Gravomagnetic Lensing by NUT Space

M. Nouri-Zonoz*, and Donald. Lynden-Bell †

Institute of Astronomy, Madingley Road, Cambridge CB3 0HA

24 March 2022

ABSTRACT

Using the fact that the null geodesics in NUT space lie on spatial cones, we consider the gravomagnetic lens effect on light rays passing a NUT deflector. We show that this effect changes the observed shape, size and orientation of a source. Compared to the Schwarzschild lens, there is an extra shear (a differential twist around the lens axis) due to the gravomagnetic field which shears the shape of the source. Gravomagnetic monopoles can thus be recognized by the spirality that they produce in the lensing pattern.

All the results obtained in this case (magnification factor, orientation of images, multiplicity of images, etc.) depend on $Q$, the strength of the gravomagnetic monopole represented by NUT metric. One recovers the results of the usual Schwarzschild lens effect by putting this factor equal to zero.

Key words: relativity - gravitation - (cosmology:) gravitational lensing.

1 INTRODUCTION

The usual gravitational lens effect is based on the bending of light rays passing a point mass $M$ (Schwarzschild lens) in Schwarzschild spacetime. The static nature of the Schwarzschild metric implies that it can only produce gravolectric fields, but in the case of NUT metric (sometimes called the generalized Schwarzschild metric), the existence of the cross term "$d\phi dt$" shows that this space has also a gravomagnetic field $B_g$ (Demiansky & Newman 1966).

A simple example illustrates what we will analyse relativistically in the following sections.

* Email: nouri@ast.cam.ac.uk
† Email: dlb@ast.cam.ac.uk
Here we consider the deflection of an almost straight ray by the overall field of a stationary gravomagnetic monopole of strength $Q$ and mass $M$. There are two fields of the following form:

$$E_g = -\frac{GM}{r^2}\hat{r} \quad \text{and} \quad B_g = -\frac{Q}{cr^2}\hat{r},$$

and the corresponding force per unit mass on the ray is

$$F = -\frac{GM}{r^2} - \frac{Q}{cr^2} v \times \hat{r},$$

where $v = c$ is the speed of the ray. The geometry is shown in Fig. 1. The ray passes the lens parallel to x-axis with a large impact parameter $b$. The normal gravitational force applies an inward radial force to the ray, but there is another force due to the gravomagnetic field along the $z$-direction.

The deflection due to the gravomagnetic force can be found by calculating the impulse applied to the ray, assuming that it is undeflected:

$$\Delta P_z = \int_{-\infty}^{+\infty} F_z dt = Qb \int_{-\infty}^{+\infty} \frac{dt}{(b^2 + c^2 t^2)^{\frac{3}{2}}} = \frac{2Q}{cb}. \quad (1)$$

The gravitational force deflects the ray in the x-y plane, but the gravomagnetic force deflects it out of the plane in the azimuthal direction so that the combined force twists the ray as the ray passes the monopole.

2 NUT SPACE AND NULL GEODESICS

The metric of the NUT space is given by (Kramer et al. 1980, Newman,Tamburino & Unti 1963, Misner 1963)

$$ds^2 = f(r)(dt - 2l\cos\theta d\phi)^2 - f(r)^{-1}dr^2 - (r^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2),$$
where
\[ f(r) = 1 - \frac{2(mr + l^2)}{(r^2 + l^2)} , \] (2)

and \( 2l = Q \) is the strength of the gravomagnetic monopole represented by the NUT metric (\( l \) is usually called the NUT factor or magnetic mass). Following the Landau and Lifshitz (1975) formulation of stationary spacetimes and using the standard variational method (i.e. \( \delta \int ds^2 = 0 \)), the authors (1996) have shown that all the geodesics in NUT space, including the null ones, lie on cones defined through the following equation:
\[ L + \varepsilon Q \hat{r} = j = \text{const.} \] (3)

where \( L, \hat{r} \) and \( \varepsilon \) are the angular momentum, unit radial vector and the energy constant of the orbiting particle and \( j \) is a constant vector along the axis of the cone. Dotting equation (3) with \( \hat{r} \) we find \( j \cdot \hat{r} = \varepsilon Q \) showing that \( \hat{r} \) lies on a cone; similarly \( L \cdot j = l^2 = j^2 - \varepsilon^2 Q^2 = \text{const.} \), so that \( L \) also moves around a cone. Later we will be concerned with the semi-angle of the cone or its complement \( \chi \). From Fig. 2 one can see that
\[ \sin \chi = \frac{\varepsilon Q}{|j|} = \frac{Q}{b(1 + \frac{Q^2}{b^2})^{\frac{3}{2}}} , \]

where \( b = \frac{l}{\varepsilon} \) is the impact parameter. In the limit \( Q \ll b \), i.e. small \( \chi \) we have
\[ \chi \approx \frac{Q}{b} . \]

† We use the natural units in which \( c = G = 1 \).
3 BENDING OF LIGHT RAYS IN NUT SPACE

The equation of the null geodesics in NUT space is (Lynden-Bell & Nouri-Zonoz 1996)

\[ f(r)^{-1} (\varepsilon^2 - \dot{r}^2) - \frac{L^2}{r^2} = 0 \quad , \quad \dot{r} \equiv \frac{dr}{d\tau}, \]

where \( \tau \) is an affine parameter.

