Modules attached to extension bundles

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Abstract
In this article, we study modules over wild canonical algebras which correspond to extension bundles (Kussin et al. Adv Math 237:194–251, 2013) over weighted projective lines. We prove that all modules attached to extension bundles can be established by matrices with coefficients related to the relations of the considered algebra. Moreover, we expand the concept of extension bundles over weighted projective lines with three weights to general weight type and establish similar results in this situation. Finally, we present a method to compute matrices for all modules attached to extension bundles using cokernels of maps between direct sums of line bundles.

Keywords  Exceptional module · Canonical algebra · Wild type · Zero-one matrix problem · Weighted projective line · Exceptional pair · Extension bundles · Frobenius category · Projective cover

Mathematics Subject Classification 16G20 · 14F05

1 Introduction

One of the problems of representation theory of finite-dimensional algebras is the classification of indecomposable modules over a given algebra. Depending on the complexity of this issue we distinguish algebras of finite, tame and wild representation type. In the case of wild algebras, the structure of the module category is rich enough that it is impossible to describe all indecomposable modules;

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however, in this situation sometimes, it is possible to describe subclasses of indecomposable modules.

In this paper, we study an important class of modules, namely the so-called extension modules for wild canonical algebras. Canonical algebras were introduced by Ringel in 1984 [16], for a definition, we refer to Sect. 2.

In the case of domestic canonical algebras, Kussin and the second author [12] described matrices for all indecomposable modules provided the characteristic of the field is different from 2. In the case of characteristic 2, matrices for the indecomposables were given in [9].

In the situation of tubular canonical algebras in [14], extending methods of [17], it was shown that the exceptional modules can be exhibited by matrices having as coefficients only 0, 1 and \(-1\) in the cases \((3, 3, 3), (2, 4, 4), (2, 3, 6)\) and \(0, 1, -1, \lambda, \lambda - 1\) in the case \((2, 2, 2, 2)\), where \(\lambda\) is the parameter appearing in the relations for the considered algebra. Based on this result in [1], an algorithm and a computer program were developed to determine a description of all exceptional modules over tubular canonical algebras. Further in [3] and [2], the problem of homogeneous modules over tubular canonical modules was studied, in particular explicit matrices for modules of integral slope were given.

For canonical algebras of wild type, it was proved by the authors in [7] that “almost all” exceptional modules can be described by matrices having coefficients \(\lambda_i - \lambda_j\), where the \(\lambda_i\) are the parameters of the canonical algebra.

In [5], Geigle and Lenzing investigated weighted projective lines to give a geometric approach to canonical algebras. More precisely, they showed that the category of coherent sheaves \(\text{coh}(\mathbb{X})\) over a weighted projective line \(\mathbb{X}\) admits a tilting bundle \(T\) which induces an equivalence of the bounded derived categories \(D^b(\text{coh}(\mathbb{X})) \cong D^b(\text{mod}(\Lambda))\), where \(\Lambda = \text{End}_\mathbb{X}(T)\) is a canonical algebra.

In 2013, Kussin, Lenzing and second author in [10] introduced the concept of extension bundles over weighted projective lines with three weights, which is important in the study of nilpotent operators with invariant subspaces (see also [11]). It was proved there, in particular, that each indecomposable vector bundle of rank two is exceptional and appears as the middle term of an exact sequence, where the other terms are line bundles with good homological properties, see Sect. 3.

The aim of this article is to study modules attached to such extension bundles in the case of canonical algebras of wild type. Those modules are called extension modules. The paper contains the following results.

1. We prove that all extension modules over a wild canonical algebra \(\Lambda\) with three arms can be described by matrices with entries 0, 1 and \(-1\). This is an improvement for those modules of the result from [7]. We will use the fact that the category of vector bundles \(\text{vect}(\mathbb{X})\) over a weighted projective line \(\mathbb{X}\) is a Frobenius category with the line bundles as the indecomposable projective-injective objects. The main tool in the proof is the fact that a vector bundle associated to a module has a line bundle, associated with a module, as a direct summand of its projective cover.
2. We extend the concept of extension bundles from [10] to the case of an arbitrary number of weights. If this number is greater than 3, then not every indecomposable vector bundle of rank two is exceptional. We present a useful characterization of exceptional modules of rank two as extension bundles with data \((L, \vec{x})\), where \(L\) is a line bundle and \(\vec{x}\) is an element of the grading group of a specific form. We also establish the projective covers and the injective hulls of those bundles.

3. We show that all extension modules for a wild canonical algebra with an arbitrary number of arms can be established by matrices with coefficients \(0\), \(\lambda_i\), \(-\lambda_i\), where the \(\lambda_i\) are the parameters of the canonical algebra.

4. We compute matrix representation for each extension module over a canonical algebra of arbitrary type. Since the method using Schofield induction for exceptional modules (see [6, 17]) is not constructive we can not proceed as in the case for tubular canonical algebras [14]. Therefore, here we present another idea. We show that each extension module appears as a cokernel of a map between direct sums of line bundles and we describe a method to calculate matrices for these cokernels.

### 2 Notations and basic concepts

Let \(k\) be an algebraically closed field. We recall the concept of a weighted projective line in the sense of Geigle and Lenzing [5]. Let \(\mathbb{L} = \mathbb{L}(p)\) be the rank one abelian group with generators \(\vec{x}_1, \ldots, \vec{x}_t\) and relations \(p_1\vec{x}_1 = \cdots p_t\vec{x}_t = \vec{c}\), where the \(p_i\) are integers greater than or equal to 2. These numbers are called *weights*. The element \(\vec{c}\) is called the *canonical element*. Recall that \(\mathbb{L}\) is an ordered group with \(\mathbb{L}_{+} = \sum_{i=1}^t \mathbb{N}\vec{x}_i\) as its set of non-negative elements. Moreover, each element \(\vec{y}\) of \(\mathbb{L}\) can be written in *normal form* \(\vec{y} = a\vec{c} + \sum_{i=1}^t a_i\vec{x}_i\) with \(a \in \mathbb{Z}\) and \(0 \leq a_i < p_i\). The polynomial algebra \(k[x_1, \ldots, x_t]\) is \(\mathbb{L}\)-graded, where the degree of \(x_i\) is \(\vec{x}_i\). Since the polynomials \(f_i = x_i^{p_i} - x_i^{p_1} - \lambda_i x_i^{p_2}\) for \(i = 3, \ldots, t\) are homogeneous, the quotient algebra \(S = k[x_1, \ldots, x_t]/\langle f_i \mid i = 3, \ldots, t\rangle\) is also \(\mathbb{L}\)-graded. Here the \(\lambda_i\) are pairwise distinct non-zero elements of \(k\), they are called the *parameters*. A **weighted projective line** \(\mathbb{X}\) is the projective spectrum of the \(\mathbb{L}\)-graded algebra \(S\). Therefore \(\mathbb{X}\) depends on a weight sequence \(p = (p_1, \ldots, p_t)\) and a sequence of parameters \(\lambda = (\lambda_3, \ldots, \lambda_t)\).

We can assume that \(\lambda_3 = 1\). The category of coherent sheaves over \(\mathbb{X}\) will be denoted by \(\text{coh}(\mathbb{X})\). Each indecomposable sheaf in \(\text{coh}(\mathbb{X})\) is a locally free sheaf, called a **vector bundle**, or a sheaf of finite length. Denote by \(\text{vect}(\mathbb{X})\) (resp. \(\text{coh}_0(\mathbb{X})\)) the subcategory of \(\text{coh}(\mathbb{X})\) consisting of all vector bundles (resp. finite length sheaves) on \(\mathbb{X}\).

The category \(\text{coh}(\mathbb{X})\) is a Hom-finite, abelian \(k\)-category. Moreover, it is hereditary that is \(\text{Ext}_{\mathbb{X}}^i(-, -) = 0\) for \(i \geq 2\) and has Serre duality in the form \(\text{Ext}_{\mathbb{X}}^1(F, G) \cong \text{DHom}_{\mathbb{X}}(G, \tau_X F)\), where the Auslander–Reiten translation \(\tau_X\) is given by the shift \(F \mapsto F(\vec{c})\), where \(\vec{c} := (t - 2)\vec{c} - \sum_{i=1}^t \vec{x}_i\) denotes the *dualizing element*. It is well known that each line bundle has the form \(O(\vec{x})\) where \(O\) is the structure
sheaf of $\mathcal{X}$ and where $\bar{x} \in \mathbb{L}$. Furthermore we have isomorphisms $\text{Hom}_{\mathcal{X}}(\mathcal{O}(\bar{x}), \mathcal{O}(\bar{y})) \cong S_{y-x}$, where $S_2$ denotes the grading component of $S$ associated to $\bar{z} \in \mathbb{L}$.

One of the main results proved in [5] is the fact that in $\text{coh}(\mathcal{X})$, there is a tilting object, which is a direct sum of line bundles $T = \bigoplus_{0 \leq i \leq \bar{z}} \mathcal{O}(\bar{x})$, such that the right derived functor of the functor $\text{Hom}_{\mathcal{X}}(T, -)$ induces an equivalence of bounded derived category $D^b(\text{coh}(\mathcal{X})) \cong D^b(\text{mod}(\Lambda))$, where $\Lambda = \text{End}_{\mathcal{X}}(T)$ is a canonical algebra, called the canonical algebra associated to the weighted projective line $\mathcal{X}$.

Originally, canonical algebras $\Lambda$ were introduced by Ringel [16] as path algebras of quivers $Q$:

$$
\begin{array}{cccccc}
\bar{x}_1 & \alpha_2^{(1)} & 2\bar{x}_1 & \ldots & \alpha_{p_1-1}^{(1)} & (p_1-1)\bar{x}_1 \\
0 & \alpha_2^{(1)} & 2\bar{x}_2 & \ldots & \alpha_{p_2-1}^{(2)} & (p_2-1)\bar{x}_1 \\
\bar{x}_2 & \alpha_2^{(1)} & 2\bar{x}_3 & \ldots & \alpha_{p_3-1}^{(3)} & (p_3-1)\bar{x}_1 \\
& \alpha_2^{(1)} & 2\bar{x}_4 & \ldots & \alpha_{p_t-1}^{(t)} & (p_t-1)\bar{x}_1 \\
& & & \ddots & & \\
& & & & \bar{x} \\
\end{array}
$$

with canonical relations

$$x_p^{(i)} \cdots x_2^{(i)} x_1^{(i)} = x_p^{(1)} \cdots x_2^{(1)} x_1^{(1)} + \lambda_i x_p^{(2)} \cdots x_2^{(2)} x_1^{(2)} \quad \text{for } i = 3, 4, \ldots, t,$$

where the $\lambda_i$ are parameters from $\mathbb{A}$ and $p_i$ are weights from $\mathbb{P}$ as before. We call $t$ the number of arms of $\Lambda$. Concerning the complexity of the module category over $\Lambda$ there are three types of canonical algebras, domestic, tubular and wild ones. Recall that $\Lambda$ is of domestic (respectively, tubular, wild) type if the Euler characteristic $\chi_\Lambda = (2 - t) + \sum_{i=1}^t 1/p_i$ is positive (respectively, zero, negative).

Denote by $Q_0$ the set of vertices and by $Q_1$ the set of arrows of the quiver $Q$. Then each finitely generated right module over $\Lambda$ is given by finite dimensional vector spaces $M_i$ for each vertex $i$ of $Q_0$ and by linear maps $M_a : M_j \to M_i$ for each arrows $a : i \to j$ of $Q_1$ such that the canonical relations are satisfied. The vector $(\dim M_i)_{i \in Q_0}$ is called dimension vector. For each fixed dimension vector one can fix basis in the vertices and then each one of these modules have representations whose matrices. The category of finite generated right modules we denote by $\text{mod}(\Lambda)$.

For coherent sheaves there are well-known invariants the rank, the degree and the determinant, which are given by linear forms on the Grothendieck group $\text{rk}, \deg : K_0(\mathcal{X}) \to \mathbb{Z}$ and $\det : K_0(\mathcal{X}) \to \mathbb{L}(\mathbb{P})$. Since $K_0(\mathcal{X}) \cong K_0(\mathcal{L})$, we have also the concept of the rank, the degree and the determinants for $\Lambda$-modules. In particular, the rank of a $\Lambda$-module is defined by the formula $\text{rk}_M := \dim_k \text{M}_0 - \dim_k \text{M}_\mathcal{L}$. We denote by $\text{mod}_+(\Lambda)$ (respectively, $\text{mod}_-(\Lambda)$, $\text{mod}_0(\Lambda)$) the full subcategory consisting of all $\Lambda$-modules, whose indecomposable summands in the decomposition into a direct sum have positive (respectively, negative or zero) rank. Further, by
Let $X$ be a weighted projective line of a triple type $(p_1, p_2, p_3)$. The concept of extension bundles was introduced in [10] in the study of stable vector bundle categories. In particular, it was shown that stable vector bundle categories of weighted projective lines of triple type admit tilting objects, being direct sums of categories. In particular, stable vector bundle categories of the middle term of a non-split exact sequence

$$
\eta_{L, \bar{x}} : 0 \to L(\bar{o}) \to E \to L(\bar{x}) \to 0
$$

for some line bundle $L$ and some element $\bar{x}$ of $\mathbb{L}$ such that $0 \leq \bar{x} \leq \bar{\delta}$, where $\bar{\delta} := \bar{c} + 2\bar{o}$ is the dominant element. Because in this case the vector space $\text{Ext}^1_X(L(\bar{x}), L(\bar{o}))$ is one dimensional, the bundle $E$ is uniquely determined up to isomorphism. It is called the extension bundle given by the pair $(L, \bar{x})$. It is easy to check, that the pair $(L(\bar{x}), L(\bar{o}))$ is exceptional and orthogonal. Therefore, if $L(\bar{x})$ and

We say in these cases that the module $\text{Hom}_X(T, E)$ (respectively, $\text{Ext}^1_X(T, E)$) is attached to $E$. For simplicity, we will often identify a sheaf $E$ in $\text{coh}_+(\mathbb{X})$ or $\text{coh}_0(\mathbb{X})$ with the corresponded $\Lambda$-module $\text{Hom}_X(T, E)$.

