On largest volume simplices and sub-determinants

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Abstract

We show that the problem of finding the simplex of largest volume in the convex hull of \( n \) points in \( \mathbb{Q}^d \) can be approximated with a factor of \( O(\log d)^{d/2} \) in polynomial time. This improves upon the previously best known approximation guarantee of \( d^{(d-1)/2} \) by Khachiyan.

On the other hand, we show that there exists a constant \( c > 1 \) such that this problem cannot be approximated with a factor of \( c^d \), unless \( P = NP \). Our hardness result holds even if \( n = O(d) \), in which case there exists a \( \bar{c}^d \)-approximation algorithm that relies on recent sampling techniques, where \( \bar{c} \) is again a constant.

We show that similar results hold for the problem of finding the largest absolute value of a subdeterminant of a \( d \times n \) matrix.

1 Introduction

Many techniques in convex geometry begin with approximating a geometric shape by a simpler one. The maximum volume ellipsoid, or John ellipsoid (see, e.g., [16, 32]), for example, is a prominent such simplification with many applications in discrete and continuous optimization.

Simplices are, next to ellipsoids, among the most primitive convex sets. We are interested here in the problem of approximating a given \( V \)-polytope by a contained simplex of largest volume. More precisely, we investigate the approximability and hardness of the following problem.

Maximum Volume Simplex (MVS)

Given \( n \) points \( a_1, \ldots, a_n \in \mathbb{Q}^d \), find a simplex of maximum volume that is contained in the convex hull \( \text{conv}\{a_1, \ldots, a_n\} \) of these points.

We assume here, without loss of generality, that the convex hull of the points \( a_1, \ldots, a_n \) is full-dimensional. As is the case for ellipsoids, the largest volume simplex in a convex body has attracted a lot of attention in the computer science and optimization literature, see, e.g., [23, 13, 14, 36, 37].

The volume \( \text{vol}(\Sigma) \) of a full-dimensional simplex \( \Sigma = \text{conv}\{v_0, v_1, v_2, \ldots, v_d\} \subseteq \mathbb{R}^d \) is

\[
\text{vol}(\Sigma) = \frac{|\det(A)|}{d!},
\]

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where $A$ is the matrix with columns $v_1 - v_0, v_2 - v_0, \ldots, v_d - v_0$. Thus MVS can be reduced to $n$ instances of the problem of finding the largest absolute value of a $d \times d$ subdeterminant of a $d \times (n-1)$ matrix. This motivates the second problem that is central to our study.

### Maximum Subdeterminant (MVS)

Given a matrix $A \in \mathbb{Q}^{d \times n}$ of full row-rank, determine a basis $B \subseteq \{1, \ldots, n\}$ of $A$ for which $|\det(A_B)|$ is maximum.

Here a basis of a $d \times n$ matrix $A$ is a maximal subset of the column indices $B \subseteq \{1, \ldots, n\}$ such that the corresponding columns are linearly independent, and $A_B$ is the matrix consisting of the columns indexed by $B$. Khachiyan [23] has shown that there exists a $((1 + \varepsilon) \cdot d)^{(d-1)/2}$ approximation algorithm for MVS, and thus also for MVS, with running time polynomial in $n, d$ and $1/\varepsilon$.

Our main contributions are as follows.

(i) We show that there exists an algorithm for MVS, with running time polynomial in $n, d$ and $1/\varepsilon$, that computes a simplex $\Sigma$ with

$$\text{vol}(\Sigma) \cdot \left(\varepsilon \ln (1 + \varepsilon) d\right)^{d/2} \geq \text{Opt},$$

where Opt denotes the maximum volume of a simplex contained in $\text{conv}\{a_1, \ldots, a_n\}$. To achieve this, we significantly tighten the analysis of Khachiyan’s algorithm [23]. We also show that our analysis is essentially tight by describing instances where the approximation ratio of Khachiyan’s algorithm is $(\alpha \ln(d))^{d/2}$ with $\alpha \geq 0.748$.

(ii) We show that there exists a constant $c > 1$ such that MVS cannot be approximated within a factor of $c^d$, unless $P = NP$. This improves the previous best 1.09-inapproximability of Packer [37].

(iii) Our hardness result (ii) holds for instances with $n = \Theta(d)$. A recent sampling technique [8] immediately yields a $\tilde{c}^d$-approximation for such instances (for another constant $\tilde{c}$), showing that the hardness is essentially tight in this case.

(iv) These results (i), (ii) and (iii) also hold for MSD.

### 1.1 Related work

The literature on topics related to MVS and MSD is extensive. In order to put our results in perspective, we provide an overview of a selection of related papers.

