Persistent Homology of Weighted Visibility Graph from Fractional Gaussian Noise

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In this paper, we utilize persistent homology technique to examine the topological properties of the visibility graph constructed from fractional Gaussian noise (fGn). We develop the weighted natural visibility graph algorithm and the standard network in addition to the global properties in the context of topology, will be examined. Our results demonstrate that the distribution of eigenvector and betweenness centralities behave as power-law decay. The scaling exponent of eigenvector centrality and the moment of eigenvalue distribution, \( M_n \), for \( n \geq 1 \) reveal the dependency on the Hurst exponent, \( H \), containing the sample size effect. We also focus on persistent homology of \( k \)-dimensional topological holes incorporating the filtration of simplicial complexes of associated graph. The dimension of homology group represented by Betti numbers demonstrates a strong dependency on the Hurst exponent. More precisely, the scaling exponent of the number of \( k \)-dimensional topological holes appearing and disappearing at a given threshold, depends on \( H \) which is almost not affected by finite sample size. We show that the distribution function of lifetime for \( k \)-dimensional topological holes decay exponentially and corresponding slope is an increasing function versus \( H \) and more interestingly, the sample size effect is completely disappeared in this quantity. The persistence entropy logarithmically grows with the size of visibility graph of system with almost \( H \)-dependent prefactors.

Keywords: Topological Data Analysis, Persistent Homology, Fractional Gaussian Noise, Weighted Natural Visibility Graph, Topological Persistence, Persistence Entropy

I. INTRODUCTION

A powerful approach to study different types of data sets ranging from point cloud data (PCD), scalar field to complex network (graph), particularly a high-dimensional data is called topological data analysis (TDA) \[\text{[1–6]}\]. TDA as an application of algebraic topology \[\text{[7–9]}\] and a branch of computational topology \[\text{[10]}\], analyzes the shape of high-dimensional complex data in terms of global features (number of connected components, loops, voids, etc.) of topological space underlying the data set. In the persistent homology (PH) technique, as of a main part of TDA, the topological approximation of phase space of any type of data sets which is called simplicial complex is assigned to the associated data, then topological invariants are computed.

The PH aims to capture topological evolution of data set by varying scale (parameter), and extracts topological invariants of data set in each scale summarizing them in different representations, persistence barcode (PB) \[\text{[11, 12]}\], persistence diagram (PD) \[\text{[13, 14]}\], persistence landscape (PL) \[\text{[15]}\], persistence image (PI) \[\text{[16]}\], persistence surface (PS) and \( \beta \)-curve, which reveal topological information of data set. Being robust to noise, PH is able to show us the essential features of the systems with high internal degrees of freedom and is capable to classify underlying data sets \[\text{[17, 18]}\]. The PH technique has attracted much attention due to its vast applications on analyzing complex networks \[\text{[19–21]}\]. Also it has been used in various systems (see e.g. \[\text{[22–31]}\] and references therein).

There are many algorithms to assign a network (graph) to different types of data sets. As an illustration, the Mapper algorithm constructing the Reeb graphs (topological networks) from high-dimensional PCD \[\text{[32–35]}\]. The visibility graph technique makes a network so-called visibility graph (VG) for a typical time-series (one-dimensional scalar field). The idea of VG, being a complex network constructed by considering the visibility algorithm, proposed by Lacasa et al as a novel way to analyze time-series in terms of complex networks language \[\text{[36]}\]. The associate networks can be examined by various methods \[\text{[1, 37]}\]. The advantage of this idea is that, one can apply many well-known techniques in networks and even in TDA such as PH for a time-series, helping for classification, discrimination and looking for exotic features hidden in the underlying time-series which could be robust in the presence of noise, trends and irregularities \[\text{[38–40]}\]. The notion of mapping the time-series to a network has been applied on different topics (see \[\text{[41–51]}\] and references therein).

As a model containing correlations tuned by one parameter (the Hurst exponent \( H \)), the fractional Gaussian noise (fGn) time-series has been intensely investigated by many methods. To quantify the properties of a given self-similar data set or a generic series whose power spectrum behaves as power-law in frequency (wavelength) domain, many methods have been proposed concerning the trends and noises which may affect the observed time-series. Many preceding methods are implemented either in time domain or frequency space. A well-studied method is multi-fractal detrended fluctuation analysis (MF DFA) \[\text{[52, 53]}\], implemented in various areas (see \[\text{[54–63]}\] and references therein).

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method to estimate results for a typical fGn data \cite{41, 42, 91}. An alternative series into a network is able to represent some interesting on the accurate estimation of Hurst exponent by some of On the other hand, the finite size of time-series impacts several robust methods have been proposed \cite{64, 86–89}. In order to eliminate the effect of trend as much as possible, trends and finite size effect have not been diminished introduced \cite{82}. In spite of many advantages brought by trended cross-correlation analysis (MFDXA) has been introduced \cite{75–81}. Taking into account the cross-correlation has also been established and utilized to eliminate the effect of trend as much as possible, higher-order detrended covariance, the multi-fractal de-termining whether an input data is a fGn signal or not associated with a self-similar series.

