Physicists’ $d = 3 + 1, \, N = 1$ superspace-time and supersymmetric QFTs from a tower construction in complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry and a purge-evaluation/index-contracting map

Chien-Hao Liu and Shing-Tung Yau

Abstract

The complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry aspect of a superspace(-time) $\hat{X}$ in Sec. 1 of D(14.1) [arXiv:1808.05011 [math.DG]) together with the Spin-Statistics Theorem in Quantum Field Theory, which requires fermionic components of a superfield be anticommuting, lead us to the notion of towered superspace(-time) $\hat{X}^{\oplus}$ and the built-in purely even physics sector $X^{\text{physics}}$ from $\hat{X}^{\oplus}$. We use this to reproduce the $d = 3 + 1, \, N = 1$ Wess-Zumino model and the $d = 3 + 1, \, N = 1$ supersymmetric $U(1)$ gauge theory with matter — as in, e.g., Chap. V and Chap. VI & part of Chap. VII of the classical Supersymmetry & Supergravity textbook by Julius Wess and Jonathan Bagger — and, hence, recast physicists’ two most basic supersymmetric quantum field theories solidly into the realm of (complexified $\mathbb{Z}/2$-graded) $C^\infty$-Algebraic Geometry. Some traditional differential geometers’ ways of understanding supersymmetric quantum field theories are incorporated into the notion of a purge-evaluation/index-contracting map $P : C^\infty(X^{\text{physics}}) \to C^\infty(\hat{X})$ in the setting. This completes for the current case a $C^\infty$-Algebraic Geometry language we sought for in D(14.1), footnote 2, that can directly link to the study of supersymmetry in particle physics. Once generalized to the nonabelian case in all dimensions and extended $N \geq 2$, this prepares us for a fundamental (as opposed to solitonic) description of super D-branes parallel to Ramond-Neveu-Schwarz fundamental superstrings

Key words: supersymmetry, Spin-Statistics Theorem; superspace, complexified super $C^\infty$-scheme; tower construction, towered superspace, purge-evaluation map; chiral superspace, antichiral superspace, Wess-Zumino model; principal sheaf, vector superfield, Wess-Zumino gauge, supersymmetry in Wess-Zumino gauge, supersymmetric $U(1)$ gauge theory with matter; super D-brane

MSC number 2010: 58A50, 81T60; 14A22, 17A70, 46L87, 81Q60, 81T30

Acknowledgements. We thank Andrew Strominger and Cumrun Vafa for influence to our understanding of strings, branes, and gravity. C.-H.L. thanks in addition Girma Hailu for his topic course on supersymmetry and discussions, fall 2018, that lead to the current work (cf. footnote 1 Special acknowledgments); Cumrun Vafa for consultation; Denis Auroux, Ashvin Vishwanath, Goran Radanovic/Haifeng Xu/Brian Plancher for other inspiring topic courses, fall 2018; Francesca Pei-Jung Chen for the recording of explanation/demonstration of J.S. Bach’s sonata that accompanies the typing of the current work; Jing Gu, Jia-Chi Lee for sharing their life stories; Ling-Miao Chou for comments that improve the illustration and moral support. The project is supported by NSF grants DMS-9803347 and DMS-0074329.
Chien-Hao Liu dedicates this note
to the loving memory of
Rev. & Mrs. R. Campbell Willman (1925-2014) and Barbara M. Willman (1927-1999).

(From C.H.L.) There are memories that are too abundant to condense and too cherished and personal to reveal. A seemingly accidental encounter in my teenage years, which turned out to have profound impact on me. The motherly love to me from Mrs. Willman and the friendship this family, including Ann and Lisa, had provided me with turned a rebellious, unresting teenager from an almost high-school dropout to a researcher. The fact that Rev. Willman, graduated from University of California at Berkeley with a degree in biology, gave up a more earthly pursuit to answer what he regarded as a higher call made a great impact on my mind. There is not a word with which I can express my gratitude to this family just like there is not a word I can use to express my gratitude to my own parents and family. The current work has equal weight to D-project as D(14.1) and, together, they completed our first step toward dynamical supersymmetric D-branes along the line of Ramond-Neveu-Schwarz superstring. It is thus dedicated to this family.
0. Introduction and outline

In Section 1 of [L-Y1] (D(14.1), arXiv:1808.05011 [math.DG]) a $d = 3 + 1$, $N = 1$ superspace-time $\hat{X} = (X, \hat{O}_X)$ was constructed from the aspect of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry. In this work, we extend the construction ibidem so that both the nature of a superspace-time from complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry and the matching of spin and statistics required of fields from Quantum Field Theory can be consistently taken into account and harmoniously built into the construction. While, as complexified super $C^\infty$-schemes, the towered superspace $\hat{X} \boxtimes$ fibers over $\hat{X}$ in [L-Y1] (D(14.1)) with a canonical section and a canonical flat connection, it has a new sector, i.e. the physics sector $X^{\text{physics}}$, whose structure sheaf $\mathcal{O}_X^{\text{physics}}$ comes from superfields in the sense of physicists, e.g., Abdus Salam & John Strathdee [S-S] (1978), and Julius Wess & Jonathan Bagger [W-B] (1992), polished to take into account the unavoidable mathematical consequences of nilpotency of component fields once the spinorial component fields are assumed to be anticommuting and hence nilpotent. As a locally-ringed space, $X^{\text{physics}}$ turns out to be a complexified $C^\infty$-scheme in its own right; in particular, it is purely even. For this reason, the parity of many physically relevant objects on the towered superspace $\hat{X} \boxtimes$ become purely even. In such situations, the sign-factor issues one has to deal with in [L-Y1] (D(14.1)) due to the $\mathbb{Z}/2$-grading are all gone and classical formulae in physics literature on supersymmetry become valid when suitably interpreted via a purge-evaluation map from $\mathcal{O}_X^{\text{physics}}$ to $\hat{O}_X$. To demonstrate the validity of above tower construction from complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry to describe particle physicists’ notion of superspaces and supersymmetric quantum field theories, the notion of $d = 3 + 1$, $N = 1$ chiral superfields and antichiral superfields and the construction of $d = 3 + 1$, $N = 1$ supersymmetric chiral matter theory and $U(1)$ gauge theory in Chap. V and Chap. VI of the standard textbook [W-B] of Wess & Bagger are re-done on $\hat{X} \boxtimes$. This completes for the current case a $C^\infty$-Algebraic Geometry language we sought for in [L-Y1; footnote 2] (D(14.1)) that can directly link to the study of supersymmetry and supersymmetric quantum field theory in particle physics.

Once generalized to the nonabelian case in all dimensions with extended $N \geq 2$ supersymmetry, this prepares us for a fundamental (as opposed to solitonic) description of super D-branes parallel to Ramond-Neveu-Schwarz fundamental superstrings

Convention. Same as [L-Y1] (D(14.1)). In particular, references for standard notations, terminology, operations and facts are (1) algebraic geometry: [Hart]; $C^\infty$-algebraic geometry: [Joy]; (2) spinors and supersymmetry (mathematical aspect): [Ch], [De], [D-F1], [D-F2], [Fr], [Harv], [S-W]; (3) supersymmetry (physical aspect, especially $d = 4$, $N = 1$ case): [W-B], [G-G-R-S], [We]; also [Argu], [Argy], [Bi], [St], [S-S].

---

Special acknowledgements from C.H.L. It’s a pure coincidence, and a God-given luck to us, that in fall semester 2018 Girma Hailu gave once again the topic course Physics 253cr Quantum Field Theory III on supersymmetry [Hai] at Department of Physics, Harvard University, just after we had completed [L-Y1] (D(14.1)) in August. This is the second time I sat in his topic courses on supersymmetry, (the first time in fall semester 2013). With half of the story (i.e. the complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry side) more solidly in mind as the reference point, we have some slight intellectual leisure to pay more attention to what we had still missed from the physics side. It is his lectures on supersymmetry, two after-class long discussions, and one email communication that propelled us to re-examine what we had taken as the foundation, i.e. superspaces, for the construction of a theory of fermionic D-branes along the line of the Ramond-Neveu-Schwarz superstrings and, in the end, led to the tower construction of superspaces in this work, which upgrades and simplifies [L-Y1] (D(14.1)) tremendously. Despite his generosity to stay behind the scene, we regard him as a hidden coauthor who nourished the current work. Sec. 1.2 is particularly attributed to him.
Outline

0. Introduction
1. The $d = 3 + 1, N = 1$ towered superspace-time and its physics sector
   1.1 Supermanifolds in the sense of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry
   1.2 What should the function ring of a superspace be: naturality from $C^\infty$-Algebraic Geometry aspect vs. naturality from Quantum Field Theory aspect
   1.3 The $d = 3 + 1, N = 1$ towered superspace-time $\hat{X}^\oplus_N$
   1.4 The physics sector $X^{\text{physics}}$ of $\hat{X}^\oplus_N$
   1.5 Purge-evaluation maps and the Fundamental Theorem on supersymmetric action functionals
2. The chiral/antichiral theory on $X^{\text{physics}}$ and Wess-Zumino model
   2.1 More on the chiral and the antichiral sector of $X^{\text{physics}}$
   2.2 Wess-Zumino model on $X$ in terms of $X^{\text{physics}}$
3. Supersymmetric $U(1)$ gauge theory with matter on $X$ in terms of $X^{\text{physics}}$
   3.1 The bundle/sheaf context underlying a supersymmetric $U(1)$ gauge theory with matter built from $X^{\text{physics}}$
   3.2 Pre-vector superfields in Wess-Zumino gauge
   3.3 Supersymmetry transformations of a pre-vector superfield in Wess-Zumino gauge
   3.4 From pre-vector superfields to vector superfields
   3.5 Supersymmetric $U(1)$ gauge theory with matter on $X$ in terms of $X^{\text{physics}}$

Appendix Notations, conventions, and identities in spinor calculus
1 The $d = 3 + 1, N = 1$ towered superspace-time and its physics sector

1.1 Supermanifolds in the sense of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry

The notion of supermanifolds in the sense of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry and a few basic objects we need are recalled in this subsection for the introduction of terminology and notations. Details are referred to [L-Y1: Sec. 1] D(14.1).

Definition 1.1.1. [supermanifold/superscheme] Given a (real smooth) manifold (in general, $C^\infty$-scheme) $M$, denote its structure sheaf of smooth functions by $O_M$ and its complexification $O^C_M := O_M \otimes_\mathbb{R} \mathbb{C}$ (i.e. the sheaf of complex-valued smooth functions on $M$). For any $O^C_M$-module $F$ of finite rank, one can construct a complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme (i.e. supermanifold)

$$\hat{M} := (M, \hat{O}_M)$$

from $F$ by taking the new structure sheaf on $M$ to be the exterior $O^C_M$-algebra generated by $F$:

$$\hat{O}_M := \bigwedge^\bullet O^C_M F.$$ 

This a locally-ringed space with the underlying topology $M$. We shall call $F$ the generating sheaf of the supermanifold. This is a $\mathbb{Z}/2$-graded $O^C_M$-algebra with the even part and the odd part given respectively by

$$\hat{O}^\text{even}_M := \bigwedge^\bullet O^C_M F, \quad \hat{O}^\text{odd}_M := \bigwedge^\bullet O^C_M F.$$ 

The $C^\infty$-hull of $\hat{O}_M$ is given by

$$C^\infty\text{-hull}(\hat{O}_M) := O_M \oplus \bigwedge^\bullet O^C_M \geq 2 \subset \hat{O}_M.$$ 

$C^\infty\text{-hull}(\hat{O}_M)$ is a sheaf of $C^\infty$-rings. This, by definition, gives a partial $C^\infty$-ring structure on $\hat{O}_M$. The set $\Gamma(\bullet)$ of global sections of these structure sheaves $\langle \bullet \rangle$ on $M$ are denoted by

$$C^\infty(M), \quad C^\infty(M)^C, \quad C^\infty(\hat{X}), \quad C^\infty\text{-hull}(C^\infty(\hat{X}))$$

respectively. Each carries a corresponding complexified, $\mathbb{Z}/2$-graded, $C^\infty$-or-partial-$C^\infty$-algebraic (whichever applicable) structure.

Definition 1.1.2. [(left) derivation of $C^\infty(\hat{M})$] (Cf. [L-Y1: Definition 1.3.2, footnote 7] (D(14.1)).) A (left) derivation of $C^\infty(\hat{M})$ over $\mathbb{C}$ is a $\mathbb{Z}/2$-graded $\mathbb{C}$-linear operation

$$\xi : C^\infty(\hat{M}) \rightarrow C^\infty(\hat{M})$$

on $C^\infty(\hat{M})$ that satisfies the $\mathbb{Z}/2$-graded Leibniz rule

$$\xi(fg) = (\xi f)g + (-1)^p(\xi)p(f)f(\xi g)$$

when in parity-homogeneous situations. The set $\text{Der}_C(\hat{M}) := \text{Der}_C(C^\infty(\hat{M}))$ of derivations of $C^\infty(\hat{M})$ is a (left) $C^\infty(\hat{M})$-module, with $(a\xi)(\cdot) := a(\xi(\cdot))$ and $p(a\xi) := p(a) + p(\xi)$ for $a \in C^\infty(\hat{M})$ and $\xi \in \text{Der}_C(\hat{M})$. 

3
The element of $\hat{\Omega}$ can convert $a, b$ for objects $c, h, d$

Lemma 1.1.5. [evaluation of $\Omega$ Eq.’s (12.2), (12.3) of Wess & Bagger.]

Convention d (D(14.1)). We treat elements $f$ for footnote 11

The specification

Footnote 11 (D(14.1).)) The specification

Definition 1.1.3. [differential of $C^\infty(\hat{M})$] (Cf. [L-Y1: Definition 1.3.6, footnote 10] (D(14.1)).) The bi-$C^\infty(\hat{M})$-module $\hat{\Omega} := \Omega_{C^\infty(\hat{M})}$ of differentials of $C^\infty(\hat{M})$ over $C$ is the quotient of the free bi-$C^\infty(\hat{M})$-module generated by $d(f), f \in C^\infty(\hat{M})$, by the bi-$C^\infty(\hat{M})$-submodule of relators generated by

1. [C-linearity] $d(c_1 f_1 + c_2 f_2) - c_1 d(f_1) - c_2 d(f_2), \text{ for } c_1, c_2 \in C, f_1, f_2 \in C^\infty(\hat{M})$;
2. [Leibniz rule] $d(f_1 f_2) - (d(f_1)) f_2 - f_1 d(f_2), \text{ for } f_1, f_2 \in C^\infty(\hat{M})$;
3. [chain-rule identities from the $C^\infty$-hull structure]

$$d(h(f_1, \cdots, f_l)) - \sum_{k=1}^{l} (\partial_k h)(f_1, \cdots, f_l) d(f_k)$$

for $h \in C^\infty(\mathbb{R}^l), f_1, \cdots, f_l \in C^\infty(\hat{M}) \subset C^\infty(\hat{M})$; here, $\partial_k h$ is the partial derivative of $h \in C^\infty(\mathbb{R}^l)$ with respect to the $k$-th argument.

The element of $\hat{\Omega}$ associated to $d(f), f \in C^\infty(\hat{M})$, is denoted by $df$. Using Relators (2), one can convert $\hat{\Omega}$ to either solely a left $C^\infty(\hat{M})$-module or solely a right $C^\infty(\hat{M})$-module.

A differential of $C^\infty(\hat{M})$ is also called synonymously a 1-form on $\hat{M}$.

By construction, there is a built-in map $d : C^\infty(\hat{M}) \to \hat{\Omega}$ defined by $f \mapsto df$.

Convention 1.1.4. [cohomological degree vs. parity] (Cf. [L-Y1: Convention 1.3.5, footnote 9] (D(14.1)).) We treat elements $f$ of $C^\infty(\hat{M})$ as of cohomological degree 0 and the exterior differential operator $d$ as of cohomological degree 1 and even. In notation, $c.h.d(f) = 0$ and $c.h.d(d) = 1, p(d) = 0$. Under such $(\mathbb{Z} \times (\mathbb{Z}/2))$-bi-grading,

$$ab = (-1)^{c.h.d(a) \cdot c.h.d(b)} (-1)^{p(a) \cdot p(b)} ba$$

for objects $a, b$ homogeneous with respect to the bi-grading. Here, $a$ and $b$ are not necessarily of the same type. This is the convention that matches with the sign rules in [W-B: Chap. XII, Eq.’s (12.2), (12.3)] of Wess & Bagger.

Lemma 1.1.5. [evaluation of $\hat{\Omega}$ on $\text{Der}_C(\hat{M})$ from the right] (Cf. [L-Y1: Lemma 1.3.7, footnote 11] (D(14.1)).) The specification

$$(df)(\xi) := (\xi)^*(df) := \xi(f)$$

for $f \in C^\infty(\hat{M})$ and $\xi \in \text{Der}_C(\hat{M})$, defines an evaluation of $\hat{\Omega}$ on $\text{Der}_C(\hat{M})$ from the right: for $\varpi = \sum_{i=1}^{k} a_i df_i \in \hat{\Omega}$, with $a_i$ parity-homogeneous, and $\xi \in \text{Der}_C(\hat{M})$ parity-homogeneous,

$$\varpi(\xi) := (\xi)^* \varpi := \sum_{i=1}^{k} (-1)^{p(\xi) p(a_i)} a_i \xi(f_i).$$

This evaluation is (left) $C^\infty(\hat{M})$-linear: $\varpi(a \xi) = a \varpi(\xi)$, for $a \in C^\infty(\hat{M})$.

Higher tensors, in particular $k$-forms, on $\hat{M}$ can also be defined. See [L-Y1: Sec. 1.3] (D(14.1)) for more details on the differential calculus on $\hat{M}$.
1.2 What should the function-ring of a superspace be: naturality from $C^\infty$-Algebraic Geometry aspect vs. naturality from Quantum Field Theory aspect

Given the 4-dimensional Minkowski space-time $X = \mathbb{R}^{3+1}$. Let

- $P$ be the Lorentzian frame bundle over $X$ with the (flat) Levi-Civita connection,
- $S_C = S' \oplus S''$ be the complexified Dirac-spinor bundle from the spinor representation of the Lorentz group, $S_C^{\vee} = S'^{\vee} \oplus S''^{\vee}$ be the dual of $S_C$. The corresponding sheaves are denoted by $S_C$, $S'$, $S''$ and $S_C^{\vee}$, $S'^{\vee}$, $S''^{\vee}$ respectively. These spinor bundles and sheaves are all equipped with a flat connection induced from that on $P$.

A ‘$d = 3+1, N = 1$ superspace’ is meant to be a supermanifold/superscheme in the sense of Definition 1.1.1 with the underlying topology $M = X$ and the generating sheaf $\mathcal{F}$ “coming from” one copy of $S_C$, or $S_C^{\vee}$ to provide “one set of fermionic/anticommuting coordinates $(\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2)$ on the superspace”. In this subsection, we shall re-examine the notion of ‘superspace’ based on this definition and guided by the question

Q. [guiding/key] What should the function ring of a $d = 3+1, N = 1$ superspace be?

with care not only from the complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry but also from the Quantum Field Theory.

Naturality from the complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry aspect

In [L-Y1: Sec. 1.2] (D(14.1)) a $d = 3+1, N = 1$ superspace $\hat{X}$ was constructed as a super $C^\infty$-scheme with complexification. There we took $M = X$, $\mathcal{F} = S_C^{\vee}$. A tuple of constant sections $(\theta, \bar{\theta}) := (\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2)$ from $S^{\vee} \oplus S''^{\vee}$ with respect the built-in flat connection was chosen to serve as the fermionic coordinate functions on the superspace $\hat{X}$. Together with the standard coordinate functions $x := (x^0, x^1, x^2, x^3)$ on $X$, they generate (in the sense of complexified $\mathbb{Z}/2$-graded $C^\infty$-ring ) the function ring

$$C^\infty(\hat{X}) = C^\infty(X)^C[\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2]^\text{anti-c}$$

of $\hat{X}$. Here $[\cdots]^\text{anti-c}$ means ‘polynomial ring (with coefficients in $C^\infty(X)^C$) in anticommuting variables · · ·’. Mathematically from pure algebra, whatever ring that contains both $C^\infty(X)^C$ and $\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2$ must contain $C^\infty(X)^C[\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2]^\text{anti-c}$, despite the fact that as $P$-modules, the corresponding structure sheaf contains not just Lorentz scalars:

$$\hat{O}_X := \bigwedge^\bullet_{\hat{X}}(S'^{\vee} \oplus S''^{\vee}) 
\approx O_{\hat{X}}^C \oplus (S'^{\vee} \oplus S''^{\vee}) \oplus (O_{\hat{X}}^C \oplus S'^{\vee} \otimes O_{\hat{X}}^C) (S''^{\vee} \otimes O_{\hat{X}}^C) \oplus (S'^{\vee} \oplus S''^{\vee}) \oplus O_{\hat{X}}^C.$$

\footnote{See [L-Y1: footnote 4 in Sec.1.2] for an explanation of $S_C$ vs. $S_C^{\vee}$. In ibidem, to avoid carrying the dual $^{\vee}$ everywhere and enough for the purpose there, we actually chose $\mathcal{F} = S_C$ instead of $S_C^{\vee}$ for the simplicity of notation. For the current work, such distinction matters and we resume what it should be; cf. [L-Y9] (D(11.4.1)) in Reference of ibidem. It turns out that the choice $\mathcal{F} = S_C^{\vee}$ also matches better the convention in [W-B: Appendix A] of Wess & Bagger since we write fermionic/anticommuting coordinates with upper spinor index $(\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2)$, rather than lower spinor index.}
Note also that an \( f \in C^\infty(\hat{X}) \) in the \((\theta, \bar{\theta})\)-expansion\(^3\)

\[
f = f_0 + \sum_\alpha \theta^\alpha f_\alpha + \sum_\beta \bar{\theta}^\beta f_{\bar{\beta}} + \theta^1 \theta^2 f_{(12)} + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta f_{(\alpha \beta)} + \bar{\theta}^1 \bar{\theta}^2 f_{(12 \bar{\beta})} + \sum_\beta \theta^1 \theta^2 \bar{\theta}^\beta f_{(12 \bar{\beta})} + \sum_\alpha \theta^\alpha \theta^1 \bar{\theta}^\beta f_{(\alpha \bar{\beta})}
\]

has all its coefficients \( f_\bullet \in C^\infty(X)^C \) and, hence, \textit{commuting}.

**Naturality from the Quantum Field Theory aspect**

For distinction, denote the would-be superspace by \(?\hat{X}\) and its function ring by \(C^\infty(?\hat{X})\). Then it follows from the discussion in the previous theme that

\[
C^\infty(?\hat{X}) \supset C^\infty(X)^C[\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2]_{\text{anti-c}}.
\]

If we insist that \(?\hat{X}\) be an \(N = 1\) superspace, i.e. the tuple \((\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2)\) remains to serve as a maximal tuple of fermionic coordinates on \(?\hat{X}\), then it is natural to assume that every \(\tilde{f} \in C^\infty(?\hat{X})\) remains to have an expansion in \((\theta, \bar{\theta})\):

\[
\tilde{f} = \tilde{f}_0 + \sum_\alpha \theta^\alpha \tilde{f}_\alpha + \sum_\beta \bar{\theta}^\beta \tilde{f}_{\bar{\beta}} + \theta^1 \theta^2 \tilde{f}_{(12)} + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \tilde{f}_{(\alpha \beta)} + \bar{\theta}^1 \bar{\theta}^2 \tilde{f}_{(12 \bar{\beta})} + \sum_\beta \theta^1 \theta^2 \bar{\theta}^\beta \tilde{f}_{(12 \bar{\beta})} + \sum_\alpha \theta^\alpha \theta^1 \bar{\theta}^\beta \tilde{f}_{(\alpha \bar{\beta})}.
\]

The question is now

**Q. [coefficients in \((\theta, \bar{\theta})\)-expansion]** If a coefficient \(\tilde{f}_\bullet\) of an \(\tilde{f} \in C^\infty(?\hat{X})\) does not lie in \(C^\infty(X)^C\), then in where should it lie?

In constructing \(\hat{X}\) as a complexified \(\mathbb{Z}/2\)-graded \(C^\infty\)-scheme in [L-Y1: Sec. 1.2], we have used all what mathematics can offer. The answer to the above question thus has to come from insights from physics.

From the Quantum Field Theory aspect, a physics-relevant element (i.e. \textit{physics superfield}) \(\tilde{f}\) in \(C^\infty(?\hat{X})\) must satisfy the following two basic requirements

1. [Lorentz scalar] \(\tilde{f}\) be a Lorentz scalar; i.e. \(\tilde{f}\) be a section of a trivial \(P\)-module over \(X\).
2. [Spin-Statistics Theorem] Bosons be commuting and fermions be anticommuting.

Since we take \(\theta^\alpha \in S^\vee\) and \(\bar{\theta}^\beta \in S''\vee\), Condition (1) suggests that

- \(\tilde{f}_\alpha \in S'\) and \(\tilde{f}_{\bar{\beta}} \in S''\); and \(\tilde{f}_{(\alpha \bar{\beta})}\) lie in some isomorphic copy of \(S', S''\) respectively.

Condition (2) then comes along to imply that

\(^3\)Here, since \(f_\bullet\) are commuting, one can write the coefficients either on the right or on the left. We choose to write them on the right in order to match better with the later setting when we allow anticommuting coefficients as well.
· The coefficients \( \tilde{f}_{(\alpha)} \), \( \tilde{f}_{(\alpha \beta)} \), \( \tilde{f}_{(12\beta)} \) in the \((\theta, \bar{\theta})\)-expansion of \( \tilde{f} \) are themselves anti-commuting. In other words, they take values on some Grassmann numbers as well. This renders \( \tilde{f} \) purely even(!). Furthermore, as the notation suggests, we require that

\[
\frac{\partial}{\partial \theta^\alpha} \tilde{f}(\bullet) = \frac{\partial}{\partial \bar{\theta}^\beta} \tilde{f}(\bullet) = 0
\]

for all component fields \( f(\bullet) \) in the expansion. In particular, while the fermionic component fields take values on Grassmann numbers, they are independent of the existing fermionic coordinates \((\theta, \bar{\theta})\).

Clearly, such an \( \tilde{f} \) cannot lie in \( C^\infty(\hat{X}) \): The sought-for \( C^\infty(\hat{X}) \) is definitely larger than \( C^\infty(\hat{X}) \) constructed.

Q. \([\mathbb{C}-\mathbb{Z}/2-C^\infty-AG + QFT=?] \) Can one take the naturality from Quantum Field Theory aspect also into account in the construction of a superspace?

(Cf. \([L-Y1: \text{footnote 2}] (D(14.1))\).)

In the next subsection, we will take \( \hat{X} \) as the foundation and the starting point — since it must be there mathematically — and extend over it to answer the above question affirmatively.

1.3 The \( d = 3 + 1, N = 1 \) towered superspace-time \( \hat{X}^{\oplus_l} \)

We now proceed in two steps to answer Question \([\mathbb{C}-\mathbb{Z}/2-C^\infty-AG + QFT=?] \) in Sec. 1.2. For the first step, physicists should be aware that when one tries to extend \( C^\infty(\hat{X}) \) to another complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-ring \( C^\infty(\hat{X}) \) by adding ‘something’, not only ‘something’ appears in the new ring but also all those objects from taking \( C^\infty \)-closure come in as well. No surprise that not all the elements in \( C^\infty(\hat{X}) \) have physical meaning, but their inclusion is a mathematical must. And, hence, we have to allow them. This is completed in this subsection. Then comes the second step: the identification of those elements in \( C^\infty(\hat{X}) \) that are physical and the justification that they are even and do form a complexified \( C^\infty \)-ring. This is completed in the next subsection.

The \( d = 4, N = 1 \) towered superspace \( \hat{X}^{\oplus_l} \) with \( l \) field-theory levels

Recall the Weyl-spinor sheaves \( S' \) and \( S'' \) and their dual \( S'^\vee, S''^\vee \) at the beginning of Sec. 1.2.

**Definition 1.3.1.** \([d = 4, N = 1 \text{ towered superspace } \hat{X}^{\oplus_l} \text{ with } l \text{ field-theory levels}] \) The complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-scheme

\[
\hat{X}^{\oplus_l} := (X, \hat{\mathcal{O}}^{\oplus_l}_X) := (X, \wedge^\bullet_{\mathcal{O}_X^F})
\]

with

\[
\mathcal{F} := (S'^{\vee}_{\text{coordinates}} \oplus S''^{\vee}_{\text{coordinates}}) \oplus (S'^{\vee}_{\text{parameter}} \oplus S''^{\vee}_{\text{parameter}}) \oplus \bigoplus_{i=1}^l (S'_{\text{field},i} \oplus S''_{\text{field},i})
\]

is called the \( d = 4, N = 1 \) towered superspace with \( l \) field-theory levels. Here, all \( S'_i \) (resp. \( S''_i, S'^{\vee}_i, S''^{\vee}_i \)) are copies of \( S' \) (resp. \( S'', S'^{\vee}, S''^{\vee} \)). When \( l \) is implicit in the problem, we will denote \( \hat{X}^{\oplus_l} \) simply by \( \hat{X}^{\oplus} \).

\(^4\)Mathematically this means that \( S'^{\vee}_{\text{coordinates}} \) is isomorphic to \( S'^{\vee} \) with a fixed isomorphism; and similarly for all other spinor sheaves that appear as direct summands of \( \mathcal{F} \).
Note that, as an $\mathcal{O}_X^\mathbb{C}$-modules,

$$
\hat{\mathcal{O}}_X = \bigwedge^{\bullet}_{\mathcal{O}_X^\mathbb{C}} \left( S'_\text{coordinates} \oplus S''_\text{coordinates} \right)
\oplus \mathcal{O}_X^\mathbb{C} \bigwedge^{\bullet}_{\mathcal{O}_X^\mathbb{C}} \left( S'_\text{parameter} \oplus S''_\text{parameter} \right) \otimes \mathcal{O}_X^\mathbb{C} \bigwedge_{i=1}^l \mathcal{O}_X^\mathbb{C} \left( S'_{\text{field},i} \oplus S''_{\text{field},i} \right).
$$

**Definition/Explanation 1.3.2. [levels of $\tilde{X}_{\mathbb{C}^4}$]**

1. The $d = 4$, $N = 1$ superspace $\tilde{X}$ constructed in [L-Y1: Definition 1.2.5] is given by

$$
\tilde{X} := (X, \hat{\mathcal{O}}_X := \bigwedge^{\bullet}(S'_\text{coordinates} \oplus S''_\text{coordinates}))
$$

in the current context; (cf. [L-Y1: footnote 4] (D(14.1))). Any collection of complex conjugate generating constant global sections, denoted collectively as $(\theta, \bar{\theta}) := (\theta^1, \theta^2; \bar{\theta}^1, \bar{\theta}^2)$, of $S'_\text{coordinates} \oplus S''_\text{coordinates}$ serves as the fermionic coordinate functions on $\tilde{X}$, ([L-Y1: Definition 1.2.5] (D(14.1))). This explains the subscript ‘coordinates’ in $S'_\text{coordinates} \oplus S''_\text{coordinates}$. $\tilde{X}$ is called the fundamental level or the ground level of $\tilde{X}_{\mathbb{C}^4}$.

