Integral Action in Output Feedback for multi-input multi-output nonlinear systems
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Abstract—We address a particular problem of output regulation for multi-input multi-output nonlinear systems. Specifically, we are interested in making the stability of an equilibrium point and the regulation to zero of an output, robust to (small) unmodelled discrepancies between design model and actual system in particular those introducing an offset. We propose a novel procedure which is intended to be relevant to real life systems, as illustrated by a (non academic) example.

Index Terms—Robust regulation, nonlinear control, output feedback, semi-global stabilization, integral action, observability, high-gain observer, forwarding, non-minimum phase systems, uncertain dynamic system.

I. INTRODUCTION

For a controlled dynamical system, it is of prime importance in real world applications to be able to design an output feedback control law which achieves asymptotic regulation of a given output while keeping the solutions in some prescribed set, in presence of (constant) uncertainties. We refer to this as the problem of robust output regulation by output feedback.

The problem has been completely solved in the linear framework by Francis and Wonham in the 70’s (see [44]). Important efforts have been done in order to extend this result to the nonlinear case (see, for instance [9], [21]) and many different solutions have been proposed (see among others [10], [14], [23], [34], [4] Chapter 7.2), [11], [18], [24], [21], [37], [28]). Nevertheless we are still far from having a complete solution to the problem of output regulation in the nonlinear multi-input-multi-output framework similar to what we have in the linear case. Indeed most of the works require a good knowledge of the effects of the disturbances on the system, or they rely on “structural properties” as, for example, normal forms, minimum phase assumption, matched uncertainties or relative degree uniform in the disturbances. In particular, for single-input single-output minimum-phase nonlinear systems which possess a well defined relative degree preserved under the effect of disturbances, a complete solution has been given in [24], further improved to the output feedback case in [36]. Under the same assumptions, this work has been successfully extended in [37] to square multi-input multi-output systems for which the notion of relative degree indices and observability indices coincides. Further, with the technique of the auxiliary system introduced in [20], the minimum-phase assumption has been removed in [28] allowing the zero-dynamics to be unstable. However, as far as we know, a general solution is still unknown when these structural properties do not hold.

The approach to nonlinear output regulation followed in this paper is motivated by the linear context developed in its full generality in the milestone paper [12] that we find useful to briefly recall here. Consider the linear system

\[ \dot{x} = A_0 x + B_0 u , \quad y = C_0 x , \]

where the state \( x \) is in \( \mathbb{R}^n \), the control \( u \) is in \( \mathbb{R}^m \) and the measured output \( y \) is in \( \mathbb{R}^p \). The output \( y \) is decomposed as \( y = (y_r, y_e) \) where \( y_r \), in \( \mathbb{R}^r \), \( r \leq m \), is the output to be regulated to zero (without loss of generality). When the system above is supposed to be only an approximation of a process given by

\[ \dot{x} = Ax + Bu + Pw , \quad y = Cx + Qw , \]

where \( w \) is an unknown constant signal to be either rejected or tracked, the well posed regulator problem with internal stability (addressed by Wonham for linear systems as shown for instance in [44] Chapter 8]) is that of finding an output feedback law based on the model such that, for all triplets \( \{A, B, C\} \) close enough to \( \{A_0, B_0, C_0\} \), and for all matrices pairs \( \{P, Q\} \), the regulation-stabilization problem is solved, i.e. the system admits a stable equilibrium point on which the output to be regulated is equal to zero. According to [9 Proposition 1.6], for example, this problem is solvable if and only if the following 3 conditions are satisfied:

(a) the pair \( \{A_0, C_0\} \) is detectable;
(b) the pair \( \{A_0, B_0\} \) is stabilizable;
(c) the matrix \( \begin{pmatrix} A_0 & B_0 \\ C_0, r \\ 0 \end{pmatrix} \) is right invertible.

Precisely, under the above 3 conditions, it is always possible to design an output feedback law of the form

\[ \dot{z} = y_r , \quad \dot{\eta} = F\eta + Ly , \quad u = K\eta + Mz + Ny \]

which solves the regulation problem provided \( F, L, K, M, \) and \( N \) are chosen such that the following matrix

\[ \begin{pmatrix} A + BNC & BK & BM \\ LC & F & 0 \\ C_r & 0 & 0 \end{pmatrix} \]

is Hurwitz for all triplets \( \{A, B, C\} \) close enough to \( \{A_0, B_0, C_0\} \), and for all matrices pairs \( \{P, Q\} \). Note that in this linear framework no structural properties are needed.

Merging all the tools in literature that are at our disposal, we try to recover the same result as in the linear case, asking for possibly minimal assumptions but at the same time paying particular attention to proposing a design truly workable in
applications. For example, minimality implies not to ask for any specific structural properties whereas applicability forbids nonlinear changes of coordinates when no expression is known for their inverse. Our answer to the problem uses “bricks” which can be found in other publications (as [59, 33, 5]) that we glue together. But for making this gluing process efficient we have to address some (new) specific problems.

As in the linear framework, we extend the system with an integral action. Then, as in [33], we rely on forwarding to design a stabilizing state feedback for the extended system. Next, for transforming this state feedback into an output feedback, it is sufficient to apply the techniques which have been proposed for asymptotic stabilization by output feedback. A lot of effort has been devoted to this question and many results have accumulated (see for instance the survey [2]). In particular the transformation is done by replacing the actual state by a state estimate provided by a tunable observer (i.e. an observer whose dynamics can be made arbitrarily fast). Stability of the overall closed loop system is established via the common separation principle [39, 6], and output regulation follows from the integral action embedded in the control law.

The tunable observer we propose is, as in [5] (previously inspired by [11] and [29]), a high-gain observer written in the original coordinates and appropriate for our multi-input multi-output (possibly non-square) case. We propose a new set of sufficient conditions which guarantees the existence of such an observer. As opposed to what we have found in the literature (see for instance [8, 17, 15]) our conditions can be verified in the original coordinates and they do not need the explicit knowledge of the inverse of nonlinear change of coordinates (which may be very hard to find). Also in looking for minimal assumptions, we do not ask for global observability or global uniformity with respect to the inputs. The latter impacts the state feedback design and we show how to address this point (in [33] only a global solution is proposed).

Finally, we show that the proposed solution guarantees robust regulation. Robustness is here with respect to unmodelled effects, not in the system state dimension, but in the approximations of the functions which define its dynamics and measurements. This has been done already in [33] but for the state feedback case and with an assumption on the closed loop system. Here we show that if the model is close enough (in a C¹ sense), in open loop, to the process, then output regulation is achieved by our output feedback design. However, as opposed to the linear case, where the result is global with respect to the magnitude of the disturbances, an unfortunate consequence of being in our less restrictive context is that we need the perturbations to be small enough.

In this work, to simplify, we restrict our attention to systems affine in the input. The extension to the non affine case is made possible by considering the system controls as state and their derivatives as fictitious controls. See [5] for example.

The paper is organized as follows. Section II is devoted to show the main assumption and results of this work. In Section III and IV we present respectively the state feedback design and the observer design. The proofs of the main propositions are given in Section V. Finally, in Section VI we illustrate the proposed design with a non-academic example inspired from a concrete case study in aeronautics (the regulation of the flight path angle of a simplified longitudinal model of a plane).

Notations
For a set $S$, $\hat{S}$ denotes its interior, $\partial S$ denotes its boundary and $d(x, S)$ denote the distance function of a point $x$ to the set $S$. When $S$ is a subset of $A \times B$ whose points are denoted $(a, b)$, $(S)_a$ denotes the set \{ $a \in A : \exists b \in B : (a, b) \in S$ \}. For a function $h$ and a vector field $f$, $L_f$ denotes the Lie derivative of $h$ along $f$, given coordinates $x$, $L_f h(x) = \frac{\partial h}{\partial x}(x)f(x)$. To any strictly positive real number $v$, we associate a “saturation” function $\text{sat}_v$ defined as a $C^1$ function bounded by $v$ and satisfying

$$\text{sat}_v(s) = s \quad \text{if} \quad |s| \leq \frac{v}{1 + \varsigma},$$

where $\varsigma$ is a (small) strictly positive real number.

II. ROBUST REGULATION BY OUTPUT FEEDBACK
A. Problem Statement and Assumptions
For a process, we have at our disposal the following dynamical model

$$\dot{x} = f(x) + g(x)u, \quad y = h(x) = (h_r(x), h_e(x))$$

where the state $x$ is in $\mathbb{R}^n$, the control $u$ is in $\mathbb{R}^m$, the measured output $y$ is in $\mathbb{R}^p$ and the functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{nm}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are smooth enough and $f$ and $h$ are zero at the origin. We investigate the problem of regulating at zero the part $y_r$ of the output decomposed as $y = (y_r, y_e)$ with $y_r \in \mathbb{R}^r$ and $r \leq m$ and this while stabilizing an equilibrium for $x$. But, being aware that the triplet $(f, g, h)$ gives only an approximation of the dynamics of the process, we would like the above regulation-stabilization property to hold not only for this particular triplet but also for any other one in a neighborhood.

The real process is described by equations of the form

$$\dot{x} = \xi(x, u), \quad y = \zeta(x, u),$$

where the functions $\xi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\zeta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are assumed continuously differentiable ($C^1$). These functions are unknown but we assume that they are close enough to $f + gu$ and $h$ respectively in the sense that the discrepancies

$$|\xi(x, u) - f(x) - g(x)u| + |\zeta(x, u) - h(x)|$$

and

$$\left| \begin{array}{c}
\frac{\partial \xi}{\partial x}(x, u) - \frac{\partial f}{\partial x}(x) - \frac{\partial g}{\partial x}(x)u \\
\frac{\partial \zeta}{\partial x}(x, u) - \frac{\partial h}{\partial x}(x)
\end{array} \right|$$

are small enough as made precise later on.

Mimicking the 3 necessary and sufficient conditions for the linear case given in the introduction, we consider the following (sufficient) assumptions that we discuss after their formal statement.

Assumption 1 There exists an open set $\mathcal{O}$ of $\mathbb{R}^n$ containing the origin and an open star-shaped subset $\mathcal{U}$ of $\mathbb{R}^m$, with the origin as star-center, such that, for any strictly positive real number $a$ and any compact subset $\mathcal{C}$ of $\mathcal{O}$, there exist an integer $d$, a compact subset $\mathcal{E}$ of $\mathcal{O}$, a real number $\mathcal{U}$ and a class-$\mathcal{K}^\infty$ function $\eta$ such that, for each integer $\kappa$, we can find $C^1$ functions $\partial_{\kappa} : \mathbb{R}^m \times \mathbb{R}^p \times \mathcal{O} \to \mathcal{O}$, $U_{\kappa} : \mathcal{O} \times \mathcal{O} \to \mathbb{R}_{\geq 0}$,
a continuous function \( L_{\theta_z} : \mathcal{O} \to \mathbb{R}_{\geq 0} \) and a strictly positive real number \( \sigma_n \), such that:

1. for any function \( t \to u(t) \) with values in \( \mathcal{U}(\bar{u}) \) defined as

\[
\mathcal{U}(\bar{u}) = \{ u \in \mathcal{U} : |u| \leq \bar{u} \} \tag{4}
\]

and any bounded function \( t \to y(t) \), the set \( \hat{\mathcal{C}} \) is forward invariant by the flow generated by the following observer

\[
\hat{x}_n = \vartheta_n(y, \hat{x}_n, u) ; \tag{5}
\]

2. \( \forall (x, \hat{x}) \in \mathcal{O} \times \mathcal{O} \), \( U_n(x, \hat{x}) = 0 \iff x = \hat{x} ; \tag{6} \)

3. \( \sigma_n^{-1} \alpha(|x - \hat{x}|) \leq U_n(x, \hat{x}) \leq \sigma_n \alpha \forall x \in \mathcal{C}, \forall \hat{x} \in \hat{\mathcal{C}} ; \tag{7} \)

4. \( \lim_{\kappa \to \infty} \sigma_n = +\infty ; \tag{8} \)

5. \( \frac{\partial U_n}{\partial \hat{x}}(x, \hat{x}_n)[f(x) + g(x)u] + \frac{\partial U_n}{\partial x}(x, \hat{x}_n) \vartheta_n(h(x), \hat{x}_n, u) \leq -\sigma_n U_n(x, \hat{x}_n) \forall u \in \mathcal{U}(\bar{u}), \forall (x, \hat{x}_n) \in \mathcal{C} \times \hat{\mathcal{C}}. \tag{9} \)

6. For all \( (y_a, y_b, \hat{x}_n, u) \) in \( \mathbb{R}^{2p} \times \mathcal{O} \times \mathcal{U}(\bar{u}) \),

\[
|\vartheta_n(y_a, \hat{x}_n, u) - \vartheta_n(y_b, \hat{x}_n, u)| \leq L_{\theta_z}(\hat{x}_n)|y_a - y_b| \tag{10} \]

Assumption 2 There exist an open subset \( \mathcal{S} \) of \( \mathbb{R}^n \) and a continuous function \( \beta : \mathcal{S} \to \mathcal{U} \) which is zero at the origin and such that the origin of \( \mathcal{O} \) with \( u = \beta(x) \), is an asymptotically and locally exponentially stable equilibrium point with \( \mathcal{S} \) as domain of attraction.

