SYMMETRIES OF SECOND ORDER ODES

BORIS KRUGLIKOV

Abstract. It is demonstrated that point symmetry algebras of general analytic second order ODEs, not necessary of principal type, can have all dimensions between 0 and 8 except for 7. For the symmetry dimension 8 the ODE must be locally trivializable.

1. Introduction and the main result

Second order scalar ordinary differential equations \( y'' = f(x, y, y') \) have been studied from the geometric viewpoint by Sophus Lie. He formulated the problem of their local classification with respect to the diffeomorphism group of the space \( \mathbb{R}^2(x, y) \) of independent and dependent variables (point transformations) that was solved by his student Arthur Tresse \[9\], see also \[7\]. In this work Tresse also shows that the symmetry algebra of such an ODE can have dimensions only 8, 3, 2, 1 or 0; see also \[8\] for an independent and more rigorous approach.

More general equations \( F(x, y, y', y'') = 0 \) not resolved with respect to the second derivative arise in the study of path geometries with singularities and are important in applications to special functions. Indeed, Bessel, Legendre and hypergeometric differential equation due to Euler and Gauss are special instances of this type. All Painlevé transcendentals are also particular cases. To our knowledge the symmetry analysis of this general class has never been performed, and the main goal of the present note is to address it.

In what follows we assume \( F \) to be analytic and we consider the algebra \( \mathfrak{g} \) of analytic point symmetries of this implicit ODE. The statements below apply to the case, when \( \mathfrak{g} \) consists of vector fields defined in a fixed connected neighborhood \( U \subset \mathbb{R}^2(x, y) \), also with a specified connected neighborhood \( \hat{U} \subset \mathcal{E} = \{ F = 0 \} \) over it, as well as to the case for germs of the fields from \( \mathfrak{g} \) at a fixed point \((x_0, y_0) \in U \).

Theorem. Let \( F(x, y, y', y'') = 0 \) be an analytic ODE of the second order, meaning that \( F_{y''} \neq 0 \). Then the point symmetry algebra \( \mathfrak{g} \) can only have dimensions 8, 6, 5, 4, 3, 2, 1, 0 and all of those are realizable.

Moreover \( \dim \mathfrak{g} > 6 \) implies that the equation is locally trivializable, i.e. equivalent to \( y'' = 0 \) in a neighborhood of any point.
Even if the ODE domain \( U \subset \mathbb{R}^2 \) is connected and simply-connected, a global trivialization may not exist, as the straightening diffeomorphism \( U \to \mathbb{R}^2 \) can only be an immersion but not an embedding.

**Remark.** If \( F_{y''} \equiv 0 \) we get an ODE of the first order. As a rule such equations have an infinite-dimensional symmetry algebra. Indeed, this is always the case with ODEs of the principal type \( y' = h(x, y) \); for instance the trivial equation \( y' = 0 \) has the infinite-dimensional symmetry algebra consisting of the vector fields \( \{ a(x, y) \partial_x + b(y) \partial_y \} \subset \mathcal{D}(\mathbb{R}^2) \). However equations not resolved with respect to the derivative can have smaller symmetry algebras; for example, the ODE \( y = (y')^m \) has two symmetries \( \partial_x, \frac{m-1}{m} x \partial_x + y \partial_y \) for \( m \geq 3 \).

Let us remind that second order ODEs of the principal type are instances of parabolic geometries of type \((SL_3, P_{1,2})\), see [1]. A complex relative of this is the Levi-nondegenerate CR-geometry in dimension 3, which has parabolic type \((SU_{1,2}, P_{1,2})\). Indeed, these two geometries have the same complexifications, and the passage between them is as follows: for a real hypersurface \( \varphi(z, \bar{z}) = 0 \) in \( \mathbb{C}^2 \) the Segre family \( Q_\zeta = \{ z \in \mathbb{C}^2 : \varphi(z, \zeta) = 0 \} \) defines a 2-parametric family of solutions to a second order ODE (obtained by elimination of \( \zeta \)). In [2] E. Cartan used his equivalence method to establish, in particular, that symmetry algebras of CR-structures in 3D have dimensions 8, 3, 2, 1 or 0, which corresponds to the Tresse result mentioned above.

