Superconductor vortex spectrum from Fermi arc states in Weyl semimetals

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Vortices in type-II superconductors host discrete energy levels that carry critical information about the normal state and pairing symmetry. For example, ordinary metals produce equally spaced levels with finite zero-point energy whereas massless Dirac metals yield exotic Majorana fermions at precisely zero energy. Weyl semimetals are gapless topological materials defined by accidental band intersections or Weyl nodes in the bulk and a bizarre surface metal composed of open Fermi arcs instead of closed Fermi surfaces. We ask, “what is the vortex spectrum of a superconductor that descends from a Weyl semimetal?” Restricting to non-magnetic Weyl semimetals and superconductivity that is fully gapped when uniform, we show that the vortex spectrum is – contrary to previous results – generically gapped. At low-energies, it is determined by the total Berry phase accrued by a wavepacket traversing closed orbits consisting of Fermi arcs on opposite surfaces connected by oneway bulk conduits, and is expected to produce periodic oscillations in physical properties such as the specific heat as the vortex is tilted. Miraculously, tilting the vortices also transmutes them between bosonic, fermionic and supersymmetric, with the last category hosting a pair of nonlocal Majorana fermions. The spectrum becomes immune to the slab thickness at tilts that we dub “magic angles”. In many models and materials, nonlocal Majorana fermions and supersymmetry exist precisely at these tilts.

A major milestone in the history of superconductivity was the discovery that it is broadly of two types: type-I superconductors expel magnetic fields completely while type-II superconductors allow magnetic field above a threshold to thread through vortices – topological defects whose winding is unaffected by smooth deformations of the superconducting order parameter. While superficially analogous to vortices in a classical spinning fluid, superconductor vortices are fundamentally quantum mechanical entities with discrete energy levels whose structure encodes properties of the parent superconductor as well as the normal state metal that the superconductor condenses from. In 1964, Caroli-deGennes-Matricon derived the vortex spectrum and proved it to be gapped if the parent metal is an ordinary Fermi gas and superconductivity is conventional [1], which triggered several decades of activity on vortex spectra in other types of metals and superconductors. In contrast, vortices in two dimensional (2D) spinless \( p + ip \) superconductors [2] and s-wave superconductors that descend from a 2D Dirac fermion [3] were predicted to host zero energy states that behave like Majorana fermions (MFs). MFs are exotic particles that are their own anti-particles and were first believed to be realized by neutrinos. Their condensed matter realizations are touted for diverse potential applications, ranging from quantum computing [4–10] and topological order [11] to supersymmetry (SUSY) [12–16], quantum chaos and holographic blackholes [17, 18], making them highly coveted in solid state systems. In recent years, Fe-based superconductors have emerged as versatile platforms that not only exhibit strongly type-II superconductivity with both gapped and gapless vortices, some of which traps MFs, but whose normal states can also be tuned into various metallic and semimetallic phases in both 2D and 3D [19–32]. Thus, given a metal that can potentially yield type-II superconductivity, a timely and pertinent goal is to determine the vortex spectrum theoretically.

The past decade has seen the rise of gapless topological matter, spearheaded by Weyl semimetals (WSMs) [33–40]. WSMs are 3D systems whose band structure contains accidental crossings between non-degenerate bands at arbitrary points in momentum space. Near such crossings or Weyl nodes, quasiparticles are gapless and mimic relativistic Weyl fermions. The Weyl nodes carry a chirality or handedness that prevents them from being destroyed perturbatively provided translational symmetry of the lattice persists, and endows them with various topological responses to electromagnetic fields. Powerful no-go theorems stipulate that generic and inversion symmetric WSMs contain an even number of Weyl nodes with half of each chirality, whereas in time-reversal (\( T \)) symmetric WSMs (TWSMs), each Weyl node is accompanied by a time-reversed partner of the same chirality which leads to quadruplets of Weyl nodes.

A striking feature of WSMs is the presence of surface Fermi arcs (FAs) – strings of zero energy surface states that connect the surface projections of pairs of Weyl nodes of opposite chirality. A FA resembles a broken segment of a 2D Fermi surface and forms a closed loop with a FA on the opposite surface of a finite slab. The penetration depth of a FA into the bulk depends strongly on the surface momentum and in fact diverges at the Weyl node projections, thus making the surface inseparable from the bulk. Thus, the Fermi “surface” of WSMs consists of FAs on the surface of the material and bulk Fermi points at the Weyl nodes (or small Fermi pockets if the Weyl nodes are doped). Such strange Fermiology cannot be captured by a purely surface or purely bulk theory; yet, a basic physical question remains: “what is the spectrum of a superconductor vortex in a WSM?”
1. GENERAL VORTEX SPECTRUM

In recent work, we addressed this question by focusing only on the bulk states and concluded that the vortex is gapless [41]. Here, we show using a powerful semiclassical approach that eliminates the need for a Hamiltonian, that the FAs fundamentally modify the spectrum and produce a generically gapped vortex instead. We restrict to TWSMs, as $\mathcal{T}$ symmetry is generically needed for a weak pairing instability in the first place, and propose that the vortex spectrum is:

$$E_n^\pm = \pm \frac{\Delta_0}{\xi_{\text{FA}}} \left( n + \frac{1}{2} + \frac{\Phi_B + \Phi_S - \Phi_Q}{2\pi} \right)$$

This is our main result. Here, $l_{\text{FA}}$ is of order the length of the FA, $\Delta_0$ is the pairing amplitude far from the vortex, $\xi$ is the superconductor coherence length and $n \in \mathbb{Z}$. Additionally, $\Phi_S$ is total Berry phase of a “classical” path defined by the FAs on both surfaces that ignores their bulk penetration. The net phase acquired by a wavepacket traversing the bulk is defined as $\Phi_R$. In the simplest case, depicted in Fig. 1, where FAs on opposite surfaces connect the same pairs of Weyl nodes, $\Phi_B = \Delta K \cdot R_v$ with $\Delta K$ connecting these nodes in momentum space and $R_v$ connecting opposite ends of the vortex in real space. Henceforth, we parameterize $R_v = (a_x \hat{x} + a_y \hat{y} + \hat{z}) L_z \equiv (a_\perp + \hat{z}) L_z$, where $L_z$ is the slab thickness and $\hat{z}$ is the surface normal. Finally, $\Phi_Q$ is a quantum correction arising from the penetration of FA states into the bulk, given in this parameterization by

$$\Phi_Q = 2d\Delta K_\perp \cdot a_\perp$$

where $d$ is the average bulk penetration depth of the FA states. Thus, Eq. 1 predicts a generically non-degenerate spectrum with equally spaced energy levels with spacing $\varepsilon = \Delta_0/\xi_{\text{FA}}$, while the zero-point energy is determined by Berry phase of the FAs, Weyl node locations, sample thickness and vortex orientation.

