Abstract: For $G = \text{PSL}(2,p^f)$ denote by $ZG$ the integral group ring over $G$ and by $V(ZG)$ the group of units of augmentation 1 in $ZG$. Let $r$ be a prime different from $p$. Using the so called HeLP-method we prove that units of $r$-power order in $V(ZG)$ are rationally conjugate to elements of $G$. As a consequence we prove that subgroups of prime power order in $V(ZG)$ are rationally conjugate to subgroups of $G$, if $p = 2$ or $f \leq 2$.

Let $G$ be a finite group and $ZG$ the integral group ring over $G$. Denote by $V(ZG)$ the group of units of augmentation 1 in $ZG$. We say that a finite subgroup $U$ of $V(ZG)$ is rationally conjugate to a subgroup $W$ of $G$, if there exists a unit $x \in \mathbb{Q}G$ such that $x^{-1}Ux = W$. The question if some, or even all, finite subgroups of $V(ZG)$ are rationally conjugate to subgroups of $G$ was proposed by H. J. Zassenhaus in the ’60s and published in [Zas74]. This so called Zassenhaus Conjectures motivated a lot of research. E.g. A. Weiss proved the strongest version, that all finite subgroups of $V(ZG)$ are rationally conjugate to subgroups of $G$, provided $G$ is nilpotent [Wei88] [Wei91]. K. W. Roggenkamp and L. L. Scott obtained a counterexample [Rog91] to this strong conjecture. The version, which asks whether all finite cyclic subgroups of $V(ZG)$ are rationally conjugate to subgroups of $G$, the so called First Zassenhaus Conjecture, is however still open, see e.g. [Her08a], [CMdR13]. Though mostly solvable groups were considered when studying such questions, there are some results available for non-solvable series of groups. E.g. a work on the symmetric groups [Pet76] or for Lie-groups of small rank [Ble99]. The groups $\text{PSL}(2,q)$, which are also the object of study in this paper, found also some special attention in [Wag95], [Her07], [HHK09] or in [BK11]. In this paper we
will limit our attention to finite $p$-subgroups of $V(ZG)$.

One could ask, what a Sylow-like theorem could mean for $V(ZG)$. One variation, lets say a **weak Sylow theorem**, would be that every finite $p$-subgroup of $V(ZG)$ is isomorphic to some subgroup of $G$. A stronger result, say a **strong Sylow theorem**, would be, if every finite $p$-subgroup of $V(ZG)$ is even rationally conjugate to a subgroup of $G$. First Sylow-like results for integral group rings were obtained in [KR93]. Later M. A. Dokuchaev and S. O. Juriaans proved a strong Sylow theorem for special classes of solvable groups [DJ96] and M. Hertweck, C. Höfert and W. Kimmerle proved a weak Sylow theorem for $\text{PSL}(2, p^f)$, where $p = 2$ or $f \leq 2$. The results of this article are as follows:

**Proposition 1:** Let $G = \text{PSL}(2, p^f)$, let $r$ be a prime different from $p$ and let $u$ be a torsion unit in $V(ZG)$ of $r$-power order. Then $u$ is rationally conjugate to a group element.

**Theorem 2:** Let $G = \text{PSL}(2, p^f)$ such that $f \leq 2$ or $p = 2$. Then a strong Sylow theorem holds in $V(ZG)$.

### 1 HeLP-method and known results

Let $G$ be a finite group. A very useful notion to study rational conjugacy of torsion units are partial augmentations: Let $u = \sum_{g \in G} a_g g \in ZG$ and $x^G$ be the conjugacy class of the element $x \in G$ in $G$. Then $\varepsilon_x(u) = \sum_{g \in x^G} a_g$ is called the **partial augmentation** of $u$ at $x$. This relates to rational conjugacy via:

**Lemma 1.1** ([MRSW87], Th. 2.5). Let $u \in V(ZG)$ be a torsion unit. Then $u$ is rationally conjugate to a group element if and only if $\varepsilon_x(u^k) \geq 0$ for all $x \in G$ and all powers $u^k$ of $u$.

