New Results for the Degree/Diameter Problem

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Abstract

The results of computer searches for large graphs with given (small) degree and diameter are presented. The new graphs are Cayley graphs of semidirect products of cyclic groups and related groups. One fundamental use of our “dense graphs” is in the design of efficient communication network topologies.

1 Introduction

The design of interconnection networks comes up against two fundamental constraints: the number of connections which can be attached to any one node is limited (e.g. by lack of space) and the number of intermediate nodes on the communication route between two nodes must be small (e.g. to meet timing tolerances). While observing such constraints, one wishes to maximize the number of nodes which can participate in such a network. In the language of graph theory, this is the

Degree/Diameter Problem: find graphs with maximal number of vertices with given constraints of maximum degree \( \Delta \) and diameter \( D \).

All our graphs are undirected and we follow the definitions and notations of [11]. For convenience some basic definitions related to this paper are now presented.
The degree of a vertex is the number of incident edges to that vertex. The distance between two vertices is the length of a shortest path between them. The diameter of a graph is the maximum distance over all pairs of vertices. A graph with maximum degree $\Delta$ and diameter $D$ is called a $(\Delta, D)$-graph. The number of vertices in a graph is called the order of the graph.

The only known bound on the order $n$ of a graph with degree $\Delta$ and diameter $D$ is the Moore bound [4]:

$$n \leq 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1}$$

$$= \begin{cases} 
\frac{\Delta(\Delta - 1)^D - 2}{\Delta - 2} & \text{if } \Delta \neq 2, \\
2D + 1 & \text{if } \Delta = 2.
\end{cases}$$

Graphs whose order attains this bound are called Moore graphs.

It was proved in [24, 18, 5] that there do not exist any Moore graphs apart from the following:

| $\Delta$ | $D$ | $n$ | Description |
|---------|-----|-----|-------------|
| 2       | $D$ | $2D + 1$ | $(2D + 1)$-gon |
| 3       | 2   | 10   | Petersen    |
| 7       | 2   | 50   | Hoffman-Singleton |
| 57      | 2   | 3250 | ?           |

It is not known if there exists a graph with $\Delta = 57$, $D = 2$ and $n = 3250$.

In the absence of realistic bounds, one resorts to constructions of largest possible $(\Delta, D)$-graphs. Descriptions of new constructions and corresponding listings of largest known $(\Delta, D)$-graphs have been published periodically by members of the mathematical and engineering communities, e.g. [26, 7, 20, 15, 8, 16, 9, 13, 12, 17, 23], beginning with a paper of Elspas [21]. Apart from the Moore graphs, the only graphs known to be optimal are $(3, 3)$, $(4, 2)$, and $(5, 2)$-graphs with 20, 15, and 24 vertices, respectively. This result goes back to Elspas [21], confirmed later by a general theorem [22, 6] which states that except for the
case of a cycle on four vertices the number of vertices in a \((\Delta, D)\)-graph never misses the Moore bound by one. In the current table of largest known graphs there is still a significant gap between the Moore bound and the orders of known graphs.

Some of the previously known largest \((\Delta, D)\)-graphs were found as Cayley graphs, e.g. for classical groups [12, 16]. Others, e.g. cube-connected cycles, were quickly recognized as Cayley graphs [2]. According to a theorem of Sabidussi [25] every vertex-transitive graph can be realized as Cayley coset graph of some group. Cayley graphs have many desirable properties for engineering applications, e.g. vertex transitivity with associated ease of routing, and fault tolerance [1, 2, 15]

Considerable progress resulted from the first explicit use of semidirect products of cyclic groups [19]. These groups should be seen as capturing the essential features of the Borel groups used by [16] to establish their longstanding records for certain degree 4 graphs which are only now superseded by our results. The present paper continues this work and considers some additional types of groups. One of the major advantages of the semidirect products of cyclic groups is their abundance, together with the fact that they are easy to code on a computer.

