On odd unitary Steinberg group

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Abstract

Let $\mathcal{R}$ be a ring with pseudo-involution, $\mathcal{L}$ be an odd form parameter, $\text{U}(2n, \mathcal{R}, \mathcal{L})$ be an odd hyperbolic unitary group, $\text{EU}(2n, \mathcal{R}, \mathcal{L})$ be its elementary subgroup and $\text{StU}(2n, \mathcal{R}, \mathcal{L})$ be an odd unitary Steinberg group (see [15, 16]). We compute the Schur multipliers of $\text{StU}(2n, \mathcal{R}, \mathcal{L})$ and $\text{EU}(\mathcal{R}, \mathcal{L})$.

MSC: 19C09

Keywords: odd unitary group, Schur multiplier, non-stable K-theory.

Acknowledgement: the author gratefully acknowledges the support of the
1) State Financed research task 6.38.74.2011 “Structure theory and geometry of algebraic groups and their applications in representation theory and algebraic K-theory” at the SPbSU,
2) RFBR project 13-01-00709 “Study of algebraic groups over rings by localization methods”,
3) Chebyshev Laboratory (Department of Mathematics and Mechanics, SPbSU) under RF Government grant 11.G34.31.0026.

Introduction

Let $\text{StU}(2n, \mathcal{R}, \mathcal{L})$ denote the analog of the Steinberg group for an odd unitary group defined by Petrov in [15]. The main goal of the present paper is to prove that $\text{StU}(2n, \mathcal{R}, \mathcal{L})$ is centrally closed.

Let $\mathcal{R}$ be an associative ring with 1, $n \geq 3$. Denote by $\text{E}(n, \mathcal{R})$ the usual elementary group, i.e. the subgroup of the group of invertible $n \times n$ matrices, generated by elementary transvections $t_{ij}(a)$. It is obviously that Steinberg relations

$$t_{ij}(a)t_{ij}(b) = t_{ij}(a + b), \quad (S1)$$
$$[t_{ij}(a), t_{kh}(b)] = 1 \text{ for } k \neq j, \ h \neq i, \quad (S2)$$
$$[t_{ij}(a), t_{jk}(b)] = t_{ik}(ab) \quad (S3)$$

hold for transvections. The abstract group defined by generators $\{x_{ij}(a) \mid 1 \leq i \neq j \leq n, \ a \in \mathcal{R}\}$ and relations $S1$–$S3$ with $x_{ij}(a)$ instead of $t_{ij}(a)$
is called the (linear) Steinberg group $\text{St}(n, R)$. It follows from S3 that the commutator subgroup $[\text{St}(n, R), \text{St}(n, R)]$ of the Steinberg group coincides with it and thus $H_1(\text{St}(n, R), \mathbb{Z})$ is trivial. For such groups (called perfect) the kernel of the so called universal central extension coincides with the second homology group $H_2(\text{St}(n, R), \mathbb{Z})$ and in this context is often called the Schur multiplier (see [14], [19]). It is known that for $n \geq 5$ the Schur multiplier of $\text{St}(n, R)$ is trivial, i.e. this group is centrally closed (or superperfect), see [14], [19].

Parallel results are known in the cases of other classical\(^1\) Chevalley groups (see [17, 13, 12, 18]), Bak’s quadratic groups $GQ(2n, R, \Lambda)$ (see [1, 2, 3, 5, 6, 7, 8, 9, 10, 20]) and Hermitian groups $\text{GH}(2n, R, a_1, \ldots, a_r)$ (see [21, 4]). In the present paper we show that the Schur multiplier of Petrov’s odd unitary Steinberg group is trivial when $n \geq 5$, which is a common generalization of all above results. The condition $n \geq 5$ can not be relaxed, since the orthogonal Steinberg group which is the special case of an odd unitary Steinberg group is not necessary centrally closed for $n = 4$ (see [13]). Using these results one can obtain that “in the limit” the Schur multiplier of elementary odd unitary group $\text{EU}(R, \mathfrak{L})$ coincides with the kernel $K_2U(R, \mathfrak{L})$ of the natural epimorphism $\text{StU}(R, \mathfrak{L}) \twoheadrightarrow \text{EU}(R, \mathfrak{L})$:

$$K_2U(R, \mathfrak{L}) = H_2(\text{EU}(R, \mathfrak{L}), \mathbb{Z}).$$

The paper is organized as follows. In Section 1 we recall the definition of the Schur multiplier and prove several elementary facts about it, in Section 2 we define an odd unitary Steinberg group following [15], in Section 3 we prove the main technical lemma and in the last section we obtain the main results.

I would like to express my thanks to Professor Nikolai Vavilov for his supervision of this work and Dr. Sergey Sinchuk for helpful discussions.

1 Central extensions

**Definition.** An epimorphism of abstract groups $\epsilon : H \twoheadrightarrow G$ is called a central extension (of $G$) if its kernel is contained in the center of $H$.

**Lemma 1** (Steinberg central trick). Let $\epsilon : H \twoheadrightarrow G$ be a central extension. Then for any elements $u_1, u_2, v_1, v_2 \in H$ such that $\epsilon(u_1) = \epsilon(u_2)$ and $\epsilon(v_1) = \epsilon(v_2)$ one has $[u_1, v_1] = [u_2, v_2]$.

**Hint.** Use that $u_1u_2^{-1}, v_1v_2^{-1} \in \ker(\epsilon) \subseteq \text{Cent}(H)$.

\(^1\)One can also find parallel results for exceptional Chevalley groups in [17, 13, 18], but they are not generalized in the present paper.
Definition. Let $\epsilon : H \rightarrow G$ be a central extension. Then for $x, y \in G$ denote by $[\epsilon^{-1}x, \epsilon^{-1}y]$ the commutator of any $u, v \in H$ such that $\epsilon(u) = x$ and $\epsilon(v) = y$. This definition is correct by Lemma 1.

Definition. Group $G$ is called perfect if it coincides with its commutator subgroup $[G, G]$.

Lemma 2. Let $\epsilon : H \rightarrow G$ be a central extension. Then $H$ is perfect if and only if for any central extension $\zeta : H' \rightarrow G$ there exists no more than one homomorphism making the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\eta} & H' \\
\downarrow \epsilon & & \downarrow \zeta \\
G & \xrightarrow{\theta} & \zeta
\end{array}
\]

commutative.

