Models for Quantum Memory Channels

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Abstract. We discuss a couple of models aimed to describe memory effects in quantum
channels. We start from a rather intuitive model and arrive to a more abstract one which
turns out to be more general.

1. Introduction
Any quantum information process, be it storage or transfer, can be represented as a quantum
channel; that is a completely positive and trace preserving map that transforms states (density
operators) on the sender’s end of the channel into states on the receiver’s end [1].

The majority of research on quantum channels has focused on memoryless channels, which
are characterized by the requirement that successive channels inputs are acted on independently.
However, in many real world applications the assumption of having uncorrelated noise channels
cannot be justified, and memory effects need to be taken into account.

Examples of quantum channels which naturally acquire a memory are common in quantum
information processing. Recently an unmodulated spin chain has been proposed as a model for
short distance quantum communication [2]. In such a scheme the state to be communicate over
the channel is placed in one end of the chain, propagates for a specific amount of time, and is
then received at the other end. It is generally assumed that a reset of the spin chain occurs after
each signal (resulting in a memoryless channel). However, a continuous operation without reset
leads to higher transmission rates and corresponds to quantum memory channel. Another model
of a quantum channel with memory is the so called ‘micromaser’ [3]. In such a device excited
atoms interact with the photon field inside a optical cavity. If the photons inside the cavity have
sufficiently long lifetime, atoms entering the cavity will feel the effect of the preceding atoms,
introducing correlations between consecutive signal states.

There have been some important results obtained for quantum channels with correlated noise
operators, or more general quantum channels [4]. An intriguing aspect of such channels is the
argued possibility to enhance the classical capacity by means of entangled inputs, contrarily to
what happen in the memoryless case [1].

Recently, we have proposed a couple of models for quantum channels with memory that can
consistently define quantum channels with correlated noise [5, 6]. In this paper we analyze and
compare these models. Let us start by reviewing some basic concepts.
1.1. Memoryless Channels

A quantum channel is defined as a completely positive, trace preserving map from the set of density operators to itself. Any such map may be represented as a unitary operation between the system state and an environment with a known initial state [7].

Memoryless quantum channels act on each input state independently of the previous input or output states. For a single channel use the output state is given by,

$$\mathcal{L} \rho_Q = \text{Tr}_E \left[ U_{QE} (\rho_Q \otimes |0\rangle_E \langle 0|) U_{QE}^\dagger \right]$$

with $\rho_Q$ the input state, $|0\rangle_E \langle 0|$ the initial state of the environment, $U_{QE}$ a unitary operation between input $Q$ and environment $E$. For a sequence of transmissions through a memoryless channel, the output state is given by

$$\mathcal{L}^{(n)} \rho_Q = \text{Tr}_E \left[ U_{Q_n E_n} ... U_{Q_1 E_1} (\rho_Q \otimes \sigma_E) U_{Q_1 E_1}^\dagger ... U_{Q_n E_n}^\dagger \right]$$

where the state $\rho_Q$ now represents a (possibly entangled) input state across the $n$ channel uses, the unitary operations $U_{Q_k E_k}$ are all identical, and the environment state is a product state $\sigma_E \equiv |0\rangle_E \langle 0| \otimes ... \otimes |0\rangle_{E_n} \langle 0|$. Thus we may write $\mathcal{L}^{(n)} \rho_Q \equiv \mathcal{L}^\otimes n \rho_Q$. Furthermore, by virtue of Ref.[7] we know that the action of the channel can also be ‘decomposed’ as

$$\mathcal{L} \rho_Q = \sum_i A_i \rho_Q A_i^\dagger$$

where $A_i \equiv E \langle i | U | 0 \rangle_E$ are operators matrix elements between the initial environment state and orthonormal environment vectors $|i\rangle_E$ of the basis used for the partial trace over $E$ (Kraus operators). They satisfy the condition $\sum_i A_i^\dagger A_i = I$. An interesting class of channels is the one for which $A_i \mapsto \sqrt{p_i} A_i$ where $p_i$ represents a probability. Then, for $n$ uses of the channel we have

$$\mathcal{L}^{(n)} \rho_Q = \sum_{i_1, \ldots, i_n} p_{i_n} p_{i_{n-1}} \cdots p_{i_1} \times (A_{i_n} \otimes \cdots \otimes A_{i_1}) \rho_Q \left( A_{i_n}^\dagger \otimes \cdots \otimes A_{i_1}^\dagger \right)$$

with $p_{i_k}$ independent probabilities. From the Kraus representation theorem [7] we also know that for any block of length $n$ then any channel acting on the $n$ states may be modeled with an environment of dimension at most $d^{2n}$, for $d$ the dimension of the channel. However, the unitary operation may not be factorable into a product of operators acting in the form of Eq.(2).

