On a Shallow Water Wave Equation

Peter A. Clarkson and Elizabeth L. Mansfield

Department of Mathematics, University of Exeter, Exeter, EX4 4QE, U.K.

Submitted to: Nonlinearity

Date: 1 April 2022

Abstract.

In this paper we study a shallow water equation derivable using the Boussinesq approximation, which includes as two special cases, one equation discussed by Ablowitz et al [Stud. Appl. Math., 53 (1974) 249–315] and one by Hirota and Satsuma [J. Phys. Soc. Japan, 40 (1976) 611–612]. A catalogue of classical and nonclassical symmetry reductions, and a Painlevé analysis, are given. Of particular interest are families of solutions found containing a rich variety of qualitative behaviours. Indeed we exhibit and plot a wide variety of solutions all of which look like a two-soliton for $t > 0$ but differ radically for $t < 0$. These families arise as nonclassical symmetry reduction solutions and solutions found using the singular manifold method. This example shows that nonclassical symmetries and the singular manifold method do not, in general, yield the same solution set. We also obtain symmetry reductions of the shallow water equation solvable in terms of solutions of the first, third and fifth Painlevé equations.

We give evidence that the variety of solutions found which exhibit “nonlinear superposition” is not an artefact of the equation being linearisable since the equation is solvable by inverse scattering. These solutions have important implications with regard to the numerical analysis for the shallow water equation we study, which would not be able to distinguish the solutions in an initial value problem since an exponentially small change in the initial conditions can result in completely different qualitative behaviours.
1 Introduction.

In this paper we discuss the generalised shallow water wave (GSWW) equation
\[
\Delta \equiv u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \tag{1.1}
\]
where \(\alpha\) and \(\beta\) are arbitrary, nonzero, constants. This equation, together with several variants, can be derived from the classical shallow water theory in the so-called Boussinesq approximation \([1]\). There are two special cases of this equation which have been discussed in the literature; (i), if \(\alpha = \beta\)
\[
\Delta = u_{xxxt} + \beta u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \tag{1.2}
\]
which we shall call the swwi equation, and (ii) if \(\alpha = 2\beta\)
\[
\Delta = u_{xxxt} + 2\beta u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \tag{1.3}
\]
which we shall call the swwii equation. These equations are often written in the nonlocal form (set \(u_x = v\))
\[
v_{xxxt} + \beta vv_t - \beta v_x \partial_x^{-1} v_t - v_t - v_x = 0, \tag{1.2}^*
\]
where \((\partial_x^{-1} f)(x) = \int_x^\infty f(y) \, dy\), which was discussed by Hirota and Satsuma \([2]\), and
\[
v_{xxxt} + 2\beta vv_t - \beta v_x \partial_x^{-1} v_t - v_t - v_x = 0, \tag{1.3}^*
\]
which was discussed by Ablowitz et al \([3]\) who showed that it is solvable by inverse scattering (see §4.2). Furthermore Ablowitz et al \([3]\) remark that (1.3\(^*\)) reduces to the celebrated Korteweg-de Vries (KdV) equation
\[
u_t + u_{xxx} + 6uu_x = 0, \tag{1.4}
\]
which also is solvable by inverse scattering \([4]\), in the long wave, small amplitude limit. Equation (1.3\(^*\)) also has the desirable properties of the regularized long wave (RLW) equation \([5,6]\]
\[
v_{xxxt} + vv_t - v_t - v_x = 0, \tag{1.5}
\]
sometimes called the Benjamin-Bona-Mahoney equation, in that it responds feebly to short waves. However, in contrast to (1.3\(^*\)), the RLW equation (1.5) is thought not to be solvable by inverse scattering (cf., \([7]\)).

The GSWW equation (1.1) is discussed by Hietarinta \([8]\) who shows that it can be expressed in Hirota’s bilinear form \([9]\) if and only if either (i), \(\alpha = \beta\), when it reduces to (1.2), or (ii), \(\alpha = 2\beta\), when it reduces to (1.3). Furthermore, as shown below, the GSWW equation (1.1) satisfies the necessary conditions of the Painlevé tests due to Ablowitz et al \([10,11]\) (see §2) and Weiss et al \([12]\) (see §4.1) to be completely integrable if and only if \(\alpha = \beta\) or \(\alpha = 2\beta\). We show in §4.2 that the GSWW equation (1.1) is solvable by inverse scattering techniques in these two special cases. These results strongly suggest that the GSWW equation is completely integrable if and only if it has one of the two special forms (1.2) or (1.3).

The SWWI equation (1.2) and SWWII equation (1.3) arise as a reduction of several higher-dimensional partial differential equations which have been discussed in the literature. The SWWI equation (1.2) arises as a reduction of:

1. The \(2 + 1\)-dimensional equation
\[
u_{yt} + u_{xxxx} - 3u_x u_y - 3u_x u_{xy} = 0, \tag{1.6}
\]
which reduces to the KdV equation (1.4) if \(y = x\). Boiti et al \([13]\) developed an inverse scattering scheme to solve the Cauchy problem for (1.6), for initial data decaying sufficiently rapidly at infinity; this was formulated as a nonlocal Riemann-Hilbert problem.
2. The $3 + 1$-dimensional equation

$$u_{yt} + u_{xxxx} - 3u_{xx}u_y - 3u_xu_{xy} - u_{xx} = 0,$$  

(1.7)

which was introduced by Jimbo and Miwa [14] as the second equation in the so-called Kadomtsev-Petviashvili hierarchy of equations; however (1.7) is not completely integrable in the usual sense (see [15]).

3. The $2 + 1$-dimensional equation

$$u_{tt} - u_{xx} - u_{yy} + u_xu_{xt} + u_yu_{yt} - u_{xxtt} - u_{yytt} = 0,$$  

(1.8)

which was introduced by Yajima et al [16] as a model of ion-acoustic waves in plasmas; Kako and Yajima [17] have studied “soliton interactions” for (1.8).

The \texttt{swwii} equation (1.3) arises as a reduction of the $2 + 1$-dimensional equation

$$u_{xt} + u_{xxxx} - 2u_{xx}u_y - 4u_xu_{xy} = 0,$$  

(1.9)

which, like (1.6), reduces to the \texttt{kdv} equation (1.4) if $y = x$, though note that the term $u_{yt}$ in (1.6) is replaced by $u_{xt}$ in (1.9). Bogoyaviemskii [18,19] discusses the inverse scattering method of solution for (1.9).

In §§2 and 3, we find first the classical Lie group of symmetries and associated reductions of (1.1) and then nonclassical symmetries and reductions of (1.1). The classical method for finding symmetry reductions of partial differential equations is the Lie group method of infinitesimal transformations (cf., [20,21]). Though this method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which can become virtually unmanageable if attempted manually, and so symbolic manipulation programs have been developed, for example in \texttt{macsyma}, \texttt{maple}, \texttt{mathematica}, \texttt{mumath} and \texttt{reduce}, to facilitate the calculations. A survey of the different packages presently available and a discussion of their strengths and applications is given by Hereman [22]. In this paper we use the \texttt{macsyma} program \texttt{symmgrp.max} [23].

In recent years the nonclassical method due Bluman and Cole [24] (in the sequel referred to as the \textit{nonclassical method}, see §3 for further details), which is also known as the “method of conditional symmetries” [25] or the “method of partial symmetries of the first type” [26], and the direct method of Clarkson and Kruskal [27] have been used to generate many new symmetry reductions and exact solutions for several physically significant partial differential equations that are not obtainable using the classical Lie method, which represents important progress (see for example [28,29] and references therein). Since solutions of partial differential equations asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions will illustrate important physical phenomena. In particular, exact solutions arising from symmetry methods can often be effectively used to study properties such as asymptotics and “blow-up” (cf. [30,31]). Furthermore, explicit solutions (such as those found by symmetry methods) can play an important role in the design and testing of numerical integrators; these solutions provide an important practical check on the accuracy and reliability of such integrators (cf. [32,33]).

There is much current interest in the determination of symmetry reductions of partial differential equations which reduce the equations to ordinary differential equations. Often one then checks if the resulting ordinary differential equation is of Painlevé type, i.e., its solutions have no movable singularities other than poles. It appears to be the case that whenever the ordinary differential equation is of Painlevé type then it can be solved explicitly, leading to exact solutions to the original equation. Conversely, if the resulting ordinary differential equation is not of Painlevé type, then often one is unable to solve it explicitly.

The method used to find solutions of the determining equations for the infinitesimals in both the classical and nonclassical case is that of Differential Gröbner Bases (DGBs), defined to be a basis $\mathcal{B}$ of the differential ideal generated by the system such that every member of the ideal pseudo-reduces to zero with respect to $\mathcal{B}$. This method provides a systematic framework for finding integrability
and compatibility conditions of an overdetermined system of partial differential equations. It avoids
the problems of infinite loops in reduction processes, and yields, as far as is currently possible, a
“Triangulation” of the system from which the solution set can be derived more easily [34–37]. In a
sense, a DGB provides the maximum amount of information possible using elementary differential
and algebraic processes in a finite time.

In pseudo-reduction, one is allowed to multiply the expression being reduced by differential,
that is, non-constant, coefficients of the highest derivative terms of the reducing equations. The
reason one must do this is that on nonlinear systems, the algorithms for calculating the differential
analogue of a Gröbner Basis will not terminate if only strict reduction is allowed. What this means
in practice is that such coefficients are assumed to be nonzero. To obtain solutions of the system
that evaluate to zero one of these coefficients, one needs to include it with the system from the start
of the calculation. Such a solution is called a singular integral for the obvious reason (cf., [38]).

