Dynamical crossover in invasion percolation

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Abstract
The dynamical properties of the invasion percolation on the square lattice are investigated with an emphasis on the geometrical properties on the growing cluster of infected sites. The exterior frontier of this cluster forms a critical loop ensemble (CLE), whose length ($l$), the radius ($r$) and also roughness ($w$) fulfill the finite-size scaling hypothesis. The dynamical fractal dimension of the CLE defined as the exponent of the scaling relation between $l$ and $r$ is estimated to be $D_f = 1.76 \pm 0.04$. By studying the autocorrelation functions of these quantities we show importantly that there is a crossover between two time regimes, in which these functions change behavior from power-law at the small times, to exponential decay at long times. In the vicinity of this crossover time, these functions are estimated by log-normal functions. We also show that the increments of the considered statistical quantities, which are related to the random forces governing the dynamics of the observables undergo an anticorrelation/correlation transition at the time that the crossover takes place.

Keywords: invasion percolation, crossover, fractal dimension, autocorrelation functions

(Some figures may appear in colour only in the online journal)

1. Introduction

Multiphase flow phenomena in porous media are of great importance in many fields of science and industry. A very promising and highly-used strategy for analyzing this is the cavity lattice method, used commonly in statistical models, expressing the porous media parameters as the local fields, which enables one to use the concepts of the percolation theory in fluid flow modeling. Among the models that aim to describe two-phase flow in porous media, invasion percolation (IP) has been subjected to intense studies due to its relatively simple structure (making it suitable for statistical investigations) and its success in capturing most relevant physical processes, and also showing a plenty interesting behaviors, like self-organizing in a critical state [1]. IP is a standard model to study the dynamics of two immiscible phases (commonly denoted by wet and non-wet phases) in a porous medium [2, 3]. During this process, the wet phase invades the non-wet phase, and the front separating the two fluids advances by invading the pore throat at the front with the lowest threshold [3]. Many aspects of IP are known, involving various fractal dimensions [1, 3–5], its properties in three dimensions [6], its fractal growth [7], its dependence on the coordination number [8], the effects of long-range correlations [9, 10], and its dynamics in correlated porous media [11, 12]. It has also many applications in the reservoir engineering [13], and also geoscience [14]. Furthermore, the ideas of IP have been moved to other fields, like the frustrated models, in which the dynamics are due to invaded clusters [15, 16]. The dynamic studies on IP show also a rich structure. For example, the spatiotemporal properties of this model were studied in [17], where some dynamical scaling behaviors were found. Mapping to a fractal growth process is another interesting strategy, using which a mass fractal dimension is estimated to fourth-order to be $D = 1.8872$ that should be compared with the exact value $\frac{41}{20} = 1.8958 \ldots$ [7] as an important test for fixed-scale transformation method. Also in a recent work [18], a novel fractal dimension 1.8644(7) is established on the square lattice for the history-dependent percolation by large-scale Monte Carlo simulations. For a good review see [19] and [20].

Despite its simple rules and structure, the IP model occasionally surprises us with some new novel features and properties. For a long time, we know that the fractal properties of the IP are similar to the ordinary critical percolation [21–24]. However, such analysis is concerning the interfaces of the ‘snapshots’ of IP. The temporal properties of this model were almost ignored in the literature. Actually we show here
that IP changes its behaviors in some characteristic times which forms the aim of the present paper. Here we focus on its temporal dynamical properties and uncover a new dynamical crossover. The temporal dynamics of IP based on the burst dynamics were first proposed and analyzed in [25], where using its connection to the ordinary percolation theory various exponents were derived [17] was the first to propose a scaling between length and time in IP by investigating the distribution function $P(r, t)$, $r$ being the distance between two sites added to the invaded cluster and separated by a time interval of $t$, and found that $P(r, t) \sim r^{-1/\phi}(r^0/t)$, where $D \approx 1.82$ refers to the fractal dimension of the invaded cluster [17, 25], as was also obtained for IPs in the low coordination number regime [8]. Here we obtain the dynamical fractal dimensions of the growing clusters. We also report on a dynamical crossover, which has not been seen in the previous studies. We show that the temporal autocorrelations of the statistical observables change behavior at a crossover time $t_{crossover}$ from power-law to exponential decay. We also show that the effective random forces that govern the growth of the invading-phase cluster also change dramatically at this point. Important the autocorrelations change sign, showing that the system undergoes an anticorrelation/correlation crossover.

