Notice: Signature Page Not Included

This thesis has been signed and approved by the appropriate parties.

The Signature page has been removed from this digital version for privacy reasons.

The Signature page is maintained as part of the official versions of the thesis in print that is kept in Special Collections of Reed Library at SUNY Fredonia.
A Study of Students’ Preference Towards Different Methods of Finding Solutions to Quadratic Equations

By

Joshua E. Zebracki

A Thesis
Submitted in Partial Fulfillment
Of the Requirements for the Degree of
Master of Science in Education-Mathematics
Department of Mathematical Sciences
At the State University of New York at Fredonia
Fredonia, New York
July 2011
Table of Contents

I. Introduction .............................................................................................................. 1

II. Literature Review .................................................................................................... 3
   A. Historical Approaches and Perspectives on Quadratics .................................... 3
   B. Current Textbook Approaches to Solving Quadratic Equations ....................... 8
      1. Solving Quadratic Equations by Factoring ................................................. 9
      2. Solving Quadratic Equations Graphically ................................................. 11
      3. Solving Quadratic Equations by Completing the Square ......................... 13
      4. Solving Quadratic Equations Using the Quadratic Formula ...................... 16

III. Experimental Design .......................................................................................... 20
   A. Subjects ............................................................................................................. 20
   B. Design ................................................................................................................. 22
      1. Development of the Quadratic Equations Assessment ............................... 22
      2. Development of the Survey ................................................................. 24

IV. Methods of Analysis ........................................................................................... 26
   A. Analysis of the Quadratic Equations Assessment ........................................ 26
   B. Analysis of the Survey ....................................................................................... 27

V. Results .................................................................................................................... 28
   A. Result 1: Students Alternating Methods ....................................................... 28
   B. Result 2: No Students Used Completing the Square .................................... 30
   C. Result 3: Most Frequent Solution Method Used ........................................... 30
   D. Result 4: Preferred Solution Methods of Students ........................................ 32
   E. Quadratic Equation Assessment Results ...................................................... 34
1. Problem 1: \( x^2 - 4x - 5 = 0 \) ................................................................. 34
2. Problem 7: \( 9x - 14 = x^2 \) ................................................................. 37
3. Problem 8: \( 4x^2 - 64 = 0 \) ................................................................. 38
4. Problems 2 Through 6 .............................................................................. 40

F. Survey Results ...................................................................................... 41

1. Question 1: Knowledge of Solution Methods ........................................ 41

VI. Implications for Teaching ...................................................................... 43
    A. Implication 1: Multiple Representations ............................................ 43
    B. Implication 2: Correct Vocabulary and Terminology .......................... 44
    C. Closing Remarks ............................................................................. 44

VII. References ............................................................................................ 46

VIII. Appendix ............................................................................................. 48
     A. Appendix A: Student Consent Form ................................................. 48
     B. Appendix B: Quadratic Equation Assessment .................................... 51
     C. Appendix C: Follow-Up Survey ....................................................... 53
     D. Appendix D: Student Sample Problem 7 ......................................... 54
Introduction

This research explores students’ methods of preference when finding solutions to quadratic equations. The four methods discussed are solving quadratic equations by factoring, graphically, by utilizing the method of completing the square, and applying the quadratic formula. The use of multiple representations is a key component in any classroom, and the more alternatives a student can choose from when solving a problem, the better off they will be. Given any quadratic equation, students can choose any of the four approaches stated above to find the roots of the equation. But what most students do not realize is that some quadratics are more easily solved utilizing one approach over another.

I was interested in this topic because of the multiple approaches that are available for students when solving quadratic equations. During my teaching experiences, when presenting the quadratic formula and the method of completing the square, I would often wonder which approach was faster and more efficient. I would often pit student against student, one using the quadratic formula and the other using the completing the square method, where each student must solve the same quadratic equation. Generally, after numerous attempts, neither method would be victorious by a large margin. This is when I questioned if students realized that different quadratic equations were solved more efficiently by one approach over the others. During these contests, I focused primarily on the two methods stated above and did not concentrate too much on solving quadratic equations by factoring or by finding the roots graphically.

Being confined to one approach when multiple methods are available undermines the purpose of evolving the students’ problem solving abilities. The ability to realize that there are some techniques that are more effective and efficient when solving problems than others is a
great component to problem solving and allows the student to overcome challenges with less effort.

This is the basis behind my study and hypothesis. My study focuses on the two approaches for solving quadratic equations that can be utilized for any given equation: the method of completing the square and the quadratic formula.

*It is hypothesized that when solving quadratic equations, college students will utilize only one method to find the solutions to the quadratic equation instead of alternating approaches when given different types of quadratic equations, even when a problem favors one of the approaches.*

I tested this hypothesis by administering a quadratic equation assessment to students enrolled in Survey of Calculus I and University Calculus I. The quadratic equation test was made up of eight quadratic equations that varied in types. Following the assessment, students were given a survey which had the students record their preferred method of solving quadratic equations and any other method they had knowledge of. I analyzed the data by comparing the percentage of how much each method was used, and also the accuracy of each method. Furthermore, the survey was taken into consideration by observing what opinions the participants had on the given methods of finding solutions to quadratic equations and the number of approaches they were able to list.
Literature Review

The purpose of this literature review is to investigate the topics that are associated with quadratic equations. More specifically, this review examines the different methods that are utilized to find solutions to quadratic equations. The first section delves into the history of quadratic equations, which dates back to the Babylonian era. Next, the current textbook approaches to introducing the different techniques for solving quadratics are analyzed, along with the perspectives of various educators and researchers. There is a section for four of the methods that are taught during a student’s secondary education: solving quadratic equations by factoring, solving quadratic equations graphically, using the method of completing the square, and finally solving quadratic equations by applying the quadratic formula. But first, it is important to understand the evolution of the methods to solving quadratic equations.

Historical Approaches and Perspectives on Quadratics

Methods for solving quadratics date back to as early as the Babylonians. Others, such as the Chinese and Arabs, also had methods for finding solutions to quadratics (Allaire, 2001; Burton, 2007; Katz, 2004; Radford & Guerette, 2000). These methods used by the early mathematicians would pave the way for the methods used today, mainly completing the square and the quadratic formula. In order to know how to use the formulas, it is essential to explore the history.

The first account of quadratic equations can be dated back to the Babylonian and their approach for solving the problems that they were challenged with. Allaire’s research (2001) cites a problem that the Babylonians usually solved when solving quadratics “If the sum of two
numbers is 20 and the product is 96, then what are these numbers?” (p. 311) In order to complete this task, they would apply their own algorithm. According to Allaire, the Babylonians would follow these certain steps to find an answer to any quadratic equation. Figure 1 displays the certain steps that the Babylonians took to find solutions to quadratic equations.

**Figure 1. Babylonian Algorithm for Finding Solutions to Quadratic Equations.**

| Step | Formula |
|------|---------|
| 1. Divide the sum $S$ in half. | $S/2$ |
| 2. Square the answer from part 1. | $(S/2)^2$ |
| 3. Subtract the product $A$ from the result in part 2. | $(S/2)^2 - A$ |
| 4. Take the square root of the result in part 3. | $\sqrt{(S/2)^2 - A}$ |
| 5. Add the answer in part 4 to the answer in part 1 to determine one dimension of the field. | $S/2 + \sqrt{(S/2)^2 - A}$ |
| 6. Subtract the answer to part 4 from the answer to part 1 to determine the other dimension of the field. | $S/2 - \sqrt{(S/2)^2 - A}$ |

Figure 1. Algorithm that the Babylonians utilized to solve any quadratic equation. Adapted from “Geometric Approaches to Quadratic Equations from Other Times and Places,” by P. Allaire, 2001, *Mathematics Teacher, 94*(4), p. 311.

By analyzing the process the Babylonians utilized when finding solutions to quadratic equations, we can see the presence of the completing the square method because of the fact that some number is divided in half, then squared, which is how one would start to complete the square. Allaire also states that, unlike what numerous educators do, the Babylonians did not utilize any form of geometry for their solution methods. They simply used the above algorithm to find the solutions (Allaire, 2001).

There is a slight recognition of the completing the square method in the method used by
the Babylonians, but there is also a very early form of the quadratic formula. Burton (2007) states

The Babylonian instructions amount to use a formula equivalent to the familiar rule
\[ x = \left( \frac{a}{2} \right)^2 + b - \left( \frac{b}{2} \right), \]
for solving quadratic equation \( x^2 + ax = b \). Although the Babylonian mathematician had no 'quadratic formula' that would solve all quadratic equations, the instructions in these concrete examples are so systematic that we can be pretty certain they were intended to illustrate a general procedure. (p. 67)

Not only are there early signs of the completing the square method in the Babylonian era, but there are also traces of an ancient procedure that resembles the quadratic formula.

