Quadratically integrable geodesic flows on the torus and on the Klein bottle

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Abstract

1. If the geodesic flow of a metric $G$ on the torus $T^2$ is quadratically integrable then the torus $T^2$ isometrically covers a torus with a Liouville metric on it.

2. The set of quadratically integrable geodesic flows on the Klein bottle is described.

§1. Introduction

Let $M^2$ be a smooth close surface with a Riemannian metric $G$ on it. The metric allows to canonically identify the tangent and the co-tangent bundles of the surface $M^2$. Therefore we have a scalar product and a norm in every co-tangential plane.

Definition 1 Hamiltonian system on the co-tangent plane with the Hamiltonian $H \equiv |p|^2$ is called the geodesic flow of the metric $G$.

It is known that the trajectories of the geodesic flow project (under the natural projection $\pi$, $\pi(x, p) \overset{\text{def}}{=} x$) in the geodesics.

Definition 2 A geodesic flow is called integrable if it is integrable as the Hamiltonian system. That is there exists a function $F : T^*M^2 \rightarrow \mathbb{R}$ such that:

- $F$ is constant on the trajectories,
- $F$ is functionally independent with $H$.

The function $F$ is called an integral.

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Recall that two functions \((F\) and \(H\) in our case) are functionally independent if the differentials \(dF\) and \(dH\) are linear independent almost everywhere.

**Definition 3** A geodesic flow is called **linear integrable** if there exists an integral \(F : T^*M^2 \to \mathbb{R}\) such that in a neighborhood of any point the integral \(F\) is given by the formula \(F(x, y, p_x, p_y) = a(x, y)p_x + b(x, y)p_y\), where \(x, y\) are coordinates on the surface, \(p_x, p_y\) are the correspondent momenta, and \(a, b\) are smooth functions of two variables.

**Definition 4** A geodesic flow is called **quadratically integrable** if it is not linear integrable and if there exists an integral \(F : T^*M^2 \to \mathbb{R}\) such that in a neighborhood of any point the integral \(F\) is given by the formula \(F(x, y, p_x, p_y) = a(x, y)p_x^2 + b(x, y)p_xp_y + c(x, y)p_y^2\), where \(x, y\) are coordinates on the surface, \(p_x, p_y\) are the correspondent momenta, and \(a, b, c\) are smooth functions of two variables.

The set of linear and quadratically integrable geodesic flows on the closed oriented surfaces was completely described in [2], [3], [5], and [7]. In particular, in [3] it was proved that there are no such geodesic flows on the surfaces of genus \(g > 1\) (see also [4]). Linear and quadratically integrable geodesic flows on the sphere were completely described in [5]. Quadratically integrable geodesic flow on the torus were described in [2]. Linear integrable geodesic flows on the torus were described in [7].

The aim of the present paper is to give another description of the set of the quadratic integrable geodesic flows on the torus and to describe the set of quadratically integrable geodesic flows on the Klein bottle.

First recall briefly (following [2]) the description of quadratically integrable geodesic flows on the torus.

Suppose \(L\) is a positive number. Let \(S_1, S_L\) be circles, supplied with smooth parameters \(x \in \mathbb{R}(\text{mod } 1)\) and \(y \in \mathbb{R}(\text{mod } L)\) respectively.

**Definition 5** The metric \((f(x) + h(y))(dx^2 + dy^2)\) on the torus \(S_1 \times S_L\), where \(f : S_1 \to \mathbb{R}\) and \(h : S_L \to \mathbb{R}\) are positive smooth non-constant functions, is called a **Liouville metric**.

**Definition 6** A metric \(G\) on the torus \(T^2\) is called **pseudo-liouville** if there exists a Liouville metric \(G_{\text{Liouv}}\) on the torus \(S_1 \times S_L\) (for an appropriate \(L\)) and a covering \(\rho : S_1 \times S_L \to T^2\) such that \(G_{\text{Liouv}} = \rho^*(G)\).
Theorem 1 ([2]) The geodesic flow of a metric is quadratically integrable iff the metric is pseudo-liouville.

The theorem describes the set of quadratically integrable geodesic flows on the torus.

We would like to suggest one more description.

Theorem 2 A metric $G$ on the torus $T^2$ is pseudo-liouville iff there exists a Liouville metric $G_{liuv}$ on the torus $S_1 \times S_l$ (for an appropriate $l$) and a covering $\chi : T^2 \to S_1 \times S_L$ such that $G = \chi^*(G_{liuv})$.

