Global existence of weak solutions to the compressible quantum Navier-Stokes equations with degenerate viscosity

Boqiang Lü∗ Rong Zhang† Xin Zhong‡

Abstract

We study the compressible quantum Navier-Stokes (QNS) equations with degenerate viscosity in the three dimensional periodic domains. On the one hand, we consider QNS with additional damping terms. Motivated by the recent works [Li-Xin, arXiv:1504.06826] and [Antonelli-Spirito, Arch. Ration. Mech. Anal., 203(2012), 499–527], we construct a suitable approximate system which has smooth solutions satisfying the energy inequality and the BD entropy estimate. Using this system, we obtain the global existence of weak solutions to the compressible QNS equations with damping terms for large initial data. Moreover, we obtain some new a priori estimates, which can avoid using the assumption that the gradient of the velocity is a well-defined function, which is indeed used directly in [Vasseur-Yu, SIAM J. Math. Anal., 48 (2016), 1489–1511; Invent. Math., 206 (2016), 935–974]. On the other hand, in the absence of damping terms, we also prove the global existence of weak solutions to the compressible QNS equations without the lower bound assumption on the dispersive coefficient, which improves the previous result due to [Antonelli-Spirito, Arch. Ration. Mech. Anal., 203(2012), 499–527].

Keywords: compressible quantum Navier-Stokes equations; global weak solutions; degenerate viscosities; vacuum.

Math Subject Classification: 35Q35; 76N10

1 Introduction

The quantum Navier-Stokes equations with damping terms which read as follows:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - 2\nu\text{div}(\rho D u) + \nabla P - 2\kappa^2 \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + r_0 u + r_1 \rho |u|^2 u &= 0.
\end{align*}
\]

Here, \( x \in \Omega \subset \mathbb{R}^3, t > 0, \rho \) is the density, \( u = (u_1, u_2, u_3) \) is the velocity field, \( D u = \frac{1}{2}(\nabla u + (\nabla u)^T) \) is the symmetric part of the velocity gradient, \( P(\rho) = a \rho^\gamma (a > 0, \gamma > 1) \) is the pressure. Without loss of generality, it is assumed that \( a = 1 \). The positive constants \( \nu \) and \( \kappa \) are the viscosity and the dispersive coefficients, respectively. The constants \( r_0 \) and \( r_1 \) in the damping terms are all positive. Let \( \Omega = \mathbb{T}^3 \) be the three dimensional torus, we consider the system (1.1) with periodic boundary conditions. The initial conditions are imposed as

\[
\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = m_0(x).
\]
When \( r_0 = r_1 = 0 \), i.e., there is no damping terms, the system (1.1) is a special case of the Navier-Stokes-Korteweg (NSK) equations, which reads as

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P - \text{div}S - \text{div}K &= 0.
\end{aligned}
\]  

(1.3)

The viscosity stress tensor \( S \) and the capillarity (dispersive) term \( K \) are defined by

\[
S \triangleq h D u + g \text{div} u I
\]

(1.4)

and

\[
K \triangleq \left( \rho \text{div}(k(\rho) \nabla \rho) - \frac{1}{2} (\rho k'(\rho) - k(\rho)) |\nabla \rho|^2 \right) I - k(\rho) \nabla \rho \otimes \nabla \rho,
\]

(1.5)

where \( I \) is the identical matrix, and \( h, g \) satisfy the physical restrictions

\[ h > 0, \quad h + 3g \geq 0. \]

Indeed, choosing

\[ h(\rho) = 2\nu \rho, \quad g(\rho) = 0, \quad k(\rho) = \frac{\kappa^2}{\rho}, \]

(1.6)

the NSK equations (1.3) becomes the QNS one (1.1) without damping terms. For more detailed derivation of the QNS equations, please refer to \([24]\). In particular, the QNS equations without viscosity \((\nu = 0)\) is the Quantum Hydrodynamics model for superfluids (see \([27]\)), whose global weak solutions with finite energy was studied in \([2,3]\). It is well known that the NSK equations reduce to the Navier-Stokes (NS) equations when there is no capillarity (dispersive) term \( K \). One of the main difficulties in studying the compressible NS (or QNS, NSK) equations with degenerate viscosity coefficients is to estimate the gradient of the velocity field in the vacuum region, please refer to \([2–5,7–12,14–23,25,28–35,37]\) and the references therein.

For the one dimensional space, the global existence of weak solutions for the QNS equations was proved by Jüngel \([23]\). Then, for weak solutions required a special choice of the test function \( \rho \phi \) with \( \phi \) smooth and compactly supported, he \([22]\) also obtained the global weak solutions to the three dimensional QNS equations in the case \( \kappa > \nu \) and \( \gamma > 3 \). Very recently, for \( \gamma, \kappa, \) and \( \nu \) satisfying

\[
\begin{aligned}
1 < \gamma, \quad \kappa < \nu, & \quad \Omega = \mathbb{T}^2, \\
1 < \gamma < 3, \quad \kappa < \nu < \frac{3\sqrt{7}}{2} \kappa, & \quad \Omega = \mathbb{T}^3,
\end{aligned}
\]

(1.7)

Antonelli-Spirito \([4]\) proved the global existence of finite energy weak solutions, which is the first result of global existence for finite energy weak solutions to NSK equations in high dimensional space. As mentioned in \([4]\), one of the key ideas in \([4]\) is to construct proper smooth approximating solutions, which is motivated by the parabolic regularization methods owing to Li-Xin \([29]\). Indeed, Li-Xin \([29]\) proved the global existence of finite energy weak solutions to the compressible NS equations with general degenerate viscosity coefficients in two or three dimensional periodic domains or whole spaces, which in particular solved an open problem proposed by Lions \([30]\).

Furthermore, there are many works considering the compressible NS (or QNS, NSK) equations by considering the system with some additional terms, such as a cold pressure term, the damping terms or other source terms (please see \([8,11,12,15,25,34]\) and the references therein). In particular, Vasseur-Yu \([34]\) considered global existence of finite energy weak solutions of the QNS equations with damping terms (1.1). Then, using the global weak solutions to system (1.1) obtained in \([34]\), by different methods from those in Li-Xin \([29]\), Vasseur-Yu \([35]\) studied the global weak solutions to the compressible NS equations (1.3)-(1.6) with \( \kappa = 0 \). The key issues in Vasseur-Yu \([34,35]\) rely crucially on the assumptions that \( \nabla u \) is a well-defined function and that \( \sqrt{\rho} \nabla u \in L^2((0,T) \times \Omega) \), which are confused for us (see Remark (1.1) below for more details). Indeed, it seems impossible to define \( \nabla u \) as functions without enough regularity of \( u \) due to the high degenerate viscosity at vacuum. Hence,
in this paper, we will reconstruct suitable approximate system to obtain the global existence of weak solutions to system \(1.1\). Moreover, the weak solutions are more regular than those obtained by Vasseur-Yu \(34\) and can be used to obtain the global weak solutions to the compressible NS equations with degenerate viscosity. This will be shown in a forthcoming paper \(32\). Furthermore, we also improve the restriction on the range of \(\kappa\) in \(4\) by removing the lower bound \(\frac{4}{3\sqrt{2}}\nu\).

Now, we explain the notations and conventions used throughout this paper. For \(\Omega = \mathbb{T}^3\), set
\[
\int \cdot \, dx \triangleq \int_{\Omega} \cdot \, dx.
\]
Moreover, for \(1 \leq r \leq \infty\) and \(k \geq 1\), the standard Lebesgue and Sobolev spaces are defined as follows:
\[
L^r = L^r(\Omega), \quad W^{k,r} = W^{k,r}(\Omega), \quad H^k = W^{k,2}.
\]
We will consider the problem \(1.1\,–\,1.2\) with the initial data \(\rho_0, m_0\) satisfying that
\[
\begin{aligned}
\rho_0 \geq 0 \text{ a.e. in } \Omega, \quad \rho_0 \neq 0, \quad \rho_0 \in L^1 \cap L^\gamma, \quad \nabla \sqrt{\rho_0} \in L^2, \\
m_0 \in L^1, \quad m_0 = 0 \text{ a.e. on } \Omega_0, \quad \rho_0^{-1} m_0^2 \in L^1, \\
-\tau_0 \log_+ \rho_0 \in L^1, \quad \text{with } \log_+ g \triangleq \log \min\{1, g\},
\end{aligned}
\quad (1.8)
\]
where \(\Omega_0\) is the vacuum set of \(\rho_0\), defined by
\[
\Omega_0 \triangleq \{ x \in \Omega | \rho_0(x) = 0 \}.
\quad (1.9)
\]
Next, we give the definition of a weak solution to \(1.1\,–\,1.2\).

**Definition 1.1** Let \(\Omega = \mathbb{T}^3\), \((\rho, u)\) is said to be a weak solution to \(1.1\,–\,1.2\) if
\[
\begin{aligned}
0 \leq \rho &\in L^\infty(0, T; L^1 \cap L^\gamma), \\
\nabla \sqrt{\rho} &\in L^2(0, T; L^2), \\
\nabla \sqrt{\rho}, \sqrt{\rho}u &\in L^\infty(0, T; L^2), \\
\nabla (\sqrt{\rho}u), (\sqrt{\rho}u) - u \otimes \nabla \sqrt{\rho} &\in L^2(0, T; L^2), \\
r_1^{1/4} \rho^{1/4} u &\in L^4(0, T; L^4), \\
\kappa \nabla^2 \sqrt{\rho}, \kappa \sqrt{\rho} \nabla^2 \log \rho &\in L^2(0, T; L^2),
\end{aligned}
\quad (1.10)
\]
with \((\rho, \sqrt{\rho}u)\) satisfying
\[
\begin{aligned}
\rho_t + \text{div}(\sqrt{\rho} \sqrt{\rho}u) = 0, \\
\rho(x, t = 0) = \rho_0(x),
\end{aligned}
\quad (1.11)
\]
and if the following equality holds for all smooth test function \(\phi(x, t)\) with compact support such that \(\phi(x, T) = 0\):
\[
\begin{aligned}
\int m_0 \cdot \phi(x, 0) dx + \int_0^T \int \left( \sqrt{\rho} \sqrt{\rho}u \cdot \phi_t + \sqrt{\rho} \sqrt{\rho}u \otimes \sqrt{\rho}u : \nabla \phi + \rho \gamma \text{div}\phi \right) dx dt \\
- \nu \int_0^T \int \sqrt{\rho} \left( (\nabla (\sqrt{\rho}u) - u \otimes \nabla \sqrt{\rho}) : \nabla \phi + (\nabla^T (\sqrt{\rho}u) - \nabla \sqrt{\rho} \otimes u) : \nabla \phi \right) dx dt \\
= \int_0^T \int \left( r_0 u \cdot \phi + r_1 \rho |u|^2 u \cdot \phi + 4 \kappa^2 \Delta \sqrt{\rho} \sqrt{\rho}u \cdot \phi + 2 \kappa^2 \Delta \sqrt{\rho} \phi \text{div}\phi \right) dx dt.
\end{aligned}
\quad (1.12)
\]
Our first result reads as follows:

**Theorem 1.1** Suppose that \(\gamma \in (1, 3)\) and \(11 \kappa \leq \nu\). Moreover, assume that the initial data \((\rho_0, m_0)\) satisfy \(1.8\). Then, there exists a global weak solution \((\rho, u)\) to the problem \(1.1\,–\,1.2\) satisfying
\[
\sup_{0 \leq t \leq T} \int \left( \rho |u|^2 + \rho \gamma \right) dx + \int_0^T \int \left( r_0 |u|^2 + r_1 \rho |u|^4 \right) dx dt \leq C,
\quad (1.13)
\]
Remark 1.2 If $T$ is a well-defined function, and the lower bound of dispersive coefficient $\kappa > 0$ is crucial to deduce the key Mellet-Vasseur type estimate in [35], it requires essentially that $\nabla u$ make sense in the presence of vacuum. In particular, in the proof of [35, Lemma 4.2], which is mentioned in [25], it still does not allow to define the gradient of velocity $\nabla u$.

Compared with [4], our Theorem 1.2 succeeds in removing their assumption on the complete new regularity estimate. Combining this fact with

\[
\sqrt{\nu} p_{\nu} = \nu \nabla (\rho u) - 2\nu \sqrt{\rho} u \otimes \nabla \sqrt{\rho},
\]

where $C$ is a positive generic constant depending only on the initial data, but independent of $\kappa$, $r_0$, and $r_1$.

A few remarks are in order:

Remark 1.1 It should be noted that the arguments in Vasseur-Yu [34, 35] rely crucially on the assumption that the gradient of velocity field $\nabla u$ is a well-defined function, which indeed does not make sense in the presence of vacuum. In particular, in the proof of [35, Lemma 4.2], which is crucial to deduce the key Mellet-Vasseur type estimate in [32], it requires essentially that $\nabla u$ is a well-defined function.

Very recently, Lacroix-Violet & Vasseur [25] also study the QNS equations and consider a new assumption that the gradient of velocity field $\nabla u$ satisfies

\[
(\rho, \sqrt{\rho} u) \in L^2(0, T; H^1(\Omega)),
\]

which will be shown in our another paper [32].

More precisely, they use the function $T_n$ to give a new understanding of $\nabla u$. However, as mentioned in [25], it still does not allow to define the gradient of velocity $\nabla u$ as a function.

Remark 1.2 If $\kappa > 0$ and $r_1 > 0$, Theorem 1.1 shows that $\sqrt{\rho} u \in L^2(0, T; H^1(\Omega))$, which is a complete new regularity estimate. Combining this fact with $\sqrt{\rho} \in L^2(0, T; H^1(\Omega))$ shows that

\[
\nabla (\rho u) = \sqrt{\rho} \nabla (\sqrt{\rho} u) + \nabla \sqrt{\rho} \otimes \sqrt{\rho} u,
\]

holds rigorously in the sense of function. This new observation is helpful for further studies on the weak solutions of compressible Navier-Stokes equations, which will be shown in our another paper [32].

Next, we also obtain the global weak solutions to system (1.1) without damping terms.

Theorem 1.2 Suppose that $r_0 = r_1 = 0$, $\gamma \in (1, 3)$, and $11\kappa \leq \nu$. Moreover, assume that the initial data $(\rho_0, m_0)$ satisfy (1.8)$_1$, (1.8)$_2$, and

\[
\sqrt{\rho_0} \in L^{2+\eta}, \quad \sqrt{\rho_0} u_0 \in L^{2+\eta},
\]

for any $\eta > 0$. Then the problem (1.1)-(1.2) admits a global weak solution $(\rho, u)$ satisfying (1.10)$_1$-(1.10)$_3$. Moreover, $(\rho, \sqrt{\rho} u)$ satisfy (1.11) and

\[
\begin{align*}
\int m_0 \cdot \phi (x, 0) dx + \int_0^T \int \left( \sqrt{\rho} \sqrt{\rho} u \cdot \phi_t + \sqrt{\rho} u \otimes \sqrt{\rho} u : \nabla \phi + \rho^2 \nabla \phi \cdot \nabla \phi \right) dx dt \\
- 2\nu \int_0^T \left( \sqrt{\rho} u \otimes \sqrt{\rho} u : \nabla \phi \right) dx dt - 2\nu \int_0^T (\nabla \sqrt{\rho} \otimes \sqrt{\rho} u) : \nabla \phi dx dt \\
+ \nu \int_0^T \left( \sqrt{\rho} \sqrt{\rho} u \cdot \Delta \phi dx dt + \nu \int_0^T \left( \nabla \sqrt{\rho} \otimes \sqrt{\rho} u : \nabla \phi \right) dx dt \\
- 4\kappa^2 \int_0^T \left( \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} \right) : \nabla \phi dx dt + 2\kappa^2 \int_0^T \left( \sqrt{\rho} \nabla \sqrt{\rho} \cdot \nabla \phi \nabla \phi dx dt = 0,
\end{align*}
\]

where $\phi (x, t)$ is a smooth test function with compact support satisfying $\phi (x, T) = 0$.

Remark 1.3 Compared with [4], our Theorem 1.2 succeeds in removing their assumption on the lower bound of dispersive coefficient $\kappa > \frac{1}{3\sqrt{2}} \nu$.  

