Acceleration of particles by black holes—a general explanation

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Abstract

We give a simple and general explanation for the effect of unbound acceleration of particles by black holes. It is related to the fact that the scalar product of a timelike vector of the four-velocity of an ingoing particle and the lightlike horizon generator tends to zero in some special cases, so the condition of ‘motion forward in time’ is marginally satisfied. In this sense, an ingoing particle with a special relation between parameters imitates the property of infinite redshift typical of any outgoing particle near the future horizon of a black hole. We check this assertion using the Reissner–Nordström and rotating axially symmetric metrics as examples.

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1. Introduction

Recently, an interesting observation was made in [1] about acceleration of particles near the horizon of a rotating black hole to unlimited energies $E_{\text{cm}}$ in the centre of mass frame. In this sense, a black hole can act as a cosmic supercollider, so in its vicinity new physics is expected at the Planck scale. A series of papers followed where details of this process were studied [2–11] and its generalization [12] and extension to charged nonrotating black holes [13] were suggested. The goal of this work is to give a general and comprehensive explanation to this interesting effect. Rather surprisingly, it turns out that such an explanation is very simple and relies not on the details of theory but on the mutual properties of particles and a light cone near the future horizon of a black hole. Thus, we generalize previous observations and elucidate the underlying reason for the effect for the variety of metrics previously considered.
2. Basic formulas

It would seem that the effect connected with acceleration of particles requires necessarily detailed analysis of their equations of motion, which is the approach developed in previous works [1–13]. Instead, in this work we focus attention on what happens to the four-velocity of a particle with respect to its local light cone in the immediate vicinity of the horizon. Let us consider the collision of two particles near the future horizon of a black hole. In doing so, one should clearly distinguish between two different cases: (1) particles move in the opposite directions (towards the horizon and away from it), (2) both particles move towards the horizon. Actually, the first case was discussed in [14] (although the corresponding condition was not explicitly pronounced there) a long time ago. The second case is discussed in the series of aforementioned papers [1–13].

We will use the following geometric construction. Let us introduce in the point P under consideration and its vicinity the tetrad with lightlike vectors $l^\mu$, $N^\mu$ and spacelike vectors $a^\mu$, $b^\mu$ orthogonal to them. Here, the vectors $l^\mu$, $N^\mu$ are normalized, say, as $l_\alpha N^\alpha = -1$. Then,

$$g_{\alpha\beta} = -l_\alpha N^\beta - l_\beta N^\alpha + \sigma_{\alpha\beta}$$  \hspace{1cm} (1)

where $\sigma_{\alpha\beta} = a_\alpha b_\beta + a_\beta b_\alpha$, $N^\mu l_\alpha = N^\alpha = 0$ (see, for example, the textbook [15]). We assume that it is the vector $l^\mu$ that becomes the generator of the future horizon. In general, we can use the decomposition of the four-velocity $u^\mu$ in the form

$$u_\mu = \frac{l_\mu}{\alpha_i} + \beta_i N^\mu + s_\mu^i, \quad s_\mu^i = A_i a^\mu + B_i b^\mu$$  \hspace{1cm} (2)

where $i = 1, 2$ labels the particles and $\alpha_i$, $\beta_i$, $A_i$ and $B_i$ are coefficients. The time-like vector $u^\mu$ is normalized as usual, $(uu) = -1$; hereafter, the symbol $(\ldots)$ denotes the scalar product. Then, it follows from (2) that

$$\beta_i = -(u_i l), \quad \alpha_i = -\frac{1}{2}(u_i N)^{-1}.$$  \hspace{1cm} (3)

$$\alpha_i > 0, \quad \beta_i > 0$$ (motion ‘forward in time’). The normalization condition entails

$$s_\mu^i s_\mu^i = \frac{\beta_i}{\alpha_i} = 1.$$  \hspace{1cm} (5)

The case $\beta_i = \alpha_i, s_\mu^i = 0$ corresponds to pure radial motion (see below).

