HEAT KERNEL BOUNDS FOR A LARGE CLASS OF MARKOV PROCESS WITH SINGULAR JUMP

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Abstract. Let $Z = (Z^1, \ldots, Z^d)$ be the $d$-dimensional Lévy processes where $Z^i$’s are independent 1-dimensional Lévy processes with jump kernel $J^{\phi,1}(u, w) = |u - w|^{-\phi(|u - w|)}$ for $u, w \in \mathbb{R}$. Here $\phi$ is an increasing function with weak scaling condition of order $\alpha$, $\alpha \in (0, 2)$. Let $J(x, y) \cong J^{\phi}(x, y)$ be the symmetric measurable function where

$$J^{\phi}(x, y) = \begin{cases}
J^{\phi,1}(x^i, y^i) & \text{if } x^i \neq y^i \text{ for some } i \text{ and } x^j = y^j \text{ for all } j \neq i \\
0 & \text{if } x^i \neq y^i \text{ for more than one index } i.
\end{cases}$$

Corresponding to the jump kernel $J$, we show the existence of non-isotropic Markov processes $X := (X^1, \ldots, X^d)$ and obtain two-sided heat kernel estimates for the transition density functions, which turn out to be comparable to that of $Z$.

1. Introduction

Suppose that $X$ is a symmetric Markov process in $\mathbb{R}^d$, with transition density $p(t, x, y)$ and generator $\mathcal{L}$. It is known that $p(t, x, y)$ is the fundamental solution to $\partial_t u = \mathcal{L}u$, hence it is also called the heat kernel of $\mathcal{L}$. Since most heat kernels do not have explicit expressions, establishing sharp two-sided heat kernel estimate is a fundamental problem and it has received intensive attention in the theory of analysis as well as that of probability, and there are researches on diverse types of Markov processes. In this paper, we consider a large class of non-isotropic pure jump Markov processes and investigate the extended version of the conjecture formulated in [14] as follows:

Conjecture: Let $L_t$ be a Lévy process (a non-degenerate $\alpha$-stable process) in $\mathbb{R}^d$ with Lévy measure $\mu$. Let $M_t$ be a symmetric Markov process whose Dirichlet form has a symmetric jump intensity $j(x, dy)$ that is comparable to the one of $L_t$, i.e., $j(x, dy) \cong \mu(x - dy)$. Then the heat kernel of $M_t$ is comparable to the one of $L_t$.

We use the notation $f \asymp g$ if the quotient $f/g$ is comparable to some positive constants.

There is a long history of studies on the above conjecture. In [1], the uniformly elliptic operator in divergence form, $\mathcal{A} := \sum \partial_i(a_{i,j}(x)\partial_j)$, is related to the Laplacian $-\Delta$, that is, the fundamental solutions of $\partial_t u = \mathcal{A}u$ and $\partial_t u = -\Delta u$ are comparable. In [5] and [9], they obtain analogous results for non-local operators related to $\alpha$-stable processes and non-degenerate jump processes with the jump kernel $j(x, y) \asymp |x - y|^{-d-\alpha}$ for $\alpha \in (0, 2)$. Very recently, there are researches on the non-isotropic case by [16, 14] with the Lévy
measure $\mu(dh) := \sum_{i=1}^{d} |h^i|^{-1-\alpha} dh^i \prod_{j \neq i} \delta_{\{0\}}(dh^j)$ and the jump intensity $j(x, dy) \asymp \mu(x-dy)$. Here, $\delta_{\{0\}}$ is the Dirac measure at $\{0\}$.

In [10], Chen-Kumagai introduce a large class of non-local symmetric Markov process of variable order using an increasing function with weak scaling conditions. Motivated by their research, we consider the following non-isotropic Markov processes.

For any $0 < \underline{\alpha} \leq \overline{\alpha} < 2$, let $\phi : [0, \infty) \rightarrow [0, \infty)$ be an increasing function with the following condition: there exist positive constants $\underline{\phi} \leq 1$ and $\overline{\phi} \geq 1$ such that

$$\frac{\underline{\phi}(R)}{\phi(r)} \leq \frac{\phi(R)}{\overline{\phi}(r)} \leq \frac{\phi(R)}{\phi(r)} \leq \frac{\overline{\phi}(R)}{\phi(r)} \leq \frac{\overline{\phi}(R)}{\underline{\phi}(r)} \quad \text{for } 0 < r \leq R.$$  

(WS)

Using this $\phi$, define $\nu^1(r) := (r\phi(r))^{-1}$ for $r > 0$. Then (WS) implies

$$\int_{\mathbb{R}} (1 \wedge |s|^2) \nu^1(|s|)ds \leq c \left( \int_{0}^{1} r^{-\underline{\alpha}+1} dr + \int_{1}^{\infty} r^{-\overline{\alpha}-1} dr \right) < \infty,$$

so $\nu^1(ds) := \nu^1(|s|)ds$ is a Lévy measure. Consider a non-isotropic Lévy process $Z$ in $\mathbb{R}^d$ defined by $Z = (Z^1, \ldots, Z^d)$, where each coordinate process $Z^i$ is an independent one-dimensional symmetric Lévy process with Lévy measure $\nu^1(ds)$.

Then the corresponding Lévy measure $\nu$ of $Z$ is represented as

$$\nu(dh) = \sum_{i=1}^{d} \nu^1(|h^i|)dh^i \prod_{j \neq i} \delta_{\{0\}}(dh^j),$$

Intuitively, $\nu$ only measures the sets containing the line which is parallel to one of the coordinate axes. The corresponding Dirichlet form $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ on $L^2(\mathbb{R}^d)$ is given by

$$\mathcal{E}^\phi(u, v) = \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} \int_{\mathbb{R}} (u(x + e^i \tau) - u(x)) (v(x + e^i \tau) - v(x)) J^\phi(x, x + e^i \tau) d\tau \right) dx,$$

$$\mathcal{F}^\phi = \{u \in L^2(\mathbb{R}^d) | \mathcal{E}^\phi(u, u) < \infty\},$$

(1.1)

with the jump kernel

$$J^\phi(x, y) := \begin{cases} |x^i - y^i|^{-1} \phi(|x^i - y^i|)^{-1} & \text{if } x^i \neq y^i \text{ for some } i; \ x^j = y^j \text{ for all } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

By [10, Theorem 1.2], the transition density $p^Z(t, x^i, y^i)$ of $Z^i$ has the following estimates

$$p^Z(t, x^i, y^i) \asymp \left( |\phi^{-1}(t)|^{-1} \wedge t \nu^1(|x^i - y^i|) \right),$$

where $a \wedge b := \min\{a, b\}$. Since $Z^i$’s are independent, it is easy to obtain the upper and lower bounds for the transition density $p^Z(t, x, y)$ of $Z$, that is,

$$p^Z(t, x, y) \asymp \prod_{i=1}^{d} \left( |\phi^{-1}(t)|^{-1} \wedge t \nu^1(|x^i - y^i|) \right).$$

(1.2)

Now we consider a symmetric measurable function $J$ on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ such that for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag},$

$$\Lambda^{-1} J^\phi(x, y) \leq J(x, y) \leq \Lambda J^\phi(x, y),$$

(1.3)
for some $\Lambda > 1$, and define $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d)$ as follows:

$$
\mathcal{E}(u, v) := \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{\mathbb{R}} (u(x + e^i \tau) - u(x))(v(x + e^i \tau) - v(x)) J(x, x + e^i \tau) d\tau \right) dx,
$$

$$
\mathcal{F} := \{ u \in L^2(\mathbb{R}^d) \mid \mathcal{E}(u, u) < \infty \}. \tag{1.4}
$$

Then we state our main result of this paper.

**Theorem 1.1.** Assume the condition (1.3) holds. Then there exists a conservative Feller process $X = (X^1, \ldots, X^d)$ associated with $(\mathcal{E}, \mathcal{F})$ that starts from every point in $\mathbb{R}^d$. Moreover, $X$ has a jointly continuous transition density function $p(t, x, y)$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, which enjoys the following estimates. There exist positive constants $c_1, c_2 > 0$ such that for any $t > 0, x, y \in \mathbb{R}^d$,

$$
c_1[\phi^{-1}(t)]^{-d} \prod_{i=1}^d \left( 1 + \frac{t\phi^{-1}(t)}{|x^i - y^i|\phi(|x^i - y^i|)} \right) \leq p(t, x, y) \leq c_2[\phi^{-1}(t)]^{-d} \prod_{i=1}^d \left( 1 + \frac{t\phi^{-1}(t)}{|x^i - y^i|\phi(|x^i - y^i|)} \right), \tag{1.5}
$$

Comparing (1.5) with (1.2), we give the answer that the **Conjecture** holds for a large class of non-isotropic Markov processes. The value of our research is that we cover diverse types of Markov processes, even including processes of which each coordinate $M^i$ in $M = (M^1, \ldots, M^d)$ is not the same.

**Remark 1.2.** We give some examples of $L := (L^1, \ldots, L^d)$ and $M := (M^1, \ldots, M^d)$ in the **Conjecture** which our study covers.

1. Let $M^i := M^{\alpha_i}$ be a 1-dimensional $\alpha_i$-stable-like process having the jump kernel $j^i(h) \sim |h|^{-1-\alpha_i}, h \in \mathbb{R}$ with possibly different $\alpha_i \in (0, 2)$. Consider an increasing function $\phi(r), r > 0$ satisfying (WS) with $\overline{\alpha} := \min_{i \in \{1, \ldots, d\}} \alpha_i$ and $\underline{\alpha} := \max_{i \in \{1, \ldots, d\}} \alpha_i$. Then,

$$
\overline{\alpha}[|h|^{\alpha_i}] \leq \overline{\phi}(|h|/\phi(1)) \leq \overline{C}|h|^\overline{\alpha} \leq \overline{C}|h|^{\alpha_i} \quad \text{for } |h| \geq 1,
$$

$$
\underline{\alpha}[|h|^{-\alpha_i}] \leq \underline{\phi}(1)/\underline{\phi}(|h|) \leq \underline{C}|h|^{-\underline{\alpha}} \leq \underline{C}|h|^{-\alpha_i} \quad \text{for } |h| \leq 1,
$$

and so that $\mu^1(h) := (|h|\phi(|h|))^{-1} \sim j^i(h)$. Our result **Theorem 1.1** offers a type of two-sided heat kernel estimates for $M = (M^{\alpha_1}, \ldots, M^{\alpha_d})$.

2. Let $m_i > 0$ and $\alpha \in (0, 2)$. A 1-dimensional $m_i$- relativistic $\alpha$-stable process $L^i$ with the characteristic function $\mathbb{E}^0[\exp (i \xi \cdot L^i)] = \exp (t(m_i^\alpha - (\xi^2 + m_i^2)^{\alpha/2}))$. By [11, (3.9)–(3.10)], the corresponding Lévy kernel $\mu^i(h)$ of $M^i$ satisfies that

$$
\mu^i(h) \sim \Phi(m_i|h|)/|h|^{1+\alpha},
$$

where $\Phi(r) \sim e^{-r}(1 + r^{\alpha/2})$ near $r = \infty$ and $\Phi(r) = 1 + (\Phi)''(0)r^2/2 + o(r^4)$ near $r = 0$. Therefore we consider $d$-dimensional Markov process $M$ with the kernel $j^i(h) \sim \mu^i(h)$ where $\mu^1(h) \sim |h|^{-1-\alpha}$.

3. For $\alpha_i \in (0, 2)$, the independent sum of 1-dimesional $\alpha_i$-stable processes $L^1 := L^\alpha := L^{\alpha_1} + \ldots + L^{\alpha_d}$ and of relativistic $\alpha_i$-stable processes $L^1 := L^m := L^{m,\alpha_n} +$
Subsection 2.2, we prove the existence of a conservative Hunt process \( X \) associated to Dirichlet form \((\mathcal{E}, \mathcal{F})\) defined in (1.4), and obtain off-diagonal upper heat kernel estimates for \( p(t, x, y) \) with the help of Nash-type inequality. Also, we discuss the scaled process \( Y_t^{(\kappa)} = \kappa^{-1}X_{\kappa(t)} \), \( \kappa > 0 \) which gives a clue of the exit time estimates for \( X \) displayed in Corollary 2.10. In Section 3, we aim to construct off-diagonal upper bound estimates. However, unlike isotropic jump processes, Meyer’s decomposition method does not work here for our non-isotropic case. Therefore, we present a strategy of the proof for off-diagonal upper bound estimates for \( p(t, x, y) \) with the main technical result Proposition 3.9. The proof of Proposition 3.9 is established in Appendix. We adopt the main idea shown in [14] for the upper bound estimates in Section 3 and Appendix. However, to control the diverse orders induced by \( \phi \), we give a new refined exponent \( \{n_l : l \in \{0, 1, \ldots, d - 1\}\} \) of \((H^1_{\phi+n_l})\) (see, (3.1)), and an effective function \( \delta(\delta), \delta \in \mathbb{Z} \) in (3.7) to obtain the estimates in an accurate way. The Hölder continuity for the bounded harmonic function is shown in Section 4, and it implies our process \( X \) is Feller. Moreover, we obtain the Hölder continuity of \( p(t, x, y) \), with which we are able to refine the process \( X \) to start from everywhere. The matching lower bound heat kernel estimate for \( p(t, x, y) \) is obtained in Section 4.

Notations. For \( k \geq 0 \), let \( C^k_b(\mathbb{R}^d) \) and \( C^k(\mathbb{R}^d) \) denote the spaces of \( C^k(\mathbb{R}^d) \) functions with compact support, and with vanishing at infinity, respectively. We use \( \langle \cdot, \cdot \rangle \) for the inner product in \( \mathbb{R}^d \), \( || \cdot ||_r \) to denote the \( L^r(\mathbb{R}^d) \) norm, and \( |A| \) to denote the Lebesgue measure of \( A \subset \mathbb{R}^d \). For \( x \in \mathbb{R}^d, r > 0 \), we use \( B(x, r) \) to denote the ball centered at \( x \) with the radius \( r \), and \( Q(x, r) \) to denote the cube centered at \( x \) with the side length \( r \). The letter \( c = c(a, b, \ldots) \) will denote a positive constant depending on \( a, b, \ldots \), and it may change at each appearance. The labeling of the constants \( c_1, c_2, \ldots \) begins anew in the proof of each statement.

2. Preliminaries

In this section, we will discuss the existence of the process \( X \) and the transition density \( p(t, x, y) \), as well as the distribution of exit times for \( X \). In Subsection 2.1, we argue that \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form, hence there exists a Hunt process that starts from almost every point, according to Theorem 7.2.1 in [13]. In Subsection 2.2, we discuss the Nash’s inequality and obtain on-diagonal upper heat kernel estimates of \( p(t, x, y) \). In Subsection 2.3, we introduce scaled and truncated processes to study the distributions of exit times.

2.1. The Dirichlet form.

Definition 2.1. We say that \((\mathcal{G}, \mathcal{H})\) is a symmetric Dirichlet form on \( L^2(\mathbb{R}^d) \) if \( \mathcal{G} \) is a closed Markovian symmetric form, that is,

\( (\mathcal{G}.1) \) (\( \mathcal{G} \) is closed): if \( \mathcal{H} \) is complete with respect to \( \mathcal{G}_1 := \mathcal{G} + \| \cdot \|^2_2 \) metric.
(G.2) \( (G \text{ is Markovian}) \): for each \( \varepsilon > 0 \), there exists a function \( \eta_\varepsilon(t) \in [-\varepsilon, 1 + \varepsilon] \) for \( t \in \mathbb{R} \) such that
\[
\eta_\varepsilon(t) = t \text{ for } t \in [0, 1]; \quad 0 \leq \eta_\varepsilon(t) - \eta_\varepsilon(s) \leq t - s \text{ for } s < t;
\]
for any \( u \in \mathcal{H} \), \( \eta_\varepsilon \circ u \in \mathcal{H} \) and \( G(\eta_\varepsilon \circ u, \eta_\varepsilon \circ u) \leq G(u, u) \).

A subset \( C \) of \( \mathcal{H} \cap C_c(\mathbb{R}^d) \) is called a core of a symmetric form \( G \) if it is dense in \( \mathcal{H} \) with \( G_1 \) norm and dense in \( C_c(\mathbb{R}^d) \) with uniform norm, and \( G \) is called regular if

(G.3) \( G \) possesses a core.

**Theorem 2.2.** Let \( J \) be the symmetric measurable function defined in (1.3). Then \( (\mathcal{E}, \mathcal{F}) \) defined in (1.4) is a regular Dirichlet form on \( L^2(\mathbb{R}^d) \).

**Proof.** Recall that \( C^1_c(\mathbb{R}^d) \) is the space of \( C^1 \) functions on \( \mathbb{R}^d \) with compact support. Since \( C^1_c(\mathbb{R}^d) \subset \mathcal{F} \) and \( C^1_c(\mathbb{R}^d) \) is a dense subspace in \( L^2(\mathbb{R}^d) \), \( \mathcal{F} \) is clearly dense in \( L^2(\mathbb{R}^d) \). Also the linearity and symmetricity of \( \mathcal{E} \) are clear. We show that \( (\mathcal{E}, \mathcal{F}) \) is a regular Dirichlet form on \( L^2(\mathbb{R}^d) \) by the following 3 steps.

(1) \( \mathcal{E} \) is closed. Let \( \{u_n\}_{n \geq 1} \) be a \( \mathcal{E}_1 \)-Cauchy sequence in \( \mathcal{F} \). Since \( \mathcal{F} \) is dense in \( L^2(\mathbb{R}^d) \), there exists \( u \in L^2(\mathbb{R}^d) \) with \( \|u - u_n\|_2 \to 0 \) as \( n \to \infty \). It remains to show \( u \in \mathcal{F} \) and \( \mathcal{E}(u_n - u, u_n - u) \to 0 \) as \( n \to \infty \). For any \( n \in \mathbb{N} \), denote \( v_n = u_n - u \). By (WS), we have that for any \( \varepsilon > 0 \)
\[
\int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{|\tau| > \varepsilon} (v_n(x + e^i\tau) - v_n(x))^2 J(x, x + e^i\tau) d\tau \right) dx \\
\leq \frac{\Lambda C}{\phi(1)} \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{|\tau| \leq 1} \frac{(v_n(x + e^i\tau) - v_n(x))^2}{|\tau|^{1+\alpha}} d\tau \right) dx \\
+ \frac{\Lambda}{\phi(1)} \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{|\tau| > 1} \frac{(v_n(x + e^i\tau) - v_n(x))^2}{|\tau|^{1+\alpha}} d\tau \right) dx \\
\leq \frac{\Lambda C}{\phi(1)} \sum_{i=1}^d \int_{\varepsilon < |\tau| \leq 1} \int_{\mathbb{R}^d} \frac{(v_n(x + e^i\tau) - v_n(x))^2}{|\tau|^{1+\alpha}} dx d\tau + \frac{4d\Lambda}{\phi(1)\alpha} \|v_n\|^2_2, \tag{2.1}
\]
and the right hand side of (2.1) converges to 0 as \( n \to \infty \). Since the left hand side of (2.1) converges to 0 as \( n \to \infty \) for any arbitrary \( \varepsilon > 0 \), \( (\mathcal{E}, \mathcal{F}) \) is closed.

(2) \( \mathcal{E} \) is Markovian. For any \( \varepsilon > 0 \), let
\[
\tilde{\eta}_\varepsilon(x) = \begin{cases} 
1 + \varepsilon & \text{if } x \in (1 + \varepsilon, \infty); \\
x & \text{if } x \in [-\varepsilon, 1 + \varepsilon]; \\
-\varepsilon & \text{if } x \in (-\infty, -\varepsilon),
\end{cases} \quad \text{and} \quad \psi_\varepsilon(x) = \begin{cases} 
c \varepsilon^{-1/(\varepsilon^2-x^2)} & \text{if } x \in [-\varepsilon, \varepsilon]; \\
0 & \text{otherwise},
\end{cases}
\]
where \( c \) is a constant such that \( \int_{\mathbb{R}} \psi_\varepsilon dx = 1 \). Then \( \eta_\varepsilon = \tilde{\eta}_\varepsilon \ast \psi_\varepsilon \) has the following properties:
- \( \eta_\varepsilon(t) = t \) for all \( t \in [0, 1] \);
- \( \eta_\varepsilon(t) \in [-\varepsilon, 1 + \varepsilon] \) for all \( t \in \mathbb{R} \);
- \( 0 \leq \eta_\varepsilon(t) - \eta_\varepsilon(s) \leq t - s \) if \( s < t \).
Thus for any $u \in \mathcal{F}$,

$$\mathcal{E}(\eta_\varepsilon(u), \eta_\varepsilon(u)) = \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{\mathbb{R}} (\eta_\varepsilon(u(x + e^i \tau)) - \eta_\varepsilon(u(x)))^2 J(x, x + e^i \tau) d\tau \right) dx$$

$$\leq \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{\mathbb{R}} (u(x + e^i \tau) - u(x))^2 J(x, x + e^i \tau) d\tau \right) dx = \mathcal{E}(u, u),$$

and hence $\mathcal{E}$ is Markovian.