It is well known that if $u \neq 1$ is a torsion unit in $V(ZG)$, then $\varepsilon_1(u) = 0$ by the so called Berman-Higman Theorem [Seh93, Prop. 1.4]. If $\varepsilon_x(u) \neq 0$, then the order of $x$ divides the order of $u$ [MRSW87, Th. 2.7], [Her06, Prop. 3.1]. Moreover the exponent of $G$ and of $V(ZG)$ coincide [CL65]. We will use this facts in the following without further mentioning.
Let \( u \) be a torsion unit in \( V(\mathbb{Z}G) \) of order \( n \) and \( \zeta \) an \( n \)-th root of unity in some field \( K \), whose characteristic does not divide \( n \). Let \( \xi \) be an (not necessarily primitive) \( n \)-th root of unity in \( K \) and let \( \varphi \) be a \( K \)-representation of \( G \). It was first obtained by Luthar and Passi for \( K \) having characteristic 0 [LP89] and later generalized by Hertweck for positive characteristic [Her07] that the multiplicity of \( \xi \) as an eigenvalue of \( \varphi(u) \), which we denote by \( \mu(\xi, u, \varphi) \) and which is of cause a non-negative integer, may be computed as

\[
\mu(\xi, u, \varphi) = \frac{1}{n} \sum_{d|n, d \neq 1} \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\varphi(u^d)\xi^{-d}) + \frac{1}{n} \sum_{x \in \mathbb{C}^*} \varepsilon(x) \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\varphi(x)\xi^{-1}),
\]

where as usual \( \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(x) = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(x) \).

If \( u \) is of prime power order \( p^k \) for the first sum in the expression above we obtain

\[
\frac{1}{n} \sum_{d|n, d \neq 1} \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\varphi(u^d)\xi^{-d}) = \frac{1}{p} \mu(\xi^p, u^p, \varphi).
\]

Using these formulas to find possible partial augmentations for torsion units in integral group rings of finite groups is today called HeLP-method. For a diagonalizable matrix \( A \) we will write \( A \sim (a_1, \ldots, a_n) \), if the eigenvalues of \( A \), with multiplicities, are \( a_1, \ldots, a_n \).

All subgroups of \( G = \text{PSL}(2, p^f) \) were first known to Dickson [Dic01, Theorem 620]. Let \( d = \gcd(2, p - 1) \). There are cyclic groups of order \( p, \frac{p^f - 1}{d} \) and \( \frac{p^f - 1}{d} \) in \( G \) and every element of \( G \) lies in a conjugate of such a group. The \( p \)-Sylow subgroups are elementary-abelian, the Sylow subgroups for all other primes, which are odd, are cyclic and if \( p \neq 2 \) the 2-Sylow subgroup is dihedral or a Kleinian four-group. There are \( d \) conjugacy classes of elements of order \( p \). If \( g \in G \) is not of order \( p \) or 2 its only distinct conjugate in \( \langle g \rangle \) is \( g^{-1} \). Especially there is always only one conjugacy class of involutions. We denote by \( a \) a fixed element of order \( \frac{p^f - 1}{d} \) and by \( b \) a fixed element of order \( \frac{p^f - 1}{d} \).

The modular representation theory of \( \text{PSL}(2, q) \) in defining characteristic is well known. All irreducible representations were first given by R. Brauer and C. Nesbitt [BN41]. The explicit Brauer table of \( \text{SL}(2, q) \), which contains the Brauer table of \( \text{PSL}(2, q) \), may be found in [Sti64]. However, I was not able to find the following Lemma in the literature, except, without proof, in Hertwecks preprint [Her07], so a short proof is included.
Lemma 1.2. Let \( G = \text{PSL}(2, p^f) \) and \( d = \gcd(2, p - 1) \). There are \( p \)-modular representations of \( G \) given by \( \varphi_0, \varphi_1, \varphi_2, \ldots \) such that there is a \( \frac{p^f - 1}{d} \)-th primitive root of unity \( \alpha \) and a \( \frac{p^f + 1}{d} \)-th primitive root of unity \( \beta \) satisfying

\[
\varphi_k(b) \sim (1, \beta, \beta^{-1}, \beta^2, \beta^{-2}, \ldots, \beta^k, \beta^{-k})
\]

\[
\varphi_k(a) \sim (1, \alpha, \alpha^{-1}, \alpha^2, \alpha^{-2}, \ldots, \alpha^k, \alpha^{-k})
\]

for every \( k \in \mathbb{N}_0 \).

Proof: The group \( \text{SL}(2, q) \) acts on the vector space spanned by the homogenous polynomials in two commuting variables \( x, y \) of some fixed degree \( e \) extending the natural operation of the 2-dimensional vector space spanned by \( x, y \), see e.g. [Alp86, p. 14-16]. Since

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\]

\( x^i y^j = (-1)^{i+j} x^i y^j \) this action affords a \( \text{PSL}(2, q) \)-representation if and only if \( e \) is even and \( p \) is odd or \( p = 2 \). so let from now on \( e \) be even for odd \( p \).