**Approximating convex bodies**

Brieden, Gritzmann and Klee [3] have shown that one can compute a simplex $\Sigma$ in a convex body with $\text{vol}(\Sigma)(d+1)^d \geq \text{Opt}$ if the convex body is equipped with a weak separation oracle. This is similar to the problem of computing a maximum volume ellipsoid. Computing the John-ellipsoid is in general NP-hard, even if $K$ is a $V$-polytope, i.e., a polytope represented by its vertices. However, one can compute an approximation of the John-ellipsoid in polynomial time. Grötschel, Lovász and Schrijver [10] have shown that, if a convex set $K \subseteq \mathbb{R}^d$ is given by a
weak separation oracle, then one can compute in polynomial time an ellipsoid in $K$ that, if scaled by a factor of roughly $d^{3/2}$, contains $K$. Thus there is an approximation algorithm for the problem of computing a maximum volume ellipsoid in convex sets with an approximation guarantee of $d^{3/2}$. When $K$ is an $H$-polytope (i.e., a polytope described through a system of linear inequalities), then a nearly optimal algorithm is known [25].

Variants where the dimension of the solution can be restricted, e.g., finding a maximum volume $j$-dimensional simplex, have been considered for MSD and MSD [14, 3, 37, 5].

**Hardness of approximation**

Packer [37] has shown that MSD is inapproximable within a constant factor smaller than 1.09. This implies the same hardness for MSD, which was shown earlier to be NP-hard by Papadimitriou [38].

Koutis [26] considered the problem of finding the maximum volume $j$-dimensional simplex in a $V$-polytope. Note that one obtains problem MSD when $j = d$. Cevir and Magdon-Ismail [6] considered the following problem that is related to MSD. Given a matrix $A \in \mathbb{Q}^{d \times n}$ and an integer $j$, select a subset $J \subseteq \{1, \ldots, n\}$ of cardinality $j$ such that $\sqrt{\det(A^T_JA_J)}$ is maximized. Note that one obtains problem MSD when $j = d$.

In both cases, the authors show that there exist a constant $c > 1$ and a function $j(d)$ such that it is NP-hard to approximate the respective problem with factor less than $c^{j(d)}$. Here, $j(d)$ is linear in $d$, but with constant dependence strictly less than one, thus these results do not cover the case $j = d$.

**Subdeterminants in optimization and combinatorics**

An integer matrix $A \in \mathbb{Z}^{d \times n}$ is **totally unimodular** if the largest absolute value $\Delta_k$ of a $k \times k$ subdeterminant of $A$ is at most one for each $k \in \{1, \ldots, d\}$. This is the case if and only if the optimum value of MSD is one for the matrix $(A \mid I_d)$, where $I_d$ is the identity matrix of size $d \times d$. Seymour [41] provided a polynomial-time algorithm that tests whether a matrix is totally unimodular. Integer programs $\max \{c^T x: Ax \leq b, x \geq 0, x \in \mathbb{Z}^n\}$ defined by totally unimodular matrices $A \in \mathbb{Z}^{d \times n}$ can be solved in polynomial time. Constraint matrices with small subdeterminant also play an important role in convex (integer) optimization [20].

Subdeterminants are also fundamental in discrepancy theory. The **discrepancy** of a matrix $A \in \mathbb{R}^{n \times d}$ is defined as $\text{disc}(A) = \max_{x\in\{-1,1\}^d} \|Ax\|_\infty$, see, e.g., [32, 17]. The hereditary discrepancy of $A$ is $\max_{S \subseteq [d]} \text{disc}(A_S)$. Very recently there have been several breakthroughs in the field of approximation algorithms related to discrepancy. Bansal [1] has shown how to find a coloring that respects Spencer’s bound [32]. The concept of hereditary discrepancy is closely related to LP rounding [29] and important in the area of approximation algorithms. Rothvoß [39] recently improved the long-standing $O((\log n)^2)$ additive error of Karmarkar and Karp [21] using techniques from discrepancy theory.

The **subdeterminant lower bound for hereditary discrepancy** is $\max_k \sqrt[3]{\Delta_k}$. Recently Matousek [33] has shown that the subdeterminant bound is tight up to a polynomial factor in $\log d$ and $\log n$. In a recent series of papers by Nikolov et al. [33, 34] it was shown how to approximate the hereditary discrepancy, and thus the subdeterminant bound $\max_k \sqrt[3]{\Delta_k}$, within a polynomial factor in $\log d$ and $\log n$. Our result provides a $O(\log d)$-approximation to $\sqrt[3]{\Delta_k}$. It is an interesting problem whether a polynomial time approximation algorithm for the subdeterminant bound with a guarantee that is polynomial in $\log d$ exists.
2 A tight analysis of Khachiyan’s algorithm

We now come to the main algorithmic result of our paper and show the following theorem. Recall that $A_B$ is the matrix corresponding to the columns of $A$ indexed by $B$. We denote the set of bases of $A$ by $\mathcal{B}$.