(3) $\mathcal{E}$ is regular. Our claim is to prove $\mathcal{F} = \overline{C^1_c(\mathbb{R}^d)}^{\mathcal{E}}$. Clearly $C^1_c(\mathbb{R}^d) \subset \mathcal{F}$, and using the similar arguments as in (1), we have $\overline{C^1_c(\mathbb{R}^d)}^{\mathcal{E}} \subset \mathcal{F}$. On the other hand, to obtain $\mathcal{F} \subset \overline{C^1_c(\mathbb{R}^d)}^{\mathcal{E}}$, since $(\mathcal{E}, \mathcal{F})$ is comparable to $(\mathcal{E}^\varnothing, \mathcal{F}^\varnothing)$ defined in (1.1) with $\mathcal{F} = \mathcal{F}^\varnothing$, it is enough to show that $\mathcal{F}^\varnothing \subset \overline{C^1_c(\mathbb{R}^d)}^{\mathcal{E}^\varnothing}$. Let $U^Z_\lambda u(x)$ be the $\lambda$-resolvent for $Z$ defined as

$$U^Z_\lambda u(x) := \mathbb{E}^x \int_0^\infty e^{-\lambda t} u(Z_t) dt = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} u(y) p^Z(t, x, y) dy dt,$$

where $p^Z(t, x, y)$ is the transition density of $Z$. Since $Z^1, Z^{(2)}, \ldots, Z^d$ are independent, the transition density $p^Z(t, x, y)$ of $Z$ has the following estimates as in (1.2): for any $t > 0, x, y \in \mathbb{R}^d$,

$$p^Z(t, x, y) \leq \prod_{i=1}^d \left( [\phi^{-1}(t)]^{-1} \wedge \frac{t}{\phi(|x_t - y_t|)|x_t - y_t|} \right).$$

Since $U^Z_\lambda(C_c(\mathbb{R}^d))$ is dense in $\mathcal{F}^\varnothing$ with respect to $\mathcal{E}^\varnothing$ metric, where $\mathcal{E}^\varnothing(C_c(\mathbb{R}^d)) := \mathcal{E}^\varnothing(u, u) + \|u\|_2^2$. It remains to show $U^Z_\lambda(C_c(\mathbb{R}^d)) \subset \overline{C^1_c(\mathbb{R}^d)}^{\mathcal{E}}$. For any $f \in U^Z_\lambda(C_c(\mathbb{R}^d))$, there exists $u \in C_c(\mathbb{R}^d)$ such that $\text{supp}[u] \subset B(0, M)$ for some $M > 0$ and $f = U^Z_\lambda u$. Then for any $t_0 > 0$,

$$|f(x)| = |U^Z_\lambda u(x)| \leq \int_{t_0}^{\infty} \mathbb{E}^x [e^{-\lambda t} |u(Z_t)|] dt$$

$$= \int_{t_0}^{\infty} e^{-\lambda t} \mathbb{E}^x [u(Z_t)] dt + \int_{t_0}^{\infty} \mathbb{E}^x [e^{-\lambda t} |u(Z_t)|] dt$$

$$\leq \frac{1 - e^{-\lambda t_0}}{\lambda} \|u\|_\infty + \int_{t_0}^{\infty} \int_{B(0, M)} e^{-\lambda t} |u(y)| p^Z(t, x, y) dy dt.$$

For any $\varepsilon > 0$, we choose $t_0$ small enough so that

$$\frac{1 - e^{-\lambda t_0}}{\lambda} \|u\|_\infty < \frac{\varepsilon}{2}.$$

Also by (1.2), there exists $M_1$ large enough such that for all $x$ with $|x| > M_1$,

$$\int_{t_0}^{\infty} \int_{B(0, M)} e^{-\lambda t} |u(y)| p^Z(t, x, y) dy dt < \frac{\varepsilon}{2}.$$

Therefore, we have $f \in L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. Let $a \vee b := \max\{a, b\}$. For any $\delta > 0$ and $\varepsilon > 0$, let

$$f^\delta := f - [(\varepsilon - \delta) \vee f] \wedge \delta$$

and

$$f^\varepsilon(x) := \int_{B(0, \varepsilon)} \psi_\varepsilon(|y|) f^\delta(x - y) dy,$$
where $\psi_\varepsilon$ is given in (2.2). By this mollification, we see that

$$f_\varepsilon^\delta \in C^1_c(\mathbb{R}^d) \quad \text{and} \quad \|f_\varepsilon^\delta\|_2 \to \|f^\delta\|_2.$$ 

Moreover, we have

$$\mathcal{E}(f_\varepsilon^\delta, f_\varepsilon^\delta) = \int_{\mathbb{R}^d} \sum_{i=1}^d \int_{B_0(\varepsilon,\varepsilon)} (f_\varepsilon^\delta(x + e^i \tau) - f_\varepsilon^\delta(x))^2 \mathcal{J}(x, x + e^i \tau) d\tau dx$$

$$\leq \int_{\mathbb{R}^d} \sum_{i=1}^d \int_{B_0(\varepsilon,\varepsilon)} (f(\varepsilon + e^i \tau - z) - f_\varepsilon^\delta(x - z))^2 \mathcal{J}(x, x + e^i \tau) \psi_\varepsilon(|z|) dz d\tau dx$$

$$= \int_{B_0(\varepsilon,\varepsilon)} \psi_\varepsilon(|z|) \int_{\mathbb{R}^d} \sum_{i=1}^d \int_{B_0(\varepsilon,\varepsilon)} (f(\varepsilon + e^i \tau) - f_\varepsilon^\delta(x))^2 \mathcal{J}(x, x + e^i \tau) d\tau dx dz = \mathcal{E}(f_\varepsilon^\delta, f_\varepsilon^\delta).$$

We now fix $\delta$. Note that for any $v \in L^2(\mathbb{R}^d)$,

$$\mathcal{E}(f_\varepsilon^\delta, U_1^Z v) = \langle f_\varepsilon^\delta, v \rangle \to \langle f^\delta, v \rangle = \mathcal{E}(f_1^\delta, U_1^Z v) \quad \text{as} \quad \varepsilon \to 0.$$ 

Since $U_1^Z(L^2(\mathbb{R}^d))$ is dense in $\mathcal{F}$ with respect to $\mathcal{E}_1^\phi$ and $f^\delta \in \mathcal{F}^\phi$ by [13, Theorem 1.4.2 (iv)], we have that $\mathcal{E}_1^\phi(f_\varepsilon^\delta, f_\varepsilon^\delta) \to \mathcal{E}_1^\phi(f^\delta, f^\delta)$ as $\varepsilon \to 0$, therefore,

$$\mathcal{E}_1^\phi(f_\varepsilon^\delta - f_\varepsilon^\delta, f_\varepsilon^\delta - f_\varepsilon^\delta) \leq 2\mathcal{E}_1^\phi(f_\varepsilon^\delta, f_\varepsilon^\delta) - 2\mathcal{E}_1^\phi(f_\varepsilon^\delta, f_\varepsilon^\delta) \xrightarrow{\varepsilon \to 0} 0.$$ 

Hence, $f_\varepsilon^\delta \to f^\delta$ with respect to $\mathcal{E}_1^\phi$ norm as $\varepsilon \to 0$. Since $f^\delta \to f$ with respect to $\mathcal{E}_1^\phi$ norm as $\delta \downarrow 0$ by [13, Theorem 1.4.2 (iv)], we conclude that $f \in C^1_c(\mathbb{R}^d)^{\Phi}. \quad \blacksquare$

**Remark 2.3.** By Theorem 2.2, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$. Therefore, there exists $\mathcal{N}_0 \subset \mathbb{R}^d$ having zero capacity with respect to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ and there exists a Hunt process $(X, \mathbb{P}_x)$ that can start from any point in $\mathbb{R}^d \setminus \mathcal{N}_0$. For more details on this, refer to [13]. Also the following Lévy system for $X$ holds:

$$\mathbb{E}^x \left[ \sum_{s \in S} f(s, X_{s-}, X_s) \right] = \mathbb{E}^x \left[ \int_0^S \left( \sum_{i=1}^d \int_{\mathbb{R}} f(s, X_s, X_s + e^i h) \mathcal{J}(X_s, X_s + e^i h) dh \right) ds \right],$$

(2.3)

where $f$ is non-negative, vanishing on the diagonal, and $S$ is a stopping time (with respect to the filtration of $X$). For the Lévy system, refer to [9] and [10] for the proof. Moreover, we will show $X$ is a strong Feller process in Remark 4.8, after obtaining Hölder continuity of resolvents.

### 2.2. Nash’s inequality and on-diagonal upper heat kernel estimates

**Proposition 2.4.** There exist constants $c_1, c_2$ such that for any $f \in \mathcal{F}$ with $\|f\|_1 = 1$, we have

$$\|f\|_2^2 \leq c_1 \mathcal{E}(f, f) \phi(c_2 \|f\|_2^{-2/d}).$$

**Proof.** For $f \in \mathcal{F} = \mathcal{F}^\phi$, denote by $f_t = P^Z_t f$ where $P^Z_t$ is the semigroup associated with $Z$. By (1.2), there exists a constant $c_1 > 0$ such that

$$\|f_t\|_\infty \leq c_1 \|f\|_1 [\phi^{-1}(t)]^{-d}.$$
Let $A^2$ be the generator of $P^2_t$ so that $f_t = f + \int_0^t A^2 f_s ds$. Then for any $t > 0$ and for $f \in \mathcal{F}$ with $\|f\|_1 = 1$, we have that

$$\|f\|_2^2 = \langle f, f_t \rangle - \int_0^t \langle f, A^2 f_s \rangle ds \leq \|f_t\|_\infty \|f\|_1 + t \mathcal{E}^\phi(f, f) \leq c_1[\phi^{-1}(t)]^{-d} + t \mathcal{E}^\phi(f, f). \quad (2.4)$$

We want to minimize the right hand side with respect to $t$. Since $\phi$ is increasing, there exists a unique $t_0 > 0$ such that $\mathcal{E}^\phi(f, f) = t_0^2[\phi^{-1}(t_0)]^{-d}$ which implies $\|f\|_2^2 \leq c_2[\phi^{-1}(t_0)]^{-d}$ and $t_0 \leq \phi(c_2^{-1/2}v|f|^2_{-2/d})$. Therefore, $\|f\|_2^2$ is bounded above at $t = t_0$ in (2.4) so that

$$\|f\|_2^2 \leq (c_1 + 1)t_0 \mathcal{E}^\phi(f, f) \leq (c_1 + 1)\phi(c_2^{-1/2}v|f|^{2/d}) \mathcal{E}^\phi(f, f).$$

Since $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ are comparable, we get our assertion.

Let $P_t$ be the transition semigroup for $X$ associated with $(\mathcal{E}, \mathcal{F})$. From the Nash-type inequality shown in Proposition 2.4, we know that for almost every $x \in \mathbb{R}^d$, $P_t$ has a kernel $p(t, x, y)$. In the following, we obtain the on-diagonal upper bound estimates for $p(t, x, y)$ almost everywhere. The exceptional set $N_0$ of next proposition was introduced in Remark 2.3.

**Proposition 2.5.** There is a properly exceptional set $N \supset N_0$ of $X$ and a positive symmetric kernel $p(t, x, y)$ defined on $(0, \infty) \times (\mathbb{R}^d \setminus N) \times (\mathbb{R}^d \setminus N)$ of $P_t$ satisfying that

$$p(t + s, x, y) = \int_{\mathbb{R}^d} p(t, x, z)p(s, z, y)dz \text{ for every } x, y \in \mathbb{R}^d \setminus N \text{ and } t, s > 0.$$ 

Also there exists a positive constant $c > 0$ such that

$$p(t, x, y) \leq c[\phi^{-1}(t)]^{-d}, \quad \text{for any } t > 0, x, y \in \mathbb{R}^d \setminus N.$$ 

Moreover, there is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ of compact sets so that $N = \mathbb{R}^d \setminus (\bigcup_{k=1}^\infty F_k)$ and that for every $t > 0$ and $y \in \mathbb{R}^d \setminus N$, $x \rightarrow p(t, x, y)$ is continuous on each $F_k$.

**Proof.** For any $f \geq 0$, consider the semigroup $P_t f(x) := \mathbb{E}^x[f(X_t)]$ for $x \in \mathbb{R} \setminus N_0$. According to [12, Proposition II.1], the Nash-type inequality in Proposition 2.4 implies

$$\|P_t f\|_\infty \leq m(t)\|f\|_1, \quad \text{for any } t > 0 \text{ and } f \in L^1(\mathbb{R}^d),$$

where $m(t)$ is the inverse function of $h(t)$ given in the following equation:

$$h(t) := \int_t^\infty c_2 \phi(c_1^{-1/d}x^{-1/d}) dx.$$ 

Then by (WS),

$$h(t) = c_3 \int_0^{c_1 t^{-1}} \frac{\phi(y^{1/d})}{y} dy \leq c_3 \sum_{k=0}^\infty \frac{\phi(2^{-k/d}c_1^{1/d}t^{-1/d})}{2^{-k+1}} \cdot (2^{-k}c_1 t^{-1} - 2^{-(k+1)}c_1 t^{-1})$$

$$\leq c_3 \phi^{-1}(c_1^{1/d}t^{-1/d}) \sum_{k=0}^\infty 2^{-k/d} \leq c_4 \phi(c_1^{1/d}t^{-1/d}).$$

1 For the definition of $\mathcal{E}$-nest, see e.g., [13, p.69].
Since $\phi$ is increasing, the inverse function $m(t) \leq c_1 [\phi^{-1}(t/c_4)]^{-d}$ and hence
\[
\|P_t f\|_\infty \leq \frac{c_1}{[\phi^{-1}(t/c_4)]^{d}} \|f\|_1, \quad \text{for any } t > 0 \text{ and } f \in L^1(\mathbb{R}^d).
\]
The rest of the proof follows from [2, Theorem 3.1] with (WS).

2.3. Scaled process $Y^{(\kappa)}$. For any $\kappa > 0$, we first introduce a $\kappa$-scaled process $Y^{(\kappa)}_t := \kappa^{-1}X_{\phi(\kappa)t}$ with the transition density
\[
q^{(\kappa)}(t, x, y) = \kappa^d p(\phi(\kappa)t, \kappa x, \kappa y).
\]
Let $Q^{(\kappa)}_t$ be a semigroup and
\[
\mathcal{E}^{(\kappa)}_t(f, f) := t^{-1} \langle f - Q^{(\kappa)}_t f, f \rangle
\]
be a bilinear form corresponding to $Y^{(\kappa)}_t$ and $q^{(\kappa)}(t, x, y)$.

**Lemma 2.6.** The Dirichlet form $(\mathcal{W}, \mathcal{D}(\mathcal{W}))$ of $Y^{(\kappa)}$ in $L^2(\mathbb{R}^d)$ is given by
\[
\mathcal{W}(u, u) := \int (u(x) - u(y))^2 J^{(\kappa)}(x, y) dxdy,
\]
\[
\mathcal{D}(\mathcal{W}) := \{ u \in C_c(\mathbb{R}^d) : \mathcal{W}(u, u) < \infty \}^{W_1},
\]
where
\[
J^{(\kappa)}(x, y) := \kappa \phi(\kappa) J(\kappa x, \kappa y)
\]
for every $x, y \in \mathbb{R}^d$,
and $\mathcal{W}_1(u, u) := \mathcal{W}(u, u) + \|u\|_2^2$. The Dirichlet form $(\mathcal{W}, \mathcal{D}(\mathcal{W}))$ is regular in $L^2(\mathbb{R}^d)$.

**Proof.** For any $t > 0$, consider a bilinear form
\[
\mathcal{E}_t(f, f) := \frac{1}{2t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 p(t, x, y) dydx.
\]
By (2.5)–(2.6), using the change of variables $w = \kappa x$ and $v = \kappa y$ with $\phi(z) = f(\kappa^{-1}z)$, we have that
\[
\mathcal{E}_t^{(\kappa)}(f, f) = \frac{1}{2t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 \kappa^d p(\phi(\kappa)t, \kappa x, \kappa y) dydx
\]
\[
= \frac{\kappa^{-d}}{2t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (g(v) - g(w))^2 p(\phi(\kappa)t, w, v) dvdw = \phi(t) \kappa^{-d} \mathcal{E}_{\phi(\kappa)t}^{(\kappa)}(g, g).
\]
By the spectral theorem, $\mathcal{E}_{\phi(\kappa)t}^{(\kappa)}(g, g) \to \mathcal{E}(g, g)$ as $t \to 0$ where
\[
\mathcal{E}(g, g) \asymp \mathcal{E}^{(\kappa)}(g, g) = \int_{\mathbb{R}^d} \sum_{i=1}^{d} \int_{\mathbb{R}} \frac{(w + e^i \xi) - g(w))}{|\xi| \phi(|\xi|)} d\xi dw
\]
\[
= \int_{\mathbb{R}^d} \sum_{i=1}^{d} \int_{\mathbb{R}} \frac{(f(\kappa^{-1}(w + e^i \xi)) - f(\kappa^{-1}w))}{|\xi| \phi(|\xi|)} d\xi dw
\]
\[
= \kappa^{1+d} \int_{\mathbb{R}^d} \sum_{i=1}^{d} \int_{\mathbb{R}} \frac{(f(x + e^i u)) - f(x)}{|\kappa u| \phi(|\kappa u|)} du dx.
\]
Therefore,
\[ E_t^{(c)} (f, f) \to W(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 J^{(c)}(x, y) dx dy, \quad \text{as } t \to 0, \]
where the jump kernel \( J^{(c)}(x, y) := \kappa \phi(\kappa) J(\kappa x, \kappa y) \). This follows that \( (W, D(W)) \) is in the class of \( (\mathcal{E}, \mathcal{F}) \) corresponding to \( Y^{(c)} \), and therefore we obtain our assertion. \( \blacksquare \)

**Remark 2.7.** By the definition of \( J^{(c)}(x, y) = \kappa \phi(\kappa) J(\kappa x, \kappa y) \),
\[
J^{(c)}(x, y) \sim \begin{cases} |x^i - y^i|^{-1} \phi^{(c)}(|x^i - y^i|)^{-1} & \text{if } x^i \neq y^i \text{ for some } i; \\ 0 & \text{otherwise}, \end{cases} \tag{2.7}
\]
where \( \phi^{(c)}(r) := \phi(\kappa r)/\phi(\kappa) \) satisfies \( (WS) \). So Proposition 2.5 with (2.5) yields
\[
q^{(c)}(t, x, y) \leq \frac{cK^d}{[\phi^{-1}((\kappa(t)))]^d} = \frac{c}{[(\phi^{(c)})^{-1}(t)]^d} \quad \text{for } t > 0, \quad x, y \in \mathbb{R}^d \setminus \kappa^{-1} \mathcal{N},
\]
where \( (\phi^{(c)})^{-1} \) is the inverse of \( \phi^{(c)} \).

We also introduce the truncated processes. Let \( \lambda > 0 \). Consider the jump kernel \( J_\lambda(x, y) := J(x, y) 1_{|x-y| \leq \lambda} \), and the bilinear form
\[
\mathcal{E}^\lambda(u, v) := \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{\mathbb{R}} (u(x + e^i h) - u(x))(v(x + e^i h) - v(x)) J_\lambda(x, x + e^i h) dh \right) dx.
\]
Then \( (WS) \) implies
\[
0 \leq \mathcal{E}(u, u) - \mathcal{E}^\lambda(u, u) = \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_{|h| \geq \lambda} (u(x + e^i h) - u(x))^2 \frac{1}{|h| \phi(|h|)} dh \right) dx \\
\leq \frac{c_1}{\phi(\lambda)} \| u \|_2^2 \int_\lambda^\infty |h|^{-1-\alpha} dh \leq \frac{c_1}{\alpha \phi(\lambda)} \| u \|_2^2, \tag{2.8}
\]
and so
\[
\mathcal{E}^\lambda_1(u, u) \leq \mathcal{E}_1(u, u) \leq (1 + c_2) \mathcal{E}^\lambda_1(u, u) \quad \text{for every } u \in \mathcal{F}. \tag{2.9}
\]
It follows that \( (\mathcal{E}^\lambda, \mathcal{F}) \) is a regular Dirichlet form on \( L^2(\mathbb{R}^d) \), and so there is a Hunt processes \( X^\lambda \) corresponding to \( (\mathcal{E}^\lambda, \mathcal{F}) \). In addition, Proposition 2.4 with (2.9) implies \( X^\lambda \) has a transition density function \( p_\lambda(t, x, y) \). By the similar proof of the near diagonal upper bounds in Proposition 2.5 and (WS), there exists an exceptional set \( \mathcal{N} \) and a constant \( c > 0 \) such that
\[
p_\lambda(t, x, y) \leq c [\phi^{-1}(t)]^{-d} \quad \text{for } t > 0, \quad x, y \in \mathbb{R}^d \setminus \mathcal{N}. \tag{2.10}
\]

Now we give the proof of the conservativeness for \( X \) in Theorem 1.1.

