WAKAMATSU’S EQUIVALENCE REVISITED

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Abstract. For a certain Wakamatsu-tilting bimodule over two artin algebras $A$ and $B$, Wakamatsu constructed an explicit equivalence between the stable module categories over the trivial extension algebra of $A$ and that of $B$. We prove that Wakamatsu’s functor is a triangle functor, thus a triangle equivalence.

1. INTRODUCTION

Let $A$ and $B$ be two artin algebras. Denote by $T(A)$ and $T(B)$ their trivial extension algebras. For a certain Wakamatsu-tilting bimodule $A_T B$, Wakamatsu constructed in [19] an explicit equivalence between the stable module categories $T(A)$-mod and $T(B)$-mod.

Wakamatsu’s construction is parallel to the one in [15], where the bimodule $A_T B$ is assumed to have projective dimension at most one on both sides; also see [13] [12]. The forerunners of the work [15] are [13], [1] and [16]. On the other hand, if the bimodule $A_T B$ is tilting of finite projective dimension, a triangle equivalence between these stable module categories was obtained in [11]. Indeed, the result in [11] is more general, which claims that such a triangle equivalence exists provided that $A$ and $B$ are derived equivalent. However, the equivalence in [11] is less explicit but with the advantage of being a triangle equivalence.

It is natural to ask whether Wakamatsu’s equivalence is a triangle equivalence. The aim of this paper is to answer this question affirmatively.

In Section 2, we recall basic facts on cotorsion pairs and $\partial$-functors. We prove that Wakamatsu’s functor is a triangle functor in Section 3. In the last section, we recall the setting of [19], where Wakamatsu’s functor becomes a triangle equivalence.

We fix a commutative artinian ring $R$. Denote by $D = \text{Hom}_R(-, E)$ the Matlis duality, where $E$ is the minimal injective cogenerator of $R$. For an artin $R$-algebra $A$, we denote by $A$-mod the category of finitely generated left $A$-modules. Any full subcategory of $A$-mod is assumed to be closed under isomorphisms. We identify right $A$-modules as left $A^{\text{op}}$-modules, where $A^{\text{op}}$ is the opposite algebra.

2. COTORSION PAIRS AND $\partial$-FUNCTORS

In this section, we recall basic facts on cotorsion pairs and $\partial$-functors. We study special envelopes of short exact sequences. The main references on cotorsion pairs are [2] [5].
2.1. Cotorsion pairs. Let $A$ be an artin $R$-algebra. Let $X = _AX$ be an $A$-module. For a full subcategory $C$ of $A$-mod, a $C$-precover of $X$ means a morphism $f : C \to X$ with $C \in C$ such that any morphism $t : C' \to X$ with $C' \in C$ factors through $f$, that is, $t = f \circ t'$ for some morphism $t' : C' \to C$. The subcategory $C$ is said to be contravariantly finite, if any module has a $C$-precover. Dually, one has the notions of a $C$-preenvelop and a covariantly finite subcategory.

We say that a subcategory $C$ is finite, if $C = \text{add}Y$ for some module $A$. Here, $\text{add}Y$ denotes the full subcategory formed by direct summands of finite direct sums of copies of $Y$. Observe that a finite subcategory is both contravariantly finite and covariantly finite.

Let $V, W$ be two full subcategories of $A$-mod. We denote by $\perp V$ the full subcategory formed by those modules $X$ satisfying $\text{Ext}^i_A(X, V) = 0$ for all $V \in V$.

By a special $V$-preenvelop of an $A$-module $X$, we mean a monomorphism $\alpha : X \to \text{V}$ with $V \in V$ and its cokernel contained in $\perp V$. Then $\alpha$ is indeed a $V$-preenvelop of $X$. This is obtained by applying $\text{Hom}_A(\cdot, V')$ to the exact sequence $0 \to X \to V \to \text{Cok} \to 0$ for each $V' \in \mathcal{V}$. Dually, one has the notion of a special $W$-precover.

A cotorsion pair $(W, V)$ in $A$-mod consists of two full subcategories satisfying $W = \perp V$ and $V = W^\perp$, in which case, both $W$ and $V$ are closed under direct summands and extensions. A cotorsion pair $(W, V)$ is complete if every $A$-module has a special $V$-preenvelop, which is equivalent to the condition that each module has a special $W$-precover; see [2, Lemma 2.2.6]. A cotorsion pair $(W, V)$ is hereditary if $\text{Ext}^i_A(W, V) = 0$ for each $i \geq 1$, $W \in W$ and $V \in V$. In this case, we have $W = \perp V$ and $V = W^\perp$.

The first part of the following result is due to [2, Proposition 3.6]. We include a proof for completeness.

