An effective field theory of QCD at high density

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Abstract

We derive a (Wilsonian) effective field theory of QCD at finite density by integrating out the states in the Dirac sea when the chemical potential $\mu \gg \Lambda_{\text{QCD}}$. The quark-gluon coupling is effectively (1+1)-dimensional and the theory contains four-quark operators which become relevant as we approach to the Fermi sea. By calculating the one-loop vacuum polarization tensor in the effective theory, we find the electric gluons have a screening mass, $M \sim g_s \mu$, while the static magnetic gluons are unscreened. We then investigate the gap equations for color anti-triplet Cooper pairs by including both gluon-exchange interactions and the marginal four-quark interactions in the effective theory.

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Quantum chromodynamics (QCD) has become a undisputed theory for the strong interaction of quarks and gluons. However, its direct experimental evidences are still limited to the high energy region where quarks and gluons are weakly coupled. At low energy where they couple strongly, we have only indirect evidences due to the calculational incapability that QCD describes the strong interaction.

Recently, QCD at high density has been studied intensively, since it not only offers a nonperturbative test but also exhibits rich novel phases which might be accessible in the core of neutron stars or in a quark-gluon plasma in heavy ion collisions [1–6]. It has been argued that at a sufficiently high density the color symmetry is spontaneously broken, leading to color superconductivity.

In this letter, we derive a (Wilsonian) effective theory of QCD at high density in loop expansion, after integrating out the irrelevant degrees of freedom at low energy. The effective theory includes new quark-gluon couplings generated by the exchange of states in the Dirac sea, which induce marginal four-fermion operators at matching one-loop amplitudes, as in the effective theory of QED under external magnetic fields [7]. The marginal four-quark operators become relevant as we scale down to the Fermi sea. In the effective theory, the gluon-quark couplings are effectively (1+1)-dimensional and the electric gluons, \( A_0^a(x) \), are screened in the static limit. At energy below the screening mass, the only relevant interactions for quarks are the four-quark operators with opposite momenta and the interactions with magnetic gluons. We then solve the gap equations for Cooper pairs in the color anti-triplet channel by long-range color magnetic interactions, screened color electric interactions, and the marginal four-quark interactions.

A system of degenerate quarks with a fixed baryon number is described by the QCD Lagrangian density with a chemical potential \( \mu \),

\[
\mathcal{L}_{\text{QCD}} = \bar{\psi} i D \psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \mu \bar{\psi} \gamma_0 \psi, \tag{1}
\]

where the covariant derivative \( D_\mu = \partial_\mu + ig_s A_\mu^a T^a \) and we neglect the mass of quarks for simplicity. (Later, we will consider the effect of quark mass.)

For a sufficiently high density such that \( \mu \gg \Lambda_{\text{QCD}} \), one can derive systematically a (Wilsonian) effective theory of QCD, where \( \Lambda_{\text{QCD}}/\mu \) provides a useful expansion parameter. At an energy scale just below the chemical potential \( \mu \) \( (\gg \Lambda_{\text{QCD}}) \), quarks are almost free and their energy spectrum is given by an eigenvalue equation,

\[
(\vec{\alpha} \cdot \vec{p} - \mu) \psi_\pm = E_\pm \psi_\pm, \tag{2}
\]

where \( \vec{\alpha} = \gamma_0 \gamma \) and \( \psi_\pm \) denote the energy eigenfunctions with eigenvalues \( E_\pm = -\mu \pm |\vec{p}| \), respectively. At low energy \( E < \mu \), the states \( \psi_+ \) near the Fermi surface, \( |\vec{p}| \sim \mu \), are easily excited but \( \psi_- \), which correspond to the states in the Dirac sea, are completely decoupled due to the presence of the energy gap \( \mu \) provided by the Fermi sea. Therefore the right degrees of freedom below \( \mu \) consist of gluons and \( \psi_+ \) only.

