Three-dimensional Gaussian fluctuations of non-commutative random surface growth with a reflecting wall

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Abstract

We consider the multi-time correlation and covariance structure of a random surface growth with a wall introduced in [10]. It is shown that the correlation functions associated with the model along space-like paths have determinantal structure, which yields the convergence of height fluctuations to that of a Gaussian free field. We also construct a continuous-time non-commutative random walk on $U(\mathfrak{so}_{N+1})$, which matches the random surface growth when restricting to the Gelfand-Tsetlin subalgebra of $U(\mathfrak{so}_{N+1})$. As an application, we prove the convergence of moments to an explicit Gaussian free field and get the covariance functions of the associated random point process along both the space-like paths and time-like paths. In particular, it does not match the three-dimensional Gaussian field from spectra of overlapping stochastic Wishart matrices in [16] even along the space-like paths.

1 Introduction

As an approach to study the Anisotropic Kardar–Parisi–Zhang (AKPZ) equation, many models in the AKPZ universality class were studied over the past decades (e.g. [5, 19, 21]). There has been lots of progress in understanding large time asymptotics of driven interacting particle systems on the $2+1$ dimensions random growth models in the AKPZ universality class (e.g. [3, 5, 7, 8]). For example, in [5], the authors constructed a class of two-dimensional random surface growth models, which can be interpreted as random point processes. It was shown that along space-like paths, the point processes are determinantal. The authors also established Gaussian fluctuations of one specific growing random surface along space-like paths by computing the correlation kernel and taking asymptotics. Later in [13], a continuous-time non-commutative random walk on $U(\mathfrak{gl}_N)$ (the universal enveloping algebra of $\mathfrak{gl}_N$) was introduced, which matches the random surface growth model introduced in [8] when restricting to the Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_N)$. As an application, the convergence to the Gaussian fluctuations along time-like paths was proved, completing the entire three-dimensional Gaussian field. In particular, it was also shown that this three-dimensional Gaussian field matches the one for eigenvalues of stochastic Wigner matrices in [2].

Another model of interest is the random surface growth with a reflecting wall constructed in [10], which can be viewed as an one-parameter family of Plancherel measures for the infinite-dimensional orthogonal group. It is believed that this model also belongs to the AKPZ universality class with certain boundary condition. Shown in Figure 1, the random surface growth is equivalent to a particle process in the quarter plane.

![Random surface growth with a wall](image)

Figure 1: random surface growth with a wall

In [10], the determinantal formula for the correlation functions at any finite time moment was derived. However, the authors did not provide the correlation functions along space-like paths since the evolution of

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measure does not match the general L-ensembles (see e.g. [4, 9, 11]). More specifically, this model has \( n \) particles at both \( (2n - 1/2 \pm 1/2) \)-th levels, thus the the corresponding L-ensemble should only have virtual variables for even levels. The Gaussian free field fluctuations for the height functions at any fixed time moment were later proved in [13]. In this paper, we first introduce a generalized L-ensemble, which implies the determinantal property and the formula for the correlation function. We then apply this generalization to extend the determinantal formula for the correlation functions to multi-time moments. Later we introduce a continuous-time non-commutative random walk on the universal enveloping algebra of the Lie group \( \mathfrak{so}_{N+1} \) (denoted \( U(\mathfrak{so}_{N+1}) \)), which is an analog of the non-commutative random walks on \( U(\mathfrak{gl}_N) \) (see e.g. [1] [12] [13]). It can be proved that this random walk matches the random surface growth. With the help of the non-commutative random walk, we obtain the three-dimensional Gaussian fluctuations and the covariance formulas are given.

Outline of the paper. In section 2, we review some facts about the algebras and the random surface growth. In section 3, we introduce a generalized L-ensemble to show that our model is a determinantal random process along space-like paths. Moreover, we compute the multi-time correlation kernel which extends the non-commutative random walk in section 4. In section 5, we show that moments of the random surface growth converge to Gaussian free fields along both space-like paths and time-like paths. In addition, the explicit covariance formulas are given.

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2 Preliminaries

2.1 Representation theory

We first review some useful results from representation theory (see e.g [10, 17, 18]). In this paper, for any positive integer \( N \) such that \( N = 2n + a - \frac{1}{2} \) with \( a \in \{ \pm \frac{1}{2} \} \), we abuse the notation for \( N \) and the corresponding pair \((n, a)\).

Let \( O(N+1) \) denote the group of \((N+1) \times (N+1)\) real-valued orthogonal matrices. The special orthogonal group \( SO(N+1) \) is the subgroup of \( O(N+1) \) consisting of matrices with determinant 1. The associated Lie algebra is denoted by \( \mathfrak{so}_{N+1} \), which consists of \((N+1) \times (N+1)\) skew-symmetric square matrices with Lie bracket the commutator. Last, let \( \Omega(\infty) = \bigcup_{N=0}^{\infty} O(N+1) \) and \( SO(\infty) = \bigcup_{N=0}^{\infty} SO(N+1) \).

Recall that if \( O \in SO(N+1) \), then the spectrum of \( O \) is of the form \( \{z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}\} \) if \( N+1 = 2n + 1 \), while when \( N+1 = 2n \), the spectrum of \( O \) is of the form \( \{z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}, 1\} \), where \( z_1 \) are roots of unity in both cases. Let \( \Omega \) be the set of all \( \omega = (\alpha, \beta, \delta) \) such that

\[
\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0) \in \mathbb{R}^{\infty}, \beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0) \in \mathbb{R}^{\infty}, \delta \in \mathbb{R}
\]

and

\[
\sum_{i=1}^{\infty}(\alpha_i + \beta_i) \leq \delta.
\]

For any \( \omega = (\alpha, \beta, \delta) \in \Omega \), we define a function

\[
\chi^{\omega}(O) = \prod_{j=1}^{n} E_{\omega} \left( \frac{z_j + z_j^{-1}}{2} \right),
\]

where

\[
E_{\omega}(x) = e^{(\delta - \sum_{i=1}^{\infty}(\alpha_i + \beta_i))(x-1)} \prod_{i=1}^{\infty} \frac{1 - \beta_i(1 - x) + \beta_i^2(1 - x)/2}{1 - \alpha_i(1 - x) + \alpha_i^2(1 - x)/2}.
\]
if we set \( x = (z + z^{-1})/2 \). Then each \( \omega \in \Omega \) identifies an extreme character \( \chi^\omega \) of \( O(\infty) \).

When \( n \in \mathbb{N} \), it is a classical result that the set of all irreducible representations of \( SO(2n + 1) \) over \( \mathbb{C} \) is parameterized by partitions of length \( \leq n \), which is a sequence of nonincreasing nonnegative integers \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0) \). For each partition \( \lambda \), denote the corresponding character of the irreducible representation of \( SO(2n + 1) \) by \( \chi^\lambda_{SO(2n+1)} \), whose dimension is denoted \( \dim_{SO(2n+1)} \lambda \). Similarly, the set of all irreducible representations of \( SO(2n) \) over \( \mathbb{C} \) is parameterized by sequences of integers \( \lambda = (\lambda_1, \ldots, \lambda_n) \) satisfying \( \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq |\lambda_n| \). Again, for each \( \lambda \), we denote the corresponding character of the irreducible representation of \( SO(2n) \) by \( \chi^\lambda_{SO(2n)} \) with dimension \( \dim_{SO(2n)} \lambda \).

Let \( J_n \) denote the set of all partitions of length \( \leq n \) and \( J_{n,a} \), with \( a \in \{\pm 1/2\} \) be two copies of \( J_n \). In what follows, we slightly abuse notations to interchange the use of \( \mathbb{C} \) matrices by the indices \( \{\pm 1/2\} \).

The commutation relation in \( F \) Denote by \( \tilde{\rho}^\omega_{n,a} \) the centre of \( U \) so that for any \( \rho^\omega_{n,a} \), the restriction of \( \tilde{\rho}^\omega_{n,a} \) to any \( SO(M) \) defines two measures \( P^\omega_{n,a} \) on \( J_{n,a} \) such that

\[
\chi^\omega|_{SO(2n+1)} = \sum_{\lambda \in J_n} P^\omega_{n,a}(\lambda) \frac{\chi^\lambda_{SO(2n+1)}}{\dim_{SO(2n+1)} \lambda},
\]

\[
\chi^\omega|_{SO(2n)} = \sum_{\lambda \in J_n} P^\omega_{n,a}(\lambda) \frac{\chi^\lambda_{SO(2n)} + \chi^\lambda_{SO(2n)}}{2 \dim_{SO(2n)} \lambda},
\]

where \( \lambda^* = (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n) \).

Let \( J^k_{a,b} \) denote the \( k \)-th Jacobi polynomial with parameters \( a, b \) (see e.g. [20]), define the constant \( c_k \) to be

\[
c_k = \begin{cases} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^{k+1}} & \text{if } k > 0, \\ 1 & \text{if } k = 0, \end{cases}
\]

and let \( J^k_{a,b} = J^k_{a,b}/c_k \).

Then for any \( \lambda \in J_{n,a} \), and \( \omega \in \Omega \), the measure \( P^\omega_{n,a} \) provided in [10] is as follows:

\[
P^\omega_{n,a}(\lambda) = C_{n,a} \times \det \left[ \psi_{n,a}^{i,j}(\lambda_i - i + n) \right]_{1 \leq i,j \leq n} \times \dim_{SO(2n+1/2+a)} \lambda,
\]

where

\[
\psi_{n-i}^{i,a}(k) = \frac{W^{(a-1/2)}(k)}{\pi} \int_{-1}^{1} E^\omega(x)(x-1)^{n-i}f^k_{a-1/2}(x)(1-x)^a(1+x)^{-1/2}dx,
\]

\[
W^{(a,b)}(k) = \begin{cases} 2, & \text{if } k > 0, \ a = b = -1/2, \\ 1, & \text{if } k = 0, \ a = b = -1/2, \\ 1, & \text{if } k \geq 0, \ a = 1/2, \ b = -1/2, \end{cases}
\]

and \( C_{n,a} \) is the normalization term for \( P^\omega_{n,a} \). In particular, \( P^{(0,0)}_{n,a} \) is the delta measure at the point \( (1, 2, \ldots, n) \).

For \( N + 1 = 2n + 1 \) or \( N + 1 = 2n \) respectively, we number the rows and columns of \( (N+1) \times (N+1) \) matrices by the indices \( \{-n, \ldots, -1, 0, 1, \ldots, n\} \) and \( \{-n, \ldots, -1, 1, \ldots, n\} \).

The Lie algebra \( \mathfrak{so}_{N+1} \) is spanned by the generators

\[
F_{ij} = E_{ij} - E_{j,-i}, \quad -n \leq i,j \leq n, \ i \neq -j.
\]

The commutation relation in \( \mathfrak{so}_{N+1} \) is

\[
[F_{ij}, F_{kl}] = \delta_{kj}F_{il} - \delta_{il}F_{kj} - \delta_{i,-k}F_{-j,l} + \delta_{-i,j}F_{k,-l}.
\]

The coproduct \( \Delta : U(\mathfrak{so}_{N+1}) \to U(\mathfrak{so}_{N+1}) \otimes U(\mathfrak{so}_{N+1}) \) is given by \( \Delta(F_{ij}) = F_{ij} \otimes 1 + 1 \otimes F_{ij} \).

Let \( Z(U(\mathfrak{so}_{N+1})) \) be the centre of \( U(\mathfrak{so}_{N+1}) \), we introduce the generators of \( Z(U(\mathfrak{so}_{N+1})) \) (see e.g. chapter 7 of [17]). Denote by \( F^{(m)} \) the matrix with entries \( F_{ij} \), where \( 1 \leq m \leq n \) and \( i,j = -m, -m+1, \ldots, m \). Denote by \( F^{(m)} \) the submatrix of \( F^{(m)} \) obtained by removing the row and column enumerated by \( -m \). Let \( \rho_i = -\rho_{-i} = -i + 1 \) if \( N + 1 \) is even, while \( \rho_i = -\rho_{-i} = -i + \frac{1}{2} \) if \( N + 1 \) is odd.
Let $\Lambda^{(m)}_k, \Phi^{(m)}_k, \tilde{\Lambda}^{(m)}_k, \tilde{\Phi}^{(m)}_k$ denote the noncommutative symmetric functions defined as the coefficients in the expansion of the following quasideterminant (see Definition 1.10.1 in [17]):

$$
1 + \sum_{k=1}^{\infty} \Lambda^{(m)}_k q^k = | 1 + q (F^{(m)} + \rho_m) |_{mm},
$$

$$
\sum_{k=1}^{\infty} \Phi^{(m)}_k q^{k-1} = - \frac{1}{ma} \log | 1 - q (F^{(m)} + \rho_m) |_{mm},
$$

$$
1 + \sum_{k=1}^{\infty} \tilde{\Lambda}^{(m)}_k q^k = | 1 + q (-F^{(m)} - \rho_m) |_{mm},
$$

$$
\sum_{k=1}^{\infty} \tilde{\Phi}^{(m)}_k q^{k-1} = - \frac{1}{ma} \log | 1 - q (-F^{(m)} - \rho_m) |_{mm}.
$$

In addition, there is a graphical construction of $\Lambda^{(m)}_k, \Phi^{(m)}_k, \tilde{\Lambda}^{(m)}_k, \tilde{\Phi}^{(m)}_k$.

For a $M \times M$ matrix $A$, we consider the complete oriented graph $A$ with vertices $1, 2, \ldots, M$ and label the arrow from vertex $i$ to $j$ by $a_{i,j}$. For each directed path of length $k$ in graph $A$ which starts from vertex $i$ and ends at vertex $j$, we define a monomial of the form $A_{i, i_1} A_{i_1, i_2} \cdots A_{i_{k-1}, j}$.

Let $A = p^{(m)} + \rho_m$, then $(-1)^{k-1} \Lambda^{(m)}_k$ is the sum of all monomials labeling simple path in $A$ of length $k$ going from $m$ to $m$. $\Phi^{(m)}_{2k}$ is the sum of all monomials labeling path in $A$ of length $k$ going from $m$ to $m$, the coefficient of each monomial being the ratio of $k$ to the number of roots to $m$. Similarly, $\Lambda^{(m)}_k$ and $\Phi^{(m)}_k$ are defined with $A = -F^{(m)} - \rho_m$.

Now, we define $\Lambda^{N+1}_{2n} = \prod_{m=1}^{n} \tilde{\Lambda}^{(m)}_1 \Lambda^{(m)}_1$ and $\Phi^{N+1}_{2k} = \sum_{m=1}^{n} \left( \Phi^{(m)}_{2k} + \tilde{\Phi}^{(m)}_{2k} \right)$.

**Example 1.** (1)

$$
\Phi^{N+1}_{2k} = 2 \sum_{m=1}^{n} \left( (F_{mm} + \rho_m)^2 + 2 \sum_{-m < i < m} F_{mi} F_{im} \right).
$$

(2)

$$
\Phi^{N+1}_{4k} = 2 \sum_{m=1}^{n} \left( (F_{mm} + \rho_m)^4 + 2 \sum_{-m < i < m} F_{mi} F_{im} F_{mj} F_{jm} \right.