Lemma 2.1. Let $(W, V)$ be a cotorsion pair which is complete and hereditary. Let $\xi : 0 \to X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \to 0$ be an exact sequence of modules. Take any special $V$-preenvelop $\alpha_1$ and $\alpha_3$ of $X_1$ and $X_3$, respectively. Then there is a commutative diagram with exact rows

$$
\begin{array}{cccccccc}
\xi : & 0 & \to & X_1 & \xrightarrow{f} & X_2 & \xrightarrow{g} & X_3 & \to & 0 \\
\xi_V : & 0 & \to & V_1 & \xrightarrow{fV} & V_2 & \xrightarrow{gV} & V_3 & \to & 0,
\end{array}
$$

where $\alpha_2$ is a special $V$-preenvelop of $X_2$. Moreover, given any morphism $t : V_1 \to V$ in $V$ satisfying $t \circ \alpha_1 = 0$, there exists a morphism $t' : V_2 \to V$ satisfying $t = t' \circ f_V$ and $t' \circ \alpha_2 = 0$.

We might call the exact sequence $\xi_V$ a special $V$-preenvelop of $\xi$.

Proof. By a pushout of $\xi$ along $\alpha_1$, we have the following commutative exact diagram

$$
\begin{array}{cccccccc}
0 & \to & X_1 & \xrightarrow{f} & X_2 & \xrightarrow{g} & X_3 & \to & 0 \\
\alpha_2 & \downarrow & \alpha_1 & \downarrow & \alpha_3 & \downarrow & \alpha_3 & \\
0 & \to & V_1 & \xrightarrow{f'} & V_2 & \xrightarrow{g'} & V_3 & \to & 0
\end{array}
$$

Then $\alpha$ is a monomorphism with $\text{Cok}a = \text{Cok}\alpha_1$. Consider the exact sequence $0 \to X_3 \xrightarrow{\alpha_3} V_3 \to W_3 \to 0$. Since $\alpha_3$ is a special $V$-preenvelop of $X_3$, its cokernel $W_3$
lies in $W$. By $\text{Ext}_A^2(W_3, V_1) = 0$, we have the following commutative exact diagram

![Diagram](image)

Then $V_2$ lies in $\mathcal{V}$. Put $\alpha_2 = a' \circ a$, which is a special $\mathcal{V}$-preenvelop, since its cokernel lies in $W$.

For the last statement, we consider the exact sequence $0 \to W_1 \xrightarrow{f_W} W_2 \to W_3 \to 0$ of the cokernels of $\alpha_i$’s. Then $t = t \circ \pi_1$ for some morphism $t: W_1 \to \mathcal{V}$, where $\pi_1: V_1 \to W_1$ is the canonical projection. Since $\text{Ext}_A^2(W_3, V) = 0$, we infer that $t$ factors through $f_W$, that is, $t = t'' \circ f_W$ for some morphism $t'': W_2 \to \mathcal{V}$. Set $t' = t'' \circ \pi_2$ with $\pi_2: V_2 \to W_2$ the canonical projection. Then we are done.$\Box$

The following result indicates that taking the special $\mathcal{V}$-preenvelop of a short exact sequence is partially functorial.

**Lemma 2.2.** Let $(W, \mathcal{V})$ be a cotorsion pair which is complete and hereditary. Assume that we are given the top of the following diagram, which is commutative with rows being short exact sequences. Consider their special $\mathcal{V}$-preenvelopes as in the previous lemma. Here, the morphisms $\alpha_i: X_i \to V_i$ and $\alpha_i': Y_i \to V_i'$ are the special $\mathcal{V}$-preenvelops. Then the dotted morphisms exist, which make the diagram commute.

![Diagram](image)

**Proof.** By the special $\mathcal{V}$-preenvelop $\alpha_1$, we have a morphism $a_V: V_1 \to V_1'$ satisfying $\alpha_1' \circ a = a_V \circ \alpha_1$. For the same reason, we have $b_V': V_2 \to V_2'$ satisfying $\alpha_2' \circ b = b_V' \circ \alpha_2$. But, in general, $a_V \circ a_V \neq b_V' \circ f_V$. By a diagram-chasing, we do have $(h_V \circ a_V - b_V' \circ f_V) \circ \alpha_1 = 0$. By Lemma 2.1 there is a morphism $t': V_2 \to V_2'$ such that $t' \circ f_V = h_V \circ a_V - b_V' \circ f_V$ and $t' \circ \alpha_2 = 0$. Set $b_V = t' + b_V'$. Then there is a unique morphism $c_V$ such that $c_V \circ g_V = k_V \circ b_V$. By a diagram-chasing, we obtain $\alpha_1' \circ c = c_V \circ \alpha_3$. Then we are done.$\Box$

2.2. **Stable categories and $\partial$-functors.** Let $\mathcal{A}$ be an abelian category. Recall that it is a Frobenius category provided that it has enough projectives and enough injectives such that the class of projective objects coincides with the class of injective objects. The stable category $\mathcal{A}$ modulo projectives is defined as follows: the objects
are the same as \( \mathcal{A} \); for two objects \( X, Y \), the Hom group, denoted by \( \text{Hom}_A(X,Y) \), is defined to be the quotient group \( \text{Hom}_A(X,Y)/P(X,Y) \), where \( P(X,Y) \) denotes the subgroup formed by morphisms that factor through projectives; the composition of morphisms is induced from \( \mathcal{A} \). For a morphism \( f: X \to Y \) in \( \mathcal{A} \), we denote by \( f: X \to Y \) the corresponding morphism in \( \mathcal{A} \).

