A generic approach to thermal machines with quantum gases

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It has been shown in literature that a single quantum particle can be realized as a thermodynamic machine as an artefact of energy quantization and hence bears no classical analogue. Yet its behaviour can be substantially different if we consider a collection of non-interacting massive indistinguishable quantum particles. Such a thermodynamic machine depends on the statistics of the particles, the chemical potential and the dimensionality of the system. Our detailed analysis demonstrates the fundamental features of quantum Stirling cycles from the viewpoint of particle-statistics and system dimensions that helps us to realize the desired quantum heat engines and refrigerators by exploiting the role of quantum statistical mechanics. In particular, we have shown that distinctive behaviour of particle statistics is quite pertinent in one dimension than in higher dimensions, indicating the conspicuous role of quantum thermodynamic signature in lower dimensions.

I. INTRODUCTION

The study of quantum thermodynamics comprises the basis for the analysis of heat engines and refrigerators at the microscopic level [1–4]. Although thermodynamics is quite successful in the classical regime, its applications towards quantum system must be reconciled in view of energy quantization, degeneracies and most importantly particle statistics [5–15]. A fundamental departure from classical mechanics is therefore imperative for analyzing the thermodynamic behaviour of quantum systems. As a result, one might expect new thermodynamic effects would emerge in the quantum domain without having any classical correspondence. Most of the cases that have been analyzed for understanding generic thermodynamic features in quantum thermal machines, do not exhibit any proper evidence of quantum advantage over their classical counterparts [16–20]. In recent years, several proposals for constructing quantum mechanical versions of thermal machines such as Otto, Carnot, Stirling and Diesel [21–32] with different working media like particle in a box, harmonic oscillator, spin systems [30, 33–42] have been presented. Among them only a few analyzes the quantum enhanced machines specifically from the viewpoint of a single particle within a potential well [5, 6, 43, 44].

The role of quantum statistics comes into play if the system comprises an ensemble of identical particles in contact with a pair of low temperature baths. In this limit, the dynamics of the system is governed by the principle how a single particle state is occupied. This distinguishes the behaviour of two different types of particles, viz. fermions and bosons. The chemical potential that depends on the dimensionality of the system in addition to its statistics, can play a huge role in determining its behaviour. Therefore, collective behaviour of quantum particles is interlinked with the dimensionality and hence the degeneracy of the energy levels.

In this article, we elaborate, rigorously and methodically study the dependence of particle statistics and dimensionality of the system in the context of a quantum Stirling cycle based on an infinite potential well. In the present scenario, when the potential is distorted by the introduction of an infinite barrier in the middle of the box, the odd levels shift upwards and overlap with the immediately next even energy levels, creating a new energy level structure with degeneracies [45, 46]. In quantum Stirling cycles, this distortion of the potential may constitute a certain amount of work and heat exchange between the baths without having any classical analog [47, 48]. Depending on the behaviour of the cycle, whether work is extracted or not, one can construct a desired Stirling heat engine or refrigerator just by exploiting the quantum nature of the identical particles. Since quantum dots, wires and wells are considered to be promising candidates for realizing quantum heat machines in recent years [17, 29, 34, 49, 50], our approach is interesting both from pedagogical and practical viewpoints.

This paper is organized as follows: In Sec. II, we present the basic formulation with a brief overview of the quantum Stirling cycle, followed by a detailed characteristics of the working medium, and the fundamental principles behind quantum statistics of identical particles. In Sec. III, we summarizes the main results and their interpretations together with the analytical approach in certain limiting cases. Finally, we conclude in Sec. IV.

II. FORMULATION

A. Quantum Stirling cycle

A quantum Stirling cycle [6] consists of two configurations, governed by two different Hamiltonians $H$ and $H'$ respectively and each configuration is kept in equilibrium with two thermal baths. Hence, the cycle has four stages, each connected to its preceding and succeeding stages through two isothermal and two isochoric processes as shown schematically in Fig 1(a). Initially at stage $A$, the system is in equilibrium with a hot bath at a temperature $T_h$. In the first step ($A \rightarrow B$), the potential is distorted quasi-statically and isothermally to change the configurations of the energy levels and the wavefunctions, while the system is still coupled to the hot bath [Fig 1(b)]. In
In what follows in Sec. II C, we explicitly evaluate the thermodynamics of isothermal processes are given by

\[ W = (Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA}) \]

\[ = -k_B \left( T_h \ln \frac{Z_B}{Z_A} - T_c \ln \frac{Z_C}{Z_D} \right), \]

\[ Q_h = -(Q_{AB} + Q_{DA}) = k_B T_h \ln \frac{Z_B}{Z_A} + U_B - U_D, \]

\[ Q_c = -(Q_{BC} + Q_{CD}) = -k_B T_c \ln \frac{Z_C}{Z_D} - U_B + U_D. \]

By the principle of conservation of energy or the first law of thermodynamics, one can check that \( Q_h + Q_c + W = 0 \). In our convention, if the quantities \( Q_h \) or \( Q_c \) are positive, the heat is flowing into the system, similarly, if work \( W \) is positive, work is done on the system. In conformity with the second law of thermodynamics, only four modes of operations are possible. The four possible modes can be identified by the signs of \( W \), \( Q_h \) and \( Q_c \), as depicted in Table I.

| Modes of operation | \( Q_h \) | \( Q_c \) | \( W \) |
|--------------------|---------|---------|-------|
| Engine             | > 0     | < 0     | < 0   |
| Refrigerator       | < 0     | > 0     | > 0   |
| Accelerator        | > 0     | < 0     | > 0   |
| Heater             | < 0     | < 0     | > 0   |

Table I. Different modes of operation of a quantum thermodynamic cycle.

