LINEAR DEGREE GROWTH IN LATTICE EQUATIONS

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To Reinout Quispel on his 66th birthday, in friendship and with gratitude.

Abstract. We conjecture recurrence relations satisfied by the degrees of some linearizable lattice equations. This helps to prove linear degree growth of these equations. We then use these recurrences to search for lattice equations that have linear growth and hence are linearizable.

1. Introduction. Discrete integrable systems have been a topic of many studies for the last three decades, where integrable refers to possession of one or more signature properties that imply ‘low complexity’ of the dynamics. Complexity of a rational map or a lattice equation can be measured through the so-called algebraic entropy. It has been used as an integrability detector [3, 7, 14, 26, 27, 29]. If a discrete map or lattice equation has vanishing entropy, i.e. degrees of iterations of the map (or equation) in terms of initial variables grow sub-exponentially, then this can be taken as a definition of being integrable and heuristically it will be accompanied by other signature properties of integrability. A type of sub-exponential growth is polynomial growth. It is noted that all known discrete maps or lattice equations with sub-exponential degree growth have, in fact, polynomial growth [4, 5, 20].

Algebraic entropy of a map or a lattice equation is often calculated as follows. One can introduce projective coordinates for each variable and write a map or an equation as a rule in these coordinates. By iteratively looking at the degrees of the projective coordinates at each vertex, expressed as functions of the initial conditions after canceling common factors, we obtain the degree sequence of the rule. The next step is to infer a generating function for this sequence and hence extrapolate to calculate algebraic entropy for the rule. It seems that the underlying reason for many integrable maps and lattice equations to have vanishing entropy has not been focused on until recently [14, 15, 23, 20, 29]. In [23] we were able to prove polynomial degree growth of a large class of lattice equations (autonomous and non-autonomous) subject to a conjecture. The key ingredient for our approach was to find a recursive formula for the greatest common factor of the coordinate functions

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and then derive a linear recurrence relation for the actual degrees. The results for lattice equations could then be applied to mappings obtained as reductions of the lattice equations (with some exceptions). In general, many known integrable lattice equations were shown to share the same universal linear recurrence for the actual degrees (see eq. (10) below) which led to quadratic growth.

We note that there is a sub-class of integrable lattice equations which are linearizable, i.e. equations can be brought to linear equations or systems after some rational transformations cf. [2, 16, 17, 18, 19, 22, 24]). One quick test for linearization is using the algebraic entropy test. Linear growth of degrees of an equation indicates that this equation can be linearized. One then can use the symmetry approach given in [17] to transform it to a linear equation or a system of linear equations. Similarly to [23], the first question that arises here is what are the recurrence relations for the actual degrees of some known linearizable lattice equations. If there is such a recurrence, is it related to the recurrence that we have found previously? On the other hand, starting from the recurrence relation that shows quadratic growth, can we find some specialization of it that implies linear growth? Can we then search for candidate equations that satisfy these relations?

In this paper, we try to answer some of these questions. This paper is organized as follows. In section 2, we briefly set up all the notations that will be used for the paper. In section 3, we present some recurrence relations for the actual degrees to have linear growth, i.e. (i) $H$, (ii) $H$ and (iii) $H$ of Theorem 3.1. In section 4, we show that the recursive formulas for the greatest common factor at each vertex of the discrete Liouville equation [2], Ramani-Joshi-Grammaticos-Tamizhmani equation [22] and the $(3, -1)$ reduction of pKdV [14] give us these degree recurrence relations. We then use the associated linear equations of the lattice equations mentioned in section 4 to prove linear growth of the original equations in section 5. A search for examples of equations on quad-graphs that satisfy those relations is carried out in section 6. This is done by constructing recursive relations for the common factors based on the linear recurrences given in section 3. The Ramani-Joshi-Grammaticos-Tamizhmani equation is obtained though this search. A nice property of algebraic entropy is that it is invariant under birational transformation but need not be under non-rational transformations, a point highlighted in the Appendix.

### 2. Setting

In this section, we give a procedure to compute degrees of lattice equations defined on a square. This procedure was presented in detail in [23]. We consider a multi-affine equation (i.e. linear in each variable) on the quad-graph

$$ Q(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = Q(u, u_1, u_2) = 0, $$

(1)

where in the second function, we highlight the shorthand we sometimes use: $u = u_{l,m}$ and subscripts 1, 2 denote the shifts in the $l$ and $m$ directions, respectively. One can solve uniquely for each vertex variable of this equation. Suppose that we solve for the top right vertex variable $u_{12}$. By introducing projective coordinates $u = x/z$ etc., we obtain the rule at the top right vertex in projective form:

$$ x_{12} = f^{(1)}(x, x_1, x_2, z, z_1, z_2), $$

$$ z_{12} = f^{(2)}(x, x_1, x_2, z, z_1, z_2). $$

The functions $f^{(1)}$ and $f^{(2)}$ are homogeneous of degree 3 in their variables, with one term from each vertex. Initial or boundary values in the literature are typically
given either on the boundary of the first quadrant ‘corner initial values’:
\[ I_1 = \{(l, m) \in \mathbb{Z} \times \mathbb{Z} \mid l \cdot m = 0, \ l, \ m \geq 0\} \]  
(2)
or on the \((1, -1)\) ‘staircase’:
\[ I_2 = \{n(1, -1) \mid n \in \mathbb{Z}\} \cup \{n(1, -1) + (1, 0) \mid n \in \mathbb{Z}\} \]
\[ = \{(n, -n) \mid n \in \mathbb{Z}\} \cup \{(n + 1, -n) \mid n \in \mathbb{Z}\}. \]  
(3)

They are given by blue paths in the following figure. The staircase boundary condition leads to an evolution of degrees on parallel staircases, the degrees being a function of one index labelling the staircase, a case that has been largely studied [7, 12, 13, 27]. The corner initial values lead to the degree being a function of the 2 coordinates \((l, m)\) on the lattice and is a more general situation. Our results for it should apply (via a linear transformation) to other wedge-shaped boundaries [23, 26].

![Figure 1. Initial values \(I_1\) (left) and \(I_2\) (right) for lattice equations.](image-url)

We choose initial values for \(x\) and \(z\) as polynomials in a variable \(w\) of the same degree. Using the rule for the top right vertex \(u_{12}\), one can find all the points on the right hand side of these boundaries as polynomials in \(w\).

Let \(\gcd_{i,m}(w) = \gcd(x_{i,m}(w), z_{i,m}(w))\) and write
\[ \bar{x}_{i,m}(w) := \frac{x_{i,m}(w)}{\gcd_{i,m}(w)} \]  
(4)
\[ \bar{z}_{i,m}(w) := \frac{z_{i,m}(w)}{\gcd_{i,m}(w)}. \]  
(5)

We denote
\[ d_{i,m} = \deg(x_{i,m}) = \deg(z_{i,m}), \ \bar{d}_{i,m} = \deg(\bar{x}_{i,m}) = \deg(\bar{z}_{i,m}), \ g_{i,m} = \deg(\gcd_{i,m}). \]  
(6)

It is easy to see that
\[ d_{i+1,m+1} = d_{i,m} + d_{i+1,m} + d_{i,m+1} \text{ and } d_{i,m} = g_{i,m} + \bar{d}_{i,m}. \]  
(7)

Moreover, we also know that \(\gcd_{i,m}(w) \gcd_{i+1,m+1}(w) \mid \gcd_{i,m+1}(w) \gcd_{i+1,m}(w)\); therefore, we introduce the spontaneous \(\gcd\)
\[ \frac{\gcd_{i+1,m+1}}{\gcd_{i,m} \gcd_{i+1,m} \gcd_{i,m+1}}. \]  
(8)
This implies
\[ \bar{g}_{l+1,m+1} := \deg(\gcd_{l+1,m+1}) = g_{l+1,m+1} - g_{l,m} - g_{l+1,m} - g_{l,m+1}. \] (9)

In [23] we conjectured a recursive formula for gcd_{l,m} which led to a recurrence relation for the actual degrees \( \bar{d}_{l,m} \). For all integrable lattice equations considered in that paper, the recurrence relation is
\[ \bar{d}_{l-1,m-1} + \bar{d}_{l,m+1} + \bar{d}_{l+1,m} - \bar{d}_{l+1,m+1} - \bar{d}_{l-1,m} + \bar{d}_{l,m-1}. \] (10)

As indicated in Figure 2 the recurrence relation (10) guarantees that the sum of the degrees on the blue vertices and on the red vertices are the same.