H

The VG approach which is derived by mapping a time-series into a network is able to represent some interesting results for a typical fGn data \cite{41, 42, 91}. An alternative method to estimate \( H \), can be devoted to graph theoretical algorithm \cite{11}. Despite of huge literature, very little attention has been paid to topological properties of VG associated with a self-similar series.

In this regard, knowing a given time-series belongs to a fGn class, some relevant questions can be raised: i) Does the topological aspect of associated VG depend on the Hurst exponent? ii) What are the effects of sample size, trends and irregularity of fGn signal on the topological properties of VG? iii) How is the multifractality expressed by persistence homology? Motivated by mention questions, we focus on the persistent homology of weighted visibility graph constructed from a typical fGn by using the filtration process and considering higher-order connections (\( k \)-cliques ; \( k > 2 \)) in all thresholds (weights) \cite{92, 93}, to examine the dependency of relevant results to \( H \) and footprint of finite size effect. We show that, in contrast to local statistical features considered in this paper which are less-sensitive to auto-correlation of fGns, the topological features are sensitive to the value of corresponding Hurst exponent and are size-independent.

The contribution of trends and irregularity and even determining whether an input data is a fGn signal or not are beyond the scope of current study.

This paper is organized as follows: In the next section we present the network concepts to be used in the paper. The Sec. \( \text{III} \) is devoted to how VGs are obtained for a time-series. The weight functions are introduced in this section. The numerical results are presented in Sec. \( \text{IV} \) which contains two subsections: local statistical properties, and topological properties. We close the paper with a summary and conclusion.

II. NETWORK ANALYSIS

In this paper, we aim to analyze the complex network of the visibility graph constructed from a fractional Gaussian noise (fGn), with an emphasis on the topological aspects. Besides this, we also compute some conventional statistical properties of mentioned signal. Therefore, it is worthy to introduce and present a short review of these analyzes, referring the interested readers to the relevant references such as \cite{94, 95}.

A. Statistical Analysis

In this subsection, we introduce some conventional quantities, would be used in the following sections, with a main focus on various centrality measures. Inspired by social network science, the centralities play an important role in identifying the key elements in a typical network, such as the most effective agents (the degree centrality or the eigenvector centrality), the easiest access agent (the closeness centrality), and the betweenness centrality.

Suppose that a network (graph) is represented by \( G = (V, E, w) \). Here, \( V \equiv \{v_i\}_{i=1}^N \) is node (vertex) set, \( E = V \times V \equiv \{e_{ij} = (v_i, v_j) \mid v_i, v_j \in V\}_{i,j=1}^N \) is link (edge) set and \( w : E \rightarrow \mathbb{R} \) is a weight function (threshold). Subsequently, the \textit{degree} of \( i \)-th node, \( v_i \), is the number of nodes straightly connected with underlying node by non-zero weight, and it is denoted by \( k_i \equiv \sum_j (1 - \delta_{0, w_{ij}}) \). The \textit{degree centrality}, \( c_i^C \), is defined by \( k_i - 1 \), which is apparently related directly to how important the underlying node is, since it is the number of agents that have connection with it. An important function concerning this quantity is the degree distribution, \( p(k) \), showing the probability distribution function for degree of all nodes in the network

\[
p(k) = \frac{1}{N} \sum_{i=1}^N \delta_{k, k_i} \tag{1}\]

The \textit{eigenvector centrality}, \( c_i^E \), is also defined via the eigenvalue equation

\[
\lambda c_i^E = \sum_{j=1}^N w_{ij} c_j^E \tag{2}\]

where \( \lambda \) is the eigenvalue. The maximum value of the \( \lambda \) spectrum, i.e. \( \lambda_{\text{max}} \) plays the dominant role in the network properties. The corresponding eigenvectors of which are denoted by \( c_i^{E, \text{max}} \), showing the importance of the nodes. To define the \textit{closeness centrality}, let us denote the shortest distance between nodes \( v_i \) and \( v_j \) by \( d_{ij} \) which is assumed to be \( N \) when there is no path connecting them (disconnected graphs). Then the \textit{closeness centrality} is defined by

\[
c_i^C = \frac{N - 1}{\sum_{j=1}^N d_{ij}} \tag{3}\]

and the \textit{betweenness centrality} is as follows

\[
c_i^B = \frac{1}{(N-1)(N-2)} \sum_{j=1,j\neq i}^N \sum_{k=1,k\neq i,j}^N \frac{n_{jk}(i)}{n_{jk}} \tag{4}\]
the \(n_{jk}\) is the number of geodesics from \(v_j\) to \(v_k\), and \(n_{jk}(i)\) is the number of geodesics from \(v_j\) to \(v_k\) which passing through node \(v_i\). Finally, another interesting quantity which enables us to assess the statistical properties of a network is known as \textit{clustering coefficient} of node \(v_i\). This shows how the first neighbors of node \(v_i\) are connected together

\[
C^CC_i = \frac{2!(k-2)!}{k!} \sum_{j=1}^{N} \sum_{k=1}^{N} (1 - \delta_{0,w_i})(1 - \delta_{0,w_j})(1 - \delta_{0,w_k}),
\]

which represents the fraction of available triangles in the network. In the upcoming subsection, we will give a brief introduction about algebraic topology.