For a Frobenius category \( \mathcal{A} \), its stable category \( \mathcal{A}^\text{st} \) has a natural triangulated structure. For the translation functor \( \Sigma \), we fix each object \( X \) an exact sequence \( 0 \to X \xrightarrow{i} I(X) \xrightarrow{d_X} \Sigma(X) \to 0 \) with \( I(X) \) injective. Any exact sequence \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) in \( \mathcal{A} \) yields an exact triangle \( X \xrightarrow{\xi} Y \xrightarrow{\omega} Z \xrightarrow{\psi} \Sigma(X) \), where \( \omega \) is given by the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & X \\
\dfrac{f}{\xi} & \to & Y \\
\dfrac{g}{d_X} & \to & Z \\
\dfrac{\omega}{\psi} & \to & \Sigma(X) \\
\end{array}
\]

Here, we use the injectivity of \( I(X) \). The morphism \( \omega \) is not unique, but its image \( \omega \) in \( \mathcal{A}^\text{st} \) is unique. In particular, for a selfinjective algebra \( A \), its stable module category \( A\text{-mod} \) becomes a triangulated category. For details, we refer to \([3, I.2]\).

Let \( F: \mathcal{A} \to \mathcal{T} \) be an additive functor from an abelian category to a triangulated category. The translation functor on \( \mathcal{T} \) is denoted by \( \Sigma \). Following \([2, \text{Section 1}]\), we say that \( F \) is a \( \partial \)-functor provided that for each short exact sequence \( \xi: 0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0 \) in \( \mathcal{A} \), there is a chosen morphism \( \omega_\xi: F(Z) \to \Sigma(FX) \), which fits into an exact triangle \( F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\omega_\xi} \Sigma(FX) \). Moreover, the chosen morphism \( \omega_\xi \) is functorial in \( \xi \). More precisely, for each commutative exact diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{a} & \swarrow{b} & \searrow{c} \\
X' & \xrightarrow{f'} & Y' \\
0 & \to & 0
\end{array}
\]

there is a morphism between exact triangles

\[
\begin{array}{c}
F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\omega_\xi} \Sigma(FX) \\
\downarrow{F(a)} & \downarrow{F(b)} & \downarrow{F(c)} & \downarrow{\Sigma(Fa)} \\
F(X') \xrightarrow{F(f')} F(Y') \xrightarrow{F(g')} F(Z') \xrightarrow{\omega_\xi'} \Sigma(FX').
\end{array}
\]

Indeed, it suffices to verify that the rightmost square commutes. We observe that for a Frobenius category, the canonical functor \( \mathcal{A} \to \mathcal{A}^\text{st} \) is a \( \partial \)-functor.

The following fact is well known.

**Lemma 2.3.** (\([3, \text{Lemma 2.5}]\)) Let \( \mathcal{A} \) be a Frobenius category and \( F: \mathcal{A} \to \mathcal{T} \) be a \( \partial \)-functor which vanishes on projective objects. Then the induced functor \( F: \mathcal{A}^\text{st} \to \mathcal{T} \) is a triangle functor.

### 3. Wakamatsu’s functor

In this section, we first recall from \([19]\) the construction of Wakamatsu’s functor. We will prove in Theorem 3.1 that it is a triangle functor.
3.1. The construction. Let $A$ and $B$ be two artin $R$-algebras. Let $A^* B$ be an $A$-$B$-bimodule, on which $R$ acts centrally.

We denote by $\varepsilon$ and $\eta$ the counit and unit of the adjoint pair $(T \otimes_B -, \text{Hom}_A(T, -))$ on $A$-mod and $B$-mod, respectively. More precisely, for each $A$-module $X$, the map $\varepsilon_X: T \otimes_B \text{Hom}_A(T, X) \rightarrow X$ is defined by $\varepsilon_X(t \otimes f) = f(t)$; for each $B$-module $Y$, the map $\eta_Y: Y \rightarrow \text{Hom}_A(T, T \otimes_B Y)$ is given by $\eta_Y(y)(t) = t \otimes y$.