Call this representation \( \varphi_z \). Let \( \gamma \) be an eigenvalue of an element in \( \text{SL}(2, q) \) mapping onto \( a \) under the natural projection from \( \text{SL}(2, q) \) to \( \text{PSL}(2, q) \). Then \( \varphi_z(a) \) has the same eigenvalues as \( \varphi_z \left( \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \right) \). Now

\[
\begin{pmatrix}
\gamma & 0 \\
0 & \gamma^{-1}
\end{pmatrix}
\]

\( x^i y^j = \gamma^{i-j} x^i y^j \), so the eigenvalues are \( \{ \gamma^{i-j} | 0 \leq i, j \leq d, \ i + j = e \} = \{ (\gamma^d)^t \ | \ \frac{e}{d} \leq t \leq \frac{e}{2} \} \). Thus setting \( \alpha = \gamma^d \) proves the first part of the claim.

Now let \( \delta \) be an eigenvalue of an element in \( \text{SL}(2, q) \) mapping onto \( b \) under the natural projection from \( \text{SL}(2, q) \) to \( \text{PSL}(2, q) \). The action of \( \text{SL}(2, q) \) may of course be extended to \( \text{SL}(2, q^2) \). So \( \varphi_z(b) \) has the same eigenvalues as \( \varphi_z \left( \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} \right) \), where the matrix may be seen as an element in \( \text{SL}(2, q^2) \). Then doing the same calculations as above and setting \( \beta = \delta^d \) proves the Lemma.

Using the HeLP-method R. Wagner [Wag95] and Hertweck [Her07] obtained already some results about rational conjugacy of torsion units of prime power order in \( \text{PSL}(2, q) \). Part of Wagners result was published in [BHK04].

Lemma 1.3. Let \( G = \text{PSL}(2, p^f) \) and \( f \leq 2 \). If \( u \) is a unit of order \( p \) in \( V(\mathbb{Z}G) \), then \( u \) is rationally conjugate to a group element.

Remark: The HeLP-method does not suffice to prove rational conjugacy of units of order \( p \) in \( V(\mathbb{Z} \text{PSL}(2, p^f)) \) if \( p \) is odd and \( f \geq 3 \). There is also no other method or idea
around how one could e.g. obtain, if units of order 3 in \(V(\mathbb{Z} \text{PSL}(2, 27))\) are rationally conjugate to group elements or not.

**Lemma 1.4.** [Her07, Prop. 6.4] Let \(G = \text{PSL}(2, p^f)\) and let \(r\) be a prime different from \(p\). If \(u\) is a unit of order \(r\) in \(V(\mathbb{Z}G)\), then \(u\) is rationally conjugate to an element of \(G\).

**Lemma 1.5.** [Her07, Prop. 6.5] Let \(G = \text{PSL}(2, p^f)\), let \(r\) be a prime different from \(p\) and \(u\) a torsion unit in \(V(\mathbb{Z}G)\) of order \(r^n\). Let \(m < n\) and denote by \(S\) a set of representatives of conjugacy classes of elements of order \(r^m\) in \(G\). Then \(\sum_{x \in S} \varepsilon_x(u) = 0\). If moreover \(g\) is an element of order \(r^n\) in \(G\), then \(\mu(1, u, \varphi) = \mu(1, g, \varphi)\) for every \(p\)-modular Brauer character \(\varphi\) of \(G\).

If one is interested not only in cyclic groups the following result is very useful. It may be found e.g. in [Sch93, Lemma 37.6] or in [Val94, Lemma 4].

**Lemma 1.6.** Let \(G\) be a finite group, \(U\) a finite subgroup of \(V(\mathbb{Z}G)\) and \(H\) a subgroup of \(G\) isomorphic to \(U\). If \(\sigma : U \to H\) is an isomorphism such that \(\chi(u) = \chi(\sigma(u))\) for all \(u \in U\) and all irreducible complex characters \(\chi\) of \(G\), then \(U\) is rationally conjugate to \(H\).

## 2 Proof of the results

We will first sum up some elementary number theoretical facts. The notation \(a \equiv b \,(c)\) will mean, that \(a\) is congruent \(b\) modulo \(c\).