3. **Conditions for linear growth.** In this section, we show that specializations of equation (10) imply linear degree growth.

The equation (10) is equivalent to any of the rearranged equations below:
\[ \bar{d}_{l,m} - \bar{d}_{l-1,m-1} + \bar{d}_{l-1,m+1} = \bar{d}_{l+1,m+1} - \bar{d}_{l+1,m} - \bar{d}_{l,m}, \] (11)
\[ \bar{d}_{l+1,m} - \bar{d}_{l,m} - \bar{d}_{l-1,m+1} = \bar{d}_{l+1,m+1} - \bar{d}_{l,m+1} - \bar{d}_{l-1,m}, \] (12)
\[ \bar{d}_{l,m+1} - \bar{d}_{l-1,m} + \bar{d}_{l-1,m-1} = \bar{d}_{l+1,m+1} - \bar{d}_{l+1,m} - \bar{d}_{l,m+1} + \bar{d}_{l,m-1}. \] (13)

where the right-hand side in each case is the shift of the left-hand side in the diagonal, vertical and horizontal directions, respectively. These equations are illustrated in Figure 3 where the ± denote the coefficients on either side of each equation and again the weighted sum of the blue vertices equals to the weighted sum of the red vertices (where 'weight' refers to the sign of the coefficient).
Proposition 1. Equation (10) is equivalent to any of the following equivalent statements

(i) \( \bar{d}_{l+1,m+1} - \bar{d}_{l,m+1} - \bar{d}_{l+1,m} + \bar{d}_{l,m} = k(l - m) \),
(ii) \( \bar{d}_{l+1,m+1} - \bar{d}_{l,m} - \bar{d}_{l,m+1} + \bar{d}_{l-1,m} = k(l) \),
(iii) \( \bar{d}_{l+1,m+1} - \bar{d}_{l+1,m} - \bar{d}_{l,m} + \bar{d}_{l,m-1} = k(m) \),

where \( k \) is a function that only depends on \( l - m \), \( l,m \), respectively.

Proof. We prove that equation (10) is equivalent to (i). We denote the right-hand side of (11) as \( k_{l,m} \). We have \( k_{l-1,m-1} = k_{l,m} \). It implies that \( k_{l,m} = k_{l,m,0} \) if \( l \geq m \) or \( k_{l,m} = k_{0,m-l} \) if \( l < m \). This means that \( k_{l,m} \) depends only on \( l - m \) i.e. \( k_{l,m} = k(l - m) \). On the other hand, if \( k_{l,m} = k(l - m) \), we have \( k_{l-1,m-1} = k_{l,m} \).

This is exactly equation (11) which is equivalent to equation (10).

Similarly, one can prove that equation (10) is equivalent to (ii) and (10) is equivalent to (iii). Hence (i), (ii), (iii) are all equivalent themselves.

It has been proved in [23, Theorem 9] that \( \bar{d}_{l,m} \) has quadratic growth along the diagonals with unit slope when corner initial values are affine in \( w \). Therefore, in general equations (i), (ii) and (iii) of Proposition 1 give such quadratic growth. However, there are special cases that give us linear growth. For example, the (1,1) staircase version, or the \( I_2 \) initial boundary condition (3) of Proposition 1 (10), (i), (ii), (iii) are

\[
\begin{align*}
\bar{d}_{n+4} - 2\bar{d}_{n+3} + 2\bar{d}_{n+1} - \bar{d}_n &= 0, \\
\bar{d}_{n+2} - 2\bar{d}_{n+1} + \bar{d}_n &= k_n, \\
\bar{d}_{n+3} - 2\bar{d}_{n+2} - \bar{d}_{n+1} + \bar{d}_n &= k_n,
\end{align*}
\]

where \( k_n = \bar{d}_{n+2} \) for the second equation and \( k_n = \bar{d}_{n+1} \) for the third equation.

The first equation gives us the solution \( \bar{d}_n = c_1 (-1)^n + c_2 + c_3 n + c_4 n^2 \) which is quadratic. Thus, in order to get linear growth we need to have \( c_4 = 0 \). Suppose that the initial values for equation (14) are \( \bar{d}_{i_0}, \bar{d}_{i_0+1}, \bar{d}_{i_0+2} \) and \( \bar{d}_{i_0+4} \). Solving the system of linear equations for \( c_i \), we get \( c_4 = 0 \) if and only if \( \bar{d}_{i_0} + \bar{d}_{i_0+3} - \bar{d}_{i_0+1} - \bar{d}_{i_0+2} = 0 \), which is equation (10) when \( k_n = 0 \).

For the last two equations if \( k_n \neq 0 \) then \( \bar{d}_n \) has quadratic growth as \( \bar{d}_{n+1} - \bar{d}_n \) is linear. However, if \( k_n = 0 \), it is easy to see that \( \bar{d}_n \) grows linearly as the characteristic equations for the last two equations are \( (\lambda - 1)^2 = 0 \) and \( (\lambda - 1)^2 (\lambda + 1) = 0 \).

On the other hand, with corner boundary conditions (2) in Proposition 1 (i) or slight variants of them to reflect the geometries in (ii) and (iii) (see Figure 3), we have the following Theorem for the homogeneous cases of (i), (ii) and (iii).

Theorem 3.1. Let (i)H, (ii)H and (iii)H be the associated homogeneous versions of equations (i), (ii) and (iii), respectively, of Proposition 1.

1. For equation (i)H, if initial values \{\( \bar{d}_{i_0,m}, \bar{d}_{i_0,m}, l \geq i_0, m \geq m_0 \)\} are linear in \( m,l \), respectively, then \( \bar{d}_{l,m} \) grows linearly in both horizontal and vertical directions.
2. For equation (ii)H, if initial values \{\( \bar{d}_{i_0-1,m}, \bar{d}_{i_0,m}, \bar{d}_{i_0,m}, l \geq i_0, m \geq m_0 \)\} are linear in \( m,l \), respectively, then \( \bar{d}_{l,m} \) grows linearly along the horizontal direction.
3. For equation (iii)H, if initial values \{\( \bar{d}_{i_0,m}, \bar{d}_{i_0,m}, \bar{d}_{i_0,m}, l \geq i_0, m \geq m_0 \)\} are linear in \( m,l \) respectively then \( \bar{d}_{l,m} \) grows linearly along the vertical direction.
Proof. 1. For the homogeneous equation (i), we have \( \ddot{d}_{l,m} = \ddot{d}_{l_0,m} + \ddot{d}_{l,m_0} - \ddot{d}_{l_0,m_0} \). Therefore, if \( \ddot{d}_{l_0,m} \) and \( \ddot{d}_{l,m_0} \) are linear functions in \( m \) and \( l \) respectively, then \( \ddot{d}_{l,m} \) grows linearly.

2. For \( k \geq 1, j \geq 0 \), we have
\[
\ddot{d}_{l_0-1+j+k,m_0+j} - \ddot{d}_{l_0-1+j+k-1,m_0+j} = \ddot{d}_{l_0-1+k,m_0} - \ddot{d}_{l_0-1+k-1,m_0}.
\]
Thus we have
\[
\ddot{d}_{l_0-1+j+k,m_0+j} = \ddot{d}_{l_0-1+k,m_0} - \ddot{d}_{l_0-1,m_0} + (\ddot{d}_{l_0,m_0+1} - \ddot{d}_{l_0-1,m_0+1}) + \ldots + (\ddot{d}_{l_0,m_0+j-1} - \ddot{d}_{l_0-1,m_0+j-1}) + \ddot{d}_{l_0,m_0+j}.
\]
It shows that for each fixed \( j \), \( \ddot{d}_{l,m_0+j} \) grows linearly along the horizontal direction.

3. For the third statement, we just need to swap \( l \) and \( m \) and then it becomes the second statement. \( \square \)

Recall that the \((q,-p)\) reduction of a lattice equation, where \( q, p \) are positive co-prime integers, gives us an ordinary difference equation of order \((p + q)\) for the variable \( V_n := u_{l,m} \), where \( n = lp + mq + 1 \) cf. [21, 23].

**Corollary 1.** For \( \gcd(q,p) = 1 \) and for \( q, p > 0 \), the \((q,-p)\) reductions of the homogeneous equations (i)\( H \), (ii)\( H \) and (iii)\( H \) provide linear growth for \( \ddot{d}_n \).