Theorem 2 will be proved in §2.

In other words, for every metric $G$ (on the torus $T^2$) with the quadratically integrable geodesic flow there exists a composition covering $\rho : S_1 \times S_L \to T^2$, $\chi : T^2 \to S_1 \times S_1$ that takes first a Liouville metric to the metric $G$ and then the metric $G$ to a Liouville metric.

How to specify a metric with the quadratically integrable geodesic flow? A Liouville metric is specified by the triple $(L, f, h)$. It is known that the finite coverings of the torus are specified by $2 \times 2$ integer non-degenerate matrices. Two matrices $(A$ and $B)$ give equivalent coverings if $A = XB$ for an integer matrix $X$, $\det(X) = \pm 1$. It is easy to prove that each integer non-degenerate $2 \times 2$ matrix is equivalent (i.e., gives the equivalent covering) to the appropriate matrix $\left( \begin{array}{cc} 1 & k \\ 0 & m \end{array} \right)$, where $0 < k \leq m$. Two matrixes $\left( \begin{array}{cc} 1 & k_1 \\ 0 & m_1 \end{array} \right)$ and $\left( \begin{array}{cc} 1 & k_2 \\ 0 & m_2 \end{array} \right)$ $(0 < k_1 < m_1, 0 < k_2 < m_2)$ are equivalent if they are equal.

Thus a metric with the quadratically integrable geodesic flow is specified by the quadruple $(L, f, h, \frac{k}{m})$.

In §3 we describe quadratically integrable geodesical flows on the Klein bottle.

Let $L$ be positive number, $f, h : R \to R$ be smooth functions such that function $f$ is non-constant and for any $x, y$ $f(x + \frac{1}{2}) = f(x), h(y + L) = h(y) = h(-y)$.

For every triple $(L, f, h)$ we shall construct the metric $G_{(L,f,h)}$ on the Klein bottle $K_L$. The geodesic flow of the metric $G_{(L,f,h)}$ is quadratically integrable. If the geodesic flow of a metric on the Klein bottle $K$ is quadratically integrable then there exists a diffeomorphism $K \to K_L$ that preserves metric.

§2. Proof of Theorem 2.
Lemma 1 Let \( f : R \to R, h : R \to R \) be bounded smooth functions. If for vector \( \bar{u} = (u_1, u_2) \) and for any \( x, y \) \( f(x + u_1) + h(y + u_2) = f(x) + h(y) \), then \( f(x + u_1) = f(x) \) and \( h(y + u_2) = h(y) \).

Proof. We have \( f(x + u_1) - f(x) = h(y) - h(y + u_2) \). Therefore for the appropriate constant \( C \) we have \( f(x + u_1) - f(x) = C = h(y) - h(y + u_2) \). Let us show that \( C = 0 \). Actually, \( f(x + nu_1) = f(x) + nC \). If \( C \not= 0 \), then function \( f \) is not bounded function. Proved.

Let a metric \( G \) on the torus \( T^2 \) is pseudo-liouville. That is there exists a covering \( \rho : S_1 \times S_L \to T^2 \) such that the metric \( \rho^*G \) is given by the formula \( (f(x) + h(y))(dx^2 + dy^2) \).

Consider the standard plane \( R^2 \) with the standard coordinates \( x, y \). Denote by \( \xi \) the action of the group \( Z \times Z \), generated by shifts along the vectors \((1,0)\) and \((0,L)\). We may consider the torus \( T_L \) as the factor-space of \( R^2 \) by \( \xi \). Since the action \( \xi \) preserve coordinate lines we may use the same notation for the coordinates on \( R^2 \) and \( T_L \).

Consider the universal covering \( U : R^2 \to T_L \) that is dual to the action \( \xi \). The mapping \( Ug : R^2 \to T^2 \) is a universal covering or \( T^2 \). The metric \( (Ug)^*(G) \) on the plane \( R^2 \) is given by the formula \( (f(x) + h(y))(dx^2 + dy^2) \).

Consider the action of the fundamental group \( \pi_1(T^2) \) on the plane \( R^2 \). Since the action is free and since it preserves angles and orientation, we see that the group \( \pi_1(T^2) \) acts by shifts.

Evidently, the group \( \pi_1(T^2) \) is isomorphic to \( Z \times Z \).