Then,

$$-(u_1 u_2) = \frac{1}{2} \left( \frac{\beta_1}{\alpha_2} + \frac{\beta_2}{\alpha_1} \right) - (s_1 s_2).$$  \hspace{1cm} (6)

The energy in the centre of mass frame [1–13] is equal to $E_{c.m.}^2 = m_1^2 + m_2^2 - 2m_1 m_2 (u_1 u_2)$ ($m_i$ are rest masses of particles), so

$$E_{c.m.}^2 = m_1^2 + m_2^2 + m_1 m_2 \left[ \frac{\beta_1}{\alpha_2} + \frac{\beta_2}{\alpha_1} - 2(s_1 s_2) \right].$$  \hspace{1cm} (7)
3. Ingoing versus outgoing particles in the vicinity of the horizon: general approach

3.1. Case 1

Let particle 1 be going from the immediate vicinity of the horizon in the outward direction. We are dealing with the future horizon of a black hole, the vector \( l^\mu \) becoming its generator when the horizon is approached. Meanwhile, this particle, by assumption, cannot cross the horizon and does not penetrate the region inside. Therefore, near the horizon it does not move in the direction of \( N^\mu \), so it moves almost in the direction of the horizon generator \( l^\mu \). Hence, the component of the four-velocity in the direction of \( N^\mu \) should vanish, so it follows from (2) that

\[
\beta_1 \to 0.
\]  

(8)

Now, it is worth noting that by construction, the vector \( s^\mu \) is spacelike, so \((ss) > 0\) for \( s^\mu \neq 0\). Then, it follows from (5) that (8) also entails

\[
\alpha_1 \to 0.
\]  

(9)

Meanwhile, \( \alpha_2 \) is an arbitrary positive quantity. Then, it is seen from (7) that \( E_{2c} \rightarrow \infty \). One can say that this is just a direct consequence of infinite redshift near the horizon. In the examples below it is checked that equation (8) is indeed satisfied.

3.2. Case 2

This case (both particles move towards the horizon) is much more interesting since the frame of the centre of mass falls down with both particles [1], so the possible effect of unbound acceleration is not direct manifestation of the redshift. In general, as it is seen from (7), \( E_{2c} \) remains finite even in the vicinity of the horizon for any nonzero \( \alpha_1, \alpha_2 \). Basically, the simple point here is that, for unbounded collision energies to occur in this case, certain conditions on the parameters of the particle need to be satisfied (as discovered by previous works), and these conditions are equivalent to requiring that \( \alpha_1 \) vanish as the particle approaches the horizon. Indeed, let us now assume that (8) and, hence, (9) hold now (in case 3.1 they were satisfied automatically). In other words, an ingoing particle imitates the property of infinite redshift (8), (9) typical of an outgoing particle near the horizon. Then, again it follows from (7)–(9) that \( E_{2c} \rightarrow \infty \). This is just the effect discovered in [1] and studied in [2–13]. Thus, in case 2 the special condition (8) is needed. It relates the parameters of a particle like the energy and angular momentum or the energy and electric charge, etc (see the examples below).

The above observations can also be reformulated as follows. Consider the vector \( \xi^\mu \) which is timelike in the region where particles approach the horizon, \( N^2 \equiv -(\xi \xi) > 0 \):

\[
\xi^\mu = \frac{1}{2} l^\mu + N^2 N^\mu.
\]  

(10)

We can easily deduce two additional properties.

1. Let, in the near-horizon limit, condition (8) for some particle be satisfied, and let \((\xi u)\) be finite (otherwise the particle is arbitrary). Then, the vector \( \xi^\mu \) becomes lightlike in this limit.