+ 2 \sum_{-m < i, j < m} (F_{mm} + \rho_m) F_{mi} (F_{ij} + \delta_{ij} \rho_m) F_{jm} + 2 \sum_{-m < i, j < m} F_{mi} (F_{ij} + \delta_{ij} \rho_m) F_{jm} (F_{mm} + \rho_m)

+ 4 \sum_{-m < i, j < m} F_{mi} (F_{ij} + \delta_{ij} \rho_m) (F_{ji} + \delta_{ij} \rho_m) F_{im} + 2 \sum_{-m < i < m} F_{mi} F_{im} F_{-mi} F_{im} \right)

+ \frac{4}{3} \sum_{-m < i < m} \left( F_{mi} F_{im} (F_{mm} + \rho_m)^2 + (F_{mm} + \rho_m)^2 F_{mi} F_{im} + (F_{mm} + \rho_m) F_{mi} F_{im} (F_{mm} + \rho_m) \right).
$$

In Chapter 7 of [17], the explicit generators for $Z(U(\mathfrak{so}_{N+1}))$ were identified using the Harish-Chandra isomorphism with the ring of shifted symmetric polynomials.

**Theorem 2.1.** [17] When $N = 2n$, $Z(U(\mathfrak{so}_{N+1}))$ is generated by 1 and $\{ \Phi^{N+1}_{2k} \}_{k=1}^{n}$. When $N = 2n - 1$, $Z(U(\mathfrak{so}_{N+1}))$ is generated by 1, $\{ \Phi^{N+1}_{2k} \}_{k=1}^{n-1}$ together with $\sqrt{\Lambda^{N+1}_{2n}}$. Images of $\Phi^{N+1}_{2k}$ and $\Lambda^{N+1}_{2n}$ under the Harish-Chandra isomorphism are

$$
\Phi^{N+1}_{2k} \overset{\phi}{\rightarrow} l_1^{2k} + \cdots + l_n^{2k}, \quad \Lambda^{N+1}_{2n} \overset{\phi}{\rightarrow} (-1)^n l_1^{l_1} \cdots l_n^{l_n},
$$

where $l_i = \lambda_i + \rho_i$.

Thus, any $Y \in Z(U(\mathfrak{so}_{N+1}))$ acts as a multiplication by a scalar $p_Y (\lambda_1, \ldots, \lambda_n)$ in $V_\lambda$. When $N = 2n$, $p_Y (\lambda_1, \ldots, \lambda_n)$ is symmetric in the variables $l_1^2, \ldots, l_n^2$. When $N = 2n - 1$, $p_Y (\lambda_1, \ldots, \lambda_n)$ is the sum of a symmetric polynomial in $l_1^2, l_1^4$ and $l_1 \cdots l_n$ times a symmetric polynomial in $l_1^2, \ldots, l_n^2$.

### 2.2 Random surface growth with a wall

In this subsection, we provide more details about the model.
Consider the two-dimensional lattice $Z_{\ge 0} \times Z_+$. On each horizontal level $Z_+ \times \{N\}$, there are $n = \lfloor \frac{N+1}{2} \rfloor$ particles.

Denote the horizontal coordinates of all particles with vertical coordinate $N$ by $y_1^N > y_2^N > \cdots > y_n^N$. The reflecting wall forces $y_i^N \ge a + \frac{1}{2}$. We have particle configurations

$$\{y_k^N \in Z_{\ge 0} | k = 1, 2, \ldots, n; N = 1, 2, \ldots\}.$$  

Additionally, the particles satisfy the interlacing property $y_k^{N+1} < y_k^N < y_k^{N+1}$ for all meaningful values of $k$ and $N$.

The densely packed initial condition is given by the configuration where $y_k^N = N - 2k + 1$ for all $k$ and $N$, which means all the particles are as much to the left as possible. Each particle has two exponential clocks of rate $\frac{1}{2}$, one is responsible for left jump and another for right jump. All clocks are independent. When the clock for particle $y_k^N$ rings, the particle tries to jump by $1$ in the corresponding direction. Right jumps are blocked if $y_k^N + 1 = y_k^{N-1}$ and left jumps are blocked if $y_k^N - 1 = y_k^{N-1}$. If the particle is against the wall (i.e. $y_i^N = 0$) and the left jump clock rings, the particle is reflected and tries to jump to the right.

When $y_k^N$ tries to jump to the right and the jump is not blocked, we find the largest non-negative $r$ such that $y_k^{N+r} = y_k^N + i$ for $0 \le i \le r$, and all particles $\{y_i^N\}_{i=0}^r$ jump to right by $1$ simultaneously. If $y_k^N$ tries to jump to the left and is not blocked, we find the largest $l$ such that $y_k^{N-l} = y_k^N - j$ for $0 \le j \le l$, and all particles $\{y_i^N\}_{i=0}^l$ jump to left by $1$.

In words, in order to maintain the interlacing property, the jump of one particle is blocked by particles below it, but if the jump is not blocked, the particle pushes particles above it to jump together. Additionally, jumps are reflected against the wall.

We then recall several equivalent ways to define the particle configurations of our model.

It is worth mentioning that the particle configurations of the above point process are obtained as an equivalent interpretation of partitions of partitions \[10\].

When $\lambda \in \mathcal{J}_{n-}$ and $\mu \in \mathcal{J}_{n,+}$, we write $\lambda \prec \mu$ if $0 \le \lambda_n \le \mu_n \le \cdots \le \lambda_1 \le \mu_1$. When $\lambda \in \mathcal{J}_{n,+}$ and $\mu \in \mathcal{J}_{n+1,-}$, we write $\lambda \prec \mu$ if $0 \le \mu_{n+1} \le \lambda_n \le \cdots \le \lambda_1 \le \mu_1$. Next, set $\mathcal{J} = \bigcup_{n \ge 1} (\mathcal{J}_{n,+} \cup \mathcal{J}_{n,-})$. A path in $\mathcal{J}$ is a sequence $\lambda = (\lambda(1)^-, \prec \lambda(1)^+, \prec \lambda(2)^-, \prec \ldots)$ such that $\lambda(i) \in \mathcal{J}_{i, \pm}$. Last, let $\mathcal{J}^N$ be the set of finite paths in $\mathcal{J}$ of length $N = 2n - 1/2 + a$.

Set $\mathcal{X} = Z_{\ge 0} \times Z_{\ge 0} \times \{\pm \frac{1}{2}\}$ and $\mathcal{Q} = Z_{\ge 0} \times Z_{> 0} \cdot \mathcal{Q}$ is equivalent to $\mathcal{X}$ via the bijection sending $(x, n, a) \in \mathcal{X}$ to $(2x + a + \frac{1}{2}, 2n + a - \frac{1}{2}) \in \mathcal{Q}$.

To any path $\lambda = (\lambda(1)^-, \prec \lambda(1)^+, \prec \lambda(2)^-, \prec \ldots) \in \mathcal{J}$, we associate point configurations in $\mathcal{X}$ and $\mathcal{Q}$ as follows:

$$\mathcal{L}_\mathcal{X}(\lambda) = \{(x_k^{n,a}, n, a) : 1 \le k \le n, a \in \{\pm 1/2\}, n \ge 1\},$$

$$\mathcal{L}_\mathcal{Q}(\lambda) = \{(y_k^{2n-1/2+a}, 2n - 1/2 + a) : 1 \le k \le n, a \in \{\pm 1/2\}, n \ge 1\},$$

where $x_k^{n,a} = \lambda_k^{n,a} + n - k$ and $y_k^{2n-1/2+a} = 2(\lambda_k^{n,a} + n - k) + a + 1/2$. In what follows, we interchange the use of $x_k^{n,a}$ and $y_k^{2n-1/2+a}$ for ease of notation.

We also adopt the notations and definitions from \[10\] as follows.

Define $T_{n,a}^{\mathcal{X}}$ on $\mathcal{J}_n \times \mathcal{J}_n$ by

$$T_{n,a}^{\mathcal{X}}(\mu, \lambda) = \det [I_a^{\mathcal{X}}(\mu_i - i + n, \lambda_j - j + n)]_{1 \le i, j \le n} \frac{\dim_{SO(2n+1/2+a)} \lambda}{\dim_{SO(2n+1/2+a)} \mu},$$

where $I_a^{\mathcal{X}}$ is given by

$$I_a^{\mathcal{X}}(l, i) = \frac{W(a, -\frac{1}{2})}{\pi} \int_{-1}^1 J_a^{(a, -\frac{1}{2})}(x) J_i^{(a, -\frac{1}{2})}(x) \varphi_l(x)(1 - x)^a(1 + x)^{-1/2} dx.$$
Define the matrix \( T_{n-1,+}^{n},(\mu, \lambda) \) on \( J_{n,-} \times J_{n,-} \) by

\[
T_{n-1,+}^{n}(\mu, \lambda) = \det \left[ \phi_{+}^{\mu}(\mu_{i} - i + n, \lambda_{j} - j + n - 1) \right]_{1 \leq i, j \leq n} \frac{\dim SO(2n-1) \lambda}{\dim SO(2n) \mu},
\]

where

\[
\phi_{+}(x, y) = \begin{cases} 
1 & x > y, \\
0 & x \leq y.
\end{cases}
\]

For each measure \( P_{n,a}^{\omega} \) in \( J_{n} \), we define the measure

\[
P_{n,a}^{\omega} = P_{n,a}^{\omega} \left( \lambda(n,a) \right) T_{n-1,+}^{n}(\mu(n,a), \mu(n-1/2+a,a)) \ldots T_{1,+}^{n}(\lambda(1,1/2), \lambda(1,1/2)) T_{1,-}^{n}(\lambda(1,-1/2), \lambda(1,-1/2))
\]

on \( J_{n} \). When \( \omega = (0, 0, 0) \), \( P_{n,a}^{\omega} \) is the delta measure at the densely packed initial condition, i.e. \( x_{k}^{m,a} = m - k \) for all \( 1 \leq k \leq m \) and \( 1 \leq m \leq n \).

Define a stochastic matrix \( A_{n}^{\omega} \) on \( J_{n} \times J_{n} \) by

\[
A_{n}^{\omega}(\lambda, \mu) = \frac{T_{n,a}^{\omega}(\lambda(n,a), \mu(n,a)) T_{n-1,a}^{\omega}(\mu(n,a), \mu(n-1/2+a,a)) \ldots T_{1,a}^{\omega}(\lambda(1,1/2), \mu(1,1/2)) T_{1,-}^{\omega}(\lambda(1,-1/2), \mu(1,-1/2))}{T_{1,+}^{\omega}(\lambda(1,1/2), \mu(1,1/2)) T_{1,-}^{\omega}(\lambda(1,-1/2), \mu(1,-1/2))}.
\]

If we let \( Q_{n} \) be the generator of the random surface growth with a wall restricted to \( J_{n} \), then the continuous-time Markov chain can be characterized by matrix \( A_{n}^{Q_{n}} \).

**Theorem 2.2.** (Theorem 3.12 in [10]) Let \( \varphi_{t}(x) = e^{t(x-1)} \), then

\[
e^{tQ_{n}} P_{n}^{\omega} = A_{n}^{Q_{n}} P_{n}^{\omega}.
\]

### 3 Correlations along space-like paths

In this section, we compute the multi-time correlation functions for the Markov process on \( J_{n} \).

**Proposition 3.1.** Consider the evolution of the measure \( P_{n,a}^{\omega} \) on \( J_{n} \) defined in \( (2.6) \) under the Markov chain \( A_{n}^{\omega} \), and denote by \( (x^{1,-1/2}(t), \ldots, x^{n,a}(t)) \) the result after time \( t \geq 0 \). Then for any

\[
0 = t_{0}^{1} \leq \cdots \leq t_{0}^{N} = N \leq \cdots \leq t_{0}^{N-2} \leq \cdots \leq t_{0}^{2} = t_{0}^{1} \leq \cdots \leq t_{0}^{1},
\]

where \( c(i) \) are arbitrary non-negative integers, the joint distribution of

\[
x^{1,a}(t_{0}^{1}), \ldots, x^{n,a}(t_{0}^{N}), x^{n-1/2+a,-a}(t_{0}^{N-1}) \ldots, x^{n-1/2+a,-a}(t_{0}^{N-1}), \ldots, x^{3/2-a,a}(t_{0}^{2}), x^{1/2}(t_{0}^{1}), \ldots, x^{1/2}(t_{0}^{1}), \ldots, x^{1/2}(t_{0}^{N}), \ldots, x^{1/-2}(t_{0}^{1})
\]

coincides with the stochastic evolution of \( P_{n,a}^{\omega} \) under transition matrices

\[
T_{n,a}^{\varphi_{c(N)}^{N}}, \ldots, T_{n,a}^{\varphi_{c(1)}^{N}}, T_{n-1,a}^{\varphi_{c(N-1)}^{N}}, \ldots, T_{n-1,a}^{\varphi_{c(1)}^{N}}, T_{n-2,a}^{\varphi_{c(N-2)}^{N}}, \ldots, T_{n-2,a}^{\varphi_{c(1)}^{N}}, \ldots, T_{1,+}^{\varphi_{c(1)}^{N}}, \ldots.
\]

**Proof.** See Proposition 2.5 of [8] and Proposition 3.6 of [10].

**Definition 3.1.** For any \( M \geq 1 \), pick \( M \) points

\[
\zeta_{j} = (n_{j}, a_{j}, t_{j}, s_{j}) \in \mathbb{Z}_{>0} \times \{ \pm \frac{1}{2} \} \times \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0}.
\]

Given the Markov process \( x(t) \), The \( M \)-th correlation function \( \rho_{M} \) at \( (\zeta_{1}, \ldots, \zeta_{M}) \) is defined as

\[
\rho(\zeta_{1}, \ldots, \zeta_{M}) = \text{Prob}(\text{For each } 1 \leq j \leq M, \text{ there exists a } k_{j}, 1 \leq k_{j} \leq n_{j}, \text{ such that } x_{k_{j}+a_{j}}^{n_{j}}(t_{j}) = s_{j}).
\]
where $\varphi$ takes the form

$$
\gamma
$$

Remark 3.1. The correlation kernel in Theorem 3.1 degenerates to the kernel in Theorem 4.1 of [10] when

$$
\text{Remark 3.1.}
$$

any zeroes of $E_u$ do not occur. Then

$$
\rho(x_1, \ldots, x_M) = \det[K(x_i, x_j)]_{i,j=1}^M,
$$

where $K(x_i, x_j)$ is the correlation kernel given by

$$
K(x_i, x_j) = \frac{W(a_{1,1}, -\frac{1}{2})(s_1)}{\pi} \left( \frac{1}{2\pi i} \int_{C} J_{s_1, \frac{a_{1,1}}{2}}(y) J_{s_2, \frac{a_{1,1}}{2}}(u) E^\omega(y) e^{\delta(x-1)}(y - 1)^{-1/2}(1 + y)^{-1/2} y - u \right) dy
$$

$$
+ \frac{1}{\pi} \frac{W(a_{1,1}, -\frac{1}{2})(s_1)}{\pi} \left( \frac{1}{2\pi i} \int_{C} J_{s_1, \frac{a_{1,1}}{2}}(y) J_{s_2, \frac{a_{1,1}}{2}}(u) E^\omega(y) e^{\delta(x-1)}(y - 1)^{-1/2}(1 + y)^{-1/2} dy \right)
$$

The $u$-contour $C$ is a positively oriented simple loop that encircles the interval $[-1, 1]$ but does not encircle any zeroes of $E^{\omega}$.