2. When physicists working on supersymmetry introduce ‘Grassmann number’ parameter $(\eta, \bar{\eta}) := (\eta^1, \eta^2; \bar{\eta}^1, \bar{\eta}^2)$ in their computation, these ‘Grassmann number’ parameter are meant to be independent of anything else. Thus, they should be thought of as constant sections of another copy of $S'_\text{coordinates} \oplus S''_\text{coordinates}$. This explains the subscript ‘parameter’ in $S'_\text{parameter} \oplus S''_\text{parameter}$. We say that $S'_\text{parameter} \oplus S''_\text{parameter}$ contributes to the Grassmann parameter level of $\tilde{X}_{\mathbb{C}^4}$.

3. The fermionic coefficients of a physical superfield are themselves anticommuting and must correspond to sections of a copy of spinor sheaves. Physical superfields that are associated to different types/classes/generations of particles should be thought of as sections of Grassmann algebra generated by different copies of spinor sheaves. This explains the subscript ‘field’ in $S'_{\text{field},i} \oplus S''_{\text{field},i}$. We say that $S'_{\text{field},i} \oplus S''_{\text{field},i}$ contributes to the $i$-th field-theory level of $\tilde{X}_{\mathbb{C}^4}$. The total level number $l$ is the number of distinct types of (particle, its superpartner) in a $d = 3 + 1$, $N = 1$ supersymmetric field theory one wants to construct. It can be different theory by theory.

By construction, there is a commutative diagram of $\mathbb{Z}/2$-grading-preserving $\mathcal{O}_X^\mathbb{C}$-algebra-homomorphisms

$$
\begin{array}{ccc}
\hat{\mathcal{O}}_X^{\mathbb{C}} & \xrightarrow{i^*} & \hat{\mathcal{O}}_X \\
\bigwedge^{\bullet}_{\mathcal{O}_X^\mathbb{C}} & \xrightarrow{\pi^*} & \bigwedge^{\bullet}_{\mathcal{O}_X^\mathbb{C}} \\
\end{array}
$$

Algebraic-Geometry-oriented readers may find something uncomfortable here. In Algebraic Geometry one usually begins with a choice of a ground field (in the sense of algebra) $\mathbb{k}$, for example, $\mathbb{k} = \mathbb{C}$ when studying Calabi-Yau spaces in string theory. Naively, here since we are dealing with complexified $\mathbb{Z}/2$-graded geometry, one would choose $k = \mathbb{C}[\eta, \bar{\eta}]^{\text{anti-c.}}$ to begin with and all the rings are $k$-algebras. However, while this has nothing wrong mathematically, it is misleading from the perspective of the physics side. That has to do with how such Grassmann numbers/parameters are used in physics. Most often they are to be paired with supersymmetry generators. In such case, supersymmetry generators are realized as a spinor representation of Lorentz group; thus these parameters have to be realized as dual spinors. It is for this reason we choose $(\eta, e^a)$ from constant sections of (a copy of) $S'_\text{coordinates} \oplus S''_\text{coordinates}$. In particular, though one intends to think of $\mathbb{C}[\eta, \bar{\eta}]^{\text{anti-c.}}$ as the ground ring in the problem, these parameters themselves are not Lorentz scalars. As we will see, nor is the function ring $C^\infty(\tilde{X}_{\mathbb{C}^4})$ physics of the physics sector a $\mathbb{C}[\eta, \bar{\eta}]^{\text{anti-c.}}$-algebra since the former is purely even.

For a topologically twisted theory, the twisted supersymmetry generators become Lorentz scalars. To match with this, the Grassmann numbers/parameters become Lorentz scalars as well. Only in such cases, the mathematical notion of $\mathbb{C}[\eta, \bar{\eta}]$ as the ground ring and physicists’ use of $(\eta, \bar{\eta})$ match well.

This is why we take $X$ as the ground level and the Grassmann number/parameter level comes next over it, not the other round.
that respect the partial \( C^\infty \)-ring structures. This gives a commutative diagram of morphisms of complexified super \( C^\infty \)-schemes

\[
\begin{array}{ccc}
\hat{X} & \overset{i}{\to} & \hat{X}_l \\
\downarrow & & \downarrow \hat{\pi} \\
\hat{X} & = & \hat{X}
\end{array}
\]

The built-in flat connection on the \( S', S'' \) and their dual induces a flat connection on \( \hat{X}_l \) over \( \hat{X} \). This defines a canonical inclusion

\[
\text{Der}_C(\hat{X}) \hookrightarrow \text{Der}_C(\hat{X}_l).
\]

And any flow on \( \hat{X} \) lifts canonically to a flow on \( \hat{X}_l \).

**Definition 1.3.3.** (derivation on \( \hat{X} \) applied to \( C^\infty(\hat{X}_l) \)) Let \( \xi \in \text{Der}_C(\hat{X}) \) be a derivation on \( \hat{X} \) and \( \hat{f} \in C^\infty(\hat{X}_l) \). Then we define \( \xi \hat{f} \in C^\infty(\hat{X}_l) \) via the canonical inclusion \( \text{Der}_C(\hat{X}) \hookrightarrow \text{Der}_C(\hat{X}_l) \).

**Definition 1.3.4.** (complex conjugation vs. twisted complex conjugation) The complex conjugation \( \bar{\cdot} : \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\hat{X}} \) and \( S' \to S'' \), \( S'' \to S' \), of Weyl spinors extends canonically to a complex conjugation

\[
\bar{\cdot} : \hat{\mathcal{O}}_{\hat{X}} \to \hat{\mathcal{O}}_{\hat{X}},
\]

by setting

1. \( \bar{\hat{f} + \hat{g}} = \bar{\hat{f}} + \bar{\hat{g}} \);
2. \( \bar{\hat{f} \hat{g}} = \bar{\hat{g}} \bar{\hat{f}} \).

and a twisted complex conjugation

\[
\dagger : \hat{\mathcal{O}}_{\hat{X}} \to \hat{\mathcal{O}}_{\hat{X}},
\]

by setting

1. \( (0') \dagger = - : \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\hat{X}} \); \( S' \to S'' \), \( S'' \to S' \);
2. \( (1') \ (\hat{f} + \hat{g})\dagger = \hat{f}\dagger + \hat{g}\dagger \);
3. \( (2') \ (\hat{f}\hat{g})\dagger = \bar{\hat{g}}\dagger \bar{\hat{f}}\dagger \).

Caution that the order of multiplication is preserved under the complex conjugate \( \bar{\cdot} \) but is reversed under the twisted complex conjugate \( \dagger \).

**Definition 1.3.5.** (standard coordinate functions on \( \hat{X}_l \)) The standard coordinate functions \((x, \theta, \bar{\theta})\) on \( \hat{X} \) extends uniquely to a tuple of coordinate functions

\[
(x^\mu, \theta^\alpha, \bar{\theta}^\beta; \eta^\alpha, \bar{\eta}^\beta; g^1_{\gamma_1}, \bar{g}^1_{\delta_1}; \ldots; g^l_{\gamma_l}, \bar{g}^l_{\delta_l}) =: (x, \theta, \bar{\theta}, \eta, \bar{\eta}, \vartheta, \bar{\vartheta})
\]
on $\hat{X}_{\hat{\bar{b}}}$ via the $\varepsilon$-tensor $\varepsilon : S' \otimes_{\hat{O}_{\hat{X}}} S' \to \hat{O}_{\hat{X}}$, $S'' \otimes_{\hat{O}_{\hat{X}}} S'' \to \hat{O}_{\hat{X}}$, and the fixed isomorphisms $S' \simeq S'$, $S'' \simeq S''$.

Explicitly, regard $S'_{\text{parameter}}$ as a copy of $S'_{\text{coordinates}}$, $S''_{\text{parameter}}$ as a copy of $S''_{\text{coordinates}}$, $S'_{\text{field},i}$ as a copy of $(S'_{\text{coordinates}})^{\dagger} = S'_{\text{coordinates}}$, and $S''_{\text{field},i}$ as a copy of $(S''_{\text{coordinates}})^{\dagger} = S''_{\text{coordinates}}$ under the fixed isomorphisms. Then, $(\eta^{\beta}, \bar{\eta}^{\beta}) = (\theta^{\alpha}, \bar{\theta}^{\beta})$ and $(\vartheta_{\alpha}, \bar{\vartheta}_{\beta}) = (\theta_{\alpha}, \bar{\theta}_{\beta})$ for all $i$, where $\theta_{\alpha} = \sum_{\gamma} \varepsilon_{\alpha\gamma} \theta^{\gamma}, \bar{\theta}_{\beta} = \sum_{\delta} \varepsilon_{\delta\beta} \bar{\theta}^{\delta}$.

We shall call $(x, \theta, \bar{\theta}, \eta, \bar{\eta}, \vartheta, \bar{\vartheta})$ the standard coordinate functions on $\hat{X}_{\hat{\bar{b}}}$.

In terms of this, $C^{\infty}(\hat{X}_{\hat{\bar{b}}}) = C^{\infty}(X)^{\mathbb{C}}[-\theta, \bar{\theta}, \eta, \bar{\eta}, \vartheta, \bar{\vartheta}]^{\text{anti-c}}$

and an $\tilde{f} \in C^{\infty}(\hat{X}_{\hat{\bar{b}}})$ has a $(\theta, \bar{\theta})$-expansion

\[
\tilde{f} = \tilde{f}^{(0)} + \sum_{\alpha} \theta^{\alpha} \tilde{f}_{(\alpha)} + \sum_{\beta} \bar{\theta}^{\beta} \tilde{f}^{(\beta)} + \theta^{1} \theta^{2} \tilde{f}^{(12)} + \sum_{\alpha, \beta} \theta^{\alpha} \bar{\theta}^{\beta} \tilde{f}^{(\alpha\beta)} + \bar{\theta}^{1} \bar{\theta}^{2} \tilde{f}^{(12)} + \sum_{\beta} \theta^{1} \bar{\theta}^{2} \tilde{f}^{(12\beta)} + \sum_{\alpha} \theta^{\alpha} \theta^{1} \theta^{2} \tilde{f}^{(12\alpha)} + \theta^{1} \theta^{2} \theta^{1} \theta^{2} \tilde{f}^{(1212)}
\]

with coefficients $\tilde{f}^{(\bullet)} \in C^{\infty}(X)^{\mathbb{C}}[\eta, \bar{\eta}, \vartheta, \bar{\vartheta}]^{\text{anti-c}}$.

The chiral, the antichiral, and the self-twisted-conjugate sector of $\hat{X}_{\hat{\bar{b}}}$

Recall from [L-Y1: Sec.1.4] (D(14.1)) the standard infinitesimal supersymmetry generators

\[
Q_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} - \sqrt{-1} \sum_{\mu=0}^{3} \sum_{\beta=1}^{2} \sigma_{\alpha\beta}^{\mu} \frac{\partial}{\partial x^{\mu}} \quad \text{and} \quad \bar{Q}_{\beta} = -\frac{\partial}{\partial \bar{\theta}^{\beta}} + \sqrt{-1} \sum_{\mu=0}^{3} \sum_{\alpha=1}^{2} \theta^{\alpha} \sigma_{\alpha\beta}^{\mu} \frac{\partial}{\partial x^{\mu}}
\]

and derivations that are invariant under the flow that generate supersymmetries

\[
e_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} + \sqrt{-1} \sum_{\mu=0}^{3} \sum_{\beta=1}^{2} \sigma_{\alpha\beta}^{\mu} \frac{\partial}{\partial x^{\mu}} \quad \text{and} \quad e_{\beta} = -\frac{\partial}{\partial \bar{\theta}^{\beta}} - \sqrt{-1} \sum_{\mu=0}^{3} \sum_{\alpha=1}^{2} \theta^{\alpha} \sigma_{\alpha\beta}^{\mu} \frac{\partial}{\partial x^{\mu}}.
\]

Since

\[
\xi \eta^{\alpha} = \xi \bar{\eta}^{\beta} = \xi \vartheta_{\gamma} = \xi \bar{\vartheta}_{\delta} = 0,
\]

for all $\xi \in \text{Der}_C(\hat{X})$, the notion of chiral functions, antichiral functions, properties of chiral function ring, and antichiral function ring studied in [L-Y1: Sec.1.4] (D(14.1)) for $C^{\infty}(\hat{X})$ can be generalized immediately to parallel notions/objects for $C^{\infty}(\hat{X}_{\hat{\bar{b}}})$ with $f^{(\bullet)}$ ibidem replaced by $\tilde{f}^{(\bullet)}$ in the $(\theta, \bar{\theta})$-expansion for $f \in C^{\infty}(\hat{X}_{\hat{\bar{b}}})$.

**Definition/Lemma 1.3.6.** [chiral sector $\hat{\hat{X}}_{\hat{\hat{b}}}, \text{ch}$ of $\hat{X}_{\hat{\bar{b}}}$] (1) $\tilde{f} \in C^{\infty}(\hat{X}_{\hat{\bar{b}}})$ is called chiral if $e_{1}\tilde{f} = e_{2}\tilde{f} = 0$. The set of chiral functions on $\hat{X}_{\hat{\bar{b}}}$ is a $\mathbb{C}$-subalgebra of $C^{\infty}(\hat{X}_{\hat{\bar{b}}})$, called the chiral function-ring of $\hat{X}_{\hat{\bar{b}}}$, denoted by $C^{\infty}(\hat{X}_{\hat{\bar{b}}})^{\text{ch}}$.

(2) Replacing $\hat{X}$ by $\hat{U}$ for $U \subset X$ open and $e_{1\nu}, e_{2\nu}$ by $e_{1\nu}|\hat{U}$, $e_{2\nu}|\hat{U}$ in Item (1), one obtains the sheaf of chiral functions $\hat{O}_{\hat{X}_{\hat{\bar{b}}}}^{\text{ch}} \subset \hat{O}_{\hat{X}}$, also called the chiral structure sheaf of $\hat{X}_{\hat{\bar{b}}}$; Denote the locally-ringed space $(X, O_{\hat{X}}^{\hat{\hat{b}}, \text{ch}})$ by $\hat{X}_{\hat{\hat{b}}}, \text{ch}$.
(3) The $C^\infty$-hull of $C^\infty(\hat{X}^{\text{ach}})$ restricts to the $C^\infty$-hull of $C^\infty(\hat{X}^{\text{ach}})^{\text{ch}}$ and similarly for the localized version. This renders $\hat{X}^{\text{ach},\text{ch}}$ a complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme.

(4) Similar to Definition 1.3.5 under the fixed isomorphisms of spinor sheaves and the $\varepsilon$-tensor, the standard chiral coordinate functions $(x', \theta, \bar{\eta}, \bar{\theta}, \bar{\vartheta})$ on $\hat{X}$, where

$$x'^\mu = x^\mu + \sqrt{-1} \sum_{\alpha, \beta} \theta^\alpha \sigma^\mu_{\alpha \beta} \bar{\theta}^\beta,$$

extends canonically to the standard chiral coordinate functions $(x', \theta, \bar{\eta}, \bar{\theta}, \bar{\vartheta})$ on $\hat{X}^{\text{ach}}$. In terms of the standard chiral coordinate functions, an $\tilde{f} \in C^\infty(\hat{X}^{\text{ach}})^{\text{ch}}$ can be expressed as

$$\tilde{f} = \tilde{f}'(x', \theta, \bar{\eta}, \bar{\vartheta}) = \tilde{f}'(0)(x', \eta, \bar{\vartheta}) + \sum_\alpha \theta^\alpha \tilde{f}'(\alpha)(x', \eta, \bar{\vartheta}) + \bar{\theta}^1 \bar{\theta}^2 \tilde{f}'(12)(x', \eta, \bar{\vartheta}),$$

where $f'(\alpha)(x', \eta, \bar{\vartheta})$ are polynomials in $\eta^\alpha$'s, $\bar{\vartheta}$'s, $\bar{\theta}^\alpha$'s, $\bar{\theta}^\beta$'s with coefficients smooth functions in $x'^\mu$'s.

Definition/Lemma 1.3.7. [antichiral sector $\hat{X}^{\text{ach}}$ of $\hat{X}$] (1) $\tilde{f} \in C^\infty(\hat{X}^{\text{ach}})$ is called antichiral if $e_1 \tilde{f} = e_2 \tilde{f} = 0$. The set of antichiral functions on $\hat{X}^{\text{ach}}$ is a $\mathbb{C}$-subalgebra of $C^\infty(\hat{X}^{\text{ach}})$, called the antichiral function-ring of $\hat{X}^{\text{ach}}$, denoted by $C^\infty(\hat{X}^{\text{ach}})^{\text{ach}}$.

(2) Replacing $\hat{X}$ by $\hat{U}$ for $U \subset X$ open and $e_1$, $e_2$ by $e_1|_{\bar{U}}$, $e_2|_{\bar{U}}$ in Item (1), one obtains the sheaf of antichiral functions $\hat{O}_X^{\text{ach}} \subset \hat{O}_X$, also called the antichiral structure sheaf of $\hat{X}^{\text{ach}}$.

Denote the locally-ringed space $(X, \hat{O}_X^{\text{ach}})$ by $\hat{X}^{\text{ach}}$.

(3) The $C^\infty$-hull of $C^\infty(\hat{X}^{\text{ach}})$ restricts to the $C^\infty$-hull of $C^\infty(\hat{X}^{\text{ach}})^{\text{ch}}$ and similarly for the localized version. This renders $\hat{X}^{\text{ach},\text{ch}}$ a complexified $\mathbb{Z}/2$-graded $C^\infty$-scheme.

(4) Similar to Definition 1.3.5 under the fixed isomorphisms of spinor sheaves and the $\varepsilon$-tensor, the standard antichiral coordinate functions $(x'', \theta, \bar{\eta}, \bar{\vartheta})$ on $\hat{X}$, where

$$x''^\mu = x^\mu - \sqrt{-1} \sum_{\alpha, \beta} \theta^\alpha \sigma^\mu_{\alpha \beta} \bar{\theta}^\beta,$$

extends canonically to the standard antichiral coordinate functions $(x'', \theta, \bar{\eta}, \bar{\vartheta})$ on $\hat{X}^{\text{ach}}$. In terms of the standard antichiral coordinate functions, an $\tilde{f} \in C^\infty(\hat{X}^{\text{ach}})^{\text{ach}}$ can be expressed as

$$\tilde{f} = \tilde{f}''(x'', \theta, \bar{\eta}, \bar{\vartheta}) = \tilde{f}''(0)(x'', \eta, \bar{\vartheta}) + \sum_\beta \bar{\theta}^\beta \tilde{f}''(\beta)(x'', \eta, \bar{\vartheta}) + \bar{\theta}^1 \bar{\theta}^2 \tilde{f}''(12)(x'', \eta, \bar{\vartheta}),$$

where $f''(\beta)(x'', \eta, \bar{\vartheta})$ are polynomials in $\eta^\alpha$'s, $\bar{\vartheta}$'s, $\bar{\theta}^\alpha$'s, $\bar{\theta}^\beta$'s with coefficients smooth functions in $x''^\mu$'s.

We refer readers to ibidem for details.
By construction, one has the following diagram of inclusions of complexified $\mathbb{Z}/2$-graded $C^\infty$-rings

\[
\begin{array}{ccc}
C^\infty(\hat{X}^c) & \xrightarrow{ch} & C^\infty(\hat{X}) \\
\downarrow & & \downarrow \\
C^\infty(\hat{X})_{ch} & & C^\infty(\hat{X})_{ach}
\end{array}
\]

Which gives rise to the following diagram of dominant morphisms of complexified $\mathbb{Z}/2$-graded $C^\infty$-schemes

\[
\begin{array}{ccc}
\hat{X}^c & \xrightarrow{ch} & \hat{X} \\
\downarrow & & \downarrow \\
\hat{X} & & \hat{X}_{ach}
\end{array}
\]

The following sector of $\hat{X}^c$ is introduced for the purpose of studying supersymmetric $U(1)$ gauge theory on $X$ in Sec. 3. In some sense, it is the "real sector" of $\hat{X}^c$:

**Definition 1.3.8.** [self-twisted-conjugate sector of $\hat{X}^c$] An element $\hat{f} \in C^\infty(\hat{X}^c)$ is called self-twisted-conjugate if $\hat{f}^\dagger = \hat{f}$. The set of all self-twisted-conjugate elements in $C^\infty(\hat{X}^c)$ is denoted by $C^\infty(\hat{X}^c)_{stc}$. Note that this is only a $C^\infty(X)$-module, not a $C^\infty(X)$-algebra.

### 1.4 The physics sector $X^{physics}$ of $\hat{X}^c$

We now proceed to identify the physical elements in $C^\infty(\hat{X}^c)$ and construct the physics sector $X^{physics}$ under $\hat{X}^c$.

The guide from [W-B] of Wess & Bagger to identify the physical elements in each field-theory level of $C^\infty(\hat{X}^c)$

Assume that $\hat{X}^c$ has only one field-theory level (i.e. $l = 1$) and suppress the parameter level. We begin with the following question:

**Q.** What should the chiral functions in the sought-for physics sector be? Similarly, for antichiral functions?

The answer, if taken from [W-B: Chap. V, Eq. (5.3) ] Wess & Bagger, would be

\[
\Phi = A(y) + \sqrt{2}\psi(y) + \theta\theta F(y),
\]

\footnote{Notations and conventions here follow [W-B] of Wess & Bagger as much as we can for the convenience of the illuminations.}
where \( y = x + \sqrt{-1}\theta\sigma\bar{\theta} \) is the chiral coordinate. In this expression, \( A \) and \( F \) are scalar functions (and hence are commuting) while \( \psi \) is a two-component spinor (whose components are thus anticommuting — which renders \( \theta\psi \) and hence \( \Phi \) even). So far so good, until one starts to think deeper. Chiral functions are required to form a ring. In particular, the multiplication of two chiral functions should be also chiral: (cf. [W-B: Chap. V, Eq. (5.7)])

\[
\tilde{\Phi} := \Phi_1\Phi_2 = \begin{array}{c}
A_1(y)A_2(y) + \sqrt{2}\theta(\phi_1(y)A_2(y) + A_1(y)\psi_2(y)) \\
+\theta\left(A_1(y)F_2(y) + A_2(y)F_1(y) - \psi_1(y)\psi_2(y)\right)
\end{array} =: \tilde{A}(y) + \sqrt{2}\theta\tilde{\psi}(y) + \theta\tilde{F}(y).
\]

When physicists say that \( F \) is a scalar, it is referred to the fact that \( F \) is a Lorentz scalar (i.e. sections of the associated bundle from the trivial representation of the principal Lorentz-group bundle). This says nothing about the nilpotency of \( F \). The coefficient \( \tilde{F} \) in this product reveals something peculiar:

- The summand \( \psi_1(y)\psi_2(y) \) in the coefficient \( \tilde{F} \) of the product \( \Phi_1\Phi_2 \) is nilpotent.

If we want to include \( \Phi_1\Phi_2 \) in the chiral ring, then we must allow \( \tilde{F} \) to have nilpotent summand. Now, if \( F_1 \) and \( F_2 \) themselves are not nilpotent, then this poses an issue:

- \( \tilde{F} \) now has a non-nilpotent summand \( A_1(y)F_2(y) + A_2(y)F_1(y) \) and a nilpotent summand \( -\psi_1(y)\psi_2(y) \). These two summands are not like terms and hence should be treated as different degrees of freedom.\(^7\)

But the chiral multiplet from the representation of \( d = 3 + 1, N = 1 \) supersymmetry algebra requires that the coefficient \( F \) of \( \theta\theta \) contribute as same degree of freedom as the total-\( \theta \)-degree-zero term \( A \). Thus, the only way to avoid such a contradiction is demand that

- \( F \) must be a nilpotent Lorentz scalar.

It is in this way, one deduces that physical chiral functions must be of the form (in our notation)

\[
\hat{f} = f^{(0)}_0(x') + \sum_\alpha \theta^\alpha(\vartheta_\alpha f^{(\alpha)}_0(x')) + \theta^1\theta^2(\dot{\vartheta}_1 \vartheta_2 f^{(12)}_0(x')) ,
\]

where \((x', \theta, \bar{\theta})\) are chiral coordinate functions on \( \hat{X} \). Such expressions are now closed under multiplications and hence form a ring. They now agree with the chiral multiplet representation of the \( d = 3 + 1, N = 1 \) supersymmetry algebra.

Similar reasoning for antichiral functions implies that physical antichiral functions must be of the form (in our notation)

\[
\hat{g} = g^{(0)}_0(x'') + \sum_\beta \vartheta^\beta(\bar{\vartheta}_\beta g^{(\beta)}_0(x'')) + \bar{\vartheta}^1\bar{\vartheta}^2(\bar{\vartheta}_1 \vartheta_2 g^{(12)}_0(x'')) ,
\]

where \((x'', \theta, \bar{\theta})\) are antichiral coordinate functions on \( \hat{X} \).

The physical function-ring must contain both the physical chiral function ring and the physical antichiral function ring, and hence their product as well. Multiplying a physical chiral \( \hat{f} \) and a physical antichiral \( \hat{g} \) as given above and re-expressing the product \( \hat{f}\hat{g} \) in the standard coordinate

---

\(^7\)Note for physicists An element \( r \neq 0 \) of a (either commutative or noncommutative) ring \( R \) is called nilpotent if \( r^{l+1} = 0 \) for some \( l \geq 1 \). The minimal such \( l \) is called the nilpotency of \( r \). In particular, any odd element of a \( \mathbb{Z}/2 \)-graded ring is nilpotent.

\(^8\)Note for physicists This is completely analogous to the statement that the expression \( f + \sqrt{-1}g, f,g \in C^\infty(\mathbb{R}^n) \), has two degrees of freedom or that the ring \( \mathbb{C}[t]/(t^{l+1}) \) is \((l + 1)\)-dimensional as a \( \mathbb{C} \)-vector space.
functions \((x, \theta, \bar{\theta})\) on \(\hat{X}\) leads to the notion of physical superfields in Definition 1.4.1 in the next theme.

From the above illumination, one sees that the construction is a minimal one: We only include those that are required to make a ring, beginning with the following two demands

1. Chiral functions must from a ring; so does antichiral functions.

2. A chiral superfield must match the chiral multiplet and an antichiral superfield must match antichiral multiplet in representations of \(d = 3 + 1, N = 1\) supersymmetry algebra.

Thus, as long as physical relevance is concerned, \(C^\infty(X^{\text{physics}})\) in Definition 1.4.1 is unique.

The above reasoning and construction works field-theory level by field-theory level. Once the physical elements of \(C^\infty(\hat{X})\) from each field-theory level are identified, the complexified \(C^\infty\)-subring in \(C^\infty(\hat{X})\) generated by them should be the sought-for \(C^\infty(\hat{X})^{\text{physics}}\).

The purely even physical structure sheaf \(\mathcal{O}_X^{\text{physics}}\) on \(X\)

For the simplicity of notations, we assume that \(\hat{X}\) has only one field-theory level (i.e. \(l = 1\)) and suppress the parameter level of \(\hat{X}\).

Recall then the standard coordinate functions

\[
(x^0, x^1, x^2, x^3; \theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2; \vartheta_1, \vartheta_2, \bar{\vartheta}_1, \bar{\vartheta}_2)
\]
on \(\hat{X}\), denoted collectively by \((x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) or \((x^\mu, \theta^\alpha, \bar{\theta}^\dot{\beta}, \vartheta, \bar{\vartheta})\)\(\mu, \alpha, \dot{\beta}, \gamma, \delta\).

