Analog gravity from electrodynamics in non-linear media

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Abstract

Working with electrodynamics in the geometrical optics approximation we derive the expression representing an effectively curved geometry which guides the propagation of electromagnetic waves in material media whose physical properties depend on an external electric field. The issue of birefringence is addressed, and the trajectory of the extraordinary ray is explicitly worked out. Quite general curves are obtained for the path of the light ray by suitably setting an electric field.

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I. INTRODUCTION

The theory of general relativity lead us to the conception of a generally curved geometry of the spacetime whose source encompass all matter and energy content of the universe. The comparative study between the kinematic aspects of general relativity and other kinds for gravitation has recently been called as analog models for gravitation. Indeed, it have extensively been examined, not only with respect to electrodynamics (non-linear) as well as in moving dielectric fluids but also for acoustic perturbations. Long ago, Gordon shown that the refractive index of a medium can be reinterpreted by means of an effective metric $g_{\text{Gordon}} = \eta^\mu\nu - (1 - c\mu)\nu^\mu\nu$, where $\nu^\mu$ represents the velocity 4-vector of an arbitrary observer. The paths of light rays in non-linear electrodynamics can be obtained in terms of the geodesic equations of an effective geometry representing the non-linearities. These works assumed electrodynamics as governed by a quite general non-linear Lagrangian which is a function of the two invariants of the Maxwell field.

Inside material media, Maxwell equations must be supplemented by constitutive laws that relate the electromagnetic excitations and the field strengths by means of quantities characterizing each medium the waves propagate into. In this context, the electromagnetic interaction was geometrized for the case of linear constitutive relations. More recently, specific non-linear constitutive relations were also geometrized. The above results lead us to conclude that electromagnetic waves propagate inside material media as if immersed in a curved spacetime. This fact allows one to make an analogy between wave propagation in non-linear media and gravitational phenomena.

This work explicitly presents, in the context of geometrical optics, the construction of an effective geometry which describes small perturbations of the electromagnetic field inside substances whose electric susceptibility depend on the electric field. A description of birefringence observed for this class of materials is presented, and the problem of the trajectory of the light ray is explicitly presented in a similar way as the bending of light due to gravitational interaction.

We work in Minkowski spacetime employing a Cartesian coordinate system. The background metric will be represented by $\eta_{\mu\nu}$, which is defined by $\text{diag}(+1, -1, -1, -1)$. Units are such that $c = 1$.

II. FIELD EQUATIONS

$F^{\mu\nu}$ and $\mu^{\mu\nu}$ are the tensors representing the total electromagnetic field, which are expressed in terms of the strengths and the excitations of the electric and magnetic fields as

$$F^{\mu\nu} = \nu^\mu E^\nu - \nu^\nu \nu^\mu - \eta^{\mu\nu} + \alpha_{\mu\nu} V^\alpha B^\beta$$

(1)

$$\mu^{\mu\nu} = \nu^\mu D^\nu - \nu^\nu \nu^\mu - \eta^{\mu\nu} + \alpha_{\mu\nu} V^\alpha H^\beta$$

(2)

where the Levi-Civita tensor introduced is defined such that $\eta^{0123} = +1$.

In general the properties of the media are determined by the tensors $\epsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$ which relate the electromagnetic excitation and the field strength by the generalized constitutive laws,

$$D^\alpha = \epsilon^{\alpha\beta}(E^\beta, H^\beta)$$

(3)

$$B^\alpha = \mu^{\alpha\beta}(E^\beta, H^\beta)$$

(4)

In the absence of sources Maxwell’s theory can be summarized by the equations

$$P^{\mu\nu}_{\mu} = 0$$

(5)

$$F^{\mu\nu}_{\mu} = 0$$

(6)

together with the constitutive laws (3) and (4), where we have introduced the dual electromagnetic field tensor

$$F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F^{\alpha\beta}.$$
Eq. (3) could be alternatively obtained by the Bianchi identity. Now, rewriting the above field equations in terms of the electric and magnetic fields it results

\[ V^\mu D^\nu,\nu - V^\nu D^\mu,\nu - \eta^{\mu\alpha\beta}V_\alpha \beta_h,\nu = 0 \quad (8) \]

\[ V^\mu B^\nu,\nu - V^\nu B^\mu,\nu + \eta^{\mu\alpha\beta}V_\alpha \beta_e,\nu = 0. \quad (9) \]

