HIGH ENERGY PHOTON DEEP INELASTIC SCATTERING AT SMALL AND LARGE $Q^2$
WITH SOFT PLUS HARD POMERON

K. Adel,
and
F. J. Ynduráin*

Departamento de Física Teórica, C-XI
Universidad Autónoma de Madrid
Canto Blanco, 28049-Madrid

ABSTRACT.

We show how the sum of a hard singularity, $F_{2H}(x, Q_0^2) \sim x^{-\lambda}$ and a soft Pomeron $F_{2P}(x, Q_0^2) \sim \text{Const.}$ for the singlet piece of the structure function $F_{2S} = F_{2H} + F_{2P}$ for $Q_0^2 \sim$ a few GeV$^2$, plus a saturating expression for the strong coupling, $\hat{\alpha}_s(Q^2) = 4\pi/\beta_0 \log[(Q^2 + \Lambda^2)/\Lambda^2]$ give an excellent description of experiment i) For small $Q^2$, $0 \lesssim Q^2 \lesssim 8.5$ GeV$^2$, and ii) For large $Q^2$, $10 \lesssim Q^2 \lesssim 1500$ GeV$^2$ if evolved with QCD. The $x$ range is $6 \times 10^{-6} \lesssim x \lesssim 0.04$. The description for low $Q^2$ implies self-consistent values for the parameters in the exponents of $x$ both for singlet and nonsinglet. One has to have $\alpha_p(0) = 0.48$ and $\lambda = 0.470$ [$\alpha_P(0) = 1.470$], in uncanny agreement with other determinations of these parameters, and in particular the results of the large $Q^2$ fits. The fit to data is so good that we may look for signals of a “triple Pomeron” vertex, for which some evidence is found.

* e-mail: fjy@delta.ft.uam.es
\section{Introduction.} The HERA collaborations\cite{1,2,3} have produced in the last years data with increasing precision for the ep deep inelastic scattering (DIS) structure function $F_2(x,Q^2)$ at small $x$, and in a wide range of momenta, from $Q^2 \sim 0$ (including real photoproduction at $Q^2 = 0$) to thousands of GeV$^2$. This allows us to study in detail not only the low $x$ behaviour at momenta sufficiently large\cite{4} that perturbative QCD be applicable, which was done in ref. 5 in a full NLO (next to leading order) calculation, but also enables us to address the important issue of the connection between the perturbative regime ($Q^2 > 10$ GeV$^2$, say) and the region ($Q^2 < 10$ GeV$^2$) where nonperturbative effects are determinant.

In ref. 5, where we send for more details, we studied fits of the large momentum region with the hard singularity hypothesis, i.e., under the assumption that the singlet component of $F_2$, $F_S$, behaves as a negative power of $x$ as $x \to 0$ so that, as discussed first in ref. 6, QCD implies the detailed behaviour

$$F_S(x,Q^2) \simeq B_S(\alpha_s(Q^2)) d_s(1+\lambda)x^{-\lambda}$$

(1)

to leading order (LO). The soft Pomeron dominance hypothesis was also considered. Here one assumes\cite{7} that at a given $Q_0^2 \sim 1$ GeV$^2$ one has a standard Pomeron (constant) behaviour, and then evolves with QCD so that, to LO,

$$F_S \simeq c_0 \frac{9 \xi \tau}{4 \pi \tau^2 (33 - 2 n_f)} \exp \left\{ \sqrt{d_0 \xi \tau - d_1 \tau} \right\},$$

(2)

with

$$\xi \equiv |\log x|, \tau \equiv \log \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}.$$

In Eqs.(1), (2) the $d_+ , d_1 , d_0$ are known quantities related to the singlet anomalous dimension. The full NLO corrections to (1), (2) may be found in ref. 5. There it was shown that both (1) and (2) could be used to give reasonable fits to the large $Q^2$ data, with a $\chi^2$/d.o.f. of the order of unity if fitting data with $x \leq 10^{-2}$ or $\chi^2$/d.o.f. $\sim 2$ if including points up to $x = 3.2 \times 10^{-2}$. The hard singularity expression gave more satisfactory results for small $x$, but tended to disagree with experiment for “large” values $x \gtrsim 10^{-2}$.

Because of this feature it was suggested in ref. 5 that one could perhaps conclude that $F_S$ should contain both a hard and a soft piece; since both (1) and (2) solve the QCD evolution equations, so will the sum. In the strict $x \to 0$ limit, (1) dominates over (2); but if the coefficients are such that $c_0 \gg B_S$ then there will exist regions of relatively large values of $x$ where both contributions will be comparable. In fact, it was pointed out that a situation similar to that occurs for real photoproduction ($Q^2 = 0$) where it has been shown\cite{9,10} that an expression\cite{2} for the cross section

$$\sigma_{\gamma}(Q^2 = 0)p(s) = C_H s^\lambda + C_P + C_p s^{-\rho},$$

(3)

provides an excellent fit to the data, up to the very high HERA energies\cite{4}; and formulas similar to (3) are also known to work well in the small $Q^2$ region\cite{11}.

