THE $\mathbb{Z}^d$ ALPERN MULTI-TOWER THEOREM FOR RECTANGLES: A TILING APPROACH

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ABSTRACT. We provide a proof of the Alpern multi-tower theorem for $\mathbb{Z}^d$ actions. We reformulate the theorem as a problem of measurably tiling orbits of a $\mathbb{Z}^d$ action by a collection of rectangles whose corresponding sides have no non-trivial common divisors. We associate to such a collection of rectangles a special family of generalized domino tilings. We then identify an intrinsic dynamic property of these tilings, viewed as symbolic dynamical systems, which allows for a multi-tower decomposition.

1. INTRODUCTION

The Rohlin Lemma is one of the fundamental results of ergodic theory. In [1] Alpern proved a generalization of the Rohlin Lemma; he showed that given a free, measure preserving transformation $T$ of a Lebesgue space $X$, one can decompose $X$ into $k$ towers provided that $k \geq 2$ and the heights of the towers do not have a non-trivial common divisor. He also showed that each tower can be made to take up any proportion of the space. A later, and shorter, proof was given in [5]. Alpern’s Multi-tower Theorem has also played an important role in the proof of many recent important results in ergodic theory (see [1] for the original application; see [7] and [5] for a survey of more recent applications) and has been the subject of some recent research activity (see for example [2] and [4]).

The Rohlin Lemma for $\mathbb{Z}^d$ actions for towers with rectangular shape was first shown by Conze [3] and Katznelson and Weiss [6]. Their results were subsumed by the later work of Ornstein and Weiss [8] who have generalized the Rohlin Lemma to free actions of countable amenable groups. It follows from their work that for finite subsets $R$ of $\mathbb{Z}^d$ there is a Rohlin Lemma with towers of shape $R$ if and only if $R$ tiles $\mathbb{Z}^d$.

The Alpern Multi-tower Theorem was generalized to $\mathbb{Z}^d$ for rectangular towers whose corresponding dimensions do not have a non-trivial common divisor by Prikhodko [9] in 1999. In this paper we give a different proof of the same result obtained independently by the author in May, 2005. The
author gave multiple presentations of the argument in this paper before learning from V. S. Prasad about Prikhodko’s article in October, 2005.

The approach given here can be viewed as playing the role of the “simple” proof of the higher dimensional result analogous to [5]. In particular, establishing the desired distribution of the towers is an easy part of our argument, whereas it constitutes the bulk of the work in [9].

The more significant difference in our approach lies in that we identify an intrinsic and dynamic property of a family of tilings which allows for a multi-tower decomposition with those tile shapes. This formulation provides a fruitful direction for investigating a more general multi-tower theorem.

More specifically, the \( \mathbb{Z}^d \) Alpern Multi-tower Theorem for rectangles can be reformulated as a problem of tiling orbits of an action. It states that for a.e. \( x \in X \), the orbit of \( x \) under the action \( T \) can be measurably tiled by \( k \geq 2 \) rectangles whose corresponding dimensions have do not have a common divisor. Further, these tilings can be constructed so that each rectangular tile has a previously prescribed probability distribution. Tilings of \( \mathbb{Z}^d \) by rectangles can be viewed as tilings of \( \mathbb{R}^d \) by dominos where each domino has integer dimensions, and whose vertices lie on the integer lattice. 

In the proof we present here we associate to each collection of required tower shapes a domino tiling of \( \mathbb{R}^d \) and we translate the problem of decomposing the space into multiple towers into the problem of finding a factor of the given system which is an invariant measure on the associated domino tiling system. This point of view allows us to identify a fairly innocuous mixing property of the tiling system (having a uniform filling set) as the key ingredient necessary to construct such a factor.

There are many other well studied and interesting tilings which have this property including lozenge tilings and square ice, (see [10]) and are a natural place to start investigating the possibility of a general multi tower theorem. Namely, given a collection of shapes which tile \( \mathbb{Z}^d \) can one establish necessary and sufficient conditions for there to be an Alpern Theorem with towers of these shapes?

We establish some notation to state the multi-tower theorem more formally. For \( \vec{w} = (w_1, \cdots, w_d) \in \mathbb{N}^d \) we set \( R_{\vec{w}} = \prod_{i=1}^{d} [0, w_i - 1] \cap \mathbb{Z} \). We call \( R_{\vec{w}} \) a rectangle in \( \mathbb{Z}^d \). Given a \( \mathbb{Z}^d \) action \( T \) on a space \( X \), a subset \( \tau = \bigcup_{\vec{v} \in R_{\vec{w}}} T^{\vec{v}} F \) of \( X \) is called a Rohlin tower of shape \( R_{\vec{w}} \) with \( \vec{w} \in \mathbb{N}^d \) if there exists a set \( F \in \mathcal{M} \) with the property that \( T^{\vec{v}} F \cap T^{\vec{u}} F = \emptyset \) for all \( \vec{v}, \vec{u} \in R_{\vec{w}} \). We call the set \( F \) the base of the tower.

