Small frequency approximation of (causal) dissipative pressure waves

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Abstract:
In this paper we discuss the problem of small frequency approximation of the causal dissipative pressure wave model proposed in Kowar et al. (2010). We show that for appropriate situations the Green function $G$ of the causal wave model can be approximated by a noncausal Green function $G^\text{pl}$ that has frequencies only in the small frequency range $[-M, M]$ ($M \leq 1/\tau_0$, $\tau_0$ relaxation time) and obeys a power law. For such cases, the noncausal wave $G^\text{pl}$ contains partial waves propagating arbitrarily fast but the sum of the noncausal waves is small in the $L^2$-sense.

Keywords: Attenuation, Dispersion, Wave equations, Causality

1. INTRODUCTION: FREQUENCY FRAMEWORK
In the first section we summarize the general framework of frequency driven wave dissipation.

From the mathematical point of view, in case of a homogeneous and isotropic medium, dissipative pressure waves can be modeled by (cf. e.g. Kowar et al. (2010))

$$p(\mathbf{x}, t) = (G \ast_{\mathbf{x}, t} f)(\mathbf{x}, t) \quad \mathbf{x} \in \mathbb{R}^3, \ t \in \mathbb{R}$$

with

$$p(\cdot, t) = 0 \quad \text{and} \quad \frac{\partial p}{\partial t}(\cdot, t) = 0 \quad \text{for} \quad t < 0, \quad (1)$$

where $G$ denotes a distribution (Green function), $\ast_{\mathbf{x}, t}$ denotes the space-time convolution and $f$ denotes a forcing term (source term) which models wave generation. We note that $f$ is the same forcing term as in the absence of dissipation as long as (1) holds. (This is not true if $p(\mathbf{x}, t)$ is calculated for $t \geq t_0$ ($t_0 > 0$) from initial data at $t_0$.)

1.1 The Green function
The Green function can be modeled by

$$G(\mathbf{x}, \omega) = \frac{e^{-\beta_\omega(|\mathbf{x}|, \omega)}}{4 \pi |\mathbf{x}|} e^{i \frac{\omega}{c_0} |\mathbf{x}|} \quad \mathbf{x} \in \mathbb{R}^3, \ \omega \in \mathbb{R}, \quad (2)$$

where $\Re(\beta_\omega(|\mathbf{x}|, \omega)) > 0$ is even in $\omega$ and $\Im(\beta_\omega(|\mathbf{x}|, \omega))$ is odd in $\omega$. Here $G(\mathbf{x}, \omega)$ denotes the Fourier transform of $G(\mathbf{x}, t)$ with respect to time $t$. The last two conditions guarantee that $G(\mathbf{x}, t)$ is real-valued. We focus on the cases

$$\beta_\omega(|\mathbf{x}|, \omega) = \alpha_\omega(\omega)|\mathbf{x}|. \quad (3)$$

We call $\alpha_\omega = \alpha_\omega(\omega)$ and $\alpha = \Re(\alpha_\omega(\omega))$ the attenuation-dispersion law and the attenuation law, respectively.

1.2 The wave equation
The above dissipative wave model satisfies the following integro-differential equation

$$\nabla^2 p(\mathbf{x}, t) - \left(D_* + \frac{1}{c_0} \frac{\partial p}{\partial t}\right)^2 p(\mathbf{x}, t) = -f(\mathbf{x}, t), \quad (4)$$

where $D_*$ denotes the time convolution operator

$$\mathcal{F}\{D_*(g)\} = \frac{1}{\sqrt{2\pi}} \alpha_\omega(\omega) \mathcal{F}\{g\}$$

defined for appropriate functions $g = g(t)$. Here $\mathcal{F}\{g\}$ denotes the Fourier transform of $g$ with respect to time.

More details about attenuation-dispersion laws and respective wave equations can be found in e.g. Nachman et al. (1990), Szabo (1995), Waters et al. (2000), Chen et al. (2004), Patch et al. (2006), Kelly et al. (2008) and Kowar et al. (2012).