Since at very low energy \( (E \ll \mu) \) quarks are almost on-shell and move at the Fermi momentum \( \vec{p}_F = \mu \vec{v}_F \), we may decompose the momentum of quarks into the Fermi momentum and a residual momentum as

\[
p_\mu = \mu v_\mu + l_\mu, \tag{3}
\]
where the residual momentum $|l_\mu| < \mu$ and $\nu^\mu = (0, \vec{v}_F)$. Now, we decompose the quark fields as

$$
\psi(x) = \sum_{\vec{v}_F} \left[ e^{i\vec{\alpha}_F \cdot \vec{x}} \psi_+(\vec{v}_F, x) + e^{i\vec{\alpha}_F \cdot \vec{x}} \psi_-\bar{\psi}_+(\vec{v}_F, x) \right],
$$

(4)

where $\vec{\alpha} \cdot \vec{v}_F \psi_\pm(\vec{v}_F, x) = \pm \psi_\pm(\vec{v}_F, x)$. The quark Lagrangian in Eq. (1) then becomes

$$
\bar{\psi} \left[ iD + \mu \gamma^0 \right] \psi = \sum_{\vec{v}_F} \left[ \bar{\psi}_+(\vec{v}_F, x)i\gamma_\parallel \mu \psi_+(\vec{v}_F, x) + \bar{\psi}_-(\vec{v}_F, x)\gamma_\parallel \mu \psi_-(\vec{v}_F, x) \right]
$$

(5)

$$
+ \sum_{\vec{v}_F} \left[ \bar{\psi}_-(\vec{v}_F, x)iD_\perp \psi_+(\vec{v}_F, x) + \text{h.c.} \right]
$$

where $\gamma_\parallel \mu \equiv (\gamma^0, \vec{v}_F \cdot \vec{\gamma})$, $\gamma_\parallel \mu = \gamma^\mu - \gamma_\parallel \mu$, $D_\parallel = \vec{V}D$ with $\vec{V}\mu = (1, -\vec{v}_F)$, and $D_\perp = \gamma_\parallel \mu D_\mu$.

At low energy, since the fast modes $\psi_-$ are decoupled, we integrate out all the fast modes $\psi_-$ to derive the low energy effective Lagrangian by matching all the one-light particle irreducible amplitudes containing gluons and $\psi_+$ in loop expansion. The effects of fast modes will appear in the quantum corrections to the couplings of low energy interactions. At tree-level, the matching is equivalent to eliminating $\psi_-$ in terms of equations of motion;

$$
\psi_-(\vec{v}_F, x) = -\frac{i\gamma_0}{2\mu + iD_\parallel} D_\perp \psi_+(\vec{v}_F, x) = -\frac{i\gamma_0}{2\mu} \sum_{n=0}^{\infty} \left( -\frac{iD_\parallel}{2\mu} \right)^n D_\perp \psi_+(\vec{v}_F, x).
$$

(6)

Therefore, the tree-level Lagrangian for $\psi_+$ becomes, using $(1 - 2\vec{\alpha} \cdot \vec{v}_F)/2\gamma^\mu(1 + \vec{\alpha} \cdot \vec{v}_F)/2 = \gamma^0 V\mu(1 + \vec{\alpha} \cdot \vec{v}_F)/2$ with $V\mu = (1, \vec{v}_F)$,

$$
L_{\text{eff}}^0 = \sum_{\vec{v}_F} \left[ \psi_+^\dagger V^\mu D_\mu \psi_+ - \frac{1}{2\mu} \psi_+^\dagger \left( D_\perp \right)^2 \psi_+ + \cdots \right],
$$

(7)

where the ellipsis denotes terms with higher derivatives. We see that in the high density limit the quark-gluon couplings are effectively $(1+1)$-dimensional: In the leading order, to quarks of Fermi velocity $\vec{v}_F$ only the $V \cdot A$ component of gluons couple and $A_\mu = (0, \vec{A})$ with $\vec{v}_F \cdot \vec{A} = 0$ does not couple to the color charge of $\psi_+$ but to its color magnetic moment, which is suppressed by $1/\mu$.