**Theorem 1.1.** (The \( \mathbb{Z}^d \) Alpern Multi-tower Theorem for rectangles) Let \( \vec{w}^1, \vec{w}^2, \cdots \in \mathbb{N}^d \), and \( p_1, p_2, \cdots \in \mathbb{R}^+ \) satisfy

\[ \begin{aligned} & (1) \quad \text{for all } i = 1, \cdots, d \quad \gcd(w_i^1, w_i^2, \cdots) = 1 \quad \text{and} \\
& (2) \quad \sum_{j=1}^{\infty} p_j = 1. \end{aligned} \]
Then given any free and measure preserving $\mathbb{Z}^d$ action $T$ of a Lebesgue probability space $(X, \mathcal{M}, \mu)$ for all $j \geq 1$, there are Rohlin towers $\tau^j$ of shape $R_{\vec{w}^j}$ with

\begin{align}
\mu(\tau^j) &= p_j, \\
\tau^j \cap \tau^{j'} &= \emptyset \text{ if } j \neq j', \\
\text{and } \bigcup_{j=1}^{\infty} \tau^j &= X.
\end{align}

In the case where $d = 1$ Theorem 1.1 is Alpern’s original result. The numbers $w_1, w_2, \ldots$ represent the heights of the towers and condition (11) states that their greatest common divisor is 1. This is clearly a necessary assumption since a non-trivial common divisor of the heights of the towers implies non-ergodicity of some power of the transformation $T$. A similar argument can be made in the higher dimensional case, where a non-trivial common divisor of corresponding dimensions $\vec{e}_i$ implies the non-ergodicity of the group element $T^{\vec{e}_i}$, where we let $\vec{e}_1, \ldots, \vec{e}_d$ denote the standard basis of $\mathbb{Z}^d$. It is worth noting that the higher dimensional result does not require additional constraints on the relationship between the dimensions of the towers in directions $\vec{e}_i$ and $\vec{e}_j$ if $i \neq j$. Namely condition (1) is a sufficient condition for all dimensions $d \geq 1$.

The organization of the paper is as follows. We begin by considering Theorem 1.1 in the case of finitely many rectangles. This case allows us to highlight the connection to symbolic tiling spaces, the mixing condition on these systems necessary to prove the result, and the ease with which we can guarantee the correct distribution of tiles. Section 2 introduces the tiling spaces and contains the reduction of the proof of Theorem 1.1 to finding an invariant measure on the tiling system for the case of finitely many rectangles (Theorem 2.1). Section 3 contains necessary definitions and results related to mixing properties of shifts of finite type. Section 4 contains the proof of Theorem 2.1 and Section 5 contains the proof of Theorem 1.1 in the case of countably infinite rectangles.

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2. Theorem 1.1 in the finite case and generalized domino tiling shift spaces

We begin by defining generalized domino tiles and their associated symbolic systems. Fix $k \in \mathbb{N}$, $k \geq 2$ and vectors $\vec{w}^1, \ldots, \vec{w}^k \in \mathbb{N}^d$ satisfying the conditions of Theorem 1.1. Define the vector $\vec{W} \in \mathbb{N}^d$ with $W_i = \prod_{j=1}^{k} w_i^j$. 
We define $k + 1$ domino tiles: the dominoes
\[ \tau_j = \prod_{i=1}^{d} (0, w_j^i - 1) \]
for $j = 1, \cdots k$ and an additional domino
\[ \tau^P = \prod_{i=1}^{d} (0, W_i - 1) \]
which we will refer to as the large domino.

Consider all tilings of $\mathbb{R}^d$ with these dominos subject to the condition that the vertices of the dominos lie on the integer lattice. We refer to such tilings as $(\vec{w}^1, \cdots, \vec{w}^k)$ domino tilings. These tilings can easily be coded into a one-step $\mathbb{Z}^d$ shift of finite type $Y(\vec{w}^1, \cdots, \vec{w}^k)$ which we call the $\vec{w}^1, \cdots, \vec{w}^k$ domino tiling shift. We will be working extensively with this shift and thus we will give some details of one such coding to establish necessary notation and terminology.

Define the alphabet of the shift by $A = \{ \tau^j_\vec{v}: 1 \leq j \leq k, \vec{v} \in R_{\vec{w}^j}\}$. The set of forbidden blocks of $Y$ are defined first to ensure that for each $1 \leq j \leq k$ the symbols $\tau^j_\vec{v}$ can be placed together so that there is a block $\tau^j$ in $A_{R_{\vec{w}^j}}$ which is a symbolic representation of the domino $\tau^j$. In the symbolic domino $\tau^j$ the symbol $\tau^j_\vec{0}$ is taken to be the origin and appears in the lower left hand vertex of the block. Then for each $\vec{v} \in R_{\vec{w}^j}$, the symbol $\tau^j_\vec{v}$ appears in position $\vec{v}$ relative to the origin. We define the symbolic block corresponding to the domino $\tau^P$ analogously. A two dimensional example of the alphabet of a generalized domino tiling shift and the symbolic domino blocks are shown in Figure 1.