Eq. 1 is inspired by results in Refs. [42], [43] and [41]. Ref. [43] showed that quasiparticle dynamics in inhomogeneous superconductors can be faithfully captured by quantizing the semiclassical action for wavepackets traveling in closed orbits in real space. The action, which appears as a phase in the relevant path integral, was shown to consist of three terms: (i) a Bohr-Sommerfeld phase $\oint k_{\text{classical}} \cdot dr_{\text{classical}}$ for the classical orbit, (ii) a Berry phase due to rotation of the Nambu spinor, and (iii) a $\pi$ phase if a unit vortex is encircled. Within a complementary momentum space picture, Ref. [42] proved that, for an arbitrary smooth Fermi surface in the normal state and arbitrary pairing symmetry that ensures a fully gapped state when superconductivity is uniform, the superconductor vortex spectrum is given by $\epsilon_n^\pm = \pm \frac{\Delta_0}{l_{\text{FS}}} \left( n + \frac{1}{2} + \frac{\Phi_{\text{FS}}}{2\pi} \right)$ for $l_{\text{FS}}\xi \gg 1$, where $l_{\text{FS}}$ and $\Phi_{\text{FS}}$ are the Fermi surface perimeter and Fermi surface Berry phase, respectively. The normal state is assumed to be $\mathcal{T}$-symmetric, which leads to a pair of Fermi surfaces with opposite Berry phases in the normal state that produce eigenstates related by particle-hole symmetry inside the vortex.

To propose Eq. 1 for a TWSM, we first note that the Bohr-Sommerfeld phase, the $\pi$ phase from the vortex and the Nambu-Berry phase can be viewed as yielding $n$, $1/2$ and $\Phi_{\text{FS}}/2\pi$, respectively in $\varepsilon_n$. Then, we recall that a Weyl node with chirality $\chi = \pm 1$ produces a chiral MF in the vortex core with chirality $\chi w$, where $w = \pm 1$ is the winding number of the vortex [41]. Thus, for $w = 1$, a right-(left-)handed Weyl node produces a chiral MF inside the vortex with upward (downward) group velocity. For a smooth vortex, defined by $|\Delta K\xi| \gg 1$, these chiral modes allow wavepackets to travel between FAs on opposite surfaces without scattering. The smoothness also ensures that a wavepacket on the surface travels along a single FA without scattering into other FAs. Thus, the semiclassical orbit naturally involves travel along a FA on the top surface, tunneling through the bulk via a downward chiral MF, FA traversal on the bottom surface followed by tunneling up the bulk via an upward chiral MF. Since a TWSM contains quadruplets of Weyl nodes and even number of FAs on each surface related by $\mathcal{T}$, such orbits appear in $\mathcal{T}$-related pairs but with opposite energies in the vortex to ensure that the overall spectrum is particle-hole symmetric. This picture inspires the generalization of $\Phi_{\text{FS}}$ to $\Phi_B + \Phi_S - \Phi_Q$, the total phase acquired by a wavepacket traversing a closed orbit in mixed real-and-momentum space, as depicted in Fig. 1.

The three components of $\Phi_{\text{FS}}$ can be understood as follows. The bulk phase $\Phi_B$ can be viewed as a phase jump during inter-surface tunneling through the bulk along the chiral Majorana modes. Equivalently, it is an additional contribution to the Bohr-Sommerfeld phase rather than to $\Phi_{\text{FS}}$. Since the chiral modes are tied to the Weyl nodes, $\Phi_B$ equals the optical path difference, $K_1 \cdot R_v - K_2 \cdot R_v = \Delta K \cdot R_v$, between semiclassical wavepackets centered at Weyl nodes at $K_1$ and $K_2$ traversing the bulk portions of the orbit. The natural analog of $\Phi_{\text{FS}}$, which captures travel around a 2D Fermi surface, is the Berry phase along the FAs, $\Phi_{\text{FA}}$. While $\Phi_{\text{FA}}$ can be calculated if the exact FA wavefunctions are known, greater insight is obtained by writing $\Phi_{\text{FA}} = \Phi_S + 2d\Delta K_z$, where $\Phi_S$ is the Berry phase along the FA acquired by states projected onto the surface and $2d\Delta K_z$ is the optical path length due to the penetration of FA wavefunctions into the bulk. The penetration also effectively reduces the thickness seen by a wavepacket traveling along the vortex axis by $2d$, as such a wavepacket can tunnel between the vortex core and the surface when it is within distance $\sim d$ from the surface. These two effects of the penetration can be conveniently combined into a quantum correction $\Phi_Q$, given by Eq. (2), which captures the optical path difference between the classical and quantum orbits.

