On a singular value decomposition of the normal Radon transform operator acting on 3D 2-tensor fields

A P Polyakova and I E Svetov
Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail: apolyakova@math.nsc.ru, svetovie@math.nsc.ru

Abstract. We consider the problem of reconstructing the potential part of a symmetric 2-tensor field defined in a unit ball by its known values of the normal Radon transform. To solve the problem, a singular value decomposition of the normal Radon transform operator is constructed. The basis functions and fields using the Jacobi, Gegenbauer polynomials and spherical harmonics are constructed.

1. Introduction

The classical integral geometry operator acting on the vector and the 2-tensor fields is the ray transform [1]. In the two-dimensional case, for the full reconstruction of the vector field, it is necessary to know values of two ray transforms (the longitudinal and transverse ray transforms) [2], since each of them has a nonzero kernel. In order to completely restore the symmetric 2-tensor field in \( \mathbb{R}^2 \), the values of three ray transforms must be known (for more details see [3], [4]). In the three-dimensional case, only the solenoidal part of the vector or the symmetric 2-tensor field can be reconstructed by the longitudinal ray transform (see, for example, [5], [6]).

To restore the potential part of the vector and the tensor fields, we need to have data of another type. One of the operators, which allows reconstructing the potential part of a vector and a symmetric 2-tensor field, is the normal Radon transform. We mention the report [7], in which the inversion of the Radon transform of a symmetric 2-tensor field in the space \( \mathbb{R}^3 \) and, in particular, the normal Radon transform, is investigated.

In this paper, we construct a singular value decomposition of the normal Radon transform operator acting on the three-dimensional symmetric 2-tensor fields. The singular value decompositions of the operators of the Radon transform [8]–[10] and the ray transform [11] acting on the scalar fields in \( \mathbb{R}^3 \) are well known. At the same time, singular value decompositions of the integral operators acting on the vector and the symmetric 2-tensor fields have been obtained relatively recently. In particular, in the space \( \mathbb{R}^2 \), the singular value decompositions of the ray transforms of the vector [12] and tensor fields [13], [14] were constructed. In the space \( \mathbb{R}^3 \), the singular value decompositions were constructed for the operators acting on the vector fields: for the ray transform operator [15] and for the normal Radon transform operator [16], [17]. We note the papers devoted to the development and research of algorithms for the numerical solution of the vector and the tensor tomography problems in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) using the above singular value decompositions [18]–[20].
The paper [21] proposes an algorithm for the numerical solution of the 2-tensor tomography problem for reconstructing the three-dimensional potential symmetric 2-tensor field by its known values of the normal Radon transform. The algorithm is based on the method of truncated singular value decomposition. It should be noted that [21] deals with the numerical solution of the three-dimensional tomography problem. At the same time, the theoretical justification of the algorithm proposed is not fully made. Namely, the orthogonality in the main space was verified by using the Wolfram Mathematica 9 software only for basis fields of degrees not exceeding 50. In the present paper, this statement is theoretically justified for the basis fields of an arbitrary degree. Thus, a singular value decomposition of the normal Radon transform operator acting on a three-dimensional symmetric 2-tensor field is constructed.

2. Definitions

We use the following notations: \( B = \{ x \in \mathbb{R}^3 \mid |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 \} \) is the unit ball, \( \partial B = \{ x \in \mathbb{R}^3 \mid |x| = 1 \} \) is the unit sphere, \( Z = \{(s, \xi) \mid \xi \in \mathbb{R}^3, |\xi| = 1, s \in (-1, 1)\} \) is the cylinder.

For functions we use the notations \( f \) and \( g \). Denote the potentials by \( \phi(x), \psi(x), \ldots \) We give some definitions for an arbitrary rank \( m \) of tensor fields, but only \( m = 0 \) (functions), \( m = 1 \) (vector fields) and \( m = 2 \) (symmetric 2-tensor fields) are used. A set of symmetric \( m \)-tensor fields \( w(x) = (w_{i_1 \ldots i_m}(x)), u(x) = (u_{i_1 \ldots i_m}(x)), v(x) = (v_{i_1 \ldots i_m}(x)), i_1, \ldots, i_m = 1, 2, 3, \) in \( B \) is denoted by \( S^m(B) \). The inner product in \( S^m(B) \) is defined by the formula

\[
\langle u(x), v(x) \rangle = \sum_{i_1, \ldots, i_m=1}^3 u_{i_1 \ldots i_m}(x)v_{i_1 \ldots i_m}(x).
\]

We use the spaces of the square-integrable symmetric \( m \)-tensor fields \( L_2(S^m(B)) \). The inner product in the space \( L_2(S^m(B)) \) is defined by

\[
\langle u, v \rangle_{L_2(S^m(B))} = \int_B \langle u(x), v(x) \rangle \, dx.
\]

