Multifractal analysis for the historic set in topological dynamical systems

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Abstract
In this paper the historic set is divided into different level sets and we use topological pressure to describe the size of these level sets. We give an application of these results to dimension theory. Our primary focus is using topological pressure to describe the relative multifractal spectrum of ergodic averages and to give a positive answer to the conjecture posed by Olsen (2003 \textit{J. Math. Pures Appl.} 82 1591–649).

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1. Introduction

Barreira \textit{et al} [4] introduced the general concept of the multifractal spectrum:

Fix a metric space $X$ and a set $Y$ and let $\varphi : X \to Y$ be a map. Recently the following problem, known as the multifractal analysis of the map $\varphi$, has attracted considerable interest. What is the Hausdorff dimension or the topological entropy or ... of the level sets of $\varphi$, i.e., what is the Hausdorff dimension or the topological entropy or ... of the following so-called multifractal decomposition sets of $\varphi$?

$$E(t) = \{ x \in X : \varphi(x) = t \}, \quad t \in Y.$$ (1.1)

For a topological dynamical system $(X, d, T)$ (or $(X, T)$ for short) consisting of a compact metric space $(X, d)$ and a continuous map $T : X \to X$, let $\varphi(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(T^ix)$
for some continuous function $\psi : X \to \mathbb{R}$. Then there are the fruitful results regarding descriptions of the structure (Hausdorff dimension or topological entropy or topological pressure) of the level sets of $\psi$ in topological dynamical systems. Early studies of the level sets addressed their dimensions and topological entropy. See Barreira and Saussol [5], Barreira et al [6], Oliver [14, 15], Fan and Feng [12], Takens and Verbitskiy [28], and Pfister and Sullivan [25]. Recently, the topological pressures of the level sets have also been investigated. See Thompson [30], Pei and Chen [22], Yamamoto [33] and Climenhaga [10].

An orbit $\{x, T(x), T^2(x), \cdots\}$ has historic behaviour if for some continuous function $\psi : X \to \mathbb{R}$, the average

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(T^i(x))$$

does not exist. This terminology was introduced by Ruelle in [26]. If this limit does not exist, it follows that the ‘partial averages’ $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(T^i x)$ change considerably so that their values give information about the epoch to which $n$ belongs. The problem of whether there are persistent classes of smooth dynamical systems such that the set of initial states which give rise to orbits with historic behaviour has ‘positive Lebesgue measure’ was discussed by Ruelle [26]. Takens also investigated the problem in a review paper [27]. The developments in the study of historic sets are similar to those in the research of multifractal decomposition sets. In 2000, Barreira and Schmeling [7] showed that the historic set of ‘non-typical’ points has full topological entropy and full Hausdorff dimension. Later, Fan et al [13] studied the historic set of the ergodic limit of a continuous function in a topologically mixing subshift of finite type on an alphabet consisting of finitely many symbols. After that, Barreira et al [6] established a higher-dimensional version of historic sets for several classes of hyperbolic dynamical systems. For general topological dynamical systems with some mild mixing conditions, Chen et al [8] proved that the entropy of the historic set can be equal to that of the whole space. Recently, Thompson [29] extended this to topological pressure. Very recently, Zhou and Chen [34] computed the topological pressure of historic sets for $\mathbb{Z}^d$-actions.

All these investigations revealed that the concept of multifractal analysis plays an important role in the study of dynamical systems. The reader is referred to [2, 3] and references therein for new developments in multifractal analysis.

Now, we present the framework introduced by Olsen. Denote by $M(X)$, $M(X, T)$, and $E(X, T)$, the set of all Borel probability measures on $X$, the collection of all $T$-invariant Borel probability measures, and the set of all ergodic $T$-invariant Borel probability measures, respectively. It is well known that $M(X)$ and $M(X, T)$ are both convex, compact spaces endowed with weak* topology. For $\mu, \nu \in M(X)$, define a compatible metric $\rho$ on $M(X)$ as follows:

$$\rho(\mu, \nu) := \sum_{k \geq 1} \left| \frac{1}{2^k} \int_X f_k \, d\mu - \int_X f_k \, d\nu \right|,$$

where \{f_1, f_2, \cdots\} is countable and dense in $C(X, [0, 1])$. Note that $\rho(\mu, \nu) \leq 1$, for any $\mu, \nu \in M(X)$. This paper uses an equivalent metric on $X$, still denoted by $d$,

$$d(x, y) := \rho(\delta_x, \delta_y),$$

for convenience. For $n \in \mathbb{N}$, let $L_n : X \to M(X)$ be the $n$th empirical measure, i.e.,

$$L_n x = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x}.$$
where $\delta_x$ denotes the Dirac measure at $x$. Let $\Xi$ be a continuous affine map from $M(X)$ to a vector space $Y$ with a linear compatible metric $d'$. $(Y, \Xi)$ is called a deformation of $L_n$. Let $A(x_n)$ be the set of accumulation points of $\{x_n\}$ and let $D(T, \Xi)$ be the set consisting of the points $x$ such that $\lim_{n \to \infty} \Xi L_n x$ does not exist. $D(T, \Xi)$ is called the historic set for $(X, T)$.

This paper is devoted to investigating the structure of $D(T, \Xi)$ via the following framework, introduced and developed by Olsen [16–19] and Olsen and Winter [20].

More precisely, for a subset $C$ of $Y$, this paper uses the topological pressure to describe the size of the so-called sup set, equ set, and sub set:

$$
\Delta_{sup}(C) = \{ x \in X : A(\Xi L_n x) \subset C \},
$$

$$
\Delta_{equ}(C) = \{ x \in X : A(\Xi L_n x) = C \},
$$

$$
\Delta_{sub}(C) = \{ x \in X : A(\Xi L_n x) \supset C \}.
$$

Such sets together give us a complete description of the dynamics of the historic set and provide the basis for a substantially better understanding of the underlying geometry of the historic set. More generally, for $S_1, S_2 \subset Y$, putting

$$
\Delta_1(S_1, S_2) = \{ x \in X : S_1 \subset A(\Xi L_n x) \subset S_2 \},
$$

we have

$$
\Delta(\emptyset, C) = \Delta_{sup}(C);
$$

$$
\Delta(C, C) = \Delta_{equ}(C);
$$

$$
\Delta(C, Y) = \Delta_{sub}(C).
$$

Obviously, multifractal analysis is a special case of this framework. For example, for any $\phi \in C(X, \mathbb{R})$, choose $Y = \mathbb{R}$, and define $\Xi : M(X) \to \mathbb{R}$ by $\Xi : \mu \mapsto \int \phi \, d\mu$. Then for $C = \{a\} \subset \mathbb{R}$, it follows that

$$
\Delta_{equ}(C) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T_k x) = a \right\},
$$

(1.2)

and

$$
D(T, \Xi) = \left\{ x \in X : \text{the limit } \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T_k x) \text{ does not exist} \right\}. \quad (1.3)
$$

At first, $D(T, \Xi)$ was considered of little interest in dynamical systems and geometric measure theory due to the fact that $\mu(D(T, \Xi)) = 0$ for any $\mu \in M(X, T)$. However, recent research [7, 8, 29, 32, 34, 35] has changed such attitudes. The Hausdorff dimension or topological entropy or topological pressure of (1.3) can be large enough, even equal to that of the whole space. This illustrates that the historic set has rich information. Hence, it is meaningful to divide the historic set into different level sets and investigate these level sets. A series of results in symbolic space and iterated function systems can be found in [1, 9, 16–19, 32].

This investigation divides $D(T, \Xi)$ into different level sets $\Delta_{equ}(\cdot)$ and $\Delta_{sub}(\cdot)$ and uses the topological pressure to describe $\Delta_{equ}(\cdot)$, $\Delta_{sub}(\cdot)$ and so on. Topological pressure is a powerful tool and is not only a generalization of topological entropy but also closely related to the Hausdorff dimension. The present paper discusses the dynamical systems satisfying the g-almost product property and the uniform separation property that were introduced by Pfister and Sullivan [24, 25]. These two properties are strictly weaker than the specification property and the positive expansive property. (For example, all $\beta$-shifts have the g-almost product property and the set of all $\beta$-shifts with the specification property has Lebesgue measure 0 and the uniform separation property is true for expansive and more generally asymptotically h-expansive maps.)
As an application of our results, we study symbolic spaces and iterated function systems. We stress that the metric in the symbolic space here can be ultrametric.

For $\varphi \in C(X, \mathbb{R})$, define

$$
\Lambda(y, \varphi) = \begin{cases} 
\sup_{\mu \in \mathcal{M}(X, T)} \left\{ h(T, \mu) + \int \varphi \, d\mu \right\}, & \text{for } y \in \Xi(M(X, T)) \\
-\infty, & \text{otherwise}.
\end{cases}
$$

(1.4)

Next, we state our main theorems.

**Theorem 1.1.** Suppose that $(X, T, \Xi, L_n, Y)$ satisfies the g-almost product property and the uniform separation property (see remark 3.3) and that $\varphi \in C(X, \mathbb{R})$.

(1) If $C \subset Y$ is not a compact and connected subset of $\Xi(M(X, T))$, then

$$
\{ x \in X : \Lambda(\Xi L_n x) = C \} = \emptyset.
$$

(2) If $C \subset Y$ is a compact and connected subset of $\Xi(M(X, T))$, then

$$
P(\Delta_{equ}(C), \varphi) = \inf_{y \in C} \sup_{\mu \in \mathcal{M}(X, T)} \left\{ h(T, \mu) + \int \varphi \, d\mu \right\} = \inf_{y \in C} \Lambda(y, \varphi).
$$

**Theorem 1.2.** Suppose that $(X, T, \Xi, L_n, Y)$ is as before, and that $\varphi \in C(X, \mathbb{R})$.

(1) If $C \subset Y$ is not a subset of $\Xi(M(X, T))$, then

$$
\{ x \in X : C \subset \Lambda(\Xi L_n x) \} = \emptyset.
$$

(2) If $C \subset Y$ is a subset of $\Xi(M(X, T))$, then

$$
P(\Delta_{sub}(C), \varphi) = \inf_{y \in C} \sup_{\mu \in \mathcal{M}(X, T)} \left\{ h(T, \mu) + \int \varphi \, d\mu \right\} = \inf_{y \in C} \Lambda(y, \varphi).
$$

**Theorem 1.3.** Suppose that $(X, T, \Xi, L_n, Y)$ is as before and that $\varphi \in C(X, \mathbb{R})$. Now fix $S_1 \subset \Xi(M(X, T)), S_2 \subset Y$.

