Bivariate Exponentiated Modified Weibull Extension Distribution

A. El-Gohary, M. El-Morshedy.
Mathematics Department,
Faculty of Science, Mansoura University, Mansoura, Egypt.

Abstract

In this paper, we introduce a new bivariate distribution we called it bivariate exponentiated modified Weibull extension distribution (BEMWE). The model introduced here is of Marshall-Olkin type. The marginals of the new bivariate distribution have exponentiated modified Weibull extension distribution which proposed by Sarhan et al. (2013). The joint probability density function and the joint cumulative distribution function are in closed forms. Several properties of this distribution have been discussed. The maximum likelihood estimators of the parameters are derived. One real data set are analyzed using the new bivariate distribution, which show that the new bivariate distribution can be used quite effectively in fitting and analyzing real lifetime data.

Key words: Joint probability density function, Conditional probability density function, Maximum likelihood estimators, Fisher information matrix.

1 Introduction

Recently, Sarhan et al. (2013) has defined a new four-parameter distribution referred to as exponentiated modified Weibull extension (EMWE) distribution. Sarhan et al. (2013) defined the (EMWE) distribution by exponentiating the new modified Weibull extension (MWE) distribution which discussed by Xie et al. (2002) as was done for the exponentiated Weibull (EW) distribution by Mudholkar et al. (1995). They observed that exponential distribution, generalized exponential distribution (1999), Gompertz distribution (1824), generalized Gompertz (GG) distribution (2013), exponentiated Weibull (EW) distribution (1995), Weibull extension model of Chen (2000), modified Weibull extension (MWE) distribution (2002) and etc distribution can be obtained as special cases of the (EMWE) distribution.

The objective of this paper is to provide a new bivariate distribution, whose marginals are (EMWE) distributions which referred to as bivariate exponentiated modified Weibull extension (BEMWE) distribution. It is obtained using a method similar to that used to obtain Marshall-Olkin bivariate exponential model Marshall and Olkin (1967).
The paper is organized as follows. Section 2 presents the shock model yielding the (BEMWE) distribution. Also, the joint cumulative distribution function, the joint probability density function, the marginal probability density functions and the conditional probability density functions of (BEMWE) distribution is derived in Section 2. In Section 3 sum reliability studies are obtained. Section 4 presents the the marginal expectation of the (BEMWE) distribution. Section 5 obtains the parameter estimation using MLE. In section 6 a numerical result are obtained using real data. Finally, a conclusion for the results is given in Section 7.

2 Bivariate exponentiated modified Weibull extension distribution

In this section we introduce the BEMWE distribution using a method similar to that which was used by Marshall and Olkin (1967) to define the Marshall Olkin bivariate exponential (MOBE) distribution. We start with the joint cumulative function of the proposed bivariate distribution and so used it to derive the corresponding joint probability density function. Finally The marginal probability density functions and conditional probability density functions of this distribution are also derived. Let X be a random variable has univariate EMWE distribution with parameters \( \gamma, \alpha, \beta, \lambda > 0 \), then the corresponding cumulative distribution function (CDF) is given by

\[
F(x) = \left[ 1 - e^{-\lambda \alpha (e^{x/\alpha} - 1) - 1} \right]^{\gamma}, \quad x \geq 0,
\]

and the probability density function (PDF) takes the following form

\[
f(x) = \gamma \lambda \beta e^{(x/\alpha)\beta} \left( \frac{x}{\alpha} \right)^{\beta-1} e^{-\lambda \alpha (e^{x/\alpha} - 1)} \left[ 1 - e^{-\lambda \alpha (e^{x/\alpha} - 1)} \right]^{\gamma-1}, \quad x \geq 0.
\]

2.1 Joint cumulative distribution function

Suppose that \( U_i \) \((i = 1, 2, 3)\) are three independent random variables such that \( U_i \sim\text{EMWE} \) \((\gamma_i, \alpha, \beta, \lambda)\). Define \( X_1 = \max\{U_1, U_3\} \) and \( X_2 = \max\{U_2, U_3\} \). Then we say that the bivariate vector \((X_1, X_2)\) has a bivariate exponentiated modified Weibull extension distribution, with parameters \((\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda)\) and we denote it by BEMWE\((\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda)\).

The following interpretation can be provided for the BEMWE model.

Shock model: Assume that there exists a three independent sources of shocks. Suppose these shocks are affecting a system with two components. It is assumed that the shock from source 1 reaches the system and destroys component 1 immediately, the shock from source 2 reaches the system and destroys component 2 immediately, while if the shock from source 3 hits the system it destroys both the components immediately. Let \( U_i \) denote the inter-arrival times, between the shocks in source \( i, i = 1, 2, 3 \), which follow the distribution EMWE. If \( X_1, X_2 \) denote the survival times of the components, then the bivariate vector \((X_1, X_2)\) follows the BEMWE model.

We now study the joint cumulative distribution function of the bivariate random vector \((X_1, X_2)\) in the following lemma.
Lemma 2.1. The joint CDF of \((X_1, X_2)\) is

\[
F_{BEMWE}(x_1, x_2) = \left[1 - e^{-\lambda_1(e^{x_1/\alpha})^\beta - 1}\right]^{\gamma_1} \left[1 - e^{-\lambda_2(e^{x_2/\alpha})^\beta - 1}\right]^{\gamma_2} \left[1 - e^{-\lambda_3(e^{z/\alpha})^\beta - 1}\right]^{\gamma_3},
\]

where \(z = \text{min} \,(x_1, x_2)\).