The Sobolev spaces of the symmetric \( m \)-tensor fields are denoted by \( H^k(S^m(B)) \). The spaces of the symmetric 2-tensor fields from \( H^k(S^m(B)) \) that vanish at the boundary of the domain together with all their derivatives up to \((k - 1)\)th order are denoted by \( H^k_0(S^m(B)) \). Also, we use the weighted \( L_2 \)-space \( L_2(Z, \rho) \), where the weight function \( \rho(s) > 0 \) is defined on \( Z \). The inner product of the functions \( f \) and \( g \) in \( L_2(Z, \rho) \) is given by

\[
\langle f, g \rangle_{L_2(Z, \rho)} = \int_Z f(y)g(y)\rho(y) \, dy.
\]

We use the following differential operators:

1) The inner derivation operator \( d : H^k(S^m(B)) \to H^{k-1}(S^{m+1}(B)) \) acts on the potential \( \psi \) and the vector field \( v \) as follows:

\[
(d\psi)_i = \frac{\partial \psi}{\partial x_i}, \quad (dv)_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
\]

2) The curl operator \( \text{curl} : H^k(S^1(B)) \to H^{k-1}(S^1(B)) \) acts on the vector field \( w \) by the rule

\[
\text{curl} w = \left( \frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_3}, \frac{\partial w_1}{\partial x_3} - \frac{\partial w_3}{\partial x_1}, \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right).
\]
3) The divergence operator \( \text{div} : H^k(S^{m+1}(B)) \rightarrow H^{k-1}(S^m(B)) \) acts on the symmetric \( m \)-tensor field \( w \) by the formula

\[
(\text{div} \, w)_{i_1 \ldots i_m} = \sum_{j=1}^{3} \frac{\partial w_{i_1 \ldots i_mj}}{\partial x_j}.
\]

We need to recall that the symmetric \( m \)-tensor field \( u \in H^k(S^m(B)) \) is potential if there exists a symmetric \((m-1)\)-tensor field \( v \in H^{k+1}(S^{m-1}(B)) \) (a potential) such that \( u = \text{div} \, v \) and the field \( w \in H^k(S^m(B)) \) is solenoidal if \( \text{div} \, w = 0 \in H^{k-1}(S^{m-1}(B)) \). Obviously, the vector field \( w = \text{curl} \, u \) is a solenoidal field. Analogously, the symmetric 2-tensor field \( w \) is solenoidal if \( (w_{i_1}, w_{i_2}, w_{i_3}) = \text{curl} \, v^i \), \( i = 1, 2, 3 \) for some vector fields \( v^i \).

It is well known [1] that every symmetric \( m \)-tensor field \( v \in L_2(S^m(B)) \) can be uniquely represented as the sum

\[
v = w + d(u), \quad w \in H^1(S^m(B)), \quad \text{div} \, w = 0, \quad u \in H_0^1(S^{m-1}(B)).
\]

In [21], it is shown that every symmetric 2-tensor field can be uniquely decomposed as the sum

\[
v = w + d(\text{curl} \, u) + d^2 \phi,
\]

where

\[
w \in H^1(S^2(B)), \quad \text{div} \, w = 0, \quad u \in H^2(S^1(B)), \quad \text{curl} \, u \in H_0^1(S^1(B)), \quad \phi \in H_0^2(B).
\]

The plane \( P_{\xi,s} \in \mathbb{R}^3 \) is defined by the normal equation \( \langle \xi, x \rangle - s = 0 \) for \( x = (x_1, x_2, x_3) \) and \( \xi = (\xi_1, \xi_2, \xi_3) \), \( |\xi| = 1 \). Here \(|s|\) is a distance from the plane \( P_{\xi,s} \) to the origin and \( \xi \) is the normal vector of the plane.

The Radon transform \( \mathcal{R} f : L_2(B) \rightarrow L_2(Z, \rho) \) of the function \( f(x) \) is defined by the formula

\[
[\mathcal{R} \, f](s, \xi) = \int_{P_{\xi,s} \cap B} f(x) \, dx.
\]

The normal Radon transform \( \mathcal{R}^\perp_2 : L_2(S^2(B)) \rightarrow L_2(Z, \rho) \) acts on the symmetric 2-tensor field \( u(x) \) by the rule

\[
[\mathcal{R}^\perp_2 \, u](s, \xi) = \int_{P_{\xi,s} \cap B} \langle u(x), \xi^2 \rangle \, dx.
\]

Further, we formulate the properties of the normal Radon transform \( \mathcal{R}^\perp_2 \) (for details see [21]).

**Statement 1.** For any function \( \psi \in H^2_0(B) \) the equality

\[
[\mathcal{R}^\perp_2 (d^2 \psi)](s, \xi) = \frac{\partial^2}{\partial s^2} [\mathcal{R} \psi](s, \xi)
\]

holds.

