Uniqueness in weighted Lebesgue spaces for a class of fractional parabolic and elliptic equations

Fabio Punzo* and Enrico Valdinoci †

Abstract

We investigate uniqueness, in suitable weighted Lebesgue spaces, of solutions to a class of fractional parabolic and elliptic equations.

1 Introduction

We are concerned with uniqueness of solutions to the following linear nonlocal Cauchy problem:

\[
\begin{cases}
\rho \partial_t u + (-\Delta)^s u = 0 & \text{in } \mathbb{R}^N \times (0, T] =: S_T \\
u = 0 & \text{in } \mathbb{R}^N \times \{0\},
\end{cases}
\]

where the coefficient \( \rho \), usually referred to as a variable density, is a positive function only depending on the space variable \( x \) and \((-\Delta)^s\) denotes the fractional Laplace operator of order \( s \in (0, 1) \). Note that when \( s = 1 \) problem (1.1) has been extensively investigated, and we shall now briefly recall the basics of the classical theory. In particular, it is well-known that problem

\[
\begin{cases}
\rho \partial_t u - \Delta u = 0 & \text{in } S_T \\
u = u_0 & \text{in } \mathbb{R}^N \times \{0\},
\end{cases}
\]

where \( u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), admits at most one bounded solutions if \( \rho(x) \to 0 \) slowly enough as \( |x| \to \infty \) (see, e.g., [6], [9], [10] for precise statements). Furthermore, uniqueness of solutions to problem (1.2), belonging to suitable weighted Lebesgue spaces, can be deduced from general results in [2]. In fact, for any \( \phi \in C(\bar{S}_T), \phi > 0, p \geq 1 \), set

\[
L^p_{\phi}(S_T) := \left\{ u : S_T \to \mathbb{R} \text{ measurable } \int_0^T \int_{\mathbb{R}^N} |u|^p \phi(x,t)dx dt < \infty \right\}.
\]

(a) Summary of known results in the local case. In [2], the operator

\[
Lu := \sum_{i,j=1}^{N} \frac{\partial^2 [a_{ij}(x,t)u]}{\partial x_i \partial x_j} - \sum_{i=1}^{N} \frac{\partial [b_i(x,t)u]}{\partial x_i} + c(x,t)u - u_t
\]
is considered; the coefficients of $L$, together with all their derivatives which appear, are locally bounded functions in $S_T$. Furthermore, the matrix $A \equiv (a_{ij})$ is assumed to be positive semidefinite in $S_T$. A function $u \in C(S_T)$, such that all of its derivatives which appear in $L$ exist and are locally integrable in $S_T$, is a solution to problem

$$
\begin{aligned}
L u &= 0 \quad \text{in } S_T \\
u &= 0 \quad \text{in } \mathbb{R}^N \times \{0\},
\end{aligned}
$$

if equalities in (1.3) are satisfied pointwise. Suppose that

$$
|a_{ij}(x,t)| \leq K_1(1 + |x|^2)^{\frac{\alpha}{2}} ,
$$

$$
|b_i(x,t)| \leq K_2(|x|^2 + 1)^{\frac{\alpha}{2}} ,
$$

$$
|c(x,t)| \leq K_3(|x|^2 + 1)^{\frac{\alpha}{2}}
$$

for almost every $(x,t) \in S_T$, for some constants $\lambda \geq 0, K_i > 0 \ (i = 1, 2, 3)$. In [2, Theorem 1] it is shown that if $u$ is a solution to problem (1.3) and $u \in L^1_0(S_T)$, with

$$
g(x) = (|x|^2 + 1)^{-\alpha_0} \quad (x \in \mathbb{R}^N) \quad \text{if } \lambda = 0,
$$

or

$$
g(x) = e^{-\alpha_0(|x|^2 + 1)^{\frac{\alpha}{2}}} \quad (x \in \mathbb{R}^N) \quad \text{if } \lambda > 0,
$$

for some $\alpha_0 > 0$, then

$$
u \equiv 0 \quad \text{in } S_T.
$$

Let us mention that a crucial step in the proof of this result consists in the construction of a positive supersolution $\phi \in C^2(S_T)$ to the adjoint equation

$$
\sum_{i,j=1}^N a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i \frac{\partial \phi}{\partial x_i} + c \phi + \partial_t \phi = 0 \quad \text{in } S_T.
$$

Clearly, as a consequence of the previous result one can immediately deduce uniqueness in $L^1_0(S_T)$ for solutions to problem

$$
\begin{aligned}
L u &= f \quad \text{in } S_T \\
u &= u_0 \quad \text{in } \mathbb{R}^N \times \{0\},
\end{aligned}
$$

where $f$ and $u_0$ are given functions defined in $S_T$ and $\mathbb{R}^N$, respectively.

Now, suppose that $\rho \in C^2(\mathbb{R}^N)$. If we choose

$$
a_{ij} = \frac{1}{\rho} \delta_{ij} , \quad b_i = 2 \frac{\partial}{\partial x_i} \left( \frac{1}{\rho} \right), \quad c = \Delta \left( \frac{1}{\rho} \right),
$$

then, it is easily checked that $u$ is a solution of (1.3) if and only if it is a solution to (1.1). Assume that

$$
\rho(x) \geq K_1(1 + |x|^2)^{-\frac{\alpha}{2}},
$$

$$
\left| \nabla \left( \frac{1}{\rho} \right) \right| \leq K_2(1 + |x|^2)^{\frac{\alpha}{2}}, \quad \left| \Delta \left( \frac{1}{\rho} \right) \right| \leq K_3(1 + |x|^2)^{\frac{2-\alpha}{2}}
$$

for every $x \in \mathbb{R}^N$, for some $\alpha \leq 2$. By [2, Theorem 1] recalled above, with $\lambda = 2 - \alpha$, $u \equiv 0$ is the unique solution to problem (1.2) in $L^1_0(S_T)$ with $g$ is defined as in (1.4)—(1.5).

Moreover, the same conclusion remains true, if we only suppose $\rho \in C(\mathbb{R}^N)$ instead of $\rho \in C^2(\mathbb{R}^N)$, and we remove assumption (1.5). In order to see this, only minor changes in the proof of [2, Theorem 1] are needed. More precisely, the operator $Lu := \Delta u - \rho \partial_1 u$ can play
the same role as $L$; furthermore, we use the fact that, indeed, the supersolution $\phi$ to equation (2.4), constructed in [2], is also a supersolution to equation
$$
\rho \partial_t \phi + \Delta \phi = 0 \quad \text{in } S_T.
$$

When $\rho \equiv 1$ the results established in [2] are in accordance with those in [18], where, not surprisingly, the same uniqueness class is obtained mainly using the heat kernel
$$
q(x, y) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}} \quad (x \in \mathbb{R}^N, t > 0).
$$

(b) Summary of known results in the fractional case. Observe that also for problem (1.1) some uniqueness results are known. In fact, in [16] it is proved that problem
$$
\begin{cases}
\rho \partial_t u + (-\Delta)^su = 0 \quad \text{in } S_T \\
u = u_0 \quad \text{in } \mathbb{R}^N \times \{0\},
\end{cases}
$$
$(m \geq 1)$ admits at most one bounded nonnegative solution, provided condition (2.5) is satisfied, for some $K_1 > 0, \alpha < 2s$. Consequently, in particular, $u \equiv 0$ is the unique solution if $u_0 \equiv 0, m = 1$. Note that the hypothesis that $\alpha < 2s$, when $s = 1$ is in agreement with that made above for problem (1.2), although for problem (1.2) $\alpha = 2$ was permitted, too. Moreover, in [3] it is shown that every nonnegative solution to equation
$$
\partial_t u + (-\Delta)^su = 0 \quad \text{in } S_T
$$
can be written as
$$
u(x, t) = \int_{\mathbb{R}^N} p(x - y, t)u(y, 0)dy,
$$
where $p$ is the fractional heat kernel defined by
$$
p(x, t) := \frac{1}{t^{N/2}} P \left( \frac{x}{t^{1/2}} \right) \quad (x \in \mathbb{R}^N, t > 0),
$$
$$
P(x) := \int_{\mathbb{R}^N} e^{ix \cdot \xi - |\xi|^{2s}} d\xi \quad (x \in \mathbb{R}^N).
$$
Note that, for some $C > 1$, the following two-sided heat kernel estimate holds (see [4]):
$$
\frac{1}{C} \min \left\{ \frac{1}{t^{N/2}}, \frac{t}{|x|^{N+2s}} \right\} \leq p(x, t) \leq C \min \left\{ \frac{1}{t^{N/2}}, \frac{t}{|x|^{N+2s}} \right\} \quad (x \in \mathbb{R}^N, t > 0). \quad (1.10)
$$
Some uniqueness results for nonlocal parabolic equations have been recently obtained in [1], [11], [12]. These works deal in fact with quite general integro-differential operators, but they require that there exist two constants $C_2 > C_1 > 0$, such that
$$
C_1 \leq \rho \leq C_2 \quad \text{in } \mathbb{R}^N. \quad (1.11)
$$

(c) Outline of our results. The main results of this paper will be given in detail in the forthcoming Theorems 2.5, 2.8, 2.11. We give here a sketchy outline of these results, describing motivations, techniques of proofs and differences with the existing literature.

As pointed out for problem (1.2), we can expect that the uniqueness class for problem (1.1) with $\rho \equiv 1$ is related to the fractional heat kernel $p$, and so to its bounds given in (2.7). In fact, suppose that condition (2.5) is satisfied for some $\alpha \leq 2s$. We shall prove that the solution to problem (1.1) is unique in the class $L^p_v(S_T)$ with $p \geq 1$ and
$$
\psi(x) := (1 + |x|^2)^{-\frac{\beta}{2}} \quad (x \in \mathbb{R}^N), \quad (1.12)
$$
for properly chosen $\beta > 0$. To be specific, when $\alpha = 0$, we can choose $\beta = N + 2s$, in agreement with (2.14). Furthermore, if $0 < \alpha \leq 2s$, then $0 < \beta < N + 2s$, thus the uniqueness class is smaller; roughly speaking, this is due to the fact that, in this case, the coefficient $\rho(x)$ makes the diffusion stronger as $|x| \to \infty$ (see Theorem 2.5 for the precise statement). Observe that, clearly, from such results we can deduce also uniqueness in $L^p_c(S_T)$ of solutions to

$$\begin{cases}
\rho \partial_t u + (\Delta)^s u = f & \text{in } S_T \\
u = u_0 & \text{in } \mathbb{R}^N \times \{0\}.
\end{cases}$$  \tag{1.13}

In order to prove such a uniqueness result we construct a positive supersolution to equation

$$-(\Delta)^s \phi - \rho \partial_t \phi = 0 \quad \text{in } S_T.$$ \tag{1.14}

Indeed, the weight function $\psi$ mentioned above is related to such a supersolution.

Observe that we also establish similar uniqueness results for the linear elliptic nonlocal equation

$$(\Delta)^s u + \rho cu = 0 \quad \text{in } \mathbb{R}^N,$$ \tag{1.15}

where $c$ is a nonnegative function defined in $\mathbb{R}^N$ (see Theorems 2.8 and 2.11). Similar results are stated in [13] and [15] for local elliptic equations in bounded domains, with coefficients that can be degenerate or singular at the boundary of the domain.