Additionally, the electromagnetic excitation is related to the field strength by means of the constitutive relations (3) and (4), whose derivatives with respect to the coordinates can be presented as

\[ D^\alpha,\tau = \varepsilon^{\alpha\beta}E^\beta,\tau + \frac{\partial \varepsilon^{\alpha\beta}}{\partial E^\mu} E^\beta H^\mu,\tau \quad (10) \]

\[ B^\alpha,\tau = \mu^{\alpha\beta}H^\beta,\tau + \frac{\partial \mu^{\alpha\beta}}{\partial H^\mu} H^\beta E^\mu,\tau. \quad (11) \]

### III. GEOMETRICAL OPTICS

In order to determine the propagation of waves associated to the electromagnetic field, we will consider the method of field discontinuities [21]. We define a surface of discontinuity \( \Sigma \) by \( z(x^\mu) = 0. \) Whenever \( \Sigma \) is a global surface, it divides the spacetime in two distinct regions \( U^- \) for \( z(x^\mu) < 0 \), and \( U^+ \) for \( z(x^\mu) > 0. \) The discontinuity of an arbitrary function \( f(x^\alpha) \) on \( \Sigma \) is given by

\[ [f(x^\alpha)]_{\Sigma} = \lim_{(P^z)\to P^\pm} [f(P^+) - f(P^-)] \quad (12) \]

with \( P^+, P^- \) and \( P \) belonging to \( U^+, U^- \) and \( \Sigma \), respectively. The electric and magnetic fields are continuous when crossing the surface \( \Sigma \). However, their derivatives behave as

\[ [E^\mu],\Sigma = e^\mu K_\nu; \quad [H^\mu],\Sigma = h^\mu K_\nu \quad (13) \]

where \( e^\mu \) and \( h^\mu \) represent the discontinuities of the fields on the surface \( \Sigma \) and

\[ K_\lambda = \frac{\partial \Sigma}{\partial x^\lambda} \quad (14) \]

is the wave vector. Applying these conditions to the field equations (8) and (9), we obtain

\[ \varepsilon^{\alpha\beta}K_\alpha e^\beta + \frac{\partial \varepsilon^{\alpha\beta}}{\partial E^\mu} E^\beta K_\alpha h^\mu + \frac{\partial \varepsilon^{\alpha\beta}}{\partial H^\mu} E^\beta K_\alpha h^\mu = 0 \quad (15) \]

\[ \mu^{\alpha\beta}K_\alpha h^\beta + \frac{\partial \mu^{\alpha\beta}}{\partial E^\mu} H^\beta K_\alpha e^\mu + \frac{\partial \mu^{\alpha\beta}}{\partial H^\mu} H^\beta K_\alpha e^\mu = 0 \quad (16) \]

\[ \left( e^\mu e^\beta + \frac{\partial e^\mu}{\partial E^\alpha} E^\alpha e^\beta + \frac{\partial e^\mu}{\partial H^\alpha} E^\alpha e^\beta \right) (KV) + \eta^{\mu\alpha\beta}K_\alpha V^\alpha h^\beta = 0 \quad (17) \]

\[ \left( h^\mu h^\beta + \frac{\partial h^\mu}{\partial E^\alpha} H^\alpha e^\beta + \frac{\partial h^\mu}{\partial H^\alpha} H^\alpha e^\beta \right) (KV) - \eta^{\mu\alpha\beta}K_\alpha V^\alpha e^\beta = 0 \quad (18) \]

where we have defined \( (KV) \equiv K^\mu V_\mu \). Eqs. (13)-(16) came from the zeroth component of Eqs. (8)-(11), and can be obtained from Eqs. (17)-(18) upon contraction with \( K_\mu \) whenever \( (KV) \neq 0 \). Then, it follows that the coupled system (17)-(18) completely describe the propagation of light rays.

\[ \left[ \left( \mu(KV)^2 \right) \varepsilon^{\mu\beta} + \frac{\partial \varepsilon^{\mu\beta}}{\partial E^\alpha} E^\alpha \right] + (KV) \frac{\partial \varepsilon^{\mu\beta}}{\partial H^\alpha} \eta^{\alpha\sigma\beta} E^\rho K_\sigma V_\rho - K^\mu K_\beta + (KV) V^\mu K_\beta + K^2 \delta^\mu_\beta - (KV)^2 \delta^\mu_\beta \right] e^\beta = 0 \quad (20) \]

The Fresnel equation represents non-trivial solutions of Eq. (20) and is given by the determinant of the term in brackets. The solution of the Fresnel equation for linear constitutive relations is derived in reference [18].

