Analysis of Network Robustness for Finite Sized Wireless Sensor Networks

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Abstract—Studying network robustness for wireless sensor networks (WSNs) is an exciting topic of research as sensor nodes often fail due to hardware degradation, resource constraints, and environmental changes. The application of spectral graph theory to networked systems has generated several important results. However, previous research has often failed to consider the network parameters, which is crucial to study the real network applications. Network criticality is one of the effective metrics to quantify the network robustness against such failures and attacks. In this work, we derive the exact formulas of network criticality for WSNs using \( r \)-nearest neighbor networks and we show the effect of nearest neighbors and network dimension on robustness using analytical and numerical evaluations. Furthermore, we also show how symmetric and static approximations can wrongly designate the network robustness when implemented to WSNs.

Index Terms—Wireless sensor networks, \( r \)-nearest neighbor networks, Network Robustness, Network Criticality, Spectral graph theory

I. INTRODUCTION

WIRELESS sensor networks play a significant role in applications such as healthcare monitoring, environmental sensing, fire detection, disaster prevention, etc. However, nodes in WSNs prone to failures due to hardware degradation, resource constraints, and environmental changes [1]. The reliability of WSNs is rely on continuing its operations when a fraction of nodes are damaged [2], [3]. Structural robustness of a network is defined as the ability of a network to maintain its connectivity against failures or attacks [4], [5]. Studies of network robustness under intentional attacks and failures have received a growing interest in the recent years (see, e.g., [6]-[7]). In [6], authors argued that node connectivity is the most suitable graph theoretic metric to study the robustness in the face of node failures. Based on different heuristics, various measures have been proposed to quantify the network robustness, such as natural connectivity [8], network criticality [9], algebraic connectivity [10], etc. Although, these measures are useful to quantify the network robustness, they cannot be applied to large-scale networks due to high computational complexity. This also implies that these measures are of no great practical use within the context of WSNs. Many studies have been discussed to optimize the network robustness. In [11], a tabu search algorithm has been proposed to optimize the network robustness by rewiring the links. Altering the edge connections between low degree nodes can improve the network robustness against intentional attacks [12]. Nonetheless, these approaches are only suitable for small to medium-scale networks. To capture the small topological changes in network caused by network component’s failures, a new measure has been proposed based on information theory [7]. A measure for evaluating network robustness for time varying networks has been proposed in [13] and it has been shown that temporal robustness gives more realistic results over static approaches. Although there have been several studies on network robustness, there appear to be inadequate analytical tools to study the robustness for WSN scenarios. In our work, we derive the exact formulas of network criticality to study the network robustness for WSNs. We adopt the network criticality metric to quantify network robustness and derive a new measure of robustness in terms of network parameters. To study the network robustness of WSNs, we use \( r \)-nearest neighbor networks [14], [15], a well known class of distance-regular networks with a varying number of direct neighbors. Lattice networks (see, e.g., [16], [17], [18], and [19]) and \( r \)-nearest neighbor networks are simple structures which allow theoretical analysis that incorporates important parameters like connectivity, scalability, network size, and node failures. These structures are quite useful for measuring and monitoring purposes in sensor networks [20]. Our analytic approach drastically decreases the computational complexity over the existing graph-theoretic metrics. These kind of analyses play a vital role in the initial stage of wireless networks design and also more reliable than simulation based evaluations. Furthermore, we use the probability switching [21] and weight design approaches to study the effects of node dynamics and asymmetric weights on network robustness respectively. The structure of a WSN is highly dynamic, as the WSNs are subject to node or link failures due to the low-power batteries of sensors or environmental changes. Hence, assuming WSN as a static network cannot model the applications which involve mobility [1], [13].

II. NETWORK ROBUSTNESS FOR WIRELESS SENSOR NETWORKS

A. Network Criticality

Network criticality is one of the effective metrics to quantify the network robustness against failures using analytical and numerical evaluations. Furthermore, we use the probability switching and weight design approaches to study the effects of node dynamics and asymmetric weights on network robustness respectively. The structure of a WSN is highly dynamic, as the WSNs are subject to node or link failures due to the low-power batteries of sensors or environmental changes. Hence, assuming WSN as a static network cannot model the applications which involve mobility [1], [13].

III. DERIVATION OF NETWORK CRITICALITY

In this section, we derive the exact formulas of network criticality for WSNs using \( r \)-nearest neighbor networks and we show the effect of nearest neighbors and network dimension on robustness using analytical and numerical evaluations. Furthermore, we also show how symmetric and static approximations can wrongly designate the network robustness when implemented to WSNs.

