5D Schwarzschild-like Spacetimes with Arbitrary Magnetic Field

by

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Abstract. We find a new class of exact solutions of the five-dimensional Einstein equations whose corresponding four-dimensional spacetime possesses a Schwarzschild-like behavior. The electromagnetic potential depends on a harmonic function and can be chosen to be of a monopole, dipole, etc. field. The solutions are asymptotically flat and for vanishing magnetic field the four metrics are of the Schwarzschild solution. The spacetime is singular in $r = 2m$ for higher multipole moments, but regular for monopoles or vanishing magnetic fields in this point. The scalar field possesses a singular behavior.

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In the last years there has been a great interest in the study of exact solutions of actions of type

\[ S = \int d^4x \sqrt{-g} \left[ -R + 2(\nabla \Phi)^2 + e^{-2\alpha \Phi} F^2 \right], \]  

(1)
because it reduces to the four-dimensional low energy Lagrangian for string theory for \( \alpha = 1 \), to the Einstein-Maxwell-Scalar theory for \( \alpha = 0 \) and it also reduces to five-dimensional gravity, for \( \alpha = \sqrt{3} \), after dimensional reduction. Some exact solutions of the field equations for charged bodies of this action are known [1]. It seems that the properties of electrically charged solutions depend on the value of \( \alpha \), but they are only different for the extrem case \( \alpha = 0 \) [2]. In this latter we want to show that magnetic fields do not alter the properties of the spacetime for static bodies, at least for \( \alpha = \sqrt{3} \). We present a set of exact static solutions of this Lagrangian where the magnetic field depend on a harmonic map, and can be chosen to be of a monopole, dipole, quadripole, etc.

Einstein theory is a good model for describing gravitational interactions in the universe. Nevertheless there are some phenomena in cosmos where gravitation is interacting with electromagnetism. Such is the case for ex-
ample, in planets and stars possessing a magnetic field as the earth or the sun. Our galaxy possesses also a magnetic field and there is not yet a convincing explanation for it. One would wait that Einstein-Maxwell theory should give such an explanation by means of a simple exact solution possessing a magnetic field like the celestial bodies. There is an exact solution of the Einstein-Maxwell equations containing magnetic dipole moment satisfying the required stationary and static limits [3]. But it is not simple at all. 5-dimensional (5D) theory is an alternative model for understanding gravitational and electromagnetical interactions together. In this work we want to show that there exist a class of very simple exact solutions of the 5D Einstein equations possessing magnetic fields which 4-dimensional metric behaves like the Schwarzschild solution. In a past work [4] we developed a method for generating exact solutions of the five-dimensional Einstein equations with a $G_3$ group of motion, putting the solutions in terms of two harmonic maps $\lambda$ and $\tau$. These solutions can be also interpreted as solutions of Lagrangian (1) for the case $\alpha = \sqrt{3}$. We separated the solutions in five tables (tables III-VII) for the one- and two- dimensional (abelian and nonabelian) subgroups of SL(2,R), in the spacetime and the potential space, and demonstrated that many of the well-known solutions are contained in these tables. In this letter we want
to present a set of new solutions which belong to the class of solutions $i, j$ and $k$ of table VI in reference [4], specialising the harmonic maps, because it represents a class of very good behaved solutions, if we choose the harmonic maps $\lambda$ and $\tau$ conveniently. In terms of the five potentials [5] the solutions are

\[ i) \quad \chi = \frac{a_1 e^{q\tau} + a_2 e^{-q\tau}}{g_{22}} \quad g_{22} = be^{q\tau} + ce^{-q\tau} \quad b + c = \frac{1}{I_0} \]

\[ j) \quad \chi = \frac{a_1 \tau + a_2}{g_{22}} \quad g_{22} = b\tau + \frac{1}{I_0} \]

\[ k) \quad \chi = \frac{a_1 e^{iq\tau} + \bar{a}_1 e^{-iq\tau}}{g_{22}} \quad g_{22} = be^{iq\tau} + \bar{b}e^{-iq\tau} \quad b + \bar{b} = \frac{1}{I_0} \]

(here we have set $\alpha + \beta = 0$ in table VI of ref. [4]). The gravitational potential and the scalar potential are the same for all cases given by

\[ f = \frac{e^{\wedge \lambda}}{\sqrt{I_0 g_{22}}} \quad , \quad I^2 = \frac{I_0 e^{2\wedge \lambda}}{g_{22}} \]

where $a_1, a_2, q, b, c, I_0$ and $\wedge$ are constants restricted by $bcq^2 = I_0 \delta \neq 0$, while the electrostatic and rotational potentials vanish, i.e. $\psi = \epsilon = 0$. Now it is easy to write the spacetime metric. Let us write it in Boyer-Lindquist coordinates

\[ \rho = \sqrt{r^2 + 2mr \sin \theta} \quad , \quad z = (r - m)\cos \theta. \]
In this coordinates the five-metric reads

\[
\begin{align*}
\text{d}S^2 &= \frac{1}{I} \left\{ \frac{1}{f^2} \left[ 1 - \frac{2m}{r} + \frac{m^2 \sin^2 \theta}{r^2} \right] \left[ \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 \right] \\
&\quad + \frac{1}{f} \left( 1 - \frac{2m}{r} \right) r^2 \sin^2 \theta d\varphi^2 - f dt^2 \right\} + I^2 (A_3 d\varphi + dx^5)^2
\end{align*}
\]

