THE COHOMOLOGICAL BRAUER GROUP OF A TORSION
$\mathbb{G}_m$-GERBE

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Abstract. Let $S$ be a scheme and let $\pi : \mathcal{G} \rightarrow S$ be a $\mathbb{G}_{m,S}$-gerbe corresponding to a torsion class $[\mathcal{G}]$ in the cohomological Brauer group $\text{Br}'(S)$ of $S$. We show that the cohomological Brauer group $\text{Br}'(\mathcal{G})$ of $\mathcal{G}$ is isomorphic to the quotient of $\text{Br}'(S)$ by the subgroup generated by the class $[\mathcal{G}]$. This is analogous to a theorem proved by Gabber for Brauer-Severi schemes.

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1. Introduction

For an algebraic stack $\mathcal{X}$, let

$$\text{Br}'\mathcal{X} := H^2_{\text{et}}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}})_{\text{tors}}$$

denote the cohomological Brauer group of $\mathcal{X}$. Gabber in his thesis described the cohomological Brauer group of a Brauer-Severi scheme $X \rightarrow S$ as a quotient of the cohomological Brauer group of the base scheme $S$. More precisely, he proved the following:

Theorem 1.1. [Gab78, Theorem 2] Let $S$ be a scheme, let $\pi_X : X \rightarrow S$ be a Brauer-Severi scheme. Then the sequence

$$H^0_{\text{et}}(S, \mathbb{Z}) \rightarrow \text{Br}'S \xrightarrow{\pi_X^*} \text{Br}'X \rightarrow 0$$

is exact, where the first map sends $1 \mapsto [X]$.

In this paper we prove an analogue of the above theorem for torsion $\mathbb{G}_m$-gerbes.
Theorem 1.2. Let $S$ be a scheme, let $\pi_G : \mathcal{G} \to S$ be a $G_{m,S}$-gerbe corresponding to a torsion class $[\mathcal{G}] \in \text{Br}' S$. Then the sequence

$$H^0_{\acute{e}t}(S, \mathbb{Z}) \to \text{Br}' S \xrightarrow{\pi_G^*} \text{Br}' \mathcal{G} \to 0$$

(1.2.1)
is exact, where the first map sends $1 \mapsto [\mathcal{G}]$.

Remark 1.3. Let $\mathcal{A}$ be an Azumaya $O_S$-algebra and let $\pi^X : X \to S$ and $\pi^G : \mathcal{G} \to S$ be the associated Brauer-Severi scheme and $G_{m,S}$-gerbe of trivializations, respectively. By [Qui73] §8, 4 there exists a finite locally free $O_X$-module $J$ and an $O_X$-algebra isomorphism $\pi^X : \mathcal{A} \simeq \text{End}_{O_X}(J)^{op}$. There is an $O_X$-algebra isomorphism $\text{End}_{O_X}(J)^{op} \simeq \text{End}_{O_X}(J')$ sending $\varphi \mapsto \varphi'$. Since $\mathcal{G}$ is the gerbe of trivializations of $\mathcal{A}$, we have an $S$-morphism $f : X \to \mathcal{G}$; this induces a commutative triangle $\pi^X = f^* \pi^G$ on the cohomological Brauer groups of $S, G, X$. Since $[\mathcal{G}] \in \ker \pi^G_* \subseteq \ker \pi^X_*$ and $\ker \pi^X_*$ is generated by $[\mathcal{G}]$ by Theorem 1.1, we have exactness of (1.2.1) at $\text{Br}' S$. The difficulty of Theorem 1.2 is in showing that $\pi^G_*$ is surjective.

Remark 1.4. By Theorem 1.2 and Remark 1.3, the pullback map

$$f^* : \text{Br}' \mathcal{G} \to \text{Br}' X$$
is an isomorphism, in other words the cohomological Brauer groups of a Brauer-Severi scheme and its associated $G_{m,S}$-gerbe are isomorphic.

Remark 1.5. For an algebraic stack $\mathcal{X}$, let $\text{Br} \mathcal{X}$ denote the (Azumaya) Brauer group of $\mathcal{X}$, namely the abelian group of Morita equivalence classes of Azumaya $O_{\mathcal{X}}$-algebras. There is a functorial injective homomorphism

$$\alpha_{\mathcal{X}} : \text{Br} \mathcal{X} \to \text{Br}' \mathcal{X}$$
called the Brauer map of $\mathcal{X}$, which sends an Azumaya $O_{\mathcal{X}}$-algebra $\mathcal{A}$ to the associated $G_{m,\mathcal{X}}$-gerbe of trivializations of $\mathcal{A}$. Then Theorem 1.2 provides a class of algebraic stacks $\mathcal{G}$ for which the Brauer map $\alpha_{\mathcal{G}}$ is surjective. Indeed, if $S$ is a scheme for which $\alpha_S$ is surjective and $\pi_G : \mathcal{G} \to S$ is a torsion $G_{m,S}$-gerbe, then $\alpha_{\mathcal{G}}$ is surjective by Theorem 1.2 and functoriality of the Brauer map.

1.6. We outline the proof of Theorem 1.2. As in [Gab73], the exact sequence (1.2.1) comes from the Leray spectral sequence for the map $\pi_G$ and sheaf $G_{m,S}$. One step in the proof of Theorem 1.2 is to show the vanishing of the higher pushforwards $R^p \pi_G_* G_{m,S}$. The stalk of $R^2 \pi_G_* G_{m,S}$ at a geometric point $\mathfrak{p}$ of $S$ is isomorphic to $H^2_{\acute{e}t}(B G_{m,A}, G_{m,BG_{m,A}})$ where $A = O^h_{S,\mathfrak{p}}$ is the strict henselization of $S$ at $\mathfrak{p}$. We compute $H^2_{\acute{e}t}(B G_{m,A}, G_{m,BG_{m,A}})$ using the descent spectral sequence associated to the covering $\xi : \text{Spec} A \to B G_{m,A}$, whose $q$th row is the Čech complex associated to the cosimplicial abelian group obtained by applying the functor $H^2_{\acute{e}t}(\_, G_{m,A})$ to the simplicial $A$-scheme $\{G_{m,A}^{p,q}\}_{p \geq 0}$ obtained by taking fiber products of $\xi$. In Section 3 and Section 4 we show that the $E^{1,1}_2$ and $E^{2,0}_2$ terms of this spectral sequence vanish, respectively. It is harder to show that $E^{2,1}_2 = 0$, which comes down to showing that

$$m^* - p_1^* - p_2^* : \text{Pic}(A[t^\pm]) \to \text{Pic}(A[t^\pm_1, t^\pm_2])$$
is injective, where $m, p_1, p_2 : A[t^\pm] \to A[t^\pm_1, t^\pm_2]$ are the $A$-algebra maps sending $t \mapsto t_1 t_2, t_1, t_2$ respectively. If $A$ is a normal domain, then $\text{Pic}(A) \simeq \text{Pic}(A[t^\pm]) \simeq \text{Pic}(A[t^\pm_1, t^\pm_2])$ so the result is trivial. In case $A$ is not normal, we use the Units-Pic
sequence associated to the Milnor square of the normalization \( A \to \overline{A} \). We view Lemma 3.6 as the key lemma of this paper.

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2. Gerbes and the transgression map

The purpose of this section is to prove Lemma 2.10 a description of the higher pushforward \( R^1\pi_*G_m,\mathcal{G} \) for a gerbe \( \pi : \mathcal{G} \to S \), and Proposition 2.11 a description of the differential \( d_2^{0,1} : E^2_{0,1} \to E^2_{2,0} \) in the Leray spectral sequence associated to \( \pi \) in terms of torsors and gerbes. This map \( d_2^{0,1} \) is called the transgression map \cite{Giraud} V, §3.2.

We first recall some background on gerbes. The standard reference is \cite{Giraud}.

**Definition 2.1.** Let \( S \) be a site, and let \( \pi : \mathcal{G} \to S \) be a category fibered in groupoids. We view \( \mathcal{G} \) as a site with the Grothendieck topology inherited from \( S \) \cite[III, 3.1]{AGV72}. For any object \( U \in S \), let \( \mathcal{G}(U) \) denote the fiber category of \( \mathcal{G} \) over \( U \).

The **inertia stack** of \( \pi : \mathcal{G} \to S \) is the 2-fiber product \( I_{\mathcal{G}/S} := \mathcal{G} \times_{\Delta_{\mathcal{G}/S}} \mathcal{G} \) of the diagonal \( \Delta_{\mathcal{G}/S} : \mathcal{G} \to \mathcal{G} \times S \mathcal{G} \) with itself. The inertia stack \( I_{\mathcal{G}/S} \) is fibered in sets over \( \mathcal{G} \) via either projection \( I_{\mathcal{G}/S} \to \mathcal{G} \), hence we may identify \( I_{\mathcal{G}/S} \) with the sheaf of groups on \( \mathcal{G} \) associating \( x \mapsto \text{Aut}_x(x) \).

We say that \( \pi \) is a **gerbe** if the following conditions are satisfied:

(i) The fibered category \( \mathcal{G} \) is a stack over \( S \).

(ii) For any \( U \in S \), there exists a covering \( \{ U_i \to U \}_{i \in I} \) such that \( \mathcal{G}(U_i) \neq \emptyset \) for all \( i \in I \).