From now on, we assume that the $A$-$B$-bimodule $T$ is faithfully balanced, that is, the structure maps $A \rightarrow \text{End}_{B^{op}}(T)$ and $B^{op} \rightarrow \text{End}_A(T)$ are both isomorphisms. In this case, we have two canonical bimodule isomorphisms

$$\delta: DT \otimes_A T \cong DB,$$

$$\delta': T \otimes_B DT \cong DA,$$

which are given by $\delta(f \otimes t)(b) = f(tb)$ and $\delta'(t \otimes f)(a) = f(at)$. Here, $DT$ has the induced $B$-$A$-bimodule structure.

Recall that $T(A) = A \oplus DA$ is the trivial extension of $A$; it is a symmetric algebra, and thus selfinjective. A $(T(A))$-module is identified with a pair $(X, \phi)$, where $X$ is an $A$-module and the structure map $\phi: DA \otimes X \rightarrow X$ is an $A$-module morphism satisfying $\phi \circ (DA \otimes \phi) = 0$. We sometimes suppress $\phi$ and denote the pair by $X$. Similar notation applies to $(T(B))$-modules.

For an $A$-module $A_V$, we consider the $B$-module

$$L(V) = \text{Hom}_A(T, V) \oplus (DT \otimes_A V),$$

whose elements are viewed as column vectors. Then $L(V)$ becomes a $T(B)$-module via the structure map

$$\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}: DB \otimes_B L(V) \rightarrow L(V),$$

where $*$ is given by the composition $(DT \otimes \varepsilon_V) \circ (\delta^{-1} \otimes \text{Hom}_A(T, V))$. We observe that $L(T)$ is isomorphic to the regular module $T(B)$.

For a $T(A)$-module $X = (X, \phi)$, the $B$-module $DT \otimes_A X$ becomes a $T(B)$-module via the structure map $DB \otimes_B (DT \otimes_A X) \rightarrow DT \otimes_A X$, which is given by the composition

$$-(DT \otimes \phi) \circ (DT \otimes \delta' \otimes X) \circ (\delta^{-1} \otimes DT \otimes_A X).$$

Here, the minus sign is needed in the following construction.

We assume that $(W, V)$ is a complete cotorsion pair in $A$-mod such that $W \cap V = \text{add} T$. For each $A$-module $X$, we fix a special $V$-preenvelop $\alpha_X: X \rightarrow V(X)$ once and for all.

We observe that $DT \otimes \alpha_X$ is always injective by

$$\text{Tor}^A_1(DT, \text{Cok} \alpha_X) \simeq D\text{Ext}^1_A(\text{Cok} \alpha_X, T) = 0.$$

For a morphism $f: X \rightarrow X'$, there is a morphism $f_V: V(X) \rightarrow V(X')$ satisfying $\alpha_{X'} \circ f = f_V \circ \alpha_X$. Note that the morphism $f_V$ is not unique.

We recall from [19] Section 1] the construction of Wakamatsu’s functor

$$S: T(A)\text{-mod} \rightarrow T(B)\text{-mod}.$$

By [19] Lemma 1.1] and [15] Proposition 1.5], for each $T(A)$-module $X = (X, \phi)$, we have the following injective $T(B)$-module homomorphism

$$\Delta_X: DT \otimes_A X \rightarrow L(V(X)) = \text{Hom}_A(T, V(X)) \oplus (DT \otimes_A V(X)),$$

where $\Delta_X$ is given by the composition

$$\text{Hom}_A(T, - \alpha_X \circ \phi \circ (\delta' \otimes X)) \circ \eta_{DT \otimes_A X}.$$
We define $S(X)$ to be the cokernel of this monomorphism. This notation is somehow sloppy, since $S(X)$ depends on $(X, \phi)$, not just the underlying $A$-module $X$.

For a morphism $f: (X, \phi) \to (X', \phi')$ of $T(A)$-modules, we take any morphism $f_V: V(X) \to V(X')$ satisfying $\alpha_{X'} \circ f = f_V \circ \alpha_X$. Then the left square in the following diagram commutes.

\[
\begin{array}{cccccc}
0 & \longrightarrow & DT \otimes_A X & \longrightarrow & L(V(X)) & \longrightarrow & S(X) & \longrightarrow & 0 \\
& & DT \otimes f & & L(f_V) & & S(f) & \\
0 & \longrightarrow & DT \otimes_A X' & \longrightarrow & L(V(X')) & \longrightarrow & S(X') & \longrightarrow & 0
\end{array}
\]

Then there is a unique morphism $S(f)$ making the diagram commute. However, the morphism $S(f)$ depends on the choice of $f_V$, but its image $S(f)$ in the stable category $T(A)$-mod is independent of the choice. This completes the construction of Wakamatsu’s functor $S$: see [19, Lemma 1.2]. Since the functor $S$ depends on the cotorsion pair $(W, V)$, we will say that $S$ is associated to $(W, V)$.

In what follows, when we write $S(f)$, we mean the corresponding morphism in $T(B)$-mod. We have to keep in mind that $S(f)$ depends on the choice of $f_V$, not just $f$.