4
We now sketch some main ideas used in our analysis. The main point of this paper is to construct smooth approximate solutions satisfying the energy inequality and the BD entropy estimate. Thanks to Li-Xin [29], we first propose to approximate (1.1) by a parabolic equation (1.20). Next, on the one hand, some similar regularization in (1.20) as those in [29] are considered accordingly with respect to the parabolic regularization in (1.20). On the other hand, the third order capillarity term will bring us some new difficulties. Motivated by [4, 5] (see also [22]), by using the effective velocity $w = u + \mu \nabla \log \rho$ with $\mu = \nu - \sqrt{\nu^2 - \kappa^2}$ to handle the third order capillarity term, we thus need some additional regularization terms of $\nabla \log \rho$ in (1.20). As a result, we consider the following approximate system

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0},
\rho u_t + \rho \cdot \nabla u - 2\nu \text{div}(\rho D u) + \nabla P - 2\kappa^2 \rho \nabla \left( \frac{\Delta v}{v} \right) + r_0 u + r_1 \rho |u|^2 u \\
&= \sqrt{\varepsilon} \text{div}(\rho \nabla u) + \sqrt{\varepsilon} \mu \text{div}(\rho^2 \log \rho) + \varepsilon |\nabla v|^2 \nabla v \cdot \nabla \rho + \varepsilon \mu |\nabla v|^2 \nabla v \cdot \nabla (\nabla \log \rho) \\
&\quad - \varepsilon \rho^{-p_0} u - \varepsilon \frac{1}{2} |\rho|^3 u - \varepsilon \mu \nabla \rho^{-p_0} - \varepsilon \mu \text{div}(\text{div}(|\nabla v|^2 \nabla v)) + \varepsilon \mu \text{div}(|\nabla v|^2 \nabla v) \nabla \log \rho,
\end{align*}
$$

(1.20)

where $v \triangleq \sqrt{\rho}$. First of all, following the similar arguments as those in [29], the smooth solutions to the approximate system (1.20) satisfy both the energy inequality and the BD entropy estimates. Thanks to the compactness results due to [5, 8, 11], we can obtain the global existence of the weak solutions to (1.20). However, due to the third order dispersive term, we thus need some additional regularization terms of $\nabla \log \rho$ in (1.20). As a result, we consider the following approximate system

$$
\rho_t + \text{div}(\rho u) = \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0},
$$

(2.1)

with $\mu = \nu - \sqrt{\nu^2 - \kappa^2}$ and $11\kappa \leq \nu$, we consider the following approximate system

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0},
\rho u_t + \rho \cdot \nabla u - 2\nu \text{div}(\rho D u) + \nabla P - 2\kappa^2 \rho \nabla \left( \frac{\Delta v}{v} \right) + r_0 u + r_1 \rho |u|^2 u \\
&= \varepsilon \text{div}(\rho \nabla u) + \varepsilon \mu \text{div}(\rho^2 \log \rho) + \varepsilon |\nabla v|^2 \nabla v \cdot \nabla u + \varepsilon \mu |\nabla v|^2 \nabla v \cdot \nabla (\nabla \log \rho) \\
&\quad - \varepsilon \rho^{-p_0} u - \varepsilon \frac{1}{2} |\rho|^3 u - \varepsilon \mu \nabla \rho^{-p_0} - \varepsilon \mu \text{div}(\text{div}(|\nabla v|^2 \nabla v)) + \varepsilon \mu \text{div}(|\nabla v|^2 \nabla v) \nabla \log \rho,
\end{align*}
$$

(2.2)

2 A priori estimates

Let $v \triangleq \rho^{1/2}$ and

$$
w \triangleq u + \mu \nabla \log \rho
$$

(2.1)

with $\mu = \nu - \sqrt{\nu^2 - \kappa^2}$ and $11\kappa \leq \nu$, we consider the following approximate system

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0},
\rho u_t + \rho \cdot \nabla u - 2\nu \text{div}(\rho D u) + \nabla P - 2\kappa^2 \rho \nabla \left( \frac{\Delta v}{v} \right) + r_0 u + r_1 \rho |u|^2 u \\
&= \varepsilon \text{div}(\rho \nabla u) + \varepsilon \mu \text{div}(\rho^2 \log \rho) + \varepsilon |\nabla v|^2 \nabla v \cdot \nabla u + \varepsilon \mu |\nabla v|^2 \nabla v \cdot \nabla (\nabla \log \rho) \\
&\quad - \varepsilon \rho^{-p_0} u - \varepsilon \frac{1}{2} |\rho|^3 u - \varepsilon \mu \nabla \rho^{-p_0} - \varepsilon \mu \text{div}(\text{div}(|\nabla v|^2 \nabla v)) + \varepsilon \mu \text{div}(|\nabla v|^2 \nabla v) \nabla \log \rho,
\end{align*}
$$

(2.2)
where the constants \( p_0 \) and \( \varepsilon \) satisfying
\[
p_0 = 50, \quad 0 < \varepsilon \leq 10^{-10}.
\]
The initial conditions of the system (2.2) are imposed as:
\[
(r, u)(x, 0) = (\rho_{0e}, u_{0e}),
\]
where smooth \( \Omega \)-periodic functions \( \rho_{0e} \) > 0 and \( u_{0e} \) satisfying
\[
\| r_0 \log - \rho_{0e} \|_{L^1} + \| \rho_{0e} \|_{L^1(\Omega)} + \| \nabla \rho_{0e} \|_{L^2} + \varepsilon \| \nabla \rho_{0e} \|_{L^4}^4 + \varepsilon \| \rho_{0e}^{-p_0} \|_{L^1} \leq C
\]
and
\[
\int \rho_{0e} |u_{0e}|^2 dx \leq C
\]
for some constant \( C \) independent of \( \varepsilon \).

Some alternative ways of the third order tensor term are stated as follows
\[
2\rho \nabla \left( \frac{\Delta v}{v} \right) = \text{div}(\rho \nabla^2 \log \rho) = \nabla \Delta \rho - 4\text{div}(\nabla v \otimes \nabla v).
\]
Let \( T > 0 \) be a fixed time and \( (\rho, u) \) be a smooth solution to (2.2)–(2.3) on \( \Omega \times (0, T] \). Then, we will establish some necessary a priori bounds for \( (\rho, u) \). The first one is the energy-type inequality.

**Lemma 2.1** Suppose that \( 11 \kappa \leq \nu \), then there exists some generic constant \( C \) independent of \( \varepsilon, r_0, r_1, \) and \( \kappa \) such that

\[
\sup_{0 \leq t \leq T} \int (\rho |u|^2 + \rho + \rho^\dagger + \varepsilon \rho^{-p_0} + (2\kappa^2 + 2\mu \sqrt{\varepsilon}) |\nabla v|^2 + \varepsilon \mu |\nabla v|^4) \, dx
\]

\[
+ \nu \int_0^T \int \rho |\nabla u|^2 dx dt + r_0 \int_0^T \int |u|^2 dx dt + r_1 \int_0^T \int \rho |u|^4 dx dt
\]

\[
+ \sqrt{\varepsilon} \int_0^T \int \rho |\nabla u|^2 dx dt + \varepsilon \int_0^T \int (|\nabla v|^4 + |\nabla v|^4 |u|^2 + \rho^{-p_0} |u|^2 + \varepsilon^{1/2} \rho |\nabla |^2 |u|^2) \, dx dt
\]

\[
+ (2\kappa^2 + 2\mu \sqrt{\varepsilon}) \varepsilon \int_0^T \int (|\nabla v|^2 |\nabla^2 v|^2 + |\nabla |^4 |v|^2 + (2p_0 + 1) |\nabla v|^2 v^{-2p_0-1} \, dx dt
\]

\[
+ \varepsilon^2 \int_0^T \int (\mu |\nabla v|^4 |\nabla^2 v|^2 + \mu |\nabla v|^4 |\nabla v|^2 + (2p_0 + 1) \nu |\nabla v|^4 v^{-2p_0-2} + \rho^{-2p_0-1} \, dx dt \leq C.
\]

**Proof.** First, integrating (2.2)1 over \( \Omega \) yields
\[
\left( \int \rho dx \right)_t + \varepsilon \int |\nabla v|^4 dx = \varepsilon \int \rho^{-p_0} dx.
\]

Next, multiplying (2.2)2 by \( u \) and integrating the resulting equations by parts, we obtain after using (2.2)1 that
\[
\frac{1}{2} \left( \int \rho |u|^2 dx \right)_t + 2\nu \int \rho |\nabla u|^2 dx + \sqrt{\varepsilon} \int \rho |\nabla u|^2 dx + \frac{\varepsilon}{2} \int \rho^{-p_0} |u|^2 dx
\]

\[
+ \varepsilon^{3/2} \int \rho |u|^3 |u|^2 dx + r_0 \int |u|^2 dx + r_1 \int \rho |u|^4 dx + \int u \cdot \nabla \rho^\dagger dx + \varepsilon \mu \int u \cdot \nabla \rho^{-p_0} dx
\]

\[
= \frac{\varepsilon}{2} \int v \text{div}(|\nabla v|^2 \nabla v)|u|^2 dx + \varepsilon \int \nabla v^2 \nabla v \cdot \nabla u \cdot u \, dx
\]

\[
+ \varepsilon \mu \int v |\nabla v|^2 \nabla v \cdot \nabla (\nabla \log \rho) \cdot u dx + 2(\kappa^2 + \sqrt{\varepsilon} \mu) \int \rho \nabla \left( \frac{\Delta v}{v} \right) \cdot u dx
\]

\[
+ \varepsilon \mu \int v |\nabla v|^2 \nabla v \text{div}(\nabla \log \rho) \cdot u dx - \varepsilon \mu \int \nabla (v \text{div}(|\nabla v|^2 \nabla v)) \cdot u dx
\]

\[
= \sum_{i=1}^{6} I_i.
\]
Integration by parts gives

\[ I_1 + I_2 = -\frac{\varepsilon}{2} \int |\nabla v|^4 |u|^2 dx. \]  

(2.10)

Since \( \nabla \log \rho = 2v^{-1} \nabla v \), one has

\[ I_3 = \varepsilon \mu \int v |\nabla v|^2 \nabla v \cdot \nabla (\log \rho) \cdot u dx \]

\[ = 2\varepsilon \mu \int |\nabla v|^2 \nabla v \cdot \nabla v \cdot u dx - 2\varepsilon \mu \int v^{-1} |\nabla v|^4 \nabla v \cdot u dx \]

\[ \leq \frac{\varepsilon}{4} \int |\nabla v|^{4} |u|^2 dx + 8\varepsilon \mu^2 \int |\nabla v|^2 |\nabla^2 v|^2 dx + 8\varepsilon \mu^2 \int v^{-2} |\nabla v|^6 dx \]

\[ \leq \frac{\varepsilon}{4} \int |\nabla v|^{4} |u|^2 dx + 24\varepsilon \mu^2 \int |\nabla v|^2 |\nabla^2 v|^2 dx + 64\varepsilon \mu^2 \int |\nabla |\nabla v|^2|^2 dx, \]

(2.11)

where in the last inequality one has used the following fact

\[ \int v^{-2} |\nabla v|^6 dx \leq 2 \int |\nabla v|^2 |\Delta v|^2 dx + 8 \int |\nabla |\nabla v|^2|^2 dx. \]

(2.12)

Indeed, integration by parts together with some directly calculations show that

\[ \int v^{-2} |\nabla v|^6 dx = \int v^{-2} |\nabla v|^4 \nabla v \cdot \nabla v dx \]

\[ = \int v \nabla v^{-2} |\nabla v|^4 \cdot \nabla v dx - \int v^{-1} \nabla |\nabla v|^4 \cdot \nabla v dx - \int v^{-1} |\nabla v|^4 \Delta v dx \]

\[ = 2 \int v^{-2} |\nabla v|^6 dx - 2 \int v^{-1} |\nabla v|^2 |\nabla v|^2 \cdot \nabla v dx - \int v^{-1} |\nabla v|^4 \Delta v dx, \]

(2.13)

that is

\[ \int v^{-2} |\nabla v|^6 dx = 2 \int v^{-1} |\nabla v|^2 |\nabla v|^2 \cdot \nabla v dx + \int v^{-1} |\nabla v|^4 \Delta v dx \]

\[ \leq \frac{1}{2} \int v^{-2} |\nabla v|^6 dx + \int |\nabla v|^2 |\Delta v|^2 dx + 4 \int |\nabla |\nabla v|^2|^2 dx. \]

(2.14)

This yields (2.12) directly.

For the term \( I_4 \), it deduces from (2.21) and integration by parts that

\[ I_4 = -2(2\alpha^2 + \sqrt{\varepsilon}) \int \frac{\Delta v}{v} \text{div}(\rho u) dx \]

\[ = -2(2\alpha^2 + \sqrt{\varepsilon}) \int \frac{\Delta v}{v} \left( -2vv_t + \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0} \right) dx \]

\[ = -2(2\alpha^2 + \sqrt{\varepsilon}) \frac{d}{dt} \int |\nabla v|^2 dx \]

\[ \quad - 2(2\alpha^2 + \sqrt{\varepsilon}) \varepsilon \int \left( |\nabla v|^2 |\nabla^2 v|^2 + \frac{1}{2} |\nabla |\nabla v|^2|^2 + (2p_0 + 1) |\nabla v|^2 v^{-2p_0-2} \right) dx \]

(2.15)

owing to the following fact (with \( r \geq 0 \))

\[ \int \text{div}(|\nabla v|^r \nabla v) \text{div}(|\nabla v|^2 \nabla v) dx \]

\[ = \int \partial_j (|\nabla v|^r \partial_j v) \partial_i (|\nabla v|^2 \partial_j v) dx \]

\[ = \int \partial_j |\nabla v|^r \partial_i \partial_j v |\nabla v|^2 \partial_j v dx + \int |\nabla v|^r \partial_i \partial_j v |\nabla v|^2 \partial_j v dx \]

\[ \quad + \int |\nabla v|^r \partial_i v |\nabla v|^2 \partial_j v dx + \int |\nabla v|^r \partial_i |\nabla v|^2 \partial_j v dx \]

\[ = \int (2r(\nabla v \cdot |\nabla v|^2)^2 |\nabla v|^r + (r + 2)|\nabla |\nabla v|^2|^2 |\nabla v|^{r+2} + |\nabla v|^{r+2} |\nabla^2 v|^2) dx. \]

(2.16)
Next, we have

\[ I_5 = 2\varepsilon \mu \int |\nabla v|^2 \Delta v \nabla v \cdot u dx + 2\varepsilon \mu \int |\nabla v|^2 \cdot \nabla v \nabla v \cdot u dx \]

\[ \leq 32\varepsilon \mu^2 \int |\nabla v|^2 |\Delta v|^2 dx + 32\varepsilon \mu^2 \int |\nabla |\nabla v|^2| dx + \frac{\varepsilon}{16} \int |\nabla v|^4 |u|^2 dx. \quad (2.17) \]

Notice that

\[ \int \rho (\text{div}u)^2 dx \leq 3 \int \rho |Du|^2 dx, \quad (2.18) \]

this combined with Hölder inequality gives

\[ I_6 \leq 3\mu \int \rho |Du|^2 dx + \frac{\varepsilon^2 \mu}{4} \int (\text{div}(|\nabla v|^2 \nabla v))^2 dx. \quad (2.19) \]

In order to control the last term of (2.19), we recall that \( v \) satisfies

\[ 2v_t - \varepsilon \text{div}(|\nabla v|^2 \nabla v) = -2u \cdot \nabla v - \text{div}uv + \varepsilon v^{-2p_0-1}. \quad (2.20) \]

Multiplying (2.20) by \( \mu \varepsilon \text{div}(|\nabla v|^2 \nabla v) \) and integrating the resulting equality over \( \Omega \) lead to

\[ \frac{\mu \varepsilon}{2} \left( \int |\nabla v|^4 dx \right) + \mu \varepsilon \int (\text{div}(|\nabla v|^2 \nabla v))^2 dx + \mu (2p_0 + 1) \varepsilon \int v^{-2p_0-2} |\nabla v|^4 dx \]

\[ = \mu \varepsilon \int \text{div}(|\nabla v|^2 \nabla v) \text{div}udx + 2\mu \varepsilon \int (\nabla |\nabla v|^2 \cdot \nabla v + |\nabla v|^2 \Delta v) u \cdot \nabla v dx \]

\[ \leq \frac{\varepsilon^2 \mu}{4} \int (\text{div}(|\nabla v|^2 \nabla v))^2 dx + \mu \int (\text{div}u)^2 dx + \frac{\varepsilon}{16} \int |\nabla v|^4 |u|^2 dx \]

\[ + 32\varepsilon \mu^2 \int |\nabla v|^2 |\nabla v|^2 dx + 32\varepsilon \mu^2 \int |\nabla v|^2 |\nabla v|^2 dx. \quad (2.21) \]

Submitting (2.10), (2.11), (2.15), (2.17), and (2.19) into (2.9), then adding the resulting inequality together with (2.21), one has

\[ \frac{d}{dt} \left[ \frac{1}{2} \int \rho |u|^2 dx + 2(\kappa^2 + \varepsilon \mu) \int |\nabla v|^2 dx + \frac{\mu \varepsilon}{2} \int |\nabla v|^4 dx \right] \]

\[ + 2\nu \int \rho |Du|^2 dx + \frac{\sqrt{\varepsilon}}{2} \int \rho |\nabla u|^2 dx + \frac{\varepsilon}{8} \int |\nabla v|^4 |u|^2 dx + \frac{\varepsilon}{2} \int \rho^{-p_0} |u|^2 dx \]

\[ + 2(\kappa^2 + \sqrt{\varepsilon} \mu) \varepsilon \int \left( |\nabla v|^2 |\nabla v|^2 + \frac{1}{2} |\nabla |\nabla v|^2|^2 + (2p_0 + 1)|\nabla v|^2 v^{-2p_0-2} \right) dx \]

\[ + \frac{1}{2} \mu \varepsilon^2 \int (\text{div}(|\nabla v|^2 \nabla v))^2 dx + \mu (2p_0 + 1) \varepsilon^2 \int v^{-2p_0-2} |\nabla v|^4 dx \]

\[ + \varepsilon^{3/2} \int \rho |u|^3 |u|^2 dx + r_0 \int |u|^2 dx + r_1 \int \rho |u|^4 dx + \int u \cdot \nabla \rho^q dx + \varepsilon \mu \int u \cdot \nabla \rho^{-p_0} dx \]

\[ \leq 6\mu \int \rho |Du|^2 dx + 128\varepsilon \mu^2 \int |\nabla |\nabla v|^2|^2 dx + 88\varepsilon \rho^2 \int |\nabla v|^2 |\nabla v|^2 dx. \quad (2.22) \]