Definition 1.4.1. [superfield in physical sector/physical superfield] An \(\tilde{f} \in C^\infty(\hat{X})\) is called a superfield in the physical sector of \(\hat{X}\) if, as a polynomial in the anticommuting coordinate-functions \((\theta, \bar{\theta}, \vartheta, \bar{\vartheta})\), it is of the following form

\[
\tilde{f} = \tilde{f}(0) + \sum_{\alpha} \theta^\alpha f^{(\alpha)}(0) + \sum_{\beta} \bar{\theta}^\beta \bar{f}^{(\beta)}(0) + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta f^{(\alpha\beta)}(0) + \sum_{\alpha} \theta^\alpha \bar{\theta}^\beta \vartheta_1 \bar{\vartheta}_2 f^{(12)}(12)
\]

\[
+ \sum_{\alpha} \theta^\alpha \vartheta_1 \bar{\vartheta}_2 f^{(12)}(12) + \sum_{\alpha} \theta^\alpha \vartheta_1 \bar{\vartheta}_2 f^{(12)}(12) + \sum_{\alpha} \theta^\alpha \bar{\vartheta}_1 \bar{\vartheta}_2 f^{(12)}(12)
\]

\[
\quad + \theta^1 \theta^2 \vartheta_1 \bar{\vartheta}_2 f^{(12)}(12) + \sum_{\alpha, \beta, \gamma, \delta} \theta^\alpha \bar{\theta}^\beta \vartheta_1 \bar{\vartheta}_2 f^{(12)}(12) + \sum_{\alpha, \beta, \gamma, \delta} \theta^\alpha \bar{\theta}^\beta \vartheta_1 \bar{\vartheta}_2 f^{(12)}(12)
\]

where \(\alpha = 1, 2; \beta = 1, 2; \mu = 0, 1, 2, 3\); and the thirty-three coefficients \(f^*\) of the \((\theta, \bar{\theta}, \vartheta, \bar{\vartheta})\)-monomial summands of \(\tilde{f}\) are complex-valued functions on \(X\):

\[
f^{(0)}(0); f^{(\alpha)}(0); f^{(\beta)}(0); f^{(12)}(12); f^{(\alpha\beta)}(12); f^{(\alpha\gamma)}(12); f^{(\alpha\delta)}(12); f^{(12)}(12); f^{(\alpha\beta\gamma\delta)}(12); f^{(12)}(12); f^{(12)}(12); f^{(12)}(12); f^{(12)}(12) \in C^\infty(X)^C
\]

For simplicity, such an \(\tilde{f}\) is also called a physical superfield on \(\hat{X}\).
Lemma 1.4.2. [physical sector of \( \tilde{X}^\mathbb{H} \)] The collection of physical superfields on \( \tilde{X}^\mathbb{H} \) as defined in Definition 1.4.1 is an even subring of the complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-ring \( C^\infty(\tilde{X}^\mathbb{H}) \). Denote this subring (also a \( C^\infty(X)^\mathbb{C} \)-subalgebra of \( C^\infty(\tilde{X}^\mathbb{H}) \)) by \( C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}} \). Then, the \( C^\infty \)-hull of \( C^\infty(\tilde{X}^\mathbb{H}) \) restricts to the \( C^\infty \)-hull of \( C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}} \), which is given by

\[
C^\infty\text{-hull}(C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}}) = \{ \tilde{f} \in C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}} | \tilde{f}(0) \in C^\infty(X) \}.
\]

Proof. From the \((\theta, \bar{\theta}, \vartheta, \bar{\vartheta})\)-expansion of a physical superfield \( \tilde{f} \), one concludes that it is even. That the set \( C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}} \) of physical superfields is a subring of \( C^\infty(\tilde{X}^\mathbb{H}) \) follows from the observation that as a \( C^\infty(X)^\mathbb{C} \)-module, \( C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}} \) is generated by thirty-three monomials,\(^9\)

\[
\left\{ 1, \theta^\alpha \partial_\alpha, \bar{\theta}^\beta \bar{\partial}_\beta, \theta^1 \theta^2 \partial_1 \partial_2, \theta^\alpha \bar{\theta}^\beta \partial_\alpha \bar{\partial}_\beta, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \partial_\beta, \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \bar{\partial}_\beta, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \partial_\beta, \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \partial_2, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \bar{\partial}_\beta, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \partial_\beta \right\}_{\alpha=1,2; \beta=1,2}
\]

and, up to a sign factor, this set is closed under multiplications. Finally, since these monomials are even, they commute with each other. Furthermore, except the monomial 1, they are all nilpotent. This implies that the \( C^\infty \)-hull of \( C^\infty(\tilde{X}^\mathbb{H}) \) restricts to the \( C^\infty \)-hull of \( C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}} \) and that the latter is given by

\[
C^\infty\text{-hull}(C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}}) = \{ \tilde{f} \in C^\infty(\tilde{X}^\mathbb{H})_{\text{physics}} | \tilde{f}(0) \in C^\infty(X) \}.
\]

This completes the proof. \( \square \)

By localizing all the constructions and discussions to open sets of \( X \), one obtains a new complexified \( C^\infty \)-scheme supported on \( X \):

Definition 1.4.3. [\( X_{\text{physics}} \) as (purely even) complexified \( C^\infty \)-scheme] Let \( O^\text{physics}_X \) be the sheaf on \( X \) associated to the assignment \( U \mapsto C^\infty(\tilde{U}^\mathbb{H})_{\text{physics}} \) for open sets \( U \) of \( X \). This is a sheaf of complexified \( C^\infty \)-rings on \( X \). Denote by \( X^\text{physics} \) the associate complexified \( C^\infty \)-scheme \((X, O^\text{physics}_X)\). Note that \( X^\text{physics} \) is purely even.\(^10\)

By construction, one has the following commutative diagram of dominant morphisms (of

---

\(^9\)A characterization of these \((\theta, \bar{\theta}, \vartheta, \bar{\vartheta})\)-monomials is given as follows. First, define a balanced monomial to be one whose \((\theta, \bar{\theta})\)-factor matches exactly with the \((\vartheta, \bar{\vartheta})\)-factor. There are fifteen of them:

\[
1, \theta^\alpha \partial_\alpha, \bar{\theta}^\beta \bar{\partial}_\beta, \theta^1 \theta^2 \partial_1 \partial_2, \theta^\alpha \bar{\theta}^\beta \partial_\alpha \bar{\partial}_\beta, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \partial_\beta, \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \bar{\partial}_\beta, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \partial_\beta, \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \partial_2, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \bar{\partial}_\beta, \theta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha \partial_\beta.
\]

\(\alpha = 1, 2; \beta = 1, 2\). Then, reduce them by dropping a \( \vartheta, \bar{\vartheta} \)-factor until there are no more such factors. For example,

\[
\theta^\alpha \bar{\theta}^\beta \partial_\alpha \partial_\beta \leadsto \theta^\alpha \bar{\theta}^\beta \partial_\beta \quad \text{and} \quad \theta^1 \theta^2 \partial_1 \partial_2 \bar{\theta}_1 \bar{\theta}_2 \leadsto \theta^1 \theta^2 \partial_1 \partial_2 \partial_\alpha \partial_\beta \leadsto \theta^1 \theta^2 \partial_1 \partial_2.
\]

\(^10\)Here, we denote this scheme by \( X^\text{physics} \), rather than \( \tilde{X}^\text{physics} \), to emphasize that it is purely even.
complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-schemes, where both the odd part of \( X^{\text{physics}} \) and \( X^C \) are zero)\)

\[
\begin{align*}
\begin{array}{c}
\mathcal{X} \\
\hat{\mathcal{X}} \oplus \mathcal{X}^{\text{physics}} \\
\mathcal{X}^C
\end{array}
\end{align*}
\]

The chiral sector of \( \hat{\mathcal{X}} \) restricts to the chiral sector of \( \mathcal{X}^{\text{physics}} \) and the antichiral sector of \( \hat{\mathcal{X}} \) restricts to the antichiral sector of \( \mathcal{X}^{\text{physics}} \). We will look at them more closely in Sec. 2.1.

For \( l \geq 2 \), as a \( C^\infty(X)^C \)-module, the complexified \( C^\infty \)-subring \( C^\infty(\hat{\mathcal{X}})^{\text{physics}} \) of \( C^\infty(\hat{\mathcal{X}}) \) is generated by elements from the product of the sets from each field-theory level of \( \hat{\mathcal{X}} \):

\[
\begin{align*}
\begin{cases}
1, \; \theta^\alpha \bar{\vartheta}_i, \; \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i, \; \theta^\alpha \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i \bar{\vartheta}_j, \; \theta^\alpha \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i, \; \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i \\
\theta^1 \theta^\alpha \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i \bar{\vartheta}_j, \; \theta^1 \theta^\alpha \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i \bar{\vartheta}_j, \; \theta^1 \theta^\alpha \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i \bar{\vartheta}_j, \; \theta^1 \theta^\alpha \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i \bar{\vartheta}_j, \; \theta^1 \theta^\alpha \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i \bar{\vartheta}_j, \; \theta^1 \theta^\alpha \bar{\vartheta}^{\dot{\beta}} \bar{\vartheta}_i \bar{\vartheta}_j
\end{cases}
\end{align*}
\]

\[i = 1, \ldots, l\]. Its \( C^\infty \)-hull remains of the form

\[
\begin{align*}
C^\infty \text{-hull}(C^\infty(\hat{\mathcal{X}})^{\text{physics}}) = \{ \hat{\mathcal{f}} \in C^\infty(\hat{\mathcal{X}})^{\text{physics}} | \mathcal{f}(0) \in C^\infty(X) \}.
\end{align*}
\]

However, clearly it becomes extremely messy to express a \((\theta, \bar{\vartheta}, \vartheta, \bar{\vartheta})\)-expansion of a general element \( \hat{\mathcal{f}} \in C^\infty(\hat{\mathcal{X}})^{\text{physics}} \). For the purpose of this work, we assume therefore for the rest of the work that \( l = 1 \) for the simplicity of notations though there is no technical difficulty to generalize to the \( l \geq 2 \) case. Cf. Figure 1-4-1; note that all the schemes involved \( \hat{\mathcal{X}}, \hat{\mathcal{X}}, X^{\text{physics}} \) have the same underlying topology \( X \), which is indicated by the “foggy/fuzzy” nature of the fermionic cloud carried by these schemes in the illustration.

Also, recall that the Grassmann parameter level is introduced to serve the need when a discussion/computation/expression requires (or becomes more convenient to present with) the Grassmann parameter \((\eta, \bar{\eta})\). This level will thus be suppressed from now on unless it is needed.

### 1.5 Purge-evaluation maps and the Fundamental Theorem on supersymmetric action functionals

In this subsection, we recast a fundamental theorem on supersymmetric action functionals (e.g. [Bi: Sec. 4.3]) into a general form under the current (complexified, \( \mathbb{Z}/2 \)-graded) \( C^\infty \)-Algebraic Geometry setting.

**The need to get rid of nilpotency**

So far so good. But when one presses on to construct a supersymmetric action functional using the fermionic integration of the form

\[
\int_{\hat{\mathcal{X}}} d^4x d\theta^\dot{3} d\bar{\vartheta}_i d\theta^2 d\bar{\vartheta}_i \hat{\mathcal{f}}
\]
Figure 1-4-1. The space-time coordinate functions \(x^\mu, \mu = 0, 1, 2, 3\), and the fermionic coordinate functions \(\theta^\alpha, \bar{\theta}^\dot{\beta}, \alpha = 1, 2, \dot{\beta} = 1, 2\), generate the function ring of the fundamental superspace \(\hat{X}\) as a complexified \(\mathbb{Z}/2\)-graded \(C^\infty\)-scheme. Over it sits a supertower with Grassmann-number level and other field-theory levels that are needed for the construction of supersymmetric quantum field theories. From the direct-sum expression of the generating sheaf

\[
\mathcal{F} := \left( S^\vee_{\text{coordinates}} \oplus S^\vee_{\text{coordinates}} \right) \oplus \left( S^\vee_{\text{parameter}} \oplus S^\vee_{\text{parameter}} \right) \\
\oplus \bigoplus_{i=1}^l \left( S^\vee_{\text{field},i} \oplus S^\vee_{\text{field},i} \right)
\]

of the structure sheaf \(\mathcal{O}_{\hat{X}}\) of \(\hat{X}\), one may think of each field-theory level as contributing a floor-[\(i\)]

\[
\hat{X}^{\text{Double}}_{[i]} := \left( X, \bigwedge_{\hat{X}}^* \left( S^\vee_{\text{coordinates}} \oplus S^\vee_{\text{coordinates}} \oplus S^\vee_{\text{field},i} \oplus S^\vee_{\text{field},i} \right) \right)
\]

of \(X\) and these field-theory floors are glued by the \(\mathbb{Z}/2\)-graded version of fibered product over \(X\) to give \(\hat{X}\). Each field-theory floor \(\hat{X}^{\text{Double}}_{[i]}\) has its own physics sector \(X^{\text{physics}}_{[i]}\) that is purely even. They generate the physics sector \(X^{\text{physics}}\) of \(\hat{X}\) that is also purely even. This physics sector is where most of physics-relevant superfields lie.
for $\tilde{f} \in C^\infty(\tilde{X}^\parallel)^{\text{physics}}$ a derived physical superfield constructed from more basic physical superfields in the problem, the result is

$$\int_X d^4x \left( f_{(12\bar{1})}^{(0)} + \sum_{\alpha,\beta} \vartheta_1 \vartheta_\beta \tilde{f}_{(12\bar{1})(\alpha\beta)} + \vartheta_1 \vartheta_2 \tilde{f}_{(12\bar{1})}^{(12\bar{1})} \right).$$

Mathematically, there is nothing wrong: The above integral is nothing but

$$\int_X d^4x f_{(12\bar{1})}^{(0)} + \sum_{\alpha,\beta} \vartheta_1 \vartheta_\beta \int_X d^4x f_{(12\bar{1})}^{(\alpha\beta)} + \vartheta_1 \vartheta_2 \int_X d^4x f_{(12\bar{1})}^{(12\bar{1})}.$$ 

Thus, for example, when one applies calculus of variations to it to derive the equations of motions of the component fields on $X$, since the six integrals are not like terms, each variation has to be set to zero:

$$\delta \int_X d^4x \left( f_{(12\bar{1})}^{(0)} + \sum_{\alpha,\beta} \vartheta_1 \vartheta_\beta \tilde{f}_{(12\bar{1})(\alpha\beta)} + \vartheta_1 \vartheta_2 \tilde{f}_{(12\bar{1})}^{(12\bar{1})} \right) = 0 \quad \Rightarrow \quad \delta \int_X d^4x f_{(12\bar{1})}^{(0)} = 0, \quad \delta \int_X d^4x f_{(12\bar{1})}^{(\alpha\beta)} = 0, \quad \delta \int_X d^4x f_{(12\bar{1})}^{(12\bar{1})} = 0.$$ 

But this is not what physicists do! In order to match what physicists do, the nilpotency — though necessary from the perspective of complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry — has to be “purged” away in the end. So that after applying such a purge to $f_{(12\bar{1})}^{(0)} + \sum_{\alpha,\beta} \vartheta_1 \vartheta_\beta \tilde{f}_{(12\bar{1})(\alpha\beta)} + \vartheta_1 \vartheta_2 \tilde{f}_{(12\bar{1})}^{(12\bar{1})}$ all the nilpotency disappears and the result is in $C^\infty(X)^C$. Calculus of variation is then applied to one integral, rather than six independent integrals.

### Purge-evaluation maps and the Fundamental Theorem

Let $\mathcal{M}_{\text{field}} := \wedge^\bullet C^\infty_X(\mathcal{S}_{\text{field}}' \oplus \mathcal{S}_{\text{field}}'') \subset \hat{\mathcal{O}}^\parallel_X$.

**Definition 1.5.1.** [Purge-evaluation map] A $(\theta, \bar{\theta})$-degree-preserving $\mathcal{O}_X^C$-module homomorphism $\mathcal{P} : \hat{\mathcal{O}}^\parallel_X \to \hat{\mathcal{O}}_X$ that restricts to the identity map $Id_{\hat{\mathcal{O}}_X} : \hat{\mathcal{O}}_X \to \hat{\mathcal{O}}_X$ and takes the fifteen $(\theta, \bar{\theta})$-monomials $\vartheta_\alpha \bar{\vartheta}_\bar{\alpha}, \vartheta_1 \vartheta_2, \vartheta_\alpha \bar{\vartheta}_{\bar{\alpha}1}, \vartheta_\alpha \bar{\vartheta}_{\bar{\alpha}2}, \vartheta_1 \vartheta_2 \bar{\vartheta}_{\bar{\alpha}1}, \vartheta_\alpha \bar{\vartheta}_{\bar{\alpha}2}$ and $\vartheta_1 \vartheta_2 \bar{\vartheta}_{\bar{\alpha}1} \bar{\vartheta}_{\bar{\alpha}2}$ in each component of fixed $(\theta, \bar{\theta})$-degree to constants in $\mathbb{C}$ is called a **purge-evaluation map**.

Explicitly, for

$$\tilde{f} = f_{(0)} + \sum_{\alpha} \theta_\alpha \tilde{f}_{(\alpha)} + \sum_{\bar{\alpha}} \bar{\theta}_{\bar{\alpha}} \tilde{f}_{(\bar{\alpha})} + \theta^1 \theta^2 \tilde{f}_{(12)} + \sum_{\alpha,\beta} \theta_\alpha \bar{\theta}_{\bar{\beta}} \tilde{f}_{(\alpha\beta)} + \bar{\theta}_1 \bar{\theta}_2 \tilde{f}_{(12\bar{1})}$$

$$+ \sum_{\bar{\alpha}} \theta^1 \theta^2 \bar{\theta}_{\bar{\alpha}} \tilde{f}_{(12\bar{1})} + \sum_{\alpha} \theta_\alpha \bar{\theta}_1 \bar{\theta}_2 \tilde{f}_{(\alpha12)} + \theta^1 \theta^2 \bar{\theta}_1 \bar{\theta}_2 \tilde{f}_{(1212)}$$

11See Sec. 2.2 and Sec. 3.5 for concrete examples.

12Remark on the naming and the formulation $\mathcal{P}$ has the effect of removing the second set of fermionic coordinate-functions $(\theta, \bar{\theta})$ on $\tilde{X}^\parallel$, hence the name ‘purge’. Note that the coefficients of each $(\theta, \bar{\theta})$-monomial summand of a function on $\tilde{X}^\parallel$ are grouped into sections of sheaves/bundles associated to some irreducible Lorentz representations. There are built-in pairings of these sheaves/bundles in the problem and these pairings define accordingly various natural evaluation maps that in practice either specify or are incorporated into $\mathcal{P}$ to obtain Lorentz-invariant expressions, hence the name ‘evaluation’.

Also note that for the construction of supersymmetric gauge theories, the purge-evaluation map is defined not on $\mathcal{O}_{X^\text{physics}}^\parallel$, but another $\mathcal{O}_X^C$-submodule of $\hat{\mathcal{O}}_X^\parallel$. The formulation given here is meant to be as general as possible in order to cover all situations since the proof of the Fundamental Theorem goes the same.
Lemma 1.5.2. [property of $\mathcal{P}$] A purge-evaluation map $\mathcal{P}: \hat{\mathcal{O}}_{\hat{X}} \to \hat{\mathcal{O}}_X$ satisfies the following properties: (1) Let $\mathcal{P}$ be uniform and let $\xi \in \text{Der}_C(\hat{X})$ be a derivation on $\hat{X}$. Then $\mathcal{P}(\xi \tilde{f}) = \xi \mathcal{P}(\tilde{f})$ for $\tilde{f} \in \hat{\mathcal{O}}_{\hat{X}}$. (2) $\int d\theta^2 d\bar{\theta}^1 d\theta^1 \mathcal{P}(\tilde{f}) = \int d\bar{\theta} d\theta^1 d\theta^1 d\bar{\theta}^1 \mathcal{P}(\bar{\mathcal{O}}_{\bar{X}})(\tilde{f})$ for $\tilde{f} \in \hat{\mathcal{O}}_{\hat{X}}$. (3) For $\tilde{f} \in \hat{\mathcal{O}}_{\hat{X}}$ chiral (resp. antichiral), $\int d\theta^2 d\theta^1 \mathcal{P}(\tilde{f}) = \int d\bar{\theta} d\theta^1 \mathcal{P}(\bar{\mathcal{O}}_{\bar{X}})(\tilde{f})$ (resp. $\int d\bar{\theta} d\theta^1 \mathcal{P}(\tilde{f}) = \int d\theta^2 d\theta^1 \mathcal{P}(\bar{\mathcal{O}}_{\bar{X}})(\tilde{f})$).

Proof. Statement (1) follows from the fact that $\xi \in \text{Der}_C(\hat{X})$ has no $(\theta, \bar{\theta})$-dependence while the uniform $\mathcal{P}$ applies to $\tilde{f}$ $(\theta, \bar{\theta})$-degree by $(\theta, \bar{\theta})$-degree with $\mathcal{P}(\bar{\theta}^1 \bar{\theta}^2 \bar{\theta}^1 \bar{\theta}^2)$ constant, $\epsilon_i = 0$ or 1, and hence $\xi(\mathcal{P}(\tilde{f}(\mathcal{P}))) = \mathcal{P}(\xi \tilde{f}(\mathcal{P}))$. Statement (2) and Statement (3) follow from the definition of fermionic integration on $\hat{X}$. 

Theorem 1.5.3. [fundamental: supersymmetric functional] Let $\mathcal{P}$ be a uniform purge-evaluation map. Then, up to a boundary term on $X$, (1) $S_1(\tilde{f}) := \int_X d^4 x d\theta^2 d\bar{\theta}^1 d\theta^1 d\bar{\theta}^1 \mathcal{P}(\tilde{f})$ is a functional on $C^\infty(\hat{\mathcal{O}}_{\hat{X}})$ that is invariant under supersymmetries; (2) $S_2(\tilde{f}) := \int_X d^4 x d\theta^2 d\theta^1 \mathcal{P}(\tilde{f})$ (resp. $S_3(\tilde{f}) := \int_X d^4 x d\theta^2 d\bar{\theta}^1 \mathcal{P}(\tilde{f})$) is a functional on $C^\infty(\hat{\mathcal{O}}_{\hat{X}})^{\text{ch}}$ (resp. $C^\infty(\hat{\mathcal{O}}_{\hat{X}})^{\text{ach}}$) that is invariant under supersymmetries.

Proof. For Statement (1), since $Q_\alpha, Q_\beta \in \text{Der}_C(\hat{X})$, it follows the invariance of $d^4 x d\theta^2 d\bar{\theta}^1 d\theta^1 d\bar{\theta}^1$, and $d^4 x d\theta^2 d\bar{\theta}^1$ under the flow that generates supersymmetries, Lemma 1.5.2, and basic calculus that

$$
\delta_{Q_\alpha} S_1(\tilde{f}) := \int_X d^4 x d\theta^2 d\bar{\theta}^1 d\theta^1 d\bar{\theta}^1 \mathcal{P}(Q_\alpha \tilde{f}) = \int_X d^4 x d\theta^2 d\bar{\theta}^1 d\theta^1 d\bar{\theta}^1 Q_\alpha \mathcal{P}(\tilde{f})
$$

$$
= -\sqrt{-1} \int_X d^4 x d\theta^2 d\bar{\theta}^1 d\theta^1 d\bar{\theta}^1 \sum_{\beta, \mu} \sigma^\mu_{\alpha\beta} \partial_\mu (\mathcal{P}(\tilde{f}))
$$

$$
= -\sqrt{-1} \int_X d^4 x \sum_\mu \partial_\mu \left( \int d\theta^2 d\bar{\theta}^1 d\theta^1 d\bar{\theta}^1 \sum_{\beta} \sigma^\mu_{\alpha\beta} \mathcal{P}(\tilde{f}) \right) = -\sqrt{-1} \int_X d\mathcal{B}_\alpha,
$$
where $B_\alpha = B_\alpha^0 dx^1 \wedge dx^2 \wedge dx^3 - B_\alpha^1 dx^0 \wedge dx^2 \wedge dx^3 + B_\alpha^2 dx^0 \wedge dx^1 \wedge dx^3 - B_\alpha^3 dx^0 \wedge dx^1 \wedge dx^2$

is a 3-form on $X$ with

$$B_\alpha^\mu = \int d\theta^2 d\bar{\theta}^1 d\theta^1 d\bar{\theta}^\mu \sum_{\beta} \sigma^\mu{}_{\alpha\beta} \bar{\theta}^\beta \mathcal{P}(\bar{f}).$$

The proof that $\delta_Q S(\bar{f})$ is also a boundary term is similar.

For Statement (2), note that for $\bar{f}$ chiral, $\delta_Q S_2(\bar{f}) = 0$ always, for $\alpha = 1, 2$, and, thus one only needs to check the variation $\delta_Q S_2(\bar{f})$:

$$\delta_Q S_2(\bar{f}) \ := \ \int_X d^4x \, d\theta^2 d\theta^1 \, \mathcal{P}(\bar{Q}_\beta \bar{f}) \ = \ \int_X d^4x \, d\theta^2 d\theta^1 \, \mathcal{P}(\bar{Q}_\beta \bar{f}) = \ \int \bar{\theta}^1 \, \mathcal{P}(\bar{f})$$

where $C_\beta = C_\beta^0 dx^1 \wedge dx^2 \wedge dx^3 - C_\beta^1 dx^0 \wedge dx^2 \wedge dx^3 + C_\beta^2 dx^0 \wedge dx^1 \wedge dx^3 - C_\beta^3 dx^0 \wedge dx^1 \wedge dx^2$ is a 3-form on $X$ with

$$C_\beta^\mu = \int d\theta^2 d\theta^1 \sum_{\alpha} \theta^\alpha \sigma^\mu{}_{\alpha\beta} \mathcal{P}(\bar{f}).$$

For $\bar{f}$ antichiral, $\delta_Q S_3(\bar{f}) = 0$ always, for $\bar{\beta} = \bar{1}, \bar{2}$, and the variation $\delta_Q S_3(\bar{f})$, $\alpha = 1, 2$, can be computed similarly to show that it is a boundary term on $X$.

This completes the proof.

Remark 1.5.4. [Lorentz invariance and $\mathcal{P}$-dependence of variations of components under supersymmetry] (1) For applications to particle physics, one takes the real-part of the complex-valued functional $S(\bullet)$ (if it is not already real) to get the action functional for the component fields $\bar{f}$ of $\bar{f}$ and requires in addition that the action functional be Lorentz-invariant, which is usually automatic when $\mathcal{P}$ comes from natural evaluation maps built-into the problem.

(2) Note also that from the equalities $Q_\alpha \, \mathcal{P}(\bar{f}) = \mathcal{P}(Q_\alpha \bar{f})$, $\bar{Q}_\beta \, \mathcal{P}(\bar{f}) = \mathcal{P}(\bar{Q}_\beta \bar{f})$, $\alpha = 1, 2$, $\beta = \bar{1}, \bar{2}$, for a uniform purge-evaluation map $\mathcal{P}$, the variation under supersymmetry of component fields of $\bar{f}$ for a physics model depends on the choice of $\mathcal{P}$ as well.

2 The chiral/antichiral theory on $X^{\text{physics}}$ and Wess-Zumino model

Having made the effort to build a platform from complexified $\mathbb{Z}/2$-graded $C^\infty$-Algebraic Geometry that incorporates basic requirements from Quantum Field Theory, one would like to know whether all the well-established supersymmetric quantum field theories in physics fit into the setting. In this work, we make a humble start to re-look at two earliest constructed, most basic supersymmetric quantum field theories in physics: the Wess-Zumino model (current section) and the supersymmetric $U(1)$ gauge theory with matter (the next section) to justify the validity.
2.1 More on the chiral and the antichiral sector of \( X^{\text{physics}} \)

As a preparation to study the Wess-Zumino model, some further details of the chiral sector and the antichiral sector of \( X^{\text{physics}} \) are given in this subsection.

The chiral sector of \( X^{\text{physics}} \)

**Definition 2.1.1. [chiral function-ring & chiral structure sheaf of \( X^{\text{physics}} \)]** (1) An \( f \in C^\infty(X^{\text{physics}}) \) is called chiral if \( e_1 f = e_2 f = 0 \). (2) As the addition and the multiplication of two chiral functions remain chiral, the set of chiral functions on \( X^{\text{physics}} \) form a ring, called the chiral function-ring of \( X^{\text{physics}} \), denoted by \( C^\infty(X^{\text{physics}})^c \). (3) Localizing Item (1) and Item (2) to open sets of \( X \), one obtains a sheaf \( O_X^{\text{ch}} \) of chiral functions on \( X^{\text{physics}} \) from the assignment \( U \to C^\infty(U^{\text{physics}})^c \) for \( U \) open sets of \( X \). \( O_X^{\text{ch}} \) is called the chiral structure sheaf of \( X^{\text{physics}} \). By construction, \( O_X^{\text{ch}} \subset O_X^{\text{ch}} \) as \( O_X^{\text{ch}} \)-algebras.

**Lemma 2.1.2. [\( C^\infty\)-hull of \( C^\infty(X^{\text{physics}})^c \)]** The \( C^\infty\)-hull \( C^\infty\)-hull\( (C^\infty(X^{\text{physics}})^c) \) of \( C^\infty(X^{\text{physics}}) \) restricts to the \( C^\infty\)-hull \( C^\infty\)-hull\( (C^\infty(X^{\text{physics}})^c) \) of \( C^\infty(X^{\text{physics}})^c \). Thus, \( C^\infty(X^{\text{physics}})^c \) is a complexified \( C^\infty\)-ring.

**Proof.** Let \( h \in C^\infty(\mathbb{R}^\ell) \) and \( \hat{f}, \cdots, \hat{f} \in C^\infty(X^{\text{physics}})^c \cap C^\infty\)-hull\( (X^{\text{physics}}) \). Then it follows from Definition/Lemma 1.3.6 that in terms of the chiral coordinate-functions \( (x', \theta', \bar{\theta}', \vartheta', \bar{\vartheta}') \) on \( \hat{X}^{\mathbb{M}} \),

\[
\hat{f} = f^{(0)}_{1,1}(x') + \sum_{\alpha} \theta^\alpha \bar{\theta}^\alpha f^{(1)}_{1,1}(x') + \theta^1 \vartheta^2 \vartheta_2 f^{(12)}_{1,1}(x'),
\]

for \( i = 1, \ldots, l \), and hence

\[
h(\hat{f}, \cdots, \hat{f}) = C^\infty\)-hull\( (C^\infty(X^{\text{physics}})) \) can be expressed as

\[
h(f^{(0)}_{1,1}(x'), \cdots, f^{(0)}_{1,1}(x'))
\]

\[
+ \sum_{k=1}^l \left( \partial_1 h(f^{(0)}_{1,1}(x'), \cdots, f^{(0)}_{1,1}(x')) \cdot \left( \sum_{\alpha} \theta^\alpha \bar{\theta}^\alpha f^{(1)}_{k,1}(x') + \theta^1 \vartheta^2 \vartheta_2 f^{(12)}_{k,1}(x') \right) \right)
\]

\[
- \theta^1 \vartheta^2 \vartheta_2 \sum_{k_1, k_2=1}^l \left( \partial_{k_1} \partial_{k_2} h(f^{(0)}_{1,1}(x'), \cdots, f^{(0)}_{1,1}(x')) \cdot f^{(1)}_{k_1,1}(x') f^{(2)}_{k_2,2}(x') \right),
\]

which is chiral. Here, the following computation and a change of dummy indices \( k_1 \) and \( k_2 \) are used to obtain the last term

\[
\sum_{\alpha, \gamma} \theta^\alpha \bar{\theta}^\alpha \vartheta^\gamma \vartheta_2 f^{(1)}_{k,1}(x') f^{(2)}_{k,1}(x') = \theta^1 \vartheta^2 \vartheta_2 \sum_{\alpha, \gamma} e^\alpha \gamma_\alpha \gamma f^{(0)}_{k_1,1}(x') f^{(1)}_{k_1,2}(x') f^{(2)}_{k_2,2}(x')
\]

\[
- \theta^1 \vartheta^2 \vartheta_2 \left( f^{(1)}_{k_1,1}(x') f^{(2)}_{k_2,2}(x') + f^{(2)}_{k_1,1}(x') f^{(1)}_{k_2,2}(x') \right)
\]

This proves the lemma.

\[\square\]

**Lemma 2.1.3. [chiral function on \( X^{\text{physics}} \) in terms of \( (x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta}) \)]** In terms of the standard coordinate functions \( (x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta}) \) on \( \hat{X}^{\mathbb{M}} \), a chiral function \( \hat{f} \) on \( X^{\text{physics}} \) is determined by the four components \( f^{(0)}_{(0)}, f^{(1)}_{(\alpha)}, \) and \( f^{(12)}_{(12)}, \alpha = 1, 2, \) of \( \hat{f} \) via the following formula

\[
\hat{f} = f^{(0)}_{(0)}(x) + \sum_{\alpha} \theta^\alpha \bar{\theta}^\alpha f^{(0)}_{(\alpha)}(x) + \theta^1 \vartheta^2 \vartheta_2 f^{(12)}_{(12)}(x) + \sqrt{-1} \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \sum_{\mu} \sigma^\mu_{\alpha, \beta} f^{(0)}_{(0)}(x)
\]

\[
+ \sqrt{-1} \sum_{\beta, \mu} \theta^1 \vartheta^2 \vartheta_2 \left( \vartheta_1 \sigma^\alpha_{\beta, 23} \partial_\mu f^{(1)}_{(1)}(x) - \vartheta_2 \sigma^\mu_{13} \partial_\mu f^{(2)}_{(2)}(x) \right) - \theta^1 \vartheta^2 \vartheta_2 \square f^{(0)}_{(0)}(x),
\]

21
where $\square := \sum_\mu \partial^\mu \partial_\mu = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$.