**Theorem 2.8.** The process \( X \) is conservative, that is, \( X \) has infinite lifetime.

**Proof.** By [15, Theorem 2.4], the corresponding process \( X^1 \) to \( (\mathcal{E}^1, \mathcal{F}^1) \) which is conservative. Since \( J_1(x, y) \leq J(x, y) \), and we denote
\[
\tilde{J}_1(x, y) := J(x, y) - J_1(x, y) \tag{2.11}
\]
then \( \int_{\mathbb{R}^d} J_1(x, y) dy \leq c \int_{|u| \geq 1} \frac{1}{|u|^{\alpha(|u|)}} du < \infty \). By the Meyer’s construction (see, [10, Section 4.1]), the processes \( X \) can be constructed from \( X^1 \). Therefore, the conservativeness of \( X \) comes from the conservativeness of \( X^1 \).

In the following, we first obtain the distribution of exit times for the \( \kappa \)-scaled process \( Y^{(\kappa)} \), and then present that of exit times for \( X \) in Corollary 2.10. For any Borel set \( A \subset \mathbb{R}^d \), we denote \( \tau_A^{(\kappa)} := \inf\{ t > 0 : Y^{(\kappa)}_t \notin A \} \) for the first exit time of \( Y^{(\kappa)} \) from \( A \) and the following proposition gives the distribution of \( \tau^{(\kappa)} \).

**Proposition 2.9.** Let \( T > 0 \). Then there exists a positive constant \( c = c(T, \phi) > 0 \) such that

\[
\mathbb{P}^x \left( \tau^{(\kappa)}_{B(x, r)} < 1 \right) \leq \frac{c}{\phi^{(\kappa)}(r)}
\]

for all \( r \in [T, \infty) \) and \( x \in \mathbb{R}^d \setminus \kappa^{-1} \mathcal{N} \).

**Proof.** First, we consider the truncated jump kernel \( J^{(\kappa)}_A(x, y) := J^{(\kappa)}(x, y)1_{\{|x-y| \leq \lambda\}} \) for \( Y^{(\kappa)} \). Using the similar argument as in (2.8) with \( \mathcal{W}^{(\kappa)}(u, u) := \int (u(x) - u(y))^2 J^{(\kappa)}_A(x, y) dx dy \), we obtain

\[
0 \leq \mathcal{W}(u, u) - \mathcal{W}^{(\kappa)}(u, u) \leq \frac{c_1}{\alpha(\phi^{(\kappa)}(\lambda)} \|u\|^2_2,
\]

so that \( \mathcal{W}^{(\kappa)}_t(u, u) \geq \mathcal{W}_t(u, u) \) for \( u \in \mathcal{D}^{(\kappa)}(\mathcal{W}) \). Therefore, \((\mathcal{W}^{(\kappa)}, \mathcal{D}(\mathcal{W}))\) is a regular Dirichlet form and there is a Hunt process \( Y^{(\kappa), \lambda} \) corresponding to \((\mathcal{W}^{(\kappa)}, \mathcal{D}(\mathcal{W}))\). In addition, by (2.12) and the same observation of Proposition 2.4 corresponding to \( \mathcal{W} \), that is the Nash’s inequality, there exists \( q^{(\kappa)}_\lambda(t, x, y) \). Also by (2.5), (2.10) and (WS) condition of \( \phi^{(\kappa)}(r) \), we have that for any \( x, y \in \mathbb{R}^d \setminus \kappa^{-1} \mathcal{N} \) and \( t \in (0, 2] \),

\[
q^{(\kappa)}_\lambda(t, x, y) \leq c_1[\phi^{(\kappa)}(\lambda)]^{-d} \leq c_2(\phi) t^{-d/\alpha}. \tag{2.13}
\]

Let

\[
\Gamma_\lambda(v)(\xi) := \int_{\{|x-y| \leq \lambda\}} (v(\xi) - v(\eta))^2 J^{(\kappa)}(\xi, \eta) d\eta, \quad \xi \in \mathbb{R}^d,
\]

\[
\Gamma(\chi)^2 := \|e^{-2\chi} \Gamma_\lambda(e\chi)\|_{\infty} \vee \|e^{-2\chi} \Gamma_\lambda(e^{-\chi})\|_{\infty}.
\]

Then (2.13) together with [2, Theorem 3.2] and [8, Theorem 3.25] implies that there exist constants \( c_3, c_4 > 0 \) such that

\[
q^{(\kappa)}_\lambda(t, x, y) \leq c_3 t^{-d/\alpha} \exp \left( -|\chi(y) - \chi(x)| \right) + c_4 \Gamma(\chi)^2 t \tag{2.14}
\]

for all \( t \in (0, 2] \) and \( x, y \in \mathbb{R}^d \setminus \kappa^{-1} \mathcal{N} \) where \( \chi(\xi) := \frac{\xi}{2} |\xi - x| \wedge |x - y| \). We will choose the constant \( s > 0 \) later. Since \( |\chi(y) - \chi(x)| \leq (s/3)|\eta - \xi| \) for any \( \xi, \eta \in \mathbb{R}^d \), by (2.7) and (WS) for \( \phi^{(\kappa)} \), we have that for \( \xi \in \mathbb{R}^d \),

\[
(e^{-2\chi} \Gamma_\lambda(e\chi)) (\xi) = c_5 \int_{|\eta - \xi| \leq \lambda} (1 - e^{\chi(\eta) - \chi(\xi)} J^{(\kappa)}(\xi, \eta) d\eta
\]

\[
\leq c_6 \sum_{i=1}^d \int_{|h| \leq \lambda} (\chi(\xi + he^i) - \chi(\xi))^2 e^{2|h(\xi + he^i) - \xi(\eta)|} J^{(\kappa)}(\xi, \xi + he^i) dh
\]

\[
\leq c_7 \left( \frac{s}{3} \right)^2 2 e^{2s3/\alpha} \int_{|h| \leq \lambda} \frac{|h|}{\phi^{(\kappa)}(|h|)} dh \leq c_8 e^{2s3/\alpha} \frac{(s\lambda)^2}{\phi^{(\kappa)}(\lambda)} \leq c_9 e^{s\lambda} \frac{1}{\phi^{(\kappa)}(\lambda)}. \]
Similarly, we obtain the upper bound of $(e^{2\lambda \Gamma (\chi^{-\lambda}))} (\xi)$. Let $s := (C_s|x-y|)^{-1} \log \left( \frac{\phi(\lambda)|x-y|}{\lambda} \right)$ and $\lambda = C_s|x-y|$ for some $C_s \leq 1$. Then

$$-|\chi(y) - \chi(x)| + c_4 \Gamma(\chi)^2 t \leq \frac{-s|x-y|}{3} + c_{10}t \frac{e^{\lambda}}{\phi(\lambda)(\lambda)}$$

$$\leq \log \left( \frac{t}{\phi(\lambda)(|x-y|)} \right)^{1/(3C_s)} + c_{11}. \quad (2.15)$$

Therefore, (2.14) and (2.15) with $C_s = \frac{2}{3(d+\alpha)}$ imply that for $x, y \in \mathbb{R}^d \setminus \kappa^{-1}N$ and $t \in (0, 2],$

$$q^{(\lambda)}(t, x, y) \leq c_{12}t^{-d/\alpha} \left( \frac{t}{\phi(\lambda)(|x-y|)} \right)^{\frac{d}{\alpha} + 1} = \frac{c_{13}t}{\phi(\lambda)(|x-y|)^{\frac{d}{\alpha} + 1}}.$$  

For the function $J^{(\lambda)}(x, y) := J^{(\lambda)}(x, y) - J^{(\lambda)}(x, y)$, there exists a constant $c_{14} > 0$ which is independent of $\lambda, \kappa$ such that

$$\|J^{(\lambda)}\|_{\infty} \leq \frac{c_{14}}{\lambda^{d/\phi(\lambda)(\lambda)}}.$$  

Let $T > 0$. According to Lemma 3.1 in [3], we have for any $t \in (0, 2]$ and $x, y \in \mathbb{R}^d \setminus \kappa^{-1}N$ with $|x-y| \geq T$,

$$q^{(\lambda)}(t, x, y) \leq q^{(\lambda)}(t, x, y) + t||J^{(\lambda)}||_{\infty}$$

$$\leq \frac{c_{13}t}{\phi(\lambda)(|x-y|)^{\frac{d}{\alpha} + 1}} + \frac{c_{14}t}{\lambda^{d/\phi(\lambda)(\lambda)}} \leq \frac{c_{15}t}{|x-y|^{d/\phi(\lambda)(|x-y|)}}.$$  

The last inequality uses the fact that $|x-y| \geq T$ and (WS) for $\phi^{(\lambda)}$. Now let $\zeta^{(\lambda)}$ be the lifetime of $Y^{(\lambda)}$. Since $Y^{(\lambda)}$ is a scaled process of $X$, by Theorem 2.8, $\mathbb{P}^{x}(\zeta^{(\lambda)} \leq r) = 0$ for any $r > 0$. So for any $x \in \mathbb{R}^d \setminus \kappa^{-1}N, t \in (0, 2]$ and $r \in [T, \infty),$

$$\mathbb{P}^{x}(|Y^{(\lambda)}_t - x| \geq r) = \int_{B(x,r)} q^{(\lambda)}(t, x, y)dy + \mathbb{P}^{x}(\zeta^{(\lambda)} \leq r)$$

$$\leq \int_{B(x,r)} \frac{c_{15}t}{|x-y|^{d/\phi(\lambda)(|x-y|)}}dy \leq \frac{c_{16}t}{\phi^{(\lambda)}(r)}.$$  

Let $\sigma^{(\lambda)} := \inf\{ s \geq 0 : |Y^{(\lambda)}_s - Y^{(\lambda)}_0| > r \}$, and in the following, we write $\sigma_r = \sigma^{(\lambda)}$ for simplicity. Then the strong Markov property implies that

$$\mathbb{P}^{x}(\sigma_r < 1, |Y^{(\lambda)}_2 - x| \leq r/2) \leq \mathbb{P}^{x}(\sigma_r < 1, |Y^{(\lambda)}_2 - Y^{(\lambda)}_\sigma| > r/2)$$

$$\leq \mathbb{E}^{x} \left[ \mathbb{1}_{\{\sigma_r < 1\}} \mathbb{P}^{Y^{(\lambda)}_{\sigma r}}(|Y^{(\lambda)}_2 - Y^{(\lambda)}_0| > r/2) \right]$$

$$\leq \sup_{y \in \{ z \in \mathbb{R}^d : |x-z| > r \}} \mathbb{P}^{y}(|Y^{(\lambda)}_2 - y| > r/2).$$  

Altogether, we have for any $x \in \mathbb{R}^d \setminus \kappa^{-1}N$ and $r \in [T, \infty)$

$$\mathbb{P}^{x}(r^{(\lambda)}_{B(x,r)} < 1) \leq \mathbb{P}^{x}(\sup_{s \leq 1} |Y^{(\lambda)}_s - Y^{(\lambda)}_0| > r) = \mathbb{P}^{x}(\sigma_r < 1)$$

$$\leq \mathbb{P}^{x}(\sigma_r < 1, |Y^{(\lambda)}_2 - x| \leq r/2) + \mathbb{P}^{x}(|Y^{(\lambda)}_2 - x| \geq r/2)$$

$$\leq \mathbb{P}^{x}(\sigma_r < 1, |Y^{(\lambda)}_2 - x| \leq r/2) + \mathbb{P}^{x}(|Y^{(\lambda)}_2 - x| \geq r/2).$$
Let \( \tau_A := \inf \{ t > 0 : X_t \notin A \} \) be the first exit time of \( X \) from \( A \). Using the relation between \( Y^{(\kappa)} \) and \( X \), we have the following distribution for \( \tau \).

**Corollary 2.10.** Let \( T > 0 \). Then there exists \( c = c(T, \phi) > 0 \) such that for any \( x \in \mathbb{R}^d \setminus \mathcal{N} \), \( r > T \) and \( t > 0 \) with \( \kappa = \phi^{-1}(t) \),

\[
\mathbb{P}^x(\tau_{B(x,r\phi^{-1}(t))} < t) \leq \frac{ct}{\phi(\phi^{-1}(t)r)}.
\]

**Proof.** For any \( \kappa > 0 \), let \( \tau^{(\kappa)} \) and \( \tau \) be the first exit times for \( Y^{(\kappa)} \) and \( X \), respectively. Then for any \( x \in \mathbb{R}^d \setminus \kappa^{-1}\mathcal{N} \) and \( r > 0 \), we have that

\[
\mathbb{P}^x(\tau^{(\kappa)}_{B(x,r)} < 1) = \mathbb{P}^x\left(\sup_{\kappa \leq 1} |Y^{(\kappa)}_s - x| > r\right) = \mathbb{P}^x\left(\sup_{u \leq \phi(\kappa)} |\kappa Y^{(\kappa)}_{u/\phi(\kappa)} - \kappa x| > r\kappa\right)
\]

\[
= \mathbb{P}^{w}\left(\sup_{u \leq \phi(\kappa)} |X_u - w| > r\kappa\right) = \mathbb{P}^{w}(\tau_{B(w,\kappa r)} < \phi(\kappa))
\)

for \( w \in \mathbb{R}^d \setminus \mathcal{N} \). Hence, with \( \kappa = \phi^{-1}(t) \) and \( r > T \), applying Proposition 2.9 and (2.16), we have

\[
\mathbb{P}^x(\tau_{B(x,r\phi^{-1}(t))} < t) = \mathbb{P}^x(\tau^{(\phi^{-1}(t))}_{B(x,r)} < 1) \leq \frac{c}{\phi(\phi^{-1}(t)r)} = \frac{ct}{\phi(\phi^{-1}(t)r)}
\]

for some \( c = c(T, \phi) \). \( \blacksquare \)

**Remark 2.11.** In the following sections, lower and upper heat kernel estimates for \( p(t, x, y) \) will be given for \( x, y \in \mathbb{R}^d \setminus \mathcal{N} \). In Theorem 4.9, we will show that \( p(t, x, y) \) is Hölder continuous in \((x, y)\). Hence, it can be extended continuously to \( \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \), which completes the proof of Theorem 1.1. In the remaining sections, for the convenience of discussions, we will directly write \( \mathbb{R}^d \) instead of \( \mathbb{R}^d \setminus \mathcal{N} \).

### 3. Upper Bound Estimates

In this section, we introduce the upper bound estimates and give the sketch of the proof with a technical result Proposition 3.9. The proof of Proposition 3.9 is similar to [14, Proposition 3.3] with some modifications, we postpone it to Appendix.

#### 3.1. Upper bounds and sketch of the proof.

**Theorem 3.1.** There is a positive constant \( c \) such that for all \( t > 0 \) and \( x, y \in \mathbb{R}^d \) the following estimate holds:

\[
p(t, x, y) \leq C[\phi^{-1}(t)]^{-d} \prod_{i=1}^d \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|\phi(|x^i - y^i|)} \right).
\]

First, we explain the sketch of the proof for Theorem 3.1. For any \( q > 0 \) and \( l \in \{1, \ldots, d-1\} \), consider the following conditions.
\( (H_q^0) \) There exists a positive constant \( C_0 = C_0(q, \Lambda, d, \phi) \) such that for all \( t > 0 \) and \( x, y \in \mathbb{R}^d \),
\[
p(t, x, y) \leq C_0[\phi^{-1}(t)]^{-d} \prod_{i=1}^{d} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|} \phi(|x^i - y^i|) \right)^q.
\]

\( (H_q^l) \) There exists a positive constant \( C_l = C_l(l, q, \Lambda, d, \phi) \) such that for all \( t > 0 \) and all \( x, y \in \mathbb{R}^d \) with \( |x^i - y^i| \leq \ldots \leq |x^d - y^d| \), the following holds true:
\[
p(t, x, y) \leq C_l[\phi^{-1}(t)]^{-d} \times \prod_{i=1}^{d-l} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|} \phi(|x^i - y^i|) \right)^q \prod_{i=d-l+1}^{d} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|} \phi(|x^i - y^i|) \right). \tag{3.1}
\]

Then we have the following Remark easily.

**Remark 3.2.**

1. The assertion of Theorem 3.1 is equivalent to \( (H_q^l) \) for any \( l \in \{0, \ldots, d-1\} \).
2. For \( q < q' \), \( (H_q^l) \) implies \( (H_{q'}^l) \).
3. For \( q \in [0, 1] \) and \( l \leq l' \), \( (H_q^{l'}) \) implies \( (H_q^l) \).

By the similar arguments as in [14, Section 5], we have the following Lemma.

**Lemma 3.3.** Assume condition \( (H_q^l) \) holds true for some \( q \in [0, 1] \). Then with the same constant \( C_l = C_l(l, q, \Lambda, d, \phi) \), for every \( t > 0 \) and \( x, y \in \mathbb{R}^d \), and any permutation \( \sigma \) satisfying \( |x^{\sigma(1)} - y^{\sigma(1)}| \leq \ldots \leq |x^{\sigma(d)} - y^{\sigma(d)}| \), the following holds true:
\[
p(t, x, y) \leq C_l[\phi^{-1}(t)]^{-d} \prod_{i=1}^{d-l} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^{\sigma(i)} - y^{\sigma(i)}|} \phi(|x^{\sigma(i)} - y^{\sigma(i)}|) \right)^q \times \prod_{i=d-l+1}^{d} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^{\sigma(i)} - y^{\sigma(i)}|} \phi(|x^{\sigma(i)} - y^{\sigma(i)}|) \right) \leq C_l[\phi^{-1}(t)]^{-d} \prod_{i=1}^{d-l} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|} \phi(|x^i - y^i|) \right)^q \prod_{i=d-l+1}^{d} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|} \phi(|x^i - y^i|) \right).
\]

**Lemma 3.3** tells us that the condition \( (H_q^l) \) with \( q \in [0, 1] \) implies condition \( (H_q^l)' \) with \( q \in [0, 1] \), which is defined as follows:

\( (H_q^l)' \) Given \( q > 0 \) and \( l \in \{1, \ldots, d-1\} \), there exists a positive constant \( C_l = C_l(l, q, \Lambda, d, \phi) \) such that for all \( t > 0 \) and \( x, y \in \mathbb{R}^d \),
\[
p(t, x, y) \leq C_l[\phi^{-1}(t)]^{-d} \prod_{i=1}^{d-l} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|} \phi(|x^i - y^i|) \right)^q \prod_{i=d-l+1}^{d} \left( 1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|} \phi(|x^i - y^i|) \right).
\]

Therefore, it is enough to consider the special points \( x, y \in \mathbb{R}^d \) such as in \( (H_q^l) \).

Next we introduce the following three lemmas.

**Lemma 3.4.** Assume condition \( (H_q^l) \) holds true for some \( l \in \{0, \ldots, d-2\} \) and \( q \in [0, \frac{1}{1+2l}] \). Then there is \( \theta_l > 0 \) such that \( (H_q^{l+\theta_l}) \) holds true.
Lemma 3.5. Assume condition $\left( H^l_q \right)$ holds true for some $l \in \{0, \ldots, d-2\}$ and $q \in \left( \frac{1}{1+2}, 1 \right)$. Then $\left( H^{l+1}_0 \right)$ holds true.

Lemma 3.6. 
(i) Assume condition $\left( H^{d-1}_q \right)$ holds true for some $q \in \left[ 0, \frac{1}{1+2} \right]$. Then there is $\theta_l > 0$ such that $\left( H^{d-1}_{q+\theta_{d-1}} \right)$ holds true.

(ii) Assume condition $\left( H^{d-1}_q \right)$ holds true for some $q \in \left( \frac{1}{1+2}, 1 \right)$. Then $\left( H^{d-1}_1 \right)$ holds true.

Assuming the above Lemmas, we finalize our assertion $\left( H^{d-1}_1 \right)$ in the following order:

$$
\begin{align*}
&\left( H^0_0 \right) \leftrightarrow \left( H^0_{\theta_0} \right) \leftrightarrow \left( H^0_{2\theta_0} \right) \ldots \leftrightarrow \left( H^0_{N_0\theta_0} \right) \\
&\leftrightarrow \left( H^1_0 \right) \leftrightarrow \left( H^1_{\theta_1} \right) \leftrightarrow \ldots \ldots \leftrightarrow \left( H^1_{N_1\theta_1} \right) \\
&\ldots \\
&\leftrightarrow \left( H^{d-2}_{\theta_{d-2}} \right) \leftrightarrow \left( H^{d-2}_{\theta_{d-1}} \right) \leftrightarrow \ldots \ldots \leftrightarrow \left( H^{d-2}_{N_{d-1}\theta_{d-1}} \right) \leftrightarrow \left( H^{d-1}_1 \right)
\end{align*}
$$

where $N_l := \inf \{ n \in \mathbb{N} : n\theta_l \geq \frac{1}{1+2} \}$ for $l \in \{0, \ldots, d-1\}$. Note that $\left( H^0_0 \right)$ is proved in Proposition 2.5. We will give the detailed proofs of Lemma 3.4, Lemma 3.5 and Lemma 3.6 in Subsection 3.2 with specific definitions of $\theta_l$ and $\theta_{d-1},$

$$
\begin{align*}
\theta_l &= \frac{\alpha}{2 + \alpha + \alpha} \left( \sum_{i=1}^{d-1} \left( \frac{\alpha + 1}{\alpha} \right)^i \right)^{-1} \text{ for } l \in \{0, 1, \ldots, d-2\}\\
\theta_{d-1} &= \frac{\alpha}{\alpha + 1}.
\end{align*}
$$

In the following, we introduce the concrete condition of two points $x_0, y_0 \in \mathbb{R}^d$ according to the time variable $\kappa := \phi^{-1}(t)$.