We should observe that the techniques used in the proofs of this paper have several conceptual and technical differences with respect to the classical case of the local equations. On the one hand, we borrow from the classical case the idea of dealing with the adjoint operator. On the other hand, the classical case relies on explicit computations based on differentiating some appropriate barriers and cut-off functions and integrating by parts, which are not available in our case. For this reason we have to perform some ad-hoc integral computations in our case, based on appropriately chosen covering of $\mathbb{R}^N \times \mathbb{R}^N$, some local and global remainder estimates that rely on some careful parameter adjustments (see Lemmas 3.1, 3.2). Furthermore, while in the local case the supersolution to the adjoint equation is defined using the function $g$ defined in (1.4) or (1.5), in the present situation it is related to the function $\psi$ defined in (1.12) (see the proof of Theorem 2.6). Moreover, to show that it is indeed a supersolution, clearly, we cannot make explicit computations based on differentiation; instead, we use some properties of the hypergeometric function $\,_2F_1(a, b, c, s) \equiv F(a, b, c, s)$, with $a, b \in \mathbb{R}, c > 0, s \in \mathbb{R} \setminus \{1\}$ (see [14] Chapters 15.2, 15.4). Similar computations, for different purposes, when $c > a + b$ have been made in [8]: however, we also consider the cases $c = a + b$ and $c > a + b$, that present some differences.

We should note that whereas the local counterpart of our results established in [2] only regard the weighted Lebesgue space $L^p_y(S_T)$, we can address $L^p_c(S_T)$, for each $p \geq 1$, thus such uniqueness results in $L^p_c(S_T)$ are new also for $s = 1$. Moreover, to the best of our knowledge our results for elliptic equations (see Theorems 2.8 and 2.11) have not corresponding results in the literature concerning the local case in the whole $\mathbb{R}^N$; some results are only available in bounded domains (see [13], [15]). Moreover, Theorem 2.8 is proved similarly to the parabolic case, while the proof of Theorem 2.11 is completely new (it does not have a corresponding argument in [13] or in [15]). Moreover, it relies on Lemma 4.2, which is rather technical.

Even if, in general, we do not require that the solutions are bounded, as a particular consequence of our uniqueness results it follows that (see Corollary 2.9) the solution to problem (1.1) is unique in $L^\infty(S_T)$. This generalizes the results in [16] for linear problems, since in [16] it was also requested that the solution is bounded and nonnegative; instead, now we do not need any sign condition on the solutions. Moreover, there is a substantial difference with uniqueness results in [1], [8], [11], [11]. In fact, on the one hand, in [8] only the case $\rho \equiv 1$ is addressed; moreover, the methods used are different form those of the present paper. On the other hand, differently from [11], [11], [11] we do not make the assumption (1.1), thus.
our density is allowed to vanish or to be singular as $|x| \to \infty$; moreover, we use completely different techniques. It is worth mentioning that degeneracy or singularity at infinity of the density is very important for the applications, e.g., see for instance, for the local case, [2], [6], [9], [13]. Clearly, the same model with singular or degenerate density occurs when considering nonlocal diffusion, in case, for instance, of rarefied media subject to non-Gaussian stochastic processes.

The paper is organized as follows. In Section 2 we recall some preliminaries about fractional Laplacian and we give the notion of solutions we shall deal with. Then we state our main results concerning both parabolic and elliptic problems. Section 3 is devoted to the proof of results for parabolic problems, instead those about elliptic equations are proved in Section 4.

## 2 Mathematical framework and results

The fractional Laplacian $(-\Delta)^s$ can be defined by Fourier transform. Namely, for any function $g$ in the Schwartz class $\mathcal{S}$, we say that

$$(-\Delta)^{s/2} g = h,$$

if

$$\hat{h}(\xi) = |\xi|^s \hat{g}(\xi).$$

(2.1)

Here, we used the notation $\hat{h} = \mathcal{F}h$ for the Fourier transform of $h$. Furthermore, consider the space

$$\mathcal{L}^s(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } \mid \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < \infty \right\},$$

dominated with the norm

$$\|u\|_{\mathcal{L}^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{n+2s}} dx.$$

If $u \in \mathcal{L}^s(\mathbb{R}^N)$ (see [17]), then $(-\Delta)^s u$ can be defined as a distribution, i.e., for any $\varphi \in \mathcal{S}$,

$$\int_{\mathbb{R}^N} \varphi(-\Delta)^s u dx = \int_{\mathbb{R}^N} u(-\Delta)^s \varphi dx.$$

In addition, suppose that, for some $\gamma > 0$, $u \in \mathcal{L}^s(\mathbb{R}^N) \cap C^{2s+\gamma}((\mathbb{R}^N)$ if $s < \frac{1}{2}$, or $u \in \mathcal{L}^s(\mathbb{R}^N) \cap C_{loc}^{1,2s+\gamma-1}((\mathbb{R}^N)$ if $s \geq \frac{1}{2}$. Then we have

$$(-\Delta)^s u(x) = C_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \quad (x \in \mathbb{R}^N),$$

(2.2)

where

$$C_{N,s} = \frac{2^{2s-1} 2 \Gamma((N+2s)/2)}{\pi^{N/2} \Gamma(1-s)},$$

$\Gamma$ being the Gamma function; moreover, $(-\Delta)^s u \in C(\mathbb{R}^N)$. Note that the constant $C_{N,s}$ satisfies the identity

$$(-\Delta)^s u = \hat{\mathcal{H}}^{-1}(|\xi|^{2s} \hat{u}), \quad \xi \in \mathbb{R}^N, u \in \mathcal{S},$$

so (see [5])

$$C_{N,s} = \left( \int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1}.$$

Concerning the coefficients $\rho$ and $c$, we always make the following assumptions:

\begin{equation}
(H_0) \begin{cases}
(i) & \rho \in C(\mathbb{R}^N), \rho(x) > 0 \quad \text{for all } x \in \mathbb{R}^N; \\
(ii) & \text{there exist } K > 0, \alpha \in \mathbb{R} \text{ such that } \\
& \rho(x) \geq K (1 + |x|^2)^{-\alpha} \quad \text{for all } x \in \mathbb{R}^N;
\end{cases}
\end{equation}
\( (H_1) \quad c \in C(R^N), \, c(x) \geq 0 \) for all \( x \in R^N \).

Now we can give the definition of solution to problem \((1.1)\) and to equation \((1.15)\).

**Definition 2.1** We say that a function \( u \) is a solution to equation
\[
\rho \partial_t u + (-\Delta)^s u = 0 \quad \text{in} \quad S_T, \tag{2.3}
\]
if
\[\text{(i) } u \in C(S_T), \text{ for each } t \in (0, T) \text{ } u(\cdot, t) \in \mathcal{L}^s(R^N) \cap C^{2s+\gamma}(R^N) \text{ if } s < \frac{1}{2}, \text{ or } u(\cdot, t) \in \mathcal{L}^s(R^N) \cap C^{1,2s+\gamma}(R^N) \text{ if } s \geq \frac{1}{2}, \text{ for some } \gamma > 0, \partial_t u \in C(S_T);\]
\[\text{(ii) } \rho(x)\partial_t u + C_{N,s}P.V. \int_{R^N} \frac{u(x,t) - u(y,t)}{|x-y|^{N+2s}} dy = 0 \quad \text{for all} \quad (x, t) \in S_T.\]
Furthermore, we say that \( u \) is a supersolution (subsolution) to equation \((2.3)\), if in \(\text{(ii)}\) instead of \(\equiv \) we have \(\geq \) \((\leq \).

**Definition 2.2** We say that a function \( u \) is a solution to problem \((1.1)\) if
\[\text{(i) } u \in C(S_T), \partial_t u \in L^1_{loc}(S_T), u \in L^1((0, T), \mathcal{L}^s(R^N));\]
\[\text{(ii) } u \text{ is a solution to equation } (2.3);\]
\[\text{(iii) } u(x,0) = 0 \quad \text{for all} \quad x \in R^N.\]

**Definition 2.3** We say that a function \( u \) is a solution to equation \((1.15)\) if
\[\text{(i) } u \in \mathcal{L}^s(R^N) \cap C^{2s+\gamma}(R^N) \text{ if } s < \frac{1}{2}, \text{ or } u \in \mathcal{L}^s(R^N) \cap C^{1,2s+\gamma}(R^N) \text{ if } s \geq \frac{1}{2}, \text{ for some } \gamma > 0;\]
\[\text{(ii) } C_{N,s}P.V. \int_{R^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy + \rho(x)c(x)u(x) = 0 \quad \text{for all} \quad x \in R^N.\]
Furthermore, we say that \( u \) is a supersolution (subsolution) to equation \((1.15)\), if in \(\text{(ii)}\) instead of \(\equiv \) we have \(\geq \) \((\leq \).

### 2.1 Parabolic equations: results

Next we prove a general criterion for uniqueness of nonnegative solutions to problem \((1.1)\) in \(L^1_p(S_T)\), where \(p\) is defined as in \((1.12)\) for some constant \(\beta > 0\).

**Proposition 2.4** Let assumption \((H_0) - (i)\) be satisfied. Let \( u \) be a solution to problem \((1.1)\) with \(|u(\cdot, t)|^p \in \mathcal{L}^s(R^N)\) for some \(p \geq 1\), for each \(t > 0\). Assume that there exists a positive supersolution \( \phi \in C^2(S_T) \) to equation
\[
-(-\Delta)^s \phi + \rho \partial_t \phi = 0 \quad \text{in} \quad S_T, \tag{2.4}
\]
such that
\[
\phi(x, t) + |
abla \phi(x, t)| \leq C\psi(x) \quad \text{for all} \quad (x, t) \in S_T, \tag{2.5}
\]
for some constants \(C > 0\) and \(\beta > 0\). If \( u \in L^p_p(S_T)\), then
\[
u \equiv 0 \quad \text{in} \quad S_T.\]

After having exhibited such a supersolution \( \phi \), as a consequence of Proposition \(2.4\) we show the following uniqueness theorem.
Theorem 2.5 Let assumption \((H_0)\) be satisfied. Let \(u\) be a solution to problem \((1.1)\) with \(|u(\cdot,t)|^p \in \mathcal{L}^s(\mathbb{R}^N)\), for some \(p \geq 1\), for each \(t > 0\). If \(u \in L^p_\psi(S_T)\), then
\[
u \equiv 0 \quad \text{in } S_T,\]
provided that one of the following conditions holds true:

(i) \(0 < \beta \leq N - 2s, \alpha \in \mathbb{R}\);

(ii) \(N - 2s < \beta < N, \alpha \leq 2s\);

(iii) \(\beta = N, \alpha < 2s\), or instead of \((H_0) - (ii)\), there holds: \(\rho(x) \geq K(1 + |x|^2)^{-s} \log(1 + |x|^2)\)
for all \(x \in \mathbb{R}^N\), for some \(K > 0\);

(iv) \(\beta > N, \alpha + \beta \leq 2s + N\).

