An orthogonal basis for the $B_N$-type Calogero model

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Abstract

We investigate algebraic structure for the $B_N$-type Calogero model by using the exchange-operator formalism. We show that the set of the Jack polynomials whose arguments are Dunkl-type operators provides an orthogonal basis.

1 Introduction

Among quantum integrable models in one dimension, Calogero-Sutherland type models catch renewed interests owing to the relation to fractional statistics. An example of such models is the Calogero model with harmonic potential \cite{1, 2}:

$$H_A = \frac{1}{2} \sum_{j=1}^{N} \left( -\frac{\partial^2}{\partial x_j^2} + x_j^2 \right) + \sum_{j<k} \frac{\beta(\beta-1)}{(x_j-x_k)^2}. \tag{1}$$

The subscript "A" signifies that this Hamiltonian is invariant under the action of the symmetric group $S_N$, i.e. the $A_{N-1}$-type Weyl group. There also exist Calogero-type models associated with other types of the Weyl groups \cite{3}. The $B_N$-invariant counterpart of the Hamiltonian (1) is the following \cite{4, 5}:

$$H_B = \frac{1}{2} \sum_{j=1}^{N} \left\{ -\frac{\partial^2}{\partial x_j^2} + x_j^2 + \frac{\gamma(\gamma-1)}{x_j^2} \right\} + \sum_{j<k} \left\{ \frac{\beta(\beta-1)}{(x_j-x_k)^2} + \frac{\beta(\beta-1)}{(x_j+x_k)^2} \right\}. \tag{2}$$

We remark that the model associated with the $C_N$-type Weyl group is equivalent to the $B_N$-case, and $D_N$-type model is obtained by setting $\gamma = 0$. The ground state wavefunction for this model is \cite{4, 5}:

$$\psi_0^{(B)}(x_1, \ldots, x_N) = \prod_{j<k} |x_j^2 - x_k^2|^\beta \prod_{j=1}^{N} |x_j|^\gamma \prod_{j=1}^{N} \exp(-x_j^2/2). \tag{3}$$

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Wavefunctions of the excited states are written as products of \( \psi_0^{(B)} \) and some symmetric polynomials. Baker and Forrester obtained an orthogonal basis of such polynomials and named “generalized Laguerre polynomials” [6, 7]. It should be noted that the properties of such polynomials have been studied also by van Diejen [8]. In [6], the proof of the orthogonality is based on the orthogonality of another set of polynomials which they call “generalized Jacobi polynomials”. They obtained the orthogonality of the generalized Laguerre polynomials via some limiting procedure.

Here we make a kind of gauge-transformations on the Hamiltonian:

\[
\tilde{H}_B = (\phi_0^{(B)})^{-1} \circ H_B \circ \phi_0^{(B)} = \frac{1}{2} \sum_{j=1}^{N} \left( -\frac{\partial^2}{\partial x_j^2} + x_j^2 - \frac{2\gamma}{x_j} \frac{\partial}{\partial x_j} \right) - \beta \sum_{k \neq j} \frac{1}{x_j^2 - x_k^2} \left( \frac{x_j}{\partial x_j} - \frac{x_k}{\partial x_k} \right),
\]

where \( \phi_0^{(B)} \) is defined by

\[
\phi_0^{(B)}(x_1, \ldots, x_N) = \prod_{j<k} |x_j^2 - x_k^2|^\beta \prod_{j=1}^{N} |x_j|^\gamma.
\]

To construct eigenstates of \( \tilde{H}_B \), exchange-operator formalism is also available [8]. One can construct an analogue of creation operators \( A_j^\dagger \) (for definition, see (19) below) and show that the wavefunctions of the form, \( f \left( (A_1^\dagger)^2, \ldots, (A_N^\dagger)^2 \right) \prod_{j=1}^{N} \exp(-x_j^2/2) \), become eigenstates of \( \tilde{H}_B \) if \( f(x_1, \ldots, x_N) \) are homogeneous polynomials. However naive choice of the polynomial does not create the orthogonal states. In previous work [9], we have shown that the set of the Jack polynomials whose arguments are Dunkl-type operators provides an orthogonal basis for the \( A_{N-1} \)-type Calogero model. The aim of this paper is to investigate the \( B_N \)-case. We shall show that the Jack polynomials appear also in the \( B_N \)-case.