**Proof.** It follows from (2), (8)–(10) that in this limit \((\xi u) \approx N^2 (Nu) = -\frac{N^2}{2\alpha_1}.\) As this quantity is finite, it follows from (8) that also \( N \to 0 \).

\[\square\]

2. Let us, instead of (8), assume that \((\xi u) \to 0 \). Then, (8) is satisfied and the vector \( \xi^\mu \) becomes lightlike in this limit.
Proof. Multiplying (10) by $u_\mu$, we observe that both terms are negative. Therefore, each of them vanishes separately in this limit, so $\alpha \to 0$, $N^2 \to 0$. As a consequence, $E_{\text{cm}} \to \infty$. □

The situation where the vector $\xi^\mu$ is timelike in some region but becomes lightlike on some hypersurface is typical of Killing horizons. However, we would like to emphasize that nowhere did we use Killing equations. It is also worth noting that in the formulation of statements (1) and (2) we relied on one particle with the four-velocity $u^\mu$, so these statements are not related to the collision of two particles directly.

The results under discussion can be reexpressed in another way with the help of Kruskal-like coordinates. Let, for simplicity, the metric be written in the form

$$d\tilde{s}^2 = -C dU dV + \gamma_{ab} dx^a dx^b$$

(11)

where $a = 1, 2$ and the metric coefficients are regular functions of the coordinates $U$ and $V$ (this is certainly possible for the nonrotating black holes). Here, the coordinates, $x^a$ have the meaning of angular coordinates in the spherically symmetric case. On the horizon $U = 0$ or $V = 0$. Then, repeating the above arguments, we see that it follows from (8) that, say, near the horizon $U = 0$ the component of the four-velocity $u^U \sim \beta \to 0$. Taking into account the regularity of the metric, we can write that $\alpha \sim \beta \sim U$, whence we have

$$\frac{dU}{d\tau} \sim U, \quad (12)$$

so

$$\tau \sim -\ln U \to \infty \quad (13)$$

in accordance with previous results for the Kerr [2, 10] or Reissner–Nordström [13] black holes.

Let us now illustrate these general properties by two examples.

4. Examples

4.1. Radial motion in Reissner–Nordström black hole

$$d\tilde{s}^2 = -dr^2 N^2 + \frac{dr^2}{N^2} + r^2 d\omega^2.$$  

(14)

Equivalently, the metric can be rewritten in the form

$$d\tilde{s}^2 = -dr^2 N^2 + dn^2 + r^2 d\omega^2.$$  

(15)

Here $n$ has the meaning of the proper distance, $d\omega^2 = \sin^2 \theta d\phi^2 + d\theta^2$, $N^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ where $M$ is the black hole mass and $Q$ is its charge. The event horizon lies at $r = r_H = M + \sqrt{M^2 - Q^2}$. Consider the radial motion of a particle having the charge $q$ and rest mass $m$. From the equations of motion one finds the components of the four-velocity for a pure radial motion:

$$u^0 = \frac{X}{N^2 m}, \quad u^1 = \frac{\varepsilon Z}{m N}$$

(16)

where $\varepsilon = -1$ for the direction towards the horizon and $\varepsilon = +1$ for the opposite direction of motion:

$$X = E - \frac{q Q}{r}, \quad Z = \sqrt{X^2 - m^2 N^2};$$

(17)

the coordinates are $x^0 = t$, $x^1 = n$, $x^2 = \theta$, $x^3 = \phi$. 

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Here, \( E \) is the conserved energy, dot denotes differentiation with respect to the proper time \( \tau \) and \( u^\mu \) is the four-velocity. The quantity \( X_H = E - \frac{qQ}{r_H} \geq 0 \), so it is positive for all \( r > r_H \) (motion 'forward in time'). Then, the vector (10) has the components \( \xi^\mu = (1, 0, 0, 0) \) and coincides with the Killing vector. Let us also introduce two lightlike vectors

\[
l^\mu = (1, N, 0, 0), \quad N^\mu = \frac{1}{2} \left( \frac{1}{N^2}, -\frac{1}{N}, 0, 0 \right).
\]

(18)

\((Nl) = -1\). The vectors \( a_\mu \) and \( b_\mu \) have nonzero components \( a_\theta = r \), \( b_\phi = r \sin \theta \). One can check that equality (1) is satisfied.