From Theorem 2.5 we deduce the following

Corollary 2.6 Let assumption \((H_0)\) be satisfied. Let \(u\) be a solution to problem \((1.1)\). Suppose that \(\alpha < 2s\). If
\[
|u(x,t)| \leq C(1 + |x|^2)^{\frac{\sigma}{2}} \quad \text{for all } x \in S_T,\]
for some \(\sigma \in (0, 2s - \alpha)\) and \(C > 0\), then
\[
u \equiv 0 \quad \text{in } S_T.\]

In order to prove Corollary 2.6(i) it suffices to apply Theorem 2.5 with \(\beta = N + 2s - \alpha > N\) and \(p = 1\).

2.2 Elliptic equations: results

Now we prove a general criterion for uniqueness of nonnegative solutions to equation \((1.15)\) in \(L^1_\zeta(\mathbb{R}^N)\). We suppose that there exists a positive function \(\zeta \in C^2(\mathbb{R}^N)\), which solves
\[
-(\Delta)^s \zeta + \rho c \zeta < 0 \quad \text{in } \mathbb{R}^N.\tag{2.6}\]
Such inequality is meant in the sense that in Definition 2.3(ii), instead of “= ” we have “<”.

Proposition 2.7 Let assumptions \((H_0)-(i), (H_1)\) be satisfied. Let \(u\) be a solution to equation \((1.13)\) with \(|u|^p \in \mathcal{L}^s(\mathbb{R}^N)\), for some \(p \geq 1\). Assume that there exists a positive function \(\zeta \in C^2(\mathbb{R}^N)\), which solves \((1.8)\), and satisfies
\[
\zeta(x) + |\nabla \zeta(x)| \leq C\psi(x) \quad \text{for all } x \in \mathbb{R}^N,\tag{2.7}\]
for some constants \(C > 0, \beta > 0\). If \(u \in L^p_\psi(\mathbb{R}^N)\), then
\[
u \equiv 0 \quad \text{in } \mathbb{R}^N.\]

After having exhibited such a supersolution \(\zeta\), as a consequence of Propositions 2.7 we show the following uniqueness theorem.

Theorem 2.8 Let assumptions \((H_0)-(H_1)\) be satisfied. Let \(u\) be a solution to equation \((1.13)\) with \(|u|^p \in \mathcal{L}^s(\mathbb{R}^N)\), for some \(p \geq 1\). Suppose that, for some \(c_0 > 0\),
\[
c(x) \geq c_0 \quad \text{for all } x \in \mathbb{R}^N.\tag{2.8}\]
If \(u \in L^p_\psi(\mathbb{R}^N)\), then
\[
u \equiv 0 \quad \text{in } \mathbb{R}^N,\]
provided \(\alpha, \beta\) satisfy the same conditions as in Theorem 2.5 and \(pc_0K\) is large enough when (ii) or (iii) or (iv) holds true.
Analogously to Corollary 2.6 we have the following

**Corollary 2.9** Let assumptions \((H_0) - (H_1)\) be satisfied. Let \(u\) be a solution to equation \((1.13)\) with \(|u|^p \in L^s(\mathbb{R}^N)\), for some \(p \geq 1\). Suppose that \(pc_0K\) is large enough and \(\alpha < 2s\).

(i) If
\[
|u(x)| \leq C(1 + |x|^2)^{\frac{\alpha}{2}} \quad \text{for all} \quad x \in \mathbb{R}^N,
\]
for some \(\sigma \in (0, 2s - \alpha)\) and \(C > 0\), then
\[
u \equiv 0 \quad \text{in} \quad \mathbb{R}^N.
\]

**Remark 2.10** The hypothesis \(pc_0K\) large enough made in Theorem 2.8 and in Corollary 2.9 will be specified in the proof of Theorem 2.8.

Let us now introduce the Riesz kernel of the \(s\)-Laplacian:
\[
I_{2s} := \frac{k_{N,s}}{|x|^{N-2s}} \quad \forall x \in \mathbb{R}^N,
\]
where \(k_{N,s}\) is a suitable positive constant only depending on \(s\) and \(N\).

Let \(F \in C^\infty(\mathbb{R}^N)\) with \(F \geq 0\), \(F \neq 0\). Define
\[
\phi = I_{2s} * F \quad \text{in} \quad \mathbb{R}^N. \quad \text{(2.9)}
\]

Clearly, we have that
\[
(-\Delta)^s \phi = F \quad \text{in} \quad \mathbb{R}^N. \quad \text{(2.10)}
\]

Furthermore, it is easily checked that \(\phi \in C^\infty(\mathbb{R}^N)\) and, for some \(0 < C_0 < C_1\),
\[
\frac{C_0}{1 + |x|^{N-2s}} \leq \phi(x) + |\nabla \phi(x)| \leq \frac{C_1}{1 + |x|^{N-2s}} \quad \text{for all} \quad x \in \mathbb{R}^N. \quad \text{(2.11)}
\]

If \(u \in L^p_{c, p0}(\mathbb{R}^N)\), then we can drop the request \(pc_0K\) big enough made in Theorem 2.8. This is the content of the next result, which will be proved by different methods from those used to prove Theorem 2.8.

**Theorem 2.11** Let assumptions \((H_0) - (H_1)\) be satisfied. Let \(u\) be a solution to equation \((1.13)\). Suppose that condition \((2.8)\) is satisfied for some \(c_0 > 0\). If \(u \in L^p_{c, p0}(\mathbb{R}^N)\) for some \(p \geq 1\), then
\[
u \equiv 0 \quad \text{in} \quad \mathbb{R}^N.
\]

As a consequence of Theorem 2.11 and (2.11) we immediately get the next result.

**Corollary 2.12** Let assumptions \((H_0) - (H_1)\) be satisfied. Let \(u\) be a solution to equation \((1.13)\). Suppose that \(c \in L^\infty(\mathbb{R}^N)\) and that condition \((2.8)\) is satisfied for some \(c_0 > 0\). If \(u \in L^p_{1 + |x|^{N-2s+\alpha}}(\mathbb{R}^N)\), then
\[
u \equiv 0 \quad \text{in} \quad \mathbb{R}^N.
\]

### 3 Parabolic equations: proofs

#### 3.1 Preliminary results

This Subsection is devoted to some preliminary results that will be used in the sequel. To begin with, let us observe that if \(f, g \in L^s(\mathbb{R}^N) \cap C^{2s+\gamma}(\mathbb{R}^N)\) if \(s < \frac{1}{2}\), or \(f, g \in L^s(\mathbb{R}^N) \cap C^{1, 2s+\gamma - 1}_{loc}(\mathbb{R}^N)\) if \(s \geq \frac{1}{2}\), for some \(\gamma > 0\), and \(fg \in L^s(\mathbb{R}^N)\), then it is easily checked that
\[
(-\Delta)^s [f(x)g(x)] = f(x)(-\Delta)^s g(x) + g(x)(-\Delta)^s f(x) - B(f, g)(x) \quad \text{for all} \quad x \in \mathbb{R}^N, \quad \text{(3.1)}
\]
where $\mathcal{B}(f, g)$ is the bilinear form given by

$$
\mathcal{B}(f, g)(x) := C_{N, s} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)||g(x) - g(y)|}{|x - y|^{N+2s}} \, dy \quad \text{for all } x \in \mathbb{R}^N.
$$

Take a cut-off function $\gamma \in C^\infty([0, \infty)), 0 \leq \gamma \leq 1$ with

$$
\gamma(r) = \begin{cases} 
1 & \text{if } 0 \leq r \leq \frac{1}{2} \\
0 & \text{if } r \geq 1 
\end{cases}
$$

for any $R > 0$ let

$$
\gamma_R(x) := \gamma \left( \frac{|x|}{R} \right) \quad \text{for all } x \in \mathbb{R}^N.
$$

For any $\tau \in (0, T)$ let

$$
S_{\tau} := \mathbb{R}^N \times (0, \tau).
$$

We shall use next

**Lemma 3.1** Let $\tau \in (0, T), \phi \in C^2(\bar{S}_{\tau}), \phi > 0$; suppose that \( (2.5) \) is satisfied. Let $u \in L^1(\psi(\bar{S}_{\tau}))$. Then

$$
\int_0^\tau \int_{\mathbb{R}^N} |u(x, t)| \phi(x, t) |(-\Delta)^s \gamma_R(x)| \, dx \, dt + \int_0^\tau \int_{\mathbb{R}^N} |\mathcal{B}(\phi, \gamma_R)(x)| \, dx \, dt \rightarrow 0
$$
as $R \to \infty$.

Observe that a similar result was obtained in the proof Theorem 2.1 in [3]. However, in [3] different hypotheses were made. To be specific, it was assumed that $u \in L^1((0, T); L^s(\mathbb{R}^N))$, and moreover that, for some $C > 0$,

$$
\phi(x, t) + |\nabla \phi(x, t)| \leq \frac{C}{1 + |x|^{N+2s}} \quad \text{for all } (x, t) \in S_{\tau}.
$$

Observe that our proof requires various quite important changes. In particular, we need to use a convenient covering of $\mathbb{R}^N \times \mathbb{R}^N$ that is a little different from that in [3]; moreover, we shall use different estimates in some regions of $\mathbb{R}^N \times \mathbb{R}^N$.

**Proof.** Observe that, for all $x \in \mathbb{R}^N$,

$$
|(-\Delta)^s \gamma_R(x)| = R^{-2s} \left| ((-\Delta)^s \gamma) \left( \frac{x}{R} \right) \right| \leq CR^{-2s}.
$$

Since $u \in L^1(\psi(S_{\tau})$ and $(2.5)$ holds, from (3.4) it follows that

$$
\int_0^\tau \int_{\mathbb{R}^N} |u(x, t)| \phi(x, t) |(-\Delta)^s \gamma_R(x)| \, dx \, dt \leq CR^{-2s} \int_0^\tau \int_{\mathbb{R}^N} |u(x, t)| \phi(x, t) \, dx \, dt \leq CR^{-2s},
$$

so

$$
\lim_{R \to \infty} \int_0^\tau \int_{\mathbb{R}^N} |u(x, t)| \phi(x, t) |(-\Delta)^s \gamma_R(x)| \, dx \, dt = 0.
$$

Now we are going to estimate

$$
I(R) := \int_0^\tau \int_{\mathbb{R}^N} |u(x, t)| |\mathcal{B}(\phi, \gamma_R)(x)| \, dx \, dt
$$

To do this, we cover $\mathbb{R}^N \times \mathbb{R}^N$ with six domains. In fact,

$$
\mathbb{R}^{2N} = \left( \bigcup_{k=1}^5 A_k \right) \cup C,
$$
where

\[ A_1 := \{(x, y) : |x| > R/2, |y| \leq R/8\}, \quad A_2 := \{(x, y) : |x| \leq R/8, |y| > R/2\}, \]

\[ A_3 := \{(x, y) : |x| \geq 2R, R/8 < |y| < R\}, \quad A_4 := \{(x, y) : R/8 < |x| < R, |y| \geq 2R\}, \]

\[ A_5 := \{(x, y) : R/8 < |x| < 2R, R/8 < |y| < 2R\} \]

and

\[ C := \{(x, y) : |x| \leq R/2, |y| \leq R/2\} \cup \{(x, y) : |x| \geq R, |y| \geq R\}. \]

This covering of \( \mathbb{R}^N \times \mathbb{R}^N \) is represented (for \( N = 1 \)) in the following picture:

From (3.2) and (3.3), we have that \( \gamma_R(x) - \gamma_R(y) = 0 \) if \( (x, y) \in C \), and so

\[
I(R) = \int_0^t \int_{\mathbb{R}^N} \left| u(x, t) \int_{\mathbb{R}^N} \frac{|\phi(x, t) - \phi(y, t)||\gamma_R(x) - \gamma_R(y)|}{|x - y|^{N+2s}} \right| dy \, dx \, dt
\]

\[
\leq \sum_{k=1}^5 I^{A_k}(R),
\]

where

\[
I^{A_k}(R) = \int_0^t \int_{A_k} \left| u(x, t) \int_{\mathbb{R}^N} \frac{|\phi(x, t) - \phi(y, t)||\gamma_R(x) - \gamma_R(y)|}{|x - y|^{N+2s}} \right| dy \, dx \, dt,
\]

for \( k = 1, \ldots, 5 \).

