\textbf{N-wave interactions related to simple Lie algebras.}  
\textit{\textit{Z}}_2- \textit{r}eductions and soliton solutions

V. S. Gerdjikov\textsuperscript{†}, G. G. Grahovski\textsuperscript{†}, R. I. Ivanov\textsuperscript{‡}\textsuperscript{§} and N. A. Kostov\textsuperscript{#}

\textsuperscript{†} Institute for Nuclear Research and Nuclear Energy,  
Bulgarian Academy of Sciences, 72 Tsarigradsko chaussee, 1784 Sofia, Bulgaria  
\textsuperscript{‡} Department of Mathematical Physics,  
National University of Ireland-Galway, Galway, Ireland  
\textsuperscript{#} Institute of Electronics, Bulgarian Academy of Sciences,  
72 Tsarigradsko chaussee, 1784 Sofia, Bulgaria

\textbf{Abstract.}  
The reductions of the integrable N-wave type equations solvable by the inverse scattering method with the generalized Zakharov-Shabat systems $L$ and related to some simple Lie algebra $\mathfrak{g}$ are analyzed. The Zakharov-Shabat dressing method is extended to the case when $\mathfrak{g}$ is an orthogonal algebra. Several types of one soliton solutions of the corresponding N-wave equations and their reductions are studied. We show that to each soliton solution one can relate a (semi-)simple subalgebra of $\mathfrak{g}$. We illustrate our results by 4-wave equations related to $\mathfrak{so}(5)$ which find applications in Stokes-anti-Stokes wave generation.

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1. Introduction

One of the important nonlinear models with numerous applications to physics that appeared at the early stages of development of the inverse scattering method (ISM), see [references], is the 3-wave resonant interaction model described by the equations:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial x} &= i \varepsilon \bar{u}_2 u_3(x,t), \\
\frac{\partial u_2}{\partial t} + v_2 \frac{\partial u_2}{\partial x} &= i \varepsilon \bar{u}_1 u_3(x,t), \\
\frac{\partial u_3}{\partial t} + v_3 \frac{\partial u_3}{\partial x} &= i \varepsilon u_1 u_2(x,t).
\end{align*}
\]

(1.1)

Here \(v_i\) are the group velocities and \(\varepsilon\) is the interaction constant.

The 3-wave equations can be solved through the ISM due to the fact that Eq. (1.1) allows Lax representation (see eq. (2.1) below). The main result of the pioneer papers [references] is the 3-wave resonant interaction model described by the equations:

\[
\begin{align*}
L_1(t)\psi(x,t,\lambda) &= \left( i \frac{d}{dx} + q(x,t) - \lambda J \right) \psi(x,t,\lambda) = 0, \\
q(x,t) &= \begin{pmatrix} 0 & q_{12} & q_{13} \\ q_{12}^* & 0 & q_{23} \\ q_{13}^* & q_{23}^* & 0 \end{pmatrix}, \quad J = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix},
\end{align*}
\]

(1.2)

are isospectral. Here \(J_1 > J_2 > J_3\), \(q_{12}(x,t) = u_1(x,t)\sqrt{J_1-J_2}\), \(q_{13}(x,t) = u_3(x,t)\sqrt{J_1-J_3}\) and \(q_{23}(x,t) = u_2(x,t)\sqrt{J_2-J_3}\). If we denote by \(T_1(\lambda,t)\) the transfer matrix of \(L_1(t)\) then \(T_1(\lambda,t)\) evolves in time according to the linear equation:

\[
\begin{align*}
\frac{i}{\lambda} \frac{dT_1}{dt} - J[I, T_1(\lambda,t)] &= 0, & I &= \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.
\end{align*}
\]

(1.3)

The group velocities \(v_k\) are expressed through the constants \(J_k\) and \(I_k\) by \(v_1 = -v_{12}\), \(v_2 = -v_{23}\) and \(v_3 = -v_{13}\) where \(v_{jk} = (I_j - I_k)/(J_j - J_k)\). Thus the problem of solving the 3-wave equation (1.3) for a given initial condition \(u_k(x,t=0) = u_{k,0}(x)\) can be performed in three steps:

a) Insert \(u_{k,0}(x)\) as potential coefficients in \(L_1(0)\) and solve the scattering problem for \(L_1(0)\) determine the initial value of transfer matrix \(T_1(\lambda,0)\);

b) Solve (1.3) and determine \(T_1(\lambda,t)\) for \(t > 0\);

c) Solve the inverse scattering problem for \(L_1(t)\), i.e. reconstruct the potential \(q(x,t)\) corresponding to \(T_1(\lambda,t)\) and, recover the solution \(u_k(x,t)\) of (1.3).

Note that step b) is solved trivially; steps a) and c) reduce to linear problems. Thus the nonlinear three-wave interaction model is reduced to a sequence of linear problems. In step a) one must solve the direct scattering problem for \(L_1(0)\) while in step c) one should solve the inverse scattering problem for the operator \(L_1(t)\). Step c) for the operator \(L_1(t)\) in the class of potentials vanishing fast enough for \(x \to \pm \infty\) was first solved by Zakharov and Manakov [2] by deriving the analog of the Gel’fand-Levitan-Marchenko equation for \(L_1(t)\). This was rather tedious procedure but soon Shabat [6] proved that step c) can be reduced to a local Riemann-Hilbert problem (RHP) for the fundamental analytic solutions of \(L_1(t)\). This fact was used in [8] to...
simplify greatly the derivation of the soliton solutions of the 3-wave system by reducing it to a simple purely algebraic procedure known now as the Zakharov-Shabat dressing method.

Quite naturally the 3-wave interaction model was generalized to N-wave equations which can be written in the form:

$$i[J, Q_1] - i[J, Q_2] + [I, Q], [J, Q] = 0,$$  \hspace{1cm} (1.4)

where $Q(x, t) = -Q^T(x, t)$ is an off-diagonal $n \times n$ matrix-valued function (i.e. $Q_{jj} = 0$) tending fast enough to 0 for $x \to \pm \infty$. The potential $q(x, t)$ in $L$ is replaced by $[J, Q(x, t)]$ and $I$ and $J$ are constant diagonal matrices:

$I = \text{diag}(I_1, I_2, \ldots, I_n), \quad J = \text{diag}(J_1, J_2, \ldots, J_n), \quad J_1 > J_2 > \ldots > J_n$,  \hspace{1cm} (1.5)

satisfying $\text{tr} I = \text{tr} J = 0$. In today’s literature the system (1.4) is known as the N-wave system with $N = n(n - 1)/2$. From algebraic point of view the 3-wave system can be related to the algebra $sl(3)$ while the N-wave system is related to $sl(n)$.

Due to the comparatively simple structure of the underlying algebra $sl(n)$ it was rather straightforward to generalize not only the ISM but also the RHP approach and the Zakharov-Shabat dressing method \cite{1,2,3}. Soon after that in \cite{4} it was proved that the transition from the potential $q(x, t)$ to the corresponding transfer matrix $T(\lambda, t)$ can be viewed as a generalized Fourier transform which allows one to analyze the hierarchy of Hamiltonian structures of the N-wave systems. The next stage, namely the proof of the complete integrability of these systems and the derivation of their action-angle variables was done first in \cite{4} and later by different method in \cite{10}.

The next generalization, also rather natural from algebraic point of view, restricts $Q(x, t)$, $I$ and $J$ in (1.4) to be elements of a (semi-)simple Lie algebra $\mathfrak{g}$; then $N = |\Delta_+|$ – the number of positive roots of $\mathfrak{g}$. Though algebraically simple, this restriction makes both the construction of the fundamental analytic solution (FAS) and the dressing method more difficult, see \cite{5} where some preliminary results on the form of $u(x, \lambda)$ are reported. The difficulties are due to the fact that both methods require explicit construction and/or factorizing of certain group elements which even for the symplectic and orthogonal algebras is not trivial.

We should mention also the papers by Zakharov and Mikhailov \cite{12} in which they generalized the dressing method and derived the soliton solutions for a number of field theory models related to the orthogonal and symplectic algebras. Their analysis is based on Lax operators more complicated than (1.4) and needs to be modified in order to apply it to $L$ (1.4).

The $N$-wave equation (1.4) related to $\mathfrak{g}$ are solvable by the ISM \cite{4} applied to the generalized system of Zakharov–Shabat type:

$$L(\lambda)\Psi(x, t, \lambda) = \left(i \frac{d}{dx} + [J, Q(x, t)] - \lambda J\right)\Psi(x, t, \lambda) = 0, \quad J \in \mathfrak{h},$$  \hspace{1cm} (1.6)

$$q(x, t) \equiv [J, Q(x, t)] = \sum_{\alpha \in \Delta_+} (\alpha, \tilde{J})(Q_\alpha(x, t)E_\alpha - Q_{-\alpha}(x, t)E_{-\alpha}) \in \mathfrak{g}\backslash \mathfrak{h},$$  \hspace{1cm} (1.7)

where $\mathfrak{h}$ is the Cartan subalgebra and $E_\alpha$ are the root vectors of the simple Lie algebra $\mathfrak{g}$. Here and below $r = \text{rank} \mathfrak{g}$, $\Delta_+$ is the set of positive roots of $\mathfrak{g}$ and $\tilde{J} = \sum_{k=1}^r J_k e_k$, $\tilde{I} = \sum_{k=1}^r I_k e_k \in \mathfrak{h}'$ are vectors corresponding to the Cartan elements $J, I \in \mathfrak{h}$.