Now, for the last two terms on the left hand side of (2.22), it holds that for \( q \neq 1, \)

\[ \int u \cdot \nabla \rho^q dx = -\frac{q}{q-1} \int \rho^{q-1} \text{div}(\rho u) dx \]

\[ = -\frac{q}{q-1} \int \rho^{q-1} (-\rho_t + \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0}) dx \quad (2.23) \]

\[ = \frac{1}{q-1} \left( \int \rho^q dx \right) - \frac{q(2q-1)}{q-1} \int \rho^{q-1} |\nabla v|^4 dx - \frac{q}{q-1} \int \rho^{q-1-p_0} dx. \]
Choosing \( q = -p_0 \) in (2.23), one gets
\[
\frac{1}{6(p_0 + 1)} \left( \int \rho^{-p_0} \,dx \right)_t + \frac{p_0(2p_0 + 1)\varepsilon^2}{6(p_0 + 1)} \int \rho^{-p_0} |\nabla v|^4 \,dx + \frac{p_0\varepsilon^2}{6(p_0 + 1)} \int \rho^{-1-2p_0} \,dx \\
= \frac{\varepsilon}{6} \int \rho^{-p_0} \text{div} \, dx \\
\leq \frac{p_0\varepsilon^2}{12(p_0 + 1)} \int \rho^{-1-2p_0} \,dx + \frac{1}{2} \int \rho|Du|^2 \,dx.
\]
Finally, choosing
\[
11\kappa \leq \nu
\]
such that
\[
20\mu < \nu, \quad 400\mu^2 < \kappa^2,
\]
multiplying (2.21) by \( \nu \) and \( 6\mu \), respectively, then adding the resulting inequalities, (2.8) and (2.22) together, we thus obtain (2.7) after using (2.23), (2.16), (2.26), Gronwall’s inequality, and the following simple fact
\[
\rho^{-p_0+\gamma-1} \leq \rho + \rho^{-p_0}.
\]
Hence, the proof of Lemma 2.1 is finished.

Next, with the same spirit of the BD entropy estimates due to Bresch-Desjardins [7,9,11], we have the following estimates in Lemma 2.2.

**Lemma 2.2** There exists some generic constant \( C \) independent of \( \varepsilon, r_0, r_1, \) and \( \kappa \) such that
\[
\sup_{0 \leq t \leq T} \int (|\nabla v|^2 + \varepsilon|\nabla v|^4 - r_0 \log \rho) \,dx + \int_0^T \int \left( |\nabla u|^2 + |\nabla (\sqrt{\rho}u) - u \otimes \nabla \sqrt{\rho}|^2 + \rho^{-2} |\nabla \rho|^2 \right) \,dx\,dt \\
+ (\kappa^2 + \sqrt{\varepsilon}\mu) \int_0^T \int \rho|\nabla^2 \log \rho|^2 \,dx\,dt + \varepsilon \nu \int_0^T \int \left( |\nabla v|^2 |\nabla^2 v|^2 + |\nabla v|^2 |\nabla |\nabla v|^2 | + \rho^{-p_0-1} |\nabla v|^2 \right) \,dx\,dt \\
+ \varepsilon^2 \int_0^T \int (|\nabla v|^4 |\nabla^2 v|^2 + |\nabla v|^4 |\nabla |\nabla v|^2 |^2 + \rho^{-p_0-1} |\nabla v|^4) \,dx\,dt + \varepsilon \mu \int_0^T \int v^{-2} |\nabla v|\,dx\,dt \\
+ r_0 \varepsilon \int_0^T \int \left( v^{-2} |\nabla v|^4 + \rho^{-p_0-1} \right) \,dx\,dt \leq C + Cr_0 + Cr_1.
\]
Furthermore, it holds that
\[
\varepsilon^2 \int_0^T \int (|w|^5 + \rho|u|^5) \,dx\,dt + \varepsilon \int_0^T \int \left( v^{-2} |\nabla v|^6 + \rho^{-3} |\nabla v|^5 \right) \,dx\,dt \leq C + Cr_0 + Cr_1.
\]

**Proof.** First, set
\[
G \triangleq \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0},
\]
multiplying (2.2) by \( \rho^{-1} \) and applying gradient to the resulting equality lead to
\[
(\nabla \log \rho)_t + u \cdot \nabla \log \rho + \nabla u \cdot \nabla \log \rho + \nabla \text{div} u = \nabla (\rho^{-1} G).
\]
Thus, multiplying (2.29) by \( \nabla \rho \), we obtain after using integration by parts and (2.22) that
\[
\frac{1}{2} \left( \int \rho^{-1} |\nabla \rho|^2 \,dx \right)_t + \int \rho^{-1} \nabla \rho \cdot \nabla u \cdot \nabla \rho \,dx + \int \nabla \rho \cdot \nabla \text{div} u \,dx \\
+ \int \rho^{-1} G \left( \Delta \rho - \frac{1}{2} \rho^{-1} |\nabla \rho|^2 \right) \,dx = 0.
\]
Then, multiplying (2.2) by $\nabla \log \rho = \rho^{-1} \nabla \rho$ and integrating by parts yield
\[
\int u_t \cdot \nabla \rho dx + \int u \cdot \nabla u \cdot \nabla \rho dx - 2\nu \int \text{div}(\rho Du) \cdot \nabla \log \rho dx - \sqrt{\varepsilon} \int \text{div}(\rho \nabla u) \cdot \nabla \log \rho dx
\]
\[
+ \int P'(\rho)|\nabla \rho|^2 dx + 2(\kappa^2 + \sqrt{\varepsilon} \mu) \int \rho|\nabla^2 \log \rho|^2 dx
\]
\[
= \varepsilon \int v|\nabla v|^2 \nabla v \cdot \nabla \log \rho dx + \varepsilon \mu \int v|\nabla v|^2 \nabla v \cdot (\nabla \log \rho) \cdot \nabla \log \rho dx
\]
\[
- \varepsilon \int \rho^{-\nu_0} u \cdot \nabla \log \rho dx - \varepsilon^{3/2} \int \rho |u|^3 \cdot \nabla \log \rho dx
\]
\[
- r_0 \int u \cdot \nabla \log \rho dx - r_1 \int \rho |u|^2 \cdot \nabla \log \rho dx - \varepsilon \mu \int \nabla(\text{div}(|\nabla v|^2 \nabla v)) \cdot \nabla \log \rho dx
\]
\[
+ \varepsilon \mu \int \text{div}(|\nabla v|^2 \nabla \log \rho) \nabla \log \rho dx - \varepsilon \mu \int \nabla \rho^{-\nu_0} \cdot \nabla \log \rho dx
\]
\[
\triangleq \sum_{i=1}^{9} \tilde{I}_i,
\]
where the first term on the left hand of (2.31) can be handled as follows
\[
\int u_t \cdot \nabla \rho dx = \left( \int u \cdot \nabla \rho dx \right)_t - \int u \cdot \nabla u \cdot \nabla \rho dx
\]
\[
- 2 \int \rho Du : \nabla u dx + \int \rho |\nabla u|^2 dx + \int \text{div} u G dx.
\]
Adding (2.30) multiplied by $2\nu + \sqrt{\varepsilon}$ to (2.31) and using (2.32), one has
\[
\frac{2\nu + \sqrt{\varepsilon}}{2} \left( \int \rho^{-1} |\nabla \rho|^2 dx \right)_t + \left( \int u \cdot \nabla \rho dx \right)_t + \int \rho |\nabla u|^2 dx + 2(\kappa^2 + \sqrt{\varepsilon} \mu) \int \rho |\nabla^2 \log \rho|^2 dx
\]
\[
+ \int P'(\rho)|\nabla \rho|^2 dx + (2\nu + \sqrt{\varepsilon}) \int \rho^{-1} G \left( \Delta \rho - \frac{1}{2} \rho^{-1} |\nabla \rho|^2 \right) dx
\]
\[
= - \int G \text{div} u dx + 2 \int \rho Du : \nabla u dx + \sum_{i=1}^{9} \tilde{I}_i.
\]
Since
\[
\Delta \rho - \frac{1}{2} \rho^{-1} |\nabla \rho|^2 = 2v \Delta v,
\]
the last term on the left-hand side of (2.33) can be calculated as
\[
\int \rho^{-1} G \left( \Delta \rho - \frac{1}{2} \rho^{-1} |\nabla \rho|^2 \right) dx
\]
\[
= 2\varepsilon \int \text{div}(|\nabla v|^2 \nabla v) \Delta v dx + 2\varepsilon \int \rho^{-\nu_0 - 1/2} \Delta v dx
\]
\[
= 2\varepsilon \int |\nabla v|^2 |\nabla^2 v|^2 dx + \varepsilon \int |\nabla v|^2 |\nabla^2 v|^2 dx + 2(2\nu_0 + 1)\varepsilon \int \rho^{-\nu_0 - 1} |\nabla v|^2 dx,
\]
where we have used (2.16) with $r = 0$.

Now, we will estimate each term on the righthand side of (2.33) in the following way. First, with the same arguments as those in [29], one has
\[
- \int \text{div} u G dx + 2 \int \rho Du : \nabla u dx + \tilde{I}_1 + \tilde{I}_3
\]
\[
\leq \frac{\varepsilon^2}{8} \int (\text{div}(|\nabla v|^2 \nabla v))^2 dx + \frac{\mu}{2} \varepsilon \int |\nabla v|^2 |\nabla^2 v|^2 dx + \frac{1}{4} \int \rho |\nabla u|^2 dx
\]
\[
+ C \varepsilon^2 \int \rho^{-2\nu_0 - 1} dx + C(\nu) \varepsilon \int |\nabla v|^4 |u|^2 dx + C \int \rho |Du|^2 dx.
\]
Next, it holds
\[
\tilde{I}_2 = -\frac{1}{2} \varepsilon \mu \int \text{div}(v|\nabla v|^2 \nabla v) |\nabla \log \rho|^2 dx
\]
\[
= -2\varepsilon \mu \int v^{-2} |\nabla v|^2 \left( |\nabla v|^4 + v \nabla |\nabla v|^2 \cdot \nabla v + v |\nabla v|^2 \Delta v \right) dx
\]
\[
= -2\varepsilon \mu \int v^{-2} |\nabla v|^6 dx - 2\varepsilon \mu \int |\nabla v|^2 |\nabla v|^2 \cdot \nabla v dx - 2\varepsilon \mu \int v^{-1} |\nabla v|^4 \Delta v dx
\]
\[
\leq -\varepsilon \mu \int v^{-2} |\nabla v|^6 dx + 2\varepsilon \mu \int |\nabla v|^2 |\nabla v|^2 dx + 2\varepsilon \mu \int |\nabla v|^2 |\Delta v|^2 dx.
\]
(2.37)

Recalling the definition of \(w\) and using Young’s inequality, one gets
\[
\tilde{I}_4 = -2\varepsilon^{3/2} \int \rho^{1/2} |w|^3 u \cdot \nabla v dx
\leq \varepsilon^{3/2} \int \rho |w|^3 |u|^2 dx + \varepsilon^{3/2} \int |w|^3 |\nabla v|^2 dx
\]
\[
\leq C(\nu)\varepsilon^{3/2} \int \rho |w|^3 |u|^2 dx + C(\nu)\varepsilon^{3/2} \int v^{-3} |\nabla v|^5 dx,
\]
(2.38)

where in the last inequality we have used the following fact:
\[
\varepsilon^{3/2} \int |w|^3 |\nabla v|^2 dx
\leq \frac{1}{16\rho^2} \varepsilon^{3/2} \int \rho |w|^3 |u + \mu \nabla \log \rho|^2 dx + C(\nu)\varepsilon^{3/2} \int \rho^{-3/2} |\nabla v|^5 dx
\]
\[
\leq \frac{1}{8\nu^2} \varepsilon^{3/2} \int \rho |w|^3 |u|^2 dx + \frac{1}{2}\varepsilon^{3/2} \int |w|^3 |\nabla v|^2 dx + C(\nu)\varepsilon^{3/2} \int v^{-3} |\nabla v|^5 dx.
\]
(2.39)

The last term on the left hand of (2.38) can be handled as follows:
\[
\int v^{-3} |\nabla v|^5 dx = \int v^{-3} |\nabla v|^3 v \cdot \nabla v dx
\]
\[
= 3 \int v^{-3} |\nabla v|^5 dx - \int (\nabla |\nabla v|^3 \cdot \nabla v + |\nabla v|^3 \Delta v) dx,
\]
(2.40)

which along with (2.16) and Young’s inequality shows
\[
C(\nu)\varepsilon^{3/2} \int v^{-3} |\nabla v|^5 dx = C(\nu)\varepsilon^{3/2} \int v^{-2} (\nabla |\nabla v|^3 \cdot \nabla v + |\nabla v|^3 \Delta v) dx
\]
\[
\leq \frac{1}{8} \varepsilon^2 \int (|\nabla v|^4 |\nabla |\nabla v|^2|^2 + |\nabla v|^4 |\nabla^2 v|^2) dx + C(\nu)\varepsilon \int \rho^{-2} |\nabla v|^2 dx
\]
\[
\leq \frac{\varepsilon^2}{8} \int (\text{div}(\nabla |\nabla v|^2 \nabla v))^2 dx + \varepsilon \int |\nabla v|^4 dx
\]
\[
+ C(\nu)\varepsilon \int \rho dx + C(\nu)\varepsilon \int \rho^{-p_0} dx.
\]
(2.41)

Combined this with (2.38) yields that
\[
\tilde{I}_4 \leq C(\nu)\varepsilon^{3/2} \int \rho |w|^3 |u|^2 dx + \varepsilon^2 \int (\text{div}(\nabla |\nabla v|^2 \nabla v))^2 dx
\]
\[
+ C(\nu) \int \rho dx + C(\nu)\varepsilon \int \rho^{-p_0} dx + \varepsilon \int |\nabla v|^4 dx.
\]
(2.42)

The terms \(\tilde{I}_5 - \tilde{I}_8\) can be handled by some directly calculations:
\[
\tilde{I}_5 = -r_0 \int \frac{u \cdot \nabla \rho}{\rho} dx = r_0 \int \rho \varepsilon \Delta v - \varepsilon \rho i v \text{div}(\nabla |\nabla v|^2 \nabla v) - \varepsilon \rho^{-p_0} dx
\]
\[
= r_0 \left( \int \log \rho dx \right) - r_0 \varepsilon \int v^{-2} |\nabla v|^4 dx - r_0 \varepsilon \int \rho^{-p_0 - 1} dx,
\]
(2.43)
\[ \tilde{I}_6 = r_1 \int |u|^2 \text{div} u \rho dx + 2r_1 \int u \cdot \nabla u \cdot u \rho dx \leq Cr_1^2 \int \rho |u|^4 dx + \frac{1}{4} \int \rho |\nabla u|^2 dx. \] (2.44)

and

\[ \tilde{I}_7 + \tilde{I}_8 + \tilde{I}_9 \leq \frac{\varepsilon^2}{4} \int |\text{div}((|\nabla v|^2 \nabla v)|^2) dx + \frac{3 \mu^2}{2} \int \rho |\nabla^2 \rho|^2 dx + \frac{\varepsilon^2}{2} \int \rho^{-2p_0-1} dx + \frac{\varepsilon^2}{2} \int \varepsilon^2 |\nabla v|^6 dx + 8 \varepsilon \mu \int |\nabla |\nabla v|^2|^2 dx \] (2.45)

Substituting (2.33)–(2.37) and (2.32)–(2.34) into (2.33), we obtain after using (2.25) and (2.26) that

\[ \frac{2 \nu + \sqrt{\varepsilon}}{2} \left( \int \rho^{-1} |\nabla \rho|^2 dx \right) + \left( \int u \cdot \nabla \rho \rho dx \right) + \frac{1}{2} \int \rho |\nabla u|^2 dx \]

\[ + \left( \frac{\kappa^2}{2} + \sqrt{\varepsilon} \mu \right) \int \rho |\nabla^2 \rho|^2 dx + \int P'(\rho) \rho^{-1} |\nabla \rho|^2 dx + \frac{1}{2} \varepsilon \mu \int \varepsilon^2 |\nabla u|^6 dx \]

\[ + (2 \nu + 2 \sqrt{\varepsilon}) \varepsilon \left( \int |\nabla v|^2 |\nabla^2 v|^2 dx + \frac{1}{2} \int |\nabla |\nabla v|^2|^2 dx \right) + (2p_0 + 1) \int \rho^{-p_0-1} |\nabla v|^2 dx \]

\[ + r_0 \varepsilon \int \varepsilon^2 |\nabla v|^4 dx + r_0 \varepsilon \int \rho^{-p_0-1} dx \] (2.46)