Now, if heat is absorbed from the hot bath at the higher temperature by the system, converts a fraction of it into work and rejects the rest to the bath at lower temperature, we term it as a heat engine mode. The system works as a heat engine if \( W \) is negative, i.e., work can be extracted from the system. The efficiency \( \eta_E \) of the engine, defined as the ratio of the work extracted to the total heat absorbed by the system is

\[ \eta_E = -\frac{W}{Q_h}. \]

In contrast, if heat is absorbed from the cold bath and rejected into the hot bath with an external work done on the system, we term it a refrigerator mode. Here, \( W \) is positive, i.e., work is done on the system. The coefficient of performance of a refrigerator, the ratio of the heat extracted from the cold bath to the work done on the system is given as,

\[ \eta_R = \frac{Q_c}{W}. \]

Apart from the two modes discussed, there can be two other modes. In the accelerator mode, there is a free flow of heat from the hot bath to the cold bath and work is done on the system. On the other hand in the heater mode, a fraction of the work done on the system, is rejected to the hot bath and the rest to the cold bath.

The Carnot engine, theoretical proposition of Carnot, provides the maximum possible efficiency for a given pair of thermal baths. Since the Carnot cycle, being a quasi-static process, takes infinite time to complete one cycle and hence...
only a theoretical construct. But we will be presenting here the efficiency and coefficient of performance of the quantum cycle scaled by the same to its Carnot counterpart in order to provide an estimate of its performance. The efficiency and the coefficient of performance of the Carnot cycle are given respectively as

$$\eta_{E}^{\text{max}} = 1 - \frac{T_{c}}{T_{h}}, \quad \text{and} \quad \eta_{R}^{\text{max}} = \frac{T_{c}}{T_{h} - T_{c}}.$$  
(8)

The work done during a cycle is primarily dependent on the potential deformation that changes the wavefunction of the working substance and shifts the energy eigenvalues along with the bath temperatures. As discussed, quasi-static deformation of the potential can cause extraction of work from the system and hence serve as a quantum heat engine or a refrigerator depending on whether heat is transferred from a hot bath to a cold bath and vice versa. As a practical example of the aforementioned situations, we consider a quantum thermal machine that uses a gas of quantum particles in a hard potential box as its working medium [5, 6] and discuss the details of its properties.

### B. The working medium

We consider $N$ non-interacting, massive, indistinguishable particles of mass $M$ confined in a $d$ dimensional potential box with length $2L$ in each dimension as our working medium of the thermal machine. The Hamiltonian of the system is given as

$$H = \sum_{i=1}^{d} \frac{p_{x_{i}}^{2}}{2M}, \quad \text{for} \ |x_{i}| < L.$$  
(9)

The potential is distorted by introducing $d$ impenetrable barriers at $x_{i} = 0$, one in each dimension, that changes the energy levels and distort the wavefunctions. In this context, to highlight the quantum advantage over its classical counterpart, we note that introducing delta function barriers does not change the volume of the classical system and thereby contributes to zero work. In addition to that, if the box becomes very large, gaps between the energy levels go to zero forming free particles with continuum of energy levels.

The Hamiltonian of the system in the distorted configuration, with $d$ impenetrable barriers, is given by

$$H' = \sum_{i=1}^{d} \frac{p_{x_{i}}^{2}}{2M} + \lambda \delta(x_{i}), \quad \text{for} \ |x_{i}| < L,$$  
(10)

where, $\lambda$ is the strength of the delta function barrier. The single particle eigen energies of the Hamiltonians $H$ and $H'$, labeled by a set of integer quantum numbers $n_{x_{i}}$, are given by the following expressions

$$E(n_{x_{1}}, n_{x_{2}}, ..., n_{x_{d}}) = \frac{\pi^{2} \hbar^{2}}{8ML^{2}} \sum_{i=1}^{d} n_{x_{i}}^{2}, \quad n_{x_{i}} = 1, 2, 3, ...$$

$$E'(n_{x_{1}}, n_{x_{2}}, ..., n_{x_{d}}) = \frac{\pi^{2} \hbar^{2}}{8ML^{2}} \sum_{i=1}^{d} \left[ n_{x_{i}} + \frac{\varepsilon(n_{x_{i}})}{2} (1 - (-1)^{n_{x_{i}}}) \right]^{2}; \quad n_{x_{i}} = 1, 2, 3, ...$$  
(11)