The results in \cite{4} were generalized to any semi-simple Lie algebra in \cite{13}. The important question addressed and answered there concerned the construction of the FAS for each of the fundamental representations of $\mathfrak{g}$. To this end one needs the
explicit formulae for the Gauss decompositions of the scattering matrix $T(\lambda, t)$ which in turn requires knowledge of the fundamental representations of $\mathfrak{g}$. The next step that could be done with the FAS is the explicit construction of the resolvent of $L(t)$ (see [4]) and the generalized Fourier expansions [3] which underlie all basic properties of the nonlinear evolution equations (NLEE).

However all results in [3] were derived under the assumption that $L(t)$ has no discrete eigenvalues. The problem is to find a correct ansatz for the Zakharov-Shabat dressing factor $u(x, \lambda)$ whose form depends on the choice of the representation of $\mathfrak{g}$.

The numbers $|\Delta_+|$ for the simple Lie algebras given in the table below

| $\mathfrak{g}$ | $A_r$ | $B_r, C_r$ | $D_r$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|----------------|-------|------------|-------|-------|-------|-------|-------|-------|
| $|\Delta_+|$ | $r(r+1)/2$ | $r^2$ | $r(r-1)$ | 6 | 24 | 36 | 63 | 120 |

grow rather quickly with the rank of the algebra $r$. If we disregard the constraint $Q^\dagger = -Q$ the corresponding generic NLEE are systems of $2|\Delta_+|$ equations for $2|\Delta_+|$ independent complex-valued functions. They are solvable for any $r$ but their possible applications to physics for large $r$ do not seem realistic. However one still may extract new integrable and physically useful NLEE by imposing reductions on $L(t)$, i.e. algebraic restrictions on $Q(x, t)$ which diminish the number of independent functions in them and the number of equations [3]. Of course such restrictions must be compatible with the dynamics of the NLEE. One of the simplest and best known reductions that we already mentioned $Q^\dagger = -Q$ diminishes the number of fields by a factor of 2.

Another famous class of reductions found long ago [15, 16] led to the integrability of the 2-dimensional Toda chains. The related group of reductions is isomorphic to $D_h$ where $h$ is the Coxeter number of the corresponding Lie algebra $\mathfrak{g}$. As a result the number of independent real-valued fields becomes equal to the rank of the algebra.

Although the two reductions outlined above have been known for quite a time, comparatively little is known about the other ‘intermediate’ types of reductions. One of the aims of this paper is to outline how this gap could be bridged. The ingredients that we need are the well known facts about the simple Lie algebras and their representations. Another aim is to extend the Zakharov-Shabat dressing method for linear systems related to orthogonal algebras. Several types of one soliton solutions of the corresponding $N$-wave equations and their reductions are analyzed.

Section 2 contains preliminaries from the ISM, the reduction group [15], the theory of simple Lie algebras and the scattering theory for $L$. In Section 3 we extend the Zakharov-Shabat dressing method for linear systems [15] related to the orthogonal algebras. We provide the general form of the one-soliton solutions of the corresponding $N$-wave equations. In Section 4 we formulate the effect of $G_R$ on the scattering data of the Lax operator and analyze two types of $\mathbb{Z}_2$-reductions. In Section 5 we analyze the consequences of the $\mathbb{Z}_2$-reductions on the soliton parameters and exhibit several types of 1-soliton solutions related to subalgebras $sl(2)$, $so(3)$ and $sl(3)$ of $\mathfrak{g}$. The 4-wave equations related to $so(5)$ are shown to have applications to physics. In the last Section 6 we briefly discuss the hierarchies of the Hamiltonian structures.

2. Preliminaries and general approach

2.1. Lax representations and reductions

Indeed the $N$-wave equations [14] as well as the other members of the hierarchy possess Lax representation of the form:

$$[L(\lambda), M_P(\lambda)] = 0,$$ (2.1)
where \( L(\lambda) \) is provided by \([1.4]\) and
\[
M_P(\lambda)\Psi(x, t, \lambda) = \left( \frac{d}{dt} + \sum_{k=0}^{P-1} V_k(x, t) - f_P \lambda^P I \right) \Psi(x, t, \lambda) = 0, \quad I \in \mathfrak{h}.
\]
(2.2)

Eq. (2.1) must hold identically with respect to \( \lambda \). A standard procedure generalizing the AKNS one \([17]\) allows us to evaluate \( V_k \) in terms of \( Q(x, t) \) and its \( x \)-derivatives. Here and below we consider only the class of smooth potentials \( Q(x, t) \) vanishing fast enough for \( |x| \to \infty \) for any fixed value of \( t \). Then one may also check that the asymptotic value of the potential in \( M_P \), namely \( f(P)(\lambda) = f_P \lambda^P I \) may be understood as the dispersion law of the corresponding NLEE. The \( N \)-wave equations \((1.4)\) is obtained in the simplest nontrivial case with \( P = 1, f_P = 1 \) and \( V_0(x, t) = [I, Q(x, t)] \).

The consistent approach to the reduction problem is based on the notion of the reduction group \( G_R \) introduced in \([13]\) and further developed in \([18, 12, 19]\). Since we impose finite number of algebraic constraints on the potentials \( U \) and \( V_P \)
\[
U(x, t, \lambda) = [J, Q(x, t)] - \lambda J, \quad V_P(x, t, \lambda) = \sum_{k=0}^{P-1} V_k(x, t) \lambda^k - f_P \lambda^P I,
\]
(2.3)
of the Lax pair it is natural to choose as \( G_R \) finite group which must allow for two realizations: i) as finite subgroup of the group \( \text{Aut}(\mathfrak{g}) \) of automorphisms of the algebra \( \mathfrak{g} \); and ii) as finite subgroup of the conformal mappings \( \text{Conf} \mathbb{C} \) on the complex \( \lambda \)-plane.

Obviously to each reduction imposed on \( L \) and \( M \) there corresponds a reduction of the space of fundamental solutions \( S_\mathfrak{g} \equiv \{ \Psi(x, t, \lambda) \} \) to \((1.4)\) and \((2.2)\). Some of the simplest \( \mathbb{Z}_2 \)-reductions (involutions) of \( N \)-wave systems have been known for a long time (see \([13]\)) and are related to outer automorphisms of \( \mathfrak{g} \) and \( \Phi \), namely:
\[
C_1 (\Psi(x, t, \lambda)) = A_1 \Psi^T(x, t, \kappa_1(\lambda)) A_1^{-1} = \Psi^{-1}(x, t, \lambda), \quad \kappa_1(\lambda) = \pm \lambda^*,
\]
(2.4)
where \( A_1 \) belongs to the Cartan subgroup of the group \( \Phi \):
\[
A_1 = \exp(\pi i H_1).
\]
(2.5)

Here \( H_1 \in \mathfrak{h} \) is such that \( \alpha(H_1) \in \mathbb{Z} \) for all roots \( \alpha \in \Delta \) in the root system \( \Delta \) of \( \mathfrak{g} \). Note that the reduction condition relates one fundamental solution \( \Psi(x, t, \lambda) \in \Phi \) of \((1.4)\) and \((2.2)\) to another \( \Psi(x, t, \lambda) \) which in general differs from \( \Psi(x, t, \lambda) \).

Another class of \( \mathbb{Z}_2 \) reductions are related to automorphisms of the type:
\[
C_2 (\Psi(x, t, \kappa_2(\lambda))) = A_2 \Psi^T(x, t, \kappa_2(\lambda)) A_2^{-1} = \Psi^{-1}(x, t, \lambda), \quad \kappa_2(\lambda) = \pm \lambda,
\]
(2.6)
where \( A_2 \) may be of the form \((2.3)\) or corresponds to a Weyl group automorphism. The best known examples of NLEE obtained with the reduction \((2.6)\) are the sine-Gordon and the MKdV equations which are related to \( \mathfrak{g} \simeq \mathfrak{sl}(2) \). For higher rank algebras such reductions to our knowledge have not been studied.