**Remark 2.1** Recall from [4, Theorem 9.1.1] that the standard duality $\text{Hom}_\Lambda(-, k)$ defines an equivalence of the categories $\text{mod}(\Lambda)$ and $\text{mod}(\Lambda^\text{op})$. Under this equivalence, $\text{mod}_-(\Lambda)$ corresponds to $\text{mod}_+(\Lambda^\text{op})$. Since $\Lambda \simeq \Lambda^\text{op}$ in many considerations, it is sufficient to consider $\Lambda$-modules of positive rank and of rank zero. In particular, modules of negative rank can be obtained from those of positive rank by reversing the arrows and transposing the matrices.

Recall that a coherent sheaf $E$ over $\mathbb{X}$ is called exceptional if $\text{End}_X(E) = k$ and $\text{Ext}^1_X(E, E) = 0$. A pair $(X, Y)$ in $\text{coh}(\mathbb{X})$ is called an exceptional pair if $X, Y$ are exceptional and $\text{Hom}_X(Y, X) = 0 = \text{Ext}^1_X(Y, X)$. Furthermore, an exceptional pair is orthogonal if addition $\text{Hom}_X(X, Y) = 0$. Finally, a $\Lambda$-module $M$ is called exceptional if $\text{End}_\Lambda(M) = k$ and $\text{Ext}^1(\Lambda, M) = 0$ for $i \geq 1$. 

**3 Extension bundles for weighted projective lines with three weights**

Let $\mathbb{X}$ be a weighted projective line of a triple type $(p_1, p_2, p_3)$. The concept of extension bundles was introduced in [10] in the study of stable vector bundle categories. In particular, it was shown that stable vector bundle categories of weighted projective lines of triple type admit tilting objects, being direct sums of extension bundles, such that their endomorphism algebras is isomorphic to the incidence algebra of the poset $[1, p_1 - 1] \times [1, p_2 - 1] \times [1, p_3 - 1]$ of cubical shape, where $[1, n]$ denotes the linear ordered set $\{1, 2, \ldots, n\}$.

From [10, Theorem 4.2], each indecomposable vector bundle $E$ can be obtained as the middle term of a non-split exact sequence

$$
\eta_{L, \bar{x}} : 0 \to L(\bar{o}) \to E \to L(\bar{x}) \to 0
$$

for some line bundle $L$ and some element $\bar{x}$ of $\mathbb{L}$ such that $0 \leq \bar{x} \leq \bar{\delta}$, where $\bar{\delta} := \bar{c} + 2\bar{o}$ is the dominant element. Because in this case the vector space $\text{Ext}^1_X(L(\bar{x}), L(\bar{o}))$ is one dimensional, the bundle $E$ is uniquely determined up to isomorphism. It is called the extension bundle given by the pair $(L, \bar{x})$. It is easy to check, that the pair $(L(\bar{x}), L(\bar{o}))$ is exceptional and orthogonal. Therefore, if $L(\bar{x})$ and
$L(\tilde{o})$ are $\Lambda$-modules, both in $\text{mod}_+(\Lambda)$ or in $\text{mod}_-(\Lambda)$, then they can be described by
$0, \pm \lambda_i$ matrices, as rank one modules (see [14]) and it follows that $E$ also can be
described by matrices with the same coefficients (see Proposition 7.1 and the remark
after its proof in [7]).

Following the introduction from [10], we will establish the Frobenius structure on
the category $\text{vect}(\mathbb{X})$ of vector bundles over $\mathbb{X}$. Recall that the short exact sequence
$0 \to X \to E \to Y \to 0$ in $\text{vect}(\mathbb{X})$ is called $\textbf{distinguished exact}$ if for each line
bundle $L$, the following sequence $0 \to \text{Hom}_{\mathbb{X}}(L, X) \to \text{Hom}_{\mathbb{X}}(L, E) \to \text{Hom}_{\mathbb{X}}(L, Y) \to 0$
is exact. The distinguished exact sequences give a rise to exact
structures in the sense of Quillen (see [8, 15]) on the category $\text{vect}(\mathbb{X})$, where the line
bundles are the only indecomposable projective–injective objects. Note that the
sequence $\eta_{L,\mathbb{X}}$ determining an extension bundle $E_L(\tilde{x})$ is not distinguished,
and in not-split, even though it has line bundle as the first and last terms.

For each vector bundle $E$, there is a projective cover $\Psi(E)$ and an epimorphism
$\pi_E : \Psi(E) \to E$ that satisfy the following properties:

(i) $\Psi(E)$ is a direct sum of line bundles;
(ii) each map $f : L \to E$ from line bundle $L$ to $E$ is factorised by $\pi_E$;
(iii) $\Psi(E)$ is minimal that satisfies the above properties.

Moreover, each vector bundle has an injective hull, that can be described in the
similar way. Therefore, the category $\text{vect}(\mathbb{X})$ is a Frobenius category.

Let us denote that concept of projectivity from Frobenius structure on $\text{vect}(\mathbb{X})$ is
not related with projectivity of modules over canonical algebra $\Lambda$. For example, only
line bundle of the form $O(\tilde{x})$ for $0 \leq \tilde{x} \leq \tilde{c}$ correspondence to indecomposable
projective $\Lambda$-modules. In the following lemma, we will prove that if vector bundle $E$
correspondence to $\Lambda$-module, then at least one direct summand of projective cover
$\Psi(E)$ in $\text{vect}(\mathbb{X})$ also correspondence to $\Lambda$-module. However, it is usually not a
projective module.

\textbf{Lemma 3.1} Let $E$ be a non-zero vector bundle on a weighted projective line $\mathbb{X}$ of
type $(p_1, p_2, p_3)$ with projective cover $\Psi(E)$. If $\text{Ext}^1_{\mathbb{X}}(T, E) = 0$ for the canonical
bundle $T$, then there is a line bundle $L$ in the decomposition $\Psi(E)$ nto a direct sum of
line bundles, such that $\text{Ext}^1_{\mathbb{X}}(T, L) = 0$.

\textbf{Proof} Assume that $\text{Ext}^1_{\mathbb{X}}(T, E) = 0$. Then $\text{Hom}_{\mathbb{X}}(T, E) \neq 0$ because $T$ is a tilting
bundle and $E$ is non-zero. Therefore, there is an element $\tilde{x}$, such that $0 \leq \tilde{x} \leq \tilde{c}$ and
$\text{Hom}_{\mathbb{X}}(O(\tilde{x}), E) \neq 0$. Each non-zero morphism $f : O(\tilde{x}) \to E$ factors through
$\pi_E : \Psi(E) \to E$, so there is morphism $0 \neq \alpha : O(\tilde{y}) \to \Psi(E)$ such that $\pi_E \circ \alpha = f$.
Hence, there is a direct summand $L = O(\tilde{y})$ of $\Psi(E)$ such that
$\text{Hom}_{\mathbb{X}}(O(\tilde{x}), O(\tilde{y})) \neq 0$.

We will show that $L$ has the desired property. Writing $\tilde{y}$ in normal form
$\tilde{y} = \alpha c + \alpha_1 \tilde{x}_1 + \alpha_2 \tilde{x}_2 + \alpha_3 \tilde{x}_3$, with $0 \leq \alpha_i \leq p_i - 1$, we obtain $\alpha \geq 0$ and
\((\Delta)\) \( c + \alpha - \bar{y} = (2 - \alpha)c - \sum_{i=1}^{\bar{z}}(\alpha_i + 1)x_i < 0.\)

Assume that \( \text{Ext}^1_\mathbb{P}(T, \mathcal{O}(\bar{y})) \neq 0 \) and let \( \bar{z} \) satisfy \( \text{Ext}^1_\mathbb{P}(\mathcal{O}(\bar{z}), \mathcal{O}(\bar{y})) \neq 0 \) and \( 0 \leq \bar{z} \leq \bar{c} \). Using Serre duality we obtain \( \bar{z} + \alpha - \bar{y} \geq 0 \). Therefore,
\[
\bar{c} + \alpha - \bar{y} = (\bar{c} - \bar{z}) + (\bar{z} + \alpha - \bar{y}) \geq 0,
\]
a contradiction with \( (\Delta) \). Thus, \( \text{Ext}^1_\mathbb{P}(T, \mathcal{O}(\bar{y})) = 0 \)

In the following lemma, we prove that each extension bundle defined by a short exact sequence \( \eta_{L,x} \) appears in addition as an extension bundle for three different pairs. In particular, the elements \( \bar{x} \) defining the extension bundle \( \eta_{L,x} \) is not unique.

**Lemma 3.2** Let \( \mathbb{P} \) be a weighted projective line of type \( (p_1, p_2, p_3) \) and let \( E \) be an extension bundle given by a pair \( (L, \bar{x}) \), where \( \bar{x} = l_1\bar{x}_1 + l_2\bar{x}_2 + l_3\bar{x}_3 \). Then \( E \) is also an extension bundle determined by the following pairs:

\[
\begin{align*}
L(\bar{x} - (1 + l_i)\bar{x}_i)(-\bar{\alpha}),
\end{align*}
\]

where \( L(\bar{x} - (1 + l_i)\bar{x}_i) \) are direct summands of \( \Psi(E) \). In particular, for each \( i \in \{1, 2, 3\} \), there is an exact short sequence

\[
0 \longrightarrow L(\bar{x} - (1 + l_i)\bar{x}_i) \longrightarrow E \longrightarrow L((1 + l_i)\bar{x}_i + \bar{\alpha}) \longrightarrow 0.
\]

**Proof** The projective cover of \( E \) has the form

\[
\Psi(E) = L(\bar{\alpha}) \oplus \bigoplus_{i=1}^{3} L(\bar{x} - (1 + l_i)\bar{x}_i), \quad [10, \text{Theorem 4.6}].
\]

Then there are exact sequences

\[
\eta_i : 0 \longrightarrow L(\bar{x} - (1 + l_i)\bar{x}_i) \longrightarrow E \longrightarrow \widehat{L}_i \longrightarrow 0, \quad \text{for} \quad i = 1, 2, 3,
\]

where from [10, Proposition 3.8] the sheaf \( \widehat{L}_i \) is a line bundle, for \( i = 1, 2, 3 \). By comparison of the determinants we obtain that \( \widehat{L}_i = L((1 + l_i)\bar{x}_i + \bar{\alpha}) \) is a direct summand of the injective hull of \( E \). Therefore, the sequence \( \eta_i \) can be presented as follows:

\[
0 \longrightarrow L(\bar{x} - (1 + l_i)\bar{x}_i - \bar{\alpha})(\bar{\alpha}) \longrightarrow E \longrightarrow L(\bar{x} - (1 + l_i)\bar{x}_i - \bar{\alpha})(\bar{y}_i) \longrightarrow 0,
\]

where \( \bar{y}_i = (1 + l_i)\bar{x}_i + 2\bar{\alpha} - \bar{x} = l_i\bar{x}_i + \sum_{j \neq i}(p_j - l_j - 2)\bar{x}_j \). Since \( 0 \leq l_j \leq p_j - 2 \), we have \( 0 \leq p_j - l_j - 2 \leq p_j - 2 \), and consequently \( 0 \leq \bar{y}_i \leq \bar{\delta} \).

**Theorem 1** Let \( \Lambda \) be a canonical algebra with three arms. Then each indecomposable \( \Lambda \)-module of rank two can be described by matrices having coefficients \( 0, 1, -1 \).
In this section, we will deal with a weighted projective line \( X \). Extension bundles in the case of \( t \) will be considered.

**Proof** Let \( M \) be a \( \Lambda \)-module of rank 2, attached to an indecomposable vector bundle \( E \) over the weighted projective line \( \mathcal{X} \) associated to \( \Lambda \). Then \( M \) is in \( \text{mod}_+(\Lambda) \). We will show that there is an exact sequence

\[
\eta : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0,
\]

where \( A, B \in \text{mod}_+(\Lambda) \) and \((B, A)\) is an orthogonal exceptional pair in \( \text{coh}(\mathcal{X}) \). Then the result follows from [7, Proposition 7.1].