An immediate experimentally testable prediction of
our result is periodic peaks in the density of states,
\[ D(E) = \sum_{n,\lambda} \delta(E - E_n^\lambda), \]
as the vortex is tilted, which appear whenever \( E_n^\pm = 0 \). These peaks will induce oscillations in the specific heat, \( C = k_B \sum_n \left[ \frac{\hbar^2}{m^2 T} \text{sech} \left( \frac{K_n^\pm}{m^2 T} \right) \right] \), and other transport and thermodynamic quantities at temperatures below the mini-gap, \( T \lesssim \epsilon/k_B \). In particular, noting that \( \Phi_S \) does not depend on the vortex orientation, \( D(0) \) peaks whenever \( \Delta K_\perp \cdot a_\perp (L_z - 2d) \) equals a half-integer. Thus, the tilt parameters for two successive peaks obey
\[ \Delta K_\perp \cdot \left[ a_\perp^{(j)} - a_\perp^{(j+1)} \right] = \frac{2\pi}{L_z - 2d} \]
so \( D(0) \) and \( C \) are periodic in \( a_\perp \) with a period determined by the Weyl node locations and the effective thickness of the slab. These oscillations are reminiscent of quantum oscillations due to FAs in WSMs [44–46]. There, a magnetic field \( B \) induces cyclotron orbits involving surface FAs and bulk chiral modes and the oscillation period in \( 1/B \) equals the area enclosed by FAs on opposite surfaces while the slab thickness enters the oscillation phase as an optical path length. At the quantum level, the magnetic field produces discrete Landau levels whose spacing relies on the area enclosed by the FAs and zero-point energy captures the bulk path length. Similarly, for a superconductor vortex, the bulk portion of the orbit induces a path length that affects the zero-point energy. However, the level spacing is governed by the FA length rather than the area enclosed. Moreover, particle-hole symmetry of the superconductor forbids oscillations in \( D(0) \) as \( \Delta_0 \) or \( \xi \) are varied. Instead, oscillations occur as \( \Phi_B \) and \( \Phi_Q \) are manipulated via the vortex orientation.

We emphasize several other striking aspects of Eq. 1. Firstly, Eq. 1 predicts a gapped spectrum with discrete energy levels for generic parameters even in the thermal-dynamic limit whereas a pure bulk analysis leads to the prediction of a gapless, continuous spectrum [41]. Pure bulk analysis does yield a discrete spectrum on a finite slab due to quantum confinement. However, the discretization predicted by Eq. 1 is reminiscent of Landau level formation in a 2D electron gas under a magnetic field rather than ordinary finite-size quantization. Secondly, Eq. 1 involves the slab thickness in a highly non-trivial way, namely, within a phase term \( \Phi_B \) that governs the zero-point energy, while \( \Phi_S \) and \( \Phi_Q \) are thickness-independent surface terms. The form \( \Phi_B = \Delta K \cdot R_v \) leads to a remarkable prediction: when the vortex is tilted to a “magic angle” such that \( \Delta K \perp \cdot R_v \) becomes independent of the thickness. Under mild conditions, we show shortly that the spectrum at the magic tilt is critical and contains a pair of highly nonlocal MFs.

Besides the FA and chiral mode contribution given by Eq. 1, the vortex spectrum will contain bulk states too. In particular, the bulk dispersion due to a single isotropic undoped Weyl node with velocity \( v \) is given by
\[ E_{\text{bulk}}^\pm (n_x, n_y, q) = \pm \hbar v \sqrt{\left( \frac{2\Delta_0}{\mu} \right)^2 + q^2} \] while doping to a chemical potential \( \mu \) such that \( |\mu| \gg \Delta_0 \) yields
\[ E_{\text{bulk}}^\pm (n_x, n_y, q) = \pm \hbar v \sqrt{\left( \frac{n_x \Delta_0}{\mu} \right)^2 + q^2}. \] Here, \( n_x, n_y, n \in \mathbb{Z} \geq 0, n_x + n_y \geq 1, n \geq 1, q \) is the momentum along the vortex axis measured relative to the Weyl node [41, 42]. The cases \( n_x = n_y = 0 \) and \( n = 0 \) correspond to the chiral MF mentioned earlier for an undoped or a doped Weyl node, respectively, and is the only gapless mode stemming from each Weyl node. The presence of such modes na\textasciitilde{}vely leads to the prediction of a gapless vortex [41]. Here, we find that the gapless modes conspire with the FAs to produce a gapped vortex. Moreover, non-chiral modes lie above the bulk
gap $E_g = \min \left( \sqrt{2\hbar v \Delta_0 / \xi}, \hbar \Delta_0 / \xi |\mu| \right)$: clearly, $E_g \gg \varepsilon$ if $l_{F,A} \gg \max \left( \sqrt{\Delta_0 / \hbar v \xi}, |\mu| / \hbar v \right) \sim \xi^{-1}$, assuming standard Ginzburg-Landau theory. In other words, the smooth vortex limit of $l_{F,A} \xi \gg 1$, where Eq. 1 is expected to hold, also ensures that non-chiral bulk modes are at parametrically higher energies. Thus, Eq. 1 describes the vortex spectrum for a smooth vortex at low energies.

We now substantiate our claims via exact diagonalization of the lattice model described in Methods. Fig. 2(top left) shows the FAs and Weyl nodes in a minimal TWSM with four Weyl nodes located at $\pm K^1$ and $\pm K^2$. We chose parameters such that all nodes are at different $k_z$ and $|\Delta K_z| < |\Delta K_y|$ where $\Delta K = K^1 - K^2$. Calculating the vortex spectrum is computationally costly; this limits us to relatively small systems and short $\xi$ which, in turn, causes level spacings to be unequal for modest values of $n$ in Eq. 1. We circumvent this limitation by studying the spectrum as a function of vortex orientation and fitting the tilt angles where zero modes occur to their expected locations based on Eq. 1. Besides yielding remarkable agreement between the expectations and the observations, the tilting protocol exposes sharp, periodic oscillations in $D(0)$ and $C$.

Fig. 2(top right) shows the vortex spectrum for a finite slab when a vortex, initially along $\hat{z}$, is tilted separately towards the $x$- and the $y$-axis. Tilting towards the positive $y$-axis ($a_y > 0$) produces numerous level crossings, which is consistent with $\Phi_B = (\Delta K_y a_y + \Delta K_z) L_z$ changing by many multiples of $2\pi$ as $a_y$ varies. In contrast, the spectrum varies weakly when the vortex is tilted towards the $x$-axis, which is consistent with $\Phi_B = \Delta K_x a_x L_z$ varying negligibly with $a_x$ since $\Delta K_z$ itself is small. In Fig. 2(bottom left), we plot the wavefunctions of a pair of levels with equal and opposite energies in $(k_x, k_y, z)$ space. The levels, which are related by particle-hole symmetry of the superconductor, are clearly localized around semiclassical orbits related by $T$. This confirms the picture that motivated Eq. 1, namely, that the vortex spectrum follows from quantizing semiclassical orbits in mixed real-and-momentum space, and that semiclassical orbits related by $T$ turn into pairs of particle-hole conjugate quantum eigenstates. Finally, 2(bottom right) shows sharp, periodic peaks in $D(0)$ at $a_y$ values where the spectrum is gapless. The oscillations are inherited by $C$ at low temperatures but get washed out at temperatures exceeding the minigap.