Since our aim is to preserve the form of the Lax pair we consider only automorphisms preserving the Cartan subalgebra \( \mathfrak{h} \). This condition is obviously fulfilled if: a) \( A_k, \ k = 1, 2 \) is in the form \((2.3)\); b) \( A_k, \ k = 1, 2 \) are Weyl group automorphisms. Most of our results below concern the orthogonal algebras which we define by \( X + S X^T S^{-1} = 0 \) with
\[
S = \sum_{k=1}^{r} (-1)^{k+1}(E_{kk} + E_{kk}) + (-1)^{r} E_{r+1,r+1},
\]

\[
k = N + 1 - k, \quad N = 2r + 1 \quad \mathfrak{g} \simeq \mathfrak{B}_r,
\]
(2.7)
\[
S = \sum_{k=1}^{r} (-1)^{k+1}(E_{kk} + E_{kk}), \quad N = 2r, \quad k = N + 1 - k, \quad \mathfrak{g} \simeq \mathfrak{D}_r,
\]
Here $E_{kn}$ is an $N \times N$ matrix whose matrix elements are $(E_{kn})_{ij} = \delta_{jk}\delta_{nj}$ and $N$ is the dimension of the typical representation of the corresponding algebra.

The corresponding reduction conditions for the Lax pair are as follows [15]:

$$C_k(U(\Gamma_k(\lambda))) = U(\lambda), \quad C_k(V_P(\Gamma_k(\lambda))) = V_P(\lambda), \quad (2.8)$$

where $C_k \in Aut \mathfrak{g}$ and $\Gamma_k(\lambda)$ are the images of $g_k$. Since $G_R$ is a finite group then for each $g_k$ there exist an integer $N_k$ such that $g_k^{N_k} = 1$; if $k = 1$ and $N_1 = 2$ then $G_R \simeq \mathbb{Z}_2$.

2.2. Cartan-Weyl basis and Weyl group

Here we fix up the notations and the normalization conditions for the Cartan-Weyl generators of $\mathfrak{g}$. The commutation relations are given by [20][21]:

$$[h_k, E_\alpha] = (\alpha, c_k)E_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha,$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha, \beta}E_{\alpha + \beta} & \text{for } \alpha + \beta \in \Delta, \\ 0 & \text{for } \alpha + \beta \notin \Delta \cup \{0\}. \end{cases} \quad (2.9)$$

If $J$ is a regular real element in $\mathfrak{h}$ then we may use it to introduce an ordering in $\Delta$ by saying that the root $\alpha \in \Delta_+$ is positive (negative) if $(\alpha, J) > 0$ ($(\alpha, J) < 0$ respectively). The normalization of the basis is determined by:

$$E_{-\alpha} = E^T_{\alpha}, \quad (E_{-\alpha}, E_\alpha) = \frac{2}{(\alpha, \alpha)}; \quad N_{-\alpha, -\beta} = -N_{\alpha, \beta}, \quad N_{\alpha, \beta} = \pm(p + 1), \quad (2.10)$$

where the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \ldots, p$ and $\alpha + (p+1)\beta \notin \Delta$. The root system $\Delta$ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections $S_\alpha$; on the vectors $\vec{y} \in \mathbb{E}'$ they act as

$$S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha, \alpha)}\alpha, \quad \alpha \in \Delta. \quad (2.11)$$

All Weyl reflections $S_\alpha$ form a finite group $W_\mathfrak{g}$ known as the Weyl group. The Weyl group has a natural action of the on the Cartan-Weyl basis:

$$S_\alpha(H_\beta) = A_\alpha H_\beta A^{-1}_\alpha = H_{S_\alpha \beta}, \quad S_\alpha(E_\beta) = n_{\alpha, \beta}E_{S_\alpha \beta}, \quad n_{\alpha, \beta} = \pm 1. \quad (2.12)$$

which shows how it can be understood as a group of inner automorphisms of $\mathfrak{g}$ preserving the Cartan subalgebra $\mathfrak{h}$. The same property is possessed also by $Ad_\mathfrak{h}$ automorphisms; indeed, choosing $K = \exp(\pi i H_{\vec{e}})$ and from (2.9) one finds

$$K H_\alpha K^{-1} = H_\alpha, \quad KE_\alpha K^{-1} = e^{\pi i (\alpha, \vec{e})} E_\alpha, \quad (2.13)$$

where $\vec{e} \in \mathbb{E}'$ is the vector corresponding to $H_{\vec{e}} \in \mathfrak{h}$. Then the condition $K^2 = 1$ means that $(\alpha, \vec{e}) \in \mathbb{Z}$ for all $\alpha \in \Delta$. If $\omega_k$ are the fundamental weights of $\mathfrak{g}$ then $H_{\vec{e}}$ must be such that $\vec{e} = \sum_{k=1}^{r} 2\epsilon_k \omega_k / (\alpha_k, \alpha_k)$ with $\epsilon_k$ integer.

We will need also the element $w_0 \in W(\mathfrak{g})$, $w_0^2 = 1$ which maps the highest weight of each irreducible representation $\Gamma(\omega)$ to the corresponding lowest weight, i.e.,

$$w_0(E_\alpha) = n_\alpha E_{w_0(\alpha)}, \quad w_0(H_k) = H_{w_0(\epsilon_k)}, \quad \alpha \in \Delta_+, \quad n_\alpha = \pm 1; \quad (2.14)$$

The action of $w_0$ on Cartan-Weyl basis is given by (2.12) with $S_\alpha$ replaces by $S$ (2.7).

On the root space $w_0$ is defined by $w_0(\epsilon_k) = -\epsilon_k$ for all $k = 1, \ldots, r$ if $\mathfrak{g} \simeq B_r$ or $D_{2r}$; for $D_{2r+1}$ we have $w_0(\epsilon_k) = -\epsilon_k$ for $k = 1, \ldots, r - 1$ and $w_0(\epsilon_r) = \epsilon_r$, see [21][21].
2.3. The inverse scattering problem for $L$

The direct scattering problem for the Lax operator (1.6) is based on the Jost solutions

$$\lim_{x \to \infty} \psi(x, \lambda)e^{i\lambda J x} = 1, \quad \lim_{x \to -\infty} \phi(x, \lambda)e^{i\lambda J x} = 1,$$

and the scattering matrix

$$T(\lambda) = (\psi(x, \lambda))^{-1} \phi(x, \lambda). \quad (2.16)$$

The FAS $\chi^\pm(x, \lambda)$ of $L(t)$ are analytic functions of $\lambda$ for $\text{Im} \lambda \geq 0$ are related to the Jost solutions by:

$$\chi^\pm(x, \lambda) = \phi(x, \lambda)S^\pm(\lambda) = \psi(x, \lambda)T^\pm(\lambda)D^\pm(\lambda) \quad (2.17)$$

where by "hat" above we denote the inverse matrix $\hat{T} \equiv T^{-1}$. Here $S^\pm(\lambda)$, $D^\pm(\lambda)$ and $T^\pm(\lambda)$ are the factors in the Gauss decomposition of the scattering matrix

$$T(\lambda) = T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)D^-(-\lambda)\hat{S}^-(\lambda), \quad (2.18)$$

$$S^\pm(\lambda) = \exp \left( \sum_{\alpha \in \Delta_+} s^\pm_\alpha(\lambda)E_{\pm\alpha} \right), \quad T^\pm(\lambda) = \exp \left( \sum_{\alpha \in \Delta_+} t^\pm_\alpha(\lambda)E_{\pm\alpha} \right), \quad (2.19)$$

$$D^+(\lambda) = \exp \left( \sum_{j=1}^r \frac{2d^+_j(\lambda)}{(\alpha_j, \alpha_j)} H_j \right), \quad D^-(\lambda) = \exp \left( \sum_{j=1}^r \frac{2d^-_j(\lambda)}{(\alpha_j, \alpha_j)} H^-_j \right), \quad (2.20)$$

where $H_j \equiv H_{\alpha_j}$, $H^-_j = w_0(H_j)$. The proof of the analyticity of $\chi^\pm(x, \lambda)$ for any semi-simple Lie algebra and real $J$ is given in [3]. The superscripts + and − in $D^\pm(\lambda)$ shows that $D^+_j(\lambda)$ and $D^-_j(\lambda)$:

$$D^+_j(\lambda) = \langle \omega^+_j|D^+(\lambda)|\omega^+_j \rangle = \exp \left( d^+_j(\lambda) \right), \quad \omega^-_j = w_0(\omega^+_j), \quad (2.21)$$

are analytic functions of $\lambda$ for Im $\lambda > 0$ and Im $\lambda < 0$ respectively. Here $\omega^+_j$ are the fundamental weights of $g$ and $|\omega^+_j \rangle$ and $|\omega^-_j \rangle$ are the highest and lowest weight vectors in these representations. On the real axis $\chi^+(x, \lambda)$ and $\chi^-(x, \lambda)$ are related by

$$\chi^+(x, \lambda) = \chi^-(x, \lambda)G(\lambda), \quad G(\lambda) = S^+(\lambda)\hat{S}^-(\lambda), \quad (2.22)$$

and the sewing function $G(\lambda)$ may be considered as a minimal set of scattering data provided the Lax operator (1.6) has no discrete eigenvalues. The presence of discrete eigenvalues $\lambda_1^\pm$ means that one (or more) of the functions $D^\pm_1(\lambda)$ will have zeroes at $\lambda_1^\pm$, for more details see [3]. If we introduce $\xi^\pm(x, \lambda) = \chi^\pm(x, \lambda)e^{i\lambda J x}$ then eq. (2.22) can be cast in the form:

$$\xi^+(x, \lambda) = \xi^-(x, \lambda)G(x, \lambda), \quad G(x, \lambda) = e^{-i\lambda J x}G(\lambda)e^{i\lambda J x}, \quad \lambda \in \mathbb{R}. \quad (2.23)$$

This relation together with the normalization condition

$$\lim_{\lambda \to \infty} \xi^\pm(x, \lambda) = 1, \quad (2.24)$$

can be interpreted as a RHP with canonical normalization.