By the tree-level matching shown in Fig. 1, we obtain a vertex of two gluons and two quarks, generated by the exchange of $\psi_-$,

$$
L_{\text{int}}^\text{eff} = -\frac{g_2^2}{2\mu} \sum_{\vec{v}_F} \bar{\psi}_+ A_{1\perp} \gamma_0 A_1 \psi_+ + \cdots
$$

(8)

where the ellipses denote terms containing more powers of gluons and derivatives. (From now on $\psi$ will denote the slow modes $\psi_+$.)

Now, consider the one-loop matching of a four-quark amplitude. (See Fig. 2.) In QCD, the amplitude is ultraviolet (UV) finite but infrared (IR) divergent, while it is both UV and IR divergent in the effective theory. Since the IR divergence is same in both theories, we need a UV counter term in the effective theory to match the amplitude, which is a four-quark operator. The one-loop four-quark amplitude in the effective theory is
where \( l \) is the external momentum transfer, \( \Lambda \) is a renormalization point, and c.t. denotes the counter terms. \( u, s, t, v \) are color indices and \( \delta^{\text{s}}_{\text{uts}tv} = (\delta_{ut} \delta_{ts} + \delta_{ut} \delta_{st})/\sqrt{2}, \delta^{\text{A}}_{\text{uts}tv} = (\delta_{ut} \delta_{st} - \delta_{ut} \delta_{st})/\sqrt{2}. \) We find that the new quark-gluon coupling generates effective four-quark operators at one-loop:

\[
S_{\text{eff}} \equiv \frac{1}{2\mu^2} \int \prod_{i=1}^{4} \frac{d^4p_i}{(2\pi)^4} (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) \left[ \psi^\dagger_i(p_3) \psi_s(p_1) \psi^\dagger_i(p_4) \psi_u(p_2) \left( g_3 \delta^{\text{A}}_{\text{uts}tv} - g_0 \delta^{\text{s}}_{\text{uts}tv} \right) ight. \\
+ \left. \psi^\dagger_i(p_3) \gamma_5 \psi_s(p_1) \psi^\dagger_i(p_4) \gamma_5 \psi_u(p_2) \left( h_3 \delta^{\text{A}}_{\text{uts}tv} - h_0 \delta^{\text{s}}_{\text{uts}tv} \right) \right]. \tag{10}
\]

At the matching scale \( \Lambda = \mu \), the couplings are of order of \( \alpha_s^2 \). As in the BCS superconductivity \([8]\), this four-quark operator is marginal under the scale transformation toward the Fermi surface, \( l_\parallel \to s l_\parallel \) with \( s < 1 \), only when the incoming quarks have opposite Fermi momenta, \( p_1 + p_2 = 0 + \mathcal{O}(l) \), where we write the quark momenta \( p_i^\mu = \mu v^\mu_i + l_i^\mu \) as before. Therefore, if this four-quark operator leads to a condensate, it must be in the s-channel, \( \langle \psi_s(p_1) \psi_u(p_2) \rangle \neq 0 \) or \( \langle \bar{\psi}_s(\vec{v}_F, x) \psi_u(-\vec{v}_F, x) \rangle \neq 0 \). Note also that we do not have marginal operators containing \( \bar{\psi}_i(\vec{v}_F, x) \psi_s(\vec{v}_F, x) \) because it vanishes identically for the slow modes \( \psi_\pm \). Therefore, one can easily understand in the effective theory that the chiral symmetry will be restored as the chemical potential becomes large, if any translationally invariant parameter for chiral symmetry breaking should contain the fast modes \( \psi_\pm \).