2. ATTENUATION-DISPERSION LAWS AND CAUSALITY

2.1 What do we mean by causality?
Let $c_F < \infty$ denote the (constant) speed of the wave front of the wave $G$. Causality requires that the wave front initiated at the origin $0$ at time $t = 0$ arrives at position $\mathbf{x} \neq 0$ not before the time period $T(|\mathbf{x}|) = \frac{|\mathbf{x}|}{c_F}$ is over. Mathematically this is equivalent to the following causality condition: for each $c_1 \geq c_F$ we have

$$G \left( \mathbf{x}, t + \frac{|\mathbf{x}|}{c_1} \right) = 0 \quad \text{if} \quad t < 0 \quad \text{and} \quad |\mathbf{x}| \neq 0. \quad (5)$$

Various dissipative wave models are analysed with respect to causality in Kowar et al. (2012). In particular, it is shown that (3) implies $c_F = \text{const}.$ if $c_F < \infty$. See also Kelly et al. (2008).

2.2 Attenuation-dispersion laws ($\gamma \in (1, 2]$)
In this paper we consider the attenuation-dispersion laws
\[ \alpha_c^\omega(\omega) = \frac{\alpha_1(-i\omega)}{c_0 \sqrt{1 + (-i\tau_0\omega)^\gamma}} \]  
for \( \omega \in \mathbb{R} \) with \( \gamma \in (1, 2] \), \( c_0 \in (0, \infty) \) and \( \alpha_1, \tau_0, a_1 > 0 \). \( \alpha_c^\omega(\omega) \) is called the frequency power law.

Let
\[ a_1 = \frac{\alpha_1 \tau_0^{-1}}{2c_0} |\cos(\gamma \pi/2)| \quad \text{and} \quad a_2 = \frac{\alpha_1}{c_0}, \]

then it follows from
\( (-i\omega)^\gamma = |\omega|^\gamma \left\{ \cos(\gamma \pi/2) - i \sin(\gamma \pi/2) \sgn(\omega) \right\} \)
that
\[ \alpha_c^\omega(\omega) = \Re(\alpha_c^\omega(\omega)) \approx a_1 |\omega|^\gamma = \alpha_c^\omega(\omega), \]
\[ \Im(\alpha_c^\omega(\omega)) \approx -a_1 \tan \left( \frac{\pi}{2} \right) \sgn(\omega) |\omega|^\gamma - a_2 \omega \]
for \( |\tau_0\omega|^{-1} << 1 \). We see that \( \alpha_c^\omega(\omega) \) is a good approximation of \( \alpha_c(\omega) \) for sufficiently small frequencies. For the case \( \tau_0 = 10^{-6} \mu s \) (liquid), the small frequency range condition \( |\tau_0\omega|^{-1} \leq 0.1 \) is visualized in Table 1.

| \( \gamma \) | \( |\omega| \) range | bound \( M \) |
|---|---|---|
| 1.1 | \( |\omega| \leq 10^{-4} \) MHz | \( 10^{-3} \) MHz |
| 1.5 | \( |\omega| \leq 10^{-4} \) MHz | \( 10^{-3} \) MHz |
| 2.0 | \( |\omega| \leq 10^{-5} \) MHz | \( 10^{-4} \) MHz |

2.3 Two results on causality

In Kowar et al. (2012), it is proven that the speed \( c_F \) of the wave front of \( G(x, t) \) with \( \alpha_c^\omega \) defined as in (6) satisfies \( c_F \leq c_0 \) and in Kowar (2010) it is proven that \( c_F = c_0 \). Moreover, it is shown that \( G(x, t) \) with \( \alpha_c^\omega \) defined as in (7) with \( a_2 = 0 \) has not a bounded wave front speed. The same result is true for \( a_2 \neq 0 \).