In fact, one can easily see this result via characteristic equations of these reduced equations. The \((q,-p)\) reductions of homogenous equations (i)\( H \), (ii)\( H \) and (iii)\( H \) give us the following ordinary linear difference equations and their characteristic equations
\[
\ddot{d}_{n+p+q} - \ddot{d}_{n+p} - \ddot{d}_{n+q} + \ddot{d}_n = 0 \iff \lambda^{p+q} - \lambda^p - \lambda^q + 1 = 0, \quad (17)
\]
\[
\ddot{d}_{n+2p+q} - \ddot{d}_{n+p+q} - \ddot{d}_{n+p} + \ddot{d}_n = 0 \iff \lambda^{2p+q} - \lambda^{p+q} - \lambda^p + 1 = 0, \quad (18)
\]
\[
\ddot{d}_{n+2q+p} - \ddot{d}_{n+p+q} - \ddot{d}_{n+q} + \ddot{d}_n = 0 \iff \lambda^{p+2q} - \lambda^{p+q} - \lambda^q + 1 = 0. \quad (19)
\]
Each of the characteristic equations has 1 as a double root and other roots which are distinct roots of unity.

4. **Gcd recursions and linear growth of some linearizable equations.** Referring back to the method given in [23], we will provide in this section recursive formulas for the gcd of some examples of linearizable equations previously identified in the literature. As a result of these recurrences, the actual degrees of these equations satisfy the homogeneous equation (i)\( H \), and a reduction of (ii)\( H \) from Theorem 3.1. We note that these recurrences have been checked using Maple on lattice squares of size given below. Some of these recurrences will be proved rigorously in section 5.

4.1. **Liouville equation.** In this section, we study growth of degrees of the discrete Liouville equation. The discrete Liouville equation is given as follows
\[
u_{l,m} u_{l+1,m+1} (u_{l+1,m+1} + 1)(u_{l,m+1} + 1) - u_{l+1,m} u_{l,m+1} = 0. \quad (20)
\]
The discrete Liouville equation was first introduced by Adler and Startsev [2]. It is known that this equation is Darboux integrable and linearizable. Therefore, it should have linear growth. In fact, this equation is equivalent to Equation 22 in [13] which has been checked to have linear growth with staircase initial values \( I_2 \) of
In terms of projective coordinates, we can rewrite (20) as
\[ x_{l+1,m+1} = x_{l+1,m} x_{l,m+1} z_{l,m}, \quad (21) \]
\[ z_{l+1,m+1} = x_{l,m}(x_{l+1,m} + z_{l+1,m})(x_{l,m+1} + z_{l,m+1}). \quad (22) \]

Taking arbitrary corner boundary conditions (2), we find by direct calculation that \( \gcd_{2,1} = x_{1,0} \), \( \gcd_{1,2} = x_{0,1} \) and \( \gcd_{2,2} = x_{1,0}^2 x_{0,1}^2 z_{0,0} = x_{1,1} x_{1,0} x_{0,1} \). More generally, this holds for any \( 2 \times 1, 1 \times 2 \) and \( 2 \times 2 \) blocks. However, if we extend to a \( 2 \times 3 \) or a \( 3 \times 2 \) block, we have an additional factor \( x_{1,1} + z_{1,1} \) for the top right vertex, i.e. \( (x_{1,1} + z_{1,1}) \) is a divisor of \( \gcd_{2,3} \) and \( \gcd_{3,2} \). This implies that for \( l, m \geq 3 \) we get \( z_{l-1,m-1} \) \( \gcd_{l,m} \). We also find that
\[ \gcd_{3,2} = \frac{\gcd_{2,2} \gcd_{3,1} x_{2,1}(x_{1,1} + z_{1,1})}{\gcd_{1,1}}, \]
\[ \gcd_{2,3} = \frac{\gcd_{2,2} \gcd_{1,3} x_{1,2}(x_{1,1} + z_{1,1})}{\gcd_{1,1}}, \]
\[ \gcd_{3,3} = \frac{\gcd_{2,2} \gcd_{3,2} x_{2,2} z_{2,2}}{\gcd_{2,2}}. \]

Thus, for \( l, m \geq 1 \), we build the following recurrence
\[ G_{l+1,m+1} = \begin{cases} 
\frac{\gcd_{l+1,m+1} G_{l+1,m+1} G_{l,m+1} x_{l,m}(x_{l,m-1} + z_{l,m-1})}{G_{l,m-1}} & \text{if } l < 1 \text{ or } m < 1 \text{ or } l = 1, m = 1 \\
\frac{G_{l+1,m+1} G_{l+1,m+1} x_{l,m}(x_{l,m-1} + z_{l,m-1})}{G_{l-1,m}} & \text{if } l > 1, m = 1, \\
\frac{G_{l+1,m+1} G_{l+1,m+1} x_{l,m} x_{l,m-1}}{G_{l,m}} & \text{if } l > 1, m > 1.
\end{cases} \quad (23) \]

Taking corner initial values as random polynomials of degree 1 in \( w \) with integer coefficients, we have checked numerically for \( l, m \leq 12 \) that \( \gcd_{l,m} = G_{l,m} \) (up to a multiplicative constant). Assuming this to be true for all \( l, m \) and taking degrees of both sides of (23) for \( l, m > 1 \) (noting (6)), we have that \( g_{l,m} \) satisfies
\[ g_{l+1,m+1} = g_{l+1,m} + g_{l,m+1} + 2d_{l,m} - g_{l,m}, \]
\[ = d_{l+1,m} + d_{l+1,m+1} + d_{l,m} - d_{l+1,m} - d_{l,m+1} + d_{l,m}, \]
\[ = d_{l+1,m+1} - d_{l+1,m} + d_{l,m+1} + d_{l,m}, \]
where we have used (7). This leads to the homogeneous equation \((i)H\) of Theorem 3.1:
\[ d_{l+1,m+1} = d_{l+1,m} + d_{l,m+1} + d_{l,m}. \quad (24) \]

Using (23) and (7) we have \( d_{1,m} = m + 2 = d_{m,1} \) for \( m \geq 1 \) and \( d_{2,m} = d_{m,2} = 2(m + 2) \). Using the recurrence relation (24) we get \( d_{l,m} = 2(l + m) \), for \( l, m \geq 2 \). It means \( d_{l,m} \) grows linearly along the diagonal, horizontal and vertical directions.

4.2. Ramani-Joshi-Grammaticos-Tamizhmani equation (RJGT). The second linearizable lattice equation that we consider here is the non-autonomous equation given in [22]. This equation is the non-autonomous version of a CAC equation found by Hietarinta [11] (after a homographic transformation). The RJGT equation is given by
\[ (u_{l+1,m+1} + r_{l,m+1})u_{l+1,m}(s_{l,m} u_{l,m} + t_{l,m}) = (u_{l+1,m} + r_{l,m}) u_{l,m} (s_{l,m+1} u_{l,m+1} + t_{l,m+1}), \quad (25) \]
where parameters \( r, s, t \) are free functions in \( l, m \). In terms of projective coordinates, we have

\[
x_{l+1,m+1} = r_{l,m} s_{l,m+1} x_{l,m} x_{l,m+1} z_{l+1,m} + r_{l,m} t_{l,m+1} x_{l,m} z_{l+1,m},
\]

\[
- r_{l,m+1} s_{l,m} x_{l+1,m} z_{l,m+1} + r_{l,m} t_{l,m+1} x_{l,m} z_{l,m+1} + s_{l+1,m} x_{l+1,m} + t_{l,m+1} x_{l,m} z_{l+1,m} \tag{26}
\]

\[
z_{l+1,m+1} = z_{l,m+1} x_{l+1,m} (s_{l,m} x_{l,m} + t_{l,m} z_{l,m}). \tag{27}
\]

Using the same method described in section 4.1, one finds that \( \gcd_{2,1} = x_{1,0} \) and \( \gcd_{1,2} = s_{0,1} x_{0,1} + t_{0,1} z_{0,1} \). Therefore, we can build the recurrence

\[
G_{l+1,m+1} = \frac{G_{l+1,m} G_{l,m+1} x_{l,m} (s_{l,m} x_{l,m} + t_{l,m} z_{l,m})}{G_{l,m}}, \tag{28}
\]

for \( l, m \geq 2 \).

