FREE NILPOTENT AND $H$-TYPE LIE ALGEBRAS. COMBINATORIAL AND ORTHOGONAL DESIGNS

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Abstract. The aim of our paper is to construct pseudo $H$-type algebras from the covering free nilpotent two-step Lie algebra. In particular, we give a proof that pseudo $H$-type algebras have rational structural constants, i.e., pseudo $H$-type groups admit lattices. Our approach substantially uses combinatorics and reveals the interplay of pseudo $H$-type algebras with combinatorial and orthogonal designs. One of the key tools is the family of Hurwitz-Radon orthogonal matrices.

1. Introduction

Our main goal is to construct explicitly the quotient maps from the free two-step nilpotent Lie algebra to pseudo $H$-type Lie algebras. We use a combinatorial approach and the Hurwitz-Radon families of orthogonal matrices and their generalization. We prove that pseudo $H$-type algebras admit rational structural constants, and therefore, the corresponding pseudo $H$-type Lie groups admit lattices. This allows to construct compact nilmanifolds where harmonic analysis can be developed. Crandall and Dodziuk [28], and Eberlein [20] dealing with classical $H$-type algebras [36] proved the existence of rational structural constants. We generalized their results to pseudo $H$-type Lie algebras giving a different proof from [23] revealing relations with combinatorics and orthogonal designs. The structure of the paper is as follows. In Section 2, we provide an overview of one of the main tools, the Hurwitz-Radon family of orthogonal matrices, and necessary background from graph theory. As a motivation, we attract reader’s attention to interrelations of our results and methods with harmonic analysis. We also define uniform lattices on Lie groups and describe relations between them and the presence of rational structural constants for the corresponding Lie algebras. We give precise definitions and preliminaries in Section 3. We prove the main results on construction of quotient maps and the existence of rational structural constants in Section 4. Section 4.3 is dedicated to the classical case of $H$-type algebras with a positive definite metric and Section 4.4 generalizes it to pseudo $H$-type algebras. Section 5 gives applications of the main results to combinatorial and orthogonal designs.

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2. Interrelations and motivations

The problem of composition of positive definite quadratic forms was proposed by Adolf Hurwitz \[30\] as early as in 1898. He \[31\] and Johann Radon \[49\] reformulated this problem in 1922 to the problem of existence of a family of orthogonal \(n \times n\)–matrices \(E_1, \ldots, E_s\) satisfying the conditions

\[
E_j^2 = -I, \quad E_i E_j + E_j E_i = 0, \quad \text{for } j = 1, \ldots, s \text{ and } i \neq j,
\]

where \(I\) stands for the identity \((n \times n)\)–matrix. This family is now known as the Hurwitz-Radon (HR) family. One of their main results states that the maximal number of the matrices in the HR family is \(\rho(n) - 1\), where \(\rho(n)\) is the Hurwitz-Radon function defined as \(\rho(n) := 8\alpha + 2\beta\), where \(n\) is uniquely represented by \(n = u2^{\alpha+\beta}\) with \(u\) odd, \(\beta = 0, 1, 2\) or 3. Later in 1943, Eckmann \[21\] simplified the Radon’s proof using representation theory and group characters. The maximal number is achievable even by integer matrices, see \[25\].

The Hurwitz-Radon result is used in a bunch of pure and applied problems. In particular, the composition of quadratic forms exists if and only if there exists a representation of the Clifford algebras \[13\], such that the representation space admits a quadratic form with some special property. It is used in the problem of existence of linearly independent vector fields constructed on a sphere in \(n\)-dimensional Euclidean space \[1\], in quantum mechanics as well as in in electronics \[15\], in space-time block coding (STBC) and orthogonal design \[55\], in signal processing and computer vision \[34\].

An important development in quantum computing, STBC and wireless communication happened after Wolfe \[56\] generalized Hurwitz-Radon result to non-positive definite quadratic forms and matrices from an HR\((s, t)\) family such that \(s\) first matrices satisfy the condition \(E_j^2 = -I\) but \(t\) following matrices satisfy the condition \(E_j^2 = I\). In our paper we essentially use this generalization as one of the main tools.

Another group of problems related to our presentation is of graph theory and combinatorics nature. It refers to a relationship between three concepts. The first one is Hamiltonian graphs, i.e., graphs containing a spanning cycle. The second one is a square 1-factorization of a graph. A 1-factorization of a graph is a decomposition of all the edges of the graph into 1-factors, the sets of \(k\) independent edges (without common vertices). A graph admits a square 1-factorization if the union of any two distinct 1-factors is a disjoint union of 4-cycles. The third concept is that of a vertex transitive graph, i.e., a graph such that the automorphism group acts transitively upon the vertices. If a connected graph has a square 1-factorization, then it is vertex transitive and Hamiltonian. Ihrig \[32\] proved that a graph admits
a square 1-factorization if and only if it is a Cayley graph with the group of auto-
morphisms \((\mathbb{Z}_2)^n\) for some \(n\). Since Cayley graphs of abelian groups are known
to be Hamiltonian, graphs that admit square 1-factorization are Hamiltonian Cayley. Kobayashi and Nakamura [41] proved that a complete graph \(K_{2k}\) admits a square 1-factorization if and only if \(k = 2^n, n \geq 1\). As a side result, we generalize this
stating that there exists a 1-factorization of a complete graph \(K_{2k}\) such that at least \(\rho(2k) - 1\) one-factors satisfy the condition of square 1-factorization. Relation with
combinatorics and space-time block coding are discussed in the last two sections of
the paper.

Another perspective is coming from Fourier analysis and its multivariate analogue
on the torus \(\mathbb{T}^n := \mathbb{Z}^n \setminus \mathbb{R}^n\). If \((M, g)\) is a smooth compact Riemannian manifold,
then the Laplace operator \(\Delta\) defined by \(\Delta f = -\text{div}\(\text{grad}(f)\), \(f \in L^2(M, g)\), pos-
sesses a discrete spectrum \(0 = \lambda_0 \leq \lambda_1 \leq \cdots \rightarrow \infty\). This simple fact naturally
implies many interesting problems related by a general question: how much of the
geometry of the manifold is determined by this spectrum. One of the seminal pa-
pers in this direction appeared in 1966 by Mark Kac [35]. He asked whether the
spectrum of the Laplacian on a compact planar domain with a boundary, acting
on smooth functions vanishing on the boundary, determined its shape. This ques-
tion was negatively answered [27] in 1992. Jeff Cheeger established the existence
of a discrete spectrum to Riemannian manifolds with conical singularities in a se-
ries of papers about 1980, see, e.g., [12], which was further extended [47] to the
Riemannian manifolds with cusps. Another type of singular manifolds is provided
by sub-Riemannian geometry. An interpretation of sub-Riemannian geometry can
by thought of as follows. Take a Riemannian manifold \((M, g)\) and an orthonor-
mal frame \((X_1, \ldots, X_n)\) with respect to \(g\). Define a family of Riemannian metrics
g_\epsilon, \epsilon > 0, in \(M\) by requiring that \((X_1, \ldots, X_m, \epsilon X_{m+1}, \ldots, \epsilon X_n)\) is an orthonormal
frame. Then the Gromov-Hausdorff limit of \((M, g_\epsilon)\) as \(\epsilon \rightarrow 0\) is a sub-Riemannian
manifold (the inverse statement, in general, is not true). Equivalently it can be
understood that, the Riemannian fiber metric \(g_0\) is given only on a sub-bundle in
\(H \subset TM\), and then the triplet \((M, H, g_0)\) is called a sub-Riemannian manifold.
Correspondingly, the Laplacian operator \(\Delta\) can be changed to the sub-Laplacian
one \(\Delta_{SR}\), where the divergence and gradient are properly defined. There exists
a large amount of literature developing sub-Riemannian geometry. Typical general
references are [45, 53, 54]. The simplest example of a sub-Riemannian manifold is
the Heisenberg group \(\mathbb{H}^1\) which is topologically \(\mathbb{R}^3\), and the distribution is given by

\[
H = \ker \omega = \{(v_1, v_2, v_3), v_3 = \frac{1}{2}(xv_2 - yv_1) = 0\} \subset \mathbb{R}^3.
\]

Equivalently, one defines \(H_p = \text{span}(X_p, Y_p), p \in \mathbb{H}^1\), where

\[
X_p = (\partial_x - \frac{1}{2}y \partial_z), \quad Y_p = (\partial_y + \frac{1}{2}x \partial_z), \quad p = (x, y, z),
\]

and \(H\) is a subbundle of the tangent bundle \(T\mathbb{R}^3\). We have \([X, Y] = Z = \partial_z\), and
\(\text{span}(X, Y, [X, Y])_p = \mathbb{R}^3\), a condition called bracket generating or Hörmander, which
 guarantees the sub-Laplacian operator \(\Delta_{SR} = -\frac{1}{2}(X^2 + Y^2)\) to be hypoelliptic.
Next we discuss nilmanifolds which are left cosets \( L \setminus G \) of a nilpotent Lie group \( G \) where \( L \) is a uniform lattice, see Definition 3, Section 3.4.

Let \( \Delta_{SR}^G \) be the left-invariant sub-Laplacian on a group \( G \) and \( L \setminus G \) be a nilmanifold. The quotient map \( G \to L \setminus G \) defines a Grusin-type operator \( \Delta_{SR}^{L,G} \) on the nilmanifold \( L \setminus G \). The operator \( \Delta_{SR}^{L,G} \) has a discrete spectrum \( \Lambda \subset [0, \infty) \) analogously to the case of \( \mathbb{T}^n := \mathbb{Z}^n \setminus \mathbb{R}^n \). Spectral geometry of such kind of nilmanifolds and operators was developed in \([5, 6, 7]\).

Mal’cev criterion \([14]\) guarantees the existence of a uniform lattice \( L \subset G \) provided rational structural constants for the nilpotent Lie algebra \( \mathfrak{g} \) of the Lie group \( G \). In this connection, the study of the existence of the rational structural constants for \( H \)-type algebras is important for developing spectral theory of Grusin-type operators on nilmanifolds generated by \( H \)-type groups. The \( H \)-type Lie algebras were introduced by Kaplan in \([36]\) and were widely studied, see, for instance \([9, 10, 18, 37, 42]\). In works \([14, 17, 26]\) the analogues of the classical \( H \)-type Lie algebras, pseudo \( H \)-type algebras, where the positive definite metrics were replaced by non-degenerate metrics with arbitrary signatures, were introduced and studied. In particular, the existence of rational structural constants in pseudo \( H \)-type algebras was proved in \([23]\).

3. Preliminaries

3.1. Clifford algebras and their representations. Let \( V \) be a real vector space endowed with a non-degenerate quadratic form \( Q(v), v \in V \), which defines a symmetric bilinear form \( \mathfrak{q}(u,v) = \frac{1}{2}(Q(u+v) - Q(u) - Q(v)) \) by polarization. Here and further on by saying scalar product we mean non-degenerate symmetric real bilinear form and by inner product a positive definite one. The Clifford algebra \( \text{Cl}(V, \mathfrak{q}) \), named after the English geometer William Kingdon Clifford \([16]\), is an associative unital algebra freely generated by \( V \) modulo the relations

\[
v^2 = -Q(v) \mathbbm{1} = -\mathfrak{q}(v,v) \mathbbm{1} \quad \text{for all } v \in V \text{ or } uv+vu = -2\mathfrak{q}(u,v) \mathbbm{1} \quad \text{for all } u, v \in V.
\]

For an introductory text, one may look at \([24]\). Every non-degenerate quadratic form on the \( n \)-dimensional vector space \( V \) is equivalent to the standard diagonal form

\[
Q_{p,q}(v) = v_1^2 + v_2^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2, \quad n = p + q.
\]

Using isomorphism \((V, Q) \simeq \mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, Q_{p,q})\) we will write \( \text{Cl}(V, \mathfrak{q}) = \text{Cl}_{p,q} \).

Starting with an orthonormal basis \( \ell_1, \ldots, \ell_n \) in \( \mathbb{R}^{p,q} \) one defines a basis of \( \text{Cl}_{p,q} \) by the sequence

\[
1, \ldots, (\ell_{k_1} \cdots \ell_{k_j}), 1 \leq k_1 < k_2 < \cdots < k_j \leq n, \quad j = 1, 2, \ldots, n.
\]

It follows that the dimension of \( \text{Cl}_{p,q} \) is \( 2^n, n = p + q \). We have the following algebra isomorphisms: \( \text{Cl}_{0,0} \simeq \mathbb{R}, \text{Cl}_{1,0} \simeq \mathbb{C}, \text{Cl}_{0,1} \simeq \mathbb{R}^2 \) where the product in \( \mathbb{R}^2 \) is defined componentwise. Furthermore, \( \text{Cl}_{2,0} \simeq \mathbb{H}, \) where \( \mathbb{H} \) denotes quaternions, and \( \text{Cl}_{1,1} \simeq \text{Cl}_{0,2} \simeq \mathbb{R}(2), \) where \( \mathbb{R}(2) \) denotes \( 2 \times 2 \) matrices with real entries.

Clifford algebras enjoy the isomorphisms

\[
\text{Cl}_{p+1,q+1} = \text{Cl}_{p,q} \otimes \mathbb{R}(2), \quad \text{Cl}_{p+4,q} = \text{Cl}_{p,q+4}.
\]
and the Cartan-Bott periodicity theorem \cite{4, 11} implies that
\[ Cl_{p+8,q} = Cl_{p+4,q+4} = Cl_{p,q+8} = Cl_{p,q} \otimes \mathbb{R}(16). \]
If the signature satisfies \( p - q = 1 \,(\text{mod } 4) \), then \( Cl_{p+4,q} = Cl_{p,q+k} \).

A Clifford module for \( Cl_{p,q} \) is a representation of a Clifford algebra given by a
finite-dimensional real space \( U \) and a linear map \( \rho: Cl_{p,q} \to \text{End}(U) \), satisfying
the Clifford relation \( \rho^2(v) = -q(v,v)\text{id}_U \) or \( \rho(u)\rho(v) + \rho(v)\rho(u) = -2(u,v)\text{id}_U \) for
\( u, v \in \mathbb{R}^{p,q} \). An abstract theory of Clifford modules was founded in \cite{4}.