Using the angle \( \phi \) defined around the surface of the cone we have

\[ r^2 \dot{\phi} = L. \]

Now using (5) we can integrate (4) and obtain

\[ \Delta \phi = \int \frac{L dr}{r^2 (\varepsilon^2 - \frac{L^2 f(r)}{r^2})^{\frac{1}{2}}}. \]

We perform the above integral in the limit where \( r^2 \gg l^2 + m^2 \), i.e. far away from the event horizon (which is the case for the observed gravitational lenses) and where we have

\[ \Delta \phi = \int_{r_{\text{min}}}^{\infty} \frac{L dr}{r^2 [\varepsilon^2 - \frac{L^2}{r^2} (1 - \frac{2m}{r})]^{\frac{1}{2}}}. \]

Changing to the variable \( u = \frac{1}{r} \) we have

\[ \Delta \phi = - \int_{0}^{u_{\text{max}}} \frac{du}{(\varepsilon^2 - L^2 u^2 + 2mL^2 u^3)^{\frac{1}{2}}}. \]

Calculating this integral to the first order in \( mu \) one can see that the deflection angle on the cone is

\[ \alpha = 2\Delta \phi - \pi = 4m \varepsilon \frac{L}{b} = \frac{4m}{b}. \]

which is the same as in the Schwarzschild case but with the difference that now \( b = \frac{L}{\varepsilon} \) is the impact parameter defined on the cone.

4 GEOMETRY OF LENSING IN NUT SPACE

The gravomagnetic lensing configuration is shown in Figs 3-5. A NUT lens is located at distances \( D_d \) and \( D_{ds} \) from an observer O and a source S respectively. The plane of the sky (or the lens plane) in which all the observations are made is the plane normal to \( D_d \) at N.

For small deflection angle \( \alpha \), \( b \ll D_d \) and \( D_{ds} \), so from figure 3 we have

\[ \beta \simeq \frac{b}{D_d} \quad , \quad \beta_s \simeq \frac{b}{D_{ds}}. \]
Using these two angles one can see that

\[ \phi = \pi + \alpha - \beta - \beta_s = \pi + \frac{4m}{b} \beta - \frac{b}{D_d} (1 + \frac{D_d}{D_{ds}}), \]  

and

\[ \eta = \frac{\phi}{\cos \chi} = \frac{[\pi + \alpha - \frac{b}{D_d} (1 + \frac{D_d}{D_{ds}})]}{\cos \chi}, \]

where \( \phi \) is the angle between generators \( D_d \) and \( D_{ds} \) around the surface of the cone and \( \eta \) is the projection of \( \phi \) onto the plane normal to the axis of the cone at its apex.

5 LENS EQUATION

The lens equation is basically the mapping of the source position to the image position on the lens plane. To find this mapping in the NUT case, we first need to find the angle between position vectors of the source and image (\( r \) and \( r' \)) on the lens plane (see Fig. 5).

As can easily be seen from the geometry in Fig. 5, the positions of the source and image on the lens plane are the intersections of this plane with lines OS and OI respectively (Points F and G). One should note that the point G is not on the cone but on the plane normal to the axis of the cone at its apex.

From Fig. 5 one can see that the angle between the position vectors of the source and image

\[ \angle \text{between vectors} = \pi + \alpha - \beta - \beta_s - \frac{4m}{b} \beta + \frac{b}{D_d} (1 + \frac{D_d}{D_{ds}}). \]
is the same as the angle between vectors \( r' \) and \( \mathbf{NM} \). Where \( \mathbf{NM} \) is the component of \( \mathbf{D}_{ds} \) on the lens plane. Therefore, representing the vector \( \mathbf{NM} \) by \( \mathbf{R} \), we have

\[
\mathbf{r}' . \mathbf{R} = \mathbf{r}' . [\mathbf{D}_{ds} - (\mathbf{D}_{ds} . \hat{\mathbf{D}}_d) \hat{\mathbf{D}}_d] = \mathbf{r}' . \mathbf{D}_{ds}.
\] (9a)

On the other hand

\[
\mathbf{r}' = \mathbf{r}' \left( \frac{\mathbf{j} \times \mathbf{D}_d}{jD_d \cos \chi} \right),
\] (9b)

\[
R = D_{ds} \sin \theta.
\] (9c)

where \( \theta \) is the angle between generators \( D_d \) and \( D_{ds} \) in the triangle ONS. Substituting equations (9b,c) in (9a) and using a spherical coordinate system centered at N one can show that

\[
D_{ds} \sin \theta \cos \psi = \frac{(\mathbf{D}_d \times \mathbf{D}_{ds}) . \mathbf{j}}{jD_d \cos \chi}.
\]

But

\[
(\mathbf{D}_d \times \mathbf{D}_{ds}) . \mathbf{j} = |j|D_d D_{ds} \cos^2 \chi \sin \eta.
\]

Therefore

\[
\cos \psi = \frac{\cos^2 \chi \sin \eta}{\sin \theta}.
\] (10)