**Statement 2.** The kernel of the operator of the normal Radon transform \( \mathcal{R}^\perp_2 \) consists of any linear combinations of the two types of symmetric 2-tensor fields:

1) solenoidal symmetric 2-tensor fields \( w \) such that \( (w_{i_1}, w_{i_2}, w_{i_3}) = \text{curl} \, u^i \), \( i = 1, 2, 3 \), where \( u^i \in H^2(S^1(B)) \cup H_0^1(S^1(B)) \), \( i = 1, 2, 3 \);

2) potential symmetric 2-tensor fields \( w = d(\text{curl} \, v) \) such that \( v \in H^2(S^1(B)) \cup H_0^1(S^1(B)) \).

From the Statements it follows that by the known values of the normal Radon transform of a symmetric 2-tensor field, only its potential part of the form \( d^2 \psi \), \( \psi \in H^2(B) \), can be restored.
The purpose of this paper is to solve the problem of reconstruction of the potential 2-tensor fields of the form \(d^2\phi\), \(\phi \in H^2_0(B)\), by the known values of the normal Radon transform \([R^\perp_2(d^2\phi)]\). For this a singular value decomposition of the normal Radon transform operator \(R^\perp_2\) is constructed. The basis functions and fields using the Jacobi, Gegenbauer polynomials and spherical harmonics are constructed.

Recall the definitions and some properties of the orthogonal polynomials. 

The Jacobi polynomial \(P_n^{(p,q)}(t)\) of degree \(n\) with the indices \((p, q)\) and \(t \in [0, 1]\) is defined by the formula

\[
P_n^{(p,q)}(t) = 1 + \sum_{k=1}^{n} \frac{(-1)^k C_n^k (p+n)(p+n+1)\ldots(p+n+k-1)}{q(q+1)\ldots(q+k-1)} t^k,
\]

where \(C_n^k\) is the binomial coefficient. On the segment \([0, 1]\) the Jacobi polynomials are orthogonal with the weight \(t^{p+1}(1-t)^{q-1}\), i.e. we have

\[
\int_0^1 t^{p-1}(1-t)^{q-1} P_n^{(p,q)}(t) P_m^{(p,q)}(t) \, dt = \frac{n!\Gamma(q)\Gamma(p+q+n+1)}{\Gamma(q+n)\Gamma(p+n)(p+2n)} \delta_{nm},
\]

where \(\Gamma(\alpha)\) is the gamma function and \(\delta_{nm}\) is the Kronecker symbol. The first and the second derivatives of the Jacobi polynomial \(P_n^{(p,q)}\) are calculated as

\[
\begin{align*}
(P_n^{(p,q)})'(t) &= -\frac{n(n+p)}{q} P_{n-1}^{(p+2,q+1)}(t), \\
(P_n^{(p,q)})''(t) &= \frac{n(n-1)(n+p)(n+p+1)}{q(q+1)} P_{n-2}^{(p+4,q+2)}(t).
\end{align*}
\]

The Gegenbauer polynomial \(C_n^{(\mu)}(t)\) of degree \(n\) with the index \(\mu\) is given by

\[
C_n^{(\mu)}(t) = \sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(n-k+\mu)}{\Gamma(\mu) k! (n-2k)!} (2t)^{n-2k},
\]

where \([\cdot]\) denotes the integer part of the number. The Gegenbauer polynomials are orthogonal on \([-1, 1]\) with the weight \((1-t^2)^{\mu-1/2}\). In other words we have

\[
\int_{-1}^{1} C_n^{(\mu)}(t) C_m^{(\mu)}(t) (1-t^2)^{\mu-1/2} \, dt = \frac{\pi 2^{1-2\mu} \Gamma(n+2\mu) \delta_{nm}}{n!(n+\mu)\Gamma^2(\mu)}.
\]

The Legendre polynomial \(L_k(t)\) of degree \(k\) is a particular case of the Gegenbauer polynomial: \(L_k(t) = C_k^{(0.5)}(t)\). These polynomials are orthogonal on \([-1, 1]\), and the norm can be calculated by the formula

\[
||L_k||^2 = \int_{-1}^{1} L_k^2(t) \, dt = \frac{2}{2k+1}.
\]

The associated Legendre polynomial \(L_{kl}(t)\) of degree \(k\) with the integer index \(l = 0, \ldots, k\) is defined via the Legendre polynomial

\[
L_{kl}(t) = (1-t^2)^{l/2} \frac{d^l}{dt^l} L_k(t).
\]
These polynomials are orthogonal on $[-1, 1]$.

The spherical function $Y_{kl}$ of order $k$ with the integer index $l = -k, \ldots, k$ is defined by the rule

$$Y_{kl}(\theta, \varphi) = L_{k|l}(|\cos \theta|) \cdot \begin{cases} \cos l\varphi, & l \geq 0, \\ \sin |l|\varphi, & l < 0. \end{cases} \tag{4}$$

The spherical functions are orthogonal on the unit sphere, and the norm is calculated by the formula

$$||Y_{kl}||^2 = \begin{cases} \frac{4\pi}{2k + 1}, & l = 0, \\ \frac{2\pi}{2k + 1} (k + |l|)!, & l \neq 0. \end{cases}$$

The harmonic polynomial $H_{kl}(x)$ of degree $k$ with the integer index $l = -k, \ldots, k$ in the spherical coordinate system has the form $H_{kl}(r, \theta, \varphi) = r^k Y_{kl}(\theta, \varphi)$.