(1) If $S_1 = \emptyset$, then

$$
P(\Delta(S_1, S_2), \varphi) = \sup_{x \in S_2} \Lambda(x, \varphi).
$$

(2) If $S_1 \neq \emptyset$ and $S_1$ is contained in a connected component of $S_2$, then

$$
\sup_{Q \in \mathcal{Q}(S_2)} \inf_{x \in Q} \Lambda(x, \varphi) \leq P(\Delta(S_1, S_2), \varphi) \leq \inf_{x \in S_1} \Lambda(x, \varphi).
$$

(3) If $S_1 \neq \emptyset$ and $S_1$ is not contained in a connected component of $S_2$, then

$$
\{ x \in X : S_1 \subset \Lambda(\Xi L_n x) \subset S_2 \} = \emptyset.
$$

**Theorem 1.4.** Suppose that $(X, T, \Xi, L_n, Y)$ is as before and that $\varphi \in C(X, \mathbb{R})$. Fix $S_1 \subset \Xi(M(X, T)), S_2 \subset Y$.

(1) If $S_1 = \emptyset$, then

$$
P(\Delta(S_1, S_2), \varphi) = \sup_{x \in S_2} \Lambda(x, \varphi).
$$

(2) If $S_1 \neq \emptyset$ and $\tau(S_1)$ and the closed convex hull of $S_1$ is contained in a connected component of $S_2$, then

$$
P(\Delta(S_1, S_2), \varphi) = \inf_{x \in S_1} \Lambda(x, \varphi).$$
3. If \( S_1 \neq \emptyset \) and \( S_1 \) is not contained in a connected component of \( S_2 \), then
\[
\{x \in X : S_1 \subset A(\mathcal{E}L_n x) \subset S_2\} = \emptyset.
\]

The paper is organized as follows:

In section 2, some notations and definitions are given. In section 3, we compute the upper and lower bounds for \( P(G_{K'}^r, \psi) \). The proofs of the theorems are presented in section 4. In section 5, we give some applications of our theorems and answer the conjecture posed by Olsen [17] positively.

2. Preliminaries

In this section, we first present some notation to be used in the paper. Then a weak specification property and a weak expansive property and topological pressure are introduced. Lastly, some properties of \( M(X) \) are also discussed.

A remark about the notation is presented here for convenience.

**Remark 2.1.** Let \((X, T)\) be a topological dynamical system.

- If \( F \subset M(X) \) be a neighbourhood, set \( X_{n,F} := \{x \in X : L_n x \in F\} \).
- Given \( \delta > 0 \) and \( \epsilon > 0 \), two points \( x \) and \( y \) are \((\delta, n, \epsilon)\)-separated if \( \#\{j : d(T^j x, T^j y) > \epsilon, 0 \leq j \leq n - 1\} \geq \delta n \). A subset \( E \) is \((\delta, n, \epsilon)\)-separated if any pair of different points of \( E \) are \((\delta, n, \epsilon)\)-separated.

- Let \( F \subset M(X) \) be a neighbourhood of \( x \), and \( \epsilon > 0 \), and set \( N(F; n, \epsilon) := \) maximal cardinality of an \((n, \epsilon)\)-separated subset of \( X_{n,F} \);

- Given \( x \in X \), set \( B_n(x, \epsilon) := \{y \in X : d_n(x, y) \leq \epsilon\} \), where \( d_n(x, y) = \max_{i=0,\ldots,n-1} d(T^i x, T^i y) \).

- A point \( x \in X \), \( \epsilon \)-shadows a sequence \( \{x_0, x_1, \ldots, x_k\} \) if \( d(T^j x, x_j) \leq \epsilon, j = 0, 1, \ldots, k \).

- Let \( g : \mathbb{N} \to \mathbb{N} \) be a given non-decreasing unbounded map with the properties \( g(n) < n \) and \( \lim_{n \to \infty} \frac{g(n)}{n} = 0 \). The function \( g \) is called a blow-up function. Given \( x \in X \) and \( \epsilon > 0 \), the \( g \)-blow-up of \( B_n(x, \epsilon) \) is the closed set \( B_n(g; x, \epsilon) := \{y \in X : \exists \Lambda \subset \Lambda_n, \#(\Lambda_n \setminus \Lambda) \leq g(n) \) and \( \max\{d(T^j x, T^j y) : j \in \Lambda\} \leq \epsilon\} \), where \( \Lambda_n = \{0, 1, \ldots, n - 1\} \).

- (i) Given \( K \subset M(X, T) \), set \( G_K := \{x \in X : A(L_n(x)) = K\} \).

- (ii) Given \( K' \subset \mathbb{Z}(X, T) \), set \( G_{K'} := \{x \in X : A(\mathcal{E}L_n(x)) = K'\} \).

- (iii) Given \( K \subset M(X) \), set \( K G := \{x \in X : A(L_n(x)) \cap K \neq \emptyset\} \).

- (iv) Given \( K' \subset Y \), set \( K' G^* := \{x \in X : A(\mathcal{E}L_n(x)) \cap K' \neq \emptyset\} \).

**Definition 2.1 ([25]).** The dynamical system \((X, d, T)\) has the \( g \)-almost product property with blow-up function \( g \) if there exists a non-increasing function \( m : \mathbb{R}^+ \to \mathbb{N} \) such that for any \( k \in \mathbb{N} \), any \( x_1, x_2, \ldots, x_k \in X \), any positive \( \epsilon_1, \epsilon_2, \ldots, \epsilon_k \), and any integers \( n_1 \geq m(\epsilon_1), \ldots, n_k \geq m(\epsilon_k) \),
\[
\bigcap_{j=1}^{k} T^{-M_{j-1}} B_{n_j}(g; x_j, \epsilon_j) \neq \emptyset,
\]
where \( M_0 = 0, M_i = n_1 + n_2 + \cdots + n_i, i = 1, 2, \ldots, k - 1 \).
Definition 2.2 ([25]). The dynamical system \((X, d, T)\) has the uniform separation property if for any \(\eta\), there exist \(\delta^* > 0\) and \(\epsilon^* > 0\) such that for \(\mu \geq n_{F,\mu,\eta}^*\), \(N(F; \delta^*, \epsilon^*) \geq \exp(n(h(T, \mu) - \eta))\), where \(h(T, \mu)\) is the metric entropy of \(\mu\).

Proposition 2.2 ([25]). Suppose that \((X, d, T)\) has the \(g\)-almost product property. Let \(x_1, \ldots, x_k \in X, \epsilon_1 > 0, \ldots, \epsilon_k > 0, \) and \(n_1 \geq m(\epsilon_1), \ldots, n_k \geq m(\epsilon_k)\) be given. Assume that \(L_{n_j}(x_j) \in B(\nu_j, \zeta_j), 0 \leq j \leq k.\) Then for any \(y \in \bigcap_{i=1}^k T^{-M-1}B_{n_i}(g; x_i, \epsilon_i)\) and any probability measure \(\alpha, \rho(L_{M_k}(y), \alpha) \leq k \sum_{j=1}^{nj} M_k(\zeta'_j + \rho(\nu_j, \alpha))\), where \(M_j = n_1 + \cdots + n_j, \zeta'_j = \zeta_j + \epsilon_j + \frac{g(n_j)}{n_j}, j = 1, \ldots, k.\)

Definition 2.3 ([23]). Suppose \(Z \subset X, \varphi \in C(X, \mathbb{R}).\) Let \(\Gamma_n(Z, \epsilon)\) be the collection of all finite or countable covers of \(Z\) by sets of the form \(B(m, \epsilon), m \geq n.\) Let \(S_n \varphi(x) := \sum_{i=0}^{n-1} \varphi(T^ix).\) Set \(M(Z, t, \varphi, \epsilon) := \inf_{C \in \Gamma_n(Z, \epsilon)} \left\{ \sum_{B_m(x, \epsilon) \in C} \exp(-tm + \sup_{y \in B_m(x, \epsilon)} S_m \varphi(y)) \right\}\), and \(M(Z, t, \varphi, \epsilon) = \lim_{n \to \infty} M(Z, t, \varphi, \epsilon).\) Then there exists a unique number \(P(Z, \varphi, \epsilon)\) such that \(P(Z, \varphi, \epsilon) = \inf \{t : M(Z, t, \varphi, \epsilon) = 0\} = \sup \{t : M(Z, t, \varphi, \epsilon) = \infty\}.\)

Proposition 2.3. Assume that \(d'\) is a linearly compatible metric in \(Y.\) Put \(V(\Xi, \epsilon) := \sup_{\mu, \nu \in M(X)} \rho(\mu, \nu) < \epsilon d'(\Xi \mu, \Xi \nu)\). Then \(V(\Xi, \epsilon) \to 0\) as \(\epsilon \to 0.\)

Proof. This is because \(\Xi : M(X) \to Y\) is continuous and \(M(X)\) is compact. \(\square\)

Proposition 2.4. For any \(x \in X\) and any \(\epsilon > 0,\) there exists \(N\) sufficiently large such that for all \(n > N\) we have \(\rho(L_n x, L_{n+1} x) \leq \epsilon.\)
Proof. Choose $N$ sufficiently large so that $\frac{1}{N+1} < \epsilon$. Then
\[
\rho(L_n x, L_{n+1} x) = \sum_{k \geq 1} 2^{-k} \left| \int f_k \, dL_n x - \int f_k \, dL_{n+1} x \right|
\]
\[
= \sum_{k \geq 1} 2^{-k} \left| \int f_k \, dL_n x - \int \frac{n}{n+1} f_k \, dL_n x - \int \frac{1}{n+1} f_k \, d\delta T^n x \right|
\]
\[
= \sum_{k \geq 1} 2^{-k} \frac{1}{n+1} \left| \int f_k \, dL_n x - \int f_k \, d\delta T^n x \right|
\]
\[
\leq \frac{1}{n+1} \rho(L_n x, \delta T^n x)
\]
\[
\leq \frac{1}{n+1} \leq \frac{1}{N+1} \leq \epsilon.
\]

Proposition 2.5. For any $x \in X$ and any $\epsilon > 0$, there exists $N$ sufficiently large so that for all $n > N$ we have that $x' \in B_\epsilon(g, x, \epsilon)$ implies $\rho(L_n x, L_n x') < 2\epsilon$.

