Continuum limit and symmetries of the periodic $g\ell(1|1)$ spin chain

A.M. Gainutdinov$^a$, N. Read$^b$ and H. Saleur$^{a,c}$

$^a$ Institut de Physique Théorique, CEA Saclay, Gif Sur Yvette, 91191, France
$^b$ Physics Department, Yale University, New Haven, CT, USA
$^c$ Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089, USA

Abstract

This paper is the first in a series devoted to the study of logarithmic conformal field theories (LCFT) in the bulk. Building on earlier work in the boundary case, our general strategy consists in analyzing the algebraic properties of lattice regularizations (quantum spin chains) of these theories. In the boundary case, a crucial step was the identification of the space of states as a bimodule over the Temperley–Lieb (TL) algebra and the quantum group $U_q\ell(2)$. The extension of this analysis in the bulk case involves considerable difficulties, since the $U_q\ell(2)$ symmetry is partly lost, while the TL algebra is replaced by a much richer version (the Jones–Temperley–Lieb - JTL - algebra). Even the simplest case of the $g\ell(1|1)$ spin chain – corresponding to the $c = -2$ symplectic fermions theory in the continuum limit – presents very rich aspects, which we will discuss in several papers.

In this first work, we focus on the symmetries of the spin chain, that is, the centralizer of the JTL algebra in the alternating tensor product of the $g\ell(1|1)$ fundamental representation and its dual. We prove that this centralizer is only a subalgebra of $U_q\ell(2)$ at $q = i$ that we dub $U_{\text{odd}} q\ell(2)$.

We then begin the analysis of the continuum limit of the JTL algebra: using general arguments about the regularization of the stress energy-tensor, we identify families of JTL elements going over to the Virasoro generators $L_n, \bar{L}_n$ in the continuum limit. We then discuss the $s\ell(2)$ symmetry of the (continuum limit) symplectic fermions theory from the lattice and JTL point of view.

The analysis of the spin chain as a bimodule over $U_{\text{odd}} q\ell(2)$ and $JTL_N$ is discussed in the second paper of this series.

1 Introduction

There are often striking similarities between the properties of (not necessarily integrable) lattice models and their conformally invariant continuum limit in two dimensions. The origin - and mathematically more precise formulation - of these similarities is partly understood, and related with the presence of common algebraic structures such as quantum groups centralizers [1, 2, 3]. Nevertheless, many features remain unexplored in this field, chief among them the relation between representations of the Virasoro algebra and various lattice objects – Temperley Lieb algebras, RSOS paths [4, 5, 6], etc.

The similarities between lattice models and conformal field theories (CFT) can be a powerful – albeit non rigorous yet – tool to infer the continuum limit of some models which are too hard to solve analytically. This idea has been exploited recently to deepen our understanding of Logarithmic CFTs (LCFTs). Indeed, models based on representations of associative algebras such as the Temperley–Lieb (TL) algebra exhibit [7], from a representation theoretic point of view, and in finite size, strong similarities with the chiral algebras in LCFTs. The structure of indecomposable modules and fusion
rules carried out sometimes with great difficulty in the Virasoro setting \[8, 9, 10\] can then be predicted from a more manageable algebraic analysis of the lattice models \[11, 12, 13\]. A rigorous reformulation of the similarities in representation theories for lattice and continuum sides requires some categorical statements like equivalence of tensor categories. Tensor structure or fusion data on the lattice part is essentially an induction (bi)functor associated with two lattices of arbitrary sizes joined to each other. Then, a construction of direct limits of ‘tensor’ categories of modules over the lattice algebras, e.g. TL-modules, should give the desired equivalence with a tensor category of modules over the chiral algebra.

It has also turned out that, beyond the abstract structure of indecomposable modules, the matrix elements of Virasoro generators themselves can also be obtained from the lattice models, although this time an extrapolation to infinite sizes and restriction to low energy part of the spectrum have to be implemented \[14\]. Indecomposability parameters characterizing Virasoro action in large families of boundary LCFTs have recently been obtained in this fashion \[15, 16\].

While the case of boundary LCFTs is thus slowly getting under control, the understanding of the bulk case remains in its infancy. The main problem here, from the continuum point of view, is the expected double indecomposability of the modules over the product of the left and right Virasoro algebras. From the lattice point of view, the necessarily periodic geometry of the model leads to more complicated algebras \[17, 18\], and to a more intricate role of the quantum group \[1\], whose symmetry is partly lost. A relative understanding of bulk LCFTs has only been gained in the rational case \[19, 20\] based on chiral W-algebras \[21, 22\], and Wess–Zumino models on supergroups which, albeit very simple as far as LCFTs go, provide interesting lessons on the coupling of left and right sectors \[23\].

The present paper is the first in our investigation of bulk LCFTs using lattice models and algebras. We shall mostly deal with super-spin chains, which are now well understood in the open case \[7\], and whose spectrum in the bulk was determined as early as 2001 \[24\]. This spectrum exhibits intricate patterns such as conformal weights covering all the rationals (modulo integers), and large degeneracies given by complicated, arithmetic formulas. To understand these patterns, and to extract the structure of the left-right Virasoro representations, what is required is a more thorough study of the lattice algebras present in this case. While difficult, this study should not be impossible, thanks in part to recent progress on the side of mathematics \[17, 18, 25, 26, 27, 28\].

Before launching into abstract algebra, it seems important to gain a better understanding of the potentially simplest case, that is closed \(g\ell(1|1)\) spin chain, whose continuum limit is expected to be described by the ubiquitous symplectic fermion theory \[29\]. Our goal is to understand this case thoroughly, in order to delineate a general strategy which we will be able to extend to other cases – such as the \(g\ell(2|1)\) spin chain – in subsequent papers. Unfortunately, even the \(g\ell(1|1)\) case is rather complicated, and will occupy us for a while.

Recall that a fundamental technical step in our approach is to analyze the Hilbert space of the system as a (bi)module over the two algebras – the algebra of Hamiltonian densities which, for the models in \[24, 12\] is the periodically extended Temperley–Lieb algebra, and its centralizing symmetry algebra. We will restrict in this first paper to the analysis of the symmetries, postponing the full bimodule discussion to our second paper \[30\]. We begin with definitions of our closed spin-chains and their relations with XX spin-chains in section \[2\] where we also remind about the continuum limit in the open case. In the closed \(g\ell(1|1)\) case we shall see in section \[3\] that the symmetry algebra is only a subalgebra of the symmetry \(U_q s\ell(2)\) of the boundary theory. The resulting object for periodic
conditions – called $\mathcal{U}_{q}^{\text{odd}}\mathfrak{sl}(2)$, with $q = i$ below – is realized as a subalgebra in $\mathcal{U}_{q}\mathfrak{sl}(2)$ which involves the use of the Lusztig limit $q \to i$ of particular polynomials of odd degree in the quantum group generators while the $\mathfrak{sl}(2)$ subalgebra is given by polynomials of even degree and realizes the symmetry for antiperiodic conditions. More rigorous statements are presented in Thm. 3.3.3 and Thm. 3.4.1.

A crucial feature of the product $\mathcal{V}(2) = \mathcal{V}(2) \otimes \overline{\mathcal{V}(2)}$ of left and right Virasoro algebras that appear in the continuum limit symplectic fermion theory is the presence of a global $\mathfrak{sl}(2)$ symmetry (the ‘symplectic’ symmetry of the theory). It turns out however that the lattice centralizer of JTL, $\mathcal{U}_{q}^{\text{odd}}\mathfrak{sl}(2)$, does not contain the subalgebra $\mathfrak{sl}(2)$. What happens to this ‘extra symmetry’ in the continuum limit will turn out to be a crucial aspect of the problem of connecting algebraic features of the lattice models with those of LCFTs. To understand this better, we spend some time in section 4 analyzing the scaling limit of the spin chain. Using general ideas about the lattice version of the stress energy tensor, we identify particular ‘local’ elements in the JTL algebra (such as the generators $e_{i}$, or the commutators $[e_{i}, e_{i+1}]$) whose long wavelength Fourier modes have a well-defined convergence to the left and right Virasoro modes $L_{n}$ and $\bar{L}_{n}$ in the logarithmic theory of symplectic fermions at $c = -2$. The fate of the $\mathfrak{sl}(2)$ symmetry in the $g\ell(1|1)$ case is then discussed in section 5.

A note on style: some of the results below – roughly, all that concerns algebraic aspects of the finite dimensional spin chain, as presented in section 3 and the three appendices - are rigorous, and presented accordingly in the form of propositions, theorems, etc. While we believe the rest of the paper could be turned into fully rigorous statements (at the price of dwelling into analysis), we have chosen not to do so, and to remain instead close to the style of physics literature.

Finally, we note that a lattice model going over in the continuum limit to symplectic fermions with periodic boundary conditions has been studied from a related but different point of view in [31, 32].

1.1 Notations

To help the reader navigate through this paper, we provide a partial list of notations (common to this paper and its sequels)

- $\mathcal{TL}_{N}$ — the (ordinary) Temperley–Lieb algebra,
- $\mathcal{TL}_{N}^{a}$ — the periodic Temperley–Lieb algebra,
- $\mathcal{JTL}_{N}$ — the Jones–Temperley–Lieb algebra,
- $\mathcal{Z}_{\mathcal{JTL}}$ — the centralizer of $\mathcal{JTL}_{N}$,
- $\pi_{\mathcal{JTL}}$ — the spin-chain representation of $\mathcal{JTL}_{N}$,
- $\mathcal{U}_{q}\mathfrak{sl}(2)$ — the full quantum group,
- $\mathcal{E, F, K}_{\pm 1}$ — the standard quantum group generators,
- $e, f$ — the renormalized powers,
- $\rho_{\mathcal{JTL}}$ — the spin-chain representation of the quantum group $\mathcal{U}_{q}\mathfrak{sl}(2)$,
- $\mathcal{V}(2)$ — the left Virasoro algebra with $c = -2$,
- $\overline{\mathcal{V}(2)}$ — the product of the left and right Virasoro algebras,
3v — the centralizer of V(2),
sℓ(2) — Kausch’s sℓ(2) symmetry.

2 Preliminaries

2.1 The gℓ(1|1) super-spin chain

The gℓ(1|1) super-spin chain [12] is the tensor product \( H_N = \bigotimes_{j=1}^{N} V_j \), with \( V_j \cong \mathbb{C}^2 \), which consists of \( N = 2L \) sites labelled by \( j = 1, \ldots, 2L \) with the fundamental representation of \( gℓ(1|1) \) on even sites and its dual on odd sites. The open \( gℓ(1|1) \) spin chain provides a free fermion representation based on operators \( f_j \) and \( f_j^\dagger \) which obey

\[
\{ f_j, f_{j'} \} = 0, \quad \{ f_j^\dagger, f_{j'}^\dagger \} = 0, \quad \{ f_j, f_{j'}^\dagger \} = (-1)^j \delta_{jj'}.
\]

(2.1)

The most general nearest-neighbour ‘Heisenberg’ coupling

\[
e_{j}^{g ℓ} = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \quad 1 \leq j \leq N - 1,
\]

(2.2)

is just a projector on the \( gℓ(1|1) \)-invariant in the product of two neighbour tensorands. It can be expressed in terms of a representation of the Temperley–Lieb algebra \( TL_{2L}(m) \) generated by \( e_j \)’s together with the identity, subject to the usual relations

\[
e_{j}^2 = m e_j, \quad e_j e_{j\pm 1} e_j = e_j, \quad e_j e_k = e_k e_j \quad (j \neq k, k \pm 1),
\]

(2.3)

where \( j = 1, \ldots, N - 1 \). The operators \( e_j^{g ℓ} \) in (2.2) satisfy the Temperley–Lieb algebra relations with \( m = 0 \) (in general for the models of [12], the parameter \( m \) is the superdimension of the fundamental representation). The open \( gℓ(1|1) \) spin-chain described by the coupling (2.2) and the Hamiltonian 

\[-\sum_{j=1}^{N-1} e_{j}^{g ℓ} \]

provides a faithful representation of \( TL_{2L}(0) \).

The closed (periodic) spin-chain is obtained simply by adding a coupling between the sites with \( j = 2L \) and \( j = 1 \), that is by adding a generator

\[
e_{2L}^{g ℓ} = (f_{2L} + f_1)(f_{2L}^\dagger + f_1^\dagger),
\]

(2.4)

which corresponds to the periodic boundary condition \( f_{2L+1}^{(i)} = f_1^{(1)} \) on the lattice fermions. The operators \( e_j^{g ℓ} \), with \( 1 \leq j \leq 2L \), satisfy the relations (2.3) with \( m = 0 \) where the indices are now interpreted modulo \( N \) (the abstract algebra generated by \( e_j \) with these relations as the defining relations is a quotient of the affine Hecke algebra of \( A \)-type and is also known as the periodic Temperley–Lieb algebra [18, 17].)

The critical Hamiltonian for our model is then expressed as

\[
H = -\sum_{j=1}^{2L} e_{j}^{g ℓ}
\]

(2.5)
(note that for this model the sign of \( H \) is irrelevant, as the algebra obeyed by \( e_j \)'s and \(-e_j \)'s are identical. This is of course not the case for other values of \( m \)). We note that the Hamiltonian is self-adjoint, \( H = H^\dagger \), with respect to an indefinite non-degenerate inner product \( \langle \cdot, \cdot \rangle \) such that \( \langle f_j x, y \rangle = \langle x, f_j^\dagger y \rangle \) for any \( x, y \in \mathcal{H}_{2L} \).

In the periodic case, we also consider the generators \( u^2 \) and \( u^{-2} \) of translations by two sites to the right and to the left, respectively. The following additional relations are then obeyed,

\[
\begin{align*}
  u^2 e_j u^{-2} &= e_{j+2}, \\
  u^2 e_{N-1} &= e_1 \cdots e_{N-1}.
\end{align*}
\] (2.6)

The expressions for the \( e_j \) defined in (2.2) and (2.3) together with the translations \( u^{\pm 2} \) of the periodic spin-chain provides a representation of the so-called Jones–Temperley–Lieb (JTL) algebra \( JTL_{2L}(m = 0) \) which we denote by \( \pi_{gL} : JTL_{2L}(0) \rightarrow \text{End}_\mathbb{C}(\mathcal{H}_N) \). The representation \( \pi_{gL} \) is known to be non-faithful and non-semisimple [12]. In the following we usually suppress all reference to \( m \) and suppose \( m = 0 \).

### 2.2 A relation with XX spin-chains

It will be useful in what follows to observe that the \( g\ell(1|1) \) spin-chain representation \( \pi_{gL} \) is equivalent to a twisted XX spin-chain representation \( \pi_{XX} \) of \( JTL_{2L} \). The expression of the Temperley–Lieb generators in this case is well known for the open chain [1],

\[
\pi_{XX}(e_j) \equiv e_j^{XX} = -\frac{1}{2} \left[ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - i(\sigma_j^z - \sigma_{j+1}^z) \right],
\] (2.7)

where \( \sigma_j^x, \sigma_j^y \) and \( \sigma_j^z \) are usual Pauli matrices acting on a \( j \)th tensorand,

\[
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (2.8)

We also use the notations \( \sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y) \) in what follows.

To get equivalence in the closed case we need to set in the expression for \( e_j^{XX} \) the following:

\[
\sigma^\pm_{2L+1} = (-1)^S \sigma^\pm_1, \quad \text{with} \quad S^z = \frac{1}{2} \sum_{j=1}^{2L} \sigma_j^z.
\] (2.9)

This means that a periodic \( g\ell(1|1) \) (alternating) spin-chain corresponds to a periodic XX spin-chain for odd values of the spin operator \( S^z \) and to an antiperiodic XX spin chain for even values of the spin.

To prove this – and for later computational simplicity – it is useful to reformulate everything in terms of ordinary fermions \( c_j^{(1)} \) obeying anticommutatoin relations \( \{ c_j^{(1)}, c_{j'}^{(1)} \} = 0, \{ c_j, c_{j'}^{\dagger} \} = \delta_{jj'} \). Starting from the XX representation \( \pi_{XX} \) and using the Jordan–Wigner transformation

\[
\begin{align*}
  c_j^{\dagger} &= i^{j-1} i^{\sigma_1^{\dagger} + \cdots + \sigma_{j-1}^{\dagger}} \otimes \sigma_j^+, \\
  c_j &= i^{-j+1} i^{-\sigma_1^{\dagger} - \cdots - \sigma_{j-1}^{\dagger}} \otimes \sigma_j^-
\end{align*}
\] (2.10)

(in each case, both \( i \) and \(-i = i^{-1}\) can indeed be used interchangeably, as the whole prefactor is real), one obtains

\[
e_j^{XX} = c_j c_{j+1} + c_{j+1} c_j^\dagger + i \left( c_j^\dagger c_j - c_{j+1}^\dagger c_{j+1} \right), \quad c_{2L+1}^{(1)} = (-1)^L c_1^{(1)}, \quad 1 \leq j \leq 2L.
\] (2.11)
Meanwhile, we can also reexpress the $f_j^{(t)}$’s from the $g\ell(1|1)$ chain in terms of these ordinary fermions:

$$f_j^{\dagger} = i^j c_j^{\dagger}, \quad f_j = i^j c_j$$

leading to the identification

$$e_j^{g\ell} = i(-1)^j \left[ c_{j+1} c_j^{\dagger} + c_j c_{j+1}^{\dagger} + i(c_j^{\dagger} c_j - c_{j+1}^{\dagger} c_{j+1}) \right] = i(-1)^j e_j^{XX}$$

which gives an isomorphism of $\pi_{g\ell}$ with the representation of $JTL_{2L}$ (2.11) obtained in the XX chain (the factor $i(-1)^j$ leaving the cubic relation invariant). We note also that our periodic $g\ell(1|1)$ chain corresponds to periodic ordinary fermions $c_j^{(1)}$ if $L$ is even, and antiperiodic fermions if $L$ is odd.

### 2.3 The continuum limit and the importance of the symmetry algebra.

The continuum limit of the $g\ell(1|1)$ spin chain (2.5) is well known [31, 12], and corresponds to the symplectic fermions logarithmic CFT at $c = -2$ [29]. It also describes the long distance properties of dense polymers. Less well known are the associated algebraic features like lattice construction of left and right Virasoro modes $L_n$, $\bar{L}_n$ based on $JTL_{2L}$, as well as the centralizer of $JTL_{2L}$, which are the main topic of this paper. We recall here briefly that for an algebra $A$ and its representation space $\mathcal{H}_N$, the centralizer of $A$ is an algebra $\mathcal{Z}_A$ of all commuting operators $[\mathcal{Z}_A, A] = 0$, i.e., the centralizer is defined as the algebra of intertwiners $\mathcal{Z}_A = \text{End}_A(\mathcal{H}_N)$.

In the open case, the $g\ell(1|1)$ spin chain exhibits a large symmetry algebra dubbed $A_{1|1}$ in [12]. This algebra is the centralizer $\mathcal{Z}_{TL}$ of $TL_{2L}(0)$ and is generated by the identity and the five generators

$$F_{(1)} = \sum_j f_j,$$

$$F_{(1)}^{\dagger} = \sum_j f_j^{\dagger},$$

$$F_{(2)} = \sum_{j<j'} f_j f_{j'},$$

$$F_{(2)}^{\dagger} = \sum_{j<j'} f_j^{\dagger} f_{j'}^{\dagger},$$

$$N = \sum_j (-1)^j f_j^{\dagger} f_j - L,$$

where the fermions-number operator $N$ should not be confused with the notation for a number of sites $N$. The operators $F_{(1)}$, $F_{(1)}^{\dagger}$ generate the subalgebra $ps\ell(1|1)$ while $F_{(2)}$, $F_{(2)}^{\dagger}$, and $N$ generate an $s\ell(2)$ Lie subalgebra, with respect to which $F_{(1)}$ and $F_{(1)}^{\dagger}$ transform as a doublet. The resulting Lie superalgebra $A_{1|1}$ is the semi-direct product of these two algebras. It turns out to coincide with the full quantum group representation $\rho_{q\ell}(U_q s\ell(2))$, for $q = i$ (see Sec. 3 for definitions).