Proof. Similar to the proof [L-Y1: Lemma 1.4.14], one can prove the statement in two ways. The first proof is slick while the second proof can be generalized to the situation that involves sections of a bundle with a connection when one studies supersymmetric D-branes.

(a) First proof

In terms of the chiral coordinate functions $(x', \theta', \bar{\theta}', \bar{\theta}^\prime)$ := $(x + \sqrt{-1} \theta \sigma \bar{\theta}', \theta, \bar{\theta}, \bar{\theta}^\prime)$ on $\mathbb{R}^4$, a chiral function $\tilde{f}$ on $X^{\text{physics}}$ can be written as

$$\tilde{f} = f^{(0)}_{(0)}(x') + \sum_\alpha \theta^\alpha \tilde{f}_{(0)}^{(\alpha)}(x') + \theta^1 \theta^2 \theta^1 \theta^2 f^{(12)}(x')$$

where $f_{(0)}^{(0)}, f_{(0)}^{(\alpha)}, f_{(12)}^{(12)} \in C^\infty(\mathbb{R}^4)$. [L-Y1: Lemma 1.1.3] applied to the real and the imaginary component of $f_{(0)}^{(0)}, f_{(0)}^{(\alpha)}, f_{(12)}^{(12)}$ gives

$$f_{(\bullet)}^{(\bullet)}(x') = f_{(\bullet)}^{(\bullet)}(x + \sqrt{-1} \theta \sigma \bar{\theta}^\prime) = f_{(\bullet)}^{(\bullet)}(x) + \sqrt{-1} \sum_\mu (\partial_\mu f_{(\bullet)}^{(\bullet)}(x')) (\theta \sigma^\mu \bar{\theta}^\prime) - \frac{1}{2} \sum_{\mu, \nu} (\partial_\mu \partial_\nu f_{(\bullet)}^{(\bullet)}(x)) (\theta \sigma^\mu \bar{\theta}^\prime) (\theta \sigma^\nu \bar{\theta}^\prime).$$

The claim follows from applying the expansion to $f_{(0)}^{(0)}(x'), f_{(0)}^{(\alpha)}(x'),$ and $f_{(12)}^{(12)}(x')$, collecting like terms in $(\theta, \bar{\theta})$, simplifying them via spinor calculus, and re-denoting $f_{(\bullet)}^{(\bullet)}(x)$ by $f_{(\bullet)}^{(\bullet)}(x)$.

(b) Second proof

This is done by directly solving the system of differential equations obtained from the chiral condition on $\tilde{f}$.

Let

$$\tilde{f} = f_{(0)}^{(0)} + \sum_\alpha \theta^\alpha f_{(0)}^{(\alpha)} + \sum_\beta \bar{\theta}^\beta f_{(0)}^{(\beta)} + \theta^1 \theta^2 \bar{\theta}^2 f_{(12)}^{(12)}$$

$$+ \sum_\alpha \theta^1 \theta^2 \bar{\theta}^2 f_{(12)}^{(12)} + \sum_\alpha \theta^0 \bar{\theta}^0 f_{(12)}^{(12)} + \bar{\theta}^1 \bar{\theta}^1 f_{(12)}^{(12)}$$

$$= f_{(0)}^{(0)} + \sum_\alpha \theta^\alpha \partial_\alpha f_{(0)}^{(\alpha)} + \sum_\beta \bar{\theta}^\beta \bar{\partial}_\beta f_{(0)}^{(\beta)}$$

$$+ \theta^1 \theta^2 \bar{\theta}^2 f_{(12)}^{(12)} + \sum_\alpha \theta^0 \bar{\theta}^0 f_{(12)}^{(12)} + \bar{\theta}^1 \bar{\theta}^1 f_{(12)}^{(12)}$$

$$\in C^\infty(X^{\text{physics}}).$$

Then a straightforward computation with applications of basic identities in spinor calculus (cf. [L-Y1: Lemma 1.4.14], we define the Laplacian $\square$ as $\sum_\mu \partial^\mu \partial_\mu = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$. Here, we recover the convention used in [W-B] of Wess & Bagger.)
Appendix) gives

\[-e_{\beta\nu}\tilde{f} = \left( \frac{\partial}{\partial\bar{\theta}^\beta} + \sqrt{-1} \sum_{\alpha\mu} \theta^\alpha_{\alpha\beta} \sigma^\mu_{\alpha\beta} \partial_\mu \right) \tilde{f} \]

\[= \tilde{f}_{\beta(\beta)} + \sum_\alpha \theta^\alpha \left( -\tilde{f}_{(\alpha\beta)} + \sqrt{-1} \sum_\mu \sigma^\mu_{\alpha\beta} \partial_\mu \tilde{f}_{(0)} - \sum_\delta \bar{\theta}^\delta \varepsilon_{\beta\delta} \tilde{f}_{(12)} \right) \]

\[+ \theta^1 \theta^2 \left( \tilde{f}_{(12)} + \sqrt{-1} \sum_{\alpha,\gamma,\mu} \varepsilon^\alpha_{\alpha\beta} \sigma^\gamma_{\alpha\beta} \partial_\mu \tilde{f}_{(\gamma)} \right) + \sum_{\alpha,\gamma,\mu} \theta^\alpha \bar{\theta}^\gamma \left( \varepsilon_{\delta\gamma} \tilde{f}_{(\delta)} + \sqrt{-1} \sum_\mu \sigma^\mu_{\alpha\beta} \partial_\mu \tilde{f}_{(\delta)} \right) \]

\[+ \sum_{\alpha,\gamma,\mu} \theta^1 \theta^2 \bar{\theta}^\gamma \left( -\varepsilon_{\beta\delta} \tilde{f}_{(12)} + \sqrt{-1} \sum_{\alpha,\gamma,\mu} \varepsilon^\alpha_{\alpha\beta} \sigma^\gamma_{\alpha\beta} \partial_\mu \tilde{f}_{(\gamma)} \right) + \sqrt{-1} \sum_{\alpha,\mu} \theta^\alpha \bar{\theta}^\gamma \bar{\theta}^\delta \sigma^\mu_{\alpha\beta} \partial_\mu \tilde{f}_{(12)} \]

\[+ \sqrt{-1} \theta^1 \theta^2 \bar{\theta}^\gamma \bar{\theta}^\delta \sum_{\alpha,\gamma,\mu} \varepsilon^\alpha_{\alpha\beta} \sigma^\gamma_{\alpha\beta} \partial_\mu \tilde{f}_{(12)}, \]

\[\beta'' = 1''', 2'''\). Similar to [L-Y1: proof of Lemma 1.4.14], solving the overdetermined system of equations on components \(f^*\) of \(\tilde{f}\) obtained by setting \(e_1\tilde{f} = e_2\tilde{f} = 0\), now with applications of basic identities from spinor calculus (cf. Appendix), proves the statement.

\[\square\]

The antichiral sector of \(X^{\text{physics}}\)

Similar arguments to the previous theme give the results for the antichiral sector of \(X^{\text{physics}}\).

**Definition 2.1.4.** [antichiral function-ring & antichiral structure sheaf of \(X^{\text{physics}}\)]\(^{(1)}\) An \(\tilde{f} \in C^\infty(X^{\text{physics}})\) is called antichiral if \(e_1\tilde{f} = e_2\tilde{f} = 0\). \(^{(2)}\) As the addition and the multiplication of two antichiral functions remain antichiral, the set of antichiral functions on \(X^{\text{physics}}\) form a ring, called the antichiral function-ring of \(X^{\text{physics}}\), denoted by \(C^\infty(X^{\text{physics}})^{\text{ach}}\). \(^{(3)}\) Localizing Item (1) and Item (2) to open sets of \(X\), one obtains a sheaf \(O_X^{\text{physics,ach}}\) of antichiral functions on \(X^{\text{physics}}\) from the assignment \(U \mapsto C^\infty(U^{\text{physics}})^{\text{ach}}\) for \(U\) open sets of \(X\). \(O_X^{\text{physics,ach}}\) is called the antichiral structure sheaf of \(X^{\text{physics}}\). By construction, \(O_X^{\text{physics,ach}} \subset O_X^{\text{physics}}\) as \(O_X^{\text{C-}}\) algebras.

**Lemma 2.1.5.** [\(C^\infty\)-hull of \(C^\infty(X^{\text{physics}})^{\text{ach}}\)] The \(C^\infty\)-hull \(C^\infty\text{-hull}(C^\infty(X^{\text{physics}})^{\text{ach}})\) of \(C^\infty(X^{\text{physics}})\) restricts to the \(C^\infty\)-hull \(C^\infty\text{-hull}(C^\infty(X^{\text{physics}})^{\text{ach}})\) of \(C^\infty(X^{\text{physics}})^{\text{ach}}\). Thus, \(C^\infty(X^{\text{physics}})^{\text{ach}}\) is a complexified \(C^\infty\)-ring.

**Lemma 2.1.6.** [antichiral function on \(X^{\text{physics}}\) in terms of \((x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\)] In terms of the standard coordinate functions \((x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) on \(\tilde{X}^{\text{ach}}\), an antichiral function \(\tilde{f}\) on \(X^{\text{physics}}\) is determined by the four components \(f^{(0)}_0, f^{(\beta)}_0, f^{(12)}_{(\beta)}\), \(\beta = 1, 2\), of \(\tilde{f}\) via the following formula

\[\tilde{f} = f^{(0)}_0(x) + \sum_{\beta} \theta^\beta \bar{\theta}^\beta f^{(\beta)}_0(x) + \bar{\theta}^1 \bar{\theta}^2 \vartheta_1 \vartheta_2 f^{(12)}_{(\beta)}(x) - \sqrt{-1} \sum_{\alpha,\beta} \theta^\alpha \bar{\theta}^\beta \sum_\mu \sigma^\mu_{\alpha\beta} \partial_\mu f^{(0)}_0(x) \]

\[+ \sqrt{-1} \sum_{\alpha,\mu} \theta^\alpha \bar{\theta}^\beta \bar{\theta}^\gamma \left( \vartheta_1 \sigma^\mu_{\alpha\beta} \partial_\mu f^{(1)}_{(\beta)}(x) - \vartheta_2 \sigma^\mu_{\alpha\beta} \partial_\mu f^{(2)}_{(\beta)}(x) \right) - \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \square f^{(0)}_0(x), \]

where \(\square := -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2\).
Proof. This follows from either the expansion of
\[
\tilde{f} = f^{(0)}_{(0)}(x''') + \sum_{\beta} \bar{\theta}^{\alpha\beta} \bar{\theta}_\alpha f^{(\beta)}_{(\beta)}(x''') + \bar{\theta}^{\alpha\beta} \bar{\theta}_\alpha f^{(12)}_{(12)}(x''),
\]
where \((x''', \theta''', \bar{\theta}'', \bar{\theta}'') := (x - \sqrt{-1} \theta \sigma \bar{\theta}', \theta, \bar{\theta}, \bar{\theta})\) is the antichiral coordinate-functions on \(\hat{X}^\mathbb{R}\), via the complexified \(C^\infty\)-ring structure of \(C^\infty(X^{\text{physics}})^{\text{ach}}\) or solving the overdetermined system of equations on components \(f^*_\alpha\) of \(\tilde{f}\) from setting to zero
\[
e_{\alpha'} \tilde{f} = \left( \frac{\partial}{\partial \theta^{\alpha}} + \sqrt{-1} \sum_{\beta, \mu} \sigma^{\alpha\beta}_{\mu} \bar{\theta}_\beta \partial_\mu \right) \tilde{f} = f_{(\alpha)} - \sum_{\gamma} \theta^\gamma \varepsilon_{\alpha\gamma} \tilde{f}_{(12)} + \sum_{\beta} \bar{\theta}^\beta \left( f_{(\alpha\beta)} + \sqrt{-1} \sum_{\mu} \sigma^{\alpha\beta}_{\mu} \partial_\mu \tilde{f}_{(0)} \right) - \sum_{\gamma, \beta} \theta^\gamma \bar{\theta}^\beta \left( \varepsilon_{\alpha\gamma} \tilde{f}_{(12\beta)} + \sqrt{-1} \sum_{\mu} \sigma^{\alpha\beta}_{\mu} \partial_\mu \tilde{f}_{(\gamma)} \right) + \sqrt{-1} \sum_{\beta, \mu} \sigma^{\beta}_{\mu} \partial_\mu \tilde{f}_{(12\beta)}
\]
for \(\alpha' = 1', 2'\).

\(C^\infty(X^{\text{physics}}) \subset C^\infty(X^{\mathbb{R}})\) under the twisted complex conjugation

Lemma 2.1.7. \([C^\infty(X^{\text{physics}}) \subset C^\infty(X^{\mathbb{R}})\) under twisted complex conjugation]\n
The twisted complex conjugation \((\cdot)^\dagger\) on \(C^\infty(X^{\mathbb{R}})\) leaves \(C^\infty(X^{\text{physics}})\) invariant and takes \(C^\infty(X^{\text{physics}})^{\text{ch}}\) to \(C^\infty(X^{\text{physics}})^{\text{ach}}\) and vice versa.

Proof. Up to a \((-1)\)-factor, the set
\[
\left\{ 1, \theta^\alpha \partial_\alpha, \bar{\theta}^\beta \bar{\partial}_\beta, \theta^1 \theta^2 \partial_1 \partial_2, \theta^\alpha \bar{\theta}^\beta, \theta^\alpha \bar{\theta}^\beta \partial_\alpha \partial_\beta, \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2, \theta^\alpha \theta^\beta \partial_1 \partial_2, \theta^\alpha \bar{\theta}^\beta \partial_\alpha \partial_\beta, \theta^\alpha \bar{\theta}^\beta \partial_1 \partial_2, \theta^\alpha \bar{\theta}^\beta \partial_\alpha \partial_\beta, \theta^\alpha \bar{\theta}^\beta \partial_1 \partial_2, \theta^\alpha \bar{\theta}^\beta \partial_\alpha \partial_\beta, \theta^\alpha \bar{\theta}^\beta \partial_1 \partial_2, \theta^\alpha \bar{\theta}^\beta \partial_\alpha \partial_\beta, \theta^\alpha \bar{\theta}^\beta \partial_1 \partial_2 \right\}_{\alpha, \beta}
\]
of generating \((\theta, \bar{\theta}, \partial, \bar{\partial})\)-monomials for \(C^\infty(X^{\text{physics}})\) in \(C^\infty(X^{\mathbb{R}})\) as a sub-\(C^\infty(X)^{\mathbb{C}}\)-module is closed under the twisted complex conjugation \((\cdot)^\dagger\). This implies that \(C^\infty(X^{\text{physics}})\) is \(\dagger\)-invariant in \(C^\infty(X^{\mathbb{R}})\).

Since a chiral function on \(X^{\text{physics}}\) is of the form
\[
\tilde{f} = f_{(0)}^{(0)}(x + \sqrt{-1} \theta \sigma \bar{\theta}') + \sum_{\alpha} \theta^\alpha \partial_\alpha f_{(\alpha)}^{(0)}(x + \sqrt{-1} \theta \sigma \bar{\theta}') + \theta^1 \theta^2 \partial_1 \partial_2 \tilde{f}_{(12)}^{(12)}(x + \sqrt{-1} \theta \sigma \bar{\theta}')
\]
for some \(f_{(0)}^{(0)}, f_{(\alpha)}^{(0)}, f_{(12)}^{(12)} \in C^\infty(X)^{\mathbb{C}}, \alpha = 1, 2\), its twisted complex conjugate must be of the form
\[
\tilde{f}^\dagger = f_{(0)}^{(0)}(x - \sqrt{-1} \theta \sigma \bar{\theta}') + \sum_{\alpha} (\theta^\alpha \partial_\alpha)^\dagger f_{(\alpha)}^{(0)}(x - \sqrt{-1} \theta \sigma \bar{\theta}') + (\theta^1 \theta^2 \partial_1 \partial_2)^\dagger f_{(12)}^{(12)}(x - \sqrt{-1} \theta \sigma \bar{\theta}')
\]

24
which is antichiral. Similarly for the converse \((\bullet)^\dagger : C^\infty(X^{\text{physics}})_{ach} \to C^\infty(X^{\text{physics}})_{ch}\).

This proves the lemma.

\[\square\]

### 2.2 Wess-Zumino model on \(X\) in terms of \(X^{\text{physics}}\)

We now proceed to construct the Wess-Zumino model ([W-Z]; also [W-B: Chap. V]) under the current setting.

**Relevant basic computations/formulae**

Let

\[
\tilde{f} = f_{(0)}^0(x) + \sum_\alpha \theta^\alpha \partial_\alpha f^{(\alpha)}_{(0)}(x) + \theta^1 \theta^2 \partial_1 \partial_2 f^{(12)}_{(12)}(x) + \sqrt{-1} \sum_{\alpha,\beta} \theta^{\alpha} \bar{\theta}^{\beta} \sum_{\mu} \sigma^\mu_{\alpha \beta} \partial_\mu f^{(0)}_{(0)}(x)
\]

\[+ \sqrt{-1} \sum_{\beta,\mu} \theta^1 \theta^2 \bar{\partial}^{\beta} \left( \bar{\partial}_1 \sigma^{\mu}_{2\beta} \partial_\mu f^{(1)}_{(1)}(x) - \bar{\partial}_2 \sigma^{\mu}_{1\beta} \partial_\mu f^{(2)}_{(2)}(x) \right) - \theta^1 \theta^2 \bar{\theta}^{\dagger} \bar{\partial}^2 \square f^{(0)}_{(0)}(x), \]

be a chiral function on \(X^{\text{physics}}\), determined by \((f^{(0)}_{(0)}, f^{(\alpha)}_{(\alpha)}, f^{(12)}_{(12)})\). It follows from Lemma 2.1.7 that its twisted complex conjugate \(\tilde{f}^\dagger\) is the antichiral function on \(X^{\text{physics}}\) determined by \((\tilde{f}^{(0)}_{(0)}, \tilde{f}^{(\alpha)}_{(\alpha)}, \tilde{f}^{(12)}_{(12)})\), where \(\tilde{f}^{(\bullet)}\) is the complex conjugate of \(f^{(\bullet)} \in C^\infty(X)^C\). Explicitly,

\[
\tilde{f}^\dagger = \tilde{f}^{(0)}_{(0)}(x) - \sum_\beta \bar{\theta}^\beta \bar{\partial}^{\beta} \tilde{f}^{(\beta)}_{(\beta)}(x) + \bar{\theta}^1 \bar{\theta}^2 \bar{\partial}_1 \bar{\partial}_2 \tilde{f}^{(12)}_{(12)}(x) - \sqrt{-1} \sum_{\alpha,\beta} \bar{\theta}^\alpha \bar{\partial}^{\beta} \sum_{\mu} \sigma^\mu_{\alpha \beta} \partial_\mu \tilde{f}^{(0)}_{(0)}(x)
\]

\[- \sqrt{-1} \sum_{\alpha,\mu} \theta^\alpha \partial^{\mu} \left( \bar{\partial}_1 \sigma^{\mu}_{\alpha 2} \partial_\mu \tilde{f}^{(1)}_{(1)}(x) - \bar{\partial}_2 \sigma^{\mu}_{\alpha 1} \partial_\mu \tilde{f}^{(2)}_{(2)}(x) \right) - \theta^1 \theta^2 \bar{\theta}^{\dagger} \bar{\theta}^2 \square \tilde{f}^{(0)}_{(0)}(x).\]
Consequently, (recall that $\Box := -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$)

\[ \vec{f}^4 = \vec{f} \vec{f} = f_{(0)}^{(0)}(x) f_{(0)}^{(0)}(x) + \sum_{\alpha} \theta^\alpha \partial_{\alpha} f_{(0)}^{(0)}(x) f_{(\alpha)}^{(0)}(x) - \sum_{\beta} \theta^\beta \partial_\beta f_{(\beta)}^{(1)}(x) f_{(0)}^{(0)}(x) + \theta^1 \theta^2 \partial_1 \partial_2 f_{(0)}^{(0)}(x) f_{(12)}^{(12)}(x) \]

\[ + \sum_{\alpha, \beta} \theta^\alpha \theta^\beta \left( \sqrt{-1} \sum_{\mu} \sigma_{\alpha \beta} \partial_{\mu} f_{(0)}^{(0)}(x) \right) \]

\[ + \sum_{\beta} \theta^1 \theta^2 \theta^3 \left( \sqrt{-1} \sum_{\mu} \left( \theta^2 \sigma_{\alpha 2} \partial_{\mu} f_{(1)}^{(1)}(x) \right) f_{(0)}^{(0)}(x) - f_{(1)}^{(1)}(x) \right) \]

\[ + \sum_{\alpha} \theta^\alpha \theta^1 \theta^2 \left( \theta^2 \theta^3 \partial_1 f_{(12)}^{(12)}(x) \right) \]

\[ \Theta \theta^2 \theta^3 \left( \left( - \Box f_{(0)}^{(0)}(x) \right) \cdot f_{(0)}^{(0)}(x) - f_{(1)}^{(1)}(x) \right) \]

\[ + \sum_{\alpha, \beta} \partial_\alpha \partial_\beta f_{(0)}^{(0)}(x) \]

\[ \vec{f} = f_{(0)}^{(0)}(x)^2 + 3 \sum_{\alpha} \theta^\alpha \partial_{\alpha} f_{(0)}^{(0)}(x) f_{(\alpha)}^{(0)}(x) + 2 \theta^1 \theta^2 \theta_1 \theta_2 \left( f_{(0)}^{(0)}(x) f_{(12)}^{(12)}(x) - f_{(1)}^{(1)}(x) f_{(2)}^{(2)}(x) \right) \]

\[ + \left( \text{terms of } \theta \text{-degree } \geq 1 \right) ; \]

\[ \vec{f}^3 = f_{(0)}^{(0)}(x)^3 + 3 \sum_{\alpha} \theta^\alpha \partial_{\alpha} f_{(0)}^{(0)}(x) f_{(\alpha)}^{(0)}(x) + 3 \theta^1 \theta^2 \theta_1 \theta_2 \left( f_{(0)}^{(0)}(x)^2 f_{(12)}^{(12)}(x) - 2 f_{(0)}^{(0)}(x) f_{(1)}^{(1)}(x) f_{(2)}^{(2)}(x) \right) \]

\[ + \left( \text{terms of } \theta \text{-degree } \geq 1 \right) ; \]

\[ \left( \vec{f}^4 \right)^2 = f_{(0)}^{(0)}(x)^2 \]

\[ + \left( \text{terms of } \theta \text{-degree } \geq 1 \right) ; \]

\[ \left( \vec{f}^4 \right)^3 = f_{(0)}^{(0)}(x)^3 - 3 \sum_{\beta} \theta^\beta \theta_1 \theta_2 f_{(0)}^{(0)}(x)^2 f_{(\beta)}^{(1)}(x) + \theta^1 \theta^2 \theta_1 \theta_2 \left( f_{(0)}^{(0)}(x)^2 f_{(12)}^{(12)}(x) - 2 f_{(0)}^{(0)}(x) f_{(1)}^{(1)}(x) f_{(2)}^{(2)}(x) \right) \]

\[ + \left( \text{terms of } \theta \text{-degree } \geq 1 \right) . \]
Basic identities in spinor calculus used in the above computations to render the spinor indices paired up more elegantly are all collected in the Appendix; see, e.g., [W-B: Chap. III, Exercises (1) & (2); Appendices A & B] for a more complete list.

The standard purge-evaluation/index-contracting map $\mathcal{P}$ with respect to $(\theta, \tilde{\theta}, \vartheta, \tilde{\vartheta})$

Given a uniform purge-evaluation map $\mathcal{P} : C^\infty(X^\text{physics}) \to C^\infty(\tilde{X})$, let $\tilde{f} \in C^\infty(X^\text{physics})$. Then the variation of $\mathcal{P}(\tilde{f})$ under the infinitesimal supersymmetry generators $Q_\alpha$'s and $\tilde{Q}_\beta$ takes the form

$$Q_\alpha \mathcal{P}(\tilde{f}) := \left( \frac{\partial}{\partial \theta^\alpha} - \sqrt{-1} \sum_{\beta, \mu} \sigma^\mu_{\alpha \beta} \bar{\theta}^\beta \partial_\mu \right) \mathcal{P}(\tilde{f})$$

$$- \mathcal{P}(\theta_\alpha f^{(\alpha)}_\mu) \mathcal{P}(\tilde{\theta}_\mu f^{(\alpha)}_\beta) + \mathcal{P}(\bar{\theta}_\beta \bar{f}^{(\beta)}_\mu) \mathcal{P}(\bar{\tilde{\theta}}^\mu f^{(\beta)}_\alpha)$$

$$+ \sum_{\gamma, \delta} \theta^\gamma \tilde{\theta}^\delta \left( - \varepsilon_{\alpha \gamma} \left( \sum_{\gamma'} \mathcal{P}(\theta_{\gamma'} f^{(\gamma')}_{(12)\delta}) \right) + \mathcal{P}(\bar{\theta}_1 \bar{\theta}_2 \bar{\theta}^\gamma f^{(\gamma')}_{(12)\delta}) \right)$$

$$- \sum_{\beta, \mu} \theta^{\beta} \theta^\mu \mathcal{P}(\bar{\theta}_1 \bar{\theta}_2) \mathcal{P}(\tilde{\theta}_\mu f^{(\beta)}_{(12)})$$

$$+ \sum_{\gamma} \theta^{\gamma} \tilde{\theta}^\gamma \mathcal{P}(\tilde{\theta}^\gamma f^{(\gamma)}_{(12)\beta})$$

$$- \bar{\theta}^{\gamma} \bar{\theta}^\gamma \left( - \varepsilon_{\gamma \delta} \left( \sum_{\beta, \mu} \mathcal{P}(\bar{\theta}_\beta \bar{\theta}^\delta f^{(\beta)}_{(12)\mu}) \right) + \mathcal{P}(\bar{\theta}_1 \bar{\theta}_2 \bar{\theta}^\gamma f^{(\gamma)}_{(12)\beta}) \right)$$

$$- \bar{\theta}^{\gamma} \bar{\theta}^\gamma \mathcal{P}(\tilde{\theta}^{\gamma} f^{(\gamma)}_{(12)\beta})$$

$$+ \sum_{\beta, \mu} \tilde{\theta}^{\beta} \tilde{\theta}^\mu \mathcal{P}(\tilde{\theta}^{\beta} f^{(\beta)}_{(12)\mu})$$

$$+ \sum_{\beta, \mu} \varepsilon_{\beta, \mu} \sigma^\mu_{\alpha \beta} \mathcal{P}(\theta_\alpha \theta^\mu f^{(\beta)}_{(12)})$$

$$+ \sum_{\beta, \mu} \varepsilon_{\beta, \mu} \sigma^\mu_{\alpha \beta} \mathcal{P}(\tilde{\theta}_\beta \tilde{\theta}^\mu f^{(\beta)}_{(12)})$$

$$- \varepsilon_{\beta, \mu} \sigma^\mu_{\alpha \beta} \mathcal{P}(\bar{\theta}_1 \bar{\theta}_2 \bar{\theta}^\gamma f^{(\gamma)}_{(12)})$$

$$- \varepsilon_{\beta, \mu} \sigma^\mu_{\alpha \beta} \mathcal{P}(\bar{\theta}_1 \bar{\theta}_2 \bar{\theta}^\gamma f^{(\gamma)}_{(12)})$$
\[ Q_\beta P(\tilde{f}) := \left( -\frac{\partial}{\partial \theta^\beta} + \sqrt{-1} \sum_{\alpha,\mu} \theta^\alpha \sigma^\mu_{\alpha \beta} \partial_\mu \right) P(f) \]

\[ = -P(\tilde{\theta}^\beta) f^{(\beta)} + \sum_\alpha \theta^\alpha \left( \sum_\mu \sigma^\mu_{\alpha \beta} (f^{(0)}_\mu + \sqrt{-1} \partial_\mu f^{(0)}(x)) + P(\partial_\alpha \tilde{\theta}^\beta) f^{(\alpha \beta)}(x) \right) \]

\[ + \tilde{\theta}^\beta P(\tilde{\theta}^\gamma \tilde{\theta}^\delta) f^{(12)}(x) \]

\[ + \theta^\gamma \theta^\delta \left( \sum_\gamma \sum_\delta \sum_\mu \epsilon^{\alpha \gamma \sigma}_{\alpha \beta} \left( \sum_\nu \sigma^\nu_{\gamma \delta} \partial_\nu f^{(0)}(x) + \sum_\nu P(\partial_\nu \tilde{\theta}^\gamma \tilde{\theta}^\delta) f^{(\gamma \delta)}(x) \right) \right) \]

\[ + \theta^\gamma \theta^\delta \cdot \sqrt{-1} \sum_\alpha \sum_\gamma \sum_\delta \sum_\mu \epsilon^{\alpha \gamma \sigma}_{\alpha \beta} \left( \sum_\nu P(\partial_\nu \tilde{\theta}^\gamma \tilde{\theta}^\delta) f^{(\gamma \delta)}(x) \right) \]

\[ + \theta^\gamma \theta^\delta \cdot \sqrt{-1} \sum_\alpha \sum_\gamma \sum_\delta \sum_\mu \epsilon^{\alpha \gamma \sigma}_{\alpha \beta} \left( \sum_\nu P(\partial_\nu \tilde{\theta}^\gamma \tilde{\theta}^\delta) f^{(\gamma \delta)}(x) \right). \]