From [10, Theorem 4.2], the vector bundle \( E \) appears as an extension bundle given by a pair \((L, x)\), this means that there is a short exact sequence

\[
\eta_{L,x} : 0 \rightarrow L(\bar{o}) \rightarrow E \rightarrow L(x) \rightarrow 0,
\]

where \((L(x), L(\bar{o}))\) is an orthogonal exceptional pair. Since the vector bundle \( E \) is attached to the \( \Lambda \)-module \( M \), we have \( \text{Ext}^1_{\mathcal{X}}(T, E) = 0 \). Applying the functor \( \text{Hom}_{\mathcal{X}}(T, -) \) to the sequence \( \eta_{L,x} \), we obtain that \( \text{Ext}^1_{\mathcal{X}}(T, L(x)) = 0 \), thus \( L(x) \) is in \( \text{mod}_+(\Lambda) \). If in addition \( \text{Ext}^1_{\mathcal{X}}(T, L(\bar{o})) = 0 \), we are done. Otherwise, we will replace the exact sequence \( \eta_{L,x} \) by another one.

To do so we recall that \( L(\bar{o}) \) is a direct summand of the projective cover \( \mathfrak{P}(E) \) and from Lemma 3.1 there is a line bundle \( \widehat{L} \), which is a direct summand of \( \mathfrak{P}(E) \) and satisfies \( \text{Ext}^1_{\mathcal{X}}(T, \widehat{L}) = 0 \) thus \( \widehat{L} \) is in \( \text{coh}_+(\mathcal{X}) = \text{mod}_+(\Lambda) \). From [10, Theorem 4.6], the line bundle \( \widehat{L} \) has the form \( L(x - (1 + l_i)\bar{x}_i) \) for some \( i \in \{1, 2, 3\} \). Thus, using Lemma 3.2, we get an exact sequence

\[
0 \rightarrow L(x - (1 + l_i)\bar{x}_i) \rightarrow E \rightarrow L((1 + l_i)\bar{x}_i + \bar{o}) \rightarrow 0.
\]

Applying the functor \( \text{Hom}_{\mathcal{X}}(T, -) \) to the sequence above we conclude that \( \text{Ext}^1_{\mathcal{X}}(E, L((1 + l_i)\bar{x}_i + \bar{o})) = 0 \) and therefore \( L((1 + l_i)\bar{x}_i + \bar{o}) \) is in \( \text{mod}_+(\Lambda) \). Moreover, it is easily checked that \((L(x - (1 + l_i)\bar{x}_i), L((1 + l_i)\bar{x}_i + \bar{o}))\) form an orthogonal exceptional pair in \( \text{coh}(\mathcal{X}) \). Thus, we get an exact sequence \( \eta \) of the desired form and the theorem is proved.

**Remark 3.3** Using Remark 2.1, we get the same result for exceptional modules of rank \(-2\) from \( \text{mod}_-(\Lambda) \).

## 4 Extension bundles in the case of \( t \) numbers of weights

In this section, we will deal with a weighted projective line \( \mathcal{X} \) of the type \((p_1, \ldots, p_t)\), where \( t \) is greater than than equal to \( 3 \).

**Theorem 2** Let \( \mathcal{X} \) be a weighted projective line of a type \((p_1, \ldots, p_t)\). Then each indecomposable vector bundle of rank two occurs as the middle term of a non-split exact sequence.
\[ \eta : 0 \to L(\bar{\omega}) \xrightarrow{\iota} E \xrightarrow{\pi} L(\bar{x}) \to 0, \]

where \( 0 \leq \bar{x} \leq \bar{\delta} := 2\bar{\omega} + \bar{c} = (t - 3)\bar{c} + \sum_{i=1}^{t}(p_i - 2)\bar{x}_i. \) Moreover, the following conditions are equivalent:

(i) The vector bundle \( E \) is exceptional.

(ii) The pair \((L(\bar{x}), L(\bar{\omega}))\) is an orthogonal exceptional pair with \( \text{Ext}^1_{\mathbb{X}}(L(\bar{x}), L(\bar{\omega})) = k. \)

(iii) \( \bar{x} = \sum_{i=1}^{t}l_i\bar{x}_i \) with \( 0 \leq l_i \leq p_i - 1 \) and there are exactly \( t - 3 \) numbers \( l_i \) equal to \( p_i - 1. \)

(iv) \( \text{Ext}^1_{\mathbb{X}}(L(\bar{\omega}), L(\bar{x})) = \text{Hom}_{\mathbb{X}}(E, L(\bar{\omega})) = \text{Ext}^1_{\mathbb{X}}(E, L(\bar{\omega})) = \text{Hom}_{\mathbb{X}}(L(\bar{x}), E) = \text{Ext}^1_{\mathbb{X}}(L(\bar{x}), E) = 0. \)

**Proof** The proof of the existence of the sequence \( \eta \) is almost the same as in the case of a triple weight type, we refer the reader to [10, Theorem 4.2].

(i) \( \Rightarrow \) (ii). Assume that \( E \) is exceptional and \( 0 \leq \bar{x} \leq \bar{\delta}. \) Then \( \bar{x} \) can be written in normal form \( \bar{x} = \bar{c} + \sum_{i=1}^{l}l_i\bar{x}_i, \) with \( l \geq 0 \) and \( 0 \leq l_i \leq p_i - 1 \) and \( \bar{x} \leq \bar{\delta}. \) Because \( \bar{x} - \bar{\omega} \leq \bar{\delta} - \bar{\omega} = \bar{c} + \bar{\omega} < 0 \) we have \( \text{Hom}_{\mathbb{X}}(L(\bar{\omega}), L(\bar{x})) \cong S_{\bar{x} - \bar{\omega}} = 0. \) Similarly, the vector space \( \text{Hom}_{\mathbb{X}}(L(\bar{x}), L(\bar{\omega})) \) also vanishes. From Serre duality, we get

\[ \text{Ext}^1_{\mathbb{X}}(L(\bar{x}), L(\bar{\omega})) \cong \text{Hom}_{\mathbb{X}}(L(\bar{\omega}), L(\bar{x} + \bar{\omega})) \cong S_{\bar{x}} \cong k^{l+1}. \]

Now, we have \([E] = [L(\bar{\omega})] + [L(\bar{x})]\) in the Grothendieck group \( K_0(\mathbb{X}) \) and applying the Euler form \( \langle -, - \rangle_{\mathbb{X}} : K_0(\mathbb{X}) \times K_0(\mathbb{X}) \to \mathbb{Z} \), we obtain

\[ \begin{align*}
1 = \langle[E], [E] \rangle_{\mathbb{X}} &= \langle[L(\bar{\omega})], [L(\bar{x})] \rangle + \langle[L(\bar{\omega})], [L(\bar{x})] \rangle + \langle[L(\bar{x})], [L(\bar{x})] \rangle \\
&= 2 + \dim_k \text{Hom}_{\mathbb{X}}(L(\bar{\omega}), L(\bar{x})) - \dim_k \text{Ext}^1_{\mathbb{X}}(L(\bar{\omega}), L(\bar{x})) \\
&+ \dim_k \text{Hom}_{\mathbb{X}}(L(\bar{x}), L(\bar{\omega})) - \dim_k \text{Ext}^1_{\mathbb{X}}(L(\bar{x}), L(\bar{\omega})) \\
&= 2 - (l + 1) - \dim_k \text{Ext}^1_{\mathbb{X}}(L(\bar{x}), L(\bar{x})).
\end{align*} \]

Therefore, \( l = \dim_k \text{Ext}^1_{\mathbb{X}}(L(\bar{\omega}), L(\bar{x})) \) and it follows that \( l \geq 0. \) Consequently, \( \dim_k \text{Ext}^1_{\mathbb{X}}(L(\bar{\omega}), L(\bar{x})) = 0 \) and \( \text{Ext}^1_{\mathbb{X}}(L(\bar{x}), L(\bar{\omega})) = k. \)

(ii) \( \Rightarrow \) (iii). Assume that \((L(\bar{x}), L(\bar{\omega}))\) is an orthogonal exceptional pair, such that \( \text{Ext}^1_{\mathbb{X}}(L(\bar{x}), L(\bar{\omega})) \cong k. \) The element \( \bar{x} \) can be written in normal form \( \bar{x} = \bar{c} + \sum_{i=1}^{l}l_i\bar{x}_i, \) with \( l \geq 0 \) and \( 0 \leq l_i \leq p_i - 1. \) From Serre duality, we obtain that \( \text{Ext}^1_{\mathbb{X}}(L(\bar{x}), L(\bar{\omega})) \cong k^{l+1}. \) Hence, \( l = 0. \) Moreover,

\[ 0 = \dim_k \text{Ext}^1_{\mathbb{X}}(L(\bar{\omega}), L(\bar{x})) \cong D\text{Hom}_{\mathbb{X}}(L(\bar{x}), L(2\bar{\omega})) \cong S_{\bar{2}\bar{\omega} - \bar{x}}, \]

where \( 2\bar{\omega} - \bar{x} = (t - 4)\bar{c} + \sum_{i=1}^{l}(p_i - 2 - l_i)\bar{x}_i < 0. \) Therefore, at least \( t - 3 \) numbers \( l_i \) have to be equal to \( p_i - 1. \) Furthermore, because \( \bar{x} \leq \bar{\delta} \) at most \( t - 3 \) numbers \( l_i \) can be equal to \( p_i - 1. \) This implies that exactly \( t - 3 \) numbers \( l_i \) are equal to \( p_i - 1. \)
(iii) ⇒ (iv). Assume that the element $\bar{x}$ has normal form $\sum_{i=1}^{t} l_i \bar{x}_i$ with $0 \leq l_i \leq p_i - 1$ and there are exactly $t - 3$ numbers $l_i$ equal $p_i - 1$. Therefore, by Serre duality, we get

$$\text{Ext}^1_{\mathcal{X}}(L(\bar{\omega}), L(\bar{x})) \cong D\text{Hom}_{\mathcal{X}}(L(\bar{x}), L(2\bar{\omega})) \cong S_{2\bar{\omega} - \bar{x}} = 0,$$

$$\text{Ext}^1_{\mathcal{X}}(L(\bar{x}), L(\bar{\omega})) \cong D\text{Hom}_{\mathcal{X}}(L(\bar{\omega}), L(\bar{x} + \bar{\omega})) \cong S_{\bar{\omega}} \cong k.$$  

Applying the functor $\text{Hom}_{\mathcal{X}}(L(\bar{x}), -)$ to $\eta$ we get an exact sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{X}}(L(\bar{x}), L(\bar{\omega})) \longrightarrow \text{Hom}_{\mathcal{X}}(L(\bar{x}), E) \longrightarrow \text{Hom}_{\mathcal{X}}(L(\bar{x}), L(\bar{x})) \longrightarrow$$

$$\delta : \text{Ext}^1_{\mathcal{X}}(L(\bar{x}), L(\bar{\omega})) \longrightarrow \text{Ext}^1_{\mathcal{X}}(L(\bar{x}), E) \longrightarrow \text{Ext}^1_{\mathcal{X}}(L(\bar{x}), L(\bar{x})) \longrightarrow 0.$$

Now, $\delta(\mathbb{1}_{L(\bar{x})}) = \eta \neq 0$ because $\eta$ does not split and consequently $\delta$ is isomorphism. Therefore,

$$\text{Hom}_{\mathcal{X}}(L(\bar{x}), F) = 0 = \text{Ext}^1_{\mathcal{X}}(L(\bar{x}), F).$$

The long exact sequence $\text{Hom}_{\mathcal{X}}(\eta, L(\bar{\omega}))$ has the form

$$0 \longrightarrow \text{Hom}_{\mathcal{X}}(L(\bar{x}), L(\bar{\omega})) \longrightarrow \text{Hom}_{\mathcal{X}}(E, L(\bar{\omega})) \longrightarrow \text{Hom}_{\mathcal{X}}(L(\bar{\omega}), L(\bar{\omega})) \longrightarrow$$

$$\text{Ext}^1_{\mathcal{X}}(L(\bar{x}), L(\bar{\omega})) \longrightarrow \text{Ext}^1_{\mathcal{X}}(E, L(\bar{\omega})) \longrightarrow \text{Ext}^1_{\mathcal{X}}(L(\bar{\omega}), L(\bar{\omega})) \longrightarrow 0.$$

Let $u : E \longrightarrow L(\bar{\omega})$ be a non-zero morphism. Then $u \circ i : L(\bar{\omega}) \longrightarrow L(\bar{\omega})$ is the zero map. Indeed, if $u \circ i$ is non-zero, it is an isomorphism and so $\eta$ splits which is impossible. Hence, $u \in \ker(- \circ i) = \text{Im}(- \circ \pi) = 0$, because $\text{Hom}_{\mathcal{X}}(L(\bar{x}), L(\bar{\omega})) = 0$. Then $\text{Hom}_{\mathcal{X}}(E, L(\bar{\omega})) = 0$ and by comparing dimensions in the sequence $\text{Hom}_{\mathcal{X}}(\eta, L(\bar{\omega}))$ we obtain that $\text{Ext}^1_{\mathcal{X}}(E, L(\bar{\omega})) = 0$.