#### 2.3.1 The continuum (scaling) limit

It is time here to discuss a bit more precisely what is meant by the continuum limit, first in the general case. It is always possible [12] to consider a $N \rightarrow \infty$ limit (or so-called projective/inductive limit) of the algebraic structures in the spin-chains, especially the centralizer of the TL algebra and
its modules, and the modules over the TL algebra as well, from a purely algebraic point of view. But for our purpose more is required. We have chosen a Hamiltonian $H$ for the spin-chain (such as (2.5)), which is an element of (the representation of) an algebra like $TL_N$ or $JTL_N$ called generally as the “hamiltonian densities” algebra. Physically, we focus on low-energy (and long-wavelength) properties in a $N \to \infty$ limit. We can for instance introduce a lattice spacing between sites and consider the limit as taken with a lattice spacing distance tending to zero as $N \to \infty$, such that the length of the chain remains constant in the limit, equal to 1, say (hence the term continuum limit), and also with the Hamiltonian $H$ rescaled by $N$. Then, low energies and long wavelengths mean excitation energies and wavevectors of order 1 in these units. We are especially interested in cases where this continuum limit is a non-trivial conformal field theory, which in these units implies that excited states at energies of order 1 above the ground state do exist. Note that in practice, it is equivalent and more convenient to keep the lattice spacing constant as $N \to \infty$. In this case, low energies and long wavelengths mean excitation energies and wavevectors of order $1/N$. To get finite results to be compared with those of the CFT one must, for instance, rescale then the gaps by $N$, hence the name scaling limit, which we will use equivalently.

It is not entirely clear how the limit can be taken in a mathematically rigorous way, but roughly we want to take the eigenvectors of $H$ that have low-energy eigenvalues only, and we expect that the inner products among these vectors can be made to tend to some limits. Further, if we focus on long wavelength Fourier components of the set of local generators of the hamiltonian densities algebra, we expect their limits to exist, and their commutation relations to tend to those of the Virasoro generators $L_n$ (or $L_n + L_{-n}$ in the closed chain case), in the sense of strong convergence of operators in this basis of low-energy eigenvectors. Then, the modules over the (J)TL algebra restricted to the low-energy states become in the scaling limit modules over the universal enveloping algebra of the Virasoro algebra (the product of left and right Virasoro algebras in the closed chain case), or possibly even a larger algebra.

An advantage in using the centralizer is that it gives a control on representation theory of the “Hamiltonian densities” algebra on a finite chain and even on fusion rules, as was demonstrated in [7]. It is clear that the centralizer of the Hamiltonian densities is a symmetry of the low-lying spectrum of the Hamiltonian for any finite $N$. The symmetry (centralizer) algebra in the scaling limit, which commutes with the Virasoro algebra (the product of left and right Virasoro algebras in the closed chain case), must be thus at least as large as that in the finite-$N$ chains. For example, the decomposition of the open $g\ell(1|1)$ spin-chain as a (bi)module over the pair $(TL_N, A_{1|1})$ of mutual centralizers goes over in the scaling limit to a semi-infinite (‘staircase’) (bi)module [7] over the Virasoro algebra $\mathcal{V}(2)$, with the central charge $c = -2$, and (the scaling limit of) $A_{1|1}$, which is just an infinite-dimensional representation of $U_i s\ell(2)$. In this case, we thus have essentially the same centralizer for lattice and continuum models.

While the scenario described above can not be fully established analytically for general models, it is confirmed a posteriori by the validity of the results obtained in [7]. Of course, in some special cases such as free theories, much more can be said, and we will go back to the question, and a more rigorous reformulation, of the scaling limit for the closed $g\ell(1|1)$ spin-chains and the associated symplectic fermions CFT in the following sections.

In the periodic $g\ell(1|1)$ spin-chains, while the $g\ell(1|1)$ symmetry remains, the equivalent of the

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1See a more precise reformulation in Sec. 4.3 in the case of the periodic $g\ell(1|1)$ spin-chain.
generators $F_{(2)}$ and $F_{(2)}^\dagger$ introduced in (2.14) disappears, since the summation, extended around the chain, vanishes by anticommutation of the $f_j$’s. Meanwhile, the Temperley–Lieb algebra is replaced by $JTL_N$. What replaces the appealing symmetry algebra known to exist in the open case when one turns to periodic systems is the subject of the following section.

3 Symmetries for the spin chain

3.1 Quantum group results

We find it convenient here to start with some notations and results about quantum groups when the deformation parameter $q$ is a root of unity. The full quantum group $U_q\mathfrak{sl}(2)$ with $q = e^{i\pi/p}$, for $p \geq 2$, is generated by $E$, $F$, $K^{\pm 1}$, and $e$, $f$, $h$. The first three generators satisfy the standard quantum-group relations

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

with additional relations

$$E^p = F^p = 0, \quad K^{2p} = 1,$$

and the divided powers $f \sim F^p/[p]!$ and $e \sim E^p/[p]!$ satisfy the usual $s\ell(2)$-relations:

$$[h,e] = e, \quad [h,f] = -f, \quad [e,f] = 2h.$$

The full list of relations with comultiplication formulae are borrowed from [3] and listed in App. A where we also give relations to more usual (in the spin-chain literature) quantum group generators $S^\pm$, $S^z$ and $q^{S^z}$.

For applications to $g\ell(1|1)$ spin-chains, we consider only the case $p = 2$ and set in what follows $q \equiv i$. As a module over $U_q\mathfrak{sl}(2)$, the spin chain $\mathcal{H}_N$ is a tensor product of two-dimensional irreducible representations such that $E \rightarrow \sigma^+$, $F \rightarrow \sigma^-$, $K \rightarrow q\sigma^z$, and $e = f = 0$, where $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ and the Pauli matrices $\sigma^x,y,z$ are from (2.8). Using the $(N - 1)$-folded comultiplications (A11), (A13), and (A14) together with the Jordan-Wigner transformation (2.10), we obtain the representation $\rho_{g\ell} : U_q\mathfrak{sl}(2) \rightarrow \operatorname{End}_{\mathbb{C}}(\mathcal{H}_N)$ (usual fermionic expressions)

$$\rho_{g\ell}(E) \equiv \Delta^{N-1}(E) = \sum_{1 \leq j \leq N} q^{j_1c_j^\dagger} \rho_{g\ell}(K) = F_{(1)}^\dagger \rho_{g\ell}(K),$$

$$\rho_{g\ell}(F) \equiv \Delta^{N-1}(F) = \sum_{1 \leq j \leq N} q^{-1}c_j = q^{-1}F_{(1)},$$

$$\rho_{g\ell}(K) \equiv \Delta^{N-1}(K) = (-1)^{\rho_{g\ell}(2h)},$$

and

$$\rho_{g\ell}(e) \equiv \Delta^{N-1}(e) = \sum_{1 \leq j_1 < j_2 \leq N} (-1)^{j_1+j_2} q^{-1}c_{j_1}^\dagger c_{j_2}^\dagger = q^{-1} \sum_{1 \leq j_1 < j_2 \leq N} f_{j_1}^\dagger f_{j_2}^\dagger = q^{-1}F_{(2)},$$

$$\rho_{g\ell}(f) \equiv \Delta^{N-1}(f) = \sum_{1 \leq j_1 < j_2 \leq N} q^{j_1+j_2-1} c_{j_1} c_{j_2} = q \sum_{1 \leq j_1 < j_2 \leq N} f_{j_1} f_{j_2} = qF_{(2)},$$

$$\rho_{g\ell}(2h) \equiv [\rho_{g\ell}(e), \rho_{g\ell}(f)] = \sum_{1 \leq j \leq N} (-1)^j f_j^\dagger f_j - L,$$
where we also set the correspondence with the generators \((2|14)\) of the \(TL\)-centralizer \(3_{TL} = A_{1|1}\).

As it is noted above, the symmetry algebra \(A_{1|1}\) of the open spin-chain \([12]\) coincides with the representation of the full quantum group \(\rho_{gl}(U_q\mathfrak{sl}(2))\), for \(q = i\). The \(gl(1|1)\) (in fact \(ps\mathfrak{sl}(1|1)\) completed with \((-1)^N\)) meanwhile corresponds to the representation of \(\text{restricted} \) quantum group \(U_q\mathfrak{sl}(2)\) generated by \(E, F, \) and \(K^{\pm 1}\), with \(E : \mathcal{H}_{[n]} \rightarrow \mathcal{H}_{[n+1]}\) and \(F : \mathcal{H}_{[n]} \rightarrow \mathcal{H}_{[n-1]}\) (satisfying \(F^2 = E^2 = 0\)) and \(\mathcal{H}_{[n]}\) denotes the subspace with \(2h = S^z = n\). The statement that the representation \(\pi_{gl}\) of the \(JTL_N\) algebra obtained from the periodic \(gl(1|1)\) spin-chain \([2,2]-[2,5]\) does exhibit the \(gl(1|1)\) symmetry corresponds to an inclusion \(\rho_{gl}(U_q\mathfrak{sl}(2)) \subset 3_{JTL}\). The question is whether there are more generators in the centralizer \(3_{JTL}\) of \(JTL_N\).

### 3.2 Fourier transforms

It is convenient in the following to use Fourier transforms, and introduce, for \(1 \leq m \leq N\) (recall that we set \(N = 2L\)),

\[
\theta_{pm} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{-ikpm} c_k, \quad \theta_{pm}^\dagger = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{ikpm} c_k^\dagger \tag{3.3}
\]

with the set of allowed momenta

\[
p_m = \begin{cases} \frac{2m}{N}, & L - \text{even}, \\ \frac{(2m-1)\pi}{N}, & L - \text{odd}, \end{cases} \quad 1 \leq m \leq N, \tag{3.4}
\]

and with the usual anti-commutation relations

\[
\{\theta_{p_1}, \theta_{p_2}^\dagger\} = \delta_{p_1,p_2}, \quad \{\theta_{p_1}, \theta_{p_2}\} = \{\theta_{p_1}^\dagger, \theta_{p_2}^\dagger\} = 0.
\]

#### 3.2.1 Quantum group generators

We then find using a direct calculation that

\[
\rho_{gl}(E) = \sqrt{N} \theta_{\pi/2}^\dagger (-1)^{S^z}, \quad \rho_{gl}(F) = q^{-1} \sqrt{N} \theta_{3\pi/2}, \tag{3.5}
\]

and the renormalized powers are

\[
\rho_{gl}(e) = -q \sum_{p \neq \frac{\pi}{2}} \frac{e^{i(\pi-p)/p} \theta_p \theta_{\pi-p} + 2 \theta_p^\dagger \theta_{\pi/2}}{e^{i(\pi-p)/p} + 1} = \sum_{p = \frac{\pi}{2} + \frac{2\pi}{N}, \text{step } = \frac{2\pi}{N}} \tan \frac{1}{2} \left( \frac{\pi}{2} + p \right) \theta_p \theta_{\pi-p} - 2i \sum_{p \neq \frac{\pi}{2}} \frac{\theta_p \theta_{\pi/2}}{e^{i(\pi-p)/p} + 1}, \tag{3.6}
\]

and

\[
\rho_{gl}(f) = q \sum_{p \neq \frac{3\pi}{2}} \frac{e^{i(3\pi-p)/p} \theta_p \theta_{p-3\pi/2} - 2 \theta_p \theta_{3\pi/2}}{e^{i(3\pi-p)/p} - 1} = -\sum_{p = \frac{\pi}{2} + \frac{2\pi}{N}, \text{step } = \frac{2\pi}{N}} \cot \frac{1}{2} \left( \frac{\pi}{2} + p \right) \theta_p \theta_{p-3\pi/2} - 2i \sum_{p \neq \frac{3\pi}{2}} \frac{\theta_p \theta_{3\pi/2}}{e^{i(3\pi-p)/p} - 1}. \tag{3.7}
\]

These results agrees with ones established before in [33] in a slightly different basis.

\(^2\)A similar observation was made in [1] for \(e^{\pi i/2}\) replaced by \(H^{1+(S^z \text{mod } 2)}\), where \(H^{0,1}\) denotes the periodic (resp. antiperiodic) XX spin-chain Hamiltonian.
3.2.2 JTL generators in the Fourier transforms

Finally, we can reexpress the generators $e_j$ of $JTL_N$ themselves:

$$
e_{j}^g = (-1)^j \frac{i}{N} \sum_{p_1, p_2} e^{i(p_2 - p_1)}(i - e^{-ip_1})(1 + ie^{ip_2})\theta_{p_1}^\dag \theta_{p_2}, \quad 1 \leq j \leq N; \tag{3.8}$$

where the sum is taken over all possible momenta defined in (3.4). In what follows, we use simply the notation $e_j$ for the representation $e_{j}^g$ in (3.8).

In order to translate the $JTL_N$ generators $e_j$, we demand

$$u^2 f^{(i)}_j u^{-2} = f^{(i)}_{j+2} \tag{3.9}$$

which means, in terms of the Fourier modes, that

$$u^2 \theta_{pm} u^{-2} = -e^{2ip_m} \theta_{pm}, \quad u^2 \theta_{pm}^\dag u^{-2} = -e^{-2ip_m} \theta_{pm}^\dag. \tag{3.10}$$

It is then convenient to express the generator $u^2$ in terms of these Fourier modes. For this, we observe that, if $\theta$ and $\theta^\dag$ are a conjugate pair of fermions, $\theta^2 = (\theta^\dag)^2 = 0$, we have

$$e^{\lambda \theta} \theta e^{-\lambda \theta} = e^{-\lambda \theta},$$

$$e^{\lambda \theta^\dag} \theta^\dag e^{-\lambda \theta^\dag} = e^{\lambda \theta^\dag},$$

from which we can finally write the coherent state representation

$$u^2 = \exp \left[ \sum_{m=1}^{N} (i\pi - 2ip_m)\theta_{pm}^\dag \theta_{pm} \right]. \tag{3.11}$$

We can then easily check the only linear combinations of fermions which commute with $e_j$, where $1 \leq j \leq N$, and $u^2$ are $\theta_{3\pi/2}$ and $\theta_{\pi/2}^\dag$. So, we have

$$[JTL_N, U_q s\ell(2)] = 0,$$

as was mentioned above. To find additional generators in the centralizer $Z_{JTL}$, we expect to realize them as elements of the centralizer $Z_{TL}$ for the subalgebra $TL_N \subset JTL_N$ – the quantum group $U_q s\ell(2)$, which is additionally (with respect to $U_q s\ell(2)$) generated by the renormalized powers $e$ and $f$, and the Cartan $h = S^z/2$. It turns out that the centralizer of $JTL_N$ is identified with the Lusztig limit ($q \to i$) of appropriate polynomials in generators of $U_q s\ell(2)$ which we explicitly describe below.

3.3 The centralizer of $JTL_N$

Next, using (3.6), (3.7) and (3.8), we calculate the commutators between $e_j$ and the renormalized powers,

$$[e_j, f] = \begin{cases} 0, & 1 \leq j \leq N - 1, \\ 2i \sum_{p' \neq \frac{3\pi}{2}} (e^{i p'} - i) \theta_{p'} \theta_{3\pi/2}, & j = N, \end{cases} \tag{3.12}$$

$$[e_j, e] = \begin{cases} 0, & 1 \leq j \leq N - 1, \\ 2i \sum_{p' \neq \frac{\pi}{2}} (e^{-i p'} - i) \theta_{p'}^\dag \theta_{\pi/2}^\dag, & j = N. \end{cases} \tag{3.13}$$

---

$^3$We sometimes simplify expressions omitting more bulky and pedantic notations like $[\pi_{gel}(JTL_N), \rho_{gel}(U_q s\ell(2))] = 0.$
We thus see that the renormalized powers $e$ and $f$ are not contained in the centralizer $\mathfrak{z}_{\text{JTL}}$ unless $L = 1$ because of the last Temperley–Lieb generator $e_N$ making the system periodic. We note also that for a finite chain the only powers of $e$ and $f$ that commute with $\mathfrak{JTL}_N$ are $e^{N/2}$ and $f^{N/2}$. They are the highest non-zero powers and just mix the two $\mathfrak{JTL}_N$-invariants – the states with the all spins up or down.

We can then modify the $e$ and $f$ by elements from the respective annihilators of the commutators (3.12) and (3.13). The first obvious candidates for the modifying elements are $E \sim \theta_{\pi/2}^I$ and $F \sim \theta_{3\pi/2}$ (see (3.5)), respectively:

$$[e_j, f F] = [e_j, f] F = 0, \quad [e_j, e E] = [e_j, e] E = 0, \quad 1 \leq j \leq N.$$  

Moreover, there are many other elements in $U_q\mathfrak{sl}(2)$ commuting with all $e_j$'s:

$$[e_j, f^n F] = [e_j, f] F^n = 0, \quad [e_j, e^m E] = [e_j, e] E^m = 0, \quad 1 \leq j \leq N, \quad n, m \geq 1, \quad (3.14)$$

which can be easily proved by induction.

In particular, we have the equality in $U_q\mathfrak{sl}(2)$,

$$f^n F e^m E = f^n e^m F E,$$

which follows from the relations

$$[F, e^m] = m \frac{K + K^{-1}}{2} e^{m-1} E, \quad [E, f^n] = n \frac{K + K^{-1}}{2} f^{n-1} F,$$

where the first two are obtained using (A5).

**Definition 3.3.1.** We now introduce an associative algebra $U_q^{\text{odd}}\mathfrak{sl}(2)$. The algebra $U_q^{\text{odd}}\mathfrak{sl}(2)$ is generated by $F_n$, $E_n$ ($n, m \in \mathbb{N} \cup \{0\}$), $K^{\pm 1}$, and $h$ with the following defining relations

$$KE_m K^{-1} = q^2 E_m, \quad K F_n K^{-1} = q^{-2} F_n, \quad K^4 = 1, \quad (3.15)$$

$$[E_m, F_n] = \sum_{r=1}^{\min(n,m)} P_r(h) F_{n-r} E_{m-r}, \quad (3.16)$$

$$E_m E_n = E_n E_m = 0, \quad F_m F_n = F_n F_m = 0, \quad [K, h] = 0, \quad (3.17)$$

$$[h, E_m] = (m + \frac{1}{2}) E_m, \quad [h, F_n] = -(n + \frac{1}{2}) F_n, \quad (3.18)$$

where $P_r(h)$ are polynomials on $h$ from the usual $\mathfrak{sl}(2)$ relation $[e^m, f^n] = \sum_{r=1}^{\min(n,m)} P_r(h) f^{n-r} e^{m-r}$, and we assume that $\sum_{r=1}^{\min(n,m)} P_r(h) f^{n-r} e^{m-r}$.

The algebra $U_q^{\text{odd}}\mathfrak{sl}(2)$ has the PBW basis $E_n F_m h^k K^l$, with $n, m, k \geq 0$ and $0 \leq l \leq 3$. The positive Borel subalgebra is generated by $h$ and $E_n$ while the negative subalgebra – by $h$ and $F_n$, for $n \geq 0$.