IV. NUMERICAL EVALUATIONS

In this section, we perform numerical evaluations to study the effect of nearest neighbors and network dimension on robustness using analytical and numerical evaluations. Furthermore, we also show how symmetric and static approximations can wrongly designate the network robustness when implemented to WSNs.

V. TABU SEARCH ALGORITHM

In this section, we propose a tabu search algorithm to optimize the network robustness by rewiring the links. Altering the edge connections between low degree nodes can improve the network robustness against intentional attacks. Nonetheless, these approaches are only suitable for small to medium-scale networks. To capture the small topological changes in network caused by network component’s failures, a new measure has been proposed based on information theory [7]. A measure for evaluating network robustness for time varying networks has been proposed in [13] and it has been shown that temporal robustness gives more realistic results over static approaches. Although there have been several studies on network robustness, there appear to be inadequate analytical tools to study the robustness for WSN scenarios. In our work, we derive the exact formulas of network criticality to study the network robustness for WSNs. We adopt the network criticality metric to quantify network robustness and derive a new measure of robustness in terms of network parameters. To study the network robustness of WSNs, we use \( r \)-nearest neighbor networks [14], [15], a well known class of distance-regular networks with a varying number of direct neighbors. Lattice networks (see, e.g., [16], [17], [18], and [19]) and \( r \)-nearest neighbor networks are simple structures which allow theoretical analysis that incorporates important parameters like connectivity, scalability, network size, and node failures. These structures are quite useful for measuring and monitoring purposes in sensor networks [20]. Our analytic approach drastically decreases the computational complexity over the existing graph-theoretic metrics. These kind of analyses play a vital role in the initial stage of wireless networks design and also more reliable than simulation based evaluations. Furthermore, we use the probability switching [21] and weight design approaches to study the effects of node dynamics and asymmetric weights on network robustness respectively. The structure of a WSN is highly dynamic, as the WSNs are subject to node or link failures due to the low-power batteries of sensors or environmental changes. Hence, assuming WSN as a static network cannot model the applications which involve mobility [1], [13].

VI. CONCLUSION

In conclusion, we propose a tabu search algorithm to optimize the network robustness by rewiring the links. Altering the edge connections between low degree nodes can improve the network robustness against intentional attacks. Nonetheless, these approaches are only suitable for small to medium-scale networks. To capture the small topological changes in network caused by network component’s failures, a new measure has been proposed based on information theory [7]. A measure for evaluating network robustness for time varying networks has been proposed in [13] and it has been shown that temporal robustness gives more realistic results over static approaches. Although there have been several studies on network robustness, there appear to be inadequate analytical tools to study the robustness for WSN scenarios. In our work, we derive the exact formulas of network criticality to study the network robustness for WSNs. We adopt the network criticality metric to quantify network robustness and derive a new measure of robustness in terms of network parameters. To study the network robustness of WSNs, we use \( r \)-nearest neighbor networks [14], [15], a well known class of distance-regular networks with a varying number of direct neighbors. Lattice networks (see, e.g., [16], [17], [18], and [19]) and \( r \)-nearest neighbor networks are simple structures which allow theoretical analysis that incorporates important parameters like connectivity, scalability, network size, and node failures. These structures are quite useful for measuring and monitoring purposes in sensor networks [20]. Our analytic approach drastically decreases the computational complexity over the existing graph-theoretic metrics. These kind of analyses play a vital role in the initial stage of wireless networks design and also more reliable than simulation based evaluations. Furthermore, we use the probability switching [21] and weight design approaches to study the effects of node dynamics and asymmetric weights on network robustness respectively. The structure of a WSN is highly dynamic, as the WSNs are subject to node or link failures due to the low-power batteries of sensors or environmental changes. Hence, assuming WSN as a static network cannot model the applications which involve mobility [1], [13]. Wireless channels in low power wireless networks (such as WSNs) are known to be time-varying, unreliable, and asymmetric (see, e.g., [22], [23], [24], and [25]). Hence, modeling WSN as a undirected graph may wrongly designate the network’s performance. To show the effects of asymmetric link weights, we consider ring topology for the ease of evaluation.

In summary, our contributions can be summarized as follows:

1) In Section III, we derive the exact formulas of network criticality using \( r \)-nearest neighbor networks to compute network criticality in terms of network parameters.
2) In Section IV, we study the effect of link failures on network robustness by means of probability switching.
3) In Section V, we introduce the parameter \( \epsilon \) and derive the network criticality expression for asymmetric ring network.
4) In Section VI, we verify our analytical expressions with the extensive simulations in MATLAB and propose a...
optimization framework to minimize the network robustness while limiting the power consumption.