If \( l = 0 \) or \( m = 0 \) or \((l, m) = (1, 1)\), we take \( G_{l,m} = 1 \). If \( l = 1, m > 1 \) we take \( G_{l,m} = G_{l-1,m-1} s_{l-1,m-1} x_{l-1,m-1} + t_{l-1,m-1} z_{l-1,m-1} \). If \( l > 1, m = 1 \), we take \( G_{l,m} = G_{l-1,m} x_{l-1,m} \).

We have checked with random integers for \( r, s, t \) at each edge and random initial values as polynomials of degree 1 in \( w \) that \( G_{l,m} = \gcd_{l,m} \) (up to a constant factor) for \( l, m \leq 12 \). Assuming that this holds in general for all \( l, m > 1 \), the analysis of the previous subsection shows that equation (28) implies equation (24), or (i) \( H \) for the degrees. Moreover, it is easy to see that \( d_{l,m} = l+m+1 \) for \( l, m \geq 1 \). Hence, \( d_{l,m} \) grows linearly.

4.3. The \((3, -1)\)-reduction of \( H_1 \). Recall that equation \( H_1 \) or discrete potential Korteweg-de Vries (pKdV) is given by cf. [1]

\[
(u_{i,n} - u_{i+1,n+1} + u_{i+1,n} - u_{i,n+1}) = \alpha. \tag{29}
\]

Using the \((3, -1)\)-reduction of this equation, we obtained

\[
(u_{n} - u_{n+4} + u_{n+3} - u_{n+1}) = \alpha \tag{30}
\]

This gives us \( u_{n+4} = u_n + \alpha/(u_{n+3} - u_{n+1}) \).

This reduction is an exceptional case which was shown to be linearizable [14].

We can introduce projective coordinates \( u_n = x_n/z_n \), and obtain the following rules

\[
x_{n+4} = -\alpha z_{n+3} x_{n+3} z_{n+1} + x_n x_{n+1} z_{n+3} - x_n x_{n+3} z_{n+1}, \tag{31}
\]

\[
z_{n+4} = z_n (x_n x_{n+1} z_{n+3} - x_{n+3} z_{n+1}). \tag{32}
\]

We start with \((x_i, z_i)\) for \( 1 \leq i \leq 4 \) as initial values. By using these rules, we can calculate \((x_n, z_n)\) for \( n \geq 5 \) as functions of initial values. We can easily see that they are polynomials in \( x_1, x_2, x_3, x_4, z_1, z_2, z_3, z_4 \). We assume that \( \deg(x_i) = \deg(z_i) = 1 \) for \( 1 \leq i \leq 4 \). Denote \( d_n = \deg(x_n) = \deg(z_n) \).

We have \( d_{n+4} = d_n + d_{n+1} + d_{n+3} \). By direct calculation, we find that \( \gcd(x_8, z_8) = x_2 z_4 - x_4 z_2 \) and \( (x_2 z_4 - x_4 z_2)^2 (x_5 z_3 - z_5 x_3) \). Therefore, we denote \( \gcd_n = \gcd(x_n, z_n) \) and we write \( x_n = \gcd_n x_n \) and \( z_n = \gcd_n z_n \). Let \( d_n = \deg(x_n) = \deg(z_n) \) and \( g_n = \deg(\gcd_n) \). We know that \( \gcd_n \gcd_{n+1} \gcd_{n+3} | \gcd_{n+4} \).

Denote \( \gcd_{n+4} = \gcd_{n+4}/(\gcd_n \gcd_{n+1} \gcd_{n+3}) \) and \( A_n = x_{n-5} z_{n-7} - z_{n-5} x_{n-7} \). For \( n \geq 5 \), we have

\[
A_{n+4} = (A_{n+4} A_{n+5} | \gcd_{n+4}).
\]

We can see that

\[
\gcd(A_{n+4} A_{n+5}, \gcd_n \gcd_{n+1} \gcd_{n+3}) = g_n g_{n-3} g_n g_{n-2} A_{n+4} B_{n+4}.
\]
Therefore for \( n \geq 5 \), we have
\[
\frac{A_{n+4}^2 A_{n+5} \gcd_n \gcd_{n+1} \gcd_{n+3}}{\gcd(A_{n+4}^2 A_{n+5}, \gcd_n \gcd_{n+1} \gcd_{n+3})} = \frac{A_{n+4} A_{n+5} \gcd_{n+3} \gcd_{n+1} \gcd_{n+3}}{\gcd_{n-1} B_{n+4}}.
\]
It suggests that we should try the following recursive formula
\[
G_{n+4} = \frac{A_{n+4} A_{n+5} G_{n+3} G_n}{G_{n-1}}
= \frac{(x_{n-1} z_{n-3} - z_{n-1} x_{n-3})(x_n z_{n-2} - z_n x_{n-2}) G_{n+3} G_n}{G_{n-1}}
\]
for \( n > 4 \) and \( G_n = \gcd_n \) for \( n \leq 8 \).

We now take initial values as random polynomials of degree 1 in \( w \). We have checked for \( n \leq 40 \) that \( G_n = \gcd_n \) (up to a constant factor). Thus, we conjecture that \( \gcd_n = G_n \). Taking degrees of both sides of (33), we obtain
\[
g_{n+4} = d_{n-1} + d_{n-3} + d_n + d_{n-2} + g_n + g_{n+3} + g_n - g_{n-1}
= \bar{d}_{n-1} + d_{n+1} + d_{n+3} + d_n - \bar{d}_n - \bar{d}_{n+3}
= \tilde{d}_{n-1} + d_{n+4} - \tilde{d}_n - \tilde{d}_{n+3}.
\]
This implies that
\[
\tilde{d}_{n+4} = \tilde{d}_n + \tilde{d}_{n+3} - \tilde{d}_{n-1},
\]
which is equation (18) with \((q, p) = (3, -1)\), i.e. a reduction of (ii)H of Theorem 3.1. The characteristic equation for this linear equation is
\[
\lambda^5 - \lambda^4 - \lambda + 1 = (\lambda - 1)^2(\lambda + 1)(\lambda^2 + 1) = 0.
\]
This equation has the following roots: 1 (double root), \(-1, i, -i\). This means \( \tilde{d}_n \) grows linearly for \( n \geq 5 \). On the other hand, we know that \( \bar{d}_i = 1 \) for \( 1 \leq i \leq 4 \), and \( \bar{d}_i = 2i - 7 \) for \( 5 \leq i \leq 8 \). It is easy to prove that \( \bar{d}_n = 2n - 7 \) for \( n \geq 5 \). It again confirms that the sequence \( \bar{d}_n \) has linear growth. We remark that a direct proof of this also follows from the linearized version of (30) presented in [14].

5. Proof of linear growth of certain lattice equations via their corresponding linearizations. It is clear that a linear lattice equation has linear growth. However, it is not trivial to prove linear growth of some linearizable equations. In this section, we provide rigorous proofs of linear growth of some equations given in section 4 by proving their gcd recurrences. Our method is based on their associated linear equations and the existence of rational transformations that transform them to these linear equations. Non-rational linearizing transformations need not preserve linear growth properties – see Appendix for some examples.

5.1. Growth of a linear lattice equation. We consider the following equation
\[
a_{l,m} u_{l,m} + b_{l,m} u_{l+1,m} + c_{l,m} u_{l,m+1} + e_{l,m} = u_{l+1,m+1},
\]
where coefficients \( a, b, c, e \) might or might not depend on \( l, m \). In terms of projective coordinates, we have
\[
x_{l+1,m+1} = a_{l,m} x_{l,m} z_{l+1,m} z_{l,m+1} + b_{l,m} x_{l+1,m} z_{l,m} z_{l,m+1}
+ c_{l,m} x_{l+1,m} z_{l,m} z_{l,m+1} + e_{l,m} z_{l,m} z_{l,m+1},
\]
\[
z_{l+1,m+1} = z_{l,m} z_{l+1,m} z_{l,m+1}.
\]
Theorem 5.1. For \( l, m \geq 1 \), we have
\[
\gcd_{l+1, m+1} = \gcd_{l, m} \gcd_{l, m+1} \gcd_{l+1, m} \hat{z}_{l, m}^{2}
\]  
for \( l, m \geq 1 \) with \( \hat{z}_{l, m} \) given by (4).  