The paper is organized as follows. In the next section we give some preliminary comments regarding Cayley graphs. The following section describes the types of groups that were used in finding dense Cayley graphs. We end the paper with a detailed listing of our new \((\Delta, D)\)-graph records. For an overview of our results, see the “DH” entries in Table 1. This listing is maintained by J.-C. Bermond, C. Delorme and J.-J. Quisquater and available on request (e-mail: cd@lri.fr).

2 Cayley graphs

Let \(G\) be a finite group, \(S\) a subset of \(G\) which generates \(G\) and does not contain the identity. The \textit{Cayley graph of \(G\) with respect to \(S\)} is the directed graph whose vertices are the elements of \(G\) and whose edge set is \(E = \{(x, y) : y = xs \text{ for some } s \in S\}\). If \(S\) is closed under inverses,
| \( \Delta \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 3 | \( P \) | \( C_3 \ast F_4 \) | \( \nu C \) | \( \nu C \) | \( GFS \) | \( CR^* \) | \( CR^* \) | \( 2\nu \) | \( 2\nu \) |
| 4 | \( K_3 \ast C_3 \) | Aller | \( C_3 \ast C_{19} \) | \( H_3^\# \) | \( H_3(K_3) \) | \( DH \) | \( DH^* \) | \( DH \) | \( DH \) |
| 5 | \( K_3 \ast X_5 \) | Lent | \( Q_4(K_3) \) | \( H_3^\# \) | \( H_3(K_3) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 6 | \( K_4 \ast X_5 \) | \( C_5 \ast C_{21} \) | \( DH^* \) | \( DH^* \) | \( H_4(K_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 7 | H5 | \( DH^* \) | \( DH^* \) | \( H_4(K_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 8 | \( p_1^d \) | \( DH \) | \( DH^* \) | \( H_4(K_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 9 | \( p_1^d \) | \( q_5 \) | \( DH \) | \( DH^* \) | \( H_4(K_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 10 | \( p_9 \) | \( q_4^d \) | \( DH \) | \( DH^* \) | \( H_4(K_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 11 | \( p_9 \) | \( q_4^d \) | \( q_7(T_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 12 | \( p_9 \) | \( q_4^d \) | \( q_7(T_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 13 | \( p_9 \) | \( q_4^d \) | \( q_7(T_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 14 | \( p_9 \) | \( q_4^d \) | \( q_7(T_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 15 | \( p_9 \) | \( q_4^d \) | \( q_7(T_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |
| 16 | \( p_9 \) | \( q_4^d \) | \( q_7(T_4) \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) | \( DH \) |

Table 1: Table of largest known \((\Delta, D)\)-graphs
i.e. $S = S \cup S^{-1}$, then $(x, y) \in E$ if and only if $(y, x) \in E$. In this case the edges can be considered as undirected. In the present paper we deal only with this situation.

An undirected Cayley graph is $\Delta$-regular (i.e. each vertex has degree $\Delta$) where $\Delta$ is the number of generators in $S = S \cup S^{-1}$. Since Cayley graphs are obviously vertex transitive [25], the computation of the diameter $D$ requires only the determination of the distances from the identity element to all other elements.

It is obvious that for the generation of large $(\Delta, D)$-graphs one has to avoid abelian groups: the relation $ab = ba$ prevents the desired Moore bound even at distance two. It is rather surprising that good results can be achieved with the following nonabelian groups which are easily constructed from cyclic (abelian) groups.

### 3 Description of the groups

#### 3.1 Semidirect product of $Z_m$ with $Z_n$

Let $Z_n$ be the cyclic group of integers under addition modulo $n$. Every automorphism of $Z_n$ can be represented by a unit of the ring $Z_n$. If the multiplicative order of the unit $a \in Z_n$ divides $m$ a semidirect product of $Z_m$ with $Z_n$ can be defined by

$$[x, y][u, v] = [x + u \mod m, ya^u + v \mod n].$$

In Table 1 groups of this kind are identified by DH; our detailed listing uses the symbol $m \times_a n$, e.g. $52 \times_2 53$.