Clifford modules \((U, \rho)\) and \((U', \rho')\) for \( Cl_{p,q} \) are isomorphic if there is a linear
map \( \phi: U \to U' \), such that \( \phi \rho(v) = \rho'(v) \phi \) for all \( v \). A linear subspace \( W \subset U \) is
submodule if it is invariant under \( \rho \). A Clifford module \((U, \rho)\) is irreducible if it has
exactly two submodules: \( U \) and \( \{0\} \).

3.2. Free nilpotent Lie algebras. Let us introduce the notion of a free nilpotent
Lie algebra appeared following, e.g., \cite{51}. Let \( U \) be a real vector space of dimension
\( m \) with a basis \( \{e_1, \ldots, e_m\} \). Let us consider the tensor products \( U, U \otimes U, \ldots, \)
\( U \otimes \cdots \otimes U \) (\( n \)-times) and let us construct a space
\[ U = U \oplus (U \otimes U) \oplus \cdots \oplus (U \otimes \cdots \otimes U). \]

We introduce a distributive, non-associative and non-commutative operation \( \times : U \times U \to U \) iteratively as follows. If \( e_k \in U \), then
\[ e_i \times (e_{j_1} \otimes \cdots \otimes e_{j_k}) = \begin{cases} e_i \otimes e_{j_1} \otimes \cdots \otimes e_{j_k}, & \text{if } k \leq n - 1, \\ 0, & \text{if } k = n. \end{cases} \]

Next iteratively, if \( e \in U \) and \( a \times b \) is already defined for a fixed \( a \in U \) and for every
\( b \in U \), then
\[ (e \otimes a) \times b = e \otimes (a \times b) - a \times (e \otimes b). \]
Then the following non-associative equality holds by induction
\[ (a \times b) \times c = a \times (b \times c) - b \times (a \times c), \]
where \( a, b \) and \( c \) are from \( U \). Let us denote by \( \mathcal{W} \) the left ideal of the elements
represented as \( v \times v \), for \( v \in U \). Let \( N = U/\mathcal{W} \). Then \( N \) carries the structure of a
nilpotent Lie algebra with the Lie product
\[ [\langle a \rangle, \langle b \rangle] := \langle a \times b \rangle, \]
where \( \langle a \rangle = \langle a \rangle_{\mathcal{W}} \) denotes the equivalence class of the element \( a \in U \) modulo \( \mathcal{W} \).
Indeed, \( \mathcal{W} \ni (a+b) \times (a+b) = a \times a + b \times b + a \times b + b \times a \), and \( \langle a \times b \rangle = -\langle b \times a \rangle \).

Jacobi identity holds and nilpotentness trivially follows from \( N^{n+1} = 0 \). Abusing
notations we shall write simply \( a \) instead of \( \langle a \rangle \) in what follows. Let us call \( N \) the
free nilpotent Lie algebra of length \( n \). Along with the free nilpotent Lie algebra of
length \( n \) we consider nilpotent Lie algebras of length \( n \) with \( m \) generators.

**Theorem 1.** \cite{51} Let \( n \) be a nilpotent Lie algebra of length \( n \) generated by \( m \) linearly
independent elements. Then there exists an ideal \( A \) of \( N \) such that \( n \simeq N/A \).

We say that \( N \) covers \( n \) and the map \( \pi: N \to n \) is the projection. In particular
we are interested in free nilpotent Lie algebras of length 2.
3.3. Pseudo $H$-type algebra. It is convenient for us to split the definition of a pseudo $H$-type algebra in two parts.

Definition 1. We say that a Lie algebra $\mathfrak{n} \equiv (\mathfrak{n}, [\cdot,\cdot], (\cdot,\cdot))$ is a two-step nilpotent metric Lie algebra if it satisfies the following properties:

- $[[\mathfrak{n},\mathfrak{n}],\mathfrak{n}] = \{0\};$
- the scalar product $(\cdot,\cdot)$ is non-degenerate;
- $\mathfrak{n} = \mathfrak{h} \oplus_1 \mathfrak{z}$ with respect to $(\cdot,\cdot)$, where $\mathfrak{z}$ is the center of $\mathfrak{n};$
- the restriction $(\cdot,\cdot)_\mathfrak{z}$ of $(\cdot,\cdot)$ to $\mathfrak{z}$ is non-degenerate.

Definition 2. A two-step nilpotent metric Lie algebra $\mathfrak{n}$ is called pseudo $H$-type algebra if the operator $J : \mathfrak{z} \times \mathfrak{h} \to \mathfrak{h}$ defined by

\begin{equation}
(J_\mathfrak{z}u, v)_\mathfrak{h} = (z, [u, v])_\mathfrak{z},
\end{equation}

satisfies the following orthogonality condition

\begin{equation}
(J_\mathfrak{z}u, J_\mathfrak{z}v)_\mathfrak{h} = (z, z)_\mathfrak{z}(u, v)_\mathfrak{h}.
\end{equation}

Given a metric two-step nilpotent Lie algebra $\mathfrak{n}$, we call the operator $J$ defined by (1) satisfying (2), a pseudo $H$-type structure or simply an $H$-type structure in the case of a positive definite metric.

This definition for an inner product (positive definite), is equivalent to that found in [36], and for a non-degenerate scalar product, in [14, 26]. The definition (1) immediately implies the following properties of the operator $J$.

- The operator $J : \mathfrak{z} \times \mathfrak{h} \to \mathfrak{h}$ is bilinear.
- The operator $J_\mathfrak{z} \in \text{End}(\mathfrak{h})$ for any fixed $z \in \mathfrak{z}$ is skew symmetric

\begin{equation}
(J_\mathfrak{z}u, v)_\mathfrak{h} = -(u, J_\mathfrak{z}v)_\mathfrak{h},
\end{equation}

and

\begin{equation}
J_\mathfrak{z}^2 = -(z, z)_\mathfrak{z}\text{id}_\mathfrak{h};
\end{equation}

- The operator $J_\mathfrak{z} : \mathfrak{h} \to \mathfrak{h}$ is an isometry for all $z \in \mathfrak{z}$ with $(z, z)_\mathfrak{z} = 1$;
- The operator $J_\mathfrak{z} : \mathfrak{h} \to \mathfrak{h}$ is an anti-isometry for all $z \in \mathfrak{z}$ with $(z, z)_\mathfrak{z} = -1$.

Remark 1. Any two of (2), (3), and (1) imply the third property.

Proposition 1. [14 Proposition 2.2] Let $\mathfrak{n}$ be a pseudo $H$-type algebra with a non-degenerate scalar product $(\cdot,\cdot)$. If the scalar product $(\cdot,\cdot)_\mathfrak{h}$ is non-positive definite, then necessarily, the signature of the scalar product $(\cdot,\cdot)_\mathfrak{h}$ is neutral.

Proposition 2. Let $\mathfrak{n}$ be a two-step nilpotent metric Lie algebra, and let $z_1, \ldots, z_p$ be an orthogonal non-null basis of $\mathfrak{z}$. Let the operator $J$ be defined by (1) on the basis vectors of $\mathfrak{z}$. The operator $J_\mathfrak{z}$, $z \in \mathfrak{z}$, satisfies condition (2), or equivalently, $\mathfrak{n}$ is a pseudo $H$-type algebra, if and only if, $J_\mathfrak{z}^2 = -(z_k, z_k)_\mathfrak{z}\text{id}_\mathfrak{h}$ and $J_{z_i}J_{z_j} = -J_{z_j}J_{z_i}$.

Proof. The necessary condition is trivial. Let us focus ourselves on the sufficient part. Indeed, the definition (1) of the operator $J$ can be extended from the basis of $\mathfrak{z}$ to the whole $\mathfrak{z}$ by linearity, and it implies the skewsymmetry $(J_\mathfrak{z}u, v)_\mathfrak{h} = -(u, J_\mathfrak{z}v)_\mathfrak{h}$ and linearity with respect to $z$ and $u$. Therefore,

\begin{equation}
(J_{\alpha z_1 + \beta z_2}^2 u, v)_\mathfrak{h} = -(J_{\alpha z_1 + \beta z_2}u, J_{\alpha z_1 + \beta z_2}v)_\mathfrak{h} = -\alpha^2(J_{z_1}u, J_{z_1}v)_\mathfrak{h} - \beta^2(J_{z_2}u, J_{z_2}v)_\mathfrak{h}
\end{equation}
is defined by (1) satisfying (4) for every 

\[ z \in J \]

for any \( u, v \) from \( J \) and for any basis vectors \( z_1 \) and \( z_2 \) from \( J \). Hence, the equality \( J_z^2 = -(z, z) i d_h \) is true for any vector \( z \in J \) and the orthogonality condition follows from Remark 1. \[ \square \]

Let us formulate how the pseudo \( H \)-type algebras are related to representations of Clifford algebras. Given a pseudo \( H \)-type algebra \( n = (\mathfrak{z} \oplus 1, [\cdot, \cdot]) \), the operator \( J_z \) is defined by (1) satisfying (4) for every \( z \in \mathfrak{z} \), and therefore, it defines a representation \( J : Cl(\mathfrak{z}, q) \to \text{End}(\mathfrak{h}) \) over the space \( \mathfrak{h} \). Here the quadratic form \( q \) is defined by the scalar product \( \langle \cdot, \cdot \rangle \). Moreover in this case, the scalar product \( \langle \cdot, \cdot \rangle \) on the representation space is such that the operator \( J_z \) is skew symmetric with respect to this scalar product, see [13]. Following [14] we call such a pair \( (h, \langle \cdot, \cdot \rangle_h) \) an admissible representation. Now let us assume that a representation \( \rho : Cl(Z, q) \to \text{End}(V) \) of the Clifford algebra \( Cl(Z, q) \) is given and let us suppose also that the representation space \( V \) admits a scalar product \( \langle \cdot, \cdot \rangle_v \) such that the map \( J = \rho|_Z \), the restriction of \( \rho \) to \( Z \), is skew symmetric with respect to \( \langle \cdot, \cdot \rangle_v \), in other words \( (V, \langle \cdot, \cdot \rangle_v) \) is an admissible representation. Then we define the Lie brackets \([\cdot, \cdot]\) by formula (1), where the scalar product \( \langle \cdot, \cdot \rangle_Z \) is the polarization of the quadratic form \( q \). The Lie algebra \( N = (Z \oplus V, [\cdot, \cdot]) \) will be a pseudo \( H \)-type Lie algebra with the center \( Z \) and with the orthogonal decomposition \( Z \oplus V \) with respect to the scalar product \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_Z + \langle \cdot, \cdot \rangle_v \).

3.4. Lattices and nilmanifolds. Let \( G \) be a nilpotent Lie group, and let \( \mathfrak{g} \) be its Lie algebra. We denote the Lie exponent and logarithm by \( \exp : \mathfrak{g} \to G \) and \( \log : G \to \mathfrak{g} \).

**Definition 3.** A subgroup \( L \) of \( G \) is called a lattice if \( L \) is discrete and the left co-set \( L \backslash G \) possesses a finite measure which is invariant under the action of \( G \), and inherited from the Haar measure on \( G \). The lattice is called uniform (or co-compact) if \( L \backslash G \) is compact. The space \( L \backslash G \) is called compact nilmanifold.

**Theorem 2** (Mal’cev criterion [14]). The group \( G \) admits a lattice \( K \) if and only if the Lie algebra \( \mathfrak{g} \) admits a basis \( B = \{b_1, \ldots, b_n\} \) with rational structural constants \( [b_i, b_j] = \sum_{k=1}^n c_{ij}^k b_k \), \( c_{ij}^k \in \mathbb{Q} \).

Given a lattice \( K \), one can construct the corresponding basis \( B \) as follows. Set \( \mathfrak{g}_Q = \text{span}_Q \log K \), which is a Lie algebra over the field \( \mathbb{Q} \). Denote by \( B_Q \) a \( \mathbb{Q} \)-basis in \( \mathfrak{g}_Q \). Then it is also an \( \mathbb{R} \)-basis in \( \mathfrak{g} \).

Reciprocally, given a basis \( B \) defined as in Theorem 2, let \( \Lambda \) be a vector lattice in \( \mathfrak{g} \), such that \( \Lambda \subset \text{span}_Q B \). Then the lattice \( K \) is generated by elements \( \exp \Lambda \) and \( \text{span}_Q (\log K) = \text{span}_Q B \).

Let \( K_1 \) and \( K_2 \) be two lattices in \( G \). Then \( \text{span}_Q (\log K_1) = \text{span}_Q (\log K_2) \), if and only if, \((K_1 \cap K_2) \setminus K_1 \) and \((K_1 \cap K_2) \setminus K_1 \) are finite.

One of the aims of our paper is to prove that pseudo \( H \)-type groups admit lattices, or equivalently, the corresponding pseudo \( H \)-type algebras admit a basis with rational structural constants.
4. H-TYPE ALGEBRAS FROM FREE ALGEBRAS

The main idea is to construct the pseudo H-type algebra by means of the free algebra step by step restricting the latter by satisfying necessary conditions and enriching it with additional structures.

