Polyakov’s spin factor and new algorithms for efficient perturbative computations in QCD

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Abstract

Polyakov’s spin factor enters as a weight in the path-integral description of particle-like modes propagating in Euclidean space-times, accounting for particle spin. The Fock-Feynman-Schwinger path integral applied to QCD accommodates Polyakov’s spin factor in a natural manner while, at the same time, it identifies Wilson line (loop) operators as sole agents of interaction dynamics among matter and gauge field quanta. A direct application of such a separation between spin content and dynamics is the emergence of master expressions for the perturbative series involving either open or closed fermionic lines which provide new, comprehensive approaches to perturbative QCD.
1. Introductory remarks. The Fock[1]-Feynman[2]-Schwinger[3] path integral, also known as worldline formalism [4-7], constitutes a tool by which one is able to study a quantum field theoretical system in terms of the propagation of its particle quanta. Polyakov’s path integral [8], on the other hand, amounts to a direct formulation regarding the (Euclidean) space-time propagation of particle-like entities. Its novel ingredient is the so-called spin factor which enters the path integral as a weight accounting for the spin of a particle that traverses a given closed contour. Polyakov’s scheme is distinguished by its geometrical mode of description and serves, via its extension to two-dimensional worldsheets, as a prototype for discussing the propagation of quantum strings.

Inspired by Polyakov’s work we have been able to reformulate [10] the Fock-Feynman-Schwinger path integral for gauge field systems with spin-1/2 matter fields so that the spin factor, associated with the propagation of the latter, explicitly makes its entrance. At the same time, the dynamics operating on the matter particles is carried by Wilson line (loop) operators.

Let us display relevant expressions corresponding to the ‘effective’ action functional \( W[A] \) and the full fermionic propagator \( G(x, y|A) \) in the presence of background gauge fields. They read, respectively, as follows:

\[
W[A] = -\int_{0}^{\infty} \frac{dT}{T} e^{-Tm^{2}} \int_{x(0)=x(T)} D\dot{x}(\tau) e^{-\frac{1}{4} \int_{0}^{T} d\tau \dot{x}^{2}(\tau)} P \exp \left\{ \frac{i}{4} \int_{0}^{T} d\tau \sigma \cdot \omega \right\} \\
\times Tr e^{-\frac{1}{4} \int_{0}^{T} d\tau \dot{x}^{2}(\tau)} P \exp \left[ i g \int_{0}^{T} d\tau \dot{x}(\tau) \cdot A(x(\tau)) \right],
\]

(1)

where \( tr(T_{\gamma}) \) denotes trace over \( \gamma \)-matrix (color) and \( P \) stands for path ordering, and

\[
iG(x, y|A) = \int_{0}^{\infty} dT e^{-Tm^{2}} \int_{x(0)=x}^{x(T)=y} D\dot{x}(\tau) \left[ m - \frac{1}{2} \gamma \cdot \dot{x}(\tau) \right] e^{\frac{-1}{4} \int_{0}^{T} d\tau \dot{x}^{2}(\tau)} \\
\times P \exp \left[ i \frac{1}{4} \int_{0}^{T} d\tau \sigma \cdot \omega \right] P \exp \left[ i g \int_{0}^{T} d\tau \dot{x}(\tau) \cdot A(x(\tau)) \right].
\]

(2)

In the above formulae colour indices have been supressed as \( A_{\mu} \equiv A_{\mu} a T^{a} \), with the generators \( T^{a} \) in the fundamental representation, while \( \sigma \cdot \omega \) actually stands for \( \sigma_{\mu \nu} \omega_{\mu \nu} \), with \( \sigma_{\mu \nu} \equiv \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}] \) and

\[
\omega_{\mu \nu}(x) = \frac{T}{2} (\dot{x}_{\mu} \dot{x}_{\nu} - \dot{x}_{\nu} \dot{x}_{\mu}).
\]

(3)

We normalize our expressions so that, for vanishing gauge potentials, \( W[A] \) is unity and \( G(x, y|A) \) coincides with the free fermionic propagator.

We shall refer to the quantity

\[
\Phi(C) \equiv P \exp \left\{ \frac{i}{4} \int_{0}^{T} d\tau \sigma \cdot \omega \right\},
\]

(4)

Polyakov actually addressed himself to the case of spin-1/2 entities. Subsequently, Korchemsky and Korchemskaya [9] discussed extensions to higher spins.
with $C$ some path parametrized by $\tau \in [0,T]$, as spin factor expression or, simply, spin factor, even though the latter characterization strictly holds for a closed path, as in (1), and is given by $\text{tr}\Phi(C)$. It serves to properly account for the reentry of the contour by a spin-$1/2$ matter particle through the involvement of the geometrical entity of torsion. The exact same expression enters the Green’s function, only now it pertains to an open path $L$. As it turns out [11], its impact is exclusively associated with points at which a four-momentum is conveyed to the fermionic contour through the emission or absorption of a gauge field quantum.

As already mentioned, the sole agent of the dynamics which involves the matter particles is the Wilson line (loop) operator whose corresponding contours coincide with the spin-$1/2$ particle paths. One might view the overall situation as if the $\bar{\psi}\gamma_\mu \psi A^\mu$ term in the field Lagrangian has been re-organized into a geometrical and a dynamical factor operating independently (albeit at the same points) on the aforementioned paths. The full dynamical treatment of gauge fields implicates the corresponding functional integration weighted by the Yang-Mills action along with whatever else it entails (gauge fixing terms, etc.).