To resolve the Berry phase dependence of the zero point energy, we note that Eq. 1 predicts a pair of zero modes, as seen for the $y$-leaning vortex, whenever $\Phi_{\text{tot}} = \Phi_B + \Phi_S - \Phi_Q$ equals an odd multiple of $\pi$. In this geometry, $\Phi_B = (\Delta K_y a_y + \Delta K_z) L_z$, $\Phi_S$ is determined by the band structure and is $a_y$- and $L_z$-independent, and $\Phi_Q \propto a_y$ and $L_z$-independent. An immediate prediction that follows is that $\Phi_{\text{tot}}$ becomes independent of $L_z$ at the “magic angle”, $\theta = -\tan^{-1}(\Delta K_z / \Delta K_y)$. Interestingly, we see in Fig. 3(top) that zero modes exist precisely at the magic tilt, $a_y \approx 0.424$, implying that $\Phi_S - \Phi_Q = \pi \text{ mod } 2\pi$ at this tilt. We proceed to fit $\Phi_{\text{tot}}$ to a suitable function of $a_y$ and $L_z$ and extract the values of $\Delta K_x$, $\Delta K_z$ and $\Phi_S$. These values show good agreement with the values determined directly from the normal state band structure, as indicated in Fig. 3(bottom). Further details of the fitting procedure are in Methods.

**II. NONLOCAL MAJORANA FERMIONS**

MFs are exotic particles that are their own anti-particles and were first proposed— but are yet to be found—in the context of neutrino physics. In recent years, the interplay of band topology, spin-orbit coupling and superconductivity has paved a new route to these elusive particles. Following realizations in semiconductor nanowire-superconductor heterostructures [47–49], MFs were recently seen for the first time in a 3D system—at the ends of vortices in the bulk superconductor FeSe0.45Te0.55 [19, 20, 23, 27]. In addition, they have been predicted to occur inside superconductor vortex cores on the surfaces of topological insulators [3], metals with strong spin-orbit coupling [29, 42], and recently, TWSMs proximate to a topological insulators [41]. Besides their fundamental importance, MFs have been shown to form building blocks for fault-tolerant quantum computation when weakly coupled [4–10] and maximally quantum-chaotic systems dual to a blackhole when strongly coupled [17, 18], as well as realize the elusive supersymmetry on a tabletop [12–16]. This inspires a continuing search for systems which host them. While MFs in all the above materials appear as topologically protected localized zero energy bound states trapped in topological defects such as superconductor vortices and domain walls [2, 3, 5–7, 10, 24, 42, 48–59], we now show that the semiclassical orbits described in this work give rise to a novel class of MFs that are nonlocal in mixed real-and-momentum space.

As depicted in Fig. 1(right) and computed numerically in Fig. 2(right), semiclassical orbits in a TWSM appear in pairs related by $T$. Heuristically, since energy is conjugate to time, the two orbits in a pair yield quantum eigenstates $|n\rangle$ and $|\bar{n}\rangle$ with opposite energies $E_n^\lambda$ and $E_{\bar{n}}^\lambda$ that guarantee a particle-hole symmetric spectrum. More precisely, they satisfy

$$\mathcal{C}|n, \lambda\rangle = |n, -\lambda\rangle, \ H_v|n, \lambda\rangle = E_n^\lambda |n, \lambda\rangle; \lambda = \pm$$

where $\mathcal{C}$ denotes charge conjugation and $H_v$ is the vortex Bogoliubov-de Gennes Hamiltonian. This behavior is a mixed real-and-momentum space generalization of the correspondence between particle-hole conjugate states in the vortex spectrum and Kramer’s conjugate Fermi surfaces in the normal state [42].

A striking behavior arises whenever $E_n^\lambda = 0$. At these points, “cat” superpositions of the eigenstates are simultaneous eigenstates of $H_v$ and $\mathcal{C}$ since

$$\mathcal{C} \left( |n, +\rangle \pm |n, -\rangle \right) = \pm \left( |n, +\rangle \pm |n, -\rangle \right)$$

(5)
Figure 2. Numerical verification of the main results. Top left: Normal state band structure showing four bulk Weyl nodes (red and blue spheres), all at different \( k_z \), and surface FAs connecting them. The four nodes lie on the green plane, which is clearly not parallel to the surface. Top right: Vortex spectrum of a \( L_x \times L_y \times L_z = 23 \times 23 \times 34 \) system as the vortex is tilted separately towards the \( x \)-axis and the \( y \)-axis, where \( \tan^{-1} a_i \) \((i = x, y)\) is the tilt angle towards the \( i \)-axis. Bottom left: Net probability density of the two lowest energy wavefunctions in \((k_x, k_y, z)\) space at \( a_y = 1.25 \), marked ‘X’ in the middle figure, obtained by Fourier transforming the 3D real space wavefunctions with respect to \( x \) and \( y \). Bottom right: Suitable normalized density of states \( D(0) \) and specific heat \( C \) at different temperatures versus \( a_y \). Zero-modes in the spectrum lead to sharp peaks in \( D(0) \) at periodic intervals of \( a_y \) and induce oscillations in \( C \) at low \( T \) that get smeared out at high \( T \). Plots are drawn for the band parameter \( u = 1.2 \) which yields \( \Delta K_{\text{calc}} = \{0.029, 0.428, 0.181\} \times 2\pi \), and superconducting parameters \( \Delta_0 = 0.50 \), \( \xi = 2.0 \) which yield \( \varepsilon = \Delta_0/\xi \approx 0.08 \). See Methods for model details. We approximate \( D(E) = \frac{1}{\pi} \text{Im} \left( \sum_n \frac{1}{(E - E_n) + i\Gamma} \right) \) with \( \Gamma = 0.0075 \).