3. The singular value decomposition of the normal Radon transform operator $\mathcal{R}_2^+$

The basis potential symmetric 2-tensor fields in $L_2(S^2(B))$ are constructed using the method of potentials. Namely, we choose a basis system of functions in $H^2_0(B)$ and generate a 2-tensor potential basis in $L_2(S^2(B))$ applying the operator $d^2$.

We introduce the following notation:

$$\Phi_{kln}(x) = (1 - |x|^2)^2 H_{kl}(x) P_n^{(k+3.5,k+1.5)}(|x|^2), \quad k, n = 0, 1, 2, \ldots, \quad l = -k, \ldots, k,$$

As the initial basis system of potentials, we choose the polynomials

$$\tilde{\Phi}_{kln}(x) := \lambda_{kln} \Phi_{kln}(x), \quad k, n = 0, 1, 2, \ldots, \quad l = -k, \ldots, k,$$

where

$$\lambda_{kln} = \frac{\Gamma(n + k + 1.5)}{(n + 2)!\Gamma(k + 1.5)||Y_{kl}||} \sqrt{\frac{2n + k + 3.5}{8}}.$$

In the spherical coordinate system $(x = r \cos \varphi \sin \theta, \quad y = r \sin \varphi \sin \theta, \quad z = r \cos \theta)$ we have

$$\tilde{\Phi}_{kln}(r, \theta, \varphi) = \lambda_{kln} (1 - r^2)^2 r^k P_n^{(k+3.5,k+1.5)}(r^2) Y_{kl}(\theta, \varphi).$$

We apply the operator $d^2$ to the potentials $\Phi_{kln}$ and $\tilde{\Phi}_{kln}$ to obtain the families of basis potential symmetric 2-tensor fields

$$T_{kln}(x) = d^2 \Phi_{kln}(x), \quad \tilde{T}_{kln}(x) = d^2 \tilde{\Phi}_{kln}(x), \quad k, n = 0, 1, 2, \ldots, \quad l = -k, \ldots, k.$$

In [21], the work on the construction of a singular value decomposition of the normal Radon transform operator $\mathcal{R}_2^+$ was begun. Namely,

1) the images $[\mathcal{R}_2^+ T_{kln}](s, \xi)$ of the fields $T_{kln}(x)$ were calculated;

2) the orthogonality of the functions $[\mathcal{R}_2^+ T_{kln}](s, \xi)$ in the space $L_2(Z, (1 - s^2)^{-1})$ was proved;

3) the orthogonality of the fields $T_{kln}(x)$ in the space $L_2(S^2(B))$ for $2n + k + 2 \leq 50$ using Wolfram Mathematica 9 software was numerically verified.

Thus, the theoretical justification of the orthogonality of the fields $T_{kln}(x)$ in the space $L_2(S^2(B))$ for arbitrary $n, k, l$ still remains open. We formulate the results obtained in [21] as statements. The notation for $\xi(\theta, \varphi)$ in the spherical coordinate system

$$\xi_1 = \cos \varphi \sin \theta, \quad \xi_2 = \sin \varphi \sin \theta, \quad \xi_3 = \cos \theta,$$
is used.

**Statement 3.** The images $[\mathcal{R}_{k,n}^{2} \bar{T}_{kln}](s, \xi) = [\mathcal{R}_{k,n}^{2} \bar{T}_{kln}](s, \theta, \varphi)$ of the fields $\bar{T}_{kln} = d^{2}\Phi_{kln}$, $k, n = 0, 1, 2, \ldots$, $l = -k, \ldots, k$ have the form

$$[\mathcal{R}_{k,n}^{2} \bar{T}_{kln}](s, \theta, \varphi) = \frac{(-1)^{n}4\pi}{(2n + k + 3)(2n + k + 4)} \| Y_{kl} \| \sqrt{\frac{2n + k + 3.5}{2}} (1 - s^{2})C^{(1,5)}_{2n+k+2}(s)Y_{kl}(\theta, \varphi).$$

**Statement 4.** The system of functions

$$G_{kln}(s, \theta, \varphi) = \frac{(-1)^{n} \sqrt{2n + k + 3.5}}{(2n + k + 3)(2n + k + 4)} \| Y_{kl} \| (1 - s^{2})C^{(1,5)}_{2n+k+2}(s)Y_{kl}(\theta, \varphi),$$

$k, n = 0, 1, 2, \ldots$, $l = -k, \ldots, k,$
is the orthonormal system in the space $L_{2}(Z, (1 - s^{2})^{-1})$.

The main result of the present paper is the following theorem.

**Theorem 1.** The system of potential symmetric 2-tensor fields $\bar{T}_{kln} = d^{2}\Phi_{kln}$ is the orthonormal system in the space $L_{2}(S^{2}(B))$.