![Figure 1](image_url)

**Figure 1.** The alphabet and domino tiles of $Y(\vec{w}^1, \vec{w}^2)$ with $\vec{w}^1 = (3, 2)$ and $\vec{w}^2 = (2, 3)$.

In addition, the forbidden blocks are defined so that the words in $Y$ correspond to a tiling of $\mathbb{R}^d$ by the dominos $\tau^j$ and $\tau^P$. For example, symbols
of the form $\tau_j^{(w_1^j, \ldots)}$ can only be followed horizontally by symbols of the form $\tau_j^{(0, \ldots)}$, and so forth.

For $y \in Y$ and $\vec{v} \in \mathbb{Z}^d$, let $y[\vec{v}] \in A$ denote the symbol occurring in the point $y$ at position $\vec{v}$. We will prove the following result about domino tiling shifts.

**Theorem 2.1.** Fix $k \in \mathbb{N}, k \geq 2$. Let $\vec{w}^1, \ldots, \vec{w}^k$, and $p_1, \ldots, p_k$ satisfy the conditions of Theorem 1.1. Set $Y = Y(\vec{w}^1, \ldots, \vec{w}^k)$ and let $A$ denote the alphabet of $y$.

Then given any free, measure preserving $\mathbb{Z}^d$ action $T$ on $(X, \mathcal{M}, \mu)$ there is a partition $\phi : X \to A$ so that for a.e. $x \in X$ there is a point $y_x \in Y$ with

$$\phi(T^\vec{v}(x)) = y_x[\vec{v}]$$

and for $j = 1, \ldots, k$

$$\mu(\phi^{-1}(\tau^j)) < \min_{i=1, \ldots, k} p_i.$$

We obtain the following immediate corollary.

**Corollary 2.2.** Theorem 1.1 holds in the case of finitely many towers.

**Proof.** Notice that (6) guarantees that for all $j = 1, \ldots, k$ the set $\phi^{-1}(\tau^j)$ is a Rohlin tower of shape $R_{\vec{w}^j}$ and

$$\phi^{-1}(\tau^j) \cap \phi^{-1}(\tau^{j'}) = \emptyset$$

when $j \neq j'$. Similarly $\phi^{-1}(\tau^P)$ is a Rohlin tower of shape $R_{\vec{w}}$ and

$$\phi^{-1}(\tau^P) \cap \phi^{-1}(\tau^j) = \emptyset$$

for all $j$.

By (7) each tower $\phi^{-1}(\tau^j)$ takes up less than its required distribution. In particular, the numbers $\alpha_j = (p_j - \mu(\phi^{-1}(\tau^j)))$ are all positive. We will construct additional towers of each shape $R_{\vec{w}^j}$ with measure $\alpha_j$ by decomposing the large tower $\phi^{-1}(\tau^P)$.

Clearly $\mu(\tau^P) = \Sigma \alpha_j$. We partition $E_P = \phi^{-1}(\tau^P)$, the base of the large tower, into $k$ measurable subsets, $E_j^P$, each of measure $\alpha_j$. Each $E_j^P$ is the base of a tower of shape $R_{\vec{w}}$. By definition $\frac{W_i}{w_i} \in \mathbb{N}$ for all $i = 1, \ldots, d$ so for each $j$ we can think of $R_{\vec{w}}$, as consisting of a disjoint union rectangles of shape $R_{\vec{w}^j}$. In particular, for $0 \leq j_i \leq \frac{W_i}{w_i}$, sets of the form

$$T^{j_1 w_1^j \vec{e}_1 + j_2 w_2^j \vec{e}_2 + \cdots + j_d w_d^j \vec{e}_d}(E_j^P)$$

are all bases of pairwise disjoint Rohlin towers of shape $R_{\vec{w}^j}$. For each $j$ we define $E_j$ to be the union of all these base sets with $\phi^{-1}(\tau^j)$.

The sets $E_j$ are now bases of Rohlin towers of shape $R_{\vec{w}^j}$. Denoting the resulting tower by $\tau^j$ we now have $\mu(\tau^j) = p_j$. It is clear from (5), (6) and our construction that (4) and (5) are also satisfied. □
3. **Good Brick Walls and Uniform Filling Sets of Domino Tiling Shifts**

   The proof of Theorem 2.1 is an application of the ideas in [10]. There (Corollary 1.3 of Theorem 1.1) it is shown that given any free, measure preserving, and ergodic $\mathbb{Z}^d$ action $T$ on a Lebesgue space $(X, \mathcal{M}, \mu)$ conclusion (6) of Theorem 2.1 holds for any shift of finite type $Y$ which has a uniform filling set, defined below. Here we modify that proof, first to eliminate the need for ergodicity in the argument, and second to also obtain conclusion (7) of Theorem 2.1.