**proof:** Since the joint CDF of the random variables \(X_1\) and \(X_2\) is defined as

\[
F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)
\]

\[
= P(\max\{U_1, U_3\} \leq x_1, \max\{U_2, U_3\} \leq x_2)
\]

\[
= P(U_1 \leq x_1, U_2 \leq x_2, U_3 \leq \text{min} \,(x_1, x_2)).
\]

As the random variables \(U_i\ (i = 1, 2, 3)\) are mutually independent, we directly obtain

\[
F_{BEMWE}(x_1, x_2) = P(U_1 \leq x_1) \, P(U_2 \leq x_2) \, P(U_3 \leq \text{min} \,(x_1, x_2))
\]

\[
= F_{EMWE}(x_1; \gamma_1, \alpha, \beta, \lambda) \, F_{EMWE}(x_2; \gamma_2, \alpha, \beta, \lambda) \, F_{EMWE}(z; \gamma_3, \alpha, \beta, \lambda)
\]

Substituting from (1) into (4), we obtain (3), which completes the proof of the lemma 2.1.

### 2.2 Joint probability density function

The following theorem gives the joint PDF of the \(X_1\) and \(X_2\) which is the joint PDF of BEMWE \((\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda)\).

**Theorem 2.1.** If the joint CDF of \(X_1\) and \(X_2\) is as in (3), then the joint PDF of \(X_1\) and \(X_2\) takes the form

\[
f_{BEMWE}(x_1, x_2) = \begin{cases} 
  f_1(x_1, x_2) & \text{if } x_1 < x_2 \\
  f_2(x_1, x_2) & \text{if } x_2 < x_1 \\
  f_0(x, x) & \text{if } x_1 = x_2 = x
\end{cases}
\]

where

\[
f_1(x_1, x_2) = f_{EMWE}(x_2; \gamma_2, \alpha, \beta, \lambda) \, f_{EMWE}(x_1; \gamma_1 + \gamma_3, \alpha, \beta, \lambda)
\]

\[
= \gamma_2 \left(\gamma_1 + \gamma_3\right) \lambda^2 \beta^2 e^{(x_2/\alpha)^\beta} \left(\frac{x_2}{\alpha}\right)^{\beta-1} e^{-\lambda_1(e^{x_1/\alpha})^\beta - 1} \left[1 - e^{-\lambda_2(e^{x_2/\alpha})^\beta - 1}\right]^{\gamma_2-1}
\]

\[
\times e^{(x_1/\alpha)^\beta} \left(\frac{x_1}{\alpha}\right)^{\beta-1} e^{-\lambda_3(e^{z/\alpha})^\beta - 1} \left[1 - e^{-\lambda_2(e^{x_1/\alpha})^\beta - 1}\right]^{\gamma_1 + \gamma_3 - 1},
\]

\[
f_2(x_1, x_2) = f_{EMWE}(x_1; \gamma_1, \alpha, \beta, \lambda) \, f_{EMWE}(x_2; \gamma_2 + \gamma_3, \alpha, \beta, \lambda)
\]

\[
= \gamma_1 \left(\gamma_2 + \gamma_3\right) \lambda^2 \beta^2 e^{(x_1/\alpha)^\beta} \left(\frac{x_1}{\alpha}\right)^{\beta-1} e^{-\lambda_1(e^{x_1/\alpha})^\beta - 1} \left[1 - e^{-\lambda_2(e^{x_2/\alpha})^\beta - 1}\right]^{\gamma_1-1}
\]

\[
\times e^{(x_2/\alpha)^\beta} \left(\frac{x_2}{\alpha}\right)^{\beta-1} e^{-\lambda_3(e^{z/\alpha})^\beta - 1} \left[1 - e^{-\lambda_2(e^{x_2/\alpha})^\beta - 1}\right]^{\gamma_2 + \gamma_3 - 1}
\]

\[
f_0(x, x) = \gamma_1 \gamma_2 \gamma_3 \lambda^3 \beta^3 e^{(x/\alpha)^\beta} \left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\lambda_1(e^{x/\alpha})^\beta - 1} \left[1 - e^{-\lambda_2(e^{x/\alpha})^\beta - 1}\right]^{\gamma_1 + \gamma_2 + \gamma_3 - 1}
\]

One can find that

\[
\begin{align*}
\int_0^\infty f_3(x, x) dx &= \frac{\gamma_3}{\gamma_1 + \gamma_2 + \gamma_3} f_{EMWE}(x; \gamma_1 + \gamma_2 + \gamma_3, \alpha, \beta, \lambda) \\
&= \frac{\gamma_3 \lambda \beta e^{(x/\alpha)\beta}}{(\frac{\lambda \beta e^{(x/\alpha)\beta}}{\alpha})^{\beta-1} e^{-\lambda \alpha (e^{(x/\alpha)\beta} - 1)}} \left[ 1 - e^{-\lambda \alpha (e^{(x/\alpha)\beta} - 1)} \right]^{\gamma_1 + \gamma_2 + \gamma_3 - 1}.
\end{align*}
\]