## Dunkl operators and Jack polynomials

In this section, we briefly review the definition of the symmetric and non-symmetric Jack polynomials. In physical context, the Jack polynomials appear as polynomial part of wavefunctions for the Sutherland \((1/\sin^2\)-interaction) model.

We first introduce the Cherednik operators [10, 11]:

\[
\hat{D}_j^{(A)} = z_j \frac{\partial}{\partial z_j} + \beta \sum_{k(<j)} \frac{z_k}{z_j - z_k} (1 - s_{jk}) + \beta \sum_{k(>j)} \frac{z_j}{z_j - z_k} (1 - s_{jk}) + \beta (j - 1)
\]

where \( s_{jk} \) are elements of the symmetric group \( S_N \) (the \( A_{N-1} \)-type Weyl group). An element \( s_{ij} \) acts on functions of \( z_1, \ldots, z_N \) as an operator which permutes arguments \( z_i \) and \( z_j \). Since the operators \( \hat{D}_j^{(A)} \) commute each other, they are diagonalized simultaneously by suitable choice of bases of \( \mathbb{C}[x_1, \ldots, x_N] \) [11, 12]. Such basis is called non-symmetric.
We denote the eigenvalues of the operators $\hat{D}_j^{(A)}$, where we have used the notation $x^\lambda_w = x_{w(1)}^\lambda \cdots x_{w(N)}^\lambda$. To define the ordering $(\mu, w') < (\lambda, w)$, we use the dominance ordering $<_D$ for partitions [13], and the Bruhat ordering $<_B$ for the elements of $S_N$ [14]. Using these, we define the ordering as follows:

$$(\mu, w') < (\lambda, w) \iff \begin{cases} (i) & \mu <_D \lambda, \\ (ii) & \text{if } \mu = \lambda \text{ then } w' <_B w. \end{cases} \quad (7)$$

We denote the eigenvalues of $\hat{D}_j^{(A)}$ as $\epsilon_j(\lambda, w)$:

$$\hat{D}_j^{(A)} J^\lambda_w(x) = \epsilon_j(\lambda, w) J^\lambda_w(x). \quad (8)$$

The eigenvalues $\epsilon_j(\lambda, w)$ are all obtained by permuting the components of the multiplet $\{\lambda_{N-j+1} + \beta(j-1)\}_{j=1,\ldots,N}$.

Using the operator $\hat{D}_j^{(A)}$, we introduce generating function of symmetric commuting operators [11]:

$$\hat{\Delta}_s^{(A)}(u) = \prod_{j=1}^N (u + \hat{D}_j^{(A)}). \quad (9)$$

Since $\hat{\Delta}_s^{(A)}(u)$ is symmetric in $\hat{D}_j$, symmetric eigenfunctions are obtained by symmetrizing $J^\lambda_w(x)$, which are nothing but the Jack symmetric polynomials $J_\lambda(x)$. Eigenvalues of $\hat{\Delta}_s^{(A)}(u)$ are then given by

$$\hat{\Delta}_s^{(A)}(u) J_\lambda(x) = \prod_{j=1}^N \{u + \lambda_{N-j+1} + \beta(j-1)\} J_\lambda(x). \quad (10)$$

We note that all the eigenvalues of $\hat{\Delta}_s^{(A)}(u)$ are distinct for generic value of $u$.

We then introduce the $B_N$-type Dunkl operators [3, 13]:

$$D_j^{(B)} = \frac{\partial}{\partial x_j} + \beta \sum_{k(\neq j)} \left( \frac{1-s_{jk}}{x_j-x_k} + \frac{1-t_j t_k s_{jk}}{x_j+x_k} \right) + \gamma \frac{1-t_j}{x_j} \quad (11)$$

where $s_{jk}$ and $t_j$ are elements of the $B_N$-type Weyl group. An element $s_{ij}$ acts as same as in the $A_{N-1}$-case and $t_j$ acts as sign-change, i.e., replaces the coordinate $x_j$ by $-x_j$.