Proof. Let $H$ be a perfect group, $\zeta$ be a central extension of $G$ and $\eta, \theta$ be homomorphisms making the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\eta} & H' \\
\downarrow \epsilon & & \downarrow \zeta \\
G & \xrightarrow{\theta} & \zeta
\end{array}
\]

commutative. Then for any two elements $x, y$ from $H$ by lemma 1 we have that $[\eta(x), \eta(y)] = [\zeta^{-1}(\epsilon(x)), \zeta^{-1}(\epsilon(y))] = [\theta(x), \theta(y)]$, i.e. $\eta([x, y]) = \theta([x, y])$. But $H$ is perfect, so $\eta = \theta$.

Now suppose that $H$ is not perfect. Then there is a nontrivial homomorphism $\alpha : H \rightarrow A = \frac{H}{[H, H]}$ onto abelian group. Using the fact that the morphism $\zeta : H \times A \rightarrow G$ defined by $\zeta(u, a) = \epsilon(u)$ is a central extension, we obtain distinct homomorphisms $\eta(u) = (u, 1)$ and $\theta(u) = (u, \alpha(u))$, making the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\eta} & H \times A \\
\downarrow \epsilon & & \downarrow \zeta \\
G & \xrightarrow{\theta} & \zeta
\end{array}
\]

commutative. \hfill \square

Definition. A central extension $\pi : U \rightarrow G$ is called a universal central extension of $G$ if for any central extension $\epsilon : H \rightarrow G$ there is a unique
group homomorphism \( \eta \) making the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\eta} & H \\
\downarrow{\pi} & & \downarrow{\epsilon} \\
G & & 
\end{array}
\]

commutative.

Remark. Lemma 2 implies that the domain (and thus the image) of a universal central extension is perfect. So the universal central extension can only exist for a perfect group.

**Lemma 3.** Any perfect group admits a universal central extension.

**Proof.** Fix an epimorphism \( \phi : F \to G \) with \( F \) free and \( R = \ker \phi \). Obviously \( \phi \) factors through the factorization epimorphism \( \tau : F \to \overline{F} = \frac{F}{[R, F]} \).

\[
\begin{array}{ccc}
F & \xrightarrow{\phi} & G \\
\downarrow{\tau} & \downarrow{\phi} & \\
F, F & \xrightarrow{[R, F]} & 
\end{array}
\]

Restricting the factored morphism \( \phi \) to the commutator subgroup we obtain the morphism \( \pi : \frac{[F, F]}{[R, F]} \to [G, G] = G \) with \( \ker \pi = \frac{R \cap [F, F]}{[R, F]} \). Obviously \( \varphi \) and \( \pi \) are central extensions of \( G \) and below we will show that \( \pi \) is a universal central extension.

Fix a central extension \( \epsilon : H \to G \). Applying the universal property of \( F \) one can obtain a group morphism \( \theta : F \to H \) making the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\theta} & H \\
\downarrow{\phi} & & \downarrow{\epsilon} \\
G & & 
\end{array}
\]

commutative. Thus \( [\theta(R), \theta(F)] \subseteq [\ker \epsilon, \theta(F)] = 1 \) so \( \theta \) factors through \( \tau \).

\[
\begin{array}{ccc}
F & \xrightarrow{\tau} & H \\
\downarrow{\phi} & \downarrow{\theta} & \\
F & \xrightarrow{\varphi} & G \\
\downarrow{\epsilon} & \downarrow{\epsilon} & \\
G & & 
\end{array}
\]
Restricting the factored morphism \( \vartheta \) to the commutator subgroup we obtain morphism \( \eta : [F, F] \to H \) making the diagram

\[
\begin{array}{c}
[F, F] \xrightarrow{\eta} H \\
\pi \downarrow \downarrow \downarrow \downarrow \downarrow \\
G \\
\end{array}
\]

commutative. Finally, observe that for any \( x, y \in F \) there are \( u, v \in [F, F] \) such that \( \varphi(x) = \pi(u) \) and \( \varphi(y) = \pi(v) \) (since \( \pi \) is a surjection) but \( \varphi \) is a central extension so that it follows from Lemma 1 that \( [x, y] = [u, v] \) and thus \( [F, F] \) is a perfect group. Now one can use Lemma 2 to finish the proof.

**Definition.** If \( \pi \) is a universal central extension of \( G \) then its kernel is called the **Schur multiplier of \( G \)** and denoted by \( \text{M}(G) = \text{Ker}(\pi) \). This definition is obviously correct, i.e. if \( \pi \) and \( \varpi \) are two universal central extensions of \( G \), their kernels are isomorphic, \( \text{Ker}(\pi) \cong \text{Ker}(\varpi) \).

**Remark.** Lemma 3 implies that the Schur multiplier is defined for all perfect groups (and only for them).

**Definition.** A perfect group \( G \) is called **centrally closed** if \( 1 \xrightarrow{} G \xrightarrow{} G \) is the universal central extension. Obviously this is equivalent to \( \text{M}(G) = 1 \).

**Definition.** A central extension \( \epsilon : H \rightarrow G \) is called **split** if there is a homomorphism \( \sigma : G \rightarrow H \) such that \( \epsilon \sigma = 1_G \), i.e. the short exact sequence

\[
1 \longrightarrow \text{Ker} \epsilon \longrightarrow H \xrightarrow{\epsilon} G \longrightarrow 1
\]

is (right) splitting.

**Lemma 4.** Let \( \epsilon : H \rightarrow G \) be a split central extension with \( H \) perfect. Then \( H \) and \( G \) are isomorphic.

**Proof.** Let \( \sigma : G \rightarrow H \) be a homomorphism such that \( \epsilon \sigma = 1_G \). Then we have a commutative diagram

\[
\begin{array}{c}
H \xrightarrow{\sigma \epsilon} H \\
\epsilon \downarrow \downarrow \downarrow \downarrow 1_H \\
G \\
\end{array}
\]

with \( H \) perfect. By Lemma 2 it follows that \( \sigma \epsilon = 1_H \) and thus \( D \cong G \).

**Lemma 5.** Let \( \epsilon : H \rightarrow G \) be a central extension with \( H \) centrally closed. Then \( \epsilon \) is the universal central extension of \( G \).
Proof. $G = \epsilon(H)$ is perfect so that one can consider the universal central extension $\pi : U \to G$ and a group homomorphism $\eta : U \to H$ making the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\eta} & H \\
\pi \downarrow & & \downarrow \epsilon \\
G & & 
\end{array}
\]

commutative. Thus by Lemma 1 $\eta(U) = [H, H] = H$ so $\eta : U \to H$ is a central extension. But $H$ is centrally closed so using the universal property of $1_H$ we obtain that $\eta$ splits. Now use Lemma 4 to finish the proof. □

# 2 Odd unitary Steinberg group

**Definition.** Let $R$ be an associative ring with identity. An additive map $\sigma : R \to R$ such that $1$ is invertible, $\sigma(a) = a$ and $ab = b\sigma^{-1}a$ for any $a, b \in R$ is called a pseudo-involution. In this paper $R$ will always denote an associative ring with identity and pseudo-involution on it.