These groups can be described in terms of generators and relations in the following way:

$$m \times_a n = \langle x, y \mid x^m = y^n = x^{-1}yx^{-a} = 1 \rangle. \quad (1)$$
3.2 Semidirect product of $Z_m$ with $Z_n \times Z_n$

A construction which brought several improvements at the low end of the table are semidirect products of a cyclic group $Z_m$ with a direct sum $Z_n \times Z_n$. An automorphism $\sigma$ of $Z_n \times Z_n$ is determined by the images of the generators $\sigma([1,0]) = [x,y]$ and $\sigma([0,1]) = [z,t]$. If the order of $\sigma$ divides $m$ we can define a multiplication on $Z_m \times Z_n \times Z_n$ by:

$$[c,d,e][f,g,h] = [c + f \mod m, \begin{bmatrix} x & y \\ z & t \end{bmatrix}^f + [g,h] \mod n].$$

In Table 1 groups of this kind are identified by $\text{DH}^*$; our detailed listing uses the symbol $m \times_{\sigma} n^2$, e.g. $40 \times_{\sigma} 3^2$; the action of the cyclic group is not encoded into this symbol and is specified separately.

These groups can be described in terms of generators and relations in the following way ($x, y, z, t$ have the same meaning as before):

$$m \times_{\sigma} n^2 = \langle a, b, c \mid a^m = b^n = c^n = bc^{-1}b^{-1} = a^{-1}bace^{-1}b^{-x} = a^{-1}cace^{-t}b^{-z} = 1 \rangle. \quad (2)$$

3.3 Semidirect product of $Z_m \times_a Z_n$ with itself

The next step in these explorations was to double up on our first construction: let $G = m \times_{\sigma} n$ and consider the semidirect product $G \times_{\sigma} G$ where $G$ acts on itself by conjugation. Explicitly, the multiplication of quadruples is given by

$$[x_1, x_2, x_3, x_4][y_1, y_2, y_3, y_4] = \left[\begin{bmatrix} x_1 \end{bmatrix}, \begin{bmatrix} x_2 \end{bmatrix}, \begin{bmatrix} y_1 \end{bmatrix}, \begin{bmatrix} y_2 \end{bmatrix}^{-1} \begin{bmatrix} x_3 \end{bmatrix} \begin{bmatrix} y_3 \end{bmatrix} \begin{bmatrix} y_4 \end{bmatrix} \right] = \left[\begin{bmatrix} x_1 + y_1 \end{bmatrix}, \begin{bmatrix} x_2 a^{y_2} + y_2 \end{bmatrix}, \begin{bmatrix} x_3 + y_3 \end{bmatrix}, \begin{bmatrix} x_4 a^{y_3 + y_4} + y_4 \end{bmatrix} \right].$$

Multiplication of pairs is the multiplication in $m \times_{\sigma} n$. In Table 1 groups of this kind are identified by $\text{DH}^{**}$; our detailed listing uses the symbol $[m \times_{\sigma} n]^2_{\sigma}$, e.g. $[5 \times_{\sigma} 11]^2_{\sigma}$.

These groups can be described in terms of generators and relations in the following way:

$$[m \times_{\sigma} n]^2_{\sigma} = \langle r, s, t, u \mid r^m = s^n = r^{-1}sr^{-s} = t^n = u^n = t^{-1}utu^{-s} = r^{-1}ltr^{-1} = s^{-1}tsu^{-1}t^{-1}u = r^{-1}uru^{-s} = s^{-1}usu^{-1} = 1 \rangle. \quad (3)$$
4 Final Remarks

Approximately two to four days of computer time was consumed for each \((\Delta, D)\) entry of Table 1. Primarily Sun-4s were used in our search for dense graphs with one million nodes or less while Los Alamos National Laboratory’s Cray Y-MPs tackled most of the diameter computations for the larger Cayley graphs. For a fixed target group, several thousand sets of random generators were tried – with fewer combinations checked as the order of the groups grew.

