A CONTACT GEOMETRIC PROOF OF
THE WHITNEY–GRAUSTEIN THEOREM

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ABSTRACT. The Whitney–Graustein theorem states that regular closed curves
in the 2-plane are classified, up to regular homotopy, by their rotation number.
Here we give a simple proof based on contact geometry.

1. INTRODUCTION

A regular closed curve in the 2-plane is a continuously differentiable map
\( \gamma : [0, 2\pi] \to \mathbb{R}^2 \) with the following properties:
(i) \( \gamma(0) = \gamma(2\pi), \quad \gamma'(0) = \gamma'(2\pi) \),
(ii) \( \gamma'(s) \neq 0 \) for all \( s \in [0, 2\pi] \).

If we identify the circle \( S^1 \) with \( \mathbb{R}/2\pi\mathbb{Z} \), we may think of \( \gamma \) as a continuously
differentiable map \( S^1 \to \mathbb{R}^2 \).

The rotation number \( \text{rot}(\gamma) \) of \( \gamma \) is the degree of the map
\[
S^1 \to \mathbb{R}^2 \setminus \{0\}, \quad s \mapsto \gamma'(s).
\]
In other words, \( \text{rot}(\gamma) \) is simply a signed count of the number of complete turns of
the velocity vector \( \gamma' \) as we once traverse the closed curve \( \gamma \), see Figure 1.

A regular homotopy between two such regular closed curves \( \gamma_0, \gamma_1 \) is a con-
tinuously differentiable homotopy via regular closed curves \( \gamma_t : S^1 \to \mathbb{R}^2, \ t \in [0, 1] \).
The rotation number clearly stays invariant under regular homotopies. The follow-
ing theorem is commonly known as the Whitney–Graustein theorem. It was first
proved in a paper by H. Whitney [1], who writes: 'This theorem, together with its
proof, was suggested to me by W. C. Graustein.' For alternative presentations see
[1] Chapter 6 or [3] p. 47 et seq.]

Theorem 1. Regular homotopy classes of regular closed curves \( \gamma : S^1 \to \mathbb{R}^2 \) are
in one-to-one correspondence with the integers, the correspondence being given by
\( [\gamma] \mapsto \text{rot}(\gamma) \).
Whitney’s proof is elementary, but not without intricacies. Here I want to present a nonelementary proof — based on contact geometry — where the geometric ideas are actually quite simple.

**Remark.** The modern terminology ‘regular homotopy’ describes what Whitney called a ‘deformation’ of regular closed curves. He seems to suggest, erroneously, that it is enough to require that \( \gamma_t(s) \) be continuous in \( s \) and \( t \) and a regular closed curve for each fixed \( t \), but in the course of his argument it becomes clear that he wants \( \gamma'_t(s) \) to depend continuously on \( t \) as well. Figure 2 shows a homotopy of regular closed curves (first traverse the big circle counter-clockwise, then the small circle) with \( \text{rot}(\gamma_t) = 2 \) for \( t \in [0, 1) \), but \( \text{rot}(\gamma_1) = 1 \).

![Figure 2](image.png)

**Figure 2.** A homotopy through regular closed curves with rot not invariant.

*Acknowledgement.* The idea for the proof presented here was inspired by a conversation with Yasha Eliashberg.

## 2. Legendrian curves

The **standard contact structure** \( \xi \) on \( \mathbb{R}^3 \), see Figure 3 (produced by Stephan Schönenberger), is the 2-plane field \( \xi = \ker(dz + x
dy) \). For a brief introduction to contact geometry see [2]. No knowledge of contact geometry beyond the concepts that I shall introduce explicitly will be required for the argument that follows.

![Figure 3](image.png)

**Figure 3.** The contact structure \( \xi = \ker(dz + x
dy) \).

A regular closed, continuously differentiable curve \( \gamma: S^1 \to (\mathbb{R}^3, \xi) \) is called **Legendrian** if it is everywhere tangent to \( \xi \), that is, \( \gamma'(s) \in \xi_{\gamma(s)} \) for all \( s \in S^1 \). When we write \( \gamma \) in terms of coordinate functions as \( \gamma(s) = (x(s), y(s), z(s)) \), the
condition for \( \gamma \) to be Legendrian becomes \( z' + xy' \equiv 0 \). The \textbf{front projection} of \( \gamma \) is the planar curve

\[ \gamma_F(s) = (y(s), z(s)) \]

its \textbf{Lagrangian projection}, the curve

\[ \gamma_L(s) = (x(s), y(s)) \].

