SUPER W-SYMMETRIES, COVARIANTLY CONSTANT FORMS AND DUALITY TRANSFORMATIONS

Byungbae Kim
Institute for Theoretical Physics,
State University of New York at Stony Brook,
Stony Brook, NY 11794 USA

Abstract

On a supersymmetric sigma model the covariantly constant forms are related to the conserved currents that are generators of a super W-algebra extending the superconformal algebra. The existence of covariantly constant forms restricts the holonomy group of the manifold. Via duality transformation we get new covariantly constant forms, thus restricting the holonomy group of the new manifold.
1 Introduction

One of the most intriguing properties of string theory is the existence of a duality symmetry that relates strings propagating on different backgrounds to each other (see, e.g., [1] and references therein). Duality transformations play an important role in the study of moduli spaces of string vacua, and are presumably part of a large and fundamental symmetry of string theory. The purpose of this note is to investigate how duality acts on $N=1$ sigma models with torsion possessing an extended algebra of conserved currents, i.e., a super $W$-algebra. As was shown in [2], conserved chiral currents can in a natural way be associated to covariantly closed forms on the target manifold. Since covariantly closed forms imply certain constraints on the holonomy, the manifolds will in general have reduced holonomy. A simple example of such a manifold is a Calabi Yau manifold, which has reduced holonomy SU(3) and whose covariantly constant complex structure enhances the superconformal algebra to an $N=2$ algebra. In this letter we show explicitly how conserved currents associated to covariantly constant differential forms are under duality mapped to another set of conserved currents on the dual manifold. It will turn out that these latter currents are also associated to covariantly constant forms on the dual manifold, for which we present explicit expressions. From this we conclude that duality does, in general, leave the holonomy of the manifold invariant.

2 Super $W$-Algebra and Covariantly Constant Forms

Given a supersymmetric nonlinear $\sigma$-model lagrangian

$$S = -\frac{1}{2} \int D^2(g_{\mu\nu} D_+ X^\mu D_- X^\nu),$$

for $J_\omega = \omega_{\sigma_1 \cdots \sigma_p} D_+ X^{\sigma_1} \cdots D_+ X^{\sigma_p}$

\footnote{Actually, as we shall see below, duality transformations make use of an isometry of the manifold, and the holonomy is preserved only if the isometry is compatible with the (covariantly constant) forms.}
\[ D_- J_\omega = \partial_\mu \omega_1 \cdots \sigma_p D_- X^\mu D_+ X^{\sigma_1} \cdots D_+ X^{\sigma_p} \]
\[ + p \omega_{\sigma_1 \sigma_2 \cdots \sigma_p} D_- D_+ X^{\sigma_1} D_+ X^{\sigma_2} \cdots D_+ X^{\sigma_p} \]
\[ = \partial_\mu \omega_{\sigma_1 \cdots \sigma_p} D_- X^\mu D_+ X^{\sigma_1} \cdots D_+ X^{\sigma_p} \]
\[ - p \omega_{\sigma_1 \sigma_2 \cdots \sigma_p} \Gamma_{\rho \sigma}^{\sigma_i} D_- X^\rho D_+ X^{\sigma_1} D_+ X^{\sigma_2} \cdots D_+ X^{\sigma_p} \]
\[ = (\nabla_\mu \omega_{\sigma_1 \cdots \sigma_p}) D_- X^\mu D_+ X^{\sigma_1} \cdots D_+ X^{\sigma_p} \]

(2)

(where we used the equation of motion \( D_- D_+ X^\nu + \Gamma_\rho^\nu D_- X^\rho D_+ X^\sigma = 0 \)).

So \( D_- J_\omega = 0 \) if the form \( \omega \) is covariantly constant, i.e., \( \nabla_\mu \omega = 0 \). In this case \( J_\omega \) is a conserved current that is a generator of a super \( W \)-algebra via Poisson Brackets, extending the superconformal algebra [2].

The existence of such covariantly constant forms is related to the holonomy group of the target manifold. The following table shows the existence of covariantly constant forms and the holonomy group of the manifold (for the nonsymmetric spaces) [3].

