Weak Coupling Casimir Energies for Finite Plate Configurations

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Abstract. We derive and use an extremely simplified formula for the interaction Casimir energy for two separate bodies in the weak coupling regime for massless scalar fields. We derive closed form solutions for a general arrangement of two δ-function plates finite in one direction and infinite in another. We examine the situation of two parallel plates finite in both transverse directions.

1. Introduction

Recently, Emig et al. [1, 2] rederived the multiple scattering formalism and used it to calculate new results for Casimir energies between disjoint bodies. Similar techniques have been employed for many years, perhaps starting with Renne in 1971 [3], who rederived the Lifshitz formula [4]. More recently many papers have used multiple scattering techniques to examine the correction to the proximity force approximation for different situations [5, 6, 7, 8, 9, 10].

Casimir calculations have conventionally been carried out assuming that parallel plates had infinite extent. Gies and Klingmüller have found corrections to the infinite-size approximation for the cases of perfectly conducting boundary conditions [11]. We will here use the multiple scattering formalism to examine δ-function plate geometries with finite lengths for scalar fields subject to very weakly coupled boundaries.

The weak coupling regime is interesting to study because the formula for the Casimir energy simplifies greatly, and in many cases is amenable to closed form solutions. This introduction gives a very quick derivation of the weak-coupling form of the Casimir energy for massless scalar fields. For a more complete derivation see Kenneth and Klich [12, 13], or Milton and Wagner [10].

The vacuum expectation value for the action of a scalar field can be given by the standard formula

$$W = -\frac{i}{2} \text{Tr} \ln \frac{\mathcal{G}}{\mathcal{G}_0},$$

where $\mathcal{G}$ is the Green’s function that satisfies the wave equation of the scalar field, including any interactions with any objects or potentials. $\mathcal{G}_0$ is the free Green’s function that satisfies the wave equation with the same boundary conditions at infinity as the full Green’s function but without any interaction with background potentials. For a time-independent system we can use
the condition that \( W = - \int dt E \) to identify the energy as

\[
E = \frac{i}{2} \int \frac{d\omega}{2\pi} \text{Tr} \ln \frac{G}{G_0},
\]

where \( G \) is the Fourier-transformed Green’s function given by

\[
G(x, x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G(x, x'),
\]

and the trace is over spatial coordinates. Given two nonoverlapping potentials such that \( V(x) = V_1(x) + V_2(x) \), we can define an interaction energy as the energy of the full system less the energy of each potential acting alone, \( E_{\text{int}} = E(V_1 + V_2) - E(V_1) - E(V_2) \). This simplified expression can be written as

\[
E_{\text{int}} = -\frac{i}{2} \int \frac{d\omega}{2\pi} \text{Tr} \ln (1 - G_0 T_1 G_0 T_2),
\]

where \( T_i \) is the scattering matrix for the \( i \)th potential defined as

\[
T_i = V_i (1 - G_0 V_i)^{-1}.
\]

By formally expanding out the logarithm in (4), we can think about the formula as describing successively more scattering events between the two objects. The weak coupling expansion keeps only the first term of the expansion of the logarithm, essentially describing only a single scattering between the objects. Additionally, in the weak-coupling regime the scattering matrix can be approximated simply by the potential, \( T \approx V \). This results in a very simplified weak-coupling single-scattering approximation to the energy \( (\omega \to i\zeta) \),

\[
E_{\text{int}} = -\frac{1}{4\pi} \int d\zeta \text{Tr} G_0 V_1 G_0 V_2.
\]