As a result, \(|\chi_{n,\pm}\rangle = \frac{1}{\sqrt{2}} (|n, +\rangle \pm |n, -\rangle) \equiv \chi_{n,\pm}|\phi\rangle\) are MFs, where \(|\phi\rangle\) denotes particle vacuum and \( \chi_{n,\pm}\) are Majorana operators that obey \( \chi_{n,\lambda}^\dagger = \chi_{n,\lambda} \) and \( \chi_{n,\lambda}, \chi_{n,\lambda'} = \delta_{\lambda\lambda'} \). These MFs have a peculiar non-local structure as they are composed of \(|n, \pm\rangle\), which themselves have a nonlocal structure in mixed real-and-momentum space. They are not protected by symmetry; rather, they appear at a series of critical points as a function of the vortex tilt. We can viewed them as analogs of the critical MF wires that occur at topological phase transitions between trivial and topological vortices which, respectively, lack and host surface MFs. This interpretation, however, raises another question: what topological phases do \(|\chi_{n,\pm}\rangle\) separate?

For general band and vortex parameters, the vortex belongs to class D in the Altland-Zirnbauer classification as \( C^2 = +1 \) while \( T \) symmetry is broken. Since the finite thickness is crucial to the physics described here, the vortex is effectively a 0D superconductor and must be viewed as a large superconducting quantum dot. This is in contrast to previous studies of superconductor vortices in metals [29, 42], semimetals [41, 60–62] and insulator surfaces [3, 63], where they are viewed as 1D superconductors. Class D superconductors in 0D are characterized by a \( \mathbb{Z}_2 \) topological invariant, \( \nu \in \{0, 1\} \), where the trivial and topological phases correspond to even and odd
fermion parity [64]. Thus, the “phases” separated by the MFs $|\chi_{n,\pm}\rangle$ carry even and odd numbers of fermions.

### III. Tunable Supersymmetry and Vortex Statistics

SUSY is a symmetry between matter (fermionic) and force (bosonic) particles that was originally proposed as an extension of the Standard Model of particle physics. It predicted every particle to have a superpartner with the same mass but opposite exchange statistics, while physical particles sans their superpartners were understood as a consequence of spontaneous SUSY breaking. Unfortunately, SUSY has received little experimental support in particle physics. However, several condensed matter systems have been shown to exhibit SUSY including the 1D Ising model at the tricritical point [65], boundaries [12, 13] and defects [66] in topological superconductors and chains and arrays of interacting MFs [14–16]. Unfortunately, these proposals face serious practical challenges such as the inability to tune the necessary parameters dynamically and the absence of materials that realize the parent topological phases, thus rendering SUSY experimentally elusive.

We now argue that the critical points discussed earlier, remarkably, exhibit SUSY, and the fundamental exchange statistics of the vortices – bosonic vs fermionic – can be toggled by simply tilting them across these critical points. Intuitively, a vortex is bosonic (fermionic) if it is effectively a 0D superconductor with even (odd) fermion parity. In this context, SUSY is essentially the degeneracy between bosonic and fermionic vortices at criticality. In particular, the pair of MFs in a critical vortex form a complex fermion state that can be either occupied or unoccupied, resulting in vortices with distinct fermion parity at the same many-body energy. For a given tilt away from criticality, no local measurement will be able to distinguish between bosonic and fermionic vortices, and clever interferometric methods that effectively braid the vortices will be necessary. Nonetheless, the current realization should be more experimentally accessible than previous proposals as it relies on existing phases of matter, namely, TWSMs and conventional type-II superconductors, and a simple tuning parameter, namely, magnetic field direction. Interestingly, in any real material, natural variations in vortex orientations will likely result in both bosonic and fermionic vortices, making it the only system to the best of our knowledge where the same type of excitation appears as both bosons and fermions.

To see the SUSY explicitly, we write the vortex Hamiltonian and many-body ground state in second quantized form as $H_v = \sum_m E_m^+ c_{m,+}^\dagger + c_{m,-} - \mathcal{E}$ and $|G\rangle = \prod_m c_{m,\text{sgn}(E_m)}^\dagger |\phi\rangle$, respectively, where the fermion operators obey $c_{m,+}^\dagger = c_{m,-}$ in the physical Hilbert space and $\mathcal{E} = -\sum_m |E_m^+|$ ensures $H_v$ is non-negative definite. When one of the single-particle energies vanishes,
\( E^2_\perp = 0 \), the corresponding fermion operators get promoted to symmetries: \( [H_v, c_{n,+}] = [H_v, c_{n,+}^\dagger] = 0 \). This allows us to introduce operators
\[
Q = c_{n,+} + \sqrt{H_v}, \quad Q^\dagger = c_{n,+}^\dagger + \sqrt{H_v}
\]
that obey the superalgebra
\[
\{Q, Q^\dagger\} = H_v, \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0
\]
and hence, define an \( N = 2 \) SUSY in 0D.

IV. MAJORANA FERMIONS AND SUPERSYMMETRY AT THE MAGIC ANGLE

In real materials, the detailed shape and connectivity of FAs depends on the local boundary conditions. Nonetheless, significant insight can be obtained by considering a slab of a material exposed to vacuum on either side. This approach is routinely adopted in theoretical treatments of topological phases and is pertinent when the topological material either couples weakly to the substrates or is sandwiched between identical substrates. In WSMs, a common feature that appears in this limit is exact overlap between FAs on the top and the bottom surfaces. We now show that magic angle vortices in materials and models with this feature are critical, provided an additional symmetry condition, detailed below, is satisfied. This result should facilitate the search for parameters and materials where the vortex is critical, and therefore carries nonlocal MFs and exhibits SUSY.