The major problems with the DGB method in practice are its poor complexity and expression
swell. However, on systems where the process can be completed within reasonable limits, by which
is meant that the length of the expressions obtained is small enough to be meaningful, the output is
extremely useful. Comparing the determining equations for classical symmetries and a triangulation
for that system illustrates this point; see (2.7) and (2.8) below. For nonlinear systems, DGBs have
been used effectively to solve the determining equations for nonclassical symmetries [37,39], using
various strategies which address the complexity problem and which minimize the number of singular
integral cases to be considered, i.e. which minimise the differential coefficients used in the pseudo-
reduction processes.

A much older method of finding a basis for the ideal of a system from which formal solutions may
be derived, due to Janet, has been implemented for linear systems [40,41]. Also for linear systems
(and linear differential-difference systems), the differential analogue of Buchberger’s algorithm [42]
for calculating an algebraic Gröbner Basis has been implemented [43]. For orthonomic systems,
those whose members are solvable for their leading derivative term, the Reid-Wittkopf Differential
Algebra package [44] will calculate the Standard Form of the system, the number of arbitrary
constants and functions a formal solution depends on (see also [40]), and the formal power series
solution to any order [35]. This package handles equations with nontrivial coefficients of the leading
derivative terms provided MAPLE can solve the expression (algebraically) for the leading term. One
can then systematically go through the singular integrals using the divpivs command.

The triangulations of the systems of determining equations for infinitesimals arising in the
classical and nonclassical methods in this article were all performed using the MAPLE package
diffrgroeb2 [45]. This package was written specifically to handle fully nonlinear equations. All
calculations are strictly “polynomial”, that is, there is no division. Implemented there are the
Kolchin-Ritt algorithm, the differential analogue of Buchberger’s algorithm using pseudo-reduction
instead of reduction, and extra algorithms needed to calculate a DGB (as far as possible using the
current theory), for those cases where the Kolchin-Ritt algorithm is not sufficient [36]. Designed to
be used interactively as well as algorithmically, the package has proved useful for solving some fully
nonlinear systems. As yet, however, algorithmic methods for finding the most efficient orderings,
the best method of choosing the sequence of pairs to be cross-differentiated, for deciding when to
integrate and read off coefficients of independent functions in one of the variables, for finding the
best change of coordinates, and so on, are still the subject of much investigation.

The nonclassical symmetry reductions obtained for (1.2) generate a wide variety of interesting
exact analytical solutions of the equations which we plot (using MAPLE) in the Figures. In §4 we
apply the Painlevé test due to Weiss et al [12] to (1.1), and then obtain another family of solutions
of (1.2) using the singular manifold method [12,46]. We then discuss the scattering problems for
(1.2) and (1.3) and show how the arbitrary functions in the solutions obtained would appear in a
solution of (1.2) obtained by the inverse scattering method. Finally in §5 we discuss our results.
2 Classical Symmetries.

To apply the classical method to the GSWW equation (1.1) we consider the one-parameter Lie group of infinitesimal transformations in \((x, t, u)\) given by

\[
\begin{align*}
\tilde{x} &= x + \varepsilon \xi(x, t, u) + O(\varepsilon^2), \\
\tilde{t} &= t + \varepsilon \tau(x, t, u) + O(\varepsilon^2), \\
\tilde{u} &= u + \varepsilon \phi(x, t, u) + O(\varepsilon^2),
\end{align*}
\]

where \(\varepsilon\) is the group parameter. Then one requires that this transformation leaves invariant the set

\[
\mathcal{S}_\Delta \equiv \{u(x, t) : \Delta = 0\},
\]

of solutions of (1.1). This yields an overdetermined, linear system of equations for the infinitesimals \(\xi(x, t, u)\), \(\tau(x, t, u)\) and \(\phi(x, t, u)\). The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

\[
v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u}.
\]

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equations

\[
\frac{dx}{\xi(x, t, u)} = \frac{dt}{\tau(x, t, u)} = \frac{du}{\phi(x, t, u)},
\]

which is equivalent to solving the invariant surface condition

\[
\psi \equiv \xi(x, t, u)u_x + \tau(x, t, u)u_t - \phi(x, t, u) = 0.
\]

The set \(\mathcal{S}_\Delta\) is invariant under the transformation (2.1) provided that

\[
\text{pr}^{(4)} v \big|_{\Delta=0} = 0,
\]

where \(\text{pr}^{(4)} v\) is the fourth prolongation of the vector field (2.3), which is given explicitly in terms of \(\xi, \tau\) and \(\phi\) (cf., [21]). This yields the following fourteen determining equations,

\[
\begin{align*}
\tau_u &= 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad \xi_t = 0, \\
\phi_{uu} &= 0, \quad \phi_{tu} = 0, \quad \phi_{xu} - \xi_{xx} = 0, \quad \phi_u + \xi_x = 0, \\
\beta \phi_t - \tau_t - \xi_x &= 0, \quad 2\beta \phi_x + \alpha \phi_{xx} - \beta \xi_{xx} = 0, \\
\beta \phi_{xx} + \phi_{xxx} - \phi_{xu} &= 0, \quad \alpha \phi_{xt} - 2\phi_x + \xi_{xx} = 0, \\
\phi_{xxxx} - \phi_{xx} - \phi_{xt} &= 0, \quad \alpha \phi_x + 3\phi_{xxu} - \xi_{xxx} - 2\xi_x = 0.
\end{align*}
\]

These equations were calculated using the MACSYMA package symmgrp.max [23].

A triangulation or standard form [34–37] (see also [40,41]) of the determining equations (2.7) for classical symmetries of the GSWW equation (1.1) is the following system of eight equations,

\[
\begin{align*}
\xi_u &= 0, \quad \xi_t = 0, \quad \xi_{xx} = 0, \quad \tau_u = 0, \quad \tau_x = 0, \\
\alpha \phi_x - 2\xi_x &= 0, \quad \beta \phi_t - \tau_t - \phi_x = 0, \quad \phi_u + \xi_x = 0.
\end{align*}
\]

from which we easily obtain the following infinitesimals,

\[
\xi = \kappa_1 x + \kappa_2, \quad \tau = g(t), \quad \phi = -\kappa_1 \left[ u - \frac{2x}{\alpha - \frac{t}{\beta}} \right] + \frac{g(t)}{\beta} + \kappa_3,
\]

where \(\kappa_1, \kappa_2, \kappa_3\) are constants.
where $g(t)$ is an arbitrary function and $\kappa_1, \kappa_2$ and $\kappa_3$ are arbitrary constants. The associated vector fields are:

$$\mathbf{v}_1 = x \frac{\partial}{\partial x} - \left( u - \frac{2x}{\alpha} - \frac{t}{\beta} \right) \frac{\partial}{\partial u}, \quad \mathbf{v}_2 = \frac{\partial}{\partial x}, \quad \mathbf{v}_3 = \frac{\partial}{\partial u}, \quad \mathbf{v}_4(g) = g(t) \left( \frac{\partial}{\partial t} + \frac{1}{\beta} \frac{\partial}{\partial u} \right).$$

We remark that $\mathbf{v}_4(g)$ shows that (1.1) is invariant under the following variable coefficient “Galilean transformation”

$$\tilde{x} = x, \quad \tilde{t} = g(t), \quad \tilde{u} = u + [g(t) - t]/\beta,$$

i.e., if $u(x, t)$ is a solution of (1.1), then so is $\tilde{u}(\tilde{x}, \tilde{t})$. Solving (2.4), or equivalently solving (2.5), we obtain two canonical symmetry reductions.

**Case 2.1** $\kappa_1 \neq 0$. In this case we set

$$\frac{1}{g(t)} = \frac{1}{f(t)} \frac{df}{dt}, \quad \kappa_1 = 1 \quad \text{and} \quad \kappa_2 = \kappa_3 = 0$$

and obtain the symmetry reduction

$$u(x, t) = f(t)w(z) + \frac{x}{\alpha} + \frac{t}{\beta}, \quad z = xf(t), \quad (2.11)$$

where $w(z)$ satisfies

$$z \frac{d^4w}{dz^4} + 4 \frac{d^3w}{dz^3} + (\alpha + \beta)z \frac{dw}{dz} \frac{d^2w}{dz^2} + \beta w \frac{d^2w}{dz^2} + 2\alpha \left( \frac{dw}{dz} \right)^2 = 0. \quad (2.12)$$

It is straightforward to show using the algorithm of Ablowitz et al [11] that this equation is of Painlevé-type only if either (i), $\alpha = \beta$ or (ii), $\alpha = 2\beta$; in the Appendix it is shown that in these two special cases (2.12) is solvable in terms of solutions of the third Painlevé equation [47]

$$\frac{d^2y}{dz^2} = \frac{1}{y} \left( \frac{dy}{dz} \right)^2 - \frac{1}{x} \frac{dy}{dx} + ay^3 + \frac{by^2 + c}{x} + \frac{d}{y}, \quad (2.13)$$

and the fifth Painlevé equation,

$$\frac{d^2y}{dz^2} = \left\{ \frac{1}{2y} + \frac{1}{y-1} \right\} \left( \frac{dy}{dz} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{(y-1)^2}{x^2} \left\{ ay + \frac{b}{y} \right\} + \frac{cy}{x} + \frac{dy(y+1)}{y-1}, \quad (2.14)$$

with $a, b, c$ and $d$ constants. Hence the Painlevé Conjecture [10,11] predicts that a necessary condition for (1.1) to be completely integrable is that $(\alpha - \beta)(\alpha - 2\beta) = 0$, i.e., only if (1.1) has one of the two special forms (1.2) or (1.3). We remark that the occurrence of the third and fifth Painlevé equations is slightly surprising since for the Boussinesq equation

$$u_{xx} + 3(u^2)_{xx} + u_{xxxx} = u_{tt}, \quad (2.15)$$

symmetry reductions reduce the equation to the first, second and fourth Painlevé equations [27].