2. 2+1 Spatial Dimensions

If the potentials are independent of the \( z \) direction then we can further simplify the interaction energy. By dividing out the infinite length in the \( z \) direction we obtain an energy per unit length, which we will represent in this paper by the fraktur symbol \( \mathfrak{E} \). By Fourier transforming in the \( z \) direction we get

\[
\mathfrak{E} = -\frac{1}{2} \int \frac{d\zeta}{2\pi} \int \frac{dk_z}{2\pi} \text{Tr} g_0 V_1 g_0 V_2,
\]

where \( g_0 \) is given by

\[
G_0(r - r'; \zeta) = \int \frac{dk_z}{2\pi} e^{ik_z(z-z')} g_0(r_\perp - r'_\perp ; \kappa),
\]

and \( \kappa \) is defined by \( \kappa^2 = \zeta^2 + k_z^2 \). The two dimensional Green’s function is explicitly

\[
g_0(r_\perp - r'_\perp ; \kappa) = \frac{1}{2\pi} K_0(\kappa|r_\perp - r'_\perp|),
\]

yielding a weak-coupling form for the energy per unit length of

\[
\mathfrak{E} = -\frac{1}{32\pi^3} \int d\zeta dk_z \int d^2r \int d^2r' K_0^2(\kappa|r_\perp - r'_\perp|) V_1(r_\perp; \zeta) V_2(r'_\perp; \zeta).
\]

In the case that the potentials are independent of the imaginary frequency \( \zeta \) this simplifies even further to

\[
\mathfrak{E} = -\frac{1}{32\pi^3} \int d^2r \int d^2r' V_1(r_\perp) V_2(r'_\perp) \frac{1}{|r_\perp - r'_\perp|^2}.
\]
To demonstrate the simplicity of this formula we will rederive the Casimir energy for two cylinders \[10\]. Assume that the two cylinders have radii \(a\) and \(b\), and their centers are separated by a distance \(R\) as shown in figure 1. This situation can be represented by potentials \(V_1 = \lambda_1 \delta(r - a)\) and \(V_2 = \lambda_2 \delta(r' - b)\) where \(r\) and \(r'\) are radial coordinates in cylindrical polar coordinate systems centered on the respective cylinders. Using \[10\], these potentials yield an energy per unit length

\[
E = -\frac{\lambda_1 \lambda_2 ab}{32\pi^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \frac{1}{R^2 + a^2 + b^2 - 2aR \cos \theta + 2bR \cos \theta' - 2ab \cos(\theta - \theta')}.
\]  

With a simple change in angular coordinates to \(u = \theta - \theta'\) and \(v = \frac{\theta + \theta'}{2}\) this expression can be integrated to yield the exact closed form \[10\]

\[
E = -\frac{\lambda_1 \lambda_2 ab}{8\pi} \frac{1}{\sqrt{(R^2 - (a - b)^2)} (R^2 - (a + b)^2)}.
\]  

### 2.1. General Configuration of Plates

Two finite plates of a general configuration as shown in figure 2 can be represented by the potentials in cylindrical coordinates, with origin at the left edge of plate 1,

\[
V_1 = \lambda_1 \delta(\theta - \varphi) \frac{\Theta(L_1 - r)}{r},
\]  

\[
V_2 = \lambda_2 \delta(y + a) \Theta(x + d) \Theta(L_2 - d - x).
\]  

Here \(\Theta(x)\) is the step function,

\[
\Theta(x) = \begin{cases} 
1, & x > 0, \\
0, & x < 0.
\end{cases}
\]  

![Figure 1. Two cylinders of radii \(a\) and \(b\), their centers separated by a distance \(R\).](image1)

![Figure 2. Two finite plates of length \(L_1\) and \(L_2\) in a general configuration. In a coordinate system centered at the edge of the top plate \(\varphi\) is the relative angle between the plates, \(a\) is the perpendicular distance the lower plate is shifted down, and \(d\) is the lateral distance the edge of the lower plate is shifted to the left.](image2)
Using these potentials with (10), we get the following expression for the energy per unit length,

$$E = -\frac{\lambda_1 \lambda_2}{32 \pi^2} \int_0^{L_1} dr \int_{-d}^{L_2-d} dx \frac{1}{(x - r \cos \varphi + a + r \sin \varphi)^2}. \quad (15)$$