We first describe the conditions that protect the coincidence of FAs on opposite surfaces in generic TWSMs. Suppose the coinciding FAs are related by an operation \( \mathcal{P} \):
\[
\mathcal{P}|t(k)\rangle = e^{i\eta(k)}|b(k)\rangle; \quad \mathcal{P}|b(k)\rangle = e^{i\eta'(k)}|t(k)\rangle
\]
where \(|t(k)\rangle, |b(k)\rangle\) denote the exact FA states on the top and bottom surfaces respectively including their spatial profile in \( z \), \( k = (k_x, k_y) \) is the in-plane momentum and \( \eta(k), \eta'(k) \) are phases. \( \mathcal{P} \) must change \( z \to -z \) to interchange the FAs, must preserve \( k \) so that it can protect overlap between FAs of arbitrary shape, and must not change \( k_z \) to preserve the locations of the bulk Weyl nodes. This restricts \( \mathcal{P} \) to be anti-unitary and of the form \( \mathcal{P} = \mathcal{T} \mathcal{I} \), where \( \mathcal{I} \) denotes spatial inversion followed by a local unitary transformation within a unit cell. Next, suppose \( [\mathcal{P}, \hat{H}(k)] = 0 \), where \( \hat{H}(k) \) is the Bloch Hamiltonian matrix at \( k \). \( \mathcal{P} \) would then conserve energy and cause the FAs it relates to disperse in the same direction. However, this leads to a contradiction as each point on the FA contour constitutes an edge state of a 2D Chern insulator defined on a momentum-space sheet that encloses a Weyl node and edge states of Chern insulators must necessarily disperse in opposite directions. If \( [\mathcal{P}, \hat{H}(k)] = 0 \) instead, \( \mathcal{P} \) can protect the coincidence of FAs at zero energy and allow them to disperse in opposite directions. Thus, protected overlap between FAs on opposite surfaces in generic TWSMs requires a particle-hole symmetry at each \( k \). We now show that if \( \mathcal{P}^2 = -1 \), so that \( \eta'(k) - \eta(k) = \pi \) and \( \hat{H} \in \text{CII} \) in the Altland-Zirnbauer classification, a striking phenomenon occurs: zero modes exist in the vortex spectrum precisely at the magic angle, as seen in Fig. 3(left) at \( a_\theta \approx 0.424 \).

At the magic angle, \( \Delta \mathcal{K} \cdot (a_\perp + \hat{z}) = 0 \) implies \( -\Phi_Q = 2d\Delta K_z \), the optical path due to penetration of the FA states into the bulk. As a result, \( \Phi_S - \Phi_Q = \Phi_{FA} \), the total Berry phase from the FA states. We now calculate \( \Phi_{FA} \) directly instead of splitting it as \( \Phi_{FA} = \Phi_S + 2\Delta K \cdot d \) and show that \( \Phi_{FA} = \pi \) if \( \mathcal{P}^2 = -1 \). Explicitly,
\[
\Phi_{FA} = \int k_1 \left( \langle t(k)|\partial_{k_{FA}} t(k)\rangle - \langle b(k)|\partial_{k_{FA}} b(k)\rangle \right) dk_{FA}
\]
\[
= \eta(K_{\perp 2}^1) - \eta(K_{\perp 1}^1)
\]
using Eq. (8), where \( k_{FA} \) is the momentum along the FA, \( K_{\perp 1}^1 = (K_{\perp 1}^1, K_{\perp 1}^2) \). A Berry phase generically is gauge-invariant only for closed paths whereas the FAs are open contours. Thus, \( \Phi_{top} = \int_{K_{\perp 1}^1}^{K_{\perp 2}^1} i \langle t(k)|\partial_{k_{FA}} t(k)\rangle dk_{FA} \)
\[
\Phi_{bottom} = \int_{K_{\perp 1}^1}^{K_{\perp 2}^1} i \langle b(k)|\partial_{k_{FA}} b(k)\rangle dk_{FA}
\]
are not gauge-invariant, which makes \( \Phi_{FA} \) naively ambiguous. To resolve this paradox, we note that each Chern insulator can be understood as an edge state of a Chern insulator. While a single edge of a Chern insulator violates gauge invariance and exhibits a 1D anomaly, opposite edges together respect gauge invariance, so \( \Phi_{FA} \) is indeed gauge invariant.

To determine \( \eta(K_{\perp 2}^1) - \eta(K_{\perp 1}^1) \), we consider the action of \( \mathcal{P} \) on the bulk states. Since Weyl nodes at \( K^1 \) and \( K^2 \) have opposite chiralities, an electron Fermi surface around \( K^1 \), carries the same Chern number as a hole Fermi surface that encloses \( K^2 \). As a result, a smooth set of unitary transformations exists that deform Bloch states on the former into Bloch states on the latter. Shrinking these Fermi surfaces to vanishing volume around the Weyl nodes then reduces the unitary transformations to a pure phase, \( e^{i\alpha} \), which implies that an upward dispersing chiral mode at \( K^1 \) has the same Bloch ket as a downward dispersing chiral mode at \( K^2 \), and vice versa. Now, the FA states smoothly merge with the bulk states at the Weyl nodes, so the upward chiral modes at both \( K^1 \) and \( K^2 \) are simply the end-points of the bottom FA, \( |b(K^1)| \) and \( |b(K^2)| \), respectively, while the downward modes are \( |t(K^1)| \) and \( |t(K^2)| \), or vice-versa. Therefore, \( \langle t(K^1)| = e^{i\alpha} \langle b(K^2)| \) and \( |t(K^2)| = e^{i\alpha} |b(K^1)| \), which yields
\[
e^{i\eta(K_{\perp 2}^1)} = \langle b(K_{\perp 1}^1)| \mathcal{P} |t(K_{\perp 2}^1)|
\]
\[
= \langle t(K_{\perp 1}^1)| \mathcal{P} |b(K_{\perp 1}^1)|
\]
\[
= -e^{i\eta(K_{\perp 1}^1)}
\]
(10)
or \( \Phi_{FA} = \eta(K_{\perp 2}^1) - \eta(K_{\perp 1}^1) = \pi \), where we have used Eq. (8) with \( \eta'(k) = \eta(k) + \pi \), which follows from \( \mathcal{P}^2 = -1 \).
V. METHODS