where $0 \leq \varepsilon(n_{x_{i}}) \leq 1$. The value of $\varepsilon(n_{x_{i}})$ depends on the barrier strength $\lambda$ and can be obtained from the graphical solution of the transcendental equation (see [45, 46] for details). Since we consider here an infinitely strong barrier, i.e., $\lambda \to \infty$, the quantity $\varepsilon(n_{x_{i}}) = 1$. Inserting an impenetrable barrier shifts each odd energy eigenstate towards its next even one and thus creates twice degenerate energy levels. The single particle energy levels and eigenfunctions are schematically shown in Fig. 1(b). In a system with $d$ dimensions, one can show that there will be $2^{d}$ degenerate energy levels [45]. For example, in two dimension, the energy levels given by Eq. (11) are denoted by a pair of quantum numbers $(n_{x_{1}}, n_{x_{2}})$. Now, without the barriers the pair $(n_{x_{1}}, n_{x_{2}})$ takes values $(1, 1), (1, 2), (2, 1), (2, 2), (1, 3), (2, 3), (1, 4), (2, 4), ...$ and so on. After introducing two impenetrable barriers at $x_{1} = 0$ and $x_{2} = 0$ the same set changes to $(2, 2), (2, 2), (2, 2), (2, 4), (2, 4), (2, 4), (2, 4), ...$ and thus introduces a degeneracy of 4.

### C. Statistics of quantum particles

Quantum statistical mechanics is the foundation of understanding the low temperature behaviour of multi-particle physical systems consist of identical particles. In quantum mechanics, all identical particles are classified into two categories; the class of particles with half integer spin, known as fermions, is described by Fermi-Dirac statistics and the other class with integer spins, known as bosons, is described by Bose-Einstein statistics. The fundamental difference between these two categories arises entirely from the principle how a single particle state is occupied. Two fermions can never occupy a single particle state; on the other hand, multiple bosons can occupy the same state.

Quantum statistics dominates only when the interparticle spacing becomes smaller than the thermal de Broglie wavelength under low temperature, which can be termed as “quantum regime”. On the contrary, both the statistics can be ap-
proximated by Maxwell-Boltzmann distribution in the classical regime, where quantum effects are negligible. The thermal partition functions $Z_T^+$ and $Z_T$ for $N$ non-interacting fermions and bosons at a temperature $T$ will respectively be

$$Z_T^+ = \prod_{n=1}^{\infty} \left( \sum_{j=0}^{N} e^{-j(\tilde{E}_n - \tilde{\mu})/T} \right)^{g_n} = \prod_{n=1}^{\infty} \left( 1 + e^{-\tilde{E}_n \tilde{\mu}/T} \right)^{g_n} = \prod_{n=1}^{\infty} \left( 1 - e^{-(\tilde{E}_n - \tilde{\mu})/T} \right)^{g_n},$$

$$Z_T = \prod_{n=1}^{\infty} \left( \sum_{j=0}^{\infty} e^{-j(\tilde{E}_n - \tilde{\mu})/T} \right)^{g_n} = \prod_{n=1}^{\infty} \left( 1 - e^{-\tilde{E}_n \tilde{\mu}/T} \right)^{g_n} = \prod_{n=1}^{\infty} \left( 1 - e^{-(\tilde{E}_n - \tilde{\mu})/T} \right)^{g_n},$$

where $g_n$ is the degeneracy of the single particle $n$-th energy level and $\tilde{\mu}$ is the scaled chemical potential. For the sake of simplicity, we express all the energies and the chemical potentials in the units of $k_B$, i.e., $\tilde{E} = \tilde{E}/k_B$ and $\tilde{\mu} = \mu/k_B$ in Eq. (12).

As we will see in the next section, chemical potential of the system plays an essential role in determining the properties of the thermal machine. Now, to estimate the value of the chemical potential for fermions, it is important to define the Fermi energy $E_F$, the energy of the highest filled state at $T = 0$, which is given as $E_F = \frac{k_F^2}{2M}$ where $k_F$ is the Fermi wave vector. In $d$-dimension, the relation $[51–54] C_d k_F^d = N$ yields $k_F = 2\pi (\frac{2M}{\sigma})^{1/d}$. Hence the dimensionless Fermi energy in terms of the volume of a $d$-dimensional unit sphere $[51, 55] C_d = \frac{2}{\Gamma(d/2)\pi^{d/2}}$ and the particle density $\sigma = \frac{N}{2\pi} \frac{1}{\pi^{d/2}} \Gamma(d/2)$ can be expressed as

$$\tilde{E}_F = \frac{E_F}{k_B} = 2\alpha L^2 \left( \frac{\sigma}{C_d} \right)^{2/d} = \frac{\alpha N^2/d}{2C_d^{2/d}}.$$

For example,

$$\tilde{E}_F^{(d=1)} = \frac{\alpha L^2 \sigma^2}{2} = \frac{\alpha N^2}{2},$$
$$\tilde{E}_F^{(d=2)} = \frac{\alpha L^2 \sigma}{\pi} = \frac{\alpha N}{2\pi},$$
$$\tilde{E}_F^{(d=3)} = \left( \frac{6}{\pi} \right)^{2/3} \frac{\alpha L^2 \sigma^{2/3}}{2} = \left( \frac{6}{\pi} \right)^{2/3} \frac{\alpha N^{2/3}}{8}.$$

For the convenience of our calculations, here, we define the dimensionless parameter $\alpha = \frac{k_F^2}{\pi^d M \pi^T}$, such that the low temperature condition are scaled w.r.t $\alpha$ as $T_h, T_c \propto 1$. We reemphasize here as the bath temperatures $T_h, T_c$ become greater than $\alpha$, the energy levels become continuous and the quantum advantage is gradually lost. In order to get an estimate of the different parameters related to the system, let us consider a system of electrons trapped in a potential well. In terms of the fundamental constants, $\alpha = 1$ corresponds to a box length $L \sim 100$ nm. Since $\alpha$ is the characteristics temperature associated with the system, the low temperature limit in this case reduces to $T_h, T_c \lesssim 1K$. From the expression of $\alpha$, it is evident with the decrease of the size of the box, i.e., more confined the quantum system becomes, the constraint on the two bath temperatures become more relaxed.