A comparison of \( \tilde{f} \tilde{f} \) (resp. \( \tilde{f}^2, \tilde{f}^3 \), and \( Q_\alpha P(f) \& Q_\beta P(f) \)) with \( \text{[W-B: Chap. V: Eq. (5.9)]} \)

(resp. \[\text{[ibidem: Eq. (5.7)]}, \text{[ibidem: Eq. (5.8)]}, \text{[ibidem: Eq. (3.10)]}\]) of Julius Wess and Jonathan Bagger motivates the following definition:

**Definition 2.2.1. [standard purge-evaluation/index-contracting map]** The purge-evaluation map \( P : C^\infty(X^{\text{physics}}) \to C^\infty(\tilde{X}) \) that takes

\[ \tilde{f} = f^{(0)} + \sum_\alpha \theta^\alpha \tilde{\gamma}_\alpha f^{(\alpha)}(x) + \sum_\beta \tilde{\gamma}_\beta \tilde{\gamma}_\beta f^{(\beta)}(x) \]

\[ + \theta^\gamma \theta^\delta \tilde{\gamma}_\gamma \tilde{\gamma}_\delta f^{(12)}(x) \]

\[ + \theta^\gamma \theta^\delta \left( \sum_\alpha \sum_\gamma \sum_\delta \sum_\mu \epsilon^{\alpha \gamma \sigma}_{\alpha \beta} \left( \sum_\nu P(\partial_\nu \tilde{\gamma}_\gamma \tilde{\gamma}_\delta) f^{(\gamma \delta)}(x) \right) \right) \]

\[ + \theta^\gamma \theta^\delta \cdot \sqrt{-1} \sum_\alpha \sum_\gamma \sum_\delta \sum_\mu \epsilon^{\alpha \gamma \sigma}_{\alpha \beta} \left( \sum_\nu P(\partial_\nu \tilde{\gamma}_\gamma \tilde{\gamma}_\delta) f^{(\gamma \delta)}(x) \right) \]

\[ + \theta^\gamma \theta^\delta \left( \sum_\alpha \sum_\gamma \sum_\delta \sum_\mu \epsilon^{\alpha \gamma \sigma}_{\alpha \beta} \left( \sum_\nu P(\partial_\nu \tilde{\gamma}_\gamma \tilde{\gamma}_\delta) f^{(\gamma \delta)}(x) \right) \right). \]
\[\mathcal{P}(\tilde{f}) = f^{(0)}_{(0)} + \sum_{\alpha} \theta^\alpha f^{(\alpha)}_{(\alpha)} + \sum_{\beta} \bar{\theta}^\beta f^{(\bar{\beta})}_{(\bar{\beta})} + \theta^1 \bar{\theta}^2 f^{(12)}_{(12)} + \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \left( \sum_{\mu} \sigma^\mu_{\alpha\beta} f^{(0)}_{(\mu)} + f^{(\alpha\bar{\beta})}_{(\alpha\bar{\beta})} \right) + \tilde{\theta}^1 \bar{\theta}^2 f^{(12)}_{(12)} + \sum_{\beta} \theta^\alpha \bar{\theta}^\beta \tilde{\theta}^\alpha \tilde{\theta}^\beta (f^{(1)}_{(12\beta)} + f^{(2)}_{(12\bar{\beta})}) + \sum_{\alpha} \theta^\alpha \bar{\theta}^\beta \tilde{\theta}^\alpha \tilde{\theta}^\beta (f^{(1)}_{(\alpha\bar{\beta})} + f^{(\alpha\bar{\beta})}_{(\alpha\bar{\beta})}) \]

\[+ \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 (f^{(0)}_{(12\bar{\beta})} + f^{(0)}_{(12\beta)} + f^{(12)}_{(12\bar{\beta})} + f^{(12)}_{(12\beta)}) \]

is called the standard purge-evaluation map with respect to \((\theta, \bar{\theta}, \bar{\phi}, \bar{\phi})\)\(^{15}\). Note that in this process, the lower fermionic indices of \((\bar{\phi}, \bar{\phi})\) in each \((\theta, \bar{\theta}, \bar{\phi}, \bar{\phi})\)-monomial are contracted out\(^{16}\) with the same upper fermionic indices of the coefficient \(f^{(*)}_{(\alpha)}\) and we call \(\mathcal{P}\) also the standard index-contracting map.

**A supersymmetric action functional for chiral multiplets via the Fundamental Theorem**

A chiral function \(\tilde{f}\) on \(X^{\text{physics}}\) contains four independent component \(f^{(0)}_{(0)}, f^{(\alpha)}_{(\alpha)}, f^{(12)}_{(12)} \in C^\infty(X)^C\), \(\alpha = 1, 2\). It follows from Theorem 1.5.3 that

\[S_1(f^{(0)}_{(0)}, f^{(1)}_{(1)}, f^{(2)}_{(2)}, f^{(12)}_{(12)}) = S_1(f^{(0)}_{(0)}, f^{(1)}_{(1)}, f^{(2)}_{(2)}; f^{(12)}_{(12)}) := \int_X d^4 x d\theta^2 d\bar{\theta}^2 d\theta^1 d\bar{\theta}^1 \mathcal{P}(\tilde{f} \lambda \tilde{f}) + \frac{1}{2} \int_X d^4 x d\theta^2 d\bar{\theta}^2 d\theta^1 d\bar{\theta}^1 \mathcal{P}(\tilde{f} \lambda \tilde{f}^2 + \frac{1}{2} \tilde{m} \tilde{f} + \frac{1}{3} \tilde{g} \tilde{f}^3)\]

\[+ \int_X d^4 x d\theta^2 d\bar{\theta}^2 d\theta^1 d\bar{\theta}^1 \mathcal{P}(\lambda \tilde{f}^1 + \frac{1}{2} \tilde{m} \lambda \tilde{f} + \frac{1}{3} \tilde{g} \lambda \tilde{f}^3 + \frac{1}{3} \tilde{g} (\tilde{f}^1)^3)\]

\(^{15}\)Here, for example, we use \(f^{(12)}_{(12\beta)}, f^{(1)}_{(12\beta)}, f^{(2)}_{(12\beta)}\) to distinguish \(\mathcal{P}(\tilde{\theta}_1 f^{(12)}_{(12\beta)})\), \(\mathcal{P}(\tilde{\theta}_2 f^{(2)}_{(12\beta)})\), \(\mathcal{P}(\tilde{\theta}_1 \tilde{\theta}_2 \tilde{\phi} \tilde{\phi} f^{(12)}_{(12\beta)})\). Since we use no further detail of this sum in this work, we leave it unspecified which is which. Similarly for \(f^{(12)}_{(12\beta)}, f^{(12)}_{(12\beta)}\), \(f^{(12)}_{(12\beta)}\).

\(^{16}\)The fermionic indices of a coefficient (e.g. \(f^{(12)}_{(12)}\)) behave like the indices of a tensor. When performing the standard purge-evaluation map \(\mathcal{P}\), paired lower-and-upper fermionic indices (e.g. the lower 1 and 2 in \(\tilde{\theta}_1 \tilde{\theta}_2\) versus the upper 1 and 2 in \(f^{(12)}_{(12)}\)) should drop out from the index structure via evaluation of a dual spinor on a spinor.

The true index structure should be the leftover indices. (E.g. \(\tilde{\theta}_1 \tilde{\theta}_2 f^{(12)}_{(12)} \rightarrow f^{(12)}_{(12)} := f^{(12)}_{(12)}\) )
gives a functional of chiral multiplets \( (f^{(0)}; f^{(1)}; f^{(2)}; f^{(12)}) \) that is invariant under supersymmetries up to a boundary term on \( X \).

Explicitly, up to boundary terms on \( X \), this is the action functional (cf. [W-B: Chap. V, Eq. (5.11)])

\[
S_1(f^{(0)}, f^{(1)}, f^{(2)}, f^{(12)}) = \int_X d^4x \left( 4 \sum_\mu \partial_\mu f^{(0)}(x) \partial^\mu f^{(0)}(x) + \mu \sum_{\alpha, \beta} \left( \bar{f}^{(\beta)}(x) \cdot \sqrt{-1} \sum_\mu \bar{\sigma}^{\mu, \beta \alpha} \partial_\mu f^{(\alpha)}(x) + f^{(\alpha)}(x) \cdot \sqrt{-1} \sum_\mu \sigma^{\mu, \beta \alpha} \partial_\mu \bar{f}^{(\beta)}(x) \right) + f^{(12)}(x) f^{(12)}(x) \right) + \int_X d^4x \left( 4 f^{(0)}(x) f^{(12)}(x) - f^{(1)}(x) f^{(2)}(x) \right) + \lambda f^{(12)}(x) + m \left( f^{(0)}(x) f^{(12)}(x) - f^{(1)}(x) f^{(2)}(x) \right) + g \left( f^{(0)}(x)^2 f^{(12)}(x) - 2 f^{(0)}(x) f^{(1)}(x) f^{(2)}(x) \right) + \text{complex conjugate} \).
\]

The index structure of this explicit expression implies that this functional is indeed Lorentz invariant.

This is what underlies [W-B: Chap. V] of Wess & Bagger from the aspect of \( C^\infty \)-Algebraic Geometry.

3 **Supersymmetric \( U(1) \) gauge theory with matter on \( X \) in terms of \( X^\text{physics} \)**

In this section, we reproduce the supersymmetric \( U(1) \) gauge theory with matter in [W-B: Chap. VI & \( U(1) \) part of Chap. VII] of Wess & Bagger from the (complexified \( \mathbb{Z}/2 \)-graded) \( C^\infty \)-Algebraic Geometry setting in Sec. 1.

3.1 **The bundle/sheaf context underlying a supersymmetric \( U(1) \) gauge theory with matter built from \( X^\text{physics} \)**

On the mathematics side, a gauge theory usually begins with the setup of a principal bundle, associate bundles from representations, connections and their curvature tensor, e.g. [D-K: Sec. 2.1]. Such a geometric setup remains there for supersymmetric gauge theories, only that there are rarely mentioned or brought to front in physics literature. When mathematicians (or mathematics-oriented physicists) attempt to set such geometry up, the precise setting depends on how the notion of ‘superspace’ is defined in their context.

In this subsection, we give such a geometric setup for supersymmetric \( U(1) \)-gauge theory with matter in the language of (complexified \( \mathbb{Z}/2 \)-graded) \( C^\infty \)-Algebraic Geometry. As nilpotent objects (e.g. \( \theta, \bar{\theta}, \vartheta, \bar{\vartheta} \)) are everywhere in our problem, it is more convenient to use the language of sheaves, rather than bundles. This is the sheaf theory in (complexified \( \mathbb{Z}/2 \)-graded) \( C^\infty \)-Algebraic Geometry behind the scene for [W-B: Chap. VI & \( U(1) \) part of Chap. VII] of Wess & Bagger. It can be generalized to the higher rank, nonabelian case.
The built-in principal bundle/sheaf and all that

The multiplicative group of invertible elements of \(O_X^{\text{physics}}\) defines a principal sheaf \(O_X^{\text{physics}, \times}\) over \(X^{\text{physics}}\). It corresponds to the sheaf of sections of a tautological principal \(\mathbb{C}^\times\)-bundle \(P^{\mathbb{C}^\times}\) over \(X^{\text{physics}}\). Note that an \(\tilde{f} \in O_X^{\text{physics}}\) is invertible if and only if its \((\theta, \bar{\theta}, \vartheta, \bar{\vartheta})\)-degree-zero component is invertible, i.e. \(f_{(0)}^{(0)} \in O_X^{\mathbb{C}^\times}\). Thus,

\[
O_X^{\text{physics}, \times} = \{ \tilde{f} \in O_X^{\text{physics}} \mid f_{(0)}^{(0)} \in O_X^{\mathbb{C}^\times} \}.
\]

This is the tautological principal \(\mathbb{C}^\times\)-sheaf on \(X^{\text{physics}}\), where \(\mathbb{C}^\times := (\mathbb{C} - \{0\}, \times)\) is the multiplicative group of \(\mathbb{C}\). Since \(\mathbb{C}^\times\) is abelian, the adjoint representation of \(\mathbb{C}^\times\) on \(X^{\text{physics}}\), where \(\mathbb{C}^\times\) is the sheaf of group of invertible elements in \(O_X^{\text{physics}}\), is given explicitly by

\[
\left(\exp f \right)_{x} = \exp f_{(0)}^{0} \cdot \sum_{l=0}^{\infty} \frac{1}{l!} (f_{(0)}^{0})^{l}.
\]

Here, \(\tilde{f} = f_{(0)}^{0} + \tilde{f}_{(\geq 1)}\), with \(f_{(0)}^{0} \in O_X^{\mathbb{C}}\) and \(\tilde{f}_{(\geq 1)}\) the nilpotent part of \(\tilde{f}\), and note that \((f_{(0)}^{0})_{(\geq 1)}^{5} = 0\) for \(\tilde{f} \in O_X^{\text{physics}}\). Note also that when \(f_{(0)}^{0}\) is real, this is compatible with the \(C^\infty\)-hull structure of \(O_X^{\text{physics}}\). Its local inverse near the identity section \(1 \in O_X^{\text{physics}, \times}\) defines the Log map:

\[
\text{Log} : O_X^{\text{physics}, \times} \longrightarrow O_X^{\text{physics}}
\]

\[
\tilde{f} \longmapsto \log f_{(0)}^{0} + \sum_{l=1}^{\infty} \frac{(-1)^l}{l} (f_{(0)}^{0})^{-1} f_{(\geq 1)}^{l}.
\]

This is also compatible with the \(C^\infty\)-hull structure of \(O_X^{\text{physics}} \supset O_X^{\text{physics}, \times}\) when \(f_{(0)}^{0}\) is real.

**Definition 3.1.1.** [physics-related principal subsheaves in \(O_X^{\text{physics}, \times}\)] There are four physics-related principal sheaves of subgroups in the tautological principal \(\mathbb{C}^\times\)-sheaf \(O_X^{\text{physics}, \times}\) over \(X^{\text{physics}}\). Each is characterized by its associated sheaf of Lie subalgebras in \(O_X^{\text{physics}}\):

1. **[tautological chiral principal \(\mathbb{C}^\times\)-sheaf \(O_X^{\text{physics}, \times, ch}\)]** Note that

\[
O_X^{\text{physics}, \times, ch} := O_X^{\text{physics}} \cap O_X^{\text{physics}, ch} = O_X^{\text{physics}, ch, \times},
\]

where \(O_X^{\text{physics}, ch, \times}\) is the sheaf of group of invertible elements in \(O_X^{\text{physics}, ch}\). This defines the tautological chiral principal \(\mathbb{C}^\times\)-sheaf on \(X^{\text{physics}}\), whose associated sheaf of Lie algebras is \(O_X^{\text{physics}, ch}\), which is the same as \(\sqrt{-1} \cdot O_X^{\text{physics, ch}}\). The exponential map \(\text{Exp}\) restricts to \(e = \text{Exp} : O_X^{\text{physics}, ch} \rightarrow O_X^{\text{physics}, ch, \times}\).

2. **[tautological antichiral principal \(\mathbb{C}^\times\)-sheaf \(O_X^{\text{physics}, ach, \times}\)]** Note that

\[
O_X^{\text{physics}, ach, \times} := O_X^{\text{physics}} \cap O_X^{\text{physics}, ach} = O_X^{\text{physics}, ach, \times},
\]

where \(O_X^{\text{physics}, ach, \times}\) is the sheaf of group of invertible elements in \(O_X^{\text{physics}, ach}\). This defines the tautological antichiral principal \(\mathbb{C}^\times\)-sheaf on \(X^{\text{physics}}\), whose associated sheaf of Lie algebras is \(O_X^{\text{physics}, ach}\), which is the same as \(\sqrt{-1} \cdot O_X^{\text{physics}, ach}\). The exponential map \(\text{Exp}\) restricts to \(e = \text{Exp} : O_X^{\text{physics}, ach} \rightarrow O_X^{\text{physics}, ach, \times}\).
Let

\[ O_X^{\text{physics},b,U(1)} : \{ f \in O_X^{\text{physics}} \mid \tilde{f} \text{ is of the form} \ (in \ the \ standard \ coordinate \ functions \ (x, \theta, \bar{\theta}, \bar{\vartheta}) \ on \ \hat{X}) \}
\]

\[ f_{(0)} + \sum \theta^\alpha \partial_\alpha f_{(0)}^{(\alpha)} + \sum \bar{\theta}^\beta \bar{\partial}_\beta f_{(\bar{\beta})} + \theta^\alpha \partial_\alpha \bar{\theta}^\beta f_{(12)} \]

\[ + \sum \bar{\theta}^\beta \bar{\theta}^\beta \sum \theta^\alpha \partial_\alpha f_{(12(\bar{\beta})} + \sum \theta^\alpha \partial_\alpha \bar{\theta}^\beta \sum \bar{\theta}^\beta f_{(12(\bar{\beta})} \]

\[ + \theta^\alpha \partial_\alpha \bar{\theta}^\beta \bar{\theta}^\beta f_{(12)} ; \]

namely, \( f_{(\alpha\bar{\beta})} = f_{(12\bar{\beta})} = f_{(12\bar{\beta})} = f_{(12\bar{\beta})} = 0 \)

and

\[ O_X^{\text{physics},b,\text{stc}} := \{ \tilde{s} \in O_X^{\text{physics}} \mid \tilde{s} = \tilde{s} \} \subset O_X^{\text{physics}} \]

as a sub-\( O_X \)-module. The image of the restriction of \( \text{Exp} \) to \( \sqrt{-1} \cdot O_X^{\text{physics},b,\text{stc}} \) is a sheaf of subgroups in \( O_X^{\text{physics},x} \), whose restriction to \( X^\mathbb{C} \) is a sheaf of sections of a principal \( U(1) \)-bundle over \( X \). Denote this image in \( O_X^{\text{physics},x} \) by \( O_X^{\text{physics},b,U(1)} \) and call it the \textit{tautological principal \( U(1) \)-sheaf on \( X^{\text{physics}} \)}. This corresponds to the principal \( U(1) \)-bundle in the gauge theory we are going to study. The construction realizes \( \sqrt{-1} \cdot O_X^{\text{physics},b,\text{stc}} \) as the associated sheaf of Lie algebras of \( O_X^{\text{physics},b,U(1)} \). One can impose further \( \mathbb{R} \)-linear constraints on \( O_X^{\text{physics},b,\text{stc}} \) to obtain \textit{descendants} of \( O_X^{\text{physics},b,U(1)} \) via the restriction of the exponential map \( \text{Exp} \) on \( \sqrt{-1} \cdot O_X^{\text{physics},b,\text{stc}} \).

(4) \text{[tautological principal \( \mathbb{R}^x \)-sheaf \( O_X^{\text{physics},b,\mathbb{R}^x} \) and its descendants]} \quad The image of the restriction of \( \text{Exp} \) to \( O_X^{\text{physics},b,\text{stc}} \) is a sheaf of subgroups in \( O_X^{\text{physics},x} \), whose restriction to \( X^\mathbb{C} \) is a sheaf of sections of a principal \( \mathbb{R}^x \)-bundle over \( X \). Denote this image in \( O_X^{\text{physics},x} \) by \( O_X^{\text{physics},b,\mathbb{R}^x} \) and call it the \textit{tautological principal \( \mathbb{R}^x \)-sheaf on \( X^{\text{physics}} \)}. The construction realizes \( O_X^{\text{physics},b,\text{stc}} \) as the associated sheaf of Lie algebras of \( O_X^{\text{physics},b,\mathbb{R}^x} \). One can impose further \( \mathbb{R} \)-linear constraints on \( O_X^{\text{physics},b,\text{stc}} \) to obtain \textit{descendants} of \( O_X^{\text{physics},b,\mathbb{R}^x} \) via the restriction of the exponential map \( \text{Exp} \) on \( O_X^{\text{physics},b,\text{stc}} \).

\textbf{Definition 3.1.2.} \( O_X^{\text{physics},ch}, O_X^{\text{physics},ach}, O_X^{\text{physics},b,\mathbb{R}^x} \) as \( O_X^{\text{physics},ch,x} \)-modules For each \( \epsilon_m \in \mathbb{R}^1 \), we shall consider the following action of the tautological principal \( \mathbb{C}^x \)-sheaf \( O_X^{\text{physics},ch,x} \) on \( O_X^{\text{physics},ch}, O_X^{\text{physics},ach}, \), and \( O_X^{\text{physics},b,U(1)} \). This turns them into \( O_X^{\text{physics},ch,x} \)-modules.

(1) \( O_X^{\text{physics},ch} \) Left multiplication in \( O_X^{\text{physics}} \) by the section: \( e^{\sqrt{-1} \epsilon_m \Lambda} \cdot O_X^{\text{physics},ch} \). Note that this leaves \( O_X^{\text{physics},ch} \subset O_X^{\text{physics}} \) invariant.

(2) \( O_X^{\text{physics},ach} \) Right multiplication in \( O_X^{\text{physics}} \) by the twisted complex conjugate of the section: \( O_X^{\text{physics},ach} \cdot e^{-\sqrt{-1} \epsilon_m \Lambda^t} \). Note that this leaves \( O_X^{\text{physics},ach} \subset O_X^{\text{physics}} \) invariant.

\textbf{Note for mathematicians} \quad The number \( \epsilon_m \) is the electric charge of the chiral matter fields realized as global sections of \( O_X^{\text{physics},ch} \) in the supersymmetric \( U(1) \) gauge theory with matter.

\textbf{Caution} \quad \( O_X^{\text{physics},ach} \) is only an \( O_X \)-submodule, not an \( O_X \)-subalgebra, of \( O_X^{\text{physics}} \). It is not closed under the multiplication of sections.

\textbf{Here, stc stands for self-twisted-complex-conjugate}

\textbf{Note for mathematicians} \quad The number \( \epsilon_m \) is the electric charge of the chiral matter fields realized as global sections of \( O_X^{\text{physics},ch} \) in the supersymmetric \( U(1) \) gauge theory with matter.
Left multiplication in $O^\text{physics}_X$ by the inverse of the twisted complex conjugate of the section in accompany with right multiplication in $O^\text{physics}_X$ by the inverse of the section:

$$e^{\sqrt{-1}e_m\lambda^\dagger} \cdot O^\text{physics}_X \cdot e^{-\sqrt{-1}e_m\lambda}.$$  

Note that this leaves $O^\text{physics}_X, \lrcorner, \mathbb{R} \times X$ invariant.

Here a section of $O^\text{physics, ch}_X$ is expressed in terms of its associated sheaf of Lie algebras as $e^{\sqrt{-1}\tilde{\Lambda}}$ with $\tilde{\Lambda} \in O^\text{physics, ch}_X$ and a section of $O^\text{physics, stc}_X$ is expressed in terms of its associated sheaf of Lie algebras as $e^{\sqrt{-1}\tilde{V}}$ with $\tilde{V} \in O^\text{physics, stc}_X$. From the gauge-theoretical aspect, $O^\text{physics, ch}_X$ plays the role of the sheaf of gauge symmetries in the problem.

For all the discussions below until the last theme ‘A supersymmetric action functional for $U(1)$ gauge theory with matter on $X$’ of Sec. 3.5, we will set $e_m = 1$ so that we don’t have to carry the symbol all along. By replacing $\tilde{\Lambda}$ with $e_m\tilde{\Lambda}$, we recover the charge $e_m$ case.

**Definition 3.1.3.** $[O^\text{physics}_X \& \hat{O}^\hat{\otimes}_X$ as $O^\text{physics, ch, x}_X$-modules] The same three operations (1), (2), (3) in Definition 3.1.2 realize both $O^\text{physics}_X$ and $\hat{O}^\hat{\otimes}_X$ as left (cf. (1)), right (cf. (2)), bi- (cf. (3)) $O^\text{physics, ch, x}_X$-modules respectively.

**Even left connections and their curvature tensor**

The notion of ‘connection’ in [L-Y1: Sec. 2.1] can be adapted here. However, there are two opposing factors ahead of us:

(+) Since physics focuses on $O^\text{physics}_X$, which is purely even, all the complication due to the $\mathbb{Z}/2$-grading that we have to address in ibidem is gone. Thus, one only needs to consider even left connections.

(−) Since a connection is a generalization of the exterior differential operator $d$ and $\text{Der}_C(\hat{\mathcal{X}})$ does not leave $O^\text{physics}_X$ invariant, one cannot just consider a connection on an $O^\text{physics}_X$-module alone.

Based on the physical applications in practice, with both of the above two factors taken into account, one is led to consider even left connections $\nabla$ on a full $\hat{O}^\hat{\otimes}_X$-module $\hat{E}$. It won’t necessarily leave an $O^\text{physics}_X$-submodule $\mathcal{F}$ of $\hat{E}$ invariant but this is okay as long as we know where $\nabla_\xi s$ is in $\hat{E}$ for all $s \in \mathcal{F}$ and $\xi \in \text{Der}_C(\hat{\mathcal{X}})$.

**Definition 3.1.4.** [even left connection on $\hat{O}^\hat{\otimes}_X$-module] (Cf. [L-Y1:Definition 2.1.2].) Let $\hat{E}$ be an $\hat{O}^\hat{\otimes}_X$-module. An even left connection $\hat{\nabla}$ on $\hat{E}$ is a $\mathbb{C}$-bilinear pairing

$$\hat{\nabla} : \mathcal{T}_{\hat{\mathcal{X}}} \times \hat{E} \rightarrow \hat{E}$$

$$\langle \xi, s \rangle \mapsto \hat{\nabla}_\xi s$$

such that

1. [\hat{O}_X-linearity in the $\mathcal{T}_{\hat{\mathcal{X}}}$-argument]

$$\hat{\nabla}_{f_1\xi_1 + f_2\xi_2}s = f_1\hat{\nabla}_{\xi_1}s + f_2\hat{\nabla}_{\xi_2}s,$$

for $f_1, f_2 \in \hat{O}^\hat{\otimes}_X$, $\xi_1, \xi_2 \in \mathcal{T}_{\hat{\mathcal{X}}}$, and $s \in \hat{E}$;
(2) [C-linearity in the $\hat{\nabla}$-argument]

\[ \hat{\nabla}_\xi(c_1s_1 + c_2s_2) = c_1\hat{\nabla}_\xi s_1 + c_2\hat{\nabla}_\xi s_2, \]

for $c_1, c_2 \in \mathbb{C}$, $\xi \in T_{\hat{\mathcal{E}}}\hat{\mathbb{H}}$, and $s_1, s_2 \in \hat{\mathcal{E}}$;

(3) [\(\mathbb{Z}/2\)-graded Leibniz rule in the $\hat{\nabla}$-argument]\footnote{In \[L-Y1: \]Definition 2.1.2, a left connection on $\hat{\mathcal{E}}$ is required to satisfy the generalized $\mathbb{Z}/2$-graded Leibniz rule in the $\hat{\nabla}$-argument: $\hat{\nabla}_\xi(f(s)) = (\xi f)s + (-1)^{p(f)p(\xi)}f \cdot \hat{\nabla}_\xi s$, for $f \in \hat{\mathcal{O}}\hat{\mathbb{H}}$, $\xi \in T_{\hat{\mathcal{E}}}\hat{\mathbb{H}}$ parity homogeneous and $s \in \hat{\mathcal{E}}$, where $\hat{\gamma}(\hat{\nabla})$ is the parity-conjugation of $\hat{\nabla}$ induced by $f$; i.e., $\hat{\gamma}(\hat{\nabla}) = \hat{\nabla}$, if $f$ is even, or $\hat{\gamma}(\hat{\nabla}) := (\text{even part of } \hat{\nabla}) - (\text{odd part of } \hat{\nabla})$ if $f$ is odd; (cf. \[L-Y1: \]Definition 1.3.1)). When $\hat{\nabla}$ is even, $\hat{\gamma}(\hat{\nabla}) = \hat{\nabla}$ always and the general $\mathbb{Z}/2$-graded Leibniz rule reduces to the $\mathbb{Z}$-graded Leibniz rule.}

\[ \hat{\nabla}_\xi(f(s)) = (\xi f)s + (-1)^{p(f)p(\xi)}f \cdot \hat{\nabla}_\xi s, \]

for $f \in \hat{\mathcal{O}}\hat{\mathbb{H}}$, $\xi \in T_{\hat{\mathcal{E}}}\hat{\mathbb{H}}$ parity homogeneous and $s \in \hat{\mathcal{E}}$.

As an operation on the pairs $(\xi, s)$, a connection $\nabla$ on $\hat{\mathcal{E}}$ is applied to $\xi$ from the right while applied to $s$ from the left\footnote{In the $\mathbb{Z}/2$-graded world, it is instructive to denote $\hat{\nabla}_\xi s$ as $\xi \hat{\nabla}s$ or $\xi \hat{\nabla}s$ (though we do not adopt it as a regularly used notation in this work). In particular, from $f(\hat{\nabla}s)$ to $f(\xi \hat{\nabla}s)$, $f$ and $\hat{\nabla}$ do not pass each other.} cf. \[L-Y1: \]Lemma 1.3.7 & Remark 1.3.8].

Note that since $\hat{\nabla}$ is even, the parity of $\hat{\nabla}_\xi$ is the same as that of $\xi$.

