in the regime $\tau_k \gg \tau_c$.

With momentum relaxation, two collective mode branches stem from equations (33) and (61); they still correspond to the plasmon-polariton and shear-polariton analyzed in section VI A, however their dispersion is affected by $\tau_k$. Formally, we have

$$
q^2 = \frac{\omega^2}{2c^2} + \frac{i\omega}{2v(\omega)} \left\{ 1 \pm \sqrt{\frac{i}{\sqrt{\tau_k} \left[ \omega(c^2 + i v(\omega)\omega)^2 - 4i c^2 v(\omega)(\omega_p)^2 \right] + i \left[ c^2 + i v(\omega)\omega \right]^2}} \right\},
$$

(66)

which reduces to equation (37) for $\tau_k \to +\infty$. Furthermore, for any ratio $\tau_c/\tau_k$ the collective modes are still given by equation (66) upon substitution of $v(\omega)$ with $v(\omega)$ given by equation (57b).

Remarkably, the dispersion of the shear-polariton is robust against momentum relaxation, as it is negligibly affected by the value of $\tau_k$: it still looks as in figure 4(a), and it emerges from the continuum as in the relaxationless case described in section IV C. Significant difference with respect to the relaxationless case only emerge for $\tau_k \ll \tau_c$, where the calculation already converges to the Drude limit in accordance with equation (64). Instead, $\tau_k$ significantly influences the plasmon-polariton dispersion, as illustrated in figure 7. Therefore, the shear- and plasmon-polariton
dispersions are mainly governed by quasiparticle collisions and momentum relaxation, respectively. The character of the two modes is swapped at the bifurcation point \( \omega \tau_k > 1 \), as already noticed in reference 54.

Two distinct refractive indexes \( n_1(\omega) \) and \( n_2(\omega) \) for radiation correspond to the two polariton solutions in the propagating shear regime, as found from the relation between the polariton dispersion (66) and the definition of the refractive index \( q = \frac{c}{\pi} n(\omega) \). In the same way, the two refractive indexes satisfy

\[
[n_i(\omega)]^2 = \epsilon_r \left[ \frac{\omega}{c} n_i(\omega), \omega \right], \quad i = \{1, 2\} \tag{67}
\]

where we use equation (57a) or (35) with or without momentum relaxation respectively. This is consistent with the semiclassical phenomenology of viscous charged fluids, stemming from the combination of Maxwell and Navier-Stokes equations. Therefore, in the propagating shear regime we can calculate the optical properties of the charged Fermi liquid by utilizing the results of reference 54. Such task is performed in section IX, while in the next section we discuss the high-momentum regime \( v_F^s \) for equation (67) that is similar to the discussion in section IX.

### B. Anomalous skin effect regime

The results of section (IV D) are also sensitive to the presence of momentum relaxation. In fact, expanding equation (54) at leading order in \( s \to 0^+ \) at finite \( \omega \tau_k \) and \( \omega \tau_c \), we achieve

\[
\sigma_\tau(q, \omega) = \frac{i}{\omega + \frac{i}{\tau_k} \frac{1 - i \omega \tau_m}{1 + \omega \tau_c}} \frac{3 \pi}{\tau_m} \frac{\epsilon_0(\omega_F)^2 \omega}{qv_F}. \tag{68}
\]

Through equation (29), the optical conductivity (54) corresponds to the dielectric function

\[
\epsilon_{\tau}(q, \omega) = \frac{q^2 c^2}{\omega^2} = 1 + \frac{(\omega_F)^2}{\omega + \frac{i}{\tau_k} \frac{1 - i \omega \tau_m}{1 - i \omega \tau_c}} \frac{3 \pi}{\tau_m} \frac{1}{qv_F}, \tag{69}
\]

from which the dispersion of collective modes descends. In section VIII, we will see that \( \tau_k \gg \tau_c \) in Fermi liquids at low temperatures. In this case, we have \( \tau_m \approx \tau_c \) and equation (68) gives

\[
\sigma_\tau(q, \omega) = \frac{1}{\omega + i \frac{1}{\tau_c}} \frac{3 \pi}{\tau_m} \frac{\epsilon_0(\omega_F)^2 \omega}{qv_F}. \tag{70}
\]

Conversely, in the high-temperature regime \( \tau_c \gg \tau_k \), so that \( \tau_m \approx \tau_c \) and equation (41) becomes

\[
\sigma_\tau(q, \omega) = \frac{1}{\omega + i \frac{1}{\tau_c}} \frac{3 \pi}{\tau_m} \frac{\epsilon_0(\omega_F)^2 \omega}{qv_F}. \tag{71}
\]

Assuming either \( \omega \tau_k \to +\infty \) in equation (70) or \( \omega \tau_c \to +\infty \) in equation (71) yields the relaxationless optical conductivity (41), so that all results or section IV D hold. In the opposite cases, i.e. \( \omega \tau_k \to 0 \) for equation (70) or \( \omega \tau_c \to 0 \) for equation (71), the optical conductivity \( \sigma_\tau(q, \omega) \) is zero, so that radiation completely decouples from the Fermi liquid and propagates freely as in vacuum.

Moreover, taking the limit \( \zeta \to 0^+ \), i.e. \( s \to 0^+ \), \( q v_F^s \tau_c \to +\infty \), \( q v_F^s \tau_k \to +\infty \) in equation (54) yields

\[
\sigma_\tau(q, \omega) = \frac{i}{\omega + \frac{1}{\tau_k} \frac{1 - i \omega \tau_m}{1 - i \omega \tau_c}} \omega \tau_c \frac{3 \pi}{\tau_m} \frac{\epsilon_0(\omega_F)^2 \omega}{qv_F}. \tag{72}
\]

The dielectric function (69) entails three polariton branches, as in the relaxationless case of section IV D. Following the arguments of section VI A, such three polaritons correspond to three frequency-degenerate refractive indexes for radiation. However, due to the connection of the conductivity (68) with anomalous skin effect, it is convenient to adopt the alternative approach of deducing the refractive index from the surface impedance in anomalous skin effect regime, as we will do in section IX C 2 – see also the discussion in section IX A.

### VII. DC CONDUCTIVITY AND TRANSPORT

The zero-frequency limit of the Fermi-liquid optical conductivity (26) gives the momentum-dependent DC conductivity \( \sigma_{DC}(q) = \lim_{\omega \to 0} \sigma_\tau(q, \omega) \), measured in transport experiments. As discussed in sections VI VI B, the result is different depending on the ratio \( \omega / q = v_F^s \).
along which the zero-frequency limit is taken.
Taking the latter at finite momentum $q$ and finite $\omega \tau_c$ implies $s \to 0$, therefore we can employ the results of section VI B in this case: for either $\tau_c \gg \tau_k$ or $\tau_k \gg \tau_c$ we obtain the frequency-independent conductivity (41), linked to anomalous skin effect, while for both finite $\tau_c$ and $\tau_k$ we have $\sigma(0,0) = 0$. As discussed in section V, the above limits are appropriate for Fermi liquids at $T = 0$, where the vanishing frequency limit is equivalent to $s \to 0$.

When both $\omega$ and $q$ are small, it is convenient to take the zero-frequency limit along radial lines in the $(\omega, \text{Re } q)$ plane at fixed $s$ as done in section V in the relaxationless case. As remarked in section V, this limit is appropriate for Fermi liquids at finite temperature and vanishing frequency. When $\tau_k \gg \tau_c$, the vanishing-frequency limit for the optical conductivity is equation (63) with $\omega \to 0$, which reads

$$\sigma_{DC}(q) = \epsilon_0 \frac{(\omega_p)^2}{(\tau_k)^{-1} + \nu(0)q^2}. \quad (73)$$

Equation (73) shows that the DC transport will be determined by the competition between the momentum-relaxing term $i(\tau_k)^{-1}$ and the spatially nonlocal, collision-dependent term $i\nu(0)q^2$. When the former dominates, we have

$$\sigma_{DC}(q) \equiv \sigma_{DC} = \epsilon_0 (\omega_p)^2 \tau_k$$

$$= \frac{m e^2 \tau_k}{\hbar}, \quad (74)$$

which is the standard DC Drude conductivity for an Ohmic conductor.\textsuperscript{65} On the contrary, for negligible momentum relaxation we obtain

$$\sigma_{DC}(q) = \frac{\epsilon_0 (\omega_p)^2}{(v_F^* q)^2 \tau_c q^2}$$

$$= \frac{m e^2 \tau_c}{3 \pi^2 \hbar^2} \left[ 1 + \frac{F_S^2}{3} \right], \quad (75)$$

which depends on momentum as well as on the collision time $\tau_c$: the DC conductivity (75) is limited by the Fermi-liquid viscosity $\nu(0)$. Once Fourier-transformed into real-space coordinates $r$, equation (75) governs the diffusive transport properties in viscous electron fluids flowing through narrow channels, with consequences like a size-dependent resistivity controlled by viscosity.\textsuperscript{67,68}

VIII. COLLISION TIME AND MOMENTUM RELAXATION TIME FROM MATTHEISSEN’S RULE

In order to qualitatively estimate the interplay of momentum-conserving collisions and momentum-relaxing scattering under realistic conditions for solid-state systems, we have to consider the microscopic nature of the different available scattering channels. Reference 77 compares the phenomenological Drude model with the microscopic optical conductivity descending from the Kubo formula and a local Fermi-liquid self-energy.\textsuperscript{7} While such a microscopic analysis transcends the scope of this paper, in the following I take the example of acoustic phonons, impurities and Umklapp processes as independent relaxation channels.

For the electron-electron collision time $\tau_c$, here I employ the standard Fermi-liquid result stemming from quasiparticle phase-space restriction,\textsuperscript{63} the derivation of which is recalled in appendix F. This gives rise to the quasiparticle collision time $\tau_c \propto \hbar / (\hbar^2 + (\pi k_B T)^2)^{-1}$.

The electron-phonon scattering time $\tau_{e-ph}(\omega, T)$ is calculated from the many-body self-energy in section G, and I assume a constant impurity scattering rate $\tau_{i} \in \mathbb{R}$ in first Born approximation.\textsuperscript{53} For Umklapp scattering I follow reference 54 in assuming a proportionality constant between the Fermi-liquid collision time $\tau_c$ and the Umklapp contribution to $\tau_U$, as $\tau_c = \alpha_{U} \tau_{U}$, with $\alpha_{U} \in (0, 1)$ Umklapp...
efficiency. Physically, this means that at each collision a relative ratio $\alpha_\lambda$ of momentum is dumped into the lattice, which works satisfactorily in transition metals.\textsuperscript{28}

To sum aforementioned contributions to the relaxation rate, I adopt the empirical Mattheissen’s rule, which assumes that the total rate $\frac{1}{\tau_k(\omega, T)}$ is the sum of the individual rates for Umklapp, electron-phonon and impurity channels:

$$\frac{1}{\tau_k(\omega, T)} = \frac{1}{\tau_U(\omega, T)} + \frac{1}{\tau_{e-ph}(\omega, T)} + \frac{1}{\tau_i}.$$  \hspace{1cm} (76)

Equation (76) assumes that the summed collision rates are independent from each other. We calculate the quasiparticle contribution from equation (F8) and the electron-phonon contribution from equation (G16). Figure 8 shows the total momentum-relaxation time $\tau_k(\omega, T)$ from Mattheissen’s rule as a function of frequency $\omega$ normalized to the plasma frequency $\omega_p$. The impurity scattering time is $\tau_i \rightarrow +\infty$. Solid curves correspond to $\tau_U(\omega, T)$, at temperatures $T = 300K$ and $T = 1K$ for panels (a) and (b). Dashed curves show the individual contributions from Umklapp and electron-phonon scattering, using the same parameters as in figures 16 and 18 respectively. Panel (a) illustrates that the phonon contribution to $\tau_k$ at room temperature prevails at low frequency, while the Umklapp component takes over in the high-frequency regime. Conversely, the low-temperature result in panel (b) shows that the Umklapp contribution limits $\tau_k$ in the low- and high-frequency limits, separated by an intermediate frequency window dominated by phonon scattering. The addition of a finite impurity scattering time $\tau_i$ further constrains the static limit of $\tau_k$ when $\tau_i$ becomes the smallest timescale in equation (76): we will further comment on this feature in section IX C 4.

IX. NONLOCAL OPTICAL PROPERTIES OF CHARGED FERMI LIQUIDS

The results obtained in sections III - VIII equip us with the necessary tools to investigate observables linked to Fermi-surface rigidity in optical spectroscopy experiments. In the following, we focus on two canonical experimental setups, namely the surface impedance and the transmission through a thin film. These configurations require to consider how radiation is reflected, absorbed and transmitted at interfaces between different dielectric media. Such problem is well-defined in the standard Drude model through Maxwell equations alone, however it becomes underdetermined when more than one optical mode is propagating in the material, as in the propagating shear case of section VI A. Multiple approaches are available to overcome this difficulty, and to those approaches the next section is dedicated for completeness.

A. Constitutive relations for electromagnetic fields at interfaces

Physically, reflection and transmission coefficients at interfaces between different dielectric media depend on how electrons at the boundary react to the incident electromagnetic radiation. When the electronic response is spatially local, it is sufficient to consider the boundary conditions stemming directly from Maxwell equations, namely the continuity of the electric field and of its derivative at the boundary for normal incidence.\textsuperscript{46} However, when more than one polarization mode propagates into the material, Maxwell equations alone are insufficient to close the problem, which is underdetermined. The need for additional boundary conditions (ABCs)\textsuperscript{79,80} originates from the fact that, rigorously, a dielectric function $\varepsilon(q, \omega)$ depending on a single wave vector $q$, or equivalently on the difference $|r - r'|$ between two coordinates $r$ and $r'$, is valid only for translational-invariant systems.\textsuperscript{58} However, a surface or interface necessarily breaks translation invariance, so that the nonlocal dielectric function $\varepsilon(r, r', \omega)$ should depend separately on $r$ and $r'$ at the boundary. To avoid complications implied by a more rigorous model of the surface,\textsuperscript{81,82} a common alternative is to retain the bulk expression $\varepsilon(q, \omega)$ even at the boundary, at the cost of introducing ABCs deduced from the specific properties of the electrons that react to radiation.\textsuperscript{83,84} We retrieve notable examples of this approach in the treatment of light-exciton coupling,\textsuperscript{83,84} longitudinal plasmons,\textsuperscript{85-87} and recently viscous charged liquids.\textsuperscript{54} The latter reference employed constitutive relations for the current density at boundaries stemming from fluid dynamics,\textsuperscript{88} and expressed in terms of the slip length $\lambda_s \in (0, +\infty)$: $\lambda_s$ is the ratio between the shear dynamical viscosity of the moving fluid and the tangential friction per unit area exerted by the fluid on the boundary of the solid.\textsuperscript{54} Since in this model the friction force is proportional to the fluid velocity $v$ at the boundary, a nonzero slip length implies a finite fluid velocity at the interface.\textsuperscript{54} with the models of Pekar\textsuperscript{90} and Ting-Frenkel-Birman\textsuperscript{90} as limiting cases for $\lambda_s = 0$ and $\lambda_s \rightarrow +\infty$ respectively.

On the other hand, textbook treatments of anomalous skin effect in metals pursue the route of a microscopic model for the interface, in the form of specular of diffuse scattering of electrons at the boundary\textsuperscript{47,73} as we will see in sections IX B 2 - IX B 3 and IX B 5: in this approach, one assumes that a portion $p \in [0, 1]$ of quasiparticles experiences specular scattering at the sample boundaries, while a portion $1 - p$ undergoes diffuse scattering. The approaches in terms of $\lambda_s$ and $p$ give qualitatively consistent results, although non-negligible differences appear at the quantitative level. In the following sections, we adopt both approaches and compare their results for the surface impedance.

A potential reconciliation of the ABC and surface-modeling approaches is offered by reference 91, which reports a microscopic calculation of $\lambda_s$ in two- and three-dimensional Fermi liquids for specular and diffusive scattering. In three dimensions and in terms of the kinematic
viscoelastic coefficient \((36)\), we have
\[
\lambda_s^i = \frac{1}{\hbar^2(h')^2(k_F)^4} \frac{45\pi^2}{(k_F)^4 \hbar n m v(0)}
\]
\[
= \frac{3}{h^2(h')^2(k_F)^2} v_F^r \tau_c, \quad (77a)
\]
\[
\lambda_s^d = \frac{8\pi^2}{(k_F)^4 \hbar} n m v(0) = \frac{8}{15} v_F^r \tau_c, \quad (77b)
\]
where \(n\) is the 3D electron density, \(k_F = (3\pi^2 n)^{1/3} = m v_F / \hbar\) is the 3D Fermi wave vector, and \((\lambda_s^i, \lambda_s^d)\) refer to specular and diffuse scattering, respectively. Notice that the slip length becomes itself a function of frequency and temperature.