   We begin by defining a uniform filling set for a shift of finite type. Let $|A|$ and $A^c$ denote the cardinality and complement of a set $A$ respectively. Given a rectangle $R_{\vec{w}} \subset \mathbb{Z}^d$ let $R_{\vec{w}}^\ell$ denote the rectangle $R_{\vec{w}}$ together with an outer filling collar of uniform width $\ell$. More formally,

   $$R_{\vec{w}}^\ell = \{ \vec{v} \in \mathbb{Z}^d : -\ell \leq v_i \leq w_i - 1 + \ell, 1 \leq i \leq d \},$$

   **Definition 3.1.** A shift of finite type $Y$ has a uniform filling set if there is a translation invariant subset $Z$ of $Y$ with $|Z| > 1$ and an integer $\ell > 0$ with the property that given any $\vec{w} \in \mathbb{N}^d$ and $z, z' \in Z$, there is a point $y \in Y$ with the property that

   $$y[R_{\vec{w}}] = z[R_{\vec{w}}]$$

   and

   $$y[(R_{\vec{w}}^\ell)^c] = z'[(R_{\vec{w}}^\ell)^c].$$

   We call $\ell$ the filling length of the set $Z$.

   **Definition 3.2.** Let $\vec{w}_1, \ldots, \vec{w}_k \in \mathbb{N}^d$, with $k \geq 2$ and let $Y = Y(\vec{w}_1, \ldots, \vec{w}_k)$. Let $\tau$ be a domino of shape $R_{\vec{w}}$. The good brick wall with domino $\tau$ is the periodic word $y \in Y$ with the property that for all $\vec{v} \in \mathbb{Z}^d$

   $$y[\vec{v}] = \tau(v_1 \mod w_1, \ldots, v_d \mod w_d).$$

   In our arguments we will be using the good brick wall with the special domino $\tau^P$. Figure 2 shows part of such a good brick wall $y^P$ for a two-dimensional example.

   **Theorem 3.3.** Let $Y$ be the $\vec{w}_1, \ldots, \vec{w}_k$ domino tiling shift for $\vec{w}_1, \ldots, \vec{w}_k \in \mathbb{N}^d, k \geq 2$ which satisfy the conditions of Theorem 2.1. Let $y^P \in Y$ be the good brick wall with domino $\tau^P$. Then the set

   $$Z = \bigcup_{\vec{v} \in \mathbb{Z}^d} T^{\vec{v}}(y^P)$$

   is a uniform filling set for $Y$.

   **Proof.** For notational convenience we give the proof for the case $d = 2$. The argument for higher dimensions is analogous. Since the vectors $\vec{w}_1, \ldots, \vec{w}_k$ satisfy (1) of Theorem 2.1 there exists $R \in \mathbb{N}$ with the property that for
all integers $r \geq R$ there exist non-negative integers $a_j, b_j$ with $j = 1, \ldots, k$ such that

$$r = \sum_{j=1}^{k} a_j w_1^j = \sum_{j=1}^{k} b_j w_2^j. \tag{10}$$

Let $\ell = R + 2W_1 + 2W_2$, where as before $\vec{W} = (W_1, W_2) = \left( \prod_{j=1}^{k} w_1^j, \prod_{j=1}^{k} w_2^j \right)$. We will show that $Z$ is a uniform filling set with filling length $\ell$.

Let $z, z' \in Z$, and a rectangle $B\vec{u} + \vec{v} \subset \mathbb{Z}^2$ be given, with $\vec{u} \in \mathbb{N}^2$ and $\vec{v} \in \mathbb{Z}^2$. Let

$$\mathbf{a} = z[B\vec{u} + \vec{v}] \quad \text{and} \quad \mathbf{b} = z'[\vec{B}^{\ell}_{\vec{u}} + \vec{v}]^c.$$

First notice that $\mathbf{a}$ might contain incomplete dominos (see Figure 3). These incomplete copies of $r^P$ can be completed by using at most $W_1 - 1$ columns on either side of $\mathbf{a}$ and at most $W_2 - 1$ rows on the top and bottom of $\mathbf{a}$. Similarly, any incomplete dominos that might occur in $\mathbf{b}$ can be completed by using at most another $W_1 - 1$ rows and $W_2 - 1$ columns of the filling collar.

The resulting new filling collar has width greater than $R$. Notice also that each piece of its boundary is the union of the boundaries of complete copies.
of the domino $\tau^P$. Thus, the filling collar can be decomposed into four rectangles with the property that each rectangle either has width greater than $R$ and height a multiple of $W_2$, or has height greater than $R$ and width a multiple of $W_1$ (see Figure 3).

We show how to fill in the tiling for the rectangle labelled Region 1 in Figure 3; the argument for the other regions is identical. Using (10) we obtain non-negative integers $a_j$ so that the width of the rectangle can be written as $\sum_{j=1}^{k} a_j w_1^j$. Working from left to right, we start tiling Region 1 by first using $a_1$ copies of the tile $\tau_1$ to tile a strip of width $a_1 w_1^1$. Since $\tau_1$ has height $w_1^2$, and the height of Region 1 is a multiple of $W_2$, and therefore of $w_2^1$, we can tile this strip without gaps or overlap with the tile $\tau_1$ without overlapping the completion of block $b$ at all. Next, we tile another strip of width $a_2 w_2^2$ using dominos of type $\tau^2$, etc.

