INTERIOR ESTIMATES
FOR THE \( n \)-DIMENSIONAL ABREU'S EQUATION

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ABSTRACT. We study the Abreu’s equation in \( n \)-dimensional polytopes and derive interior estimates of solutions under the assumption of the uniform \( K \)-stability.

1. Introduction

The primary goal of this paper is to study a nonlinear fourth-order partial differential equation for an \( n \)-dimensional convex function \( u \) of the form

\[
\sum_{i,j=1}^{n} \frac{\partial^2 u_{ij}}{\partial \xi_i \partial \xi_j} = -A.
\]

Here, \( A \) is a given function and \( (u^{ij}) \) is the inverse of the Hessian matrix \( (u_{ij}) \).

The equation (1.1) was introduced by Abreu [1] in the study of the scalar curvature of toric varieties, in which case the domain of \( u \) is a bounded convex polytope in \( \mathbb{R}^n \) and \( A \) is the scalar curvature of toric varieties. Abreu proved that its solution \( u \) yields an extremal metric on toric varieties when \( A \) is a linear function in \( \xi \). Guillemin [15] observed that \( u \) is required to have prescribed boundary behavior near the boundary of the polytope. As is well known now, the solvability of the equation (1.1) is closely related to certain stability conditions.

Tian [19] first introduced \( K \)-stability and proved that it is a necessary condition for the existence of a Kähler-Einstein metric with positive scalar curvature. The sufficiency on Fano manifolds was recently established by Tian [22], and by Chen, Donaldson and Sun [9]. This provides an affirmative answer to the Yau-Tian-Donaldson conjecture on Fano manifolds.

Donaldson [10] generalized the notion of \( K \)-stability by giving an algebro-geometric definition of the Futaki invariant. In particular, he formulated \( K \)-stability for polytopes and conjectured that it is equivalent to the existence of Kähler metrics of constant scalar curvature (cscK metrics) on toric varieties.

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Donaldson \cite{Donaldson2004} also considered a stronger version of stability which we call *uniform K-stability* in this paper.

Under the assumption of the uniform K-stability, Donaldson \cite{Donaldson2004} derived interior estimates for solutions of the Abreu’s equation \eqref{AbreuEquation} satisfying Guillemin’s boundary conditions in polytopes in the case of dimension 2. Donaldson \cite{Donaldson2006} subsequently solved \eqref{AbreuEquation} when $A$ is constant in the 2-dimensional case, and hence proved the existence of metrics with constant scalar curvature on 2-dimensional toric varieties. Recently, Chen, Li and Sheng \cite{Chen2008}, \cite{Chen2009} generalized this result and proved the existence of metrics with prescribed scalar curvature on 2-dimensional toric varieties.

These works suggest that the uniform K-stability is the correct notion of the stability associated with the existence of metrics with prescribed scalar curvature on toric varieties. Indeed, Chen, Li and Sheng \cite{Chen2010} proved that the uniform K-stability is a *necessary* condition of the existence of solutions of \eqref{AbreuEquation} satisfying Guillemin’s boundary conditions. It is natural to ask whether such a uniform K-stability is a *sufficient* condition. Results by Donaldson \cite{Donaldson2006} and by Chen, Li and Sheng \cite{Chen2008}, \cite{Chen2009} answered this question affirmatively in the 2-dimensional case.

It remains an open problem to study the existence of metrics with prescribed scalar curvature in higher dimensional toric varieties.

Recently, there have been several results on the pure PDE aspects of the Abreu’s equation. Feng and Székelyhidi \cite{Feng2011} studied periodic solutions of the Abreu’s equation and proved the existence of a smooth periodic solution of \eqref{AbreuEquation} if $A$ is periodic and has a zero average. Chen, Li and Sheng \cite{Chen2008} studied the Abreu’s equation in bounded, smooth and strictly convex domains and proved the existence of smooth solutions of \eqref{AbreuEquation} for a class of prescribed boundary values.

In order to relate solutions of \eqref{AbreuEquation} to metrics with prescribed scalar curvature on toric varieties, the equation \eqref{AbreuEquation} is required to hold in polytopes and its solutions satisfy the Guillemin’s boundary conditions. This is a major difficulty associated with \eqref{AbreuEquation}. As the first step of studying the Abreu’s equation \eqref{AbreuEquation}, we discuss interior estimates of its solutions in polytopes satisfying Guillemin’s boundary conditions. Following Donaldson \cite{Donaldson2004}, we will keep the differential geometry in the background.