B. Surface impedance

An ideal probe of spatial nonlocality in the electrodynamics response of metals is the surface impedance \(Z = Z(\omega)\). In general, it is defined as the ratio of the electric field \(E\) normal to the surface of a metal to the total current density \(J\) induced in the bulk of the semi-infinite sample \((35,46)\). The relationship between the surface rugosity, parametrized by the height \(\delta_s\) and characteristic length \(\lambda\), and the surface impedance is given by \((35,46)\)
\[
Z(\omega) = \frac{E(z, \omega)_{z=0^+}}{\int_0^\infty J(z, \omega) dz}. \quad (78)
\]
Here, we defined the orthogonal coordinate \(z\) with respect to the metallic surface \(z = 0\), and \(J(z, \omega)\) is the current density per unit area, which decays with increasing distance \(z\) from the surface. Physically, were the electrodynamics response local, the current density induced by the surface electric field \(E(0^+, \omega)\) would be also located exclusively at the surface \(J(z, \omega) \equiv J(0^+, \omega)\). Conversely, any degree of spatial nonlocality generates a current response extending at \(z > 0\). Hence, the surface impedance is directly sensitive to nonlocal electrodynamics effects and measurable using state-of-the-art optical spectroscopy equipment. For instance, there is a useful relation between \(Z(\omega)\) and the reflection coefficient \(r(\omega)\) at the boundary between vacuum and a semi-infinite dielectric medium: \((35,46)\)
\[
r(\omega) = \frac{Z(\omega) - Z_0}{Z(\omega) + Z_0}, \quad (79)
\]
where \(Z_0 = \mu_0 \varepsilon_0\) is the vacuum surface impedance.

\[
\delta_s(\omega) = \sqrt{\frac{2}{\mu_0 \varepsilon_0 \sigma_{DC}}} \propto \frac{1}{\sqrt{\omega}}. \quad (80)
\]
where \(\sigma_{DC}\) is the Drude DC conductivity \((74)\). We will recall the derivation of equation \((80)\) in section IXB1. When \(l_{MFP} \ll \delta_s\), electrons respond locally to the incident electromagnetic field, and the latter gets exponentially damped inside the metal within a characteristic length scale \(\delta_s\), as illustrated in figure 9(a). The latter hand, for \(l_{MFP} \gg \delta_s\), the response becomes spatially nonlocal, and
bulk transverse currents arise in the metal, as sketched in figure 9(b). This allows for a much larger penetration depth of the electric field inside the metal, up to distances of the order of \( l_{\text{MFP}} \), which affects the optical properties. Hence, we can define the qualitative criterion \( \delta_s \ll l_{\text{MFP}} \) for anomalous skin effect regime, while for \( \delta_s \gg l_{\text{MFP}} \) we retrieve the results of the Drude model, as described in sections IX B 1 and IX B 2. The ratio \( \delta_s/l_{\text{MFP}} \propto (\tau_\xi)^{-\frac{1}{2}} \) is governed by momentum relaxation. In a clean Fermi liquid with no relaxing processes, \( \delta_s/l_{\text{MFP}} \to 0 \) and spatial nonlocality dominates the dielectric response at all frequencies – see appendix H. However, in crystalline solids phonons provide strong momentum damping at room temperature, so that \( \delta_s \gg l_{\text{MFP}} \) and local electrodynamics occurs: this is why anomalous skin effect is observed in clean samples at low temperatures, such that the mean free path increases due to freezing of phonon scattering channels.92,93

Textbook derivation of anomalous skin effect are usually performed in the regime \( \omega \ll v_1^* \text{Re} \{q\} \). However, since the generalized shear modulus \( \nu(\omega) \) entails spatial nonlocality, it produces a phenomenology analogous to anomalous skin effect in the propagating shear regime \( \omega \gg v_1^* \text{Re} \{q\} \), where the response would be entirely local for \( \nu(\omega) = 0 \). Therefore, observing the characteristic phenomenology of anomalous skin effect in the regime \( \omega \gg v_1^* \text{Re} \{q\} \) is a direct evidence for a propagating shear mode in the solid-state Fermi liquid.

From a macroscopic standpoint, Maxwell’s equations yield the spatial profile of the electric field and the generated current density for given sample geometry and boundary conditions, which in turn determines the surface impedance (78). Here we follow Dressel and Grüner46 in sketching the derivation of anomalous skin effect in the absence of an external magnetic field. The electromagnetic wave equation, for fields in the \( xy \) plane and without free charge density, depends on the current density \( J(z, \omega) \) induced inside the metal:

\[
\frac{\partial^2 E(z, \omega)}{\partial z^2} + \frac{\omega^2}{c^2} \int dz' \varepsilon_T(z-z', \omega) E(z, \omega) = -\mu_0 \epsilon \omega J(z, \omega) + \alpha_b,
\]  

where \( \alpha_b \) is a factor depending on the quality of boundary scattering. The integral on the left-hand side of equation (81) takes into account spatial nonlocality in the dielectric function \( \varepsilon_T(z-z', \omega) \), which depends on the difference between the probe and reaction coordinates \( z \) and \( z' \) in the translational-invariant case. In the following we assume specular scattering at the metal surface, which implies \( \frac{\partial E(z, \omega)}{\partial z} \big|_{z=0^+} = \frac{\partial E(z, \omega)}{\partial z} \big|_{z=0^-} \) and therefore \( \alpha_b = 2 \frac{\partial E(z, \omega)}{\partial z} \big|_{z=0^+} \). Other types of boundary scattering lead to quantitative but not qualitative modifications of the results here outlined.46 Fourier-transforming equation (81) to reciprocal space of momenta \( q \) yields

\[
-q^2 E(q, \omega) + \frac{\omega^2}{c^2} \varepsilon_T(\omega) E(q, \omega) \bigg|_{A} = -\mu_0 \epsilon \omega J(q, \omega) + 2 \frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0^+}. \tag{82}
\]

The term \( A \) in equation (82) stems from the displacement current, while conduction currents generate the term \( B \). The general solution of equation (82) satisfies Chambers’ formula.46,95 Assuming that the collision time \( \tau_c \) does not depend on \( q \), one can also utilize the Boltzmann kinetic equation to calculate the current density \( J(q, \omega) \), similarly to what we did in section III B. In an even simpler approximation, one can utilize Ohm’s law

\[
J(q, \omega) = \sigma_T(q, \omega) E(q, \omega) \tag{83}
\]

knowing the transverse conductivity \( \sigma_T(q, \omega) \). In this one-dimensional configuration, the surface impedance then follows as

\[
Z(\omega) = i \mu_0 \epsilon_0 \frac{E(0, \omega)}{\frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0^+}}. \tag{84}
\]

We first recall the analysis of ‘normal’ skin effect for the local Drude model, and we sketch the derivation of anomalous skin effect in the standard regime \( \omega \ll v_1^* q \). Then, we compare these results to the analogous phenomenon occurring when the shear polariton propagates in the Fermi liquid.54

1. Normal’ skin effect in the Drude model

The standard derivation of the skin effect neglects the displacement current term \( A \) in equation (82) with respect to the conduction current \( B \). This approximation holds for frequencies up to \( \omega \approx \sqrt{\frac{\sigma_T(q, \omega)}{\epsilon_0 \tau_c}} \), which lie in the near ultraviolet region for most metals,73 i.e. at \( \omega \approx \omega_p \) when the metal become transparent. Inserting the generalized Ohm’s law (83) in the wave equation (82), we obtain the momentum-dependent electric field

\[
E(q, \omega) = 2 \frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0^-} \frac{1}{\mu_0 \epsilon_0 \omega \sigma_T(q, \omega) - q^2}. \tag{85}
\]

We now employ the spatially local Drude conductivity with momentum relaxation,

\[
\sigma_T(\omega) = -i \epsilon_0 \omega \varepsilon_T(\omega) - 1 = i \epsilon_0 \omega \frac{\omega^2}{\omega^2 + i\omega \frac{1}{\tau_c}}, \tag{86}
\]
which corresponds to the regime $|i\omega\nu(\omega)q^2| \ll |i\omega(\tau_k)^{-1}|$ in equation (63), i.e. the limit in which momentum relaxation dominates over spatial nonlocality. After inserting equation (86) into equation (85), we transform back the latter to real space, and at the interface $z = 0$ we have

$$E(0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq E(q, \omega) e^{-iqz} = -\frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} \frac{c}{\omega p} \sqrt{1 + \frac{i}{\omega\tau_k}}. \quad (87)$$

This allows us to calculate the Drude surface impedance from equation (84):

$$Z(\omega) = -i\mu_0 \omega \frac{c}{\omega p} \sqrt{1 + \frac{i}{\omega\tau_k}}, \; \omega \ll \omega p. \quad (88)$$

Expanding equation (88) in the Hagen-Rubens regime $\omega\tau_k \ll 1$, we retrieve the well-known result for 'normal' skin effect:

$$Z(\omega) = \frac{\mu_0 c \sqrt{\omega}}{\omega p \sqrt{\tau_k}} \frac{1-i}{\sqrt{2}} = \mu_0 \omega \delta_s(\omega) \frac{1-i}{2}, \quad (89)$$

where $\delta_s$ is the classical skin depth (80), i.e. the characteristic damping length of the electromagnetic field inside the Drude metal. In the opposite regime $\omega\tau_k \gg 1$ equation (88) gives a purely imaginary result

$$Z(\omega) = -i\mu_0 \omega \frac{c}{\omega p}. \quad (90)$$

Notice that one can also derive the same results above from the relation (79) between the surface impedance and the reflection coefficient: the reflection coefficient between vacuum and a metal is

$$r(\omega) = \frac{1-n(\omega)}{1+n(\omega)} \quad (91)$$

where $n(\omega) = \sqrt{\varepsilon_T(\omega)}$ is the refractive index. Using the Drude dielectric function $\varepsilon_T(\omega) = 1 - \frac{(\omega_p)^2}{\omega^2 + i\omega/\tau_k}$ and equation (79), we obtain

$$Z(\omega) = \frac{\mu_0 c}{n(\omega)} = \mu_0 c \omega \sqrt{\frac{1 + \frac{i}{\omega\tau_k}}{\omega p^2 + i\omega/\tau_k}}. \quad (92)$$

which reduces to equation (88) in the Hagen-Rubens regime.\(^{46}\)

2. Anomalous skin effect regime

The results of section IX.B.1 are modified in the high-momentum regime $\omega \ll \nu_c Re(|q|)$, where the momentum dependence of the transverse conductivity $\sigma_T(q, \omega)$ leads to an anomalous damping of the electromagnetic field inside the metal, known as anomalous skin effect. To see this, we revert to equation (85) and we insert the momentum-dependent conductivity (41), which we derived from the Fermi-liquid kinetic equation, consistently with semiclassical Boltzmann theory.\(^{46}\) Fourier-transforming back to real space implies

$$E(0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq E(q, \omega) e^{-iqz} = \frac{1}{\pi} \frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} \frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} \int_{-\infty}^{+\infty} dq \left[ -i\omega \left( \frac{3}{4} \frac{\sigma_{DC}}{|q|^2} - q^2 \right) \right]^{-1} = \frac{1}{\pi} \frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} \int_{-\infty}^{+\infty} dq \left[ i\mu_0 \omega \frac{3}{4} \frac{\sigma_{DC}}{|q|^2} - q^2 \right]^{-1} = \frac{1}{\pi} \frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} \int_{-\infty}^{+\infty} dq \left[ \frac{1}{i\delta_s^aN(q) - q^2} \right]^{-1}, \quad (93)$$

where we have defined the skin depth in anomalous regime:

$$\delta_s^a = \left[ \frac{4}{3} \frac{c^2}{\omega p} \right]^{-\frac{1}{2}} \left[ \frac{4}{3} \frac{l_{MF}}{\sigma_{DC}} \right]^{-\frac{1}{2}} \propto \omega^{-\frac{1}{2}}. \quad (94)$$

The $q$-integration in equation (93) gives

$$E(0, \omega) = \frac{1}{\pi} \frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} \frac{2\pi}{3} \frac{\delta_s^a}{\sqrt{3}} \left( 1 - \frac{i}{\sqrt{3}} \right). \quad (95)$$

Inserting equation (95) in equation (84), we obtain the surface impedance

$$Z(\omega) = i\mu_0 \omega \delta_s^a \frac{2}{3} \left( 1 - \frac{i}{\sqrt{3}} \right) \left[ \frac{4}{3} \frac{l_{MF}}{\sigma_{DC}} \right]^{\frac{1}{2}} \left( \frac{1}{\sqrt{3}} + i \right). \quad (96)$$

Equation (96) coincides with the asymptotic value of the surface impedance for specular boundary scattering, found for $l_{MF} \gg \delta_s$ from the more general theory by Reuter and Sondheimer.\(^{47,73}\) The value for diffuse boundary scattering is obtained by multiplying equation (96) by $\frac{2}{3}$.\(^{36,73}\) Figure 10(a) shows the asymptotic specular limit (96) and its diffractive counterpart as horizontal blue and red dashed lines, respectively.

In anomalous skin effect regime, the electric field penetrating into the metal is exponentially damped as a function of coordinate $z$ – see equation (95) – as for normal skin effect – cfr. equation (87): the majority of the field is confined within a length scale of the order of the skin depth (94). However, the frequency dependence $\delta_s^a \propto \omega^{-\frac{1}{2}}$ found in anomalous regime is different from the Hagen-Rubens expression $\delta_s \propto \omega^{-\frac{1}{2}}$: this is a manifestation of the nonlocal character of the optical conductivity (41) in anomalous regime. Furthermore, although the electric field is mostly confined within $\delta_s \ll l_{MF}$, electrons respond coherently at distances up to $l_{MF}$ from the sample surface.
An accurate calculation of the surface impedance at finite \( \tau_k \) requires to go beyond the leading-order high-momentum expansion (41). This task has been performed by G. E. H. Reuter and E. H. Sondheimer, by solving the inhomogeneous Boltzmann equation for free electrons in relaxation-time approximation,\(^{47,73}\) Such solution provides a quantitatively precise description of anomalous skin effect for any value of \( \tau_k \), including the crossover to the Drude model, while it neglects quasiparticle interactions and momentum-conserving collisions. In principle, one could also calculate the surface impedance for any ratio \( \tau_c/\tau_k \) from the general form of the transverse susceptibility (51) complemented by boundary conditions at the vacuum-Fermi liquid interface, and compare the results with Reuter-Sondheimer theory in the regime \( \tau_c/\tau_k \rightarrow +\infty \). Such detailed comparison is left for future work, while here we quote that in the microwave spectral region the Reuter-Sondheimer surface impedance can be written as\(^{47,73}\)

\[
Z(\omega) = -i\sqrt{8\alpha/3}A_{\text{MFP}}^{1/3} \frac{1}{\delta_s} \left. \frac{E(0, \omega)}{\partial z} \right|_{z=0},
\]

\[A = \frac{\sqrt{6}}{2\pi} \left( \frac{mv_F}{3ne^2} \right)^{1/2} (\mu_0\omega)^{3/2}, \]

\[\alpha = \frac{3}{2} \left( \frac{l_{\text{MFP}}}{\delta_s} \right)^2 = \frac{3}{4} \left( \frac{v_F}{c} \right)^2 \omega \tau_k (\tau_k)^2 (\omega p)^{3/2},\]

where \( \delta_s \) is the classical skin depth (80). Notice that the parameter \( \alpha \) is precisely the ratio between mean free path and skin depth, which determines the crossover between the Drude/local and anomalous/nonlocal regimes, as mentioned at the beginning of section IX.B.

The solid blue (red) curve in figure 10(a) shows the inverse surface reactance from equations (97) in the specular (diffuse) scattering case, as a function of \( \alpha^{1/2} \propto \sqrt{\tau_k} \), as commonly done after Pippard.\(^{92,93,96}\) This highlights the transition from the Drude result (dashed gray curve) to the asymptotic saturation (96) as the momentum-relaxation time \( \tau_k \) increases. Appendix H considers the ratio \( \delta_s/l_{\text{MFP}} \) for a Fermi liquid, employing the total relaxation time from section VIII.