This integral can be done exactly, yielding a closed form for the general configuration,

$$E = -\frac{\lambda_1 \lambda_2}{32 \pi^2 \sin \varphi} \left[ \text{Ti}_2 \left( \frac{L_2 - d}{a}, \cot \varphi \right) - \text{Ti}_2 \left( \frac{L_2 - d - L_1 \cos \varphi}{a + L_1 \sin \varphi}, \cot \varphi \right) \right. \\
- \left. \text{Ti}_2 \left( \frac{-d}{a}, \cot \varphi \right) + \text{Ti}_2 \left( \frac{-d - L_1 \cos \varphi}{a + L_1 \sin \varphi}, \cot \varphi \right) \right], \quad (16)$$

where $\text{Ti}_2$ is the generalized inverse tangent integral defined by

$$\text{Ti}_2 (x, a) = \int_0^x dy \frac{\text{ArcTan} y}{y + a}. \quad (17)$$

2.2. Torque

![Figure 3. A finite plate of length $L$, above an infinite plate, The center of the finite plate is a perpendicular distance $b$ above the infinite plate, and the finite plate makes an angle $\varphi$ with respect to the infinite plate.](image)

It would be desirable to examine the rotational stability of parallel plates. This can be done simply by looking at the sign of the torque for small angular displacements, while leaving the center of mass stationary. Although one can get a general form for the torque from (10), it is much simpler to restructure the problem to isolate the torque as shown in figure 3. This is the same as the “Casimir pendulum” problem studied by Scardicchio et al., who used the optical approximation [15].

Given a situation of a tilted plate of length $L$ over an infinite plate ($L_1 \to L$, $L_2 \to \infty$, $d \to -\infty$ in figure 2), we can very easily isolate the torque that the finite plate experiences. The energy per unit length is given by

$$E = -\frac{\lambda_1 \lambda_2}{32 \pi^2 \sin \varphi} \ln \left( \frac{b + L_1/2 \sin \varphi}{b - L_1/2 \sin \varphi} \right). \quad (18)$$

The torque per unit length $\mathcal{T}$ found by taking the negative derivative of the energy with respect to the tilt angle. This gives an expression for the torque on the plates as

$$\mathcal{T} = -\frac{\lambda_1 \lambda_2 \cos \varphi}{32 \pi^2 \sin \varphi} \left( \frac{1}{\sin \varphi} \ln \left( \frac{b + L_1/2 \sin \varphi}{b - L_1/2 \sin \varphi} \right) - \frac{Lb}{b^2 - L_1^2 \sin^2 \varphi} \right). \quad (19)$$

1 The generalized inverse tangent integral is related to the dilogarithm function, and much information about it can be found in [14].
From the expression for the torque we can see some clear qualitative features. The torque has a zero value at $\phi = 0$, and a quick evaluation shows that the first derivative is positive, of value
\[
\left. \frac{\partial T}{\partial \phi} \right|_{\phi=0} = \frac{\lambda_1 \lambda_2}{192\pi^2} \frac{L^3}{b^3},
\]signifying an unstable equilibrium. For values of $L$ such that $L > 2b$, the torque diverges as $\sin \phi$ approaches $2b/L$. This is simply the result of the fact that the plates would touch in this situation. If $L < 2b$ the torque has another zero at $\phi = \frac{\pi}{2}$, with a first derivative of
\[
\left. \frac{\partial T}{\partial \phi} \right|_{\phi=\frac{\pi}{2}} = -\frac{\lambda_1 \lambda_2}{32\pi^2} \left( \frac{Lb}{b^2 - \frac{L^2}{4}} - \ln \left( \frac{b + \frac{L}{2}}{b - \frac{L}{2}} \right) \right),
\]
which is negative for all $L < 2b$, meaning a stable equilibrium. Therefore a finite flat plate suspended above another infinite plate will tend to orient itself perpendicular to the plate if left to rotate about its center of mass. Physically, the reason for this is clear: because the magnitude of the Casimir force decreases with distance, to minimize the energy for a fixed center of mass, the smaller plate wants to rotate so as to place the shortest side closest to the infinite plate. (For the situation when the thickness of the plate is finite, see [16].)

