Scattering by magnetic and spin-orbit impurities and the Josephson current in superconductor-ferromagnet-superconductor junctions.

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We analyze the Josephson current in a junction consisting of two superconductors (S) and a ferromagnetic layer (F) for arbitrary impurity concentration. In addition to non-magnetic impurities, we consider also magnetic ones and spin-orbit scattering. In the limit of weak proximity effect we solve the linearized Eilenberger equation and derive an analytical expression for the Josephson critical current valid in a broad range of parameters. This expression enables us to obtain not only known results in the dirty and clean limits but also in an intermediate region of the impurity concentration, which may be very important for comparison with experimental data.

I. INTRODUCTION

Negative Josephson coupling has been predicted almost three decades ago \cite{1,2}. Bulaevskii et al \cite{1} obtained this coupling from a model of a tunnel Josephson junction containing magnetic impurities. By tunneling from one superconductor to the other, electrons are scattered by these impurities. Explicit calculations led the authors of Ref.\cite{1} to the conclusion that under certain conditions the Josephson critical current $I_c$ might change its sign.

Later on, Buzdin et al. suggested that a similar behavior might take place provided the insulating barrier was substituted by a ferromagnetic metallic layer (F) \cite{3}. Since then, the 0-π transition, i.e. the change of sign of $I_c$, has been the focus of study of many theoretical works (see \cite{4,5,6} and references therein).

However, in spite of the theoretical progress, the 0-π transition has been observed only recently in SFS junctions with characteristic thicknesses of the F layer of the order 100 Å or less \cite{7,8,9,10,11,12,13,14}. Although a qualitative explanation for the observed drops of $I_c$ as a function of thickness or temperature could be given in the framework of the known theories, a quantitative description is, in many cases, still lacking. This is due to the fact that the authors of the theoretical works considered either the pure ballistic case \cite{3,15,16,17,18} or the dirty limit using the Usadel equation \cite{19,20}. In the pure ballistic case, the critical current $I_c$ oscillates with the period $v_F/2h$ as a function of the thickness $2d$ of the F layer, where $v_F$ is the Fermi velocity and $h$ the exchange field in the ferromagnet. In the diffusive limit, both the period of oscillation and the decaying length of the function $I_c(2d)$ are equal to $\sqrt{D/2h}$, where $D = v_F^2\tau/3$ is the classical diffusion coefficient and $\tau$ is the momentum relaxation time due to the impurity scattering.

The diffusive limit is realized provided either the exchange energy $h$ is not too large or the mean free path $l$ is short enough. More precisely, the condition $h\tau << 1$ should be fulfilled. However, in many experiments this is not the case. In particular, in those performed with strong ferromagnets \cite{9,12,14}. Theory considering an arbitrary value of $h\tau$ and $l$ has been presented in Ref. \cite{21}, where the Eilenberger equation for the condensate has been solved in the weak proximity effect limit. In particular, it was shown that in a quasiballistic case (or in case of strong ferromagnets), i.e., when the condition

$$h\tau >> 1$$

is fulfilled, the critical current oscillates with the period $v_F/2h$ and decreases exponentially over the mean free path $l = v_F\tau$.

In all theoretical works mentioned above the ferromagnet was modeled as a normal metal with an exchange field acting on the spin of the conduction electrons, while depairing factors such as spin-dependent (SD) scattering on magnetic impurities, condensate flow (due to an internal magnetic field in F) and spin-orbit (SO) scattering have not been taken into account, although they may play an important role in the proximity effect.

The effect of the spin-orbit scattering on the critical Josephson current in SFS junctions in the diffusive limit has been studied in Ref. \cite{22,23} and in a recent work \cite{24}, where it was shown that the SO interaction affects the
characteristic length of the decay of the critical current with increasing the thickness of F. Also in the diffusive limit the effect of the scattering on magnetic impurities was considered in Refs. [24, 25].

Less attention has been paid though to the study of depairing effects in the quasiballistic regime when the condition is fulfilled. An attempt to solve the Eilenberger equation with account for the SO and SD scattering and for arbitrary $l$ was undertaken in Ref. [26]. However, the solution suggested by the authors of the latter work is valid only in the diffusive limit. As we will see, in the limit determined by Eq. (1), the expression obtained in Ref. [26] is a small part of the total solution. The SD scattering was also considered in a recent work [27] in the case of a weak proximity effect and arbitrary mean free path $l$. Unlike Ref. [21], the authors suggested an approximate solution for the Eilenberger equation and neglected the SO scattering. Therefore, the problem of calculating the Josephson current $I_J$ in a general case of arbitrary mean free path $l$ taking into account different depairing mechanisms remained unsolved.