Assumption 3 The matrix

\[
\begin{pmatrix}
\frac{\partial L(0)}{\partial x} & g(0) \\
\frac{\partial L(0)}{\partial x} & 0
\end{pmatrix}
\]

is right invertible.

Assumption 1 is aimed at being a counter-part of the detectability condition (a). But we have to face here problems specific to this nonlinear framework:

- In our construction we shall rely on the so called separation principle. For nonlinear systems (see \[39\] for example), it asks for an observer with a tunability property, i.e. an observer the speed of convergence of which can be made arbitrary fast (see \[7\]). This property is provided here by the family of observers \( \vartheta \) satisfying \( \alpha \) and \( \beta \).

- Observability may depend on the input. This explains why we impose the control to belong to the set \( \mathcal{U} \).

- The tuning of observers for non linear systems may depend on the local Lipschitz constant of the non lineairities. This explains why the family of observers depends on the bound \( \bar{u} \) of the input.

On the other hand, to reduce the restrictiveness, Assumption 1 is imposed only for system states belonging to an open subset \( \mathcal{O} \) of \( \mathbb{R}^n \). In Section \[17\] we shall see how the family of observers in this assumption can be designed as observers based on high-gain techniques.

Assumption 2 is the counter-part of the stabilizability condition (b) and claims the existence of a state feedback law which asymptotically stabilizes the system \( \mathcal{O} \). Actually it assumes that a preliminary design step can be done. For it any tool – Lyapunov design, feedback (partial) linearization, passivity, use of structure of uncertainties in combination with gain assignment techniques, etc. – can be exploited. However, because Assumption 1 imposes the control be in \( \mathcal{U} \), we propagate this restriction here, asking the stabilizing control \( \beta \) to be made arbitrary fast (see \[7\]).

Finally Assumption 3 corresponds to the non-resonance condition (c) and states that the first order approximation at the origin of the system \( \mathcal{O} \) does not have any zero at 0.

B. Main results

Assumptions 1 to 3 are sufficient to guarantee the existence of an output feedback law solving the regulation-stabilization problem for the model.

Proposition 1 Suppose Assumptions 1, 2 and 3 hold. There exists an open subset \( \mathcal{S} \) of \( (\mathcal{S} \cap \mathcal{O}) \times \mathbb{R}^r \) such that, for any of its compact set \( \mathcal{C}_{\kappa z} \), there exist an integer \( \kappa \), a compact subset \( \mathcal{C}_z \) of \( \mathcal{O} \), a real number \( \mu \) and \( C^1 \) functions \( k : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^r \) and \( \psi_{sat} : \mathbb{R}^n \times \mathbb{R}^r \to \mathcal{U}(\mu) \), such that the origin of the model \( \mathcal{O} \), in closed-loop with the dynamic output feedback

\[
\dot{z} = k(\dot{x}, h_r(x)), \quad \dot{x} = \vartheta_n(y, \hat{x}, u), \quad u = \psi_{sat}(\hat{x}, z) \tag{11}
\]

with \( \kappa \geq \mathcal{C}_z \) is asymptotically stable with a domain of attraction \( \mathcal{A} \) containing the set \( \mathcal{C}_{\kappa z} \times \mathcal{C}_{\kappa z} \).

\[\text{Proof: See Section V-A.}\]

In the case where \( \mathcal{S} \) and \( \mathcal{O} \) are the full space \( \mathbb{R}^n \), this result would be a semi-global regulation-stability result. It claims the existence of a dynamic output feedback which asymptotically stabilizes the origin of the model \( \mathcal{O} \). Such a result is not new per se. It is in line with many results related to the separation principle as those in \[39\], \[6\] or \[19\], Chapter 12.3.

But as written in the introduction, we do not state only “existence” but instead we propose an explicit and workable design. We refer the reader to Section III for the definition of the set \( \mathcal{S} \), the real number \( \mu \) and the functions \( k \) and \( \psi_{sat} \) and to Section IV-A for the definition of the integer \( \kappa \) and the set \( \mathcal{C}_{\kappa z} \).

In the following propositions, under the Assumptions 1, 2 and 3 and knowing the result of Proposition 1 holds, we study the process (5) in closed-loop with the control law (11) designed for the model \( \mathcal{O} \).

Proposition 2 Let \( \mathcal{C} \) be an arbitrary compact subset of the domain of attraction \( \mathcal{A} \), given by Proposition 1 which admits the equilibrium as an interior point and is forward invariant for the closed-loop system (11). For any open neighborhood \( N_{\mu z} \) of the boundary set \( \partial \mathcal{C} \), contained in \( \mathcal{A} \), there exists a

1 See respectively \[33\] and \[55\] for \( \mathcal{S} \), \[55\] for \( \mu \), \[40\] and \[35\] for \( k \) and \( \psi_{sat} \).

2 See successively \[62\] and \[54\].
strictly positive real number \( \delta \) such that, for any pair \((\xi, \zeta)\) of \(C^1\) functions which satisfies
\[
|\xi(x, u) - (f(x) + g(x)u) + \zeta(x, u) - h(x)| \leq \delta \quad \forall (x, u) \in \mathcal{N}_C \times \mathcal{U}(\mu)
\]
the closed-loop system \([3], (11)\) has equilibria and at any such point the output \(y_r\) is zero.

**Proof:** See Section [V-B].

If the domain of attraction were the full space, this result would follow from \([38, \text{Section 12}]\). It says that, when the evaluation, on a "spherical shell"-like set, of the model and process functions are close enough, equilibria where output regulation occurs do exist. If this closeness is everywhere in the domain of attraction, then we have even a solution to the basin of attraction \([38, \text{Section 12}]\). It says that, when the domain of attraction were the full space, this result satisfying \(\eta(0) = 0\) and
\[
|y_r| \leq [1 + |x| + |y_r|^2] \eta(|k(x, y_r)|) \quad \forall (x, y_r) \in \mathbb{R}^n \times \mathbb{R}^p.
\]
Actually the function \(k\) used in the output feedback \([11]\) is the modified version given later in \([39]\).

**B. Design the function \(\psi\) via forwarding**

Let us consider the extended system
\[
\dot{x} = f(x) + g(x)u, \quad \dot{z} = k(x, h_r(x)).
\]
With Assumption 2 we are left with modifying the given state feedback \(\beta\) to obtain a state feedback stabilizing asymptotically the origin for the extended system \([20]\). Fortunately it has the so-called feedforward form which has been extensively studied in the 90’s with in particular the introduction of the forwarding techniques based on saturations as in \([40]\) or on Lyapunov design with coordinate change as in \([20]\) or coupling term as in \([22]\). We recall briefly these techniques. They differ on the available knowledge they require. Specifically, Assumption 2 has two consequences:

1. With the converse Lyapunov theorem of \([27]\), we know there exists a \(C^1\) function \(V: \mathcal{S} \to \mathbb{R}^+\) which is positive definite and proper on \(\mathcal{S}\) and such that the function \(x \mapsto \frac{\partial V}{\partial x}(f(x) + g(x)\beta(x))\) is negative definite on \(\mathcal{S}\) and upperbounded by a negative definite quadratic form of \(x\) in a neighborhood of the origin.

2. Since the origin of the system \([2]\) in closed-loop with \(\beta(x)\) is locally exponentially stable, there exists (see \([30, \text{Lemma IV.2}]\)) a \(C^1\) function \(H: \mathcal{S} \to \mathbb{R}^+\) satisfying
\[
\frac{\partial H}{\partial x}(f(x) + g(x)\beta(x)) = k(x, h_r(x)), \quad H(0) = 0.
\]

Depending on whether or not we know the function \(V\) and/or the function \(H\) or only its first order approximation at the origin leads to different designs.

a) **Forwarding with \(V\) and \(H\) known**

When \(L_0 L_i h_r(x) = 0\), for \(i \in \{0, \ldots, \rho\}\), \([16]\) can be relaxed in \(\{k(x, h_r(x)) = 0, L_0 L_1 h_r(x) = \ldots = L_0^{\rho-1} h_r(x) = 0\} \Rightarrow h_r(x) = 0\). See \([23]\) for example.

The function \(\eta\) is a smoothed version of \(s \to \sup\{s(y_r); k(x, y_r) \leq s, \frac{|y_r|}{1 + |x| + |y_r|^2}\}.

**III. STATE FEEDBACK DESIGN**

**A. Adding an integral action: design of the function \(k\)**

To solve the problem of regulating \(y_r\) to 0 we follow the very classical idea of adding an integral action. To do so we first select a \(C^1\) function \(k: \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^r\) satisfying\(^3\) for all \(x \in \mathbb{R}^n\) and all \((y_r, y_{r}^0) \in \mathbb{R}^{2r},\)
\[
k(x, y_r) = 0 \iff y_r = 0,
\]
\[
|k(x, y_{r}^0) - k(x, y_r^0)| \leq L_k(x)|y_r^0 - y_r^0|,
\]
where \(L_k : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\) is a continuous function. Of course the function \(k\) can be simply \(h_r\). But, in its choice, we can take advantage of the properties of the physical system under consideration and it can simplify the feedback design or its implementation. An example is given in section [VI].
When both $V$ and $H$ are known, a stabilizer $\psi$ for the system (20), is
\[
\psi(x, z) = \beta(x) - J \left( x, [L_0 V(x) - (z - H(x))^\top L_0 H(x)]^\top \right),
\]
with $H$ defined by (21), and with $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ any continuous function satisfying, for all $x \in \mathbb{R}^n$,
\[
v^\top J(x, v) > 0 \quad \forall v \neq 0, \quad \det \left( \frac{\partial J}{\partial v}(x, 0) \right) \neq 0. \tag{23}
\]
Following [30] this can be established under Assumptions 2 and 3 with the function $V_e : \mathcal{S} \times \mathbb{R}^r \to \mathbb{R}_{\geq 0}$ defined as
\[
V_e(x, z) = V(x) + \frac{1}{2} (z - H(x))^\top (z - H(x)). \tag{24}
\]

**Remark 1** If $V$ is from known from the design of $\beta$, it may not be proper on $\mathcal{S}$. To make it proper we first define $v_S$ as
\[
v_S = \inf_{x \in \mathcal{S}} V(x)
\]
and we replace $V(x)$ by $\frac{V(x)}{v_S - V(x)}$. See [32]. Unfortunately in doing so, the domain of definition of this new function $V$ may be a strict subset of $\mathcal{S}$. In the following, we still call $\mathcal{S}$ this domain on which $V$ is proper.

b) **Forwarding with $V$ unknown but $H$ known**

When $V$ is unknown, but $H$ is known, there exists a function $\gamma : \mathcal{S} \to \mathbb{R}_{\geq 0}$ with strictly positive values such that a state feedback for the system (20) is
\[
\psi(x, z) = \beta(x) + \gamma(x) L_0 H(x)^\top J(x, z - H(x)), \tag{25}
\]
with $H$ defined by (21), and $J : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^r$ bounded and satisfying (23). This can be established with the Lyapunov function (24).

c) **Forwarding with $V$ unknown and $H$ approximated**

Instead of solving the partial differential equation (21) for $H$, and using (25), we pick
\[
\psi(x, z) = \beta(x) + \gamma(x) g(x)^\top H_0^\top J(x, z - H_0 x), \tag{26}
\]
where $H_0$ is obtained as
\[
H_0 = \frac{\partial k}{\partial y_r}(0, 0) \frac{\partial h_r}{\partial x}(0) \left[ \frac{\partial f}{\partial x}(0) + g(0) \frac{\partial \beta}{\partial x}(0) \right]^{-1}. \tag{27}
\]
The corresponding Lyapunov function is
\[
V_e(x, z) = d(V(x)) + \sqrt{1 + \frac{1}{4} (z - H_0 x)^\top (z - H_0 x) - 1},
\]
where $d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $C^1$ function with strictly positive derivative, to be chosen large enough (see [30]). In the case where the system
\[
\dot{x} = f(x) + g(x) (\beta(x) + v), \tag{28}
\]
with $v$ as input is input to state stable with restriction, i.e. provided $|v|$ is bounded by some given strictly positive real number $\Delta$, then following [40], the state feedback can be chosen as
\[
\psi(x, z) = \beta(x) + \epsilon J \left( x, \frac{g(0)^\top H_0^\top (z - H_0 x)}{\epsilon} \right), \tag{29}
\]
with $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ bounded and satisfying (23) and $\epsilon$ is a small enough strictly positive real number.