Let the generators of the group \( Z^2 \) act by shifts along vectors \( \bar{v} = (v_1,v_2) \) and \( \bar{u} = (u_1,u_2) \). Using lemma 1, \( f(x + u_1) = f(x + v_1) = f(x) \) and \( h(y + u_2) = h(y + v_2) = h(y) \). Let us prove that the fractions \( \frac{v_1}{u_1} \) and \( \frac{v_2}{u_2} \) are rational numbers.

Assume the converse. Then for any positive number \( \epsilon \) there exists the pair of integer numbers \( n_1, n_2 \) such that \( 0 < n_1v_1 + n_2v_2 < \epsilon \). Therefore the derivative of \( f \) is equal to zero in every point. Hence, \( f \equiv const. \) Contradiction.

Thus there exist the numbers \( \alpha_1, \alpha_2 \) and the \( 2 \times 2 \) matrix \( A \) of integers such that \( (\alpha_1,0) = uA, (0,\alpha_2) = vA \). Therefore, \( f(x) = f(x + \alpha_1), h(y) = h(y + \alpha_2) \). We see that shifts along the vectors \((\alpha_1,0)\) and \((0,\alpha_2)\) preserve the metric \( (f(x) + h(y))(dx^2 + dy^2) \).

Consider the action of the group \( Z \times Z \), generated by shifts along vectors \((\alpha_1,0)\) and \((0,\alpha_2)\). Denote by \( T_l \) the factor-space of \( R^2 \) by this action. Since
the action preserves the metric \((f(x) + h(y))(dx^2 + dy^2)\), we see that the metric \((f(x) + h(y))(dx^2 + dy^2)\) induces the metric on \(T_i\). It is clear that the metric on the torus \(T_i\) is Liouville, and the induced mapping \(T^2 \to T_i\) is covering. The theorem is proved.

§3. Quadratically integrable geodesic flows on the Klein bottle.

Suppose \(L\) is a positive number, \(f, h : \mathbb{R} \to \mathbb{R}\) are smooth positive functions such that \(f\) is non-constant and periodic with the period \(\frac{1}{2}\), \(h\) is periodic with period \(L\) and even. Consider the metric \(ds^2 = (f(x) + h(y))(dx^2 + dy^2)\) on the plane \(R^2\). Consider the vectors \(u = (1, 0), v = (0, L)\).

Denote by \(s\) the "slipping reflection" \((x, y) \mapsto (x + \frac{1}{2}, -y)\). Denote by \(\Gamma\) the group, generated by shifts along \(u, v\), and by \(s\). The group \(\Gamma\) acts freely and preserves the metric.

Consider the factorspace \(R^2/\xi\). It is homeomorphic to the Klein bottle. Actually, since \(s\) changes the orientation, we see that \(R^2/\xi\) is nonorientable.

Since the action preserves the vector field \((1, 0)\), we see that on \(R^2/\xi\) there exists a vector field without critical points. Hence the Euler characteristic of \(R^2/\xi\) is equal to zero.

The metric \((f(x) + h(y))(dx^2 + dy^2)\) induces the metric (we denote the induced metric by \(G_{(L,f,h)}\)) on the Klein bottle \(K^2/\xi\).

**Theorem 3** The geodesic flow of a metric \(G\) on the Klein bottle \(K^2\) is quadratically integrable iff for the appropriate triple \((L, f, h)\) there exists a diffeomorphism \(K^2 \to R^2/\xi\) that takes the metric \(G\) to the metric \(G_{(L,f,h)}\).

**Remark** Two different triples can specify the same metric.

Consider the set of triples \((L, f, h)\).

Consider the following operation on the set of triples.

\[
\alpha_\varepsilon(L, f, h) \overset{\text{def}}{=} (L, \hat{f}, h), \quad \text{where} \quad \hat{f}(x) = f(x + \varepsilon)
\]

\[
\beta(L, f, h) \overset{\text{def}}{=} (L, f, \hat{h}), \quad \text{where} \quad \hat{h}(y) = h(y + \frac{L}{2})
\]

\[
\gamma(L, f, h) \overset{\text{def}}{=} (L, \hat{f}, h), \quad \text{where} \quad \hat{f}(x) = f(-x)
\]

\[
\delta(L, f, h) \overset{\text{def}}{=} (L, \hat{f}, \hat{h}), \quad \text{where} \quad \hat{f}(x) = f(x) + \text{const}, \quad \hat{h}(y) = h(y) - \text{const}
\]

**Theorem 4** There exists a diffeomorphism \(\chi : R^2/\xi \to R^2/\hat{\xi}\) that takes the metric \(G_{(L,f,h)}\) to the metric \(G_{(\hat{L},\hat{f},\hat{h})}\) iff the triple \((L, f, h)\) could be transformed to the triple \((\hat{L}, \hat{f}, \hat{h})\) by operations \(\alpha, \beta, \gamma\) and \(\delta\).
§4. The proof of the classification theorems for quadratically integrable geodesic flows on the Klein bottle.