Finally, we note that in the low temperature limit, the chemical potential for fermions can be taken to be equal to the Fermi energy; as the difference is extremely small even at room temperature $[51, 56]$. In case of bosons, the chemical potential is slightly negative in the low temperature limit and therefore in all practical situations, it can be taken as zero $[51, 56]$.

III. RESULTS

It is often useful to express the work done in terms of relative partition functions, i.e., the ratios of the partition functions for the adjacent stages, i.e., $\zeta(T_h) = \frac{Z}{Z_0}$ and $\zeta(T_c) = \frac{Z_c}{Z_D}$. Then the work done given by Eq. (5) can be rewritten as

$$W = -T_h \ln \zeta(T_h) - T_c \ln \zeta(T_c).$$

Now we use the expressions derived in Sec. II C for fermions and bosons to evaluate the expressions for work and heat exchanged by a quantum Stirling cycle.

Before we discuss the quantum regime we briefly review the well known results of the classical regime for the sake of completeness. In the classical regime, i.e., $T \gg \alpha$, the thermodynamics of the particles are governed by Maxwell-Boltzmann statistics. The single particle thermal partition function in this regime is given by

$$Z_T = \sum_{n=1}^{\infty} e^{-\tilde{E}_n/T}.$$  

In terms of the eigenenergies given in Eq. (11), one readily finds the expression for the relative partition function

$$\zeta(T) = \frac{\sum_{n=1}^{\infty} e^{-\alpha \tilde{E}_n^2/8T}}{2 \sum_{n=1}^{\infty} e^{-\alpha \tilde{E}_n^2/2T}} \approx \frac{1}{2} \frac{2\pi T}{\sqrt{2\pi T}} = 1.$$

In view of Eqs. (15) and (17), we obtain that the work done (extracted) by (from) the system is identically zero in the classical limit. Therefore, one can easily see that the counterpart of such a quantum cycle operating between two thermal baths, is an incompressible classical engine/refrigerator with zero efficiency $[5, 47, 48]$. This paradigm breaks down in the quantum domain, where the discreteness in the energy levels and
the inhomogeneous shift of the population distribution can
lead to efficient quantum thermal machines with no classical
analog [5, 6].

\[
\zeta^+(T) = \prod_{n=1}^{\infty} \frac{1 + e^{-(\alpha(2n^2)/8-\bar{\mu})/T}}{1 + e^{-(\alpha(2n-1)^2/8-\bar{\mu})/T}},
\]

(18)

and

\[
\zeta^-(T) = \prod_{n=1}^{\infty} \frac{1 - e^{-N(\alpha(2n^2)/8-\bar{\mu})/T}}{1 - e^{-N(\alpha(2n-1)^2/8-\bar{\mu})/T}} \cdot \frac{1 - e^{-(\alpha(2n^2)/8-\bar{\mu})/T}}{1 - e^{-(\alpha(2n-1)^2/8-\bar{\mu})/T}}.
\]

(19)

The expressions for the internal energies at \( B \) and \( D \) are
respectively given by

\[
U^+_B = \sum_{n=1}^{\infty} \frac{\alpha n^2/2 - \bar{\mu}}{1 + e^{(\alpha n^2/2-\bar{\mu})/T}},
\]

\[
U^+_D = \sum_{n=1}^{\infty} \frac{\alpha n^2/8 - \bar{\mu}}{1 + e^{(\alpha n^2/8-\bar{\mu})/T}},
\]

(20)

for fermions and

\[
U^-_B = \sum_{n=1}^{\infty} 2N \frac{\alpha n^2/2 - \bar{\mu}}{1 - e^{N(\alpha n^2/2-\bar{\mu})/T}} - \sum_{n=1}^{\infty} \frac{2 \alpha n^2/2 - \bar{\mu}}{1 - e^{(\alpha n^2/2-\bar{\mu})/T}};
\]

\[
U^-_D = \sum_{n=1}^{\infty} N \frac{\alpha n^2/8 - \bar{\mu}}{1 - e^{(\alpha n^2/8-\bar{\mu})/T}} - \sum_{n=1}^{\infty} \frac{\alpha n^2/8 - \bar{\mu}}{1 - e^{(\alpha n^2/8-\bar{\mu})/T}},
\]

(21)

for bosons.