(iv) ⇒ (i). Assume that the vector spaces $\text{Ext}^1_{\mathcal{X}}(L(\bar{\omega}), L(\bar{x}))$, $\text{Hom}_{\mathcal{X}}(E, L(\bar{\omega}))$, $\text{Ext}^1_{\mathcal{X}}(E, L(\bar{\omega}))$, $\text{Hom}_{\mathcal{X}}(L(\bar{x}), E)$ and $\text{Ext}^1_{\mathcal{X}}(L(\bar{x}), E)$ vanish. We apply the functor $\text{Hom}_{\mathcal{X}}(L(\bar{\omega}), -)$ to $\eta$ and obtain a long exact sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{X}}(L(\bar{\omega}), L(\bar{\omega})) \longrightarrow \text{Hom}_{\mathcal{X}}(L(\bar{\omega}), E) \longrightarrow \text{Hom}_{\mathcal{X}}(L(\bar{\omega}), L(\bar{x})) \longrightarrow$$

$$\text{Ext}^1_{\mathcal{X}}(L(\bar{\omega}), L(\bar{\omega})) \longrightarrow \text{Ext}^1_{\mathcal{X}}(L(\bar{\omega}), E) \longrightarrow \text{Ext}^1_{\mathcal{X}}(L(\bar{\omega}), L(\bar{x})) \longrightarrow 0.$$

Hence, $\text{Hom}_{\mathcal{X}}(L(\bar{\omega}), E) \cong k$ and $\text{Ext}^1_{\mathcal{X}}(L(\bar{\omega}), E) = 0$. Finally, we apply the functor $\text{Hom}_{\mathcal{X}}(-, E)$ to $\eta$ and obtain a long exact sequence
\[
0 \longrightarrow \text{Hom}_E(L(\bar{x}), E) \longrightarrow \text{Hom}_E(E, E) \longrightarrow \text{Hom}_E(L(\bar{\omega}), E) \longrightarrow \\
\longrightarrow \text{Ext}^1_E(L(\bar{x}), E) \longrightarrow \text{Ext}^1_E(E, E) \longrightarrow \text{Ext}^1_E(L(\bar{\omega}), E) \longrightarrow 0.
\]

Therefore, \(\text{Hom}_E(E, E) = k\) and \(\text{Ext}^1_E(E, E) = 0\), so the vector bundle \(E\) is exceptional.

For an exceptional bundle \(E\), the non-split sequence \(\eta\) uniquely determines \(E\), and in this case we will say that the extension bundle is given by the pair \((L, \bar{x})\) and we will denote it by \(E_L(\bar{x})\).

**Theorem 3** Let \(E\) be an indecomposable vector bundle over \(\mathbb{X}\) such that there is a short exact sequence

\[
\eta: \ 0 \longrightarrow L(\bar{\omega}) \longrightarrow E \longrightarrow L(\bar{x}) \longrightarrow 0,
\]

where \(\bar{x} = \sum_{i=1}^I l_i \bar{x}_i\), with \(0 \leq l_i \leq p_i - 1\) and \(0 \leq \bar{x} \leq \bar{\delta}\). Moreover, let \(I = \{i \mid l_i \neq p_i - 1\}\). Then

\[
\mathfrak{P}(E_L(\bar{x})) = L(\bar{\omega}) \oplus \bigoplus_{j \in I} L(\bar{x} - (1 + l_j) \bar{x}_j)
\]

\[
\mathfrak{S}(E_L(\bar{x})) = L(\bar{x}) \oplus \bigoplus_{j \in I} L((1 + l_j) \bar{x}_j + \bar{\omega}).
\]

Further, the line bundle summands of \(\mathfrak{P}(E_L(\bar{x}))\) (resp. \(\mathfrak{S}(E_L(\bar{x}))\)) are mutually Hom-orthogonal.

**Proof** Observe that the condition \(0 \leq \bar{x} \leq \bar{\delta}\) implies that \(\#I \geq 3\). We will consider the case of injective hulls, the arguments for projective covers are dual.

From Serre duality, we obtain that \(\text{Ext}^1_E(L(\bar{x}), L(\bar{\omega} + (1 + l_j) \bar{x}_j)) = 0\) for \(j \in I\). Hence, applying the functor \(\text{Hom}_E(-, L(\bar{\omega} + (1 + l_j) \bar{x}_j))\) for \(j \in I\) to \(\eta\) we see that there are morphisms \(x_j^{l+1}: E \longrightarrow L(\bar{\omega} + (1 + l_j) \bar{x}_j)\) such that \(x_j^{l+1} \circ x_i = x_i^j\), where \(x_j^0: L(\bar{\omega}) \longrightarrow L(\bar{\omega} + (1 + l_j) \bar{x}_j)\). We will show that \(j_E = \left(\pi_1, \left(x_j^{l+1}\right)_{j \in I}\right): E \longrightarrow L(\bar{x}) \oplus \bigoplus_{j \in I} L(\bar{\omega} + (1 + l_j) \bar{x}_j)\) is an injective hull of the bundle \(E\). For this, we will prove that each morphism \(E \longrightarrow L'\), where \(L'\) is a line bundle, factors through \(L(\bar{x}) \oplus \bigoplus_{j \in I} L(\bar{\omega} + (1 + l_j) \bar{x}_j)\). For simplicity we can write \(L'\) as \(L(\bar{\omega} + \bar{z})\) for some \(\bar{z} \in \mathbb{L}\). Remark, that for \(j \not\in I\) the space \(\text{Ext}^1_E(L(\bar{x}), L(\bar{\omega} + (1 + l_j) \bar{x}_j)) \neq 0\), and there are no maps \(x_j^{l+1}\) for \(j \not\in I\).

First, we show that

\[
\text{Hom}_E(E, L(\bar{\omega} + \bar{z})) = 0 \quad \text{for} \quad 0 \leq \bar{z} \leq \bar{x}.
\]

Indeed, let \(\bar{z} = \sum_{j=1}^I a_j \bar{x}_j\) be an element of \(\mathbb{L}\) with \(0 \leq a_j \leq l_j\) for all \(j\) and let \(z = x_1^{a_1} x_2^{a_2} \cdots x_I^{a_i}\) be a morphism form \(L(\bar{\omega})\) to \(L(\bar{\omega} + \bar{z})\). Applying the functor
Hom\(_X(E, -)\) to the sequence \(0 \rightarrow L(\tilde{o}) \xrightarrow{\sim} L(\tilde{o} + \tilde{z}) \rightarrow S \rightarrow 0\) we obtain that \(\text{Ext}_X^1(E, L(\tilde{o})) = 0\). Next, applying the functor \(\text{Hom}_X(-, L(\tilde{o} + \tilde{z}))\) to \(\eta\) we obtain a long exact sequence

\[
0 \rightarrow \text{Hom}_X(E, L(\tilde{o} + \tilde{z})) \rightarrow \text{Hom}_X(L(\tilde{o}), L(\tilde{o} + \tilde{z})) \rightarrow \\
\text{Ext}_X^1(L(\tilde{x}), L(\tilde{o} + \tilde{z})) \rightarrow \text{Ext}_X^1(E, L(\tilde{o} + \tilde{z})) \rightarrow \cdots
\]

By comparing dimensions we get \(\text{Hom}_X(E, L(\tilde{o} + \tilde{z})) = 0\).

Let \(h : E \rightarrow L(\tilde{o} + \tilde{z})\) be a non-zero morphism for some \(z \in \mathbb{L}\). Because \(\eta\) is a non-split exact sequence, the map \(h \circ i : L(\tilde{o}) \xrightarrow{i} E \xrightarrow{h} L(\tilde{o} + \tilde{z})\) is a non-isomorphism. If \(h \circ i\) is the zero map, then \(h\) factors through by \(\pi\) and we are done. Suppose now that \(h \circ i \neq 0\). Then \(\tilde{z} \geq 0\), because \(\text{Hom}_X(L(\tilde{o}), L(\tilde{o} + \tilde{z})) \neq 0\). Moreover, from property (\(\star\)) we have \(\tilde{z} \notin \tilde{x}\), and so \(\tilde{x} - \tilde{z} \notin \tilde{0}\). Now, we prove that there are maps \(h_j : L((l_j + 1)\tilde{x}_j + \tilde{o}) \rightarrow L(\tilde{o} + \tilde{z})\) for \(j \in I\) such that \(h \circ i = \sum_{j \in I} h_j \circ x_j^{l_j+1}\). Since \(z \geq 0\) and \(\tilde{x} - \tilde{z} \notin \tilde{0}\), after standard calculation in the group \(\mathbb{L}(p)\), we see that there is an index \(j_0 \in I\) such that \(\tilde{z} - (l_{j_0} + 1)\tilde{x}_{j_0} \geq 0\). Therefore, there is a map \(h_{j_0} : L((l_{j_0} + 1)\tilde{x}_{j_0} + \tilde{o}) \rightarrow L(\tilde{o} + \tilde{z})\) such that \(h \circ i = h_{j_0} \circ x_{j_0}^{l_{j_0}+1}\). Further, we define \(h_j = 0\) for \(j \neq j_0\).

Then we have

\[
\left( h - \sum_{j \in I} h_j \circ x_j^{l_j+1} \right) \circ i = h \circ i - \sum_{j \in I} h_j \circ (x_j^{l_j+1} \circ i) = h \circ i - \sum_{j \in I} h_j \circ x_j^{l_j+1} = 0,
\]

and we conclude that \(h - \sum_{j \in I} h_j \circ x_j^{l_j+1} \in \ker(- \circ i) = \text{Im}(- \circ \pi)\). Thus, there is a map \(g : L(\tilde{x}) \rightarrow L(\tilde{o} + \tilde{z})\) such that \(g \circ \pi = h - \sum_{j \in I} h_j \circ x_j^{l_j+1}\) and hence

\[
h = g \circ \pi + \sum_{j \in I} h_j \circ x_j^{l_j+1}.
\]

The \(\text{Hom}_X\)-orthogonality is easy to check. The minimality for the map \(j_E : E \rightarrow L(\tilde{x}) \oplus \bigoplus_{j \in I} L((l_j + 1)\tilde{x}_j + \tilde{o})\) follows then from the \(\text{Hom}_X\)-orthogonality of the line bundles \(L(\tilde{x})\) and \(L((l_j + 1)\tilde{x}_j + \tilde{o})\) for \(j \in I\).

\[\square\]

**Remark 4.1** In [10], it was shown that in the case of weight type \((2, a, b)\), the suspension functor \([1]\) in the stable vector bundle category \(\text{vect}(X)\) coincides with the shift functor by \(\tilde{x}_1\). Therefore, there is a short exact sequence

\[
0 \rightarrow \mathfrak{P}(E)(-\tilde{x}_1) \rightarrow E \rightarrow \mathfrak{P}(E) \rightarrow 0
\]

for each indecomposable bundle \(E\). Hence, in this case, we have \(\text{rk}\mathfrak{P}(E) = 2\text{rk}E\). From the theorem above we see that in the case of weights with \(t > 3\) there is an indecomposable, not exceptional rank two bundle such that \(\text{rk}\mathfrak{P}(E) > 4 = 2\text{rk}E\).

For example, in the case \((2, 2, 2, 3)\) the projective cover of an indecomposable
bundle of the data \((L, \bar{x}_4)\) has rank 5. Therefore, in the case \(2, p_2, p_3, \ldots, p_t\) and \(t > 3\) the suspension functor cannot be realized by a shift with an element from \(\mathbb{L}\).

In the same way as Lemma 3.1 and Lemma 3.2 the Sect. 3, we can prove the following two lemmata.

**Lemma 4.2** Let \(E\) be a non-zero vector bundle over \(\mathbb{X}\) of a type \((p_1, p_2, \ldots, p_t)\) with a projective cover \(\Psi(E)\). If \(\text{Ext}^1_{\mathbb{X}}(T, E) = 0\) for the canonical bundle \(T\), then there is a line bundle \(L\) in the decomposition \(\Psi(E)\) into a direct sum of line bundles such that \(\text{Ext}^1_{\mathbb{X}}(T, L) = 0\).

From Theorem 2, each extension bundle can be given by a line bundle \(L\) and an element \(\bar{x} = \sum_{i \in I} l_i \bar{x}_i + \sum_{i \notin I} (p_i - 1) \bar{x}_i \in \mathbb{L}\) for some \(I \subset \{1, 2, \ldots, t\}\) with \(#I = 3\).

**Lemma 4.3** Let \(\mathbb{X}\) be a weighted projective line of type \((p_1, p_2, \ldots, p_t)\) and let \(E\) be an extension bundle given by a pair \((L, \bar{x})\), where \(\bar{x} = \sum_{i \in I} l_i \bar{x}_i + \sum_{i \notin I} (p_i - 1) \bar{x}_i\). Denote by \(\hat{L}_i\) the direct summand \(L(\bar{x} - (1 + l_i) \bar{x}_i)\) of \(\Psi(E)\).