2.3. Parallel Plates

![Figure 4. Parallel plates](image)

Parallel plates are perhaps the most interesting special case. We can compare the exact expressions for energy and force to those for infinite parallel plates, getting corrections for finite size. In addition the parallel plates case, due to its simplicity, lends itself well to studying both normal and lateral forces.

Consider the same setup as in the general case shown in figure 2, simply letting $\phi$ go to zero, as shown in figure figure 4. The energy per unit length can be derived directly from (16) by using the identity
\[
\lim_{a \to \infty} aT_{12} (x, a) = \int dx \arctan x,
\]
yielding an integral form for the energy per unit length
\[
E = -\frac{\lambda_1 \lambda_2}{32\pi^3} \left[ \int_{\frac{L_2 - d}{a}}^{\frac{d + L_1}{a}} dx \arctan x \right] + \int_{\frac{L_2 - d - L_1}{a}}^{\frac{d + L_1}{a}} dx \arctan x.
\]

Although an indefinite integral for the arctangent exists, this form is perhaps more illuminating because all the physical quantities are in the limits. The forces, which are given as derivatives of the energy, are all given in terms of arctangents.

Equation (23) yields closed forms for the normal force between the plates and the lateral force experienced by the plates by taking the negative derivative with respect to $a$ or $d$, respectively.
The general form of the normal force, defined as $N_a = -\partial E/\partial a$, is

$$N_a = -\frac{\lambda_1 \lambda_2}{32 \pi^2 a^2} \left[ (L_2 - d)\text{ArcTan} \left( \frac{L_2 - d}{a} \right) - (L_2 - d - L_1)\text{ArcTan} \left( \frac{L_2 - d - L_1}{a} \right) \right.$$

$$\left. - d\text{ArcTan} \left( \frac{d}{a} \right) - (d + L_1)\text{ArcTan} \left( \frac{d + L_1}{a} \right) \right].$$

(24)

In the limiting case of the plates getting very close together we expect to recover the result for the pressure for infinite parallel plates times the area exposed. By mathematically taking $a \to 0$, we use the large argument expansion of the inverse tangent,

$$\text{ArcTan} (x) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3} \frac{1}{x^3} + \cdots, \quad \text{for } x \to \infty,$$

(25)

to recover the expected result plus corrections to that result. Because the limiting form of the arctangent depends on the sign of the argument, the single general equation can give several different answers depending on the size and position of the plates. For the situation shown in figure 4 the limiting form is

$$N_a = -\frac{\lambda_1 \lambda_2}{32 \pi^2 a^2} \left( (L_2 - d) + O(a^3) \right),$$

(26)

and the first correction is zero. However, if the plates are the same size and aligned the limiting form of the force with the first correction is

$$N_a = -\frac{\lambda_1 \lambda_2}{32 \pi^2 a^2} \left( L - \frac{1}{\pi} 2a + O(a^3) \right).$$

(27)

If we let one end of both plates extend off into infinity then we can get the edge correction for two aligned plates. This correction is

$$\frac{N_a/N_0 - 1}{2a} = \frac{1}{\pi}.$$

(28)

The general form of the lateral force, similarly defined as $N_d = -\partial E/\partial d$, is

$$N_d = -\frac{\lambda_1 \lambda_2}{32 \pi^3 a^3} \left[ \text{ArcTan} \left( \frac{L_2 - d - L_1}{a} \right) - \text{ArcTan} \left( \frac{L_2 - d}{a} \right) \right.$$

$$\left. - \text{ArcTan} \left( \frac{d}{a} \right) + \text{ArcTan} \left( \frac{d + L_1}{a} \right) \right].$$

(29)

From the exact expression for the lateral force, we find there is only one equilibrium position, occurring at $d = \frac{L_1 - L_2}{2}$, where the derivative of the force is negative:

$$\frac{\partial N_d}{\partial d} \bigg|_{d=\frac{L_1 - L_2}{2}} = -\frac{\lambda_1 \lambda_2}{16 \pi^3} \frac{L_1 L_2}{\left( a^2 + \left( \frac{L_1 + L_2}{2} \right)^2 \right)^2}.$$