We now generalize the result given in the previous section
for a quantum gas in a \( d \)-dimensional box, with a barrier in
each dimension. As discussed in Sec. II B, this will render an
ergy level structure with degeneracy of \( 2^d \). The expressions
for the relative partition functions of the particles are modified
according to Eq. (11) as

\[
\zeta^+(T) = \prod_{n_{x_1}=1}^{\infty} \prod_{j_{x_1}=0}^{1} \frac{1 + e^{-(\alpha \sum_{i} (2n_{x_i})^2/8-\bar{\mu})/T}}{1 + e^{-(\alpha \sum_{i} (2n_{x_i}-j_{x_i})^2/8-\bar{\mu})/T}},
\]

(22)

for fermions and

\[
\zeta^-(T) = \prod_{n_{x_1}=1}^{\infty} \prod_{j_{x_1}=0}^{1} \frac{1 - e^{-N(\alpha \sum_{i} (2n_{x_i})^2/8-\bar{\mu})/T}}{1 - e^{-N(\alpha \sum_{i} (2n_{x_i}-j_{x_i})^2/8-\bar{\mu})/T}} \cdot \frac{1 - e^{-(\alpha \sum_{i} (2n_{x_i})^2/8-\bar{\mu})/T}}{1 - e^{-(\alpha \sum_{i} (2n_{x_i}-j_{x_i})^2/8-\bar{\mu})/T}},
\]

(23)

for bosons, respectively. The extension for the \( d \)-dimensional

internal energies at \( B \) and \( D \), for a system of fermions and
bosons are respectively given by

\[
U^+_B = \sum_{n_{x_1}=1}^{\infty} 2^d \frac{\alpha \sum_{i} (2n_{x_i})^2/8 - \bar{\mu}}{1 + e^{(\alpha \sum_{i} (2n_{x_i})^2/8-\bar{\mu})/T}},
\]

\[
U^+_D = \sum_{n_{x_1}=1}^{\infty} \frac{\alpha \sum_{i} n_{x_i}^2/8 - \bar{\mu}}{1 + e^{(\alpha \sum_{i} n_{x_i}^2/8-\bar{\mu})/T}};
\]

(24)

and

\[
U^-_B = \sum_{n_{x_1}=1}^{\infty} 2^d N \frac{\alpha \sum_{i} (2n_{x_i})^2/8 - \bar{\mu}}{1 - e^{N(\alpha \sum_{i} (2n_{x_i})^2/8-\bar{\mu})/T}} - \frac{2^d \alpha \sum_{i} (2n_{x_i})^2/2 - \bar{\mu}}{1 - e^{(\alpha \sum_{i} (2n_{x_i})^2/2-\bar{\mu})/T}};
\]

\[
U^-_D = \sum_{n_{x_1}=1}^{\infty} N \frac{\alpha \sum_{i} n_{x_i}^2/8 - \bar{\mu}}{1 - e^{(\alpha \sum_{i} n_{x_i}^2/8-\bar{\mu})/T}} - \frac{\alpha \sum_{i} n_{x_i}^2/8 - \bar{\mu}}{1 - e^{(\alpha \sum_{i} n_{x_i}^2/8-\bar{\mu})/T}}.
\]

(25)

To see the contrasting role of the chemical potential on the
relative partition functions for fermions and bosons, the tempera-
ture dependence of \( \zeta^+(T) \) have been plotted in Fig. 2
following Eqs. (22) and (23) for \( d = 1, 2 \) and 3. It is evident
from Fig. 2(a) that the dimension of the system of fermions deci-
des the qualitative behaviour of \( \zeta^+(T) \). The value of \( \zeta^+(T) \)
is zero at \( T = 0 \) irrespective of the dimension but shows dif-
f erent behaviour depending on the dimension owing to the
particle number dependence of \( \bar{\mu} \). In \( T \alpha \ll 1 \) regime,
the value of \( \zeta^+(T) \) does not change appreciably from zero for
\( d = 1 \), increases for \( d = 2 \) and \( d = 3 \). However, the value
of \( \zeta^+(T) \) for \( d = 3 \) is much smaller compared to \( d = 2 \). On
the other hand, for bosons, the qualitative behaviour of \( \zeta^-(T) \)
does not change with the dimension as seen in Fig. 2(b). The
value of \( \zeta^-(T) \) is unity at \( T = 0 \) and then decays to zero
quickly irrespective of the dimension. It is interesting to ob-
serve that \( \zeta^-(T) \) is unity at \( T = 0 \) as the lowest energy level
is filled with all the particles. This means the bosons are most 'classical' at $T = 0$ when compared with Eq. (17).

It is further evident from Fig. 2 that low temperature behavior of fermions and bosons would be completely opposite. The relative partition functions for fermions and bosons $\zeta^\pm(T)$ and $\zeta^-(T)$ tend to zero and unity respectively in the low temperature limit. It is expected that the behavior of fermions will strongly depend on dimensionality of the system whereas for bosons, the qualitative behavior would be more or less similar. This corollary solely follows from the particle statistics and is independent of the specific form of the cycle. Therefore it captures the generic nature of the low temperature behaviour of any quantum thermal machines with quantum gases. We will see in the next subsection that our results with Stirling cycle are consistent with these observations.

Given the two baths with two different temperatures and a system with arbitrary dimensions and number of particles, it is not straightforward to simplify the above expressions to predict the behaviour of the cycle. One can easily plot the expressions to observe the characteristics of the system for a given pair of baths set at two arbitrary temperatures. However, in the next subsection, we elaborate that it is actually possible to look at the extremely low temperature of the system in $N \to \infty$ limit analytically and show the contrasting behaviour of the system with fermions and bosons in different dimensions.