Using the same spherical coordinates and equations (8a,b) one can show that

\[
\cos \theta = \cos^2 \chi \cos \left\{ \frac{(\pi + \alpha) - \beta [1 + (D_d / D_{ds})]}{\cos \chi} \right\} + \sin^2 \chi.
\] (11)

Equations (10) and (11) together determine \( \psi \) the angle between the position vectors of the source and image in the lens plane. In what follows we will be concerned with the small angle (i.e. \( \alpha, \beta, \chi \ll 1 \)) approximation to the expressions (10) and (11) which are

\[
\cos \theta \cong -1 + 2 \chi^2 + \frac{1}{2} [\alpha - \beta (1 + \frac{D_d}{D_{ds}})],
\] (12a)

and
\[
\cos \psi \approx \frac{\beta(1 + \frac{D_d}{D_{ds}}) - \alpha - \frac{\pi \chi^2}{2}}{\left[4\chi^2 + (\alpha - \beta(1 + \frac{D_d}{D_{ds}}))^2\right]^{1/2}}.
\]

(12b)

The following three limiting cases are worth considering:

a) \(\chi = 0\) for which we get \(\psi = 0\) for \(\alpha > \beta\) and \(\psi = \pi\) for \(\alpha < \beta\),

b) \(\alpha = \beta = 0\) and \(\chi \ll 1\) for which we find that \(\psi = \frac{\pi}{2}\),

and finally,

c) \(\beta \gg \alpha \& \chi\) from which we have \(\psi = 2/\bar{\beta}(q/b)\) where \(\bar{\beta} = \beta(1 + \frac{D_d}{D_{ds}})\).
The above three results are all in agreement with what one expects from simple geometrical considerations.

Having found the angle $\psi$, the lens equation which maps the position of the source to that of the image, can be written as follows:

$$ r = \kappa Ar' \quad \kappa = \frac{r}{r'} $$

or

$$ \begin{cases} x = \kappa(x'\cos\psi - y'\sin\psi) \\ y = \kappa(x'\sin\psi + y'\cos\psi) \end{cases} $$

(13)

where $\kappa$ is given in equation (17) and $r$ and $r'$ denote the position vectors of the source and image in the lens plane with the coordinates $(x,y)$ (see Fig. 5).

6 THE MAGNIFICATION FACTOR

To see what happens to a bundle of rays in passing a NUT lens, we consider the magnification of an infinitesimal circular source centered at $r$; i.e. (Fig. 6)

$$ (X - x)^2 + (Y - y)^2 = \delta x^2 + \delta y^2 = s^2 \quad s \ll 1. $$

(14)

Now by varying equations (13) and using (14) we have

$$ \delta x^2 + \delta y^2 = s^2 = (\delta\kappa)^2(x'^2 + y'^2) + \kappa^2(\delta x'^2 + \delta y'^2) + \kappa^2(\delta\psi)^2 r'^2 + 
+ 2\kappa(\delta\kappa)(x'\delta x' + y'\delta y') + 2\kappa^2(x'\delta y' - y'\delta x')\delta\psi $$

(15)

On the other hand, using the geometry of lensing as shown in Fig. 5 for small angles, we have

$$ r' = \beta D_d \approx b, $$

(16a)

and

$$ r = \gamma D_d = D_d \frac{\{4\chi^2 + [\alpha - \beta(1 + \frac{D_d}{D_{ds}})]^2\}^{\frac{1}{2}}}{(1 + \frac{D_d}{D_{ds}})}. $$

(16b)

where in finding the angle $\gamma$ we used equation (12a) and the fact that (fig.5)

$$ \gamma D_d = \xi D_{ds}, \quad \gamma + \xi = \pi - \theta. $$

© 0000 RAS, MNRAS 000, 1–?
Figure 6. Small circular source at \( r \) lensed to an ellipse at \( r' \)

So

\[
\kappa = \frac{r}{r'} = \frac{\gamma}{\beta} = \frac{[4\chi^2 + (\alpha - \bar{\beta})^2]^{1/2}}{\beta},
\]

(17)

and

\[
\delta\kappa = \left( \frac{\delta b}{b\beta} \right) \frac{(2\alpha\bar{\beta} - 8\kappa^2 - 2\alpha^2)}{[4\chi^2 + (\alpha - \bar{\beta})^2]^{1/2}},
\]

(18)

where

\[
\bar{\beta} \equiv \beta(1 + \frac{D_d}{D_{ds}}),
\]

\[
\delta b \approx \delta r' = \frac{(x'\delta x' + y'\delta y')}{r'},
\]

(19)

and we have used the fact that for small angles \( \alpha, \beta \) and \( \chi \)

\[
\delta \beta = \beta(\frac{\delta b}{b}) \quad \delta \alpha = -\alpha(\frac{\delta b}{b}) \quad \delta \chi = -\chi(\frac{\delta b}{b}).
\]

Using equations (19) and (12) one can show after some manipulation that

\[
\delta \psi = -\frac{4\beta\chi}{[4\chi^2 + (\alpha - \bar{\beta})^2]^{1/2}} \left( \frac{\delta b}{b} \right).
\]

(20)