**Lemma 2.1.** Let \(t\) and \(s\) be natural numbers such that \(s\) divides \(t\) and denote by \(\zeta_t\) and \(\zeta_s\) a primitive complex \(t\)-th root of unity and \(s\)-th root of unity respectively. Then

\[
\text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\zeta_s) = \mu(s) \frac{\varphi(t)}{\varphi(s)},
\]

where \(\mu\) denotes the Möbius function and \(\varphi\) Euler’s totient function. So for a prime \(r\) and natural numbers \(n, m\) with \(m \leq n\) we have

\[
\text{Tr}_{\mathbb{Q}(\zeta_{r^n})/\mathbb{Q}}(\zeta_{r^m}) = \begin{cases} 
 r^{n-1}(r-1), & m = 0 \\
 -r^{n-1}, & m = 1 \\
 0, & m > 1 
\end{cases}
\]
Let moreover \( i \) and \( j \) be integers prime to \( r \), then

\[
\text{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_{r^m}^i \zeta_{r^m}^{j}) = \begin{cases} 
   r^{n-1}(r - 1), & i \equiv j \ (r^m) \\
   -r^{n-1}, & i \not\equiv j \ (r^m), \ i \equiv j \ (r^{m-1}) \\
   0, & i \not\equiv j \ (r^{m-1})
\end{cases}
\]

Proof of Lemma 2.1: Let \( s = p_1^{f_1} \cdots p_k^{f_k} \) be the prime factorisation of \( s \). For a natural number \( l \) let \( I(l) = \{ i \in \mathbb{N} \mid 1 \leq i \leq l, \ \gcd(i, l) = 1 \} \). As is well known, \( \text{Gal}(\mathbb{Q}(\zeta_i)/\mathbb{Q}) = \{ \sigma_i : \zeta_i \mapsto \zeta_i^i \mid i \in I(t) \} \). From this the case \( s = 1 \) follows immediately. Otherwise we have

\[
\text{Tr}_{\mathbb{Q}(\zeta_i)/\mathbb{Q}}(\zeta_i) = \sum_{i \in I(t)} \zeta_i = \frac{\varphi(t)}{\varphi(s)} \sum_{i \in I(s)} \zeta_i = \frac{\varphi(t)}{\varphi(s)} \prod_{j=1}^{k} \sum_{i \in I(p_j^{f_j})} \zeta_i^{f_j}.
\]

Now \( \sum_{i \in I(p_j^{f_j})} \zeta_i^{f_j} = \begin{cases} 
   -1, & f_j = 1 \\
   0, & f_j > 1
\end{cases} \) and this gives the first formula. The other formulas are special cases of this general formula since \( \varphi(r^n) = (r - 1)(r^{n-1}) \).

Proof of Proposition 1: Let \( G = \text{PSL}(2, p^l) \), \( r \) be a prime different from \( p \) and let \( u \) be a torsion unit in \( V(\mathbb{Z}G) \) of order \( r^m \). Let \( \zeta \) be an \( r^m \)-th primitive complex root of unity and set \( \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}} = \text{Tr} \). If \( n = 1 \), then by Lemma 1.4 \( u \) is rationally conjugate to an element in \( G \), so assume \( n \geq 2 \). Assume further that by induction \( u^r \) is rationally conjugate to an element in \( G \). Let \( m \) be a natural number such that \( m < n \).

We will proceed by induction on \( m \) to show that \( \varepsilon_x(u) = 0 \), if the order of \( x \) is \( r^m \). If \( m = 0 \) this is the Berman-Higman Theorem and if \( r = 2 \) and \( m = 1 \) this follows from Lemma 1.5. So assume we know \( \varepsilon_x(u) = 0 \) for \( o(x) < r^m \). Let \( l = \frac{r^m - 1}{2} \) if \( r \) is odd and \( l = \frac{r^m - 2}{2} \) if \( r = 2 \). Let \( \{ x_i \mid 1 \leq i \leq l, \ \gcd(i, r) = 1 \} \) be a full set of representatives of conjugacy classes of elements of order \( r^m \) in \( G \) such that \( x_i^r = x_i \) (this is possible by the group theoretical properties of \( G \) given above).

We will prove by induction on \( k \) that \( \varepsilon_{x_i}(u) = \varepsilon_{x_j}(u) \) for \( i \equiv \pm j \ (r^{m-k}) \). This is certainly true for \( k = 0 \) and once we establish it for \( k = m \), if \( r \) is odd, and \( k = m - 1 \), if \( r = 2 \), it will follow from Lemma 1.5 that \( \varepsilon_{x_i}(u) = 0 \) for all \( i \). So assume \( \varepsilon_{x_i}(u) = \varepsilon_{x_j}(u) \) for \( i \equiv \pm j \ (r^{m-k}) \). Since \( u^r \) is rationally conjugate to a group element, there exists a primitive \( r^{n-1} \)-th root of unity \( \zeta_{r^{n-1}} \) such that

