Supersymmetric waves in Bose-Fermi mixtures

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We re-examine the symmetries of a simple model of an interacting Bose-Fermi mixture, and find a fermionic (super)symmetry when bosons and fermions in the mixture have equal masses. This symmetry is spontaneously broken in the ground state of the mixture, leading to a novel Goldstone mode with fermionic statistics and quadratic dispersion. When the symmetry is not exact and the system is allowed to deviate from the symmetric point, we find that the Goldstone mode acquires an energy gap. We show that the excitations manifest themselves in the non-analyticity of the pressure.

1. Introduction. Recent years have witnessed outstanding progress in high precision experiments with cold atomic gases. By exploiting Feshbach resonances \textsuperscript{1}, experimentalists have achieved a high degree of control over the collective behavior of strongly interacting atomic systems. In particular, mixtures of bosonic and fermionic atoms in the quantum degenerate regime have begun to attract much attention. These systems possess a very rich phase diagram, and a plethora of different phenomena are possible \textsuperscript{2,3}, including a collapse, phase separation \textsuperscript{4,5}, depletion of the BEC \textsuperscript{6} and quantum phase transitions \textsuperscript{7}.

A special case of a Bose-Fermi mixture where two isotopes of the same atom are combined is known as an isotopic mixture. These systems are of particular experimental interest. So far, mixtures of the isotopes of K, Li, Rb, Sr and Yb have been realized experimentally \textsuperscript{8}. When the mass difference of the isotopes is sufficiently small, there is an enhanced symmetry that leads to novel exotic phenomena and even integrability \textsuperscript{9,10}.

Perhaps surprisingly, cold gas systems are an exciting test ground for some abstract high energy field theories. Recent realizations of synthetic gauge fields \textsuperscript{11} in cold atoms have attracted great interest, as they may help to understand strongly interacting gauge theories such as QCD. In that same vein, we suggest that some relics of supersymmetry can be observed in Bose-Fermi mixtures. In particle physics supersymmetry is a high energy symmetry that assigns to every elementary particle a superpartner - a particle with the same mass, but the opposite statistics \textsuperscript{12}. This symmetry - if it exists - appears to be broken in our world; the superpartners must be very heavy, otherwise they would have been found in particle collision experiments. In addition, supersymmetry is necessarily broken at any finite temperature, because it exchanges particle statistics while leaving the thermal distribution functions unchanged. This implies by Goldstone’s theorem that there must be a massless fermionic collective mode \textsuperscript{13}, known as a Goldstino.

In this Letter we study the simplest one channel model for an isotopic mixture in the broad resonance limit. We find that when the masses of the isotopes are large there is an approximate supersymmetry. While the model is not integrable in more than one dimension, the enhanced symmetry still has profound consequences. In particular, this symmetry is spontaneously broken by the ground state, and there is a fermionic Goldstone mode - a Goldstino - in the spectrum. This mode has quadratic dispersion and carries a close resemblance to spin waves or magnons. The difference is that Goldstino is a fermionic collective mode. It has been mentioned previously in Ref. \textsuperscript{9}, in connection with a related model.

Since in any real situation this symmetry can only be approximate, the Goldstino mode is gapped - that is, it acquires a negative chemical potential - and the gap is set by the mass and chemical potential differences of the two atomic species. We will derive the dispersion relation for the Goldstino mode in a completely symmetric theory, and in the presence of various sources of explicit symmetry breaking.

When the supersymmetry is broken explicitly by tuning the interspecies interaction strength, the Goldstino gap can be controlled via the Feshbach resonance. When the gap is tuned through a critical value, we find a non-analyticity in the thermodynamic pressure which we predict can be observed in experiment. We discuss the possibility to observe a gapped Goldstino mode in realizable isotopic mixtures of Sr and Yb.

