FRACTOCONVEX STRUCTURES

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ABSTRACT. We define a new structure on a space endowed with convexities, and call it a fractoconvex structure (or, a space with fractoconvexity). We introduce two operations on a set of fractoconvexities and in a special case we show that they satisfy the laws for a distributive lattice. We establish a connection between fractoconvex sets and convex sets using the concept of independent convexities, based on the possibility of representing a fractoconvex set as the intersection of its convex hulls. Finally, we consider some examples of fractoconvexities on the 2-sphere and on \( \mathbb{Z} \).

1. INTRODUCTION

The concept of a convexity plays an important role in many topics of mathematics. In each of them the properties associated with convexity appear on an appropriate abstract level. The theory that deals with convexity and its applications from a general point of view was formed in the 1960–1980s and was called the theory of convex structures. A fairly complete exposition of the theory of convex structures can be found in the monographs [1, 2, 3]. Following [1], we recall the most general viewpoint of the notion of convexity.

Let \( X \) be a set. A convexity \( G \subset 2^X \) on \( X \) is defined, as a rule, in two equivalent ways. In the first way, a collection \( G \subset 2^X \) is called a convexity on \( X \) if \( X \in G \) and \( G \) is closed under arbitrary intersections of its elements. In the second way, a convexity is generated by a convex hull operator \( g: 2^X \to 2^X \) such that, for any sets \( A, B \) satisfying the inclusions \( A \subset B \subset X \), the following conditions hold: \( A \subset gA, gA \subset gB, \) and \( g(gA) = gA \); this convexity is in turn equal to \( G = \{ A \subset X : gA = A \} \). A convexity \( G \) is called finitely defined if for every \( A \subset X \) it follows that \( gA = \bigcup \{ gB : B \subset A, |B| < \infty \} \). A convexity \( G \) is called \( n \)-ary if \( A \in G \iff \forall B \subset A (|B| \leq n \Rightarrow \exists i \in \{1, 2\} g_i B \subset A) \) whenever \( A \subset X \).

Now let the set \( X \) be endowed with a family of convexities on it. Clearly, in \( X \) we can construct new structures based on the convexities, and these structures are not necessarily convexities on \( X \), but may have similarities with them. For example, in [4] the author proposed and investigated the notions of an \( n \)-semiconvex set and an \( n \)-biconvex set. We briefly recall these notions.

Let \( G_1 \) and \( G_2 \) be convexities on a set \( X \), \( g_1 \) and \( g_2 \) be the convex hulls associated with them, respectively, and let \( n \) be a natural number. A set \( A \subset X \) is called \( n \)-semiconvex with respect to \( G_1 \) and \( G_2 \) if

\[
\forall B \subset A (|B| \leq n \Rightarrow \exists i \in \{1, 2\} g_i B \subset A). \tag{1}
\]

A set \( A \subset X \) is called \( n \)-biconvex with respect to \( G_1 \) and \( G_2 \) if

\[
\forall B \subset A (|B| \leq n \Rightarrow \forall i \in \{1, 2\} g_i B \subset A). \tag{2}
\]

From (1) and (2) (see the quantifiers in the round brackets) it follows that, for fixed \( n, G_1, \) and \( G_2 \), the family of all \( n \)-semiconvex sets is not stable for intersections, but the family of all \( n \)-biconvex sets is in turn stable for intersections. Therefore, the family of all \( n \)-biconvex sets is a convexity on \( X \). At the same time, it has been shown in [4] that

- Some special \( n \)-semiconvex sets can be represented as the intersection of their convex hulls.
- In some cases the theorems which are similar to the hyperplane separation theorems in \( \mathbb{R}^n \) can be applied to \( n \)-semi- and \( n \)-biconvex sets.

In this paper the notions of \( n \)-semi- and \( n \)-biconvex set are generalized for an arbitrary, not necessarily finite, family of convexities. We define the notions of "fractoconvexity" (fractional convexity) and "multiconvexity" and investigate their properties. We shall briefly consider the question of how fractoconvex sets can be represented by convex sets belonging to given convexities. Finally, we provide four examples to illustrate the notions and statements proposed in our paper.