Proof. By iterating the rule, it is easy to see that \( u_{l, m} \) is a linear combinations of initial values given on the first quadrant, i.e.,
\[
u_{l, m} = \sum_{i=0}^{l} k_{l, m}^{i, 0} u_{i, 0} + \sum_{j=1}^{m} k_{0, j}^{l, m} u_{0, j} + K_{l, m}.
\]  
With general parameters, it implies that the coefficients \( k_{l, m}^{i, m} \) do not vanish. Therefore, given linearity and \( u_{l, m} = x_{l, m}/z_{l, m} = \tilde{x}_{l, m}/\tilde{z}_{l, m} \), taking the least common denominator of (40) gives
\[
\tilde{z}_{l, m} = z_{l, 0} \tilde{z}_{l, 1} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 1} \tilde{z}_{l, 1} \ldots z_{l, 1} \tilde{z}_{l, 0} \ldots z_{l, 0}.
\]  
On the other hand using (38), we have
\[
z_{l, m} = r_{l, m}^{0, 0} r_{l, m}^{0, 1} \ldots r_{l, m}^{0, 1} r_{l, m}^{0, 0} \ldots r_{l, m}^{0, 0} r_{l, m}^{0, 0},
\]  
where \( r_{l, m}^{i, m} \) satisfies the recurrence for \( m, l \geq 1 \):
\[
r_{l, m}^{i, m} = r_{l, m}^{i-1, m-1} + r_{l, m}^{i-1, m} + r_{l, m}^{i-1, m}.
\]  
On the boundary, for \( (i, j) \neq (0, 0) \) we have \( r_{0, 0} = 1 \), \( r_{j, 0} = \delta_{ij} \), \( r_{0, j} = 0 \), \( r_{0, i} = \delta_{ij} \) and \( r_{0, 0} = 0 \). Therefore, for \( l, m > 1 \) we obtain
\[
\tilde{z}_{l, 1} = z_{l, 0} \tilde{z}_{l, 1} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 1} \ldots z_{l, 1} \tilde{z}_{l, 1} \ldots z_{l, 1} \tilde{z}_{l, 0} \ldots z_{l, 0}.
\]  
This means for \( l, m \geq 2 \), we have
\[
\gcd_{l, 1} = z_{l, 0} z_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} ;
\]  
\[
\gcd_{l, m} = z_{l, 0} z_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} ;
\]  
\[
\gcd_{l, m} = z_{l, 0} z_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} \tilde{z}_{l, 0} \ldots z_{l, 0} ;
\]  
Using these formulas together with (41) and (43) we obtain (39).  

Remark 1. The total degree of \( \tilde{z}_{l, m} \) or \( \tilde{x}_{l, m} \) is \( l + m + 1 \) in terms of the initial values \( \{ x_{i, 0}, z_{i, 0}, x_{0, j}, z_{0, j} \} \).

We now use this result to obtain growth of degrees of some original equations via point transformations and the Cole-Hopf transformation.  

5.2. Growth of degrees via a Möbius transformation. Suppose we can use an invertible Möbius transformation \( v_{l, m} = \frac{p_{l, m} u_{l+1, m} + q_{l, m}}{p_{l, m} u_{l+1, m} + q_{l, m}} \) to transform a non-linear lattice equation \( Q_{l, m}(u_{l, m}, u_{l+1, m}, u_{l+1, m+1}, u_{l+1, m+1}) = 0 \) to a linear lattice equation:
\[
L_{l, m}(v_{l, m}, v_{l+1, m}, v_{l, m+1}, v_{l+1, m+1}) = 0.
\]
We use the corner boundary conditions for the original equation $Q$. This yields the corner boundary condition for the projective version of the equation $L$ with $v_{l,m} := X_{l,m}/Z_{l,m}$, that is

$$X_{l,0} = p_{l,0} x_{l,0} + q_{l,0} z_{l,0}, \quad Z_{l,0} = p'_{l,0} x_{l,0} + q'_{l,0} z_{l,0},$$

$$X_{0,m} = p_{0,m} x_{0,m} + q_{0,m} z_{0,m}, \quad Z_{0,m} = p'_{0,m} x_{0,m} + q'_{0,m} z_{0,m}.$$  

It is easy to see that $\deg(x_{l,m}) = \deg(z_{l,m}) = \deg(X_{l,m}) = \deg(Z_{l,m})$. Using results in the previous section, we have $\deg(\bar{X}_{l,m}) = \deg(\bar{Z}_{l,m}) = l + m + 1$. We can write

$$u_{l,m} = \frac{x_{l,m}}{z_{l,m}} = \frac{q_{l,m} v_{l,m} - q'_{l,m} \bar{X}_{l,m}}{p_{l,m} v_{l,m} - p'_{l,m} \bar{X}_{l,m}}.$$  

This shows that $\deg(\bar{x}_{l,m}) \leq l + m + 1$ and $\deg(\bar{z}_{l,m}) \leq l + m + 1$. On the other hand, we have

$$v_{l,m} = \frac{\bar{X}_{l,m}}{\bar{Z}_{l,m}} = \frac{p_{l,m} \bar{x}_{l,m} + q_{l,m} \bar{z}_{l,m}}{p'_{l,m} \bar{x}_{l,m} + q'_{l,m} \bar{z}_{l,m}}.$$  

This implies that $\deg(\bar{X}_{l,m}) \leq l + m + 1$ and $\deg(\bar{Z}_{l,m}) \leq l + m + 1$ as $\deg(\bar{x}_{l,m}) \leq l + m + 1$ and $\deg(\bar{z}_{l,m}) \leq l + m + 1$. Therefore, we can conclude that $\deg(\bar{x}_{l,m}) = \deg(\bar{z}_{l,m}) = l + m + 1 = \deg(\bar{X}_{l,m}) = \deg(\bar{Z}_{l,m})$.

### 5.3. RJGT equation: Linear growth via a Cole-Hopf transformation

In this section, we study growth of RJGT equation (25) which is known to be linearized by a discrete version of Cole-Hopf transformation $v_{l,m} = v_{l,m+1}/u_{l,m}$ [17, 22]. By substituting this transformation into (25), we obtain

$$\frac{v_{l+1,m+2}}{s_{l+1,m+2} v_{l,m+1} + t_{l,m+1} v_{l+1,m+1}} = \frac{v_{l+1,m+1} + r_{l,m} v_{l+1,m}}{s_{l,m} v_{l+1,m} + t_{l,m} v_{l,m}}. \quad (44)$$

This gives us the following linear equation for $v$

$$v_{l+1,m+1} + r_{l,m} v_{l,m+1} = C_l (s_{l,m} v_{l,m+1} + t_{l,m} v_{l,m}), \quad (45)$$

where $C_l = \frac{v_{l+1,m} + r_{l,m} v_{l,m+1}}{s_{l,m} v_{l,m+1} + t_{l,m} v_{l,m}}$. We note that this is a complicated version of (36).

We take general corner boundary initial values for the original RJGT equation (25) , i.e. taking $\{x_{i,0}, z_{i,0}, x_{0,j}, z_{0,j}\}_{i,j \geq 0}$ as initial values. This helps to build initial values for equation (45) as follows. For $l \geq 0, m \geq 1$, we take

$$v_{l,m} = \frac{X_{l,m}}{Z_{l,m}},$$

$$X_{l,0} = z_{l,0}, \quad Z_{l,0} = 1,$$

$$X_{l,1} = x_{l,0}, \quad Z_{l,1} = 1,$$

$$X_{0,m} = x_{0,m} x_{0,1} \ldots x_{0,m-1}, \quad Z_{0,m} = z_{0,1} z_{0,2} \ldots z_{0,m-1}.$$  

We notice that some of the initial values of (45) lie on the line $y = 1$ corresponding to $m = 1$. This is because equation (45) becomes a trivial equation when $m = 1$. This boundary condition also defines the parameter $C_l = \frac{x_{l+1,0} + r_{l,0} z_{l+1,0}}{s_{l,0} x_{l,0} + t_{l,0} z_{l,0}}$. We want to prove (28) using the linear equation (45).
For \( l \geq 1, m \geq 2 \), iterating equation (45) we obtain a shifted version of equation (40) in the \( m \) direction by 1. i.e.

\[
v_{l,m} = \sum_{i=0}^{l} k_{l,m}^i v_{i,1} + \sum_{j=2}^{m} k_{0,j} v_{0,j}.
\]  

Moreover, we note that the degree of \( C_i \) in each \( k_{l,m}^i \) can not exceed 1, i.e. \( k_{l,m}^i \) is affine linear in \( C_i \). In particular, the total degree of variables \( \bar{C}_0, \ldots, \bar{C}_{l-1} \) in \( k_{l,m}^i \) is \( l \) and in \( k_{l,1}^i \) is \( l-i \). This can be done by induction on \( N = l + m \).