2. Symmetries of Bose-Fermi Mixtures. We start with the familiar one channel model of a Bose-Fermi mixture. The grand canonical Hamiltonian is

\[ H = \frac{1}{2m_B}(\partial_i \psi)^\dagger \partial_i \psi + \frac{1}{2m_B}(\partial_i \phi)^\dagger \partial_i \phi + V(\rho_B, \rho_F), \]

where \( \phi \) is a complex bosonic field and \( \psi \) is a complex fermionic (Grassmann) field. We also introduced the densities \( \rho_B = \phi^\dagger \phi \) and \( \rho_F = \psi^\dagger \psi \) of bosons and fermions respectively. The chemical potential terms are absorbed into \( V(\rho_B, \rho_F) \).

The model Eq. \textsuperscript{14} possesses a global \( U_b(1) \times U_f(1) \) symmetry

\[ \phi' = e^{i\alpha} \phi, \quad \psi' = e^{i\beta} \psi, \]

which ensures the independent conservation of the number of bosons and fermions.
When \( m_B = m_F \) and \( V(\rho_B, \rho_F) = V(\rho_B + \rho_F) \) there is an additional symmetry. Let us define a vector

\[
\Phi = (\phi, \psi)^T .
\]

It is straightforward to check that Eq. (1) is invariant under the following symmetry transformation

\[
\Phi' = e^{iN} \Phi , \quad N = \begin{pmatrix} 0 & -\bar{\eta} \\ \eta & 0 \end{pmatrix} ,
\]

where \( \eta \) is a complex Grassmann number. Roughly speaking, this symmetry exchanges bosonic and fermionic fields, taking the statistics into account. Recall that a complex Grassmann number satisfies

\[
\eta^2 = \bar{\eta}^2 = 0 ,
\]

\[
\bar{\eta} \eta = -\bar{\eta} \eta .
\]

The symmetry Eq. (4) is reminiscent of supersymmetry (SUSY) in particle physics. For this reason we will refer to Eq. (4) as exchange supersymmetry, as it acts by exchanging bosons with fermions. It differs from its relativistic cousin, however, in that it is completely decoupled from spacetime symmetries (the non-relativistic limit of spacetime supersymmetry, on the other hand, still mixes with the rotation generators [14, 15]). We defer discussion of the salient features of this algebra to the supplementary material.

According to Noether's theorem there must be a conservation law associated with the exchange supersymmetry Eq. (4). It reads

\[
\dot{Q} + \partial_i J_Q^i = 0 ,
\]

where we have defined

\[
Q = \phi^\dagger \psi \quad \text{and} \quad J_Q^i = \frac{1}{2mi} \left( \psi^\dagger \partial_i \phi - \phi^\dagger \partial_i \psi \right) .
\]

We will refer to the non-Hermitian operator \( Q \) as the supercharge density. We define the total supercharge \( Q \) operator as an integral of the local density \( Q \)

\[
Q = \int d^d x Q
\]

(see the supplemental material). In order to get some physical intuition about the supercharge we can make a linear combinations of \( Q \) and \( \bar{Q} \)

\[
\tau = \eta \bar{Q} + \bar{\eta} Q .
\]

The Hermitian operator \( \tau \) generates the exchange SUSY transformations [4]. This can be understood in analogy with an ordinary \( SU(2) \) transformation as follows. Note first that the combination \( \eta \bar{\psi} \) is Grassmann even, that is it can be thought of as a bosonic object. Then we can identify “spin up” states with \( \phi \), and “spin down” states with \( \bar{\eta} \psi \). In this analogy, the difference of bosonic and fermionic densities is mapped to the \( z \)-component of angular momentum, and \( \tau \) becomes the spin-flip operator \( L_z \).