If the potential $q(x, t)$ of the Lax operator (1.4) satisfies the $N$-wave equation

then $S^\pm(t, \lambda)$, $T^\pm(t, \lambda)$ satisfy the linear evolution equations (1.8), while the functions $D^\pm(\lambda)$ are time-independent. Therefore $D^\pm_1(\lambda)$ can be considered as the generating functions of the integrals of motion of (1.4).
2.4. Hamiltonian properties and simple Lie algebras

The interpretation of the ISM as a generalized Fourier transform and the expansions over the “squared solutions” of (1.4) were derived in [1]. All N-wave type equations are Hamiltonian and possess a hierarchy of pair-wise compatible Hamiltonian structures \( \{ H^{(k)}, \Omega^{(k)} \}, \quad k = 0, \pm 1, \pm 2, \ldots \). The phase space \( \mathcal{M} \) of these equations is the space spanned by the complex-valued functions \( \{ Q_\alpha, \alpha \in \Delta \} \), \( \dim \mathbb{C} \mathcal{M} = |\Delta| \). The corresponding NLEE as, e.g. (1.4) and its higher analogs can be formally written down as Hamiltonian equations of motion:

\[
\Omega^{(k)}(Q_t, \cdot) = dH^{(k)}(\cdot), \quad k = 0, \pm 1, \pm 2, \ldots, \quad (2.25)
\]

where both \( \Omega^{(k)} \) and \( H^{(k)} \) are complex-valued. The simplest Hamiltonian formulation of (1.4) is given by \( \{ H^{(0)}, \Omega^{(0)} \} \) where \( H^{(0)} = H_0 + H_{\text{int}} \) and

\[
H_0 = \frac{c_0}{2i} \int_{-\infty}^{\infty} dx \langle Q_x[I, Q_x] \rangle = ic_0 \int_{-\infty}^{\infty} dx \sum_{\alpha > 0} \frac{\{\vec{b}, \alpha\}}{(\alpha, \alpha)} (Q_{\alpha}Q_{-\alpha,x} - Q_{\alpha,x}Q_{-\alpha}), \quad (2.26)
\]

\[
H_{\text{int}} = \frac{c_0}{3} \int_{-\infty}^{\infty} dx \left\langle [J, Q], [Q, [I, Q]] \right\rangle = \sum_{[\alpha, \beta, \gamma] \in \mathcal{M}} \omega_{\beta, \gamma} H(\alpha, \beta, \gamma); \quad (2.27)
\]

and the symplectic form \( \Omega^{(0)} \) is equivalent to a canonical one

\[
\Omega^{(0)} = \frac{i c_0}{2} \int_{-\infty}^{\infty} dx \left\langle [J, \delta Q(x, t)] \wedge \delta Q(x, t) \right\rangle = \sum_{\alpha \in \Delta_+} 2c_0 (\vec{a}, \alpha) \int_{-\infty}^{\infty} dx \delta Q_{\alpha}(x, t) \wedge \delta Q_{-\alpha}(x, t). \quad (2.28)
\]

Here \( c_0 \) is a constant adjusted so that both \( H^{(0)} \) and \( \Omega^{(0)} \) be real, \( \langle \cdot, \cdot \rangle \) is the Killing form of \( \mathfrak{g} \) and the triple \( [\alpha, \beta, \gamma] \) belongs to \( \mathcal{M} \) if \( \alpha, \beta, \gamma \in \Delta_+ \) and \( \alpha = \beta + \gamma \).

For the \( N \)-wave equations and their higher analogs \( H^{(k)} \) depend analytically on \( Q_\alpha \). That allows one to rewrite the equation (2.25) as a standard Hamiltonian equation with real-valued \( \Omega^{(k)} \) and \( H^{(k)} \). The phase space then is viewed by the manifold of real-valued functions \( \{ \text{Re} Q_\alpha, \text{Im} Q_\alpha \}, \alpha \in \Delta, \) so \( \dim \mathbb{R} \mathcal{M} = 2|\Delta| \). Such treatment is formal and we will not explain it into more details here.

Another well known way to make \( \Omega^{(k)} \) and \( H^{(k)} \) real is to impose reduction on them involving complex or hermitian conjugation as in (1.4).

Physically to each term \( H(\alpha, \beta, \gamma) \) we relate part of a wave-decay diagram which shows how the wave associated with the root \( \alpha \) decays into \( \beta \) and \( \gamma \) waves. We assign to each root \( \alpha \) an wave with wave number \( k_\alpha \) and frequency \( \omega_\alpha \). Each of the elementary decays preserves them, i.e.

\[
k_\alpha = k_\beta + k_\gamma, \quad \omega(k_\alpha) = \omega(k_\beta) + \omega(k_\gamma).
\]

Thus the number of the different wave types and their decay modes are determined by the properties of the system of positive roots \( \Delta_+ \) of \( \mathfrak{g} \).

3. The dressing Zakharov-Shabat method

The main goal of the dressing method is, starting from a FAS \( \chi^{(+)\#}_{0}(x, \lambda) \) of \( L \) with potential \( q_{(0)} = [J, Q_{(0)}] \) to construct a new singular solution \( \chi^{(+)\#}_{1}(x, \lambda) \) of the RHP
Then vectors \( \chi^\pm_1(x, \lambda) \) will correspond to a potential \( q_{(1)} = [J, Q_{(1)}] \) of \( L(1) \) with two discrete eigenvalues \( \lambda^\pm \). It is related to the regular one by a dressing factor \( u(x, \lambda) \)
\[
\chi^\pm_1(x, \lambda) = u(x, \lambda)\chi^\pm_0(x, \lambda)u^{-1}(\lambda), \quad u_-(\lambda) = \lim_{x \to -\infty} u(x, \lambda). \tag{3.1}
\]
The sewing function in the RHP is modified to \( G_1(x, \lambda) = u_-(\lambda)G(x, \lambda)u^{-1}(\lambda) \); as we will see below \( u_-(\lambda) \) is an element of the Cartan subgroup of \( \mathfrak{g} \). Then \( u(x, \lambda) \) obviously must satisfy the equation
\[
i\frac{du}{dx} + q_{(1)}(x)u(x, \lambda) - u(x, \lambda)q_{(0)}(x) - \lambda[J, u(x, \lambda)] = 0, \tag{3.2}
\]
and the normalization condition \( \lim_{\lambda \to -\infty} u(x, \lambda) = 1 \). Besides \( \chi^\pm_1(x, \lambda), i = 0, 1 \) and \( u(x, \lambda) \) must belong to the corresponding group \( \mathfrak{g} \); in addition \( u(x, \lambda) \) by construction has poles and/or zeroes at \( \lambda^\pm \). Below all quantities related to \( L(1) \) with potential \( q_1(x) \) will be supplied by the corresponding index \( i \). Their scattering data are related by:
\[
S^\pm_{(1)}(\lambda) = u_-(\lambda)S^\pm_{(0)}(\lambda)u^{-1}(\lambda), \quad T^\pm_{(1)}(\lambda) = u_+(\lambda)T^\pm_{(0)}(\lambda)u^{-1}(\lambda),
\]
\[
D^\pm_{(1)}(\lambda) = u_+(\lambda)D^\pm_{(0)}(\lambda)u^{-1}(\lambda), \quad u_\pm(\lambda) = \lim_{x \to \pm\infty} u(x, \lambda). \tag{3.3}
\]
Since the limits \( u_\pm(\lambda) \) are \( x \)-independent and belong to the Cartan subgroup of \( \mathfrak{g} \), so \( S^\pm_{(1)}(\lambda), T^\pm_{(1)}(\lambda) \) are of the form \( (2.15, 2.20) \).