Next, we match the gluon two-point amplitude at one-loop. Since the quarks in the Dirac sea are integrated out, we need to match only the quark-loop correction to the amplitude. The quark-loop contribution to the gluon two-point function in QCD at finite density and at zero temperature is calculated by Manuel \([10]\). It consists of two parts; one due to the matter and the other due to the vacuum. In terms of the vacuum polarization tensor, \( \Pi_{\text{abfull}}^{\mu\nu}(p) = \Pi_{\text{abmat}}^{\mu\nu}(p) + \Pi_{\text{abvac}}^{\mu\nu}(p) \), where \( \Pi_{\text{abvac}}^{\mu\nu} \) is the quark-loop contribution when there is no matter, \( \mu = 0 \). For \( |p^\mu| \ll \mu \), the matter part of the vacuum polarization becomes for \( N_f \) light quarks, with \( M^2 = N_f g^2 \mu^2/(2\pi^2) \),

\[
\Pi_{\text{abmat}}^{\mu\nu} = -iM^2 \frac{1}{2} \delta_{ab} \int d\Omega_{\vec{v}_F} \frac{4\pi}{3} \left( V^\mu V^\nu + g^{\mu\nu} - \frac{V^\mu V^\nu + V^\nu V^\mu}{2} \right) \tag{11}
\]

which is transversal, \( p_\mu \Pi_{\text{abmat}}^{\mu\nu}(p) = 0 \) \([10]\).

On the other hand, in the effective theory the quark-loop correction to the vacuum polarization is given as:

\[
\Pi_{\text{ab}}^{\mu\nu}(p) = -g_5^2 \delta_{ab} \sum_{\vec{v}_F} \frac{d^4q}{(2\pi)^4} \text{Tr} \left[ \frac{1}{2} \frac{l}{l_\parallel^*} i \gamma^\mu i \gamma^\nu i \frac{l}{l_\parallel^*} + i \gamma^\nu i \gamma^\mu i \frac{l}{l_\parallel^*} \right] = -\frac{i}{8\pi} \delta_{ab} \sum_{\vec{v}_F} V^\mu V^\nu \left( 1 - \frac{p_0 + \vec{v}_F \cdot \vec{p}}{p_0 - \vec{v}_F \cdot \vec{p} + i\epsilon p_0} \right) M^2. \tag{12}
\]
To match the gluon two-point amplitudes in both theories, we therefore need to add a term in the one-loop effective Lagrangian,

$$\mathcal{L}_{\text{eff}} \supset -\frac{M^2}{16\pi} \sum_{\vec{v}_F} A^a_{\perp} A^a_{\perp\mu},$$

(13)

which also ensures the gauge invariance of the effective Lagrangian at one-loop. Since in the effective theory $\Pi^{00}_{ab} \sim -iM^2\delta_{ab}$, as $p_0 \to 0$, we see that the electric gluons, $A^a_0$, have a screening mass, but the static magnetic gluons are not screened at one-loop due to the added term Eq. (13), which holds at all orders in perturbation as in the finite temperature [11,12].

Finally, we match the quark two-point amplitudes to get a one-loop low energy (Wilsonian) effective Lagrangian density

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}(1 + a_1)\left(F^a_{\mu\nu}\right)^2 - \frac{M^2}{16\pi} A^a_{\perp} A^a_{\perp\mu} + (1 + b_1)\bar{\psi}\gamma_{\mu} D_{\mu}\psi - \frac{1}{2\mu}(1 + c_1)\psi^\dagger (\gamma_{\cdot D})^2 \psi$$

$$+ \frac{1}{2\mu^2} \left[ \left(g_3\delta^S_{us, tv} - g_6\delta^S_{us, tv}\right) \psi_s^\dagger(\vec{v}_F, x)\psi_s(\vec{v}_F, x)\psi^\dagger_s(-\vec{v}_F, x)\psi_s(-\vec{v}_F, x) \right. \right.$$  

$$+ \left. \left(h_3\delta^S_{us, tv} - h_6\delta^S_{us, tv}\right) \psi^\dagger_s(\vec{v}_F, x)\gamma_5\psi^\dagger_s(\vec{v}_F, x)\psi^\dagger_s(-\vec{v}_F, x)\gamma_5\psi^\dagger_s(-\vec{v}_F, x) \right] + \cdots,$$

(14)

where the summation over $\vec{v}_F$ is suppressed and the coefficients $a_1, b_1, c_1$ are dimensionless and of order $\alpha_s(\mu)$. The ellipsis denotes the irrelevant four-quark operators and terms with more external fields and derivatives.