3. Invariant interactions. The total density \( \rho = \rho_B + \rho_F \) is invariant under exchange supersymmetry. We use this observation to construct an invariant potential \( V(\rho_B, \rho_F) = V(\rho) \). We take

\[
V = -\mu \rho^2 + g \rho^4 = -\mu (\phi^\dagger \phi + \bar{\psi} \psi) + g|\phi|^4 + 2g|\phi|^2|\psi|^2 .
\]

Note that the anticommuting (Grassmann) nature of the fermionic field \( \psi \) forces the quartic term \( |\psi|^4 \) to vanish.

Unfortunately, the non-interacting \( (g = 0) \) version of this theory with potential [11] does not make sense, since the (bare) chemical potentials for the bosons and the fermions have to be equal in a SUSY invariant model. This theory is either unstable (for \( \mu > 0 \)) or has no fermions (\( \mu \leq 0 \) [17]). Interactions will shift the (effective) chemical potential in such a way that the ground state will have a finite density of both bosons and fermions.

The minimum of the interaction potential is at

\[
\rho_F = 0 , \quad \rho_B = \frac{\mu}{2g} ,
\]

however, a finite fermion density is stabilized in the ground state due to radiative quantum corrections [18].

In the weak coupling regime the ground state has a simple structure. The bosons form a BEC that breaks the bosonic \( U_B(1) \) symmetry. The fermions form a sharp Fermi sphere with a well-defined notion of particle number, so the fermionic \( U_F(1) \) symmetry is not broken. Clearly bosons and fermions in the ground state behave very differently, so the exchange symmetry Eq. (4) is spontaneously broken at finite chemical potential (\textit{i.e.} finite density). According to the Goldstone theorem there must be a gapless excitation. For lack of better terminology we will refer to it as a Goldstino. We discuss some details of this argument in the attached supplement.

4. Sound. Before tackling the Goldstone mode associated with the breakdown of [4] we briefly recall how one can quickly arrive at the effective action for the \( U(1) \) sound mode in a BEC; we will exploit this same technique to derive the dispersion relation for the Goldstino later.

Consider the action for a complex scalar boson

\[
S = \int dt d^d x \left[ i \phi^\dagger \partial_0 \phi - \frac{1}{2m} \partial_i \phi^\dagger \partial_i \phi + g \phi^4 \right] .
\]

We will follow the beautiful construction of [19]. Those authors noticed that \( [13] \) is Galilean invariant. Assuming that the Galilean symmetry is not spontaneously broken, the effective theory of the Goldstone (phonon) mode \( \phi(x) \) also must be Galilean invariant. Note also that the phonon field \( \phi \) is real. It is not hard to see that the most general effective Lagrangian must have the form

\[
\mathcal{L}_{eff} = P \left( \partial_0 \phi - \frac{(\partial_x \phi)^2}{2m} \right) ,
\]
where \( P(x) \) is an arbitrary polynomial. Taking the low gradient terms \( P(x) = \rho_0 x + \frac{\rho_0}{2m^* x^2} \) (\( v \) is the speed of sound) we have

\[
\mathcal{L}_{\text{eff}} = \rho_0 \left( \partial_0 \varphi - \frac{(\partial_i \varphi)^2}{2m} \right) + \frac{\rho_0}{2m^*} \left( \partial_0 \varphi - \frac{(\partial_i \varphi)^2}{2m} \right)^2 \]

\[
= \rho_0 \partial_0 \varphi + \frac{\rho_0}{2m} \left[ \frac{1}{v^2} (\partial_0 \varphi)^2 - (\partial_i \varphi)^2 \right] + \mathcal{O}(\partial^3), \tag{15}
\]

where we have absorbed the higher derivative terms into \( \mathcal{O}(\partial^3) \). This is a striking birth of emergent low energy Lorentz symmetry from the high energy Galilean symmetry. We can quickly read off the low energy linear dispersion of the Bogoliubov quasi-particles from the second term in Eq. (15) (The first term is a total divergence that fixes the number density and will not concern us here).

The requirement of Galilean symmetry also fixes some of the higher gradient corrections to the effective action. This phenomenon has been exploited in the theory of unitary Fermi gasses \( [20] \).