**Definition.** A biadditive map $B : V_R \times V_R \to R$ is called an anti-Hermitian form (on $V_R$) if it satisfies the following axioms

1) $B(ua, vb) = a^{-1}B(u, v)b$,
2) $B(u, v) = -B(v, u)$

for any $u, v \in V$ and $a, b \in R$.

**Definition.** Let $B$ be an anti-Hermitian form on $V_R$. Then the set $V \times R$ with the composition law given by

$$(u, a) \dot{+} (v, b) = (u + v, a + b + B(u, v))$$

is called the Heisenberg group $\mathfrak{h}$ of the form $B$.

**Remark.** It is clear that $\dot{+}$ is associative, $(0, 0)$ is the identity element and the inverse is given by

$$(u, a) \dot{+} (u, a) = (0, 0),$$

so Heisenberg group is actually a group.

**Definition.** Let $B$ be an anti-Hermitian form on $V_R$ and $\mathfrak{h}$ be its Heisenberg group. We can define the right action of $R$ on $\mathfrak{h}$ by

$$(u, a) \leftarrow b = (ub, b\sigma^{-1}ab).$$
Remark. It’s easy to see that
\[ \lambda \mapsto a \mapsto b = \lambda \mapsto ab \]
and
\[ (\lambda + \mu) \mapsto a = \lambda \mapsto a + \mu \mapsto a \]
for all \( \lambda, \mu \in \mathcal{H} \) and \( a, b \in R \).

**Definition.** Subgroups of a Heisenberg group \( \mathcal{H} \)

\[ L_{\text{min}} = \{ (0, a + \overline{a}) \mid a \in R \} \quad \text{and} \quad L_{\text{max}} = \{ (u, a) \mid a = \overline{a} + B(u, u) \} \]

are called the minimal and the maximal odd form parameters, respectively.

Remark. It’s clear that \( L_{\text{min}} \leq L_{\text{max}} \) and that \( L_{\text{min}} \) and \( L_{\text{max}} \) are stable under the action of \( R \).

**Definition.** A subgroup \( \mathcal{L} \) of a Heisenberg group \( \mathcal{H} \) is called an (odd) form parameter if \( L_{\text{min}} \leq \mathcal{L} \leq L_{\text{max}} \) and \( \mathcal{L} \) is stable under the action of \( R \).

The triple \((V, B, \mathcal{L})\) is called an odd quadratic space.

The orthogonal sum of two odd quadratic spaces \( V \) and \( V' \) is constructed as follows: the underlying module is \( V \oplus V' \), the anti-Hermitian form is given by \((B+B')(u+u', v+v') = B(u, v) + B'(u', v')\) and the odd form parameter consists of all pairs \((u+u', a+a')\), where \((u, a) \in \mathcal{L} \) and \((u', a') \in \mathcal{L}'\).

**Definition.** Let \( V' \) and \( V \) be two modules over \( R \) with bilinear forms \( B' \) and \( B \). A module homomorphism \( f : V' \to V \) is called an isometry if \( B(fu, fv) = B'(u, v) \) for all \( u, v \in V' \).

If \( \mathcal{L} \) is an odd form parameter for \( V \) and \( f \) and \( g \) are isometries from \( V' \) to \( V \) such that \((fv - gv, B(gv - fv, gv)) \in \mathcal{L} \) for every \( v \in V \) we say that \( f \) and \( g \) are equivalent modulo \( \mathcal{L} \) and write \( f \equiv g \mod \mathcal{L} \). One can see that it is an equivalence relation between the isometries.

The odd unitary group \( U(V, B, \mathcal{L}) \) of the odd quadratic space \( V \) is the group of all bijective isometries of \( V \) onto itself that are equivalent to the identity map modulo \( \mathcal{L} \).

**Definition.** Consider a free module \( H \) spanned on vectors \( e_1, e_{-1} \) and anti-Hermitian form \( B \) on it such that \( B(e_1, e_{-1}) = 1, B(e_1, e_1) = B(e_{-1}, e_{-1}) = 0 \). Denote \( \mathcal{L} = \{ (e_1a + e_{-1}b, \overline{a}b + c + \overline{c}) \mid a, b, c \in R \} \). One can check that \( \mathcal{L} \) is a form parameter for \( H \).

Denote by \( H^n \) the orthogonal sum of \( n \) copies of \( H \). Its basis coming from the bases of the summands will be indexed as follows: \( e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1} \).

Suppose we are given an odd quadratic space \((V, B, \mathcal{L})\). The orthogonal sum \( H^n \oplus V \) is called an odd hyperbolic unitary space of rank \( n \). The unitary
group of $H^n \oplus V$ is called the odd hyperbolic unitary group and denoted $U(2n, R, \mathcal{L})$.

Consider an odd quadratic space $(V, B, \mathcal{L})$. A pair of vectors $(u, v)$ from $V$ such that $B(u, v) = 1$, $(u, 0), (v, 0) \in \mathcal{L}$ is called a hyperbolic pair. The greatest $n$ satisfying the condition that there exist $n$ mutually orthogonal hyperbolic pairs in $V$ is called the Witt index of $V$ and denoted by $\text{ind}(V, B, \mathcal{L})$. It is easy to see that the Witt index coincides with the greatest $n$ satisfying the condition that there exists an isometry $f$ of the space $H^n$ to $V$ such that $(fu, a) \in \mathcal{L}$ for $(u, a)$ from the form parameter of $H^n$.

Suppose that the Witt index of $V$ is at least $n$. Fix an embedding of $H^n$ to $V$, i.e. fix elements $e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}$ in $V$ such that $(e_i, e_j) = 0$ for $i \neq j$, $(e_i, e_{-i}) = 1$ for $i \in \{1, \ldots, n\}$, $(e_i, 0) \in \mathcal{L}$. Define $V_0$ as the orthogonal complement to $\sum_{i=1}^{n-1} e_i R$ in $V$ (it can be defined since the restriction of $B$ to this subspace is nonsingular), $B_0$ as the restriction of $B$ to $V_0$ and $\mathcal{L}_0$ as the restriction of $\mathcal{L}$ to $V_0$. Then it is easy to see that $V$ is isometric to the odd hyperbolic space $H^n \oplus V_0$. Thus the unitary group of an odd quadratic space with Witt index at least $n$ can be identified with the odd hyperbolic unitary group $U(2n, R, \mathcal{L})$ corresponding to an appropriate odd form parameter.

