A BRST Operator for non-critical W-Strings

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Abstract

We construct the BRST operator for non-critical $W_3$-strings and discuss the tachyon-like spectrum. For $N$-punctured spheres with $N \leq 5$ we briefly describe a formal definition of the integral over $W_3$-moduli space.
1. Introduction.

During the last decade we have learnt much about the structure of two-dimensional gravity. The gravitational action can be induced by coupling to the matter system. This coupling is given by

$$\delta S = \int d^2 z \sqrt{g} \delta g^{ab} T_{ab}(z),$$

where $T_{ab}(z)$ is the stress-energy tensor of the matter. If this matter system is conformal, then gravity is governed by the Liouville action \[1\]. The fact that two-dimensional gravity is invariant under diffeomorphisms implies the existence of a nilpotent BRST charge \[2\]

$$Q_{BRST} = \oint c(z) \left[ T^{(matter)}_{zz} + T^{(Liouville)}_{zz} + \frac{1}{2} T^{(ghost)}_{zz} \right],$$

and the physical states are given by the non-trivial BRST cohomology. The $c < 1$ conformal minimal models coupled to two-dimensional gravity can have non-trivial cohomology at any ghost number \[3,4\]. For $c = 1$, one has non-trivial cohomology at a finite set of ghost numbers. The states with $q_{gh} = 0$ and $h = 0$ make up the ground ring \[5\], while states with $q_{gh} = 0$ and $h = 1$ generate the symmetry of 2d quantum gravity \[5–7\]. This symmetry group, $W_\infty$, is responsible for the solvability of the system.

If, on the other hand, the conformal matter system has a larger symmetry then it can be coupled to some extended background geometry. Thus, in addition to the well-known generalizations to super-geometry, one can also try to construct a “$W$-geometry” \[8\] in which $W$-gravity is coupled to conformal $W$-matter \[9–11\]. One expects that such a theory is governed by a Toda system that extends the Liouville theory of two-dimensional gravity (see, for example, \[12\]). In a sense, this generalization consists in replacing $SL(2)$ by some other semi-simple Lie group $G^\star$. This amounts to introducing spin $s$ “gravitons”, where the spins, $\{s\}$, are given by the degrees of independent Casimirs of $G$ (for $G = SL(n)$ one has $\{s\} = \{2, 3, \ldots, n\}$), and leads to what might be called “$W$-strings” (see, for example, \[13–15\]).

In \[16\] a BRST charge was constructed for critical $W_3$-string theory, that is, for pure $W_3$-matter; nilpotency of $Q_{BRST}$ requires that the matter central charge must satisfy: $c_M = 100$. More generally, for $W_n$ one requires

$$c_M = c^{(crit)} \equiv 4n^3 - 2n - 2, \quad n = 2, 3, \ldots .$$

\[1.2\]

\* In this letter we will mainly consider $G = SL(n)$ and we will take “$W_n$-gravity” to refer to this choice of $G$. 

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The value of \( c^{(\text{crit})} \) just balances the contribution from the ghosts \((b[i], c[i]), i = 1, 2, \ldots, n - 1\) with spins \((i + 1, -i)\). The foregoing values of \( c_M \) correspond to critical \( W_n \)-strings, where the Liouville-like degrees of freedom are expected to decouple. Matter theories with these central charges can indeed be constructed as described in [13], but these theories are slightly artificial in that the \( W \)-generators are constructed from the stress tensor of the matter by folding in a collection of auxiliary scalars.