**Proof** Let us first assume that \( x_1 < x_2 \). Then, the expression for \( f_1(x_1, x_2) \) can be simply obtained by differentiating the joint CDF \( F_{EMWE}(x_1, x_2) \) given in (3) with respect to \( x_1 \) and \( x_2 \). Similarly, we find the expression of \( f_2(x_1, x_2) \) when \( x_2 < x_1 \). But \( f_3(x, x) \) cannot be derived in a similar method. For this reason, we use the following identity to derive \( f_3(x, x) \).

\[
\int_0^\infty f_1(x_1, x_2) dx_1 dx_2 + \int_0^\infty f_2(x_1, x_2) dx_2 dx_1 + \int f_3(x, x) dx = 1
\]

Let

\[
I_1 = \int_0^\infty f_1(x_1, x_2) dx_1 dx_2 \quad \text{and} \quad I_2 = \int_0^\infty f_2(x_1, x_2) dx_2 dx_1
\]

One can find that

\[
I_1 = \int_0^\infty \frac{\gamma_2 \lambda \beta e^{(x_2/\alpha)\beta}}{(\frac{\lambda \beta e^{(x_2/\alpha)\beta}}{\alpha})^{\beta-1} e^{-\lambda \alpha (e^{(x_2/\alpha)\beta} - 1)}} \left[ 1 - e^{-\lambda \alpha (e^{(x_2/\alpha)\beta} - 1)} \right]^{\gamma_1 + \gamma_2 + \gamma_3 - 1} dx_2
\]

and

\[
I_2 = \int_0^\infty \frac{\gamma_1 \lambda \beta e^{(x_1/\alpha)\beta}}{(\frac{\lambda \beta e^{(x_1/\alpha)\beta}}{\alpha})^{\beta-1} e^{-\lambda \alpha (e^{(x_1/\alpha)\beta} - 1)}} \left[ 1 - e^{-\lambda \alpha (e^{(x_1/\alpha)\beta} - 1)} \right]^{\gamma_1 + \gamma_2 + \gamma_3 - 1} dx_1
\]

Substituting from (10) and (11) into (9) we obtain

\[
\int_0^\infty f_3(x, x) dx = 1 - I_1 - I_2
\]

Thus,

\[
f_3(x, x) = \frac{\gamma_3 \lambda \beta e^{(x/\alpha)\beta}}{(\frac{\lambda \beta e^{(x/\alpha)\beta}}{\alpha})^{\beta-1} e^{-\lambda \alpha (e^{(x/\alpha)\beta} - 1)}} \left[ 1 - e^{-\lambda \alpha (e^{(x/\alpha)\beta} - 1)} \right]^{\gamma_1 + \gamma_2 + \gamma_3 - 1},
\]

\[\text{(8)}\]
2.3 Marginal probability density functions

The following theorem gives the marginal probability density functions of \( X_i, i = 1, 2 \) is given by

\[
f_{X_i}(x_i) = f_{EMWE}(x_i; \gamma_i + \gamma_3, \alpha, \beta, \lambda), \quad x_i > 0, \ i = 1, 2
\]

\[
= (\gamma_i + \gamma_3) \lambda \beta e^{(x_i/\alpha)\beta} (\frac{x_i}{\alpha})^\gamma_i - 1 e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1} [1 - e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1}]^{(\gamma_i + \gamma_3) - 1}
\]

\( \text{Proof:} \) The marginal cumulative distribution function for \( X_i \) is

\[
F(x_i) = P(X_i \leq x_i) = P(\max\{U_i, U_3\} \leq x_i) = P(U_i \leq x_i, U_3 \leq x_i).
\]

As the random variables \( U_i \ (i = 1, 2) \) and \( U_3 \) are mutually independent, we directly obtain

\[
F(x_i) = P(U_i \leq x_i) P(U_3 \leq x_i)
\]

\[
= \left[ 1 - e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1} \right]^{\gamma_i} \left[ 1 - e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1} \right]^{\gamma_3}
\]

\[
= \left[ 1 - e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1} \right]^{\gamma_i + \gamma_3} = F_{EMWE}(x_i; \gamma_i + \gamma_3, \alpha, \beta, \lambda).
\]

From which we readily derive the pdf of \( X_i, f(x_i) = \frac{\partial}{\partial x_i} F(x_i) \), as in (12).

2.4 Conditional probability density functions

The following theorem gives the marginal probability density functions of \((X_1, X_2)\).

\[
\text{Theorem 2.3.} \quad \text{The conditional probability density function of} \quad X_i \text{ given} \quad X_j = x_j, \quad (i, j = 1, 2, i \neq j) \quad \text{is given by}
\]

\[
f_{X_i|X_j}(x_i | x_j) = \begin{cases} f_{X_i|X_j}(x_i | x_j) & \text{if} \ 0 < x_i < x_j \\ f_{X_i|X_j}(x_i | x_j) & \text{if} \ 0 < x_j < x_i \\ f_{X_i|X_j}(x_i | x_j) & \text{if} \ x_i = x_j > 0 \end{cases}
\]