Next, with similar arguments as (2.21), it holds that

\[ \left( \frac{1}{2} \int \varepsilon |\nabla v|^4 dx \right) + \varepsilon^2 \int (\text{div}((|\nabla v|^2 \nabla v)|^2) dx + (2p_0 + 1) \varepsilon^2 \int \varepsilon^2 |\nabla v|^4 dx \]

\[ = \varepsilon \int \text{div}((|\nabla v|^2 \nabla v) \text{div} u dx + 2 \varepsilon \int \text{div}((|\nabla v|^2 \nabla v) u \cdot \nabla v dx \]

\[ \leq \frac{\varepsilon^2}{2} \int |\nabla v|^2 |\nabla v|^2 dx + C \int \rho(\text{div} u)^2 dx + \frac{\nu^2}{2} \int |\nabla v|^2 |\nabla^2 v|^2 dx + C(\nu) \varepsilon \int |u|^2 |\nabla v|^4 dx. \] (2.47)

The combination of (2.46) with (2.47) yields

\[ \frac{2 \nu + \sqrt{\varepsilon}}{2} \left( \int \rho^{-1} |\nabla \rho|^2 dx \right) + \left( \int u \cdot \nabla \rho \rho dx \right) + \frac{\varepsilon^2}{2} \left( \int |\nabla v|^4 dx \right) + \frac{1}{2} \int \rho |\nabla u|^2 dx \]

\[ + \left( \frac{\kappa^2}{2} + \sqrt{\varepsilon} \mu \right) \int \rho |\nabla^2 \rho|^2 dx + \int P'(\rho) \rho^{-1} |\nabla \rho|^2 dx + \frac{1}{2} \varepsilon \mu \int \varepsilon^2 |\nabla u|^6 dx \]

\[ + (\nu + 2 \sqrt{\varepsilon}) \varepsilon \left( \int |\nabla v|^2 |\nabla^2 v|^2 dx + \frac{1}{2} \int |\nabla |\nabla v|^2|^2 dx \right) + (2p_0 + 1) \int \rho^{-p_0-1} |\nabla v|^2 dx \]

\[ + r_0 \varepsilon \int \varepsilon^2 |\nabla v|^4 dx + r_0 \varepsilon \int \rho^{-p_0-1} dx \]

\[ \leq C \varepsilon^2 \rho^{-2p_0-1} dx + C \int \rho |\nabla u|^2 dx + C(\nu) \varepsilon \int |\nabla v|^2 |\nabla v|^2 dx + C(\nu) \varepsilon^{3/2} \int \rho |\nabla v|^3 |u|^2 dx \]

\[ + C(\nu) \int \rho dx + C(\nu) \varepsilon \int \rho^{-p_0} dx + \varepsilon \int |\nabla v|^4 dx + Cr_1^2 \int \rho |u|^4 dx + r_0 \left( \int \log \rho dx \right) \]

\[ \triangleq H + r_0 \left( \int \log \rho dx \right), \]
On the one hand, one deduces from (2.7) that $H$ satisfies
\[ \int_0^T H dt \leq C + Cr_1. \tag{2.49} \]

On the other hand, recalling that $-\log_+ \rho_0 \in L^1$ and using (2.7), it holds
\[ \int_0^T r_0 \left( \int \log \rho dx \right)_t dt = r_0 \int \log \rho dx - r_0 \int \log \rho_0 dx = r_0 \int \log_+ \rho dx + r_0 \int \log_+ \rho dx - r_0 \int \log_+ \rho dx = r_0 \int \log_+ \rho dx + Cr_0, \tag{2.50} \]
where $\log_+ g \equiv \log \max\{1, g\}$.

Noting that
\[ \sqrt{p} \nabla u = \nabla (\sqrt{p} u) - u \otimes \nabla \sqrt{p}, \tag{2.51} \]
we thus deduce (2.27) directly by integrating (2.48) over $[0, T]$ and using (2.49), (2.50), (2.7), and (2.51).

Finally, some directly calculations together with Hölder inequality and (2.12) deduce that
\[ \varepsilon \int v^{-3} |\nabla v|^5 dx dt + \varepsilon \int v^{-2} |\nabla v|^6 dx \leq C \varepsilon \int v^{-2} |\nabla v|^6 dx + C \varepsilon \int \rho^{-4} dx \leq C \varepsilon \int |\nabla v|^2 |\Delta v|^2 dx + C \varepsilon \int |\nabla |\nabla v|^2|^2 dx + C \varepsilon \int (\rho + \rho^p_0) dx, \]
which along with (2.7) and (2.27) shows that
\[ \varepsilon \int_0^T \int v^{-2} |\nabla v|^6 dx dt + \varepsilon \int_0^T \int v^{-3} |\nabla v|^5 dx dt \leq C + Cr_0 + Cr_1. \tag{2.52} \]

Then it follows from (2.7), (2.27), (2.52), and Hölder inequality that
\[ \varepsilon^{3/2} \int_0^T \int (\rho |w|^5 + \rho |u|^5) dx \]
\[ = \varepsilon^{3/2} \int_0^T \int (\rho |w|^3 |u| + \mu \nabla \log \rho^2 + \rho |w - \mu \nabla \log \rho| |u|^2) dx \leq C \varepsilon^{3/2} \int_0^T \int \rho |w|^3 |u|^2 dx + C \varepsilon^{3/2} \int_0^T \int \rho |w|^3 |\nabla \log \rho|^2 dx dt + C \varepsilon^{3/2} \int_0^T \int \rho |\nabla \log \rho|^3 |u|^2 dx \]
\[ \leq C + Cr_0 + Cr_1 + C \varepsilon^{3/2} \int_0^T \int |\nabla v|^5 v^{-3} dx + \frac{1}{2} \varepsilon^{3/2} \int_0^T \int \rho |w|^5 dx + \frac{1}{2} \varepsilon^{3/2} \int_0^T \int \rho |u|^5 dx \]
\[ \leq C + Cr_0 + Cr_1 + \frac{1}{2} \varepsilon^{3/2} \int_0^T \int \rho |w|^5 dx + \frac{1}{2} \varepsilon^{3/2} \int_0^T \int \rho |u|^5 dx. \tag{2.53} \]

Thus, The combination of (2.7) and (2.53) gives (2.26). The proof of Lemma 2.2 is completed. \[ \square \]

Now, using the BD-entropy inequality obtained in Lemma 2.2, we can obtain following useful a priori estimates.

**Lemma 2.3** There exists some generic constant $C$ independent of $\varepsilon$, $r_0$, $r_1$, and $\kappa$ such that
\[ \kappa^2 \int_0^T \int \left( |\nabla \rho^\frac{1}{\kappa} |^4 + |\nabla^2 \rho^\frac{1}{\kappa} |^2 \right) dx dt + r_1 \kappa \int_0^T \int |\nabla (\sqrt{p} u)|^2 dx dt \leq C + Cr_0 + Cr_1. \tag{2.54} \]
Lemma 2.4 There exists some positive constant $C$ depending on $\varepsilon$, $r_0$, $r_1$, and $\kappa$ such that for all $(x,t) \in \Omega \times (0,T)$

$$C^{-1} \leq \rho(x,t) \leq C.$$  

Proof. First, recalling the following facts due to Jüngel [22] (see also [34, Lemma 2.1])

$$\int |\nabla \rho|^4 dx \leq 8 \int \rho |\nabla \log \rho|^2 dx, \quad \int |\nabla^2 \rho|^2 dx \leq 7 \int \rho |\nabla \log \rho|^2 dx,$$  

(2.55)

which combined with (2.27) shows that

$$\kappa^2 \int_0^T \int \left( |\nabla \rho|^4 + |\nabla^2 \rho|^2 \right) dx dt \leq 15\kappa^2 \int_0^T \int \rho |\nabla \log \rho|^2 dx dt \leq C + Cr_0 + Cr_1.$$  

(2.56)

Next, we have

$$\nabla (\sqrt{\rho} u) = \sqrt{\rho} \nabla u + u \otimes \nabla \sqrt{\rho} = \sqrt{\rho} \nabla u + 2 \rho^{1/2} u \otimes \nabla \rho^{1/2},$$

(2.57)

which along with (2.56), (2.7), and (2.27) that

$$\int_0^T \int r_1 \kappa |\nabla (\sqrt{\rho} u)|^2 dx dt \leq 2r_1 \kappa \int_0^T \int \rho |\nabla u|^2 dx dt + 8r_1 \kappa \int_0^T \int \rho^{1/2} |u|^2 |\nabla \rho^{1/2}|^2 dx dt$$

$$\leq Cr_1 \int_0^T \int \rho |\nabla u|^2 dx dt + 4r_1^2 \int_0^T \int \rho |u|^2 dx dt + 4r_1^2 \int_0^T \int |\nabla \rho^{1/2}|^4 dx dt$$

$$\leq C + Cr_0 + Cr_1.$$  

(2.58)

This combined with (2.56) gives (2.54) and thus finishes the proof of Lemma 2.4.

Proof. The proofs are similar to the arguments in Li-Xin [29, Lemma 4.4]. We sketch them here for completeness.

First, it follows from (2.27), (2.7), and Sobolev inequality that

$$\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} = \sup_{0 \leq t \leq T} \|v\|_{L^\infty}^2 \leq C \sup_{0 \leq t \leq T} (\|v\|_{L^2} + \|\nabla v\|_{L^4})^2 \leq C.$$  

(2.59)

Next, we will use a De Giorgi-type procedure to obtain the lower bound of the density. In fact, since $h \equiv v^{-1}$ satisfies

$$2h_t + 2u \cdot \nabla h - h \text{div} u + \varepsilon h^{2p_0+3} + 2 \varepsilon h^{-5} |\nabla h|^4 = \varepsilon \text{div}(h^{-4} |\nabla h|^2 \nabla h),$$

(2.60)

multiplying (2.60) by $(h - k)_+$ with $k \geq \|h(\cdot, 0)\|_{L^\infty} = \|\rho_0^{-1/2}\|_{L^\infty}$ yields that

$$\sup_{0 \leq t \leq T} \int (h - k)_+^2 dx + \varepsilon \int_0^T \int h^{-4} |\nabla (h - k)_+|^4 dx dt$$

$$\leq C \int_0^T \int h|u||\nabla (h - k)_+| dx dt + C \int_0^T \int (h - k)_+ |u||\nabla h| dx dt$$

$$\leq C \int_0^T \int \frac{1}_{\Lambda_k} \rho^{-4/3} |u|^{4/3} dx dt + \frac{\varepsilon}{2} \int_0^T \int h^{-4} |\nabla (h - k)_+|^4 dx dt,$$  

(2.61)
where $\hat{A}_k \triangleq \{(x, t) \in \Omega \times (0, T) | h(x, t) > k\}$. Denote $\hat{\nu}_k \triangleq |\hat{A}_k|$, it follows from Hölder inequality, (2.7), and (2.28) that

$$
\int_0^T \int 1_{\hat{A}_k} \rho^{-4/3} |u|^{4/3} dx dt \\
\leq \left( \int_0^T \int 1_{\hat{A}_k} \rho^{-24/11} dx dt \right)^{11/15} \left( \int_0^T \int \rho |u|^5 dx dt \right)^{4/15} \\
\leq C \left( \int_0^T \int (\rho + \rho^{-p_0}) dx dt \right)^{1/15} |\hat{A}_k|^{2/3} \\
\leq C \hat{\nu}_k^{2/3}.
$$

(2.62)

Now, submitting (2.62) into (2.61) leads to

$$
\sup_{0 \leq t \leq T} \int (h - k)^2_+ dx + \int_0^T \int h^{-4} |\nabla (h - k)_+|^4 dx dt \leq C \hat{\nu}_k^{2/3}.
$$

(2.63)

This together with (2.7) and Hölder inequality gives

$$
\int_0^T \int |\nabla (h - k)_+|^2 dx dt = \int_0^T \int 1_{\hat{A}_k} h^2 h^{-2} |\nabla (h - k)_+|^2 dx dt \\
\leq \int_0^T \left( \int \left( \int h^2 dx \right)^{1/3} \left( \int h^{12} dx \right)^{1/6} \left( \int h^{-4} |\nabla (h - k)_+|^4 dx \right)^{1/2} dt \right) \\
\leq C \hat{\nu}_k^{1/3} \left( \int_0^T \left( \int h^{12} dx \right)^{1/3} dt \right)^{1/2} \left( \int_0^T \int h^{-4} |\nabla (h - k)_+|^4 dx dt \right)^{1/2} \\
\leq C \hat{\nu}_k^{2/3} \left( \int_0^T \left( \int (\rho + \rho^{-p_0}) dx \right)^{1/3} dt \right)^{1/2} \\
\leq C \hat{\nu}_k^{2/3}.
$$

(2.64)

Hence, the Sobolev inequality combined with (2.63) and Hölder inequality derive that

$$
\|(h - k)_+\|_{L^{10/3}(\Omega \times (0, T))}^2 \leq C \sup_{0 \leq t \leq T} \int (h - k)^2_+ dx + C \int_0^T \int |\nabla (h - k)_+|^2 dx dt \leq C \hat{\nu}_k^{2/3}.
$$

(2.65)

This implies that for $\tilde{k} > k$,

$$
\hat{\nu}_k \leq C (\tilde{k} - k)^{-10/3} \hat{\nu}_k^{10/9}
$$

(2.66)

due to the following simple fact that

$$
(\tilde{k} - k)^2 |\hat{A}_k|^{3/5} \leq \|(h - k)_+\|_{L^{10/3}(\Omega \times (0, T))}^2.
$$

Finally, it follows from (2.66) and the De Giorgi-type lemma [36, Lemma 4.1.1] that there exists some positive constant $C \geq \hat{C}$ such that

$$
\sup_{(x, t) \in \Omega \times (0, T)} \rho^{-1}(x, t) \leq C,
$$

which along with (2.59) gives (2.58) and thus completes the proof of Lemma 2.4.

$\square$

In order to overcome the difficulties come from the third order tensor term in (1.1), we will use a transformation through the effective velocity $w$ which is defined in (2.1). Next lemma shows that the system of $(\rho, u)$ can be written equivalently in terms of $(\rho, w)$.
Lemma 2.5 Let \((\rho, u)\) be a smooth solution of the system \((2.2)\), then \((\rho, w)\) with \(w\) defined in \((2.1)\) will satisfy the following system

\[
\begin{align*}
\rho_t + \text{div}(\rho w) &= \mu \Delta \rho + \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-\rho_0}, \\
\rho w_t + \rho w \cdot \nabla w + \nabla P - 2(\nu - \mu)\text{div}(\rho Dw) - \mu \Delta w - \sqrt{\varepsilon} \text{div}(\rho \nabla w) &= 2\mu \rho \nabla w + \varepsilon |\nabla v|^2 \nabla v \cdot \nabla w - \varepsilon^{3/2} \rho |w|^3 u - r_0 u - r_1 |u|^2 u - \varepsilon \rho^{-\rho_0} w.
\end{align*}
\]  

\(2.67\)

Proof. First, it is easy to deduce from \((2.2)\) that

\[
\rho_t + \text{div}(\rho w) = \mu \Delta \rho + \varepsilon \text{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-\rho_0}. \tag{2.68}
\]

In order to prove \((2.67)\)_2, we recall some identities as follows:

\[
\begin{align*}
\mu (\rho \text{div}(\rho u))_t &= -\mu \text{div}(\rho u) + \varepsilon \mu \text{div}(\varepsilon |\nabla v|^2 \nabla v) + \varepsilon \mu \rho^{-\rho_0}, \\
\mu \text{div}(\rho u \otimes \nabla \rho + \rho \nabla \rho \otimes u) &= \mu \Delta (\rho u) - 2\mu \text{div}(\rho Du) + \mu \text{div}(\rho u), \\
\mu^2 \text{div}(\rho \text{div}(\rho u \otimes \nabla \rho)) &= \mu^2 \Delta (\rho \text{div}(\rho u)) - \mu^2 \text{div}(\rho \text{div}(\rho u \otimes \nabla \rho)).
\end{align*}
\]  

\(2.69\)

Furthermore, using \((2.1)\) and \((2.6)\), one can rewrite \((2.2)_2\) as

\[(\rho u)_t + \text{div}(\rho u \otimes u) - 2 \nu \text{div}(\rho Du) + \nabla P + \varepsilon \mu \rho^{-\rho_0} - \kappa \text{div}(\rho \nabla^2 \log \rho) + r_0 u + r_1 |u|^2 u = \sqrt{\varepsilon} \text{div}(\rho \nabla w) - \varepsilon \mu \text{div}(\varepsilon |\nabla v|^2 \nabla v) + \varepsilon |\nabla v|^2 \nabla v \cdot \nabla w + \varepsilon \text{div}(|\nabla v|^2 \nabla v)w - \varepsilon^{3/2} \rho |w|^3 u. \tag{2.70}\]

This combined with \((2.68)\) gives directly \((2.67)\) and finishes the proof of Lemma 2.5 \(\blacksquare\)

Next, with the estimates of \((\rho, u)\) in Lemmas 2.1–2.4 in hand, we will derive some estimates on \((\rho, u)\) following Lemma 2.6.