In the present note we pursue this line in two directions. First, we write QCD inspired phenomenological formulas that permit us to interpolate between the large $Q^2$ regime, with $F_2$ given by a combination of (1) and (2) plus a nonsinglet contribution, and the low $Q^2$ region. Here we will show that a natural consistency condition as $Q^2 \to 0$ allows us to determine the constants $\lambda = \lambda_0 , \rho = \rho_0$ finding values remarkably close to those given by the fits at large $Q^2$: $\lambda_0 \simeq 0.470, \rho_0 \simeq 0.522$. What is more, the formulas provide excellent fits to data, with $\chi^2$/d.o.f. $\sim 106^2 / 104$ for $Q^2 \leq 8.5$ GeV$^2$.

Secondly, using these values $\lambda_0 , \rho_0$, and profiting from the fact proved in refs. 6, 8 that these quantities are independent of $Q^2$, we fix them to the values found at low $Q^2$ and produce a fit to high

$^1$ We will consider the region $Q^2 \gtrsim 10$ GeV$^2$ to be that where perturbative QCD is to be trusted. This is because, at $x \to 0$, the NLO corrections are very large.

$^2$ We have added in Eq. (3) the contribution of the $\rho - A_2$ Regge trajectory, which corresponds in DIS language to the nonsinglet piece of the structure function; see below.
moments data with an expression\textsuperscript{3}

\begin{equation}
\begin{aligned}
F_2 &= \langle e_q^2 \rangle [F_S + F_N S]; \\
F_S(x, Q^2) &= B_S[\alpha_s(Q^2)]^{-d_S(1+\lambda_0)}x^{-\lambda_0} \\
&+ \frac{c_0}{\xi} \left[ \frac{9\xi}{4\pi^2(33-2n_f)} \right]^{1/2} \exp \left\{ \sqrt{d_S \xi} \tau - d_1 \tau \right\}; \\
F_N S(x, Q^2) &= B_{N S}[\alpha_s(Q^2)]^{-d_{N S}(1-\rho_0)}x^{\rho_0},
\end{aligned}
\end{equation}

\[ [d_{N S}(1-\rho_0) \text{ is the nonsinglet anomalous dimension, a known function of } 1-\rho_0]. \] This fit improves substantially those obtained with (1) or (2) separately, with the addition of a single new parameter, since we took \( \lambda_0, \rho_0 \) given by the low energy calculation. We are therefore able to give a consistent description of data, from \( Q^2 = 0 \) to thousands of GeV\(^2\), and from \( x \sim 10^{-6} \) to \( x = 0.032 \) with, as free parameters, only the constants \( c_0, B_S, B_{N S} \) and \( Q_0^2 \), plus a low energy one (see below).

\section{2. The low \( Q^2 \) region}

The expression for the virtual photon scattering cross section in terms of the structure function \( F_2 \) is

\begin{equation}
\sigma_{\gamma}(Q^2=0)_{p}(s) = \frac{4\pi\alpha}{Q^2} F_2(x, Q^2), \quad \text{with } s = Q^2/x.
\end{equation}

In the low energy region we should, as discussed, take the soft-Pomeron dominated expression (2) to be given by an ordinary Pomeron, i.e., behaving as a constant for \( x \to 0 \) (or equivalently, \( s \to \infty \)). So the expression that will, when evolved to large \( Q^2 \) yield (4) is

\begin{equation}
\begin{aligned}
F_2 &= \langle e_q^2 \rangle \left\{ B_S[\alpha_s(Q^2)]^{-d_S(1+\lambda_0)}x^{-\lambda_0} \\
&+ C + B_{N S}[\alpha_s(Q^2)]^{-d_{N S}(1-\rho_0)}x^{\rho_0} \right\}.
\end{aligned}
\end{equation}

Note that \( C \neq c_0 \) as the gluon component also intervenes in the evolution; the relation between \( C, c_0 \) may be found in refs. 8, 5, and later in the present work.

On comparing (5) and (6) we see that, as noted in ref. 10, we have problems if we want to extend (6) to very small \( Q^2 \). First of all,

\begin{equation}
\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \log Q^2/L^2}
\end{equation}

diverges when \( Q^2 \sim L^2 \). Secondly, Eq. (5) contains the factor \( Q^2 \) in the denominator so the cross section blows up as \( Q^2 \to 0 \) unless \( F_2 \) were to develop a zero there.

