Some Liouville theorems and applications

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Dedicated to Haim Brezis with high respect and friendship

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Abstract

We give exposition of a Liouville theorem established in [6] which is a novel extension of the classical Liouville theorem for harmonic functions. To illustrate some ideas of the proof of the Liouville theorem, we present a new proof of the classical Liouville theorem for harmonic functions. Applications of the Liouville theorem, as well as that of earlier ones in [5], can be found in [6, 7] and [9].

The Laplacian operator $\Delta$ is invariant under rigid motions: For any function $u$ on $\mathbb{R}^n$ and for any rigid motion $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\Delta(u \circ T) = (\Delta u) \circ T.$$
The following theorem is classical:

\[ u \in C^2, \quad \Delta u = 0 \text{ and } u > 0 \text{ in } \mathbb{R}^n \text{ imply that } u \equiv \text{ constant.} \quad (1) \]

In this note we present a Liouville theorem in [6] which is a fully nonlinear version of the classical Liouville theorem (1).

Let \( u \) be a positive function in \( \mathbb{R}^n \), and let \( \psi : \mathbb{R}^n \cup \{ \infty \} \to \mathbb{R}^n \cup \{ \infty \} \) be a Möbius transformation, i.e. a transformation generated by translations, multiplications by nonzero constants and the inversion \( x \to x/|x|^2 \). Set

\[ u_\psi := |J_\psi|^{\frac{n-2}{2n}} (u \circ \psi), \]

where \( J_\psi \) is the Jacobian of \( \psi \).

It is proved in [3] that an operator \( H(u, \nabla u, \nabla^2 u) \) is conformally invariant, i.e.

\[ H(u_\psi, \nabla u_\psi, \nabla^2 u_\psi) \equiv H(u, \nabla u, \nabla^2 u) \circ \psi \]

holds for all positive \( u \) and all Möbius \( \psi \), if and only if \( H \) is of the form

\[ H(u, \nabla u, \nabla^2 u) \equiv f(\lambda(A^u)) \]

where

\[ A^u := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I, \]

\( I \) is the \( n \times n \) identity matrix, \( \lambda(A^u) = (\lambda_1(A^u), \cdots, \lambda_n(A^u)) \) denotes the eigenvalues of \( A^u \), and \( f \) is a function which is symmetric in \( \lambda = (\lambda_1, \cdots, \lambda_n) \).

Due to the above characterizing conformal invariance property, \( A^u \) has been called in the literature the conformal Hessian of \( u \). Since

\[ \sum_{i=1}^{n} \lambda_i(A^u) = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Delta u, \]

Liouville theorem (1) is equivalent to

\[ u \in C^2, \quad \lambda(A^u) \in \partial \Gamma_1 \text{ and } u > 0 \text{ in } \mathbb{R}^n \text{ imply that } u \equiv \text{ constant,} \quad (2) \]

where

\[ \Gamma_1 := \{ \lambda \mid \sum_{i=1}^{n} \lambda_i > 0 \}. \]
Let
\[ \Gamma \subset \mathbb{R}^n \]
be an open convex symmetric cone with vertex at the origin \( \Gamma \subset \Gamma_1 \) satisfying
\[ \Gamma_n := \{ \lambda \mid \lambda_i > 0, 1 \leq i \leq n \} \subset \Gamma \subset \Gamma_1. \]
Examples of such \( \Gamma \) include those given by elementary symmetric functions. For \( 1 \leq k \leq n \), let
\[ \sigma_k(\lambda) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \]
be the \( k \)-th elementary symmetric function and let \( \Gamma_k := \{ \lambda \in \mathbb{R}^n \mid \sigma_1(\lambda), \cdots, \sigma_k(\lambda) > 0 \} \), which is equal to the connected component of \( \{ \lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0 \} \) containing the positive cone \( \Gamma_n \), satisfies (3) and (4).

For an open subset \( \Omega \) of \( \mathbb{R}^n \), consider
\[ \lambda(A^u) \in \partial \Gamma, \quad \text{in } \Omega. \]
The following definition of viscosity super and sub solutions of (5) has been given in [6].

**Definition 1** A positive continuous function \( u \) in \( \Omega \) is a viscosity subsolution [resp. supersolution] of (5) when the following holds: if \( x_0 \in \Omega, \psi \in C^2(\Omega), (u - \psi)(x_0) = 0 \) and \( u - \psi \leq 0 \) near \( x_0 \) then
\[ \lambda(A^\psi(x_0)) \in \mathbb{R}^n \setminus \Gamma. \]
[resp. if \( (u - \psi)(x_0) = 0 \) and \( u - \psi \geq 0 \) near \( x_0 \) then \( \lambda(A^\psi(x_0)) \in \Gamma \).]
We say that \( u \) is a viscosity solution of (5) if it is both a viscosity supersolution and a viscosity subsolution.