Substituting (17), (18), (19) and (20) in (15) and after a simple but long calculation one finds that to the lowest order in angles and increments

\[
s^2 = \frac{4\alpha(x'\delta x' + y'\delta y')^2}{b^2\beta} + \frac{[4\chi^2 + (\alpha - \bar{\beta})^2]}{\beta^2}[(\delta x')^2 + (\delta y')^2] + \\
\quad + \frac{8\chi}{b^2\beta}(x'\delta y' - y'\delta x')(x'\delta x' + y'\delta y'),
\]

(21)

Putting \( y' = 0 \), without loss of generality, the equation of the image becomes

\[
(A + B)X'^2 + BY'^2 + 2CX'Y' = 1,
\]

(22)
where
\[ A = \frac{4\alpha}{\beta s^2}, \quad C = \frac{4\chi}{\beta s^2}, \] (23)
\[ B = \frac{[4\chi^2 + (\alpha - \bar{\beta})^2]}{s^2 \beta^2}, \]
and we put \( \delta x' = X' \) and \( \delta y' = Y' \).

One can see that (22) is the equation of an ellipse in \( X'Y' \) coordinates. By definition the magnification \( \mu \) is the ratio of the area of the image to the area of the unlensed source, so
\[ \mu = \frac{\text{the area of the image}}{\text{the area of the source}} = \frac{1}{(\lambda_+ \lambda_-)^{\frac{1}{2}}} = \frac{1}{[(1 - \frac{\alpha^2}{\beta^2})(1 - \frac{\alpha^2}{\beta^2} - \frac{8\chi^2}{\beta^2})]^{\frac{1}{2}}}, \] (24)
where \( \lambda_s \) are the eigenvalues of the rotation matrix that transforms the ellipse into its principal axes \((x''y'')\). (Fig. 6)

As it can easily be seen for \( \chi = 0 \) one recovers the known result of the Schwarzschild lens. The orientation of the ellipse is determined by the angle \( \zeta \) shown in figure 6
\[ \tan \zeta = \frac{2\chi}{(\alpha^2 + 4\chi^2)^{\frac{1}{2}} - \alpha}, \]
or in terms of the strength of the gravomagnetic monopole \( Q \)
\[ \tan \zeta = \frac{Q}{(4m^2 + Q^2)^{\frac{1}{2}} - 2m}. \]
For \( Q = 0 \), as expected, we recover the result of the Schwarzschild lens which is \( \zeta = \frac{\pi}{2} \). So the presence of the NUT lens, not only changes the shape and size of the source but also its orientation and it is this last change which produces the spirality effect in the lensing pattern. For example for \( Q_1 = \frac{m}{2} \) and \( Q_2 = m \) we get \( \zeta_1 = 82.98 \) and \( \zeta_2 = 76.71 \). Therefore the effect of the gravomagnetic field even in these extreme cases is quite small. This twist in the observed image would not be an unexpected effect if one looks at the null tetrad formulation of null geodesics in NUT space (see the appendix).

7 MULTIPLICITY OF IMAGES

One can easily see that the limit \( \chi = 0 \) of the lens equation recovers the results of the Schwarzschild lens concerning the number of the images. In this section we will consider the number of the images for a given position of the source when \( \chi \neq 0 \).

Writing equation (17) and using the fact that to the lowest order in \( \chi \), \( r' = b \), we have
\[ \frac{r}{b} = \frac{[4Q^2 + (4m - bA)^2]^{\frac{1}{2}}}{bA}, \]
where \( A = \frac{D_{ds} + D_d}{D_{ds} D_d} \). Using the above equation one arrives at the following equation for \( b \)
\[
A^2 b'^2 + B b' + 4C = 0 \quad , \quad b' = b^2,
\]
(25)
where
\[
B = -(8mA + r^2 A^2),
\]
\[
C = 4Q^2 + 16m^2.
\]
Solutions of (25) are
\[
b = \left( \frac{4m}{A} + \frac{r^2}{2} \pm \frac{1}{2A^2} \sqrt{\Delta} \right)^{\frac{1}{2}},
\]
(26)
where
\[
\Delta = r^4 A^4 + 16mA^3 r^2 - 16A^2 Q^2.
\]
(27)
We see that, in general, for a given \( r \) there can be two solutions for \( b \) (when \( \Delta > 0 \) or equivalently \( r^2 > r_0^2 \); see below). When \( \Delta = 0 \), there is one solution for \( b \) which happens at
\[
r^2 = r_0^2 = \frac{8m}{A} \left( 1 + \frac{Q^2}{4m^2} \right)^{\frac{1}{2}} - \frac{8m}{A}.
\]
(28)
In this case using (27) and (28) one can see that for \( Q = 0 \rightarrow r = 0 \) and therefore
\[
b = \left( \frac{4m}{A} \right)^{\frac{1}{2}}.
\]
This is the solution which corresponds to the Einstein rings in the Schwarzschild lensing.