### 3. Propagating shear regime at low frequencies

We now analyze the surface impedance in the presence of the generalized shear modulus (36). In particular, we first want to demonstrate the existence of a constant asymptotic value of the surface resistance for \( \tau_k \rightarrow +\infty \), as it occurs in the anomalous skin regime of section IX.B.2. Our starting point is again equation (85), assuming a Ohmic relation

\[
\text{(83)}
\]

between current density and electric field. Using the dielectric function (57a) and the relation (29), we achieve

\[
E(0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq E(q, \omega) e^{-iqz} \bigg|_{z=0} \int_{-\infty}^{+\infty} dz \left[ -\frac{1}{A + Bz^2} - z^2 \right]^{-1},
\]

where

\[
z = \frac{qc}{\omega_p}
\]

\[
\text{(99)}
\]
and
\begin{equation}
A = \left( 1 + \frac{i}{\omega \tau_k} \right),
\end{equation}
\begin{equation}
B = iA \left( \frac{\omega_p^2}{c^2} \right) \frac{\nu(\omega)}{\omega}.
\end{equation}

We perform the momentum integration in equation (98) and we write the result in terms of the variables (100):
\begin{equation}
E(0, \omega) = -\frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} + \frac{c}{\omega_p \sqrt{A^2 - 4B + 2\sqrt{B^2 - 4B + A}}} \sqrt{2(A+\sqrt{B})},
\end{equation}
\begin{equation}
\frac{Z(\omega)}{c\mu_0} = -\frac{i}{\omega_p} \frac{\sqrt{2(A+\sqrt{B})}}{\sqrt{A^2 - 4B + 2\sqrt{B^2 - 4B + A}}}.
\end{equation}

Remarkably, the propagating-shear surface impedance (102) displays the phenomenology typical of anomalous skin effect, which is evident from figure 10(b) displaying $|\text{Re} \{Z(\omega)\}|^{-1}$ against $\sqrt{\omega_p \tau_k}$. Here, I used a renormalized Fermi velocity $v^*_F/c = 0.001$, first Landau parameter $F^*_s = 20$ and at frequency $\omega/\omega_p = 0.002$. The blue, red and golden curves show the result for $\omega_p \tau_c = \{10^3, 10^4, 10^5\}$, respectively. For consistency, the frequency $\omega/\omega_p = 0.002$ is chosen such that the shear-polariton mode is outside of the electron-hole continuum for $\omega_p \tau_c \geq 10^3$, in accordance with section IV C – see also figure 5. Indeed, in the limit $\tau_k \to +\infty$, equation (102) reaches the asymptotic value
\begin{equation}
\lim_{\tau_k \to +\infty} \frac{Z(\omega)}{c\mu_0} = -\frac{i}{\omega_p} \frac{\sqrt{2(B^* + 1)}}{\sqrt{1 - 4B^* + 2\sqrt{B^* - 4B^* + 1}}},
\end{equation}
where \begin{equation}
B' = i \left( \frac{\omega_p}{c} \right) ^2 \frac{\nu(\omega)}{\omega}.
\end{equation}

Therefore, the asymptotic limit (103) of the inverse surface resistance is controlled by the generalized shear modulus $\nu(\omega)$ through equation (104). Dashed horizontal blue, red and golden lines in figure 10(b) show the limit (103) for $\omega_p \tau_c = \{10^3, 10^4, 10^5\}$, respectively. One sees that $\nu(\omega)$ governs the propagating-shear skin effect also from the skin depth. In fact, in the $\tau_k \to \infty$ limit equation (98) can be written as
\begin{equation}
E(0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq E(q, \omega) e^{-iqz} = \frac{1}{\pi} \left. \frac{\partial E(z, \omega)}{\partial z} \right|_{z=0} + \frac{c}{\omega_p} \sqrt{\frac{2}{\sqrt{A^2 - 4B + 2\sqrt{B^2 - 4B + A}}}} \sqrt{2(A+\sqrt{B})},
\end{equation}
\begin{equation}
\cdot \int_{-\infty}^{+\infty} dq \left[ \frac{\omega_p}{c} \frac{1}{1 + i \nu(\omega)/\omega^2 - q^2} \right]^{-1}.
\end{equation}

where we identify a complex-valued length scale $\delta^\text{ps}$, analogous to the skin depths (94) and (80) in Drude and anomalous regimes:
\begin{equation}
\delta^\text{ps} = \sqrt{\frac{\nu(\omega)}{\omega}} = \sqrt{\left( 1 + \frac{F^*_s}{3} \right) \frac{(v^*_F)^2 \tau_c}{5\omega(1 - i\omega \tau_c)}}.
\end{equation}

In hydrodynamic regime $\omega \tau_c \to 0$, equation (106) specializes to
\begin{equation}
\delta^\text{vl} = \sqrt{\frac{\nu(\omega)}{\omega}} = \sqrt{\left( 1 + \frac{F^*_s}{3} \right) \frac{(v^*_F)^2 \tau_c}{5\omega}},
\end{equation}
that is, the skin depth is determined by the Fermi-liquid shear viscosity $\nu(\omega)$. Conversely, in the collisionless regime $\omega \tau_c \to +\infty$ equation (105) becomes
\begin{equation}
E(0, \omega) = \frac{1}{\pi} \left. \frac{\partial E(z, \omega)}{\partial z} \right|_{z=0} + \frac{c}{\omega_p} \sqrt{\frac{2}{\sqrt{A^2 - 4B + 2\sqrt{B^* - 4B^* + 1}}}} \sqrt{2(B^* + 1)},
\end{equation}
\begin{equation}
\cdot \int_{-\infty}^{+\infty} dq \left[ \frac{\omega_p}{c} \frac{1}{1 - \delta^\text{sh} q^2} \right]^{-1},
\end{equation}
where the skin depth is
\begin{equation}
\delta^\text{sh} = \sqrt{\frac{\nu(\omega)}{\omega}} = \left( \frac{v^*_F}{5} \right) \sqrt{\frac{1 + F^*_s}{3}} \sqrt{\frac{\omega}{\omega}},
\end{equation}
related to the reactive shear modulus (16) of the Fermi liquid. Hence, $\delta^\text{sh}$ plays the role of the skin depth in the collisionless and relaxation-free limit of the propagating shear regime. Table I summarizes the values assumed by the skin depth in all analyzed regimes.

Crossovers among the different regimes characterized by the skin depths may be envisaged on the basis of qualitative criteria analogous to the normal/anomalous condition $\delta_s(\omega) = l_{\text{MFP}}$ in section IX B 2. In particular, in relaxationless Fermi liquids at finite temperatures, a high-frequency crossover between the normal and propagating-shear skin effects may be qualitatively set at
\begin{equation}
\delta_s(\omega) < l_c = v^*_F \tau_c.
\end{equation}
where the skin depth assumes the shear-propagation value (109), consistently with the saturation of the surface resistance in figure 10. In the same way, at very low frequencies close to the DC limit, one expects a crossover between anomalous and hydrodynamic regimes for

\[ \delta_s^{\text{an}} < l, \quad (111) \]

whereby the skin depth becomes hydrodynamic, i.e. (107). Using the Fermi-liquid collision time \( \tau_F \) in equation (111), one realizes that such hydrodynamic regime occurs at frequencies close to the DC limit, \( \omega \lesssim (10^{-10} \div 10^{-15}) \omega_p \), for \( F^3_1 = (6 \div 50) \) and a representative electron density \( n = 10^{23} \text{ cm}^{-3} \) for standard metals. As such, the hydrodynamic skin effect is practically undetectable by optical means, its consequences being mainly visible in transport experiments on clean samples – see section VII.

It is also interesting to see how the saturation limit (103) for the surface impedance depends on the parameters which influence the generalized shear modulus (36), namely frequency \( \omega \) and collision time \( \tau_c \). Figure 11 shows the asymptotic value \( \lim_{\omega \to \infty} \text{Res} \{Z(\omega)\} \) as a function of normalized frequency \( \omega / \omega_p \), for different values of collision time. The purely reactive/elastic limit \( \omega \tau_c \to +\infty \) is depicted by the green curve. The vertical dashed lines mark the frequency at which the shear-polariton emerges from the continuum, at the corresponding value of \( \omega \tau_c \) – see also figure 5: the calculation is valid for frequencies greater than the threshold marked by the dashed line. We see that, in the propagating/reactive regime for the shear polariton, the asymptotic value (103) is a nearly universal function of \( \omega / \omega_p \), at fixed renormalized Fermi velocity \( v_e / c \) and first Landau parameter \( F^1_1 \). In particular, in this regime the curves overlap with the one for purely reactive/elastic-like limit \( \omega \tau_c \to +\infty \), thus demonstrating that the saturation limit of the surface reactance in propagating regime is determined by the reactive/elastic-like component of the generalized shear modulus \( \nu(\omega) \): the phenomenology of anomalous skin effect for \( \omega \gg v_e q \) reflects the dynamical reactive character of the Fermi-surface shear response to transverse excitations enforced by the incident electromagnetic field.

**TABLE I. Skin depth in various regimes for the charged Fermi liquid.** The Drude value (80) is valid for finite \( \tau_k \), while all other quoted results hold in the relaxationless regime \( \tau_k \to +\infty \).

| Drude/Ohmic  | \( \delta_s(\omega) = \sqrt{\frac{\mu}{2 \pi \omega \varepsilon_0 \varepsilon_r} \omega} \propto \frac{1}{\omega} \) (80) |
|-------------|-----------------------------------------------------------------------------------|
| Anomalous   | \( \delta_s^{\text{an}} = \sqrt{\frac{\mu}{4 \pi \varepsilon_0 \varepsilon_r} \frac{v_e^2}{\omega^2}} \omega \propto \omega^{-1} \) (94) |
| Hydrodynamic| \( \delta_s^{\text{h}} = \sqrt{\frac{\mu}{4 \pi \varepsilon_0 \varepsilon_r} \frac{c^2}{\omega^2}} \propto \frac{1}{\sqrt{\omega}} \) (107) |
| Collisionless| \( \delta_s^{\text{cl}} = \frac{\lambda}{c} = \frac{1}{\omega} \left( 1 + \frac{1}{\omega^2} \right) \propto \frac{1}{\omega} \) (109) |

**FIG. 11.** Relaxationless limit of the inverse surface resistance for a 3D Fermi liquid in the propagating shear regime as a function of normalized frequency \( \omega / \omega_p \), according to equation (103), for renormalized Fermi velocity \( v_e^* = 10^{-3} c \) and first Landau parameter \( F^1_1 = 20 \). The collision times \( \omega \tau_c = \{10^3, 10^4, 10^5\} \) give the blue, red and gold curves respectively. The green dashed curve shows the purely reactive limit \( \omega \tau_c \to +\infty \). Vertical dashed lines mark the frequency below which the shear-polariton is in the Lindhard continuum for each value of \( \tau_c \).

4. High-frequency regime in the Drude model

To contrast the results in section IXB.1, it is instructive to analyze what happens to the Drude dielectric response when \( \omega > \sqrt{\frac{\pi^2 q_c \mu}{\omega_c \tau_k}} \), that is \( \omega > \omega_p \): this amounts to neglecting the conduction term \( B \) in equation (82) in favor of the displacement current term \( A \). In this case, we have

\[ E(0, \omega) = 2 \frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} \frac{1}{\omega^2 - \varepsilon_0 c \varepsilon_r (q, \omega) - q^2}, \quad (112) \]

which corresponds to equation (85) if we substitute \( \varepsilon_r(q, \omega) \rightarrow 1 \) in the latter with \( \varepsilon_1(q, \omega) \). Using the Drude dielectric function \( \varepsilon_1(\omega) \) and Fourier-transforming back to real space, we achieve

\[ E(0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq E(q, \omega) e^{-iqz}\bigg|_{z=0} = -\frac{\partial E(z, \omega)}{\partial z} \bigg|_{z=0} \frac{c}{\omega_p} \sqrt{\frac{i + \omega \tau_k}{\omega^2 \tau_k \left[ 1 - \left( \frac{\omega}{\omega_p} \right)^2 \right] - i \left( \frac{\omega}{\omega_p} \right)^2}}. \quad (113) \]

From equation (84), we have the surface impedance:

\[ Z(\omega) = \mu_0 \omega c \frac{1}{\omega_p} \sqrt{\frac{i + \omega \tau_k}{i \left( \frac{\omega}{\omega_p} \right)^2 - \omega \tau_k + \left( \frac{\omega}{\omega_p} \right)^2 \tau_k}}, \quad (114) \]
which is equivalent to the expression (92) obtained from the Drude refractive index. For \( \omega > \omega_p \), the surface impedance quickly drops to the constant real value \( \lim_{\omega \to +\infty} Z(\omega) = \frac{c}{\omega p} \) due to the displacement current. These results are modified by the presence of a generalized shear modulus \( \lambda_s \).

5. Propagating shear regime at high frequencies

In principle, the spatial nonlocality of the dielectric function (57a) could affect the surface impedance even for \( \omega > \omega_p \). Hence, it is worth checking whether quantitative differences with respect to the Drude calculation IX B 4 emerge in such regime. In this case, equation (82) with the displacement current term \( A \) and neglecting the conduction term \( B \) gives

\[
E(q, \omega) = 2 \left. \frac{\partial E(z, \omega)}{\partial z} \right|_{z=0^+} \cdot \left[ \mu_0 c \omega \sigma_T(q, \omega) + \frac{\omega^2}{c^2} \varepsilon_T(q, \omega) \right]^{-1} = 2 \left. \frac{\partial E(z, \omega)}{\partial z} \right|_{z=0^+} \left\{ \frac{\omega^2}{c^2} \varepsilon_T(q, \omega) - q^2 \right\}^{-1}.
\] (115)

However, it turns out that the differences between the Drude surface impedance and the one stemming from equation (115) are negligible, as shown in appendix I. Therefore, the best regime to seek signatures of Fermi-surface shear reactance in the surface impedance is the low-frequency one of section IX B 3.

6. Surface impedance from the refractive indexes and slip length

The derivations in sections IX B 3 and IX B 5 illustrate how the phenomenology of anomalous skin effect emerges from the propagation of the shear-polariton in the Fermi liquid. However, so far we have considered the boundary conditions of the problem in terms of specular/diffusive scattering at the vacuum-metal interface, while other types of boundary conditions may quantitatively modify the results.\(^{57,24,51}\) An alternative way to calculate the Fermi-liquid surface impedance in propagating shear regime is to exploit the relation between \( Z(\omega) \), the reflection coefficient \( r(\omega) \), and the refractive index, similarly to equation (79) for the Drude model. As argued in section VI A, Fermi-liquid theory gives two polariton modes (37), hence two refractive indexes \( n_1(\omega) \) and \( n_2(\omega) \), for each \( \omega \) in the regime \( \omega \gg \nu_F \) \( \text{Re} (q) \) and \( \tau_c \approx \tau_k \), where the dielectric function is (62). Then, the response of the Fermi surface as a whole is equivalent to the one of a viscous charged liquid, and the constitutive relations at the interface with vacuum may be derived from fluid dynamics and expressed in terms of the slip length \( \lambda_s \). This allows us to utilize the results of reference 54 in the calculation of the surface impedance.

In terms of the refractive indexes, the reflection coefficient at normal incidence for a viscous electron liquid reads

\[
r(\omega) = t_1 + t_2 - 1,
\] (116a)

\[
t_j = \frac{2(n_j - 1)}{(n_j + 1)(n_j - n_j) 1 + (1 - n_j - n_j))\omega/c\lambda_s}, \quad \{j, j\} = \{1, 2\}.
\] (116b)

where \( \lambda_s \in (0, +\infty) \) is the slip length, introduced in section IX A. With the expression (116a), can again employ the relation (79) for the surface impedance, which yields

\[
\frac{Z(\omega)}{Z_0} = \frac{n_1 + n_2 + i\frac{\lambda_s}{c}\omega[1 - (n_1)^2 - (n_2)^2 - n_1 n_2]}{1 + n_1 n_2[1 - i\frac{\lambda_s}{c}\omega(n_1 + n_2)]}.
\] (117)

One verifies that the surface impedance (117) reduces

FIG. 12. Inverse surface resistance \( Re(\{Z(\omega)\})^{-1} \) in propagating shear regime as a function of \( \omega \tau_k \), using the slip-length constitutive relations\(^{50}\) in accordance with equation (117), at frequency \( \omega = 2 \times 10^{-3} \omega_p \), with renormalized Fermi velocity \( \nu_F^* = 10^{-3} c \), collision time \( \tau_c = 10^3 \omega_p \), and first Landau parameter \( F_s^* = 20 \). In the purple-shaded area \( \tau_c > \tau_k \) and the dielectric function (62) is no longer accurate. Panel (a) shows the opposite cases \( \lambda_s = 0 \) and \( \lambda_s \to +\infty \) as blue and red solid curves, respectively. The dashed gray curve is the Drude result (92). Panel (b) shows the results for specular and diffusive boundary scattering, stemming from equations (77), as blue and red solid curves, respectively. In the specular case, we assume a surface roughness \( h = h' = 100/k \).

to the Drude expression (91) consistently for \( \nu_F(\omega) \to 0 \): in this case, only one optical mode is propagating in the metal, that is \( n_1(\omega) \to +\infty \) or \( n_2(\omega) \to +\infty \). Figure 12(a) shows the inverse surface reactance \( Z_0/Re\{Z(\omega)\} \) as a function of momentum-relaxation time \( \omega_p \tau_k \), for renormalized Fermi velocity \( \nu_F^*/c = 10^{-3} \), first Landau parameter
Fermi liquid, Drude

\[ E(\mathbf{z}, \omega) \]

\[ \mathbf{E}_{T}(\mathbf{z}, \omega) \]

\[ \mathbf{k} \]

\[ n(\omega) \]

\[ n_1(\omega) \]

\[ n_2(\omega) \]

\[ E_1(\mathbf{z}, \omega) \]

\[ E_2(\mathbf{z}, \omega) \]

\[ \mathbf{E}(\mathbf{z}, \omega) \]

\[ \mathbf{E}_{T}(\mathbf{z}, \omega) \]

\[ \theta_1 e^{i \frac{\pi}{2} n(\omega) d} + \theta_2 e^{-i \frac{\pi}{2} n(\omega) d} \]

\[ t_{\text{film}}(\omega) = \theta_1 e^{i \frac{\pi}{2} n_1(\omega) d} + \theta_2 e^{-i \frac{\pi}{2} n_2(\omega) d} \]

\[ f_1^S = 20, \text{collision time } \tau_\omega \omega_p = 1000, \text{and at frequency } \omega/\omega_p = 2 \cdot 10^{-3} \]. Blue and red curves show the results for \( \lambda_q = 0 \) and \( \lambda_q \to +\infty \), respectively. We see that, even for constitutive relations in terms of the slip length, we retrieve the phenomenology of anomalous skin effect for \( \omega \gg v_F^* \mathbf{q} \): in particular, with increasing relaxation time \( \tau_k \) we progressively fall into the regime \( \tau_c \ll \tau_k \) of section VI A, where we can use macroscopic arguments stemming from fluid dynamics to describe the electrodynamic response of the system and the associated constitutive relations at interfaces. The slip-length parametrization may be converted into specular or diffuse interface scattering by the means of equations (77). Using the latter into equation (117), we obtain the surface impedance displayed in figure 12(b). In the specular case we assume \( h = h' = 100/k_p \), which corresponds to \( h \approx 5 \text{ nm} \) for the present parameters. Since the slip length for diffusive scattering is larger than for specular scattering, the associated red and blue curves in figure 12(b) are close to the infinite- and zero-slip length results in panel (a), respectively. Overall, quantitative differences with respect to the specular-scattering results in figure 9 emerge in the asymptotic limit \( \tau_k \to +\infty \), however the qualitative outcome remains the same using the slip-length approach. This leads to the following conclusion: the saturation of the inverse surface resistance for \( \tau_k \to +\infty \) is a robust feature of the propagating shear regime, with an asymptotic limit governed by the generalized shear modulus \( v(\omega) \), although the numerical value of such limit depends parametrically on the boundary conditions assumed for the motion of the electron fluid at the sample interface.47

C. Thin-film transmission

Another prominent spectroscopic probe of spatial nonlocality in Fermi liquids is the optical transmission through a thin film of thickness \( d \). In fact, the nonlocal response of the quasiparticles to the incident electromagnetic wave modulates the transmission coefficient \( t_{\text{film}}(\omega) \) as a function of frequency \( \omega \) in a different way with respect to the Drude model.54 In the following, we analyze the thin-film transmission for the two opposite cases of section VI B, i.e. in the anomalous skin effect regime, and of section VI A, i.e. in the propagating-shear regime. In the latter regime, the transmission modulus \( |t_{\text{film}}(\omega)| \) is significantly higher than in Drude theory at low temperatures. Furthermore, the interference between the two frequency-degenerate optical modes (37) determines characteristic oscillations of \( [t_{\text{film}}(\omega)] \),54 which may be detected in strongly interacting Fermi liquids.