**Case 2.2** $\kappa_1 = 0$. In this case we set

$$\frac{1}{g(t)} = \frac{df}{dt}, \quad \kappa_2 = 1 \quad \text{and} \quad \kappa_3 = -1/\beta$$

and obtain the symmetry reduction

$$u(x, t) = w(z) + t/\beta, \quad z = x - f(t), \quad (2.16)$$

where $w(z)$ satisfies

$$\frac{d^4w}{dz^4} + (\alpha + \beta) \frac{dw}{dz} \frac{d^2w}{dz^2} + \beta w \frac{d^2w}{dz^2} = 0. \quad (2.17)$$

Setting $W = dw/dz$ and integrating twice yields

$$\left( \frac{dW}{dz} \right)^2 + \frac{1}{3}(\alpha + \beta)W^3 = AW + B, \quad (2.18)$$

where $A$ and $B$ are constants of integration. This equation is equivalent to the Weierstrass elliptic function equation

$$\left( \frac{d\wp}{dz} \right)^2 = 4\wp^3 - g_2\wp - g_3, \quad (2.19)$$

where $g_2$ and $g_3$ are arbitrary constants [48].
3 Nonclassical Symmetries.

There have been several generalisations of the classical Lie group method for symmetry reductions. Bluman and Cole [24], in their study of symmetry reductions of the linear heat equation, proposed the so-called nonclassical method of group-invariant solutions. This method involves considerably more algebra and associated calculations than the classical Lie method. In fact, it has been suggested that for some partial differential equations, the calculation of these nonclassical reductions might be too difficult to do explicitly, especially if attempted manually since the associated determining equations are now an overdetermined, nonlinear system. For some equations such as the KDV equation (1.4), the nonclassical method does not yield any additional symmetry reductions to those obtained using the classical Lie method, while there are partial differential equations which do possess symmetry reductions not obtainable using the classical Lie group method. It should be emphasised that the associated vector fields arising from the nonclassical method do not form a vector space, still less a Lie algebra, since the invariant surface condition (2.5) depends upon the particular reduction.

In the nonclassical method one requires only the subset of $S_{\Delta}$ given by

$$S_{\Delta,\psi} = \{ u(x,t) : \Delta(u) = 0, \psi(u) = 0 \}, \quad (3.1)$$

where $S_{\Delta}$ is as defined in (2.2) and $\psi = 0$ is the invariant surface condition (2.5), is invariant under the transformation (2.1). The usual method of applying the nonclassical method (e.g., as described in [25]), to the GSW equation (1.1) involves applying the prolongation $pr^{(1)}v$ to the system of equations given by (1.1) and the invariant surface condition (2.5) and requiring that the resulting expressions vanish for $u \in S_{\Delta,\psi}$, i.e.,

$$pr^{(4)}v(\Delta)\big|_{\Delta=0,\psi=0} = 0, \quad pr^{(1)}v(\psi)\big|_{\Delta=0,\psi=0} = 0. \quad (3.2)$$

It is easily shown that

$$pr^{(1)}v(\psi) = - (\xi u_x + \tau u_t + \phi u) \psi,$$

which vanishes identically when $\psi = 0$ without imposing any conditions upon $\xi$, $\tau$ and $\phi$. However as shown in [39], this procedure for applying the nonclassical method can create difficulties, particularly when implemented in symbolic manipulation programs. These difficulties often arise for equations such as (1.1) which require the use of differential consequences of the invariant surface condition (2.5). In [39] we proposed an algorithm for calculating the determining equations associated with the nonclassical method which avoids many of the difficulties commonly encountered; we use that algorithm here.

In the canonical case when $\tau \neq 0$ we set $\tau = 1$. We omit the special case $\tau \equiv 0$; in that case one obtains a single condition for $\phi$ with 424 summands, and even considering the subcase $\phi_u = 0$ leads to an equation more complex than the one we are studying. Eliminating $u_t$, $u_{xt}$ and $u_{xxt}$, in (1.1) using the invariant surface condition (2.5) yields

$$\tilde{\Delta} \equiv \phi u_{xxxx} + 3\phi u_{uu} u_{xx} + 3\phi u_x u_{xxx} + \phi u_{uuu} u_x + 3\phi u_{uuu} u_x^2 + 3\phi u_{ux} u_{xx} + \phi_{xxx}$$

$$- \xi u_{xxx} u_x - 3\xi u_{xx} u_{xx} - 3\xi u_{xxx} u_x - \xi u_{uu} u_x^4 - 3\xi u_{uuu} u_x^3 - 6\xi u_{uu} u_x^2 u_{xx}$$

$$- 3\xi u_{uu} u_x^2 - 9\xi u_{u} u_{xx} u_x - \xi (4u_{x} u_{xxx} + 3u_{xx}^2) - \xi u_{xxxx}$$

$$+ (4u_{x} - 1) [\phi_x + \phi u_{xx} - \xi u_{xx} - \xi u_x^2 - \xi u_{xx}] + u_{xx} [\beta (\phi - \xi u_x) - 1] = 0, \quad (3.3)$$

with $t$ a parameter, which involves the infinitesimals $\xi$ and $\phi$ that are to be determined. Now we apply the classical Lie algorithm to this equation using the fourth prolongation $pr^{(4)}v$ and eliminate $u_{xxxx}$ using (3.3). This yields the following overdetermined, nonlinear system of equations for $\xi$
These equations were calculated using the MACSYMA package symgrp.max [23]. We then used the method of DGBS as outlined in [37,39] to solve this system.

**Case 3.1** $\alpha + \beta \neq 0$. In this case it is straightforward to obtain the condition

$$\xi_x \xi^2 (\alpha + \beta)(3\beta - 2\alpha) = 0.$$  

The case $3\beta - 2\alpha = 0$ leads to no solutions different from those obtained using the classical method. **Subcase 3.1.1** $\xi_x \neq 0$. This is the generic case which has the solution

$$\xi = [\kappa_1 x + \kappa_2] f(t), \quad \phi = -\kappa_1 f(t) \left[ u - \frac{2x}{\alpha} + \frac{\kappa_2 - t}{\beta} \right] + \frac{1}{\beta},$$

where $f(t)$ is an arbitrary function and $\kappa_1 \neq 0$ and $\kappa_2$ are arbitrary constants. These are equivalent to the infinitesimals (2.9) obtained using the classical method. **Subcase 3.1.2** $\xi_x = 0$. In this case it is easy to obtain the condition

$$\phi_x \xi^3 (3 - \beta)(\alpha + \beta) = 0.$$  

There are two subcases to consider.

(i) $\alpha \neq \beta, \xi_x = 0$. In this case the solution is

$$\xi = f(t), \quad \phi = \kappa_3 f(t) + 1/\beta,$$

where $f(t)$ is an arbitrary function and $\kappa_3$ is an arbitrary constant, which is equivalent to the infinitesimals (2.9) obtained using the classical method in the case when $\kappa_1 = 0$.

(ii) $\alpha = \beta, \xi_x = 0$. In this case, we obtain the following DGB for $\xi, \phi$

$$\xi_u = 0, \quad \phi_u = 0, \quad (\alpha + \beta)(\phi_u + \xi_x) = 0, \quad (3.4i)$$

$$\xi \phi_{tu} + 3\xi^2 \xi_{xx} + 3\xi \xi_x - 3\xi \xi_{xt} - 3\xi^2 \phi_{xx} - \xi_t \phi_u = 0, \quad (3.4ii)$$

$$\alpha \xi \phi_u^2 - \alpha \xi_t \phi_u + \alpha \xi \phi_{tu} - 2\beta \xi^2 \phi_{xx} - \alpha \xi^2 \phi_{xx} + \beta \xi \phi_{xx} - \alpha \xi \xi_x - \alpha \xi \xi_{xt} = 0, \quad (3.4iii)$$

$$3\xi \phi_{ux} \phi_{xx} + \beta \xi \phi_{xx} - \xi_t \phi_{xx} - 3\xi \xi_x \phi_{xx} - \xi \xi_{xx} + \alpha \phi^2_x + 3\xi \phi_{ux} \phi_x - \xi \xi_{xx} \phi_x$$

$$- 2\xi x \phi_x + \xi_t \phi_x + \xi \phi_{xx} \phi_{xx} - \xi \phi_{xx} \phi_{xx} - \xi \phi_{xx} = 0, \quad (3.4iv)$$

$$\phi_{xx} = 0 \text{ leads to no solutions different from those obtained using the classical method.}$$

Thus $\xi$ is an arbitrary function of $t$, and so we set $\xi(t) = df/dt$. It is easiest to integrate (3.7) first using the method of characteristics which yields

$$\phi = 2V(\xi \frac{df}{dt} + \frac{1}{\beta}), \quad \xi = x + f(t). \quad (3.8)$$
Equation (3.6) can be integrated twice with respect to $x$. This yields
\[ \frac{df}{dt} \phi_{xx} + \frac{1}{2} \beta \phi^2 - \left[ \frac{df}{dt} + 1 \right] \phi = x \lambda(t) + \mu(t), \] (3.9)
for some arbitrary functions $\lambda(t)$ and $\mu(t)$. Substituting (3.8) into (3.9) yields
\[ 2 \left( \frac{df}{dt} \right)^2 \left[ \frac{d^2V}{d\zeta^2} + \beta V^2 - V \right] = \lambda(t)[\zeta - f(t)] + \mu(t) + \frac{1}{\beta} \frac{df}{dt} + \frac{1}{2\beta} = 0. \]
We obtain an ordinary differential equation for $V(\zeta)$ if we set
\[ \lambda(t) = 2 \kappa_4 \left( \frac{df}{dt} \right)^2, \quad \mu(t) = 2 \kappa_4 f(t) \left( \frac{df}{dt} \right)^2 - \frac{1}{\beta} \frac{df}{dt} - \frac{1}{2\beta} + \kappa_5, \] where $\kappa_4$ and $\kappa_5$ are arbitrary constants, yielding
\[ \frac{d^2V}{d\zeta^2} + \beta V^2 - V = \kappa_4 \zeta + \kappa_5. \] (3.10)
This equation is equivalent to the first Painlevé equation [47]
\[ \frac{d^2y}{dx^2} = 6y^2 + x, \] (3.11)
if $\kappa_4 \neq 0$, otherwise it is equivalent to the Weierstrass elliptic function equation (2.19). Therefore we obtain the infinitesimals
\[ \xi = \frac{df}{dt}, \quad \phi = 2V(\zeta) \frac{df}{dt} + \frac{1}{\beta}, \] (3.12)
where $\zeta = x + f(t)$, $f(t)$ is an arbitrary function and $V(\zeta)$ satisfies (3.10).