(i) There is an exact sequence

\[
0 \longrightarrow \hat{L}_i \longrightarrow E \longrightarrow L((1 + l_i) \bar{x}_j + \bar{\omega}) \longrightarrow 0,
\]

where \(L((1 + l_i) \bar{x}_j + \bar{\omega})\) is a direct summand of \(\mathfrak{Z}(E)\).

(ii) The extension bundle \(E\) can be determined by the following pairs:

\[
\left( L(\bar{x} - (1 + l_i) \bar{x}_i)(-\bar{\omega}), \quad 2\bar{\omega} + 2(1 + l_i) \bar{x}_i - \bar{x} \right) \quad \text{for} \quad i \in I.
\]

(iii) The element \(2\bar{\omega} + 2(1 + l_i) \bar{x}_i - \bar{x}\) in normal form has exactly \(t - 3\) coefficients equal to \(p_i - 1\) for \(i \notin I\).

**Proof** The statements (i) and (ii) are proved in the same way as in the case of three weights. The part (iii) follows from calculations in the group \(\mathbb{L}\) and is left to the reader.

The next result is a generalization of Theorem 1 and can be proved analogously using Lemma 4.2 and Lemma 4.3.

**Theorem 4** Let \(\Lambda = \Lambda(p, \lambda)\) be a canonical algebra with \(t\) arms. Then each exceptional \(\Lambda\)-module of rank two can be described by matrices having entries 0, \(\lambda_i\) and \(-\lambda_i\).

5 Exceptional cokernels

In this section, we will deal with cokernels of maps of the form
\[
[x^b_i]_{i \in I} : \mathcal{O}(\vec{y}) \to \bigoplus_{i \in I} \mathcal{O}(\vec{y} + b_i \vec{x}_i),
\]

where \( I \subset \{1, 2, \ldots, t \}, 0 < b_i < p_i - 1 \) for \( i \in I \) and \( \vec{y} \in \mathbb{L}_+ \). We will prove that such cokernels are exceptional modules. Moreover, every exceptional module of rank two, can be obtain in this way. Finally, by the cokernel construction, we compute matrices for each exceptional module of rank two.

**Lemma 5.1** Consider an exact sequence

\[
(\star) \quad 0 \longrightarrow F \xrightarrow{f} G \xrightarrow{\pi} E \longrightarrow 0
\]

in \( \text{coh}(\mathcal{X}) \) such that the following conditions are satisfied.

1. **(C1)** \( F \) is exceptional,
2. **(C2)** \( \text{Hom}_{\mathcal{X}}(G, F) = 0 = \text{Ext}^1_{\mathcal{X}}(G, F) \),
3. **(C3)** \( \text{Ext}^1(\mathcal{X}, G) = 0 \),
4. **(C4)** The map \( - \circ f : \text{End}_{\mathcal{X}}(G) \to \text{Hom}_{\mathcal{X}}(F, G) \) is an isomorphism.

Then the following properties holds:

1. **(i)** \( \text{Ext}^1_{\mathcal{X}}(E, F) \cong k \) and \( \text{Hom}_{\mathcal{X}}(E, G) = 0 = \text{Ext}^1_{\mathcal{X}}(E, G) \),
2. **(ii)** \( E \) is exceptional,
3. **(iii)** Up to an isomorphism \( E \) does not depend on the map \( f \).

**Proof** (i). Applying the functor \( \text{Hom}_{\mathcal{X}}(-, F) \) to the exact sequence \((\star)\) we obtain a long exact sequence

\[
\cdots \longrightarrow \text{Hom}_{\mathcal{X}}(G, F) \longrightarrow \text{End}_{\mathcal{X}}(F) \longrightarrow \text{Ext}^1_{\mathcal{X}}(E, F) \longrightarrow \text{Ext}^1_{\mathcal{X}}(G, F) \longrightarrow \cdots
\]

Therefore, \( \text{Ext}^1_{\mathcal{X}}(E, F) \cong \text{End}_{\mathcal{X}}(F) \cong k \).

Furthermore, applying the functor \( \text{Hom}_{\mathcal{X}}(-, G) \) to \((\star)\) we obtain a long exact sequence

\[
0 \longrightarrow \text{Hom}_{\mathcal{X}}(E, G) \xrightarrow{- \circ \pi} \text{End}_{\mathcal{X}}(G) \xrightarrow{- \circ f} \text{Hom}_{\mathcal{X}}(F, G) \xrightarrow{\delta} \text{Ext}^1_{\mathcal{X}}(E, G) \longrightarrow \text{Ext}^1_{\mathcal{X}}(G, G) \longrightarrow \text{Ext}^1_{\mathcal{X}}(F, G) \longrightarrow 0.
\]

Because \( - \circ f \) is an isomorphism, we get \( \text{Hom}_{\mathcal{X}}(E, G) = 0 \) and \( \text{Ext}^1_{\mathcal{X}}(E, G) = 0 \)

(ii). Applying the functor \( \text{Hom}_{\mathcal{X}}(E, -) \) to the exact sequence \((\star)\), we have a long exact sequence
We conclude that $\text{Ext}^1_{\mathcal{X}}(E, E) = 0$, and using (i) also that $\text{End}_{\mathcal{X}}(E) \cong \text{Ext}^1_{\mathcal{X}}(E, F) \cong k$.

(iii). Suppose that we have exact sequences

$$0 \longrightarrow F \xrightarrow{f} G \longrightarrow E \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F \xrightarrow{\widehat{f}} G \longrightarrow \widehat{E} \longrightarrow 0.$$ 

Then we have $[E] = [G] - [F] = [\widehat{E}]$ in the Grothendieck group $K_0(\mathcal{X})$. From (ii), we know that the sheaves $E$ and $\widehat{E}$ are exceptional. We infer from [13, Proposition 4.4.1] that $E \cong \widehat{E}$.

We will study the following cases of maps satisfying conditions (C1)–(C4) of the previous proposition.

a. A map $[x^b_1, \ldots, x^b_t]^T : \mathcal{O} \longrightarrow \bigoplus_{i=1}^t \mathcal{O}(b_i \tilde{x}_i)$, where $1 \leq b_i \leq p_i - 1$.

b. For $J \subseteq \{1, \ldots, t\}$ we consider $[x^b_i]_{i \in J} : \mathcal{O} \longrightarrow \bigoplus_{i \in J} \mathcal{O}(b_i \tilde{x}_i)$, where $1 \leq b_i \leq p_i - 1$ for $i = 1, \ldots, t$.

c. If $f : F \longrightarrow G$ satisfies the conditions C1–C4. then for each $\tilde{x} \in \mathbb{L}$ the map $f(\tilde{x}) : F(\tilde{x}) \longrightarrow G(\tilde{x})$ also satisfies these conditions.

Note that if $G$ is in mod$_+ (\Lambda)$ in the sequence (★), then $E$ is also in mod$_+ (\Lambda)$. Therefore, in the cases a and b above the cokernels are exceptional $\Lambda$-modules.

Let $\mathcal{X} = \mathcal{X}(p_1, p_2, \ldots, p_t)$ be a weighted projective line and let $b_i, i = 1, \ldots, t$ are natural number such that $1 \leq b_i \leq p_i - 1$. Denote $I = \{i : b_i < p_i - 1\}$ and assume that $\# I = 3$.

**Proposition 5.2** The cokernel $E$ of the exact sequence

$$0 \longrightarrow L^f = \left[\begin{array}{c} \vdots \\ \vdots \end{array}\right]_{i \in I} \bigoplus_{i \in I} L(b_i \tilde{x}_i) \xrightarrow{\pi} E \longrightarrow 0$$

is an extension bundle with data

$$\left(L(b_{i_0} \tilde{x}_{i_0} - \bar{\omega}), \quad \bar{\omega} + \sum_{i \in I} b_i \tilde{x}_i - 2b_{i_0} \tilde{x}_{i_0} \right) \quad \text{for each} \quad i_0 \in I.$$ 

Moreover, for each $i_0 \in I$ the line bundle $L(b_{i_0} \tilde{x}_{i_0})$ is a direct summand of the projective cover $\mathfrak{P}(E)$.

**Proof** From Lemma 5.1, we conclude

$$\text{Hom}_{\mathcal{X}}(E, L(b_{i_0} \tilde{x}_{i_0})) = 0 = \text{Ext}^1_{\mathcal{X}}(E, L(b_{i_0} \tilde{x}_{i_0})) \quad \text{for each} \quad i_0 \in I.$$ 

Applying the functor $\text{Hom}_{\mathcal{X}}(L(b_{i_0} \tilde{x}_{i_0}), -)$ to (★) we obtain a long exact sequence.
Proposition 5.3 Let $E = E_L(\vec{x})$ be an extension bundle with $\vec{x} = \sum_{i \in I} l_i \vec{x}_i + \sum_{j \not\in I} (p_j - 1) \vec{x}_j$ and $\#I = 3$. Then $E$ is the cokernel in the following exact sequence:
Then $x ightarrow L(\bar{x} - \bar{c}) \rightarrow \bigoplus_{i \in I} L(\bar{x} - (1 + l_i)\bar{x}_i) \rightarrow E \rightarrow 0$.

Moreover, the line bundles $L(\bar{x} - (1 + l_i)\bar{x}_i)$ are direct summands of $\Psi(E)$ and $L(\bar{x} - \bar{c})$ is direct summand of $\mathfrak{A}(E)(-\bar{c})$.

**Proof** Consider the map $[x_i^{p_i - h} - 1]_{i \in I} : L(\bar{x} - \bar{c}) \rightarrow \bigoplus_{i \in I} L(\bar{x} - (1 + l_i)\bar{x}_i)$.

This map is a monomorphism and from the Proposition 5.2 the cokernel of the map $[x_i^{p_i - h} - 1]_{i \in I}$ is the extension bundle with data $(L, \bar{x})$.

Hence coker $[x_i^{p_i - h} - 1]_{i \in I} \cong E_L(\bar{x})$. The second claim follows from the form of the projective cover and the injective hull of the extension bundle $E_L(\bar{x})$.

**Lemma 5.4** A line bundle $L$ is in mod$_+^+(\Lambda)$ if and only if $\det L \geq 0$.

**Proof** Assume first that $L$ is in mod$_+^+(\Lambda)$, i.e. $\operatorname{Ext}^1_{\mathfrak{X}}(T, L) = 0$. Then $\operatorname{Hom}_{\mathfrak{X}}(T, L) \neq 0$, so there is an element $\bar{x} \in \mathfrak{L}$ such that $0 \leq \bar{x} \leq \bar{c}$ and $\operatorname{Hom}_{\mathfrak{X}}(\mathcal{O}(\bar{x}), L) \neq 0$. Therefore, $\det L - \bar{x} \geq 0$, so $\det L = \det L - \bar{x} + \bar{x} \geq 0$.

Now, assume that $\det L = n\bar{c} + \sum_{i=1}^t a_i \bar{x}_i \geq 0$. Let $\bar{x} \in \mathfrak{L}$ satisfy that $0 \leq \bar{x} \leq \bar{c}$.

Then $\bar{x} + \bar{c} - \det L = \bar{x} + \sum_{i=1}^t (p_i - a_i - 1)\bar{x}_i - (n + 2)\bar{c} \neq 0$. Therefore, $\operatorname{Ext}^1_{\mathfrak{X}}(\mathcal{O}(\bar{x}), L) \cong D\operatorname{Hom}_{\mathfrak{X}}(L, \mathcal{O}(\bar{x} + \bar{c})) = 0$, and consequently the line bundle $L$ is in mod$_+^+(\Lambda)$.

**Proposition 5.5** Let $E$ be an extension bundle.

(i) For each direct summand $\hat{L}$ of $\mathfrak{A}(E)$ there is a short exact sequence

$$
\eta^{-}_L : 0 \rightarrow \hat{L}(-\bar{c}) \rightarrow \bigoplus_{i=1}^3 L_i \rightarrow E \rightarrow 0,
$$

where the $L_i$ are pairwise distinct direct summands of projective cover $\Psi(E)$.

(ii) If $E$ is a $\Lambda$-module from mod$_+^+(\Lambda)$, then for at least one direct summand $\hat{L}$ of $\mathfrak{A}(E)$ the line bundle $\hat{L}(-\bar{c})$ is in mod$_+^+(\Lambda)$ and $\eta^{-}_L$ is a sequence of $\Lambda$-modules.

**Proof** The statement (i) is a consequence of Lemma 4.3 and Proposition 5.3.

(ii) Since $E$ is an extension bundle there is an exact sequence

$$
0 \rightarrow L(\bar{c}) \rightarrow E \rightarrow L(\bar{x}) \rightarrow 0,
$$

of $\Lambda$-modules, where $\bar{x} = \sum_{i \in I} l_i \bar{x}_i + \sum_{j \in I} (p_j - 1)\bar{x}_j$ with $\# I = 3$ and $0 \leq l_i \leq p_i - 2$. Recall form Lemma 5.4 that $\det L(\bar{c}) \geq 0$ and $\det L(\bar{x}) \geq 0$.

