Regularity of limit sets of AdS quasi-Fuchsian groups

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Abstract

Limit sets of AdS-quasi-Fuchsian groups of $\text{PO}(n, 2)$ are always Lipschitz submanifolds. The aim of this article is to show that they are never $C^1$, except for the case of Fuchsian groups. As a byproduct we show that AdS-quasi-Fuchsian groups that are not Fuchsian are Zariski dense in $\text{PO}(n, 2)$.

1 Introduction

The study of various notions of convex cocompact groups in semi-simple Lie groups has gain considerable interest the last decade, thanks to its relation with Anosov representations. A particularly nice setting is for subgroups of $\text{PO}(p, q)$ where the quadratic form helps to construct invariant domains of discontinuity, see [DGK18].

In a previous paper, we studied the metric properties of limit sets for such representations [GM] and proved a rigidity result for quasi-Fuchsian representations in $\text{PO}(2, 2)$. Recently Zimmer [Zim18] showed a $C^2$ rigidity result for Hitchin representations in $\text{PSL}_n(\mathbb{R})$ ($C^\infty$ rigidity was known from the work of Potrie-Sambarino [PS17]).

In this paper, we study the $C^1$ regularity of such a limit set and prove a rigidity result for quasi-Fuchsian subgroups $\text{PO}(n, 2)$. They are examples of AdS-convex cocompact groups, as defined by [DGK18].

Given the standard quadratic form $q_{n, 2}$ of signature $(n, 2)$ on $\mathbb{R}^{n+2}$, we define $\partial \AdS^{n+1}$ as the subset of $\mathbb{RP}^{n+1}$ consisting of negative lines for $q_{n, 2}$. Its boundary $\partial \AdS^{n+1}$ is the set of $q_{n, 2}$-isotropic lines.

Definition 1.1. [DGK18] A discrete subgroup $\Gamma$ of $G = \text{PO}(n, 2)$ is AdS-convex cocompact if it acts properly discontinously and cocompactly on some properly convex closed subset $\mathcal{C}$ of $\partial \AdS^{n+1}$ with nonempty interior whose ideal boundary $\partial \mathcal{C} := \overline{\mathcal{C}} \setminus \mathcal{C}$ does not contain any nontrivial projective segment.

Any infinite convex-cocompact group contains proximal elements, i.e. elements that have a unique attractive fixed point in $\partial \AdS^{n+1}$. For $\Gamma$ a discrete
subgroup of \( \text{PO}(n,2) \), the \textit{proximal limit set} of \( \Gamma \) is the closure \( \Lambda_\Gamma \subset \mathbb{R}P^{n,2} \) of the set of attracting fixed points of proximal elements of \( \Gamma \). Since \( \Gamma \) acts properly discontinuously on a convex set \( \mathcal{C} \), the proximal limit set coincides with the ideal boundary of \( \mathcal{C} \). It is shown in [DGK18] that this notion of limit set coincides with the closure of orbits in the boundary.

**Definition 1.2.** A discrete group of \( \text{PO}(n,2) \) is AdS-quasi-Fuchsian if it is AdS-convex cocompact and its proximal limit set is homeomorphic to a \( n-1 \) dimensional sphere.

If moreover, the group preserves a totally geodesic copy of \( \mathbb{H}^n \), it is called AdS-Fuchsian.

The limit set of an AdS-Fuchsian group is a geometric sphere, hence a \( C^1 \)-submanifold of \( \partial \text{AdS} \). The principal aim of this article is to show that the converse holds:

**Theorem 1.3.** Let \( \Gamma \subset \text{PO}(n,2) \) be AdS quasi-Fuchsian. If \( \Lambda_\Gamma \) is a \( C^1 \) submanifold of \( \partial \text{AdS}^{n+1} \), then \( \Gamma \) is Fuchsian.

The proof is based on the following result which is interesting on its own:

**Proposition 1.4.** Let \( \Gamma \subset \text{PO}(n,2) \) be AdS quasi-Fuchsian. If \( \Gamma \) is not AdS-Fuchsian, then it is Zariski dense in \( \text{PO}(n,2) \).