(iii) For any \( U \in S \) and \( x_1, x_2 \in \mathcal{G}(U) \), there exists a covering \( \{ U_i \to U \}_{i \in I} \) such that for all \( i \in I \) there exists an isomorphism \( x_1|_{U_i} \simeq x_2|_{U_i} \) in \( \mathcal{G}(U_i) \).

Let \( \mathbb{A} \) be an abelian sheaf on \( S \). We say that a gerbe \( \pi \) is an **\( \mathbb{A} \)-gerbe** if it is equipped with an isomorphism \( \iota : \mathbb{A}_{G} \to I_{G/S} \) of sheaves of groups on \( \mathcal{G} \).

If \( S \) is equipped with a sheaf of rings such that \( (S, \mathcal{O}_S) \) is a locally ringed site, we set \( \mathcal{O}_G := \pi^{-1}\mathcal{O}_S; \) then the pair \( (\mathcal{G}, \mathcal{O}_G) \) is a locally ringed site. \( \square \)

For the remainder of this section, we will assume the following setup:

**Setup 2.2.** Let \( S \) be a locally ringed site, let \( \mathbb{A} \) be an abelian sheaf on \( S \), let \( \pi : \mathcal{G} \to S \) be an \( \mathbb{A} \)-gerbe.

**Definition 2.3** (Inertial action, eigensheaves, twisted sheaves) \cite{LiebA, LiebB} Let \( \mathcal{F} \) be an \( \mathcal{O}_G \)-module. For an object \( x \in \mathcal{G} \) and \( a \in \Gamma(x, \mathbb{A}_{G}) \), let \( \iota(a)^* : \Gamma(x, \mathcal{F}) \to \Gamma(x, \mathcal{F}) \) be the restriction map of the sheaf \( \mathcal{F} \) via the automorphism \( \iota(a) : x \to x; \) such \( x \) and \( a \) defines an \( \mathcal{O}_{\mathcal{G}/x} \)-linear automorphism of \( \mathcal{F}|_{\mathcal{G}/x} \) by \( y \mapsto \iota(a_y)^* \); thus we have a homomorphism \( \mathbb{A}_{G} \to \text{Aut}_{\mathcal{O}_G}(\mathcal{F}) \) of group sheaves on \( \mathcal{G} \) corresponding to an \( \mathcal{O}_G \)-linear \( \mathbb{A}_{G} \)-action on \( \mathcal{F} \), called the **inertial action**.
Let
\[ \hat{\mathbb{A}} := \text{Hom}_{\text{Ab}(S)}(\mathbb{A}, G_{m,S}) \]
denote the group of characters of \( \mathbb{A} \). Given an \( O_G \)-module \( \mathcal{F} \) and a character \( \chi \in \hat{\mathbb{A}} \), the \( \chi \)th eigensheaf is the subsheaf \( \mathcal{F}_\chi \subseteq \mathcal{F} \) defined as follows: for all objects \( x \in G \), a section \( f \in \Gamma(x, \mathcal{F}) \) is contained in \( \Gamma(x, \mathcal{F}_\chi) \) if for any morphism \( y \rightarrow x \) and any \( a \in \Gamma(y, A_G) \) we have
\[ (\iota(a)^*)(f|_y) = \chi_G(a) \cdot f|_y \] (2.3.1)
in \( \Gamma(y, \mathcal{F}) \). The \( O_G \)-module \( \mathcal{F} \) is called \( \chi \)-twisted if the inclusion \( \mathcal{F}_\chi \subseteq \mathcal{F} \) is an equality. If \( \chi \) is the trivial character, the eigensheaf \( \mathcal{F}_\chi \) is denoted \( \mathcal{F}_0 \) and \( \chi \)-twisted sheaves are called 0-twisted. For any \( O_S \)-module \( M \), the pullback \( \pi^* M \) is 0-twisted; in particular the structure sheaf \( O_G \) is 0-twisted. \( \square \)

**Definition 2.4** (Category of \( \chi \)-twisted modules). For a character \( \chi \in \hat{\mathbb{A}} \), let
\[ \text{Mod}(G, \chi) \]
denote the full subcategory of \( \text{Mod}(G) \) consisting of \( \chi \)-twisted \( O_G \)-modules. \( \square \)

**Remark 2.5.** Given two \( O_G \)-modules \( \mathcal{F} \) and \( \mathcal{G} \), any \( O_G \)-linear morphism \( \varphi : \mathcal{F} \rightarrow \mathcal{G} \) restricts to an \( O_G \)-linear morphism \( \varphi_\chi : \mathcal{F}_\chi \rightarrow \mathcal{G}_\chi \); the assignment \( \mathcal{F} \rightarrow \mathcal{F}_\chi \) defines a functor \( \text{Mod}(G) \rightarrow \text{Mod}(G, \chi) \) which is right adjoint (and a retraction) to the inclusion \( \text{Mod}(G, \chi) \rightarrow \text{Mod}(G) \).

**Remark 2.6** (Modules on trivial gerbes). We say that an \( \mathbb{A} \)-gerbe \( G \) is trivial if there is an isomorphism \( G \simeq B\mathbb{A} \). In this case we have the usual equivalence of categories between sheaves on \( G \) and sheaves on \( S \) equipped with an \( \mathbb{A} \)-action. For a sheaf \( \mathcal{F} \in \text{Sh}(B\mathbb{A}) \), the pushforward \( \pi_* \mathcal{F} \) is identified with the subsheaf of \( \mathcal{F} \) of sections invariant under the action of \( \mathbb{A} \). For any sheaf \( M \in \text{Sh}(S) \), the inverse image \( \pi^{-1} M \in \text{Sh}(B\mathbb{A}) \) corresponds to the sheaf \( M \) equipped with the trivial \( \mathbb{A} \)-action. If \( s : S \rightarrow B\mathbb{A} \) is the section of \( \pi \) corresponding to the trivial \( \mathbb{A} \)-torsor, then \( s^{-1} : \text{Sh}(B\mathbb{A}) \rightarrow \text{Sh}(S) \) is the functor forgetting the \( \mathbb{A} \)-action.

**Remark 2.7.** For any \( O_G \)-module \( \mathcal{F} \), the counit map
\[ \pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F} \]
is injective and its image coincides with \( \mathcal{F}_0 \). Indeed, this can be checked locally on \( S \), in which case we may assume \( G \) is the trivial gerbe and use [Remark 2.6].

**Lemma 2.8.** Let \( S \) be a locally ringed site, let \( \mathbb{A} \) be an abelian sheaf on \( S \), and let \( \pi : G \rightarrow S \) be an \( \mathbb{A} \)-gerbe. The pullback functor
\[ \pi^* : \text{Mod}(S) \rightarrow \text{Mod}(G, 0) \]
is an equivalence of categories with quasi-inverse \( \pi_* \). If \( P \) is a property of modules preserved by pullback via arbitrary morphisms of sites (e.g. quasi-coherent, flat, locally of finite type, locally of finite presentation, locally free), an \( O_S \)-module \( M \) has \( P \) if and only if the \( O_G \)-module \( \pi^* M \) has \( P \).

**Proof.** For the first assertion, it suffices to show that for any \( O_S \)-module \( M \) the unit map
\[ M \rightarrow \pi_* \pi^* M \]
is an isomorphism, and that for any 0-twisted \( O_G \)-module \( \mathcal{F} \) the counit map
\[ \pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F} \]
is an isomorphism. Both of these claims are local on \( S \), hence we may assume that \( \mathcal{G} \) is trivial, in which case the claims follow from Remark 2.6 and Remark 2.7. The second assertion is also local on \( S \), hence we may assume that \( \mathcal{G} \) is trivial, in which case there is a section \( s : S \to \mathcal{G} \) of \( \pi \). For any 0-twisted \( \mathcal{O}_\mathcal{G} \)-module \( \mathcal{F} \), we have \( \pi_* \mathcal{F} \simeq s^* \mathcal{F} \) by the discussion in Remark 2.6. □

In order to describe the higher pushforwards \( R^1 \pi_* \mathbb{G}_m \), we will use the following result on the Picard group of \( A \)-gerbes.

**Remark 2.9** (Picard group of \( A \)-gerbes). Assume the setup of Lemma 2.8. By [Bro09, 5.3.4], for any invertible \( \mathcal{O}_\mathcal{G} \)-module \( \mathcal{L} \), there exists a unique character \( \chi_\mathcal{L} \in \hat{A} \) such that the diagram

\[
\begin{array}{ccc}
\mathbb{A}_{\mathcal{G}} \times \mathcal{L} & \longrightarrow & \mathcal{L} \\
\pi^* \chi_\mathcal{L} \times \text{id}_\mathcal{L} & \downarrow & \text{id}_\mathcal{L} \\
\mathbb{G}_{m, \mathcal{G}} \times \mathcal{L} & \longrightarrow & \mathcal{L}
\end{array}
\]  