Statement 3 and the definition of the functions $G_{kln}$ imply the equalities:

$$[\mathcal{R}_{k,n}^{2} \bar{T}_{kln}](s, \theta, \varphi) = \sigma_{kn} G_{kln}(s, \theta, \varphi), \quad k, n = 0, 1, 2, \ldots$, $l = -k, \ldots, k.$

The numbers

$$\sigma_{kn} = \frac{2\sqrt{2\pi}}{(2n + k + 3)(2n + k + 4)}$$

are the singular values of the normal Radon transform operator. Thus, the following theorem holds.

**Theorem 2.** The singular value decomposition of the normal Radon transform operator

$$\mathcal{R}_{k,n}^{2} : L_{2}(S^{2}(B)) \to L_{2}(Z, (1 - s^{2})^{-1})$$

has the form

$$g(s, \theta, \varphi) := [\mathcal{R}_{k,n}^{2} w](s, \theta, \varphi) = \sum_{k, n = 0}^{\infty} \sum_{l = -k}^{k} \sigma_{kn}(w, \bar{T}_{kln})_{L_{2}(S^{2}(B))} G_{kln}(s, \theta, \varphi),$$

for a potential symmetric 2-tensor field $w = d^{2}\phi$, $\phi \in H_{0}^{2}(B)$ and a value of the inverse operator can be calculated by the formula

$$w(x) = \left( (\mathcal{R}_{k,n}^{2})^{-1} g \right)(x) = \sum_{k, n = 0}^{\infty} \sum_{l = -k}^{k} \sigma_{kn}^{-1}(g, G_{kln})_{L_{2}(Z, (1 - s^{2})^{-1})} \bar{T}_{kln}(x).$$

4. The proof of Theorem 1

The proof of Theorem 1 present difficulties, so we give only the main steps of the proof. We introduce the following notations

$$P_{k,n} := P_{k,n}^{(k+3.5,k+1.5)}(r^{2}), \quad P'_{k,n} := \frac{\partial P_{k,n}^{(k+3.5,k+1.5)}}{\partial (r^{2})}(r^{2}), \quad P''_{k,n} := \frac{\partial^{2} P_{k,n}^{(k+3.5,k+1.5)}}{\partial (r^{2})^{2}}(r^{2}),$$

$$Y_{kl} := Y_{kl}(\varphi, \theta), \quad Y'_{kl,\varphi} := \frac{\partial Y_{kl}}{\partial \varphi}(\varphi, \theta), \quad Y'_{kl,\theta} := \frac{\partial Y_{kl}}{\partial \theta}(\varphi, \theta),$$

$$Y''_{kl,\varphi\varphi} := \frac{\partial^{2} Y_{kl}}{\partial \varphi^{2}}(\varphi, \theta), \quad Y''_{kl,\varphi\theta} := \frac{\partial^{2} Y_{kl}}{\partial \varphi \partial \theta}(\varphi, \theta), \quad Y''_{kl,\theta\theta} := \frac{\partial^{2} Y_{kl}}{\partial \theta^{2}}(\varphi, \theta).$$
Lemma 1. The values of the basis fields components can be calculated by the formulas

\[
(T_{\text{kin}})_{ij}(r, \theta, \varphi) = \left( 2A_{ij}r^{k+2} - 2B_{ij}(1-r^2)r^k + C_{ij}(1-r^2)^2r^{k-2} \right) P_n \\
+ \left( -4A_{ij}(1-r^2)r^{k+2} + B_{ij}(1-r^2)^2r^k \right) P'_n + A_{ij}(1-r^2)^2r^{k+2} P''_n,
\]

for \( i, j = 1, 2, 3 \), where

\[
A_{11} = Y_{kl}(4 \cos^2 \varphi \sin^2 \theta), \\
B_{11} = Y_{kl}(4k \cos^2 \varphi \sin^2 \theta + 2) + Y'_{kl, \varphi}(-2 \sin 2\varphi) + Y'_{kl, \theta}(2 \cos^2 \varphi \sin 2\theta), \\
C_{11} = Y_{kl}(k(k-2) \cos^2 \varphi \sin^2 \theta + k) + Y''_{kl, \varphi} \sin(2\varphi(\cot^2 \theta - k + 1)) \\
+ Y''_{kl, \theta} \sin^2(1 + \cot^2 \theta) + Y'_{kl, \theta}((k-1) \cos^2 \varphi \sin 2\theta + \sin^2 \varphi \cot \theta) \\
+ Y''_{kl, \theta, \theta}(-\sin 2\varphi \cot \theta) + Y''_{kl, \theta, \theta}(\cos^2 \varphi \sin^2 \theta),
\]