Hence solving the characteristic equations (2.4) yields the nonclassical symmetry reduction
\[ u(x, t) = v(\zeta) + w(z) + t/\beta, \quad \zeta = x + f(t), \quad z = x - f(t), \] (3.13)
where $f(t)$ is an arbitrary function and $v(\zeta) = \int_{-\infty}^{\zeta} V(\zeta_1) d\zeta_1$ and $w(z)$ satisfy
\[ \frac{d^4v}{d\zeta^4} + 2\beta \frac{dv}{d\zeta} \frac{d^2v}{d\zeta^2} - \frac{d^2v}{d\zeta^2} = -\lambda, \] (3.14a)
and
\[ \frac{d^4w}{dz^4} + 2\beta \frac{dw}{dz} \frac{d^2w}{dz^2} - \frac{d^2w}{dz^2} = \lambda, \] (3.14b)
respectively, with $\lambda$ a “separation” constant. Integrating (3.14) and setting $V = dv/d\zeta$ and $W = dw/dz$, yields
\[ \frac{d^2V}{d\zeta^2} + \beta V^2 - V = -\lambda \zeta + \mu_1, \] (3.15a)
and
\[ \frac{d^2W}{dz^2} + \beta W^2 - W = \lambda z + \mu_2, \] (3.15b)
respectively, where $\mu_1$ and $\mu_2$ are arbitrary constants. If $\lambda \neq 0$ then these equations are equivalent to the first Painlevé equation (3.11), whilst if $\lambda = 0$ then they are equivalent to the Weierstrass elliptic function equation (2.19).
In particular, if $\lambda = 0$, then equations (3.15) possess the special solutions

$$V(\zeta) = \frac{6\kappa^2}{\beta}\sech^2(\kappa_1\zeta) + \frac{1 - (1 + 4\mu_1\beta)^{1/2}}{2\beta}, \quad W(z) = \frac{6\kappa^2}{\beta}\sech^2(\kappa_2z) + \frac{1 - (1 + 4\mu_2\beta)^{1/2}}{2\beta},$$

where $\kappa_1 = \frac{1}{2}(1 + 4\mu_1\beta)^{1/4}$ and $\kappa_2 = \frac{1}{2}(1 + 4\mu_2\beta)^{1/4}$. Hence we obtain the exact solution of (1.2) given by

$$u(x, t) = \frac{6\kappa_1}{\beta}\tanh\{\kappa_1[x + f(t)]\} + \frac{6\kappa_2}{\beta}\tanh\{\kappa_2[x - f(t)]\} + \frac{x(1 - 2\kappa^2_1 - 2\kappa^2_2)}{\beta} + \frac{2f(t)(\kappa^2_2 - \kappa^2_1)}{\beta} + \frac{t}{\beta}, \quad (3.16)$$

where $f(t)$ is an arbitrary function.

If $\mu_1 = \mu_2 = 0$ then $\kappa_1 = \kappa_2 = \frac{1}{2}$ and (3.16) simplifies to

$$u(x, t) = \frac{3}{\beta}\tanh\{\frac{1}{2}[x + f(t)]\} + \frac{3}{\beta}\tanh\{\frac{1}{2}[x - f(t)]\} + \frac{t}{\beta}. \quad (3.17)$$

In Figures 1 and 2 we plot $u_x$ with $u$ given by (3.17) for various choices of the arbitrary function $f(t)$. This is one of the simplest, nontrivial family of solutions of (1.1) with $\alpha = \beta$, using this reduction. In Figure 1, $f(t)$ is chosen so that $f(t) \sim t + t_0$, as $t \to \infty$, where $t_0$ is a constant. Consequently all the solutions plotted in Figure 1 have a similar asymptotic behaviour as $t \to \infty$. However the asymptotic behaviours as $t \to -\infty$ are radically different.

In the special case when $f(t) = ct$, then choosing $\kappa_1 = \frac{1}{2}(1 + 1/c)^{1/2}$ and $\kappa_2 = \frac{1}{2}(1 - 1/c)^{1/2}$ in (3.16) yields the two-soliton solution of (1.2) given by

$$u(x, t) = \frac{3}{\beta}\left\{\left(\frac{c + 1}{c}\right)^{1/2}\tanh\left(\left(\frac{c + 1}{4c}\right)^{1/2}(x + ct)\right) + \left(\frac{c - 1}{c}\right)^{1/2}\tanh\left(\left(\frac{c - 1}{4c}\right)^{1/2}(x - ct)\right)\right\}. \quad (3.18)$$

This solution is of special interest since such two-soliton solutions are normally associated with so-called Lie-Bäcklund transformations (cf., [49]) whereas (3.18) has arisen from a Lie point symmetry, albeit nonclassical. A plot of (3.18) for $c = 2$ is given in Figure 3a where it is compared to the so-called resonant two-soliton solution obtained using the singular manifold method in §4.1 below.

We remark that this “decoupling” of the nonclassical symmetry reduction solution (3.13) into a function of $\zeta = x + f(t)$ and a function of $z = x - f(t)$ occurs for the GSW equation (1.1) only in this special case when $\alpha = \beta$.

**Case 3.2** $\alpha + \beta = 0$. Substituting $\phi = u\theta(x, t) + \sigma(x, t)$ it is easy to find the condition

$$\theta_\theta_{xx} - \theta_x^2 = 0.$$

Thus either $\theta_x = 0$ or $\theta(x, t) = \exp\{x\lambda_1(t) + \lambda_2(t)\}$, where $\lambda_1(t)$ and $\lambda_2(t)$ are arbitrary functions. In fact, unless $\theta_x = 0$ there are no solutions. This can be shown by substituting into the equations the expressions $\theta_x = \theta\lambda_1$ and $\theta_t = (x\lambda_{1,t} + \lambda_{2,t})\theta$, to obtain, using the usual DGB techniques,

$$\xi_{xx} + \lambda_{1,t} - \xi_x\lambda_1 = 0.$$

This can be integrated; substituting into the equations the solution, along with $\theta = \exp\{x\lambda_1(t) + \lambda_2(t)\}$, and reading off coefficients in the independent functions, the exponentials in $x$, $2x$, and so on, yields various equations which lead to an inconsistency. Thus we need only consider the cases $\theta = 0$ and $\theta_x = 0$, i.e., $\phi_u = 0$ and $\phi_{xu} = 0$. 

Nonclassical symmetries and exact solutions of a shallow water wave equation
Subcase 3.2.1 \( \phi_u = 0 \). Substituting \( \phi_u = 0 \) into the determining equations we obtain the following DGB for the system:

\[
\begin{align*}
\xi_u &= 0, \\
\xi_x &= 0, \\
\phi_u &= 0, \\
\beta \phi_{xx} - \phi_{xx} - \beta \phi_x^2 - \xi \phi_{xxxx} + \xi \phi_{xx} &= 0, \\
\xi_t (1 - \beta \phi) + \beta \xi^2 \phi_x + \beta \xi \phi_t &= 0.
\end{align*}
\]

(3.19) (3.20) (3.21)

Thus \( \xi \) is an arbitrary function of \( t \) and so, as in Case 3.1.2(ii), we set \( \xi(t) = df/dt \). We integrate (3.21) using the method of characteristics to obtain

\[
\phi = \frac{df}{dt} \eta(\zeta) + \frac{1}{\beta}, \quad \zeta = x - f(t).
\]

(3.22)

Substituting (3.22) into (3.20) yields

\[
\beta \left[ \eta \frac{d^2 \eta}{d\zeta^2} - \left( \frac{d\eta}{d\zeta} \right)^2 \right] - \frac{d^4 \eta}{d\zeta^4} + \frac{d^2 \eta}{d\zeta^2} = 0,
\]

(3.23)

which is not of Painlevé type. Hence we obtain the infinitesimals

\[
\xi = \frac{df}{dt}, \quad \phi = \frac{df}{dt} \eta(\zeta) + \frac{1}{\beta},
\]

(3.24)

where \( \zeta = x - f(t) \) and \( \eta(\zeta) \) is a solution of (3.23). These yield the (classical) symmetry reduction (2.16) with \( z \equiv \zeta = x - f(t) \).