Definition 3.7. Let $x_0, y_0 \in \mathbb{R}^d$ satisfy $|x_0^i - y_0^i| \leq |x_0^{i+1} - y_0^{i+1}|$ for every $i \in \{1, \ldots, d-1\}$. For any $t > 0$, set $\kappa := \phi^{-1}(t)$. For $i \in \{1, \ldots, d\}$, define $n_i \in \mathbb{Z}$ and $R_i > 0$ such that

$$
\frac{5}{2} 2^{n_i} \kappa \leq |x_0^i - y_0^i| < \frac{10}{2} 2^{n_i} \kappa \quad \text{and} \quad R_i = 2^{n_i} \kappa. \quad (3.2)
$$

Then $n_i \leq n_{i+1}$ and $R_i \leq R_{i+1}$. We say that condition $\mathcal{R}(i_0)$ holds if

$$
n_1 \leq \ldots \leq n_{i_0-1} \leq 0 < 1 \leq n_{i_0} \leq \ldots \leq n_d. \quad \mathcal{R}(i_0)
$$

We say that condition $\mathcal{R}(d+1)$ holds if $n_1 \leq \ldots \leq n_d \leq 0 < 1$.

The following Lemma tells us that in specific geometric situations, Theorem 3.1 follows directly.

Lemma 3.8. Let $t > 0$ and $x_0, y_0$ be two points in $\mathbb{R}^d$ satisfying condition $\mathcal{R}(i_0)$ for $i_0 \in \{d - l + 1, \ldots, d+1\}$. Assume (3.1) holds for some $l \in \{0, \ldots, d-1\}$ and $q \geq 0$. Then

$$
p(t, x_0, y_0) \leq c[\phi^{-1}(t)]^{-d} \prod_{i=1}^{d} \left( 1 + \frac{t\phi^{-1}(t)}{|x_0^i - y_0^i|\phi(|x_0^i - y_0^i|)} \right) \quad (3.3)
$$

for some constant $c > 0$ that is independent of $t$ and $x_0, y_0$. 


Proposition 2.5, Lemma 3.5, Lemma 3.6, Proposition 3.9, Lemma 3.8, Lemma 3.4.

Using (3.3) > (3.3) with non-negative Borel functions.

Now we apply Lemma 3.10.

Proof. For $t > 0$, let $\kappa := \phi^{-1}(t)$. (i) The case $R(d + 1)$ (that is, $|x_0^d - y_0^d| < \frac{10}{4}\kappa^{-1}$) is simple since Proposition 2.5 implies $p(t, x_0, y_0) \leq c\kappa^{-d}$. Estimate (3.3) follows directly. (ii) Assume condition $R(i_0)$ holds true for some $i_0 \in \{d - l + 1, \ldots, d\}$. Then there exists $C = C_{i_0}$ independent of $d$, and $\tau = \tau_{B(x_0, 2^{n_0}c/4)}$. Hence, (3.1) implies

$$p(t, x_0, y_0) \leq C_1[\phi^{-1}(t)]^{-d} \prod_{i=1}^{d-l} \left( \frac{t\phi^{-1}(t)}{|x_0^d - y_0^d|} \phi(|x_0^d - y_0^d|) \right)^q \prod_{i=d-l+1}^{d} \left( \frac{t\phi^{-1}(t)}{|x_0^d - y_0^d|} \phi(|x_0^d - y_0^d|) \right)^1 \times \left[ \phi^{-1}(t) \right]^{-d} \prod_{i=1}^{d} \left( \frac{t\phi^{-1}(t)}{|x_0^d - y_0^d|} \phi(|x_0^d - y_0^d|) \right)^1 .$$

By Lemma 3.8, we only need consider the cases $R(i_0)$ for $i_0 \in \{1, \ldots, d - l\}$. Here is our main technical result.

**Proposition 3.9.** Assume that $(H^l_q)$ holds true for some $l \in \{0, 1, \ldots, d - 1\}$ and $q \in [0, 1)$. For any $t > 0$, set $\kappa = \phi^{-1}(t)$ and consider $x_0, y_0 \in \mathbb{R}^d$ satisfying the condition $R(i_0)$ for some $i_0 \in \{1, \ldots, d - l\}$. Let $j_0 \in \{i_0, \ldots, d - l\}$ and define an exit time $\tau = \tau_{B(x_0, 2^{n_0}c/8)}$. Let $f$ be a non-negative Borel function on $\mathbb{R}^d$ supported in $B(y_0, \frac{c}{8})$. Then there exists $C_{3.9} > 0$ independent of $x_0, y_0$ and $t$ such that for every $x \in B(x_0, \frac{c}{8})$,

$$\mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \leq C_{3.9} \|f\|_1[\phi^{-1}(t)]^{-d} \prod_{j=d-l+1}^{d} \left( \frac{t\phi^{-1}(t)}{|x_0^d - y_0^d|} \phi(|x_0^d - y_0^d|) \right)^1 \left( \frac{t\phi^{-1}(t)}{|x_0^d - y_0^d|} \phi(|x_0^d - y_0^d|) \right)^q \left( \frac{t\phi^{-1}(t)}{|x_0^d - y_0^d|} \phi(|x_0^d - y_0^d|) \right)^1 .$$

3.2. **Proofs of Lemma 3.4, Lemma 3.5 and Lemma 3.6.** In this subsection, we give the proofs of Lemma 3.4, Lemma 3.5 and Lemma 3.6 using Proposition 3.9. We first recall the following result.

**Lemma 3.10.** [3, Lemma 2.1] Let $U$ and $V$ be two disjoint non-empty open subsets of $\mathbb{R}^d$ and $f, g$ be non-negative Borel functions on $\mathbb{R}^d$. Let $\tau = \tau_U$ and $\tau' = \tau_V$ be the first exit times from $U$ and $V$, respectively. Then, for all $a, b, t \in \mathbb{R}_+$ such that $a + b = t$, we have

$$\langle Pf, g \rangle \leq \mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \|f\|_1[\phi^{-1}(t)]^{-d} \prod_{j=d-l+1}^{d} \left( \frac{t\phi^{-1}(t)}{|x_0^d - y_0^d|} \phi(|x_0^d - y_0^d|) \right)^q \left( \frac{t\phi^{-1}(t)}{|x_0^d - y_0^d|} \phi(|x_0^d - y_0^d|) \right)^1 .$$

Now we apply Lemma 3.10 with non-negative Borel functions $f, g$ on $\mathbb{R}^d$ supported in $B(y_0, \frac{c}{8})$ and $B(x_0, \frac{c}{8})$, respectively, and subsets $U := B(x_0, s), V := B(y_0, s)$ for some $s > 0$, $a = b = t/2$ and $\tau = \tau_U, \tau' = \tau_V$. The first term of the right hand side of (3.5) is

$$\langle \mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right], g \rangle = \int_{B(x_0, \frac{c}{8})} \mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] g(x) dx ,$$

and a similar identity holds for the second term.
Proof of Lemma 3.6. By Lemma 3.8, for any $t > 0$ we only consider the case where $x_0, y_0$ satisfy condition $\mathcal{R}(1)$. Recall that $\kappa = \phi^{-1}(t)$ and $R_1 = 2^{n_1} \kappa$. Applying Proposition 3.9 with $i_0 = j_0 = 1$, for $x \in B(x_0, \frac{\kappa}{2})$ and $\tau = \tau_{B(x_0, R_1/8)}$, we obtain

$$\mathbb{E}^x \left[ \mathbf{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \leq C_{3.9} [\phi^{-1}(t)]^{-d} \|f\|_1 K_1(t, x_0, y_0)$$

where

$$K_1(t, x_0, y_0) := \prod_{j=2}^d \left( \frac{t \phi^{-1}(t)}{|x_j^0 - y_0| \phi(|x_j^0 - y_0|)} \right)^{\frac{\alpha}{2}} \left( \frac{t \phi^{-1}(t)}{|x_j^0 - y_0| \phi(|x_j^0 - y_0|)} \right)^{1 - \frac{\alpha}{2}}$$

and by (3.6),

$$\mathbb{E}^x \left[ \mathbf{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right], g \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 g \|1\| K_1(t, x_0, y_0).$$

Similarly we obtain the second term of right hand side in (3.5) and therefore,

$$\langle P_t f, g \rangle \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 g \|1\| K_1(t, x_0, y_0).$$

Since $P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy$ with the continuous function $p(t, x, y)$ (see, Proposition 2.5), we obtain the following estimate for $t > 0$ and $x_0, y_0$ satisfying the condition $\mathcal{R}(1)$,

$$p(t, x_0, y_0) \leq c[\phi^{-1}(t)]^{-d} K_1(t, x_0, y_0)$$

$$\asymp [\phi^{-1}(t)]^{-d} \prod_{j=2}^d \left( \frac{t \phi^{-1}(t)}{|x_j^0 - y_0| \phi(|x_j^0 - y_0|)} \right) \left( \frac{t \phi^{-1}(t)}{|x_j^0 - y_0| \phi(|x_j^0 - y_0|)} \right)^{1 - \frac{\alpha}{2}}$$

and therefore,

$$\langle P_t f, g \rangle \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 g \|1\| K_1(t, x_0, y_0).$$

This proves Lemma 3.6.

Proof of Lemma 3.5. Let $l \in \{0, 1, \ldots, d - 2\}$, $t > 0$ and $x_0, y_0$ satisfy the condition $\mathcal{R}(i_0)$ for some $i_0 \in \{1, \ldots, d - l\}$. Recall that $\kappa = \phi^{-1}(t)$ and $R_1 = 2^{n_1} \kappa$. Applying Proposition 3.9 with $j_0 = d - l$, for $x \in B(x_0, \frac{\kappa}{2})$ and $\tau = \tau_{B(x_0, R_{d-l}/8)}$, we obtain

$$\mathbb{E}^x \left[ \mathbf{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \leq C_{3.9} [\phi^{-1}(t)]^{-d} \|f\|_1 \prod_{j=d-l}^d \left( \frac{t \phi^{-1}(t)}{|x_j^0 - y_0| \phi(|x_j^0 - y_0|)} \right).$$

By the similar argument used in Lemma 3.6, we obtain for $t > 0$ and for a.e. $(x, y) \in B(x_0, \frac{\kappa}{2}) \times B(y_0, \frac{\kappa}{2})$,

$$p(t, x, y) \leq c[\phi^{-1}(t)]^{-d} \prod_{j=d-l}^d \left( \frac{t \phi^{-1}(t)}{|x_j^0 - y_0| \phi(|x_j^0 - y_0|)} \right),$$

therefore, for $t > 0$ and $x_0, y_0$ satisfying the condition $\mathcal{R}(i_0)$,

$$p(t, x_0, y_0) \leq c[\phi^{-1}(t)]^{-d} \prod_{j=d-l}^d \left( \frac{t \phi^{-1}(t)}{|x_j^0 - y_0| \phi(|x_j^0 - y_0|)} \right) \leq c[\phi^{-1}(t)]^{-d} \prod_{j=d-l}^d \left( \frac{t \phi^{-1}(t)}{|x_j^0 - y_0| \phi(|x_j^0 - y_0|)} \right).$$

This implies $(\mathcal{H}_0^{l+1})$ for $l \in \{0, 1, \ldots, d - 2\}$ by Lemma 3.8 and hence we have proved Lemma 3.5.
The proof of Lemma 3.4 is more complicated. We introduce
\[ \mathcal{M}(\delta) := \frac{\phi(\kappa)}{2^\delta \phi(2^\delta \kappa)} \quad \text{for } \delta \in \mathbb{Z}. \] (3.7)
Then (WS) implies that there exists \( c = c(\phi) \geq 1 \) such that for any \( \delta \in \mathbb{Z}_+ \),
\[ c^{-1} 2^{-\delta(\alpha+1)} \leq \mathcal{M}(\delta) \leq c 2^{-\delta(\alpha+1)}. \] (3.8)

**Proof of Lemma 3.4.** Let \( l \in \{0, 1, \ldots, d-2\} \). For any \( t > 0 \), consider \( x_0, y_0 \) satisfying the condition \( \mathcal{R}(i_0) \) for some \( i_0 \in \{1, \ldots, d-l\} \). Assume (H\(_d^l\)) holds for \( q \in [0, \frac{1}{1+\alpha}] \). Recall that \( \kappa = \phi^{-1}(t) \) and \( R_j = 2^{\alpha+j} \kappa \approx |x_0^j - y_0^j| \) for \( j \in \{i_0, \ldots, d\} \). Applying Proposition 3.9 with \( j_0 \in \{i_0, \ldots, d-l\} \), we obtain that for \( x \in B(x_0, \frac{\delta}{2}) \) and \( \tau = \tau_B(x_0, R_{j_0}/8) \),
\[ \mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \leq C_{3.9}[\phi^{-1}(t)]^{-d} \|f\|_1 \mathcal{G}(l), \]
where
\[ \mathcal{G}_{j_0}(l) := \mathcal{M}(n_{j_0}) \frac{\phi(-\kappa)}{\phi(\kappa)} \prod_{j=j_0+1}^{d-l} \mathcal{M}(n_j)^q \prod_{j=d-l+1}^{d} \mathcal{M}(n_j). \] (3.9)

By a similar argument used in Lemma 3.6, we obtain for \( t > 0 \) and for a.e. \( (x, y) \in B(x_0, \frac{\delta}{2}) \times B(y_0, \frac{\delta}{2}) \),
\[ p(t, x, y) \leq c[\phi^{-1}(t)]^{-d} \mathcal{G}(l) \quad \text{for } j_0 \in \{i_0, \ldots, d-l\}, \]
hence, for \( t > 0 \) and \( x_0, y_0 \) satisfying the condition \( \mathcal{R}(i_0) \),
\[ p(t, x_0, y_0) \leq c[\phi^{-1}(t)]^{-d} \mathcal{G}(l)_{i_0} \wedge \mathcal{G}(l)_{i_0+1} \wedge \ldots \wedge \mathcal{G}(l)_{d-l}(l). \] (3.10)
Given \( l \in \{0, 1, \ldots, d-2\} \), in order to obtain \( \theta_l \) in Lemma 3.4, we first define \( \theta_l^{i_0} \) inductively for \( i_0 \in \{1, \ldots, d-l\} \). Let \( t > 0 \), and suppose that \( x_0, y_0 \) satisfy \( \mathcal{R}(d-l) \), since (3.10) implies \( p(t, x_0, y_0) \leq c[\phi^{-1}(t)]^{-d} \mathcal{G}(l) \), we have that
\[ p(t, x_0, y_0) \leq c[\phi^{-1}(t)]^{-d} \mathcal{M}(n_{d-l}) \frac{\phi(-\kappa)}{\phi(\kappa)} \prod_{j=d-l+1}^{d} \mathcal{M}(n_j) \times \left[ \phi^{-1}(t) \right]^{-d} \prod_{j=1}^{d-l} \left( \frac{t \phi^{-1}(t)}{|x_0^j - y_0^j| \phi(|x_0^j - y_0^j|)} \wedge 1 \right) \right]^{\theta_l^{d-l+1} + q} \prod_{j=d-l+1}^{d} \left( \frac{t \phi^{-1}(t)}{|x_0^j - y_0^j| \phi(|x_0^j - y_0^j|)} \wedge 1 \right), \] (3.11)
with \( \theta_l^{d-l} := \frac{\alpha}{\alpha + 1} \). Now we consider the case \( x_0, y_0 \) satisfying \( \mathcal{R}(i_0) \) for some \( i_0 \in \{1, \ldots, d-l-1\} \). Let
\[ \theta_l^{i_0} := \frac{\alpha}{2 + \alpha + \alpha} \left( \sum_{i=1}^{d-l-i_0} \left( \frac{\alpha + 1}{\alpha} \right)^i \right)^{-1}. \]
Recall that for \( a, b \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), the notation \( \mathcal{N}(\cdot, \cdot) \) is defined as follows:
\[ \mathcal{N}(a, b) := \sum_{j=a}^{b} n_j \quad \text{if } a \leq b; \quad \text{otherwise}, \quad \mathcal{N}(a, b) := 0. \]
For the convenience of the notation, let \( b_0 := \frac{\alpha}{2+1} \). Suppose that
\[
\theta_{i_0}^i \leq \frac{n_{j_0}(\alpha + 1)b_0 - (\alpha + 1)N(i_0, j_0 - 1)q}{(\alpha + 1)N(i_0, d - l)}
\]
for some \( j_0 \in \{i_0, \ldots, d - l - 1\} \). (3.12)

Since \( b_0 \geq \theta_{i_0}^i \), (3.8) and (3.12) imply that
\[
\left( \mathcal{M}(n_{j_0})^{b_0+q} \prod_{j=j_0+1}^{d-l} \mathcal{M}(n_j)^q \right) \left( \prod_{j=i_0}^{j_0-1} \mathcal{M}(n_j)^{q+\theta_{i_0}^i} \right)^{-1} \prod_{j=i_0}^{j_0-1} \mathcal{M}(n_j)^{q+\theta_{i_0}^i} = \prod_{j=j_0+1}^{d-l} \mathcal{M}(n_j)^{q+\theta_{i_0}^i} \\
\geq c \frac{2^{-(\alpha+1)N(i_0,j_0-1)(q+\theta_{i_0}^i)}}{2^{(\alpha+1)n_{j_0}(b_0-\theta_{i_0}^i)2-(\alpha+1)N(j_0+1,d-l)\theta_{i_0}^i}} \mathcal{M}(n_{j_0})^{\theta_{i_0}^i} \\
\geq c \frac{2^{-(\alpha+1)N(i_0,j_0-1)q}2^{-\alpha+1}N(i_0,d-l-1)\theta_{i_0}^i}}{2^{n_{j_0}(\alpha+1)b_0}} \geq c,
\]
and therefore,
\[
\mathcal{M}(n_{j_0})^{b_0+q} \prod_{j=j_0+1}^{d-l} \mathcal{M}(n_j)^q \leq c \prod_{j=i_0}^{d-l} \mathcal{M}(n_j)^{q+\theta_{i_0}^i}.
\]

Suppose that
\[
\theta_{i_0}^i > \max_{j_0 \in \{i_0, \ldots, d - l - 1\}} \frac{n_{j_0}(\alpha + 1)b_0 - (\alpha + 1)N(i_0, j_0 - 1)q}{(\alpha + 1)N(i_0, d - l)}.
\]

Let \( K_0 := \frac{\alpha+1}{\alpha+1} \). For \( i \in \{0, \ldots, d - l - 1 - i_0\} \), the lower bounds of \( 1 + \frac{q}{K_0b_0} \theta_{i_0}^i \) could be obtained as follows;
\[
\begin{align*}
\theta_{i_0}^i &> \frac{K_0b_0n_{i_0+1}}{N(i_0,d-l)} - qn_{i_0} \quad \Longrightarrow \quad (1 + \frac{q}{K_0b_0})\theta_{i_0}^i > \frac{K_0b_0n_{i_0+1}}{N(i_0,d-l)} \\
\theta_{i_0}^i &> \frac{K_0b_0n_{i_0+2}}{N(i_0,d-l)} - \frac{q(n_{i_0+1}+n_{i_0+1})}{N(i_0,d-l)} \quad \Longrightarrow \quad (1 + \frac{q}{K_0b_0})^2\theta_{i_0}^i > \frac{K_0b_0n_{i_0+2}}{N(i_0,d-l)} \quad : \quad : \\
\theta_{i_0}^i &> \frac{K_0b_0n_{d-l-1}}{N(i_0,d-l)} - \frac{qN(i_0,d-l-2)}{N(i_0,d-l)} \quad \Longrightarrow \quad (1 + \frac{q}{K_0b_0})^{d-l-1-i_0}\theta_{i_0}^i > \frac{K_0b_0n_{d-l-1}}{N(i_0,d-l)}.
\end{align*}
\]

Hence,
\[
\theta_{i_0}^i \sum_{i=0}^{d-l-i_0-1} (1 + \frac{q}{K_0b_0})^i > \frac{K_0b_0 \cdot N(i_0, d - l - 1)}{N(i_0, d - l)}.
\]

Now our claim is that
\[
\mathcal{M}(n_{d-l})^{b_0+q} \leq c \prod_{j=i_0}^{d-l} \mathcal{M}(n_j)^{q+\theta_{i_0}^i}.
\]

For this, we only need to prove that
\[
-(\alpha + 1)N(i_0, d - l - 1)(q + \theta_{i_0}^i) + (\alpha + 1)n_{d-l}(b_0 - \theta_{i_0}^i) \geq 0.
\]