(2.9.1)

commutes, where the top row is the inertial action and the bottom row is the restriction of the \( \mathcal{O}_\mathcal{G} \)-module structure on \( \mathcal{L} \). The condition that (2.9.1) commutes is equivalent to the condition (2.3.1), in other words \( \mathcal{L} \) is a \( \chi_\mathcal{L} \)-twisted sheaf. For two invertible \( \mathcal{O}_\mathcal{G} \)-modules \( \mathcal{L}_1, \mathcal{L}_2 \) we have \( \chi_{\mathcal{L}_1} \otimes \chi_{\mathcal{L}_2} = \chi_{\mathcal{L}_1 \cdot \mathcal{L}_2} \) by [Bro09, 5.3.6 (2)], hence the assignment \( \mathcal{L} \mapsto \chi_\mathcal{L} \) defines a group homomorphism

\[
\beta_\mathcal{G} : \text{Pic}(\mathcal{G}) \to \hat{A}
\]

of abelian groups. By [Bro09, 5.3.6 (3)], we have that \( \chi_\mathcal{L} = 0 \) if and only if \( \mathcal{L} \) is of the form \( \pi^* M \) for an invertible \( \mathcal{O}_S \)-module \( M \); in other words there is an exact sequence

\[
0 \to \text{Pic}(S) \overset{\pi^*}{\longrightarrow} \text{Pic}(\mathcal{G}) \overset{\beta_\mathcal{G}}{\longrightarrow} \hat{A}
\]  

(2.9.2)

where injectivity of \( \pi^* \) follows from Lemma 2.8. The sequence (2.9.2) is functorial on \( S \) in the following sense: if \( p : T \to S \) is a morphism of locally ringed sites and \( \pi_T : \mathcal{G}_T \to T \) is the \( A_T \)-gerbe obtained by pullback, then the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Pic}(S) \\
\| & p^* & p^* \\
0 & \longrightarrow & \text{Pic}(T)
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & p^* & \downarrow p^* \\
\text{Pic}(\mathcal{G}) & \longrightarrow & \text{Pic}(\mathcal{G}_T) \\
\downarrow & \beta_{\mathcal{G}} & \downarrow \beta_{\mathcal{G}_T} \\
\hat{A} & \longrightarrow & \hat{A}_T
\end{array}
\]

commutes.

In case \( \mathcal{G} := BA \), by [Bro09, 5.3.7] the map \( \beta_\mathcal{G} \) is surjective and the sequence (2.9.2) is split; the map \( \beta_\mathcal{G} \) admits a natural section \( \hat{A} \to \text{Pic}(\mathcal{G}) \) taking a character \( \chi \) to the trivial \( \mathcal{O}_S \)-module equipped with the \( A \)-action corresponding to \( \chi \) via the isomorphism \( \mathbb{G}_{m, S} \simeq \text{Aut}_{\mathcal{O}_S}(\mathcal{O}_S) \).

**Lemma 2.10.** Let \( S \) be a locally ringed site, let \( A \) be an abelian sheaf on \( S \), let \( \pi : \mathcal{G} \to S \) be an \( A \)-gerbe. There is a natural isomorphism

\[
R^1 \pi_* \mathbb{G}_{m, \mathcal{G}} \simeq \text{Hom}_{\text{Ab}(S)}(A, \mathbb{G}_{m, S})
\]  

(2.10.1)

of abelian sheaves on \( S \).
Proof. Let \( U \in \mathcal{S} \) be an object. Taking \( T := \mathcal{S}/U \) and \( p : \mathcal{S}/U \to \mathcal{S} \) the inclusion of categories, we obtain an exact sequence

\[
0 \to \text{Pic}(\mathcal{S}/U) \xrightarrow{\pi_{\mathcal{S}/U}} \text{Pic}(\mathcal{G}_{\mathcal{S}/U}) \xrightarrow{\beta_{\mathcal{G}_{\mathcal{S}/U}}} \hat{\mathcal{A}}_{\mathcal{S}/U} \tag{2.10.2}
\]

of abelian groups. Letting \( U \) range over the objects of \( \mathcal{S} \), we obtain an exact sequence of abelian presheaves whose value on \( U \) is \([2.10.2] \) and sheafifying this sequence gives the desired isomorphism. \( \square \)

We specialize to the case \( A = \mathbb{G}_{m,S} \).

**Proposition 2.11.** Let \( \mathcal{S} \) be a locally ringed site and let \( \pi : \mathcal{G} \to \mathcal{S} \) be a \( \mathbb{G}_{m,S} \)-gerbe. Let

\[
d_2^{0,1} : H^0(\mathcal{S}, \mathbb{R}^1 \pi_* \mathbb{G}_{m,G}) \to H^2(\mathcal{S}, \mathbb{R}^0 \pi_* \mathbb{G}_{m,G}) \tag{2.11.1}
\]

be the differential in the Leray spectral sequence associated to the map \( \pi \) and sheaf \( \mathbb{G}_{m,G} \). Under the identification \([2.10.1] \), the differential \( d_2^{0,1} \) sends the identity \( id_{\mathcal{G}_{m,S}} \in \text{Hom}_{\text{Ab}(\mathcal{S})}(\mathcal{G}_{m,S}, \mathcal{G}_{m,S}) \) to the class \([\mathcal{G}] \in H^2(\mathcal{S}, \mathcal{G}_{m,S}) \).

Proof. Let \( c \in H^0(\mathcal{S}, \mathbb{R}^1 \pi_* \mathbb{G}_{m,G}) \) be the class corresponding to the identity section \( \chi := id_{\mathcal{G}_{m,S}} \in \text{Hom}_{\text{Ab}(\mathcal{S})}(\mathcal{G}_{m,S}, \mathcal{G}_{m,S}) \) via the isomorphism \([2.10.1] \). Let \( D(c) \to \mathcal{S} \) denote the category fibered in groupoids whose fiber category \((D(c))(U)\) for an object \( U \in \mathcal{S} \) consists of the invertible \( \mathcal{O}_{\mathcal{G}_{\mathcal{S}/U}} \)-modules whose image under the map

\[
H^1(\mathcal{G}_{\mathcal{S}/U}, \mathcal{G}_{m,S}) \to H^0(U, \mathbb{R}^1 \pi_* \mathbb{G}_{m,G})
\]

is equal to the image of \( c \) under the restriction map

\[
H^0(\mathcal{S}, \mathbb{R}^1 \pi_* \mathbb{G}_{m,G}) \to H^0(U, \mathbb{R}^1 \pi_* \mathbb{G}_{m,G})
\]

of the sheaf \( \mathbb{R}^1 \pi_* \mathbb{G}_{m,G} \). By \([\text{Gir71} \text{, II, 3.2.1}] \), the category \( D(c) \) is a \( \mathbb{G}_{m,S} \)-gerbe, and the assignment \( c \mapsto [D(c)] \) coincides with the differential \([2.11.1] \). By the above description of \( D(c) \) and the definition of the isomorphism \([2.10.1] \) as the one obtained by sheafifying the maps \( \beta_{\mathcal{G}_{\mathcal{S}/U}} \) in \([2.10.2] \), we have that an invertible \( \mathcal{O}_{\mathcal{G}_{\mathcal{S}/U}} \)-module \( \mathcal{L} \) is contained in \((D(c))(U)\) if and only if it is \( \chi_{\mathcal{S}/U} \)-twisted. By \([\text{Lie04} \text{, Proposition 2.1.2.5}] \), we have that there is an isomorphism \( \mathcal{G} \simeq D(c) \) of \( \mathbb{G}_{m,S} \)-gerbes. \( \square \)

3. **Picard groups of (Laurent) polynomial rings**

In this section we prove \( \text{Lemma 3.11} \). For us, the main difficulty is that there are rings \( A \) for which the pullback map \( \text{Pic}(A) \to \text{Pic}(A[t]) \) is not an isomorphism. The ring \( A \) is called *seminormal* \([\text{Swa80} \text{, p. 210}], \) \([\text{Wei13} \text{, p. 29}] \) if for every \( b, c \in A \) satisfying \( b^3 = c^2 \) there exists \( a \in A \) such that \( a^2 = b \) and \( a^3 = c \). Seminormal rings are automatically reduced \([\text{Lam06} \text{, VIII, §7}] \). By Traverso's theorem \([\text{Tra70} \text{, Theorem 3.6}], \) \([\text{Wei13} \text{, Theorem 3.11}] \), the map \( \text{Pic}(A) \to \text{Pic}(A[t]) \) is an isomorphism if and only if the reduction \( A_{\text{red}} \) is a seminormal ring. Taking the strict henselization of the cuspidal cubic \( k[x, y]/(y^2 = x^3) \) at the cusp gives an example of a reduced strictly henselian local ring \( A \) which is not seminormal; by \( \text{Remark 3.9} \) in this case we also have \( \text{Pic}(A[t, t^{-1}]) = 0 \).

Throughout this section and \( \text{Section 4} \) we will use \( \text{Notation 3.1} \) and \( \text{Notation 3.2} \).
Notation 3.3 (\(\Delta, C^*G, h^n(C^*G)\)). Let \(\Delta\) be the category with objects \([n] := \{0, \ldots, n\}\) for each nonnegative integer \(n \geq 0\) and whose morphisms \(\varphi : [m] \rightarrow [n]\) correspond to nondecreasing maps \(\varphi : \{0, \ldots, m\} \rightarrow \{0, \ldots, n\}\). For \(n \geq 0\) and \(0 \leq i \leq n + 1\), we denote \(\delta^n_i : [n] \rightarrow [n + 1]\) the injective nondecreasing map whose image does not contain \(i\). For \(n \geq 0\) and \(0 \leq i \leq n\), we denote \(\sigma^n_i : [n + 1] \rightarrow [n]\) the surjective nondecreasing map satisfying \((\sigma^n_i)^{-1}(i) = (i, i + 1)\). A cosimplicial set (resp. abelian group, resp. ring) is a covariant functor from \(\Delta\) to \((\text{Set})\) (resp. \((\text{Ab})\), resp. \((\text{Ring})\)).