II. SPECTRAL GRAPH THEORY

Let $G = (V, E)$ be an undirected graph with the set of nodes $V = \{1, 2, \ldots, n\}$, edge set $E \subseteq V \times V$, and an adjacency matrix $A$ consists of non-negative elements $a_{ij}$. The degree matrix $D$ is expressed as $D = \text{diag}[d_i]$, where $d_i = \sum_{j=1}^{n} a_{ij}$.

The Laplacian matrix $L$ is a $n \times n$ symmetric matrix defined as $L = D - A$, where each entry in $L$ is expressed as

$$l_{ij} = l_{ji} = \begin{cases} \text{deg}(v_i) & \text{if } j = i, \\ -a_{ij} & \text{if } j \neq i. \end{cases}$$

(1)

Inspiring from Darwin’s survival value, the theory of network criticality has been developed in [9]. A survival value quantifies the adaptability of network to unexpected variations. Network criticality is defined as the average random walk betweenness of a link or node normalized by its weight. Random Walk Betweenness measures how many times a node appears on random walks between all node-pairs in the graph. A random-walk starts from a source node $s$, chooses one of its neighbors with equal probabilities and continues traveling until it reaches the destination $d$. So the betweenness $b_{st}(d)$ of a node $t$ for source-destination pair $s$-$d$ is the expected number of times that a packet passes via node $t$ in its journey from source $s$ to destination $d$. Node criticality $\eta_k$ is defined as the random-walk betweenness of node over the weight of the node.

$$\eta_k = \frac{b_k}{w_k},$$

where $b_k$, $w_k$ are the betweenness and weight of node $k$. Similarly, link betweenness $\eta_{ij}$ is defined as the betweenness of the link over its weight.

$$\eta_{ij} = \frac{b_{ij}}{w_{ij}},$$

where $b_{ij}$, $w_{ij}$ are the betweenness and weight of link $(i,j)$. The parameter weight captures link quality and models the QoS parameters like Bandwidth, Packet loss etc.

The probability of passing node or link $k$ in the next step is expressed as

$$p_{st}(d) = \left\{ \begin{array}{ll} 0 & \text{if } s = d \\ \frac{w_{st}}{\sum_{q \in A(s)} w_{sq}} & \text{Otherwise}, \end{array} \right.$$

where $A(s)$ is the direct neighbor nodes of $s$ and $w_{st}$ is the weight of link $(s,t)$.

Definition 1: Network criticality ($\tau$) [9] quantifies the network robustness that captures the effect of environmental changes in communication networks. It is expressed as

$$\tau = 2n \text{Tr}(L^+),$$

(2)

where $\text{Tr}(L^+)$ represents trace of the Moore-Penrose inverse of Laplacian matrix.

III. $r$-NEAREST NEIGHBOR NETWORKS

In $r$-nearest neighbor networks, a communication link exists between every pair of nodes that are $r$ hops away. In this section, we derive the network criticality expressions for $r$-nearest neighbor cycle, $r$-nearest neighbor torus, and $m$ dimensional $r$-nearest neighbor torus networks.

Definition 2: The $(j + 1)^{th}$ eigenvalue $\lambda_j$ of a circulant matrix $\text{circ}(a_1, a_2, \ldots, a_n)$ is defined as

$$\lambda_j = a_1 + a_2 \omega^j + \ldots + a_n \omega^{(n-1)j},$$

where $\omega = e^{\frac{2\pi j}{n}}$ and $\{a_i\}_{i=1}^{n}$ are row entries of circulant matrix.

A. $r$-Nearest Neighbor Cycle

The one dimensional wireless sensor network topology can be modeled by $r$-nearest neighbor cycle $C_r^n$. Lemma 1: The $(j + 1)^{th}$ eigenvalue of a Laplacian matrix $L$ of $C_r^n$ is

$$\lambda_j(L) = 2r - 2 \sum_{j=1}^{r} \cos \frac{2\pi j}{n},$$

(4)

where $j = 0, 1, 2, \ldots, (n-1)$.

Proof: The Laplacian matrix of $C_r^n$ can be written as

$$L = \text{circ}(2r-1, -1, 1, \ldots, 0, \ldots, -1, 1),$$

(5)

Using equation (3) and (5), we obtain equation (4).