Remark 3.1. The correlation kernel in Theorem 3.1 degenerates to the kernel in Theorem 4.1 of [10] when $t_1 = t_2$, but with slightly different notations: In [10], the time was implicitly written in the parameter $\omega$, where they introduced a parameter $\gamma = \delta - \sum_{i=1}^{\infty} (\alpha_i + \beta_i)$ in the definition of function $E^\omega$ and let it be the time parameter. In this paper, the function $E^\omega$ defined in (2.2) is fixed as time changes.

### 3.1 Determinantal structure of the correlation functions

The initial conditions for the Markov process are the distributions $P^{N, \omega}$. Proposition 3.1 implies that the joint distribution of

$$
x^{n_1, n_2, \ldots, n_{N-1}}(t_0^{(N)}) \cdot x^{n_1-2+a_{1,1}, -a_{1,1}}(t_0^{(N-1)}) \cdot \ldots \cdot x^{n_1-2+a_{1,1}, -a_{1,1}}(t_0^{(N-1)}) \cdot x^{n_1-3/2+a_{1,1}, -a_{1,1}}(t_0^{(N-2)}) \cdot \ldots \cdot x^{1, -1/2}(t_0^{(2)}) \cdot x^{1, -1/2}(t_0^{(1)})
$$

takes the form

$$
\text{const} \times \prod_{m=1}^{n} \left[ \det \left[ \phi_+ \left( x_i^{m_1, -a_{1,1}}(t_0^{(m_1-1)}), x_i^{m_1+1, (t_0^{(m_1-1)})} \right) \right]_{1 \leq k, l \leq m} \times \prod_{b=1}^{c(2m-1)} \det \left[ \tau_{t_0^{(2m-1)}}(x_b^{m_1, -a_{1,1}}(t_0^{(2m-1)}), x_b^{m_1+1, (t_0^{(2m-1)})}) \right]_{1 \leq k, l \leq m} \times \prod_{b=1}^{c(2m)} \det \left[ \psi_{n_1, a_{1,1}}(x_b^{n_1, a_{1,1}}(t_0^{(2m)})) \right]_{1 \leq k, l \leq n} \right] \quad (3.2)
$$

where $T_{t_0^{(2m-1)}}$ and $x_i^{m_1, -a_{1,1}}$ is set to be $-1$, we refer to this variable as virt. If $a = -1/2$, then the last determinant of $\phi_+ \left( x_i^{m_1, -a_{1,1}}(t_0^{(2m-1)}), x_i^{m_1+1, (t_0^{(2m-1)})} \right)$ and the product of determinants of $\tau_{t_0^{(2m-1)}}(x_b^{n_1, a_{1,1}}(t_0^{(2m)}))$ do not occur.

With the above formula for the joint distribution, we apply an analogue of Theorem 4.2 of [6] to compute the correlation kernel. Let $*$ denote convolution:

$$
f * g(x, y) = \sum_{z \geq 0} f(x, z)g(z, y).
$$
For any \( n, a \) and any two time moments \( t^{2n-1/2+a}_i < t^{2n-1/2+a}_j \), define
\[
\mathcal{T}^{n,a}_{t^{2n-1/2+a}_i t^{2n-1/2+a}_j} = \mathcal{T}^{n,a}_{t^{2n-1/2+a}_{i+1} t^{2n-1/2+a}_{i+2} \cdots t^{2n-1/2+a}_{j-1} t^{2n-1/2+a}_j}.
\]
and
\[
\mathcal{T}^{n,a} = \mathcal{T}^{n,a}_{t^{2n-1/2+a}_{i} t^{2n-1/2+a}_{i+1}}.
\]

For any time moments \( t^{2n-1/2+a}_b < t^{2n-1/2+a}_a \) with \( n_1 \leq n_2 \), we denote the convolution over all transitions between them by \( \phi^{n_1,a}_{t^{2n-1/2+a}_b t^{2n-1/2+a}_a} \).

For example, when \( a_1 = -a_2 = 1/2 \),
\[
\phi^{n_1,a}_{t^{2n-1/2+a}_b t^{2n-1/2+a}_a} = \mathcal{T}^{n_1,a}_{t^{2n-1/2+a}_{b_1} t^{2n-1/2+a}_{b_2}} \ast \phi^{n_1,a}_{t^{2n-1/2+a}_{b_1} t^{2n-1/2+a}_{b_2}} \ast \cdots \ast \phi^{n_1,a}_{t^{2n-1/2+a}_{b_1} t^{2n-1/2+a}_{b_2}}.
\]

For any \( n, a \) and any two time moments \( t^{2n-1/2+a}_b < t^{2n-1/2+a}_a \), let
\[
\phi^{n_1,a}_{t^{2n-1/2+a}_b t^{2n-1/2+a}_a} = 0.
\]

For \( a = \pm \frac{1}{2} \), we define matrices \( M^{n,a} = \{M^{n,a}_{k,l}\}_{k,l=1}^n \) with entries

\[
M^{n,a}_{k,l} = \psi^{n,a}_{k-l} \ast \mathcal{T}^{n,a} \ast \phi^{n,a}_{k-l} \ast \mathcal{T}^{n-1/2+a,-a} \ast \cdots \ast \mathcal{T}^{n-1/2+a,-a} \ast \phi^{n,a}_{k-l} (\text{virt}).
\]

where \( \text{virt} \) is the virtual variable for \( \phi^{n,a}_{-a} \), which is equal to \(-1\).

Last, for \( N = 2n-1/2+a \) and \( l \leq n \), define
\[
\psi^{n,a}_{k-l} = \psi^{n,a}_{k-l} \ast \phi^{n,a}_{k-l} \ast \mathcal{T}^{n,a}_{t^{2n-1/2+a}_j t^{2n-1/2+a}_i}.
\]

**Lemma 3.1.** If \( M^{n,a} \) is upper triangular and invertible, there exist functions \( \Phi^{k,j}_{n} \) such that

(1) \( \left\{ \Phi^{k,j}_{n} \right\}_{j=1}^k \) is a basis of the linear span of
\[
\left\{ \mathcal{T}^{j,j+1}_{t^{2k-1/2+a}_j t^{2k-1/2+a}_j} \ast \cdots \ast \mathcal{T}^{j,j+1}_{t^{2k-1/2+a}_j t^{2k-1/2+a}_j} \ast \phi_{+}^{j,j+1}(s, \text{virt}) \right\}_{j=1}^k.
\]

(2) For \( 0 \leq j_1, j_2 \leq k - 1 \),
\[
\sum_{s \geq 0} \Phi^{k,j_1}_{j_2} \ast \mathcal{T}^{j_1,j_2}_{t^{2k-1/2+a}_j t^{2k-1/2+a}_j} \ast \phi_{+}^{j_1,j_2}(s) (s) = \delta_{j_1,j_2}.
\]

Then the correlation kernel for two comparable pairs \((n_1, a_1, t_1)\) and \((n_2, a_2, t_2)\) with \(2n_1 - 1/2 + a_1 \leq N\) is given by

\[
K(n_1, a_1, t_1, n_2, a_2, t_2) = -\phi^{2n_1-1/2+a_1}_{t^{2n_1-1/2+a_1}_1 t^{2n_1-1/2+a_1}_1} (s_1, s_2) + \sum_{k=1}^{n_2} \psi^{2n_1-1/2+a_1}_{t^{2n_1-1/2+a_1}_1 t^{2n_1-1/2+a_1}_1} (s_1, s_2) \Phi^{2n_1-1/2+a_1}_{n_2-k} (s_2),
\]

where \( t^{2n_1-1/2+a_1}_b = t_i, i = 1, 2.\)
Lemma 3.2. \[ \text{Recall that } \varphi \text{ and } \] \[ \text{3.2 Calculating } \varphi \text{ from section 3.3–3.4, we combine all the functions to get the correlation kernel } K \] \[ \text{Proof.} \text{ The proof is similar to the proof of Theorem 4.2 in [11] and Lemma 3.4 of [9].} \]

First we define a matrix \( L^{n,a} \) such that the measure in (3.2) is proportional to a suitable symmetric minor of \( L^{n,a} \).

\[
L^{n,a} = \begin{pmatrix}
0 & E_1 & 0 & 0 & 0 & E_2 & \cdots & E_n & 0 & 0 & 0 \\
0 & 0 & -T^{1,-} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -T^{1,+} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -T^{2,-} & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -F_{n-1,+} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -F_{n,+} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.
\]

If \( a = -1/2 \), the last two columns do not occur. The matrix blocks in \( L^{n,a} \) have the following entries:

\[
[y^{n,a}]_{k,l} = y^{n,a}_{n-l} \left( x_k^{n,a} \left( t_0^{n} \right) \right), \quad 1 \leq k, l \leq n.
\]

\[
[E_m]_{k,l} = \begin{cases}
\phi_+(x_k^{m,-} \left( t_0^{2m-1} \right), x_k^{m-1,+} \left( t_0^{2m-2} \right)), & k = m, \ 1 \leq l \leq m, \\
0, & 1 \leq k \leq n, \ k \neq m, \ 1 \leq l \leq m,
\end{cases}
\]

\[
[F_{m,+}]_{k,l} = \phi_+(x_k^{m,+} \left( t_0^{2m} \right), x_k^{m-1,-} \left( t_0^{2m-1} \right)), \quad 1 \leq k, l \leq m,
\]

\[
[F_{m,-}]_{k,l} = \phi_-(x_k^{m,-} \left( t_0^{2m-1} \right), x_k^{m-1,+} \left( t_0^{2m-2} \right)), \quad 1 \leq k \leq m-1, \ 1 \leq l \leq m,
\]

and \( \Psi_{m,=} \) is the matrix made of blocks

\[
\Psi_{m,=} = \begin{pmatrix}
I_{1 \times 1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & I_{m,=} & 0 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]

where

\[
[\Psi_{m,=}]_{k,l} = \begin{pmatrix}
I_{m,=}^{m,-1/2+1/2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & I_{m,=}^{m-1/2+1/2} & 0 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]

\[
1 \leq k, l \leq m, \ 1 \leq b \leq c(2m - 1/2 \pm 1/2).
\]

The rest of the proof is along the same lines as that of Lemma 3.4 in [9].

In the following sections, we compute each term that appears in Lemma 3.1. First we calculate the function \( \Phi^{k-1/2+a_1}_{k-1/2+a_2, N} \) in section 3.3. Then we calculate the matrix \( M^{n,a} \) and the function \( \Phi^{k-1/2+a_1}_{n-k} \) in section 3.3. In section 3.4, we compute \( \Phi^{k-1/2+a_1}_{k-1/2+a_2, N} \) and \( \sum_{k=1}^{n} \psi_{n-k}^{n-1/2+a_1, N} (s_1) \psi_{n-k}^{n-1/2+a_2, N} (s_2) \). Last, in section 3.5, we combine all the functions obtained from section 3.3–3.4 to get the correlation kernel \( K \) in Theorem 3.4.

3.2 Calculating \( \Phi^{k-1/2+a_1}_{k-1/2+a_2, N} \)

Recall that \( \varphi_t(x) = e^{t(x-1)} \), for ease of notation, let \( \varphi_{n,+}(x) = \varphi_{n-1/2}^{x}, \varphi_{n,-}(x) = \varphi_{n-1/2}^{-x} \) and \( \varphi_{n,=}(x) = \varphi_{n,+}(x) \varphi_{n,-}(x) = \varphi_{n-1/2}^{x} \varphi_{n-1/2}^{-x} \). Let’s recall some useful identities from Lemma 2.2–2.7 in [10].

Lemma 3.2. \[ \text{Let } T(x) \in C^1([-1, 1]), \text{ the following identities hold:} \]

9
Proposition 3.2. (1)
\[
\sum_{r=0}^{s} W\left(-\frac{1}{2}-\frac{1}{2}\right)(r) J_{r}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = J_{s}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x),
\]

(2)
\[
\frac{1}{\pi} \int_{-1}^{1} J_{r}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)(1 - x)^{-1/2}(1 + x)^{-1/2} dx = 1,
\]

(3)
\[
\sum_{r=0}^{\infty} W\left(-\frac{1}{2}-\frac{1}{2}\right)(r) \frac{1}{\pi} \int_{-1}^{1} J_{r}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) T(x)(1 - x)^{-1/2}(1 + x)^{-1/2} dx = T(1),
\]

(4)
\[
\sum_{r=s+1}^{\infty} W\left(-\frac{1}{2}-\frac{1}{2}\right)(r) \frac{1}{\pi} \int_{-1}^{1} J_{r}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) T(x)(1 - x)^{-1/2}(1 + x)^{-1/2} dx \\
= \frac{1}{\pi} \int_{-1}^{1} J_{s}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) T(1) - T(x) (1 - x)^{-1/2}(1 + x)^{-1/2} dx,
\]

(5)
\[
\sum_{r=s}^{\infty} W\left(-\frac{1}{2}-\frac{1}{2}\right)(r) \frac{1}{\pi} \int_{-1}^{1} J_{r}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) T(x)(1 - x)^{1/2}(1 + x)^{-1/2} dx \\
= \frac{1}{\pi} \int_{-1}^{1} J_{s}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) T(x)(1 - x)^{-1/2}(1 + x)^{-1/2} dx,
\]

(6) for \( a = \pm \frac{1}{2} \), \(-1 \leq \zeta \leq 1\),
\[
\sum_{k=0}^{\infty} W\left(a,-\frac{1}{2}\right)(k) \frac{1}{\pi} \int_{-1}^{1} J_{k}^{\left(a,-\frac{1}{2}\right)}(x) J_{k}^{\left(a,-\frac{1}{2}\right)}(\zeta) T(x)(1 - x)^{\nu}(1 + x)^{-\nu/2} dx = T(\zeta).
\]

We start by computing some basic convolutions that will be useful later.