The following subtlety has to be clarified; compare the treatment in the third paragraph of [19, p.19]. Assume that we are given a special $V$-preenvelope $\alpha'_X: X \to V'(X)$, which might not equal the fixed $\alpha_X$. Replacing $V(X)$ by $V'(X)$ and $\alpha_X$ by $\alpha'_X$ in (3.1), we obtain the cokernel $S'(X)$. Then we have a canonical isomorphism in $T(B)$-mod:

\[\text{can}: S'(X) \xrightarrow{\sim} S(X).\]

Indeed, there is a morphism $s: V'(X) \to V(X)$ satisfying $\alpha_X = s \circ \alpha'_X$. Then a similar diagram as (3.2) defines the above isomorphism, which is independent of the choice of $s$. Consider the previous morphism $f: (X, \phi) \to (X', \phi')$. There is a morphism $f_V: V'(X) \to V(X')$ satisfying $f_V \circ \alpha'_X = \alpha_{X'} \circ f$. Then by replacing $f_V$ by $f_{V'}$ in (3.2), we obtain a morphism

\[S'(f): S'(X) \longrightarrow S(X'),\]

which depends on the choice of $f_{V'}$. We observe the following fact

\[S'(f) = S(f) \circ \text{can}.\]

This fact enables us to abuse $S'(X)$ with $S(X)$, $S'(f)$ with $S(f)$. The notation $S'(f)$ also applies, if the range $X'$ of $f$ has taken a special $V$-preenvelop, different from the fixed one.

### 3.2. The $\partial$-functor

The above recalled Wakamatsu’s functor $S$ vanishes on projective $T(A)$-modules; see [19, Lemma 1.3]. Then it induces an additive functor from the stable module category of $T(A)$ to that of $T(B)$. Our main result claims that the induced functor is a triangle functor, provided that the cotorsion pair $(W, V)$ is in addition hereditary.

**Theorem 3.1.** Let $A T_B$ be a faithfully balanced $A$-$B$-bimodule. Assume that $(W, V)$ is a complete hereditary cotorsion pair in $A$-mod satisfying $W \cap V = \text{add} T$. Then Wakamatsu’s functor $S: T(A)$-mod $\to T(A)$-mod associated to $(W, V)$ is a $\partial$-functor. In particular, it induces a triangle functor $T(A)$-mod $\to T(B)$-mod.

**Proof.** The second statement follows from Lemma 3.3. For the first statement, we take an exact sequence $\xi: 0 \to (X_1, \phi_1) \xrightarrow{f} (X_2, \phi_2) \xrightarrow{g} (X_3, \phi_3) \to 0$ in $T(A)$-mod.
Recall the fixed special $V$-preenvelops $\alpha_{X_1}: X_1 \rightarrow V(X_1)$ and $\alpha_{X_3}: X_3 \rightarrow V(X_3)$. Applying Lemma 2.1, we obtain the following commutative exact diagram

$$
\begin{array}{c}
0 \rightarrow X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{0} \\
\downarrow\alpha_{X_1} \downarrow\alpha'_{X_2} \downarrow\alpha_{X_3} \\
0 \rightarrow V(X_1) \xrightarrow{f'_{V'}} V'(X_2) \xrightarrow{g'_{V'}} V(X_3) \rightarrow 0.
\end{array}
$$

Here, $\alpha'_{X_2}$ is a special $V'$-preenvelop, which might not equal the fixed $\alpha_{X_2}$. For this reason, we use the notation $f'_{V'}$ and $g'_{V'}$, instead of $f_{V'}$ and $g_{V'}$.

We have the following commutative diagram in $T(B)$-mod with exact rows.

$$
\begin{array}{c}
0 \rightarrow DT \otimes_A X_1 \xrightarrow{DT \otimes f} L(V(X_1)) \xrightarrow{L(f_{V'})} S(X_1) \rightarrow 0 \\
\downarrow DT \otimes f \downarrow L(f_{V'}) \downarrow S(f) \\
0 \rightarrow DT \otimes_A X_2 \xrightarrow{DT \otimes g} L(V'(X_2)) \xrightarrow{L(g_{V'})} S'(X_2) \rightarrow 0 \\
\downarrow DT \otimes g \downarrow L(g_{V'}) \downarrow S'(g) \\
0 \rightarrow DT \otimes_A X_3 \xrightarrow{DT \otimes 1} L(V(X_3)) \xrightarrow{S'(g)} S(X_3) \rightarrow 0
\end{array}
$$

Here, for the notation $S'(f)$ and $S'(g)$, we refer to the last paragraph in the previous subsection.