On the other hand, the prototype \( W_n \)-matter theories are the \( W_n \)-minimal models, \( \mathcal{M}_{p,q} \), with central charges [14]

\[
c_{p,q}^{(\text{matter})} \equiv c(\mathcal{M}_{p,q}) = (n - 1) \left(1 - n(n + 1) \frac{(p - q)^2}{pq}\right).
\]

(1.3)

It has been noted [13,18] that there is a natural pairing of \( W_n \)-Liouville (Toda) theory (denoted by \( \mathcal{M}_{p,-q} \)) with these matter models in the following sense:

\[
c_{p,q}^{(\text{matter})} + c_{p,q}^{(\text{Liouville})} = c^{(\text{crit})}
\]

(here \( c_{p,q}^{(\text{Liouville})} \equiv c(\mathcal{M}_{p,-q}) \)). These theories appear to be dual in the sense described in [3], and this suggests that the tensor product models

\[
\mathcal{W}^{(n)}_{p,q} \equiv \mathcal{M}^{(W_n, \text{matter})}_{p,q} \otimes \mathcal{M}^{(W_n, \text{Liouville})}_{p,-q} \otimes \{b[i], c[i]\}
\]

(1.4)

describe non-critical \( W_n \)-strings [19]. These models are conceivably solvable and are thus, for us, the most interesting ones. A further observation [13,18] is that there is a similar natural pairing in the construction of tachyonic states: if \( V_{r_i,s_i} \) denotes a vertex operator in either the Liouville or matter sector in the usual parametrization, then

\[
h(V_{r_i,-s_i}^{(\text{Liouville})}) + h(V_{r_i,s_i}^{(\text{matter})}) = h_{\text{crit}} \equiv \frac{1}{6} n(n^2 - 1) \quad \forall r_i, s_i.
\]

(1.5)

This is just the maximal dimension that can be compensated by all of the \( c \)-ghosts, and suggests that at least the following, tachyonic operators should be physical:

\[
T_{r_i,s_i} = X(c) V_{r_i,-s_i}^{(\text{Liouville})} V_{r_i,s_i}^{(\text{matter})}, \quad h(T_{r_i,s_i}) \equiv 0,
\]

\[
X(c) \equiv \prod_{i=1}^{n-1} \left[ c[i] \partial c[i] \partial^2 c[i] \ldots \partial^{i-1} c[i] \right].
\]

(1.6)
While these observations, and some other pieces of circumstantial evidence, suggest a natural formulation of non-critical $W$-strings, the construction of a proper BRST charge has been lacking. Without such a charge, the discussion of the physical states can only be rather speculative. The problem in constructing the BRST charge lies in the non-linearity of the $W$-algebra, and so the $W$-matter models apparently cannot be coupled to $W$-gravity using the BRST operator of [16].

The main purpose of this letter is to construct explicitly the BRST charge for $W_3$-matter coupled to $W_3$-gravity, thus demonstrating that this tensor product actually does yield a sensible theory (and strongly suggesting that this is true for all $n$). Furthermore, we show that the tachyonic operators (1.6) are indeed BRST invariant, as expected. We also discuss some issues related to $W$-moduli. In addition, we make some brief remarks on the extra states of the associated $c_M = 2$ model (corresponding to $p = q$ above).

2. The BRST operator

The construction of $Q_{BRST}$ for $W_3$-Liouville coupled to $W_3$-matter proceeds in the simplest possible manner and closely parallels the computation of [16]. Following [17] we take

$$W(z)W(w) = \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2} [2b^2 \Lambda(w) + \frac{3}{10} \partial^2 T(w)] + \frac{1}{(z-w)} [b^2 \partial \Lambda + \frac{1}{15} \partial^3 T(w)]$$

(2.1)

