Constraints on Cosmic Ray Transport in Galaxy Clusters from Radio and Gamma Ray Observations

Joshua Wiener\(^1\) & Ellen G. Zweibel\(^{1,2}\)

\(^1\) Department of Astronomy, University of Wisconsin-Madison, Madison, WI 53706, USA.
\(^2\) Department of Physics, University of Wisconsin-Madison, Madison, WI 53706, USA.

ABSTRACT

The nature of cosmic rays (CRs) and cosmic ray transport in galaxy clusters is probed by a number of observations. Radio observations reveal the synchrotron radiation of cosmic ray electrons (CRe) spiraling around cluster magnetic fields. \(\gamma\)-ray observations reveal hadronic reactions of cosmic ray protons (CRp) with ambient gas nuclei which produce pions. To date, no such cluster-wide \(\gamma\)-ray signal has been measured, putting an upper limit on the density of CRp present in clusters. But the presence of CRe implies some source of CRp, and consequently there must be some CRp loss mechanism. In this paper we quantify the observational constraints on this loss mechanism assuming that losses are dominated by CR transport, ultimately deriving a minimum diffusion coefficient of \(\sim 10^{31}\) cm\(^2\) s\(^{-1}\) in the Coma cluster. This lower limit on transport may help illuminate some unknown properties of the cluster field topology. Conversely, measurements of cluster field tangling scales can constrain other model parameters, such as the relative acceleration efficiency of protons to electrons. To be consistent with the Coma observations, protons cannot be accelerated more than 15 times more efficiently than electrons of the same energy.

1 INTRODUCTION

Galaxy clusters are the largest (\(\sim 10^{14} – 10^{15} M_\odot\)) gravitationally bound objects in the Universe and are host to a plethora of physical processes ranging orders of magnitude in scale length and energy. The energy budget of clusters includes several non-thermal components, such as magnetic fields and cosmic rays (CRs), which are potentially important in cluster dynamics. For instance, wave heating by CRs has been proposed as a heating mechanism to prevent cooling catastrophes in cool core (CC) clusters (Loewenstein et al. (1991); Guo & Oh (2008)).

Some information about the magnetic and CR content of clusters can be determined from radio observations, which in some clusters reveal large scale diffuse synchrotron emission from cosmic ray electrons (CRe). These giant radio halos provide insight into the nature of cluster-wide magnetic fields and CR transport. The Coma radio halo in particular has been the target of several radio observations and studies (see Deiss et al. (1997), Thierbach et al. (2003), Brown & Rudnick (2011), and Brunetti et al. (2012) for just a few examples).

Cosmic ray protons (CRp) can be independently detected from their hadronic reactions - neutral pions produced in high energy hadronic collisions decay into \(\gamma\)-rays (charged pions also produced in these collisions decay into CR electrons and positrons, referred to as ‘CR secondaries’). Since \(\gamma\)-radiation can also be produced as inverse Compton emission from high energy electrons up-scattering cosmic microwave background (CMB) photons, any detection of \(\gamma\)-rays is only an upper bound on the rate of hadronic reactions. However, while \(\gamma\)-radiation has been seen in several galaxies, no diffuse cluster-wide \(\gamma\)-ray emission has yet been detected despite deep searches with \(\gamma\)-ray telescopes such as the Fermi Large Area Telescope (see Huber et al. (2013) and Ackermann et al. (2016) for just two examples). These non-detections put strict upper limits on the CRp content of galaxy clusters.

In this paper we describe a theory of CRs which combines a given radio halo detection with a \(\gamma\)-ray flux upper limit to derive a minimum CRp transport speed. In simple terms, if CRs are being accelerated in a cluster at a rate consistent with its radio emission, they must escape the cluster on short enough time scales in order to bring the CRp density low enough to explain the lack of \(\gamma\)-rays. We will specify to the Coma cluster, but the analysis is general.

It should be noted that ours is far from the first study of cosmic ray transport in galaxy clusters. Enßlin et al. (2011) assumed radial transport at the local sound speed. Wiener et al. (2013) and Wiener et al. (2018) assumed radial transport at a speed determined by equating the rate at which cosmic rays excite the waves that confine them through the streaming instability to the rate at which the waves are damped in the cluster plasma (see Zweibel (2017) for a review). These studies are appropriate when the magnetic field is relatively well ordered and the cosmic rays stream down their density gradient at all times. Here we consider a wider class of models in which the magnetic field is very tangled and we do not ascribe a physical origin to streaming.

In §2 we review the connection between synchrotron...
emission and the underlying CRe population. In §3 we review the connection between γ-ray emission and CRp to derive constraints on the CRp population from the upper limits on the γ-ray flux. We combine the analysis of §2 to derive lower limits on CR transport under two limiting assumptions about the origin of the CRe. In §2.4.1, we assume that the CRe are all primaries (i.e. accelerated by the same processes that produce CRp) and we consider two transport models, bulk advection or streaming at speed $v_s$, and diffusion with diffusivity $\kappa$. This assumption is pictorialized in the simplified schematic of CR processes shown in figure 1b. In §2.4.2, we assume the CRe are all secondary particles. Under this assumption, there are no constraints on the transport per se, but other conditions must be satisfied to make the radio detection and γ-ray nondetection compatible. This assumption is pictorialized in the simplified schematic of CR processes shown in figure 1c. In §4 we apply the analysis in §§2 and 3 to the Coma cluster. In §5 we check some of the assumptions made in the analysis for consistency. We find that the minimum transport rates derived for CRp in the primary model are so slow that appreciable secondaries would be generated before the CRp escape.

In §6 we summarize and draw conclusions. Our main result is that if the cosmic ray electrons are primary particles and the magnetic field is nearly radial, then with standard assumptions about the magnetic field strength and spatial profile in Coma, and Milky Way like assumptions about the ratio of proton to electron cosmic ray sources, the streaming speeds calculated in Wiener et al. (2013) and Wiener et al. (2018) are high enough to prevent proton cosmic rays from generating a detectable γ-ray signal. But, if the magnetic field is dominated by turbulence on 10s of kpc scales, as suggested by the observations of Bonafede et al. (2010), cosmic rays build up in intensity to a level that should produce a detectable γ-ray signal. And, if the cosmic ray electrons are all secondaries, the radio detections and γ-ray nondetections require a rather strong magnetic field.

2 PROPERTIES DETERMINED FROM RADIO EMISSION

2.1 CRe Profile

As per Rybicki & Lightman (1979), an isotropic, spherically symmetric, power law distribution of electrons

$$f_e(\alpha, \gamma) = \frac{1}{2}C_e(r)^{\gamma - \alpha_e}$$

in the presence of a magnetic field $B(r)$ will emit synchrotron radiation with combined power per unit volume per unit frequency given by:

$$P_{\text{tot}}(\omega, r) = C_e(r) \frac{\sqrt{3\pi}}{4\pi(\alpha_e + 1)} \frac{e^3 B(r)}{m_e c^2} \frac{(\omega m_e c)}{3eB(r)}^{-(\alpha_e - 1)/2} \times \Gamma \left( \frac{\omega^2}{\omega_m^2} + \frac{12}{17} \right) \Gamma \left( \frac{\alpha_e}{\omega_m c} - \frac{1}{17} \right) \Gamma \left( \frac{\alpha_e}{\omega} + \frac{3}{2} \right) \Gamma \left( \frac{\omega^2}{\omega_m^2} + \frac{2}{17} \right) \Gamma \left( \frac{\alpha_e}{\omega_m c} + \frac{3}{2} \right)$$

