PAPER

Minimal analytical model for undular tidal bore profile; quantum and Hawking effect analogies*

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Abstract

Waves travelling up-river, driven by high tides, often consist of a smooth front followed by a series of undulations. A simple approximate theory gives the rigidly travelling profile of such ‘undular hydraulic jumps’, up to scaling, as the integral of the Airy function; applying self-consistency fixes the scaling. The theory combines the standard hydraulic jump with ideas borrowed from quantum physics: Hamiltonian operators and zero-energy eigenfunctions. There is an analogy between undular bores and the Hawking effect in relativity: both concern waves associated with horizons.

1. Introduction

An ocean tide enters the wide mouth of a river, gets concentrated while travelling upstream as the river narrows, and eventually forms a wall of water that can be several metres high, reversing the river’s normal downstream flow. This dramatic phenomenon is a tidal bore [1–7], visible on some of the world’s rivers at times of particularly high tides, and illustrated in figure 1. One of several common types of bore takes the form of a smooth front followed by a series of waves: an ‘undular bore’, that can travel up-river for tens of kilometres, with approximately unchanging profile.

My main aim here is to obtain a simple approximate analytic form for this profile. The theory is based on a heuristic application of the dispersion relation for long waves on shallow water, with a self-consistent extension, incorporating hydraulic jump theory, to include weak nonlinearity in an approximate way. The dispersion relation is interpreted as a Hamiltonian, with the steady profile as its zero frequency eigenfunction, whose form, up to scaling, is the indefinite integral of the Airy function; self-consistency fixes the scaling.

One of my subsidiary aims is to use tidal bores as an illustration of Hamiltonian theory (borrowed from quantum mechanics), that physicists (at least geographically fortunate ones) can experience directly. Another aim is to present one more illustration of the ubiquity of the Airy function.

A final subsidiary aim is to draw attention to an analogy between the classical physics of undular bores and horizons in relativity [8]. This is based on the fact that the front of a tidal bore is a horizon, separating the downstream region, where the water flows more slowly than surface waves, from the upstream region where the water flows faster. Hawking radiation concerns waves associated with horizons, and with tidal bores these are the undulations downstream and the evanescent waves leaking upstream, whose description is our main interest.

The analogy is the subject of numerous detailed studies [9] of waves on inhomogeneous flows. My purpose is to point out that undular tidal bores are natural phenomena where a classical counterpart of one aspect of Hawking radiation—the horizon—can be seen outside the laboratory. (Like all analogies, this one is partial: in tidal bores, the analogy is with white holes, not black holes [10, 11], and nothing corresponds to particle creation.)

Existing theory for tidal bores [12–14] describes the development of the undulations over time, using numerical solution of nonlinear equations (for example Korteweg de Vries). This yields profiles resembling an undular bore, but not the steady profile and not an explicit analytic form. The waves following the front have been modelled as sinusoids, or their nonlinear generalisation: cnoidal waves (Jacobi elliptic functions), where

* ‘Physics is not just Concerning the Nature of Things, but Concerning the Interconnectedness of all the Natures of Things’ (Sir Charles Frank, retirement speech 1976).
the crests are sharper [4]. Sophisticated analytical investigations of flows in connection with an analogy with relativistic horizons [8, 9, 15–17] could surely be applied to the steady profile of tidal bores but to my knowledge have not been.

The theory presented here gives simple expressions for the shape of the bore, the wavelength and amplitude of its undulations, and its leading slope, in terms of a single parameter characterising the downflowing river and the incoming tide.

A disclaimer: this theory is not intended to give precise predictions for observed undular bores (whose profiles depend on tidal conditions, vary from river to river, at different locations on each river, and across the front of the bore at a given place and time). Rather, it is a ‘minimal model’ [18, 19], that ‘most economically caricatures the essential physics’, not claiming quantitative accuracy. In this context it is worth repeating an old opinion [4], that remains valid: ‘none of the commonly used wave theories (of tidal bores) are in exceptional agreement with data’.
The paper is organised as follows. Section 2 is a reprise of standard hydraulic jump theory, deriving relations between the depths and speeds of the water before and after the bore passes, and describing the horizon. Section 3 introduces the Hamiltonian theory, leading to an expression for the bore profile, though with some quantities undetermined. In section 4, elementary application of self-consistency fixes these quantities, and derives the simple expressions already mentioned. Section 5 discusses the order-of-magnitude agreement with some measurements on real bores. The concluding section 6 describes some related phenomena. Appendix A outlines a possible extension of the approximate theory, incorporating Lagrangian manifolds and Maslov asymptotics, and envisaging a stronger application of self-consistency. The physics of tidal bores requires a particular operator ordering, explained in appendix B.