**Definition.** Let $U(2n, R, \mathcal{L})$ be an unitary group of odd hyperbolic space $V$, denote by $\Omega_+, \Omega_-$ the sets $\{1, \ldots, n\}, \{-n, \ldots, -1\}$ respectively, set $\Omega = \Omega_+ \cup \Omega_-$. Set $\varepsilon_i = \mathbf{T}^{-1}$ if $i \in \Omega_+$ and $\varepsilon_i = -1$ if $i \in \Omega_-$. Define $e_i = \mathbf{T}^{-1}$ if $i \in \Omega_+$ and $e_i = -1$ if $i \in \Omega_-$.

For $i \in \Omega$, $j \in \Omega \setminus \{\pm j\}$, $a \in R$, $(u, b)$ denote by $T_{ij}(a)$ linear transformation of $V$ to itself

$$T_{ij}(a) : w \mapsto w + e_{-j} \varepsilon_{-j} \mathbf{a}^{-1} (e_i, w) - e_i a \varepsilon_j (e_{-j}, w)$$

and by $T_i(u, b)$ transformation

$$T_i(u, b) : w \mapsto w - e_i \varepsilon_i (u, w) - e_i \varepsilon_i b \varepsilon_{-i} (e_i, w) + u \varepsilon_{-i} (e_i, w).$$

The transformations $T_{ij}(a)$ and $T_i(u, b)$ are called odd unitary elementary transvections. One can check that elementary transvections lie in $U(2n, R, \mathcal{L})$ (see [15]). The subgroup of the hyperbolic unitary group generated by elementary transvections is called an odd hyperbolic elementary group and denoted $EU(2n, R, \mathcal{L})$.

The linear Steinberg group is defined by “elementary” relations between linear transvections. Now we will define the unitary Steinberg group by the relations between unitary transvections.

**Definition.** Let $n \geq 3$. The odd unitary Steinberg group $\text{StU}(2n, R, \mathcal{L})$ is the group defined by generators $\{X_{ij}(a) \mid i, j \in \Omega, i \notin \{\pm j\}, a \in R\} \cup \{X_i(\xi) \mid i \in \Omega, \xi \in \mathcal{L}\}$ and relations
\[X_{ij}(a) = X_{-j,-i}(\varepsilon_{-j} \xi_i),\] (R0)
\[X_{ij}(a)X_{ij}(b) = X_{ij}(a + b),\] (R1)
\[X_i(\xi)X_i(\zeta) = X_i(\xi + \zeta),\] (R2)
\[[X_{ij}(a), X_{hk}(b)] = 1, \text{ for } h \notin \{j, -i\}, k \notin \{i, -j\},\] (R3)
\[[X_i(\xi), X_{jk}(a)] = 1, \text{ for } j \neq -i, k \neq i,\] (R4)
\[[X_{ij}(a), X_{jk}(b)] = X_{ik}(ab),\] (R5)
\[[X_i(u, a), X_j(v, b)] = X_{i,-j}(\varepsilon_i b(u, v)), \text{ for } i \notin \{\pm j\},\] (R6)
\[[X_i(u, a), X_i(v, b)] = X_i(0, B(u, v) - B(v, u)),\] (R7)
\[[X_i(u, a), X_{-i,j}(b)] = X_{ij}(\varepsilon_i ab)X_{-j}((u, -b) \rightsquigarrow b),\] (R8)
\[[X_{ij}(a), X_{j,-i}(b)] = X_i(0, -\varepsilon_{-i} \xi_0 + b \xi_b^{-1} \xi_i),\] (R9)

where commutators are left-normed.

**Remark 1.** Relation R1 implies that \(X_{ij}(0)^2 = X_{ij}(0),\) i.e. \(X_{ij}(0) = 1\) and thus that \(X_{ij}(-a) = X_{ij}(-a)X_{ij}(a)X_{ij}(a)^{-1} = X_{ij}(a)^{-1}.\) Similarly, R2 implies that \(X_i(0, 0) = 1\) and \(X_i(-u, a)) = X_i(u, a)^{-1}.

**Remark 2.** Relation R7 is the direct consequence of the relation R2. It is listed to emphasize that \(X_i(u, a)\) and \(X_i(v, b)\) do not commute in general.

One can check the following result.

**Lemma 6.** Relations R0–R9 hold for elementary transvections \(T_i(a)\) and \(T_i(u, a).\) Thus when \(n \geq 3\) there is a natural epimorphism from \(\text{StU}(2n, R, \xi)\) to \(\text{EU}(2n, R, \xi)\) sending the generators of the Steinberg group to the corresponding elementary transvections.

**Definition.** Define \(\text{StU}_1(2n, R, \xi)\) to be a subgroup of \(\text{StU}(2n, R, \xi)\) generated by \(\{X_{n,i}(a) \mid i \in \Omega \setminus \{\pm n\}, a \in R\} \cup \{X_n(\xi) \mid \xi \in \xi\}\) and \(\text{EU}_1(2n, R, \xi)\) to be its image in the \(\text{EU}(2n, R, \xi)\).

**Lemma 7.** \(\text{StU}_1(2n, R, \xi) \cong \text{EU}_1(2n, R, \xi).\)

**Proof.** First observe that \([X_n(\xi), X_{n,i}(a)] = [X_{n,i}(a), X_{n,j}(b)]\) for \(i \in \Omega \setminus \{\pm n\}, j \in \Omega \setminus \{\pm n, \pm i\}\) and \([X_{n,i}(a), X_{n,-i}(b)] = X_n(\xi)\) for some \(\xi \in \xi\) so that any \(x \in \text{StU}_1(2n, R, \xi)\) can be decomposed as
\[x = X_n(\xi) \cdot X_{n,1}(a_1) \cdot X_{n,2}(a_2) \cdot \ldots \cdot X_{n,-1}(a_{-1}).\]

Now we will check that this decomposition is unique. Let
\[X_n(\xi) \cdot X_{n,1}(a_1) \cdot X_{n,2}(a_2) \cdot \ldots \cdot X_{n,-1}(a_{-1}) = X_n(\xi) \cdot X_{n,1}(b_1) \cdot X_{n,2}(b_2) \cdot \ldots \cdot X_{n,-1}(b_{-1}).\]
Then

\[ 1 = X_n(\zeta) \cdot X_{n,1}(a_1) \cdot \ldots X_{n,-1}(a_{-1}) \cdot X_{n,-1}(-b_{-1}) \cdot \ldots X_{n,1}(-b_1) \cdot X_n(\xi) = \]
\[ = X_n(\eta) \cdot X_{n,1}(a_1 - b_1) \cdot \ldots \cdot X_{n,-1}(a_{-1} - b_{-1}). \]

Then also \( T_n(\eta) \cdot T_{n,1}(a_1 - b_1) \cdot \ldots \cdot T_{n,-1}(a_{-1} - b_{-1}) = 1 \) thus \( a_i = b_i \) for all \( i \in \Omega \setminus \{\pm n\} \) and thus \( \zeta = \xi \). Now the claimed result is obvious. \( \square \)

**Definition.** One can define

\[ \text{St}_U^{-1}(2n, R, \mathfrak{L}) = \langle X_{-n,i}(a), X_{-n}(\zeta) \mid i \in \Omega \setminus \{\pm n\}, a \in R, \zeta \in \mathfrak{L} \rangle, \]
its image \( \text{EU}_U^{-1}(2n, R, \mathfrak{L}) \) and check that they are isomorphic.

