HITTING PROBABILITIES FOR GENERAL
GAUSSIAN PROCESSES

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Abstract. For a scalar Gaussian process $B$ on $\mathbb{R}_+$ with a prescribed general variance function $\gamma^2(r) = \text{Var}(B(r))$ and a canonical metric $E[(B(t) - B(s))^2]$ which is commensurate with $\gamma^2(t-s)$, we estimate the probability for a vector of $d$ iid copies of $B$ to hit a bounded set $A$ in $\mathbb{R}^d$, with conditions on $\gamma$ which place no restrictions of power type or of approximate self-similarity, assuming only that $\gamma$ is continuous, increasing, and concave, with $\gamma(0) = 0$ and $\gamma'(0+) = +\infty$. We identify optimal base (kernel) functions which depend explicitly on $\gamma$, to derive upper and lower bounds on the hitting probability in terms of the corresponding generalized Hausdorff measure and non-Newtonian capacity of $A$ respectively. The proofs borrow and extend some recent progress for hitting probabilities estimation, including the notion of two-point local-nondeterminism in Biermé, Lacaux, and Xiao [4]. Other techniques, such as a reliance on classical Gaussian path regularity theory, or quantitative estimates based on Hölder continuity or indexes, must be entirely abandoned because they cannot provide results which are sharp enough. Instead, all calculations are intrinsic to $\gamma$, and we use new density estimation techniques based on the Malliavin calculus in order to handle the probabilities for scalar processes to hit points and small balls. We apply our results to the probabilities of hitting singletons and fractals in $\mathbb{R}^d$, for a two-parameter class of processes. This class is fine enough to narrow down where a phase transition to point polarity (zero probability of hitting singletons) might occur. Previously, the transition between non-polar and polar singletons had been described as the single point where a process is $H$-Hölder-continuous in the mean-square with $H = 1/d$; now we can see how a range of logarithmic corrections affects this transition.

1. Introduction

1.1. Background and motivation. In this paper we will assume throughout that $B = (B(t), t \in \mathbb{R}_+)$ is a centered continuous Gaussian process in $\mathbb{R}$ such that for some constant $\ell \geq 1$, some continuous strictly increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_0 \gamma = 0$, and for all $s, t \in \mathbb{R}_+$,

$$
(1/\ell)\gamma^2(|t-s|) \leq E[(B(t) - B(s))^2] \leq \ell \gamma^2(|t-s|).
$$

The above condition yields, with $s = 0$, that $\text{Var}B(t)$ is commensurate with $\gamma^2(t)$; in addition, we also assume throughout the paper that $\gamma$ and $\text{Var}B(\cdot)$ actually coincide: for all $t \in \mathbb{R}_+$,

$$
\text{Var}B(t) = \gamma^2(t).
$$

Note that $\gamma$ does not define the law of $B$ since distinct processes with the same variance function $\gamma$ may satisfy (1.1). For economy of notation, we also use the letter $B$ to
A number of Gaussian processes satisfy (1.1) and (1.2), including each fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, in which case, $\ell = 1$ and $\gamma(t) = t^H$; in other words, the inequalities in (1.1) are equalities, and they reflect fBm’s self-similarity and stationarity; Brownian motion $W$ is included therein, by taking $H = 1/2$. The so-called Riemann-Liouville fractional Brownian motion (RL-fBm) with parameter $H$, defined via standard Brownian motion $W$ by $B^{RL,H}(t) := \sqrt{2H} \int_0^t (t - s)^{H-1/2} dW(s)$ is also covered, with $\gamma(t) = t^H$ just like with fBm, but with $\ell = 2$: this process has non-stationary increments, see [10]. The fBm and the RL-fBm are both self-similar with parameter $H$, both correspond to $\gamma(t) = t^H$, and therefore exhibit continuity properties that have traditionally been evaluated within the Hölder scale. This pattern of concentrating on the value of $H$ has permeated research on the estimation of hitting probability properties, see for e.g. [3] [18] [22] [23]. Another important class of self-similar examples with non-stationary increments, which still satisfy (1.1) and (1.2), are the solutions of the stochastic heat equation with additive noise whose space behavior is of Riesz-kernel type, as described in the preprint to appear [17], or solutions of stochastic heat equations can also easily lose the self-similarity property: see those studied in [13] in the context of hitting probabilities, and the preprint [20]. Solutions of stochastic heat equations can also easily lose the self-similarity property: see those studied in [13] in the context of hitting probabilities, where the Hölder scale is still the dominant yardstick; in those examples, (1.1) and (1.2) are still satisfied, for a function $\gamma(t)$ which is equivalent to $t^H$ as $t \to 0$ for some $H \in (0, 1)$, hence the Hölder property. Any process satisfying (1.1) and (1.2) where $\gamma(t)$ is commensurate with $t^H$ for small $t$ will still live in this Hölder scale.

One motivation of the present paper is to avoid such restrictions, by letting the function $\gamma$, whether it be commensurate with a power function or not, tell us how hitting probabilities behave. Classical results of R. Dudley and others from Gaussian continuity theory (see [2]) tell us that under (1.1), if $\gamma(r) = o\left(\log^{-1/2}(1/r)\right)$ for $r$ near zero, then $B$ is almost-surely continuous, and the function $h : r \mapsto \gamma(r)\log^{1/2}(1/r)$ is, up to a deterministic constant, a uniform modulus of continuity for $B$; i.e.

$$\sup_{0 \leq s < t \leq T} \frac{|B(t) - B(s)|}{h(t - s)} < \infty \quad \text{a.s.} \quad (1.3)$$

Such precise quantitative results depending only on a general $\gamma$ should also be available for hitting probability questions, and this is the question we try to address in this paper. In the case of moduli of continuity for Gaussian processes, while these results are known to be sharp in some cases (see a rather general treatment for stationary processes on the unit circle in [19]), there are generally no lower-bound results based on $\gamma$. In this article we will try to go one step further for hitting probabilities, as we will derive upper and lower bounds which both relate to the function $\gamma$.

To illustrate what kinds of Gaussian processes are covered by our study, consider the following. One extends the RL-fBm via the class of so-called Gaussian Volterra processes defined as the family containing one process $B^\gamma$ for each function $\gamma$, via a standard Brownian motion $W$ and the formula

$$B^\gamma(t) := \int_0^t \sqrt{\left(\frac{d\gamma^2}{dt}\right)(t - s)} dW(s). \quad (1.4)$$

If $\gamma^2$ is of class $C^2$ on $\mathbb{R}_+ \setminus 0$, $\lim_{s \uparrow 0} \gamma^2 = 0$, and $\gamma^2$ is increasing and concave ($d\gamma^2/dr$ is non-increasing), then $B^\gamma$ satisfies (1.1) and (1.2) for the fixed function $\gamma$, with $\ell = 2$. 

designate a vector of $d$ iid copies of the scalar version of $B$. Whether $B$ needs to represent a scalar or vector version should be clear from the context.
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See [10] and [11]. In addition, it is a simple matter to produce processes that satisfy the same conditions (1.1) and (1.2), with possibly different constants \( \ell \); in the definition of \( B^\gamma \) in (1.4), replace \( (d\gamma^2/dt)(t-s) \) by any function of the pair \((s,t)\) which is bounded above and below by multiples of \((d\gamma^2/dt)(t-s)\). All of these processes have the added bonus that they are adapted to a Brownian filtration. None of them have stationary increments. When \( \gamma \) is not commensurate with a power function near 0, the resulting \( B^\gamma \) is far from being self-similar.

1.2. Summary of results. This paper’s main effort is devoted to proving upper and lower bounds for the probability that a \( d \)-dimensional process \( B \) with iid coordinates satisfying conditions (1.1) and (1.2) hits a given Borel set \( A \) in \( \mathbb{R}^d \), in terms of certain \( \gamma \)-dependent Hausdorff measure and capacity of the set \( A \), respectively. In this subsection, we provide a summary of our main results, including precise statement of all corresponding bounds, and indications of the main articulations of their proofs.

In Section 2 we show (see Theorem 2.5) that if the function \( \gamma \) in (1.1) is concave in a neighborhood of zero, then for all \( 0 < a < b \) and \( M > 0 \), there exists a constant \( C > 0 \) depending only on \( a, b, M \) and the law of \( B \), such that for any Borel set \( A \subset [-M,M]^d \)

\[
CC_K(A) \leq P(B([a,b]) \cap A \neq \emptyset), \tag{1.5}
\]

where \( C_K(A) \) denotes the capacity of the set \( A \) with respect to the potential kernel

\[
K(x) := \max \{ 1; v(\gamma^{-1}(x)) \}, \quad v(r) := \int_r^{b-a} ds/\gamma^d(s) \tag{1.6}
\]

(see Section 2 for the definition of capacity). In particular, from (1.5) it follows that if \( \gamma \) is concave near zero and \( 1/\gamma^d \) is integrable at 0 (i.e. \( v \) is bounded), then the process \( B \) hits points (singletons) with positive probability.

In order to prove (1.5), one of the key ideas comes from [4], where the authors obtain a lower bound in terms of capacity for the hitting probabilities of a general class of multi-parameter anisotropic Gaussian random fields within a power scale (see also [18]). We first establish a two-point local nondeterminism property, which serves as a lower bound for the conditional variance of the random variable \( B(t) \) given \( B(s) \) (see Lemma 2.4). Then for the proof of (1.5) we use a second-moment argument, which is the classical way to prove lower capacity bounds for hitting probabilities. But unlike in [4], the proof of our results and the computation of the capacity, only use the function \( \gamma \) in assumptions (1.1) and (1.2), not an identified Hölder exponent or index. This is how we are able to provide a capacity kernel in (1.6) that is intrinsic to \( \gamma \), not index-dependent.

Next, in preparation for proving upper Hausdorff measure bounds for the hitting probabilities in Section 4, we establish results in Section 3 for hitting probabilities in dimension 1; these new results are interesting in their own right. The main result in Section 3 (see Proposition 3.1 and Corollary 3.2) is the following upper estimate in terms of the function \( \gamma \) for the probability that each one-dimensional component of our process \( B \), also denoted by \( B \) for economy of notation as mentioned above, hits points in dimension 1: if there exists \( k, y_0 > 0 \) such that

\[
\int_0^{1/2} \gamma(xy) \left( \frac{dy}{y \sqrt{\log(1/y)}} \right) \leq k \gamma(x), \tag{1.7}
\]
for all \( x \in [0, y_0] \), then for some constant \( L \) depending only on \( \gamma \) and \( y_0 \), for all \( z \in \mathbb{R} \) and for all \( a, b \) such that \( 0 < a < b \) and \( b - a \leq t_0 \),
\[
P(B([a, b]) \ni z) \leq \frac{L\sqrt{t}}{\gamma(a)^2} \gamma(b - a).
\]

Condition (1.7) is satisfied for all classes of examples described in the first part of this introduction, and many other non-stationary non-self-similar processes, some of which are the subject of Section 6. We also show (see Lemma 3.3) that any \( \gamma \) with positive index also satisfies Condition (1.7).

The bound (1.8) is then the basis for our Hausdorff measure upper bound in Section 4. In that section, we first establish, in Proposition 4.1, an upper bound for the probability that \( B \) hits a small ball \( B(z, \varepsilon) \) of radius \( \varepsilon > 0 \) and center \( z \in \mathbb{R}^d \): under assumption (1.7), for all \( 0 < a < b < \infty \), with \( b - a \) small enough,
\[
P(B([a, b]) \cap B(z, \varepsilon) \neq \emptyset) \leq \left( \varepsilon \frac{2\kappa \gamma(b)}{\gamma^2(a)} \left( 1 + \frac{1}{F(|z|)} \right) + \frac{L\sqrt{t}}{\gamma(a)} \gamma(b - a) \right)^d. \tag{1.9}
\]

Here, we defined \( \kappa := P(\inf_{[a, b]} B > \gamma(b)) \), \( F(z) := 1 \) for \( z \leq \gamma(b) \) and \( F(z) := 1 - e^{-2\text{arctanh}(\gamma(b)/z^2)} \) for \( z > \gamma(b) \); note that \( \kappa > 0 \) because \( B \) is Gaussian and \( a > 0 \), and also note that, since \( \text{arctanh}(x) \) is equivalent to \( x \) for \( x \) small, for large \( z \), \( 1/F(z) \) is equivalent to \( z^2/(2\gamma^2(b)) \).

We reverting to the case \( d = 1 \) without loss of generality to prove (1.9), still denoting the scalar process by \( B \). Note that the event \( \{B([a, b]) \cap B(z, \varepsilon) \neq \emptyset\} \) is equal to \( \{Z \leq \varepsilon\} \) where the random variable \( Z \) is defined as \( Z := \inf_{s \in [a, b]} |B(s) - z| \). In order to derive (1.9), it is important to note that since the scalar \( B \) hits points with positive probability, the r.v. \( Z \) has an atom at 0. Thus we have
\[
P \{B([a, b]) \ni B(z, \varepsilon) \neq \emptyset\} = P \{0 < Z \leq \varepsilon\} + P \{B([a, b]) \ni z\},
\]
where, indeed, \( \{B([a, b]) \ni z\} = \{Z = 0\} \) is the atom.

Since the probability \( P \{B([a, b]) \ni z\} \) can be estimated using (1.8), in order to obtain (1.9), it is sufficient to establish that
\[
P(0 < Z \leq \varepsilon) \leq C\varepsilon,
\]
for some constant \( C \). To this aim, it would be sufficient to show that \( Z \) has a bounded density on \((0, +\infty)\). For this, we will use the criterion established in [14] Theorem 3.1, that provides a quantitative estimate of the density of a random variable which is differentiable in the Malliavin sense, using techniques of the Malliavin calculus that will be recalled in Section 4. Strictly speaking, this criterion does not apply to random variables with atoms, but in our case, as it turns out, \( Z \) has a single atom, at the edge of its support, and we are able to adapt the method of [14] to our situation (see Proposition 4.1).

Finally, using a covering argument similarly as in [6] Theorem 3.1, we ultimately show in Theorem 4.6 that if (1.7) is satisfied and the function
\[
\varphi(s) := s^d/\gamma^{-1}(s) \tag{1.10}
\]
is right-continuous and non-decreasing near 0 with \( \lim_{s \uparrow \infty} \varphi = 0 \), then for all \( 0 < a < b < \infty \) and \( M > 0 \), there exists a constant \( C > 0 \) depending only on \( a, b \), the law of \( B \), and \( M \), such that for any Borel set \( A \subset [-M, M]^d \),
\[
P(B([a, b]) \cap A \neq \emptyset) \leq C\mathcal{H}_\varphi(A). \tag{1.11}
\]
Recall that in the case of fBm, the upper Hausdorff measure bound (1.11) applies when 
\( d > 1/H \) with \( \varphi(r) = r^{d-1/H} \), and the capacity lower bound (1.10) uses the Newtonian kernel \( K = K_{d-1/H} \) defined in Section 2, which is equal to \( 1/\varphi(r) = r^{1/H-d} \) for \( d > 1/H \), meaning that \( K = 1/\varphi \) in the fBm case. To understand this phenomenon further, at the end of Section 4, we study whether our potential kernel \( K \) given in (1.6) is commensurate to \( 1/\varphi \) near 0, with \( \varphi \) as in (1.10). For this, we assume that the assumptions that lead to (1.3) and (1.11) hold true. That is, \( \gamma \) is concave near zero, \( \varphi \) is right-continuous and non-decreasing near 0 with \( \lim_{0+} \varphi = 0 \), and assumption (1.7) holds true. First observe that the condition on \( \varphi \) implies that \( v \) in (1.6) is unbounded. Then we show in Proposition 4.7 that if \( \lim_{r \to 0} \frac{\varphi(r)}{r^\gamma(r)} \) exists, the functions \( K \) and \( 1/\varphi \) are commensurate if and only if

\[
d > \lim_{r \to 0} \frac{\gamma(r)}{r^\gamma(r)}.
\]

**Section 5** is devoted specifically to questions on hitting points. We first show that if \( 1/\gamma^d \) is integrable around zero, our process \( B \) in \( \mathbb{R}^d \) hits points with positive probability (see Theorem 5.3), using a classical argument with local times. Secondly, as an application of our upper bound (1.11), we show in Theorem 5.4 that if \( \gamma \) is such that \( \gamma(r) = o(r^{1/d}) \) near 0, \( \varphi(s) = s^d/\gamma^{-1}(s) \) is non-decreasing near 0, and (1.7) holds, then almost surely, \( B \) does not hit points. In the Appendix, we will apply a classical method based on the modulus of continuity to our class of processes \( B \) used in [8] for the Brownian sheet case. This technique provides a sufficient condition for \( B \) not to hit points in \( \mathbb{R}^d \) that is stronger than that of Theorem 5.3. In other words, Theorem 5.4 implies a sharper result than the classical method; we have included the result via the classical method in the Appendix to illustrate this point only.