Proof. Since $\lim_{n \to \infty} \frac{x(n)}{n} = 0$, we have that for any $\epsilon > 0$, there exists a sufficiently large $N \in \mathbb{N}$ such that $\frac{x(n)}{n} < \epsilon$ whenever $n > N$. Then
\[
\rho(L_n x, L_n x') = \sum_{k \geq 1} 2^{-k} \left| \int f_k \, dL_n x - \int f_k \, dL_n x' \right|
\]
\[
\leq \sum_{k \geq 1} 2^{-k} \frac{g(n)}{n} + \frac{n - g(n)}{n} \epsilon
\]
\[
\leq 2\epsilon.
\]

Proposition 2.6 ([32]). For any $x \in X$, $A(\Sigma L_n x)$ is a compact and connected subset of $Y$.

3. Upper and lower bounds for $P(G_K^*, \varphi)$

This section finds the upper and lower bounds for $P(G_K^*, \varphi)$.

Proposition 3.1 ([22]). Let $(X, d, T)$ be a dynamical system.

(i) If $K \subseteq M(X, T)$ is a closed subset, then
\[
P(K, \varphi) \leq \sup \left\{ h(T, \mu) + \int \varphi \, d\mu : \mu \in K \right\}.
\]

(ii) If $\mu \in M(X, T)$, then
\[
P(G_\mu, \varphi) \leq h(T, \mu) + \int \varphi \, d\mu.
\]

(iii) If $K \subseteq M(X, T)$ is a non-empty closed set, then
\[
P(G_K, \varphi) \leq \inf \left\{ h(T, \mu) + \int \varphi \, d\mu : \mu \in K \right\}.
\]

By the above proposition, we get the following theorem.
**Theorem 3.1.** Let $(X, d, T)$ be a dynamical system.

1. If $K' \subset Y$ is a closed subset, then
   \[ P(K^*_{G^*}), \varphi) \leq \sup \{ \Lambda(y, \varphi) : y \in K' \} . \]

2. If $K' \subset Y$ is a non-empty closed set, then
   \[ P(G^*_{K'}, \varphi) \leq \inf \{ \Lambda(y, \varphi) : y \in K' \} . \]

**Proof.**

1. If $K' \subset Y$ is a closed subset, then
   \[ K' G^* = \left\{ x \in X : A(L_n x) \cap K' \neq \emptyset \right\} = \left\{ x \in X : \Xi (A(L_n x)) \cap K' \neq \emptyset \right\} = \left\{ x \in X : (A(L_n x)) \cap \Xi^{-1} K' \neq \emptyset \right\} = \Xi^{-1} K' \cap M(X, T) G . \]
   Hence,
   \[ P(K^*_{G^*}), \varphi) = P(\Xi^{-1} K' \cap M(X, T) G, \varphi) \leq \sup \left\{ h(T, \mu) + \int \varphi d\mu : \mu \in \Xi^{-1} K' \cap M(X, T) \right\} = \sup \{ \Lambda(y, \varphi) : y \in K' \} . \]

2. If $K' \not\subseteq \Xi M(X, T)$, then $G^*_{K'} = \emptyset$. In this case, there exists $y \in K' \setminus \Xi M(X, T)$ such that $\Lambda(y, \varphi) = -\infty$. If $K' \subseteq \Xi M(X, T)$, then for any $y \in K'$ we have
   \[ G^*_{K'} = \left\{ x \in X : A(\Xi L_n x) = K' \right\} \leq \left\{ x \in X : A(\Xi L_n x) \cap \{ y \} \neq \emptyset \right\} = \{ y \} G^* . \]
   Then,
   \[ P(G^*_{K'}, \varphi) \leq P(\{ y \} G^*), \varphi) \leq \Lambda(y, \varphi), \]
   for any $y \in K'$. In summary,
   \[ P(G^*_{K'}, \varphi) \leq \inf \{ \Lambda(y, \varphi) : y \in K' \} . \]

To obtain the lower bound of $P(G^*_{K'}, \varphi)$, we need to suppose that the dynamical system satisfies some mild conditions.

**Proposition 3.2 ([25]).** Assume that $(X, d, T)$ has the $g$-almost product property and the uniform separation property. For any $\eta$, there exists $\delta^*$ and $e^* > 0$ such that for $\mu \in M(X, T)$ and any neighbourhood $F \subset M(X)$ of $\mu$, there exists $n_{F, \mu}^{e^*}$ such that
   \[ N(F; \delta^*, e^*) \geq \exp(n(h(T, \mu) - \eta)) . \]
   whence $n \geq n_{F, \mu}^{e^*}$. Furthermore, for any $\mu \in M(X, T)$,
   \[ h(T, \mu) \leq \lim \inf_{\epsilon \to 0} \lim_{\delta \to 0} \inf_{F \ni \mu} \lim_{n \to \infty} \frac{1}{n} \log N(F; \delta, n, \epsilon) . \]
Lemma 3.1 ([25]). If \( K' \) is a connected, non-empty and compact subset of \( \Omega(M(X, T)) \), then there exists a sequence \( \{a''_j, a''_{j+1}, \ldots\} \) in \( K' \) such that

\[
\{a''_j : j \in \mathbb{N}, j > n\} = K',
\]

for any \( n \in \mathbb{N} \), and \( \lim_{j \to \infty} d'(a''_j, a''_{j+1}) = 0 \).

Theorem 3.2. Suppose \((X, d, T)\) is a dynamical system with the uniform separation and \( g\)-almost product property and let \( \varphi \in C(X, \mathbb{R}) \). If \( K' \) is a connected, non-empty and compact subset of \( \Omega(M(X, T)) \), then

\[
\inf_{y \in K'} \Lambda(y, \varphi) \leq P(G^*_K, \varphi).
\]

Proof. Let \( \eta > 0 \) and \( h^* := \inf_{y \in K'} \Lambda(y, \varphi) - \eta \). For any \( s < h^* \), set \( h^* - s := 2\delta > 0 \). Given a sequence \( \{a''_j\} \) as in lemma 3.1, we construct a subset \( G \) such that for each \( x \in G, \{\Xi L_{n_k} x\} \) has the same limit-point set as the sequence \( \{a''_j\} \). and \( P(G, \varphi) \geq h^* \). For \( \frac{\eta}{2} \) and \( \alpha''_k \in K' \), there exists \( \alpha_k \in \Xi^{-1}\{a''_k\} \subseteq M(X, T) \) such that \( \Lambda(\alpha''_k, \varphi) \leq h(T, \alpha_k) + \int \varphi \, d\alpha_k + \frac{\eta}{2} \). By proposition 3.2, it is easy to see that for \( \frac{\eta}{2} > 0 \), there exist \( \delta^* > 0 \) and \( \varepsilon^* > 0 \) such that for any neighbourhood \( F'' \subset \Xi(M(X)) \) of \( \alpha''_k \) (choose \( F'' = B(\alpha''_k, \varphi') \)) there exist \( B(\alpha_k, \zeta_k) \subseteq \Xi^{-1} F'' \) and \( n^*_{B(\alpha_k, \zeta_k), \alpha_k, \varphi} \) satisfying

\[
N(B(\alpha_k, \zeta_k); \delta^*, n, \varepsilon^*) \geq \exp \left( n \left( h(T, \alpha_k) - \frac{\eta}{2} \right) \right),
\]

whence \( n \geq n^*_{B(\alpha_k, \zeta_k), \alpha_k, \varphi} \) and \( \zeta_k, \zeta''_k \) will be determined later.

Choose three strictly decreasing sequences \( \{\zeta_k\}, \{\zeta''_k\}_k \) and \( \{\epsilon_k\}_k \), such that

(i) \( \lim_{k \to \infty} \zeta_k = 0 \), \( \lim_{k \to \infty} \zeta''_k = 0 \) and \( \lim_{k \to \infty} \epsilon_k = 0 \),

(ii) \( \epsilon_1 < \varepsilon^* \) and \( \int \varphi \, d\alpha_k - \int \varphi \, d\mu \leq \frac{\delta}{5} \forall \mu \in B(\alpha_k, \zeta_k + 2\epsilon_k) \).