**B. Topological Analysis: Algebraic Topology**

In addition to the conventional statistical analysis, introduced in the previous subsection, complex networks exhibit some interesting topological properties. Such properties enable us in classifying the underlying network in a feasible approach. It may utilize to recognize exotic features in data sets. Topology is generally refers to the global features in contrast to geometrical invariants of underlying objects or sets. Having two spaces represented by \(X\) and \(Y\), and they have same local (geometric) features if any relevant features are invariant under \textit{congruence}. While, mentioned spaces are topologically equivalent, if associated features are invariant under \textit{homeomorphisms}. In other words, they are homeomorphic. Homology theory plays a crucial role in the mathematical description of the relevant building block of a typical topological space, and reveal the connectedness of underlying space \[94–96\]. Based on such properties, for the sake of clarity, we will give a brief review on the building blocks of algebraic topology which are useful to set up homology groups.

**Simplex:** A \(k\)-simplex, \(\sigma_k\), is a convex-hull of any geometrically independent subset, accordingly \(\sigma_k \equiv \{x_0, x_1, ..., x_k\} \subseteq \mathbb{R}^D\). By this definition, a 0-simplex is a point, a 1-simplex is a segment of a line, a 2-simplex is a filled triangle, a 3-simplex is a filled tetrahedron and so on (Fig. 1).

**Face:** A \(l\)-simplex which is denoted by \(\sigma_l\), is a subset of \(k\)-simplex \((\sigma_l \subseteq \sigma_k)\) and it is so-called the \(l\)-face of \(k\)-simplex.

**Simplex:** A simplicial complex, \(\psi\), is a collection of simplices such that: any \(l\)-face of any \(k\)-simplex of a typical complex \((0 < l < k)\), is a member of complex. In addition, the non-empty intersection of any two simplices, \(\sigma_k\) and \(\sigma_m\), from complex, is a \(l\)-face of both simplices. Dimension of a complex is the maximum dimension of all simplices of the complex. According to the definition of complex, one can define \(k\)-ordered subcollection of complex \(\psi\) as follows

\[
\Sigma_k(\psi) \equiv \{ \sigma \in \psi | \text{dim}(\sigma) = k \}
\]

**Chain:** For a given simplicial complex, a \(k\)-chain \((k\text{-dimensional chain})\) is a linear combination of \(k\)-simplices of \(\psi\), defined by

\[
c_k \equiv \sum_{i=1}^{\lvert \Sigma_k(\psi) \rvert} a_i \sigma_k(i) \ ; \ \sigma_k(i) \in \Sigma_k(\psi), a_i \in \mathbb{F}
\]

**Boundary operator:** For the simplices in any dimension, the boundary operator \(\partial_k\) is an operator mapping \(\sigma_k\) to its boundary according to

\[
\partial_k(\sigma_k) \equiv \sum_{j=0}^{k} (-1)^j [x_0, x_1, ..., x_j-1, x_{j+1}, ..., x_k] \subseteq \sigma_k
\]

**Boundary:** A \(k\)-chain which is the boundary of a \((k+1)\)-chain, is called \(k\)-boundary, denoted by \(b_k\). The \(k\)-boundary group is the collection of all \(k\)-boundaries in complex \(\psi\)

\[
B_k(\psi) \equiv \left\{ c_k \in C_k(\psi) \left| \exists c_{k+1} \in C_{k+1}(\psi); \partial_{k+1}(c_{k+1}) = c_k \right. \right\} = \left\{ \{b^{(i)}_k\}_i \subseteq C_k(\psi) \right\}
\]

**Cycle:** A \(k\)-chain that has no boundary, is called a \(k\)-cycle denoted by \(z_k\) as

\[
\partial_k(z_k) = \emptyset
\]

The \(k\)-cycle group is defined as the collection of all \(k\)-cycles in complex \(\psi\)

\[
Z_k(\psi) \equiv \left\{ c_k \in C_k(\psi) \left| \partial_k(c_k) = \emptyset \right. \right\} = \left\{ \{z^{(i)}_k\}_i \subseteq C_k(\psi) \right\}
\]

Since "Boundaries have no boundary," therefore, we have

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**Face:** A \(l\)-simplex which is denoted by \(\sigma_l\), is a subset of \(k\)-simplex \((\sigma_l \subseteq \sigma_k)\) and it is so-called the \(l\)-face of \(k\)-simplex.
Among growing applicability of complex networks in many fields and interdisciplinary branches in science [100], a technique has been suggested which converts a time-series to a network, so-called visibility graph which takes time-value quantity representations of PH. As an illustration, k-dimensional persistence diagram of weighted complex \( \psi(w) \) which any complex with a distinct weight is a subcollection of any complex with higher weight

\[
\phi(\psi(w)) = \left( \psi(w) \left| \forall w' < w'' : \psi(w') \subseteq \psi(w'') \right. \right)_{w_{\max} \ldots w_{\min}}
\]

