Improved non-overlapping domain decomposition algorithms for the eddy current problem

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Abstract

A domain decomposition method is proposed based on carefully chosen impedance transmission operators for a hybrid formulation of the eddy current problem. Preliminary analysis and numerical results are provided in the spherical case showing the potential of these conditions in accelerating the convergence rate. To cite this article: Y. Boubendir, H. Haddar and M.K. Riahi, C. R. Acad. Sci. Paris, Ser. I 340 (2015).

Résumé

Nous proposons une méthode de décomposition de domaine basée sur un choix particulier de l’écriture des conditions de transmission pour une formulation hybride du modèle courant de Foucault. Nous donnons des résultats analytiques et numériques préliminaires dans le cas sphérique qui montrent le potentiel de ces conditions dans l’accélération de la vitesse de convergence d’une résolution itérative du problème. Pour citer cet article : Y. Boubendir, H. Haddar et M.K. Riahi, C. R. Acad. Sci. Paris, Ser. I 340 (2015).

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Le modèle de courant de Foucault s’écrit pour le champ électrique \(\mathbf{E}\) et le champ magnétique \(\mathbf{H}\)

\[
\begin{aligned}
curl \mathbf{H} - \sigma \mathbf{E} &= 0 \quad \text{dans } \Omega \\
curl \mathbf{E} - i\omega \mu \mathbf{H} &= 0 \quad \text{dans } \Omega \\
\mathbf{n} \times \mathbf{H} &= \mathbf{n} \times \mathbf{J}_e \quad \text{sur } \partial \Omega.
\end{aligned}
\] (1)

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où la perméabilité magnétique $\mu$ est une fonction à valeurs réelles qui peut dépendre de l’espace, et $\mathbf{J}_e$ représente un terme source. La conductivité électrique $\sigma$ est nulle en dehors de la partie conductrice $\Omega_C$. Ces équations sont complétées par les conditions d’interface traduisant la continuité des composantes tangentiels de $\mathbf{E}$ et $\mathbf{H}$ à travers $\Gamma$, l’interface entre $\Omega_C$ et le vide $\Omega_I := \Omega \setminus \Omega_C$.

Le champ magnétique $\mathbf{H}$ est à rotationnel nul dans $\Omega_I$. Ceci implique, lorsque ce dernier est simplement connexe, que $\mathbf{H} = \nabla p$ (voir par exemple [3]), et le modèle s’écrit sous la forme

\[
\begin{align*}
\text{curl } \text{curl } \mathbf{E} - \kappa^2 \mathbf{E} &= 0 \quad \text{dans } \Omega_C \\
\nabla \cdot (\mu \nabla p) &= 0 \quad \text{dans } \Omega_I
\end{align*}
\]

avec $\kappa^2 := i \omega \mu \sigma$. Les conditions de continuité à l’interface $\Gamma$ s’écrivent

\[
\text{curl } \mathbf{E} \times \mathbf{n} = i \omega \mu \text{curl}_\Gamma p \quad \frac{\partial p}{\partial \mathbf{n}} = \frac{1}{i \omega \mu} \text{curl}_\Gamma \mathbf{E} \quad \text{sur } \Gamma
\]

où $\mathbf{n}$ est un vecteur normal unitaire sur $\Gamma$ qui pointe vers l’extérieur de $\Omega_C$ et où $\text{curl}_\Gamma p := \nabla_\Gamma p \times \mathbf{n}$ et $\text{curl}_\Gamma \mathbf{E} := \text{div}_\Gamma (\mathbf{E} \times \mathbf{n})$ avec $\nabla_\Gamma$ et $\text{div}_\Gamma$ désignant respectivement le gradient et la divergence surfaciques. Nous remarquons que le cas non simplement connexe pourrait également être traité au moyen de rajouter à $p$ les contributions d’un nombre fini de fonctions harmoniques ([3]). Dans ce travail nous proposons une méthode de décomposition de domaine basé sur la décomposition de $\Omega$ en partie conductrice et le vide mais en remplaçant les conditions d’interface naturelle par les conditions de continuité suivantes