It turns out that there is a simple way to solve both difficulties at the same time. It has been conjectured\textsuperscript{12} that the expression (7) for \( \alpha_s \) should be modified for values of \( Q^2 \) near \( L^2 \) in such a way that it saturates, producing in particular a finite value for \( Q^2 \sim L^2 \). To be precise, one alters (7) according to

\begin{equation}
\alpha_s(Q^2) \to \frac{4\pi}{\beta_0 \log(Q^2 + M^2)/L^2},
\end{equation}

where \( M \) is a typical hadronic mass, \( M \sim m_{\rho} \sim A(n_f = 2) \ldots \); the value \( M = 0.96 \text{ GeV} \) has been suggested on the basis of lattice calculations.

Here we will simply set \( M = A = A_{\text{eff}} \), to avoid a proliferation of parameters. For the Pomeron term [the constant in Eq. (6)] we merely replace \( C \to Q^2/(Q^2 + A_{\text{eff}}^2) \). The expression we will use for low \( Q^2 \) is thus,

\begin{equation}
\begin{aligned}
F_2 &= \langle e_q^2 \rangle \left\{ B_S[\tilde{\alpha}_s(Q^2)]^{-d_S(1+\lambda)}Q^{-2\lambda}s^\lambda \\
&+ CQ^2/Q^2 + A_{\text{eff}}^2 + B_{N S}[\tilde{\alpha}_s(Q^2)]^{-d_{N S}(1-\rho)}Q^{2\rho}s^{-\rho} \right\},
\end{aligned}
\end{equation}

where

\begin{equation}
\tilde{\alpha}_s(Q^2) = \frac{4\pi}{\beta_0 \log(Q^2 + A_{\text{eff}}^2)/A_{\text{eff}}^2}
\end{equation}

\textsuperscript{3} We of course include NLO corrections to (4) in the actual fits.
and we have changed variables, \((Q^2, x) \rightarrow (Q^2, s = Q^2/x)\).

We have still not solved our problems: given Eq. (5) it is clear that a finite cross section for \(Q^2 \rightarrow 0\) will only be obtained if the powers of \(Q^2\) match exactly. This is automatic by construction for the Pomeron term, but for the hard singlet and the nonsinglet piece it will only occur if we have consistency conditions satisfied. With the expression given in (8b) for \(\tilde{\alpha}_s\) it diverges as \(\text{Const.}/Q^2\) when \(Q^2 \rightarrow 0\): so we only get a matching of zeros and divergences if \(\lambda = \lambda_0, \rho = \rho_0\) with

\[
d_s(1 + \lambda_0) = 1 + \lambda_0, \quad d_{NS}(1 - \rho_0) = 1 - \rho_0.
\]

The solution to these expressions depends very little on the number of flavours; for \(n_f = 2\), probably the best choice at the values of \(Q^2\) we will be working with, one finds \(\lambda_0 = 0.470, \rho_0 = 0.522\). The second is in uncanny agreement with the value obtained with either a Regge analysis in hadron scattering processes, or by fitting structure functions in DIS. The first is slightly larger than the value obtained in fits to DIS\(^{[5,6]}\) which give \(\lambda = 0.32\) to 0.38. Nevertheless, these figures were obtained in fits omitting the soft Pomeron-dominated piece; when adding it the favoured value (see below) is \(\lambda = 0.43\), and, as we will see, an essentially as good fit may be obtained provided \(\lambda \leq 0.5\).

We are perfectly aware that we are pushing QCD well below its region of applicability, and that the condition of matching at \(Q^2 \rightarrow 0\) is at best only of phenomenological value. Nevertheless, the fact that we get such reasonable predictions for \(\lambda_0, \rho_0\) probably indicates that our procedure represents, \textit{grosso modo}, the actual situation, which is also justified by the quality of the fit Eq. (9) provides. If we take all H1 and Zeus data\(^{[2,3]}\) for \(0.31 \text{ GeV}^2 \leq Q^2 < 8.5 \text{ GeV}^2\), and we include 10 neutrino \(xF_3\) data\(^4\) we find

\[
\Lambda_{\text{eff}} = 0.87 \text{ GeV}, \quad \langle e^2_q B_S \rangle = 5.28 \times 10^{-3}, \quad \langle e^2_q B_{NS} \rangle = 0.498, \quad \langle e^2_q \rangle C = 0.486,
\]

for a \(\chi^2/\text{d.o.f.} = \frac{106.2}{104.4}\). The value of \(\Lambda_{\text{eff}}\) we have obtained lies somewhere inside the expected bracket, \(\Lambda(n_f = 2) \simeq 0.35 \text{ GeV}\) and the value found in the quoted lattice calculation, 0.96 GeV. Clearly, the fit gives a compromise, phenomenological quantity.