That is, a power law CRe distribution of index $\alpha_e$ emits a power law radio spectrum of index $s = (\alpha_e - 1)/2$. Inverting this relationship, an observed power law radio spectrum of index $s$ implies a CRe index $\alpha_e = 2s + 1$. Let us rewrite the above in terms of $s$, and also switch to frequency $\nu$ instead of $\omega$ by multiplying by $d\omega/d\nu = 2\pi$.

$$P_{\text{tot}}(\nu, r) = N C_e(r) \frac{e^3 B(r)}{m_e c^2} \left( \frac{2\pi \nu m_e c}{3eB(r)} \right)^s$$

To get the observed radio surface brightness at frequency $\nu$ we would take a line integral of $P_{\text{tot}}$ along a line of

$$N = \frac{\sqrt{3\pi}}{4s + 4} \Gamma \left( \frac{s}{2} + \frac{11}{17} \right) \Gamma \left( \frac{s}{2} + \frac{1}{17} \right) \Gamma \left( \frac{s}{2} + rac{3}{2} \right) \Gamma \left( \frac{s}{2} + \frac{2}{17} \right)$$

1 Although we formally integrate over all $\gamma_e$ (i.e. all CRe energies) to obtain this expression, only a certain range of energies will contribute significantly. Namely, because of the exponential behavior of $F(x)$ at large $x$, there is an exponential cutoff of the integrand at low $\gamma_e$. The location of this cutoff depends on the frequency $\nu$. This means we don’t have to worry about the fact that our assumed CRe distribution formally contains some electrons that violate the assumption in Rybicki & Lightman (1979) that $\beta \approx 1$. This also means we only need the CRe spectrum to follow a power law in this energy range to be able to use (1).
sight through the cluster. This would involve a non-analytic integral at every point on the sky that we wanted to compare to observations. Let us instead consider the total radio power emitted by the entire cluster. Denote the surface brightness as a function of position on the sky (an observed quantity with units Jy sr\(^{-1}\)) by \(S_c(\Omega)\). The power per unit volume in radio (an intrinsic quantity with units erg cm\(^{-3}\) s\(^{-1}\) Hz\(^{-1}\)) is \(P_{\text{tot}}(\nu, r)\) as given above in (2). Then the total observed flux can be expressed in two ways:

\[
\int d^2\Omega S_c(\Omega) = \frac{1}{4\pi D^2} \int d^3V P_{\text{tot}}(\nu, r)
\]

\[
\int_0^{\theta_{\text{max}}} 2\pi \sin \theta d\theta S_c(\theta) = \int_0^{r_{\text{max}}} dr \frac{r^2}{D^2} P_{\text{tot}}(\nu, r)
\]

(3)

Here we have assumed the cluster is spherically symmetric, is a distance \(D\) away, and has some maximum extent on the sky \(\theta_{\text{max}} = r_{\text{max}} / D\). We have also neglected geometric effects of the size of the cluster by approximating the distance to every point in the cluster as \(D\). The left hand side of this equation encases the observations, while the right hand side, via equation (2), contains parts of our cluster model \(B(r)\) and \(C_c(r)\).

Let us plug in (2) for \(P_{\text{tot}}\) in (3), scaling to the central magnetic field \(B_0\) and central value of \(C_c(r)\), denoted \(C_{c0}\). We have

\[
\int_0^{\theta_{\text{max}}} 2\pi \sin \theta d\theta S_c(\theta) = NC_{c0} e^2 B_0 \left( \frac{2\pi \nu m_e c}{3eB_0} \right)^s \int_0^{r_{\text{max}}} dr \frac{r^2}{D^2} C_c(r) \left( \frac{B(r)}{B_0} \right)^{s+1}
\]

Thus, for a given model of the magnetic field \(B(r)\) and a model for the spatial dependence of \(C_c(r)\), we can use the observations \(S_c\) to determine the normalization of \(C_c(r)\).

Let us do this by defining a CRE shape function

\[
\eta_e(r) = C_c(r) / C_{c0},
\]

with the inherent normalization \(\eta_e(0) = 1\). Denoting the total integrated radio flux per frequency bin at frequency \(\nu\) (the left-hand side of (3)) by \(S_\nu\) and solving for \(C_{c0}\), we arrive at

\[
C_{c0} = \frac{S}{N L_1 \frac{m_e c^2 D^2}{e^3 B_0}}
\]

(5)

where

\[
S = S_c \left( \frac{2\pi \nu m_e c}{3eB_0} \right)^s
\]

(6)
is a single constant that encompasses the radio observations and

\[
L_1 = \int_0^{r_{\text{max}}} dr r^2 \eta_e(r) \left( \frac{B(r)}{B_0} \right)^{s+1}
\]

(7)
is an integral containing information about the cluster model.

### 2.2 CRE Loss and Source Rate

In the relevant CRE energy range that is probed by the radio observations as described above, energy losses are dominated by synchrotron and inverse Compton (IC) losses. We will assume that transport losses are negligible, an assumption we will check after the fact in §3. The photon field responsible for the IC losses is itself dominated by cosmic microwave background (CMB) photons. The energy loss rate per electron is then

\[
\dot{\gamma}_{\text{loss}}(\text{per electron}) = 4 \sigma_T c^2 \left( \frac{\varepsilon_{\text{IC}}(r) + \varepsilon_{\text{cmb}}}{m_e c^2} \right) = \frac{1}{2\pi} \frac{\sigma_T c^2}{m_e c^2} (B^2(r) + B_{\text{cmb}}^2)
\]

(8)

where \(B_{\text{cmb}} \approx 3.24 \mu G\) is the magnetic field with equivalent energy density as the CMB.

A CRE distribution function \(f_e(r, \gamma_e)\) that suffers from energy losses \(\dot{\gamma} < 0\) and is replenished by some number source function \(s_e(r, \gamma_e)\) has the steady state solution

\[
f_e(r, \gamma_e) = \frac{1}{\dot{\gamma}_{\text{tot}}} \int_0^{\gamma_{\text{max}}} s_e(r, \gamma_e') d\gamma_e'
\]

(9)

For our power law distribution, \(f_e(r, \gamma_e) = C_e \eta_e(r) \gamma_e^{-\alpha_e}\). Since \(\dot{\gamma}_{\text{loss}}\) goes as \(\gamma_e^2\), the source function’s energy dependence must be \(\gamma_e^{1-\alpha_e}\). Let us then write the source function in the form \(s_e(r, \gamma_e) = G_e(r) \gamma_e^{1-\alpha_e}\). Then

\[
C_e \eta_e(r) \gamma_e^{-\alpha_e} = \frac{6\pi m_e c^2}{\sigma_T c^2 (B^2(r) + B_{\text{cmb}}^2)} G_e(r) \int_{\gamma_e}^{\infty} \gamma_e^{1-\alpha_e} d\gamma_e'
\]

Inverting this relationship gives us the source function \(G_e(r)\) required to give a steady state CRE distribution \(C_e \eta_e(r)\):

\[
G_e(r) = \frac{\alpha_e - 2}{6\pi} \frac{\sigma_T c}{m_e c^2} C_e \eta_e(r) (B^2(r) + B_{\text{cmb}}^2)
\]

(10)

If we use our formula for \(C_{c0}\) based on the radio emission (5), we get

\[
s_e(r, \gamma_e) = G_e(r) \gamma_e^{-2\alpha_e},
\]

\[
G_e(r) = \frac{(2s - 1) S \sigma_T c D^2}{6\pi \eta_0} \frac{e^3 B_0}{S^2} \eta_e(r) (B^2(r) + B_{\text{cmb}}^2)
\]

(11)

Put simply, for a given cluster model, the spatial shape of the source function (assuming the CREs are in steady state) is uniquely determined, and the radio observations determine its normalization.