Figure 2(a) illustrates the bore as seen from the river bank (the ‘land frame’), moving upstream with speed $U$: the downstream velocity is $-U e_x$. In this frame, the downstream flow velocity of the river before the bore passes is $u_0 e_x$, with $u_0 > 0$. After the bore passes, the water velocity of the incoming tide is $u_1 e_x$, with $u_1 < 0$: the river flows backwards. The depth of the river before the bore passes is $d_0$, and the depth after the bore passes is $d_1$; of course, $d_1 > d_2$. Specification of $d_1$ is an approximation, because the depth continues to increase after the bore passes and the tide continues to rise; here $d_1$ refers to the depth soon after the main undulations have passed, before the subsequent rise.

Figure 2(b) illustrates the more convenient ‘bore frame’, in which the bore profile is stationary. In this frame, the velocity of the water before the bore passes ($x \ll 0$) is $v_0 e_x$, and after it has passed ($x \gg 0$) the velocity is $v_1 e_x$, with $v_0$ and $v_1$ both positive. The relations between the speeds in the two frames are

$$v_0 = u_0 + U, \quad v_1 = u_1 + U. \quad (1.1)$$

Our aim will be to determine the bore profile $d(x)$.

2. Review of hydraulic jump theory [4, 21]

Water is assumed incompressible, and the width of the river channel is assumed to be the same before and after the bore passes. The currents through vertical sections before and after the bore passes must be equal, so continuity, in the bore frame, gives

$$v_0 d_0 = v_1 d_1. \quad (2.1)$$

An immediate application, using (1.1), gives the bore speed in terms of quantities describing the downflowing river and the incoming tide:
We assume that friction is negligible. Then the difference between the pressure forces on a volume of water before and after must equal the rate of change of momentum in the volume. This application of Newton’s law is the ‘momentum equation’:

$$\frac{1}{2} \rho (d_0^2 - d_1^2) = d_1 v_1^2 - d_0 v_0^2.$$  

(2.3)

Convenient dimensionless parameters, extensively used in the following, are the ratio of depths and the initial bore frame Froude number, defined as

$$r \equiv \frac{d_1}{d_0}, \quad Fr_0 \equiv \frac{v_0}{\sqrt{gd_0}}.$$  

(2.4)

Continuity and the momentum equation connect these parameters [4]:

$$Fr_0 = \sqrt{\frac{1}{2}r (r + 1)}, \quad i.e. \quad r = \frac{1}{2}\left(1 + 8Fr_0^2 - 1\right).$$  

(2.5)

Bores correspond to \( r > 1 \) and \( Fr_0 > 1 \), and undular bores correspond to the approximate range \( 1 < Fr_0 < 1.5 \), i.e. \( 1 < r < 1.35 \)[4]. (In this range, the relations between \( r \) and \( Fr_0 \) are approximately linear: \( r \approx 1 + 1.354(Fr_0 - 1) \) reproduces the second equation in (2.5) with error less than 0.15%.) For larger values of \( r \) and \( Fr_0 \), the bore front can be turbulent.

Equation (2.5) imply

$$v_0 = \sqrt{gd_0\left(\frac{1}{2}r(r + 1)\right)} > \sqrt{gd_1} \text{ (supercritical),}$$

$$v_1 = \sqrt{gd_1\left(\frac{1}{2}(r + 1)/r^2\right)} < \sqrt{gd_1} \text{ (subcritical).}$$  

(2.6)

An important consequence [4] follows from the additional fact that the maximum speed of long waves on still water of depth \( d \) is \( \sqrt{gd} \); waves on the river before the bore arrives, travelling in either direction relative to the water, will travel into the bore and be swallowed by it; similarly, waves travelling upstream on the tide after the bore has passed will also travel into it. This contributes to the stability of the bore under small disturbances.