Figure 4 shows the front and Lagrangian projection of a Legendrian unknot in \( \mathbb{R}^3 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{legendrian_unknot.png}
\caption{A Legendrian unknot.}
\end{figure}

Notice that a Legendrian curve \( \gamma \) can be recovered from its front projection \( \gamma_F \), since

\[ x(s) = -\frac{z'(s)}{y'(s)} = -\frac{dz}{dy} \]

is simply the negative slope of the front projection. (Of course this only makes sense for \( y'(s) \neq 0 \). Generically, the zeros of the function \( y'(s) \) are isolated, corresponding to isolated cusp points where \( \gamma_F \) still has a well-defined slope.) Since \( x(s) \) is always finite, \( \gamma_F \) does not have any vertical tangencies, and we can sensibly speak of left and right cusps. These cusps are ‘semi-cubical’; a model is given by \( (x(s), y(s), z(s)) = (s, s^2/2, -s^3/3) \).

Likewise, \( \gamma \) can be recovered from its Lagrangian projection \( \gamma_L \) (unique up to translation in the \( z \)-direction), for the missing coordinate \( z \) is given by

\[ z(s_1) = z(s_0) - \int_{s_0}^{s_1} x(s) y'(s) \, ds. \]

Observe that the integral \( \int xy' \, ds = \int x \, dy \), when integrating over a closed curve, measures the oriented area enclosed by that curve. Moreover, the Lagrangian projection \( \gamma_L \) of a regular Legendrian curve \( \gamma \) is always regular: if \( y'(s) = 0 \), the Legendrian condition forces \( z'(s) = 0 \), and then the regularity of \( \gamma \) gives \( x'(s) \neq 0 \).

The idea for the proof of Theorem 1 is now the following. Given a (regular closed) Legendrian curve \( \gamma \) in \((\mathbb{R}^3, \xi)\), one can assign to it an invariant (under Legendrian regular homotopies, i.e. regular homotopies via Legendrian curves). This invariant is likewise called ‘rotation number’. In fact, the rotation number of \( \gamma \) will be seen to equal the rotation number of its Lagrangian projection \( \gamma_L \). Alternatively, the rotation number of \( \gamma \) can be computed from its front projection \( \gamma_F \), where it becomes a simple combinatorial quantity (a count of cusps). Now, given two regular closed curves \( \gamma_0, \gamma_1 \) in the plane with equal rotation number, we can consider their lifts to Legendrian curves \( \gamma_0, \gamma_1 \) (still with equal rotation number), and in the front projection we can now ‘see’, in a combinatorial way, a Legendrian regular homotopy between them. The Lagrangian projection of this Legendrian regular homotopy will give us the regular homotopy between \( \gamma_0 \) and \( \gamma_1 \).
3. The rotation number

The plane field $\xi$ is spanned by the globally defined vector fields $e_1 = \partial_x$ and $e_2 = \partial_y - x \partial_z$. In terms of the trivialisation of $\xi$ defined by these vector fields, we may regard the map $\gamma'$ (coming from a regular closed Legendrian curve $\gamma$) as a map

$$S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$$

$$s \mapsto \gamma'(s).$$

The rotation number $\text{rot}(\gamma)$ of a Legendrian curve $\gamma$ is the degree of that map. This means that $\text{rot}(\gamma)$ counts the number of rotations of the velocity vector $\gamma'$ relative to the oriented basis $e_1, e_2$ of $\xi$ as we go once around $\gamma$. The rotation number is clearly an invariant of Legendrian regular homotopies.

Under the projection $(x, y, z) \mapsto (x, y)$, each 2-plane $\xi_{\gamma(s)}$ maps isomorphically onto $\mathbb{R}^2$, and the basis $e_1, e_2$ for $\xi_{\gamma(s)}$ is mapped to the standard basis $\partial_x, \partial_y$ for $\mathbb{R}^2$. So the following is immediate from the definitions.

**Proposition 2.** The rotation number of a (regular closed) Legendrian curve in $(\mathbb{R}^3, \xi)$ equals the rotation number of its Lagrangian projection. \qed

A little more work is required to read off $\text{rot}(\gamma)$ from the front projection $\gamma_F$. This, however, is well worth the effort, because it turns the rotation number into a simple combinatorial quantity.

**Proposition 3.** Let $\gamma$ be a (regular closed) Legendrian curve in $(\mathbb{R}^3, \xi)$. Write $\lambda_+$ or $\lambda_-$, respectively, for the number of left cusps of the front projection $\gamma_F$ oriented upwards or downwards; similarly we write $\rho_\pm$ for the number of right cusps with one or the other orientation. Finally, we write $c_\pm$ for the total number of cusps oriented upwards or downwards, respectively. Then the rotation number of $\gamma$ is given by

$$\text{rot}(\gamma) = \lambda_- - \rho_+ = \rho_- - \lambda_+ = \frac{1}{2}(c_- - c_+).$$

**Proof.** The rotation number $\text{rot}(\gamma)$ can be computed by counting (with sign) how often the velocity vector $\gamma'$ crosses $e_1 = \partial_x$ as we travel once along $\gamma$.