It is possible to fix the coefficient in front of \( (\partial_i \varphi)^2 \) of (15) directly from the action \( [13] \). We perform a \( \Phi \) finite \( U(1) \) symmetry transformation with the parameter \( \varphi(x,t) \) that smoothly depends on space and time.

\[
\mathcal{L}[e^{i\psi} \Phi] - \mathcal{L}[\Phi] = \rho_0 \left[ i \partial_0 \varphi - \frac{1}{2m} (\partial_i \varphi)^2 \right], \tag{16}
\]

where we have “integrated out” the absolute value \( |\phi|^2 \) \( (\text{i.e. replaced it by the mean field value } \rho_0) \). Eq. (16) fixes the aforementioned coefficient, but does not fix the sound velocity \( v \).

5. Fermionic sound. We will first use symmetry arguments to fix the form of the Goldstino effective action, just like in the previous section. Then we will use the trick \( [16] \) to determine the entire long distance effective action. It will be convenient to work in the Lagrangian formalism. We re-write \( [11] \) for \( m_B = m_F \) with potential \( [11] \) in a matrix form as

\[
\mathcal{L} = i \Phi^\dagger \partial_0 \Phi - \frac{1}{2m} \partial_i \Phi^\dagger \partial_i \Phi - \mu \Phi^\dagger \Phi + g \Phi^\dagger \Phi^2. \tag{17}
\]

The Lagrangian (17) is clearly Galilean invariant. Since the transformation parameter of (11) is a complex Grassmann number we also expect that the Goldstino mode will be described by a complex Grassmann scalar field \( \eta(x) \). The most general Galilean invariant effective Lagrangian for a complex Grassmann scalar is \( [10] \)

\[
\mathcal{L}_{\text{eff}} = P \left( \frac{i}{2} \eta^\dagger (\partial_0 \eta) - (\partial_0 \eta) \eta - \frac{1}{2m^*} (\partial_i \eta^\dagger)(\partial_i \eta) \right), \tag{18}
\]

where \( P \) is again an arbitrary polynomial. Notice that contrary to (14) the time derivative term must be quadratic in \( \eta \) so that the action is real. Taking \( P(x) = Cx \) we get the low energy effective Lagrangian for the Goldstino mode:

\[
\mathcal{L}_{\text{eff}} = C \left( m^* \eta^\dagger (\partial_0 \eta) - \frac{1}{2m^*} (\partial_i \eta^\dagger)(\partial_i \eta) \right), \tag{19}
\]

where \( C \) and \( m^* \) are constants to be determined form the microscopic model \( [17] \). We have also integrated by parts in the time derivative term in (19).

We will now employ a trick similar to (16) to fix the constants \( C \) and \( m^* \). Namely, we will perform a \( \Phi \) finite and spatially varying symmetry transformation \( [14] \) and integrate out \( \Phi \), replacing it by its mean field value. At this level of approximation, we then have \( \Phi^\dagger \Phi = \rho_B + \rho_F = \rho \), where \( \rho_B \) and \( \rho_F \) are the mean field values of the densities. This mean-field ground state has been shown to be stable for reasonable values of the system parameters \( [18] \). After some algebra we get for the low-energy Lagrangian

\[
\mathcal{L}[\Phi^\dagger] - \mathcal{L}[\Phi] = \rho \left[ m \eta^\dagger (\partial_0 \eta) - \frac{1}{2m} \frac{\rho_B - \rho_F}{\rho} (\partial_i \eta^\dagger)(\partial_i \eta) \right]. \tag{20}
\]

This fixes the coefficients in the effective action Eq. (19) to be

\[
C = \rho, \tag{21}
\]

\[
m^* = \frac{m \rho}{\rho_B - \rho_F}. \tag{22}
\]

The Goldstino dispersion relation is then given by

\[
\omega = \frac{1}{2m^*} k^2. \tag{23}
\]

We see that the Goldstino is a gapless fermionic excitation with quadratic dispersion. The dispersion relation also shows immediately that SUSY is unbroken only if \( \rho_B = \rho_B \) in the ground state, so that the Goldstino mode becomes dispersionless (something similar happens with the ordinary \( U(1) \) phonon mode when the condensate density \( \rho_0 \) vanishes). The full low energy effective action of the model \( [17] \) is given by sum of (15) and (19).