Since the bore profile is smooth, the depth-averaged water velocity \( v(x) \) must change from \( v_0 \) to \( v_1 \) as the bore passes, and so must pass through \( \sqrt{gd_0} \) near the front of the bore where the depth is \( d_0 \) (to be determined later). Therefore the bore is a horizon, upstream of which no waves can propagate away; only evanescent waves (decaying upstream) can exist there. This is the basis of the connection with the Hawking effect associated with horizons in relativity. Undular tidal bores provide a natural example, complementing similar flow horizons that have been extensively investigated [9]. I am grateful for a referee for pointing out that, as already mentioned, the analogy is with white, not black, holes [10, 11]. The region upstream of the bore front corresponds to the interior of the hole; in this region, gravity waves cannot travel upstream (away from the horizon, deeper into the hole), and can only travel into the horizon and thence to the region outside (downstream, i.e. on the incoming tide)—as in a white hole.

3. Hamiltonian theory for steady bore profile

The profile will be built from the dispersion relation for long waves on the surface of shallow water [22]. The relation, assumed approximately applicable when the depth \( d(x) \) is varying, gives frequency as a function of wavenumber \( k \):

$$\Omega_0(x, k) = \sqrt{\frac{gk}{d} \tanh(d(x)k)}.$$  

(3.1)

The corresponding group velocity \( \partial\Omega_0(x, k)/\partial k \) decreases from a maximum \( \sqrt{gd(x)} \) at \( k = 0 \), representing the longest waves.

We will apply (3.1) in an unusual way, by regarding \( d(x) \) not as the depth of an undisturbed profile decorated by small disturbances, but as the total depth including the undulations. (One insight from this ‘poor person’s nonlinearity’ is a simple explanation of wave-breaking (see Jeffreys in [6]): \( \sqrt{gd} \) is greater for crests than troughs, so crests travel faster than troughs and eventually overtake them.)

In (3.1), \( \Omega_0 \) represents waves in water at rest, but for a bore the water is moving, and moreover moving nonuniformly: the flow speed depends on \( x \), and varies (in the bore frame: figure 1(b)) from \( v_0 \) to \( v_1 \). To transform \( \Omega_0 \) to the bore frame, we identify the front of the bore with the place where the depth is \( d_b \), with flow speed \( v_0 \); both values will be determined later. The transformation, obtained via the Doppler effect, or alternatively by regarding \( \Omega_0(x, k) \) as a Hamiltonian and using the standard boost formula [23], with the
velocities away from the front being determined by continuity, is
\[ \Omega(x, k) = \sqrt{gk \tanh(d(x)k)} - \frac{v_b d_b}{d(x)}. \] (3.2)

The further assumption here is that the boost formula is approximately applicable for nonuniform flows.

We will find that near the front of the bore the undulations are much longer than the depth, and observations of undular bores confirm this. Therefore we can expand for small \( k \):
\[ \Omega(x, k) \approx \omega(x, k) = \left( \sqrt{g d(x)} - \frac{v_b d_b}{d(x)} - \frac{1}{6} \sqrt{g d(x)k^2} \right)k. \] (3.3)

Also, we expand \( d(x) \) near the front of the bore, since we are interested in the undulations there. The front is chosen as the location \( x = 0 \) where the water speed \( v_b \) is related to the depth \( d_b \) in order to make the first two terms in \( \omega \) cancel to leading order:
\[ d(x) = d_b + s_b x + \cdots (s_b = \tan \theta_b), \quad v_b = \sqrt{gd_b}. \] (3.4)

Here \( s_b \) is the slope of the profile at the front, whose value will also be determined later, and the slope angle is \( \theta_b \). Thus the dispersion relation becomes
\[ \omega(x, k) = \sqrt{gd_b} \left( \frac{3s_b x}{2d_b} - \frac{d_b^2 k^2}{6} \right)k. \] (3.5)