In this paper, we attack this problem calculating the current $I_J$ through an SFS structure in the general case of an arbitrary mean free path $l$ taking into account diverse depairing mechanisms. Following the method presented in Ref. [21] we solve in the next section the Eilenberger equation for the case of the spin dependent scattering on magnetic impurities. As in Refs. [21, 26, 27], we assume that the proximity effect is weak, i.e., that the amplitude of the condensate function induced in the ferromagnet is small. This assumption is reasonable even for low temperatures due to the large mismatch of the electronic parameters of the superconductor and ferromagnet resulting in a strong reflection at the S/F interfaces. Using the exact solution obtained for the condensate function we derive a general expression for the Josephson critical current through the SFS system. This expression can be used for calculating the current at arbitrary impurity concentration. In particular, we calculate $I_c$ in the diffusive and the quasi-ballistic limit. In section III we consider the effect of the SO scattering on the critical current $I_c$ for these two cases.

II. SOLUTION FOR THE EILENBERGER EQUATION AND DERIVATION OF THE JOSEPHSON CURRENT

We consider an SFS layered structure. The thickness of the F layer is $2d$ and the F/S interfaces are located at $x = \pm d$. The thickness of the S layers is assumed to be infinite. Scattering of electrons by magnetic impurities in a bulk superconductor has first been studied by Abrikosov and Gor’kov [28] using microscopic Green’s functions. For non-homogeneous finite systems it is more convenient to use quasiclassical Green’s functions determined by the equations derived by Eilenberger, Larkin and Ovchinnikov [29, 30]. In order to justify the applicability of the Eilenberger equation, we also assume that the distance between the superconductors, i.e. the thickness of the F layer, is larger than the mean free path $l$ (see Refs. [31, 32]). We consider the case of a weak proximity effect, i.e. when the amplitude of the elements of the condensate matrix function $\hat{f}$ is assumed to be small: $|\hat{f}| < < 1$. In this case one can linearize the Eilenberger equation that in the presence of the scattering on non-magnetic and magnetic impurities takes the form [5, 6, 21, 33]

$$\text{sgn}\omega \hat{\tau}_3 \hat{e} \nabla \hat{f}_\pm + (\kappa \hat{f})_\pm = (1 - 2\lambda_z - \lambda_\perp)(\hat{f}_\pm) + \lambda_\perp (\hat{f}_\mp)$$

(2)

where $\hat{\tau}_3$ is the $z$-component of the Pauli matrices $\hat{\tau}$, $\hat{e} = v_F/v_F$ is unit vector in the direction of the Fermi velocity, $\kappa_{\pm} = 1 + 2(|\omega| \pm i\hbar \omega)\hat{\tau}_1$, $\omega \equiv \omega_n = \pi T (2n + 1)$ is the Matsubara frequency $\hbar \omega = h \text{sgn}\omega$, $\tau_{1}^{-1} = \tau_{N}^{-1} + \tau_{M}^{-1}$ is the total scattering rate and $\tau_N$ is the momentum relaxation time due to scattering by nonmagnetic impurities. The rate $\tau_{M}^{-1} = \tau_{M}^{-1}/(1 + \alpha_z + \alpha_\perp)$ is the total scattering rate due to scattering by magnetic impurities. The parameters $\alpha_z$, $\alpha_\perp$ characterize the spin-dependent scattering. The potential of interaction with magnetic impurities can be written in the form

$$U(\mathbf{r}) = U_o(\mathbf{r}) + U_s(\mathbf{r}) \mathbf{S} \cdot \sigma / S,$$

where the potential $U_o(\mathbf{r})$ describes interaction with a spin-independent part of magnetic impurities. The impurity spin $\mathbf{S}$ is assumed to be classical. The coefficients $\alpha_z$ and $\alpha_\perp$ are given by $\alpha_z = [\langle U_S^2 \rangle / \langle U_o^2 \rangle] \langle S_z^2 \rangle / S^2$ and $\alpha_\perp = [\langle U_S^2 \rangle / \langle U_o^2 \rangle] \langle S_\perp^2 \rangle / S^2$. These coefficients are related to a spin-dependent scattering rate $\tau_{M}^{-1}$ used in Ref. [24]. For example, $\alpha_z = 2(\tau_{M} / \tau_{M}) \langle S_z^2 \rangle / S^2$. The angle brackets denote averaging over angles. The coefficients $\lambda_z$, $\lambda_\perp$ are defined as $\lambda_z = (\alpha_z / \tau_{M}) / (\tau_{M}^{-1} + \tau_{N}^{-1})$ and $\lambda_\perp = (\alpha_\perp / \tau_{M}) / (\tau_{M}^{-1} + \tau_{N}^{-1})$. Note that $\alpha_z, \lambda_z << 1$.

In the general case the condensate function $\hat{f}$ is a $4 \times 4$ matrix in the particle-hole (Gor’kov-Nambu)⊗spin space. However, in the case of a homogeneous magnetization considered here and in the absence of the spin-orbit scattering (the SO scattering will be taken into account in the next section), the function $\hat{f}$ is diagonal in the spin space. Then, the function $\hat{f}_+$ in Eq. (2) is defined as $\hat{f}_+ = \hat{f}_{\alpha \alpha}$ with $\alpha = 1$ ($\alpha$ is the spin index), while the other diagonal element
is given by \( \hat{J}_\pm = \hat{f}_{22} = -\hat{f}_+(x) \). In the present case of a planar geometry, the function \( \hat{f} \) depends only on the coordinate \( x \).