We are going to estimate each of these five integral separately. For all \( (x, y) \in A_1 \) we get

\[
|x - y| \geq C|x|,
\]

and

\[
|x - y| \geq R/2 - R/8 \geq R/4 + |y|.
\]

Hence

\[
\frac{1}{|x - y|^{N+2s}} \leq \frac{C}{|x|^\beta \left( \frac{R}{2} + |y| \right)^{N+2s-\beta}}.
\]

Moreover, from (2.5) it follows that, if \( (x, y) \in A_1 \), then

\[
|\phi(x, t)| + |\phi(y, t)| \leq \frac{C}{1 + |y|^\beta}.
\]
Inequalities (3.10) and (3.11) yield, for any $R > 1$,  
\[ I_{A_1}(R) \leq C \int_0^T \int_{|x| > R/2} \frac{|u(x,t)|}{|x|^\beta} \int_{|y| \leq R/8} \frac{1}{1 + |y|^{N + 2s}} \, dy \, dx \, dt \]
\[ \leq C \int_0^T \int_{|x| > R/2} \frac{|u(x,t)|}{|x|^\beta} \, dx \, dt. \]  
(3.12)

For all $(x, y) \in A_2$ we have
\[ |\phi(x,t)| + |\phi(y,t)| \leq \frac{C}{1 + |x|^\beta}, \]  
(3.13)

and
\[ |x - y| \geq C|y|. \]  
(3.14)

In view of (3.13) and (3.14), we obtain
\[ I_{A_2}(R) \leq \int_0^T \int_{|x| \leq R/8} \frac{|u(x,t)|}{1 + |x|^\beta} \int_{|y| > R/2} \frac{C}{|y|^{N + 2s}} \, dy \, dx \, dt \]
\[ \leq CR^{-2s} \int_0^T \int_{|x| \leq R/8} \frac{|u(x,t)|}{1 + |x|^\beta} \, dx \, dt. \]  
(3.15)

Also, for all $(x, y) \in A_3$ we have that (3.8), (3.11) and (3.14) hold true. From (3.8) and (3.14) we get
\[ \frac{1}{|x - y|^{N + 2s}} \leq \frac{C}{|x|^\beta |y|^{N + 2s - \beta}}. \]  
(3.16)

So, due to (3.11) and (3.16), we obtain
\[ I_{A_3}(R) \leq C \int_0^T \int_{|x| \geq 2R} \frac{|u(x,t)|}{|x|^\beta} \int_{|y| < |y| < R} \frac{1}{|y|^{N + 2s}} \, dy \leq \]
\[ \leq CR^{-2s} \int_0^T \int_{|x| \geq 2R} \frac{|u(x,t)|}{|x|^\beta} \, dx \, dt. \]  
(3.17)

For all $(x, y) \in A_4$, we have that (3.13) and (3.14) hold true. Hence,
\[ I_{A_4}(R) \leq \int_0^T \int_{R/8 \leq |x| \leq R} \frac{|u(x,t)|}{1 + |x|^\beta} \int_{|y| > 2R} \frac{C}{|y|^{N + 2s}} \, dy \, dx \, dt \]
\[ \leq CR^{-2s} \int_0^T \int_{R/8 \leq |x| \leq R} \frac{|u(x,t)|}{1 + |x|^\beta} \, dx \, dt. \]  
(3.18)

Then, using the Monotone Convergence Theorem, since $u \in L_1^1(S_T)$, from (3.12), (3.15), (3.17) and (3.18) it follows that
\[ \lim_{R \to \infty} I_{A_1}(R) = \lim_{R \to \infty} I_{A_2}(R) = \lim_{R \to \infty} I_{A_3}(R) = \lim_{R \to \infty} I_{A_4}(R) = 0. \]  
(3.19)

To estimate $I_{A_5}(R)$ we will consider separately the cases $s \in (0, \frac{1}{2})$ and $s \in [\frac{1}{2}, 1)$.

Let $s \in (0, \frac{1}{2})$ and $(x, y) \in A_5$. Since in $A_5$ the roles of $x$ and $y$ are symmetric, from (2.5) we can infer that
\[ |\phi(x,t)| + |\phi(y,t)| \leq \frac{C}{|x|^\beta}. \]  
(3.20)

Furthermore,
\[ |\gamma_R(x) - \gamma_R(y)| \leq \frac{C}{R}|x - y|. \]  
(3.21)
Hence

\[ I^{A_5}(R) \leq \frac{C}{R} \int_0^T \int_{\frac{R}{4} \leq |x| \leq 2R} \frac{|u(x, t)|}{|x|^\beta} \int_{\frac{R}{4} \leq |\tilde{y}| \leq 2R} \frac{1}{|x-y|^{N+2s-1}} \, d\tilde{y} \, dx \, dt. \]  

(3.22)

By the change of variables \( \tilde{y} := x - y \), since \( s \in \left(0, \frac{1}{2}\right) \), from (3.22) it follows that

\[ I^{A_5}(R) \leq \frac{C}{R} \int_0^T \int_{\frac{R}{4} \leq |x| \leq 2R} \frac{|u(x, t)|}{|x|^\beta} \int_{\frac{R}{4} \leq |\tilde{y}| \leq 2R} \frac{1}{|\tilde{y}|^{N+2s-1}} \, d\tilde{y} \, dx \, dt.

\leq \frac{C}{R} \int_0^T \int_{\frac{R}{4} \leq |x| \leq 2R} \frac{|u(x, t)|}{|x|^\beta} \int_{|\tilde{y}| \leq 4R} \frac{1}{|\tilde{y}|^{N+2s-1}} \, d\tilde{y} \, dx \, dt.

\leq CR^{-2s} \int_0^T \int_{\frac{R}{4} \leq |x| \leq 2R} \frac{|u(x, t)|}{|x|^\beta} \, dx \, dt. \]  

(3.23)

Therefore, since \( u \in L^1_v(S_T) \), we conclude that

\[ \lim_{R \to \infty} I^{A_5}(R) = 0, \quad \text{when } s \in \left(0, \frac{1}{2}\right). \]  

(3.24)

Now, let \( s \in \left[\frac{1}{2}, 1\right) \). By (3.20), we get

\[ |\phi(x, t) - \phi(y, t)| \leq \frac{C}{1 + |z|^\beta}|x - y|, \]  

(3.25)

for some \( z \) in the segment joining \( x \) and \( y \). For any \( R > 0 \) let

\[ Q_R \equiv Q := \left\{ (x, y) \in A_5 : |x - y| \leq \frac{R}{100} \right\}. \]

Note that, if \( (x, y) \in Q \) then every point \( z \) lying on the segment from \( x \) to \( y \) verifies \( |z| \geq C|x| \).

Hence, since \( s \in \left[\frac{1}{2}, 1\right) \), (3.25) and (3.21) yield

\[ \int_0^T \int_{(x, y) \in Q} \left| \frac{(\gamma R(x) - \gamma R(y)) (\phi(x, t) - \phi(y, t))}{|x - y|^\beta} \right| \, dy \, dx \, dt \]

\[ \leq \int_0^T \int_{(x, y) \in Q} \frac{|u(x, t)|}{R|x|^\beta|x - y|^{N+2s-2}} \, dy \, dx \, dt \]

\[ \leq \frac{C}{R} \int_0^T \int_{\frac{R}{4} \leq |x| \leq 2R} \frac{|u(x, t)|}{|x|^\beta} \int_{\frac{R}{4} \leq |y| \leq 2R} \frac{1}{|x - y|^{N+2s-2}} \, dy \, dx \, dt \]

\[ \leq \frac{C}{R} \int_0^T \int_{\frac{R}{4} \leq |x| \leq 2R} \frac{|u(x, t)|}{|x|^\beta} \int_{|y| \leq 4R} \frac{1}{|y|^{N+2s-2}} dy \, dx \, dt \]

\[ \leq CR^{1-2s} \int_0^T \int_{\frac{R}{4} \leq |x| \leq 2R} \frac{|u(x, t)|}{|x|^\beta} \, dx \, dt. \]  

(3.26)

On the other hand, if \( (x, y) \in A_5 \setminus Q \) we have that

\[ |x - y| > \frac{R}{100} \geq C|y|. \]  

(3.27)
Then, by (3.21) and (3.27),
\[
\begin{align*}
&\int_0^T \int_{(x,y) \in A_\delta \setminus Q} |u(x,t)| \frac{|(\gamma_R(x) - \gamma_R(y))(\phi(x,t) - \phi(y,t))|}{|x - y|^{N+2s}} \, dy \, dx \, dt \\
&\leq \frac{C}{R} \int_0^T \int_{(x,y) \in A_\delta \setminus Q} |u(x,t)| \frac{|(\phi(x,t) - \phi(y,t))|}{|x - y|^{N+2s-1}} \, dy \, dx \, dt \\
&\leq \frac{C}{R} \int_0^T \int_{(x,y) \in A_\delta \setminus Q} |u(x,t)| \frac{1}{|x|^{\beta}} \frac{1}{|y|^{N+2s-1}} \, dy \, dx \, dt \\
&\leq CR^{-2s} \int_0^T \int_{\frac{R}{2} \leq |x| \leq 2R} \frac{|u(x,t)|}{|x|^\beta} \, dx \, dt.
\end{align*}
\]
Therefore, from (3.26) and (3.28) we have
\[
I^{A_\delta}(R) \leq \int_0^T \int_{(x,y) \in Q} |u(x,t)| \frac{|(\gamma_R(x) - \gamma_R(y))(\phi(x,t) - \phi(y,t))|}{|x - y|^{N+2s}} \, dy \, dx \, dt \\
+ \int_0^T \int_{(x,y) \in A_\delta \setminus Q} |u(x,t)| \frac{|(\gamma_R(x) - \gamma_R(y))(\phi(x,t) - \phi(y,t))|}{|x - y|^{N+2s-1}} \, dy \, dx \, dt \\
\leq CR^{-2s} \int_0^T \int_{\frac{R}{2} \leq |x| \leq 2R} \frac{|u(x,t)|}{|x|^\beta} \, dx \, dt \\
+ CR^{-2s} \int_0^T \int_{\frac{R}{2} \leq |x| \leq 2R} \frac{|u(x,t)|}{|x|^\beta} \, dx \, dt.
\]
Then, since \( u \in L^1_\psi(S_T) \), using the Monotone Convergence Theorem we obtain
\[
\lim_{R \to \infty} I^{A_\delta}(R) = 0, \quad \text{when } s \in \left[\frac{1}{2}, 1\right).
\]
That is, by (3.23) and (3.30), we get
\[
\lim_{R \to \infty} I^{A_\delta}(R) = 0, \quad \text{whenever } s \in (0, 1).
\]
Putting together (3.7), (3.19) and (3.31) it follows that
\[
\lim_{R \to \infty} I(R) = 0, \quad \text{when } s \in (0, 1).
\]
From (3.3), (3.6) and (3.32) the conclusion follows.