1. Drude regime

Whenever a single polariton mode propagates into the system, the standard Fresnel equations are sufficient to determine the optical transmission and reflection coefficients at interfaces. This is true for the Drude regime – cfr. equations (64), (65) and section IXB 1 – where the only optical mode is the plasmon-polariton. In this case, a method to obtain the thin-film transmission coefficient \( t_{\text{film}}(\omega) \), including Fabry-Perot internal reflections inside the slab, is to consider the electric field inside the film as the superposition of two counterpropagating rays with refractive index \( n(\omega) \), as depicted in panel (a) of figure 13. The continuity of the electric field and of its derivative at the slab boundaries give rise to a system of four equations, from which the slab transmission amplitude results:

\[ t_{\text{film}}(\omega) = \theta_1 e^{i \frac{\pi}{2} n(\omega) d} + \theta_2 e^{-i \frac{\pi}{2} n(\omega) d} \]
\[
\theta_t = -\frac{2[n(\omega) + 1]}{e^{2i\pi n(\omega)d} [n(\omega) - 1]^2 - [n(\omega) + 1]^2}. \tag{118b}
\]
\[
\theta_s = \frac{2[n(\omega) - 1]}{e^{-2i\pi n(\omega)d} [n(\omega) + 1]^2 - [n(\omega) - 1]^2}. \tag{118c}
\]

Equations (118) produce the dashed gray lines in figures 14 and 15, for the parameters there specified.

2. Anomalous skin effect regime

In anomalous skin regime, at leading order in \(\omega/(v_F^{\ast}Re \{q\}) \to 0\) there are three polariton branches, each associated to a frequency-degenerate refractive index, as mentioned in sections IV D and VI B. Hence, in principle one could adopt the approach of the previous section IX C 1 to calculate the thin-film transmission, i.e. considering constitutive relations at the slab boundaries for each of the three refractive indexes. However, as mentioned in section IX A, Fresnel equations leave the system underdetermined, and one would require ABCs to form a closed problem. In section IX C 3, we will see that suitable ABCs can be mutuated from fluid dynamics in the regime \(\omega \gg v_F^{\ast}Re \{q\}\), since there the Fermi liquid responds analogously to a viscous charged fluid. Instead, in the regime \(\omega/(v_F^{\ast}Re \{q\}) \to 0\) an alternative convenient approach is to deduce an effective single refractive index \(n(\omega)\) from the surface impedance \(Z(\omega)\), written in terms of the specularity coefficient \(p\) for boundary scattering: this builds directly upon the results of section IX B 2. In fact, one can still utilize standard optical formulae like (118) in anomalous regime, provided that the refractive index is deduced from \(Z(\omega)\)\(^{-1}\), i.e.

\[
n(\omega) = [Z(\omega)]^{-1}, \tag{119}
\]
as verified by combining equations (79) and (91). In the absence of momentum relaxation \(Z(\omega)\) is given by equation (96). The latter, together with equations (119) and (118), determines \(t_{\text{film}}\). At finite momentum relaxation and for \(\tau_e \to +\infty\) one retrieves Reuter-Sondheimer theory of anomalous skin effect, expressed by equations (97) in the microwave regime.

Based on equations (97), the ratio (H1) between the mean free path and the Drude skin depth determines the crossover from the Drude regime \(l_{\text{MFP}} \lesssim \delta_e\) to anomalous skin effect regime \(l_{\text{MFP}} \gg \delta_e\). \(\tau_k\) from Umklapp and acoustic-phonons scattering, as determined through equation (76), gives the following standard qualitative picture, substantiated by appendix H: at room temperature, the skin depth is larger than, or comparable to, the mean free path at all frequencies, so that the results from the Drude model are expected to hold. At cryogenic temperatures, there is an extended low-frequency window in which \(\delta_e/l_{\text{MFP}} \ll 1\), so that anomalous skin effect develops. The case including both finite \(\tau_k\) and \(\tau_e\) as well as quasiparticle interactions requires a generalization of Reuter-Sondheimer theory, which in principle can be obtained from the Fermi-liquid optical conductivity (26) and deserves future work. The low-temperature regime in which \(\tau_e \gg \tau_k\) is the most interesting, since a finite collision rate may modify the optical conductivity and the surface impedance with respect to Reuter-Sondheimer theory, as exemplified by equation (68) for \(\omega/(v_F^{\ast}q) \to 0\). The low-frequency blue (red) solid curves in the green-shaded areas of figure 14 show the thin-film transmission in anomalous skin regime for specular (diffusive) boundary scattering, for particle density \(n = 10^{12} \text{cm}^{-3}\), first Landau parameter \(F_1 = 20\) and \(\tau_k \to +\infty\). In this case, the optical conductivity is (41). Notice that the latter is independent from \(\tau_e\), which makes \(t_{\text{film}}(\omega)\) temperature-independent in the absence of relaxation: this is why panels (a) and (b) of figure 14 report identical results in anomalous skin regime.

Since equation (41) holds for \(\omega \ll v_F^{\ast}Re \{q\}\), a qualitative constraint on the frequency domain where equation (41) holds stems from the condition \(Re \{q\} = \frac{\omega}{v_F}\), or

\[
Re \{n(\omega)\} > n_{\min} = \frac{c}{v_F}. \tag{120}
\]

Using equations (119) and (120), we obtain the frequency \(\omega_{\min}\) such that \(Re \{n(\omega_{\min})\} = n_{\min}\): this frequency corresponds to the blue (red) vertical dashed line in figure 14 for specular (diffusive) boundary scattering.

Finally, it is conceptually important to mention the presence of a hydrodynamic skin effect regime in finite-temperature charged Fermi liquids with \(\tau_k \gg \tau_e\), for frequencies close to the DC limit as indicated by the criterion (111). As discussed in section IX B 3, this regime is practically beyond the detective power of standard optical techniques, but it influences transport experiments.

3. Propagating shear regime

In propagating shear regime, the two frequency-degenerate polariton modes (66) give rise to the two refractive indexes (67) and hence to two counterpropagating rays for each mode inside the slab – see figure 13(b). In this case, we are able to identify appropriate ABCs from the analogy between Fermi-liquid electrodynamics for \(\omega \gg v_F^{\ast}Re \{q\} \to +\infty\) and the electromagnetic response of viscous charged fluids,\(^{54}\) which makes the computation of the thin-film transmission a closed problem. Such approach follows section II B of reference 54; in essence, the continuity of the electric field and of its derivative, together with the constitutive relation stemming from the linearized Navier-Stokes equation, generates a system of six linear equations. Here we quote the final result:

\[
\frac{t_{\text{film}}(\omega)}{t_{s}} = e^{-\frac{i\pi d}{2}} [t_1 e^{-\frac{i\pi}{2}n_{s}(\omega)d} + \theta_1 e^{-\frac{i\pi}{2}n_{i}(\omega)d} + t_2 e^{-\frac{i\pi}{2}n_{i}(\omega)d} + \theta_2 e^{-\frac{i\pi}{2}n_{s}(\omega)d}]. \tag{121}
\]

The coefficients \((t_1, \theta_1, t_2, \theta_2)\) in equation (121), which describe the contributions of two counter-propagating
waves inside the slab for each of the two degenerate optical modes, are determined numerically through equation (9) of reference 54.

In the following, we use the relations (77) between the slip length $\lambda$, and the generalized shear modulus $\nu(\omega)$, as done in section IX B 6, to convert the slip-length parametrization into diffuse/specular boundary scattering. The blue and red solid curves in the orange-shaded areas of figure 14 show the modulus of the thin-film transmission coefficient $|t_{film}(\omega)|$ as a function of frequency $\omega$ in the relaxationless limit $\tau_k \rightarrow +\infty$, for specular and diffuse boundary scattering respectively. Particle density is fixed to $n = 10^{23}$ cm$^{-3}$ and the first Landau parameter is $F_1^S = 20$. The Drude result for the same parameters is shown by the dashed gray line for comparison.

The vertical dashed orange line shows the frequency $\omega_{VE}$ corresponding to the frequency limit $\omega \gg \omega_{VE}$ from section IV B, for the phenomenology of equation (35) applies for $\omega \gg \omega_{VE}$. Such constraint translates into

$$\text{Re} \{n_i(\omega)\} < \frac{c}{\nu_p}$$

(122)

for the refractive index, where $n_i(\omega)$ is the refractive index of the shear-polariton ($n_1(\omega)$ and $n_2(\omega)$ swap character at $\omega \tau_k = 1$).

Figure 14(a) suggests a higher absolute value of the thin-film transmission coefficient in propagating shear regime even at $T = 300$ K without momentum relaxation, with respect to the Drude model. The room-temperature wiggles as a function of $\omega$ in panel (a) are amplified into sizable oscillations in the low-temperature case of panel (b), and stem from the mutual interference of the two polaritons (66). The amplification of the oscillations with decreasing temperature originates from the increase in $\tau_k$, which brings the system towards the collisionless regime $\omega \tau_k \gg 1$. In the latter condition, the Fermi surface resonates with a dominant dissipationless reactive response, as analyzed in section IV A, and radiation couples to such reactive resonance by exchanging energy and momentum: the less dissipative the Fermi-surface resonance is, the more efficient is its coupling to photons, and the more its traces in the optical transmission are visible. However, as noted in sections VI and VI B, in a realistic solid-state system Galilean invariance is inevitably broken by the crystalline lattice and impurities, which cause the relaxation of quasiparticle momentum. Hence, we have to address the question of whether some traces of spatial nonlocality persist in the transmission spectrum at finite $\tau_k$.

The next section deals with such question, employing the results of section VIII.

4. Role of impurity and phonon scattering

In the presence of momentum relaxation, the dielectric function in the propagating shear regime becomes (62). Here I assume a momentum-relaxation time given by Mathiessen’s rule (76), with independent contributions stemming from acoustic phonons, impurities and Umklapp scattering as described in section VIII. As in the momentum-conserving case, the two roots of equation (62) give the frequency-degenerate refractive indexes to use in equation (121). The resulting absolute value of the transmission coefficient $|t(\omega)|$ is displayed in figure 15: panel (a) shows the case at $T = 300$ K with acoustic phonon and Umklapp scattering but without impurities, panel (b) illustrates the case at $T = 1$ K with acoustic phonons and Umklapp scattering but no impurities, and panel (c) displays the calculation at $T = 1$ K with phonons, Umklapp scattering and different values of impurity scattering time $\tau_i$. In all
The blue and red solid curves in panel (c) additionally include shear-mode propagation at room temperature: the curves give the vertical dashed red and blue lines in panel (c), for accordance with the constraint (122). The same constraint persists at low temperatures in the presence of relaxation. A constraint on sample purity descends from panel (c): decreasing the impurity scattering time $\tau_i$ effectively abates nonlocal effects by decreasing $\tau_k$, until the calculation from the propagating-shear-model becomes practically indistinguishable from the Drude result at $\tau, \omega_p = 100$. For a typical electron density $n \approx 10^{23} \text{ cm}^{-3}$, this means $\tau_i \approx 5.6 \times 10^{-15}$ s, while typical Drude scattering rates of metals lie in the range $\tau_k \approx (10^{-13} \div 10^{-11})$ s at $T = 77 \text{ K}$. Therefore, a sufficiently low impurity concentration is essential to unveil phenomena connected to the Fermi-surface shear rigidity in low-temperature optical spectra.

FIG. 15. Transmission modulus $|t_{f/fin}(\omega)|$ from equation (121) including impurity and electron-phonon scattering through equation (76). The blue (red) curve refers to specular (diffuse) boundary scattering. Panels (a) and (b) show the results with acoustic phonons and no impurities, at $T = 300 \text{ K}$ and $T = 1 \text{ K}$ respectively. The blue and red solid curves in panel (c) additionally include impurity scattering using $\tau_i, \omega_p = \{100, 1000\}$, respectively, assuming specular boundary scattering. Dashed curves show the Drude calculation for the same $\tau_k$. All other parameters are fixed as in figure 14.

panels, dashed lines show the result from the standard Drude model, using the same relaxation time $\tau_k$ as in the propagating-shear calculation. An orange vertical dashed line in panels (a) and (b) marks the frequency below which the shear-polariton enters into the Lindhard continuum, and the viscoelastic model is no longer valid, in accordance with the constraint (122). The same constraint gives the vertical dashed red and blue lines in panel (c), for $\tau_{imp}, \omega_p = \{100, 1000\}$ respectively.

The calculations in figure 15(a) confirm the educated guess that momentum relaxation conceals any traces of shear-mode propagation at room temperature: the curves for both specular and diffuse scattering are very close to the Drude transmission, even for a first Landau parameter $F_1^S = 20$. This is because the relaxation time $\tau_k \ll \tau_c$ is severely limited by phonons at room temperature, such that momentum-relaxing processes dominate over the nonlocal term $i \omega \nu(\omega)q^2$ in equation (62). A lower $F_i^S > 12$ increases $\tau_c$, which in principle favors shear propagation; however, it also rapidly reduces the frequency window in which the viscoelastic model holds, because it pushes the emergence of the shear-polariton from the continuum to higher frequency, until the shear-polariton never exits from the continuum for $F_i^S \rightarrow 12$ – see also discussion in section IV C. However, qualitative differences between the Drude and propagating-shear models emerge in the low-temperature calculation of panel (b), even in the presence of acoustic phonons and Umklapp scattering. In this case, $\tau_k$ increases due to the progressive freezing of phonon scattering, and oscillations induced by $\nu(\omega)$ emerge for $\omega/\omega_p = (10^{-4} \div 10^{-3})$. These are the same oscillations of figure 14, which are seen to partially persist at low temperatures in the presence of relaxation.

In summary, the transverse shear response of an electrically charged Fermi liquid is spatially nonlocal, due to correlations among quasiparticles entailed by interactions and collisions. The nature of this response depends on the frequency $\omega$ and on the momentum $q$ of the external probe, as realized by calculating the transverse susceptibility in three dimensions in the kinetic equation approach of Landau, Abrikosov and Khalatnikov (section III). The regime $\omega \ll \nu_f \text{Re} \{q\}$ corresponds to incoherent dissipative electron-hole excitations in the Lindhard continuum (section IV D). In the regime $\omega \gg \nu_f \text{Re} \{q\}$ quasielectrons composing the Fermi surface collectively respond to dynamical perturbations as a macroscopic substance endowed with a frequency-dependent generalized shear modulus $\nu(\omega)$ (section IV A) in the collisional/hydrodynamic regime $\omega \tau_c \ll 1$ the response is predominantly dissipative, akin to that of a highly viscous liquid, as the collision time $\tau_c$ provides the smallest timescale in the system and allows for local thermodynamic equilibrium. More generally, the hydrodynamic regime $\omega \tau_c \rightarrow 0$ holds in the low-frequency, low-momentum region of the $(\omega, \text{Re} \{q\})$ plane at any finite $\omega/\nu_f \text{Re} \{q\}$ ratio (section V). In the opposite collisionless/elastic-like regime $\omega \tau_c \gg 1$, with $\omega \gg \nu_f \text{Re} \{q\}$ and for sufficiently
strong interactions $F^1_1 > 6$, the Fermi surface responds with a dominant dissipationless reactive contribution: over short time scales, the Fermi liquid reacts similarly to an elastic solid.

To probe such rigidity of the Fermi sea to shear deformations, the photon represents an ideal instrument since electromagnetic waves are transversely polarized and couple to the electric charge of quasiparticles, thus inducing a shear response. However, this requires to consider spatial nonlocality in the Fermi-liquid dielectric function $\epsilon_F(q, \omega)$ in the presence of collisions (section IV) and including momentum relaxation (section VI). In particular, in the propagating shear regime $\omega / (v_F^2 \text{Re} \{q\}) \to +\infty$ the leading nonlocal correction to standard Drude electrodynamics can be written in terms of a generalized shear modulus $\tilde{\gamma} (\omega)$ (section IV A), even in the presence of weak momentum relaxation (section VI A).