Theorem 1: Network criticality $\tau$ of a $r$-nearest neighbor cycle $C_r^n$ is given by

$$\tau(C_r^n) = \sum_{j=1}^{n-1} \frac{2n}{2r + 1} - \frac{\sin \left(\frac{2(r+1)\pi}{n}\right)}{\sin \left(\frac{\pi}{n}\right)}. $$

(6)

Proof: Substituting the equation (4) in (2), we get

$$\tau(C_r^n) = \sum_{j=1}^{n-1} \frac{2n}{2r - 2} \cos \frac{2\pi j}{n}.$$  

(7)

Definition 3: Dirichlet kernel is defined as

$$1 + 2 \sum_{j=1}^{r} \cos(jx) = \frac{\sin \left(x + \frac{1}{2}\right) x}{\sin \left(\frac{x}{2}\right)}.$$  

(8)

We obtain (6), by substituting equation (8) in (7).
Theorem

Proof

Without loss of generality, by observing (13) and (9), we can where $r$-nearest neighbor torus is as shown in Fig. 2.

For the Laplacian matrix of $T_{k_1,k_2}$ is

$$\lambda_{j_1,j_2}(L_{k_1,k_2}^r) = 4r - 2 \sum_{i=1}^{r} \cos \left( \frac{2\pi j_1 i}{k_1} \right) - 2 \sum_{i=1}^{r} \cos \left( \frac{2\pi j_2 i}{k_1} \right),$$

where $j_1 = 0, 1, 2, \ldots, (k_1-1)$, $j_2 = 0, 1, 2, \ldots, (k_2-1)$.

Proof: The Laplacian matrix of $T_{k_1,k_2}$ can be written as

$$L = circ\{L_{k_1} + 2r I_{k_1}, I_{k_1}, \ldots, I_{k_1}, I_{k_1}, \ldots, I_{k_1}\}$$

From (3) and (10), we obtain

$$\lambda_{j_1,j_2}(L_{k_1,k_2}^r) = \lambda_{j_1}(L_{k_1}^r) + 2r - 2 \sum_{i=1}^{r} \cos \left( \frac{2\pi j_2 i}{k_2} \right)$$

By substituting equation (4) into (11), we obtain equation (9).

C. m-dimensional r-nearest neighbor torus

Lemma 3: The generalized expression for eigenvalue of Laplacian matrix of $m$-dimensional $r$-nearest neighbor torus is

$$\lambda_{j_1,j_2,\ldots,j_m}(L_{k_1,k_2,\ldots,k_m}^r) = 2mr - 2 \sum_{j=1}^{m} \sum_{i=1}^{r} \cos \left( \frac{2\pi j_i}{k_i} \right).$$

where $j_i = 0, 1, 2, \ldots, (k_i-1)$.

Proof: The $(j_1+j_2+j_3+1)^{th}$ eigenvalue of Laplacian matrix $L$ for $3$-dimensional $r$-nearest neighbor torus $T_{k_1,k_2,k_3}$ is

$$\lambda(L_{k_1,k_2,k_3}^r) = 6r - 2 \sum_{i=1}^{3} \sum_{j=1}^{r} \cos \left( \frac{2\pi j_i}{k_i} \right).$$

Without loss of generality, by observing (13) and (9), we can write (12) for variable $m$.

Theorem 2: The network criticality of $r$-nearest neighbor torus $T_{k_1,k_2}$ between every arbitrary pair of nodes is

$$\tau(T_{k_1,k_2}^r) = \sum_{j_1=0}^{k_1-1} \sum_{j_2=0}^{k_2-1} \left( \frac{2n}{(4r+2)\sin \frac{2\pi j_1}{k_1} \sin \frac{2\pi j_2}{k_2}} \right).$$

Proof: Using (2) and (12), the network criticality of $T_{k_1,k_2}^r$ can be written as

$$\tau(T_{k_1,k_2}^r) = \sum_{j_1=0}^{k_1-1} \sum_{j_2=0}^{k_2-1} \left( \frac{2n}{(4r+2)\sin \frac{2\pi j_1}{k_1} \sin \frac{2\pi j_2}{k_2}} \right).$$

To get (14), we further simplify equation (15) using (8).

Without loss of generality, we write the network criticality of $T_{k_1,k_2,\ldots,k_m}^r$ as

$$\tau(T_{k_1,k_2,\ldots,k_m}^r) = \sum_{j_1=0}^{k_1-1} \sum_{j_2=0}^{k_2-1} \ldots \sum_{j_m=0}^{k_m-1} \left( \frac{2n}{(2r+1)m- \sum_{i=1}^{m} \frac{2\pi j_i}{k_i}} \right).$$

D. Computational Complexity

$O(n)$ denotes the asymptotic upper bound and it says that the limit of a function when the argument tends towards infinity. Time complexity for calculating $\tau$ using equation (2) is $O(n^3)$, which is prohibited for large scale WSNs. Specifically, our approach overcomes this disadvantage and also effective to study the network robustness for large sized networks.