+ regular terms,

where $c$ is the central charge and $b^2 \equiv \frac{16}{5c+2}$. The composite operator on the right-hand side is defined by $\Lambda(z) = (TT)(z) - \frac{3}{10} \partial^2 T(z)$. (Normal ordering of two operators $A(z)$ and $B(z)$ is defined by $(AB)(z) = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta - z} A(\zeta) B(z)$.) Now consider a theory that consists of a tensor product of $W_3$-matter coupled to $W_3$-Liouville $\ast$, and let $W_L(z), W_M(z), T_L(z), T_M(z), c_L$ and $c_M$ denote the currents and central charges. The stress tensor of the $j$-th ($j = 1, 2$) ghost system is given by: $T^{[j]}_{gh} = -(j +$
We find that the BRST current \( J(z) \) has the form (up to total derivatives):
\[
J(z) = c^{[2]}(z) [\hat{W}_L(z) \pm i\hat{W}_M(z)] + c^{[1]}(z) [T_L(z) + T_M(z) + \frac{1}{2}T^{[1]}_{gh}(z) + T^{[2]}_{gh}(z)] \\
+ \left[ T_L(z) - T_M(z) \right] b^{[1]}(z) \left( c^{[2]}(z) \right) + \mu \left( \partial_z b^{[1]}(z) \right) \left( \partial^2_z c^{[2]}(z) \right) \\
+ \nu b^{[1]}(z) \left( c^{[2]}(z) \partial^3_z c^{[2]}(z) \right),
\]
where \( \hat{W} = \frac{1}{5} W \) and \( \mu = \frac{3}{5} \nu = \frac{1}{10b_L^2} (1 - 17b_L^2) \). The choice of the first two terms is rather natural, and the form of the entire current, \( J(z) \), is similar to that of \([10]\). To calculate \((Q_{BRST})^2 \), where \( Q_{BRST} = \oint J(z) dz \), one computes the simple pole terms in \( J(z)J(w) \), and discards all total derivatives. Rather than recount the calculation in detail we simply note the following features of the calculation:

(i) The contribution of the operator product of the second term, \( c^{[1]}[T_L + T_M + \ldots] \), with itself is the standard BRST computation and gives rise to the condition \( c_L + c_M = 100 \).

(ii) Essentially because the first term is a primary field of weight one, its operator product with the second term yields simple pole terms that are purely total derivatives.

(iii) The third term has been chosen so that its operator product with the second term gives rise to a factor of the form: \( \frac{1}{(z-w)} [(T_L T_L) - (T_M T_M)] c^{[2]}(z) \), which cancels against a similar term arising from \( c^{[2]}(\hat{W}_L \pm i\hat{W}_M)(z) c^{[2]}(\hat{W}_L \pm i\hat{W}_M)(w) \).

(iv) The coefficients \( \mu \) and \( \nu \) are (over)determined by the requirement that all simple pole terms proportional to \( (\partial^m T)(\partial^n c^{[2]})(\partial^p c^{[2]})) \) are total derivatives.

(v) The many other terms that arise in computing \((Q_{BRST})^2 \) then vanish as a consequence of the choice of coefficients and the requirement that \( c_L + c_M = 100 \).

As a result of the computation, we find that \( J(z) \) yields a nilpotent BRST charge. The only ambiguity is the choice of sign in front of \( i\hat{W}_M \). In fact, the BRST current is almost invariant under the interchange of the matter and Liouville systems, modulo conjugation. That is, rescaling the ghosts \( c^{[2]} \to \mp i c^{[2]} \) and \( b^{[2]} \to \pm i b^{[2]} \) one effectively flips the matter and Liouville systems, provided that one has:
\[
\frac{1}{10b_L^2} (1 - 17b_L^2) = -\frac{1}{10b_M^2} (1 - 17b_M^2).
\]
However, this identity is equivalent to \( c_L + c_M = 100 \).
3. Physical states

We now look for physical, tachyonic states that are the counterparts of the Distler-Kawai states \[20\] of ordinary matter and gravity. Specifically, we seek physical states of the form

\[
|v\rangle = |v_L\rangle \otimes |v_M\rangle \otimes |\tilde{0}\rangle_{gh},
\]

(3.1)

where \(|v_L\rangle\) and \(|v_M\rangle\) are pure momentum states in the Liouville and matter sectors, and \(|\tilde{0}\rangle_{gh}\) is the ghost vacuum states of the form (3.1), one finds that the physical state condition \(Q_{BRST} |v\rangle = 0\) reduces to

\[
h_L + h_M = 4, \quad \tilde{w}_L \pm i\tilde{w}_M = 0.
\]