From (3.5), we deduce the existence of \( n_k \) and a \( (\delta^*, n_k, \varepsilon^*) \)-separated subset \( \Gamma_k \subseteq X_{n_k, B(\alpha_k, \zeta_k)} \subseteq X_{n_k, \Xi^{-1} F''} \) with

\[
\#\Gamma_k \geq \exp \left( n_k \left( h(T, \alpha_k) - \frac{\eta}{2} \right) \right).
\]

Assume that \( n_k \) satisfies

\[
\delta^* n_k > 2g(n_k) + 1 \quad \text{and} \quad \frac{g(n_k)}{n_k} \leq \epsilon_k.
\]

The orbit-segments \( \{x, T x, \ldots, T^{n_k - 1} x\}, x \in \Gamma_k \), are the building-blocks for the construction of the points of \( G \). By propositions 2.2 and (3.7), we obtain

\[
x \in \Gamma_k \quad \text{and} \quad y \in B_{n_k}(g; x, \epsilon_k) \Rightarrow \rho(\alpha_k, L_{n_k} y) \leq \zeta_k + 2\epsilon_k
\]

\[
\Rightarrow d'(\alpha''_k, \Xi L_{n_k} y) \leq V(\Xi, \zeta_k + 2\epsilon_k).
\]

Choose a strictly increasing sequence \( \{N_k\} \), with \( N_k \in \mathbb{N} \), such that

\[
n_{k+1} \leq \zeta_k \sum_{j=1}^k n_j N_j
\]

and

\[
\sum_{j=1}^{k-1} n_j N_j \leq \zeta_k \sum_{j=1}^k n_j N_j.
\]
Finally define the (stretched) sequences \( \{n_j'\}, \{\epsilon_j'\} \) and \( \{\Gamma_j'\} \) by setting
\[
n_j' := n_k \quad \epsilon_j' := \epsilon_k \quad \Gamma_j' := \Gamma_k,
\]
for \( j = N_1 + \cdots + N_{k-1} + q \) with \( 1 \leq q \leq N_k \).
\[
G_k := \bigcap_{j=1}^k \left( \bigcup_{x_j \in \Gamma_j'} T^{-M_{j-1}} B_{n_j'} (g; x_j, \epsilon_j') \right),
\]
where \( M_j := \sum_{i=1}^j n_i \). \( G_k \) is a non-empty closed set. Label each set obtained by developing this formula by the branches of a labelled tree of height \( k \). A branch is labelled by \( (x_1, \ldots, x_k) \), with \( x_j \in \Gamma_j' \). Theorem 3.2 is proved by proving lemma 3.2. \( \square \)

**Lemma 3.2.** Let \( \epsilon \) be such that \( 4\epsilon = \epsilon^{*} \), and let
\[
G := \bigcap_{k \geq 1} G_k.
\]
(i) Let \( x_j, y_j \in \Gamma_j' \) with \( x_j \neq y_j \). If \( x \in B_{n_j'} (g; x_j, \epsilon_j') \) and \( y \in B_{n_j'} (g; y_j, \epsilon_j') \), then
\[
\max \{ d(T^m x, T^m y) : m = 0, \ldots, n_j' - 1 \} > 2\epsilon.
\]
(ii) \( G \) is a closed set and is the disjoint union of non-empty closed sets \( G(x_1, x_2, \cdots) \) labelled by \( (x_1, x_2, \cdots) \) with \( x_j \in \Gamma_j' \). Two different sequences label two different sets.
(iii) \( G \subset G_{k'} \).
(iv) \( P(G, \varphi) \geq h^{*} \).

**Proof.** (i) and (ii) can be seen in [25] for details.
(iii) Define the stretched sequence \( \{a_m'\} \) by \( a_m' := a_k \) if \( \sum_{j=1}^{k-1} n_j N_j + 1 \leq m \leq \sum_{j=1}^{k} n_j N_j \).
The sequence \( \{a_m'\} \) has the same limit-point set as the sequence \( \{a_k\} \). \( \{a_m''\} \) has the same limit-point set as the sequence \( \{\Xi a_k\} \). If
\[
\lim_{n \to \infty} d' (\Xi L_n y, \Xi a_m'') = 0,
\]
then the two sequences \( \{\Xi L_n y\} \) and \( \{a_m''\} \) have the same limit-point set. Because of (3.9) and the definition of \( \{a_m''\} \), it is sufficient to show that
\[
\lim_{k \to \infty} d' (\Xi L_{M_k} (y), \alpha_{M_k}^m) = 0.
\]
Suppose that \( \sum_{i=1}^{j} n_i N_i < M_k \leq \sum_{i=1}^{j+1} n_i N_i \); hence \( \alpha_{M_k} = \alpha_{j+1} \). \( M_k \) can be written as
\[
M_k = \sum_{i=1}^{j} n_i N_i + n_j N_j + q n_{j+1}, \quad \text{where} \quad 1 \leq q \leq N_{j+1}.
\]
Since
\[
\Xi \left( L_{\sum_{i=1}^{j} n_i N_i + q n_{j+1}} (y) \right)
\]
\[
= \frac{1}{M_k} \left( \sum_{i=1}^{j-1} n_i N_i L_{\sum_{i=1}^{j-1} n_i N_i} (y) + \sum_{i=0}^{N_j-1} L_{n_j} (T^{\sum_{i=1}^{j} n_i N_i} y) + \sum_{i=0}^{q-1} L_{n_j} (T^{\sum_{i=1}^{j} n_i N_i} y) \right)
\]
\[
= \frac{\sum_{i=1}^{j-1} n_i N_i \Xi (L_{\sum_{i=1}^{j-1} n_i N_i} (y)) + \sum_{i=0}^{N_j-1} \Xi L_{n_j} (T^{\sum_{i=1}^{j} n_i N_i} y)}{M_k} + \frac{\sum_{i=0}^{q-1} \Xi L_{n_j} (T^{\sum_{i=1}^{j} n_i N_i} y)}{M_k}
\]
\[ \alpha''_m = \frac{\sum_{k=1}^{j-1} n_1 \alpha''_m}{M_k} + \frac{\sum_{k=1}^{N_j-1} \alpha''_m}{M_k} + \frac{\sum_{k=1}^{q-1} \alpha''_m}{M_k}, \]

we have

\[
d' \left( \mathbb{E} \left( L_{\sum_{k=1}^{j-1} n_1 N_j} \right), \alpha''_m \right) \leq \frac{\sum_{k=1}^{j-1} n_1 N_j d' \left( \mathbb{E} \left( L_{\sum_{k=1}^{N_j-1} n_1 N_j} \right), \alpha''_m \right)}{M_k}
+ \frac{\sum_{k=1}^{q-1} d' \left( \mathbb{E} L_{\Delta_j n_1 n_1 + n_1 + n_1}, \Xi \alpha_{j+1} \right) \mathbb{E} \left( /Xi_1 \alpha_j, /Xi_1 \alpha_j \right) + V \left( \mathbb{E}, \zeta_j + 2 \epsilon_j \right) + V \left( \mathbb{E}, \zeta_j + 2 \epsilon_j \right)}{M_k}
\leq \zeta_j V \left( \Xi, 1 \right) + d' \left( \Xi \alpha_j, \Xi \alpha_{j+1} \right) + V \left( \Xi, \zeta_j + 2 \epsilon_j \right) + V \left( \Xi, \zeta_j + 2 \epsilon_j \right).
\]

Since \( \lim \zeta_j = 0, \lim \epsilon_j = 0 \Rightarrow \lim d' \left( \Xi \alpha_j, \Xi \alpha_{j+1} \right) = 0 \), this proves (iii).

(iv) From the choice of \( \left\{ N_k \right\} \) we can get \( \lim_{n \to \infty} \frac{M_n}{M_{n+1}} = 1 \), where \( M_j = n_1 + \cdots + n_j \). There exist \( n_k \in \mathbb{N} \) and a \( (\delta^*, n_k, \epsilon^*) \)-separated subset \( \Gamma_k \) of \( X_{n_1 \times \cdots \times n_j} \) such that

\[ \# \Gamma_k \geq \exp \left( n_k (h(T, \alpha_k) - \eta/2) \right). \]

And for any \( x \in \Gamma_k \), we have \( L_n(x) \in B(\alpha_k, \zeta_k) \). So

\[ \left| \int \psi dL_n(x) - \int \psi \, d\alpha_k \right| = \frac{1}{n} S_n \psi(x) - \int \psi \, d\alpha_k \leq \frac{\delta}{6}. \]

Thus

\[ \# \Gamma_k \geq \exp \left( n_k h(T, \alpha_k) + \int \psi \, d\alpha_k - \eta/2 - S_n \psi(x) - n_k \frac{\delta}{6} \right) \]
\[ \geq \exp \left( n_k h^* - S_n \psi(x) - n_k \frac{\delta}{6} \right). \]

Since \( G \) is a compact set we can just consider the finite covers \( C \) of \( G \) with the property that if \( B_m(x, \epsilon) \subseteq C \), then \( B_m(x, \epsilon) \cap G \neq \emptyset \) \( \forall B_m(x, \epsilon) \subseteq C \). For each \( C \in \Gamma_k \), we define the cover \( C' \) in which each ball \( B_m(x, \epsilon) \) is replaced by \( B_{M_m}(x, \epsilon) \) when \( M_m \leq m < M_{m+1} \). Then

\[ M \left( G, s, \psi, n, \epsilon \right) = \inf_{C \in \Gamma_k} \sum_{B_m(x, \epsilon) \subseteq C} \exp \left( -sm + \sup_{y \in B_m(x, \epsilon)} S_m \psi(y) \right) \]
\[ \geq \inf_{C \in \Gamma_k} \sum_{B_m(x, \epsilon) \subseteq C, z \in B_m(x, \epsilon) \cap B_{M_m}(x, \epsilon) \cap G} \exp (-sm + S_m \psi(z) \]

Consider a specific \( C' \) and let \( m \) be the largest value of \( p \) such that there exists a \( B_{M_m}(x, \epsilon) \subseteq C' \). Set

\[ \mathcal{W}_k := \prod_{i=1}^{k} \Gamma'_i, \quad \mathcal{W}_m := \bigcup_{k=1}^{m} \mathcal{W}_k. \]

Each \( z \in B_{M_m}(x, \epsilon) \cap G \) corresponds to a point in \( \mathcal{W}_m \). Lemma 3.2 (i) implies that this point is uniquely defined. For \( 1 \leq j \leq k \), the word \( v = (v_1, \ldots, v_j) \in \mathcal{W}_j \) is a prefix of
$w = (w_1, \ldots, w_k) \in W_k$ if $v_i = w_i, i = 1, \ldots, j$. Note that each $w \in W_k$ is the prefix of exactly $\#W_k/\#W$ words of $W_m$. If $W \subset W$ contains a prefix of each word of $W_m$, then

$$\sum_{k=1}^{m} \#(W \cap W_k) \#W_m/\#W_k \geq \#W_m.$$ 

So if $W$ contains a prefix of each word of $W_m$, then

$$\sum_{k=1}^{m} \#(W \cap W_k)/\#W_k \geq 1.$$ 

Since $C'$ is a cover, each point of $W_m$ has a prefix associated with some $B_{M_p}(x, \epsilon) \in C'$. Hence,

$$\#W_p \geq \exp \left[ M_p h^* - \sum_{i=1}^{p} (S_{\alpha'} \varphi'(x) + n'_\delta/6) \right],$$

where $\varphi' x \in \Gamma'$. So

$$\sum_{B_{M_p}(x, \epsilon) \in C'} \exp \left[ - M_p h^* + \sum_{i=1}^{p} (S_{\alpha'} \varphi'(x) + n'_\delta/6) \right] \geq 1.$$ 

Next, we want to prove

$$M_p h^* - \sum_{i=1}^{p} (S_{\alpha'} \varphi'(x) + n'_\delta/6) - sm + S_m \varphi(z) = m(h^* - s) + \sum_{i=1}^{p} (S_{\alpha'} \varphi(T_{M_i-1} z) - S_{\alpha'} \varphi'(z)) + n'_\delta/6 + S_{M_p} \varphi(T_{M_0} y) - (m - M_p) h^* > 0.$$ 

Since $z \in G$, by the construction of $G$ we know there exists a closed subset

$$G(x_1, x_2, \ldots) = \bigcap_{i=0}^{\infty} T^{-M_{i-1}} B_{\alpha'}(g; x_j, \epsilon'_j)$$

such that $T_{M_i-1} z \in B_{\alpha'}(g; x_j, \epsilon'_j)$.