**Lemma 8.** Steinberg group \( \text{St}_U(2n, R, \mathfrak{L}) \) is generated by \( \text{St}_U^{-1}(2n, R, \mathfrak{L}) \) and \( \text{St}U(2n, R, \mathfrak{L}) \).

**Hint.** Use R5 and R8.

**Definition.** Define \( K_2U(2n, R, \mathfrak{L}) \) to be the kernel of natural epimorphism of the Steinberg group \( \text{St}_U(2n, R, \mathfrak{L}) \) onto \( \text{EU}(2n, R, \mathfrak{L}) \).

\[ K_2U(2n, R, \mathfrak{L}) \twoheadrightarrow \text{St}_U(2n, R, \mathfrak{L}) \twoheadrightarrow \text{EU}(2n, R, \mathfrak{L}) \]

**Lemma 9.** Consider a natural mapping \( \phi_n : \text{St}_U(2n, R, \mathfrak{L}) \rightarrow \text{St}_U(2n + 2, R, \mathfrak{L}) \), sending \( X_{ij}(a) \) to \( X_{ij}(a) \) and \( X_i(\zeta) \) to \( X_i(\zeta) \). Then

\[ \phi_n(K_2U(2n, R, \mathfrak{L})) \subseteq \text{Cent}(\text{St}_U(2n + 2, R, \mathfrak{L})). \]

**Proof.** Fix \( x \in K_2U(2n, R, \mathfrak{L}) \) and \( y \in \text{St}_U^{-1}(2n + 2, R, \mathfrak{L}) \). Steinberg relations imply that \( \phi_n(x) \cdot y \cdot \phi_n(x)^{-1} \in \text{St}_U^{-1}(2n + 2, R, \mathfrak{L}) \). But \( \phi_n(x) \in K_2U(2n + 2, R, \mathfrak{L}) \) so images of \( \phi_n(x) \cdot y \cdot \phi_n(x)^{-1} \) and \( y \) coincide in \( \text{EU}_U^{-1}(2n + 2, R, \mathfrak{L}) \) and thus by lemma 7 \( \phi_n(x) \cdot y \cdot \phi_n(x)^{-1} = y \). Similarly, for any \( z \in \text{St}_U^{-1}(2n + 2, R, \mathfrak{L}) \) one has \( [\phi_n(x), z] = 1 \). Now use Lemma 8. \( \square \)

**Remark.** Centrality of \( K_2U(2n, R, \mathfrak{L}) \) in \( \text{St}_U(2n, R, \mathfrak{L}) \) is not so easy to obtain (see [11, 22] for the linear case).

**Definition.** Define \( \text{St}_U(\infty, R, \mathfrak{L}) = \text{St}_U(R, \mathfrak{L}), \text{EU}(R, \mathfrak{L}) \) and \( K_2U(R, \mathfrak{L}) \) as direct limits of corresponding sequences.

\[
\begin{align*}
\ldots & \rightarrow K_2U(2n, R, \mathfrak{L}) \rightarrow K_2U(2n + 2, R, \mathfrak{L}) \rightarrow \ldots \\
\ldots & \rightarrow \text{St}_U(2n, R, \mathfrak{L}) \xrightarrow{\phi_n} \text{St}_U(2n + 2, R, \mathfrak{L}) \rightarrow \ldots \\
\ldots & \rightarrow \text{EU}(2n, R, \mathfrak{L}) \xrightarrow{\mathfrak{L}} \text{EU}(2n + 2, R, \mathfrak{L}) \rightarrow \ldots
\end{align*}
\]
Remark. Lemma 9 implies that
\[ K_2 U(R, \mathcal{L}) \hookrightarrow StU(R, \mathcal{L}) \rightarrow EU(R, \mathcal{L}) \]
is a central extension.

Lemma 10. An odd unitary Steinberg group \( StU(2n, R, \mathcal{L}) \) is perfect.

Hint. Use the fact that \( n \geq 3 \) and Relations R5 and R8.

We will need the results of the following section to compute the Schur multiplier of \( StU(2n, R, \mathcal{L}) \).

3 Main lemma

Main lemma. Let \( n \) be integer such that \( n \geq 4 \) or \( n = \infty \), \( \epsilon \) be a central extension of \( StU(2n, R, \mathcal{L}) \) such that property \( \dagger \) holds:
\[ [\epsilon^{-1} X_{ij}(a), \epsilon^{-1} X_{kh}(b)] = 1, \quad (\dagger) \]
where \( a, b \) are elements of \( R \) and \( i, j, k, h \) are indices from \( \Omega \) such that \( \text{Card}\{i, -i, j, -j, k, -k, h, -h\} = 8 \), i.e. any two of these four indices neither coincide nor have a zero sum. Then \( \epsilon \) splits.

In this section \( n \) and \( \epsilon \) will be always as in the Main lemma, \( \Omega \) will denote \( \{1, \ldots, n, -n, \ldots, -1\} \) for integer \( n \) and \( \{1, \ldots, n, -n, \ldots, -1\} \) for \( n = \infty \).

The idea of the proof is to find elements \( S_{ij}(a) \in \epsilon^{-1} X_{ij}(a) \) and \( S_i(u, a) \in \epsilon^{-1} X_i(u, a) \) such that relations R0–R9 hold for these elements. It will follow from this fact that there is a homomorphism \( \sigma \) from \( StU(2n, R, \mathcal{L}) \) to the group spanned on these elements sending generators to generators what will immediately imply that \( \epsilon \) splits. Now we will give a detailed proof of the Main lemma, but indeed all lemmas proved in this section correspond to one of the relations R0–R9.

The following commutation identities will be essentially used throughout this section.