**Remark 3.3.2.** We note there is an injective homomorphism $U_q^{\text{odd}}\mathfrak{sl}(2) \to U_q\mathfrak{sl}(2)$:

$$E_m \mapsto e^m E \frac{K^2 + 1}{2}, \quad F_n \mapsto f^n F \frac{K^2 + 1}{2}.$$  

Thus defined subalgebra in $U_q\mathfrak{sl}(2)$ can be realized as the limit $q \to i$ of the renormalized odd-powers of the $E$ and $F$ in $U_q\mathfrak{sl}(2)$ at generic $q$:

$$\frac{E^{2m+1}}{[2m+1]!} \xrightarrow{q \to i} e^m E, \quad \frac{F^{2n+1}}{[2n+1]!} \xrightarrow{q \to i} f^n F, \quad n, m \geq 0,$$

up to some irrelevant coefficients.
We now ready to formulate the main result of this section about the centralizer of the image of \( JTL_{2L}(0) \) under the representation \( \pi_{\text{JTL}} \).

**Theorem 3.3.3.** Let \( \mathfrak{z} \) be the subalgebra in \( \rho_{\text{JTL}}(U_q\mathfrak{sl}(2)) \) generated by \( U_q^{\text{odd}}\mathfrak{sl}(2) \) and \( t^L, e^L \). On the alternating periodic \( \mathfrak{gl}(1|1) \) spin chain \( \mathcal{H}_{2L} \), the centralizer \( \mathcal{Z}_{\text{JTL}} \) of the image of Jones–Temperley–Lieb algebra \( \pi_{\text{JTL}}(JTL_{2L}(0)) \) is the associative algebra \( \mathfrak{z} \), where \( \pi_{\text{JTL}} \) is defined in (2.2) and (2.4).

The full proof of this statement is too long and has been relegated to App. B.

### 3.3.4 Fermion expression for the centralizer \( \mathcal{Z}_{\text{JTL}} \)

We note here that generators of the \( JTL_{2L} \) centralizer on the \( \mathfrak{gl}(1|1) \) spin-chain stated in the theorem 3.3.3 just above have a simple fermionic expression, for \( n \geq 0 
\begin{align*}
F_{(2n+1)} &= \sum_{1 \leq j_1 < j_2 < \cdots < j_{2n+1} \leq 2L} f_{j_1}f_{j_2} \cdots f_{j_{2n+1}}, \\
F_{(2n+1)}^\dagger &= \sum_{1 \leq j_1 < j_2 < \cdots < j_{2n+1} \leq 2L} f_{j_1}^\dagger f_{j_2}^\dagger \cdots f_{j_{2n+1}}^\dagger, \\
F_{(2L)} &= f_1f_2 \cdots f_{2L}, \\
F_{(2L)}^\dagger &= f_1^\dagger f_2^\dagger \cdots f_{2L}^\dagger, \\
N &= \sum_{1 \leq j \leq 2L} (-1)^j f_1^\dagger f_j - L,
\end{align*}
which is to be compared with the generators of the centralizer \( A_{1|1} \) in the open case (2.14). The correspondence with the generators of \( U_q^{\text{odd}}\mathfrak{sl}(2) \) is \( F_{(2n+1)} = \frac{2^{n+1}}{n!} \rho_{\mathfrak{gl}}(F_n) \), \( F_{(2n+1)}^\dagger = \frac{2^n}{n!} \rho_{\mathfrak{gl}}(E_nK^{-1}) \), with \( n > 0 \), while \( n = 0 \) correspondence is given in (3.1), and \( N \) is proportional to \( \rho_{\mathfrak{gl}}(h) = S^z/2 \).

In our second paper [30], we rely on representation theory of the \( JTL_N \)-centralizer \( \mathcal{Z}_{\text{JTL}} \) in order to study the decomposition of the periodic spin-chain into indecomposable \( JTL_N \)-modules.

### 3.4 A note on the twisted model

We can also consider the antiperiodic model for the \( \mathfrak{gl}(1|1) \) chain, obtained by setting \( f_{2L+1}^{(\dagger)} = -f_1^{(\dagger)} \). The generators \( e_j \), for \( 1 \leq j \leq 2L - 1 \), have the same representation (2.2) while the last generator is then given by
\[ e_{2L} = (f_{2L} - f_1)(f_{2L}^\dagger - f_1^\dagger), \]
to be compared with (2.14). This does not provide more a representation of the \( JTL_N \) algebra but rather a representation of an abstract algebra generated by \( e_j \) and \( u^{\pm 2} \) with the relations (2.3) for \( 1 \leq j \leq N \) and (2.16), among others. We will call the corresponding algebra \( JTL_N^{tw} \), see a precise definition in our second paper [30]. The corresponding XX spin chain now is periodic for even spin, and antiperiodic for odd spin. Note that the action of \( JTL_N^{tw} \) does not commute with \( \mathfrak{gl}(1|1) \) generators \( F_{(1)} \) and \( F_{(1)}^\dagger \) (or \( F \) and \( E \), equivalently) defined in (2.14). The Hamiltonian densities algebra does not thus have \( \mathfrak{gl}(1|1) \) symmetry in this case.

We next study the centralizer of the representation of \( JTL_N^{tw} \). It turns out that the choice of “even” subalgebra in \( U_q\mathfrak{sl}(2) \) at generic \( q \), i.e., generated by the renormalized even-powers of the \( E \)
and $F$ gives in the limit $q \to i$ the centralizer for the representation of $JTL_N^{tw}$ on the spin-chain with.
the opposite twist — the usual $U(s\ell(2))$ generated by the $e$ and $f$. We give a proof below.

**Theorem 3.4.1.** On the alternating antiperiodic $g\ell(1|1)$ spin chain, the centralizer of the image of
the representation of the algebra $JTL_N^{tw}$ is the associative algebra $\rho_{gL}(U\ell(2))$.

**Proof.** We first check using expressions (3.22) for the $U\ell(2)$ generators in terms of $f_j$ and $f_j^\dagger$ fermions
that the action of $U\ell(2)$ indeed commutes with the additional generator $e_N = (f_N - f_1)(f_N^\dagger - f_1^\dagger)$;
that the generators $e_j$, for $1 \leq j \leq N - 1$, commute with the $U\ell(2)$ is obvious because the centralizer of
the $TL_N$ contains $\rho_{gL}(e)$ and $\rho_{gL}(f)$. Next, a simple calculation using again the $f_j$ and $f_j^\dagger$ fermions
shows that the $e_N$ does not commute with the operators $\rho_{gL}(e^{n\ell m h^{k \ell} F})$, $\rho_{gL}(e^{n\ell m h^{k \ell} E})$, $\rho_{gL}(e^{n\ell m h^{k \ell} F E})$, for $n, m, k \geq 0$. To show that there are no linear combinations of the operators in the centralizer, we
go to the Fourier transforms as in Sec. 3.2 introducing $\theta_\ell$ and $\theta_\ell^\dagger$ with the same formal expression (3.3)
but now the momenta $p_n$ takes values $\frac{2m}{N}$ for $L$ odd and $\frac{2m-1}{N}$ for $L$ even. Then, we repeat similar
calculations in the proof of Thm. 3.3.1 which are mainly presented in Lem. 3.4. The only care should
be taken for fermionic expressions for the $U_q\ell(2)$ generators in terms of $\theta_\ell$ and $\theta_\ell^\dagger$ which are different from the ones in (3.5)-(3.7). This way, we prove that the centralizer of the algebra generated by $e_j$, for $1 \leq j \leq N$, in the antiperiodic spin-chain is given by $\rho_{gL}(U\ell(2))$.

Finally, we show that the generators $u^{\pm 2}$ commute with the action of $U\ell(2)$. The $u^2$ acts on the
fermions $f_j$ and $f_j^\dagger$ formally in the same way (3.9) as in the periodic model but it changes sign in front
of $f_{j+2-N}$ whenever the position $j+2$ is greater than $N$ due to the antiperiodic conditions. We then obtain

$$u^2 \rho_{gL}(f)u^{-2} = q \sum_{1 \leq j_1 < j_2 \leq N} f_{j_1+2} f_{j_2+2} = q \sum_{1 \leq j_1 < j_2 \leq N} f_{j_1} f_{j_2} = \rho_{gL}(f)$$

and similarly for $\rho_{gL}(e)$. This finishes the proof. \qed

We emphasize that this antiperiodic $g\ell(1|1)$ spin chain does not have $g\ell(1|1)$ symmetry any longer.
We will come back briefly to this twisted case in other subsections – the main text meanwhile is only
devoted to the periodic case.

## 4 The scaling limit of the closed $g\ell(1|1)$ chains

In this Section, we discuss how to proceed from the $JTL_N$ generators to get Virasoro modes in the
non-chiral logarithmic conformal field theory of symplectic fermions: we show that the combinations

$$H(n) = - \sum_{j=1}^{N} e^{-i q j} e_j^{gL}, \quad P(n) = \frac{i}{2} \sum_{j=1}^{N} e^{-i q j} [e_j^{gL}, e_{j+1}^{gL}] , \quad q = \frac{n\pi}{L}, \quad (4.1)$$

of the (representation of) $JTL_N$ generators converge in a certain sense (the scaling limit) as $L \to \infty$
to the well-known symplectic fermions representation of the left and right Virasoro generators

$$\frac{L}{2\pi} H(n) \mapsto L_n + \bar{L}_{-n}, \quad \frac{L}{2\pi} P(n) \mapsto L_n - \bar{L}_{-n}.$$

For convenience, we begin with studying the $g\ell(1|1)$-Hamiltonian spectrum on a finite lattice in
Sec. [4.11] where we also introduce our technical notations for more suitable lattice fermions, and Sec. [4.2]
Then we give a formal definition of the scaling limit procedure in Sec. 4.3 and show the convergence of
the whole family of lattice higher Hamiltonians (with their Fourier transformations) to all generators
of the product \( \mathcal{V}(2) = \mathcal{V}(2) \otimes \overline{\mathcal{V}}(2) \) of the left and right Virasoro algebras with the central charge
\( c = -2 \). An important result of this section is that the scaling limit respects algebraic relations which
is discussed in Sec. 4.5.

### 4.1 The Hamiltonian and \( \chi-\eta \) fermions

We now go back to the periodic \( g\ell(1|1) \) spin-chain with the following \( JT \mathcal{L}_N \)-representation:

\[
e^{g\ell}_j = (f_j + f_{j+1}) \left( f_j^\dagger + f_{j+1}^\dagger \right), \quad f_{N+1} = f_1, \quad f_{N+1}^\dagger = f_1^\dagger, \quad 1 \leq j \leq N,
\]

which is discussed above in Sec. 2.1 and Sec. 3.2. We abuse the notation for the representation of the
\( JT \mathcal{L}_N \) generators in what follows and write simply \( e_j \) instead of \( e^{g\ell}_j \). Setting

\[
f_j^\dagger = i^j c_j^\dagger, \quad f_j = i^j c_j
\]

we get as well

\[
e_j = i(-1)^j \left[ c_j^\dagger c_{j+1}^\dagger - c_j^\dagger c_{j+1} + i(c_j^\dagger c_j - c_{j+1}^\dagger c_{j+1}) \right], \quad 1 \leq j \leq 2L,
\]

(4.2)

We find it more convenient to use Fourier transforms of the fermions \( c_j \) and \( c_j^\dagger \), and set

\[
c_j = \frac{1}{\sqrt{N}} \sum_{p_m} e^{ijp_m} \theta_{p_m}, \quad c_j^\dagger = \frac{1}{\sqrt{N}} \sum_{p_m} e^{-ijp_m} \theta_{p_m}^\dagger \tag{4.3}
\]

where the sums are taken over all the momenta \( p_m \) introduced in (3.4). We obtain then the Hamiltonian

\[
H = - \sum_{j=1}^{2L} e_j = 2 \sum_p (1 + \sin p) \theta_p^\dagger \theta_{p+\pi},
\]

(4.4)

which can be rewritten in (almost) diagonal form:

\[
H = 2 \sum_{p=\epsilon}^{\pi-\epsilon} \sin p \left( \chi_p^\dagger \chi_p - \eta_p^\dagger \eta_p \right) + 4 \chi_0^\dagger \eta_0,
\]

(4.5)

where \( \epsilon = \frac{2\pi}{N} \) and we use the notations

\[
\chi_p = \frac{1}{\sqrt{2}} \left( \sqrt{\cot \frac{p}{2}} \theta_p - \frac{1}{2}, \cot \frac{p}{2} \theta_p^\dagger + \frac{1}{2} \right), \quad \chi_0 = \frac{1}{\sqrt{2}} \left( \sqrt{\cot \frac{p}{2}} \theta_0 - \frac{1}{2}, \cot \frac{p}{2} \theta_0^\dagger + \frac{1}{2} \right),
\]

\[
\eta_p = \frac{1}{\sqrt{2}} \left( \sqrt{\cot \frac{p}{2}} \theta_p - \frac{1}{2}, \cot \frac{p}{2} \theta_p^\dagger + \frac{1}{2} \right), \quad \eta_0 = \frac{1}{\sqrt{2}} \left( \sqrt{\cot \frac{p}{2}} \theta_0 - \frac{1}{2}, \cot \frac{p}{2} \theta_0^\dagger + \frac{1}{2} \right),
\]

(4.6)

with momenta \( p \) shifted by \( \pi/2 \) and taking thus values \( p = p_n = \epsilon n \), where \( 1 \leq n \leq L - 1 \), for
even and odd \( L \). The normalization has been chosen to ensure relativistic excitations and satisfy the
anti-commutation relations

\[
\{ \chi_p^\dagger, \chi_p' \} = \{ \eta_p^\dagger, \eta_p' \} = \delta_{p,p'}, \quad \{ \chi_p, \eta_p' \} = \{ \chi_p^\dagger, \eta_p' \} = \{ \eta_p^\dagger, \chi_p' \} = 0.
\]

14
For convenience, we also give expressions for \( \theta^{(i)} \)s in terms of \( \chi^{(i)} \)s and \( \eta^{(i)} \)s,

\[
\begin{align*}
\theta_{p'}^+ & = \sqrt{\cot \left( \frac{p'}{2} \right)} \left( \chi_{p'}^+ + \eta_{p'}^+ \right), \\
\theta_{p'}^- & = \sqrt{\tan \left( \frac{p'}{2} \right)} \left( \chi_{p'}^- + \eta_{p'}^- \right), \\
\epsilon & \leq p' \leq \pi - \epsilon. 
\end{align*}
\] (4.7)

### 4.2 Hamiltonian spectrum and Jordan blocks

We now study the spectrum of the Hamiltonian (4.5) and analyze the Jordan blocks appearing on a finite lattice – the zero-mode term \( \chi_0^\dagger \eta_0 \) in the Hamiltonian is proportional to the Casimir operator of the quantum group \( U_q \mathfrak{su}(2) \) which implies the existence of non-trivial Jordan blocks in the spectrum.

We first note that the diagonal part \( H^{(d)} \) of the Hamiltonian has the eigenvectors

\[
v(\{p_k\}, \{p'_j\}) = \prod_{\{p_k\}} \eta_{p_k} \prod_{\{p'_j\}} \chi_{p'_j} | \uparrow \ldots \uparrow\rangle,
\] (4.8)

where \( | \uparrow \ldots \uparrow \rangle \) is the state with all spins up, with the eigenvalues

\[
2 \sum_{p \in \{p_k\}} \sin p - 2 \sum_{p \in \{p'_j\}} \sin p,
\] (4.9)

where the sets \( \{p_k\} \) and \( \{p'_j\} \) are any subsets in the set \( \{p_n = \pi n / L, \ 1 \leq n \leq L - 1\} \) of allowed momenta. We thus immediately find the four ground states

\[
\phi^2 = \prod_{p=\epsilon \text{ step}=\epsilon}^{\pi-\epsilon} \chi_p | \uparrow \ldots \uparrow\rangle, \quad \phi^1 = \chi_0 \eta_0 \phi^2, \quad \Omega = \eta_0 \phi^2, \quad \omega = \chi_0 \phi^2,
\] (4.10)

where the two fermionic states \( \phi^2 \) and \( \phi^1 \) belong to the sectors with \( S^z = +1 \) and \( S^z = -1 \), respectively, and the two bosonic states \( \Omega \) and \( \omega \) have \( S^z = 0 \).

What is crucial for logarithmic CFT is to know structure of Jordan blocks. The Hamiltonian we study has the off-diagonal part \( \chi_0^\dagger \eta_0 \) which generates Jordan blocks of rank 2. For example, the space of ground states has the following structure:

\[
\begin{array}{ccc}
\chi_0^\dagger & \quad & \eta_0 \\
\phi^2 \quad \downarrow \quad \quad \downarrow \quad \quad \omega \\
\phi^1 \quad \quad \quad \Omega \\
\end{array}
\] (4.11)

where the vacuum \( \Omega \) and the state \( \omega \) form a two-dimensional Jordan cell of the lowest eigenvalue for \( H \). We also show the action of \( F \sim \eta_0 \) and \( E \sim \chi_0^\dagger \) in (4.11).

The whole space of states \( \mathcal{H}_{2L} \) is generated from one cyclic vector \( \omega \) by the algebra of creation modes (including the zero modes generating the vacuum subspace)

\[
\mathcal{A} = \{ \chi_p^\dagger, \eta_p; \ p = \pi n / L, \ 0 \leq n < L \}.
\] (4.12)

The annihilation modes are

\[
\chi_p \Omega = \eta_p^\dagger \Omega = \chi_0^\dagger \Omega = \eta_0 \Omega = 0, \quad p \in \{\pi n / L, \ 1 \leq n \leq L - 1\}.
\] (4.13)
4.3 Emergence of the left and right Virasoro algebras

In this section, we study the scaling limit properties of the periodic spin-chain in detail. Recall that an essential ingredient in the general definition of the scaling limit sketched in Sec. 2.3.1 is the low-lying eigenstates of the Hamiltonian $H$. In order to study the action of JTL elements on these eigenstates in the limit $L \to \infty$ (recall $N = 2L$) we first truncate each $\mathcal{H}_2L$, keeping only eigenspaces up to an energy level $M$, for each positive number $M$. Each such truncated space turns out to be finite-dimensional in the limit, i.e., it depends on $M$ but not $L$. Then, keeping matrix elements of JTL elements that correspond to the action only within these truncated spaces of scaling states, we obtain well-defined operators in the limit $L \to \infty$. The corresponding operators acting on all scaling states of the CFT can be finally obtained (if they exist) in the second limit $M \to \infty$.

To put things a little more formally, we define the scaling limit denoted simply by ‘→’ as a limit over graded spaces of coinvariants with respect to smaller and smaller subalgebras in the creation modes algebra $A$ introduced in (4.12), i.e., along the following lines:

1. we consider a family of subalgebras $A[M] \subset A$ generated by the creation modes $\chi_p^\dagger$ and $\eta_p$ in the range $M < p < \pi - M\epsilon$, where $0 \leq M \leq L'/2$ and we set $L' = L - L \mod 2$ and recall $\epsilon = \pi/L$; we thus have a tower of subalgebras

$$0 = A[L'/2] \subset A[L'/2 - 1] \subset \cdots \subset A[2] \subset A[1] \subset A[0] \subset A; \quad (4.14)$$

2. we consider then vector-spaces $\mathcal{H}_2L/A[M]\mathcal{H}_2L$ of coinvariants graded by the Hamiltonian $H$, for any finite $M$. Note also that for each fixed $M$ these graded spaces are stabilized after some $L = L_0$ and they are finite-dimensional at $L \to \infty$; each of these stabilized spaces we denote as $C_M$. In physical terms, we keep only the low energy modes, which are those close to 0 and $\pi$.