On the other hand, from the Fermi-liquid dielectric function in the regime $\omega \ll v_F^2 \text{Re} \{q\}$ one retrieves anomalous skin effect, a well-known manifestation of spatial nonlocality in the current response of metals (sections IV D and VI B).

Furthermore, in the low-frequency and low-momentum regime at finite temperature, the charged Fermi liquid manifests a hydrodynamic response when momentum relaxation is negligible (section V): this occurs for very low frequencies, close to the DC limit (section VII).

The surface impedance of semi-infinite samples (section IX B) provides a suitable experimental test for the different aforementioned regimes of Fermi-liquid electrodynamics. In fact, the characteristic skin depth for penetration of electromagnetic fields in the Fermi liquid is sensitive to Fermi-surface shear stresses (see table 1), and it exhibits qualitatively distinct frequency dependencies in different regimes.

Moreover, for $\omega \gg v_F^2 \text{Re} \{q\}$ the saturation of the surface resistance at vanishing $\gamma_\perp$ gives a phenomenology analogous to anomalous skin effect but with a different skin depth, which can be interpreted in terms of Fermi-liquid shear propagation (section IX B 3).

Fermi-liquid shear modes can also be probed by the optical transmission of thin films (section IX C). In propagating shear regime (section IX C 3), thin-film transmission modulus significantly increases with respect to its Drude counterpart (section IX C 1), with pronounced oscillations due to the interference of two mutually coherent polariton modes. Such oscillations are amplified in the collisionless regime $\omega \tau_\perp \gg 1$, reflecting the coupling of radiation to the reactive shear mode of the Fermi surface, which resonates like an elastic membrane. Momentum relaxation, considered in a simple model with acoustic phonons, Umklapp scattering and impurities as independent relaxation channels, suppresses the effects of shear viscoelasticity at room temperature, while residual traces of spatial nonlocality may persist at cryogenic temperatures in strongly-interacting Fermi liquids, with a high effective mass $m^* = m\left(1 + \frac{S}{\tau}\right) \gg 1$ and a low impurity concentration (section IX C 4).

Assessing the feasibility of observing shear-propagation effects in real three-dimensional solids requires further careful analysis of the assumptions made in this paper. For instance, I assumed a single parabolic band, while non-parabolicity destroys the proportionality between current density $J$ and quasiparticle momentum $k$: in the latter situation, the low-momentum transverse response will not simply be connected to a generalized shear modulus $\gamma (\omega)$, as new dissipative and/or nondissipative transport coefficient may arise.

Likewise, I considered nearly-isotropic Fermi liquids, in which translational invariance is broken only on the microscopic scale of the crystalline lattice. This allows for a simplification of the viscosity tensor, with the bulk and shear viscosity coefficients as the only independent components. The number of components increases with lower symmetry of the system, as in the case of graphene which is neither Galilean-, nor Lorentz-invariant.

Moreover, I considered the simplest parametrization of the transverse shear response in terms of the first Landau parameter $F^1_1$. Considering more Landau parameters modifies quantitatively the results: for instance, transverse zero sound propagates even for $F^1_1 < 6$ if the second Landau parameter is $F^2_2 > 0$. The quantitative impact of momentum relaxation on the calculations depends on the microscopic scattering sources considered in the model. In particular, Matthiessen’s rule for adding scattering rates can be justified for independent relaxation channels. A more microscopic understanding of the interplay between spatial nonlocality and relaxation deserves future work, even for 3D Fermi liquids.

Within the assumptions of this paper, the main requirements which a candidate 3D material must satisfy to observe Fermi-liquid shear propagation are dictated by the frequency-dependent coefficient $\gamma (\omega)$: one needs a heavy effective mass $m^* \gg 1$, or equivalently a high first Landau parameter $F^1_1 \gg 1$, and very clean samples at cryogenic temperatures. Fermi liquids formed by heavy-fermion systems, or the low-density doped perovskyte SrTiO$_3$ might provide appropriate platforms to investigate the electrodynamic shear response of the Fermi surface. A more detailed discussion of suitable materials will be reported in a companion paper.

More generally, the macroscopic approach of continuum mechanics applied to electron ensembles, together with appropriate microscopic models for transport and electrodynamics coefficients, may open new avenues in the investigation of low-temperature correlations in metals: the reactive coefficients describing low-momentum electrodynamics persist even in the presence of pairing correlations, while the dissipative components disappear in the superconducting state. The evolution of the reactive shear coefficients across a superconducting transition, experimentally probed by optical spectroscopy, could offer valuable insight into the finite-momentum dynamics of Cooper pairing in unconven-
tional superconductors. 101,102

XI. ACKNOWLEDGMENTS

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Appendix A: Derivation of the Fermi liquid kinetic equation

In this appendix we report a detailed derivation of the kinetic equation for the nonequilibrium distribution function of a Fermi liquid. Our main references for this section will be the works of Nozières and Pines 1 and Abrikosov and Khalatnikov. 2 The essence of the kinetic equation approach is to consider how the nonequilibrium distribution function $N_p(r, t)$ of Fermi liquid electron-hole quasiparticles evolves in space $r$ and time $t$ in response to perturbations. As such, the kinetic equation is equivalent to a quantum Boltzmann equation for the system distribution function, here describing the statistical distribution of electron-hole elementary excitations of the Fermi liquid. Following Nozières and Pines, 1 given a total energy of the Fermi liquid $E_F(n, t)$ which is a function of electron density $n$, the local excitation energy of a quasiparticle is $\tilde{E}_n[p(r, t), \sigma] = \frac{\partial N_p(L(n), t)}{\partial \delta n_p(r, t)} = E_n[p(r, t), \sigma] + \sum_{\sigma', \sigma''} f_{p, \sigma, p', \sigma'} \delta n_{p', \sigma'}(r, t)$. This means that the local excitation energy is the sum of the energy of the quasiparticle $E_n[p(r, t), \sigma]$, with momentum $p$ and spin $\sigma$, and the local interaction energy of that quasiparticle interacting with other quasiparticles in its surroundings; the latter term is similar to an effective mean field of the Fermi liquid, given by the reaction of all other quasielectrons-quasiholes when we perturb the energy of a given quasiparticle of momentum $p$. The short-range interaction matrix elements between quasiparticles of momenta $\{p, p'\}$ and spin $\{\sigma, \sigma'\}$ are $f_{p, \sigma, p', \sigma'}$. Landau considered the excitation energy $E_n[p(r, t), \sigma]$ as an effective Hamiltonian for the Fermi liquid, in order to describe the transport properties of the system. For electrically charged quasiparticles, we write the microscopic Hamiltonian as

$$\hat{H}_{qp}(r, p, \sigma) = E_n[p + eA(r, t), \sigma] + \sum_{p', \sigma', \sigma''} f_{p, \sigma, p', \sigma'} \delta n_{p', \sigma'}(r, t) - e\phi_{\sigma}(r, t). \quad (A1)$$

In equation (A1), we recognize the quasiparticle energy eigenvalues $E_n[p + eA(r, t), \sigma]$, modified by the presence of the vector potential $A(r, t)$, as well as by the short-range interaction matrix elements $f_{p, \sigma, p', \sigma'}$. The quasiparticle velocities are defined from the crystalline momentum $v_p = \nabla_p E_n(p) = \frac{p}{\hbar}$. The nonequilibrium distribution function $N_p(r, t) = \frac{\partial N_{p, \sigma}(r, t)}{\partial \delta n_p(r, t)} + N_0(E_n[p + eA(r, t), \sigma])$ contains the departure of the quasiparticle statistics from local equilibrium $N_0(E_n[p + eA(r, t), \sigma])$, and can be calculated from particle number conservation and gauge invariance.2,10 We also include a scalar potential $\phi_{\sigma}(r, t)$.

Equation (A1) is valid in the regime $k_BT \ll E_F$ and $\hbar\omega \ll E_F$, with $E_F = \frac{\eta(N_p)^2}{2m}$ Fermi energy, with quasiparticles of large lifetime $\tau_c \gg \frac{1}{\omega}$ and a well-defined Fermi surface and relative excitation spectrum. Physically, these conditions signify that thermal agitation and the external perturbation are not strong to drive quasielectrons completely out of equilibrium, thus destroying the Fermi surface. From the Hamiltonian (A1), one derives a linearized kinetic equation for quasiparticles of velocity $v_{p, \sigma}$ on the Fermi surface, which participate to the collective mode thus changing the Fermi-surface distribution. The response to the perturbations on the quasiparticles, which are parameterized by Landau parameters $F_i^{SA}$, and on the angle $\theta$ between the excitation wave vector $q$ and the quasiparticle wave vector $k$.

Starting from the Hamiltonian (A1), we linearize the changes in the quasiparticle energies $E_n[p + eA(r, t), \sigma]$ and in the distribution function $\delta n_{p, \sigma}(r, t)$, with respect to the vector potential $A(r, t)$, as in

$$\delta N_{p, \sigma}(r, t) = N_{p, \sigma}(r, t) - N_0(E_n[p + eA(r, t), \sigma])$$

$$\approx N_{p, \sigma}(r, t) - N_0(E_n[p]) - \frac{dN_0(E_n[p])}{dE_n(p, \sigma)} \cdot \frac{m^*}{\hbar^2} E_n[p, \sigma] \cdot \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial p} \cdot \delta n_{p, \sigma}(r, t). \quad (A2)$$

The nonequilibrium quasiparticle distribution function $N_{p, \sigma}(r, t)$ descends directly from Liouville’s equation, for a classical flow characterized by volume-conserving evolution in phase space. Therefore, we can write a quasi-classical mean-field kinetic equation for $N_{p, \sigma}(r, t)$, which is

$$\frac{\partial N_{p, \sigma}(r, t)}{\partial t} + \frac{1}{\hbar} \frac{\partial \hat{H}_{qp}(r, p, \sigma)}{\partial p} \cdot \frac{\partial \delta n_{p, \sigma}(r, t)}{\partial r} - \frac{1}{\hbar} \frac{\partial \hat{H}_{qp}(r, p, \sigma)}{\partial r} \cdot \frac{\partial \delta n_{p, \sigma}(r, t)}{\partial p} = \mathcal{J}_{\text{coll}}(r, t). \quad (A3)$$

In equation (A2), the collisional term $\mathcal{J}_{\text{coll}}(r, t) = \frac{\partial n_{p, \sigma}(r, t)}{\partial t} \cdot _{\text{coll}}$ describes damping by collision processes that are not included into the mean-field Hamiltonian $\hat{H}_{qp}(r, p, \sigma)$. In fact, the Fermi liquid phenomenology relies on the existence of low-energy nearly-independent quasiparticles, which do not collide, and additional collisions must be introduced by an external damping term $\mathcal{J}_{\text{coll}}(r, t)$. This collisional integral makes quasielectrons and quasiholes acquire a finite lifetime, $\tau_{\text{qel}} < +\infty$ and $\tau_{\text{qh}} < +\infty$ respectively: as common in quantum mechanics, we are treating a time-dependent problem with the eigenstates of a stationary configuration, in this case the one for free fermions, but including time-dependent scattering as a finite lifetime for the stationary states. We emphasize that collisions redistribute momentum and energy among the quasiparticles,
but the total momentum and the total energy of the ensemble is conserved in collisions. For the true thermodynamic equilibrium, we know that the distribution function is the Fermi-Dirac statistics \( f_{FD}(E_n(p, \sigma)) \). Since this is the configuration for global thermodynamic equilibrium, it is a stationary state and we must have a vanishing collision term \( \frac{\partial f_{FD}(E_n(p, \sigma))}{\partial t} \) for any \( p \) and \( \sigma \).

In principle, excitations can alter the energies, with respect to the Fermi-Dirac distribution at first order in the quasiparticle perturbation.\(^{A4} \) Inserting the latter term into equation (A2), we arrive at

\[
\frac{\partial \Xi_{k,\sigma}(r, t)}{\partial t} + v_{k,\sigma} \cdot \frac{\partial \Xi_{k,\sigma}(r, t)}{\partial r} + v_{k,\sigma} \cdot \vec{F}_s(r, t) \delta [E_n(p, \sigma) - \mu] = \mathcal{J}_{\text{coll}}(r, t). \quad (A4)
\]

In equation (A4), we have defined the total quasi-classical force acting on the individual quasiparticles

\[
F_s(r, t) = -\nabla_r [-e\phi_s(r, t)] + \sum_{k', \sigma'} f_{k,\sigma',k',\sigma} \Xi_{k,\sigma}(r, t) + \frac{\hbar k}{m} \cdot \mathbf{A}(r, t). \quad (A5)
\]

The linearized kinetic equation (A4) is manifestly gauge-invariant, and it satisfies the continuity equation for the flow of quasiparticles:\(^2\) This is the starting point for many applications of Landau phenomenology of Fermi liquids, for example the study of collective modes in the absence of perturbations.

Now, we concentrate on density-density collective modes in the Fermi liquid. Collective modes propagate even in the absence of perturbations, so we set the electromagnetic potentials to zero as \( \phi_s(r, t) = 0, \mathbf{A}(r, t) = 0 \). Furthermore, collective modes have wave-like evolution, therefore the deviation of the distribution function with respect to global equilibrium will be periodic in space and time, i.e., \( \Xi_{k,\sigma}(r, t) \sim e^{i(q \cdot r - \omega t)} \), with \( q \) transferred wave vector for the excitation. Inserting this wave-like form into equation (A4), without electromagnetic potentials and collisions, leads to

\[
(q \cdot v_{k,\sigma} - \omega) \Xi_{k,\sigma}(r, t) + q \cdot v_{k,\sigma} \delta [E_n(k, \sigma) - \mu] \cdot \sum_{k', \sigma'} f_{k,\sigma',k',\sigma} \Xi_{k,\sigma}(r, t) = \mathcal{J}_{\text{coll}}(r, t). \quad (A6)
\]

We assume small deviations with respect to thermodynamic equilibrium: physically, we consider changes in the distribution function at first order in the quasiparticle energies, with respect to the Fermi-Dirac distribution at global equilibrium. In principle, excitations can alter the quasiparticle energies \( E_n(k, \sigma) \) either by changing the wave vector \( k \) or by flipping the spin \( \sigma \); in the following, having in mind sound-like collective modes, we assume that alterations of \( E_n(k, \sigma) \) stem from variations in the quasiparticle momentum \( k \). Under these hypothesis, expanding the distribution deviations to first order in the energies \( E_n(q, \sigma) \), we have \( \Xi_{k,\sigma}(q, \omega) = \epsilon_k(q, \omega) \Xi_{k,\sigma}(0, \mu) \), with \( \epsilon_k(q, \omega) \equiv -\delta [E_n(k, \sigma) - \mu] \) for the Fermi-Dirac distribution.\(^3\)\(^4\)\(^10\) If we had wanted to analyze magnetic modes in the Fermi liquid, like spin waves, we could also have assumed a distribution function change of the kind \( \Xi_{k,\sigma}(q, \omega) = s_q(q, \omega) \delta [E_n(k, \sigma) - \mu] \), with the quasiparticle energies modified by spin flipping.\(^2\) Finally, let us note that a Taylor series expansion at first order of the change in the distribution function selects quasiparticles at the chemical potential \( \mu \): essentially only excitations around \( \mu \) contribute to collective modes, since states at \( E < \mu \) deep down into the Fermi sea are occupied and blocked by Pauli exclusion principle.

This way, the kinetic equation (A6) in local equilibrium, written in reciprocal space of momenta \( q \) and frequency \( \omega \), becomes

\[
(q \cdot v_{k,\sigma} - \omega) \epsilon_k(q, \omega) + q \cdot v_{k,\sigma} \delta [E(k, \sigma) - \mu] \cdot \sum_{k', \sigma'} f_{k,\sigma',k',\sigma} \delta [E_n(k', \sigma') - \mu] = \mathcal{J}_{\text{coll}}(q, \omega), \quad (A7)
\]

where \( \mathcal{J}_{\text{coll}}(q, \omega) \) is the Fourier transform of the collision term \( \mathcal{J}_{\text{coll}}(r, t) \) with respect to space and time. As previously mentioned, the delta functions \( \delta [E_n(k, \sigma) - \mu] \) and \( \delta [E_n(k', \sigma') - \mu] \) fix the value of \( k \) and \( k' \), respectively: therefore, the character of the collective mode, including its polarization direction, will be determined by the respective orientation of the excitation wave vector \( q \) with respect to \( k \) and \( k' \).

For long wavelengths \( q \to 0 \), we expect sound waves with acoustic dispersion \( \omega_s(q) = v_s q \), with \( v_s \) sound velocity for the Fermi liquid. Setting the latter relation into equation (A7), we have in scalar form

\[
(q \cdot v_{k,\sigma} - \omega_s) \epsilon_k(q, \omega) + v_{k,\sigma} \cos \theta \delta [E(k, \sigma) - \mu] \cdot \sum_{k', \sigma'} f_{k,\sigma',k',\sigma} \epsilon_{k'}(q, \omega) \delta [E_n(k', \sigma') - \mu] = \mathcal{J}_{\text{coll}}(q, \omega), \quad (A8)
\]

where \( \theta = \frac{\arccos(q \cdot v_{k,\sigma})}{q \cdot v_{k,\sigma}} \) is the angle between the wave vector \( q \) and the quasiparticle velocity \( v_{k,\sigma} \).