\[
\varphi_{r^k}(u^r) \sim (1, \zeta_{r^{n-1}}, \zeta_{r^{n-1}}^{-1}, \zeta_{r^{n-1}}^2, \zeta_{r^{n-1}}^{-2}, \ldots, \zeta_{r^{n-1}}^{r^k}, \zeta_{r^{n-1}}^{-r^k}).
\]
Now all $p$-modular Brauer characters of $G$ are real valued and thus we obtain that
\[
\varphi_r(u) \sim (1, a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1}),
\]
where for every $i$ we have $a_i$ a root of unity such that $a_i^{r^{m-k}} \neq 1$. So for every primitive $r^{m-k}$-th root of unity $\zeta_{r^{m-k}}$ we have $\mu(\zeta_{r^{m-k}}, u, \varphi_r) = 0$. Let $\zeta_m$ be a primitive $r^m$-th root of unity such that we have $\varphi_r(x_1) \sim (1, \zeta_m, \zeta_m^{-1}, \ldots, \zeta_m^{k}, \zeta_m^{-k})$ and set $\xi = \zeta_m^k$. Let $S$ be a set of representatives of elements of $G$ of $r$-power order not greater than $r^n$ containing $x_1, \ldots, x_l$ and let moreover $\alpha$ be a natural number prime to $r$ such that $1 \leq \alpha \leq l$. Thus $\mu(\xi^\alpha, u, \varphi_r) = 0$ and $\varepsilon_x(u) = 0$ for $o(x) < r^m$. From here on a sum over $i$ will always mean a sum over all defined $i$, that will be $1 \leq i \leq l$ and $r \nmid i$. Then using the HeLP-method we get
\[
0 = \mu(\xi^\alpha, u, \varphi_r) = \frac{1}{r^n} \mu(\xi^\alpha r, u^r, \varphi_r) + \frac{1}{r^n} \sum_{x \in S} \varepsilon_x(u) \text{Tr}(\varphi_r(x)\xi^{-\alpha})
\]
\[
= \frac{1}{r^n} \mu(\xi^\alpha r, u^r, \varphi_r) + \frac{1}{r^n} \sum_{x \in S} \varepsilon_x(u) \text{Tr}(\varphi_r(x)\xi^{-\alpha}) + \frac{1}{r^n} \sum_i \varepsilon_x(u) \text{Tr}(\varphi_r(x_i)\xi^{-\alpha})
\]
\[
= \frac{1}{r^n} \mu(\xi^\alpha r, u^r, \varphi_r) + \frac{1}{r^n} \sum_{x \in S} \varepsilon_x(u) \text{Tr}(\xi^{-\alpha}) + \frac{1}{r^n} \sum_i \varepsilon_x(u) \text{Tr}((\xi^i + \xi^{-i})\xi^{-\alpha})
\]
\[
= \frac{1}{r^n} \mu(\xi^\alpha r, u^r, \varphi_r) + \frac{\text{Tr}(\xi^{-\alpha})}{r^n} + \frac{1}{r^n} \sum_i \varepsilon_x(u) \text{Tr}((\xi^i + \xi^{-i})\xi^{-\alpha}). \tag{1}
\]
In the third line we used that if $\tilde{\zeta}$ is a root of unity of $r$-power order such that $\tilde{\zeta}^{r^{m-k}} \neq 1$, then $\tilde{\zeta}\xi$ has the same order as $\zeta$ and so $\text{Tr}(\tilde{\zeta}\xi) = 0$ by Lemma 2.4. Note that as $i$ is prime to $r$ the congruence $i \equiv \alpha \pmod{r^{m-k}}$ implies $-i \not\equiv \alpha \pmod{r^{m-k}}$ for $r^{m-k} \notin \{1, 2\}$ and these exceptions don’t have to be considered by our assumptions on $m$ and $k$.

There are now two cases to consider. First assume $k < m - 1$, so $\xi$ is at least of order $r^2$. Then we have $\mu(\xi^\alpha r, u^r, \varphi_r) = 0$ and using Lemma 2.1 in (1) we obtain
\[
0 = \frac{1}{r^n} \sum_i \varepsilon_x(u) \text{Tr}((\xi^i + \xi^{-i})\xi^{-\alpha})
\]
\[
= \frac{1}{r^n} \sum_{i \equiv \pm \alpha r^{m-k}} \varepsilon_x(u) (r^{n-1}(r - 1)) + \frac{1}{r^n} \sum_{i \equiv \pm \alpha r^{m-k-1}} \varepsilon_x(u) (-r^{n-1})
\]
\[
= \sum_{i \equiv \pm \alpha r^{m-k}} \varepsilon_x(u) - \frac{1}{r} \sum_{i \equiv \pm \alpha r^{m-k-1}} \varepsilon_x(u). \tag{2}
\]
So

\[ r \sum_{i \equiv \pm \alpha(r^{m-k})} \varepsilon_{x_i}(u) = \sum_{i \equiv \pm \alpha(r^{m-k-1})} \varepsilon_{x_i}(u). \]