To proof Theorem 3 and Theorem 4, we need several lemmas. Let $R^2$ be the standard plane with the standard coordinates $x, y$, let $\lambda : R^2 \to R$ be a smooth positive function.

**Lemma 2** Suppose the geodesic flow of the metric $\lambda(x, y)(dx^2 + dy^2)$ on the plane $R^2$ admits a quadratical in momenta integral. Let the function $\lambda$ and the coefficients of the integral are bounded functions. Then in the appropriate (global) coordinates $\hat{x}, \hat{y}$ the metric $\lambda(x, y)(dx^2 + dy^2)$ is given by the formula $(f(\hat{x})+ h(\hat{y}))(d\hat{x}^2 + d\hat{y}^2)$, and the integral is given by the formula 

$$
\int (f(\hat{x})- C_0)\hat{p}_x^2 - (h(\hat{y})+ C_0)\hat{p}_y^2, \\
$$

where $C_0$ is the appropriate constant.

Proof. The Hamiltonian of the geodesical flow is the function $H(x, y, p_x, p_y) = \frac{p_x^2 + p_y^2}{\lambda(x, y)}$. Let the integral is given by the formula $F(x, y, p_x, p_y) = a(x, y)p_x^2 + b(x, y)p_xp_y + c(x, y)p_y^2$. Consider the complex-valued function $R_{x, y}(z) = a(x, y) - c(x, y) + ib(x, y)$ of the complex variable $z = x + iy$. Suppose the coordinates $(\hat{x}, \hat{y})$ as functions of coordinates $(x, y)$ are given by the formula $\hat{z} = Z_0 z$, where $Z_0$ is the appropriate complex constant, $z \defeq x + iy$, and $\hat{z} \defeq \hat{x} + i\hat{y}$. Let in the coordinates $(\hat{x}, \hat{y})$ the integral $F$ is given by the formula 

$$
F(\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y) = \hat{a}(\hat{x}, \hat{y})\hat{p}_x^2 + \hat{b}(\hat{x}, \hat{y})\hat{p}_x\hat{p}_y + \hat{c}(\hat{x}, \hat{y})\hat{p}_y^2, \\
$$

where $\hat{p}_x$ and $\hat{p}_y$ are canonically correspondent to $\hat{x}$, $\hat{y}$ momenta. Consider the function $R_{\hat{x}, \hat{y}}$, $R_{x, y}(z) \defeq \hat{a}(\hat{x}, \hat{y}) - \hat{c}(\hat{x}, \hat{y}) + i\hat{b}(\hat{x}, \hat{y})$. In [5], V. Kolokoltzov proved that functions $R_{x, y}$ and $R_{\hat{x}, \hat{y}}$ are holomorphic functions of the complex variable $z$, and satisfy the formula $R_{x, y} = Z_0^2 R_{\hat{x}, \hat{y}}$.

Since function $R_{x, y}$ is bounded, it follows that it is constant. We shall prove that $R_{x, y} \neq 0$.

Assume the converse. We have $\{H, F\} = 0$. Therefore we have

$$
\{H, F\} = H_{p_x} F_x + H_{p_y} F_y - F_{p_x} H_x - F_{p_y} H_y = \\
= 2\lambda^{-2} (a_x \lambda + a_x \lambda_x)p_x^3 + (a_y \lambda + a_y \lambda_x)p_x^2 p_y + (a_x \lambda + a_y \lambda_x)p_x p_y^2 + (a_y \lambda + a_y \lambda_x)p_y^3 = 0.
$$

Since the left part is the uniform polynom in momenta, we obtain the system

$$
(a \lambda)_x = 0, \\
(a \lambda)_y = 0.
$$

Thus for the appropriate constant $D$ we have $a = \frac{D}{\lambda(x, y)}$.
If we replace $a$ by $\frac{D}{\lambda(x,y)}$ in the formula for the integral $F$, we obtain $F = DH$. The last formula contradicts the functional independence of $H$ and $F$.

