The generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative

Agnieszka B. Malinowska\textsuperscript{1,2} Delfim F. M. Torres\textsuperscript{1}
abmalinowska@ua.pt delfim@ua.pt

\textsuperscript{1}Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
\textsuperscript{2}Faculty of Computer Science, Bialystok University of Technology, 15-351 Bialystok, Poland

Abstract

This paper presents necessary and sufficient optimality conditions for problems of the fractional calculus of variations with a Lagrangian depending on the free end-points. The fractional derivatives are defined in the sense of Caputo.

Keywords: fractional Euler–Lagrange equation; fractional derivative; Caputo derivative.

Mathematics Subject Classification: 49K05; 26A33.

1 Introduction

Fractional calculus is one of the generalizations of the classical calculus and it has been used successfully in various fields of science and engineering — see, e.g., [11, 12, 15, 19, 20, 23, 24, 26, 27, 30, 34, 35]. In recent years, there has been a growing interest in the area of fractional variational calculus and its applications [1–8, 13, 14, 16–18, 22, 28, 32]. Applications include classical and quantum mechanics, field theory, and optimal control. In the papers cited above, the problems have been formulated mostly in terms of two types of fractional derivatives, namely Riemann–Liouville and Caputo. The natural boundary conditions for fractional variational problems, in terms of the Riemann–Liouville and the Caputo derivative, are presented in [1, 2]. Here we develop further the theory by proving necessary optimality conditions for more general problems of the fractional calculus of variations with a Lagrangian that may also depend on the unspecified end-points $y(a), y(b)$. More precisely, the problem under our study consists to minimize a functional which is defined in terms of the Caputo derivative and having no constraint on $y(a)$ and/or $y(b)$. The novelty is the dependence of the integrand $L$ on the free end-points $y(a), y(b)$. This class of problems is motivated by applications in the field of economics [10].

The paper is organized as follows. Section [2] presents the necessary definitions and concepts of the fractional calculus; our results are formulated, proved, and illustrated through an

*Accepted (19 February 2010) for publication in Computers and Mathematics with Applications.
example in Section 3. Main results of the paper include necessary optimality conditions with
the new natural boundary conditions (Theorem 3.1) that become sufficient under appropriate
convexity assumptions (Theorem 3.3).

2 Fractional Calculus

In this section we review the necessary definitions and facts from fractional calculus. For
more on the subject we refer the reader to [21,29,31,33].

Let \( f \in L^1([a, b]) \) and \( 0 < \alpha < 1 \). We begin with the left and the right Riemann–Liouville
Fractional Integrals (RLFI) of order \( \alpha \) of function \( f \) which are defined as: the left RLFI

\[
a I^\alpha_x f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x \in [a, b],
\]

(1)

the right RLFI

\[
x I^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x \in [a, b],
\]

(2)

where \( \Gamma(\cdot) \) represents the Gamma function. Moreover, \( a I^0_x f = x I^0_b f = f \) if \( f \) is a continuous
function.

Let \( f \in AC([a, b]) \), where \( AC([a, b]) \) represents the space of absolutely continuous func-
tions on \([a, b]\). Then using equations (1) and (2), the left and the right Riemann–Liouville and
Caputo derivatives are defined as: the left Riemann–Liouville Fractional Derivative (RLFD)

\[
a D^\alpha_x f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)dt = \frac{d}{dx} a I^{1-\alpha}_x f(x), \quad x \in [a, b],
\]

(3)

the right RLFD

\[
x D^\alpha_b f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)dt = \left( -\frac{d}{dx} \right) x I^{1-\alpha}_b f(x), \quad x \in [a, b],
\]

(4)

the left Caputo Fractional Derivative (CFD)

\[
C a D^\alpha_x f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} \frac{d}{dt} f(t)dt = a I^{1-\alpha}_x \frac{d}{dx} f(x), \quad x \in [a, b],
\]

(5)

the right CFD

\[
x D^\alpha_b f(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} \frac{d}{dt} f(t)dt = x I^{1-\alpha}_b \left( -\frac{d}{dx} \right) f(x), \quad x \in [a, b],
\]

(6)

where \( \alpha \) is the order of the derivative.