A. Analytical approach for extremely low temperature & $N \to \infty$ limit of the cycle

We are specifically interested in the behaviour of the Stirling cycle in the quantum regime, i.e., in the extremely low temperatures, given by $T_h \to 0, T_c \to 0$ with $\Delta T = (T_h - T_c) > 0$. In this limit the expression for work [Eq. (15)] reduces to

$$W = \lim_{T \to 0, \Delta T \to 0} -((T + \Delta T) \ln \zeta((T + \Delta T)) + T \ln \zeta(T)).$$

(26)

Now, let us define the following function of temperature,

$$\omega(T) = T \ln \zeta(T).$$

(27)

Slope of this function solely determines, whether at low temperature, work can be extracted from the system or not. Work can be extracted from the system only if the slope of the function, i.e., $\lim_{T \to 0} \frac{d}{dT} \omega(T)$ as $T \to 0$ is positive and work is done on the system if the same is negative. Now, in the thermodynamic limit, i.e., $N \to \infty$, one can analytically evaluate the sign of the above quantity and thereby decide the nature of the system.

I. Fermions

Let us first consider the case of non interacting fermions in a one dimensional potential well. The function $\omega(T)$ takes the form of [Cf. Eq. (18)]

$$\omega(T) = T \ln \prod_{n=1}^{\infty} \frac{1 + e^{-\alpha((2n)^2-N^2)/8T}}{1 + e^{-\alpha((2n-1)^2-N^2)/8T}}. \quad (28)$$

In the aforementioned limiting cases, the quantities $\lim_{T \to 0} \lim_{N \to \infty} e^{\frac{\alpha}{8T}((2n)^2-N^2)}$ and $\lim_{T \to 0} \lim_{N \to \infty} e^{\frac{\alpha}{8T}((2n-1)^2-N^2)}$ both are very small positive numbers and

$$e^{\frac{\alpha}{8T}((2n)^2-N^2)} > e^{\frac{\alpha}{8T}((2n-1)^2-N^2)} \quad \forall n.$$

Hence, expanding the terms of $\frac{d}{dT} \omega(T)$ we get

$$\frac{d}{dT} \omega(T) = \frac{\alpha}{8T} \sum_{n=1}^{\infty} \left[ (N^2 - 4n^2 + 1) e^{\frac{\alpha}{8T}((2n)^2-N^2)} - e^{\frac{\alpha}{8T}((2n-1)^2-N^2)} ight] + (1 - 4n) e^{\frac{\alpha}{8T}((2n-1)^2-N^2)}. \quad (29)$$
Clearly, for $N \to \infty$, the quantity $\frac{d}{dT} \omega(T)$ is positive. This implies that work can be extracted from a system with large number of non-interacting fermions in low temperature limit. Therefore from Table I one concludes that the system behaves exclusively as a heat engine.

The efficiency of the heat engine is then given as

$$\eta_E = \frac{W}{Q_h} = 1 - \frac{T_h \ln \zeta^+(T_h)}{U - U_h}.$$  

(30)

In the aforementioned limit, the quantities $T_h \ln \zeta^+(T_h) \to -\infty$ and $T_h \ln \zeta^+(T_h) \to -\infty$ and $T_h \ln \zeta^+(T_h) \to -\infty$. Therefore, efficiency $\eta_E \to 1 - \frac{T_h}{T_b}$, i.e., the Carnot limit. The above equation connotes that a large number of non-interacting fermions entrapped in a one-dimensional potential well behave like a heat engine and its efficiency tends to Carnot limit as the bath temperatures approach absolute zero. Here we restate that it is purely a quantum effect with no classical correspondence, yet the machine still works as highest possible efficiency abide by the second law of thermodynamics. This is the first important result of our analysis exhibiting true quantum signature at a macroscopic scale.

But for fermions at higher dimensions, the behaviour of the system is entirely different because of the different number dependence of the chemical potential. The function $\omega(T)$ in this case is given by [Cf. Eq. (22)]

$$\omega(T) = T \ln \prod_{n=1}^{\infty} \frac{1 + e^{-\alpha(\sum_{i=1}^{n} (2n_{i})^2/2\tilde{\mu})/T}}{1 + e^{-\alpha(\sum_{i=1}^{n} (2n_{i} - j_{i})^2/2\tilde{\mu})/T}}.$$  

(31)

We know that in the low temperature limit, the chemical potential $\tilde{\mu}$ can be taken equal to be the Fermi energy $\tilde{E}_F$. It can be seen from Eq. (13) that the chemical potential in higher dimensions varies in a sub-quadratic fashion with $N \propto \gamma^2 < 2$, whereas the energy dispersion is still quadratic as given in Eq. (11). As a result, in the macroscopic thermodynamic limit $N \to \infty$, the energy always outgrows the chemical potential. Now, in the limit $N \to \infty$ and $T \to 0$, the expression $\frac{1}{T} (\alpha \sum_{i=1}^{n} (2n_{i})^2/2\tilde{\mu})$ tends to $\infty$ or $-\infty$ depending on the values of $\tilde{\mu}$. If $\frac{1}{T} (\alpha \sum_{i=1}^{n} (2n_{i})^2/2\tilde{\mu})$ tends to $\infty$, then $e^{-\alpha(\sum_{i=1}^{n} (2n_{i})^2/2\tilde{\mu})}$ tends to zero, on the other hand if $\frac{1}{T} (\alpha \sum_{i=1}^{n} (2n_{i})^2/2\tilde{\mu})$ tends to $-\infty$, then $e^{-\alpha(\sum_{i=1}^{n} (2n_{i})^2/2\tilde{\mu})}$ tends to $0$. So, we split up the sum over $n_{i}, j_{i}$ in two parts and denote them by $\sum_{1}$ and $\sum_{2}$ for these two cases respectively. Hence

