Angular Momentum Mixing in Crystalline Color Superconductivity

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Abstract

In crystalline color superconductivity, quark pairs form at non-zero total momentum. This crystalline order potentially enlarges the domain of color superconductivity in cold dense quark matter. We present a perturbative calculation of the parameters governing the crystalline phase and show that this is indeed the case. Nevertheless, the enhancement is modest, and to lowest order is independent of the strength of the color interaction.

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While Quantum Chromodynamics (QCD) has now been firmly established as the theory of strong interactions, many important questions, such as the nature of confinement and chiral symmetry breaking, still remain. Attempts to address such issues have led to recent interest in exploring the nature of nuclear matter under extreme conditions of temperature and density. New phases of nuclear matter have long been sought after, both theoretically and experimentally. Under the right conditions, nuclear matter is expected to undergo a phase transition to a quark-gluon plasma. At cold temperatures, however, recent application of QCD suggests a much richer phase structure of quark matter than originally anticipated. Perhaps the most prominent of such phases, and one which may play an important role in the core of neutron stars, is that of color superconductivity [1,2].

Color superconductivity is essentially the quark analog of BCS superconductivity. Because of attractive quark-quark interactions in QCD, the Fermi sphere of quarks becomes unstable against the formation of Cooper pairs and the system becomes superconducting at sufficiently low temperatures. Quantitative results may be obtained at ultra-high baryon densities where asymptotic freedom ensures the validity of the weak coupling approximation. In this regime, the main contribution arises from one-gluon exchange in the color antisymmetric channel. The resulting transition temperature, $T_c$, is given in terms of the running QCD coupling $g(\mu)$ by [3–11]

$$k_B T_c = \frac{512}{\pi^3} \frac{2}{N_f^3} \frac{\mu^5}{g^5} e^{-\frac{1}{6} \frac{N_c}{N_c+1} \frac{e^2}{g^2} - \frac{\pi^2+4}{16} (N_c-1)}, \quad (1)$$

where $\mu$ is the chemical potential and $\gamma = 0.5772\ldots$ is the Euler constant.

This result that $T_c \sim e^{-\kappa/g}$ is remarkable for its non-BCS behavior (where BCS theory instead predicts $T_c(\text{BCS}) \sim e^{-\lambda/g^2}$). The relation between $T_c$ and the zero temperature gap energy $\Delta_0$, on the other hand, remains BCS-like: $\Delta_0 = \pi e^{-\gamma} k_B T_c$. This non-BCS scaling is driven by the poor screening of the long range color magnetic interaction. The same mechanism is also responsible for non-Fermi liquid behavior which suppresses $T_c$ significantly [8–10].

Since the main attractive channel is between different species of quarks, search for a realistic color superconductivity phase must take into consideration the masses of the light quarks. This effect, not accounted for in Eq. (1), is especially important when considering the color flavor-locked state [2] involving $u$, $d$ and $s$ quarks. Although not strictly identical, this situation is well modeled by separating the Fermi surfaces of massless quarks that participate in the pairing. This case was recently investigated in Refs. [12–15], where it was shown that a new crystalline superconductivity state may arise for an appropriate window of separation between Fermi surfaces.

Crystalline superconductivity was originally investigated in BCS theory, where ferromagnetic impurities separate the Fermi surfaces of each electron spin [16–18], so that $\mu_{1,2} = \mu \pm \delta$. When the system is cooled from the normal phase, the transition temperature is the maximum pairing temperature, allowing for a possible net momentum $2\vec{q}$ for a pair. For a sufficiently small separation, $\delta \leq \delta_c$, zero momentum pairing wins. However, for $\delta$ in the range $\delta_c < \delta < \delta_{\text{max}}$, pairing with non-zero momentum is favored. The net momentum provides a region of phase space where both quarks may sit on their respective Fermi surfaces. However it breaks rotational invariance and leads to crystalline order. This state is
also known as the LOFF state, after the work of Refs. [16,17]. Finally, for $\delta > \delta_{\text{max}}$, the transition temperature drops to zero, and superconductivity is lost.