The operators \((1) - (6)\) are obviously linear. We now present the rule of fractional integra-
tion by parts for RLFI (see for instance [9]). Let \( 0 < \alpha < 1, p \geq 1, q \geq 1, \) and \( 1/p+1/q \leq 1+\alpha \).
If \( g \in L_p([a, b]) \) and \( f \in L_q([a, b]) \), then

\[
\int_a^b g(x) a I^\alpha_x f(x)dx = \int_a^b f(x) x I^\alpha_b g(x)dx.
\]

(7)
Remark 1. We are assuming that the admissible functions exist on the closed interval $[a, b]$.

In the discussion to follow, we will also need the following formulae for fractional integrations by parts:

$$
\int_a^b g(x) \frac{C}{x} D_x^\alpha f(x) \, dx = f(x) x^{1-\alpha} g(x) \bigg|_{x=a}^{x=b} + \int_a^b f(x) x D_a^\alpha g(x) \, dx,
$$

$$
\int_a^b g(x) \frac{C}{x} D_b^\alpha f(x) \, dx = -f(x) x^{1-\alpha} g(x) \bigg|_{x=a}^{x=b} + \int_a^b f(x) x D_x^\alpha g(x) \, dx.
$$

They can be derived using equations (4)–(6), the identity (7) and performing integration by parts.

3 Main Results

Let us consider the following problem:

$$
\mathcal{J}(y) = \int_a^b L(x, y(x), \frac{C}{x} a D_x^\alpha y(x), \frac{C}{x} b D_b^\beta y(x), y(a), y(b)) \, dx \rightarrow \text{extr}
$$

$$
(y(a) = y_a, \quad y(b) = y_b).
$$

Using parentheses around the end-point conditions means that the conditions may or may not be present. We assume that:

(i) $L(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \in C^1([a, b] \times \mathbb{R}^5; \mathbb{R})$;

(ii) $x \to \partial_3 L(x, y(x), \frac{C}{a} D_x^\alpha y(x), \frac{C}{b} D_b^\beta y(x), y(a), y(b))$ has continuous right RLFI of order $1-\alpha$ and right RLFD of order $\alpha$, where $\alpha \in (0, 1)$;

(iii) $x \to \partial_4 L(x, y(x), \frac{C}{a} D_x^\alpha y(x), \frac{C}{b} D_b^\beta y(x), y(a), y(b))$ has continuous left RLFI of order $1-\beta$ and left RLFD of order $\beta$, where $\beta \in (0, 1)$.

Remark 1. We are assuming that the admissible functions $y$ are such that $\frac{C}{a} D_x^\alpha y(x)$ and $\frac{C}{b} D_b^\beta y(x)$ exist on the closed interval $[a, b]$.

Along the work we denote by $\partial_i L$, $i = 1, \ldots, 6$, the partial derivative of function $L(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot)$ with respect to its $i$th argument.

3.1 Necessary Optimality Conditions

Next theorem gives necessary optimality conditions for the problem (9).

**Theorem 3.1.** Let $y$ be a local extremizer to problem (9). Then, $y$ satisfies the fractional Euler–Lagrange equation

$$
\partial_2 L(x, y(x), \frac{C}{a} D_x^\alpha y(x), \frac{C}{b} D_b^\beta y(x), y(a), y(b)) + x D_a^\alpha \partial_3 L(x, y(x), \frac{C}{a} D_x^\alpha y(x), \frac{C}{b} D_b^\beta y(x), y(a), y(b))
$$

$$
+ a D_x^\alpha \partial_4 L(x, y(x), \frac{C}{a} D_x^\alpha y(x), \frac{C}{b} D_b^\beta y(x), y(a), y(b)) = 0
$$

(10)
for all $x \in [a, b]$. Moreover, if $y(a)$ is not specified, then

$$
\int_a^b \partial_5 L(x, y(x), \frac{C}{x} D_x^a y(x), \frac{C}{x} D_x^b y(x), y(a), y(b)) \, dx
$$

$$
= \left[ a I_b^{1-\alpha} \partial_3 L(x, y(x), \frac{C}{x} D_x^a y(x), \frac{C}{x} D_x^b y(x), y(a), y(b)) \\
- a I_x^{1-\beta} \partial_4 L(x, y(x), \frac{C}{x} D_x^a y(x), \frac{C}{x} D_x^b y(x), y(a), y(b)) \right]_{x=a}
$$

