SINICULAR DEGENERATE PROBLEMS AND APPLICATIONS
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ABSTRACT

The boundary value problems for linear and nonlinear singular degenerate differential-operator equations are studied. We prove a well-posedness of linear problem and optimal regularity result for the nonlinear problem which occur in fluid mechanics, environmental engineering and in the atmospheric dispersion of pollutants.

Key Words: differential-operator equations, Semigroups of operators, Banach-valued function spaces, separability, fredholmness, interpolation of Banach spaces, Atmospheric dispersion of pollutants.

1. Introduction, notations and background

The maximal regularity properties for boundary value problems (BVPs) for linear differential-operator equations (DOEs) have been studied extensively by many researchers (see e.g. [1–10] and the references therein). The main objective of the present paper is to discuss BVPs for the following nonlinear singular degenerate DOE

\[-x^{2\alpha} \frac{\partial^2 u}{\partial x^2} - y^{2\beta} \frac{\partial^2 u}{\partial y^2} + a(x, u, u_x, u_y) u = F(x, u, u_x, u_y)\]

on the rectangular domain \(G = (0, a) \times (0, b)\).

Several conditions for the uniform separability and the resolvent estimates for the corresponding linear problem are given in abstract \(L_p\)-spaces. Especially, we prove that the linear differential operator is positive and is a generator of an analytic semigroup. Moreover, the existence and uniqueness of maximal regular solution of the above nonlinear problem are obtained. One of the important characteristics of these DOEs are that the degeneration process are taking place at different speeds at boundary, in general. Maximal regularity properties of regular degenerated nonlinear DOEs are studied e.g. in [1, 8, 10]. Unlike to these we consider here the singular degenerate DOEs. In applications maximal regularity properties of infinite systems of singular degenerate PDE are studied.

Let \(\gamma = \gamma(x), x = (x_1, x_2, ..., x_n)\) be a positive measurable function on a domain \(\Omega \subset \mathbb{R}^n\). Let \(L_{p,\gamma}(\Omega; E)\) denote the space of strongly measurable \(E\)-valued functions that are defined on \(\Omega\) with the norm

\[\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega; E)} = \left(\int \|f(x)\|^p_{E \gamma(x)} \, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.\]
For $\gamma(x) \equiv 1$, $L_{p,\gamma}(\Omega;E)$ will be denoted by $L_p = L_p(\Omega;E)$. Let $C$ be the set of the complex numbers and

$$S_\varphi = \{ \lambda: \lambda \in C, |\arg \lambda| \leq \varphi \} \cup \{0\}, \ 0 \leq \varphi < \pi.$$  

A linear operator $A$ is said to be $\varphi$-positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and $\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \leq M (1 + |\lambda|)^{-1}$ for any $\lambda \in S_\varphi$, $0 \leq \varphi < \pi$, where $I$ is the identity operator in $E$ and $B(E)$ is the space of bounded linear operators in $E$.

The $\varphi$-positive operator $A$ is said to be $R$-positive in a Banach space $E$ if the set $L_A = \{ \xi (A + \xi I)^{-1}: \xi \in S_\varphi \}$, $0 \leq \varphi < \pi$ is $R$-bounded (see e.g. [5]). Let $G$ be a domain in $R^n$. Let $W_{p,\gamma}^m = W_{p,\gamma}^m(G;E(A),E)$ and $W_{p,\gamma}^{[m]} = W_{p,\gamma}^{[m]}(0,a;E(A),E)$ are $E$-valued weighted function spaces defined in [8].

2. Linear degenerate DOEs

Consider the BVP for the singular degenerate differential-operator equation

$$-x^{2\alpha} \frac{\partial^2 u}{\partial x^2} - x^{2\beta} \frac{\partial^2 u}{\partial y^2} + Au + x^\alpha A_1 \frac{\partial u}{\partial x} + y^\beta A_2 \frac{\partial u}{\partial y} + \lambda u = f(x,y), \quad (1)$$

$$L_1 u = \sum_{i=0}^{m_1} \delta_{i1} u_x^{[i]}(a,y) = 0, \quad L_2 u = \sum_{i=0}^{m_2} \delta_{i2} u_y^{[i]}(x,b) = 0,$$

on the domain $G = (0,a) \times (0,b)$, where $u = u(x,y)$, $u_x^{[i]} = \left[x^\alpha \frac{\partial}{\partial x}\right]^i u(x,y)$, $u_y^{[i]} = \left[y^\beta \frac{\partial}{\partial y}\right]^i u(x,y)$, $m_1, m_2 \in \{0,1\}$; $\delta_{ijk}$ are complex numbers, $\lambda$ is a complex parameter, $A$ and $A_i = A_i(x,y)$ are linear operators in a Banach space $E$.