This filling process clearly results in a $\vec{w}^1, \ldots, \vec{w}^k$ domino tiling of the plane, and thus a point in $Y$. \hfill $\square$

4. PROOF OF THEOREM 2.1

We first establish some notation. For $s > 0$ and $\vec{u} \in \mathbb{N}^d$ we have already in Section 3 defined $R_{\vec{u}}^s \subset \mathbb{Z}^d$ to be $R_{\vec{u}}$ together with an outer collar of uniform width $s$. We denote the outer collar itself by:

$$\partial^s(R_{\vec{u}}) = R_{\vec{u}}^s \setminus R_{\vec{u}}$$
and refer to it as the outer $s$-collar of $R_{\vec{u}}$. Still assuming $s > 0$, we define the $s$-interior of $R_{\vec{u}}$ by $$R_{\vec{u}}^{-s} = \{ \vec{v} \in R_{\vec{u}} : s \leq v_i \leq u_i - 1 - s, \ i \leq d \}$$ and we call $$\partial^{-s}(R_{\vec{u}}) = R_{\vec{u}} \setminus R_{\vec{u}}^{-s}$$ the inner $s$-collar of $R_{\vec{u}}$.

For notational convenience we will refer to squares of size $n$, namely $R_{\vec{u}}$ with $\vec{u} = (n, \cdots, n)$, as $R_n$.

Our argument will rely heavily on the following gluing property of shifts of finite type with uniform filling sets. If a shift of finite type $Y$ has a uniform filling set $Z$, one can glue blocks from other points $y$ in the shift into points from $Z$, as long as the block in $y$ has sufficient overlap with a block from $Z$. The next lemma is a formal statement of this fact.

**Lemma 4.1.** (Lemma 3.5 from [10]). Suppose $Y$ is a shift of finite type with step size $s$ and filling set $Z$ with filling length $\ell$. Fix a rectangle $R_{\vec{u}} \subset \mathbb{Z}^d$, with $\vec{u} \in \mathbb{N}^d$. Suppose $y \in Y$ is such that there exists $z \in Z$ with

$$y[\partial^{-s}[R_{\vec{u}}]] = z[\partial^{-s}R_{\vec{u}}].$$

Then given $z' \in Z$ we can find a point $y' \in Y$ such that

$$y'[R_{\vec{u}}] = y[R_{\vec{u}}] \quad \text{and} \quad y'[((R_{\vec{u}})^c) \cap z'[((R_{\vec{u}})^c)].}$$

We refer the reader to [10] for the proof.

Given a Rohlin tower of shape $R \subset \mathbb{Z}^d$ with base $F$ and $x \in X$ with the property that $T^{\vec{n}}x \in F$ for some $\vec{n} \in \mathbb{Z}^d$ we call $T^{\vec{n}+R}(x)$ an occurrence of the tower $T^R(F)$, and we call $y = T^{\vec{n}}(x) \in F$ the base point of this occurrence. The set $T^R(x)$ will be called the slice of the tower based at $x$.

### 4.1. The parameters of the construction.

We will construct the function $\phi$ of Theorem 2.3 as a limit of a sequence of functions $\phi_i$ defined on levels of Rohlin towers of increasing measures. We let $Z$ be as in Theorem 3.3. We denote the filling length of $Z$ by $\ell$, and fix $z \in Z$. Let $\epsilon_i$ be a sequence of positive numbers with the property that for all $i$

$$\epsilon_i < \frac{1}{4i}.$$  

Let $n_i$ be a sequence of positive integers increasing to infinity with the property that

$$\frac{2d(\ell + 2 + n_{i-1})}{n_i} < \frac{1}{4i}$$

and

$$2d \sum_{i=1}^{\infty} \frac{1}{n_i} < \min_{i=1,\cdots,k} p_i.$$
Finally we apply the $\mathbb{Z}^d$ Rohlin Lemma to obtain sets $F_i \in \mathcal{M}$ which are bases of a Rohlin tower of shape $R_{n_i}$ and error sets $B_i$ with

$$\mu B_i < \epsilon_i.$$  

4.2. Constructing the maps $\phi_i$. The function $\phi_1$ will be defined on $R_{n_1}^{-(\ell+1)}$. For $\vec{v} \in R_{n_1}^{-(\ell+1)}$ we set $\phi_1(T^\vec{v}x) = z[\vec{v}]$.

For ease of notation we describe the construction of $\phi_2$. This step encompasses all the details of the general inductive step. We will define $\phi_2$ on $T^{R_{n_2}^{-(\ell+1)}}(F_2)$. For each $x \in F_2$ we define the set

$$B(x) = \{ \vec{b}_1 \in R_{n_2}^{-(\ell+2+n_1)} : T^{\vec{b}_1}x \in F_1 \}.$$  

Note that these are base points of occurrences of $T^{R_{n_1}}(F_1)$ in $T^{R_{n_2}}(x)$

which are completely contained in $T^{R_{n_2}^{-(\ell+2)}}(x)$. We refer to these as \textit{good occurrences} of the first stage tower.