(11)

if $y(b)$ is not specified, then

$$
\int_a^b \partial_6 L(x, y(x), \frac{C}{x} D_x^a y(x), \frac{C}{x} D_x^b y(x), y(a), y(b)) \, dx
$$

$$
= \left[ a I_b^{1-\alpha} \partial_3 L(x, y(x), \frac{C}{x} D_x^a y(x), \frac{C}{x} D_x^b y(x), y(a), y(b)) \\
- a I_x^{1-\beta} \partial_4 L(x, y(x), \frac{C}{x} D_x^a y(x), \frac{C}{x} D_x^b y(x), y(a), y(b)) \right]_{x=b}.
$$

(12)

Proof. Suppose that $y$ is an extremizer of $J$. We can proceed as Lagrange did, by considering the value of $J$ at a nearby function $\tilde{y} = y + \varepsilon h$, where $\varepsilon \in \mathbb{R}$ is a small parameter, $h$ is an arbitrary admissible function. We do not require $h(a) = 0$ or $h(b) = 0$ in case $y(a)$ or $y(b)$, respectively, is free (it is possible that both are free). Let

$$
\varphi(\varepsilon) = \int_a^b L(x, y(x) + \varepsilon h(x), \frac{C}{x} D_x^a (y(x) + \varepsilon h(x)), \frac{C}{x} D_x^b (y(x) + \varepsilon h(x)), y(a) + \varepsilon h(a), y(b) + \varepsilon h(b)) \, dx
$$

Since $\frac{C}{x} D_x^a$ and $\frac{C}{x} D_x^b$ are linear operators, it follows that

$$
\frac{C}{x} D_x^a (y(x) + \varepsilon h(x)) = \frac{C}{x} D_x^a y(x) + \varepsilon \frac{C}{x} D_x^a h(x)
$$

$$
\frac{C}{x} D_x^b (y(x) + \varepsilon h(x)) = \frac{C}{x} D_x^b y(x) + \varepsilon \frac{C}{x} D_x^b h(x).
$$

A necessary condition for $y$ to be an extremizer is given by

$$
\left. \frac{d\varphi}{d\varepsilon} \right|_{\varepsilon=0} = 0
$$

$$
\Leftrightarrow \int_a^b \left[ \partial_2 L(\cdots) h(x) + \partial_3 L(\cdots) \frac{C}{x} D_x^a h(x) + \partial_4 L(\cdots) \frac{C}{x} D_x^b h(x) + \partial_5 L(\cdots) h(a) + \partial_6 L(\cdots) h(b) \right] \, dx
$$

$$
= 0,
$$

(13)

where $(\cdots) = (x, y(x), \frac{C}{x} D_x^a y(x), \frac{C}{x} D_x^b y(x), y(a), y(b))$. Using formulae (S) for integration by parts, the second and the third integral can be written as

$$
\int_a^b \partial_3 L(\cdots) \frac{C}{x} D_x^a h(x) \, dx = \int_a^b h(x) \frac{C}{x} D_x^b \partial_3 L(\cdots) \, dx + \int_a^b \frac{C}{x} I_b^{1-\alpha} \partial_3 L(\cdots) h(x) \bigg|_{x=a},
$$

$$
\int_a^b \partial_4 L(\cdots) \frac{C}{x} D_x^b h(x) \, dx = \int_a^b h(x) \frac{C}{x} D_x^a \partial_4 L(\cdots) \, dx - \int_a^b \frac{C}{x} I_x^{1-\beta} \partial_4 L(\cdots) h(x) \bigg|_{x=a}.
$$