More precisely, PH technique enumerates \( k \)th Betti number of any subcomplex in \( \phi \) and assigns an ordered tuple \( w^{(h_k)} \equiv (w^{(h_k)}_{\text{birth}}, w^{(h_k)}_{\text{death}}) \) to existed \( k \)-dimensional topological hole. Here \( w^{(h_k)}_{\text{birth}} \) and \( w^{(h_k)}_{\text{death}} \) are the thresholds for which \( h_k \) appears (birth) and disappears (death) respectively. Since \( w^{(h_k)}_{\text{birth}} < w^{(h_k)}_{\text{death}} \), we can define the positive-value quantity \( \ell^{(h_k)} \equiv w^{(h_k)}_{\text{death}} - w^{(h_k)}_{\text{birth}} \) as persistency (lifetime) of \( k \)-dimensional hole. Persistence barcode (PB) or equivalently persistence diagram (PD) are the famous representations of PH. As an illustration, k-dimensional persistence diagram of weighted complex \( \psi(w) \) is a multiset \( PD_k\left(\phi(\psi(w))\right) \equiv (M, N) \), where \( M \equiv \left\{ w^{(h_k)} \right\}, h_k \in H_k(\psi(w)), \psi(w) \in \phi \) and \( N: M \rightarrow \mathbb{N} \) is the count function. Inspired by Shannon entropy for a typical state probability, one can define the persistence entropy (PE) of kth PD (PB). To this end, we construct the probability for lifetime of homology classes as

\[
p(\ell^{(h_k)}) = \frac{\ell^{(h_k)}}{L} ; \quad L \equiv \sum_{w^{(h_k)} \in M(PD_k)} \ell^{(h_k)}
\]

Therefore, the PE for \( k \)-dimensional persistent homology is defined by [97, 99]:

\[
P E_k = - \sum_{w^{(h_k)} \in M(PD_k)} p(\ell^{(h_k)}) \log p(\ell^{(h_k)})
\]

Relying on previous quantities, we try to characterize the synthetic fGn series.

III. VISIBILITY GRAPH

Among growing applicability of complex networks in many fields and interdisciplinary branches in science [100], a technique has been suggested which converts a time-series to a network, so-called visibility graph...
(VG) \(36\). Generally, suppose \(\{x\} \equiv \{x(t_i), i = 1, ..., N\}\) represents a real-valued time-series. One can construct a network, so-called visibility graph, denoted by \(G = (V,E,w)\), the \(V \equiv \{v_i\}^N_{i=1}\) is again node (vertex) set, and \(E \equiv \{e_{ij}\}^N_{i,j=1}\) is link (edge) set. The VG is defined by using the bijection as follows

\[
f : V \equiv \{v_i\}^N_{i=1} \leftrightarrow T \equiv \{t_i\}^N_{i=1} ; \ f(v_i) = t_i \quad (8)
\]

and the connections are constructed according to visibility condition between the nodes, i.e. the nodes \(v_i\) and \(v_j\) are connected if the node \(v_j\) is visible from the node \(v_i\) and vice versa, therefore the resulting graph is undirected (for more details on the properties of VGs, see \(36\)). In general, there are two ways to construct a network (graph) from a time-series: the horizontal visibility graph (HVG) \(39\ 101\ 102\) and the natural visibility graph (NVG) \(103\ 105\), the former is more sparse than the latter case and in this work we focus on the NVG. In Fig. 4 we show how a HVG (left panel) and a NVG (right panel) for a synthetic fGn series can be constructed. In a binary set up, the corresponding visibility graph chooses the range of the weights from a binary set, \(w^{(B)} : E \rightarrow \{0,1\}\), e.g. for a binary NVG (BNVG) the weight function can be written according to following relation

\[
w^{(BN)}_{ij} = \begin{cases} 1 & : \ |f(v_i) - f(v_j)| = 1 \\ \prod_{k=i+1}^{j-1} \Theta\Big(s_{ij} - s_{ik}\Big) & : \ |f(v_i) - f(v_j)| > 1 \end{cases}
\]

(9)

where \(\Theta\) is the step function, and \(s_{ij} \equiv \frac{\pi(f(v_j)) - \pi(f(v_i))}{f(v_j) - f(v_i)}\). The argument of the \(\Theta\) function being positive guarantees that the node \(v_j\) is visible from the node \(v_i\) and vice versa.

Since the edge in a BNVG has the weight 0 or 1, consequently, making it unsuitable for continuous filtering. Instead, we suggest the weighted version of the natural visibility graph (WNVG), by considering the weight function as follows

\[
w^{(WN)}_{ij} \equiv \begin{cases} \frac{s_{ij}}{j-i-1} & : \ |f(v_i) - f(v_j)| = 1 \\ \prod_{k=i+1}^{j-1} \Theta\Big(s_{ij} - s_{ik}\Big) \bigg|s_{ij} - s_{ik}\bigg|^{1/(j-i-1)} & : \ |f(v_i) - f(v_j)| > 1 \end{cases}
\]

(10)

There are two factors inside the product. In the second branch of Eq. (10), one is the step function just like the binary graph, and the other is the weight which is proportional to “how visible is the site \(j\) from \(i\) and vice versa”, i.e. the more distinguishable the data points are in the original time-series, the higher the corresponding weight is in the constructed network. The term \(\frac{1}{j-i-1}\) is necessary to make the weights reasonable numbers for comparison reasons. In the absence of this exponent, the more the distance between the nodes are, the higher the corresponding weights are. For both statistical and the topological analysis, we use this weight function which admits continuous filtering.