For the cases \( \varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta}(E^\mu, H^\mu) \) and \( \mu^{\alpha\beta} = \mu(\eta^{\alpha\beta} - V^\alpha V^\beta) \) with \( \mu \) a constant, Eqs. (17)-(18) reduces to

\[ h^\alpha = \frac{1}{\mu(KV)} \eta^{\alpha\beta\rho\sigma} K_\beta V_\sigma e_\rho \quad (19) \]

and the eigen-vector equation

\[ e^\mu = \epsilon(E^\lambda) \left( \eta^{\rho\beta} - V^\mu V^\beta \right) - \alpha(E^\lambda) E^\mu E^\beta. \quad (21) \]
Each particular medium is characterized by the parameters $\alpha$ and $\epsilon$. Considering Eq. (21), the eigen-vector problem reduces to

\[
\left\{ (K^2 - (KV)^2 + \mu(KV)^2(\epsilon + \alpha E^2))\right. \delta_\beta^\alpha - K^\mu K_\beta + \left. (KV)V^\mu V_\beta - E^\mu \left( 2\alpha E_\beta - \frac{\partial \epsilon}{\partial E^\beta} - E_\beta^2 \frac{\partial \alpha}{\partial E^\beta} \right) \right\} e^\beta = 0 \quad (22)
\]

where we denoted $E^2 = -E^\alpha E_\alpha$. Expanding the polarization vector in a suitable basis

\[
e^\beta = aE^\beta + bH^\beta + cK^\beta + dV^\beta \quad (23)
\]

one gets

\[
a \left[ \frac{K^2}{\mu(KV)^2} - \frac{1}{\mu} + \epsilon + 3\alpha E^2 + E^\beta \left( \frac{\partial \epsilon}{\partial E^\beta} + E_\beta^2 \frac{\partial \alpha}{\partial E^\beta} \right) \right] + b \left[ -2\alpha E^\beta H_\beta + H^\beta \left( \frac{\partial \epsilon}{\partial E^\beta} + E_\beta^2 \frac{\partial \alpha}{\partial E^\beta} \right) \right] = 0 \quad (24)
\]

\[
b \left[ K^2 - (KV)^2 + \mu(KV)^2(\epsilon + \alpha E^2) \right] = 0 \quad (25)
\]

\[
a E^\mu K_\mu + b H^\mu K_\mu + c(KV)^2 \left[ 1 - \mu(\epsilon + \alpha E^2) \right] + d(KV) = 0 \quad (26)
\]

\[
a(KV)E^\mu K_\mu + b(KV)H^\mu K_\mu + c(KV)K^2 + d \left[ K^2 + \mu(KV)^2(\epsilon + \alpha E^2) \right] = 0. \quad (27)
\]

There are two distinct solutions of the above system. First, by fixing $b \neq 0$, it follows that $a = c = d = 0$, and we obtain the light cone condition

\[
K^2 = (KV)^2 \left[ 1 - \frac{\Omega}{E} \right] \quad (28)
\]

where $\Omega \equiv (E(\epsilon + \alpha E^2))$. Conversely, assuming $b = 0$ and $a \neq 0$, it follows that $c = [E E^\beta K_\beta/\mu\Omega(KV)^2]a$ and $d = -[E E^\beta K_\beta/\mu\Omega(KV)]a$, and results in the light cone condition

\[
K^2 = (KV)^2 \left[ 1 - \frac{E^\beta \frac{\partial \Omega}{\partial E^\beta}}{E} - \frac{E E^\alpha K_\alpha K_\beta}{\Omega} \frac{\partial}{\partial E^\beta} \left( \frac{\Omega}{E} \right) \right] \quad (29)
\]

Therefore, there are different light-cone conditions for each polarization eigenvectors $e^\beta$. Corresponding to Eq. (29), one has the polarization mode

\[
e^\beta_+ = b H^\beta \quad (30)
\]

traveling with phase velocity

\[
v^2_+ = \frac{E}{\mu \Omega}. \quad (31)
\]