Lemma 2.6 There exists some constant \(C\) depending on \(\varepsilon, r_0, r_1, \) and \(\kappa\) such that

\[\sup_{0 \leq t \leq T} (||w||_{L^2} + ||\nabla v||_{L^2}) \leq C. \tag{2.72}\]

\(2.72\)

Proof. First, it follows from \((2.65), (2.71), (2.27), \) and \((2.28)\) that

\[\sup_{0 \leq t \leq T} (||w||_{L^2} + ||\nabla v||_{L^2}) + \int_0^T (|||\nabla v|^2 |\nabla^2 v|^2 + |\nabla w|^2 + |\nabla v|^2) \, dx \, dt \leq C. \tag{2.73}\]

Then it follows from \((2.67)_1\) and \((2.24)\) that \(v\) satisfies

\[2v_t - 2\mu \Delta v - \varepsilon \text{div}(|\nabla v|^2 \nabla v) = -\varepsilon \text{div} w - 2w \cdot \nabla v + 2\mu \nu^{-1} |\nabla v|^2 + \varepsilon v^{-2p_0-1}. \tag{2.74}\]

This yields that

\[2v_t - \varepsilon \text{div}((2\varepsilon^{-1} + |\nabla v|^2) \nabla v) = -\text{div}(w v + \nabla g) - \frac{1}{|\Omega|} \int (w \cdot \nabla v - \varepsilon v^{-2p_0-1} - 2\mu \varepsilon^{-1} |\nabla v|^2 \, dx, \tag{2.75}\]

16
where \( g(\cdot, t) \) (with \( t > 0 \)) is the unique solution to the following problem

\[
\begin{cases}
\Delta g = w \cdot \nabla v - \varepsilon v^{-2p_0-1} - 2\mu v^{-1} |\nabla v|^2 - \frac{1}{|\Omega|} \int (w \cdot \nabla v - \varepsilon v^{-2p_0-1} - 2\mu v^{-1} |\nabla v|^2) dx, \quad x \in \Omega, \\
\int g dx = 0.
\end{cases}
\]

(2.76)

Since (2.73) implies

\[
\left| \int w \cdot \nabla v dx \right| \leq C \|w\|_{L^2} \|\nabla v\|_{L^2} \leq C,
\]

(2.77)

we obtain that \( \nabla g \) satisfies for any \( p > 2 \),

\[
\|\nabla g\|_{L^p} \leq C \|\Delta g\|_{L^{3p/(p+3)}} \leq C(p) \|w\|_{L^p} \|\nabla v\|_{L^3} + C(p) \|\nabla v\|_{L^p} \|\nabla v\|_{L^3} + C(p) \leq C(p) \|w\|_{L^p} + C(p) \|\nabla v\|_{L^p} + C(p),
\]

(2.78)

due to (2.76), (2.73), and (2.58).

Setting

\[
\tilde{v}(x, t) \triangleq v(x, t) + \frac{1}{2|\Omega|} \int_0^t \int (w \cdot \nabla v - \varepsilon v^{-2p_0-1} - 2\mu v^{-1} |\nabla v|^2) dx dt,
\]

one deduces from (2.76) that

\[
\begin{cases}
2\tilde{v}_t - \varepsilon \text{div}(|\nabla \tilde{v}|^2 \nabla \tilde{v}) = \text{div} \tilde{f}, \\
\tilde{v}(x, 0) = v(x, 0),
\end{cases}
\]

(2.79)

with \( \tilde{f} \triangleq 2\mu \nabla \tilde{v} - wv - \nabla g \).

Thus, applying the \( L^p \)-estimates [1, Theorem 1] (see also [6, 13]) to (2.79) with periodic data yields that for any \( p \geq 4 \)

\[
\int_0^T \|\nabla v\|_{L^p}^{3p} dt = \int_0^T \|\nabla \tilde{v}\|_{L^p}^{3p} dt
\]

\[
\leq C(p) \left( 1 + \int_0^T \|\tilde{f}\|_{L^p}^p dt \right)^2
\]

\[
\leq C(p) \left( 1 + \int_0^T \|w\|_{L^p}^p dt \right)^2 + C(p) \left( \int_0^T \|\nabla \tilde{v}\|_{L^p}^p dt \right)^2
\]

\[
\leq C(p) + C(p) \left( \int_0^T \|w\|_{L^p}^p dt \right)^2 + \frac{1}{2} \int_0^T \|\nabla v\|_{L^p}^{3p} dt,
\]

(2.80)

where we have used (2.78), (2.58), and (2.73). The combination of (2.73) with (2.80) gives

\[
\int_0^T \int |\nabla v|^3 dx dt \leq C.
\]

(2.81)

Next, it follows from (2.64) that

\[
w_t - (\nu + \sqrt{\varepsilon}) \Delta w - (\nu - \mu) \nabla \text{div} w = F
\]

(2.82)

with

\[
F \triangleq - w \cdot \nabla w - \rho^{-1} \nabla P + (\nu + \mu + \sqrt{\varepsilon}) \rho^{-1} \nabla \rho \cdot \nabla w + (\nu - \mu) \rho^{-1} \nabla w \cdot \nabla \rho + \varepsilon \rho^{-1} v |\nabla v|^2 \nabla v \cdot \nabla w - \varepsilon \rho^{-1} |\nabla v|^{3/2} w - r_0 \rho^{-1} u - r_1 |u|^2 u - \varepsilon \rho^{-p_0-1} w.
\]

(2.83)
Multiplying (2.82) by $|w|^2 w$ and integrating the resulting equality by parts, it holds that

\[
\frac{1}{4} \frac{d}{dt} ||w||^4_{L^4} + (\nu + \sqrt{\varepsilon}) \int (|\nabla w|^2 |w|^2 + \frac{1}{2} |\nabla |w|^2|^2) \, dx \\
+ (\nu - \mu) \int |\text{div}w|^2 |w|^2 \, dx + \varepsilon \int \rho^{-\mu - 1} |w|^4 \, dx
\]

\[
= - (\nu - \mu) \int \text{div}w \nabla |w|^2 \cdotwdx - \int w \cdot \nabla w \cdot w |w|^2 \, dx - \int \rho^{-\mu - 1} \nabla P \cdot w |w|^2 \, dx \\
+ (\nu + \mu + \sqrt{\varepsilon}) \int \rho^{-\mu} \nabla \rho \cdot \nabla w \cdot w |w|^2 \, dx + (\nu - \mu) \int \rho^{-\mu} |w|^2 w \cdot \nabla w \cdot \nabla \rho \, dx \\
+ \varepsilon \int \rho^{-\mu} w |\nabla v|^2 \nabla v \cdot w |w|^2 \, dx - \varepsilon^{3/2} \int |w|^3 u \cdot w |w|^2 \, dx \\
- r_0 \int \rho^{-1} u \cdot w |w|^2 \, dx - r_1 \int |u|^2 u \cdot w |w|^2 \, dx
\]

\[
\triangleq \sum_{i=1}^{9} J_i.
\]

The straight arguments together with (2.83), (2.86), and (2.81) derive the estimates on each $J_i (i = 1, 2, \cdots, 10)$ as follows:

\[
J_1 \leq \frac{3(\nu - \mu)}{4} \int |\text{div}w|^2 |w|^2 \, dx + \frac{\nu - \mu}{3} \int |\nabla |w|^2|^2 \, dx,
\]

\[
J_2 + J_3 + J_8 \leq \int |\nabla w||w|^4 \, dx + C \int |\nabla v||w|^3 \, dx + C \int |u||w|^3 \, dx \\
\leq \delta \int |\nabla w|^2 |w|^2 \, dx + C \int |w|^6 \, dx + C ||\nabla v||_{L^2}^4 + C ||u||_{L^2}^4
\]

\[
J_4 + J_5 + J_6 \leq C \int |\nabla w||\nabla v||w|^3 \, dx + C \int |\nabla w||\nabla v|^3 |w|^3 \, dx \\
\leq \delta \int |\nabla w|^2 |w|^2 \, dx + C \int |\nabla v|^2 |w|^4 \, dx + C \int |\nabla w|^4 \, dx
\]

\[
J_7 = -\varepsilon^{3/2} \int |w|^5 (w - \mu \nabla \log \rho) \cdot wdx \\
= -\varepsilon^{3/2} \int |w|^7 \, dx + \varepsilon^{3/2} \mu \int |w|^5 \nabla \log \rho \cdot wdx \\
\leq -\varepsilon^{3/2} \int |w|^7 \, dx + C ||\nabla v||_{L^2}^4 + C
\]

\[
J_9 = -\frac{r_1}{2} \int |u|^2 (w - \mu \nabla \log \rho) \cdot w |w|^2 \, dx - \frac{r_1}{2} \int |u|^2 u \cdot (u + \mu \nabla \log \rho) |w|^2 \, dx
\]

\[
= -\frac{r_1}{2} \int |u|^2 |w|^4 \, dx - \frac{r_1}{2} \int |u|^4 |w|^2 \, dx + \frac{r_1}{2} \int |u|^2 |\nabla \log \rho|^2 |w|^2 \, dx
\]

\[
\leq -\frac{r_1}{2} \int |u|^2 |w|^4 \, dx - \frac{r_1}{2} \int |u|^4 |w|^2 \, dx + \frac{r_1}{4} \int |u|^4 |w|^2 \, dx + C \int |\nabla v|^4 |w|^2 \, dx
\]

\[
\leq -\frac{r_1}{2} \int |u|^2 |w|^4 \, dx - \frac{r_1}{2} \int |u|^4 |w|^2 \, dx + \frac{r_1}{4} \int |u|^4 |w|^2 \, dx + C ||\nabla v||_{L^2}^5 + C ||\nabla v||_{L^2}^5 + C
\]
Substituting (2.85)–(2.89) into (2.84) and choosing $\delta$ suitably small enough, we get

$$\frac{1}{4} \frac{d}{dt} \| w \|_{L^4}^4 + \frac{\nu + \sqrt{\epsilon}}{4} \int |\nabla w|^2 |w|^2 dx + \frac{\nu - \mu}{6} \int |\text{div} w|^2 |w|^2 dx$$

$$+ \frac{\epsilon^{3/2}}{4} \int |w|^7 dx + \epsilon \int \rho^{-p_0 - 1} |w|^4 dx + \frac{r_1}{2} \int |u|^2 |w|^4 dx + \frac{r_1}{4} \int |u|^4 |w|^2 dx$$

$$\leq C \| w \|_{L^5}^5 + C \| \nabla v \|_{L^{15}}^{15} + C,$$  

which together with (2.73), (2.81), and (2.80) gives that

$$\sup_{0 \leq t \leq T} \| w \|_{L^4}^4 + \int_0^T \int (|\nabla w|^2 |w|^2 + |\text{div} w|^2 |w|^2 + |w|^7 + |\nabla v|^2) \, dx \, dt$$

$$+ \int_0^T \int (\rho^{-p_0 - 1} |w|^4 + |u|^2 |w|^4 + |u|^4 |w|^2) \, dx \, dt \leq C.$$

Hence, (2.72) is deduced directly from (2.73) and (2.91). The proof of Lemma 2.6 is finished. □

In order to obtain the global strong solutions of problem (2.2)–(2.3), we still need to derive some necessary higher order estimates on $(\rho, w)$ in the following lemma.

**Lemma 2.7** For any $p > 2$, there exists some constant $C$ depending on $\epsilon, r_0, r_1, \kappa, \nu$ and $p$ such that

$$\int_0^T \left( \|(\rho_1, \nabla \rho_1, u_1, w_1)\|_{L^p}^p + \|(\rho, \nabla \rho, u, w)\|_{L^{2p}}^p \right) \, dt \leq C.$$

**Proof.** Multiplying (2.82) by $-2\Delta w$ and integrating the resulting equality over $\Omega$ lead to

$$\frac{d}{dt} \| \nabla w \|_{L^2}^2 + \int 2 \left( (\nu + \sqrt{\epsilon}) |\Delta w|^2 + (\nu - \mu) |\text{div} w|^2 \right) \, dx$$

$$= -2 \int (-\Delta w \cdot \nabla w - \rho^{-1} \nabla P + (\mu + \nu + \sqrt{\epsilon}) \rho^{-1} \nabla \rho \cdot \nabla w + (\nu - \mu) \rho^{-1} \nabla w \cdot \nabla \rho$$

$$+ \epsilon \rho^{-1} \nabla v \nabla \cdot \nabla w - \epsilon^{3/2} |w|^3 u - r_0 \rho^{-1} u - r_1 |u|^2 u - \epsilon \rho^{-p_0 - 1} w) \cdot \Delta w \, dx$$

$$\triangleq \sum_{i=1}^{9} \tilde{J}_i.$$

Using (2.58) and (2.72), the terms $\tilde{J}_i (i = 1, 2, \ldots, 9)$ in (2.93) can be estimated as follows:

$$\tilde{J}_1 + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5 \leq C \| \nabla w \|_{L^2} \left( \| w \|_{L^5} + \| \nabla v \|_{L^5} + \| \nabla v \|_{L^{15}}^3 \right) \| \nabla w \|_{L^{10/3}}$$

$$\leq C \| \nabla w \|_{L^2} \left( \| w \|_{L^5} + \| \nabla v \|_{L^5} + \| \nabla v \|_{L^{15}}^3 \right) \| \nabla w \|_{L^2}^{2/5} \| \nabla w \|_{L^{12}}^{3/5}$$

$$\leq \delta \| \Delta w \|_{L^2}^2 + C \left( \| w \|_{L^5}^2 + \| \nabla v \|_{L^5}^2 + \| \nabla v \|_{L^{15}}^{15} \right) \| \nabla w \|_{L^2}^2,$$  

$$\tilde{J}_7 + \sum_{i=7}^{9} \tilde{J}_i \leq C \int (|\nabla v| + |u| + |u|^3 + |w|) \| \Delta w \| \, dx$$

$$\leq \delta \| \Delta w \|_{L^2}^2 + C \| \nabla v \|_{L^2}^2 + C \| u \|_{L^2}^2 + C \| w \|_{L^2}^2 + C \| u \|_{L^6}^6$$

$$\leq \delta \| \Delta w \|_{L^2}^2 + C + C \| w \|_{L^6}^6 + C \| \nabla v \|_{L^6}^6$$

$$\leq \delta \| \Delta w \|_{L^2}^2 + C + C \| w \|_{L^2}^7 + C \| \nabla v \|_{L^{15}}^{15},$$
and

\[ J_6 = 2 \varepsilon^{3/2} \int |w|^3 (w - \mu \nabla \log \rho) \cdot \Delta w \, dx \]

\[ = -2 \varepsilon^{3/2} \int |w|^3 |\nabla w|^2 \, dx - 6 \varepsilon^{3/2} \int |\nabla |w|^2| |w|^3 \, dx - 2 \varepsilon^{3/2} \mu \int |w|^3 \nabla \log \rho \cdot \Delta w \, dx \]

\[ \leq -2 \varepsilon^{3/2} \int |w|^3 |\nabla w|^2 \, dx - 6 \varepsilon^{3/2} \int |\nabla |w|^2| |w|^3 \, dx + C \|w\|_{L^7}^7 + C \|\nabla v\|_{L^{14}}^{14} + \delta \|\Delta w\|_{L^2}^2 \]

\[ \leq -2 \varepsilon^{3/2} \int |w|^3 |\nabla w|^2 \, dx - 6 \varepsilon^{3/2} \int |\nabla |w|^2| |w|^3 \, dx + C \|w\|_{L^7}^7 + C \|\nabla v\|_{L^{15}}^{15} + C + \delta \|\Delta w\|_{L^2}^2. \]

(2.96)

Submitting (2.94)–(2.96) into (2.93), one gets after choosing \( \delta \) suitably small enough that

\[ \left( |\nabla w|^2 \right) + \int \left( (\nu + \sqrt{\varepsilon}) |\Delta w|^2 + (\nu - \mu) |\nabla \text{div} w|^2 \right) \, dx \]

\[ + 2 \varepsilon^{3/2} \int |w|^3 |\nabla w|^2 \, dx + 6 \varepsilon^{3/2} \int |\nabla |w|^2| |w|^3 \, dx \]

\[ \leq C \left( \|w\|_{L^5}^5 + \|\nabla v\|_{L^5}^5 + \|\nabla v\|_{L^{15}}^{15} \right) \|\nabla w\|_{L^2}^2 + C \|w\|_{L^7}^7 + C \|\nabla v\|_{L^{15}}^{15} + C, \]

which together with (2.72) and Gronwall’s inequality yields

\[ \sup_{0 \leq t \leq T} \|\nabla w\|_{L^2}^2 + \int_0^T \|\nabla^2 w\|_{L^2}^2 \, dt \leq C. \]