**Holonomy Groups and Covariantly Constant Forms on Nonsymmetric spaces [4]**

| dim | holonomy group | covariantly constant forms (generators of the algebra) |
|-----|----------------|-------------------------------------------------------|
| \( n \) | \( SO(n) \) | \( \omega_n \) = volume form |
| \( 2n \) | \( U(n) \) | \( (\omega_2)^k \) \( k = 1, \ldots n; \omega_2 \) Kähler form |
| \( 2n \) | \( SU(n) \) | \( (\omega_2)^k \) \( k = 1, \ldots n; \omega_n = (vol)^\frac{2}{n} \) complex volume form, \( \omega_n \) Kähler manifold |
| \( 4n \) | \( Sp(n) \) | \( \omega = a \omega_1 + b \omega_2 + c \omega_3 \), \( a^2 + b^2 + c^2 = 1 \) Hyperkähler |
| \( 4n \) | \( Sp(n)Sp(1) \) | \( \Omega_4 = \omega_1^2 + \omega_2^2 + \omega_3^2 \) Quaternionic Kähler |
| \( 7 \) | \( G_2 \) | \( \omega_3, \omega_4 \) |
| \( 8 \) | \( Spin(7) \) | \( \omega_4 \) |

\( ^3 \)The product of the algebra involves not only wedge products of forms but also contractions of two or more forms using the metric.
For each holonomy group, there is an associated covariantly constant totally antisymmetric tensor, and this implies the existence of an associated symmetry of the corresponding supersymmetric sigma model \([2]\). These currents generate a super W-algebra via Poisson Brackets and these algebras are extensions of the superconformal algebra.

### 3 Covariantly Constant Forms and Duality Transformation

Now given a supersymmetric nonlinear \(\sigma\)-model lagrangian (we allow torsion for more generality) with covariantly constant \(p\)-forms \(\omega^\pm\) (see below), we have the action

\[
S = -\frac{1}{2} \int D^2((g_{\mu\nu} + b_{\mu\nu})D_+ X^\mu D_- X^\nu) .
\]

Suppose further there is an isometry generated by a vector field \(Y\) with \(\mathcal{L}_Y(db) = 0\) and \(\mathcal{L}_Y \omega^\pm = 0\); then one can choose (local) coordinates such that \(Y = \partial/\partial X^0\) and the metric \(g\), the torsion potential \(b\), and the \(p\)-forms \(\omega^\pm\) are independent of \(X^0\).

Under the transformations

\[
(\delta X^\mu) g_{\mu\nu} = \epsilon \omega^+_{\nu\sigma_1 \ldots \sigma_{p-1}} D_+ X^{\sigma_1} \ldots D_+ X^{\sigma_{p-1}} ,
\]

\(i.e.,\)

\[
\delta X^\mu = \epsilon g^\mu\nu \omega^+_{\nu\sigma_1 \ldots \sigma_{p-1}} D_+ X^{\sigma_1} \ldots D_+ X^{\sigma_{p-1}}
\]

(when \(p = 2,\)

\[
\delta X^\mu = \epsilon g^\mu\nu \omega^+_{\nu\sigma} D_+ X^\sigma \\
= \epsilon J^+_\sigma D_+ X^\sigma
\]
where $\omega^+ = g_{\nu\rho} J^+_{\sigma}$, the 2 form associated with the complex structure $J^+$, the variation of the Lagrangian $L$ is

$$
\delta L = 2 g_{\mu\nu} \delta X^\mu [D_- D_+ X^\nu + \Gamma_{\rho\sigma}^+ D_- X^\rho D_+ X^\sigma]
$$

$$
= 2 \epsilon \omega^+_\rho_\sigma_1 ... \sigma_{p-1} D_+ X^{\sigma_1} ... D_+ X^{\sigma_{p-1}} [D_- D_+ X^\rho + \Gamma_{\mu\nu}^+ D_- X^\mu D_+ X^\nu]
$$

$$
= 2 \epsilon [D_-(\omega^+_\rho_\sigma_1 ... \sigma_{p-1}) D_+ X^\rho D_+ X^{\sigma_1} ... D_+ X^{\sigma_{p-1}}]
$$