Let $\delta_{1m_1}, \delta_{2m_2} \neq 0$, $k = 1, 2$. The main result is the following

**Theorem 1.** Let $E$ be an UMD space space (see [11]), $A$ be a $R$-positive operator in $E$, $A_i A^{-1}(t^{-\mu})(t^{-\mu}) \in L_\infty(G;B(E))$ for $\mu \in \left(0, \frac{1}{2}\right)$ and $1 + \frac{1}{p} < \alpha, \beta < \frac{1}{2}$. Then the problem (1) has a unique solution $u \in W_{p,\alpha,\beta}^{[2]}(G;E(A),E)$ for all $f \in L_p(G;E)$ and $|\arg \lambda| \leq \varphi$ with sufficiently large $|\lambda|$ and the following coercive uniform estimate holds

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left[ \left\| x^{i\alpha} \frac{\partial^i u}{\partial x^i} \right\|_{L_p} + \left\| y^{i\beta} \frac{\partial^i u}{\partial y^i} \right\|_{L_p} \right] + \|Au\|_{L_p} \leq M \|f\|_{L_p}. \quad (2)$$

For proving the main theorem, consider at fist, BVPs for the singular degenerate DOE

$$(L + \lambda) u = -u^{[2]}(x) + (A + \lambda) u(x) = f, \quad (3)$$

$$L_1 u = \sum_{i=0}^{m_k} \delta_i u^{[i]}(a) = 0,$$
where \( u^{[i]} = \left[ x^{\alpha \frac{d}{dx}} \right]^i u(x) \), \( m_k \in \{0, 1\} \); \( \delta_i \) are complex numbers and \( A \) is a linear operator in \( E \), \( \delta_{m_k} \neq 0 \).

In a similar way as in \([9, \text{Theorem } 5.1]\) we obtain

**Theorem A**. Suppose \( E \) is an \( UMD \) space, \( A \) is an \( R \) positive in \( E \), \( 1 + \frac{1}{p} < \alpha < \frac{(p-1)}{2} \). Then the problem \((3)\) has a unique solution \( u \in W_p^{[2]} \) for all \( f \in L_p(0, a; E) \), \( p \in (1, \infty) \). Moreover for \( |\arg \lambda| \leq \phi \) and sufficiently large \( |\lambda| \) the following uniform coercive estimate holds

\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{p}} \left\| u^{[i]} \right\|_{L_p(0, a; E)} + \| Au \|_{L_p(0, a; E)} \leq C \| f \|_{L_p(0, a; E)}.
\]

Let \( B \) denote the operator in \( L_p(0, a; E) \) generated by problem \((2)\), i.e.

\[ D(B) = \left\{ u : u \in W_p^{[2]}, L_k u = 0 \right\}, \quad Bu = -u^{[2]} + Au. \]

In a similar way as in \([8, \text{Theorem } 3.1]\) we obtain

**Theorem A**. Let all conditions of Theorem A are satisfied. Then, the operator \( B \) is \( R \)-positive in \( L_p(0, a; E) \).

Theorem A implies that the operator \( B \) is positive and is a generator of analytic semigroups in \( L_p(0, a; E) \).

Consider now the following degenerate DOEs with the boundary conditions

\[-x^{2\alpha} u^{(2)}(x) + (A + \lambda) u(x) = f, \quad L_k u = 0.\]

**Theorem A**. Let all conditions of Theorem A are satisfied. Then the problem \((5)\) has a unique solution \( u \in W_p^{[2]} \) for all \( f \in L_p(0, a; E) \). Moreover for \( |\arg \lambda| \leq \phi \) and sufficiently large \( |\lambda| \) the following coercive estimate holds

\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{p}} \left\| x^{\alpha \frac{d}{dx}} u^{(i)} \right\|_{L_p(0, a; E)} + \| Au \|_{L_p(0, a; E)} \leq C \| f \|_{L_p(0, a; E)}.
\]