**Remark 1** If a positive \( u \) in \( C^{1,1}(\Omega) \) satisfies \( \lambda(A^u) \in \partial \Gamma \) a.e. in \( \Omega \), then it is a viscosity solution of (5).

Here is the Liouville theorem.

**Theorem 1** ([6]) For \( n \geq 3 \), let \( \Gamma \) satisfy (3) and (4), and let \( u \) be a positive locally Lipschitz viscosity solution of
\[ \lambda(A^u) \in \partial \Gamma \quad \text{in } \mathbb{R}^n. \]
Then \( u \equiv u(0) \) in \( \mathbb{R}^n \).
Remark 2 It was proved by Chang, Gursky and Yang in [1] that positive $C^{1,1}(\mathbb{R}^4)$ solutions to $\lambda(A^u) \in \partial \Gamma_2$ are constants. Aobing Li proved in [2] that positive $C^{1,1}(\mathbb{R}^3)$ solutions to $\lambda(A^u) \in \partial \Gamma_2$ are constants, and, for all $k$ and $n$, positive $C^3(\mathbb{R}^n)$ solutions to $\lambda(A^u) \in \partial \Gamma_k$ are constants. The latter result for $C^3(\mathbb{R}^n)$ solutions is independently established by Sheng, Trudinger and Wang in [8]. Our proof is completely different.

Remark 3 Writing $u = w - n^{-2}$, then

$$A^u \equiv A_w := w \nabla^2 w - \frac{1}{2} |\nabla w|^2 I.$$ 

Theorem 1, with $\lambda(A^u) \in \partial \Gamma$ being replaced by $\lambda(A_w) \in \partial \Gamma$, holds for $n = 2$ as well. See [6].

In order to illustrate some of the ideas of our proof of Theorem 1 in [6], we give a new proof of the classical Liouville theorem (1). We will derive (1) by using the

**Comparison Principle for $\Delta$:** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ containing the origin $0$. Assume that $u \in C^2_{\text{loc}}(\Omega \setminus \{0\})$ and $v \in C^2(\Omega)$ satisfy

$$\Delta u \leq 0 \quad \text{in} \quad \Omega \setminus \{0\} \quad \text{and} \quad \Delta v \geq 0 \quad \text{in} \quad \Omega,$$

and

$$u > v \quad \text{on} \quad \partial \Omega.$$

Then

$$\inf_{\Omega \setminus \{0\}} (u - v) > 0.$$

It is easy to see from this proof of the Liouville theorem (1) that the following Comparison Principle for locally Lipschitz viscosity solutions of (5), established in [5, 6], is sufficient for a proof of Theorem 1.

**Proposition 1** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ containing the origin $0$, and let $u \in C^{0,1}_{\text{loc}}(\Omega \setminus \{0\})$ and $v \in C^{0,1}(\Omega)$. Assume that $u$ and $v$ are respectively positive viscosity supersolution and subsolution of (5), and

$$u > v > 0 \quad \text{on} \quad \partial \Omega.$$

Then

$$\inf_{\Omega \setminus \{0\}} (u - v) > 0.$$
For the proof of Proposition 1 and Theorem 1, see [5, 6]. In this note, we give the
Proof of Liouville theorem (1) based on the Comparison Principle for $\Delta$. Let

$$v(x) := \frac{1}{2} \min_{|y|=1} u(y) |x|^{2-n}, \quad v_1(x) := \frac{1}{|x|^{n-2}} v \left( \frac{x}{|x|} \right), \quad u_1(x) := \frac{1}{|x|^{n-2}} u \left( \frac{x}{|x|} \right).$$

Since $u_1$ and $v_1$ are still harmonic functions, an application of the Comparison Principle for $\Delta$ on $\Omega := \text{the unit ball}$ yields

$$\lim \inf_{|y| \to \infty} |y|^{n-2} u(y) > 0. \quad (7)$$

**Lemma 1** For every $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that

$$u_{x,\lambda}(y) := \frac{\lambda^{n-2}}{|y-x|^{n-2}} u \left( x + \frac{\lambda^2 (y-x)}{|y-x|^2} \right) \leq u(y) \quad \forall \ 0 < \lambda < \lambda_0(x), |y-x| \geq \lambda. \quad (8)$$