At the intuitive level the Goldstino mode can be thought of as an analogue of a spin wave or a magnon. Just like a spin wave is a disturbance propagating on top of an ordered antiferromagnetic ground state, the Goldstino is a disturbance propagating on top of the “ordered” Bose-Fermi ground state. Similarly, both collective excitations have quadratic dispersion (at low momentum). The average value of the generator \( \hat{Y} = \frac{1}{2}(\phi^\dagger \phi - \psi^\dagger \psi) \) plays the role of “magnetization” and is sensitive to the symmetry breaking. The mass difference is similar to a magnetic field in that it will open a gap for the Goldstino (c.f. a Bose-Bose mixture with two “pseudospin” states).

There are, of course, other examples of the fermionic quasi-particles. One might suspect that the Goldstino mode is simply a glorified electron dressed with the condensate atoms (known in solid state physics as polaron).
On the contrary, the Goldstino is a true collective excitation and require a finite density of both bosons and fermions in order to propagate. The clearest evidence for this is that the Goldstino dispersion relation depends on the Fermi energy only through the effective mass $m^*$.

6. Explicit symmetry breaking. In any realistic system, it is unreasonable to expect the exchange supersymmetry to be exact. In fact, the spin-statistics theorem tells us immediately that the boson mass $m_B$ cannot be equal to the fermion mass $m_F$ for neutral isotopes of the same element. Additionally, the boson-boson interaction strength $g_{BB}$ will in general differ slightly from the boson-fermion interaction strength $g_{BF}$, although these parameters are tunable somewhat. For the experimentally interesting case of Strontium mixtures, this mass difference is small - approximately one part in one hundred - and the interaction strengths can be tuned to be small via the Feschbach resonance. Finally, one can engineer a difference in chemical potential for the bosonic and fermionic species, which also breaks the symmetry.

We can incorporate these forms of explicit symmetry breaking directly into our mean-field theory with only minor modifications. Introducing the symmetry breaking parameters

$$
\delta m = m_B - m_F, \quad \delta g = g_{BB} - g_{BF}, \quad \delta \mu = \mu_B - \mu_F,
$$

along with the average mass

$$
m_0 = \frac{1}{2} (m_B + m_F), \quad (24)
$$

we find after repeating the derivation above that the fermionic sound dispersion relation is modified to

$$
\omega = \left( \frac{\rho_B - \rho_F}{2m_0\rho} + \frac{\delta m}{4m_0} \right) k^2 - \delta \mu, \quad (25)
$$

where $\mu_\eta$ is an effective chemical potential given by

$$
\mu_\eta = -\frac{E_K}{\rho} \delta m - 2\rho_B \left( 1 + \frac{\rho_F}{\rho} \right) \delta g - 2\delta \mu. \quad (26)
$$

Here $E_K$ is the total average kinetic energy of the system, which at low temperatures we expect to be dominated by the kinetic energy of the fermions. We see then that there are two main effects on the fermionic sound mode due to the explicit symmetry breaking. The first is a modification of the inverse effective mass (the coefficient of the $k^2$ term in the dispersion) by an amount proportional to the mass difference $\delta m$. The second - and perhaps more interesting - modification is the appearance of a chemical potential $\mu_\eta$ for the Goldstino mode. Note that by tuning $g_{BB}$ relative to $g_{BF}$ we can change the effective chemical potential $\mu_\eta$ from positive to negative. This can in principle be done experimentally by exploiting techniques such as the Feshbach resonance.