E. Asymptotic Results

In this section, we derive the network criticality expressions for $n \to \infty$.

Theorem 3: The network criticality of $C_n^r$ when $n \to \infty$ is

$$\tau(C_n^r) \approx \frac{n^3}{2r(r+1)(2r+1)}.$$}

Proof: Proof is technical and deferred to Appendix A.

Theorem 4: The network criticality of $T_{k_1,k_2}^r$ when $r \ll n$ is

$$\tau(T_{k_1,k_2}^r) \approx \frac{3n^3\Theta(\log n)}{8r(r+1)(2r+1)^2\pi^2}.$$}

Proof: Proof is technical and deferred to Appendix B.

IV. Dynamic Network Analysis

To study the effect of link failures and switching neighbors, we use the probability switching method proposed in [21]. Here, we consider a $n$ node ring network, where every time instant, graph $G_t$ is selected with probability $p_t$. Since at each time instant, graph topology is selected identically distributed and independent of the previous topologies, the average of the stochastic matrices will be evaluated.

A. Random Communication Links

In a $n$-node ring network, if each link exists between any two nodes with probability $q$, then the link is selected independently. Then adjacency matrix $\tilde{A}$ of a random ring network is written as

$$\tilde{A} = circ\{0 q q \ldots \ldots q\},$$

(19)
Using equation (25) and equation (26), \( \mathcal{L} \) is expressed as

\[
\mathcal{L} = \text{circ}\{q(n-1) - q \, \ldots \, - q\}. \tag{21}
\]

From equations (3) and (21), \((j+1)^{th}\) eigenvalue can be written as

\[
\lambda_j = q \left( (n-1) - \sum_{k=1}^{(n-1)/2} e^{\frac{2\pi i j}{n}} \right). \tag{22}
\]

Thus, substituting equation (22) in equation (2) gives

\[
\tau = \sum_{j=1}^{n-1} \frac{1}{q} \left( (n-1) - \sum_{k=1}^{(n-1)/2} e^{\frac{2\pi i j}{n}} \right). \tag{23}
\]

For \( n \) odd, equation (23) can be further simplified as

\[
\tau = \sum_{j=1}^{n-1} \frac{2n}{q} \left( n - \sin \frac{n \pi j}{n} \right). \tag{24}
\]

**B. Identically Independent Link Losses due to Communication Failures**

To study the link failures in WSNs, we assume that each link in a ring network fails with a probability \( p \). These link failures occur independently with respect to other link failures. Then adjacency matrix \( \overline{A} \) can be written as

\[
\overline{A} = \text{circ}\{0, (1-p), 0 \, \ldots \, 0, (1-p)\}. \tag{25}
\]

Degree matrix \( \overline{D} \) is given by

\[
\overline{D} = \text{circ}\{2(1-p), 0 \, \ldots \, 0\}. \tag{26}
\]

Using equation (25) and equation (26), we get \( \overline{L} \) as

\[
\overline{L} = \text{circ}\{2(1-p), 0 \, \ldots \, 0\}. \tag{27}
\]

Using equation (3) and equation (27), we obtain \((j+1)^{th}\) eigenvalue as

\[
\lambda_j = 2(1-p) \left( 1 - \cos \frac{2\pi j}{n} \right). \tag{28}
\]

Substituting the equation (28) in equation (2), results in

\[
\tau = \sum_{j=1}^{n-1} \frac{n}{(1-p) \left( 1 - \cos \frac{2\pi j}{n} \right)}. \tag{29}
\]

**C. Topology switch due to changing neighbors**

The frequent topology changes due to node failures is quite often in wireless sensor network operations. To study the effect of change in neighbors, we consider a \( n \)-node ring network with \( n \)-time slots. Every time each node chooses two neighbors bidirectionally with equal probability. In this case, adjacency matrix \( \overline{A} \) is

\[
\overline{A} = \text{circ}\{0, \frac{2}{n-1}, \frac{2}{n-1} \, \ldots \, \frac{2}{n-1}\}. \tag{30}
\]

Hence, the degree matrix \( \overline{D} \) is given by

\[
\overline{D} = \text{circ}\{2, 0 \, \ldots \, 0\}. \tag{31}
\]

Using equations (30) and (31), we get Laplacian matrix \( \overline{L} \) as

\[
\overline{L} = \text{circ}\{2 - \frac{2}{n-1} \, \ldots \, - \frac{2}{n-1}\}. \tag{32}
\]