The direct summands of the injective hull $\mathfrak{A}(E)$ are as follows:
\[ L(\vec{x}), \quad L \left( \vec{\omega} + (1 + l_i) \vec{x}_i \right) \text{ for } i \in I. \]

If the line bundle \( L(\vec{x} - \vec{c}) \) is in \( \text{mod}_+(\Lambda) \), we put \( \widehat{L} = L(\vec{x}) \) and the claim holds. Assume now that \( L(\vec{x} - \vec{c}) \) does not belong to \( \text{mod}_+(\Lambda) \), then \( \det L(\vec{x} - \vec{c}) \neq 0 \). We write \( \det L \) in normal form \( \det L = n\vec{c} + \sum_{i=1}^t a_i \vec{x}_i \) and define two numbers \( m \) and \( m_I \) as follows

\[ m := \# \left\{ i \mid i \notin I \land a_i > 0 \right\}, \quad m_I := \# \left\{ i \mid i \in I \lor a_i + l_i > p_i \right\}. \]

Then we can write \( \det L(\vec{x}) \) in normal form

\[
\det L(\vec{x}) = n\vec{c} + \sum_{i=1}^t a_i \vec{x}_i + \sum_{i \in I} l_i \vec{x}_i + \sum_{i \notin I} (p_i - 1) \vec{x}_i
\]

\[
= n\vec{c} + \sum_{i \in I} (a_i + l_i) \vec{x}_i + \sum_{i \notin I} (p_i - 1 + l_i) \vec{x}_i
\]

\[
= (n + m + m_I) \vec{c} + \sum_{i \in I} b_i \vec{x}_i + \sum_{i \notin I} c_i \vec{x}_i \geq 0,
\]

where

\[
b_i = \begin{cases} a_i + l_i - p_i & \text{if } a_i + l_i \geq p_i \\ a_i + l_i & \text{if } a_i + l_i < p_i \end{cases} \quad \text{and} \quad c_i = \begin{cases} p_i - 1 & \text{if } l_i = 0 \\ l_i - 1 & \text{if } l_i > 0 \end{cases}
\]

Since

\[
\det L(\vec{x} - \vec{c}) = (n + m + m_I - 1) \vec{c} + \sum_{i \in I} b_i \vec{x}_i + \sum_{i \notin I} c_i \vec{x}_i \neq 0,
\]

we have \( n + m + m_I = 0 \), hence \( (\star) \ n + m = -m_I \). Similarly, we compute the determinant for the line bundle \( L(\vec{\omega}) \). We have

\[
\det L(\vec{\omega}) = n\vec{c} + \sum_{i=1}^t a_i \vec{x}_i + (t - 2) \vec{c} - \sum_{i=1}^t \vec{x}_i = (n + t - 2) \vec{c} + \sum_{i=1}^t (a_i - 1) \vec{x}_i,
\]

where \( \sum_{i=1}^t (a_i - 1) \vec{x}_i = -(t - 3 - m) \vec{c} + \sum_{i \in I} (a_i - 1) \vec{x}_i + \sum_{i \notin I} \vec{x}_i \). We denote by \( d_i \) the number \( p_i - 1 \) if \( a_i = 0 \) or \( a_i - 1 \) if \( a_i > 0 \). Then

\[
\det L(\vec{\omega}) = (n + m + 1) \vec{c} + \sum_{i \in I} (a_i - 1) \vec{x}_i + \sum_{i \notin I} d_i \vec{x}_i
\]

\[
= (1 - m_I) \vec{c} + \sum_{i \in I} (a_i - 1) \vec{x}_i + \sum_{i \notin I} d_i \vec{x}_i \geq 0.
\]

Therefore \( m_I \) is equal to 0 or 1. Moreover if \( m_I = 1 \) then \( a_i > 0 \) for all \( i \in I \), and if \( m_I = 0 \), then at most one of the numbers \( a_i \) for \( i \in I \) is 0.

In the case \( m_I = 1 \) there is an index \( i_0 \in I \) such that \( a_{i_0} + l_{i_0} \geq p_{i_0} \). Then
\[
\det L\left( \tilde{o} + (1 + l_{i_0})\tilde{x}_{i_0} - \tilde{c} \right) = -\tilde{c} + \sum_{i \in I} (a_i - 1)\tilde{x}_i + (1 + l_{i_0})\tilde{x}_{i_0} + \sum_{i \in I} d_i\tilde{x}_i \\
= (a_{i_0} + l_{i_0} - p_{i_0})\tilde{x}_{i_0} + \sum_{i \in I, i \neq i_0} (a_i - 1)\tilde{x}_i + \sum_{i \in I} d_i\tilde{x}_i \geq 0.
\]

Therefore, \( L\left( \tilde{o} + (1 + l_{i_0})\tilde{x}_{i_0} - \tilde{c} \right) \) is in \( \text{mod}_+ (\Lambda) \) and we put \( \widehat{L} = L\left( \tilde{o} + (1 + l_{i_0})\tilde{x}_{i_0} \right) \).

In the case that \( m_I = 0 \) if \( a_i > 0 \) for all \( i \in I \) then each line bundle \( L\left( \tilde{o} + (1 + l_i)\tilde{x}_i - \tilde{c} \right) \) is in \( \text{mod}_+ (\Lambda) \) and each of those line bundles gives us the claim. If \( a_{i_0} = 0 \) for some \( i_0 \in I \), then only \( L\left( \tilde{o} + (1 + l_{i_0})\tilde{x}_{i_0} - \tilde{c} \right) \) is in \( \text{mod}_+ (\Lambda) \) and we put \( \widehat{L} = L\left( \tilde{o} + (1 + l_{i_0})\tilde{x}_{i_0} \right) \).

\[ \square \]

As a conclusion of the previous proposition, we obtain an improvement of Lemma 3.1.

**Corollary 1** If an extension bundle \( E \) is a \( \Lambda \)-module, then at least three direct summands of \( \Psi (E) \) are also \( \Lambda \)-modules. \[ \square \]

Recall that we work with a map of the form
\[
f_{y}^{b_1, b_2, b_3} = [x_i^b]_{i \in I} : \mathcal{O}(\tilde{y}) \longrightarrow \bigoplus_{i \in I} \mathcal{O}(\tilde{y} + b_i\tilde{x}_i),
\]
where \( I = \{i_1, i_2, i_3 \mid i_1 < i_2 < i_3 \} \) is a subset of \( \{1, 2, \ldots, t \} \), \( 0 < b_i < p_i - 1 \) for \( i \in I \) and \( \tilde{y} \in \mathbb{L}_+ \). If we write the element \( \tilde{y} \) in normal form
\[
\tilde{y} = n\tilde{c} + \sum_{i=1}^t a_i\tilde{x}_i, \quad n \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad 0 \leq a_i \leq p_i - 1 \quad \text{for} \quad i = 1, 2, \ldots, t,
\]
then we distinguish the following 8 cases:

**A** \( a_j + b_j < p_j \) for all \( j \in I \),

**B**\( ^1 \) \( a_i_1 + b_i_1 \geq p_i_1 \) and \( a_j + b_j < p_j \) for \( j \in I - \{i_1\} \),

**B**\( _2 \) \( a_i_2 + b_i_2 \geq p_i_2 \) and \( a_j + b_j < p_j \) for \( j \in I - \{i_2\} \),

**B**\( _3 \) \( a_i_3 + b_i_3 \geq p_i_3 \) and \( a_j + b_j < p_j \) for \( j \in I - \{i_3\} \),

**C**\( _1 \) \( a_i_1 + b_i_1 < p_i_1 \) and \( a_j + b_j \geq p_j \) for \( j \in I - \{i_1\} \),

**C**\( _2 \) \( a_i_2 + b_i_2 < p_i_2 \) and \( a_j + b_j \geq p_j \) for \( j \in I - \{i_2\} \),

**C**\( _3 \) \( a_i_3 + b_i_3 < p_i_3 \) and \( a_j + b_j \geq p_j \) for \( j \in I - \{i_3\} \),

**D** \( a_i + b_i \geq p_i \) for all \( i \in I \).

In the following lemma, we prove that it is sufficient to study the cases **A** or **B**\( _3 \).

**Lemma 5.6** Each extension module can be obtained as a cokernel of the map \( f_{y}^{b_1, b_2, b_3} \) in the case **A** or **B**\( _3 \).
Proof We proof that the cokernels in the cases $B_1, B_2$ and $D$ are isomorphic to cokernels of the case $B_3$. Moreover, the cokernels in the cases $C_1, C_2$ and $C_3$ are isomorphic to cokernels of the case $A$.

Let $E$ be an extension module in the case $B_2$, thus $E$ is the cokernel of a map $f_{p_i}^{h_i}$, where $a_i + b_i < p_i$, $a_i + b_i = p_i$, $a_i + b_i < p_i$.

From Lemma 5.2, applied to $i_1$, we infer that $E$ is an extension bundle $E_L(\tilde{x})$ with data $(L, \tilde{x})$ such that

$$
\det L = \tilde{y} + b_i \tilde{x}_i - \tilde{c} = n\tilde{c} + (a_i + b_i)\tilde{x}_i + \sum_{j \neq i} a_j \tilde{x}_j - \tilde{c},
$$

$$
\tilde{x} = \tilde{c} + b_i \tilde{x}_i + b_i \tilde{x}_i - b_i \tilde{x}_i,
$$

Consider the map

$$
f_{z}^{d_1, d_2, d_3} = \begin{bmatrix}
\begin{pmatrix}
d_1 \\
x_{i_1}
\end{pmatrix} \\
\begin{pmatrix}
d_2 \\
x_{i_2}
\end{pmatrix} \\
\begin{pmatrix}
d_3 \\
x_{i_3}
\end{pmatrix}
\end{bmatrix} : \mathcal{O}(z) \rightarrow \bigoplus_{i=1}^{3} \mathcal{O}(z + b_i \tilde{x}_i),
$$

where

$$d_1 = b_i, \quad d_2 = p_i - b_i, \quad d_3 = p_i - b_i$$

and

$$\tilde{z} = \tilde{y} + b_i \tilde{x}_i + b_i \tilde{x}_i - \tilde{c} = n\tilde{c} + (a_i + b_i)\tilde{x}_i + (a_i + b_i)\tilde{x}_i + \sum_{j \neq i} a_j \tilde{x}_j.$$

Then from Lemma 5.2, applied to $i_2$, we conclude that the cokernel of the map $f_{z}^{d_1, d_2, d_3}$ is an extension bundle with data $(\tilde{L}, \tilde{x})$, such that

$$
\det \tilde{L} = \tilde{y} + d_i \tilde{x}_i - \tilde{c} = n\tilde{c} + (a_i + b_i)\tilde{x}_i + \sum_{j \neq i} a_j \tilde{x}_j - \tilde{c},
$$

$$
\tilde{x} = \tilde{c} + d_i \tilde{x}_i + d_i \tilde{x}_i + d_i \tilde{x}_i = \tilde{c} + b_i \tilde{x}_i - (p_i - b_i)\tilde{x}_i + (p_i - b_i)\tilde{x}_i
$$

$$= \tilde{c} + b_i \tilde{x}_i + b_i \tilde{x}_i - b_i \tilde{x}_i.
$$

Therefore, the cokernels of the maps $f_{p_i}^{h_i}$ and $f_{z}^{d_1, d_2, d_3}$ are isomorphic. Furthermore, we have that

$$d_i + a_i = a_i + b_i < p_i, \quad d_i + (a_i + b_i - p_i) = a_i < p_i,$$

$$d_i + (a_i + b_i) = a_i + p_i \geq p_i,$$

hence, $E$ is isomorphic to an extension module in the case of $B_3$.