\[
\begin{align*}
\frac{\partial p}{\partial \mathbf{n}} + \beta_I \text{curl}_\Gamma \mathbf{n} \rightarrow p & = \frac{1}{i \omega \mu} \text{curl}_\Gamma (\mathbf{E} + \beta_I \text{curl } \mathbf{E} \times \mathbf{n}) \\
\text{curl } \mathbf{E} \times \mathbf{n} + \beta_C \text{curl}_\Gamma (\text{curl } \mathbf{E} \cdot \mathbf{n}) &= i \omega \mu \text{curl}_\Gamma p + \beta_C \frac{\partial p}{\partial \mathbf{n}}
\end{align*}
\]

où $\beta_I$ et $\beta_C$ sont des nombres complex appropriés. On montre que ces conditions sont équivalentes aux conditions originales sous certaines conditions sur $\beta_C$ et $\beta_I$ (Lemme 1.1). Notre algorithme itératif est décrit par (14)-(15). Dans la cas sphérique, on montre que asymptotiquement, l’opérateur d’itération (16) est une contraction plus rapide que celle correspondant au choix $\beta_I = \beta_C = 0$ (Proposition 2.2). Nous concluons cette note par un exemple numérique 3D montrant une meilleure vitesse de convergence pour des valeurs de $\beta_C$ et $\beta_I$ différentes de 0.

1. Introduction

The eddy current approximation of the Maxwell equations, for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ reads

\[
\begin{align*}
\text{curl } \mathbf{H} - \sigma \mathbf{E} &= 0 \quad \text{in } \Omega \\
\text{curl } \mathbf{E} - i \omega \mu \mathbf{H} &= 0 \quad \text{in } \Omega \\
\mathbf{n} \times \mathbf{H} &= \mathbf{n} \times \mathbf{J}_e \quad \text{on } \partial \Omega.
\end{align*}
\]

where the magnetic permeability $\mu$ is a real valued function that may depend on space and $\mathbf{J}_e$ stands for the source excitations. The electric conductivity $\sigma$ has support only in the conductive material $\Omega_C$. These equations are supplemented with the continuity of tangential components of $\mathbf{E}$ and $\mathbf{H}$ across $\Gamma$, the interface between conductive and non conductive regions.

The field $\mathbf{H}$ is curl free in the insulating region $\Omega_I := \Omega \setminus \Omega_C$. This implies in the case of a simply connected topology that the magnetic field is a gradient of a harmonic scalar field i.e. $\mathbf{H} = \nabla p$ (see [3] and references therein), which leads to the problem

2
\[
\begin{align*}
\text{curl } \text{curl } E - \kappa^2 E &= 0 \quad \text{in } \Omega_C \\
\nabla \cdot (\mu \nabla p) &= 0 \quad \text{in } \Omega_I.
\end{align*}
\]

(7)

where \( \kappa^2 := i\omega\mu\sigma \). The interface continuity conditions across \( \Gamma \) lead to

\[
\text{curl } E \times n = i\omega\mu \text{curl}_\Gamma p \quad \frac{\partial p}{\partial n} = \frac{1}{i\omega\mu} \text{curl}_\Gamma E \quad \text{on } \Gamma
\]

(8)

where \( n \) is a unitary normal vector on \( \Gamma \) pointing toward the exterior of the conductor. We use \( \text{curl}_\Gamma p := \nabla_\Gamma p \times n \) and \( \text{curl}_\Gamma E := \text{div}_\Gamma (E \times n) \) with \( \nabla_\Gamma \) and \( \text{div}_\Gamma \) respectively being the surface gradient and the surface divergence. To simplify the presentation we also assume that \( \partial \Omega \subset \partial \Omega_I \) and close the problem by imposing a boundary condition \( p = f \) on \( \partial \Omega \). Let us already remark that the case of non simply connected domain can also be treated in a similar way but with additional technicalities related to the incorporation of (finite number of) divergence and curl free functions.