We end this paper with **Section 6**, where we apply our results of Sections 2-5 to the family of Gaussian processes satisfying (1.1) and (1.2) with, for every \( r \) in a closed interval in \( [0,1) \),

\[
\gamma(r) = \gamma_{H,\beta}(r) := r^H \log^{\beta}(\frac{1}{r}),
\]

for some \( \beta \in \mathbb{R} \), \( H \in (0,1) \), or \( H = 1, \beta > 0 \), or \( H = 0, \beta < -1/2 \). Specifically, we prove in Theorem 6.2 and Corollary 6.4 that, in this case, the bounds (1.3) and (1.11) are as follows:

- If \( d > 1/H \), for all 0 < \( a < b < 1 \) and \( M > 0 \), there exist constants \( C_1, C_2 > 0 \) such that for any Borel set \( A \subset [-M, M]^d \),

\[
C_1 \mathcal{C}_{1/\varphi}(A) \leq P(B([a,b]) \cap A \neq \emptyset) \leq C_2 \mathcal{H}_\varphi(A),
\]

where \( \varphi(x) = x^{d-1/H} \log^{\beta/H}(1/x) \).
- If \( d = 1/H \) and \( \beta < 0 \), the upper bound still holds, and \( \varphi(x) = \log^{\beta/H}(1/x) \).
- If \( d = 1/H \), for \( \beta < 1/d \), the lower bound holds with \( \varphi(x) = \log^{\beta/H-1}(1/x) \).
- If \( d = 1/H \) and \( \beta \geq 1/d \), or if \( d < 1/H \), the lower bound holds with \( \varphi \equiv 1 \).
- If \( H = 1 \) and \( \beta > 0 \), for all \( d > 1 \), both bounds hold, and \( \varphi(x) = x^{d-1} \log^{\beta}(1/x) \).
- If \( H = 0 \) and \( \beta < -1/2 \), the lower bound holds with \( \varphi = 1 \).

This set of results is particularly revealing in the so-called “critical” case, where \( H = 1/d \). Unlike in the Hölder scale, where this critical case is represented only by fBm or similar processes, here we have an entire scale of processes as \( \beta \) ranges in all of \( \mathbb{R} \), the fBm corresponding to \( \beta = 0 \). Our results imply that if \( \beta < 0 \) (processes that are more regular than critical fBm, while being indistinguishable from it in the Hölder
scale) then a.s. \( B \) does not hit points, while if \( \beta \geq 1/d \) (processes that are less regular than fBm, while also being indistinguishable from it in the Hölder scale), then \( B \) hits points with positive probability. The gap corresponding to the range \( \beta \in [0,1/d) \), which is also visible in the case \( \beta < 0 \) via the discrepancy of order \( \log (1/x) \) between the two expressions for \( \varphi \) in the upper and lower bounds, indicates that there is presumably a slight inefficiency in at least one of the two estimation methods for hitting probability (Hausdorff measure and capacity). This thought is not new; what is more important here is that our theorems give a precise quantification of the inefficiency: it is of no more than a log order, which, in practice, could be considered difficult to detect. Arguably, this provides support for continuing to study both estimation methods in the future. We finish Section 6 with applications to estimating probabilities of hitting Cantor sets for various combinations of parameters when \( \gamma = \gamma_{H,\beta} \).

2. LOWER CAPACITY BOUND FOR THE HITTING PROBABILITIES

Recall that \( B = (B(t), t \in \mathbb{R}_+) \) is a centered continuous Gaussian process in \( \mathbb{R}^d \) with variance \( \gamma^2(t) \) for some continuous strictly increasing function \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( \lim_0 \gamma = 0 \), such that for some constant \( \ell \geq 1 \) and for all \( s,t \geq 0 \),

\[
(1/\ell)\gamma^2(|t-s|) \leq E[|B(t) - B(s)|^2] \leq \ell\gamma^2(|t-s|).
\]

The aim of this section is to obtain a lower bound for the probability that the process \( B \) hits a set \( A \) in terms of the capacity of \( A \), a concept from potential theory. In this context, it is customary to say that any measurable function \( F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (0,\infty] \) can serve as a so-called potential kernel. In this article, we will focus on the case where \( F(x,y) = K(|x-y|) \), and \( K \) is a positive, non-increasing, continuous function in \( \mathbb{R}_+ \setminus 0 \) with \( \lim_0 K \leq +\infty \).

The capacity of a Borel set \( A \subset \mathbb{R}^d \) with respect to a potential kernel \( K \) is defined as

\[
C_K(A) := \left[ \inf_{\mu \in \mathcal{P}(A)} E_K(\mu) \right]^{-1},
\]

where \( \mathcal{P}(A) \) denotes the set of probability measures with support in \( A \), and \( E_K(\mu) \) denotes the energy of a measure \( \mu \in \mathcal{P}(A) \) with respect to the kernel \( K \), which is defined as

\[
E_K(\mu) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} K(|x-y|) \mu(dx) \mu(dy).
\]

By convention for computing \( C_K(A) \), we let \( 1/\infty := 0 \). The Newtonian \( \beta \)-kernel is the potential kernel defined as

\[
K_\beta(r) := \begin{cases} 
  r^{-\beta} & \text{if } \beta > 0, \\
  \log \left( \frac{r}{\beta} \right) & \text{if } \beta = 0, \\
  1 & \text{if } \beta < 0.
\end{cases}
\]

The \( \beta \)-capacity of a Borel set \( A \subset \mathbb{R}^d \), denoted \( C_\beta(A) \), and the \( \beta \)-energy of a measure \( \mu \in \mathcal{P}(A) \), denoted \( E_\beta(\mu) \), are the capacity and the energy with respect to the Newtonian \( \beta \)-kernel \( K_\beta \).

Let us consider the following additional hypotheses on \( \gamma \).

**Hypothesis 2.1.** Recall the constant \( \ell \) in \( (1.1) \). The increasing function \( \gamma \) in \( (1.1) \) is concave in a neighborhood of the origin, and for all \( 0 < a < b < \infty \), there exists \( \varepsilon > 0 \) such that \( \gamma'(\varepsilon+) > \sqrt{b} \gamma'(a-) \).
Hypothesis 2.2. Recall the constant $\ell$ in (1.1). For all $0 < a < b < \infty$, there exists $\varepsilon > 0$ and $c_0 \in (0, 1/\sqrt{\ell})$, such that for all $s, t \in [a, b]$ with $0 < t - s \leq \varepsilon$,

$$\gamma(t) - \gamma(s) \leq c_0 \gamma(t - s).$$

Since a concave function has a derivative almost everywhere with finite left and right limits for this derivative everywhere, the strict inequality in Hypothesis 2.1 is simply saying that $\gamma$ is strictly concave near the origin. In all the examples that have been mentioned up to now, and the ones which we consider later in this article (see Section 6, in particular), we have $\gamma'(0+) = +\infty$, from which Hypothesis 2.1 reduces to simply requiring concavity near the origin. That concavity is satisfied in all examples we look at.

To construct a Gaussian process which fails to satisfy Hypothesis 2.1 one has to resort to processes which are somewhat pathological, and whose smoothness are such that hitting probabilities become trivial. For instance, the fBm $B^1$ with parameter $H = 1$ satisfies $\mathbb{E} \left[ (B^1(t) - B^1(s))^2 \right] = |t - s|^2$ which implies that there exists a standard normal rv $Z$ such that $B^1(t) = tZ$, and we have $\gamma(r) = r$ in (1.1).

Hypothesis 2.2 implies a lower bound for the conditional variance of the random variable $B(t)$ given $B(s)$, provided in Lemma 2.4 below. This property of the conditional variance can be referred as a two-point local nondeterminism (see [4]). We first prove that Hypothesis 2.1 implies Hypothesis 2.2 and that under the stronger assumption (satisfied in all our examples) that $\gamma'(0+) = +\infty$, the constant $c_0$ in Hypothesis 2.2 can be chosen arbitrarily small.

Lemma 2.3. Hypothesis 2.1 implies Hypothesis 2.2. If moreover $\gamma'(0+) = +\infty$, then for all $0 < a < b < \infty$, and all $c_0 > 0$, there exists $\varepsilon > 0$ such that for all $s, t \in [a, b]$ with $0 < t - s \leq \varepsilon$, $\gamma(t) - \gamma(s) \leq c_0 \gamma(t - s)$.

Proof. By concavity of $\gamma$, and the fact that $\gamma(0) = 0$, for any $s, t \geq a$ such that $0 < t - s \leq \varepsilon$

$$\frac{\gamma(t) - \gamma(s)}{t - s} \leq \gamma'(a-)$$

which proves the conclusion of Hypothesis 2.2 by taking $c_0 := \gamma'(a-) / \gamma'(\varepsilon+)$ which is strictly less than $1/\sqrt{\ell}$ by Hypothesis 2.1. The second statement of the lemma follows using the same proof by noting that for any fixed $a > 0$, one can make $\gamma'(a-) / \gamma'(\varepsilon+)$ as small as desired by choosing $\varepsilon$ small enough. $\square$

Lemma 2.4. Assume Hypothesis 2.2. Then for all $0 < a < b < \infty$, there exists $\varepsilon > 0$ and a positive constant $c(a, b)$ depending only on $a, b$ and the law of $B$, such that for all $i \in \{1, \ldots, d\}$ and all $s, t \in [a, b]$ with $|t - s| \leq \varepsilon$,

$$\text{Var}(B_i(t)|B_i(s)) \geq c(a, b) \gamma^2(|t - s|).$$

More specifically, assume simply that $\gamma$ is concave in a neighborhood of the origin and $\gamma'(0+) = +\infty$; then the above conclusion holds for any $c(a, b) < \gamma^4(a) / (2 \ell \gamma^4(b))$, for some $\varepsilon > 0$ small enough.

Proof. Recall that

$$\text{Var}(B_i(t)|B_i(s)) = \gamma^2(t)(1 - \rho^2(s, t)),$$

where $\rho(s, t)$ denotes the correlation coefficient between $B_i(s)$ and $B_i(t)$, that is,

$$\rho(s, t) := \frac{\sigma(s, t)}{\gamma(s)\gamma(t)},$$

with parameter $H = 1$ satisfies $\mathbb{E} \left[ (B^1(t) - B^1(s))^2 \right] = |t - s|^2$ which implies that there exists a standard normal rv $Z$ such that $B^1(t) = tZ$, and we have $\gamma(r) = r$ in (1.1).
and \( \sigma(s,t) \) denotes the covariance of \( B_i(s) \) and \( B_i(t) \), that is,

\[
\sigma(s,t) := E[B_i(s)B_i(t)].
\]

None of these functions depend on \( i \).

Hence, as \( \gamma \) is increasing, \( \gamma^2(t) \geq \gamma^2(a) \) and it suffices to find a lower bound for \( 1 - \rho^2(s,t) \). Using the lower bound in (1.1), we have that

\[
1 - \rho(s,t) = \frac{\gamma(s)\gamma(t) - \frac{1}{2}(\gamma^2(t) + \gamma^2(s) - \delta^2(s,t))}{\gamma(s)\gamma(t)}
\]

\[
= \frac{\delta^2(s,t) - (\gamma(t) - \gamma(s))^2}{2\gamma(s)\gamma(t)}
\]

\[
\geq \frac{(1/\ell)\gamma^2(|t-s|) - (\gamma(t) - \gamma(s))^2}{2\gamma(s)\gamma(t)}
\]

where \( \delta^2(s,t) := E[(B_i(t) - B_i(s))^2] \).

Next, appealing to Hypothesis 2.2 and the fact that \( \gamma \) is increasing, we get that for all \( s,t \in [a,b] \) with \( |t-s| \leq \varepsilon \),

\[
1 - \rho(s,t) \geq \frac{\frac{1}{2} - \frac{c_0^2}{\gamma^2(s)}}{2\gamma(s)\gamma(t)} \gamma^2(|t-s|)
\]

\[
\geq \frac{1}{2\gamma^2(b)} \gamma^2(|t-s|).
\]

On the other hand, by (1.1) and the fact that \( \gamma \) is increasing, for all \( s,t \in [a,b] \) with \( |t-s| \leq \varepsilon \),

\[
1 + \rho(s,t) = \frac{(\gamma(t) + \gamma(s))^2 - \delta^2(s,t)}{2\gamma(s)\gamma(t)}
\]

\[
\geq \frac{2\gamma^2(a)}{\gamma^2(b)} - \frac{\ell}{2\gamma^2(a)} \gamma^2(\varepsilon).
\]

Since \( \lim_0 \gamma = 0 \), we can choose \( \varepsilon \) sufficiently small such that the last displayed line above is bounded below by \( \gamma^2(a) / \gamma^2(b) \). Therefore, for such \( \varepsilon \), and all \( |t-s| < \varepsilon \), \( s,t \in [a,b] \), we have

\[
1 - \rho^2(s,t) = (1 + \rho(s,t))(1 - \rho(s,t)) \geq \frac{\gamma^2(a)}{\gamma^2(b)} \frac{1 - \varepsilon^2}{2\gamma^2(b)} \gamma^2(|t-s|),
\]

which concludes the proof of the lemma’s first statement. The lemma’s second statement also follows because, by Lemma 2.3 and the discussion following the introduction of Hypotheses 2.1 and 2.2, we can choose \( c_0 \) above arbitrarily small. \( \square \)

We next obtain a lower bound for the hitting probabilities of \( B \) in terms of capacity. Our proof employs a strategy developed in [1] Theorem 2.1 where the authors obtain a lower bound in terms of capacity, for the hitting probabilities of a general class of multi-parameter anisotropic Gaussian random fields within a power scale (see also [13]). Unlike in [1] Theorem 2.1 our result is not formulated using – and our proofs are not based on – proximity in law to a process with self-similar increments (the power scale), or on related concepts such as Hölder continuity or function index. The capacity kernel identified in the next result is intrinsic to the law of the Gaussian process \( B \), since it is computed using only the function \( \gamma \) in assumptions (1.1) and (1.2), not an exogenously identified Hölder exponent or index.
Theorem 2.5. Let \( K(x) := \max \{1; v \left( \gamma^{-1}(x) \right) \} \) where \( v(r) := \int_{-a}^{b-r} ds/\gamma^d(s) \). Assume Hypothesis 2.1 holds. Then for all \( 0 < a < b < \infty \) and \( M > 0 \), there exists a constant \( C > 0 \) depending only on \( a, b, M \) and the law of \( B \), such that for any Borel set \( A \subseteq [-M, M]^d \)

\[
CC_K(A) \leq P(B(\mathbb{[}a, b\mathbb{])} \cap A \neq \emptyset).
\]

Remark 2.6. Recall that Hypothesis 2.1 holds as soon as \( \gamma \) is concave near 0 with \( \gamma'(0+) = +\infty \).

Remark 2.7. When \( 1/\gamma^d \) is integrable at 0, \( K \) is bounded by \( K_\infty := \max \left\{1; \int_0^{b-a} 1/\gamma^d\right\} \), and therefore, replacing \( C \) by \( C/K_\infty \), we may replace \( C_K(A) \) by \( C_1(A) \) in Theorem 2.6. As \( C_1(A) = 1 \) for every non-empty set \( A \), Theorem 2.6 shows that if \( 1/\gamma^d \) is integrable at 0, then the process \( B \) hits points with positive probability, so we recover the statement of Theorem 2.3 below, but with the additional Hypothesis 2.1. Theorem 2.6’s statement is thus an improvement on Theorem 2.3 for non-singleton sets only.

Proof of Theorem 2.6. \( \text{Step 1: Setup and general strategy.} \)

Note that \( K \), as defined in the statement of the theorem, is a bonafide univariate potential kernel since it is non-increasing and continuous on \( \mathbb{R}_+ \) with \( \lim_0 K = K_\infty = \max \left\{1; \int_0^{b-a} 1/\gamma^d\right\} \leq +\infty \). Assume that \( C_K(A) > 0 \) otherwise there is nothing to prove. This implies the existence of a probability measure \( \mu \in \mathcal{P}(A) \) such that

\[
\mathcal{E}_K(\mu) \leq \frac{2}{C_K(A)} \tag{2.2}
\]

Consider the sequence of random measures \( (\nu_n)_{n \geq 1} \) on \( [a, b] \) defined as

\[
\nu_n(dt) = \int_{\mathbb{R}^d} (2\pi n)^{d/2} \exp \left( -\frac{n|B(t) - x|^2}{2} \right) \mu(dx) dt.
\]

By the Fourier inversion theorem

\[
\nu_n(dt) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp \left( -\frac{||\xi||^2}{2n} + i \langle \xi, B(t) - x \rangle \right) d\xi \mu(dx) dt.
\]

Denote the total mass of \( \nu_n \) by \( |\nu_n| := \nu_n([a, b]) \). We claim that

\[
E(|\nu_n|) \geq c_1, \quad \text{and} \quad E(|\nu_n|^2) \leq c_2 \mathcal{E}_K(\mu), \tag{2.3}
\]

where the constants \( c_1 \) and \( c_2 \) are independent of \( n \) and \( \mu \).