3.2 Proof of Proposition 2.4

The next lemma will be used.

Lemma 3.2 Let \( G \in C^2(\mathbb{R}; \mathbb{R}) \) be a convex function. Let \( u \in L^s(\mathbb{R}^N) \cap C^{2s+\gamma}(\mathbb{R}^N) \) if \( s < \frac{1}{2} \), or \( u \in L^s(\mathbb{R}^N) \cap C^{1,2s+\gamma-1}_loc(\mathbb{R}^N) \) if \( s \geq \frac{1}{2} \), for some \( \gamma > 0 \). Suppose that \( G(u) \in L^s(\mathbb{R}^N) \). Then
\[
(-\Delta)^s[G(u)] \leq G'(u)(-\Delta)^s u \quad \text{in } \mathbb{R}^N.
\]

Proof. We can choose, by a suitable convolution, a sequence \( \{u_n\} \subset \mathcal{S} \) uniformly bounded in \( C^{2s+\gamma}_loc(\mathbb{R}^N) \), if \( s < \frac{1}{2} \), or in \( C^{1,2s+\gamma-1}_loc(\mathbb{R}^N) \) if \( s \geq \frac{1}{2} \), for some \( \gamma > 0 \), with \( u_n \to u \) as \( n \to \infty \).
both in $L^s(\mathbb{R}^N)$ and locally uniformly in $\mathbb{R}^N$. Since $G \in C^2(\mathbb{R}; \mathbb{R})$ and $G(u) \in L^s(\mathbb{R}^N)$, analogously to the proof of [17, Proposition 2.1.4] we have that
\[(\Delta)^su_n \to (\Delta)^su, (\Delta)^s[G(u_n)] \to (\Delta)^sG(u) \quad \text{as } n \to \infty,
\]locally uniformly in $\mathbb{R}^N$. From [7, Lemma 4.1] we have
\[(-\Delta)^s[G(u_n)] \leq G'(u_n)(-\Delta)^s u_n \quad \text{in } \mathbb{R}^N.
\]So, passing to the limit as $n \to \infty$ we get (3.33).

**Proof of Proposition 2.3.** Let $\tau \in (0,T)$. Take a nonnegative function $v \in C^2(\bar{S}_T)$ with $supp\ v(\cdot, t)$ compact for each $t \in [0, \tau]$. Moreover, take a function $w \in C(\bar{S}_T)$ such that for each $t \in (0, \tau], w(\cdot, t) \in L^s(\mathbb{R}^N) \cap C^{2s+\gamma}(\mathbb{R}^N)$ if $s < \frac{1}{2}$, or $w(\cdot, t) \in L^s(\mathbb{R}^N) \cap C^{12s+\gamma-1}_{loc}(\mathbb{R}^N)$ if $s \geq \frac{1}{2}$, for some $\gamma > 0, w \in L^1((0, \tau); L^s(\mathbb{R}^N))$. For any $\epsilon \in (0, \tau)$, integrating by parts we have:
\[
\int_0^\tau \int_{\mathbb{R}^N} v[-(-\Delta)^s w - \rho \partial_t w] \, dx dt = \int_0^\tau \int_{\mathbb{R}^N} \left\{ v[-(-\Delta)^s v + \rho \partial_t v] \, dx dt \right. \\
- \int_{\mathbb{R}^N} \rho(x)v(x, \tau)w(x, \tau)dx + \int_{\mathbb{R}^N} \rho(x)v(x, \epsilon)w(x, \epsilon)dx.
\]Let $p \geq 1$. For any $\alpha > 0$, set
\[G_\alpha(r) := (r^2 + \alpha)^\frac{p}{2} \quad \text{for all } r \in \mathbb{R}.
\]It is easily seen that
\[G''_\alpha(r) \geq 0 \quad \text{for all } r \in \mathbb{R}.
\]By (1.1),
\[\rho \partial_t[G_\alpha(u)] = G'_\alpha(u)\partial_t u = -G'_\alpha(u)(-\Delta)^s u \quad \text{for all } (x, t) \in S_T.
\]From (3.36), (3.37) and Lemma 3.2 we obtain
\[\rho \partial_t[G_\alpha(u)] + (-\Delta)^s[G_\alpha(u)] \leq 0 \quad \text{in } S_T.
\]So, from (3.34) with $w = G_\alpha(u)$ and (3.37) we obtain
\[
\int_{\mathbb{R}^N} \rho(x)G_\alpha[u(x, \tau)]v(x, \tau)dx \leq \int_0^\tau \int_{\mathbb{R}^N} G_\alpha(u)\left[ -(-\Delta)^s v + \rho \partial_t v \right]dx dt \\
+ \int_{\mathbb{R}^N} \rho(x)v(x, \epsilon)G_\alpha[u(x, \epsilon)]dx.
\]Letting $\epsilon \to 0^+$ in (3.39), by the dominated convergence theorem,
\[
\int_{\mathbb{R}^N} \rho(x)G_\alpha[u(x, \tau)]v(x, \tau)dx \leq \int_0^\tau \int_{\mathbb{R}^N} G_\alpha(u)\left[ -(-\Delta)^s v + \rho \partial_t v \right]dx dt \\
+ \alpha \int_{\mathbb{R}^N} \rho(x)v(x, 0)dx.
\]Now, letting $\alpha \to 0^+$, by the dominated convergence theorem,
\[
\int_{\mathbb{R}^N} \rho(x)|u(x, \tau)|p v(x, \tau)dx \leq \int_0^\tau \int_{\mathbb{R}^N} |u|p\left[ -(-\Delta)^s v + \rho \partial_t v \right]dx dt
\]For any $R > 0$, we can choose
\[v(x, t) := \phi(x, t)\gamma_R(x) \quad \text{for all } (x, t) \in \bar{S}_T.
\]
From (3.41), using the fact that \( \phi \) is a supersolution to equation (2.4) and \( \gamma_R \geq 0 \), we obtain

\[
-(-\Delta)^s v + \rho \partial_t v = \gamma_R \left[ -(-\Delta)^s \phi + \rho \partial_t \phi \right] - \phi(-\Delta)^s \gamma_R + B(\phi, \gamma_R) \leq -\phi(-\Delta)^s \gamma_R + B(\phi, \gamma_R) \quad \text{in } S_\tau.
\] (3.42)

Since \( |u|^p \geq 0 \), by (3.41) and (3.42) we conclude that

\[
\int_{\mathbb{R}^N} \rho(x)|u(x, \tau)|^p \phi(x, \tau) \gamma_R(x) dx \leq \int^\tau_0 \int_{\mathbb{R}^N} |u|^p \left[ -\phi(-\Delta)^s \gamma_R + B(\phi, \gamma_R) \right] dx dt \leq \int^\tau_0 \int_{\mathbb{R}^N} |u|^p \left[ |\phi| \left|(-\Delta)^s \gamma_R \right| + |B(\phi, \gamma_R)| \right] dx dt.
\] (3.43)

Hence, from Lemma 3.1 with \( v = |u|^p \) and the monotone convergence theorem, sending \( R \to \infty \) in (3.43) we get

\[
\int_{\mathbb{R}^N} \rho(x)|u(x, \tau)|^p \phi(x, \tau) dx \leq 0.
\] (3.44)

From (3.44), (H0) – (i), since \( \phi > 0 \) in \( S_\tau \) and \( |u|^p \geq 0 \) we infer that \( u \equiv 0 \) in \( S_\tau \). This completes the proof. \( \square \)

### 3.3 Proof of Theorem 2.5

Before proving Theorem 2.5, we need some preliminary results.

**Proposition 3.3** Let \( \tilde{w} \in C^2([0, \infty)) \cap L^\infty((0, \infty)) \). Let

\[
w(x) := \tilde{w}(|x|) \quad \text{for all } x \in \mathbb{R}^N.
\]

Set \( r \equiv |x| \). If

\[
\tilde{w}''(r) + \frac{N - 2s + 1}{r} \tilde{w}'(r) \geq 0,
\] (3.45)

then \( w \) is a supersolution to equation

\[
(-\Delta)^s w = 0 \quad \text{in } \mathbb{R}^N.
\] (3.46)

Observe that in Proposition 3.3 \( w \) is a supersolution to equation (3.46) in the sense of Definition 2.3 with \( c \equiv 0 \).

**Proof.** From [8] Theorems 1.1, 1.2, and remarks after Theorem 1.2, due to (3.45) we have:

\[
-(-\Delta)^s w \geq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N),
\]
i.e.,

\[
\int_{\mathbb{R}^N} w(-\Delta)^s \zeta dx \leq 0
\] (3.47)

for any \( \zeta \in C^\infty_c(\mathbb{R}^N) \), \( \zeta \geq 0 \). Since \( w \in L^s(\mathbb{R}^N) \cap C^{2s+\gamma}_c(\mathbb{R}^N) \) for some \( \gamma > 0 \), from (3.47) we can infer that

\[
\int_{\mathbb{R}^N} \zeta(-\Delta)^s w dx \leq 0
\] (3.48)

for any \( \zeta \in C^\infty_c(\mathbb{R}^N) \), \( \zeta \geq 0 \). Inequality (3.48) immediately yields the thesis. \( \square \)

In the sequel we shall use the next well-known result, concerning the hypergeometric function \( {}_2F_1(a, b, c, s) \equiv F(a, b, c, s) \), with \( a, b, c > 0 \), \( s \in \mathbb{R} \setminus \{1\} \) (see [14] Chapters 15.2, 15.4).

**Lemma 3.4** The following limits hold true:
(i) if $c > a + b$, then
\[ \lim_{s \to 1^-} F(a, b, c, s) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}; \]
(ii) if $c = a + b$, then
\[ \lim_{s \to 1^-} \frac{F(a, b, c, s)}{-\log(1 - s)} = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)}; \]
(iii) if $c < a + b$, then
\[ \lim_{s \to 1^-} \frac{F(a, b, c, s)}{(1 - s)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}. \]

For further references, observe that
\[ \Gamma(t) > 0 \text{ for all } t > 0, \quad \Gamma(t) < 0 \text{ for all } t \in (-1, 0). \] (3.49)

**Proof of Theorem 2.5.** Let $\psi = \psi(|x|)$ be defined as in (1.12), where $\beta > 0$ is a constant to be chosen. Also, let $\alpha$ be as in $(H_0) - (ii)$. Set $r \equiv |x|$. We have:
\[ \psi'(r) = -\beta r(1 + r^2)^{-(\frac{d}{2}+1)} \text{ for all } r > 0, \]
\[ \psi''(r) = \beta (1 + r^2)^{-(\frac{d}{2}+2)}[-1 + (\beta + 1)r^2] \text{ for all } r > 0. \] (3.50) (3.51)

For any $\lambda > 0$ define
\[ \phi(x, t) := e^{-\lambda t} \psi(r) \text{ for all } (x, t) \in \bar{S}_T. \]

At first observe that (2.5) is satisfied.