In figures 7 and 8 a horizontal line of point sources and its images for the two cases \( \alpha < \beta \) and \( \alpha > \beta \) are shown and Figs 9 - 13 show randomly distributed small circular sources ( for both \( m = Q = 1 \) and \( m = 0 \), \( Q = 1 \) cases with \( D_{ds} \rightarrow \infty \) ) in the plane of the sky and their lensing pattern. One should note that when \( m = 0 \), according to equatin (6), \( \alpha = 0 \) to the first order in \( \frac{m}{r} \) but by performing integral (5) to the lowest order in \( \frac{1}{b} \) (for \( m = 0 \)) we can show that \( \alpha = \frac{\pi l^2}{2b^2} \) (Appendix B). To have a better idea of lensing, some of the point sources as well as the circular ones and their corresponding images are labeled. For \( \alpha < \beta \) all the circles are magnified but for \( \alpha > \beta \) ( Figs 11 and 13 ) all the circles are reduced. The spirality of the lensing pattern can be seen more clearly near to the center of the deflection.

What the observer will actually see is the superposition of the two cases \( \alpha < \beta \) and \( \alpha > \beta \) ( Figs 10 and 12 ).

As is clear from equation (26), for \( \Delta < 0 \) (or equivalently \( r^2 < r_0^2 \) ) there are no solutions for \( b \), in other words an observer at a given position will not be able to see the source, if it lies inside a circle of radius less than \( r_0 \). (point source \( a \) of figure 7 and circular source \( b \) of Fig. 9 are absent in the lensing pattern of the Figs 8 and 12 because of this and the same
reason applies to the incomplete images in Figs 10 - 13). This does not mean that there is a gap or a hole (in the case of a uniform distribution of point sources) in the observed image, because as is shown in Figs 10 and 12, the reduced images (along with the images of the sources nearer to the lens which have smaller $r_0$ by equation (28)) will fill in the expected hole area.

7.1 Part of the sky which cannot be seen by a specified observer

As we mentioned in previous section equation (25) has no solutions for $r < r_0$. This means that for an observer with a given distance $D_d$ from the lens a part of sky is unseeable. Expanding equation (28) in powers of $\frac{Q^2}{m^2}$ and keeping the first order term we have

$$r_0^2 = \frac{Q^2}{Am}$$

Figures 14 and 15 show $r_0$ as a function of $D_{ds}$, the distance between the source and the lens.

7.2 Critical orbits and the dark area of NUT black hole

To find the area at the NUT lens where due to the trapped null geodesics the observer sees (looking directly at NUT hole) a small black sphere we need to find the critical orbits which are specified by

$$\frac{dr}{d\phi} = 0 \quad \frac{d^2r}{d\phi^2} = 0$$

Using (5) one can see that the above two equations are equivalent to the following two equations

$$G(r) = \frac{1}{r^2 + l^2} \left( \frac{dr}{d\phi} \right)^2 = 0 \equiv (r^2 + l^2)^2 - (r^2 - l^2 - 2mr)b^2 = 0 \quad (30)$$

$$\frac{d^2r}{d\phi^2} = 0 \equiv G'(r) = r^4 - l^4 - l^2b^2 - mrb^2 = 0 \quad (31)$$

Now substituting for $r^4$ in (31) from (30) we have

$$r^2(2l^2 - b^2) + 3mb^2r + 2l^2(b^2 + l^2) = 0$$

Solving this for $r$ we find

$$r = \frac{3mb^2 + \sqrt{9m^2b^4 + 8l^2(b^2 + l^2)(b^2 - 2l^2)}}{2(b^2 - 2l^2)} \quad (32)$$

Now to find the impact parameter which is basically the radius of the dark area we need $r$
in the two limiting cases in which a) \( l \ll m \) and b) \( m \ll l \).

For \( l \ll m \) we find from (32)

\[
r = 3m\left(1 + \frac{2}{9} \frac{l^2}{m^2}\right)
\]

(substituting this into equation (31) and solving for \( b^2 \) we have)

\[
b^2 = 27m^2\left[1 + \frac{1}{3}\left(\frac{l^2}{m^2}\right)\right]
\]

It is clear that these equations give the right answer for the Schwarzschild case i.e. for \( l = 0 \)

\[
r = 3m, \quad b = 3\sqrt{3}m
\]

but because we took the limit in which \( l \ll m \), the above results do not give the exact value for the pure NUT case (\( m=0 \)), for this we need to repeat the above procedure for the \( m \ll l \) limit where from (32) we have

\[
r = \sqrt{3}l \left[1 + \frac{m}{l}\left(\frac{6}{27}D + \frac{2}{\sqrt{3}} - \frac{2}{3}\right)\right]
\]

\[
b = 2\sqrt{2}l \left[1 + D(m/l)\right]
\]

in which \( D = \frac{5\sqrt{2} + 48}{24\sqrt{3}} \). These equations are in agreement with the results of the pure NUT case which are

\[
r = \sqrt{3}l, \quad b = 2\sqrt{2}l
\]

An interpolation formula between equations (33a,b) and (35a,b) which are good when either \( m \) or \( l \) is small is given by

\[
r^2_N = \frac{m^2(9m^2 + 13l^2) + 3l^3 \left[l + 2m\left(\frac{6}{27}D + \frac{2}{\sqrt{3}} - \frac{2}{3}\right)\right]}{l^2 + m^2}
\]

\[
b^2_N = 9m^2(3m^2 + 4l^2) + 8l^3(l + Dm)
\]

\[
(m^2 + l^2)
\]

Equations (37) show that a circular orbit of radius \( r_N \) is an unstable null geodesic and null geodesics arriving from infinity with an impact parameter less than \( b_N \) cross the event horizon and get trapped. Therefore \( b_N \) is the radius of the small black sphere the observer sees when he looks directly at the NUT black hole.
As we saw in the last section gravomagnetic lensing by NUT space can produce multiple images of a source. Different images correspond to different impact parameters and path lengths and therefore to different travel times.