If so, using (3.8) with \( b_0 \geq \theta_{i_0}^i \), we have that
\[
\mathcal{M}(n_{d-l})^{-b_0-q} \prod_{j=i_0}^{d-l} \mathcal{M}(n_j)^{q-\theta_{i_0}^i} \geq c \frac{2^{-(\alpha+1)N(i_0,d-l-1)(q+\theta_{i_0}^i)}}{2^{(\alpha+1)n_{d-l}(b_0-\theta_{i_0}^i)}} \geq c.
\]
By (3.14) and the fact that \( K_0 \leq 1 \), we observe that

\[
\begin{align*}
\frac{N(i_0, d - l - 1)}{N(i_0, d - l)} (\overline{\alpha} + 1)(q + \theta_l^{i_0}) - \left( 1 - \frac{N(i_0, d - l - 1)}{N(i_0, d - l)} \right) (\overline{\alpha} + 1)(b_0 - \theta_l^{i_0}) \\
\leq \frac{N(i_0, d - l - 1)}{N(i_0, d - l)} (\overline{\alpha} + 1)(b_0 + q) - (\overline{\alpha} + 1)(b_0 - \theta_l^{i_0}) \\
< \theta_l^{i_0} \sum_{i=0}^{d-l-i_0-1} \left( 1 + \frac{q}{K_0 b_0} \right)^i (\overline{\alpha} + 1) \left( \frac{1}{K_0} + \frac{q}{K_0 b_0} \right) - (\overline{\alpha} + 1)(b_0 - \theta_l^{i_0}) \\
\leq \theta_l^{i_0} \sum_{i=1}^{d-l-i_0} \left( \frac{1}{K_0} + \frac{q}{K_0 b_0} \right)^i (\overline{\alpha} + 1) - (\overline{\alpha} + 1)(b_0 - \theta_l^{i_0}). \tag{3.18}
\end{align*}
\]

The definitions of \( b_0 \) and \( \theta_l^{i_0} \) yield that \( b_0 - \theta_l^{i_0} \geq \frac{\overline{\alpha} + 1}{2+\overline{\alpha}+\overline{\beta}} b_0 \) and hence,

\[
\theta_l^{i_0} \sum_{i=1}^{d-l-i_0} \left( \frac{\overline{\alpha} + 1}{\overline{\beta}} \right)^i (\overline{\alpha} + 1) - (\overline{\alpha} + 1)(b_0 - \theta_l^{i_0}) \leq 0. \tag{3.19}
\]

Since \( \frac{1}{K_0} + \frac{q}{K_0 b_0} \leq \frac{\overline{\beta} + 1}{\overline{\alpha}} \) for \( q \leq (1 + \overline{\alpha})^{-1} \), (3.18)–(3.19) yield (3.16). By (3.13) and (3.15), we obtain the upper bounds of \( G_{j_0}(l) \) for \( j_0 \in \{i_0, \ldots, d - l\} \) in (3.9), and therefore, (3.10) implies that for any \( t > 0 \) and \( x_0, y_0 \) satisfying \( R(i_0) \), \( i_0 \in \{1, \ldots, d - l - 1\} \),

\[
\begin{align*}
p(t, x_0, y_0) &\leq c[\phi^{-1}(t)]^{-d} \prod_{j=i_0}^{d-l} \mathcal{N}(n_j)^q + \theta_l^{i_0} \prod_{i=d-l+1}^{d} \mathcal{N}(n_i) \\
&\leq \left[ \phi^{-1}(t) \right]^{-d} \prod_{i=1}^{d-l} \left( \frac{t \phi^{-1}(t)}{|x_{i_0} - y_0|} \wedge 1 \right)^q + \theta_l^{i_0} \prod_{i=d-l+1}^{d} \left( \frac{t \phi^{-1}(t)}{|x_{i_0} - y_0|} \wedge 1 \right), \tag{3.20}
\end{align*}
\]

where \( \theta_l^{i_0} := \frac{\overline{\alpha}}{2+\overline{\alpha}+\overline{\beta}} \left( \sum_{i=1}^{d-l-i_0} \left( \frac{\overline{\beta} + 1}{\overline{\alpha}} \right)^i \right)^{-1} \).

By (3.11) and (3.20) in connection with Lemma 3.8, we have \( (H_{q+\theta_l}^l) \) with \( \theta_l := \min_{i_0 \in \{1, 2, \ldots, d - l\}} \theta_l^{i_0} = \frac{\overline{\alpha}}{2+\overline{\alpha}+\overline{\beta}} \left( \sum_{i=1}^{d-l-1} \left( \frac{\overline{\beta} + 1}{\overline{\alpha}} \right)^i \right)^{-1} \) for \( l \in \{0, 1, \ldots, d - 2\} \). Hence, the proof of Lemma 3.4 is complete. \( \blacksquare \)

## 4. Hölder Continuity and Lower Bound Estimates

In this section, we prove the Hölder continuity of heat kernel \( p(t, x, y) \). Consequently, the process \( X \) can be refined to start from every point in \( \mathbb{R}^d \), and the estimates for the transition density functions hold for every \( x, y \in \mathbb{R}^d \). As indicated in [6, Remark 3.6], the Hölder continuity of transition densities can be derived from the boundedness of transition densities plus the Hölder continuity of bounded harmonic functions. According to which, we also conclude that \( X \) is a strong Feller process. At the end, we give the sharp lower bound for \( p(t, x, y) \).

### 4.1. Hölder Continuity
Proposition 4.1. For each $A > 0$ and $B \in (0, 1)$, there exists $\varrho = \varrho(A, B, \phi) \in (0, 2^{-1})$ such that for every $r > 0$ and $x \in \mathbb{R}^d$, 
\[ \mathbb{P}^x (\tau_{B(x, Br)} < \varrho \phi(r)) = \mathbb{P}^x \left( \sup_{s \leq \varrho \phi(r)} |X_s - X_0| > Ar \right) \leq B. \]

Proof. Let $\zeta$ be the life time of $X$. By Theorem 2.8, $\mathbb{P}^x (\zeta \leq u) = 0$ for any $u > 0$. So for any $x \in \mathbb{R}^d$ and $t, u > 0$, by the upper bound estimates in Theorem 3.1 of $p(t, x, y)$, and (WS) condition, we have that 
\[ \mathbb{P}^x (|X_t - x| \geq u) = \int_{B(x, u)^c} p(t, x, y) dy + \mathbb{P}^x (\zeta \leq u) \]
\[ \leq c_1 \prod_{i=1}^d \int_{y_i : |x_i - y_i| \geq \frac{u}{\sqrt{d}}} \frac{t}{|x_i - y_i|^{\varphi(|x_i - y_i|)}} dy_i \leq c_1 t^d \prod_{i=1}^d \int_{\frac{u}{\sqrt{d}}}^{\infty} (w \phi(w))^{-1} dw \]
\[ \leq c_2 \left( \frac{t}{\phi(u)} \right)^d \prod_{i=1}^d \int_{\frac{u}{\sqrt{d}}}^{\infty} w^{-\frac{d}{2}} dw \leq c_3 \left( \frac{t}{\phi(u)} \right)^d. \]

By the similar proof as in Proposition 2.9, for $\sigma_u := \inf \{ s \geq 0 : |X_s - X_0| > u \}$, we have 
\[ \mathbb{P}^u (\sigma_u < t, |X_{2t} - x| \leq u/2) \leq \sup_{y \in \{ z \in \mathbb{R}^d : |x - z| > u \}} \mathbb{P}^y (|X_{2t} - y| > u/2), \]
and for any $x \in \mathbb{R}^d$ and $u, t > 0$,
\[ \mathbb{P}^x \left( \sup_{s \leq t} |X_s - X_0| > u \right) \leq \mathbb{P}^x (\sigma_u < t, |X_{2t} - x| \leq u/2) + \mathbb{P}^x (|X_{2t} - x| \geq u/2) \]
\[ \leq \sup_{y \in \{ z \in \mathbb{R}^d : |x - z| > u \}} \mathbb{P}^y (|X_{2t} - y| > u/2) + c_4 \left( \frac{t}{\phi(u)} \right)^d \leq c_3 \left( \frac{t}{\phi(u)} \right)^d. \tag{4.1} \]

For any $A, B \in (0, 1)$, (4.1) and (WS) imply that 
\[ \mathbb{P}^x \left( \sup_{s \leq \varrho_1 \phi(r)} |X_s - X_0| > Ar \right) \leq c_5 \left( \frac{\varrho_1 \phi(r)}{\phi(Ar)} \right)^d \leq c_5 \left( \varrho_1 C A^{-\frac{d}{2}} \right)^d \leq B \tag{4.2} \]
for some constant $\varrho_1 := C^{-1} A^{\frac{d}{2}} (c_5^{-1} B)^{1/d} \wedge 2^{-1}$, and it proves our assertion in this case.

Now we consider the case $A \geq 1$ and $B \in (0, 1)$. In the similar way to obtain (4.2), using (4.1),
\[ \mathbb{P}^x \left( \sup_{s \leq \varrho_2 \phi(Ar)} |X_s - X_0| > Ar \right) \leq c_5 \left( \frac{\varrho_2 \phi(Ar)}{\phi(Ar)} \right)^d \leq B \]
for some $\varrho_2 := (c_5^{-1} B)^{1/d} \wedge 2^{-1}$, and therefore we have our assertion. \hfill \blacksquare

Proposition 4.2. For $r > 0$, there exist $a_i = a_i(\phi) > 0, i = 1, 2, 3$ such that 
\[ a_1 \phi(r) \leq \mathbb{E}^x [\tau_{B(x, r)}] \leq a_2 \phi(r) \quad \text{and} \quad \mathbb{E}^x [\tau^2_{B(x, r)}] \leq a_3 \phi(r)^2. \]

Proof. We write $\tau = \tau_{B(x, r)}$ for simplicity. By Proposition 4.1 with $A = 1, B = \frac{1}{2}$, there exists $\varrho = \varrho(1, \frac{1}{2}, \phi)$ such that 
\[ \mathbb{E}^x [\tau] \geq \varrho \phi(r) \mathbb{P}^x (\tau \geq \varrho \phi(r)) = \varrho \phi(r) [1 - \mathbb{P}^x (\tau < \varrho \phi(r))] \geq \frac{\varrho}{2} \phi(r). \]
On the other hand, the Lévy system in (2.3) and (WS) imply that for any \( t > 0 \),

\[
\mathbb{P}^x(\tau \leq t) \geq \mathbb{E}^x \left[ \sum_{s \leq t \wedge \tau} 1_{|X_s - x_{s-}| > 2r} \right] = \mathbb{E}^x \left[ \int_0^{t \wedge \tau} \sum_{i=1}^d \int_{|h| > 2r} J(X_s, X_s + c^1 h) dh ds \right] \\
\geq \mathbb{E}^x \left[ \int_0^{t \wedge \tau} \sum_{i=1}^d \int_{|h| > 2r} \frac{\Lambda^{-1}}{|h| \phi(|h|)} dh ds \right] \\
\geq c_1 [\phi(r)]^{-1} \mathbb{E}^x[t \wedge \tau] \geq c_1 (\phi)[\phi(r)]^{-1} t \mathbb{P}^x(\tau > t).
\]

Thus, \( \mathbb{P}^x(\tau > t) \leq 1 - c_1 [\phi(r)]^{-1} t \mathbb{P}^x(\tau > t) \). Choose \( t = c_1^{-1} \phi(r) \) so that \( \mathbb{P}^x(\tau > t) \leq 1/2 \).

Using the Markov property at time \( mt \) for \( m = 1, 2, \ldots \),

\[
\mathbb{P}^x(\tau > (m+1)t) \leq \mathbb{E}^x[\mathbb{P}^{X^m}(\tau > t); \tau > mt] \leq \frac{1}{2} \mathbb{P}^x(\tau > mt).
\]

By induction, we obtain that \( \mathbb{P}^x(\tau > mt) \leq 2^{-m} \) with \( t = c_1^{-1} \phi(r) \), and hence

\[
\mathbb{E}^x[\tau] \leq \sum_{m=0}^{\infty} t \mathbb{P}^x(\tau > mt) \leq \sum_{m=0}^{\infty} t 2^{-m} \leq a_2 \phi(r).
\]

Similarly, with \( t = c_1^{-1} \phi(r) \),

\[
\mathbb{E}^x[\tau^2] \leq \sum_{m=0}^{\infty} 2(m+1)t^2 \mathbb{P}^x(\tau > mt) \leq \sum_{m=0}^{\infty} (m+1)2^{-(m-1)}t^2 \leq a_3 \phi(r)^2.
\]

In order to prove the bounded harmonic functions associated with \( X \) are Hölder continuous, we need a support theorem stated in the following.

**Theorem 4.3** (Support theorem). Let \( \psi : [0, t_1] \rightarrow \mathbb{R}^d \) be continuous with \( \psi(0) = x \) and the image of \( \psi \) be contained in \( B(0, 1) \). For any \( \varepsilon > 0 \), there exists a constant \( c = c(\psi, \varepsilon, t_1) \) such that

\[
\mathbb{P}^x \left( \sup_{s \leq t_1} |X_s - \psi(s)| \leq \varepsilon \right) > c.
\]

**Proof.** This follows from the proof of [16, Theorem 4.9] with the fact that

\[
\frac{c_1}{\phi(\delta)} \leq \int_{\mathbb{R}^d} \mathcal{J}_\delta(x, y) dy \leq \frac{c_2}{\phi(\delta)} \quad x \in \mathbb{R}^d
\]

for some constant \( c_1, c_2 > 0 \). Here \( \mathcal{J}_\delta(x, y) \) is the function defined in (2.11) with \( \delta \) instead of 1.

For any Borel set \( A \subset \mathbb{R}^d \), we denote by \( T_A := \inf\{ t > 0 : X_t \in A \} \) the hitting time of \( X \) from \( A \).

**Corollary 4.4.** For \( r \in (0, 1) \), let \( x \in Q(0, 1) \) with \( \text{dist}(x, \partial Q(0, 1)) > r \). If \( Q(z, r) \subset Q(0, 1) \), then

\[
\mathbb{P}^x(T_{Q(z, r)} < \tau_{Q(0, 1)}) \geq c
\]

for some positive constant \( c = c(r) > 0 \).
Proof. Let $\psi$ be the line segment from $x$ to $z$ with $\psi(0) = x$ and $\psi(1) = z$. Since $\text{dist}(x, \partial Q(0,1)) > r$ and $Q(z, r) \subset Q(0,1)$, by Theorem 4.3, we have that
\[
\mathbb{P}^x(T_{Q(z, r)} < \tau_{Q(0,1)}) \geq \mathbb{P}^x \left( \sup_{s \leq 1} |X_s - \psi(s)| \leq \frac{r}{2} \right) > c(r).
\]

\begin{proposition}
There exists a nondecreasing function $\varphi : (0, 1) \to (0, 1)$ such that for any open set $D \subset Q(0,1)$, $|D| > 0$, and for any $x \in Q(0, \frac{1}{2})$,
\[
\mathbb{P}^x(T_D < \tau_{Q(0,1)}) \geq \varphi(|D|).
\]
\end{proposition}

Proof. Let $D \subset Q(0,1)$ be an open set such that $|D| > 0$ and $x \in Q(0, \frac{1}{2})$. We first show that
\[
\mathbb{P}^x(T_D < \tau_{Q(0,1)}) \geq c \text{ when } |Q(0,1) \setminus D| \text{ is small enough.}
\] (4.3)

For simplicity, we write $\tau$ for $\tau_{Q(0,1)}$. By Proposition 4.2, we observe that
\[
a_1 \leq \mathbb{E}^x[\tau] = \mathbb{E}^x[\tau; T_D < \tau] + \mathbb{E}^x \left[ \int_0^\tau 1_{D^c}(X_s)ds \right]
\]
\[
\leq (\mathbb{E}^x[\tau^2])^{1/2}(\mathbb{P}^x(T_D < \tau))^{1/2} + \mathbb{E}^x \left[ \int_0^\tau 1_{D^c}(X_s)ds \right]
\]
\[
\leq a_1^{1/2}(\mathbb{P}^x(T_D < \tau))^{1/2} + \mathbb{E}^x \left[ \int_0^\tau 1_{D^c}(X_s)ds \right].
\] (4.4)

By Proposition 4.2 again, since for $t_0 > 0$,
\[
\mathbb{E}^x[\tau - \tau \wedge t_0] \leq \mathbb{E}^x[\tau; \tau > t_0] \leq \left( \frac{\mathbb{E}^x[\tau^2]\mathbb{E}^x[\tau]}{t_0} \right)^{1/2} \leq \frac{c_4}{\sqrt{t_0}}
\]
for some $c_4 > 0$, we can choose $t_0$ large enough such that $\mathbb{E}^x[\tau - \tau \wedge t_0] \leq \frac{4\varphi}{4} \leq t_0$. Then
\[
\mathbb{E}^x \left[ \int_0^\tau 1_{D^c}(X_s)ds \right] = \mathbb{E}^x \left[ \int_0^\tau 1_{Q(0,1) \setminus D}(X_s)ds \right] \leq \frac{a_1}{4} + \mathbb{E}^x \left[ \int_0^{t_0} 1_{Q(0,1) \setminus D}(X_s)ds \right]
\]
\[
= \frac{a_1}{4} + \int_0^{t_0} \mathbb{P}^x(X_s \in Q(0,1) \setminus D)ds
\]
\[
= \frac{a_1}{4} + \int_0^{t_0} a_1 \mathbb{P}^x(X_s \in Q(0,1) \setminus D)ds + \int_{t_0}^{t_0} \int_{Q(0,1) \setminus D} p(s, x, y)dyds
\]
\[
\leq \frac{a_1}{2} + |Q(0,1) \setminus D| \int_{t_0}^{t_0} c_2[\phi^{-1}(s)]^{-d}ds.
\]
The last inequality comes from Proposition 2.5. Since $\int_{t_0}^{t_0} c[\phi^{-1}(s)]^{-d}ds$ is bounded by a constant depending on $t_0$ and $a_1$, letting $|Q(0,1) \setminus D|$ small enough so that $\mathbb{E}^x \left[ \int_0^\tau 1_{D^c}(X_s)ds \right] \leq 3a_1/4$ with (4.4), we have that
\[
\mathbb{P}^x(T_D < \tau) \geq a_2^2/16a_2,
\]
which implies (4.3). Let $\varphi(\varepsilon) := \inf\{\mathbb{P}^y(T_D < \tau_{Q(0,1)}) : y \in Q(0, \frac{1}{2}), D \subset Q(0,1), |D| \geq \varepsilon|Q(0,1)|\}$. By (4.3), $\varphi(\varepsilon) > 0$ for $\varepsilon$ sufficiently close to 1. By [4, Proposition V.7.2] together with Corollary 4.4, one can follow the proof in [4, Theorem V.7.4] to show that $\varepsilon_0 = \inf\{\varepsilon : \varphi(\varepsilon) > 0\} = 0$. Therefore, we conclude our assertion with $\varphi$. 
\end{proof}
A function $h$, bounded in $\mathbb{R}^d$, is harmonic with respect to $X$ in a domain $\mathcal{F}$ if $h(X_{t\wedge \tau_D})$ is a martingale with respect to $\mathbb{P}^x$ for every $x$ in the domain.

**Theorem 4.6.** For any $x_0 \in \mathbb{R}^d$ and $r \in (0, 1)$, suppose that $h$ is harmonic in $B(x_0, r)$ with respect to $X$ and bounded in $\mathbb{R}^d$. Then there exist constants $c, \beta > 0$ such that

$$|h(x) - h(y)| \leq c \left( \frac{|x - y|}{r} \right)^\beta \|h\|_\infty, \quad \text{for } x, y \in B(x_0, r/2).$$

**Proof.** The proof is similar to that of [4, Theorem V.7.5]. For the reader’s convenience, we give the modified proof here. Define $\text{Osc}_D h := \sup_{x \in D} h(x) - \inf_{x \in D} h(x)$. To prove the result, it suffices to show there exists $\rho < 1$ such that for all $z \in B(x_0, r/2)$ and $s \leq r/4$,

$$\text{Osc}_{B(z,s/2)} h \leq \rho \text{Osc}_{B(z,s)} h. \tag{4.5}$$

Without loss of generality, we can assume $\inf_{B(z,s)} h = 0$ and $\sup_{B(z,s)} h = 1$, otherwise we can do a linear transform of $h$. Let $D = \{x \in B(z, s/2) : h(x) \geq 1/2\}$. We can assume $|D| \geq \frac{1}{2} |B(z, s/2)|$, otherwise we replace $h$ by $1 - h$. Since $h(X_{t\wedge \tau_{B(z_0, r)}})$ is a martingale,

$$h(x) = \mathbb{E}^x [h(X_{\tau_{B(z,s)\wedge \tau_D}})] \geq \frac{1}{2} \mathbb{P}^x (T_D < \tau_{B(z,s)}),$$

and by Proposition 4.5 with the scaling, $D_s := s^{-1}D$, we have

$$\mathbb{P}^x (T_D < \tau_{B(z,s)}) = \mathbb{P}^{s^{-1}x} (T_{D_s} < \tau_{B(s^{-1}z, 1)}) \geq \varphi (|D_s|) \geq \varphi (2^{-1}|B(z, 1/2)|) \geq \varphi (2^{-(d+1)}).$$

Taking $\rho = 1 - \varphi (2^{-(d+1)})/2$ proves (4.5).