Loosely speaking, causality restricts the growth of \( \alpha(\omega) \) and consequently \( \omega \mapsto \tilde{G}(x, \omega) \) must not decrease too fast for fixed \( x \in \mathbb{R} \). If \( \omega \mapsto \tilde{G}(x, \omega) \) decreases too fast, then it behaves like the truncated Green function
\[ \tilde{G}_M(x, \omega) := \tilde{G}(x, \omega) \chi_{[-M,M]}(\omega), \]
for which causality condition (5) cannot hold. Here
\[ \chi_{[-M,M]}(\omega) := \begin{cases} 
1 & \text{if } \omega \in [-M, M] \\
0 & \text{otherwise}
\end{cases} \]
denotes the characteristic function of \( \omega \) on \([-M, M]\).

3. SMALL FREQUENCY APPROXIMATION

We now derive two theorems that permit to estimate the quality of small frequency approximations. Let
- \( G^c(x, t) \) denote the Green function with \( \alpha^\omega_c \) defined as in (6),
- \( \tilde{G}^M_M(x, \omega) \) denote the Green function with \( \alpha^\omega_c \) defined as in (7),
- \( \tilde{G}_M(x, \omega) \) be defined as in (9) and
- \( G_M(x, t) := F^{-1}\{\tilde{G}_M\}(x, t) \).

Theorem 1. Let \( M > 0 \) and \( A_0, A_1, A_3 > 0 \) be such that
\[ \alpha(M) + A_0 |\omega - M| \leq \alpha^c(\omega) \leq A_1 \omega^2 + A_2 |\omega| \]
holds for \( |\omega| > M \). For \( |x| \neq 0 \) it follows that
\[ \frac{||G^c(x, \cdot) - \tilde{G}^c_M(x, \cdot)||}{||G^c(x, \cdot)||} \leq \sqrt{\frac{2A_1}{\pi A_0^2}} \frac{e^{-\frac{(\alpha(M)+\frac{2\pi}{A_0})}{\sqrt{\pi} |x|}}}{\sqrt{\pi} |x|^2}. \]

Proof. \( \omega \mapsto \tilde{G}^c(x, \omega) \) and \( \omega \mapsto \tilde{G}_M^c(x, \omega) \) are square integrable and thus their inverse Fourier transforms \( t \mapsto G^c(x, t) \) and \( t \mapsto G^c_M(x, t) \) are square integrable, too. Because of the Plancherel-Parseval equality, it follows
\[ \frac{||G^c(x, \cdot) - \tilde{G}^c_M(x, \cdot)||}{||G^c(x, \cdot)||} \leq \frac{||G^c(x, \cdot) - \tilde{G}^c_M(x, \cdot)||}{||G^c(x, \cdot)||^2} \]
From (11) and
\[ \int_{-\infty}^{\infty} \exp(-\omega^2/2) \, d\omega = \sqrt{2\pi}, \]
it follows
\[ ||G^c(x, \cdot) - \tilde{G}^c_M(x, \cdot)|| \leq e^{-\frac{2A_1 |\omega - M| |x|}{4\pi |x|^2}} \int_{-\infty}^{\infty} e^{-2A_0 |\omega - M| |x|} \, d\omega. \]
and
\[ ||G^c(x, \cdot)|| \leq \frac{e^{-2A_0 |\omega - M| |x|}}{(4\pi |x|^2)} \]
\[ \frac{||G^c(x, \cdot)||}{||G^c(x, \cdot)||^2} \leq \frac{\delta}{4\pi |x|^2 \sqrt{2A_1 |x|}}. \]
The theorem follows from (12) and the last two results.

Remark. Because \( \omega \mapsto \tilde{G}^c_M(x, \omega) \) vanishes on a nonempty interval, causality condition (5) is not satisfied for \( G^c_M \). However, Theorem 1 shows that \( G^c_M(x, t) \) is a good \( L^2 \)-approximation of \( G^c(x, t) \) if \( M \) is sufficiently large.