(14)
Substituting (14) into (13), we get
\[
\int_a^b \left[ \partial_2 L(\cdots) + x D_x^\alpha \partial_3 L(\cdots) + a D_x^\beta \partial_4 L(\cdots) \right] h(x) + x \left. I_b^1 - \alpha \partial_3 L(\cdots) h(x) \right|_{x=a} - \left. a I_x^1 - \beta \partial_4 L(\cdots) h(x) \right|_{x=a} + \left. \int_a^b (\partial_5 L(\cdots) h(a) + \partial_6 L(\cdots) h(b)) \, dx = 0. \tag{15}
\]
We first consider functions \( h(t) \) such that \( h(a) = h(b) = 0 \). Then, by the fundamental lemma of the calculus of variations, we deduce that
\[
\partial_2 L(\cdots) + x D_x^\alpha \partial_3 L(\cdots) + a D_x^\beta \partial_4 L(\cdots) = 0
\]
for all \( x \in [a, b] \). Therefore, in order for \( y \) to be an extremizer to the problem (9), \( y \) must be a solution of the fractional Euler–Lagrange equation. But if \( y \) is a solution of (16), the first integral in expression (15) vanishes, and then the condition (13) takes the form
\[
h(b) \left\{ \int_a^b \partial_5 L(\cdots) \, dx - \left. \left[ a I_x^1 - \beta \partial_4 L(\cdots) x I_b^1 - \alpha \partial_3 L(\cdots) \right] \right|_{x=b} \right\} + h(a) \left\{ \int_a^b \partial_6 L(\cdots) \, dx - \left. \left[ x I_b^1 - \alpha \partial_3 L(\cdots) - a I_x^1 - \beta \partial_4 L(\cdots) \right] \right|_{x=a} \right\} = 0.
\]
If \( y(a) = y_a \) and \( y(b) = y_b \) are given in the formulation of problem (9), then the latter equation is trivially satisfied since \( h(a) = h(b) = 0 \). When \( y(a) \) is free, then
\[
\int_a^b \partial_5 L(\cdots) \, dx - \left[ x I_b^1 - \alpha \partial_3 L(\cdots) - a I_x^1 - \beta \partial_4 L(\cdots) \right] \big|_{x=a} = 0,
\]
when \( y(b) \) is free, then
\[
\int_a^b \partial_6 L(\cdots) \, dx - \left[ a I_x^1 - \beta \partial_4 L(\cdots) - x I_b^1 - \alpha \partial_3 L(\cdots) \right] \big|_{x=b} = 0
\]
since \( h(a) \) or \( h(b) \) is, respectively, arbitrary.

**Remark 2.** Conditions (10)–(12) are only necessary for an extremum. The question of sufficient conditions for an extremum is considered in Subsection 3.2.

In the case \( L \) does not depend on \( y(a) \) and \( y(b) \), by Theorem 3.1 we obtain the following result.

**Corollary 1** (Theorem 1 of [2]). If \( y \) is a local extremizer to problem
\[
J(y) = \int_a^b L(x, y(x), \alpha D_x^\alpha y(x), \beta D_x^\beta y(x)) \, dx \rightarrow extr,
\]
then \( y \) satisfies the fractional Euler–Lagrange equation
\[
\partial_2 L(x, y(x), \alpha D_x^\alpha y(x), \beta D_x^\beta y(x)) + x D_x^\alpha \partial_3 L(x, y(x), \alpha D_x^\alpha y(x), \beta D_x^\beta y(x)) + a D_x^\beta \partial_4 L(x, y(x), \alpha D_x^\alpha y(x), \beta D_x^\beta y(x)) = 0
\]
for all $x \in [a, b]$. Moreover, if $y(a)$ is not specified, then
\[
\left[ x^\alpha \partial_3 L(x, y(x), C_a D_0^\alpha y(x), C D_b^\beta y(x)) \right]_{x=a} = 0,
\]
if $y(b)$ is not specified, then
\[
\left[ a x^\alpha \partial_4 L(x, y(x), C_a D_0^\alpha y(x), C D_b^\beta y(x)) \right]_{x=b} = 0.
\]

We note that the generalized Euler–Lagrange equation contains both the left and the right fractional derivative. The generalized natural conditions contain also the left and the right fractional integral. Although the functional has been written only in terms of the CFDs, necessary conditions (10)–(12) contain Caputo fractional derivatives, Riemann–Liouville fractional derivatives and Riemann–Liouville fractional integrals.

Observe that if $\alpha$ goes to 1, then the operators $C_a D_0^\alpha$ and $C D_0^\beta$ can be replaced with $\frac{d}{dx}$ and the operators $C x D_0^\alpha$ and $x D_0^\beta$ can be replaced with $-\frac{d}{dx}$ (see [31]). Thus, if the $\frac{C}{x} D_0^\beta$ term is not present in (9), then for $\alpha \rightarrow 1$ we obtain a corresponding result in the classical context of the calculus of variations [10] (see also [25 Corollary 1]).