The construction of \( u(x, \lambda) \) will be based on an appropriate ansatz specifying explicitly the form of its \( \lambda \)-dependence which crucially depends on the choice of \( \mathfrak{g} \) and its representation. Here we will consider separately the classical series of simple Lie algebras: \( A_r \simeq sl(r+1), B_r \simeq so(2r+1) \) and \( D_r \simeq so(2r) \). The simplest nontrivial case \( \mathfrak{g} \simeq A_r \) is solved in the classical papers \( \mathfrak{g} \) with the ansatz
\[
u(x, \lambda) = 1 + (c_1(\lambda) - 1)P_1(x), \quad c_1(\lambda) = \frac{\lambda - \lambda^+_1}{\lambda - \lambda^-_1}, \tag{3.4}
\]
where the rank 1 projector \( P_1(x) \) is of the form:
\[
P_1(x) = \frac{|n(x)\rangle \langle m(x)|}{\langle m(x)|n(x)\rangle}, \quad |n(x)\rangle = \chi^+_0(x, \lambda^+_1)|n_0\rangle, \quad \langle m(x)| = \langle m_0|\chi^-_0(x, \lambda^-_1). \tag{3.5}
\]
where \( |n_0\rangle \) and \( \langle m_0| \) are constant vectors. It can be easily checked that \( u(x, \lambda) \) \( (3.4) \)
corresponds to a potential
\[
q_{(1)}(x) = q_{(0)}(x) + \lim_{\lambda \to -\infty} (J - u(x, \lambda)Ju^{-1}(x, \lambda)). \tag{3.6}
\]

In fact \( u(x, \lambda) \) \( (3.4) \) belongs not to \( SL(r+1) \), but to \( GL(r+1) \). It is not a problem to multiply \( u(x, \lambda) \) by an appropriate scalar and thus to adjust its determinant to 1. Such a multiplication easily goes through the whole scheme outlined above.

**Theorem. 1** Let \( \mathfrak{g} \simeq B_r \) or \( D_r \) and let the dressing factor \( u(x, \lambda) \) be of the form:
\[
u(x, \lambda) = 1 + (c_1(\lambda) - 1)P_1(x) + (c^-_1(\lambda) - 1)P^-_1(x), \quad P^-_1 = SP^1S^{-1}, \tag{3.7}
\]
where \( S \) is introduced in \( (3.3) \) and \( P_1(x) \) is a rank 1 projector \( (3.4) \). Let the constant vectors \( |n_0\rangle \) and \( \langle m_0| \) satisfy the condition
\[
(m_0|S|n_0) = (n_0|S|m_0) = 0. \tag{3.8}
\]
Then \( u(x, \lambda) \) \( (3.4) \) satisfies the equation \( (3.2) \) with a potential
\[
q_{(1)}(x) = q_{(0)}(x) - (\lambda^+_1 - \lambda^-_1)[J, p(x)], \quad p(x) = P_1(x) - P^-_1(x). \tag{3.9}
\]
Proof. Due to the fact that $\chi^\pm_0(x, \lambda)$ take values in the corresponding orthogonal group we find that from (3.3) it follows $(m|S|m) = 0$, $(m|JS|m) = 0$ and analogous relations for the vector $|n\rangle$. As a result we get

$$P_1(x)P_{-1}(x) = P_{-1}(x)P_1(x) = 0, \quad P_1(x)JP_{-1}(x) = P_{-1}(x)JP_1(x) = 0. \quad (3.10)$$

Let us now insert (3.7) into (3.2) and take the limit of the r.h.side of (3.2) for $\lambda \to \infty$. This immediately gives eq. (3.9). In order that Eq. (3.2) be satisfied identically with respect to $\lambda$ we have to put to 0 also the residues of its r.h.side at $\lambda \to \lambda^+_1$ and $\lambda \to \lambda^-_1$. This gives us the following system of equation for the projectors $P_1(x)$ and $P_{-1}(x)$:

$$i \frac{dP_1}{dx} + q_1(x) - P_1(x)q_0(x) - \lambda^-_1 [J, P_1(x)] = 0, \quad (3.11)$$

$$i \frac{dP_{-1}}{dx} + q_1(x) - P_{-1}(x)q_0(x) - \lambda^+_1 [J, P_{-1}(x)] = 0, \quad (3.12)$$

where we have to keep in mind that $q_1(x)$ is given by (3.9). Taking into account (3.10) and the relation between $P_1(x)$ and $P_{-1}(x)$ eq. (3.11) reduces to:

$$i \frac{dP_1}{dx} + [q_0(x), P_1(x)] + \lambda^-_1 P_1(x)J - \lambda^+_1 J P_1(x) - (\lambda^-_1 - \lambda^+_1)P_1(x)JP_1(x) = 0. \quad (3.13)$$

One can check by a direct calculation that (3.7) satisfies identically (3.13). The theorem is proved.

The explicit form of the dressing factor $u(x, \lambda)$ (3.7) can be viewed as an extension of the results in [11] where only the structure of the singularities in $\lambda$ of $u(x, \lambda)$ or rather of $\chi^\pm_0(x, \lambda)$ was derived for each irreducible representation of $g$.

**Corollary.** 1 The dressing factor (3.7) can be written in the form

$$u(x, \lambda) = \exp \left( \ln c(\alpha)p(x) \right), \quad (3.14)$$

where $p(x) \in g$, and consequently $u(x, \lambda)$ belongs to the corresponding orthogonal group.

Let us consider now the purely solitonic case, i.e. $q_0(x) = 0$ and $\chi^\pm_0(x, t, \lambda) = \exp(-i\lambda(Jx + It))$. The condition (3.8) which the vector $|n\rangle$ must satisfy goes into

$$\sum_{k=1}^r 2(-1)^kn_0kn_0, = (-1)^r (n_0, r+1)^2, \quad (3.15)$$

and an analogous one for the vector $|m_0\rangle$. The condition $\lim_{|x| \to \infty} Q(x, t) = 0$ can be satisfied only if $n_0k = 0$ whenever $m_0k = 0$ and vice versa. Making use of the explicit form of the projectors $P_{\pm 1}(x)$ valid for the typical representations of $B_r$ we write down:

$$p(x) = \frac{2}{(m|n)} \left( \sum_{k=1}^r h_k(x)H_{e_\alpha} + \sum_{\alpha \in \Delta^+} (P_\alpha(x)E_\alpha + P_{-\alpha}(x)E_{-\alpha}) \right), \quad (3.16)$$

where

$$h_k(x, t) = n_0, m_0, ke^{2\nu_y t} - n_0, m_0, e^{-2\nu_y t},$$

$$P_\alpha = \begin{cases} P_{ks} & \text{for } \alpha = e_k - e_s, \\ P_{ks} & \text{for } \alpha = e_k + e_s, \\ P_{k, r+1} & \text{for } \alpha = e_k, \end{cases} \quad P_{-\alpha} = \begin{cases} P_{sk} & \text{for } \alpha = -(e_k - e_s), \\ P_{sk} & \text{for } \alpha = -(e_k + e_s), \\ P_{r+1, k} & \text{for } \alpha = -e_k, \end{cases}$$
In order to determine the singularities of the functions $D^b$ have $D^b$ where the integers $k$ if we assume that the ansatz (3.7) for the dressing factor $B$ leads to additional pole in the region of analyticity. This happens if we have $k ≃ r$ for all $n$ we put $⟨m|n⟩ = \sum_{k=1}^{r} (n_0,km_0,k e^{2\nu_1 y_k} + n_0,km_0,k e^{-2\nu_1 y_k}) + n_0,r+1m_{0,r+1}$.

The corresponding result for the $D_r$-series is obtained formally if in the above expressions (3.17) we put $n_0,r+1 = m_{0,r+1} = 0$; this means that $P_k,r+1 = P_{r+1,k} = 0$ for all $k \leq r$. Besides in the expression for $⟨m|n⟩$ (3.18) the last term in the right hand side will be missing.