Subcase 3.2.2 \( \phi_{ux} = 0 \). In this case we obtain the solution

\[
\xi = [x + \kappa_6] f(t), \quad \phi = -f(t) [u + (2x + t + \kappa_7)/\beta] + 1/\beta,
\]

where \( f(t) \) is an arbitrary function and \( \kappa_6 \) an \( \kappa_7 \) are arbitrary constants. This is the same as the general case, with \( \alpha = -\beta \).

4 The integrability of the shallow water wave equation (1.1).

In this section, we give first the Painlevé analysis of (1.1), and use the singular manifold method to find another family of solutions similar, but not equivalent, to the family (3.16), for the special case \( \alpha = \beta \). We show that both (1.2) and (1.3) satisfy the Painlevé property and are solvable by the inverse scattering method, suggesting that the solutions found using the nonclassical and singular manifold methods do not arise because there exists some transformation that linearises the equation (1.2). In fact, it can be seen that the arbitrary function \( f(t) \) that occurs in the families of solutions found arises naturally during the inverse scattering method of solution.

4.1 The Painlevé Tests. We apply the Painlevé PDE test due to Weiss et al [12] to the GSWW equation (1.1). The Painlevé Conjecture (or Painlevé ODE test) as formulated by Ablowitz et al [10,11] asserts that every ordinary differential equation which arises as a symmetry reduction of a completely integrable nonlinear partial differential equation is of Painlevé type, though perhaps only after a transformation of variables. Ablowitz et al [11] and McLeod and Olver [7] have given proofs of the Painlevé ODE test under certain restrictions.

Subsequently, Weiss et al [12] proposed the Painlevé PDE test as a method of applying the Painlevé ODE test directly to a given partial differential equation without having to consider symmetry reductions (which might not exist). As for the Painlevé ODE test, at present there is no rigorous proof of the Painlevé PDE test, though a partial proof can be inferred from the partial proof of
the Painlevé ODE test due to McLeod and Olver [7]. Despite being no means foolproof the Painlevé tests appear to provide a useful criterion for the identification of completely integrable partial differential equations. In addition to providing a valuable first test for whether a given partial differential equation is completely integrable, other important information can be obtained by use of Painlevé analysis such as Bäcklund transformations, Lax pairs, Hirota’s bilinear representation, special and rational solutions for completely integrable equations and special and rational solutions for nonintegrable equations (see, for example, [50,51] and the references therein).

To apply the Painlevé PDE test to the GSWW equation (1.1) we seek a solution in the form

\[ u(x, t) = \sum_{k=0}^{\infty} u_k(t) \phi^{k+p}(x, t), \quad \phi = x + \psi(t), \]

(4.1)

where \( \psi(t) \) is an arbitrary analytic function and \( u_k(t), k = 0, 1, 2, \ldots, \) analytic functions such that \( u_0 \neq 0, \) in the neighbourhood of an arbitrary, non-characteristic movable singularity manifold defined by \( \phi(x, t) = 0, \) and \( p \) is a constant to be determined. By leading order analysis we find that \( p = -1 \) and \( u_0(t) = 12/((\alpha + \beta)), \) provided that \( \alpha + \beta \neq 0. \) In the case when \( \alpha = -\beta \) it is routine to show that (1.1) is non-Painlevé. We now substitute (4.1) into (1.1) to obtain from the coefficient of \( \phi^{k-4}, \)

\[(k + 1)(k - 1)(k - 4)(k - 6) u_k = H_k \left( u_{k-1}, u_{k-2}, \ldots, u_0, \psi \right), \]

(4.2a)

where

\[ H_k = (k - 3)(k - 4) \left( 1 + \frac{d\psi}{dt} \right) u_{k-2} + (k - 4) \frac{du_{k-3}}{dt} - (k - 2)(k - 3)(k - 4) \frac{du_{k-1}}{dt} \]

\[ - \sum_{j=1}^{k-1} (k - j - 1)(j - 1) [(j - 2)\alpha + (k - j - 2)\beta] u_j u_{k-j} \frac{d\psi}{dt} \]

\[ - \sum_{j=1}^{k} (k - j - 1) [(j - 2)\alpha + (k - j - 2)\beta] u_{k-j} \frac{du_{j-1}}{dt} \]

(4.2b)

for \( k = 0, 1, 2, \ldots \) and where we define \( u_k = 0 \) for \( k < 0. \) This defines \( u_k \) unless \( k = 1, k = 4 \) or \( k = 6 \) which are the so-called resonances. At each positive resonance there is a compatibility condition which must be identically satisfied for the expansion (4.2) to be valid, i.e., we require that \( H_1 \equiv 0, H_4 \equiv 0 \) and \( H_6 \equiv 0 \) for (1.1) to have a solution of the form (4.2). The compatibility condition \( H_1 \equiv 0 \) is identically satisfied which implies that \( u_1(t) \) is arbitrary. Equations (4.2) with \( k = 2 \) and \( k = 3 \) yield

\[ u_2 = \frac{1}{(\alpha + \beta)} \left\{ \frac{d\psi}{dt} + 1 - \beta \frac{dv_1}{dt} \right\} \left( \frac{d\psi}{dt} \right)^{-1} \]

and

\[ u_3 = \frac{(\alpha - 2\beta)}{2(\alpha + \beta)^2} \left\{ \left( \beta \frac{dv_1}{dt} - 1 \right) \frac{d^2\psi}{dt^2} - \beta \frac{d\psi}{dt} \frac{d^3v_1}{dt^3} \right\} \left( \frac{d\psi}{dt} \right)^{-3} \]

respectively. The compatibility condition \( H_4 \equiv 0 \) yields

\[ \frac{12(\alpha - \beta)(\alpha - 2\beta)}{(\alpha + \beta)^3} \left\{ \left( \beta \frac{dv_1}{dt} - 1 \right) \left[ \frac{d^3\psi}{dt^3} - 3 \left( \frac{d^2\psi}{dt^2} \right)^2 \right] + 2\beta \frac{d^2v_1}{dt^2} \frac{d^2\psi}{dt^2} - \beta \frac{d^3v_1}{dt^3} \left( \frac{d\psi}{dt} \right)^2 \right\} = 0. \]

Since \( \psi \) is an arbitrary function this implies that

\[(\alpha - \beta)(\alpha - 2\beta) = 0, \]

(4.3)
is a necessary condition for (1.1) to have a solution of the form (4.2). The compatibility condition $H_6 \equiv 0$ is also satisfied if and only if (4.3) is satisfied. Therefore we conclude that (1.1) has a solution in the form (4.1) if either (i), $\alpha = \beta$ or (ii), $\alpha = 2\beta$. These are the same conditions for (2.12), which arises in the classical reduction, to be of Painlevé-type and for the GSWV equation (1.1) to expressible in Hirota’s bilinear form. If $(\alpha - \beta)(\alpha - 2\beta) \neq 0$, then it is necessary to introduce a $v_4(t)\phi^3(x,t)$ term, where $v_4(t)$ is to be determined, into the expansion (4.1) and at higher orders of $\phi(x,t)$, higher and higher powers of $\ln \phi(x,t)$ are required; a strong indication of non-Painlevé behaviour. Hence the Painlevé PDE test suggests that $\alpha = \beta$ and $\alpha = 2\beta$ are the only integrable cases of the GSWV equation (1.1).

Exact solutions of the SWWI equation (1.2) can be obtained using the so-called singularity manifold method which uses truncated Painlevé expansions [12,46]. If we seek a solution of (1.2) in the form

$$u(x,t) = \frac{6}{\beta} \frac{\phi_x(x,t)}{\phi(x,t)},$$  

and then equate coefficients of powers of $\phi$ to zero, we find that $\phi(x,t)$ satisfies the overdetermined system

$$\begin{align*}
\phi_{xxxx} - \phi_{xx} - \phi_{xt} &= 0, \\
\phi_t \phi_{xx} - 3\phi_{xt} \phi_{xx} - \phi_x^2 + \phi_x (3\phi_{xxx} - \phi_t) &= 0.
\end{align*}$$

(A DGB analysis of this system leads to some very complex expressions. Although it does yield some ordinary differential equations in $x$ for $\phi$ in the various subcases they appear difficult to solve.)

Now suppose we seek a solution of these equations in the form

$$\phi(x,t) = a_1 \exp \{\kappa_1 x + \mu_1 t\} + a_2 \exp \{\kappa_2 x + \mu_2 t\} + a_0,$$

where $a_0, a_1, a_2, \kappa_1, \kappa_2, \mu_1$ and $\mu_2$ are constants. It is straightforward to show that equations (4.5) have a solution of the form (4.6) provided that $\mu_1 = \kappa_1/(\kappa_1^2 - 1), \mu_2 = \kappa_2/(\kappa_2^2 - 1)$ and $\kappa_1$ and $\kappa_2$ satisfy the constraint

$$\kappa_1^2 - \kappa_1 \kappa_2 + \kappa_2^2 = 3.$$  

Thus we obtain the following exact solution of the SWWI equation (1.2) given by

$$u(x,t) = \frac{6}{\beta} \left[\frac{a_1 \kappa_1 \exp \{\kappa_1 x + \frac{\kappa_1 t}{\kappa_1^2 - 1}\} + a_2 \kappa_2 \exp \{\kappa_2 x + \frac{\kappa_2 t}{\kappa_2^2 - 1}\}}{a_1 \exp \{\kappa_1 x + \frac{\kappa_1 t}{\kappa_1^2 - 1}\} + a_2 \exp \{\kappa_2 x + \frac{\kappa_2 t}{\kappa_2^2 - 1}\}} + a_0\right],$$

provided $\kappa_1$ and $\kappa_2$ satisfy (4.7).