Not all of our attempts were crowned with success. We note the following two types of groups which did not stand a chance in the competition.

The full automorphism group of the cyclic group \(Z_n\) is isomorphic to the group \(U_n\) of units modulo \(n\). Like every finite abelian group, \(U_n\) can be decomposed:

\[
U_n = C_1 \oplus C_2 \oplus \cdots \oplus C_k,
\]

where each \(C_i\) is cyclic of order \(d_i\) and \(d_1 | d_2 | \cdots | d_k\). The groups \(m \times_n n\) were concerned with \(C_k\) or a subgroup thereof. Letting \(C_{k-1} \oplus C_k\) act on \(Z_n\) did not lead to any large graphs.

Another natural step after the groups \(m \times_n n^2\) would be a cyclic group \(Z_m\) operating on \(Z_n \times Z_n \times Z_n\). Again, these constructions did not produce good graphs.

5 Acknowledgments

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## Groups and Generators

For an explanation of the entries in the column headed ‘Group’ refer to equations (1), (2), (3) in section 3.

| $(\Delta, D)$ | Order | Group | Generators | Inverses | Order of Generator |
|---------------|-------|-------|------------|----------|-------------------|
| $(4, 7)$      | 1155  | $15 \times_4 77$ | [6 2] [10 9] | 9 5 | 35 |
| $(4, 8)$      | 3025  | $[5 \times_4 11]_5^2$ | [0 5 1 1] [4 5 1 7] | 0 6 4 7 | 55 |
| $(4, 9)$      | 7550  | $25 \times_{171} 302$ | [8 156] [10 31] | 17 82 | 25 |
| $(4, 10)$     | 16555 | $35 \times_{256} 473$ | [8 342] [3 60] | 27 343 | 35 |
| $(4, 11)$     | 42861 | $39 \times_{16} 1099$ | [19 863] [28 466] | 20 783 | 39 |
| $(4, 12)$     | 95634 | $66 \times_2 1449$ | [16 1289] [13 1253] | 50 1270 | 99 |
| $(4, 13)$     | 140868| $84 \times_2 1677$ | [37 462] [44 814] | 47 1212 | 84 |
| $(5, 7)$      | 5334  | $42 \times_2 127$  | [8 50] [27 15] [21 0] | 34 10 15 99 | 21 |
| $(5, 8)$      | 15532 | $44 \times_{207} 353$ | [25 50] [43 41] [22 0] | 19 200 1 338 | 44 |
| $(5, 9)$      | 49932 | $36 \times_{25} 1387$ | [7 440] [28 105] [18 0] | 29 462 8 1113 | 36 |
| $(5, 10)$     | 145584| $48 \times_{2300} 3033$ | [6 2477] [17 2951] [18 0] | 42 1303 31 2786 | 72 |
| $(6, 4)$      | 360   | $40 \times_\sigma 3^2$ | [39 1 2] [14 2 0] [32 0 2] | 1 0 2 26 2 1 8 0 1 | 40 |
| $(6, 5)$      | 1230  | $15 \times_{37} 82$ | [7 60] [4 30] [3 37] | 8 68 11 38 12 23 | 15 |
| $(6, 7)$      | 18775 | $25 \times_{481} 751$ | [9 556] [2 570] [3 22] | 16 291 23 386 22 605 | 25 |
| $(\Delta, D)$ | Order  | Group       | Generators | Inverses | Order of Generator |
|------------|--------|-------------|------------|----------|-------------------|
| (6, 8)     | 69540  | $60 \times_8 1159$ | [42 500]  | [18 545] | 190               |