3. we compute Fourier transforms of $e_j$’s and $[e_j, e_{j+1}]$’s corresponding to finite modes on the finite-dimensional graded vector-spaces of coinvariants $C_M$ in the limit $L \to \infty$ (physically, we keep only long wave-length contribution to low-lying excitations over the ground states). By computing in the limit $L \to \infty$ we mean here showing strong convergence of the sequence of operators (the Fourier transforms) parametrized by $L$ towards a particular operator acting on $C_M$.

4. we finally take a limit with respect to smaller and smaller subalgebras $A[M]$ in the tower (4.14), i.e., we take the second limit $M \to \infty$.

Note that we could equivalently consider the same construction/definition of the scaling limit based on a slightly different tower of subalgebras $\tilde{A}[M] \subset A$ which generate all eigenstates between the energy level $M + 1$ and the maximum one. But then a definition of each $\tilde{A}[M]$ is more complicated: it is generated by all monomials $\prod_{(p_k)} \eta_{p_k} \prod_{(p'_j)} \chi_{p'_j}^\dagger$ such that $2 \sum_{p \in (p_k)} \sin p + 2 \sum_{p' \in (p'_j)} \sin p > 2 \sin M \epsilon$ (recall the eigenvalues in (4.9)). This choice is probably more natural, in view of the discussion in the beginning of this subsection, but the first choice (4.14) of the tower of the subalgebras $A[M]$, which is much simpler technically, is enough for the purposes of this paper.

\footnote{Here, $A[M]\mathcal{H}_N$ means the image of the action of the whole algebra $A[M]$ on $\mathcal{H}_N$.
Then, coinvariants by definition are elements of the quotient-space $\mathcal{H}_N/A[M]\mathcal{H}_N$.}

\footnote{The strong convergence of operators requires a normed vector space, or positive-definite inner product. One can introduce this inner product here using the fact that a finite-dimensional vector space with non-degenerate indefinite inner product is a Krein space and, therefore, can be turned into a positive-definite inner product space \cite{33}. This applies to $\mathcal{H}_{2L}$ endowed with non-degenerate indefinite inner product $(\cdot, \cdot)$ such that $(f_jx, y) = (x, f_j^\dagger y)$ for any $x, y \in \mathcal{H}_{2L}$.}
4.3.1 The scaling limit of the Hamiltonian

Following the lines 1.-4. in the definition above, we first study the scaling limit of the Hamiltonian (4.5). We rewrite it in the normal-ordered form as

\[ H = 2 \sum_{p=\text{step}=\epsilon}^{\pi-\epsilon} \sin p (\chi_p^\dagger \chi_p + \eta_p \eta_p^\dagger) + 4 \chi_0 \eta_0 - 2 \sum_{p=\epsilon}^{\pi-\epsilon} \sin p, \]  

(4.15)

where we explicitly extracted the ground-state energy in the last sum. We can now linearize the dispersion relation around \( p = 0 \) and \( p = \pi \) in the first limit \( N \to \infty \) introducing the left-moving modes \( \chi^{(t)}_p = \chi_{\pi-p} \) and \( \eta^{(t)}_p = \eta_{\pi-p} \). The excitations over the Dirac sea are thus described by

\[ H = \frac{4\pi}{N} \sum_{m>0} m \left( \chi_p^\dagger \chi_p + \chi_p^\dagger \chi_p + \eta_p \eta_p^\dagger + \overline{\eta}_p \overline{\eta}_p^\dagger \right) + 4 \chi_0 \eta_0 + \langle \text{vac}|H|\text{vac} \rangle, \quad p \equiv \frac{m\pi}{L}. \]  

(4.16)

The ground-state energy has the following leading asymptotics in the large-\( N \) limit,

\[ \langle \text{vac}|H|\text{vac} \rangle = -2 \sum_{m=1}^{L-1} \sin \frac{m\pi}{L} = -2 \cot \frac{\pi}{N} = -\frac{2N}{\pi} + \frac{2\pi}{3N} + o(1/N), \]

where we used the trigonometric identity

\[ \sum_{k=1}^{n} \sin k\alpha = \frac{\sin \left( \frac{(n+1)\alpha}{2} \right)}{\sin \frac{\alpha}{2}}. \]

We thus obtain the expansion

\[ H = H^{(d)}(\eta) + H^{(n)} = -\frac{2N}{\pi} + \frac{4\pi}{N} \left( (L_0 + \bar{L}_0)^{(d)} + (L_0 + \bar{L}_0)^{(n)} - \frac{c}{12} \right) + o(1/N), \]

(4.17)

with the central charge \( c = -2 \), the diagonal part of the Hamiltonian in the scaling limit is

\[ (L_0 + \bar{L}_0)^{(d)} = \sum_{m>0} m \left( \chi_p^\dagger \chi_p + \chi_p^\dagger \chi_p + \eta_p \eta_p^\dagger + \overline{\eta}_p \overline{\eta}_p^\dagger \right), \quad p \equiv \frac{m\pi}{L}, \]

and the non-diagonal part is

\[ (L_0 + \bar{L}_0)^{(n)} = \frac{N}{\pi} \chi_0^\dagger \chi_0. \]

Finally, we introduce some new notation convenient for the scaling limit, for \( m > 0 \) and \( p = m\pi/L \)[6]

\[ \psi_m^1 = \sqrt{m} \chi_p, \quad \psi_m^2 = \sqrt{m} \eta_p, \quad \overline{\psi}_m^1 = \sqrt{m} \chi_p^\dagger, \quad \overline{\psi}_m^2 = -\sqrt{m} \eta_p^\dagger, \quad \psi_0^2 = \psi_0^2 = \sqrt{L/\pi} \chi_0^\dagger = \sqrt{L/\pi} \theta_0^\dagger, \]  

\[ \psi_{-m}^1 = -\sqrt{m} \eta_p, \quad \psi_{-m}^2 = \sqrt{m} \chi_p^\dagger, \quad \overline{\psi}_{-m}^1 = \sqrt{m} \eta_p^\dagger, \quad \overline{\psi}_{-m}^2 = -\sqrt{m} \chi_p^\dagger, \quad \psi_0^1 = \psi_0^1 = \sqrt{L/\pi} \eta_0 = \sqrt{L/\pi} \theta_0^\dagger, \]  

(4.18)

One has now the anti-commutation relations

\[ \{ \psi_m^\alpha, \psi_{m'}^\beta \} = mJ^{\alpha\beta} \delta_{m+m',0}, \quad \alpha, \beta \in \{1, 2\}, \]

[6] The distinction between \( L \) even and odd disappears as the new moments are defined with respect to \( \Theta_c \).
with the symplectic form $J^{12} = -J^{21} = 1$. So, we get the scaling limit

$$\frac{L}{2\pi}(H + \frac{2N}{\pi}) \mapsto L_0 + \tilde{L}_0 - \frac{c}{12} = \sum_{m>0} (\psi^2_m \psi^1_m - \psi^1_{-m} \psi^2_{m} + \psi \to \tilde{\psi}) + 2\psi^2_0 \psi^1_0 - \frac{c}{12}$$

$$= \sum_{m \in \mathbb{Z}} :\psi^2_{-m} \psi^1_{m}: + (\psi \to \tilde{\psi}) - \frac{c}{12}. \quad (4.19)$$

This expression of $L_0 + \tilde{L}_0$ is well known and appears in the theory of symplectic fermions [29]

$L_0 = \psi^2_0 \psi^1_0 + \sum_{m>0} (\psi^2_{-m} \psi^1_{m} - \psi^1_{-m} \psi^2_{m}) = \sum_{m \in \mathbb{Z}} :\psi^2_{-m} \psi^1_{m} :,$

$L_0 = \psi^2_0 \psi^1_0 + \sum_{m>0} (\tilde{\psi}^2_{-m} \tilde{\psi}^1_{m} - \tilde{\psi}^1_{-m} \tilde{\psi}^2_{m}) = \sum_{m \in \mathbb{Z}} :\tilde{\psi}^2_{-m} \tilde{\psi}^1_{m} :,$

where $L_0$ and $\tilde{L}_0$ have a common part, made of $\psi^2_0 \psi^1_0$.

### 4.3.2 The momentum operator

We next obtain the conformal spin operator $L_0 - \tilde{L}_0$ using lattice calculations. The general mapping [14] between anisotropic transfer matrices and evolution operators in CFT suggests that a lattice analogue of $T_{xy}$, the off-diagonal component of the stress tensor, is a momentum operator

$$P = \frac{1}{2} \sum_{j=1}^{N} [e_j, e_{j+1}]. \quad (4.20)$$

Straightforward calculations for the $g\ell(1|1)$ spin chain give

$$[e_j, e_{j+1}] = -i \left[ c_j c_{j+1}^\dagger - c_j^\dagger c_{j+1} - c_j + c_{j+1} c_{j+2}^\dagger + c_{j+1}^\dagger c_{j+2} + i (c_j c_{j+2}^\dagger - c_{j+2} c_j) \right], \quad (4.21)$$

so the momentum reads, in terms of fermion Fourier variables

$$P = \sum_{p=\epsilon}^{\pi-\epsilon} \sin 2p (\chi^\dagger_p \chi_p + \eta^\dagger_p \eta_p). \quad (4.22)$$

The scaling limit of the rescaled operator $\frac{L}{2\pi} P$ gives the conformal spin operator $L_0 - \tilde{L}_0$, keeping only the leading term:

$$\frac{L}{2\pi} P \mapsto \sum_{m \in \mathbb{Z}} :\psi^2_{-m} \psi^1_{m}: + (\psi \to \tilde{\psi}) = L_0 - \tilde{L}_0.$$

We also note that the generator $u^2$ of translations is simply related with the momentum in the continuum limit. Going to the $\eta$ and $\xi$ modes in (3.11) and using repeatedly that $e^{2i\pi} = 1$ to shift summation leads to

$$u^2 = \exp \left[ -2i \sum_{p=\epsilon}^{\pi-\epsilon} p (\chi^\dagger_p \chi_p + \eta^\dagger_p \eta_p) \right], \quad \epsilon = \frac{2\pi}{N}, \quad (4.23)$$

with the step $\epsilon$ in the sum. The term in the exponential is a linearized version of the momentum $P$. 

18
4.3.3 Higher Virasoro modes

It is interesting to obtain expressions for all other modes \( L_n \) and \( \tilde{L}_n \) of the stress tensor by sticking to the lattice some more. We take the Fourier transform of \( e_j \),

\[
H(n) = - \sum_{j=1}^{N} e^{-i q j} e_j = \sum_p \left[ 1 + e^{i q} + i e^{-i p} - i e^{i(p+q)} \right] \bar{\theta}_p \theta_{p+q+\pi}
\]

\[
= 4e^{i q/2} \left( \sum_{p=\pi}^{\pi-\epsilon} \left( \sin \frac{p+q}{2} \sin \frac{p}{2} \bar{\theta}_p \theta_{p+q+\pi} + [p \to p + \pi] \right) + \cos \frac{2}{2} \theta_p \theta_{q-\pi} \right), \quad q = \frac{n \pi}{L}, \quad (4.24)
\]

where \( n \) is integer. This sum can be split into the two sums \( \sum_{\epsilon}^{\pi-\epsilon} \) and \( \sum_{\pi-\epsilon}^{\pi} \) to be sure that the subscript \( p \) in the terms \( \theta_{p+q} \) takes values between \( \epsilon \) and \( \pi - \epsilon \) which is necessary to use the notations \( (4.7) \). We first consider the case \( 0 < n < L \). Using the formulas \( (4.7) \) expressing the \( \theta^{(i)} \)s in terms of the \( \chi^{(i)} \)s and \( \eta^{(i)} \)s, the \( H(n) \) can be rewritten as

\[
H(n) = 2e^{i q/2} \left( \sqrt{\sin q} \sum_{p=\pi}^{\pi-\epsilon} \chi_p \eta_q + \sum_{p=\pi}^{\pi-\epsilon} \sqrt{\sin p} \sin (p+q) \left( \chi_p \chi_{p+q} - \eta_p \eta_{p+q} \right) \right.
\]

\[
+ \left. \sum_{p=\epsilon}^{\pi-\epsilon} \sqrt{\sin (q-p)} \left( \chi_{\pi-p} \eta_{q-p} + \eta_{\pi-p} \chi_{q-p} \right) + \sqrt{\sin q} \left( \chi_{\pi-q} + \eta_{\pi-q} \right) \right). \quad (4.25)
\]

Using the transformation \( (4.18) \) to the fermions \( \psi^{1,2} \) and linearizing the dispersion relation, we thus have in the scaling limit (keeping the low- and high-\( p \) terms which have momenta close to \( 0 \) or \( \pi \), following the lines 1.-4. in the definition in Sec. 4.3), with a finite mode \( n \),

\[
\frac{L}{2\pi} H(n) \mapsto \psi_0^2 (\psi_n^1 + \bar{\psi}_n^1) + \sum_{m>0} (\psi_{-m}^2 \psi_{m+n}^1 + \psi_{m+n}^2 \psi_{-m}^1 + \bar{\psi}_{m}^2 \bar{\psi}_{-m-n}^1 + \bar{\psi}_{-m-n}^2 \psi_{m}^1)
\]

\[
+ \sum_{m=1}^{n-1} (\psi_m^2 \psi_{n-m}^1 + \psi_{n-m}^2 \psi_m^1) + (\psi_n^2 + \bar{\psi}_n^2) \psi_0^1.
\]

We finally obtain the contribution corresponding to low-lying excitations over the ground state,

\[
\frac{L}{2\pi} H(n) \mapsto \sum_{m \in \mathbb{Z}} \psi_{-m}^2 \psi_{m+n}^1 + \sum_{m \in \mathbb{Z}} \bar{\psi}_{-m}^2 \bar{\psi}_{m-n}^1 = L_n + \tilde{L}_n, \quad n > 0. \quad (4.26)
\]

This expressions are in agreement with \( (29) \) where the right-moving Virasoro generators for a non-zero integer \( n \) are expressed as

\[
L_n = \sum_{m \in \mathbb{Z}} \psi_{n-m}^2 \psi_m^1 \quad (4.27)
\]

and the generators for the left-moving part are

\[
\tilde{L}_n = \sum_{m \in \mathbb{Z}} \bar{\psi}_{n-m}^2 \bar{\psi}_m^1. \quad (4.28)
\]
The left and right Virasoro algebras of course commute, and the vacuum is annihilated by all non-negative modes.

Similarly, we can show that the scaling limit of $H(n)$ for $n < 0$ gives also the sum $L_n + \bar{L}_{-n}$ of left and right Virasoro generators. To cover the full Virasoro, we still need to get $L_n - \bar{L}_{-n}$.

It turns out that the corresponding lattice analogue of $L_n - \bar{L}_{-n}$ is the Fourier image of the momentum operator $P$ in (4.20):

$$P(n) = \frac{i}{2} \sum_{j=1}^{N} e^{-i q j} [e_j, e_{j+1}], \quad q = \frac{n \pi}{L}. $$

We first obtain expression for the commutator in terms of $\theta$-fermions,

$$[e_j, e_{j+1}] = \frac{1}{N} \sum_{p_1, p_2} e^{i p_2 (p_2 - p_1)} (e^{-i p_1} - e^{i p_2}) (i e^{-i p_1} (1 + i e^{i p_2}) \theta^\dagger_{p_1} \theta_{p_2}, \quad 1 \leq j \leq N, $$

which we use to get

$$P(n) = 4 e^{i q} \left( \sum_{p, q \epsilon \text{step}=\epsilon} \left( \cos p + \frac{q}{2} \right) \sin \frac{p + q}{2} \cos \frac{p + q}{2} \theta^\dagger_{p+q} \theta_{p+q} + [p \to p+\pi] \right) + \cos \frac{q}{2} \sin \frac{q}{2} \theta^\dagger_{q} \theta_{q+\pi}, \quad q = \frac{n \pi}{L}. $$

We consider the case $0 < n < L$. Using the formulas (4.7) expressing the $\theta$-fermions in terms of the $\chi$, we rewrite the $P(n)$ as

$$P(n) = 2 e^{i q} \left( \cos \frac{q}{2} \sqrt{\sin q} \chi^\dagger_0 (\chi_q - \eta_0) + \sum_{p \epsilon \text{step}=\epsilon} \left( \cos p + \frac{q}{2} \right) \sin (p + q) \sin (p + q) \left( \chi^\dagger_p \chi_{p+q} + \eta^\dagger_p \eta_{p+q} \right) \right. $$

$$\left. - \sum_{p \epsilon \text{step}=\epsilon} \cos (p - \frac{q}{2}) \sqrt{\sin (q-p)} \sin (q-p) \left( \chi^\dagger_{p-q} \eta_{p-q} - \eta^\dagger_{p-q} \chi_{p-q} \right) - \cos \frac{q}{2} \sqrt{\sin q} \left( \chi^\dagger_{n-q} - \eta^\dagger_{n-q} \right) \eta_0 \right) $$

which finally gives in the scaling limit (for any finite mode $n$)

$$\frac{L}{2 \pi} P(n) \mapsto \sum_{m \epsilon \mathbb{Z}} \psi^2_{-m+n} \psi^1_m - \sum_{m \epsilon \mathbb{Z}} \bar{\psi}^2_{-m-n} \bar{\psi}^1_m = L_n - \bar{L}_{-n}, \quad n > 0. $$

We can similarly show that the scaling limit of $P(n)$ for $n < 0$ gives also $L_n - \bar{L}_{-n}$.

### 4.4 The twisted model

We can perform the same analysis in the model with antiperiodic $gl(1|1)$ fermions discussed in Sec. 3.3. This requires the introduction of a new set of momenta replacing (3.4):

$$p_m = \begin{cases} \frac{2 \pi m}{2(N - 1) + m}, & L - \text{odd}, \\ \frac{2 \pi m}{2(N - 1) - m}, & L - \text{even}, \end{cases} \quad m = 1, \ldots, N \quad (4.29)$$

with, as before, $1 \leq m \leq N$, while the formal expression (3.3) for the fermions $\theta_{p_m}$ and $\theta^\dagger_{p_m}$ is the same. Proceeding, we now find the Hamiltonian $H_{a.p.}$ in the antiperiodic model as

$$H_{a.p.} = 2 \sum_p (1 + \sin p) \theta^\dagger_p \theta_{p+\pi} \quad (4.30)$$
which is the same formal expression as for the periodic model. The difference is that now the momenta run over a different set. As a result, the values $p = \frac{\pi}{2}, \frac{3\pi}{2}$ are not allowed, and there are no zero modes. The ground state in this model is non-degenerate, and we find

$$\langle \text{vac}|H_{a,p}|\text{vac}\rangle = -2\cot\frac{\pi}{2N} - 2\cot\frac{\pi}{3N} = -\frac{2N}{\pi} - \frac{\pi}{3N}$$  \hspace{1cm} (4.31)$$

which corresponds to an effective central charge $c_{\text{eff}} = 1 = -\frac{2}{-24 \times \frac{1}{8}}$. We introduce $\chi_p^{(\dagger)}$ and $\eta_p^{(\dagger)}$ fermions generating Hamiltonian eigenstates from the vacuum by the same formal definition (4.6) but now momenta takes values $\frac{\epsilon}{2} \leq p \leq \pi - \frac{\epsilon}{2}$ with the step $\epsilon = \pi/L$. The normal ordered Hamiltonian then reads

$$H_{a,p} = 2 \sum_{p=\epsilon/2}^{\pi-\epsilon/2} \sin p (\chi_p^{\dagger} \chi_p + \eta_p^{\dagger} \eta_p) + \langle \text{vac}|H_{a,p}|\text{vac}\rangle$$ \hspace{1cm} (4.32)$$

with $\epsilon = \frac{2\pi}{N}$, and the momenta are of the form $p = \frac{(2m-1)\pi}{N}$, with $1 \leq m \leq L$. Introducing exactly the same dictionary for modes (4.18), with $p_m = (m-1/2)\frac{\pi}{L}$, gives the scaling limit

$$\frac{L}{2\pi} \left( H_{a,p} + \frac{2N}{\pi} \right) \mapsto L_0 + \bar{L}_0 - \frac{1}{12}$$ \hspace{1cm} (4.33)$$

with the representation of the Virasoro modes now

$$L_0 + \bar{L}_0 = \sum_{m \in \mathbb{Z}} : \psi_{-m+1/2}^{2} \psi_{m-1/2}^{1} : + (\psi \rightarrow \bar{\psi}).$$  \hspace{1cm} (4.34)$$

Similar analysis of the Hamiltonians $H(n)$ and of the momenta modes $P(n)$ provides the expected formulas for $L_n$ and $\bar{L}_n$ in this model as well.