To continue, we need to expand the angular dependence of the interaction matrix elements \( f_{k,\sigma,k',\sigma'} \), which determines the spatial polarization of the perturbation. Being an angular distribution in 3-dimensional space, a suitable expansion basis for \( f_{k,\sigma,k',\sigma'} \) is given by spherical harmonics, in terms of Legendre polynomials.\(^2\)\(^4\) We recall that Legendre polynomials are \( P_l^m(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left( (x^2 - 1)^l \right) \), and the associated Legendre polynomials satisfy \( P_l^m(x) = (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} \left( P_l^m(x) \right) \). The latter
are used in the definition of spherical harmonics $Y_{l,m}(\theta, \phi) = \frac{(-1)^m}{\sqrt{4\pi (2l+1)}} P_l^m(\cos \theta) \text{e}^{im\phi}$. With these definitions, we can act on the term between $k_S$ and $\kappa$, where $\kappa$ remains of the sum is an integration over angular variables, and the delta function distinguishes between symmetric and antisymmetric Landau parameters. The former kind is associated to the charge or subtractive combination of $f_k \pm f_{k'}$, which represents the respective magnitudes of quasiparticle residual interactions, and therefore differs from $N_0^\alpha(0)$ the magnetic spin-spin response. The density of states per unit volume $N_0^\alpha(0)$ includes renormalization effects due to quasiparticle residual interactions, and therefore differs from the free fermions one $N_0^\alpha(0)$. We concentrate on the problem of finding the dispersion relation of the first Fermi-surface collective mode at the Fermi surface, i.e. for all spherical harmonics.

The first term at the right-hand side of equation (B2) stems from the derivative expansion (A10) of the interaction matrix elements leads to equation (1) quoted in the main text.

### Appendix B: Derivation of the dispersion relation for transverse sound with collisions

To find the dispersion relation of the first Fermi-surface transverse mode with $m = 1$ in the presence of collisions, we have to study the following kinetic equation, first obtained by Lea et al.:

\[
\begin{align*}
(\cos \theta - \xi) e^S(\theta) + & \left[3e^S(\theta)\sin \theta \right]_{\omega r} \sin \theta \left( \frac{F_S^1}{3} \cos \theta - \beta \xi \right) = 0, \\
\end{align*}
\]

where we have defined $\xi = \frac{1 - \frac{1}{2} \cos \theta}{1 - \cos \theta}$ and $\beta = \frac{1}{\tau_c(\xi)}$, as in equations (9), and we have used the parametrization (7) for the collision integral in terms of a single collision time $\tau_c$. From equation (B1), we immediately deduce the Fermi surface displacement $e^S(\theta) \propto \frac{3 \sin(\frac{3}{2} \cos \theta - \beta \xi)}{\xi - \cos \theta}$.

The first term at the right-hand side of equation (B2) stems from short-ranged quasiparticle interactions, and it is present also in the absence of collisions; the second term at the right-hand side of equation (B2) arises only at finite $\tau_c$. Integrating over the angle $\theta$ in equation (B2), we obtain

\[
\begin{align*}
[3e^S(\theta)\sin \theta]_{\omega r} &= 3 \int_0^\pi \left( \sin \theta \right)^2 d\theta \frac{F_S^1}{4} \sin \theta \cos \theta - \beta \int_0^\pi \left( \sin \theta \right)^2 d\theta \frac{3 \sin(\frac{3}{2} \cos \theta - \beta \xi)}{\xi - \cos \theta}.
\end{align*}
\]
Using \( \varepsilon^\beta(\theta) \propto \frac{\sin^{\beta/2} \cos \theta - \beta \xi}{\xi - \cos \theta} \), the collective mode kinetic equation becomes

\[
\sin \theta \left( \frac{F_1^S}{3} \cos \theta - \beta \xi \right) - \frac{[3\varepsilon^\beta(\theta) \sin \theta]_{av} \sin \theta \left( \frac{F_1^S}{3} \cos \theta - \beta \xi \right) = 0,
\]

which means

\[
-1 + 3 \varepsilon^\beta(\theta) \sin \theta \sin \theta = 0. \tag{B4}
\]

Inserting the explicit expression of \([3\varepsilon^\beta(\theta) \sin \theta]_{av}\) from equation (B3), we have

\[
3 \left( \frac{F_1^S}{3} - \beta \right) \left[ -1 + \frac{\xi^2}{2} - \frac{\xi}{4} (\xi^2 - 1) \ln \left( \frac{\xi + 1}{\xi - 1} \right) \right] - \beta = 1. \tag{B5}
\]

Rearranging equation (B5), we finally achieve

\[
-1 + \frac{\xi^2}{2} - \frac{\xi}{4} (\xi^2 - 1) \ln \left( \frac{\xi + 1}{\xi - 1} \right) = \frac{1 + \beta}{F_1^S - 3\beta},
\]

in other words

\[
\xi^2 - \frac{\xi}{2} (\xi^2 - 1) \ln \left( \frac{\xi + 1}{\xi - 1} \right) - 1 = 2 \left[ \frac{3 + 3\beta}{3F_1^S - 9\beta} + \frac{1}{3} \right] - 1 = \frac{-F_1^S + 6 + 9\beta}{3F_1^S - 9\beta},
\]

which gives equation (8) reported in the main text.

**Appendix C: Derivation of the noninteracting transverse susceptibility without collisions**

We start from the kinetic equation (A4) with the total quasiparticle force (A5). This includes the effect of the transverse vector potential \( A(q, \omega) = A_s(q, \omega) \). Linearizing the change in the quasiparticle distribution function with respect to global equilibrium, similarly to Appendix A, we obtain

\[
\left( q \cdot \nu_{k,\sigma} - \omega \right) \varepsilon_k(q, \omega) + q \cdot \nu_{k,\sigma} \delta \left[ E(k, \sigma) - \mu \right]
\]

\[
\cdot \left\{ \sum_{k', \sigma'} J_{k, \sigma, k', \sigma'} \varepsilon_{k'}(q, \omega) \delta \left[ E_n(k', \sigma') - \mu \right] \right\}
\]

\[
= \mathcal{J}(q, \omega) - \frac{e}{mv_F} \left( q \cdot \nu_{k,\sigma} \right) \left[ k \cdot A(q, \omega) \right]. \tag{C1}
\]

We transform the sum over \( k' \) and \( \sigma' \) in an integral over angular coordinates at the Fermi wave vector \( k_F \), similarly to the procedure in appendix A. This fixes the quasiparticle velocity to the renormalized Fermi velocity, \( \nu_{k, \sigma} \equiv v_F \). Then, equation (C1) becomes

\[
(\cos \theta - s) \varepsilon_k(q, \omega) + \cos \theta \cdot \left[ \int \frac{d\mathcal{Q}}{4\pi} \sum_{l=0}^{\infty} F_l^{SA} \varepsilon_{k_l}(q, \omega) + \frac{e}{mv_F} k \cdot A(q, \omega) \right]
\]

\[
= \frac{1}{qv_F} \mathcal{J}(q, \omega), \tag{C2}
\]

which is just equation (2) with the addition of the driving term that depends on the applied vector potential \( A(q, \omega) \) (recall that \( s = \omega/(qv_F^3) \)). From this point, we perform the same operations on the kinetic equation (C2) as we have done for equation (2): we expand the Fermi surface displacement function \( \varepsilon_k(q, \omega) \) in spherical harmonics using equation (3), we consider the first transverse mode with \( m = 1 \) in the density, i.e. symmetric, interaction channel, and we truncate the infinite sum over \( l \) to \( l = 1 \), so that the interaction becomes \( \sum_{l=0}^{\infty} F_l^{SA} \varepsilon_l(q, \omega) \equiv F_0^S + F_1^S \cos \alpha \), and the Fermi surface displacement can be written as \( \varepsilon_k(q, \omega) = \sum_{l=0}^{\infty} \varepsilon_{k_l}(q, \omega) \equiv \varepsilon_0(q, \omega) \equiv \varepsilon^0(\theta), \quad \varepsilon^0(\theta) \equiv e^0(\theta)e^{i\phi} \), where \( e^0(\theta) \) collects the \( \theta \)-dependent portion of the displacement. We also neglect collisions in this section, i.e. \( \mathcal{J}(q, \omega) = 0 \). The resulting kinetic equation is

\[
(\cos \theta - s) e^0(\theta)e^{i\phi} + \cos \theta \cdot \int_0^{2\pi} \frac{d\phi'}{4\pi} \sin \theta' d\phi' \left( F_0^S + F_1^S \cos \alpha \right) e^0(\theta')e^{i\phi'} + \frac{e}{mv_F} \cos \theta k \cdot A(q, \omega) = 0 \tag{C3}
\]

By kinetics the angle \( \alpha \) is such that \( \cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' (\phi - \phi') \). If we assume no interactions - i.e. \( F_0^S = 0 \) and \( F_1^S = 0 \) - the Fermi surface displacement \( e^0(\theta)e^{i\phi} \) follows immediately:

\[
e^0(\theta)e^{i\phi} = \frac{e}{mv_F} \cos \theta k \cdot A(q, \omega) \tag{C4}
\]

The paramagnetic current density is:\(^2,^9\)

\[
J(q, \omega) = \frac{1}{\gamma} \sum_{k, \sigma} \frac{k}{m} v_F^2 \delta(\xi_k) \varepsilon_k(q, \omega)
\]

\[
= \frac{2}{\gamma} \frac{e}{mv_F} \sum_{k} \delta(\xi_k) \cos \theta \frac{k \cdot A(q, \omega)}{s - \cos \theta} = \sum_{k} \chi^0_{\mu\nu}(q, \omega) \varepsilon_k(q, \omega) \tag{C5}
\]

where the last line defines the noninteracting paramagnetic susceptibility tensor in Landau theory:

\[
\chi^0_{\mu\nu}(q, \omega) = \frac{2}{\gamma} \frac{1}{m^2} \sum_{k} \delta(\xi_k) \frac{\cos \theta}{s - \cos \theta} k_{\mu} k_{\nu} \tag{C6}
\]
In the transverse case, for polarization along $x$ we take $\mu = \nu = x$ and we obtain

$$\begin{align*}
X_0^0(q, \omega) &= X_{x,x}^0(q, \omega) = \frac{2}{m} \frac{1}{m^2} \sum_k \delta(\xi_k) \cos \frac{q \cdot \mathbf{k}_x}{s - \cos \theta} (k_x)^2 \\
&= \frac{2}{m^2} N_0^+(E_F) \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\sin \theta d\theta}{4\pi} \cos \frac{q \cdot \mathbf{k}_x}{s - \cos \theta} \left( \frac{k_x^2 \sin \theta}{2} \right)^2 \\
&= \frac{2}{m^2} N_0^+(E_F)(k_x)^2 \frac{1}{8} \int_0^{\pi} \sin \theta^2 \cos \theta d\theta \\
&= \frac{3/4 n_m^*}{m^2} \left[ -\frac{4}{3} + 2s^2 + \frac{2}{4} (1-s^2) \ln \left( \frac{s+1}{s-1} \right) \right] \\
&= 3 n_m^* \mathcal{I}(s)
\end{align*}$$

where we have defined the integral

$$\mathcal{I}(s) = \frac{1}{4} \int_0^{\pi} \frac{\sin \theta^2 \cos \theta}{s - \cos \theta} d\theta \\
= -\frac{1}{3} + \frac{1}{2}s^2 + \frac{2}{4} (1-s^2) \ln \left( \frac{s+1}{s-1} \right).$$

In conclusion we have

$$X_0^0(q, \omega) \equiv X_x^0(s) = 3 n_m^* \mathcal{I}(s)$$

$$= \frac{3 n_m^*}{m^2} \left[ -\frac{4}{3} + 2s^2 + \frac{2}{4} (1-s^2) \ln \left( \frac{s+1}{s-1} \right) \right]$$

where the effective mass of Landau quasiparticles is $m^* = m \left( 1 + \frac{E_F}{q} \right)$. Equations (C7) and (C8) coincide with equations (19) and (18) quoted in the main text.

**Appendix D: Derivation of the interacting transverse susceptibility with collisions**

In this section, we detail the derivation of the interacting transverse susceptibility of a Fermi liquid in the presence of collisions. Starting from the kinetic equation (22), we calculate the paramagnetic current density in a Fermi liquid using the definition

$$J_p(q, \omega) = \frac{2}{\nu} \sum_k \frac{1}{m} v_x^i \delta(\xi_k) \epsilon_{k,x}(q, \omega).$$

The Fermi surface displacement $\epsilon_k(q, \omega)$ can be deduced directly from equation (22) if we solve the latter equation for $\epsilon^s(\theta)$:

$$\epsilon^s(\theta) = \frac{\left[ 3e^{S}(\theta) \sin \theta \right]_{av} \sin \theta}{\xi - \cos \theta} \left( \frac{m}{m^*} k \cdot \mathbf{A}(q, \omega) \cos \theta \right)$$

$$\equiv \frac{2}{\nu} \sum_k \delta(\xi_k) k_x \frac{1}{m} e A_x(q, \omega)$$

Inserting equation (D2) in equation (D1), we have

$$J_p(q, \omega) = \frac{2}{\nu} \sum_k \delta(\xi_k) \frac{k_x}{m}$$

$$\left[ 3e^{S}(\theta) \sin \theta \right]_{av} \sin \theta \left( \frac{m}{m^*} k \cdot \mathbf{A}(q, \omega) \cos \theta \right)$$

$$\equiv \frac{2}{\nu} \sum_k \left[ 3e^{S}(\theta) \sin \theta \right]_{av} \sin \theta \left( \frac{m}{m^*} k \cdot \mathbf{A}(q, \omega) \cos \theta \right) + X_0^0(q, \omega)$$

where $X_0^0(q, \omega) = \left[ \frac{m}{m^*} k \cdot \mathbf{A}(q, \omega) \cos \theta \right] / (\xi - \cos \theta)$ is the noninteracting paramagnetic current density, in accordance with the results of section IIIA. In the last step of equation (D3), we have defined the paramagnetic interacting transverse susceptibility $X_{p,xx}^0(q, \omega)$ as the ratio between the total paramagnetic current density component $J_p^0(q, \omega)$ and the vector potential component $A_x(q, \omega)$. We take the transverse component $\mu = \nu = x$ of the transverse current density (D3), and we find

$$X_p^0(q, \omega) = X_{p,xx}^0(q, \omega) = \frac{2}{\nu} \sum_k \delta(\xi_k) k_x \frac{1}{m} e A_x(q, \omega)$$

$$\left[ 3e^{S}(\theta) \sin \theta \right]_{av} \sin \theta \left( \frac{m}{m^*} k \cdot \mathbf{A}(q, \omega) \cos \theta \right)$$

$$\equiv \frac{2}{\nu} \sum_k \delta(\xi_k) k_x \frac{1}{m} e A_x(q, \omega)$$

where $X_0^0(q, \omega) \equiv X_0^0(\xi)$ is the noninteracting paramagnetic transverse susceptibility, in accordance with the result (18) in section IIIA. Now, from the kinetic equation, not counting the linear term in the vector potential which is already included in $X_0^0(\xi)$, we expect a Fermi surface displacement $\epsilon^s(\theta) \propto \sin \theta \left( \frac{m}{m^*} k \cdot \mathbf{A}(q, \omega) \cos \theta \right)$ as found previously. Therefore, we can write the term multiplying $\left[ 3e^{S}(\theta) \sin \theta \right]_{av}$ in equation (D4) as

$$\frac{2}{\nu} \sum_k \delta(\xi_k) k_x \frac{1}{m} e A_x(q, \omega)$$

$$\left[ 3e^{S}(\theta) \sin \theta \right]_{av} \sin \theta \left( \frac{m}{m^*} k \cdot \mathbf{A}(q, \omega) \cos \theta \right)$$

$$\equiv \frac{2}{\nu} \sum_k \delta(\xi_k) k_x \frac{1}{m} e A_x(q, \omega)$$

Hence, equation (D4) becomes

$$X_p^0(q, \omega) = X_{p,xx}^0(q, \omega) \left[ 3e^{S}(\theta) \sin \theta \right]_{av} + X_0^0(q, \omega).$$
We are left with the calculation of \[ 3e^S(\theta) \sin \theta \] _av, which we perform as follows:

\[
\begin{align*}
3e^S(\theta) \sin \theta & = 3 \int_{0}^{\pi} \frac{\sin \theta}{3} \frac{d^2 \theta}{4} F_1^S \frac{\sin \theta \cos \theta}{\xi - \cos \theta} \\
& - \beta \int_{0}^{\pi} \frac{\sin \theta}{3} \frac{d^2 \theta}{4} \frac{\xi \sin \theta}{\xi - \cos \theta} \\
& = 3 \left( \frac{F_1^S}{3} \right) \left[ -\frac{1}{3} + \frac{\xi^2}{2} - \frac{\xi}{4} (\xi^2 - 1) \ln \left( \frac{\xi + 1}{\xi - 1} \right) \ight] \\
- 3\beta \left[ -\frac{\xi^2}{2} + \frac{\xi}{4} (\xi^2 - 1) \ln \left( \frac{\xi + 1}{\xi - 1} \right) \right] \\
& = 3 \left( \frac{F_1^S}{3} - \beta \right) \left[ -\frac{m^2}{3nm^r} X_0^q(\xi) \right] - \beta.
\end{align*}
\]

Inserting the last result into equation (D5), we finally obtain

\[
X^p_i(q, \omega) \left\{ 1 - 3 \left( \frac{F_1^S}{3} - \beta \right) \left[ -\frac{m^2}{3nm^r} X_0^q(\xi) + \beta \right] \right\} = X^q_i(\xi).
\] (D6)

When we solve equation (D6) for \( X^q_i(q, \omega) \), we obtain equation (23) quoted in the main text.