Many methods, such as potential and hybrid formulations \([4,2,9]\), have been developed to deal with this type of problem. An extensive overview of these methods can be found in \([3]\). Generally speaking, these methods fall in the class of direct formulations gathering the problem in \( \Omega_C \) and \( \Omega_I \) into the same linear system to solve. A nice discussion of the advantages/disadvantages of such methods and others can be also found in \([2]\), where the authors propose a new numerical discretization to overcome these difficulties, mainly related to the size of the linear systems. Although the approach in \([2]\) represents an important contribution in solving these difficulties, it remains strongly dependent on the use of preconditioners because of the ill-conditioning which is further worsened by the high contrast created by the conductivity in \( \Omega_C \).

Domain decomposition methods \([8,10]\) are well suited for this problem since they allow the problems in \( \Omega_C \) and \( \Omega_I \) to be solved separately with appropriate approaches. Up to our knowledge, the only domain decomposition algorithm developed for formulation (7) is the given in \([3]\) and exploits transmission conditions (8).

We here propose to improve this method by modifying these conditions under the form

\[
\text{curl } E \times n + \beta_C \text{curl}_\Gamma (\text{curl } E \cdot n) = i\omega\mu \text{curl}_\Gamma \left( p + \beta_C \frac{\partial p}{\partial n} \right)
\]

(9)

\[

\text{curl } E \times n + \beta_I \text{curl}_\Gamma \text{curl}_\Gamma p = \frac{1}{i\omega\mu} \text{curl}_\Gamma (E + \beta_I \text{curl } E \times n)
\]

(10)

where \( \beta_I \) and \( \beta_C \) denote appropriate (complex) numbers. For similar ideas in different contexts we refer the reader to \([5,7,6]\).

We establish in this part a consistency property for the above impedance boundary condition.

**Lemma 1.1** The following conditions

\[
\text{Re } \{ -\beta_C \beta_I \} \geq 0, \quad \text{or} \quad \text{Im } \{ \beta_C \beta_I \} \neq 0
\]

(11)

ensure consistency between the original conditions (8) and the new ones (9)-(10).

**Proof.** Let us define the following quantities on \( \Gamma \)

\[
\Xi_C := \text{curl } E \times n - i\omega\mu \text{curl}_\Gamma p, \quad \Xi_I := \frac{\partial p}{\partial n} - \frac{1}{i\omega\mu} \text{curl}_\Gamma E,
\]

(12)

which are zero if interface conditions (8) hold. Interface conditions (10) and (9) can be written as

\[
\Xi_C + i\omega\mu \beta_C \text{curl}_\Gamma \Xi_I = 0, \quad \Xi_I + \frac{1}{i\omega\mu} \beta_I \text{curl}_\Gamma \Xi_C = 0.
\]

(13)

Therefore, \( \Xi_C - \beta_C \beta_I \text{curl}_\Gamma \text{curl}_\Gamma \Xi_C = 0 \). With implies, using a variational form, that \( \| \Xi_C \|_{L^2(\Gamma)}^2 \) -
Then we have

\[
\beta \Re \beta I \| \nabla \times \xi C \|_{L^2(\Gamma)}^2 = 0. \text{ If } (11) \text{ is satisfied then } \xi C = 0 \text{ which also implies } \xi I = 0 \text{ and one recovers interface conditions } (8). \]

The problem at hand is then reduced to the following iterative algorithm where the two problems are solved separately (with appropriate variational formulations and discretization)

\[
\begin{cases}
\Delta p^{(k+1)} = 0 \quad \text{in } \Omega_I \\
\frac{\partial p^{(k+1)}}{\partial n} + \beta_I \nabla \times \mathbf{E}^{(k+1)} = \frac{1}{i \omega \mu} \nabla \times \mathbf{E}^{(k)} + \beta_I \nabla \times \mathbf{E}^{(k)} \quad \text{on } \Gamma
\end{cases}
\]