(2.98)

It thus follows from (2.98) and Sobolev inequality that

\[ \|w\|_{L^{10}(\Omega \times (0,T))} + \|\nabla w\|_{L^{10/3}(\Omega \times (0,T))} \leq C. \]

This along with (2.80)–(2.83) and (2.98) gives

\[ \|w_t\|_{L^2(\Omega \times (0,T))} + \|\nabla^2 w\|_{L^2(\Omega \times (0,T))} + \|F\|_{L^{5/2}(\Omega \times (0,T))} \leq C. \]

(2.99)

Using (2.99) and applying the standard \( L^p \)-estimates to (2.82) (2.83) (2.3) with periodic data yield that for any \( p \geq 2 \),

\[ \|w_t\|_{L^p(\Omega \times (0,T))} + \|\nabla^2 w\|_{L^p(\Omega \times (0,T))} \leq C(p) + C(p) \|F\|_{L^p(\Omega \times (0,T))}. \]

(2.100)

In particular, the combination of (2.99) with (2.100) shows

\[ \|w_t\|_{L^{5/2}(\Omega \times (0,T))} + \|\nabla^2 w\|_{L^{5/2}(\Omega \times (0,T))} \leq C. \]

This combined with (2.72) and the Sobolev inequality ( (26 Chapter II (3.15)) yields that for any \( q > 2 \),

\[ \|w\|_{L^q(\Omega \times (0,T))} + \|\nabla w\|_{L^q(\Omega \times (0,T))} \leq C(q), \]

which along with (2.80) and (2.83) gives

\[ \|F\|_{L^{5/2}(\Omega \times (0,T))} \leq C. \]

Combining this with (2.100) leads to

\[ \|w_t\|_{L^{5/2}(\Omega \times (0,T))} + \|\nabla^2 w\|_{L^{5/2}(\Omega \times (0,T))} \leq C, \]

which together with the Sobolev inequality ( (26 Chapter II (3.15)) shows

\[ \|w\|_{L^q(\Omega \times (0,T))} + \|\nabla w\|_{L^4(\Omega \times (0,T))} \leq C. \]
Thus, we get
\[ \|F\|_{L^{40}(\Omega \times (0,T))} \leq C, \]
which along with (2.100) gives
\[ \|w_t\|_{L^{40}(\Omega \times (0,T))} + \|\nabla^2 w\|_{L^{40}(\Omega \times (0,T))} \leq C. \]
The Sobolev inequality ([26, Chapter II (3.15)]) thus implies
\[ \|\nabla w\|_{L^{\infty}(\Omega \times (0,T))} \leq C. \]
Then, it holds that for any \( p > 2 \),
\[ \|w_t\|_{L^p(\Omega \times (0,T))} + \|\nabla^2 w\|_{L^p(\Omega \times (0,T))} \leq C(p). \tag{2.101} \]
With (2.101) in hand, one can deduce easily from (2.74) and (2.3) that for any \( p > 2 \),
\[ \|\rho_t\|_{L^p(0,T,W^{1,p}(\Omega))} + \|\nabla^2 \rho\|_{L^p(0,T,W^{1,p}(\Omega))} \leq C(p). \tag{2.102} \]
Recalling the definition of \( w \) in (2.1), the combination of (2.101) with (2.102) yields
\[ \|u_t\|_{L^p(\Omega \times (0,T))} + \|\nabla^2 u\|_{L^p(\Omega \times (0,T))} \leq C(p), \tag{2.103} \]
which together with (2.101)–(2.102) gives the desired estimate (2.92), and thus finishes the proof of Lemma 2.7.

3 Compactness results

Let
\[ \sigma_0 \triangleq 10^{-10}, \tag{3.1} \]
we choose
\[ 0 \leq \tilde{\rho}_0 \in C^\infty(\Omega), \quad \|\nabla \sqrt{\tilde{\rho}_0}\|_{L^4} \leq \varepsilon^{-4\sigma_0} \]
satisfying
\[ \|r_0 \log - \tilde{\rho}_0 - r_0 \log - \rho_0\|_{L^1} + \|\tilde{\rho}_0 - \rho_0\|_{L^1} + \|\tilde{\rho}_0 - \rho_0\|_{L^2} + \|\nabla (\sqrt{\tilde{\rho}_0} - \sqrt{\rho_0})\|_{L^2} < \varepsilon. \]
Set
\[ \rho_{0e} = \left(\tilde{\rho}_0 + \varepsilon^{24\sigma_0}\right)^{\frac{1}{2}}, \]
it is easy to check that
\[ \lim_{\varepsilon \to 0} \|\rho_{0e} - \rho_0\|_{L^1} = 0 \tag{3.2} \]
and that there exists some constant \( C \) independent of \( \varepsilon \) such that (2.4) holds. Furthermore, we choose \( m_{0e} \) such that
\[ \|m_{0e} - \rho_0^{-1/2} m_0\|_{L^2} \leq \varepsilon. \]
Then, define \( u_{0e} \) as follows,
\[ u_{0e} = \rho_0^{-1/2} m_{0e}, \tag{3.3} \]
we thus have
\[ \lim_{\varepsilon \to 0} \|\rho_{0e} u_{0e} - m_0\|_{L^1} = 0. \tag{3.4} \]
Moreover, it is easy to check that (2.5) is still valid for \( (\rho_{0e}, u_{0e}) \).
Extending \((\rho_{0\varepsilon}, w_{0\varepsilon})\) \(\Omega\)-periodically to \(\mathbb{R}^3\), we will consider the problem \((2.67)\) with the initial data \((\rho_{0\varepsilon}, w_{0\varepsilon})\) for \(w_{0\varepsilon} \equiv u_{0\varepsilon} + \mu \log \rho_{0\varepsilon}\). The standard parabolic theory \([26]\) together with Lemmas \(2.3\) and \(2.6\) \(2.7\) illustrates that there is a unique strong solution \((\rho_{\varepsilon}, u_{\varepsilon}) \in C([0, T), W^{2,p}(\Omega))\) for any \(T > 0\) and any \(p > 2\). Then, one deduces from \((3.9), (3.10), \) Hölder and Sobolev inequalities that in particular, it holds

Moreover, all estimates obtained in Lemmas \(2.1\) and \(2.2\) still hold for the solution \((\rho_{\varepsilon}, u_{\varepsilon})\) to the problem \((2.2), (2.3)\).

Letting \(\varepsilon \to 0^+\), we will prove that \((\rho_{\varepsilon}, \sqrt{\rho_{\varepsilon}} u_{\varepsilon})\) converges, up to the extraction of subsequences, to the limit \((\rho, \sqrt{\rho u})\) in some sense. These convergences, see Lemmas \(3.1, 3.2\) are crucial to show that \((\rho, \sqrt{\rho u})\) is a weak solution to \((1.1), (1.2)\). The proof of Lemmas \(3.1, 3.2\) are similar as those in Li-Xin \([29]\) (see also partially in \([31], [34]\)), which are sketched here for completeness.

We begin with the following strong convergence of \(\sqrt{\rho_{\varepsilon}}\) and \(\rho_{\varepsilon}\).

**Lemma 3.1** There exists a function \(\rho \in L^\infty(0, T; L^1 \cap L^\gamma)\) such that up to a subsequence,

\[
\sqrt{\rho_{\varepsilon}} \to \sqrt{\rho} \text{ strongly in } L^2(0, T; H^1),
\]

\[
\rho_{\varepsilon} \to \rho \text{ strongly in } L^\gamma(\Omega \times (0, T)),
\]

\[
\nabla^2 \sqrt{\rho_{\varepsilon}} \to \nabla^2 \sqrt{\rho} \text{ weakly in } L^2(\Omega \times (0, T)).
\]

In particular, it holds

\[
\rho_{\varepsilon} \to \rho \text{ almost everywhere in } \Omega \times (0, T).
\]

**Proof.** First, for \(v_{\varepsilon} \equiv \sqrt{\rho_{\varepsilon}}\), it follows from \((2.7), (2.27), (2.28), (2.54)\) that there exists some generic positive constant \(C\) independent of \(\varepsilon\) such that

\[
\sup_{0 \leq t \leq T} \int (|v_{\varepsilon}|^2 + |v_{\varepsilon}|^4) \, dx + \int_0^T \int \nabla^2 v_{\varepsilon}^2 \, dx \, dt \leq C,
\]

\[
\int_0^T \int (|v_{\varepsilon}|^2 + |v_{\varepsilon}|^4) \, dx \, dt \leq C,
\]

\[
\int_0^T \int (|\nabla v_{\varepsilon}|^2 + |\nabla v_{\varepsilon}|^4) \, dx \, dt \leq C,
\]

\[
\int_0^T \int (|\nabla \rho_{\varepsilon}|^2 + |\nabla \rho_{\varepsilon}|^4) \, dx \, dt \leq C,
\]

Then, one deduces from \((3.9), (3.10)\), Hölder and Sobolev inequalities that

\[
\varepsilon^\frac{3}{4} \int_0^T \|v_{\varepsilon}\|_{L^6}^6 \, dt \leq C \varepsilon \int_0^T \|v_{\varepsilon}\|_{L^4}^4 \, dt \leq C \varepsilon \int_0^T \|v_{\varepsilon}\|_{L^2}^2 \, dt \leq C.
\]

Since \(\rho_{\varepsilon}\) satisfies

\[
(\rho_{\varepsilon})_t + \text{div}(\rho_{\varepsilon} u_{\varepsilon}) = \varepsilon v_{\varepsilon} \text{div}(\nabla v_{\varepsilon}^2) + \varepsilon \rho_{\varepsilon}^{p_0},
\]

(3.12)
by assuming }\rho_\varepsilon > 0\text{, we may rewrite (3.12) as follows}

\begin{equation}
2(\sqrt{\rho_\varepsilon})_t = -2\text{div}(\sqrt{\rho_\varepsilon}u_\varepsilon) + \sqrt{\rho_\varepsilon}\text{div}u_\varepsilon + \varepsilon\text{div}(|\nabla v_\varepsilon|^2\nabla v_\varepsilon) + \varepsilon\rho_\varepsilon^{-\frac{1}{2}}. \tag{3.13}
\end{equation}

It follows from (3.9), (3.10), and (3.11) that

\begin{equation}
\varepsilon^2 \int_0^T \int |\nabla v_\varepsilon|^6 \, dx \, dt \leq C\varepsilon^{\frac{2}{3}} \tag{3.14}
\end{equation}

and

\begin{equation}
\int_0^T \int (\rho_\varepsilon|u_\varepsilon|^2 + \rho_\varepsilon(\text{div}u_\varepsilon)^2) \, dx \, dt + \varepsilon^2 \int_0^T \int \rho_\varepsilon^{-2\rho_0 - 1} \, dx \, dt \leq C. \tag{3.15}
\end{equation}

The combination of (3.13)–(3.15) implies that

\begin{equation}
\|((\sqrt{\rho_\varepsilon})_t)\|_{L^2(0,T;H^{-1})} \leq C. \tag{3.16}
\end{equation}

Furthermore, it is easy to derive from (3.9) and (3.10) that

\begin{equation}
\|\sqrt{\rho_\varepsilon}\|_{L^2(0,T;H^2)} \leq C, \tag{3.17}
\end{equation}

which combined with (3.16) and Aubin-Lions lemma yields (3.18).

Next, we claim that for }\gamma \in (1, 3),

\begin{equation}
\|\rho_\varepsilon^\gamma\|_{L^\frac{5}{3}((0,T) \times \Omega)} \leq C. \tag{3.18}
\end{equation}

This along with (3.5) yields directly the desired (3.6) and (3.8). Furthermore, the convergence (3.7) is deduced directly from (3.14) and (3.6).

Now, it remains to prove (3.18). It is easy to deduce from (3.10) that

\begin{equation}
\|\nabla \rho_\varepsilon^\gamma\|_{L^2((0,T) \times \Omega)} \leq C, \tag{3.19}
\end{equation}

which together with Sobolev’s embedding theorem gives

\begin{equation}
\|\rho_\varepsilon^\gamma\|_{L^1(0,T;L^3)} \leq C. \tag{3.20}
\end{equation}

Note that (3.17) implies that

\begin{equation}
\|\rho_\varepsilon^\gamma\|_{L^\infty(0,T;L^1)} \leq C,
\end{equation}

which combined with (3.19) yields (3.18). The proof of Lemma 3.1 is finished. \qed

\textbf{Lemma 3.2} There exists a function }m(x,t) \in L^2(0,T;L^\frac{5}{3})\text{ such that up to a subsequence,}

\begin{equation}
\rho_\varepsilon u_\varepsilon \to m \text{ in } L^2(0,T;L^p) \tag{3.20}
\end{equation}

for all }p \in [1, \frac{5}{3}). \text{ Moreover, there exists a function } u \text{ in } L^2((0,T) \times \Omega) \text{ such that up to a subsequence}

\begin{equation}
u_\varepsilon \to u \text{ weakly in } L^2((0,T) \times \Omega). \tag{3.21}
\end{equation}

\text{And, it holds that}

\begin{equation}
\rho_\varepsilon u_\varepsilon \to \rho u \text{ almost everywhere } (x,t) \in \Omega \times (0,T). \tag{3.22}
\end{equation}

\textbf{Proof.} First, it follows from Hölder inequality, (3.9), and (3.10) that

\begin{equation}
\int_0^T \|\nabla(\rho_\varepsilon u_\varepsilon)\|^2_{L^1} \, dt \leq C \int_0^T (\|\rho_\varepsilon\|_{L^1}\|\sqrt{\rho_\varepsilon}\nabla u_\varepsilon\|^2_{L^2} + \|\sqrt{\rho_\varepsilon}u_\varepsilon\|^2_{L^2}\|\nabla\sqrt{\rho_\varepsilon}\|^2_{L^2}) \, dt \leq C \tag{3.23}
\end{equation}
and

\[ \sup_{0 \leq t \leq T} \| \rho_\varepsilon u_\varepsilon \|_{L^1} \leq C \sup_{0 \leq t \leq T} \left( \| \rho_\varepsilon \|_{L^1} + \| \rho_\varepsilon |u_\varepsilon|^2 \|_{L^1} \right) \leq C. \]

Hence, one has

\[ \| \rho_\varepsilon u_\varepsilon \|_{L^2(0,T;W^{1,1})} \leq C. \]

Next, the straight calculations show that

\begin{align*}
(r_\varepsilon u_\varepsilon)_t + \div (\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) - 2\nu \div (\rho_\varepsilon D u_\varepsilon) - \kappa^2 \div (\rho_\varepsilon \nabla^2 \log \rho_\varepsilon) + \nabla P(\rho_\varepsilon) \\
= \varepsilon \div (v_\varepsilon \nabla v_\varepsilon^2 \nabla v_\varepsilon \otimes u_\varepsilon) - \varepsilon |\nabla v_\varepsilon|^4 u_\varepsilon - \varepsilon \mu \nabla \rho_\varepsilon^\nu + \sqrt{\varepsilon} \div (\rho_\varepsilon \nabla u_\varepsilon) \\
- \varepsilon \frac{\kappa}{2} \rho_\varepsilon |u_\varepsilon|^3 u_\varepsilon - r_0 u_\varepsilon - r_1 \rho_\varepsilon |u_\varepsilon|^2 u_\varepsilon + \sqrt{\varepsilon} \mu \div (\rho_\varepsilon \nabla^2 \log \rho_\varepsilon) - \varepsilon \mu |\nabla v_\varepsilon|^4 \nabla \log \rho_\varepsilon \\
- \varepsilon \mu \div (\varepsilon v_\varepsilon \div (|\nabla v_\varepsilon|^2 \nabla v_\varepsilon)) + \varepsilon \mu \div (v_\varepsilon |\nabla v_\varepsilon|^2 \nabla v_\varepsilon \otimes \nabla \log \rho_\varepsilon). \tag{3.24}
\end{align*}

For the terms on the left-hand side of (3.24), one has

\[ \int_0^T \int_0^T \rho_\varepsilon |u_\varepsilon|^2 \, dx \, dt \leq C, \tag{3.25} \]

\[ \int_0^T \int_0^T \rho_\varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt \leq C \int_0^T \int_0^T \rho_\varepsilon |\nabla u_\varepsilon|^2 \, dx \, dt + C \int_0^T \rho_\varepsilon \, dx \, dt \leq C, \tag{3.26} \]

\[ \int_0^T \int_0^T \rho_\varepsilon |\nabla^2 \log \rho_\varepsilon|^2 \, dx \, dt \leq C \int_0^T \int_0^T \rho_\varepsilon \, dx \, dt + C \int_0^T \rho_\varepsilon |\nabla^2 \log \rho_\varepsilon|^2 \, dx \, dt \leq C. \tag{3.27} \]