Proposition 3.2. (1)
\[
\mathcal{T}^{n,a}_{\nu}\left(\phi^{\pm}_{n}(m,l) = W\left(-\frac{1}{2}-\frac{1}{2}\right)(l) \frac{1}{\pi} \int_{-1}^{1} J_{l}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) J_{m}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) \phi_{n}(x)(1 - x)^{-1/2}(1 + x)^{-1/2} dx.
\]

(2)
\[
\mathcal{T}^{n,+}\phi^{\pm}_{n}(m,l) = W\left(-\frac{1}{2}-\frac{1}{2}\right)(l) \frac{1}{\pi} \int_{-1}^{1} J_{l}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) J_{m}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) \phi_{n}(x)(1 - x)^{-1/2}(1 + x)^{-1/2} dx.
\]

(3)
\[
\mathcal{T}^{n,-}\phi^{\pm}_{n}(m,l) =
\begin{cases}
\frac{W\left(-\frac{1}{2}-\frac{1}{2}\right)(l)}{\pi} \int_{-1}^{1} J_{l}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) \left( J_{m}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1) \phi_{n,-}(1) - J_{m}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) \phi_{n,-}(x) \right) x (1 - x)^{-1/2}(1 + x)^{-1/2} dx, & m, l \geq 0, \\
J_{m}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1) \phi_{n,-}(1), & m \geq 0, l = \text{virt}.
\end{cases}
\]
\( T^{n,+} \ast \phi_\pm^+ \ast T^{n,-} \ast \phi_\pm^-(m,l) = \)
\[
\left\{ \begin{array}{l}
\frac{1}{\pi} \int_{-1}^{1} \mathcal{J}_m^{(\frac{1}{2} - \frac{1}{2})}(y) \mathcal{J}_l^{(\frac{1}{2} - \frac{1}{2})}(y) \varphi_{n,\pm}(y)(1 - y)^{-1/2}(1 + y)^{-1/2}dy \\
+ \mathcal{J}_m^{(\frac{1}{2} - \frac{1}{2})}(1) \varphi_{n,\pm}(1),
\end{array} \right. 
\]
\[
m \geq 0, \ l \geq 0,
\]
\[
\sum_{k=0}^{\infty} W^{(-\frac{1}{2}, -\frac{1}{2})}(k) \int_{-1}^{1} \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \varphi_{n,\pm}(x)(1 - x)^{-1/2}(1 + x)^{-1/2}dx \times
\]
\[
\int_{-1}^{1} \mathcal{J}_l^{\frac{1}{2} - \frac{1}{2}} (y) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (y) \varphi_{n,\pm}(y)(1 - y)^{-1/2}(1 + y)^{-1/2}dy.
\]

Proof. Results (1)–(3) follow directly from Lemma 3.2. We only show the computation for (4). By definition, when \( m, l \geq 0 \),
\[
T^{n,+} \ast \phi_\pm^+ \ast T^{n,-} \ast \phi_\pm^-(m,l) = \]
\[
\frac{1}{\pi} \sum_{k=0}^{\infty} W^{(-\frac{1}{2}, -\frac{1}{2})}(k) \int_{-1}^{1} \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \varphi_{n,\pm}(x)(1 - x)^{-1/2}(1 + x)^{-1/2}dx \times
\]
\[
\int_{-1}^{1} \mathcal{J}_l^{\frac{1}{2} - \frac{1}{2}} (y) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (y) \varphi_{n,\pm}(y)(1 - y)^{-1/2}(1 + y)^{-1/2}dy.
\]
According to Lemma 3.2 (2),
\[
\frac{1}{\pi} \int_{-1}^{1} \mathcal{J}_l^{\frac{1}{2} - \frac{1}{2}} (y) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (1) \varphi_{n,\pm}(1)(1 - y)^{-1/2}(1 + y)^{-1/2}dy = \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (1) \varphi_{n,\pm}(1),
\]
thus we have
\[
T^{n,+} \ast \phi_\pm^+ \ast T^{n,-} \ast \phi_\pm^-(m,l) = \]
\[
- \frac{1}{\pi} \sum_{k=0}^{\infty} W^{(-\frac{1}{2}, -\frac{1}{2})}(k) \int_{-1}^{1} \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \varphi_{n,\pm}(x)(1 - x)^{-1/2}(1 + x)^{-1/2}dx \times
\]
\[
\int_{-1}^{1} \mathcal{J}_l^{\frac{1}{2} - \frac{1}{2}} (y) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (y) \varphi_{n,\pm}(y)(1 - y)^{-1/2}(1 + y)^{-1/2}dy
\]
\[
+ \frac{1}{\pi} \sum_{k=0}^{\infty} W^{(-\frac{1}{2}, -\frac{1}{2})}(k) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (1) \varphi_{n,\pm}(1) \int_{-1}^{1} \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \varphi_{n,\pm}(x)(1 - x)^{-1/2}(1 + x)^{-1/2}dx.
\]
Apply Lemma 3.2 (3) to the second summation in (3.3)
\[
\frac{1}{\pi} \sum_{k=0}^{\infty} W^{(-\frac{1}{2}, -\frac{1}{2})}(k) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (1) \varphi_{n,\pm}(1) \int_{-1}^{1} \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (x) \varphi_{n,\pm}(x)(1 - x)^{-1/2}(1 + x)^{-1/2}dx
\]
\[
= \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (1) \varphi_{n,\pm}(1).
\]
For the first summation, we apply Lemma 3.2 (6),
\[
\frac{1}{\pi} \sum_{k=0}^{\infty} W^{(-\frac{1}{2}, -\frac{1}{2})}(k) \int_{-1}^{1} \mathcal{J}_l^{\frac{1}{2} - \frac{1}{2}} (y) \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (y) \varphi_{n,\pm}(y)(1 - y)^{-1/2}(1 + y)^{-1/2}dy
\]
\[
+ \frac{1}{\pi} \int_{-1}^{1} \mathcal{J}_m^{\frac{1}{2} - \frac{1}{2}} (y) \mathcal{J}_l^{\frac{1}{2} - \frac{1}{2}} (y) \varphi_{n,\pm}(y)(1 - y)^{-1/2}(1 + y)^{-1/2}dy.
\]
When \( m \geq 0 \) and \( l = \text{virt} \), the proof is the same. □
Proposition 3.3. For \( k \geq 0 \),
\[
\mathcal{T}_{n-} \ast \phi_{n-1,+}^+(m,l) = \begin{cases} 
\frac{1}{2\pi i} \oint_{\mathcal{C}} J_m(n,\frac{1}{2},-\frac{1}{2} ) (x) \frac{1}{2\pi i} \oint_{\mathcal{C}} J_m(n,\frac{1}{2},-\frac{1}{2} ) (u) \frac{u}{u-x} du, \\
\frac{1}{2\pi i} \oint_{\mathcal{C}} J_m(n,\frac{1}{2},-\frac{1}{2} ) (u) \frac{u}{u-x} du,
\end{cases}
\]
where the \( n \)-contour is a positively oriented simple closed curve that encircles point \( x \). Then, if we enlarge the \( n \)-contour such that it encircles the interval \([-1,1] \),
\[
\mathcal{T}_{n-} \ast \phi_{n-1,+}^+(m,l) = \begin{cases} 
\frac{1}{2\pi i} \oint_{\mathcal{C}} J_m(n,\frac{1}{2},-\frac{1}{2} ) (x) \frac{1}{2\pi i} \oint_{\mathcal{C}} J_m(n,\frac{1}{2},-\frac{1}{2} ) (u) \frac{u}{u-x} du, \\
\frac{1}{2\pi i} \oint_{\mathcal{C}} J_m(n,\frac{1}{2},-\frac{1}{2} ) (u) \frac{u}{u-x} du,
\end{cases}
\]
Following the same computations, we compute iterative convolutions of the functions.

Proposition 3.4. Suppose \( t_{b_1}^{2n_1-1/2+a_1} < t_{b_2}^{2n_2-1/2+a_2} \), if \( n_1 = n_2 \),
\[
\phi_{b_2}^{2n_2-1/2+a_2} \ast t_{b_1}^{2n_1-1/2+a_1} (s_1, s_2) = \frac{W(s_1, -\frac{1}{2})}{\pi} \int_{-1}^{1} J_{s_1}(a_1, -\frac{1}{2}) (x) \frac{J_{s_2}(a_2, -\frac{1}{2}) (x) x + \varphi_{t_{b_1}^{2n_1-1/2+a_1} - t_{b_2}^{2n_2-1/2+a_2}}(x) dx; \]
if \( n_1 < n_2 \),
\[
\phi_{b_2}^{2n_2-1/2+a_2} \ast t_{b_1}^{2n_1-1/2+a_1} (s_1, s_2) = \frac{W(s_1, -\frac{1}{2})}{\pi} \int_{-1}^{1} J_{s_1}(a_1, -\frac{1}{2}) (x) \frac{J_{s_2}(a_2, -\frac{1}{2}) (x) x + \varphi_{t_{b_1}^{2n_1-1/2+a_1} - t_{b_2}^{2n_2-1/2+a_2}}(x) dx; \]
where
\[ \phi_{n_1+1,n_2}(x) = \frac{1}{(2\pi i)^{n_2-n_1}} \times \int \cdots \int J_{2^k(2^{n_2-n_1} - n_2)}^2(u_{n_2}) \varphi_{n_2-1,\pm}(u_{n_2-1}) \cdots \varphi_{n_1+1,\pm}(u_{n_1+1}) \frac{du_{n_2} \cdots du_{n_1+1}}{(u_{n_2} - 1)(u_{n_2} - u_{n_2-1}) \cdots (u_{n_1+1} - 1)(u_{n_1+1} - x)}. \]

**Proof.** Convoluting the leftover terms with the expression for \( \mathcal{T}^{n^+, *}_{n} \cdots \mathcal{T}^{n-k^-, *}_{n} \) \( \phi_{n-k-1,+,n} \) in Proposition 3.3 finishes the proof. \( \square \)

### 3.3 The matrix \( M^{n,a} \) and \( \psi_{k-l}^{2k-1/2+a_1,N} \)

**Proposition 3.5.** When \( k \geq l \),
\[ \psi_{k-l}^{2k-1/2+a_1,N}(s) = \frac{W(a_1, -\frac{k}{2})}{\pi} \int_{-1}^{1} \mathcal{E}^{\omega}(y) T_{a}^{(a_1, -\frac{k}{2})}(y) \varphi_{b}^{2k-1/2+a_1-N}(y) \times (y - 1)^{k-l}(1 - y)^{a_1}(1 + y)^{-1/2} dy, \]

When \( k < l \),
\[ \psi_{k-l}^{2k-1/2+a_1,N}(s) = \frac{W(a_1, -\frac{k}{2})}{\pi} \frac{1}{(2\pi i)^{l-k}} \int_{-1}^{1} J_{a}^{(a_1, -\frac{k}{2})}(x) \times \int \cdots \int \mathcal{E}^{\omega}(u_1) \varphi_{b_1-1,\pm}(u_1) \varphi_{b_2-1,\pm}(u_2) \cdots \varphi_{b_k-1,\pm}(u_k) \frac{du_1 \cdots du_k}{(u_1 - 1)(u_1 - u_2) \cdots (u_k - 1)(u_k - x)} \varphi_{b_{l-k}}^{2k-1/2+a_1-N}(x)(1 - x)^{a_1}(1 + x)^{-1/2} dx, \]

where the \( u_i \)-contour \( (k+2 \leq i \leq l) \) is a positively oriented simple loop that encircles point 1 and \( u_{l-1} \)-contour, the \( u_{k+1} \)-contour is a positively oriented simple loop that encircles interval \([-1, 1]\).

**Proof.** The proof can be done by direct calculation of convolutions, or by the induction method as in the proof of Theorem 4.4 of [10]. \( \square \)

**Lemma 3.3.** The matrix \( M^{n,a} \) is upper triangular and invertible.

**Proof.** We first show that when \( k > l \), \( M^{n,a}_{k,l} = 0 \). Recall that
\[ M^{n,a}_{k,l} = \psi_{n-l}^{n,a} * \mathcal{T}^{n,a} * \varphi_{a}^{a_1} * \mathcal{T}^{n-1/2+a_1,-a} * \cdots * \mathcal{T}^{b_1} * \varphi_{b}^{(virt)}. \]

So,
\[ M^{n,a}_{k,l} = \frac{1}{(2\pi i)^{n-k+1}} \int \cdots \int \mathcal{E}^{\omega}(u_1) \varphi_{b_1-1,\pm}(u_1) \varphi_{b_2-1,\pm}(u_2) \cdots \varphi_{b_k-1,\pm}(u_k) \frac{du_1 \cdots du_k}{(u_1 - 1)(u_1 - u_2) \cdots (u_k - 1)} = 0. \]

When \( k \leq l \),
\[ M^{n,a}_{k,l} = \frac{1}{(2\pi i)^{l-k+1}} \int \cdots \int \mathcal{E}^{\omega}(u_1) \varphi_{b_1-1,\pm}(u_1) \varphi_{b_2-1,\pm}(u_2) \cdots \varphi_{b_k-1,\pm}(u_k) \frac{du_1 \cdots du_k}{(u_1 - 1)(u_1 - u_2) \cdots (u_k - 1)} = 0. \]

Thus the diagonal elements of \( M^{n,a} \) are nonzero and \( M^{n,a} \) is invertible. \( \square \)

### 3.4 \( \Phi_{k-j}^{2k-1/2+a_1,N}(s) \) and \( \sum_{k=1}^{n_2} \psi_{n_1-k}^{2n_1-1/2+a_1,N}(s_1) \Phi_{n_2-k}^{2n_2-1/2+a_2,N}(s_2) \)

**Lemma 3.4.** For any \( 1 \leq j \leq k \leq N \), define
\[ \Phi_{k-j}^{2k-1/2+a_1,N}(s) = \frac{1}{2\pi i} \int \mathcal{E}^{\omega}(u) \varphi_{b_1-1,\pm}(u) \frac{1}{(u - 1)^{k-j+1}} du. \]

Then,
(1) \( \left\{ \Phi_{k-j}^{2^{2k-1/2+a_1}N}(s) \right\}_{j=1}^k \) is a basis of the linear span of
\[
\left\{ \mathcal{T}_{t_0}^{k,a_1} t_{2^{2k-1/2+a_1} t_{2^{2k-1/2+a_1}}} \cdots \mathcal{T}_{t_0}^{j_{j_1-1},t_{(2j_1-1)}} \Phi_{k-j}^-(s,\text{virt}) \right\}_{j=1}^k.
\]

(2) For \( 0 \leq j_1, j_2 \leq k - 1 \),
\[
\sum_{s \geq 0} \Phi_{j_1}^{2^{2k-1/2+a_1}N}(s) \Phi_{j_2}^{2^{2k-1/2+a_1}N}(s) = \delta_{j_1,j_2}.
\]

Proof. To prove (1), we use the fact that \( \left\{ \frac{\partial}{\partial u} J_k(\frac{a_1}{k}-\frac{1}{2}) (u) \right\}_{u=1}^{k-1} \) is a set of polynomials in variable \( s \) of degree \( 2j + 1/2 + a_1 \), which is a linear basis of both the linear span of
\[
\left\{ \mathcal{T}_{t_0}^{k,a_1} t_{2^{2k-1/2+a_1} t_{2^{2k-1/2+a_1}}} \cdots \mathcal{T}_{t_0}^{j_{j_1-1},t_{(2j_1-1)}} \Phi_{k-j}^-(s,\text{virt}) \right\}_{j=1}^k
\]
and \( \left\{ \Phi_{k-j}^{2^{2k-1/2+a_1}N}(s) \right\}_{j=1}^k \).