By (3.8) and the fact that $\varphi' x \in \Gamma'_{\alpha'}$, we get $L_{\alpha'}(T_{M_i-1} z) \in B(\alpha', \zeta_i + 2\epsilon'_i)$ and $L_{\alpha'}(\varphi' x) \in B(\alpha', \zeta_i)$. Thus,

$$\left| \int \varphi dL_{\alpha'}(T_{M_i-1} z) - \int \varphi dL_{\alpha'}(\varphi' x) \right| \leq \left| S_{\alpha'} \varphi(T_{M_i-1} z) - S_{\alpha'} \varphi'(z) \right| \leq n'_\delta/2.$$ 

So,

$$M_p h^* - \sum_{i=1}^{p} (S_{\alpha'} \varphi'(z) + n'_\delta/6) - sm + S_m \varphi(z)$$

$$\geq m(h^* - s) - \sum_{i=1}^{2n'_\delta/3 - n'_p} + \frac{n_p}{6} \geq 2\delta M_p - m(h^* - n'_p + \frac{n_p}{6})$$

Since $\lim_{p \to \infty} \frac{n_p}{2\delta} = 0$, it is possible to choose a sufficiently large $p$ such that $M_p \delta - n'_p + \frac{n_p}{6} > 0$. Then

$$\sum_{B_{\alpha'}(x, \epsilon) \in C'} \exp \left( - sm + \sup_{y \in B_{\alpha'}(x, \epsilon)} S_m \varphi(y) \right)$$

$$\geq \sum_{B_{\alpha'}(x, \epsilon) \in C'} \exp \left[ - M_p h^* + \sum_{i=1}^{p} (S_{\alpha'} \varphi'(z) + n'_\delta/6) \right].$$

This implies $M(G, s, \varphi, n, \epsilon) \geq 1$, i.e., $s \leq P(G, \varphi, \epsilon)$. Letting $s \to h^*$, we complete the proof of lemma 3.2.
Remark 3.3. That a quintuple \((X, T, Y, \Xi, L_n)\) satisfies the g-almost product property and the uniform separated condition means that

(a) \(X\) is a compact metric space and \(T : X \to X\) is a continuous map satisfying the g-almost product property and the uniform separated condition.

(b) \(Y\) is a vector space and \(\Xi : M(X) \to Y\) is a continuous and affine map.

(c) \(L_n x : X \to M(X)\), where \(L_n x = \sum_{i=0}^{n-1} \delta_{T^i x}\).

For \(y \in Y\), set

\[
\Delta(y) = \{x \in X : \{y\} = A(\Xi L_n x)\}, \quad \tilde{\Delta}(y) = \{x \in X : y = A(\Xi L_n x)\}.
\]

It is easy to obtain the following corollary from the above two theorems.

Corollary 3.1. For any \(\varphi \in C(X, \mathbb{R})\), if \((X, T, Y, \Xi, L_n)\) satisfies the g-almost product property and the uniform separated condition, then

\[
P(\Delta(y), \varphi) = P(\tilde{\Delta}(y), \varphi).
\]

4. Proofs of the theorems

This aim of this section is to prove the theorems.

Proof of theorem 1.1.

(1) This follows from proposition 2.6.

(2) Since \(C \subset Y\) is a compact and connected subset of \(\Xi(M(X, T))\), then \(P(\Delta_{\text{eq}}(C), \varphi) = P(G^*_C, \varphi) = \inf_{y \in C} \Lambda(y, \varphi)\).

Proof of theorem 1.2.

(1) This is obvious.

(2) This is proved by means of several lemmas.

Lemma 4.1. Let \((X, T, Y, \Xi, L_n)\) satisfy the g-almost product property and the uniform separated condition. If \(C \subset Y\) and \(\varphi \in C(X, \mathbb{R})\), then

\[
\inf_{y \in C} \Lambda(y, \varphi) = \inf_{y \in \text{co}(C)} \Lambda(y, \varphi),
\]

where \(\text{co}(C)\) is the convex hull of \(C\).

Proof. The direction \(\geq\) is obvious. As to the other direction, for any \(y \in \text{co}(C)\), fix \(\epsilon > 0\). Since \(y \in \text{co}(C)\), there exist \(y_1, y_2, \ldots, y_n \in C\) and \(\lambda_1, \ldots, \lambda_n \geq 0\) with \(\sum \lambda_i = 1\), such that \(\sum \lambda_i y_i = y\). For each \(y_i\), choose \(\mu_i \in M(X, T)\) such that \(\Xi \mu_i = y_i\) and \(h(T, \mu_i) + \int \varphi d\mu_i \geq \Lambda(y_i, \varphi) - \epsilon\). Since the entropy function is affine and \(\sum \lambda_i \mu_i \in M(X, T)\) satisfies \(\Xi(\sum \lambda_i \mu_i) = \sum \lambda_i \Xi \mu_i = \sum \lambda_i y_i = y\), we get

\[
\Lambda(y, \varphi) \geq \sup_{\mu \in M(X, T), \Xi \mu = y} \left\{ h(T, \mu) + \int \varphi d\mu \right\} \geq h\left(T, \sum \lambda_i \mu_i\right) + \int \varphi d\left(\sum \lambda_i \mu_i\right) = \sum \lambda_i \left(h(T, \mu_i) + \int \varphi d\mu_i\right).
\]
\[ \sum_i \lambda_i \Lambda(y, \varphi) - \epsilon \geq \inf_{y \in C} \Lambda(y, \varphi) - \epsilon. \]

Thus,
\[ \inf_{y \in C} \Lambda(y, \varphi) \leq \inf_{y \in \text{co}(C)} \Lambda(y, \varphi). \]

**Lemma 4.2.** Let \((X, T, Y, \Sigma, L_n)\) satisfy the g-almost product property and the uniform separated condition. If \(C \subset Y\) and \(\varphi \in C(X, \mathbb{R})\), then
\[ \inf_{y \in C} \Lambda(y, \varphi) = \inf_{y \in \text{co}(C)} \Lambda(y, \varphi). \]

**Proof.** The direction \(\geq\) is obvious. The other direction follows from the fact that \(y \rightarrow \Lambda(y, \varphi)\) is upper semi-continuous. Pfister and Sullivan [25] proved that the entropy map on \(M(X, T), \mu \rightarrow h(T, \mu)\), is upper semi-continuous under the g-almost product property and uniformly separation property. \(\forall \gamma > 0, \forall y \in \overline{C}, \exists \{y_n\} \subset C\), such that \(y_n \rightarrow y\), as \(n \rightarrow \infty\) and there exists \(\mu_n \in M(X, T) \cap \Xi^{-1}y_n\), such that \(\Lambda(y_n, \varphi) \leq h(T, \mu_n) + \int \varphi \, d\mu_n + \gamma\). Assume that \(\mu\) is a limit point of \(\{\mu_n\}\). Then \(\Xi\mu = y\). So
\[ \limsup_{n \rightarrow \infty} \Lambda(y_n, \varphi) \leq \limsup_{n \rightarrow \infty} h(T, \mu_n) + \int \varphi \, d\mu_n + \gamma \leq h(T, \mu) + \int \varphi \, d\mu + 3\gamma \leq \Lambda(y, \varphi) + 3\gamma. \]
The conclusion of lemma 4.2 follows.

Now, continue the proof of (2). It suffices to show that for any nonempty \(C \subset \Xi(M(X, T))\),
\[ P(\Delta_{\text{equ}}(\text{co}(C)), \varphi) = P(\Delta_{\text{sub}}(C), \varphi). \]
Since \(\Delta_{\text{equ}}(\text{co}(C)) = \{x \in X|A(\Xi L_n x) = \text{co}(C)\} \subset \{x \in X|C \subset A(\Xi L_n x)\} = \Delta_{\text{sub}}(C)\), it is obvious that \(P(\Delta_{\text{sub}}(C), \varphi) \geq P(\Delta_{\text{equ}}(\text{co}(C)), \varphi)\). On the other hand, by corollary 3.1, if \(\Delta(y) \neq \emptyset\), then \(P(\Delta(y), \varphi) = P(\overline{\Delta}(y), \varphi)\). So for any \(y \in C\),
\[ P(\Delta_{\text{sub}}(C), \varphi) \leq P(\{x \in X|[y] \subset A(\Xi L_n x)\}, \varphi) = P(\overline{\Delta}(y), \varphi) = P(\Delta(y), \varphi) = \Lambda(y, \varphi). \]
Hence,
\[ P(\Delta_{\text{sub}}(C), \varphi) \leq \inf_{y \in C} \Lambda(y, \varphi) = \inf_{y \in \text{co}(C)} \Lambda(y, \varphi) = P(\Delta_{\text{equ}}(\text{co}(C)), \varphi). \]
So,
\[ P(\Delta_{\text{sub}}(C), \varphi) = \inf_{y \in C} \sup_{\mu \in M(X, T)} \left\{ h(T, \mu) + \int \varphi \, d\mu \right\} = \inf_{y \in C} \Lambda(y, \varphi). \]

**Proof of theorem 1.3.** First, we show the following proposition.
Proposition 4.1. Suppose that $(X, T, Y, \Xi, L_n)$ is as before, and suppose $\varphi \in C(X, \mathbb{R})$. If $C \subseteq Y$, then

\[ P(\Delta_{sup}(C), \varphi) = P(\Delta_{sup}(C \cap \Xi(M(X, T))), \varphi) = \sup_{y \in C} \Lambda(y, \varphi). \]

Proof. $P(\Delta_{sup}(C), \varphi) \leq P(C^* G^*, \varphi) \leq \sup_{y \in C} \Lambda(y, \varphi).$

On the other hand, $\forall \epsilon > 0, \exists y' \in C \cap \Xi(M(X, T)), \text{ such that } \sup_{y \in C} \Lambda(y, \varphi) \leq \Lambda(y', \varphi) + \epsilon,$

and $\Lambda(y', \varphi) = P(G_{[y']}^1, \varphi) \leq P(\Delta_{sup}(C), \varphi).$

So,

\[ P(\Delta_{sup}(C), \varphi) + \epsilon \geq \sup_{y \in C} \Lambda(y, \varphi). \]

Thus,

\[ P(\Delta_{sup}(C), \varphi) = \sup_{y \in C} \Lambda(y, \varphi). \]

Now, we continue the proof of theorem 1.3.