$$
- (\nabla^+_\mu \omega^+_\rho_\sigma_1 ... \sigma_{p-1}) D_- X^\mu D_+ X^\rho D_+ X^{\sigma_1} ... D_+ X^{\sigma_{p-1}}],
$$

where

$$
\Gamma_{\rho\sigma}^+ = \Gamma_{\rho\sigma}^\nu + T_{\rho\sigma}^\nu
$$

and $T_{\mu\nu\rho} = \frac{1}{2} (b_{\mu\nu,\rho} + b_{\nu\rho,\mu} + b_{\rho\mu,\nu})$. Similarly,

$$
\delta L = 2 \eta [D_-(\omega^-_{\rho_\sigma_1 ... \sigma_{p-1}} D_- X^\rho D_+ X^{\sigma_1} ... D_+ X^{\sigma_{p-1}})
$$

$$
- (\nabla^-_\mu \omega^-_{\rho_\sigma_1 ... \sigma_{p-1}}) D_+ X^\mu D_- X^\rho D_- X^{\sigma_1} ... D_- X^{\sigma_{p-1}}]
$$

(\text{where } \nabla^- = \partial + \Gamma^- \text{ and } \Gamma^-_{\rho\sigma} = \Gamma_{\rho\sigma}^\nu - T_{\rho\sigma}^\nu) \text{ under the transformation}

$$
(\delta X^\mu) g_{\mu\nu} = \eta \omega^-_{\nu\sigma_1 ... \sigma_{p-1}} D_- X^{\sigma_1} ... D_- X^{\sigma_{p-1}},
$$

\text{i.e.,}

$$
\delta X^\mu = \eta g^{\mu\nu} \omega^-_{\nu\sigma_1 ... \sigma_{p-1}} D_- X^{\sigma_1} ... D_- X^{\sigma_{p-1}}.
$$

So the action is invariant if we choose the form $\omega$ covariantly constant, \text{i.e., } \nabla^\pm_\mu \omega^\pm_{\rho_\sigma_1 ... \sigma_{p-1}} = 0.

In this case

$$
J^+_{\omega^+} = \omega^+_\rho_\sigma_1 ... \sigma_{p-1} D_+ X^\rho D_+ X^{\sigma_1} ... D_+ X^{\sigma_{p-1}}
$$

and

$$
J^-_{\omega^-} = \omega^-_{\rho_\sigma_1 ... \sigma_{p-1}} D_- X^\rho D_- X^{\sigma_1} ... D_- X^{\sigma_{p-1}}
$$

are conserved (Noether) currents that are generators of a super $W$-algebra extending the superconformal algebra.\[4, 5\]
Now one can get a new target manifold with new metric \( \tilde{g} \) and torsion potential \( \tilde{b} \) by dualizing as in [6, 7, 8]. Gauging the symmetry with gauge fields \( V_\pm \) (replace \( D_\pm X^0 \) with \( D_\pm X^0 + V_\pm \) and choosing \( X^0 = 0 \) gauge), we get the first order lagrangian

\[
S = -\frac{1}{2} \int D^2(e_{00}V_+V_+ + e_{i0}D_+X^iV_+ + e_{0i}V_+D_-X^i + e_{ij}D_+X^iD_-X^j \\
+ \phi(D_+V_+ + D_-V_+))
\]  

where \( \{\mu, \nu\} = 0, 1, \ldots, 2n - 1 \), \( \{i, j\} = 1, \ldots, 2n - 1 \), \( e_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu} \), and \( \phi \) is the lagrange multiplier whose variation gives \( V_\pm = D_\pm X^0 \) and gives back the original action (3).