CR-structures in 3D without Levi-nondegeneracy constraint were considered in [6], and it was proved that if CR-manifold \( M^3 \) is not Levi-flat then its symmetry algebra has dimension 8, 5, 4, 3, 2, 1 or 0. This was done by exploiting the above Segre correspondence and studying the respective Fuchsian type second order equations. A much simpler proof of what Kossovskiy and Shafikov called the Poincaré dimension conjecture was presented in [4]. It used only a combination of some fundamental facts of Lie theory and CR-geometry.

In this paper we prove our main result in the same spirit. Note however that second order ODEs and CR-structures in 3D are analogous when non-degenerate, but not in general. In fact, CR-geometry is given by a complex structure on a 2-distribution that can fail to be contact in general, but degenerations of path geometries in 3D consist of collapsing two line fields, spanning a contact distribution at regular points, to one line at non-resolvable points of implicitly given ODEs, and also one or both of these lines can blow up to a plane at singular points.

2. **The proof**

Geometrically, the ODE is encoded as 3-dimensional submanifold \( \mathcal{E} = \{ F(x, y, y_1, y_2) = 0 \} \) in the space of 2-jets \( J^2(\mathbb{R}) = \mathbb{R}^4(x, y, y_1, y_2) \). The latter is equipped with the Cartan distribution \( \mathcal{C}_2 = \text{Ann}(dy_1 - y_1 dx, dy_2 - y_2 dx) \). This induces line field with singularities \( \mathcal{C}_{\mathcal{E}} = \mathcal{C}_2 \cap T\mathcal{E} \) on \( \mathcal{E} \). Another line field with singularities \( V \) is tangent to the fiber of the projection \( \mathcal{E} \to J^0(\mathbb{R}) = \mathbb{R}^2(x, y) \).
At singular points the line $\langle \partial_u \rangle$ is contained in both $C_E$ and $V$, and each of those can become a 2-plane. These two fields span the distribution $\Pi = C_E + V$ on $E$. At the regular points, where the map $\pi_{2,1} : E \to J^1(\mathbb{R})$ is a submersion, i.e. $E = \{y_2 = f(x, y, y_1)\}$, this distribution pulls back the contact distribution $C_1 = \text{Ann}(dy_1 - y_2 dx)$ on $J^1(\mathbb{R}) = \mathbb{R}^3(x, y, y_1)$. Point symmetries of the ODE are bijective with symmetries of the pair of line fields with singularities $(C_E, V)$ on the equation-manifold $E$.

Let us note that due to the assumption $F_{y_2} \not\equiv 0$ the set of points, where the equation has principal type is open and dense. Near each of them $(E, C_E, V)$ has at most 8-dimensional symmetry algebra, and so $g$ satisfies $\dim g \leq 8$. Moreover by the result of Tresse mentioned above [9, 7, 8] the inequality $\dim g > 3$ implies $g \subset sl_3$.

We begin with the proof of the second part of the theorem, namely that if dimension of the symmetry algebra is at least 7, then the ODE is locally trivializable. Since $sl_3$ has no subalgebras of dimension 7, we have $g = sl_3$ in this case. Thus for regular points $a \in E$, where our ODE is of principal type, it is flat, i.e. trivializable. In these points the isotropy algebra $g_a = p_{1,2}$ is the minimal parabolic (Borel) subalgebra in $sl_3$, and a neighborhood of $a$ is homogeneous of the type $SL_3/P_{1,2}$.

Clearly the isotropy has dimension $\dim g_a \in [5, 8]$ and the case 5 yields a locally homogeneous space. Consider the other cases. If $\dim g_a = 8$, then $a$ is a fixed point of the simple Lie algebra $g = sl_3$ and so, by the Guillemin-Sternberg theorem [3], the algebra linearizes at $a$. However for the standard representation of $sl_3$ on $\mathbb{R}^3$ the isotropy of $b \neq 0 = a$ is $g_b = sl_2 \ltimes \mathbb{R}^2$, which is not Borel. Alternatively, we can observe that the isotropy representation of $g_b$ on $\mathbb{R}^3 = g/g_b$ has no invariant 2-plane (and only 1 invariant line). Thus this case is impossible.