Using (3.9) and (3.10), we can estimate each term on the right-hand side of (3.24) as follows:

\[ \varepsilon \int_0^T \int_0^T (v_\varepsilon \nabla v_\varepsilon^3 |u_\varepsilon| + |\nabla v_\varepsilon|^4 |u_\varepsilon|) \, dx \, dt \]

\[ \leq C \varepsilon \int_0^T \| v_\varepsilon u_\varepsilon \|_{L^2} \| \nabla v_\varepsilon \|_{L^6} \, dt + C \left( \varepsilon \int_0^T \| \nabla v_\varepsilon \|_{L^6}^2 |u_\varepsilon|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \varepsilon \int_0^T \| \nabla v_\varepsilon \|_{L^6}^4 \, dx \, dt \right)^{\frac{1}{2}} \]

\[ \leq C \varepsilon \left( \int_0^T \| \nabla v_\varepsilon \|_{L^6}^6 \, dx \, dt \right)^{\frac{1}{2}} + C \left( \varepsilon \int_0^T \| \nabla v_\varepsilon \|_{L^6}^6 \, dx \, dt \right)^{\frac{1}{2}} \]

\[ \leq C \varepsilon^{\frac{1}{2}}, \]

where in the last inequality one has used (3.11). Moreover, it holds

\[ \varepsilon \int_0^T \int_0^T \rho_\varepsilon^{-p_0} \, dx \, dt \leq \varepsilon \frac{1}{2p_0+1} \left( \varepsilon^2 \int_0^T \int_0^T \rho_\varepsilon^{-2p_0-1} \, dx \, dt \right)^{\frac{p_0}{2p_0+1}} \leq C \varepsilon \frac{1}{2p_0+1}, \tag{3.29} \]

and

\[ \int_0^T \int_0^T (|u_\varepsilon| + \rho_\varepsilon |u_\varepsilon|^3) \, dx \, dt \]

\[ \leq C \left( \int_0^T \int_0^T |u_\varepsilon|^2 \, dx \, dt \right)^{\frac{1}{2}} + C \left( \int_0^T \int_0^T \rho_\varepsilon |u_\varepsilon|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_0^T \rho_\varepsilon |u_\varepsilon|^4 \, dx \, dt \right)^{\frac{1}{2}} \]

\[ \leq C. \tag{3.30} \]
The Hölder inequality together with (3.9) and (3.10) yields

\[
\varepsilon^{3} \int_{0}^{T} \int \rho_{\varepsilon} |w_{\varepsilon}|^{3} |u_{\varepsilon}| \|dxdt \\
\leq C \varepsilon^{2} \left( \int_{0}^{T} \int \rho_{\varepsilon} |w_{\varepsilon}|^{5} \|dxdt \right)^{3/5} \left( \int_{0}^{T} \int \rho_{\varepsilon} |u_{\varepsilon}|^{5/2} \|dxdt \right)^{2/5} \\
\leq C \varepsilon^{3} \left( \int_{0}^{T} \int \rho_{\varepsilon} |w_{\varepsilon}|^{5} \|dxdt \right)^{3/5} \left( \int_{0}^{T} \int \rho_{\varepsilon} |u_{\varepsilon}|^{5/2} \|dxdt \right)^{2/5} \\
\leq C \varepsilon^{5/2}. \tag{3.31}
\]

It follows from (3.9), (3.10), (3.11), Hölder and Sobolev inequalities that

\[
\varepsilon \int_{0}^{T} \int v_{\varepsilon} \text{div}(|\nabla v_{\varepsilon}|^{2} \nabla v_{\varepsilon}) \|dxdt \leq C \varepsilon \int_{0}^{T} \int v_{\varepsilon} |\nabla v_{\varepsilon}|^{2} |\nabla^{2} v_{\varepsilon}| \|dxdt \\
\leq C \varepsilon \int_{0}^{T} \|v_{\varepsilon}\|_{L^{6}} \|\nabla^{2} v_{\varepsilon}\|_{L^{6}} \|\nabla^{2} v_{\varepsilon}\|_{L^{2}} \|dt \\
\leq C \sup_{0 \leq t \leq T} \left( \int_{0}^{T} \|v_{\varepsilon}\|_{L^{6}}^{4} \|dt \right)^{1/2} \left( \int_{0}^{T} \|\nabla^{2} v_{\varepsilon}\|_{L^{2}}^{2} \|dt \right)^{1/2} \\
\leq C \varepsilon^{5/3} \left( \int_{0}^{T} \|\nabla v_{\varepsilon}\|_{L^{6}}^{6} dt \right)^{1/3} \\
\leq C \varepsilon^{5/3}. \tag{3.32}
\]

Finally, we deduce from Hölder inequality and (3.10) that

\[
\varepsilon \int_{0}^{T} \int \|\nabla v_{\varepsilon}\|^{4} \|\nabla \log \rho_{\varepsilon}\| \|dxdt + \varepsilon \int_{0}^{T} \int v_{\varepsilon} |\nabla v_{\varepsilon}|^{3} \|\nabla \log \rho_{\varepsilon}\| \|dxdt \leq C \varepsilon \int_{0}^{T} \int v_{\varepsilon} |\nabla v_{\varepsilon}|^{5} \|dxdt + C \varepsilon \int_{0}^{T} \int |\nabla v_{\varepsilon}|^{4} \|dxdt \leq C \varepsilon^{7/6} \int_{0}^{T} \int v_{\varepsilon}^{-1} |\nabla v_{\varepsilon}|^{5} \|dxdt + C \varepsilon^{5/6} \int_{0}^{T} \int |\nabla v_{\varepsilon}|^{4} \|dxdt \\
\leq C \varepsilon^{1/6} + C \varepsilon^{5/6} \int_{0}^{T} \|\nabla v_{\varepsilon}\|_{L^{2}} \|\nabla v_{\varepsilon}\|_{L^{6}}^{3} \|dt \\
\leq C \varepsilon^{1/6}. \tag{3.33}
\]

The combination of (3.9)–(3.10) with (3.21)–(3.33) leads to

\[
\| (\rho_{\varepsilon} u_{\varepsilon})_{t} \|_{L^{1}(0,T;W^{-1,1})} \leq C. \tag{3.34}
\]

Hence, (3.21) is deduced from Aubin-Lions lemma, (3.26), and (3.34).

Next, it’s noted that \( u_{\varepsilon} \) is uniformly bounded in \( L^{2}((0,T) \times \Omega) \), which yields directly (3.21).

Now, it follows from (3.21) that

\[
\rho_{\varepsilon} u_{\varepsilon} \to m \text{ almost everywhere } (x, t) \in \Omega \times (0, T). \tag{3.35}
\]

On the one hand, (3.35) and (3.8) show that

\[
u_{\varepsilon} = \frac{\rho_{\varepsilon} u_{\varepsilon}}{\rho_{\varepsilon}} \to \frac{m}{\rho} \text{ almost everywhere } \{(x, t) \in \Omega \times (0, T) | \rho(x, t) > 0\}, \tag{3.36}
\]

which together with (3.21) gives that for \( \rho > 0 \),

\[
m = \rho u.
\]
On the other hand, it follows from Fatou’s lemma and (3.9) that
\[
\int_0^T \int \liminf_{\varepsilon \to 0^+} \frac{\rho \varepsilon u_\varepsilon^2}{\rho \varepsilon} \, dx \, dt = \int_0^T \int \liminf_{\varepsilon \to 0^+} \rho \varepsilon |u_\varepsilon|^2 \, dx \, dt \leq \liminf_{\varepsilon \to 0^+} \int_0^T \rho \varepsilon |u_\varepsilon|^2 \, dx \, dt \leq C.
\]
This implies that if \( \rho = 0 \), it has
\[
m = 0.
\]
Then, (3.22) is proved. The proof of Lemma 3.2 is completed. \( \square \)

With Lemmas 3.1 and 3.2 in hand, we are now in a position to prove the strong convergence of \( \sqrt{\rho \varepsilon} u_\varepsilon \). This is crucial for deriving the global existence of the weak solution.

**Lemma 3.3** Up to a subsequence, it holds
\[
\sqrt{\rho \varepsilon} u_\varepsilon \to \sqrt{\rho} u \text{ strongly in } L^2(0, T; L^2),
\]
(3.37)
with
\[
\sqrt{\rho} u \in L^\infty(0, T; L^2).
\]
(3.38)
Moreover, it holds that
\[
\sqrt{\rho \varepsilon} u_\varepsilon \to \sqrt{\rho} u \text{ almost everywhere } (x, t) \in \Omega \times (0, T).
\]
(3.39)

**Proof.** For any \( M > 0 \), the straight calculation shows that
\[
\begin{aligned}
\int_0^T \int |\sqrt{\rho \varepsilon} u_\varepsilon - \sqrt{\rho} u|^2 \, dx \, dt &
\leq 2 \int_0^T \int |\sqrt{\rho \varepsilon} u_\varepsilon 1_{(|u_\varepsilon| \leq M)} - \sqrt{\rho} u 1_{(|u| \leq M)}|^2 \, dx \, dt \\
&+ 2 \int_0^T \int |\sqrt{\rho \varepsilon} u_\varepsilon 1_{(|u_\varepsilon| \geq M)}|^2 \, dx \, dt + 2 \int_0^T \int |\sqrt{\rho} u 1_{(|u| \geq M)}|^2 \, dx \, dt \\
&\leq 2 \int_0^T \int |\sqrt{\rho \varepsilon} u_\varepsilon 1_{(|u_\varepsilon| \leq M)} - \sqrt{\rho} u 1_{(|u| \leq M)}|^2 \, dx \, dt + \frac{2}{M^2} \int_0^T \int \rho |u_\varepsilon|^4 + \rho |u|^4 \, dx \, dt.
\end{aligned}
\]
(3.40)

First, it follows from (3.32) and (3.8) that
\[
\sqrt{\rho \varepsilon} u_\varepsilon \to \sqrt{\rho} u \text{ almost everywhere in } \{(x, t) \in \Omega \times (0, T) | \rho(x, t) > 0 \}.
\]
(3.41)
Moreover, since
\[
\sqrt{\rho \varepsilon} |u_\varepsilon 1_{(|u_\varepsilon| \leq M)}| \leq M \sqrt{\rho \varepsilon}
\]
(3.42)
and
\[
\rho \varepsilon \to \rho \text{ almost everywhere in } \{(x, t) \in \Omega \times (0, T) | \rho(x, t) = 0 \},
\]
(3.43)
we have
\[
\sqrt{\rho \varepsilon} u_\varepsilon 1_{(|u_\varepsilon| \leq M)} \to \sqrt{\rho} u 1_{(|u| \leq M)} \text{ almost everywhere in } \Omega \times (0, T),
\]
which, together with (3.42) and (3.43), implies
\[
\lim_{\varepsilon \to 0^+} \int_0^T \int |\sqrt{\rho \varepsilon} u_\varepsilon 1_{(|u_\varepsilon| \leq M)} - \sqrt{\rho} u 1_{(|u| \leq M)}|^2 \, dx \, dt = 0.
\]
(3.44)

Next, Lemma 2.1 yields that there exists some constant \( C \) independent of \( \varepsilon \) such that
\[
\int_0^T \int \rho \varepsilon |u_\varepsilon|^4 \, dx \, dt \leq C;
\]
(3.45)
which, together with (3.22), (3.8), and Fatou’s lemma, gives
\[ \int_0^T \int \rho |u|^4 \, dx \, dt \leq \int_0^T \int \liminf_{\varepsilon \to 0^+} \rho_{\varepsilon} |u_{\varepsilon}|^4 \, dx \, dt \leq \liminf_{\varepsilon \to 0^+} \int_0^T \rho_{\varepsilon} |u_{\varepsilon}|^4 \, dx \, dt \leq C. \] (3.46)

Substituting (3.44)–(3.46) into (3.40) yields that up to a subsequence
\[ \limsup_{\varepsilon \to 0^+} \int_0^T \int |\sqrt{\rho_{\varepsilon}} u_{\varepsilon} - \sqrt{\rho} u|^2 \, dx \, dt \leq \frac{C}{M^2}, \text{ for any } M > 0. \] (3.47)

We thus obtain (3.37) by taking \( M \to \infty \) in (3.47). The combination of (3.9) with (3.37) gives (3.38). The proof of Lemma 3.3 is finished. \( \square \)

Similar to the proof of Lemma 3.3 we can establish the following convergence of the damping terms.

**Lemma 3.4** Up to a subsequence, it holds
\[ \rho_{\varepsilon} |u_{\varepsilon}|^2 u_{\varepsilon} \to \rho |u|^2 u \text{ strongly in } L^1(0, T; L^1). \] (3.48)

**Proof.** The direct calculation shows that for any \( M > 0 \),
\[ \int_0^T \int |\rho_{\varepsilon} u_{\varepsilon}|^2 u_{\varepsilon} - \rho |u|^2 u \, dx \, dt \leq \int_0^T \int |\rho_{\varepsilon} u_{\varepsilon}|^2 u_{\varepsilon} 1_{(|u_{\varepsilon}| \leq M)} - \rho |u|^2 u 1_{(|u| \leq M)} \, dx \, dt + 2 \int_0^T \int \rho |u|^3 1_{(|u| \geq M)} \, dx \, dt. \] (3.49)

First, it follows from (3.36) and (3.8) that
\[ \rho_{\varepsilon} |u_{\varepsilon}|^2 u_{\varepsilon} \to \rho |u|^2 u \text{ almost everywhere in } \{ (x, t) \in \Omega \times (0, T) | \rho(x, t) > 0 \}. \] (3.50)

Moreover, since
\[ \rho_{\varepsilon} |u_{\varepsilon}|^2 u_{\varepsilon} 1_{(|u_{\varepsilon}| \leq M)} \leq M^3 \rho_{\varepsilon}, \] (3.51)
which together with (3.49) implies that
\[ \rho_{\varepsilon} |u_{\varepsilon}|^2 u_{\varepsilon} 1_{(|u_{\varepsilon}| \leq M)} \to \rho |u|^2 u 1_{(|u| \leq M)} \text{ almost everywhere in } \Omega \times (0, T). \]

Then, it holds that
\[ \int_0^T \int |\rho_{\varepsilon} u_{\varepsilon}|^2 u_{\varepsilon} 1_{(|u_{\varepsilon}| \leq M)} - \rho |u|^2 u 1_{(|u| \leq M)} \, dx \, dt \to 0 \text{ as } \varepsilon \to 0^+. \] (3.52)

Next, it follows from (3.35) and (3.46) that
\[ \int_0^T \int (\rho_{\varepsilon} |u_{\varepsilon}|^3 1_{(|u_{\varepsilon}| \geq M)} + \rho |u|^3 1_{(|u| \geq M)}) \, dx \, dt \leq \frac{1}{M} \int_0^T \int (\rho_{\varepsilon} |u_{\varepsilon}|^4 + \rho |u|^4) \, dx \, dt \leq \frac{C}{M}. \] (3.53)

Substituting (3.52) and (3.53) into (3.49) yields that up to a subsequence
\[ \limsup_{\varepsilon \to 0^+} \int_0^T \int |\rho_{\varepsilon} u_{\varepsilon}|^2 u_{\varepsilon} - \rho |u|^2 u \, dx \, dt \leq \frac{C}{M}, \text{ for any } M > 0. \] (3.54)

We thus obtain (3.38) by taking \( M \to \infty \) in (3.54). The proof of Lemma 3.4 is completed. \( \square \)

Moreover, we can show the following lemma, which shows that \( \nabla(\sqrt{\rho u}) - u \otimes \nabla \sqrt{\rho} \) is indeed a function in \( L^2((0, T) \times \Omega) \) and is the limit of \( \nabla(\sqrt{\rho_{\varepsilon} u_{\varepsilon}}) - u_{\varepsilon} \otimes \nabla \sqrt{\rho_{\varepsilon}} \) in the sense of distribution.
Lemma 3.5 \textit{Up to a subsequence, it holds that}
\begin{align}
\nabla(\sqrt{\rho_e}u_e) - u_e \otimes \nabla \sqrt{\rho_e} & \rightarrow \nabla(\sqrt{\rho}u) - u \otimes \nabla \sqrt{\rho} \text{ in } D'((0, T) \times \Omega), \\
\nabla^{tr}(\sqrt{\rho_e}u_e) - \nabla \sqrt{\rho_e} \otimes u_e & \rightarrow \nabla^{tr}(\sqrt{\rho}u) - \nabla \sqrt{\rho} \otimes u \text{ in } D'((0, T) \times \Omega).
\end{align}

Furthermore, it holds
\begin{align}
\int_0^T \int |\nabla(\sqrt{\rho}u) - u \otimes \nabla \sqrt{\rho}|^2 \, dx \, dt \leq C + Cr_0 + Cr_1.
\end{align}

\textbf{Proof.} It is easy to deduce from (3.5) and (3.21) that
\[ u_e \otimes \nabla \sqrt{\rho_e} \rightarrow u \otimes \nabla \sqrt{\rho} \text{ in } D'((0, T) \times \Omega), \]
which together with (3.37) gives (3.55) and thus (3.56). Furthermore, (3.57) is obtained directly from (3.56) and (3.55). This finishes the proof of Lemma 3.5. \hfill \Box

\section{Proof of Theorem 1.1}

We will follow the arguments in [29, Section 2.3] to prove that the limit (in some sense) \((\rho, \sqrt{\rho}u)\) of \((\rho_e, \sqrt{\rho_e}u_e)\) (up to a subsequence) is a weak solution to (1.1)–(1.2).