We first have

\[
E(|\nu_n|) = \int_a^b \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp \left( -\frac{||\xi||^2}{2} \left( \frac{1}{n} + \gamma^2(t) \right) - i \langle \xi, x \rangle \right) d\xi \mu(dx) dt
\]

\[
\geq \int_a^b \int_{\mathbb{R}^d} \left( \frac{2\pi)^{d/2}}{(1 + \gamma^2(t))^{d/2}} \right) \exp \left( -\frac{|x|^2}{2\gamma^2(t)} \right) \mu(dx) dt
\]

\[
\geq \int_a^b \left( \frac{2\pi)^{d/2}}{(1 + \gamma^2(b))^{d/2}} \exp \left( -\frac{dM^2}{2\gamma^2(a)} \right) \right) dt =: c_1,
\]

where the second inequality follows because \( A \subseteq [-M, M]^d \). This proves the first inequality in (2.3).
We next prove the second inequality in (2.3). We have that
\[
E(|\nu_n|^2) = \int_a^b \int_a^b \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} e^{-i(\xi,\eta)} e^{-\frac{i}{2} (\xi,\eta) \Gamma_n(s,t)(\xi,\eta)^T} d\xi d\eta \mu(dx) \mu(dy) ds dt,
\]
where \(\Gamma_n(s,t)\) denotes the \(2d \times 2d\) matrix
\[
\Gamma_n(s,t) = n^{-1} I_{2d} + \text{Cov}(B_s(B_t, B(t)), I_{2d} \text{ denotes the } 2d \times 2d \text{ identity matrix, and Cov}(B_s(B_t, B(t)) \text{ denotes the } 2d \times 2d \text{ covariance matrix of } (B(s), B(t))).
\]

Observe that \(B(t) = (B_1(t), ..., B_d(t))\) where the \(B_i(t)\)'s are the \(d\) independent coordinate processes of \(B(t)\). Hence,
\[
E(|\nu_n|^2) = \int_a^b \int_a^b \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} e^{-i \sum_{j=1}^d (\xi_j x_j + \eta_j y_j)}
\]
\[
\times \exp \left( -\frac{1}{2} \sum_{j=1}^d \left( (\gamma^2(s) + \frac{1}{n}) x_j^2 + 2\sigma(s,t) x_j y_j + (\gamma^2(t) + \frac{1}{n}) y_j^2 \right) \right) d\xi d\eta \mu(dx) \mu(dy) ds dt
\]
\[
= \int_a^b \int_a^b \prod_{j=1}^d \left( \int_{\mathbb{R}^2} e^{-i (\xi_j x_j + \eta_j y_j)} \times \exp \left( -\frac{1}{2} \left( (\gamma^2(s) + \frac{1}{n}) x_j^2 + 2\sigma(s,t) x_j y_j + (\gamma^2(t) + \frac{1}{n}) y_j^2 \right) \right) d\xi \right) \mu(dx) \mu(dy) ds dt,
\]
where recall that \(\sigma(s,t)\) denotes the covariance of \(B_j(s)\) and \(B_j(t)\) which does not depend on \(j\).

Next observe that integral inside the product in (2.4) is equal to
\[
\int_{\mathbb{R}^2} e^{-i (\xi x_j + \eta y_j)} \exp \left( -\frac{1}{2} (\xi_j, \eta_j) \Phi_n(s,t)(\xi_j, \eta_j)^T \right) d\xi d\eta
\]
where \(\Phi_n(s,t)\) denotes the \(2 \times 2\) matrix
\[
\Phi_n(s,t) = n^{-1} I_2 + \text{Cov}(B_j(s), B_j(t)),
\]
which is independent of \(j\).

Since \(\Phi_n(s,t)\) is positive definite, we have
\[
\int_{\mathbb{R}^2} e^{-i (\xi x_j + \eta y_j)} \exp \left( -\frac{1}{2} (\xi_j, \eta_j) \Phi_n(s,t)(\xi_j, \eta_j)^T \right) d\xi d\eta
\]
\[
= \frac{2\pi}{\sqrt{\det(\Phi_n(s,t))}} \exp \left( -\frac{1}{2} (x_j, y_j) \Phi_n^{-1}(s,t)(x_j, y_j)^T \right).
\]

**Step 2: using the method of [4] near the diagonal.** We now follow the proof of [4, Lemma 11], in order to show that for all \(s, t \in [a, b]\) with \(|t-s| \leq \varepsilon\)
\[
(x_j, y_j) \Phi_n^{-1}(s,t)(x_j, y_j)^T \geq c_3 \frac{(x_j - y_j)^2}{\det(\Phi_n(s,t))},
\]
for some constant \(c_3 > 0\) depending only on \((a, b)\), and \(\varepsilon > 0\) as in Lemma 2.4.
First remark that
\[(x_j, y_j) \Phi^{-1}_n(s, t)(x_j, y_j)^T \geq \frac{1}{\det(\Phi_n(s, t))} E\{(x_j B_j(t) - y_j B_j(s))^2\}.
\]

Thus, in order to show (2.7), it suffices to show that
\[E\{(x_j B_j(t) - y_j B_j(s))^2\} \geq c_3 (x_j - y_j)^2.\] (2.7)

Next observe that in order to prove (2.6), it suffices to prove that
\[E((B_j(t))^2) \geq c_4,\] (2.8)

and that
\[\frac{\det(\text{Cov}(B_j(s), B_j(t)))}{E((B_j(s) - B_j(t))^2)} \geq c_5,\] (2.9)

for some constants \(c_4, c_5 > 0\) only depending on \((a, b)\).

Observe that (2.8) holds by (1.2) for all \(t \in [a, b]\) as \(\gamma^2\) is increasing. On the other hand, by (1.1), the denominator in (2.9) is bounded by \(t \gamma^2(\gamma - s)\). Moreover, by Hypothesis 2.1 and Lemmas 2.3 and 2.4, the numerator in (2.9) satisfies that for all \(s, t \in [a, b]\) with \(|t - s| \leq \epsilon\)
\[\det(\text{Cov}(B_j(s), B_j(t))) = \gamma^2(s) \text{Var}(B_j(t)|B_j(s)) \geq c_6 \gamma^2(\gamma - s),\]

for some \(c_6 > 0\) depending only on \((a, b)\). Therefore, (2.9) holds for all \(s, t \in [a, b]\) with \(|t - s| \leq \epsilon\). Thus, (2.6) holds true.

Now, replacing (2.6) into (2.5), returning to the computation in (2.4), and appealing to Fubini’s theorem, we get that
\[E(|\nu_n|^2) \leq I_1 + I_2,\]

where
\[I_1 := \int_{\mathbb{R}^d} \int_{D(\epsilon)} \frac{(2\pi)^d}{\sqrt{\det(\Phi_n(s, t))}} \exp\left(-\frac{c_3}{2} \frac{|x - y|^2}{\det(\Phi_n(s, t))}\right) dsdt \mu(dx)\mu(dy),\]
\[I_2 := \int_{\mathbb{R}^d} \int_{[a, b]^2 \setminus D(\epsilon)} \frac{(2\pi)^d}{\sqrt{\det(\Phi_n(s, t))}} dsdt \mu(dx)\mu(dy),\]

and \(D(\epsilon) = \{(s, t) \in [a, b]^2 : |t - s| \leq \epsilon\}\).

Step 3: Bounding the off-diagonal contribution to \(E(|\nu_n|^2)\).

We start by bounding \(I_2\). Observe that
\[\det(\Phi_n(s, t)) = \gamma^2(s) \gamma^2(t) - \sigma^2(s, t) + \frac{\gamma^2(s)}{n} + \frac{\gamma^2(t)}{n} + \frac{1}{n^2} \geq \gamma^2(s) \gamma^2(t) - \sigma^2(s, t).\]

By the Cauchy-Schwartz inequality, the function \((s, t) \mapsto \gamma^2(s) \gamma^2(t) - \sigma^2(s, t)\) is non-negative, and since \(\gamma(r) = 0 \iff r = 0\), this function is strictly positive and continuous away from the diagonal \(\{s = t\}\). Therefore, for all \(s, t \in [a, b]\) with \(|t - s| > \epsilon\),
\[\det(\Phi_n(s, t)) \geq c_7,\]

for some constant \(0 < c_7(a, b)\). Hence, we get that
\[\int_{[a, b]^2 \setminus D(\epsilon)} \frac{(2\pi)^d}{\sqrt{\det(\Phi_n(s, t))}} dsdt \leq c_8,\]
where the constant $c_8$ only depends on $(a,b)$. Therefore from the definition of $I_2$ in step 2, and since $K$ is non-increasing, we get

$$I_2 \leq c_8 \int_{\mathbb{R}^d} \mu(dx)\mu(dy) = c_8 \int_{\mathbb{R}^d} \mu(dx)\mu(dy) \frac{K(|x-y|)}{K(|x-y|)} \leq \frac{c_8}{K(2M)} E_K(\mu) \leq c_8 E_K(\mu).$$

This proves the part of claim (2.3) corresponding to $I_2$, i.e. $I_2 \leq \text{const} \cdot E_K(\mu)$.

Step 4: Bounding the diagonal contribution to $E(|\nu_n|^2)$.

We next treat $I_1$. Observe that if $\det(\Phi_n(s,t)) < |x-y|^2$, using the inequality $x^{d/2}e^{-ct} \leq c'$ valid for all $x > 0$, it yields that

$$\frac{(2\pi)^d}{(\sqrt{\det(\Phi_n(s,t))})^d} \exp\left(-\frac{c_3}{2} \frac{|x-y|^2}{\det(\Phi_n(s,t))}\right) \leq \frac{c_9}{|x-y|^d}.$$

Therefore, by Lemma 2.3 we conclude that

$$\int \int_{D(\varepsilon)} \left(\frac{(2\pi)^d}{(\sqrt{\det(\Phi_n(s,t))})^d}\right)^d \exp\left(-\frac{c_3}{2} \frac{|x-y|^2}{\det(\Phi_n(s,t))}\right) dsdt \leq c_{10} \int_a^b \int_a^b \frac{1}{\max(\gamma^d(|t-s|), |x-y|^d)} dsdt.$$

We next break the last integral into the regions $\{(s,t) \in [a,b]^2 : \gamma(|t-s|) \leq |x-y|\}$ and $\{(s,t) \in [a,b]^2 : \gamma(|t-s|) > |x-y|\}$ and denote them by $J_1$ and $J_2$, respectively. We first have that

$$J_1 = \int_a^b \int_{\{t \in [a,b] : \gamma(|t-s|) \leq |x-y|\}} \frac{1}{|x-y|^d} dsdt \leq c_{11} |x-y|^{-d} \gamma^{-1}(|x-y|), \quad (2.10)$$

where in the last inequality we have used the fact that $\gamma^{-1}$ is strictly increasing.

On the other hand, using the change of variable $u = t - s$, we have that

$$J_2 = \int_a^b \int_{\{u \in [0,b-a] : \gamma(u) > |x-y|\}} \frac{1}{\gamma^d(u)} du \leq c_{12} \int_{\{u \in [0,b-a] : \gamma(u) > |x-y|\}} \frac{1}{\gamma^d(u)} du \quad (2.11)$$

where we again used the fact that $\gamma^{-1}$ is strictly increasing. Integrating the last expression above against $\mu(dx)\mu(dy)$ over $D(\varepsilon)$ yields the upper bound $c_{12} \cdot E_K(\mu)$, which is what is required for the corresponding portion of the proof that $I_1 \leq \text{const} \cdot E_K(\mu)$.

To finish the proof that $I_1 \leq \text{const} \cdot E_K(\mu)$, it is sufficient to show that

$$J_1/v \left(\gamma^{-1}(|x-y|)\right)$$

is bounded. To prove this, given the upper bound in (2.10), and reverting to $r = \gamma^{-1}(|x-y|)$ as our variable, it is sufficient to prove that the function defined on $(0, b-a]$ by

$$f(r) := \frac{r/\gamma^d(r)}{v(r)}.$$

is bounded. Since the functions in this ratio are all continuous, it is sufficient to prove that $\lim_{0+} f < +\infty$. For this, let us consider two different cases depending on whether
$v$ is bounded or not. First, when $v$ is bounded, as $1/\gamma^d$ is integrable at 0 and $1/r$ is not, we have that $\lim_{0+} 1/\gamma^d(r)/1/r < +\infty$, and hence $\lim_{0+} f < +\infty$. We now assume that $v$ is unbounded. We may then invoke the following elementary one-sided extension of l'Hôpital's rule for functions $g, h$ whose derivatives have left and right limits everywhere: if $\lim_{0+} h = +\infty$ and $\limsup_{0+} \max\{g'(r-); g'(r+)\}/\min\{h'(r+); h'(r-)\} < +\infty$, then $\limsup_{0+} g(r)/h(r) < +\infty$. With the corresponding $g$ and $h$ from the definition of $f$ above, we find

$$
\frac{\max\{g'(r-); g'(r+)\}}{\min\{h'(r+); h'(r-)\}} = \frac{g'(r-)}{h'(r)} = \frac{-d\gamma^{-d-1}(r) \gamma'(r-) r + 1/\gamma^d(r)}{-1/\gamma^d(r)} = -1 + d\gamma'(r-)/\gamma(r).
$$

By our concavity assumption in Hypothesis 2.4, since $\gamma(0) = 0$, we get that for every $r$ close enough to 0, $\gamma(r)/r \geq \gamma'(r-)$. This proves that the last expression above is bounded above by $-1 + d$. This finishes the proof that $\lim_{0+} f < +\infty$, and therefore that $I_1 \leq \text{const} \cdot E_K(\mu)$.

Step 5: Conclusion.

By Step 2, and the final estimates from Steps 3 and 4, the claim (2.3) is justified. Using (2.3) and the Paley-Zygmund inequality (cf. [7, p.8]), one can check that there exists a subsequence of the sequence $(\nu_n)_{n \geq 1}$ that converges weakly to a finite measure which is positive with positive probability, satisfies (2.3), and is supported on $[a, b] \cap (B)^{-1}(A)$. Therefore, using again the Paley-Zygmund inequality, we conclude that

$$
P(B([a, b]) \cap A \neq \emptyset) \geq P(|\nu| > 0) \geq \frac{E(|\nu|^2)}{E(|\nu|^2)} \geq \frac{c_1^2}{c_2 E_K(\mu)}.
$$

Finally, (2.2) finishes the proof of the Theorem. □

3. The Probability for a scalar Gaussian process to hit points

Recall that $B = (B(t), t \in \mathbb{R}_+)$ is a centered continuous Gaussian process in $\mathbb{R}^d$ with variance $\gamma^2(t)$ for some continuous strictly increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_0 \gamma = 0$, such that for some constant $\ell \geq 1$ and for all $s, t \geq 0$,

$$
(1/\ell)^2(|t - s|) \leq E[|B(t) - B(s)|^2] \leq \ell^2(|t - s|).
$$

In this section we find a sharp estimate of the probability for $B$ to hit points when $d = 1$. Since $B$ is almost surely continuous and Gaussian, we know that there is a positive probability to hit any point $z \in \mathbb{R}$ in any time interval $[a, b]$ with $0 < a < b$. We have the following quantitative estimate.

**Proposition 3.1.** There exists a universal positive constant $c_u$ and a positive constant $t_0$ depending only on the law of $B$, such that for all $z \in \mathbb{R}$, for all $a, b$ such that $0 < a < b$ and $b - a \leq t_0$,

$$
P(B([a, b]) \ni z) \leq \frac{c_u \sqrt{\ell}}{\gamma(a)} f_\gamma(b - a).
$$
where \( \ell \) is the constant in (1.1), and

\[
f_\gamma(x) := \gamma(x) \sqrt{\log 2} + \int_0^{1/2} \frac{\gamma(xy)}{y \sqrt{\log(1/y)}} \, dy.
\]

In any specific situation, the function \( f_\gamma \) can be computed more explicitly. In most cases, both terms in \( f_\gamma \) are commensurate. We state such a situation as follows.

**Corollary 3.2.** Assume there exists \( k, y_0 > 0 \) such that

\[
\int_0^{1/2} \frac{\gamma(xy)}{y \sqrt{\log(1/y)}} \, dy \leq k \gamma(x)
\]

for all \( x \in [0,y_0] \). Then, for some constant \( L \) depending only on \( \gamma \) and \( y_0 \), for all \( z \in \mathbb{R} \) and for all \( a,b \) such that \( 0 < a < b \) and \( b - a \leq t_0 \),

\[
P(B ([a,b]) \ni z) \leq \frac{L \sqrt{\ell}}{\gamma(a)} \gamma(b - a).
\]

The corollary’s assumption (3.1) is satisfied, and the corollary’s conclusion holds, for instance, for \( \gamma(r) = \gamma_{H,\beta} (r) := r^H \log^\beta (1/r) \) for some \( \beta \in \mathbb{R} \) and \( H \in (0,1) \), or \( \beta \leq 0 \) and \( H = 1 \). We will see what this implies in Section 6. In examples such as these, particularly when \( \beta = 0 \), i.e. \( \gamma(r) = r^H \), we can use the same method of proof to show that up to a multiplicative constant not dependent on \( b - a \), \( \gamma(b - a) \) is also a lower bound for \( P(B ([a,b]) \ni z) \), justifying our terminology of “sharp” mentioned above. Since this lower bound would not be used in this article, we omit further discussion of how to establish it.