If we consider the time variable fixed then (3.16) and (3.19) namely, they provide us $p(x,t)$ and $q(x,t)$ in any representation of $g$. They also allow us to evaluate explicitly the singularities of both the FAS and the functions $D^r_\gamma(\lambda)$ for each of the different types of solitons. Each type of soliton solution is determined by the set of nonvanishing components of the vectors $\langle n_0| \rangle$ and $\langle m_0| \rangle$. In the general case when all components $n_0,s$, $m_0,s$ for $k_1 ≤ s ≤ k_2$ are nonzero from (3.3) we obtain

$$u_\gamma(\lambda)= \exp[\ln c_1(\lambda)(H_{\gamma_1} - H_{\gamma_2})].$$

By $\gamma_{1,2}$ we denote the weights of the typical representation of the $B_r$-algebra. Thus if we assume that $k_1 < r$ and $k_2 ≤ r$ then $\gamma_1 = k_1$ and $\gamma_2 = k_2$; for $k_2 = r + 1$ we have $\gamma_{r+1} = 0$ and for $k_2 > r + 1$ (therefore $k_2 = 2r + 2 - k_2 ≤ r$) we have $\gamma_2 = -e_{k_2}$. In order to determine the singularities of the functions $D^r_\gamma(\lambda) = \exp(d^r_\gamma(\lambda))$ we have to use (2.20). The general formula for that is

$$D^r_{1,j}(\lambda) = (c_1(\lambda))^{b_1} D^r_{0,j}(\lambda),$$

where the integers $b_1 = (\gamma_1 - \gamma_2, \omega^j)$. Note that in some cases $\beta = \gamma_1 - \gamma_2$ is not a root. Indeed if $k_1 = k$ and $k_2 = k$ then $\beta = 2e_k$ and $b_1 = \begin{cases} 0 & \text{for } s < k, \\ 2, & \text{for } k ≤ s ≤ r, \end{cases}$ $\begin{cases} 0 & \text{for } s < k, \\ 1, & \text{for } s = r - 1, r, \end{cases}$

From (3.22) we see that in the majority of cases adding a soliton leads to additional zeroes of the functions $D^r_{\gamma}(\lambda)$ of order 2 or 1 in their regions of analyticity; the order depends on the choices of $k_2$. There is however one special situation when adding a soliton leads to additional pole in the region of analyticity. This happens if we have $\lambda \simeq D_r$ and $k_1 = r$, $k_2 = r$. Then from (3.22) we get

$$D^r_{1,r-1}(\lambda) = (c_1(\lambda))^{-1} D^r_{0,r-1}(\lambda), \quad D^r_{1,r}(\lambda) = c_1(\lambda) D^r_{0,r}(\lambda),$$

i.e., $D^r_{1,r-1}(\lambda)$ acquires a first-order pole in $C_+$. Let us analyze the types of singularities and zeroes of the FAS $\chi^\pm(x,t,\lambda)$. Since the ansatz (3.7) for the dressing factor $g(x,\lambda)$ contains both positive and negative
powers of $c_1(\lambda)$ the dressed FAS have both zeroes and poles in their regions of analyticity. This is one of the important differences between the analytic properties of the FAS related to the $A_r$ and $B_r$, $D_r$ series. Obviously the order of the poles and zeroes of FAS will depend on the soliton type and on the representation of $\mathfrak{g}$ chosen.

The final remark in this section is about the structure of the different types of soliton solutions. They are determined by the non-vanishing components of $|n_0\rangle$ and $|m_0\rangle$. This in fact picks up a subset of weights $\Gamma_0 \subset \Gamma(\omega_1)$ in the typical representation of $\mathfrak{g}$. We may relate a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ whose typical representation is realized on $\Gamma_0$. From (3.16) we see, that $q_\alpha$ will be non-zero only if $\alpha \in \Delta_0$ – the root system of $\mathfrak{g}_0$. In Section 4 we give several examples related to different subalgebras of $\mathfrak{g}$.

4. Scattering data and the $\mathbb{Z}_2$-reductions.

Let us address now the question of how the soliton solutions are influenced by the reductions. To be more specific we consider two important $\mathbb{Z}_2$ reductions on $U(x, \lambda)$ (2.3), namely:

1) $KU^\dagger(x, \lambda^*)K^{-1} = U(x, \lambda)$, $K^2 = 1$, (4.1)
2) $SU(x, -\lambda)S^{-1} = U(x, \lambda)$, (4.2)

where $K$ (2.13) is an element of the Cartan subgroup of $\mathfrak{g}$ and $S$ is given by (2.7). Such reductions on $U(x, \lambda)$ will reflect on $Q(x, t)$ and $J$, and also on the FAS and the scattering data of $L(t)$ as follows:

1) $s_\alpha Q^*_{-\alpha}(x, t) = -Q_{\alpha}(x, t)$, $J^* = J$, (4.3)
2) $Q_{-\alpha}(x, t) = Q_{\alpha}(x, t)$, (4.4)

1) $S^+(\lambda) = K \left( \hat{S}^-(\lambda^*) \right)^\dagger K^{-1}$, $T^+(\lambda) = K \left( \hat{T}^-(\lambda^*) \right)^\dagger K^{-1}$,
2) $D^+(\lambda) = K \left( \hat{D}^-(\lambda^*) \right)^\dagger K^{-1}$, $F(\lambda) = K \left( F(\lambda^*) \right)^\dagger K^{-1}$, (4.5)

As a minimal set of scattering data we may consider $T^\pm(\lambda)$ or $S^\pm(\lambda)$. The functions $D^\pm(\lambda)$ or equivalently $d_j^\pm(\lambda)$ can be reconstructed by making use of their integral representations; in the case of absence of discrete eigenvalues we have [9]:

$$D_j(\lambda) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \ln\langle \omega_j^+ | \hat{T}^+(\mu) T^-(\mu) | \omega_j^+ \rangle,$$ (4.7)

where $\omega_j^+$ and $|\omega_j^+\rangle$ are the $j$-th fundamental weight of $\mathfrak{g}$ and the highest weight vector in the corresponding fundamental representation $\Gamma(\omega_j^+)$ of $\mathfrak{g}$. The function $D_j(\lambda)$ as a fraction-analytic function of $\lambda$ is equal to:

$$D_j(\lambda) = \begin{cases} d_j^+(\lambda), & \text{for } \lambda \in \mathbb{C}_+ \\ (d_j^+(\lambda) - d_{j'}^-(\lambda))/2, & \text{for } \lambda \in \mathbb{R}, \\ -d_{j'}^-(\lambda), & \text{for } \lambda \in \mathbb{C}_-, \end{cases}$$ (4.8)

where $d_j^+(\lambda)$ were introduced in (2.20) and $d_j^-((\lambda)$ and the index $j'$ is related to $j$ by $\omega_0(\alpha_j) = -\alpha_{j'}$. The functions $D_j(\lambda)$ can be viewed also as generating functions of the integrals of motion. Indeed, if we expand

$$D_j(\lambda) = \sum_{k=1}^{\infty} D_{j,k} \lambda^{-k},$$ (4.9)
and take into account that $D_k(\lambda)$ are time independent we find that $dD_{j,k}/dt = 0$ for all $k = 1, \ldots, \infty$ and $j = 1, \ldots, r$. Moreover it can be checked that $D_{j,k}$ is local in $Q(x, t)$, i.e. depends only on $Q$ and its derivatives with respect to $x$.

From (4.7) and (4.8), (4.9) we easily obtain the effect of the reductions on the set of integrals of motion; namely, for the reduction (4.5):

$$D_j(\lambda) = -D_j^*(\lambda^*), \quad \text{i.e.,} \quad D_{j,k} = -D_{j,k}^*,$$

and for (4.9)

$$D_j(\lambda) = -D_j(-\lambda), \quad \text{i.e.,} \quad D_{j,k} = (-1)^{k+1}D_{j,k}.$$

From (4.11) it follows that all integrals of motion with even $k$ become degenerate. The reduction (4.10) means that the integrals $D_{j,k}$ are purely imaginary; if we impose in addition also (4.11) then $D_{j,2k} = 0$. The simplest local integrals of motion $D_{j,1}$ and $D_{j,2}$ can be expressed as functionals of the potential $Q$ of (1.6) as follows (see [9]):

$$D_{j,1} = -\frac{i}{4} \int_{-\infty}^{\infty} dx \langle [J, Q], [H^\gamma_j, Q]\rangle,$$

$$D_{j,2} = -\frac{1}{2} \int_{-\infty}^{\infty} dx \langle Q, [H^\gamma_j, Q_x]\rangle - \frac{i}{3} \int_{-\infty}^{\infty} dx \langle [J, Q], [Q, [H^\gamma_j, Q]]\rangle,$$

where $\langle H^\gamma_j, H_k \rangle = \delta_{j,k}$. The fact that $D_{j,1}$ are integrals of motion for $j = 1, \ldots, r$, can be considered as natural analog of the Manley–Rowe relations [1, 9]. In the case when the reduction is of the type (2.4), i.e. $Q_{-\alpha} = s_\alpha Q_{\alpha}$ then (4.12) is equivalent to

$$\sum_{\alpha > 0} \frac{2(J, \alpha)(\omega^\gamma_\alpha, \alpha)}{(\alpha, \alpha)^2} \int_{-\infty}^{\infty} dx s_\alpha |Q_\alpha(x)|^2 = \text{const},$$

and can be interpreted as relations between the wave densities $|Q_\alpha|^2$.