Theorem 2. Let \( B_1 := \Re(\alpha^\omega_c - \alpha^\omega_c^\gamma) \), \( B_2 := \Im(\alpha^\omega_c - \alpha^\omega_c^\gamma) \),
\[ C := \left| 1 - 2e^{-B_1 |\omega|} \cos(B_2 |\omega|) \right| + e^{-2B_1 |\omega|} \right| \]
and \( M_\delta \in (0, M] \) be such that
\[ ||\tilde{G}^c_M(x, \omega)||^2 \leq \frac{(1 - \delta)||\tilde{G}^c(x, \omega)||^2}{L^2} \]
for some \( 0 < \delta < 1 \). For \( |x| \neq 0 \) it follows that
\[ \frac{||\tilde{G}^c_M(x, \omega) - \tilde{G}^c_M(x, \omega)||}{||G^c(x, \omega)||^2} \]
with \( D_j(|x|) := \max_{\omega \in I_j} C(|x|, \omega)^2 \) for \( I_1 := [-M_\delta, M_\delta] \) and \( I_2 := \mathbb{R} \setminus I_1 \).
Table 2. Visualization of error $\epsilon_M$ for different distances $L$ (in cm).

| $L$   | $10^{-6}$ | $10^{-5}$ | $10^{-4}$ | $10^{-3}$ | $10^{-2}$ |
|-------|-----------|-----------|-----------|-----------|-----------|
| $\epsilon_M$ | 7.62 · $10^{-8}$ | 7.35 · $10^{-5}$ | 4.46 · $10^{-4}$ | 7.13 · $10^{-5}$ |          |

$G^pl_M(x,t)$ are good approximations of $G^c(x,t)$ as long as $M \geq 100 \, MHz \quad \text{and} \quad |x| \geq 10^{-6} \, cm$.

In particular, this means that the wave can be predicted for $t \geq \frac{10^{-6} \, cm}{c_0} = 6.67 \cdot 10^{-6} \, \mu s$.

Remark on causality. Because the wave $G^c$ has the finite wave front speed $c_F = c_0$ and $G^pl_M(x,t)$ is a good $L^2$-approximation of $G^c(x,t)$, it follows that $G^pl_M$ is in "some sense" the same wave front speed. In "some sense" means that $G^pl_M$ contains partial waves propagating arbitrarily fast but their sum is small in the $L^2$-sense.

Remark on large frequencies. We note that the power attenuation law $\alpha^pl(\omega)$ increases much faster than $\alpha^c(\omega)$ for large frequencies and that the respective phase speed $c^pl(\omega)$ has a singularity at $\omega_1 \approx 7.950959 \cdot 10^6 \, MHz$ and at $\omega_2 = -\omega_1$ (cf. Fig 2), respectively. In contrast to the latter fact, the phase speed $c^c(\omega)$ of the causal model has no singularity.

5. CONCLUSIONS

We showed that - for appropriate situations - the causal wave $G^c$ defined by (2), (3) and (6) can be approximated by the noncausal wave $G^pl_M$ defined by (9) (2), (3), (7) and (8). In words, the wave $G^pl_M$ contains partial waves propagating arbitrarily fast but the sum of the noncausal waves is small in the $L^2$-sense. This result fits in with the results in Kelly et al. (2008).

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Fig. 1. Numerical comparison of the attenuation laws and phase speeds. The dashed lines correspond to model $\alpha^c(\omega)$ and the solid lines correspond to model $\alpha^{pl}(\omega)$.

Fig. 2. Numerical comparison of the attenuation laws and phase speeds for large frequencies The dashed lines correspond to model $\alpha^c(\omega)$ and the solid lines correspond to model $\alpha^{pl}(\omega)$.

Fig. 3. Visualization of functions $M_0 \rightarrow g(M_0) := \|G_{M_0}(x, \cdot)\|_{L^2}$ and $\omega \rightarrow C(|x|, \omega)$ for $|x| = 1 cm$. 