As stated previously, the completing the square method could be recognized in the Babylonian approach to solving quadratics, but it became much more prominent in the ninth century. Identified by Allaire (2001), the Islamic mathematician Al-Khwarizmi wrote two methods for finding solutions to quadratic equations in his book, *The Calculation of al-Jabr and al-Muqabala*. Unlike the Babylonians, Al-Khwarizmi utilized geometry to illustrate why the algebraic method produced the correct solutions to quadratic equations, which was demonstrated in the book written by Al-Khwarizmi (Allaire, 2001). The starting point for Al-Khwarizmi was to construct a figure composed of a square and two rectangles. In order to find a solution, his procedure calls to complete the figure to form a square (Burton, 2007). Figure 2, shown below, displays the process that Al-Khwarizmi took to solve the quadratic equation \( x^2 + 10x = 39 \).

Burton (2007) states

The starting point is a figure composed of a square of side \( x \) (and area \( x^2 \)) and two rectangles, each having length \( x \) and width \( 10/2 \). Because the area of each rectangle is \( x(10/2) \), the area of the entire figure is \( x^2 + 2(10/2)x \). To complete the figure so as to form a square, it is necessary to add a new square of area \( (10/2)^2 \). The area of the completed square is \( (x + 10/2)^2 \). (p. 243)
Figure 2. An Example of a Geometric Representation of Al-Khwarizmi’s Solution Method to Quadratic Equations.

Figure 2. The completing the square method to solving quadratic equations represented geometrically. Adapted from D. Burton, 2007, The History of Mathematics: An Introduction (6th Edition), p. 243. Copyright 2007 by The McGraw-Hill Companies, Inc.

From this point, the quadratic equation would be \( x^2 + 2(10/2)x + (10/2)^2 = 64 \), which then can be written as \((x + 10/2)^2 = 64\). After solving this quadratic equation, one would receive the answer of \( x = 3 \). Islamic mathematicians during the era of Al-Khwarizmi did not consider solutions, or even coefficients, to be negative. Therefore the only linear equation that is necessary to solve from the above quadratic equation is \( x + 10/2 = 8 \). According to Allaire, one can hypothesize that the procedure that is used today, the completing the square method, was utilized as early as the ninth century.

Another prominent mathematician who provided an alternate approach to solving quadratic equations was René Descartes. Allaire (2001) states that Descartes was able to find solutions to quadratic equations in the form of \( x^2 + bx = c \) geometrically. But negative roots are not produced when solving these equations because Descartes did not utilize Cartesian axes. Figure 3 is the geometric construction that Descartes produced in order to solve quadratic equations.
Figure 3. Geometric Construction Used for Descartes’ Method to Solve Quadratic Equations.

Figure 3. Using Descartes’ approach to solving quadratic equations, the steps involved will lead one to produce this geometric construction in order to solve for $x$ geometrically. Adapted from “Geometric Approaches to Quadratic Equations from Other Times and Places,” by P. Allaire, 2001, Mathematics Teacher, 94(4), p. 312.

The first step that Descartes followed was to construct a segment $AB$ such that the length was equal to $\sqrt{c}$. Once this is completed, a perpendicular must be erected to the segment $AB$ at point $A$, labeled line $AC$ with a length of $b/2$. Next, one is to construct a circle that has a center of $C$ and a radius of $b/2$, otherwise labeled as line $AC$. Finally, a line must be constructed between points $B$ and $C$. This segment will intersect the circle at points $E$, which is between points $C$ and $B$, and $D$. Finally, it can be concluded that $x = BE$, according to Figure 3, above. To reinforce this statement, one may observe Figure 4, a proof by using the Pythagorean Theorem. The only value for segment $BE$ that will produce $x^2 + bx = c$ after using the Pythagorean Theorem is $x$. 
**Figure 4.** Proof of Descartes’ Method to Solving Quadratic Equations Geometrically Using the Pythagorean Theorem.

\[
CB^2 = AC^2 + AB^2,
\]

\[
\left(x + \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 + (\sqrt{c})^2,
\]

\[
x^2 + bx + \frac{b^2}{4} = \frac{b^2}{4} + c,
\]

\[
x^2 + bx = c.
\]

Figure 4. The proof of Descartes’ geometric approach to solving quadratic equations utilizing the Pythagorean Theorem. Adapted from “Geometric Approaches to Quadratic Equations from Other Times and Places,” by P. Allaire, 2001, *Mathematics Teacher, 94*(4), p. 313.

The history of solving quadratic equations illustrates various methods that mathematicians utilized to produce solutions. Some of these methods aided in the approaches that the current textbooks present to students today, such as the completing the square method and the quadratic formula.

**Current Textbook Approaches to Solving Quadratic Equations**

Commonly, the different methods used to solve quadratics are taught at distinct times. Students learn to solve quadratic equations as early as ninth grade, where they learn the factoring and graphing method (Gantert, 2007). But the method of completing the square and the quadratic formula are not taught until Algebra 2, Trigonometry, and Precalculus (Blitzer, 2007; Blitzer, 2010; Esty, 1997; Hungerford, Jovell & Mayberry, 2004; Keenan & Gantert, 1991;
There are also numerous articles that describe how educators go about introducing these methods in their classrooms. Some prefer to use technology (McCallum, 2001; Phelps & Edwards, 2010), while others use a different method, such as derivation of the formula (Karp, 2007; Kotelawala, 2010; Norton, 2008; Vaiyavutjamai & Clements, 2006; Ward, 2003). And a number of educators seem to be entirely against some methods of solving quadratics (Bosse & Nandakumar, 2005). It suffices to analyze these methods separately in the order in which they are introduced in the textbooks.

**Solving Quadratic Equations by Factoring**

Typically, the first method that students are taught to find solutions to quadratic equations is by factoring, which is introduced in the ninth grade (Gantert, 2007). In this text, there are precise steps that Gantert states for the students to follow. They are to first rewrite the quadratic equation in standard form, factor the expression on the left-hand side, set each factor containing the variable equal to 0, solve each of the resulting linear equations, and finally the students must check their answers (p. 504). But throughout this text, the author does not reproduce the steps necessary to factor a polynomial of the second degree, which is typically how one would solve a quadratic equation.

The Algebra 2 and Trigonometry textbooks present the factoring method before the introduction of the approach by completing the square and the quadratic formula. Shults, Ellis, Hollowell, and Kennedy (2001) delve into the factoring method in detail, ranging from how to factor a greatest common factor, to using Algebra Tiles. They also state how to factor any quadratic expression that are in the form \( ax^2 + bx + c \): "look for integers \( r \) and \( s \) such that \( rs = c \) and \( r + s = b \). Then factor the expression." (p. 291) The factored form of this quadratic
expression will resemble \((x + r)(x + s)\). Also, the authors display a method to factor the difference of two perfect squares and perfect-square trinomials so the student has even more guidelines to follow. Figures 5 and 6, which follow, display the techniques that the authors use to present this topic.

**Figure 5.** Factoring Quadratic Equations Using Algebra Tiles.

You can model a quadratic expression that can be factored with algebra tiles, as shown below.

The rectangular region formed by algebra tiles above illustrates that the total area, \(x^2 + 7x + 10\), can be represented as the product \((x + 5)(x + 2)\).

Figure 5. Algebra Tiles being used to display how to factor a quadratic expression. Adapted from J.E. Schultz, W. Ellis, K.A. Hollowell, and P.A. Kennedy, 2001, *Algebra 2*, p. 291. Copyright 2001 by Holt, Rinehart, and Winston.

**Figure 6.** Technique Used for Finding the Factored Forms of Quadratic Equations

**Look for a pattern.** Notice how the sums and products of the constants in the binomial factors are related to the last two terms in the unfactored expression.

\[
x^2 + 7x + 10 = (x + 5)(x + 2)
\]

\[
\begin{align*}
5 + 2 &= 7 \\
5 \times 2 &= 10 \\
-5 - 2 &= -7 \\
-5 \times (-2) &= 10
\end{align*}
\]

\[
x^2 + 3x - 10 = (x + 5)(x - 2)
\]

\[
\begin{align*}
5 - 2 &= 3 \\
5 \times (-2) &= -10 \\
-5 + 2 &= -3 \\
-5 \times 2 &= -10
\end{align*}
\]

The patterns shown above suggest a rule for factoring quadratic expressions of the form \(x^2 + bx + c\).