**Theorem 1.** There exists an algorithm that given a matrix $A \in \mathbb{Q}^{d \times n}$ and $\varepsilon > 0$, identifies a basis $B \subseteq \{1, \ldots, n\}$ such that

$$|\det(A_B)| \cdot \left(\varepsilon \ln((1 + \varepsilon)d)\right)^{d/2} \geq \max_{B' \in \mathcal{B}} |\det(A_{B'})|.$$ 

The algorithm runs in time polynomial in $n, d$, and $1/\varepsilon$. Thus MSD and MVS can be approximated within the factor above.

We first review Khachiyan’s algorithm [23] for MVS and his analysis. Suppose we are given a matrix $A \in \mathbb{Q}^{d \times n}$ of full row rank whose columns are $a_1, \ldots, a_n$ respectively. Consider the symmetric polytope $A = \text{conv}\{\pm a_1, \ldots, \pm a_n\}$. The largest $d \times d$ subdeterminant of $A$, in absolute value, corresponds to the largest volume simplex in $A$ with one vertex being the origin. The algorithm begins by rounding the polytope $A$.

Khachiyan [24] showed that, if $K$ is a symmetric $V$-polytope that is explicitly given by its vertices, then it is possible to compute an ellipsoid $E$ such that $E \subseteq K \subseteq \sqrt{(1 + \varepsilon)d} E$ in time polynomial in the number of vertices of $K$, $d$ and $1/\varepsilon$. This applies to the polytope $A$.

The rounding step is now as follows. Compute an approximation of the ellipsoid $E$ with $E \subseteq A \subseteq \sqrt{(1 + \varepsilon)d} E$. Now, there exists a non-singular matrix $T \in \mathbb{R}^{d \times d}$ such that the image of $E$ is the $d$-dimensional unit ball $B_d$. The rounded instance of MSD is then $T \cdot A$. Clearly, this transformation is approximation preserving, since $\det((T \cdot A)_B) = \det(T) \cdot \det(A_B)$ for every basis $B$ of $A$. Assume we have performed this rounding step. Then

$$B_d \subseteq A \subseteq \sqrt{(1 + \varepsilon)d} B_d$$

holds. From there, the algorithm proceeds in a greedy fashion, see Figure 1.

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**Input:** Matrix $A \in \mathbb{Q}^{d \times n}$ with columns $a_1, \ldots, a_n$ and $\varepsilon > 0$.

1. Round the instance such that $B_d \subseteq A \subseteq \sqrt{(1 + \varepsilon)d} B_d$, where $A = \text{conv}\{\pm a_1, \ldots, \pm a_n\}$ and $B_d$ is the $d$-dimensional unit ball.

2. For $i = 1, \ldots, d$:
   1. Pick $v_i$ as the vector from $\{a_1, \ldots, a_n\}$ with largest norm;
   2. Replace vectors $a_1, \ldots, a_n$ with their projections onto the orthogonal complement of $v_i$.

3. Return the original vectors corresponding to $\{v_1, \ldots, v_d\}$.

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**Figure 1:** Khachiyan’s algorithm for MSD.

Note that $v_1, \ldots, v_d$ correspond to the Gram-Schmidt orthogonalization of the vectors returned by the greedy procedure. Thus, after rounding, the absolute value of the determinant induced by the chosen vectors is $\|v_1\| \ldots \|v_d\|$. 

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2.1 Khachiyan’s analysis

We now review Khachiyan’s analysis showing that his algorithm is a factor \(((1 + \varepsilon)d)^{(d-1)/2}\) approximation algorithm for \(\text{MSD}\). Let us denote the Euclidean lengths of the picked vectors \(v_i\) by \(\rho_i = \|v_i\|\) for \(i = 1, \ldots, d\), and define \(\Delta_{\text{max}} = \max_{B \in \mathcal{S}} |\det(A_B)|\). The claimed bound follows from the following two facts:

(i) \(\rho_d \geq \Delta_{\text{max}}\),
(ii) \(\rho_i \geq 1\) for \(i = 1, \ldots, d\).

Fact (i) is the well known Hadamard bound, see, e.g., [40], while (ii) follows from the fact that the (lower dimensional) unit ball continues to be included in the convex hull of the projection of the input vectors to a lower-dimensional space. The output of the algorithm is the subdeterminant \(\Delta_{\text{out}} = \rho_1 \cdots \rho_d\). By combining (i) and (ii) with \(\rho_1 \leq \sqrt{(1 + \varepsilon)d}\) one obtains \(\Delta_{\text{out}} \cdot \left(\sqrt{(1 + \varepsilon)d}\right)^{d-1} \geq \Delta_{\text{max}}\), which is the claimed approximation ratio.

Theorem 2 (Khachiyan [23]). There is a polynomial-time \(((1 + \varepsilon)d)^{(d-1)/2}\)-approximation algorithm for problems \(\text{MSD}\) and \(\text{MVS}\).