We now show the Hölder continuity of $\lambda$-resolvent $U_\lambda$ for the process $X$, that is,

$$U_\lambda f(x) = \mathbb{E}^x \left[ \int_0^\infty e^{-\lambda t} f(X_t) dt \right] = \int_0^\infty e^{-\lambda t} P_t f(x) dt,$$

where $P_t$ is the corresponding semigroup operator defined in Subsection 2.2.

**Proposition 4.7.** If $f$ is bounded, there exists $c = c(\lambda)$ and $\zeta > 0$ such that

$$|U_\lambda f(x) - U_\lambda f(y)| \leq c |x - y|^{\zeta} \|f\|_\infty, \quad \text{for } |x - y| < 1.$$

**Proof.** For any $x, y \in \mathbb{R}^d$ with $|x - y| < 1$, consider $x_0 \in \mathbb{R}^d$ and $r := |x - y|^{1/2} \in (0, 1)$ so that $x, y \in B(x_0, r/2)$. Write $\tau_x = \tau_{B(x,r)}$ for simplicity. By the strong Markov property, we have that

$$U_\lambda f(x) = \mathbb{E}^x \left[ \int_0^\tau_x e^{-\lambda t} f(X_t) \right] dt + \mathbb{E}^x \left[ e^{-\lambda \tau_x} - 1 \right] U_\lambda f(X_{\tau_x}) + \mathbb{E}^x \left[ \lambda \tau_x \right].$$

Using the similar expression where $x$ is replaced by $y$, Proposition 4.2 and the mean value theorem imply that

$$|U_\lambda f(x) - U_\lambda f(y)| \leq (\mathbb{E}^x[\tau_x] + \mathbb{E}^y[\tau_y]) (\|f\|_\infty + \lambda \|U_\lambda f\|_\infty) + \mathbb{E}^x \left[ U_\lambda f(X_{\tau_x}) - \mathbb{E}^y [U_\lambda f(X_{\tau_y})] \right]$$

$$\leq 2\lambda \phi(r) (\|f\|_\infty + \lambda \|U_\lambda f\|_\infty) + \mathbb{E}^x \left[ U_\lambda f(X_{\tau_x}) - \mathbb{E}^y [U_\lambda f(X_{\tau_y})] \right].$$

Since $z \rightarrow \mathbb{E}^z \left[ U_\lambda f(X_{\tau_x}) \right]$ is bounded and harmonic in $B(x_0, r)$, applying Theorem 4.6, we have that

$$\mathbb{E}^x \left[ U_\lambda f(X_{\tau_x}) - \mathbb{E}^y [U_\lambda f(X_{\tau_y})] \right] \leq c_1 \left( \frac{|x - y|}{r} \right)^\beta \|U_\lambda f\|_\infty.$$
for some $c_1, \beta > 0$. Since $r = |x - y|^{1/2} < 1$, (WS) yields $\phi(r) \leq c^{-1}\phi(1)|x - y|^{\alpha/2}$. Therefore, using the fact that $\|U_\lambda f\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty$, we have that

$$|U_\lambda f(x) - U_\lambda f(y)| \leq c_2 \left(\phi(r) + \left(\frac{|x - y|}{r}\right)^\beta\right)\|f\|_\infty$$

$$\leq c_3 (|x - y|^{\alpha/2} + |x - y|^{\beta/2})\|f\|_\infty.$$

\[\Box\]

**Remark 4.8.** By Proposition 4.7 with [7, (2.16) in page 77], we conclude that $X$ is a strong Feller process.

According to the spectral theorem, there exist projection operators $E_\mu$ on $L^2(\mathbb{R}^d)$ such that

$$f = \int_0^\infty dE_\mu(f), \quad P_t f = \int_0^\infty e^{-\mu t} dE_\mu(f) \quad \text{and} \quad U_\lambda f = \int_0^\infty \frac{1}{\lambda + \mu} dE_\mu(f).$$

**Theorem 4.9.** For $f \in L^2(\mathbb{R}^d)$, $P_t f$ is equal a.e. to a function that is Hölder continuous. Hence, we can refine $p(t, x, y)$ to be jointly continuous for any $t > 0$ and $x, y \in \mathbb{R}^d$.

**Proof.** Let $\lambda, \mu$ and $t > 0$. For any $f \in L^2(\mathbb{R}^d)$, define

$$h := h(f) := \int_0^\infty (\lambda + \mu)e^{-\mu t} dE_\mu(f). \quad (4.6)$$

Then $h \in L^2(\mathbb{R}^d)$ since $\sup_\mu (\lambda + \mu)^2 e^{-2\mu t} \leq c$ so that

$$\|h\|^2_2 = \int_0^\infty (\lambda + \mu)^2 e^{-2\mu t} d\langle E_\mu(f), E_\mu(f)\rangle$$

$$\leq c_1 \int_0^\infty d\langle E_\mu(f), E_\mu(f)\rangle = c_1 \|f\|^2_2 < \infty.$$  

For any $g \in L^2(\mathbb{R}^d)$, note that $\|P_t g\|_1 \leq \|g\|_1$ and $\|P_t g\|_\infty \leq c_2 \|g\|_1$ since $p(t, x, y)$ is bounded. Then it follows that $\|P_t g\|_2 \leq c_3 \|g\|_1$. By Cauchy-Schwartz inequality,

$$\langle h, g \rangle = \int_0^\infty (\lambda + \mu)e^{-\mu t} d\langle E_\mu(f), E_\mu(g)\rangle$$

$$\leq \left(\int_0^\infty (\lambda + \mu)e^{-\mu t} d\langle E_\mu(f), E_\mu(f)\rangle\right)^{1/2} \left(\int_0^\infty (\lambda + \mu)e^{-\mu t} d\langle E_\mu(g), E_\mu(g)\rangle\right)^{1/2}$$

$$\leq c_4 \left(\int_0^\infty d\langle E_\mu(f), E_\mu(f)\rangle\right)^{1/2} \left(\int_0^\infty e^{-\mu t/2} d\langle E_\mu(g), E_\mu(g)\rangle\right)^{1/2}$$

$$= c_4 \|f\|_2 \|P_{t/4} g\|_2 \leq c_5 \|f\|_2 \|g\|_1.$$  

Thus, we have $\|h\|_\infty \leq c_5 \|f\|_2$ by taking the supremum for $g \in \{u \in L^1(\mathbb{R}^d) : \|u\|_1 \leq 1\}$. Since

$$U_\lambda h = \int_0^\infty e^{-\mu t} dE_\mu(f) = P_t f \quad \text{a.e.,} \quad (4.7)$$

we have our first assertion.
Fix $y$ and let $\tilde{f}(z) = p(t/2, z, y)$. By Proposition 2.5, since $\|\tilde{f}\|_1 = 1$ and $\|\tilde{f}\|_\infty \leq \frac{c_6}{[\phi^{-1}(t)]^\alpha}$, we have that
$$\|\tilde{f}\|_2 \leq \|\tilde{f}\|_\infty \|\tilde{f}\|_1 \leq \frac{c_6}{[\phi^{-1}(t)]^\alpha},$$
that is, $\tilde{f} \in L^2(\mathbb{R}^d)$. Since Proposition 2.5 again implies that
$$p(t, x, y) = \int_{\mathbb{R}^d} p(t/2, x, z)p(t/2, z, y)dz,$$
by Proposition 4.7 and (4.7), we have that
$$|p(t, x, y) - p(t, z, y)| = |P_{t/2} \tilde{f}(x) - P_{t/2} \tilde{f}(z)| = |U_\lambda \tilde{h}(x) - U_\lambda \tilde{h}(z)| \leq c|x - z|^\alpha \|\tilde{h}\|_\infty \quad \text{for } |x - z| < 1,$$
where $\tilde{h} = \tilde{h}(\tilde{f})$ defined in (4.6) satisfies $\|\tilde{h}\|_\infty < c$. Hence, $p(t, x, y)$ is Hölder continuous with constants independent of $x$ and $y$, and the symmetry of $p(t, x, y)$ gives the joint continuity of $p(t, x, y)$.

4.2. Lower bounds. For the lower bound of $p(t, x, y)$, we first obtain the following on-diagonal estimate.

Proposition 4.10. There exists $c = c(\phi) \in (0, 1)$ such that
$$p(t, x, x) \geq \frac{c}{[\phi^{-1}(t)]^d} \quad \text{for } t > 0, \quad x \in \mathbb{R}^d.$$

Proof. By Proposition 4.1, there exists $\varrho_0 \in (0, 2^{-1})$ such that for any $x \in \mathbb{R}^d$ and $r > 0$
$$\mathbb{P}^x(\tau_{B(x, r)} \leq \varrho_0 \phi(r)) \leq \frac{1}{2}.$$
For any $t > 0$, let $r_1 := \phi^{-1}\left(\frac{t}{2\varrho_0}\right)$ and $r_2 := \phi^{-1}(t)$. Because $2\varrho_0 \leq 1$ and $\phi^{-1}$ is increasing, we note that $r_2 \leq r_1$. Therefore, (WS) implies
$$c\left(\frac{r_1}{r_2}\right)^\alpha \leq \frac{\phi(r_1)}{\phi(r_2)} = 2\varrho_0 \leq C\left(\frac{r_1}{r_2}\right)^\alpha,$$
so that $r_2 \leq r_1 \leq c_0 r_2$ for some $c_0 := (2\varrho_0/C)^{1/\alpha}$. Since
$$\int_{\mathbb{R}^d \setminus B(x, c_0 \phi^{-1}(t))} p(t/2, x, y)dy \leq \mathbb{P}^x(\tau_{B(x, c_0 r_2)} \leq \frac{1}{2}) \leq \mathbb{P}^x(\tau_{B(x, r_1)} \leq \varrho_0 \phi(r_1)) \leq \frac{1}{2},$$
we have that
$$p(t, x, x) = \int_{\mathbb{R}^d} p^2(t/2, x, y)dy \geq \int_{B(x, c_0 \phi^{-1}(t))} p^2(t/2, x, y)dy$$
$$\geq |B(x, c_0 \phi^{-1}(t))|^{-1} \left(\int_{B(x, B(x, c_0 \phi^{-1}(t)))} p(t/2, x, y)dy\right)^2 \geq \frac{c_1}{[\phi^{-1}(t)]^d}. \quad \bl$$
Using the scaling method in Subsection 2.3, we then obtain near diagonal lower estimate.

Proposition 4.11. There exist positive constants $c_1, c_2$ such that
$$p(t, x, y) \geq \frac{c_1}{[\phi^{-1}(t)]^d} \quad \text{for } t > 0, \quad |x - y| \leq c_2 \phi^{-1}(t).$$
Proof. Because of (2.5), it is enough to show that
\[ q^{(\phi^{-1}(t))}(1, x, y) \geq c_1 \]
for \( |x - y| \leq c_2 \)
for some \( c_1, c_2 > 0 \). Let \( \kappa := \phi^{-1}(t) \). With the same discussion as in the proof of Theorem 4.9, there exist \( c, \varrho > 0 \) such that for \( \lambda > 0 \)
\[ |q^{(\kappa)}(1, x, z) - q^{(\kappa)}(1, y, z)| = |Q^{(\kappa)}_{1/2} f(x) - Q^{(\kappa)}_{1/2} f(y)| \]
\[ = |U^{(\kappa)}_\lambda h(x) - U^{(\kappa)}_\lambda h(y)| \leq c|x - y|^\varsigma. \]
Here \( f(z) := q^{(\kappa)}(1/2, z, y) \) and \( h(z) := \int_0^\infty (\lambda + \mu) e^{-\mu t} dE^{(\kappa)}_\mu(f) \) where \( \mu > 0 \) and \( E^{(\kappa)}_\mu \)
is the projection operators in \( L^2(\mathbb{R}^d) \) related to \( Q^{(\kappa)} \) and \( U^{(\kappa)} \). Therefore, we have our assertion by the symmetry of \( q^{(\kappa)}(t, x, y) \) and Proposition 4.10.

We finally obtain the lower bound estimates of \( p(t, x, y) \) for any \( t > 0 \) and \( x, y \in \mathbb{R}^d \) using the similar method as in [16, Theorem 4.21].

**Theorem 4.12.** There exists a positive constant \( c = c(\phi) > 0 \) such that for any \( t > 0, x, y \in \mathbb{R}^d \),
\[ p(t, x, y) \geq c[\phi^{-1}(t)]^{-d} \prod_{i=1}^{d} \left( 1 + \frac{t \phi^{-1}(t)}{|x^i - y^i| \phi(|x^i - y^i|)} \right). \]

**Proof.** We first note that by Proposition 4.1, there exists \( \varrho_0 \in (0, 2^{-1}) \) such that for any \( w \in \mathbb{R}^d \) and \( r > 0 \),
\[ \mathbb{P}^x(\tau_{B(w,4^{-d}r)} \geq \varrho_0 \phi(r)) \geq 1/2. \] (4.8)
Let \( \varrho := 2\varrho_0 \in (0, 1) \). By Proposition 4.11, it is enough to show that there exists \( c_1 > 0 \) such that for any \( t > 0 \) and \( x, y \in \mathbb{R}^d \) satisfying \( |x^i - y^i| \geq \phi^{-1}(\varrho^{-1} t) \) for each \( i \in \{1, \ldots, d\} \),
\[ p(t, x, y) \geq c_1[\phi^{-1}(t)]^{-d} \prod_{i=1}^{d} \frac{t \phi^{-1}(t)}{|x^i - y^i| \phi(|x^i - y^i|)}. \]
Consider points \( \xi(0), \xi(1), \ldots, \xi(d) \in \mathbb{R}^d \) tracing from \( x \) to \( y \) where the only difference of \( \xi(i-1) \) and \( \xi(i) \) is the value of \( i \)-th coordinate, that is,
\[ \xi(i) = x \text{ if } i \leq k, \]
\[ \xi(i) = y \text{ if } i > k, \]
For each \( k = 1, 2, \ldots, d \), consider cubes \( Q_k := Q(\xi(k), r_k) \) centered at \( \xi(k) \) with side length \( r_k := \phi^{-1}(\varrho^{-1} t)/4^{d-k} \). Since \( p(t, w, y) \geq c_0[\phi^{-1}(t)]^{-d} \) for \( w \in Q_d \) by Proposition 4.11, the semigroup property implies that
\[ p(d \cdot t, x, y) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} p(t, x, \eta(1)) p(t, \eta(1), \eta(2)) \cdots p(t, \eta(d), y) \ d\eta(1) \cdots d\eta(d) \]
\[ \geq \int_{Q_d} \cdots \int_{Q_d} p(t, x, \eta(1)) p(t, \eta(1), \eta(2)) \cdots p(t, \eta(d), y) \ d\eta(1) \cdots d\eta(d) \]
\[ \geq c_2[\phi^{-1}(t)]^{-d} \mathbb{P}^x(\tau_{Q_1} \leq d-1) \prod_{k=1}^{d-1} \inf_{\eta(k) \in Q_k} \mathbb{P}^{\eta(k)}(X_t \in Q_{k+1}). \] (4.9)
Let \( r_0 := \phi^{-1}(q^{-1}t) \). Then \( r_k/2 \geq 4^{-d}r_0 \), and (4.8) with the fact that \( q_0 = q/2 \) implies that for any \( w \in \mathbb{R}^d \),
\[
\mathbb{P}^w(\tau(w,r_k/2) \geq t/2) \geq \mathbb{P}^w(\tau(w,4^{-d}r_0) \geq t/2) = \mathbb{P}^w(\tau(w,4^{-d}r_0) \geq q_0\phi(r_0)) \geq 1/2. \tag{4.10}
\]
Let \( \overline{Q}_k := Q(\xi(k),r_k/2) \). For any \( t > 0 \), define \( \mathcal{M}^k_t := \{ \omega : X_t(\omega) \text{ hits } \overline{Q}_k \text{ by } t \} \). The strong Markov property with (4.10) and Lévy system in (2.3) imply that for any \( \eta \in Q_k \),
\[
\mathbb{P}^\eta(X_t \in Q_{k+1}) \geq 2^{-1}\mathbb{P}^\eta(\mathcal{M}^{k+1}_{t/2}) \geq c_3 \mathbb{E}^\eta \left[ \int_0^{t/2 \wedge \tau(Q_{\eta,r_k})} \int_{\Omega_{k+1}} \frac{1}{|X_s-u|\phi(|X_s-u|)} m(du)ds \right] \tag{4.11}
\]
where \( m(du) \) is the measure on \( \sum_{i=1}^d \mathbb{R} \) restricted only on each coordinate. Let \( k \in \{1, \ldots, d-1\} \). For \( w \in Q(\xi(k),2r_k) \) and \( u \in Q(\xi(k+1),2r_k) \), since \( \xi(k) = x^{k+1}, \xi(k+1) = y^{k+1} \) and \( |x^{k+1} - y^{k+1}| \geq r_k \), we have that
\[
\begin{align*}
|w^{k+1} - u^{k+1}| &\leq |x^{k+1} - y^{k+1}| + |x^{k+1} - w^{k+1}| + |y^{k+1} - u^{k+1}| \\
&= |x^{k+1} - y^{k+1}| + |\xi(k) - w^{k+1}| + |\xi(k+1) - u^{k+1}| \leq 5|x^{k+1} - y^{k+1}|.
\end{align*}
\]
Also (4.10) implies
\[
\mathbb{E}^\eta \left[ \frac{t}{2} \wedge \tau(Q_{\eta,r_k}) \right] \geq \int_0^{t/2} \int_{\Omega_{k+1}} \frac{1}{|X_s-u|\phi(|X_s-u|)} m(du)ds \geq 2^{-1}t. \tag{4.12}
\]
Therefore, (4.11)–(4.12) with (WS) and the fact that \( \phi^{-1}(q^{-1}t)/\phi^{-1}(t) \approx 1 \) imply that for any \( \eta \in Q_k \) and \( k \in \{1, \ldots, d-1\} \),
\[
\mathbb{P}^\eta(X_t \in Q_{k+1}) \geq c_4 \mathbb{E}^\eta \left[ \frac{t}{2} \wedge \tau(Q_{\eta,r_k}) \right] \frac{\phi^{-1}(t)}{|x^{k+1} - y^{k+1}|\phi(|x^{k+1} - y^{k+1}|)} \geq \frac{c_4}{2} \left( \frac{t\phi^{-1}(t)}{|x^{k+1} - y^{k+1}|} \right). \tag{4.13}
\]
In a similar way to obtain (4.13), we have that
\[
\mathbb{P}^\eta(X_t \in Q_1) \geq c_5 \left( \frac{t\phi^{-1}(t)}{|x^1 - y^1|} \right). \tag{4.14}
\]
Hence we have the lower bound for \( p(t,x,y) \) by plugging (4.13) and (4.14) into (4.9).

5. Appendix: Proof of Proposition 3.9

In this section, we present the proof of Proposition 3.9. For the convenience of notations, we use constants \( c \) instead of \( c_i, i = 1, 2, \ldots \) in proofs even the values are changed.

5.1. Preliminary estimates. We first give the definition of \( D_k \subset \mathbb{R}^d \) as in [14].

Definition 5.1.

(0) Define \( D_0 = \bigcup_{i=1, \ldots, d} \{|x^i| < 1\} \cup (-2,2)^d \).

(1) Given \( k \in \mathbb{N}, \gamma := (\gamma^1, \ldots, \gamma^d) \in \mathbb{N}_0^d \) with \( \sum_{i=1}^d \gamma^i = k \) and \( \epsilon \in \{-1, 1\}^d \), define a box (hyper-rectangle) \( D_k^\gamma \) by
\[
D_k^\gamma = \epsilon^1[2^{\gamma^1}, 2^{\gamma^1+1}] \times \epsilon^2[2^{\gamma^2}, 2^{\gamma^2+1}] \times \ldots \times \epsilon^d[2^{\gamma^d}, 2^{\gamma^d+1}].
\]
(2) Given \( k \in \mathbb{N} \) and \( \gamma \in \mathbb{N}_0^d \) with \( \sum_{i=1}^d \gamma^i = k \), define
\[
D_k^\gamma = \bigcup_{\epsilon \in \{-1,1\}^d} D_{k,\gamma,\epsilon}.
\]

(3) Given \( k \in \mathbb{N} \), define
\[
D_k = \bigcup_{\gamma \in \mathbb{N}_0^d, \sum_{i=1}^d \gamma^i = k} D_k^\gamma.
\]

Then we have the following Remark by [14, Lemma 4.2].

**Remark 5.2.** Let \( k \in \mathbb{N}_0, \gamma \in \mathbb{N}_0^d \) and \( \epsilon \in \{-1,1\}^d \).

1. Given \( k \in \mathbb{N}, \gamma \) with \( \sum_{i=1}^d \gamma^i = k \), there are \( 2^d \) sets of the form \( D_k^\gamma, \epsilon \), and \( |D_k^\gamma,\epsilon| = \prod_{i=1}^d 2^{\gamma^i} = 2^k \).