It is convenient to represent \( \hat{f} \) as a sum of a symmetric and antisymmetric part with respect to the momentum direction: \( \hat{f}_\pm(x) = \hat{s}_\pm(x) + \hat{a}_\pm(x) \). As follows from Eq. (2), the antisymmetric part \( \hat{a}(x) \) is related to the symmetric one by the expression
\[
\hat{a}_\pm = -\text{sgn}\omega (\mu l / \kappa_\pm) \tau_3 \partial \hat{s}_\pm / \partial x
\] (3)
while the symmetric part \( \hat{s}(x) \) in the ferromagnetic region \((-d < x < d)\) obeys the equation
\[
\mu^2 \partial^2 \hat{s}_\pm / \partial x^2 - \kappa_\pm^2 \hat{s}_\pm = -\kappa_\pm (h) \left[ (1 - 2\lambda_\pm - \lambda_\perp) \langle \hat{s}_\pm \rangle + \lambda_\perp \langle \hat{s}_\mp \rangle \right]
\] (4)
where \( l = v_F \tau_\perp \) is the mean free path and \( \mu = v_x / v_F = \cos \theta \). These two equations should be complemented by the boundary condition \([34]\)
\[
\hat{a} \mid_{x=\pm d} = \mp \gamma(\mu) \text{sgn}\omega \left( \tau_3 \hat{s}_s \right), \mu > 0
\] (5)
where \( \gamma(\mu) = T(\mu)/4 \) and \( T(\mu) \) is the transmission coefficient. The latter is assumed to be small and therefore the matrix \( \hat{f} \) in the superconductor has its bulk form \( \hat{f}_s(\pm d) = f_s(\tau_\perp \cos(\varphi/2) \pm \tau_\parallel \sin(\varphi/2)) \), where \( f_s = \Delta / \sqrt{\omega^2 + \Delta^2} \).

Note that the boundary condition Eq. (5) does not take into account spin-flip processes at the interface. Boundary conditions for magnetically active interfaces were derived in Ref. [35] and used in Ref. [36] by calculating the supercurrent through a pure ballistic half-metallic layer.

In order to derive an expression for the Josephson critical current avoiding straightforward but cumbersome calculations we first neglect the terms proportional to \( \lambda_\perp \). This term will be included again at the end of this section where we will present the final expression for the current. The way how to obtain the solution of Equations (3) was presented in Ref. [21]. One formally extends the solution over the whole \( x \)-axis, performs a Fourier transformation and obtains finally the transformed function \( \hat{s}_k \) from an algebraic equation. Following this scheme we obtain an expression describing the spatial dependence of the condensate function
\[
\hat{s}_\pm(x) = \pm \int \frac{dk}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}_{s,n} \exp(ikd(2n + 1) - ikx) \left[ \kappa_\pm (1 - 2\lambda_\pm) N_{\mp}^{-1} \left( \frac{\mu \gamma(\mu)}{M_{k \pm}^{-1}(\mu)} \right) + \mu \gamma \right]
\] (6)
with \( \hat{f}_{s,n} = f_s(\tau_\perp \cos(\varphi/2) + (-1)^n \tau_\parallel \sin(\varphi/2)) \), \( M_k(\mu) = (k d)^2 + \kappa_\pm^2 \) and \( N_{\mp} = 1 - (1 - 2\lambda_\pm) \kappa_\pm \langle M_{k \pm}^{-1}(\mu) \rangle \).

The average \( \kappa \langle M_k^{-1}(\mu) \rangle \) can easily be calculated (we drop the subindex \( \pm \))
\[
\kappa \langle M_k^{-1}(\mu) \rangle = \frac{1}{k l} \tan^{-1} \frac{k l}{\kappa} = \frac{1}{2 k l} \ln \left( \frac{i k - k l}{i k + k l} \right)
\] (7)
The \( x \)-dependence of the condensate function, in particular, its exponential decay is determined by the singular points of the integrand in the complex \( k \) plane of Eq. (6). From Eq. (7) one can immediately see, that the function in the integrand has poles at \( kl = \pm i \kappa / |\mu| \) and branch points at \( kl = \pm i \kappa_\pm \). The poles lead to the exponential decay of \( \hat{s}(x) \) like \( s_{\text{pol}}(x) \sim C_{\text{pol}}(\mu) \exp(-\kappa(\mu) / l |\mu|) \) (in the vicinity of the left superconductor), whereas the branch points lead to terms in the solution that decay as \( s_{\text{br}}(x) \sim C_{\text{br}}(\mu) \exp(-\kappa(x) / l) \) (where \( -d < x \)). This means that in the limit \( T \tau < 1 \) the terms \( s_{\text{pol}}(x) \) exponentially decrease over an angle-dependent distance of the order of the mean free path \( l \). On the other hand, the terms \( s_{\text{br}}(x) \) decay exponentially over an angle-independent distance of the order \( l \). Both the terms, \( s_{\text{pol}}(x) \) and \( s_{\text{br}}(x) \), oscillate with the periods \( h / v_F |\mu| \) and \( h / v_F \), respectively. The amplitudes \( C_{\text{pol}}(\mu) \) and \( C_{\text{br}}(\mu) \) depend on the parameters of the system.