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2. Fractoconvexities

Let $\Lambda \neq \emptyset$ be an index set, $G_\lambda, \lambda \in \Lambda$ be convexities on a set $X, \{M_i, |M_i| \geq 1\}$ be a partition of $\Lambda$ such that

$$\forall i (\lambda_{1,2} \in M_i, \lambda_1 \neq \lambda_2 \Rightarrow G_{\lambda_1} \neq G_{\lambda_2}).$$

and, for every $i$, let $m_i$ be a cardinal number such that $1 \leq m_i \leq |M_i|$. (Here and in what follows, by $|\cdot|$ we denote the cardinality of a set).

We say that a set $A$ is an $n$-ary fractoconvex set of the type $\{(M_i, m_i)\}$ in $X$ with respect to the convexities $G_\lambda$ (briefly, $(n)-\bigwedge_i m_i G_{\lambda_i}$ -fractoconvex set in $X$), if

$$\forall B \subset A \left(|B| \leq n \Rightarrow \exists \Omega \subset M_i, |\Omega| = m_i : \bigcup_{\lambda \in \Omega} g_\lambda B \subset A \right).$$

Similarly, we say that a set $A$ is an $n$-ary multiconvex set in $X$ with respect to the convexities $G_\lambda$ (briefly, $(n)-\{G_\lambda, \lambda \in \Lambda\}$-multiconvex set in $X$), if

$$\forall B \subset A \left(|B| \leq n \Rightarrow \bigcup_{\lambda \in \Lambda} g_\lambda B \subset A \right).$$

The collection of all $(n)-\bigwedge_i m_i G_{\lambda_i}$-fractoconvex (resp., $(n)-\{G_\lambda, \lambda \in \Lambda\}$-multiconvex) sets in $X$ will be called a fractoconvexity (resp., multiconvexity) and will be denoted by $(n)-\bigwedge_i m_i G_{\lambda_i}$ (resp., $(n)-\{G_\lambda, \lambda \in \Lambda\}$). The pair $(X, (n)-\bigwedge_i m_i G_{\lambda_i})$ will be called a fractoconvex structure (or, a space with fractoconvexity).

From these definitions we obtain that a multiconvexity is a particular case of a fractoconvexity, i.e. any $(n)-\{G_\lambda, \lambda \in \Lambda\}$-multiconvex set is an $(n)-\bigwedge_i m_i G_{\lambda_i}$-fractoconvex set. Also, a set that is $n$-semiconvex with respect to $G_1$ and $G_2$ is an $(n)-\frac{1}{\{G_1, G_2\}}$-fractoconvex set, a set that is $n$-biconvex with respect to $G_1$ and $G_2$ is an $(n)-\frac{2}{\{G_1, G_2\}}$-fractoconvex set and is an $(n)-\{G_1, G_2\}$-multiconvex set, a set that is convex with respect to an $n$-ary convexity $G_1$ is an $(n)-\frac{1}{G_1}$-fractoconvex set and is an $(n)-\{G_1\}$-multiconvex set.

Let $\Psi(n)(X)$ be the collection of all $n$-ary convexities on $X$. By $\Psi(n)(X)$ (resp., $\Psi_{\text{fin}}(n)(X)$) denote the collection of all $(n)$-fractoconvexities on $X$ with respect to all (resp., all finite) subsets of $\Psi(n)(X)$. It is obvious that $\Psi_{\text{fin}}(n)(X) \subset \Psi(n)(X)$ and $\Psi(n-1)(X) \subset \Psi(n)(X)$. If not otherwise stated, we assume that all convexities $G_\lambda, \lambda \in \Lambda$ considered below are $n$-ary; additionally, the prefix $(n)$- for the fracto- and multiconvexities will be omitted.