Whatever design route a), b) or c) we follow, we obtain the following lemma.

**Lemma 1** Under Assumptions 2 and 3 the function $V_e$ is positive definite and proper on $\mathcal{S} \times \mathbb{R}^r$. Its derivative along the extended system (20) in closed-loop with $u = \psi(x, z)$ is negative definite on $\mathcal{S} \times \mathbb{R}^r$ and upperbounded by a negative definite quadratic form of $(x, z)$ in a neighborhood of the origin. Consequently, for the corresponding closed loop system, the origin is asymptotically stable with $\mathcal{S} \times \mathbb{R}^r$ as domain of attraction (without forgetting Remark 2) and locally exponentially stable.

**Proof:** Since $V$ is positive definite and proper on $\mathcal{S}$, $V_e$ is positive definite and proper on $\mathcal{S} \times \mathbb{R}^r$. Also the derivative of $V_e$, along the solutions of the closed loop system is negative definite in $(x, \psi(x, z))$ and upperbounded by a negative definite quadratic form of $(x, \psi(x, z))$ in a neighborhood of the origin (see [30], [40] for example). With this, to complete the proof, it is sufficient to show the existence of a real number $c$ such that
\[
|z| \leq c |\psi(0, z)|.
\]
Since we have
\[
\psi(0, z) = J(0, L_0 H(0) z), \quad \text{respectively} \quad = J(0, H_0 g(0) z)
\]
where the function $J$ satisfies (23), the above inequality holds if $L_0 H(0)$, respectively $H_0 g(0)$, is right invertible. But, by differentiating (21) which holds at least in a neighborhood of the origin, using (18) and (27), and since $f$ and $\beta$ are zero at the origin, we have
\[
\frac{\partial H}{\partial x}(0) = H_0.
\]
Assume the matrix $H_0 g(0)$ is not right invertible, i.e. the exists a vector $v$ in $\mathbb{R}^r$ such that
\[
v^\top H_0 g(0) = 0.
\]
Then we have
\[
\left( v^\top H_0 - v^\top \frac{\partial k}{\partial y_r}(0, 0) \frac{\partial f}{\partial x}(0), g(0) \right) = 0,
\]
which contradicts Assumption 3.

**Remark 2**

- Because the set $\mathcal{U}$ in Assumption 7 is star-shaped, while satisfying (24), the function $J$ can always be chosen such that the function $\psi$ above defined takes values in $\mathcal{U}$.
- A drawback of the integral action is the possible wind-up. To prevent this phenomenon, in all the above, $\dot{z}$ can be modified in
\[
\dot{z} = k(x, y_r) + \omega \left[ \text{sat}_{\pm}(z + H(x)) - (z + H(x)) \right] \tag{30}
\]
with $H(x)$ replaced by $H_0 x$ when needed and where the saturation function is defined in (1), $\omega$ is any strictly positive real number and $\dot{z}$ should be chosen large enough to allow the $z$-dynamics to converge to the right equilibrium point. This modification does not change anything to the asymptotic stability which can be established with the same Lyapunov functions.
C. Definitions of $SO$ and $\mu$ and saturation of $\psi$ to get the function $\psi_{sat}$

If we were to design a state feedback, we could stop here. But the output feedback we design is based on the previous state feedback and augmented with an observer. Since the estimated state may make no sense during some transient periods, we need a mechanism to prevent any bad closed-loop effects during these periods. As proposed in [23], we use saturation.

First we define the set $SO$ where we would like the state to remain. For this, let $S$ be given by Assumption 2, maybe modified as explained in Remark 1 above. Similarly, let $O$ be given by Assumption 1 (maybe modified later as in [54]). Let also the function $V_e$, positive definite and proper on $S \times \mathbb{R}^r$. Then, by the above design of the state feedback or a converse Lyapunov theorem [27] satisfying

$$V_e(x, z) = \frac{\partial V_e}{\partial x}(x, z)\{f(x) + g(x)\psi(x, z)\} + \frac{\partial V_e}{\partial z}(x, z)k(x, h_r(x)) = -W_e(x, z) \tag{31}$$

where the function $W_e$ defined here is positive definite on $S \times \mathbb{R}^r$. Then, if $S$ is not a subset of $O$, we let $v_\infty$ be the real number defined as

$$v_\infty = \inf_{(x, z) \in (S \times \mathbb{R}^r) \backslash (O \times \mathbb{R}^r)} V_e(x, z) \tag{32}$$

If not let formally $v_\infty$ be infinity. We define the open set

$$SO = \{(x, z) \in S \times \mathbb{R}^r : V_e(x, z) < v_\infty\} \tag{33}$$

This set in non empty since it contains the origin.

In the same way, to each real number $v$ in $[0, v_\infty)$ we associate the set

$$\Omega_v = \{(x, z) \in S \times \mathbb{R}^r : V_e(x, z) < v\} \tag{34}$$

It is a compact subset of $SO$. Also, from Lemma 1 it is forward invariant for the extended system [20] in closed-loop with $u = \psi(x, z)$. On the other hand, for any $C_{\infty}$ compact subset of $SO$, we can find real numbers $v_1 < v_2$ satisfying

$$C_{\infty} \subseteq \Omega_{v_1} \subseteq \Omega_v \subsetneq \Omega_{v_2} \subseteq SO \tag{35}$$

Then, with $\mu$ the real number defined as

$$\mu = (1 + \epsilon) \max_{(x, z) \in \Omega_{v_2}} |\psi(x, z)| \tag{36}$$

with $\epsilon$ a small number as in [1], we consider the subset $U(\mu) \subset U$ (see (3)). As $U$ in Assumption 1 it is star-shaped with the origin as a star-center. Let then the function $\psi_{sat} : \mathbb{R}^n \times \mathbb{R}^r \rightarrow U(\mu)$ be

$$\psi_{sat}(x, z) = \text{sat}_\mu(\psi(x, z)) \tag{37}$$

It is bounded and Lipschitz and, as $\psi$, it is $C^1$ on a neighborhood of the origin. Similarly, we modify the function $k$ (defined in [15]) by saturating its argument $x$. Namely we replace

$$k(x, h(x)) \quad \text{by} \quad k(\text{sat}_\bar{x}(x), h_r(x)) \tag{38}$$

where

$$\bar{x} = (1 + \epsilon) \max_{(x, z) \in \Omega_{v_2}} |x| \tag{39}$$

IV. Observer Design

In the Assumption [1] we ask for the knowledge of the family of observers [5]. Fortunately it can be obtained as a high gain observer. A lot of attention has been devoted to this type of observers and many results are available at least for the single output case. See for example the survey [25] and the references therein. We are interested here in some specific aspects as (a) the possibility of writing the dynamics of the observer in the original coordinates; (b) the multi-output case; as far as we know at the time we write this text, the study of tunable observers in the multi-output case is far from being conclusive. Only some sufficient conditions are known (see, for instance [7], [26], [41], [17], [8], [15], [6]); (c) the fact that observability holds only on $O$, a (possibly) strict subset of the full space $\mathbb{R}^n$.

To introduce them, we find useful to start with a very brief reminder on single output high gain observers.

A. Reminder on high gain observers in the single output case

It is known (see [13] Theorem 3.4.1) for example) that, for a single-input single-output system of the form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \quad x \in \mathbb{R}^n, u, y \in \mathbb{R}, \quad (40)$$

which is observable uniformly with respect to the input and is differentially observable of order $n_o$, there exists an injective immersion $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^{n_o}$, obtained as

$$\phi = \Phi(x) = \left(h(x), L_1 h(x), \ldots, L_f^{n_o-1} h(x)\right)^\top, \tag{41}$$

which puts the system (40) into the so called observability (triangular) normal form

$$\dot{\phi} = A_n \phi + B_n b(\phi) + D_{n_o}(\phi) u, \quad y = C_{n_o} \phi \tag{42}$$

where

$$A_n = \begin{pmatrix} 0_{n_o-1 \times 1} & I_{n_o-1 \times n_o-1} \\ 0 & 1_{n_o-1 \times 1} \end{pmatrix}, \quad B_n = \begin{pmatrix} 0_{n_o-1 \times 1} \\ 1 \end{pmatrix}, \quad C_{n_o} = \begin{pmatrix} 1 & 0_{1 \times n_o-1} \end{pmatrix}, \quad D_{n_o}(\phi) = (d_1(\phi_1), \ldots, d_i(\phi_1, \ldots, \phi_i), \ldots, d_{n_o}(\phi))^\top \tag{43}$$

and where $b(\cdot)$, $d_i(\cdot)$ are locally Lipschitz function. An observer for the system (40) is

$$\dot{\hat{\phi}} = A_n \hat{\phi} + B_n b(\hat{\phi}) + D_{n_o}(\hat{\phi}) u + K_n L_n(\ell) (y - C_{n_o} \hat{\phi}), \quad (44)$$

$$\dot{x} = \Phi^{\ell-\text{inv}}(\hat{\phi}), \quad (45)$$

where $K_n$ is such that $(A_n - K_n C_{n_o})$ is Hurwitz, $L_n(\ell) = \text{diag}(\ell, \ldots, \ell^{n_o})$ and $\Phi^{\ell-\text{inv}}$ is any locally Lipschitz left inverse function of $\Phi$ satisfying

$$\Phi^{\ell-\text{inv}}(\Phi(x)) = x \quad \forall x \in \mathbb{R}^n.$$

In the $\phi$-coordinates it is a standard high gain observer the dynamics of which can be made arbitrary fast by increasing the high-gain parameter $\ell$ (see for instance [8]).
B. On the possibility of writing the dynamic of the observer in the original coordinates

As already observed in [29], a main issue in implementing the observer (44) is about the function \(\Phi^{\ell \rightarrow \nu \nu}\) for which we have typically no analytical expression, meaning that we have to solve on-line a minimization problem as

\[
\hat{x} = \text{argmin}_x |\phi(x) - \hat{\phi}|.
\]

Fortunately as noticed in [11] and proposed also in [29], this difficulty can be rounded when \(\Phi\) is a diffeomorphism. Indeed in this case \(\phi\) is simply another set of coordinates for \(x\) and the observer (44) can be simply rewritten in the original \(x\) coordinates as

\[
\dot{x} = f(\hat{x}) + g(\hat{x})u + \left( \frac{\partial \Phi}{\partial x}(\hat{x}) \right)^{-1} K_n L_n(\ell)(y - h(\hat{x})). \tag{45}
\]

As a consequence there is no need to find the inverse mapping of the function \(\Phi\) but, (infinitely) more simply, only to invert the matrix \(\frac{\partial \Phi}{\partial x}(\hat{x})\). But for \(\Phi\) to be a diffeomorphism, we need \(n_0\) to be equal to \(n\), i.e. to have the (full order) observer to have the smallest possible dimension.

C. High gain observer in the multi-output case

As shown in [41], in the multi-input multi-output case (2), a typical expression for \(\Phi\) is

\[
\Phi(x) = \left( \Phi_1(x) \ldots \Phi_p(x) \right)^\top, \\
\Phi_i(x) = \left( h_i(x) \ L_i h_i(x) \ldots \ L_i^p h_i(x) \right)^\top, \tag{46}
\]

where \(h_i\) is the \(i\)-th component of \(h\) and \(p_i\) are integers called the observability indexes and \(\sum_{i=1}^p p_i \geq n\). The dynamics of system (2) expressed in these coordinates is

\[
\dot{\phi} = A\phi + \bar{b}(\phi) + D(\phi)u, \qquad y = C\phi \tag{47}
\]

where

\[
A = \text{blkdiag}(A_{p_1}, \ldots, A_{p_p}), \\
B = \text{blkcol}(B_{p_1}, \ldots, B_{p_p}), \\
C = \text{blkrow}(C_{p_1}, \ldots, C_{p_p}), \\
\bar{b}(\phi) = (b_1(\phi), \ldots, b_p(\phi))^\top, \\
D(\phi) = \text{blkcol}(D_{p_1}(\phi), \ldots, D_{p_p}(\phi)),
\]

where \(\bar{b}(\phi)\) and \(D(\phi)\) are locally Lipschitz functions. However, even when the system is observable uniformly in the input, the functions \(\bar{b}\) and \(D\) may not have the triangular structure we need for the design of a high-gain observer. Conditions under which we do get triangular dependence for \(\bar{b}(\phi)\) and \(D(\phi)\) have been studied for instance in [8] and [17]. Going on along this route and imposing \(\Phi\) to be a diffeomorphism (in order to write the observer in the original coordinates, as done in (45)), an alternative condition under which we do have an appropriate structure is given by the following (technical) assumption, for which we do not need to know the inverse of \(\Phi\).