### 4.5 From $JTL_N$ to $\mathcal{V}(2) \otimes \mathcal{V}(2)$?

It is possible to calculate the scaling limit of more complicated expressions. In particular, it is known that the scaling limit of the logarithm of the transfer matrix itself involves only $L_0$ and $\bar{L}_0$. Expanding this transfer matrix in powers of the spectral parameter shows that there is an infinity of lattice Hamiltonians (see below for more details) and momenta with identical scaling limits [14]. For instance, instead of taking $H \propto -\sum e_i$ we could take the next Hamiltonian $H \propto \sum [e_j, [e_{j+1}, e_{j+2}]]$, which should also give $L_0 + \bar{L}_0$ when acting on low energy states. This shows that the correspondence between $JTL_N$ elements and elements in the product $\mathcal{V}(2)$ of left and right Virasoro algebras is certainly not a bijection.

While most of the foregoing results (such as the existence of expressions in JTL generators which converge in the scaling limit to Virasoro generators) are expected to hold for more general models, how this precisely occurs is not fully understood in general, because of our only partial control on the eigenstates of the Hamiltonian and matrix elements of generators (through the algebraic Bethe ansatz). Even the fact that the Fourier modes of the local density of energy and momentum give, when restricting to low lying energy states, the modes of the stress energy tensor, can only be established analytically in free fermionic models - the Ising chain [14], and the $g\ell(1|1)$ chain here. 

\footnote{Like in the periodic model, the difference with $\frac{\pi}{2}$ in the notation for $\chi_p$, $\eta_p$ fermions and $\theta_p$ fermions makes both cases $L$ even and odd similar.}
Indeed, a major difficulty in studying the correspondence between lattice algebras and \( \mathcal{V}(2) \) is that the lattice algebra acts on all the states of the lattice model, including a priori the high energy states which disappear in the continuum limit. As a result, it is not clear on general grounds how to relate the structure of \( JTL_N \) modules and \( \mathcal{V}(2) \) modules: for instance, we could have two \( JTL_N \) modules in the spin chain mapped by some words in \( JTL_N \) generators, but in such a way that this connection involves only highly excited states, and disappears when we restrict to excitations at small momentum and energy. On the other hand, it is tempting to speculate in general that low and high energy states are not special in an algebraic sense, so that, if a mapping exists between two modules, it will still be present when restricting to the scaling limit.

Of course, for \( g\ell(1|1) \) things are particularly simple: a look at \( H(n) \) (4.25) for instance shows that, for any finite \( n \) as \( L \) becomes large, it only connects low energy states to low energy states and high energy states to high energy states. This implies that the continuum limit of products of \( H(n) \)'s should coincide with the product of their continuum limits — in particular, we can easily compute the commutators

\[
[H(n), H(-n)] = -4 \sin q \sum_p \sin(2p)\theta_p^\dagger \theta_p = -4 \sin(q)P, \quad q = \frac{n\pi}{L},
\]

using the finite-chain fermionic expression (4.24), and their scaling limit \( \left[ \frac{L}{2\pi} H(n), \frac{L}{2\pi} H(-n) \right] \rightarrow 2n(L_0 - \bar{L}_0) \). On the other hand, the commutator of the scaling limits (4.26) of \( \frac{L}{2\pi} H(\pm n) \) gives the same expression. One can then for instance obtain the central charge directly from the commutator \([H(n), P(-n)]\). Indeed, a long calculation gives

\[
\left[ \frac{L}{2\pi} H(n), \frac{L}{2\pi} P(-n) \right] = \frac{L^3}{2\pi^2} e^{-iQ/2} \sin(q/2) \left( \sin^2 \frac{qL}{2} [H[\cos^2] - \cos^2 \frac{qL}{2} (H - 3H[\sin^2])] \right), \quad (4.35)
\]

where the Hamiltonian \( H \) is given in (4.4) and we use the notation for modified (by a weight \( f(p) \)) Hamiltonians \( H[f] = \sum_p f(p)(1 + \sin p)\theta_p^\dagger \theta_{p+\pi} \), for any periodic function \( f(p + 2\pi) = f(p) \). For \( f(p) = \cos^2(p) \), we have the normal-ordered expression (in terms of the \( \chi\eta \) fermions introduced above)

\[
H[\cos^2] = \sum_{p=\epsilon} \sin^3 p \left( \chi_p^\dagger \chi_p + \eta_p \eta_p^\dagger \right) - 2 \sum_{p=\epsilon} \sin^3 p, \quad (4.36)
\]

where we have extracted the ground-state value of \( H[\cos^2] \) in the second sum, compare with (4.15), which has the leading asymptotic for large \( L \)

\[
\langle \text{vac}|H[\cos^2]|\text{vac}\rangle = -2 \sum_{m=1}^{L-1} \sin^3 \frac{m\pi}{L} = -\frac{4N}{3\pi} + o(1/N), \quad (4.37)
\]

with an \( N \)-linear contribution canceled. We then note \( H[\sin^2] = H - H[\cos^2] \) and that the first sum in (4.36) give a contribution of order \( 1/L^2 \) to the Hamiltonian \( H \) in (4.35) which has to be neglected in the scaling limit. We thus keep only the vacuum value (4.37) to obtain finally the scaling limit of (4.35)

\[
\left[ \frac{L}{2\pi} H(n), \frac{L}{2\pi} P(-n) \right] \rightarrow 2n(L_0 - \bar{L}_0) + \xi_6 n(n^2 - 1) = [L_n + \bar{L}_{-n}, L_{-n} - \bar{L}_n].
\]

A similar calculation using the fermions shows that all other products also commute with the scaling limit, so that in particular the scaling limit of a commutator is the commutator of the scaling limits.

22
4.5.1 Higher Hamiltonians and their Fourier images

It is also interesting (and we will use these results in our subsequent papers) to consider the scaling limit of the whole family of higher Hamiltonians in the periodic $gl(1|1)$ spin-chain. These can be obtained using the underlying integrable structure, and building the family of commuting diagonal-to-diagonal transfer matrices $T_l(u)$. An expansion of (the logarithm of) $T_l(u)$ in powers of $u$ produce an infinite of commuting operators $H_l(0)$, with $l \geq 0$, see [14] and references therein. To explore the properties of these $H_l(0)$, we first compute multiple commutators of the $JTL_N$-generators $e_j$

$$E_{j,l} = [e_j, \ldots, e_{j+l-2}, e_{j+l-1}, e_{j+l}]$$, \quad 1 \leq j \leq N. \quad (4.38)

By an induction, we prove the following, for $1 \leq j \leq N$ and $l > 0$,

$$E_{j,l} = \begin{cases} (-1)^{j} \frac{i}{N} \sum_{p_1,p_2} e^{ij(p_2-p_1)}(e^{ip_2} + e^{-ip_1})(i - e^{-ip_1})(1 + i e^{ip_2})\theta_{p_1}\theta_{p_2}, & l - even, \\ -\frac{i}{N} \sum_{p_1,p_2} e^{ij(p_2-p_1)}(e^{ip_2} - e^{-ip_1})(i - e^{-ip_1})(1 + i e^{ip_2})\theta_{p_1}\theta_{p_2}, & l - odd, \end{cases} \quad (4.39)$$

where the sums are taken over all allowed momenta $p_1, p_2$ from the set $\{3,4\}$. Then, the integrable Hamiltonians $H_l(0)$ are given by the sums of the $E_{j,l}$ over all sites. In particular, the operators $H_0(0) = H$ and $H_1(0) = P$ were studied above in Sec. 4.3 where we also studied their Fourier images. To find fermionic expressions for Fourier images of all the higher Hamiltonians

$$H_l(n) = \frac{-1}{2}\frac{e^{-il\frac{p}{L}}}{N} \sum_{j=1}^{N} e^{-iqj}E_{j,l}, \quad q = \frac{n\pi}{L} \quad and \quad l \in \mathbb{N}, \quad n \in \mathbb{Z},$$

we repeat all the previous steps in the study of $H(n)$ and $P(n)$ in Sec. 4.3 and get

$$H_l(n) = 4e^{iq(l+1)/2} \sum_{p=0}^{2\pi-\epsilon} \cos(l(p + q/2)) \cos\frac{P}{2} \begin{cases} \cos\frac{p + q}{2} \theta_{p+q}, & l - even, \\ \sin\frac{p + q}{2} \theta_{p+q}, & l - odd, \end{cases}$$

which we rewrite in the $\chi-\eta$ fermions, for integer $0 \leq n < L$, as

$$H_l(n) = 2e^{iq(l+1)/2} \left( \sum_{p=\epsilon}^{\pi-\epsilon} \cos l(p + \frac{q}{2}) \sqrt{\sin(p)} \sin(p + q) \left( \chi^\dagger_p \chi_{p+q} - (-1)^l \eta^\dagger_p \eta_{p+q} \right) \right. \right.$$

$$+ \left. \sum_{p=\epsilon}^{q-\epsilon} \cos l(p - \frac{q}{2}) \sqrt{\sin(q)} \sin(q - p) \left( \eta^\dagger_{p-q} \chi_{p-q} + (-1)^l \chi^\dagger_{p-q} \eta_{p-q} \right) \right.$$

$$+ \left. \cos\frac{iq}{2} \sqrt{\sin(q)} \left( \chi^\dagger_0 \chi_q - (-1)^l \chi^\dagger_q \eta_0 + \eta^\dagger_{-q} \eta_0 + (-1)^l \chi^\dagger_{-q} \eta_0 \right) \right) + \delta_{n,0} (1 + (-1)^l \chi^\dagger_0 \eta_0).$$

This finally gives in the scaling limit (for finite $n$ and $l$) the left and right Virasoro generators:

$$\frac{L}{2\pi} H_l(n) \mapsto \sum_{m \in \mathbb{Z}} \psi^2_{m+n} \psi^1_m + (-1)^l \sum_{m \in \mathbb{Z}} \bar{\psi}^2_{m-n} \bar{\psi}^1_m = L_n + (-1)^l \bar{L}_{-n}, \quad l \geq 0, \quad (4.40)$$

which does not depend on $l$, only on its value modulo 2. We note that this result is obtained to taking the leading term in the expansion $\cos l(p + \frac{q}{2}) = 1 - (m+\frac{n}{2})^2 \pi^2 l^2 + \ldots$ only. It is interesting to explore
the content of the higher order terms in the scaling limit, and their relation with conserved quantities in the conformal field theory. We leave this problem for a future work \cite{35}. A very similar calculation gives the same scaling limit \eqref{4.40} for all negative modes $n < 0$ as well.

To examine further the relation between $JTL_N$ and $\mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$, it is possible to compare the modules over these two algebras present in the spin chain and the continuum limit respectively. This will be discussed in our second paper \cite{30}. But before launching into representation theory, a lot can be learned from the analysis of the lattice symmetries, to which we now return.

5 Symmetries and the scaling limit

The expectation that the natural equivalent of the $JTL_N$ algebra in the continuum, would be the product of the left and right Virasoro algebras encounters difficulties when we consider the centralizer $Z_{\mathcal{V}}$ of $\mathcal{V}(2) = \mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$. While for finite chains, the centralizer of $JTL_N$ is $U_{\text{odd}}^{s\ell}(2)$, in the continuum limit, it is well known that $\mathcal{V}(2)$ commutes at least with $g^{\ell}(1|1)$ and the $s\ell(2)$ symmetry discovered by Kausch. The situation in the boundary and periodic cases is thus quite different. Several questions thus arise, the most obvious being, what happens to $U_{\text{odd}}^{s\ell}(2)$ and how it is related with the continuum $s\ell(2)$. This is what we consider first.

We first introduce the continuum fermions via the mode expansion in the complex plane \cite{29}

$$\Phi^\alpha(z, \bar{z}) = \phi^\alpha_0 - i \psi^\alpha_0 \ln(z \bar{z}) + \sum_{m \neq 0}^{\infty} \frac{\psi^\alpha_m}{m} z^{-m} + \frac{\bar{\psi}^\alpha_m}{m} \bar{z}^{-m}, \quad \alpha, \beta \in \{1, 2\}, \quad (5.1)$$

where the modes have the anti-commutation relations

$$\{\psi^\alpha_n, \psi^\alpha_{m'}\} = mJ^{\alpha\beta} \delta_{m+m',0}, \quad \{\phi^1_0, \psi^2_0\} = i, \quad \{\phi^2_0, \psi^1_0\} = -i,$n

with the symplectic form $J^{12} = -J^{21} = 1$. Then, the generators of the global $s\ell(2)$ in the symplectic-fermion theory are

$$Q^\alpha = d^\alpha_{\alpha\beta} \left( i \phi^\alpha_0 \psi^\beta_0 + \sum_{n=1}^{\infty} \left( \frac{\psi^\alpha_n \bar{\psi}^\alpha_n}{n} + \frac{\bar{\psi}^\alpha_n \psi^\alpha_n}{n} \right) \right) \quad (5.2)$$

with

$$d^0_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d^1_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d^2_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (5.3)$$

with $[Q^\alpha, Q^\beta] = f^{ab}_{\gamma} Q^\gamma$ and $f^{01}_2 = -1$.

A superficial look at the model would suggest that this $s\ell(2)$ should somehow ‘emerge’ from the lattice symmetry $U_q^{s\ell}(2)$. In the open case indeed, the lattice model has the full quantum group $U_q^{s\ell}(2)$ symmetry, and the $s\ell(2)$ part in that case coincides with the $s\ell(2)$ of the continuum limit. While in the periodic case, the lattice model has less symmetry, the degeneracies remain of the form $4j$, see \cite{30}, and would suggest that the non-commutation with Temperley–Lieb of the ‘even part’ of $U_q^{s\ell}(2)$ is a lattice effect disappearing in the continuum limit. A more careful look at the model shows that this expectation is not correct at all. Maybe the quicker is simply to work out the scaling limit of the generators.
5.1 Scaling limit of the (lattice) $\mathfrak{sl}(2)$ generators $e$ and $f$

Here, we consider the scaling limit of the $\mathfrak{sl}(2)$ generators – the renormalized powers $e$ and $f$. Recall first the fermionic formula (3.6) for the operator $e$,

$$ e = -\sum_{\substack{p=\epsilon \text{ step} = \epsilon \text{ step}}}^{\pi - \epsilon} \cot \frac{p}{2} \theta_{p+\frac{\epsilon}{2}} \theta_{p-\frac{\epsilon}{2}} - 2i \sum_{p \neq \pi} \theta_{p+\frac{\pi}{2}} \theta_{p-\frac{\pi}{2}} e^{ip + 1} $$

which can be rewritten in the $\chi$-$\eta$ notation as

$$ e = \phi^0_0 \psi^2_0 + \sum_{p=\epsilon}^{\pi - \epsilon} \eta^+_p \chi^+_p - \sum_{p=\epsilon}^{\pi - \epsilon} \eta^+_p \chi^+_p e^{-ip/2}( (1 - i) \chi^+_p - (1 + i) \eta^+_p ) \chi^+_0, $$

where we introduce the operators $\phi^{1,2}_0$ conjugated to $\psi^{1,2}_0$,

$$ \phi^0 = -i \sqrt{\pi/L} \theta^+_{3\pi/2}, \quad \phi^1 = i \sqrt{\pi/L} \theta^+_{3\pi/2} $$

Going to the $\psi^n$-fermions defined in (4.18) gives the scaling limit

$$ e \mapsto \phi^0_0 \psi^2_0 + \sum_{m>0} \left( \frac{\psi^2_m \psi^{-2}_m}{m} - \frac{\psi^2_{-m} \psi^{-2}_{-m}}{m} \right) + \sum_{m \neq 0} \frac{1}{m} \left( (i + 1) \psi^2_m \psi^{-2}_0 + (i - 1) \psi^2_{-m} \psi^{-2}_0 \right). $$

The scaling limit for $f$ is given by similar formula with the substitution $\psi^2 \to \psi^1$, $\phi^2 \to \phi^1$.

As we can see, the scaling limit of the renormalized powers $e$ and $f$ describes a different $\mathfrak{sl}(2)$ than the global $\mathfrak{sl}(2)$ we have in the symplectic fermions theory mostly because of the second sum. Mainly for these reasons, the four-dimensional space of the ground states (4.11), spanned by the vacuum $\Omega$, the state $\omega$ and the two fermionic states $\phi^{1,2}$, is not invariant under the action of $e$ and $f$ on a finite lattice. Indeed, it is easy to check using (5.5) that the vacuum $\Omega$ is the $\mathfrak{sl}(2)$-invariant while its logarithmic partner $\omega$ is not an invariant,

$$ e(\omega) = \sum_{m=1}^{L-1} (-1)^{m-1} (1 - i)^{m-1} \frac{e^{-\pi m^2/L}}{\sqrt{\sin m \pi}} \prod_{j=1}^{L-1} \chi_{\frac{j}{L}} \uparrow \cdots \uparrow \equiv \omega' \quad \text{and} \quad e(\phi^1) = i \phi^2 + \eta \omega', $$

which is not surprising because the Hamiltonian on a finite lattice does not commute with the $\mathfrak{sl}(2)$ generated by the $e$ and $f$.

5.2 Scaling limit of $U_q^{\text{odd}}\mathfrak{sl}(2)$ in the periodic model

For a finite chain, the additional elements $f^L$ and $e^L$ which do not belong to $U_q^{\text{odd}}\mathfrak{sl}(2)$ and commute with $JTL_{2L}$ have no meaning at the scaling limit $L \to \infty$ and we thus suppress them and make no difference between $3JTL$ and $U_q^{\text{odd}}\mathfrak{sl}(2)$.
In the scaling limit, the centralizer $U_{\text{odd}} s\ell(2)$ gives rise to the zero modes $F \mapsto \psi_0^1$ and $E \mapsto \psi_0^2$ and products of these with the renormalized even powers. We thus get the limit of the generators

$$E_n = e^n E \mapsto \left[ \sum_{m>0} \left( \frac{\psi_m \psi_{-m}}{m} - \frac{\bar{\psi}_m \bar{\psi}_{-m}}{m} \right) \right]^n \psi_0^2, \quad F_n = f^n F \mapsto \left[ \sum_{m>0} \left( \frac{\psi_m \psi_{-m}}{m} - \frac{\bar{\psi}_m \bar{\psi}_{-m}}{m} \right) \right]^n \psi_0^1, \quad (5.9)$$

and the representation of the Cartan element $h$ is given on a finite lattice by

$$2h = S^z = \sum_p \theta_p^1 \theta_p^1 - L = \sum_{p=0}^{\pi-\epsilon} \left( \chi_p \chi_p - \eta_p \eta_p \right) \quad (5.10)$$

and has the limit

$$2h \mapsto -i \left( \psi_0 \phi_0^1 + \psi_0 \phi_0^2 \right) + \sum_{m>0} \frac{1}{m} \left( \psi_m^2 \psi_m^1 + \psi_m^1 \psi_m^2 + [\psi \to \bar{\psi}] \right) \quad (5.11)$$

while the generator $K = (-1)^{2h}$. Note that the value of $S^z$ on the lattice is twice the value of the third component of the $s\ell(2)$ isospin in the continuum. The case $S^z$ even (odd) corresponds to bosonic (fermionic) states, so the continuum isospin is integer (respectively, half integer).