\(^4\) We choose to include these because the function \(xF_3\) is pure nonsinglet and its presence thus helps stabilize the fits: see ref. 5 for details.
The agreement between phenomenology and experiment, shown graphically in Fig.1, is unlikely to be trivial; $x$ varies between $6 \times 10^{-6}$ and $4 \times 10^{-2}$, and $F_2$ changes by almost one order of magnitude. To see this more clearly, we replace the hard singularity by an evolved soft Pomeron, with a saturated $\alpha_s$. That is, we now fit with the expression

$$F_2 = \langle c_q^2 \rangle \left\{ \frac{c_0}{\xi} \left[ \frac{9}{4\pi^2(33-2n_f)} \frac{9}{4\pi^2(33-2n_f)} \right] \right\} \left\{ \exp \left( \frac{d_0}{d_0} \left[ \frac{\log \tilde{\alpha}_s(Q^2)}{\tilde{\alpha}_s(Q^2)} - \frac{d_1}{d_1} \log \tilde{\alpha}_s(Q^2) \right] \right) \right\} + C \frac{Q^2}{Q^2 + A_{\text{eff}}^2} + B_{NS}[\tilde{\alpha}_s(Q^2)]^{-d_{NS}(1-\rho)Q^{2\rho}S^{1-\rho}} \right\},$$

$\tilde{\alpha}_s$ as before. Then we find

$$A_{\text{eff}} = 0.41 \text{ GeV, } \langle c_q^2 \rangle c_0 = 0.094, \langle c_q^2 \rangle C = 0.253, \langle c_q^2 \rangle B_{NS} = 0.41$$

and a $\chi^2$/d.o.f. $= 250/(104-4)$. We consider these results as convincing proof of the necessity of a hard component, which will be henceforth assumed.

§3. The high $Q^2$ region. We next discuss the large $Q^2 \gtrsim 10 \text{ GeV}^2$ region, under various hypotheses for the small $Q^2$ region, which we then evolve with QCD. We will consider moderately large values of $x$, $x \leq 0.032$ because we will be interested not only on the leading behaviour as $x \rightarrow 0$, given almost certainly by a hard singularity, but also on the subleading corrections. We will, because of the success of the small $Q^2$ fits just discussed, assume for the leading piece a hard Pomeron term with a critical value for $\lambda = \lambda_0$. This will be also justified a posteriori both by the quality of the fits and by the fact that if, in the best alternative (which will turn out to be a hard plus a soft Pomeron), we leave $\lambda$ as a free parameter, its optimum falls very close to $\lambda_0$; see below. So we assume that at a certain, fixed $Q_0^2 \sim 1 \text{ GeV}^2$, one has

$$F_S(x, Q_0^2) \simeq B_S x^{-\lambda_0} + F_{\text{corr}}(x, Q_0^2).$$

For the correction term, $F_{\text{corr}}(x, Q_0^2)$, we consider the following possibilities: a soft Pomeron,

$$F_{\text{corr}}^P(x, Q_0^2) \simeq \text{constant};$$

and a $P^\prime$ Regge pole,$^5$

$$F_{\text{corr}}^{P^\prime}(x, Q_0^2) \simeq \text{constant} \times x^1 - \alpha_{P^\prime}(0), \alpha_{P^\prime}(0) \sim 0.5.$$

We also consider the possibility of adding a triple Pomeron-induced term,

$$F_{\text{TP}}(x, Q_0^2) \simeq \text{constant} \times x^{-2\lambda_0}.$$

Once assumed the behaviours given in (10), and taking for simplicity that the gluon structure function behaves like the quark singlet one, we evolve with QCD for higher $Q^2$. Following the methods of refs. 5, 6, 8 we find, to NLO,

$$F_S(x, Q^2) \simeq B_S \left\{ 1 + \frac{c_S(1 + \lambda_0)\alpha_s}{4\pi} \right\} e^{Q^2(1 + \lambda_0)\alpha_s/4\pi} [\alpha_s(Q^2)]^{-d_+(1 + \lambda_0) x^{-\lambda_0} + F_{\text{corr}}(x, Q^2)},$$

and the values of the quantities $d_+, c_S, q_S$ may be found in ref. 5. Then, and depending on the low $Q_0^2$ hypothesis we make we find the correction terms,

$$F_{\text{corr}}^P(x, Q_0^2) \simeq \left\{ 1 + 2 \left[ \frac{\xi}{d_0\tau} \left[ k_1 \frac{\alpha_s(Q^2)}{4\pi} - k \frac{\alpha_s(Q_0^2)}{4\pi} \right] \right] \right\} \right\} \times C_0 \left\{ \frac{9\xi\tau}{4\pi^2(33-2n_f)} \right\}^{\frac{1}{2}} \exp \left\{ \sqrt{d_0\xi \tau - d_1 \tau} \right\},$$