(14)

\[
\begin{align*}
\nabla \times \nabla \times \mathbf{E}^{(k+1)} - \kappa^2 \mathbf{E}^{(k+1)} &= 0 \quad \text{in } \Omega_C \\
\nabla \times \mathbf{E}^{(k+1)} \times \mathbf{n} + \beta_C \nabla \times (\nabla \times \mathbf{E}^{(k+1)} \cdot \mathbf{n}) &= i \omega \mu \nabla \times (p^{(k)} + \beta_C \frac{\partial p^{(k)}}{\partial n}) \quad \text{on } \Gamma.
\end{align*}
\]

(15)

It order to ensure well posed problems for \( p^{(k+1)} \) and \( \mathbf{E}^{(k+1)} \), a variational study of problems (14) and (15) show that sufficient conditions are respectively \( \Re \beta I \leq 0 \) and \( \Re \beta_C \geq 0 \), \( \Im \beta_C \leq 0 \).

2. Iteration operator in the case of concentric spheres

This section is dedicated to the computation of the eigenvalues of the iteration operator, denoted by \( \mathcal{T} \), in the case of concentric spheres. The main goal is to study the dependence of these eigenvalues on \( \beta_I \) and \( \beta_C \) and show that non zero values of \( \beta_I \) and \( \beta_C \) improve the convergence of the iterative procedure. In the case \( f = 0 \) (meaning the solution is 0) better behavior correspond with eigenvalues of modulus closer to 0 in the case of a successive iterative procedure. Consider the sphere \( \Omega = B_R \subset \mathbb{R}^3 \) with radius \( R > 1 \). The insulating and conducting regions are respectively given by \( B_R \setminus B_1 \) and \( B_1 \), where \( B_1 \) is the unit sphere (the case of a sphere of radius \( r < R \) can be easily deduced using an appropriate scaling). Assume that \( g_C \) (resp. \( g_I \)) represents the right side on \( \Gamma \) in (15) (resp. (14)), and let us define \( \mathbf{g} = (g_C, g_I)^T \).

Performing one iteration consists in computing

\[
\mathbf{g}^{(n+1)} = \mathcal{T} \mathbf{g}^{(n)} = \begin{pmatrix} 0 & \mathcal{T}_C \\ \mathcal{T}_I & 0 \end{pmatrix} \begin{pmatrix} g_C^{(n)} \\ g_I^{(n)} \end{pmatrix}
\]

(16)

with

\[
\mathcal{T}_I \left( g_C^{(n)} \right) := \frac{1}{i \omega \mu} \nabla \times \left( \mathbf{E}^{(n+1)} + \beta_I \nabla \times \mathbf{E}^{(n+1)} \times \mathbf{n} \right), \quad \mathcal{T}_C \left( g_I^{(n)} \right) := i \omega \mu \nabla \times \left( p^{(n+1)} + \beta_C \frac{\partial p^{(n+1)}}{\partial n} \right)
\]

(17)

Let \( Y_n^m, n = 0, 1, \ldots, m \leq m \leq n \) denote the spherical harmonics and set \( U_n^m := \frac{1}{\sqrt{n(n+1)}} \nabla_{\mathbb{S}^2} Y_n^m \) and \( V_n^m(\mathbf{\hat{x}}) := \mathbf{\hat{x}} \times U_n^m(\mathbf{\hat{x}}), \mathbf{\hat{x}} \in \mathbb{S}^2 \). We remark from the expression of \( g_C \) that this field belongs to span \( \{ V_n^m, n = 1, 2, \ldots, m \leq m \leq n \} \). We also denote by \( j_n \) the spherical Bessel function of the first kind of order \( n \).