If \(G\) is a cosimplicial abelian group, we denote by

\[ C^*G \]

the cochain complex where \(C^nG := G([n])\) for \(n \geq 0\) and where the \(n\)th differential \(d^n_G : C^nG \rightarrow C^{n+1}G\) is the alternating sum \(\sum_{i=0}^{n+1} G(\delta^n_i)\). We denote by

\[ h^n(C^*G) \]

the cohomology of \(C^*G\) at \(C^nG\).

Notation 3.2 (\(L_A, P_A\)). Let \(A\) be a ring, let \(\pi : B\mathbb{G}_{m,A} \rightarrow \text{Spec} A\) be the trivial \(\mathbb{G}_{m,A}\)-gerbe, let \(\xi : \text{Spec} A \rightarrow B\mathbb{G}_{m,A}\) be the section of \(\pi\) corresponding to the trivial \(\mathbb{G}_{m,A}\)-torsor. Taking 2-fiber products of \(\xi\), we obtain a cosimplicial \(A\)-algebra

\[ L_A : \Delta \rightarrow (A\text{-alg}) \]

where

\[ L_A([p]) := A[t_1^{\pm}, \ldots, t_p^{\pm}] \]

is the Laurent polynomial ring in \(p\) indeterminates over \(A\) (where by convention \(L_A([0]) := A\)). For \(p \geq 0\) and \(0 \leq i \leq p + 1\), the \(i\)th degeneracy map \(L_A(\delta^n_i) : L_A([p]) \rightarrow L_A([p + 1])\) is the \(A\)-algebra map sending \((t_1, \ldots, t_p) \mapsto (t_1, \ldots, t_it_{i+1}, \ldots, t_{p+1})\) where by abuse of notation we write “\(t_0\)” and “\(t_{p+2}\)” to mean “\(1\)” (in the cases \(i = 0\) and \(i = p + 1\) respectively). For \(p \geq 0\) and \(0 \leq i \leq p\), the \(i\)th face map \(L_A(\sigma^n_i) : L_A([p + 1]) \rightarrow L_A([p])\) is the \(A\)-algebra map sending \((t_1, \ldots, t_{p+1}) \mapsto (t_1, \ldots, t_i, t_{i+1}, \ldots, t_p)\).

We also have the cosimplicial \(A\)-algebra

\[ P_A : \Delta \rightarrow (A\text{-alg}) \]

where

\[ P_A([p]) := A[t_1, \ldots, t_p] \]

is the polynomial ring in \(p\) indeterminates over \(A\), viewed as the subalgebra of \(L_A([p])\), and for which the \(A\)-algebra map \(P_A(\varphi) : P_A([m]) \rightarrow P_A([n])\) is obtained by restricting \(L_A(\varphi) : L_A([n]) \rightarrow L_A([m])\).

We make the formulas \(P_A(\delta^n_i)\) and \(P_A(\sigma^n_i)\) explicit for \(p = 0, 1\). For \(0 \leq i \leq 1\), the \(A\)-algebra map \(P_A(\delta^0_i) : A \rightarrow A[t_1]\) is the unique one. For \(0 \leq i \leq 2\), the \(A\)-algebra map \(P_A(\delta^1_i) : A[t_1] \rightarrow A[t_1, t_2]\) sends \(t_1\) to \(t_1, t_1t_2, t_2\) respectively. For \(0 \leq i \leq 0\), the \(A\)-algebra map \(P_A(\sigma^0_i) : A[t_1] \rightarrow A\) sends \(t_1\) to \(1\). For \(0 \leq i \leq 1\), the \(A\)-algebra map \(P_A(\sigma^1_i) : A[t_1, t_2] \rightarrow A[t_1]\) sends \(t_1, t_2\) to \((1, t_1), (t_1, 1)\) respectively.

Notation 3.3 (\(N_xF, N_{x_1,x_2}F\)). Given a functor

\[ F : (\text{Ring}) \rightarrow (\text{Ab}) \]

we define new functors

\[ N_xF, N_{x_1,x_2}F : (\text{Ring}) \rightarrow (\text{Ab}) \]

(3.3.1)
by
\[ N_x F(A) := \ker(F(x = 1) : F(A[x]) \to F(A)) \]
\[ N_{x_1, x_2} F(A) := \ker((F(x_2 = 1), F(x_1 = 1)) : F(A[x_1, x_2]) \to F(A[x_1]) \oplus F(A[x_2])) \]
for any ring \( A \), where \( x, x_1, x_2 \) are indeterminates. The notation “\( N_x F \)” was defined by Weibel in [Wei91, §1].

The operation “\( N_x \)” can be iterated, for example if \( x_1, x_2 \) are indeterminates, then \( N_{x_1}(N_{x_2} F) \) is a functor \((\text{Ring}) \to (\text{Ab})\).

**Lemma 3.4.** In Notation 3.3, we have
\[ N_{x_1, x_2} F(A) = N_{x_1}(N_{x_2} F)(A) = N_{x_2}(N_{x_1} F)(A) \]
for any ring \( A \).

**Proof.** The claim follows from considering the commutative diagram

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & N_{x_1, x_2} F(A) & N_{x_1} F(A[x_2]) & N_{x_1} F(A) \\
0 & N_{x_2} F(A[x_1]) & F(A[x_1, x_2]) & F(A[x_1]) \\
0 & N_{x_2} F(A) & F(A[x_2]) & F(A) \\
0 & 0 & 0 & 0
\end{array}
\]

where each row and column is (split) exact. \( \square \)

**Lemma 3.5.** Assume Notation 3.1, Notation 3.2, and Notation 3.3. We have
\[ d_{1,P_A}^1(N_{t_1} F(A)) \subset N_{t_1, t_2} F(A) \]
for any ring \( A \).

**Proof.** For \( 0 \leq i \leq 2 \), the composition \( P_A(\sigma_0^i)P_A(\delta_1^i) \) correspond to the \( A \)-algebra maps \( A[t_1] \to A[t_1] \) sending \( t_1 \mapsto 1, t_1, t_1 \), respectively; thus \( F(P_A(\sigma_0^i))(d_{1,P_A}^1(N_{t_1} F(A))) = 0 \). By a similar argument, we have \( F(P_A(\sigma_1^i))(d_{1,P_A}^1(N_{t_1} F(A))) = 0 \). \( \square \)

**Lemma 3.6.** Assume Notation 3.1, Notation 3.2, and Notation 3.3. We have
\[ h^1(C^\bullet(\text{Pic} P_A)) = 0 \]
for any ring \( A \).
Proof. Since $P_A(\delta^0_0) = P_A(\delta^0_1)$, the differential $d^0_{\text{Pic}P_A}: \text{Pic}(A) \to \text{Pic}(A[t])$ is the 0 map. Hence it suffices to show that
$$d^1_{\text{Pic}P_A}: \text{Pic}(A[t]) \to \text{Pic}(A[t_1, t_2])$$
is injective.

We have that $A$ is the filtered colimit of subrings of $A$ which are finite type $\mathbb{Z}$-algebras, hence by e.g. [Sta18, 0B8W] we may reduce to the case when $A$ is a finite type $\mathbb{Z}$-algebra. In particular $A$ has finite Krull dimension. We proceed by induction on $\dim A$. Since the Picard group of a ring is invariant under nilpotent thickenings, we may assume that $A$ is reduced. If $\dim A = 0$, then $A$ is a finite product of fields, hence $\ker d^1_{\text{Pic}P_A} = 0$ (since in fact $\text{Pic}(A[t]) = 0$ in this case).

Suppose $\dim A > 0$ and let $\alpha \in \text{Pic}(A[t])$ be a class such that $d^1_{\text{Pic}P_A}(\alpha) = 0$. We have a direct sum decomposition $\text{Pic}(A) \oplus N_{t_1} \text{Pic}(A) \simeq \text{Pic}(A[t_1])$, and $P_A(\delta^0_0), P_A(\delta^0_1), P_A(\delta^0_2)$ are $A$-algebra maps, so in fact $\alpha \in N_{t_1} \text{Pic}(A)$. Let $Q(A)$ denote the total ring of fractions of $A$, and let $A^{sn} \subset Q(A)$ denote the seminormalization [Swa80, Lemma 2.2] of $A$ in $Q(A)$.

Write
$$A^{sn} = \varprojlim_{\lambda \in \Lambda} A_{\lambda}$$
where each $A \subset A_{\lambda} \subset A^{sn}$ is a finitely generated subextension of $A^{sn}$; then $A \subset A_{\lambda}$ is a finite extension of rings since it is an integral extension. Thus
$$N_{t_1} \text{Pic}(A^{sn}) \simeq \varprojlim_{\lambda \in \Lambda} N_{t_1} \text{Pic}(A_{\lambda})$$
by e.g. [Sta18, 0B8W]. By [Swa80, Corollary 3.4], we have that $A^{sn}$ is seminormal, thus $N_{t_1} \text{Pic}(A^{sn}) = 0$ by Traverso’s theorem [Wei13, Theorem 3.11]. Hence there exists some $\lambda \in \Lambda$ for which $\alpha$ lies in the kernel of $N_{t_1} \text{Pic}(A) \to N_{t_1} \text{Pic}(A_{\lambda})$.