From equations (3) and (32), we obtain \((j+1)^{th}\) eigenvalue as

\[
\lambda_j = 2 - \frac{2}{n-1} \sum_{i=1}^{n-1} e^{\frac{2\pi i j}{n}}. \tag{33}
\]

Substituting equation (33) in equation (2) results in

\[
\tau = \sum_{j=1}^{n-1} \frac{n}{1 - \frac{1}{n-1} \sum_{i=1}^{n-1} e^{\frac{2\pi i j}{n}}}. \tag{34}
\]

For \( n \) odd, \( \tau \) can be further simplified as

\[
\tau = \sum_{j=1}^{n-1} \frac{n}{1 - \frac{1}{n-1} \sin \frac{\pi j}{2n}} \sin \frac{\pi j}{n}. \tag{35}
\]

**V. ASYMMETRIC WEIGHT ANALYSIS**

In this section, we compute the network criticality of an asymmetric ring network. Asymmetric ring network can be modeled as a directed graph, where we assume that forward link weight is ‘1’ and the backward link weight is ‘\( \epsilon \)’ as shown in Fig. 3.

*Theorem 5* : The network criticality \( \tau \) of asymmetric ring network is given by

\[
\tau = \sum_{j=2}^{n-1} \frac{1}{1 + \frac{1}{\epsilon} - (1 + \frac{1}{\epsilon}) \cos \frac{2\pi j}{n} - i(1 - \frac{1}{\epsilon}) \sin \frac{2\pi j}{n}}. \tag{36}
\]
Proof: Laplacian matrix $L$ of a asymmetric ring network is

$$L = \text{circ}(1 + \varepsilon) - 1 \frac{0 0 \ldots 0 - \varepsilon}{n-3}$$  \ (37)$$

Using equation (3) and equation (37), $(j+1)^{th}$ eigenvalue can be expressed as

$$\lambda_j(L) = 1 + \varepsilon - (1 + \varepsilon) \cos \frac{2\pi j}{n} - i(1 + \varepsilon) \sin \frac{2\pi j}{n}$$  \ (38)$$

Substituting equation (38) in (2), proves the theorem.

VI. NUMERICAL RESULTS AND DISCUSSION

In this section, we present the analytical results to study the effect of network parameters, link losses, topological changes, random links, and asymmetric links on the network robustness. The effectiveness of analytical expressions have been verified with the extensive simulations using MATLAB. We have plotted $\tau$ versus $n$, for $r=1$ in Fig. 4. We observe that $\tau$ increases exponentially from $n=25$. So it has to be noted that large scale sensor networks exhibit less robustness to topology changes. In Fig. 5, $\tau$ versus $r$, has been plotted and we can observe a rapid decrease of $\tau$ values at $r=5$. Because the node or link betweenness decreases with the nearest neighbors when $n$ is constant. From $r=5$, $\tau$ values are too low and almost constant despite the increase in $r$ values. This result is justified as the $r$ values increase the network connectivity and a fully connected network always ensures high robustness to topology changes. Similarly, to understand the network criticality of a torus network, we have plotted $\tau$ versus $k_1$ and $k_2$ as shown in Fig. 6. We observe that, $\tau$ increases exponentially with the number of nodes in two dimensions. For $k_1=k_2=300$, $\tau$ versus $r$ plot can be seen in Fig. 7, and we find that for $r$-nearest neighbor torus network, a significant decay of $\tau$ values for $r=5$. We have also noticed that from $r = 5$, $\tau$ values are almost constant. To investigate the effect of network dimension $m$ on $\tau$, we have plotted the Fig. 8, for $k_1=16$, $k_2=18$, $k_3=20$, and $k_4=22$. We have noticed that WSN applications work in multiple dimensions with high $r$ values exhibit high robustness to topology changes. In Fig. 9, we have compared the static ring network with the topologies discussed in Section IV for $p=q=0.2$. We have observed that network robustness is drastically increases with the topology switching between nodes and random WSNs exhibits more robustness than static networks. In Fig. 10, we have plotted the $\tau$ against $n$ for $p=q=0.7$ to see the effect of communication link probabilities $p$ and $q$ on $\tau$. We have observed that increase in probability $q$ of link existence results in increase of network robustness, whereas probability of link failures $p$ reduces the network robustness exponentially. As shown in Fig. 11, we have plotted $\tau$ versus $n$, varying $\varepsilon$ values from 0 to 1. We observe that $\tau$ values increases with the $\varepsilon$, which reveals that network robustness increases with asymmetric link weights.