**Corollary 2.** If $y$ is a local extremizer for
\[
J(y) = \int_a^b L(x, y(x), y'(x), y(a), y(b)) \, dx \rightarrow \text{extr}
\]
\[
(y(a) = y_a), \quad (y(b) = y_b),
\]
then
\[
\frac{d}{dx} \partial_3 L(x, y(x), y'(x), y(a), y(b)) = \partial_2 L(x, y(x), y'(x), y(a), y(b))
\]
for all $x \in [a, b]$. Moreover, if $y(a)$ is free, then
\[
\partial_3 L(a, y(a), y'(a), y(a), y(b)) = \int_a^b \partial_5 L(x, y(x), y'(x), y(a), y(b)) \, dx;
\]
if $y(b)$ is free, then
\[
\partial_3 L(b, y(b), y'(b), y(a), y(b)) = -\int_a^b \partial_5 L(x, y(x), y'(x), y(a), y(b)) \, dx.
\]

### 3.2 Sufficient Conditions

In this section we prove sufficient conditions that ensure the existence of minimum (maximum). Similarly to what happens in the classical calculus of variations, some conditions of convexity (concavity) are in order.

**Definition 3.2.** Given a function $L$, we say that $L(x, y, z, t, u, v)$ is jointly convex (concave) in $(y, z, t, u, v)$, if $\partial_i L$, $i = 2, \ldots, 6$, exist and are continuous and verify the following condition:

\[
L(x, y + y_1, z + z_1, t + t_1, u + u_1, v + v_1) - L(x, y, z, t, u, v) \geq (\leq) \partial_2 L(\bullet) y_1 + \partial_3 L(\bullet) z_1 + \partial_4 L(\bullet) t_1 + \partial_5 L(\bullet) u_1 + \partial_6 L(\bullet) v_1
\]

for all $(x, y, z, t, u, v)$, $(x, y + y_1, z + z_1, t + t_1, u + u_1, v + v_1) \in [a, b] \times \mathbb{R}^5$, where $(\bullet) = (x, y, z, t, u, v)$. 
Theorem 3.3. Let $L(x, y, t, u, v)$ be jointly convex (concave) in $(y, z, t, u, v)$. If $y_0$ satisfies conditions (10)–(12), then $y_0$ is a global minimizer (maximizer) to problem (9).

Proof. We shall give the proof for the convex case. Since $L$ is jointly convex in $(y, z, t, u, v)$ for any admissible function $y_0 + h$, we have

$$J(y_0 + h) - J(y_0)$$

$$= \int_a^b \left[ L(x, y_0(x) + h(x), C_x D_x^\alpha(y_0(x) + h(x)), C_x D_x^\beta(y_0(x) + h(x)), y_0(a) + h(a), y_0(b) + h(b)) - L(x, y_0(x), C_x D_x^\alpha y_0(x), C_x D_x^\beta y_0(x), y_0(a), y_0(b)) \right] dx$$

$$\geq \int_a^b \left[ \partial_2 L(*) h(x) + \partial_3 L(*) C_x D_x^\alpha h(x) + \partial_4 L(*) C_x D_x^\beta h(x) + \partial_5 L(*) h(a) + \partial_6 L(*) h(b) \right] dx$$

where $(*) = (x, y_0(x), C_x D_x^\alpha y_0(x), C_x D_x^\beta y_0(x), y_0(a), y_0(b))$. We can now proceed analogously to the proof of Theorem 3.1. As the result we get

$$J(y_0 + h) - J(y_0) \geq \int_a^b \left[ \partial_2 L(*) + C_x D_x^\alpha \partial_3 L(*) + C_x D_x^\beta \partial_4 L(*) \right] h(x)$$

$$+ h(b) \left\{ \int_a^b \partial_5 L(*) dx - \left[ a I_1^{1-\alpha} \partial_3 L(*) - x I_1^{1-\alpha} \partial_3 L(*) \right]_{x=b} \right\}$$

$$+ h(a) \left\{ \int_a^b \partial_6 L(*) dx - \left[ x I_1^{1-\alpha} \partial_3 L(*) - a I_1^{1-\alpha} \partial_4 L(*) \right]_{x=a} \right\} = 0.$$

Since $y_0$ satisfy conditions (10)–(12), we obtain $J(y_0 + h) - J(y_0) \geq 0$. □

3.3 Example

We shall provide an example in order to illustrate our main results.