$$\frac{d}{dT} \omega(T) = \sum_{1} \left[ e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i})^2 - \tilde{\mu})} - e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i} - j_{i})^2 - \tilde{\mu})} \right]
+ \frac{1}{T} \sum_{1} \left[ \left( \frac{\alpha}{8} \sum_{i=1}^{n} (2n_{i})^2 - \tilde{\mu} \right) e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i})^2 - \tilde{\mu})} - \left( \frac{\alpha}{8} \sum_{i=1}^{n} (2n_{i} - j_{i})^2 - \tilde{\mu} \right) e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i} - j_{i})^2 - \tilde{\mu})} \right]
- \frac{1}{T} \sum_{2} \left[ \left( \frac{\alpha}{8} \sum_{i=1}^{n} (2n_{i})^2 - \tilde{\mu} \right) e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i})^2 - \tilde{\mu})} - \left( \frac{\alpha}{8} \sum_{i=1}^{n} (2n_{i} - j_{i})^2 - \tilde{\mu} \right) e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i} - j_{i})^2 - \tilde{\mu})} \right].$$  

(32)

Using the following inequalities for each set of integers $(n_{i}, n_{j}, \ldots)$ and $j_{i} = 0, 1$,

$$e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i})^2 - \tilde{\mu})} \geq e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i} - j_{i})^2 - \tilde{\mu})},$$

$$e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i} - j_{i})^2 - \tilde{\mu})} \geq e^{-\frac{1}{T} (\sum_{i=1}^{n} (2n_{i})^2 - \tilde{\mu})},$$  

(33)

one finds that the first and the third terms are negative and positive respectively and the second term can be both positive and negative. Therefore, $\frac{d}{dT} \omega(T)$ can be both positive and negative depending on the actual functional forms of the chemical potential and dimension. Thus the non-interacting fermions at low temperature in more than one dimension, can behave as a heat engine or a refrigerator, accelerator and heater depending on the sign of $Q_h$ and $Q_c$. Hence, it can be seen that dimensionality of the system also dictates the behaviour of fermions. This is the second important observation of our analysis.

2. Bosons

Now, we consider the case of non-interacting bosons in a one dimensional box. The function $\omega(T)$ takes the form of

$$\omega(T) = T \ln \prod_{n=1}^{\infty} \frac{1 - e^{-\alpha(2n-1)^2/8T}}{1 - e^{-\alpha(2n)^2/8T}},$$  

(34)

in the $N \to \infty$ limit [Cf. Eq. (19)]. Using $e^{-\frac{\pi}{8}4n^2} < e^{-\frac{\pi}{8}4n^2}$ for each integer $n$, we find the following quantity

$$\frac{d}{dT} \omega(T) = \frac{\alpha}{8T} \sum_{n=1}^{\infty} (4n^2 + 1)(e^{-\frac{\pi}{8}4n^2} - e^{-\frac{\pi}{8}(2n-1)^2})$$

$$+ (4n - 1)e^{-\frac{\pi}{8}(2n-1)^2},$$  

(35)

is clearly negative. This indicates that a system with large number of non-interacting bosons in the low temperature limit,
behaves as a refrigerator or accelerator or a heater depending on the signs of $Q_h$ and $Q_c$. Further the expression for $Q_h$ in this case is

$$Q_h = T_h \ln \zeta^- (T_h) + U_B^- (T_h) - U_D^- (T_c).$$  \hspace{1cm} (36)$$

The first term in the above expression is negative as seen from Fig. 3(b) and $\frac{U_B^- - U_D^-}{T_h \ln \zeta^- (T_h)} \ll 1$. Therefore in the limit we are interested in, $Q_h$ is negative and thereby implying the system behaves as a refrigerator or a heater.

In this limit the coefficient of performance of the refrigerator is given as,

$$\eta_R = \frac{Q_h}{W} = \frac{-U_B^- - U_D^-}{T_h \ln \zeta^- (T_h)} - \frac{1}{1 - \frac{T_h \ln \zeta^- (T_h)}{T_c \ln \zeta^- (T_c)}}. \hspace{1cm} (37)$$

In the aforementioned limit, the quantities $(T_h \ln \zeta^- (T_h))/(T_c \ln \zeta^- (T_c)) \gg \frac{U_B^- - U_D^-}{T_h \ln \zeta^- (T_h)}$. Therefore coefficient of performance $\eta_R \rightarrow 0$. This result is consistent with the second law of thermodynamics since the coefficient of performance of the refrigerator don’t exceed the Carnot bound which is zero in this said limit. Later on, we show in Fig. 4(b) that the performance of the refrigerator can be optimized for a certain low temperature that has no classical analog.