Lemma/Definition 3.1.5. [curvature tensor of (even left) connection] (Cf. \[L-Y1: \]Lemma/Definition 2.1.9). Continuing Definition 3.1.4. Let $\hat{\nabla}$ be an even left connection on $\hat{\mathcal{E}}$. Then the correspondence

\[ F^{\hat{\nabla}} : (\xi_1, \xi_2; s) \longmapsto \big( [\hat{\nabla}_{\xi_1}, \hat{\nabla}_{\xi_2}] - \hat{\nabla}_{[\xi_1, \xi_2]} \big) s, \]

for $\xi_1, \xi_2 \in \text{Der}_\mathbb{C}(\hat{\mathcal{X}}\hat{\mathbb{H}})$ parity-homogeneous and $s \in \hat{\mathcal{E}}$, defines an $\text{End}_{\hat{\mathcal{O}}\hat{\mathbb{H}}}(\hat{\mathcal{E}})$-valued 2-tensor on $\hat{\mathcal{X}}\hat{\mathbb{H}}$. We shall call $F^{\hat{\nabla}}$ thus defined the curvature tensor on $\hat{\mathcal{X}}\hat{\mathbb{H}}$ associated to the even left connection $\hat{\nabla}$ on $\hat{\mathcal{E}}$.

Proof. This is a special case of \[L-Y1: \]Lemma/Definition 2.1.9 with the odd part of $\hat{\nabla}$ vanishes. Using the $\mathbb{Z}/2$-graded Leibniz rule, one can show straightforwardly that, for $f$, $\xi_1$, $\xi_2$ parity-homogeneous,

\[ \big( [\hat{\nabla}_{f\xi_1}, \hat{\nabla}_{\xi_2}] - \hat{\nabla}_{[\xi_1, \xi_2]} \big) s = f \cdot \big( [\hat{\nabla}_{\xi_1}, \hat{\nabla}_{\xi_2}] - \hat{\nabla}_{[\xi_1, \xi_2]} \big) s, \]

\[ \big( [\hat{\nabla}_{\xi_1}, \hat{\nabla}_{f\xi_2}] - \hat{\nabla}_{[\xi_1, \xi_2]} \big) s = (-1)^{p(f)p(\xi_1)} f \cdot \big( [\hat{\nabla}_{\xi_1}, \hat{\nabla}_{\xi_2}] - \hat{\nabla}_{[\xi_1, \xi_2]} \big) s, \]

\[ \big( [\hat{\nabla}_{\xi_1}, \hat{\nabla}_{\xi_2}] - \hat{\nabla}_{[\xi_1, \xi_2]} \big) (f s) = (-1)^{p(f)(p(\xi_1)+p(\xi_2))} f \cdot \big( [\hat{\nabla}_{\xi_1}, \hat{\nabla}_{\xi_2}] - \hat{\nabla}_{[\xi_1, \xi_2]} \big) s. \]

This proves the lemma.

Since in this work, we only address even left connections, we will simply call them connections.
Definition 3.1.6. [connection on $\mathcal{O}^\text{physics}_X$-submodule - abuse] Though in general a connection $\hat{\nabla}$ on an $\hat{\mathcal{O}}^\oplus_X$-module $\hat{E}$ does not leave a $\mathcal{O}^\text{physics}_X$-submodule $F$ invariant and hence does not restrict to a connection on $F$, for $\xi \in \text{Der}_C(\hat{X}^\oplus)$ and $s \in F$ one does know where $\hat{\nabla}_\xi s$ goes in $\hat{E}$. Furthermore, in all our applications, $\xi \in \text{Der}_C(\hat{X})$ and hence

$$\hat{\nabla}_\xi : F \rightarrow \hat{O}_X \cdot F$$

in $\hat{E}$. For the convenience of terminology, we will still call $\hat{\nabla}$ a connection on $F$ with the understanding that it may not take values in $F$ alone.

Pre-vector superfields and their associated (even left) connections

Definition 3.1.7. [pre-vector superfield] A global section

$$\hat{V} \in \Gamma(\mathcal{O}^\text{physics}_X^{\flat,\text{stc}}) =: C^\infty(\mathbb{R}^\text{physics}_X^{\flat,\text{stc}})$$

is called a pre-vector superfield on $X^\text{physics}$.

For physicists working on supersymmetric gauge theories, the following class of even left connections (adapted to the current $U(1)$ case) is the major concern.

Definition 3.1.8. [(even left) connection associated to pre-vector superfield] With the above setting, let $\hat{V} \in C^\infty(\mathbb{R}^\text{physics}_X^{\flat,\text{stc}})$ be a pre-vector superfield on $X^\text{physics}$. Then, one can define an (even left) connection $\hat{\nabla}\hat{V}$ on $\hat{\mathcal{O}}^\oplus_X$ (as a left $\mathcal{O}^\text{physics}_X$-module) associated to $\hat{V}$ as follows.

1. Firstly, we acquire the compatibility with the chiral structure on $\mathcal{O}^\text{physics}_X$ by setting

$$\hat{\nabla}_{\hat{e}_\beta'} \hat{V} := e^{\beta'}.$$  

2. Secondly, we set

$$\hat{\nabla}_{\hat{e}_\alpha'} \hat{V} := e^{-\hat{V}} \circ e_{\alpha'} \circ e^{\hat{V}} = e_{\alpha'} + e^{-\hat{V}}(e_{\alpha'} e^{\hat{V}}).$$

Thus, in a way $V$ is an indication of the twisting of the original antichiral structure of $\mathcal{O}^\text{physics}_X$ to the one selected by $\hat{\nabla}_{\hat{e}_\alpha'}$.  

3. Finally, we set

$$\hat{\nabla}_{\hat{e}_\mu} \hat{V} = \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta} \hat{\sigma}_\mu^{\alpha\beta} \{ \hat{\nabla}_{\hat{e}_{\alpha'}}, \hat{\nabla}_{\hat{e}_{\beta'}} \},$$

where $\hat{\sigma}_\mu = (\hat{\sigma}_\mu^{\alpha\beta})_{\alpha\beta}$ with

$$\hat{\sigma}_0 := \frac{1}{2} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{\sigma}_1 := \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_2 := \frac{1}{2} \begin{bmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}, \quad \hat{\sigma}_3 := \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This is indeed a flatness condition on the curvature of $\hat{\nabla}\hat{V}$ in the fermionic directions $(e_{\alpha'}, e_{\beta'})$. (Cf. Lemma 3.1.9 for the precise statement.)

---

22The choice of using whether $e^{-\hat{V}} \circ e_{\alpha'} \circ e^{\hat{V}}$ or $e^{\hat{V}} \circ e_{\alpha'} \circ e^{-\hat{V}}$ as the definition of $\hat{\nabla}_{\hat{e}_{\alpha'}}$ is dictated by how one would construct the action functional for the gauge-invariant kinetic term for the chiral superfield in the supersymmetric $U(1)$-gauge theory with matter. The former is consistent with the setting in Sec. 3.5 while the latter isn’t. Cf. Lemma 3.2.6 vs. Sec. 3.5, theme ’Explicit computations/formulae’.  

35
Since $\tilde{V}$ is even, $\hat{\nabla} \tilde{V}$ as defined is even as well. In this way a pre-vector superfield $\tilde{V} \in C^\infty(X^{\text{physics}})^{\Lambda,\text{stc}}$ determines an even left connection $\hat{\nabla} \tilde{V}$ on $\hat{O}_X$. $\hat{\nabla} \tilde{V}$ is called the connection on $\hat{O}_X$ associated to $\tilde{V}$; cf. Definition 3.1.6. For simplicity of notations, we often denote $\hat{\nabla} \tilde{V}$ by $\hat{\nabla}$, keeping $\tilde{V}$ implicit.

Lemma 3.1.9. [flatness of $\hat{\nabla} \tilde{V}$ along fermionic directions] Let $\hat{\nabla} = \hat{\nabla} \tilde{V}$ be the connection on $\hat{O}_X$ associated to a pre-vector superfield $\tilde{V}$. Let $F_{\hat{\nabla}}$ be the curvature 2-tensor of $\hat{\nabla}$ and denote $F_{\hat{\nabla}}(e_{\alpha'}, e_{\beta'})$ (resp. $F_{\hat{\nabla}}(e_{\alpha''}, e_{\beta''})$, $F_{\hat{\nabla}}(e_{\alpha'}, e_{\beta''})$) by $F_{\hat{\nabla}}^\alpha{}_{\beta'}$ (resp. $F_{\hat{\nabla}}^\alpha{}_{\beta''}$, $F_{\hat{\nabla}}^\alpha{}_{\beta''}$). Then with respect to the supersymmetrically invariant coframe $(e^1)$ on $\hat{X}$, the components of the curvature tensor $F_{\hat{\nabla}}$ of $\hat{\nabla}$ in purely fermionic directions all vanish: for $\alpha', \beta' = 1', 2'$ and $\alpha'', \beta'' = 1'', 2''$,

$$ F_{\hat{\nabla}}^\alpha{}_{\beta'} = F_{\hat{\nabla}}^\alpha{}_{\beta''} = F_{\hat{\nabla}}^\alpha{}_{\beta''} = 0. $$

Proof. $F_{\hat{\nabla}}^\alpha{}_{\beta'} = \{e_{\alpha''}, e_{\beta''}\} = 0$. $F_{\hat{\nabla}}^\alpha{}_{\beta'} = e^{-\tilde{V}} \circ \{e_{\alpha''}, e_{\beta''}\} \circ e^{\tilde{V}} = 0$. And $F_{\hat{\nabla}}^\alpha{}_{\beta''} = \{\nabla_{e_{\alpha'}}, \nabla_{e_{\beta''}}\} - \nabla_{\{e_{\alpha'}, e_{\beta''}\}} = 0$ by tautology since $\nabla_{\{e_{\alpha'}, e_{\beta''}\}} = -2\sqrt{-1} \sum_{\mu} \sigma_{\alpha'\beta'} \tilde{\nabla}_{e_{\mu}}$ and the design of $\tilde{\nabla}_{e_{\mu}}$ as a $\mathbb{C}$-combination of $\hat{\nabla}_{(e_{\alpha'}, e_{\beta''})}$'s comes exactly from solving $F_{\hat{\nabla}}^\alpha{}_{\beta''} = 0$.

With the super version of the standard geometry for a $U(1)$ gauge theory $X^{\text{physics}}$ provided, we can now redo [W-B: Chap. VI] of Wess & Bagger to construct a supersymmetric $U(1)$ gauge theory on $X$. Readers are referred also to, e.g., [Argu: Sec. 4.3] of Riccardo Argurio for a very detailed physicists' treatment of the topics in the next two subsections. By comparison, one can see that the notion of the physics sector $X^{\text{physics}}$ of $\hat{X}$ from (complexified $\mathbb{Z}$/2-graded) $C^\infty$-Algebraic Geometry and the purge-evaluation map $\mathcal{P} : C^\infty(X^{\text{physics}}) \to C^\infty(\hat{X})$ together really fits particle physicists' language of and ways of playing with supersymmetries.

3.2 Pre-vector superfields in Wess-Zumino gauge

Most of the discussions in [W-B: Chap. VI] of Wess & Bagger for vector superfields hold for pre-vector superfields as well.

Gauge transformations of a pre-vector superfield

Recall from Definition 3.1.2 that under a gauge transformation specified by a chiral superfield $\tilde{\Lambda} \in C^\infty(X)^{\Lambda,\text{ch}}$, a pre-vector superfield $\tilde{V}$ transforms as

$$ \tilde{V} \quad \rightarrow \quad \tilde{V} + \delta_{\tilde{\Lambda}} \tilde{V} := \tilde{V} - \sqrt{-1}(\Lambda - \Lambda^\dagger). $$

Explicitly, let

$$ \begin{align*}
\tilde{V} &= V^{(0)} + \sum_{\alpha} \theta^\alpha \partial_{\alpha} V^{(\alpha)} - \sum_{\beta} \bar{\theta}^\beta \partial_{\beta} \overline{V^{(\beta)}} \\
&\quad + \theta^1 \theta^2 \vartheta_1 \partial_{2,1} V^{(12)} + \sum_{\alpha, \beta} \theta^\alpha \vartheta_1 \sum_{\mu} \sigma_{\alpha\beta} \vartheta_{\mu} V^{(0)}_{\mu} + \bar{\theta}^1 \bar{\theta}^2 \bar{\vartheta}_1 \bar{\vartheta}_2 \overline{V^{(12)}} \\
&\quad + \sum_{\beta} \theta^1 \theta^2 \overline{\vartheta}_1 \sum_{\alpha} \partial_{\alpha} V^{(\alpha)} + \sum_{\alpha} \theta^\alpha \vartheta_1 \theta^2 \sum_{\beta} \partial_{\beta} \overline{V^{(\beta)}}_{12} + \theta^1 \theta^2 \vartheta_1 \vartheta_2 V^{(0)}_{12} \\
&\quad \in C^\infty(X^{\text{physics}})^{\Lambda,\text{stc}}
\end{align*} $$

\footnote{From now on, we keep the $x$-dependence of $f^\Lambda \in C^\infty(X)^C$ implicit to declutter the notations.}

36
\[ \dot{\bar{\Lambda}} = \Lambda^{(0)}_0 + \sum_\alpha \theta^\alpha \partial_\alpha \Lambda^{(\alpha)}_0 + \theta^1 \partial^2 \bar{\theta}_1 \partial_2 \Lambda^{(12)}_0 + \sqrt{-1} \sum_\alpha \theta^\alpha \bar{\theta}_\beta \sum_\mu \sigma^\mu_{\alpha \beta} \partial_\mu \Lambda^{(0)}_0 - \sqrt{-1} \sum_\mu \theta^\alpha \bar{\theta}_\beta \sum_\gamma \varepsilon^{\alpha \gamma} \sigma^\mu_{\alpha \beta} \partial_\mu (\partial_\gamma \Lambda^{(\gamma)}_0) + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 - \Box \Lambda^{(0)}_0 \]

\[ \in C^\infty(X^{\text{physics}})_{\text{ch}} \]

where \( \Box := -\partial^2_0 + \partial^2_1 + \partial^2_2 + \partial^2_3 \). The twisted complex conjugate \( \dot{\bar{\Lambda}} \) of \( \dot{\Lambda} \) is given by

\[ \dot{\bar{\Lambda}}' = \overline{\Lambda^{(0)}_0} - \sum_\beta \bar{\theta}^\beta \partial_\beta \Lambda^{(\beta)}_0 + \theta^1 \theta^2 \bar{\theta}_1 \partial_2 \dot{\bar{\Lambda}}^{(12)}_0 - \sqrt{-1} \sum_\alpha \theta^\alpha \bar{\theta}_\beta \sum_\mu \sigma^\mu_{\alpha \beta} \partial_\mu \Lambda^{(0)}_0 + \sqrt{-1} \sum_\alpha \theta^\alpha \bar{\theta}_\beta \sum_\delta \varepsilon^{\alpha \beta} \sigma^\mu_{\alpha \beta} \partial_\mu (\partial_\delta \Lambda^{(\delta)}_0) - \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \Box \Lambda^{(0)}_0 \]

\[ \in C^\infty(X^{\text{physics}})_{\text{ach}}. \]

Then,

\[ \dot{V} + \delta_{\dot{\Lambda}} \dot{V} := \dot{V} - \sqrt{-1} (\dot{\Lambda} - \dot{\Lambda}') \]

\[ = \left( V^{(0)}_0 + \delta_{\Lambda} V^{(0)}_0 \right) + \sum_\alpha \partial_\alpha \left( V^{(\alpha)}_0 + \delta_{\Lambda} V^{(\alpha)}_0 \right) - \sum_\beta \bar{\theta}^\beta \partial_\beta \left( V^{(\beta)}_0 + \delta_{\Lambda} V^{(\beta)}_0 \right) + \theta^1 \theta^2 \bar{\theta}_1 \partial_2 \left( V^{(12)}_0 + \delta_{\Lambda} V^{(12)}_0 \right) + \sum_\alpha \theta^\alpha \bar{\theta}_\beta \sum_\mu \sigma^\mu_{\alpha \beta} \partial_\mu \left( V^{(0)}_0 + \delta_{\Lambda} V^{(0)}_0 \right) + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \left( V^{(0)}_0 + \delta_{\Lambda} V^{(0)}_0 \right) \]

\[ \in C^\infty(X^{\text{physics}})_{\text{ach}}. \]

A comparison of \( \delta_{\Lambda} V^{(12)}_0 \) against \( \delta_{\Lambda} V^{(0)}_0 \) and \( \delta_{\Lambda} V^{(\alpha)}_0 \) against \( \delta_{\Lambda} V^{(\alpha)}_0 \) implies that if one expresses a pre-vector superfield \( \dot{V} \in C^\infty(X^{\text{physics}})_{\text{ach}} \) in the following shifted form (cf. [W-B:

24] In Sec. 2.1 we express the coefficient of the \( \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \)-term (resp. the \( \theta^\alpha \bar{\theta}_\beta \)-term) of a chiral superfield (resp. antichiral superfield) in an expanded form. Here, it is more convenient to express them in the summation form.
Lemma 3.2.2. [pre-vector superfield in Wess-Zumino gauge]

An $\mathbb{R}$-linear combination\(^25\) of pre-vector superfields in the shifted expression is also a pre-vector superfield in the shifted expression.

It follows that, for a given pre-vector superfield $\tilde{V}$ in the shifted expression, if one chooses $\tilde{\Lambda}$ with

$$Im \Lambda^{(0)} = -\frac{1}{2} V^{(0)}_0, \quad \Lambda^{(\alpha)} = -\sqrt{-1} V^{(\alpha)}_0, \quad \Lambda^{(12)} = -\sqrt{-1} V^{(12)}_0,$$

which always exists, then after the gauge transformation specified by $\tilde{\Lambda}$, $\tilde{V}$ becomes

$$\tilde{V}' = \sum_{\alpha, \beta} \theta^\alpha \tilde{\theta}^\beta \sum_{\mu} \sigma^\mu_{\alpha \beta} \left( V^{(0)}_{[\mu]} + 2 \partial_{\mu} \text{Re} \Lambda^{(0)}_0 \right) + \sum_{\beta} \tilde{\theta}^\beta \tilde{\theta}^\beta \sum_{\alpha} \theta^\alpha \tilde{\theta}^\beta V^{(\alpha)}_{(12 \beta)} + \sum_{\alpha} \theta^\alpha \tilde{\theta}^\alpha \tilde{\theta}^\beta V^{(\alpha)}_{(12 \beta)} + \theta^1 \tilde{\theta}^2 \tilde{\theta}^2 V^{(0)}_{(1212)}.$$  

We summarize the above discussion into the following definition and lemmas:

Definition/Lemma 3.2.3. [pre-vector superfield in Wess-Zumino gauge]

We call a pre-vector superfield $\tilde{V} \in \mathbb{C}^\infty(X^{\text{phys}})^{\text{stc}}$ that is in the following form

$$\tilde{V} = \sum_{\alpha, \beta} \theta^\alpha \tilde{\theta}^\beta \sum_{\mu} \sigma^\mu_{\alpha \beta} V^{(0)}_{[\mu]} + \sum_{\beta} \theta^1 \tilde{\theta}^\beta \tilde{\theta}^\beta \sum_{\alpha} \theta^\alpha V^{(\alpha)}_{(12 \beta)} + \sum_{\alpha} \theta^\alpha \tilde{\theta}^\beta \sum_{\alpha} \theta^\alpha \tilde{\theta}^\beta V^{(\beta)}_{(12 \alpha)} + \theta^1 \tilde{\theta}^2 \tilde{\theta}^2 V^{(0)}_{(1212)}$$ a pre-vector superfield in Wess-Zumino gauge.

Given any pre-vector $\tilde{V}$, there exists a unique chiral superfield $\tilde{\Lambda}$ depending on $\tilde{V}$ with $\text{Re} \Lambda^{(0)}_0 = 0$ such that the gauge transformation specified by $\tilde{\Lambda}$ takes $\tilde{V}$ to a pre-vector superfield in Wess-Zumino gauge.

\(^25\)However, caution that a $\mathbb{C}^\infty(X)$-linear combination of pre-vector superfields in general is not directly a pre-vector superfield in the shifted expression. One has to convert it accordingly.
Lemma 3.2.4. [naturality] (1) The set of pre-vector superfields in Wess-Zumino gauge is a \( C^\infty(X)^{\text{phys}} \)-submodule of \( C^\infty(X^{\text{phys}})^{\hat{\Lambda},\text{stc}} \). (2) If a pre-vector superfield \( \tilde{V} \) expressed in terms of the standard coordinate functions \((x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) on \( \tilde{X}^{\hat{\Lambda}} \) is in Wess-Zumino gauge, then it remains in Wess-Zumino gauge when re-expressed in terms of the chiral coordinate functions \((x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) or the antichiral coordinate functions \((x'', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) on \( \tilde{X}^{\hat{\Lambda}} \).

Proof. Statement (1) is clear. We focus on Statement (2).

Recall that \( x' = x + \sqrt{-1}\theta\sigma\bar{\theta}^t \). When in Wess-Zumino gauge, a pre-vector superfield \( \tilde{V} \) in terms of the standard coordinate functions \((x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) is written as

\[
\tilde{V} = \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \sum_{\mu} \sigma_{\alpha \beta} V^{(0)}_{\mu}(x) + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \sum_{\alpha} \vartheta_\alpha V^{(12\beta)}_{(1)}(x)
+ \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \sum_{\beta} \bar{\vartheta}_\beta V^{(12\beta)}_{(2)}(x) + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 V^{(12\beta)}_{(12)}(x)
\]

To re-express \( \tilde{V} \) in terms of the chiral coordinate functions \((x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) on \( \tilde{X}^{\hat{\Lambda}} \), one substitutes \( x \) in \( V^{\star}(x) \) by \( x' - \sqrt{-1}\theta\sigma\bar{\theta}^t \) and use the \( C^\infty \)-hull structure of \( C^\infty(X^{\hat{\Lambda}}) \) to expand it in \( x' \). Due to the product structure of \( \theta^\alpha, \bar{\theta}^\beta \), this will only influence the coefficient of \( \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \) and, hence, keep the pre-vector superfield in Wess-Zumino gauge. Explicitly, the result after collecting like terms is

\[
\tilde{V} = \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \sum_{\mu} \sigma_{\alpha \beta} V^{(0)}_{\mu}(x') + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \sum_{\alpha} \vartheta_\alpha V^{(12\beta)}_{(1)}(x')
+ \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \sum_{\beta} \bar{\vartheta}_\beta V^{(12\beta)}_{(2)}(x') + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \left( V^{(12\beta)}_{(12)}(x') + 2\sqrt{-1} \partial^\mu V^{(0)}_{\mu}(x') \right).
\]

Similar argument goes when re-expressing \( \tilde{V} \) in terms of the antichiral coordinate functions \((x'', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\), with \( x'' = x - \sqrt{-1}\theta\sigma\bar{\theta}^t \), \( \theta, \bar{\theta} \), on \( \tilde{X}^{\hat{\Lambda}} \). The explicit expression is given by

\[
\tilde{V} = \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \sum_{\mu} \sigma_{\alpha \beta} V^{(0)}_{\mu}(x'') + \sum_{\beta} \theta^1 \theta^2 \bar{\theta}^\beta \sum_{\alpha} \vartheta_\alpha V^{(12\beta)}_{(1)}(x'')
+ \sum_{\alpha} \theta^\alpha \bar{\theta}^1 \bar{\theta}^2 \sum_{\beta} \bar{\vartheta}_\beta V^{(12\beta)}_{(2)}(x'') + \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \left( V^{(12\beta)}_{(12)}(x'') - 2\sqrt{-1} \partial^\mu V^{(0)}_{\mu}(x'') \right).
\]

This completes the proof. \[\square\]

Lemma 3.2.5. [representative in Wess-Zumino gauge] Any pre-vector superfield \( \tilde{V} \in C^\infty(X^{\text{phys}})^{\hat{\Lambda},\text{stc}} \) can be transformed to a pre-vector superfield in Wess-Zumino gauge by a gauge transformation.

In the shifted expression, once a pre-vector superfield is rendered a pre-vector superfield \( \tilde{f} \) in Wess-Zumino gauge, a gauge transformation specified by \( \hat{\Lambda} \) with

\[
Im \Lambda^{(0)}_{(0)} = \Lambda^{(1)}_{(0)} = \Lambda^{(12)}_{(1)} = 0,
\]

(i.e.,

\[
\hat{\Lambda} = \Lambda^{(0)}_{(0)} + \sqrt{-1} \sum_{\alpha, \beta} \theta^\alpha \bar{\theta}^\beta \sum_{\mu} \sigma_{\alpha \beta} \partial^\mu \Lambda^{(0)}_{(0)} - \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \Box \Lambda^{(0)}_{(0)}
\]

39
with $\Lambda^{(0)}_{(0)}$ real-valued) will send $\tilde{f}$ to another $\tilde{f}'$ still in Wess-Zumino gauge with only the components $V_{[\mu]}^{(0)}$ of $\tilde{V}$ transformed, by

$$V_{[\mu]}^{(0)} \rightarrow V_{[\mu]}^{(0)} + 2 \partial_\mu \Lambda^{(0)}_{(0)}.$$  

**Lemma 3.2.6.** [restriction $\nabla\tilde{V}$ of $\nabla\tilde{V}$ to $X^C$] Let $\tilde{V}$ be a pre-vector superfield in Wess-Zumino gauge. Then the restriction $\nabla\tilde{V}$ of $\nabla\tilde{V}$ to $X^C$ is given by $\partial_\mu - \sqrt{-1} V_{[\mu]}^{(0)}$, $\mu = 0, 1, 2, 3$.

**Proof.** Denote $\nabla\tilde{V}$ by $\hat{\nabla}$. Since $X^C \subset \hat{X}^\bar{\theta}$ is described by the ideal in $\mathcal{C}^\infty(\hat{X}^\bar{\theta})$ generated by $\theta, \bar{\theta}, \vartheta, \bar{\vartheta}$, the restriction of $\hat{\nabla}$ to $X^C$ is simply the $(\theta, \bar{\theta}, \vartheta, \bar{\vartheta})$-degree-zero part of $\hat{\nabla}$ when expressed in terms of the standard coordinate functions $(x, \theta, \vartheta, \bar{\theta}, \bar{\vartheta})$ on $\hat{X}^\bar{\theta}$. This in turn is a $\mathbb{C}$-linear combination of the $(\theta, \bar{\theta}, \vartheta, \bar{\vartheta})$-degree-zero part of $\{\nabla\alpha, \nabla\beta\}$ in the standard coordinate functions $(x, \theta, \vartheta, \bar{\theta}, \bar{\vartheta})$.

From the definition of $\hat{\nabla}$ and the $\mathbb{Z}/2$-graded Leibniz rule, one has

$$\{\nabla\alpha, \nabla\beta\} = e_{\alpha'} + e^{-\hat{V}}(e_{\alpha'} e^\hat{V}), \quad e_{\beta''} = e_{\alpha'} e^\hat{V} + e^{-\hat{V}}(e_{\beta''} e_{\alpha'} e^\hat{V}).$$

The $(\theta, \bar{\theta}, \vartheta, \bar{\vartheta})$-degree-zero terms of $\{\nabla\alpha, \nabla\beta\}$ thus come from $\{e_{\alpha'}, e_{\beta''}\}$, which equals $-2\sqrt{-1} \sum_{\nu} \sigma_{\alpha'\beta}^{\nu} \partial_{\nu}$, and the $(\theta, \bar{\theta}, \vartheta, \bar{\vartheta})$-degree-zero terms of the summand $e_{\beta''} e_{\alpha'}$ from the expansion of $e^{-\hat{V}}(e_{\beta''} e_{\alpha'} e^\hat{V}) = (1 - \hat{V} + \frac{1}{2} \hat{V}^2)(e_{\beta''} e_{\alpha'} (1 + \frac{1}{2} \hat{V}^2))$, which is $-\sum_{\nu} \sigma_{\alpha'\beta}^{\nu} V_{[\nu]}^{(0)}$. It follows that

$$\nabla_{\mu} := \nabla_{\mu}|_{X^C} = \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta} \bar{\sigma}_{\alpha\beta}^{\nu} \sum_{\nu} \sigma_{\alpha'\beta}^{\nu} (\frac{1}{2} - \sqrt{-1} \partial_{\nu} - V_{[\nu]}^{(0)}) = \partial_{\mu} - \frac{\sqrt{-1}}{2} V_{[\mu]}^{(0)}.$$  

Here, the identity $\sum_{\alpha, \beta} \bar{\sigma}_{\alpha\beta}^{\nu} \sigma_{\alpha'\beta}^{\nu} = \delta_{\mu}^{\nu}$ is used. This proves the lemma.