**Appendix E: Derivation of the transverse susceptibility with collisions and momentum relaxation**

The transverse susceptibility with momentum relaxation stemming from the kinetic equation (49) can be related to the momentum-conserving solution (23) as follows. If we parametrize the relaxation integral \( \mathcal{F}_i(q, \omega) \) in the relaxation-time approximation (50), it is convenient to define a ‘dynamic correction to the displacement function’ \( e^D(\theta) \):

\[
e^D(\theta) = e^S(\theta) - \frac{i/\tau_k}{\omega + i/\tau_k} e^e(\theta),
\] (E1)

where \( e^e(\theta) \) is the 'locally relaxed' equilibrium displacement function for the relaxing processes (50). We also define the associated modified vector potential \( A_D(q, \omega) \),

\[
A_D(q, \omega) = A(q, \omega) - \frac{i/\tau_k}{\omega + i/\tau_k} A_i(q, \omega) \\
\equiv \frac{\omega}{\omega + i/\tau_k} A(q, \omega),
\] (E2)

where \( A_i(q, \omega) \equiv A(q, \omega) \) for processes that do not conserve current. Inserting equations (E1) and (E2) into the kinetic equation (49), and using the equilibrium condition for \( e_e(\theta) \)

\[
e^e(\theta) = \frac{1}{\tau_q} \int_{0}^{\pi} \frac{\sin \theta}{3} \frac{d^2 \theta}{4} F_1^S \frac{\sin \theta \sin \theta' \pi e^e(\theta')}{4\pi} \\
- \frac{e}{mv_F} k \cdot A_D(q, \omega) = 0,
\] (E3)

we see that the momentum-relaxing kinetic equation (49) for \( e^S(\theta) \) is equivalent to the kinetic equation without relaxation for \( e^D(\theta) \), in the presence of the vector potential \( A_D(q, \omega) \) and with modified frequency \( \xi_k = s \left( 1 + \frac{1}{\omega \tau_q} \right) \):

\[
(\cos \theta - \xi_k e^D(\theta) + \cos \theta) \left[ 1 - 3 \left( \frac{F_1^S}{3} - \beta \right) \left[ -\frac{m^2}{3nm^r} X_0^q(\xi) + \beta \right] \right] = X^q_i(\xi).
\]

Therefore, from this point on we can follow appendices C and D, with a result formally equivalent to (23) upon substituting \( \xi \) with \( \xi_k = s \left( 1 + \frac{1}{\omega \tau_q} \right) + \frac{\omega}{\omega + i/\tau_k} \) and further multiplied by \( \frac{\omega}{\omega + i/\tau_k} \) due to the definition (E2) of \( A_D(q, \omega) \). The final result is equation (51) quoted in the main text.

**Appendix F: Derivation of Landau quasiparticle scattering time**

In a Fermi liquid, short-range quasiparticle residual interactions at the Fermi surface are usually expressed in terms of the symmetric and antisymmetric Landau parameters \( A(9) \). These quantities parametrize the bare interactions among quasiparticles. The many-body polarization of the medium renormalizes these interactions, producing renormalized Landau parameters, and with modified frequency \( \xi_k = s \left( 1 + \frac{1}{\omega \tau_q} \right) \):

\[
A_D = \frac{F_1^S}{1 + \frac{F_1^S}{2 \tau_D}},
\] (F1)

The quantities (F1) represent the scattering amplitudes between quasiparticles and can mediate Cooper pairing in superconducting materials. They also enter into the Fermi liquid collision time, due to phase-space limitation of quasiparticle scattering, in accordance with

\[
\frac{1}{\tau_{QP}(\omega, T)} = \frac{(m^*)^3 \alpha_F}{12 \pi^2 \hbar^2} \left[ (\hbar \omega)^2 + (\pi k_F T)^2 \right] \left( \frac{W(\theta, \phi)}{\cos \left( \frac{\theta}{2} \right)} \right),
\] (F2)

where \( W(\theta, \phi) \) is the transition probability governing inelastic scattering at the Fermi surface, and \( \{ \theta, \phi \} \) are the angles between the Fermi momentum \( k_F \) and the excitation momentum \( q \) in 3D space. The brackets \( \langle \rangle \) represent an average over the solid angle in momentum space. In s-p approximation, we consider short-range quasiparticle interactions only in the angular channels \( l = \{0, 1\} \),
comprising both symmetric and antisymmetric parts. Then, the angular average in equation (F2) is performed on the transition probability

$$W(\theta, \phi) = \frac{\pi}{2\hbar} \left[ A_s(\theta, \phi) + A_t(\theta, \phi) \right]^2 + \frac{\pi}{2\hbar} \left[ A_s(\theta, \phi) \right]^2,$$

where

$$A_s(\theta, \phi) = \frac{1}{N_0^2(0)} \left[ (A_0^0 - 3A_0^1) + (A_0^1 - 3A_1^0 \cos \theta) \right]$$

and

$$A_t(\theta, \phi) = \frac{1}{N_0^2(0)} \left[ (A_0^0 + A_0^1) + (A_0^1 + A_1^0 \cos \theta) \right] \cos \phi$$

are the scattering amplitudes for singlet and triplet state, respectively.\(^{63,65}\) Averaging over the angular coordinates \(\theta\) and \(\phi\) gives

$$\frac{W(\theta, \phi)}{\cos \frac{\theta}{2}} = \frac{\sin \theta d\theta d\phi}{4\pi} \cos \frac{\theta}{2} W(\theta, \phi) = 12(\lambda_s)^2 \left( \frac{\pi^5 \hbar^5}{(\hbar^2)^2} \right)$$

where \(12(\lambda_s)^2\) contains contributions from the Landau interaction parameters \(\{A_0^0, A_0^1, A_1^0, A_1^1\}\) with numerical coefficients, in accordance with equation (B.7) of reference.\(^{63}\) In evaluating the quasiparticle collision time (F2) we have to consider that Fermi-liquid quasiparticles cannot scatter on an energy scale larger than the Fermi energy \(E_F\). This sets a lower bound for the scattering time

$$\tau_{min} = \frac{\hbar}{E_F}.$$  \hspace{1cm} (F7)

Inserting equation (F6) into equation (F2) finally yields a quadratic energy-temperature dependence of the collision time,\(^{63,77}\)

$$\tau_{FL}(\omega, T) = \tau_{qF}(\omega, T) = \frac{\pi}{\hbar E_F} \left( \frac{\lambda_s^2}{(\hbar \omega)^2 + (\pi \kappa_0 \hbar T)^2} \right)^{-1},$$

where \(\hbar \omega\) is the excitation frequency, \(\kappa_0 T\) is the thermal energy at temperature \(T\), \(E_F=\left[ k^2(F_0)^2 \right] / (2m^*)\) is the renormalized Fermi level, and \((\lambda_s)^2\) stems from the angular integration of the scattering probability and depends on the Landau parameters \(\{F_0^S, F_1^S\}\). When we retain only \(F_0^S\) and \(F_1^S\) in the spherical harmonics expansion of the quasiparticle short-ranged interactions, \((\lambda_s)^2\) becomes

$$(\lambda_s)^2 = \frac{1}{12} \left( \frac{7}{24} (A_0^0)^2 + \frac{5}{8} (A_0^1)^2 - \frac{5}{12} A_0^1 A_1^0 \right).$$

(F9)

From equations (F8) and (F9), we see that the collisionless limit \(\tau_{FL} \to +\infty\) is reached only when both \(\omega = 0\) and \(T = 0\), or \((\lambda_s)^2 = 0\) which implies \(\{F_0^S, F_1^S\} = 0\) that is the noninteracting Fermi gas limit. Figure 16 shows the Fermi-liquid scattering time (F8) in s-p approximation as a function of frequency \(\omega\), at temperatures \(T = 300\) K and \(T = 1\) K and for the following material parameters: renormalized Fermi velocity \(\frac{\lambda_s}{c} = 10^{-3}\); first Landau parameters \(F_0^S = \{2, 20\}\); and \(F_1^S = 0\).

**Appendix G: Electron-phonon collision time for Debye phonon spectrum**

A relaxation time approximation for the Boltzmann equation is justified for acoustic phonon scattering.\(^{103}\) In the following, we derive simple expressions for the electronic relaxation rate due to acoustic phonons from many-body theory.

The self-energy of electrons due to one-phonon exchange processes satisfies

$$\Sigma(k, i\omega_n) = -\frac{k_0 T}{\hbar^2} \sum_{q, \Omega_n} V_{\rightarrow ph}(q, i\Omega_n) G(k-q, i\omega_n - i\Omega_n),$$

where \(G(k, \tau)\) is the Matsubara form of the electron propagator \(G(k, \tau) = \langle \tau \vert \left[ c_k(\tau) c_k^\dagger(0) \right] \rangle\) with \(c_k, c_k^\dagger\) electronic field operators.\(^{58,65,66}\) The electron-phonon interaction potential 

$$V_{\rightarrow ph}(q, i\Omega_n) = |g_3(q)|^2 \tilde{G}_3(q, i\Omega_n).$$

(G2)
In equation (G2), \( \mathcal{S}_\lambda(q, i\Omega_n) \) is the Matsubara form of the phonon propagator \( \mathcal{S}_\lambda(q, \tau) = \langle \mathcal{T} \left[ B_{q, \lambda}(\tau) B_{q, \lambda}^*(0) \right] \rangle \) for branch \( \lambda \), and \( g_{\lambda}(q) \) is the electron-phonon vertex. We also defined the linear combination \( B_{q, \lambda}^+ = b_{\lambda}^+(q) + b_{\lambda}^-(q) \) of the phonon field operators \( b_{\lambda}^+(q) \), \( b_{\lambda}^-(q) \).

We assume the free fermion propagator
\[
\mathcal{S}_\lambda^{0}(q, i\omega_n) = \frac{1}{i\omega_n - \xi_k} \tag{G3}
\]
and the free phonon propagator
\[
\mathcal{S}_\lambda^{0}(q, i\Omega_n) = \frac{1}{i\Omega_n - \omega_{\lambda}(q)} - \frac{1}{i\Omega_n + \omega_{\lambda}(q)}. \tag{G4}
\]
Inserting equations (G3) and (G4) into equation (G1), we obtain
\[
\Sigma(k, i\omega_n) = -\frac{k_BT}{\mathcal{V}} \sum_{q, \lambda} |g_{\lambda}(q)|^2 \left[ \frac{1}{i\Omega_n - \omega_{\lambda}(q)} - \frac{1}{i\Omega_n + \omega_{\lambda}(q)} \right] \omega_n - i\Omega_n - \xi_{k-q}.
\]
\[
\Sigma(k, i\omega_n) = -\frac{1}{\mathcal{V}} \sum_{q, \lambda} |g_{\lambda}(q)|^2 \left\{ \frac{-f_{BE}[\omega_{\lambda}(q)]}{i\omega_n - \omega_{\lambda}(q) - \xi_{k-q}} - \frac{-f_{BE}[-\omega_{\lambda}(q)]}{i\omega_n + \omega_{\lambda}(q) - \xi_{k-q}} \right. \\
\left. + f_{BE}(i\omega_n - \xi_{k-q}) \right\} \left[ \frac{1}{i\omega_n - \omega_{\lambda}(q) - \xi_{k-q}} - \frac{1}{i\omega_n - \omega_{\lambda}(q) + \omega_{\lambda}(q)} \right]. \tag{G5}
\]
Using \( f_{BE}(-\omega) = 1 - f_{BE}(\omega) = f_{BE}(\omega_n - \xi) = -f_{FD}(-\xi) \), and \( f_{FD}(-\xi) = 1 - f_{FD}(\xi) \), we have
\[
\Sigma(k, i\omega_n) = -\frac{1}{\mathcal{V}} \sum_{q, \lambda} |g_{\lambda}(q)|^2 \left\{ \frac{f_{FD}(\xi_{k-q}) + f_{BE}[\omega_{\lambda}(q)]}{i\omega_n - \omega_{\lambda}(q) - \xi_{k-q}} + \frac{f_{FD}(-\xi_{k-q}) + f_{BE}[\omega_{\lambda}(q)]}{i\omega_n + \omega_{\lambda}(q) - \xi_{k-q}} \right\}. \tag{G6}
\]

We now overlook any anisotropy in the electron-phonon matrix element \( g_{\lambda}(q) \), which allows us to pass from the sum over momenta \( q \) in equation (G7) to an integral over energy \( \Omega \), making use of the phonon density of states (PDOS) per unit volume \( N_{ph}(\Omega) = \frac{1}{\mathcal{V}} \sum q, \lambda \delta [\Omega - \omega_{\lambda}(q)] \). We identify the electron-phonon spectral function with the product of the electron-phonon matrix element times the PDOS,
\[
\alpha^2 F(\Omega) = |g(\Omega)|^2 N_{ph}(\Omega). \tag{G8}
\]

With the definition (G8), equation (G7) becomes
\[
\Sigma(k, i\omega_n) = -\int_{-\infty}^{+\infty} d\Omega \alpha^2 F(\Omega) \left\{ \frac{f_{FD}(-\xi_{k-q}) + f_{BE}[\omega_{\lambda}(q)]}{i\omega_n - \omega_{\lambda}(q) - \xi_{k-q}} + \frac{f_{FD}(\xi_{k-q}) + f_{BE}[\omega_{\lambda}(q)]}{i\omega_n + \omega_{\lambda}(q) - \xi_{k-q}} \right\}. \tag{G9}
\]
We can achieve a momentum-independent self-energy with an appropriate average of equation (G7) over momenta \( k \):
\[
\langle \Sigma(\omega_n) \rangle_k = \int_{-\infty}^{+\infty} d\xi N_0(\xi) \Sigma(k, i\omega_n) = -\int_{-\infty}^{+\infty} d\Omega \alpha^2 F(\Omega) \left\{ \frac{f_{FD}(\xi) + f_{BE}[\omega_{\lambda}(q)]}{i\omega_n - \omega_{\lambda}(q) - \xi} + \frac{f_{FD}(-\xi) + f_{BE}[\omega_{\lambda}(q)]}{i\omega_n + \omega_{\lambda}(q) - \xi} \right\}, \tag{G10}
\]
where we have used the electron density of states \( N_0(\xi) = \frac{1}{\mathcal{V}} \sum q, \lambda \delta (E - \xi_{q}) \). In the following, we assume a constant density of states within a bandwidth \( \mathcal{W} \) much larger than any other energy scale of the problem. Formally,
\[
N_0(\xi) = \begin{cases} N_0(0), & |\xi| < \mathcal{W}, \\ 0, & |\xi| > \mathcal{W} \end{cases} \tag{G11}
\]
If we are interested in the energy and temperature dependence of the scattering rate due to electron-boson interactions, we can use the identity \( \text{Im}\{\frac{1}{\mathcal{V} \omega^{\alpha} \tau_{\alpha}}\} = -\pi\delta(x) \) to find the imaginary part of the momentum-averaged self-energy (G10). The integral over the quasiparticle energies \( \xi \) is convergent in the limit \( \mathcal{W} \to +\infty \). Employing the symmetries \( \alpha^2 F(-\Omega) = -\alpha^2 F(\Omega) \), \( f_{FD}(\xi) = 1 - f_{FD}(\xi) \), and \( f_{BE}(-\Omega) = 1 - f_{BE}(\Omega) \), we obtain
\[
-\text{Im}\{\langle \Sigma(\omega) \rangle_k\} = -\pi N_0(0) \int_{0}^{+\infty} d\Omega \alpha^2 F(\Omega) \left[ f_{FD}(\Omega - \omega) + f_{FD}(\Omega + \omega) + 2f_{BE}(\Omega) \right]. \tag{G12}
\]
It is customary to define the strength of the electron-phonon coupling by a coupling constant, which descends from equation (G10):
\[
\lambda = \left. \frac{d\text{Re}\{\langle \Sigma(\omega) \rangle_k\}}{d\omega} \right|_{\omega=0, T=0} = 2 \int_{0}^{+\infty} \frac{d\Omega}{\Omega} \alpha^2 F(\Omega). \tag{G13}
\]
where in the last step we employed the property $\alpha^2 F(-\Omega) = -\alpha^2 F(\Omega)$.