Remark that this proposition and Zimmer’s result [Zim18, Corollary 1.48] imply that the limit set is not \( C^2 \).

### 2 Background on AdS-quasi-Fuchsian groups.

We introduce the results needed for the proofs of Theorem 1.3 and Proposition 1.4. Most of this section follows directly from the work of [BM12] and [DGK18], except maybe the characterization of Fuchsian groups as subgroups of \( \text{O}(n,1) \) in Proposition 2.6.

First, let us define the anti-de Sitter space. We denote by \( \langle \cdot | \cdot \rangle_{n,2} \)

**Definition 2.1.** The anti-de Sitter space is defined by

\[
\text{AdS}^{n+1} := \{ [x] \in \mathbb{R}P^{n+1} | \langle x|x \rangle_{n,2} < 0 \}.
\]

Its boundary is

\[
\partial \text{AdS}^{n+1} := \{ [x] \in \mathbb{R}P^{n+1} | \langle x|x \rangle_{n,2} = 0 \}.
\]

Two points \([x], [y] \in \partial \text{AdS}^{n+1}\) are called transverse if \( \langle x|y \rangle_{n,2} \neq 0 \).

We now give a brief review of the proximal limit set:
Definition 2.2. Given $\gamma \in \text{PO}(n,2)$, we denote by $\lambda_1(\gamma) \geq \lambda_2(\gamma) \geq \cdots \geq \lambda_{n+2}(\gamma)$ the logarithms of the moduli of the eigenvalues of any of its representants in $O(n,2)$. We say that $\gamma$ is proximal if $\lambda_1(\gamma) > \lambda_2(\gamma)$.

Remark that an element of $\text{PO}(n,2)$ has not always a lift in $SO(n,2)$. However since it is the quotient of $O(n,2)$ by $\pm \text{Id}$, the set of moduli of eigenvalues of a lift is well defined. If $\gamma \in \text{PO}(n,2)$ is proximal, it has a unique lift $\hat{\gamma} \in O(n,2)$ which has $e^{\lambda_1(\gamma)}$ as an eigenvalue.

Notice that we always have $\lambda_3(\gamma) = \cdots = \lambda_n(\gamma) = 0$, as well as $\lambda_1(\gamma) + \lambda_{n+2}(\gamma) = \lambda_2(\gamma) + \lambda_{n+1}(\gamma) = 0$.

Definition 2.3. If $\gamma \in \text{PO}(n,2)$ is proximal, we denote by $\gamma_+ \in \mathbb{RP}^{n+1}$ its attractive fixed point, i.e. the eigendirection for the eigenvalue of modulus $e^{\lambda_1(\gamma)}$ of a lift of $\gamma$ to $O(n,2)$. We also set $\gamma_- = (\gamma^{-1})_+$.

Note that $\gamma_+$ is necessarily isotropic, i.e. $\gamma_+ \in \partial \text{AdS}^{n+1}$.

Proposition 2.4 (Proposition 5 in [Fra05]). If $\gamma \in \text{PO}(n,2)$ is proximal, then $\lim_{n \to +\infty} \gamma^n(\xi) = \gamma_+$ for all $\xi \in \partial \text{AdS}^{n+1}$ which is transverse to $\gamma_-$ (i.e. such that $\langle \xi \mid \gamma_- \rangle_{n,2} \neq 0$).

Recall that the proximal limit set of a discrete subgroup $\Gamma \subset \text{PO}(n,2)$ is the closure $\Lambda_\Gamma$ in $\mathbb{RP}^{n+1}$ of the set of all attractive fixed points of proximal elements of $\Gamma$, it is therefore a subset of $\partial \text{AdS}^{n+1}$. If additionally $\Gamma$ is AdS-convex cocompact, then it is word-hyperbolic and the action of $\Gamma$ on its proximal limit set is conjugated to the action on its Gromov boundary [DGK18]. As a consequence, we have:

Proposition 2.5. [Gai18] If $\Gamma \subset \text{PO}(n,2)$ is AdS-convex cocompact, the action of $\Gamma$ on the limit set $\Lambda_\Gamma$ is minimal, i.e. all orbits are dense.