(3.3)

If \(|0\rangle\) denotes the \(SL(2, \mathbb{R})\) invariant ghost vacuum with \(c_n^{[j]} |0\rangle = 0, n \geq (j + 1), b_n^{[j]} |0\rangle = 0, n \geq -j,\) then \(|\tilde{0}\rangle_{gh} = c^{[2]}(0)\partial c^{[2]}(0)c^{[1]}(0) |0\rangle \equiv X(c) |0\rangle\). On states of the form (3.1), one can parameterize such \(|0\rangle\) by introducing free field realizations of the currents:

\[
T_M(z) = -\frac{1}{2}(\partial \phi_M)^2 - i\alpha_0 \rho \cdot \partial^2 \phi_M
\]

(3.4)

\[
T_L(z) = -\frac{1}{2}(\partial \phi_L)^2 - \beta_0 \rho \cdot \partial^2 \phi_L,
\]

where \(\phi_M(z)\) and \(\phi_L(w)\) are both two-component vectors of bosons with \(\phi_M^a(z)\phi_M^b(w) \sim \phi_L^a(z)\phi_L^b(w) \sim -\delta^{ab}\ln(z - w).\) The vector, \(\rho,\) is the Weyl vector of \(SU(3),\) and satisfies \(\rho^2 = 2.\) The background charge parameters, \(\alpha_0\) and \(\beta_0,\) are real, and the condition \(c_L + c_M = 100\) is equivalent to \(\beta_0^2 - \alpha_0^2 = 4.\) One can parameterize such a \(\beta_0\) and \(\alpha_0\) by introducing a real, positive parameter, \(t,\) and setting \(\alpha_0 = \sqrt{t} - 1/\sqrt{t}\) and \(\beta_0 = \sqrt{t} + 1/\sqrt{t}.\) (For the models \(\mathcal{W}_{p,q}^{(3)}\) in (1.4) one has \(t = q/p.\)) In this representation, the \(W\)-generator takes the form \[17\]

\[
\tilde{W}(z) = -\frac{i}{12} \left[ (\partial \psi_2)^3 - 3(\partial \psi_1)^2(\partial \psi_2) + 3\gamma (\partial^2 \psi_1)(\partial \psi_2) 
+ 9\gamma (\partial \psi_1)(\partial^2 \psi_2) - 6\gamma^2 (\partial^3 \psi_2) \right],
\]

(3.5)
where $\tilde{W}(z) \equiv \frac{1}{6} W(z)$. For the matter sector one takes $\gamma = i\alpha_0 = i(\sqrt{7} - 1/\sqrt{7})$ and $\psi_1 = -(\alpha_1 + \alpha_2) \cdot \phi_M$, $\psi_2 = \frac{1}{\sqrt{3}}(\alpha_1 - \alpha_2) \cdot \phi_M$, where $\alpha_1$ and $\alpha_2$ are the simple roots of $SU(3)$. For the Liouville sector one takes $\gamma = \beta_0 = \sqrt{\frac{2}{7} + 1/\sqrt{7}}$ and $\psi_1 = -(\alpha_1 + \alpha_2) \cdot \phi_L$, $\psi_2 = \frac{1}{\sqrt{3}}(\alpha_1 - \alpha_2) \cdot \phi_L$. Introduce vertex operators

$$V_M(a_1, a_2) = \exp[i(a_1 \lambda_1 + a_2 \lambda_2) \cdot \phi_M]$$
$$V_L(b_1, b_2) = \exp[(b_1 \lambda_1 + b_2 \lambda_2) \cdot \phi_L],$$

where $\lambda_{1,2}$ are the fundamental weights of $SU(3)$ with $\alpha_i \cdot \lambda_j = \delta_{ij}$. From (3.5) and (3.4), a simple computation shows that for these vertex operators one has:

\begin{align}
    h_M(a_1, a_2) &= \frac{1}{12} [3(a_1 + a_2 + 2\alpha_0)^2 + (a_1 - a_2)^2 - 12\alpha_0^2] \\
    \tilde{w}_M(a_1, a_2) &= -\frac{1}{9\sqrt{3}}(a_1 - a_2)(2a_1 + a_2 + 3\alpha_0)(a_1 + 2a_2 + 3\alpha_0) \\
    h_L(b_1, b_2) &= -\frac{1}{12} [3(b_1 + b_2 + 2\beta_0)^2 + (b_1 - b_2)^2 - 12\beta_0^2] \\
    \tilde{w}_L(b_1, b_2) &= -\frac{i}{9\sqrt{3}}(b_1 - b_2)(2b_1 + b_2 + 3\beta_0)(b_1 + 2b_2 + 3\beta_0),
\end{align}

(3.7)

From this it is easy to see that if $b_1 = a_1 + \alpha_0 - \beta_0$ and $b_2 = a_2 + \alpha_0 - \beta_0$ then the physical state condition (3.3) is satisfied. There are, however, further solutions to (3.3) that can be generated by the action of the Weyl group on the foregoing obvious solution. A simple way to see this is to introduce the orthonormal basis: $e_i$, $i = 1, 2, 3$, in which the simple roots can be written $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$. Now consider the vector

$$u \equiv b_1 \lambda_1 + b_2 \lambda_2 + \beta_0 \rho = \frac{1}{7}(2b_1 + b_2 + 3\beta_0)e_1 - \frac{1}{3}(b_1 - b_2)e_2 - \frac{1}{3}(b_1 + 2b_2 + 3\beta_0)e_3,$$

and observe that $\tilde{w}_L(b_1, b_2)$ is a constant multiple of the product of the components of $u$. Moreover, one has $h_L(b_1, b_2) = -\frac{1}{2} u^2 + \beta_0^2$. Since the Weyl group acts by permuting the $e_i$, it follows that the vertex operator $\exp[(\sigma(b_1 \lambda_1 + b_2 \lambda_2 + \beta_0 \rho) - \beta_0 \rho) \cdot \phi_L]$ has the same values of $h_L$ and $\tilde{w}_L$ for any element $\sigma$ of the Weyl group of $SU(3)$. It follows that the operators

$$T_{\Lambda, \sigma}(z) = c^{[2]}(z) \partial c^{[2]}(z) c^{[1]}(z) \exp[i\Lambda \cdot \phi_M(z) + (\sigma(\Lambda + \alpha_0 \rho) - \beta_0 \rho) \cdot \phi_L(z)],$$

(3.8)

for any $\Lambda$, and for any choice of $\sigma$ in the Weyl group of $SU(3)$, create physical states on the $SL(2, \mathbb{R})$ invariant vacuum of the theory. Note that because of the Weyl rotations in (3.8), each matter field has six possible Liouville dressings. We expect that there exists some analogue of Seiberg’s condition $[\mathbb{P}]$ that selects one choice of dressing. In particular it seems reasonable that if $\Lambda$ is in the fundamental Weyl chamber, then one should take $\sigma = 1$. This corresponds to the operators (1.6) mentioned in the introduction.
4. Additional comments

In analogy with two-dimensional gravity \([3]\), we expect that there will be other physical states in addition to the tachyons (3.8). In particular, we anticipate that there will be physical states with different ghost numbers. The \(W\)-minimal models coupled to \(W\)-gravity clearly should have physical states for any ghost numbers, while the physical states of the \(c = 2\) theory will appear at a finite set of ghost numbers \(*\). The operators at different ghost numbers might be constructed by using the BRST current (2.2) and the corresponding descent equations similar to those employed in \([3,22]\). The whole structure will be more complicated than for ordinary gravity precisely because of the larger number of extra states and more involved structure of the descent equations. The ghost number zero, dimension zero operators should certainly form a ground ring. We will discuss these states and ground rings in a later paper \([23]\), and here we will simply make some brief remarks about the theory with \(c_M = 2\) at the \(SU(3)\) symmetric point.