A. Numerical calculation of vortex spectrum

We support our general claims of Eq. 1 with numerics on an orthorhombic lattice model of a tilted superconducting vortex on a TWSM defined by \(11\). The lattice model contains two layers per unit cell in the \(z\)-direction, assumed to be the surface normal, and is described by the bulk Bloch Hamiltonian and spectrum

\[
H_0 (k, k_z) = \tau \cdot d (k, k_z) - \mu
\]

\[
\varepsilon_{\pm}^2 (k, k_z) = \left( v_x^2 \sin^2 k_x + v_y^2 \sin^2 k_y \pm \ell \right)^2 + d_z^2 (k, k_z)
\]

where \( k = (k_x, k_y) \), \( \tau_z = \pm 1 \) for the two layers of the bilayer, \( d_x (k, k_z) = m_0 + \beta_x \cos k_x + \beta_y \cos k_y + \beta_z \cos k_z \), \( d_y (k, k_z) = u_x \sin k_x + u_y \sin k_y - u_z \sin k_z \), \( d_z (k, k_z) = |d_y (k, k_z)|^2 + |d_x (k, k_z)|^2 \) and \( d_z (k, k_z) \equiv \delta_z (k) = \sigma_x v_x \sin k_x + \sigma_y v_y \sin k_y - \ell \) denotes purely in-plane hopping that captures spin-orbit coupling through spin Pauli matrices \( \sigma_y \) and \( \sigma_z \) symmetry breaking through the term \( \propto \ell \). \( H_0 (k, k_z) \) can be understood as two copies – one for each spin – of the bilayer prescription for WSMs described in Ref. [67]. It preserves \( T \) symmetry (\( T = \sigma_y \mathbb{I} \)) but breaks all spatial symmetries for general \( (u_x, u_y, u_z) \). It preserves a chiral symmetry, \( \mathcal{I} = \mathcal{I}_0 \sigma_z \otimes (r \to -r) \), which is better understood as spatial inversion about a point between the layers of the bilayer, \( \tau_x \sigma_z \otimes (r \to -r) \), followed by a gauge transformation \( \psi \to e^{i\tau_z \sigma_z} \psi \). The resulting particle-hole symmetry, \( \mathcal{P} = \mathcal{T} \mathcal{I} \), causes FAs on opposite surfaces to coincide.

The prescription yields bulk Weyl nodes when \( \varepsilon_{-\text{sgn}(\ell)} (k) = 0 \), while surface FAs occur between projections of the Weyl nodes along curves where \( d_z (k) \) has zero eigenvalues. For \( u_x = u_y = 0 \), all the nodes lie at either \( k_z = 0 \) or \( \pi \), while non-zero \( u_x, u_y \) place the Weyl nodes to distinct \( k_z \). In our calculations, we choose band parameters \( \{v_x, v_y, \ell\} = \{3.53, 2.48, 3.00\}, \{m_0, \beta_x, \beta_y, \beta_z\} = \{1.000, -0.939, 0.371, 0.652\}, \{u_x, u_y, u_z\} = \{u \cos \pi/5, u \sin \pi/5, 1\} \), and tune \( u \) to create different Weyl node and FA configurations.

To determine the vortex spectrum, we diagonalize the Bogoliubov-deGennes Hamiltonian

\[
H = \pi_z H_0 (k, k_z) + H_v (r - a z)
\]

in real space, where \( \pi_i \) are Pauli matrices in Nambu space and

\[
H_v (r - a z) = \Delta (|r - a z|) \times \left( \pi_x \cos \Theta (r - a z) + \pi_y \sin \Theta (r - a z) \right)
\]

captures the vortex. Here, \( \Delta (r - a z) = \Delta_0 \tanh (|r - a z|/\xi) \) and the vortex orientation is parametrized as \( R_v = L_z (a_x, a_y, 1) \).

B. Fitting zero modes to Eq. 1

Direct numerical verification of Eq. 1 involves diagonalizing \( H_v \) in real space and comparing the resulting energy spectrum with Eq. 1. However, the lack of translation invariance in all directions limits us to relatively small \( \xi \), which causes departure from the semiclassical limit and hence, from Eq. 1 at modest energies. We bypass this limitation by varying the vortex orientation and comparing the locations of the zero modes with the predictions of Eq. 1. This way, we always probe only the lowest few energy levels, which are less sensitive to departure from the semiclassical limit.

For fixed band parameters in the normal state, we set \( a_x = 0 \) and vary \( a_y \) between \( \pm L_y/L_z \); for larger \( a_y \), the vortex enters and exits from the side surfaces. Zero modes exist at regular intervals of \( a_y \), which we expect to correspond to \( \Phi_B + \Phi_S - \Phi_Q \) sweeping past an odd multiple of \( \pi \). We repeat an analogous procedure for \( a_x \) with \( a_y = 0 \), but find no zero modes, which is consistent with the prediction of Eq. 1 for \( \Delta K_x \approx 0 \).

The results for \( a_y \neq 0 \), \( a_x = 0 \) are shown in Fig. 4. Defining \( j = \Phi_{\text{tot}}/2\pi \), we assign consecutive half-integer \( j \) values to the zero modes for fixed \( L_z \). For each \( L_z, \Phi_{\text{tot}} (L_z) \) fits excellently to separate straight lines for \( a_y > 0 \) (right tilt) and \( a_y < 0 \) (left tilt):

\[
\Phi_{\text{fit}} (L_z, a_y) = m (L_z) a_y + c (L_z)
\]

The slope and intercept, \( m (L_z) \) and \( c (L_z) \), are each found to be almost perfect straight lines functions of \( L_z \). The upshot is that \( \Phi_{\text{fit}} \) is of the form

\[
\Phi_{\text{fit}} (L_z, a_y) = A + B a_y + C L_z + D a_y L_z
\]

while we expect

\[
\Phi_{\text{tot}} (L_z, a_y) = \Phi_S + \Delta K_y a_y + \Delta K_z L_z + \Delta K_y a_y L_z
\]

Comparing (15) and (16), we extract the values of \( \Delta K_y, \Delta K_z, \Phi_S \) and \( d \). The first three parameters match remarkably well with values calculated directly in the normal state while \( d \) gives \( \Phi_S + 2d \Delta K_z \approx \pi \) at the magic angle as expected.