**Explicit formulae for $\nabla\tilde{V}$**

The full expression of $\nabla\tilde{V}$ for $\tilde{V}$ in Wess-Zumino gauge is given here for the completeness of the discussion. It’s a curious feature that the curvature of $\nabla := \nabla\tilde{V}|_{X^C}$ is somehow already captured in the $(\theta, \bar{\theta})$-degree-$0$ terms of $\nabla\tilde{V}$'s.

$$\nabla_{\mu} := \partial_{\mu} + \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta} \bar{\sigma}_{\alpha\beta}^{\nu} \Theta_{\alpha\beta}^{\nu},$$
where
\[ \Theta_{\alpha \beta} = (e_{\beta \nu} e^{-\bar{V}})(e_{\alpha \nu} e^{\bar{V}}) + e^{-\bar{V}}(e_{\beta \nu} e_{\alpha \nu} e^{\bar{V}}) \]
\[ = - \sum_{\nu} \sigma_{\alpha \beta}^{\nu} V^{(0)}_{[\nu]} - \sum_{\gamma} \theta_{\nu} \varepsilon_{\alpha \gamma} (\sum_{\gamma} V^{(\gamma)}_{(12\beta)}) + \sum_{\delta} \theta^{\delta} \varepsilon_{\alpha \delta} (\sum_{\delta} \bar{V}_{(12\alpha)}^{(\delta)}) \]
\[ + \sum_{\gamma, \delta} \theta_{\gamma} \bar{\theta}_{\delta} (\varepsilon_{\alpha \gamma} \varepsilon_{\beta \delta} V^{(0)}_{(12\gamma \delta)} + \sqrt{-1} \sum_{\beta', \delta'} \varepsilon_{\beta \delta} \varepsilon_{\gamma \delta'} \sigma_{\alpha \beta'} \gamma_{\delta'} \alpha_{\beta} V^{(0)}_{[\nu]} - \sqrt{-1} \sum_{\mu, \nu} \sigma_{\gamma \beta}^{\nu} \varepsilon_{\alpha \delta} \partial_{\gamma} V^{(0)}_{[\nu]} \]
\[ + \sum_{\alpha \beta} \sigma_{\alpha \beta}^{\nu} \varepsilon_{\beta \delta} \sigma_{\alpha \delta}^{\nu} \Theta_{\nu} (\sum_{\nu, \nu} \bar{V}_{(12\alpha \beta)}^{(\nu)}) \]
\[ + \sqrt{-1} \sum_{\gamma} \varepsilon_{\gamma} (\sum_{\gamma} \varepsilon_{\beta \delta} \sigma_{\alpha \delta}^{\nu} \Theta_{\nu} \partial_{\gamma} V^{(0)}_{[\nu]} \partial_{\mu} V^{(0)}_{[\nu]} \]
\[ - \sqrt{-1} \sum_{\mu, \nu} \sigma_{\gamma \beta}^{\nu} \varepsilon_{\alpha \delta} \sigma_{\gamma \delta} \alpha_{\beta} V^{(0)}_{[\nu]} \partial_{\mu} V^{(0)}_{[\nu]} \]
\[ + \sum_{\gamma, \delta} \varepsilon_{\gamma} (\sum_{\delta} \varepsilon_{\gamma} \varepsilon_{\beta \delta} \sigma_{\alpha \delta}^{\nu} \Theta_{\nu} \partial_{\gamma} V^{(0)}_{[\nu]} \partial_{\mu} V^{(0)}_{[\nu]} \]
\[ - \sqrt{-1} \sum_{\mu, \nu} \sigma_{\gamma \beta}^{\nu} \varepsilon_{\alpha \delta} \sigma_{\gamma \delta} \alpha_{\beta} V^{(0)}_{[\nu]} \partial_{\mu} V^{(0)}_{[\nu]} \]
\[ + 4 (\sum_{\gamma} \varepsilon_{\gamma} \varepsilon_{\beta \delta} \sigma_{\alpha \delta}^{\nu} \Theta_{\nu} \partial_{\gamma} V^{(0)}_{[\nu]} \partial_{\mu} V^{(0)}_{[\nu]}) \] .

Observe that \( \Theta_{\alpha \beta} \) involves at most terms quadratic in components of \( \bar{V} \). Terms cubic in components of \( \bar{V} \) do appear in the immediate steps but they cancel each other in the end. There are terms that contain the factor \( \sigma_{\alpha \beta}^{\nu} \). They give rise to terms in \( \hat{\nabla}_{\mu} \bar{V} \) of the following form:
\[ \hat{\nabla}_{\mu} \bar{V} = \partial_{\mu} - \frac{\sqrt{-1}}{2} V^{(0)}_{[\mu]} + \sum_{\gamma, \delta} \varepsilon_{\gamma} (\sum_{\delta} \varepsilon_{\beta \delta} \sigma_{\alpha \delta}^{\nu} \Theta_{\nu} \partial_{\gamma} V^{(0)}_{[\nu]} + \frac{1}{2} \sum_{\gamma} \varepsilon_{\gamma} \varepsilon_{\beta \delta} \sigma_{\alpha \delta}^{\nu} \Theta_{\nu} \partial_{\gamma} V^{(0)}_{[\nu]} \partial_{\mu} (\sum_{\gamma} \partial_{\gamma} V^{(\gamma)}_{(12\beta)}) \]
\[ + \sqrt{-1} \sum_{\gamma} \varepsilon_{\gamma} (\sum_{\delta} \varepsilon_{\gamma} \varepsilon_{\beta \delta} \sigma_{\alpha \delta}^{\nu} \Theta_{\nu} \partial_{\gamma} V^{(0)}_{[\nu]} \partial_{\mu} V^{(0)}_{[\nu]} \]
\[ + \sum_{\gamma, \delta} \varepsilon_{\gamma} (\sum_{\delta} \varepsilon_{\gamma} \varepsilon_{\beta \delta} \sigma_{\alpha \delta}^{\nu} \Theta_{\nu} \partial_{\gamma} V^{(0)}_{[\nu]} \partial_{\mu} V^{(0)}_{[\nu]} \partial_{\mu} (\sum_{\nu} V^{(0)}_{[\nu]}) \] ,

where all terms in ( \( \cdots \cdots \) ) have the total (\( \theta, \bar{\theta} \))-degree \( \geq 1 \).

### 3.3 Supersymmetry transformations of a pre-vector superfield in Wess-Zumino gauge

Readers are recommend to read this subsection along with [Argu: Sec. 4.3, pp. 76, 77] of Argurio and [G-G-R-S: Sec. 4.2.1, from before Eq. (4.2.7) to after Eq. (4.2.13)] of Gates, Grosaru, Roček, & Siegel for comparison.

Recall the Grassmann parameter level of \( \hat{X} \). The setup, discussions, and results in Sec. 3.1 and Sec. 3.2 can be generalized straightforwardly to the case where the Grassmann parameter

41
level is turned on. We shall use this to understand how supersymmetries act on pre-vector superfields in Wess-Zumino gauge.

To preserve the self-twisted-conjugate condition \( \bar{\hat{V}} = \hat{V} \) of a pre-vector superfield \( \bar{\hat{V}} \), let \((\eta, \bar{\eta}) := (\eta^1, \eta^2; \eta^1, \eta^2)\) be a conjugate pair of constant sections of \( S^\vee_{\text{parameter}} \oplus S^\vee_{\text{parameter}} \subset \hat{O}_{X}^\vee \) and consider the infinitesimal supersymmetry transformation

\[
\delta_{(\eta, \bar{\eta})} \hat{V} := (\eta Q + \bar{\eta} \bar{Q}) \hat{V} := \left( \sum_\alpha \eta^\alpha Q^\alpha - \sum_\beta \bar{\eta}^\beta \bar{Q}_\beta \right) \hat{V}
\]

of \( \hat{V} \). Then \((\delta_{(\eta, \bar{\eta})} \hat{V})^\dagger = \delta_{(\eta, \bar{\eta})} \hat{V}\). However, for \( \hat{V} \) in Wess-Zumino gauge, \((\eta Q + \bar{\eta} \bar{Q}) \hat{V}\) remains a pre-vector superfield but in general no longer in Wess-Zumino gauge. This can be remedied by a gauge transformation: (e.g., [Argu: Sec. 4.3.1], [G-G-R-S: Sec. 4.2.a.1], [W-B: Chap. VII, Exercise (8)], and [We: Sec. 15.3, Eq. (15.78)])

Lemma 3.3.1. [uniqueness of correcting gauge transformation] Let \( \hat{V} \) be a pre-vector superfield in Wess-Zumino gauge. Then there is a unique chiral superfield \( \Lambda_{(\eta, \bar{\eta}; \bar{\hat{V}})} \) depending \( \mathbb{C} \)-multilinearly on \((\eta, \bar{\eta})\) and \( \bar{\hat{V}} \) such that the gauge transformation

\[
(\eta Q + \bar{\eta} \bar{Q}) \hat{V} - \sqrt{-1}(\Lambda_{(\eta, \bar{\eta}; \bar{\hat{V}})} - \Lambda_{(\eta, \bar{\eta}; \bar{\hat{V}})}^\dagger)
\]

of \((\eta Q + \bar{\eta} \bar{Q}) \hat{V}\) is in Wess-Zumino gauge.

Proof. When \( \bar{\hat{V}} \) is in Wess-Zumino gauge, \((\ast)_{(0)}\)-component of \((\eta Q + \bar{\eta} \bar{Q}) \hat{V}\) is always zero. It follows from the explicit computation in the previous theme that leads to Lemma 3.2.5 that there is a unique chiral superfield \( \Lambda \) associated to \((\eta Q + \bar{\eta} \bar{Q}) \hat{V}\) with \( \Lambda_{(0)} = 0 \) such that \((\eta Q + \bar{\eta} \bar{Q}) \hat{V} - \sqrt{-1}(\Lambda - \Lambda^\dagger)\) is in Wess-Zumino gauge. The same explicit computation implies also that this unique \( \Lambda \) depends \( \mathbb{C} \)-multilinearly on \((\eta, \bar{\eta})\) and \( \bar{\hat{V}} \). This proves the lemma.

Definition 3.3.2. [supersymmetry in Wess-Zumino gauge] Set

\[
(\eta Q + \bar{\eta} \bar{Q}) \hat{V} - \sqrt{-1}(\Lambda_{(\eta, \bar{\eta}; \bar{\hat{V}})} - \Lambda_{(\eta, \bar{\eta}; \bar{\hat{V}})}^\dagger) = \sum_\alpha \eta^\alpha Q^w_{\alpha} \hat{V} - \sum_\beta \bar{\eta}^\beta \bar{Q}^w_{\beta} \hat{V}.
\]

This defines \( (\text{infinitesimal}) \) supersymmetry transformations in Wess-Zumino gauge \( Q^w_{\alpha}, Q^w_{\beta} \) that take a superfield in Wess-Zumino gauge to another in Wess-Zumino gauge.

Explicitly, let

\[
\hat{V} = \sum_{\gamma, \delta, \nu} \theta^\gamma \theta^\delta \theta^\nu \hat{V}^{(0)}_{[\nu]} + \sum_{\delta} \theta^1 \theta^2 \theta^\delta \sum_{\gamma'} \theta_{\gamma'} \hat{V}^{(\gamma')}_{(12\delta)} + \sum_{\gamma} \theta^1 \theta^2 \theta^1 \theta^2 \sum_{\delta'} \bar{\theta}_{\delta'} \hat{V}^{(\delta')}_{(12\gamma)} + \theta^1 \theta^2 \theta^1 \theta^2 \hat{V}^{(0)}_{(1212)}
\]
be a pre-vector superfield in Wess-Zumino gauge. Then,
\[
\delta_{\eta Q + \bar{\eta} \bar{Q}} \tilde{V} := (\eta Q + \bar{\eta} \bar{Q}) \tilde{V} \\
= \left( \sum_{\alpha} \eta^\alpha \frac{\partial}{\partial \theta^\alpha} - \sqrt{-1} \sum_{\alpha, \beta, \mu} \eta^\alpha \sigma^\mu_{\alpha \beta} \hat{\theta}^\beta \partial_\mu \right) \tilde{V} + \left( \sum_{\beta} \bar{\eta}^\beta \frac{\partial}{\partial \bar{\theta}^\beta} + \sqrt{-1} \sum_{\alpha, \beta, \mu} \theta^\alpha \sigma^\mu_{\alpha \beta} \hat{\bar{\theta}}^\beta \partial_\mu \right) \tilde{V} \\
= \sum_{\gamma} \theta^\gamma \sum_{\beta, \mu} \bar{\eta}^\beta \sigma^\nu_{\gamma \beta} V^{(0)}_{[\nu]} + \sum_{\delta} \bar{\theta}^\delta \cdot (\sqrt{-1}) \sum_{\alpha, \nu} \eta^\alpha \sigma^\nu_{\alpha \beta} V^{(0)}_{[\nu]} + \theta^1 \theta^2 \sum_{\beta} \bar{\eta}^\beta (\sum_{\gamma} \hat{\theta}^\gamma V^{(\gamma)}_{(12\bar{\beta})}) \\
+ \sum_{\gamma} \theta^\gamma \bar{\theta}^\delta (\sum_{\alpha} \eta^\alpha \varepsilon_{\alpha \gamma} \sum_{\gamma} \hat{\theta}^\gamma V^{(\gamma)}_{(12\bar{\beta})}) + \sum_{\beta} \bar{\eta}^\beta \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu V^{(0)}_{[\nu]} \\
+ \sum_{\gamma} \theta^\gamma \bar{\theta}^\delta (\sum_{\alpha} \eta^\alpha \varepsilon_{\alpha \gamma} \sum_{\gamma} \hat{\theta}^\gamma V^{(\gamma)}_{(12\bar{\beta})}) \cdot (\sqrt{-1}) \sum_{\beta} \bar{\eta}^\beta \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu V^{(0)}_{[\nu]} \\
+ \theta^1 \theta^2 \bar{\eta}^\beta \cdot (\sqrt{-1}) \sum_{\beta, \alpha, \gamma, \mu} \eta^\alpha \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu (\sum_{\bar{\gamma}} \hat{\theta}^\gamma V^{(\bar{\gamma})}_{(12\bar{\beta})}).
\]

Let $\hat{A}$ be the unique chiral superfield in $\hat{O}_\chi$ with
\[
\hat{A}_{(0)} = 0, \quad \hat{A}_{(\gamma)} = -\sqrt{-1} \sum_{\beta, \nu} \bar{\eta}^\beta \sigma^\nu_{\gamma \beta} V^{(0)}_{[\nu]}, \quad \hat{A}_{(12)} = -\sqrt{-1} \sum_{\beta} \bar{\eta}^\beta (\sum_{\gamma} \hat{\theta}^\gamma V^{(\gamma)}_{(12\bar{\beta})}).
\]

I.e.
\[
\hat{A} = -\sqrt{-1} \sum_{\gamma} \theta^\gamma \sum_{\beta, \nu} \bar{\eta}^\beta \sigma^\nu_{\gamma \beta} V^{(0)}_{[\nu]} - \sqrt{-1} \theta^1 \theta^2 \sum_{\beta} \bar{\eta}^\beta (\sum_{\gamma} \hat{\theta}^\gamma V^{(\gamma)}_{(12\bar{\beta})}) \\
- \theta^1 \theta^2 \bar{\eta}^\beta \sum_{\beta, \alpha, \gamma, \mu} \eta^\alpha \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu V^{(0)}_{[\nu]}.
\]

Then,
\[
\delta_{\eta Q + \bar{\eta} \bar{Q}} \tilde{V} + \delta_{\hat{A}} \tilde{V} = (\eta Q + \bar{\eta} \bar{Q}) \tilde{V} - \sqrt{-1} (\hat{A} - \hat{A}^l)
= \sum_{\gamma, \delta} \theta^\gamma \bar{\theta}^\delta \left( -\sum_{\alpha} \eta^\alpha \varepsilon_{\alpha \gamma} \sum_{\gamma} \hat{\theta}^\gamma V^{(\gamma)}_{(12\bar{\beta})} \right) + \sum_{\beta} \bar{\eta}^\beta \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu F^{(\nu)}_{\beta} \\
+ \sum_{\gamma} \theta^\gamma \bar{\theta}^\delta \sum_{\alpha} \eta^\alpha \varepsilon_{\alpha \gamma} \sum_{\gamma} \hat{\theta}^\gamma V^{(\gamma)}_{(12\bar{\beta})} \cdot (\sqrt{-1}) \sum_{\beta, \alpha, \gamma, \mu, \nu} \eta^\alpha \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu F^{(\nu)}_{\beta} \\
+ \theta^1 \theta^2 \bar{\eta}^\beta \cdot (\sqrt{-1}) \sum_{\beta, \alpha, \gamma, \mu, \nu} \eta^\alpha \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu F^{(\nu)}_{\beta} \\
= \sum_{\alpha} \eta^\alpha Q^{\mu \nu}_{\alpha} \tilde{V} - \sum_{\beta} \bar{\eta}^\beta Q^{\mu \nu \beta}_{\alpha} \tilde{V},
\]

where $F^{\mu \nu}_{\beta} := \partial_\mu V^{(0)}_{[\beta]} - \partial_\beta V^{(0)}_{[\mu]}$; now resumes in Wess-Zumino gauge.

From this, one reads off
\[
Q^{\mu \nu \beta}_{\alpha} \tilde{V} = -\sum_{\gamma, \delta} \theta^\gamma \bar{\theta}^\delta \varepsilon_{\alpha \gamma} \sum_{\gamma} \hat{\theta}^\gamma V^{(\gamma)}_{(12\bar{\beta})} \cdot (\sqrt{-1}) \sum_{\beta, \alpha, \gamma, \mu, \nu} \eta^\alpha \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu F^{(\nu)}_{\beta} \\
+ \theta^1 \theta^2 \theta^\delta \tilde{V} \cdot (\sqrt{-1}) \sum_{\delta, \beta, \alpha, \gamma, \mu, \nu} \eta^\alpha \varepsilon_{\beta \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \delta} \partial_\mu F^{(\nu)}_{\beta},
\]

43
\[
\bar{Q}_{\dot{\beta}}^{WZ} \dot{V} = - \sum_{\gamma, \delta} \theta^\gamma \bar{\theta}^\delta \varepsilon_{\beta \delta} (\sum_{\dot{\gamma}} \bar{\theta}_{\dot{\gamma}} V^{(\dot{\gamma})}_{(12)}(1)) + \sum_{\delta} \theta^1 \theta^2 \bar{\theta}^\delta \cdot \left( \varepsilon_{\beta \delta} V^{(0)}_{(1212)}(1) + \sqrt{-1} \sum_{\alpha, \gamma, \mu, \nu} \varepsilon^{\alpha \gamma} \sigma^\mu_{\alpha \beta} \sigma^\nu_{\gamma \dot{\delta}} F_{\mu \nu} \right) \\
+ \theta^1 \theta^2 \bar{\theta}^\delta \cdot \sqrt{-1} \sum_{\alpha, \gamma, \mu} \varepsilon^{\alpha \gamma} \sigma^\mu_{\alpha \beta} \partial_\mu (\sum_{\dot{\gamma}} \bar{V}^{(\dot{\gamma})}_{(12)}(1)) .
\]

The supersymmetry algebra generated by \( \bar{Q}_{\alpha}^{WZ} \)'s, \( \bar{Q}_{\dot{\beta}}^{WZ} \)'s, and \( \partial_\mu \)'s is now closed only up to a gauge transformation.

### 3.4 From pre-vector superfields to vector superfields

The discussion in Sec. 3.2 is mathematically perfectly fine. However, when compared to the vector multiplet in representations of \( d = 3 + 1, N = 1 \) supersymmetry algebra, there are two redundant degrees of freedom in a pre-vector superfield \( \dot{V} \) due to that for each \( \dot{\beta} \in \{ 1, 2 \} \), the coefficient of the \( \theta^1 \theta^2 \bar{\theta}^\beta \)-term contains \( \partial_1 V^{(1)}_{(12\dot{\beta})} + \partial_2 V^{(2)}_{(12\dot{\beta})} \), which has two component-functions \( V^{(1)}_{(12\dot{\beta})}, V^{(2)}_{(12\dot{\beta})} \in C^\infty(X)^C \), instead of one. Such redundancies can be removed easily\(^{26}\) as follows:

- For a pre-vector superfield \( \dot{V} \) in the shifted expression, for each \( \dot{\beta} = 1, 2 \), introduce an \( \mathbb{R} \)-linear constraint on \( \sum_\gamma \partial_\gamma V^{(\gamma)}_{(12\dot{\beta})} \), which then induces simultaneously the same \( \mathbb{R} \)-linear constraint on its complex conjugate \( \sum_\delta \bar{\partial}_{\delta} \bar{V}^{(\delta)}_{(12\dot{\beta})} \) (equivalently, for each \( \alpha = 1, 2 \), introduce an \( \mathbb{R} \)-linear constraint on \( \sum_\delta \bar{\partial}_{\delta} V^{(\delta)}_{(a12)} \)), which then induces simultaneously the same \( \mathbb{R} \)-linear constraint on its complex conjugate \( \sum_\gamma \partial_\gamma \bar{V}^{(\gamma)}_{(12\dot{\alpha})} = \sum_\gamma \partial_\gamma V^{(\gamma)}_{(12\dot{\alpha})} \) to remove a redundant degree of freedom.

- Denote this set of constrained pre-vector superfields by \( C^\infty(X^{physics})^{\gamma, stc}_{(constrained)} \).

Since for a pre-vector superfield in the shifted expression, the component-functions \( V^{(\alpha)}_{(12\dot{\beta})} \) and \( \bar{V}^{(\alpha)}_{(12\dot{\beta})} \) are fixed under gauge transformations, the \( C^\infty(X) \)-submodule \( C^\infty(X^{physics})^{\gamma, stc}_{(constrained)} \) of \( C^\infty(X^{physics})^{\gamma, stc} \) is invariant under gauge transformations. And all the discussion in Sec. 3.2 on \( C^\infty(X^{physics})^{\gamma, stc} \) applies to \( C^\infty(X^{physics})^{\gamma, stc}_{(constrained)} \) as well. In particular, once using a \( \mathbb{R} \)-linear constraints to remove the redundancies and bringing the constrained pre-vector superfield to be in Wess-Zumino gauge, the remaining redundancy are the gauge transformations on \( V_{[\mu]}^{(0)} \), cf. the discussion after Lemma 3.2.5. Once modding out this last class of gauge symmetries, the remaining degrees of freedom of a pre-vector superfield in Wess-Zumino gauge match exactly with the vector multiplet of the representations of the \( d = 3 + 1, N = 1 \) supersymmetry algebra.

To fix the notion and for the simplicity of the notation, before proceeding to the next subsection, we choose the following most simple \( \mathbb{R} \)-linear constraint\(^{27}\) on pre-vector superfields \( \dot{V} \)

\(^{26}\)Note that, unlike the set of chiral functions or the set of antichiral functions on \( X^{physics} \), the set of vector superfields on \( X^{physics} \) is only required to be a \( C^\infty(X) \)-module, rather than a \( C^\infty(X) \)-algebra. Naively this is what makes it easy. However, gauge transformations act on this module and preferably one wants to remove the redundancy in a gauge-invariant way. Usually this may not be always easy. That, when in the shifted expressions, these terms are themselves gauge-fixed comes to the rescue.

\(^{27}\)Since the underlying topology \( \mathbb{R}^3 \) of the space-time \( X \) is contractible, different choices of the linear constraints would bear no significant consequences mathematically and likely so also physically.
in the shifted expression (and in standard coordinate functions \((x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\)):

\[
V^{(2)}_{(12\beta)}(x) = V^{(2)}_{(\alpha12)}(x) = 0.
\]

**Definition 3.4.1. [vector superfield]** A pre-vector superfield \(\tilde{V}\) that satisfies the above constraint when in the shifted expression is called a vector superfield. The set of vector superfields on \(X^{\text{physics}}\) is a \(C^\infty(X)\)-module, denoted by \(C^\infty(X^{\text{physics}})^{\text{vb, etc.}}\).

It follows that

**Lemma 3.4.2. [from pre-vector superfield to vector superfield]** Definition 3.2.1, Lemma 3.2.2, Definition/Lemma 3.2.3, Lemma 3.2.4, Lemma 3.2.5, Lemma 3.3.1, and Definition 3.3.2 in Sec. 3.2 and Sec. 3.3 remain valid with ‘pre-vector superfield’ replaced by ‘vector superfield’.

**Lemma 3.4.3. [independence of coordinate functions chosen when in Wess-Zumino gauge]** When in Wess-Zumino gauge, the conditions \(V^{(2)}_{(12\beta)} = V^{(2)}_{(\alpha12)} = 0\) on a pre-vector superfield \(\tilde{V}\) remain to hold whether one expresses \(\tilde{V}\) in terms of the chiral coordinate functions \((x', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\) or antichiral coordinate functions \((x'', \theta, \bar{\theta}, \vartheta, \bar{\vartheta})\).

**Proof.** This follows from the proof of Lemma 3.2.4 in Sec. 3.2.

\[
\square
\]

### 3.5 Supersymmetric \(U(1)\) gauge theory with matter on \(X\) in terms of \(X^{\text{physics}}\)

With the preparations in Sec. 3.1, Sec. 3.2, and Sec. 3.4, we are now ready to construct a supersymmetric \(U(1)\) gauge theory with matter on \(X\) in terms of \(X^{\text{physics}}\).

Two basic derived superfields: gaugino superfield and kinetic-term superfield

Unlike chiral or antichiral superfields, a vector superfield \(\tilde{V}\) contains no components that involve space-time derivatives. For that reason, to construct a supersymmetric action functional for components of \(\tilde{V}\), one needs to work out appropriate derived superfields from \(\tilde{V}\) first.

---

28 **Note for mathematicians.** We are in no position to illuminate the original physical insight in the construction of the supersymmetric action functional for \(U(1)\) gauge theory with matter. As in Sec. 2, our attempt here is only to demonstrate that there is a precise (complexified, \(Z/2\)-graded) \(C^\infty\)-Algebraic Geometry tower construction in which everything in Chap. VI and the part of Chap. VII on the \(U(1)\) case of [W-B] by Julius Wess & Jonathan Bagger is accounted for and mathematically harmoniously interpreted. Nevertheless, mathematicians are highly recommended to try [Argu: Sec. 4.3.2] of Riccardo Argurio, [G-G-R-S: Sec. 4.2.a.1] of James Gates, Jr., Marcus Grisaru, Martin Roček, Warren Siegel, [We: Sec. 15.2] of Peter West to at least get some sense of why and from where some quantities in this subsection are considered.

29 **Here, we are not using the term ‘derived’ in any deeper sense.** We only mean that such superfields arise from the combination of more basic superfields such as chiral superfields and vector superfields. For example, the superpotential is a polynomial (or more generally holomorphic function) of chiral superfields and thus can be regarded as a “derived” superfield. Caution that these derived superfields may go beyond \(C^\infty(X^{\text{physics}})\) and lie only in \(C^\infty(\hat{X}^{\mathbb{R}})\).
Lemma/Definition 3.5.1. [gaugino superfield] ([W-B: Chap. VI, Eq. (6.7)])

Let $\tilde{V} \in C^\infty (X^{\text{physics}})^{\bar{\beta}, \text{stc}}$ be a vector superfield. Define

$$W_{\alpha} := e^{\dot{2}}e_{1\nu}e_{\dot{\alpha}}\tilde{V} \quad \text{(resp. } \tilde{W}_{\dot{\beta}} := e_{1\nu}e_{2\nu}\tilde{V})$$

$\alpha = 1, 2$, $\dot{\beta} = \dot{1}, \dot{2}$. Then (1) $W_{\alpha}$ (resp. $\tilde{W}_{\dot{\beta}}$) is chiral (resp. antichiral). (2) $W_{\alpha}$ and $\tilde{W}_{\dot{\beta}}$ are invariant under gauge transformations on $\tilde{V}$.

$W_{\alpha}$, $\tilde{W}_{\dot{\beta}}$ are called the gaugino superfields associated to the vector superfield $\tilde{V}$.

Proof. For Statement (1),

$$e_{1\nu}W_{\alpha} = -e_{2\nu}(e_{1\nu})^2e_{\dot{\alpha}}\tilde{V} = 0,$$
$$e_{2\nu}W_{\alpha} = (e_{2\nu})^2e_{1\nu}e_{\dot{\alpha}}\tilde{V} = 0$$

since $(e_{1\nu})^2 = (e_{2\nu})^2 = 0$. Similarly for the antichirality of $\tilde{W}_{\dot{\beta}}$.

For Statement (2), under a gauge transformation $\tilde{V} \to \tilde{V} - \sqrt{-1}(\tilde{A} - \tilde{A}^\dagger)$ on $\tilde{V}$ specified by a chiral superfield $\tilde{A}$,

$$W_{\alpha} \to e^{\dot{2}}e_{1\nu}e_{\dot{\alpha}}(\tilde{V} - \sqrt{-1}(\tilde{A} - \tilde{A}^\dagger)) = W_{\alpha} - \sqrt{-1}e_{2\nu}e_{1\nu}e_{\dot{\alpha}}\tilde{A}$$
$$= W_{\alpha} - \sqrt{-1}(e_{1\nu}e_{\dot{\alpha}}e_{2\nu} - e_{2\nu}e_{\dot{\alpha}}e_{1\nu})\tilde{A} = W_{\alpha}$$

since $\tilde{A}^\dagger$ is antichiral (thus, $e_{\dot{\alpha}}\tilde{A}^\dagger = 0$) and $\tilde{A}$ is chiral (thus, $e_{1\nu}\tilde{A} = e_{2\nu}\tilde{A} = 0$). Similarly for $\tilde{W}_{\dot{\beta}}$.

It follows that in the construction of a supersymmetric $U(1)$-gauge theory with matter, one may assume that the vector superfield $\tilde{V}$ is in Wess-Zumino gauge, which encodes the component fields $V^{(\alpha)}_{[\mu]}$, $V^{(1)}_{(12\dot{\beta})}$, and $V^{(0)}_{(121\dot{2})}$ on $X$. Here, $\mu = 0, 1, 2, 3$, $\alpha = 1, 2$, $\dot{\beta} = \dot{1}, \dot{2}$. For $\tilde{V}$ in Wess-Zumino gauge, $\tilde{V}^3 = 0$ and its exponential $e^{\tilde{V}}$ is simply the polynomial $1 + \tilde{V} + \frac{1}{2}\tilde{V}^2$ in $\tilde{V}$.

Lemma/Definition 3.5.2. [gauge-invariant kinetic term for chiral superfield] Let $\tilde{V} \in C^\infty (X^{\text{physics}})^{\bar{\beta}, \text{stc}}$ be a vector superfield and $\tilde{\Phi}$ be a chiral superfield on $X^{\text{physics}}$. Then the product

$${\tilde{\Phi}}^\dagger e^{\tilde{V}} \tilde{\Phi}$$

is gauge-invariant. Since the expression of the product in $(x, \theta, \bar{\theta}, \bar{\vartheta}, \vartheta)$ involves space-time derivatives $(\partial_\mu, \mu = 0, 1, 2, 3)$ of components of $\tilde{\Phi}$, this product is called the gauge-invariant kinetic term for the chiral superfield $\tilde{\Phi}$.

Proof. By construction, under the gauge transformation specified by a chiral superfield $\tilde{A}$,

$$\tilde{\Phi}^\dagger e^{\tilde{V}} \tilde{\Phi} \to (\tilde{\Phi}^\dagger e^{-\sqrt{-1}\tilde{A}^\dagger}) e^{\sqrt{-1}\tilde{A}} (e^{\sqrt{-1}\tilde{A}} \tilde{\Phi})$$
$$= \tilde{\Phi}^\dagger e^{-\sqrt{-1}\tilde{A}^\dagger} + \sqrt{-1} (\tilde{A} - \tilde{A}^\dagger) + \sqrt{-1} \tilde{A} \tilde{\Phi} = \tilde{\Phi}^\dagger e^{\tilde{V}} \tilde{\Phi}$$

\footnote{The design here is made so that $W_{\alpha} = \tilde{V}_{(\alpha \dot{1} \dot{2})} + \text{terms of } (\theta, \bar{\theta})\text{-degree } \geq 1$ and $\tilde{W}_{\dot{\beta}} = \tilde{V}_{(12\dot{\beta})} + \text{terms of } (\theta, \bar{\theta})\text{-degree } \geq 1$. Caution that, while $e_{\dot{\alpha}} = \partial/\partial \theta^\alpha + \cdots$, $e^{\alpha} = -\partial/\partial \bar{\theta}^\beta + \cdots$.}
Note that in general $W_\alpha, \bar{W}_\beta$ only lie in $C^\infty(\hat{X})$, not in $C^\infty(X^{\text{physics}})$, while $\Phi^\dagger V \Phi$ always lies in $C^\infty(X^{\text{physics}})$.