In what follows we shall apply the above formalism to perturbative QCD illustrating the role played by the spin factor in arriving at general expressions which provide alternative ways of forming perturbative expansions. Two different viewpoints will be taken. The first, to be considered in the next section, entails carrying the path integral from the very beginning with the spin factor surviving as a momentum dependent quantity. In this way one is able to bring each term of the perturbative expansion into a form that can be directly compared with the Feynman diagrammatic logic, albeit through a different and hopefully more economical organization. A relevant analytic computation to low order, will be conducted, cf. section 3, nevertheless the real usefulness of our results can be fully assessed once numerical methodologies are applied towards the evaluation of higher order perturbative contributions. A second possibility offered by our approach leads to a perturbative expansion which is characterized by its space-time perspective. The basic computational task, in this case, reduces to determining correlators with respect to a particle-like, i.e. one dimensional, action over bosonic and fermionic variables into which the spin factor is incorporated. The relevant analysis will be presented in section 4.

2. Master expressions for Feynman-type perturbative expansions. Relations (1) and (2), as well as similar ones for higher order Green’s functions, amount to a recasting of the given quantum field theoretical system by which nothing of its original content has been lost\(^2\). The question of interest, then, is whether the Fock-Feynman-Schwinger reformulation of the quantum field theory, QCD in particular, presents advantages at a practical level. As already indicated, our focus, in this paper, will lie on perturbation theory algorithms. For closed fermionic contours it is already known, from the work of Strassler [6], that the worldline framework reproduces the string theory-inspired rules of Bern and Kosower [12] which promote efficient perturbative computations in QCD. For open contours, on the other hand, the situation is somewhat more demanding on account of qualitatively different boundary conditions that one needs to accommodate. It is to this case that we shall direct our attention throughout this paper, restricting ourselves to situations where a single fermionic line is

\(^2\)This is so because terms in the field theoretical action that have been integrated over are the ones involving the spin-$1/2$ fields, i.e. they are of Gaussian form.
involved. A point of reference is furnished by expression (2) for the quark propagator in a background of gluonic fields, even though we shall eventually generalize our considerations so as to include gluons as dynamical entities in both virtual and external states. The specific methodology that will characterize our work in this section aims at direct comparisons with the Feynman diagrammatic logic.

Expanding the Wilson (line) exponential in (2), we readily obtain a series of parametric integrations of the form

$$\left(ig\right)^M \int_0^T d\tau_M \ldots \int_0^T d\tau_1 \theta(\tau_M, \ldots, \tau_1) A(x(\tau_M)) \cdot \dot{x}(\tau_M) \ldots A(x(\tau_1)) \cdot \dot{x}(\tau_1)$$

which find themselves immersed into larger expressions. Here, \(\theta(\tau_M, \ldots, \tau_1) \equiv \theta(\tau_M - \tau_{M-1}) \ldots \theta(\tau_2 - \tau_1)\).

Let us now go to the full theory by (functionally) integrating over the gauge potentials and treating the interaction terms in the Yang-Mills Lagrangian as perturbations. Correlators, with respect to the quadratic terms in the action, of a given \(A_\mu(x(\tau_m))\) entering (5) with another gauge field produce gluonic propagators attached to fermionic lines. In our specific case, pairing two \(A_\mu(x(\tau_m))\)'s leads to a situation where a gluon is emitted and subsequently absorbed by the fermionic line. Pairing with an external field operator produces an attached external gluon. Finally, a correlator with a field entering an interaction term in the gluonic Lagrangian density connects the attachment with a three or four point vertex\(^3\). Typical examples are displayed in Fig. 1.

We isolate the following generic expression, after transcribing to momentum representation for the gauge fields,

$$i\Delta^{(M)}(x,y|A) = \left(ig\right)^M \int_0^\infty dTe^{-\frac{1}{4}m^2} \left( \prod_{n=M}^{1} \int_0^T d\tau_n \right) \theta(\tau_M, \ldots, \tau_1) \int_{x(0)=x}^{x(T)=y} \mathcal{D}x(\tau) \left[ m - \frac{1}{2} \gamma \cdot \dot{x}(T) \right]$$

$$\times P \exp \left[ \frac{i}{4} \int_0^T d\tau \sigma \cdot \omega \right] \dot{A}(k_M) \cdot \dot{x}(\tau_M) \ldots \dot{A}(k_1) \cdot \dot{x}(\tau_1)$$

$$\times \exp \left[ -\frac{1}{4} \int_0^T d\tau \dot{x}^2(\tau) - i \sum_{n=1}^{M} k_n \cdot x(\tau_n) \right],$$

which makes its appearance once we expand the Wilson line.