As we scale further down, the effective four-quark operators will evolve together with other operators, which can be seen by further integrating out the high frequency modes, $s\mu < |l_\mu| < \mu$. The scale dependence of the four-quark operators has three pieces. One is from the one-loop matching condition for the four-quark amplitudes and the other two are from the loop corrections to the four-quark operators, shown in Fig. 3. Putting all contribution together, we find the one-loop renormalization group equations for the four-quark operators to be

$$s \frac{\partial}{\partial s} \tilde{g}_i = -\gamma_i\alpha_s^2 - \frac{1}{4\pi^2}\tilde{g}_i^2 - \frac{\ln 2}{12\pi}\delta_i\tilde{g}_i\alpha_s,$$

(15)

where $i = (6, 3)$, $\tilde{g}_0 = -g_6$, $\tilde{g}_3 = -g_3$, and $\gamma_i = (\sqrt{2}/9)(13/4, 1/2)$ and $\delta_i = (-1, 2)$. Since in the high density limit the quark-gluon coupling is $(1+1)$-dimensional, the quarks do not contribute to the running of the strong coupling. The one-loop $\beta$ function for the strong coupling constant at high density is $\beta(\alpha_s) = -11/(2\pi)\alpha_s^2$. Integrating the RG equation (15), we find for $\mu \gg \Lambda$

$$\tilde{g}_i(\Lambda) \approx \frac{2\pi}{11}\alpha_s(\Lambda) \left[ \gamma_i - \frac{(\ln 2)^2}{144}\delta_i^2 \right].$$

(16)

At a scale much less than the chemical potential, $\Lambda \ll \mu$, $g_3(\Lambda) \simeq 0.04\alpha_s(\Lambda)$ and $g_6(\Lambda) \simeq 0.29\alpha_s(\Lambda)$. Similarly for $h_i$, $\gamma_i = \sqrt{2}/2(2, 1)$ and $\delta_i = (-1, 2)$ and we get $h_3(\Lambda) \simeq 0.4\alpha_s(\Lambda)$ and $h_6(\Lambda) \simeq 0.8\alpha_s(\Lambda)$.

At a scale below the screening mass, we further integrate out the electric gluons, which will generate four-quark interactions. For quarks moving with opposite Fermi momenta, the electric-gluon exchange four-quark interaction is given as
\[ \mathcal{L}_{1g} \equiv -\frac{g_s^2(M)}{2M^2} \sum_{\vec{v}_F} \bar{\psi} \gamma^0 T_a \psi(\vec{v}_F, x) \bar{\psi} \gamma_0 T_a \psi(-\vec{v}_F, x). \] (17)

Using \( T_{tu} T_{vs} = 1/2 \delta_{tu} \delta_{uv} - 1/6 \delta_{tu} \delta_{ve} \), we find that the four-quark couplings are shifted as \( g_3(M) \rightarrow g_3(M) + 2\sqrt{2}g_s^2(M)/3 \approx 0.95g_s^2(M) \) and \( g_6(M) \rightarrow g_6(M) + \sqrt{2}g_s^2(M)/3 \approx 0.49g_s^2(M) \), while \( h_i \)'s are unchanged.