Commutation relations of the $B_N$-type Dunkl operators are

$$[D_i^{(B)}, D_j^{(B)}] = 0,$$

$$[D_i^{(B)}, x_j] = \delta_{ij} \left( 1 + \beta \sum_{k(\neq j)} (s_{jk} + t_j t_k s_{jk}) + 2 \gamma t_j \right) - (1 - \delta_{ij}) \beta (s_{ij} - t_j t_i s_{ij}),$$

$$s_{ij} D_j^{(B)} = D_j^{(B)} s_{ij}, \quad s_{ij} D_k^{(B)} = D_k^{(B)} s_{ij} \quad (k \neq i, j),$$

$$t_j D_j^{(B)} = -D_j^{(B)} t_j, \quad t_j D_k^{(B)} = D_k^{(B)} t_j \quad (k \neq j). \quad (12)$$
We denote the algebra generated by the elements \( x_j, D_j^{(B)} \), \( s_{ij} \) and \( t_j \) as \( \mathcal{A}_s^{(B)} \). We introduce an \( \mathcal{A}_s^{(B)} \)-module \( \mathcal{F}_s^{(B)} \) ("Fock space" for \( \mathcal{A}_s^{(B)} \)) generated by the vacuum vector \( |0\rangle_s = 1 \):

\[
\mathcal{F}_s^{(B)} = \mathbb{C}[x_1^2, \ldots, x_N^2]|0\rangle_s.
\]

The elements \( D_j^{(B)} \) of \( \mathcal{A}_s^{(B)} \) annihilate the vacuum vector, and \( s_{ij}, t_j \) preserve \( |0\rangle_s \):

\[
D_j|0\rangle_s = 0, \quad s_{ij}|0\rangle_s = |0\rangle_s, \quad t_j|0\rangle_s = |0\rangle_s.
\]

We then define Cherednik-type commuting operators associated with (11):

\[
\hat{D}_j^{(B)} = x_j D_j^{(B)} + \beta \sum_{k(<j)} (s_{jk} + t_k s_{jk})
\]

\[
= x_j \frac{\partial}{\partial x_j} + \beta \sum_{k(<j)} \left\{ \frac{x_k}{x_j - x_k}(1 - s_{jk}) - \frac{x_k}{x_j + x_k}(1 - t_k s_{jk}) \right\}
\]

\[
+ \beta \sum_{k(>j)} \left\{ \frac{x_j}{x_j - x_k}(1 - s_{jk}) + \frac{x_j}{x_j + x_k}(1 - t_k s_{jk}) \right\} + 2\beta(j - 1) + \gamma(1 - t)\tilde{\mathfrak{h})}
\]

We introduce the notation \( \text{Res}^{(t)}(X) \) which means the action of the operator \( X \) is restricted to the functions with the symmetry \( t_j f(x) = f(x) \). Under this restriction, the action of the operator \( \hat{D}_j^{(B)} \) is reduced to the following form:

\[
\text{Res}^{(t)}(\hat{D}_j^{(B)}) = x_j \frac{\partial}{\partial x_j} + 2\beta \sum_{k(<j)} \frac{x_k^2}{x_j^2 - x_k^2}(1 - s_{jk})
\]

\[
+ 2\beta \sum_{k(>j)} \frac{x_j^2}{x_j^2 - x_k^2}(1 - s_{jk}) + 2\beta(j - 1).
\]

Comparing (10) with (11), we find that \( \text{Res}^{(t)}(\hat{D}_j^{(B)}) \) is equivalent to \( 2\hat{D}_j^{(A)} \) if we make a change of the variables \( z_j = x_j^2/2 \). If we define the operator \( \hat{\Delta}_s^{(B)}(u) \) as

\[
\hat{\Delta}_s^{(B)}(u) = \prod_{j=1}^N (u + \hat{D}_j^{(B)}),
\]

we have the following equation by using the correspondence between \( \text{Res}^{(t)}(\hat{D}_j^{(B)}) \) and \( 2\hat{D}_j^{(A)} \):

\[
\hat{\Delta}_s^{(B)}(u) J_\lambda \left( x_1^2/2, \ldots, x_N^2/2 \right)
\]

\[
= \prod_{j=1}^N \left\{ u + 2\lambda_{N-j+1} + 2\beta(j - 1) \right\} J_\lambda \left( x_1^2/2, \ldots, x_N^2/2 \right).
\]