\section{A. Synthetic Fractional Gaussian Noise}

In order to modeling the stochastic fractal processes, Mandelbrot and Van Ness introduced the fractional Brownian motion (fBm) and fractional Gaussian noise (fGn) \(106\). The mathematical generalization of the classical random walk and Brownian motion is given by the theory of fBm \(106\ 108\). A 1-dimensional fBm is represented by \(B \equiv \{B(t) : t \geq 0\}\), with power-law variance is the fractional Brownian motion (fBm), for which \(\text{Var}(B(at)) \equiv \text{Var}(a^{2H}B(t)) = a^{2H}\text{Var}(B(t))\), where \(H \in (0,1)\) is called Hurst exponent. For this random force, the Markov property and the stationarity are violated (note that when we have domain Markov property, stationarity and continuity for a time-series, then it should be proportional to a 1-dimensional Brownian motion). A model for generating fBm (denoted by \(B_H\) to emphasis on its Hurst exponent \(H\)) is a generalization of the Brownian motion which is non-stationary and non-Markov \(109\ 110\), is given by the Holmgren-Riemann-Liouville fractional integral

\[
B_H(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}} dB(s) \quad (11)
\]

where \(\Gamma\) is Gamma function, \(dB(s) = dB(s + ds) - B(s)\) is the increment of 1-dimensional Brownian motion and it has the following covariance:

\[
\langle B_H(t)B_H(s) \rangle = \frac{\sigma^2}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right), \quad (12)
\]

where \(\sigma^2 \equiv \langle B(0) \rangle\) and also \(\langle B_H \rangle = 0\). The common fBm is retrieved by setting \(H = \frac{1}{2}\). The increments, \(x_H(t) = \delta_t (B_H(t + \delta t) - B_H(t))\) are known as fractional Gaussian noise (fGn). For \(H > \frac{1}{2}\) (\(H < \frac{1}{2}\)) the corresponding fGn is positively (negatively) correlated, called the superdiffusive (subdiffusive) regime. Throughout this paper, we simulated various fGn with different
IV. RESULTS

Here we focus on the statistical (next subsection) and topological (the other subsection) properties of the VGs constructed from the fGn time-series, and investigate their behavior with respect to the Hurst exponent H. The networks of sizes N = 2^7, 2^8, 2^9, 2^10, 2^11, and 2^12 (that a desktop with 128 GB memory is capable to perform matrix operations) are considered and the ensemble averages are performed over 10^4 realizations. The Python toolbox “NetworkX”[111] is employed for the matrix operations on the graphs. In the topological analysis we especially focus on the Betti-0 (represented by the β_0 defined as the number of connected components of the network) and Betti-1 (represented by β_1 defined as the number of loops) features, which are extracted by using "Dionysus" Python package[112]. The persistence statistics, containing the lifetime (the interval between birth and death) of the topological features, and its Shannon entropy are also analyzed.

Each exponent has been estimated by Bayesian statistics accordingly, the {D} and {Υ} reveal the data and model free parameters, respectively. The posterior function is defined by

\[ P(Υ|D) = \frac{L(D|Υ)P(Υ)}{\int L(D|Υ)P(Υ)dΥ} \]  \hspace{1cm} (13)

where L is the likelihood and P(Υ) is the prior probability function containing all information concerning model parameters. Here we adopt the top-hat function for prior function whose window’s size depends on expected range of corresponding exponent. Taking into account the central limit theorem, the functional form of likelihood becomes multivariate Gaussian, i.e. \( L(D|Υ) \sim \exp(-\chi^2/2) \). The \( \chi^2 \) for determining the best-fit value for the scaling exponent reads as

\[ \chi^2(Υ) \equiv \Delta^\top C^{-1} \Delta \]  \hspace{1cm} (14)

where \( \Delta \) is a column vector whose elements are determined by difference between computed value and theoretical form for each measure and \( C \) is the corresponding covariance matrix. Finally, the best fit value of considering exponent is computed by maximizing the likelihood probability distribution and associated error-bar is given by

\[ 68.3\% = \int_{-\sigma_Υ}^{+\sigma_Υ} L(D|Υ)dΥ \]  \hspace{1cm} (15)

Subsequently, we report the best value of the scaling exponent at a 1σ confidence interval as \( Υ_{-\sigma_Υ}^{+\sigma_Υ} \).

A. Local Statistical Properties

By local properties, we mean the properties that are node-dependent, and are not necessarily globally defined. It has been confirmed that, the distribution function of the node degree of VGs of the fBms and fGns is power-law \( P(k) \propto k^{-\gamma} \) with the exponent \( \gamma(H) = 3 - 2H \) and \( \gamma(H) = 5 - 2H \), respectively[36,41]. In this subsection, we perform our calculations for the WNVG, introduced for the first time in this paper.
The probability distribution function of eigenvector \((c^B)\) and betweenness \((c^E)\) centralities are indicated in Fig. 5 in terms of \(H\) for WN VGs. The upper and lower right panels in this plot show the exponents of \(p(y) \sim y^{-\gamma_y}\), where \(y \equiv c^E, c^B\) computed for the proper range in which the probability distribution function reveals the scaling behavior (noticed by dashed-dot line). The \(\gamma_{cE}\) is almost constant for the negatively correlated regime \((0 < H < \frac{1}{2})\) and is a decreasing function of \(H\) for the positively correlated regime \((\frac{1}{2} < H < 1)\). The \(\gamma_{cE}\) is however constant for all \(H\) values, showing that the betweenness centrality is not affected by changing \(H\).