In the general case their calculation is rather complicated and a numerical analysis is needed. However, as we will see in the next paragraphs, in the quasi-ballistic and diffusive cases one can obtain explicit expressions for the condensate function \( \hat{s}(x) \). If one is interested in an intermediate case it is convenient for numerical calculations to perform the integration over momentum in Eq. (6) taking into account the relation (see, for example [37])
\[
\sum_{n=-\infty}^{\infty} \exp(i2nkd) = \frac{\pi}{d} \sum_{m=-\infty}^{\infty} \delta(k - k_m),
\] (8)
where \( k_m = \pi m / d \).
Writing the condensate matrix as $s_{\pm} = s_{1}^{\pm} \tilde{r}_{1} + s_{2}^{\pm} \tilde{r}_{2}$ one finally obtains

$$s_{1}^{\pm} = \pm \frac{i \kappa_{\pm}}{d} \sum_{m=-\infty}^{\infty} \exp[i(x/d - 1)(2m + 1)\pi/2] \left[ \kappa_{\pm}(1 - 2\lambda_{z})(N_{2m+1}^{\pm})^{-1} \left( \frac{\mu_{\gamma}(\mu)}{M_{2m+1}^{\pm}(\mu)} + \mu_{\gamma} \right) f_{s} \sin \varphi/2 \right] (9)$$

$$s_{2}^{\pm} = \pm \frac{i \kappa_{\pm}}{d} \sum_{m=-\infty}^{\infty} \exp[i(x/d - 1)m\pi] \left[ \kappa_{\pm}(1 - 2\lambda_{z})(N_{2m}^{\pm})^{-1} \left( \frac{\mu_{\gamma}(\mu)}{M_{2m}^{\pm}(\mu)} + \mu_{\gamma} \right) f_{s} \cos \varphi/2 \right] (10)$$

where $M_{m}^{\pm} = (\mu \pi m/2d)^{2} + \kappa_{\pm}^{2}$ and $N_{m}^{\pm} = 1 - (1 - 2\lambda_{z})\kappa_{\pm}(1/M_{m}^{\pm})$. Knowing the condensate function induced in the ferromagnetic region one can calculate the Josephson dc current $I_{c}$ through the SFS junction. This current is given by

$$I = (\pi i T/4)(e^{2}/h)(k_{F}^{2}S/\pi^{2})Tr \left[ \tilde{r}_{3} \otimes \sigma_{0} \sum_{\omega=-\infty}^{\infty} \langle \mu_{\gamma}[\tilde{s}(d), \tilde{f}_{s}(d)] \rangle \right] (11)$$

where $[\tilde{s}, \tilde{f}_{s}]_{-} = \tilde{s}\tilde{f}_{s} - \tilde{f}_{s}\tilde{s}$, $\sigma_{0}$ is the unit matrix and the symbol $T r$ stands for the trace over the $4 \times 4$ matrices (see [21]).

Using Eq. (6), one can write the current in the form $I = I_{c} \sin \varphi$ with

$$I_{c} = A(2\pi T)Re \sum_{m=-\infty, \omega \geq 0}^{\infty} f_{s}^{2}(-1)^{m} \mu_{\gamma}(\mu) \frac{i \kappa_{+}}{M_{m}(\mu)} \left[ \kappa_{m}^{+} \frac{\kappa_{m}^{+}}{N_{m}^{+}}(1 - 2\lambda_{z}) \left( \frac{\mu_{\gamma}(\mu')}{M_{m}(\mu')} + \mu_{\gamma}(\mu) \right) \right]_{\mu'} (12)$$

where $A = (e^{2}/h)(k_{F}^{2}S/\pi^{2})$ and $M_{m}(\mu) \equiv M_{m}^{+}(\mu)$. Eq. (12) is the most general expression for the Josephson current in terms of the solution of the Eilenberger equation. In the next sections we give expressions for the critical current $I_{c}$ in the quasiballistic and diffusive limits.

**The quasiballistic case: $|\kappa| >> 1$**

The quasiballistic case corresponds either to a strong ferromagnet ($h > \tau^{-1}$) or to a clean sample ($T \tau > 1$). As follows from Eq. (7), in this case $N \approx 1$ and the second term in the square brackets in Eq. (6) is much larger than the first one. Calculating the residue at the pole $kl = i\kappa/|\mu|$, we obtain for $\tilde{s}(x) \equiv \tilde{s}_{+}(x)$

$$\tilde{s}(x) = \gamma(\mu)f_{s} \left[ \frac{\cosh(x/L_{qF}(\mu))}{\sinh(d/L_{qF}(\mu))} \cos(\varphi/2) + \frac{\sinh(x/L_{qF}(\mu))}{\cosh(d/L_{qF}(\mu))} \sin(\varphi/2) \right] (13)$$

with the length $L_{qF}(\mu)$ characterizing the quasiballistic case defined as $L_{qF}(\mu) = l/\mu_{\gamma}$.