In order to achieve direct comparisons with the Feynman diagram perturbative logic we must first execute the path integral while rendering the spin factor independent from the details of the specific path. This can be achieved by employing the functional derivative operator \([13,14]\)

$$\frac{\delta}{\delta s_{\mu\nu}(t)} \equiv \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} d\sigma \frac{\delta^2}{\delta x_\mu(t + \sigma/2) \delta x_\nu(t - \sigma/2)}.$$ 

One easily determines \([11]\)

$$\frac{\delta}{\delta s_{\mu\nu}(t)} \exp \left[ -\frac{1}{4} \int_0^T d\tau \dot{x}^2(\tau) \right] = \frac{1}{2} \omega_{\mu\nu}(x(t)) \left[ -\frac{1}{4} \int_0^T d\tau \dot{x}^2(\tau) \right].$$

\(^3\)It is important to keep in mind that the fermionic line has a space-time and not a Feynman diagrammatic interpretation.
Introducing the anticommuting \( \xi \) variables by \( \bar{A}_n \cdot \dot{x} (\tau_n) = \int d\xi_n d\tilde{\xi}_n e^{\xi_n \bar{A}_n \cdot \dot{x} (\tau_n) \xi_n} \) and setting \( \hat{k}_\mu (\tau_n) \equiv k_{n\mu} + + g \xi_n \xi_\mu \bar{A}_{n\mu} \frac{\partial}{\partial \tau_n} \) we obtain

\[
i \Delta^{(M)} (x, y | A) = (i g)^M \int_0^\infty dT e^{-T m^2} \left( \prod_{n=M}^1 \int_0^T d\tau_n \right) \theta (\tau_M, \ldots, \tau_1) \left( \prod_{n=M}^0 \int d\xi_n d\tilde{\xi}_n \right) 
\times \int_{x(0) = x}^{x(T) = y} D x (\tau) \left[ m - \frac{1}{2} \gamma \cdot \dot{x} (\tau) \right] e^{\left[ - \frac{i}{2} \int_0^T d\tau \dot{x}^2 (\tau) - i \sum_{n=1}^M \hat{k}_n x (\tau_n) \right] \Phi [\hat{k}],} \]

where

\[
\Phi [\hat{k}] = \exp \left[ i \sum_{n=1}^M \hat{k}_n \cdot x (\tau_n) \right] \times \exp \left[ \frac{i}{2} \int_0^T d\sigma_{\mu \nu} \frac{\delta}{\delta \sigma_{\mu \nu}} \right] \exp \left[ - i \sum_{i=1}^M \hat{k}(\tau_i) \cdot x (\tau_i) \right]
\]

is the spin-factor expression in the "\( \hat{k} \)-representation".

The integral over paths can now be easily performed as the particle-like action functional \( S [x] = \frac{1}{2} \int_0^T d\tau \dot{x}^2 (\tau) - i \sum_{n=1}^M \hat{k}_n \cdot x (\tau_n) \) does not contain non-linear terms. The procedure has been deliberated in Ref. [11] and will be carried out, in this paper, for the more complex 'particle action functionals' that will arise in section 4. Here, we give the final result which reads, in Fourier conjugate space,

\[
i \bar{\Delta}^{(M)} (p, p' | A) = (2 \pi)^4 (i g)^M \delta (p + p' - \sum_{n=1}^M k_n) \int_0^\infty dT e^{-T m^2} \left( \prod_{n=M}^1 \int_0^T d\tau_n \right) \theta (\tau_M, \ldots, \tau_1) 
\times \left( \prod_{n=M}^0 \int d\xi_n d\tilde{\xi}_n \right) \left[ m - i \gamma \cdot p - i g \sum_{n=1}^M \bar{\xi}_n \xi_n \gamma \cdot \bar{A}_n \delta (\tau_n - T) \right] 
\times \exp \left[ - \sum_{n=1}^{M+1} \sum_{m=1}^{M+1} \hat{k}(\tau_n) \cdot \hat{k}(\tau_m) \min (\tau_n, \tau_m) \right] \Phi [\hat{k}] \]

with \( k_{M+1, \mu} \equiv p'_{\mu}, \tau_{M+1} = T \) and \( \bar{A}_{M+1, \mu} \equiv 0_{\mu} \).

The spin factor expression assumes a form, for a given \( M \), which explicitly displays its path independence. It reads as follows

\[
\Phi [\hat{k}] = \text{Pexp} \left[ - i g \sum_{n=1}^M \sigma_{\mu_n \nu_n} f_{\mu_n \nu_n} (\hat{k}) \right] 
\text{def} \equiv 1 + (-i g) \sum_{n=1}^M \sigma_{\mu_n \nu_n} f_{\mu_n \nu_n} + (-i g)^2 \sum_{n=1}^{M-1} \sum_{m=1}^{M-1} \sigma_{\mu_n \nu_n} \sigma_{\mu_m \nu_m} f_{\mu_n \nu_n} f_{\mu_m \nu_m} + \ldots, \]

where\(^{[4]}\)

\[
f_{\mu_n \nu_n} \equiv \bar{\xi}_n \xi_n (\bar{A}_{n\mu_n} k_{n\nu_n} - \bar{A}_{n\nu_n} k_{n\mu_n}) 
- i g \bar{\xi}_n \xi_n \xi_{n+1} \xi_{n+1} \xi_n (\bar{A}_{n\mu_n} k_{n+1, \nu_n} - \mu \leftrightarrow \nu) \delta (\tau_{n+1} - \tau_n). \]

\(^{[4]}\)It is evident that, due to the presence of the \( \xi \)-variables, the sums entering (12) have, at most, \( M \) terms. One should also take note of the fact that the subscript \( n \) implicitly incorporates group indices.
It can be easily verified that $\Phi[k]$ provides the numerator resulting from the string of fermionic propagators for $M$ gluonic insertions on the the quark (world) line, with the corresponding denominators cast in the form $p^2 - m^2 + i\epsilon$.