Also Fig. 6 illustrates the probability distribution function of closeness centrality \((c^C)\) and eigenvalue for various \(H\) exponent for WN VGs with different sizes. As shown in this plot, the \(p(c^C)\) does not depend on \(H\) (upper left panel of Fig. 6). The full spectrum of the eigenvalues (Eq. (2)) is illustrated in the lower left panel of Fig. 6. It shows the activity, or equivalently the strength of the weights of the network \([113]\). We see that the impact of \(H\) is changing the range of spectrum and by increasing Hurst exponent the range of spectrum for WN VGs become tight. This phenomenon can be understood recalling that, as a well-known fact, correlations (obtained by increasing \(H\)) smooths the underlying time-series, causing the corresponding network has more link with low weight. For more smoothed time-series, the typical slopes for associated WN VG become low, leading to lower weights according to Eq. (10), and equivalently making shorter range for the distribution of As. The right panels in Fig. 6 indicate different moments for \(p(c^C)\) (upper panel) and \(p(\lambda)\) (lower panel). The solid and dashed lines correspond to \(M_2\) and \(M_3\), respectively. Let us summary the results of this section:

1. The distribution of the eigenvector and betweenness centralities behave as power-law. The scaling exponent
for \( p(c^B) \) is almost independent of \( H \), while the \( \gamma_c, c \) behaves as decreasing function in terms of Hurst exponent. Precisely, for \( H < 0.5 \) the corresponding scaling exponent is almost constant versus \( H \) and it decreases by increasing Hurst exponent for \( H > 0.5 \).

2- Increasing correlation (i.e. increasing \( H \)) makes the time-series smooth, leading to more dense networks and shifts the weights to lower values, making the width of the distribution function of the eigenvalues smaller.

3- Fig. 5 confirm that the local observables are weakly dependent on the value of \( H \), but in the next subsection, we will show that the dependence of the topological observables on \( H \) are more considerable and more interestingly, the impact of sample size is almost diminished.

### B. Topological Properties

In this subsection, we rely on the statistics of topological measures to examine the structures embedded in the networks constructed by visibility graph mapping. For any given value of Hurst exponent, the associated BNVGs of fGns are topological tree, therefore, the BNVGs, are topologically equivalent to each others. In another word, various BNVGs are homeomorphic for different \( H \) according to homeomorphism theorem. Consequently, we only focus on topological aspects of WNVGs.

Generally, in the homology theory, the computable algebraic invariants of topological space are introduced. The Betti numbers are among the topological invariants counting the number of \( k \)-homology class. Such quantities represent the number of \( k \)-dimensional topological holes of associated simplicial complex produced by different methods such as clique simplicial complex strat-
In principle, the Betti numbers can be modified and become as a more simple measures known as Euler characteristics. Our purpose here is that, by computing topological invariants of underlying sets we attempt to distinguish between various series. To this end, we carry out the filtration approach on the WNVGs produced for each synthetic fGn and then, we will compute the set of Betti numbers, $\beta_k$ ($k = 0$ corresponds to connected components, $k = 1$ is associated with loops and so on). The filtration at (continuous) weight, $w$, is simply performed as follows: we connect nodes $v_i$ and $v_j$ if $w_{ij} < w$. It is worth noting that all values reported for relevant quantities have been determined by ensemble average as mentioned before.

For the first step, we compute the normalized number of simplices of weighted clique simplicial complex at the threshold for the WNVGs associated with fGns. Fig. 7 shows the normalized numbers of links (1-simplices), $|\Sigma_1|$, and triangles (2-simplices), $|\Sigma_2|$, for different Hurst exponents. By increasing $H$, the abundance of links and triangles grow at low thresholds, on the contrary, for $w > w^*$ regime, the abundance of links or triangles are reducing when the value of Hurst exponent increases. At this threshold, the value of $|\Sigma_1|(w^*)$ is nearly independent to $H$, while for $|\Sigma_2|$, we have a small range of $w$, such crossing happens. Nevertheless, we interpret $w^*$ as the crossover point, the scale which separates the low and high weights.

Now, we are going to evaluate topological measures. The upper panel of Fig. 8 indicates the $\beta_0$ (the number of connected components) as a function of filtration parameter (threshold) for clique complexes of WNVGs produced for various time-series with different values of Hurst exponent. To reduce the effect of sample size, we divide the Betti numbers by the samples size and called it as normalized Betti numbers. By increasing the value of Hurst exponent, the normalized $\beta_0$ versus threshold decrease. Namely, the number of connected components in the network decreases with increasing $H$ (the graphs become more steep). Interestingly, the slope of normalized $\beta_0$ as a function of $w$ is different for different $H$, consequently, the WNVGs of fGn sets reaches to connected regime (path-connected), $\beta_0 = 1$, (in the upper left panel of Fig. 8 we take $N = 1024$) by different rates and at different thresholds, $w_0$. The upper right panel of Fig. 8 represents the value of $w_0$ as a function of Hurst exponent for different networks size. As depicted in mentioned panel, the value of $w_0$ behaves as a sample size dependent quantity and by increasing $H$, such dependency becomes negligible.