As we approach further to the Fermi surface, closer than the screening mass \( M \), the four-quark operators in the color anti-triplet channel become stronger, because the \( \beta \) function for the attractive four-quark operators is negative, \( \beta(g_3) = -\frac{g_3}{4\pi^2} \). If the four-quark interaction is dominant at low energy, it leads to vacuum instability in the infrared region by forming a color anti-triplet condensate or Cooper pair, \( \epsilon_l^{tsu} \langle \psi^\alpha_s(\vec{v}_F, x) \psi^{\beta_j}(\vec{v}_F, x) \rangle \neq 0 \), where flavor indices \( \alpha, \beta \) and Dirac indices \( i, j \) are restored. The size of the condensate driven by the four-quark interaction is determined by the RG invariant scale,

\[ \Delta \approx M \exp \left( -\frac{4\pi^2}{0.95g_s^2(M)} \right). \] (18)

But, since the long-range color-magnetic interactions become also strong at low energy, we need to consider both interactions to determine the Cooper-pair gap.

To describe the Cooper-pair gap equation, we introduce a charge conjugate field,

\[ (\bar{\psi}_c)_{\alpha} = C_{\alpha \beta} \bar{\psi}_\beta(x), \] (19)

where \( \alpha \) and \( \beta \) are Dirac indices and the matrix \( C \) satisfies \( C^{-1} \gamma_\mu C = -\gamma_\mu^T \). Then, we may write the inverse propagator for \( \Psi(\vec{v}_F, x) = (\psi(\vec{v}_F, x), \bar{\psi}_c(-\vec{v}_F, x))^T \) as

\[ S^{-1}(-\vec{v}_F, l) = \gamma^0 \begin{pmatrix} l \cdot V & \Delta(l) \\ \Delta^\dagger(l) & l \cdot V \end{pmatrix}. \] (20)

The gap equation for \( \Delta \) in the color-antitriplet channel is then given in the hard dense loop (HDL) approximation as (see Fig. 4)

\[ \Delta(p) = (-ig_s)^2 \int \frac{d^4l}{(2\pi)^4} D_{\mu \nu}(p - l) \left\{ T^a \frac{\Delta(l)}{l^2 - \Delta^2(l)} V^\nu + i \frac{g_3}{\mu^2} \int \frac{d^4l}{(2\pi)^4} \frac{\Delta(l)}{l^2 - \Delta^2(l)} \right\}. \] (21)

where \( D_{\mu \nu} \) is the gluon propagator.

The gluon propagator is given in the HDL approximation as following the notations used by Schäfer and Wilczek [13],

\[ iD_{\mu \nu}(k) = \frac{P_{\mu \nu}^T}{k^2 - G} + \frac{P_{\mu \nu}^L}{k^2 - F} - \xi \frac{k_\mu k_\nu}{k^4}, \] (22)

where \( \xi \) is the gauge parameter and the projectors are defined by

\[ \footnotesize 1 \text{In the Schwinger-Dyson equation the loop momentum should take the whole range up to the ultraviolet cutoff, which is the chemical potential \( \mu \) in the case of high density effective theory. Hence the gluon propagator includes both magnetic and electric gluons.} \]
\[ P^T_{ij} = \delta_{ij} - \frac{k_i k_j}{|k|^2}, \quad P^T_{00} = 0 = P^T_{0i} \]

\[ P^L_{\mu\nu} = -g_{\mu\nu} + \frac{k_{\mu} k_{\nu}}{k^2} - P^T_{\mu\nu}. \]

In the weak coupling limit, \(|k_0| \ll |\vec{k}|\) and thus

\[ F(k_0, \vec{k}) \simeq M^2, \quad G(k_0, \vec{k}) \simeq \frac{\pi}{4} M^2 \frac{k_0}{|\vec{k}|}. \]

Since the gap has to be fully antisymmetric in color indices, we get

\[ T^a_{\mu\nu} \Delta_{\mu\nu}(T^a)^T_{\mu\nu} = \left( \frac{1}{2} \delta_{tv} \delta_{us} - \frac{1}{6} \delta_{tu} \delta_{vs} \right) \Delta_{uv} = -\frac{2}{3} \Delta_{ts} \]