**Proof.** Since \( \alpha > 1 \), by \([9, \text{Theorem } 2.3]\) we get that there is a small \( \varepsilon > 0 \) and \( C(\varepsilon) \) such that

\[
\left\| \alpha x^{\alpha-1} u^{[1]} \right\|_{L_p(0, a; E)} \leq \varepsilon \left\| u \right\|_{W_p^{[2]}(0, a; E(A), E)} + C(\varepsilon) \left\| u \right\|_{L_p(0, a; E)}. \]

Then in view of \((6)\), \((7)\) and due to positivity of operator \( B \), we have the following estimate

\[
\left\| \alpha x^{\alpha-1} u^{[1]} \right\|_{L_p(0, a; E)} \leq \varepsilon \left\| Bu \right\|_{L_p(0, a; E)}. \]

Since \(-x^{2\alpha} u^{(2)} = -u^{[2]} + \alpha x^{\alpha-1} u^{[1]}\), the assertion is obtained from Theorem A and the estimate \((8)\).

In this stage we can show the proof of Theorem 1.

**Proof of Theorem 1.** Consider at first the principal part of the problem \((1)\) i.e.
\[-x^{2\alpha} \frac{\partial^2 u}{\partial x^2} - x^{2\beta} \frac{\partial^2 u}{\partial y^2} + Au + \lambda u = f(x, y), \quad L_1 u = 0, \quad L_2 u = 0. \quad (9)\]

Since \( L_p(0, b; L_p(0, a; E)) = L_p(G; E) \) then, this BVP can be express as:

\[-y^{2\beta} \frac{\partial^2 u}{\partial y^2} + (B + \lambda) u(y) = f(y), \quad L_2 k u = 0. \quad (10)\]

By virtue of [1, Theorem 4.5.2] \( F = L_p(0, b; E) \in UMD \) provided \( E \in UMD, p \in (1, \infty) \). By Theorem A2, the operator \( B \) is \( R \)-positive in \( F \). Then by virtue of Theorem A3, for \( f \in L_p(0, a; F) = L_p(G; E) \) problem (9) has a unique solution \( u \in W^{2, \alpha, \beta}_{\gamma} \) and the operator \( Q \) generated by problem (9) has a bounded inverse from \( L_p(G; E) \) to \( W^{2, \alpha, \beta}_{\gamma} \). Moreover by using embedding theorems in \( W^{2, \alpha, \beta}_{\gamma} \) (see e.g. [9, Theorem 2.3] we get the following estimate

\[
\left\| x^\alpha A_1 \frac{\partial u}{\partial x} \right\|_{L_p(G; E)} + \left\| y^\beta A_2 \frac{\partial u}{\partial y} \right\|_{L_p(G; E)} \leq \varepsilon \|Qu\|_{L_p(G; E)}, \quad \varepsilon < 1.
\]

By virtue the above estimate and by using perturbation properties of linear operators we obtain the assertion.

**Remark 1.** Note that, by using the similar techniques similar to those applied in Theorems 1, 2, we can obtained the same results for differential-operator equations of the arbitrary order.

### 3. Singular degenerate BVPs with small parameters

Consider the BVP for the parameter dependent degenerate differential-operator equation

\[ Lu = -t u^{[2]}(x) + (A + \lambda) u(x) = f, \quad x \in (0, a) \quad (11)\]

\[ L_1 u = \sum_{i=0}^{1} t^{\sigma_i} \alpha_i u[i](a) = f_1, \]

where \( u[i] = \left[ x^{-\gamma} \frac{d}{dx} \right]^i u(x) \), \( \sigma_i = \frac{i}{2} + \frac{1}{2(1-\gamma)p} \), \( \gamma > 1 + \frac{1}{p} \); \( \alpha_i \) are complex numbers \( t \) is a small positive and \( \lambda \) is a complex parameter, \( A \) is a linear operator in a Banach space \( E \) and \( f_1 \in E_1 = ((E(A), E)_\theta), \quad \theta = \frac{1}{2} \left( 1 + \frac{1}{(1-\alpha)p} \right) \).

A function \( u \in W^{2, \alpha, \beta}_{\gamma}(0, a; E(A), E) \) satisfying the equation (1) a.e. on \( (0, 1) \) is said to be the solution of the equation (1) on \( (0, 1) \).