The paper has been organized as follows: In the next section, we shortly introduce the model. The section 3 has been devoted to the numerical details and results. We close the paper with a conclusion 4.

2. The model

In this model, one starts by slow injection of a wetting fluid into a horizontal cell saturated with a denser nonwetting fluid, often called the defender. The invading fluid moves preferentially through the pores with the least resistance. In the IP with (without) trapping, and the defender is supposed to be an incompressible (compressible) fluid so that if a bubble is surrounded by an invader, it becomes impenetrable (permeable) to the invading phase [26]. In this work, we consider no trapping.

Let us describe the invasion percolation on a $L \times L$ square lattice. We assign uncorrelated uniform random numbers ($r$) in the interval $[0, 1]$ to all lattice sites and choose the site $r_0 = \left(\frac{r_0}{L}, \frac{r_0}{L}\right)$ as the seed of the growth, i.e. for injecting fluid. The random numbers represent the resistance of the pores so that the fluid moves preferentially through the regions with the least resistance, i.e. minimum $r$. The dynamics start with injecting fluid to $r_0$. At each time step, a unique neighbor site with the smallest associated random number $r$ is occupied. In case of degeneracy, i.e. there is a set of sites with the same (lowest) $r$, we choose a site from the set randomly. As time goes on, a connected cluster of occupied sites forms, which we call occupied sites cluster (OSC). In each sample, the dynamics are represented by a time parameter, defined by $t \equiv \left[\frac{m}{\mu}\right]$, where $m$ is the mass (the number of occupied sites) of the growing cluster at that time, and $[\cdot]$ is the integer part. $t_{perc}$ is defined as the time in which two opposite boundaries are connected by a giant cluster when the process stops. In this time, OSC is a giant cluster that spans the lattice (not all sites), namely spanning OSC (SOSC). There is however another time scale in between, denoted by $t_{BH}$, in which the invading cluster hits one of the boundaries for the first time. Its importance is in the fact that for the times larger than it, the geometrical properties of OSC changes due to the boundary. The periodic boundary condition is considered for one direction, and open boundary condition is imposed for the boundaries in the perpendicular direction.

At each time gyration radius ($r_g$), the mass gyration radius $r_m$ and the length ($l$) and the roughness ($w$) of the external perimeter of the OSC are recorded for each time. These quantities are defined as $r_g \equiv \sqrt{\sum_{i=1}^{L^2} (x_i^2 + y_i^2)}$, $r_m \equiv \sqrt{\sum_{i=1}^{L^2} (x_i^2 + y_i^2)}$ where the sums run over the sites on the boundary ($r_d$) and total sites ($r_t$) of the OSC, and $X_i \equiv (x_i, y_i)$, and $x_i$ and $y_i$ are the Cartesian coordinates of the site $i$, and

$$w^2 = \frac{1}{l} \sum_{i=1}^{L} (r_i - \bar{r})^2$$

(1)

\[ D_f = 1.336 \pm 0.002 \]

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where again the sum runs over the external perimeter of OSC, $r_i \equiv (x_i^2 + y_i^2)^{1/2}$ and $r \equiv \frac{1}{2} \sum_{i=1}^{L^2} (x_i^2 + y_i^2)^{1/2}$. The external perimeter of an OSC, may be closed loop (for finite OSCs) or be open (for SOSC or boundary-hit OSC). For the former case the fractal dimension is defined as $\langle \log l \rangle = D_f \langle \log r_g \rangle + \ln (\bar{r})$ (being the ensemble average), whereas for the latter case we use the box-counting scheme to find $D_f^I$. The holes of the system are obtained using the Hoshen Kopelman (HK) algorithm [27]. Importantly, using HK algorithm we extracted and analyzed the largest hole at the $t_{perc}$ for which the fractal dimension was obtained to be $D_f = 1.33 \pm 0.01$, as depicted in figure 1. It is worth mentioning that this result is not new and is known long time ago [21–24, 28, 29]. It is consistent with the fractal dimension of the self-avoiding walks $D_f^{AW} = \frac{4}{3}$ [30, 31], which is related to the fractal dimension of the interfaces of percolation
\(D_{\text{perc}} = \frac{1}{4} \sqrt{15} - 1(15^{1/4})\) where \(D_{\text{perc}}\) is the fractal dimension of the original traces, and \(D_{\text{perc}}\) is the fractal dimension of their hull. This is also understood in terms of Schramm-Loewner evolution (SLE) duality, stating that the diffusivity parameter \(\kappa\) of a SLE class is dual to another SLE with the diffusivity parameter \(\kappa = \frac{16}{\kappa}\) [33–35]. This operation does not change the CFT universality class of the model, as a well-known result for IP [3].