Partition $F_2$ into subsets $F_2^1, \ldots, F_2^{k_2}$ such that the set of indices $B(x)$ is constant on each subset. For each $i = 1, \ldots, k_2$ we refer to the relevant constant set of indices by $B(t)$. Fix such a $t$. For all $x \in F_2^t$ we set $\phi_2(T^\vec{v}x) = z[\vec{v}]$ for the following choices of $\vec{v}$:

(17) \[ \vec{v} \in \left( \bigcup_{\vec{b} \in B(t)} \vec{b} + R_{n_1} \right)^c \cap T^{R_{n_2}^{-(\ell+1)}}(F_2^t) \]

(18) \[ \vec{v} \in \partial^{-1}[\vec{b} + R_{n_1}] \text{ where } \vec{b} \in B(t) \]

Locations specified in (17) puts symbols from $z[R_{n_2}]$ onto locations in the $(\ell + 1)$-interior of $R_{n_2}$ which are not covered by good copies of the first stage tower. We call this the \textit{ambient word}. Locations specified in (18) extend the ambient picture from $z[R_{n_2}]$ to the interior 1-collar around good first stage towers.

For $\vec{v} \in \vec{b} + R_{n_1}^{-(\ell+1)}$ where $\vec{b} \in B(t)$, namely in locations that lie in the domain of $\phi_1$, we set

$$\phi_2(T^\vec{v}x) = \phi_1(T^\vec{v}x).$$

We note that each good occurrence of a first stage tower now sees

(19) \[ \text{symbols from } z[R_{n_1}] \text{ in positions } \vec{b} + R_{n_1}^{-(\ell+1)} \text{ and } \]

symbols from $z[R_{n_2}]$ in locations $\vec{b} + \partial^{-1}(R_{n_1})$.

where $\vec{b} \in B(t)$.

Note that $\phi_2$ is as yet undefined on a collar of width $\ell$ in all good occurrences of the first stage tower. We call this the \textit{filling collar} of the tower. Since the collars of width one immediately before and after the filling collar have been assigned symbols from points in $Z$ we can now interpolate between them and assign symbols to the filling collar to obtain a block from
\( Y \). More formally, using (19) and the fact that \( Z \) is a uniform filling set we can now find a point \( z_2^t \in Y \) such that for all \( \bar{b} \in B(t) \)
\[
  z_2^t[\bar{b} + R_{n_1}^{-\ell + 1}] = z[R_{n_1}^{-\ell + 1}]
\]
\[
  z_2^t[\bar{b} + \partial^{-1}(R_{n_1})] = z[\bar{b} + \partial^{-1}(R_{n_1})].
\]

We set
\[
  \phi_2(\bar{v}) = z_2^t[\bar{v}]
\]
for all \( \bar{v} \in \bar{b} + \partial^t(R_{n_1}^{-\ell + 1}) \).

For the inductive step we need to make two observations. Since \( z_2^t \in Y \), \( \phi_2(T^{R_{n_2}^{-\ell + 1}}(F_2^t)) \) is a block which occurs in \( Y \). In addition, (16) and (17) guarantee that
\[
(20) \quad \text{locations } \bar{v} \in \partial^1(R_{n_2}^{-\ell + 2}) \text{ are assigned the symbol } z[\bar{v}].
\]
Thus, even though \( \phi_2(T^{R_{n_2}^{-\ell + 1}}(F_2^t)) \) is not a block from \( Z \), we will be able to glue it into blocks from \( Z \) in the next stage of the construction by using Lemma 4.1.

Repeating the process for all \( t = 1, \ldots, k_2 \) we define \( \phi_2 \) on \( T^{R_{n_2}^{-\ell + 1}}(F_2) \).

4.3. Constructing \( \phi \). We first identify those points \( x \in X \) for which \( \lim_{i} \phi_i(x) \) does not exist. These are exactly those points who lie in the error sets \( B_i \) for infinitely many \( i \) or lie in the inner \( (\ell + 1 + n_{i-1}) \)-collar of the \( i \)-th tower for infinitely many \( i \). For each \( i \)
\[
  \mu(B_i \cup T^{\partial^{-(\ell+1+n_{i-1})}(R_{n_1})}(F_i)) < \epsilon_i + \frac{2d(\ell + 2 + n_{i-1})}{n_i}
\]
so by (12) and (13) we can use the easy direction of the Borell-Cantelli Lemma to conclude that for a.e. \( x \in X \) the requisite limit exists.

To obtain (3) we argue as follows. This argument is identical to the convergence argument given in (11), we include it here for completeness' sake. Let \( G_0 \) be the subset of \( X \) consisting of points \( x \) for which \( \lim_{i \to \infty} \phi_i(x) \) exists and who land in the middle ninth of a tower infinitely often. Set \( G_1 = \cup_{\bar{v} \in \mathbb{Z}^d} T^{\bar{v}}G_0 \). \( G_1 \) is clearly a measurable and invariant set and \( \mu(G_1) = \alpha > \frac{2}{3d} \).