Figure 6. A tip for finding the factored forms of quadratic equations. Adapted from J.E. Schultz, W. Ellis, K.A. Hollowell, and P.A. Kennedy, 2001, *Algebra 2*, p. 291. Copyright 2001 by Holt, Rinehart, and Winston.
A different approach to teach the factoring method is commonly taken in precalculus textbooks (Blitzer, 2010; Esty, 1997; Hungerford, Jovell & Mayberry, 2004). The authors assume that the students are able to factor a trinomial when they reach this certain level of mathematics. Blitzer (2010) provides steps for the students to follow and only renders a couple of examples for the student to complete.

Some educators feel that the factoring method is a circumstantial and insufficient method for solving quadratic equations. For example, Bosse (2005) states that the probability of being able to factor a trinomial to solve a quadratic equation is very low, considering the different coefficients that may be present in the problem. He also states that the student only chooses to use the method of completing the square or the quadratic formula after frustration has risen from trying to factor the quadratic, which implies that those approaches are a last resort when it comes to finding solutions.

**Solving Quadratic Equations Graphically**

Traditionally, along with factoring, solving quadratic equations by the use of a graphing calculator is one of the first methods that students start to utilize when they enter the ninth grade. The graphing calculator can be an extremely powerful piece of technology when solving quadratics (Phelps & Edwards, 2010; Taylor & Mittag, 2001), on the other hand, it can also be used as a crutch for students in their mathematics class (McCallum, 2001). In various textbooks, authors do not have specific sections for solving quadratic equations graphically. Instead, tutorials and examples are dispersed throughout every section to guide the reader with multiple representations (Blitzer, 2007; Esty, 1997; Hungerford, Jovell & Mayberry, 2004; Shults, Ellis, Hollowell & Kennedy, 2001). Unfortunately, using the graphing calculator as a method for
finding solutions of quadratic equations appears to be omitted in certain textbooks (Blitzer, 2010).

Frequently, the approach used to introduce solving quadratic equations by using a graphing calculator is comparable among different authors. For example, Blitzer (2007) explores the use of graphing calculators by first explaining that the real solutions, or zeros, of the quadratic equation are the x-intercepts of the quadratic function. Along with explanations, visuals of the graphs are also present to allow the reader to actually see what the quadratic equation and its solutions appear to be. Figure 7, shown below, illustrates how the author presents the material. Throughout the sections dealing with methods for finding solutions of quadratic equations, the author continues to provide examples and figures so that the students can witness multiple representations of each problem given.

**Figure 7.** Approach Educators Utilize for Presenting Solution Methods to Solving Quadratic Equations

**Technology**

You can use a graphing utility to check the real solutions of a quadratic equation. The real solutions of \( ax^2 + bx + c = 0 \) correspond to the x-intercepts of the graph of \( y = ax^2 + bx + c \). For example, to check the solutions of \( 2x^2 + 7x - 4 = 0 \), graph \( y = 2x^2 + 7x - 4 \). The cuplike U-shaped graph is shown on the right. Note that it is important to have all nonzero terms on one side of the quadratic equation before entering it into the graphing utility. The x-intercepts are \(-4\) and \(2\), and the graph of \( y = 2x^2 + 7x - 4 \) passes through \((-4, 0)\) and \((2, 0)\). This verifies that \( \{-4, 2\} \) is the solution set of \( 2x^2 + 7x - 4 = 0 \).

Figure 7. A display of how some authors present the method of solving quadratic equations graphically. Adapted from R. Blitzer, 2007, *Algebra and Trigonometry* (3rd Edition), p. 132. Copyright by Prentice Hall, Inc.
Solving Quadratic Equations by Completing the Square

Generally, the method of completing the square is taught prior to the quadratic formula. The depth of the sections and the presentation of this approach to solving quadratic equations vary among authors and educators. Some textbooks present the method of completing the square briefly and then delve into the quadratic formula (Blitzer, 2010; Esty, 1997; Hungerford, Jovell & Mayberry, 2004). Whereas other textbooks and educators provide visual examples that include pictorial representations and the use of manipulatives, such as Algebra Tiles (Blitzer, 2007; Gardella, 2000; Kotelawala, 2010; Phelps & Edwards, 2010; Picciotto, 2008; Schultz, Ellis, Hollowell & Kennedy, 2001; Yarema & Hendricks, 2010). Shockingly, the completing the square method is entirely absent from certain literature e.g., (Keenan & Gantert, 1991). It is apparent that mathematics teachers, researchers, and authors have opposing viewpoints on how to introduce this topic.

Certain textbooks do not add a visual reinforcement when introducing the method of completing the square. For example, Hungerford, Jovell, and Mayberry (2004) provide minimal coverage of this technique used to solve quadratic equations. Located in their textbook, is a procedure consisting of four steps which the authors assume the students are able to follow to complete the square. The authors explain that the reader must

Write the equation in the form of \( x^2 + bx = c \). Add \((b/2)^2\) to both sides so that the left side is a perfect square and the right side is a constant. Take the square root of both sides. Simplify. (p. 91)

Once this is stated, the author then proceeds to guide the audience through one example using the completing the square method, then continues onto deriving the quadratic formula. No other examples or explanations of completing the square are present in this textbook. However, the
author states that it is important for students to understand the procedure of completing the square because of the appearance of this technique in later chapters.

An approach taken by some authors and educators is to provide visualizations to aid students' understanding of the completing the square technique. Blitzer (2007) follows this procedure when introducing the completing the square method. By observing Figure 8 located on the following page, one can see how Blitzer goes about explaining this method geometrically to enable the student to see how to geometrically complete the square. One can notice that Blitzer introduces a square, and then four rectangles attached to each side. This diagram represents the quadratic expression $x^2 + 8x$, which is the area of this figure. The author continues to illustrate that one must add 16 smaller squares with a side length of 1 to transform the object into a square. Therefore, this new figure represents $x^2 + 8x + 16 = (x + 4)^2$. To conclude this section, Blitzer provides numerous examples with guided solutions for the students to try on their own and check their mathematics.

Educators feel that manipulatives are useful when students are attempting to understand how to complete the square (Picciotto, 2008). In their article, Yarema and Hendricks (2010) describe a set of activities designated for students to use virtual and physical Algebra Tiles to help participants understand how to complete the square, along with the derivation of the quadratic formula. The students are given certain problems and are then asked to use Algebra Tiles to represent the quadratic equations geometrically. Yarema and Hendricks stated that

After seeing the geometric representation for completing the square, students quickly generalize the geometric procedure to an algebraic one: $\frac{1}{2}$ of the coefficient of the x term and square it. Following a similar set of instructions, students who physically manipulate algebra tiles come to the same generalized procedure. (p. 32)
Figure 8. Visualization Used to Display the Completing the Square Method for Better Understanding.

![Figure 8](image-url)

Figure 8. A geometric representation of the completing the square method to solving quadratic equations. Adapted from R. Blitzer, 2007, *Algebra and Trigonometry* (3rd Edition), p. 134. Copyright by Prentice Hall, Inc.

So after working with these visuals, students are able to derive for themselves how to complete a square algebraically. Yarema and Hendricks also state that the students who participated in these exercises demonstrated a firmer understanding of the quadratic formula and were able to reproduce the derivation that was investigated in this activity.

When introducing the technique of completing the square, the topic is generally taught by introducing visuals. The students are given Algebra Tiles to manipulate by themselves and to figure out what quantity they require to complete the square. It is apparent that this type of approach helps the students better understand the concepts of completing the square and the
quadratic formula. On the other hand, some authors and educators barely cover, or entirely omit, the use of the method of completing the square from their units. Instead, the authors delve into the quadratic formula by deriving it through completing the square.

**Solving Quadratic Equations Using the Quadratic Formula**

The quadratic formula is traditionally the last method to solve quadratic equations presented in the selected textbooks. When demonstrating the quadratic formula, authors seem to use the same approach. Before the students begin to apply the quadratic formula, they are presented with the derivation of the quadratic formula by using the completing the square method (Blitzer, 2007; Blitzer, 2010; Esty, 1997; Hungerford, Jovel & Mayberry, 2004; Schultz, Ellis, Hollowell & Kennedy, 2001). Often, educators and researchers believe that having students discover the quadratic formula before applying it is essential to student understanding (Kotelawala, 2010; Yarema & Hendricks, 2010). However, some mathematics teachers find that deriving the quadratic formula from the method of completing the square does not help the students to understand the quadratic formula at all (Picciotto, 2008).