Lemma 11. Let \( G \) be a group, \( x, y, z, y_1, \ldots, y_m \) be elements of \( G \). For any \( a, b \in G \) we denote \( aba^{-1} \) by \( ^a b \) and left-normed commutator \( aba^{-1}b^{-1} \) by \([a, b]\). Then straightforward calculation shows that
\[
[xy, z] = ^x[y, z] \cdot [x, z], \quad (C1)
\]
\[
[x, yz] = [x, y] \cdot ^y [x, z], \quad (C2)
\]
\[
[x, y_1 \cdots y_m] = [x, y_1] \cdot ^{y_1} [x, y_2] \cdot ^{y_1y_2} [x, y_3] \cdots \cdot ^{y_1 \cdots y_{m-1}} [x, y_m], \quad (C3)
\]
\[
[x, y] \cdot [x, z] = [x, yz] \cdot [y, [x, z]], \quad (C4)
\]
\[
^y [x, [y^{-1}, z]] \cdot ^z [y, [z^{-1}, x]] \cdot ^z [z, [x^{-1}, y]] = 1 \quad (C5)
\]
\[
^z [y, [z^{-1}, x]] = ^z [y, [x, z]]. \quad (C6)
\]
The next lemma is a stronger version of the Property †.

**Lemma 12.** Let \( i \in \Omega, \ j \in \Omega \setminus \{\pm i\}, \ k \in \Omega \setminus \{-i, j\}, \ h \in \Omega \setminus \{i, -j, \pm k\} \). Then for any \( a, b \in R \)

\[
[e^{-1}X_{ij}(a), e^{-1}X_{kh}(b)] = 1.
\]

**Proof.** If \( \text{Card}\{\pm i, \pm j, \pm k, \pm h\} \neq 8 \) then using the fact that \( n \geq 4 \) we can fix \( l \in \Omega \setminus \{\pm i, \pm j, \pm k, \pm h\} \) and \( x \in e^{-1}X_{ij}(a), \ y \in e^{-1}X_{kl}(b), \ z \in e^{-1}X_{lh}(1) \) (for \( n = \infty \) we should work in the \( \text{StU}(2m, \ R, \ L) \) with \( m \) large enough). Relation R5 implies that \( [y, z] \in e^{-1}X_{kh}(b) \) and Relation R3 implies that \( [x, y], [x, z] \in \text{Ker}(e) \subseteq \text{Cent} \left( \text{Dom}(e) \right) \). Thus using Identity C2 we have

\[
1 = [x, y^{-1}y] = [x, y^{-1}] \cdot [x, y],
\]
i.e. \( [x, y^{-1}] = [x, y]^{-1} \) (so it is central) and the same about \( [x, z^{-1}] \). Now using Lemma 1 and C3 we obtain that

\[
[e^{-1}X_{ij}(a), e^{-1}X_{kh}(b)] = [x, [y, z]] = [x, y] \cdot [x, z] \cdot [x, y^{-1}] \cdot [x, z^{-1}] = 1.
\]

If \( \text{Card}\{\pm i, \pm j, \pm k, \pm h\} = 8 \) we can just use the Property †. \( \square \)

**Remark.** It is easy to see that if \( n \geq 5 \) or \( n = \infty \) then Property † holds for every central extension of \( \text{StU}(2m, \ R, \ L) \). Indeed, if \( n \geq 5 \) then we can fix \( l \notin \{\pm i, \pm j, \pm k, \pm h\} \) in the proof above even if \( \text{Card}\{\pm i, \pm j, \pm k, \pm h\} = 8 \).

**Lemma 13.** Let \( i \in \Omega, \ j \in \Omega \setminus \{\pm i\}, \ k \in \Omega \setminus \{i, \pm j\} \). Then for any \( \lambda \in L, \ a \in R \)

\[
[e^{-1}X_i(\lambda), e^{-1}X_{jk}(a)] = 1.
\]

**Hint.** Like previous lemma.

**Lemma 14.** Let \( i, \ j, \ k, \ h \) be indices from \( \Omega \) such that \( \text{Card}\{\pm i, \pm j, \pm k, \pm h\} = 8 \). Then for any \( a, b \in R \)

\[
[e^{-1}X_{ki}(a), e^{-1}X_{jh}(b)] = [e^{-1}X_{kj}(ab), e^{-1}X_{jh}(1)].
\]

**Proof.** Fix \( x \in e^{-1}X_{ki}(a), \ y \in e^{-1}X_{ij}(-b), \ z \in e^{-1}X_{jh}(1) \). By Lemma 12 \( [z^{-1}, x] = 1 \) and thus identity C5 implies that

\[
y[x, [y^{-1}, z]] = x[[x^{-1}, y], z].
\]

But using R5 we have that \( [y^{-1}, z] \in e^{-1}X_{kh}(b), \ [x^{-1}, y] \in e^{-1}X_{kj}(ab) \) and \( [x, [y^{-1}, z]], [[x^{-1}, y], z] \in e^{-1}X_{kh}(ab) \) and thus commute with \( x \) and \( y \) by Lemma 12. \( \square \)
Definition. For $a \in R$, $k, h \in \Omega$ such that $k \notin \{\pm h\}$ we will denote the commutator $[\epsilon^{-1} X_{ki}(a), \epsilon^{-1} X_{ih}(1)]$ by $S_{kh}(a)$, where $i \in \Omega \setminus \{\pm k, \pm h\}$. This definition does not depend on the choice of $i$ by Lemma 14.

Remark. Lemma 14 implies that $[\epsilon^{-1} X_{ki}(a), \epsilon^{-1} X_{ih}(b)] = S_{kh}(ab)$.

We want to find $S_{kh}(a) \in \epsilon^{-1} X_{kh}(a)$ such that Relations R0–R9 would hold for them. In particular, Relation R5 should hold but central trick implies that this relation is equivalent to the identity in the remark above. So it was natural to define right hand side of this identity as its left hand side.

Lemma 15. For any $i \in \Omega$, $j \in \Omega \setminus \{\pm i\}$, $a, b \in R$

\[ S_{ij}(a)S_{ij}(b) = S_{ij}(a+b). \]

**Proof.** Fix $l \in \Omega \setminus \{\pm i, \pm j\}$, $x \in \epsilon^{-1} X_{il}(1)$, $y \in \epsilon^{-1} X_{ij}(a)$, $z \in \epsilon^{-1} X_{lj}(b)$.