### 2.3 CRE Shape Function

Before moving on it will be useful to come up with an educated guess for what the CRE shape function \(\eta_e(r)\) should be. To do this, consider the more complicated problem of looking at the radio emission as a function of position. This involves a line of sight integral

\[
S_\nu(\theta) = \frac{1}{2\pi} \int_0^{\infty} dl P_{\text{tot}}(\nu, r(l) = \sqrt{D^2 + l^2})
\]

with \(P_{\text{tot}}\) given by equation (2).

Recall that the normalization of \(\eta_e(r)\) is arbitrary - we are only interested in its spatial dependence. If we just want to find this dependence, we can ignore all factors except those which depend on \(r\). We are left with the relationship
between the radio brightness $S_\nu(\theta)$ and a line integral of the shape function:

$$S_\nu(\theta) \propto \int_{0}^{\infty} dl \ n_e(r(l)) \left( \frac{B(r(l))}{B_0} \right)^{\alpha_B + 1}.$$

From here on we will model the Coma cluster specifically, but the analysis can be applied more generally. We model the B field of the Coma cluster as one that goes as a power $\alpha_B$ of the thermal electron density,

$$B(r) = B_0 \left( \frac{n_e(r)}{n_0} \right)^{\alpha_B}.$$

Let us consider shape functions of the same form, going as the electron density to some power $\alpha_n$. The electron density itself can be modeled with a beta profile:

$$n_e(r) = n_0 \left( 1 + \left( \frac{r}{r_c} \right)^2 \right)^{-3\beta/2}$$

with core radius $r_c = 294$ kpc and $\beta = 0.75$ for the specific case of the Coma cluster determined from X-ray surface brightnesses in Briel et al. (1992). The determination of the shape function thus reduces to determining the power $\alpha_n$ from

$$S_\nu(\theta) \propto \int_{0}^{\infty} dl \ \left( 1 + \frac{l^2 + D^2 \theta^2}{r_c^2} \right)^{-3\beta/2(\alpha_B(s+1) + \alpha_n)}.$$

The primary dependence on $\theta$ (at least for large $\theta$) appears to be $\theta^{-3\beta(\alpha_B(s+1) + \alpha_n)}$. We can compare this to the $\theta$ dependence of $S_\nu(\theta)$ at large $\theta$ which (very roughly) is about the same as that of the X-ray surface brightness, $S_X(\theta) \sim \theta^{1-6\beta}$. We end up with

$$1 - 6\beta = -3\beta(\alpha_B(s+1) + \alpha_n)$$

$$\rightarrow \alpha_n = 2 - \frac{1}{3\beta} - \alpha_B(s+1) \quad (12)$$

For $\beta = 0.75$, $\alpha_B = 0.5$, and $s = 1.35$, we get $\alpha_n = 0.381$.

Since the extent of the radio halo may depend on frequency and data analysis techniques (Brown & Rudnick 2011) find a more extended halo in the Coma cluster than Deiss et al. (1997) even at the same frequency), we will investigate the effects of changing $\alpha_n$, using the above formula as a fiducial value.

### 2.4 Cosmic Ray Proton (CRp) Density

The connection between CRe and cosmic ray protons (CRp) can be complicated, but we can consider two limiting cases. In one limit, CRe secondaries produced by hadronic interactions are negligible in number compared to CRp primaries directly accelerated by the plasma. In the other limit, direct acceleration is negligible, and the CRe are entirely sourced by hadronic reactions. We consider both limits in turn, and check for consistency as possible in defining quantities as functions of $\gamma$ or $E$.

The only case where we switch between variables for the same function is the electron source function $s_e$. For ease of notation we don’t denote $s_e(r, \gamma_e)$ and $s_e(r, E_e)$ with different symbols, since the arguments distinguish them. These are related by

$$s_e(r, E_e) = \gamma_e^2 (m_e c^2)^2 = \gamma_e^2 \left( \frac{dE_e}{\gamma_e} \right) = \frac{1}{m_e c^2} s_e(r, \gamma_e)$$

### 2.4.1 Primary limit

In this limit we simply assume some relative acceleration efficiency of protons to electrons $\zeta \approx 100$, and the CRp source function is just our result from the previous section times $\zeta$:

$$s_p(r, E_p) = \zeta s_e(r, E_e = E_p)$$

$$= \frac{\zeta}{m_e c^2} s_e \left( \gamma_e = \frac{E_p}{m_e c^2} \right)$$

$$= \frac{\zeta}{m_e c^2} G_e(r) \left( \frac{E_p}{m_e c^2} \right)^{-2s}.$$

If we want to find a steady state distribution function $f_p(r, E_p)$, we need to balance the above source function with some loss terms. Cooling times for CRp are comparatively long, so let us suppose that CRp losses are dominated by transport. We consider two transport models in turn, streaming and diffusion.

We first consider streaming. Let us characterize the transport by some bulk speed $v_l$ which may in principle vary with position and CRp energy. The steady state equation is then

$$\nabla \cdot (f_p(r, E_p) v_l(r, E_p)) = s_p(r, E_p),$$

which has the solution, assuming the net flow is always outward,

$$4\pi r^2 f_p(r, E_p) v_l(r, E_p) = \int_{r}^{\infty} 4\pi r'^2 s_p(r', E_p) dr'.$$

Supposing the functional form of the transport speed $v_l$ is known, this gives us a way to determine the CRp distribution from the radio emission. Namely,

$$f_p(r, E_p) = \frac{\zeta}{r^2 v_l(r, E_p)m_e c^2} \left( \frac{E_p}{m_e c^2} \right)^{-2s} \int_{r}^{\infty} r'^2 G_e(r')dr'$$

$$= \frac{\zeta \sigma_T \epsilon B_0 D^2 (2s-1)\Sigma L_0(r)}{6\pi N T_1} \left( \frac{E_p}{m_e c^2} \right)^{-2s}.$$

where we have defined another moment of the model profiles

$$I_2(r) = \int_{0}^{\infty} dr' r'^2 \eta_e(r') \left( B_0^2(r') + B_{\text{mb}}^2 \right). \quad (14)$$

If we assume our transport speed is independent of energy, then we have a power law CRp distribution$^2$
Constraints on Cosmic Ray Transport in Galaxy Clusters from Radio and Gamma Ray Observations

\[ f_p(r, E_p) = \hat{C}_p(r)E_p^{-\alpha_p} \]

with spectral index \( \alpha_p = 2s = \alpha_e - 1 \) and spatial dependence

\[
\frac{\hat{C}_p(r)}{(m_e c^2)^{2s-1}} = \frac{\zeta \sigma_T B_0 D^2 (2s - 1) \Sigma_{\ell}(r)}{r^{\alpha} \langle r \rangle_e^{3}} \frac{6 \eta N L_1}{r} \tag{15}
\]

We next consider diffusion. We can alternatively characterize CRp transport by a diffusive process, where the CRp flux is

\[ F_{dt} = -\kappa \nabla f_p \]

for some diffusion coefficient \( \kappa \). In the following we will assume \( \kappa \) does not depend on position or CR energy, but the analysis can be generalized. The steady state requirement in spherical symmetry then becomes

\[
\frac{\partial f_p}{\partial r} = -\frac{1}{\kappa r^2} \int_0^r r'^2 s_p(r', E_p) dr'
\]

We can find \( f_p(r, E_p) \) from this by assuming the CRp density at \( r = \text{negligible and integrating inward:} \)

\[
f_p(r, E_p) = \frac{1}{\kappa} \int_0^{\max} \frac{dr''}{(r'')^2} \int_0^{r''} r'^2 s_p(r', E_p) dr'
\]