7. Experimental Implications This ability to change the sign of the chemical potential for the Goldstino mode has implications for the low-temperature thermodynamics of Bose-Fermi mixtures. In the context of our effective theory, the pressure of the mixture can be written as

$$
P = P_0 + P_\eta, \quad (27)
$$

where $P_0$ is the contribution from the bare bosons and fermions, while $P_\eta$ is the contribution from the Goldstino excitations. The key point is that far from a bulk phase transition, we expect $P_0$ to be an analytic function of the system parameters, while $P_\eta$ is strongly dependent on the sign of the Goldstino chemical potential $\mu_\eta$.

Using the fact that the Goldstino behaves as a free fermion, we have at zero temperature that

$$
P_\eta = \left\{ \begin{array}{ll}
0, & \mu_\eta < 0 \\
\frac{2}{\pi^2} \mu_\eta \bar{n}_\eta, & \mu_\eta > 0
\end{array} \right., \quad (28)
$$

where $\bar{n}_\eta \propto \mu_\eta^{d/2}$ is the ground-state Goldstino density. The nonzero pressure for $\mu_\eta > 0$ simply reflects the fact that, for a positive chemical potential, the Goldstino excitations fill a Fermi sea and exert the usual degeneracy pressure. When the chemical potential is negative, there are no excitations in the ground state, and this contribution vanishes. Thus, we predict that the total system pressure $P$ is a non-analytic function of $\delta g$. This should manifest in a discontinuity in the derivative

$$
\frac{\partial P}{\partial (\delta g)}
$$

at some critical value $\delta g^*$ of the difference in coupling that makes Eq. (26) vanish. Solving for this critical value analytically is challenging as Eq. (26) is a complicated implicit equation. On the other hand, for small values of $\delta m$ and $\delta \mu$, we expect that $\delta g^*$ is only slightly perturbed from zero. It may be possible to detect this non-analytic behavior experimentally, as the pressure may be computed from an experimental determination of the equation of state of the mixture.

Our analysis so far has been for free translationally invariant mixtures. Of course, any real experiment involves some sort of trapping potential, and care is needed to interpret our results in this case. We note first that an external confining potential can be added to our action without explicitly violating supersymmetry. For a trapping potential that varies sufficiently slowly in space, this implies that our results hold essentially unmodified, with the exception that thermodynamic quantities which up to this point have been global constants will, in the Thomas-Fermi approximation, become spatially dependent. Thus, the density difference $\rho_B - \rho_F$, the average kinetic energy $E_K$, the chemical potential difference $\delta \mu$, and the gas pressure $P$ will all become functions of spatial position $x$. Note that the interaction strengths
cloud. The quantity of interest is then $V$, where
the coupling difference must be taken to interpret our prediction Eq. (28) for
changes from point to point. In light of this, some care
that the nonrelativistic effective mass of the Goldstino
still be spatially independent constants. Because of this,
relevant average pressure pressure, however, we should look at the experimentally
modified as the size of the trap becomes large compared
to the average interparticle spacing. Thus, we suggest
examining Bose-Fermi mixtures in flat traps in order to
verify our predictions.

Finally, we should note that our approximation scheme
has neglected higher order effects which may cause the
Goldstino to acquire a finite lifetime (a special case has
recently been treated in Ref. [25]). For small values of
Goldstino to acquire a finite gap that can be tuned through
the Goldstino gap is tuned through zero.