Before stating the main result in this paper, we first introduce some notations and terminologies.

Let $\Delta$ be a bounded open polytope in $\mathbb{R}^n$, $c_k$ be a constant and $h_k$ be an affine linear function in $\mathbb{R}^n$, $k = 1, \cdots, K$. Suppose that $\Delta$ is defined by linear inequalities $h_k(\xi) - c_k > 0$, for $k = 1, \cdots, K$, where each $h_k(\xi) - c_k = 0$ defines a facet of $\Delta$. Write $\delta_k(\xi) = h_k(\xi) - c_k$ and set

$$\begin{equation}
 v(\xi) = \sum_k \delta_k(\xi) \log \delta_k(\xi).
\end{equation}$$
This function was first introduced by Guillemin [15]. It defines a Kähler metric on the toric variety defined by $\Delta$ if $\Delta$ is a Delzant polytope.

We first introduce several classes of functions. Set

$$C = \{ u \in C(\bar{\Delta}) : u \text{ is convex on } \bar{\Delta} \text{ and smooth on } \Delta \},$$

$$S = \{ u \in C(\bar{\Delta}) : u \text{ is convex on } \bar{\Delta} \text{ and } u - v \text{ is smooth on } \bar{\Delta} \},$$

where $v$ is given in (1.2). For a fixed point $p_0 \in \Delta$, we consider

$$C_{p_0} = \{ u \in C : u \geq u(p_0) = 0 \},$$

$$S_{p_0} = \{ u \in S : u \geq u(p_0) = 0 \}.$$

We say functions in $C_{p_0}$ and $S_{p_0}$ are normalized at $p_0$.

Let $A$ be a smooth function on $\bar{\Delta}$. Consider the functional

$$(1.3) \quad F_A(u) = -\int_{\Delta} \log \det(u_{ij}) d\mu + \mathcal{L}_A(u),$$

where

$$(1.4) \quad \mathcal{L}_A(u) = \int_{\partial \Delta} u d\sigma - \int_{\Delta} Aud\mu.$$ When $A$ is a constant, $F_A$ is known to be the Mabuchi functional and $\mathcal{L}_A$ is closely related to the Futaki invariants.

The Euler-Lagrangian equation for $F_A$ is

$$\sum_{i,j} \frac{\partial^2 u_{ij}}{\partial \xi_i \partial \xi_j} = -A.$$ This is the Abreu’s equation (1.1). It is known that, if $u \in S$ satisfies the equation (1.1), then $u$ is an absolute minimizer for $F_A$ on $S$.

**Definition 1.1.** Let $A$ be a smooth function on $\bar{\Delta}$. Then, $(\Delta, A)$ is called uniformly $K$-stable if the functional $\mathcal{L}_A$ vanishes on affine-linear functions and there exists a constant $\lambda > 0$ such that, for any $u \in C_{p_0}$,

$$(1.5) \quad \mathcal{L}_A(u) \geq \lambda \int_{\partial \Delta} u d\sigma.$$ We also say that $\Delta$ is $(A, \lambda)$-stable.

The conditions in Definition 1.1 are exactly the contents of Condition 1 [11], introduced by Donaldson.

The following interior estimate is the main result in this paper.

**Theorem 1.2.** Let $\Delta$ be a bounded open polytope in $\mathbb{R}^n$ and $A$ be a smooth function on $\bar{\Delta}$. Suppose $(\Delta, A)$ is uniformly $K$-stable and $u$ is a solution in $S_{p_0}$ of the Abreu’s equation (1.1). Then, for any $\Omega \subset \subset \Delta$, any nonnegative integer $k$ and any constant $\alpha \in (0, 1)$,

$$\|u\|_{C^{k+3, \alpha}(\Omega)} \leq C\|A\|_{C^k(\Delta)},$$
where $C$ is a positive constant depending only on $n$, $k$, $\alpha$, $\Omega$, and $\lambda$ in the uniform $K$-stability.