As in the usual lensing two different effects are responsible for this delay, namely the difference in path lengths and the difference in the gravitational potential traversed by different light rays. In what follows we calculate the time delay between the arrival time for an image in the presence of the NUT lens and in its absence, taking into account both of the above contributions.

### 8.1 Geometric time delay

Geometric time delay by definition is

\[(\Delta t)_{\text{geo.}} = \Delta l/c.\]  \(38\)

where \(\Delta l\) is the extra path length of the deflected light ray relative to an unperturbed null geodesic in the absence of the lens. Using Figs 3.3-3.5 and the geometrical relations defined there we see that

\[
\Delta l = (D_1 + D_2) - |D_{ds} - D_d|,
\]

where

\[
D_1 = \frac{\cos(\frac{\alpha}{2} - \beta)}{\cos \frac{\alpha}{2}} D_d = (1 + \frac{\alpha^2}{8})[1 - \frac{1}{2}(\frac{\alpha}{2} - \beta)^2] D_d.
\]

\[
D_2 = \frac{\cos(\frac{\alpha}{2} - \beta_s)}{\cos \frac{\alpha}{2}} D_{ds} = (1 + \frac{\alpha^2}{8})[1 - \frac{1}{2}(\frac{\alpha}{2} - \beta_s)^2] D_{ds},
\]

and

\[
|D_{ds} - D_d| = (D_{ds} + D_d) - \frac{D_d D_{ds}}{D_d + D_{ds}} [2\chi^2 + \frac{1}{2}(\alpha - \beta)^2],
\]

is the norm of the vector joining the source and the observer.

Therefore

\[
\Delta l = \frac{1}{2} D_{ds}(\beta_s \alpha - \beta_s^2) + \frac{1}{2} D_d(\beta \alpha - \beta^2) + \frac{D_d D_{ds}}{D_d + D_{ds}} [2\chi^2 + \frac{1}{2}(\alpha - \beta)^2].
\]

Substituting for \(\beta\) and \(\beta_s\) from equation (7) we have

\[(\Delta t)_{\text{geo.}} = \frac{\Delta l}{c} = \frac{1}{c} \left[\frac{1}{2} \frac{D_d D_{ds}}{D_d + D_{ds}} \alpha^2 + 2(\frac{D_d D_{ds}}{D_d + D_{ds}}) \chi^2\right].\]  \(39\)
8.2 Potential delay (Shapiro’s delay)

As we showed in section 1.2.5, potential delay in NUT space can be calculated by using the Fermat generalized principle for stationary spacetimes (Landau & Lifshitz 1975, Schneider, Ehlers & Falco 1992) which reads;

\[
\delta t = \delta \left( \frac{1}{c} \int \frac{dl}{[g_{00}^{-1} + (A_\alpha \frac{dx^\alpha}{dl})]^{-1}} \right) = 0 \quad \alpha = 1, 2, 3.
\]  

(40)

where \( A_\alpha = \frac{g_{0\alpha}}{g_{00}} \) and \( dl \) is the Euclidean distance element along the light trajectory.\(^5\)

Equation (49) shows that \( g_{00}^{-1} + A_\alpha \frac{dx^\alpha}{dl} \) can be thought of as the refractive index, \( n \), of the spacetime under consideration. (in this case the NUT metric). Using this fact, the potential delay will have the following form

\[
(\Delta t)_{\text{pot.}} = \int \frac{dl}{n} - \int \frac{dl}{c},
\]

or for the NUT space

\[
(\Delta t)_{\text{pot.}} = \frac{1}{c} \int_{\text{Source}}^{\text{Observer}} \left[ (1 - \frac{2m}{r})^{-1} - Q \cos \theta \frac{d\eta}{dl} \right] dl - \frac{1}{c} \int_{\text{Observer}}^{\text{Source}} dl
\]

where \( r, \theta \) and \( \eta \) are the usual spherical coordinates. Using the facts that \( m \ll r, \cos \theta = \sin \chi \) and \( \eta = \frac{\phi}{\cos \chi} \) we have

\[
(\Delta t)_{\text{pot.}} = \frac{1}{c} \int_{\text{Source}}^{\text{Observer}} \frac{2m}{r} dl + \frac{\chi^2}{c} (\pi + \alpha - \bar{\beta}) b,
\]  

(41)

where we used equation (8a) and the fact that \( \tan \chi \cong \chi = \frac{Q}{b} \). It can easily be shown that the first term to the first order in \( \frac{r}{b} \) is the potential delay for Schwarzschild metric, i.e.

\[
\frac{1}{c} \int_{\text{Source}}^{\text{Observer}} \frac{2m}{r} dl = \frac{2m}{c} \ln \frac{AD_dD_ds}{b^2}.
\]  

(42)

Therefore the total time delay in NUT lensing is

\[
(\Delta t)_{\text{tot.}} = (\Delta t)_{\text{geo.}} + (\Delta t)_{\text{pot.}} = \frac{1}{c} \left[ \frac{D_dD_ds}{2D_ds} + 2\left( \frac{D_dD_ds}{D_d + D_ds} \right)^2 \chi^2 + \frac{2m}{c} \ln \frac{AD_dD_ds}{b^2} \right] + \frac{\chi^2}{c} (\pi + \alpha - \bar{\beta}) b
\]

(43)

Again we note that (43) reduces to the Schwarzschild lensing time delay when \( \chi = 0 \).