2.2 Improving the analysis

We will now show that the approximation factor can in fact be bounded by \((e \ln((1 + \varepsilon)d))^{d/2}\). To do so, we significantly tighten the upper bound \([\mathbf{3}]\) presented in the analysis above. Key to this improvement is the following observation.

Lemma 3. Let \(a_1, \ldots, a_n \in \mathbb{R}^d\) be the input vectors and let \(v_1, \ldots, v_d\) be the picked vectors in the course of Khachiyan’s algorithm, with lengths \(\rho_1, \ldots, \rho_d\) respectively. Let \(\mathcal{E}\) be an ellipsoid with the following properties:

(a) Each \(v_i\) is on a principal axis of \(\mathcal{E}\).
(b) None of the vectors \(v_i\) is contained in the interior of \(\mathcal{E}\).
(c) There exists \(\alpha > 0\) such that each \(a_i\) is contained in \(\alpha \cdot \mathcal{E}\).

Then \(\Delta_{\text{max}} \leq \alpha^d \rho_1 \cdots \rho_d\).

Proof. We first observe that the largest \(d \times d\) subdeterminant of a matrix whose columns are in \(\mathcal{E}\) is bounded by \(\rho_1 \cdots \rho_d\). To see this, let \(p_i\) be an intersection point of the principal axis that includes \(v_i\), with the boundary of \(\mathcal{E}\). Since \(v_i\) is not in the interior of \(\mathcal{E}\) we have \(\|p_i\| \leq \rho_i\). Let \(T\) be the inverse of the matrix with columns \(p_1, \ldots, p_d\). In fact, \(T\) has rows \(p_1^T/\|p_1\|^2, \ldots, p_d^T/\|p_d\|^2\). The transformation \(x \mapsto Tx\) maps \(\mathcal{E}\) to the unit ball \(\mathcal{B}_d\). By the Hadamard bound, a selection of \(d\) vectors in \(\mathcal{B}_d\) has a determinant of at most 1 in absolute value. This implies that the largest \(d \times d\) subdeterminant of a matrix whose columns are in \(\mathcal{E}\) is bounded by \(\|p_1\| \cdots \|p_d\| \leq \rho_1 \cdots \rho_d\).

Now, since \(\alpha \cdot \mathcal{E}\) contains all the input vectors, we have \(\Delta_{\text{max}} \leq \alpha^d \rho_1 \cdots \rho_d\). \(\square\)

We can now prove our main result.

Proof of Theorem 2. The theorem follows from the existence of an ellipsoid \(\mathcal{E}\) satisfying the conditions of Lemma 3 with \(\alpha = \sqrt{e \ln((1 + \varepsilon)d)}\).
We are done once we have shown that the ellipsoid $E$ first element of $G$ otherwise the projection of $a$ be the set of indices $i$ such that $\rho_i \geq \frac{\rho_1}{\exp((1-1/2))}$. Let $t$ denote the number of such groups. Since $\rho_1/\rho_t \leq \sqrt{(1+\varepsilon)d}$, we have $t \leq \log_{\exp(1/2)} \sqrt{(1+\varepsilon)d} = \ln((1+\varepsilon)d)$. Assume that all groups $G_j$ are non-empty (discarding empty groups will decrease the number of groups and subsequently yield an improved analysis). Let $r_j$ be the largest element of $G_j$ and note that $r_j/\rho_t \leq \sqrt{e}$ for all $\rho_t \in G_j$.

Let us decompose every $x \in \mathbb{R}^d$ into $x = x^{(1)} + \cdots + x^{(t)}$, where $x^{(1)} \in \mathbb{R}^d$ is equal to $x$ in the first $|G_1|$ components and zero otherwise, $x^{(2)}$ is equal to $x$ in the next $|G_2|$ components and zero otherwise, and so forth. Let $\mathcal{E}$ be the ellipsoid $$\mathcal{E} = \left\{ x \in \mathbb{R}^d : \left\| \frac{x^{(1)}}{r_1/\sqrt{e}} \right\|^2 + \cdots + \left\| \frac{x^{(t)}}{r_t/\sqrt{e}} \right\|^2 \leq 1 \right\}.$$ We are done once we have shown that the ellipsoid $\mathcal{E}$, together with $\alpha = \sqrt{e \ln((1+\varepsilon)d)}$, satisfies the conditions [a]–[c] from Lemma 3.