where

\[
f_{X_i|X_j}(x_i | x_j) = \frac{\gamma_j (\gamma_i + \gamma_3) \lambda \beta e^{(x_i/\alpha)\beta} (\frac{x_i}{\alpha})^\gamma_i - 1 e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1} [1 - e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1}]^{(\gamma_i + \gamma_3) - 1}}{(\gamma_j + \gamma_3) [1 - e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1}]^{\gamma_i + \gamma_3 - 1}},
\]

\[
f_{X_i|X_j}(x_i | x_j) = \gamma_i \lambda \beta e^{(x_i/\alpha)\beta} (\frac{x_i}{\alpha})^\gamma_i - 1 e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1} [1 - e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1}]^{\gamma_i - 1}
\]

and

\[
f_{X_i|X_j}(x_i | x_j) = \frac{\gamma_3}{\gamma_i + \gamma_3} [1 - e^{-\lambda \alpha (e^{x_i/\alpha})^{\beta} - 1}]^{\gamma_i}.
\]
Proof: The proof follows immediately by substituting the joint probability density function of \((X_1, X_2)\) given in (6), (7) and (8) and the marginal probability density function of given in (12), using the relation
\[
f_{X_i|X_j}(x_i | x_j) = \frac{f_{X_i,X_j}(x_i,x_j)}{f_{X_i}(x_i)}, \quad (i = 1, 2).
\]

3 Reliability studies

In this section, we present the joint survival function of \((X_1, X_2)\), the CDF of the random variable \(Y = \max\{X_1, X_2\}\) and the CDF of the random variable \(W = \min\{X_1, X_2\}\).

3.1 Joint survival function

In this subsection, we derive the joint survival function of \((X_1, X_2)\) in a compact form.

Theorem 3.1. The joint survival function of \((X_1, X_2)\) is given by
\[
S_{X_1,X_2}(x_1, x_2) = \begin{cases} 
S_1(x_1, x_2) & \text{if } x_1 < x_2 \\
S_2(x_1, x_2) & \text{if } x_2 < x_1 \\
S_0(x, x) & \text{if } x_1 = x_2 = x
\end{cases},
\]
\[\text{(14)}\]

where
\[
S_1(x_1, x_2) = 1 - \left[1 - e^{-\lambda \alpha (e^{(r_2/\alpha)^\beta} - 1)}\right]^{\gamma_2 + \gamma_3} - \left[1 - e^{-\lambda \alpha (e^{(r_1/\alpha)^\beta} - 1)}\right]^{\gamma_1 + \gamma_3}
\times \left(1 - \left[1 - e^{-\lambda \alpha (e^{(r_2/\alpha)^\beta} - 1)}\right]^{\gamma_2}\right),
\]
\[
S_2(x_1, x_2) = 1 - \left[1 - e^{-\lambda \alpha (e^{(r_1/\alpha)^\beta} - 1)}\right]^{\gamma_1 + \gamma_3} - \left[1 - e^{-\lambda \alpha (e^{(r_2/\alpha)^\beta} - 1)}\right]^{\gamma_2 + \gamma_3}
\times \left(1 - \left[1 - e^{-\lambda \alpha (e^{(r_1/\alpha)^\beta} - 1)}\right]^{\gamma_1}\right)
\]

and
\[
S_0(x, x) = 1 - \left[1 - e^{\lambda \alpha (1-e^{(x/\alpha)^\beta})}\right]^{\gamma_3} \times
\left(\left[1 - e^{-\lambda \alpha (e^{(x/\alpha)^\beta} - 1)}\right]^{\gamma_1} + \left[1 - e^{-\lambda \alpha (e^{(x/\alpha)^\beta} - 1)}\right]^{\gamma_2} - \left[1 - e^{-\lambda \alpha (e^{(x/\alpha)^\beta} - 1)}\right]^{\gamma_1 + \gamma_2}\right).
\]

Proof: The joint survival function of \((X_1, X_2)\) can be obtained from the following relation
\[
S_{X_1,X_2}(x_1, x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1,X_2}(x_1, x_2).
\]
\[\text{(15)}\]
Substituting from (3) and (13) in (15), we get
\[
S_{X_1,X_2}(x_1, x_2) = 1 - \left[ 1 - e^{-\lambda_0 (e^{x_1/\alpha})^\beta - 1} \right]^{\gamma_1 + \gamma_3} - \left[ 1 - e^{-\lambda_0 (e^{x_2/\alpha})^\beta - 1} \right]^{\gamma_2 + \gamma_3} + \\
\left[ 1 - e^{-\lambda_0 (e^{x_1/\alpha})^\beta - 1} \right]^{\gamma_1} \left[ 1 - e^{-\lambda_0 (e^{x_2/\alpha})^\beta - 1} \right]^{\gamma_2} \left[ 1 - e^{-\lambda_0 (e^{x_3/\alpha})^\beta - 1} \right]^{\gamma_3}
\]
where \( z = \min(x_1, x_2) \). From (16) we can be obtained simply the expressions of \( S_1(x_1, x_2) \), \( S_2(x_1, x_2) \) and \( S_0(x_1, x_2) \) for \( x_1 < x_2 \), \( x_2 < x_1 \) and \( x_1 = x_2 = x \) respectively, which completes the proof.