The Hamiltonian of the N-wave equations (1.4) is expressed through $D_{j,2}$, namely:

$$H_{N-wave} = -\sum_{j=1}^{r} \frac{2(\alpha_j, \tilde{I})}{(\alpha_j, \alpha_j)^2} D_{j,2} = \frac{1}{2i} \left\langle \left\langle \hat{D}(\lambda), f(\lambda) \right\rangle_0 \right\rangle,$$

where $\hat{D}(\lambda) = dD/d\lambda$ and $f(\lambda) = \lambda$ is the dispersion law of the N-wave equation (1.4). In (4.17) we used just one of the hierarchy of scalar products in the Kac-Moody algebra $\mathfrak{g} \equiv \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ (see [23]):

$$\langle \langle X(\lambda), Y(\lambda) \rangle\rangle_k = \text{Res } \lambda^{k+1} \left\langle \hat{D}^+(\lambda)X(\lambda), Y(\lambda) \right\rangle, \quad X(\lambda), Y(\lambda) \in \mathfrak{g}. \tag{4.16}$$

5. Soliton solutions and examples of N-wave equations.

5.1. Examples of soliton solutions

Here we will give several examples of 1-soliton solutions of the N-wave equations subject to the reductions (4.5) and (4.9). Obviously the dressing factor $u(x, \lambda)$ must satisfy the same reduction conditions as $U(x, \lambda)$. This allows us to derive the following consequences on the vectors $|n_0\rangle$, $|m_0\rangle$ and $\lambda_1^\pm$:

1) $|n_0\rangle = K|m_0^*\rangle$, $\lambda_1^- = (\lambda_1^+)^*$

2) $|n_0\rangle = |m_0\rangle$, $\lambda_1^- = -\lambda_1^+$, \tag{5.1} \tag{5.2}
Example 1 Let \( g_0 \simeq sl(2) \) be formed by the generators \( H_\alpha \) and \( E_{\pm \alpha} \) with \( \alpha = e_k - e_s \). This is possible if only two of the components \( n_{0k}, n_{0s} \) are non-vanishing with \( 1 \leq k < s \leq r \). Then:

\[
p(x) = \frac{1}{2} H_{e_k + e_s} + \frac{1}{2 \cosh \Phi} (\sinh \Phi H_\alpha + e^{-i \Psi} E_\alpha + e^{i \Psi} E_{-\alpha}),
\]

(5.3)

\[
\Phi(x,t) = \nu_1(y_k - y_s) + \frac{1}{2} \ln \frac{n_{0,k} m_{0,k}}{n_{0,s} m_{0,s}}, \quad \Psi(x,t) = \mu_1(y_k - y_s) + \frac{i}{2} \ln \frac{n_{0,k} m_{0,k}}{n_{0,s} m_{0,s}},
\]

If we apply the reduction 1) we find that \( \Phi \) and \( \Psi \) become real provided \( K_s = K_k \); the corresponding soliton solution is regular exponentially decaying function for all values of \( t \). If \( K_k K_s = -1 \) then \( \Phi = \nu_1(y_k - y_s) + \ln(|n_{0,k} n_{0,s}| - i \pi/2) \) which effectively changes the denominators in (5.3) to \( \sinh \Phi \) thus making the soliton solution singular. Such solution require additional care.

Implying reduction 2) the arguments of \( p(x,t) \) simplify to

\[
\Phi(x,t) = \nu_1(y_k - y_s) + \frac{1}{2} \ln \frac{n_{0,k} m_{0,k}}{n_{0,s} m_{0,s}}, \quad \Psi(x,t) = \mu_1(y_k - y_s).
\]

If we apply both reduction simultaneously then the eigenvalues of \( L \) become purely imaginary \( \lambda^+_k = -\lambda^-_k = i \nu_1 \), \( \Phi(x,t) \) remains as in the last line and \( \Psi(x,t) \) vanishes since \( \mu_1 = 0 \).

We can evaluate also the solitons related to another choice of \( \alpha = e_k + e_s \); then we must assume that the non-vanishing \( n_{0k}, n_{0s} \) are those with \( 1 \leq k < s \leq r \). The explicit formulae are analogous to the ones above and we skip them.

Example 2 The solitons related to the subalgebra \( g_0 \simeq so(3) \) spanned by \( H_\alpha \) and \( E_{\pm \alpha} \) with \( \alpha = e_k \) are obtained by choosing \( n_{0k}, n_{0,r+1} \) and \( n_{0,k} \), \( k \leq r \) as the only non-vanishing components. Due to (3.14) we have \( (n_{0,r+1})^2 = 2n_{0,k} n_{0,k} \) and:

\[
p(x) = \tanh \Phi H_\alpha + \frac{e^\Phi + (-1)^{k+r} e^{-\Phi}}{2\sqrt{2 \cosh^2 \Phi}} (e^{-i \Psi} E_\alpha + e^{i \Psi} E_{-\alpha}),
\]

(5.4)

\[
\Phi(x,t) = \nu_1 y_k + \frac{1}{4} \ln \frac{n_{0,k} m_{0,k}}{n_{0,k} m_{0,k}}, \quad \Psi(x,t) = \mu_1 y_k + \frac{1}{4} \ln \frac{n_{0,k} m_{0,k}}{n_{0,k} m_{0,k}}.
\]

These solitons are not singular due to the fact that \( K_k = K_k \). The reduction 1) makes both \( \Phi(x,t) \) and \( \Psi(x,t) \) real. Under the reduction 2) \( \Psi \) vanishes as in the first example.

Example 3 The solitons related to the subalgebra \( g_0 \simeq sl(3) \) whose positive roots are \( e_i - e_s, e_k - e_j \) and \( e_i - e_j \) are obtained if we choose as the only non-vanishing components \( n_{0,i}, n_{0,k} \) and \( n_{0,j}, i < k < j \leq r \); the condition (3.13) holds identically. Here we directly write down the formulae with reduction 1) applied.

\[
p(x) = \frac{1}{3} H_{\varepsilon} + \frac{1}{3 (n^* | K | n)} \left( (2h_i - h_j - h_k)(x,t) H_{e_i - e_k} + (2h_j - h_k - h_i)(x,t) H_{e_j - e_k} \right)
\]

\[
+ \frac{1}{3 (n^* | K | n)} (N_{ik}(x,t) + N_{kj}(x,t) + N_{ij}(x,t)),
\]

(5.5)

\[
\langle n^* | K | n \rangle = (h_i + h_j + h_k)(x,t), \quad h_i(x,t) = K_i e^{\Phi_i},
\]

\[
N_{ik}(x,t) = e^{\Phi_i} \left( e^{-i \Psi_i} E_{e_i - e_k} + e^{i \Psi_i} E_{-e_i + e_k} \right), \quad \varepsilon = e_i + e_j + e_k,
\]

\[
\Phi_{ik}(x,t) = \nu_1(y_i + y_k) + \ln |n_{0,i} n_{0,k}|, \quad \Psi_{ik}(x,t) = \mu_1(y_i - y_k) + \arg n_{0,k} - \arg n_{0,i}.
\]

This result for \( K_k = K_i = K_j \) is quite analogous to the regular soliton solutions of the 3- and \( N \)-wave equations studied in detail in [4]. The situation when \( K_k = -K_i = -K_j \) again may lead to singular solitons which describe blow-up instabilities in \( \chi^3 \)-media.
Figure 1. Wave-decay diagram for the \(so(5)\) algebra. To each positive root of the algebra \(k_\alpha = k_\alpha 1 + n_\alpha 2\) we put in correspondence a wave of type \(k_\alpha\). If the positive root \(k_\alpha = k'_\alpha 1 + k''_\alpha 2\) can be represented as a sum of two other positive roots, we say that the wave \(k_\alpha\) decays into the waves \(k'_\alpha\) and \(k''_\alpha\) as shown on the diagram to the left.

In all the examples above the corresponding reflectionless potentials of \(L\) are obtained with the formula (5.19).

5.2. Real forms of \(B_2\).

Let us illustrate these general results by an example related to the \(B_2\) algebra. This algebra has two simple roots \(\alpha_1 = e_1 - e_2\), \(\alpha_2 = e_2\), and two more positive roots: \(\alpha_1 + \alpha_2 = e_1\) and \(\alpha_1 + 2\alpha_2 = e_1 + e_2 = \alpha_{\text{max}}\). When they come as indices, e.g. in \(Q_\alpha\) we will replace them by sequences of two integers: \(\alpha \rightarrow k_\alpha\) if \(\alpha = k\alpha_1 + n\alpha_2\); if \(\alpha = -(k\alpha_1 + n\alpha_2)\) we will use \(k_\alpha\).