$^5$ For Regge pole theory, and the triple (and multiple) Pomeron terms cf. respectively refs. 13, 14.
$k, k_1$ also given in ref. 5. For the $P'$ pole we find a very similar formula:

$$\begin{align*}
F_{\text{corr.}}^{P'}(x, Q_0^2) & \simeq \left\{ 1 + 2\sqrt{\frac{\xi}{d_0}} \left[ k_1 \frac{\alpha_s(Q^2)}{4\pi} - k \frac{\alpha_s(Q_0^2)}{4\pi} \right] \right\} \\
& \times \exp \left\{ \sqrt{d_0} \xi - d_1 \xi \right\}.
\end{align*}$$

(12b)

Finally, for the triple Pomeron-induced term,

$$F_{\text{TP}}^{\ast}(x, Q_0^2) \simeq B_{\text{TP}} \left\{ 1 + \frac{c_S(1 + 2\lambda_0)\alpha_s}{4\pi} \right\} e^{g_S(1 + 2\lambda_0)\alpha_s/4\pi} \frac{\alpha_s}{d_+} - d_+ (1 + 2\lambda_0) x^{-2\lambda_0}. \quad (12c)$$

As we will see, none of the three possibilities gives a really good fit in the “large” $0 < x < 0.032$ region; for the more precise Zeus data the $\chi^2$/d.o.f. is of 1.7. To remedy this we consider the possibility of softening the large $x$ region by multiplying $F_S(x, Q^2)$ by a factor $(1 - x)^{\nu/2}$: for the discussion and justification of this we send to refs. 5, 10.

The results are summarized in Table I, where we compare the fits obtained with (12a) with the fits found in ref. 5 using only the hard singularity or soft Pomeron-dominated expressions, i.e., setting respectively $c_0, B_S$ equal to zero.

### Table Ia. $n_f = 4$; H1 data. $x \leq 0.032$, $Q^2 \geq 12 \text{ GeV}^2$

| Hard singularity, large $x$ softened | $A$ | $\lambda$ | $\langle e^2 \rangle_B$ | $\langle e^2 \rangle_{BNS}$ | $\chi^2$/d.o.f. |
|-------------------------------------|-----|-----------|----------------|-----------------|----------------|
|                                     | 0.140 GeV | 0.318 | $1.78 \times 10^{-4}$ | 0.35 | $\frac{134}{110-4}$ |
| Soft Pomeron only                   | $A$ | $Q_0^2$ | $\langle e^2 \rangle_{B}$ | $\langle e^2 \rangle_{BNS}$ | $\chi^2$/d.o.f. |
|                                     | 0.165 GeV | 0.70 GeV$^2$ | 0.265 | 0.246 | $\frac{191}{189-4}$ |
| Hard + Soft Pomeron$^*$             | $\lambda$ (fixed) | $Q_0^2$ | $\langle e^2 \rangle_c$ | $\langle e^2 \rangle_{B}$ | $\langle e^2 \rangle_{BNS}$ | $\chi^2$/d.o.f. |
|                                     | 0.47 | 2.17 GeV$^2$ | 0.226 | 4.33 $\times 10^{-4}$ | 0.288 | $\frac{138}{110-4}$ |
| Hard + Soft, large $x$ softened$^*$ | $\lambda$ (fixed) | $Q_0^2$ | $\langle e^2 \rangle_c$ | $\langle e^2 \rangle_{B}$ | $\langle e^2 \rangle_{BNS}$ | $\chi^2$/d.o.f. |
|                                     | 0.47 | 2.22 GeV$^2$ | 0.397 | 3.36 $\times 10^{-4}$ | 0.353 | $\frac{165.7}{110-4}$ |

$A$ fixed at 0.230 GeV.

For the Zeus data one gets similar results, with somewhat less good $\chi^2$/d.o.f.:

### Table Ib. $n_f = 4$; Zeus data. $x \leq 0.025$, $Q^2 \geq 12.5 \text{ GeV}^2$

| Hard singularity, only $x \leq 0.01$ data | $A$ | $\lambda$ | $\langle e^2 \rangle_B$ | $\langle e^2 \rangle_{BNS}$ | $\chi^2$/d.o.f. |
|------------------------------------------|-----|-----------|----------------|-----------------|----------------|
|                                          | 0.135 GeV | 0.301 | 1.25 $\times 10^{-4}$ | 0.314 | $\frac{126}{92-4}$ |
| Soft Pomeron only                        | $A$ | $Q_0^2$ | $\langle e^2 \rangle_c$ | $\langle e^2 \rangle_B$ | $\langle e^2 \rangle_{BNS}$ | $\chi^2$/d.o.f. |
|                                          | 0.165 GeV | 0.90 GeV$^2$ | 0.282 | 0.240 | $\frac{273}{273-4}$ |
| Hard + Soft Pomeron$^*$                  | $\lambda$ (fixed) | $Q_0^2$ | $\langle e^2 \rangle_c$ | $\langle e^2 \rangle_B$ | $\langle e^2 \rangle_{BNS}$ | $\chi^2$/d.o.f. |
|                                          | 0.47 | 2.45 GeV$^2$ | 0.252 | 4.42 $\times 10^{-4}$ | 0.294 | $\frac{197}{120-4}$ |
| Hard + Soft, large $x$ softened$^*$      | $\lambda$ (fixed) | $Q_0^2$ | $\langle e^2 \rangle_c$ | $\langle e^2 \rangle_B$ | $\langle e^2 \rangle_{BNS}$ | $\chi^2$/d.o.f. |
|                                          | 0.47 | 2.72 GeV$^2$ | 0.310 | 3.417 $\times 10^{-4}$ | 0.370 | $\frac{143}{120-4}$ |

$A$ fixed at 0.230 GeV. The optimum value would correspond to $A \sim 0.45$ GeV.

In both tables the expression “large $x$ softened” means that we have multiplied the formulas for $F_S$ by a factor $(1 - x)^{\nu}$, to correct the structure functions for (relatively) large values of $x$. For the hard
singularity case, cf. ref. 5; for the Hard + Soft singularities cases, we have taken \( \nu = 10 \sim 11 \). It is also to be remarked that, if we had fitted \( \lambda \) to e.g., the Zeus data using a hard plus soft term, we would have obtained (not correcting for large \( x \), and fixing \( \Lambda = 0.23 \) GeV),

\[
\lambda = 0.429, \quad Q_0^2 = 2.40 \text{ GeV}^2, \quad \langle e_q^2 \rangle c_0 = 0.244, \quad \langle e_q^2 \rangle B_S = 3.83 \times 10^{-4}, \quad \langle e_q^2 \rangle B_{NS} = 0.30,
\]

for a \( \chi^2/\text{d.o.f.} = 195/120 \). Clearly the values of the parameters, and the \( \chi^2/\text{d.o.f.} \) are similar to those obtained in Table I \( b \) fixing \( \lambda \) to its low energy value found before,\(^6\) \( \lambda = 0.47 \), that we feel justified \( a \) posteriori in taking such value also at large \( Q^2 \).

As next possibility, we try a hard Pomeron, plus a \( P' \) Regge trajectory, Eq. (12b). If we fit the Zeus data with this, adding a softening factor \( (1 - x)^\nu \), \( \nu = 11 \), we find the results of Table II.

\[
\text{Table II.} - n_f = 4; \quad \text{Zeus data.}
\]

| Term                      | \( \lambda \) (fixed) | \( Q_0^2 \) | \( \langle e_q^2 \rangle c_{P'} \) | \( \langle e_q^2 \rangle B_S \) | \( \langle e_q^2 \rangle B_{NS} \) | \( \chi^2/\text{d.o.f.} \) |
|---------------------------|------------------------|-------------|---------------------------------|---------------------------------|---------------------------------|------------------|
| Hard + \( P' \), large \( x \) softened* | 0.47                   | 1.11 GeV\(^2\) | 0.616                           | 4.25 \times 10^{-4}             | 0.343                           | \( \frac{171}{120-4} \) |

* A fixed at 0.230 GeV.

If we had not corrected for the large \( x \) values, i.e., we had not included the factor \( (1 - x)^\nu \), we would have obtained a \( \chi^2/\text{d.o.f.} \) of 270.

We finish this section by describing the results of the fits including a triple Pomeron-induced term. That is, we consider that

\[
F_S = F_H + F_{\text{corr.}} + F_{TP}.
\]

Then we get, for the Zeus data,

\[
\text{Table III.} - n_f = 4; \quad \text{Zeus data. Triple Pomeron correction added.}
\]

| Term                      | \( \lambda \) (fixed) | \( Q_0^2 \) | \( \langle e_q^2 \rangle c_P \) | \( \langle e_q^2 \rangle B_S \) | \( \langle e_q^2 \rangle B_{TP} \) | \( \langle e_q^2 \rangle B_{NS} \) | \( \chi^2/\text{d.o.f.} \) |
|---------------------------|------------------------|-------------|---------------------------------|---------------------------------|---------------------------------|------------------|------------------|
| Hard + \( P \) +TP term*  | 0.47                   | 3.35 GeV\(^2\) | 0.201                           | 8.67 \times 10^{-4}             | -1.96 \times 10^{-4}             | 0.315            | \( \frac{185}{120-5} \) |

* A fixed at 0.230 GeV.

For H1 we would have obtained similar values for the parameters and a \( \chi^2/\text{d.o.f.} = 105/(110-5) \). The inclusion of the triple Pomeron term improves slightly the fit; but perhaps more interesting is that the coefficient of it is, as expected from multi-Pomeron theory,\(^14\) small (when compared with the hard Pomeron) and negative.