Using what we know for \( s_p \) and assuming the previously given power law form for \( f_p \) (which is valid when we assume \( \kappa \) is independent of energy), we get

\[ \hat{C}_p(r)E_p^{-\alpha_p} = \frac{\zeta}{\kappa m_e c^2} \left( \frac{E_p}{m_e c^2} \right)^{-2s} \]

\[
\times \int_0^{\max} \frac{dr''}{(r'')^2} \int_0^{r''} r'^2 G_c(r') dr'
\]

As before this implies \( \alpha_p = 2s \) and gives us the normalization of \( f_p \),

\[
\hat{C}_p(r) = \frac{\zeta \sigma_T B_0 D^2 (2s - 1) \Sigma_{\ell}(r)}{m_e c^2 \kappa r^{\alpha} \langle r \rangle_e^{3}} \frac{6 \eta N L_1}{r} \tag{16}
\]

In the above we have defined a third profile integral

\[ I_3(r) = \int_0^{\max} \frac{dr''}{(r'')^2} I_2(r'') \]

\[ = \int_0^{\max} \frac{dr''}{(r'')^2} \left( B^2(r') + B_{\text{amb}}^2 \right) \left( B^2(r') + B_{\text{amb}}^2 \right) \tag{17} \]

2.4.2 Secondary limit

In the opposite limit, particle acceleration is negligible and the CRe are entirely sourced by hadronic interactions. Schematically, we start with some distribution of CRp \( f_p(r, E_p) \) and some cluster gas density model \( n(r) \). From these we determine the pion production rate, and then the production of CRe via the decay of charged pions.

There are a variety of schemes in the literature for determining pion source functions. The delta-function approximation should work for our purposes - we just approximate that every collision of a single CRp of energy \( E_p \) with a thermal nucleus results in \( \xi \) pions of energy \( K_p T_p \) each, where \( T_p = E_p - m_p c^2 \) is the kinetic energy of the incoming proton. In other words, the pion number source function for each collision is

\[ Q_p(E_\pi, E_p) = \xi \delta(E_\pi - K_p T_p) \]

There are different conventions for choosing the values \( K_p \) and \( \xi \) to match experiments. Kelner et al. (2006) choose \( K_p \approx 0.17 \) with \( \xi = 1 \) when calculating the neutral pion source only. Pfrommer et al. (2008) use \( K_p = 0.25 \) and \( \xi = 2 \) for all pion species together. To be unambiguous, we use \( \xi \) here to refer to the total pion multiplicity across all pion species and assume that charged and neutral pions are produced approximately in the ratio 2:1.

If we have a CRp distribution \( f_p(r, E_p) \), then the rate of collisions (per unit volume) is \( R = cn_N(r) \sigma_{pp}(E_p) f_p(r, E_p) \) and so the pion source function is

\[ q_p(r, E_\pi) = \int_0^{\infty} RQ_p(E_\pi, E_p) dE_p \]

\[ = \frac{\xi}{K_p} cn_N(r) \sigma_{pp} \left( \frac{E_p}{K_p} + m_p c^2 \right) f_p \left( r, E_p = \frac{E_\pi}{K_p} + m_p c^2 \right) \]

Here, \( n_N = n_{\text{HI}} + 4 n_{\text{HII}} \) is the target nucleon density and \( \sigma_{pp} \) is the (energy-dependent) cross section for proton-proton collisions. The above expression gives the number source function for all pion types, so we assume the symmetry the charged pion source function is \( s_{e \pi} = 2s_e / 3 \).

Let us assume a power law CRp distribution \( f_p(r, E_p) = \hat{C}_p(r)E_p^{-\alpha_p} \). The charged pion source function is then

\[ s_e(r, E_\pi) = \frac{2}{3} \frac{\xi}{K_p} cn_N(r) \sigma_{pp}(E_e/K_p + m_p c^2) \hat{C}_p(r) \left( E_e/K_p + m_p c^2 \right)^{-\alpha_p} \]

For the next step, we must describe the decay of a charged pion of energy \( E_e \) into electrons and positrons. Following Pfrommer et al. (2008), we use a delta-function approximation and assume that every charged pion of energy \( E_e \) decays into exactly one electron/positron of energy \( E_e \). This means

\[ s_e(r, E_\pi) = s_{e \pi}(r, E_e = 4E_\pi) \frac{dE_e}{dE_\pi} \]

\[ = \frac{8}{3 K_p} cn_N(r) \sigma_{pp}(4E_e/K_p + m_p c^2) \hat{C}_p(r)(4E_e/K_p)^{-\alpha_p} \]

\[ s_e(r, E_\pi) = \frac{8}{3 K_p} cn_N(r) \sigma_{pp}(4E_e/K_p + m_p c^2) \hat{C}_p(r)(4E_e/K_p)^{-\alpha_p} \]

In the last step we have assumed we are looking at high energy CRe with \( E_e \gg K_p m_p c^2 / 4 \). We can also write this in terms of the Lorentz factor \( \gamma_e \):

\[ s_e(r, \gamma_e) = \frac{8}{3 K_p} cn_N(r) \sigma_{pp}(4\gamma_e m_e c^2/K_p) \gamma_e^{-\alpha_p} \frac{dE_e}{d\gamma_e} \]

\[ s_e(r, \gamma_e) = \frac{8}{3 K_p} \left( \frac{K_p}{4} \right)^{-\alpha_p} (m_e c^2)^{1-\alpha_p} cn_N(r) \sigma_{pp} \hat{C}_p(r) \gamma_e^{-\alpha_p} \]

In the above (and from now on) we will omit the energy dependence of the cross-section \( \sigma_{pp} \) with the understanding.
that it is a very weak function of energy in the relevant energy range around $E_p \sim 100$ GeV (see Kelner et al. (2006)). Pfommer et al. (2008) account for this weak dependence with an effective cross section that is a function of the CRp power spectrum, $\sigma_{pp}(\alpha_p) = 32(0.96 + e^{1.4 - 2.4\alpha_p})$ barn.

We know from §2.2 that the CRe source function must be (11) to explain the synchrotron emission. In the limit where CRp are completely sourced by hadronic interactions, we therefore have

$$
\frac{2\xi}{3} \left( \frac{4m_e c^2}{K_e} \right)^{1-2s} c n_N(r) \sigma_{pp} \tilde{C}_p(r)
$$

or

$$
\tilde{C}_p(r) \left( \frac{m_e c^2}{s} \right)^{1-2s} = \frac{(2s - 1) \frac{\pi}{4} \int_0^{\infty} \frac{dE_p}{E_p} \eta_p(r)(B^2(r) + B_{\text{emb}}^2)}{\sigma_{pp} \frac{\pi}{2} \eta_p(r)} (4s - 1)
$$

where $s$ is the synchrotron spectral index.

Recall that in the primary limit, we needed, in addition to the radio observations and a cluster model, a model for the transport (in the form of speed $v_t$ or diffusion coefficient $\kappa$) and a relative acceleration efficiency $\zeta$ to determine the spatial distribution of CRp. In the secondary limit, the radio observations and cluster model alone determine the CRp distribution.

3 PROPERTIES DETERMINED FROM $\gamma$-RAY EMISSION

No diffuse $\gamma$-ray emanation has yet been detected from galaxy clusters. At high enough energies, any potential such $\gamma$-rays are expected to be dominated by neutral pion decay. As such, the non-detection of $\gamma$-rays translates to an upper limit on the CRp density. We quantify this below, and follow with a discussion of the meaning of this upper limit in the two limiting cases (primary-dominated and secondary-dominated CRe).

3.1 CRp Density Upper Limit

The source function of $\gamma$-rays coming from neutral pion decay is a straightforward function of the pion source function:

$$
s_\gamma(E_\gamma) = \frac{2}{3} \frac{\xi}{K_e} c n_N(r) \sigma_{pp} \tilde{C}_p(r)
$$

%\textit{with $E_\gamma = E_p + m_e^2 c^2/(4E_p)$}.