The class of examples where \( \gamma = \gamma_{H,\beta} \) is a special case of continuous functions \( \gamma \) with a positive “index”: \( \text{ind} f := \inf \{ \alpha > 0 : f(x) = o(x^\alpha) \} \). Saying that a function \( f \) has a positive finite index thus simply means that it is negligible w.r.t. some power function but not all polynomials. The index of a function is a concept which was already used for estimating hitting probabilities in [1] in the context of \( \gamma_{H,0} \) (the case \( \beta = 0 \)) (see also [13]). More generally, the index of \( \gamma_{H,\beta} \) is equal to \( H \) no matter what \( \beta \) is. However, being able to compute the index of a function \( \gamma \) is not enough to be able to estimate its hitting probabilities precisely, as the last corollary above, or as the results in Section 6 show, since these results depend on the values of \( \beta \). Nevertheless, we record and sketch a proof of the reassuring fact that any function \( \gamma \) we might use with positive finite index satisfies condition (3.1).

**Lemma 3.3.** Define \( \text{ind} \gamma := \inf \{ \alpha > 0 : \gamma(x) = o(x^\alpha) \} \). Assume \( \gamma \) is continuous and increasing. Assume \( \text{ind} \gamma \in (0, \infty) \). Then \( \gamma \) satisfies condition (3.1).

**Proof.** Let \( \alpha = \text{ind} \gamma \). The lemma’s assumption implies that for any \( \varepsilon > 0 \), there exists a constant \( c \) such that if \( x \in [0,1/2] \), \( \gamma(x) \leq cx^{\alpha-\varepsilon} \). It also implies that there exists \( C > 0 \) and a sequence \( (x_n)_n \) decreasing to 0 such that \( \gamma(x_n) \geq C (x_n)^{\alpha+\varepsilon} \). To shorten this sketch of proof, we will assume that \( \gamma(x) \geq C x^{\alpha+\varepsilon} \) holds for all \( x \in [0,1/2] \); the details in the general case are left to the reader. We now only need to show that

\[
I := \int_0^{1/2} \frac{\gamma(xy)}{\gamma(x) \sqrt{\log(1/y)}} \, dy
\]
is bounded as \( x \) approaches the origin. For \( x < 1/2 \), with constants \( k \) which may change from line to line, using the bounds on \( \gamma \) and the fact that \( \gamma \) is increasing, we have

\[
0 \leq I = \int_0^x \frac{\gamma(xy)}{y \sqrt{\log(1/y)}} \, dy + \int_x^{1/2} \frac{\gamma(x)}{y \sqrt{\log(1/y)}} \, dy \leq kx^{-2\varepsilon} \int_0^x y^{\alpha - \varepsilon - 1} \, dy + k \int_x^{1/2} \frac{\gamma(x^2)}{y \sqrt{\log(1/y)}} \, dy \leq kx^{\alpha - 3\varepsilon} + kx^{\alpha - 3\varepsilon} \sqrt{\log(1/x)}.
\]

By choosing a value \( \varepsilon \in (0, \alpha/3) \), the result follows. \( \Box \)

For processes which are continuous but not H"older-continuous \( 3.1 \) may fail, as may the above corollary’s conclusion. For instance, this occurs with \( \gamma = \gamma_{0, \beta} \). To guarantee that \( B \) is continuous in that particular case, we need only assume \( \beta < -1/2 \). In this case, we get a \( \sqrt{\log} \) correction: for \( \gamma (r) = \log^3 (1/r) \),

\[
P(B([a, b]) \ni z) \leq \frac{2c_{\gamma} \sqrt{\ell}}{\gamma(a) \gamma(b-a)} \sqrt{\log \frac{1}{b-a}},
\]

as shown via a direct estimation of \( f_\gamma \) which we omit here.

**Proof of Proposition 3.1**

**Step 1: setup.** The event we must estimate is \( A = \{ B([a, b]) \ni z \} \). We clearly have

\[
A = \left\{ \min_{[a, b]} B \leq z \leq \max_{[a, b]} B \right\}.
\]

Now consider

\[
A_{a, z} := \left\{ B(a) \leq z \leq \max_{[a, b]} B \right\},
\]

\[
A_{b, z} := \left\{ B(b) \leq z \leq \max_{[a, b]} B \right\}.
\]

Since \( B(a) \geq \min_{[a, b]} B \), we have \( A_{a, z} \subset A \), and similarly \( A_{b, z} \subset A \). Let us prove that

\[
P(A) \leq P(A_{a, z}) + P(A_{b, z}) + P(A_{a, z}^c) + P(A_{b, z}^c).
\]

To lighten the notation, set \( m = \min_{[a, b]} B \), \( M = \max_{[a, b]} B \). Now consider the case \( B(a) < B(b) \), we get \( m \leq B(a) \leq B(b) \leq M \). Thus the interval \([m, M]\) is the union of the three intervals \( I = [m, B(a)] \), \( J = [B(a), B(b)] \), and \( K = [B(b), M] \). In the event \( A \cap \{ B(a) < B(b) \} \), if \( z \in I \), then \( z \in [m, B(b)] \); if \( z \in J \), then \( z \in [B(a), M] \), and finally if \( z \in K \), then \( z \in [B(a), M] \) also. This proves that

\[
[A \cap \{ B_a < B_b \}] \subset \{ [B(a) \leq z \leq M] \cup [m \leq z \leq B(b)] \}.
\]

By reversing the roles of \( B(a) \) and \( B(b) \) we get

\[
[A \cap \{ B_b < B_a \}] \subset \{ [B(b) \leq z \leq M] \cup [m \leq z \leq B(a)] \}.
\]
We immediately get

\[
P(A) \leq P(A \cap \{B_a < B_b\}) + P(A \cap \{B_b < B_a\}) \\
\leq P(\{B(a) \leq z \leq M\} \cup \{m \leq z \leq B(b)\}) \\
+ P(\{B(b) \leq z \leq M\} \cup \{m \leq z \leq B(a)\}) \\
\leq P(B(a) \leq z \leq M) + P(m \leq z \leq B(b)) \\
+ P(B(b) \leq z \leq M) + P(m \leq z \leq B(a)).
\]

The first and third terms in the last sum of four terms above are precisely \(P(A_{a,z})\) and \(P(A_{b,z})\). For the second term, we can write

\[
\{m \leq z \leq B(b)\} = \left\{ \max_{[a,b]} (-B) \geq -z \geq -B(b) \right\},
\]

and since \(B\) and \(-B\) have the same law, the second term is equal to \(P(A_{b,-z})\). Similarly the fourth term is equal to \(P(A_{a,-z})\), proving our claim (3.2).

**Step 2: Gaussian linear regression.** Here we appeal to the linear regression between \(B(a)\) and any value \(B(s)\) for \(s \in (a, b]\). Recall the notation

\[
\sigma^2(s, t) := E[B(s)B(t)],
\]

\[
\delta^2(s, t) := E[(B(s) - B(t))^2],
\]

and that \(\gamma^2(t) = \text{Var}B(t)\). For any \(s \in [a, b]\), let

\[
\rho(s) := \frac{\sigma^2(a, s)}{\gamma^2(a)}.
\]

It is elementary that there exists a centered Gaussian random variable \(R(s)\) independent of \(B(a)\) such that

\[B(s) = \rho(s)B(a) + R(s)\).

Note that this defines \(R(s) := B(s) - \rho(s)B(a)\), so that \(R\) is almost-surely continuous on \([a, b]\). It is also elementary to compute the covariance structure of \(R(s)\):

\[
E\left[(R(s) - R(t))^2\right] = E\left[(B(s) - B(t))^2\right] + (\rho(s) - \rho(t))^2 E\left[B(a)^2\right] \\
- 2(\rho(s) - \rho(t)) E[B(a)(B(s) - B(t))] \\
= \delta^2(s, t) + \frac{1}{\gamma^2(a)} (\sigma^2(a, s) - \sigma^2(a, t))^2 \\
- 2\frac{1}{\gamma^2(a)} (\sigma^2(a, s) - \sigma^2(a, t)) (\sigma^2(a, s) - \sigma^2(a, t)) \\
= \delta^2(s, t) - \frac{1}{\gamma^2(a)} (\sigma^2(a, s) - \sigma^2(a, t))^2.
\]
Step 3: Estimation of \( P ( A_{a,z} ) \) via \( R \). We can now write

\[
P ( A_{a,z} ) = P \left( B ( a ) \leq z \leq \max_{s \in [a,b]} \{ \rho ( s ) B ( a ) + R ( s ) \} \right)
\]

\[
\leq P \left( B ( a ) \leq z \leq \max_{s \in [a,b]} \{ \rho ( s ) B ( a ) \} + \max_{s \in [a,b]} R ( s ) \right)
\]

\[
= P \left( B ( a ) \leq z \leq B ( a ) + \max_{s \in [a,b]} \{ (\rho ( s ) - 1) B ( a ) \} + \max_{s \in [a,b]} R ( s ) \right)
\]

\[
\leq P \left( B ( a ) \leq z \leq B ( a ) + B ( a ) \text{sgn} ( B ( a ) ) \max_{s \in [a,b]} |\rho ( s ) - 1| + \max_{s \in [a,b]} R ( s ) \right)
\]

\[
= P \left( z - \max_{[a,b]} R \leq \frac{1}{1 + \text{sgn} ( B ( a ) ) \max_{s \in [a,b]} |\rho ( s ) - 1|} \leq B ( a ) \leq z \right),
\]

where the last inequality holds provided that \( \max_{s \in [a,b]} |\rho ( s ) - 1| < 1 \), also noting that \( \max_{[a,b]} R \geq 0 \) since \( R ( a ) = 0 \). Since \( \sigma^2 ( a, \cdot ) \) is continuous on \([a,b] \), for \( b - a \) small enough, \( \sigma^2 ( a, s ) \) can be made arbitrarily close to \( \sigma^2 ( a, a ) = \gamma^2 ( a ) \). Thus, by definition of \( \rho \), we can make \( \rho ( s ) \) close to 1. Hence, there exists \( t_0 > 0 \) such that \( 0 < b - a < t_0 \) implies \( \max_{s \in [a,b]} |\rho ( s ) - 1| \leq 1/2 \), which implies in turn that

\[
\frac{2}{3} \leq \frac{1}{1 + \text{sgn} ( B ( a ) ) \max_{s \in [a,b]} |\rho ( s ) - 1|} \leq 2.
\]

Consequently,

\[
P ( A_{a,z} ) \leq P \left( z - 2 \max_{[a,b]} R \leq B ( a ) \leq z \right)
\]

\[
= \int_{(z - 2 \max_{[a,b]} R) / \gamma ( a )}^{z / \gamma ( a )} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) dx
\]

\[
\leq \frac{2}{\gamma ( a ) \sqrt{2\pi}} \mathbb{E} \left[ \max_{[a,b]} R \right].
\]

In the last equality above, we used the independence of the process \( R \) and the random variable \( B ( a ) \).

Step 4: Estimation of \( \mathbb{E} [\max_{[a,b]} R] \). We saw in Step 2 that \( R \) is a centered Gaussian process with a canonical metric \( \mathbb{E}^{1/2} \left[ (R ( s ) - R ( t ))^2 \right] \) bounded above by \( \delta ( s, t ) \), the canonical metric of \( B \). From our assumption (1.1), we have \( \delta ( s, t ) \leq \ell \gamma ( |t - s| ) \). This means that, to cover the interval \([a,b] \) with balls of radius \( x \) in the canonical metric of \( R \), we require no more than \( N ( x ) := (b - a) / \gamma^{-1} \left( x / \sqrt{\ell} \right) \) such balls. Since \( \gamma \) is increasing, the diameter of \([a,b] \) in this canonical metric is bounded above by \( \sqrt{\ell} \gamma ( b - a ) \). We can thus apply the classical entropy upper bound of \( R \). Dudley (see [2]) to obtain, for some
universal constant $C_{\text{univ}}$,

$$
\frac{1}{C_{\text{univ}}} \mathbb{E} \left[ \max_{[a,b]} R \right] \leq \int_0^{\sqrt{\mathcal{F}_\gamma(b-a)}} \sqrt{\log N(x)} dx
= \int_0^{\sqrt{\mathcal{F}_\gamma(b-a)}} \sqrt{\log \frac{b-a}{\gamma^{-1}(x/\sqrt{k})}} dx
= \sqrt{k} \int_0^{b-a} \sqrt{\log \frac{b-a}{r}} d\gamma(r)
\leq \sqrt{k} \int_0^{(b-a)/2} \sqrt{\log \frac{b-a}{r}} d\gamma(r) + \sqrt{E} \left( \gamma(b-a) - \gamma \left( \frac{b-a}{2} \right) \right) \sqrt{\log 2}
$$

(3.3)

where we used a change of variables. By an integration by parts and another change of variables, using the fact that since $B$ is a.s. continuous, we have $\gamma(x) = o \left( \frac{1}{\sqrt{\log (1/x)}} \right)$, we get

$$
\int_0^{(b-a)/2} \sqrt{\log \frac{b-a}{r}} d\gamma(r) = \gamma \left( \frac{(b-a)/2}{2} \right) \sqrt{\log 2} + \int_0^{1/2} \gamma \left( \frac{(b-a)y}{2} \right) \frac{dy}{y \sqrt{\log (1/y)}}.
$$

(3.4)

Relations (3.3) and (3.4) now yield

$$
\mathbb{E} \left[ \max_{[a,b]} R \right] \leq C_{\text{univ}} \sqrt{E} \left( \gamma(b-a) \sqrt{\log 2} + \int_0^{1/2} \gamma \left( \frac{(b-a)y}{2} \right) \frac{dy}{y \sqrt{\log (1/y)}} \right),
$$

where we recognize the function $f_\gamma$ identified in the statement of the proposition.

Step 5: Conclusion. The results of Step 3 and Step 4 now imply, for another universal constant $C'_{\text{univ}}$

$$
P(A_{a,z}) \leq C'_{\text{univ}} \frac{\gamma}{\gamma(a)} f_\gamma(b-a).
$$

(3.5)

There is nothing in the arguments of Steps 2, 3, and 4 which prevents us from relating everything to $B(b)$ rather than $B(a)$. The only quantitative differences occur as follows: with $R$ relative to $b$, not $a$,

- at the end of Step 2, the expression for $\mathbb{E} \left[ (R(s) - R(t))^2 \right]$ involved $b$, not $a$, but this is still bounded above by $\delta^2(s,t)$, so there is no quantitative change in the application of Step 2 in Step 4, i.e. the upper bound of Step 4 remains unchanged for the new $R$;
- the expression for $\rho_s$ in Step 2 is now relative to $b$ rather than $a$, but we can still have $\max_{s \in [a,b]} |\rho(s) - 1| \leq 1/2$ for $b-a$ small enough;
- at the end of Step 3, we now get

$$
P(A_{b,z}) \leq P \left( z - 2 \max_{[a,b]} R \leq B(b) \leq z \right)
= \int_{(z-2 \max_{[a,b]} R)/\gamma(b)}^{z/\gamma(b)} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) dx
\leq \frac{2}{\gamma(b) \sqrt{2\pi}} \mathbb{E} \left[ \max_{[a,b]} R \right].
$$
Consequently, we have proved
\[
P(A_{b,z}) \leq C_{\text{univ}}^\gamma f_\gamma(b - a) .
\] (3.6)
Since the estimates (3.5) and (3.6) are uniform in \( z \), they also apply to \( P(A_{a,-}) \) and \( P(A_{b,-}) \) respectively. Since \( \gamma(b) > \gamma(a) \), (3.2) now implies the statement of the proposition, with \( c_u = (8/\sqrt{2\pi}) C_{\text{univ}} \).

\[
\text{Proposition 4.1.}\ 
\]

4. Upper Hausdorff Measure Bound for the Hitting Probabilities

Recall that \( B = (B(t), t \in \mathbb{R}_+) \) is a centered continuous Gaussian process in \( \mathbb{R}^d \) with variance \( \gamma^2(t) \) for some continuous strictly increasing function \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \lim_0 \gamma = 0 \), such that for some constant \( \ell \geq 1 \) and for all \( s, t \geq 0 \),
\[
(1/\ell) \gamma^2(|t - s|) \leq \mathbb{E}[|B(t) - B(s)|^2] \leq \ell \gamma^2(|t - s|).
\]

The aim of this section is to prove an upper bound for the probability that \( B \) hits a set \( A \) in terms of a certain Hausdorff measure of \( A \).