First, the principal axes of $\mathcal{E}$ are the coordinate directions $\{ \lambda \cdot e_i : \lambda \in \mathbb{R} \}$, which implies [a]. Second, suppose that $i \in G_j$ and recall that $v_i = \rho_i \cdot e_i$. Since $\rho_i \geq r_j/\sqrt{e}$, it follows that $v_i$ is not in the interior of $\mathcal{E}$, which implies [b]. Every input column $a$ satisfies $\|a^{(j)}/r_j\| \leq 1$, otherwise the projection of $a$ would have been picked instead of the vector corresponding to the first element of $G_j$. Consequently, each input column $a$ satisfies the constraint $$\left\| \frac{a^{(1)}}{r_1/\sqrt{e}} \right\|^2 + \cdots + \left\| \frac{a^{(t)}}{r_t/\sqrt{e}} \right\|^2 \leq e \cdot t \leq e \ln((1+\varepsilon)d),$$ which implies [c].

### 2.3 Tightness of the analysis

We now provide a family of instances of increasing dimension where Khachiyan’s algorithm achieves a ratio of $(\alpha \ln(d))^{d/2}$, where $\alpha \geq 0.748$ tends to 1 as $d$ grows. Thus, we basically match the upper bound on the approximation ratio given in the previous section.

Let $d \geq 4$ be a power of two. The instances are matrices with $d$ rows of the form

$$A = \begin{bmatrix} D & \sqrt{2-\varepsilon} & DH & E \end{bmatrix}.$$  

Here, $D$ is a $d \times d$ diagonal matrix of the form

$$D = \begin{pmatrix} e^{d-1} & e^{d-2} & \cdots & e^0 \\ \vdots & \vdots & \ddots & \vdots \\ e^{d-1} & e^{d-2} & \cdots & e^0 \end{pmatrix}$$

with $e = d^{\frac{1}{2}}$. Each column of $D$ has Euclidean norm of at most $\sqrt{d}$. The matrix $H$ is a Hadamard matrix, i.e., $H \in \{-1, 1\}^{d \times d}$ with $|\det(H)| = d^{d/2}$ (it is well-known that such
a matrix certainly exists if \( d \) is a power of two). Also the Euclidean norm of each column of \( \frac{\sqrt{c^2-1}}{c} DH \) is bounded by \( \sqrt{d} \). The matrix \( E \) is such that the norm of each column is at most \( c \) and the unit ball is contained in \( \text{conv}(E) \). Clearly, such a matrix \( E \) exists, as \( c > 1 \); see, e.g., [9] for explicit bounds.

Thus the polytope that is generated by the columns of the matrix \( E \) and their negatives is “round”, in the sense that it contains the unit ball and it is contained in the unit ball scaled by \( \sqrt{d} \). We will show below that Khachiyan’s algorithm will output a solution of value at most \( c \cdot |\det D| \). This implies the claim, as

\[
\left| \det \left( \frac{\sqrt{c^2-1}}{c} DH \right) / (c \det(D)) \right| = \left( \frac{c^2-1}{c^2} \right)^{d/2} \frac{d^{d/2}}{c}.
\]

Then using \( x - 1 \geq \ln(x) \), we deduce that the approximation ratio is

\[
d^{-\frac{2(d-1)}{(d-1)}} \left( \frac{d^{1/(d-1)} - 1}{d^{1/(d-1)} - d} \right)^{d/2} \geq \left( \frac{d^{d-2} - \frac{1}{d^{d-1}}}{d^{d-1}} \ln(d) \right)^{d/2} \geq (\alpha \ln(d))^{d/2},
\]

where \( \alpha \) is as required.

Let \( w \) be any column vector of \( \frac{\sqrt{c^2-1}}{c} DH \). The squared norm of the projection of \( w \) into the orthogonal complement of the first \( i \) column vectors of \( D \) (where \( i \in \{0, \ldots, d-1\} \)) is

\[
\frac{c^2-1}{c^2} \sum_{j=0}^{d-i-1} c^{2j} = \frac{c^2-1}{c^2} \cdot \frac{c^{2(d-i)} - 1}{c^2 - 1} = \frac{1}{c^2} \cdot (c^{2(d-i)} - 1) < c^{2(d-i-1)}.
\]

The last term is the squared norm of the \( (i+1) \)-th column vector of \( D \). Thus Khachiyan’s algorithm outputs the first \( d-1 \) columns of \( D \), and in the last step a column of norm at most \( c \) from \( E \). The determinant of the column vector selected by Khachiyan’s algorithm is then at most \( c \times |\det D| \), as claimed above.

### 3 Hardness

We now consider the hardness of approximating \( \text{MVS} \) and \( \text{MSD} \). As mentioned in Section 1.1, the best inapproximability result was due to Packer [37], who proved that \( \text{MVS} \) cannot be approximated with a factor better than 1.09, unless \( P = NP \). In this section we provide a drastic improvement showing that it is NP-hard to approximate \( \text{MVS} \) and \( \text{MSD} \) with a factor \( c^d \), where \( c > 1 \) is an explicit constant. In particular, we show that the result holds for instances where \( n = \Theta(d) \). We will also conclude that the hardness result is best possible for such instances.

Our argument is based on the connection between \( \text{MSD} \) and the following problem.