4.1. Construction. First we assign the vector space $U$ to be the horizontal space for the future pseudo H-type algebra and so we choose an even number $m = 2k$ of generators

$$e_1, \ldots, e_k, e_{k+1}, \ldots, e_{2k}.$$  

The pseudo H-type algebra is of step two, so we are interested in length two free nilpotent Lie algebra $N$. Let us introduce a scalar product $(\cdot, \cdot)_N$ in $N$ of some signature, such that the basis vectors $e_1, \ldots, e_{2k}, e_1 \times e_2, \ldots, e_{2k-1} \times e_{2k}$ are orthogonal and non-null with respect to this product. The scalar product $(\cdot, \cdot)_N$ splits as $(\cdot, \cdot)_N = (\cdot, \cdot)_U + (\cdot, \cdot)_Z$, where $Z = \text{span} \{e_1 \times e_2, e_1 \times e_3, \ldots, e_{2k-1} \times e_{2k}\}$. This way the Lie algebra $N$ splits into

$$N = U \bigoplus Z = \text{span} \{e_1, \ldots, e_{2k}, e_1 \times e_2, \ldots, e_{2k-1} \times e_{2k}\}.$$  

Define the signature index of a scalar product $(\cdot, \cdot)_U$ by

$$\varepsilon_j(s, r) = \begin{cases} 1, & \text{for } j = 1, \ldots, s; \\ -1, & \text{for } j = s + 1, \ldots, s + r. \end{cases}$$  

Let us normalize the basis vectors such that

- $(e_j, e_j)_U = \varepsilon_j(s, r)/k$ for $j = 1, \ldots, 2k$, $s + r = 2k$, where $s, r$ are not specified so far;
- $|(e_i \times e_j, e_i \times e_j)_Z| = 1/k$.

If $A \subset Z$ is an ideal of $N$, such that $N = U \bigoplus \Phi \bigoplus A$, and $n = N/A$, then $\mathfrak{h} := U(\text{mod } A)$, $\mathfrak{z} := \Phi(\text{mod } A)$, the scalar product $(\cdot, \cdot)_N = (\cdot, \cdot)_U + (\cdot, \cdot)_\Phi + (\cdot, \cdot)_A$ gives $(\cdot, \cdot)_n := (\cdot, \cdot)_U + (\cdot, \cdot)_\Phi$. Let us denote by $\tilde{u} = u + A$, and $\tilde{v} = v + A$ for $u, v \in U$. The commutator on $n$ is defined by $[\tilde{u}, \tilde{v}] = u \times v + A$.

Let us define an operator $J_\omega : U \rightarrow U$, $\omega \in Z$, by $(J_\omega u, v)_U = (\omega, u \times v)_Z$. So far it is only a definition of $J$ without requiring the orthogonality condition (2).

In order to make $n$ a pseudo H-type algebra, the operator $\tilde{J}_z$, $z \in \mathfrak{z}$, $z = \omega + A$, $\omega \in \Phi$, must satisfy

$$(\tilde{J}_z u, \tilde{v})_U = (\omega + A, u \times v + A))_Z = (\omega, u \times v)_\Phi = (J_\omega u, v)_U$$  

and the orthogonality condition (2).

The next step is to construct the ideal $A$ of $N$ as an orthogonal complement in $Z$ to $\Phi$. To define $\Phi$ we consider a family of partitions $p = \{p_l\}$, $l = 1, \ldots, (2k)!/(2k)^k$, of the set of integers $1, 2, \ldots, 2k$ in ordered pairs $(i,j)$, $i < j$. For each $p_l \in p$ we construct ordered pairs $(e_i, e_j)$, $(i,j) \in p_l$, the product $e_i \times e_j$, and the vector

$$\omega_l = \sum_{(i,j) \in p_l} \alpha_{ij}^l(e_i \times e_j) \in Z, \quad \alpha_{ij}^l \in \mathbb{R}.$$  

The operator $J_{\omega_l} : U \to U$ is defined by the equality
\begin{equation}
(J_{\omega_l}(u), v)_U = (\omega_l, u \times v)_Z, \quad u, v \in U.
\end{equation}
Let us assume that the coefficients $\alpha_{ij}^l$ are chosen so that $Z = \text{span} \{\omega_l : l = 1, \ldots, (2k)!/2^k\}$. From the definition of the operator $J_{\omega_l}$ and the product $(\times)$ it immediately follows that $J_{\omega_l}$ extends to the operator $J_{\omega} : U \to U$, $\omega \in Z$ by linearity, and
\[
(J_{\omega}(u), v)_U = -(u, J_{\omega}(v))_U.
\]

We restrict the number of partitions from $p$ assigning special values to the coefficients $\alpha_{ij}^l$ by requiring that the operators $J_{\omega_l}$ act in a special way on the basis $\{e_1, \ldots, e_{2k}\}$ of $U$.  

**Definition 4.** By signed permutation of vectors $e_1, \ldots, e_{2k}$ from $U$ by the operator $J_{\omega}$ we understand a permutation where some of resulting vectors may change orientation. That is, for any $i = 1, \ldots, 2k$, there exists $j = 1, \ldots, 2k$, such that $J_{\omega}e_i = \pm e_j$.

**Proposition 3.** For a fixed $l$, the operator $J_{\omega_l} : U \to U$ is a signed permutation of the basis $e_1, \ldots, e_{2k}$, if and only if, for every fixed $a \in \{1, \ldots, 2k\}$ there exists a unique $b \in \{1, \ldots, 2k\}$, $a \neq b$, such that $\alpha_{ab}^l = \pm 1$ if $a < b$ or $\alpha_{ba}^l = \pm 1$ if $a > b$.

**Proof.** The sufficient part is trivial. For the necessary part, fix some $a \in \{1, \ldots, 2k\}$ and $l$. Then in the partition $p_l$ there is unique index $b \in \{1, \ldots, 2k\}$ such that $(a, b) \in p_l$. For simplicity assume that $a < b$. The definition (5) of the operator $J_{\omega_l}$ immediately implies
\[
(J_{\omega_l}(e_a), e_b)_U = (\omega_l, e_a \times e_b)_Z = \pm \frac{\alpha_{ab}^l}{k} \neq 0,
\]
and
\[
(J_{\omega_l}(e_a), e_j)_U = (\omega_l, e_a \times e_j)_Z = 0 \quad \text{for all } \quad j \in \{1, \ldots, 2k\}, \quad j \neq b.
\]
since $\omega_l$ does not contain the term $e_a \times e_j$. The map $J_{\omega_l}$ acts by sign permutation of the basis, therefore $J_{\omega_l}(e_a) = \pm e_b$ and this implies also that $\alpha_{ab}^l = \pm 1$. Here the sign $\pm$ means, $+$ or $-$. \(\square\)

**Remark 2.** Observe that the map $J_{\omega_l} : U \to U$ is injective in the case of signed permutation.

The following simple statement can be found in standard texts in combinatorics and graph theory, e.g. [3].

**Proposition 4 ([3]).** The set $p$ of all $(2k)!/2^k k!$ possible partitions contains a subset
\[
p' = \{p_{l_1}, \ldots, p_{l_{2k-1}}\},
\]
such that each pair $(i, j)$, $1 \leq i < j \leq 2k$, appears only once in these partitions.

This proposition can be reformulated in terms of graph theory. Let us consider the complete graph $K_{2k}$ of $2k$ vertices at the numbers $1, \ldots, 2k$, and recall that a 1-factorization of a graph is a decomposition of all the edges of the graph into
1-factors, the sets of $k$ independent edges (without common vertices). The graph $K_{2k}$ has exactly $(2k - 1)$ 1-factors, see e.g., [29], which in our case coincide with a possible set of partitions $p' = \{p_1, \ldots, p_{2k-1}\}$.

We observe that splitting of $K_{2k}$ into $(2k - 1)$ 1-factors is not unique and may be even not isomorphic, e.g., Kirkman [39] and Steiner [52] tournaments are not isomorphic [3], see example:

\[
\begin{array}{cccc}
12 & 38 & 47 & 56 \\
13 & 24 & 58 & 67 \\
14 & 26 & 35 & 78 \\
15 & 28 & 37 & 46 \quad \text{and} \quad 15 & 24 & 36 & 78 \\
16 & 23 & 57 & 48 \\
17 & 25 & 34 & 68 \\
18 & 27 & 36 & 45
\end{array}
\]

Indeed, in the first array any two lines set together in one graph give a cycle of length 8, while in the second array this will give two disjoint cycles of length 4.

Let $\omega_{l_m}$ be constructed by $p_{l_m} \in p'$, and from now on, let the corresponding operator $J_{\omega_{l_m}}$ acts as a signed basis permutation on $U$. Fixing some $e_j$ we obtain an operator $J(e_j): Z \to U$. From now on we omit the upper index writing simply $\alpha_{ij}$ instead of $\alpha_{ij}^l$ because the pair $(i, j)$ is met only once in every $p'$.

**Proposition 5.** Let $p' = \{p_1, \ldots, p_{2k-1}\}$ be a set of partitions, where each pair $(i, j)$, $1 \leq i < j \leq 2k$, appears only once. Let the operator $J_{\omega_{l_m}}$ acts over $U$ as a signed permutation. The map

\[
J(e_j): \text{span} \{\omega_{l_m} = \sum_{(i,j) \in p_{l_m}} \alpha_{ij}(e_i \times e_j), \ m = 1, \ldots, (2k-1)\} \to \{e_j\}^\perp
\]

is bijective.

**Proof.** Without loss of generality, fix $j = 1$. Then $J_{\omega_{l_m}}(e_1) = \pm e_{j_m}$ by Proposition 3, where $m = 1, \ldots, (2k-1)$, the index $j_m \in \{2, 3, \ldots, 2k\}$, and $j_m \neq j_n$ if $m \neq n$. If $\omega = \sum_{m=1}^{2k-1} c_m \omega_{l_m}$, then $J_\omega(e_1) = \sum_{m=1}^{2k-1} \pm c_m e_{j_m}$, which vanishes if and only if, all coefficients $c_m = 0$. Therefore, $J(e_1)$ acts as an isomorphism of linear spaces span$_{1 \leq m \leq 2k-1} \{\omega_{l_m}\}$ and $\{e_1\}^\perp$. \hfill \square

The scalar project on $U$ is defined and now we fix it on $Z$ requiring $(\omega_{l_m}, \omega_{l_m})_Z = \pm 1$. This can be always achieved by the following procedure. Fix $m$ and write

\[
(\omega_{l_m}, \omega_{l_m})_Z = \sum_{(i,j) \in p_{l_m}} \langle e_i \times e_j, e_i \times e_j \rangle_Z.
\]

We choose a signature $(p, q)$ of $(\cdot, \cdot)_Z$, $p + q = 2k - 1$. Then we define $(e_i \times e_j, e_i \times e_j)_Z = 1/k$ for $(i, j) \in p_{l_m}$, where $m = 1, \ldots, p$, and $(e_i \times e_j, e_i \times e_j)_Z = -1/k$ for $m = p + 1, \ldots, p + q = 2k - 1$. We arrive at two options: the metric on $Z$ is positive definite, $q = 0$, and $(\omega_{l_m}, \omega_{l_m})_Z = 1$, and the metric on $Z$ is indefinite non-degenerate, $q \neq 0$, so that $(\omega_{l_m}, \omega_{l_m})_Z = -1$ for some of $l_m$. 


4.2. Inner product on $Z$. Since we have chosen an inner (positive definite) product on $Z$, the scalar product on $U$ may by either an inner product or a scalar negative definite product, because of the properties of the operator $J$. Both pictures are isomorphic, hence we choose an inner product on the whole $N$. In this section we study the case of both metrics $(\cdot, \cdot)_U$ and $(\cdot, \cdot)_Z$ to be positive definite. In this case

$$J^2_{\omega_1} = -\text{id}_U \quad \text{and} \quad J_{\omega_1} J_{\omega_2} = - J_{\omega_2} J_{\omega_1}. \tag{6}$$

For every $p_l \in p'$, for the corresponding vector $\omega_l$, and finally, for the operator $J_{\omega_l}$ acting by signed permutation, we associate a $2k \times 2k$ matrix of coefficients $E_l = \{ \alpha_{ij} \}$ where $\alpha_{ij} = -\alpha_{ji}$ and $\alpha_{ij} = 0$ if $(i, j) \not\in p_l$. Matrices $E_l$, $l = 1, \ldots, 2k - 1$ are orthogonal with entries $0, \pm 1$. So we constructed an injective homomorphism of the operators $J_{\omega_l}$ to the set of orthogonal matrices.

We continue with the Hurwitz-Radon-Eckmann theorem, see [21, 36].

**Theorem 3** ([21, 31, 49]). A family of $2k - 1$ real orthogonal $2k \times 2k$ matrices $E_1, \ldots, E_{2k - 1}$ admits at most $\rho(2k) - 1$ matrices satisfying $E_l^2 = -I$ and $E_i E_j = -E_j E_i$, where $\rho(\cdot)$ is the Hurwitz-Radon function.

This theorem was proved for integer matrices $E_k$ having entries among $\{0, \pm 1\}$ in [23]. Moreover, the upper bound $\rho(2k) - 1$ is achieved for integer Hurwitz-Radon matrices.

Relations (6) and Theorem 3 reveal the fact that the HR-family of orthogonal matrices gives a representation of a Clifford algebra $Cl_{p,0}$.

**Theorem 4.** Given a free two-step nilpotent Lie algebra $N = U \oplus Z$ with an inner product and $\dim(U) = 2k$, there exists an ideal $A$ such that $N/A$ is an $H$-type algebra $n = h \oplus z$, where $h = U$ and the center is $z = Z/A$. Moreover, the ideal $A$ can be chosen such that $\dim(z)$ takes every value from $\{1, \ldots, \rho(2k) - 1\}$.

**Proof.** We reformulate this statement in the language of partitions as follows. There exists a family of partitions

$$p'' = \{p''_1, \ldots, p''_{\rho(2k) - 1} \} \subset p',$$

equivalent to a subset of 1-factorization of the complete graph $K_{2k}$, such that the corresponding set of vectors $\{\omega_1, \ldots, \omega_{\rho(2k) - 1}\}$ with the coefficients $\alpha_{ij} = \pm 1$ defines $A$ as the orthogonal complement to span$\{\omega_1, \ldots, \omega_{\rho(2k) - 1}\}$ in $Z$, and $z = Z/A$ is the center of an $H$-type algebra. In what follows we describe the choice of $p''$.