The approach that is unanimous among selected textbooks on the topic of presenting the quadratic formula is by showing the derivation by completing the square. For example, Blitzer (2010) addresses the section of the quadratic formula very precisely. He starts out by providing a table that lists out each step clearly with guided comments and an example of that certain step. Figure 9, shown on the following page, displays how Blitzer explains the process of deriving the quadratic formula. Following the chart, Blitzer provides the quadratic formula with a statement of the formula in words. To end the section, similarly to the other textbooks examined in this literature review, Blitzer provides numerous examples that allow the reader to practice this method of solving quadratics.
Figure 9. The Procedure Followed to Derive the Quadratic Formula by Completing the Square.

| General Form of a Quadratic Equation | Comment | A Specific Example |
|-------------------------------------|---------|-------------------|
| $ax^2 + bx + c = 0$, $a > 0$       | This is the given equation. | $3x^2 - 2x - 4 = 0$ |
| $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ | Divide both sides by $a$ so that the coefficient of $x^2$ is 1. | $x^2 - \frac{2}{3}x - \frac{4}{3} = 0$ |
| $x^2 + \frac{b}{a}x = -\frac{c}{a}$ | Isolate the binomial by adding $-\frac{c}{a}$ on both sides of the equation. | $x^2 - \frac{2}{3}x = \frac{4}{3}$ |
| $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ | Complete the square. Add the square of half the coefficient of $x$ to both sides. | $x^2 - \frac{2}{3}x + \left(-\frac{1}{3}\right)^2 = \frac{4}{3} + \left(-\frac{1}{3}\right)^2$ |
| $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$ | Factor on the left side and obtain a common denominator on the right side. | $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$ |
| $\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$ | Add fractions on the right side. | $\left(x - \frac{1}{3}\right)^2 = \frac{4}{3} + \frac{1}{9}$ |
| $\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$ | Add fractions on the right side. | $\left(x - \frac{1}{3}\right)^2 = \frac{12 + 1}{9}$ |
| $\left(x - \frac{1}{3}\right)^2 = \frac{13}{9}$ | Add fractions on the right side. | $\left(x - \frac{1}{3}\right)^2 = \frac{13}{9}$ |
| $x = \pm \frac{1}{3} \pm \frac{\sqrt{13}}{9}$ | Apply the square root property. | $x = \frac{1}{3} \pm \frac{\sqrt{13}}{9}$ |
| $x = \pm \sqrt{\frac{b^2 - 4ac}{2a}}$ | Take the square root of the quotient, simplifying the denominator. | $x = \frac{1}{3} \pm \frac{\sqrt{13}}{3}$ |
| $x = \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ | Solve for $x$ by subtracting $\frac{b}{2a}$ from both sides. | $x = \frac{1}{3} \pm \frac{\sqrt{13}}{3}$ |
| $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$ | Combine fractions on the right side. | $x = \frac{1}{3} \pm \frac{\sqrt{13}}{3}$ |

Figure 9. The approach used to derive the quadratic formula by using the completing the square method. Adapted from R. Blitzer, *Precalculus* (4th Edition), 2010, p. 137. Copyright by

Some educators and researchers agree with the textbook approach and believe that the derivation by completing the square is the most efficient and effective way to present the quadratic formula to students. Kotelawala (2010) states the steps that occurred during a collaborative planning exercise with two other mathematics teachers who discussed how to present a unit on the quadratic formula. The intended goal for this collaborated unit was to have students derive the quadratic formula independently. Kotelawala cites that one approach the group discussed was to use visuals, such as Algebra Tiles, to guide the students along their investigation of the quadratic formula. The group noted: "Although the use of algebra tiles had limitations, team members found that the power of the visual outweighed the limits" (p. 674).
Whereas some educators mold their units around the approach stated above, some believe that it is not sufficient in helping the students truly understand the quadratic formula. Picciotto (2008) believes that by showing students the derivation of the quadratic formula by completing the square does not make anything more understandable, only more confusing. Furthermore, since graphing technology is readily available in classrooms, he investigated a derivation of the quadratic formula by observing the intersection points of a line and a hyperbola. Picciotto cites that this approach to the derivation of the quadratic formula is nontraditional; however there is another option on “how to present a core part of the high school mathematics curriculum” (p. 477).

When students are first introduced to the quadratic formula, all they see is a formula. In defense, that is exactly what the quadratic formula is. It is not a technique to solve quadratic equations; however it is only a formula. To aid in students’ understanding, it is essential to demonstrate where this formula originates from. Whether it is from the method of completing the square, where historically the quadratic formula comes from, or a nontraditional way, for example, by intersecting certain lines and hyperbolas, this demonstration will help reinforce the students’ comprehension of this mathematical topic.

Solving quadratic equations, generally, is a challenge faced by students in their secondary and post-secondary education. There are many tools at their disposal to find solutions, such as factoring the quadratic polynomial, using a graphing calculator, applying the technique of completing the square, or utilizing the quadratic formula. Each method has strengths, but at the same time, each approach has weaknesses that make it a less optimal method to choose given certain circumstances. For example, Bosse and Nandakumar (2005) state that factoring is certainly the fastest to perform, but what if the quadratic is not factorable? Also, the method of
completing the square and applying the quadratic formula will let the user “determine the solution of any quadratic expression” (p. 144), but the students’ mathematical confidence may be nonexistent to perform operations with fractions and radicals.

Below, in Figure 10, Bosse and Nandakumar (2005) cite when each method stated above is most efficient at solving different forms of quadratic equations.

**Figure 10.** Steps to Choose the Most Efficient Solution Method for Quadratic Equations.

| Is the equation in the form ...? | Yes | Use Square Root Method |
|---------------------------------|-----|------------------------|
| \( x^2 + 6 = 0 \)               |     |                        |
| \((rx)^2 + f = 0\)             |     |                        |
| \((rx^2 + h)^2 = f\)           |     |                        |
| \((rx + f)^2 = 0\)             |     |                        |

| Is the equation obviously factorable? | Yes | Factor the Equation |
|--------------------------------------|-----|---------------------|
| \( ax^2 + bx = 0 \)                  |     |                     |
| or in standard form with obvious factorization |     |                     |
| \( ax^2 + bx + c = 0 \)              |     |                     |

| Is \( a = 1 \)? | Yes | Complete the Square |
|-----------------|-----|---------------------|
|                 |     |                     |
| Quadratic Formula | or divide the equation by \( a \) and Complete the Square

Figure 10. A display of what to take into consideration when choosing a solution method for solving quadratic equations. Adapted from “The Factorability of Quadratics: Motivation for More Techniques,” by M. Bosse and N. Nandakumar, *Teaching Mathematics and Its Applications: An International Journal of the IMA*, 24(4), p. 149.

Do students follow the same chart characterized by the figure below when finding solutions to quadratic equations? Or would the students utilize the same method disregarding the type of quadratic equation that is presented? These questions directly relate to the hypothesis statement that was given previously. A more detailed discussion of these questions is presented in the sections that follow.
Experimental Design

This experiment was designed to test the hypothesis that when solving quadratic equations, college students would utilize only one method to find the solutions to the quadratic equation instead of alternating approaches when given different types of quadratic equations, even when a question favors one of the approaches. During this experiment, students were administered an eight-problem assessment that had the students solve quadratic equations of varying types. Following the quadratic equation test, the students completed a survey encompassing questions and opinions about quadratic equations and their solution methods. The survey was paired with the test for each student, which was then evaluated for correctness, the types of methods used, and the frequency of each method used.

Subjects

This study was conducted in two different educational settings at the collegiate level. Both groups of participants in this experiment were enrolled at SUNY Fredonia in Fredonia, which is a state university located in Western New York. There are approximately 5,800 students attending SUNY Fredonia, which includes individuals from different counties throughout New York State. The breakdown of the counties can be found in Figure 11.

The first group of students who participated at this site was enrolled in Survey of Calculus I. Survey of Calculus I is an introduction to differential calculus course that incorporates single variable functions that are applied to different aspects of business and economics. This course is available for students who need to fulfill a general education requirement for mathematics, or for various majors such as business administration or biology. There were exactly 31 students in this participating group. The group was predominately Caucasian and consisted of 21 males and 10 females.
Figure 11. Details of the Locations SUNY Fredonia Students Live In.

| County/Location          | Percent/Number |
|--------------------------|----------------|
| Erie County              | 23.9%          |
| Chautauqua County        | 18.4%          |
| Monroe County            | 12.0%          |
| Cattaraugus County       | 4.2%           |
| Niagara County           | 3.5%           |
| Onondaga County          | 3.0%           |
| Other U.S. States        | 113 students   |
| Other Foreign Countries  | 93 students    |

Figure 11. List of locations that students reside in who are enrolled in SUNY Fredonia during 2009. Adapted from www.fredonia.edu.