A. Network Robustness-Overhead Optimization

As shown in the Figs.5 and 7, network criticality $\tau$ increases with $r$. But the node’s power consumption $P$ \cite{17} is

$$P = \left( \frac{r}{\sqrt{n}} \right)^\alpha$$  \ (39)$$

where $\alpha$ is a path-loss exponent. Since WSNs consist of limited-battery powered nodes, it is necessary to minimize the $\tau$ without effecting the power consumption $P$. So to handle this trade off, an optimization framework has been proposed to minimize the $\tau$ subject to total power consumption $P$ constraint or minimizing the $P$ subject to $\tau$.

minimize $\tau$

subject to $r \leq r_{\text{max}}$, $P \leq P_{\text{max}}$, or

minimize $P$

subject to $\tau \leq \tau_{\text{max}}$, $r \leq r_{\text{max}}$. 

In this paper, we have derived the analytical expressions of network criticality for \( m \)-dimensional WSNs to study the network robustness. The derived analytical expressions reduce the computational complexity compared to the existing metrics based on graph-theoretic concepts. We have also studied the effect of number of nodes, nearest neighbors, and network dimension on network robustness. We have shown that network robustness decreases with the number of nodes in large scale WSNs and exponentially increases with the nearest neighbors. This result is well justified as the number of nearest neighbors or network dimension is inversely proportional to the node or link betweenness. Since the sensor node’s power consumption is increases with the nearest neighbors, WSNs face strict tradeoff between the network robustness and power consumption. We have proved that probability of link failures drastically reduces the network robustness. We have also proved that asymmetric link weights increase the network robustness. Furthermore, the proposed optimization framework can be used in network control and optimization problems.

VII. Conclusions

The proposed framework can be extended to study network criticality in asymmetric ring networks. The derived analytical expressions allow for a better understanding of network robustness in large scale WSNs. The framework can be used to optimize network control and optimization problems.
REFERENCES

[1] I. F. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci, “A survey on sensor networks,” Communications magazine, IEEE, vol. 40, no. 8, pp. 102–114, 2002.

[2] L. Paradis and Q. Han, “A survey of fault management in wireless sensor networks,” Journal of Network and systems management, vol. 15, no. 2, pp. 171–190, 2007.

[3] H. M. Ammari and S. K. Das, “Fault tolerance measures for large-scale wireless sensor networks,” ACM Transactions on Autonomous and Adaptive Systems (TAAS), vol. 4, no. 1, p. 4, 2009.

[4] A. K. Ng and J. Efstathiou, “Structural robustness of complex networks,” Physical Review, vol. 3, pp. 175–188, 2006.

[5] W. E. Whiteley and R. E. Kooij, “Graph measures and network robustness,” arXiv preprint arXiv:1311.5064, 2013.

[6] A. H. Dekker and B. D. Colbert, “Network robustness and graph topology,” in Proceedings of the 27th Australasian conference on Computer science-Volume 26. Australian Computer Society, Inc., 2004, pp. 359–368.

[7] T. A. Schieber, L. Carpi, A. C. Frey, O. A. Rosso, P. M. Pardalos, and M. G. Ravetti, “Information theory perspective on network robustness,” Physics Letters A, vol. 380, no. 3, pp. 359–364, 2016.

[8] W. Jun, M. Barahona, T. Yue-Jin, and D. Hong-Zhong, “Natural connectivity of complex networks,” Chinese physics letters, vol. 27, no. 7, p. 070902, 2010.

[9] A. Tizghadam and A. Leon-Garcia, “Survival value of communication networks,” in Infocom Workshop on Network Science for Communication, NetSciCom, 2009.

[10] M. Fiedler, “Algebraic connectivity of graphs,” Czechoslovak mathematical journal, vol. 23, no. 2, pp. 298–305, 1973.

[11] G.-s. Peng and J. Wu, “Optimal network topology for structural robustness based on natural connectivity,” Physica A: Statistical Mechanics and its Applications, vol. 443, pp. 212–220, 2016.

[12] A. Beygelzimer, G. Grinstein, R. Linsker, and I. Rish, “Improving network robustness by edge modification,” Physica A: Statistical Mechanics and its Applications, vol. 357, no. 3, pp. 593–612, 2005.

[13] S. Scellato, I. Leontiadis, C. Mascolo, P. Basu, and M. Zafer, “Evaluating temporal robustness of mobile networks,” Mobile Computing, IEEE Transactions on, vol. 12, no. 1, pp. 105–117, 2013.

[14] C.-K. Chau and P. Basu, “Analysis of latency of stateless opportunistic forwarding in intermittently connected networks,” IEEE/ACM Transactions on Networking (TON), vol. 19, no. 4, pp. 1111–1124, 2011.