We start by analysing the correspondence between the operators $J_{\omega_l}$, $l \in \{1, \ldots, 2k - 1\}$, and the orthogonal matrices $E_l$. Indeed, an orthogonal matrix $E_l$ with entries $0, \pm 1$ is a permutation of the standard basis of $\mathbb{R}^{2k}$, if an only if, each row and each column contains only one entry $\pm 1$ and others are 0. Respectively, $E_l$ is a signed permutation of the canonical basis, if and only if, each row and each column contains only one entry 1 or (-1) and others are 0. The form of the partitions $p'$ guarantees that the matrices $E_l$ constructed above with the entries $\alpha_{ij}$, $(i, j) \in p_l \in p'$ for $i < j$ and $\alpha_{ij} = -\alpha_{ji}$, are the sign permutation of the basis $\{e_1, \ldots, e_{2k}\}$, i.e., we constructed a one-to-one correspondence between the orthogonal sign permutation matrices $E_l$, satisfying the conditions of Theorem 3 and the operators $J_{\omega_l}$.
Given the set of matrices \( \{ E_{l_1}, \ldots, E_{l_{p(2k)-1}} \} \) with integer entries, representing the maximal family of matrices in Theorem 3, we set \( \mathfrak{p}'' \) to be the family of partitions \( \mathfrak{p}_k \) consisting of elements \((i, j)\) such that \( \alpha_{ij} \) is the entry in the upper triangular part of the matrix \( E_{l_k} \). According to \( \mathfrak{p}'' \), we construct the vectors \( \omega_l \), \( l = 1, \ldots, \omega_{p(2k)-1} \). Corresponding operators \( J_{\omega_l} \) satisfy relations (6) since the matrices \( E_l \) do. Now we set \( \mathfrak{h} = U \) and \( A \) to be the orthogonal complement in \( Z \) to span \( \{ \omega_{l_1}, \ldots, \omega_{p(2k)-1} \} \). Then we define \( \mathfrak{z} = Z/A \), and \( \mathfrak{h} \oplus_\perp \mathfrak{z} \) is an \( H \)-type algebra \( \mathfrak{n} \) by Proposition 2.

The following theorem was proved in [20, 23, 28] by different methods.

**Theorem 5.** The \( H \)-type algebras admit rational structure constants.

**Proof.** Let \( \mathfrak{h} = U \) and \( \mathfrak{z} = Z/A \) is chosen and is spanned by the vectors \( \omega_l = \sum_{(i, j) \in \mathfrak{p}''_l} \alpha_{ij} e_i \times e_j \) modulo the ideal \( A \). Then \([e_n, e_m] = e_n \times e_m \pmod{A} \in \mathfrak{z} \). Let \((n, m) \in \mathfrak{p}''_l \) for some \( l \). Write

\[
\omega_l = \alpha_{nm} e_n \times e_m + \sum_{(i, j) \in (\mathfrak{p}''_l \setminus (n, m))} \alpha_{ij} e_i \times e_j,
\]

and set

\[
\omega_{(l; ij)} = \alpha_{nm} e_n \times e_m - \alpha_{ij} e_i \times e_j, \quad (i, j) \in (\mathfrak{p}''_l \setminus (n, m))
\]

Then, \( \omega_l \perp \omega_{(l; ij)} \) and \( \omega_{(l; ij)} \in A \). So we write

\[
e_n \times e_m \pmod{A} = \frac{1}{k} \left( \alpha_{nm} \omega_l + \sum_{(i, j) \in (\mathfrak{p}''_l \setminus (n, m))} \omega_{(l; ij)} \right) \pmod{A} = \frac{\alpha_{nm}}{k} \omega_l \pmod{A}.
\]

\( \Box \)

**Remark 3.** Observe that \( \omega_l = \frac{1}{2} \sum_{j=1}^{2k} e_j \times E_l e_j = \frac{1}{2} \sum_{j=1}^{2k} e_j \times J \omega_l e_j \).

We are not focused on algorithms of calculation of matrices \( E_l \), vectors \( \omega_l \) and operators \( J_l \) in full generality which is the subject of computational algebra, however, we look at an illustrative example in the next subsection.

**4.2.1. The case \( k = 2 \).** Let us have a close look at the simplest example of \( U \) generated by \( \{ e_1, e_2, e_3, e_4 \} \) and

\[
\omega_1 = \alpha_{12} e_1 \times e_2 + \alpha_{34} e_3 \times e_4,
\]

\[
\omega_2 = \alpha_{13} e_1 \times e_3 + \alpha_{24} e_2 \times e_4,
\]

\[
\omega_3 = \alpha_{14} e_1 \times e_4 + \alpha_{23} e_2 \times e_3.
\]

That is we have chosen \( \mathfrak{p}'' = \mathfrak{p}' \subset \mathfrak{p} \) such that \( \mathfrak{p}'' = \{ \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \} \), where

\[
\mathfrak{p}_1 = 12 \quad 34,
\]

\[
\mathfrak{p}_2 = 13 \quad 24,
\]

\[
\mathfrak{p}_3 = 14 \quad 23.
\]
Definition of the metric implies $J^2_{\omega m} = -\text{id}_U$. In order to define the coefficients $\alpha$ we need to check anti-commutativity of $J_{\omega_1}$, $J_{\omega_2}$, and $J_{\omega_3}$ on the basis. We have

$$
e_1 \xrightarrow{J_{\omega_1}} \alpha_2 e_2 \xrightarrow{J_{\omega_2}} \alpha_1 \alpha_2 e_4, \\
e_1 \xrightarrow{J_{\omega_2}} \alpha_3 e_3 \xrightarrow{J_{\omega_1}} \alpha_3 \alpha_4 e_4, \\
e_1 \xrightarrow{J_{\omega_2}} \alpha_3 e_3 \xrightarrow{J_{\omega_3}} -\alpha_3 \alpha_2 e_2, \\
e_1 \xrightarrow{J_{\omega_3}} \alpha_4 e_4 \xrightarrow{J_{\omega_2}} -\alpha_4 \alpha_2 e_2, \\
e_1 \xrightarrow{J_{\omega_3}} \alpha_4 e_4 \xrightarrow{J_{\omega_1}} -\alpha_4 \alpha_3 e_3, \\
e_1 \xrightarrow{J_{\omega_1}} \alpha_2 e_2 \xrightarrow{J_{\omega_3}} \alpha_2 \alpha_3 e_3.$$

In order to satisfy anti-commutativity we write the homogeneous equations

$$(7) \quad \alpha_1 \alpha_2 \alpha_3 + \alpha_2 \alpha_3 \alpha_4 = 0, \quad \alpha_1 \alpha_3 \alpha_4 + \alpha_2 \alpha_3 \alpha_4 = 0, \quad \alpha_1 \alpha_2 \alpha_4 + \alpha_2 \alpha_3 \alpha_4 = 0.$$

Analogously we start with the vectors $e_2, e_3$, and $e_4$. We arrive at three additional equations

$$(8) \quad \alpha_1 \alpha_2 \alpha_3 + \alpha_2 \alpha_3 \alpha_4 = 0, \quad \alpha_1 \alpha_2 \alpha_4 + \alpha_2 \alpha_3 \alpha_4 = 0, \quad \alpha_1 \alpha_2 \alpha_4 + \alpha_2 \alpha_3 \alpha_4 = 0,$$

which we solve over the integers $(1, -1) \in \mathbb{Z}$. However, the systems $(7)$ and $(8)$ are equivalent. Indeed, multiplying the first equation in $(8)$ by $\alpha_1 \alpha_2 \alpha_4$ we obtain the first equation in $(7)$, multiplying the second equation in $(8)$ by $\alpha_2 \alpha_2 \alpha_3$ we obtain the second equation in $(7)$, and finally multiplying the third equation in $(8)$ by $\alpha_1 \alpha_2 \alpha_4$ we obtain the third equation in $(7)$. This reflects the fact that we can check anti-commutativity of $J_{\omega_1}$, $J_{\omega_2}$, and $J_{\omega_3}$ starting only with one vector, e.g., $e_1$.

The operators $J_{\omega_1}$, $J_{\omega_2}$, and $J_{\omega_3}$ define the orthogonal matrices $E_1$, $E_2$, and $E_3$ as was described in Theorem 4.

$$E_1 = \begin{pmatrix} 0 & -\alpha_{12} & 0 & 0 \\ \alpha_{12} & 0 & 0 & 0 \\ 0 & 0 & -\alpha_{34} & 0 \\ 0 & \alpha_{34} & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & -\alpha_{13} & 0 \\ 0 & 0 & 0 & -\alpha_{24} \\ \alpha_{13} & 0 & 0 & 0 \\ 0 & \alpha_{24} & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{14} \\ 0 & 0 & -\alpha_{23} & 0 \\ 0 & \alpha_{23} & 0 & 0 \\ \alpha_{14} & 0 & 0 & 0 \end{pmatrix},$$

which satisfy the conditions

$$E_j^2 = -I \quad \text{and} \quad E_i E_j = -E_j E_i.$$ 

In this case, in view of Hurwitz-Radon-Eckmann’s Theorem 8 $u = 1, \alpha = 0, \beta = 2$, and $\rho(4) - 1 = 3$.

The system $(7)$ has a non-unique solution in integers $\pm 1$. This fact can be viewed in the context of algebraic geometry and computer algebra. Let us define a symmetric square $S^2(\mathfrak{h}) = \mathfrak{h} \vee \mathfrak{h}$ of the vector space $\mathfrak{h} = \text{span} \{ (x_1, \ldots, x_m)^t \}$ of dimension $m = 2k$, and let $\mathbb{P}(S^2(\mathfrak{h}))$ be its projectivization of (projective) dimension.
\[ \frac{1}{2}(m-1)(m+2), \]
\[ \mathbb{P}(S^2(\mathfrak{h})) = \{(y_{11}, y_{12}, \ldots, y_{ij}, \ldots, y_{mm}), \quad y_{ij} \neq 0, \quad i \leq j\}. \]

The Veronese embedding \( \varphi: \mathbb{P}(\mathfrak{h}) \to \mathbb{P}(S^2(\mathfrak{h})) \) is defined by
\[ (x_1, \ldots, x_m)^t \mapsto (x_1^2, x_1x_2, \ldots, x_1x_m, x_2^2, x_2x_3, \ldots, x_m^2)^t. \]

The Veronese variety \( \mathcal{V}(\mathfrak{h}) \), see e.g., [19], is the image \( \varphi(\mathbb{P}(\mathfrak{h})) = \mathcal{V}(\mathfrak{h}) \subset \mathbb{P}(S^2(\mathfrak{h})) \).

The intersection of the Veronese variety \( \mathcal{V}(\mathfrak{h}) \) with a generic \( \frac{1}{2}(m-1)(m+2) - m+1 \)-dimensional hyperplane has \( 2^{m-1} \) points, so \( \mathcal{V}(\mathfrak{h}) \) is of order \( 2^{m-1} \).

The vector space \( S^2(\mathfrak{h}) \) and its projectivization \( \mathbb{P}(S^2(\mathfrak{h})) \) can also be realized in terms of symmetric \( m \times m \)-matrices
\[ y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{12} & y_{22} & \cdots & y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1m} & y_{2m} & \cdots & y_{mm} \end{pmatrix}. \]

Then the Veronese embedding \( \varphi: \mathbb{P}(\mathfrak{h}) \to \mathbb{P}(S^2(\mathfrak{h})) \) is defined as
\[ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \begin{pmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_m \\ x_1x_2 & x_2^2 & \cdots & x_2x_m \\ \vdots & \vdots & \ddots & \vdots \\ x_1x_m & x_2x_m & \cdots & x_m^2 \end{pmatrix}. \]

A point \( y \) is in \( \mathcal{V}(\mathfrak{h}) \) if and only if the matrix corresponding to \( y \) is of rank 1. An \( m \times m \) symmetric matrix has \( \frac{1}{2}(m-1)m^2(m+1) \) independent 2-minors, which all vanish for the rank 1 matrix \( y \). Each 2-minor defines a quadric in \( \mathbb{P}(S^2(\mathfrak{h})) \) and \( y \in \mathcal{V}(\mathfrak{h}) \) if and only if \( y \) lies in the intersection of these quadrics. The linearization algorithm can be viewed as follows, see e.g., [18].

Consider a system of \( p \) independent homogeneous quadratic equations \( f_1 = 0, \ldots, f_p = 0 \) with the projective point \( x \) as a non-trivial solution. This system is projected to the linear system of \( p \) equations \( \hat{f}_1 = 0, \ldots, \hat{f}_p = 0 \) in terms of variables \( y \). Define the projective subspace \( \mathcal{H} \subset \mathbb{P}(S^2(\mathfrak{h})) \) as the intersection of \( p \) hyperplanes defined by \( \hat{f}_1 = 0, \ldots, \hat{f}_p = 0 \). So we have that the solution \( x \) is embedded into the intersection \( \mathcal{H} \cap \mathcal{V}(\mathfrak{h}) \). Thus the solution is obtained by \( x = \varphi^{-1}(\mathcal{H} \cap \mathcal{V}(\mathfrak{h})) \). Since \( \mathcal{V}(\mathfrak{h}) \) contains no non-trivial linear subspaces, the algorithm is efficient if \( \mathcal{H} \) consists of a finite number of isolated points.