It is seen for example that in the vicinity of the left superconductor, i.e., at $1 << (x + d)/l << d/\kappa$ the condensate function $\tilde{s}(x)$ oscillates with the period $\pi L_{qF}/h|\mu|$ decaying over the angle-dependent mean free path $l/|\mu|: s(x) \sim \exp(-kz/\mu) [21]$. Thus, if the exchange energy is large, $h > \tau^{-1}$, the function $\tilde{s}(x)$ experiences many oscillations over the decay length $l$. Note that the period of oscillations and the decay length strongly depend on the angle $\mu = \cos \theta$.

This result contradicts the conclusion of Ref. [26] where the solution for Eq. (3) was taken in a form of an exponential function with an angle-independent exponent. As can be understood from Eq. (4), the solution obtained in Ref. [26] is not general and corresponds only to a contribution from the branch points, i.e., from the first term in the square brackets in Eq. (4). However, in the limit of large $\kappa$ this term is small compared to the other one (see [21]).

We see from Eq. (4) that in the quasiballistic case the depairing leads only to a renormalization of the scattering time $\tau \rightarrow \tau_{s}$.

The main contribution to the critical Josephson current in the quasiballistic regime stems from the second term in the square brackets in Eq. (12). Calculating the residues at the poles of $\kappa_{m}^{+}(\mu')$, we obtain (cf. Ref. [21])

$$I_{c} = A(2\pi T)^{2} \sum_{\omega \geq 0}^{\infty} f_{s}^{2}Re \left\langle \frac{\mu_{\gamma}^{2}(\mu')}{\sinh(2d/L_{qF}(\mu'))} \right\rangle_{\mu'} (14)$$

If the thickness of the F layer considerably exceeds the value of $L_{qF}(1)$, one obtains for the critical current $I_{c}$

$$I_{c} = A(2\pi T) \sum_{\omega \geq 0}^{\infty} f_{s}^{2}\gamma^{2}(1) \frac{\sin(4hd/v_{F})}{4hd/v_{F}} \exp[-(2d/l)(1 + 2\omega\tau_{s})] (15)$$

Eqs. (14, 15) fit well recent experimental data [6, 12].
The diffusive case: $|\kappa| \sim 1$.

In the diffusive limit the conditions $h\tau, T\tau << 1$ hold and the characteristic length of the spacial variation of $\hat{s}(x)$ is much larger than the mean free path $l$. Therefore, we obtain from the following expressions for the parameters: $N_k = \kappa_{dif}^2 + (1 - 2\lambda_\perp)(kl)^2/3$ and $k^2 \approx M_k(\mu) \approx 1$ (cf. Eq. (7)).

The behavior of $\hat{s}(x)$ at distances from the superconductors larger than the mean free path $l$ is determined by the residue of the pole in $N_k$. Equation (16) finally yields

$$\hat{s}(x) = \frac{\sqrt{3}(1 - \lambda_\perp)}{\kappa_{dif}} \langle \gamma(\mu) \rangle f_s \left[ \hat{\tau}_2 \frac{\cosh(x/L_{dif})}{\sinh(d/L_{dif})} \cos(\varphi/2) + \hat{\gamma} \frac{\sinh(x/L_{dif})}{\cosh(d/L_{dif})} \sin(\varphi/2) \right]$$  (16)

where $\kappa_{dif}^2 = \lambda_\perp + 2(\Delta) - i(h\tau_\perp)\tau_\perp$, and the characteristic length $L_{dif}$ of the condensate decay in F is $L_{dif} = l/(1 - 2\lambda_\perp)/3\kappa_{dif}^2 \approx l/(\kappa_{dif} \sqrt{3})$.

If the exchange energy $h$ is much smaller than the spin-dependent scattering rate, $h\tau_\perp << \lambda_\perp$, we obtain for the characteristic length $L_{dif} = 1/\sqrt{3\lambda_\perp} = \sqrt{D\tau_{sp}/2}$, when $\tau_{sp} \equiv \tau/\lambda_\perp$. Thus, as expected, the decay length in this case is related to the spin-dependent relaxation time (see, for example, [39]). In the opposite limit, $h >> 1/\tau_{sp}$, the characteristic length $L_{dif}$ is given by the well known expression [3] [19] [24] [23]: $L_{dif} \approx 1/(1 - \sqrt{3h\tau_\perp}) = (1/2)(1 + i)\sqrt{D/h}$. Note that in this case the condensate function both decays and oscillates on the same length $\sqrt{D/h}$.

If one takes into account an internal magnetic field $B$ inside the ferromagnet given by $B = 4\pi M$, the length $L_{dif}$ in the limit $h\tau_\perp << \lambda_\perp$ is equal to: $L_{dif} = 1/\sqrt{(2dB/\phi_0)^2 + (\tau_{sp}/2)^{-1}}$, where $M$ is the magnetization in F and $\phi_0 = \pi hc/e$ is the magnetic flux quantum (see, e.g., [40]).

In the diffusive limit, the critical Josephson current $I_c$ is determined by the first term in the square brackets of Eq. (12), i.e., by the poles of the function $N^{-1}_k$. Then, we find

$$I_c = A(2\pi T) \sum_{\omega > 0} \sqrt{3} f_{\omega}^2 \text{Re} \left[ \frac{1}{\kappa_{dif}^2 \sinh(2d/L_{dif})} \right]$$  (17)

where $L_{dif}$ is defined in Eq. (16).