Since \( v_{i,1} = x_{i,0} \), using formulas for \( C_i \) and \( v_{0,j} = X_{0,j}/Z_{0,j} \) given above, we have

\[
\bar{Z}_{l,m} = z_{0,1} \ldots z_{0,m-1}(s_{0,0} x_{0,0} + t_{0,0} z_{0,0}) \ldots (s_{l-1,0} x_{l-1,0} + t_{l-1,0} z_{l-1,0}),
\]

which is also a common divisor for all the terms in (46) and \( \bar{Z}_{l,m} \) is the denominator of \( v_{l,m} \) after cancelling some common factors of \( X_{l,m} \) and \( Z_{l,m} \). This implies that \( \deg(\bar{X}_{l,m}) = l + m \) (by adding the degree of \( v_{i,1} \) to \( \deg Z_{l,m} \)). Thus, we have

\[
u_{l,m} = \frac{v_{l,m+1}}{v_{l,m}} = \frac{\bar{X}_{l,m+1} \bar{Z}_{l,m}}{\bar{Z}_{l,m+1} \bar{X}_{l,m}} = \frac{\bar{X}_{l,m+1}}{\bar{X}_{l,m} \bar{Z}_{l,m}}.
\]

It implies that \( \deg \bar{x}_{l,m} = \deg \bar{z}_{l,m} \leq l + m \). If we take \( \bar{x}_{l,m} = \bar{x}_{l,m+1} \) then

\[
\bar{Z}_{l,m} = \bar{X}_{l,m} z_{0,0} = \bar{x}_{l,m-1} \bar{z}_{0,0}.
\]

We also know that for the RJGT equation (25), we have from (27):

\[
z_{i+1,m+1} = (s_{i,m} x_{l,m} + t_{i,m} z_{l,m}) x_{i+1,m} z_{i+1,m+1}.
\]

Using this recursive formula, we obtain

\[
z_{i+1,m+1} = x_{i,0} x_{2,0} \ldots x_{i,0} (s_{0,0} z_{0,0} + t_{0,0} z_{0,0}) \ldots (s_{i,m} x_{l,m} + t_{i,m} z_{l,m})
\]

This implies that

\[
gcd_{l+1,m+1} = x_{1,0} x_{2,0} \ldots x_{1,0} (s_{0,0} z_{0,0} + t_{0,0} z_{0,0}) \ldots (s_{i,m} x_{l,m} + t_{i,m} z_{l,m})
\]

Using this formula, we have

\[
gcd_{l,m+1} = x_{1,0} x_{2,0} \ldots x_{1,0} (s_{0,0} z_{0,0} + t_{0,0} z_{0,0}) \ldots (s_{i,m} x_{l,m} + t_{i,m} z_{l,m})
\]

Substitute this into (47), we obtain

\[
gcd_{l+1,m+1} = \frac{gcd_{l+1,m} gcd_{l+1,m} x_{l,m} (s_{l,m} x_{l,m} + t_{l,m} z_{l,m})}{gcd_{l,m}},
\]

which is the recursive formula (28) postulated previously in section 4.2 for \( gcds \) of RJGT equation (25).

5.4. Discrete Liouville equation: Linear growth via a three-point transformation. In this section we prove linear growth of the discrete Liouville equation (20). This equation is linearized by the three-point transformation given by cf. [2]

\[
u_{l,m} = \frac{-(v_{l+1,m} - v_{l,m})(v_{l,m+1} - v_{l,m})}{v_{l+1,m} v_{l,m+1}},
\]

where \( v_{l,m} \) satisfies the linear equation

\[
v_{l,m} - v_{l+1,m} - v_{l,m+1} + v_{l+1,m+1} = 0.
\]

This equation implies that \( v_{l,m} = v_{l,0} + v_{0,m} - v_{0,0} \).
Initial values for Liouville equation are given on the first quadrant which are \( \{x_{l,0}, z_{l,0}, x_{0,m}, z_{0,m}\}_{1,m \geq 0} \). We want to express \( v_{0,j} \) and \( v_{1,0} \) in terms of these initial values and \( v_{0,0}, v_{1,0}, v_{0,1} \). We denote \( K = v_{0,1} - v_{0,0} \) and \( K_1 = v_{1,0} - v_{0,0} \).

On the horizontal axis we have

\[
u_{l,0} = \frac{x_{l,0}}{z_{l,0}} = -\frac{(v_{l+1,0} - v_{l,0})(v_{l,1} - v_{l,0})}{v_{l+1,0}v_{l,1}} = -\frac{(v_{l+1,0} - v_{l,0})(v_{0,1} - v_{0,0})}{v_{l+1,0}(v_{0,0} + v_{0,1} - v_{0,0})} = -\frac{K(v_{l+1,0} - v_{l,0})}{(K + v_{0,0})v_{l+1,0}}.
\]

This yields

\[
\frac{1}{v_{l+1,0}} + \frac{1}{K} = \frac{x_{l,0} + z_{l,0}}{z_{l,0}} \left( \frac{1}{v_{l,0}} + \frac{1}{K} \right).
\]

Therefore, we obtain on iterating

\[
\frac{1}{v_{l+1,0}} + \frac{1}{K} = \frac{(x_{0,0} + z_{0,0})(x_{1,0} + z_{1,0}) \ldots (x_{l,0} + z_{l,0})}{z_{0,0} \ldots z_{l,0}} \left( \frac{1}{v_{0,0}} + \frac{1}{K} \right). \tag{52}
\]

Similarly, for the vertical axis we have

\[
\frac{1}{v_{0,m+1}} + \frac{1}{K_1} = \frac{(x_{0,0} + z_{0,0})(x_{0,1} + z_{0,1}) \ldots (x_{0,m} + z_{0,m})}{z_{0,0} \ldots z_{0,m}} \left( \frac{1}{v_{0,0}} + \frac{1}{K_1} \right). \tag{53}
\]

By choosing \( K = v_{0,0} \) and \( v_{0,0} = 1 \) and using the formulas for \( u_{0,0} \), we obtain

\[K_1 = -\frac{2x_{0,0}}{2(x_{0,0} + z_{0,0}) \ldots (x_{l-1,0} + z_{l-1,0}) - z_{0,0} z_{1,0} \ldots z_{l-1,0}} = \frac{X_{0,m}}{Z_{0,m}}\]

Therefore the boundary conditions for linear equation (51) are given by

\[
v_{l,0} = \frac{z_{0,0} z_{1,0} \ldots z_{l-1,0}}{2(x_{0,0} + z_{0,0}) \ldots (x_{l-1,0} + z_{l-1,0}) - z_{0,0} z_{1,0} \ldots z_{l-1,0}} = \frac{X_{l,0}}{Z_{l,0}}.
\]

\[
v_{0,m} = \frac{z_{0,0} z_{1,0} \ldots z_{0,m-1}}{(x_{0,0} + z_{0,0}) \ldots (x_{0,m-1} + z_{0,m-1}) - (2x_{0,0} + z_{0,0}) z_{0,1} \ldots z_{0,m-1}} = \frac{X_{0,m}}{Z_{0,m}}.
\]

For \( l, m \geq 1 \), we have from (50)

\[
u_{l,m} = \frac{x_{l,m}}{z_{l,m}} = -\frac{(v_{l+1,0} - v_{l,0})(v_{l,m+1} - v_{l,m})}{v_{l+1,0}v_{l,m+1}} = \frac{-(v_{l+1,0} - v_{l,0})(v_{0,m+1} - v_{0,m})}{(v_{l+1,0} + v_{0,m} - v_{0,0})(v_{l,0} + v_{0,m+1} - v_{0,0})}.
\]

It is easy to see that

\[
v_{l+1,0} - v_{l,0} = -\frac{2z_{0,0} \ldots z_{l-1,0}(x_{0,0} + z_{0,0}) \ldots (x_{l-1,0} + z_{l-1,0}) x_{l,0}}{Z_{l,0}Z_{l+1,0}}, \tag{54}
\]

\[
v_{0,m+1} - v_{0,m} = \frac{2x_{0,0} z_{0,1} \ldots z_{0,m-1}(x_{0,0} + z_{0,0}) \ldots (x_{0,m-1} + z_{0,m-1}) x_{0,m}}{Z_{0,m}Z_{0,m+1}}. \tag{55}
\]