**Remark 2.** Let

\[ y = \frac{x}{\int_0^x z^{-\gamma} dz}. \quad (12)\]
for all
\[ \sum \text{ the following uniform coercive estimate holds} \]
\[ u \]
we obtain that the problem (12) has a unique solution
\[ G \]
large \[ \in \] \[ u \]
L mapped isomorphically onto weighted spaces
\[ f \]
the problem (10) for all \[ A \]
is a linear operator in a Banach space \[ W \]
\[ p, \gamma \]
(0, \[ (A) \] , \[ E \] ) and
Moreover, under the substitution (11) the problem (2) is transformed into a non degenerate problem
\[ Lu = -tu^{(2)} (y) + Au(y) = f, \quad L_1u = \sum_{i=0}^{1} t^{\sigma_i} \alpha_i u^{(i)} (0) = f_1, \quad \text{(13)} \]

**Theorem 2.** Suppose \( E \) is a UMD and \( 1 + \frac{1}{p} < \gamma, 1 < p < \infty \). Then the problem (10) for all \( f \in L_p(0, a; E), f_1 \in E_1 \) has a unique solution \( u \in W^{[2]}_{p, \gamma} (0, a; E(A), E) \) and for \( \arg \lambda \leq \varphi \) and sufficiently large \( |\lambda| \) the following uniform coercive estimate holds
\[
\sum_{i=0}^{2} |\lambda|^{1 - \frac{i}{2}} t^\frac{i}{2} \left\| u^{(i)} \right\|_{L_p(0, a; E)} + \| Au \|_{L_p(0, a; E)} \leq C \left[ \| f \|_{L_p(0, a; E)} + \| f_1 \|_{E_1} + |\lambda|^{1 - \theta} \| f_1 \|_{E} \right].
\]

**Proof.** Consider the problem (12). In a similar way as in [10, Theorem 3.2] we obtain that the problem (12) has a unique solution \( u \in W^{[2]}_{p, \gamma} (-\infty, 0; E(A), E) \) for all \( f \in L_p(0, \gamma, E(A), E) \), \( f_1 \in E_1 \) and \( \arg \lambda \leq \varphi \) with sufficiently large \( |\lambda| \) the following uniform coercive estimate holds
\[
\sum_{i=0}^{2} |\lambda|^{1 - \frac{i}{2}} t^\frac{i}{2} \left\| u^{(i)} \right\|_{L_p(-\infty, 0; E)} + \| Au \|_{L_p(-\infty, 0; E)} \leq C \left[ \| f \|_{L_p(-\infty, 0; E)} + \| f_1 \|_{E_1} + |\lambda|^{1 - \theta} \| f_1 \|_{E} \right].
\]

Then in view of the Remark 2 we obtain the assertion.
Consider now the parameter dependent singular degenerate BVP
\[ -t_1 x^{2\alpha} \frac{\partial^2 u}{\partial x^2} - t_2 x^{2\beta} \frac{\partial^2 u}{\partial y^2} + Au + \lambda u = f(x, y), \quad \text{(15)} \]
\[ L_1u = \sum_{i=0}^{m_1} t_i^{\sigma_{1i}} u^{[i]}_{1}(a, y) = 0, \quad L_2u = \sum_{i=0}^{m_2} t_i^{\sigma_{2i}} u^{[i]}_{y}(x, b) = 0, \]
on the domain \( G = (0, a) \times (0, b) \), where \( \sigma_{1i} = \frac{i}{2} + \frac{1}{2p(1-\alpha)}, \sigma_{2i} = \frac{i}{2} + \frac{1}{2p(1-\beta)} \).
A is a linear operator in a Banach space \( E \) and \( t_i \) are small parameters.

**Theorem 3.** Let \( E \) be an UMD space space, \( A \) be an \( R \)-positive operator in and \( 1 + \frac{1}{p} < \alpha, \beta < \frac{(p-1)}{2} \). Then the problem (14) has a unique solution \( u \in W^{[2]}_{p, \alpha, \beta} (G; E(A), E) \) for all \( f \in L_p(G; E) \) and \( \arg \lambda \leq \varphi \), with sufficiently large \( |\lambda| \) and the following coercive uniform estimate holds
\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left[ t_1^i \left\| x^{i\alpha} \frac{\partial^i u}{\partial x^i} \right\|_{L_p} + t_2^i \left\| y^{i\beta} \frac{\partial^i u}{\partial y^i} \right\|_{L_p} \right] + \|Au\|_{L_p} \leq M \|f\|_{L_p}. \tag{16}
\]