Moreover, Eqs. (24), (26)–(27) for this mode require the following conditions: $H^\beta K_\beta = 0$, $\alpha(E^\lambda H_\lambda)(E^\beta K_\beta) = 0$, and $H^\beta(\partial/\partial E^\beta)(\Omega/E) = 0$. Similarly, from Eq. (29) the polarization mode and its phase velocity are given by

\[
e^\beta_- = c \left[ \frac{K^2}{E^\lambda K_\lambda} \right] E^\beta + K^\beta \quad (32)
\]

\[
v^2_- = \frac{E}{\mu E^\beta \frac{\partial \Omega}{\partial E^\beta}} \left[ 1 - \frac{E^\gamma \hat{k}^\gamma \hat{k}^\alpha \frac{\partial}{\partial E^\alpha} \left( \frac{\Omega}{E} \right) }{\Omega} \right] \quad (33)
\]

where we introduced the space-like unit propagation vector $\hat{k}^\alpha = (\eta^\alpha - V^\alpha V^\beta) K_\beta/\sqrt{(KV)^2 - K^2}$. Contrarily to the velocity $v_+$, the velocity $v_-$ depends on the direction of the electric field $E^\beta$ (i.e., $\hat{k}^\beta = \pm E^\beta/E$). The appearance of two different modes with different velocities points out to birefringence phenomena [22,23], in this case induced by the electric field. The modes (30) and (32) correspond to the ordinary and the extraordinary rays, respectively. The corresponding refraction indexes are given by $n_\pm = 1/v_\pm$.

**IV. ANALOG GRAVITY**

Eqs. (28)–(29) can be restated in the more appealing form

\[
g^{\mu\nu}K_\mu K_\nu = 0 \quad (34)
\]

where we defined the symmetric tensors.
null vector with respect to the effective metric tensor in terms of the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$

$$g_+^{\mu\nu} = \eta^{\mu\nu} - \left(1 - \frac{\Omega}{E}\right)V^\mu V^\nu$$ (35)

$$g_-^{\mu\nu} = \eta^{\mu\nu} - \left(1 - \frac{E^\beta}{E} \frac{\partial \Omega}{\partial E^\beta}\right)V^\mu V^\nu + \frac{E}{2\Omega} \frac{\partial}{\partial E^\beta} \left(\frac{\Omega}{E}\right)\eta^{\beta(\mu} E^{\nu)}$$ (36)

in which $a^{\mu\nu} = a^{\mu\nu} + a^{\nu\mu}$ for an arbitrary tensor $a^{\mu\nu}$. The inverse symmetric tensor $g_{\mu\nu}$ is defined in such way that $g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu\sigma}$.

Differentiation of Eq. (34) with respect to $x^\lambda$ yields

$$2K_{\nu,\lambda} K^\mu_{\nu} + K^\mu_{\nu} g^{\mu\nu}, \lambda = 0.$$ (37)

As the vector $K_{\mu}$ is a gradient, it follows that $K_{\mu, \lambda} = K_{\lambda, \mu}$. Moreover, one has the identity

$$g^{\mu\nu}, \lambda = -g^{\nu\sigma} g^{\mu\alpha} (g_{\alpha\sigma}, \lambda + g_{\alpha\lambda}, \sigma - g_{\sigma\lambda}, \alpha).$$ (38)

Thus, Eq. (37) can be presented as

$$g^{\mu\nu} K^\lambda_{\mu} \left[ K_{\lambda, \nu} - \Gamma^\sigma_{\nu\lambda} K_{\alpha} \right] = 0$$ (39)

where we defined the quantity

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \{ g^{\alpha\nu} (g_{\beta\lambda, \nu} + g_{\beta\nu, \lambda} - g_{\lambda\nu, \beta}) \}.$$ (40)

Eqs. (33) and (36) depend on the total field $\vec{E}$, which encompasses both the external field and the one associated with the wave itself. Thus, the standard treatment of birefringence by setting a frequency-dependent $\epsilon^{\beta}(\omega)$ is included in the present formalism as a particular case as $\partial/\partial \omega = (\partial E^\nu/\partial \omega) \partial E^\nu$. We will hereafter adopt the point of view that the wave fields are much smaller than their external counterparts. Thus, the fields appearing in $g^{\mu\nu}$ are independent of the light ray.

Therefore one recognizes that, for the propagation vector $K_{\nu}$, the tensor $g^{\mu\nu}$ is a metric tensor. Indeed, $K_{\nu}$ is a light-like (null) vector with respect to this metric tensor, Eq. (34), and also satisfies the geodesic equation (35) in terms of the Christoffel symbols $\Gamma^\sigma_{\nu\lambda}$ associated with this metric, Eq. (34). We thus refer to $K_{\nu}$ as a geodesic null vector with respect to the effective metric tensor $g^{\mu\nu}$.