Now come two crucial points. First, we are interested in a steady profile \( d(x) \), which corresponds to zero frequency \( \omega(x, k) \). The condition \( \omega = 0 \) for each \( x \) determines three `local waves'. One of them has \( k = 0 \), which as will soon be apparent leads to the difference in height between the river and the tide (i.e. \( d_1 - d_0 \) in figure 2). The other two are real for \( x > 0 \), i.e. in the subcritical region downstream (behind the bore), and \( k^2 \) increases with \( x \); they represent waves following the bore front \( x = 0 \). At the front—the horizon, analogous to those in relativity [17]—these two \( k \) values coalesce. This corresponds to a caustic of the associated locally sinusoidal waves. For \( x < 0 \), i.e. in the supercritical region in the river upstream of the bore, the \( k \) values are imaginary, and represent an evanescent wave (analogous to tunnelling [24]). (In the white hole analogy, these would be evanescent waves leaking away from the horizon into the hole. Moreover, waves entering the horizon and escaping outside would have finite frequency, corresponding to scattering, not the zero frequency steady profile relevant to bores.)

Second, the dispersion relation must be regarded as an operator acting on \( d(x) \), with \( k \) interpreted as \(-i\partial/\partial x\) (see also [21] for a number of applications of this idea). Thus
\[ \omega(x, \frac{i}{\partial x})d(x) = 0. \] (3.6)

In the analogy with quantum physics, \( \omega \) (multiplied by Planck’s constant \( \hbar \)) corresponds to a Hamiltonian (energy) operator, involving momentum \( \hbar k \), and \( d(x) \) is the zero-energy eigenfunction: analogous to a ‘soft mode’ [25, 26]. Because of the ordering ambiguity associated with the product of the non-commuting quantities \( x \) and \( k \) in (3.5), this differential operator is not unique. In fact, as argued in appendix B, the unique ordering that gives a profile representing a tidal bore is as written in (3.5): \( x \) to the left of \( k \), i.e. \(-ix\partial/\partial x\).

To solve (3.6), we note that with this ordering the external factor \( k \) in (3.5) generates an indefinite integration, and the function multiplying \( k \) gives the differential equation for the Airy function:
\[ \frac{d^2}{d\xi^2} \text{Ai}(\xi) = \xi \text{Ai}(\xi), \] (3.7)
in which
\[ \text{Ai}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \exp \left\{ i\left( \frac{t^3}{3} + \xi t \right) \right\} = \frac{1}{\pi} \int_0^{\infty} dt \cos \left( \frac{4t^3}{3} + \xi t \right). \] (3.8)

Thus, as can be checked directly, the solution of (3.6) with upstream and downstream boundary conditions \( d(-\infty) = d_0, \; d(+\infty) = d_1 \) is
\[ d(x) = d_0 (1 + (r - 1) \eta(x/L)), \] (3.9)
in which the dimensionless ‘undular bore function’ is
\[ \eta(\xi) = \int_{-\xi}^{\infty} du \text{Ai}(u) = \frac{1}{2\pi} \text{Im} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{dt}{t} \exp \left\{ i\left( \xi t - \frac{1}{3} t^3 \right) \right\}. \] (3.10)
and the length \( L \) is

\[
L = \frac{d_b}{(9\eta_0)^{1/3}}. \tag{3.11}
\]

The undular bore function \( \eta(\xi) \) is easily calculated numerically from either integral representation, and is illustrated in figure 3. The same integral of \( \text{Ai}(\xi) \) is described—also as the zero frequency limit—at the end of a detailed study of scattering and coupling of waves on varying flows (p 21 of [16]) with finite \( \omega \).