By Lemma 12 $[y, [z, x]] = [\epsilon^{-1} X_{ij}(a), \epsilon^{-1} X_{lj}(-b)] = 1$ and thus C4 implies that

\[ [x, y][x, z] = [x, yz]. \]

\[ \square \]

Remark. As we mentioned earlier, Lemma 15 implies that $S_{ij}(0) = 1$ and $S_{ij}(a)^{-1} = S_{ij}(-a)$.

Lemma 16. For any $i \in \Omega$, $j \in \Omega \setminus \{\pm i\}$, $a \in R$

\[ S_{ij}(a) = S_{-j,-i}(\epsilon_{-j}a). \]

**Proof.** Fix $l \in \Omega \setminus \{\pm i, \pm j\}$. Obviously $\epsilon_{l} \epsilon_{-l} = -\overline{1}$ so

\[ S_{ij}(a) = [\epsilon^{-1} X_{il}(a), \epsilon^{-1} X_{lj}(1)] = [\epsilon^{-1} X_{-l,-i}(\epsilon_{-l}a), \epsilon^{-1} X_{-j,-l}(\epsilon_{-j}a)] = \]

\[ = [\epsilon^{-1} X_{-j,-l}(\epsilon_{-j}a), \epsilon^{-1} X_{-j,-l}(\epsilon_{-j}a)]^{-1} = S_{-j,-i}(\epsilon_{-j}a) \]

\[ = S_{-j,-i}(\epsilon_{-j}a) = S_{-j,-i}(\epsilon_{-j}a). \]

\[ \square \]

Lemma 17. For any $i \in \Omega$, $j \in \Omega \setminus \{\pm i\}$, $(u, a)$, $(v, b) \in \mathcal{L}$

\[ [\epsilon^{-1} X_{i}(u), \epsilon^{-1} X_{j}(v)] = S_{i,j}(\epsilon_{i}B(u, v)). \]

**Proof.** Fix $l \in \Omega \setminus \{\pm i, \pm j\}$, $x \in \epsilon^{-1} X_{i}(u)$, $y \in \epsilon^{-1} X_{j}(v)$, $z \in \epsilon^{-1} X_{l,-j}(1)$ (note that $(-v, b) = (v, b) \leftrightarrow (-1) \in \Omega$). Using the fact that $[z^{-1}, x] = 1$ (Lemma 13) and Identity C5 we have

\[ x[x^{-1}, y], z] = y[x, [y^{-1}, z]]. \]

Now one can check that $[x^{-1}, y] \in \epsilon^{-1} X_{i}(\epsilon_{i}B(u, v))$ and $[y^{-1}, z] \in \epsilon^{-1} (X_{j}(v, b).X_{l,-j}(\epsilon_{-j}B))$ (note that $(v, b) \in \mathcal{L}$ so $y^{-1} \in \epsilon^{-1} X_{l}(v, -\overline{b}))$. C2 implies that $[x, [y^{-1}, z]] = [\epsilon^{-1} X_{i}(u, a), \epsilon^{-1} X_{j}(v, b)] \cdot 1$. Now use that $S_{i,j}(\epsilon_{i}B(u, v))$ commutes with $x$ and $y$ (by Lemma 13).
Lemma 18. For any \( i, j, k \in \Omega \), such that \( \text{Card}\{\pm i, \pm j, \pm k\} = 6 \), \((u, a) \in \mathcal{L}, b \in R\)

\[
S_{i, k}(\varepsilon \bar{b}T^{-1}ab)[\varepsilon^{-1}X_i((u, -\overline{a}) \leftrightarrow b), \varepsilon^{-1}X_{-i, -k}(1)] = S_{j, -k}(\varepsilon \overline{a}b)[\varepsilon^{-1}X_j(u, -\overline{a}), \varepsilon^{-1}X_{-j, -k}(b)].
\]

Proof. Fix elements \( x \in \varepsilon^{-1}X_j(-(u, a)), y \in \varepsilon^{-1}X_{-j, -i}(-b), z \in \varepsilon^{-1}X_{-i, -k}(1), w \in \varepsilon^{-1}X_{j-i}(\varepsilon ab) \). Using C1 we have \([\varepsilon^{-1}, y]w, z] = [\varepsilon^{-1}, y][w, z][\varepsilon^{-1}, y], z] \) and using C5 and the fact \([z^{-1}, x] = 1 \) we have \( [\varepsilon^{-1}, y], z] = y[x, [y^{-1}, z]], \)

so

\[
[x^{-1}, y]w, z] = [x^{-1}, y][w, z] \cdot x^{-1}[y, [x^{-1}, z]] \cdot [x^{-1}, [x^{-1}, z]] \cdot [x, [y^{-1}, z]].
\]

One can check that

\[
[x, [y^{-1}, z]] = [\varepsilon^{-1}X_j(u, -\overline{a}), \varepsilon^{-1}X_{-j, -k}(b)] \in \varepsilon^{-1}(X_{j-k}(\varepsilon \overline{a}b) \cdot X_k((u, a) \leftrightarrow b)),
\]

\[
([\varepsilon^{-1}, y]w, z] = [\varepsilon^{-1}X_i((u, -\overline{a}) \leftrightarrow b), \varepsilon^{-1}X_{-i, -k}(1)),
\]

\[
[x^{-1}, [x^{-1}, z]] = S_{j, -k}(\varepsilon \overline{a}b
\]

(\text{use Lemmae} 13 \text{ and } 17), \( [y, [x^{-1}, z]] = S_{i, -k}(\varepsilon \overline{a}a^{-1}b) \) (\text{use Lemmae} 13 \text{ and } 14) \text{ and } \([w, z] = S_{j, -k}(\varepsilon ab) \). Use Lemmae 12, 13 and 15 to finish the proof. \( \Box \)

Definition. For \( k \in \Omega \), \((u, a) \in \mathcal{L} \) we will denote by \( S_k(u, a) \) the element

\[
S_{i, k}(\varepsilon \overline{a}b)[\varepsilon^{-1}X_i((u, -\overline{a}) \leftrightarrow b), \varepsilon^{-1}X_{-i, -k}(1)].
\]

This definition does not depend on the choice of \( i \) by Lemma 18.

Remark. Observe that by definition \( S_k((u, a) \leftrightarrow b) \) is exactly

\[
S_{i, -k}(\varepsilon \overline{a}b)[\varepsilon^{-1}X_i((u, -\overline{a}) \leftrightarrow b), \varepsilon^{-1}X_{-i, -k}(1)].
\]

Thus, Lemma 18 implies (changing \( a \) by \(-\overline{a}\) and \( k \) by \(-k\)) that

\[
S_{jk}(\varepsilon ab)S_{j, -k}((u, -\overline{a}) \leftrightarrow b) = [\varepsilon^{-1}X_j(u, a), \varepsilon^{-1}X_{-j, -k}(b)].
\]

Again, we wanted to find \( S_i(u, a) \) such that R0–R9 would hold for them, in particular, Relation R8, i.e. precisely the identity above. So we defined left hand side of that identity as it’s right hand side.