Clearly, the best fit is obtained with the hard plus soft Pomeron, just as for the low \( Q^2 \) region. Not only the \( \chi^2/\text{d.o.f.} \) is quite good, but the values of the parameters are very reasonable. In particular, the consistency of the picture is shown by the fact that the value of \( Q_0^2 \) where \( F_{TP} \) is supposed to behave like a constant falls in the middle of the low \( Q^2 \) region where we showed that precisely this assumptions leads to a good fit to data. Because of this, we present in Table IV the parameters of the fits to, simultaneously, H1 and Zeus data on \( ep \), plus neutrino data. This gives our best set of formulas, providing an excellent fit to experiment in a very wide range of \( Q^2, x \).

\(^6\) It is of some interest to remark that the optimum \( \lambda \) found is actually practically identical to the critical value we would have obtained at low \( Q^2 \) with zero flavours, \( \lambda_{n_f=0} = 0.447 \). The difference with the value for two flavours, and the uncertainties inherent to the analysis, however, make us stick to the choice \( \lambda_{n_f=2} = 0.470 \).
In the second case (Table IVb) we do not give the fit including a triple Pomeron term as the \( \chi^2 / \text{d.o.f.} \) does not vary appreciably if including it provided \( \langle e_q^2 \rangle |B_{TP}| \lesssim 2 \times 10^{-4} \). We consider the parameters given in Table IVa to be the more reliable ones for describing low \( x \) structure functions. If we had fitted also \( \lambda \) with the whole set of data we would have obtained minima for values comprised between 0.42 and 0.49, with a variation of the chi-squared of less than two units with respect to the one obtained fixing \( \lambda = 0.470 \). Finally, if we fit the QCD parameter \( \Lambda \), the values which provide minima vary between 0.555 GeV and 0.310 GeV, and the chi-squared improves by less than five units. Because of this we consider, as stated, that it is justified to favour the fits obtained with fixed \( \lambda = 0.470 \), \( \Lambda = 0.23 \) GeV.

Figure 2a. Comparison of predictions from Table IV with H1 ep data\(^{[2]}\) for \( F_2 \).
In Figs. 2 we show the comparison of our fits, with the parameters in Table IV, with data. Note that the same values of the parameters are used in Fig. 2a and Fig. 2b. For both Figs. 2 we give the fits with the “softened” and straight formulas: the continuous lines indicate large-\(x\) softening, and the dotted lines no softening.

§4. **Gluon and longitudinal structure functions.** Detailed predictions for the gluon and longitudinal structure functions are obtained trivially by adding the soft and hard Pomeron expressions given in ref. 5. We will leave the details of this to the reader; but we would like to comment here on a particularly interesting prediction of our analysis for the growth of the cross-section \(\sigma_{\gamma p \to J/\psi p}(W)\) as a function of the c.m. energy, \(W\). In fact, this cross section may be expressed as a function of the gluon structure function \(F_G\),

\[
\sigma_{\gamma p \to J/\psi p}(W) = AF_G(\bar{x} = a \frac{M^2_{J/\psi}}{W^2}, Q^2 = M^2_{J/\psi}),
\]

and \(A, a\) are constants approximately known. In the case in which the dominant singularity is a hard one, one can obtain, parameter-free the gluon structure function as\[^6\]

\[
F_G(x, Q^2) \xrightarrow{x \to 0} B_G[\alpha_s]^{-d_+} x^{-\lambda_0}
\]

and \(B_G\) may be calculated in terms of \(\lambda_0, B_S\). So, using above formulas we have, for the logarithmic slope of the cross section,

\[
\delta = \frac{\log \sigma_{\gamma p \to J/\psi p}(W)}{\log W} \to 2\lambda_0 = 0.94,
\]

The numerical value is obtained with the \(\lambda_0\) found in the previous analyses. This is in remarkable agreement with the figure reported in a fit\[^5\] including recent HERA data\[^6\] which gives \(\delta = 0.9\). A more detailed calculation would produce the whole of \(F_G\) including corrections subleading as \(x \to 0\); this unfortunately involves a new unknown constant, the soft Pomeron component of the gluon.

§5. **Discussion.** The main outcome of our analysis in the present note is that we are able to give a unified,
consistent description of small \( x \) DIS data, both for large and small values of \( Q^2 \). Besides this, there are a number of specific points to which we would like to draw also attention.

First of all there is the matter of the dependence of our low \( Q^2 \) results on the saturation hypothesis for \( \alpha_s \). It is clear that the good quality of the fits indicates that, with suitable modifications, QCD may give a phenomenological description of the data down to very low momenta; but of course this should not be construed as a proof of saturation, in particular of the very specific form considered here. One may interpret our results, however, as showing that the saturation expression is particularly adapted to represent, in DIS, a variety of effects: higher twists, renormalons, and likely also genuine saturation.