The group $O(n,1)$ can be embedded in $\text{PO}(n,2)$ by the following map:

$$A \rightarrow \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$  

We will say that an element (respectively a subgroup) of $\text{PO}(n,2)$ is conjugate to an element (respectively to a subgroup) of $O(n,1)$ if it has a conjugate in the image of this embedding.

Note that if $\gamma \in \text{PO}(n,2)$ is proximal, we have $\lambda_2(\gamma) = 0$ if and only if $\gamma$ is conjugate to an element of $O(n,1)$.

A subgroup of $\text{PO}(n,2)$ which is conjugate to a cocompact lattice of $O(n,1)$ is AdS-Fuchsian, as it fixes a totally geodesic copy of $\mathbb{H}^n$ on which it acts properly discontinuously and cocompactly. These are the only AdS-Fuchsian groups:
Proposition 2.6. A discrete group of $\PO(n, 2)$ is AdS-Fuchsian if and only if it is conjugate to a cocompact lattice of $O(n, 1)$.

**Proof.** Let $\Gamma \subset \PO(n, 2)$ be an AdS-Fuchsian group. Let $H$ be a totally geodesic copy of $\mathbb{H}^n$ in $\text{AdS}^{n+1}$ preserved by $\Gamma$. Since the stabilizer $L \subset \PO(n, 2)$ of $H$ is conjugate to $O(n, 1)$, we only have to show that $\Gamma$ is a cocompact lattice of $L$. This will be a consequence of the fact that $\Gamma$ acts properly discontinuously and cocompactly on $H$.

Let $\gamma$ be a proximal element of $\Gamma$. Let $\xi \in \partial H$ be transverse to the repelling fixed point $\gamma^-$. The sequence $\gamma^n \xi$ lies in $H$ and converges to $\gamma^+$. Therefore, $\partial H$ contains the attracting point of $\gamma$, and it follows that $\Lambda_\Gamma \subset \partial H$. Since $\Lambda_\Gamma$ and $\partial H$ are homeomorphic to $S^{n-1}$, we have $\Lambda_\Gamma = \partial H$.

Finally since, $\Gamma$ is convex-cocompact, $\Gamma$ acts properly discontinuously and cocompactly on the convex hull of $\Lambda_\Gamma$ that is $H$ (see [DGK18]).

The boundary $\partial \text{AdS}^{n+1}$ is naturally equipped with a conformal Lorentzian structure. It is conformally equivalent to the quotient of $S^{n-1} \times S^1$ endowed with the Lorentzian conformal metric $[g_{S^{n-1}} - d\theta^2]$ (where $g_{S^{n-1}}$ is the round metric of curvature 1 on $S^{n-1}$, and $d\theta^2$ is the round metric on the circle of radius one) by the antipodal map $(x, \theta) \mapsto (-x, -\theta)$. See [BM12, paragraph 2.3] for more details.

Using the absence of segments in the limit sets of AdS-quasi-Fuchsian groups we have:

**Proposition 2.7.** The limit set $\Lambda_\Gamma \subset \partial \text{AdS}^{n+1}$ of an AdS-quasi-Fuchsian group $\Gamma \subset \PO(n, 2)$ is the quotient by the antipodal map of the graph of a distance-decreasing map $f : S^{n-1} \to S^1$ where $S^{n-1}$ and $S^1$ are endowed with the round metrics.

**Proof.** Barbot-Mérigot showed in [BM12] that the limit set of a quasi-Fuchsian group lifts to the graph of a 1-Lipschitz map. Since the limit set does not contain any non trivial segment of $\partial \text{AdS}^{n+1}$ the map strictly decreases the distance.

Finally we will need the following proposition, which in the Lorentzain vocabulary translates as the fact that the limit set is a Cauchy hypersurface:

**Proposition 2.8.** If $\Gamma \subset \PO(n, 2)$ is AdS-quasi-Fuchsian, then every isotropic geodesic of $\partial \text{AdS}^{n+1}$ intersects $\Lambda_\Gamma$ at exactly one point.