**Proof.** By reasoning as in the proof of Theorem 1, the problem (14) is reduced to the following BVP for ordinary equation

\[- t_2 u^{[2]} (y) + (B_{t_1} + \lambda) u (y) = f, \quad y \in (0, b) \tag{17}\]

where \(B_{t_2}\) is the operator in \(L_p (0, b; E)\) generated by BVP

\[- t_1 u^{[2]} (x) + (A + \lambda) u (x) = f, \quad x \in (0, a) \]

\[L_1 u = \sum_{i=0}^{1} \delta_1^i \alpha_i u^{[i]} (a) = 0, \quad k = 1, 2, \tag{18}\]

Then by applying Theorem 2 to problem (16) in \(L_p (0, a; F) = L_p (G; E), F = L_p (0, b; E)\) we obtain the assertion.

4. **Singular degenerate BVPs in moving domains**

Consider the linear BVPs in moving domain \(G_s = (0, a(s)) \times (0, b(s))\)

\[- x^{2\alpha} \frac{\partial^2 u}{\partial x^2} - x^{2\beta} \frac{\partial^2 u}{\partial y^2} + Au + du = f (x, y), \tag{19}\]

\[L_1 u = \sum_{i=0}^{1} \delta_{1i} u^{[i]} (a(s), y) = 0, \quad L_2 u = \sum_{i=0}^{1} \delta_{2i} u^{[i]} (b(s), x) = 0, \quad k = 1, 2, \]

where \(a(s)\) and \(b(s)\) are positive continues function depended on parameter \(s\) and \(A\) is a linear operator in a Banach space \(E, \delta_{k1} \neq 0.\)

**Theorem 4.** Let \(E\) be an UMD space space, \(A\) be an \(R\)-positive operator in \(E, 1 + \frac{1}{p} < \alpha, \beta < \frac{(p-1)}{2}\) and \(f_k \in E_k.\)

Then problem (10) for \(f \in L_p (G(s) ; E), f_k \in E_k, p \in (1, \infty)\) and the sufficiently large \(d > 0\) has a unique solution \(u \in W^2_{p, \alpha, \beta} (G(s) ; E(A), E)\) and the following coercive uniform estimate holds

\[\left\| x^{2\alpha} \frac{\partial^2 u}{\partial x^2} \right\|_{L_p (G(s) ; E)} + \left\| y^{2\beta} \frac{\partial^2 u}{\partial y^2} \right\|_{L_p (G(s) ; E)} + \|Au\|_{L_p (G(s) ; E)} \leq C \|f\|_{L_p (G(s) ; E)}.\]
Proof. Under the substitution $\tau = xb(s)$, $\sigma = yb(s)$ by denoting $\tau$, $\sigma$, $u(x, y)$, $f(x, \tau, y(\sigma))$ and $f(x, y)$ respectively we get the moving boundary problem (10) maps to the following BVP with parameter in the fixed domain $(0,1) \times (0,1)$

$$-\beta^{2(1-\alpha)}(x)\frac{\partial^2 u}{\partial x^2} - \beta^{2(1-\beta)}(x)\frac{\partial^2 u}{\partial y^2} + Au + du = f(x, y),$$

(19)

$L_1 u = \sum_{i=0}^{1} b^i \delta_{i, u_x} (1, y) = 0$, $L_2 u = \sum_{i=0}^{1} b^i \delta_{i, u_y} (1, 1) = 0$, $k = 1, 2$.

Then by virtue of Theorem 3 we obtain the assertion.

5. Nonlinear degenerate DOE

Consider now the following nonlinear problem

$$-x^2 \frac{\partial^2 u}{\partial x^2} - y^{2\beta} \frac{\partial^2 u}{\partial y^2} + a(u, u_x, u_y) u = F(x, y, u, u_x, u_y), \quad L_{1k} u = 0, \quad L_{2k} u = 0$$