First, it follows from (3.9), (3.10), and (3.11) that
\[ \varepsilon \int_0^T \left( v_\varepsilon |\nabla v_\varepsilon|^2 + |\nabla v_\varepsilon|^4 \right) \, dx \, dt \leq C\varepsilon \int_0^T \left( \|v_\varepsilon\|_{L^2} \|\nabla v_\varepsilon\|^2_{L^6} + \|\nabla v_\varepsilon\|_{L^6} \|\nabla v_\varepsilon\|^3_{L^6} \right) \, dt \]
\[ \leq C\varepsilon^{\frac{1}{3}} \left( \varepsilon^{\frac{1}{4}} \int_0^T \|\nabla v_\varepsilon\|^6_{L^6} \, dt \right) \]
\[ \leq C\varepsilon^{\frac{1}{4}}. \] (4.1)

Then, on the one hand, for any test function \(\psi\), multiplying (3.12) by \(\psi\), integrating the resulting equality over \(\Omega \times (0, T)\), and taking \(\varepsilon \rightarrow 0\) (up to a subsequence), one can verify easily after using (3.10), (3.37), (3.2), (3.29), and (4.1) that \((\rho, \sqrt{\rho}u)\) satisfies (1.11).

On the other hand, let \(\phi\) be a test function. Multiplying (3.24) by \(\phi\), integrating the resulting equality over \(\Omega \times (0, T)\), and taking \(\varepsilon \rightarrow 0\) (up to a subsequence), by Lemmas 3.1, 3.3, and 3.4 we obtain after using (3.20), (3.28), and (3.31)–(3.34) that \((\rho, \sqrt{\rho}u)\) satisfies (1.12).

The proof of Theorem 1.1 is completed. \hfill \Box

\section{Proof of Theorem 1.2}

The system (1.1) without damping terms is as follows:
\begin{align}
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - 2\nu \text{div}(\rho Du) + \nabla P - 2\kappa^2 \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 0.
\end{cases}
\end{align}

We will consider the system (5.1) on bounded domain \(\Omega = T^3\) with periodic boundary conditions and the initial conditions (1.2). The notion of the weak solution of problem (5.1) (1.2) is defined by \((\rho, \sqrt{\rho}u)\) satisfying (1.11) and (1.19).

We will consider the approximate system of (5.1) by choosing \(r_0 = r_1 = 0\) in (5.2), that is,
\begin{align}
\begin{cases}
\rho_t + \text{div}(\rho u) = \varepsilon \text{div}(\nabla |v|^2 \nabla v) + \varepsilon \rho^{-\pi_0}, \\
\rho u_t + \rho u \cdot \nabla u - 2\nu \text{div}(\rho Du) + \nabla P - 2\kappa^2 \rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = \sqrt{\varepsilon} \text{div}(\rho \nabla u) + \sqrt{\varepsilon} \mu \text{div}(\rho \nabla^2 \log \rho) + \varepsilon |\nabla |v|^2 \nabla v \cdot \nabla u + \varepsilon \mu |\nabla |v|^2 \nabla v \cdot \nabla (\nabla \log \rho) \\
-\varepsilon\rho^{-\pi_0} u - \varepsilon^2 \rho |w|^2 u - \varepsilon \mu \nabla \rho^{-\pi_0} - \varepsilon \mu \nabla (\text{div}(\nabla |v|^2 \nabla v)) + \varepsilon \mu \text{div}(|\nabla |v|^2 \nabla v|) \nabla \log \rho.
\end{cases}
\end{align}
In order to obtain the global existence of weak solution to the problem (5.1), the main arguments here are to ensure the smooth approximate solutions satisfying the a priori bounds in [3], where the compactness of finite weak solutions is shown clearly. Indeed, one needs to prove that the smooth solutions to system (5.2) satisfying the energy estimate, the BD entropy inequality, and the Mellet-Vasseur type estimate.

It is clear that both the energy estimate and the BD entropy inequality obtained in Lemmas 2.1–2.2 are independent of \( r_0 \) and \( r_1 \). Hence, letting \( r_0 = r_1 = 0 \) in Lemmas 2.1–2.2 we can get directly the energy and BD entropy estimates on the smooth solutions to system (5.1) as follows:

**Lemma 5.1** Suppose that \( 11\kappa \leq \nu \), there exists some generic constant \( C \) independent of \( \varepsilon \) and \( \kappa \) such that

\[
\sup_{0 \leq t \leq T} \int (\rho|u|^2 + \rho + \rho^\gamma + \varepsilon \rho^{-p_0} + \kappa^2 |\nabla v|^2 + \varepsilon \mu |\nabla v|^4) \, dx
+ \nu \int_0^T \int \rho |\nabla u|^2 \, dx \, dt + \sqrt{\varepsilon} \int_0^T \int \rho |\nabla u|^2 \, dx \, dt
+ \kappa^2 \varepsilon \int_0^T \int (|\nabla v|^2 |\nabla^2 v|^2 + |\nabla |\nabla v||^2|^2 + |\nabla v|^2 v^{-2p_0-1}) \, dx \, dt
+ \varepsilon^2 \int_0^T \int (\mu |\nabla v|^4 |\nabla^2 v|^2 + \mu |\nabla v|^4 |\nabla v||^2 + |\nabla v|^4 v^{-2p_0-2} + \rho^{-2p_0-1}) \, dx \, dt \leq C,
\]

and

\[
\sup_{0 \leq t \leq T} \int (|\nabla v|^2 + \varepsilon |\nabla v|^4) \, dx + \int_0^T \int (\rho |\nabla u|^2 + \rho^{\gamma-2} |\nabla \rho|^2) \, dx \, dt
+ \kappa^2 \int_0^T \int \rho |\nabla^2 \rho|^2 \, dx \, dt + \varepsilon \nu \int_0^T \int (|\nabla v|^2 |\nabla^2 v|^2 + |\nabla v|^2 |\nabla |\nabla v||^2 + \rho^{-p_0-1} |\nabla v|^2) \, dx \, dt
+ \varepsilon^2 \int_0^T \int (|\nabla v|^4 |\nabla^2 v|^2 + |\nabla v|^4 |\nabla |\nabla v||^2 + \rho^{-p_0-1} |\nabla v|^4) \, dx \, dt
+ \varepsilon^2 \int_0^T \int (\rho |w|^5 + \rho |u|^5) \, dx \, dt + \varepsilon \int_0^T \int (v^{-2} |\nabla v|^6 + v^{-3} |\nabla v|^5) \, dx \, dt \leq C.
\]

Now, we need only to prove the Mellet-Vasseur type estimate. Motivated by [4,5], this is obtained by considering the following equivalent transformation system of \((\rho, w)\):

\[
\begin{aligned}
\rho_t + \text{div}(\rho w) &= \mu \Delta \rho + \varepsilon \text{div}(\nabla v^2 \nabla v) + \varepsilon \rho^{-p_0}, \\
\rho w_t + \rho w \cdot \nabla w + \nabla \rho - 2(\nu - \mu) \text{div}(\rho D w) - \mu \rho \Delta w - \sqrt{\varepsilon} \text{div}(\rho \nabla w) &= 2 \mu \nabla \rho \cdot \nabla w + \varepsilon v |\nabla v|^2 \nabla v \cdot \nabla w - \varepsilon^{3/2} |\nabla w|^3 u - \varepsilon \rho^{-p_0} w,
\end{aligned}
\]

which is deduced with the same arguments as Lemma 2.5

**Lemma 5.2** Suppose that \( 11\kappa \leq \nu \), there exists some generic constant \( C \) independent of \( \varepsilon \) such that

\[
\sup_{0 \leq t \leq T} \int \rho (e + |u|^2) \ln(e + |u|^2) \, dx \leq C.
\]

**Proof.** Notice that the definition of \( w \) in (2.1), one needs only to prove

\[
\sup_{0 \leq t \leq T} \int \rho (e + |w|^2) \ln(e + |w|^2) \, dx \leq C.
\]
Multiplying (5.11) by $H \equiv (1 + \ln(e + |w|^2))w$ and integrating by parts yield
\[
\frac{1}{2} \frac{d}{dt} \int \rho(e + |w|^2) \ln(e + |w|^2) dx + \int \rho \ln(e + |w|^2) (2(\nu - \mu)|Dw|^2 + \sqrt{\varepsilon} |\nabla w|^2) dx \\
\leq \frac{1}{2} \mu \int \Delta \rho(e + |w|^2) \ln(e + |w|^2) dx + 2\mu \int \nabla \rho \cdot \nabla w \cdot H dx + \mu \int \rho \Delta w \cdot H dx \\
+ \frac{1}{2} \varepsilon \int \nu \text{div}((\nabla v)^2 \nabla v)(e + |w|^2) \ln(e + |w|^2) dx + \varepsilon \int |\nabla v|^2 \nabla v \cdot \nabla w \cdot H dx \\
+ \frac{1}{2} \varepsilon \int \rho^{-p_0}(e + |w|^2) \ln(e + |w|^2) dx - \varepsilon \int \rho^{-p_0} w \cdot H dx \\
- \int \nabla P \cdot H dx - \varepsilon^{3/2} \int \rho |w|^3 u \cdot H dx + C \int \rho |\nabla w|^2 dx \\
\triangleq \sum_{i=1}^{9} K_i + C \int \rho |\nabla w|^2 dx.
\] (5.8)

The terms $K_i (i = 1, 2, \cdots, 9)$ in (5.8) can be bounded as follows. It is easy to deduce that
\[
K_4 + K_5 = -\frac{\varepsilon}{2} \int (e + |w|^2) \ln(e + |w|^2) |\nabla v|^4 dx \leq 0,
\] (5.9)

and
\[
K_6 + K_7 \leq C \varepsilon \int \rho^{-p_0} dx.
\] (5.10)

Furthermore, integration by parts gives
\[
K_3 = \mu \int \rho \Delta w \cdot (1 + \ln(e + |w|^2)) w dx \\
= -\mu \int \nabla \rho \cdot \nabla w \cdot (1 + \ln(e + |w|^2)) w dx - \mu \int \rho \nabla \ln(e + |w|^2) \cdot \nabla w \cdot w dx \\
- \mu \int \rho |\nabla w|^2 (1 + \ln(e + |w|^2)) dx \\
= -\frac{K_2}{2} - \frac{\mu}{2} \int \rho(e + |w|^2)^{-1} |\nabla |w|^2|^2 dx - \mu \int \rho |\nabla w|^2 (1 + \ln(e + |w|^2)) dx,
\] (5.11)

and
\[
\frac{K_2}{2} = \mu \int \nabla \rho \cdot \nabla w \cdot (1 + \ln(e + |w|^2)) w dx = \frac{\mu}{2} \int \nabla \rho \cdot \nabla ((e + |w|^2) \ln(e + |w|^2)) dx \\
= \frac{\mu}{2} \int \nabla \rho \cdot \nabla ((e + |w|^2) \ln(e + |w|^2)) dx = -\frac{\mu}{2} \int \Delta \rho(e + |w|^2) \ln(e + |w|^2) dx = -K_1.
\] (5.12)

The combination of (5.11) with (5.12) gives
\[
K_1 + K_2 + K_3 = -\frac{\mu}{2} \int \rho(e + |w|^2)^{-1} |\nabla |w|^2|^2 dx - \mu \int \rho |\nabla w|^2 (1 + \ln(e + |w|^2)) dx \leq 0.
\] (5.13)

For the term $K_8$, it holds
\[
|K_8| = \left| \int \nabla P \cdot (1 + \ln(e + |w|^2)) w dx \right| \\
\leq \int \rho^{\gamma - 1/2} (1 + \ln(e + |w|^2)) \rho^{1/2} |\text{div} w| dx + \left| \int \rho^{\gamma} \nabla (1 + \ln(e + |w|^2)) \cdot w dx \right| \\
\leq C \int \rho^{\gamma - 1} \ln^2 (e + |w|^2) dx + \int \rho |\text{div} w|^2 dx + 2 \left| \int \rho^{\gamma - 1} \frac{w \cdot \nabla w \cdot w}{(e + |w|^2)} dx \right| \\
\leq C \int \rho^{\gamma - 1} \ln^2 (e + |w|^2) dx + C \int \rho |\nabla w|^2 dx \\
\leq C + C \|\nabla \rho^{\gamma/2}\|_{L^2}^2 + C \int \rho |\nabla w|^2 dx,
\] (5.14)
where in the last inequality one has used the following fact
\[
\int \rho^{2\gamma-1} \ln^2(e + |w|^2) dx \leq C \int \rho^{5\gamma/3} dx + C \int \rho \ln^{\frac{10}{5\gamma-6}}(e + |w|^2) dx
\]
\[
\leq C \|\rho\|_{L_\gamma}^{2\gamma/3} \left(\|\rho\|_{L_1}^{\gamma} + \|\nabla \rho^{\gamma/2}\|_{L_2}^2\right) + C \int \rho dx + C \int \rho |w|^2 dx
\]
\[
\leq C + C \|\nabla \rho^{\gamma/2}\|_{L_2}^2
\]
owing to Sobolev inequality and (5.3).

Finally, the term \(K_9\) can be handled as follows:
\[
K_9 = -\varepsilon^{3/2} \int \rho |w|^3 u \cdot (1 + \ln(e + |w|^2)) w dx
\]
\[
= -\varepsilon^{3/2} \int \rho(1 + \ln(e + |w|^2))|w|^3 u |w|^{\beta} \cdot \nabla \log \rho dx
\]
\[
= -\frac{1}{2}\varepsilon^{3/2} \int \rho(1 + \ln(e + |w|^2))|w|^3 u |w|^2 + C\varepsilon^{3/2} \int \rho(1 + \ln(e + |w|^2))|w|^3 \nabla \log \rho |w|^2 dx.
\]
The second term of the right hand of (5.15) holds that for any \(0 < \beta < \frac{1}{5}\),
\[
\varepsilon^{3/2} \int \rho(1 + \ln(e + |w|^2))|w|^3 \nabla \log \rho |w|^2 dx
\]
\[
\leq C\varepsilon^{3/2} \int \rho|w|^3 + |w|^{3+\beta} \rho^{-1} |\nabla v|^2 dx
\]
\[
\leq C\varepsilon^{3/2} \int \rho|w|^5 dx + \varepsilon^{3/2} \int v^{-3} |\nabla v|^6 dx + C\varepsilon^{3/2} \int \rho \rho^{\frac{5}{\beta-\gamma}} |\nabla v|^\frac{10}{5-\beta} dx
\]
\[
\leq C\varepsilon^{3/2} \int \rho|w|^5 dx + \varepsilon^{3/2} \int \rho^{-2} |\nabla v|^6 dx + C\varepsilon^{3/2} \int \rho^{-\frac{4+3\beta}{5-\beta}} |\nabla v|\frac{10-\beta}{2} dx
\]
\[
\leq C + C\varepsilon^{3/2} \int \rho|w|^5 dx + C\varepsilon \int v^{-2} |\nabla v|^6 dx + C\varepsilon^{3/2} \int \rho^{-\frac{4+3\beta}{5-\beta}} |\nabla v|\frac{10-\beta}{2} dx
\]
where one has used (5.3) and (5.4).

Submitting (5.9), (5.13), (5.16) into (5.8) yields that
\[
\frac{1}{2} \frac{d}{dt} \int \rho(e + |w|^2) \ln(e + |w|^2) dx + \int \rho \ln(e + |w|^2) (2(\nu - \mu)|\nabla w|^2 + \sqrt{\varepsilon} |\nabla w|^2) dx
\]
\[
\leq C + C \|\nabla \rho^{\gamma/2}\|_{L_2}^2 + C \int \rho |\nabla u|^2 dx + C\varepsilon^{3/2} \int \rho |\nabla \log \rho|^2 dx + C\varepsilon^{3/2} \int \rho |w|^5 dx + C\varepsilon \int v^{-2} |\nabla v|^6 dx.
\]
Integrating the upper inequality over \([0,T]\), one obtains (5.7) after suing (6.3), (6.4), and (1.18). The proof of Lemma 5.2 is completed.

Proofs of Theorem 1.2. With the energy estimate, the BD entropy inequality, and the Mellet-Vasseur type estimate obtained in Lemmas 5.1, 5.2 in hand, following the similar compactness arguments as in Section 3 (see also those in [16, 29]), one can perform the limit progress \(\varepsilon \to 0^+\) to the smooth approximation solutions and thus complete the proof of Theorem 1.2. We omit the details here.

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