In the above analysis we approximated the chemical potential to be zero for bosons. However, bosons possess a very small but a negative chemical potential [51, 56]. But as long as the quantity $E_n - \tilde{\mu}$ is positive for all energy levels, our conclusions will remain unchanged. Exactly similar analysis will show that the qualitative behaviour of bosons in higher dimensions and at low temperatures would be no different compared to that in one dimension.

In Fig. 3 we present different modes of operations for fermions and bosons in one, two and three dimensions from numerical calculations. Fermions in a one dimensional box behave strictly as a heat engine in the $T_h - T_c$ parameter space [Fig. 3(a)] as predicted from the analytical approach. The efficiency is almost constant and equal to the Carnot bound in the low temperature regime $T_h \lesssim \alpha$. On the other hand, in $d = 2$ and $d = 3$ we predicted that all the modes can exist in the $T_h - T_c$ plane in the low temperature limit. We find that all the three modes, refrigerator, heater and accelerator and all four modes, i.e., refrigerator, engine, heater and accelerator coexist for $d = 2$ and $d = 3$ respectively [Figs. 3(b) and (c)]. In contrast, as predicted from the analytical approach, the system with bosons shows qualitatively similar behaviour in all the dimensions [Figs. 3(d)-(f)]. There exist two distinct regions, heater and refrigerator, i.e., the work done on the system either goes to both the baths or transfers heat from the cold bath to the hot bath. These two regions are demarcated by a straight line in the $T_h - T_c$ plane. Coefficient of performance of the refrigerator is maximum at $T_c \rightarrow T_h$, although quite below the Carnot bound.
IV. CONCLUSIONS

To conclude, we have demonstrated some fundamental features of quantum particles in the context of quantum Stirling cycle. We analyze the role of quantum statistics and system dimension in determining the collective thermodynamic behaviour of particles. It is worth mentioning here that the efficiency considered here exhibits pure quantum signature with no classical analog and manifests only in the quantum regime. Though our analysis is focused on a specific working medium, viz., quantum gas trapped in an infinite potential well or a square well, one can trivially generalize our analysis for any system. A similar analysis with other trapping potentials would connote a qualitatively similar result as a consequence of the statistical nature of identical quantum particles.

We find that bosons behave completely opposite to fermions as a manifestation of the fundamental difference between the particle statistics in the quantum domain. Therefore, fermions and bosons are useful in a different way from the viewpoint of constructing a desirable thermodynamic machine. A Stirling cycle with fermions confined in a one dimensional potential well, when connected to two low temperature baths, behaves exclusively like a heat engine. On the other hand, a Stirling cycle with bosons behaves like a refrigerator and a heater depending on the bath temperatures.

We have also shown that the particle number or the dimension does not appreciably affect the performance for bosons. Unlike bosons, the number dependence of the chemical potential for fermions decides the behaviour of the system. As dimension of the system decides the number dependence of the chemical potential, it determines the overall thermodynamic behaviour of the system. The behaviour of fermions in a one dimensional well is completely different compared to that of two or higher dimensions. Higher the dimension, less the dependence on the fermion number and the system of fermions shows less "quantumness". We have found that increasing number of particles in a system of quantum particles can boost

B. Dependence on particle number

We have already shown that a Stirling cycle with fermions in one dimension exclusively works as an engine whereas with bosons it both behaves as a refrigerator and a heater. To explore the $N$ dependence for fermions, we plot the heat engine efficiency scaled w.r.t. the Carnot efficiency, $\eta_E/\eta_E^{\text{max}}$, in Fig. 4(a), with the number of particles $N$ and $T_h$, kept in a one dimensional box keeping the ratio $T_c/T_h$ fixed. The efficiency reaches the Carnot bound as the bath temperatures tend to zero. It is interesting to see that for a given pair of bath with non-zero temperatures, a system with larger $N$ yields better efficiency. Owing to a prominent number dependence on fermions, the engine efficiency can be boosted by adding more fermions to the system.

In Fig. 4(b), we plot the refrigerator coefficient of performance for bosons scaled w.r.t. the Carnot efficiency, $\eta_R/\eta_R^{\text{max}}$, with the number of particles $N$ and $T_h$, kept in a one dimensional box keeping the ratio $T_c/T_h$ fixed. The coefficient of performance tend to zero as the bath temperatures tend to zero irrespective of $N$ as predicted from our analysis and it also goes to zero in the classical limit slowly. It can be also understood from Fig. 2(b) that the bosons are 'classical' at $T = 0$ and naturally in the higher temperature limit. Hence no quantum advantage is expected if the bath temperatures are in these two limits. The coefficient of performance for all $N$, however, peaks at a certain temperature in between. The maximum coefficient of performance being independent of $N$, as seen from Fig. 4(b), reveals that bosons do not show any prominent dependence on $N$, unlike fermions. A system with a small number of bosons is almost as efficacious as that with a large number.

Figure 4. Dependence on the particle number $N$ for $d = 1$ dimension. Here (a) engine efficiency scaled w.r.t. Carnot efficiency $\eta_E/\eta_E^{\text{max}}$ for a fixed $T_c/T_h = 0.50$ for fermions and (b) refrigerator coefficient of performance scaled w.r.t Carnot coefficient of performance $\eta_R/\eta_R^{\text{max}}$ for a fixed $T_c/T_h = 0.75$ for bosons.
its performance in spite of engine efficiency/coefficient of performance being bounded by Carnot.

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