Note that if the calculation were to be done in R-W spacetimes the D’s would have been distances determined by the apparent size i.e. \( D_d \) is evaluated at the time when light passes the lens ( redshift \( Z_d \) ) and \( D_ds \) is evaluated when light is emitted. (redshift \( Z_ds \))

\(^5\) Note that the calculations are done in the limit where \( r^2 \gg Q^2, m^2 \) i.e far away from the horizon.
9 DISCUSSION

We have studied the lensing of light rays passing a NUT deflector in the context of gravo-electromagnetism. We have shown that there is an extra shear due to the presence of the gravomagnetic field of the NUT space which shears the shape of the source. It was shown that this effect produces a spirality in the lensing pattern of a random distribution of circular sources located at infinity. This spirality effect about a NUT lens is very characteristic and is not displayed in normal lensing and the gravomagnetic lens due to a rotating object seen pole on does not show it because the twist of the ray as it approaches such a lens is cancelled by the opposite twist as it recedes. Thus the discovery of a spiral shear field about a lens would indicate, at least in principle, the presence of a gravomagnetic lens.

The observability of gravomagnetic lenses is also discussed in the context of the constraints on magnetic masses (Mueller & Perry 1986). It was shown by Misner (Misner 1963) that one can remove the string singularity in NUT space by using two different coordinate patches to cover the northern \((0 \leq \theta \leq \frac{\pi}{2})\) and southern \((\frac{\pi}{2} \leq \theta \leq \pi)\) hemispheres at the cost of identifying the time coordinate with period \(8\pi l\). One can show that this periodicity is equivalent to the existence of closed timelike curves which in turn violates the causality. Mueller and Perry then discuss that since the Universe is not observed to be periodic on the scales less than \(10^{10}\) yr, any magnetic mass is presumably very large and at least \(10^{60}\) Planck masses. This means that the characteristic scale size for such a mass would be the Hubble scale or larger. This large value for magnetic masses makes the lensing effect the most promising way to look for them. Although the expectation must be small, such effects should be looked for by those studying gravitational lenses.

ACKNOWLEDGEMENTS

The authors would like to thank H. Ardavan for useful discussions and comments. MNZ thanks the Ministry of culture and Higher Education of Iran for financial support and is grateful to L. W. Chen and C. Rola for helping him with computer graphics.

\footnote{It was shown that this periodicity condition is a natural consequence of quantum mechanical considerations and holds for any spacetime that has magnetic mass (Dowker 1974, Hawking 1979).}
APPENDIX A: NULL TETRADS AND THE GEOMETRY OF NULL GEODESICS

Consider the family of null geodesics of a metric $g^{ab}$. Let $l^a = \frac{dx^a}{du}$ denote the null tangent vectors to these geodesics, where $u$ is an affine parameter along the geodesic. At each point one can define a complex, quasi-orthonormal null tetrad by the equation (Sachs 1961)

$$g^{ab} = 2l^{(a}n^{b)} - 2\bar{m}^{(a}m^{b)},$$

where $n^a$ is another real null vector and $\bar{m}^a$ is the complex conjugate of $m^a$. The above equations holds if and only if the quasi-orthonormality relations

$$l^a n_a = -\bar{m}^a m_a = 1,$$
$$l^a l_a = n^a n_a = m^a m_a = l^a m_a = n^a m_a = 0,$$

hold. The vectors $(l^a, n^a, m^a, \bar{m}^a)$ define what is called a null tetrad.

Using this null tetrad one can find some of the geometrical properties of the null geodesics, e.g. define the following two complex quantities (Hawking 1973)

$$\rho = l_{ab} m^a \bar{m}^b,$$
$$\sigma = l_{ab} m^a m^b.$$

The real part of $\rho$ measures the average rate of convergence of nearby null geodesics in space $g^{ab}$ and therefore is responsible for the magnification or reduction of the observed image. The imaginary part of $\rho$ measures the twist or rate of rotation of neighbouring null geodesics i.e. this is the part responsible for the spirality in the lensing pattern shown in the text. One can easily calculate $\rho$ for the Schwarzschild and NUT metrics:

$$\rho_{NUT} = \frac{-r + il}{r^2 + l^2},$$
$$\rho_{Schw.} = -\frac{1}{r}.$$

We note that the same correspondence between the two metrics also applies here i.e.

$$\rho_{Schw.} = \lim_{l \to 0} \rho_{NUT}.$$

The above relations show that in NUT case not only the size but also the orientation of the observed image changes ( $\text{Im} \rho = \frac{l}{r^2 + l^2}$) but in the Schwarzschild case there is no twist in the observed image because $\text{Im} \rho = 0$. The quantity $\sigma$ measures the rate of distortion or shear of the null geodesics. The effect of shear makes an originally small circular cross section of a bundle of rays become elliptical.
APPENDIX B: BENDING ANGLE FOR PURE NUT \((M = 0)\)