For the other cases, we use the same kind of arguments. We only put in a table the
choice of \( d_i, d_{i_2}, d_{i_3} \) and \( z \). For simplicity, in the case \( a_i + b_i \geq p_i \), we denote \( a_i + b_i - p_i \) by \( c_i \).

| Case | \( d_{i_1} \) | \( d_{i_2} \) | \( d_{i_3} \) | \( z \) |
|------|----------------|----------------|----------------|----------------|
| \( B_1 \) | \( p_{i_1} - b_{i_1} \) | \( b_{i_2} \) | \( p_{i_3} - b_{i_3} \) | \( n\tilde{c} + c_1 \tilde{x}_{i_1} + (a_{i_3} + b_{i_3})\tilde{x}_{i_3} + \sum_{j \neq i_1, i_3} a_j \tilde{x}_j \) |
| \( C_1 \) | \( b_{i_1} \) | \( p_{i_2} - b_{i_2} \) | \( p_{i_3} - b_{i_3} \) | \( (n + 1)\tilde{c} + c_1 \tilde{x}_{i_1} + c_1 \tilde{x}_{i_3} + \sum_{j \neq i_2, i_3} a_j \tilde{x}_j \) |
| \( C_2 \) | \( p_{i_1} - b_{i_1} \) | \( b_{i_2} \) | \( p_{i_3} - b_{i_3} \) | \( (n + 1)\tilde{c} + c_1 \tilde{x}_{i_1} + c_1 \tilde{x}_{i_3} + \sum_{j \neq i_1, i_3} a_j \tilde{x}_j \) |
| \( C_3 \) | \( p_{i_1} - b_{i_1} \) | \( p_{i_2} - b_{i_2} \) | \( b_{i_3} \) | \( (n + 1)\tilde{c} + c_1 \tilde{x}_{i_1} + c_1 \tilde{x}_{i_3} + \sum_{j \neq i_1, i_3} a_j \tilde{x}_j \) |
| \( D \) | \( p_{i_1} - b_{i_1} \) | \( b_{i_2} \) | \( p_{i_3} - b_{i_3} \) | \( (n + 1)\tilde{c} + c_1 \tilde{x}_{i_1} + c_1 \tilde{x}_{i_3} + \sum_{j \neq i_1, i_3} a_j \tilde{x}_j \) |

\[ \square \]

### 5.1 The cokernel construction

We consider an exact sequence of \( \Lambda \)-modules

\[ 0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0, \]

and we assume that representations \( L = \left( \{ L_x \}_{x \in Q_0}, \{ L_x \}_{x \in Q_1} \right) \), \( M = \left( \{ M_x \}_{x \in Q_0}, \{ M_x \}_{x \in Q_1} \right) \) by vector spaces and matrices and also the morphism \( f = (\tilde{f}_x)_{x \in Q_0} \) are known. Here \( Q_0 \) and \( Q_1 \) denote the set of vertices and arrows of the quiver of the canonical algebra, respectively.

We will construct a representation \( \left( \{ N_x \}_{x \in Q_0}, \{ N_x \}_{x \in Q_1} \right) \) for \( N \). The vector space \( N_{\tilde{x}} \) is the cokernel of the linear map \( f_{\tilde{x}} \), and \( g_{\tilde{x}} \) the reduction modulo \( \text{Im} f_{\tilde{x}} \). Let \( v_1, \ldots, v_m \) be a basis of \( M_{\tilde{x}} \) and let \( \dim_k \text{Im} f_{\tilde{x}} = l \). We have that \( M_{\tilde{x}} = \text{Im} f_{\tilde{x}} \oplus k v_1 \oplus \cdots \oplus k v_{m-l} \) and that the set \( v_1 + \text{Im} f_{\tilde{x}}, \ldots, v_{m-l} + \text{Im} f_{\tilde{x}} \) is a basis of the linear space \( N_{\tilde{x}} = M_{\tilde{x}}/\text{Im} f_{\tilde{x}} \). Moreover, for \( j \in \{ 1, \ldots, l \} \), we have \( v_{m-l+j} = f_{\tilde{x}}(w_j) + a_1 j v_1 + \cdots + a_{m-l} j v_{m-l} \) for some \( a_{ij} \in k \) i \( w_j \in L_{\tilde{x}} \). Then

\[ v_i \xrightarrow{g_{\tilde{x}}} v_i + \text{Im} f_{\tilde{x}}, \quad \text{for} \quad i = 1, \ldots, m-l \]

\[ v_{m-l+j} \xrightarrow{g_{\tilde{x}}} \sum_{i=1}^{m-n} a_{ij} v_i + \text{Im} f_{\tilde{x}}, \quad \text{for} \quad i = 1, \ldots, m-l. \]

Therefore, \( g_{\tilde{x}} = \begin{pmatrix} I_{m-l} & A \end{pmatrix} \), where \( A = [a_{ij}] \in M_{m-l,k} \).

Next we will determine the maps \( N_x \) for \( x \in Q_1 \). Let \( x : \tilde{x} \rightarrow \tilde{y} \) be an arrow of the quiver of the algebra \( \Lambda \). Then the following diagram
can be uniquely completed to a commutative diagram by the map \( N_a : N_{\overline{y}} \to N_{\overline{z}}, v + \text{Im} f_{\overline{y}} \mapsto M_{\overline{z}}(v) + \text{Im} f_{\overline{z}}. \)

It is easily checked that the maps \( N_a \) satisfy the canonical relations.

### 5.2 Construction of modules of type \( \Lambda \)

Let \( I \subset \{1, 2, \ldots, t\} \) with \( \#I = 3 \), and let \( b = (b_i)_{i \in I} \) with \( 1 \leq b_i \leq p_i - 1 \). For \( I \) and \( b \), we consider an exact sequence of vector bundles

\[
0 \longrightarrow L \longrightarrow \bigoplus_{i \in I} L(b_i \overline{x}_i) \longrightarrow \text{coker} \left[ x_i^{b_i} \right]_{i \in I} =: E \longrightarrow 0,
\]

where \( L \) is a line bundle, with \( \det L = n \overline{c} + \sum_{i \in I} a_i \overline{x}_i \geq 0 \) and \( a_i + b_i < p_i \) for \( i \in I \).

We denote by \( I_n \) the identity matrix of size \( n \) and by \( X_{n+m \times n}, Y_{n+m \times n}, Z_n(\lambda) \) the following matrices

\[
X_{n+m \times n} := \left[ \begin{array}{c} I_n \\ 0 \end{array} \right] \in M_{n+m \times n}(k), \quad Y_{n+m \times n} := \left[ \begin{array}{c} 0 \\ I_n \end{array} \right] \in M_{n+m \times n}(k),
\]

\[
Z_n(\lambda) := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \in M_n(k).
\]

The \( \Lambda \)-module attached to the line bundle \( L \) has the following shape:

\[
L : \quad \begin{array}{cccccccccc}
    & k^{n+1} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n+1} & \overset{L_{\alpha_{1}^{(i)}}}{\longrightarrow} & X_{n+1 \times n} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n} \\
    & 1 & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & 1 & \overset{L_{\alpha_{2}^{(i)}}}{\longrightarrow} & Y_{n+1 \times n} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & 1 \\
    k^{n+1} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n+1} & \overset{L_{\alpha_{3}^{(i)}}}{\longrightarrow} & X_{n+1 \times n} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n} \\
    k^{n+1} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n+1} & \overset{L_{\alpha_{4}^{(i)}}}{\longrightarrow} & Y_{n+1 \times n} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & 1 \\
    & k^{n+1} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & 1 & \overset{L_{\alpha_{5}^{(i)}}}{\longrightarrow} & X_{n+1 \times n} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n} \\
    Z_{n+1}(\lambda_i) & k^{n+1} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n+1} & \overset{L_{\alpha_{6}^{(i)}}}{\longrightarrow} & X_{n+1 \times n} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n} \\
    Z_{n+1}(\lambda_i) & k^{n+1} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n+1} & \overset{L_{\alpha_{7}^{(i)}}}{\longrightarrow} & X_{n+1 \times n} & \overset{1}{\longrightarrow} & \cdots & \overset{1}{\longrightarrow} & k^{n}
\end{array}
\]

where \( \mathbb{I} \) is the identity map (see [14, Proposition 3.4]). If \( a_i = 0 \) for some \( i \geq 3 \), then

\[
L_{x_{1}^{(i)}} = L_{x_{2}^{(i)}} = Z_{n+1}(\lambda_i) \cdot X_{n+1 \times n}.
\]
The modules $L(b_i x_i)$ for $i \in I$ have a similar shape, with the difference that in the $i$th arm, the jump of dimension is realized for the arrow $z_i^{(i)}_{a_i + b_i + 1}$.

First we compute matrices of maps $x_i^h : L \rightarrow L(b_i x_i)$ for each $i \in I$. The map $x_i^h$ has the following matrices:

$$
\begin{pmatrix}
\ldots & a_1 \bar{z}_i & \langle a_1 + 1 \rangle \bar{z}_i \\
\ldots & I_{n+1} & I_n \\
\ldots & a_i \bar{z}_i & \langle a_i + 1 \rangle \bar{z}_i \\
\ldots & X_{n+1} & I_n \\
\ldots & a_j \bar{z}_j & \langle a_j + 1 \rangle \bar{z}_j \\
\ldots & I_{n+1} & I_n \\
\ldots & \ldots & \ldots & X_{n+1} & I_n \\
I_{n+1} & \ldots & \ldots & I_n
\end{pmatrix},
$$

for some $\mu_i \in k$, where $i \neq 2$ and $j \neq i$. Here the captions above the frames mean the vertices of the quiver and the matrices in the frames the matrices for them. Moreover, in the case $i = 2$ we need to switch from $X_{n+1} \otimes M$ to $Y_{n+1} \otimes M$. Therefore the map $[x_i^h]_{i \in I}$ depends on the three scalars $\mu_i$, $\mu_j$ and $\mu_k$. From Lemma 5.1 (iii), we can put $\mu_i = 1 = \mu_j$ and $\mu_k = -1$.

The second step is the computation of the map $g : \bigoplus_{i \in I} L(b_i x_i) \rightarrow E$. For this purpose we will use the following lemma from linear algebra, where for simplicity, we will use notation $B_{b \times a}$ for a matrix $B \in M_{b \times a}(k)$.

**Lemma 5.7** Let $0 \rightarrow V \xrightarrow{f} W \xrightarrow{-coker(f)} \rightarrow 0$ be an exact sequence of linear maps, where $\dim V = a$, $\dim W = a + b + c$ and $-$ is the reduction modulo $\text{Im}(f)$.

1. If $f$ has a block matrix form

$$
\begin{pmatrix}
B_{b \times a} \\
C_{c \times a} \\
-I_a
\end{pmatrix},
$$

then the reduction map $-coker(f)$ has a block matrix form

$$
\begin{pmatrix}
I_b & 0 & B_{b \times a} \\
0 & I_c & C_{c \times a}
\end{pmatrix}.
$$

2. If $f$ has a block matrix form

$$
\begin{pmatrix}
I_a \\
B_{b \times a} \\
-C_{c \times a}
\end{pmatrix},
$$

then the reduction map $-coker(f)$ has a block matrix form

$$
\begin{pmatrix}
-I_b & 0 \\
-C_{c \times a} & I_b
\end{pmatrix}.
$$
Proof (i) Let $v_1, \ldots, v_a$ be a basis of $V$ and let $w_1^1, \ldots, w_b^1, w_1^2, \ldots, w_c^2, w_1^3, \ldots, w_d^3$ be a basis of $W$. Furthermore, we choose $w_1^1, \ldots, w_b^1, w_1^2, \ldots, w_c^2$ as a basis of $\text{coker}(f)$. Then the equalities

$$w_i^3 = w_i^3 + f(w_i^3) = Bw_i^3 + Cw_i^3, \quad \text{for} \quad i = 1, 2, \ldots, a$$

implies the claim.

(ii) We choose $v_1, \ldots, v_a$ as a basis of $V$ and $w_1^1, \ldots, w_a^1, w_1^2, \ldots, w_b^2, w_1^3, \ldots, w_c^3$ as a basis of $W$. Further, we choose $w_1^1, \ldots, w_b^1, w_1^2, \ldots, w_c^2$ as a basis of $\text{coker}(f)$. Then

$$w_i^1 = w_i^1 - f(w_i^1) = -Bw_i^1 + Cw_i^1, \quad \text{for} \quad i = 1, 2, \ldots, a$$

so the claim holds.

Using the lemma above, we obtain a matrix representation of the map $g = (g_i) : \bigoplus_{i \in I} L(b_i \tilde{x}_i) \to E$. Recall that $I = \{i_1, i_2, i_3\}$ is given in ascending order. Then

$$g_i = g_{i,1} = \cdots = g_{a_i, a_i} \tilde{x}_i = \begin{bmatrix} I_{n+1} & 0 & I_{n+1} \\ 0 & I_{n+1} & I_{n+1} \end{bmatrix}$$

for $j = 1, 2, \ldots, t$

$$g(a_j+1)\tilde{x}_j = \cdots = g_{a_j} \tilde{x}_j = \begin{bmatrix} I_n & 0 & I_n \\ 0 & I_n & I_n \end{bmatrix}$$

for $j \notin I$

$$g(a_i+b_i+1)\tilde{x}_i = \cdots = g_{a_i+b_i} \tilde{x}_i = \begin{bmatrix} I_n & 0 & I_n \\ 0 & I_n & I_n \end{bmatrix}$$

for $i \in I$

$$g(a_{i_1}+1)\tilde{x}_{i_1} = \cdots = g(a_{i_1+b_{i_1}}) \tilde{x}_{i_1} = \begin{bmatrix} I_{n+1} & 0 & X_{n+1 \times n} \\ 0 & I_n & I_n \end{bmatrix}$$

for $i_1 \neq 2$

$$g(a_{j_1}+1)\tilde{x}_j = \cdots = g(a_{j+b_{j}}) \tilde{x}_j = \begin{bmatrix} I_n & 0 & I_{n+1} \\ 0 & I_n & X_{n+1 \times n} \end{bmatrix}$$

for $i_2 \neq 2$

$$g(a_{i_3}+1)\tilde{x}_{i_3} = \cdots = g(a_{i_3+b_{i_3}}) \tilde{x}_{i_3} = \begin{bmatrix} -I_n & I_n & 0 \\ X_{n+1 \times n} & 0 & I_{n+1} \end{bmatrix}$$

Note that in the case $i_1 = 2$ or $i_2 = 2$ we need to switch from $X_{n+1 \times n}$, to $Y_{n+1 \times n}$ in the above block matrices. From this, we obtain
Proposition 5.8  The module of type A has the following dimensional vector

\[
\begin{pmatrix}
& 
& a_i x_i & (a_i+1)x_i & (a_i+b_i)x_i & (a_i+b_i+1)x_i \\
\cdots & 2n+2 & 2n+1 & 2n+1 & 2n & \cdots \\
2n+2 & \cdots & 2n & \cdots & \varepsilon \\
\cdots & \frac{a_j x_j}{2n+2} & \frac{(a_j+1)x_j}{2n} & \frac{(a_j+b_j)x_j}{2n} & \cdots \\
\cdots & & & & \\
\end{pmatrix}
\]

where \( i \in I \) and \( j \not\in I \). Here the captions above the frames denote the vertices of the quiver and the numbers in the frames the dimensions of the vector spaces for them.