The expression in brackets \{\ldots\}, is interpreted as the four-dimensional metric in the five-dimensional theory and corresponds to the spacetime metric of Lagrangian (1). The functions \(k\) and \(A_3\) are completely determined by the potentials \(f, \chi\) and \(\kappa^2 = I^3\)

\[
k, \zeta = \rho \left[ \frac{(f, \zeta)^2}{2f^2} + \frac{1}{2f} \frac{(\chi, \zeta)^2}{\kappa^2} + \frac{2}{3} \frac{(\kappa, \zeta)^2}{\kappa^2} \right] = \rho \left[ \frac{4}{3} \Lambda^2 (\lambda, \zeta)^2 + q^2 (\tau, \zeta)^2 \right]
\]

\[
A_{3, \zeta} = -\frac{\rho}{f \kappa^2} \chi, \zeta = -\rho \tau, \zeta
\]

\[
A_{3, \zeta} = \rho \tau, \zeta ; \quad \zeta = \rho + iz
\]

Observe that the function \(A_3\) is integrable because \(\tau\) fulfills the Laplace equation \((\rho \tau, \zeta)^2 + (\rho \tau, \zeta, \zeta) = 0\). In reference [3] a set of solutions of the Laplace equation and their corresponding magnetic potential \(A_3\) is listed. Two examples are

\[
a) \quad \tau = \tau_0 \ln \left( 1 - \frac{2m}{r} \right) \quad A_3 = 2\tau_0 m (1 - \cos \theta)
\]
\[ b) \quad \tau = \frac{\tau_0 m^2 \cos \theta}{(r - m)^2 - m^2 \cos^2 \theta}, \quad A_3 = \frac{m^2 \tau_0 (r - m) \sin^2 \theta}{(r - m)^2 - m^2 \cos^2 \theta} \]

written in Boyer-Lindquist coordinates. The magnetic potential \( a) \) and \( b) \) represents a magnetic monopole and a magnetic dipole, respectively. In general the harmonic function \( \tau \) determines the magnetic field in the solution and can be chosen to obtain monopoles, dipoles, quadripoles, etc. fields.

The harmonic function \( \lambda \) determines the gravitational potential \( f \). Let us choose \( \lambda = \lambda_0 \ln(1 - \frac{2m}{r}) \). The five metric transforms to

\[
\begin{align*}
\text{d}S^2 &= \sqrt{\frac{g_{22}}{I_0}} \left( 1 - \frac{2m}{r} \right)^{\frac{2\lambda \lambda_0}{r}} \left\{ \frac{(1 - \frac{2m}{r} + \frac{m^2 \sin^2 \theta}{r^2})^{1 - \frac{4}{3} \lambda^2 \lambda_0^2}}{(1 - \frac{2m}{r})^{\lambda \lambda_0 - \frac{2}{3} \lambda^2 \lambda_0^2}} \sqrt{I_0 g_{22} e^{2k_1}} \left( \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 \right) \right. \\
&+ \left. \left( 1 - \frac{2m}{r} \right)^{1 - \lambda \lambda_0} \sqrt{I_0 g_{22} r^2 \sin^2 \theta} d\phi^2 - \frac{1}{\sqrt{I_0 g_{22}}} \left( 1 - \frac{2m}{r} \right)^{\lambda \lambda_0} dt^2 \right) \\
&+ I^2 (A_3 d\phi + dx^5)^2
\end{align*}
\]

where \( k_1, \zeta = \frac{1}{2} q^2 \tau_0 (\tau_\zeta)^2, g_{22} \) and \( A_3 \) are determined only by the harmonic function \( \tau \). If we choose \( \tau \) to vanish for some limit \( r \gg m \), (the two examples \( a) \) and \( b) \) fulfill this condition) then the metric (2) is asymptotically flat. If \( \tau \) and \( m \) vanish, metric (2) is flat.

If we put \( \lambda = -2, \lambda_0 = -\frac{1}{2} \) in (2) we can interprete \( m \) as the mass parameter and \( \sqrt{I_0 g_{22}} \) as the contribution of the magnetic field to the metric. In this case, metric (2) reads
This metric can be interpreted as a magnetized Schwarzschild solution in five-dimensional gravity. The difference to a previous one [7] is that in metric (3) the magnetic potential can be chosen in many ways. If the magnetic field $A_3$ in (3) vanishes, the expression in brackets $\{\ldots\}$ is just the Schwarzschild metric. Therefore we can interpret $r = 2m$ as the horizon of the four metric. Observe that the presence of the magnetic field does not alter the horizon of the metric, conserving the main feature of its topology. Nevertheless the scalar field do. We can see that the scalar potential tends very fastly to $I_0$ for $r >> 2m$ and is singular for $r = 2M$. If we interpret the expression in brackets $\{\ldots\}$ as the spacetime metric, we find that its Riemannian invariant $R^{abcd}R_{abcd}$ and its Ricci invariant $R^{ab}R_{ab}$ are singular for $r = 2M$, (but not its scalar curvature $R$), when $\tau$ depends on $\theta$. This is so for the case when $A_3$ represents the magnetic field of a dipole, but when $A_3$ represents a monopole, all invariants remain regular on $r = 2M$. So, one expects that $r = 2M$ is
a coordinate singularity when $A_3$ is a monopole field, but the spacetime is really singular for higher multipole moments in this point, and are not black holes. However for geodesical trajectories around the surface $r > 2M$, the effective potential is regular for $r = 2M$ even for magnetic dipole fields, but the scalar field increases without bound for all these cases when $r$ approaches $2M$. The scalar field $I$ is topologically the radius of the fifth-dimension which is a circle. This circle has constant radius for $r >> 2M$, but tends to a line when $r$ approaches $2M$. That means that the scalar potential is really important only near of the horizon but desapears very fastly far away of it. One would suspect that the properties of the geometry change near the horizon with respect to Schwarzschild’s geometry due to the interaction of the scalar field. That means that the relevant modifications of the Schwarzshcild’s geometry is not due to the magnetic field but due to the scalar interaction. The geodesic motion in this spacetime will be publish elsewhere [8].

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