**Comment 3.1.** Basu (1971) defined the bivariate failure rate function \( h(x_1, x_2) \) for the random vector \( (X_1, X_2) \) as the following relation
\[
h_{X_1,X_2}(x_1, x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{S_{X_1,X_2}(x_1, x_2)}.
\]
(17)
We can be obtained the bivariate failure rate function \( h(x_1, x_2) \) for the random vector \( (X_1, X_2) \) by substituting from (5) and (14) in (17).

**Lemma 3.1.** The CDF of the random variable \( Y = \max\{X_1, X_2\} \) is given as
\[
F_Y(y) = \left[ 1 - e^{-\lambda_0 (e^{y/\alpha})^\beta - 1} \right]^{\gamma_1 + \gamma_2 + \gamma_3}.
\]
(18)

**Proof:** Since
\[
F_Y(y) = P(Y \leq y) = P(\max\{X_1, X_2\} \leq y) = P(X_1 \leq y, X_2 \leq y) = P(U_1 \leq y, U_2 \leq y, U_3 \leq y),
\]
where the random variables \( U_i \) \( (i = 1, 2, 3) \) are mutually independent, we directly obtain
\[
F_Y(y) = P(U_1 \leq y)P(U_2 \leq y)P(U_3 \leq y) = F_{EMWE}(y; \gamma_1, \alpha, \beta, \lambda)F_{EMWE}(y; \gamma_2, \alpha, \beta, \lambda)F_{EMWE}(y; \gamma_3, \alpha, \beta, \lambda).
\]
(19)
Substituting from (1) in (19), we get (18) which completes the proof of the lemma.3.1.

**Comment 3.2.** From lemma 3.1. we can say that, if \( X_1 \) and \( X_2 \) are independent EMWE random variables then \( \max\{X_1, X_2\} \) is also EMWE random variable.

**Lemma 3.2.** The CDF of the random variable \( W = \min\{X_1, X_2\} \) is given as
\[
F_W(w) = \left[ 1 - e^{-\lambda_0 (e^{w/\alpha})^\beta - 1} \right]^{\gamma_1} + \left[ 1 - e^{-\lambda_0 (e^{w/\alpha})^\beta - 1} \right]^{\gamma_2} + \left[ 1 - e^{-\lambda_0 (e^{w/\alpha})^\beta - 1} \right]^{\gamma_3}
\]
(20)
Proof: Since

\[ F_W(w) = P(W \leq w) = P(\min\{X_1, X_2\} \leq w) = 1 - P(\min\{X_1, X_2\} > w) \]
\[ = 1 - P(X_1 > w, X_2 > w) = 1 - S(w, w) \quad (21) \]

Substituting from (14) into (21), we get

\[ F_W(w) = F_{X_1}(w) + F_{X_2}(w) - F_{X_1, X_2}(w, w). \quad (22) \]

Substituting from (3) and (13) in (22), we get (20) which completes the proof of the lemma 3.2.

4 The marginal expectation

In this section, we derive the marginal expectation of \( X_i \) \((i = 1, 2)\). The following theorem gives the \( r \)th moments of \( X_i \) \((i = 1, 2)\) as infinite series expansion.

Theorem 3.1. The \( r \)th moment of \( X_i \) \((i = 1, 2)\) is given by:

\[ E(X_i^r) = \frac{(\gamma_i + \gamma_3)\lambda}{\alpha^{\beta - 1}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( \begin{array}{c} \gamma_i + \gamma_3 - 1 \\ j \end{array} \right) \left( \begin{array}{c} 1 \\ j \end{array} \right) (-1)^{j+k} \lambda^k \alpha^{j+k+r} (j+1)^k (k+1)^{\beta+r} \Gamma(\beta) \Gamma(\beta+1). \quad (23) \]

Proof: We will start with the known definition of the \( r \)th moment of the random variables \( X_i \) with pdf \( f(x_i) \) given by

\[ E(X_i^r) = \int_0^\infty x_i^r f(x_i) dx_i. \]

Substituting for \( f_X(x_i) \) from (12), we get

\[ E(X_i^r) = \frac{(\gamma_i + \gamma_3)\lambda\beta}{\alpha^{\beta - 1}} \int_0^\infty x_i^{r+\beta - 1} e^{(x_i/\alpha)\beta} e^{-\lambda(x_i/\alpha)\beta} \left[ 1 - e^{-\lambda(x_i/\alpha)\beta} \right]^{(\gamma_i + \gamma_3) - 1} dx_i. \quad (24) \]

Since \( 0 < e^{-\lambda(x_i/\alpha)\beta} < 1 \) for \( x > 0 \), then by using the binomial series expansion of

\[ \left[ 1 - e^{-\lambda(x_i/\alpha)\beta} \right]^{(\gamma_i + \gamma_3) - 1} \]

given by

\[ = \sum_{j=0}^{\infty} \left( \begin{array}{c} \gamma_i + \gamma_3 - 1 \\ j \end{array} \right) (-1)^j e^{-j\lambda(x_i/\alpha)\beta}. \quad (25) \]