We claim that the sequence $0 \rightarrow S(X_1) \xrightarrow{S'(f)} S'(X_2) \xrightarrow{S'(g)} S(X_3) \rightarrow 0$ is exact. For the claim, we view the columns in the above diagram as complexes. The middle complex is written as $\text{Hom}_A(T, V(X_3)) \oplus (DT \otimes_A V(X_3))$. Since $\text{Ext}_A^1(T, V(X_1)) = 0$, the subcomplex $\text{Hom}_A(T, V(X_3))$ is acyclic. The claim is equivalent to the fact that the following monomorphism

$$DT \otimes \alpha_{X_1}: DT \otimes_A X_1 \longrightarrow DT \otimes_A V(X_3)$$

is a quasi-isomorphism. However, the cokernel of $DT \otimes \alpha_{X_1}$ is isomorphic to $DT \otimes W(X_3)$, where each $W(X_i)$ is the cokernel of $\alpha_{X_i}$, respectively. Here, we abuse $W(X_2)$ with $W(X_2)$, the cokernel of $\alpha'_{X_2}$. The cokernels $W(X_i)$ belong to $W$ and the complex $W(X_3)$ is acyclic. It follows that the complex $DT \otimes W(X_3)$ is also cyclic, since $\text{Tor}_A^1(DT, W(X_3)) \cong D\text{Ext}_A^1(W(X_3), T) = 0$. From this, we infer that $DT \otimes \alpha_{X_1}$ is a quasi-isomorphism. We are done with the claim.

Thanks to the claim and (2.1), we have an exact triangle in $T(B)$-mod

$$S(X_1) \xrightarrow{S'(f)} S'(X_2) \xrightarrow{S'(g)} S(X_3) \xrightarrow{\omega} \Sigma(SX_1).$$

Identifying $S'(X_2)$ with $S(X_2)$ via the canonical isomorphism (5.3) and using (5.4), we obtain the desired triangle

$$S(X_1) \xrightarrow{S'(f)} S(X_2) \xrightarrow{S'(g)} S(X_3) \xrightarrow{\omega} \Sigma(SX_1).$$

It remains to show that $\omega$ is functorial in $\xi$. Before doing this, we notice that $\omega(\xi)$ seems to depend on our choice of $\alpha'_{X_2}$, $f'_{V'}$ and $g'_{V'}$. We claim that $\omega(\xi)$ is actually independent of the choice. This will be proved along with the functorial property of $\omega(\xi)$.

We assume that there is a commutative diagram in $T(A)$-mod with exact rows

$$\xi:\quad 0 \xrightarrow{} (X_1, \phi_1) \xrightarrow{f} (X_2, \phi_2) \xrightarrow{g} (X_3, \phi_3) \xrightarrow{} 0 \\
\downarrow a \downarrow b \downarrow c \\
0 \xrightarrow{} (Y_1, \psi_1) \xrightarrow{h} (Y_2, \psi_2) \xrightarrow{k} (Y_3, \psi_3) \xrightarrow{} 0.$$
For $\xi'$, we have the following commutative diagram

\[
\begin{array}{c}
0 \longrightarrow Y_1 \overset{h}{\longrightarrow} Y_2 \overset{k}{\longrightarrow} Y_3 \longrightarrow 0 \\
0 \overset{\alpha_{Y_1}}{\longrightarrow} V(Y_1) \overset{b_{V'}}{\longrightarrow} V'(Y_2) \overset{c_{V'}}{\longrightarrow} V(Y_3) \longrightarrow 0,
\end{array}
\]

which yields an exact sequence $0 \rightarrow S(Y_1) \overset{S(h)}{\rightarrow} S'(Y_2) \overset{S'(k)}{\rightarrow} S(Y_3) \rightarrow 0$ of $T(B)$-modules. We apply Lemma 2.2 to obtain the relevant morphisms $a_Y: V(X_1) \rightarrow V(Y_1)$, $b_{V'}: V''(X_2) \rightarrow V'(Y_2)$ and $c_{V'}: V(X_3) \rightarrow V(Y_3)$, which make the diagram commute. Then we obtain a commutative diagram between two $3 \times 3$ modules. We apply Lemma 2.2 to obtain the relevant morphisms $\omega$. This proves that $\xi'$ is independent of our choice.

\[\square\]

4. Wakamatsu-tilting bimodules

In this section, we recall from [13, 9] basic facts on Wakamatsu-tilting bimodules. For a certain Wakamatsu-tilting bimodule, Wakamatsu’s functor in Theorem 3.1 can be defined and becomes a triangle equivalence.

Let $\mathcal{A}T$ be an $A$-module satisfying $\text{Ext}^i_A(T, T) = 0$ for each $i \geq 1$. Write $T^\perp$ for the full subcategory consisting of those modules $X$ satisfying $\text{Ext}^i_A(T, X) = 0$ for each $i \geq 1$. Set $\mathcal{T}X$ to be the full subcategory formed by those modules $X$, which admit a long exact sequence $\cdots \rightarrow T^{-2} \overset{d^{-2}}{\longrightarrow} T^{-1} \overset{d^{-1}}{\longrightarrow} T^0 \rightarrow X \rightarrow 0$ with each $T^{-i} \in \text{add}T$ and each cokernel $\text{Cok}d^{-i} \in T^\perp$. In particular, $\mathcal{T}X \subseteq T^\perp$. Recall from [2, Proposition 5.1] that $\mathcal{T}X$ is closed under extensions, cokernels of monomorphisms and direct summands. Similarly, we have the subcategories $\mathcal{X}_T \subseteq T^\perp$.