As we mentioned earlier, Donaldson [11] proved Theorem 1.2 for $n = 2$. A crucial step in his proof is a derivation of lower and upper bounds of determinants of the Hessian of solutions. Donaldson’s lower bound holds for all dimensions. However, his upper bound is limited to dimension 2. A major contribution in this paper is a new upper bound of determinants of the Hessian in all dimensions. This new upper bound relates to the Legendre transforms of solutions. Once we have established upper and lower bounds of determinants of the Hessian, we can prove Theorem 1.2 with the help of estimates for linearized Monge-Ampère equations due to Caffarelli and Gutiérrez [3] and estimates for Monge-Ampère equations due to Caffarelli [2]. Legendre transforms play an important role in our arguments. In fact, we establish the interior estimates for the Legendre transform of $u$, instead of for $u$ directly.

This paper is organized as follows. In Section 2 we derive an equivalent equation for the Legendre transforms. In Section 3 we derive an upper bound of the determinants of the Hessian of solutions satisfying the Guillemin’s boundary conditions. Such an upper bound plays an important role in this paper. Finally in Section 4 we prove Theorem 1.2.

2. Preliminaries

In this section, we write the Abreu’s equation (1.1) in its equivalent form for Legendre transforms.

Let $f = f(x)$ be a smooth and strictly convex function defined in a convex domain $\Omega \subset \mathbb{R}^n$. As $f$ is strictly convex, $G_f$ defined by

$$G_f = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j = \sum_{i,j} f_{ij} dx_i dx_j$$

is a Riemannian metric in $\Omega$. The gradient of $f$ defines a (normal) map $\nabla^f$ from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$\xi = (\xi_1, \ldots, \xi_n) = \nabla^f(x) = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

The function $u$ on $\mathbb{R}^n$

$$u(\xi) = x \cdot \xi - f(x)$$

is called the Legendre transform of $f$. We write

$$u = L(f), \quad \Omega^* = \nabla^f(\Omega) \subset \mathbb{R}^n.$$
Conversely, $f = L(u)$. It is well-known that $u(\xi)$ is a smooth and strictly convex function. Corresponding to $u$, we have the metric

$$G_u = \sum_{i,j} \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} d\xi_i d\xi_j = \sum_{i,j} u_{ij} d\xi_i d\xi_j.$$ 

Under the normal map $\nabla f$, we have

$$\frac{\partial \xi_i}{\partial x_k} = \frac{\partial^2 f}{\partial x_i \partial x_k},$$

and

$$\det \left( \frac{\partial^2 f}{\partial x_i \partial x_k} \right) = \det \left( \frac{\partial^2 u}{\partial \xi_i \partial \xi_k} \right)^{-1}.$$ 

Then,

$$(\nabla f)^* (G_u) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j = G_f,$$

i.e., $\nabla f : (\Omega, G_f) \to (\Omega^*, G_u)$ is locally isometric.

Set

$$(2.1) \quad \rho = \left[ \det(f_{ij}) \right]^{-\frac{1}{n+2}},$$

and

$$(2.2) \quad \Phi = \frac{\|\nabla \rho\|_2^2}{\rho^2}.$$ 

Now we derive a formula for the Laplace-Beltrami operator $\Delta$ in terms of $x_1, \ldots, x_n$ and $f$. Recall that

$$\Delta = \frac{1}{\sqrt{\det(f_{kl})}} \sum_{i,j} \frac{\partial}{\partial x_i} \left( f^{ij} \sqrt{\det(f_{kl})} \frac{\partial}{\partial x_j} \right),$$

where $(f^{ij})$ denotes the inverse matrix of $(f_{ij})$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. By a direct calculation, we get

$$(2.3) \quad \Delta = \sum_{i,j} f^{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{n+2}{2\rho} \sum_{i,j} f^{ij} \frac{\partial \rho}{\partial x_j} \frac{\partial}{\partial x_i} + \sum_{i,j} \frac{\partial f^{ij}}{\partial x_i} \frac{\partial}{\partial x_j}.$$ 

Differentiating the equality $\sum f^{ik} f_{kj} = \delta_j^i$, we have

$$\sum_{i,k} \frac{\partial f^{ik}}{\partial x_i} f_{kj} = - \sum_{i,k} f^{ik} \frac{\partial f_{kj}}{\partial x_i} = \frac{(n+2)}{\rho} \frac{\partial \rho}{\partial x_j}.$$ 