|            |        |             | [23 1038] | [37 94]  | 60                |
|            |        |             | [57 403]  | [3 1125] | 20                |
| (6, 9)     | 275540 | $92 \times_{2202} 2995$ | [28 2233] | [64 2932] | 115               |
|            |        |             | [51 2790] | [41 325] | 92                |
|            |        |             | [42 2531] | [50 556] | 46                |
| (6, 10)    | 945574 | $238 \times_{81} 3973$ | [211 2137] | [27 216] | 238               |
|            |        |             | [30 32]   | [208 2390] | 119               |
|            |        |             | [4 460]   | [234 412] | 119               |
| (7, 3)     | 144    | $16 \times_\sigma 3^2$ | [13 0 1]  | [3 1 2]  | 16                |
|            |        |             | [5 2 2]   | [11 2 0] | 16                |
|            |        |             | [10 0 2]  | [6 2 2]  | 8                 |
|            |        |             | [8 0 0]   |          | 2                 |
| (7, 4)     | 600    | $24 \times_\sigma 5^2$ | [22 0 3]  | [2 3 1]  | 12                |
|            |        |             | [15 1 3]  | [9 0 1]  | 8                 |
|            |        |             | [18 4 2]  | [6 2 1]  | 4                 |
|            |        |             | [12 0 0]  |          | 2                 |
| (7, 5)     | 2756   | $52 \times_2 53$ | [25 45]   | [27 37]  | 52                |
|            |        |             | [30 23]   | [22 18]  | 26                |
|            |        |             | [40 39]   | [12 51]  | 13                |
|            |        |             | [26 0]    |          | 2                 |
| (7, 7)     | 47304  | $72 \times_5 657$ | [67 155]  | [5 491]  | 72                |
|            |        |             | [59 160]  | [13 100] | 72                |
|            |        |             | [66 305]  | [6 253]  | 36                |
|            |        |             | [36 0]    |          | 2                 |
| (7, 8)     | 214500 | $60 \times_2 3575$ | [50 1706] | [10 1231] | 66                |
|            |        |             | [29 3164] | [31 2428] | 60                |
|            |        |             | [56 3360] | [4 3440]  | 15                |
|            |        |             | [30 0]    |          | 2                 |
| (7, 9)     | 945574 | $238 \times_{81} 3973$ | [71 3406] | [167 1196] | 238               |
|            |        |             | [109 2984] | [129 2397] | 238               |
|            |        |             | [184 915] | [54 441]  | 119               |
|            |        |             | [119 0]   |          | 2                 |