But since by induction \( \varepsilon_{x_i}(u) = \varepsilon_{x_j}(u) \) for \( i \equiv \pm j \ (r^{m-k}) \) the summands on the left hand side are all equal and since changing \( \alpha \) by \( r^{m-k-1} \) does not change the right hand side of the equation we get \( \varepsilon_{x_i}(u) = \varepsilon_{x_j}(u) \) for \( i \equiv \pm j \ (r^{m-k-1}) \).

Now consider \( k = m - 1 \), then \( \xi \) is a primitive \( r \)-th root of unity and thus we have \( \mu(\xi^{or}, u^r, \varphi_r, \varepsilon) = 1 \). So using Lemma \( 2.1 \) in (1) we get

\[ 0 = \frac{1}{r} + \frac{1 - r^{n-1}}{r^n} + \frac{1}{r^n} \sum_{i \equiv \pm \alpha(r)} \varepsilon_{x_i}(u)(-2r^{n-1}) + \frac{1}{r^n} \sum_{i \equiv \pm \alpha(r)} \varepsilon_{x_i}(u)(r^{n-1}(r - 1) - r^{n-1}) \]

\[ = \sum_{i \equiv \pm \alpha(r)} \varepsilon_{x_i}(u) - 2 \frac{2}{r} \sum_i \varepsilon_{x_i}(u). \]

So

\[ r \sum_{i \equiv \pm \alpha(r)} \varepsilon_{x_i}(u) = 2 \sum_i \varepsilon_{x_i}(u). \]

Now by Lemma \( 1.6 \) the right side of this equation is zero and by induction all summands on the left side are equal. Hence varying \( \alpha \) gives \( \varepsilon_x(u) = 0 \) for \( \circ(x) = r^m \).

So it only remains to show that \( \varepsilon_x(u) = 1 \) for exactly one conjugacy class \( x^G \) in \( G \), where \( \circ(x) = r^n \). The arguments in this case are very close to the arguments above. Let \( k \leq n \). As in the computation above we have \( \varphi_{r^k}(u^r) \sim (1, \zeta_{r^{n-1}}, \zeta_{r^{n-1}}^{-1}, \ldots, \zeta_{r^{n-1}}^{r^k}, \zeta_{r^{n-1}}^{-r^k}) \) for some primitive \( r^{n-1} \)-th root of unity and \( \varphi_{r^k}(u^r) \sim (a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_{r^k}, a_{r^k}^{-1}) \), where \( a_i \) are roots of unity such that \( a_i^{r^k} \neq 0 \) for \( 1 \leq i \leq r^k - 1 \) and \( a_{r^k} \) is some primitive \( r^{n-k} \)-th root of unity. Set \( \xi = a_{r^k} \) and let \( l = \frac{r^{n-1}}{2} \), if \( r \) is odd, and \( l = \frac{r^{n-k}}{2} \), if \( r = 2 \).

Let \( \{ x_i | 1 \leq i \leq l, \ \gcd(i, r) = 1 \} \) be a set of representatives of conjugacy classes of elements of order \( r^n \) in \( G \) such that \( x_i = x_1^r \) and \( \varphi_i(x_1) \sim \varphi_1(u) \). Then \( x_i^r \) is rationally conjugate to \( u^r \). We will prove by induction on \( k \) that:

(i) \( \varepsilon_{x_1}(u) = 1 \) and \( \varepsilon_{x_i}(u) = 0 \) for \( i \equiv \pm 1 \ (r^{n-k}), i \neq 1 \).

(ii) \( \varepsilon_{x_1}(u) = \varepsilon_{x_j}(u) \) for \( i \equiv \pm j \ (r^{n-k}) \) and \( i \neq \pm 1 \ (r^{n-k}) \).