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[25] H. H. Lai and K. Yang, (2015). [arXiv:1504.05879]
SUPERSYMMETRY ALGEBRA

Here we summarize the (super)algebra of SUSY transformations. Recall that we have as fundamental microscopic fields a complex scalar boson \( \phi \) and a spinless complex fermion \( \psi \) which satisfy

\[
[\phi(x), \phi(y)] = \{\psi(x), \psi(y)\} = 0, \quad (1)
\]

\[
[\phi(x), \phi^\dagger(y)] = \{\psi(x), \psi^\dagger(y)\} = \delta(x - y), \quad (2)
\]

\[
[\phi(x), \psi(y)] = [\phi^\dagger(x), \psi^\dagger(y)] = 0. \quad (3)
\]

In addition, we introduce Grassmann-valued supercharge operators given by

\[
Q = \int d^4x \phi^\dagger \psi = \int d^4x Q, \quad \bar{Q} = \int d^4x \psi^\dagger \phi = \int d^4x \bar{Q},
\]

(4)

(together with two additional bosonic generators)

\[
\bar{N} = \int d^4x \phi^\dagger \phi + \psi^\dagger \psi, \quad (5)
\]

\[
\bar{Y} = \frac{1}{2} \int d^4x \phi^\dagger \phi - \psi^\dagger \psi. \quad (6)
\]

These satisfy the super-commutation relations

\[
\{Q, \bar{N}\} = \{\bar{Q}, N\} = \{\bar{Y}, \bar{N}\} = 0, \quad (7)
\]

\[
\{Q, \bar{Q}\} = \{\bar{Q}, Q\} = 0, \quad (8)
\]

\[
\{Q, \bar{Y}\} = Q, \quad (9)
\]

\[
\{\bar{Q}, \bar{Y}\} = -\bar{Q}, \quad (10)
\]

\[
\{Q, \bar{Q}\} = \bar{N}. \quad (11)
\]

Note that this is a supersymmetric generalization of the commutation relations of \( su(2) \) - in fact, this is the super-algebra known as \( u(1|1) \) [1]. That is, \( Y \) acts like the \( z \)-component of angular momentum, and \( Q \) (\( \bar{Q} \)) acts like an angular momentum raising (lowering) operator. Furthermore, the number operator \( N \) is the quadratic Casimir operator. The main difference lies in the fact that \( Q \) and \( \bar{Q} \) satisfy anticommutation relations.

When acting on the elementarybose and fermi fields of our model, these operators satisfy

\[
[Q, \phi] = -\psi, \quad (12)
\]

\[
\{Q, \psi\} = 0, \quad (13)
\]

\[
[\bar{Q}, \phi] = 0, \quad (14)
\]

\[
\{\bar{Q}, \psi\} = \phi, \quad (15)
\]

\[
[N, \phi] = \phi, \quad (16)
\]

\[
[N, \psi] = -\psi. \quad (17)
\]

\[ Y, \phi = -\frac{1}{2}\phi, \quad (18) \]

\[ Y, \psi = \frac{1}{2}\psi. \quad (19) \]

The particle exchange supersymmetry discussed in the main text is generated by the operator

\[ \hat{\tau} = (\bar{\eta} \bar{Q} + \eta Q). \quad (20) \]

Under this transformation, the elementary fields transform as

\[ \phi \to \left(1 + \frac{m}{2}\right) \phi - \bar{\eta} \psi, \quad (21) \]

\[ \psi \to \left(1 - \frac{m}{2}\right) \psi + \eta \phi. \quad (22) \]

This is an explicit form of the matrix transformation law mentioned in the main text.

For completeness, we note that this algebra admits a representation in terms of Pauli matrices:

\[ Q \to \frac{1}{2\sqrt{N}} \sigma_+, \quad (23) \]

\[ \bar{Q} \to \frac{1}{2\sqrt{N}} \sigma_-, \quad (24) \]

\[ Y \to \frac{1}{2} \sigma_z, \quad (25) \]

\[ \bar{N} \to I. \quad (26) \]

APPROXIMATELY BROKEN SYMMETRIES
AND ALMOST-GAPLESS EXCITATIONS

Here we will review some field-theoretic arguments for the existence of gapped collective excitations when a symmetry is explicitly broken. In the process, we will uncover some justification for the mean-field “trick” we used in the main text to fix the goldstino dispersion relation. Our discussion follows that of Weinberg’s text [2], with only small modifications.