As stated above, long time ago the community knows that the fractal properties of the IP are similar to the percolation. However, this analysis is for the ‘snapshots’ of IP, where the ‘interfaces’ are considered. The dynamical aspects on IP have not been completely considered in the literature, which forms the aim of the remaining of the paper. We take the ensemble averages up to \(t_{\text{perc}}\), and obtain various dynamical statistical observables. Our main focus is on the spatiotemporal properties of the model, and also the temporal structure of noises. To this end we concentrate on the temporal dynamics of the growing interfaces that separate wetting-nonwetting phases, as well as the bulk properties. The autocorrelation functions are suitable quantities to reveal the temporal structure of the observables as time series. Consider the normalized quantities

\[
f_i(t) = \frac{x(t)}{\langle x(t) \rangle} - 1
\]

where \(x = r_l, l, w\) are our statistical observables. These functions have been built in such a way that \(f_i(0) = 0\). Also let us consider the increments \(\Delta f_i(t) \equiv f_i(t) - f_i(t - 1)\), which is the noise associate with the quantity \(x\).

To understand the statistical interpretation of \(\Delta f_i\), let us suppose that the behavior of \(x(t)\) is controlled by a force, so that \(\frac{dx}{dt} = F_{\text{deterministic}} + \delta F_{\text{noise}}\), where \(F_{\text{deterministic}}\) is the deterministic force, and \(\delta F_{\text{noise}}\) is the noise part. Then

\[
\frac{1}{\delta t} \Delta f_i(t) = \langle \frac{x}{\langle x \rangle} \rangle \frac{\delta x}{\delta t} = \frac{x}{\langle x \rangle^2} \frac{\delta x}{\delta t}
\]

\[
\approx F_{\text{deterministic}} + \delta F_{\text{noise}} = F_{\text{deterministic}} + \delta F_{\text{noise}}
\]

where in the last line we used \(\frac{\delta x}{\delta t} \approx \frac{\delta (\langle x \rangle)}{\delta t} \approx F_{\text{deterministic}}\). We therefore see that \(\Delta f_i(t) = \delta (\langle x / \langle x \rangle \rangle) = \delta F_{\text{noise}}\) (note that \(\delta \equiv 1\)), i.e. \(\Delta f_i(t)\) is nothing but the random force acting on \(x\).

The autocorrelation is defined by

\[
\text{Auto}_i(t, \tau) \equiv \langle f_i(t) f_i(t + \tau) \rangle
\]

\[
\text{Auto}^{(2)}_i(t, \tau) \equiv \langle \Delta f_i(t) \Delta f_i(t + \tau) \rangle
\]

Note that for all the quantities that we considered, the process is not stationary so that \(\text{Auto}_i(t, \tau) \neq \text{Auto}_i(t - \tau)\). Therefore, the power spectrum is more complicated than the stationary case (for which the power spectrum is simply the Fourier transform with respect to \(\tau\)). For later convenience, let us also define the cumulative autocorrelation function as follows:

\[
\text{Auto}_i^{(c)}(t, \tau) = \sum_{\tau' = 1}^{\tau} \text{Auto}_i(t, \tau')
\]

For the case where \(\text{Auto}_i\) follows a power-law behavior with respect to \(\tau\), i.e. \(\text{Auto}_i(t, \tau) = c(t)\tau^{-\zeta_i} (c(t)\text{being an arbitrary function of } t)\), then one can easily show that \(\text{Auto}^{(c)}_i(t, \tau) = c(t)H(\tau, \zeta_i)\), where \(H(n, r)\) is the \(n\text{th}\) harmonic number of order \(r\). This function helps to decide about the noisy functions by killing the noise, to firstly observe if they are fitted to power-law functions, and secondly calculate the corresponding exponent. We will observe that the cumulative autocorrelation functions are properly smooth allowing us to detect and investigate the power-law behaviors.