Take arbitrary fractoconvexities $F_j = \bigvee_i m_i G_{\lambda_i} \in \Psi(n)(X), j \in \{1, 2\}$. According to the definition of a fractoconvexity, we can also define a new fractoconvexity $F_1 \vee F_2 \in \Psi(n)(X)$ and a corresponding operation $\vee : \Psi(n)(X) \times \Psi(n)(X) \to \Psi(n)(X)$ as follows:

$$A \in F_1 \vee F_2 \iff \forall B \subset A \left(|B| \leq n \Rightarrow \exists j \in \{1, 2\} \exists \Omega \subset M_i, |\Omega| = m_i : \bigcup_{\lambda \in \Omega} g_\lambda B \subset A \right).$$

It is readily seen that the operation $\vee$ on the set $\Psi_{\text{fin}}(n)(X)$ possesses the following properties (hereinafter, $k \geq 1$):

(i) $\vee$ is commutative and associative;

(ii) if $k < l$, then $\{G_1, \ldots, G_k\} \vee \{G_1, \ldots, G_l\} = \{G_1, \ldots, G_k\}$;

(iii) $\{G_1, \ldots, G_m\} = \bigvee_{1 \leq i_1 < \ldots < i_k \leq m} \{G_{i_1}, \ldots, G_{i_k}\}$.

Property (iii), in particularly, implies the equality

$$1 \{G_1, \ldots, G_k\} = \frac{1}{k} \bigvee_{i=1}^{k} \frac{1}{G_i}.$$
Since the convexities $G_i = 1$, these laws follow simply from (i)–(iii), Proposition 1, and the distributivity of the operations $\cap$ and $\lor$. 

**Proof.** Without loss of generality, we put $k = 2$. We shall show that the convexity $G_1 \cap G_2$ is $n$-ary. Indeed, let $\mathfrak{g}$ be the convex hull associated with $G_1 \cap G_2$. From the properties of the convex hull operator it follows that 

$$A \in G_1 \cap G_2 \Rightarrow \forall B \subset A (|B| \leq n \Rightarrow \mathfrak{g}B \subset A).$$

On the other hand, if $\forall B \subset A (|B| \leq n \Rightarrow \mathfrak{g}B \subset A)$, then, in view of the inclusions $g_1 B, g_2 B \subset \mathfrak{g}B$, we have 

$$\forall B \subset A (|B| \leq n \Rightarrow g_1 B \subset A, g_2 B \subset A).$$

Since the convexities $G_1$ and $G_2$ is $n$-ary, we obtain $A \in G_1 \cap G_2$, whence $G_1 \cap G_2$ is $n$-ary.

Now, in our notation, we can write $G_1 \cap G_2 = \frac{1}{2} \left( \frac{1}{G_1} \cap \frac{1}{G_2} \right)$. Since the convexities $G_1$ and $G_2$ are $n$-ary, the convexity $G_1 \cap G_2$ coincides with the set $\left( \frac{1}{G_1} \cap \frac{1}{G_2} \right)$. Hence, the second required equality has been proved.

In the same way, we obtain $A \in \frac{1}{2} \{ \frac{1}{G_1}, \frac{1}{G_2} \} \Leftrightarrow A \in G_1 \cap G_2$, whence the first equality is true. 

Thus, if the cardinality in the numerator equals the cardinality of the denominator and $G_i$ are $n$-ary for all $i = 1, \ldots, k$, $k < \infty$, then the fractoconvexity $\frac{1}{k} \{ G_1, \ldots, G_k \}$ is the $n$-ary convexity $G_1 \cap \ldots \cap G_k$. In other words, as in the case of biconvex sets, the collection of all $n$-ary multiconvex sets is an $n$-ary convexity.

