Strongly interacting mesoscopic systems of anyons in one dimension

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Using the fractional statistical properties of so-called anyonic particles, we present exact solutions for up to six strongly interacting particles in one-dimensional confinement that interpolate the usual bosonic and fermionic limits. Specifically, we consider two-component mixtures of anyons and use these to elucidate the mixing-demixing properties of both balanced and imbalanced systems. Importantly, we demonstrate that the degree of demixing depends sensitively on the external trap in which the particles are confined. We also show how one may in principle probe the statistical parameter of an anyonic system by injection a strongly interacting impurity and doing spectral or tunneling measurements.

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Introduction. In the quantum world we classify physically identical particles according to their statistical properties and typically divide them into two distinct sets. The key characteristic is that upon exchange of two such particles the total wave function changes only by a sign which is positive for bosonic and negative for fermionic particles. It came as quite a surprise to many when Leinass and Myrheim [1] (see also Refs. [2, 3]) discovered that in two dimensions (2D) one can accomodate exchange statistics that is neither bosonic nor fermionic but rather interpolates the usual possibilities and gives rise to so-called anyonic particles. Systems that display effective anyonic statistics are a topic of great current interest due to the integral role they enjoy in the field of quantum computation [4] (see Ref. [5] for an overview of recent theoretical and experimental progress).

An early breakthrough in the understanding of anyons was achieved by Haldane who generalized the 2D case and introduced the notion of 'fractional statistics' in any dimension [6]. In one dimension (1D) anyonic systems have been a research topic of great current interest due to the integral role they enjoy in the field of quantum computation [4] (see Ref. [5] for an overview of recent theoretical and experimental progress).

In this paper we will utilize the notion of anyons to elucidate the behavior of strongly interacting mesoscopic 1D systems. In particular, we will describe a general framework that can deal with mixtures of several components of anyonic particles with the important limiting cases being Fermi-Fermi, Bose-Fermi, and Bose-Bose systems. Using exact wave functions for the up to six-body systems in both box and harmonic confinement, we will show how statistics and trapping potentials are important for the tendency of two-component systems to either mix or phase separate when the particles have strong short-range repulsive interactions. This mixing-demixing transition remains a very active research area with several open questions [28–32]. The results we present quantify exactly what one should understand by demixing at the level of particle ordering in the exact wave functions that take the full trap geometry into account and thus go beyond any local density approximation based on Bethe ansatz, mean-field, or Luttinger liquid theory.

The exact wave functions are obtained based on a functional approach to systems with strong zero-range interactions (parametrized by a coupling strength $g$) that...
we have recently developed [33]. A great advantage is
that we do not rely on Bose-Fermi [34–36] or AnyonFermi mappings [10] as these techniques are not capable
of solving general multi-component systems (see Ref. [33] for
details). In constrast, our approach yields energies and
wave functions that are adiabatically connected to the
eigenstates for large by finite interaction strengths. In Fig. 1
we show an example with four particles in a
hard-wall box for different particle statistics (to be
defined below). In order to reach the strongly interact-
ing (‘hard core’) regime one typically tunes the interac-
tion strength which is always parametrized by
$g$ and $L$ in a tunneling experiment to infer the statistical proper-
ties contained in the wave functions. As a concrete ex-
ample we consider using a strongly interacting impurity
in a tunneling experiment to infer the statistical proper-
ties of the majority particles.

Model. The model Hamiltonian for our $N$-body sys-
tem has the form

$$H = \sum_i \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + V(x_i) \right] + g \sum_{i,j} \delta(x_i - x_j), \quad (1)$$

where $m$ is the mass, $V(x)$ is the external trap potential,
and $g$ is the interaction strength. Here we assume that
all particles have the same mass and the same interaction
strength which is always parametrized by $g$. The trap
potential length scale is $L$ (box size or harmonic trap
length) which is our basic unit throughout. From $L$ we
obtain $\hbar^2/mL^2$ as our unit of energy and likewise we
will measure $g$ in units of $\hbar^2/mL$. The anyonic exchange
symmetry implies that