It should be noted that (4.8) and (3.18) are fundamentally different solutions of the SWWI equation (1.2) as we shall now demonstrate. The general two-soliton solution of (1.2) is given by

$$u(x,t) = \frac{6}{\beta} \frac{1 + \kappa_1 \exp(\eta_1) + \kappa_2 \exp(\eta_2) + A_{12}(\kappa_1 + \kappa_2) \exp(\eta_1 + \eta_2)}{1 + \exp(\eta_1) + \exp(\eta_2) + A_{12} \exp(\eta_1 + \eta_2)},$$

where

$$\eta_j = \kappa_j x + \frac{\kappa_j t}{\kappa_j^2 - 1} + \delta_j, \quad j = 1,2, \quad A_{12} = \frac{(\kappa_1 - \kappa_2)^2(\kappa_1^2 - \kappa_1 \kappa_2 + \kappa_2^2 - 3)}{(\kappa_1 + \kappa_2)^2(\kappa_1^2 + \kappa_1 \kappa_2 + \kappa_2^2 - 3)}.$$
Figure 3a). On the other hand, (3.18) is the special case of (4.9) with $A_{12} = 1$ where two solitons pass through each other with no phase shift as a consequence of the interaction (see Figure 3b). Thus whereas both solutions are asymptotically equivalent as $t \to \infty$, they are qualitatively very different as $t \to -\infty$. This shows that nonclassical method and the singular manifold method do not, in general, yield the same solution set.

4.2 Inverse Scattering. The inverse scattering method, originally developed by Gardner et al [4] in order to solve the KDV equation (1.4), has led to the solution of numerous physically significant nonlinear evolution equations, such as the nonlinear Schrödinger, Sine-Gordon, Modified KDV and Boussinesq equations (cf. [54]). Nonlinear evolution equations solvable by inverse scattering are known to possess a number of remarkable properties which appear to characterise the equations, including: the existence of multi-soliton solutions, an infinite number of symmetries and conservation laws, Bäcklund transformations, a Lax pair, a bi-Hamiltonian representation, a prolongation structure, the Hirota bilinear representation, and the Painlevé property (cf. [54]). However, the precise relationship between these properties has yet to be rigorously established.

There are two special cases of the GSWW equation (1.1) which have been studied from the inverse scattering point of view, namely the SWWI equation (1.2) and the SWWII equation (1.3), or equivalently (1.2*) and (1.3*), respectively. Hirota and Satsuma [2] studied both (1.2*) and (1.3*) using Hirota’s bilinear technique [9]. Equation (1.3*) is known to be solvable by inverse scattering [3]. Several of the aforementioned properties of completely integrable equations have been derived for (1.2,1.3) [52,53,55–61].

The scattering problem for the SWWII equation (1.3) is the second order problem [3]

$$
\psi_{xx} + \frac{1}{2} \beta u_x \psi = \lambda \psi,
$$

(4.10)

with associated time-dependence

$$(4\lambda - 1) \psi_t = (1 - \beta u_t) \psi_x + \frac{1}{2} \beta u_{xt} \psi,$$

(4.11)

where $\lambda$ is the constant eigenvalue, and $\psi_{xxt} = \psi_{txx}$ if and only if $u$ satisfies (1.3) We note that (4.10) is the time-independent Schrödinger equation which is also the scattering problem for the KDV equation (1.4) [4]. In contrast, the scattering problem for the SWWI equation (1.2) is the third order problem [55,59]

$$
\psi_{xxx} + \left(\frac{1}{2} \beta u_x - 1\right) \psi_x = \lambda \psi,
$$

(4.12)

with associated time-dependence

$$
3\lambda \psi_t = (1 - \beta u_t) \psi_{xx} + \beta u_{xt} \psi_x.
$$

(4.13)

We remark that (4.12) is similar to the scattering problem

$$
\psi_{xxx} + \frac{1}{4}(1 + 6u) \psi_x + \frac{3}{2} \left[ u_x - i\sqrt{3} \partial_x^{-1}(v_t) \right] \psi = \lambda \psi
$$

(4.14)

which is the scattering problem for the Boussinesq equation

$$
u_{xxxx} + 3(u^2)_{xx} + u_{xx} = u_{tt},
$$

(4.15)

and which has been comprehensively studied by Deift et al [62].

Only the derivative $u_x$ appears in the scattering problem (4.12) and so an arbitrary function of $t$ may be added to $u$ without affecting this. This function can be fixed by the requirement that $u_t(x, t) \to 0$ as $x \to \infty$; the scattering problem (4.12) is solvable for $u$ such that $u_t(x, t) \to 0$ sufficiently rapidly as $x \to \infty$. Moreover, the associated time-dependence, (4.13), is invariant under the variable-coefficient Galilean transformation (2.10). Furthermore, note that one can integrate the SWWI equation (1.2) once with respect to $x$ and so introduce an arbitrary function of $t$. 

5 Discussion.

In this paper we have discussed the shallow water equation (1.1). In particular, for the special case of (1.1) given by the SWI equation (1.2), using the nonclassical symmetry reduction method originally proposed by Bluman and Cole [24], we obtained a family of solutions (3.16) which have a rich variety of qualitative behaviours. This is due to the freedom in the choice of the arbitrary function \( f(t) \). One can choose \( f_1(t) \) and \( f_2(t) \) such that \( |f_1(t) - f_2(t)| \) is exponentially small as \( t \to \infty \), yet \( f_1(t) \) and \( f_2(t) \) are quite different as \( t \to -\infty \), so that as \( t \to \infty \) the two solutions are essentially the same, yet as \( t \to -\infty \) they are radically different. In Figure 1 we show that by a judicious choice of \( f(t) \) we can exhibit a plethora of different solutions.

We believe that these results suggest that solving the SWI equation (1.2) numerically for initial conditions such as those in the solutions plotted in Figure 1 could pose some fundamental difficulties. An exponentially small change in the initial data yields a fundamentally different solution as \( t \to -\infty \). How can any numerical scheme in current use cope with such behaviour? Recently Ablowitz et al [65] have shown that the focusing nonlinear Schrödinger equation

\[
iu_t + u_{xx} + |u|^2u = 0, \tag{5.1}
\]

exhibits numerical chaos created by small errors on the order of roundoff. The results of Ablowitz et al together with those given in this paper suggest that numerical analysts need to take care to ensure the accuracy of their programs.

The solution (3.16) appears to be a nonlinear superposition of solutions suggesting that the SWI equation (1.2) may be linearisable through a transformation to a linear partial differential equation, analogous to the linearisation of Burgers’ equation

\[
u_t = u_{xx} + 2uu_x, \tag{5.2}
\]

which is mapped to the linear heat equation through the Cole-Hopf transformation [63,64]. If so then the solution (3.16) could be viewed as an artefact of the fact that the SWI equation (1.2) is linearisable. However as illustrated in §4, the SWI equation (1.2) can be expressed as the compatibility condition of a third order spectral problem. Further the associated scattering problem (4.12) is very similar to that for the Boussinesq equation which has been thoroughly studied by Deift et al [62]. This strongly suggests the SWI equation (1.2) is solvable by inverse scattering. Additionally, as mentioned in §4, the spatial part of the inverse scattering formalism (4.12) only defines \( u \) up to an arbitrary additive function of \( t \); this arbitrary function may be incorporated into \( u \) using the variable-coefficient Galilean transformation (2.10).

Since the generalised shallow water equation (1.1) is invariant under the variable-coefficient Galilean transformation (2.10) for all \( \alpha \) and \( \beta \), one can take any solution of the equation and using (2.10) generate some interesting solutions.

Fujioka and Espinosa [69] have discussed symmetry reductions of (1.2*) using the classical Lie method and direct method due to Clarkson and Kruskal [27]. They claim that the classical method yields no symmetry reductions and that the direct method yields symmetry reductions that are a subset of those we obtained in §2 using the classical method. When we applied the nonclassical method to (1.2) in §3, we found that \( \xi_u = 0 \), consequently the results of Olver [66] (see also [67,68]) show that the direct and nonclassical methods yield the same reductions. The difficulty Fujioka and Espinosa [69] appear to have experienced is with the nonlocal term in (1.2*); considering (1.2) rather than (1.2*) seems to be simpler.
Appendix 1. The solution of (2.12)

In this appendix we show how equation (2.12), in the special case when $\alpha = \beta$, i.e.,

$$z \frac{d^4 w}{dz^4} + 4 \frac{d^3 w}{dz^3} + 2\beta z \frac{dw}{dz} \frac{d^2 w}{dz^2} + \beta w \frac{d^2 w}{dz^2} + 2\beta \left( \frac{dw}{dz} \right)^2 = 0,$$

(A.1)

can be solved in terms of the third Painlevé equation (PIII) [47]

$$\frac{d^2 y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + ay^3 + \frac{by^2 + c}{x} + \frac{d}{y},$$

(A.2)

with $a$, $b$, $c$ and $d$ constants, and also in terms of the fifth Painlevé equation (PV)

$$\frac{d^2 y}{dx^2} = \left\{ \frac{1}{2y} + \frac{1}{y - 1} \right\} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{(y - 1)^2}{x^2} \left\{ ay + \frac{b}{y} \right\} + \frac{cy}{x} + \frac{dy(y + 1)}{y - 1},$$

(A.3)

with $a$, $b$, $c$ and $d$ constants. We also demonstrate how in the special case when $\alpha = 2\beta$, i.e.,

$$z \frac{d^4 w}{dz^4} + 4 \frac{d^3 w}{dz^3} + 3\beta z \frac{dw}{dz} \frac{d^2 w}{dz^2} + \beta w \frac{d^2 w}{dz^2} + 4\beta \left( \frac{dw}{dz} \right)^2 = 0,$$

(A.4)

(2.12) can be solved in terms of PIII.

