FROM A REEB ORBIT TRAP TO A HAMILTONIAN PLUG

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Abstract. We present a simple construction of a plug for Hamiltonian flows on hypersurfaces of dimension at least five by doubling a trap for Reeb orbits.

1. Introduction

In [17], H. Seifert remarked that ‘it is unknown if every continuous [nowhere vanishing] vector field of the three-dimensional sphere $S^3$ contains a closed integral curve.’ The supposition that the answer to this question is positive has become known as the Seifert conjecture. In its original form, this conjecture was disproved by P. A. Schweitzer [16]. Earlier, F. W. Wilson [18] had shown the existence of aperiodic flows on any compact manifold of dimension at least four and of vanishing Euler characteristic (which is the condition for the existence of a non-singular vector field). His result is based on the construction of what, following J. Harrison [7] and K. Kuperberg [12], is now known as a plug. This is a local model of an aperiodic flow that can be inserted into a flow box around an isolated periodic orbit of a given flow in order to open up that orbit. Needless to say, care has to be taken not to create new periodic orbits by this process. For a beautiful survey on constructions of aperiodic flows see [13].

In this note, we are concerned with the Hamiltonian version of the Seifert conjecture: does the Hamiltonian flow on a closed hypersurface in a symplectic manifold necessarily have a periodic orbit? In this generality, the conjecture has been disproved for hypersurfaces of dimension at least five by V. Ginzburg [3, 4] and M. Herman [8, 9]; Ginzburg’s construction was simplified by E. Kerman [10]. In dimension three, the best counterexample to date is one where the Hamiltonian function defining the hypersurface is only $C^2$-smooth, and hence the Hamiltonian vector field only $C^1$-smooth [6]. By contrast, there are many positive results under more restrictive assumptions, such as the hypersurface being of contact type, or for dense subsets of levels of a Hamiltonian function. These matters are surveyed comprehensively in [5]; see also [9] and the introductions to [6, 10].

Here we describe a simple construction of a plug for smooth Hamiltonian flows on hypersurfaces of dimension at least five, starting from a trap for Reeb orbits invented in [2].

2. Plugs

For our purposes, a plug will be a certain smooth non-singular vector field $X$ on $D \times I$, where $D = D^{m-1}_\delta$ is an $(m - 1)$-dimensional disc of radius $\delta$, and $I$ an interval $[-\varepsilon, \varepsilon]$. Write $z$ for the $I$-coordinate. This vector field is supposed to have the following properties:

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(i) \( X = \partial_z \) near the boundary \( \partial(D \times I) \).
(ii) There is a trajectory of \( X \) that enters the plug at \( D \times \{-\varepsilon\} \) and is trapped, i.e. it never leaves the plug.
(iii) The flow of \( X \) on \( D \times I \) is aperiodic, i.e. it does not have any closed orbits.
(iv) Any orbit that traverses the plug enters and exits the plug at a pair of matching points \( (x, \pm \varepsilon) \).

Inside the plug, there will be an aperiodic invariant set that serves as a trap for at least one orbit; this orbit enters the plug and becomes asymptotic to the invariant set. By inserting the plug into a flow box around a point on an isolated periodic orbit of a given flow such that the trapped trajectory matches the entrance point of the periodic orbit, one destroys this periodic orbit without creating any new closed trajectories.

In general, \( D \) may be replaced by any \((m-1)\)-dimensional manifold with boundary such that \( D \times I \) embeds into \( \mathbb{R}^m \), with \( \{p\} \times I \) mapping to a line segment parallel to the \( x_m \)-direction for all \( p \in D \). For instance, in the case \( m = 3 \) one can take \( D \) to be any orientable surface with boundary.

There are two features of our plug that make it considerably simpler than those used by Ginzburg [3, 4] and Kerman [10]. First of all, we use an irrational flow on a torus as a trap, as in Wilson’s original construction, whereas Ginzburg worked with the horocycle flow on the unit tangent bundle of a hyperbolic surface, and Kerman with the dense aperiodic subset inside the geodesic flow on the unit tangent bundle of a torus.

Secondly, the main task for both Ginzburg and Kerman is to construct a symplectic embedding of the plug into a flow box of the original Hamiltonian flow. For Ginzburg’s plug, this requires a subtle application of Gromov’s \( h \)-principle, see also [1, pp. 118–120]. In our construction, the plug comes from a deformation of a contact hypersurface in standard symplectic space, so the symplectic embedding comes for free.

A flow on \( D \times I \) satisfying properties (i) to (iii) will be called a half-plug. Our construction in [2] yields a half-plug for Reeb flows. Here we show that by taking its mirror image under \( z \mapsto -z \) and reversing the flow direction, this is still a half-plug for Hamiltonian flows. When one half-plug is put on top of the other, the matching condition (iv) will be satisfied. This is reminiscent of Wilson’s construction.