The first order action (14) is invariant under

\[
\delta \epsilon \phi = \epsilon e_{\mu0}g^{\mu\nu}[ \omega^+_{\nu0\sigma_2\ldots\sigma_{p-1}} V_+D_+X^\sigma_2 \ldots D_+X^\sigma_{p-1} \\
+ \ldots \\
+ \omega^+_{\nu\sigma_1\ldots\sigma_{p-2}0} D_+X^\sigma_1 \ldots D_+X^\sigma_{p-2}V_+ \\
+ \omega^+_{\nu\sigma_1\ldots\sigma_{p-1}k_1} D_+X^{k_1} \ldots D_+X^{k_{p-1}}],
\]

and

\[
\delta \eta \phi = -\eta e_{0\mu}g^{\mu\nu}[ \omega^-_{\nu0\sigma_2\ldots\sigma_{p-1}} V_-D_-X^\sigma_2 \ldots D_-X^\sigma_{p-1} \\
+ \ldots \\
+ \omega^-_{\nu\sigma_1\ldots\sigma_{p-2}0} D_-X^\sigma_1 \ldots D_-X^\sigma_{p-2}V_- \\
+ \omega^-_{\nu\sigma_1\ldots\sigma_{p-1}k_1} D_-X^{k_1} \ldots D_-X^{k_{p-1}}],
\]

and

\[
\delta \epsilon X^j = \epsilon g^{j\nu}[ \omega^+_{\nu0\sigma_2\ldots\sigma_{p-1}} V_+D_+X^\sigma_2 \ldots D_+X^\sigma_{p-1} \\
+ \ldots \\
+ \omega^+_{\nu\sigma_1\ldots\sigma_{p-2}0} D_+X^\sigma_1 \ldots D_+X^\sigma_{p-2}V_+ \\
+ \omega^+_{\nu\sigma_1\ldots\sigma_{p-1}k_1} D_+X^{k_1} \ldots D_+X^{k_{p-1}}],
\]

and

\[
\delta \eta X^j = \eta g^{j\nu}[ \omega^-_{\nu0\sigma_2\ldots\sigma_{p-1}} V_-D_-X^\sigma_2 \ldots D_-X^\sigma_{p-1} \\
+ \ldots \\
+ \omega^-_{\nu\sigma_1\ldots\sigma_{p-2}0} D_-X^\sigma_1 \ldots D_-X^\sigma_{p-2}V_- \\
+ \omega^-_{\nu\sigma_1\ldots\sigma_{p-1}k_1} D_-X^{k_1} \ldots D_-X^{k_{p-1}}].
\]
To find the dual model, we eliminate $V_\pm$ by the equations of motion and get
\[ V_+ = e_0^{-1}(D_+ \phi - e_0 D_+ X^i) \quad V_- = -e_0^{-1}(D_- \phi + e_0 D_- X^i) \quad (19) \]
so that
\[
\delta \phi = \epsilon e_\mu g^{\mu \nu} \quad (p - 1)e_0^{-1} \omega^+_{\nu \nu 0k_2 \ldots k_{p-1}} D_+ \phi \{ D_+ X^{k_2} \ldots D_+ X^{k_{p-1}} \}
\]
\[
\delta X^j = \epsilon g^{j \nu} \quad (p - 1)e_0^{-1} \omega^+_{\nu 0k_2 \ldots k_{p-1}} D_+ \phi \{ D_+ X^{k_2} \ldots D_+ X^{k_{p-1}} \}
\]
and
\[
\delta \phi = -\eta e_\mu g^{\mu \nu} \quad -(p - 1)e_0^{-1} \omega^-_{\nu \nu 0k_2 \ldots k_{p-1}} D_- \phi \{ D_- X^{k_2} \ldots D_- X^{k_{p-1}} \}
\]
\[
\delta X^j = \eta g^{j \nu} \quad -(p - 1)e_0^{-1} \omega^-_{\nu 0k_2 \ldots k_{p-1}} D_- \phi \{ D_- X^{k_2} \ldots D_- X^{k_{p-1}} \}
\]
and we can read off the dual covariantly constant forms $\tilde{\omega}^+$ and $\tilde{\omega}^-$:
\[
\tilde{\omega}^+_{0k_1 \ldots k_{p-1}} = g_0^{-1} \omega^+_{0k_1 \ldots k_{p-1}} \quad (24)
\]
\[
\tilde{\omega}^+_{jk_1 \ldots k_{p-1}} = \omega^+_{jk_1 \ldots k_{p-1}} - g_0^{-1} W^+_{jk_1 \ldots k_{p-1}} \quad (25)
\]
where
\[
W^+_{jk_1 \ldots k_{p-1}} = e_{j0} \omega^+_{0k_1 \ldots k_{p-1}} - e_{k_10} \omega^+_{jk_2 \ldots k_{p-1}}
+ e_{k_20} \omega^+_{0k_1 k_3 \ldots k_{p-1}}
\]
\[
\ldots + (-1)^{p-1} e_{k_{p-1}0} \omega^+_{0k_1 \ldots k_{p-2} j} \quad (26)
\]
\[ \tilde{\omega}_{0k_1\ldots k_{p-1}} = -g_{00}^{-1}\omega_{0k_1\ldots k_{p-1}} \]  
(27)