3. Results

To control the finite size effects, we simulated the model for square lattices with sizes \(L = 64, 128, 256, 512, 1024\) and measured the time dependence of \(r_m, r_l, l\) and \(w\). In figure 2 we show the time dependence of \(\langle x(t) \rangle\) in terms of \(t\), \(x = r_m, r_l, l\) and \(w\). The graphs fit nicely by power-law for early times and change behavior at some crossover time to long time regime. This crossover takes place for all studied quantities. To quantify this behavior, we used the data collapse technique based on finite-size scaling:

\[
\langle x(t, L) \rangle = L^{\beta_1} F_{x}(\frac{t}{L^{\gamma_1}})
\]

\[
= L^{\frac{\beta_1}{\gamma_1}} G_{x}(\frac{t}{L^{\gamma_1}})
\]

where \(F_x\) and \(G_x\) defined by \(F_x(y) = y^{\gamma_1} G_x(y)\) are universal functions with the asymptotic behaviors: \(\lim_{y \to 0} G_x(y) = c(y)\). For long time limit, let us suppose that \(F_x(y) \propto y^{\gamma},\) so that \(\langle x(t) \rangle \propto L^{\beta_1 - \gamma_1} t^{\gamma}\). Requiring that \(\langle x(t) \rangle \propto L^{\beta_1 - \gamma_1} t^{\gamma}\), implies that \(\beta_1 = 1\) (since OSC stops at that time). The data collapse analysis presented in the insets of figure 2 support the finite size scaling hypothesis equation (6). Note also that for small times (compared to \(t_{\text{HH}}\) and \(t_{\text{perc}}\)) \(t \sim t^\alpha\), and \(t \sim t^{\alpha_w}\), where \(\alpha_w\) should satisfy the hyperscaling relation \(\alpha_w = \frac{\beta_1}{\gamma_1}\). Note that \(\beta_1 = 0.53 \pm 0.05, \quad \frac{\beta_1}{\gamma_1} = 0.92 \pm 0.05\) and \(\gamma_1 = 0.54 \pm 0.02\) which are compatible with \(\alpha_w = 0.51 \pm 0.04\). \(\alpha_l = 0.90 \pm 0.03\) and \(\alpha_w = 0.51 \pm 0.01\) respectively. Also one can estimate the dynamic fractal dimension using the relation \(l \sim r^2\), where \(D_l = \frac{\alpha_l}{2}\) is \(1.76 \pm 0.09 \approx \frac{3\beta_1}{5\gamma_1}\), Another method to estimate the dynamic fractal dimension \(D_l\) is within investigating directly \(l\) in any time, which is done in figure 3 suggesting that \(D_l \approx 1.76 \pm 0.04\). The data collapse analysis of \(l - r\) graphs is shown in the inset of this figure, confirming the above estimation of the fractal dimension \(D_l = \frac{\beta_1}{\gamma_1} = 1.76 \pm 0.05\), which is consistent with the known fractal dimension of external perimeters of simple percolation \(D_{\text{perc}} = \frac{11}{7}\) [36, 37]. This fractal dimension should not be confused with the mass fractal dimension which is estimated to be \(\approx 1.82\) for low coordination numbers, and \(\approx 1.896\) for high coordination numbers [8]. The exponents are gathered in table 1.
To reveal the dynamical structure of the model, we consider the autocorrelation function of the quantities considered above, for which a crossover was observed. As stated in the previous section, \( f(t, \tau) \) depends both on \( t \) and \( \tau \), representing the non-stationarity of the time series. We found that in terms of \( t \) there are three dynamical regimes that are

![Graph showing the time dependence of the average of \( r_m \), \( w \), \( r_l \), and \( l \). The upper insets are the data collapse analysis, showing the corresponding \( \beta/\zeta \) exponent. The lower insets are the \( L \)-dependence of the slopes of the curves at the main panels, to be extrapolated to \( L \to \infty \).](image)

**Figure 2.** The time dependence of the average of (a) \( r_m \), (b) \( w \), (c) \( r_l \), and (d) \( l \) on time \( t \). The upper insets are the data collapse analysis showing the corresponding \( \beta/\zeta \) exponent. The lower insets are the \( L \)-dependence of the slopes of the curves at the main panels, to be extrapolated to \( L \to \infty \).