### 3 \( B_N \)-type Calogero model

We now turn to the \( B_N \)-type Calogero model. We introduce an analogue of creation and annihilation operators [13]:

\[
A_j^\dagger = \frac{1}{\sqrt{2}}(-D_j^{(B)} + x_j), \quad A_j = \frac{1}{\sqrt{2}}(D_j^{(B)} + x_j).
\]
By a direct calculation, we can show that the operator $A_j^\dagger$ is adjoint of $A_j$ with respect to the scalar product

$$
(f, g)_B = \int_{-\infty}^{\infty} f(x_1, \ldots, x_N) g(x_1, \ldots, x_N) (\phi_0^B)^2 \prod_{j=1}^{N} d x_j.
$$

We call an algebra generated by $A_j, A_j^\dagger, s_{ij}$ and $t_j$ as $\mathcal{A}_c^{(B)}$. Since the commutation relations of these operators are the same as those of $x_j$ and $D_j^{(B)}$, we can define an isomorphism of $\mathcal{A}_s^{(B)}$ to $\mathcal{A}_c^{(B)}$ as follows:

$$
\sigma(x_j) = A_j^\dagger, \quad \sigma(D_j^{(B)}) = A_j.
$$

Fock space for $\mathcal{A}_c^{(B)}$ is constructed in the same way as $\mathcal{F}_s^{(B)}$:

$$
\mathcal{F}_c^{(B)} = C[(A_1^\dagger)^2, \ldots, (A_N^\dagger)^2] |0\rangle_c,
$$

with $|0\rangle_c = \prod_{j=1}^{N} \exp(-x_j^2/2)$. The elements $A_j$ of $\mathcal{A}_c^{(B)}$ annihilate the vacuum vector, and $s_{ij}, t_j$ preserve $|0\rangle_c$:

$$
A_j |0\rangle_c = 0, \quad s_{ij} |0\rangle_c = |0\rangle_c, \quad t_j |0\rangle_c = |0\rangle_c.
$$

Comparing (23) with (14), we know that the isomorphism $\sigma$ can be extended to the isomorphism of the Fock spaces:

$$
\sigma(|0\rangle_s) = |0\rangle_c, \quad \sigma(a |v\rangle) = \sigma(a) \sigma(|v\rangle)
$$

for $a \in \mathcal{A}_s^{(B)}$ and $|v\rangle \in \mathcal{F}_s^{(B)}$.

Applying this isomorphism to (18), we get the following equation:

$$
\tilde{\Delta}_c^{(B)}(u) J_{\lambda} \left( (A_1^\dagger)^2/2, \ldots, (A_N^\dagger)^2/2 \right) |0\rangle_c
$$

$$
= \prod_{j=1}^{N} \left\{ u + 2\lambda_{N-j+1} + 2\beta(j-1) \right\} J_{\lambda} \left( (A_1^\dagger)^2/2, \ldots, (A_N^\dagger)^2/2 \right) |0\rangle_c,
$$

where we define $\tilde{\Delta}_c^{(B)}(u)$ as

$$
\tilde{\Delta}_c^{(B)}(u) = \sigma \left( \tilde{\Delta}_s^{(B)}(u) \right) = \prod_{j=1}^{N} \left( u + \hat{h}_j^{(B)} \right)
$$

with

$$
\hat{h}_j^{(B)} = \sigma \left( \hat{D}_j^{(B)} \right) = A_j^\dagger A_j + \beta \sum_{k(<j)} (s_{jk} + t_j t_k s_{jk}).
$$

We note that the operator $\hat{h}_j^{(B)}$ is self-adjoint with respect to (24). The transformed Hamiltonian $\tilde{H}_B$ is related to (26) as follows; if we denote the $(N-j)$-th coefficient of $\tilde{\Delta}_c^{(B)}(u)$ as $I_{c,j}^{(B)}$, then $\tilde{H}_B$ is obtained from $I_{c,1}^{(B)}$ after restricting to the $B_N$-invariant subspace:

$$
\text{Res} \left( I_{c,1}^{(B)} \right) = \text{Res} \left( \sum_{j=1}^{N} \hat{h}_j^{(B)} \right) = \tilde{H}_B - \frac{N}{2} - \gamma N.
$$
From (25) we find that all the eigenvalues of $\hat{\Delta}_c^{(B)}(u)$ are distinct. On the other hand, the operator $\hat{\Delta}_c^{(B)}(u)$ is self-adjoint with respect to the scalar product (24). From these facts, we conclude that the vectors,

$$|\lambda\rangle = J_\lambda \left( (A_1^\dagger)^2/2, \ldots, (A_N^\dagger)^2/2 \right) |0\rangle_c,$$

form an orthogonal basis with respect to the scalar product (20). We note that polynomial parts of the basis (29) are equivalent to the generalized Laguerre polynomials introduced by Baker and Forrester up to constant.