\[
A_{12} = A_{21} = Y_{kl}(2 \sin 2\varphi \sin^2 \theta), \\
B_{12} = B_{21} = Y_{kl}(2k \sin 2\varphi \sin^2 \theta) + Y'_{kl, \varphi}(2 \cos 2\varphi) + Y'_{kl, \theta}(\sin 2\varphi \sin 2\theta), \\
C_{12} = C_{21} = Y_{kl}(k(k-2) \sin^2 \varphi \sin^2 \theta + 2) + Y'_{kl, \varphi} \cos(2\varphi(k-1 - \cot^2 \theta)) \\
+ Y''_{kl, \varphi}(-\sin 2\varphi (1 + \cot^2 \theta)/2) + Y'_{kl, \theta}(\sin 2\varphi ((k-1) \sin 2\theta - \cot \theta)/2) \\
+ Y''_{kl, \theta} \cos(2\varphi \cot \theta) + Y''_{kl, \theta, \theta}(\sin 2\varphi \cos^2 \theta/2),
\]

\[
A_{22} = Y_{kl}(4 \sin^2 \varphi \sin^2 \theta), \\
B_{22} = Y_{kl}(4k \sin^2 \varphi \sin^2 \theta + 2) + Y'_{kl, \varphi}(2 \sin 2\varphi) + Y'_{kl, \theta}(2 \sin^2 \varphi \sin 2\theta), \\
C_{22} = C_{kl}(k(k-2) \sin^2 \varphi \sin^2 \theta + k) + Y'_{kl, \varphi} \sin(2\varphi(k-1 - \cot^2 \theta)) \\
+ Y''_{kl, \varphi} \cos(2\varphi (1 + \cot^2 \theta)) + Y'_{kl, \theta}((k-1) \sin^2 \varphi \sin 2\theta + \cos^2 \varphi \cot \theta) \\
+ Y''_{kl, \theta, \theta} \sin(2\varphi \cot \theta) + Y''_{kl, \theta, \theta}(\sin^2 \varphi \cos^2 \theta),
\]

\[
A_{13} = A_{31} = Y_{kl}(2 \cos \varphi \sin 2\theta), \\
B_{13} = B_{31} = Y_{kl}(2k \cos \varphi \sin 2\theta) + Y'_{kl, \varphi}(-2 \sin \varphi \cot \theta) + Y'_{kl, \theta}(2 \cos \varphi \cos 2\theta), \\
C_{13} = C_{31} = Y_{kl}(k(k-2) \cos \varphi \sin 2\theta/2) + Y'_{kl, \varphi}(-k \sin \varphi \cot \theta) \\
+ Y''_{kl, \theta}((k-1) \cos \varphi \cos 2\theta) + Y'_{kl, \theta, \theta} \sin \varphi + Y''_{kl, \theta, \theta}(-\cos \varphi \sin 2\theta/2),
\]

\[
A_{23} = A_{32} = Y_{kl}(2 \sin \varphi \sin 2\theta), \\
B_{23} = B_{32} = Y_{kl}(2k \sin \varphi \sin 2\theta) + Y'_{kl, \varphi}(2 \cos \varphi \cot \theta) + Y'_{kl, \theta}(2 \sin \varphi \cos 2\theta), \\
C_{23} = C_{32} = Y_{kl}(k(k-2) \sin \varphi \sin 2\theta/2) + Y'_{kl, \varphi}(k \cos \varphi \cot \theta) \\
+ Y''_{kl, \theta}((k-1) \sin \varphi \cos 2\theta) + Y'_{kl, \theta, \theta}(-\cos \varphi) + Y''_{kl, \theta, \theta}(-\sin \varphi \sin 2\theta/2),
\]

\[
A_{33} = Y_{kl}(4 \cos^2 \theta), \\
B_{33} = Y_{kl}(4k \cos^2 \theta + 2) + Y'_{kl, \theta}(-2 \sin 2\theta), \\
C_{33} = Y_{kl}(k(k-2) \cos^2 \theta + k) + Y'_{kl, \theta}(-(k-1) \sin 2\theta) + Y''_{kl, \theta, \theta}(\sin^2 \theta).
\]

The proof of the Lemma consists in direct calculation.

The first step. We consider the case \( l_1 = l_2 = l, \ k_1 = k_2 = k \) and \( n_1 \neq n_2 \). We calculate the inner product \( \langle T_{\text{kin}_1}, T_{\text{kin}_2} \rangle_{L^2(S^2(B))} \) in the spherical coordinate system. Taking into account
Lemma 1 and the orthogonality of the Jacobi polynomials (1), we obtain

\[
(T_{\kappa\kappa_1}, T_{\kappa\kappa_2})_{L_2(S^2)} = \frac{1}{2} \int_0^1 \int_0^{2\pi} \int_0^{\pi} \sum_{i,j=1}^{3} \left( B_{ij}^2 (2k + 1) - A_{ij} B_{ij} \frac{(2k + 3)(2k + 1)}{2} - A_{ij} C_{ij} \frac{(2k + 3)(2k + 1)}{2} - 2 C_{ij} B_{ij} C_{ij} (2k - 1) \right) \sin \theta d\theta d\varphi dr^2
\]