In the previous section we discussed the relationships between the CRp distribution $f_p$ and the pion source function $q_p$, as well as between $q_p$ and the secondary CRe source function $s_\gamma$.

The neutral pion source function is $s_{\pi}\pi = q_\pi/3$. In the previous section we calculated $q_\pi$ for a power law CRp distribution $f_p(r, E_p) = \tilde{C}_p(r) E_p^{-\alpha_p}$. Plugging this into the $\gamma$-ray equation above, we have

$$
s_\gamma(r, E_\gamma) = \frac{2}{3} \frac{\xi}{K_e} c n_N(r) \sigma_{pp} \tilde{C}_p(r)
$$

The above integral cannot be done analytically without some further assumptions. If we assume $E_p = E_\gamma/K_e \gg m_e^2 c^2$, we can throw away the $m_e^2 c^2$ in the numerator and the integral can be done as an incomplete Beta function. However, in this limit we also have $E_\gamma \gg m_e^2 c^2$ so the denominator would also simplify, giving us a simple power law. In this limit we also have $E_{\gamma,\text{min}} \approx E_\gamma$. The integral therefore reduces to

$$
\frac{K_\alpha}{E_\gamma} \int_0^{E_\gamma} \frac{dE_p}{E_p} \frac{E_p^{-\alpha_p}}{E_p^{1-\alpha_p} - \alpha_p}
$$

and so the $\gamma$-ray source function (just from neutral pion decay) is

$$
s_\gamma(r, E_\gamma) = \frac{2}{3} \frac{\xi}{K_e} \int_0^{E_\gamma} \frac{dE_p}{E_p} \frac{E_p^{-\alpha_p}}{E_p^{1-\alpha_p} - \alpha_p}
$$

and the predicted differential $\gamma$-ray number flux per energy bin at Earth is just the volume integral of this quantity divided by $4\pi D_r^2$. To connect to a potential observation we would then integrate above some energy $E_r$:

$$
F_\gamma(r, E_r) = \int_{E_r}^{E_\gamma} \frac{dE_p}{E_p} \frac{E_p^{-\alpha_p}}{E_p^{1-\alpha_p} - \alpha_p}
$$

We can now relate a photon number flux upper limit $F_{\gamma,\text{max}}$ to an upper limit on an integral of the CRp distribution:

$$
F_{\gamma,\text{max}}(E_r) \geq \frac{2}{3} \frac{\xi}{K_e} \int_0^{E_\gamma} \frac{dE_p}{E_p} \frac{E_p^{-\alpha_p}}{E_p^{1-\alpha_p} - \alpha_p}
$$

Encompassing the $\gamma$-ray upper limit in the following shorthand

$$
F = F_{\gamma,\text{max}}(E_\gamma) \left( \frac{E_\gamma}{m_e c^2} \right)^{\alpha_p-1}
$$

we can express this constraint as

$$
\int_0^{E_{\gamma,\text{max}}} dE_\gamma \frac{\tilde{C}_p(r)}{(m_e c^2)^{\alpha_p-1}} \leq \frac{3\alpha_p}{2\xi} \frac{\alpha_p-1}{\int_0^{E_\gamma} \frac{dE_p}{E_p} \frac{E_p^{-\alpha_p}}{E_p^{1-\alpha_p} - \alpha_p}}
$$

This inequality holds for all upper limits on $F_r$, so the smallest value of $F_{\gamma,\text{max}}(E_\gamma)$ ($E_\gamma/m_e c^2)^{\alpha_p-1}$ across all observations gives us the tightest constraint. Since we have already predicted $\tilde{C}_p(r)$ in the primary- and secondary-dominated limits, we can just plug it into the above and interpret the results.

3.1.1 Primary limit

In section 2.4.1 we derived a relationship (15) between the CRp distribution function $\tilde{C}_p(r)$ and the synchrotron power...
$S_e$ in the limit where CRs are dominated by primaries. By combining this with the above upper limit on $C_p(r)$ from $\gamma$-ray observations, again taking $\alpha_e = 2\pi$, we obtain:

$$\int_0^{r_{\text{max}}} dr \, n_N(r) = \frac{\xi C_p B_0 D^2 (2\pi - 1) S_{\gamma}(r)}{6\pi N I_1} \leq \frac{3\pi (2\pi - 1) D^2 F}{\xi K^{2-1} \sigma_{pp}}.$$  

(25)

Ultimately this amounts to a constraint on the transport speed $v_t$ for a given model and set of observations. In the very special case of a spatially-independent $v_t$ we get a simple lower limit

$$v_t \geq \frac{\xi K^{2-1} B_0 \sigma_{pp} c^2 S}{18\pi N I_1 F e^3} \int_0^{r_{\text{max}}} dr \, n_N(r) I_2(r)$$  

(26)

We evaluate this limit for different cluster models and discuss its characteristics in section 4.

Suppose instead we consider diffusive transport characterized by a constant diffusion coefficient $\kappa$ as described by equation (16). Plugging this into the upper limit (24) gives

$$\int_0^{r_{\text{max}}} dr \, n_N(r) \frac{\xi C_p B_0 D^2 (2\pi - 1) S_{\gamma}(r)}{6\pi N I_1} \leq \frac{3\pi (2\pi - 1) D^2 F}{\xi K^{2-1} \sigma_{pp}}$$

which, finally, can be rearranged into a lower limit on the diffusion coefficient:

$$\kappa \geq \frac{\xi K^{2-1} B_0 \sigma_{pp} c^2 S}{18\pi N I_1 F e^3} \int_0^{r_{\text{max}}} dr \, n_N(r) I_3(r)$$

(27)

We evaluate this limit for different cluster models and discuss its characteristics in section 4.

3.1.2 Secondary limit

In the limit of secondary CRs dominating the population, we find that the radio observations directly translate into a CRp model (20). The $\gamma$-ray upper limit then offers a consistency check - if the inequality above is violated, secondary CRs cannot dominate the CRp population. Plugging (20) into the upper limit (24), we obtain:

$$\frac{(2\pi - 1) S_{\gamma}}{4\pi N I_1} \left( \frac{4}{K_e} \right)^{2-1} \frac{\sigma_T}{\sigma_{pp}} B_0 D^2 \int_0^{r_{\text{max}}} dr \, n_N(r) \frac{B(r)^2 + B_{\text{mb}}^2}{B_0^2} \leq \frac{3\pi (2\pi - 1) D^2 F}{\xi K^{2-1} \sigma_{pp}},$$

$$\frac{I_2(f_{\text{max}})}{I_1} \leq \frac{12\pi N I_1}{4^{2-1} \sigma_T B_0 S}$$

(28)

We can of course frame this in another way, using the secondary-dominated assumption to make a prediction for the $\gamma$-ray number flux:

$$F_\gamma(> E_\gamma) = \frac{S I_2(f_{\text{max}})}{12\pi N I_1} \left( \frac{E_\gamma}{4n_c c^2} \right)^{1-2\pi} B_0 \frac{E_{\gamma}}{c^3} \sigma_T,$$

(29)

or equivalently, a prediction for the $\gamma$-ray number flux in some energy band

$$F_\gamma(E_1, E_2) = \frac{S}{E_1} \left( \frac{2\pi - 1}{2\pi - 2} \right) \left( \frac{E_2}{E_1} \right)^{2-1} - 1$$

(30)

to be directly compared with measurements such as from Ackermann et al. (2016). See section 4 for discussion on this limit.

4 RESULTS AND DEPENDENCIES

We consider here the results of our analysis in the various limiting cases in turn and discuss their dependence on different model parameters.