Substituting from (25) into (24), we get

\[ E(X_i^r) = \frac{(\gamma_i + \gamma_3)\lambda\beta}{\alpha^{\beta - 1}} \sum_{j=0}^{\infty} \left( \begin{array}{c} \gamma_i + \gamma_3 - 1 \\ j \end{array} \right) \left( -1 \right)^j \lambda^j e^{\lambda(j+1)\beta} \int_0^\infty x_i^{r+\beta - 1} e^{(x_i/\alpha)\beta} e^{-\lambda(j+1)(x_i/\alpha)\beta} dx_i. \]
Using the series expansion of $e^{-\lambda(x+1)}e^{(x_2/\alpha)}$, one gets

$$E(X_1) = \frac{(\gamma_1 + \gamma_3)\lambda\beta}{\alpha^{\beta-1}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} \lambda^k \alpha^{j+1} k!}{j!} e^{\lambda\alpha(j+1)} \times \int_0^{\infty} x_i^{\tau+\beta-1} e^{(x_i/\alpha)^\beta} dx_i.$$ 

Let $y = (k + 1)(x_i/\alpha)^\beta$ in the above integral, then we can get

$$E(X_1) = \frac{(\gamma_1 + \gamma_3)\lambda\beta}{\alpha^{\beta-1}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} \lambda^k \alpha^{j+1} k!}{j!} e^{\lambda\alpha(j+1)} \int_0^{\infty} y_i^{\tau+\beta-1} e^y dy$$ 

(26)

Since, $\Gamma(z) = \int_0^{\infty} e^{xt}t^{z-1} dt$, $z > 0$, $x > 0$, then

$$\int_0^{\infty} y_i^{\tau+\beta-1} e^y dy = \Gamma(\frac{\tau}{\beta} + 1).$$ 

(27)

Substituting from (27) into (26), we get (23). This completes the proof.

5 Maximum likelihood estimators

In this section, we use the method of maximum likelihood to estimate the unknown parameters of the BEMWE distribution. Consider constant values to the parameters $\alpha$, $\beta$, and $\lambda$ so, we want to estimate the other parameters $\gamma_1$, $\gamma_2$, and $\gamma_3$. Suppose that we have a sample of size $n$, of the form $\{(x_{11}, x_{21}), (x_{12}, x_{22}), ..., (x_{1n}, x_{2n})\}$ from BEMWE distribution. We use the following notation

$I_1 = \{x_{1i} < x_{2i}\}$, $I_2 = \{x_{1i} > x_{2i}\}$, $I_3 = \{x_{1i} = x_{2i} = x_i\}$, $I = I_1 \cup I_2 \cup I_3$, $|I_1| = n_1$, $|I_2| = n_2$, $|I_3| = n_3$, and $n_1 + n_2 + n_3 = n$.

Based on the observations, the likelihood function of the sample of size $n$ given by:

$$l(\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda) = \prod_{i=1}^{n_1} f_1(x_{1i}, x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i}, x_{2i}) \prod_{i=1}^{n_3} f_3(x_i, x).$$

The log-likelihood function can be written as

$$L(\gamma_1, \gamma_2, \gamma_3, \alpha, \beta, \lambda) = n_1 \ln(\gamma_2 (\gamma_1 + \gamma_3) \lambda^2 \beta^2) + \sum_{i=1}^{n_1} \frac{X_{1i}^2}{\alpha} - \lambda \alpha \sum_{i=1}^{n_1} (e^{(x_i/\alpha)^\beta} - 1)$$
In this subsection we consider the approximate confidence intervals of the parameters \( \gamma_1, \gamma_2, \) and \( \gamma_3 \), hence numerical technique is needed to get the MLEs. The solution of equations (29), (30) and (31) are not easy to solve, so numerical technique is needed to get the MLEs.

\[
-\lambda \alpha \sum_{i=1}^{n_1} (e^{(x_{2i}/\alpha)\beta} - 1) + (\gamma_2 - 1) \sum_{i=1}^{n_1} \ln(1 - e^{-\lambda \alpha (e^{(x_{2i}/\alpha)\beta} - 1)}) + (\beta - 1) \sum_{i=1}^{n_1} \ln\left(\frac{x_{2i} x_3}{\alpha^2}\right)
\]

\[
+ (\gamma_1 + \gamma_3 - 1) \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{1i}/\alpha)\beta} - 1)}) + \sum_{i=1}^{n_2} \left(\frac{x_{2i}}{\alpha}\right)^\beta + n_2 \ln\left(\gamma_1 (\gamma_2 + \gamma_3) \lambda^2 \beta^2\right)
\]

\[
+ \sum_{i=1}^{n_2} \left(\frac{x_{1i}}{\alpha}\right)^\beta - \lambda \alpha \sum_{i=1}^{n_2} (e^{(x_{1i}/\alpha)\beta} - 1) + \sum_{i=1}^{n_2} \left(\frac{x_{2i}}{\alpha}\right)^\beta - \lambda \alpha \sum_{i=1}^{n_2} (e^{(x_{2i}/\alpha)\beta} - 1)
\]

\[
+ (\gamma_1 - 1) \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{1i}/\alpha)\beta} - 1)}) + (\gamma_2 + \gamma_3 - 1) \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{2i}/\alpha)\beta} - 1)})
\]

\[
+ (\beta - 1) \sum_{i=1}^{n_2} \ln\left(\frac{x_1 x_2}{\alpha^2}\right) + n_3 \ln(\gamma_2 \lambda \beta) - \lambda \alpha \sum_{i=1}^{n_3} (e^{(x_{i}/\alpha)\beta} - 1) + \sum_{i=1}^{n_3} \left(\frac{x_i}{\alpha}\right)^\beta
\]

\[
+ (\beta - 1) \sum_{i=1}^{n_3} \ln\left(\frac{x_1}{\alpha}\right) + (\gamma_1 + \gamma_2 + \gamma_3 - 1) \sum_{i=1}^{n_3} \ln(1 - e^{-\lambda \alpha (e^{(x_{i}/\alpha)\beta} - 1)}).
\]