Then,

\[-(\xi u) = \frac{X}{m}, \]

and, according to (3),

\[\beta = -(ul) = \frac{X - \varepsilon Z}{m} > 0.\]

(20)

The quantity \(- (uN)N^2 = \frac{1}{2} \frac{X + \varepsilon Z}{m} > 0\) is finite for both signs of \( \varepsilon \) in agreement with the discussion in section 2, so it follows from (4) that

\[\alpha = \frac{mN^2}{X + \varepsilon Z}.\]

(21)

Bearing in mind that \( X^2 - Z^2 = m^2 N^2 \), it is easy to see that

\[\beta = \alpha\]

(22)

in accordance with what is said about pure radial motion in section 2.

4.1.1. Case 1. Let us take \( \varepsilon = +1 \) in expression (16) for \( u^1 \) that corresponds to the motion away from the horizon towards infinity. Then, it follows from (20) and (22) that

\[\alpha = \frac{X - Z}{m}.\]

(23)

Outside the horizon, \( \alpha > 0 \). In the horizon limit \( N \to 0 \) and it is seen from (17) that \( Z \to X \) in this limit, so for any particle irrespective of the relation between the parameters \( \alpha \to 0 \) in accordance with the general discussion of case 1 in section 3.1.

4.1.2. Case 2. Now, \( \varepsilon = -1 \). On the horizon \( Z = X_H \) (hereafter we use the subscript ‘H’ for the values calculated on the horizon), \( \sigma_H = \frac{2XH}{N} \geq 0, -(\xi u)_H = \frac{XH}{m} \geq 0 \). If for particle 1, \( X_H = 0, qQ = Er_H \), it follows that \( \alpha_1 = \beta_1 \to 0 \), when the horizon is approached. Then, the above consideration applies which leads to the result \( E_{c.m.}^2 \to \infty \) that agrees with the one obtained earlier [13].

4.2. Axially symmetric rotating black hole

Now, let us consider the generic metric describing an axially symmetric black hole:

\[ds^2 = -N^2 dt^2 + g_{\phi\phi} (d\phi - \omega dt)^2 + dl^2 + g_{zz} dz^2\]

(24)

that includes the Kerr and Kerr–Newman black holes. However, the configuration is more general due to the possible presence of matter (dirty black holes). We want to compare the general formalism of sections 2 and 3 with the more standard approach based on equations of motions. For metric (24), it follows from equations of motion that

\[i = u^0 = \frac{X}{N^2}, \quad X = E - \omega L\]

(25)
(for simplicity, here we assume that the rest mass \( m = 1 \))

\[
\dot{\phi} = \frac{L}{g_{\phi\phi}} + \omega X N^2, \tag{26}
\]

\[
l = \varepsilon \frac{Z}{N}, \quad Z^2 = X^2 - N^2 \left( 1 + \frac{L^2}{g_{\phi\phi}} \right) \tag{27}
\]

where \( u_0 = -E \) is the energy, \( u_\phi = L \) is the angular momentum and \( \varepsilon = \pm 1 \) has the same meaning as before. For motion ‘forward in time’, we must have \( \dot{l} > 0 \), so \( E - \omega L > 0 \).

Now, the relevant lightlike vectors are

\[
l_\mu = (-N^2, N, 0, 0) \tag{28}
\]

\[
N_\mu = \frac{1}{2N^2} (-N^2, -N, 0, 0) \tag{29}
\]

\[
(Nl) = -1. \tag{30}
\]

The vector (10) reads

\[
\xi^\mu = \xi^\mu_1 + \omega \xi^\mu_2 \tag{31}
\]

where \( \xi^\mu_1 = (1, 0, 0, 0) \) is the Killing vector that generates translations in time and \( \xi^\mu_2 = (0, 0, 1, 0) \) generates rotations. On the horizon \( N = 0 \) the vector \( \xi^\mu \) becomes lightlike.