Let $I := \{x \in A : xA_{\lambda} \subset A\} = \text{Ann}_A(A_{\lambda}/A)$ be the conductor ideal of $A \subset A_{\lambda}$; it is the largest ideal of $A_{\lambda}$ contained in $A$ so in particular it is also an ideal of $A$. We denote $U(A) := A^\times$ the group of units of $A$. By Milnor’s theorem [Bas68, IX, (5.3)], the Milnor square
$$
\begin{array}{ccc}
A & \rightarrow & A_{\lambda} \\
\downarrow & & \downarrow \\
A/I & \rightarrow & A_{\lambda}/I 
\end{array}
$$
gives an exact sequence
$$1 \to U(A) \xrightarrow{\Delta} U(A/I) \oplus U(A_{\lambda}) \xrightarrow{\delta} U(A_{\lambda}/I)$$

\text{(3.6.1)}

$$
\begin{array}{ccc}
\text{Pic}(A) & \xrightarrow{\delta_A} & \text{Pic}(A/I) \oplus \text{Pic}(A_{\lambda}) \\
\downarrow & & \downarrow \\
\text{Pic}(A_{\lambda}/I) & \xrightarrow{\Delta} & \text{Pic}(A_{\lambda}/I) 
\end{array}
$$
\text{(3.6.2)}

\footnote{Here, instead of using the limit argument, we may also use that the extension $A \subset A^{sn}$ is finite since $A$ is a Nagata ring (it is a finite type $\mathbb{Z}$-algebra) and thus has finite normalization, hence has finite seminormalization.}
of abelian groups, called the Units-Pic sequence [Wei13, I, Theorem 3.10]; here we denote by $\Delta$ the diagonal map and by $\pm$ the difference map. The boundary map $\partial$ of (3.6.2) is functorial for morphisms between Milnor squares. Hence applying $N_{t_1}$ and $N_{t_1,t_2}$ to (3.6.2) gives a commutative diagram

$$
\begin{array}{c}
N_{t_1}U(A/I) \oplus N_{t_1}U(A_\lambda) \xrightarrow{\pm t_1} N_{t_1,t_2}(A/I) \oplus N_{t_1,t_2}U(A_\lambda) \\
\downarrow \partial_1 \downarrow \downarrow \downarrow \partial_1, t_2 \\
N_{t_1}U(A_\lambda/I) \xrightarrow{d_{A_{\lambda}}^{1}} N_{t_1,t_2}(A_\lambda/I) \\
\downarrow \downarrow \downarrow \\
N_{t_1}Pic(A) \xrightarrow{d_{\operatorname{Pic}(A)}^{1}} N_{t_1,t_2}Pic(A) \\
\downarrow \downarrow \downarrow \\
N_{t_1}Pic(A/I) \oplus N_{t_1}Pic(A_\lambda) \xrightarrow{d_{\operatorname{Pic}(A/I)}^{1}} N_{t_1,t_2}Pic(A/I) \oplus N_{t_1,t_2}Pic(A_\lambda)
\end{array}
$$

where we denote by $\pm t_1, \partial_1, \Delta_1$ (resp. $\pm t_1, \partial_1, \Delta_1, t_2$) the corresponding maps in the Units-Pic sequences associated to the Milnor squares obtained by tensoring (3.6.1) with $- \otimes \mathbb{Z}[t_1]$ (resp. $- \otimes \mathbb{Z}[t_1, t_2]$), and we denote by $d_{\operatorname{Pic}(A)}^{1}$ the restriction to $N_{t_1}F(A) \rightarrow N_{t_1,t_2}F(A)$ (which makes sense by Lemma 3.5). Each column of (3.6.3) is exact. By Lemma 3.7 below, we have that $I$ contains a nonzerodivisor of $A$; hence $I$ is not contained in any minimal prime of $A$ by [AK13, Lemma (14.10)]; hence $A/I$ has smaller Krull dimension than that of $A$ (c.f. [Wei13, p. 15]); the image of $\alpha$ under $N_{t_1}Pic(A) \rightarrow N_{t_1}Pic(A/I)$ is contained in $\ker d_{\operatorname{Pic}(A)}^{1}$, which is 0 by the induction hypothesis since $\dim A/I < \dim A$. Hence $\Delta_1(\alpha) = 0$, so by exactness of the left column of (3.6.3) there exists

$$
\xi \in N_{t_1}U(A_\lambda/I)
$$

such that $\alpha = \partial_1(\xi)$. By e.g. [Wei13, I, Lemma 3.12] we have that $\xi$ is of the form

$$
\xi = 1 + \beta(t_1)
$$

where

$$
\beta \in t_1(\operatorname{nil}(A_\lambda/I)[t_1])
$$

is a polynomial with nilpotent coefficients and whose constant coefficient is zero. We have

$$
d_{\operatorname{Pic}(A_\lambda/I)}^{1}(\xi) = (1 + \beta(t_1))(1 - \beta(t_1,t_2))(1 + \beta(t_2))
$$

in $N_{t_1,t_2}U(A_\lambda/I)$. Since $\partial_1(t_2) d_{\operatorname{Pic}(A_\lambda/I)}^{1}(\xi) = d_{\operatorname{Pic}(A_\lambda)}^{1}(\partial_1, t_2) = d_{\operatorname{Pic}(A_\lambda)}^{1}(\alpha) = 0$, by the exactness of the right column of (3.6.3) there exists

$$
\gamma \in N_{t_1,t_2}U(A/I) \oplus N_{t_1,t_2}U(A_\lambda)
$$

such that $d_{\operatorname{Pic}(A_\lambda/I)}^{1}(\xi) = \pm t_1, t_2(\gamma)$. Here by [Wei13, I, Lemma 3.12] the inclusion $U(A_\lambda) \subset U(A_\lambda[t_1, t_2])$ is an equality since $A_\lambda$ is reduced, hence $N_{t_1,t_2}U(A_\lambda) = 0$. Moreover $A/I \rightarrow A_\lambda/I$ is injective (since $I$ is the largest ideal of $A_\lambda$ contained in $A$), hence (3.6.4) is in fact contained in $N_{t_1,t_2}U(A/I)$. Thus in fact

$$
\beta \in (A/I)[t_1]
$$
as can be seen for example by setting \( t_2 = 0 \) in \[3.6.4\]. In other words, we have that \( \xi \) is in the image of \( \pm \iota_{t_1} \); since \( \partial_{t_1} \circ \pm \iota_{t_1} = 0 \), we conclude \( \alpha = 0 \). \( \square \)

In the following lemma, we write out the details of a claim in [Wei13, p. 15].

**Lemma 3.7.** Let \( A \) be a ring with total ring of fractions \( Q(A) \), and let \( A \subset B \subset Q(A) \) be a subring.

(i) The inclusion \( A \subset B \) preserves nonzerodivisors, and any nonzerodivisor of \( B \) is of the form \( r/u \) where \( r, u \in A \) are nonzerodivisors of \( A \).

(ii) The total ring of fractions of \( B \) is \( Q(A) \).

(iii) If \( A \subset B \) is a finite extension, the conductor ideal \( I = \{ x \in A : x B \subset A \} = \text{Ann}_A(B/A) \) contains a nonzerodivisor of \( A \).

**Proof.** (i) If \( x \in A \) is a nonzerodivisor of \( A \), then its image in \( Q(A) \) is a nonzerodivisor of \( Q(A) \), hence its image in \( B \) is a nonzerodivisor of \( B \). An arbitrary element of \( B \) is of the form \( r/u \) where \( r, u \in A \) and \( u \) is a nonzerodivisor of \( A \). If \( x \in A \) is an element such that \( rx = 0 \) in \( A \), then \( u(r/u)x = 0 \) in \( B \) and \( u \) is a nonzerodivisor of \( B \) (by the first part) so \( (r/u)x = 0 \) in \( B \); then \( x = 0 \) since \( r/u \) is by assumption a nonzerodivisor of \( B \); hence \( r \) is a nonzerodivisor of \( A \).

(ii) Let \( \varphi : B \to S \) be a ring homomorphism such that \( \varphi \) sends nonzerodivisors of \( B \) to units of \( S \). By the first part, \( \varphi \) sends nonzerodivisors of \( A \) to units of \( S \), hence there exists a ring map \( \xi : Q(A) \to S \) such that \( \xi|_A = \varphi|_A \). By definition of \( \xi \), for any \( a/u \in B \) we have \( \xi(a/u) = \varphi(a) \cdot (\varphi(u))^{-1} = \varphi(a/u) \); hence \( \xi|_B = \varphi \).

(iii) Let \( x_1/u_1, \ldots, x_n/u_n \) be elements of \( B \) which generates \( B \) as an \( A \)-module; then \( u_1 \cdots u_n \) is a nonzerodivisor of \( A \) which is contained in \( I \). \( \square \)

**Remark 3.8.** In the proof of [Lemma 3.6] we may use the normalization instead of the seminormalization. If \( A \) is a reduced finite type \( \mathbb{Z} \)-algebra, then its normalization \( \overline{A} \subset Q(A) \) is a finite extension of \( A \); thus \( \overline{A} \) is a Noetherian reduced ring which is integrally closed in its total ring of fractions (by e.g. [Lemma 3.7] (ii)), hence it is finite product of Noetherian normal integral domains [Sta18, 030C]. It is easily checked from the definition of a seminormal ring that normal domains are seminormal.