Let us denote \( x_1 = \alpha_{12}, \ x_2 = \alpha_{24}, \ x_3 = \alpha_{13}, \ x_4 = \alpha_{34}, \ x_5 = \alpha_{23}, \ x_6 = \alpha_{14}, \) then
\[ y_{12} = \alpha_{12}\alpha_{24}, \ y_{13} = \alpha_{12}\alpha_{13}, \ y_{24} = \alpha_{24}\alpha_{34}, \ y_{34} = \alpha_{13}\alpha_{34}, \ y_{35} = \alpha_{13}\alpha_{23}, \ y_{26} = \alpha_{14}\alpha_{24}, \ y_{46} = \alpha_{14}\alpha_{34}, \ y_{15} = \alpha_{12}\alpha_{23}, \ y_{36} = \alpha_{13}\alpha_{14}, \ y_{25} = \alpha_{23}\alpha_{24}, \ y_{16} = \alpha_{12}\alpha_{14}, \ y_{45} = \alpha_{23}\alpha_{34}. \]

The equations (9) are equivalent to the linear homogeneous system \( \hat{f}_1 = 0, \ldots, \hat{f}_8 = 0 \)
\[ y_{11} - y_{22} = 0, \quad y_{11} - y_{33} = 0, \quad y_{11} - y_{44} = 0, \quad y_{11} - y_{55} = 0, \quad y_{11} - y_{66} = 0, \]
\[ y_{12} + y_{34} = 0, \quad y_{35} + y_{26} = 0, \quad y_{46} - y_{15} = 0. \]
In the projective coordinates this system has 3 solutions which we write in the matrix form as

\[
\begin{pmatrix}
1 & 1 & y_{13} & y_{14} & 1 & y_{16} \\
1 & 1 & y_{23} & y_{24} & y_{25} & -1 \\
y_{13} & y_{23} & 1 & -1 & 1 & y_{36} \\
y_{14} & y_{24} & -1 & 1 & y_{45} & 1 \\
y_{16} & -1 & y_{36} & 1 & y_{56} & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & y_{13} & y_{14} & 1 & y_{16} \\
1 & 1 & y_{23} & y_{24} & y_{25} & 1 \\
y_{13} & y_{23} & 1 & -1 & 1 & y_{36} \\
y_{14} & y_{24} & -1 & 1 & y_{45} & 1 \\
y_{16} & -1 & y_{36} & 1 & y_{56} & 1
\end{pmatrix}
\]

In the original variables we have 6 solutions in coordinates

\((α_{12}, α_{24}, α_{13}, α_{34}, α_{23}, α_{14})\)

as

\((1, 1, 1, -1, 1, -1), \ (1, 1, -1, 1, 1, 1), \ (1, 1, -1, 1, -1, -1), \ (-1, -1, 1, -1, -1, -1), \ (-1, -1, 1, -1, 1, 1).\)

The values of the remaining ‘free’ variables \(y_{13}, \ldots, y_{56}\) then follow. Fixing \(α_{12} = 1\) (or the first line of solutions) we arrive at three groups of vectors

\[(10) \quad Ω_1 = \text{span} \begin{pmatrix}
ω_{Ω_1}^{1} = e_1 \times e_2 - e_3 \times e_4 \\
ω_{Ω_1}^{2} = e_1 \times e_3 + e_2 \times e_4 \\
ω_{Ω_1}^{3} = -e_1 \times e_4 + e_2 \times e_3
\end{pmatrix},
\]

\[(11) \quad Ω_2 = \text{span} \begin{pmatrix}
ω_{Ω_2}^{1} = e_1 \times e_2 + e_3 \times e_4 \\
ω_{Ω_2}^{2} = -e_1 \times e_3 + e_2 \times e_4 \\
ω_{Ω_2}^{3} = e_1 \times e_4 + e_2 \times e_3
\end{pmatrix},
\]

and

\[(12) \quad Ω_3 = \text{span} \begin{pmatrix}
ω_{Ω_3}^{1} = e_1 \times e_2 + e_3 \times e_4 \\
ω_{Ω_3}^{2} = -e_1 \times e_3 + e_2 \times e_4 \\
ω_{Ω_3}^{3} = -e_1 \times e_4 - e_2 \times e_3
\end{pmatrix}.
\]

Recall that now we construct the ideals \(A_1, A_2,\) and \(A_3\) as the orthogonal complements to \(Ω_1, Ω_2,\) and \(Ω_3\) in \(Z:\)

\[(13) \quad A_1 = \text{span} \begin{pmatrix}
ω_{A_1}^{1} = e_1 \times e_2 + e_3 \times e_4 \\
ω_{A_1}^{2} = e_1 \times e_3 - e_2 \times e_4 \\
ω_{A_1}^{3} = -e_1 \times e_4 - e_2 \times e_3
\end{pmatrix},
\]

\[(14) \quad A_2 = \text{span} \begin{pmatrix}
ω_{A_2}^{1} = e_1 \times e_2 - e_3 \times e_4 \\
ω_{A_2}^{2} = -e_1 \times e_3 - e_2 \times e_4 \\
ω_{A_2}^{3} = e_1 \times e_4 - e_2 \times e_3
\end{pmatrix},
\]
and

\[
(15) \quad A_3 = \text{span} \left( \begin{array}{c} \omega_1^{A_3} = e_1 \times e_2 - e_3 \times e_4 \\ \omega_2^{A_3} = -e_1 \times e_3 - e_2 \times e_4 \\ \omega_3^{A_3} = -e_1 \times e_4 + e_2 \times e_3 \end{array} \right).
\]

Then, \( \mathfrak{j}_j = \Omega_j \text{(mod } A_j) = Z/A_j, \ j = 1, 2, 3. \) The corresponding \( H \)-type groups are \( \mathfrak{n}_j = \mathfrak{h}_j \oplus \mathfrak{j}_j, \) where \( \mathfrak{h}_j = U(\text{mod } A_j). \) Fixing \( a_{12} = -1 \) leads us to the inverse matrices \( E_i^{-1} = -E_i \) and the inverse operators \( J_{\omega_i}^{-1} = -J_{\omega_i} \) corresponding to \( \omega_i, \) which are obtained also from the solutions to \( (11). \)

### 4.3 Lie algebra isomorphism of solutions.

Let us remind some facts from the general theory of Lie algebras, see, e.g., [10] Page 30. Let \( \mathfrak{g} \) be a Lie algebra, and let \( \mathfrak{a} \) be an ideal in \( \mathfrak{g}. \) Then the vector space quotient \( \mathfrak{g}/\mathfrak{a} \) becomes a Lie algebra under the definition of Lie brackets \([u + \mathfrak{a}, v + \mathfrak{a}] = [u, v] + \mathfrak{a}.\) The quotient map \( q: \mathfrak{g} \to \mathfrak{g}/\mathfrak{a} \) is a Lie algebra homomorphism and \( \mathfrak{a} = \ker q. \) Let \( \pi: \mathfrak{g} \to \mathfrak{h} \) be another Lie algebra homomorphism. If \( \mathfrak{a} = \ker \pi, \) then \( p: \mathfrak{g}/\mathfrak{a} \to \mathfrak{h}, \) \( \pi = p \circ q, \) is a Lie algebra isomorphism.

In our case, \( N = U \oplus \Omega_j \oplus A_j \) and \( \mathfrak{n}_j = N/A_j. \) Let \( B: N \to N \) be a free Lie algebra isomorphism such that \( B(A_j) = A_j. \) Then, denote \( q_j: N \to N/A_j, \) \( \pi = q_i \circ B, \) and \( q = q_j. \) Then there is a Lie algebra isomorphism \( p: N/A_j \to N/A_i. \)

Let us now check the isomorphism of \( H \)-type structures on \( \mathfrak{n}_j. \) We construct the operator \( B \) on \( N \) such that

1. \( B \) is a free Lie algebra \( N \) isomorphism;
2. \( B(A_i) = A_j; \)
3. \( B \) is an isometry of \( N \) preserving the orthogonal decomposition

\[
B: U \oplus \Omega_i \oplus A_i \to U \oplus \Omega_j \oplus A_j.
\]

The condition (1) implies that \( B(u \times v) = Bu \times Bv \) for all \( u, v \in U, \) and (3) implies \( (Bu, Bv)_U = (u, v)_U \) and

\[
(16) \quad (B\omega, Bu \times Bv)_Z = (\omega, u \times v)_Z, \quad \text{for all } \omega \in Z.
\]

The condition (3) implies also that \( B^T = B^{-1}. \) Then

\[
(17) \quad (J_B \omega Bu, Bv)_U = (J_\omega u, v)_U, \quad \text{for all } \omega \in Z,
\]

and therefore,

\[
(18) \quad J_B \omega = BJ_\omega B^{-1}.
\]

Obviously, the operators \( J_\omega \) and \( BJ_\omega B^{-1} \) satisfy conditions (6) on \( \Omega_i \) and \( \Omega_j \) respectively. The conditions (1-2) imply the Lie algebra isomorphism \( p: \mathfrak{n}_i \to \mathfrak{n}_j \) induced by \( B. \) The quotient maps \( q_j: N \to N/A_j = \mathfrak{n}_j \) are the orthogonal projections, hence \( p \) is an isometry preserving the orthogonal decompositions \( p: \mathfrak{h}_i \oplus \mathfrak{j}_i \to \mathfrak{h}_j \oplus \mathfrak{j}_j. \) This will guarantee that

\[
(z, [\bar{u}, \bar{v}])_{\mathfrak{j}_i} = (pz, [p\bar{u}, p\bar{v}])_{\mathfrak{j}_j}.
\]

Then analogously to (16), (18) we deduce that the isomorphism of \( H \)-type structures is given by \( J_{pz} = pJ_{z}p^{-1}. \)
Before we prove a general result on isomorphisms we look at our illustrative example for \( k = 2 \).

4.3.1. The case \( k = 2 \). Let us realize this scheme constructing a Lie algebra isomorphism \( n_1 \to n_3 \) with the free Lie algebra isomorphism \( B \) as follows. We define \( B \) on \( U \) as

\[
B: e_1 \to e_2, \quad e_2 \to -e_1, \quad e_3 \to e_4, \text{ and } e_4 \to e_3.
\]

Then \( B(e_i \times e_j) := Be_i \times Be_j \). Then, \( \omega_l^{\Omega_1} = B(\omega_l^{\Omega_1}) \), and \( \omega_l^{A_3} = B(\omega_l^{A_3}) \) for \( l = 1, 2, 3 \). The corresponding matrix \( B \) satisfies the orthogonality condition \( B^{-1} = B^T \) on \( U \).

The condition (16) follows from the equality

\[
(B(e_a \times e_b), Be_i \times Be_j)_Z = (Be_a \times Be_b, Be_i \times Be_j)_Z = \frac{1}{k} \delta_{a,i} \delta_{b,j} = (e_a \times e_b, e_i \times e_j)_Z.
\]

Analogously, there exists an isomorphism \( C \) between \( n_1 \) and \( n_2 \) acting as a signed basis permutation

\[
C: e_1 \to e_1, \quad e_2 \to e_2, \quad e_3 \to e_4, \text{ and } e_4 \to e_3.
\]

Correspondingly, \( C(\omega_1^{\Omega_1}) = \omega_1^{\Omega_2}, C(\omega_2^{\Omega_1}) = \omega_2^{\Omega_2}, \text{ and } C(\omega_3^{\Omega_1}) = \omega_3^{\Omega_2} \).

Thus, all solutions give isomorphic \( H \)-type algebras \( n \) with 3-dimensional center \( \mathfrak{z} \) and 4-dimensional carrier space \( \mathfrak{h} \).

Of course, all 2-dimensional centers can be realized by choosing \( \mathfrak{z} \) to be generated by any two vectors \( \omega \) from each of six solutions. In order to show isomorphisms between all 18 options, it is sufficient to establish isomorphism between all possible three pairs in one of the solutions, and then, to compose with already established isomorphisms \( B \) and \( C \). For example, for \( \Omega_1 \), we have

- \((\omega_1^{\Omega_1}, \omega_2^{\Omega_1}) \simeq (\omega_3^{\Omega_1}, \omega_2^{\Omega_1})\) by permutation \( e_1 \to e_1, \quad e_2 \to -e_4, \quad e_3 \to e_3, \text{ and } e_4 \to e_2 \);
- \((\omega_1^{\Omega_1}, \omega_2^{\Omega_1}) \simeq (\omega_1^{\Omega_1}, \omega_3^{\Omega_1})\) by permutation \( e_1 \to e_1, \quad e_2 \to e_2, \quad e_3 \to -e_4, \text{ and } e_4 \to e_3 \).

Then we construct the orthogonal decompositions \( N = U \oplus_\perp \Xi_j \oplus_\perp F_j, j = 1, 2, 3 \), where \( \Xi_1 = \text{span} (\omega_1^{\Omega_1}, \omega_2^{\Omega_1}) \), \( \Xi_2 = \text{span} (\omega_2^{\Omega_1}, \omega_3^{\Omega_1}) \), \( \Xi_3 = \text{span} (\omega_1^{\Omega_1}, \omega_3^{\Omega_1}) \) and \( F_j \) is the orthogonal complement in \( Z \) to \( \Xi_j \). The above signed basis permutations give isomorphisms between the \( H \)-type Lie algebras \( N/F_j \).

Observe that dimension 1 of \( \mathfrak{z} \) can be chosen always for any even dimension of \( \mathfrak{h} \) and all such algebras will generate one and the same Heisenberg algebra.

4.3.2. Existence of isomorphism. Let us turn to the general case \( k \geq 2 \), and let us construct \( B: N \to N \) satisfying the conditions at the beginning of Section 4.3. Let \( \{E_i\}_{i=1}^{r} \) and \( \{\tilde{E}_i\}_{i=1}^{r} \), \( r = \rho(2k) - 1 \) be two different HR families of orthogonal matrices, i.e., solutions \( E_i \) to the equations \( E_j^2 = -I \) and \( E_i E_j = -E_j E_i \) with the entries \( 0, \pm 1 \), and let \( B \) be a unitary signed basis permutation of \( U \). Our method works for any \( r \leq \rho(2k) - 1 \), but for simplicity we describe the case \( r = \rho(2k) - 1 \).

In order to find an isomorphism \( B \) between \( H \)-type algebras we must find a signed basis permutation \( B \) on \( U \) satisfying the system of \( \rho(2k) - 1 \) equations

\[
\tilde{E}_i = B E_i B^{-1}
\]
Therefore, the family of matrices

\[
\{ E_l \}_{l=1}^r \quad \text{and} \quad \{ \tilde{E}_l \}_{l=1}^r, \quad r = \rho(2k) - 1
\]

be two different HR families of orthogonal matrices with entries 0, ±1. Then, there exists a signed permutations \( B \) of the basis \( U \), such that \( \tilde{E}_l = BE_l B^{-1} \) for all \( l = 1, 2, \ldots, r \).

**Proof.** If \( B \) is a signed permutation of the standard basis of \( \mathbb{R}^{2k} \), then \( B^T = B^{-1} \) and \( |\det B| = 1 \). Let \( \tilde{E}_l = BE_l B^{-1} \). We check the following properties of an HR family of matrices.