We will prove these two facts for \( k = n - 1 \). If \( r = 2 \), then the Proposition will follow from this. If \( r \) is odd, we will prove afterwards that \( \sum_{i \equiv \alpha(r)} \varepsilon_{x_i}(u) = 0 \) for \( \alpha \neq \pm 1 \ (r) \), which then also implies the Proposition.
Let $\alpha$ be a natural number prime to $r$ with $1 \leq \alpha \leq l$. Using the HeLP-method and $\varepsilon_x(u) = 0$ for $\sigma(x) < r^n$ we obtain, doing the same calculations as in (1):

$$
\mu(\xi^\alpha, u, \varphi_{r^k}) = \frac{1}{r} \mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) + \frac{1}{r^n} \sum_i \varepsilon_{x_i}(u) \text{Tr}((\xi^i + \xi^{-1})\xi^{-\alpha}).
$$

(4)

As $u^r$ is rationally conjugate to $x_1^r$ we know that $\xi^{\pm r}$ are eigenvalues of $\varphi_{r^k}(u^r)$. So we get

$$
\mu(\xi^\alpha, u, \varphi_{r^k}) = \begin{cases} 
1, & \alpha \equiv \pm 1 \ (r^{n-k}) \\
0, & \text{else}
\end{cases}
$$

and

$$
\mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) = \begin{cases} 
1, & \alpha \equiv \pm 1 \ (r^{n-k-1}) \\
0, & \text{else}
\end{cases}
$$

There are now several cases to consider: (ii) is clear for $k = 0$ and if $\alpha \not\equiv \pm 1 \ (r^{n-k})$ we can do the same computations as in (2) to obtain (ii), if $k < n - 1$. So (ii) holds for $k = n - 1$.

To obtain the base case for (i) set $k = 0$. Then from (4) we obtain (similar to the computation in (2)):

$$
1 = \frac{1}{r} + \varepsilon_{x_1}(u) - \frac{1}{r} \sum_{i \equiv \pm 1 (r^{n-1})} \varepsilon_{x_i}(u)
$$

and

$$
0 = \frac{1}{r} + \varepsilon_{x_\alpha}(u) - \frac{1}{r} \sum_{i \equiv \pm 1 (r^{n-1})} \varepsilon_{x_i}(u)
$$

for $\alpha \equiv \pm 1 \ (r^{n-1})$ and $\alpha \neq 1$. Subtracting two such equations gives

$$
1 = \varepsilon_{x_1}(u) - \varepsilon_{x_\alpha}(u)
$$

(5)

for every $\alpha \equiv \pm 1 \ (r^{n-1})$ and $\alpha \neq 1$. Let $t = \{|i \in \mathbb{N}|i \leq l, i \equiv \pm 1 \ (r^{n-1})\}|$. Then summing up the equations for all $\alpha \equiv \pm 1 \ (r^{n-1})$ gives

$$
1 = \frac{t}{r} + \sum_{i \equiv \pm 1 (r^{n-1})} \varepsilon_{x_i}(u) - \frac{t}{r} \sum_{i \equiv \pm 1 (r^{n-1})} \varepsilon_{x_i}(u) = \frac{t}{r} + (1 - \frac{t}{r}) \sum_{i \equiv \pm 1 (r^{n-1})} \varepsilon_{x_i}(u).
$$

So $\sum_{i \equiv \pm 1 (r^{n-1})} \varepsilon_{x_i}(u) = 1$ and the base case of (i) follows from (5).

So assume $1 \leq k < n - 1$. Then $\sum_{i \equiv \pm 1 (r^{n-k})} \varepsilon_{x_i}(u) = 1$ by induction and for $\alpha \equiv \pm 1 \ (r^{n-k})$
from (4) computing as in (2) we obtain

\[ 1 = 1 + \frac{1}{r} \sum_{i \equiv \pm 1 (r^{n-k})} \varepsilon_{x_i}(u) - \frac{1}{r} \sum_{i \equiv \pm 1 (r^{n-k-1})} \varepsilon_{x_i}(u) = \frac{1}{r} + 1 - \frac{1}{r} \sum_{i \equiv \pm 1 (r^{n-k-1})} \varepsilon_{x_i}(u). \]

For \( \alpha \not\equiv \pm 1 \ (r^{n-k}) \) and \( \alpha \equiv \pm 1 \ (r^{n-k-1}) \) we obtain the same way

\[ 0 = \frac{1}{r} + \sum_{i \equiv \pm \alpha (r^{n-k})} \varepsilon_{x_i}(u) - \frac{1}{r} \sum_{i \equiv \pm 1 (r^{n-k-1})} \varepsilon_{x_i}(u). \]

Thus subtracting the last equation from the one before gives

\[ 1 = 1 - \sum_{i \equiv \pm \alpha (r^{n-k})} \varepsilon_{x_i}(u). \]

The summands on the right hand side are all equal by (ii), so \( \varepsilon_{x_\alpha}(u) = 0 \), as claimed.