An alternative way to obtain Eq. (17) is to solve the Usadel equation, which was done in many publications [3] [19] [24] [25].

For an arbitrary value of the parameter $h\tau_\perp$ the critical current can be computed from Eq. (12). In Fig. 1 we show the dependence of the absolute value of $I_c$ on the thickness $2d$ of the ferromagnet for different values of $h\tau_\perp$ taking into account only the scattering by non-magnetic impurities, i.e., when $\tau_{sp}^{-1} \rightarrow 0$. One can readily observe the crossover from the diffusive case ($h\tau_\perp = 0.2$), when the decay length and the period of the oscillations are the same $\sqrt{D/h}$, to the quasiballistic case ($h\tau_\perp = 2.2$), when the period of the oscillations is $\pi v_F/h$ while the decay length is of the order of the mean free path $l$.

To simplify numerical calculations, we assume that the transmission parameter $\gamma(\mu)$ is peaked at $\mu = 1$ and replace it in Eq. (12) by a delta-function, i.e. we assume that only electrons with momentum direction perpendicular to the S/F interface are transmitted (the angle-dependence of “the transmission coefficient” $\gamma(\mu)$ depends on properties of the S/F interface). Another limit ($\gamma(\mu) = \text{const}$) was assumed in Refs. [3] [12] where the expression for $I_c$ derived in Ref. [21] in the absence of depairing mechanisms was used for comparison between theory and experimental data. In Fig. 2 we represent the critical current as a function of $2d$ for three different values of the parameter $h\tau_\perp$. One can see that in all cases an increase of $\lambda_\perp$ leads to a decrease of the amplitude of the condensate in the ferromagnet.

Finally we write down the expression for the critical current for the case when perpendicular fluctuations of the exchange field are taken into account, i.e., when $\lambda_\parallel$ is finite. In that case we obtain for the critical current a rather cumbersome formula

$$I_c = A(2\pi T) \text{Re} \sum_{n = -\infty}^{\infty} \int_{\omega > 0} f_{\omega}^2 (-1)^n \langle \mu' \gamma(\mu') \rangle \frac{2\kappa_+}{d M^n_+ (\mu') \left| N^n_+ (\mu') \right|^2 + \left| \kappa_+ \right|^2 \left| \gamma(\mu') \right|^2} \times$$

$$\times \left[ (1 - 2\lambda_\parallel - \lambda_\perp) \left( \frac{\mu' \gamma(\mu')}{M^n_+ (\mu')} \right) - \lambda_\perp \left| \kappa_+ \right|^2 \left( \langle M^n_+ (\mu') \rangle \mu' \langle \mu' \gamma(\mu') (M^n_+)^{-1} (\mu') \rangle_\mu' \right) \right]$$

$$- \left( \lambda_\perp \left[ N^n_+ \left( \frac{\mu' \gamma(\mu')}{M^n_+ (\mu')} \right) - \left| \kappa_+ \right|^2 \left( \langle (M^n_+)^{-1} (\mu') \rangle_\mu' \langle \mu' \gamma(\mu') (M^n_+)^{-1} (\mu') \rangle_\mu' \right) \right] \left( \frac{2\kappa_+}{d M^n_+ (\mu')} \mu_\mu' \right) \right)$$  (18)
FIG. 1: Dependence of the absolute value of the Josephson critical current $I_c$ on the thickness $2d$ of the ferromagnetic layer for different values of $h\tau$. Here $\Delta \tau = 0.1$ and $T\tau = 0.05$. Only scattering by non-magnetic impurities is considered.

This equation can be evaluated only numerically. We represent the function $I_c(d)$ for different values of $\lambda_\perp$ in Fig. 3. Note that, again, with increasing $\lambda_\perp$ the amplitude of the condensate decreases. This is clear from the physical point of view because any depairing factors lead to a suppression of the condensate amplitude.

III. SPIN-ORBIT SCATTERING

In this section we consider influence of the spin-orbit (SO) scattering on the Josephson current. For simplicity we neglect the spin-dependent scattering analyzed in the preceding Section. In the presence of the SO scattering, the condensate function is no longer diagonal in the spin space. Therefore, instead of the $2 \times 2$ matrix $\hat{f}$, we have to introduce a more complicated $4 \times 4$ matrix $\tilde{\mathbf{f}}$ (see for example [6]).

The Josephson current in SFS junctions in the presence of the SO interaction was analyzed in the diffusive limit in Ref. [22]. Here we focus on the opposite quasiballistic limit.