Assumption 4 There exist

i) an open set \(O \subset \mathbb{R}^n\) containing the origin and a star-shaped set \(U\) with the origin as star-center,

ii) a \(C^1\) function \(\Phi : O \rightarrow \mathbb{R}^n\),

iii) sequences of matrices \(L_\ell \in \mathbb{R}^{n \times n}\), \(M_\ell \in \mathbb{R}^{n \times n}\) and \(N_\ell \in \mathbb{R}^{n \times p}\), a matrix \(C \in \mathbb{R}^{p \times n}\),

iv) matrix functions \(u \in U \mapsto K(u) \in \mathbb{R}^{n \times p}\) and \(u \in U \mapsto A(u) \in \mathbb{R}^{n \times n}\),

v) and, for any positive real number \(\bar{u}\), there exist a positive definite symmetric matrix \(P \in \mathbb{R}^{n \times n}\) and strictly positive real numbers \(\nu\) and \(d\), such that

\[
O1) \quad \text{the function } \Phi \text{ is a diffeomorphism on the set } O \text{ and } \Phi(0) = 0, \\
O2) \quad C(\Phi(x)) = h(x), \\
O3) \quad \text{the matrices } A(u), K(u), P, C \text{ satisfy, for any } u \in U(\bar{u}), \\
\quad P(A(u) - K(u)C) + (A(u) - K(u)C)^\top P \preceq -2\nu P, \\
\quad A(u) L_\ell = L_\ell M_\ell A(u), \quad N_\ell C L_\ell = C, \\
O4) \quad \text{the matrix } M_\ell \text{ is such that } M_\ell P^{-1} \text{ is symmetric and satisfies}
\]

\[
\lim_{\ell \to +\infty} \lambda_{\text{min}}(M_\ell P^{-1}) = +\infty, \tag{50}
\]

\[
O5) \quad \lambda_{\text{max}} \left( L_\ell M_\ell P^{-1} L_\ell^\top \right) \leq \lambda_{\text{min}}(M_\ell P^{-1})^d, \\
1 \leq \lambda_{\text{min}} \left( L_\ell M_\ell P^{-1} L_\ell^\top \right) \lambda_{\text{min}}(M_\ell P^{-1})^d.
\]

Moreover, for any compact set \(\mathcal{C} \subset O\)

there exists a sequence of positive real numbers \(c_\ell\) such that

\[
O6) \quad \lim_{\ell \to +\infty} c_\ell = 0, \tag{49}
\]

\[
O7) \quad \text{the function } B : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n} \text{ defined as}
\]

\[
B(\Phi(x), u) = L_f \Phi(x) + L_g \Phi(x)u - A(\Phi(x)), \tag{48}
\]

satisfies, for all \(x_a \in \mathcal{C}, x_b \in \mathcal{C} \subset U(\bar{u})\)

\[
\left| P_{\nu}^\frac{1}{\ell} M_{\ell}^{-1} L_{\ell}^{-1} \left[ B(\Phi(x_a), u) - B(\Phi(x_b), u) \right] \right| \leq c_\ell \left| P_{\nu}^\frac{1}{\ell} L_{\ell}^{-1} [\Phi(x_a) - \Phi(x_b)] \right|. \tag{49}
\]

Remark 3

- As shown in the next Lemma, the existence of a high-gain observer for the system (2) is guaranteed if Assumption 4 holds. In particular the properties O1, O2, O3, O6 and O7 guarantee the existence of a converging observer in the original coordinates whereas properties O4 and O5 assure its tunability property.

- We remark that these conditions can be checked without need of finding formally the inverse mapping \(\Phi^{-1}\). In particular, given a system and a candidate diffeomorphism \(\Phi\) (property O1), one can immediately check properties O2 (linear dependence of the diffeomorphism on the output) Then, if this properties holds, one can fix the degrees of freedom \(K(u), M_\ell, N_\ell, L_\ell, P\) which properly defines the high-gain observer as shown later in Lemma 2 (see (50)) and check also the Lipschitz condition (49) in 07. Finally,
property O3 guarantees the convergence of the observer (see proof of Lemma 2).

The conditions of Assumption 4 are satisfied in the single-output case considered in Section [VI.A] when $n_a = n$, by choosing $\Phi$ as in (41), and picking

$$L_\ell = \text{diag} \{ 1, \ell, \ldots, \ell^{n-1} \}, \quad M_\ell = \ell, \quad N_\ell = 1$$

$A(u) = A_n, \quad B(\Phi(x), u) = B_n b(\Phi(x)) + D_n(\Phi(x)) u,$

and $C = C_n$, where the triplet $A_n, B_n, C_n$ and the functions $b(\cdot), D(\cdot)$ are given in (43). In this case, we set $L_n(\ell) = L_n M_n N_\ell$ and $K(\ell) = K_n$ in the observer (44).

In this assumption A is allowed to be input-dependent to allow a broader class of nonlinear systems. For instance it can be verified that the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad y = -x_1 + x_2 + x_2^2,$$

can not be transformed in the form (42) but it satisfies Assumption 4.

In some cases, the nonlinear terms (48) can be disregarded in the high gain observer design (usually also called dirty derivative observer). This is possible for example when the notions of observability indexes and relative degree indexes coincide (see [32] among others). In this case, these nonlinear terms act through their bound and not their Lipschitzness. Unfortunately then a very specific structure is needed because otherwise the gain between these nonlinear terms and some estimation error is increasing with the observer gain. Here we intend to consider a broader class of systems and thus we do need to have these terms present in the observer.

**Lemma 2** Under Assumption 4, for any compact set $\mathcal{C}$ and $\tilde{\mathcal{C}}$ satisfying

$$\mathcal{C} \subset \tilde{\mathcal{C}} \subset \mathcal{O},$$

the family of systems

$$\dot{x}_\ell = f(\tilde{x}_\ell) + g(\tilde{x}_\ell) u + \left( \frac{\partial \Phi}{\partial x}(\tilde{x}_\ell) \right)^{-1} L_\ell M_\ell K(\ell) N_\ell [y - h(\tilde{x}_\ell)]$$

indexed by $\ell$ in $\mathbb{R}_{>0}$ satisfies points 2 to 6 of Assumption 1.

**Proof:** We let

$$\phi = \Phi(x), \quad \hat{\phi}_\ell = \Phi(\tilde{x}_\ell), \quad \hat{\phi}_\ell = \phi - \hat{\phi}_\ell.$$  (51)

With (48) and (50), systems (2) and (50) are transformed in

$$\dot{\phi} = A(u) \phi + B(\phi, u)$$

$$\dot{\hat{\phi}} = A(u) \hat{\phi} + B(\hat{\phi}, u) + L_\ell M_\ell K(\ell) N_\ell C(\phi - \hat{\phi})$$

With Assumption 4 and the notations (51), we define the Lyapunov Function

$$U_\ell(x, \tilde{x}) = \frac{1}{2} (\phi - \hat{\phi}_\ell)^T [L_\ell M_\ell P^{-1} L_\ell^T]^{-1} (\phi - \hat{\phi}_\ell).$$

As $\Phi$, it is defined on $\mathcal{O} \times \tilde{\mathcal{C}}$ and it takes values in $\mathbb{R}_{>0}$. Also, because the matrix $L_\ell M_\ell P^{-1} L_\ell^T$ is positive definite, we have

$$\forall (x, \tilde{x}) \in \mathcal{O} \times \tilde{\mathcal{C}}, \quad U_\ell(x, \tilde{x}) = 0 \iff x = \tilde{x}.$$  (52)

So point 2 of Assumption 1 holds. Also we get

$$\dot{U}_\ell(x, \tilde{x}) = (\phi - \hat{\phi}_\ell)^T [L_\ell M_\ell P^{-1} L_\ell^T] - \frac{1}{2} L_\ell M_\ell K(\ell) N_\ell C(\phi - \hat{\phi}_\ell).$$

which, with using O3 and (49), gives, for all $(x, \tilde{x}) \in \mathcal{C} \times \tilde{\mathcal{C}},$

$$\dot{U}_\ell \leq -\nu |P_\ell^\frac{1}{2} L_\ell^{-1} \phi_\ell|^2 + c_\ell |P_\ell^\frac{1}{2} L_\ell^{-1} \hat{\phi}_\ell|^2 (1 + |u|).$$

Then, with O6, for any $|u| \leq \bar{a}$, there exists a $\ell$ such that, for any $\ell \geq \underline{\ell}$,

$$\dot{U}_\ell(x, \tilde{x}) \leq -\nu \lambda_{\min} \lambda_{\min}(M_\ell P^{-1}) - P_\ell^\frac{1}{2} L_\ell^{-1} \hat{\phi}_\ell \leq \nu \lambda_{\min} \lambda_{\min}(M_\ell P^{-1}) - 2 U_\ell(x, \tilde{x}).$$

So, with O4, points 4 and 5 of Assumption 1 hold when we choose the integer $\kappa$ as the integer part of the ratio $\ell / \underline{\ell}$ and with

$$\sigma_\kappa = \frac{\nu \lambda_{\min} \lambda_{\min}(M_\ell P^{-1})}{2}.$$

Next, we have

$$U_\ell(x, \tilde{x}) \lambda_{\min}(L_\ell M_\ell P^{-1} L_\ell^T) = \frac{\hat{\phi}_\ell^T (L_\ell M_\ell P^{-1} L_\ell^T)-1 \hat{\phi}_\ell}{\lambda_{\max}(L_\ell M_\ell P^{-1} L_\ell^T)} \leq |\phi - \hat{\phi}_\ell|^2$$

$$|\phi - \hat{\phi}_\ell|^2 \leq \frac{\hat{\phi}_\ell^T (L_\ell M_\ell P^{-1} L_\ell^T)-1 \hat{\phi}_\ell}{\lambda_{\min}(L_\ell M_\ell P^{-1} L_\ell^T)} \leq U_\ell(x, \tilde{x}) \lambda_{\max}(L_\ell M_\ell P^{-1} L_\ell^T)$$

So, with O5, we get

$$U_\ell(x, \tilde{x}) \lambda_{\min}(M_\ell P^{-1}) - d \leq |\Phi(x) - \Phi(\tilde{x})|^2 \leq U_\ell(x, \tilde{x}) \lambda_{\min}(M_\ell P^{-1}) - d.$$  (53)

But, because $\Phi$ is a diffeomorphism defined on $\mathcal{O}$, for any compact subsets $\mathcal{C}$ and $\tilde{\mathcal{C}}$ of $\mathcal{O}$, there exist real numbers $\Phi$ and $L_{\phi^{-1}}$, independent of $\ell$, such that, for all $x \in \mathcal{C}$ and $\tilde{x} \in \tilde{\mathcal{C}},$ we have

$$|x - \tilde{x}| = |\Phi^{-1}(\Phi(x)) - \Phi^{-1}(\Phi(\tilde{x}))| \leq L_{\phi^{-1}} |\Phi(x) - \Phi(\tilde{x})| \leq \Phi.$$  (54)

This gives

$$|x - \tilde{x}|^2 \leq \frac{\lambda_{\min}(M_\ell P^{-1}) - d}{L_{\phi^{-1}}} \leq U_\ell(x, \tilde{x}) \leq \Phi^2 \lambda_{\min}(M_\ell P^{-1}) - d.$$  (55)

So, with O4, point 3 of Assumption 1 holds.

Finally, point 6 of Assumption 1 holds too. Indeed, by definition of the set $U(\tilde{u})$, the matrices $K(u), M_\ell, N_\ell, L_\ell$ and the diffeomorphism $\Phi$, there exists a positive definite function $L_{\phi^{-1}}(\tilde{x})$ such that

$$\left( \frac{\partial \Phi}{\partial x}(\tilde{x}) \right)^{-1} L_\ell M_\ell K(\ell) N_\ell \leq L_{\phi^{-1}}(\tilde{x})$$

for any $\ell > 0$, $u \in U(\tilde{u})$ and $\tilde{x} \in \tilde{\mathcal{C}}$.  ■
D. Taking care of observability restricted to \( \mathcal{O} \) by an observer modification

In the above [3], we are missing point 1 of Assumption [1] namely \( \hat{\mathcal{C}} \) may not be forward invariant. The problem is that the observer (57) does not guarantee that \( \hat{x}_\ell \) remains in \( \mathcal{O} \) and therefore that \( \frac{\partial h_2}{\partial x}(\hat{x}_\ell) \) is invertible. To round this problem, as in [29], we modify this observer, here not by projection, but by considering a dummy measured output (extending the results in [5]). To make our point clear, we introduce the following assumption.