The reduction which extracts the real forms of \(B_2 \simeq so(5)\) is \(KU^1(\lambda^*)K^{-1} = U(\lambda)\) where \(K\) is an element of the Cartan subgroup: \(K = \text{diag}(s_1, s_2, 1, s_2, s_1)\) with \(s_k = \pm 1, k = 1, 2\). This means that \(J_i = J_i^*, i = 1, 2\) and \(Q_\alpha\) must satisfy:

\[
p_{10} = -s_2 s_1 Q_{10}^*, \quad p_{01} = -s_2 Q_{01}^*, \quad p_{11} = -s_1 Q_{11}^*, \quad p_{12} = -s_1 s_2 Q_{12}^*.
\]

Thus we get 4-wave system with the Hamiltonian \(H = H_0 + H_{\text{int}}\) [23]:

\[
H_0 = -\frac{i}{2} \int_{-\infty}^{\infty} dx \left[ s_1 s_2 (I_1 - I_2)(Q_{10} Q_{10,x} - Q_{10,x} Q_{10}) + 2 s_2 I_2 (Q_{01} Q_{01,x} - Q_{01,x} Q_{01}) \\
+ 2 s_1 I_1 (Q_{11} Q_{11,x} - Q_{11,x} Q_{11}) + s_1 s_2 (I_1 + I_2)(Q_{12} Q_{12,x} - Q_{12,x} Q_{12}) \right],
\]

\[
H_{\text{int}} = 2\kappa s_1 \int_{-\infty}^{\infty} dx \left[ s_2 (Q_{12} Q_{12}^* Q_{01} + Q_{12}^* Q_{11} Q_{01}) + (Q_{11} Q_{01} Q_{10}^* + Q_{11}^* Q_{01} Q_{10}) \right],
\]

where \(\kappa = J_1 I_2 - J_2 I_1\), and the symplectic 2-form:

\[
\Omega^{(0)} = i \int_{-\infty}^{\infty} dx \left[ (J_1 - J_2) \delta Q_{10} \wedge \delta Q_{10}^* + 2 J_2 \delta Q_{01} \wedge \delta Q_{01}^* \\
+ 2 J_1 \delta Q_{11} \wedge \delta Q_{11}^* + (J_1 + J_2) \delta Q_{12} \wedge \delta Q_{12}^* \right].
\]

The corresponding wave-decay diagram is shown in figure 1.

The particular case \(s_1 = s_2 = 1\) leads to the compact real form \(so(5, 0) \simeq so(5, \mathbb{R})\) of the \(B_2\)-algebra. The choice \(s_1 = -s_2 = -1\) leads to the noncompact real form.
The generic \( \mathfrak{so}(2,3) \) and \( s_1 = s_2 = -1 \) gives another noncompact one—\( \mathfrak{so}(1,4) \). If in this last case we identify \( Q_{01} = Q_{\text{pol}}, Q_{10} = -E_a, Q_{11} = E_p \) and \( Q_{12} = -E_a \), then we obtain the system studied in [23] which describes Stockes-anti-Stockes wave generation. Here \( Q_{\text{pol}} \) is the normalized effective polarization of the medium and \( E_p, E_a \) and \( E_a \) are the normalized pump, Stockes and anti-Stockes wave amplitudes respectively.

5.3. One more \( \mathbb{Z}_2 \) reduction

Let us apply a second \( \mathbb{Z}_2 \)-reduction to the already reduced system of the previous subsection. We take it of the form (4.2) which gives \( J_i = J_i^*, I_i = I_i^* \) and:

\[
Q_{10} = -s_1 s_2 Q_{10}, \quad Q_{01} = -s_2 Q_{01}, \quad Q_{11} = -s_1 Q_{11}, \quad Q_{12} = -s_1 s_2 Q_{12}.
\]

These reduction conditions allow us to make the following change of the fields \( Q_\alpha \) to the real-valued ones \( v_\alpha \) as follows:

\[
Q_{10} = i^{(1+s_1 s_2)/2} v_{10}, \quad Q_{01} = i^{(1+s_2)/2} v_{01}, \quad Q_{11} = i^{(1+s_1)/2} v_{11}, \quad Q_{12} = i^{(1+s_1 s_2)/2} v_{12}.
\]

Thus we get the following 4-wave system for 4 real-valued functions:

\[
\begin{align*}
(J_1 - J_2) v_{10}, t - (I_1 - I_2) v_{10}, x + 2 s_2 \kappa v_{11} v_{01} &= 0, \\
J_2 v_{01}, t - I_2 v_{01}, x + s_1 s_2 \kappa (v_{11} v_{12} + v_{11} v_{10}) &= 0, \\
J_1 v_{11}, t - I_1 v_{11}, x + \kappa (v_{12} v_{10} - v_{10} v_{01}) &= 0, \\
(J_1 + J_2) v_{12}, t - (I_1 + I_2) v_{12}, x - 2 s_2 \kappa v_{11} v_{01} &= 0.
\end{align*}
\]

Since \( w_0(J) = -J \) the Hamiltonian structure \( \{H^{(0)}, \Omega^{(0)} \} \) becomes degenerated.

Such reductions applied to the Zakharov-Shabat system with \( g \simeq \mathfrak{sl}(2) \) picks up the sine-Gordon and the MKdV equations which have two types of soliton solutions: i) ones related to pairs \( \pm i \nu_k \) of purely imaginary eigenvalues of \( L \) and ii) ones related to quadruplets \( \pm \lambda_1^+, \pm \lambda_1^- \) of eigenvalues of \( L \). The first type are known as ‘topological’ solitons, while the second type are known as ‘breathers’.

The same situation persists also for \( L \) related to any \( g \). One of the differences between \( g \simeq \mathfrak{sl}(2) \) and generic \( g \) consists also in the fact that in the \( \mathfrak{sl}(2) \) case the \( N \)-wave interaction is trivial, while for higher ranks of \( g \) it is non-trivial, see e.g. (5.10).

6. Hamiltonian structures of the reduced \( N \)-wave equations

The generic \( N \)-wave interactions (i.e., prior to any reductions) possess a hierarchy of Hamiltonian structures generated by the so-called generating (or recursion) operator \( \Lambda \) defined in [13] as follows:

\[
\Omega^{(k)} = \frac{i}{2} \int_{-\infty}^{\infty} dx \left\langle [J, \delta Q(x,t)] \wedge \Lambda^k \delta Q(x,t) \right\rangle,
\]

Using the spectral decomposition of \( \Lambda \) we can recalculate \( \Omega^{(k)} \) in terms of the scattering data of \( L \) with the result:

\[
\begin{align*}
\Omega^{(k)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \lambda^k \left\langle \Omega_0^+ (\lambda) - \Omega_0^- (\lambda) \right\rangle, \\
\Omega_0^\pm (\lambda) &= \left\langle \hat{D}^\pm (\lambda) \hat{T}^\pm (\lambda) \delta T^\pm (\lambda) D^\pm (\lambda) \wedge \hat{S}^\pm (\lambda) \delta S^\pm (\lambda) \right\rangle.
\end{align*}
\]

The first consequence from [6,2] is that the kernels of \( \Omega^{(k)} \) differs only by the factor \( \lambda^k \); i.e., all of them can be cast into canonical form simultaneously. This is quite compatible with the results of [1, 3, 10] for the action-angle variables.
Again it is not difficult to find how the reductions influence $\Omega^{(k)}$. Using the invariance of the Killing form, from (4.2) and (1.3) we get respectively: $\Omega^+_0(\lambda) = (\Omega^-_0(\lambda^*))^*$, and $\Omega^-_0(\lambda) = \Omega^+_0(-\lambda)$. Therefore for the reduction (1.5) we get that $i\Omega^{(k)}$ together with all the integrals $iD_{j,k}$ become simultaneously real. The other reduction (4.6) means that $\Omega^{(k)} = (-1)^{k+1}\Omega^{(k)}$; besides from (4.11) we have $D_{j,2k} = 0$. Thus 'half' of the Hamiltonian structures $\{\Omega^{(p)}, H^{(p)}\}$ with even $p$ degenerate. However the other 'half' for odd $p$ survivors. In particular this means that the canonical 2-form $\Omega^{(0)}$ is also degenerate, so the $N$-wave equations with the reduction (4.11) do not allow Hamiltonian formulation with canonical Poisson brackets. For more details see [15, 13, 24].

7. Conclusions

We end with several remarks.

1. Here we presented only examples related to the $B_2$ algebra. Many additional examples can be found in [25].

2. To all reduced systems given above we can apply the analysis in [1, 24] and derive the spectral decompositions for the corresponding recursion $\Lambda$ operator. Such analysis allows one to prove the pair-wise compatibility of the Hamiltonian structures.

3. In many cases the reduction conditions on $J$ may lead to complex values of $J_k$. The construction of the corresponding FAS is first given in [26] for $\mathfrak{sl}(n)$ and in [2] for simple $g$. It is an interesting task to extend the results of [26, 24] to systems with reductions.

4. The cases when the element $J$ is not regular requires additional care in constructing the theory of the recursion operator $\Lambda$, see [27].

5. The list of examples with soliton solutions can easily be continued. Several factors are important for the structure of the different types of solitons. These are: the rank of the projectors $P_1$ and $P_{-1}$ and the subalgebra $g_0$ which is picked up. Obviously depending on these choices we get solitons with different number of internal degrees of freedom. These questions will be discussed elsewhere. Some alternative approaches for constructing soliton solutions of reduced systems are presented in [28, 29].

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