**Odd Cycle Packing (OCP)**

Given a simple undirected graph, find a maximum family of vertex-disjoint odd cycles.

In fact, given a graph \( G \), let \( A_G \) be the node-edge incidence matrix of \( G \). Then, for every odd cycle \( C \) of \( G \), the square submatrix of \( A_G \) with rows corresponding to the nodes of \( C \) and columns corresponding to the edges of \( G \) has determinant \( \pm 2 \). Therefore any collection
of $k$ vertex-disjoint odd cycles in $G$ determines a submatrix of $A_G$ whose determinant is $2^k$ in absolute value. This implies that $\Delta_{\text{max}}(A_G) \geq 2^{\text{ocp}(G)}$, where $\text{ocp}(G)$ denotes the optimal value of OCP on $G$. Conversely, all non-zero subdeterminants of $A_G$ are powers of two (in absolute value) and indeed one can show that $\Delta_{\text{max}}(A_G) = 2^{\text{ocp}(G)}$ (see, e.g., [14]).

The overall strategy for proving hardness is the following. We first build on a hardness result by Berman and Karpinski [2] on stable sets in 3-regular graphs and show that OCP is NP-Hard to approximate with a factor $\tilde{c}$, where $\tilde{c} > 1$ is an explicit constant. Our second step is to reduce OCP to MSD, using the construction seen above. Hence the constant inapproximability for OCP leads to a $\tilde{c}$-inapproximability for MSD. Last, we reduce MSD to MVS.

Let us remark that OCP is NP-hard even when restricted to planar graphs [17] and, in that case, allows for constant factor approximations [11, 27]. Following the hardness for packing the maximum number of disjoint cycles by Friggstad and Salavatipour [12], we can deduce for OCP a constant hardness under $P \neq \text{NP}$ and a hardness of $O\left(\log^{\frac{1}{2}-\varepsilon} n\right)$ unless $\text{NP} \subseteq \text{ZPTIME}(n^{\text{polylog}(n)})$, where $n$ is the number of nodes of the graph. This result relies on the PCP-theorem. Our construction below yields a weaker hardness for OCP, but it does not rely on the PCP-theorem, it leads to an improved hardness result for the vertex-disjoint triangle packing problem, and it is much simpler to use for constructing a hardness for MSD subsequently. In particular, it enables us to easily calculate the explicit constants in the hardness for MSD.

On the positive side, Kawarabayashi and Reed [22] showed that for general graphs MSD can be approximated within a factor of $\sqrt{n}$.

### 3.1 From stable set in 3-regular graphs to OCP

We now describe our inapproximability result for OCP. We require a result of Berman and Karpinski [2] for maximum stable set on 3-regular graphs. Given a system of $2n$ linear equations modulo 2 with 3 variables per equation, Hästad [18] showed that it is NP-hard to distinguish between instances where there exists a solution satisfying $(1-\varepsilon)2n$ equations and instances where no solution satisfies more than $(1+\varepsilon)n$ equations, for any arbitrarily small $\varepsilon > 0$. Building on this result, Berman and Karpinski [2] gave a polynomial time construction of a 3-regular graph on $176n$ vertices that translates $n$ satisfied equations to a maximum stable set of size at most $97n$ and $2n$ satisfied equations to a maximum stable set of size at least $98n$. Thus, it is NP-Hard to detect if a 3-regular graph on $176n$ vertices has a stable set of size at least $(1-\varepsilon)98n$, or at most $(1+\varepsilon)97n$, for each $\varepsilon > 0$.

Let $G = (V, E)$ be the graph constructed in [2]. Intuitively, we would like to construct a new graph $H$ such that every vertex in $G$ corresponds to a triangle in $H$ and that a stable set in $G$ also corresponds to a packing of triangles in $H$. A first candidate for such a graph $H$ would be the line graph of $G$ (recall that $G$ is 3-regular), but in the line graph we might also create triangles that do not correspond to vertices in $G$.

We solve this issue by slightly changing $G$. Subdivide every edge in $G$ twice, i.e., substitute an edge $\{u, v\}$ by a path $P_{uv} = u, p_1, p_2, v$. Let $G'$ be the obtained graph. Since $G$ has $176n$ vertices, hence $\frac{3}{2}176n = 264n$ edges, $G'$ has $176n + 2 \cdot 264n = 704n$ vertices and $3 \cdot 264n = 792n$ edges. Note that $G'$ is triangle-free.