(2) follows from Lemma 3.2 (6).

Next, we calculate
\[
\sum_{k=1}^{n_2} \psi_{n_1-k}^{2^{n_2-1/2+a_1}N}(s_1) \Phi_{n_2-k}^{2^{n_2-1/2+a_2}N}(s_2). \]
When \( n_1 < n_2 \), let
\[
f^{N}_{t_0} t_{2^{n_2-1/2+a_2}}(u) = \frac{1}{N} E^\omega(u) \varphi_{t_0}^{2^{2n_2-1/2+a_2}N}(u)
\]
and
\[
h_{n_1+1,k}^{N} = \frac{1}{(2\pi i)^{k-n_1}} \oint \cdots \oint \frac{E^\omega(u_1) \varphi_{t_0}^{2^{2n_1}N}(u_1) \cdots \varphi_{n_1+1,\pm}(u_1+1)}{(u_1 - 1)(u_1 - u_2 - u_3) \cdots (u_1+1 - 1)(u_1+1 - x)} du_1 \cdots du_{n_1+1}.
\]
Then
\[
\gamma_{n_1+1,n_2}^{a_2} = \frac{1}{(2\pi i)^{n_2-n_1}} \times \oint \cdots \oint \frac{f^{N}_{t_0} t_{2^{n_2-1/2+a_2}}(u_2) E^\omega(u_2) \varphi_{t_0}^{2^{n_2}N}(u_2) \cdots \varphi_{n_2+1,\pm}(u_2+1)}{(u_2 - 1)(u_2 - u_3 - u_4) \cdots (u_2+1 - 1)(u_2+1 - x)} du_2 \cdots du_{n_2+1},
\]
which gives
\[
f^{N}_{t_0} t_{2^{n_2-1/2+a_2}}(u) = \frac{1}{2\pi i} \oint (u - 1)^{n_2-n_1} (x - u) E^\omega(x) \varphi_{t_0}^{2^{n_1}N}(x) + \gamma_{n_1+1,n_2}^{a_2}(x)
\]

Lemma 3.5. When \( n_1 < n_2 \),
\[
\sum_{k=n_1+1}^{n_2} h_{n_1+1,k}^{N} \Phi_{n_2-k}^{2^{2n_2-1/2+a_2}N} = \frac{1}{2\pi i} \oint (u - 1)^{n_2-n_1} (x - u) E^\omega(x) \varphi_{t_0}^{2^{n_1}N}(x) + \gamma_{n_1+1,n_2}^{a_2}(x).
\]

Proof. First, when \( n_2 = n_1 + 1 \), the equality holds. Now suppose this holds for fixed \( n_1 \) and \( n_2 \), we show this holds for \( n_2 \) and \( n_1 + 1 \) as well. Then, it suffices to show that
\[
\frac{1}{2\pi i} \int \frac{E^\omega(u_n)\varphi_{n(2n+1)}^{-1}(u_n)}{(u_n - 1)(u_n - x)} \, du_n \cdot \Phi_{n_2-n_1}^{2n-1/2+a_2, N} \\
+ \sum_{k=n+1}^{n_2} \int \frac{h_{n,k}(u_n)\varphi_{n_1}(u_n)}{(u_n - 1)(u_n - x)} \, du_n \cdot \Phi_{n_2-k}^{2n-1/2+a_2, N} \\
= \frac{1}{2\pi i} \int \frac{f_{n_2-n_1}^N(u)}{(u - 1)^{n_2-n_1+1}(x - u)} \, du E^\omega(x)\varphi_{n(2n+1)}^{-1}(u) + \frac{1}{2\pi i} \int \frac{g_{n_1+n_2}(u_n)\varphi_{n_1}(u_n)}{(u_n - 1)(u_n - x)} \, du_n.
\]

By induction assumption,
\[
\sum_{k=n+1}^{n_2} \left( \int \frac{h_{n,k}(u_n)\varphi_{n_1}(u_n)}{(u_n - 1)(u_n - x)} \, du_n \cdot \Phi_{n_2-k}^{2n-1/2+a_2, N} \right) = \frac{-g_{n_1+n_2}}{1-x} \left( \int \frac{f_{n_2-n_1}^N(u)}{(u - 1)^{n_2-n_1+1}(x - u)} \, du E^\omega(1)\varphi_{n(2n+1)}^{-1}(u) \right) \\
- \frac{1}{1-x} \cdot \frac{1}{2\pi i} \int \frac{f_{n_2-n_1}^N(u)}{(u - 1)^{n_2-n_1+1}(x - u)} \, du E^\omega(x)\varphi_{n(2n+1)}^{-1}(u),
\]

while
\[
\frac{1}{2\pi i} \int \frac{E^\omega(u_n)\varphi_{n(2n+1)}^{-1}(u_n)}{(u_n - 1)(u_n - x)} \, du_n \cdot \Phi_{n_2-n_1}^{2n-1/2+a_2, N} \\
= \frac{1}{2\pi i} \int \frac{f_{n_2-n_1}^N(u)}{(u - 1)^{n_2-n_1+1}(x - u)} \, du E^\omega(x)\varphi_{n(2n+1)}^{-1}(u) \\
= \frac{E^\omega(1)\varphi_{n(2n+1)}^{-1}(u)}{1-x} \cdot \frac{1}{2\pi i} \int \frac{f_{n_2-n_1}^N(u)}{(u - 1)^{n_2-n_1+1}} \, du \\
- \frac{1}{1-x} \cdot \frac{1}{2\pi i} \int \frac{f_{n_2-n_1}^N(u)}{(u - 1)^{n_2-n_1+1}(x - u)} \, du E^\omega(x)\varphi_{n(2n+1)}^{-1}(u) \\
= \frac{1}{1-x} \cdot \frac{1}{2\pi i} \int \frac{f_{n_2-n_1}^N(u)}{(u - 1)^{n_2-n_1+1}} \, du E^\omega(1)\varphi_{n(2n+1)}^{-1}(u) \\
+ \frac{1}{1-x} \cdot \frac{1}{2\pi i} \int \frac{f_{n_2-n_1}^N(u)}{(u - 1)^{n_2-n_1+1}(x - u)} \, du E^\omega(x)\varphi_{n(2n+1)}^{-1}(u).
Proposition 3.6. If $n_1 \geq n_2 \geq 1$ and $s_1, s_2 \in \mathbb{Z}_{\geq 0}$, we have

\[
\sum_{k=1}^{n_2} \psi_{n_1-k}^{2n_1-1/2+n_1,N}(s_1)\Phi_{k_2}^{2n_2-1/2+n_2,N}(s_2) =
\frac{W(a_1, -\frac{1}{2})(s_1)}{\pi} \int_{-1}^{1} \mathcal{F}_{a_1}^{2n_1-1/2+n_1}(y) \mathcal{F}_{a_2}^{2n_2-1/2+n_2}(y) \frac{E_\nu(y) \phi_{k_1}^{2n_1-1/2+n_1-t_0(y)}(u)}{E_\nu(u) \phi_{k_2}^{2n_2-1/2+n_2-t_0(y)}(u)} \times
(y - 1)^{n_1} (1 - y)^{n_1} (1 + y)^{-\frac{1}{2}} dy
\]

\[+
\frac{W(a_1, -\frac{1}{2})(s_1)}{\pi} \int_{-1}^{1} \mathcal{F}_{a_1}^{2n_1-1/2+n_1}(y) \mathcal{F}_{a_2}^{2n_2-1/2+n_2}(y) \frac{\phi_{k_1}^{2n_1-1/2+n_1-t_0(y)}(y) \phi_{k_2}^{2n_2-1/2+n_2-t_0(y)}(y)}{y - u} \left(\frac{y - 1}{y - u}\right)^{n_1} (1 - y)^{n_1} (1 + y)^{-\frac{1}{2}} dy.
\]

If $1 \leq n_1 < n_2$,

\[
\sum_{k=1}^{n_2} \psi_{n_1-k}^{2n_1-1/2+n_1,N}(s_1)\Phi_{k_2}^{2n_2-1/2+n_2,N}(s_2) =
\frac{W(a_1, -\frac{1}{2})(s_1)}{\pi} \int_{-1}^{1} \mathcal{F}_{a_1}^{2n_1-1/2+n_1}(y) \mathcal{F}_{a_2}^{2n_2-1/2+n_2}(y) \frac{E_\nu(y) \phi_{k_1}^{2n_1-1/2+n_1-t_0(y)}(u)}{E_\nu(u) \phi_{k_2}^{2n_2-1/2+n_2-t_0(y)}(u)} \times
(y - 1)^{n_1} (1 - y)^{n_1} (1 + y)^{-\frac{1}{2}} dy
\]

\[+
\phi_{k_2}^{2n_2-1/2+n_2} \phi_{k_1}^{2n_1-1/2+n_1}(s_1, s_2).
\]

Proof. The calculation for

\[
\sum_{k=1}^{\min(n_1, n_2)} \psi_{n_1-k}^{2n_1-1/2+n_1,N}(s_1)\Phi_{k_2}^{2n_2-1/2+n_2,N}(s_2)
\]

follows the same arguments as the proof for Proposition 4.6 in [10]. We only show the proof when $1 \leq n_1 < n_2$.

First,

\[
\sum_{k=1}^{n_1} \psi_{n_1-k}^{2n_1-1/2+n_1,N}(s_1)\Phi_{k_2}^{2n_2-1/2+n_2,N}(s_2) =
\frac{W(a_1, -\frac{1}{2})(s_1)}{\pi} \int_{-1}^{1} \mathcal{F}_{a_1}^{2n_1-1/2+n_1}(y) \mathcal{F}_{a_2}^{2n_2-1/2+n_2}(y) \frac{E_\nu(y) \phi_{k_1}^{2n_1-1/2+n_1-t_0(y)}(u)}{E_\nu(u) \phi_{k_2}^{2n_2-1/2+n_2-t_0(y)}(u)} \times
(y - 1)^{n_1} (1 - y)^{n_1} (1 + y)^{-\frac{1}{2}} dy.
\]

Now we only need to find $\sum_{k=n_1+1}^{n_2} \psi_{n_1-k}^{2n_1-1/2+n_1,N}(s_1)\Phi_{k_2}^{2n_2-1/2+n_2,N}(s_2)$. By Lemma 3.5

\[
\sum_{k=n_1+1}^{n_2} \psi_{n_1-k}^{2n_1-1/2+n_1,N}(s_1)\Phi_{k_2}^{2n_2-1/2+n_2,N}(s_2) =
\frac{W(a_1, -\frac{1}{2})(s_1)}{\pi} \int_{-1}^{1} \mathcal{F}_{a_1}^{2n_1-1/2+n_1}(y) \left(\sum_{k=n_1+1}^{n_2} h_{k_1}^{N}(x) \phi_{k_2}^{2n_2-1/2+n_2,N}(s_2)\right) \phi_{k_1}^{2n_1-1/2+n_1-t_0(x)}(1-x)^{n_1} (1+x)^{-1/2} dx
\]

\[=
\frac{W(a_1, -\frac{1}{2})(s_1)}{\pi(2\pi i)} \int_{-1}^{1} \mathcal{F}_{a_1}^{2n_1-1/2+n_1}(y) \left(\sum_{k=n_1+1}^{n_2} h_{k_1}^{N}(x) \phi_{k_2}^{2n_2-1/2+n_2,N}(s_2)\right) \phi_{k_1}^{2n_1-1/2+n_1-t_0(x)}(1-x)^{n_1} (1+x)^{-1/2} dy dx
\]

\[+
\frac{W(a_1, -\frac{1}{2})(s_1)}{\pi} \int_{-1}^{1} \mathcal{F}_{a_1}^{2n_1-1/2+n_1}(y) \left(\sum_{k=n_1+1}^{n_2} h_{k_1}^{N}(x) \phi_{k_2}^{2n_2-1/2+n_2,N}(s_2)\right) \phi_{k_1}^{2n_1-1/2+n_1-t_0(x)}(1-x)^{n_1} (1+x)^{-1/2} dx
\]

\[+
\frac{W(a_1, -\frac{1}{2})(s_1)}{\pi} \int_{-1}^{1} \mathcal{F}_{a_1}^{2n_1-1/2+n_1}(y) \left(\sum_{k=n_1+1}^{n_2} h_{k_1}^{N}(x) \phi_{k_2}^{2n_2-1/2+n_2,N}(s_2)\right) \phi_{k_1}^{2n_1-1/2+n_1-t_0(x)}(1-x)^{n_1} (1+x)^{-1/2} dy dx.
\]
Note that
\[
W^{(a_1, -\frac{1}{2})}(s_1) = \frac{1}{\pi} \int_{-1}^{1} J_{s_1}^{(a_1, -\frac{1}{2})}(x) g_{n_1+1,n_2}(x) \phi_{n_1, n_2, n_1}^{(a_1, 1)}(1-x)^{-1/2} dx
\]
\[
= \phi_{n_2}^{2n_2-1/2+a_2} \phi_{n_1}^{2n_1-1/2+a_1} (s_1, s_2).
\]
Thus,
\[
\sum_{k=n_1+1}^{n_2} \psi_{n_1}^{2n_1-1/2+a_1} N(s_1) \Phi_{n_2}^{2n_2-1/2+a_2, N}(s_2)
\]
\[
= \frac{W^{(a_1, -\frac{1}{2})}(s_1)}{\pi (2\pi i)} \int_{-1}^{1} J_{s_1}^{(a_1, -\frac{1}{2})}(x) f_{n_2}^{N-1/2+a_2} \frac{E_{\gamma}(x) \phi_{n_2}^{2n_2-1/2+a_1-t_0}(x)(1-x)^{-1/2} du dx}{E_{\gamma}(u) \phi_{n_2}^{2n_2-1/2+a_2-t_0}(u)(u-1)^{n_2}}
\]
\[
+ \phi_{n_2}^{2n_2-1/2+a_2} \phi_{n_1}^{2n_1-1/2+a_1} (s_1, s_2).
\]
Adding the above two summations together, we get
\[
\sum_{k=n_1+1}^{n_2} \psi_{n_1}^{2n_1-1/2+a_1} N(s_1) \Phi_{n_2}^{2n_2-1/2+a_2, N}(s_2) = \frac{W^{(a_1, -\frac{1}{2})}(s_1)}{\pi} \frac{1}{2\pi i} \int_{-1}^{1} J_{s_1}^{(a_1, -\frac{1}{2})}(y) J_{s_2}^{(a_2, -\frac{1}{2})}(u) \frac{E_{\gamma}(y) \phi_{n_2}^{2n_2-1/2+a_1-t_0}(y)(y-1)^{n_1}(1-y)^{a_1} + (1+y)^{-1/2} du dy}{E_{\gamma}(u) \phi_{n_2}^{2n_2-1/2+a_2-t_0}(u)(u-1)^{n_2}}
\]
\[
+ \phi_{n_2}^{2n_2-1/2+a_2} \phi_{n_1}^{2n_1-1/2+a_1} (s_1, s_2).
\]
\[
3.5 \text{ Computing the kernel}
\]
In this section, we apply Lemma 3.1 to derive the correlation kernel $K$. Adding Proposition 3.4 and Proposition 3.6 we get the following:

When $(n_1, a_1, t_1) \succ (n_2, a_2, t_2)$, which means $t_1 \leq t_2$, $2n_1 - 1/2 + a_1 \geq 2n_2 - 1/2 + a_2$ and $(n_1, a_1, t_1) \neq (n_2, a_2, t_2)$. Let $2n_i - 1/2 + a_i = t_i$, $i = 1, 2$. By the fact that $\phi^{2, A} = 0$ when $t_1 \leq t_2$, we have
\[
K(n_1, a_1, t_1, s_1; n_2, a_2, t_2, s_2) = \sum_{k=1}^{n_2} \psi_{n_1}^{2n_1-1/2+a_1} N(s_1) \Phi_{n_2}^{2n_2-1/2+a_2, N}(s_2)
\]
\[
= \frac{W^{(a_1, -\frac{1}{2})}(s_1)}{\pi} \frac{1}{2\pi i} \int_{-1}^{1} J_{s_1}^{(a_1, -\frac{1}{2})}(y) J_{s_2}^{(a_2, -\frac{1}{2})}(u) \frac{E_{\gamma}(y) \phi_{n_2}^{2n_2-1/2+a_1}(y-1)^{n_1}(1-y)^{a_1}(1+y)^{-1/2} du dy}{E_{\gamma}(u) \phi_{n_2}^{2n_2-1/2+a_2}(u-1)^{n_2}}
\]
\[
+ \phi_{n_2}^{2n_2-1/2+a_2} \phi_{n_1}^{2n_1-1/2+a_1} (s_1, s_2).
\]
When $(n_1, a_1, t_1) \nprec (n_2, a_2, t_2)$, which means $2n_1 - 1/2 + a_1 \leq 2n_2 - 1/2 + a_2$ and $t_1 > t_2$,
\[
K(n_1, a_1, t_1, s_1; n_2, a_2, t_2, s_2) = -\phi_{n_2}^{2n_2-1/2+a_2} \phi_{n_1}^{2n_1-1/2+a_1} (s_1, s_2) + \sum_{k=1}^{n_2} \psi_{n_1}^{2n_1-1/2+a_1} N(s_1) \Phi_{n_2}^{2n_2-1/2+a_2, N}(s_2)
\]
\[
= \frac{W^{(a_1, -\frac{1}{2})}(s_1)}{\pi} \frac{1}{2\pi i} \int_{-1}^{1} J_{s_1}^{(a_1, -\frac{1}{2})}(y) J_{s_2}^{(a_2, -\frac{1}{2})}(u) \frac{E_{\gamma}(y) \phi_{n_2}^{2n_2-1/2+a_1}(y-1)^{n_1}(1-y)^{a_1}(1+y)^{-1/2} du dy}{E_{\gamma}(u) \phi_{n_2}^{2n_2-1/2+a_2}(u-1)^{n_2}}
\]
\[
+ \phi_{n_2}^{2n_2-1/2+a_2} \phi_{n_1}^{2n_1-1/2+a_1} (s_1, s_2).
\]

4 Non-commutative random walk on $U(\mathfrak{so}_{N+1})$

In this section, we construct a non-commutative random walk on $U(\mathfrak{so}_{N+1})$, which is analogous to the non-commutative random walk on $U(\mathfrak{gl}_N)$ constructed in [15].
We take the universal enveloping algebra of the Lie group $\mathfrak{so}_{N+1}$ as the state space and define a semi-group of the non-commutative Markov operator $\{P_t\}_{t \geq 0}$ on $U(\mathfrak{so}_{N+1})$.

For each class function $\kappa \in L^2(SO(N+1))$, we can define a state $\langle \cdot \rangle_\kappa$ on $U(\mathfrak{so}_{N+1})$ by $\langle X \rangle_\kappa = D(X)\kappa(U)|_{U=I}$, where $D$ is the canonical isomorphism from $U(\mathfrak{so}_{N+1})$ to the algebra of left-invariant differential operators on $SO(N+1)$ with complex coefficients (see e.g. [23]).

It is not hard to see (see e.g. [15]) that if the class function $\kappa$ decomposes as

$$\kappa = \sum_\lambda \hat{\kappa}(\lambda) \frac{\chi^\lambda}{\dim \lambda},$$

where $\lambda$ ranges over all irreducible representations of $SO(N+1)$ and $\chi^\lambda$ are the corresponding characters, then

$$\langle X \rangle_\kappa = \sum_\lambda \hat{\kappa}(\lambda) \sum_{i=1}^{\dim \lambda} \Tr \left(X|_{Y_i}\right).$$

(4.1)

In what follows, we let $\kappa_t(O) = e^{t\Tr(O-I_d)}$ and write $\langle \cdot \rangle_t$ for $\langle \cdot \rangle_{\kappa_t}$.

If $X = F_{i_1,j_1} \cdots F_{i_k,j_k}$, then the state can be computed with the following formula (see e.g. page 101 of [22]):

$$D(X)\kappa(O) = \partial_{i_1} \cdots \partial_{i_k} \kappa(Oe^{1F_{i_1,j_1} \cdots e^{kF_{i_k,j_k}}})|_{t_1=\cdots=t_k=0},$$

(4.2)

where $e^{tF}$ is the usual exponential of matrices. In particular, we have

$$e^{tF_{i,j}} = \begin{cases} 
Id + tF_{i,j}, & i \neq \pm j, \; i, j \neq 0, \\
Id + tF_{i,j} - \frac{t^2}{2}E_{-j,j}, & i = 0, \\
Id + tF_{i,j} - \frac{t^2}{2}E_{+i,-i}, & j = 0, \\
Id + (e^{t} - 1)E_{ii} + (e^{-t} - 1)E_{-i,-i}, & i = j.
\end{cases}$$

Since (4.2) only involves linear terms in $t_j$, we can replace $e^{tF_{i,j}}$ with $Id + tF_{i,j}$. Applying Faa di Bruno formula (see e.g. [15]), we have

$$\langle F_{i_1,j_1} \cdots F_{i_m,j_m} \rangle_t = \sum_{\pi \in \Pi} t^{||\pi||} \prod_{B \in \pi, B = \{b_1, \ldots, b_k\}} \Tr \left( \prod_{b \in B, B = \{b_1, \ldots, b_k\}} F_{i_b,j_b} \right),$$

where $\Pi$ is the set of partitions of the set $\{1, 2, \ldots, m\}$ and $B \in \pi$ means that $B$ is a block in partition $\pi$.

It is not hard to see that the non-commutative Markov operator $P_t$ defined in [15] also defines a Markov operator on $U(\mathfrak{so}_{N+1})$.

**Theorem 4.1.** (Theorem 3.1 in [15]) Define $P_t = (id \otimes \langle \cdot \rangle_{\kappa_t}) \circ \Delta$, then

1. $P_t$ satisfies the semi-group property $P_{t+s} = P_t \circ P_s$.
2. $P_t$ preserves $Z(U(\mathfrak{so}_{N+1}))$, i.e. $P_t Z(U(\mathfrak{so}_{N+1})) \subset Z(U(\mathfrak{so}_{N+1}))$.
3. For all $Y \in Z(U(\mathfrak{so}_{N+1}))$, $\langle P_t Y \rangle_s = \langle Y \rangle_{s+t}$.

Since $P_t$ preserves $Z(U(\mathfrak{so}_{N+1}))$, we can expand $P_t \Phi_2^{N+1}$ in terms of generators of $Z(U(\mathfrak{so}_{N+1}))$.

**Example 2.** (1)

$$P_t \Phi_2^{N+1} = \Phi_2^{N+1} + \text{constant}.$$

(2)

$$P_t \Phi_4^{N+1} = \Phi_4^{N+1} + 16tn\Phi_4^{N+1} + \text{constant}.$$

If we define $Q_t^{n,a}$ to be the Markov operator for the point process $\{x_k^{n,a} | 1 \leq k \leq n, \; a \in \{\pm 1/2\}, \; n \geq 1\}$ projected onto $Z_{\geq 0} \times \{n\} \times \{a\}$, then $Q_t^{n,a}$ also defines a Markov operator on $Z(U(\mathfrak{so}_{N+1}))$ through the Harish-Chandra isomorphism.

**Proposition 4.1.** For any $Y \in Z(U(\mathfrak{so}_{N+1}))$, there exists a polynomial $p^{N+1}$ such that

$$\langle Y \rangle_t = \mathbb{E} \left[p^{N+1}(x_1^{n,a}(t), \ldots, x_n^{n,a}(t))\right].$$

In addition, $\langle Q_t^{n,a} Y \rangle_t = \langle Y \rangle_\frac{t}{n+\alpha}$. 

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Proof. Suppose $Y \in Z(U(\mathfrak{so}_{N+1}))$ is sent to the polynomial $P_Y^{N+1}$ by the Harish-Chandra isomorphism. When $N + 1 = 2n + 1$, by \([2.3]\), for any $O \in SO(N + 1)$,

$$e^{\frac{i}{2} \text{Tr}(O - 1)} = \sum_{\lambda} \text{Prob} \left( x_k^{\frac{1}{2}}(t) = \lambda_k + n - k, 1 \leq k \leq n \right) \frac{\chi_{SO(2n+1)}^\lambda(O)}{\dim SO(2n+1)}.$$ 

Recall that $l_k = \lambda_k - k + \frac{1}{2}$ when $N + 1 = 2n + 1$, so we define a polynomial $\overline{P}_Y^{N+1}$ with a change of variables, i.e.

$$P_Y^{N+1} \left( (l_1 + n - \frac{1}{2}, \ldots, l_n + n - \frac{1}{2}) \right) = \overline{P}_Y^{N+1}(\lambda_1, \ldots, \lambda_n).$$ 

Thus, by linearity and \([4.1]\),

$$(Y_{\mathfrak{so}}) = \sum_{\lambda} \text{Prob} \left( x_k^{\frac{1}{2}}(t) = \lambda_k + n - k, 1 \leq k \leq n \right) \frac{(Y)_{\lambda}}{\dim \lambda}$$

by \([2.4]\).

$$(Y_{\mathfrak{so}}) = \sum_{\lambda} \text{Prob} \left( x_k^{\frac{1}{2}}(t) = \lambda_k + n - k, 1 \leq k \leq n \right) \frac{\langle Y \rangle_{\lambda \lambda} + \langle Y \rangle_{\lambda \lambda^*}}{2 \dim \lambda}$$

$$= \sum_{\lambda} \text{Prob} \left( x_k^{\frac{1}{2}}(t) = \lambda_k + n - k, 1 \leq k \leq n \right) \frac{p_Y^{N+1}(\lambda_1, \ldots, \lambda_n) + p_Y^{N+1}(\lambda_1, \ldots, -\lambda_n)}{2}.$$ 

Since $l_k = \lambda_k - k + 1$ when $N + 1 = 2n$, we define $\overline{P}_Y^{N+1}$ for even $N + 1$ such that

$$P_Y^{N+1}(l_1 + n - 1, \ldots, l_n + n - 1) = \frac{p_Y^{N+1}(\lambda_1, \ldots, \lambda_n) + p_Y^{N+1}(\lambda_1, \ldots, -\lambda_n)}{2}.$$ 

Then,

$$(Y_{\mathfrak{so}}) = \sum_{\lambda} \text{Prob} \left( x_k^{\frac{1}{2}}(t) = \lambda_k + n - k, 1 \leq k \leq n \right) \frac{p_Y^{N+1}(l_1 + n - 1, \ldots, l_n + n - 1)}{2}$$

$$= \mathbb{E} \left[ \overline{P}_Y^{N+1} \left( x_1^{\frac{1}{2}}(t), \ldots, x_n^{\frac{1}{2}}(t) \right) \right].$$ 

Again, we have $(Q_t^{n, 1/2} Y)_{\mathfrak{so}} = \langle Y \rangle_{\mathfrak{so}}$.

**Theorem 4.2.** $P^t_{\mathfrak{so}} X = Q_t^{n, a} X$ for all $X \in Z(U(\mathfrak{so}_{N+1}))$. In particular, $P^t_{\mathfrak{so}}$ is the Markov operator of the process $(x_1^{n, a}(t) > \cdots > x_n^{n, a}(t))$.

**Proof.** From Theorem 4.1 and Proposition 4.1 we have $\langle P^t_{\mathfrak{so}} Y \rangle_{\mathfrak{so}} = \langle Q_t^{n, a} Y \rangle_{\mathfrak{so}} = \langle Y \rangle_{\mathfrak{so}}$. The rest of proof follows the standard arguments as that of Proposition 4.4 in \([15]\). We only sketch the proof here. To show $P^t_{\mathfrak{so}} X = Q_t^{n, a} X$ for all $X \in Z(U(\mathfrak{so}_{N+1}))$, we show that if there exists a $Y \in Z(U(\mathfrak{so}_{N+1}))$ such that $\langle Y \rangle_t = 0$ for all $t \geq 0$, then $Y = 0$ by a contradiction.
When restricting our non–commutative random walk to the Gelfand–Tsetlin subalgebra, which is the sub-algebra of $U(\mathfrak{so}_{N+1})$ generated by the centres $Z(U(\mathfrak{so}_k))$, $1 \leq k \leq N+1$, it also matches the two–dimensional particle system along space–like paths:

**Theorem 4.3.** Suppose $Y_1 \in Z(U(\mathfrak{so}_{N+1})), \ldots, Y_r \in Z(U(\mathfrak{so}_{N+1}))$ are mapped to the shifted polynomials $P_{Y_1}^{N+1}, \ldots, P_{Y_r}^{N+1}$ as in Proposition 4.1 under the Harish-Chandra isomorphism. Assume $N_1 \geq \cdots \geq N_r$ and $t_1 \leq \cdots \leq t_r$, then

$$\left\langle Y_1 \left( P_{t_{r-1}} X_2 \right) \cdots \left( P_{t_{r-2}} X_2 \right) \right\rangle = E \left\{ P_{t_{r-1}}^{N+1} (X_{t_{r-1}a_1}) \cdots P_{t_{r}}^{N+1} (X_{t_{r}a_r}) \right\},$$

where $X_{t_{r}a_r}$ is the vector of $x_{t_{r}a_r}$ with $1 \leq j \leq n_r$.

**Proof.** The proof only needs the Gibbs property of the Markov process $\eta_{[16]}$ due to the lack of a second integral. It is straightforward from inspection that the covariance for the random surface growth when $\eta_{[16]}$ follows the same, the covariance formula in Theorem 5.1.

5 Three dimensional Gaussian fluctuations

In this section, we show that certain elements of the Gelfand–Tsetlin subalgebra are asymptotically Gaussian with an explicit covariance along space–like paths and time-like paths.