A second question is the connection between low and high \( Q^2 \). For the hard piece there is no problem, as both expressions are identical up to NLO corrections. For the soft piece, if we start with a constant behaviour for \( Q^2 \sim 2 - 5 \text{ GeV}^2 \), then as \( Q^2 \) grows an expression like (2) will start to develop. the details of this will depend on what one assumes for the gluon structure function. Because the variation both with \( Q^2 \) and \( x \) of the soft piece is slower than that of the hard part, we think the best procedure is to assume constancy of the soft piece up to \( Q^2 = 8.5 \text{ GeV}^2 \), and the evolved form from there on; since a very good fit is obtained at the low momentum region already with the constant behaviour there is little point in adding frills, and a new constant (the soft component of the gluon structure function).

Finally we devote a few words to the matter of multi-Pomeron exchange. If, at a fixed \( Q^2 \) a single hard Pomeron gives\(^7\)
\[
F_{1P}(x, Q^2) \underset{s \to \infty}{\sim} b_{1P}(Q^2)s^\lambda,
\]
then an \( n \)-Pomeron term will produce the behaviour
\[
F_{nP}(x, Q^2) \underset{s \to \infty}{\sim} b_{nP}(Q^2)s^{n\lambda};
\]
the constants \( b_{nP} \) should depend on the momentum at which they are calculated.

In some approximations (e.g., of eikonal type\(^{14}\)) one has, for \( Q^2 = -M_{\text{had}}^2 \), i.e., for on-shell scattering of hadrons,
\[
b_{nP}(-M_{\text{had}}^2) = (-1)^{n+1}\frac{\delta^n}{n!}C,
\]
so we get for the sum
\[
F_S(x, -M_{\text{had}}^2) = \sum_{n=1}^\infty F_{nP}(x, Q^2 = -M_{\text{had}}^2) = C - C \exp \left[ -\delta s^\lambda \right] \underset{s \to \infty}{\sim} C.
\]
For values of \( Q^2 \) of the order of \( Q_0^2 \sim 2 - 4 \text{ GeV}^2 \), we expect that the \( b_{nP}(Q^2) \) will not change much, so if we write
\[
b_{nP}(Q_0^2) \approx b_{nP}(-M_{\text{had}}^2) + \Delta_n,
\]
we will then get,
\[
F_S(x, Q_0^2) \approx F_S(x, -M_{\text{had}}^2) + \sum \Delta_n \approx C + \Delta_1 x^{-\lambda} + \Delta_2 x^{-2\lambda} + \ldots
\]
which is the expression we have used in the text. In this respect, the evidence for a behaviour like this may be taken as supporting multi Regge theory.

\(^7\) The following discussion is rather sketchy; details and references may be found in the review of ref. 14.
ACKNOWLEDGEMENTS

The financial help of CICYT, Spain, is gratefully acknowledged. Thanks are due to F. Barreiro and A. Kaidalov for illuminating discussions.

REFERENCES

1. M. Derrick et al, Z. Phys. C65 (1995) 397; Phys. Lett. B345 (1995) 576.
2. H1 Collaboration, preprint DESY 96-039, 1996.
3. Zeus Collaboration, preprint DESY 96-076, 1996.
4. M. Derrick et al., Phys. Lett., B295 (1992) 465; ibid., Z. Phys. C, 63 (1994) 399.
5. K. Adel, F. Barreiro and F. J. Ynduráin, FTUAM 96-39 (hep-hp/9610380), in press in Z. Phys. C.
6. C. López and F. J. Ynduráin, Nucl. Phys., B171 (1980) 231.
7. A. De Rújula et al., Phys. Rev., D10 (1974) 1649.
8. F. Martin, Phys. Rev., D19 (1979) 1382.
9. C. López and F. J. Ynduráin, Phys. Rev. Lett., 44 (1980) 1118.
10. F. J. Ynduráin, Preprint FTUAM 96-12 (revised), to be published in Proc. QCD 96, Nucl. Phys. Suppl.
11. B. Badelek and J. Kwiecinski, Phys. Lett. B295 (1992) 263; Rev. Mod. Phys., 68 (1996) 445.
12. Yu. A. Simonov, Yadernaya Fizika, 58 (1995) 113, and work quoted there.
13. V. D. Barger and D. B. Cline, Phenomenological Theories of High Energy Scattering, Benjamin, 1969.
14. A. B. Kaidalov, Survey in High Energy Physics, Vol. 9 (1996) 143.
15. H. Klein, DESY 96-218 (1996).
16. M. Derrick et al., Phys. Lett., B350 (1995) 134, and work quoted there.