| (8, 3)     | 234    | $18 \times_3 13$ | [7 5]     | [11 7]   | 18                |
|            |        |             | [5 1]     | [13 10]  | 18                |
|            |        |             | [16 12]   | [2 9]    | 9                 |
|            |        |             | [14 2]    | [4 7]    | 9                 |
| (8, 4)     | 1012   | $22 \times_{25} 46$ | [1 7]     | [21 31]  | 22                |
|            |        |             | [14 33]   | [8 39]   | 22                |
|            |        |             | [18 19]   | [4 41]   | 22                |
|            |        |             | [4 44]    | [18 26]  | 11                |
| $(\Delta, D)$ | Order | Group | Generators | Inverses | Order of Generator |
|-------------|-------|-------|------------|----------|-------------------|
| $(8, 5)$    | 4704  | $96 \times_8 7^2$ | $\begin{bmatrix} 5 & 3 & 2 \\ 1 & 5 & 3 \\ 39 & 0 & 4 \\ 36 & 6 & 1 \end{bmatrix}$ | $\begin{bmatrix} 91 & 1 & 5 \\ 95 & 6 & 2 \\ 57 & 3 & 3 \\ 60 & 0 & 4 \end{bmatrix}$ | 96 |
| $(8, 7)$    | 127134| $126 \times_{993} 1009$ | $\begin{bmatrix} 73 & 719 \\ 60 & 639 \\ 48 & 998 \\ 14 & 447 \end{bmatrix}$ | $\begin{bmatrix} 53 & 953 \\ 66 & 1007 \\ 78 & 745 \\ 112 & 711 \end{bmatrix}$ | 126 |
| $(8, 8)$    | 654696| $216 \times_{625} 3031$ | $\begin{bmatrix} 160 & 1966 \\ 14 & 2452 \\ 127 & 1541 \\ 153 & 1702 \end{bmatrix}$ | $\begin{bmatrix} 56 & 816 \\ 202 & 885 \\ 89 & 1305 \\ 63 & 678 \end{bmatrix}$ | 27 |
| $(8, 9)$    | 2408704| $[16 \times_8 97]_\sigma^2$ | $\begin{bmatrix} 10 & 59 & 6 & 89 \\ 3 & 7 & 14 & 92 \\ 1 & 41 & 11 & 79 \\ 5 & 80 & 6 & 13 \end{bmatrix}$ | $\begin{bmatrix} 6 & 57 & 10 & 27 \\ 13 & 68 & 2 & 21 \\ 15 & 7 & 5 & 44 \\ 11 & 26 & 10 & 81 \end{bmatrix}$ | 776 |
| $(9, 4)$    | 1430  | $10 \times_{64} 143$ | $\begin{bmatrix} 0 & 84 \\ 7 & 54 \\ 1 & 51 \\ 7 & 121 \\ 5 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 59 \\ 3 & 80 \\ 9 & 64 \\ 3 & 121 \\ 2 \end{bmatrix}$ | 143 |
| $(9, 5)$    | 7344  | $48 \times_5 153$ | $\begin{bmatrix} 0 & 71 \\ 16 & 86 \\ 47 & 97 \\ 37 & 130 \\ 24 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 82 \\ 32 & 118 \\ 1127 \\ 1100 \\ 2 \end{bmatrix}$ | 153 |
| $(9, 7)$    | 264024| $72 \times_{1923} 3667$ | $\begin{bmatrix} 27 & 3187 \\ 1 & 1495 \\ 7 & 1659 \\ 6 & 1431 \\ 36 & 0 \end{bmatrix}$ | $\begin{bmatrix} 45 & 2969 \\ 71 & 1151 \\ 65 & 2792 \\ 66 & 661 \\ 2 \end{bmatrix}$ | 152 |
| $(9, 8)$    | 1354896| $[12 \times_6 97]_\sigma^2$ | $\begin{bmatrix} 11 & 34 & 11 & 21 \\ 5 & 75 & 0 & 39 \\ 8 & 76 & 3 & 60 \\ 6 & 22 & 10 & 48 \\ 6 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 87 & 1 & 70 \\ 7 & 62 & 0 & 40 \\ 4 & 56 & 9 & 55 \\ 6 & 22 & 2 & 85 \\ 2 \end{bmatrix}$ | 12 |