One concludes that, according to the Fock-Feynman-Schwinger path integral casting of the field theory, a perturbative computation involving gauge field attachments to a given fermionic line can be organized according to the number $M$ of points where the gauge fields “strike” the spin-1/2 particle (world) line(s). For a given Green’s function and to a given perturbative order the aforementioned number $M$ takes more than one value. On the other hand, for fixed $M$ the parametric integrations should take care of all possible rearrangements, including those of colour indices, of the struck points along the contour. An example, corresponding to a “Compton gluon scattering process”, will occupy our attention next.

3. A low order analytic computation. Consider the QCD “Compton” process depicted in Fig. 2. At second perturbative order we have to contend with an $M = 2$ and an $M = 1$ contribution.

For $M = 2$ and given that for incoming/outgoing gluons $\epsilon_i \cdot k_i = 0$, we have

$$\Phi[k] = 1 - ig\bar{\xi}_1\xi_1\gamma \cdot \epsilon_1\gamma \cdot k_1 - ig\bar{\xi}_2\xi_2\gamma \cdot \epsilon_2\gamma \cdot k_2 - g^2\bar{\xi}_2\xi_1[\gamma \cdot \epsilon_2, \gamma \cdot \epsilon_1]\delta(\tau_2 - \tau_1).$$

(14)

Substituting into the expression for $\Delta^{(2)}$, allowing for momentum conservation, choosing $\epsilon_i \cdot p = 0$ and performing the integrations over the Grassmann variables we obtain

$$i\tilde{\Delta}^{(2)}(p, p', \{\epsilon\}) = -g^2(2\pi)^4\delta(p + p' + k_1 + k_2)\int_0^\infty dTe^{-T(m^2 + p'^2)}\int_0^T d\tau_2\int_0^T d\tau_1\theta(\tau_2 - \tau_1)$$

$$\times \exp[-T(m^2 + p'^2) - 2p' \cdot k_2(\tau_2 - \tau_1)]\{(m - i\gamma \cdot p')$$

$$\times [2\gamma \cdot \epsilon_1\gamma \cdot \epsilon_2\delta(\tau_1 - \tau_2) + 2\epsilon_2 \cdot p'\gamma \cdot k_1\gamma \cdot \epsilon_1 - \gamma \cdot k_1\gamma \cdot \epsilon_1\gamma \cdot k_2\gamma \cdot \epsilon_2$$

$$+ i\gamma \cdot \epsilon_2\gamma \cdot k_1\gamma \cdot \epsilon_1\delta(\tau_2 - T)] \}.$$  

(15)

Standard manipulations involving the gamma matrices along with the restoration of (basically group) factors, omitted in $\Delta^{(2)}$, easily lead to the following $M = 2$ contribution to the amplitude

$$A^{(2)\alpha_1\alpha_2}_{Comp} = g^2(2\pi)^4\delta(p + p' + k_1 + k_2)\frac{1}{m + i\gamma \cdot p'}$$

$$\times \left[ \frac{T^{\alpha_2} \gamma \cdot \epsilon_2\gamma \cdot k_1\gamma \cdot \epsilon_1 T^{\alpha_1}}{2p' \cdot k_2} + \frac{T^{\alpha_1} \gamma \cdot \epsilon_1\gamma \cdot k_2\gamma \cdot \epsilon_2 T^{\alpha_2}}{2p' \cdot k_1} \right] \frac{1}{m - i\gamma \cdot p'}. \quad (16)$$

For the $M = 1$ contribution we have

$$\Phi[k] = P\exp[-\frac{i}{2}g\bar{\xi}\sigma_{\mu\nu}(\bar{A}(k)_{\mu}k_{\nu} - \bar{A}(k)_{\nu}k_{\mu})]$$

$$= 1 - \frac{1}{2}g\bar{\xi}[\gamma \cdot \bar{A}(k), \gamma \cdot k]$$

(17)

\[\text{Actually, if more than one fermionic lines are involved, e.g. quark-quark “scattering” process, the total number of attachments to all lines should be counted.}]}
(note that this time the attached gauge field is internal) leading to

\[ i\tilde{\Delta}^{(1)}(p, p'|\tilde{A}) = -g(2\pi)^4\delta(p + p' + k_1 + k_2) \int_0^\infty dT e^{-T(m^2 + p'^2)} \int_0^T d\tau \{(m - i\gamma \cdot p') \]
\[ \times ((p' - p) \cdot \tilde{A}(k) + \frac{1}{2} [\gamma \cdot \tilde{A}(k), \gamma \cdot k]) + i\gamma \cdot \tilde{A}(k)\delta(\tau - T) \} \]

which, after some gamma-matrix algebra routine and once taking into account the presence of free spinor wavefunctions at each end of the fermionic line, gives

\[ i\tilde{\Delta}^{(1)}(p, p'|\{k\}) = -ig \frac{1}{m + i\gamma \cdot p'} \gamma \cdot \tilde{A}(k) \frac{1}{m - i\gamma \cdot p}. \]

The remaining work leading to the amplitude contribution involves standard perturbation theory manipulations related to the ‘merging’ of the attached gluon with the three point Yang-Mills vertex. It now becomes imperative to restore the color indices for the gauge fields, so let us assign index \( a \) to the gluon coming from the Wilson line, corresponding to the attachment on the fermionic contour, and indices \( b, c, d \) to the vertex gluons. Clearly, two of the latter register as incoming/outgoing states hence they are associated with polarization vectors.