$$\Psi(x_k, x_{k+1}) = -e^{-i\theta\epsilon(x_{k+1}-x_k)} \Psi(x_{k+1}, x_k), \quad (2)$$

where we suppress the dependence on all $N$ coordinates
for simplicity and $x_k$ and $x_{k+1}$ are two adjacent iden-
tical (anyonic) particles that we exchange and $\epsilon(x) =
-\epsilon(-x) = 1 (\epsilon(0) = 0)$. For $\theta = 0$ they are fermions
(F) and for $\theta = \pi$ they are bosons (B). When we dis-
cuss two-component mixtures below it is important to
note that there are no symmetry requirements between
different components. In Ref. [33], solutions with sym-
metric exchange of different types of particles have been
discussed, but here we do not make any assumptions and
the symmetry is decided by the Hamiltonian through the
eigenstates in the $1/g \to 0$ limit. A powerful feature of
our approach to strongly interacting systems [33], is that
we obtain these states without using the representation
theory of symmetry algebras. As discussed in Refs. [3, 39] using the Bethe ansatz, the ground state energy depends
$g$ and $\theta$. This is also the case here as illustrated in Fig. 1
for the FF, BF, and BB limits. Our formalism goes be-
yond the Bethe ansatz since it can treat arbitrary external
traps. Introducing several strengths for intra- and
interspecies interactions is an interesting question that
has led to recent surprises [40] but will not be pursued
here. Furthermore, one could include also odd-parity in-
teractions [35] but we assume that these are negligible
compared to the even-parity ones in Eq. (1).

In order to find the spectrum and wave functions in the
limit where $1/g \to 0$, we employ the energy functional
method presented in Ref. [33]. The general setup is
described in full detail in Ref. [33] and we will only discuss
the modifications that occur when considering fractional
exchange statistics (the appendix below presents details
for the three-body system as an example of the anyonic
case and what happens if one treats the anyons as strictly
hard core). In the limit $1/g \to 0$, $\Psi$ must vanish when-
ever two particle coordinates overlap. The discussion in
Ref. [33] shows how $\Psi$ is obtained from a basis of totally
antisymmetric $N$-body functions. Here we consider just
the ground state with the $N$ lowest single-particle levels
occupied which we denote $\Psi_A$. By writing a linear combi-
nation of $\Psi_A$ with a generally different coefficient, $a_k$, for
each possible ordering of the $N$ particles one can write an
expression for the total energy that upon variation will
yield a set of linear equations for the $a_k$. The set of $a_k$
coefficients are restricted by exchange symmetry. Con-
sider two particles adjacent with coordinates $x_1$ and $x_2$
and assume that $a_1 \Psi_A$ is the wave function for $x_1 > x_2$
while $a_2 \Psi_A$ is the one for $x_1 < x_2$. The contribution to
the energy due to the pair is proportional to $|a_1 - a_2|^2$
[33]. Assuming two identical anyons that obey Eq. (2),
we must have $a_1 = a_2 e^{i\theta}$ (the minus sign in Eq. (2)
is obtained from the antisymmetry of $\Psi_A$). The energy
contribution is therefore $|a_1|^2 4 \sin^2(\theta/2)$. For $\theta = \pi$
we get a contribution from the cusp of the wave function for
two identical bosons as in Girardeau’s orginal work [38],
while for $\theta = 0$ this vanished as two identical fermions
must be antisymmetric under exchange and thus do not
experience a zero-range interaction. Note that a different
exponent and/or overall sign convention in Eq. (2) would
merely switch the Bose and Fermi limits but otherwise
have no impact on our results.