**Proposition 2.1** Assume that

\[
g_C = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} g_C^{n,m} V_n^m \quad \text{and} \quad g_I = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} g_I^{n,m} Y_n^m(\mathbf{\hat{x}}).
\]

Then we have

\[
\mathcal{T}_C \left( g_I \right) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (\mathcal{T}_C)^m g_I^{n,m} V_n^m \quad \text{and} \quad \mathcal{T}_I \left( g_C \right) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (\mathcal{T}_I)^m g_C^{n,m} Y_n^m(\mathbf{\hat{x}})
\]

(18)
With

\[(T_C)^m_n = -i\omega\mu \left( \frac{1}{\sqrt{n(n+1)}} \mathbb{B}_I + \beta_C n(n+1) \mathbb{A}_I \right) / (\mathbb{A}_I + \beta_I \mathbb{B}_I) \] (18)

and \(\mathbb{A}_I := -n(1 + R^{-2n})\), \(\mathbb{B}_I := n(n+1)(1 - R^{-2n})\), \(\mathbb{A}_C := -\sqrt{n(n+1)}(j_n(\kappa) + \kappa j_n'(\kappa))\) and \(\mathbb{B}_C := -(n(n+1))^\frac{3}{2} j_n(\kappa)\).

**Proof.** The solution \(p\) of problem (14) can be expressed as

\[p(x) = \sum_{n=1}^{\infty} \sum_{m=\pm n} a_n^m M_n^m(x) - \frac{i}{\kappa} b_n^m \text{curl} M_n^m(x) \]

with \(M_n^m(x) := \text{curl} (x j_n(\kappa|x|) Y_n^m(x))\). Since \(M_n^m(x) = -\sqrt{n(n+1)} j_n(\kappa|x|) V_n^m(x)\) and \(\text{curl} M_n^m(x) = -\sqrt{n(n+1)} \left( \frac{1}{|x|} j_n(\kappa|x|) + \kappa j_n'(\kappa|x|) \right) U_n^m(x)\)

we get at \(|x| = 1\),

\[\text{curl} E \times n = \sum_{n=1}^{\infty} \sum_{m=\pm n} -a_n^m \sqrt{n(n+1)} \left( j_n(\kappa) + \kappa j_n'(\kappa) \right) V_n^m(\hat{x}) + i k b_n^m \sqrt{n(n+1)} j_n(\kappa) U_n^m(\hat{x}).\]

In addition, because \(\text{curl}_T U_n^m = 0\), we obtain at \(|x| = 1\),

\[\overrightarrow{\text{curl}}_T E = \sum_{n=1}^{\infty} \sum_{m=\pm n} -a_n^m (n(n+1))^\frac{3}{2} j_n(\kappa) V_n^m(\hat{x}).\] (20)

Therefore, combination of (20), (20) and the boundary condition (10) leads to

\[\text{curl} E \times n + \beta_C \overrightarrow{\text{curl}}_T \text{curl}_T E = \sum_{n=1}^{\infty} \sum_{m=\pm n} (\mathbb{A}_C + \beta_C \mathbb{B}_C) V_n^m(\hat{x}).\] (21)

This implies in particular that

\[a_n^m = g_C^{n,m} / (\mathbb{A}_C + \beta_C \mathbb{B}_C), \quad b_n^m = 0.\] (22)

The expression \((T)^m_n\) then directly follows from evaluating \(T_I(g_C)\).

Using the structure of \(T\) one observes that the operator \(T^2\) is diagonal with eigenvalues

\[(T)^m_n := (T_I)^m_n : (T_C)^m_n.\] (23)

**Proposition 2.2** The leading term in the asymptotic expansion of the amplification coefficient for large \(n\) and any \(\beta_I, \beta_C\) is given by

\[|(T)^m_n| \sim \frac{1 - n\beta_C}{1 + n\beta_C} \text{ as } n \to \infty.\]
Proof. The asymptotic expansion for the Bessel functions of the first kind with complex argument [1, Formula 9.3.1] for a fixed complex argument \( \kappa \) and a large integer \( n \) is given by \( j_n(\kappa) \sim \frac{1}{\sqrt{2\pi n}} \left( \frac{\kappa}{2} \right)^n \) and \( j'_n(\kappa) \sim \frac{2}{n} j_n(\kappa) \). We then have the following asymptotic expansions of the coefficients appearing in Proposition 2.1.