Let $\mathcal{A}T_B$ be an $A$-$B$-bimodule. We say that $\mathcal{A}T_B$ is a Wakamatsu-tilting bimodule provided that it is faithfully balanced satisfying $\text{Ext}^i_A(T, T) = 0 = \text{Ext}^i_B(T, T)$ for each $i \geq 1$. An $A$-module $\mathcal{A}T$ is a Wakamatsu-tilting module if the natural bimodule $\mathcal{A}T_B$ is Wakamatsu-tilting with $B = \text{End}_A(T)^{op}$. In this case, the dual bimodule $\mathcal{B}(DT)_A$ is also Wakamatsu-tilting, and thus the $B$-module $\mathcal{B}(DT)$ is Wakamatsu-tilting.

We collect known facts on Wakamatsu-tilting bimodules in the following lemma.

**Lemma 4.1.** Let $\mathcal{A}T_B$ be a Wakamatsu-tilting bimodule. Then the following statements hold.

1. $(\perp_r \mathcal{X}, \mathcal{X})$ and $(\mathcal{A}T, (\mathcal{A}T)^\perp)$ are both hereditary cotorsion pairs in $A$-mod; moreover, $(\perp_r \mathcal{X})' \cap \mathcal{T}X = \text{add}T = \mathcal{X}_T \cap (\mathcal{X}_T)^\perp$ and $(\mathcal{A}T)^\perp \subseteq \mathcal{T}X$.
2. $(\mathcal{X}_{DT}, (\mathcal{X}_{DT})^\perp)$ and $(\perp_r \mathcal{X}_{DT})$ are both hereditary cotorsion pairs in $B$-mod; moreover, $\mathcal{X}_{DT} \cap (\mathcal{X}_{DT})^\perp = \text{add}DT = (\perp_r \mathcal{X})' \cap DT\mathcal{X}$ and $(\perp_r \mathcal{X}_{DT}) \subseteq \mathcal{X}_{DT}$.
(3) There are equivalences between these subcategories given by the Hom and tensor functors.

\[
T\mathcal{X} \supseteq (\mathcal{X}_T)^\perp; \quad \mathcal{X}_T \supseteq \perp(\mathcal{T}\mathcal{X})
\]

In general, the above cotorsion pairs are not complete.

**Proof.** For (1), we refer to [9, Proposition 3.1], and (2) follows from (1) applied to the dual bimodule \(B(DT)_A\). For (3), we refer to [19, Proposition 2.14]. □

Following [20], a Wakamatsu-tilting bimodule \(A_TB\) is good provided that there are cotorsion pairs \((\mathcal{W}, \mathcal{V})\) in \(A\)-mod and \((\mathcal{Y}, \mathcal{Z})\) in \(B\)-mod, respectively, which satisfy the following conditions.

(GW1) These two cotorsion pairs are complete hereditary.

(GW2) \(\mathcal{W} \cap \mathcal{V} = \text{add}T\) and \(\mathcal{Y} \cap \mathcal{Z} = \text{add}DT\).

(GW3) The adjoint pair \((T \otimes_B -, \text{Hom}_A(T, -))\) induces an equivalence \(\mathcal{V} \sim \mathcal{Y}\).

(GW4) The adjoint pair \((DT \otimes_A -, \text{Hom}_B(DT, -))\) induces an equivalence \(\mathcal{W} \sim \mathcal{Z}\).

We mention that these conditions are essentially given in [19, Hypothesis 1.4]. In the above situation, we observe that \((\mathcal{X}_T)^\perp \subseteq \mathcal{V} \subseteq \mathcal{T}\mathcal{X}\). Indeed, one proves that \(A_T\) is an Ext-projective generator for \(\mathcal{V}\) and then applies [9, Corollary 3.3]; also see [20, Proposition 3.2.2].

In the following example, we use the well-known fact: a cotorsion pair \((\mathcal{C}, \mathcal{D})\) is complete if and only if \(\mathcal{D}\) is covariantly finite, if and only if \(\mathcal{C}\) is contravariantly finite; see [2, Proposition 1.9].