One can check that equation (1) is indeed satisfied, where nonzero components of vectors \( a_\mu \) and \( b_\mu \) equal

\[
a_Z = \sqrt{g_{ZZ}}, \quad a_\phi = \sqrt{g_{\phi\phi}}, \quad a_0 = -\omega a_\phi. \tag{40}
\]

The scalar product \( (ua) \) is finite. Then, similar to what we had in the Reissner–Nordström case, one finds that \( -(u\xi) = X \) and equations (20) and (21) hold where now \( Z \) is defined in (27). Then, one can easily obtain that in this case,

\[
\frac{\beta}{\alpha} = \frac{1 + \frac{L^2}{g_{\phi\phi}}}{1} \tag{22}
\]

instead of (22). Although \( \beta \) and \( \alpha \) are not equal now, they are proportional to each other, so if \( \beta \to 0 \), also \( \alpha \to 0 \) in accordance with the general discussion in section 3.1.

### 4.2.1. Case 1.

For the motion away from the horizon, \( \varepsilon = 1 \). In the horizon limit \( N \to 0 \), one can see from (27) that \( Z \to X \), so we again obtain properties (8) and (9) for any relationship between the energy and the angular momentum of a particle.

### 4.2.2. Case 2.

Now, in the horizon limit \( \beta \to 2X_H \). The critical value is singled out by the condition \( X = 0 \) on the horizon \( (E = \omega_H L) \) that indeed coincides with (8). Then, we again obtain that \( q \omega_{c.m.} \to \infty \) in accordance with the previous discussion and [12].

For completeness, we should make a reservation. Apart from the cases \( \dot{l} > 0 (\varepsilon = +1) \) and \( \dot{l} < 0 (\varepsilon = -1) \) in both examples there exist special orbits for which \( \dot{l} = 0 \) (see [17] for the discussion of these orbits in the case of the Kerr metric). It is seen from (16), (17) or (27) that for such orbits \( Z = 0, X \sim N \). In the horizon limit, \( N \to 0 \), \( X \to 0 \) and we again return to condition (8).
5. Conclusions

Thus, we elucidated the generic nature of the effect and showed that diversity of different metrics and even classes of metric have the same underlying reason in this context. In doing so, we did not use explicitly the equations of motion of particles at all, did not rely on an explicit form of the metric, field equations from which it is obtained, etc. (We used equations of motions to compare two different approaches only.) Actually, the nature of the effect turned out to be surprisingly simple and stemming from the mutual properties of lightlike and timelike vectors in the vicinity of the future horizon. It may happen that condition (1) is not realized in some particular cases (say, for some classes of trajectories [8]). Nonetheless, if (i) the horizon exists and (ii) the condition (8) is indeed satisfied, the effect of unbound $E_{\text{c.m.}}$ can manifest itself in general. Moreover, it follows from our derivation that these reasonings apply not only to the horizons of static or stationary black holes. As a matter of fact, the effect is valid even if the aforementioned condition is obeyed for some portion of the surface only. Moreover, these portions can shrink to the point. In particular, the results of this work seem to apply to dynamic or isolated horizons [16].

The fact that the essence of the effect of infinite $E_{\text{c.m.}}$ reveals itself in so general a setting lends support to the idea that it can survive notwithstanding model-dependent factors (electromagnetic radiation, gravitational radiation, etc). However, at present, this is only a conjecture since, say, the role of gravitational radiation becomes more significant when the particle’s velocity approaches the speed of light [5, 6, 19]. Also important is whether or not the phenomenon is observable in more realistic astrophysical situations including measurements which can be done at infinity [18]. These issues deserve further study.

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