(20)
on the domain $G_0 = (0, a_0) \times (0, b_0)$, where $L_{jk}$ are boundary conditions defined by (1). Let

$$X_1 = L_p(0, a; E), \quad Y_1 = \mathcal{W}^{[2]}_{p,a}(0, a; E(A), E), \quad X_2 = L_p(0, b; E),$$

$$Y_2 = \mathcal{W}^{[2]}_{p,\beta}(0, b; E(A), E), \quad E_{ki} = (X_k, Y_k)_{\theta_{ki}, p}, \quad \theta_{1i} = \frac{p(1-\alpha) i + 1}{2p(1-\alpha)},$$

$$\theta_{2i} = \frac{p(1-\beta) i + 1}{2p(1-\beta)}, \quad E_0 = \prod_{i,k} E_{ki}, \quad i = 0, 1, k = 1, 2.$$

Condition 1. Assume the following satisfied:

(1) $E$ is an $UMD$ space, $a(x, y, U) = A(x, y)$ is a positive operator in $E$ for $x, y \in G_0$, $u_i \in E_{1_i}$, $g_i \in E_{1_i}$, $D(a(x, y, U))$ does not depend on $x, y, U$, where $U = \{u_0, u_1, g_0, g_1\}$ and $a : G_0 \times E_0 \to B(E(A), E)$ is continuous;

(2) $F : G_0 \times E_0 \to E$ be a measurable function; $F(x, y, \cdot)$ is continuous with respect to $x, y \in G_0$ and $F(x, y) = F(x, y, 0) \in X$. Moreover, for each $R > 0$ there exists $\mu_R$ such that $\|F(x, U) - F(x, \bar{U})\|_E \leq \mu_R \|U - \bar{U}\|_{E_0}$ for a.a. $x, y \in G_0$, $u_j, \bar{u}_j \in X_j$ and $\|U\|_{E_0} \leq R$, $\|\bar{U}\|_{E_0} \leq R$, $1 + \frac{1}{p} < \alpha, \beta < \left(\frac{p-1}{2}\right)$;

(3) there exist $v_j \in E_{1j}$, $v_j \in E_{2j}$ such that the operator $A(x, y, \Phi)$ for $\Phi = \{v_1, v_2, v_1, v_2\}$ is $R$-positive in $E$ uniformly with respect to $x, y \in G_0$; $A(x, y, \Phi) A^{-1}(x^0, y^0, \Phi) \in C(G_0; B(E))$;

(4) Moreover, for each $R > 0$ there is a positive constant $L(R)$ such that $\|A(x, y, U) - A(x, y, \bar{U})\|_E \leq M(R) \|U - \bar{U}\|_{E_0} \|A\|_E$ for $x, y \in G_0$, $\|U\|_{E_0}$, $\|\bar{U}\|_{E_0} \leq R$ and $v \in D(A(x, y, U))$. 

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In this section we prove the existence and uniqueness of maximal regular solution for the nonlinear problem (11).

**Theorem 3.** Let the Condition 1 holds. Then there is \( a \in (0, a_0], \ b \in (0, b_0] \) such that problem (11) has a unique solution belongs to \( W^2_{p, \alpha, \beta} ((G; E (A), E) \).

**Proof.** By Theorem , the linear problem

\[-x^{2\alpha} \frac{\partial^2 w}{\partial x^2} - y^{2\beta} \frac{\partial^2 w}{\partial y^2} + Aw (x, y) = f (x, y), \]  

is maximal regular in \( X \) uniformly with respect to \( a \in (0, a_0] \) and \( b \in (0, b_0] \) i.e. for all \( f \in X \) there is a unique solution \( w \in Y \) of the problem (31) and has a coercive estimate

\[ \|w\|_Y \leq C \|f\|_X, \]

where the constant \( C \) does not depends on \( a \in (0, a_0] \) and

\[ f (x) = F (x, 0). \]

We want to solve the problem (30) locally by means of maximal regularity of the linear problem (31) via the contraction mapping theorem. For this purpose let \( w \) be a solution of the linear BVP (31). Consider a ball

\[ B_r = \{v \in Y, \|v - w\|_Y \leq r\}. \]

Given \( v \in B_r \), solve the problem

\[-tu^{(2m)} (x) + Au (x) = F \left(x, v, v^{(1)}, ..., v^{(2m-1)}\right), \]  

where \( x \in (0, a) \). Define a map \( Q \) on \( B_r \) by \( Qv = u \), where \( u \) is a solution of the problem (38). We want to show that \( Q (B_r) \subset B_r \) and that \( L \) is a contraction operator in \( Y \), provided \( a \) is sufficiently small, and \( r \) is chosen properly. For this aim by using maximal regularity properties of the problem (37) for \( V = \{u^{(m_k)} (0) \}, k = 1, 2, ..., 2m \) we have

\[ \|Qv - w\|_Y = \|u - w\|_Y \leq C_0 \|F (x, V) - F (x, 0)\|_X. \]