Lemma 19. For any \( i \in \Omega \), \( j \in \Omega \setminus \{\pm i\} \), \( a \in R \)

\[
[\varepsilon^{-1}X_{ij}(a), \varepsilon^{-1}X_{j, -i}(b)] = S_i(0, -\varepsilon_{-i}T_{ab} + \bar{b}T^{-1}a_{\varepsilon_{i}}).
\]
Proof. Fix $t \in \Omega \setminus \{\pm i, \pm j\}$, $x \in \epsilon^{-1}X_{j,-t}(b)$, $y \in \epsilon^{-1}X_{-j,-t}(-\epsilon_{j}^{}X_{e_{j}})$, and $z \in \epsilon^{-1}X_{-t,-1}(1)$. Using C1 we have $y[y^{-1} \cdot z, [x, z]] = \epsilon y^{-1} [z, y, [x, z]] y^{-1} [x, z, y]$. Thus C5 and C6 imply that
\[
\sigma([x^{-1}, y], z) = y[x, [y^{-1}, z]] \cdot (\epsilon[y^{-1}, z, [x, z]] y^{-1} [x, z, y]).
\]
One can obtain that $[x^{-1}, y] \in \epsilon^{-1}X_{t}(0, -(-\epsilon_{j}^{}X_{e_{j}} + bT^{-1}X_{e_{j}}))$ using R0 and R9. Relations R5 and R0 imply that $[y^{-1}, z] \in \epsilon^{-1}X_{ij}(a)$ and $[x, z] \in \epsilon^{-1}X_{j,-t}(b)$. Thus we obtain that $[x, [y^{-1}, z]] = S_{t,-i}(\epsilon^{-1}b^{-1}X_{e_{j}})$ and $[x, z, y] = S_{t,-i}(\epsilon^{-1}X_{e_{j}})$. Now use Lemmas 12, 13 and 15 to finish the proof. 

Lemma 20. For any $i \in \Omega$, $(u, a), (v, b) \in \Sigma$
\[
S_{i}(u, a)S_{i}(v, b) = S_{i}((u, a) \oplus (v, b)).
\]

Proof. Fix $t \in \Omega \setminus \{\pm i\}$ and $x \in \epsilon^{-1}X_{-t,-1}(1)$, $y \in \epsilon^{-1}X_{t}(v, -b)$, $z \in \epsilon^{-1}X_{t}(u, -\overline{a})$. Identity C4 implies that
\[
[z, x][y, x] = [[z, x], y][y, x].
\]
By C2 one has $[[z, x], y] = 1 \cdot S_{t,-i}(\epsilon_{t}B(u, v))$ (use R8 and Lemma 17 and 16). Now 18 and 15 finish the proof. 

Lemma 21. For any $i \in \Omega$, $(u, a), (v, b) \in \Sigma$
\[
[S_{i}(u, a), S_{i}(v, b)] = S_{i}(0, B(u, v) - B(v, u)).
\]

Hint. Use Lemma 20.

Proof of the Main lemma. Lemmata 7–16 imply that Relations R3, R4, R5, R1, R0, R6, R8, R9, R2 and R7 respectively hold for $S_{ij}(a)$ and $S_{i}(u, a)$. Thus, there is a group homomorphism $\sigma$ from $\text{StU}(2n, R, \Sigma)$ on the group spanned on these elements, such that $\sigma(X_{ij}(a)) = S_{ij}(a)$ and $\sigma(X_{i}(u, a)) = S_{i}(u, a)$ (for $n = \infty$ we should use here the universal property of the direct limit). One can see that $\epsilon \sigma = 1_{\text{StU}(2n, R, \Sigma)}$, i.e $\epsilon$ splits. 

4 Schur multiplier of unitary Steinberg group

In this section we will obtain the main results of our paper.

Theorem 1. Let $\text{StU}(2n, R, \Sigma)$ be an odd unitary Steinberg group, where $n \geq 5$ or $n = \infty$. Then $\text{M}(	ext{StU}(2n, R, \Sigma)) = 1$. 

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Proof. $\text{StU}(2n, R, \Sigma)$ is perfect group by Lemma 10 so there is a universal central extension $\pi : U \to \text{StU}(2n, R, \Sigma)$. As we mentioned in the previous section when $n \geq 5$ property $\dagger$ from the Main lemma holds for every central extension of $\text{StU}(2n, R, \Sigma)$, in particular for $\pi$. Thus, $\pi$ is split extension. But its domain $U$ is perfect (see remark after the definition of the universal central extension) so that by Lemma 4, extension $\pi$ is in fact an isomorphism.

Theorem 2. Let $\pi : U \to \text{StU}(8, R, \Sigma)$ be a universal central extension. Then Schur multiplier $M(\text{StU}(8, R, \Sigma))$ coincides with the subgroup of $U$ generated by the elements $\{\pi^{-1}X_{ij}(a), \pi^{-1}X_{kh}(b)\} | i, j, k, h \in \Omega, \text{Card}\{\pm i, \pm j, \pm k, \pm h\} = 8, a, b \in R\}$.

Proof. Denote by $M$ the subgroup generated by $\{\pi^{-1}X_{ij}(a), \pi^{-1}X_{kh}(b)\} | i, j, k, h \in \Omega, \text{Card}\{\pm i, \pm j, \pm k, \pm h\} = 8, a, b \in R\}$. It is contained in the Ker $\pi \subseteq \text{Cent}U$ so it is normal. One has $\pi(M) = 1$, and thus $\pi$ induces the natural morphism $\varpi : U/M \to \text{StU}(8, R, \Sigma)$. Obviously, $\varpi$ is a central extension, its domain is perfect and property $\dagger$ holds for $\varpi$. Thus, by Main lemma and Lemma 4 $\text{StU}(8, R, \Sigma) \cong U/M$.

Theorem 3. Let $n \geq 5$ or $n = \infty$. Suppose that $K_2\text{U}(2n-2, R, \Sigma) \subseteq \text{Cent}(\text{StU}(2n, R, \Sigma))$ (it holds for example when $K_2\text{U}(2n-2, R, \Sigma) \to K_2\text{U}(2n, R, \Sigma)$ is surjective or $n = \infty$, see Lemma 9). Then lemma 5 implies that $K_2\text{U}(2n, R, \Sigma) = M(\text{EU}(2n, R, \Sigma))$.

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