\[ \tilde{\omega}_{jk_1\ldots k_{p-1}} = \omega_{jk_1\ldots k_{p-1}} - g_{00}^{-1}W_{jk_1\ldots k_{p-1}}, \]  
(28)

where

\[ W_{jk_1\ldots k_{p-1}} = e_{0j}\omega_{0k_1\ldots k_{p-1}} - e_{0k_1}\omega_{0j k_2\ldots k_{p-1}} + e_{0k_2}\omega_{0k_1 k_3\ldots k_{p-1}} \ldots + (-1)^{p-1}e_{0k_{p-1}}\omega_{0k_1\ldots k_{p-2}j}. \]  
(29)

Now the new currents in the dual space are (with \( X_0 = \phi \))

\[ \tilde{J}_+^\omega = \tilde{\omega}^+_{\rho \sigma_1\ldots \sigma_{p-1}}D_+X^\rho D_+X^{\sigma_1} \ldots D_+X^{\sigma_{p-1}} \]  
(30)

and

\[ \tilde{J}_-^\omega = \tilde{\omega}^-_{\rho \sigma_1\ldots \sigma_{p-1}}D_-X^\rho D_-X^{\sigma_1} \ldots D_-X^{\sigma_{p-1}} \]  
(31)

which generate a super \( W \)-algebra in the new setting.

Also substituting (19) into (14), we get the new lagrangian \[6, 9\]

\[ \tilde{S} = \frac{1}{2} \int D^2 \left( \epsilon_{00}^{-1}D_+\phi D_+\phi + \epsilon_{00}^{-1}\epsilon_{0j}D_+\phi D_-X^j - \epsilon_{00}^{-1}\epsilon_{i0}D_+X^iD_-\phi \right. \]

\[ \left. + (\epsilon_{ij} - \epsilon_{00}^{-1}\epsilon_{i0}\epsilon_{0j})D_+X^iD_-X^j \right) \]

and read off the dual metric \( \tilde{g} \) and torsion potential \( \tilde{b} \), i.e.,

\[ \tilde{g}_\mu^\nu = \begin{pmatrix} g_{00}^{-1} & g_{00}^{-1}b_{0j} \\ g_{00}^{-1}b_{0i} & g_{ij} - g_{00}^{-1}g_{i0}g_{j0} + g_{00}^{-1}b_{i0}b_{j0} \end{pmatrix}, \]  
(32)

and

\[ \tilde{b}_{\mu^\nu} = \begin{pmatrix} 0 & g_{00}^{-1}g_{0j} \\ -g_{00}^{-1}g_{0i} & b_{ij} + g_{00}^{-1}(g_{i0}b_{j0} - g_{j0}b_{i0}) \end{pmatrix} \]  
(33)

with respect to the basis \( \{d\phi, dX^1 \ldots dX^{2n-1}\} \).
4 Conclusions

The existence of covariantly constant forms restricts the holonomy group of the manifold. When the connection is the Levi-civita connection, the classification was done by Berger [3]. In the case of sigma model with a riemannian target manifold, via duality transformations we get another manifold with new metric and torsion and the existence of the covariantly constant forms will restrict the holonomy of this new manifold. The classification of holonomy groups in the case of the connections with torsion, generalizing Berger’s work, is an interesting area to be studied and the duality transformation will be very useful for that purpose.

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