4.1 Primary Limit, Advevtive Transport

In the limit of primary CRs dominating over secondaries, we found a lower limit on the advective transport speed given by equation (26). To evaluate this, we model the Coma cluster using the density model $n_e$ given in §2.3. The total mass of this model is divergent, so we must also assume a cluster extent $r_{\text{max}}$. Based on the X-ray data from Briel et al. (1992) we estimate a maximum radius of $r_{\text{max}} = 4000$ kpc. As a fiducial magnetic field model we take $\alpha_B = 0.5$ and $B_0 = 3 \mu G$. We will anchor the radio observations at 1.4 GHz with measured intensity $S_1.4\, \text{GHz} \approx 6.4 \times 10^{-24} \text{erg cm}^{-2} \text{s}^{-1} \text{Hz}^{-1}$ (Deiss et al. (1997)), and take the spectral index to be $s = 1.35$ (see Thierbach et al. (2003)).

For the $\gamma$-ray flux limits, we impose the energy flux limit in the 7.5 - 10 GeV band from Ackermann et al. (2016) of $\approx 10^{-7} \text{MeV cm}^{-2} \text{s}^{-1}$ (the exact value of the limit depends on the spatial model of the emission). We use the highest energy band as to be most in line with our assumption of high energy. For a power law spectrum of index 2, this translates into a photon number flux limit above 10 GeV of

$$F_{\gamma,\text{max}} = \frac{(2\pi - 2)10^{-24}}{2\pi - 1(1.33^{2-1} - 1)} \text{photons cm}^{-2} \text{s}^{-1}$$

(31)

This will go into the $F$ quantity in our analysis.

We assume a relative acceleration efficiency of $\zeta = 100$. This parameter is informed by observations of galactic CRs, but in principle may be highly uncertain. The acceleration mechanisms in a galaxy cluster such as Coma may be highly varied. We have purposefully avoided any treatment of specific mechanisms and encompassed all our ignorance of them into $\zeta$. Fortunately the dependence of our transport limits $v_t$ and $\kappa$ on this parameter are linear, and so any uncertainty in $\zeta$ can be propagated easily. This analysis can also be turned around to impose constraints on $\zeta$ (and consequently on the acceleration mechanisms present) if we fix all other model parameters. We discuss this possibility in §6.

3 By picking a single value in this way we are essentially assuming the nature of the acceleration is the same at all locations in the cluster. This is hardly certain, but any treatment of acceleration mechanisms beyond this simplification is outside the scope of this work.
For the above fiducial values, we derive a lower limit on bulk CRp transport of $v_t \gtrsim 42$ km s$^{-1}$. The dependence of this limit on some of the model parameters is explicit from equation (26): $v_t$ is directly proportional to the proton-to-electron relative acceleration efficiency $\zeta$ and the total radio surface brightness $S_\nu$, and inversely proportional to the upper limit on $\gamma$-ray flux $F_{\gamma,\text{max}}$. It depends on the normalization of the magnetic field as $B_0^{1-s}$ for strong fields ($B_0 \gg B_{\text{emb}}$) and as $B_0^{1-s}$ for weak fields. If we assume $\xi$ and $K_\kappa$ are related by the observational constraint that the total fraction of CRp kinetic energy in each collision, $\xi K_\kappa$, is constant, we see that $v_t \propto K_\kappa^{2-s}$. The dependencies on other parameters such as cluster extent $r_{\text{max}}$, radio spectral index $s$, and magnetic field-density dependence $\alpha_B$ are not explicit, so we try many different models to find empirical trends. The minimum speed $v_t$ for various magnetic field models is shown in figure 2. $v_t$ seems to be most sensitive to $B_0$, roughly following the relation

$$v_t \approx 42 \left( \frac{B_0}{3 \text{ G}} \right) ^{-1-s} \text{ km} \text{ s}^{-1}.$$  

(32)

### 4.2 Primary Limit, Diffusive Transport

If we describe CRp transport as a diffusive process with constant diffusion coefficient $\kappa$ instead of an advective one, we found the $\gamma$-ray upper limits put a lower limit on $\kappa$ according to (27). For the fiducial values described above, this lower limit comes out to $2.7 \times 10^{31}$ cm$^2$/s. We plot the values of $\kappa$ for different models in figure 3. Again the strongest dependence is on $B_0$, following the same scaling as $v_t$:

$$\kappa \approx 2.7 \times 10^{31} \left( \frac{B_0}{3 \text{ G}} \right) ^{-1-s} \text{ cm}^2 \text{ s}^{-1}.$$  

(33)

We can convert these limits on diffusion coefficient $\kappa$ into limits on the mean free path due to scattering seen by individual CRs $l_{\text{mfp}}$ by the simple relation $l_{\text{mfp}} = 3\kappa/c$. For our fiducial model, $l_{\text{mfp}}$ must be at least 900 pc for CRp to be able to leave the cluster fast enough to bring $\gamma$-radiation under the detection limit in Coma.

In a model where radial transport is due to scattering by magnetic inhomogeneities at the cosmic ray gyroscale, and the background magnetic field is nearly radial and straight, the advection speed $v_t$ and diffusion coefficient $\kappa$ can be combined to produce a lengthscale $R \equiv \kappa/v_t$ which is representative of the size of the system; for the values given here, $R \sim 2$ Mpc, which is reasonable for Coma (the general expression for this quantity is

$$\frac{\kappa}{v_t} = \frac{\int_0^{r_{\text{max}}} dr r^2 n_N(r) I_3(r)}{\int_0^{r_{\text{max}}} dr r n_N(r) I_2(r)},$$

where $I_2$ and $I_3$ are defined in eqns. (14) and (17)).

Alternatively, motivated by observations which show that cluster magnetic fields have strong random components (Vogt & Enßlin (2005), Bonafede et al. (2010)) we can interpret $\kappa$ in terms of a a magnetic field which is tangled on a lengthscale $l_{\text{corr}}$ which is intermediate between the size of the system and the cosmic ray gyroradius. In this case, the maximum transport rate is the one given by field line random walk (Minnle et al. (2009)): $\kappa \approx l_{\text{corr}}c/3$. This is also approximately the maximum rate of transport across magnetic fieldlines found in Desiati & Zweibel (2014). Additional scattering up and down the tangled fieldlines, which could be represented by a streaming velocity such as $v_s$, only reduces the transport rate (Rechester & Rosenbluth (1978)). Therefore, the minimum tangling length $l_{\text{corr}}$, consistent with the diffuse $\gamma$-ray upper limits is the same as the mean free path $l_{\text{mfp}}$ estimated in the preceding paragraph.

To drive this point home, we have predicted what $v_s$ would be as a function of $r$ for the 100 GeV CRp that would be responsible for (undetected) $\gamma$-radiation by taking the resulting CRp density profile from our model (16) and plugging it into the formalism in Wiener et al. (2013) and Wiener et al. (2018) for some assumed level of wave damping (we use MHD turbulence damping characterized by the length scale $l_{\text{MHD}} = 100$ kpc). This gives us $v_s$ as a function of position in the cluster, which can be translated into a field tangling length $l_{\text{corr}} = 3v_s/c$. We record the maximum $l_{\text{corr}}$ for each model and plot the results in figure 4, which also includes the range of length scales in the magnetic turbulence power spectrum inferred from Faraday rotation observations of Coma (Bonafede et al. (2010)).

We see from figure 4 that the magnetic field observed in Coma is tangled too much ($l_{\text{corr}}$ is too small) to explain the lack of $\gamma$-radiation under this set of assumptions (primary CRe dominate, CRp are transported via streaming with the assumed level of wave damping). This tension is relieved if the cluster magnetic field is in the upper range of values tested, which reduces the CRe population required to explain the radio emission, or if the relative proton-to-electron acceleration efficiency factor $\zeta$ is smaller than Milky Way values.