For a function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \), right-continuous and non-decreasing near zero with \( \lim_{0+} \varphi = 0 \), we define the \( \varphi \)-Hausdorff measure of a set \( A \subset \mathbb{R}^d \) as
\[
\mathcal{H}_\varphi(A) = \lim_{\varepsilon \to 0+} \inf \left\{ \sum_{i=1}^{\infty} \varphi(2r_i) : A \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), \sup r_i \leq \varepsilon \right\},
\]
where \( B(x_i, r_i) \) denotes the open ball of center \( x_i \) and radius \( r_i \) in \( \mathbb{R}^d \). When \( \varphi(r) = r^\beta \), \( \beta > 0 \), we write \( \mathcal{H}_\beta \) and call it the \( \beta \)-Hausdorff measure. When \( \beta \leq 0 \), we define \( \mathcal{H}_\beta \) to be infinite.

We first provide an upper bound for the probability that \( B \) hits a small ball in \( \mathbb{R}^d \). The papers [23, Lemma 7.8] or [4, Lemma 3.1] contain some analogous results in the case that \( \gamma \) is a power, but the techniques there do not seem to be applicable to the case of general \( \gamma \).

**Proposition 4.1.** For all \( 0 < a < b < \infty \), with \( b - a \) small enough, for all \( z \in \mathbb{R}^d, \varepsilon > 0 \),
\[
P(B([a, b]) \cap B(z, \varepsilon) \neq \emptyset) \leq \left( \frac{2\kappa\gamma(b)}{\gamma^2(a)} \left( 1 + \frac{1}{F(|z|)} \right) + \frac{c_u \sqrt{\ell}}{\gamma(a)} f_\gamma(b - a) \right)^d.
\]
where \( B(z, \varepsilon) \) denotes the open ball of center \( z \) and radius \( \varepsilon \) in \( \mathbb{R}^d \), \( \ell \) and \( \gamma \) are from condition (3.1), \( c_u \) and \( f_\gamma \) are defined in Proposition 3.7.

\[
F(z) = \begin{cases} 
1 & \text{for } z \leq \gamma(b), \\
1 - e^{-2\text{arctanh}(\gamma^2(a)/z^2)} & \text{for } z > \gamma(b),
\end{cases}
\]

\[
\kappa = \mathbb{P} \left[ \inf_{[a,b]} B > \gamma(b) \right].
\]

In particular, when Condition (3.1) is satisfied, with \( L \) as in Corollary 3.2,
\[
P(B([a, b]) \cap B(z, \varepsilon) \neq \emptyset) \leq \left( \frac{2\kappa\gamma(b)}{\gamma^2(a)} \left( 1 + \frac{1}{F(|z|)} \right) + \frac{L\sqrt{\ell}}{\gamma(a)} \gamma(b - a) \right)^d.
\]

**Remark 4.2.** Note that \( \text{arctanh}(x) \) is equivalent to \( x \) for \( x \) small, and therefore, for large \( z, 1/F(z) \) in the above proposition is equivalent to \( z^2/(2\gamma^2(b)) \).
Remark 4.3. Since the various components of \( B \) are independent, it is equivalent to prove this proposition with \( d = 1 \) only.

The proof of this proposition needs some preparation. We first explain why the results of Section 3 on hitting points in one dimension will be needed to prove this proposition. The event whose probability we need to estimate is

\[
D := \{ B([a, b]) \cap B(z, \varepsilon) \neq \emptyset \} = \left\{ \inf_{s \in [a, b]} |B(s) - z| \leq \varepsilon \right\} 
\]

\[
= \left\{ 0 < \inf_{s \in [a, b]} |B(s) - z| \leq \varepsilon \right\} \cup \{ B([a, b]) \ni z \} =: D_1 \cup D_2.
\]

The second event \( D_2 \) in the last line above is the one whose probability we estimated in Section 3. We choose to separate it from the remaining event \( D_1 \) because, since \( B \) hits points with positive probability during \([a, b]\), the random variable \( Z := \inf_{s \in [a, b]} |B(s) - z| \) has an atom at 0. Note that \( D_1 \) and \( D_2 \) are disjoint. In any case, Proposition 4.4 shows that to prove Proposition 4.1, it is sufficient to establish

\[
P(D_1) = P(0 < Z \leq \varepsilon) \leq C\varepsilon
\]  
(4.1)

for the appropriate constant \( C \). To prove this, it would be sufficient to show that, \( Z \) has a bounded density on \((0, +\infty)\).

To establish such a density bound, we must take a minor detour via the Malliavin calculus. A criterion was established in [14, Theorem 3.1] for proving the existence of a density and a method for estimating it quantitatively. That technique does not apply in \([16, \text{Chapters 2 and 10}]\). Let \( X \) be the Gaussian field in \([16, \text{Chapters 2 and 10}]\). Let \( D \) be the Malliavin derivative operator in the Wiener space \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) induced by the process \( B \). Let \( \mathbb{D}^{1,2} \) be the Gross-Sobolev subspace of \( L^2(\Omega) \), identified with the domain of \( D \). Let \( X \) be a centered random variable in \( \mathbb{D}^{1,2} \) and define

\[
G_X := \int_0^{+\infty} d\varepsilon \mathbb{E} \left[ \int_{\mathbb{R}^+} D_r X \ D_r^{(\varepsilon)} X^{(\varepsilon)} dr \right],
\]  
(4.2)

and

\[
g_X(x) = \mathbb{E} [G_X \mid X = x],
\]  
(4.3)

where the notation \( X^{(\varepsilon)} \) denotes a random variable with the same law as \( X \), but constructed using a copy \( B^{\varepsilon} \) of the Gaussian field \( B \) such that the correlation coefficient \( \text{Corr}(B, B^{\varepsilon}) = e^{-\varepsilon} \), and \( D^{(\varepsilon)} \) is the Malliavin derivative operator on the Wiener space induced by \( B^{\varepsilon} \). The expression in (4.2) coincides with the random variable \( \langle DX, -DL^{-1}X \rangle_B \) where \( L^{-1} \) is the pseudo-inverse of the generator of the Ornstein-Uhlenbeck semigroup on \( B \)'s Wiener space, and \( \langle \cdot, \cdot \rangle_B \) is the canonical inner product defined by \( B \)'s Gaussian Wiener integrals; this coincidence comes from the so-called Mehler formula, and we also have that for every continuously differentiable function \( f \) with bounded derivative,

\[
\mathbb{E} [X f(X)] = \mathbb{E} [g_X(X) f'(X)].
\]  
(4.4)

In the sequel, we will only need to use (1.2), (1.3), and (4.2). We have the following

Proposition 4.4. Let \( X \) be a centered random variable in \( \mathbb{D}^{1,2} \), and \( G_X \) and \( g_X \) be defined in (1.2), (1.3). The support of the law of \( X \) is an interval \([\alpha, \beta]\) with \(-\infty \leq \alpha <
\[ 0 < \beta \leq +\infty. \text{ Assume there exists } \alpha' \in (\alpha, 0) \text{ such that } g_X(x) > 0 \text{ for all } x \in [\alpha', \beta). \]

Then \( X \) has a density \( \rho \) on \([\alpha', \beta)\), and for almost every \( z \in [\alpha', \beta) \),

\[
\rho(x) = \frac{E[|X|]}{2g_X(x)} \exp \left( -\int_0^x ydy/g_X(y) \right). \tag{4.5}
\]

**Proof.** The proof of this proposition varies only slightly from that of [14, Theorem 3.1]; we provide it here for completeness. The statement about the support of \( X \) is well-known [12, Proposition 2.1.7]. Let \( A \) be a Borel set included in \([\alpha', \beta)\), and assume that its Lebesgue measure is 0. By using a monotone approximation argument, we can apply (4.3) to \( f(x) = \int_{\alpha'}^x 1_A(y) dy \). Thus

\[
0 = E[Xf(X)] = E[1_A(X)g_X(X)].
\]

Since \( A \subset [\alpha', \beta) \), by assumption, \( g_X(X) > 0 \) on the event \( \{X \in A\} \). Consequently, \( 1_A(X) = 0 \) almost surely, i.e. \( P[X \in A] = 0 \), which means the law of \( X \) restricted to \([\alpha', \beta)\) is absolutely continuous w.r.t. Lebesgue’s measure, and therefore \( X \) has a density \( \rho \) on \([\alpha', \beta)\), and note that \( \rho(X) \) is positive almost surely.

Now for any continuous function \( f \) with compact support in \([\alpha', \beta)\), and its antiderivative \( F(x) = \int_{\alpha'}^x f(y) dy \) (which is necessarily bounded), by (4.4) we have

\[
E[g_X(X)f(X)] = E[XF(X)] = \int_{\alpha'}^{\beta} \rho(y) yF(y) dy.
\]

We perform the integration by parts with parts \( F(y) \) and \( \rho(y) ydy. \) Note that \( \varphi \) is differentiable almost everywhere on \([\alpha', \beta)\), and is bounded since \( X \in L^2(\Omega) \subset L^1(\Omega) \).

Thus, with \( \varphi(x) = \int_{\alpha'}^{x} y\rho(y) dy \), we get

\[
E[g_X(X)f(X)] = \int_{\alpha'}^{\beta} f(y) \varphi(y) dy + \lim_{x \to \beta} F(x) \varphi(x) - \lim_{x \to \alpha'} F(x) \varphi(x).
\]

Here, \( \lim_{x \to \alpha'} F(x) = 0 \), by definition, and \( \lim_{x \to \beta} \varphi(x) = 0 \) since \( X \in L^1(\Omega) \). Thus

\[
E[g_X(X)f(X)] = \int_{\alpha'}^{\beta} f(y) \varphi(y) dy = E \left[ \frac{\varphi(X)}{\rho(X)} f(X) \right].
\]

This implies that on the event \( \{X \in [\alpha', \beta]\} \), \( g_X(X) = \varphi(X)/\rho(X) \) almost surely, which, since \([\alpha', \beta]\) is inside the support of \( X \), implies that for almost every \( x \in [\alpha', \beta]\),

\[
g_X(x) = \varphi(x)/\rho(x).
\]

Since by definition, \( \varphi'(x) = -x\rho(x) \), we get an ordinary differential equation for \( \varphi \), whose unique solution is identical to the relation (4.5), provided one uses the boundary condition given by \( \varphi(0) \), which equals \( E[|X|]/2 \) because \( E[X] = 0 \).

This proposition provides a convenient criterion to establish existence and upper bounds on densities: if one can show that \( g_X(x) \geq c > 0 \) for all \( x \in [\alpha', \beta) \), then (4.5) implies for all \( x \in [\alpha', \beta) \) :

\[
\rho_X(x) \leq \frac{E[|X|]}{2c}. \tag{4.6}
\]

This follows from the fact that \( g_X \) is a positive function, so that for any \( x \), whether positive or negative, the exponential in the density formula (4.5) is always less than 1. The positivity of \( g_X \) is well-known (see [15]), and can also be inferred directly from formula (4.5).

As it turns out, the random variable \( Z \) is difficult to estimate via Proposition 4.4 because the expression one finds for \( g_{Z-EZ} \) via (4.2) is an integral of a signed function.
However, an easy expansion of $D_1$ is helpful. Since in $D_1$, $Z$ is positive, and $B$ is continuous, this means that either $\inf_{s \in [a,b]} |B(s) - z|$ was attained for the whole trajectory $B([a,b])$ below the level $z$, or above it, and these two events are disjoint. Therefore

$$
P(D_1) = \mathbb{P} \left( 0 < \inf_{s \in [a,b]} (B(s) - z)_+ \leq \varepsilon \right) + \mathbb{P} \left( 0 < \inf_{s \in [a,b]} (B(s) - z)_- \leq \varepsilon \right)$$

$$= \mathbb{P} \left( 0 < \inf_{s \in [a,b]} (B(s) - z)_+ \leq \varepsilon \right) + \mathbb{P} \left( 0 < \inf_{s \in [a,b]} (-B(s) - (-z))_+ \leq \varepsilon \right)$$

$$=: D'_z + D'_z (4.7)$$

where in the last line we used the fact that $B$ has a symmetric law. According to the strategy leading to (4.6), we only need to study the random variable $G_X$ relative to

$$X := \inf_{s \in [a,b]} (B(s) - z)_+ - \mu,$$

$$\mu := \mathbb{E} \left[ \inf_{s \in [a,b]} (B(s) - z)_+ \right].$$

**Proof of Proposition 4.1** Recall that by Remark 4.3, we assume $B$ is scalar.

**Step 0:** what we must prove. The centered random variable $X$ above is supported in $[-\mu, +\infty)$. Moreover, it is a Lipschitz functional of a continuous Gaussian process, and as such, belongs to $L^{1/2}$. From Proposition 4.1 and relation (4.6), it is sufficient to prove that there is a positive constant $c$ such that for any $x > -\mu$,

$$g_X(x) \geq c.$$

**Step 1:** Computing $G_X$. To use formula (4.2), we must compute $DX$. One may always assume that, with $\mathcal{H}$ the canonical Hilbert space of the isonormal Gaussian process $W$ underlying $B$, for $s \in [a,b]$, there exists an element $f_s \in \mathcal{H}$ such that $B(s) = W(f_s)$. Note that $\langle f_s, f_t \rangle = \sigma^2(s, t) := \mathbb{E}[B(s)B(t)]$. Then, by using the same argument as in the proof of [14, Lemma 3.11], we find that on the event $\{X > -\mu\}$,

$$DX = \mathbf{1}_{B(\tau) > z} f_\tau,$$

where $\tau = \arg \min_{s \in [a,b]} (B(s) - z) = \arg \min_{s \in [a,b]} B(s)$. Note that since $B$ is continuous, this arg min is uniquely defined in the event $\{X > -\mu\}$. Thus by the Mehler-type representation formula (4.2),

$$G_X = \int_0^\infty du \ e^{-u} \mathbb{E} \left[ \left\langle \mathbf{1}_{B(\tau) > z} f_\tau; \mathbf{1}_{B(u)}(\tau(u)) > z f_{\tau(u)} \right\rangle \mathcal{F} \right]$$

$$= \int_0^\infty du \ e^{-u} \tilde{\mathbb{E}} \left[ \mathbf{1}_{B(\tau) > z} \mathbf{1}_{B(u)}(\tau(u)) > z \langle f_\tau; f_{\tau(u)} \rangle \right]$$

$$= \int_0^\infty du \ e^{-u} \tilde{\mathbb{E}} \left[ \mathbf{1}_{B(\tau) > z} \mathbf{1}_{B(u)}(\tau(u)) > z \sigma^2(\tau, \tau(u)) \right],$$

(4.8)

where $\tilde{\mathbb{E}}$ represents the expectation with respect to the randomness in the independent copy $\tilde{B}$ of $B$, and the superscripts $(u)$ mean that the corresponding random variables are relative to $B^{(u)} = e^{-u}B + \sqrt{1 - e^{-2u}}\tilde{B}$.

**Step 2:** Estimating $g_X$. We must compute $\mathbb{E}[G_X \mid X = x]$ for any $x > -\mu$. Here, a convenient simplification occurs: since we are conditioning by $\{X = x\}$, on this event,
inf_{s \in [a,b]} \{ B(s) - z \}, is strictly positive, and in fact it equals \( B_\tau - z \); therefore, \( 1_{B_\tau > z} = 1 \) almost surely on that event. Therefore, for any \( x > -\mu \),

\[
g_X(x) = \int_0^\infty du \ e^{-u} \tilde{\mathbb{E}} \left[ 1_{B(u) \tau(u)} > z \right] \sigma^2(\tau, \tau(u)) \left| X = x \right].
\]

The goal being to bound this expression uniformly from below, we note that both \( \tau \) and \( \tau(u) \) are in the non-random interval \( [a, b] \). Since \( B \) is a.s. continuous, the bivariate function \( \sigma^2(\tau, \tau(u)) \) is uniformly continuous on \( [a, b] \times [a, b] \). Since \( a > 0 \), \( \sigma^2(a, a) = \gamma^2(a) > 0 \), and by making \( b - a \) small enough, we can get \( \min_{(s, t) \in [a, b]^2} \sigma^2(s, t) \geq \gamma^2(a) / 2 \). Thus

\[
g_X(x) \geq \frac{\gamma^2(a)}{2} \int_0^\infty du \ e^{-u} \tilde{\mathbb{E}} \left[ 1_{B(u) \tau(u)} > z \right] \left| X = x \right].
\]

**Step 3: Estimating the last expectation.** We now evaluate the remaining expectation above. We have for any \( x > -\mu \), and any \( z \in \mathbb{R} \),

\[
\tilde{\mathbb{E}} \left[ 1_{B(u) \tau(u)} > z \right] | X = x = \tilde{P} \left[ e^{-u} B(\tau(u)) + \sqrt{1 - e^{-2u}} \tilde{B}(\tau(u)) > z \right] | X = x.
\]

In the conditional probability above, since \( x > -\mu \) and since \( \tau = \arg \min_{[a,b]} B \), we get \( B(\tau(u)) \geq B(\tau) = x + \mu + z \). Similarly, \( \tilde{B}(\tau(u)) \geq \tilde{B}(\tilde{\tau}) \) where \( \tilde{\tau} := \arg \min_{[a,b]} \tilde{B} \). In addition, we have that \( \tilde{B}(\tilde{\tau}) = \min_{[a,b]} \tilde{B} \) is independent of \( X \). Therefore we can write

\[
\tilde{\mathbb{E}} \left[ 1_{B(u) \tau(u)} > z \right] \left| X = x \right] \geq \tilde{P} \left[ e^{-u} (x + \mu + z) + \sqrt{1 - e^{-2u}} \tilde{B}(\tilde{\tau}) > z \right] \left| X = x \right]
\]

\[
= \tilde{P} \left[ \min_{[a,b]} \tilde{B} > z \left( 1 - e^{-u} \right) / \sqrt{1 - e^{-2u}} \right].
\]

Recall that we are trying to find a uniform lower bound on \( g_X(x) \) for all \( x + \mu > 0 \); therefore, the term \( -e^{-u} (x + \mu) \) in the probability above will not help us even though it is negative, so we simply ignore it, obtaining

\[
\tilde{\mathbb{E}} \left[ 1_{B(u) \tau(u)} > z \right] \left| X = x \right] \geq \tilde{P} \left[ \min_{[a,b]} \tilde{B} > z \sqrt{\frac{1 - e^{-u}}{1 + e^{-u}}} \right] = \tilde{P} \left[ \min_{[a,b]} \tilde{B} > z \sqrt{\tanh(u/2)} \right].
\]

**Step 4: General tail lower bound for Gaussian infimum.** We are left to find a lower bound on the last expression above. Since \( z \) is arbitrary, this is a general question about Gaussian infima, and can be solved using a strategy similar to the one we are using for this entire proof of Proposition 4.1, albeit easier because we now have reduced the problem to studying the tail of the random variable \( \inf \tilde{B} \), which does not involved positive parts, making the required Malliavin calculus computations more straightforward. We state this result as a general Lemma of independent interest, even though in the remainder of the proof of our Proposition 4.1 only the second statement of this Lemma, which is essentially trivial, is needed.