In this appendix we calculate the bending angle for a pure NUT case i.e. for \(m=0\). Writing integral (5) for pure NUT we have

\[
\Delta \phi = \int \frac{L dr}{r^2 \left( \varepsilon^2 - \frac{L^2}{r^2} (1 - \frac{2l^2}{r^2}) \right)^{\frac{3}{2}}}
\]

where \((1 - \frac{2l^2}{r^2}) = e^{-2\lambda}|_{m=0}\). Now expanding the integrand in powers of \(\frac{l^2}{r^2}\) and keeping the lowest order term we have

\[
\Delta \phi = b \int \frac{dr}{(r^4 - b^2 r^2 + 2l^2 b^2)^{\frac{3}{2}}}
\]

where \(b = L/\varepsilon\) is the impact parameter. One can write the above integral in the following form

\[
\Delta \phi = b \int_{r_{\text{min}}}^{\infty} \frac{dr}{((r^2 - a^2)(r^2 - d^2))^{\frac{3}{2}}}
\]  

(B1)

where

\[
a^2 = \frac{2l^2 b^2}{b^2/2 - b/2 \sqrt{b^2 - 8l^2}} \simeq b^2
\]  

(B2)

\[
d^2 = \frac{b^2}{2} - b/2 \sqrt{b^2 - 8l^2} \simeq 2l^2
\]  

(B3)

Now due to the fact that the condition \(r_{\text{min}} \geq a > d > 0\) is satisfied the above integral is an elliptic one of the first kind i.e. (equation 12 page 246 of Gradshteyn and Ryzhik 1980)

\[
\Delta \phi = (b/a) F(\nu, t)
\]  

(B4)

where \(\nu = \arcsin(a/r_{\text{min}})\) and \(t = d/a\). Substituting (B2) and (B3) in (B4) we have

\[
\Delta \phi = b/a F(\pi/2, \sqrt{2} \frac{1}{b}) = b/a \int -0^{\pi/2} [1 - 2(l^2/b^2) \sin^2 \theta]^{-1/2} d\theta = \pi/2 + \pi/4(l^2/b^2)
\]

where we have used the fact that \(l \ll b\). So the bending angle \(\alpha\) is

\[
\alpha = 2\Delta \phi - \pi = \pi/2(l^2/b^2).
\]

REFERENCES

Demiansky M., Newman E. T., Bulletin De l’academie Polonaise des Sciences, Serie des Sciences math., astr. et phys.-Vol. XIV, No.11, 1966.
Dowker J. S., 1974, Gen. Rel. Grav. 5, 603.
Kramer D., Stephani H., Herlt E., MacCallum M., Exact Solutions of Einstein’s Field Equations, Ed. E. Schmutzer, 1980, CUP.
Hawking S. W., The event Horizon, in Black Holes (eds. DeWitt and DeWitt), Gordon and Breach (1973).
Hawking S. W., General Relativity an Einstein Survey, edited by S. W. Hawking and W. Israel (Cambridge University Press), P. 746.
Landau L. D., Lifishitz E. M., 1975, Classical theory of fields., 4th edn. Pergamon press, Oxford.
Lynden-Bell D., Nouri-Zonoz M., gr-qc/9612049.
Misner C. W., 1963, J. Math. Phys. 4, 924.
Mueller, M and M. J. Perry, 1986, Class. Quantum Grav. 3, 65.
Newman E. T., Tamburino L., Unti T., 1963, J. Math. Phys. 4, 915.
Sachs R., Proc. R. Soc. Lond.(A) 264, 309 (1961).
Schneider P., Ehlers J., Falco E. E., 1992 , Gravitational Lenses, Springer-Verlag .
Figure 7. A horizontal line of point sources in the plane of the sky.
Figure 8. Corresponding lensed point sources for $m = Q = 1$ and for both $\alpha < \beta$ (primed ones) and $\alpha > \beta$ (double primed ones) cases. Note that the point source $a$ can not be seen by the observer. Using a much denser line of point sources one will find that the two images form a continuous curve, i.e. there is a point between sources $a$ and $b$ whose two images coincide.
Figure 9. Randomly distributed circular sources in the plane of the sky.
Figure 10. Lensed circular sources in the plane of the sky for $m=Q$. The central part which is the image for the case $\alpha > \beta$ is enlarged in the next figure.
Figure 11. Lensed circular sources in the plane of the sky for $m=Q$ and $\alpha > \beta$. 
Figure 12. Lensed circular sources in the plane of the sky for m=0 (i.e. pure NUT). Note that the source b is completely absent in this case. The central part which is the image for the case $\alpha > \beta$ is enlarged in the next figure.
Figure 13. Lensed circular sources in the plane of the sky for $m=0$ and $\alpha > \beta$. 
Figure 14. $r_0$, the radius of the unseeable area in terms of the distance of the source to the lens, $D_{ds}$ for $m = 1$, $Q = m/10$, $D_d = 1$
Figure 15. Three-dimensional version of fig. 14 for $m = 1$, $Q = m/5$, $D_d = 1$. Note the difference of the scales on the Lens plane and along $D_{ds}$. 