**Proof.** Let $f : S^{n-1} \to S^1$ be a distance-decreasing map such that the quotient by the antipodal map of its graph is $\Lambda_\Gamma$. An isotropic geodesic can be parametrized by $(c(\theta), \theta)$, where $c : \theta \to c(\theta)$ is a unit speed geodesic on $S^{n-1}$. Then the proposition is equivalent to the existence and uniqueness of

$^1$That is $\forall x \neq y, d(f(x), f(y)) < d(x, y)$.
a fixed point for the map \( f \circ c : S^1 \to S^1 \).
It is a simple exercise to show that a distance-decreasing map of a compact
metric space to itself has a unique fixed point.

\[ \Box \]

3 The Zariski closure of AdS quasi-Fuchsian groups

We prove in this section the Zariski density of AdS-quasi-Fuchsian subgroups
of \( \text{PO}(n, 2) \) which are not AdS-Fuchsian. This result, which happens to be
interesting in itself, will considerably simplify the proof of Theorem 1.3 when
we will use Benoist’s Theorem \([\text{Ben97}]\) about Jordan projections for discrete
subgroups of semi-simple Lie groups in the last section.

Lemma 3.1. Let \( \Gamma \subset \text{PO}(n, 2) \) be AdS quasi-Fuchsian. If \( \Gamma \) is reducible,
then it is Fuchsian.

Proof. Assume that \( \Gamma \) is not Fuchsian, and let \( V \subset \mathbb{R}^{n+2} \) be a \( \Gamma \)-invariant
subspace with \( 0 < \dim(V) < n + 2 \).
First, let us show that the restriction of \( \langle \cdot | \cdot \rangle_{n,2} \) to \( V \) is non degenerate.
Assume that it is not the case. Then \( \Gamma \) preserves the totally isotropic space
\( V \cap V^\perp \). It has dimension 1 or 2. If \( \dim(V \cap V^\perp) = 1 \), then \( \mathbb{P}(V \cap V^\perp) \)
is a global fixed point for the action on \( \partial \text{AdS}^{n+1} \), which cannot exist. The
case \( \dim(V \cap V^\perp) = 2 \) is impossible because it also implies the existence
of a global fixed point on \( \partial \text{AdS}^{n+1} \) (the intersection of the null geodesic
\( \mathbb{P}(V \cap V^\perp) \) of \( \partial \text{AdS}^{n+1} \) with \( \Lambda_\Gamma \)).
We can now assume that the restriction of \( \langle \cdot | \cdot \rangle_{n,2} \) to \( V \) is non degenerate.
It can have signature \((k,2), (k,1) \) or \((k,0) \) (where \( k \geq 0 \) is the number of
positive signs).
In the first case, \( \Gamma \) acts on some totally geodesic copy \( X \) of \( \text{AdS}^{k+1} \) (with
\( k < n \)) in \( \text{AdS}^{n+1} \). Then \( \partial X \cap \Lambda_\Gamma \) is a non empty closed invariant subset of
\( \Lambda_\Gamma \), hence \( \Lambda_\Gamma \subset \partial X \) and \( C(\Lambda_\Gamma) \subset X \). Since \( C(\Lambda_\Gamma) \)
has non empty interior in \( \text{AdS}^{n+1} \) (Lemma 3.13 in \([\text{BM12}]\) ), we see that \( X = \text{AdS}^{n+1}, \) i.e. \( V = \mathbb{R}^{n+2}, \)
which is absurd.
Now assume that \( V \) has Lorentzian signature \((k,1) \). Then \( \Gamma \) preserves \( X = \mathbb{P}(V) \cap \text{AdS}^{n+1} \) which is a totally geodesic copy of \( \mathbb{H}^k \). It also acts on \( X' = \mathbb{P}(V^\perp) \cap \text{AdS}^{n+1} \) which is a totally geodesic copy of \( \mathbb{H}^{k'} \) (with \( k + k' = n \)).
Considering a proximal element \( \gamma \in \Gamma \), there is a point in \( \partial X \cup \partial X' \) which
is transverse to the repelling fixed point \( \gamma_- \) of \( \gamma \) (otherwise \( \gamma_- \) would be
in \( V \cap V^\perp \)). This implies that \( \gamma_+ \in \partial X \cup \partial X', \) hence \( \Lambda_\Gamma \cap \partial X \neq \emptyset \) or
\( \Lambda_\Gamma \cap \partial X' \neq \emptyset \). The action of \( \Gamma \) on \( \Lambda_\Gamma \) being minimal, we find that \( \Lambda_\Gamma \subset \partial X \)
or \( \Lambda_\Gamma \subset \partial X' \). This is impossible because \( \Lambda_\Gamma \) is homeomorphic to \( S^{n-1} \) and
\( \partial X \) (resp. \( \partial X' \)) is homeomorphic to \( S^{k-1} \) (resp. \( S^{k'-1} \)).
Finally, if \( V \) is positive definite, then \( V^\perp \) has signature \((n-k,2) \), this case
has already been ruled out. \[ \Box \]
Corollary 3.2. If $\Gamma \subset \PO(n, 2)$ is AdS-quasi-Fuchsian but not AdS-Fuchsian, then the identity component of the Zariski closure of $\Gamma$ acts irreducibly on $\mathbb{R}^{n+2}$.