**Lemma 4.5.** Let \( B \) be a continuous scalar center Gaussian process on \([a,b] \) satisfying (1.1) and (1.2). Assume \( b - a \) is small enough to ensure that for all \( s, t \in [a,b] \),
E\[B (s) B (t)\] ≥ γ^2 (a) / 2. Define
\[ν := -E \inf_{[a,b]} B = \left| E \inf_{[a,b]} B \right| = E \sup_{[a,b]} B,\]
\[λ := E \left[ \inf_{[a,b]} B + ν \right] = E \left[ \sup_{[a,b]} B - ν \right].\]
Then for any \(y ≥ \gamma (b) - ν\),
\[P \left[ \inf_{[a,b]} B > y \right] \geq \lambda \frac{1 - e^{-1}}{y + ν} \frac{1}{4} \exp \left( -\frac{(y + ν)^2}{\gamma^2 (a)} \right).\]
Also note that for all \(y ≤ \gamma (b)\),
\[P \left[ \inf_{[a,b]} B > y \right] ≥ P \left[ \inf_{[a,b]} B > \gamma (b) \right] =: κ > 0. \] (4.9)

Note that the positive constants \(κ, λ, ν, γ (a), γ (b)\) depend only on \(a, b\) and the law of \(B\). The last statement follows trivially from the fact that \(\inf_{[a,b]} B\) has a positive density on \(\mathbb{R}\). That fact comes easily from Proposition 4.4 and the fact that \(γ^2 (a) / 2 ≤ G_{\inf_{[a,b]} B} \leq γ^2 (b)\) almost surely, which is easy to prove using the technique in Step 2. These inequalities are also useful to prove the first statement of the lemma, via a modification of the proof of [21, Corollary 4.5]. The full proof of this lemma is left to the reader.

**Step 5: Applying the lemma.**
We apply Lemma 4.5 to \(\tilde{B}\), with \(y = z \sqrt{\tanh (u/2)}\). Since \(B\) and \(\tilde{B}\) have the same law, all the constants, particularly \(κ\) in (4.5), are as in Lemma 4.5. Since \(\tanh ≤ 1\) on \(\mathbb{R}_+\), from the second part of the lemma and the conclusion of Step 3 we get for every \(z ≤ \gamma (b)\), every \(x > -μ\), and every \(u ≥ 0\),
\[\tilde{E} \left[ 1_{B^{(u)} (x)} > z \right] \geq κ. \] (4.10)

For \(z ≥ \gamma (b)\), we can still apply the second part of Lemma 4.5 for \(u\) small enough. Since \(\tanh\) is bijective and increasing on \(\mathbb{R}_+\), we get \(y = z \sqrt{\tanh (u/2)} ≤ \gamma (b)\) if and only if \(u ≤ 2 \arctanh (γ^2 (b) / z^2)\). For such \(u\) and \(z\), and every \(x > -μ\), by the conclusion of Step 3, inequality (4.10) still holds.

**Step 6. Conclusion.**
By the conclusion of Step 2, from (4.10), we can now write, for all \(z ≤ \gamma (b)\) and all \(x > -μ\),
\[g_X (x) ≥ \frac{γ^2 (a)}{2} κ.\]

By the conclusion of Step 2, we can find a lower bound on \(g_X (x)\) by integrating only over the range \(u ∈ [0, 2 \arctanh (γ^2 (b) / z^2)]\), where (4.10) is still valid: we get, for all
\[ z \geq \gamma(b) \text{ and every } x > -\mu, \]
\[ g_X(x) \geq \frac{\gamma^2(a)}{2} \kappa \int_0^{2\arctanh(\gamma^2(b)/z^2)} du \ e^{-u} \]
\[ = \frac{\gamma^2(a)}{2} \kappa \left( 1 - e^{-2\arctanh(\gamma^2(b)/z^2)} \right). \]

With \( F(z) \) as defined in the statement of Proposition 4.1, we summarize the two inequalities above as: for all \( x > -\mu \) and all \( z \in \mathbb{R} \)
\[ g_X(x) \geq F(z) \frac{\gamma^2(a)}{2} \kappa. \]

Thus by relations (4.6) and (4.7), since either \( z \) or \( -z \) is positive,
\[ P(D_1) \leq \varepsilon \frac{2\kappa E[|X|]}{\gamma^2(a)} \left( 1 + \frac{1}{F(|z|)} \right). \]

Finally, we can easily estimate \( E[|X|] \), and show that this does not depend on \( z \). Indeed, from (4.8) and Cauchy-Schwartz, we immediately get \( G_X \leq \gamma^2(b) \). Then, from (4.9), with \( f \) the identity, we get \( E[X^2] = E[G_X] \leq \gamma^2(b) \). Hence by Jensen, \( E[|X|] \leq \gamma(b) \).

Plugging this into the last estimate of \( P(D_1) \), and combining this with the estimate of \( P(D_2) \) from Theorem 3.1, finishes the proof of the proposition’s first statement. The second statement follows immediately from Corollary 3.2. \( \square \)

Using a covering argument and Proposition 4.1 one obtains the following upper bound for the hitting probabilities of \( B \) in terms of Hausdorff measure (see [6, Theorem 3.1] where a similar argument is performed).

**Theorem 4.6.** Assume that the function \( \varphi(s) = s^d/\gamma^{-1}(s) \) is right-continuous and non-decreasing near 0 with \( \lim_{s \to 0^+} \varphi = 0 \). Also assume that \( \gamma \) satisfies the condition (3.1) from Corollary 3.2. Then for all \( 0 < a < b < \infty \), any \( M > 0 \), there exists a constant \( C > 0 \) depending only on \( a, b \), the law of \( B \), and \( M \), such that for any Borel set \( A \subset [-M, M]^d \),
\[ P(B([a, b]) \cap A \neq \emptyset) \leq C \mathcal{H}\varphi(A). \]

**Proof.** For all positive integers \( n \), consider the intervals of the form
\[ I^n_j := [t^n_j, t^n_{j+1}], \quad \text{where} \quad t^n_j := j \gamma^{-1}(2^{-n}). \]
Fix \( \varepsilon \in (0, 1) \) and \( n \in \mathbb{N} \) such that \( 2^{-n-1} < \varepsilon \leq 2^{-n} \), and write, for any \( z \in A \),
\[ P(B([a, b]) \cap B(z, \varepsilon) \neq \emptyset) \leq \sum_{j: I^n_j \cap [a, b] \neq \emptyset} P(B^\gamma(I^n_j) \cap B(z, \varepsilon) \neq \emptyset). \]

The number of \( j \)'s involved in the sum is at most \( (b - a)/\gamma^{-1}(2^{-n}) \). Also note that the diameter \( \eta \) of \( I^n_j \) is \( \gamma^{-1}(2^{-n}) \), and therefore, \( \gamma(\eta) = 2^{-n} < 2\varepsilon \). Then, for \( \varepsilon \leq \varepsilon_0 \) small enough, we can apply Proposition 4.1 to each interval \( I^n_j \). The constant \( \kappa \) in this proposition must then be replaced by \( \kappa_j := P \left[ \inf_{t^n_j} B > \gamma(t^n_j) \right] \). In any case, we may use the uniform bound \( \kappa_j \leq 1 \). Hence, Proposition 4.1 implies that for all large \( n \) and
In the first inequality we used the fact that the endpoints of each interval \( I^n_j \) are bounded above by \( b \) and below by \( a \), and we appealed to the fact that \( \gamma \) is increasing; in the last inequality we used again the fact that \( \gamma \) is increasing, and \( \varepsilon \leq 2^{-n} \). Observe that

\[
C := (b - a) \left( \frac{2 \gamma(b)}{\gamma^2(a)} \left( 1 + \frac{1}{F(M)} \right) + \frac{2L\sqrt{\ell}}{\gamma(a)} \right)^d
\]

does not depend on \( n, \varepsilon \), or \( A \), except via the value \( M \). Therefore, (4.11) is valid for all \( \varepsilon \in (0, \varepsilon_0) \).

Now we use a covering argument: Choose \( \varepsilon \in (0, \varepsilon_0) \) and let \( \{ B(z_i, r_i) \}_{i=1}^\infty \) be a sequence of open balls in \( \mathbb{R}^d \) with radii \( r_i \in (0, \varepsilon] \) such that

\[
A \subseteq \bigcup_{i=1}^\infty B(z_i, r_i) \quad \text{and} \quad \sum_{i=1}^\infty \varphi(2r_i) \leq \mathcal{H}_\varphi(A) + \varepsilon,
\]

where \( \varphi(r) = r^d/\gamma^{-1}(r) \). Then, (4.11) and (4.12) together imply that

\[
P(B([a, b]) \cap A \neq \emptyset) \leq \sum_{i=1}^\infty P(B([a, b]) \cap B(z_i, r_i) \neq \emptyset) \leq C \sum_{i=1}^\infty \varphi(2r_i) \leq C(\mathcal{H}_\varphi(A) + \varepsilon).
\]

Finally, let \( \varepsilon \to 0^+ \) to deduce the desired upper bound. \( \square \)

In Section 6 we will see what our main theorems 2.5 and 4.6 mean in a specific two-parameter class of examples with highly non-stationary increments. We finish this section with a general discussion of how close our canonical kernel functions \( K \) and \( 1/\varphi \) are to each other, as identified in the Hausdorff measure upper bound (Theorem 4.6) and the capacity lower bound (Theorem 2.5). To fix ideas, recall that in the case of fBm, the relevant function for the Hausdorff measure is \( \varphi(r) = r^d/\gamma^{-1}(r) \), and that the Hausdorff measure result applies when \( d > 1/H \). In that same case, the capacity lower bound uses the Newtonian kernel \( K = K_{d-1/H} \), meaning that \( K = 1/\varphi \), or at least, since all results are given modulo multiplicative constants (depending on \( a, b \) and the law of \( B \)), and the values obtained in the bounds depend qualitatively only on the behavior of \( K \) and \( 1/\varphi \) near 0, the fBm case shows that what is important is that lower and upper bounds refer to commensurate canonical functions \( K \) and \( 1/\varphi \) near 0.

In the general case, we would like to know to what extent we still have that \( K \) and \( 1/\varphi \) are commensurate near 0. Recall that

\[
K(x) = \max \left( 1, v\left( \gamma^{-1}(x) \right) \right),
\]
where \( v(r) = \int_r^{b-a} ds / \gamma^d(s) \), and
\[
1 / \varphi(x) = \gamma^{-1}(x) / x^d.
\]

Thus, their commensurability near 0 is equivalent to that of \( v(r) \) and \( r \mapsto r / \gamma^d(r) \) for \( r \) near 0. Since all these functions are continuous and non-zero everywhere except at \( r = 0 \), we only need to investigate whether
\[
0 < \liminf_{r \to 0} \frac{r / \gamma^d(r)}{v(r)} \leq \limsup_{r \to 0} \frac{r / \gamma^d(r)}{v(r)} < +\infty.
\]

This comparison is only fair if one also takes into account the assumptions used in Theorems 4.6 and 2.5, which include the concavity of \( \gamma \), and the fact that \( \lim_0 \varphi = 0 \) and \( \varphi \) is non-decreasing. To make this presentation more elementary, we specialize to the case where \( \gamma \) is differentiable, but similar arguments can be developed in the case where differentiability holds only almost everywhere.

Since we now assume that \( \lim_0 \varphi = 0 \) and \( \varphi \) is non-decreasing, there exists an increasing sequence of constant \( c_n \) with \( \lim_n c_n = +\infty \), and a decreasing sequence of constants \( r_n \in (0, b-a) \) in with \( \lim_n r_n = 0 \) such that for every \( r \in [r_n+1, r_n] \), \( 1 / \gamma^d(r) \geq c_n / r \).

Therefore, for any integer \( N \),
\[
v(r_N) = \int_{r_N}^{b-a} ds / \gamma^d(s) \geq \sum_{n=1}^{N-1} c_n (\ln r_n - \ln r_{n+1})
\]
\[
\geq c_1 \left( \ln \left( \frac{1}{r_N} \right) - \ln \left( \frac{1}{b-a} \right) \right).
\]

Since this expression goes to \(+\infty\) with \( N \), \( v \) is unbounded. We may thus apply l'Hôpital’s rule similarly to what we did in Step 4 of the proof of Theorem 2.5 to get that
\[
\lim_{r \to 0} \frac{r / \gamma^d(r)}{v(r)} = -1 + d \lim_{r \to 0} \frac{r \gamma'(r)}{\gamma(r)}
\]
if the last limit above exists. We already know by concavity of \( \gamma \) that the last expression above is bounded above by \( d - 1 \) (see Step 4 of the proof of Theorem 2.5). Therefore, by requiring that it be bounded away from 0, we can assert the following.

**Proposition 4.7.** Under the assumptions in Theorems 4.6 and 2.5 if \( \lim_{r \to 0} \frac{r \gamma'(r)}{\gamma(r)} \) exists, then the functions \( K \) and \( 1 / \varphi \) are commensurate if and only if
\[
d > 1 / \lim_{r \to 0} \frac{r \gamma'(r)}{\gamma(r)}.
\]

The advantage of this criterion is that it separates the dimension \( d \) from the information contained in \( \gamma \) about the law of the scalar process \( B \). One can also reformulate the above proposition by assuming that \( \gamma \) is of index \( \text{ind} \gamma \) (see the discussion surrounding Lemma 3.3 for the significance of the index): \( d > 1 / \text{ind} \gamma \) implies that \( K \) and \( 1 / \varphi \) are commensurate; the proof of this fact is left to the reader.

### 5. Local times and hitting points

In this section we study the question whether the \( d \)-dimensional Gaussian process \( B \) satisfying (1.1) and (1.2) hits points in \( \mathbb{R}^d \). This question is strongly related to the existence of the local time for \( B \). We noted earlier in Remark 2.7 that our lower capacity bound already gives us results on hitting points. In this section we show that by using
local time techniques instead of general capacity bounds, one gets those same results again, but without having to use Hypothesis \([\ref{hyp:B}].\)

Recall that the local time of a process is defined as the density of its occupation measure, when it exists. Let us be more specific. Let \(0 < a < b < \infty\) be fixed. The occupation measure of the process \(B\) in the time interval \([a, b]\) is defined, for any Borel set \(A\) in \(\mathbb{R}^d\), as

\[
\nu(A) = \int_a^b 1_A(B(s)) ds.
\]

This measure describes the amount time in \([a, b]\) that the process \(B\) spends in an infinitesimal box of volume \(dx\) around the point \(x\). Thus, \(\ell(x, [a, b]) dx\) can be interpreted as the amount of time in \([a, b]\) that the process \(B\) spends in an infinitesimal box of volume \(dx\) around the point \(x\). In particular, \(\ell(x, \cdot)\) is supported on the \(x\)-level set

\[
(B)^{-1}(x) = \{t \geq 0 : B(t) = x\}.
\]

Consider the following assumption on \(\gamma\).

**Hypothesis 5.1.** The function \(1/\gamma^d(r)\) is integrable around zero.

**Theorem 5.2.** The process \(B\) has a local time \(\ell(x, [a, b])\) for all \(x \in \mathbb{R}^d\) and \(0 < a < b < \infty\), which is square integrable in \(x\) for all \([a, b]\) a.s. if and only if Hypothesis \([\ref{hyp:B}].\) holds true.