Using the symplectic fermions expressions (4.127) and (4.128) for the left and right Virasoro modes $L_n, \bar{L}_n$, we state that the scaling limit (5.9) and (5.11) of the $JTL_N$ centralizer $\mathcal{Z}_{NTL}$ does commute with the full Virasoro algebra $\mathcal{V}(2)$ (the multiplication by the zero modes suppresses all the unwanted terms in the expression (5.8)). We should also note that the limit of $\mathcal{Z}_{NTL}$ can not be obtained as the multiplication of the global $s\ell(2)$ with the zero modes. There remains a minus sign between the left and right moving components, which is at odds with the expression for the global $s\ell(2)$ in symplectic fermions.

### 5.3 How to get the (continuum) $s\ell(2)$ generators from the spin chain?

It is possible to study in more detail the scaling limit of the lattice fermions themselves, and build formulas for quantities going over to the $s\ell(2)$ generators in the continuum limit. A little trial and error suggests the introduction of

$$e_+ = \chi_0^+ \eta_0^- - \sum_{p=0}^{\pi-\epsilon} \left( \cos p \right) \eta_p^+ \chi_{p-\pi}^-, \quad f_+ = \eta_0 \chi_0^+ + \sum_{p=\pi}^{\epsilon} \left( \cos p \right)^{-1} \chi_p \eta_{p-\pi}^- \quad (5.12)$$

which are “slightly” modified (by a weight $\cos p$, compare with (5.5)) operators in $\theta$-fermions expression

$$e_+ = \frac{1}{2} \sum_p \left( \sin p \right) \theta_p \theta_{-p}, \quad f_+ = \frac{1}{2} \sum_p \left( \sin p \right)^{-1} \theta_{-p} \theta_p, \quad \text{with} \quad [e_+, f_+] = S^z \quad (5.13)$$

We now have

$$e_+ \mapsto -i \phi_0^2 \psi_0^2 + \sum_{m>0} \left( \frac{\psi_m^2 \psi_{-m}}{m} + \frac{\bar{\psi}_m^2 \bar{\psi}_{-m}}{m} \right), \quad f_+ \mapsto i \phi_0^1 \psi_0^1 + \sum_{m>0} \left( \frac{\psi_m^1 \psi_{-m}}{m} + \frac{\bar{\psi}_m^1 \bar{\psi}_{-m}}{m} \right) \quad (5.14)$$

in agreement with the expressions (5.2) of the global $s\ell(2)$ generators. It is straightforward to check that, on a finite-lattice, the $e_+$ and $f_+$ commute with the Hamiltonian (4.4) which can be easily checked using (5.13):

$$[e_+, H] = \sum_p \sin p (1 + \sin p) \theta_{p-\pi} \theta_p = 0.$$

26
Nevertheless, \( e_+ \) and \( f_+ \) do not commute with \( H(n) \) for \( n \geq 1 \) – and thus are not part of \( \mathfrak{z}_{JTL} \). The reader interested in the centralizer \( \mathfrak{z}_H \) of the Hamiltonian (but not of the whole algebra \( JTL_N \)) can find a discussion in section 5.5 below.

**Remark 5.3.1.** We could equivalently study the family of operators

\[
e_n = \chi_0^\dagger \eta_0^\dagger - \sum_{p=\epsilon}^{\pi-\epsilon} (\cos p)^n \eta_p^\dagger \chi_{\pi-p}^\dagger, \quad f_n = \eta_0 \chi_0 + \sum_{p=\epsilon}^{\pi-\epsilon} (\cos p)^{-n} \chi_p \eta_{\pi-p}, \quad n \in \mathbb{Z}, \tag{5.15}
\]

with the \( \mathfrak{s}(2) \) relations

\[
[h, e_n] = e_n, \quad [h, f_n] = -f_n, \quad [e_n, f_n] = 2h = S^z, \quad n \in \mathbb{Z}. \tag{5.16}
\]

The scaling limit of \( e_n \) and \( f_n \) for odd \( n \) are all identical with (5.2), but we stress that these operators do not commute with the \( JTL_N \). The \( e_n \) and \( f_n \) are however in the centralizer for the Hamiltonian \( H(0) \) – see section 5.5 below.

In terms of \( \theta \)-fermions expression the generators read

\[
e_n = \frac{1}{2} \sum_p (\sin p)^n \theta_p^\dagger \theta_{-p}^\dagger, \quad f_n = \frac{1}{2} \sum_{p \neq 0, \pi} (\sin p)^{-n} \theta_{-p} \theta_p. \tag{5.17}
\]

Going back to real space however leads to a strongly non local expression for one of these generators (e.g. \( f_n \) for \( n \) positive), since the pole in the Fourier transform give rise to a power law growth for the couplings between pairs of fermions \( f \).

### 5.4 The twisted model

In this model, things are a bit different. There are no zero modes, and the continuum limit of the \( \mathfrak{s}(2) \) generators reads simply

\[
\tilde{Q}^a = d^a_{\alpha\beta} \sum_{n=0}^{\infty} \left( \frac{\psi^\alpha_{n-1/2} \psi^\beta_{n+1/2}}{n + 1/2} - \frac{\bar{\psi}^\alpha_{n-1/2} \bar{\psi}^\beta_{n+1/2}}{n + 1/2} \right). \tag{5.18}
\]

Of course, in this case the continuum limit exhibits in fact two \( \mathfrak{s}(2) \)'s, left and right being fully factorized (while they remain coupled by the zero modes in the periodic model). These two \( \mathfrak{s}(2) \)'s can be combined with plus or minus sign; the lattice symmetry (see section 3.4) becomes one of them.

### 5.5 Remarks about the Hamiltonian centralizer and loop \( \mathfrak{s}(2) \) symmetry

It is interesting to consider further the centralizer \( \mathfrak{z}_H \) of the Hamiltonian \( H \) on a finite lattice. For this, we first give the quantum-group expression for the \( n = 0 \) member of the \( \mathfrak{s}(2) \) family (5.15)

\[
e_0 = \frac{1}{N}(i[eE, \tilde{E}]K^{-1} + \bar{E}E), \quad f_0 = \frac{1}{N}(i[fF, \tilde{F}]K + F\tilde{F}), \tag{5.19}
\]

\footnote{In particular, \([H(n), e_0] = -4e^{i/2} \sqrt{\sin q} \chi_0^\dagger \) which has the limit \((L \rightarrow \infty) : L/2\pi[H(n), e_0] \rightarrow -2\tilde{\psi}_{-n} \tilde{\psi}_0^\dagger \) consistent with a direct computation in the scaling limit. So, we also conclude that the scaling limit of \( e_0 \) and \( f_0 \) commutes only with the chiral Virasoro part.}
where $\tilde{E}$ and $\tilde{F}$ are generators of (representation of) $U_{q^{-1}s\ell(2)}$

$$\Delta^{N-1}(\tilde{E}) = q^{-1} \sum_{j=1}^{N} q^{-j} c_{j}^{1 \downarrow} K^{-1} = \sqrt{N}q^{j_{\pi}/2} K^{-1}, \quad \Delta^{N-1}(\tilde{F}) = \sum_{j=1}^{N} q^{-j+1} c_{j} = q^{\sqrt{N}j_{\pi}/2}. \quad (5.20)$$

Recalling also the fermionic expressions for the generators $E$ and $F$ in (3.5), we obtain

$$e_{0} = e_{\frac{1}{2} \theta^{\dagger} \theta_{\pi}, \theta^{\dagger} + \theta_{\pi} \theta^{\dagger}} = \sum_{p=\frac{-1}{2} + \epsilon}^{\frac{3}{2} - \epsilon} \tan \frac{1}{2}(p + \frac{\pi}{2}) \theta^{\dagger}_{p} \theta_{p-\pi} + \theta_{\pi}^{\dagger} \theta^{\dagger}_{p} + \frac{\sqrt{3}}{\pi} \tan \frac{1}{2}(p + \frac{\pi}{2}) \theta_{\pi}^{\dagger} \theta_{p-\pi} + \theta_{\pi}^{\dagger} \theta^{\dagger}_{p}, \quad (5.21)$$

$$f_{0} = f_{\frac{1}{2} \theta^{\dagger} \theta_{\pi}, \theta^{\dagger} + \theta_{\pi} \theta^{\dagger}} = \sum_{p=\frac{-1}{2} + \epsilon}^{3 - \epsilon} \cot \frac{1}{2}(p + \frac{\pi}{2}) \theta_{\pi}^{\dagger} \theta_{p-\pi} + \theta_{\pi}^{\dagger} \theta^{\dagger}_{p} + \frac{\sqrt{3}}{\pi} \theta^{\dagger}_{\pi} \theta_{p-\pi} + \theta_{\pi}^{\dagger} \theta^{\dagger}_{p}. \quad (5.22)$$

It is then possible to show that the $e_{0}$ and $f_{0}$ together with $e_{+}$ and $f_{+}$ – a lattice analogue of Kausch’s $s\ell(2)$ defined in (5.12) – generate a loop $s\ell(2)$ algebra. First, we note (5.13), and (5.16) and $[e_{0}, e_{+}] = [f_{0}, f_{+}] = 0$. Then, we only need to check the higher-order Serre relations

$$[e_{0}, [e_{0}, [e_{0}, f_{0}]]] = [f_{0}, [f_{0}, [f_{0}, e_{+}]]] = 0, \quad (5.23)$$

$$[e_{+}, [e_{+}, [e_{+}, f_{0}]]] = [f_{+}, [f_{+}, [f_{+}, e_{0}]]] = 0. \quad (5.24)$$

Using (5.12), we compute the double commutators $[e_{0}, [e_{0}, f_{+}]] = -2e_{-1}$, etc., which immediately give the Serre relations (5.23) and we proceed similarly to get (5.24). We also note the relations

$$[f_{0}, [e_{0}, [e_{0}, f_{+}]]] = 2[e_{0}, f_{+}].$$

Since we have seen that the $e_{n}$ and $f_{n}$ commute with the Hamiltonian, we have thus find a loop algebra symmetry of the Hamiltonian $H$ in the $g\ell(1|1)$ spin chain. This is much like the symmetry uncovered in [33, 36], but a more careful comparison shows that the sectors we are considering are different: while in [33], the loop algebra is observed for periodic (antiperiodic) XX spin chain and even (odd) spin, ours is obtained in the opposite case, corresponding to periodicity for the $g\ell(1|1)$ fermions. We stress that in contrast with the main focus of this paper, the loop algebra is only a symmetry of the Hamiltonian, and does not extend to the full $JTL_{N}$ algebra.

Of course, having observed the loop algebra on the lattice it is natural to ask what happens of it in the continuum limit. We have already seen that the scaling limit of $e_{+}$ and $f_{+}$ coincides with the $s\ell(2)$ generators (5.22). The scaling limit of $e_{0}$, $f_{0}$ gives very similar expressions, only with the opposite sign between the chiral and antichiral components in the sum. In the end, we get a (representation of the) loop $s\ell(2)$ algebra, with further additional relations like $[e_{0}, [e_{0}, f_{+}]] = -2e_{+}$ due to coincidence of $e_{+}$ with $e_{-1}$ and $f_{+}$ with $f_{-1}$ in the scaling limit.

We note however that there exists a potential for yet more symmetries of the Hamiltonian in the finite-lattice problem. While the loop $s\ell(2)$ describes intertwining operators of the Hamiltonian between sectors $\mathcal{H}_{[j]}$ and $\mathcal{H}_{[j']}$, with $|j - j'| = 0 \mod 2$ and $\mathcal{H}_{[n]}$ denotes the subspace with $2h = S^{2} = n$, there are two linearly independent copies of $U_{q^{\text{odd}}}s\ell(2)$ in $3H$ describing intertwining operators between sectors with $|j - j'| = 1 \mod 2$. One copy of (the representation of) $U_{q^{\text{odd}}}s\ell(2)$ in $3H$ is generated by $e_{n}^{\text{odd}}E$ and $f_{m}^{\text{odd}}F$, with $n, m \geq 0$, and coincides with the representation $\rho_{g\ell}(U_{q^{\text{odd}}}s\ell(2))$. The second copy is generated by $e_{n}^{\text{even}}E$ and $f_{m}^{\text{even}}F$, with $n, m \geq 0$. The two copies are coupled/intersected by the same $g\ell(1|1)$ subalgebra.
6 Conclusion

The main mathematical result of this paper is the symmetry algebra $Z_{JT L}$ found in the periodic $g\ell(1|1)$ spin-chain – the centralizer of the representation of the Jones–Temperley–Lieb algebra $JTL_N$. This symmetry algebra will be exploited in an analysis [30] of the spin-chain as a module over $JTL_N$ following earlier results in the boundary case [7].

We also discussed in this paper how to proceed from the $JTL_N$ generators to get the Virasoro modes in the non-chiral logarithmic conformal field theory of symplectic fermions: the combinations $H(n)$ and $P(n)$, introduced in (4.1), of the $JTL_N$ generators converge as $L \to \infty$ to the well-known symplectic fermions representation of the left and right Virasoro generators

$$\frac{L}{2\pi}H(n) \mapsto L_n + \bar{L}_{-n}, \quad \frac{L}{2\pi}P(n) \mapsto L_n - \bar{L}_{-n}.$$

Finally, we showed in Sec. 5 that the scaling limit of the $JTL$ centralizer $Z_{JT L}$ describes a symmetry of the left-right Virasoro algebra – that is, gives an algebra of intertwining operators respecting the left and right Virasoro. It is thus reasonable to expect module structures in the continuum for the non-chiral Virasoro algebra to be related to the ones for $JTL_N$: this will be discussed in our second [30] and mostly in our third paper [37].

The continuum theory admits a further $s\ell(2)$ symmetry, which can only lead to a refinement of the results inherited from the lattice, since this symmetry is not present in the microscopic model. In fact, if one insists in considering only, in the algebraic approach, the product $V(2) = V(2) \otimes \overline{V}(2)$, it is necessary to consider the centralizer $3_\mathcal{V}$ of $V(2)$ in the local theory. This obviously is not a simple object. It clearly contains $U_q s\ell(2)$ (at $q = i$) generated by the $s\ell(2)$

$$Q^a = d^a_{\alpha\beta} \left\{ i\phi^\alpha_0 \psi^\beta_0 + \sum_{n=1}^{\infty} \left( \frac{\psi^\alpha_n \psi^\beta_n}{n} + \frac{\bar{\psi}^\alpha_n \bar{\psi}^\beta_n}{n} \right) \right\}$$

and $g\ell(1|1)$ (generated by $\psi^1_0, \bar{\psi}^1_0$) but this subalgebra does not exhaust the centralizer: extending $g\ell(1|1)$ we have the full scaling limit of the lattice $U_q^{odd}s\ell(2)$ which we have seen is generated by

$$\left[ \sum_{m>0} \left( \frac{\psi^1_m \psi^1_{-m}}{m} - \frac{\bar{\psi}^1_m \bar{\psi}^1_{-m}}{m} \right) \right]^n \psi^1_0,$$

and the Cartan element, already present in the $s\ell(2)$ (6.1). On the other hand, it would have been natural to describe the centralizer as a quotient of $U_q s\ell(2) \otimes U_q s\ell(2) – the tensor product of the centralizers for (anti)chiral theories $\mathcal{V}(2)$ and $\overline{\mathcal{V}}(2)$. How to do this in practice is not entirely clear. We only note that the centralizer $3_\mathcal{V}$ should in particular contain two copies of $U_q^{odd}s\ell(2)$ — one is the $JTL_N$’s centralizer discussed in Sec. 5.2 and the second is obtained from similar formulas but with the opposite sign between the left and right moving components in (6.2) — coming from each of the chiral halves and coupled by the same $g\ell(1|1)$ subalgebra. However, all these subalgebras still do not exhaust the centralizer, as can be easily seen by commuting them with the $s\ell(2)$.

We believe that refining our understanding of $\mathcal{V}(2)$ and $3_\mathcal{V}$ is not the way to go. We have strong evidence – coming from the study of other models such as those based on $g\ell(2|2)$ or $g\ell(2|1)$ that, in fact, the lattice results fully represent the algebraic structure of the continuum limit. This means that the good object to consider is not just $\mathcal{V}(2)$, but a larger object, extended by fields mixing the chiral...
and antichiral sectors, whose representation theory can be directly inferred from the representation theory of JTL, and whose centralizer is only $U_q^{\text{odd}} s\ell(2)$. This will be discussed in detail in our second and third papers \cite{30,37}.

To conclude, we briefly discuss the triplet $W$-algebra of Kausch. While this algebra does not seem to play an important role in the analysis of models based e.g. on $g\ell(2|2)$ or $g\ell(2|1)$, it is nevertheless tempting to wonder if, like for the Virasoro algebra, its generators can be simply obtained from lattice considerations. A remark to that effect concerns the permutation $\Pi_{j,j+2}$ of sites at positions $j$, $j+2$.

It is easy to write this operator in terms of fermions

$$\Pi_{j,j+2} = -1 + (-1)^j (f_j - f_{j+2}) \left( f_j^\dagger - f_{j+2}^\dagger \right).$$

(6.3)

Consider now

$$\Pi \equiv \sum_{j=1}^N \Pi_{j,j+2} = 2 \sum_{\substack{\pi=\epsilon \\
\text{step}=\epsilon}} \left( \chi_p^\dagger \chi_p + \eta_p^\dagger \eta_p \right) (1 - \cos 2p).$$

(6.4)

In the scaling limit this becomes

$$\Pi \mapsto \frac{1}{L^2} \sum_{m>0} m \left( \psi_m^2 \psi_{-m}^1 + \psi_{-m}^1 \psi_m^2 \right) + [\psi \to \bar{\psi}].$$

(6.5)

We recognize the zero mode of the $W^0 + \bar{W}^0$ generator, with

$$W^0 = \partial \psi^1 \psi^2 - \psi^1 \partial \psi^2,$$

(6.6)

where $\psi^a = i \partial \Phi^a$. It is in fact possible to come up with a full lattice version of the triplet $W$-algebra; this will also be discussed in a separate paper \cite{35}.

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Appendix A: The full quantum group $U_q s\ell(2)$ at roots of unity

Here, we collect standard expressions for the quantum group $U_q s\ell(2)$ we use in the analysis of symmetries of $g\ell(1|1)$ spin-chains. We introduce standard notation for $q$-numbers $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and set $[n]! = [1][2] \ldots [n]$.  