Proof. Let $\Gamma_0 \subset G$ be a finite index subgroup. Since $\Lambda_{\Gamma_0} = \Lambda_{\Gamma}$, it cannot be Fuchsian, so it acts irreducibly on $\mathbb{R}^{n+2}$ by Lemma 3.1. 

Proposition 3.3. Let $\Gamma \subset \PO(n, 2)$ be AdS-quasi-Fuchsian. If $\Gamma$ is not AdS-Fuchsian, then it is Zariski dense in $\PO(n, 2)$.

Proof. Let $G \subset \SO_0(n, 2)$ be the pre-image by the quotient map $\SO_0(n, 2) \to \PO(n, 2)$ of the identity component of the Zariski closure of $\Gamma$, and assume that $\Gamma$ is not Fuchsian.

By Corollary 3.2, we know that $G$ acts irreducibly on $\mathbb{R}^{n+2}$. According to [DSL], the only connected irreducible subgroups of $\SO(n, 2)$ other than $\SO(n, 2)$ are $U(\frac{n}{2}, 1)$, $SU(\frac{n}{2}, 1)$, $S^1\SO_0(\frac{n}{2}, 1) \ (\text{when } n \text{ is even})$ and $\SO_0(2, 1) \ (\text{when } n = 3)$.

The first three cases are subgroups of $U(\frac{n}{2}, 1)$, which only contains elements $\gamma \in \SO(n, 2)$ satisfying $\lambda_1(\gamma) = \lambda_2(\gamma)$ so $G$ cannot be one of them (otherwise $\Gamma$ would not contain any proximal element and $\Lambda_{\Gamma} = \emptyset$).

The irreducible copy of $\SO_0(2, 1)$ in $\SO(3, 2)$ can also be ruled out because a quasi-Fuchsian subgroup of $\PO(3, 2)$ has cohomological dimension 3, so it cannot be isomorphic to a discrete subgroup of $\SO_0(2, 1) \approx \PSL(2, \mathbb{R})$.

The only possibility left is that $\Gamma$ is Zariski dense in $\PO(n, 2)$. 

4 Non differentiability of limit sets

We finally prove the main result, Theorem 1.3. The proof goes as follows: first, we prove that the tangent spaces of the limit set are space like (i.e. positive definite for the natural Lorentzian conformal structure on $\partial \AdS^{n+1}$).

Then by an algebraic argument, this shows that all proximal elements of $\Gamma$ are conjugate (by an a priori different element of $\PO(n, 2)$) to an element of $O(n, 1)$. Finally, using a famous theorem of Benoist, this implies that $\Gamma$ is not Zariski-dense, and therefore by Proposition 1.4 that the group is Fuchsian.

4.1 Spacelike points

Lemma 4.1. If $\Gamma \subset \PO(n, 2)$ is AdS quasi-Fuchsian and $\Lambda_{\Gamma}$ is a $C^1$ submanifold of $\partial \AdS^{n+1}$, then there is $\xi \in \Lambda_{\Gamma}$ such that $T_{\xi} \Lambda_{\Gamma}$ is spacelike.

Proof. Let $f : S^{n-1} \to S^1$ be a distance-decreasing map such that the quotient by the antipodal map of its graph is $\Lambda_{\Gamma}$.