3. The trap for Reeb orbits

A contact form on a \((2n-1)\)-dimensional manifold is a 1-form \( \alpha \) such that \( \alpha \wedge (d\alpha)^{n-1} \) is a volume form. An example is the standard contact form

\[
\alpha_{st} = dz + \frac{1}{2} \sum_{j=1}^{n-1} (x_j \, dy_j - y_j \, dx_j)
\]

on \( \mathbb{R}^{2n-1} \). The Reeb vector field of a contact form \( \alpha \) is the unique vector field \( R \) satisfying \( d\alpha(R, \cdot) \equiv 0 \) and \( \alpha(R) \equiv 1 \). The Reeb vector field of \( \alpha_{st} \) is \( \partial_z \).

For \( n \geq 3 \), in [2] we constructed a trap for Reeb orbits as the Reeb flow of a suitable contact form \( \alpha_{st}/H \), where \( H : \mathbb{R}^{2n-1} \to \mathbb{R}^+ \) is a smooth function that is identically 1 outside a compact set. This function may be chosen \( C^0 \)-close to 1. The map

\[
(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z) \mapsto (\lambda x_1, \lambda y_1, \ldots, \lambda x_{n-1}, \lambda y_{n-1}, \lambda^2 z),
\]

where \( \lambda \) is a large positive number.
with \( \lambda \in \mathbb{R}^+ \), pulls back \( \alpha_{st} \) to \( \lambda^2 \alpha_{st} \). This rescaling allows one to choose the support of \( H - 1 \) in an arbitrarily small neighbourhood of the origin.

We showed that a function \( H \) can be found such that the Reeb flow of \( \alpha_{st}/H \) remains aperiodic, and some Reeb orbits become asymptotic to an irrational flow on a Clifford \((n-1)\)-torus, and hence trapped. So this local model is a half-plug, but the matching condition (iv) is not satisfied. Indeed, as there are cases where the Weinstein conjecture for Reeb flows has been resolved positively, there can be no general plug construction for Reeb orbits.

4. Hamiltonian interpretation of the half-plug

Now let \((W,\omega)\) be a \(2n\)-dimensional symplectic manifold, i.e. \( \omega \) is a closed 2-form on \( W \) such that \( \omega^n \) is a volume form. Let \( K: W \to \mathbb{R} \) be a smooth function. Then the Hamiltonian vector field \( X_K \) corresponding to \( K \) is defined by

\[
\omega(X_K, \cdot) = -dK.
\]

If \( \Sigma = K^{-1}(c) \) is a regular level set of \( K \), then \( X_K \) is a tangent vector field along this smooth hypersurface. By replacing \( K \) by \( K - c \), we may always assume that \( \Sigma \) is the zero level set of \( K \). Given any other function \( K' \) with \( \Sigma \) as its regular zero level set, we have \( K' = fK \) with \( f \) some smooth nowhere zero function, and then \( X_{K'} = fX_K \). Hence, up to reparametrisation, the Hamiltonian flow is determined by \( \Sigma \) and \( \omega \) alone.

In fact, a little more is true. The 2-form \( \omega \) restricts to a 2-form \( \omega_\Sigma := \omega|_{T\Sigma} \) of maximal rank \( 2n - 2 \), and the flow lines of \( X_K \) are the characteristics of \( \omega_\Sigma \), i.e. trajectories tangent to the kernel of \( \omega_\Sigma \). By the symplectic neighbourhood theorem [15, p. 104], a neighbourhood of any oriented hypersurface \( \Sigma \) in \((W,\omega)\) is symplectomorphic to \((-\sigma,\sigma) \times \Sigma\) with symplectic form \( \omega_\Sigma + d(s\beta) \), where \( s \) is the coordinate in \((-\sigma,\sigma)\), and \( \beta \) is a 1-form on \( \Sigma \) that does not vanish in the characteristic direction. Then the vector field \( X \) in this characteristic direction (i.e. in the kernel of \( \omega_\Sigma \)) with \( \beta(X) = 1 \) is the Hamiltonian vector field corresponding to the function \( s \). In other words, the Hamiltonian flow is completely determined (up to reparametrisation) by \((\Sigma,\omega_\Sigma)\).

The symplectisation of a contact manifold \((M,\alpha)\) is the symplectic manifold \((\mathbb{R} \times M, d(e^f\alpha))\). The Hamiltonian vector field corresponding to the function \( e^f \) at the level \( e^f = 1 \) is then the Reeb vector field \( R \) of \( \alpha \). The rescaled contact form \( e^f\alpha \) on \( M \), where \( f \) is some smooth function on \( M \), can be obtained by pulling back the 1-form \( e^f\alpha \) under the embedding \( M \ni x \mapsto (f(x), x) \in \mathbb{R} \times M \). The Reeb vector field of \( e^f\alpha \), when interpreted as a vector field along that graph embedding, is the Hamiltonian vector field of the function \( e^{f-f} \) at the level 1.