2. Given \( k \in \mathbb{N}, \epsilon \in \{-1,1\}^d \), there are \( \binom{d+k-1}{d-1} \) sets \( D_{k,\gamma,\epsilon} \) with \( \sum_{i=1}^d \gamma^i = k \).

Thus, the set \( D_k \) consists of \( 2^d \binom{d+k-1}{d-1} \) disjoint boxes.

3. \( D_k \cap D_l = \emptyset \) if \( k \neq l \) and \( \bigcup_{k \in \mathbb{N}_0} D_k = \mathbb{R}^d \).

Now we give the definitions of the shifted boxes centered at \( y_0 \). For \( y_0 \in \mathbb{R}^d \), \( t > 0 \) and \( \kappa = \phi^{-1}(t) \), let \( A_0 := y_0 + \kappa D_0 \). For \( k \in \mathbb{N}, \gamma \in \mathbb{N}_0^d \) with \( \sum_{i=1}^d \gamma^i = k \) and \( \epsilon \in \{-1,1\}^d \), we define
\[
A_{k,\gamma,\epsilon} := y_0 + \kappa D_{k,\gamma,\epsilon}, \quad A_{k,\gamma} := y_0 + \kappa D_{k,\gamma} \quad \text{and} \quad A_k := y_0 + \kappa D_k.
\]

By the definition of \( D_k \), it is easy to see that \( A_k \cap A_l = \emptyset \) for \( k \neq l \) and \( \bigcup_{k=0}^\infty A_k = \mathbb{R}^d \).

For the rest of Section 5, we assume \( l \in \{0,1,\ldots,d-1\} \), \( i_0 \in \{1,\ldots,d-l\} \) and assume \( x_0, y_0 \in \mathbb{R}^d \) such that the condition \( \mathcal{R}(i_0) \) is satisfied. For \( t > 0 \), set \( \kappa = \phi^{-1}(t) \). Then as we defined in Definition 3.7, there exist \( n_i \geq 1, i \in \{i_0, \ldots, d\} \) such that
\[
\frac{5}{4} 2^{n_i} \kappa \leq |x_0^i - y_0^i| < \frac{10}{4} 2^{n_i} \kappa \quad \text{and} \quad R_i = 2^{n_i} \kappa.
\]

Since the proofs of the following results reveal the same geometric structures as shown in [14, Lemma 4.3, Remark 4.4, Lemma 4.5, Remark 4.6], we omit them.

**Lemma 5.3.** Assume \( x_0, y_0 \in \mathbb{R}^d \) satisfy condition \( \mathcal{R}(i_0) \) for some \( i_0 \). Set \( s(j_0) := \frac{R_{j_0}}{8} \) for \( j_0 \in \{i_0, \ldots, d\} \). Then the following holds true:
\[
\bigcup_{u \in B(x_0,s(j_0))} \{ u + he^i \mid h \in \mathbb{R} \} \subset y_0 + \left( \bigotimes_{j=1}^{j_0-1} \mathcal{J}_{n_j} \times \bigotimes_{j=j_0}^d \mathcal{I}_{n_j} \right) \quad \text{if} \ i < j_0,
\]
\[
\bigcup_{u \in B(x_0,s(j_0))} \{ u + he^i \mid h \in \mathbb{R} \} \subset y_0 + \left( \bigotimes_{j=1}^{j_0-1} \mathcal{J}_{n_j} \times \bigotimes_{j=j_0}^{i-1} \mathcal{I}_{n_j} \times \mathbb{R} \times \bigotimes_{j=i+1}^d \mathcal{I}_{n_j} \right) \quad \text{if} \ i \geq j_0,
\]
where \( \mathcal{I}_{n_j} := \pm [2^{n_j} \kappa, 2^{n_j+2} \kappa] \), \( \mathcal{J}_{n_{j_0}} := \pm [0, 2^{n_{j_0}+2} \kappa] \) and \( n_{j_0}, \ldots, n_d \in \mathbb{N} \).

For \( j_0 \in \{i_0, \ldots, d\} \), define
\[
A_{k}^i := A_k \cap \bigcup_{u \in B(x_0,s(j_0))} \{ u + he^i \mid h \in \mathbb{R} \} \quad \text{for} \ k \in \mathbb{N}_0, i \in \{1, \ldots, d\}.
\]
that is, $A^{i}_k$ contains all possible points in $A_k$ when the process $X$ leaves the ball $B(x_0, s(j_0))$ by a jump in the $i$-th direction.

**Remark 5.4.** Using the above notations together with Lemma 5.3, the following observation holds true:

$$A^{i}_k \neq \emptyset \quad \implies \quad \begin{cases} k = 0 \text{ or } k \geq \sum_{j \in \{j_0, \ldots, d\}} n_j & \text{if } i < j_0, \\ k = 0 \text{ or } k \geq \sum_{j \in \{j_0, \ldots, d\} \setminus \{i\}} n_j & \text{if } i \geq j_0. \end{cases}$$

**Lemma 5.5.** Let $i \in \{1, \ldots, d\}$ and $k \in \mathbb{N}_0$. For any $z \in A^{i}_k$ and $y \in B(y_0, \frac{5}{8})$, there exists $\gamma \in \mathbb{N}_0^d$ such that

$$|z^j - y^j| \in [a_4 2^{\gamma^j} \kappa, a_5 2^{\gamma^j+1} \kappa) \quad \text{if } k \in \mathbb{N} \text{ and } j \in \{1, \ldots, d\},$$

(5.1)

for some $a_4, a_5 > 0$. Moreover, for $z \in A^{i}_k$ and $y \in B(y_0, \frac{5}{8})$, the following holds true:

$$|z^j - y^j| \in [0, a_5 2^{\gamma^j_0} \kappa) \quad \text{if } j \in \{1, \ldots, j_0 - 1\} \setminus \{i\},$$

(5.2)

and

$$|z^j - y^j| \in [a_4 2^{\gamma^j} \kappa, a_5 2^{\gamma^j+1} \kappa) \quad \text{if } j \in \{j_0, \ldots, d\} \setminus \{i\},$$

(5.3)

for some $a_4, a_5 > 0$.

**Remark 5.6.** Given $A^{i}_k$ and $B(y_0, \frac{5}{8})$, $\gamma \in \mathbb{N}_0^d$ can be chosen independently of $z \in A^{i}_k$ and $y \in B(y_0, \frac{5}{8})$.

For $s(j_0) = \frac{R_{j_0}}{8}$, $x \in B(x_0, \frac{5}{8})$ and $\tau = \tau_{B(x_0, s(j_0))}$, let

$$\Psi(k) := \mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} 1_{\{X_\tau \in A_k\}} P_{t-\tau} f(X_\tau) \right], \quad k \in \mathbb{N}_0,$$

and

$$\Psi^i(k) := \mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} 1_{\{X_\tau \in A^{i}_k\}} P_{t-\tau} f(X_\tau) \right], \quad k \in \mathbb{N}, \ i \in \{1, 2, \ldots, d\}.$$

Now we decompose the left-hand side in (3.4) according to the following:

$$\mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] = \sum_{k=0}^{\infty} \Psi(k) = \sum_{k=1}^{d} \sum_{i=1}^{d} \Psi^i(k) \quad + \quad \Psi(0)$$

$$= \sum_{k=1}^{\infty} \Psi^d(k) + \sum_{k=1}^{\infty} \Psi^{d-1}(k) + \ldots + \sum_{k=1}^{\infty} \Psi^1(k) + \Psi(0).$$

Moreover, Remark 5.4 yields that the above decomposition can be separated as follows:

$$\mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right]$$

$$= \sum_{k=0}^{\infty} \Psi^d(k) + \sum_{k=0}^{\infty} \Psi^{d-1}(k) + \ldots + \sum_{k=0}^{\infty} \Psi^{j_0}(k)$$

$$+ \sum_{k=0}^{\infty} (\Psi^{j_0-1}(k) + \ldots + \Psi^1(k)) + \Psi(0)$$

(5.4)

$$= \sum_{i=j_0}^{d} S(i) + \sum_{i=1}^{j_0-1} T(i) + \Psi(0),$$

where $S(i) := \sum_{k=N(j_0,d)-n_i}^{\infty} \Psi^i(k)$ and $T(i) := \sum_{k=N(j_0,d)}^{\infty} \Psi^i(k)$. 


Recall from (3.7) and (3.8) that \( \mathcal{M}(\delta) := \frac{\phi(\tau)}{\mathbb{E}^\phi(2^\alpha)} \) for \( \delta \in \mathbb{Z} \), and there is \( c > 0 \) such that
\[
c^{-1}2^{-\delta(\tau + 1)} \leq \mathcal{M}(\delta) \leq c2^{-\delta(\tau + 1)} \quad \text{for } \delta \in \mathbb{Z}_+.
\]

**Remark 5.7.** Here we give useful results for obtaining the upper bounds of \( \Psi(0), \Psi^i(k) \), and the **Conclusion** in the next subsection. Assume \( x_0, y_0 \in \mathbb{R}^d \) satisfy the condition \( \mathcal{R}(i_0) \) for some \( i_0 \), so that \( |x_j^i - y_0^i| \geq \frac{\mathcal{R}_j}{2^i} \) for \( j \in \{i_0, \ldots, d\} \). Recall that for \( l \in \{0, \ldots, d - 1\} \), consider \( j_0 \in \{i_0, \ldots, d - l\} \), \( s(j_0) = \frac{R_j}{8} \), and \( \tau = \tau_{B(x_0, s(j_0))} \).

1. For any \( w \in B(x_0, s(j_0)) \), \( z \in \mathbb{R}^d \) satisfying \( |z^i - y_0^i| \in [2^{\gamma_i} \kappa, 2^{\gamma_i + 1} \kappa) \) with \( \gamma_i + 1 \leq n_i \) or \( |z^i - y_0^i| \in [0, 2\kappa) \cup [2^{\gamma_i - 1} \kappa, 2^{\gamma_i + 1} \kappa) \) for some \( m + 1 \leq n_i \), and \( i \geq j_0 \),
\[
|w^i - z^i| \geq |x_0^i - y_0^i| - |w^i - x_0^i| + |z^i - y_0^i| \geq 5R_i/4 - R_i/8 = 2^{\gamma_i} \kappa = R_i / 8.
\]
For \( k \geq 1 \), let \( I_k^i := \{ \ell \in \mathbb{R} : |\ell - y_0^i| \in [2^{\gamma_i} \kappa, 2^{\gamma_i + 1} \kappa), \gamma_i + 1 \leq n_i \} \). Then for any \( x \in B(x_0, \frac{\kappa}{5}) \) and \( i \geq j_0 \), (WS) implies that
\[
\mathbb{E}^x \left[ \int_0^{t/2}\! \! \int \frac{1}{|X_s^i - \ell| \phi(|X_s^i - \ell|)} \, d\ell \right] \leq \frac{ct}{R_i \phi(R_i)} \cdot 2^{\gamma_i} \kappa = c2^{\gamma_i} \mathcal{M}(n_i). \tag{5.5}
\]
For \( k = 0 \), let \( I_{0,0}^i := \{ \ell \in \mathbb{R} : |\ell - y_0^i| \in [0, 2\kappa) \} \) and \( I_{0,i,m}^i := \{ \ell \in \mathbb{R} : |\ell - y_0^i| \in [2^{m-1} \kappa, 2^{m+1} \kappa) \} \), \( m \in \{1, \ldots, n_i - 1\} \). Then for any \( x \in B(x_0, \frac{\kappa}{5}) \) and \( i \geq j_0 \), (WS) implies that
\[
\mathbb{E}^x \left[ \int_0^{t/2}\! \! \int \frac{1}{|X_s^i - \ell| \phi(|X_s^i - \ell|)} \, d\ell \right] \leq \frac{ct}{R_i \phi(R_i)} \cdot 2^{m_\kappa} = c2^{m} \mathcal{M}(n_i). \tag{5.6}
\]
for any \( m \in \{0, 1, \ldots, n_i - 1\} \).

2. Let \( x \in B(x_0, \frac{\kappa}{5}) \). For any \( y \in B(x, s(j_0)/2) \), since \( s(j_0) = 2^n \kappa \geq 2^{-2} \kappa \),
\[
|y - x_0| \leq |y - x| + |x - x_0| \leq (2^n \kappa + 8^{-1}) \kappa \leq 2^n \kappa
\]
so that \( B(x, s(j_0)/2) \subset B(x_0, s(j_0)) \). Therefore, by Corollary 2.10 with \( r := 2^n \kappa - 4 \geq 2^{-3} \) and (WS), we have that for \( x \in B(x_0, \frac{\kappa}{5}) \) and \( t > 0 \),
\[
\mathbb{P}^x(\tau \leq t/2, X_\tau \in A_k^i) \leq \mathbb{P}^x(\tau \leq t/2) \leq \mathbb{P}^x(\tau_{B(x, r\kappa)} \leq t/2) \leq \frac{ct}{\phi(2^n \kappa)} = c\mathcal{M}(n_{j_0})2^n \kappa. \tag{5.7}
\]

3. Since \( |x_j^i - y_0^i| \geq 2^n \kappa \) for \( j \in \{j_0, \ldots, d\} \), we note that
\[
\prod_{j=j_0}^{d-1} \left( \frac{t^\phi^{-1}(t)}{|x_j^i - y_0^i| \phi(|x_j^i - y_0^i|)} \right)^q \prod_{j=d-l}^{d-1} \left( \frac{t^\phi^{-1}(t)}{|x_j^i - y_0^i| \phi(|x_j^i - y_0^i|)} \right)^{-l} \leq \mathcal{F}(l)
\]
where \( \mathcal{F}(l) := \prod_{j=j_0}^{d-l} \mathcal{M}(n_j)^q \prod_{j=d-l+1}^{d} \mathcal{M}(n_j) \).

5.2. **Estimates for** \( \Psi(0), S(i), i \in \{j_0, \ldots, d\} \) **and** \( T(i), i \in \{1, \ldots, j_0 - 1\} \).

**Estimate of** \( \Psi(0) \). First, we obtain the upper bound of
\[
P_{t,x} f(z) = \int_{B(x_0, \frac{\kappa}{5})} p(t - \tau, z, y) f(y) \, dy
\]
for $z \in A_0^i$ and $t/2 \leq t - \tau \leq t$. By (WS), we note that
\[
\phi^{-1}(t) \asymp \phi^{-1}(t - \tau),
\]
and $\left(1 \wedge \frac{\phi^{-1}(t)}{1/|\phi(t)|}\right) \leq 1$. For $z \in A_0^i$ and $y \in B(y_0, \kappa/8)$, by (5.2) and (5.3), the following holds:
\[
\begin{align*}
|z^j - y^j| & \in [0, a_5 2^n \kappa) \\
|z^j - y^j| & \in [a_4 2^n \kappa, a_5 2^{n+1} \kappa)
\end{align*}
\]
for some $a_4, a_5 > 0$. Let $s_j := |z^j - y^j|$ for $j \in \{1, \ldots, d\}$. We first consider the case $d - l < i$. Since \((H_q^n)\) (to be precise, \((H_q^n)'\)) yields that for $t/2 \leq t - \tau \leq t$,
\[
p(t - \tau, z, y) \leq c[\phi^{-1}(t)]^{-d}
\]
\[
\left(\prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j)^q \prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j)\right) \quad \text{if } s_i < 2^2 \kappa,
\]
\[
\left(\prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j)^q \prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j)\right) \quad \text{if } s_i \geq 2^{n-1} \kappa.
\]
Otherwise, there exists a constant $m \in \{1, \ldots, n_i - 1\}$ such that $s_i \asymp m \kappa$ and
\[
p(t - \tau, z, y) \leq c[\phi^{-1}(t)]^{-d}
\]
\[
\left(\prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j)^q \prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j)\right) \quad \text{if } m < n_{d-l},
\]
\[
\left(\prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j)^q \prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j)\right) \quad \text{if } n_{d-l} \leq m < n_i.
\]
For the rest of cases (that is, $j_0 \leq i \leq d - l$ and $i < j_0$), we obtain the upper bounds in a similar way. If $i \leq d - l$, \((H_q^n)\) (to be precise, \((H_q^n)'\)) yields that for $t/2 \leq t - \tau \leq t$,
\[
p(t - \tau, z, y) \leq c[\phi^{-1}(t)]^{-d}
\]
\[
\left(\prod_{j \in \{1, \ldots, d-l\} \setminus \{i\}} \mathcal{H}_j(n_j)^q \prod_{j \in \{d-l+2, \ldots, d\}} \mathcal{H}_j(n_j)\right) \quad \text{if } s_i < 2^2 \kappa,
\]
\[
\left(\prod_{j \in \{1, \ldots, d-l\} \setminus \{i\}} \mathcal{H}_j(n_j)^q \prod_{j \in \{d-l+2, \ldots, d\}} \mathcal{H}_j(n_j)\right) \quad \text{if } s_i \geq 2^{n-1} \kappa,
\]
and otherwise, there exists a constant $m \in \{1, \ldots, n_i - 1\}$ such that $s_i \asymp m \kappa$ and
\[
p(t - \tau, z, y) \leq c[\phi^{-1}(t)]^{-d}
\]
\[
\prod_{j \in \{j_0, \ldots, d-l\} \setminus \{i\}} \mathcal{H}_j(n_j)^q \prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{H}_j(n_j).
\]
If \( i < j_0 \), we have that
\[
p(t - \tau, z, y) \leq c[\phi^{-1}(t)]^{-d} \prod_{j \in \{j_0, \ldots, d - l\}} \mathfrak{M}(n_j)^q \prod_{j \in \{d-l+1, \ldots, d\}} \mathfrak{M}(n_j).
\]

Recall that for \( l \in \{1, \ldots, d\} \) and \( j_0 \in \{i_0, \ldots, d - l\} \),
\[
F_{j_0}(l) = \prod_{j = j_0}^{d-l} \mathfrak{M}(n_j)^q \prod_{j = d-l+1}^{d} \mathfrak{M}(n_j).
\]

Altogether, \((H_0^i)\) yields for \( t/2 \leq t - \tau \leq t\) and \( z \in A_0^i \), we obtain the upper bound of \( P_{t-\tau} f(z) \) according to the point \( z \in A_0^i \).