Knowing that the graph of $f$ is a $C^1$-submanifold, we first want to show that
\( f \) is \( C^1 \). Using the Implicit Function Theorem, it is enough to know that the tangent space of the graph projects non trivially to the tangent space of \( \mathbb{S}^{n-1} \). This is true because \( \Lambda_\Gamma \) is acausal.

Since \( f \) satisfies \( d(f(x), f(y)) < d(x, y) \) for \( x \neq y \) [BM12], it cannot be onto, so it can be seen as a function \( f : \mathbb{S}^{n-1} \to \mathbb{R} \). At a point \( x \in \mathbb{S}^{n-1} \) where it reaches its maximum, it satisfies \( df_x = 0 \), so the tangent space to \( \Lambda_\Gamma \) at \((x, f(x))\) is \( T_x \mathbb{S}^{n-1} \times \{0\} \), which is spacelike.

**Corollary 4.2.** If \( \Gamma \subset \text{PO}(n,2) \) is AdS quasi-Fuchsian and \( \Lambda_\Gamma \) is a \( C^1 \) submanifold of \( \partial \text{AdS}^{n+1} \), then for all \( \xi \in \Lambda_\Gamma \), the tangent space \( T_\xi \Lambda_\Gamma \) is spacelike.

**Proof.** Let \( E = \{ \xi \in \Lambda_\Gamma : T_\xi \Lambda_\Gamma \) is spacelike} \}. Then \( E \) is open and \( \Gamma \)-invariant. Since the action of \( \Gamma \) on \( \Lambda_\Gamma \) is conjugate to the action on its Gromov boundary, it is minimal (i.e. all orbits are dense). It follows that \( E \) is either empty or equal to \( \Lambda_\Gamma \) and by Lemma 4.1 it is not empty.

**Remark:** Lemma 4.1 fails in general in higher rank pseudo-Riemannian symmetric spaces, i.e. for \( \mathbb{H}^{p,q} \)-quasi-Fuchsian groups. Indeed, Hitchin representations in \( \text{PO}(3,2) \) provide \( \mathbb{H}^{2,2} \)-quasi-Fuchsian groups which are not \( \mathbb{H}^{2,2} \)-Fuchsian, yet have a \( C^1 \) limit set (which is isotropic for the natural Lorentzian conformal structure on \( \partial \mathbb{H}^{2,2} \)).

### 4.2 Fixed points and Benoist’s asymptotic cone

**Lemma 4.3.** Let \( \Gamma \subset \text{PO}(n,2) \) be AdS-quasi-Fuchsian. If the limit set \( \Lambda_\Gamma \subset \partial \text{AdS}^{n+1} \) is a \( C^1 \) submanifold, then every proximal element \( \gamma \in \Gamma \) is conjugate in \( \text{PO}(n,2) \) to an element of \( \text{O}(n,1) \).

**Proof.** Let \( \gamma \in \Gamma \) be proximal, and let \( \hat{\gamma} \in \text{O}(n,2) \) be the lift with eigenvalue \( e^{\lambda_1(\gamma)} \). Let \( \gamma_+ \in \Lambda_\Gamma \) be the attractive fixed point. Then the differential of \( \gamma \) acting on \( \partial \text{AdS}^{n+1} \) at \( \gamma_+ \) preserves \( T_{\gamma_+} \Lambda_\Gamma \). It also preserves \( (T_{\gamma_+} \Lambda_\Gamma)^\perp \), which is a timelike line because of Corollary 4.1.

Lifting everything to \( \mathbb{R}^{n+2} \) and using the identification of \( T_{\gamma_+} \partial \text{AdS}^{n+1} \) with \( \gamma_+^\perp / \gamma_+ \), we see that \( \hat{\gamma} \) preserves a two-dimensional plane \( V \subset \gamma_+^\perp \) which contains \( \gamma_+ \) and a negative direction. Let \((u, v)\) be a basis of \( V \), where \( u \in \gamma_+ \) and \( \langle v \mid v \rangle_{n,2} = -1 \).