\[ A_I \sim -n, \quad B_I \sim n^2, \quad A_C \sim -n^2 j_n(\kappa), \quad B_C \sim -n^3 j_n(\kappa). \]

Plugging these asymptotics in formula (18) and (19) respectively give for large \( n \)

\[ (T_C)^m_n \sim \frac{i \omega \mu (1 - n \beta_C)}{(1 - n \beta_I)} \quad \text{and} \quad (T_I)^m_n \sim \frac{1}{i \omega \mu} \frac{(1 - \beta_I n)}{(-1 - \beta_C n)}. \]

This gives the announced result. \( \square \)

We then conclude that as long as \( \text{Re} \beta_C > 0 \) the coefficient \((T)^m_n\) has a modulus strictly smaller than 1 for large \( n \). It is surprising that the asymptotic behavior is independent from the values of \( \beta_I \) which may suggest that this parameter has less important influence in accelerating the iterations. More importantly, the asymptotic formula shows that \(|(T)^m_n(\text{Re} \beta_C > 0)| < |(T)^m_n(\text{Re} \beta_C = 0)|\) for \( n \) sufficiently large which would indicate for the cases \( \text{Re} \beta_C > 0 \) better convergence properties than for the natural choice \( \beta_I = 0 \) and \( \beta_C = 0 \). We also remark that purely imaginary values of \( \beta_C \) do not improve the convergence properties for large modes.

We plot in Figure 1 the amplification coefficient (23) of the iterative procedure for each mode \( n,m \).

We observe that the above conclusions also hold for all modes and that better behavior is observed for \( \text{Re} \beta_C > 0 \). The asymptotic behavior of \(|(T)^m_n|\) is also confirmed by Figure 1 right.

3. 3D Finite Elements preliminary experiments

Preliminary results of the new formulation are presented in this section. The insulating region is given by \( B_R \setminus B_r \) where \( R = 2 \) and \( r = 1 \), and the conduction region by \( B_r \). The electromagnetic coefficients are chosen to be; \( \sigma = 1 \) for the conductivity and \( \mu = 1 \) for the permeability. The frequency is set to \( \omega = \pi/4 \). The coefficients \( \beta_I \) and \( \beta_C \) similar to the ones used in Figure 1 (e.g. \( \beta_I, \beta_C = \{(-1.e-2,1.e-2),(-1.e-2,1.e-1)\} \) (left) and different choices of \( \beta_I = -\beta_C \) (right)).

We plot in Figure 1 the amplification coefficient (23) of the iterative procedure for each mode \( n,m \).

We observe that the above conclusions also hold for all modes and that better behavior is observed for \( \text{Re} \beta_C > 0 \). The asymptotic behavior of \(|(T)^m_n|\) is also confirmed by Figure 1 right.

Figure 1. Plots of \( n \mapsto |(T)^m_n| \) for the choice of \((\beta_I, \beta_C) = \{(-1.e-2,1.e-2),(-1.e-2,1.e-1)\}\) (left) and different choices of \( \beta_I = -\beta_C \) (right).

More advanced numerical analysis of the proposed scheme will be the subject of forthcoming work. We shall in particular numerically and theoretically discuss optimal choices for the impedance parameters.
Figure 2. Computed solution for the potential $p$ in the insulator (a) and for electric field in the conductor (b). The residual of the DDM iterations (c).

$\beta_C$ and $\beta_I$ and explore the possibility of using parameters that are operators of an appropriate negative order that would provide an asymptotic limit of $|(T^n_m)|$ strictly smaller than 1 (this is clearly not the case of constant parameters as indicated by Proposition 2.2).

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