| $(9, 9)$    | 4980696| $1288 \times_{11} 3867$ | $\begin{bmatrix} 442 & 2170 \\ 925 & 2708 \\ 1276 & 3002 \\ 408 & 2678 \\ 644 & 0 \end{bmatrix}$ | $\begin{bmatrix} 846 & 2609 \\ 363 & 3857 \\ 12 & 2002 \\ 880 & 619 \\ 2 \end{bmatrix}$ | 1932 |
| \((\Delta, D)\) | Order | Group | Generators | Inverses | Order of Generator |
|-----------------|-------|-------|------------|----------|-------------------|
| \((10, 4)\)    | 2200  | \(20 \times_3 110\) | \[
\begin{bmatrix}
17 & 1 \\
3 & 0 \\
2 & 80 \\
4 & 33 \\
8 & 6
\end{bmatrix}
\] | \[
\begin{bmatrix}
3 & 83 \\
17 & 0 \\
18 & 40 \\
16 & 77 \\
12 & 34
\end{bmatrix}
\] | 20 |
| \((10, 5)\)    | 12288 | \(48 \times_{\sigma} 16^2\) | \[
\begin{bmatrix}
25 & 7 & 15 \\
29 & 1 & 2 \\
43 & 0 & 9 \\
46 & 6 & 10 \\
44 & 10 & 15
\end{bmatrix}
\] | \[
\begin{bmatrix}
23 & 15 & 6 \\
19 & 13 & 5 \\
5 & 5 & 5 \\
2 & 14 & 12 \\
4 & 11 & 8
\end{bmatrix}
\] | 48 |
| \((10, 7)\)    | 554580| \(156 \times_2 3555\) | \[
\begin{bmatrix}
32 & 2159 \\
66 & 2090 \\
4 & 1182 \\
41 & 2287 \\
3 & 3039
\end{bmatrix}
\] | \[
\begin{bmatrix}
124 & 1096 \\
90 & 3355 \\
152 & 2148 \\
115 & 3319 \\
153 & 1842
\end{bmatrix}
\] | 585 |
| \((10, 8)\)    | 3069504| \(24 \times_{5_2 73} 73^2\) | \[
\begin{bmatrix}
19 & 36 & 22 & 45 \\
3 & 47 & 7 & 41 \\
19 & 61 & 4 & 69 \\
14 & 37 & 22 & 39 \\
18 & 3 & 10 & 26
\end{bmatrix}
\] | \[
\begin{bmatrix}
5 & 15 & 2 & 8 \\
21 & 61 & 17 & 43 \\
5 & 68 & 20 & 17 \\
10 & 61 & 2 & 63 \\
6 & 65 & 14 & 30
\end{bmatrix}
\] | 24 |
| \((10, 9)\)    | 9003000| \(3000 \times_1 3001\) | \[
\begin{bmatrix}
77 & 967 \\
1864 & 494 \\
1624 & 838 \\
2380 & 572 \\
576 & 73
\end{bmatrix}
\] | \[
\begin{bmatrix}
2923 & 395 \\
1136 & 124 \\
1376 & 104 \\
620 & 2799 \\
2424 & 1225
\end{bmatrix}
\] | 3000 |
| \((11, 5)\)    | 17458 | \(14 \times_{\tau_{29}} 1247\) | \[
\begin{bmatrix}
1 & 459 \\
1 & 134 \\
3 & 433 \\
10 & 443 \\
10 & 325 \\
7 & 0
\end{bmatrix}
\] | \[
\begin{bmatrix}
13 & 1154 \\
13 & 1228 \\
11 & 1199 \\
4 & 910 \\
4 & 175 \\
2
\end{bmatrix}
\] | 14 |
| \((11, 7)\)    | 945574| \(238 \times_{81} 3973\) | \[
\begin{bmatrix}
111 & 2465 \\
131 & 211 \\
59 & 3508 \\
32 & 3841 \\
188 & 2240 \\
119 & 0
\end{bmatrix}
\] | \[
\begin{bmatrix}
127 & 2668 \\
107 & 3540 \\
179 & 513 \\
206 & 1445 \\
50 & 3522 \\
2
\end{bmatrix}
\] | 238 |
\[\text{(\(\Delta, D\))} \quad \text{Order} \quad \text{Group} \quad \text{Generators} \quad \text{Inverses} \quad \text{Order of Generator}\
\begin{array}{|c|c|c|c|c|}
\hline
(12, 5) & 26871 & 39 \times_{16} 689 & \begin{bmatrix} 13 & 383 \\ 5 & 667 \\ 28 & 303 \\ 25 & 41 \\ 36 & 361 \\ 27 & 400 \end{bmatrix} & \begin{bmatrix} 26 & 518 \\ 34 & 235 \\ 11 & 16 \\ 14 & 86 \\ 3 & 627 \\ 12 & 81 \end{bmatrix} & 159 \\