In all, the final \( M = 1 \) contribution to the amplitude is

\[ A_{Comp}^{(1)\alpha_1\alpha_2}(p, p') = (2\pi)^4\delta(p + p' + k_1 + k_2)g^2 T^c f^{ca_1a_2} \frac{1}{m + i\gamma \cdot p'} [\gamma_{\rho} \epsilon_{\mu} \epsilon_{2\nu} \]
\[ \times [\delta_{\mu\nu}(k_2 - k_1)_{\rho} - \delta_{\rho\mu}(k_2 + k_1)_{\nu} + \delta_{\mu\nu}(k_2 + 2k_1)_{\nu} \frac{1}{(k_1 + k_2)^2} \frac{1}{m - i\gamma \cdot p}.] \]

which, combined with equ (16), yields the correct final result.

A few comments are in order at this point. First, the present example serves more as an illustration of the fact that the Fock-Feynman-Schwinger recasting of QCD, aided by the presence of the spin factor, provides a viable methodology for conducting perturbative calculations. The real test concerning possible advantages of this approach should come with respect to higher order calculations where a plethora of Feynman diagrams enters. Efforts are under way to probe such a prospect via the employment of numerical methodologies. Second, the fact that purely gluonic perturbative contributions have been treated conventionally in the above application does not preclude the possibility of furnishing the gluonic sector with a worldline description. Of special notice, in this respect, is the work in Ref [15] which, even though conducted in the string context, adopts a space-time mode of description while addressing itself to the gauge field sector. An alternative possibility within the framework of our approach is viable and will be presented next. Its basic feature is that, as a perturbative scheme, it deviates from the Feynman diagram logic by sidestepping the momentum representation and arriving at final expressions which solely involve (one-dimensional) path integrations.

4. A space-time approach to perturbative expansions. We shall now switch policy and leave the task of carrying out the path integral until after we have explicitly dealt with the gauge field sector. Insisting on the perturbative treatment let us consider a given configuration entering a process of interest contributing to some fixed order. Introducing for
each pairing of gauge fields the corresponding space-time correlator, we obtain expressions which no longer involve vector potentials.

Now, the general form of a gauge field correlator in configuration space, in the Feynman gauge, is

\[ < A^a_\mu(x) A^b_\nu(y) > = \delta^{ab} \delta^{\mu\nu} \frac{\Gamma \left( \frac{D}{2} - 1 \right)}{4\pi^{D/2} |x - y|^{D-2}}. \]  \(21\)

Clearly, if \( x = x(\tau) \) and/or \( y = y(\tau) \), then one, or both, arguments of the gauge fields lie on a fermionic particle’s contour, hence the corresponding correlator is directly fed into the path integral. Under these circumstances the latter cannot be performed in the same manner as before. One might say that the induced ‘disappearance’ of the gauge fields from our expressions calls for a readjustment on how the separation between geometry and dynamics is to be effected. Our new strategy will be to incorporate both the spin factor and the gauge field correlator into an overall expression that will define a ‘particle action functional’ with respect to which velocity-velocity correlators are to be computed. The latter arise as remnants of the expansion of the Wilson exponential(s), once the gauge fields have been removed.

In order to form the aforementioned action functional it becomes imperative to get rid of the second derivatives entering the expression for \( \omega_{\mu\nu}(x) \). To confront this task we must first devise an alternative way of displaying the \( \gamma \)-matrix path ordering in the spin factor. To this end we follow Ref [16] and introduce auxiliary anticommuting variables \( \psi_\mu(\tau) \) obeying the relations \( \{ \psi_\mu(\tau), \psi_\nu(\tau') \} = 0 \) and \( \{ \psi_\mu(\tau), \psi_\nu(\tau) \} = \frac{1}{2} \delta_{\mu\nu} \). One can then show that

\[ \text{tr} P \exp \left\{ \frac{i}{4} \int_0^T d\tau \, \sigma \cdot \omega \right\} = \frac{1}{D} \int_{\psi_\mu(0) + \psi_\mu(T) = 0} [d\psi] \exp \left[ \int_0^T d\tau \psi(\tau) \cdot \dot{\psi}(\tau) - \int_0^T d\tau \psi_\mu(\tau) \omega_{\mu\nu} \right] \]  \(22\)

relevant for closed and

\[ P \exp \left\{ \frac{i}{4} \int_0^T d\tau \, \sigma \cdot \omega \right\} = \exp \left( i \gamma \cdot \frac{\partial}{\partial \lambda} \right) \int_{\psi_\mu(0) + \psi_\mu(T) = \lambda_\mu} [d\psi] \exp \left[ \int_0^T d\tau \psi(\tau) \cdot \dot{\psi}(\tau) - \int_0^T d\tau \psi_\mu(\tau) \omega_{\mu\nu} + \psi(T) \cdot \psi(0) \right]_{\lambda = 0} \]  \(23\)

relevant for open particle contours, respectively.