Cosgrove and Scoufis [70] consider the the equation

\[ X^2 \left( \frac{d^2 Y}{dX^2} \right)^2 = -4 \left( \frac{dY}{dX} \right)^2 \left( X \frac{dY}{dX} - Y \right) + A_1 \left( X \frac{dY}{dX} - Y \right)^2 + A_2 \left( X \frac{dY}{dX} - Y \right) + A_3 \frac{dY}{dX} + A_4, \]

(A.5)

where $A_1$, $A_2$, $A_3$ and $A_4$ are constants (their equation SD-I.b (5.5)). In the general case when $A_1$ and $A_2$ are not both zero, Cosgrove and Scoufis show that (A.5) is solvable in terms of PV (A.3) though the transformation

\[ Y(X) = \frac{1}{4y} \left( \frac{x}{y - 1} \frac{dy}{dx} - y \right)^2 - \frac{1}{4} (1 - \sqrt{2a})^2 (y - 1) - \frac{1}{2} b (y - 1)^2 \]

\[ + \frac{1}{4} cx \frac{y + 1}{y - 1} + \frac{1}{2} d \frac{x^2 y}{(y - 1)^2}, \]

(A.6a)

\[ X = x, \]

(A.6b)

where

\[ A_1 = -2d, \quad A_2 = \frac{1}{4} c^2 + 2bd - d(1 - \sqrt{2a})^2, \]

\[ A_3 = bc + \frac{1}{2} c(1 - \sqrt{2a})^2, \quad A_4 = \frac{1}{16} c^2 \left[ (1 - \sqrt{2a})^2 - 2b \right] - \frac{1}{8} d \left[ (1 - \sqrt{2a})^2 + 2b \right]^2. \]

(A.7a)

In the case when $A_1 = 0$ and $A_2$ is unrestricted, Cosgrove and Scoufis show that (A.6) is solvable in terms of PIII (A.2) through the transformation

\[ Y(X) = \frac{1}{16y^2} \left( x \frac{dy}{dx} - y \right)^2 - \frac{1}{16} ax^2 y^2 - \frac{1}{8} (b + 2\sqrt{a}) xy + \frac{cx}{8y} + \frac{dx^2}{16y^2}, \quad X = x^2, \]

(A.8)
where

\[ A_2 = -\frac{1}{16} ad, \quad A_3 = \frac{1}{16} c (c + 2\sqrt{a}), \quad A_4 = \frac{1}{256} \left[ ac^2 - d (c + 2\sqrt{a})^2 \right]. \quad \text{(A.9)} \]

Therefore if \( A_1 = 0 \) and \( A_2 \neq 0 \) then (A.6) is solvable in terms of both PIII (A.2) and PV (A.3) since the special case of PV with \( d = 0 \) can always be solved in terms of solutions of PIII [71,72]; Cosgrove and Scoufis [70] remark that there are infinitely many other special cases of PV and (A.6) that are solvable in terms of solutions of PIII, e.g., if \( A_3 = 0 \) and \( A_2^2 = 4A_1A_4 \) in (A.6).

To illustrate how (A.1) and (A.4) are solvable in terms of PIII, we differentiate (A.5) with respect to \( X \) to yield

\[ X \frac{d^3 Y}{dX^3} + \frac{d^2 Y}{dX^2} = -6 \left( \frac{dY}{dX} \right)^2 + 4 \frac{dY}{X} + A_1 \left( X \frac{dY}{dX} - Y \right) + \frac{1}{2} A_2 + \frac{A_3}{2X}. \quad \text{(A.10)} \]

Integrating (A.1) once yields

\[ \frac{d^3 w}{dz^3} + 3 \frac{d^2 w}{dz^2} + 2\beta z \left( \frac{dw}{dz} \right)^2 + \beta w \frac{dw}{dz} = B_1, \quad \text{(A.11)} \]

with \( B_1 \) an arbitrary constant. Now making the transformation

\[ w(z) = \frac{Y(X)}{z} - \frac{1}{4\beta z}, \quad X = z^{3/2}, \]

and setting \( \beta = 9 \) (without loss of generality) yields (A.10) with \( A_1 = 0, A_2 = -16B_1/27 \) and \( A_3 = 0 \). Therefore (A.11) is solvable in terms of PV with

\[ a = \frac{a}{2} \left[ 1 - \frac{3}{4} \left( \frac{-3A_1}{B_1} \right)^{1/2} \right]^2, \quad b = -\frac{27A_4}{32B_1}, \quad c = \left( \frac{-64B_1}{27} \right)^{1/2}, \quad d = 0, \]

and in terms of PIII with either (i) \( a \) and \( b \) arbitrary, \( c = 0 \) and \( d = 256B_1/(27a) \), or (ii) \( a \) and \( c \) arbitrary, \( b = -2\sqrt{a} \) and \( d = 256B_1/(27a) \).

Analogously integrating (A.4) once yields

\[ z^2 \frac{d^3 w}{dz^3} + 2z \frac{d^2 w}{dz^2} - 2 \frac{dw}{dz} + \frac{3}{2} \beta \left( \frac{dw}{dz} \right)^2 + \beta \left( zw \frac{dw}{dz} - \frac{1}{2} w^2 \right) = B_2, \quad \text{(A.12)} \]

with \( B_2 \) an arbitrary constant. Then making the transformation

\[ w(z) = \frac{Y(X)}{z} - \frac{1}{2\beta z}, \quad X = z^2, \]

and setting \( \beta = 8 \) yields (A.10) with \( A_1 = 0, A_2 = 0 \) and \( A_3 = -\frac{1}{4}B_2 \). Thus (A.12) is solvable in terms of PIII with either (i) \( c \) and \( d \) arbitrary, \( a = 0 \) and \( b = -4B_2/c \), or (ii) \( a \) and \( b \) arbitrary, \( c = -4B_2/(b + 2\sqrt{a}) \) and \( d = 0 \).

We remark that (A.5), or equations that are equivalent to (A.5) through a Lie point transformation, appear in the work of Bureau [73,74], Chazy [75], Cosgrove [76,77], Jimbo [78] and Jimbo and Miwa [79] (see [70] for further details).

It is well known that PIII (A.2) possesses rational solutions and one-parameter families of solutions expressible in terms of Bessel functions (cf., [71,80–84]) and Bäcklund transformations which map solutions of PIII into new solutions for PIII but for different values of the parameters (cf., [71,72,80–85]). Starting with these known rational and one-parameter family solutions, hierarchies of solutions of PIII can be generated by means of the above Bäcklund transformations (cf., [86]). Analogously, it is well known that PV (A.3) possesses rational solutions and one-parameter
families of solutions expressible in terms of Whittaker functions (cf., [71.80–82,87,88]) and Bäcklund transformations which map solutions of the equation into new solutions with different values of the parameters (cf., [71,80,81,83,88–90]). Using these special exact solutions of PIII (A.2) and PV (A.3), one can construct exact solutions of (1.2) and (1.3), though we do pursue this further here.

Acknowledgements

It is a pleasure to thank Mark Ablowitz and Jim Curry for several illuminating discussions. We also thank the Program in Applied Mathematics, University of Colorado at Boulder and the Department of Mathematics and Statistics, University of Pittsburgh for their hospitality during our visits and Chris Cosgrove and Willy Hereman for expert suggestions and comments. The support of SERC (grant GR/H39420) is gratefully acknowledged. PAC is also grateful for support through a Nuffield Foundation Science Fellowship and NATO grant CRG 910729.