After these preliminaries, we now want to interpret our Reeb trap in this Hamiltonian setting. Consider \( \mathbb{R}^{2n} \) with the standard symplectic form

\[
\omega_{st} = dw \wedge dz + \sum_{j=1}^{n-1} dx_j \wedge dy_j.
\]

The vector field

\[
Y = \frac{1}{2} (w \partial_w + z \partial_z) + \frac{1}{2} \sum_{j=1}^{n-1} (x_j \partial_{x_j} + y_j \partial_{y_j})
\]
is a Liouville vector field for $\omega_{st}$, i.e. $L_\gamma \omega_{st} = \omega_{st}$, and $\mathbb{R}^{2n-1} \equiv \{ w = 2 \}$ is transverse to $Y$ and hence a contact type hypersurface, on which $i_Y \omega_{st}$ restricts to $\alpha_{st}$.

The symplectisation $(\mathbb{R} \times \mathbb{R}^{2n-1}, d(e^t \alpha_{st}))$ of $(\mathbb{R}^{2n-1}, \alpha_{st})$ can be identified symplectically with $(\{ w > 0 \}, \omega_{st})$ by identifying $\{ 0 \} \times \mathbb{R}^{2n-1}$ with $\{ w = 2 \}$ and mapping the flow lines of $\partial_t$ to those of $Y$. Now, replacing $\alpha_{st}$ on $\mathbb{R}^{2n-1}$ by $\alpha_{st}/H$, as in the construction of our trap, amounts to replacing $\{ 0 \} \times \mathbb{R}^{2n-1}$ by the graph of $-\log H$ in the symplectisation, and the new Reeb flow corresponds to the Hamiltonian flow on this new embedding of $\mathbb{R}^{2n-1}$. This embedding is isotopic (under a compactly supported isotopy) and $C^0$-close to the original one; the analogous statement holds for the corresponding embeddings in $(\mathbb{R}^{2n}, \omega_{st})$.

Now suppose we have an isolated periodic orbit $\Gamma$ in the Hamiltonian flow of $X_K$ on $\Sigma = K^{-1}(c)$. Since $\omega_\Sigma$ is invariant under the flow of $X_K$, Darboux’s theorem allows us to choose a flow box $B = D^{2n-2} \times I$ around a point on $\Gamma$ such that $\omega_\Sigma$ is given by

$$\sum_{j=1}^{n-1} dx_j \wedge dy_j = d\alpha_{st}$$

on the flow box. The symplectic form in a neighbourhood $(-\sigma, \sigma) \times B$ is then given by

$$\omega_\Sigma + d(s \alpha_{st}) = ds \wedge \alpha_{st} + (1 + s) d\alpha_{st}.$$  

With the substitution $s = e^t - 1$, this becomes the symplectic form on the symplectisation of $(B, \alpha_{st})$.

By the construction of the Reeb trap, we can trap the periodic orbit $\Gamma$ by a deformation of $\{ 0 \} \times B$ inside $(-\sigma, \sigma) \times B$, supported in the interior of $B$.

5. A Hamiltonian Plug

In order to build a Hamiltonian plug, we also need to take care of the matching condition (iv). To this end, we think of two boxes $B_\pm$ sitting inside $B$. With $B = D^{2n-2}_0 \times [-\varepsilon, \varepsilon]$, we take

$$B_+ = D^{2n-2}_{\delta/2} \times [-3\varepsilon/4, -\varepsilon/4]$$

and

$$B_- = D^{2n-2}_{\delta/2} \times [\varepsilon/4, 3\varepsilon/4],$$

say. On $B_+$ we perform the previous construction, so we replace the linear flow in the $z$-direction by the flow of the Reeb vector field $R$ of $\alpha_{st}/H$, realised as a Hamiltonian flow by a deformation of $\{ 0 \} \times B_+$ inside $(-\sigma, \sigma) \times B_+$, supported in the interior of $B_+$.