(1) This comes from proposition 4.1.

(2) We are given that $S_1 \subseteq Q \subseteq S_2, \varphi \subseteq \Xi(M(X, T))$ is compact and connected.

Since $\Delta_{equiv}(Q) = \{ x \in X | A(\Xi L_n x) = Q \} \subseteq \{ S_1 \subseteq A(\Xi L_n x) \subseteq S_2 \}$, we then get

\[ P(\Delta(S_1, S_2), \varphi) \geq P(\Delta_{equiv}(Q), \varphi) = \inf_{x \in Q} \Lambda(x, \varphi). \]

Since $Q$ is arbitrary, we obtain

\[ P(\Delta(S_1, S_2), \varphi) \geq \sup_{Q \subseteq \Xi(M(X, T)) \text{ is compact and connected}} \inf_{x \in Q} \Lambda(x, \varphi). \]

As to the other inequality, observe that

\[ \Delta(S_1, S_2) \subseteq \Delta(S_1, Y) = \Delta_{sub}(S_1). \]

(3) This is obvious.

Proof of theorem 1.4.

(1) This follows from proposition 4.1.

(2) Combining the fact that $S_1 \subseteq \overline{\sigma}(S_1)$ and that $\overline{\sigma}(S_1)$ is a compact and connected subset of $S_2$, and using theorem 1.3, we obtain

\[ \inf_{y \in S_1} \Lambda(y, \varphi) = \inf_{y \in \overline{\sigma}(S_1)} \Lambda(y, \varphi) \leq \sup_{Q \subseteq \Xi(M(X, T)) \text{ is compact and connected}} \inf_{y \in Q} \Lambda(y, \varphi) \leq P(\Delta(S_1, S_2), \varphi) \leq \inf_{y \in S_1} \Lambda(y, \varphi). \]

(3) This is obvious.

5. Some applications

In this section, we firstly present some spectra induced by different deformations. Secondly, we use the BS-dimension to describe some level sets. Thirdly, the relative multifractal spectrum of ergodic averages are discussed. Lastly, symbolic spaces and iterated function systems are investigated.
5.1. Some spectra

Different spectra are induced by different deformations \((Y, \Xi)\) [17].

- The spectrum of the historic set of ergodic averages. Let \(\varphi : X \to \mathbb{R}\) be continuous and define \(\Xi : \mu \mapsto \int \varphi \, d\mu\). In this case we obtain for \(S_1, S_2 \subset \mathbb{R}\),
  \[ \Delta(S_1, S_2) = \left\{ x \in X : S_1 \subset A \left( \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k x) \right) \subset S_2 \right\}. \]

- The spectrum of the historic set of empirical measures. Define \(\Xi : M(X) \to M(X)\) by \(\Xi : \mu \mapsto \mu\).
  In this case we obtain for \(S_1, S_2 \subset \mathbb{R}\),
  \[ \Delta(S_1, S_2) = \left\{ x \in X : S_1 \subset A \left( \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x} \right) \subset S_2 \right\}. \]

- The spectrum of the historic set of local Lyapunov exponents. Let \(\chi(x) = \lim_{n \to \infty} \frac{1}{n} \log |(DT^n)(x)|\). Define \(\Xi : M(X) \to \mathbb{R}\) by \(\mu \mapsto \int DT \, d\mu\).
  In this case we obtain for \(S_1, S_2 \subset \mathbb{R}\),
  \[ \Delta(S_1, S_2) = \left\{ x \in X : S_1 \subset A \left( \frac{1}{n} \log |(DT^n)(x)| \right) \subset S_2 \right\}. \]

- The mixed spectrum of the historic set of ergodic averages of arbitrary families of continuous functions. Assume that the family of maps \((M(X) \to \mathbb{R}) : \mu \mapsto \int \varphi_i d\mu \in I^\infty(I)\) is totally bounded. Define \(\Xi : M(X) \to \mathbb{R}\) by \(\Xi : \mu \mapsto \left( \int \varphi_i d\mu \right)_{\in I} \).
  In this case we obtain for \(S_1, S_2 \subset I^\infty(I)\),
  \[ \Delta(S_1, S_2) = \left\{ x \in X : S_1 \subset A \left( \frac{1}{n} \sum_{k=0}^{n-1} \varphi_i(T^k x) \right)_{\in I} \subset S_2 \right\}. \]

We only consider some (not all) of the above spectra and obtain several corollaries as examples. It is easy to get corollary 5.1 and corollary 5.2, so we omit the proofs.

**Corollary 5.1.** With \((X, T, L_n)\) as before, let \(Y = \mathbb{R}\) and suppose that \(\phi_j : X \to \mathbb{R}\) is a family of continuous functions. Assume that the family of maps \((\Xi_j : M(X) \to \mathbb{R}) : \mu \mapsto \int \phi_i d\mu \in I^\infty(I)\) is totally bounded. Fix \(S_1, S_2 \subset I^\infty(I)\), \(\psi \in C(X, \mathbb{R})\).

1. If \(S_1 = \emptyset\) and \(S_2\) is closed and convex, then
   \[ P \left( \left\{ x \in X : A \left( \frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x) \right)_{\in I} \subset S_2, \psi \right\} \right) = \sup_{x \in S_2} \Lambda(x, \psi). \]

2. If \(S_1 \neq \emptyset\) and \(\overline{\varnothing}(S_1)\) is contained in a connected component of \(S_2\), then
   \[ P \left( \left\{ x \in X : S_1 \subset A \left( \frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x) \right)_{\in I} \right\}, \psi \right) = \inf_{x \in S_1} \Lambda(x, \psi). \]
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(3) If \( S_1 \neq \emptyset \) and \( \overline{\text{co}}(S_1) \) is not contained in a connected component of \( S_2 \), then

\[
x \in X : S_1 \subset A \left( \left\{ \frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x) \right\}_{j \in I} \right) = \emptyset.
\]

**Corollary 5.2.** With \( (X, T, L_\alpha) \) as before, let \( (Y_i, \Xi_i) \) be a (possibly uncountable) family of deformations and assume that \( Y_i \) is a normed vector space and that \( \Xi_i : M(X) \to Y_i \) is affine and continuous. Define the vector spaces \( \times_i Y_i \) and \( [\times_i Y_i]^\infty \) by

\[
\times_i Y_i = \{(y_i)_i : y_i \in Y_i \forall i\},
\]

\[
[\times_i Y_i]^\infty = \{(y_i)_i \in \times_i Y_i : \sup_i \|y_i\| < \infty\},
\]

and equip \( [\times_i Y_i]^\infty \) with the norm \( \| (y_i)_i \| = \sup_i \|y_i\| \). Assume \( \sup_{\mu \in M(X), i} \|\Xi_i \mu\| < \infty \) and the map

\[
M(X) \to [\times_i Y_i]^\infty : \mu \mapsto (\Xi_i \mu)_i
\]

is continuous, and put \( \Xi = (\Xi_i)_{i \in I} \). Fix \( S_1, S_2 \subset [\times_i Y_i]^\infty \), \( \psi \in C(X, \mathbb{R}) \)

(1) If \( S_1 = \emptyset \) and \( S_2 \) is closed and convex, then

\[
P\left( \{ x \in X : S_1 \subset A((\Xi_j L_\alpha x)_{j \in I}) \subset S_2 \}, \psi \right) = \sup_{x \in S_2} \sup_{\mu \in M(X, T)} \left\{ h(T, \mu) + \int \psi \, d\mu \right\}.
\]

(2) If \( S_1 \neq \emptyset \) and \( \overline{\text{co}}(S_1) \) is contained in a connected component of \( S_2 \), then

\[
P\left( \{ x \in X : S_1 \subset A((\Xi_j L_\alpha x)_{j \in I}) \subset S_2 \}, \psi \right) = \inf_{x \in S_1} \sup_{\mu \in M(X, T)} \left\{ h(T, \mu) + \int \psi \, d\mu \right\}.
\]

(3) If \( S_1 \neq \emptyset \) and \( \overline{\text{co}}(S_1) \) is not contained in a connected component of \( S_2 \), then

\[
\{ x \in X : S_1 \subset A((\Xi_j L_\alpha x)_{j \in I}) \subset S_2 \} = \emptyset.
\]

Next, we use dimension theory to discuss \( \Delta_\text{equ}(\cdot), \Delta_\text{sub}(\cdot) \), and so on.

Let \( \psi : X \to \mathbb{R} \) be a strictly positive continuous function. For each set \( Z \subset X \) and each number \( t \in \mathbb{R} \), define

\[
N(Z, t, \psi, n, \epsilon) := \inf_{C \in \mathcal{G}_n(Z, \epsilon)} \left\{ \sum_{B_m(x, \epsilon) \in C} \exp \left( -t \sup_{y \in B_m(x, \epsilon)} S_m \psi(y) \right) \right\},
\]

where \( \mathcal{G}_n(Z, \epsilon) \) is the collection of all finite or countable covers of \( Z \) by sets of the form \( B_m(x, \epsilon) \), with \( m \geq n \).