**Corollary 1.** For any fractoconvexity $F = \bigvee_i \frac{m_i}{G_{\lambda_i} \in M_i} \in \mathfrak{g}^{(n)}_{\text{fin}}(X)$ we have 

$$F = \bigvee_{\Omega, \lambda \in \Omega} \frac{1}{G_{\lambda}} \emptyset \left( \left\{ \frac{1}{G_{\lambda}} \right\} \cap \left\{ \frac{1}{G_{\lambda}} \right\} \right).$$

**Proof.** Evidently, this equality follows from (iii) and Proposition 1.

**Proposition 2.** The operations $\lor$ and $\cap$ satisfy the distributive laws on $\mathfrak{g}^{(n)}_{\text{fin}}(X)$: $\forall F_1, F_2, F_3 \in \mathfrak{g}^{(n)}_{\text{fin}}(X)$

$$(F_1 \lor F_2) \cap F_3 = (F_1 \cap F_3) \lor (F_2 \cap F_3),$$

$$(F_1 \lor F_2) \cap F_3 = (F_1 \lor F_3) \cap (F_2 \lor F_3).$$

**Proof.** These laws follow simply from (i)–(iii), Proposition 1 and the distributivity of the operations $\lor$ and $\cap$ for the fractoconvexities $\frac{1}{G_{\lambda}} \in \mathfrak{g}^{(n)}_{\text{fin}}(X)$. The last property follows from the definition of $\lor$. For example, we show the distributivity of $\cap$ over $\lor$.

$$A \in \left( \left\{ \frac{1}{G_{\lambda}, \lambda \in \Lambda_1} \right\} \lor \left\{ \frac{1}{G_{\lambda}, \lambda \in \Lambda_2} \right\} \right) \cap \left\{ \frac{1}{G_{\lambda}, \lambda \in \Lambda_3} \right\} \Leftrightarrow$$

$$\Leftrightarrow \forall B \subset A \bigg( |B| \leq n \Rightarrow \left( \exists \lambda_1 \in \Lambda_1 : g_{\lambda_1} B \subset A \right) \lor \left( \exists \lambda_2 \in \Lambda_2 : g_{\lambda_2} B \subset A \right) \bigg) \Leftrightarrow$$

$$\Leftrightarrow \forall B \subset A \bigg( |B| \leq n \Rightarrow \left( \exists \lambda_1 \in \Lambda_1 \exists \lambda_2 \in \Lambda_2 : g_{\lambda_1} B \lor g_{\lambda_2} B \subset A \right) \bigg) \Leftrightarrow$$

$$\Leftrightarrow A \in \left( \left\{ \frac{1}{G_{\lambda}, \lambda \in \Lambda_1} \right\} \lor \left\{ \frac{1}{G_{\lambda}, \lambda \in \Lambda_2} \right\} \right) \lor \left( \left\{ \frac{1}{G_{\lambda}, \lambda \in \Lambda_3} \right\} \right).$$

In the same way, one can prove the absorption law:

$$(F_1 \lor F_2) \cap F_1 = F_1, \quad (F_1 \cap F_2) \lor F_1 = F_1.$$
3. Independent convexities

Convexities \( G_\lambda \in \mathcal{G}(n)(X) \), \( \lambda \in \Lambda \) will be called mutually \((n)\)-independent or, briefly, independent, if the following condition holds

\[
\forall A \in (n): \frac{1}{\{G_\lambda, \lambda \in \Lambda\}} = \bigcap_{\lambda \in \Lambda} g_\lambda A.
\]

If the equality does not necessarily hold for all \( A \in (n) \), then the maximal subfamily of \((n)\)-independent \( \{G_\lambda, \lambda \in \Lambda\} \) will be called the \((n)\)-independence domain of the convexities \( G_\lambda, \lambda \in \Lambda \), and will be denoted by \((n)\)-idc\((G_\lambda, \lambda \in \Lambda)\). The sets belonging to \((n)\)-idc\((G_\lambda, \lambda \in \Lambda)\) will be called the elements of \((n)\)-independence of the convexities \( G_\lambda, \lambda \in \Lambda \). In what follows, if not otherwise stated, the prefix \((n)\) will be omitted.