4. Applying self-consistency

Evaluating the bore profile (3.9) at the front \( x = 0 \), and using [27]

\[
\eta(0) = \int_0^{\infty} d\xi \text{ Ai}(\xi) = \frac{1}{3}, \tag{4.1}
\]

fixes the depth at the front:

\[
d_b = d(0) = \left(1 + \frac{1}{3}(r - 1)\right)d_0 = \frac{1}{3}(r + 2)d_0. \tag{4.2}
\]

This is the first application of self-consistency. A remark: the depth (4.2) is in apparent contradiction with (3.4) and the continuity and momentum relation (2.5), because

\[
d_b = \frac{v_0^2}{g} = \frac{v_0^2}{g} \frac{d_0^2}{} = \left(\frac{\eta_0 d_0}{\sqrt{g}}\right)^{2/3} = (\text{Fn})^{2/3}d_0 = \left(\frac{1}{2}(r + 1)\right)^{1/3}d_0. \tag{4.3}
\]

This discrepancy with (4.2) is related to the fact, ignored in our approximate theory, that near the front, where the river reverses direction (in the land frame), no unique velocity \( v(x) \) can be defined, because the bore frame flow speed at each \( x \) is different at different depths (faster near the bottom than at the surface). In any case, the two expressions for \( d_b \) differ by less than 6% over the range \( 1 \leq r \leq 1.5 \), so the discrepancy is not important.

For the second application, we fix the slope \( s_b \) at the front, by differentiating the profile there; this involves \( L \) in (3.11) which itself involves \( s_b \):

\[
s_b = \frac{d_0(r - 1)}{L} \eta'(0) = \frac{3(r - 1)(9\eta_0)^{1/3}}{(r + 2)}\text{Ai}(0)
\]

\[
= 9\sqrt{3}\left(\frac{r - 1}{r + 2}\right)^{3/2} = 3.2976\left(\frac{r - 1}{r + 2}\right)^{3/2}. \tag{4.4}
\]

The maximum slope of the undular bore function occurs not at \( \zeta = 0 \) but at the first inflection point of \( \eta(\xi) \), which is (see (3.10)) the first zero of \( \text{Ai}'(\xi) \). The ratio of slopes is

\[
\frac{\eta''_{\text{max}}}{\eta''(0)} = \frac{\text{Ai}(\xi_0)\text{Ai}'(\xi_0) = 0}{\text{Ai}(0)} = 1.50877. \tag{4.5}
\]
Therefore the maximum slope $\theta_{\text{max}}$ of the bore (figure 1(b)), is

$$\tan \theta_{\text{max}} = \frac{\eta_{\text{max}}}{\eta'(0)} = 4.9753 \left(\frac{r - 1}{r + 2}\right)^{3/2}. \quad (4.6)$$

Figure 4 shows this slope as a function of the depth ratio $r$. The slope is gentler for weaker bores ($r$ near 1), as it must be.

In the third application, we calculate the ‘wavelength’ of the bore, defined as the distance $\Delta x$ (figure 1(b)) between its first two crests. This involves the corresponding distance $\Delta \zeta$ in the undular bore function (figure 3), which is the distance between the first and third zero of $Ai(\zeta)$:

$$\Delta \zeta = 3.18245. \quad (4.7)$$

Thus the wavelength is

$$\frac{\Delta x}{d_0} \equiv \frac{L \Delta \zeta}{d_0} = \frac{\Delta \zeta (r + 2)^{3/2}}{9\sqrt{3} Ai(0)(r - 1)} = 0.3426 \frac{(r + 2)^{3/2}}{\sqrt{(r - 1)}}. \quad (4.8)$$

Figure 5 shows this wavelength ratio as a function of the depth ratio $r$. The wavelength is longer than the depth, and more so for weaker bores, supporting the interpretation in terms of long waves on shallow water.

We can also calculate the ‘amplitude’ $A(r)$ of the undulations, defined [4] as half the difference between the depth of the first crest and the first trough, divided by the depth $d_0$ of the river before the bore passes. The values $\eta_{\text{max}} = 1.2744$, and $\eta_{\text{min}} = 0.8082$ lead to

$$A(r) \equiv \frac{d_{\text{max}} - d_{\text{min}}}{2d_0} = \frac{1}{2}(\eta_{\text{max}} - \eta_{\text{min}})(r - 1) = 0.2331(r - 1). \quad (4.9)$$
Finally, the ‘steepness’ $S(r)$ of the bore [4] is the dimensionless ratio of the contrast and the ‘wavelength’, namely

$$S(r) = \frac{d_{\text{max}} - d_{\text{min}}}{2\Delta x} = \frac{9}{2} \sqrt[3]{\frac{2}{r+2}} \left( \frac{r-1}{r+2} \right)^{3/2} = 0.6803 \left( \frac{r-1}{r+2} \right)^{3/2}. \tag{4.10}$$