Set

\[
N(Z, t, \psi, \epsilon) = \lim_{n \to \infty} N(Z, t, \psi, n, \epsilon),
\]

and

\[
BS(Z, \psi, \epsilon) = \inf \{ t : N(Z, t, \psi, \epsilon) = 0 \} = \sup \{ t : N(Z, t, \psi, \epsilon) = \infty \}.
\]

Let \( BS(Z, \psi) = \lim_{\epsilon \to 0} BS(Z, \psi, \epsilon) \), and we call it the BS-dimension of \( Z \). This notation was introduced by Barreira and Schmeling [7].

**Remark 5.1.** By the definition of topological pressure and the BS-dimension, we get that for any set \( Z \subset X \), the BS-dimension of \( Z \) is the unique root of Bowen’s equation \( P(Z, -s\psi) = 0 \), i.e., \( s = BS(Z, \psi) \).
The following corollaries follow from the above theorems and remark 5.1.

**Corollary 5.3.** Suppose that \((X, T, \Xi, L_n, Y)\) satisfies the g-almost product property and the uniform separation property and suppose that \(\varphi \in C(X, \mathbb{R}^+)\).

1. If \(C \subset Y\) is not a compact and connected subset of \(\Xi(M(X, T))\), then
   \[
   \{x \in X : A(\Xi L_n x) = C\} = \emptyset.
   \]
2. If \(C \subset Y\) is a compact and connected subset of \(\Xi(M(X, T))\), then
   \[
   BS(\Delta_{\text{qua}}(C), \varphi) = \inf_{y \in C} \sup_{\mu \in M(X, T)} \left\{ \frac{h(T, \mu)}{\int \varphi \, d\mu} \right\}.
   \]

**Corollary 5.4.** With \((X, T, \Xi, L_n, Y)\) as before, let \(\varphi \in C(X, \mathbb{R}^+)\).

1. If \(C \subset Y\) is not a subset of \(\Xi(M(X, T))\), then
   \[
   \{x \in X : C \subset A(\Xi L_n x)\} = \emptyset.
   \]
2. If \(C \subset Y\) is a subset of \(\Xi(M(X, T))\), then
   \[
   BS(\Delta_{\text{sub}}(C), \varphi) = \inf_{y \in C} \sup_{\mu \in M(X, T)} \left\{ \frac{h(T, \mu)}{\int \varphi \, d\mu} \right\}.
   \]

**Corollary 5.5.** With \((X, T, \Xi, L_n, Y)\) as before, and letting \(\varphi \in C(X, \mathbb{R}^+)\), fix \(S_1 \subset \Xi(M(X, T))\), \(S_2 \subset Y\).

1. If \(S_1 = \emptyset\), then
   \[
   BS(\Delta(S_1, S_2), \varphi) = \sup_{x \in S_2} \sup_{\mu \in M(X, T)} \left\{ \frac{h(T, \mu)}{\int \varphi \, d\mu} \right\}.
   \]
2. If \(S_1 \neq \emptyset\) and \(S_1\) is contained in a connected component of \(S_2\), then
   \[
   \sup_{\mu \in M(X, T)} \inf_{Q \subset \Xi(M(X, T)) \text{ is compact and connected}} \{ h(T, \mu) \} \leq BS(\Delta(S_1, S_2), \varphi) \leq \inf_{x \in S_1} \sup_{\mu \in M(X, T)} \left\{ \frac{h(T, \mu)}{\int \varphi \, d\mu} \right\}.
   \]
3. If \(S_1 \neq \emptyset\) and \(S_1\) is not contained in a connected component of \(S_2\), then
   \[
   \{x \in X : S_1 \subset A(\Xi L_n x) \subset S_2\} = \emptyset.
   \]

**Corollary 5.6.** With \((X, T, \Xi, L_n, Y)\) as before, and letting \(\varphi \in C(X, \mathbb{R}^+)\), fix \(S_1 \subset \Xi(M(X, T))\), \(S_2 \subset Y\).

1. If \(S_1 = \emptyset\), then
   \[
   BS(\Delta(S_1, S_2), \varphi) = \sup_{x \in S_2} \sup_{\mu \in M(X, T)} \left\{ \frac{h(T, \mu)}{\int \varphi \, d\mu} \right\}.
   \]
2. If \(S_1 \neq \emptyset\) and if the closed convex hull of \(S_1, \overline{\text{co}}(S_1)\), is contained in a connected component of \(S_2\), then
   \[
   BS(\Delta(S_1, S_2), \varphi) = \inf_{x \in S_1} \sup_{\mu \in M(X, T)} \left\{ \frac{h(T, \mu)}{\int \varphi \, d\mu} \right\}.
   \]
(3) If \( S_1 \neq \emptyset \) and \( S_1 \) is not contained in a connected component of \( S_2 \), then
\[
\{ x \in X : S_1 \subset A(\Xi L_n x) \subset S_2 \} = \emptyset.
\]

**Corollary 5.7.** With \((X, T, L_n)\) as before, let \( Y = \mathbb{R} \) and suppose \( \phi_j : X \to \mathbb{R} \) is a family of continuous functions. Assume that the family of maps \((\Xi_j : M(X) \to \mathbb{R} : \mu \mapsto \int \phi_j \, d\mu)_{j \in I}\) is totally bounded, where \( \Xi = (\Xi_j)_{j \in I} \). Fix \( S_1, S_2 \subset l^\infty(I), \varphi \in C(X, \mathbb{R}^+) \).

(1) If \( S_1 = \emptyset \) and \( S_2 \) is closed and convex, then
\[
BS \left( \left\{ x \in X : S_1 \subset A \left( \left\{ \frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x) \right\} \right) \right\} \right) \subset S_2, \varphi = \sup_{x \in S_2} \sup_{\varphi_{\mu} \in M(X, T)} \left\{ h(T, \mu) \right\},
\]

(2) If \( S_1 \neq \emptyset \) and \( \overline{\mathcal{C}}(S_1) \) is contained in a connected component of \( S_2 \), then
\[
BS \left( \left\{ x \in X : S_1 \subset A \left( \left\{ \frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x) \right\} \right) \right\} \right) \subset S_2, \varphi = \inf_{x \in S_1} \sup_{\varphi_{\mu} \in M(X, T)} \left\{ h(T, \mu) \right\}.
\]

(3) If \( S_1 \neq \emptyset \) and \( \overline{\mathcal{C}}(S_1) \) is not contained in a connected component of \( S_2 \), then
\[
\{ x \in X : S_1 \subset A(\Xi L_n x) \subset S_2 \} = \emptyset.
\]

**Corollary 5.8.** With \((X, T, L_n)\) as before, let \((Y_i, \Xi_i)\) be (a possibly uncountable) family of deformations and assume that \( Y_i \) is a normed vector space and that \( \Xi_i : M(X) \to Y_i \) is affine and continuous. Define the vector spaces \( \times_i Y_i \) and \( [\times_i Y_i]^\infty \) by
\[
\times_i Y_i = \{(y_i) : y_i \in Y_i, \forall i\},
\]
\[
[\times_i Y_i]^\infty = \{(y_i) : y_i \in \times_i Y_i, \sup_i ||y_i|| < \infty\},
\]
and equip \([\times_i Y_i]^\infty\) with the norm \( ||(y_i)|| = \sup_i ||y_i|| \). Assume that \( \sup_{\mu \in M(X, I)} ||\Xi_i \mu|| < \infty \) and that the map
\[
M(X) \to [\times_i Y_i]^\infty : \mu \mapsto (\Xi_i \mu)_i
\]
is continuous. Fix \( S_1, S_2 \subset [\times_i Y_i]^\infty, \varphi \in C(X, \mathbb{R}^+) \).

(1) If \( S_1 = \emptyset \) and \( S_2 \) is closed and convex, then
\[
BS\{ x \in X : S_1 \subset A([\Xi J L_n x]_{j \in I}) \subset S_2 \}, \varphi = \sup_{x \in S_2} \sup_{(\Xi_j)_{j \in I} \in M(X, T)} \left\{ h(T, \mu) \right\},
\]

(2) If \( S_1 \neq \emptyset \) and \( \overline{\mathcal{C}}(S_1) \) is contained in a connected component of \( S_2 \), then
\[
BS\{ x \in X : S_1 \subset A([\Xi J L_n x]_{j \in I}) \subset S_2 \}, \varphi = \inf_{x \in S_1} \sup_{(\Xi_j)_{j \in I} \in M(X, T)} \left\{ h(T, \mu) \right\},
\]

(3) If \( S_1 \neq \emptyset \) and \( \overline{\mathcal{C}}(S_1) \) is not contained in a connected component of \( S_2 \), then
\[
\{ x \in X : S_1 \subset A([\Xi J L_n x]_{j \in I}) \subset S_2 \} = \emptyset.
\]
5.2. The relative multifractal spectrum of ergodic averages

This subsection investigates the relative multifractal spectrum of ergodic averages.

Let \( f, g \in C(X, \mathbb{R}) \) with \( g(x) \neq 0 \) for all \( x \in X \) and \( C \subseteq \mathbb{R} \). Define \( \Xi : M(X) \to \mathbb{R} \) by \( \Xi : \mu \mapsto \frac{\int f \, d\mu}{\int g \, d\mu} \). Note that \( \Xi \) is continuous but not affine.