**Remark 3.9.** For any ring \( R \), by [Wei91, Lemma 1.5.1] and [Wei91, Theorem 5.5] we have an exact sequence

\[
0 \to \text{Pic}(R) \xrightarrow{\partial} \text{Pic}(R[t]) \oplus \text{Pic}(R[t^{-1}]) \xrightarrow{\Sigma} \text{Pic}(R[t^\pm]) \to H^1_{\text{et}}(\text{Spec } R, \mathbb{Z}) \to 0
\]  

(3.9.1)

of abelian groups, where \( \partial \) denotes the map sending \( \alpha \mapsto (\alpha, -\alpha) \) and \( \Sigma \) denotes the addition map. For any ring \( R \), by [Wei91, Theorem 2.4] and [Wei91, Theorem 5.5] we have isomorphisms

\[
H^1_{\text{et}}(\text{Spec } R, \mathbb{Z}) \simeq H^1_{\text{et}}(\text{Spec } R[t], \mathbb{Z}) \simeq H^1_{\text{et}}(\text{Spec } R[t^\pm], \mathbb{Z})
\]

(3.9.2)

of abelian groups.

The following is stated in [Wei91]; we write out the details here.
Lemma 3.10. Let $A$ be a strictly henselian local ring. Then the canonical map
\[
\bigoplus_{(t, \diamond) \in \{1, 2\} \times \{+,-\}} \text{Pic}(A[t^\diamond_t]) \oplus \bigoplus_{(\alpha, \alpha_2) \in \{+,-\}^2} N_{\alpha_1, \alpha_2} \text{Pic}(A) \rightarrow \text{Pic}(A[t^+_1, t^-_2])
\]
induced by the inclusions $A[t^\diamond_t] \rightarrow A[t^+_1, t^+_1, t^-_2]$ and $A[t^\diamond_1, t^-_2] \rightarrow A[t^+_1, t^-_2]$ is an isomorphism.

Proof. For notational convenience, we denote $t^+ = t$ and $t^- = t^{-1}$, etc. Since $A$ is strictly henselian local, by (3.9.2) the exact sequence (3.9.1) reduces to an isomorphism
\[
\text{Pic}(A[t^+]) \oplus \text{Pic}(A[t^-]) \cong \text{Pic}(A[t^\pm])
\]
and split exact sequences
\[
0 \rightarrow \text{Pic}(A[t^+_2]) \rightarrow \text{Pic}(A[t^+_1, t^+_2]) \oplus \text{Pic}(A[t^+_1, t^-_2]) \rightarrow \text{Pic}(A[t^+_1, t^-_2]) \rightarrow 0 \quad (3.10.2)
\]
and
\[
0 \rightarrow \text{Pic}(A[t^+_1]) \rightarrow \text{Pic}(A[t^+_1, t^+_2]) \oplus \text{Pic}(A[t^+_1, t^-_2]) \rightarrow \text{Pic}(A[t^+_1, t^-_2]) \rightarrow 0 \quad (3.10.3)
\]
by taking $R := A, A[t^+_2], A[t^+_1]$ respectively for $\diamond \in \{+, -, \}$. The sequence (3.10.2) induces a natural isomorphism
\[
\text{Pic}(A[t^+_2]) \oplus N_{t^+_1} \text{Pic}(A[t^+_2]) \oplus N_{t^-_2} \text{Pic}(A[t^+_2]) \cong \text{Pic}(A[t^+_1, t^-_2])
\]
and the subgroups of elements annihilated by setting $t^\diamond_2 = 1$. The sequence (3.10.3) induces a natural isomorphism
\[
\text{Pic}(A[t^+_1]) \oplus N_{t^+_1} \text{Pic}(A[t^+_1]) \oplus N_{t^-_2} \text{Pic}(A[t^+_1]) \cong \text{Pic}(A[t^+_1, t^-_2])
\]
of abelian groups. We combine (3.10.4) and (3.10.5) and (3.10.6) (for $\diamond \in \{+, -, \}$) and Lemma 3.4 to obtain the desired result. \qed

Lemma 3.11. We have
\[
\ker(d^1_{\text{Pic}(A)}) = 0
\]
for any strictly henselian local ring $A$.

Proof. The inclusion $N_{t^1, \text{Pic}(A)} \subseteq \text{Pic}(A[t^1_1])$ is an equality since $A$ is a local ring; recall that $d^1_{\text{Pic}(A)}(N_{t^1, \text{Pic}(A)}) \subseteq N_{t^1, t^2_2} \text{Pic}(A)$ by Lemma 3.5. We have a commutative diagram
\[
\begin{array}{ccc}
(Pic(A[t^1_1]))^{op} & \xrightarrow{\cong} & \text{Pic}(A[t^+_1]) \\
(d^1_{\text{Pic}(A)})^{op} \downarrow & & \downarrow d^1_{\text{Pic}(A)} \\
(N_{t^1, t^2_2} \text{Pic}(A))^{op} & \xrightarrow{f_2} & \text{Pic}(A[t^+_1, t^-_2])
\end{array}
\]
where \( f_1 \) and \( f_2 \) are the addition maps induced on the Picard groups by the \( A \)-algebra maps \( A[t_1] \to A[t_1^\pm] \) sending \( t_1 \) to \( t_1, t_1^{-1} \) and \( A[t_1, t_2] \to A[t_1^\pm, t_2^\pm] \) sending \((t_1, t_2) \mapsto (t_1, t_2), (t_1^{-1}, t_2^{-1}) \) respectively. Here \( f_1 \) is an isomorphism by \((3.9.1)\) since \( A \) is strictly henselian local, and \( f_2 \) is injective by \textbf{Lemma 3.10}. Since \( \mathcal{d}^1_{\text{Pic} \mathbb{P}_A} \) is injective by \textbf{Lemma 3.6} we have that \( \mathcal{d}^1_{\text{Pic} \mathcal{L}_A} \) is injective.

\[\square\]

4. Unit groups of Laurent polynomial rings

The purpose of this section is to prove \textbf{Lemma 4.2}. As in \textbf{Section 3}, when it is convenient we will denote \( U(A) := A^\times \) the group of units of a ring \( A \).

**Lemma 4.1.** Let \( \{A_\lambda\}_{\lambda \in A} \) be a filtered inductive system of rings, and let

\[A := \lim_{\lambda \in A} A_\lambda\]

be the colimit ring. In the notation of \textbf{Notation 3.1} and \textbf{Notation 3.2}, the induced morphism of complexes

\[\lim_{\lambda \in A} C^\bullet(U \mathcal{L}_{A_\lambda}) \to C^\bullet(U \mathcal{L}_A)\]

is an isomorphism.

**Proof.** For any \( n \geq 0 \), the functor \( \text{Ring} \to (\text{Ab}) \) sending \( A \mapsto (A[t_1^\pm, \ldots, t_n^\pm]^\times) \) is locally of finite presentation. \( \square \)

**Lemma 4.2.** For any ring \( A \), we have \( h^2(C^\bullet(U \mathcal{L}_A)) = 0. \)

**Proof.** By writing \( A \) as the filtered colimit of subrings which are finite type \( \mathbb{Z} \)-algebras, by \textbf{Lemma 4.1} we may reduce to the case when \( A \) is a finite type \( \mathbb{Z} \)-algebra. By replacing \( \text{Spec} \; A \) by a connected component, we may assume that \( \text{Spec} \; A \) is connected. Let \( n \subset A \) be the nilradical of \( A \). By \textbf{Neh09} Corollary 6, a unit \( \xi \) of \( A[t_1^\pm, t_2^\pm] \) is of the form

\[\xi = ut_1^{e_1}t_2^{e_2} + x(t_1, t_2) \quad (4.2.1)\]

where \( u \in A^\times \) is a unit and \((e_1, e_2) \in \mathbb{Z}^{\geq 2} \) is an ordered pair of integers and \( x(t_1, t_2) \in \mathfrak{n}A[t_1^\pm, t_2^\pm] \) is a Laurent polynomial all of whose coefficients are nilpotent. We have that each unit \( u \in A^\times \subset (A[t_1^\pm, t_2^\pm])^\times \) is in the image of \( \mathcal{d}^1_{U \mathcal{L}_A} \), namely the image of the unit \( u \in A^\times \subset (A[t_1^\pm])^\times \) since \( u \cdot u^{-1} \cdot u = u \). Hence we may assume that the unit \( u \) of \((4.2.1)\) is equal to 1. Suppose \( \xi \in \ker \mathcal{d}^1_{U \mathcal{L}_A / n^s} \). By reduction \( A \to A/n \) we have that

\[\mathcal{d}^1_{U \mathcal{L}_A}(ut_1^{e_1}t_2^{e_2}) = (ut_1^{e_1}t_2^{e_2}) \cdot (ut_1^{e_1}(t_2t_3)^{e_2} - 1) \cdot (u(t_1t_2)^{e_1}t_2^{e_2}) \cdot (ut_2^{e_2}t_2^{e_2})^{-1} = t_1^{e_1}t_3^{e_2} \]

must be equal to 1, hence \( e_1 = e_2 = 0 \). This implies that \( h^2(C^\bullet(U \mathcal{L}_{A/n^s})) = 0 \). We have a sequence

\[A/n^s \to A/n^{s-1} \to \cdots \to A/n^2 \to A/n^1\]

where each map is a surjective ring map with square-zero kernel. Hence, since the complex \( C^\bullet(U \mathcal{L}_A) \) is functorial in \( A \), it suffices to show that, for any ring \( A \) and ideal \( I \subset A \) satisfying \( I^2 = 0 \), if \( h^2(C^\bullet(U \mathcal{L}_{A/I})) = 0 \) then \( h^2(C^\bullet(U \mathcal{L}_A)) = 0 \). The
quotient \( A \to A/I \) induces a morphism \( C^\bullet(UL_A) \to C^\bullet(UL_{A/I}) \) of complexes of abelian groups, part of which is a commutative diagram