30
A.1 Defining relations

The full (or Lusztig) quantum group $U_q\mathfrak{sl}(2)$ with $q = e^{i\pi/p}$, for $p \geq 2$, is generated by $E$, $F$, and $K$ satisfying the standard relations for the quantum $\mathfrak{sl}(2)$,

$$\begin{align*}
KEK^{-1} &= q^2E, \\
KFK^{-1} &= q^{-2}F, \\
[E,F] &= \frac{K-K^{-1}}{q-q^{-1}},
\end{align*}$$

with some constraints,

$$\begin{align*}
E^p &= F^p = 0, \\
K^{2p} &= 1,
\end{align*}$$

and additionally by the divided powers $f \sim F^p/[p]!$ and $e \sim E^p/[p]!$, which turn out to satisfy the usual $\mathfrak{sl}(2)$-relations:

$$\begin{align*}
[h,e] &= e, \\
[h,f] &= -f, \\
[e,f] &= 2h.
\end{align*}$$

There are also ‘mixed’ relations

$$\begin{align*}
[h,K] &= 0, \\
[E,e] &= 0, \\
[K,e] &= 0, \\
[F,f] &= 0, \\
[K,f] &= 0,
\end{align*}$$

$$\begin{align*}
[F,e] &= \frac{1}{[p-1]}K^p\frac{qK^{-1}-q^{-1}E^{p-1}}{q-q^{-1}}, \\
[E,f] &= \frac{(-1)^{p+1}}{[p-1]}F^{p-1}\frac{qK^{-1}-q^{-1}K^{-1}}{q-q^{-1}},
\end{align*}$$

$$\begin{align*}
[h,E] &= \frac{1}{2}EA, \\
[h,F] &= -\frac{1}{2}AF,
\end{align*}$$

where

$$A = \sum_{s=1}^{p-1} \frac{u_s(q^{-s-1}) - u_s(q^{s-1})K + q^{s-1}u_s(q^{s-1}) - q^{-s-1}u_s(q^{-s-1})}{(q^{s-1} - q^{-s-1})u_s(q^{s-1})u_s(q^{-s-1})} u_s(K)e_s$$

with the polynomials $u_s(K) = \prod_{n=1, n \not= s}^{p-1} (K - q^{s-1-2n})$, and $e_s$ are some central primitive idempotents [3]. The relations (A1)-(A7) are the defining relations of the quantum group $U_q\mathfrak{sl}(2)$.

The quantum group $U_q\mathfrak{sl}(2)$ has the Hopf-algebra structure with the comultiplication

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

$$\begin{align*}
\Delta(e) &= e \otimes 1 + KP \otimes e + \frac{1}{[p-1]} \sum_{r=1}^{p-1} q^{r(p-r)}[r]K^pE^{p-r} \otimes E^rK^{-r}, \\
\Delta(f) &= f \otimes 1 + KP \otimes f + \frac{(-1)^p}{[p-1]} \sum_{s=1}^{p-1} q^{-s(p-s)}[s]K^{p+s}F^s \otimes F^{p-s}.
\end{align*}$$

The antipode and counity are not used in the paper but a reader can find them, for example, in [3].

We can easily write the $(N-1)$-folded coproduct for the capital generators $E$ and $F$,

$$\begin{align*}
\Delta^{N-1}E &= \sum_{j=1}^{N} 1 \otimes \ldots \otimes 1 \otimes E \otimes K \otimes \ldots \otimes K, \\
\Delta^{N-1}F &= \sum_{j=1}^{N} K^{-1} \otimes \ldots \otimes K^{-1} \otimes F \otimes 1 \otimes \ldots \otimes 1.
\end{align*}$$

A.2 Standard spin-chain notations

We note the Hopf-algebra homomorphism

$$E \mapsto S^+k, \quad F \mapsto k^{-1}S^-, \quad \text{with} \quad k = \sqrt{K},$$

31
where we introduced the more usual (in the spin-chain literature) quantum group generators

$$S^\pm = \sum_{1 \leq j \leq N} q^{-\sigma_j^+ / 2} \otimes \ldots \otimes q^{-\sigma_{j-1}^+ / 2} \otimes \sigma_j^\pm \otimes q^{\sigma_{j+1}^+ / 2} \otimes \ldots \otimes q^{\sigma_N^+ / 2}$$  \hspace{1cm} (A12)

together with $k = q^{S^z}$ and the relations

$$k S^\pm k^{-1} = q^{\pm 1} S^\pm, \quad [S^+, S^-] = \frac{k^2 - k^{-2}}{q - q^{-1}},$$

$$\Delta(S^\pm) = k^{-1} \otimes S^\pm + S^\pm \otimes k.$$

A.2.1 The case of XX spin-chains

For $p = 2$ or “XX spin-chain” case, the $N$-folded coproduct of the renormalized powers $e$ and $f$ reads

$$\Delta^{N-1} e = \sum_{j=1}^N 1 \otimes \ldots \otimes 1 \otimes e \otimes K^2 \otimes \ldots \otimes K^2 +$$

$$+ q \sum_{t=0}^{N-2} \sum_{j=1}^{N-1-t} 1 \otimes \ldots \otimes 1 \otimes E \otimes K \otimes \ldots \otimes K \otimes E K \otimes K^2 \otimes \ldots \otimes K^2 \hspace{1cm} (A13)$$

and

$$\Delta^{N-1} f = \sum_{j=1}^N K^2 \otimes \ldots \otimes K^2 \otimes f \otimes 1 \otimes \ldots \otimes 1 +$$

$$+ q^{-1} \sum_{t=0}^{N-2} \sum_{j=1}^{N-1-t} K^2 \otimes \ldots \otimes K^2 \otimes K^{-1} F \otimes K^{-1} \otimes \ldots \otimes K^{-1} \otimes F \otimes 1 \otimes \ldots \otimes 1. \hspace{1cm} (A14)$$

These renormalized powers can also be expressed in terms of the more usual spin-chain operators, and one finds at $p = 2$

$$\Delta^{N-1}(e) = q S^{+(2)} k^2, \quad \Delta^{N-1}(f) = q^{-1} k^{-2} S^{-(2)},$$

where $q = i$ and

$$S^{\pm(2)} = \sum_{1 \leq j < k \leq N-1} q^{-\sigma_j^+} \otimes \ldots \otimes q^{-\sigma_{j-1}^+} \otimes \sigma_j^\pm \otimes 1 \otimes \ldots \otimes 1 \otimes \sigma_k^\pm \otimes q^{\sigma_{k+1}^+} \otimes \ldots \otimes q^{\sigma_N^+}. \hspace{1cm} (A15)$$

Appendix B: A proof for the centralizer of $JTL_N$

Our proof of Thm. \[3.3.3\] consists in the following three lemmas. First, in Lem. \[B.3\] we describe the two-parameter family of vector spaces $\mathcal{E}_{k,t}$ spanned by homomorphisms (respecting the open Temperley–Lieb algebra $TL_N$ generated by $e_j$ with $1 \leq j \leq N - 1$) between any two sectors $\mathcal{H}_{(k)}$ and $\mathcal{H}_{(k')}$. If $0 \leq k \leq N$ and $k - k' = 1 \ mod \ 2$; we denote by $\mathcal{H}_{(k)}$ the sector\[\hspace{1cm}^9\] with $k$ antifermions $\theta_{p,j}$. Then, in\[\hspace{1cm}^9\] we note that our convention for the spin-chain representation differs from the one in \[1\] by the change $q \rightarrow q^{-1}$.

\[\hspace{1cm}^9\]We note that the subspace $\mathcal{H}_{(k)}$ coincides with $\mathcal{H}_{(\frac{N}{2} - k)}$, where $\mathcal{H}_{[n]}$ denotes the subspace with $2h = S^z = n$. \[\hspace{1cm}^10\]We note the subspace $\mathcal{H}_{(k)}$ coincides with $\mathcal{H}_{(\frac{N}{2} - k)}$, where $\mathcal{H}_{[n]}$ denotes the subspace with $2h = S^z = n$.\[\hspace{1cm}^10\]
Lem. [B.4] we compute commutators between \( e_N \) and an intertwining operator from \( \mathcal{E}_{k,t} \) and show that all homomorphisms (between \( \mathcal{H}(k) \) and \( \mathcal{H}(k') \)) respecting the periodic Temperley–Lieb\(^{11}\) algebra \( TL^e_N \) are exhausted by elements from \( U_q^{\text{odd}}sl(2) \). In Lem. [B.5] we state that all homomorphisms between \( \mathcal{H}(k) \) and \( \mathcal{H}(k') \) (as modules over \( TL^e_N \)) for \( k-k' = 0 \mod 2 \) are also given by \( U_q^{\text{odd}}sl(2) \), together with the two operators \( e^L \) and \( f^L \) mixing the two \( TL^e_N \)-invariants on the opposite ends of the spin-chain. Then, we finally state an isomorphism between the centralizers for (the \( g\ell(1|1) \) representations of) \( TL^e_N \) and \( JTL_N \).

In what follows, we omit the notation for the spin-chain representation \( \rho_{g\ell} \) of the quantum group for brevity and simply write \( F \) or \( E \) instead of \( \rho_{g\ell}(F) \) or \( \rho_{g\ell}(E) \). We do the same for the representation \( \pi_{g\ell} \) of generators of \( JTL_N \).

**Lemma B.3.** The vector space \( \mathcal{E}_{k,t} \) of homomorphisms respecting the \( TL_N \)-action,

\[
\mathcal{E}_{k,t} = \text{Hom}_{TL_N}(\mathcal{H}(k), \mathcal{H}(k-2t-1)), \quad 0 \leq k \leq N, \quad \left\lceil \frac{k-N-1}{2} \right\rceil \leq t \leq \left\lfloor \frac{k-1}{2} \right\rfloor,
\]

has the dimension and a basis listed below.

1. For \( 0 \leq t \leq \left\lfloor \frac{k-N-1}{2} \right\rfloor \), we have
   - for \( k \leq \frac{N}{2} \), and also for \( k > \frac{N}{2} \) and \( k - \frac{N}{2} \leq t \leq \left\lfloor \frac{k-1}{2} \right\rfloor \),
     \[
     \mathcal{E}_{k,t} = \left\langle f^{n-t}e^nE, Ef^{l-t}e^{l+1}; t \leq n \leq \left\lfloor \frac{k-1}{2} \right\rfloor, t \leq l \leq \left\lfloor \frac{k-2}{2} \right\rfloor \right\rangle,
     \]
     \[
     \dim \mathcal{E}_{k,t} = k - 2t,
     \]
   - for \( k > \frac{N}{2} \) and \( 0 \leq t \leq k - \frac{N}{2} - 1 \),
     \[
     \mathcal{E}_{k,t} = \left\langle e^tE, f^{n-t}e^nE, Ef^{l-t}e^{l+1}; k - \frac{N}{2} \leq n \leq \left\lfloor \frac{k-1}{2} \right\rfloor, k - \frac{N}{2} \leq l \leq \left\lfloor \frac{k-2}{2} \right\rfloor \right\rangle,
     \]
     \[
     \dim \mathcal{E}_{k,t} = N - k + 1,
     \]
2. For \( \left\lceil \frac{k-N-1}{2} \right\rceil \leq t \leq -1 \), we have
   - for \( k \geq \frac{N}{2} \), and also for \( k < \frac{N}{2} \) and \( \left\lceil \frac{k-N-1}{2} \right\rceil \leq t \leq k - \frac{N}{2} \),
     \[
     \mathcal{E}_{k,t} = \left\langle e^{n+t+1}f^nF, Ee^{l+t+1}f^{l+1}; -t - 1 \leq n \leq \left\lfloor \frac{N-k-1}{2} \right\rfloor, -t - 1 \leq l \leq \left\lfloor \frac{N-k-2}{2} \right\rfloor \right\rangle,
     \]
     \[
     \dim \mathcal{E}_{k,t} = N - k + 2(t+1),
     \]
   - for \( k < \frac{N}{2} \) and \( k - \frac{N}{2} + 1 \leq t \leq -1 \),
     \[
     \mathcal{E}_{k,t} = \left\langle f^{-t-1}F, e^{n+t+1}f^nF, Ee^{l+t+1}f^{l+1}; \frac{N}{2} - k \leq n \leq \left\lfloor \frac{N-k-1}{2} \right\rfloor, \frac{N}{2} - k \leq l \leq \left\lfloor \frac{N-k-2}{2} \right\rfloor \right\rangle,
     \]
     \[
     \dim \mathcal{E}_{k,t} = k,
     \]

where we suppose that each basis element is multiplied by an appropriate projector on the sector \( \mathcal{H}(k) - a \) polynomial in the Cartan element \( h \).

\(^{11}\)It is the algebra generated by the \( e_j \) with \( 1 \leq j \leq N \), i.e., without the translation generator \( u^2 \), see Sec. 2.1.
where we define the submodules \( M \) has the dimension from App. C, we obtain the short exact sequences of \( TL \) module. We use the notations \( P \) using again the decompositions (B4) restricted to the subspace with multiplicities \( H \) independent.

Proof. The idea of the proof is to compute dimensions of the spaces \( E_{k,t} \) of homomorphisms using explicit decompositions over the two commuting algebras and then to check that the images (in \( H_{(k-2t-1)} \)) of the basis elements proposed in the lemma are non-isomorphic, so they are indeed linearly independent.

We recall a decomposition of the tensor-product space \( H_N \) over the two commuting algebras \( TL_N \) and \( U_q sl(2) \) (centralizing each other) in the open case [7],

\[
H_N|_{U_q sl(2)} = \bigoplus_{j=1}^L (d^0_j) \otimes P_{1,j}, \quad H_N|_{TL_N} = \bigoplus_{j=1}^L P_j \otimes X_{1,j} \oplus W_L \otimes X_{1,L+1}, \quad (B4)
\]

with multiplicities \( d^0_j = \sum_{i=j}^L (-1)^{j-i} \left( \binom{N}{L+i} - \binom{N}{L+i+1} \right) \) given by dimensions of irreducibles over \( TL_N \). We use the notations \( P_j \) and \( W_j \) for projective and standard \( TL_N \)-modules, respectively. The standard module \( W_L \) is the trivial representation denoted also by (1); a standard module \( W_j \), with \( 1 \leq j < L \), has the dimension \( \left( \binom{N}{L+i} \right) - \binom{N}{L+i+1} \) and is indecomposable with the structure of subquotients \( W_j : (d^0_j) \to (d^0_{j+1}) \). The projectives \( P_j \) are self-conjugate and described by the diagram \( W_j \to W_{j-1} \) or with simple subquotients as

\[
\begin{align*}
\mathcal{P}_j &= \begin{pmatrix}
(d^0_j) \\
(d^0_{j-1}) \\
(d^0_{j+1})
\end{pmatrix} \\
\end{align*}
\]

\[
(B5)
\]

see also more details in [7]. On the quantum-group side, the \( U_q sl(2) \)-action on the \( j \)-dimensional irreducible modules \( X_{1,j} \) is defined in (C1) and and on the projective modules \( P_{1,j} \) is defined in (C2)-(C4). We note also the the only non-trivial Hom spaces for a pair of projectives over \( TL_N \) are \( \text{Hom}_{TL_N}(\mathcal{P}_j, \mathcal{P}_j) \cong \text{Hom}_{TL_N}(\mathcal{P}_j, \mathcal{P}_{j+1}) \cong \mathbb{C} \).

Using the decomposition (B4) over \( TL_N \) restricted to sectors with \( S^z = k \) and \( S^z = k - 2t - 1 \) as well as the \( \text{Hom}_{TL_N} \) spaces for a pair of projective \( TL_N \)-modules described just above, we easily compute dimensions of the spaces \( E_{k,t} \) for all cases described in the lemma.

Next, in order to describe images of intertwining operators \( e^{mf}\mathbb{F} \) and \( e^{mf}\mathbb{E} \) in each subspace \( H_{(k)} \) we introduce “zig-zag” type \( TL_N \)-modules in Fig. [1]. These are obtained as kernels of \( \mathbb{F} \) or \( \mathbb{E} \) in the following way. We note that the spin-chain \( H_N = \bigoplus_{j=1}^L H_j \) graded by \( S^z \) defines two long exact sequences with the differentials \( \mathbb{F} \) and \( \mathbb{E} \) (we remind that \( \mathbb{F}^2 = \mathbb{E}^2 = 0 \)) with \( \mathbb{E} : H_{[n]} \to H_{[n+1]} \) and \( \mathbb{F} : H_{[n]} \to H_{[n-1]} \). The images and kernels of these differentials are \( TL_N \)-modules: for any \( j > 0 \), using again the decompositions (B4) restricted to the subspace with \( S^z = j \) and the \( U_q sl(2) \)-action from App. C, we obtain the short exact sequences of \( TL_N \)-modules

\[
0 \to \mathcal{N}_1 \to \mathcal{N}_0 \to \mathcal{N}_1^* \to 0, \\
0 \to \mathcal{M}_{j+1} \to \mathcal{H}_{[j]} \to \mathcal{N}_j \to 0, \quad j - \text{odd}, \\
0 \to \mathcal{N}_{j+1} \to \mathcal{H}_{[j]} \to \mathcal{M}_j \to 0, \quad j - \text{even},
\]

where we define the submodules \( \mathcal{M}_{j+1} \) and \( \mathcal{N}_{j+1} \) as the kernels of the quantum-group generator \( F \) on \( \mathcal{H}_{[j]} \), for odd and even \( j \), respectively. Equivalently, they are defined as the kernels of \( \mathbb{E} \) but for \( j < 0 \).
Figure 1: For even \( L \), the indecomposable “zig-zag” \( TL_N \)-modules \( M_{n-1} \) at the top, with odd \( n \), and \( N_{n-1} \) at the bottom, with even \( n \), \( k = (L - \bar{n})/2 \), and \( \bar{n} = n - (n \mod 2) \).

We note next that the modules \( M_j \) and \( N_j \) have filtrations by submodules \( M_j' \) and \( N_j' \) with appropriate \( j' > j \). We then use the explicit action of \( U_q\mathfrak{sl}(2) \) generators \( e \) and \( f \) given also in App. C to identify some terms of these filtrations with kernels/images of intertwining operators \( e^m f^n E \) and \( e^m f^n F \). This allows us to check that the images of intertwining operators proposed in the lemma for all possible pairs \((k, t)\) are given by non-isomorphic “zig-zag” type \( TL_N \)-modules (together with their duals), defined just above and described in Fig. 1. This finishes our proof.

**Lemma B.4.** All homomorphisms between \( \mathcal{H}(k) \) and \( \mathcal{H}(k') \), for \( 0 \leq k \leq N \) and \( k - k' = 1 \mod 2 \), respecting the periodic Temperley–Lieb algebra \( TL_N^0 \) are given by action of elements from \( U_{q^{odd}}\mathfrak{sl}(2) \).

**Proof.** We use the fermionic expressions (3.6)-(3.7) for the generators of \( U_q\mathfrak{sl}(2) \) and the explicit expressions for the commutators (3.12) and (3.13) to find, for \( n \leq N/2 - 1 \),

\[
[e_N, f^n] = 2^n i \sum_{p_1, \ldots, p_n \neq \frac{\pi}{2}} (e^{ip_n} - i) \prod_{j=1}^{n-1} f(p_j) \theta_{p_j} \theta_{-p_j} \theta_{p_n} \theta_{3\pi/2},
\]

\[
[e_N, e^n] = 2^n i \sum_{p_1, \ldots, p_n \neq \frac{\pi}{2}} (e^{-ip_n} - i) \prod_{j=1}^{n-1} g(p_j) \theta_{p_j} \theta_{-p_j} \theta_{p_n} \theta_{\pi/2},
\]

where

\[
f(p_j) = q \frac{e^{i(p + p_j)/2}}{e^{i(p + p_j)/2} - 1}, \quad g(p_j) = -q \frac{e^{i(p + p_j)/2}}{e^{i(p + p_j)/2} + 1}.
\]
Simplifying, we get
\[
\langle e, f^n \rangle = (-2)^n i(n - 1)! \sum_{\frac{\pi}{2} = p_1 < p_2 < \ldots < p_{n-1} \text{ such that } \epsilon = \pi/2} \sum_{p_n} (i - e^{ip_n}) \prod_{j=1}^{n-1} \cot \frac{1}{2} \left( \frac{\pi}{2} + p_j \right) \theta_{p_j} \theta_{p_j - p_{j+1}} \theta_{3\pi/2} \neq 0, \quad (B6)
\]
and, using (3.14) and (B6),
\[
\langle e, f^n \rangle = 2^n i(n - 1)! \sum_{\frac{\pi}{2} = p_1 < p_2 < \ldots < p_{n-1} \text{ such that } \epsilon = \pi/2} \sum_{p_n} (e^{-ip_n} - i) \prod_{j=1}^{n-1} \tan \frac{1}{2} \left( \frac{\pi}{2} + p_j \right) \theta_{p_j} \theta_{p_j - p_{j+1}} \theta_{3\pi/2} \neq 0, \quad (B7)
\]
where we introduce \( \epsilon = \frac{2\pi}{N} \) and all the non-zero summands are linearly independent.