**Corollary 5.9.** With \((X, T, L)\) as before, let \( f_1, g_1, \ldots, f_m, g_m \) be continuous functions such that \( f_i, g_i : X \to \mathbb{R} \) with \( g_i(x) \neq 0 \) for all \( x \in X \), \( i = 1, \ldots, m \) and \( \int g_i \, d\mu \neq 0 \) for all \( \mu \in M(X, T) \), \( i = 1, \ldots, m \). If \( C \subseteq \mathbb{R}^m \) is closed and convex and \( \psi \in C(X, \mathbb{R}) \), then

\[
P \left\{ x \in X : A \begin{pmatrix} \sum_{k=0}^{n-1} f_j(T^k x) \\ \sum_{k=0}^{n-1} g_j(T^k x) \end{pmatrix}_{j \in \{1, 2, \ldots, m\}} \subseteq C \right\}, \psi
\]

\[
= \sup \left\{ h(T, \mu) + \int \psi \, d\mu : \mu \in M(X, T), \frac{\int f_i \, d\mu}{\int g_i \, d\mu}_{i \in \{1, \ldots, m\}} \subseteq C \right\}.
\]

**Proof.** Since \( \Xi : \mu \mapsto (\int f_i \, d\mu)_{i=1, \ldots, m} \) is continuous, we have

\[
\{ x \in X : A (\Xi L_n(x)) \subseteq C \}
\]

\[
= \{ x \in X : A (\Xi(L_n(x)) \subseteq \Xi^{-1} C \}
\]

\[
\subseteq \{ x \in X : A (L_n(x)) \cap A (\Xi^{-1} C) \neq \emptyset \}
\]

\[
= \{ x \in X : A (L_n(x)) \cap (\Xi^{-1} C \cap M(X, T)) \neq \emptyset \}.
\]

It follows from part (i) of proposition 3.1 that

\[
P \left\{ x \in X : A \begin{pmatrix} \sum_{k=0}^{n-1} f_j(T^k x) \\ \sum_{k=0}^{n-1} g_j(T^k x) \end{pmatrix}_{j \in \{1, 2, \ldots, m\}} \subseteq C \right\}, \psi
\]

\[
\leq \sup \left\{ h(T, \mu) + \int \psi \, d\mu : \mu \in M(X, T), \frac{\int f_i \, d\mu}{\int g_i \, d\mu}_{i \in \{1, \ldots, m\}} \subseteq C \right\}.
\]

For the opposite inequality, we prove the case \( m = 1 \) as an example. For any \( \alpha \in C \),

\[
\left\{ x \in X : A \left( \frac{1}{n} \sum_{k=0}^{n-1} (f_1(T^k x) - \alpha g_1(T^k x)) \right) = 0 \right\}
\]

\[
\subseteq \left\{ x \in X : A \left( \frac{\sum_{k=0}^{n-1} f_1(T^k x)}{\sum_{k=0}^{n-1} g_1(T^k x)} \right) \subseteq C \right\}.
\]

Hence,

\[
\sup \left\{ h(T, \mu) + \int \psi \, d\mu : \frac{\int f_1 \, d\mu}{\int g_1 \, d\mu} = \alpha \in C, \mu \in M(X, T) \right\}
\]

\[
= \sup \left\{ h(T, \mu) + \int \psi \, d\mu : \int f_1 - \alpha g_1 \, d\mu = 0, \alpha \in C, \mu \in M(X, T) \right\}
\]

\[
= \sup_{\alpha \in C} P \left\{ x \in X : A \left( \frac{1}{n} \sum_{k=0}^{n-1} (f_1(T^k x) - \alpha g_1(T^k x)) \right) = 0 \right\}, \psi
\]

\[
\leq P \left\{ x \in X : A \left( \frac{\sum_{k=0}^{n-1} f_1(T^k x)}{\sum_{k=0}^{n-1} g_1(T^k x)} \subseteq C \right\}, \psi \right\}.
\]

Since the case \( m > 1 \) is similar to \( m = 1 \), the proof is omitted. \( \square \)
Corollary 5.10. With \((X, T, L_{\mu})\) as before, let \(f_1, g_1, \ldots, f_m, g_m\) be continuous functions with \(f_i, g_i : X \to \mathbb{R}\) and \(g_i(x) \neq 0\) for all \(x \in X, i = 1, \ldots, m\) and \(\int g_i \, d\mu \neq 0\) for all \(\mu \in M(X, T)\), \(i = 1, \ldots, m\). If \(C \subseteq \mathbb{R}^m\) is closed and convex and \(\varphi \in C(X, \mathbb{R}^\ast)\), then
\[
BS \left( \left\{ x \in X : A \left( \left( \frac{\sum_{k=0}^{n-1} f_j(T^k x)}{\sum_{j=0}^{n-1} g_j(T^k x)} \right)_{j=1,2,\ldots,m} \right) \leq C \right\}, \varphi \right) = \sup \left\{ h(T, \mu) : \mu \in M(X, T), \left( \frac{\int f_i \, d\mu}{\int g_i \, d\mu} \right)_{i=1,\ldots,m} \in C \right\}.
\]

5.3. Symbolic space and iterated function systems

This section aims at answering the conjecture in [17] positively. For convenience, we only treat the iterated function system induced by a subshift of finite type of the unilateral full shift.

Consider a subshift of finite type \(\Sigma_{A}^+\) of the unilateral full shift on \(m\) symbols \(I = \{1, 2, \ldots, m\}\) with \(m \geq 2\). Let \(A\) be the shift map, and let \(A = (a_{ij})_{1 \leq i, j \leq m}\) be the transfer matrix of zeros and ones. In this section, we assume that \(A\) is an irreducible andaperiodic stochastic matrix, that is, there is some power \(m\) such that all the entries of \(A^m\) are strictly positive. This assumption implies the specification property.

For \(x = (x_i)_{i \geq 1}\) and \(y = (y_i)_{i \geq 1}\), set \(\nu(x, y) = \inf\{i \geq 1 : x_i \neq y_i\}\). Let \(\varphi\) be a strictly positive continuous function on \(\Sigma_{A}^+\). Write \(S_n \varphi = \sum_{i=0}^{n-1} \varphi \circ \sigma^i\) for each \(n \geq 1\). For \(x \neq y \in \Sigma_{A}^+\), define
\[
d_{\varphi}(x, y) = \begin{cases} 
0, & x = y, \\
1, & x_i \neq y_i, \\
\exp \left( -\min_{v(x, z) \geq m} S_m \varphi(z) \right), & m = \nu(x, y).
\end{cases}
\]
Note that, given \(\Psi > 1\), we can choose \(\varphi \equiv \ln \Psi\), \(d_{\varphi}\) is the metric in [17].

Proposition 5.2. In \((\Sigma_{A}^+, d_{\varphi})\), for any subset \(Z \subseteq \Sigma_{A}^+\), we get \(dim_H(Z) = BS(Z, \varphi)\).

Let \(\omega_{ij}\) be a Lipschitz contraction map on \(\mathbb{R}^m\) for each nonzero \(a_{ij}\). There exists a unique array \(E = (E_1, \ldots, E_m)\) of non-empty compact subsets of \(\mathbb{R}^m\) satisfying \(E_i = \bigcup_{a_{ij} = 1} \omega_{ij}(E_j)\).

The union \(E = \bigcup_{i=1}^{m} E_i\) is called a self-similar set for the recurrent iterated function system \(\{\omega_{ij}, (a_{ij})\}\).

Let \(F\) be a compact subset of \(E\). Set \(F_i = F \cap E_i, i = 1, \ldots, m\). If \(F_i \subseteq \bigcup_{a_{ij} = 1} \omega_{ij}(E_j)\) for all \(i\), then \(F\) is called a sub-self-similar set for \(\{\omega_{ij}, (a_{ij})\}\).

Assume that:

(i) Each map \(\omega_{ij}\) is a \(C^{1+\gamma}\) diffeomorphism.
(ii) \(D\omega_j\) is always a similarity map, i.e., \(|D\omega_j|_1(v) = s_j(x) \cdot |v|\) for each \(x, v \in \mathbb{R}^n\).
(iii) \(\{\omega_{ij}, (a_{ij})\}\) satisfies the open set condition [11].

Let \(\pi : \Sigma_{A}^+ \to E\) be given by \(\pi(x) = \) the only point in \(\bigcap_{n \geq 1} \omega_{x_1,2} \omega_{x_2,3} \cdots \omega_{x_{n-1},n} (E_{x_n})\).

The scale function of \(E\) is the map \(\psi : \Sigma_{A}^+ \to \mathbb{R}\) given by \(\psi(x) = \log s_{x_1,2} (\pi \sigma x)\). Let \(\varphi(x) = -\psi(x)\), then \(\varphi\) is a positive H"older continuous function.

Proposition 5.3. In \((\Sigma_{A}^+, d_{\varphi})\), for any subset \(Z \subseteq \Sigma_{A}^+\), we get \(dim_H(\pi Z) = dim_H(Z)\).
Combining propositions 5.2 and 5.3, corollaries 5.1, 5.9, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.10, and 5.8 hold regarding the Hausdorff dimension in an iterated function system satisfying the open set condition. We take corollary 5.10 as an example.

**Corollary 5.11.** Let \( f_1, g_1, \ldots, f_m, g_m \) be continuous functions \( f_i, g_i : \Sigma_1^+ \to \mathbb{R} \) with \( g_i(x) \neq 0 \) for all \( x \in \Sigma_1^+ , i = 1, \ldots, m \) and \( \int g_i d\mu \neq 0 \), for all \( \mu \in M(\Sigma_1^+, \sigma) , i = 1, \ldots, m \).

If \( C \subseteq \mathbb{R}^m \) is closed and convex, then

\[
\dim_H \left( \pi \left\{ x \in \Sigma_1^+ : A \left( \frac{\sum_{k=0}^{n-1} f_j(\sigma^k x)}{\sum_{k=0}^{n-1} g_j(\sigma^k x)} \right) \right\} \subseteq C \right) = \sup \left\{ h(T, \mu) - \int \log s_{x|x_2}(\pi \sigma x) d\mu : \mu \in M(\Sigma_1^+, \sigma) , \left( \frac{\int f_i d\mu}{\int g_i d\mu} \right)_{i=1,\ldots,m} \in C \right\} .
\]

Corollary 5.11, in the case of the iterated function systems induced by a directed and strongly connected multi-graph, is the conjecture posed by Olsen in [17].

Note that the multi-graph can be viewed as an invariant subset of a symbolic space. Weiss [31] showed that a directed and strongly connected multi-graph satisfies the specification property and the positive expansive property is satisfied naturally. This implies that we can apply the above theorems to the model in [17]. Therefore, we give a positive answer to the conjecture in [17] (see [21] for the point of view of the Hausdorff dimension) from the point of view of topological pressure.

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