### 4.3 Secondary Limit

In the limit where the CRe are dominated by secondaries from hadronic interactions, we derived a flux prediction (30) which can be used to predict $\gamma$-ray energy fluxes in arbitrary energy bands. We compare with the highest energy band

Figure 2. Minimum net outward flow velocity of CRp required to reduce pion decay $\gamma$-ray emission below current upper limits, assuming CRe secondaries are negligible. Results shown for different cluster magnetic field models. The central Alfvén speed for each model is shown for reference.
Constraints on Cosmic Ray Transport in Galaxy Clusters from Radio and Gamma Ray Observations

Figure 3. Minimum radial diffusion coefficient required to reduce γ-ray emission below current upper limits, assuming CRe secondaries are negligible. Results shown for different cluster magnetic field models.

Figure 4. Amount of magnetic field tangling allowed (in the self-confinement regime) while staying under γ-ray emission upper limits, assuming CRe secondaries are negligible. If the field is tangled on scales less than this value, CRp cannot escape the cluster fast enough to be consistent with observations. Results shown for different magnetic field models. The dotted lines represent the range of tangling lengths (2 - 34 kpc) in the magnetic turbulence in Coma (from Bonafede et al. (2010)).

used in Ackermann et al. (2016), which is about 7.5 - 10 GeV. Ackermann et al. (2016) find an upper limit in this bin of (depending on their emission model) around $6-10 \times 10^{-11}$ GeV cm$^{-2}$ s$^{-1}$. We compare this to the prediction of our models in figure 5. Our fiducial model predicts a γ-ray flux of $1.9 \times 10^{-9}$ GeV cm$^{-2}$ s$^{-1}$, far above the Fermi upper limit. A secondary-only model for Coma’s radio halo is thus highly disfavored.

Figure 5. Predicted γ-ray flux from pion decay in the 7.5-10 GeV band, assuming secondary CRe dominate. Results shown for different cluster magnetic field models. The dotted line represents the non-detection upper limit from Ackermann et al. (2016). This model is only consistent with the γ-ray observations at the highest magnetic fields.

5 CONSISTENCY CHECKS

In this section we check some of the assumptions we have made in various regimes.

5.1 CRe Transport Losses

Throughout this work we have assumed that the CRe losses in the relevant energy range were dominated by synchrotron and IC emission. Namely, we neglected transport effects. We estimate those here and compare them to the radiative losses.

Suppose we can characterize the CRe transport with a bulk transport speed $v_t(r, \gamma_e)$, as we did for the CRp. Then the number loss (in particles per volume per energy per time) due to transport is:

$$\dot{n}_{\text{cre,trans}}(r, \gamma_e) = \nabla \cdot (f_e(r, \gamma_e) v_t(r, \gamma_e))$$

This is to be compared to the radiative loss rate,

$$\dot{n}_{\text{cre,rad}}(r, \gamma_e) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 C_e(r) v_t(r, \gamma_e))$$

$$= \gamma_e^{-\alpha_e} \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 C_e(r) v_t(r, \gamma_e))$$

$$= \gamma_e^{-\alpha_e} G_e(r) = \gamma_e^{-\alpha_e} \frac{\alpha_e}{2} \frac{\sigma T \epsilon}{m_e c^2} C_e(r) (B^2(r) + B_{\text{cmb}}^2)$$

In the above we have assumed a power law for the CRe distribution, $\gamma_{\text{loss}}$ refers to the radiative losses per electron (8), and $G_e(r)$ refers specifically to the solution from our previous analysis (10,11).

In the simple case of a constant, energy-independent transport speed $v_t(r, \gamma_e) = v_t$, the ratio of transport losses to radiative losses for CRe is

$$\frac{\dot{n}_{\text{cre,trans}}(r, \gamma_e)}{\dot{n}_{\text{cre,rad}}(r, \gamma_e)} = \frac{1}{\gamma_e \alpha_e - 2 \epsilon c B^2(r) + B_{\text{cmb}}^2}$$

$$= \left[ \frac{2}{r} \frac{C_e(r)}{C_e(r)} \right]$$
Using our estimate of the CRe shape function $n_e(r) = n_0 e^{-r^2/(r_c^2 + 3 \alpha \sigma_p \beta \gamma e)}$, we get

$$\frac{C'_e(r)}{C_e(r)} = \frac{n'_e(r)}{n_e(r)} = -\frac{3 \beta \gamma e 2 r}{r} + \left(1 + \frac{1}{r/r_c^2}\right) = -\frac{3 \beta \gamma e 2 r}{r} + \left(1 + \frac{1}{r/r_c^2}\right)$$

and so the transport to radiative loss ratio is

$$\frac{\dot{n}_{CRe,\text{rad}}(r, \gamma_c)}{\dot{n}_{CRe,\text{rad}}(r, \gamma_c)} = \frac{1}{\gamma e \sigma e - 2 c B^2(r) + B_{\text{ cmb}}^2} \int \frac{2}{\sigma_T T} \left[1 - \frac{3 \beta \gamma e 2 r^2}{2(r^2 + r_c^2)}\right]$$

The last term is a fairly weak function of $r$ of order unity, so let us discard it. Picking some scale values, the rest of the expression comes to

$$\frac{\dot{n}_{CRe,\text{rad}}(r, \gamma_c)}{\dot{n}_{CRe,\text{rad}}(r, \gamma_c)} \approx 370 \frac{v_T^2}{r_c} \frac{10^3 r_c}{5000} \frac{1}{R_{\text{ cmb}}^2} \frac{100}{\mu G} \frac{4000}{1000} \frac{1000}{100} \frac{1}{100}.$$  

Recall that $r_c = 297$ kpc for the Coma cluster. The scale field $4.4 \, \mu G$ is chosen as the effective combined field of a 3 $\mu G$ field with $R_{\text{ cmb}} = 3.24$ $\mu G$.

We see that our assumption that this quantity is much much less than one is valid except when the transport speed $v_T$ is very large, or $r$ is very small. The approximation is better for higher energies.

5.2 Estimate of CRe Secondaries

We have done our analysis in two limiting cases, where the CRe are dominated by primaries and secondaries respectively. We have already seen in §3.1.2 that the upper limits on $\gamma$-ray flux rule out the secondary-dominated limit in Coma unless the magnetic field is fairly high. However, we have yet to check how many secondaries would be produced in the primary-dominated limit. If too many are produced, the primary-dominated limit assumption is inconsistent.

The CRp distribution in the primary-dominated limit inferred from radio observations was determined to be (15). The secondary CRe source function expected from a general CRp distribution was determined to be (19). The expected secondary CRe source function in the primary-dominated limit is then

$$s_{e,\text{ sec}}(r, \gamma_c) = \frac{8 \xi}{3 K_e} \left(\frac{K_e}{4}\right)^{\alpha_p} c \sigma_{pp} n_e(r) (m_e c^2)^{\alpha_p - 1}$$

$$= \frac{8 \xi}{3 K_e} \left(\frac{K_e}{4}\right)^{2 \alpha_p} c \sigma_{pp} n_e(r) \frac{\zeta c T c B_D D^2 (2s - 1) S L(z)}{r^2 v_T(r) c^2} \frac{6 n N I_1}{10^3}$$

This must be compared with the total CRe source function given by (11). Their ratio is

$$\frac{s_{e,\text{ sec}}(r, \gamma_c)}{s_e(r, \gamma_c)} = \frac{8 \xi}{3 K_e} \left(\frac{K_e}{4}\right)^{2 \alpha_p} c \sigma_{pp} n_e(r) \frac{c}{v_T(r)}$$

$$\times \left\{\frac{L_2(r)}{r^2 v_T(r) \left(\frac{B_D^2(r) + B_{\text{ cmb}}^2}{B_0^2}\right)}\right\}$$  