For a Laurent polynomial $p(v)$, let $p(v)[v^r]$ denotes the coefficient of $v^r$ in $p(v)$. Then, the main theorem is

**Theorem 5.1.** Suppose $N_j = [\eta_j L]$, $t_j = \tau_j L$ for $1 \leq j \leq r$. Assume $\min \{\tau_1, \ldots, \tau_r \} = \tau_1$. Then as $L \to \infty,$

$$\left( \frac{\Phi^{N+1}_{\tau_1} - \langle \Phi^{N+1}_{\tau_1} \rangle}{2L^{2\tau_1}}, \ldots, \frac{\Phi^{N+1}_{\tau_{r-1}} - \langle \Phi^{N+1}_{\tau_{r-1}} \rangle}{2L^{2\tau_{r-1}}}, \frac{\Phi^{N+1}_{\tau_r} - \langle \Phi^{N+1}_{\tau_r} \rangle}{2L^{2\tau_r}} \right) \to (\xi_1, \ldots, \xi_r),$$

where the convergence is with respect to the state $\langle \cdot \rangle_{\tau_k}$ and $(\xi_1, \ldots, \xi_r)$ is a Gaussian vector with covariance

$$E[\xi_i \xi_j] = \begin{cases} 1 \frac{1}{(2\pi)^2} \int_{|v|>|u|} \left( \frac{(v+2)(\eta_j/2+\tau_i v)^2}{v} \right)^{k_i} \left( \frac{(u+2)(\eta_j/2+\tau_i u)^2}{u} \right)^{k_j} \frac{1}{(v-u)^2} du dv, & \eta_i \geq \eta_j, \tau_i \leq \tau_j, \\ \sum_{j=1}^r c_{k,i,\tau_1,\tau_2,\eta_2} \int_{|v|>|u|} \left( \frac{(v+2)(\eta_j/2+\tau_i v)^2}{v} \right)^{k_i} \left( \frac{(u+2)(\eta_j/2+\tau_i u)^2}{u} \right)^{k_j} \frac{1}{(v-u)^2} du dv, & \eta_i < \eta_j, \tau_i \leq \tau_j, \end{cases}$$

where $c_{k,i,\tau_1,\tau_2,\eta_2}$ is defined as the coefficient in the following expansion: for $r \leq -1,$

$$\sum_{i=1}^k c_{k,i,\tau_1,\tau_2,\eta_2} \left( \frac{v+2}{v} \right)^{k_i} \left( \frac{\eta_j}{v} + \tau_i v \right)^2 \frac{1}{(v-u)^2} du dv \left[ [v^r] = \left( \frac{v+2}{v} \right)^{k_i} \left( \frac{\eta_j}{v} + \tau_i v \right)^2 \right]^k [v^r].$$

**Remark 5.1.** It is straightforward from inspection that the covariance for the random surface growth when $\eta_i \geq \eta_j, \tau_i \leq \tau_j$ is different form the covariance for the spectra of overlapping stochastic Wishart matrices in [10] due to the lack of a second integral.

It is not obvious that the covariances along time-like paths are different. Heuristically, if they were the same, the covariance formula in Theorem 5.1 when $\eta_i < \eta_j, \tau_i \leq \tau_j$ could be written as

$$\frac{C_1}{(2\pi)^2} \int_{|v|>|u|} \left( \frac{(v+2)(\eta_j/2+\tau_i v)^2}{v} \right)^{k_i} \left( \frac{(u+2)(\eta_j/2+\tau_i u)^2}{u} \right)^{k_j} \frac{1}{(v-u)^2} du dv$$

$$+ C_2 \cdot \text{Residue} \left[ \frac{(v+2)(\eta_j/2+\tau_i v)^2}{v} \right] \text{Residue} \left[ \frac{(u+2)(\eta_j/2+\tau_i u)^2}{u} \right],$$

where $C_1$ and $C_2$ are constants which are independent of $k_i$ and $k_j$. However, after checking a few examples, the constants $C_1$ and $C_2$ do not exist. For example, letting $k_i = 1, 2$ and $k_j = 1$ and solving for $C_1, C_2$ yields $C_1 = \tau_i^2/\tau_j^2$ and $C_2 = 2\tau_i(\tau_2-\tau_1)/\eta_2\eta_2$. However, if $C_1 = \tau_i^2/\tau_j^2$ and $C_2 = 2\tau_i(\tau_2-\tau_1)/\eta_2\eta_2$, the covariances are not equal when $k_i = 1$ and $k_j = 2$. 

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5.1 Gaussian fluctuations along space-like paths

In this section, we focus on the space-like paths. It was proved that the random surface growth converges to a deterministic limit shape and the fluctuations around the limit shape are described by the Gaussian free field fluctuations (see e.g. [10, 13]).

Let \( G(z) = G(\nu, \eta, \tau; z) \) be the function

\[
G(\nu, \eta, \tau; z) = \nu z^{-1} + \eta \log \left( \frac{z + z^{-1}}{2} - 1 \right) - \nu \log z,
\]

and \( D \) be the connected domain consisting of all triples \((\nu, \eta, \tau)\) such that \( G(\nu, \eta, \tau; z) \) has a unique critical point in the region \( \mathbb{H} \setminus D = \{ z | 3z > 0, |z| > 1 \} \). Let \( \Upsilon \) be the map sending \((\nu, \eta, \tau) \in D\) to the critical point of \( G(\nu, \eta, \tau; z) \) in \( \mathbb{H} \setminus D \). Specifically,

\[
D = \{ (\nu, \eta, \tau) | l(\nu, \eta, \tau) < \nu < r(\nu, \eta, \tau), \nu, \eta > 0 \},
\]

where

\[
\begin{align*}
\nu & = \sqrt{\frac{7}{2} + 5\tau \eta + 1 + \frac{7}{2} (1 + \frac{4\eta}{\tau})^3}, \\
\eta & = \left\{ \begin{array}{ll}
0, & \frac{\eta}{\tau} \leq 2, \\
\sqrt{\frac{7}{2} + 5\tau \eta + 1 - \frac{7}{2} (1 + \frac{4\eta}{\tau})^3}, & \frac{\eta}{\tau} > 2.
\end{array} \right.
\end{align*}
\]

Let \( H(x, N, t) \) be the height function of \( x \), which is defined as the number of particles to the right of \((x, n, a)\) at time \( t \). We recall the limit shape of the height function \( H \) from [10].

**Theorem 5.2.** [10] For any \((\nu, \frac{\eta}{\tau}, \tau) \in D\), suppose \( \Upsilon(\nu, \frac{\eta}{\tau}, \tau) = z_0 \), then the limit shape exists,

\[
h(z_0) = \lim_{L \to \infty} \frac{1}{L} \mathbb{E} H([\nu L], [\eta L], \tau L) = 3 \left( \frac{G(z_0)}{\pi} \right).
\]

By Theorem 1.1 in [13], the height fluctuations converge to a Gaussian free field on the domain \( D \). The proof is based on the fact that the interacting particle system is a determinantal point process. Analogous to [9, 13], the determinantal structure derived in Theorem 3.1 also leads to the convergence of the moments of height fluctuations to that of a Gaussian free field along space-like paths. As a result, we could generalize Theorem 1.1 in [13] as the following.

**Theorem 5.3.** For any \( r \in \mathbb{N}^+ \), let \( \lambda_j = (\nu_j, \frac{\eta_j}{\tau_j}) \in D \) for \( 1 \leq j \leq r \). Define

\[
H_L(\nu, \eta, \tau) = \sqrt{\pi} (H(\nu L, [\eta L], \tau L) - \mathbb{E} H(\nu L, [\eta L], \tau L)),
\]

and \( \Upsilon_j = \Upsilon(\lambda_j) \). Assume \( \{\lambda_j\}_{j=1}^r \) lie on a space-like path, that is \( \eta_1 \geq \ldots \geq \eta_r \) and \( \tau_1 \leq \ldots \leq \tau_r \), then

\[
\lim_{L \to \infty} \mathbb{E} (H_L(\lambda_1) \cdots H_L(\lambda_r)) = \left\{ \begin{array}{ll}
\sum_\sigma \prod_{i=1}^{r/2} G(\Upsilon(\sigma(2i-1)), \Upsilon(\sigma(2i))), & \text{r even}, \\
\sum_\sigma \prod_{i=1}^{(r-1)/2} G(\Upsilon(\sigma(2i-1)), \Upsilon(\sigma(2i))), & \text{r odd},
\end{array} \right.
\]

where the sum is over all fixed point free involutions \( \sigma \) on \( \{1, \ldots, r\} \) and \( G \) is the function

\[
G(z, w) = \frac{1}{2\pi} \log \left| \frac{z + z^{-1} - w - w^{-1}}{z + z^{-1} - w - w^{-1}} \right|.
\]

Let \( p^{N+1}_{2k} = \sum_{i=1}^n i^2 k \), which is the image of \( \Phi_{2k}^{N+1} \) under the Harish-Chandra isomorphism. We first relate the height function and the polynomials \( p^{N+1}_{2k} \) as in [3]. The definition of the height function implies that

\[
\frac{d}{du} H(u, N) = -\sqrt{\pi} \sum_{s=1}^N \delta(u - (\lambda^N_s - s + n)).
\]

Recall that for \( N + 1 = 2n + 1/2 + a, l_i = \lambda_i - i + \frac{3}{4} - \frac{a}{2} \),

\[
\frac{d}{du} H(u, N) = -\sqrt{\pi} \sum_{s=1}^N \delta \left( u - \left( l_s + n - \frac{3}{4} + \frac{a}{2} \right) \right).
\]

(5.2)
Then, let \( u = Lx, N = [L\eta] \), \( t = L\tau \),

\[
P_{2k}^{N+1} = \int_{0}^{\infty} \left( u - n + \frac{3}{4} - \frac{a}{2} \right)^{2k} \left( \sum_{s=1}^{n} \delta \left( u - n - l_s + \frac{3}{4} - \frac{a}{2} \right) \right) du
\]

\[
= -\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \left( u - n + \frac{3}{4} - \frac{a}{2} \right)^{2k} \frac{d}{du} H(u, N) du
\]

\[
\approx \frac{1}{\sqrt{\pi}} \left( n^{2k+1} + 2k \int_{0}^{\infty} (u - n)^{2k-1} H(u, N) du \right).
\]

The relation between height function and the polynomial \( P_{2k}^{N+1} \) indicates the convergence to a Gaussian free field.

**Proposition 5.1.** Suppose \( N_j = [\eta_j L], t_j = \tau_j L \) for \( 1 \leq j \leq r \). Assume \( \eta_1 \geq \ldots \geq \eta_r \) and \( \tau_1 \leq \ldots \leq \tau_r \), then as \( L \to \infty \),

\[
\left( \frac{\Phi_{2k_1}^{N_1+1} - \langle \Phi_{2k_1}^{N_1+1} \rangle_{\tau_1}}{2L^{2k_1}}, \ldots, \frac{P_{(\tau_2 - \tau_1)/2L} \Phi_{2k_r}^{N_2+1} - \langle P_{(\tau_2 - \tau_1)/2L} \Phi_{2k_r}^{N_2+1} \rangle_{\tau_2}}{2L^{2k_r}} \right) \to (\xi_1, \ldots, \xi_r),
\]

where the convergence is with respect to the state \( \langle \cdot \rangle_{\tau_j} \) and \( (\xi_1, \ldots, \xi_r) \) is a Gaussian vector.

**Proof.** The convergence to a Gaussian vector (along both space-like and time-like paths) for elements in \( Z(U(\mathfrak{so}_N+1)) \) can be proved using combinatorial arguments analogous to Proposition 5.1 of [3]. Another way to see that is to apply Theorem 5.3 and Theorem 5.3 to show that the limit as \( L \to \infty \) of joint moments of

\[
\frac{\Phi_{2k_j}^{N_1+1} - \langle \Phi_{2k_j}^{N_1+1} \rangle_{\tau_j}}{2L^{2k_j}}
\]

satisfy the Wick’s formula, which implies the convergence to a Gaussian vector. \( \square \)

Next, we calculate the covariance structure for elements in \( Z(U(\mathfrak{so}_N+1)) \).

**Proposition 5.2.** Suppose \( \eta_1 \geq \eta_2 \) and \( \tau_1 \leq \tau_2 \),

\[
\lim_{L \to \infty} \left\langle \frac{\Phi_{2k}^{L\eta_1+1} - \langle \Phi_{2k}^{L\eta_1+1} \rangle}{2L^{2k}} \right\rangle_{\tau_1 L/2} \cdot \frac{P_{(\tau_2 - \tau_1)/2L} \Phi_{2k}^{L\eta_2+1} - \langle P_{(\tau_2 - \tau_1)/2L} \Phi_{2k}^{L\eta_2+1} \rangle}{2L^{2k}} \right\rangle_{\tau_2 L/2}
\]

\[
\quad = \frac{1}{(2\pi)^2} \int_{|v|>|u|} \left( \frac{v + 2}{v} \right)^{\frac{k}{2}} \left( \frac{u + 2}{u} \right)^{\frac{l}{2}} \frac{1}{(u - v)^2} du dv. \quad (5.3)
\]

**Proof.** By Theorem 5.3 it suffices to compute the limit of

\[
\mathbb{E} \left[ \left( \frac{p_{2k}^{L\eta_1+1}(\tau_1 L) - \mathbb{E}p_{2k}^{L\eta_1+1}(\tau_1 L)}{L^{2k+2l}} \right) \left( \frac{p_{2k}^{L\eta_2+1}(\tau_2 L) - \mathbb{E}p_{2k}^{L\eta_2+1}(\tau_2 L)}{L^{2k+2l}} \right) \right]
\]

as \( L \to \infty \). Apply Theorem 5.3 and dominated convergence theorem,

\[
\lim_{L \to \infty} \mathbb{E} \left[ \left( \frac{p_{2k}^{L\eta_1+1}(\tau_1 L) - \mathbb{E}p_{2k}^{L\eta_1+1}(\tau_1 L)}{L^{2k+2l}} \right) \left( \frac{p_{2k}^{L\eta_2+1}(\tau_2 L) - \mathbb{E}p_{2k}^{L\eta_2+1}(\tau_2 L)}{L^{2k+2l}} \right) \right]
\]

\[
= \frac{4kl}{\pi} \int_{(x, \xi_1) \in \mathcal{D} \ (y, \xi_2) \in \mathcal{D}} \int_{(x, \xi_1) \in \mathcal{D} \ (y, \xi_2) \in \mathcal{D}} \left( \frac{x - \eta_1}{2} \right)^{2k-1} \left( \frac{y - \eta_2}{2} \right)^{2l-1} \mathcal{G} \left( (y, \frac{\eta_2}{2}, \tau_2), (x, \frac{\eta_1}{2}, \tau_1) \right) dx dy. \quad (5.4)
\]

Recall that

\[
\mathcal{G}(z, w) = \frac{1}{2\pi} \log \left| \frac{z + z^{-1} - w - w^{-1}}{z + z^{-1} - w - w^{-1}} \right|
\]
and Υ : D → ℍ − ℍ is defined by sending (d, l, t) ∈ D to the solution of
\[ \begin{align*}
    z + \tau - \frac{1}{2} - \tau = -\frac{2i - 2d - t}{i}, \\
    \frac{1}{2} + \frac{1}{4} - \tau z = -\frac{2i - 2d - t}{i}.
\end{align*} \]

Now, let \( z(x, \eta_1, \tau_1) = Υ(x, \eta_1/2, \tau_1), \) w(y, η_2, τ_2) = Υ(y, η_2/2, τ_2), then \( x(z) = \eta_1 + \frac{z - 3z^2 + (\frac{\eta_1}{2} - 1)z + 1}{2} \) and \( y(w) = \eta_2 + \frac{w^3 - w^2 + (\frac{\eta_2}{2} - 1)w + 1}{w(w-1)} \).