**Assumption 5** Given the set \( \mathcal{O} \) and the diffeomorphism \( \Phi \) of Assumption [4] for any compact subset \( \mathcal{C} \) of \( \mathcal{O} \), we know of a \( C^1 \) function \( h_2 : \mathcal{O} \to \mathbb{R}_{\geq 0} \) such that:

1. the set \( \{ x \in \mathbb{R}^n : h_2(x) < 1 \} \) is a subset of \( \mathcal{O} \);
2. the function \( x \mapsto \frac{h_2(x)}{\Phi(x)} \) is continuous on \( \mathcal{O} \);
3. for any real number \( s \) in \([0,1]\), and any \( x_1 \) and \( x_2 \) in \( \mathcal{O} \) satisfying
   \[ h_2(x_1) \leq s , \quad h_2(x_2) \leq s , \]
   we have \( h_2(x) \leq s \) for all \( x \) which satisfies for some \( \lambda \) in \([0,1]\)
   \[ \Phi(x) = \lambda \Phi(x_1) + (1 - \lambda)\Phi(x_2) . \]
   This means nothing but the fact that, for any \( s \) in \([0,1]\), the image by \( \Phi \) of the set \( \{ x \in \mathbb{R}^n : h_2(x) \leq s \} \) is convex;
4. the set \( \mathcal{O}_{mod} \) defined as
   \[ \mathcal{O}_{mod} = \{ x \in \mathbb{R}^n : h_2(x) \leq 0 \} \quad (54) \]
   contains \( \mathcal{C} \) and has a non empty interior which contains the origin;
5. the set
   \[ \hat{\mathcal{C}} = \{ x \in \mathbb{R}^n : |h_2(x)| \leq \frac{1}{2} \} \]
   is compact.

**Remark 4**

- There is a systematic way to define this function \( h_2 \) when, given the compact set \( \mathcal{C} \), we know a positive definite symmetric matrix \( Q \) and a real number \( R \) satisfying
   \[ \Phi(\mathcal{C}) \subset \{ \phi \in \mathbb{R}^n : \phi^\top Q \phi \leq R \} \subset \Phi(\mathcal{O}) . \]
   Indeed, in this case we let \( \varrho \) be the number defined as
   \[ \varrho = \sup_{R : \{ \phi : \varrho^\top Q \phi \leq R \} \subset \Phi(\mathcal{O})} R . \]
   Since \( \mathcal{O} \) is a neighborhood of the origin, \( \varrho \) is strictly positive. Then we select a real number \( \epsilon \) in \((0,1)\) and let
   \[ h_2(x) = \max \left\{ \frac{\Phi(x)}{\varrho} - \epsilon , 0 \right\}^2 . \quad (55) \]
   With this choice and since \( \Phi \) is a diffeomorphism, we can check that Properties H1 to H5 are satisfied.

- If we dislike the convexity property mentioned in H3 of Assumption [3], unfortunately it is in some sense necessary. Indeed our objective with the modification \( E \) is to preserve the high-gain paradigm. This means in particular that we choose to keep an Euclidean distance in the image by \( \Phi \) as a Lyapunov function for studying the error dynamics. Also we need an infinite gain margin, as defined in Definition 2.8 in [33], since the correction term must dominate all the other ones in the expression of \( \hat{x} \) when \( h_2 \) becomes too large. Then as proved in Lemma 2.7 [33], with such constraints, the convexity assumption is necessary. This implies that, if we want to remove the convexity assumption, we have to find another class of observers.

We are interested in the function \( h_2 \) because it satisfies the property
\[ h_2(x) = 0 \quad \forall x \in \mathcal{O}_{mod} . \]
This leads us to introduce a dummy measured output
\[ y_2 = h_2(x) . \]
Indeed \( y_2 \) is zero when \( x \) is in \( \mathcal{O}_{mod} \). But \( \mathcal{O}_{mod} \) being a strict subset of \( \mathcal{O} \), we have here a stronger constraint. To deal with this restriction, we need to “reduce” the set \( \mathcal{SO} \) by modifying its definition given in (33) into
\[ v_\infty = \inf_{(x,z) \in (S \times \mathbb{R})/(\mathcal{O}_{mod} \times \mathbb{R})} V(x,z) \quad (56) \]
\[ \mathcal{SO} = \left\{ (x,z) \in (S \times \mathbb{R}) : V(x,z) < v_\infty \right\} . \]
With Assumption [5] point 1 of Assumption [1] can be established via a modification of the observer.

**Lemma 3** Assume Assumptions [5] holds. Let \( \Phi : \mathcal{O} \to \mathbb{R}^n \) be a diffeomorphism, \( \bar{u} \) be a positive real number and \( t \to u(t) \) be a continuous function with values in \( U(\bar{u}) \) and \( t \to y(t) \) be a continuous bounded function. The set \( \mathcal{C} \) given in H5 is forward invariant for any system in the family, indexed by \( \ell \) in \( \mathbb{R}_{>0} \),
\[ \dot{x}_\ell = f(\hat{x}_\ell) + g(\hat{x}_\ell)u + \left( \frac{\partial \Phi}{\partial x}(\hat{x}_\ell) \right)^{-1} L_\ell M_\ell K(u) N_\ell [y - h(\hat{x}_\ell)] + E(\hat{x}_\ell, u, y) \]
where the term \( E \) is defined as
\[ E(\hat{x}_\ell, u, y) = -\tau_\ell (\hat{x}_\ell, u, y) \times \]
\[ \left( \frac{\partial \Phi}{\partial x}(\hat{x}_\ell) \right)^{-1} L_\ell M_\ell P^{-1} L_\ell^\top \left( \frac{\partial \Phi}{\partial x}(\hat{x}_\ell) \right)^{-1} \frac{\partial h_2}{\partial x}(\hat{x}_\ell)^\top h_2(\hat{x}_\ell) \]
where \( \tau_\ell \) is a \( C^1 \) function to be chosen large (see [53]).
If all conditions of Assumption [4] hold and the model state \( x \) remains in \( \mathcal{O}_{mod} \), then all the points of Assumption [7] are satisfied.

**Proof:** First we observe that
\[ \frac{\partial h_2}{\partial x}(\hat{x}_\ell) \dot{x}_\ell = R(\hat{x}_\ell, u, y) + \tau_\ell(\hat{x}_\ell) \left( M_\ell P^{-1} \right)^{-1} L_\ell^\top \left( \frac{\partial \Phi}{\partial x}(\hat{x}_\ell) \right)^{-1} \frac{\partial h_2}{\partial x}(\hat{x}_\ell)^\top h_2(\hat{x}_\ell) \]
where we have let
\[ R(\hat{x}_\ell, u, y) = \frac{\partial h_2}{\partial x}(\hat{x}_\ell) \times \]
\[ \tau_\ell(\hat{x}_\ell, u, y) \geq \frac{8h_2(\hat{x}_\ell)^2 R(\hat{x}_\ell, u, y)}{(M_t P^{-1})^{-\frac{1}{2}} L_\ell^{-1}} \left( \frac{\partial h_2}{\partial x}(\hat{x}_\ell)^T \right)^2 \]

which can be computed on-line.

Thanks to H2, the function \( x \mapsto \tau_\ell(x) \) defined this way is continuous on \( \mathcal{O} \). So we can use \( \tau_\ell \) as long as \( \hat{x}_\ell \) is in \( \mathcal{O} \).

It implies that \( h_2(\hat{x}_\ell) \) is non-positive when \( h_2(\tilde{x}_\ell) \) is strictly larger than \( \frac{1}{2} \). With uniqueness of solutions, this implies that, for each \( s \) in \( [\frac{1}{2}, 1] \) the set \( \{ \hat{x}_\ell : h_2(\hat{x}_\ell) \leq s \} \) is forward invariant and so is the compact set \( \mathcal{C} \) in particular. This says that point 1 of Assumption \( \mathbb{H} \) hold.

On the other hand, the modification \( E \) augments \( U_\ell \) in \( (52) \) with

\[ -\tau_\ell(\hat{x}_\ell) \left[ \Phi(\hat{x}_\ell) - \Phi(x) \right]^T \left( \frac{\partial \Phi}{\partial x}(\hat{x}_\ell) \right)^{-1} \frac{\partial h_2}{\partial x}(\hat{x}_\ell) h_2(\hat{x}_\ell) . \]

But, when \( h_2(x) \) is zero which is the case when the model state \( x \) remains in \( \mathcal{O}_{\text{nod}} \) and when \( h_2(\hat{x}_\ell) \) is in \( [0, 1] \), the convexity property of \( h_2 \) in H3 gives

\[ 0 \leq \left[ \Phi(\hat{x}_\ell) - \Phi(x) \right]^T \left( \frac{\partial \Phi}{\partial x}(\hat{x}_\ell) \right)^{-1} \frac{\partial h_2}{\partial x}(\hat{x}_\ell) h_2(\hat{x}_\ell) . \]

We conclude that, when all conditions of Assumption \( \mathbb{H} \) hold, \( (53) \) holds even with the modification \( E \). Hence, from the proof of Lemma \( \mathbb{L} \) points 2 to 5 of Assumption \( \mathbb{H} \) hold. Finally, with \( (58) \) and \( (59) \), the function defined by the right hand side of \( (57) \) satisfies also the point 6 of the Assumption \( \mathbb{H} \).

**Remark 5** An important feature is that, thanks to the additional term \( E \), no other modification (as saturation) is needed. This modification, in fact, guarantees that the state estimate \( \hat{x}_\ell \) remains in a compact subset of \( \mathcal{O} \) which depends on the choice of the parameters.

### V. PROOFS OF PROPOSITIONS

#### A. Proof of Proposition \( \mathbb{I} \)

This proof follows the same lines as in \( [5] \), inspired by \( [19] \) Chapter 12.3] with no meaningful or meaningful or originality. We give it only for the sake of completeness.

We introduce the notations

\[ x_e = \begin{bmatrix} x \\ z \end{bmatrix}, \quad u_e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad f_e(x_e, u_e) = \begin{bmatrix} f(x) + g(x)u_1 \\ k(u_2, h(x)) \end{bmatrix}, \]

\[ \psi_e(x, z) = \begin{bmatrix} \psi(x, z) \\ \psi_{sat}(\hat{x}_\ell, z) \end{bmatrix}, \quad \psi_{sat}(x, z) = \begin{bmatrix} \psi_{sat}(x, z) \\ \hat{z} \end{bmatrix}, \]

where the functions \( \psi, \psi_{sat} \) and \( \text{sat}_{\hat{x}_\ell} \) are defined in Sections \( \mathbb{M} \) and \( \mathbb{C} \). The closed-loop system \( (2) \), \( (11) \) can be compactly written as

\[ \dot{x}_e = f_e(x_e, u_e), \quad u_e = \psi_e(x, z). \]

We have a function \( V_e \), positive definite and proper on \( \mathcal{S} \times \mathbb{R}^r \) which allows us to define the compact sets \( \Omega_{\epsilon_1} \) and \( \Omega_{\epsilon_2} \), as in \( (34) \) and satisfying \( (35) \). Also, with \( (31) \) and \( (37) \), the function

\[ -W_e(x, z) = \hat{V}_e(x, z) \]

\[ = \frac{\partial V_e}{\partial x}(x, z)f_e(x, u_e(\hat{x}_\ell, z)) + \frac{\partial V_e}{\partial z}(x, z)k(x, h(x)) \]

is continuous and negative definite on \( \Omega_{\epsilon_2} \).