\[
+ \frac{1}{2} \int_0^1 \int_0^{2\pi} \int_0^{\pi} \sum_{i,j=1}^{3} \left( B_{ij}^2 + A_{ij} C_{ij} (k^2 - 1/4) - B_{ij} C_{ij} (k - 1/2) \right) (1 - r^2)^2 r^{2k-1} P_{n_1} P_{n_2} \sin \theta d\theta d\varphi dr^2
\]

\[
+ \frac{1}{2} \int_0^1 \int_0^{2\pi} \int_0^{\pi} \sum_{i,j=1}^{3} \left( B_{ij}^2 - 2 A_{ij} C_{ij} - A_{ij} B_{ij} \frac{2k + 3}{2} \right) (1 - r^2)^3 r^{2k+1} P_{n_1} P_{n_2} \sin \theta d\theta d\varphi dr^2.
\]

The third term in the sum is equal to

\[
\frac{1}{2} \int_0^1 (1 - r^2)^3 r^{2k+1} P_{n_1} P_{n_2} dr^2 \cdot \int_0^{2\pi} \int_0^{\pi} \sum_{i,j=1}^{3} \left( B_{ij}^2 - 2 A_{ij} C_{ij} - A_{ij} B_{ij} \frac{2k + 3}{2} \right) \sin \theta d\theta d\varphi
\]

\[
= 4 \int_0^1 (1 - r^2)^3 r^{2k+1} P_{n_1} P_{n_2} dr^2 \cdot \int_0^{2\pi} \int_0^{\pi} \left( -k(k + 1) Y_{kl}^2 + Y_{kl,\theta} Y_{kl,\theta} + \frac{1}{\sin^2 \theta} Y_{kl,\varphi} Y_{kl,\varphi} \right) \sin \theta d\theta d\varphi.
\]

However in paper [17] it is proved, that

\[
\int_0^{2\pi} \int_0^{\pi} \left( -k(k + 1) Y_{kl}^2 + Y_{kl,\theta} Y_{kl,\theta} + \frac{1}{\sin^2 \theta} Y_{kl,\varphi} Y_{kl,\varphi} \right) \sin \theta d\theta d\varphi = 0.
\]

In other words, the third term in the sum is equal to 0.

Note that the integrands in the integrals under consideration are linearly dependent. Namely, for all \(i, j = 1, 2, 3\), we have

\[
B_{ij}^2 (2k + 1) - A_{ij} B_{ij} \frac{(2k + 3)(2k + 1)}{2} - A_{ij} C_{ij} \frac{(2k + 3)(2k + 1)}{2} - 2 C_{ij} B_{ij} C_{ij} (2k - 1)
\]

\[
= (2k + 1) \left( B_{ij}^2 - 2 A_{ij} C_{ij} - A_{ij} B_{ij} \frac{2k + 3}{2} \right) - 2 \left( C_{ij}^2 + A_{ij} C_{ij} (k^2 - 1/4) - B_{ij} C_{ij} (k - 1/2) \right).
\]

Therefore, in order to prove the orthogonality, it is sufficient to show that

\[
\int_0^{2\pi} \int_0^{\pi} \sum_{i,j=1}^{3} \left( C_{ij}^2 + A_{ij} C_{ij} (k^2 - 1/4) - B_{ij} C_{ij} (k - 1/2) \right) \sin \theta d\theta d\varphi = 0.
\]

The equality is proved by the usage of definition (4) of \(Y_{kl}\) and the properties of the associated Legendre polynomials \(L_{kl}\).

**The second step.** We consider the case when either \(l_1 \neq l_2\) or \(k_1 \neq k_2\) and \(n_1, n_2\) are arbitrary.

**Lemma 2.** The values of the basis fields components can be calculated by the formulas

\[
(T_{\kappa\kappa})_{ij}(r, \theta, \varphi) = A_{ij} r^{k+2} F_n^\alpha + B_{ij} r^k F_n^\alpha + C_{ij} r^{k-2} F_n, \ i, j = 1, 2, 3,
\]

where \(A_{ij}, B_{ij}, C_{ij}\) are defined in Lemma 1 and \(F_n = (1 - r^2)^2 F_n^{(k+3,5,k+1.5)}(r^2)\).
This fact can be proved by rearranging the terms in the representation from Lemma 1. The proof of the orthogonality of basis fields in this case consists in using in Lemma 2, direct calculation on the integrals by the usage of the orthogonality of the spherical functions $Y_{kl}$.