Given a set \( M \subset \{G_\lambda, \lambda \in \Lambda\} \), we consider the question whether the inclusion \( M \subset \text{idc}(G_\lambda, \lambda \in \Lambda) \) is true. This question can be answered in two ways. First, the condition of convexities independence is directly verified if the inclusion \( M \subset \text{idc}(G_\lambda, \lambda \in \Lambda) \) holds. Moreover, these elements are separated by a set \( g \), and will be denoted by \((\Lambda, g) \). The sets belonging to \((\Lambda, g)\)-idc\((G_\lambda, \lambda \in \Lambda)\) will be called the elements of \((\Lambda, g)\)-independence of the convexities \( G_\lambda, \lambda \in \Lambda \). In what follows, if not otherwise stated, the prefix \((\Lambda, g)\) will be omitted.

The importance of the notion of independent convexities is that, in special cases, we can obtain statements about the separation property for two certain elements of independence of convexities, and the statements are analogous to the separation theorems for two convex sets. Moreover, these elements are separated by a set represented by intersection of some \( g_\lambda \)-halfspaces. A recent investigation on this topic for semiconvex sets in \( S^2 \) has been carried out in [4] by the author.

Now we shall give the following definition, which will play an important role in Lemma 1 and in Proposition 3.

Two finitely defined (not necessarily \( n \)-ary) convexities \( G_1 \) and \( G_2 \) on \( X \), \(|X| \geq 4\), are said to be conically independent if the following condition is true

\[
\forall n \geq 4 \forall B \subset X, |B| = n - 1 \forall x_n \in X \setminus B \forall x \in g^\land(B \cup \{x_n\}) \exists y_1, y_2 \in g^\land B: x \in g_1\{y_1, x_n\} \cap g_2\{y_2, x_n\}.
\]

(Here and in what follows, we use the notation: \( g^\land A = g_1 A \cap g_2 A \).)

**Lemma 1.** Let the convexities \( G_1 \) and \( G_2 \) be conically independent, and let the set \( A \) satisfy the condition

\[
\forall x_1, x_2, x_3 \in A: g^\land\{x_1, x_2, x_3\} \subset A.
\]

Then we have \( A = g^\land A \).

**Proof.** Since the convexities \( G_1 \) and \( G_2 \) are finitely defined, we see that

\[
\forall x \in g^\land A \exists x_1, \ldots, x_n \in A: x \in g^\land\{x_1, \ldots, x_n\}.
\]

From (3) it follows that the proof is trivial if \( n < 4 \); therefore, we may put \( n \geq 4 \). Iterating the conical independence of \( G_1 \) and \( G_2 \) in (1), we get the chain of implications:

\[
\exists y_1, y_2 \in g^\land\{x_1, \ldots, x_n\}: x \in g_1\{y_1, x_n\} \cap g_2\{y_2, x_n\} \Rightarrow
\]

\[
\exists y_{11}, y_{12}, y_{21}, y_{22} \in g^\land\{x_1, \ldots, x_{n-2}\}: y_1 \in g_1\{y_{11}, x_{n-1}\} \cap g_2\{y_{12}, x_{n-1}\},
\]

\[
y_2 \in g_1\{y_{21}, x_{n-1}\} \cap g_2\{y_{22}, x_{n-1}\} \Rightarrow \ldots \Rightarrow \exists y_{11}, y_{12}, y_{13}, \ldots, y_{n-3}, \ldots, y_{n-1}, \ldots \Rightarrow
\]

\[
\exists y_{11}, y_{12}, y_{13}, \ldots, y_{n-3}, \ldots, y_{n-1}, \ldots \in g^\land\{x_1, x_2, x_3\}: y_{11} \in g_1\{y_{11}, x_{n-1}\} \cap g_2\{y_{12}, x_{n-1}\} \Rightarrow
\]