This can be found numerically, but we can get a good estimate by recognizing that the quantity in the braces is of order $r$:

$$\frac{s_{e,\text{ sec}}(r, \gamma_c)}{s_e(r, \gamma_c)} \approx \frac{8 \xi}{3 K_e} \left(\frac{K_e}{4}\right)^{2 \alpha_p} c \sigma_{pp} n_0 \frac{c}{v_T(r)} \left(1 + \frac{r^2}{r^2 + r_c^2}\right)^{-3 \beta \gamma e 2}$$

For $s$ close to 1, and a fully ionized medium, $(8/3)(\xi/K_e)(k_e/4)^{2} \zeta \mu_e$ is approximately 10, and $\sigma_{pp}$ is approximately 50 m barn. Picking scale values for $n_0$ and $r$, we have

$$\frac{s_{e,\text{ sec}}(r, \gamma_c)}{s_e(r, \gamma_c)} \sim 10^{-4} \frac{n_0}{10^{-3} \text{ cm}^{-3}} \frac{r}{100 \text{ kpc}} \frac{c}{100} \left(1 + \frac{r^2}{r^2 + r_c^2}\right)^{-3 \beta \gamma e 2}$$

Considering how large we expect $c/v_T$ to be, this suggests that our assumption of primary CRe domination is inconsistent. Even at the large end of the range of values of $v_T$ we found, around 100 km s$^{-1}$, we may expect some 50% of CRe to be secondaries. For expected relative acceleration efficiencies and CR bulk transport speeds, secondary CRe injected from hadronic processes must make up a significant fraction of the total CRe source.

What does this mean for our earlier analysis? Qualitatively, the presence of secondary CRe suggests that for the same level of radio emission, fewer primary CRe are necessary. That is, we can explain the radio emission with some primary CRe source function that is some fraction of the one we derived in §2.2. This makes it easier to get under the $\gamma$-ray upper limits, implying we can get away with lower transport speeds $v_T$. A more quantitative treatment is beyond the scope of this work.

We can alternatively derive a new, more restrictive lower limit on the transport speed $v_T$ such that secondaries are negligible. Let’s say we required that secondaries make up less than one percent of all CRe. Then the transport speed must be at least

$$v_T(r) \gtrsim \frac{n_0}{10^{-3} \text{ cm}^{-3}} \frac{r}{100 \text{ kpc}} \frac{c}{100} \left(1 + \frac{r^2}{r^2 + r_c^2}\right)^{-3 \beta \gamma e 2}.$$  

Such speeds are not expected and we can reasonably surmise that secondary CRe in the Coma cluster are non-negligible. We touch on the implications of this in the next section.

6 DISCUSSION AND CONCLUSIONS

We presented an analysis of CR populations based on radio and $\gamma$-ray observations. Given a cluster model, radio observations constrain the CRe population and, assuming steady state, the CRe source strength. $\gamma$-ray upper limits put upper limits on the CRp density present in the cluster. We then combined these results in two limiting cases. In the case where we assume secondary CRe production is negligible, we arrived at a minimum transport speed necessary to reduce the CRp density below the upper limits. In the case where we assume primary CRe production is negligible, we obtain a $\gamma$-ray flux prediction to be compared to the upper limits.

Using radio (Thierbach et al. (2003)) and $\gamma$-ray (Ackermann et al. (2016)) observations of the Coma cluster we have found that under the assumption that all synchrotron-emitting CRe are directly accelerated, we require CRp to
be transported outward at a bulk speed of 10-100 km s\(^{-1}\)
or with diffusion coefficient \(10^{31}-10^{32} \text{ cm}^2 \text{s}^{-1}\) in order to explain the lack of \(\gamma\)-ray detection. However, at such comparatively low transport speeds we expect CRe secondaries to make significant contribution to the population. We have also found that unless the magnetic field in the center of the Coma cluster is around 10 \(\mu\)G or above, much higher than observed, the CRe population cannot be dominated by secondaries or else the \(\gamma\)-ray flux prediction exceeds upper limits. This suggests neither component, primaries or secondaries, can be consistently ignored. The observations of Coma can only be explained by a hybrid population of primary CRe which are being continuously injected into the cluster and secondary CRe which arise from hadronic interactions. A more detailed treatment is therefore necessary to examine CR transport in the Coma cluster. We leave such a treatment for future work, but speculate that the inclusion of a small secondary component in the primary-dominated analysis would alleviate the limits derived here: the presence of secondaries implies less acceleration of primary CRe is needed to produce the observed radio emission, which in turn implies CR \(p\) are produced at a slower rate.

The primary-dominated assumption is still useful as an approximation for a real cluster. In section 4 we presented the dependence of our results on the assumed magnetic field model. We can conversely use observations of the magnetic field to constrain other parameters in our model, namely the relative acceleration efficiency \(\zeta\). For example, if we fix our magnetic field model using the values from Bonafede et al. (2010) \((B_0 = 5 \mu\text{G}, \alpha_B = 0.5, l_{\text{cor}} = 34 \text{ kpc})\), we find that \(\zeta\) must be less than 15. The value of this limit depends on the assumed level of wave damping. Various regions of the (extensive) parameter space can be ruled out in this way.

Finally, we reiterate that while we have specified our results to the example of the Coma cluster, the analysis is general and can be used on any individual system reasonably approximated by spherical symmetry.

We are happy to acknowledge discussions with Gianfranco Brunetti. JW and EGZ acknowledge support by NSF Grant AST-1616037, the WARF Foundation, and the Vilas Trust.

REFERENCES

Ackermann M., et al., 2016, ApJ, 819, 149
Bonafede A., Feretti L., Murgia M., Govoni F., Giovannini G., Dallacasa D., Dolag K., Taylor G. B., 2010, A&A, 513, A30
Briel U. G., Henry J. P., Boehringer H., 1992, A&A, 259, L31
Brown S., Rudnick L., 2011, MNRAS, 412, 2
Brunetti G., Blasi P., Reimer O., Rudnick L., Bonafede A., Brown S., 2012, MNRAS, 426, 956
Deiss B. M., Reich W., Lesch H., Wielebinski R., 1997, A&A, 321, 55
Desiati P., Zweibel E. G., 2014, ApJ, 791, 51
Enßlin T., Frommert C., Miniati F., Subramanian K., 2011, A&A, 527, A99
Guo F., Oh S. P., 2008, MNRAS, 384, 251
Huber B., Tchernin C., Eckert D., Farnier C., Manalaysay A., Straumann U., Walter R., 2013, A&A, 560, A64
Kelner S. R., Aharonian F. A., Bugayov V. V., 2006, Phys. Rev. D, 74, 034018
Loewenstein M., Zweibel E. G., Begelman M. C., 1991, ApJ, 377, 392
Minnie J., Matthaeus W. H., Bieber J. W., Ruffolo D., Burger R. A., 2009, Journal of Geophysical Research (Space Physics), 114, A01102
Pfrommer C., Enßlin T. A., Springel V., 2008, MNRAS, 385, 1211
Rybicki G. B., Lightman A. P., 1979, Radiative Processes in Astrophysics. New York: Wiley
Thierbach M., Klein U., Wielebinski R., 2003, A&A, 397, 53
Vogt C., Enßlin T. A., 2005, A&A, 434, 67
Wiener J., Oh S. P., Guo F., 2013, MNRAS, 434, 2209
Wiener J., Zweibel E. G., Oh S. P., 2018, MNRAS, 473, 3095
Zweibel E. G., 2017, Physics of Plasmas, 24, 055402