By writing \( \hat{\gamma} v = au + bv \), we find that \( b^2 = -\langle \hat{\gamma} v \mid \hat{\gamma} v \rangle_{n,2} = -(v \mid v)_{n,2} = 1 \).

So the matrix of the restriction of \( \hat{\gamma} \) to \( P \) in the basis \((u, v)\) has the form

\[
\begin{pmatrix}
e^{\lambda_1(\gamma)} & a \\ 0 & \pm 1
\end{pmatrix}
\]

It has \( \pm 1 \) as an eigenvalue, and the eigendirection is in \( V \) but is not \( \gamma_+ \) (because \( \lambda_1(\gamma) > 0 \)), so it is negative for \( \langle \cdot \mid \cdot \rangle_{n,2} \). This eigendirection is a point of \( \text{AdS}^{n+1} \) fixed by \( \gamma \).
Theorem 4.4. Let $\Gamma \subset \text{PO}(n,2)$ be AdS quasi-Fuchsian. If the limit set $\Lambda_\Gamma \subset \partial\text{AdS}^{n+1}$ is a $C^1$ submanifold, then $\Gamma$ is Fuchsian.

Proof. By Lemma 4.3, the Jordan projections of proximal elements of $\Gamma$ all lie in a half line in a Weyl chamber $a^+$ of $\text{PO}(n,2)$, therefore its asymptotic cone has empty interior in $a^+$. Benoist’s Theorem [Ben97] implies $\Gamma$ is not Zariski dense. Proposition 1.4 implies that $\Gamma$ is Fuchsian. \hfill \square

References

[Bar15] Thierry Barbot. Deformations of fuchsian AdS representations are quasi-fuchsian. Journal of Differential Geometry, 101(1):1–46, 2015.

[Ben97] Yves Benoist. Propriétés asymptotiques des groupes linéaires. Geom. and Funct. Anal. 7 (1997), no. 1, 1–47.

[BBZ07] Thierry Barbot, François Béguin, and Abdelghani Zeghib. Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on ads 3. Geometriae Dedicata, 126(1):71–129, 2007.

[BM12] Thierry Barbot and Quentin Mérigot Anosov AdS representations are quasi-fuchsian. Groups, Geometry, and Dynamics, 6(3):441–483, 2012.

[DGK17] Jeffrey Dancinger, Fanny Kassel, Francois Guéritaud. Convex cocompactness in real projective space. [arXiv:1704.08711]

[DGK18] Jeffrey Dancinger, Fanny Kassel, Francois Guéritaud. Convex cocompactness in pseudo-Riemannian hyperbolic spaces. Geom. Dedicata 192 (2018), p. 87-126.

[DSL] Antonio J. Di Scala and Thomas Leistner. Connected subgroups of $\text{SO}(2,n)$ acting irreducibly on $\mathbb{R}^{2,n}$ T. Isr. J. Math. (2011) 182: 103.

[Fra05] Charles Frances. Lorentzian Kleinian groups. Comment. Math. Helv. 80 (2005), no. 4, 883–910.

[GdIH] Etienne Ghys, Pierre de la Harpe. Sur les groupes Hyperboliques d’après Mikhael Gromov Birkhäuser, Boston, MA ,Springer, 1990

[GM] Olivier Glorieux, Daniel Monclair Critical exponent and Hausdorff dimension in pseudo-Riemannian hyperbolic geometry pre-print arXiv: 1606.05512.
[KK16] Fanny Kassel and Toshiyuki Kobayashi. Poincaré series for non-riemannian locally symmetric spaces. In *Advances in Mathematics*, 287, p.123-236, 2016

[Lab06] François Labourie. Anosov flows, surface groups and curves in projective space. *Inventiones Mathematicae*, 165(1):51–114, 2006.

[Mes07] Geoffrey Mess. Lorentz spacetimes of constant curvature. *Geometriae Dedicata*, 126(1):3–45, 2007.

[PS17] Raphael Potrie and Andres Sambarino. Eigenvalues and Entropy of a Hitchin representation *Inventiones Mathematicae*, 209(3):885–925, 2017

[Zim18] Andrew Zimmer. Projective Anosov representations, convex cocompact actions, and rigidity. Arxiv: 1704.0858v2