We shall look upon the factor \( \exp \left[ \int_0^T d\tau \psi(\tau) \cdot \dot{\psi}(\tau) \right] \) entering (23) as furnishing a fermionic sector to the ‘particle action’ thereby formulating a ‘super particle’ description. For future reference we note that

\[ < \psi_\mu(t) \psi_\nu(t') >_{\psi,\lambda} = \int_{\psi_\mu(0) + \psi_\mu(T) = \lambda_\mu} [d\psi] \exp \left[ \int_0^T d\tau \psi(\tau) \cdot \dot{\psi}(\tau) + \psi(0) \cdot \psi(T) \right] \psi_\mu(t) \psi_\nu(t') \\
= -\frac{1}{4} \delta_{\mu\nu} \text{sign}(t - t') \lambda_\mu \lambda_\nu \]  \(24\)

with \( \text{sign} 0 = 0 \).

The next step is to write

\[ \int_0^T d\tau \dot{\psi}_\mu(\tau) \omega_{\mu\nu} = -T \int_0^T d\tau (\psi \cdot \dot{x})(\psi \cdot \dot{x}) + T \int_0^T d\tau (\psi \cdot \dot{x})(\psi \cdot \dot{x}) + \frac{T}{4} \int_0^T d\tau \dot{x} \cdot \dot{x} \]  \(25\)
whose usefulness will become evident once we specify the correlators $<\dot{x}_\mu(t)\dot{x}_\nu(t')>$ and $<\psi_\mu(t)\psi_\nu(t')>$ in the limit where the two arguments coincide. To this end we define

$$<\dot{x}_\mu(t)\dot{x}_\nu(t)>_{x} \overset{def}{=} -\lim_{\epsilon \to 0} \frac{1}{T} \int_{-\epsilon}^{\epsilon} d\sigma \frac{\delta^2}{\delta J_\mu(t + \frac{\sigma}{2}) \delta J_\nu(t - \frac{\sigma}{2})} \times <e^{i \int_{0}^{T} d\sigma J(\tau) \dot{x}(\tau)}>_{x} |_{J=0} = \frac{2}{T} \delta_{\mu\nu},$$

(26)

reflecting the fact that we are integrating over paths with $|\dot{x}| = \text{const}$ and, similarly, $(\eta_\mu$ and $\theta_\mu$ anticommuting sources)

$$<\psi_\mu(t)\psi_\nu(t)>_{\psi} \overset{def}{=} -\lim_{\epsilon \to 0} \frac{1}{T} \int_{-\epsilon}^{\epsilon} d\sigma \frac{\delta^2}{\delta \eta_\mu(\tau + \frac{\sigma}{2}) \delta \theta_\nu(\tau - \frac{\sigma}{2})} \times <e^{i \int_{0}^{T} d\sigma (\eta \psi + \theta \dot{x})}>_{\psi} |_{\eta=\theta=0} = \frac{1}{2T} \delta_{\mu\nu}$$

(27)

whereupon we are led to the substitution rules$^6$ $\dot{x}_\mu(t)\dot{x}_\nu(t) \to 2 \frac{2}{T} \delta_{\mu\nu}, \psi_\mu(t)\psi_\nu(t) \to \frac{1}{2T} \delta_{\mu\nu}$. It, thereby, follows that (25) assumes the form

$$\int_{0}^{T} d\tau \psi_\mu \psi_\nu \omega_{\mu\nu} = -T \int_{0}^{T} d\tau (\psi \dot{x})(\dot{x} \psi) + \text{const.}$$

(28)

Designating $\frac{1}{\sqrt{T}} \psi_5(\tau) \equiv \psi(\tau) \cdot \dot{x}(\tau)$ and enforcing this definition through a delta-function constraint (enter Grassmann path variable $\chi(\tau)$) we find$^7$

$$P \exp \left\{ \frac{i}{4} \int_{0}^{T} d\tau \sigma \cdot \omega \right\} = \exp \left( i \gamma \cdot \frac{\partial}{\partial \lambda} \right) \int_{\psi_\mu(0)+\psi_\mu(T)=\lambda_\mu} [d\psi(\tau)] [d\psi_5(\tau)] [d\chi(\tau)]

\times \exp \left\{ \int_{0}^{T} d\tau \psi \dot{x} + \int_{0}^{T} d\tau \psi_5 \dot{x}_5 + \frac{i}{\sqrt{T}} \int_{0}^{T} d\tau \chi(\tau) \psi_5(\tau)

- i \int_{0}^{T} d\tau \chi(\tau) (\psi(\tau) \cdot \dot{x} + \psi(T) \cdot \psi(0)) \right\}_{\lambda=0}.$$

(29)

For the purpose of computing velocity-velocity correlators we need to extract from the expression on the right hand side that part which is relevant to forming the ‘bosonic sector’ of the ‘particle action’. This is furnished by the term $-i \int_{0}^{T} d\tau J \cdot \dot{x}$, where

$$J_\mu(\tau) \equiv \chi(\tau) \psi_\mu(\tau).$$

(30)

A further contribution to the ‘bosonic particle action’ comes from the propagator expression. Exponentiating the denominator entering (21), via the employment of (Feynman) auxiliary integration variables $\alpha_i$ we obtain

$$S_2[x] = \frac{1}{4} \int_{0}^{T} d\tau \dot{x}^2 + i \int_{0}^{T} d\tau J(\tau) \cdot \dot{x}(\tau) + \sum_{i=1}^{M} \alpha_i [x(\tau_a) - x(\tau_b)]^2$$

(31)

$^6$Note that only the $\int_{0}^{T} d\tau \dot{x}^2$ part of the bosonic particle action contributes to (26) so any additional terms are inessential as far as our definition of velocity-velocity correlators at equal times is concerned.