We start by recalling that for a quantum system with action \( S[\phi, \psi] \), we can form the generating functional (for brevity, we suppress the explicit dependence of functionals on the Hermitian conjugate of the variables. We hope this does not cause undue confusion)

\[ e^{iW[J, \theta]} = \int D\phi D\psi \exp \left( iS[\phi, \psi] + i \int d^4x J\phi + \theta \psi + \text{h.c.} \right). \quad (27) \]

\( W \) is the generating functional for connected correlation functions (cumulants) of the quantum system; that is, functional derivatives of \( W \) with respect to the source fields \( J \) and \( \theta \) give - upon setting the sources to zero -
the connected averages of products of the fields $\phi$ and $\psi$. From $W$, we can define the effective action $\Gamma$ via a Legendre transform:

$$\Gamma(\phi, \psi) = W - \int d^d x J \phi + \theta \psi + \text{h.c.} \quad (28)$$

The effective action $\Gamma$ is the quantum analogue of the classical action - the averages of the external fields in the absence of sources are given by extremal points of $\Gamma$.

The effective action has many other useful properties, although we will only need two of them here. First, the matrix of second variational derivatives of $\Gamma$ is precisely the negative of the inverse propagator. Using the superfield notation of Eq. (3), we have

$$\frac{\delta^2 \Gamma}{\delta \Phi_i(x) \delta \Phi_j(y)} = - \langle \Phi_i(x) \Phi_j(y) \rangle^{-1} \equiv \Delta^{-1}_{ij}(x-y), \quad (29)$$

when evaluated at the extrema of $\Gamma$. The second property we need is that linear symmetries of the microscopic action are automatically also symmetries of the effective action.

Since we are only interested in homogeneous configurations of our system, it is sufficient for us to consider the effective potential $V$ defined by considering only homogeneous field configurations.

Let us consider first the case when supersymmetry is an exact symmetry of the microscopic action, and ask what happens when it is spontaneously broken. Spontaneous breakdown of the symmetry means in this language that the minimum of the effective potential occurs at a nonzero value of at least one of the fields. The invariance of the microscopic action under supersymmetry implies for the effective potential that

$$0 = \frac{\partial V}{\partial \Phi_i} N_{ij} \Phi_j + \frac{\partial V}{\partial \Phi_i} N_{ij} \Phi_j \quad (30)$$

Taking an additional derivative with respect to $\Psi_j$ or $\Psi_j^\dagger$ and evaluating at the nonzero minimum of the effective potential tells us immediately that the inverse propagator $\Delta^{-1}_{ij}$ has a zero (super-)determinant at vanishing momentum (since we are looking at homogeneous configurations). We conclude that the two-point function for the matter fields has a pole at zero wavevector - there is a gapless collective excitation. This is merely a generalization of Goldstone’s theorem to the case of Grassmann-valued symmetries; it yields a fermionic collective mode at zero energy.

We would now like to ask what happens when supersymmetry is explicitly broken in the microscopic action. In this case, the RHS of Eq. (30) will no longer be zero. In fact, carrying through the same derivation as above, the right hand side, whatever it is, gives the chemical potential for the fermionic collective mode.

In general, the variation of the effective potential under a SUSY transformation is given by matrix elements of the supercurrent operator Eq.(3), which can be calculated in perturbation theory. However, this is where our mean-field trick comes to the rescue. Considering the change in the microscopic action under spatially varying SUSY transformations is precisely equivalent to calculating the relevant matrix elements to tree level! Thus, our derivation of the fermionic sound spectrum in the main text is justified whenever perturbation theory is valid.

[1] V. Kac, Commun. Math. Phys. 53, 31 (1977).
[2] S. Weinberg, The Quantum Theory of Fields, Vol. 2 (Cambridge University Press, New York, NY, 1996).