The second group of students who participated from this location were enrolled in University Calculus I. This is a course on differential calculus that provides students with an introduction to functions, limits, continuity, derivatives and antiderivatives. This class is available to the students who are enrolled in a major that relies heavily on mathematics, such as mathematics, chemistry, physics, and mathematics education. Two sections of University Calculus I created this second grouping. There were 17 students enrolled in one section and 15 students enrolled in the second section, which were taught by two different professors. The group was predominately Caucasian and consisted of 18 males and 14 females. This course requires students to have a prerequisite of either four years of mathematics in high school or University Precalculus.

The subjects who participated in this experiment varied in terms of skill levels when it came to mathematics. All of the participants in this study have graduated high school, but the length of time that they had been out of a mathematics classroom and their skill levels differed.
Typically, the students enrolled in University Calculus I are expected to be at a higher skill level mathematically than the students who are enrolled Survey of Calculus I.

**Design**

This experiment, for each group of participants, was conducted over a period of one day. The subjects were administered an eight-problem quadratic equation test and were given 15 minutes to complete this portion of the experiment.

Following the assessment, the students were given a survey to complete in class. This survey prompted the students to respond to three questions dealing with the methods used to solve quadratic equations and their opinions on which method they prefer. Figure 12, shown below, is a diagram that represents how the experiment was conducted.

**Figure 12.** Details of the Experimental Procedure.

| Day 1 |
|-------|
| Participating Group | Quadratic Equation Test | Survey |

**Development of the Quadratic Equation Assessment**

The quadratic equation test contains 8 problems that instruct the students to find the solutions to quadratic equations algebraically. The exam problems chosen fall into three categories, which are based on which method would be most efficient to use according to Bosse and Nandakumar (2005). The three categories are: utilizing the method of completing the square, applying the quadratic formula, and solving by factoring.
Problem three, \( x^2 - 6x = -11 \), is an example of a quadratic equation that would fall into the completing the square category. To use the completing the square method efficiently, the quadratic equation should not be factorable. Also, the coefficient of the quadratic term must be equal to one. If the coefficient is anything but one, then the student must divide the entire equation by that coefficient. Also, when using this approach, the coefficient in front of the \( x \) term must be divided by two, and then squared. Once this step is complete, this new number is added to both sides. Observing the above example, we can see that the leading coefficient is in fact one, and that the coefficient in front of the linear term is even, thus making it divisible by two. Therefore, the approach that the student should find most efficient would be the completing the square method.

Problem six, \( x^2 + 5x + 7 = 0 \), is an example of a quadratic equation that would fall in the category of applying the quadratic formula. This category encompasses all of the problems that do not fit in the factoring category or the completing the square category. The most effective situation for the student to be in when solving quadratic equations would be when the coefficient in front of the quadratic term cannot be divided easily into the other coefficients or constants in the quadratic equation. Also, another situation is when the coefficient in front of the linear term is odd. When this number is odd, the student would have to use algebra involving fractions if they were to use the completing the square method, which may prove difficult for numerous students.

Problem seven, \( 9x - 14 = x^2 \), falls into the final category, which is solving by factoring. Students are only able to utilize this method when the quadratic equation is factorable. This is true when two integers can be multiplied together to produce the constant term and added together to produce the coefficient of the linear term. This method is most efficient when the
coefficients are integers. Some problems, for example equation 8 in Figure 13 below, is another problem that requires the students to first divide by the leading coefficient before factoring. Also, the student may solve this quadratic equation by factoring out a common factor of four instead of dividing by the leading coefficient.

**Figure 13.** Quadratic Equation Test Used to Conduct the Experiment

| Name: | Date: |
|-------|-------|

**Quadratic Equation Assessment**

Solve the following problems algebraically showing all work necessary to find the solutions. If you use a calculator, make sure all of your work is recorded algebraically.

1. \(x^2 - 4x - 5 = 0\)  
2. \(2x^2 + 6x - 22 = 0\)

3. \(x^2 - 6x = -11\)  
4. \(4x^2 + 32x = 36\)

5. \(3(x^2 - 1) = x\)  
6. \(x^2 + 5x + 7 = 0\)

7. \(9x - 14 = x^2\)  
8. \(4x^2 - 64 = 0\)

**Development of the Survey**

The survey for this experiment contained three questions that had the students respond with their opinions about the methods for solving quadratic equations. The first prompt had the students record all of the approaches that they had knowledge of for finding quadratic equations. The second question, which had the students list their favorite method of finding
solutions to quadratic equations, was included for the sole purpose of seeing which technique they prefered. This question also had the students explain why they selected that specific approach. The follow-up survey can be observed below in Figure 14.

**Figure 14.** The Follow-Up Survey Utilized During The Experiment.

| Name: | Date: |
|------|-------|
| Survey I |       |
| 1. List as many methods for solving quadratic equations as you can. | |
| 2. What is your favorite or preferred method listed above for solving quadratic equations? | |
| 3. Did you use a calculator? If yes, what kind and for which problems? | |
Methods of Data Analysis

The data collected from this study was both quantitative and qualitative. The quantitative data included the scores of the assessments, the total number of instances that a certain method was used to solve each problem, and the percentage of students who found the correct solution for each problem using the given methods. The qualitative data was taken from the survey, which were the students' preferences about which method to utilize when solving quadratic equations.

Analysis of the Quadratic Equation Assessment

The participants completed an eight-problem quadratic equation assessment. The only limitation the students had when completing this exam is that they must solve each quadratic equation algebraically. So the first factor that was analyzed was which method each student used for each problem, which provided an idea as of which method was preferred by students.

Another way that these assessments were analyzed was whether or not the solutions that the participants provided were correct or incorrect, which allowed the possibility to observe the accuracy of each method for the selected problems. To score these problems, each problem was marked completely correct or incorrect. No partial credit was given. If a student omitted the problem, or left it blank, it was counted as an incorrect solution. Each correct solution was assigned one point.

Following the marking of the individual problems, the assessment was broken down by analyzing each problem separately. Given each question, the percent correct was calculated for each method utilized. A chart was constructed to organize the data and to easily compare the
results. A discussion of the accuracy of each method and the students' preferences when solving quadratic equations follows in the results section.

Finally, each assessment was analyzed. Certain methods were favored over others for specific problems due to the efficiency of each approach. For example, question one on the test is easily solved by factoring rather than applying the quadratic formula. So each test was examined to identify if the students used the most efficient method, which was recorded and organized in a chart. This chart was grouped by whether or not a student used the most efficient method for all of the problems, several of the problems, or for none of the problems.

Analysis of the Survey

Along with the quadratic equation assessment, a student survey was analyzed. Following the exam, the participants were given a short questionnaire to determine their opinions about and knowledge of the different methods for solving quadratic equations. Their favorite approach to solving quadratic equations that they wrote on the survey was compiled into a chart. Also, the number of approaches that the participants listed for solving quadratic equations was examined, which indicated if the students have knowledge of all available methods for finding solutions, and it also denoted whether or not the students responded with incorrect solution methods.
Results

After analyzing the eight-problem quadratic equation assessment and the follow-up survey, the following four results emerged:

- When focusing on the questions that could be solved by factoring, approximately a third of the students used the most efficient method for all three problems (problems 1, 7, and 8).
- No students utilized the completing the square method for any of the problems.
- The most frequently used solution method was the quadratic formula (57.14%).
- Survey results indicated that the top two preferred methods for solving quadratic equations were the quadratic formula and solving by factoring.

Result 1 directly relates to the hypothesis that was stated. Results two through four support the corresponding information that is associated with the first result.

Result 1: When focusing on the questions that could be solved by factoring, approximately a third of the students used the most efficient method for all three problems (problems 1, 7, and 8).

This result directly addresses the main component of the hypothesis, which corresponds with the students changing methods for efficiency when solving quadratic equations. When determining whether or not the participant fully switched approaches, only the problems that could be solved by factoring were examined. Upon analysis, it was determined that approximately a third of the students were able to differentiate between the various types of quadratic equations and use the preferred method. Figure 15 displays the results for students
alternating approaches. This was because these were the only problems in which the students were able to apply more than one solution method. Problems 3 and 4 were dismissed from this analysis because no one solved these by completing the square.