Suppose that (i) is satisfied. In view of (3.50)-(3.51), we have:
\[ \psi''(r) + \frac{N - 2s + 1}{r} \psi'(r) \]
\[ = \beta (1 + r^2)^{-(\frac{d}{2}+2)}[(\beta - N + 2s)r^2 - (N - 2s + 2)] \text{ for all } r > 0, t > 0. \] (3.52)

Since $0 < \beta \leq N - 2s$, by (3.52),
\[ \psi''(r) + \frac{N - 2s + 1}{r} \psi'(r) \leq 0 \text{ for all } r > 0. \] (3.53)

By Proposition 3.3
\[ -(-\Delta)^s \phi(x, t) = -e^{-\lambda t}(-\Delta)^s \psi(r) \leq 0 \text{ for all } (x, t) \in \bar{S}_T. \] (3.54)

From (3.54), for any $\alpha \in \mathcal{R}$, we obtain
\[ -(-\Delta)^s \phi(x, t) + \rho(x)\partial_t \phi(x, t) \leq -\lambda \rho(x)e^{-\lambda t} < 0 \text{ for all } (x, t) \in \bar{S}_T. \] (3.55)

By (3.55) and Proposition 2.4, the conclusion follows.

In order to obtain the thesis of Theorem 2.5 in cases (ii), (iii), (iv), note that (see the proof of Corollary 4.1 in [8]) we have:
\[ -(-\Delta)^s \psi(r) = -\hat{C} F(a, b, c, -r^2) \text{ for all } r > 1, \] (3.56)

where $\hat{C} > 0$ is a positive constant, and
\[ a = \frac{N}{2} + s, \quad b = \frac{\beta}{2} + s, \quad c = \frac{N}{2}. \]
By Pfaff’s transformation,
\[ F(a, b, c, -r^2) = \frac{1}{(1 + r^2)^b} F\left(c - a, b, c, \frac{r^2}{1 + r^2}\right) \text{ for all } r > 1. \quad (3.57) \]

Suppose that (ii) is satisfied. From Lemma 3.4 (i), 3.56 and 3.57, for any \( \epsilon > 0 \), for some \( R_\epsilon > 1 \), we have:
\[ -(-\Delta)^s \psi(r) \leq \tilde{\mathcal{C}}(C_1 + \epsilon)(1 + r^2)^{-\left(s + \frac{d}{2}\right)} \text{ whenever } |x| > R_\epsilon, \quad (3.58) \]
where
\[ C_1 = -\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-\beta}{2}\right)}{\Gamma\left(\frac{N+s}{2}\right)\Gamma\left(\frac{N-\beta - s}{2}\right)} > 0 \]
(see 3.49). From (3.58), since \( \alpha \leq 2s \), we obtain for all \( |x| > R_\epsilon, t \in [0, T] \)
\[ -(-\Delta)^s \phi(x, t) + \rho(x) \partial_t \phi(x, t) \leq e^{-\lambda t} \left[(C_1 + \epsilon)\tilde{\mathcal{C}}(1 + r^2)^{-\left(s + \frac{d}{2}\right)} - \lambda K(1 + r^2)^{-\frac{\alpha + \beta}{2}}\right] < 0, \quad (3.59) \]
provided
\[ \lambda > \frac{M_{\epsilon, \beta}(1 + R_\epsilon^2)^{\frac{\alpha + \beta}{2}}}{K}. \quad (3.60) \]
On the other hand, for all \( |x| \leq R_\epsilon, t \in [0, T] \)
\[ -(-\Delta)^s \phi(x, t) + \rho(x) \partial_t \phi(x, t) \leq e^{-\lambda t} \left[M_{\epsilon, \beta} - \lambda K(1 + R_\epsilon^2)^{-\frac{\alpha + \beta}{2}}\right] < 0, \quad (3.61) \]
taking
\[ \lambda > \frac{M_{\epsilon, \beta}(1 + R_\epsilon^2)^{\frac{\alpha + \beta}{2}}}{K}, \quad (3.62) \]
where
\[ M_{\epsilon, \beta} := \max_{x \in B_{R_\epsilon}} \{|-(-\Delta)^s \psi(|x|)|\}. \]
By 3.59, 3.61 the conclusion follows by Proposition 2.4.

Suppose that (iii) is satisfied. Let \( \alpha < 2s \). From Lemma 3.4 (ii) and 3.57, for any \( \epsilon > 0 \), for some \( R_\epsilon > 1 \), we have:
\[ -(-\Delta)^s \psi(r) \leq \tilde{\mathcal{C}}(C_2 + \epsilon)(1 + r^2)^{-\left(s + \frac{d}{2}\right)} \log(1 + r^2) \text{ whenever } |x| > R_\epsilon, \quad (3.63) \]
where
\[ C_2 = -\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(-s)\Gamma\left(\frac{d}{2} + s\right)} > 0 \]
(see 3.49). From (3.63) we obtain for all \( |x| > R_\epsilon, t \in [0, T] \)
\[ -(-\Delta)^s \psi(x, t) + \rho(x) \partial_t \phi(x, t) \]
\[ \leq e^{-\lambda t} \left[(C_2 + \epsilon)\tilde{\mathcal{C}}(1 + r^2)^{-\left(s + \frac{d}{2}\right)} \log(1 + r^2) - \lambda(1 + r^2)^{-\frac{\alpha + \beta}{2}}\right] < 0, \]
taking a possibly larger \( R_\epsilon > 1 \), and so the desired claim follows from (3.61).

Now, let \( \alpha = 2s \). From (3.63) again, we have for all \( |x| > R_\epsilon, t \in [0, T] \)
\[ -(-\Delta)^s \psi(x, t) + \rho(x) \partial_t \phi(x, t) \]
\[ \leq e^{-\lambda t} \left[(C_2 + \epsilon)\tilde{\mathcal{C}}(1 + r^2)^{-\left(s + \frac{d}{2}\right)} \log(1 + r^2) - \lambda(1 + r^2)^{-\left(s + \frac{d}{2}\right)} \log(1 + r^2)\right] < 0, \quad (3.64) \]
provided
\[ \lambda > (C_2 + \epsilon) \frac{\tilde{C}}{K}. \]  
(3.65)

Furthermore, (3.61) holds true, provided (3.62) holds true when \( \alpha < 2s \), or
\[ \lambda > \frac{1}{Kse} \]  
(3.66)
when \( \alpha = 2s \). From (3.64) and (3.61), the conclusion follows.

Finally, suppose that (iv) is satisfied. From Lemma 3.4 (iii) and (3.57), for any \( \epsilon > 0 \), for some \( R_\epsilon > 1 \), we have:
\[ -(-\Delta)^s \psi(r) \leq \tilde{C} (C_3 + \epsilon)(1 + r^2)^{-(s + \frac{\beta}{2})} \]  
whenever \( |x| > R_\epsilon \),
(3.67)
where
\[ C_3 = -\frac{\Gamma(\frac{N}{2}) \Gamma(\frac{\beta - N}{2})}{\Gamma(-s) \Gamma(\frac{\beta + s}{2})} > 0 \]  
(see (3.49)). From (3.67), since \( \alpha + \beta \leq 2s + N \), we obtain for all \( |x| > R_\epsilon, t \in [0, T] \)
\[ -(-\Delta)^s \phi(x, t) + \rho(x) \partial_t \phi(x, t) \]
\[ \leq e^{-\lambda t} [(C_3 + \epsilon) \tilde{C}(1 + r^2)^{-(s + \frac{\beta}{2})} - \lambda K(1 + r^2)^{-\frac{\alpha + \beta}{2}}] < 0, \]
provided
\[ \lambda > (C_3 + \epsilon) \frac{\tilde{C}}{K}. \]  
(3.69)
On the other hand, (3.61) holds true, provided (3.62) is satisfied. In view of (3.61) and (3.68), the conclusion follows by Proposition 2.4. This completes the proof.

\[ \square \]

4 Elliptic equations: proofs

4.1 Proof of Proposition 2.7

Analogously to Lemma 3.5 the next lemma can be shown.

Lemma 4.1 Let \( \phi \in C^2(\mathbb{R}^N), \phi > 0 \); suppose that (2.7) is satisfied. Let \( v \in L^1(\mathbb{R}^N) \). Then
\[ \int_{\mathbb{R}^N} |v(x)| \phi(x)|(-\Delta)^s \gamma_R(x)|dxdt + \int_{\mathbb{R}^N} |v(x)| |\mathcal{D}(\phi, \gamma_R)(x)|dx \to 0 \]
as \( R \to \infty \).

Proof of Proposition 2.7. Take a function \( v \in C^2(\mathbb{R}^N) \) with \( supp \ v \) compact. Moreover, take a function \( w \in C(\mathbb{R}^N) \) such that \( w \in L^s(\mathbb{R}^N) \cap C^{2s}(\mathbb{R}^N) \) if \( s < 1 \), or \( w \in L^s(\mathbb{R}^N) \cap C^{1,2s}(\mathbb{R}^N) \) if \( s \geq 1 \), for some \( \gamma > 0 \). Integrating by parts we have:
\[ \int_{\mathbb{R}^N} v \left[ -(-\Delta)^s w - \rho(x)c(x)w \right] dx = \int_{\mathbb{R}^N} w \left[ -(-\Delta)^s v - \rho(x)c(x)v \right] dx. \]  
(4.1)
Let \( G_\alpha \) be defined as in (3.36). Thanks to (1.13) and (3.30) we obtain
\[ (-\Delta)^s[G_\alpha(u)] + \rho cG_\alpha(u) \leq p(u^2 + \alpha)^{\frac{2s}{s+1}} u(-\Delta)^s u + \rho c G(u) \]
\[ = p(u^2 + \alpha)^{\frac{2s}{s+1}} u([-\Delta]^s u + \rho cu] + \rho c(u^2 + \alpha)^{\frac{2s}{s+1}} - p(u^2 + \alpha)^{\frac{2s}{s+1}} u^2 c \rho \]
\[ = (u^2 + \alpha)^{\frac{2s}{s+1}} \rho c(u^2 + \alpha - pu^2) = (u^2 + \alpha)^{\frac{2s}{s+1}} \rho c[(1 - p)u^2 + \alpha] \quad \text{in} \ \mathbb{R}^N. \]  
(4.2)
From (4.1) with $w = G_\alpha(u)$ and (4.2) it follows that
\[
\int_{\mathbb{R}^N} v(u^2 + \alpha)\frac{p}{p-1} \rho c [(p-1)u^2 - \alpha] \leq \int_{\mathbb{R}^N} G_\alpha(u) \left[ -(-\Delta)^s v - \rho(x) c(x) v \right] \, dx. \tag{4.3}
\]
Letting $\alpha \to 0^+$ in (4.3), by the dominated convergence theorem we get
\[
\int_{\mathbb{R}^N} |u|^p \left[ (-\Delta)^s v + \rho(x) c(x) pv \right] \, dx \leq 0. \tag{4.4}
\]
For any $R > 0$, we can choose \( v(x) := \zeta(x) \gamma_R(x) \) for all $x \in \mathbb{R}^N$.