[15] S. Dhuli, K. Gaurav, and Y. N. Singh, “Convergence analysis for regular wireless consensus networks,” IEEE Sensors Journal, vol. 15, no. 8, pp. 4522–4531, 2015.

[16] G. Barrenetxea, B. Berefull-Lozano, and M. Vetterli, “Lattice networks: capacity limits, optimal routing, and queueing behavior,” IEEE/ACM Transactions on Networking (TON), vol. 14, no. 3, pp. 492–505, 2006.

[17] S. Vanka, V. Gupta, and M. Haenggi, “Power-delay analysis of consensus algorithms on wireless networks with interference,” International Journal of Systems, Control and Communications, vol. 2, no. 1-3, pp. 256–274, 2010.

[18] X. Ma and N. Elia, “Mean square performance and robust yet fragile topology of torus networks based on torus networked average consensus,” IEEE Transactions on Control of Network Systems, vol. 2, no. 3, pp. 216–225, 2015.

[19] B. Baumierh, M. R. Jovanovic, P. Mitra, and S. Patterson, “Coherence in large-scale networks: Dimension-dependence limitations of local feedback,” IEEE Transactions on Automatic Control, vol. 59, no. 9, pp. 2235–2249, 2012.

[20] G. J. Pottie and W. J. Kaiser, “Wireless integrated network sensors,” Communications of the ACM, vol. 43, no. 5, pp. 51–58, 2000.

[21] P. Hovareshi, J. S. Baras, and V. Gupta, “Average consensus over small world networks: A probabilistic framework,” in Decision and Control, 2008, CDC 2008. 47th IEEE Conference on, IEEE, 2008, pp. 375–380.

[22] M. Z. Zamalloa and B. Krishnamachari, “An analysis of unreliability and asymmetry in low-power wireless links,” ACM Transactions on Sensor Networks (TOSN), vol. 3, no. 2, p. 7, 2007.

[23] D. Kotz, C. Newport, and C. Elliott, “The mistaken axioms of wireless network research,” 2003.

[24] M. Z. Zamalloa and B. Krishnamachari, “An analysis of unreliability and asymmetry in low-power wireless links,” ACM Transactions on Sensor Networks (TOSN), vol. 3, no. 2, p. 7, 2007.

[25] G. Zhou, T. He, S. Krishnamurthy, and J. A. Stankovic, “Models and solutions for radio irregularity in wireless sensor networks,” ACM Transactions on Sensor Networks (TOSN), vol. 2, no. 2, pp. 221–262, 2006.

APPENDIX A

PROOF OF THEOREM 3

Taylor series expansion for \( \sin(2x) = 2x - \frac{8x^3}{3} + \frac{32x^5}{15} - \frac{128x^7}{105} + \cdots \). So, we can rewrite equation (6) as

\[
\tau(T_n') = \sum_{j=2}^{n} \frac{2n}{(2r+1) - \sin \left( \frac{2\pi j}{n} \right)} \approx \frac{3n^3}{r(r+1)(2r+1)\pi^2} \sum_{j=2}^{n} \frac{1}{j^2}
\]

(40)

Substituting the below identity in equation (40) proves the theorem.

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

(41)

APPENDIX B

PROOF OF THEOREM 4

After substituting the \( k_1 = k_2 = \frac{n}{2} \) in equation (14), we obtain

\[
\tau(T_n') = \sum_{j=0}^{\frac{n}{2}-1} \sum_{j_2=0}^{\frac{n}{2}-1} \frac{3n^3}{8(r+1)(2r+1)\pi^2} \left( \frac{1}{j_1^2 + j_2^2} \right)
\]

From Taylor series expansion, we can rewrite equation (42) as

\[
\tau(T_n') = \sum_{j=0}^{\frac{n}{2}-1} \sum_{j_2=0}^{\frac{n}{2}-1} \frac{3n^3}{8(r+1)(2r+1)\pi^2} \left( \frac{1}{j_1^2 + j_2^2} \right)
\]

(43)

for \( 1 \ll n \) and \( 0 < \tan^{-1} x \leq \frac{\pi}{2} \), we get

\[
\int_{0}^{n} \int_{j_2}^{n} \frac{1}{j_1^2 + j_2^2} \, dj_1 \, dj_2 \approx \int_{0}^{n} \Theta \left( \frac{1}{j_1} \right) \, dj_1 = \Theta(\log n),
\]

(44)

So equation (44) can be approximated as

\[
\tau(T_n') \approx \frac{3n^3\Theta(\log n)}{8r(r+1)(2r+1)\pi^2}.
\]

(45)