The two terms in brackets in Eq. (33) correspond to the covariant derivative of $K_{\nu}$ and will henceforth be denoted as $K^\mu_{\nu; \lambda}$. The effective geometry $g^{\mu\nu}$ allows us to write $K^\mu = g^{\mu\nu} K_{\nu}$, for which the geodesic equation can be written in the simpler form

$$K^\mu_{\nu; \lambda} K^\lambda = 0$$ (41)

which avoids explicit dependence on the metric tensor.

V. TRAJECTORIES

The trajectories of light-rays are the geodesic lines which solve the geodesic equation (35). For the sake of simplicity, we will consider $V^\nu = \delta^\nu_\sigma$. Only the calculations for the extraordinary for the ordinary ray will be presented, as the corresponding ones for the ordinary ray follow exactly the same reasoning (being much simpler, however). In order to fit it best problems with axial symmetry, standard cylindrical coordinates $x^\mu = (t, r, \theta, z)$ will be adopted. For a configuration of electric field $E^\nu = (0, \vec{E})$ with $\vec{E} = (E_r, E_\theta, E_z) = E_z(r, \theta) \hat{z}$, and assuming $\partial \Omega/\partial E_\theta = 0 = \partial \Omega/\partial E_\theta$, the metric associated with the extraordinary ray reduces to

$$g^{\mu\nu} = \left(\begin{array}{cccc}
\mu E_r \frac{\partial \Omega}{\partial E_z} & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -\frac{1}{r} & 0 \\
0 & 0 & 0 & 0
\end{array}\right).$$ (42)

Thus, $g_{\mu\nu} = diag[f_1(r, \theta), -1, -r^2, -f_2(r, \theta)]$, where we introduce the notation

$$f_1(r, \theta) = \left(\frac{\mu E_z \frac{\partial \Omega}{\partial E_z}}{E_z \frac{\partial \Omega}{\partial E_z}}\right)^{-1}$$ (43)

$$f_2(r, \theta) = \left(\frac{E_z \frac{\partial \Omega}{\partial E_z}}{\Omega \frac{\partial \Omega}{\partial E_z}}\right)^{-1}.$$ (44)

We will also denote the components of the propagation vector as $K^\nu = (dt, dr, d\theta, dz)/dr$, where $\tau$ is an affine parameter along the integral curves of the vector $K^\nu$, and $(t, r, \theta, z)$ represent the coordinates of the points belonging to these lines. The geodesic lines for this case are given by Eq. (41) as

$$\frac{dt}{d\tau} = - \frac{1}{f_1} \frac{\partial f_1}{\partial x^2} \frac{dx}{d\tau},$$ (45)

$$\frac{dr}{d\tau} = \frac{1}{2} \left[ \frac{\partial f_2}{\partial r} \left(\frac{dz}{d\tau}\right)^2 - \frac{\partial f_1}{\partial r} \left(\frac{dt}{d\tau}\right)^2 \right] + r \left(\frac{d\theta}{d\tau}\right)^2,$$ (46)

$$\frac{d\theta}{d\tau} = \frac{1}{2r^2} \left[ \frac{\partial f_2}{\partial \theta} \left(\frac{dz}{d\tau}\right)^2 - \frac{\partial f_1}{\partial \theta} \left(\frac{dt}{d\tau}\right)^2 \right] - \frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau},$$ (47)

$$\frac{dz}{d\tau} = - \frac{1}{f_2} \frac{\partial f_2}{\partial x^2} \frac{dz}{d\tau}.$$ (48)

The solutions of Eqs. (43) and (48) are

$$\frac{dt}{d\tau} = A, \quad \frac{dz}{d\tau} = B, \quad \frac{dz}{d\tau} = \frac{A^2}{f_1} - \frac{B^2}{f_2}.$$ (51)

where $A$ and $B$ are arbitrary constants. Inserting these relations in Eqs. (48)–(47) one gets

$$\frac{d^2 \rho}{d\tau^2} - r \left(\frac{d\theta}{d\tau}\right)^2 = \frac{1}{2} \frac{\partial}{\partial r} \left[ \frac{A^2}{f_1} - \frac{B^2}{f_2} \right],$$ (52)
For \( dr/d\tau \neq 0 \neq d\theta/d\tau \) equations (51)–(52) lead to

\[
\left( \frac{dr}{d\tau} \right)^2 + v^2 \left( \frac{d\theta}{d\tau} \right)^2 = \frac{A^2}{f_1} - \frac{B^2}{f_2} + D
\]