Then, using (3.5)-(3.7), we obtain
\[
e^{m}E = m! \sqrt{N} \sum_{\frac{\pi}{2} = p_1 < p_2 < \ldots < p_{m} \text{ such that } \epsilon = \pi/2} \prod_{j=1}^{m} \tan \frac{1}{2} \left( \frac{\pi}{2} + p_j \right) \theta_{p_j} \theta_{p_j - p_{j+1}} \theta_{3\pi/2} K,
\]
and, using (3.14) and (B6),
\[
\langle e, f^n e^m \rangle = \langle e, f^n \rangle e^m E = \langle e, f^n \rangle = (-2)^n i(n - 1)! m! \sqrt{N} \sum_{\frac{\pi}{2} = p_1 < p_2 < \ldots < p_{n-1} \text{ such that } \epsilon = \pi/2} \sum_{p_n} \sum_{\frac{\pi}{2} = p_1 < p_2 < \ldots < p_{m} \text{ such that } \epsilon = \pi/2} (i - e^{ip_n}) \prod_{j=1}^{n-1} \cot \frac{1}{2} \left( \frac{\pi}{2} + p_j \right) \theta_{p_j} \theta_{p_j - p_{j+1}} \theta_{3\pi/2} \prod_{j=1}^{m} \tan \frac{1}{2} \left( \frac{\pi}{2} + p_j \right) \theta_{p_j} \theta_{p_j - p_{j+1}} \theta_{3\pi/2} K, \quad n \geq 1, \ m \geq 0, \quad (B8)
\]
and, similarly,
\[
\langle e, F f^n e^m \rangle = F f^n [e, e^m] = 2^n (-1)^n n! \sqrt{N} \sum_{\frac{\pi}{2} = p_1 < p_2 < \ldots < p_{n-1} \text{ such that } \epsilon = \pi/2} \sum_{p_n} \sum_{\frac{\pi}{2} = p_1 < p_2 < \ldots < p_{m} \text{ such that } \epsilon = \pi/2} (e^{-ip_m} - i) \prod_{j=1}^{n} \cot \frac{1}{2} \left( \frac{\pi}{2} + p_j \right) \theta_{p_j} \theta_{p_j - p_{j+1}} \theta_{3\pi/2} \prod_{j=1}^{m-1} \tan \frac{1}{2} \left( \frac{\pi}{2} + p_j \right) \theta_{p_j} \theta_{p_j - p_{j+1}} \theta_{3\pi/2}, \quad m \geq 1, \ n \geq 0, \quad (B9)
\]
where all the summands are linearly independent.

Next, we restrict the action on a sector with \( k \) antifermions \( \theta_{p_j}, \ 0 \leq k \leq N \),
\[
\mathcal{H}_{\frac{\pi}{2} - k} = \mathcal{H}(k) = \left\langle \prod_{j=1}^{k} \theta_{p_j} \uparrow \cdots \uparrow, \ p_1 > p_2 > \cdots > p_k \right\rangle,
\]

where the momenta $p_j$ belong to the set \((3.4)\), and compute commutation relations between $e_N$ and an operator from the vector space $\mathcal{E}_{k,t} = \text{Hom}_T \mathcal{L}_N(\mathcal{H}(k),\mathcal{H}(k-2t-1))$ described in Lem. [B.3] We first note that the intersection of $\mathcal{E}_{k,t}$ with $U_q^\text{odd} s\ell(2)$ is spanned by the operators $e^j \mathcal{E}$, for $t \geq 0$, and by $f^{-t-1} F$, for $t \leq -1$. Then, we show in three steps that any non-zero linear combination of all the other operators is not contained in the space $\text{Hom}_T \mathcal{L}_N(\mathcal{H}(k),\mathcal{H}(k-2t-1))$ of homomorphisms respecting the periodic Temperley–Lieb algebra $TL_N^a$. We begin with consideration of the case 1. in Lem. [B.3]

1. Using \((B8)\), we calculate the action of the commutator $[e_N, f^m e^n \mathcal{E}]$ on a vector $v_k(n) = \theta_{p_1}^{\pi} \theta_{p_2}^{\pi} \ldots \theta_{p_k}^{\pi} | \ldots \rangle$ with the specifically chosen momenta

\[
p_j'' = \frac{\pi}{2} + (n - j + 1)\epsilon, \quad 1 \leq j \leq k, \quad 1 \leq k \leq N. \quad (B10)
\]

Here, we set $m = n - t$ and the power $n$ is running the values $t + 1, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor$ in the case corresponding to \((B11)\), and for the case \((B12)\) – the values $k - \frac{N}{2}, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor$.

\[
[e_N, f^m e^n \mathcal{E}]v_k(n) = [e_N, f^m e^n \mathcal{E}] \theta_{\frac{\pi}{2} + n \epsilon}^{\pi} \theta_{\frac{\pi}{2} + (n-1)\epsilon}^{\pi} \ldots \theta_{\frac{\pi}{2} - n \epsilon}^{\pi} \ldots \theta_{\frac{\pi}{2} + (n-k+1)\epsilon}^{\pi} | \ldots \rangle
\]

\[
= a(n) \sum_{\frac{\pi}{2} + \epsilon = p_1 < p_2 < \ldots < p_m}^{\frac{\pi}{2} + \epsilon = p_1 < p_2 < \ldots < p_m} \sum_{\text{step}=\epsilon}^{m-1} (i - e^{ip_m}) \prod_{j=1}^{m-1} \cot \frac{1}{2}(\pi p_j - p_j) \theta_{p_j - p_j}^{\pi} \theta_{p_j}^{\pi} \ldots \theta_{\frac{\pi}{2} + (n-k+1)\epsilon}^{\pi} | \ldots \rangle
\]

\[
\times \prod_{l=1}^{n} \tan \frac{1}{2}(\pi + l \epsilon) \theta_{\frac{\pi}{2} + l \epsilon}^{\pi} \theta_{\frac{\pi}{2} - l \epsilon}^{\pi} \ldots \theta_{\frac{\pi}{2} - 2 \epsilon}^{\pi} \theta_{\frac{\pi}{2} + n \epsilon}^{\pi} \theta_{\frac{\pi}{2} + (n-1)\epsilon}^{\pi} \ldots \theta_{\frac{\pi}{2} + (n-k+1)\epsilon}^{\pi} | \ldots \rangle,
\]

where $a(n) = (-1)^{m-k+N/2} i 2^m n!(m-1)! \sqrt{N}$. In consequence of the condition $\frac{\pi}{2} + (n-k+1)\epsilon \geq -\frac{\pi}{2} + \epsilon$ or, equivalently, $k - n \leq N/2$, the state $v_k(n)$ contains precisely $n$ pairs of momenta $(p_l, \pi - p_l)$, where $\frac{\pi}{2} + n \epsilon \leq p_l \leq \frac{\pi}{2} + \epsilon$. Therefore, the sum over $p_l$ (on the third line) contains only one non-zero term corresponding to $p_l' = \frac{\pi}{2} + l \epsilon$, $1 \leq l \leq n$:

\[
\prod_{l=1}^{n} \tan \frac{1}{2}(\pi + l \epsilon) \theta_{\frac{\pi}{2} + l \epsilon}^{\pi} \theta_{\frac{\pi}{2} - l \epsilon}^{\pi} \ldots \theta_{\frac{\pi}{2} - 2 \epsilon}^{\pi} \theta_{\frac{\pi}{2} + n \epsilon}^{\pi} \theta_{\frac{\pi}{2} + (n-1)\epsilon}^{\pi} \ldots \theta_{\frac{\pi}{2} + (n-k+1)\epsilon}^{\pi} | \ldots \rangle
\]

where the under-braced term is annihilated by the corresponding $\theta_l$-operators. We thus obtain

\[
[e_N, f^m e^n \mathcal{E}]v_k(n) = a(n) \sum_{\frac{\pi}{2} + \epsilon = p_1 < p_2 < \ldots < p_m}^{\frac{\pi}{2} + \epsilon = p_1 < p_2 < \ldots < p_m} \sum_{\text{step}=\epsilon}^{m-1} (i - e^{ip_m}) \prod_{l=1}^{n} \tan \frac{1}{2}(\pi + l \epsilon) \prod_{j=1}^{m-1} \cot \frac{1}{2}(\pi p_j - p_j) \theta_{p_j - p_j}^{\pi} \theta_{p_j}^{\pi} \ldots \theta_{\frac{\pi}{2} + (n-k+1)\epsilon}^{\pi} | \ldots \rangle
\]

where the sum contains non-zero terms in consequence of the inequality $m < n$ and all the non-zero terms are linearly independent.
2. Using (B2), we next calculate similarly the action of the commutator \([e_N, F^m e^{n+1}]\), where \(m = n - t\) and \(t \leq n \leq \left\lfloor \frac{k-2}{2} \right\rfloor\) for the case (B1) and \(k - \frac{N}{2} \leq n \leq \left\lfloor \frac{k-2}{2} \right\rfloor\) in the case (B2), on the same vector \(v_k(n) = \theta_{p_1} \ldots \theta_{p_k} \uparrow \ldots \uparrow\) with the momenta (B10).

\[
[e_N, F^m e^{n+1}]v_k(n) = b(n) \sum_{\text{step}=\varepsilon} \prod_{j=1}^{k-2n-1} (-1)^{r-1} \prod_{j=1}^{k} \cot \frac{1}{2}(\pi + p_j) \prod_{l=1}^{n} \tan \frac{1}{2}(\pi + le) 
\times (e^{-i\frac{\pi}{2}(n+r)\varepsilon} - i) \theta_{p_j} \theta_{p_{j-1}} \ldots \theta_{p_1} \uparrow \ldots \uparrow 
\equiv \sum_{l=1}^{m} \cot \frac{1}{2}(\pi + p_j) \prod_{l=1}^{n} \tan \frac{1}{2}(\pi + le) 
\]

where \(b(n) = (-1)^{m} \cdot n! \sqrt{N}\) and the notation \(\hat{\theta}\) means the absence of the corresponding term.

3. We then note that, for \(t \leq r \leq n - 1\) in the case corresponding to (B1) and \(k - \frac{N}{2} \leq r \leq n - 1\) in the case (B2),

\[
[e_N, \alpha_n f^{n-t} e^{n} E + \beta_n F f^{n-t} e^{n+1}]v_k(r) = 0, \quad \alpha_n, \beta_n \in \mathbb{C},
\]

and

\[
[e_N, \alpha_n f^{n-t} e^{n} E + \beta_n F f^{n-t} e^{n+1}]v_k(n) \neq 0,
\]

for any non-zero complex numbers \(\alpha_n\) and \(\beta_n\). Therefore, we can prove by induction with respect to \(n\) that the only operator from \(\mathcal{E}_{k,t}\) which commutes with \(e_N\) is \(e^t E \in U_q^{\text{odd}} \mathfrak{sl}(2)\).

Similar analysis can be carried out for the case 2. of Lem. B.3. More convenient basis to express \(v_k(n)\) for this case is spanned by \(\prod_{j=1}^{k} \theta_j | \downarrow \ldots \downarrow\).

Lemma B.5. The vector space \(\text{Hom}_{TL_N}(H_{(k)}, H_{(k-2t)})\) of homomorphisms between \(H_{(k)}\) and \(H_{(k')}\), for \(k - k' = 0\) mod 2, respecting TL_N^a is spanned by elements from \(U_q^{\text{odd}} \mathfrak{sl}(2)\) and operators \(f^L, e^L\).

Proof. First, using the fermionic expression (B6) and (B7) for \([e_N, f^n]\) and \([e_N, e^n]\), we conclude that the only power of \(f\) and \(e\) that commutes with \(e_N\) is \(n = N/2 = L\).

Second, we note that (for \(t \geq 0\)) the vector space of operators intertwining the TL_N-action has the basis

\[
\tilde{\mathcal{E}}_{k,t} = \text{Hom}_{TL_N}(H_{(k)}, H_{(k-2t)}) = \left\{ p_k(h) \delta_{t,0}, f^{n-t} e^{n} F^{n'} E^{n'}; \nu \in \{0,1\}, \ t \leq n \leq \left\lfloor \frac{k-1}{2} \right\rfloor \right\},
\]

where \(\delta\) is the Kronecker symbol and we introduce polynomials \(p_k(h)\) in \(h\) projecting the tensor-product space \(H_N\) onto the subspace \(H_{(k)}\),

\[
p_k(h) = \prod_{j=-N/2; j \neq L-k}^{j=N/2} (2h - j).
\]

The intersection of \(\tilde{\mathcal{E}}_{k,t}\) with \(U_q^{\text{odd}} \mathfrak{sl}(2)\) is spanned by the operators \(f^{n-t} e^{n} F E\) (and \(p_k(h)\), for \(t = 0\)). Assume there is a non-zero operator \(O\) from \(\tilde{\mathcal{E}}_{k,t}\) which is not presented by an element from \(U_q^{\text{odd}} \mathfrak{sl}(2)\) but commutes with \(e_N\) and consider the product of the operator \(O\) with \(E\). Obviously, the operator \(OE\) is non-zero because by our assumption \(O\) is not in \(U_q^{\text{odd}} \mathfrak{sl}(2)\) and therefore it is a linear combination.
of projectors on a $TL_N$ direct summand times $e^t$ which does not belong to the kernel of $E$. Then, the assumption $[\mathcal{O}, e_N] = 0$ with the fact $[E, e_N] = 0$ imply that the non-zero homomorphism $\mathcal{O}E \in \mathcal{E}_{k,l} = \text{Hom}_{TL_N}(\mathcal{H}_{(k)}, \mathcal{H}_{(k-2l-1)})$ which is not represented by an element from $U_q^{\text{odd}} s(2)$ commutes with $e_N$. We thus get a contradiction to Lem. [3.1]. This finishes our proof for $t \geq 0$. The case $t < 0$ is considered in the same manner.

We finally note that operators $E_n f^L$ and $F_n e^L$ belongs to $\rho_\theta(U_q^{\text{odd}} s(2))$. Therefore, the multiplication of $f^L$ or $e^L$ with ‘off-diagonal’ elements from $U_q^{\text{odd}} s(2)$ does not give any new operators centralizing $TL_N^a$.

Combining all the three Lemmas we prove the following result.

**Corollary B.6.** The centralizer of the periodic Temperley–Lieb algebra $\pi_{\theta}(TL_N)$ on the spin-chain is isomorphic to the subalgebra in $\rho_\theta(U_q s(2))$ generated by $U_q^{\text{odd}} s(2)$ and $f^L$, and $e^L$.

A final comment is in order however: the algebra $JTTL_N$ contains also the generator $u^2$ expressed in terms of the $\theta$-fremions in (3.11). This generator acts on the fermion generators as

$$u^2 f_j^{(t)} u^{-2} = f_{j+2}^{(t)}$$

(B12)

while it does not leave the generators $f$ and $e$ invariant, it does leave $U_q^{\text{odd}} s(2)$ (together with $f^L$ and $e^L$) invariant. This can be seen by using fermionic expressions in Sec. 3.1 and Sec. 3.3.4. This observation together with Cor. B.6 finally prove Thm. 3.3.3

**Appendix C: Projective $U_q s(2)$-modules $P_{1,r}$**

Here, we recall $[3]$ $U_q s(2)$-action (for $q = i$) in projective modules $P_{1,r}$, for $r \in \mathbb{N}$. Their simple subquotients are $r$-dimensional irreducible modules $X_{1,r}$ spanned by $x_m$, $0 \leq m \leq r-1$, with the action$^{13}$

$$
E x_m = F x_m = 0, \quad K x_m = (-1)^{r-1} x_m, \\
\h x_m = \frac{1}{2} (r - 1 - 2m) x_m, \quad \epsilon x_m = m(r - m)x_{m-1}, \quad f x_m = x_{m+1},
$$

(C1)

where we set $x_{-1} = x_r = 0$. For $r = 0$, we also set $X_{1,0} \equiv 0$. The subquotient structure of $P_{1,r}$ is then given as

$$
P_{1,r} = X_{1,r-1} \quad X_{1,r+1} \quad X_{1,r}
$$

(C2)

For $r > 1$, the projective module $P_{1,r}$ has the basis

$$
\{ t_m, b_m \}_{0 \leq m \leq r-1} \cup \{ t_l \}_{1 \leq l \leq r-1} \cup \{ r_l \}_{0 \leq l \leq r},
$$

(C3)

$^{12}$We could equivalently consider a representation depending on a phase $\varphi$: $u^2 f_j u^{-2} = e^{i \varphi} f_{j+2}$ and $u^2 f_j u^{-2} = e^{-i \varphi} f_{j+2}$, with $\varphi = 2\pi n/L$, $n \in \mathbb{Z}$, but it has the same centralizer as the representation with $\varphi = 0$.

$^{13}$We simplify a notation used in $[3]$ assuming $X_{1,r} \equiv X^{(r)}_{1,r}$ with $\alpha(r) = (-1)^{r-1}$, and the same for $P_{1,r}$.
where \{t_m\}_{0 \leq m \leq r-1} is the basis corresponding to the top module in (C2), \{b_m\}_{0 \leq m \leq r-1} to the bottom, \{k_l\}_{1 \leq l \leq r-1} to the left, and \{r_l\}_{0 \leq l \leq r} to the right module. For \( r = 1 \), the basis does not contain \{k_l\}_{1 \leq l \leq r-1} terms and we imply \( k_l \equiv 0 \) in the action.

We set \( \alpha(r) = (-1)^{r-1} \). The \( U_q\sl(2) \)-action on \( P_{1,r} \) is then given by

\[
\begin{align*}
Kt_m &= \alpha(r)t_m, & 0 \leq m \leq r - 1, \\
Kb_m &= \alpha(r)b_m, & 0 \leq m \leq r - 1, \\
Kl_l &= -\alpha(r)l_l, & 0 \leq l \leq r, \\
Kr_l &= -\alpha(r)r_l, & 0 \leq l \leq r, \\
\text{Et}_m &= \alpha(r)\frac{r-m}{r}r_m + \alpha(r)\frac{m}{r}l_m, & 0 \leq m \leq r - 1, \\
\text{Eb}_m &= 0, & 0 \leq m \leq r - 1, \\
\text{El}_l &= \alpha(r)(l-r)b_{l-1}, & 1 \leq l \leq r - 1, \\
\text{Er}_l &= \alpha(r)r_l, & 0 \leq l \leq r, \\
\text{Fl}_l &= b_l, & 1 \leq l \leq r - 1, \\
\text{Fr}_l &= b_l, & 0 \leq l \leq r.
\end{align*}
\]

In thus introduced basis, the \( s\sl(2) \)-generators \( e, f \) and \( h \) act in \( P_{1,r} \) as in the direct sum \( X_{1,r} \oplus X_{1,r-1} \oplus X_{1,r+1} \oplus X_{1,r} \) with the action defined in (C4).

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