**Proof.** It suffice to show that \(\hat{\nu} \in L^2(\mathbb{R}^d)\) a.s., where \(\hat{\nu}\) is the Fourier transform of the occupation measure \(\nu\). Indeed, this would imply that a.s. \(\nu\) is absolutely continuous with respect to the Lebesgue measure in \(\mathbb{R}^d\), and its density (the local time) is in \(L^2(\mathbb{R}^d)\) (cf. \([\ref{book:livre} Theorem 8.6.2]\)).

Therefore, we only need to show that Hypothesis \([\ref{hyp:B}].\) holds true if and only if

\[
\int_{\mathbb{R}^d} \left| \int_a^b \exp(i\xi \cdot B(s) ) ds \right|^2 d\xi < \infty \quad \text{a.s.,}
\]

which is equivalent to the fact that the integral has a finite expectation. But the expectation is simply

\[
\int_{\mathbb{R}^d} \int_a^b \int_a^b \mathbb{E} \left[ e^{i\xi \cdot (B(t) - B(s))} \right] ds dt d\xi.
\]

As \(B\) is Gaussian and using \([\ref{exp:gauss}]\), we get that

\[
\int_{\mathbb{R}^d} \int_a^b \int_a^b e^{-|\xi|^2/2} \gamma^2(|t-s|) dldsdt d\xi \leq \int_{\mathbb{R}^d} \int_a^b \int_a^b \mathbb{E} \left[ e^{i\xi \cdot (B(t) - B(s))} \right] ds dt d\xi
\]

\[
\leq \int_{\mathbb{R}^d} \int_a^b \int_a^b e^{-|\xi|^2/2} \gamma^2(|t-s|) dldsdt d\xi.
\]

We now fix \(t\) and use the change of variables \(u = t-s\) to conclude that

\[
c_1 \int_0^{b-t} \frac{1}{\gamma^d(u)} du \leq \int_{\mathbb{R}^d} \int_a^b \int_a^b \mathbb{E} \left[ e^{i\xi \cdot (B(t) - B(s))} \right] ds dt d\xi \leq c_2 \int_0^{b-t} \frac{1}{\gamma^d(u)} du.
\]

Hence, the statement of the Theorem holds true. \(\square\)
Let us now return to the question of hitting points.

**Theorem 5.3.** If Hypothesis 5.1 holds true, then the process \( B \) hits points in \( \mathbb{R}^d \) with positive probability, that is, for all \( x \in \mathbb{R}^d \),
\[
P((B)^{-1}(x) \neq \emptyset) > 0.
\]

**Proof.** By Theorem 5.2, since Hypothesis 5.1 is in force, the local time \( \ell(x, \cdot) \) exists and is supported on \((B)^{-1}(x)\). Moreover, we can express the local time \( \ell(x, \cdot) \) as the inverse Fourier transform of \( \hat{\nu}(\xi) \), namely
\[
\ell(x, \cdot) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \hat{\nu}(\xi) d\xi.
\]

Then by (1.2), it follows that
\[
E[\ell(x, [a, b])] = \frac{1}{(2\pi)^d} \int_a^b \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{i\xi \cdot B(t)} d\xi dt.
\]

Therefore, \( \ell(x, [a, b]) > 0 \) with positive probability. Thus,
\[
P([a, b] \cap (B)^{-1}(x) \neq \emptyset) > 0.
\]

and this holds for any interval \([a, b] \subset (0, \infty) \) and any \( x \in \mathbb{R}^d \). \(\square\)

Our next aim is to find a necessary condition in order to hit points. In the Appendix, we apply a classical method that uses the modulus of continuity to our class of processes \( B \) (see Theorem A.2). However, as we will see in the examples in Section 6 (Proposition 6.1), this result is not sharp. We present here an application of our upper bound in terms of Hausdorff measure obtained in Theorem 4.6, that will give a weaker sufficient condition for not hitting points a.s.

**Theorem 5.4.** Let \( \gamma \) such that \( \gamma(r) = o(r^{1/d}) \) near 0. Also assume that \( \varphi(s) := s^d/\gamma^{-1}(s) \) is non-decreasing near 0 and Condition (3.1) holds. Then almost surely, \( B \) does not hit points.

**Remark 5.5.** Observe that Hypothesis A.1 of Theorem A.2 in the Appendix says that \( \gamma^d(r) \log^{d/2}(1/r) = o(r) \), i.e. \( \gamma(r) = o(r^{1/d} \log^{-1/2}(1/r)) \), but since \( \log^{-1/2}(1/r) \) tends to 0, this is much more restrictive than the condition \( \gamma(r) = o(r^{1/d}) \) in Theorem 5.4.

**Proof.** To prove this theorem, first note that since \( \gamma \) is continuous and strictly increasing, \( \varphi \) is a continuous function. The assumption of the theorem also says that \( \varphi \) is non-decreasing. We claim that \( \lim_{0} \varphi = 0 \), which we will proved in Lemma 5.6 below.
Assuming this is true, we can apply Theorem 4.6. Thus,

\[ \mathcal{H}_\varphi \{ \{ x \} \} = \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} \varphi(2r_i) : x \in \bigcup_{i \geq 1} B(x_i, r_i), \sup_{i \geq 1} r_i \leq \varepsilon \right\} \]

\[ = \lim_{\varepsilon \to 0^+} \inf \{ \varphi(2\varepsilon) : x \in B(x, \varepsilon) \} \]

\[ = \lim_{\varepsilon \to 0^+} \inf \varphi(2\varepsilon) = 0, \]

finishing the desired proof. \( \square \)

We are left to prove the following Lemma.

**Lemma 5.6.** Let \( \gamma \) be continuous and strictly increasing. Assume that near 0, \( \gamma(r) = o\left(r^{1/d}\right) \). Let \( \varphi(s) := s^d/\gamma^{-1}(s) \). Then \( \varphi \) is continuous and \( \lim_0 \varphi = 0 \).

**Proof.** Since \( \gamma \) is bijective, to compute its inverse, we can solve for \( r \) in \( \gamma(r) = s \), and we get \( \lim_0 \gamma^{-1} = 0 \). Thus in the relation \( \gamma(r) = o\left(r^{1/d}\right) \) we can replace \( r \) by \( \gamma^{-1}(s) \) to get, for \( s \) near 0, \( \gamma(\gamma^{-1}(s)) = o\left(\left(\gamma^{-1}(s)\right)^{1/d}\right) \), which, after taking the power of \( d \) on both sides, yields \( s^d = o\left(\gamma^{-1}(s)\right) \). Dividing by \( \gamma^{-1}(s) \) yields the result of the lemma. \( \square \)

6. Examples

We are now ready to look at a class of examples, to see what our results of Sections 2, 3, 4 and 5 imply in practice. We define a 2-parameter collection of Gaussian processes as follows: for every \( \beta \in \mathbb{R}, H \in (0, 1) \), we will use the notation \( B^{H,\beta} \) for any process \( B \) satisfying (1.1) and (1.2) with, for every \( r \) in a closed interval in \( [0, 1] \),

\[ \gamma(r) = \gamma_{H,\beta}(r) := r^H \log^\beta \frac{1}{r}. \]  

(6.1)

It should be noted that since the constants in (1.1) are not equal to 1, there is a considerable amount of flexibility in how each \( B^{H,\beta} \) is defined, that is to say, for \( H \) and \( \beta \) fixed, \( B^{H,\beta} \) represents a generic element of an entire family, constrained only by (1.1) and (1.2). Thus the notation \( B^{H,\beta} \) can be understood as a class of processes, or a representative member of this class.

To define \( B \) on a larger time interval than \( [0, 1] \), one may replace \( \log^\beta(\frac{1}{r}) \) by \( \log^\beta(\frac{1}{\gamma}) \) for some appropriately small constant \( c \), but we will not consider this extension. Nor will we consider the case where the formula for \( \gamma_{H,\beta} \) has a leading constant \( c \), nor the case where \( \gamma \) is only assumed to be commensurate with \( \gamma_{H,\beta} \) defined in (6.1). All these additional cases can be treated just as we do below using either trivial or straightforward extensions, with essentially identical results. We omit any further discussion of these cases for the sake of conciseness.

When \( \beta = 0 \), the family covers fractional Brownian motion, and is essentially the class studied in [3]. Those processes are not self-similar, but have the same behavior as fBm in terms of their hitting probabilities. Indeed, one easily checks that the results of our Theorems 1.6 and 2.6 translate into the same results as for the fBm with the corresponding \( H \): the capacity lower bound holds with potential kernel \( K = K_{d-1/H} \) for any \( d \), and the Hausdorff measure upper bound holds with the function \( \varphi (r) = r^{d-1/H} \) when \( d > 1/H \).

When \( \beta \neq 0 \), the processes in this family are highly non-self-similar. In particular, for \( H \) fixed, if \( \beta > 0 \), \( B^{H,\beta} \) is infinitely more irregular than the fBm \( B^H \), and if \( \beta < 0 \) it is infinitely more regular than \( B^H \). By the classical Dudley-Fernique-type results on
regularity of Gaussian fields (see [2]), it is easy to check that \( r \mapsto r^H \log^{\beta+1/2}(1/r) \) is an almost-sure modulus of continuity for \( B^{H,\beta} \). Thus the three processes \( B^H, B^{H,\beta} \) for \( \beta < 0 \), \( B^{H,\beta} \) for \( \beta > 0 \), share the property that they are \( \alpha \)-Hölder-continuous almost surely as soon as \( \alpha < H \). As a matter of fact, if \( \beta < -1/2 \), \( B^{H,\beta} \) is almost surely \( H \)-Hölder continuous, but not \( \alpha \)-Hölder-continuous almost surely if \( \alpha > H \).

We now explore what the theorems in this article imply for \( B^{H,\beta} \), and will see that for the most part, the potential kernel \( K = K_{d-1/H} \) and the Hausdorff measure function \( \varphi(r) = r^{d-1/H} \) need to be abandoned. We will also see that the case \( H = 1/d \) represents a critical situation, in which a transition occurs on the question of hitting points, depending on the value of \( \beta \).

Let us start with the results of Section 5 on probabilities of hitting points. As a consequence of Theorems 5.3 and 5.4 we obtain the following results on hitting points for the process \( B^{H,\beta} \) (it is also a consequence of Theorems 1.6 and 2.3).

**Proposition 6.1.** If \( d < 1/H \), or if \( d = 1/H \) and \( \beta > 1/d \), then any process \( B^{H,\beta} \) hits points with positive probability. On the other hand, if \( d > 1/H \) or if \( d = 1/H \) and \( \beta < 0 \), any process \( B^{H,\beta} \) a.s. does not hit points.

**Proof.** By Theorem 5.3 and 5.4 or by Remark 2.7, both of which are applicable for our choice of \( \gamma = \gamma_{H,\beta} \), to prove that a process \( B^{H,\beta} \) hits points with positive probability, we only needs to check the convergence of

\[
\int_0 ds/\gamma^d(s) = \int_0^{dH} \frac{ds}{sH \log^{\beta}(1/s)}
\]

which holds as soon as \( dH < 1 \) or as soon as \( dH = 1 \) and \( d\beta > 1 \).

For completeness, we check that we are allowed to apply Remark 2.7, i.e. that the assumptions of Theorem 2.3 are satisfied; we only need to check Hypothesis 2.1. Since \( \gamma'(r) \) exists and is equal to \( r^{H-1}(H \log^\beta(1/r) - \beta \log^{\beta-1}(1/r)) \), we see that for small \( r \), this is strictly positive, asymptotically equivalent to \( r^{H-1}H \log^\beta(1/r) \), and strictly decreasing. Therefore Hypothesis 2.1 holds, finishing the proof of the proposition. \( \Box \)

We next apply the results of Sections 2 and 4. As a consequence of Theorems 1.6 and 2.3 we will obtain bounds for the hitting probabilities of \( B \) for arbitrary sets.

**Theorem 6.2.** Assume \( B = B^{H,\beta} \), i.e. assume (1.1) and (1.2) hold with \( \gamma = \gamma_{H,\beta} \) as in (6.1). Then the following statements hold.

1. If \( d > 1/H \), for all \( 0 < a < b < 1 \) and \( M > 0 \), there exist constants \( C_1, C_2 > 0 \) such that for any Borel set \( A \subset [-M, M]^d \),
   \[
   C_1 C_{1/\varphi}(A) \leq P(B([a, b]) \cap A \neq \emptyset) \leq C_2 H \varphi(A),
   \]
   where \( \varphi(x) = x^{d-\frac{dH}{\beta} \log^{\beta/H}(1/x)} \).

2. If \( d = 1/H \) and \( \beta < 0 \), the upper bound still holds, with the same \( \varphi \), namely \( \varphi(x) = \log^{\beta/H}(1/x) \).

3. If \( d = 1/H \), for \( \beta < 1/d \), the lower bound holds with \( \varphi(x) = \log^{\beta/H-1}(1/x) \).

4. If \( d = 1/H \), for \( \beta \geq 1/d \), the lower bound holds with \( \varphi \equiv 1 \).

5. If \( d < 1/H < +\infty \) the lower bound holds with \( \varphi \equiv 1 \).

**Remark 6.3.** Notice that in the case \( d = 1/H \), for \( \beta < 0 \), there is a discrepancy factor equal to \( \log(1/x) \) between the two functions \( \varphi \) in the upper and lower bounds. This lack of precision at the logarithmic level is not visible in the power scales. It shows that in
the so-called “critical case” identified for fBm and other power-scale-based processes as in [4], at least one of the lower capacity bounds or the upper Hausdorff measure bounds must be inefficient.

Unlike in the power scale, our theorems allow us to look at our examples when \( H = 0 \) or 1. When \( H = 1 \), we get non-trivial (non-smooth) processes as soon as \( \beta > 0 \). When \( H = 0 \), we get continuous processes as soon as \( \beta < -1/2 \); in this case, Condition (3.1) does not hold, so care is required for the upper bound.

Corollary 6.4. Assume \( B, a, b, M, A \) are as in Theorem 6.2. If \( H = 1 \) and \( \beta > 0 \), for all \( d > 1 \), both bounds in Theorem 6.2 hold with \( \varphi(x) = x^{d-1} \log^\beta (1/x) \). If \( H = 0 \) and \( \beta < -1/2 \), then the lower bound in Theorem 6.2 hold with \( \varphi = 1 \), so that \( B \) hits points with positive probability.

We can also construct Borel sets which are polar for one process and are visited with positive probability for another, with both processes having the same Hölder continuity properties. In the next corollary, we consider the chance for a process in \( \mathbb{R}^d \) to hit a linear Cantor set (a subset of the \( x \)-axis). While our method applies to a variety of Cantor sets (see for instance the \( p \)-Cantor sets in [4]), for simplicity, we consider first the classical Cantor set with fixed ratio \( q \in (0, 1/2) \) defined as follows. Let \( A_0 = [0, 1] \); for \( n \in \mathbb{N} \), assuming \( A_n \) has been defined and is a union of \( 2^n \) intervals of length \( q^n \), we define \( A_{n+1} \) by removing a central open interval of length \( q^n (1 - 2q) \) from each interval in \( A_n \). The Cantor set is \( A := \lim_{n \to \infty} A_n = \cap_n A_n \). It is known that \( A \) has Hausdorff dimension \( d(A) = \log 2 / \log (1/q) \in (0, 1) \). It is also known that it has positive K-capacity \( C_K(A) > 0 \) if and only if \( \sum_{n=1}^{\infty} 2^{-n} K(q^n) < \infty \). See [3]. Moreover, it is easy to check that the \( d(A) \)-Hausdorff measure of \( A \) satisfies \( \mathcal{H}_{d(A)}(A) \leq 1 \); indeed \( A_n =: \sum_{j=1}^{2^n} A_{n,j} \) is a covering of \( A \) with intervals \( A_{n,j} \) of length \( q^n \), which can be made arbitrarily small, and \( \sum_{j=1}^{2^n} |A_{n,j}|^{d(A)} = \sum_{j=1}^{2^n} q^{d(A)n} = (2q^{d(A)})^n = 1 \) since \( q^{d(A)} = 2^{-1} \). With all these facts, we state the following.

Corollary 6.5. For any dimension \( d \geq 2 \), let \( H \in (1/d, 1/(d-1)) \). Assume \( A \) is a binary Cantor set on the \( x \)-axis of \( \mathbb{R}^d \) with constant ratio \( q := 2^{-1/(d-1/H)} \), so that its Hausdorff dimension is \( d - 1/H \in (0, 1) \). Then \( A \) is polar for any process in the class \( B^{H,\beta} \) with \( \beta < 0 \), i.e. with probability 1, \( B^{H,\beta} \) does not hit \( A \) during the time interval \( [a, b] \subset (0, \infty) \). On the other hand, for \( \beta' > H \), any process in the class \( B^{H,\beta'} \) hits \( A \) with positive probability during the time interval \( [a, b] \).