Since $x(s)$ equals the negative slope of the front projection, points of $\gamma$ where the (positive) tangent vector equals $\partial_x$ are exactly the left cusps oriented downwards (see Figure 5) and the right cusps oriented upwards.

At a left cusp oriented downwards, the tangent vector to $\gamma$, expressed in terms of $e_1, e_2$, changes from having a negative component in the $e_2$-direction to a positive one, i.e. such a cusp yields a positive contribution to $\text{rot}(\gamma)$. Analogously, one sees that a right cusp oriented upwards gives a negative contribution to the rotation number. This proves the formula $\text{rot}(\gamma) = \lambda_- - \rho_+$. The second expression for the rotation number is obtained by counting crossings through $-e_1$ instead; the third expression is found by averaging the first two. \qed

4. Proof of the Whitney–Graustein theorem

First we give a classification of regular closed Legendrian curves up to Legendrian regular homotopy.

**Proposition 4.** Legendrian regular homotopy classes of regular closed Legendrian curves $\gamma: S^1 \rightarrow (\mathbb{R}^3, \xi)$ are in one-to-one correspondence with the integers, the correspondence being given by $[\gamma] \mapsto \text{rot}(\gamma)$. 
Proof. With the help of either of the two foregoing propositions one can construct a regular closed Legendrian curve $\gamma$ with $\text{rot}(\gamma)$ equal to any prescribed integer. Thus, we need only show that two regular closed Legendrian curves $S^1 \to (\mathbb{R}^3, \xi)$ with the same rotation number are Legendrian regularly homotopic.

In the front projection of the Legendrian immersion $\gamma$, left and right cusps alternate. We label the up-cusps with + and the down-cusps with −. Up to Legendrian regular homotopy, $\gamma$ is completely determined by the sequence of these labels, starting at a right-cusp, say, and going once around $S^1$. This can be seen by homotoping $\gamma_F$ so that all left cusps come to lie on the line $\{y = 0\}$ and all right cusps on the line $\{y = 1\}$, say. The cusps on either line can be shuffled by further homotopies; in particular, they may be arranged in the same order along these lines in which they are traversed along the closed Legendrian curve. Figure 6 shows a standard model for a front $\gamma_F$ containing cusps of one sign only.

Moreover, a pair $+−$ or $−+$ can be cancelled from this sequence by a Legendrian regular homotopy; locally this is in fact achieved by a Legendrian isotopy, i.e. a regular homotopy not creating self-intersections: the so-called first Legendrian Reidemeister move (see Figure 7; there is an analogous move with the picture rotated by $180^\circ$).

Therefore, this sequence of labels can be reduced to a sequence containing only plus or only minus signs, or to one of the sequences $(+, −)$, $(−, +)$; see Figure 8 for an example. The formula $\text{rot}(\gamma) = (c_− − c_+)/2$ shows that there are the following possibilities: if $\text{rot}(\gamma)$ is positive (resp. negative), we must have a sequence of $2\text{rot}(\gamma)$ minus (resp. plus) signs; if $\text{rot}(\gamma) = 0$, we must have the sequence $(+, −)$ or $(−, +)$. The proof is completed by observing that these last two sequences correspond to Legendrian isotopic knots: use a first Reidemeister move as in Figure 7 followed by the inverse of the rotated move.

Remark. Self-tangencies in the front projection $\gamma_F$ correspond to self-intersections of the Legendrian curve $\gamma$, since the negative slope of $\gamma_F$ gives the $x$-component of $\gamma$. Therefore, as we pass such a self-tangency in the moves of Figure 8, we effect a crossing change. With the orientation indicated in the figure, this example has $\text{rot}(\gamma) = −1$. 

\[ \text{Figure 5. Contribution of a cusp to } \text{rot}(\gamma). \]
Proof of Theorem 1. Again we only have to show that two regular closed curves \( \gamma_0, \gamma_1 : S^1 \to \mathbb{R}^2 \) (where we think of \( \mathbb{R}^2 \) as the \((x,y)\)-plane) with \( \text{rot}(\gamma_0) = \text{rot}(\gamma_1) \) are regularly homotopic.

After a regular homotopy we may assume that the \( \gamma_i \) satisfy the area condition \( \oint_{\gamma_i} x \, dy = 0 \) and thus lift to regular closed Legendrian curves \( \gamma_i : S^1 \to (\mathbb{R}^3, \xi) \) with
— by Proposition 2 — \( \text{rot}(\gamma_i) = \text{rot}(\gamma_i') \). By the preceding proposition, \( \gamma_0 \) and \( \gamma_1 \) are Legendrian regularly homotopic. The Lagrangian projection of this homotopy gives a regular homotopy between the curves \( \tau_0 \) and \( \tau_1 \), since — as pointed out in Section 2 — the Lagrangian projection of a regular Legendrian curve is regular. □

References

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