Condition (iv) will be satisfied if the linear flow on $B_-$ is replaced by the flow of $-\Phi^* R$, where $\Phi(x, y, z) = (x, y, -z)$, i.e. the negative Reeb flow of the contact form $\Phi^* (\alpha_{st}/H)$. In other words, we simply reverse the Reeb flow in our local model, and turn the local model upside down. Notice that $\Phi^* d\alpha_{st} = d\alpha_{st}$, so, by the neighbourhood theorem for hypersurfaces, $\Phi$ extends to a symplectomorphism of a neighbourhood of $B_-$ to a neighbourhood of $B_+$ in $W$. This diffeomorphism, however, is not straightforward, since $\Phi^*$ does not pull back $\alpha_{st}$ to $-\alpha_{st}$. It is clear, however, that this extended symplectomorphism must reverse the coorientation of $B$ in $W$. 
The negative Reeb flow, too, can be interpreted as a Hamiltonian flow; in the symplectisation it would be the one corresponding to the Hamiltonian function $-e^t$ at the level 1. So the desired flow on $B_-$ can likewise be realised by a deformation of the hypersurface in the symplectic manifold.

Here, briefly, is an alternative look at this mirror construction. By the symplectic neighbourhood theorem we may write the symplectic form on a neighbourhood $(-\sigma, \sigma) \times B_+$ of $B_+$ in $W$ as $ds \wedge dz + d\alpha_{st}$; a perhaps smaller neighbourhood is symplectomorphic to a neighbourhood of $\{(0) \times B_+, \alpha_{st}\}$ in its symplectisation $(\mathbb{R} \times B_+, d(e^t \alpha_{st}))$. In this neighbourhood we perform the deformation to produce a half-plug.

The symplectic form on a neighbourhood $(-\sigma, \sigma) \times B_-$ of $B_-$ in $W$ can likewise be written as $ds \wedge dz + d\alpha_{st}$. This contains a smaller neighbourhood symplectomorphic to the symplectisation of $\{(0) \times B_-, \Phi^* \alpha_{st}\}$, where positive values of $t$ in the symplectisation correspond to negative values of $s$. The negative Reeb field of $\Phi^* \alpha_{st}$ is $\partial_z$, and after the ‘mirror’ deformation, the negative Reeb flow will be as desired.

6. APERIODIC HAMILTONIAN AND VOLUME-PRESERVING FLOWS

One can now, as in [4] and [10], construct smooth aperiodic Hamiltonian flows on hypersurfaces of dimension at least five by starting with a Hamiltonian flow having only isolated periodic orbits. Examples are ellipsoids $\{\sum_{j=1}^{n} a_j |z_j|^2 = 1\}$ in $\mathbb{C}^n$ with $a_1, \ldots, a_n$ positive rationally independent real numbers. Non-simply connected hypersurfaces with this property have been constructed by F. Laudenbach [14].

The flow of a Hamiltonian vector field $X_K$ on a $(2n-1)$-dimensional hypersurface $\Sigma = K^{-1}(c)$ preserves the volume form $\beta \wedge \omega_{\Sigma}^{n-1}$, where $\beta$ is a 1-form on $\Sigma$ with $\beta(X_K) = 1$. More generally, our plug can be inserted into a flow box of a volume-preserving flow on any manifold of dimension $2n-1 \geq 5$, with the volume form on the plug defined by the contact form. In even dimensions $2n \geq 6$, we take the product of our half-plugs in dimension $2n-1$ with an interval $[-\varepsilon, \varepsilon]$. On this interval we consider a smooth function $\psi \colon [-\varepsilon, \varepsilon] \to [0, 1]$, supported in $[-\varepsilon/2, \varepsilon/2]$ and with $\psi(u) = 1$ for $u$ near zero. Then, on the slice $B_+ \times \{u\}$, we take the Reeb flow of the contact form

$$\alpha_u := \alpha_{st} / (\psi(u) \cdot H + 1 - \psi(u)).$$

This flow preserves the volume form $\Omega$ on $B_+ \times [-\varepsilon, \varepsilon]$ given along the slice $B_+ \times \{u\}$ by $\alpha_u \wedge (d\alpha_u)^{n-1} \wedge du$. On $B_- \times [-\varepsilon, \varepsilon]$ we mirror this construction.

By [2] eqn. (H-iv), the Reeb vector field $R_u$ of $\alpha_u$ satisfies

$$dz(R_u) = \psi(u) H - \frac{\psi(u)}{2} \sum_{j=1}^{n-1} (x_j H_{x_j} + y_j H_{y_j}) + 1 - \psi(u),$$

and $H$ was chosen such that for $\psi(u) = 1$ this expression is zero on the Clifford torus and positive elsewhere. Thus, for $\psi(u) \in [0, 1)$, we have $dz(R_u) > 0$. This guarantees that we do not produce any new periodic orbits inside the half-plug.

Thus, again, starting from any volume-preserving flow in dimension at least five with isolated periodic orbits, one can produce a smooth aperiodic one. According to [5], Wilson’s plug can be chosen divergence-free in all dimensions $\geq 4$. On 3-manifolds, aperiodic volume-preserving flows of class $C^1$ have been constructed by G. Kuperberg [11].
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