References

[1] Whitham G B 1974, “Linear and Nonlinear Waves” Wiley, New York.
[2] Hirota R and Satsuma J 1976, J. Phys. Soc. Japan 40 611–612
[3] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974, Stud. Appl. Math. 53 249–315
[4] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967, Phys. Rev. Lett 19 1095–1097
[5] Peregrine H 1966, J. Fluid Mech. 25 321–330
[6] Benjamin T B, Bona J L and Mahoney J 1972, Phil. Trans. R. Soc. Lond. Ser. A 272 47–78
[7] McLeod J B and Olver P J 1983, SIAM J. Math. Anal. 14 488–506
[8] Hietarinta J 1990, in “Partially Integrable Evolution Equations in Physics” [Eds. R. Conte and N. Boccara] NATO ASI Series C: Mathematical and Physical Sciences, 310, Kluwer, Dordrecht pp459–478
[9] Hirota R 1980, in “Solitons” [Eds. R.K. Bullough and P.J. Caudrey] Topics in Current Physics, 17, Springer-Verlag, Berlin pp157–176
[10] Ablowitz M J, Ramani A and Segur H 1978, Phys. Rev. Lett. 23 333–338
[11] Ablowitz M J, Ramani A and Segur H 1980, J. Math. Phys 21 715–721
[12] Weiss J, Tabor M and Carnevale G 1983, J. Math. Phys 24 522–526
[13] Boiti M, Leon J J-P, Manna M and Pempinelli F 1986, Inverse Problems 2 271–279
[14] Jimbo M and Miwa T 1983, Publ. R.I.M.S. 19 943–1001
[15] Dorizzi B, Grammaticos B, Ramani A and Winternitz P 1986, J. Math. Phys. 27 2848–2852
[16] Yajima N, Oikawa M and Satsuma J 1978, J. Phys. Soc. Japan 44 1711–1714
[17] Kako F and Yajima N 1980, J. Phys. Soc. Japan 49 2063–2071
[18] Bogoyavlenskii O I 1990, Math. USSR Izves. 34 245–259
[19] Bogoyavlenskii O I 1990, Russ. Math. Surv. 45 1–86
[20] Bluman G W and Kumei S 1989, “Symmetries and Differential Equations” Appl. Math. Sci., 81, Springer-Verlag, Berlin.
[21] Olver P J 1993, “Applications of Lie Groups to Differential Equations” 2nd Edition, Springer Verlag, New York.
[22] Hereman W 1993, Euromath Bull. 2 to appear
[23] Champagne B, Hereman W and Winternitz P 1991, Comp. Phys. Comm. 66 319–340
[24] Bluman G W and Cole J D 1969, J. Math. Mech. 18 1025–1042
[25] Levi D and Winternitz P 1989, J. Phys. A: Math. Gen. 22 2915–2924
[26] Vorob’ev E M 1991, Acta Appl. Math. 24 1–24
[27] Clarkson P A and Kruskal M D 1989, J. Math. Phys 30 2201–2213
[28] Clarkson P A 1993, Math. Comp. Model. 18 45–68
[29] Fushchich W I 1991, Ukrain. Mat. Zh. 43 1456–1470
[30] Galaktionov V A 1990, Diff. and Int. Equns. 3 863–874
[31] Galaktionov V A, Dorodnytzin V A, Elenin G G, Kurdjumov S P and Samarskii A A 1988, J. Sov. Math. 41 1222–1292
[32] Ames W F 1992, Appl. Num. Math. 10 235–259
[33] Shokin Yu I 1983, “The Method of Differential Approximation” Springer-Verlag, New York.
[34] Reid G J 1990, J. Phys. A: Math. Gen. 23 L853–L859
[35] Reid G J 1991, Europ. J. Appl. Math. 2 293–318
[36] Mansfield E and Fackerell E 1992, “Differential Gröbner Bases”, preprint 92/108, Macquarie University, Sydney, Australia
[37] Clarkson P A and Mansfield E L 1994, Physica D 70 250–288
[38] Zwilinger D 1992, “Handbook of Differential Equations” Second Edition, Academic, Boston.
[39] Mansfield E and Fackerell E 1994, SIAM J. Appl. Math. to appear
[40] Schwarz F 1992, Computing 49 95–115
[41] Topunov V L 1989, Acta Appl. Math. 16 191–206
[42] Buchberger B 1988, in “Mathematical Aspects of Scientific Software” [Ed. J. Rice] Springer Verlag pp59–87
[43] Pankrat’ev E V 1989, Acta Appl. Math. 16 167–189
[44] Reid G J and Wittkopf A 1993, “A Differential Algebra Package for maple”, ftp 137.82.36.21 login: anonymous, password: your email address, directory: pub/standardform
[45] Mansfield E 1993, “dolgrob2: A symbolic algebra package for analysing systems of PDE using Maple”, ftp 137.82.36.21 login: anonymous, password: your email address, directory: pub/maths/Maple, files: dolgrob2.src.tar.Z, dolgrob2.man.tex.Z
[46] Weiss J 1983, J. Math. Phys 24 1405–1413
[47] Ince E L 1956, “Ordinary Differential Equations” Dover, New York.
[48] Whittaker E E and Watson G M 1927, “Modern Analysis” 4th Edition, C.U.P., Cambridge.
[49] Anderson R L and Ibragimov N H 1979, “Lie-Bäcklund Transformations in Applications” SIAM, Philadelphia.
[50] Newell A C, Tabor M and Zeng Y B 1987, Physica 29D 1–68
[51] Weiss J 1990, in “Solitons in Physics, Mathematics and Nonlinear Optics” [Eds P.J. Olver and D.H. Sattinger] IMA Series, 25, Springer-Verlag, Berlin pp175–202
[52] Musette M, Lambert F and Decuyper J C 1987, J. Phys. A: Math. Gen. 20 6223–6235
[53] Hirota R and Ito M 1983, J. Phys. Soc. Japan 52 744–748
[54] Ablowitz M J and Clarkson P A 1991, “Solitons, Nonlinear Evolution Equations and Inverse Scattering” L.M.S. Lect. Notes Math., 149, C.U.P., Cambridge.
[55] Conte R and Musette M 1991, J. Math. Phys 32 1450–1457
[56] Hietarinta J 1987, J. Math. Phys 28 1732–1742
[57] Hu X B and Li Y 1983, J. Phys. A: Math. Gen. 24 1979–1986
[58] Matsuno Y 1990, J. Phys. Soc. Japan 59 3093–3100
[59] Musette M 1987, in “Painlevé Transcendents: Their Asymptotics and Physical Applications” [Eds. D. Levi and P. Winternitz] NATO ASI Series B: Physics, 278, Plenum, New York pp197–209
[60] Tagami Y 1989, Phys. Lett. 141A 116–120
[61] Weiss J 1985, J. Math. Phys 26 2174–2180
[62] Deift P, Tomei C and Trubowitz E 1982, Commun. Pure Appl. Math. 35 567–628
[63] Cole J D 1951, Quart. Appl. Math. 9 225–236
[64] Hopf E 1950, Commun. Pure Appl. Math. 3 201–250
[65] Ablowitz M J, Schober C and Herbst B M 1993, Phys. Rev. Lett. 71 2683–2686
[66] Olver P J 1993, “Direct reduction and differential constraints”, preprint, Department of Mathematics, University of Maryland, College Park, MD
[67] Arrigo D J, Broadbridge P and Hill J M 1993, *J. Math. Phys* **34** 4692–4703
[68] Pucci E 1992, *J. Phys. A: Math. Gen.* **25** 2631–2640
[69] Fujioka J and Espinosa A 1980, *J. Phys. Soc. Japan* **60** 4071–4075
[70] Cosgrove C M and Scoufis G 1993, *Stud. Appl. Math.* **88** 25–87
[71] Fokas A S and Ablowitz M J 1983, *J. Math. Phys* **23** 2033–2042
[72] Gromak V I 1975, *Diff. Eqns.* **11** 285–287
[73] Bureau F 1972, *Ann. Mat. Pura Appl. (IV)* **91** 163–281
[74] Bureau F, Garcet A and Goffar J 1972, *Ann. Mat. Pura Appl. (IV)* **92** 177–191
[75] Chazy J 1911, *Acta Math.* **34** 317–385
[76] Cosgrove C M 1977, *J. Phys. A: Math. Gen.* **10** 2093–2105
[77] Cosgrove C M 1978, *J. Phys. A: Math. Gen.* **11** 2405–2430
[78] Jimbo M 1982, *Publ. RIMS, Kyoto Univ.* **18** 1137–1161
[79] Jimbo M and Miwa T 1981, *Physica* **D2** 407–488
[80] Airault H 1979, *Stud. Appl. Math.* **61** 31–53
[81] Gromak V I 1978, *Diff. Eqns.* **14** 1510–1513
[82] Lukashevich N A 1965, *Diff. Eqns.* **1** 561–564
[83] Mugan U and Fokas A S 1992, *J. Math. Phys* **33** 2031–2045
[84] Okamoto K 1987, *Funkcial. Ekvac.* **30** 305–332
[85] Lukashevich N A 1967, *Diff. Eqns.* **3** 994–999
[86] Milne A E and Clarkson P A 1993, in “Applications of Analytic and Geometric Methods to Nonlinear Differential Equations” [Editor P.A. Clarkson] NATO ASI Series C: Mathematical and Physical Sciences, Kluwer, Dordrecht pp341–352
[87] Kitaev A V, Law C K and McLeod J B 1993, *J. Diff. Int. Eqns.* to appear
[88] Okamoto K 1987, *Jap. J. Math.* **13** 47–76
[89] Gromak V I 1976, *Diff. Eqns.* **12** 740–742
[90] Lukashevich N A 1968, *Diff. Eqns.* **4** 1413–1420
Figure captions

**Figure 1.** The solution (3.17) where

(i), \( f(t) = \frac{1}{2}t \),
(ii), \( f(t) = \frac{1}{2}t + \exp(-t/10) \),
(iii), \( f(t) = \frac{1}{2}t + [1 - \tanh t] \sin t \),
(iv), \( f(t) = \frac{1}{2}t + 2\text{Ai}(2t) \),
(v), \( f(t) = \frac{1}{2}t + 2\exp(-t^2/20) \sin t \),
(vi), \( f(t) = \frac{1}{2}t + \exp(-t^2/100) \sin t \),
(vii), \( f(t) = \frac{1}{2}t + 2\pi^{-1} \tan^{-1} t \),
(viii), \( f(t) = \frac{1}{2}(t + 1/t) \),
(ix), \( f(t) = \frac{1}{4}t(1 + \tanh t) \),

where \( \text{Ai}(z) \) is the Airy function which is the solution of \( \text{Ai}''(z) - z\text{Ai}(z) = 0 \), satisfying \( \text{Ai}(z) \sim \frac{1}{2}\pi^{-1/2}z^{-1/4}\exp\left(-\frac{2}{3}z^{3/2}\right) \) as \( z \to \infty \) and \( \text{Ai}(z) \sim \pi^{-1/2}|z|^{-1/4}\cos\left(\frac{2}{3}|z|^{3/2} + \frac{1}{4}\pi\right) \) as \( z \to -\infty \).

**Figure 2** “Breather” solutions. The solution (3.17) where

(i), \( f(t) = 2 \sin t + 10 \),
(ii), \( f(t) = 2 \sin t + 1 \),
(iii), \( f(t) = 5 \sin t + 3 \),
(iv), \( f(t) = 5 \sin t + 1 \).

**Figure 3.** (a) The solution (3.18) with \( c = 2 \) and (b), the solution (4.8) with \( \kappa = 1.2 \) and \( a_0 = a_1 = a_2 = 1 \).