2. An intuitive memory model

A memory channel is characterized by nontrivial correlations between the environment actions on the different channel uses which cannot be accounted for by Eq.(2). We may simply model this situation by replacing the separable state $\sigma_E$ with a correlated state given by

$$\omega_E = \Omega_E \sigma_E \Omega_E^\dagger$$

where $\Omega_E$ is a unitary introducing correlations between $|0\rangle_{E_k}$. For example we can consider $\Omega_E$ as generating some entanglement in the environment

$$\Omega_E = \exp \left[ \sum_{k,k'} \left( \xi_{k,k'}^* \sigma_k \sigma_{k'} - \xi_{k,k'} \sigma_k^\dagger \sigma_{k'}^\dagger \right) \right]$$
where \( \sigma_k, \sigma_k^\dagger \) are ladder operators in \( E_k \) and \( \xi_{k,k'} \in \mathbb{C} \).

As a consequence of Eq.(5), the unitary representation of the channel becomes

\[
\mathcal{L}(n) \rho_Q = \text{Tr}_E \left[ U_Q^n E_n \cdots U_{Q1} E_1 (\rho_Q \otimes \omega_E) U_{Q1}^\dagger E_1 \cdots U_{Qn}^\dagger E_n \right] \tag{7}
\]

Figure 1 illustrates the action of the unitary operators on the input and correlated environment states representing the channel. In such a case it will generally be \( \mathcal{L}(n) \rho_Q \neq \mathcal{L}^\otimes n \rho_Q \).

This model has first been presented by considering continuous alphabets (infinite dimensional systems, \( d = \infty \)) [6]. Specifically the analogous of Eq.(6) has been studied and the possibility of improving the classical information transmission rate by means of entangled inputs shown [8].

3. The memory aside
More generally we can consider the memory as a system itself, aside the environment and the channel input. Then, the model of a quantum memory channel is where each state going through the channel acts with a unitary interaction on the same channel memory state, as well as an independent environment. The backaction of the channel state on the message state therefore gives a memory to the channel. The general model thus includes a channel memory \( M \), and the independent environments \( E_k \) for each input. Hence,

\[
\mathcal{L}(n) \rho_Q = \text{Tr}_M \left[ U_Q^n M_{E_n} \cdots U_{Q1} M_{E_1} (\rho_Q \otimes \sigma_E \otimes \mu_M) U_{Q1}^\dagger M_{E_1} \cdots U_{Qn}^\dagger M_{E_n} \right] \tag{8}
\]

where \( \mu_M \) is the initial memory state, \( \rho_Q \) and \( \mathcal{L}(n) \rho_Q \) are the input and the output states of the channel, respectively, and the trace is over all environment states and memory state. Figure 2 illustrates the action of the unitary operators on the input, memory and environment states representing the channel.

If the unitaries factor into independent unitaries acting on the memory and the combined state and environment, that is, \( U_{Qk} M E_k = U_{Qk} E_k U_M \), then the memory traces out and we have a memoryless channel. If instead the unitaries reduce to \( U_{Qk} M \), we can call it a perfect memory channel, as no information is lost to the environment [9].

Within this model, it will generally be \( \mathcal{L}(n) \rho_Q \neq \mathcal{L}^\otimes n \rho_Q \), however \( \mathcal{L}(n) \rho_Q = \text{Tr}_M \left[ \mathcal{L}^\otimes n (\rho_Q \otimes \mu_M) \right] \), that is to say the channel can be expressed as tensor product of a map acting on a larger space (which includes the memory).