For \( z \in A_0^i \) with \( i \geq j_0 \) and \( |z^i - y_0^i| < 2\kappa \), by (5.10) and (5.12),
\[
P_{t-\tau} f(z) = \int_{B(y_0, \frac{\tau}{2})} p(t - \tau, z, y) f(y) dy \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0}(l)
\]
\[
\leq c[\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0}(l) \begin{cases}
\mathfrak{M}(n_{d-l})^{-q+1}\mathfrak{M}(n_i)^{-1} \mathbb{1}_{\{z^1 - y_0^1 \leq 2\kappa\}} & \text{if } d - l < i, \\
\mathfrak{M}(n_i)^{-q} \mathbb{1}_{\{z^1 - y_0^1 \leq 2\kappa\}} & \text{if } j_0 \leq i \leq d - l.
\end{cases}
\] (5.14)

If \( z \in A_0^i \) with \( i \geq j_0 \) and \( 2\kappa \leq |z^i - y_0^i| < 2^m \kappa \), there exists \( m \in \{1, \ldots, n_i - 1\} \) such that 
\( 2^{m-1}\kappa \leq |z^i - y_0^i| < 2^m \kappa \). So (5.11) and (5.13) imply that
\[
P_{t-\tau} f(z) = \int_{B(y_0, \frac{\tau}{2})} p(t - \tau, z, y) f(y) dy \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 \cdot F_{j_0}(l). \] (5.16)

For the rest of cases, that is, if \( z \in A_0^i \) with \( i < j_0 \),
\[
P_{t-\tau} f(z) = \int_{B(y_0, \frac{\tau}{2})} p(t - \tau, z, y) f(y) dy \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 \cdot F_{j_0}(l). \] (5.17)

Let us estimate \( \Psi(0) \). By (2.3), (5.6) and (5.14), we first observe that
\[
\mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_\tau \in \mathcal{A}_0^i, i \geq j_0, |X_\tau^i - y_0^i| \leq 2\kappa\}} P_{t-\tau} f(X_\tau) \right] \]
\[
\leq c[\phi^{-1}(t)]^{-d} \|f\|_1 \cdot F_{j_0}(l) \begin{cases}
\mathfrak{M}(n_{d-l})^{-q+1} & \text{if } d - l < i, \\
\mathfrak{M}(n_i)^{-q+1} & \text{if } j_0 \leq i \leq d - l.
\end{cases}
\]

The last inequality holds since \( q \leq 1 \) and \( \delta \to \mathfrak{M}(\delta) \) is decreasing. Note that \( 2^m \leq \mathfrak{M}(m)^{-1+2\kappa} \) by (3.8). Then (2.3), (5.6) and (5.15) imply that for any \( m \in \{1, \ldots, n_i - 1\},
\[
\mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_\tau \in \mathcal{A}_0^i, i \geq j_0, 2^{m-1}\kappa \leq |X_\tau^i - y_0^i| \leq 2^m \kappa\}} P_{t-\tau} f(X_\tau) \right] \]
\[
\begin{align*}
N \leq c[\phi^{-1}(t)]^{-d} \|f\|_{1, F_{j_0}(l)} \begin{cases} 
\mathcal{N}(m)^{-\frac{1}{m}} \mathcal{N}(n_{d-i})^{-q+1} & \text{if } d - l < i \text{ and } m < n_{d-l}, \\
\mathcal{N}(m)^{-1} \mathcal{N}(n_i)^{-q+1} & \text{if } j_0 \leq i \leq d - l, \\
\mathcal{N}(m)^{-\frac{1}{m}} \mathcal{N}(n_{d-i})^{-q+1} & \text{if } d - l < i \text{ and } m < n_{d-l}, \\
\mathcal{N}(n_{j_0})^q \mathcal{N}(n_{d-i})^{-q+1} & \text{if } d - l < i \text{ and } n_{d-l} \leq m < n_i, \\
\mathcal{N}(m)^{-\frac{1}{m}} \mathcal{N}(n_{j_0})^{q} \mathcal{N}(n_i)^{-q+1} & \text{if } j_0 \leq i \leq d - l.
\end{cases}
\end{align*}
\]

For the rest of cases, by (3.8), (5.7) and (5.16), we have that
\[
\begin{align*}
\mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_{\tau} \in A_0^i, i \geq j_0, |X_{\tau} - y_0| < 2m^{1+\kappa}\}} \mathcal{P}_{t-\tau} f(X_\tau) \right] \\
\quad \leq c[\phi^{-1}(t)]^{-d} \|f\|_{1, F_{j_0}(l)} \begin{cases} 
\mathcal{N}(n_{j_0})^{\frac{1}{m}+q} & \text{if } q \in \left[ 0, \frac{1}{1+\sqrt{q}} \right), \\
\mathcal{N}(n_{j_0}) & \text{if } q \in \left( \frac{1}{1+\sqrt{q}}, 1 \right).
\end{cases}
\end{align*}
\] (5.18)

Therefore, by (5.17)–(5.19), we obtain
\[
\begin{align*}
\Psi(0) = \sum_{i=1}^{d} \Psi^i(0) \leq c[\phi^{-1}(t)]^{-d} \|f\|_{1, F_{j_0}(l)} \begin{cases} 
\mathcal{N}(n_{j_0})^{\frac{1}{m}+q} & \text{if } q \in \left[ 0, \frac{1}{1+\sqrt{q}} \right), \\
\mathcal{N}(n_{j_0}) & \text{if } q \in \left( \frac{1}{1+\sqrt{q}}, 1 \right).
\end{cases}
\end{align*}
\] (5.20)

Estimates of \(S(i) := \sum_{k=N(j_0,d)-n_i}^{k} \Psi^i(k)\) for \(i \in \{j_0, \ldots, d\}\). Let \(l \in \{0, 1, \ldots, d - 1\}\), \(i_0 \in \{1, \ldots, d - l\}\) and \(j_0 \in \{i_0, \ldots, d - l\}\). We follow the strategy of the estimate for \(\Psi(0)\). For \(z \in A_k^i\) and \(y \in B(y_0, \kappa/8)\), let \(s_j := |z^j - y^j|\) for \(j \in \{1, \ldots, d\}\). Note that
\[
s_j \asymp 2^{\gamma^j} \kappa \text{ for } j \in \{1, \ldots, j_0 - 1\} \cup \{i\} \text{ and } s_j \asymp 2^{\gamma^j} \kappa \text{ for } j \in \{j_0, \ldots, d\} \setminus \{i\}\] (5.21)
by (5.1) and (5.3).

The index \(\gamma = (\gamma^1, \ldots, \gamma^d) \in \mathbb{N}_0^d\) of \(A_k^i\) is determined when the jump at \(\tau\) from the ball \(B(x_0, s(j_0))\) and it is independent of the choice of the elements \(z \in A_k^i\) and \(y \in B(y_0, \kappa/8)\), as mentioned in Remark 5.6. Hence, \((\mathcal{H}_q^i)\), together with (5.9) and (5.21), yields that for \(t/2 \leq t - \tau \leq t\) and \(i \leq d - l\),
\[
\begin{align*}
p(t - \tau, z, y) & \leq c[\phi^{-1}(t)]^{-d} \prod_{j \in \{1, \ldots, d - l\}} \frac{t \phi^{-1}(t)}{s_j \phi(s_j)} \prod_{j \in \{d - l+1, \ldots, d\}} \frac{t \phi^{-1}(t)}{s_j \phi(s_j)} \\
& \leq c[\phi^{-1}(t)]^{-d} \prod_{j \in \{1, \ldots, j_0 - 1\} \cup \{i\}} \left( \frac{\phi(\kappa)}{2^{\gamma^j} \phi(2^{\gamma^j} \kappa)} \right)^q \prod_{j \in \{j_0, \ldots, d - l\} \setminus \{i\}} \left( \frac{\phi(\kappa)}{2^{n_j} \phi(2^{n_j} \kappa)} \right)^q \\
& \quad \cdot \prod_{j \in \{d - l+1, \ldots, d\}} \left( \frac{\phi(\kappa)}{2^{n_j} \phi(2^{n_j} \kappa)} \right) \\
& = c[\phi^{-1}(t)]^{-d} \prod_{j \in \{1, \ldots, j_0 - 1\} \cup \{i\}} \mathcal{N}(\gamma^j)^q \prod_{j \in \{j_0, \ldots, d - l\} \setminus \{i\}} \mathcal{N}(n_j)^q \prod_{j \in \{d - l+1, \ldots, d\}} \mathcal{N}(n_j). \tag{5.22}
\end{align*}
\]
Similarly, for $t/2 \leq t - \tau \leq t$, the case $i > d - l$ can be dealt with accordingly,

$$p(t - \tau, z, y) \leq c[\phi^{-1}(t)]^{-d} \prod_{j \in \{1, \ldots, j_{j_{0-1}}\}} \mathcal{N}(\gamma^j)^q \prod_{j \in \{j_{0-1}, \ldots, l\}} \mathcal{N}(n_j)^q \cdot \mathcal{N}(\gamma^i) \prod_{j \in \{d-l+1, \ldots, d\}\{i\}} \mathcal{N}(n_j).$$

Since $\delta \to \mathcal{N}(\delta)$ is decreasing, for $t/2 \leq t - \tau \leq t$ and $z \in A_k^i \cap A_{k,\gamma}$, we have that

$$P_{t-\tau}f(z) = \int_{B(y_0, \frac{t}{\tau})} p(t - \tau, z, y) f(y) dy \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 \cdot F_{j_{0-1}}(l) \prod_{j \in \{1, \ldots, j_{j_{0-1}}\}} \mathcal{N}(\gamma^j)^q \left\{ \begin{array}{ll}
\mathcal{N}(\gamma^i)^q \mathcal{N}(n_i)^{-q} & \text{if } \gamma^i < n_i, \\
\mathcal{N}(\gamma^i)^q \mathcal{N}(n_i)^{-1} & \text{if } \gamma^i \geq n_i.
\end{array} \right. \quad (5.23)$$

The last inequality is due to $q < 1$ and the fact that $\delta \to \mathcal{N}(\delta)$ is decreasing.

For any $a, b \in N_0$, we define

$$\Upsilon(a, b) := \sum_{j=a}^{b} \gamma^j \text{ if } a \leq b, \text{ and otherwise } \Upsilon(a, b) := 0.$$  

Recall the definition of $A_k$ so that $k = \sum_{j=1}^{d} \gamma^j$ and for $z \in A_k^i$, there exists $\gamma \in N_0^d$ such that $z \in A_{k,\gamma}$ with $\gamma^j = n_j$ or $n_j + 1$ for $j \in \{j_0, \ldots, d\}\{i\}$ (see Lemma 5.3). Therefore,

$$\Upsilon(1, j_0 - 1) + \mathcal{N}(j_0, d) - n_i + \gamma^i \leq k \leq \Upsilon(1, j_0 - 1) + \mathcal{N}(j_0, d) - n_i + \gamma^i + d,$$

and so that

$$\Upsilon(1, j_0 - 1) + \gamma^i - n_i \geq k - \mathcal{N}(j_0, d) - d \quad \text{and} \quad \gamma^i - n_i \leq k - \mathcal{N}(j_0, d). \quad (5.24)$$

Now decompose $S(i)$ as follows:

$$S(i) = \sum_{k=\mathcal{N}(j_0, d) - n_i}^{\infty} \Psi(k) \mathbb{1}_{\{\gamma^i < n_i\}} + \sum_{k=\mathcal{N}(j_0, d) - n_i}^{\infty} \Psi(k) \mathbb{1}_{\{\gamma^i \geq n_i\}} =: I + II.$$

For $I$, by (2.3), (5.5) and (5.23),

$$\mathbb{E}^x \left[ \mathbb{1}_{\{\gamma \geq t/2\}} \mathbb{1}_{\{X_{\tau} \in A_k^i\}} P_{t-\tau}f(X_{\tau}) \mathbb{1}_{\{\gamma^i < n_i\}} \right] \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0}(l) \cdot \mathcal{N}(\gamma^i)^q \mathcal{N}(\gamma^i)^q \mathcal{N}(n_i)^{-q} \mathbb{E}^x \left[ \int_0^{t/2} \int_{I_k^i} \frac{\mathbb{1}_{\{\gamma^i < n_i\}}}{|X^i_s - z^i| \phi(|X^i_s - z^i|)} dz^i ds \right] \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0}(l) \prod_{j \in \{1, \ldots, j_0-1\}} \mathcal{N}(\gamma^j)^q \mathcal{N}(n_i)^{-q} \mathcal{N}(n_i)^{1-q} \gamma^i. \quad (5.25)$$

By (3.8) and (5.24), we have that

$$\prod_{j \in \{1, \ldots, j_0-1\}} \mathcal{N}(\gamma^j)^q \mathcal{N}(\gamma^i)^q \mathcal{N}(n_i)^{1-q} \mathcal{N}(n_i)^{1-q} \gamma^i \leq c2^{-((\Upsilon(1, j_0 - 1) + \gamma^i)(d+1)q)} \gamma^i \mathcal{N}(n_i)^{1-q} \leq c2^{-((k - \mathcal{N}(j_0, d) + n_i)(d+1)q)} \gamma^i \mathcal{N}(n_i)^{1-q} \leq c2^{-((k - \mathcal{N}(j_0, d) + n_i)((d+1)q-1)) \mathcal{N}(n_i)^{1-q}}.$$
where

\[
\sum_{k=N(j_0,d)-n_i}^{N(j_0,d)-1} 2^{-k-N(j_0,d)+n_i((\alpha+1)q-1)} \mathcal{M}(n_i)^{1-q}
\]

\[
\leq c \begin{cases} 
2n_0(1-(\alpha+1)q) \mathcal{M}(n_i)^{1-q} & \text{if } q \in [0, \frac{1}{1+\alpha}) \\
\mathcal{M}(n_i)^{1-q} & \text{if } q \in (\frac{1}{1+\alpha}, 1). 
\end{cases}
\]

Similarly, by (3.8) and (5.24), we have that

\[
\prod_{j \in \{1, \ldots, j_0-1\}} \mathcal{M}(\gamma^j)^q \mathcal{M}(n_i)^{1-q} 2^{\gamma^j} \mathcal{M}(n_i)^{1-q} 
\leq c 2^{-(k-N(j_0,d)+n_i)((\alpha+1)q-1)} \mathcal{M}(n_i)^{1-q},
\]

where

\[
\sum_{k=N(j_0,d)}^{\infty} 2^{-(k-N(j_0,d)+n_i)((\alpha+1)q-1)} \mathcal{M}(n_i)^{1-q}
\]

\[
\leq c \begin{cases} 
\mathcal{M}(n_i)^{(\alpha+1)-1-(\alpha+1)q} \cdot \mathcal{M}(n_i)^{1-q} = \mathcal{M}(n_i)^{\frac{2n_0}{\gamma^j}} & \text{if } q \in [0, \frac{1}{1+\alpha}), \\
\mathcal{M}(n_i)^{1-q} & \text{if } q \in (\frac{1}{1+\alpha}, 1). 
\end{cases}
\]

Therefore, by (5.25), (5.26) and (5.27), we conclude that

\[
1 \leq \left( \sum_{k=N(j_0,d)-n_i}^{N(j_0,d)-1} + \sum_{k=N(j_0,d)}^{\infty} \right) \mathbb{E}^x \left[ 1_{\{\tau \geq t/2\}} 1_{\{X_\tau \in A^t_k\}} P_{t-\tau} f(X_\tau) \right] 1_{\{\gamma^i < n_i\}} 
\leq c [\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0} (l) \begin{cases} 
\mathcal{M}(n_i)^{\frac{2n_0}{\gamma^j}} & \text{if } q \in [0, \frac{1}{1+\alpha}), \\
\mathcal{M}(n_i)^{1-q} & \text{if } q \in (\frac{1}{1+\alpha}, 1). 
\end{cases}
\]

Now we consider the case II. By (2.3), (5.7) and (5.23), we have that

\[
\mathbb{E}^x \left[ 1_{\{\tau \geq t/2\}} 1_{\{X_\tau \in A^t_k\}} P_{t-\tau} f(X_\tau) \right] 1_{\{\gamma^i \geq n_i\}} 
\leq c [\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0} (l) \prod_{j \in \{1, \ldots, j_0-1\}} \mathcal{M}(\gamma^j)^q \mathcal{M}(n_i)^{1-q} \cdot \mathbb{P}^x (\tau \leq t/2, X_\tau \in A^t_k) 1_{\{\gamma^i \geq n_i\}} 
\leq c [\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0} (l) \prod_{j \in \{1, \ldots, j_0-1\}} \mathcal{M}(\gamma^j)^q \mathcal{M}(n_i)^{-q} \mathcal{M}(n_{j_0}) 2^{n_{j_0}}.
\]

By (3.8) and (5.24), we have that

\[
\prod_{j \in \{1, \ldots, j_0-1\}} \mathcal{M}(\gamma^j)^q \mathcal{M}(n_i)^{-q} \mathcal{M}(n_{j_0}) 2^{n_{j_0}} \leq c 2^{-(\gamma(1,j_0-1)+\gamma(2)+1)q} \mathcal{M}(n_i)^{-q} \mathcal{M}(n_{j_0}) 2^{n_{j_0}}
\]

\[
\leq c 2^{-(k-N(j_0,d)+n_i)((\alpha+1)q-1)} \mathcal{M}(n_i)^{-q} \mathcal{M}(n_{j_0}) 2^{n_{j_0}},
\]

where

\[
\sum_{k=N(j_0,d)}^{\infty} 2^{-(k-N(j_0,d)+n_i)((\alpha+1)q-1)} \mathcal{M}(n_i)^{-q} \mathcal{M}(n_{j_0}) 2^{n_{j_0}}
\]
\[ \leq c2^{-n_1(\zeta+1)q} \mathcal{N}(n_i)^{-q} \mathcal{N}(n_{j_0}) 2^{n_{j_0}} \]
\[ \leq c2^{n_{j_0}(1-\zeta)q} \mathcal{N}(n_i)^{-q} \mathcal{N}(n_{j_0}) \]
\[ \leq c \left\{ \begin{array}{ll}
\mathcal{N}(n_{j_0})^{-(\zeta+1)(1-\zeta)q} \mathcal{N}(n_i)^{-q} \mathcal{N}(n_{j_0}) & \text{if } q \in \left[0, \frac{1}{\zeta+2}\right), \\
\mathcal{N}(n_i)^{1-q} & \text{if } q \in \left(\frac{1}{\zeta+2}, 1\right). 
\end{array} \right. \]

For the last inequality, we have used the fact that \( \mathcal{N}(n_{j_0}) \leq \mathcal{N}(n_i) \). Therefore,
\[
\Pi \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0}(l) \left\{ \begin{array}{ll}
\mathcal{N}(n_{j_0})^{\frac{\zeta}{\zeta+1}} & \text{if } q \in \left[0, \frac{1}{\zeta+2}\right), \\
\mathcal{N}(n_i)^{1-q} & \text{if } q \in \left(\frac{1}{\zeta+2}, 1\right). 
\end{array} \right. \tag{5.29} \]
Since \( j \to n_j \) is increasing and \( q < 1 \), (5.28) and (5.29) imply that for any \( i \in \{j_0, \ldots, d-l-1, \ldots, d\} \),
\[
S(i) \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0+1}(l) \left\{ \begin{array}{ll}
\mathcal{N}(n_{j_0})^{\frac{\zeta}{\zeta+1}+q} & \text{if } q \in \left[0, \frac{1}{\zeta+2}\right), \\
\mathcal{N}(n_{j_0}) & \text{if } q \in \left(\frac{1}{\zeta+2}, 1\right). 
\end{array} \right. \tag{5.30} \]

**Estimates of \( T(i) \) for \( i \in \{1, 2, \ldots, j_0 - 1\} \).** Let \( l \in \{0, 1, \ldots, d-1\}, i_0 \in \{1, \ldots, d-l\} \) and \( j_0 \in \{i_0, \ldots, d-l\} \). Analogous to the proof of (5.22), we apply (5.1)–(5.3) together with \( \langle H_q^l \rangle \) (to be precise \( \langle H_q^l \rangle \)) in order to prove that for \( t/2 \leq \tau \leq t \), \( y \in B(y_0, \frac{\mathcal{N}}{\tau}) \) and \( z \in \Lambda_k \),
\[ p(t-\tau, z, y) \leq c[\phi^{-1}(t)]^{-d} \prod_{j \in \{1, \ldots, j_0 - 1\}} \mathcal{N}(\gamma_j)^q \prod_{j \in \{j_0, \ldots, d-l\}} \mathcal{N}(n_j)^q \prod_{j \in \{d-l+1, \ldots, d\}} \mathcal{N}(n_j). \]
Hence
\[
P_{t-\tau}f(z) = \int_{B(y_0, \frac{\mathcal{N}}{\tau})} p(t-\tau, z, y) f(y) dy \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0}(l) \prod_{j \in \{1, \ldots, j_0 - 1\}} \mathcal{N}(\gamma_j)^q, \tag{5.31} \]
where \( F_{j_0}(l) \) is defined in (5.8). Regarding the definition of \( A_k \), recall \( k = \sum_{j=1}^d \gamma_j \). For \( z \in \Lambda_k \), there exists \( \gamma \in \mathcal{N}_0^d \) such that \( z \in A_k, \gamma \) with \( \gamma_j = n_j \) or \( n_j + 1 \) for \( j \in \{j_0, \ldots, d\} \) (see Lemma 5.3), hence,
\[ \mathcal{N}(1, j_0 - 1) + \mathcal{N}(j_0, d) \leq k \leq \mathcal{N}(1, j_0 - 1) + \mathcal{N}(j_0, d) + d. \]
By (3.8), since
\[ \prod_{j \in \{1, \ldots, j_0 - 1\}} \mathcal{N}(\gamma_j)^q \leq c2^{-\gamma(1, j_0 - 1)(\zeta+1)q} \leq c2^{(k-\mathcal{N}(j_0, d))(\zeta+1)q}, \]
combining (5.7) and (5.31), we have that
\[ T(i) = \sum_{k \geq \mathcal{N}(j_0, d)} \mathbb{E}^z \left[ 1_{\{\tau \leq t/2\}} 1_{\{X_{\tau} \in \Lambda_k\}} P_{t-\tau}f(X_{\tau}) \right] \]
\[ \leq c[\phi^{-1}(t)]^{-d} \|f\|_1 F_{j_0}(l) \mathcal{N}(n_{j_0}) 2^{n_{j_0}} \sum_{k \geq \mathcal{N}(j_0, d)} c2^{(k-\mathcal{N}(j_0, d))(\zeta+1)q} \]
\[ \leq c[\phi^{-1}(t)]^{-d} \| f \|_1 F_{j_0}(l) \mathfrak{N}(n_{j_0}) 2^{n_{j_0}} \]
\[ \leq c[\phi^{-1}(t)]^{-d} \| f \|_1 F_{j_0+1}(l) \mathfrak{N}(n_{j_0}) \left( \frac{d}{\alpha - 1} + q \right). \]  

(5.32)

The last inequality holds by (3.8) with \( n_{j_0} \geq 1 \).

**Conclusion.** Finally, by the estimates (5.20), (5.30) and (5.32) in the representation (5.4), we obtain the upper bound of (5.4) as follows:

\[ \mathbb{E}^x \left[ 1_{\{ \tau \leq t/2 \}} P_{t-\tau} f(X_\tau) \right] = \sum_{i=1}^{j_0-1} T(i) + \sum_{i=j_0}^d S(i) + \Phi(0) \]
\[ \leq c[\phi^{-1}(t)]^{-d} \| f \|_1 F_{j_0+1}(l) \begin{cases} \mathfrak{N}(n_{j_0}) \left( \frac{d}{\alpha - 1} + q \right) & \text{if } q \in [0, \frac{1}{1+\alpha}) , \\ \mathfrak{N}(n_{j_0}) & \text{if } q \in (\frac{1}{1+\alpha}, 1). \end{cases} \]

This proves Proposition 3.9 by (WS), (3.2), (3.7) and Remark 5.7 (3).

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