**Example 4.2.** Let \(A_TB\) be a Wakamatsu-tilting bimodule.

(1) If both \(A_T\) and \(T_B\) have finite projective dimension, then \(A_TB\) is called a **tilting bimodule**. This coincides with the tilting module of finite projective dimension in [10, 4, 6]. In this case, the cotorsion pairs \((\perp(\mathcal{T}\mathcal{X}), \mathcal{T}\mathcal{X})\) and \((\mathcal{X}_DT, (\mathcal{X}_DT)^\perp)\) are complete; see [19, Theorem 2.17]. In this case, we have \(\mathcal{T}\mathcal{X} = T^\perp\); see [2, Theorem 5.4]. Hence, by Lemma 4.1 a tilting bimodule is a good Wakamatsu-tilting bimodule.

If both \(A_T\) and \(T_B\) have finite injective dimension, then \(A_TB\) is called a **cotilting bimodule**. By duality, we observe that a cotilting bimodule is a good Wakamatsu-tilting bimodule.

(2) Following [20], the Wakamatsu-tilting bimodule \(A_TB\) is said to be of **finite type**, if either the subcategory \(\perp(\mathcal{T}\mathcal{X})\) or \((\mathcal{X}_T)^\perp\) of \(A\)-mod is finite. This happens when \(A\) or \(B\) is of finite representation type; for an explicit example, see [19, Example 3.1]. Then a Wakamatsu-tilting bimodule of finite type is good.

Indeed, if \((\perp(\mathcal{T}\mathcal{X}))\) is finite, so is \((\mathcal{X}_DT)^\perp\) by Lemma 4.1(3). Then both cotorsion pairs \((\perp(\mathcal{T}\mathcal{X}), \mathcal{T}\mathcal{X})\) and \((\mathcal{X}_DT, (\mathcal{X}_DT)^\perp)\) are complete. Similar argument applies if \((\mathcal{X}_T)^\perp\) is finite.

We now reformulate Wakamatsu’s equivalence as follows, which combines [19, Theorem 1.5] and Theorem 3.1.

**Theorem 4.3.** (Wakamatsu) Let \(A_TB\) be a good Wakamatsu-tilting bimodule with the relevant cotorsion pairs \((\mathcal{W}, \mathcal{V})\) and \((\mathcal{Y}, \mathcal{Z})\) as above. Then the Wakamatsu’s functor

\[
S: T(A)\text{-mod} \rightarrow T(B)\text{-mod}
\]

associated to \((\mathcal{W}, \mathcal{V})\) is a triangle equivalence. □
Remark 4.4. We keep the assumptions in Theorem 4.3.

(1) If the given Wakamatsu-tilting bimodule $A_T B$ is tilting, there is a triangle equivalence between $T(A)-\text{mod}$ and $T(B)-\text{mod}$ obtained in [11, Theorem 3.1]. It would be of interest to compare these two triangle equivalences. If the tilting module has projective dimension at most one, these two triangle equivalences might coincide in view of [14, Theorem 8].

(2) Consider the category $T(A)-\text{Mod}$ of arbitrary $T(A)$-modules. Using filtered colimits and [8, Theorem 2.4], we obtain a cotorsion pair in $T(A)-\text{Mod}$ and a cotorsion pair in $T(B)-\text{Mod}$, which still satisfy (GW1)-(GW4). Here, we have to replace “add” by “Add” in (GW2). Then we obtain a triangle functor

$$S: T(A)-\text{Mod} \rightarrow T(B)-\text{Mod},$$

which is an equivalence by Theorem 4.3 and infinite dévissage.

(3) We view $T(A) = A \oplus DA$ as a $\mathbb{Z}$-graded algebra with $\deg A = 0$ and $\deg DA = 1$. Then the category $T(A)-\text{gr}$ of graded $T(A)$-modules is equivalent to the module category of the repetitive algebra of $A$; in particular, it is a Frobenius category. By [6, Theorem II.4.9], there is a triangle full embedding from the bounded derived category $D^b(A-\text{mod})$ of $A-\text{mod}$ to the stable category $T(A)-\text{gr}$.

A graded $T(A)$-module $(X, \phi)$ consists of a graded $A$-module $X$ with a structure map $\phi : DA \otimes X \rightarrow X$ of degree one, which satisfies $\phi \circ (DA \otimes \phi) = 0$. Then a parallel argument as in [19, Section 1] carries over to graded modules, and thus we obtain a triangle equivalence

$$S: T(A)-\text{gr} \rightarrow T(B)-\text{gr}.$$

The construction of $S$ is similar to the one in [17, Section 2], where the grading shift by one appears naturally. For the details, we refer to [20, Section 4]. We might call the above equivalence $S$ a repetitive equivalence between the algebras $A$ and $B$. It seems that a good Wakamatsu-tilting module plays a similar role for repetitive equivalence as a tilting module for derived equivalences.

We observe that the above repetitive equivalence $S$ usually will not restrict to a derived equivalence, that is, an equivalence between $D^b(A-\text{mod})$ and $D^b(B-\text{mod})$; see the explicit example in [19, Examples 3.1 and 3.2], where the two algebras are not derived equivalent.

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