**The third step.** We need to calculate the norm of the basis fields:

$$\|T_{kln}\|^2 = \int_0^1 \int_0^{2\pi} \int_0^\pi \left( \sum_{i,j=1}^3 (T_{kln})_{ij}^2 (r, \theta, \varphi) \right) r^2 \sin \theta \ d\theta \ d\varphi \ dr.$$ 

Using Lemma 1 and the formulas

$$\|P_n\|^2 = \int_0^1 r^{2k+1}(1-r^2)^2 P_n P_n \ dr^2,$$
$$\|P'_n\|^2 = \int_0^1 r^{2k+3}(1-r^2)^3 P'_n P'_n \ dr^2,$$
$$\|P''_n\|^2 = \int_0^1 r^{2k+5}(1-r^2)^4 P''_n P''_n \ dr^2,$$
$$\|Y_{kl}\|^2 = \int_0^{2\pi} \int_0^\pi Y_{kl} Y_{kl} \sin \theta \ d\theta \ d\varphi,$$

we come to

$$\|T_{kln}\|^2 = 16(k + 1.5)(k + 2.5)\|P_n\|^2\|Y_{kl}\|^2 + 32(k + 2.5)\|P'_n\|^2\|Y_{kl}\|^2 + 8\|P''_n\|^2\|Y_{kl}\|^2.$$ 

Further, we calculate the norms of the Jakobi polynomials and the norms of their derivatives, using formulas (1)–(3), and obtain

$$\|T_{kln}\|^2 = \frac{8((n + 2)!)^2(\Gamma(k + 1.5))^2\|Y_{kl}\|^2}{(\Gamma(n + k + 1.5))^2(k + 2n + 3.5)}.$$ 

In other words, we have proved, that

$$\|\tilde{T}_{kln}\|^2 = 1.$$ 

Theorem 1 is proven.

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**References**

[1] Sharafutdinov V A 1994 *Integral Geometry of Tensor Fields* (Utrecht: VSP)

[2] Svetov I E, Derevtsov E Yu, Volkov Yu S and Schuster T 2014 A numerical solver based on $B$-splines for 2D vector field tomography in a refracting medium *Math. Comput. Simul.* 97 207–23

[3] Svetov I E 2014 Properties of the Ray Transforms of Two-Dimensional 2-Tensor Fields Defined in the Unit Disk *J. Appl. Industr. Math.* 8 106–14

[4] Derevtsov E Yu and Svetov I E 2015 Tomography of tensor fields in the plain *Eurasian J. Math. Comput. Appl.* 3 2 24–68

[5] Vertgeim L 2000 Integral geometry problems for symmetric tensor fields with incomplete data *J. Inverse Ill-posed Problems* 8 353–62

[6] Sharafutdinov V 2007 Slice-by-slice reconstruction algorithm for vector tomography with incomplete data *Inverse Problems* 23 2003–27

[7] Defrise M and Gullberg G T 2005 3D reconstruction of tensors and vectors (Technical Report No. LBNL–54936) (Berkeley: LBNL)

[8] Davison M E 1981 A singular value decomposition for the Radon transform in n-dimensional Euclidean space *Numer. Funct. Anal. Optimization* 3 321–40

[9] Louis A K 1984 Orthogonal function series expansions and the null space of the Radon transform *SIAM J. Math. Anal.* 15 621–33
[10] Quinto E T 1985 Singular value decomposition and inversion methods for the exterior Radon transform and a spherical transform J. Math. Anal. Appl. 95 437–48
[11] Maass P 1987 The X-ray transform: Singular value decomposition and resolution Inverse problems 3 727–41
[12] Derevtsov E Yu, Efimov A V, Louis A K and Schuster T 2011 Singular value decomposition and its application to numerical inversion for ray transforms in 2D vector tomography J. Inverse Ill-posed Problems 19 689–715
[13] Kazantsev S G and Bukhgeim A A 2004 Singular value decomposition for the 2D fan-beam Radon transform of tensor fields J. Inverse Ill-posed Problems 12 245–78
[14] Derevtsov E Yu and Polyakova A P 2014 Solution of the integral geometry problem for 2-tensor fields by the singular value decomposition method J. Math. Sci. 202 50–71
[15] Derevtsov E Yu, Kazantsev S G and Schuster T 2007 Polynomial bases for subspaces of vector fields in the unit ball. Method of ridge functions J. Inverse Ill-posed Problems 15 19–55
[16] Polyakova A 2013 Reconstruction of potential part of 3D vector field by using singular value decomposition Journal of Physics: Conference Series 410 012015
[17] Polyakova A P 2015 Reconstruction of a vector field in a ball from its normal radon transform J. Math. Sci. 205 418–39
[18] Svetov I E and Polyakova A P 2013 Comparison of two algorithms for the numerical solution of the two-dimensional vector tomography Siberian Electron. Math. Rep. 10 90–108 (in Russian)
[19] Svetov I E and Polyakova A P 2015 Approximate solution of two-dimensional 2-tensor tomography problem using truncated singular value decomposition Siberian Electron. Math. Rep. 12 480–99 (in Russian)
[20] Polyakova A P and Svetov I E 2015 Numerical solution of the problem of reconstructing a potential vector field in the unit ball from its normal Radon transform J. Appl. Industr. Math. 9 547–58
[21] Polyakova A P and Svetov I E 2016 Numerical solution of reconstruction problem of a potential symmetric 2-tensor field in a ball from its normal Radon transform Siberian Electron. Math. Rep. 13 154–74 (in Russian)