It follows that

$$(2.4) \quad \sum_{i} \frac{\partial f^{ik}}{\partial x_i} = \frac{(n+2)}{\rho} \sum_{j} f^{jk} \frac{\partial \rho}{\partial x_j}.$$
Inserting (2.4) into (2.3), we obtain (cf. [17])

\[
\Delta = \sum_{i,j} f^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + n + 2 \rho \sum_{i,j} f^{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial}{\partial x^j}.
\]

In particular,

\[
\Delta f = n + \frac{n + 2}{2 \rho} \langle \nabla \rho, \nabla f \rangle,
\]

and

\[
\Delta \left( \sum_k x_k^2 \right) = 2 \sum_k f^{kk} + \frac{n + 2}{2 \rho} \left\langle \nabla \rho, \nabla \sum_k x_k^2 \right\rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is with respect to the metric \( G_f \). Similarly in terms of coordinates \( \xi_1, \ldots, \xi_n \), we have

\[
\Delta = \sum_{i,j} u^{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j} - \frac{n + 2}{2 \rho} \sum_{i,j} u^{ij} \frac{\partial \rho}{\partial \xi_j} \frac{\partial}{\partial \xi_i},
\]

and hence,

\[
\Delta u = n - \frac{n + 2}{2 \rho} \langle \nabla \rho, \nabla u \rangle,
\]

and

\[
\Delta \left( \sum_k \xi_k^2 \right) = 2 \sum_i u^{ii} - \frac{n + 2}{2 \rho} \left\langle \nabla \rho, \nabla \sum_k \xi_k^2 \right\rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is with respect to the metric \( G_u \).

**Lemma 2.1.** The Abreu’s equation (1.1) is equivalent to any of the following two equations:

\[
\sum_{i,j} f^{ij} \frac{\partial^2}{\partial x^i \partial x^j} (\log \det (f_{kl})) = -A,
\]

and

\[
\Delta \rho = \frac{n + 4}{2} \frac{\| \nabla \rho \|^2}{\rho} + \frac{\rho A}{n + 2},
\]

where and later \( \Delta \) and \( \| \cdot \| \) are with respect to the metric \( G_f \).

**Proof.** The equivalence between (1.1) and (2.8) is well-known. We now prove the equivalence between (2.8) and (2.9). Note that

\[
- \frac{\partial^2}{\partial x_i \partial x_j} (\ln \det (f_{kl})) = (n + 2) \left( \frac{\rho_{ij}}{\rho} - \frac{\rho_i \rho_j}{\rho^2} \right),
\]

where \( \rho_i = \frac{\partial \rho}{\partial x_i} \) and \( \rho_{ij} = \frac{\partial^2 \rho}{\partial x_i \partial x_j} \). It is easy to see that (2.8) is equivalent to

\[
\frac{\rho A}{n + 2} = \sum_{i,j} f^{ij} \rho_{ij} - \frac{1}{\rho} \sum_{i,j} f^{ij} \rho_i \rho_j,
\]
which is equivalent to (2.9) by (2.5).

3. Estimates of the Determinant

Let $u \in S$ be a solution of the Abreu’s equation (1.1). In this section, we derive a global upper bound of the determinant of the Hessian of $u$. Recall that the classes $S$ and $S_{po}$ were introduced in Section 1.

The following two lemmas were proved by Donaldson [11]. Refer to Theorem 5 and Theorem 6 [11].

**Lemma 3.1.** Suppose that $u \in S$ satisfies the Abreu’s equation (1.1). Then,

$$\det(u_{ij}) \geq C_1 \text{ in } \Delta,$$

where $C_1$ is a positive constant depending only on $n$, $\max_{\Delta} |A|$ and $\text{diam}(\Delta)$.

In fact, we can take

$$C_1 = (4n^{-1} \max_{\Delta} |A| \text{diam}(\Delta)^2)^{-n}.$$

We point out that we can take $\alpha = 0$ in Theorem 5 [11].

**Lemma 3.2.** Suppose that $u \in S_{po}$ satisfies the Abreu’s equation (1.1). Assume that the section

$$\tilde{S}_u(p_o, C) = \{\xi \in \Delta : u(\xi) \leq C\}$$

is compact and that there is a constant $b > 0$ such that

$$\sum_{k=1}^{n} \left( \frac{\partial u}{\partial \xi_k} \right)^2 \leq b \text{ on } \tilde{S}_u(p_o, C).$$

Then,

$$\det(u_{ij}) \leq C_2 \text{ in } S_u(p_o, C/2),$$

where $C_2$ is a positive constant depending on $n$, $C$ and $b$.