$^7$From hereon we restrict our attention to open line expressions.
representing all double-ended gauge field attachments and

\[ S_1[x] = \frac{1}{4} \int_0^T d\tau \dot{x}^2 + i \int_0^T d\tau J(\tau) \cdot \dot{x}(\tau) + \sum_{i=1}^M \alpha_i [x(\tau_i) - z_i]^2 \]  

for single-end attachments to the matter field contour.

Clearly, unattached gauge field propagators are ‘averaged’ with a ‘particle action’ that does not contain the third term. Moreover, one should not forget that there are fermionic path integrals which need to be performed. In what follows we shall focus our efforts on determining velocity-velocity correlators for each of the above two action functionals.

Consider, first, the case of a single attachment, i.e. functional \( S_1[x] \). The ‘classical equations of motion’ read

\[ \ddot{x}_\mu(\sigma) = -2i\dot{J}_\mu(\sigma) + 4 \sum_{i=1}^M \alpha_i [x(\tau_i) - z_i] \delta(\tau_i - \sigma). \]  

(33)

Making a variable change specified by \( x_\mu(\sigma) = w_\mu(\sigma) + \frac{(y-x)_\mu}{T} \sigma + x_\mu \), which obeys the, simpler, boundary conditions \( w_\mu(0) = w_\mu(T) = 0 \) the equations of motion assume the form

\[ \ddot{w}_\mu(\sigma) - b(\sigma) w_\mu(\sigma) = f_\mu(\sigma), \]  

(34)

where \( f_\mu(\sigma) \equiv -2i\dot{J}_\mu(\sigma) + 4 \sum_{i=1}^M \alpha_i [\frac{(y-x)_\mu}{T} \tau_i + x_\mu - z_{i\mu}] \delta(\tau_i - \sigma) \) and \( b(\sigma) \equiv 4 \sum_{i=1}^M \alpha_i \delta(\tau_i - \sigma) \).

Introducing the Green’s function \( K^{(N)}(\sigma, \sigma') \) by

\[ \left[ \frac{d^2}{d\sigma^2} - b(\sigma) \right] K^{(N)}(\sigma, \sigma') = \delta(\sigma - \sigma'), \]  

(35)

obeying boundary conditions \( K^{(N)}(0, \sigma') = K^{(N)}(T, \sigma') = 0 \), we write

\[ w_\mu(\sigma) = \int_0^T d\sigma' K^{(N)}(\sigma, \sigma') f_\mu(\sigma'). \]  

(36)

Manipulations involved in the determination of the Green’s function \( K^{(N)}(\sigma, \sigma') \) proceed along the lines employed in Ref [17], adopted also in Ref [11], leading to the result

\[ K^{(N)}(\sigma, \sigma') = -\Delta(\sigma, \sigma') + \sum_{i,j=1}^N \hat{\Delta}(\sigma, \tau_i) \hat{\Delta}(\tau_i, \tau_j) \hat{\Delta}(\tau_j, \sigma'), \]  

(37)

where

\[ \Delta(\sigma, \sigma') = - \langle \sigma' | \partial^{-2} | \sigma \rangle = \frac{\theta(T - \sigma') - \theta(\sigma' - \sigma) + \theta(T - \sigma) - \theta(\sigma - \sigma')}{T} \]  

(38)

while the \( \hat{\Delta} \)’s are related to the \( \Delta \)’s through a multiplication of the latter by a factor \( \sqrt{\alpha_i} \) for each \( \tau_i \) in the argument.

\[ ^8 \text{Recall that the corresponding conditions for } x_\mu(\sigma) \text{ are } x_\mu(0) = x_\mu \text{ and } x_\mu(T) = y_\mu. \]
Consider, finally, the velocity-velocity correlator
\[
< \dot{x}_\mu(t_1) \dot{x}_\nu(t_2) > S_1 = -\frac{\delta^2}{\delta J_\mu(t_1) \delta J_\nu(t_2)} \int_{x(0)=x, x(T)=y} \mathcal{D}x(\tau) e^{-S_1[x]}
\]
\[
= -\mathcal{N}(\alpha) \frac{\delta^2}{\delta J_\mu(t_1) \delta J_\nu(t_2)} e^{\exp\{-S_1[x]\}}.
\]  

where
\[
\mathcal{N}(\alpha) = \int_{x(0)=x(T)=0} \mathcal{D}x(\tau) e^{\exp\{-\frac{1}{4} \int_0^T d\tau \dot{x}^2 - \sum_{i=1}^N \alpha_i x^2(\tau_i)\}} = \frac{1}{(4\pi T)^2} \det^{-2N} (1 + 4\hat{\Delta}).
\]  

Straight forward manipulations lead to the result
\[
< \dot{x}_\mu(t_1) \dot{x}_\nu(t_2) > S_1 = \mathcal{N}(\alpha) \left[ \dot{x}_\mu^{cl}(t_1) \dot{x}_\nu^{cl}(t_2) - 2\delta_{\mu\nu} \frac{\partial^2}{\partial t_1 \partial t_2} K^{(N)}(t_1, t_2) \right] e^{-S_1[x^{cl}]} \]  

with all the ‘classical’ quantities readily calculable.