The linearized Eilenberger equation for $\tilde{\mathbf{f}}$ has the form

$$\text{sgn} \omega_3 \mathbf{e} \nabla \tilde{\mathbf{f}} + \hat{\kappa} \tilde{\mathbf{f}} = \rho(\tilde{\mathbf{f}}) - \lambda_{so}(\tilde{\mathbf{f}})_{so},$$

In Eq. (19), $l = v_F \tau_1$, $\tau_1^{-1} = \tau^{-1} + (2/3)\tau_{so}^{-1}$, where $\tau$ and $\tau_{so}$ are the momentum relaxation time due to a potential scattering and the spin orbit scattering time, $\rho = \tau_1/\tau \approx 1$, $\lambda_{so} = \tau_1/\tau_{so} << 1$. The matrix $\hat{\kappa} = \kappa(h) = 1 + 2(|\omega| + i\hbar \omega \hat{\sigma}_3)\tau_1$, which means that $\kappa_+$ in Eq.(2) is the $(\hat{\kappa})_{11}$ element of the matrix $\hat{\kappa}$. The angle brackets mean the angle averaging:

$$\langle \tilde{\mathbf{f}} \rangle = (1/4\pi) \int d\Omega \tilde{\mathbf{f}}(\Omega), \langle \tilde{\mathbf{f}} \rangle_{so} = (1/4\pi) \int d\Omega' e'_k(\tilde{\mathbf{S}} \times \mathbf{e})_i\tilde{\mathbf{f}}(\Omega')(\tilde{\mathbf{S}} \times \mathbf{e})_k,$$

where the vector $\tilde{\mathbf{S}}$ has components $S_i$: $\tilde{\mathbf{S}} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3 \otimes \hat{\tau}_3)$.

As before, we represent $\tilde{\mathbf{f}}$ as a sum of the antisymmetric and symmetric parts: $\tilde{\mathbf{f}} = \tilde{\mathbf{a}} + \tilde{\mathbf{s}}$, where the antisymmetric part $\tilde{\mathbf{a}}$ is expressed in terms of the symmetric function as $\tilde{\mathbf{a}} = -\hat{\kappa}^{-1}\text{sgn} \omega_3 e_{1l} \partial \delta / \partial x$ and the symmetric part obeys the equation

$$- e_1^2 l^2 \partial^2 \tilde{\mathbf{s}} / x^2 + \hat{\kappa}^2 \tilde{\mathbf{s}} = \hat{\kappa}[\rho(\tilde{\mathbf{f}}) - \lambda_{so}(\tilde{\mathbf{f}})_{so} + 2(e_{1l})_\gamma \sum_{n=-\infty}^{\infty} \tilde{f}_{s,n} \delta(x - d(2n + 1))]$$
FIG. 2: Dependence of the absolute value of Josephson critical current $I_c$ on the thickness $2d$ of the ferromagnetic layer for different values of $h\tau$ and $\lambda_z$. Here $\Delta \tau_t = 0.1$, $T\tau_t = 0.05$ and $\lambda_x = 0$. Without loss of generality we have set $\tau_N^{-1} \to 0$. 

[Graph showing the dependence of $|I_c/\sqrt{2\pi A}]$ on $2d/l$ for $h\tau_t = 0.2$, $h\tau_t = 1.2$, and $h\tau_t = 2.2$.]
Taking into account the structure of the $\tilde{f}_s$ matrix in the spin space we see that the matrix $\tilde{f}_{s,n}$ coincides with the one presented above, $\tilde{f}_{s,n} = \hat{\sigma}_3 \otimes \tilde{f}_{s,n}$.

Our task now is to solve Eq. (21). The presence of the term of the SO scattering makes this task more difficult than previously. In order to simplify the problem, we use the usual smallness of $\lambda_{so}$ [28]. An additional simplification comes from using the quasiballistic case when the value of $|\kappa|$ is large.

In order to find the solution of Eq. (21) we represent the Fourier transform $\tilde{s}_k$ in a form of an expansion in the small parameter $\lambda_{so}$: $\tilde{s}_k = \tilde{S}_k + \delta \tilde{S}_k$.

In zero order approximation in $\lambda_{so}$ and in the main approximation in the parameter $|\kappa|$ we obtain

$$\tilde{S}_k = 2ke_1 \gamma \tilde{M}_k^{-1}(e_1) \sum_{n=-\infty}^{\infty} \tilde{f}_{s,n} \exp(ikd(2n + 1)),$$

where $\tilde{M}_k(e_1)$ is a $2 \times 2$-matrix in spin-space with component $(1,1)$ equals to $M_{k+}(e_1)$ and component $(2,2)$ to $M_{k+}(e_1)$. Although the matrix $\tilde{S}_k$ is diagonal in the spin space, the correction $\delta \tilde{S}_k$ is not. It is equal to

$$\delta \tilde{S}_k = -\kappa \tilde{M}_k^{-1}(\Omega) \lambda_{so} \langle \tilde{S}_k(\Omega') \rangle_{so}$$