(30)

signifying a stable equilibrium. The position and qualitative behavior is as expected, the plate have an stable equilibrium when they are symmetrically aligned.
We are also interested in how the lateral force behaves if the plates are very close together.
To study that we simply take the limit as \(a \to 0\). Assuming without loss of generality that \(L_2 > L_1\), to lowest order the force is

\[
\mathcal{F}_d = \begin{cases} \frac{\lambda_1 \lambda_2}{16 \pi^2 a} & \text{for } d > 0 \text{ and } d > L_2 - L_1 \\ 0 & \text{for } d > 0 \text{ and } 0 < d < L_2 - L_1 \\ -\frac{\lambda_1 \lambda_2}{16 \pi^2 a} & \text{for } d < 0 \end{cases}
\]  

(31)

This is what we would expect if we approximated the energy simply as the energy per area between the two infinite plates times the area exposed between the two plates, and took the derivative of this very simple approximation as the force.

3. Three spatial dimensions
Until now we have worked in 2+1 dimensions, meaning that the potentials had infinite length in one direction. So the finite plates considered in the previous section were more like two infinite ribbons. In this section we will work with plates of finite area.

We start by working out the \(TGTG\) formula (5) in three dimensions. In three dimensions the form of the Green’s function is even easier to work with than in two dimensions,

\[
G_0(r, r') = \frac{1}{4 \pi} \frac{e^{-|\zeta||r-r'|}}{|r - r'|}.
\]  

(32)

This gives a form of the energy

\[
E = -\frac{1}{64 \pi^3} \int_{-\infty}^{\infty} d\zeta \int d^3 r \int d^3 r' \frac{e^{-2|\zeta||r-r'|} V_1(r; \zeta)V_2(r'; \zeta)}{|r - r'|^2},
\]  

(33)

and if the potentials are independent of \(\zeta\) then the expression simplifies further to

\[
E = -\frac{1}{64 \pi^3} \int d^3 r \int d^3 r' V_1(r)V_2(r') \frac{1}{|r - r'|^3}.
\]  

(34)

If we restrict ourselves to parallel plate \(\delta\)-function potentials of any general shape the energy can written in an even simpler form,

\[
E = -\frac{\lambda_1 \lambda_2}{64 \pi^3} \int_{A_1} d^2 r_\perp \int_{A_2} d^2 r'_\perp \left[a^2 + (r_\perp - r'_\perp)^2\right]^{-3/2},
\]  

(35)

where \(A_1\) and \(A_2\) are the areas of the two plates, and \(a\), again, is the separation between the plates. If we let one of the areas, let it be \(A_2\), tend to infinity, then the energy for a single finite plate above an infinite plate is

\[
E = -\frac{\lambda_1 \lambda_2}{32 \pi^2 a} A_1,
\]  

(36)

exactly the energy per area from the Lifshitz formula times the area of the finite plate. For weak coupling plates, if one of the plates is infinite, and the other finite, then there is no correction to the Lifshitz formula. This is not unexpected, and is a result of the fact that the weak coupling approximation is the same as pairwise summation.
3.1. Rectangular Parallel Plates

For two rectangular parallel plates, as shown in figure 5, the interaction energy is given by the integral

$$E = -\frac{\lambda_1 \lambda_2}{64\pi^3} \int_{d_x}^{L_{1x} + d_x} \int_0^{L_{2x}} dx \int_{d_y}^{L_{1y} + d_y} \int_0^{L_{2y}} dy \int_{d_y}^{L_{1y} + d_y} dy' \left[ a^2 + (x - x')^2 + (y - y')^2 \right]^{-3/2}. \quad (37)$$

This expression can be partially integrated and rewritten as

$$E = -\frac{\lambda_1 \lambda_2 a}{64\pi^3} \left[ \int_{L_{1x} + d_x}^{L_{1x} + d_x + d_x} \int_{L_{1y} + d_y}^{L_{1y} + d_y + d_y} dx \int_{d_y}^{L_{1y} + d_y} dy \int_{d_y}^{L_{1y} + d_y} dy' \arctan \left( \frac{xy}{\sqrt{1 + x^2 + y^2}} \right) \right]. \quad (38)$$