5. Comparison with observation

The very useful collection of data for natural and artificial undular bores in [4] indicates good agreement with the standard hydraulic jump theory reproduced in section 2 (see figure 2.13 of [4]). For the wave phenomena of interest here, the situation is less clear. Measurements on particular properties give widely spread results, frustrating systematic comparison with theory. Nevertheless, a few remarks can be made.

Figure 6 shows data for the steepness $S(r)$, together with the theoretical prediction (4.10). Evidently the agreement is as good as can be expected with these data. For other properties, for example the maximum wave height $\eta_{\text{max}}$ and the amplitude $A(r)$, for which data are also collected in [4], the agreement is less satisfactory: in these cases, the theoretical predictions are substantially smaller than most of the measurements.

6. Concluding remarks

The foregoing theory identifies the profiles of undular tidal bores as classical counterparts of zero-energy eigenfunctions and soft modes in quantum physics, and as classical horizons analogous to those in relativity (associated with white, not black, holes). It involves essentially a single parameter (the after/before depth ratio $r = d_1/d_0$), and is based simply on the dispersion relation for long waves on shallow water, with nonlinearity incorporated heuristically. The theory generates a dimensionless profile $\eta(\xi)$ (equation (3.10)) that depends on no parameters at all, and resembles the shapes of real undular bores.

But the theory is not, and is not intended to be, quantitatively accurate. As explained in the previous section, it gives a reasonable representation for some data, but reproduces others only within an order of magnitude. Moreover, it fails to reproduce a feature sometimes seen in real bores: the first maximum is not always the highest. This may be associated with the steady increase in depth as the tide continues to rise after the bore has passed. Within the framework presented here, the theory could be improved as outlined in the appendix, by removing some of the approximations made in section 3. This would still be no substitute for more accurate theories, based more firmly on nonlinear fluid mechanics.
Although the Airy function enters in an unusual way, through its integral, its occurrence was not unexpected. This is because the maximum of the dispersion relation (3.1) corresponds to the simplest type of caustic, on which focusing occurs where two locally sinusoidal waves coalesce; the caustic is the envelope of the associated rays. Caustics are common features of waves of all kinds [28], and, as understood long ago by Airy [29], the associated wave that decorates them has a special form, described by his eponymous function (3.8).

In optics, an Airy function that everyone can see describes ‘supernumerary rainbows’, which are wave oscillations decorating the directional caustic [30, 31] resulting from focusing of sunlight by raindrops. Currently much studied [32, 33], also in optics, are ‘Airy beams’: propagating along curved paths without spreading. And on a cosmic scale, gravitational lensing of light is dominated by caustics [34, 35], but the associated Airy functions are either too small to see, or obscured by decoherence [36].

In the waves on water that are our main interest here, several Airy functions occur. Tsunamis, whose arrival at coasts cause probably the worst natural disasters, are caustics in spacetime [37, 38], travelling across oceans, with Airy-dominated forms determined by (3.1); further focusing can result from variations in the ocean depth [39]. The V shaped waves generated by ships and swimming ducks are caustics in deep-water waves [40] (see also section 36.13 of [27]); people in a small boat that gets rocked as the V from a large ship passes by feel the oscillations of the Airy function in their bodies. For ripples that are dominated by surface tension, the generalisation of (3.1) [22] implies a minimum group velocity rather than a maximum, leading to tiny oscillations ahead the front, rather than behind it. These can sometimes be seen in a sink, just inside the circular hydraulic jump caused by water radiating from the point struck by a jet from a tap above; the theory is close to that presented here for undular bores, with the difference that the sign of the coefficient of $k^2$ in (3.3) is positive rather than negative.