We now prove that every subdivision of an edge in $G$ augments the stable sets by exactly one vertex. Consider a stable set $S$ in $G$. By choosing $p_2$ when $u \in S$ or $p_1$ otherwise we obtain an induced stable set in $G'$ of size $|S| + 264n$. Conversely, let $S'$ be a stable set in $G'$. Modify $S'$ such that for each $P_{uv}$ not both $u$ and $v$ are in $S'$: if both are in $S'$, then the stable set $S' \setminus \{v\} \cup \{p_2\}$ has the same cardinality and only includes one of $\{u, v\}$. Then we can obtain a stable set in $G$ of size at most $|S'| - 264n$. In particular, a stable set of size $97n$ (resp., $98n$) in
Figure 2: Construction of graph $H$ (solid) from $G'$ (dashed) and definition of $H_{uv}$ (dotted).

$G$ translates into a stable set of size $97n + 264n = 361n$ (resp., $362n$) in $G'$.

Now let $H$ be constructed as follows: starting from the line graph of $G'$, for each $P_{uv}$ add two vertices and connect them as to obtain the graph $H_{uv}$ depicted in Figure 2. The number of vertices in $H$ is $792n + 2 \cdot 264n = 1320n$ and the number of edges is $\frac{1}{2} \cdot 4 \cdot 792n + 2 \cdot 2 \cdot 264n = 2112n$, as every vertex belonging to the line graph of $G'$ has degree four and the two additional vertices in each $H_{uv}$ have degree two.

As $G'$ is triangle-free, there is a one-to-one correspondence between triangles in $H$ and vertices in $G'$. Moreover, two vertices in $G'$ are adjacent if and only if their corresponding triangles in $H$ have a common vertex. Thus a maximum stable set of size $361n$ (resp., $362n$) in $G'$ translates into a maximum number of vertex-disjoint triangles in $H$. It is known that finding the maximum number of vertex-disjoint triangles in a graph is APX-hard [4]. However, no explicit lower bound was known.

**Theorem 4.** It is NP-hard to approximate the maximum number of vertex-disjoint triangles in a graph with a factor of $(\frac{362}{361} - \varepsilon)$ for arbitrarily small constant $\varepsilon > 0$. The result holds even for graphs with maximum degree four.

Now, consider OCP in $H$, i.e., finding the maximum number of vertex-disjoint odd cycles. We prove that there is always an optimal solution consisting of triangles only. Assume the contrary, i.e., the optimal solution includes a cycle $C$ of length at least 5. We distinguish three cases. First, $C$ does not contain any vertex other than those of type $x$ and $y$ (see Figure 2). Then, $C$ must be a triangle. Second, $C$ is fully contained in some $H_{uv}$. One easily checks that also in this case $C$ must be a triangle. Third, $C$ is not contained in any $H_{uv}$. Then $C$ has to pass through the node $z$ of some $H_{uv}$ and hence the solution does not include any triangle in such $H_{uv}$. Substitute $C$ by any of the two triangles in $H_{uv}$ to obtain another optimal solution to OCP. Hence there is an optimal solution to OCP in $H$ only containing triangles. Therefore we have the following hardness result.

**Corollary 5.** It is NP-hard to approximate OCP with a factor of $(\frac{362}{361} - \varepsilon)$ for arbitrarily small constant $\varepsilon > 0$. The result holds even for graphs with maximum degree four.

### 3.2 From OCP to MSD and MVS

Consider the node-edge incidence matrix $A$ of $H$: this is a $1320n \times 2112n$ matrix. From what was argued above, the maximum number of vertex-disjoint odd cycles of $361n$ (resp., $362n$) translates into subdeterminants of size $2^{361n}$ (resp., $2^{362n}$). This allows us to prove a hardness of $2^n$ when the dimension is $d = 1320n$.

**Theorem 6.** It is NP-hard to approximate MSD with a factor of $(2^{1/1320} - \varepsilon)^d$ for arbitrarily small constant $\varepsilon > 0$. The hardness even holds when restricted to node-edge incidence matrices of graphs with maximum degree four.
Proof. Due to the above construction of $H$, it is NP-hard to distinguish between the case when $\text{ocp}(H) \leq (1 + \varepsilon)361n$ and $\text{ocp}(H) \geq (1 - \varepsilon)362n$ for arbitrarily small constant $\varepsilon > 0$. Hence, for $\text{MSD}$ on $\alpha$ we obtain a gap of

$$2^{(1 - \varepsilon)362n - (1 + \varepsilon)361n} = 2^{n(1 - 723\varepsilon)} = \left(2^{1/1320} - \varepsilon''\right)^d,$$

where $\varepsilon' = 723\varepsilon$ and $\varepsilon'' = 2^{1/1320} \left(1 - 2^{-1/1320}\right) > 0$. \qed

We now derive a similar inapproximability result for $\text{MVS}$.

Corollary 7. It is NP-hard to approximate $\text{MVS}$ with a factor of $\left(2^{1/1320} - \varepsilon\right)^d$ for arbitrarily small constant $\varepsilon > 0$.