From (1.15) we obtain
\[
-(-\Delta)^s v - p \rho c v = \gamma_R \left[ -(-\Delta)^s \zeta - p \rho \zeta \right] - \zeta(-\Delta)^s \gamma_R + \mathcal{B}(\zeta, \gamma_R) \quad \text{in} \quad \mathbb{R}^N. \tag{4.5}
\]
By (4.4), (4.5),
\[
\int_{\mathbb{R}^N} |u(x)|^p \left[ (-\Delta)^s \zeta + p \rho \zeta \right] \, dx \leq \int_{\mathbb{R}^N} |u(x)|^p \left[ -\zeta(-\Delta)^s \gamma_R + \mathcal{B}(\zeta, \gamma_R) \right] \, dx
\leq \int_{\mathbb{R}^N} |u(x)|^p \left[ |\zeta(-\Delta)^s \gamma_R| + |\mathcal{B}(\zeta, \gamma_R)| \right] \, dx. \tag{4.6}
\]
Hence, from Lemma 4.1 and the monotone convergence theorem, sending $R \to \infty$ in (4.6) we get
\[
\int_{\mathbb{R}^N} |u(x)|^p \left[ (-\Delta)^s \zeta + p \rho \zeta \right] \, dx \leq 0. \tag{4.7}
\]
From (4.7) and (1.8), since $|u|^p \geq 0$, we can infer that $u \equiv 0$ in $\mathbb{R}^N$. This completes the proof. \( \square \)

### 4.2 Proof of Theorem 2.8

**Proof of Theorem 2.8.** Let $\psi$ be defined by (1.12). From the same arguments as in the proof of Theorem 2.5 we can infer that $\psi$ solves (3.36), for properly chosen $\beta > 0$. Note that to do this, we require that by hypothesis $pc_0$ satisfies the same conditions as $\lambda$ in the proof of Theorem 2.8. To be specific, we require that, for some $\epsilon > 0$,
\[
pc_0K > \max \left\{ (C_1 + \epsilon) \tilde{C}, M_{\epsilon, \beta}(1 + R^2_\epsilon)^{\frac{\alpha + \beta}{2}} \right\}, \tag{4.8}
\]
when (ii) is satisfied (see (3.60), (3.62));
\[
pc_0K > \max \left\{ (C_2 + \epsilon) \tilde{C}, M_{\epsilon, \beta}(1 + R^2_\epsilon)^{\frac{\alpha + \beta}{2}} \right\}, \tag{4.9}
\]
when (iii) is satisfied and $\alpha < 2s$ (see (3.65), (3.62)), while
\[
pc_0K > \max \left\{ (C_2 + \epsilon) \tilde{C}, \frac{1}{s\epsilon} \right\}, \tag{4.10}
\]
if $\alpha = 2s$ (see (3.65), (3.66));
\[
pc_0K > \max \left\{ (C_3 + \epsilon) \tilde{C}, M_{\epsilon, \beta}(1 + R^2_\epsilon)^{\frac{\alpha + \beta}{2}} \right\}. \tag{4.11}
\]
when (iv) is satisfied (see (3.69), (3.62)). Thus, by Proposition 2.7 the conclusion follows. \( \square \)
4.3 Proof of Theorem 2.11

In order to prove Theorem 2.11 we need the next lemma.

**Lemma 4.2** Let assumption \((H_0)\) - (i), (ii) be satisfied with \(\alpha < s\). Let \(N > -2s + \alpha\). Then there exist constants \(C > 0\) and \(\sigma > 0\) such that, for any \(R > 1\),

\[
R^\sigma \left[ \phi(x)(-\Delta)^s \gamma_R(x) + |\mathcal{B}(\phi, \gamma_R)(x)| \right] \leq C \rho(x) \phi(x) \quad \text{for all } x \in \mathbb{R}^N; \quad (4.12)
\]

here \(\phi\) is defined as in (2.9).

For any \(0 < r < R\) we set \(B_R := \{x \in \mathbb{R}^N : |x| < R\}, B_R^c := \mathbb{R}^N \setminus B_R, A_{r,R} := B_R \setminus \bar{B}_r\).

**Proof.** Now we estimate \(|\mathcal{B}(\phi, \gamma_R)|\). To do this, we cover \(\mathbb{R}^N \times \mathbb{R}^N\) as in the proof of Lemma 3.1. For any \(R > 1\) set \(f_R(x, y) := \frac{|\phi(x, t) - \phi(y, t)|}{|x - y|^{N + 2s}}\) for all \(x, y \in \mathbb{R}^N, x \neq y\).

Due to (3.2)-(3.3) we have that for all \((x, y) \in B_R^c\)

\[
\int_{B_R^c} f_R(x, y) dy = 0; \quad (4.13)
\]

furthermore, for all \(x \in B_R^c\)

\[
\int_{B_R^c} f_R(x, y) dy = 0. \quad (4.14)
\]

Note that in the sequel we shall denote by the same \(C\) different positive constants independent of \(R\). Let \(0 < \beta < N, 0 < \sigma < \min\{\beta, 2s\}\) be two parameters to be chosen later. For all \((x, y) \in A_1\) we get

\[
|x - y| \geq C|x|, \quad (4.15)
\]

and

\[
|x - y| \geq \frac{R}{2} - \frac{R}{8} \geq \frac{R}{4} + |y|. \quad (4.16)
\]

Hence,

\[
\frac{1}{|x - y|^{N + 2s}} \leq \frac{C}{|x|^{\beta(\frac{R}{4} + |y|)^{N - 2s}}}. \quad (4.17)
\]

Hence, from (2.11) it follows that, if \((x, y) \in A_1\), then, for any \(R > 1\),

\[
|\phi(x)| + |\phi(y)| \leq \frac{C}{1 + |y|^{N - 2s}}. \quad (4.18)
\]

Inequalities (4.17) and (4.18) yield, for any \(x \in B_R^c\),

\[
R^\sigma \int_{B_R^c} f_R(x, y) dy \leq C \frac{R^\sigma}{|x|^\sigma} \int_{B_R^c} \frac{dy}{1 + |y|^{2N - \beta}} \leq \frac{C}{1 + |x|^{N - 2s}}. \quad (4.19)
\]

For all \((x, y) \in A_2\) we have

\[
|\phi(x)| + |\phi(y)| \leq \frac{C}{1 + |x|^{N - 2s}}. \quad (4.20)
\]

and

\[
|x - y| \geq C|y|. \quad (4.21)
\]
In view of (4.20) and (4.21), we obtain for any \( x \in B_{\frac{1}{\delta}} \)

\[
R^\sigma \int_{B_{\frac{1}{\delta}}} f_R(x,y)dy \leq \frac{C R^\sigma}{1 + |x|^N - 2s} \int_{B_{\frac{1}{\delta}}} \frac{dy}{|y|^N + 2s} \leq \frac{C R^{-2s+\sigma}}{1 + |x|^N - 2s} \leq \frac{C}{1 + |x|^N - \sigma}.
\] (4.22)

Also, for all \((x,y) \in A_3\) we have that (4.15), (4.18) and (4.21) hold true. From (4.15) and (4.21) we get

\[
1 \leq \frac{|x-y|^{N+2s}}{|x|^N |y|+2s}.
\] (4.23)

So, due to (4.18) and (4.23), we obtain for any \( x \in B_{\frac{1}{R}} \)

\[
R^\sigma \int_{A_{\frac{1}{R},R}} f_R(x,y)dy \leq \frac{C}{1 + |x|^N - 2s} \int_{A_{\frac{1}{R},R}} \frac{R^\sigma}{|y|^{2N-\beta}}dy \leq \frac{C}{1 + |x|^{\beta-\sigma}}.
\] (4.24)

For all \((x,y) \in A_4\), we have that (4.20) and (4.21) hold true. Hence, for any \( x \in A_{\frac{1}{R},R} \)

\[
R^\sigma \int_{B_{\frac{1}{R}}} f_R(x,y)dy \leq \frac{C}{1 + |x|^N - 2s} \int_{B_{\frac{1}{R}}} \frac{R^\sigma}{|y|^{N+2s}}dy \leq \frac{C R^{-2s+\sigma}}{1 + |x|^N - 2s} \leq \frac{C}{1 + |x|^N - \sigma}.
\] (4.25)

Now, let \((x,y) \in A_5\). We shall distinguish the cases \(s \in (0, \frac{1}{2})\) and \(s \in (\frac{1}{2}, 1)\). To begin with, take any \(s \in (0, \frac{1}{2})\). Since in \(A_5\) the roles of \(x\) and \(y\) are symmetric, from (2.11) we can infer that

\[
|\phi(x)| + |\phi(y)| \leq \frac{C}{|x|^{N-2s}}.
\] (4.26)

Furthermore,

\[
|\gamma_R(x) - \gamma_R(y)| \leq \frac{C}{R|x-y|}.
\] (4.27)

Thus, for any \(0 < \delta < \frac{N-\beta}{N+1-2s}\), for all \((x,y) \in A_5\),

\[
|\phi(x) - \phi(y)| = |\phi(x) - \phi(y)|^\delta |\phi(x) - \phi(y)|^{1-\delta} \leq \frac{C|x-y|^\delta}{|x|^{(N-2s)(1-\delta)}}.
\]

Hence, for any \( x \in A_{\frac{1}{R},R} \)

\[
R^\sigma \int_{A_{\frac{1}{R},R}} f_R(x,y)dy \leq \frac{C R^{\sigma-1}}{|x|^{(N-2s)(1-\delta)}} \int_{A_{\frac{1}{R},R}} \frac{1}{|x-y|^{N+2s-1-\sigma}}dy.
\] (4.28)

By the change of variables \(\tilde{y} := x - y\), since \(s \in (0, \frac{1}{2})\), from (4.28) it follows that, for all \( x \in A_{\frac{1}{R},R} \)

\[
R^\sigma \int_{A_{\frac{1}{R},R}} f_R(x,y)dy \leq \frac{C R^{\sigma-1}}{|x|^{(N-2s)(1-\delta)}} \int_{\frac{1}{R} \leq |x-y| \leq 2R} \frac{1}{|\tilde{y}|^{N+2s-1-\sigma}}|\tilde{y}| \leq \frac{C R^{-2s+\sigma+\delta}}{|x|^{(N-2s)(1-\delta)}} \leq \frac{C}{1 + |x|^{\beta-\sigma}}.
\] (4.29)

Now, let \(s \in (\frac{1}{2}, 1)\). By (2.11), we get

\[
|\phi(x) - \phi(y)| \leq \frac{C}{1 + |x|^{N-2s}}|x - y|,
\] (4.30)
for some \( z \) in the segment joining \( x \) and \( y \). For any \( R > 0 \) let

\[
Q_R \equiv Q := \left\{ (x, y) \in A_5 : |x - y| \leq \frac{R}{100} \right\}.
\]