Note that processes in the classes \( \{ B^{H,\beta} : -1/2 \leq \beta < 0 \} \) and \( \{ B^{H,\beta'} : \beta' \geq H \} \) share the same Hölder continuity in the sense that they are \( \alpha \)-Hölder-continuous a.s. if and only if \( \alpha < H \).

The previous corollary shows that our results help us construct processes that hit classical Cantor sets, and others that do not even though their path regularity is very similar to the first ones. This is an improvement over Hölder-scale tools such as those in [4]; Corollary 6.5 cannot be established with those tools, since it is known that a Cantor set with constant ratio \( q = 2^{-1/(d-1/H)} \) has null \((d - 1/H)\)-capacity, so that capacity lower bounds for hitting probabilities are inconclusive; similarly, a positive \((d - 1/H)\)-Hausdorff measure does not help prove whether a set is non-polar.

On the other hand, our results leave an open gap in the analysis: the case \( \beta \in [0, H] \) is not covered. For instance, our results are not fine enough to tell whether classical fBm (a member of the class \( B^{H,0} \), i.e. with \( \beta = 0 \)) in \( \mathbb{R}^d \) with Hurst parameter \( H \) hits a
classical linear Cantor set with dimension $d - 1/H$ on the $x$-axis. However, our results are fine enough to show us how to construct a generalized Cantor set with the same dimension $d - 1/H$, which fBm does hit. For such a construction, referring again to \([5]\), consider a sequence $(q_n)_{n \in \mathbb{N}}$ of scalars in $(0, 1/2)$, and construct a Cantor set $\tilde{A} = \cap_n A_n$ like we did above for $A$, except that from levels $n$ to $n + 1$, we remove the central open interval of length $(\prod_{i=1}^{n} q_i) (1 - 2q_{n+1})$. Then it is known that for a given capacity kernel $K$, the $K$-capacity $C_K(A)$ is strictly positive if and only if $\sum_n 2^{-n}K(\prod_{i=1}^{n} q_i)$ is a convergent series. We now devise a sequence $(q_n)_{n}$ such that the resulting Cantor $\tilde{A}$ set is slightly bigger than the classical set for which $q_n \equiv q = 2^{-1/(d-1/H)}$, just big enough for us to guarantee a positive $(d - 1/H)$-capacity, and therefore for any process in the class $B^{H,0}$, including classical fBm, to hit $\tilde{A}$ with positive probability.

**Corollary 6.6.** Let $c > 1$. For any dimension $d \geq 2$, let $H \in (1/d, 1/(d-1))$. Assume $\tilde{A}$ is a generalized Cantor set on the $x$-axis of $\mathbb{R}^d$ with level-$n$ ratio $q_n \geq (2^{-1} (1 - c/n))^{1/(d-1/H)}$, constructed as described above. Then any process in the class $B^{H,0}$, including fBm, hits $\tilde{A}$ with positive probability during the time interval $[a, b] \subset (0, \infty)$.

If the inequality on $q_n$ is replaced by an equality, then the Hausdorff dimension of $\tilde{A}$ is $d - 1/H \in (0, 1)$.

**Proof of Theorem 6.2.** Step 0: checking the assumptions of the theorems.

Throughout this entire proof, we will use the fact that the conclusions of Theorems 4.6 and 2.5 can be stated for any functions that are commensurate with the $\varphi$ and $K$ defined therein.

In the proof of Proposition 6.3, we already checked that $\gamma_{H,\beta}$ satisfies Hypothesis 2.1, so that we may apply Theorem 2.5. To be allowed to apply Theorem 4.6, which we only need to check that $\gamma_{H,\beta}$ satisfies Hypothesis 2.1, we only need to check the following:

(i): $\varphi(x) = x^d/\gamma^{-1}(x)$ is right-continuous and non-decreasing near $0$ with $\lim_{0+} \varphi = 0$.

(ii): $\gamma$ satisfies the condition \((5.1)\).

Condition (ii) is easy for $\beta \geq 0$ since for $x, y < 1/e$, we have $\log (1/x), \log (1/y) > 1$ and thus

$$
\gamma(x) \gamma(y) = (x^H) \log^\beta (1/x) \log^\beta (1/y) \geq (x^H)^{2-\beta} (\log (1/x) + \log (1/y))^\beta = 2^{-\beta} \gamma(xy),
$$

so that we get $\int_0^{1/2} \gamma(xy) \frac{dy}{y \sqrt{\log (1/y)}} \leq 2^\beta \gamma(x) \int_0^{1/2} \frac{dy}{y^{1-\beta}} = \gamma(x) 2^{H+\beta}/H$. When $\beta < 0$, on the other hand, since for $x < 1$, $\log (1/(xy)) \geq \log (1/x)$, we get

$$
\int_0^{1/2} \gamma(xy) \frac{dy}{y \sqrt{\log (1/y)}} \leq x^H \int_0^{1/2} \frac{dy}{y^{1-H} \sqrt{\log (1/y)}} \leq \gamma(x) 2^{H}/2.
$$

To prove condition (i), we start by noting that for small $x$, $\gamma^{-1}(x)$ is commensurate with $x^{d-1/H} \log^{-\beta}/H(1/x)$. Since we may replace $\varphi$ by a constant multiple of it to apply to Theorem 4.6 we only need to check that $x \mapsto x^{d-1/H} \log^\beta(1/x)$ is right-continuous
and non-decreasing near 0. Since we assume that $d > 1/H$ or $\{d = 1/H; \beta < 0\}$, this is immediately true.

**Step 1: Proof of case 1.** By Theorem 4.6, the upper bound in case 1 holds with $ϕ(x) = x^d/γ^{-1}(x)$. Since $γ^{-1}(x)$ is commensurate with $x^{1/H} \log^{-β/H}(1/x)$, the upper bound conclusion of case 1 holds. By Theorem 2.5, the lower bound in case 1 holds with the capacity $C_K$ where $K = v \circ γ^{-1}$. By Proposition 4.7, the lower bound conclusion of case 1 will hold as soon as we can show that $d > 1/\lim_{r→0} rγ'(r)/γ(r)$. We have $γ'(r) = r^{H-1}(H \log^β(1/r) - β \log^{β-1}(1/r))$ so that the limit above computes as $1/H$, finishing the proof of case 1.

**Step 2: Proof of case 2.** This case follows using the argument and computation in Step 1, since the original $ϕ(x)$ in Theorem 4.6 is commensurate with $\log^{-β/H}(1/x)$.

**Step 3: Proof of case 3 and 4.** If $d = 1/H$, for any $β$, we may use Theorem 2.5 with $K = \max(1, v \circ γ^{-1})$, but one may check that Proposition 4.7 is not applicable, so we need to compute $v$ directly. Since everything only needs to be determined up to multiplicative constants, we do not keep track of them. We compute

$$v(r) = \int_r^{b-a} s^{-1} \log^{-β/H}(1/s) \, ds$$

$$= \frac{1}{1 - βH} \left( \log^{1-β/H}(1/r) - \log^{1-β/H}(1/b-a) \right)$$

$$\asymp \log^{1-β/H}(1/r).$$

Since, as we said above, $γ^{-1}(x) \asymp x^{1/H} \log^{-β/H}(1/r)$, it is then easy to get

$$v \circ γ^{-1}(x) \asymp \log^{1-β/H}\left(x^{-1/H} \log^{β/H}(1/r)\right)$$

$$\asymp \log^{1-β/H}(x^{-1}).$$

Therefore, for $β ≥ H = 1/d$ (case 4), we see that $v \circ γ^{-1}(x)$ is bounded, and thus we use $K = 1$, while for $β < H = 1/d$, $v \circ γ^{-1}(x)$ is unbounded, and we thus use $K(x) = \log^{1-β/H}(x^{-1})$.

**Step 4: Proof of case 5.** For $dH < 1$, let $ε > 0$ such that $dH + ε < 1$. Using the same argument as in the previous step, and using the fact that $s^ε \log^{-β/H}(1/s)$ is bounded above by some constant $M$, we compute

$$v(r) = \int_r^{b-a} s^{-dH} \log^{-β/H}(1/s) \, ds$$

$$= \int_r^{b-a} s^{-dH-ε} s^ε \log^{-β/H}(1/s) \, ds$$

$$≤ M \int_r^{b-a} s^{-dH-ε} \, ds$$

$$≤ \frac{(b-a)^{1-dH-ε} M}{1 - dH - ε}.$$
Proof of Corollary 6.4. The result for $H = 1$ is immediate. For $H = 0$, noting that $v(r) := \int_r^{b-a} \log^{-\beta}(1/s) \, ds$ is bounded, by Theorem 2.5 the lower bound holds with $\varphi = 1$. □

Proof of Corollary 6.5. First note that from the definitions, since a Hausdorff measure is computed using scalar diameters, and a capacity is computed using measures supported on the set, these quantities relative to a subset of $\mathbb{R}$ are invariant when the subset is immersed in $\mathbb{R}^d$.

Therefore, as mentioned in the paragraph following Corollary 6.4, our Cantor set $A$ has finite Hausdorff measure $\mathcal{H}_{d-1/H}(A) \leq 1$. This is the Hausdorff measure $\mathcal{H}_\psi(A)$ with $\psi(x) = x^{d-1/H}$. By the definition of Hausdorff measure, it is then immediate that, for the function $\varphi(x) = x^{d-1/H} \log^{\beta}(1/x)$ with $\beta < 0$, the Hausdorff measure $\mathcal{H}\varphi(A) = 0$. Theorem 6.2, part 1, upper bound, implies that $A$ is polar for $B^H,\beta$.

Now to show that, for $\beta' \geq H$, $B^{H,\beta'}$ hits $A$ with positive probability, by Theorem 6.2, part 1, lower bound, it is sufficient to show that, with $\bar{\varphi}(x) = x^{d-\frac{\beta}{H}} \log^{\beta'/H}(1/x)$, $C_{1,\bar{\varphi}}(A) > 0$. Using the classical criterion mentioned above (see [3]), it is sufficient to show that $\sum_{n=1}^{\infty} 2^{-n} (1/\bar{\varphi}(q^n)) < \infty$. Since we chose $q$ such that $q^{1/H-d} = 2$, we compute
\[
\sum_{n=1}^{\infty} 2^{-n} (1/\bar{\varphi}(q^n)) = \sum_{n=1}^{\infty} \log^{-\beta/H} (q^n) = \log (1/q) \sum_{n=1}^{\infty} n^{-\beta/H} < \infty.
\]
This finishes the proof of the corollary. □

Proof of Corollary 6.6. Similarly to the previous proof, thanks to Theorem 6.2, part 1, lower bound, and thanks to the characterization of capacity positivity described above, it is sufficient to show that, with $\tilde{\varphi}(x) = x^{d-\frac{\beta}{H}} \log^{\beta'/H}(1/x)$, $C_{1,\tilde{\varphi}}(A) > 0$. By assumption, this series is bounded above by $\sum_{n=1}^{\infty} \prod_{i=1}^{n} (1 - \frac{c_i}{c})$. Using $\log (1 - u) \leq -u$ for all $u > 0$, the product in this series is bounded above as
\[
\prod_{i=1}^{n} (1 - \frac{c_i}{c}) = \exp \left( \sum_{i=1}^{n} \log (1 - c_i/i) \right) \leq \exp \left( -c \sum_{i=1}^{n} i^{-1} \right)
\]
where $\gamma$ is Euler’s constant. For any $c > 1$, this is the general term of a converging series, finishing the proof of the corollary, modulo the statement about the Hausdorff dimension of $A$ which is well known (see [3]) and elementary, and can serve as an exercise for the interested reader. □

Appendix A. Necessary Condition to Hit Points, via Gaussian Supremum Estimates

We present here a partial converse of Theorem 5.3, that is, we seek a necessary condition in order for the process $B$ to hit points in $\mathbb{R}^d$. For this, we will use a strategy defined in [3]. Our point here is that this strategy does not seem to yield results that are as sharp as the techniques leading to Theorems 4.5 and 2.5 as evidenced by Example A.3 below. We include this short section to illustrate this point only.

Hypothesis A.1. The following limit holds
\[
\lim_{\varepsilon \to 0} \frac{\hat{h}^d(\varepsilon)}{\varepsilon} = 0,
\]
where recall that $h(r) = \gamma(r) \log^{1/2} \left( \frac{1}{r} \right)$ is the uniform modulus of continuity of $B$.

**Theorem A.2.** Hypothesis A.1 implies that the process $B$ does not hits points in $\mathbb{R}^d$ a.s., that is,

$$P((B)^{-1}(x) \neq \emptyset) = 0.$$ 

**Proof.** Let $0 < a < b$ fixed. It suffices to show that

$$P([a, b] \cap (B)^{-1}(x) \neq \emptyset) = 0.$$ 

We start covering the interval $[a, b]$ with closed non-overlapping intervals $I_j(\varepsilon)$ of side $\varepsilon > 0$, $1 \leq j \leq n(\varepsilon)$. Then there exist constants $C_i(a, b)$, $i = 1, 2$ such that

$$C_1 \varepsilon \leq n(\varepsilon) \leq C_2 \varepsilon.$$ 

Let us now consider

$$v(\varepsilon) = \sup_{s, t \in [a, b]: |t - s| \leq \varepsilon} \left| \frac{|B(t) - B(s)|}{h(\varepsilon)} \right|,$$

where $h(\varepsilon) = \gamma(\varepsilon) \log^{1/2} \left( \frac{1}{\varepsilon} \right)$ is the uniform modulus of continuity of $B$.

Observe that

$$\max_{1 \leq j \leq n(\varepsilon)} \sup_{s, t \in I_j(\varepsilon)} \left| \frac{|B(t) - B(s)|}{h(\varepsilon)} \right| \leq v(\varepsilon). \quad (A.1)$$

Consider the Gaussian field

$$B(s, t) = \frac{B(t) - B(s)}{h(\varepsilon)}, \quad s, t \in [a, b].$$

Then, $v(\varepsilon) = \sup \{|B(s, t)| : |t - s| \leq \varepsilon\}$. By Borell-Sudakov-Tsirelson’s inequality, for all $u \geq 0$,

$$P(v(\varepsilon) - E[v(\varepsilon)] > u) \leq \exp \left( -\frac{u^2}{2\ell \log \left( \frac{1}{\varepsilon} \right)} \right),$$

where $\ell$ is the constant in (1.1). Taking $u^2 = 4\ell$ and $\varepsilon = \frac{1}{m}$, $m \in \mathbb{N}$, and using Borel-Cantelli’s lemma, it yields that there exists $m_0(\omega)$ such that for all $m > m_0$

$$v \left( \frac{1}{m} \right) \leq E \left[ v \left( \frac{1}{m} \right) \right] + 4\ell =: C_3. \quad (A.2)$$

Consider now the random process

$$N(\varepsilon) = \sum_{j=1}^{n(\varepsilon)} 1_{\{3 \in I_j(\varepsilon); B(s) = x\}}.$$ 

It follows from (A.1) and (A.2) that for all $\varepsilon$ small enough,

$$N(\varepsilon) \leq M(\varepsilon) := \sum_{j=1}^{n(\varepsilon)} 1_{\{|B(t)| \leq 2C_3 h(\varepsilon), \forall t \in I_j(\varepsilon)\}}.$$ 

Hence, to conclude the proof of the theorem, is suffices to show that

$$\lim_{\varepsilon \to 0} E[M(\varepsilon)] = 0.$$ 

Note that

$$1_{\{|B(t)| \leq 2C_3 h(\varepsilon), \forall t \in I_j(\varepsilon)\}} \leq 1_{\{|B(b_j(\varepsilon))| \leq 2C_3 h(\varepsilon)\}},$$
where \( b_j(\varepsilon) \) is the center of \( I_j(\varepsilon) \). Hence,

\[
E[M(\varepsilon)] \leq \sum_{j=1}^{n(\varepsilon)} P[\|B(b_j(\varepsilon))\| \leq 2C_3 h(\varepsilon)].
\]

Let \( p_t(y) \) denote the (Gaussian) density of \( B(t) \) at \( y \in \mathbb{R}^d \). Then,

\[
\sup_{y \in \mathbb{R}^d} \sup_{t \in [a,b]} p_t(y) \leq C_4(a,b).
\]

Thus, as \( n(\varepsilon) \leq C_2/\varepsilon \),

\[
E[M(\varepsilon)] \leq C_5 \frac{h^d(\varepsilon)}{\varepsilon}.
\]

By Hypothesis A.1 we conclude that

\[
\lim_{\varepsilon \to 0} E[M(\varepsilon)] \leq C_5 \lim_{\varepsilon \to 0} \frac{h^d(\varepsilon)}{\varepsilon} = 0,
\]

and the proof is completed. \( \Box \)

**Example A.3.** If we apply this theorem to our functions \( \gamma_{H,\beta} \) of Section 6 in the critical case \( d = \frac{1}{H} \), we obtain that \( B \) does not hit points a.s. if \( h(r)^d r^{-1} = \log \frac{d+3}{2} (1/r) \) tends to 0 as \( r \to 0 \), i.e. if \( \beta < -1/2 \), which is less sharp than the result obtained in Proposition 6.1.

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