Proof. We show that, if we were able to solve $\text{MVS}$ on $\alpha$ with an approximation factor $\alpha(d)$, then we would be able to solve $\text{MSD}$ with input $A \in \mathbb{Q}^{d \times n}$ with an approximation factor of $\alpha(d) \cdot (d + 1)^2$. As we proved that $\text{MSD}$ is inapproximable up to a factor $\varepsilon^d$ for some constant $c > 1$ unless $P = NP$, we conclude an inapproximability for $\text{MVS}$ of $\varepsilon^d/(d + 1) = \Omega(\varepsilon^d)$ for any $\varepsilon < c$, unless $P = NP$.

Without loss of generality, let $a_1, \ldots, a_d$ be an optimal solution to $\text{MSD}$. Consider the $\text{MVS}$ instance with input $0, a_1, \ldots, a_n$, and let $S$ be the simplex output by the algorithm. Note that $\text{conv}\{0, a_1, \ldots, a_n\}$ is a feasible solution to $\text{MVS}$, hence $|\det(a_1, \ldots, a_d)|/d! \leq \alpha(d) \cdot \text{vol}(S)$. Now consider the triangulation of $\text{conv}\{S, \{0\}\}$ into simplices $S_1, \ldots, S_{d+1}$ containing the origin and $d$ of the vertices of $S$. We obtain

$$\text{vol}(S) \leq \text{vol}(\text{conv}\{S, \{0\}\}) = \sum_{i=1}^{d+1} \text{vol}(S_i) \leq (d + 1) \text{vol}(S'),$$

where $S'$ is the simplex among $S_1, \ldots, S_{d+1}$ of maximum volume. Note moreover that the submatrix $A'$ associated to the non-zero vertices of $S'$ is a feasible solution to the original $\text{MSD}$ problem. We deduce

$$\frac{|\det(a_1, \ldots, a_d)|}{d!} \leq \alpha(d) \cdot \text{vol}(S) \leq \alpha(d) \cdot (d + 1) \cdot \text{vol}(S') = \alpha(d) \cdot (d + 1) \frac{|\det(A')|}{d!}.$$

Hence, we can output $A'$ and obtain the required approximation. \qed

Remark 1. We remark that the construction of $G'$ and $H$ in Section 3.1 can be improved. In fact, we do not need to subdivide every edge of $G$ twice. It is sufficient to subdivide edges so that for every vertex of $G$, two of its incident edges are subdivided twice. Hence, we can leave a maximum matching of $G$ untouched. Since we can compute a maximum matching in $G$ in polynomial time and every 3-regular graph of $\ell$ vertices has a matching of size $\frac{7}{16} \ell$, we obtain the following slight improvements:

For every $\varepsilon > 0$, it is NP-Hard to approximate

- the maximum number of vertex-disjoint triangles in a graph and $\text{OCP}$ within a factor of $\left(\frac{285}{284} - \varepsilon\right)$;

- $\text{MSD}$ and $\text{MVS}$ with a factor of $\left(2^{1/1012} - \varepsilon\right)^d$. 

\footnote{It is not difficult to prove that $d + 1$ can be replaced by $d$. As this is not crucial for our proof, we leave the details to the interested reader.}
3.3 Tightness for instances with \( n = O(d) \)

If the number of columns is linear in the number of rows, then a better approximation result than that of Theorem 1 is possible for MVS and MSD. The next theorem is a consequence of a recent result of Despande and Rademacher [8]. These authors have shown how to randomly sample from the set of bases of a given matrix \( A \) such that the probability of sampling a particular basis \( B \) is proportional to \( \det(A^T_B A_B) \). In fact, their algorithm is more general as it can also handle \( k \)-subsets of linearly independent columns. Assume now that \( n \leq \alpha \cdot d \) with some \( \alpha \in \mathbb{N} \). Then the number of bases \( |\mathcal{B}_d| \) is bounded by \( \binom{\alpha \cdot d}{d} \leq (e \cdot \alpha)^d \). We deduce

\[
\mathbb{E}(|\det(A_B)|) \cdot (\alpha e)^d \geq \Delta_{\text{max}}.
\]

The claimed result then follows by repeated sampling and picking the largest basis.

**Theorem 8.** If \( n = O(d) \), then there exists a randomized algorithm for MSD with approximation ratio \( \bar{c}^d \) for some constant \( \bar{c} \) that depends on the constant in the \( O \)-notation.

An alternative proof of this assertion relies on the fact that a random point in the zonotope \( \{ y = A \cdot x : 0 \leq x \leq 1 \} \) can be efficiently sampled [10, 28, 30]. It is folklore that a zonotope can be partitioned into parallelepipeds generated by the bases of \( A \), i.e., each parallelepiped can be mapped one-to-one to a basis. We can then identify the parallelepiped where the sampled point resides. This results in sampling each basis with a probability proportional to its determinant. We then obtain an estimation of \( \mathbb{E}(|\det(A_B)|) \), as required.

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