For the appropriate constant $Z_0$ the function $R_{\hat{x},\hat{y}} \equiv 1$. Therefore, $\hat{b}(\hat{x}, \hat{y}) \equiv 0$ and $\hat{a}(\hat{x}, \hat{y}) = \hat{c}(\hat{x}, \hat{y}) + 1$.

The condition $\{H, F\} = 0$ implies the following system: \[
\begin{cases}
(\hat{a}(\hat{x}, \hat{y})\lambda)_{\hat{x}} = 0 \\
(\hat{a}(\hat{x}, \hat{y})\lambda)_{\hat{y}} = \lambda_{\hat{y}}.
\end{cases}
\]

Hence, $(a\lambda)_{xy} = \lambda_{xy} = (a\lambda)_{yx} = 0$. Therefore, $\lambda = f(x) + h(y)$. Substituting $f(x) + h(y)$ for $\lambda$ in the system, we obtain $\hat{a}(\hat{x}, \hat{y}) = \frac{f(\hat{x})-C_0}{f(\hat{x})+h(\hat{y})}$, $\hat{b} \equiv 0$, and $\hat{c}(\hat{x}, \hat{y}) = -\frac{h(\hat{y})-C_0}{f(\hat{x})+h(\hat{y})}$.

Thus, $F(\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y) = \frac{(f(\hat{x})-C_0)p_y^2-(h(\hat{y})+C_0)p_x^2}{f(\hat{x})+h(\hat{y})}$. Proved.

**Lemma 3** Suppose the function $\lambda$ is non-constant, bounded, and $\lambda_{xy} = \lambda_{yx} = 0$. If coordinates $(\hat{x}, \hat{y})$ as functions of coordinates $(x, y)$ are given by either formula $\hat{z} = Z_0 z + Z_1$ or $\hat{z} = Z_0 \hat{z} + Z_1$, then the number $Z_0$ is pure real or pure imaginary number.

Proof. Suppose coordinates $(\hat{x}, \hat{y})$ are given by the formula $\hat{z} = Z_0 z$. Suppose $Z_0$ is equal to $r e^{i\phi}$. Then, $\hat{x} = r(x \cos \phi + y \sin \phi)$, $\hat{y} = r(-x \sin \phi + y \cos \phi)$. Therefore, $\lambda_{\hat{x}\hat{y}} = r^2 \cos \phi \sin \phi (\lambda_{xx} - \lambda_{yy}) = 0$. We shall prove that $\lambda_{xx} - \lambda_{yy} \neq 0$. Assume converse. Since $\lambda_{xy} = 0$, we see that $\lambda_{xx}$ depends only on $x$, $\lambda_{yy}$ depends only on $y$. Therefore, $\lambda_{xx} = \lambda_{yy} = \text{const}$. Since $\lambda$ is bounded, we see that $\lambda = \text{const}$. Contradiction.

We have $r^2 \cos \phi \sin \phi (\lambda_{xx} - \lambda_{yy}) = 0$. Hence, $r^2 \cos \phi \sin \phi = 0$. Proved.

**Corollary** Suppose $\lambda : R^2 \rightarrow R$ is bounded non-constant function, diffeomorphism $\chi : R^2 \rightarrow R^2$ preserves the metric $\lambda(x,y)(dx^2 + dy^2)$, and the geodesic flow of the metric admits a quadratic in momenta integral $F$. Then for any integral $F_1$ for the appropriate constants $C_0, C_1 \neq 0$ $F_1 = C_0 H + C_1 F$.

Let us prove Theorem 3. Suppose the geodesic flow of the metric $G_K$ on the Klein bottle $K^2$ admits a quadratic in momenta integral $F_K$. Consider the simple-connected covering $U : R^2 \rightarrow K^2$. Denote by $G$ the metric $U^*(G_K)$ on the plane $R^2$; by $F$ we denote the function $U^*(F_K)$. Evidently, $F$ is a quadratic in momenta integral of the geodesical flow of the metric $G$.

It is known that in the appropriate coordinates $x, y$ the metric $G$ is given by the formula $\lambda(x,y)(dx^2 + dy^2)$, where $\lambda : R^2 \rightarrow R$ is positive function.
Since the Klein bottle is a compactum set, we see that the function $\lambda$ and the coefficients of the integral $F$ are bounded.