In the BCS case, both $\delta_c$ and $\delta_{\text{max}}$ as well as the pair momentum $q_{\text{max}}$ (at $\delta = \delta_{\text{max}}$) are proportional to the initial (i.e. $\delta = 0$) zero temperature gap energy, with the constants of proportionality being independent of the details of the interaction. In particular, $\delta_c^{(\text{BCS})} \approx 0.606 \Delta_0$, while $\delta_{\text{max}}^{(\text{BCS})} \approx 0.754 \Delta_0$. As this indicates, the window for BCS crystalline superconductivity is fairly narrow.

In this letter, we explore the crystalline superconductivity state for color superconductivity. Our approach follows the perturbative method developed in Refs. [8,9] which allows a systematic approach to determining $T_c$ (but not directly the zero-temperature gap). We find that for color superconductivity, while $\delta_c$ remains unchanged, $\delta_{\text{max}}$ is modestly enhanced, so that there is a widening of the region of crystalline order compared to the BCS case. As in the behavior of $T_c$ itself in Eq. (1), the resulting non-BCS behavior of the color LOFF state also traces its origin to the long range nature of the magnetic gluons.

We begin with a brief review of the perturbative method developed in Refs. [8,9] for a normal-phase determination of the transition temperature, however now applied to pairing with total momentum $2\vec{q}$ and shifted chemical potentials $\mu \pm \delta$. The proper vertex function for quark pair scattering in the color antisymmetric channel is given by a Dyson-Schwinger equation

$$\Gamma_{\vec{q},\delta}^{(\text{PI})}(P_f|P_i) = \Gamma_{\vec{q},\delta}^{(\text{PI})}(P_f|P_i) + \frac{1}{\beta} \sum_\nu \int \frac{d^3 \vec{p}}{(2\pi)^3} K_{\vec{q},\delta}(P_f|P) \Gamma_{\vec{q},\delta}(P|P_i), \quad (2)$$

where the initial and final momenta are $\vec{q} \pm \vec{p}_i$ and $\vec{q} \pm \vec{p}_f$. Here $P$ stands for Euclidean four-momentum, $(\vec{p}, -\nu)$ with $\nu$ being the Matsubara energy, and the four-momentum of the pair is $Q = (2\vec{q}, 0)$. Eq. (2) is of the Fredholm type with kernel

$$K_{\vec{q},\delta}(P'|P) = \Gamma_{\vec{q},\delta}^{(\text{PI})}(P'|P) S(Q + P, \mu + \delta) S(Q - P, \mu - \delta), \quad (3)$$

where $\Gamma_{\vec{q},\delta}^{(\text{PI})}$ denotes the two quark irreducible part of the vertex and $S(P, \mu)$ the full quark propagator with chemical potential $\mu$.

The pairing instability may be probed by examining the spectrum of the kernel, given by the eigenvalue equation

$$E f(P) = \sum_\nu \int \frac{d^3 \vec{p}'}{(2\pi)^3} K_{\vec{q},\delta}(P'|P') f(P'), \quad (4)$$

for eigenfunctions $f(P)$. At weak coupling, all eigenvalues are less than one for sufficiently high temperature. Then, as the temperature is lowered, a point is reached where the largest eigenvalue reaches unity, $E_{\text{max}}(T, \mu, \delta, q) = 1$, and the solution to Eq. (4) becomes singular, corresponding to the onset of the pairing instability.

We have indicated explicitly the dependence of $E_{\text{max}}$ on the temperature, mean chemical potential, Fermi surface separation and the pair momentum. The condition $E_{\text{max}} = 1$ may thus be solved to obtain a pairing temperature as a function of $q$ for given $\mu$ and $\delta$; this provides a determination of $T_c(\mu, \delta)$ when maximized over $q$. Ordinary and crystalline order corresponds to $q = 0$ and $q \neq 0$, respectively, at this maximum point. At weak coupling, the
relevant \( \delta \) and \( q \) are both much smaller than the mean chemical potential \( \mu \), and the onset of LOFF order can be treated as a perturbation from ordinary pairing at \( \delta = q = 0 \). Thus we look for an expansion of the eigenvalue \( E_{\text{max}} \) in ascending powers of \( g \) (with coefficients that may depend on \( \ln g \)). Through Eq. (3), this is equivalent to an expansion in descending powers of \( \ln \mu/(k_B T^0_c) \), where \( T^0_c \) is the \( \delta = 0 \) value. Quantities such as \( k_B T^0_c/\mu, \delta/\mu \) and \( q/\mu \) are much smaller than any finite power of \( g \) and can be neglected.