Example 1. Consider the following problem:

$$J(y) = \frac{1}{2} \int_0^1 \left[ \left( C_0 D_x^\alpha y(x) \right)^2 + \gamma y^2(0) + \lambda y(1) - 1 \right]^2 \, dx \rightarrow \min$$

where $\gamma, \lambda \in \mathbb{R}^+$. For this problem, the generalized Euler–Lagrange equation and the natural boundary conditions (see Theorem 3.1) are given, respectively, as

$$x D_1^\alpha \left( C_0 D_x^\alpha y(x) \right) = 0,$$

$$\int_0^1 \gamma y(0) dx = x I_1^{1-\alpha} \left( C_0 D_x^\alpha y(x) \right) |_{x=0},$$

$$\int_0^1 \lambda (y(1) - 1) dx = -x I_1^{1-\alpha} \left( C_0 D_x^\alpha y(x) \right) |_{x=1}.$$

Note that it is difficult to solve the above fractional equations. For $0 < \alpha < 1$ numerical method should be used. When $\alpha$ goes to 1 problem (17) tends to

$$J(y) = \frac{1}{2} \int_0^1 \left[ (y'(x))^2 + \gamma y^2(0) + \lambda (y(1) - 1)^2 \right] \, dx \rightarrow \min$$
and equations (18)–(20) could be replaced with

\begin{align}
  y''(x) &= 0, \\
  \gamma y(0) &= y'(0), \\
  \lambda (y(1) - 1) &= -y'(1).
\end{align}

Solving equations (22)–(24) we obtain that

\begin{equation}
  \bar{y}(x) = \frac{\gamma \lambda}{\gamma \lambda + \lambda + \gamma} x + \frac{\lambda}{\gamma \lambda + \lambda + \gamma}
\end{equation}

is a candidate for minimizer. Observe that problem (17) satisfies assumptions of Theorem 3.3. Therefore \( \bar{y} \) is a global minimizer to problem (21).

Acknowledgments

The authors were partially supported by the R&D unit CEOC, via FCT and the EC fund FEDER/POCI 2010, and the research project UT Austin/MAT/0057/2008. The first author was also supported by Białystok University of Technology, via a project of the Polish Ministry of Science and Higher Education “Wsparcie międzynarodowej mobilności naukowców”.

We are grateful to two anonymous reviewers for their comments.

References

[1] O. P. Agrawal, Fractional variational calculus and the transversality conditions, J. Phys. A 39 (2006), no. 33, 10375–10384.

[2] O. P. Agrawal, Generalized Euler-Lagrange equations and transversality conditions for FVPs in terms of the Caputo derivative, J. Vib. Control 13 (2007), no. 9-10, 1217–1237.

[3] O. P. Agrawal, Fractional variational calculus in terms of Riesz fractional derivatives, J. Phys. A 40 (2007), no. 24, 6287–6303.

[4] R. Almeida, A. B. Malinowska and D. F. M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string, J. Math. Phys. 51 (2010), DOI:10.1063/1.3319559 [arXiv:1001.2722]

[5] R. Almeida and D. F. M. Torres, Calculus of variations with fractional derivatives and fractional integrals, Appl. Math. Lett. 22 (2009), no. 12, 1816–1820. [arXiv:0907.1024]

[6] T. M. Atanacković, S. Konjik and S. Pilipović, Variational problems with fractional derivatives: Euler-Lagrange equations, J. Phys. A 41 (2008), no. 9, 095201, 12 pp.

[7] D. Baleanu and Om. P. Agrawal, Fractional Hamilton formalism within Caputo’s derivative, Czechoslovak J. Phys. 56 (2006), no. 10-11, 1087–1092.

[8] D. Baleanu, S. I. Muslih and E. M. Rabei, On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative, Nonlinear Dynam. 53 (2008), no. 1-2, 67–74.
[9] R. Brunetti, D. Guido and R. Longo, Modular structure and duality in conformal quantum field theory, Comm. Math. Phys. 156 (1993), no. 1, 201–219.