![Log-log plot of \( l - r_l \) graph for all times, whose slope is the dynamical fractal dimension. In the inset the fractal dimension is shown in terms of \( l/L \), and in the upper inset the data collapse of the data is presented, from which the exponents \( \beta_l \) and \( \zeta_l \) are extracted.](image)

**Figure 3.** Log-log plot of \( l - r_l \) graph for all times, whose slope is the dynamical fractal dimension. In the inset the fractal dimension is shown in terms of \( l/L \), and in the upper inset the data collapse of the data is presented, from which the exponents \( \beta_l \) and \( \zeta_l \) are extracted.

**Table 1.** The critical exponents \( \beta, \zeta, \beta_l, \zeta_l \), and \( \alpha \) of \( r_m, r_m, l \) and \( w \). The \((x,y)\) shows that data collapse analysis of the quantity \( x \) in terms of \( y \). The exponents \( \beta \) and \( \zeta \) have been calculated using the data collapse method, and \( \alpha \) was directly estimated by linear fitting of the quantities in terms of \( t \) for all sizes, and extrapolating \( L \to \infty \).

|      | \((r_m, t)\) | \((r_m, t)\) | \((l, t)\) | \((w, t)\) | \((l, r_l)\) |
|------|--------------|--------------|------------|------------|-------------|
| \( \beta \) | 0.97(9)      | 1.00(2)      | 1.65(8)    | 1.00(2)    | 1.70(5)     |
| \( \zeta \) | 1.83(1)      | 1.85(1)      | 1.79(9)    | 1.85(8)    | 0.98(5)     |
| \( \beta_l \) | 0.53(5)      | 0.54(1)      | 0.92(5)    | 0.54(2)    | 1.76(5)     |
| \( \zeta_l \) | 0.51(4)      | 0.52(0)      | 0.90(3)    | 0.51(1)    | 1.76(4)     |

To reveal the dynamical structure of the model, we consider the autocorrelation function of the quantities considered above, for which a crossover was observed. As stated in the previous section, \( f(t, \tau) \) depends both on \( t \) and \( \tau \), representing the non-stationarity of the time series. We found that in terms of \( t \) there are three dynamical regimes that are
depicted in figure 4. These regimes are separated and identified by a crossover time \( t_{\text{crossover}} \). For the early times \( t \ll t_{\text{crossover}} \), the autocorrelations are power-law in terms of \( \tau \) as shown in the main panel of figure 4(a). For \( t \approx t_{\text{crossover}} \), the autocorrelations are best fitted by log-normal functions, i.e. \( \text{Auto}_x \propto \exp \left[ -a_x (\log \tau)^2 \right] \) with non-universal exponent pre-factor \( a_x \), and for large times \( t \gg t_{\text{crossover}} \) the autocorrelation functions decay exponentially with \( \tau \). The crossover time is estimated to be \( t_{\text{crossover}} \approx 10 \) for all lattice sizes, however, its precise determination needs some additional works, and is beyond our analysis since we just looked at the autocorrelation functions. Indeed our computational error is too large to detect the \( L \) dependence of \( t_{\text{crossover}} \).

As is evident in figure 4(a), although the power-law fits are acceptable, they are noisy, and need another test that kills the noise. In the lower inset of this graph, we show the accumulated autocorrelation function in terms of \( \tau \) for various \( t \) values, which contains considerable lower noise, and fit properly to Harmonic functions. The exponent of the power-laws \( \zeta \) is shown in the upper inset which fixes at small times to 0.9 ± 0.01 and decreases monotonically as time decreases. The fitting is valid up to \( t \approx t_{\text{crossover}} \) at which the power-law behavior is completely destroyed, and a new regime begins, that is shown in the lower and upper insets of figure 4(b), where \( \text{logAuto}_x \) is shown as a function of \( (\log \tau)^2 \). The finite size dependence at \( t = t_{\text{crossover}} \) is shown in the upper inset, exhibiting a linear behavior as claimed. For long enough times (the main panel of figure 4(b)) the dependence is exponential. The same features were also observed for the roughness, as depicted in figures 4(c) and (d). We see that the quantities \( f_{\Delta \xi} x, w \) uncover a dynamical crossover structure of IP, which should also be reflected in their increments, i.e. \( \Delta f_{\Delta \xi} \) which in fact are the corresponding noises, whose correlations (\( \text{Auto}^2_{\Delta \xi} \)) are important to realize the dynamical structure of the model.