Further, for \( x \in G_1 \) since the \( n_i \) grow to infinity we have that for any \( \bar{v} \in \mathbb{Z}^d \), there exist infinitely many \( i \) such that \( x \) and \( T^{\bar{v}}x \) are in the same slice of the \( i \)-th stage tower. For such \( i \) clearly
\[
  \phi_i(T^{\bar{v}}x) = S^{\bar{v}}(\phi_i(x))
\]
so \( \lim_{i \to \infty} \phi_i(T^{\bar{v}}x) \) must exist and (3) must hold.

If \( \alpha < 1 \) we can use the same procedure, adjusting the measures \( p_1, \ldots, p_k \) as necessary, to obtain a set \( G_2 \subset G_1 \) such that \( \mu(G_1 \cup G_2)^c < (1 - \alpha)^2 \). Countably many such steps yields a countable union of invariant measurable sets \( G_n \) with the property that \( \mu(\cup G_n) = 1 \) and on each set \( \phi \) is defined and satisfies (3).
To see that (7) holds, we note that our tiling argument guarantees that the only part of the space that is not tiled by the domino $\tau^P$ is the filling collars of the towers. Thus by (14) we have

$$\mu(\phi^{-1}(\tau^P)) > 1 - \sum_{i=1}^{\infty} \frac{2d\ell}{n_i} > 1 - \lim_{i=1,\ldots,k} p_i.$$ 

5. **Proof of Theorem 2.1 in the case of countably many towers**

Our construction here will give a measurable tiling of the orbits of the action by finitely many actual size dominos and infinitely many large dominos. As before, the actual size small dominos will take up less than their allotted measure of space and we will use the large dominos to carve out the required size small towers of appropriate measures.

There are several new issues we must address. First, we must ensure that we have infinitely many large domino towers, since a tower of size $\tau^j$ can only be constructed out of a large domino whose dimensions are the product of at least the first $j$ dimensions. Our tiling construction then has to be modified to incorporate large dominos of growing scales.

Second, a large domino whose dimensions are a product of the first $j$ dimensions must be entirely used up when we construct small towers of size $1,\ldots,j$, since it will be too small to use in later stages. We thus have to bound the measures of the larger dominos appropriately.

Now for the details. Choose a sequence $n_k \in \mathbb{N}$ so that for all $i = 1,\ldots,d$,

$$\gcd(w_1^n,\ldots,w_d^n) = 1,$$

(21)

$$\varepsilon_k = \sum_{j=n_k+1}^{\infty} p_j < \frac{1}{8^k}, \quad \text{and}$$

$$\sum_{j=n_k+1}^{n_k+1} p_j > \sum_{j=n_k+1}^{\infty} p_j.$$  \hspace{1cm} (22)

Following the notation established in Section 3 we let $Y_k = Y(\vec{w}^1,\ldots,\vec{w}^m)$ and $Z_k$ denote the uniform filling set of $Y_k$ generated by the large domino $\tau^{P_k}$. We let $Y_\infty$ denote the shift on the infinite alphabet required to define all tiles $\tau^1,\ldots$. As in the finite case we will first prove that we can construct a factor map $\phi : X \to Y_\infty$ satisfying conditions (6) and (7).

We prove that any block from $Z_k$ can be glued into a word from $Z_1$ using a collar whose size depends only on $k$ and not on the size of the block itself. This is the key ingredient which will allow us to tile with growing sizes of large dominos.

**Corollary 5.1.** (of Theorem 3.3). Given $k \in \mathbb{N}$, there exists $l_k \in \mathbb{N}$ such that for all $\vec{w} \in \mathbb{N}^d$, $z \in Z_k$ and $z' \in Z_1$, there exists a point $y \in Y_k$ such
that

\[ y[R_w] = z[R_w] \]
\[ y[(R_w^c)] = z'[(R_w^c)] \]

with the property that \( y[R_w \setminus R_w] \) is a union of tiles only of type \( \tau^1, \ldots, \tau^n \).

**Proof.** The case for \( k = 1 \) is exactly Theorem 3.3. For \( k > 1 \) we note that the extra width in the collars provides enough space to complete any necessary tiles of shape \( \tau P_k \) in \( z[R_w] \), resulting in an inner rectangle whose boundaries have dimensions of completed rectangles from \( Z_k \), and therefore from \( Z_1 \). The remaining filling collar is now large enough so that together with (??) the filling algorithm from the proof of Theorem 3.3 can be used verbatim to fill the collar using only tiles of type \( \tau^1, \ldots, \tau^n \). \( \square \)

There is also an analogous extension of Lemma 4.1 which we omit stating.