Using the symmetry of the integrand, the right-hand-side of (5.4) can be written as
\[
\frac{4kl}{(2\pi i)^2} \int_{\gamma_1} \int_{\gamma_2} \left( x(z) - \eta_1 \right)^{2k-1} \left( y(w) - \eta_2 \right)^{2l-1} \log(z + w - w^{-1}) \frac{d(x(z))}{dz} \frac{d(y(w))}{dw} \, dz \, dw,
\]
where for each fixed \( \eta_i \) and \( \tau_i \), the integration contour \( \gamma_i \) is a positively orientated path given by all points in the set \( \{ \gamma(x, \eta_i, \tau_i) \} \) (see figure 2).

Integrate by part in \( z \) and \( w \) and let \( v = \frac{z + z^{-1}}{2} - 1, \ u = \frac{w + w^{-1}}{2} - 1 \), we have
\[
\frac{1}{(2\pi i)^2} \int_{\gamma_1} \int_{\gamma_2} \left( \frac{v + 2}{v} \left( \eta_1/2 + \tau_1 v \right)^2 \right)^k \left( \frac{u + 2}{u} \left( \eta_2/2 + \tau_2 u \right)^2 \right)^l \frac{1}{(v-u)^2} \, dv \, du.
\]

Here \( \gamma_1 \) and \( \gamma_2 \) are images of \( \gamma_1 \) and \( \gamma_2 \) under the change of variable. Last, modify \( \gamma_1 \) and \( \gamma_2 \) to two concentric circles centered at 0, then the double integral could be written as
\[
\frac{1}{(2\pi i)^2} \int_{|v| > |u|} \left( \frac{v + 2}{v} \left( \eta_1/2 + \tau_1 v \right)^2 \right)^k \left( \frac{u + 2}{u} \left( \eta_2/2 + \tau_2 u \right)^2 \right)^l \frac{1}{(v-u)^2} \, dv \, du.
\]

Note that
\[
\frac{(v + 2)(\eta_1/2 + \tau_1 v)^2}{v} = \frac{\eta_1^2}{2v} + \tau_1^2 v^2 + (2\tau_1^2 + \eta_1 \tau_1)v + \frac{\eta_1^2}{4} + 2\eta_1 \tau_1,
\]
then the double integral can be computed using residue theorem and the Taylor series
\[
(v - u)^{-2} = v^{-2} \left( 1 + 2u v + 3v^2 + \cdots \right).
\]
Example 3. Recall \( \Phi_2^{N+1} = 2 \sum_{m=1}^{n} (F_{mm} + \rho_m)^2 + 2 \sum_{m<j<m} F_{mj}F_{jm} \). When \( k = l = 1 \), (5.3) equals \( \tau_1^2 \eta_2^2 + 1/2\tau_1 \eta_1 \eta_2^2 \).

By direct computation,
\[
P_{\tau_2-\tau_1} L \Phi_2^{N+1} = \Phi_2^{N+1} + 2 \sum_{m=1}^{n} \left( 2(\tau_2 - \tau_1) + \sum_{m<j<m} 2(\tau_2 - \tau_1) \right)
\]
and
\[
\langle \Phi_2^{N+1} \rangle_{\tau_1 L} = 2 \sum_{m=1}^{n} \left( 2\tau_1 L + \rho_m^2 + 2 \sum_{m<j<m} 2\tau_1 L \right),
\]
thus
\[
P_{\tau_2-\tau_1} L \Phi_2^{N+1} = \Phi_2^{N+1} - \langle \Phi_2^{N+1} \rangle_{\tau_1 L},
\]
moreover,
\[
\lim_{L \to \infty} \frac{\langle \Phi_2^{[\eta L]+1} \rangle_{\tau_1 L/2} - \langle \Phi_2^{[\eta L]+1} \rangle_{\tau_1 L/2}}{2 L^{2k}} = \lim_{L \to \infty} \frac{\langle P_{\tau_2-\tau_1} L \Phi_2^{[\eta L]+1} \rangle_{\tau_1 L/2} - \langle P_{\tau_2-\tau_1} L \Phi_2^{[\eta L]+1} \rangle_{\tau_1 L/2}}{2 L^{2k}} = \tau_1^2 \eta_2^2 + 1/2\tau_1 \eta_1 \eta_2^2,
\]
which matches (5.3).

5.2 Gaussian fluctuations along time-like paths

Next, we prove an analogy of Proposition 5.3 in [15], which provides an expression for the covariance along the time-like paths.

Given a partition \( \rho = (\rho_1, \ldots, \rho_l) \), let \( \Phi_\rho = \prod_{i=1}^{l} \Phi_{\rho_i} \), denote \( |\rho| = 2(\rho_1 + \cdots + \rho_l) \) and \( \text{wt}(\rho) = |\rho| + l \).

Proposition 5.3. Let \( \eta, \tau > 0 \), set \( N = [\eta L] \), and \( t = \tau L \), then
(1) \( \langle \Phi_\rho^{N+1} \rangle_t = \Theta(L^{\text{wt}(\rho)}) \).

(2) There exist constants \( c_{k,\rho}(\tau, \eta) \) such that
\[
P_{\tau L} \Phi_\rho^{N+1} = \sum_{\rho} \left( c_{k,\rho}(\tau, \eta) + o(1) \right) L^{2k+1-\text{wt}(\rho)} \Phi_\rho^{N+1},
\]
where the sum is over \( \rho \) such that \( \text{wt}(\rho) \leq 2k + 1 \).

(3) For any \( \tau_2 \geq \tau_1 \), there exist constants \( c_{k,j}(\tau_2, \tau_1, \eta_2) \) such that
\[
\lim_{L \to \infty} \frac{\langle \Phi_{[\eta L]+1} \rangle_{\tau_2 L} - \langle \Phi_{[\eta L]+1} \rangle_{\tau_1 L}}{L^{2k}} = \lim_{L \to \infty} \sum_{j=1}^{k} c_{k,j}(\tau_2, \tau_1, \eta_2) \langle \Phi_{[\eta L]+1} \rangle_{\tau_2 L} - \langle \Phi_{[\eta L]+1} \rangle_{\tau_1 L}.
\]

Proof. We can prove all the results following the same arguments in [15]. Another intuitive way to see that (1) is true is to use Proposition 4.3, Theorem 5.2 and Theorem 5.3. Recall the limit of height function in frozen region is given by \((2 - x)\), and in liquid region \( D \) is expected to be the function \( h(T(x, \frac{\eta}{2}, \tau)) \) defined in (5.1).
Note that $l\left(\frac{q}{2},\tau\right) < \frac{q}{2} < r\left(\frac{q}{2},\tau\right)$ when $\eta > 0$. Let $N = \lfloor\eta L\rfloor$, $t = \tau L$, then interchange the expectation and integration, we have
\[
\lim_{L \to \infty} \left(\frac{\Phi_{2k}^{N+1}}{2L^{2k+1}}\right)_{\frac{\tau}{2}} = \lim_{L \to \infty} \mathbb{E}\left[p_{2k}^{N+1}\right] = \frac{1}{\sqrt{\pi}} \left(2k\int_{0}^{\left(\frac{x}{2},\tau\right)} \left(x - \frac{\eta}{2}\right)^{2k-1} h \left(\left(x,\frac{\eta}{2},\tau\right)\right) dx \right.
- 2k \int_{0}^{\left(\frac{x}{2},\tau\right)} \left(x - \frac{\eta}{2}\right)^{2k} dx + \left(\frac{\eta}{2}\right)^{2k+1}\right).
\]
(5.6)

The right-hand-side of (5.6) is strictly bounded below by
\[
\frac{1}{\sqrt{\pi}} \left(2k\int_{0}^{\left(\frac{x}{2},\tau\right)} \left(x - \frac{\eta}{2}\right)^{2k-1} \left(\frac{\eta}{2} - l\left(\frac{\eta}{2},\tau\right)\right) dx - 2k \int_{0}^{\left(\frac{x}{2},\tau\right)} \left(x - \frac{\eta}{2}\right)^{2k} dx + \left(\frac{\eta}{2}\right)^{2k+1}\right)
= \left(\frac{\eta}{2}\right)^{2k+1} - \frac{\left(\frac{\eta}{2} - l\left(\frac{\eta}{2},\tau\right)\right)^{2k+1}}{\sqrt{\pi}(2k + 1)}.
\]
Thus, when $\eta > 0$, the right-hand-side of (5.6) is strictly larger than 0 and is finite, which means $\left(\Phi_{2k}^{N+1}\right)_{\tau} = \Theta(L^{2k+1})$. Also by Theorem 5.1 we know that
\[
\mathbb{E}\left[\Phi_{2k}^{N+1}\right] = \Theta(L^{2(2k+1)}).
\]

Thus by Cauchy-Schwartz inequality, $\langle \Phi_{\rho} \rangle_{\tau} = O(L^{w(\rho)})$.

To get a lower bound, one notice that $\lim_{L \to \infty} \mathbb{E}\left[p_{2k}^{N+1}\right] > 0$ a.e. with respect to $\mathbb{E}$,
\[
\lim_{L \to \infty} \left(\prod_{i=1}^{l} \Phi_{2i}^{N+1}\right)_{\frac{\tau}{2}} = \lim_{L \to \infty} \mathbb{E}\left(\prod_{i=1}^{l} \Phi_{2i}^{N+1}\right)_{\frac{\tau}{2}} > 0.
\]
(2) follows from the fact that $\langle \Phi_{2k}^{N+1}\rangle_{\tau L} = \Theta(L^{2k+1})$, thus only $w(\rho) \leq 2k + 1$ terms have nonzero coefficients.

To prove (3), we first apply (2) to the left-hand-side. Note that $\frac{\Phi_{2k}^{N+1}}{L^{2k+1}}$ corresponds to the moment of the fluctuation of height function, thus converges to a Gaussian random variable. Heuristically, $\Phi_{2k}^{N+1} \approx \langle \Phi_{2k}^{N+1}\rangle_{\tau} + \xi L^{2k}$, where $\xi$ is a Gaussian random variable.
\[
\prod_{i=1}^{l} \Phi_{2i}^{N+1} - \left(\prod_{i=1}^{l} \Phi_{2i}^{N+1}\right)_{\tau L} = \sum_{j=1}^{l} \langle \Phi_{2i}^{N+1}\rangle_{\tau L} \cdots \langle \Phi_{2i}^{N+1}\rangle_{\tau L} \cdots \langle \Phi_{2i}^{N+1}\rangle_{\tau L} \left(\Phi_{2i}^{N+1} - \langle \Phi_{2i}^{N+1}\rangle_{\tau L}\right) + \text{smaller order terms}
\]
Thus in the asymptotic limit, we could replace $\Phi_{\rho}$ with a linear combination of $\Phi_{2k}$ with $2k + 1 \leq w(\rho)$. □

Expand $(v - u)^{-2}$ as $v^{-2} \left(1 + 2u + 3u^2 + \cdots\right)$ in Proposition 5.2 and take residues, one obtains
\[
\sum_{j=1}^{k} c_{kj}(\tau_2, \tau_1, \eta_2) \left(\frac{(v + 2) (\eta_2 + \tau_1 v)}{v}\right)^j \left[v^r\right] = \left(\frac{(v + 2) (\eta_2 + \tau_1 v)}{v}\right)^k \left[v^r\right]
\]
for $r \leq -1$. We make use of the expansion (5.7) to compute the coefficients $c_{k,j}$, which provides a formula for the covariance along the time-like paths.

**Example 4.** When $k = 2$, solving (5.7) for $r = -1,-2$, we have $c_{2,2} = 1$ and $c_{2,1} = 4\eta_2(\tau_2 - \tau_1)$, which agrees with the expansion of $P_t\Phi_4^{N+1}$ derived in Example 2.

**Example 5.** (1) When $k = 3$, 
\[
c_{3,3} = 1, \\
c_{3,2} = -6\eta_2(\tau_1 - \tau_2), \\
c_{3,1} = -\frac{3}{2} \left(3\eta_2 \tau_1 - 6\eta_2^2 \tau_1^2 - 3\eta_2 \tau_2 + 16\eta_2^2 \tau_1 \tau_2 - 10\eta_2^2 \tau_2^2\right).
\]

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Proposition 5.4. Suppose $c_{4,4} = 1$, $c_{4,3} = 8n_2(\tau_1 - \tau_2)$, $c_{4,2} = -2(n_2^2\tau_1 - 10n_2^2\tau_1^2 - n_3^2\tau_2 + 4\eta_2^2\tau_1\tau_2 - 14n_2^2\tau_2^2\tau_2)$, $c_{4,1} = -\frac{1}{2}(-n_2^2\tau_1 + 12n_2^2\tau_1^2 - 32n_2^2\tau_1^3 + n_3^2\tau_2 - 40n_2^2\tau_1\tau_2 + 144n_2^2\tau_1^2\tau_2 + 28n_2^2\tau_2^2 - 224n_2^2\tau_1\tau_2^2 + 12n_2^2\tau_3^2).$

Last, we briefly show the convergence to a Gaussian vector along time-like paths.

**Proposition 5.4.** Suppose $N_j = \lfloor \eta_j L \rfloor$, $t_j = \tau_j L$ for $1 \leq j \leq r$. Assume $\eta_1 \leq \ldots \leq \eta_r$ and $\tau_1 \leq \ldots \leq \tau_r$, then as $L \to \infty$,

\[
\left( \frac{\Phi_{2k_1}^{N_1} - \left\langle \Phi_{2k_1}^{N_1} \right\rangle}{2L^{2k_1}}, \ldots, \frac{P_{t_j-t_1}^{N_j} - \left\langle P_{t_j-t_1}^{N_j} \right\rangle}{2L^{2k_r}} \right) \to (\xi_1, \ldots, \xi_r),
\]

where the convergence is with respect to the state $\langle \cdot \rangle_{\frac{1}{2}}$ and $(\xi_1, \ldots, \xi_r)$ is a Gaussian vector.

**Proof.** Same as the proof for space-like paths in Proposition 5.1, we could apply Theorem 4.3, 5.3 and Proposition 5.3 to derive the explicit formula for the joint moments, which satisfies the Wick’s formula and implies the convergence to a Gaussian vector.

\[\square\]

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