Here, we write Theorem 6 [11] in the form of Lemma 3.2 for convenience of applications in this paper.

We point out that Theorem 6 [11] holds for all dimensions. However, in Donaldson’s application of this result, additional information on “modulus of convexity” of $u$ is required. Such a modulus of convexity was verified only for the 2-dimensional case. Refer to Section 5 [11] for details. It is not clear whether the required modulus of convexity holds for higher dimensions.

In the following, we derive a global estimate for the upper bound of $\det(D^2 u)$, which plays a key role in this paper. This new upper bound relates to the Legendre transforms of solutions.

For any point $p$ on $\partial \Delta$, there is an affine coordinate $\{\xi_1, ..., \xi_n\}$, such that, for some $1 \leq m \leq n$, a neighborhood $U \subset \Delta$ of $p$ is defined by $m$ inequalities

$$\xi_1 \geq 0, \quad ..., \quad \xi_m \geq 0,$$
with \( \xi(p) = 0 \). Then, \( v \) in (1.2) has the form
\[
v = \sum_{i=1}^m \xi_i \log \xi_i + \alpha(\xi),
\]
where \( \alpha \) is a smooth function in \( \bar{U} \). By Proposition 2 in [11], we have the following result.

**Lemma 3.3.** There holds
\[
\det(v_{ij}) = \left[ \xi_1 \xi_2 \ldots \xi_m \beta(\xi) \right]^{-1} \quad \text{in } \Delta,
\]
where \( \beta(\xi) \) is smooth up to the boundary and \( \beta(0) = 1 \).

Denote by \( d_E(p, \partial \Delta) \) the Euclidean distance from \( p \) to \( \partial \Delta \). By Lemma 3.3, we have
\[
(3.1) \quad \det(v_{ij}) \leq \frac{C}{[d_E(p, \partial \Delta)]^n} \quad \text{in } \Delta,
\]
where \( C \) is a positive constant.

Recall that \( p_o \in \Delta \) is the point we fixed for \( S_{p_o} \). Now we choose coordinates \( \xi_1, \ldots, \xi_n \) such that \( \xi(p_o) = 0 \). Set
\[
x_i = \frac{\partial u}{\partial \xi_i}, \quad f = \sum_i x_i \xi_i - u.
\]

**Lemma 3.4.** Suppose that \( u \in S_{p_o} \) satisfies the Abreu’s equation (1.1). Assume, for some positive constants \( d \) and \( b \),
\[
\frac{1 + \sum x_i^2}{(d + f)^2} \leq b \quad \text{in } \mathbb{R}^n.
\]
Then,
\[
\exp \left\{ -C_3 f \right\} \frac{\det(u_{ij})}{(d + f)^n} \leq C_4 \quad \text{in } \Delta,
\]
where \( C_3 \) is a positive constant depending only on \( n \) and \( \Delta \), and \( C_4 \) is a positive constant depending only on \( n, d, b \) and \( \max_{\Delta} |A| \).

**Proof.** Let \( v \) be given as in (1.2). By adding a linear function, we assume that \( v \) is also normalized at \( p_o \). Denote \( g = L(v) \). By (3.1), it is straightforward to check that there exists a positive constant \( C_1 \) such that
\[
\det(v_{ij})e^{-C_1 g} \to 0 \quad \text{as } p \to \partial \Delta.
\]
Since \( u = v + \phi \) for some \( \phi \in C^\infty(\bar{\Delta}) \), then
\[
(3.2) \quad \det(u_{ij})e^{-C_1 f} \to 0 \quad \text{as } p \to \partial \Delta.
\]
Consider, for some constant \( \varepsilon \) to be determined,
\[
F = \exp \left\{ -C_2 f + \varepsilon \frac{1 + \sum x_i^2}{(d + f)^2} \right\} \frac{\rho}{(d + f)^{\frac{2n}{n+2}}},
\]
where \( C_2 = \frac{C_1}{|\nabla f|^2} \) and \( \rho \) is defined in (2.1). By (3.2), \( F \to 0 \) as \( p \in \partial \Delta \). Assume \( F \) attains its maximum at an interior point \( p^* \). Then at \( p^* \), we have