\[
\begin{array}{ccc}
(A[t^\pm]) & \xrightarrow{d^1_{UL_A}} & (A[t^\pm, t^\pm_1]) \\
\pi^1 \downarrow & & \downarrow \\
(A/I)[t^\pm] & \xrightarrow{d^2_{UL_A}} & (A/I)[t^\pm, t^\pm_1] \\
\pi^2 \downarrow & & \downarrow \\
(A/I)[t^\pm_2] & \xrightarrow{d^2_{UL_A/I}} & (A/I)[t^\pm_1, t^\pm_2, t^\pm_3] \\
\pi^3 \downarrow & & \\
(A/I)[t^\pm_3] & & 
\end{array}
\] (4.2.2)

where each vertical arrow \( \pi^1, \pi^2, \pi^3 \) is surjective since \( I \) is square-zero. By a diagram chase on (4.2.2) to show that the top row is exact it suffices to show that every element of \( (\ker d^1_{UL_A}) \cap (\ker \pi^2) \) is in the image of \( d^1_{UL_A} \). We have \( \ker \pi^2 = 1 + IA[t^\pm_1, t^\pm_2] \); moreover, since \( I \) is square-zero, the (multiplicative) condition that \( 1 + x(t_1, t_2) \in \ker d^1_{UL_A} \) is equivalent to the (additive) condition that the element

\[
x(t_1, t_2) - x(t_1, t_2, t_3) + x(t_1 t_2, t_3) - x(t_2, t_3)
\]

of \( A[t^\pm_1, t^\pm_2, t^\pm_3] \) is equal to zero. Let

\[
\begin{align*}
H_0 &: = \{ e_3 = 0 \} \\
H_1 &: = \{ e_2 = e_3 \} \\
H_2 &: = \{ e_1 = e_2 \} \\
H_3 &: = \{ e_1 = 0 \}
\end{align*}
\]

be hyperplanes of \( \mathbb{Z}^{\otimes 3} = \{ (e_1, e_2, e_3) \} \) defined by the equations corresponding to the maps \( L_A(p^2_0), L_A(p^2_1), L_A(p^2_2), L_A(p^2_3) \) in the sense that the image of \( \mathbb{Z}^{\otimes 2} \) under \( L_A(p^2_i) \) is \( H_i \subset \mathbb{Z}^{\otimes 3} \). Then the pairwise intersections

\[
\begin{align*}
H_0 \cap H_1 &= \mathbb{Z}(1, 0, 0) & H_1 \cap H_2 &= \mathbb{Z}(1, 1, 1) \\
H_0 \cap H_2 &= \mathbb{Z}(1, 1, 0) & H_1 \cap H_3 &= \mathbb{Z}(0, 1, 1) \\
H_0 \cap H_3 &= \mathbb{Z}(0, 1, 0) & H_2 \cap H_3 &= \mathbb{Z}(0, 0, 1)
\end{align*}
\]

are all distinct. Let

\[
x_{e_1, e_2} \in I
\]

be the coefficient of \( t^\pm_1 t^\pm_2 \) in \( x(t_1, t_2) \). Then if \( (e_1, e_2) \in \mathbb{Z}^{\otimes 2} \) is an ordered pair for which \( x_{e_1, e_2} \neq 0 \), then we must have

\[
(e_1, e_2) \in \mathbb{Z}(1, 0) \cup \mathbb{Z}(1, 1) \cup \mathbb{Z}(0, 1)
\]
in \( \mathbb{Z}^{\otimes 2} \). Moreover, saying that (4.2.3) is equal to zero translates to the collection of equations

\[
\begin{align*}
x_{e,0} - x_{e,0} &= 0 & x_{e,e} - x_{e,e} &= 0 \\
x_{e,e} + x_{e,0} &= 0 & x_{0,e} + x_{e,e} &= 0 \\
x_{0,e} - x_{e,0} &= 0 & x_{0,e} - x_{0,e} &= 0
\end{align*}
\]

for all \( e \in \mathbb{Z} \), which simplifies to

\[
x_{e,0} = x_{0,e} = -x_{e,e}
\]

for all \( e \in \mathbb{Z} \). Then

\[
1 + x(t_1, t_2) = d^1_{UL_A}(1 - \sum_{e \in \mathbb{Z}} x_{e,e} t^e_1)
\]

so we have the desired result. \( \square \)
5. Proof of the main theorem

In this section we prove Theorem 1.2.

Our argument, in outline, is that of the proof of [Gab78, II, Lemma 1']. Namely, we compute $H^2_{\text{et}}(G, G_{m,G})$ using the Leray spectral sequence associated to the map $\pi$ and sheaf $G_{m,G}$, which is of the form

$$E^{p,q}_{2} = H^p_{\text{et}}(S, R^q\pi_*G_{m,G}) \Rightarrow H^{p+q}_{\text{et}}(G, G_{m,G})$$

with differentials $d^{p,q}_{2} : E^{p,q}_{2} \to E^{p+2,q-1}_{2}$. The stalks of $R^q\pi_*G_{m,G}$ are described by Lemma 5.2.

Setup 5.1 (Descent spectral sequence for $BG_{m}$). Let $A$ be a ring and let $\xi : Spec A \to BG_{m,A}$ be the smooth cover associated to the trivial $G_{m,A}$-torsor. The cohomological descent spectral sequence associated to $\xi$ gives a spectral sequence

$$E^{p,q}_{0} = H^q_{\text{et}}(G_{m,A}^\times, G_m) \Rightarrow H^{p+q}_{\text{et}}(BG_{m,A}, G_m)$$

where the $q$th row $E^{p,q}_{1} = H^q_{\text{et}}(G_{m,A}^\times, G_m)$ can be realized as the complex $C^*F(LA)$ where the functor $F : (\text{Ring}) \to (\text{Ab})$ is defined as $F(R) := H^0_{\text{et}}(\text{Spec } R, G_m)$. The lower-left part of the $E_1$-page of the spectral sequence (5.1.1) is

$$H^3_{\text{et}}(G_{m,A}^0, G_m) \to H^3_{\text{et}}(G_{m,A}^1, G_m) \to H^3_{\text{et}}(G_{m,A}^2, G_m) \to H^3_{\text{et}}(G_{m,A}^3, G_m)$$

$$H^2_{\text{et}}(G_{m,A}^0, G_m) \to H^2_{\text{et}}(G_{m,A}^1, G_m) \to H^2_{\text{et}}(G_{m,A}^2, G_m) \to H^2_{\text{et}}(G_{m,A}^3, G_m)$$

$$H^1_{\text{et}}(G_{m,A}^0, G_m) \to H^1_{\text{et}}(G_{m,A}^1, G_m) \to H^1_{\text{et}}(G_{m,A}^2, G_m) \to H^1_{\text{et}}(G_{m,A}^3, G_m)$$

$$H^0_{\text{et}}(G_{m,A}^0, G_m) \to H^0_{\text{et}}(G_{m,A}^1, G_m) \to H^0_{\text{et}}(G_{m,A}^2, G_m) \to H^0_{\text{et}}(G_{m,A}^3, G_m)$$

where $d^{0,q}_{1}$ is the zero map for all $q \geq 0$ since $BG_{m,A}$ is the quotient of $\text{Spec } A$ by the trivial action of $G_{m,A}$.

Lemma 5.2. Assume the setup of Setup 5.1. For any strictly henselian local ring $A$, we have $H^2_{\text{et}}(BG_{m,A}, G_m) = 0$.

Proof. We have $E^{0,q}_{1} = H^q_{\text{et}}(\text{Spec } A, G_m) = 0$ for any $q \geq 1$ since $A$ is strictly henselian. We have $E^{1,1}_{2} = 0$ by Lemma 3.11 and $E^{2,0}_{2} = 0$ by Lemma 4.2.

\[\square\]