In the following, we consider the simple case of an acoustic phonon spectrum, characterized by the dispersion relation $\omega_\lambda(q) = v_\lambda q$ where $v_\lambda$ is the acoustic phonon sound velocity. Given a density of lattice ions $n_{ions}$, the Debye energy is then $\hbar \Omega_D = \hbar (6\pi^2 n_{ions})^{1/3} v_{\text{ac}}$. The density of states for such acoustic phonon spectrum is then

$$N_D(E) = \frac{3E^2}{2(v_\lambda)^3 \pi^3} \Theta(\hbar \Omega_D - E) \quad (G14)$$

with $\Theta(x)$ Heavyside step function. Assuming a constant electron-phonon matrix element $g(\Omega) \equiv g_\alpha \forall \Omega$, we can write the electron-phonon spectral function $(G8)$ of the acoustic spectrum as

$$\alpha^2 F_\alpha(\Omega) = \frac{\lambda}{\hbar \Omega_D} \Theta(\hbar \Omega_D - \Omega), \quad (G15)$$

where $\lambda = \frac{3}{\pi} |g_\alpha|^2 \frac{6\pi^2 n_{ions}}{\hbar \Omega_D}$. Inserting the Debye-spectrum electron-phonon spectral function (G15) into equation (G12), we obtain the imaginary part of the self-energy due to exchange of acoustic phonons

$$-\text{Im} \langle \Sigma(i\omega_n) \rangle_k = 2\pi x \hbar \xi \Omega_D \mathcal{S}(\omega, k_B T \frac{\hbar q}{\hbar \Omega_D}), \quad (G16)$$

where we have defined the function

$$\mathcal{S}(x, t) = t \int_0^t du \left( \frac{1}{e^{u^2/2} + 1} + \frac{1}{e^{u^2/2} + 1} + \frac{2}{e^{u^2/2} - 1} \right), \quad (G17)$$

which possesses an analytical solution in terms of polylogarithms. For $t \gg 1$, we have $t^{-1} \to 0$ and we can expand the integrand in equation (G17) in powers of $u$: this gives $2u + o(u^2)$, from which $\lim_{t \to \infty} \mathcal{S}(x, t) = t$. Hence we can write

$$-\text{Im} \langle \Sigma(\omega) \rangle_k = 2\pi x \hbar \xi \Omega_D \mathcal{S}(\omega, k_B T \frac{\hbar q}{\hbar \Omega_D} \gg 1). \quad (G18)$$

Consider now the opposite limit $t \to 0$. We can focus on $x > 0$ since the function $\mathcal{S}(-x, t) = \mathcal{S}(x, t) \forall x$. The first term in brackets in equation (G17) becomes the step function

$$\lim_{t \to 0} e^{-u^2/2} + 1 = \begin{cases} 1, & u < \frac{x}{e} \\ 0, & u > \frac{x}{e} \end{cases},$$

which cuts the superior limit of the integral in equation (G17) to $\min(\frac{x}{e}, \frac{1}{e})$. The second term in the integrand vanishes and the third one is negligible in the limit $t \to 0$ due to the $t^3$ prefactor in equation (G17). All in all, we obtain $\lim_{t \to 0} \mathcal{S}(x, t) = \min(\frac{x^3}{e}, 1)$, so that

$$-\text{Im} \langle \Sigma(\omega) \rangle_k = 2\pi x \hbar \xi \Omega_D \min\left( \frac{\omega}{\hbar \Omega_D}, 1 \right) \mathcal{S}(\omega, k_B T \frac{\hbar q}{\hbar \Omega_D} \to 0. \quad (G19)$$

To calculate the real part of the momentum-averaged self-energy $(\Sigma(\omega))_k$, we can employ the Kramers-Kronig transformation on equation (G10) with the electronic DOS (G11). We must keep $\mathcal{W} < +\infty$ until the very end of the calculation to ensure the convergence of the $\xi$ integral. We have

$$\text{Re} \langle \Sigma(\omega) \rangle_k = -\int_0^{+\infty} d\omega \alpha^2 F(\Omega) \mathcal{P} \int_{-\mathcal{W}}^{\mathcal{W}} d\xi \left[ \frac{f_{FD}(-\xi) + f_{BE}(\Omega)}{\omega - \Omega - \xi} + \frac{f_{FD}(\xi) + f_{BE}(\Omega)}{\omega + \Omega - \xi} \right] \quad (G20)$$

where the symbol $\mathcal{P}$ denotes the Cauchy principal value of the integral. Equation (G20) splits into $\text{Re} \langle \Sigma(\omega) \rangle_k = A + B$, depending on $f_{BE}(\Omega)$ and $f_{FD}(\xi)$ respectively. Integrating the first term gives

$$A = -\int_0^{+\infty} d\omega \alpha^2 F(\Omega) \mathcal{P} \int_{-\mathcal{W}}^{\mathcal{W}} d\xi \left[ \frac{f_{BE}(\Omega)}{\omega - \Omega - \xi} + \frac{f_{BE}(\Omega)}{\omega + \Omega - \xi} \right] = \int_0^{+\infty} d\omega \alpha^2 F(\Omega)f_{BE}(\Omega) \ln \left( \frac{(\mathcal{W} + \omega)^2 - \Omega^2}{(\mathcal{W} - \omega)^2 - \Omega^2} \right) \quad (G21)$$

so that $\lim_{\mathcal{W} \to +\infty} A = 0$. In that limit, we are left with term $B$, which is

$$\text{Re} \langle \Sigma(\omega) \rangle_k = -\int_0^{+\infty} d\omega \alpha^2 F(\Omega) \mathcal{P} \int_{-\mathcal{W}}^{\mathcal{W}} d\xi \left[ \frac{f_{FD}(-\xi)}{\omega - \Omega - \xi} + \frac{f_{FD}(\xi)}{\omega + \Omega - \xi} \right]$$

$$= \int_0^{+\infty} d\omega \alpha^2 F(\Omega) \left[ \mathcal{P} \int_{-\mathcal{W}}^{\mathcal{W} + \omega} d\epsilon \frac{f_{FD}(\epsilon - \omega)}{\epsilon - \Omega} + \mathcal{P} \int_{-\mathcal{W}}^{-\mathcal{W}} d\epsilon \frac{f_{FD}(\epsilon + \omega)}{-\epsilon + \Omega} \right] \quad (G22)$$

In the limit $\mathcal{W} \gg |\omega| \forall \omega$, up to errors of order $|\omega|/\mathcal{W}$ we have

$$\text{Re} \langle \Sigma(\omega) \rangle_k = -\int_0^{+\infty} d\omega \alpha^2 F(\Omega) \mathcal{P} \int_{-\mathcal{W}}^{\mathcal{W}} d\epsilon \frac{f_{FD}(\epsilon - \omega) - f_{FD}(\epsilon + \omega)}{\epsilon - \Omega} + \mathcal{O}\left(\frac{|\omega|}{\mathcal{W}}\right). \quad (G23)$$

Now the integrand over $\epsilon$ in equation (G23) is explicitly cut by the Fermi-Dirac distributions to the range of $\pm(\omega + \alpha)$, where $\alpha$ is a few times the thermal energy $k_B T$. This allows us to safely take the limit $\mathcal{W} \to +\infty$. Further using the
Re \(\Sigma(\omega)\) k as a function of normalized temperature \(k_B T/(\hbar \omega_D)\) for scattering off acoustic phonons. The blue, red, gold and green curves are calculated at frequency \(\omega/\omega_D = 0, 0.5, 1, 10\) respectively. Solid lines in panel (a) show the real part of the self-energy according to equation (G27), with dashed lines indicating the corresponding high-temperature expansion (G29). Solid lines in panel (b) show the imaginary part of the self-energy for equation (G16). The gray dashed line is \(k_B T/(\hbar \omega_D)\).

Matsubara representation of the Fermi-Dirac distribution, we have

\[
\text{Re} \left\langle \Sigma(\omega) \right\rangle_k = - \int_0^{+\infty} d\Omega \alpha^2 F(\Omega) \cdot \int_{-\infty}^{+\infty} d\epsilon \frac{f_{\text{FD}}(\epsilon - \Omega) - f_{\text{FD}}(\epsilon + \Omega)}{\epsilon - \Omega + i0^+} \]

\[
\cdot \text{Re} \left( k_B T \sum_{\omega_n} \int_{-\infty}^{+\infty} d\epsilon \left( \frac{1}{i\omega - \epsilon - \Omega} \right) \left( \frac{1}{\epsilon - \Omega + i0^+} \right) \right) 
\]

(G24)

The integral over \(\epsilon\) in equation (G24) can be solved by the residues theorem, by closing the contour on the lower half of the complex plane, thus enclosing the pole at \(\epsilon = \Omega - i0^+\). This yields

\[
\int_{-\infty}^{+\infty} d\epsilon \left( \frac{1}{i\omega - \epsilon - \Omega} \right) \left( \frac{1}{\epsilon - \Omega + i0^+} \right) = 2\pi i \Theta(\omega_n) \frac{2\omega}{(i\omega - \epsilon)^2 - \omega^2} 
\]

(G25)

The remaining Matsubara sum in equation (G24) is convergent and can be written in terms of the Digamma function, \(\psi(z) = \lim_{M \to +\infty} \left( \ln M - \frac{1}{2\pi} \right)\). The final result is

\[
\text{Re} \left\langle \Sigma(\omega) \right\rangle_k = - \int_0^{+\infty} d\Omega \alpha^2 F(\Omega) \cdot \left[ \psi \left( \frac{1}{2} - i \frac{\omega - \Omega}{2\pi k_B T} \right) - \psi \left( \frac{1}{2} + i \frac{\omega + \Omega}{2\pi k_B T} \right) \right] 
\]

(G26)

The result (G26) applies to any isotropic phonon spectrum \(\alpha^2 F(\Omega)\). Specializing the calculation to the acoustic phonon spectrum (G15) yields

\[
\text{Re} \left\langle \Sigma(\omega) \right\rangle_k = - \lambda \hbar \omega_D \alpha^2 \left( \frac{\omega}{\hbar \omega_D} \right)^2 \left( \frac{1}{2} \right) 
\]

\[
\approx 16.83 \lambda \hbar \omega_D \left( \frac{\hbar \omega_D}{k_B T} \right) 
\]

(G27)

Figure 17 shows the real and imaginary parts of the acoustic-phonons self-energy as a function of normalized temperature \(k_B T/(\hbar \omega_D)\), calculated at different frequencies according to equations (G18) and (G27), with dashed lines indicating the high-temperature expansion (G29) for \(\text{Re} \left\langle \Sigma(\omega) \right\rangle_k\).

The scattering rate of electrons due to the exchange of acoustic phonons satisfies

\[
\left[ \tau_{\text{e-ph}}(\omega, T) \right]^{-1} = -2Z(T) \text{Im} \left\langle \Sigma(\omega) \right\rangle_k 
\]

(G30)

where the quasiparticle spectral weight is defined as

\[
[Z(T)]^{-1} = 1 - \frac{\partial \text{Re} \left\langle \Sigma(\omega) \right\rangle_k}{\partial \omega} \bigg|_{\omega = 0}. 
\]

(G31)

For the acoustic phonon spectrum, utilizing equation (G27) and calculating the limit

\[
\lim_{x \to -\infty} \frac{1}{\hbar \omega_D} \text{Re} \left\{ \psi \left( \frac{1}{2} - i \frac{x - u}{k_B T} \right) - \psi \left( \frac{1}{2} + i \frac{x + u}{k_B T} \right) \right\}
\]

\[
= \frac{1}{\hbar \omega_D} \left\{ \frac{\pi t}{2\pi i} \right\}
\]

(G32)
where \( \psi'(z) \) is the first derivative of the Digamma function, we have

\[
[Z(T)]^{-1} = 1 - \frac{\lambda \hbar \omega_D}{\pi k_B T} \varphi \left( \frac{k_B T}{\hbar \omega_D} \right), \tag{G33}
\]

where we have defined the function

\[
\varphi(t) = \int_0^t du u^2 \text{Im} \left\{ \psi' \left( \frac{1}{2} + i \frac{u}{2 \pi t} \right) \right\}. \tag{G34}
\]

At zero temperature, we have \( \lim_{T \to 0} \varphi(t) = -1 \), so that we retrieve

\[
[Z(0)]^{-1} = 1 + \lambda, \tag{G35}
\]

which holds at \( T = 0 \) for any spectrum of bosons exchanged by electrons. Inserting equations (G31), (G32) and (G16) into equation (G30), we obtain an explicit expression for the scattering rate due to acoustic phonon exchange:

\[
\frac{1}{\tau_{e-ph}(\omega, T)} = \frac{2}{1 - \frac{\lambda \hbar \omega_D}{\pi k_B T} \varphi \left( \frac{k_B T}{\hbar \omega_D} \right)} \frac{\pi \lambda \hbar \omega_D}{\hbar \omega_D} \varphi \left( \frac{\omega}{\hbar \omega_D}, \frac{k_B T}{\hbar \omega_D} \right). \tag{G36}
\]

Figure 18(a) illustrates the temperature evolution of the quasiparticle weight (G33): blue, red and gold curves show the results for electron-phonon coupling constants \( \lambda = \{0.1, 0.5, 1\} \), respectively. Figure 18(b) shows the electron-phonon collision time \( \tau_{e-ph}(\omega, T) \) as a function of frequency \( \omega \) normalized to the plasma frequency \( \omega_p \), resulting from equation (G36), for an electron-phonon coupling constant \( \lambda = 0.1 \) and at temperatures \( T = \{1, 30, 300\} \) K.

Appendix H: Momentum-relaxation dependence of the skin depth for the Fermi liquid

The ratio between the Drude skin depth \( \delta_s \) and the mean free path \( l_{\text{MFP}} \) is

\[
\frac{\delta_s}{l_{\text{MFP}}} = \frac{c}{v_F} \sqrt{\frac{2}{\omega(\omega_p)^2(\tau_{\text{MFP}})^3}}, \tag{H1}
\]

from equations (80) and (48). Therefore, such ratio depends on the momentum-relaxation time \( \tau_{\text{MFP}} \) at a given frequency \( \omega \). According to Reuter-Sondheimer theory, such ratio determines the crossover from local electrodynamics to anomalous skin effect – see equation (97).

Figure 19 shows the ratio \( \delta_s/l_{\text{MFP}} \) as a function of frequency \( \omega \) normalized to the plasma frequency \( \omega_p \), for a momentum-relaxation time \( \tau_{\text{MFP}} = \tau_e/\alpha_U \) resulting from Umklapp processes, according to section F, and acoustic phonons, in accordance with section G. The parameters used are: electron density \( n = 10^{29} \) cm\(^{-3}\), Umklapp efficiency \( \alpha_U = 0.5 \), Debye temperature \( T_D = 500 \) K, electron-phonon coupling constant \( \lambda = 0.1 \), zero-order Landau parameter \( F_0 = 1 \) (its inclusion does not qualitatively modify the results). The different curves in all panels refer to first symmetric Landau parameter \( F_1 = \{1, 5, 10, 20\} \). In the green-coloured region \( \delta_s < l_{\text{MFP}} \), so the electrodynamic response is nonlocal giving anomalous skin effect, while in the blue-coloured region \( \delta_s > l_{\text{MFP}} \) and local electrodynamics is at play, yielding normal skin effect. The gray dashed line marks the condition \( \delta_s = l_{\text{MFP}} \). Panels (a) and (b) neglect the phonon contribution, and only consider Umklapp processes at \( T = 300 \) K and \( T = 1 \) K respectively: in both cases, there is an extended frequency window in which \( \delta_s < l_{\text{MFP}} \). Upon adding the contribution from acoustic phonons, we obtain panel (c) and (d) at \( T = 300 \) K and \( T = 1 \) K respectively: we see that typically \( \delta_s/l_{\text{MFP}} \gg 1 \) at all frequencies for the room-temperature case, so that electrodynamics is local and normal skin effect.
The analysis here outlined is further modified by the inclusion of the collision time \( \tau_s \), in the low-temperature calculation, there is a low-frequency range in which \( \delta_{lf}/l_{MFP} \ll 1 \), where anomalous skin effect develops.

The analysis here outlined is further modified by the inclusion of the collision time \( \tau_s \): new regimes arise for the skin depth, as discussed in section IX B 3.

**Appendix I: Surface impedance in the high-frequency propagating shear regime**

The high-frequency surface impedance for the propagating shear regime stems from the real-space electric field profile \( E(z, \omega) \) that results from inserting equation (57a) into the expression (115), and Fourier-transforming back to real space according to

\[
E(z, \omega) = \frac{1}{\pi} \left. \frac{\partial E(z, \omega)}{\partial z} \right|_{z=0^+} \cdot \int_{-\infty}^{+\infty} dq \left\{ \frac{\omega^2}{c^2} \left[ 1 - \frac{(\omega_p)^2}{\omega^2 + i \frac{\omega}{\tau_k} + i \left( \omega + \frac{1}{\omega_p} \right) \gamma(\omega) q^2} \right] \right\}^{-1}
\]

where the variables \( A \) and \( B \) are given by equation (100) and the integration variable is (99). Performing the momentum integration in equation (11), one obtains a lengthy analytic expression for \( E(z, \omega) \) in terms of \( A \) and \( B \), which translates into the surface impedance by the means of equation (84).

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In the hydrodynamic/collisional regime, we find the transverse collective mode satisfying equation (8) in the regime $\omega \tau \ll 1$, the transverse collective mode is damped, i.e. it has a complex sound velocity $s = \frac{\omega}{\tau} \in \mathbb{C}$ with equal real and imaginary parts. In such conditions, the transverse mode is called relaxed mode in part of the literature, distinguishing this regime from the one of propagating transverse sound. Other references employ the label damped transverse sound in referring to damped transverse waves in the collisional regime. In this paper, we will employ the nomenclature relaxational mode and damped transverse sound as synonyms, meaning a transverse collective mode satisfying equation (8) in the regime $\omega \tau \ll 1$. Generic solutions of equation (8), without specifying whether we are

...
in collisional/collisionless regime, will be named shear mode or transverse sound in this paper.

In this document we will refer to real solutions of equation (8) in the collisionless limit $\omega \tau_c \to +\infty$ as transverse zero sound or propagating shear.

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