**Proof.** Without loss of generality we may take $x = 0$, and we use $u_{\lambda}$ to denote $u_{0,\lambda}$. By the positivity and the Lipschitz regularity of $u$, there exists $r_0 > 0$ such that

$$r^{\frac{n-2}{2}} u(r,\theta) < s^{\frac{n-2}{2}} u(s,\theta), \quad \forall \ 0 < r < s < r_0, \ \theta \in S^{n-1}.$$

The above is equivalent to

$$u_\lambda(y) < u(y), \quad 0 < \lambda < |y| < r_0. \quad (8)$$

We know from (7) that, for some constant $c > 0$,

$$u(y) \geq c |y|^{2-n}, \quad |y| \geq r_0.$$

Let

$$\lambda_0 := \left( \frac{c}{\max_{|z| \leq r_0} u(z)} \right)^{\frac{1}{n-2}}.$$ 

Then

$$u_\lambda(y) \leq \left( \frac{\lambda_0}{|y|} \right)^{n-2} (\max_{|z| \leq r_0} u(z)) \leq c |y|^{2-n} \leq u(y), \quad \forall \ 0 < \lambda < \lambda_0, |y| \geq r_0. \quad (9)$$

It follows from (8) and (9) that

$$u_\lambda(y) \leq u(y), \quad \forall \ 0 < \lambda < \lambda_0, |y| \geq \lambda.$$ 

Lemma 1 is established.

Because of Lemma 1, we may define, for any $x \in \mathbb{R}^n$ and any $0 < \delta < 1$, that

$$\bar{\lambda}_\delta(x) := \sup \{ \mu > 0 \mid u_{x,\lambda}(y) \leq (1 + \delta) u(y), \ \forall \ 0 < \lambda < \mu, |y-x| \geq \lambda \} \in (0, \infty].$$
Lemma 2 For any \( x \in \mathbb{R}^n \) and any \( 0 < \delta < 1 \), \( \bar{\lambda}_\delta(x) = \infty \).

Proof. We prove it by contradiction. Suppose the contrary, then, for some \( x \in \mathbb{R}^n \) and some \( 0 < \delta < 1 \), \( \bar{\lambda}_\delta(x) < \infty \). We may assume, without loss of generality, that \( x = 0 \), and we use \( u_\lambda \) and \( \bar{\lambda}_\delta \) to denote respectively \( u_{0,\lambda} \) and \( \bar{\lambda}_\delta(0) \). Since the harmonicity is invariant under conformal transformations and multiplication by constants, and since

\[
\inf_{0 < |y| < \bar{\lambda}_\delta} [(1 + \delta)u_{\bar{\lambda}_\delta}(y) - u(y)] > 0.
\]

Namely, for some constant \( c > 0 \),

\[
(1 + \delta)u(y) - u_{\bar{\lambda}_\delta}(y) \geq c|y|^{2-n}, \quad \forall |y| \geq \bar{\lambda}_\delta.
\]

By the uniform continuity of \( u \) on the ball \( \{z \mid |z| < \bar{\lambda}_\delta\} \), there exists \( 0 < \epsilon < \bar{\lambda}_\delta \) such that for all \( \bar{\lambda}_\delta \leq \lambda \leq \bar{\lambda}_\delta + \epsilon \), and for all \( |y| \geq \lambda \), we have

\[
(1 + \delta)u(y) - u_\lambda(y) \geq (1 + \delta)u(y) - u_{\bar{\lambda}_\delta}(y) + [u_{\bar{\lambda}_\delta}(y) - u_\lambda(y)]
\]

\[
\geq c|y|^{2-n} - |y|^{2-n}|\lambda|^{n-2} - \bar{\lambda}_\delta^{n-2}u(\frac{\lambda^2y}{|y|^2}) - \bar{\lambda}_\delta^{n-2}u(\frac{\bar{\lambda}_\delta^2y}{|y|^2}) \geq \frac{c}{2}|y|^{2-n}.
\]

This violates the definition of \( \bar{\lambda}_\delta \). Lemma 2 is established.

By Lemma 2, \( \bar{\lambda}_\delta \equiv \infty \) for all \( 0 < \delta < 1 \). Namely,

\[
(1 + \delta)u(y) \geq u_{x,\lambda}(y), \quad \forall 0 < \delta < 1, x \in \mathbb{R}^n, |y - x| \geq \lambda > 0.
\]

Sending \( \delta \) to 0 in the above leads to

\[
u(y) \geq u_{x,\lambda}(y), \quad \forall x \in \mathbb{R}^n, |y - x| \geq \lambda > 0.
\]

This easily implies \( u \equiv u(0) \). Liouville theorem (1) is established.
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