First observe that \(c_M = 2\) corresponds to \(\alpha_0 = 0\). We can therefore introduce the \(SU(3)\) currents

\[
J^\pm\alpha_j(z) = e^{\pm i \alpha_j \cdot \phi_M(z)},
\]

(4.1)

where \(\alpha_j, j = 1, 2, 3\) are the positive roots of \(SU(3)\). One can verify that the zero modes of these currents commute with \(T_M(z)\) and \(\tilde{W}_M(z)\). Consequently, if \(\Lambda\) is a weight of \(SU(3)\), we may use the generators (4.1) on a tachyonic state \(T_{\Lambda,\sigma}(z)\) so as to obtain an entire \(SU(3)\) multiplet of physical states of the \(W\)-string. Note that the “corners” of a weight diagram will correspond to tachyon states (3.8), while the rest of the multiplet corresponds to “extra states” that are descendents of vertex operator states. Such \(SU(3)\) discrete states have already been discussed in \([24]\). We therefore find that this part of the structure of \(W_3\)-gravity coupled to \(W_3\)-matter is a relatively straightforward generalization of \(c = 1\) matter coupled to ordinary gravity.

Having remarked upon the similarities between \(W\)-strings and ordinary strings, we now would like to note some fundamental differences. The basic problem now is how to construct correlation functions, or more generally, how to understand \(W\)-geometry and the integration over \(W\)-moduli. For example, the \(c^{[2]}\) ghost number

\(*\) The \(c_M = 2\) model corresponds to a very intriguing 4-dimensional string theory. The target space has (2,2) signature and Lorentz symmetry is broken by the dilaton background.
anomaly on the sphere is equal to 5, but three insertions of $X(c)$ have $c^{[2]}$-ghost charge equal to 6. This implies that even the three-punctured sphere has a non trivial $W$-modulus, that is, in contrast to usual gravity, the sphere is not “rigid” in $W$-gravity. The $N$-punctured sphere has $(N - 3) + (2N - 5)$ moduli, and therefore a $N$-point correlation function should involve additional $2N - 5$ integrations over the $W$-moduli.

One cannot thus employ only the standard tachyons (3.8) to make non-vanishing correlation functions in $W_n$-models for $n > 2$; one also needs physical operators at different ghost numbers. In analogy to ordinary strings, we expect that each physical state can be represented in different “$W$-pictures” at various ghost numbers. Such pictures can probably have all possible permutations of ghost content at each fixed ghost number.

We suggest the following partial resolution of this problem. Let us first define the total stress-energy tensor and the total spin-3 current*

$$\mathcal{T} = [Q_{BRST}, b^{[1]}], \quad \mathcal{W} = [Q_{BRST}, b^{[2]}].$$

Each tachyonic vertex operator (3.8) can be mapped to representatives, or “avatars”, in other $W$-pictures. Each tachyon will have avatars with at least ghost numbers 0, 1, 2 and 3. Here we present only those avatars with ghost numbers 1, 2 and 3 that are relevant for the discussion below:

$$\Phi_{\Lambda}^{(3)} = c^{[1]}c^{[2]}\partial c^{[2]}V_{\Lambda} \equiv T_{\Lambda},$$
$$\Phi_{\Lambda,1}^{(2)} = c^{[2]}\partial c^{[2]}V_{\Lambda}, \quad \Phi_{\Lambda,2}^{(2)} = c^{[1]}\partial c^{[2]}V_{\Lambda} + \ldots$$
$$\Phi_{\Lambda}^{(1)} = \partial c^{[2]}V_{\Lambda} + \ldots.$$