It is important to perform the fitting separately for \( a_y > 0 \) and \( a_y < 0 \). This is because the semiclassical orbits for the two cases encircle the vortex in opposite directions and acquire equal and opposite \( \Phi_S \), but yield the same values for the other parameters. Moreover, zero modes near \( a_y = 0 \) must be ignored because they involve interference between clockwise and counterclockwise orbits around the vortex, which causes deviations from the semiclassical limit. We also ignore zero modes for large \( |a_y| \), when the vortex ends are near the edge of the lattice.

C. \( \Phi_S \) as a Berry phase on a Bloch sphere

The lattice model defined by Eq. (11) admits an elegant analytical determination of \( \Phi_S \), which facilitates comparison with the vortex numerics.
Figure 4. Detailed numerical fits. Top left: Zero modes indices \((j)\) vs \(a_y\) fit well to straight lines for each \(L_z\), but separately for \(a_y > 0\) (“right”) and \(a_y < 0\) (“left”). Fitting the slopes and intercepts to straight lines in \(L_z\), as shown on the top right, allows extracting \(\Phi_s\), \(\Delta K_y\) and \(\Delta K_\ell\) numerically, which match well with their values calculated from the band structure. Bottom left: The calculated and fitted values of \(\Delta K_{y,z}\) are similar and in good agreement and for both \(a_y > 0\) and \(a_y < 0\). Bottom right: The calculated and fitted values of \(\Phi_s\) are in good agreement, and are equal and opposite mod \(2\pi\) for \(a_y > 0\) and \(a_y < 0\).

| Segment                  | Behavior in \(\tau \otimes \sigma\) | Behavior in \(S\) |
|--------------------------|---------------------------------------|-------------------|
| Top surface FA           | Rotation of \(\langle \sigma \rangle\) about \(\sigma_z\) by \(\phi\), with fixed \(\langle \tau \rangle = (0, 0, 1)\). | Rotation of \(\langle S \rangle\) about \(S_z\) by \(\phi\), with fixed \(\langle S_z \rangle = 1/2\). |
| Downward bulk travel     | \(\pi\) rotation of \(\langle \tau \rangle\) from \((0, 0, 1)\) to \((0, 0, -1)\) with fixed \(\langle \sigma \rangle\). | Rotation of \(\langle S \rangle\) from \((S_z) = +1/2\) to \((S_z) = -1/2\) with fixed \((S_z)\). |
| Bottom surface FA        | Rotation of \(\langle \sigma \rangle\) about \(\sigma_z\) by \(-\phi\), with fixed \(\langle \tau \rangle = (0, 0, -1)\). | Rotation of \(\langle S \rangle\) about \(\sigma\) by \(\phi\), with fixed \(\langle S_z \rangle = -1/2\). |
| Upward bulk travel       | \(\pi\) rotation of \(\langle \tau \rangle\) from \((0, 0, -1)\) to \((0, 0, 1)\) with fixed \(\langle \sigma \rangle\). | Rotation of \(\langle S \rangle\) from \((S_z) = -1/2\) to \((S_z) = +1/2\) with fixed \((S_z)\). |

Table I. Description of the four segments of a semiclassical orbit in bilayer-spin space, \(\tau \otimes \sigma\), and in terms of total spin, \(S = (\tau + \sigma)/2\). The angle \(\phi\) is given in Eq. (17).

As stated in Sec. V A, FAs occur along curves where \(d_z(k)\) has zero eigenvalues. Assuming \(L > 0\), \((k_x, k_y)\) satisfy \(v_x^2 \sin^2 k_x + v_y^2 \sin^2 k_y = \ell\) along such a curve while the spin part of the wavefunction is an eigenstate of \(\sigma_x v_x \sin k_x + \sigma_y v_y \sin k_y\) with eigenvalue \(+\ell\). As a result, the surface projection of a FA state at \(k\), \(|\psi_{k,\lambda}^{FA}\rangle\), satisfies

\[
\langle \psi_{k,\lambda}^{FA} | \sigma_z | \psi_{k,\lambda}^{FA} \rangle = v_z \sin k_z / \ell, \quad \text{where } \lambda = \pm 1 \text{ denote FAs on the top (bottom) surface, while the spin rotates by an angle}
\]

\[
\phi = \arg \left( \frac{v_x \sin K_x^2 + i v_y \sin K_y^2}{v_x \sin K_x^2 + i v_y \sin K_y^2} \right) \quad (17)
\]

along a FA that connects surface projections of Weyl nodes at \(K^1\) and \(K^2\). Moreover, \(|\psi_{k,\lambda}^{FA}\rangle\) satisfy

\[
[\tau_x d_x(k, -i \partial_z) + \tau_y d_y(k, -i \partial_z)] |\psi_{k,\lambda}^{FA}\rangle = 0 \quad (18)
\]

which immediately implies that \(|\psi_{k,\lambda}^{FA}\rangle\) are Jackiw-Rebbi zero modes in bilayer space, spanned by \(\tau_z\). In particular, they are eigenstates of \(\tau_z\) with opposite eigenvalues on
Figure 5. Semiclassical orbit (yellow curve) on the $S = 1$ Bloch sphere that results from the triplet combination of the spin ($\sigma$) and bilayer pseudospin ($\tau$) degrees of freedom. Horizontal (vertical) arms of the orbit capture motion along the FAs (through the bulk) and yield a Berry phase equal to the solid angle enclosed by the orbit.

On the Bloch sphere for total spin, these segments define a patch with area $\phi \times \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \phi$, as shown in Fig. 5. This patch induces a Berry phase $S\phi$, which vanishes in the singlet sector ($S = 0$) and equals $\phi$ in the triplet sector ($S = 1$). Thus, the total Berry phase acquired through the above rotations is

$$\Phi_S = \phi$$

(19)

where $\phi$ is given in Eq. (17).

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