To the accuracy of Eq. (1), the contribution of antiparticles to the scattering amplitude can be ignored and the two quark irreducible part of the vertex \( \Gamma^{2\Pi I} \) is well approximated by single gluon exchange:

\[
\Gamma_{q,\delta}(P|P) \approx -\frac{i}{g^2}(1 + \frac{1}{N})[D_M(\vec{k},\omega)N_M(\vec{p},\vec{p}',\vec{k}) + D_E(\vec{k},\omega)N_E(\vec{p},\vec{p}',\vec{k})],
\]

(5)

where \( \vec{k} = \vec{p}' - \vec{p} \) and \( \omega = \nu' - \nu \). Here \( N_M \) and \( N_E \) are kinematic factors and \( D_M(\vec{k},\omega) \) and \( D_E(\vec{k},\omega) \) are magnetic and electric gluon propagators with hard dense loop (HDL) resummation. While the one-loop quark self energy contributes to the prefactor of Eq. (1), it will not interfere with the perturbative contribution of nonzero \( \delta \) and \( q \). Consequently we replace \( S \) with the free quark propagator, \( S(\nu,\vec{p};\mu) = i/(i \nu - p + \mu) \), throughout.

The eigenvalue problem, Eq. (3), was previously analyzed within each angular momentum channel for \( \delta = q = 0 \) [4]. The highest eigenvalue with angular momentum \( J \) reads

\[
E_J = 1 + \frac{2}{\ln \frac{1}{\epsilon}} \left( 3c_J + \ln \frac{T_c}{T^0_c} \right) + O \left( \ln^{-2 \frac{1}{\epsilon}} \right),
\]

(6)

where

\[
c_J = \int_{-1}^{1} dx \frac{P_J(x) - 1}{1 - x} = \left\{ \begin{array}{ll}
0, & \text{if } J = 0; \\
-2 \sum_{j=1}^{J} \frac{1}{j}, & \text{if } J \neq 0,
\end{array} \right.
\]

(7)

and \( \epsilon = k_B T^0_c/\hat{\mu} \) where \( \hat{\mu} = 256 \pi^3 g^{-5} (2/N_f)^{3/2} \mu \) is a rescaled chemical potential. The corresponding eigenfunction is

\[
f_{JM}(P) = \frac{2\pi}{\sqrt{\ln \frac{T_c}{T^0_c}}} \sin \left( \frac{\pi \ln \frac{1}{\epsilon}}{2 \ln \frac{T_c}{T^0_c}} \right) \frac{Y_{JM}(\hat{\nu})}{\hat{\nu}},
\]

(8)

where \( \hat{\nu} = \nu/\hat{\mu} \) and \( Y_{JM}(\hat{\nu}) \) are spherical harmonics. Since the level spacing for different \( J \) is of the same order as the perturbation to be introduced, a degenerate perturbation method has to be employed.

Nonzero \( \delta \) and \( q \) modify the pole structure of the quark pair propagator of Eq. (3) in a similar manner as in the BCS case. In addition, a nonzero \( \delta \) will shift the Debye mass in the HDL resummed denominators of the vertex, and the numerators of the one-gluon vertex, \( N_M \) and \( N_E \), depends on the total momentum \( \vec{q} \). Both corrections to the vertex function are of the order of \( (\delta/\mu)^2 \) or \( (q/\mu)^2 \) and may be ignored.