Without loss of generality, consider the points \( u = y_{11}, v_1 = y_{11}, v_2 = y_{12} \). From (3) it follows that \( x_4, v_1, v_2 \in A \); hence, again from (3), \( g^\land\{x_4, v_1, v_2\} \subset A \). Obviously, if \( u \in \{x_4, v_1, v_2\} \), then \( u \in A \). Let \( u \notin \{x_4, v_1, v_2\} \). Taking into account the properties of a convex hull operator, for each \( i \in \{1, 2\} \), we obtain

\[
g_1\{v_1, x_4\} \cap g_2\{v_2, x_4\} \subset g_i\{v_i, x_4\} \subset g_i\{x, v_1, v_2\}.
\]
whence \( g_1\{x_1, x_4\} \cap g_2\{x_2, x_4\} \subseteq g^{(2)}\{x_1, x_4, v_1, v_2\} \). Since \( u \in g_1\{x_1, x_4\} \cap g_2\{x_2, x_4\} \), we see again that \( u \in A \).

Moving back along the chain of implications (4) in which the previous procedure applies to all points \( y \) and \( x \), we obtain \( g_1\{y_1, x_n\} \cap g_2\{y_2, x_n\} \subseteq g^{(3)}\{x_n, y_1, y_2\} \). From the arbitrariness of \( x \), we conclude that \( g_1 A \cap g_2 A \subseteq A \). The reverse inclusion is evident. \( \square \)

**Proposition 3.** Let the convexities \( G_1, G_2 \in \mathfrak{G}^{(3)}(X) \) be conically independent; then \( G_1 \) and \( G_2 \) are independent.

**Proof.** Consider an arbitrary set \( A \in (3) \frac{1}{\{G_1,G_2\}} \). The 3-arity of \( G_1 \) and \( G_2 \) imply that for any \( x_1, x_2, x_3 \in A \) there exists an \( i \in \{1, 2\} \) for which \( g_i\{x_1, x_2, x_i\} \subseteq A \). Since \( g^{(2)}\{x_1, x_2, x_3\} \subseteq g_i\{x_1, x_2, x_3\} \), it follows that \( A \) satisfy (2). Thus, by Lemma (1) we have \( A = g^{(3)}A \). \( \square \)

4. Examples of spaces with fractoconvexity

To describe the fractoconvexities in Examples 1–3, we need the following construction [4].

Let \( S \) be the 2-sphere in \( \mathbb{R}^3 \) with center at the origin, \( B \) be the closed ball bounded by \( S \), \( C \) be an arbitrary fixed set in the interior of \( B \), and the symbol \( [ , ] \) denotes the line segment operator in \( \mathbb{R}^3 \). The convexities \( G(c), c \in C \) on \( S \) are defined analogously to the convexity in the sense of Robinson [5] but with respect to the points \( c \in C \), respectively. This means that a set \( A \subseteq S \) is \( G(c) \)-convex if, for two distinct points \( x_1, x_2 \) such that the straight line determined by them does not pass through the point \( c \), the set cut out by the 2-dimensional cone with vertex \( c \) and base \( [x_1, x_2] \) is entirely contained in \( A \). It is readily seen that all convexities \( G(c), c \in C \) are binary, and the segment \( g_i\{x_1, x_2\} \) joining two points \( x_1, x_2 \in S \) coincides with the subset of \( S \) mentioned above if \( x_1, x_2 \), and \( c \) are non-collinear and equals \( \{x_1, x_2\} \) otherwise.

**Example 1.**

Let \( C = \{c_0, c_1\}; \) then the fractoconvexity \( F_1 = (2) \frac{1}{\{G(c_0), G(c_1)\}} \) is the family of all 2-semiconvex sets with respect to \( G(c_0) \) and \( G(c_1) \). Among these 2-semiconvex sets, the so-called \( g_{01} \)-regular 2-semiconvex sets are very important. (A set \( A \) is called \( g_{01} \)-regular if there exists an open halfspace \( H \) in \( \mathbb{R}^3 \) such that \( c_0, c_1 \in H \) and \( A \subset H^c \).) It has been shown in [4] that every \( g_{01} \)-regular 2-semiconvex set belongs to the independence domain of \( G(c_0) \) and \( G(c_1) \).