These properties imply the following:

1. there exists a positive real number \( W \) and a continuous function \( \alpha : \mathbb{R}_{\geq 0} \rightarrow [0, \infty) \) such that

\[ \frac{\partial V_e}{\partial x}(x, z)f_e(x, u_e(\hat{x}_\ell, z)) - f_e(x, \psi_e(x, z)) \]

\[ \leq \alpha(|\hat{x} - x|) \quad \forall (x, z) \in \Omega_{\epsilon_2}, \quad \forall \hat{x} \in \mathbb{R}^n ; \]

2. there exists a strictly positive real number \( W_0 \) such that

\[ \frac{\partial V_e}{\partial x}(x, z)f_e(x, u_e(\hat{x}_\ell, z)) \leq -W \]

\[ \forall (x, z) : (x, z) \in \Omega_{\epsilon_2} \setminus \Omega_{\epsilon_1}, \quad |\hat{x} - x| \leq \delta_{xw} , \]

where \( \delta_{xw} \) is the strictly positive real number defined as

\[ \delta_{xw} = \alpha^{-1} \left( \frac{1}{2} \min_{(x, z) \in \Omega_{\epsilon_2} \setminus \Omega_{\epsilon_1}} W_e(x, z) \right) . \]

By collecting this, we obtain

\[ \frac{\partial V_e}{\partial x}(x, z)f_e(x, u_e(\hat{x}_\ell, z)) \]

\[ \leq -W_e(x, z) + \alpha(|\hat{x} - x|) \leq W \]

\[ \forall (x, z) \in \Omega_{\epsilon_2}, \quad \forall \hat{x} \in \mathbb{R}^n , \]

\[ \leq -W \]

\[ \forall (x, z) \in \Omega_{\epsilon_2} \setminus \Omega_{\epsilon_1}, \quad \forall \hat{x} : |\hat{x} - x| \leq \delta_{xw} . \]

On another hand, let \( \mathcal{C} = (\Omega_{\epsilon_2})^{\mathbb{R}} \). From \( (59) \), it is a compact subset of \( \mathcal{O} \). With this set \( \mathcal{P} \) and \( \mu \) defined in \( (36) \) we can invoke Assumption \( \mathbb{I} \). It gives us in particular the sequence \( \sigma_\kappa \), the integer \( d \), the real number \( \mathcal{U} \) and the function \( \kappa \). This allows us to define the integer \( \kappa \) as the smallest one satisfying

\[ \sigma_\kappa^{d} \exp \left( -\frac{\sigma_\kappa \mathcal{V}_2 - \mathcal{V}_1}{2W} \right) \mathcal{U} \leq \Omega(\delta_{xw}) \quad \forall \kappa \geq \kappa \ . \]

From now on, we fix \( \kappa \) arbitrarily but larger than \( \kappa \).

Assumption \( \mathbb{I} \) gives us also the functions \( \vartheta_\kappa \) and \( \mathcal{U}_\kappa \) and the forward invariant compact subset \( \mathcal{C} \) of \( \mathcal{O} \). Then, because \( \mathcal{U}_\kappa \) is continuous and satisfies the point 2 of Assumption \( \mathbb{I} \) there exists a class-\( \mathcal{K} \) function \( \mathbb{K} \) satisfying

\[ \mathcal{U}_\kappa(x, \hat{x}) \leq \mathbb{K} \left( |x - \hat{x}| \right) \quad \forall (x, \hat{x}) \in \mathcal{C} \times \hat{\mathcal{C}} . \]

From \( \mathcal{C} \) we define \( \mathbb{C}_\kappa \) and \( \Gamma \) as the sets

\[ \mathbb{C}_\kappa = \mathcal{C} , \quad \Gamma = \mathcal{C} \times \Omega_{\epsilon_2} . \]

\( \mathbb{C}_\kappa \) is a compact subset of \( \mathcal{O} \) and \( \Gamma \) is a compact subset of \( \mathcal{O} \times \Omega_{\epsilon_2} \). Since all the functions are Lipschitz on \( \Gamma \), the solutions of the closed-loop system \( (2) \), \( (11) \) are well defined and unique as long as they are in the interior set \( \Gamma \). Moreover their values satisfy inequalities \( (60) \) and \( (66) \). Also since \( \mathbb{C}_\kappa = \mathcal{C} \) is forward invariant, the \( x \)-component of this solution cannot reach the boundary of this set in finite time.

**Stability**

\[ 6 \quad \mathbb{K} \] can be constructed from \( s \mapsto \max_{(x, \hat{x}) \in \mathcal{C} \times \hat{\mathcal{C}} : |x - \hat{x}| \leq \delta_s} U(x, \hat{x}) .\]
Let \( N_\alpha \), contained in \( \hat{\Gamma} \), be an open neighborhood of the origin whose points \((\hat{x}, x, z)\) satisfy
\[
V_e(x_e) + \alpha \left( \alpha^{-1}(\sigma^d_\alpha \tau_n(|x - \hat{x}|)) \right) < \frac{v_1}{2}.
\]
Consider a solution of the closed-loop system, starting from an arbitrary point \((\hat{x}, x, z)\) in \( N_\alpha \). Let \([0, T]\) be its right maximal interval of definition when it takes its value in the open set \( \hat{\Gamma} \). To simplify the notation we add \((t)\) to denote those variables which are evaluated along this solution.

With (6) and (63) we have
\[
U(x(t), \hat{x}(t)) \leq \tau_n(|x(0) - \hat{x}(0)|) \quad \forall t \in [0, T],
\]
\[
\alpha(|x(t) - \hat{x}(t)|) \leq \sigma^d_{\alpha} U(x(t), \hat{x}(t)) \quad \forall t \in [0, T].
\]
This implies
\[
|x(t) - \hat{x}(t)| \leq \alpha^{-1}(\sigma^d_\alpha \tau_n(|x(0) - \hat{x}(0)|)) \quad \forall t \in [0, T].
\]
This inequality and (60), where \( W_e \) is non-negative, give
\[
V_e(x(t), z(t)) \leq V_e(x(0), z(0)) + \alpha(\alpha^{-1}(\sigma^d_\alpha \tau_n(|x(0) - \hat{x}(0)|))) < \frac{v_1}{2},
\]
\[
\forall t \in [0, T].
\]
Thus, if the initial condition \((\hat{x}(0), x(0), z(0))\) is in \( N_\alpha \), the solution remains inside a strict subset of \( \hat{\Gamma} \). Hence \( T \) is infinite and from (65) and (66) we can conclude that the origin is stable.

**Attractiveness**

Consider now a solution of the closed-loop system with initial condition \((\hat{x}, x, z)\) in \( \tilde{\Gamma} \times \Omega_v \), which, according to (65), contains \( \tilde{\Gamma} \times C_{\alpha} \). Let \([0, T]\) be the right maximal interval of definition of this solution when it takes its values in \( \hat{\Gamma} \). With (8), (3) and (60), we have, for all \( t \) in \([0, T]\),
\[
U(x(t), \hat{x}(t)) \leq \exp(-\sigma_\alpha^d t) U(x(0), \hat{x}(0)) \leq \exp(-\sigma_\alpha^d t) \sigma^d_{\alpha} \tau_n,
\]
\[
V_e(x(t), z(t)) \leq V_e(x(0), z(0)) + \frac{W}{t} \leq \frac{v_1}{2} + \frac{W}{t}.
\]
Since the \( \hat{x} \)-component of the solution cannot reach the boundary of \( \hat{\Gamma} \) in finite time and since \( V_e(x(t), z(t)) \) is smaller than \( v_2 \), we must have
\[
T \geq \frac{v_2 - v_1}{W},
\]
and
\[
V_e(x \left( \frac{v_2 - v_1}{2W} \right), z \left( \frac{v_2 - v_1}{2W} \right)) \leq \frac{v_2 + v_1}{2} < v_2.
\]
Then, because \( \kappa \) satisfies (62), (67) and (6) give, for all \( t \) in \([\frac{v_2 - v_1}{2W}, T]\),
\[
\alpha(|x(t) - \hat{x}(t)|) \leq \sigma^d_{\alpha} U(x(t), \hat{x}(t)) \leq \alpha(\delta_{xw})
\]
and therefore
\[
|x(t) - \hat{x}(t)| \leq \delta_{xw} \quad \forall t \in [\frac{v_2 - v_1}{2W}, T].
\]

Then, with (61) and (68), we obtain
\[
\max \left\{ V_e(x(t), z(t)) \right\}, v_1 \leq v_2.
\]
for all \( t \) in \([\frac{v_2 - v_1}{2W}, T]\). Since \( \tilde{\Gamma} \) is forward invariant, this establishes that the solution cannot reach the boundary of \( \hat{\Gamma} \) on \([0, T]\). This implies that \( T \) is infinite and that the solution remains in \( \hat{\Gamma} \) for all \( t \) in \( \mathbb{R}_{\geq 0} \). So inequalities (69) and (70) and therefore inequalities (62), (6) and (8) hold for all \( t \) larger than \( \frac{v_2 - v_1}{2W} \). With LaSalle invariance principle, we conclude
\[
\lim_{t \to +\infty} V_e(x(t), z(t)) = 0
\]
and thus that the solution of the closed-loop system converges to the origin provided its initial condition \((\hat{x}(0), x(0), z(0))\) is in \( \tilde{\Gamma} \times \Omega_\alpha \supset \tilde{\Gamma} \times C_{\alpha} \).

**B. Proof of Proposition 2**

We denote
\[
x = (x, z, \hat{x}),
\]
\[
\varphi_m(x) = \left( f(x) + g(x)v_\text{sat}(\hat{x}, z) \right),
\]
\[
\varphi_p(x) = \left( k(\hat{z}, h_\text{sat}(\hat{x}, z)) \psi_\text{sat}(\hat{x}, z), \hat{z}, \psi_\text{sat}(\hat{x}, z)) \right).
\]

A first elementary remark is that, if \( x_e = (x_e, z_e, \hat{x}_e) \) is an equilibrium point of \( \varphi_p \), then we have in particular
\[
0 = \dot{z}_{x = x_e} = k(\hat{x}_e, h_\text{sat}(x_e)).
\]
With (16) this implies \( h_\text{sat}(x_e) \) is zero.

To prove the existence of \( x_e \), we use [16] Theorem 8.2 which says that a forward invariant set which is homeomorphic to the closed unit ball of \( \mathbb{R}^n \) contains an equilibrium. The Proposition[11] gives us a forward invariant set, which may not be homeomorphic to the closed unit ball. So our next task is to build another set satisfying the required properties.

The equilibrium of
\[
\dot{x} = \varphi_m(x)
\]
being asymptotically attractive and interior to \( \overline{\mathbb{C}} \) which is forward invariant, \( \overline{\mathbb{C}} \) is attractive. It is also stable due to the continuity of solutions with respect to initial conditions uniformly on compact time subsets of the domain of definition. So it is asymptotically stable with the same domain of attraction \( \mathcal{A} \) as the equilibrium. It follows from [43] Theorem 3.2 that there exist \( C^\infty \) functions \( V : \mathcal{A} \to \mathbb{R}_{\geq 0} \) and \( V_\overline{\mathbb{C}} : \mathcal{A} \to \mathbb{R}_{\geq 0} \) which are proper on \( \mathcal{A} \) and a class \( K_\infty \) function \( \alpha \) satisfying
\[
\alpha(|x|) \leq V(x), \quad V(0) = 0,
\]
\[
\alpha(d(x, \overline{\mathbb{C}})) \leq V_\overline{\mathbb{C}}(x), \quad V_\overline{\mathbb{C}}(x) = 0 \quad \forall x \in \overline{\mathbb{C}},
\]
\[
\frac{\partial V}{\partial x} (x) \varphi_m(x) \leq -V(x) \quad \forall x \in \mathcal{A},
\]
\[
\frac{\partial V_\overline{\mathbb{C}}}{\partial x} (x) \varphi_m(x) \leq -V_\overline{\mathbb{C}}(x) \quad \forall x \in \mathcal{A}.
\]
Since \( \overline{C} \) is compact and \( \mathcal{N}_{\partial \overline{C}} \) is a neighborhood of its boundary, there exists a strictly positive real number \( \overline{d} \) such that the set
\[
\{ x \in A : d(x, \overline{c}) \in (0, \overline{d}] \}
\]is a subset of \( \mathcal{N}_{\partial \overline{C}} \). Then, with the notations
\[
v_{\overline{C}} = \sup_{x \in A: d(x, \overline{c}) \leq \overline{d}} V(x), \quad \varpi = \frac{\alpha(\overline{d})}{2v_{\overline{C}}},
\]and since \( \alpha \) is of class \( \mathcal{C}_\infty \), we obtain the implications
\[
V_{\overline{C}}(x) + \varpi V(x) = \alpha(\overline{d}) \Rightarrow \alpha(d(x, \overline{c})) \leq V_{\overline{C}}(x) \leq \alpha(\overline{d}),
\]
\[
d(x, \overline{c}) \leq \overline{d},
\]
\[
V(x) \leq v_{\overline{C}}.
\]
With our definition of \( \varpi \), this yields also
\[
\alpha(\overline{d}) - \varpi V(x) = V_{\overline{C}}(x) \Rightarrow 0 < \frac{\alpha(\overline{d})}{\varpi} \leq V_{\overline{C}}(x),
\]
\[
0 < d(x, \overline{c}) \leq \overline{d},
\]
\[
x \in \mathcal{N}_{\partial \overline{C}} \setminus \overline{C}. \quad (74)
\]
On the other hand, with the compact notation
\[
V(x) = V_{\overline{C}}(x) + \varpi V(x),
\]
we have
\[
\frac{\partial V}{\partial x}(x) \varphi_m(x) < -V(x) \quad \forall x \in A. \quad (75)
\]
All this implies that \( V \) is a Lyapunov Function for (72) on \( A \) in the sense of [42 Page 324] and that the sublevel set \( \{ x \in A : V(x) \leq \alpha(\overline{d}) \} \) is contained in \( \mathcal{N}_{\partial \overline{C}} \cup \overline{C} \). It follows from [42 Corollary 2.3] that the level set \( \{ x \in A : V(x) = \alpha(\overline{d}) \} \) is homeomorphic to the unit sphere. But, with the fact that the origin is asymptotically stable and the arguments used in the proof of [42 Theorem 1.2], this implies that the sublevel set \( \{ x \in A : V(x) \leq \alpha(\overline{d}) \} \) is homeomorphic to the closed unit ball.