Remark 5.3. We show that, in the proof of Lemma 5.2, it is possible to reduce to the case when $A$ is a reduced ring; the reducedness assumption simplifies the proof of Lemma 4.2. By standard limit arguments, we may assume that $A$ is a finite type $\mathbb{Z}$-algebra. Then the reduction $A \to A_{\text{red}}$ can be factored as a finite sequence of square-zero thickenings. Thus we reduce to showing that if $A \to A_0$ is a surjection of rings whose kernel $I$ is square-zero, then the reduction map

$$H^2_{\text{et}}(BG_{m,A}, G_m) \to H^2_{\text{et}}(BG_{m,A_0}, G_m)$$

is an isomorphism. Set $\mathcal{X} := BG_{m,A}$ and $\mathcal{X}_0 := BG_{m,A_0}$ and let $i : \mathcal{X}_0 \to \mathcal{X}$ be the closed immersion. We may use either the big étale site (Sch/\mathcal{X})_ét or the
lisse-étale site \( \text{Lis-Et}(\mathcal{X}) \) to compute cohomology on \( \mathcal{X} \), since the inclusion functor of sites

\[
u : \text{Lis-Et}(\mathcal{X}) \to (\text{Sch}/\mathcal{X})_{\text{ét}}
\]

induces a restriction functor on abelian sheaves

\[
u^{-1} : \text{Ab}((\text{Sch}/\mathcal{X})_{\text{ét}}) \to \text{Ab}(\text{Lis-Et}(\mathcal{X}))
\]

which is exact and admits an exact left adjoint \( u \) (see \[\text{Sta}18\] 0788 (1)). There is an exact sequence

\[
1 \to 1 + I \to \mathbb{G}_m, \mathcal{X} \to i_*\mathbb{G}_m, \mathcal{X}_0 \to 1
\]
of abelian sheaves on \( \text{Lis-Et}(\mathcal{X}) \); here left exactness follows from the fact that for any scheme \( X \) and smooth morphism \( X \to \mathcal{X} \), the composition \( X \to \mathcal{X} \to \text{Spec} \, A \) is flat. We have an induced long exact sequence

\[
\cdots \to H^p_{\text{ét}}(\mathcal{X}, I) \to H^p_{\text{ét}}(\mathcal{X}, \mathbb{G}_m, \mathcal{X}) \to H^p_{\text{ét}}(\mathcal{X}, i_*\mathbb{G}_m, \mathcal{X}_0) \to H^{p+1}_{\text{ét}}(\mathcal{X}, I) \to \cdots
\]
in cohomology. We have \( H^p_{\text{ét}}(\mathcal{X}, i_*\mathbb{G}_m, \mathcal{X}_0) \simeq H^0_{\text{ét}}(\mathcal{X}_0, \mathbb{G}_m, \mathcal{X}_0) \) for \( p \geq 0 \) since pushforward along a closed immersion in the étale topology is exact (using e.g. \[\text{Sta}18\] 04E3). It suffices now to show that if \( \mathcal{X} \) is any quasi-coherent \( \mathcal{O}_\mathcal{X} \)-module then \( H^p_{\text{ét}}(\mathcal{X}, \mathcal{F}) = 0 \) for all \( p > 0 \). The category of quasi-coherent \( \mathcal{O}_\mathcal{X} \)-modules corresponds to the category \( C \) of \( \mathcal{A} \)-modules. Denoting by \( \pi : \mathcal{X} \to \text{Spec} \, A \) the structure map, the pushforward functor \( \pi_* : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(\mathcal{A}) \) corresponds to sending a \( \mathcal{A} \)-graded module \( M_\bullet = \bigoplus_{n \in \mathbb{Z}} M_n \) to the degree zero component \( M_0 \). Since this is an exact functor, we have that \( \pi \) is cohomologically affine \[\text{Alp}13\] Definition 3.1. Since \( \pi \) has affine diagonal, we have the desired result by \[\text{Alp}13\] Remark 3.5.

5.4 (Proof of Theorem 1.2). For any strictly henselian local ring \( A \), we have

\[
H^2_{\text{ét}}(B\mathbb{G}_m, A, \mathbb{G}_m, B\mathbb{G}_m, A) = 0
\]

by \[\text{Lemma } 5.2\] hence

\[
\mathbf{R}^2\pi^*_{\mathcal{G}, \mathcal{X}}\mathbb{G}_m, \mathcal{X} = 0
\]

since its stalks vanish. By \[\text{Lemma } 2.10\] we have

\[
\mathbf{R}^1\pi^*_{\mathcal{G}, \mathcal{X}}\mathbb{G}_m, \mathcal{X} \simeq \text{Hom}_{\text{Ab}(\mathcal{S})}((\mathbb{G}_m, \mathcal{S}), (\mathbb{G}_m, \mathcal{S})) = \mathbb{Z}
\]

so the Leray spectral sequence \[(5.0.1)\] gives an exact sequence

\[
H^0_{\text{ét}}(\mathcal{S}, \mathbb{Z}) \xrightarrow{\iota} H^0_{\text{ét}}(\mathcal{S}, \mathbb{G}_m, \mathcal{S}) \xrightarrow{\pi^*_{\mathcal{G}}} H^2_{\text{ét}}(\mathcal{G}, \mathbb{G}_m, \mathcal{G}) \to H^1_{\text{ét}}(\mathcal{S}, \mathbb{Z})
\]

(5.4.1)

where by \[\text{Proposition } 2.11\] the first map \( \iota \) sends \( 1 \mapsto [\mathcal{G}] \). By \[\text{Lemma } A.3\] the last term \( H^1_{\text{ét}}(\mathcal{S}, \mathbb{Z}) \) is a torsion-free abelian group. If \( \alpha \in H^2_{\text{ét}}(\mathcal{S}, \mathbb{G}_m, \mathcal{S}) \) is a class such that \( \pi^*_G(\alpha) \) is \( n \)-torsion for some \( n \in \mathbb{Z}_{\geq 0} \), then \( \pi^*_G(n\alpha) = 0 \), hence \( n\alpha = m[G] \) for some \( m \in \mathbb{Z}_{\geq 0} \); since by assumption \( [\mathcal{G}] \) is a torsion class, we have that \( \alpha \) is torsion; in other words the restriction \( \pi^*_G : H^2_{\text{ét}}(\mathcal{S}, \mathbb{G}_m, \mathcal{S})_{\text{tors}} \to H^2_{\text{ét}}(\mathcal{G}, \mathbb{G}_m, \mathcal{G})_{\text{tors}} \) is surjective. Hence we have the desired result.

\[\square\]

Remark 5.5. As pointed out to me by Siddharth Mathur, in \[\text{Theorem } 1.2\] the restriction map

\[
\pi^*_G : \text{Br}'(\mathcal{S}) \to \text{Br}'(\mathcal{G})
\]

is not necessarily surjective if \([\mathcal{G}] \in H^2_{\text{ét}}(\mathcal{S}, \mathbb{G}_m, \mathcal{S})\) is a nontorsion class. Let \( S \) be a scheme for which \( H^2_{\text{ét}}(\mathcal{S}, \mathbb{G}_m, \mathcal{S}) \) is not a torsion group; let \( \alpha \in H^2_{\text{ét}}(\mathcal{S}, \mathbb{G}_m, \mathcal{S}) \) be a nontorsion element, and let \( \pi_G : \mathcal{G} \to S \) be the \( \mathbb{G}_m, \mathcal{S} \)-gerbe corresponding to the class \( 2\alpha \in H^2_{\text{ét}}(\mathcal{S}, \mathbb{G}_m, \mathcal{S}) \). Then \( \pi^*_G(\alpha) \) is a 2-torsion class of \( H^2_{\text{ét}}(\mathcal{G}, \mathbb{G}_m, \mathcal{G}) \).
We show that there does not exist any torsion element \( \beta \in H^2_\et(S, G_{m,S}) \) such that \( \pi^*G(\alpha) = \pi^*G(\beta) \). If so, then \( \alpha - \beta = n[G] = 2n\alpha \) for some \( n \), which means \( (2n-1)\alpha \) is torsion, which contradicts our assumption that \( \alpha \) is nontorsion. Taking as our \( S \) above the normal surface of Mumford \([\text{Gro}68, \text{Remarques 1.11, b}]\) for which \( H^2_\et(S, G_{m,S}) \) is not a torsion group, we obtain an example of a \( G_{m,S} \)-gerbe \( \pi_G : \mathcal{G} \to S \) for which the restriction 

\[
\pi^*_G : H^2_\et(S, G_{m,S}) \to H^2_\et(G, G_{m,G})
\]

is surjective (by \([5.4.1]\) using that \( H^1_\et(S, \mathbb{Z}) = 0 \) by \([\text{CRR}72, \text{VIII, Prop. 5.1}]\) since \( S \) is geometrically unibranch) but the restriction to the torsion subgroups is not surjective.

**Appendix A. Torsors under torsion-free abelian groups**

In \([\text{Wei}91, \text{Corollary 7.9.1}]\), it is proved that \( H^1_\et(S, \mathbb{Z}) \) is a torsion-free abelian group if \( S \) is a quasi-compact quasi-separated scheme. In this section, we record a different proof which works over an arbitrary site. This argument is from \([\text{Sta}18, 093J]\).

Let \( S \) be a site. For any set \( S \), let \( \underline{S} \) denote the constant sheaf on \( S \) associated to \( S \).

**Lemma A.1.** Let \( f : S \to T \) be a surjective function between sets. Then the induced map

\[
\Gamma(S, f) : \Gamma(S, \underline{S}) \to \Gamma(S, \underline{T})
\]

is surjective.

**Proof.** Choose a function \( g : T \to S \) satisfying \( fg = \text{id}_T \). By functoriality of the “constant sheaf” functor, we have \( \Gamma(S, f) \circ \Gamma(S, g) = \text{id}_{\Gamma(S, \underline{T})} \).

**Lemma A.2.** Let \( 0 \to A \to B \to C \to 0 \) be an exact sequence of abelian groups. Then the induced map

\[
H^1(S, A) \to H^1(S, B)
\]

is injective.

**Proof.** As part of the long exact sequence in cohomology, we obtain an exact sequence

\[
\Gamma(S, B) \to \Gamma(S, C) \to H^1(S, A) \to H^1(S, B)
\]

where the first arrow is surjective by **Lemma A.1** hence the third arrow is injective.

**Lemma A.3.** Let \( A \) be a torsion-free abelian group. Then \( H^1(S, A) \) is a torsion-free abelian group.

**Proof.** Let \( n \) be a positive integer. Applying **Lemma A.2** to the exact sequence

\[
0 \to A \xrightarrow{\times n} A \to A/\text{na} \to 0
\]

implies that the multiplication-by-\( n \) map on \( H^1(S, A) \) is injective.
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