**Example 2.**

Let \( C = \{c_\lambda := \lambda c_0 + (1 - \lambda)c_1 \mid \lambda \in [0, 1]\} \) and let us introduce the convexities \( G'(c_\lambda), c_\lambda \in C \), which are the restriction of \( G(c_\lambda), c_\lambda \in C \) to the subfamilies of the sets that are \( g_{01} \)-regular with respect to some fixed open halfspace \( H \) parallel to the segment \( [c_0, c_1] \). Considering the fractoconvexity \( F_2 = (2) \frac{1}{\{G'(c_\lambda), c_\lambda \in C\}} \), by analogy with the previous example, one can prove that the convexities \( G'(c_\lambda), c_\lambda \in C \) are independent.

**Example 3.**

Let \( C \) be as in Example 2. Consider the multiconvexities

\[ G_1 = (2) \{ G'(c_\lambda) \mid c_\lambda \in [c_0, (c_0 + c_1)/2]\}, \quad G_2 = (2) \{ G'(c_\lambda) \mid c_\lambda \in [(c_0 + c_1)/2, c_1]\}, \]

and put \( F_3 = (2) \frac{1}{\{G_1, G_2\}} \). Obviously, these multiconvexities are binary. By definition of an \( n \)-ary convexity, they are 3-ary as well.

Consider the convexity \( G'((c_0 + c_1)/2) \). By \( \tilde{g} \) denote the corresponding convex hull operator. It is easily seen that

\[ \forall n \geq 2 \forall x_1, \ldots, x_n \in X, x_i \neq x_j (i \neq j) \forall x \in \tilde{g}\{x_1, \ldots, x_n\} \exists y \in \tilde{g}\{x_1, \ldots, x_{n-1}\} : x \in \tilde{g}\{y, x_n\}. \quad (6) \]

Moreover, the operators \( g_1, g_2, \) and \( \tilde{g} \) are related to each other by the condition

\[ \forall n \in \mathbb{N} \forall x_1, \ldots, x_n \in X \quad g^{(n)}\{x_1, \ldots, x_n\} = \tilde{g}\{x_1, \ldots, x_n\}. \quad (7) \]

Combining (6) and (7), we see that the 3-ary convexities \( G_1 \) and \( G_2 \) are conically independent; therefore, considering the fractoconvexity \( F_3 = (2) \frac{1}{\{G_1, G_2\}} \) and using Proposition 3 we conclude that these convexities are independent.

**Example 4.**

Suppose that the binary convexities \( G_1 \) and \( G_2 \) and the fractoconvexity \( F_4 \) are defined on \( Z \) as follows. Let \( G_1 \) be the collection of all sets \( A \cap Z \), where \( A \subseteq \mathbb{R} \) is a standard convex set. Suppose \( f \) is a bijective function
from \( \mathbb{Z} \) to itself. Using \( f \), for any \( x_1, x_2 \in \mathbb{Z} \) we define the segment \( g_2 \{x_1, x_2\} \) by the formula

\[
g_2 \{x_1, x_2\} := f(g_1 \{f^{-1}(x_1), f^{-1}(x_2)\}).
\]

We put \( G_2 = \{A \subset \mathbb{Z} \mid \forall x_1, x_2 \in A \ g_2 \{x_1, x_2\} \subset A\} \).

If a set \( A \subset \mathbb{Z} \) is bounded, then the operators \( g_1 \) and \( g_2 \) satisfy the following equalities:

1) \( g_1 A = \mathbb{Z} \cap \{a_1 \lambda + b_1 (1 - \lambda) \mid \lambda \in [0, 1], a_1 = \min A, b_1 = \max A\};

2) \( g_2 A := f(g_1 \{a_2, b_2\}) = g_2 \{f(a_2), f(b_2)\}, \) where \( a_2 = \min f^{-1}(x), b_2 = \max f^{-1}(x) \).

Consider the fractoconvexity \( F_4 = (2) \frac{1}{(G_1, G_2)} \). The following proposition is valid.