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**Appendix A. Lagrangian manifold and Maslov asymptotics**

This outlines a way to eliminate two of the approximations in section 3: small $k$ in the dispersion relation, and small $x$ in the bore profile. The first step is to write the condition for zero frequency, using (3.2) and $v_b = \sqrt{g d_b}$, as

$$\Omega(x, k) = \left(\frac{g \tanh(d(x)k)}{k} - \sqrt{g d_b} \frac{d_b}{d(x)}\right) k = 0. \quad (A.1)$$

With $k = -i \partial / \partial x$, this represents an infinite-order differential operator, whose exact eigenfunction seems analytically inaccessible. However, an asymptotic approximation can be built from the curve $x_c(k)$ ('Lagrangian manifold') in the phase space $(x, k)$, obtained from (A.1).

The curve is conveniently represented in the $(K, D)$ plane, where

$$K \equiv kd_b, \quad D \equiv \frac{d(x)}{d_b}. \quad (A.2)$$

Thus (A.1) becomes

$$\frac{D \tanh KD}{K} = 1, \quad (A.3)$$

illustrated in figure A1, together with the small $K$, small $D - 1$ limiting form

$$K^2 = 6(D - 1), \quad (A.4)$$

corresponding to the approximations in section 3. As in more general situations [41], the caustic (here $x = 0$) is the singularity of the projection of the manifold ‘down’ $k$.

To build an asymptotic approximation from the Lagrangian manifold, we use the method of Maslov [20] (explained simply in section 5 of [42]). This represents the profile as a Fourier transform, incorporating the WKB approximation in $k$ space. The result is
Appendix B. Operator ordering

After scaling, and with an inessential change of sign, the dispersion relation (3.5) can be written

\[ \omega(x, k) = k^3 + xk. \]  

(A.6)

The most general ordering, parameterised by an angle \( \theta \), is

\[ xk \Rightarrow xk \cos^2 \theta + kx \sin^2 \theta = -ix \frac{\partial}{\partial x} - \sin^2 \theta. \]  

(A.7)

The differential equation for the profile \( d(x) \) that this generates is

\[ d''''(x) = xd''(x) + \sin^2 \theta \ d'(x). \]  

(A.8)

in which primes denote derivatives.

For the choice \( \theta = 0 \), the general solution is

\[ d(x) = \int_{x_0}^{x} dx' (CAi(x') + DBi(x')). \]  

(A.9)

This represents a tidal bore if \( D = 0 \) and \( x_0 = -\infty \), giving the integrated Airy profile in the main text.

Another exactly solvable case is \( \theta = \pi/4 \), i.e. \( \sin^2 \theta = 1/2 \), for which the operator in (A.7) is formally Hermitian. The general solution for this case is

\[ d(x) = CAi\left(\frac{x}{2^{2/3}}\right)^2 + DAi\left(\frac{x}{2^{2/3}}\right)Bi\left(\frac{x}{2^{2/3}}\right) + EBi\left(\frac{x}{2^{2/3}}\right)^2. \]  

(A.10)

This ordering is excluded on physical grounds, because no combination of \( C, D, E \) gives a profile that interpolates between different heights at \( x = +\infty \) and \( x = -\infty \), as is required to represent a tidal bore.

For general \( \theta \), the solution can be expressed in terms of three \( 1F2 \) generalised hypergeometric functions, which seem not to be able to be represented more simply. However, taking the Fourier transform in (A.8) leads to the following integral representation for the profile:

\[ \begin{align*}
&\text{Figure A1. Lagrangian manifold for zero frequency. Thick curve: exact, from ((A.1) to (A.3)), Dashed curve: approximation (A.4).} \\
\end{align*} \]
There is a branchpoint singularity at \( k = 1 \), whose resolution distinguishes the different solutions. If \( \theta = 0 \) the singularity is integrable, so there is no secular jump between \( x = +\infty \) and \( x = -\infty \), and therefore no bore solution. Only if \( \theta = 0 \) is there a singularity leading different depths for \( x = +\infty \) and \( x = -\infty \), and therefore to a bore.

This situation, where a particular ordering is uniquely selected because it supplies a qualitative feature that the physical phenomenon requires, is unusual. More familiar ordering ambiguities, for example in quantum mechanics, simply differ by higher-order (e.g. semiclassical) corrections.

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