The operators at different ghost numbers are related to each other by descent equations similar to those employed in ordinary gravity:

$$[Q_{BRST}, \Phi^{(3)}] = 0,$$
$$[Q_{BRST}, \Phi_{1}^{(2)}] = \mathcal{L}_{-1}\Phi^{(3)},$$
$$[Q_{BRST}, \Phi_{2}^{(2)}] = -\mathcal{W}_{-2}\Phi^{(3)},$$
$$[Q_{BRST}, \Phi^{(1)}] = \mathcal{W}_{-2}\Phi_{1}^{(2)} + \mathcal{L}_{-1}\Phi_{2}^{(2)}.$$

* $\mathcal{T}$ and $\mathcal{W}$ do not appear to generate a $W$-algebra.
These descent equations should involve only $W_{-2}$ and $L_{-1}$. The complete structure of $W$-avatars is more complicated. It is crucial that $W_{-2}$ and $L_{-1}$ commute with each other and therefore they may be realized as derivations: $L_{-1} = \partial_z$ and $W_{-2} = \partial_\xi$. It is natural to define a “$W$-field” as follows:

$$\mathcal{P}(z, \xi) = e^{\xi W_{-2}} \Phi(z).$$

The new coordinate $\xi$ corresponds to the $W$-modulus. One can also define $\mathcal{P}^{(3)}, \mathcal{P}_1^{(2)}, \mathcal{P}_2^{(2)}$ and $\mathcal{P}^{(1)}$, which are related to each other by descent equations (4.3). Now it is almost obvious that

$$\int d\xi \mathcal{P}_2^{(2)}(z, \xi), \quad \int d\xi dz \mathcal{P}^{(1)}(z, \xi)$$

are (at least formally) BRST invariant operators\(^\dagger\). Using these operators one can construct 2-, 3-, 4- and 5-point correlation functions. The failure of this approach to yield arbitrary $N$-point correlation functions is due to the complicated structure of the $W$-moduli for $N > 5$. To construct the general correlation functions one has to utilize the properties of the complete collection of $W$-avatars. We intend to discuss the properties of $W$-avatars in one of our next publications.

Finally, we would like to point out that it is the appearance of a holomorphic structure that is responsible for various remarkable properties of this class of theories. For example, the fact that the dimension of the tachyon operator (3.8) vanishes for all $\Lambda$, rests crucially on the holomorphic\(^*\) combination $\Lambda (\phi_L + i\phi_M)$. Similarly, physical states satisfy the holomorphic condition $\tilde{w}_L + i\tilde{w}_M = 0$. These features are reminiscent and actually related to similar properties of topological, twisted $N = 2$ superconformal theories. Indeed, one can show that for the topological models, $\mathcal{W}_p^{(n)}_{p=1, q=n+k}$, the spectrum contains operators that have an algebraic structure analogous to the chiral primary fields of $N = 2$ coset models of $SU(n)_k/U(n-1)$. More specifically, for these models the tachyons (1.6) can be viewed as $T_{\Lambda} = e^{\Lambda \Phi P}$, where

\(^\dagger\) Note also that $z$-integrals over the screening operators are BRST-invariant. These operators for the Liouville sector (which appear in the Toda action) are thus the simplest examples for discrete states with (for $W$-gravity) non-standard ghost number.

\(^*\) With “holomorphic” we mean holomorphic up to Weyl transformations $\sigma$ in (3.8). We can always choose a representative in each Weyl orbit such that only the holomorphic combination $\Lambda (\phi_L + i\phi_M)$ appears.
\[ \Phi = (\phi + i\varphi) \] plays the role of the bottom component of a chiral superfield and its exponentials correspond to a free superfield realization of the primary chiral fields of the coset models. The remaining non-holomorphic piece, \( P \), is the \( W \)-gravity puncture operator. Similarly, the ground rings of ghost-neutral operators seem to contain the chiral rings of the coset models, together with their \( W \)-gravitational extensions. Thus, the models \( \mathcal{W}_{1,n+k}^{(n)} \) seem also to describe topological \( W_n \)-gravity coupled to topological minimal \( W_n \)-matter models at level \( k \), in correspondence to the results (for \( n = 2 \)) of \([26]\). We will present a detailed analysis of the various ring structures in \([28]\).

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