An important class of channels that may be represented by this memory model are channel with Markovian correlated noise. A Markovian correlated quantum noise channel of length \( n \) is...
Figure 2. Diagram of the model for a quantum memory channel. The initial memory state interacts with each transmitted quantum state $Q_k$ and environment $E_k$. The correlations between the error operators on each state $Q_k$ are determined by the unitary operation $U_k$ and the memory state at each stage of the channel evolution.

of the form

$$L^{(n)} \rho_Q = \sum_{i_0, \ldots, i_n} p_{i_n|i_{n-1}} p_{i_{n-1}|i_{n-2}} \cdots p_{i_2|i_1} p_{i_1}$$

$$\times (A_{i_n} \otimes \cdots \otimes A_{i_1}) \rho_Q (A_{i_1}^\dagger \otimes \cdots \otimes A_{i_n}^\dagger)$$

(9)

where the $A_{i_k}$ are Kraus operators for single uses of the channel on the $k$th state. It results a straightforward generalization of Eq.(4) with $p_{i_k|i_{k-1}}$ nearest inputs conditional probabilities. This is because as a typical feature of Markovian chains we have $p_{i_k|i_{k-1}} = p_{i_k|i_{k-1}i_{k-2} \cdots i_{k'}}$ for all $k' < k$. An interesting example is provided by a channel with $d = 2$ and noise operators $A_{0_k} = I^{(k)}$, $A_{1_k} = \sigma_X^{(k)}$, $A_{2_k} = \sigma_Y^{(k)}$ and $A_{3_k} = \sigma_Z^{(k)}$ and the transition matrix elements defined as $p_{i_k|i_{k-1}} = (1 - \theta)p_{i_k} + \theta \delta_{i_k,i_{k-1}}$ where $\theta$ is a correlation parameter [5]. This channel shows improvement of classical information transmission rate by using entangled inputs [4].

4. Models comparison

We have considered the model of Section 3 as a generalization of that of Section 2, however it is not straightforward that the latter is (logically) included in the former. To convince ourself it suffices to redefine the environments of Section 3, so that each one includes the memory system, i.e. $\tilde{E}_i = E_i \otimes M$. Then, we can immediately obtain the model of Section 2. In fact, on one hand we have $U_{Q_i,M,E_i} \rightarrow U_{Q_i,E_i}$ like in Eq.(7). On the other hand, since the environments $\tilde{E}_i$ have a common subspace $M$ (of finite dimension [5]), we can always write a correlated state (like $\omega_{E}$ of Eq.(5)) as a sequence of unitaries $U_{E_i} \equiv U_{M,E_i}$ acting on each of them.

The generality of the model of Section 3 has allowed to pointed out that the standard mutual information bound (Holevo bound) on the classical channel capacity as well as the coherent information bound on the quantum capacity can be extended to quantum channels with memory [5]. In fact, these bounds ultimately depend only on the mutual information between Alice’s input register and Bob’s output register, and are independent of the internal structure of the quantum channels that links both parties.

By looking at Figure 2 it comes out the possibility of defining different capacities (classical and quantum) depending on whether the sender (Alice), or the receiver (Bob), or eventually a third malicious party (Eve), has access to the memory. This may lead to explore connections with cryptography and private information transfer [10]. This is not conceivable within the model of Section 2 as it is evident by simply comparing Figures 1 and 2.
5. Conclusions
Summarizing, we have moved from a rather intuitive model to a more abstract one to account for the memory effects in a quantum channel. The latter model turns out to be the most general for this purpose.

The presented models reveal a constructive approach to quantum channel with memory [11]. It is certainly the appropriate framework when the physical realization of the memory is known. However, in many applications of information theory only the input-output behavior of a channel is of interest. From this point of view the memory would be part of the internal workings of the channel, and would not be made part of the description. This way of describing channels belongs to the axiomatic approach which has been developed in Ref. [11]. It takes a channel as a transformation turning infinite strings of input systems to infinite strings of output, with only two basic assumptions: translational invariance and causality.

Taking a ’causal’ channel and representing it as a channel with memory amounts to reconstructing a model of the channel and its internal memory states and dynamics. This is a non trivial task, but a formal reconstruction can always be given [11].

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