Figure 15. The Percentage of Students Who Alternated Methods When Finding Solutions to Quadratic Equations.

| Alternated Methods          | Percentage |
|-----------------------------|------------|
| Fully Alternated            | 33.33%     |
| Partially Alternated        | 30.16%     |
| Did Not Alternate           | 36.51%     |

As stated before, problems 1, 7, and 8 can be solved by factoring, taking the square root of both sides of the equation, and by applying the quadratic formula. But the most efficient method for these three problems is to solve by factoring (or for problem 8, one can take the square root just as easily). So when deciding if the student fully alternated approaches, did he/she use the factoring method for all three problems? If the participant partially alternated methods, then they either used the most efficient method for one or two of these problems, but not for the third. If they did not alternate at all, they did not utilize the more efficient approach.

Figure 15 indicates that approximately 36.51% of the participants did not use the more efficient method available for finding a solution. The majority of those students who did not alternate methods either omitted the problem completely, or used the quadratic formula. The problems that were omitted were counted as did not alternate. Even though the percentages were fairly similar, when added together, two thirds of the students could not differentiate between the
various types of quadratics. It can be concluded that these participants were unable to apply the most efficient solution method.

**Result 2: No students utilized the completing the square method for any of the problems**

Upon analysis of the quadratic equation assessment, it was observed that out of 63 students, none used the completing the square method to find a solution to a quadratic equation. So out of the 504 problems that were solved, none of these were done so by completing the square.

One is able to complete the square easily when the coefficient of the quadratic term is equal to one. One may also complete the square if $a \neq 1$, but the first step would be to divide the quadratic equation by that coefficient. It is more efficient when the coefficient of the linear term is divisible by two so there is no fraction present during operations. Given the quadratic equation assessment that was used in this experiment, problem 3, $x^2 - 6x = -11$, is easily solved by completing the square. For problem 4, $4x^2 + 32x = 36$, one may use the completing the square method once the leading coefficient of the quadratic term is divided through the rest of the equation, or factored out. Also, during the process of completing the square, the constant term must be added or subtracted to the other side of the equation. These two problems already have the constant term in that location.

**Result 3: The most frequently used method was the quadratic formula (57.14%)**

The different methods that were utilized during this experiment were examined. Upon analysis of the 504 problems that were solved, the quadratic formula was used the majority of the
time, 57.14%. Figure 16, located on the next page, displays the remaining methods (n = 504).

**Figure 16.** The Frequency of Use of Different Methods During the Experiment.

| Method                  | Percentage |
|-------------------------|------------|
| Quadratic Formula       | 57.14%     |
| Solve by Factoring      | 23.16%     |
| Omit                    | 13.69%     |
| Unknown Method          | 2.78%      |
| Taking the Square Root  | 2.44%      |
| Taking the Limit        | 0.79%      |

The quadratic formula can be applied to every quadratic equation to find a solution (also by completing the square), whereas the other methods cannot be used on all problems. When observing the solutions to the quadratic equation assessment, problems numbered 1, 7, and 8 could be solved by factoring, 3 and 4 could be solved by completing the square, and 2, 5, and 6 could be solved by utilizing the quadratic formula. Problem 8 could also be solved by taking the square root of both sides of the equation. A more comprehensive analysis of individual problems can be found on page 30.

Since no student solved a quadratic equation by completing the square, then the only other way to solve problems 2, 5, and 6 is by applying the quadratic formula. Figure 17 includes the percentage of students who used the various methods only for the problems that could be solved by factoring. These problems, after excluding the possibility of completing the square, are the only problems located in the test that can be solved by using different approaches. Figure 17 displays the frequency of use of methods when participants solved problems 1, 7, and 8.
Figure 17. Frequency of Use of Different Methods the Problems Which Could be Solved by Factoring.

| Method               | Percentage |
|----------------------|------------|
| Solve by Factoring   | 41.27%     |
| Quadratic Formula    | 38.10%     |
| Omit                 | 11.11%     |
| Unknown Method       | 2.65%      |
| Taking the Square Root | 6.34%  |
| Taking the Limit     | 0.53%      |

By observing the figure above, it can be concluded that when the quadratic formula is not the only choice for finding a solution to the quadratic equation, solving by factoring is the most used method, with 41.27%. It is evident that when the participants had different options to choose from when solving the quadratic equations, they chose the more efficient method in the general case. A more specific analysis will follow in the results section for the quadratic equation test.

**Result 4: Survey results indicated that the top two preferred methods were the quadratic formula and solving by factoring**

Upon analysis of the survey, it was determined that the participants preferred the quadratic formula over any other method of solving quadratic equations. Figure 18 displays the top three results from the survey (n = 63). The three methods listed received the highest number
of votes. The other methods that were stated on the survey are not listed due to the fact that they received a small number of votes.

**Figure 18.** The Most Preferred Method by Students when Solving Quadratic Equations.

| Method             | Frequency (Number of Students) |
|--------------------|--------------------------------|
| Quadratic Formula  | 31                             |
| Solve by Factoring | 16                             |
| FOIL               | 4                              |

Along with stating their preferred method for solving quadratics, the students were directed to also state why they chose that specific approach. The majority of explanations that were given for the quadratic formula were that it was the easiest to remember, that it works for every quadratic equation, it is much easier to plug in the numbers and solve, and in some cases, it was the only method that the participants could remember. For example, one student stated “I memorized the quadratic formula a long time ago and works whether or not the equation can be factored.” It is evident that once the students learned the quadratic formula, it is the only one they utilized because of the fact that it works for every possible quadratic. This result directly relates to Result 3, which was that the most used method was the quadratic formula.

Along with the quadratic formula and solving by factoring, the FOIL method received the third highest number of votes. But the FOIL method (First Outside Inside Last) is not used for finding solutions to quadratic equations. It is actually used to multiply two binomials together.
So this result brings into question whether or not the students have knowledge of the correct vocabulary and terminology.

Some of the students responded with answers that were circumstantial. For example, one of the students stated “Depends on the equation. Factoring is usually easier, but sometimes the quadratic equation is more effective.” So instead of directly applying the quadratic formula, some students would analyze the quadratic equation first and decide if it is factorable.

**Quadratic Equation Assessment Results**

In this section, the problems that could be solved by factoring are examined, which also were the quadratic equations that the participants solved using more than one solution method. Each problem is examined separately, followed by a more generalized conclusion that covers all three problems.

**Problem 1: \( x^2 - 4x - 5 = 0 \)**

Figure 19 displays the percentage of students who found the correct answer and an incorrect solution to problem 1 given the certain methods that were utilized.

**Figure 19.** Percentage of Correct and Incorrect Solutions Given Certain Methods for Problem 1.

| Solution Method               | Solve by Factoring | Quadratic Formula | Total   |
|-------------------------------|--------------------|-------------------|---------|
| Percentage                   |                    |                   |         |
| Correct                       | 80.65%             | 50%               | 61.90%  |
| Incorrect                     | 19.35%             | 50%               | 38.10%  |
| Frequency                     | 31                 | 28                | 59      |
One student did not complete this problem, one student took the limit of the quadratic equation, and two students used an unknown method that was incorrect. It can be observed that when the students found a solution by factoring, there was a much higher success rate than when they applied the quadratic formula. Below, Figure 20 displays a response from a student who applied the quadratic formula and from a student who solved by factoring.

Figure 20. Student Solution Samples for Applying the Quadratic Formula and Solving by Factoring for Problem 1.
After observing the student work samples, one can quickly notice that there is a significantly less amount of work to produce a correct answer when the student solved by factoring. Also, it is noticeable that there are more steps that must be taken to solve this quadratic equation by applying the quadratic formula. Students are more prone to make mistakes when the procedure is longer. Figure 21, located below, displays a common error that was made during this experiment.