(53)

where \( D \) is an arbitrary constant. Introducing Eqs. (50) and (53) in the light-cone constraint

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu} K^\mu K^\nu dr^2 = 0
\]

(54)

it follows that \( D = 0 \). Thus, Eq. (53) yields \( dr/d\tau \) in terms of \( r \) which is given by

\[
\frac{d^2(r^2)}{dr^2} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r^2 \left( \frac{A^2}{f_1} - \frac{B^2}{f_2} \right) \right].
\]

(55)

### Cylindrical symmetry

For the simple case \( f_1 = f_1(r) \) and \( f_2 = f_2(r) \) it follows from Eq. (52) that

\[
\frac{d\theta}{d\tau} = \frac{C}{r^2}
\]

with an arbitrary constant \( C \), from which Eq. (53) yields

\[
\frac{dr}{d\tau} = \pm \sqrt{\frac{A^2}{f_1} - \frac{B^2}{f_2} - \frac{C^2}{r^2}}.
\]

(57)

The trajectories of the light ray is then found from the ratios \((dz/dr)/(dr/d\tau)\) and \((d\theta/d\tau)/(dr/d\tau)\), from which one obtains \( z = z(r) \) and \( \theta = \theta(r) \) as

\[
z = \pm B \int_r^r \frac{dr}{f_2 \sqrt{\frac{A^2}{f_1} - \frac{B^2}{f_2} - \frac{C^2}{r^2}}} \quad (58)
\]

\[
\theta = \pm C \int_r^r \frac{dr}{r^2 \sqrt{\frac{A^2}{f_1} - \frac{B^2}{f_2} - \frac{C^2}{r^2}}} \quad (59)
\]

Eqs. (50), (51), (52) and (53) encompass in this way a complete description for the path of the light ray with arbitrary \( \Omega(E_z) = E[\varepsilon(E_z) + \alpha(E_z) E^2] \) as a function of \( E_z(r, \theta) \) in terms of the arbitrary constants \( A, B, C \). Thus, quite general curves can be obtained for the path of the light ray by suitably setting the electric field \( E_z \) at the given medium. From \( v^2 = (dx^i/d\tau)/(dt/d\tau) \) it follows that

\[
v^2 = \frac{E}{\mu E_z \frac{\partial \Omega}{\partial E_z}} + \left( \frac{B^2}{2\mu A^2} \right)^2 \left[ \frac{\Omega}{E_z} \right]^{-2} - \left( \frac{\partial \Omega}{\partial E_z} \right)^{-2}
\]

(60)

which is to be compared with Eq. (53). Birefringence then occurs for this model whenever \( \partial \Omega/\partial E_z \neq \Omega/E_z \), which can be ascribed either to a non zero \( \alpha \) parameter or to a varying \( \varepsilon(E) \) in equation (21), or even to both of them.

### VI. CONCLUSION

Maxwell linear equations together with the constitutive relations for non-linear materials lead to an effective non-linear theory of electrodynamics. The problem of the propagation of electromagnetic waves was here dealt with under the approximation of geometrical optics. The path of light rays thus belong to the characteristic surfaces which depend on both the external fields and the propagation fields. For the case of a constant scalar magnetic permeability a Fresnel problem is obtained [20]. The propagating modes and their velocities were obtained for the class of electric permittivity \( \varepsilon^{\mu\nu}(\vec{E}) \). Weak waves were then described as light rays propagating along null geodesics of an effectively deformed geometry which depends on the external electric field, and birefringence phenomena were found to occur if \( \partial \varepsilon^{\mu\nu}/\partial \varepsilon^{\lambda\lambda} \neq 0 \). Explicit calculations for the path and velocity of the rays were presented for the particular \( \varepsilon^{\mu\nu} = \varepsilon(\vec{E})(\eta^{\mu\nu} - V^{\mu\nu}) - \alpha(\vec{E}) E^{\mu\nu} \) case for cylindrically symmetric media, illustrating the usefulness of the geometrical approach. The results of this investigation may be useful to test kinematic aspects of general relativity in the laboratory. Indeed there are many works in the recent literature dealing with such theme, not only in the context of light propagation in dielectric media; see for instance the review in Ref. [2].

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