Using Eqs. (22), (20), one can represent the average $\langle \tilde{S}_k(\Omega') \rangle_{so}$ as

$$\langle \tilde{S}_k(\Omega') \rangle_{so} = \langle \tilde{S}_{k0}(\Omega') \otimes \sigma_0 A_0 - \tilde{S}_{k3}(\Omega') \otimes \sigma_3 A_3 + 2\tilde{\sigma}_3 \tilde{S}_{k0}(\Omega') \otimes i(\hat{\sigma}_1 A_1 - \hat{\sigma}_2 A_2) \rangle$$

where $A_0 = A_0(\Omega, \Omega') = e_1^2(e_3^2 - e_2^2) + e_2^2(e_3^2 - e_1^2) + e_3^2(e_1^2 + e_2^2); A_3 = A_3(\Omega, \Omega') = e_1^2(e_3^2 + e_2^2) + e_2^2(e_3^2 + e_1^2) + e_3^2(e_1^2 + e_2^2); A_1 = e_1^2 e_2 e_3; A_2 = e_1^2 e_2 e_3$. The matrices $\tilde{S}_{k0,3}$ are defined with the help of the relation $\tilde{S}_k = \tilde{\sigma}_0 \otimes \tilde{S}_{k0} + \tilde{\sigma}_3 \otimes \tilde{S}_{k3}$.

Using Eq. (22) we obtain

$$\tilde{S}_{k0} = 2te_1 \gamma i \tilde{F}_s \text{Im} \frac{\kappa^+}{M_{k+}}, \tilde{S}_{k3} = 2te_1 \gamma i \tilde{F}_s \text{Re} \frac{\kappa^+}{M_{k+}}$$

where $\tilde{F}_s = \sum_{n=-\infty}^{\infty} \tilde{f}_{s,n} \exp(ikd(2n + 1))$ and the matrix $\tilde{f}_{s,n}$ has been introduced in Eq. (6).

Note that due to the last two terms in Eq. (24), the condensate matrix $\delta \tilde{S}_k$ contains not only the triplet component with the zero projection on the $z$-axis but also the triplet components of the type $\uparrow \uparrow, \downarrow \downarrow$. However, in the lowest order in $\lambda_{so}$, these components do not contribute to the Josephson current because they are odd functions with respect to the inversion $e_{1,2} \Rightarrow -e_{1,2}$.
The correction $\delta I_c$ to the Josephson current due to spin orbit scattering is given by the expression

$$\delta I_c = A(2\pi i T) Tr \left[ \hat{\sigma}_3 \otimes \hat{\sigma}_0 \sum_{\omega = -\infty}^{\infty} \int \frac{dk}{2\pi} \langle e_i \gamma [\delta \hat{S}_k(\Omega), \hat{f}_s(d)] \rangle \exp(-ikd) \right]$$

Finally, we obtain the correction $\delta I_c$ to the critical current originating from the SO scattering

$$\delta I_c = A(2\pi T) \frac{l}{d} \sum_{n,\omega = -\infty}^{\infty} f^2 \int \langle e_i e_i' \gamma(\Omega) \gamma(\Omega') [-A_0 \text{Im} \frac{\kappa_+}{M^+_2(\Omega)} \text{Im} \frac{\kappa_+}{M^+_2(\Omega')} + A_0 \text{Re} \frac{\kappa_+}{M^+_2(\Omega)} \text{Re} \frac{\kappa_+}{M^+_2(\Omega')} \rangle \Omega \rangle$$

The structure of this equation is similar to the one of the contribution of the first term in the square brackets in Eq.(12) to the Josephson current. This contribution is also small in comparison with the current $I_c$ determined by Eq.(14) with $L_{\text{eff}}(\mu) = |\mu|/\kappa$ and $l = v_F \tau$, where $\tau_t^{-1} = \tau^{-1} + (2/3) \tau_{so}^{-1}$. This small correction to the critical current $I_c$ can be calculated numerically. In the diffusive case the SO scattering has been studied in Refs.[5, 19, 24, 25].

### IV. CONCLUSIONS

Assuming a weak proximity effect we have derived the exact expression for the Josephson current through an SFS junction for arbitrary impurity concentration and in the presence of spin-dependent scattering. In the quasiballistic and diffusive limits this expression takes a simple form. In the former case, the parameter $1/(h\tau)$ is small. In the main approximation the expression for the critical Josephson current is reduced to Eq.(14) that agrees with the corresponding equation of Ref.[21] provided the momentum relaxation time $\tau$ is replaced as: $\tau^{-1} \Rightarrow \tau_t^{-1} = (1 + \lambda_3)\tau^{-1} + (2/3)\tau_{so}^{-1}$. Therefore, the depairing leads only to a renormalization of the mean free path $l$ that determines the decay of the condensate function in F and of the Josephson critical current $I_c$. Oscillations of these quantities have the period $\pi v_F / h$.

In the diffusive case the oscillation period and the decay of the critical current $I_c$ are determined by the value of the product $h\tau_{dep}$ where the time $\tau_{dep}$ is defined as $\tau_{dep} = \min \{\tau_{sp}, \tau_{so}\}$. If the exchange energy lies in the interval $\tau_{dep} < h < \tau^{-1}$, then the period of the oscillations of $I_c$ is $2\pi \sqrt{D\tau_{dep}/(h\tau_{dep})}$ and the decay length is $\sqrt{D\tau_{dep}}$. In the limit $h < \tau^{-1}$, the period of oscillations and the decay length are determined by the value $\sqrt{D}/h$. The general expression, Eq.(12), may serve for numerical calculation of the critical current in the intermediate region of parameters.

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