Finally we compute the matrices of the the module \( E \), by completing the following square:

\[
\begin{array}{ccc}
G_{\beta} & G_{\alpha}=1 & G_{\beta} \\
g_{\beta} & g_{\beta} & \\
E_{\beta} & E_{\beta} &
\end{array}
\]

to a commutative diagram, for each arrow \( \alpha : x \longrightarrow y \). Since the maps \( G_\beta \) are monomorphism, each of these squares can be complete only in one way. If for some arrow \( \alpha : x \longrightarrow y \), the maps \( L(b_i x_i)_x \) are identities for each \( i \in I \) and \( g_\beta = g_y \), then \( E_\alpha \) is the identity map. Therefore, we need only to determine matrices for the following arrows:

\[
\begin{align*}
\chi_{i,j}^{(l)} & \quad \text{for } j = 3, 4, \ldots, t \\
\chi_{a_j+1}^{(l)} & \quad \text{for } j = 1, 2, \ldots, t \\
\chi_{a_i+b_i+1}^{(l)} & \quad \text{for } i \in I.
\end{align*}
\]

In the case of the arrows of the \( j \)th arm, with \( j \neq i_3 \) (and arrow \( \chi_{i_3}^{(l)} \)), we deal with the following commutative diagram
Then from the commutativity of the diagram above we get $E_\alpha = \begin{pmatrix} D_{b_1 \times b_2} & 0 \\ 0 & D_{c_1 \times c_2} \end{pmatrix}$.

In the case of arrow $\alpha_{a_1+1}$ we deal with a commutative diagram of the form

Then $E_\alpha = \begin{pmatrix} 0 & B_{b_1 \times a_1} \\ D_{c_1 \times c_2} & B_{c_1 \times a_1} \end{pmatrix}$.

In the case of the arrow $\alpha_{a_1+b_1+1}$ we deal with a commutative diagram of the form

Then $E_\alpha = \begin{pmatrix} -B_{c_1 \times b_1} \cdot D_{b_1 \times b_2} & D_{c_1 \times c_2} \\ B_{a_1 \times b_1} \cdot B_{b_1 \times b_2} & 0 \end{pmatrix}$.
Theorem 5  The extension module of Type A can be established by the following vector spaces and matrices.

\[
\begin{array}{cccccc}
& E_{a_1(1)} & \cdots & E_{a_1(i)} & \cdots & E_{a_1(3)} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
E_{a_1(1)} & \cdots & E_{a_1(i)} & \cdots & E_{a_1(3)} & \vdots \\
\end{array}
\]

where

\[
\begin{align*}
E_{a_1(1)} & = E_{a_1(2)} = I, \\
E_{a_1(i+1)} & = E_{a_1(i+2)} = \begin{pmatrix}
I_{n+1} & 0 \\
0 & X_{n+1 \times n}
\end{pmatrix}, \\
E_{a_3(i_3)} & = \begin{pmatrix}
0 & I_{n+1} \\
X_{n+1 \times n} & I_{n+1}
\end{pmatrix}, \\
E_{a_3(i_3)} & = \begin{pmatrix}
X_{n+1 \times n} & 0 \\
0 & X_{n+1 \times n}
\end{pmatrix}
\end{align*}
\]

for \( j > 2 \) and \( j \notin \{i_1, i_2, i_3\} \).

In the case of the second arm we need to switch from the matrices \( X_{s \times s} \) to \( Y_{s \times s} \). Moreover, if \( a_i = 0 \), then the arrow \( \alpha_1(i) \) coincides with the arrow \( \alpha_{a_i+1} \), and in this case in the place of \( \alpha_1(i) \) we put the composition of the above matrices \( E_{\alpha_1(i)} \) and \( E_{\alpha_{a_i+1}} \).

5.3 The modules of the type B_3

This case is similar to the previous computation. We will point out the differences using the same notations as before. In the case of type B_3, we assume that

\[ a_{i_1} + b_{i_3} \geq p_{i_1} \quad \text{and} \quad a_i + b_i < p_i \quad \text{for} \quad i = i_1, i_2. \]

Let \( c_{i_3} := a_{i_1} + b_{i_3} - p_{i_1} \), then \( 0 \leq c_{i_3} < a_{i_1} \).

The maps \( x^{b_i}_{i_3} : \mathcal{O}(\bar{y}) \to \mathcal{O}(\bar{y} + b_{i_3} \bar{x}_i) \) for \( i = i_1 \) or \( i = i_2 \) are the same as before.

The map \( x^{b_i}_{i_3} : \mathcal{O}(\bar{y}) \to \mathcal{O}(\bar{y} + b_{i_3} \bar{x}_i) \) has the form \( \mu_{i_3} : (h_x)_{0 \leq \bar{x} \leq \bar{c}} \), where for \( j \neq i_3 \) we have

\[
\begin{align*}
h_0 &= h_{\bar{x}_j} = \cdots = h_{a_j \bar{x}_j} = Z_{n+2 \times n+1}(-\lambda_{i_3}) \\
h_{(a_j+1) \bar{x}_j} &= \cdots = h_{\bar{c}} = Z_{n+1 \times n}(-\lambda_{i_3})
\end{align*}
\]

and for \( i_3 \) holds
\[ h_0 = h_{\alpha_1} = \cdots = h_{\alpha_{i_2}} = Z_{n+2 \times n+1}(-\lambda_{i_2}) \]
\[ h_{(\alpha_{i_2}+1)\alpha_1} = \cdots = h_{\alpha_{i_3}} = Z_{n+1}(-\lambda_{i_2}) \]
\[ h_{(\alpha_{i_3}+1)\alpha_1} = \cdots = h_{\alpha_i} = Z_{n+1 \times n}(-\lambda_{i_3}). \]

Then the map \( g = (g_{\alpha_1}x)_{0 \leq \alpha_1 \leq \alpha_i} : G \rightarrow E \) has the following shape:

\[
g_{\alpha_1} = g_{\alpha_2} = \cdots = g_{\alpha_2} = \begin{bmatrix}
-I_{n+1} & I_{n+1} & 0 \\
0 & I_{n+1} & I_{n+1}
\end{bmatrix}
\text{ for } j \neq i_3
\]

\[
g_{(\alpha_{i_1}+1)x} = \cdots = g_{(\alpha_{i_1}+\beta_1)x} = \begin{bmatrix}
-I_{n+1} & -X_{n+1 \times n} & 0 \\
0 & Z_{n+1 \times n}(\lambda_{i_2}) & I_{n+1}
\end{bmatrix}
\text{ for } i_1 \neq 2
\]

\[
g_{(\alpha_{i_2}+1)x} = \cdots = g_{(\alpha_{i_2}+\beta_2)x} = \begin{bmatrix}
-I_{n+1} & -X_{n+1 \times n} & 0 \\
0 & Z_{n+1 \times n}(\lambda_{i_3}) & I_{n+1}
\end{bmatrix}
\text{ for } i_2 \neq 2
\]

\[
g_{\alpha_1} = g_{\alpha_2} = \cdots = g_{\alpha_{i_3}} = \begin{bmatrix}
-I_{n+1} & I_{n+1} & 0 \\
0 & I_{n+1} & I_{n+1}
\end{bmatrix}
\text{ for } \alpha_1 \in \{i_1, i_2\}
\]

\[
g_{(\alpha_{i_2}+1)x} = \cdots = g_{(\alpha_{i_2}+\beta_2)x} = \begin{bmatrix}
-I_{n+1} & I_{n+1} & 0 \\
0 & I_{n+1} & I_{n+1}
\end{bmatrix}
\text{ for } \alpha_2 \in \{i_1, i_2\}
\]

\[
g_{(\alpha_{i_3}+1)x} = \cdots = g_{(\alpha_{i_3}+\beta_3)x} = \begin{bmatrix}
-I_{n+1} & I_{n+1} & 0 \\
0 & I_{n+1} & I_{n+1}
\end{bmatrix}
\text{ for } \alpha_3 \in \{i_1, i_2\}
\]

Lets remark, that if \( i_k = 2 \) for same \( k = 1, 2 \) or 3, then we need to switch matrices \( X_{*,*} \) to \( Y_{*,*} \) in \( i_k \)-arm.

**Proposition 5.9** The module of type \( B_3 \) has the following dimension vector:

\[
\begin{pmatrix}
\begin{array}{cccccccccc}
\alpha_{i_1}x & \alpha_{i_2}x & \alpha_{i_3}x & (\alpha_{i_1}+1)x & (\alpha_{i_2}+\beta_1)x & (\alpha_{i_2}+\beta_2)x & (\alpha_{i_2}+\beta_3)x & (\alpha_{i_3}+1)x & (\alpha_{i_3}+\beta_3)x & (p_{i_3}-1)x \\
2n+3 & 2n+3 & 2n+3 & 2n+3 & 2n+2 & 2n+2 & 2n+2 & 2n+2 & 2n+1 & 2n+1
\end{array}
\end{pmatrix}
\]

where \( i \in \{i_1, i_2\} \) and \( j \notin I \). Here the captions above the frames denote the vertices of the quiver and the numbers in the frames the dimensions of the vector spaces for them.

Proceeding as in the case before we get representations for a module of type \( B_3 \).

**Theorem 6** The extension module \( E \) of type \( B_3 \) can be exhibited by the following vector spaces and matrices.
for $j \notin \{i_1, i_2, i_3\}$, where

$$E^{(j)}_{\alpha_{j}+1} = \begin{pmatrix} Z_{n+1}(\lambda_j) & 0 \\ 0 & Z_{n+2}(\lambda_j) \end{pmatrix} \quad \text{for } j > 2$$

$$E^{(1)}_{\alpha_1} = E^{(2)}_{\alpha_1} = \mathbb{1},$$

$$E^{(i_1)}_{\alpha_{i_1+1}} = \begin{pmatrix} -I_{n+1} \\ Z_{n+2}(\lambda_{i_1}) \end{pmatrix} \begin{pmatrix} X_{n+2n+1} \\ 0 \end{pmatrix}$$

$$E^{(i_2)}_{\alpha_{i_2+1}} = \begin{pmatrix} I_{n+1} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ X_{n+2n+1} \end{pmatrix}$$

$$E^{(i_3)}_{\alpha_{i_3+1}} = \begin{pmatrix} I_{n+1} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ X_{n+2n+1} \end{pmatrix}$$

$$E^{(j)}_{\alpha_{j+1}} = \begin{pmatrix} X_{n+1n} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ X_{n+2n+1} \end{pmatrix} \quad \text{for } j \notin \{i_1, i_2, i_3\}.$$  

In the case of the second arm, we need to switch from the matrices $X_{s \times s}$ to $Y_{s \times s}$. Moreover, if $a_i = 0$, then the arrow $z^{(i)}_{a_{i}+1}$ coincide with the arrow $z^{(i)}_{a_{i}+1}$ in this case in the place of $z^{(i)}_{a_{i}+1}$ we put composition of matrices $E^{(i)}_{a_{i}+1}$ and $E^{(j)}_{a_{j}+1}$.

### 6 Exceptional modules of the higher rank

In this paper, we have focused on the case of $\Lambda$-modules of rank two. We remark that the presented construction of them by cokernels can be applied also for exceptional modules of higher rank. For this, we have to consider exact sequences of the form

$$0 \to L \xrightarrow{f} \bigoplus_{j \in J} L(b_j \bar{x}_j) \to E \to 0,$$

for $J \subset \{1, 2, \ldots , t\}$. In this case, the cokernel $E$ is exceptional of rank $|J| - 1$. Therefore, here we obtain exceptional modules of rank from 2, to $t - 1$. It is an open question whether in this way we get all exceptional modules of rank $r$, for $3 \leq r \leq t$. 

\[ \text{Birkhäuser} \]
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