In the figure 5 we show \( \text{Auto}^2_{\Delta \xi}(t, \tau) \) (\( x = r_l, l, w \)) in terms of \( \tau \) for various amounts of \( t \) for the maximum system size \( L = 1024 \). For example let us focus on the \( x = r_l \) (figure 5(a)), for which an interesting change of sign takes place at small \( \tau_s \), signaling an anticorrelation-correlation (AC) crossover. This AC that occurs in the vicinity of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure_4.png}
\caption{\( \tau \) dependence of \( \text{Auto}_x \) (a) and (b), and \( \text{Auto}_w \) (c) and (d)). In (a) and (c) the plots are in log-log scale for \( L = 1024 \), and the upper insets show the accumulated autocorrelation functions, and in the lower insets, the exponents (\( \zeta \)) are reported in terms of \( t \) in log-log scale. In (b) and (d) the main panels show the same in semi-log scale for very long times for \( L = 1024 \), and the insets show the mid-time behavior of \( \text{logAuto} \) in terms of \( (\log \tau)^2 \) for \( L = 1024 \) for various times (lower insets) and system sizes for \( t = 10 \) (upper insets). The dashed lines are for eye guide for comparing the graphs with linear ones.}
\end{figure}
t = t_{\text{crossover}} has been observed for all system sizes that we considered in this work and seems to survive at the thermodynamic limit, although some larger size simulations are necessary to make this hypothesis more precise and reliable. For t ≈ t_{\text{crossover}} the correlations are positive. These behaviors are more evident in the zoomed graphs in the insets of these figures. The same features are also observed for l (figure 5(b)) and w (figure 5(c)). Another interesting behavior is that all autocorrelation functions become negligibly small (more precisely become $1/t_{\text{crossover}}$) at $t \approx t_{\text{crossover}}$, showing that this is an intrinsic time scale in IP. Therefore $t_{\text{crossover}}$ is the average decay time (in terms of $\tau$) of the autocorrelations for all $t$.

4. Conclusion

Two-dimensional invasion percolation (IP) was considered in this work with a focus on the dynamical aspects of the occupied sites cluster (OSC). This problem was viewed as an isotropic growth process starting from an injection point in the bulk. The dynamical properties of some geometrical statistical observables for the external frontier of OSCs were analyzed, like the gyration radius, the loop length, and the roughness, the scaling relations between which give important exponents. After re-generating the well-known fractal dimension of the external perimeter of the largest hole ($D_{\text{fr}} \approx 1.76 \pm 0.05$ which is consistent with the fractal dimension of the external perimeters in the percolation model. We also reported the other growth exponents that are collected in 1.

The random force was argued to be proportional to the increments of $f_x(t)$ (i.e. the increments $\Delta f_x(t)$, see equation (3)) was shown to undergo an anticorrelation/correlation crossovers.
background random force changes character at this time, i.e. for \( t \lesssim t_{\text{crossover}} \) (contrary to the \( t \gg t_{\text{crossover}} \) case) the background compensates the fluctuations of the forces [meaning that a force larger (lower) than the average leads to a lower (larger) random force in the next time]. The fingerprints of this time scale also was seen in the analysis of \( \tau \), where \( \tau_0 \approx t_{\text{crossover}} \) was observed to be an average decay time of the corresponding autocorrelations.

We think that this crossover has the potential to be investigated more deeply in the community to find its other aspects. Importantly one may address the question whether these properties arise for the small/large scales or it is purely dynamical. Also does the IP model defined on top of an imperfect support (which is more realistic in realizing the flow propagation in porous media) show the similar properties? These form our future investigations on the topic.

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