After Wick-rotating into Euclidean space, the gap equation becomes

\[ \Delta(p_\parallel) = \int \frac{d^4q}{(2\pi)^4} \left[ -\frac{2}{3} g_s^2 \left\{ \frac{V \cdot P^T \cdot \bar{V}}{(p-q)_\parallel + q_\perp^2 + \frac{\pi}{4} M^2 |p_0 - q_0|/|\vec{p} - \vec{q}|} \right. \right. \]

\[ \left. - \frac{1}{(p-q)_\parallel + q_\perp^2 + M^2} - \frac{\xi (p-q)_\parallel^2}{(p-q)^4} \right\} + \frac{g_3}{\mu^2} \right] \frac{\Delta(q_\parallel)}{q_\parallel^2 + \Delta^2(q_\parallel)}. \]

Note that the main contribution to the integration comes from the loop momenta in the region \(q_\parallel^2 \sim \Delta^2\) and \(|q_\perp| \sim M^{2/3} \Delta^{1/3}\). Therefore, we find that the leading contribution is by the first term due to the Landau-damped magnetic gluons. For this momentum range, we can take \(|\vec{p} - \vec{q}| \sim |\vec{q}_\perp|\) and

\[ V \cdot P^T \cdot \bar{V} = -v_F^i v_F^j \left( \delta_{ij} - \hat{k}_i \hat{k}_j \right) = -1 + O \left( \frac{\Delta^{4/3}}{M^{4/3}} \right). \]

We also note that the term due to the four-Fermi operator is negligible, since \(g_3 \sim g_s^4\) at the matching scale \(\mu\).

Neglecting \((p-q)_\parallel^2\) in the denominator, the gap equation becomes at the leading order in the weak coupling expansion and \(1/\mu\) expansion

\[ \Delta(p_\parallel) = \frac{2g_s^2}{3} \int \frac{d^4q}{(2\pi)^4} \left[ \frac{1}{q_\perp^2 + \frac{\pi}{4} M^2 |p_0 - q_0|/|\vec{q}_\perp|} \frac{1}{q_\perp^2 + M^2} + \frac{\xi (p-q)_\parallel^2}{|\vec{q}_\perp|^4} \right] \frac{\Delta(q_\parallel)}{q_\parallel^2 + \Delta^2(q_\parallel)}. \]

The \(q_\perp\) integration can now be performed easily to get

\[ \Delta(p_\parallel) = \frac{g_s^2}{9\pi} \int \frac{d^2q_\perp}{(2\pi)^2 q_\parallel^2 + \Delta^2} \left[ \ln \left( \frac{\mu^3}{\frac{\pi}{4} M^2 |p_0 - q_0|} \right) + \frac{3}{2} \ln \left( \frac{\mu^2}{M^2} \right) + \frac{3}{2} \xi \right]. \]

\[ \text{2 The hard dense loop approximation is therefore consistent since } \Delta \ll M. \text{ I thank Steve Hsu for explaining this point.} \]
We see that in this approximation $\Delta(p_0) \simeq \Delta(p_0)$. Then, we can integrate over $\vec{v}_F \cdot \vec{q}$ to get

$$\Delta(p_0) = \frac{g_s^2}{36\pi^2} \int_{-\mu}^{\mu} dq_0 \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta^2}} \ln \left( \frac{\tilde{\Lambda}}{|p_0 - q_0|} \right) \tag{31}$$

where $\tilde{\Lambda} = 4\mu/\pi \cdot (\mu/M)^5 e^{3/2\xi}$. If we take $\Delta \simeq \Delta(0)$ for a rough estimate of the gap,

$$1 = \frac{g_s^2}{36\pi^2} \left[ \ln \left( \frac{\tilde{\Lambda}}{\Delta} \right) \right]^2 \quad \text{or} \quad \Delta \simeq \tilde{\Lambda} \exp \left( -\frac{6\pi}{g_s} \right). \tag{32}$$