We define the $\beta_k^{(\text{birth})}(w)$ and $\beta_k^{(\text{death})}(w)$ as the number of $k$-dimensional holes that born and die at a given threshold, $w$, respectively. It turns out that $\beta_0^{(\text{birth})} = 0$.
for $w > 0$, since at $w = 0$ the underlying network has $N$ connected components, therefore, all connected components born at $w = 0$. The lower left panel of Fig. 8 illustrates the $\beta_0^{(\text{death})}/N$ as a function of $w^2$ in the semi-log plot. As shown in this plot, one of the proper fitting function to describe the normalized $\beta_0^{(\text{death})}$ is given by $\beta_0^{(\text{death})}(w) \sim \exp(-\alpha_0^{(\text{death})}w^2)$ for $2 \lesssim w^2 \lesssim w_{\text{max}}^2$, where the value of $w_{\text{max}}$ depends on the $H$ value. The $\alpha_0^{(\text{death})}$ depends on Hurst exponent as increasing function and it behaves as an almost independent function on size of series (lower right panel of Fig. 8).

The upper left panel of Fig. 8 is devoted to $\beta_1/N$ for various Hurst exponents. As we expect, for trivial threshold, $w = 0$, we have $N$ connected components ($\beta_0 = N$) and therefore the number of loops is identically zero. By increasing the threshold, the higher value of Hurst exponent leads to more rapid in the increasing rate of $\beta_1$. On the other hand, for the high enough value of threshold, again the underlying data set behaves as a topological tree without any topological loops. Therefore, the normalized $\beta_1$ goes asymptotically to zero and such descending is more rapid for higher $H$. We also determine the lowest non-trivial threshold for which, there is no loop in underlying network denoted by $w_1$, and depict this threshold versus $H$ for different samples size in the upper right panel of Fig. 8. The $w_1$ is also size-dependent quantity. Comparing the $\beta_0/N$ and $\beta_1/N$ demonstrate that the by increasing threshold value, the WNVGs of $fGn$ series reach to loop-less regime ($\beta_1 = 0$) before appearing the connected regime (for which $\beta_0 = 1$), irrespective to Hurst exponent, i.e. $w_0(H) > w_1(H)$. The quantities $\beta_1^{(\text{birth})} \propto \exp(-\alpha_1^{(\text{birth})}w)$, $\beta_1^{(\text{death})} \propto \exp(-\alpha_1^{(\text{death})}w)$ in semi-log plots versus thresholds and corresponding slopes are illustrated in the middle and the lower panels of Fig. 8, respectively. The $\alpha_1^{(\text{birth})}$ and $\alpha_1^{(\text{death})}$ are
FIG. 9. Upper left panel shows the normalized $\beta_1$-curve for clique complexes of WNVGs associated with fGns of various Hurst exponent versus threshold, while, upper right panel indicates the $w_1$ as a function of $H$ for different length of data set. The distribution of persistence diagram versus birth-axis as a function of threshold. The middle right panel represents the corresponding coefficient of $\beta_1^{(\text{birth})}$ in terms of threshold known as $\alpha_1^{(\text{birth})}$ as a function of Hurst exponent. The lower panels are the same as middle just for dying 1-hole statistics. In this plot for computing $\beta_1$, we took $N = 1024$. 
The probability distribution of lifetime for 1-dimensional holes is depicted in left panel. Different symbols correspond to various $H$. This diagram has been obtained by doing ensemble average. Right panel is associated with the $\alpha_1^{\text{(lifetime)}}$ exponent for different sizes represented by different symbols.

Another interesting property to assess is the probability distribution of topological 1-dimensional in a 1-homology class. The left panel of Fig. 10 shows the probability distribution of lifetime for 1-holes which is different between the death and birth thresholds of a typical measure $\beta_1$. This result confirms that, $\alpha_1^{\text{(lifetime)}}$ can be considered as a robust measure for determining the Hurst exponent of fGn signal which is not affected by sample size even compared to $\alpha_0^{\text{(death)}}$, $\alpha_1^{\text{(birth)}}$ and $\alpha_1^{(\text{death})}$.

The trajectory of $\beta$-vector ($\beta$-curve) of weighted clique simplicial complex of WNVGs associated with fGns is also a sensitive criterion to reveal $H$-dependency. The overall behavior of mentioned trajectory does not depend on sample size during filtration process (the left panel of Fig. 11). By increasing $H$ the peak positions move to left showing that higher $H$ values favor smaller $\beta_0$s and higher $\beta_1$s as discussed above.

To make more complete our investigation, regarding the comparison of different topological measures, we de-
pict \(|\Sigma_1|\) (1-simplex), \(|\Sigma_2|\) (2-simplex), \(\beta_0\) (connected components) and \(\beta_1\) (loops) as a function of threshold, for \(H = 0.5\), in the right panel of Fig. 11. Each symbol has been computed by ensemble average on different sample size. The maximum value of \(|\Sigma_2|\) and \(\beta_1\) take place almost at the same threshold, while the peak of \(|\Sigma_1|\) is different, taking place at \(w\) which is approximately half the one for \(|\Sigma_2|\) and \(\beta_1\). We verify that mentioned treatments are almost independent to Hurst exponent.