Then, since the set
\[
C = \{ x \in \mathcal{N}_{\partial \overline{C}} : d(x, \overline{c}) \in [0, \overline{d}] \}
\]is a compact subset of \( \mathcal{N}_{\partial \overline{C}} \subset A \), the real number
\[
G = \sup_{x \in C} \left| \frac{\partial V}{\partial x}(x) \right| \quad (76)
\]is well defined and strictly positive. We get, for all \( x \in C \),
\[
\frac{\partial V}{\partial x}(x) \varphi_p(x) = \frac{\partial V}{\partial x}(x) \varphi_m(x) + \frac{\partial V}{\partial x}(x) [\varphi_p(x) - \varphi_m(x)] ,
\]
\[
\leq -V(x) + G \sup_{x \in C} |\varphi_p(x) - \varphi_m(x)| .
\]
So, if \( \varphi_p \) satisfies
\[
|\varphi_p(x) - \varphi_m(x)| \leq \frac{\inf_{x \in C} V(x)}{2G} \quad \forall x \in \mathcal{N}_{\partial \overline{C}}, \quad (77)
\]we have, for all \( x \) in \( \{ x \in A : V(x) = \alpha(\overline{d}) \} \)
\[
\frac{\partial V}{\partial x}(x) \varphi_p(x) \leq -\frac{1}{2} V(x).
\]
This implies the compact sublevel set \( \{ x : V(x) \leq \alpha(\overline{d}) \} \) is homeomorphic to the closed unit ball and forward invariant for the system [31]. With [16 Theorem 8.2], we conclude that this sublevel set contains an equilibrium of this system.

Finally, from points 1 and 6 of Assumption [1] we know that, even when the observer in (11) is fed with \( y = \zeta(x, u) \) and not with \( h(x) \), it admits a forward invariant compact subset \( \overline{C} \) of \( \mathcal{O} \). So with
\[
L = \sup_{x \in \overline{C}} \{ L_{\varphi_m}(\hat{x}), L_k(\hat{x}) \}
\]with \( L_{\varphi_m} \) given by (9) and \( L_k(\hat{x}) \) given by (17), we have, for all \( (x, z, \hat{x}, u) \) in \( \mathbb{R}^n \times \mathbb{R}^r \times \mathcal{C} \times \mathcal{U} \),
\[
|\varphi_p(x) - \varphi_m(x)| \leq |\xi(x, u) - [f(x) + g(x)u] + 2L \zeta(x, u) - h(x)| .
\]
Hence (77) holds when (12) is satisfied with
\[
\delta = \frac{1}{1 + 2L} \inf_{x \in \mathcal{C}} \left| \frac{\partial V}{\partial x}(x) \right| .
\]

C. Proof of Proposition [3]

We start with the following Lemma which combines total stability and hyperbolicity and is a variation of [33 Theorem 6].

Lemma 4 Let a \( C^1 \) function \( \varphi_m : \mathbb{R}^n \to \mathbb{R}^n \) be given such that the origin is an exponentially stable equilibrium point of:
\[
x = \varphi_m(x)
\]with \( A \) as domain of attraction. For any compact sets \( \mathcal{C} \) and \( \overline{C} \), the latter being forward invariant for the above system, which satisfy
\[
\{0\} \subsetneq \mathcal{C} \subsetneq \overline{C} \subsetneq A ,
\]there exists a strictly positive real number \( \delta \) such that, for any \( C^1 \) function \( \varphi_p : \mathbb{R}^n \to \mathbb{R}^n \) which satisfies:
\[
|\varphi_p(x) - \varphi_m(x)| \leq \delta, \quad \forall x \in \overline{C},
\]
\[
\left| \frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x) \right| \leq \delta, \quad \forall x \in \mathcal{C},
\]
there exists an exponentially stable equilibrium point \( x_e \) of:
\[
x = \varphi_p(x) ,
\]the basin of attraction of which contains the compact set \( \overline{C} \).

Proof: Let \( \Pi \) be a positive definite symmetric matrix and \( a \) a strictly positive real number satisfying
\[
\Pi \frac{\partial \varphi_m}{\partial x}(0) + \frac{\partial \varphi_m}{\partial x}(0) \mathcal{P} \leq -a \Pi , \quad \lambda_{\text{min}}(\Pi) = 1 ,
\]where \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) respectively stand for max and min eigenvalues. By continuity there exists a strictly positive real number \( p_0 \) such that we have, for all \( x \) satisfying \( x^\top \Pi x \leq p_0 \),
\[
\Pi \frac{\partial \varphi_m}{\partial x}(x) + \frac{\partial \varphi_m}{\partial x}(x) \mathcal{P} \leq -\frac{a}{2} \Pi
\]and
\[
x^\top \Pi \varphi_m(x) \leq -\frac{a}{4} x^\top \Pi x.
\]
Let \( \varphi_p : \mathbb{R}^n \to \mathbb{R}^n \) be any \( C^1 \) function satisfying

\[
|\varphi_p(x) - \varphi_m(x)| \leq \frac{a}{4} \sqrt{\frac{p_0}{12\lambda_{\max}(\Pi)}} \quad \forall x : x^T \Pi x = \frac{p_0}{6}.
\]

We obtain

\[
x^T \Pi \varphi_p(x) = x^T \Pi \varphi_m(x) + x^T \Pi [\varphi_p(x) - \varphi_m(x)],
\]

and therefore

\[
x^T \Pi \varphi_p(x) \leq -\frac{a}{16} x^T \Pi x, \quad \forall x : x^T \Pi x = \frac{p_0}{6}.
\]

In this condition, it follows from [16, Theorem 8.2] that, for each function \( \varphi_p \) satisfying (82), there exits a point \( x_e \) satisfying

\[
\varphi_p(x_e) = 0, \quad (x_e)^T \Pi x_e \leq \frac{p_0}{6}.
\]

Assume further that \( \varphi_p \) satisfies

\[
\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x) \leq \frac{a}{8\lambda_{\max}(\Pi)} \quad \forall x : x^T \Pi x \leq p_0.
\]

In this case, we have, for all \( x \) satisfying \( x^T \Pi x \leq p_0 \),

\[
\Pi \frac{\partial \varphi_p}{\partial x}(x) + \frac{\partial \varphi_p}{\partial x}(x)^T \Pi = \left[ \frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x) \right]^T \Pi
\]

\[
+ \Pi \frac{\partial \varphi_m}{\partial x}(x) + \frac{\partial \varphi_m}{\partial x}(x)^T \Pi = \Pi \left[ \frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x) \right]
\]

\[
\leq -\frac{a}{4} \Pi.
\]

Note also that we have

\[
[x_e + s(x - x_e)]^T \Pi [x_e + s(x - x_e)] \leq p_0\]

\[
\forall (x, x_e, s) : s \in [0, 1], (x_e)^T \Pi x_e \leq \frac{p_0}{6}, x^T \Pi x \leq \frac{p_0}{3}.
\]

Then, with

\[
\varphi_p(x) = \varphi_p(x) - \varphi_p(x_e) = \int_0^1 \frac{\partial \varphi_p}{\partial x}(x_e + s(x - x_e)) ds [x - x_e],
\]

and (84), we get, for all \( x \) satisfying \( x^T \Pi x \leq \frac{p_0}{3} \),

\[
[x - x_e]^T \Pi \varphi_p(x) = \int_0^1 \left( [x - x_e]^T \Pi \frac{\partial \varphi_p}{\partial x}(x_e + s(x - x_e))[x - x_e] \right) ds,
\]

\[
\leq -\frac{a}{4} [x - x_e]^T \Pi [x - x_e].
\]

Let

\[
\delta_1 = \min \left\{ \frac{a}{4} \sqrt{\frac{p_0}{12\lambda_{\max}(\Pi)}}, \frac{a}{8\lambda_{\max}(\Pi)} \right\},
\]

and reduce \( p_0 \) if necessary to have that \( x \) satisfying \((x_e)^T \Pi x_e \leq p_0 \) is in \( \mathbb{C} \). Then (79) and (80) with \( \delta = \delta_1 \) implies (82) and therefore (84). We have established that the system (81) has an exponentially stable equilibrium with basin of attraction containing the compact set \( \{ x \in \mathbb{R}^n : x^T \Pi x \leq \frac{p_0}{3} \} \).

Now, with \( \mathbb{C} \) and \( \mathbb{V} = V_{\mathbb{C}} + \pi V \) as defined in the proof of Proposition 2, we let \( m \) be a strictly positive real number such that we have

\[
x^T \Pi x \leq \frac{p_0}{3} \quad \forall x \in \mathbb{A} : \mathbb{V}(x) \leq \mathbb{V}
\]

Let also

\[
C = \{ x \in \mathbb{A} : \mathbb{V}(x) \leq \mathbb{V} \}, \quad d(x, \mathbb{C}) \in [0, \mathbb{C}]
\]

It is a compact subset of \( \mathbb{N}_{\mathbb{C}} \subset \mathbb{A} \). By mimicking the same steps as in the proof of Proposition 2 we can obtain that, if \( \varphi_p \) satisfies

\[
|\varphi_p(x) - \varphi_m(x)| \leq \inf_{x \in \mathbb{C}} \mathbb{V}(x), \quad \forall x \in \mathbb{C}
\]

we have

\[
\frac{\partial \mathbb{V}}{\partial x}(x) \varphi_p(x) \leq -\frac{1}{2} \mathbb{V}(x) \quad \forall x \in C.
\]

This implies the compact set \( \{ x \in \mathbb{A} : \mathbb{V}(x) \leq \mathbb{V} \} \) is asymptotically stable for the system (81) with basin of attraction \( B \) containing the compact set \( \{ x \in \mathbb{A} : \mathbb{V}(x) \leq \alpha(\mathbb{C}) \} \) which contains \( \mathbb{C} \). Since, with (86), we have

\[
\{ x \in \mathbb{A} : \mathbb{V}(x) \leq \mathbb{V} \} \subset \{ x \in \mathbb{R}^n : x^T \Pi x \leq \frac{p_0}{3} \}
\]

with (82), (85), and (87) we have established our result with \( \delta \) given as

\[
\delta = \min \left\{ \frac{a}{4} \sqrt{\frac{p_0}{12\lambda_{\max}(\Pi)}}, \frac{a}{8\lambda_{\max}(\Pi)} \left( \frac{\inf_{x \in \mathbb{C}} \mathbb{V}(x)}{2 \sup_{x \in \mathbb{C}} \frac{\partial \mathbb{V}}{\partial x}(x)} \right) \right\}
\]

**Proof of Proposition 3** In view of the above Lemma and [19], Proposition 3 holds if (13) and (14) imply (79) and (80). At the end of the proof of Proposition 2 we have seen that (13) implies (79). So we are left with proving that (13) and (14) imply (80). By using again the notations (71) and dropping the arguments we see that

\[
\left| \frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x) \right| \\
\leq \left| \Delta_k \vartheta (\Delta_p - \Delta_m) \Delta_u \right| + \left| \Delta_2 + \Delta_2 \vartheta \Delta_u \right| |\Delta_y|,
\]

where

\[
\Delta_k \vartheta = \begin{pmatrix} I & 0 & 0 \\ 0 & \frac{\partial \varrho}{\partial y} & 0 \\ \frac{\partial \varrho}{\partial y} & \frac{\partial \varrho}{\partial y} & \frac{\partial \varrho}{\partial y} \end{pmatrix}, \quad \Delta_u = \begin{pmatrix} I & 0 & 0 \\ 0 & \frac{\partial \psi_{\text{sat}}}{\partial z} & 0 \\ \frac{\partial \psi_{\text{sat}}}{\partial z} & 0 & \frac{\partial \psi_{\text{sat}}}{\partial z} \end{pmatrix},
\]

\[
\Delta_p = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial u} & \frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial x} & \frac{\partial g}{\partial u} & \frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial x} & \frac{\partial g}{\partial u} & \frac{\partial g}{\partial u} \end{pmatrix}, \quad \Delta_m = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial x} & \frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial x} & \frac{\partial g}{\partial u} \end{pmatrix},
\]

\[
\Delta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial \varrho}{\partial y} & 0 \\ \frac{\partial \varrho}{\partial y} & \frac{\partial \varrho}{\partial y} & \frac{\partial \varrho}{\partial y} \end{pmatrix}, \quad \Delta_2 \vartheta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\Delta_y = \zeta(x, \psi_{\text{sat}}(x, z)) - h(x).
\]