**Proposition 4.** If a set \( A \in F_4 \) is bounded, then \( A \in \text{idc}(G_1, G_2) \).

**Proof.** Since \( A \subset g_1 A \cap g_2 A \), it suffices to verify the reverse inclusion. If \( g_1 A \subset A \) or \( g_2 A \subset A \), then proof is trivial. Therefore, we assume that \( g_1 A \not\subset A \) and \( g_2 A \not\subset A \).

From invertibility of the function \( f \) and from the definition of the points \( a_2 \) and \( b_2 \) it follows that the points \( f(a_2) \) and \( f(b_2) \) belong to the set \( A \). The set \( A \) is semiconvex: hence, from the relations \( g_1 \{a_1, b_1\} = g_1 A \not\subset A \) and \( g_2 \{f(a_2), f(b_2)\} = g_2 A \not\subset A \), we obtain

\[
g_2 \{a_1, b_1\} = f(g_1 \{f^{-1}(a_1), f^{-1}(b_1)\}) \subset A,
\]

\[
g_1 \{f(a_2), f(b_2)\} \subset A.
\]

Suppose that the set \( (g_1 A \cap g_2 A) \setminus A \) is nonempty, that is,

\[
\exists x \in (g_1 \{a_1, b_1\} \cap g_2 \{f(a_2), f(b_2)\}) \setminus A.
\]

Under this assumption and by (8), we see that \( x \notin g_1 \{f(a_2), f(b_2)\} \subset A \). But from (9) it follows that \( x \in g_1 \{a_1, b_1\} \); hence, we have the following possible locations of the point \( x \): either \( x \in g_1 \{a_1, f(\cdot)\} \) or \( x \in g_1 \{f(\cdot), b_1\} \), where \( f(\cdot) \) belongs to \( \{f(a_2), f(b_2)\} \) and is determined depending on the mutual position of \( f(a_2) \) and \( f(b_2) \) (see below). Both these cases are investigated equally. For example, consider only the first of them. We have two subcases:

- either \( x \in g_1 \{a_1, f(a_2)\} \) if \( f(a_2) \leq f(b_2) \); or \( x \in g_1 \{a_1, f(b_2)\} \) if \( f(a_2) > f(b_2) \).

(Here we have taken into account that \( f(a_2) = f(\min f^{-1}(x)) \geq \min A = a_1 \).) Without loss of generality it can be investigated one subcase, for example, \( f(a_2) \leq f(b_2) \).

Since \( x \notin A \) and \( x \in g_1 \{a_1, f(a_2)\} \), we obtain

\[
g_1 \{a_1, f(a_2)\} \not\subset A.
\]

Therefore, we have \( g_2 \{a_1, f(a_2)\} \subset A \), since \( A \) is semiconvex. Hence, by the definition of \( g_2 \), we get

\[
f(g_1 \{a_2, f^{-1}(a_1)\}) \subset A.
\]

From (8) it follows that \( x \in g_2 \{f(a_2), f(b_2)\} \), in other words, that \( x \in f(g_1 \{a_2, b_2\}) \). The definitions of the points \( a_2 \) and \( b_2 \) imply that \( f^{-1}(a_1) \in g_1 \{a_2, b_2\} \). Combining these remarks with (11) and \( x \notin A \), we obtain \( x \in f(g_1 \{f^{-1}(a_1), b_2\}) \not\subset A \). At the same time, \( f(g_1 \{f^{-1}(a_1), b_2\}) = g_2 \{a_1, f(b_2)\} \); hence, \( g_1 \{a_1, f(b_2)\} \subset A \), since \( A \) is semiconvex.

By virtue of the assumption \( f(a_2) \leq f(b_2) \), we have \( g_1 \{a_1, f(a_2)\} \subset g_1 \{a_1, f(b_2)\} \), whence \( g_1 \{a_1, f(a_2)\} \subset A \). But the last contradicts inclusion (10). Thus, \( (g_1 A \cap g_2 A) \setminus A = \emptyset \), and since \( A \subset g_1 A \cap g_2 A \), the last equality is true iff \( A = g_1 A \cap g_2 A \). 

\[\square\]

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