- Obviously \( \tilde{E}_l \) has entries 0, ±1;
- \( \tilde{E}_l^{-1} = (BE_l B^{-1})^{-1} = BE_l^{-1} B^{-1} = BE_l^{-1} B = (BE_l B) T = (BE_l B T) T = \tilde{E}_l^T \);
- \( \tilde{E}_l^2 = (BE_l B^{-1})^2 = BE_l B^{-1} BE_l B^{-1} = BE_l B = -I \);
- \( \tilde{E}_l \tilde{E}_j = BE_l B^{-1} BE_j B^{-1} = BE_l E_j B^{-1} = -BE_l E_j B^{-1} = -\tilde{E}_j \tilde{E}_l \).

Therefore, the family of matrices \( \{ \tilde{E}_l \}_{l=1}^{\rho(2k) - 1} \) is an HR-family for a fixed \( B \).

Reciprocally, given two HR-families \( \{ E_l \}_{l=1}^{\rho(2k) - 1} \) and \( \{ \tilde{E}_l \}_{l=1}^{\rho(2k) - 1} \), we have to find a solution \( B \) to the overdetermined system of equations \( \tilde{E}_l = BE_l B^{-1} \) for all \( l = 1, 2, \ldots, r \).

Let us use the Kronecker matrix product identity

\[
\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B),
\]

where \( \otimes \) stands for the **Kronecker matrix product** and \( \text{vec}(V) \) for the **vectorization** of a matrix \( V \), a linear transformation which converts the matrix into a column vector. The Kronecker matrix product possesses several nice properties, some necessary of them we list below.

- Bilinearity and associativity:
  - \( A \otimes (B + C) = A \otimes B + A \otimes C \),
  - \( (B + C) \otimes A = B \otimes A + C \otimes A \),
  - \( A \otimes (B \otimes C) = (A \otimes B) \otimes C \);
- Transposition \( (A \otimes B)^T = A^T \otimes B^T \);
- For \( r \times r \) matrices the determinant is \( \det(A \otimes B) = (\det A)^r (\det B)^r \);
In order to compute the rank of $E$, which is a system of $(2^k)$ equations
(19)
$$\begin{align*}
\tilde{E}_l = BE_l B^{-1} & \iff \tilde{E}_l B = BE_l \iff \tilde{E}_l BI = IBE_l.
\end{align*}$$
So the equation $\tilde{E}_l B = BE_l$ is transformed to the equation
\begin{align*}
(E_l^T \otimes I - I \otimes \tilde{E}_l)\text{vec}(B) &= 0,
\end{align*}
which is a system of $(2k)^2$ equations for each $l$.

Let us analyze the matrix of our system
\[\mathcal{E} = \begin{pmatrix}
E_1^T \otimes I - I \otimes \tilde{E}_1 \\
E_2^T \otimes I - I \otimes \tilde{E}_2 \\
\vdots \\
E_r^T \otimes I - I \otimes \tilde{E}_r
\end{pmatrix}.
\]
In order to compute the rank of $\mathcal{E}$, we use the equality $\text{rank } \mathcal{E} = \text{rank } \mathcal{E}^T \mathcal{E}$, and calculate
\begin{align*}
\mathcal{F} := \mathcal{E}^T \mathcal{E} &= \begin{pmatrix}
E_1 \otimes I - I \otimes \tilde{E}_1 \\
E_2 \otimes I - I \otimes \tilde{E}_2 \\
\vdots \\
E_r \otimes I - I \otimes \tilde{E}_r
\end{pmatrix} \cdot \begin{pmatrix}
E_1^T \otimes I - I \otimes \tilde{E}_1 \\
E_2^T \otimes I - I \otimes \tilde{E}_2 \\
\vdots \\
E_r^T \otimes I - I \otimes \tilde{E}_r
\end{pmatrix} \\
&= (E_1 \otimes I)(E_1^T \otimes I) + (I \otimes \tilde{E}_1)(I \otimes \tilde{E}_1) - (E_1 \otimes I)(I \otimes \tilde{E}_1) - (I \otimes \tilde{E}_1)(E_1^T \otimes I) + \ldots \\
&+ (E_r \otimes I)(E_r^T \otimes I) + (I \otimes \tilde{E}_r)(I \otimes \tilde{E}_r) - (E_r \otimes I)(I \otimes \tilde{E}_r) - (I \otimes \tilde{E}_r)(E_r^T \otimes I) = \\
&\quad = 2 \sum_{j=1}^r (I - (E_j \otimes \tilde{E}_j)) = 2 \sum_{j=1}^r ((I \otimes I) - (E_j \otimes \tilde{E}_j)),
\end{align*}
where we distinguish the dimension of the matrix $I$ in $\mathcal{F}$ and in its blocks. We use that $(E_i \otimes \tilde{E}_i)(E_j \otimes \tilde{E}_j) = (E_i E_j) \otimes (\tilde{E}_i \tilde{E}_j) = (E_j E_i) \otimes (\tilde{E}_j \tilde{E}_i)$. Notice that if we denote $A_i = E_i \otimes \tilde{E}_i$, then $A_i^2 = I$, $A_i A_j = A_j A_i$, and they are elements of a commutative ring and satisfy Lemma 1. So we write
\[\mathcal{F}(I + A_1 + \cdots + A_r) \equiv 2(r I - A_1 - \cdots - A_r) \prod_{i=1}^r (I + A_j) = 0.
\]
The matrix $I + A_1 + \cdots + A_r$ is non-vanishing (it has all 1 in the principal diagonal), therefore, $\det \mathcal{F} = 0$, and the rank of the matrix $\mathcal{E}$ is smaller than $(2k)^2$. Thus, a solution $B$ exists. We also observe that the matrix $\mathcal{E}$ in the left hand side of equation (19) contains only two non-zero entries in each row and column and these entries equal $\pm 1$. Thus the system reduces to the form $b_{ij} \pm b_{ab} = 0$ and the orthogonality $B^T = B^{-1}$ follows from the form the equation. Therefore, choosing free variables $b_{ij}$ equal 0, $\pm 1$, we always can find a solution $B$ consisting of 0, $\pm 1$. \hfill \Box

We visualise this method in our example. Denote by $E_i^{\Omega_j}$ the matrix which corresponds to the vector $\omega_i^{\Omega_j}$. Then we choose, for instance, $\tilde{E}_1 = E_1^{\Omega_3} = B E_1^{\Omega_1} B^{-1}$ and write equation (19).
The matrix $B$ which is solution of (20) is the following

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
b_{11} \\
b_{12} \\
b_{13} \\
b_{14}
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix} = 0.
\]

Check $E_{2}^{\Omega_{3}} = BE_{2}^{\Omega_{3}}B^{-1}$

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
and $E_3^{\Omega_3} = BE_3^{\Omega_1}B^{-1}$

$$
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
$$

We see that the matrix

$$
B_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

gives also a solution to the equation $E_1^{\Omega_2} = B_1E_1^{\Omega_1}B_1^{-1}$ which is the same as (20) because $\omega_1^{\Omega_2} = \omega_1^{\Omega_3}$:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= 0
$$

Check $E_3^{\Omega_2} = B_1E_2^{\Omega_1}B_1^{-1}$

$$
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
$$
We see that different basis permutation matrices $B$ and $B_1$ which are the solutions to (20) give isomorphisms of $H$-type algebras.

4.4. Non-degenerate metric. In this section we study the case when the metric $(\cdot, \cdot)_U$ is neutral, $(\cdot, \cdot)_Z$ is non-positive, and both metrics are non-degenerate, see Proposition 1.

4.4.1. Construction of pseudo $H$-type algebras. We need to prove an analogue of Theorem 4 and we need auxiliary technical lemmas. Let us start reminding that for the inner (positive definite) product, if $\omega_l = \cdots + \alpha^\ell_{ij}(e_i \times e_j) + \cdots$ for some $p_l$, where $i < j$, then $J_{\omega_l}e_i = \alpha^\ell_{ij}e_j$, where $\alpha^\ell_{ij}$ is $(+1)$ or $(-1)$. In the case of a non-degenerate scalar product the situation is more complicated, and we describe it in the following proposition. For definiteness, we assume that $(e_i, e_i)_U = 1/k$ for $i = 1, 2, \ldots, k$ and $(e_i, e_i)_U = -1/k$ for $i = k + 1, k + 2, \ldots, 2k$.

**Proposition 7.** Let $\omega_l = \sum_{(i,j) \in p_l} \alpha^\ell_{ij}(e_i \times e_j)$, $p_l \in p'$, $\alpha^\ell_{ij} = \pm 1$. Denote by $\beta := k\alpha^\ell_{ij}(e_i \times e_j, e_i \times e_j)_Z$ for an arbitrary chosen pair $(i, j) \in p_l$. Then

- If $1 \leq i < j \leq k$ and $i < j < k$, then $J_{\omega_l}e_i = \beta e_j$, and $J_{\omega_l}e_j = -\beta e_i$.
- If $1 \leq i < k$ and $k + 1 \leq j \leq 2k$, then $J_{\omega_l}e_i = -\beta e_j$, and $J_{\omega_l}e_j = \beta e_i$.
- If $1 \leq i < k$ and $1 \leq j \leq 2k$, then $J_{\omega_l}e_i = -\beta e_j$, and $J_{\omega_l}e_j = -\beta e_i$.

**Proof.** Recall that $J_{\omega_l}(e_i) = \pm e_j$, $(i, j) \in p_l$. Let $1 \leq i < j \leq k$. Then using the definition of the operator $J_{\omega_l}$ we obtain

$$J_{\omega_l}e_i, e_j)_U = (\omega_l, e_i \times e_j)_Z = \alpha^\ell_{ij}(e_i \times e_j, e_i \times e_j)_Z = \beta \frac{1}{k} = \beta(e_j, e_j)_U.\tag{21}$$

The integer $i$ appears in pairs of $p_l$ only in the pair $(i, j)$, therefore the equality $(J_{\omega_l}e_i, e_m)_U = \beta(e_j, e_m)_U = 0$ trivially holds for all $m = 1, \ldots, 2k$, $m \neq j$, hence $J_{\omega_l}e_i = \beta e_j$ as in the formulation of Proposition 7. At the same time, $(e_j, e_j)_U = (e_i, e_i)_U$, and by skew symmetry of $J_{\omega_l}$ we have $(J_{\omega_l}e_i, e_j)_U = -(J_{\omega_l}e_j, e_i)_U$. So $(J_{\omega_l}e_j, e_i)_U = -\beta(e_i, e_i)_U$, from (21), which implies $J_{\omega_l}e_j = -\beta e_i$. Analogously we argue for $k + 1 \leq i, j \leq 2k$.

For $1 \leq i \leq k$ and $k + 1 \leq j \leq 2k$, arguing as in (21), we obtain $(J_{\omega_l}e_i, e_j)_U = -\beta(e_j, e_j)_U$. Since $(e_j, e_j)_U = -(e_i, e_i)_U$, and $(J_{\omega_l}e_j, e_i)_U = -\beta(e_i, e_i)_U$, by skew symmetry of $J_{\omega_l}$, we proof the third statement of Proposition 7.

Both, $\beta$ and $\alpha^\ell_{ij}$, admit values in $\{-1, 1\}$, observe that $|(e_i \times e_j, e_i \times e_j)_Z| = \frac{1}{k}$, and the choice of the sign of $\beta$ depends both on $\alpha^\ell_{ij}$ and on the signature of $(\cdot, \cdot)_Z$. 

Assume that the family of 1-factors $p' \subseteq p'$ satisfies the property that any two factors set together form the union of disjoint cycles.

**Proposition 8.** If $p_\ell \in p'$ defines $\omega_\ell$ satisfying the condition $J_{\omega_\ell}^2 = -(\omega_\ell, \omega_\ell)_{\mathbb{Z}}id_U$, then we have two options. The pairs $(i, j) \in p_\ell$ are of the following types:

- either $1 \leq i, j \leq k$ or $k + 1 \leq i, j \leq 2k$ for all $(i, j) \in p_\ell$;
- $1 \leq i \leq k$ and $k + 1 \leq j \leq 2k$ for all $(i, j) \in p_\ell$.

**Proof.** Indeed, assume that for $p_\ell$, there exist two pairs $(i_1, j_1) \in p_\ell$ and $(i_2, j_2) \in p_\ell$, such that $1 \leq i_1, j_1 \leq k$ but $1 \leq i_2 \leq k$ and $k + 1 \leq j_2 \leq 2k$. Specify indices for $\beta$ from Proposition 7 as $\beta_{ij} := k\alpha_{ij}(e_i \times e_j, e_i \times e_j)_{\mathbb{Z}}$. Then

$$J_{\omega_\ell}e_{i_1} = \beta_{i_1j_1}e_{j_1}, \quad J_{\omega_\ell}e_{i_2} = -\beta_{i_2j_2}^2e_{i_1} = -e_{i_1},$$

by Proposition 7 and necessarily $(\omega_\ell, \omega_\ell)_3 = 1$. But

$$J_{\omega_\ell}e_{i_2} = -\beta_{i_2j_2}e_{j_2}, \quad J_{\omega_\ell}e_{i_2} = \beta_{i_2j_2}^2e_{i_2} = e_{i_1},$$

and $(\omega_\ell, \omega_\ell)_3 = -1$ at the same time. This contradiction brings us to the statement. The same proof holds for $k + 1 \leq i_1, j_1 \leq 2k$. 