We also have

\[
v_{l+1,0} + v_{0,m} - v_{0,0} = \frac{A}{Z_{l+1,0}Z_{0,m}},
\]

\[
v_{l,0} + v_{0,m+1} - v_{0,0} = \frac{B}{Z_{l,0}Z_{0,m+1}}.
\]
where
\[ A = X_{l+1,0}Z_{0,m} + X_{0,m}Z_{l+1,0} - Z_{l+1,0}Z_{0,m}, \quad B = X_{l,0}Z_{0,m+1} + X_{0,m+1}Z_{l,0} - Z_{0,m+1}Z_{l,0}. \]

Substituting \( x_{0,0} = -z_{0,0} \) into \( A \) and \( B \) we get \( A = B = 0 \). This implies that \( A \) and \( B \) both have a factor \((x_{0,0} + z_{0,0})\). Thus, we can write \( A = A_1(x_{0,0} + z_{0,0}) \) and \( B = B_1(x_{0,0} + z_{0,0}) \), where \( A_1 \) and \( B_1 \) are polynomials of initial values \( \{x_{l,0}, z_{l,0}, x_{0,m}, z_{0,m}\}, l, m \geq 0 \). Since \( \deg X_{l,0} = \deg Z_{l,0} = l \), \( \deg X_{0,m} = \deg Z_{0,m} = m \), \( \deg A_1 \leq l + m \) and \( \deg B_1 \leq l + m \). By cancelling \((x_{0,0} + z_{0,0})\) in both (54), (55) we have
\[ u_{l,m} = \frac{\bar{x}_{l,m}}{\bar{z}_{l,m}}, \quad (56) \]
where \( \bar{z}_{l,m} = A_1B_1 \) and
\[ \bar{x}_{l,m} = x_{0,0}z_{0,0} \cdots z_{l-1,0}z_{0,1} \cdots z_{0,m-1}x_{l,0}z_{0,m}(x_{1,0} + z_{1,0}) \cdots (x_{l-1,0} + z_{l-1,0}) \]
\[ (x_{0,1} + z_{0,1}) \cdots (x_{0,m-1} + z_{0,m-1}). \quad (57) \]

It is easy to check that \( \bar{x}_{l,m} \) and \( \bar{z}_{l,m} \) do not have any common factors as \( \bar{z}_{l,m} \) does not vanish at any zero of \( \bar{x}_{l,m} \). This implies that \( \deg u_{l,m} = 2(l + m) \).

Moreover, using the quad-rule for the Liouville equation, we get \( x_{l+1,m+1} = x_{l+1,m}x_{l,m+1}z_{l,m} \). Since \( \gcd_{l,m} = x_{l,m}/\bar{x}_{l,m} \), we obtain
\[ \gcd_{l+1,m+1} = \gcd_{l+1,m} \gcd_{l,m+1} \bar{x}_{l+1,m+1}z_{l,m+1}. \quad (58) \]

Substituting formulas for bar variables using (57), for \( l, m > 1 \) we have
\[ \gcd_{l+1,m+1} = \gcd_{l+1,m} \gcd_{l,m+1} x_{l,m+1}z_{l,m}x_{0,0}z_{0,0} \cdots z_{l-1,0}z_{0,1} \cdots z_{0,m-1}x_{l,0}z_{0,m} \]
\[ (x_{1,0} + z_{1,0}) \cdots (x_{l-1,0} + z_{l-1,0})(x_{0,1} + z_{0,1}) \cdots (x_{0,m-1} + z_{0,m-1}) \]
\[ = \gcd_{l+1,m} \gcd_{l,m+1} z_{l,m} \bar{x}_{l,m} \]
\[ = \gcd_{l+1,m} \gcd_{l,m+1} \gcd_{l,m} \bar{z}_{l,m} \bar{x}_{l,m}, \]
i.e. we have just proved the recursive formula (23), postulated previously in section 4.1.

6. Searching for lattice equations with linear growth. The previous sections have given confidence that the homogeneous equations of Theorem 3.1 characterise linear growth and linearisability. In this last section, we use the homogeneous equations given in Theorem 3.1 to search for examples of lattice equations with linear growth. Our search recreates an autonomous version of a known equation given in [22].

We start with a general form of a multi-affine equation on quad-graphs
\[ Q : a_{15}u_{1}u_{2}u_{12} + a_{11}u_{1}u_{1}u_{2} + a_{12}u_{1}u_{1}u_{12} + a_{13}u_{1}u_{2}u_{12} + a_{14}u_{1}u_{2}u_{12} + a_{5}u_{1}u_{1} + a_{6}u_{2} + a_{8}u_{12} + a_{9}u_{12} + a_{10}u_{2}u_{12} + a_{1}u + a_{2}u_{2} + a_{3}u_{2} + a_{4}u_{12} + a_{0} = 0, \quad (59) \]
where we use the shorthand notation for \( u = u_{l,m} \), \( u_2 = u_{l,m+1} \) etc. and similarly for the variable \( v \) (see below). As the simplest case, we can search for autonomous equations that satisfy the relation \((i)H \) and work on one quad square as per Figure 3 (left). Going backwards from this equation, we obtain
\[ g_{l+1,m+1} = g_{l+1,m} + g_{l,m+1} + 2d_{l,m} - g_{l,m}. \]
This suggests that

\[ \gcd_{l+1,m+1} = \frac{\gcd_{l+1,m} \gcd_{l,m+1} A_{l+1,m+1}}{\gcd_{l,m}}, \tag{60} \]

where \( \deg(A_{l+1,m+1}) = 2d_{l,m} \) (this relation over one lattice square should be compared to the \( \gcd \) relation over four lattice squares – see Conjecture 7 of [23]). The obvious choice is

\[ A_{l+1,m+1} = f^{(1)}x_{l,m}^2 + f^{(2)}x_{l,m}z_{l,m} + f^{(3)}z_{l,m}^2 = (t_1x_{l,m} + t_2z_{l,m})(s_1x_{l,m} + s_2z_{l,m}). \]

We also assume that at \((2,1)\) and \((1,2)\) we have a common factor \(t_1x_{1,0} + t_2z_{1,0}\) and \(s_1x_{0,1} + s_2z_{0,1}\), respectively. The search algorithm can be broken down to the following steps.

1. Write the rule in projective coordinates and calculate \(x\) and \(z\) at vertices \((2,1)\) and \((1,2)\) as polynomials in initial values.
2. At the point \((2,1)\) and \((1,2)\), substitute \(x_{1,0} = -t_2z_{1,0}/t_1\) and \(x_{0,1} = -s_2z_{0,1}/s_1\) into \(x\) and \(z\) if \(t_1,s_1 \neq 0\). If \(t_1 = 0\) we substitute \(z_{1,0} = 0\) and if \(s_1 = 0\) we substitute \(z_{0,1} = 0\). We then collect all the coefficients of initial values.
3. Set these coefficients to 0 as \(x_{1,2} = z_{1,2} = x_{2,1} = z_{2,1} = 0\) for the above substitutions. We then solve for \(a_0, a_1, \ldots, a_{15}\).
4. Substitute solutions back to \(Q\) and choose only irreducible equations, i.e. equations that cannot be factored.
5. For the ‘surviving equations’, check the recurrence relation (60).