It is clear that representation of the fundamental group $\pi_1(K^2)$ is the following: $\langle a, b | abab^{-1} = 1 \rangle$. We shall describe the action of the group $\pi_1(K^2)$ on $R^2$.

It is easy to prove that the action of an arbitrary element $c \in \pi_1(K^2)$ is given either by formula $Az + B$ or $A\bar{z} + B$, where $A$ and $B$ are the appropriate complex constants, and $z \overset{\text{def}}{=} x + iy$ is the complex coordinate. Indeed, any $a \in \pi_1(K^2)$ acts by conformal auto-diffeomorphism $R^2 \to R^2$.

Let us show that the complex constant $A$ is equal to 1. First since the action preserves the volume, we have $|A| = 1$. Secondly since the action is free, we see that $A = 1$.

Thus, the oriented elements of $\pi_1(K^2)$ act by shifts, non-oriented elements act by slipped reflections.

Consider the subgroup $\langle a, b^2 \rangle$ of $\pi_1(K^2)$, generated by the elements $a$ and $b^2$. The elements $a$ and $b^2$ are oriented. Therefore they act by shifts. Denote by $\bar{a}$, $\bar{b}$ the shifts, correspondent to $a$ and $b^2$. Let us prove that the vector $\bar{a}$ is not parallel to the vector $\bar{b}$. Assume the converse. Then the factorspace of $R^2$ by the action of the subgroup $\langle a, b^2 \rangle$ is not homeomorphic to the torus. Contradiction.

Let us show that the element $b$ acts by the slipping reflection with the directing vector $\bar{b}$. Since element $b$ is non-oriented, we see that it acts by a slipping reflection. Since $b^2$ acts by shift along the vector $\bar{b}$, we see that the directing vector is $\frac{\bar{b}}{2}$.

We shall prove that the vector $\bar{a}$ is orthogonal to the vector $\bar{b}$. We have $abab^{-1} = 1$. Hence, $bab^{-1} = a^{-1}$. Let the element $b$ acts by composition of shift along $\frac{\bar{b}}{2}$ and a reflection $s_1$. Since $s_1b = bs_1$ and since $ab = ba$, we have $s_1as_1^{-1} = a^{-1}$. Therefore the axis of the reflection $s_1$ is orthogonal to the vector $\bar{a}$. Proved.

Using lemma 2, there exists a coordinate system $x, y$, in which the metric is given by the formula $(f(x) + h(y))(dx^2 + dy^2)$.

Let we prove that the vector $\bar{b}$ is parallel to a coordinate axis of this coordinate system. Using lemma 3, we have two cases. 1. The vector $\bar{b}$ is parallel to an axis. 2. The angles between the vector $\bar{b}$ and the axes are equal to $\frac{\pi}{2}$.

We prove that the second case is impossible. Using lemma 3, we have $f(x + k_b) = h(x)$. Since $F$ is giving by the formula $\frac{(f(x) - C_0)p_y^2 - (h(y) + C_0)p_x^2}{f(x) + h(y)}$,
we see that $b$ does not preserves the integral $F$. Proved.

Thus the Klein bottle $K^2$ can be considered as the factorspace of $R^2$ by the action of the group, generated by a shift along the first coordinate axis and a slipped reflection along the second coordinate axis.

To complete the proof we shall prove that $f \equiv \text{const}$ iff the geodesical flow of metric $G_{(L,f,h)}$ is linear integrable.

It is clear that if $f \equiv \text{const}$ then $F \overset{\text{def}}{=} p_x$ is an integral. Let us prove that for $f \neq \text{const}$ the geodesic flow of the metric $G_{(L,f,h)}$ is not linear integrable. Suppose a linear integral is given by the formula $a(x,y)p_x + b(x,y)p_y$. Using lemma 2, we have

$$(a(x,y)p_x + b(x,y)p_y)^2 = \frac{(f(x) - C_0)p_y^2 - (h(y) + C_0)p_x^2}{f(x) + h(y)}.$$  

Therefore, $2(f(x) - C_0)(h(y) + C_0) = 0$. By assumption, $f \neq \text{const}$. Hence, $h \equiv C_0$. Then, $b \equiv \text{const}$, $a \equiv 0$. But since the slipping reflection $(x,y) \rightarrow (x+\frac{1}{2}, -y)$ changes the sign of the function $p_y$, we see that const $\equiv 0$. The theorem is proved.

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