\[
(3.3) \quad \frac{\rho_i}{\rho} - C_2 f, i - \frac{2n}{n + 2} \frac{f, i}{d + f} + \varepsilon \frac{1 + \sum x_i}{1 + \sum x_i^2} \left[ \frac{(\sum x_i^2), i}{1 + \sum x_i^2} - 2 \frac{f, i}{d + f} \right] = 0,
\]

and

\[
(3.4) \quad \frac{n + 2}{2} \Phi + \frac{A}{n + 2} - C_2 \Delta f - \frac{2n}{n + 2} \frac{\Delta f}{d + f} + \frac{2n}{n + 2} \frac{||\nabla f||^2}{d + f^2} + \varepsilon \frac{1 + \sum x_i^2}{(d + f)^2} \left[ \frac{\Delta (\sum x_i^2)}{1 + \sum x_i^2} - \frac{||\nabla (\sum x_i^2)||^2}{(d + f)^2} - \frac{2 \Delta f}{d + f} + \frac{2||\nabla f||^2}{(d + f)^2} \right].
\]

where we used (2.9) for \( \Delta \rho \). By (2.6) and (2.7), we get

\[
\frac{1 + \sum x_i^2}{(d + f)^2} \left[ \frac{2 \sum f, i^2}{1 + \sum x_i^2} - \frac{4(\nabla \sum x_i^2, \nabla f)}{(d + f)^2} \right] - \frac{n + 2}{2} \frac{\rho_i}{\rho} \left[ \frac{C_2 f, i}{n + 2} + \frac{2n}{n + 2} \frac{f, i}{d + f} \right]
\]

\[
+ \frac{n + 2}{2} \Phi + \frac{2n}{n + 2} \frac{||\nabla f||^2}{d + f^2} - \frac{2n^2}{n + 2} \frac{1}{d + f^2} + \frac{A}{n + 2} - C_2 n \leq 0.
\]

By inserting (3.3) into (3.5) and by the definition of \( \Phi \) in (2.2), we obtain

\[
\frac{1 + \sum x_i^2}{(d + f)^2} \left[ \frac{2 \sum f, i^2}{1 + \sum x_i^2} - \frac{4(\nabla \sum x_i^2, \nabla f)}{(d + f)^2} \right] + \frac{2n}{n + 2} \frac{||\nabla f||^2}{d + f^2} - \frac{2n^2}{n + 2} \frac{1}{d + f^2} + \frac{A}{n + 2} - C_2 n \leq 0.
\]

By the Schwarz inequality, we have

\[
\frac{4(\nabla \sum x_i^2, \nabla f)}{(1 + \sum x_i^2)(d + f)} \leq \frac{||\nabla \sum x_i^2||^2}{4(1 + \sum x_i^2)^2} + \frac{16||\nabla f||^2}{(d + f)^2}.
\]

Hence,

\[
\frac{4(\nabla \sum x_i^2, \nabla f)}{(1 + \sum x_i^2)(d + f)} \leq \frac{\sum f, i^2}{(1 + \sum x_i^2)^2} + \frac{16||\nabla f||^2}{(d + f)^2}.
\]

Combining (3.6) and (3.7) yields

\[
\frac{1 + \sum x_i^2}{(d + f)^2} \left[ \frac{\sum f, i^2}{1 + \sum x_i^2} - \frac{2n}{d + f} \frac{1 - 10||\nabla f||^2}{(d + f)^2} \right] + \frac{2n}{n + 2} \frac{||\nabla f||^2}{d + f^2} - \frac{2n^2}{n + 2} \frac{1}{d + f^2} + \frac{A}{n + 2} - C_2 n \leq 0.
\]
By choosing $\varepsilon > 0$ such that $10\varepsilon b \leq 1$, we have
\[
\varepsilon \sum_{i,j} f^{ij} \leq \frac{A}{n+2} - C_3 \leq 0.
\]
By the relation between the geometric mean and the arithmetic mean, we get
\[
\frac{\rho}{(d+f)^{\frac{2n}{n+2}}} = \frac{(\det(f^{ij}))^{\frac{1}{n+2}}}{(d+f)^{\frac{2n}{n+2}}} \leq C_4.
\]
Therefore, $F(p^*) \leq C_5$, and hence $F \leq C_5$ everywhere. The definition of $F$ implies
\[
\exp\{-C_2 f\} \frac{\rho}{(d+f)^{\frac{2n}{n+2}}} \leq C_5.
\]
This is the desired estimate. □

4. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 1.2. We first introduce a notation.