This, for example, means that the partition when $p_\ell = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$ is allowed but $p_\ell = \{(1, 2), (3, 7), (4, 8), (5, 6)\}$ for $k = 4$ is forbidden. Proposition 8 yields that the operators $J_{\omega_\ell}$ have a matrix representation in the block form

$$E_\ell = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the matrices $A, D$ are skew-symmetric, and $B = C^T$. For example, in the case $k = 2$, and the signature $(1, 2)$ we have

$$E_1 = \begin{pmatrix} 0 & \alpha_{12} & 0 & 0 \\ -\alpha_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{34} \\ 0 & 0 & -\alpha_{34} & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & \alpha_{13} & 0 \\ 0 & 0 & 0 & \alpha_{24} \\ \alpha_{13} & 0 & 0 & 0 \\ 0 & \alpha_{24} & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & \alpha_{14} \\ 0 & 0 & \alpha_{23} & 0 \\ 0 & \alpha_{23} & 0 & 0 \\ \alpha_{14} & 0 & 0 & 0 \end{pmatrix}.$$ 

Proposition 8 has a clear meaning in the matrix form, i.e., all non-zero elements $\alpha_{ij}$, $i, j \in p_\ell$ are either in $A$ and $D$ or in $B$ and $C$ blocks of the matrix $E_\ell$.

4.4.2. **Construction of partitions.** An analogue of Theorem 8 for generalized HR family of type $(s, t)$ was proved by Wolfe [56] in 1976. Let us define a family of real matrices $HR(s, t)$ by the following conditions

- $E_0 = I$, $E_k^2 = -I$ for $k = 1, \ldots, s$; $E_k^2 = I$ for $k = s + 1, \ldots, s + t$;
- $E_k = -E_k^2$ for $k = 1, \ldots, s$; $E_k = E_k^2$ for $k = s + 1, \ldots, s + t$;
- $E_k E_j + E_j E_k = 0$ for $k \neq j$ and $k, j = 1, \ldots, s + t$. 


**Definition 5.** Let $n$, $s$ be positive integers and $t \geq 0$. We define
\[
\rho_t(n) = \max \{ s : \text{Cl}_{s-1,t} \text{ has an irreducible } (n \times n) \text{ matrix representation over } \mathbb{R} \},
\]
\[
\sigma_s(n) = \max \{ t : \text{Cl}_{s,t-1} \text{ has an irreducible } (n \times n) \text{ matrix representation over } \mathbb{R} \}.
\]

Wolfe [56] completely characterised $\rho_t(n)$ and $\sigma_s(n)$. In particular,
- $\rho_t(2) = \rho_2(2) = 2$, $\rho_5(8) = 1$;
- $\rho_t(2n) = \rho_{t-1}(n) + 1$;
- $\rho_t(n) = \rho_{t+8}(16n)$;
- $\sigma_1(2) = 3$, $\sigma_3(4) = \sigma_5(8) = \sigma_6(8) = \sigma_7(8) = 1$;
- $\sigma_s(2n) = \sigma_{s-1}(n) + 1$;
- $\sigma_s(n) = \sigma_{s+8}(16n)$.

Both $\rho_t(n)$ and $\sigma_s(n)$ are completely characterised by $\rho(n)$ and the above relations as $\rho_0(n) = \rho(n)$ and $\sigma_0(2n) = \rho(n) + 2$. Define the next function $\tau(n) = \max \{ \rho_t(n) + t : t \geq 0 \}$. If $n = 2^r$, then $\tau(n) = 2(r + 1)$. Given positive integer $n$, if there exists an HR$(s,t)$ family in order $n$, then [56],
- $s + t \leq \tau(n) - 1$;
- $s \leq \rho_t(n) - 1$ and $t \leq \sigma_s(n) - 1$.

Moreover, the limits are achievable and the matrices can be assumed with integer entries. The dimension of a Clifford algebra is a power of 2 and $\rho_t(u2^r) = \rho_t(2^r)$ for odd $u$, hence it is sufficient to reduce the dimension of the representation space to the closest power of 2. In order to come up with an analogue of Theorem 4 we need the following result proved by Ciatti in [14].

**Theorem 6 ([14]).** The following triplets $(s,t,r)$ guarantee that the Clifford algebra $\text{Cl}_{s,t} \text{ has irreducible and admissible representation over the representation space of minimal dimension } 2^r$.
- $s - t \equiv 0, 6(\text{mod } 8)$ for $r \equiv 0, 6(\text{mod } 4)$;
- $s - t \equiv 2, 4(\text{mod } 8)$ for all values of $r$;
- $s - t \equiv 1, 5(\text{mod } 8)$ for $r \equiv 0, 2, 3(\text{mod } 4)$;
- $s - t \equiv 3(\text{mod } 8)$ for $r \equiv 0, 2(\text{mod } 4)$;
- $s - t \equiv 7(\text{mod } 8)$ for $r \equiv 3(\text{mod } 4)$.

All irreducible representations of Clifford algebras $\text{Cl}_{s,0}$ are admissible with an inner product, and the representation space is of dimension $2^r$, where $s = \rho(2^r) - 1$. In the case $t > 0$, if $(s, t)$ does not satisfy the conditions of Theorem 6 then we need to double the representation space and the corresponding representation map has to be redefined in a proper way, see [14].

Modifying reasonings given in the previous section and applying the Wolfe’s existence theorem [56, Theorem 2.8] we can formulate the following theorem.

**Theorem 7.** Let $N = U \oplus_{\perp} Z$ be a free nilpotent Lie algebra with a scalar product $\langle \cdot, \cdot \rangle_N = \langle \cdot, \cdot \rangle_U + \langle \cdot, \cdot \rangle_Z$ which is neutral on $U$ and of an arbitrarily fixed signature $(s,t)$, $t \geq 1$, on $Z$, with $\dim(U) = 2^r$, $r \geq 1$. There exists an ideal $A$ such that $N/A$ is an $H$-type algebra $n = h \oplus_{\perp} z$, where $h = U$ and the center is $z = Z/A$. Given an
integer $t \geq 1$, the ideal $A$ can be chosen such that $\dim(j) = s + t$ and the signature $(s, t)$ are such that $s$ takes every value from

- $0 \leq s \leq \rho_t(2^r) - 1$ in the case when the triplet $(\rho_t(2^r) - 1, t, r)$ satisfies one of the conditions of Theorem [6].
- Otherwise, $0 \leq s \leq \rho_t(2^r) - m$, where $1 < m \leq \rho_t(2^r)$ is the maximal positive integer for which either the triplet $(\rho_t(2^r) - m, t, r)$ satisfies the conditions of Theorem [6] or $\rho_t(2^r) - m = \rho_t(2^{r-1}) - 1$.

**Proof.** Similarly to the proof of Theorem [4] we consider a family of partitions

$$p' = \{p'_1, \ldots, p'_{2k-1}\}.$$ 

It is equivalent to a 1-factorization of the complete graph $K_{2k}$. We recall the isomorphism between the operators $J_{\omega_l}$, $l \in 1, \ldots, 2k - 1$, and the orthogonal matrices $E_l$ constructed in the proof of Theorem [4]. Next we choose a family of partitions $p'' = \{p''_1, \ldots, p''_{s+t}\} \subset p'$ such that the set of matrices

$$\{E_{l_1}, \ldots, E_{l_s}, E_{l_{s+1}}, \ldots, E_{l_{s+t}}\}$$

representing a family with the maximal value of $s$ forms an HR($s, t$) family.

Now we set $h = U$ and $A$ to be the orthogonal complement in $Z$ to

$$\text{span}\{\omega_1, \ldots, \omega_s, \omega_{s+1}, \ldots, \omega_{l_{s+1}}\}.$$ 

Due to the correspondence between the operators $J_{\omega_m}$, $m = 1, \ldots, s + t$ and the matrices from the HR($s, t$) family, the operators $J_{\omega_m}$ satisfy the conditions of Proposition [2] and therefore, the orthogonality condition [2]. Thus, we define $j = Z/A$, and $h \oplus_\perp j$ is a pseudo $H$-type algebra $n$. □

We illustrate Theorem 7 in Figure 1 for all possible triplets $(s, t = t_{\text{fix}}, r)$ satisfying the conditions of Theorem 6 which we denote by $(\ast)$ in the diagrams. The first line corresponds to the situation when the triplet $(\rho_t(2^r) - 1, t, r)$ satisfies one of the conditions of Theorem 6 and then, the maximal value of $s$ is chosen. Otherwise, we analize one step down $s = \rho_t(2^r) - 2$. Then the representation space can be either of dimension $2^{r-1}$ (line 2 and 3) or of dimension $2^r$ (line 3 and 4). In the first case (line 2), we may have that the triplet $(\rho_t(2^r) - 2, t, r - 1)$ satisfies the conditions of Theorem 6 and we choose the maximal value of $s$ to be $\rho_t(2^r) - 2$. Otherwise (line 3), the triplet $(\rho_t(2^r) - 2, t, r - 1)$ does not satisfy the conditions of Theorem 6 however $\rho_t(2^r) - 2 = \rho_t(2^{r-1}) - 1$ and we choose the maximal value of $s$ to be $\rho_t(2^r) - 2$. If the representation space for $s = \rho_t(2^r) - 2$ is of dimension $2^r$, then we may have the triplet $(\rho_t(2^r) - 2, t, r)$ satisfying the conditions of Theorem 6 and then we choose the maximal value of $s$ to be $\rho_t(2^r) - 2$. Otherwise, we continue to step down $s \rightarrow s - 1$ and repeat the previous procedure over again. Positivity of $s$ guarantees finiteness of this algorithm. Normally, it takes at most 3 steps to finish the choice of the maximal value of $s$.

Analogously we formulate a theorem of existence of $H$-type algebras for given value of $s$.

**Theorem 8.** Let $N = U \oplus Z$ be a free nilpotent Lie algebra with a scalar product $(\cdot, \cdot)_N = (\cdot, \cdot)_U + (\cdot, \cdot)_Z$ which is neutral on $U$ and of arbitrarily fixed signature $(s, t)$,
\[
s = \rho_t(2^r) - 2 \quad s = \rho_t(2^r) - 1
\]

\[
t_{\text{fix}} \quad \cdots \quad *2^r
\]

\[
s = \rho_t(2^r) - 2 \quad s = \rho_t(2^r) - 1
\]

\[
t_{\text{fix}} \quad *2^{r-1} \quad 2^r
\]

\[
s = \rho_t(2^r) - 2 \quad s = \rho_t(2^r) - 1
\]

\[
t_{\text{fix}} \quad 2^{r-1} \quad 2^r
\]

\[
s = \rho_t(2^r) - 2 \quad s = \rho_t(2^r) - 1
\]

\[
t_{\text{fix}} \quad *2^r \quad 2^r
\]

\[
s = \rho_t(2^r) - 2 \quad s = \rho_t(2^r) - 1
\]

\[
t_{\text{fix}} \quad 2^r \quad 2^r
\]

\[\, \, \text{continue choosing } m\]

Figure 1. The diagram illustrates the algorithm of the choice of maximal s in Theorem 7. Filled nodes mean the chosen position of s.

\[\, \, \, t \geq 1, \, \, \text{on } Z, \, \, \text{with } \dim(U) = 2^r, \, r \geq 1. \, \, \text{There exists an ideal } A \, \, \text{such that } N/A \, \, \text{is an } H\text{-type algebra } n = \mathfrak{h} \oplus \mathfrak{z}, \, \, \text{where } \mathfrak{h} = U \, \, \text{and the center is } \mathfrak{z} = Z/A. \, \, \text{Given an integer } s \geq 0, \, \, \text{the ideal } A \, \, \text{can be chosen such that } \dim(\mathfrak{z}) = s + t \, \, \text{and the signature } (s, t) \, \, \text{are such that } t \, \, \text{takes every value from}
\]

- \[\, \, \, 0 \leq t \leq \sigma_s(2^r) - 1 \, \, \text{in the case the triplet } (s, \sigma_s(2^r) - 1, r) \, \, \text{satisfies one of the conditions of Theorem 6}.
\]

- \[\, \, \, \text{Otherwise, } 0 \leq t \leq \sigma_s(2^r) - m, \, \, \text{where } 1 < m \leq \sigma_s(2^r) \, \, \text{is the maximal positive integer for which either the triplet } (s, \sigma_s(2^r) - m, r) \, \, \text{satisfies the conditions of Theorem 6} \, \, \text{or } \sigma_s(2^r) - m = \sigma_s(2^{r-1}) - 1.
\]

Below we collect in Table 1 the first maximal values of the parameter s with a given dimension of \(\mathfrak{h}\) and a given value of \(t\); and in Table 2 the first maximal values of the parameter t with a given dimension of \(\mathfrak{h}\) and a given value of s such that the \(H\)-type algebra \(n = \mathfrak{h} \oplus \mathfrak{z}\) exists.

If an \(H\)-type algebra corresponds to signature \((s^*, t^*)\), then there exist all \(H\)-type algebras with the signatures \((0, t^*)\), \ldots, \((s^* - 1, t^*)\); \((s^*, 0)\), \ldots, \((s^*, t^* - 1)\). The space \(\mathfrak{h}\) of these \(H\)-type algebras can be the minimal dimensional admissible representation of the corresponding Clifford algebra, but can be of a bigger dimension. It follows from Tables 1 and 2 that, for example, given the dimension 4 of \(\mathfrak{h}\) the following
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\[
\begin{array}{c|ccccc}
2^r & 0 & 1 & 2 & 3 & 4 \\
\hline
2 & 1 & 0 & - & - & - \\
4 & 3 & 1 & 1 & - & - \\
8 & 7 & 3 & 3 & 3 & 3 \\
16 & 8 & 7 & 5 & 4 & 4 \\
32 & 9 & 8 & 7 & 5 & 5 \\
\end{array}
\]

Table 1. Maximal values of $s$

\[
\begin{array}{c|ccccc}
2^r & 0 & 1 & 2 & 3 & 4 \\
\hline
2 & 1 & 0 & - & - & - \\
4 & 2 & 2 & 0 & 0 & - \\
8 & 4 & 4 & 4 & 4 & 0 \\
16 & 8 & 6 & 5 & 5 & 4 \\
32 & 9 & 8 & 6 & 6 & 5 \\
\end{array}
\]

Table 2. Maximal values of $t$

$H$-type algebras exist with the signatures of $\mathfrak{z}$ equal to

\[(0, 2), (1, 2), (2, 0), (3, 0), (1, 1), (1, 2),\]

as well as with the signatures $(0, 1)$ and $(1, 0)$, but do not exist with the signatures $(2, 1)$ and $(0, 3)$.