Baker and Forrester also introduced “non-symmetric generalized Laguerre polynomials” [4]. In our formulation, such polynomials are related to joint eigenfunctions of the operators $\hat{h}_j^{(B)}$, which are of the form, $\mathcal{J}_w^\lambda \left( (A_1^\dagger)^2/2, \ldots, (A_N^\dagger)^2/2 \right) |0\rangle_c$. The non-symmetric generalized Laguerre polynomials are polynomial parts of these eigenfunctions.

In conclusion, we have constructed operator expression of the orthogonal basis for the $B_N$-type Calogero model by using the Jack polynomials whose arguments are the Dunkl-type creation and annihilation operators. We stress that our proof of orthogonality is algebraic and does not make use of the limiting procedure.

**Appendix**

In this appendix, we investigate the one-variable case in more detail to clarify the relationship to the Laguerre polynomials.

For $N = 1$ case, Hamiltonian (2) is reduced to

$$\hat{H} = \frac{1}{2} \left\{ -\frac{d^2}{dx^2} + x^2 + \frac{\gamma(\gamma - 1)}{x^2} \right\}.$$  \hspace{1cm} (A1)

The ground state wavefunction is $\psi_0(x) = |x|^{\gamma} \exp(-x^2/2)$ whose eigenvalue is $1/2 + \gamma$ (we omit the normalization constant).

On the other hand, the creation and annihilation operators (13) are reduced to

$$A^\dagger = \frac{1}{\sqrt{2}} \left\{ -\frac{d}{dx} + x - \frac{\gamma}{x} (1 - \hat{t}) \right\}, \quad A = \frac{1}{\sqrt{2}} \left\{ \frac{d}{dx} + x + \frac{\gamma}{x} (1 - \hat{t}) \right\},$$  \hspace{1cm} (A2)

where $\hat{t}$ is the reflection operator $\hat{t} f(x) = f(-x)$. The Hamiltonian (A1) and the operators (A2) are related as follows:

$$\hat{H} = |x|^\gamma \circ \frac{1}{2} \text{Res}(A^\dagger A + AA^\dagger) \circ |x|^{-\gamma},$$

where $\text{Res} X$ means that action of the operator $X$ is restricted to even functions.

Wavefunctions for excited states can be constructed by using gauge-transformed version of (A2), i.e.

$$\hat{A}^\dagger = |x|^\gamma \circ A^\dagger \circ |x|^{-\gamma} = \frac{1}{\sqrt{2}} \left\{ -\frac{d}{dx} + x + \frac{\gamma}{x} \hat{t} \right\},$$
The creation and annihilation operators obey the commutation relations
\[ [\hat{H}, \hat{A}^\dagger] = \hat{A}^\dagger, \quad [\hat{H}, \hat{A}] = -\hat{A}. \]  

(A3)

It should be noted that these creation and annihilation operators have been introduced by Yang [16].

To preserve the symmetry \( \psi(x) = \psi(-x) \), we apply \( \hat{A}^\dagger \) by even times to the ground state wavefunction \( \psi_0(x) \):
\[ \psi_{2n}(x) = (\hat{A}^\dagger)^{2n}\psi_0(x). \]

This formula is \( N = 1 \) counterpart of (29). From (A3), it follows that \( \psi_{2n}(x) \) are also eigenfunctions of \( \hat{H} \):
\[ \hat{H}\psi_{2n}(x) = (2n + 1/2)\psi_{2n}(x). \] 

(A4)

The wavefunctions \( \psi_{2n}(x) \) are expressed as product of some polynomials \( f_{2n}(x) \) and the ground state wavefunction \( \psi_0(x) \). Rewriting (A4), one can obtain the following differential equation for \( f_{2n}(x) \):
\[ \frac{d^2 f_{2n}}{dx^2} - \left( 2x - \frac{2\gamma}{x} \right) \frac{df_{2n}}{dx} + 4nf_{2n} = 0. \]