[10] P. A. F. Cruz, D. F. M. Torres and A. S. I. Zinober, A non-classical class of variational problems, Int. J. Mathematical Modelling and Numerical Optimisation 1 (2010), no. 3, 227–236. arXiv:0911.0353

[11] L. Debnath, Recent applications of fractional calculus to science and engineering, Int. J. Math. Math. Sci. 2003, no. 54, 3413–3442.

[12] K. Diethelm and A.D. Freed, On the Solution of Nonlinear Fractional-Order Differential Equations Used in the Modeling of Viscoelasticity. In F. Keil, W. Mackens, H. Voß, J. Werther (eds.): Scientific Computing in Chemical Engineering II. Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties. Springer-Verlag, Heidelberg, 1999, 217–224.

[13] R. A. El-Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order $(\alpha, \beta)$, Math. Methods Appl. Sci. 30 (2007), no. 15, 1931–1939. arXiv:math-ph/0702099

[14] R. A. El-Nabulsi and D. F. M. Torres, Fractional actionlike variational problems, J. Math. Phys. 49 (2008), no. 5, 053521, 7 pp. arXiv:0804.4500

[15] N. M. Fonseca Ferreira, F. B. Duarte, M. F. M. Lima, M. G. Marcos and J. A. Tenreiro Machado, Application of fractional calculus in the dynamical analysis and control of mechanical manipulators, Fract. Calc. Appl. Anal. 11 (2008), no. 1, 91–113.

[16] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether’s theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. 334 (2007), no. 2, 834–846. arXiv:math.OC/0701187

[17] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynam. 53 (2008), no. 3, 215–222. arXiv:0711.0609

[18] G. S. F. Frederico and D. F. M. Torres, Fractional Noether’s theorem in the Riesz-Caputo sense, Appl. Math. Comput. (2010), in press. DOI:10.1016/j.amc.2010.01.100 arXiv:1001.4507

[19] R. Hilfer, Fractional diffusion based on Riemann-Liouville fractional derivatives, J. Phys. Chem. B 104 (2000), no. 16, 3914–3917.

[20] R. Hilfer, Applications of fractional calculus in physics, World Sci. Publishing, River Edge, NJ, 2000.

[21] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam, 2006.

[22] M. Klimek, Stationarity-conservation laws for fractional differential equations with variable coefficients, J. Phys. A 35 (2002), no. 31, 6675–6693.

[23] V. V. Kulish and J. L. Lage, Application of fractional calculus to fluid mechanics, J. Fluids Eng. 124 (2002), no. 3, 803–806.
[24] R. Magin, Fractional calculus in bioengineering. Part 1-3, Critical Reviews in Bioengineering, 32 (2004).

[25] A. B. Malinowska and D. F. M. Torres, Natural boundary conditions in the calculus of variations, Math. Methods Appl. Sci., DOI:10.1002/mma.1289 arXiv:0812.0705

[26] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, J. Chem. Phys. 103 (1995), 7180–7186.

[27] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley, New York, 1993.

[28] S. I. Muslih and D. Baleanu, Fractional Euler-Lagrange equations of motion in fractional space, J. Vib. Control 13 (2007), no. 9-10, 1209–1216.

[29] K. B. Oldham and J. Spanier, The fractional calculus, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1974.

[30] A. Oustaloup, V. Pommier and P. Lanusse, Design of a fractional control using performance contours. Application to an electromechanical system, Fract. Calc. Appl. Anal. 6 (2003), no. 1, 1–24.

[31] I. Podlubny, Fractional differential equations, Academic Press, San Diego, CA, 1999.

[32] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, Phys. Rev. E (3) 53 (1996), no. 2, 1890–1899.

[33] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

[34] J. A. Tenreiro Machado and R. S. Barbosa, Introduction to the special issue on “Fractional Differentiation and its Applications”, J. Vib. Control 14 (2008), no. 9-10, 1253.

[35] G. M. Zaslavsky, Hamiltonian chaos and fractional dynamics, Reprint of the 2005 original, Oxford Univ. Press, Oxford, 2008.