By assumption Condition1 and in view of Remark1 we have

\[ \|F (x, V) - F (x, 0)\|_E \leq \]

\[ \|F (x, V) - F (x, W)\|_E + \|F (x, W) - F (x, 0)\|_E \leq \]

\[ MR [\|V - W\|_{E_0} + \|W\|_{E_0}] \]
\[ M_R C_1 \left[ \| v - w \|_Y + \| w \|_Y \right] \leq MC_1 \left[ r + \| w \|_Y \right], \]

where \( R = M_R C_1 \left[ r + \| w \|_Y \right] \) is a fixed number such that \( R \leq \frac{1}{a_0} \). In view of Condition C and the above estimates for sufficiently small \( a \in [0; a_0) \) we have

\[ \| Qv - w \|_Y \leq C_0 R \leq r \]

i.e.

\[ Q(B_r) \subset B_r. \]

In a similar way for \( \bar{V} = \{ \bar{v}^{(m)}(0) \} \) we obtain

\[ \| Qv - Q\bar{v} \|_Y \leq C_0 \left[ \| F(x, V) - F(x, \bar{V}) \|_X \right] \leq C_0 M_R \| (v - \bar{v}) \|_Y. \]

Therefore for \( C_0 M_R < 1 \) the operator \( Q \) becomes a contraction mapping. Eventually, the contraction mapping principle implies a unique fixed point of \( Q \) in \( B_r \) which is the unique strong solution \( u \in Y = W^{2m}_p(0, a; E(A), E) \).

4. Singular degenerate boundary value problems for infinite systems of equations

Consider the infinite system of BVPs

\[ -x^{2\alpha} \frac{\partial^2 u_m}{\partial x^2} - x^{2\beta} \frac{\partial^2 u_m}{\partial y^2} + d_m u_m + \sum_{j=1}^{\infty} x^\alpha a_{m_j}(x, y) \frac{\partial u_j}{\partial x} + \sum_{j=1}^{\infty} y^\beta b_{m_j}(x, y) \frac{\partial u_j}{\partial y} + \lambda u = f_m(x, y), \quad L_{1k} u = 0, \quad L_{2k} u = 0, \]

where \( L_{ik} \) are defined by (1). Let

\[ D = \{ d_m \}, \quad d_m > 0, \quad u = \{ u_m \}, \quad Du = \{ d_m u_m \}, \quad m = 1, 2, \ldots, \]

\[ l_q(D) = \{ u: u \in l_q, \quad \| u \|_{l_q(D)} = \left( \sum_{m=1}^{\infty} |d_m u_m|^q \right)^\frac{1}{q} < \infty, q \in (1, \infty) \}. \]

From Theorem 1 we obtain

**Theorem 3.** Assume \( a_{m_j}, b_{m_j} \in L_{\infty}(G) \). For \( 0 < \mu < \frac{1}{2} \) and for all \( x, y \in (G) \)

\[ \sup_m \sum_{j=1}^{\infty} a_{m_j}(x) d_j^{-(\frac{1}{2} - \mu)} < M, \quad \sup_m \sum_{j=1}^{\infty} b_{m_j}(x) d_j^{-(\frac{1}{2} - \mu)}. \]
Then for all \( f(x) = \{f_m(x)\}_1^\infty \in L_p((G);l_q) \), \( p,q \in (1,\infty) \), \( |\arg \lambda| \leq \varphi \), \( 0 \leq \varphi < \pi \) and for sufficiently large \( |\lambda| \) problem (12) has a unique solution \( u = \{u_m(x)\}_1^\infty \) that belongs to space \( W^2_{p,a,b}(G,l_q(D);l_q) \) and

\[
\sum_{i=0}^{2} |\lambda|^{i-\frac{1}{2}} \left[ \left\| x^{i\alpha} \frac{\partial^i u}{\partial x^i} \right\|_{L_p(G;l_q)} + \left\| y^{i\beta} \frac{\partial^i u}{\partial y^i} \right\|_{L_p(G;l_q)} \right] + \|Du\|_{L_p(G;l_q)} \leq M \|f\|_{L_p(G;l_q)}.
\]

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