The associated persistence entropies (PEs) defined by Eq. (7) are obtained using the persistence diagram. The upper and middle panels of Fig. 13 depict the \(PE_0\) and \(PE_1\), as a function of sample size in semi-log scale, respectively. We compute persistence entropy for all available Hurst exponent values represented by different symbols. Our results demonstrate that \(PE_k = A_k(H) \log N\) for \(k = 0, 1\). The behavior of pre-factor, \(A_k\) versus \(H\) is represented in the lower panel of Fig. 13. The \(A_0\) is almost an increasing function versus Hurst exponent, while the \(A_1\) becomes constant.

V. SUMMARY AND CONCLUDING REMARKS

In this paper, we used the persistent homology technique to examine the visibility graph (VG) constructed from fractional Gaussian noise (fGn), characterized by Hurst exponent \(H\). To this end, we developed a method to derive weighted visibility graph associated with a time-series. The statistical properties of the model were analyzed using the standard network measures. Our results revealed power-law behaviors for probability distribution function of \(eigenvector\) and \(betweenness centralities\) in the proper range of corresponding quantities (left panels of Fig. 5). The scaling exponent of \(betweenness centrality\) did not depend on \(H\), while the \(\gamma_c,H\) depends on Hurst exponent for \(H > 0.5\) and it almost remained constant for anti-correlated regime (right panels of Fig. 5).

The probability distribution function of \(closeness centrality\) and associated moments were independent from \(H\). Increasing the correlation results in to have more dense networks and shifts the weights to lower values, making the width of the distribution function of the eigenvalues to be more tight. The \(M_n(\lambda)\) behaved as \(H\)-dependent quantity. The higher order of moments showed a weak dependency on sample size (Fig. 6).

In the second part, we considered the topological properties of synthetic fGns. The normalized number of simplices (1-simplex and 2-simplex) for data sets with different \(H\) the abundance of links and triangles grow at low thresholds when we increase the \(H\). The threshold for which the \(|\Sigma_1|\) and \(|\Sigma_2|\) reach to their maximum is weakly dependent to Hurst exponent. We also defined a characteristic threshold for separating “low” from “high” weights. The value of \(|\Sigma_1|/(w^*)\) was nearly independent on \(H\), while for \(|\Sigma_2|\), we has a small range of \(w\), such crossing happens. The decreasing rate of Betti-0 which is a representative for the number of connected components with respect to threshold increase.
by increasing $H$. The threshold value for which the underlying WNNG reaches to connected regime (path-connected), $w_0$ indicated a $H$-dependency. Our result confirmed that, the value of $\alpha_0^{(\text{death})}$ which is the coefficient of $\beta_0^{(\text{death})}(w) \sim \exp(-\alpha_0^{(\text{death})} w^2)$ depends on Hurst exponent and interestingly it is almost size independent (Fig. 8). The statistics of 1-holes analyzed and we showed that the $w_1$ showing vanishing threshold for $\beta_1$ was $H$-dependent and it contained the sample size effect. The $\alpha_1^{(\text{birth})}$ and $\alpha_1^{(\text{death})}$ revealed the proper criteria for measuring Hurst exponent (Fig. 9).

The probability distribution of lifetime of 1-hole confirmed that the corresponding coefficient represented by $\alpha_1^{(\text{lifetime})}$ is an increasing function versus $H$ emphasizing that, the sample size effect is completely diminished in this quantity. Consequently, for a self-similar time-series in the absent of trends, the $\alpha_1^{(\text{lifetime})}$ can be a reliable measure for estimating Hurst exponent even for small sample size irrespective to value of $H$ (Fig. 10).

The persistence pairs (PPs) in persistence diagram (PD) and persistence barcode (PB) for weighted clique complex of WNNG which are indicators of persistent homology have been computed and by increasing the value of Hurst exponent are shrink to origin of coordinate (Fig. 12). We also computed persistence entropies (PEs) for 0-homology and 1-homology groups. Both quantities depend on sample size as expected and the corresponding slopes in semi-log scale were almost $H$-dependent (Fig. 13).

Finally, we emphasize that, the exponents of the local (statistical) observables depend weakly on $H$, whereas the exponents of global (topological) observables are almost strongly $H$-dependent. The footprint of sample size on the $\alpha_1^{(\text{lifetime})}$ is completely diminished.

In this paper, we have not verified that whether the persistent homology (PH) technique is capable to recognize the underlying time-series is a self-similar set or not. Indeed, this purpose is beyond the scope of this paper. In addition, it could be interesting to examine the effect of trends and irregularity which may occur in the wide range of events in the nature in the context of TDA and more precisely via persistent homology approach and we left them for future research. Above analysis can be done on different phenomena ranging from cosmology, astrophysics, economy to biology [73, 114–117].

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![FIG. 13. The upper panel indicates the persistence entropy for 0-homology group, $PE_0$, while, the middle is associated with $PE_1$ for 1-hole as a function of sample size in the semi-log scale. Different symbols correspond to various values of Hurst exponent. The lower panel shows the pre-factor of persistence entropy as a function of $H$ computed by ensemble average.](image-url)
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