The two-dimensional indefinite integral in the equation is given by

$$\int dx \int dy \arctan \left( \frac{xy}{\sqrt{1 + x^2 + y^2}} \right) = xy \arctan \left( \frac{xy}{\sqrt{1 + x^2 + y^2}} \right) + x \ln \left( x + \sqrt{1 + x^2 + y^2} \right) + y \ln \left( y + \sqrt{1 + x^2 + y^2} \right) - \frac{1}{2} x \ln(1 + y^2) - \frac{1}{2} y \ln(1 + x^2) - \sqrt{1 + x^2 + y^2}. \quad (39)$$

The final closed-form expression for the energy of the two rectangular parallel plates is somewhat messy, consisting of the above indefinite integral evaluated at 16 different combinations of variables.

The normal and lateral forces can again be given by the derivatives of the energy with respect to the separation $a$ or to the displacement (this time either $d_x$ or $d_y$).

The lateral force from the plates has a stable equilibrium when the centers of the two plates are aligned. However, first derivatives of the force can be different for displacements from the equilibrium position in the $x$ and $y$ directions depending on the geometry.

Perhaps the most interesting property of this system as in section 2.3 to examine what happens to the attractive force between the plates as the plates get very close together. For very small separations we should get an expression for the force as a power series in $a$ where the first term is the pressure given by the Lifshitz formula times the area between the plates,

$$F_a = -\frac{\lambda_1 \lambda_2}{32\pi^2 a^2} A (1 + c_1 a + c_2 a^2 \cdots). \quad (40)$$
Figure 6. Two coaxial disks, separated by a distance $a$. The radii for the two disks are $R_1$ and $R_2$.

Using the large argument expansion for the arctangent, it is possible to get such an expression for the two plate arrangement, although the expressions for the area and the correction terms depend on the layout of the plates. For a situation in which the upper plate is completely above the lower plate, with none of the edges aligned, the area is given as $A = L_xL_y$ and the first correction term is $c_1 = 0$. For a situation where both plates are the same size, and they are exactly aligned ($d_x = d_y = 0$) then the area is $A = L_xL_y = L_1L_{1y} = L_{2x}L_{2y}$ and

$$c_1 = -\frac{1}{\pi} \frac{2(L_x + L_y)}{L_xL_y}.$$  

3.2. Parallel Disks

Instead of asking how two parallel rectangular plates attract, we could just have easily asked how two disks attract. The interaction Casimir energy for two coaxial disks separated by a distance $a$ is given by

$$E = -\frac{\lambda_1\lambda_2}{32\pi^2} \int_0^{R_1} dr \int_0^{R_2} dr' \int_0^{2\pi} d\theta \frac{rr'}{[a^2 + r^2 + r'^2 - 2rr'\cos\theta]^{3/2}}.$$  

This expression can be integrated term by term in a series expansion in powers of $r$ and $r'$. The energy can then be expressed as

$$E = -\frac{\lambda_1\lambda_2}{32\pi} \frac{R_1R_2^2}{a^4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n} \left(\frac{R_1}{a}\right)^{2(m-n)} \left(\frac{R_2}{a}\right)^{2n},$$

where

$$A_{m,n} = \frac{1}{2} \left(\frac{-1}{4}\right)^m \binom{2(m+1)}{m+1} \left[\binom{m}{n}^2 - \binom{m}{n+1}\binom{m}{n-1}\right].$$

The power series is convergent, so we can simply take the derivative of each term to get the force between the plates. By using the asymptotics of the power series in the limit as $a \to 0$ we recover the expected result and get corrections to the infinite plate result. For two different sized disks, if $R_1 < R_2$ then $A = \pi R_2^2$ and $c_1 = 0$. For two equal sized disks where $R_1 = R_2 = R$, $A = \pi R^2$ and

$$c_1 = -\frac{1}{\pi} \frac{2\pi R}{\pi R^2}.$$  

4. Conclusions

The weak-coupling regime greatly simplifies Casimir calculations, and even more so the interaction energy between two bodies. The simplicity allows us to obtain closed-form solutions.
to the energy and force between some nontrivial geometries. These closed-form solutions can help us in understanding the mechanics of these systems, such as the torque on two finite plates studied here. Also these solutions can help us in understanding the limits of certain approximations, such as the proximity force approximation or the correction to the attractive force between plates of finite size.