Recall that by construction the functions \( \psi_{\text{sat}} \), \( k \) and \( \vartheta \) are
C^{1}. Hence, by letting (where the arguments are dropped for compactness)
\[ L_{k\theta} = \sup_{(x,z) \in \mathcal{L}} \left\{ \frac{\partial k}{\partial r}, \frac{\partial k}{\partial y} \right\}, \quad L_h = \sup_{x \in \mathcal{L}} \left\{ \frac{\partial h}{\partial x} \right\}, \quad L_u = \sup_{(x,z) \in \mathcal{L}} \left\{ \frac{\partial \mu_{out}}{\partial z}, \frac{\partial \mu_{out}}{\partial x} \right\}, \quad L_{2k} = \sup_{(x,z) \in \mathcal{L}} \left\{ \frac{\partial^2 k}{\partial z^2}, \frac{\partial^2 k}{\partial x^2} \right\}, \quad L_{2\theta} = \sup_{(x,z) \in \mathcal{L}} \left\{ \frac{\partial^2 \theta}{\partial z^2}, \frac{\partial^2 \theta}{\partial x^2}, \frac{\partial^2 \phi}{\partial z^2} \right\}, \quad \text{and} \quad L_2 = \max\{L_{2k}, L_{2\theta}\}, \]
and \[ \frac{\partial \phi}{\partial x}(x) - \frac{\partial \phi_m(x)}{\partial x} \leq 4 L_u L_{k\theta} |\Delta_p - \Delta_m| + 2 L_2 (1 + L_u + L_h) |\Delta_p| \].

The proof can be completed by using (13) and (14) in place of the solutions when \( \Delta = 0 \).

The problem is to regulate \( \theta \) at 0, with \( e \) remaining close to a prescribed cruise speed \( v_{\text{cruise}} \), using the pitch rate \( q \) and the thrust \( e \) as controls, and with \( \gamma \) and \( \theta \) as only measurements. So here, by using the notation introduced in Section II
\[ x = (\theta, \gamma, e), \quad u = (c, q), \quad y = (\theta, \gamma), \quad y_r = \gamma. \]

A. Choice of the function \( k \) in the integral action
We select \[ k(x, h(x)) = v \sin(\gamma). \]
The motivation is that, then the integrator state \( z \) has the same dynamics as the altitude of the vehicle (not taken into account in this illustration).

B. State feedback design
To design the state feedback \( \psi \) and the associated Lyapunov function \( V \), we start by noting that the so called phugoid mode is conservative (see for instance [32, Section VII.4]). Precisely we have that the following function remains constant along the solutions when \( e = 0 \) and \( \sin(\theta - \gamma) = \frac{2v}{k^2} \)
\[ \mathcal{I}(v, \gamma) = \frac{v^3}{3v_0^3} - \frac{v}{v_0} \cos(\gamma) \]
This can be checked by looking the time derivative of \( \mathcal{I} \). Also the open sublevel set of \( \mathcal{I} \)
\[ \mathcal{S} = \left\{ (v, \gamma) : \frac{v^3}{3v_0^3} - \frac{v}{v_0} \cos(\gamma) < 0 \right\} \]
is the largest sublevel set not containing a point of the type \((0, \gamma)\). Namely it is the largest sublevel set of \( \mathcal{I} \) where the model (88) is well defined. Moreover in this set the \( \gamma \)-component of any point is in \((-\frac{\pi}{2}, \frac{\pi}{2})\). Also \( \mathcal{I} \) is positive definite in \( v - v_0 \) and \( \gamma \) on \( \mathcal{S} \). We conclude that \( \mathcal{I} + \frac{k^2}{2} \) restricted to \( \mathcal{S} \) is a candidate for playing the role of a Lyapunov function. Also forwarding with the functions \( V \) and \( H \) known is possible since when \( e = 0 \), the function
\[ z - H(v) = z + \frac{v^2}{2g} \]
remains constant along the solutions of the following \((z, v)\)-subsystem
\[ \dot{z} = v \sin(\gamma), \quad \dot{v} = e - g \sin(\gamma). \]

Finally we can complete the design of a state feedback by applying backstepping from the fact that \( \theta \) given as
\[ \theta = \gamma + \arcsin\left(\frac{g}{L v_0^3}\right) \]
is stabilizing for the \((z, v, \gamma)\)-subsystem. All this leads to the following (weak) Control Lyapunov function
\[ V(z, v, \gamma) = \frac{v^3}{3v_0^3} + \frac{2}{3} \frac{v}{v_0} \cos(\gamma) + k_1 \left( 2gz + v^2 - v_0^2 \right) \]
\[ + \frac{k_2}{2} \left[ \theta - \gamma - \arcsin\left(\frac{g}{L v_0^3}\right) \right]^2, \quad (89) \]
where the dimensionless numbers \( k_1 \) and \( k_2 \) are arbitrary but strictly positive, and the following feedback law
\[ e = -\text{sat}_{k_e}\left( k_3 \left[ \frac{v^2}{v_0^2} - \cos(\gamma) + k_1 \frac{2gz + v^2 - v_0^2}{v_0^2} \right] \right), \]
\[ q = -\left( \frac{L v_0^3 \sin(\theta - \gamma) - g v^2}{\theta - \gamma - \arcsin\left(\frac{g}{L v_0^3}\right)} \right) \frac{v - \cos(\gamma)}{v} \]
\[ -L v \sin(\theta - \gamma) + k_4 \left[ \theta - \gamma - \arcsin\left(\frac{g}{L v_0^3}\right) \right], \]
where \( k_3 \) and \( k_4 \) are dimensionless arbitrary strictly positive real numbers and \( k_e \) and \( k_q \) are arbitrary strictly positive saturation levels. With LaSalle invariance principle it is possible to prove that \( (v, \gamma, \theta) = \left( \frac{v_0}{3}, 0, \arcsin\left(\frac{v_0}{L v_0^3}\right) \right) \) is the only asymptotically stable equilibrium point of the system. In this simple illustration we have chosen the simpler Lyapunov Function (89), but it does not give enough degrees of freedom to improve performance and increase the domain of attraction. More appropriate designs are possible by choosing different Lyapunov functions (see [32]). Finally, according to Section
\[ ^9 \text{Its derivative along the solutions may be only non positive.} \]
III-C for its use in the output feedback, the state feedback law \( q \) above has to be modified by adding a saturation (see in particular the function \( \psi_{sat} \) in (37)).

C. Design of the high-gain observer

To obtain an observer we check that the conditions of Assumptions 3 are satisfied. Let \( \gamma_{dot} \) be defined as the following function

\[
\gamma_{dot}(\theta, \gamma, v) = L v \sin(\theta - \gamma) - \frac{g \cos(\gamma)}{v}
\]

Then let

\[
\Phi((\theta, \gamma, v)) = \Phi(x) = (\phi_1, \phi_2, \phi_3) = (\theta, \gamma, \gamma_{dot}(\theta, \gamma, v)).
\]

It is defined on the set

\[
\mathcal{O} = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \times (0; +\infty),
\]

and \((\theta, \gamma, v)\) can be recovered from its values \((\phi_1, \phi_2, \phi_3)\) in the following subset of \(\Phi(\mathcal{O})\)

\[
\Xi = \left\{ \phi \in \mathbb{R}^3 : \phi_1 \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right), \phi_2 \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right), \phi_3 < -2\sqrt{gL} |\phi_1 - \phi_2| \right\}.
\]

Note also that \(\partial \Phi/\partial x\) is always non-singular on the set \(\mathcal{O}\) because \(\partial \gamma_{dot}/\partial v\) cannot be equal to 0 when \(\phi \in \Xi\). Hence the function \(\Phi\) is a diffeomorphism satisfying Assumption O1.

Then, with \(C\) defined as

\[
C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]

Assumption O2 also holds.

Now let \(A, B, L, M, \) and \(N\) be defined as

\[
A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L = \text{diag}(1,1,\ell), \quad M = \text{diag}(\ell,\ell,\ell), \quad N = \text{diag}(1,1),
\]

\[
B(\Phi(x), u) = \text{col} \left( u_1, 0, \frac{\partial \gamma_{dot}}{\partial \theta} u_1 + \frac{\partial \gamma_{dot}}{\partial v} u_2 + \frac{\partial \gamma_{dot}}{\partial \gamma} \gamma_{dot} - \frac{\partial \gamma_{dot}}{\partial v} g \sin(\gamma) \right).
\]

Also, given any strictly positive number \(\nu\), let \(P\) be a symmetric positive definite matrix defined as

\[
P = \begin{pmatrix} * & * & * \\ * & * & p_{23} \\ * & p_{23} & p_{33} \end{pmatrix}
\]

where \(2\nu_{p_{33}} \leq -\nu_{p_{23}}\). Then there exists a real number \(\rho\) such that we have

\[
PA + A^T P - \rho C^T C \leq -\nu P.
\]

This implies the existence of a real number \(\nu_k\) such that, for any \(\nu_k \geq \nu\), with

\[
K = \nu_k P^{-1} C^T
\]

assumptions O3 to O7 are satisfied.

We use \(|\phi_1 - \phi_2|\) to upper bound \(\cos \phi_2 \sin(\phi_1 - \phi_2)\).

D. Design of the correction term

Following Section IV, the function \(h_2(x)\) can be defined as

\[
h_2(x) = h_2^1(x) + h_2^2(x) + h_2^3(x) + h_2^4(x)
\]

with

\[
h_2^1(x) = \max \left\{ \frac{\nu \gamma^2}{\pi^2} - \varepsilon_1; 0 \right\}, \quad h_2^2(x) = \max \left\{ \frac{\nu \gamma^2}{\pi^2} - \varepsilon_2; 0 \right\}, \quad h_2^3(x) = \max \left\{ \varepsilon_3 (\theta - \gamma) - \gamma_{dot} - \varepsilon_4; 0 \right\}, \quad h_2^4(x) = \max \left\{ \frac{\gamma_{dot}}{\gamma_{dot max}} - \varepsilon_5; 0 \right\},
\]

where \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\) and \(\gamma_{dot max}\) are constants to be properly chosen. The functions \(h_{2,1}\) and \(h_{2,2}\) take care respectively of \(\theta\) and \(\gamma\) to stay in the set \(\Xi\) as showed in Figure 1 whereas functions \(h_{2,3}\) and \(h_{2,4}\) take care of \(f(\theta, \gamma, v)\) as in Figure 2.

The correction term \(E\) is defined as in Lemma 3. Finally the functions \(U_{\kappa}\) and \(\sigma_{\kappa}\) can be defined as in the proof of Lemma 2.

\[
h_2^1(x) = 0, \quad h_2^2(x) = 0, \quad h_2^3(x) = 0, \quad h_2^4(x) = 0
\]

Fig. 1: Design of the functions \(h_2^1, h_2^2\).

\[
h_2^1(x) = 0, \quad h_2^2(x) = 0
\]

Fig. 2: Design of the functions \(h_2^3, h_2^4\).

VII. CONCLUSIONS

Robust asymptotic output regulation by output feedback has been investigated. Our design technique follows the very usual approach of stabilizing the origin of the model augmented with integrators of the output errors. To do so we assume we have already a stabilizing state feedback for the model but not asking for any specific structure nor for normal form nor for minimum phase. For the augmented model we redesign the
state feedback by applying forwarding. The output feedback is obtained by introducing a high-gain observer expressed in the original coordinates. The output regulation is shown to be robust to any small enough (in a $C^1$ sense) unstructured discrepancy between model and process in open loop.

In establishing our main propositions we obtained new results, which may have their own interest. They concern high-gain observers for multi-output systems (Lemma 2) and persistence of equilibria under small perturbations (Proposition 2).

The design we propose is illustrated by the regulation of the flight path angle for a simplified longitudinal model of a plane.

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