**Explicit computations/formulae**

The explicit expression of these derived superfields and some related products can be computed via spinor calculus. The results are listed below.

**Notations**
In the expressions below, the Minkowski space-time metric tensor ($-, +, +, +$) is used to raise or lower the space-time index $\mu, \nu, \cdots$ while the $\varepsilon$-tensor is used to raise or lower the spinor index $\alpha, \gamma, \cdots, \beta, \delta, \cdots$; $F_{\mu\nu} := \partial_\mu V^{(0)} - \partial_\nu V^{(0)}$; $\delta^\ast$ is the Kronecker $\delta$ (when $\delta$ not served as a spinor index); $\varepsilon_{\mu\nu\mu'}$ indicates the standard volume-form on the Minkowski space-time $X$ with $\varepsilon_{0123} = -1$; all the summations $\sum_{\cdots}$ are written explicitly.

- $W_\alpha$: (in chiral coordinate functions $(x', \theta, \bar{\theta}, \tilde{\theta}, \tilde{\bar{\theta}})$ on $\hat{X}$)
  
  \[ W_\alpha = \bar{\theta}_1 V^{(1)}_{(12\alpha)} + \sum_\gamma \theta_\alpha \left( \delta^{\alpha}_\gamma V^{(0)}_{(12\beta)} \right) + \sqrt{-1} \sum_{\mu, \nu} (\sigma^\mu \bar{\sigma}^\nu)_{\alpha \gamma} \left( \partial_\mu V^{(0)}_{[\nu]} - \partial_\nu V^{(0)}_{[\mu]} \right) \]
  
  \[ + \theta^\dagger \theta^2 \left( 2 \sqrt{-1} \sum_{\beta, \delta} \varepsilon^{\beta \delta} \sum_\mu (\sigma^\mu \bar{\sigma}^\nu)_{\alpha \beta} (\partial_\mu (\bar{\theta}_1 V^{(1)}_{(12\beta)})) \right). \]

- $(W_1 W_2)|_{\theta^1 \theta^2}$: (in coordinate functions $x$ on $X$)
  
  \[ (W_1 W_2)|_{\theta^1 \theta^2} = - \frac{1}{2} \sum_{\mu, \nu} F^{\mu \nu} F_{\mu \nu} + \frac{\sqrt{-1}}{4} \sum_{\mu, \nu, \mu', \nu'} \varepsilon_{\mu \nu \mu' \nu'} F^{\mu \nu} F^{\mu' \nu'} \]
  
  \[ - \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta, \mu} (\partial_\mu V^{(1)}_{(12\alpha)}) \bar{\sigma}^\mu \bar{\sigma}^\nu (\partial_\mu (\bar{\theta}_1 V^{(1)}_{(12\beta)})) + \frac{1}{4} V^{(0)}_{(12\beta)}^2. \]

- $\bar{W}_\beta$: (in antichiral coordinate functions $(x'', \theta, \bar{\theta}, \tilde{\theta}, \tilde{\bar{\theta}})$ on $\hat{X}$)
  
  \[ \bar{W}_\beta = \bar{\theta}_1 V^{(1)}_{(12\beta)} + \sum_\delta \theta^\dagger (\delta^{\beta}_\delta V^{(0)}_{(12\gamma)} - \sqrt{-1} \sum_\delta \varepsilon_{\beta \delta} (\sigma^\mu \bar{\sigma}^\nu)_{\delta \gamma} (\partial_\mu V^{(0)}_{[\nu]} - \partial_\nu V^{(0)}_{[\mu]})) \]
  
  \[ + \bar{\theta}^\dagger \bar{\theta}^2 \left( - 2 \sqrt{-1} \sum_{\beta, \gamma} \varepsilon^{\beta \gamma} (\sigma^\mu \bar{\sigma}^\nu)_{\delta \beta} (\partial_\mu (\bar{\theta}_1 V^{(1)}_{(12\beta)})) \right). \]

- $(\bar{W}_1 \bar{W}_2)|_{\bar{\theta}^1 \bar{\theta}^2}$: (in coordinate functions $x$ on $X$)
  
  \[ (\bar{W}_1 \bar{W}_2)|_{\bar{\theta}^1 \bar{\theta}^2} = - \frac{1}{2} \sum_{\mu, \nu} F^{\mu \nu} F_{\mu \nu} - \frac{\sqrt{-1}}{4} \sum_{\mu, \nu, \mu', \nu'} \varepsilon_{\mu \nu \mu' \nu'} F^{\mu \nu} F^{\mu' \nu'} \]
  
  \[ + \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta, \mu} \partial_\mu (\bar{\theta}_1 V^{(1)}_{(12\alpha)}) \bar{\sigma}^\mu \bar{\sigma}^\nu (\partial_\mu (\bar{\theta}_1 V^{(1)}_{(12\beta)})) + \frac{1}{4} V^{(0)}_{(12\beta)}^2. \]

- $(\Phi^\dagger V \Phi)|_{\theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2}$: (in coordinate functions $x$ on $X$)
  
  \[ \Phi^\dagger V \Phi = \Phi^\dagger \Phi + \Phi^\dagger 
  \]
Term (I) is computed in Sec. 2.2.

\[
\text{Term (III)} = \theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2 \cdot \Phi^{(0)}(0) \sum_{\mu} V_{\mu}^{(0)} V_{\mu}^{(0)}.
\]

Term (II)|_{\theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2}

\[
\begin{align*}
= & -\Phi^{(0)}(0) \Phi^{(0)}(0) - \Phi^{(0)}(0) \Phi^{(0)}(0) + 2 \sum_{\mu} \partial_{\mu} \Phi^{(0)}(0) \partial_{\mu} \Phi^{(0)}(0) + (\partial_1 \partial_2 \Phi^{(12)}) (\bar{\partial}_1 \bar{\partial}_2 \Phi^{(12)}) \\
+ & 2 \sqrt{-1} \sum_{\mu} V_{\mu}^{(0)} \Phi^{(0)}(0) \Phi^{(0)}(0) - 2 \sqrt{-1} \sum_{\mu} \partial_{\mu} \Phi^{(0)}(0) V_{\mu}^{(0)} \Phi^{(0)}(0) + (\sum_{\mu} V_{\mu}^{(0)} V_{\mu}^{(0)}) \Phi^{(0)}(0) \\
- & \sqrt{-1} \sum_{\mu, \alpha, \beta} (\partial_\beta \Phi^{(\beta)}(\bar{\beta})) (\bar{\partial}_\alpha \Phi^{(\alpha)}(\bar{\alpha})) + \sqrt{-1} \sum_{\mu, \alpha, \beta} \partial_{\mu} (\partial_\beta \Phi^{(\beta)}(\bar{\beta})) (\bar{\partial}_\alpha \Phi^{(\alpha)}(\bar{\alpha})) \\
- & \sum_{\mu, \alpha, \beta} V_{\mu}^{(0)} (\partial_\beta \Phi^{(\beta)}(\bar{\beta})) (\bar{\partial}_\alpha \Phi^{(\alpha)}(\bar{\alpha}))
\end{align*}
\]

Altogether

\[
(\Phi^1 \epsilon \Phi^1) \big|_{\theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2}
\]

\[
\begin{align*}
= & -\Phi^{(0)}(0) \Phi^{(0)}(0) - \Phi^{(0)}(0) \Phi^{(0)}(0) + 2 \sum_{\mu} \partial_{\mu} \Phi^{(0)}(0) \partial_{\mu} \Phi^{(0)}(0) + (\partial_1 \partial_2 \Phi^{(12)}) (\bar{\partial}_1 \bar{\partial}_2 \Phi^{(12)}) \\
+ & 2 \sqrt{-1} \sum_{\mu} V_{\mu}^{(0)} \Phi^{(0)}(0) \Phi^{(0)}(0) - 2 \sqrt{-1} \sum_{\mu} \partial_{\mu} \Phi^{(0)}(0) V_{\mu}^{(0)} \Phi^{(0)}(0) + (\sum_{\mu} V_{\mu}^{(0)} V_{\mu}^{(0)}) \Phi^{(0)}(0) \\
- & \sqrt{-1} \sum_{\mu, \alpha, \beta} (\partial_\beta \Phi^{(\beta)}(\bar{\beta})) (\bar{\partial}_\alpha \Phi^{(\alpha)}(\bar{\alpha})) + \sqrt{-1} \sum_{\mu, \alpha, \beta} \partial_{\mu} (\partial_\beta \Phi^{(\beta)}(\bar{\beta})) (\bar{\partial}_\alpha \Phi^{(\alpha)}(\bar{\alpha})) \\
+ & \Phi^{(0)}(0) \sum_{\beta, \delta} \varepsilon^{\beta \delta} (\partial_1 V_{(1)}^{(12)}) (\bar{\partial}_\beta \Phi^{(\beta)}(\bar{\beta})) - \Phi^{(0)}(0) \sum_{\alpha, \gamma} \varepsilon^{\alpha \gamma} (\partial_1 V_{(12)}^{(1)}) (\bar{\partial}_\gamma \Phi^{(\gamma)}(\bar{\gamma})) + V_{(12)}^{(1)} \Phi^{(0)}(0) \\
= & -\sum_{\mu} \partial_{\mu} \Phi^{(0)}(0) \partial_{\mu} \Phi^{(0)}(0) + 4 \sum_{\mu} \nabla_{\mu} \Phi^{(0)}(0) \nabla_{\mu} \Phi^{(0)}(0) + (\partial_1 \partial_2 \Phi^{(12)}) (\bar{\partial}_1 \bar{\partial}_2 \Phi^{(12)}) \\
- & \sqrt{-1} \sum_{\mu, \alpha, \beta} (\partial_\beta \Phi^{(\beta)}(\bar{\beta})) (\bar{\partial}_\alpha \Phi^{(\alpha)}(\bar{\alpha})) + \sqrt{-1} \sum_{\mu, \alpha, \beta} \partial_{\mu} (\partial_\beta \Phi^{(\beta)}(\bar{\beta})) (\bar{\partial}_\alpha \Phi^{(\alpha)}(\bar{\alpha})) \\
+ & \Phi^{(0)}(0) \sum_{\beta, \delta} \varepsilon^{\beta \delta} (\partial_1 V_{(1)}^{(12)}) (\bar{\partial}_\beta \Phi^{(\beta)}(\bar{\beta})) - \Phi^{(0)}(0) \sum_{\alpha, \gamma} \varepsilon^{\alpha \gamma} (\partial_1 V_{(12)}^{(1)}) (\bar{\partial}_\gamma \Phi^{(\gamma)}(\bar{\gamma})) + V_{(12)}^{(1)} \Phi^{(0)}(0)
\end{align*}
\]

where

\[
\nabla_{\mu} \Phi^{(0)} := (\partial_{\mu} - \frac{\sqrt{-1}}{2} V_{[\mu]}^{(0)}) \Phi^{(0)}(0), \quad \nabla_{\mu} \Phi^{(0)} := (\partial_{\mu} + \frac{\sqrt{-1}}{2} V_{[\mu]}^{(0)}) \Phi^{(0)}(0),
\]

\[
\nabla_{\mu} \Phi^{(0)} := (\partial_{\mu} - \frac{\sqrt{-1}}{2} V_{[\mu]}^{(0)}) \Phi^{(0)}(0), \quad \nabla_{\mu} \Phi^{(0)} := (\partial_{\mu} + \frac{\sqrt{-1}}{2} V_{[\mu]}^{(0)}) \Phi^{(0)}(0)
\]

are the covariant derivatives defined by \( \bar{V} \) on components of \( \Phi \). Note that this is consistent with Lemma 3.2.6; cf. footnote 22.

A supersymmetric action functional for \( U(1) \) gauge theory with matter on \( X \)

Now restore the electric charge \( e_m \) in the discussion. Then the gauge-invariant kinetic term for the matter chiral superfield \( \Phi \) becomes

\[
\hat{\Phi}^1 e_m \hat{V} \Phi.
\]
Thus, replacing $\bar{\Lambda}$ with $e_m\bar{\Lambda}$ and $\bar{V}$ with $e_m\bar{V}$ in the above discussion and computations, we recover the charge $e_m$ case we well.

Recall the standard purge-evaluation/index-contracting map $\mathcal{P} : \mathcal{O}_{\mathcal{X}}^{\text{physics}} \rightarrow \tilde{\mathcal{O}}_{\mathcal{X}}$ from Sec. 2.2 and redenote:

$$
\Phi^{(0)} := \Phi^{(0)}, \quad \Phi^{(\alpha)} := \mathcal{P}(\vartheta_\alpha \Phi^{(\alpha)}), \quad \Phi^{(12)} := \mathcal{P}(\vartheta_1 \vartheta_2 \Phi^{(12)}),
$$

$$
V_{[\mu]} := V^{(0)}_{([\mu]}), \quad V_{(\alpha 12)} := \mathcal{P}(\vartheta_1 V^{(1)}_{(\alpha 12)}), \quad V_{(12 \alpha 2)} := V^{(0)}_{(12 \alpha 2)}.
$$

Then, it follows from Theorem 1.5.3 that

$$
S_2(V_0, V_{[1]}, V_{[2]}, V_{[3]}, V_{(112)}, V_{(1212)}, \Phi^{(0)}, \Phi^{(1)}, \Phi^{(2)}, \Phi^{(12)}),
$$

$$
:= \frac{1}{2g_{\text{gauge}}} \int_{\mathcal{X}} d^4x \, d\theta^2 d\bar{\theta}^1 \mathcal{P}(W_1 W_2) + \frac{1}{2g_{\text{gauge}}} \int_{\mathcal{X}} d^4x \, d\bar{\theta}^2 d\bar{\theta}^1 \mathcal{P}(\bar{W}_1 \bar{W}_2) + \frac{1}{4} \int_{\mathcal{X}} d^4x \, d\theta^2 d\bar{\theta}^1 d\theta^1 \, \mathcal{P}\left(\Phi^4 e^{-m} \bar{\Phi}\right)
$$

$$
+ \int_{\mathcal{X}} d^4x \, d\theta^2 d\bar{\theta}^1 \, \mathcal{P}\left(\lambda \Phi + \frac{1}{2} m \bar{\Phi}^2 + \frac{1}{3} \bar{\Phi}^3\right) + \int_{\mathcal{X}} d^4x \, d\bar{\theta}^2 d\bar{\theta}^1 \, \mathcal{P}\left(\bar{\lambda} \Phi + \frac{1}{2} \bar{m} (\Phi^4)^2 + \frac{1}{3} \bar{\Phi}^3(\bar{\Phi}^3)^2\right)
$$

where $g_{\text{gauge}}$ is the gauge coupling constant, gives a functional of the component fields $(\Phi^{(0)}; \Phi^{(\alpha)}; \Phi^{(12)})_{\alpha=1,2}$ of $\Phi$ (cf. chiral matter) and $(V_{[\mu]}; V_{(\alpha 12)}; V_{(12 \alpha 2)})_{\mu=0,1,2,3; \alpha=1,2}$ of $\bar{V}$ (cf. gauge field) on $\mathcal{X}$ that is invariant under supersymmetries up to boundary terms on $\mathcal{X}$.

Explicitly, up to boundary terms on $\mathcal{X}$, this is the action functional (cf. [W-B: Chap. VI, Eq. (6.13) & Chap. VII, Eq. (7.10)], with mild adjustment, cf. footnote 13.)

$$
S_2(V_0, V_{[1]}, V_{[2]}, V_{[3]}, V_{(112)}, V_{(1212)}, \Phi^{(0)}, \Phi^{(1)}, \Phi^{(2)}, \Phi^{(12)}),
$$

$$
= \int_{\mathcal{X}} d^4x \left(-\frac{1}{2g_{\text{gauge}}} \sum_{\mu, \nu} F_{\mu \nu} F^{\mu \nu} + \frac{\sqrt{-1}}{2g_{\text{gauge}}} \sum_{\alpha, \beta, \mu} \partial_\mu V_{(\alpha 12)} \bar{\sigma}^{\mu, \beta, \alpha} V_{(\beta 12)} + \frac{1}{4g_{\text{gauge}}} V_{(1212)}^2 \right)
$$

$$
+ \sum_\mu \nabla_\mu \Phi^{(0)} \nabla^\mu \Phi^{(0)} - \frac{\sqrt{-1}}{2} \sum_{\mu, \alpha, \beta} \Phi^{(\alpha)} \bar{\sigma}^{\mu, \beta, \alpha} \nabla_\mu \Phi^{(\alpha)} + \frac{1}{4} \Phi^{(12)} \Phi^{(12)}
$$

$$
+ \frac{1}{4} \Phi^{(0)} \epsilon^{\beta \delta} V_{(12 \beta)} \bar{\Phi}^{(\delta)} - \frac{1}{4} \bar{\Phi}^{(0)} \epsilon^{\gamma \alpha} V_{(\alpha 12)} \Phi^{(\gamma)} + \frac{1}{4} V_{(1212)} \bar{\Phi}^{(0)} \Phi^{(0)}
$$

$$
+ \int_{\mathcal{X}} \left(\lambda \Phi^{(12)} + m (\Phi^{(0)} \Phi^{(12)} - \Phi^{(1)} \Phi^{(2)}) + g (\Phi^{(0)} \Phi^{(12)} - 2 \Phi^{(0)} \Phi^{(1)} \Phi^{(2)}) + \text{ complex conjugate} \right).
$$

where

$$
\nabla_\mu \Phi^{(0)} := (\partial_\mu - \frac{\sqrt{-1}}{2} \bar{e}_m V^{(0)}_{[\mu]} \Phi^{(0)}) \Phi^{(0)}, \quad \nabla_\mu \bar{\Phi}^{(0)} := (\partial_\mu + \frac{\sqrt{-1}}{2} e_m V^{(0)}_{[\mu]} \bar{\Phi}^{(0)}),
$$

$$
\nabla_\mu \Phi^{(\alpha)} := (\partial_\mu - \frac{\sqrt{-1}}{2} \bar{e}_m V^{(0)}_{[\mu]} \Phi^{(\alpha)}) \Phi^{(\alpha)}, \quad \nabla_\mu \bar{\Phi}^{(\beta)} := (\partial_\mu + \frac{\sqrt{-1}}{2} e_m V^{(0)}_{[\mu]} \bar{\Phi}^{(\beta)}).
$$

---

**Note for mathematicians** The coefficients are chosen to make the kinetic term of the complex scalar field $\Phi^{(0)}$ in the standard/normalized form: $\sum_\mu \partial_\mu \Phi^{(0)} \partial^\mu \Phi^{(0)}$ and the kinetic term of the gauge field $V_{[\mu]}$ in the standard form $-\frac{1}{2g_{\text{gauge}}} \mathrm{Tr} \sum_{\mu, \nu} F_{\mu \nu} F^{\mu \nu}$ of the Yang-Mills theory with gauge coupling $g_{\text{gauge}}$. One may also consider

$$
\frac{\tau}{2} \int_{\mathcal{X}} d^4x \, d\theta^2 d\bar{\theta}^1 \mathcal{P}(W_1 W_2) + \frac{\tau}{2} \int_{\mathcal{X}} d^4x \, d\bar{\theta}^2 d\bar{\theta}^1 \mathcal{P}(\bar{W}_1 \bar{W}_2)
$$

for the pure gauge term, where $\tau := \frac{1}{g_{\text{gauge}}} - \sqrt{-1} \frac{\alpha}{4\pi}$ is the complexified coupling constant. This will keep the topological term $\int_{\mathcal{X}} F \wedge F$ to the action functional.
are the covariant derivatives defined by $\tilde{V}$ on components of $\tilde{\Phi}$. The index structure of this explicit expression implies that this functional is indeed Lorentz invariant.

This is what underlies [W-B: Chap. VI & U(1) part of Chap. VII] of Wess & Bagger from the aspect of (complexified $\mathbb{Z}/2$-graded) $C^\infty$-Algebraic Geometry. Together with Sec. 2, physicists’ two most basic supersymmetric quantum field theories are now recast solidly into the realm of (complexified $\mathbb{Z}/2$-graded) $C^\infty$-Algebraic Geometry.

The same tower construction can be applied to superspace(-time)s of all other space(-time) dimensions with either simple (i.e. $N = 1$) or extended (i.e. $N \geq 2$) supersymmetries, with necessary modifications dictated by the specifics of spinors in each dimension and signature. A redo of [L-Y1] (D(14.1)) along this new setting should give a fundamental (as opposed to solitonic) description of super D-branes parallel to Ramond-Neveu-Schwarz fundamental superstrings.
Appendix  Notations, conventions, and identities in spinor calculus

Notations, conventions, and identities in spinor calculus that are used in the current work are collected below. See [W-B: Appendix A & Appendix B] and [Argu] for a more complete list.

- **Minkowski metric:** \( \eta_{\mu\nu} = (-+++). \)

- **With the identification** \( \text{Spin}(3,1;\mathbb{R}) \simeq SL(2;\mathbb{C}) \), two-component spinors with upper or lower, dotted or undotted indices transform as follows \((\alpha,\gamma = 1,2; \beta,\dot{\beta} = \dot{1},\dot{2})\)

\[
\begin{align*}
\psi'_\gamma &= \sum_\alpha m^\gamma_\alpha \psi_\alpha, \\
\bar{\psi}'_\delta &= \sum_\beta \bar{m}^\delta_\beta \bar{\psi}_\beta,
\end{align*}
\[
\psi'_\gamma &= \sum_\alpha m^{-1}_\alpha \psi_\alpha, \\
\bar{\psi}'_\delta &= \sum_\beta \bar{m}^{-1}_\beta \bar{\psi}_\beta,
\]

where \( m \in SL(2;\mathbb{C}) \) as \(2 \times 2\) matrices, \( \bar{m} \) the complex conjugate of \( m \).

- **The \( \varepsilon \)-tensor:**

  - \( \varepsilon^{12} = 1 = \varepsilon^{\dot{1}\dot{2}}, \varepsilon^{\alpha\gamma} = -\varepsilon^{\gamma\alpha}, \varepsilon^{\beta\dot{\beta}} = -\varepsilon^{\dot{\beta}\beta}, \) for \( \alpha,\gamma = 1,2 \) and \( \beta,\dot{\beta} = \dot{1},\dot{2} \).

  - Use of \( \varepsilon \)-tensor to raise and lower spinor indices of the same chirality:

\[
\begin{align*}
\psi^\gamma &= \sum_\alpha \varepsilon^{\gamma\alpha} \psi_\alpha, & \psi_\gamma &= \sum_\alpha \varepsilon^{\gamma\alpha} \psi^\alpha, & \bar{\psi}^\delta &= \sum_\beta \varepsilon^{\delta\beta} \bar{\psi}_\beta, & \psi_\delta &= \sum_\beta \varepsilon^{\delta\beta} \bar{\psi}^\alpha.
\end{align*}
\]

  - \( \theta^\alpha\theta^\gamma = \varepsilon^{\alpha\gamma}\theta^1\theta^2, \theta_\alpha\theta_\gamma = -\varepsilon_{\alpha\gamma}\theta_1\theta_2, \theta^\beta\theta^\dot{\beta} = \varepsilon^{\beta\dot{\beta}}\theta^1\theta^2, \theta_\beta\theta_\dot{\beta} = -\varepsilon_{\beta\dot{\beta}}\theta_1\theta_2. \)

- **Pauli matrices** \( \bar{\sigma}^\mu \) and \( \bar{\sigma}^\mu \): \( \bar{\sigma}^{\mu,\dot{\gamma}} := \sum_{\alpha,\beta} \varepsilon^{\delta\dot{\beta}} \varepsilon^{\gamma\alpha} \sigma^{\mu}_{\alpha\beta}, \) where

\[
\begin{align*}
\sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma^2 &= \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

  - Explicitly, \( \sigma^0 = \bar{\sigma}^0, \sigma^\mu = -\bar{\sigma}^\mu \) for \( \mu = 1,2,3 \); in entries, \( \sigma^0_{\alpha\beta} = \bar{\sigma}^{0,\dot{\alpha}\dot{\beta}}, \sigma^\mu_{\alpha\beta} = -\bar{\sigma}^{\mu,\dot{\alpha}\dot{\beta}}. \)

  - Special relations: \( \sigma^\mu_{11} = \bar{\sigma}^{\mu,22} \) and \( \sigma^\mu_{22} = \bar{\sigma}^{\mu,11}. \)

  - \( \theta_{\alpha\dot{\beta}} = \frac{1}{2} \sum_{\mu,\gamma,\delta} \theta^\gamma \sigma_{\mu,\gamma\delta} \bar{\theta}^{\delta} \sigma^\mu_{\alpha\dot{\beta}}, \) \( \theta^\alpha\bar{\theta}^\beta = \frac{1}{2} \sum_{\mu,\gamma,\delta} \theta^\gamma \sigma_{\mu,\gamma\delta} \bar{\theta}^\delta \bar{\sigma}^{\mu,\dot{\alpha}\dot{\beta}}. \)

  - \( \text{Tr}((\sum_\mu p_\mu \bar{\sigma}^\mu))((\sum_\nu p_\nu \bar{\sigma}^\nu)) = -2 \langle p,p' \rangle. \)

- **Let** \( S', S' \) **be the Weyl-spinor representations and** \( V \) **be the vector/fundamental representation of** \( SO(3,1) \). Then, \( S'V \otimes_{\mathbb{C}} S'' \otimes_{\mathbb{R}} \mathbb{C} \simeq V \otimes_{\mathbb{R}} \mathbb{C} \) **as** \( SO(3,1) \)-modules.
# References

[R] R. Argurio, *Introduction to supersymmetry*, lecture notes for PHYS-F-417, October, 2017.

[Arg] P. Argyres, *Introduction to supersymmetry*, Physics 661 lecture notes, Cornell University, fall, 2000.

[Bi] A. Bilal, *Introduction to supersymmetry*, arXiv:hep-th/0101055

[Ch] C. Chevalley, *The algebraic theory of spinors*, Columbia Univ. Press, 1954.

[De] P. Deligne, *Notes on spinors*, in *Quantum fields and strings: a course for mathematicians*, vol. 1, P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, and E. Witten eds., 99-135, American Math. Soc., 1999.

[D-F1] P. Deligne and D.S. Freed, *Supersolutions*, in *Quantum fields and strings: a course for mathematicians*, vol. 1, P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, and E. Witten eds., 227-355, American Math. Soc., 1999.

[D-F2] ———, *Sign manifesto*, in *Quantum fields and strings: a course for mathematicians*, vol. 1, P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, and E. Witten eds., 357-363, American Math. Soc., 1999.

[D-K] S.K. Donaldson and P.B. Kronheimer, *The geometry of four manifolds*, Oxford Univ. Press, 1990.

[Ei] D. Eisenbud, *Commutative algebra – with a view toward algebraic geometry*, GTM 150, Springer, 1994.

[E-H] ———, *The geometry of schemes*, GTM 197, Springer, 2000.

[Fr] D.S. Freed, *Five lectures on supersymmetry*, Amer. Math. Soc., 1999.

[F-Z] S. Ferrara and B. Zumino, *Supergauge invariant Yang-Mills theories*, Nucl. Phys. B79 (1974), 413–421.

[G-G-R-S] ———, *Superspace – one thousand and one lessons in supersymmetry*, Frontiers Phys. Lect. Notes Ser. 58, Benjamin/Cummings Publ. Co., Inc., 1983.

[Hai] G. Hailu, *Quantum field theory III: Supersymmetry*, course Physics 253cr given at Department of Physics, Harvard University, fall 2018.

[Hart] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer, 1977.

[Harv] F.R. Harvey, *Spinors and calibrations*, Pers. Math. 9, Academic Press, 1990.

[Jo] D. Joyce, *Algebraic geometry over C∞-rings*, arXiv:1001.0023 [math.AG].

[Ko] S. Kobayashi, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan 15, Princeton Univ. Press, 1987.

[K-N] ———, *Foundations of differential geometry*, vol. I & vol. II, Interscience Publ., John Wiley & Sons, 1963 and 1969.

[L-Y1] C.-H. Liu and S.-T. Yau, *N = 1 fermionic D3-branes in RNS formulation I. C∞-Algebgeometric foundations of d = 4, N = 1 supersymmetry, SUSY-rep compatible hybrid connections, and ˜D-chiral maps from a d = 4 N = 1 Azumaya/matrix superspace*, arXiv:1808.05011 [math.DG]. (D(14.1))

[L-Y2] ———, manuscript in preparation.

[Ma] Y.I. Manin, *Gauge field theory and complex geometry*, Springer, 1988.

[P-S] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley Publ. Co., 1995.

[St] M.J. Strassler, *An unorthodox introduction to supersymmetric gauge theory*, lectures given at TASI 2001, arXiv:hep-th/0309149

[S-S] A. Salam and J.A. Strathdee, *Supersymmetry and superfields*, Fort. Phys. 26 (1978), 57–142.

[S-W] S. Shnider and R.O. Wells, Jr., *Supermanifolds, super twister spaces and super Yang-Mills fields*, Séminaire Math. Supér. 106. Press. Univ. Montréal, 1989.

[We] P. West, *Introduction to supersymmetry and supergravity*, extended 2nd ed., World Scientific, 1990.

[W-B] J. Wess and J. Bagger, *Supersymmetry and supergravity*, 2nd ed., Princeton Univ. Press, 1992.

[W-Z1] J. Wess and B. Zumino, *A Lagrangian model invariant under supergauge transformations*, Phys. Lett. 49B (1974), 52–54.

[W-Z2] ———, *Super gauge invariant extension of quantum electrodynamics*, Nucl. Phys. B78 (1974), 1–13.

chienhao.liu@gmail.com, chienliu@cmsa.fas.harvard.edu;
yau@math.harvard.edu