As mentioned before, the case $\dim g_a = 7$ is impossible too, so consider the last possibility $\dim g_a = 6$. There is only one subalgebra of dimension 6 up to an outer automorphism (or two up to an inner automorphism): the maximal parabolic subalgebra $p_1$ (or $p_2$). Let $O_a$ be the local 2-dimensional orbit through $a$, it has homogeneous geometry of the type $SL_3/P_1$. For every point $b \in O_a$ the isotropy in $p_1$ contains a simple Lie subalgebra $h_b \simeq sl_2$, analytically depending on $b$, which, by an application of the Guillemin-Sternberg theorem, is linearizable. Thus we get a curve $\ell_b$ of fixed points of $h_b$ passing through $b \in O_a$. The union $\bigcup_{b \in O_a} \ell_b$ is an open set and hence contains a regular point $c \in \ell_b$ arbitrary close to $a$. The algebra $h_b$ belongs to the isotropy subalgebra $g_c$ and so should leave invariant a 2-plane $\Pi_c \subset T_c E$ split in two invariant lines. However there is only one invariant plane on which $h_b$ acts via the standard representation, so no invariant lines exist.

This contradiction shows that the points in a neighborhood of $a$ are regular and hence all of them are flat with respect to the canonical Cartan connection; in other words, the ODE is locally trivializable, as required.
To prove the first part of the theorem it suffices to demonstrate that every integer between 0 and 6 is realizable as the symmetry dimension of some ODE. Dimensions 0 to 3 are realizable via ODEs of principal type \([9]\), so we restrict to dimensions 6, 5, 4. Below are the examples realizing these dimensions (we indicate only the symmetry fields on \(J^0(\mathbb{R}) = \mathbb{R}^2(x, y)\), their prolongations to \(J^2(\mathbb{R})\) are straightforward).

\[ y'y'' = 2(y')^2 : \quad g = \langle \partial_x, x\partial_x, y\partial_y, x^2\partial_x - xy\partial_y, y^2\partial_y, xy^2\partial_y \rangle, \]
\[ xy'' = 2y(xy' + y) : \quad g = \langle x\partial_x, y\partial_y, y^2\partial_y, x^3y^2\partial_y, x^4\partial_x - 3x^3y\partial_y \rangle, \]
\[ y'y'' = (y')^2 : \quad g = \langle \partial_x, x\partial_x, y\partial_y, xy\partial_y \rangle. \]

This finishes the proof. \(\square\)

**Remark.** The subalgebra of dimension 6 in \(\mathfrak{sl}_3\) is unique up to outer automorphisms. However the submaximal symmetric model is not unique. For every integer \(k > 1\) the ODE \(y'y'' = k(y')^2\) has 6-dimensional symmetry algebra \(g = \langle \partial_x, x\partial_x, y\partial_y, (k - 1)x^2\partial_x - xy\partial_y, y^k\partial_y, xy^k\partial_y \rangle\). The form of vector fields shows that these models are not mutually equivalent. This is similar to the effect observed in \([5]\) for CR-structures.

Note also that the above ODE is of Fuchsian type \(y'' = -ky'/x\) after the interchange of variables \(x \leftrightarrow y\).

**Acknowledgement.** The symbolic package MAPLE (DifferentialGeometry) was used in computing symmetries.

**References**

[1] A. Čap, J. Slovák, *Parabolic Geometries I: Background and General Theory*, Mathematical Surveys and Monographs 154, Amer. Math. Soc. (2009).

[2] É. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes*: I. Ann. Math. Pura Appl. 11, no.4, 17-90 (1932); II. Ann. Scuola Norm. Sup. Pisa 1, 333-354 (1932).

[3] V. Guillemin, S. Sternberg, *Remarks on a paper of Hermann*, Trans. Amer. Math. Soc. 130, 110-116 (1968).

[4] A. Isaev, B. Kruglikov, *A short proof of the dimension conjecture for real hypersurfaces in \(\mathbb{C}^2\)*, Proc. A.M.S. 144, no.10, 4395-4399 (2016).

[5] A. Isaev, B. Kruglikov, *On the symmetry algebras of 5-dimensional CR-manifolds*, Adv. Math. 322, 530-564 (2017).

[6] I. Kossovskiy, R. Shafikov, *Analytic differential equations and spherical real hypersurfaces*, J. Differential Geom. 102, no. 1, 67126 (2016).

[7] B. Kruglikov, *Point Classification of Second Order ODEs: Tresse Classification Revisited and Beyond*, Abel Symposia 5, 199-221, Springer (2009).

[8] B. Kruglikov, D. The, *The gap phenomenon in parabolic geometries*, Journal für die Reine und Angewandte Mathematik (Crelle’s Journal) 723 (2017), no. 723, 153–216.

[9] A. Tresse, *Détermination des invariants ponctuels de l’équation différentielle ordinaire du second ordre \(y'' = \omega(x, y, y')\)*, Leipzig (1896).