For any $u \in C_{p_0}$, we set
\[
\|u\|_b = \int_{\partial \Delta} u d\sigma.
\]  
An important consequence of the uniform $K$-stability is the following result. See Corollary 2 [11].

Lemma 4.1. Suppose $(\Delta, A)$ is uniformly $K$-stable and $u \in S_{p_0}$ is a solution of the Abreu’s equation (1.1). Then,
\[
\|u\|_b \leq C,
\]
where $C$ is a positive constant depending only on $n$ and $\lambda$.

Donaldson pointed out that Lemma 4.1 implies interior gradient estimates of solutions of the Abreu’s equation. See Corollary 3 [11]. As a consequence, a sequence of normalized solutions $\{u^{(k)}\} \subset S_{p_0}$ with uniformly bounded $\|u^{(k)}\|_b$ is locally uniformly convergent to a convex function $u$ in $\Delta$. (See Section 5 [10])

Now, we are ready to prove the following result.

Theorem 4.2. Suppose that $(\Delta, A)$ is uniformly $K$-stable and that $\{A^{(k)}\}$ is a sequence of smooth functions in $\Delta$ such that $A^{(k)}$ converges to $A$ smoothly in $\Delta$. Assume $u^{(k)} \in S_{p_0}$ is a sequence of solutions of the Abreu’s equation
\[
\sum_{i,j} \frac{\partial^2 (u^{(k)})^{ij}}{\partial \xi_i \partial \xi_j} = -A^{(k)} \quad \text{in} \; \Delta.
\]
Then there is a subsequence, still denoted by $u^{(k)}$, such that $u^{(k)}$ converges to $u$ smoothly in any compact set $\Omega \subset \Delta$, for some smooth and strictly convex function $u$ in $\Delta$. 

We note that Theorem 4.2 is equivalent to Theorem 1.2.

Proof of Theorem 4.2. Since $(\Delta, A)$ is uniformly $K$-stable and $A^{(k)}$ converges to $A$ smoothly in $\Delta$, then $(\Delta, A_k)$ is uniformly $K$-stable for large $k$, i.e., $\Delta$ is $(A_k, \lambda)$-stable for some constant $\lambda > 0$ independent of $k$. Since $u^{(k)}$ satisfies the Abreu’s equation (4.2), then

$$L_{A_k}(u^{(k)}) = \int_{\Delta} \sum_{i,j} (u^{(k)})_{ij}(u^{(k)})_{ij} d\mu = n\text{Area}(\Delta),$$

and hence,

$$\|u^{(k)}\|_b \leq \lambda^{-1} L_{A_k}(u^{(k)}) = \lambda^{-1} n\text{Area}(\Delta).$$

It follows that $u^{(k)}$ locally and uniformly converges to a convex function $u$ in $\Delta$.

Claim. For any point $\xi \in \Delta$ and any $B_\delta(\xi) \subset \Delta$, there exists a point $\xi_o \in B_\delta(\xi)$ such that $u$ has second derivatives and is strictly convex at $\xi_o$. Here, $B_\delta(\xi)$ denotes the Euclidean ball centered at $\xi$ with radius $\delta$.

The proof of the claim is the same as in [4]. For convenience, we present the proof here. Since $u$ is convex, it has second order derivatives almost everywhere. Let $G \subset B_\delta(\xi)$ be the set where $u$ has second order derivatives. Then, $|B_\delta(\xi) \setminus G| = 0$. Let $O$ be an open subset of $B_\delta(\xi)$ such that $B_\delta(\xi) \setminus G \subset O$ with $|O| \leq \epsilon$. We choose $\epsilon$ so small that

$$|B_\delta(\xi) \setminus O| > \frac{1}{2}|B_\delta(\xi)|.$$

By Lemma 3.1 and the weak convergence of Monge-Ampère measures, we have

$$(4.3) \quad \int_{B_\delta(\xi) \setminus O} \det(u_{kl})d\mu > \frac{1}{2} C_1 |B_\delta(\xi)|.$$

Hence, there exists a point $\xi_o \in B_\delta(\xi) \setminus O$ such that

$$\det(u_{kl})(\xi_o) \geq \frac{C_1 |B_\delta(\xi)|}{2|B_\delta(\xi) \setminus O|}.$$

The claim is proved.