For the three cases of parallel plates studied here, the 2+1 D parallel plates, the rectangular parallel plates, and the parallel co-axial disks, we get very similar results in the limit as the plates get very close together. If the edges of the plates do not align then the first correction term $c_1 = 0$ and the area is the area of overlap of the plates. This fact can be reconciled with the fact that for a finite plate over an infinite plate the exact result for the attractive force is simply the pressure from the Lifshitz formula times the area of the plate. We would expect this to be a good approximation if one plate were much larger than the other, which corresponds to the first correction being zero. If one plate is even slightly larger, in the limit as the separation goes to zero the difference in the size of the plates is still large in comparison to the separation.

In the case of the edges of the two plates aligning exactly (that is, plates of the same size and shape directly above one another) then the area is simply the area of the plates and the first correction term takes the form

$$c_1 = -\frac{1}{\pi} \frac{\text{Perimeter}}{\text{Area}}.$$  \hfill (46)

This is a general property of this system, and can be visualized by realizing that in the limit as $a$ goes to zero locally along any edge the system will appear to be two semi-infinite plates with their edges aligned. Therefore we might expect the correction to be a proportional to the perimeter of the plates, with the constant of proportionality given by $2\pi$.

In addition, these closed-form results will act as simple test cases for numerical studies.

Acknowledgments

We thank K.V. Shajesh and S.A. Fulling for helpful remarks. This work is supported in part by the US National Science Foundation and the US Department of Energy.

References

[1] Emig T and Jaffe R L 2008 *J. Phys. A: Math. Theor.* 41 164001 (*Preprint* 0710.5104)
[2] Emig T, Graham N, Jaffe R L and Kardar M 2008 *Phys. Rev. D* 77 025005 (*Preprint* 0710.3084)
[3] Renne M J 1971 *Physica* 56 125
[4] Lifshitz E M 1955 *JETP* 29 94 [Sov. Phys. JETP 1955 2 74]
[5] Bulgac A, Magierski P and Wirzba A 2006 *Phys. Rev. D* 73 025007 (*Preprint* hep-th/0611056)
[6] Wirzba A, Bulgac A and Magierski P 2006 *J. Phys. A: Math. Gen.* 39 6815 (*Preprint* quant-ph/0511057)
[7] Bordag M 2007 *Phys. Rev. D* 75 065003 (*Preprint* quant-ph/0611243)
[8] Bordag M 2006 *Phys. Rev. D* 73 125018 (*Preprint* hep-th/0602295)
[9] Bordag M, Geyer B, Klimchitskaya G L and Mostepanenko V M 2006 *Phys. Rev. B* 74 205431
[10] Milton K A and Wagner J 2008 *J. Phys. A: Math. Theor.* 41 155402 (*Preprint* 0712.3811)
[11] Gies H and Klingmuller K 2006 *Phys. Rev. Lett.* 97 220405 (*Preprint* quant-ph/0606235)
[12] Kenneth O and Klich I 2006 *Phys. Rev. Lett.* 97 160401 (*Preprint* quant-ph/0601011)
[13] Kenneth O and Klich I 2007 (*Preprint* 0707.4017)
[14] Lewin L 1981 *Polylogarithms and associated functions.* (New York, Oxford: North Holland)
[15] Scardicchio A and Jaffe R L 2005 *Nucl. Phys. B* 704 552 (*Preprint* quant-ph/0408041)
[16] Milton K A, Parashar P and Wagner J 2008 (*Preprint* 0806.2880)