Computing the first partial derivatives of (28) with respect to \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) and setting the results equal zeros, we get the likelihood equations as in the following form

\[
\frac{\partial L}{\partial \gamma_1} = \frac{n_1}{\gamma_1 + \gamma_3} + \sum_{i=1}^{n_1} \ln(1 - e^{-\lambda \alpha (e^{(x_{1i}/\alpha)\beta} - 1)}) + \frac{n_2}{\gamma_1} \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{2i}/\alpha)\beta} - 1)})
\]

\[
+ \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{1i}/\alpha)\beta} - 1)}), \tag{29}
\]

\[
\frac{\partial L}{\partial \gamma_2} = \frac{n_1}{\gamma_2} + \sum_{i=1}^{n_1} \ln(1 - e^{-\lambda \alpha (e^{(x_{2i}/\alpha)\beta} - 1)}) + \frac{n_2}{\gamma_2 + \gamma_3} \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{2i}/\alpha)\beta} - 1)})
\]

\[
+ \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{2i}/\alpha)\beta} - 1)}), \tag{30}
\]

and

\[
\frac{\partial L}{\partial \gamma_3} = \frac{n_1}{\gamma_1 + \gamma_3} + \sum_{i=1}^{n_1} \ln(1 - e^{-\lambda \alpha (e^{(x_{1i}/\alpha)\beta} - 1)}) + \frac{n_2}{\gamma_2 + \gamma_3} \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{2i}/\alpha)\beta} - 1)})
\]

\[
+ \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda \alpha (e^{(x_{1i}/\alpha)\beta} - 1)}).
\]

To get the MLEs of the parameters \( \gamma_1, \gamma_2 \) and \( \gamma_3 \), we have to solve the above system of three non-linear equations. The solution of equations (29), (30) and (31) are not easy to solve, so numerical technique is needed to get the MLEs.

### 5.1 Asymptotic confidence bounds

In this subsection we consider the approximate confidence intervals of the parameters \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) by using variance covariance matrix \( I_0^{-1} \) see Lawless (2003), where \( I_0^{-1} \) is the inverse
of the observed information matrix

\[ I_0^{-1} = \left( \begin{array}{ccc}
\frac{\partial^2 L}{\partial \gamma_1^2} & \frac{\partial^2 L}{\partial \gamma_1 \partial \gamma_2} & \frac{\partial^2 L}{\partial \gamma_1 \partial \gamma_3} \\
\frac{\partial^2 L}{\partial \gamma_2 \partial \gamma_1} & \frac{\partial^2 L}{\partial \gamma_2^2} & \frac{\partial^2 L}{\partial \gamma_2 \partial \gamma_3} \\
\frac{\partial^2 L}{\partial \gamma_3 \partial \gamma_1} & \frac{\partial^2 L}{\partial \gamma_3 \partial \gamma_2} & \frac{\partial^2 L}{\partial \gamma_3^2}
\end{array} \right)^{-1} = \left( \begin{array}{ccc}
Var(\hat{\gamma}_1) & Cov(\hat{\gamma}_1, \hat{\gamma}_2) & Cov(\hat{\gamma}_1, \hat{\gamma}_3) \\
Cov(\hat{\gamma}_2, \hat{\gamma}_1) & Var(\hat{\gamma}_2) & Cov(\hat{\gamma}_2, \hat{\gamma}_3) \\
Cov(\hat{\gamma}_3, \hat{\gamma}_1) & Cov(\hat{\gamma}_3, \hat{\gamma}_2) & Var(\hat{\gamma}_3)
\end{array} \right). \tag{32}
\]

The derivatives in \( I_0^{-1} \) are given as follows

\[
\begin{align*}
\frac{\partial^2 L}{\partial \gamma_1^2} &= -\frac{n_1}{(\gamma_1 + \gamma_3)^2} - \frac{n_2}{\gamma_1^2}, \\
\frac{\partial^2 L}{\partial \gamma_1 \partial \gamma_2} &= -\frac{n_1}{\gamma_2} - \frac{n_2}{(\gamma_2 + \gamma_3)^2}, \\
\frac{\partial^2 L}{\partial \gamma_1 \partial \gamma_3} &= 0, \\
\frac{\partial^2 L}{\partial \gamma_2 \partial \gamma_1} &= 0, \\
\frac{\partial^2 L}{\partial \gamma_2^2} &= -\frac{n_2}{(\gamma_2 + \gamma_3)^2}, \\
\frac{\partial^2 L}{\partial \gamma_2 \partial \gamma_3} &= -\frac{n_2}{(\gamma_2 + \gamma_3)^2}.
\end{align*}
\]

and

\[
\frac{\partial^2 L}{\partial \gamma_3^2} = -\frac{n_1}{(\gamma_1 + \gamma_3)^2} - \frac{n_2}{(\gamma_2 + \gamma_3)^2} - \frac{n_3}{\gamma_3^2}.
\]

We can derive the \((1 - \delta)100\%\) confidence intervals of the parameters \(\hat{\gamma}_1, \hat{\gamma}_2\) and \(\hat{\gamma}_3\) by using variance covariance matrix as in the following forms

\[
\hat{\gamma}_i \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\gamma}_i)}, \quad i = 1, 2, 3.
\]

where \(Z_{\frac{\delta}{2}}\) is the upper \((\frac{\delta}{2})\)th percentile of the standard normal distribution.