Making a change of the variable \( y = x^2 \), we obtain
\[ y\frac{d^2 f_{2n}}{dy^2} + \left( \frac{1}{2} + \gamma - y \right) \frac{df_{2n}}{dy} + nf_{2n} = 0, \]

which coincides with the differential equation for the Laguerre polynomials. Hence we conclude that \( f_{2n}(x) \) can be written by using the Laguerre polynomials:
\[ f_{2n}(x) = n!(\gamma^{1/2}L_n^{(\gamma-1/2)}(x^2)). \]

Since \( \psi_{2n}(x) \) are even functions, the operators \((\hat{A}^\dagger)^2\) and \(\hat{A}^2\) act equivalently to
\[ B^+ = \text{Res}((\hat{A}^\dagger)^2) = \frac{1}{2} \left\{ \frac{d^2}{dx^2} - 2x \frac{d}{dx} + x^2 - 1 - \frac{\gamma(\gamma - 1)}{x^2} \right\}, \]
\[ B^- = \text{Res}(\hat{A}^2) = \frac{1}{2} \left\{ \frac{d^2}{dx^2} + 2x \frac{d}{dx} + x^2 + 1 - \frac{\gamma(\gamma - 1)}{x^2} \right\}, \]

respectively. We remark that the operators \( B^+ \) and \( B^- \) have been introduced by Perelomov [17].

The operators \( B^+ \) and \( B^- \) give the recursion relations for the wavefunctions:
\[ B^+\psi_{2n} = \psi_{2n+2}, \quad B^-\psi_{2n} = 4n \left( n - \frac{1}{2} + \gamma \right) \psi_{2n-2}, \] 

(A5)
where the constant factor of the second relation is determined by comparing the coefficient of $x^{2n-2}\psi_0(x)$. One can obtain recursion relations for the Laguerre polynomials by rewriting (A3) [17]:

\[
\left\{ y \frac{d^2}{dy^2} + \left( \frac{1}{2} + \gamma - 2y \right) \frac{d}{dy} + 2y - \frac{1}{2} - \gamma \right\} L_n^{(\gamma-1/2)}(y) = -(n+1)L_{n+1}^{(\gamma-1/2)}(y),
\]

\[
\left\{ y \frac{d^2}{dy^2} + \left( \frac{1}{2} + \gamma \right) \frac{d}{dy} \right\} L_n^{(\gamma-1/2)}(y) = -\left( n - \frac{1}{2} + \gamma \right)L_{n-1}^{(\gamma-1/2)}(y).
\]

References

[1] F. Calogero: J. Math. Phys. 12 (1971) 419-36.
[2] B. Sutherland: Phys. Rev. A4 (1971) 2019-21; Phys. Rev. A5 (1972) 1372-6.
[3] M.A. Olshanetsky and A.M. Perelomov: Phys. Rep. 94 (1983) 313-404.
[4] T. Yamamoto: Phys. Lett. A208 (1995) 293-302.
[5] T. Yamamoto and O. Tsuchiya: J. Phys. A29 (1996) 3977-84.
[6] T.H. Baker and P.J. Forrester: “The Calogero-Sutherland Model and Generalized Classical Polynomials”, preprint [solv-int/9608004].
[7] T.H. Baker and P.J. Forrester: “The Calogero-Sutherland Model and Polynomials with Prescribed Symmetry”, preprint [solv-int/9609010].
[8] J.F. van Diejen: “Confluent Hypergeometric Orthogonal Polynomials Related to the Rational Quantum Calogero System with Harmonic Confinement”, preprint [q-alg/9609032].
[9] S. Kakei: J. Phys. A29 (1996) L619-24.
[10] I. Cherednik: Invent. Math. 106 (1991) 411-32.
[11] D. Bernard, M. Gaudin, F.D.M. Haldane and V. Pasquier: J. Phys. A26 (1993) 5219-36.
[12] E.M. Opdam: Acta Math. 175 (1995) 75-121.
[13] I. G. Macdonald: “Symmetric Functions and Hall Polynomials”, 2nd edition, Clarendon Press, Oxford, 1995.
[14] J.E. Humphreys: “Reflection groups and Coxeter groups” (Cambridge Studies in Advanced Mathematics 29), Cambridge University Press, 1990.

[15] C.F. Dunkl: Trans. Amer. Math. Soc. 311 (1989) 167-83.

[16] L.M. Yang: Phys. Rev. 84 (1951) 788-90.

[17] A.M. Perelomov: Theor. Math. Phys. 6 (1971) 263-82.