Choose an increasing sequence of integers satisfying:

\[
\frac{2d(\ell_k + 2 + n_k)}{n_k} < \frac{1}{4^k}
\]

(24)

and

\[
\sum_{k=1}^{\infty} \frac{2d\ell_k}{n_k} < \min_{j=1, \ldots, n_1} p_j
\]

(25)

We fix \( z_i \in Z_i \), and choose \( R_i \), \( F_i \) and \( B_i \) as before. For \( i > 1 \) we divide \( F_i \) into two subsets \( F_{i\text{main}} \) and \( F_{i\text{tail}} \) with the property that

\[
\mu(T_{R_{n_i}}(F_{i\text{tail}})) = \sum_{k=n_i+1}^{n_i+1-1} p_k.
\]

(26)

5.1. **Constructing** \( \phi \). We begin by constructing \( \phi_1 \) exactly as before. We define \( \phi_2 \) as follows. On \( F_{2\text{main}} \) we proceed exactly as in the finite case, using \( z_1 \) to obtain the ambient word. For \( x \in F_{2\text{tail}} \) we fix \( z_2 \in Z_2 \) and we set

\[
\phi_2(T\vec{v}x) = z_2[\vec{v}]
\]

for \( \vec{v} \in R_{n_2}^{-\ell_2+1} \).

To see all the necessary ingredients for the general induction step we also need to describe the construction of \( \phi_3 \). Fix \( z_3 \in Z_3 \) and for \( x \in F_{3\text{tail}} \) we assign

\[
\phi_3(T\vec{v}x) = z_3[\vec{v}]
\]

for \( \vec{v} \in R_{n_3}^{-\ell_3+1} \). For \( x \in F_{3\text{main}} \) we proceed as in the finite case using \( z_1 \) to provide an ambient word from \( Z_1 \), noting that any slices of previous stage towers that appear in \( TR_3(x) \) come with sufficiently large filling collars that can be filled either by Theorem 3.3 or by Corollary 5.1. We note that the resulting shapes can be interpolated into a word from \( Z_1 \) using the modified version of Lemma 4.1.

To define \( \phi : X \to Y_\infty \) we again begin by identifying points \( x \in X \) for which \( \lim \phi_i(x) \) does not exist. In addition to points that lie in the error
sets $B_i$ infinitely often or in the filling collars of infinitely many towers we must now also include points that land in $P_{i}^{\text{tail}}$ for infinitely many $i$. For each $i$ this is a set of measure less than

$$
\epsilon_i + \frac{2d(\ell_i + 2 + n_{i-1})}{n_i} + \sum_{k=n_i}^{n_{i+1}-1} p_k,
$$

which by (22) and (24) is less than $\frac{3}{8}$. We can thus proceed to define the map $\phi$ exactly as in the finite case.

At the end of this construction we obtain a decomposition of $X$ into towers of shapes $\tau^1, \ldots, \tau^{n_1}, \tau^{P_1}, \tau^{P_2}, \ldots$. As before, by (25) none of the first set of small towers take up more than their allotted part of space. In addition, we have by our choice of $\epsilon_i$, (22), (23), and (26) that $0 < \mu(\tau^{P_i}) < \sum_{j=n_i}^{\infty} p_j$. So if we supplement the measures of the towers $\tau^i$, or create such towers in the case that $i > n_1$, in increasing order of index we are guaranteed that we will decompose all of $\tau^{P_i}$ into required shape towers for all $i \geq 1$, ending with only towers of the desired shapes and measures.

References

[1] Steve Alpern, Generic properties of measure preserving homeomorphisms, Ergodic theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1978), Lecture Notes in Math., vol. 729, Springer, Berlin, 1979, pp. 16–27.

[2] S. Bezuglyi, A. H. Dooley, and K. Medynets, The Rokhlin lemma for homeomorphisms of a Cantor set, Proc. Amer. Math. Soc. 133 (2005), no. 10, 2957–2964 (electronic). MR MR2159774

[3] J. P. Conze, Entropie d’un groupe abélien de transformations, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 25 (1972/73), 11–30.

[4] S. Eigen, A. B. Hajian, and V. S. Prasad, Universal skyscraper templates for infinite measure preserving transformations, Discrete Contin. Dyn. Syst. 16 (2006), no. 2, 343–360.

[5] S. J. Eigen and V. S. Prasad, Multiple Rokhlin tower theorem: a simple proof, New York J. Math. 3A (1997/98), no. Proceedings of the New York Journal of Mathematics Conference, June 9–13, 1997, 11–14 (electronic). MR MR1604573 (99h:28032)

[6] Yitzhak Katznelson and Benjamin Weiss, Commuting measure-preserving transformations, Israel J. Math. 12 (1972), 161–173.

[7] Isaac Kornfeld, Some old and new Rokhlin towers, Chapel Hill Ergodic Theory Workshops, Contemp. Math., vol. 356, Amer. Math. Soc., Providence, RI, 2004, pp. 145–169.

[8] D. S. Ornstein and B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, J. Analyse Math. 48 (1987), 1–141.

[9] A. A. Pirkhod’ko, Partitions of the phase space of a measure-preserving $\mathbb{Z}^d$-action into towers, Mat. Zametki 65 (1999), no. 5, 712–725. MR MR1716239 (2001a:37007)

[10] E. Arthur Robinson, Jr. and Ayse A. Sahin, On the existence of Markov partitions for $\mathbb{Z}^d$ actions, J. London Math. Soc. (2) 69 (2004), no. 3, 693–706.

[11] D. J. Rudolph, Markov tilings of $\mathbb{R}^n$ and representations of $\mathbb{R}^n$ actions, Contemp. Math. 94 (1989), 271–290.