6 Data analysis

In this section we present the analysis of a data set and we consider a constant value to the parameters \(\alpha, \beta\) and \(\lambda\) which take the values 0.1, 0.3 and 0.05 respectively. The data set has been represent the American Football (National Football League) League data and they are obtained from the matches played on three consecutive weekends in 1986. The data were first published in ‘Washington Post’ and they are also available in Csorgo and Welsh (1989). It is a bivariate data set, and the variables \(X_1\) and \(X_2\) are as follows: \(X_1\) represents the ‘game time’ to the first points scored by kicking the ball between goal posts, and \(X_2\) represents the ‘game time’ to the first points scored by moving the ball into the end zone. These times are of interest to a casual spectator who wants to know how long one has to wait to watch a touchdown or to a spectator who is interested only at the beginning stages of a game.

The data (scoring times in minutes and seconds) are represented in Table 1. Here also all the data points are divided by 100 just for computational purposes. The variables have the following structure: (i) \(X_1 < X_2\) means that the first score is a field goal, (ii) \(X_1 > X_2\), means the first score is an unconverted touchdown or safety, (iii) \(X_1 = X_2\) means the first score is a converted touchdown.
From this data, we find that the values of the unknown parameters \( \hat{\gamma}_1, \hat{\gamma}_2 \) and \( \hat{\gamma}_3 \) are 0.0416, 0.253 and 0.52 respectively and the log-likelihood equals (-250.28). By substituting the MLE of unknown parameters in (32), we get estimation of the variance covariance matrix as

\[
I_0^{-1} = \begin{pmatrix}
0.000842 & 0.0000395 & -0.000299 \\
0.0000395 & 0.00394 & -0.000299 \\
-0.000299 & -0.000299 & 0.00711
\end{pmatrix}.
\]

The 95% confidence intervals of \( \hat{\gamma}_1, \hat{\gamma}_2 \) and \( \hat{\gamma}_3 \) are (0,0.098), (0.130,0.376) and (0.355, 0.685) respectively.

7 Conclusions

In this paper we have introduced the bivariate exponentiated modified Weibull extension distribution whose marginals are exponentiated modified Weibull extension distributions. We discussed some statistical properties of the new bivariate distribution. Maximum likelihood estimates of the new distribution are discussed and we provided the observed Fisher information matrix. One real data set are analyzed using the new distribution.

References

[1] Al-Khedhairi, A. and El-Gohary, A. (2008). "A new class of bivariate Gompertz distributions" International Journal of Mathematics Analysis, 2(5), 235 – 253.
[2] Basu, A.P.(1971)."Bivariate failure rate". American Statistics Association, 66,103-104.
[3] B. Gompertz, (1824) On the nature of the function expressive of the law of human mortality and on the new mode of determining the value of life contingencies, Philosophical Transactions of Royal Society A115, pp. 513-580.
[4] Chen, Z. (2000). "A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function". Statistics and Probability Letters, 49, 155–161.
[5] Csorgo, S. and Welsh, A.H. (1989). "Testing for exponential and Marshall-Olkin distribution". Journal of Statistical Planning and Inference, vol. 23, 287-300.

[6] El-Gohary, A., Alshamrani, A. and Al-Otaibi, A. N. (2013). "The Generalized Gompertz Distribution". Journal of Applied Mathematical Modelling, 37(1-2), 13-24.

[7] El-Sherpieny, E. A., Ibrahim, S. A., and Bedar, R. E. (2013). "A new bivariate generalized Gompertz distribution". Asian Journal of Applied Sciences, 1-4, 2321 – 0893.

[8] Johnson, N.L. and Kotz, S. (1975). "A vector valued multivariate hazard rate". Journal of Multivariate Analysis, 5, 53-66.

[9] Kundu, D., Gupta, K. (2013)." Bayes estimation for the Marshall–Olkin bivariate Weibull distribution". Journal of Computational Statistics and Data Analysis, 57(1), 271–281.

[10] Kundu, D. and Gupta, R. D. (2009). "Bivariate generalized exponential distribution". Journal of Multivariate Analysis, 100(4), 581-593.

[11] Kundu, D., Gupta, K. (2013)." Bayes estimation for the Marshall–Olkin bivariate Weibull distribution". Journal of Computational Statistics and Data Analysis, 57(1), 271–281.

[12] Lawless, J. F. (2003), "Statistical Models and Methods for Lifetime Data". John Wiley and Sons, New York, 20, 1108-1113.

[13] Sarhan, A. and Balakrishnan, N. (2007). "A new class of bivariate distributions and its mixture". Journal of the Multivariate Analysis, 98, 1508-1527.

[14] Xie, M., Tang, Y., and Goh, T. N. (2002). "A modified Weibull extension with bathtub-shaped failure rate function". Reliability Engineering and System Safety, 76, 279–285.