To diagonalize the kernel, Eq. (3), with nonzero \( \delta \) and \( q \) to \( O(\ln^{-1 \frac{1}{\epsilon}}) \), we evaluate \( K \) between the \( O(1) \) eigenfunction \( f_{JM}(P) \) and its adjoint

\[
\overline{f}_{JM}(P) = S(P)S(-P)f_{JM}(P) \simeq \frac{f_{JM}(P)}{\nu^2 + (p - \mu)^2},
\]

(9)
and define

\[
E_{J'M',JM} = \frac{1}{\beta^2 \Omega^2} \sum_{P,P'} \overline{f}_{J'M'}(P') K_{\overline{q},\delta}(P'|P) f_{JM}(P).
\]

Using the eigenvalue equation, Eq. (4), for \(\delta = q = 0\) and to leading order in \(g\), we find

\[
E_{J'M',JM} = \delta_{JJ'} \delta_{MM'} + \frac{2}{\ln \alpha} \langle J'M'|h_{op}|JM \rangle.
\]

The operator \(h_{op}\) is given by \(3c_{op} + v_{op}\), with \(c_{op}\) diagonal in the angular momentum representation, \(\langle J'M'|c_{op}|JM \rangle = 3c_{JJ'} \delta_{MM'}\), and \(v_{op}\) diagonal in the coordinate (angle) representation,

\[
v_{op} = \ln \frac{T}{T^0} + \psi\left(\frac{1}{2}\right) - \text{Re} \psi\left(\frac{1}{2} - \frac{i}{2\pi} \beta (\delta - q \cos \theta)\right),
\]

where \(\psi(\zeta) = \frac{d}{d\zeta} \ln \Gamma(\zeta)\). Here \(\theta\) is the polar angle with respect to momentum \(\vec{q}\). The azimuthal quantum number \(M\) remains conserved. Both \(c_{op}\) and \(v_{op}\) are Hermitian and \(c_{op}\) is non-positive. For \(q = 0\), the rotation symmetry is intact and the largest eigenvalue of \(h_{op}\) still corresponds to \(J = 0\). Here we recover the BCS result

\[
E_0 = 1 + \frac{2}{\ln \alpha} \left[ \ln \frac{T^0}{T} + \psi\left(\frac{1}{2}\right) - \text{Re} \psi\left(\frac{1}{2} - \frac{i}{2\pi} \beta \delta\right) \right].
\]

With \(\vec{q} \neq 0\), different angular momentum components start to mix and this widens the LOFF window with respect to \(\delta\). In what follows, we shall focus on the component \(M = 0\) since the additional sign change introduced by nonzero \(M\) will not be the most favorable. On returning to a coordinate representation, \(Y_{JM}(\hat{\rho}) \to u(x \equiv \cos \theta)\), the eigenvalue equation for \(h_{op}\) becomes

\[
3 \int_{-1}^{1} dx' \frac{u(x') - u(x)}{|x - x'|} + v_{op}(x)u(x) = \lambda u(x),
\]

where the first term on the left hand side is the coordinate representation of \(c_{op}\), following from the integral representation of \(c_J\) in Eq. (7).

To explore the upper threshold of the LOFF window, \(\delta_{\text{max}}\), we take the limit \(T \to 0\), whereupon the function \(v_{op}(x)\) reduces to

\[
v_{op}(x) = \psi\left(\frac{1}{2}\right) - \ln \frac{\delta_{\text{max}}}{2\pi k_B T^0} - \ln |1 - \frac{x}{x_c}|,
\]

with \(x_c = \delta/q\). The only possibility for obtaining a diverging \(\delta_{\text{max}}\) is to take full advantage of the logarithmic singularity of \(v_{op}\) as \(x \to x_c\). Consider, e.g., a trial function