We now choose coordinates such that $\xi_o = 0$. By adding linear functions, we assume that all $u^{(k)}$ and $u$ are normalized at 0. Since $u$ is strictly convex at 0, there exist constants $\epsilon' > 0$, $d_2 > d_1 > 0$ and $b' > 0$, independent of $k$, such that, for large $k$,

$$B_{d_1}(0) \subset \tilde{S}_{u^{(k)}}(0, \epsilon') \subset B_{d_2}(0) \subset \Delta,$$

and

$$\sum_i \left( \frac{\partial u^{(k)}}{\partial \xi_i} \right)^2 \leq b' \quad \text{in} \quad S_{u^{(k)}}(0, \epsilon').$$
By Lemma 3.1 and Lemma 3.2 we have
\begin{equation}
C_1 \leq \det(u_{ij}^{(k)}) \leq C_2 \quad \text{in } S_{u(k)}(0, \frac{1}{2} \epsilon'),
\end{equation}
where $C_1 < C_2$ are positive constants independent of $k$.

By an estimate due to Caffarelli and Gutiérrez [3], there is a uniform interior $C^{\alpha}$-bound of $\det(u_{ij}^{(k)})$. Caffarelli and Gutiérrez originally proved this result for homogeneous linearized Monge-Ampère equations. Trudinger and Wang [23] pointed out that such a result can be extended to the Abreu’s equation (4.2) if $A_k \in L^\infty(\Delta)$, under the assumption (4.4). (See also [11].)

By $C^{2,\alpha}$ estimates for Monge-Ampère equation due to Caffarelli [2], we have, for any $\Omega^* \subset B_{d_1}(0)$,
\begin{equation}
\|u^{(k)}\|_{C^{2,\alpha}(\Omega^*)} \leq C_2.
\end{equation}
Then, we employ the Schauder estimate to conclude that $\{u^{(k)}\}$ converges smoothly to $u$. Therefore, $u$ is a smooth and strictly convex function in $S_{u}(0, \epsilon'/2)$.

Let $f^{(k)}$ be the Legendre transform of $u^{(k)}$. Then, $\{f^{(k)}\}$ locally uniformly converges to a convex function $f$ defined in the whole $\mathbb{R}^n$. Furthermore, in a neighborhood of 0, $f$ is a smooth and strictly convex function such that its Legendre transform $u$ satisfies the Abreu’s equation.

By the convexity of $f^{(k)}$ and the local and uniform convergence of $\{f^{(k)}\}$ to $f$, we conclude, for any $k$,
\begin{equation}
\frac{1 + \sum_i x_i^2}{(d + f^{(k)})^2} \leq b \quad \text{in } \mathbb{R}^n,
\end{equation}
and, for any $C > 1$,
\begin{equation}
B_r(0) \subset S_{f^{(k)}}(0, C) \subset B_{RC}(0),
\end{equation}
for some positive constants $d$, $b$, $r$ and $RC = R(C) > 0$.

By Lemma 3.1 and Lemma 3.4 we have
\begin{equation}
\exp\{-C_3 C\} \frac{1}{(d + C)^{2n}} \leq \det(f_{ij}^{(k)}) \leq C_1.
\end{equation}
We note that each $f^{(k)}$ satisfies (2.3), with $f$ and $A$ there replaced by $f^{(k)}$ and $A^{(k)}$. By the $C^\alpha$-estimates due to Caffarelli and Gutiérrez and the $C^{2,\alpha}$-estimates due to Caffarelli as above, we conclude that $\{f^{(k)}\}$ uniformly and smoothly converges to $f$ in $S_f(0, C/2)$. Since $C$ is arbitrary, $f$ is a smooth and strictly convex function in $\mathbb{R}^n$, and the sequence $\{f^{(k)}\}$ locally and smoothly converges to $f$. By Legendre transforms, we obtain that $u$ is a smooth and strictly convex function in $\Delta$ and that the sequence $\{u^{(k)}\}$ locally and smoothly converges to $u$. This completes the proof of Theorem 4.2.
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