Balanced systems. In Fig. 1 we show the $N = 4$ case
with two $A$ and two $B$ particles. Panels a), b), and c)
show the slopes of the energy around $1/g = 0$ as the
statistics changes from Fermi-Fermi (FF) a), across Bose-
Fermi (BF) b), and onto the Bose-Bose (BB) mixture
case in c). Notice the totally antisymmetric state in a) which
is the horizontal line and corresponds to $a_k = 1$
for all orderings. It is only a solution in the FF case.
The slopes have a distinct evolution with statistics which
could in principle be observed by energy measurements in
strongly interacting systems. Fig. 1 assumes a box trap
but only minute quantitative changes occurs if one uses harmonic confinement. There has been a lot of recent interest in strongly interacting Fermi-Fermi [33, 41–46] and Bose-Bose mixtures [40, 47–49], and extended focus on the spatial configuration that such systems display for strong interactions. In Fig. 1d) we present the exact results for the ground state configurations in the limit for strong interactions. In Fig. 1d) we present the exact results for the ground state configurations in the limit for strong interactions. In Fig. 1d) we present the exact results for the ground state configurations in the limit for strong interactions. In Fig. 1d) we present the exact results for the ground state configurations in the limit for strong interactions.

The numerical DMRG results in Ref. [31] seems to agree with a mixed state for very large interaction strengths which hints at an underlying issue with applying DMRG to strongly repulsive particles. It is intrinsically variational and will therefore have great difficulties with the (quasi)-degenerate many-body spectrum for strong interactions unless one uses exact solutions as a guide here. However, we notice that for large but not extreme values of the repulsive coupling strength Ref. [31] does indeed find the demixed ground state that is perfectly consistent with the exact result presented here.

For larger systems the story is similar as we show in Fig. 2 with the antiferromagnetic dominance being taken over by mixed configurations as one goes from FF to BF limits. Note that we only show the configurations carrying the largest part of the total probability. We omit the results as \( \theta_A \to \pi \) (BB limit) as they are similar to the four-body case in Fig. 1d). However, in Fig. 2 we show results for both a box and a harmonic trap which indeed demonstrates that the trap can have decisive influence on system configuration for \( 1/g \to 0 \) both quantitatively and qualitatively. In particular, mixed configurations dominate the antiferromagnet in the BF limit for harmonic but not for box traps. This shows how trap engineering can become state engineering as first discussed in Ref. [33].

**Imbalanced systems.** We now explore the interplay of statistics and imbalanced in our strongly interacting mixed systems. The two upper panels in Fig. 3 show the cases with three (left) and two (right) fermions (A) mixed with anyons (B). Again the FF limit is clearly antiferromagnetic, while in the BF limit it depends on which particle is in majority. With three B particles, the BF system is dominated by the (phase separated) \( ABBA \) configuration, while with only two B particles the system remains mainly antiferromagnetically ordered \( ABABA \). The differences due to the box or harmonic confinement that is a box (solid lines) or a harmonic trap (dashed lines). The configurations as indicated above each set of lines. For simplicity we show only the configurations with the largest probabilities.

**FIG. 2:** Same as in Fig. 1d) but for a balanced six-body case where the A particles are fermions \( (\theta_A = \pi) \) assuming an external confinement that is a box (solid lines) or a harmonic trap (dashed lines). The configurations as indicated above each set of lines. For simplicity we show only the configurations with the largest probabilities.

**FIG. 3:** Imbalanced five- (upper row) and six-body (low row) systems. All A particles are fermions \( (\theta_A = 0) \). The configurations follow the lines in the \( \theta_B = 0 \) limit (left-hand side) from top to bottom. Solid lines are for box and dashed lines for harmonic trapping. As in Fig. 2 we show only the dominant configurations.
trap are merely quantitative in this case. In contrast, for the six-body systems in the two lower panels of Fig. 3 we do see some qualitative changes with external trap, where a harmonic trap enhances the configuration $AABBAA$ in the BF limit (lower left panel). Similarly for the lower right panel, we see enhancement of the phase separated $ABBBBA$ and $ABBBAB/BABBBBA$ configurations, and again some qualitative dependence on the trap. We conclude that the tendency for phase separation for larger systems in the BF limit discussed in the introduction seems to be there but that we identify a crucial dependence on the confinement which makes the local density approximation questionable for smaller systems. An outstanding problem is to extrapolate the results obtained here to larger system sizes and match the few- and many-body limits.