\[
u(x) = \begin{cases} \frac{1}{x_c^{2\alpha}} & \text{for } x_c - \alpha < x < x_c + \alpha; \\ 0 & \text{otherwise}. \end{cases}
\]
In the limit $\alpha \to 0$, we find $u^\dagger v_{\text{op}} u \to \ln \frac{1}{\alpha}$ but $u^\dagger c_{\text{op}} u \to -\ln \frac{1}{\alpha}$, so that $u^\dagger h_{\text{op}} u \to -2 \ln \frac{1}{\alpha}$. Therefore the operator $h_{\text{op}}$ is bounded from above, and the widening of the LOFF window cannot grow indefinitely. A numerical analysis reveals that the largest eigenvalue of the integral equation, Eq. (14), with the limiting $v_{\text{op}}$ of Eq. (15) for a fixed $\delta$ is $\lambda \simeq -\ln \frac{\delta}{2\pi k_B T^0_c} - 1.304$, which occurs at $q \simeq 1.16 \delta$. The corresponding $u(x)$ is shown in Fig. 1, demonstrating the peak at $x = x_c$. For a second order phase transition, the eigenvalue equation coincides with the limit of the gap equation as $T \to T_c$. An anisotropy will develop for the order parameter in the superphase as well. Applying the pairing condition, $E_{\text{max}} = 1$, and the relation between $T^0_c$ and $\Delta_0$, we find

$$\delta_{\text{max}} \simeq 1.67 k_B T^0_c \simeq 0.96 \Delta_0,$$

(17)

to be compared with the BCS case, $\delta_{\text{max}}^{(\text{BCS})} \simeq 0.754 \Delta_0$.

The determination of the onset, $\delta_c$, is numerically more tedious, but an upper bound can be readily obtained. For very small $\delta$, we have $v_{\text{op}} \simeq \ln \frac{T^0_c}{2\pi^2} - \frac{7\zeta(3)}{2\pi^2} \beta^2 (\delta - q \cos \theta)^2$. The last term is too small to mix different $J$, and we have $J = 0$ for pairing. Since $(\frac{\partial E_0}{\partial q})_{\delta,\beta} < 0$ at $q = 0$, the LOFF state is not favored. Then if we assume the onset of the LOFF state starts from $q_c = 0$ for sufficiently large $\delta$, the critical value of $\delta$ follows from the condition $(\frac{\partial E_0}{\partial q})_{\delta,\beta} = 0$. Applying Eq. (13) and the pairing condition $E_{\text{max}} = 1$, we obtain

$$\delta_c \simeq 0.606 \Delta_0,$$

(18)

which is identical to the BCS result. Although we have not been able to verify it analytically, numerical results indicate that pairing indeed starts at $q = 0$. The numerically computed phase diagram is shown in Fig. 2.
FIG. 2. The transition temperature, $T$, as a function of $\delta$. The BCS transition is shown for comparison.

We note that Ref. [14] suggested that there would be an indefinite widening of the window for the color LOFF state by drawing an analog with a $1 + 1$ dimensional system on account of the forward singularity of the scattering vertex. The present results, however, demonstrate that while the window is indeed widened over that of the BCS case, the widening is fairly modest, as indicated in Fig. 2. This result may be understood since the reduction of the available phase space for pairing with increasing $\delta$ in higher dimensions dominates over any enhancement arising from a $1 + 1$ dimensional analog. Furthermore, unlike the isotropic BCS case, different angular momenta have to be considered simultaneously towards a determination of $\delta_{\text{max}}$, and the long range order parameter itself would not be spherical. To our knowledge, such a mixing effect is entirely new and the approximations we have used are well controlled.

This angular momentum mixing in the color LOFF order is quite unique to the dynamically screened magnetic interaction. On writing the highest eigenvalue of the kernel, Eq. (3), with angular momentum $J$, $E_J = E_J^{(0)} + O(\ln^{-1} \frac{1}{\epsilon})$, we have $E_J^{(0)} = 1$ for all $J$ here. On the other hand, for a BCS model with a pointlike interaction, pairing only occurs in the $s$-wave channel, and $E_J^{(0)} = \delta_{J,0}$. Even for a Yukawa type of interaction, say the screened electric interaction alone, $E_J^{(0)}$ would be different for different $J$. In both the pointlike and Yukawa cases, the angular momentum only mixes in higher order corrections, and would not lead to the enhancement seen in the color LOFF state.
ACKNOWLEDGMENTS

We wish to thank K. Rajagopal for providing useful insight and comments. This work is supported in part by the US Department of Energy under grants DE-FG02-91ER40651-TASKB and DE-FG02-95ER40899-TASKG.
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