\hline
(12, 8) & 1007820 & 2580 \times_{5} 3879 & \begin{bmatrix} 428 & 151 \\ 678 & 3117 \\ 276 & 717 \\ 1375 & 2266 \\ 735 & 3686 \\ 1665 & 1590 \end{bmatrix} & \begin{bmatrix} 2152 & 2690 \\ 1902 & 3102 \\ 2304 & 3621 \\ 1205 & 2426 \\ 1845 & 3830 \\ 915 & 2058 \end{bmatrix} & 1935 \\
\hline
(12, 9) & 48532122 & 6966 \times_{5} 6967 & \begin{bmatrix} 4643 & 2316 \\ 5284 & 3171 \\ 6987 & 6154 \\ 1812 & 5557 \\ 3987 & 6878 \\ 6880 & 2878 \end{bmatrix} & \begin{bmatrix} 2323 & 4128 \\ 1682 & 2937 \\ 6897 & 6154 \\ 5154 & 3426 \\ 2979 & 1122 \\ 86 & 54 \end{bmatrix} & 6966 \\
\hline
(13, 5) & 37056 & 64 \times_{125} 579 & \begin{bmatrix} 23 & 254 \\ 27 & 422 \\ 9 & 486 \\ 43 & 278 \\ 8 & 124 \\ 12 & 156 \\ 32 & 0 \end{bmatrix} & \begin{bmatrix} 41 & 161 \\ 37 & 44 \\ 55 & 456 \\ 21 & 116 \\ 56 & 428 \\ 52 & 159 \\ 86 & 54 \end{bmatrix} & 64 \\
\hline
(13, 8) & 15027252 & 3876 \times_{2} 3877 & \begin{bmatrix} 2785 & 2526 \\ 3739 & 2740 \\ 790 & 430 \\ 2560 & 3681 \\ 1716 & 2226 \\ 1520 & 223 \\ 1938 & 0 \end{bmatrix} & \begin{bmatrix} 1091 & 3330 \\ 137 & 1997 \\ 3086 & 1901 \\ 1316 & 3504 \\ 2160 & 3789 \\ 2356 & 1970 \\ 1938 & 0 \end{bmatrix} & 3876 \\
\hline
(13, 9) & 72598920 & 8520 \times_{13} 8521 & \begin{bmatrix} 110 & 233 \\ 294 & 989 \\ 1492 & 3266 \\ 4967 & 2870 \\ 51 & 1308 \\ 6648 & 4785 \\ 4260 & 0 \end{bmatrix} & \begin{bmatrix} 8410 & 7302 \\ 8226 & 4335 \\ 7028 & 5171 \\ 3553 & 5684 \\ 8469 & 361 \\ 1872 & 8005 \\ 4260 & 0 \end{bmatrix} & 852 \\
\hline
\end{array}
| $(\Delta, D)$ | Order | Group   | Generators | Inverses | Order of Generator |
|-------------|-------|---------|------------|----------|-------------------|
| $(14, 5)$   | 53955 | $45 \times_{234} 1199$ | 36 972 [2 533 [32 288 [4 529 [1 508 [34 850 [16 1187 | 9 336 [43 1119 [13 617 [41 481 [44 377 [11 618 [29 565 | 545 45 45 45 45 45 45 |
| $(15, 5)$   | 69972 | $84 \times_{81} 833$   | 80 62 [40 365 [3 565 [45 636 [15 453 [50 644 [35 219 [42 0 | 4 312 [44 502 [81 492 [39 652 [69 520 [34 168 [49 468 | 357 357 196 196 196 42 12 |
| $(15, 7)$   | 7100796 | $1884 \times_{45} 3769$ | 943 2746 [1402 2317 [34 1747 [1131 40 [1040 3137 [1568 145 [342 393 [942 0 | 941 1979 [482 2419 [1850 2321 [753 3119 [844 2872 [316 2312 [1542 461 | 1884 942 942 628 471 471 314 |
| $(15, 8)$   | 38471006 | $6202 \times_{2} 6203$ | 4159 3195 [1486 1096 [3916 3469 [4333 2871 [5215 3672 [2968 321 [5726 2428 [3101 0 | 2043 3638 [4716 3827 [2286 3796 [1869 276 [987 4456 [3234 5123 [476 4180 | 6202 3101 3101 886 886 443 443 |

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