Note that, if \((x, y) \in Q\) then every point \( z \) lying on the segment from \( x \) to \( y \) verifies \( |z| \geq C|x|\). Hence, since \( s \in (\frac{1}{2}, 1) \), (4.30) and (4.27) yield, for all \( x \in A_{\frac{R}{2}R} \),

\[
R^s \int_{\|y\| < 2R, |x-y| < \frac{R}{100}} f_R(x,y) dy \leq \int_{\|y\| < 2R, |x-y| < \frac{R}{100}} \frac{CR_1^{s-1}}{|x|^{N-2s}} |x-y|^{N+2s-2} dy \leq \frac{CR_1^{s-1}}{|x|^{N-2s}} \int_{B_{\frac{R}{2}R}} \frac{dy}{|y|^{N+2s-2}} \leq \frac{C}{1 + |x|^{N-\sigma}}.
\]

(4.31)

On the other hand, if \((x, y) \in A_5 \setminus Q\) we have that

\[
|x - y| > \frac{R}{100} \geq C|y|.
\]

(4.32)

Then, by (4.26) and (4.32), for all \( x \in A_{\frac{R}{2}R} \),

\[
R^s \int_{\|y\| < 2R, |x-y| > \frac{R}{100}} f_R(x,y) dy \leq \frac{C}{1 + |x|^{N-\sigma}}.
\]

(4.33)

Therefore, from (4.29), (4.31) and (4.33), we have for all \( x \in A_{\frac{R}{2}R} \), for each \( s \in (0, 1) \),

\[
R^s \int_{A_{\frac{R}{2}R}} f_R(x,y) dy \leq \frac{C}{1 + |x|^{N-\sigma}}.
\]

(4.34)

From (4.13), (4.14), (4.19), (4.22), (4.24), (4.25) and (4.34) we obtain, for all \( x \in \mathbb{R}^N \),

\[
R^s \int_{\mathbb{R}^N} f_R(x,y) dy \leq \frac{C}{1 + |x|^{N-\sigma}}.
\]

(4.35)

Since \( N - 2s + 2\alpha > 0 \), we can choose \( 0 < \beta < N - 2s + 2\alpha \), and then

\[
0 < \sigma < \beta - N + 2s - 2\alpha.
\]

(4.36)

Thus

\[
\beta - \sigma > 2\alpha + N - 2s.
\]

(4.37)

Note that by \((H_0) - (ii)\) and (2.11),

\[
\rho(x)\phi(x) \geq \frac{C}{1 + |x|^{2\alpha + N - 2s}} \quad \text{for all } x \in \mathbb{R}^N.
\]

(4.38)

In view of (4.36), (4.37) and (4.38) we can infer that

\[
R^s |B(\phi, \gamma_R)(x)| \leq C \rho(x)\phi(x) \quad \text{for all } x \in \mathbb{R}^N,
\]

(4.39)

with \( \sigma \) as in (4.36).

Observe that, for all \( x \in \mathbb{R}^N \),

\[
\left| (-\Delta)^s \gamma_R(x) \right| = R^{-2s} \left| (-\Delta)^s \gamma \left( \frac{x}{R} \right) \right| \leq CR^{-2s}.
\]

(4.40)
Take \( \sigma > 0 \) satisfying (4.36). For any \( R > 1, x \in B_R \) we get
\[
R^\sigma |(-\Delta)^s \gamma_R(x)| \leq \frac{CR^\sigma}{R^{2s}} \leq \frac{C}{1 + R^{2s - \sigma}} \leq \frac{C}{1 + |x|^{2s - \sigma}}. \tag{4.41}
\]
Note that for all \( x \in \mathbb{R}^N \)
\[
|(-\Delta)^s \gamma_1(x)| \leq \frac{C}{1 + |x|^{N+2s}},
\]
so
\[
|(-\Delta)^s \gamma_R(x)| \leq \frac{CR^{-2s}}{1 + \left(\frac{|x|}{R}\right)^{N+2s}}. \tag{4.42}
\]
This implies that for any \( x \in B_R^c \) we have
\[
R^\sigma |(-\Delta)^s \gamma_R(x)| \leq \frac{CR^{N+2s}}{R^{2s}(R^{N+2s} + |x|^{N+2s})} \leq \frac{C}{1 + |x|^{2s - \sigma}}. \tag{4.43}
\]
Choose \( \sigma > 0 \) so that
\[
0 < \sigma < 2s - 2\alpha. \tag{4.44}
\]
Thus,
\[
2\alpha < 2s - \sigma. \tag{4.45}
\]
In view of (4.41), (4.43) and (4.45) we obtain
\[
R^\sigma |(-\Delta)^s \gamma_R(x)| \leq \frac{C}{1 + |x|^{2s}} \text{ for all } x \in \mathbb{R}^N. \tag{4.46}
\]
By (4.40) and (H0) \(- (ii)\),
\[
R^\sigma \phi(x)|(-\Delta)^s \gamma_R(x)| \leq C\phi(x)\rho(x) \text{ for all } x \in \mathbb{R}^N. \tag{4.47}
\]
We can select \( \sigma > 0 \) such that both (4.36) and (4.44) hold true. From (4.39) and (4.47) we get (4.12). This completes the proof. \( \square \)

Let us prove Theorem 2.11.

Proof of Theorem 2.11 Let \( u \) be any solution to equation (1.13). We can repeat the proof of Proposition 2.7 in order to obtain (4.6). Then we choose \( \zeta = \phi \). This combined with (2.10) yields
\[
\int_{\mathbb{R}^N} |u(x)|^p \gamma_R(x) F(x) dx + p \int_{\mathbb{R}^N} |u(x)|^p \phi(x) \gamma_R(x) c(x) \rho(x) dx
\leq - \int_{\mathbb{R}^N} |u(x)|^p \left\{ \phi(x)(-\Delta)^s [\gamma_R(x)] - B(\phi, \gamma_R)(x) \right\} dx =: \mathcal{I}(R). \tag{4.48}
\]
Since \( \gamma_R \geq 0, \phi > 0, |u|^p \geq 0, \rho > 0 \), due to (4.48) we obtain
\[
\int_{\mathbb{R}^N} |u(x)|^p \gamma_R(x) F(x) dx dt \leq \mathcal{I}(R) \text{ for all } R > 1. \tag{4.49}
\]
We claim that
\[
\liminf_{R \to \infty} \mathcal{I}(R) = 0. \tag{4.50}
\]
In fact, suppose by contradiction that \( \kappa := \liminf_{R \to \infty} \mathcal{I}(R) > 0 \). So, there exists \( R_0 > 0 \) such that
\[
\mathcal{I}(R) \geq \frac{\kappa}{2} \text{ for all } R > R_0. \tag{4.51}
\]
From (4.12), (2.8) and (4.51) we have, for all $R > R_0$,

$$p \int_{\mathbb{R}^N} |u(x)|^p c(x) \rho(x) \phi(x) dx$$

$$\geq \frac{c_0}{C} R^\sigma \int_{\mathbb{R}^N} |u(x)|^p \left[ |\phi(x)(-\Delta)^{\gamma_0} R(x)| + |B(\phi, \gamma_0)(x)| \right] dx$$

$$\geq \frac{c_0}{C} R^\sigma I(R) \geq \frac{c_0 \kappa}{2C} R^\sigma . \quad (4.52)$$

Sending $R \to \infty$ in (4.52), we deduce that $\int_{\mathbb{R}^N} c(x) \rho(x) |u(x)|^p \phi(x) dx = \infty$. This is in contrast with the hypothesis $u \in L^p_{c \rho \phi}(\mathbb{R}^N)$. Thus, $\kappa = 0$, and the Claim is proved.

From (4.49) and (4.50), by Fatou’s Lemma,

$$p \int_{\mathbb{R}^N} |u(x)|^p c(x) \rho(x) F(x) dx \leq 0 .$$

Since $|u|^p \geq 0, c > 0, \rho > 0,$ and $F \geq 0$ were arbitrary we can infer that $u \equiv 0$ in $\mathbb{R}^N$. This completes the proof. □

References

[1] H. Abels, M. Kassmann, The Cauchy problem and the martingale problem for integro-differential operators with non-smooth kernels, Osaka J. Math., 46 (2009), 661–683.

[2] D.G. Aronson, P. Besala, Uniqueness of solutions to the Cauchy problem for parabolic equations, J. Math. Anal. Appl. 13 (1966), 516–526.

[3] B. Barrios, I. Peral, F. Soria, E. Valdinoci, A Widder’s type theorem for the heat equation with nonlocal diffusion, preprint (2013), available on-line at http://arxiv.org/abs/1302.1786.

[4] R.M. Blumenthal, R.K. Getoor, Some theorems on stable processes, Trans. Amer. Math. Soc. 95 (1960), 260–273.

[5] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), 521–573.

[6] S. D. Eidelman, S. Kamin and F. Porper, Uniqueness of solutions of the Cauchy problem for parabolic equations degenerating at infinity, Asympt. Anal., 22 (2000), 349–358.

[7] F. Ferrari, B. Franchi, I.E. Verbitsky, Hessian inequalities and the fractional Laplacian, J. Reine Angew. Math. (to appear).

[8] F. Ferrari, I. E. Verbitsky, Radial fractional Laplacian operators and hessian inequalities, preprint (2012), available on-line at http://arxiv.org/abs/1203.3149.

[9] A. M. Il’in, A. S. Kalashnikov, O. A. Oleinik, Linear equations of the second order of parabolic type, Russian Math. Surveys 17 (1962), 1–144.

[10] S. Kamin, M.A. Pozio, A. Tesei, Admissible conditions for parabolic equations degenerating at infinity, St. Petersburg Math. J. 19 (2008), 239–251.

[11] R. Mikulevicius, H. Pragarauskas, On the Cauchy problem for integro-differential operators in Holder classes and the uniqueness of the martingale problem, Potential Anal. (to appear) doi: 10.1007/s11118-013-9359-4.

[12] R. Mikulevicius, H. Pragarauskas, On the Cauchy problem for integro-differential operators in Sobolev classes and the martingale problem, arXiv:1112.4467 (2011).

[13] O. A. Oleinik and E. V. Radkevic, “Second Order Equations with Nonnegative Characteristic Form”, Amer. Math. Soc., Plenum Press, New York - London, 1973.
[14] F.W.J. Olver, D. W. Lozier, R.F. Boisvert, C.W. Clark (eds.), "NIST Handbook of Mathematical functions", Cambridge University Press, New York, NY, 2010, available on-line at [http://dlmf.nist.gov].

[15] F. Punzo, Uniqueness of solutions to degenerate parabolic and elliptic equations in weighted Lebesgue spaces, Math. Nachr. 286 (2013), 1043–1054.

[16] F. Punzo, G. Terrone, On the Cauchy problem for a general fractional porous medium equation with variable density, Nonlin. Anal. 8 (2014), 27–47.

[17] L. Silvestre, “Regularity of the obstacle problem for a fractional power of the Laplace operator” (PhD Thesis). The University of Texas at Austin (2005).

[18] A. N. Tihonov, Théorèmes d’unicité pour l’équation de la chaleur, Mat. Sb. 42 (1935), 199–215.