**Figure 21.** Student Work Sample from Problem 1 That Depicts an Error While Using the Quadratic Formula.

\[ x^2 - 4x - 5 = 0 \]

\[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[ \frac{-4 \pm \sqrt{(-4)^2 - 2(1)(-5)}}{2(1)} \]

\[ x = 9, 10, -1, 10 \]

A common error, such as the one shown above in Figure 21, that appeared in this problem was the use of an incorrect version of the quadratic formula. Many students wrote the discriminant as \( b^2 + 4ac \) rather than \( b^2 - 4ac \). Also, the arithmetic proved to be somewhat difficult for numerous students. When squaring a negative number, some participants obtained a negative answer, when the answer should have been positive. Those who found a solution that was incorrect by factoring recorded the following \((x - 5)(x - 1) = 0\) as the factored form of the quadratic equation.
Problem 7: $9x - 14 = x^2$

Figure 22 displays the percentage of students who found the correct solution and an incorrect solution to problem 7 given certain methods that were utilized during the experiment.

|                  | Solve by Factoring | Quadratic Formula | Total  |
|------------------|--------------------|-------------------|--------|
| Percentage       | Correct            | 90.48%            | 65.10% |
|                  | Incorrect          | 9.52%             | 34.90% |
| Frequency        | 21                 | 30                | 51     |

Ten of the students who participated in this study did not attempt to find a solution for this problem, one student took the limit of the quadratic equation, and one student utilized an unknown method that was incorrect. Upon further examination, it can be noted that there is a higher rate of obtaining a correct solution when the student solves by factoring rather than applying the quadratic formula. Before finding a solution, students had to move all terms to one side of the equation. This step led to a common error when students used the quadratic formula. Since they did not have one side equal to zero, the numbers they substituted into their formula were incorrect, thus giving them an incorrect solution. The results from problem 1 also apply for problem 7. A copy of student work for problem 7 can be found in the Appendix.
Problem 8: $4x^2 - 64 = 0$

Figure 23, on the following page, displays the percentage of students who found the correct answer and an incorrect solution to problem 8 given the different methods students used.

**Figure 23.** Percentage of Correct and Incorrect Solutions Given Certain Methods for Problem 8.

| Solution Method          | Percentage | Frequency |
|--------------------------|------------|-----------|
| Solve by Factoring       | 69.23%     | 26        |
| Quadratic Formula        | 57.14%     | 14        |
| Taking the Square Root   | 36.36%     | 11        |
| Total                    | 49.20%     | 51        |

Ten of the students did not attempt to find a solution to this quadratic equation, one student took the limit of the quadratic equation, and one student used an unknown method that was incorrect. Problem 8 is different than the previous two problems examined because it has the possibility of being solved by taking the square root of both sides after the problem is rewritten. But it also provides more opportunities for errors to occur.

Shown in Figure 23, the students who took the square root of both sides provided an incorrect solution more often than a correct solution. This is a direct result that is associated with common errors in taking the square root of a number. The students who provided an incorrect solution either wrote $x = 4$ or $x = -4$. Figure 24, located on the following page, displays a student work example of this error.

When taking the square root of both sides of an equation, there is a positive answer and a negative answer, but students sometimes forget this fact. This was the only mistake made when taking the square root of both sides for this quadratic equation. Again, the students who found a
solution by factoring had a higher success rate than the students who applied the quadratic formula.

**Figure 24.** Student Work Sample Involving an Error When Taking the Square Root of Both Sides of an Equation.

![Image of a quadratic equation problem with a student's work sample showing an error in taking the square root of both sides of the equation.]

There were some common errors that the solutions of problems 1, 7, and 8 shared. Numerous students who obtained an incorrect solution when applying the quadratic formula did so because they wrote down the formula incorrectly. Some students recorded \( b^2 + 4ac \) as the discriminant, and others divided by 4a instead of 2a. The other mistakes that were made when applying the quadratic formula were done by making arithmetic errors.

Furthermore, these mistakes lead to a lower success rate when using the quadratic formula compared to finding a solution by factoring. Examining all three problems, the percentages of students who received a correct solution by factoring were 80.65%, 90.48% and 69.23%, whereas the percentage of correct solutions when applying the quadratic formula was 50.00%, 73.33%, and 57.14%, respectively. When utilizing the quadratic formula, it may be less complex to just plug in numbers, but there are more steps where mistakes can be made. Whereas
when a student solves by factoring, there are fewer steps, so there are fewer chances for the student to make a mistake.

**Problems 2 Through 6**

Problems 2 through 6 are grouped together because the only method that the students used to receive a correct solution was the quadratic formula. Questions 3 and 4 could be solved by completing the square, but no student used this approach. Since these five problems were only solved correctly by using one method, students were not able to alternate methods and produce a solution. So these five problems will not be discussed as thoroughly as the problems that could be solved by factoring. Figure 25 and Figure 26 present the different methods that students tried to use, but received an incorrect answer and the percentage of incorrect and correct solutions for each problem.

**Figure 25.** Frequency of Solution Methods for Problems 2 through 6.

| Problem # | Solve by Factoring | Quadratic Formula | Did Not Attempt |
|-----------|--------------------|-------------------|-----------------|
| 2         |        5          |       45         |          9      |
| 3         |        5          |       44         |          12     |
| 4         |       13          |       43         |          4      |
| 5         |       12          |       41         |          8      |
| 6         |        3          |       43         |          15     |
After analyzing Figure 25, it is apparent that the most frequently omitted problem is number six, which is \(x^2 + 5x + 7 = 0\). Also, the students realized that these 5 problems were not factorable, so the majority applied the quadratic formula to find a solution. Examining Figure 26, we can conclude that the question with the highest percentage of incorrect solutions is problem #6, which was \(3(x^2 - 1) = x\). This quadratic equation requires the most work before being able to produce a solution. One must use the distributive property and move all terms to one side of the equation.

**Figure 26.** Percentage of Correct and Incorrect Solutions for Problems 2 Through 6.

| Problem # | Correct   | Incorrect |
|-----------|-----------|-----------|
| 2         | 50.80%    | 49.20%    |
| 3         | 49.20%    | 50.80%    |
| 4         | 44.44%    | 55.56%    |
| 5         | 38.10%    | 61.90%    |
| 6         | 46.00%    | 54.00%    |

**Survey Results**

Analyzing the survey was just as important as examining the quadratic assessment. The second question of the survey, which was "What is your favorite or preferred method listed above for solving quadratic equations? Why?", was already analyzed in the beginning of the results section. In this section, responses to question 1 are explored and discussed.
Question 1: List as many methods for solving quadratic equations as you can.

Figure 27, located below, represents how many times a method was listed by the participants for question 1 of the survey.

Figure 27. Frequency of Responses That Included the Different Solution Methods to Quadratic Equations.

| Method                          | Frequency |
|--------------------------------|-----------|
| Quadratic Formula              | 48        |
| Solve by Factoring             | 29        |
| Completing the Square          | 19        |
| Graphically                    | 8         |
| FOIL                           | 6         |
| Taking the Square Root         | 1         |

After analyzing the figure above, it is evident that the quadratic formula is the more memorable solution method, and finding a solution by factoring is second. This result ties in directly with the previous analysis, where the two most used and preferred methods were applying the quadratic formula and solving by factoring. Nineteen of the students stated that completing the square was a method of finding a solution to quadratic equations, but none of the students utilized this method during the assessment.

When examining student responses, the use of the correct terminology was rare for some cases. The quadratic formula was often referred to as the quadratic equation, or the student wrote out the formula. Also, the term factor was commonly called the reverse FOIL method. Some students went as far as saying algebraically is their preferred method (this answer was not included in Figure 27). A question stems from this analysis: Do the students believe that the FOIL method is a correct approach, or do they mean to find a solution by factoring?
Implications for Teaching

It was hypothesized that students would not alternate methods for solving quadratic equations given that some problems were more efficiently solved by one approach than another. This experiment indicates that numerous college students enrolled in Calculus I courses are not comfortable with all of the available methods to find solutions to quadratic equations.

**Implication 1: Educators need to demonstrate efficient use of various methods for solving problems in the classroom.**

Being able to solve quadratic equations is an important skill that high school students must possess to enhance their mathematical skills. Luckily, there are numerous methods for solving quadratic equations. But what most students do not realize, along with numerous educators, is that these equations are not all in the same form. The different approaches can be used more efficiently when solving certain quadratic equations rather than others. These differences need to be made apparent to students.

Time needs to be spent during lecture to allow the students to examine which method to use and when. If an equation can easily be solved by factoring, why would we want our students to use the quadratic formula? The inverse of this statement is also valid, that we do not want our students trying to factor a quadratic equation when it cannot be factored. Altering lessons by indicating the various types of quadratic equations would benefit the students immensely. They would be able to observe when to utilize certain approaches, and why other methods are inferior in specific situations.

Not only must the different types of quadratic equations be communicated clearly, but also the teacher must allow enough time for the students to become comfortable with each method. Students often believe that if one method is good enough, then they do not need to
know the others. This information is just redundant to them. But being able to utilize each approach would positively impact their test taking ability. Students are given a time limit on the standardized tests that are administered at the end of the year. It is our job to make sure they use their time wisely. This can be accomplished by making sure these various methods are not to be forgotten, but instead, to be used efficiently.

**Implication 2:** Teaching and using correct vocabulary and terminology during instruction is critical for students’ mathematical success.

Students do not realize the importance of vocabulary in a mathematics classroom. Understanding what each term means and being able to communicate effectively is imperative to their success in mathematics. It is the educator’s job to demonstrate correct usage of the terminology and hold the students responsible for knowing what each formula represents and what each definition is.