As was done by Son [13], one can convert the Schwinger-Dyson gap equation (31) into a differential equation to take into account the energy dependence of the gap. Approximating the logarithm in the gap equation as

$$\ln |p_0 - q_0| \simeq \begin{cases} \ln |p_0| & \text{if} \ |p_0| > |q_0|, \\ \ln |q_0| & \text{otherwise}, \end{cases} \tag{33}$$

we get

$$p \Delta''(p) + \Delta'(p) + \frac{2\alpha_s}{9\pi} \frac{\Delta(p)}{\sqrt{p^2 + \Delta^2}} = 0, \tag{34}$$

where $p \equiv p_0$. The solution to the differential equation (34) is found to be at $p = \Delta(p)$

$$\Delta \simeq \tilde{\Lambda} \exp \left( -\frac{\pi}{\nu} + 1 + O(\nu^2) \right), \tag{35}$$

where $\nu = \sqrt{8\alpha_s/(9\pi)}$. The gap is therefore given as at the leading order in the weak coupling expansion

$$\Delta = c \cdot \frac{\mu}{g_s^5} \exp \left( -\frac{3\pi^2}{2\sqrt{2}g_s} \right), \tag{36}$$

where $c = 2^7\pi^4 N_f^{-5/2} e^{3/2\xi} e^{1}$. This agrees with the RG analysis done by Son [13] (see also [14]) and also with the Schwinger-Dyson approach in full QCD [16,15,17]. The $1/g_s$ behavior of the exponent of the gap at high density is due to the double logarithmic divergence in the gap equation (27), similarly to the case of chiral symmetry breaking under external magnetic fields [18,19]. In addition to the usual logarithmic divergence in the quark propagator as in the BCS superconductivity, there is another logarithmic divergence due to the long-range gluon exchange interaction, which occurs when the gluon loop momentum is colinear to the incoming quark momentum ($q_\perp \to 0$).

The color anti-triplet condensate has an interesting flavor and Lorentz structure as discussed in [2,3,6,12]. So far we have neglected quark masses, since they are irrelevant at

\[ \text{3 The gauge-parameter dependent term is subleading in the gap equation (27). Since the gap has to be gauge-independent, the gauge parameter dependence in the prefactor will disappear if one includes the higher order corrections.} \]
scales higher than the quark masses, namely at $\Lambda > m_s$, the mass of strange quark in the three flavor case. But, if $g_\bar{3}$ or $g_s$ do not become strong enough to form a condensate at the scale $\Lambda = m_s$, the strange quark becomes irrelevant in the low energy effective theory and decouples from the condensate. Furthermore, if the condensate does not form at low energy scale where the instanton effects are important, one may expect the instanton contribution to the condensate will be quite significant. But, as analyzed in [3] in detail, the instanton generated operators are not dominant when the chemical potential is quite large.

In conclusion, we have derived a (Wilsonian) low energy effective theory of QCD at high density in loop expansion by integrating out the antiparticles. The effective theory contains marginal four-quark operators which become relevant as we approach to the Fermi sea. We also calculate the screening mass for gluons and find that electric gluons have screening mass, $M \sim g_s \mu$, while the magnetic gluons are not screened in the static limit. Finally, in the effective theory, we solve the gap equation for the Cooper pairs in the color anti-triplet channel in the hard dense loop approximation. We find that the gap is given as $\Delta \sim \mu g_s^{-5} \exp\left[-3\pi^2/(\sqrt{2}g_s)\right]$ up to a numerical constant in the prefactor, verifying recent results.

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FIGURES

FIG. 1. The tree-level matching condition. Wiggly lines denote gluons; solid lines, states near the Fermi surface; and double solid line, states in the Dirac sea.

FIG. 2. The one-loop matching condition for a four-quark amplitude.
FIG. 3. One-loop corrections to four-quark operators

FIG. 4. The gap equation for the color anti-triplet Cooper pairs.