For example, we obtain the following equation which grows linearly

\[ Q_7: \ u_1u_2u_3u_4u_5u_6 + u_1u_5u_6u_7u_8u_9u_10 + (a_1^2 s_2 + a_6 a_{12}) u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 u_{10} + (a_6 a_{11} s_2 + a_6 a_{12} t_2 u_{12}) u + (a_1^2 t_2 u_1 u_2 + a_{11} a_{12} s_2 u_1 u_{12}) + (a_1^2 s_2 t_2 + a_6 a_{12} s_2) u_1 + a_{11} t_2 u_2 + a_{11} a_{12} s_2 t_2 u_{12} + a_6 a_{11} s_2 t_2 + a_6 a_{12} s_2 + a_6 a_{12} t_2 t_2 = 0. \]

This equation can be written as

\[ a_{11} (u + t_2)(u_2 + s_2)(a_{11} u_1 + a_6) = -a_{12} (u_1 + t_2)(u + s_2)(a_{11} u_1 + a_6), \tag{61} \]

or equivalently

\[ \frac{(a_{11} u_1 + a_6)}{(a_{11} u_1 + a_6)} \frac{(u + s_2)}{(u_2 + s_2)} = -\frac{a_{11} (u + t_2)}{a_{12} (u_1 + t_2)}. \tag{62} \]

By using the Cole-Hopf transformation \(u = v_2/v - t_2\), equation (62) can be rewritten as the following equation

\[ \frac{S_2(f(v, v_1, v_2))}{f(v, v_1, v_2)} = -\frac{a_{11}}{a_{12}}, \]

where \(S_2\) denotes the shift in the vertical direction and

\[ f(v, v_1, v_2) = \frac{(a_{11} u_1 + a_6) v_1}{(u + s_2)v} = \frac{a_{11} v_1 - t_2 v_1 + a_6 v_1}{v_2 - t_2 v + s_2 v}. \]

It implies that

\[ f(v, v_1, v_2) = c_1 \left( \frac{-a_{11}}{a_{12}} \right)^m. \]

This equation can be written in a linear form as follows

\[ a_{11} v_1 - t_2 v_1 + a_6 v_1 = c_1 \left( \frac{-a_{11}}{a_{12}} \right)^m (v_2 - t_2 v + s_2 v). \]
We note that this linearization process also extends to a non-autonomous version of (61) if $s_2(l,m), a_0(l,m)/a_{11}(l,m)$ depend on $m$ only and $a_{11}/a_{12} = -k$ and $t_2$ are constant. This non-autonomous equation has the form:

$$\frac{(a_{11}(l,m+1)u_{12} + a_0(l,m+1))}{(a_{11}(l,m)u_1 + a_0(l,m))} \frac{(u + s_2(l,m))}{(u_2 + s_2(l,m+1))} = \frac{k(u + t_2)}{(u_1 + t_2)}. \quad (63)$$

If $k = 1$, this equation is actually the RJGT equation (25) we considered above, which is known to be linearizable [22, 17].

If we replicate the above search algorithm for the relation (ii)H, we get

$$g_{l+1,m+1} = g_{l,m} + g_{l,m+1} + g_{l,m+1} + (l_{l-1,m} + d_{l+1,m})$$

$$= g_{l,m} + g_{l,m+1} + d_{l+1,m} + d_{l-1,m} - g_{l-1,m}.$$

We can use the similar argument for relation (iii)H. Thus, we can take respectively for (ii)H and (iii)H:

$$\gcd_{l+1,m+1} = \gcd_{l,m+1} \gcd_{l,m},$$

$$\gcd_{l+1,m+1} = \gcd_{l+1,m} \gcd_{l,m+1} \gcd_{l,m},$$

where

$$\deg(A_{l+1,m+1}) = d_{l+1,m} + d_{l-1,m} = (d_{l-1,m} + d_{l,m-1}) + (d_{l,m} + d_{l+1,m-1}), \quad (64)$$

$$\deg(A_{l+1,m+1}) = d_{l,m+1} + d_{l,m-1} = (d_{l,m-1} + d_{l-1,m}) + (d_{l,m} + d_{l-1,m+1}). \quad (65)$$

For the former case (64), one can try $A_{l+1,m+1} = B_{l,m+1} B_{l+1,m+1}$, where

$$\deg(B_{l+1,m+1}) = (d_{l,m} + d_{l+1,m-1}).$$

One choice is

$$B_{l+1,m+1} = x_{l+1,m} (t_3 x_{l,m} + t_2 z_{l,m}) = z_{l+1,m} - (s_1 x_{l,m} + s_2 z_{l,m}). \quad (66)$$

We assume that this common factor $B_{2,1}$ first appears at vertex (2,1). In addition, we try to mimic the behaviour that we got with the (3, −1) reduction of $H_1$; that is we assume that $B_{2,1}^c A_{2,1}$. We follow the steps that were given above to search for equations, however in this case we do not obtain any non-degenerate equations.

For the case (65), one can try $A_{l+1,m+1} = B_{l+1,m} B_{l+1,m+1}$, where

$$\deg(B_{l+1,m+1}) = (d_{l,m} + d_{l-1,m+1}).$$

We can do similarly by swapping $l$ with $m$ and we do not obtain any non-degenerate equations either.

7. Conclusion. In this paper, based on the recurrence relation (10) found previously for the degrees of many integrable lattice equations [23], we derived some linear recurrences (i)H, (ii)H and (ii)H of Theorem 3.1 that are special cases of (10) and imply linear degree growth. We then used these recurrences to build recursive formulas for the gcds. Thus, we were able to search for examples of linearizable equations with certain gcd patterns, for example the RJGT equation and discrete Burgers equation. A symmetry method given in [17] was used to linearize the RJGT equation. Moreover, we have proved linear degree growth of certain equations such as the Liouville equation, RJGT equation and lattice equations linearizable via Möbius transformations (for example see [17]) by using their associated
linear equations. We have also noted in the Appendix that there are linearizable equations with exponential growth [9, 19]. This is because the transformations to bring these equations to linear equations are not rational.

We have also noticed that some other linearizable equations such as Equation 15 in [12] where \( p_3 = 0 \) and another form of Liouville equation given in [16] satisfy the recurrence (60) and hence satisfy \((i)H\). Furthermore, we have not found a lattice equation whose gcd relations give us the homogeneous equations \((ii)H\) or \((iii)H\) of Theorem 3.1 directly. Moreover, it is known that there are also other linearizable equations (for example see [6, 10]) that we have not studied in this paper. It would be interesting to study their gcd recurrences in the future.

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Appendix: Examples of linearizable equations via non-rational transformations. We comment on two more examples of linearizable equations that have occurred in the literature.

Example 1. In [9] the authors gave the following ordinary difference equation which is chaotic and linearizable (under the transformation \( x_n = \tan(y_n) \))

\[
x_{n+1} = \frac{3x_n - x_n^3 - x_{n-1}(1 - 3x_{n-1}^2)}{1 - 3x_n^2 + (2x_n - x_{n-1})x_{n-1}},
\]

(67)

In fact, (67) can be seen as the \((1, -1)\) reduction \( u_1 = u_2 \) of two associated nonlinear lattice equations transformable via \( u = \tan(v) \) to the linear equations

\[
v + av_1 + bv_2 + cv_{12} = p\pi,
\]

(68)

where \( p \in \mathbb{Z} \) and \( a = -1, b = -2, c = 1 \) or \( a = -2, b = -1, c = 1 \). The growth of the lattice equation in \( u \) variables will not be linear because of the non-rational change of coordinates. In [19], the authors give two other examples of lattice equations in \( u \) variables with exponential degree growth, that can also be transformed to particular cases of (68) with \( a = -3, b = 3, c = 1 \) and \( a = -2, b = c = 1 \).

Example 2. Consider the following two QRT-type lattice equations cf. [16, 19]

\[
\begin{align*}
u_{12} &= \frac{u_1 + u_2 - (1 - u_1u_2)u}{1 - u_1u_2 + (u_1 + u_2)u}, \\
u_{12} &= \frac{u_1 - u_2 + (1 + u_1u_2)u}{1 + u_1u_2 - (u_1 - u_2)u}.
\end{align*}
\]

(69) (70)

We find that the gcd in the projective versions of these equations also behaves as in (28) by replacing \( x_{l,m}(s_{l,m}x_{l,m} + t_{l,m}z_{l,m}) \) with \( x_{l,m}^2 + z_{l,m}^2 \). So equation (24) or \((i)H\) also holds here for the degree growth of either equation but this time we obtain the constant degree solution \( \bar{d}_{l,m} = 3 \) for \( l, m \geq 1 \).

By using the transformation \( u = \tan(v) \) (similar to the previous example) these equations were shown to be linearizable [19]. They can be brought to the linear lattice equation (68) with, respectively, \( a = 1, b = c = -1 \) and \( a = b = -1, c = 1 \). This prompts the question whether there might be rational transformations that bring equations (69) and (70) to linear lattice equations. We have found that these
equations are similar to equation $L_4$ given in [8]. Therefore, the transformations for (69) and (70) are given by

$$v_{l,m} = 1 + \frac{u_{l,m} u_{l+1,m}}{u_{l,m} - u_{l+1,m}}, \quad \text{and} \quad v_{l,m} = 1 - \frac{u_{l,m} u_{l+1,m}}{u_{l,m} + u_{l+1,m}},$$

respectively. The corresponding linear equations are then $v_{l,m} = v_{l,m+1}$ for both equations (69) and (70).

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