Just because the students are not in an English classroom does not mean that vocabulary tests are out of the question. Incorporating vocabulary exams in a mathematics classroom will hold the students responsible for knowing the definitions. In doing so, they will not get terms confused, such as mentioning the FOIL method as a means to solving quadratic equations, which students did during this experiment.

**Closing Remarks**

The motivation behind this study was to determine if students were able to differentiate between the various types of quadratic equations and apply the most efficient method to find the solution. Based on the results, it appears that college students are unable to differentiate between
the different forms of quadratic equations. And as a result, they are unable to apply the method that could produce the solution with the fewest number of steps. It is critical that students understand and feel comfortable with various methods of finding solutions, and not just for quadratic equations.
References

Allaire, P. (2001). Geometric Approaches to Quadratic Equations from Other Times and Places. *Mathematics Teacher, 94*(4), 308.

Blizer, R. (2007). *Algebra and Trigonometry* (3rd Edition). Upper Saddle River, New Jersey: Prentice-Hall, Inc.

Blitzer, R. (2010). *Precalculus* (4th Edition). Upper Saddle River, New Jersey: Prentice-Hall, Inc.

Bosse, M., & Nandakumar, N. (2005). The Factorability of Quadratics: Motivation for More Techniques. *Teaching Mathematics and Its Applications: An International Journal of the IMA, 24*(4), 143-153.

Burton, D. M. (2007). *The History of Mathematics: An Introduction* (6th Edition). New York, New York: McGraw-Hill Companies, Inc.

Esty, W. W. (1997). *Precalculus Concepts* (Preliminary Edition). Upper Saddle River, New Jersey: Prentice Hall.

Gantert, A.X. (2007), Integrated Algebra 1. New York, New York: Amsco School Publications, Inc.

Gardella, F. (2000). Algebraic Squares: Complete and Incomplete. *Mathematics Teacher, 93*(4), 283-84.

Hungerford, T. W., Jovell, I. S., & Mayberry, B. (2004). *Precalculus: A Graphing Approach*. Austin, Texas: Holt, Rinehart, and Winston.

Karp, A. (2007). Once more about quadratic trinomial…. On the formation of methodological skills. *Journal of Mathematics Teacher Education, 10*(4-6), 405-414. doi: 10.1007/s10857-007-9037-9.

Katz, V. J. (2004). *A History of Mathematics* (Brief Edition). Boston, MA: Pearson Education.

Keenan, E. P., & Gantert, A. X. (1991). *Integrated Mathematics* (Second Edition). New York, NY: Amsco School Publications, Inc.

Kotelawala, U. (2010). Collaborative Planning for a Unit on the Quadratic Formula. *Mathematics Teacher, 103*(9), 669-674.

McCallum, W. G. (2001). Thinking out of the box. *Computer Algebra Systems in Secondary School Mathematics Education*, 73-86.

Phelps, S., & Edwards, T. (2010). New life for an old topic: completing the square using technology. *Mathematics Teacher, 104*(3), 230-236.
Picciotto, H. (2008). A New Path to the Quadratic Formula. *Mathematics Teacher*, 101(6), 473-478.

Radford, L., & Guerette, G. (2000). Second degree equations in the classroom: a Babylonian approach. *Using History to Teach Mathematics: An International Perspective.* (pp. 69-75)

Schultz, J.E., Ellis, W., Hollowell, K. A., & Kennedy, P. A. (2001). *Algebra 2.* Austin, Texas: Holt, Rinehart, and Winston.

Taylor, S., & Mittag, K. (2001). Seven Wonders of the Ancient and Modern Quadratic World. *Mathematics Teacher*, 94(5), 349-50, 361.

Vaiyavutjamai, P., & Clements, M. (2006). Effects of Classroom Instruction on Students' Understanding of Quadratic Equations. *Mathematics Education Research Journal*, 18(1), 47-77.

Vinogradova, N. (2007). Solving Quadratic Equations by Completing Squares. *Mathematics Teaching in the Middle School*, 12(7), 403-405.

Ward, A. (2003). A Unified Approach to Teaching Quadratic and Cubic Equations. *International Journal of Mathematical Education in Science and Technology*, 34(1), 158-60.

Yarema, C., & Hendricks, T. (2010). Using a Square to Complete the Algebra Student: Exploring Algebraic and Geometric Connections in the Quadratic Formula. *MathAMATYC Educator*, 1(3), 30-35.
Appendix A

TO: Students in Survey of Calculus I / University Calculus I
FROM: Mr. Zebracki
DATE: 
RE: Consent Form

Purpose, Procedure, and Benefits

➢ The purpose of this study is to determine which solution method to solving quadratic equations is preferred by you, the students.

➢ In two consecutive days, you will be given a quadratic equation assessment and a survey. On the first day, you will be asked to complete a 10 question quadratic equation assessment. On the second day, you will be asked to fill out a survey with questions pertaining to opinions on which method is the most effective or favorite.

➢ The goal of this study is to improve mathematics instruction in the classroom. This is an important study because of the potential benefits it may hold, not just for you, but for mathematics education in its entirety. This study will allow educators to alter their lessons to better accommodate the needs of the students when it comes to quadratic equations.

Related Information

➢ You are being asked to participate in this study.

➢ To maintain confidentiality, your name will not be used in any way, shape, or form. Any name or identification will not be used with any materials related to the study.

➢ There is no cost (nor any compensation) to participate in this study.

➢ Participation in this study is voluntary. You may withdraw at any time. No penalty can or will be assessed to yourself for declining participation. Please bear in mind that the assessment used to analyze data is the standard test that you would normally receive throughout regular instruction (whether you choose to participate or decline). Therefore, all students will be required to take this assessment. Declining to participate will prevent your score from being recorded and analyzed for the purposes of the study.

➢ Only minimal risks (if any) are anticipated to yourself. This study deals primarily with routine practices, so there should be no undue stress or strain placed on you. If you wish, you may be removed from the study at any time.
The potential benefits to you will be to receive more effective teaching strategies and an overall improvement in performance within the math classroom.

You may contact Mr. Zebracki at 716-673-4811 or zebr4929@fredonia.edu. Or you may contact Dr. Howard at 716-673-3873 or keary.howard@fredonia.edu.
Student Consent Form

SUNY Fredonia

Thank you for being a part of this study. Please print and sign your name in the space provided to show that you agree to participate. Remember that signing the form allows Mr. Zebracki to use your data for the research project. All students must participate in class whether they sign this form or not.

Voluntary Consent: I have read this memo and I am fully aware of all that this study involves. My signature below shows that I freely agree to participate in this study. I understand that there will be no penalty for not participating. I understand that I may withdraw from the study at any time, also without penalty. I understand that my name and any personal information will be kept out of the study. I understand that if I have any questions about this study, I may contact Mr. Zebracki at 716-673-4811 or zebr4929@fredonia.edu. I may also contact Dr. Howard at 716-673-3873 or keary.howard@fredonia.edu.

Please return this original, completed consent form as soon as possible. Thank you for your cooperation.

Student Name (please print): ________________________________________

Student Signature: ________________________________________________

Date: ____________________
Appendix B

Quadratic Equation Assessment

Solve the following 8 problems algebraically showing all work necessary to find the solutions. If you use a calculator, make sure all of your work is recorded algebraically.

1. \( x^2 - 4x - 5 = 0 \)  
2. \( 2x^2 + 6x - 22 = 0 \)

3. \( x^2 - 6x = -11 \)  
4. \( 4x^2 + 32x = -36 \)
5. \( 3(x^2 - 1) = x \)

6. \( x^2 + 5x + 7 = 0 \)

7. \( 9x - 14 = x^2 \)

8. \( 4x^2 - 64 = 0 \)
Survey 1

1. List as many methods for solving quadratic equations as you can.

2. What is your favorite or preferred method listed above for solving quadratic equations? Why?

3. Did you use a calculator? If yes, what kind and for which problems?
Appendix D

7. \(9x - 14 = x^2\)

\[
\begin{align*}
\frac{-9x}{-14} &= x^2 - 9x + 14 \\
X &= 2, 7 \\
X^2 - 9x + 14 &= 0 \\
X &= \frac{9 \pm \sqrt{81 - 4(1)(14)}}{2(1)} \\
X &= \frac{9 \pm \sqrt{65}}{2} \\
X &= \frac{9 + 5}{2} \quad \text{or} \quad \frac{9 - 5}{2} \\
X &= 7 \quad \text{or} \quad 2
\end{align*}
\]