The metric $(\cdot, \cdot)_Z$ on $N$, see Section 3.1, is now normalized so that $(e_i \times e_j, e_i \times e_j) = \varepsilon_{ij}(s, t)/k$, where

\[
\varepsilon_{ij}(s, t) = \begin{cases} 
1, & \text{for } (i, j) \in \mathfrak{p}''_t, \ell = 1, \ldots, s; \\
-1, & \text{for } (i, j) \in \mathfrak{p}''_t, \ell = s + 1, \ldots, s + t. 
\end{cases}
\]

Remark 4. For the space $\mathfrak{h} = U$ of dimension $u2^r$, where $u$ is odd, Theorems 7 and 8 are applied as for the dimension $2^r$.

Analogously to Theorem 5 we prove the following statement.

Theorem 9. The pseudo $H$-type algebras admit rational structure constants.

4.4.3. Existence of solution and isomorphisms. Let us analyse the equation $E_1E_2 + E_2E_1 = 0$. If the matrix $E_1E_2 + E_2E_1$ has a non-zero element at a place $(i, j)$, then it is of the form $\alpha_{ik_1}\alpha_{k_1j} + \alpha_{ik_2}\alpha_{k_2j}$, which is the result of the product and sum of the elements at the places $(i, k_1), (k_1, i), (k_2, j)$, and $(j, k_2)$ in $E_1$, and $(i, k_2)$, $(k_2, i)$, $(k_1, j)$, and $(j, k_1)$ in $E_2$. Without loss of generality we assume that $i < k_1 < k_2 < j$. Otherwise, we consider the upper triangles in the matrices $E_1$ and $E_2$. The difference between symmetric and skew-symmetric matrices is the change of sign at
the elements $\alpha_{ik_1}$ and $\alpha_{k_2j}$ or at the elements $\alpha_{ik_2}$ and $\alpha_{k_1j}$ which does not influence the equation $\alpha_{ik_1}\alpha_{k_1j} + \alpha_{ik_2}\alpha_{k_2j} = 0$. Therefore, the existence of solutions for $\alpha_k$ is the same as for the positive definite metric. Indeed, we split this problem in two. Given a set $p''_\ell$ of pairs $(i, j)$, and therefore, a skew-symmetric matrix $E_\ell = \{\alpha_{ij}^{(\ell)}\}$ with $\alpha_{ij}^{(\ell)} = \pm 1$ for all $(i, j) \in p''_\ell$, $\alpha_{ij}^{(\ell)} = 0$ for all $(i, j) \not\in p''_\ell$, and $\alpha_{ji}^{(\ell)} := -\alpha_{ij}^{(\ell)}$, let us introduce the diagonal matrix $B_\ell$ having 1 at $(i, i)$ place and -1 at $(j, j)$ place, $(i, j) \in p''_\ell$, $i < j$, and zeros otherwise. Then $B_\ell^2 = I$, and $E_\ell B_\ell$ is a symmetric matrix which is obtained from $E_\ell$ by multiplication by -1 of the lower triangle part. Now

- We are looking for the orthogonal matrices $E_\ell$, which are the solutions to the equations $E_\ell^2 = -I$ and $E_\ell E_i = -E_j E_\ell$ for $(i, j) \in p''_\ell$, $\ell = 1, \ldots, s + t$ as for the positive definite metric;
- Change $E_\ell \rightarrow \tilde{E}_\ell := E_\ell B_\ell$ for $\ell = s + 1, \ldots, s + t$.

Then $E_i E_j = -E_j E_i$, $E_i \tilde{E}_j = -\tilde{E}_j E_i$, $\tilde{E}_i \tilde{E}_j = -\tilde{E}_j \tilde{E}_i$, $E_j^2 = -I$, and $\tilde{E}_j^2 = I$. This gives the existence of solutions for $\alpha_{ij}$ and the existence of isomorphism of the resulting pseudo $H$-type algebras for different solutions.

## 5. Applications to combinatorial and orthogonal designs

### 5.1. Square 1-factorization of a complete graph.**

Given a set of partitions $p$ of $2k$ numbers into pairs we want to check all possible subset of partitions $p' \subseteq p$ in which we look for $p'' \subseteq p' \subseteq p$ of maximal possible length satisfying the following condition: any two partitions $p_{m_\alpha} \in p''$ and $p_{m_\beta} \in p''$ set together give disjoint cycles of length 4. This can be reformulated in terms of graphs as in Section 4.1. Given a complete graph with $2k$ vertices at the numbers $\ldots, 2k$ we look at the maximal number of 1-factors from a 1-factorization defined by $p'$ satisfying the property: any two 1-factors set together give disjoint cycles of length 4. In particular, the Steiner tournament for a complete graph $K_8$ of 8 vertices guarantees the existence of maximal $p'' = p'$ of length $p = 7$.

The general case of higher dimensions represents a hard problem of combinatorial design related to so-called 1-factorization conjecture, i.e., the problem of counting the number of non-isomorphic 1-factorizations of a complete graph $K_{2n}$. For small $n$ the answer is known only for $n \leq 7$. The unique factorization exists for $n = 1, 2, 3$. In the case $n = 4$, there are 6 non-isomorphic factorizations, two of which, Kirkman and Steiner were mentioned before. Other non-trivial non-isomorphic factorizations are known for $n = 5, 6, 7$, see [38].

The existence of $p''$ of maximal length is achieved for $K_{2k}$ if $k = 2^n$. This problem is known as the square or 4-semiregular 1-factorization of a complete graph and the complete answer to this question is given in the following theorem.

**Lemma.** [41] Kobayashi and Nakamura** There exists a 1-factorization of a complete graph $K_{2k}$ such that any two 1-factors set together form the union of disjoint cycles of length 4, if and only if, $k = 2^n$, $n \geq 1$.

As an application to combinatorial design, let us formulate a proposition.
Proposition 9. There exists a 1-factorization of a complete graph $K_{2k}$ such that at most $\rho(2k) - 1$ one-factors satisfy the condition: being set together pairwise they form the union of disjoint cycles of length 4.

Remark. In general, $\rho(2^n) \leq 2^n$ with the equality only for $n = 0, 1, 2,$ and $3$.

Proof. In order to prove Proposition 9 let us start with a statement on interplay between the properties of the operator $J$ defined by (1) and the cycle properties of the partitions $p'$.

Let the vectors $\omega_m = \sum_{(i,j) \in p_m} (e_i \times e_j)$, $p_m \in p'$ for $m = m_1, \ldots, m_p$, for some $1 < p \leq 2k - 1$, and let $J_{\omega_\alpha} J_{\omega_\beta} = -J_{\omega_\beta} J_{\omega_\alpha}$ for every $\alpha, \beta \in \{1, \ldots, p\}$ and $\alpha \neq \beta$. Then any two 1-factors $p_{\omega_\alpha}$ and $p_{\omega_\beta}$ set together give disjoint cycles of length 4.

Indeed, by the condition on $J_{\omega_\alpha}$ and $J_{\omega_\beta}$, and from the fact that both operators act as signed permutations of the basis of $U$, we have

$e_k \xrightarrow{J_{\omega_\alpha}} \beta_1 e_s \xrightarrow{J_{\omega_\beta}} \pm e_\ell$

$e_k \xrightarrow{J_{\omega_\beta}} \beta_2 e_r \xrightarrow{J_{\omega_\alpha}} \mp e_\ell$

where $\beta_1, \beta_2$ are +1 or -1. Thus the vertices $k, \ell, s, r$ together with edges $ks, kr, sl, rl$ define a separate cycle in the graph $p_{\omega_\alpha} \cup p_{\omega_\beta}$. The statement of Proposition 9 follows from Theorem 4 with $J_{\omega_m}$, where $\omega_1, \ldots, \omega_p$ is an orthonormal basis of the center $z$. □

5.2. Space-time block codes, orthogonal designs, and wireless communication. Surprisingly, the above results have applications in wireless networks with multiple transmit antennas, in which encoded signals undergo fading under local scattering, intersymbol interference under multipath propagation, the channels are time varying due to mobile motions, and co-channel interference due to cellular spectrum reuse. However, we use pocket communicators which must remain relatively simple commercially accessible and reliable in different type of environments. The idea boiled down to space-time block coding for communication over Rayleigh fading channels. Data is encoded using space-time block coding and then splits into $n$ streams, which are then simultaneously transmitted via $n$ transmit antennas. The first two-branch transmit diversity scheme was proposed by Alamouti [2] in 1998, and then, this scheme was generalized to $n$-branch scheme by Tarokh, Jafarkhani, and Calderbank [55]. The proposed construction was inductive, i.e., the bigger size block matrices were constructed using block matrices of smaller size. The direct method was proposed by Morier-Genoud and Ovsienko [46] in 2013.

Transmit diversity is provided by the simplest linear processing orthogonal designs preserving the following properties

- No loss in bandwidth, i.e., maximum possible transmission rate at full diversity;
- Maximum likelihood decoding algorithm at the receiver.

A model of transmitting process with space-time block coding can be formulated as follows. We want to transmit $n$ signals from $n$ transmitting antennas during the
time instances $t_1, \ldots, t_T$ achieving maximal likelihood. Without loss of generality we assume that the receiving antenna is only one. Otherwise, the maximum ratio combining is applied, see e.g., [33]. Let us assume that the path gains from transmit antennas to the receive antenna are $c = (c_1, \ldots, c_T)$, respectively. Then, the decoder receives signals $r^j = (r_{1j}, \ldots, r_{nj})^T$ at each time $t_j$, $j = 1, \ldots, T$ as

$$r^j = cW_j + \eta^j,$$

where $\eta^j$ is the Gaussian noise sample vector of the receive antenna and $W_j$ is a space time codeword, i.e., a $n \times T$ matrix with the entries $x_1, \ldots, x_n$. Assume at this moment that $T = n$. In order to satisfy the above property of full diversity and maximal likelihood, the matrices $W_j$ must form a real orthogonal design, see e.g., [33, Section 4.4].

**Definition 6.** A linear processing real orthogonal design in order $n$ of type $u_1, \ldots, u_n$ on commuting variables $x_1, \ldots, x_n$ is an $n \times n$ matrix $W$ with the entries \{0, $\pm x_1, \ldots, \pm x_n$\}, such that

$$WW^T = \left( \sum_{k=1}^{n} u_kx_k^2 \right) I,$$

where $I$ is the $n \times n$ unit matrix.

A linear processing real orthogonal design in order $n$ exists if and only if there exists linear processing orthogonal design $\mathcal{L}$ in order $n$, such that

$$\mathcal{L}\mathcal{L}^T = \mathcal{L}^T\mathcal{L} = \left( \sum_{k=1}^{n} x_k^2 \right) I,$$

see [55, Theorem 3.4.1]. Let

$$\mathcal{L} = \sum_{k=1}^{n} x_kX_k.$$

Then the matrices $X_k$ are such that

- all entries are \{0, $\pm 1$\};
- the Hadamard (entrywise) product $X_k \circ X_j = 0$ whenever $j \neq k$;
- $X_kX_k^T = I$;
- $X_kX_j^T + X_jX_k^T = 0$.

Denote $E_k = X_k^T X_k$, $k = 1, \ldots, n$. Then, $E_1 = I$, and $E_2, \ldots, E_n$ form the HR family. The Hurwitz-Radon-Eckmann Theorem [3] implies that the real orthogonal design exists if and only if $n = 2^r$ and $r = 1, 2, 3$.

However, we may ask a question: how many non-zero entries $x_1, \ldots, x_m$ one can transmit by making use of a linear processing real orthogonal design in order $n$?

**Proposition 10.** A linear processing real orthogonal design in order $n = 2k$ exists for at most $m = \rho(n)$ non-zero entries.
Proof. Indeed, assume that $e_1, \ldots, e_{2k}$ is the basis of a vector space $U$ and $x_1, \ldots, x_m$ are non-zero entries of the matrix $W$. Then the matrices $X_1, \ldots, X_m$ of dimension $(2k \times 2k)$ defined above act on $U$ as well as the matrices $E_2, \ldots, E_m$ that form the HR family and represent the basis of the center of an $H$-type algebra. Theorem 4 implies that $m \leq \rho(n)$ and the upper bound is achieved. □

Definition 7. Two orthogonal designs $X$ and $Y$ of the same order are called amicable if $XY^T = YX^T$.

Suppose that $X = \sum_{k=1}^s x_k X_k$ and $Y = \sum_{k=1}^t y_k Y_k$ are two amicable orthogonal designs of order $n$ over variables $x_1, \ldots, x_s$ and $y_1, \ldots, y_t$ of type $u_1, \ldots, u_s$ and $v_1, \ldots, v_t$ respectively. So in addition to the matrices $X_k, k = 1, \ldots, s$ we operate with matrices $Y_j, j = 1, \ldots, t$, such that

- all entries are $\{0, \pm 1\}$;
- the Hadamard product $Y_k \circ Y_j = 0$ whenever $j \neq k$;
- $Y_k Y_k^T = -I_n$;
- $Y_k Y_j^T + Y_j Y_k^T = 0$;
- $X_k Y_j^T = Y_j X_k^T$.

Denote $E_k = X_k^T X_k, k = 1, \ldots, s$ and $E_k = Y_k^T Y_k, k = s + 2, \ldots, s + t$. Then, the matrices $E_2, \ldots, E_s, E_{s+2}, \ldots, E_{s+t}$ form the HR$(s-1, t-1)$ family. An analogue of Proposition 10 for amicable orthogonal designs is the following statement.

Proposition 11. Given non-zero variables $y_1, \ldots, y_t$ of an amicable orthogonal design of order $2n$, Theorem 7 gives the maximal number of variables $x_1, \ldots, x_s$. Analogously, given non-zero variables $x_1, \ldots, x_s$ of an amicable orthogonal design of order $2n$, Theorem 8 gives the maximal number of variables $y_1, \ldots, y_t$.

The proof is analogues to the proof of Proposition 10 using Theorem 7 and 8.

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