Probing statistics with an impurity. Finally, we address the case of a single impurity that is strongly interacting with a number of anyons, $N_B$. As discussed above the statistic of the anyons will in general influence the energy spectrum and the configurations in the system. Measuring the 'fan' of states shown in Fig. 1 could therefore provide insights into the statistics by comparing to the theoretical prediction given the trap shape and the number of particles. The dependence of the slopes on statistics has also been identified within the Bethe ansatz approach for single-component anyons [33].

A different approach which can access more information about the system is to use tunneling experiments as done recently for an FF mixture [51]. In the limit $1/g \to 0$ where the particles become impenetrable, one can use a simple picture when opening the trap by lowering the trap on one side [31]. Here we may assume that only the particle located immediately next to the lowered barrier can tunnel. The probability that this is the impurity can then be approximated by the configurational probabilities that we have discussed above. In the lower panels of Fig. 4 we show this probability for different $\theta_B$ in a box (upper) or harmonic (lower) trap. We see clear variation with $\theta_B$ and with $N_B$ which implies that this could be used to detect the statistics of 1D anyonic systems. While the precise way in which the trap is lowered to allow for tunneling is except to have a minor quantitative effect, we do not expect qualitative differences. Alternatively, it may be possible to use single site/single atom resolution quantum gas microscopy [52, 53] to probe the 1D system locally [54, 55]. Here one could probe the probability of finding the impurity in the center of the trap which is also very sensitive to statistics as shown on the right-hand panels in Fig. 4. While the experiments cited here have an optical lattice on top of the external confinement, this will not qualitatively change our predictions. It may change the geometric factors from the confinement which can be computed using the formulas presented in Ref. [33].

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It can be taken outside for the parity invariant box of these factors in the result [33], but for three-body systems it can be taken outside, and consists of the six pieces \( K \) where \( \theta \) is a statistical parameter (an impurity).

The most general wave function with the correct boundary condition is found by taking \( \Psi_A \) as a function of \( x \) and consists of the six pieces.

The full wave function, \( \Psi \), is a function of \( x_1, x_2, x_3 \) and consists of the six pieces \( a_i \) with different coefficients, \( a_i \), in the six regions in Fig. 9. This basis is complete and we may expand the solutions in a different basis. As shown in Fig. 6 all solutions are degenerate in energy when \( 1/g = 0 \). We now use the fact that as \( 1/g \to 0 \), the ground state (for \( g > 0 \)) has the maximum slope of the energy as a function of \( g \). Using linear perturbation theory in \( 1/g \) or the Hellmann-Feynman theorem, we have \( E = E_0 - K/g \) where

\[
K = \lim_{g \to \infty} g \sum_{i,j} \int |\Psi|^2 \delta(x_i - x_j) \prod_{k=1}^3 dx_k \\
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and harmonic potentials we work with here. By variation of $K$ with respect to $a_1$, $a_2$, and $a_3$, one can obtain the eigenstates that are adiabatically connected to the eigenstate for large but finite $g$ as well as the slope of the energy to linear order in $1/g$.

As discussed, the decisive quantity that determines the wave functions that are adiabatically connected eigenstates in the limit $1/g = 0$ is the slope of the energy, $K$. If we describe the anyons as strictly hard core particles this means they are to be regarded as ideal fermions and will make no contribution to $K$. This reduces the problem to that of two identical fermions and an impurity that was discussed previously in Ref. [33], and this is true no matter what value the anyonic exchange parameter, $\theta$, takes. In turn one would not be able to recover the correct Bose-Fermi mixture limit discussed in detail for the four-body system in the main text. This implies that one needs to consider the anyons when calculating the energies even in the 'hard core' limit $1/g \to 0$ in order to have a model that matches the behavior in the known limiting cases where $\theta = 0$ or $\theta = \pi$. Our approach is therefore closely related to the work using the Bethe ansatz in Refs. [9, 39], and we also obtain a strong coupling expansion of the energy which depends on $\theta$ (see Eq. (10) of Ref. [39]). However, our formalism goes beyond the Bethe ansatz in being able to handle arbitrary confining geometries without resorting to the local density approximation.