The case with \( S_2[x] \), even though technically somewhat more complex, proceeds along parallel lines. It becomes convenient to introduce \([17]\) the kernel
\[
B(\sigma_1, \sigma_2) \equiv \sum_{i=1}^N \alpha_i [\delta(\sigma_1 - \tau_{a_i}) - \delta(\sigma_1 - \tau_{b_i})] [\delta(\sigma_2 - \tau_{a_i}) - \delta(\sigma_2 - \tau_{b_i})]
\]  

which allows us to cast the third term on the right hand side of equ (31) in the form \( \int_0^T d\sigma_1 \int_0^T d\sigma_2 B(\sigma_1, \sigma_2) x(\sigma_1) \cdot x(\sigma_2) \). Accordingly, the relevant Green’s function \( \Delta^{(N)}(\sigma, \sigma') \) now satisfies the equation
\[
\int_0^T d\sigma_1 \left[ \delta(\sigma - \sigma_1) \frac{\partial^2}{\partial \sigma_1^2} - 4B(\sigma_1, \sigma) \right] \Delta^{(N)}(\sigma_1, \sigma') = \delta(\sigma - \sigma'),
\]  

where
\[
\Delta^{(N)}(\sigma, \sigma') = -\Delta(\sigma, \sigma') + \sum_{i,j=1}^N [\hat{\Delta}(\sigma, \tau_{a_i}) - \hat{\Delta}(\sigma, \tau_{b_i})] [\hat{D}_{ij} \hat{\Delta}(\sigma, \tau_{a_j}) - \hat{\Delta}(\sigma, \tau_{b_j})]
\]  

with
\[
\hat{D}_{ij} \equiv \hat{\Delta}(\tau_{a_i}, \tau_{a_j}) + \hat{\Delta}(\tau_{b_i}, \tau_{b_j}) - \hat{\Delta}(\tau_{a_i}, \tau_{b_j}) - \hat{\Delta}(\tau_{b_i}, \tau_{a_j}).
\]  

The final result for the velocity-velocity correlator acquires the corresponding form
\[
< \dot{x}_\mu(t_1) \dot{x}_\nu(t_2) > S_2 = \hat{\mathcal{N}}(\alpha) \left[ \dot{x}_\mu^{cl}(t_1) \dot{x}_\nu^{cl}(t_2) - 2\delta_{\mu\nu} \frac{\partial^2}{\partial t_1 \partial t_2} \Delta^{(N)}(t_1, t_2) \right] e^{-S_2[x^{cl}]} \]

where
\[
\hat{\mathcal{N}}(\alpha) = \frac{1}{(4\pi T)^2} \det^{-2N} (1 + 4\hat{D}).
\]  

Path integrals over the fermionic variables are much simpler, as can be witnessed from the relevant manipulations performed in Ref [11]. The basic result is displayed by equ (24).
5. **Summary.** We have applied the Fock-Feynman-Schwinger path integral to perturbative expansions in QCD for processes involving an open fermionic line. The novel feature of the formalism is that it contains a spin factor and a dynamical factor (Wilson line) whose distinguishable roles facilitate the emergence of comprehensive expressions for the perturbative series. Two alternative approaches have been discussed. According to the first one the path integral is performed right away while the spin factor is recast into a momentum representation form which communicates with the gauge sector. In this way one achieves direct comparison with the Feynman diagrammatic expansion as the gauge fields enter through conventional momentum space propagators while the spin factor expression provides the contribution coming from a fermionic line that has a given number of gauge field attachments. All this enters a master expression with parametric integrations which account for all possible rearrangements of the points on the fermionic line where the attachments occur. The second approach has as its starting point the removal (in pairs) of the gauge fields in favor of correlator expressions in a space-time representation. The spin factor, on the other hand, is now transcribed into a form that contributes to a superparticle-like (one-dimensional) action given that it involves anticommuting coordinates as well. The result of this procedure is the emergence of a space-time picture for the perturbative series which boils down to computing correlators of the type $<\dot{x}_\mu(t)\dot{x}_\nu(t')>_x$ and $<\psi_\mu(t)\psi_\nu(t')>_\psi$ in a particle-based representation of the theory (QCD). All that remains is to perform integrations over the Feynman parameters. Non-trivial applications of the proposed scheme are currently in pursuit.

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Figure Captions

Fig. 1: Gluonic field attachments on a fermionic (world)line. The Wilson line, which carries the dynamics between matter and gauge field quanta in the Fock-Feynman-Schwinger path integral approach to QCD, is responsible for the emergence of master expressions organized according to the number $M$ of such attachments. The spin factor, on the other hand, provides the numerator of the string of fermionic propagators corresponding to a given number of insertions.

Fig. 2: Configurations entering the ‘QCD Compton Scattering amplitude’, to the lowest order, organized according to number of attachments onto the fermionic (world)line.