Finally let \( r \) be odd, \( k = n - 1 \) and \( \alpha \not\equiv \pm 1 \ (r) \). Then \( \mu(\xi^{\alpha}, u^r, \varphi_{r^k}) = \mu(1, u^r, \varphi_{r^k}) = 3 \).

So from (4) computing as in (3) we obtain

\[ 0 = \frac{3}{r} + \frac{-r^{n-1}}{r^n} - \frac{2}{r} \sum_{i} \varepsilon_{x_i}(u) + \sum_{i \equiv \pm \alpha (r)} \varepsilon_{x_i}(u) = \sum_{i \equiv \pm \alpha (r)} \varepsilon_{x_i}(u). \]

As by (ii) all summands in the last sum are equal, we get \( \varepsilon_{x_\alpha}(u) = 0 \) and the Proposition is finally proved.

**Proof of Theorem 2:** Let \( G = \text{PSL}(2, p^f) \) such that \( f \leq 2 \) or \( p = 2 \). Assume first that \( r \) is an odd prime, which is not \( p \), and \( R \) is an \( r \)-subgroup of \( V(ZG) \). As every \( r \)-subgroup of \( G \) is cyclic so is \( R \) by [Her08b, Theorem A] and thus \( R \) is rationally conjugate to a subgroup of \( G \) by Proposition 1. If \( p \neq 2 \) and \( R \) is a 2-subgroup of \( V(ZG) \), then \( R \) is either cyclic or dihedral or a Kleinian four group by [HHK09, Theorem 2.1]. If \( R \) is cyclic, then it is rationally conjugate to a subgroup of \( G \) by Proposition 1. If \( R \) is dihedral or a Kleinian four group let \( S = \langle s \rangle \) be a maximal cyclic subgroup of \( R \). Then \( s \) is rationally conjugate to an element \( g \in G \) by Proposition 1. Moreover \( R \) is isomorphic to some subgroup of \( H \) of \( G \), such that the maximal cyclic subgroup of \( H \) is generated by \( g \). As there is only one conjugacy class of involutions in \( G \) every isomorphism \( \sigma \) between \( R \) and \( H \) mapping \( s \) to \( g \) satisfies \( \chi(\sigma(u)) = \chi(u) \) for every irreducible complex character of \( G \). Thus \( R \) is rationally conjugate to \( H \) by Lemma 1.6.

If \( p = 2 \) and \( P \) is a 2-subgroup of \( V(ZG) \) then all non-trivial elements of \( P \) are in-
volutions, so $P$ is elementary abelian. As there is again only one conjugacy class of involutions in $G$ every isomorphism $\sigma$ between $P$ and a subgroup of $G$ isomorphic with $P$ satisfies $\chi(\sigma(u)) = \chi(u)$ for every irreducible complex character of $G$. So $P$ is rationally conjugate to a subgroup of $G$ by Lemma 1.6. Finally assume that $p$ is odd and $P$ is a $p$-subgroup of $V(ZG)$. If $P$ is of order $p$ it is rationally conjugate to a subgroup of $G$ by Lemma 1.3. If $P$ is of order $p^2$, it is elementary abelian. Let $c$ and $d$ be generators of $P$, then they are rationally conjugate to group elements by Lemma 1.3. But there are only two conjugacy classes of elements of order $p$ and to whichever elements $c$ and $d$ are conjugate, it is possible to pick some, which generate an elementary abelian subgroup of $G$ of order $p^2$. Then again we obtain an isomorphism $\sigma$ preserving character values.

**Remark:** Let $G = \text{PSL}(2, p^f)$ and let $n$ be a number prime to $p$. The structure of the Brauer table of $G$ in defining characteristic yields immediately, that if we can prove that a unit $u \in V(ZG)$ of order $n$ is rationally conjugate to an element in $G$ applying the HeLP-method to the Brauer table, then this calculations will hold over any $\text{PSL}(2, q)$, if $n$ and $q$ are coprime. In this sense it would be interesting, and seems actually achievable, to determine a subset $A_{p^f}$ of $\mathbb{N}$ such that we can say: The HeLP-method proves that a unit $u \in V(ZG)$ of order $n$ is rationally conjugate to an element in $G$ if and only if $n \in A_{p^f}$. Test computations yield the conjecture that $A_{p^f}$ actually contains all odd numbers prime to $p$. If this turned out to be true this would yield, using the results in [Her07], the First Zassenhaus Conjecture for the groups $\text{PSL}(2, p)$, where $p$ is a Fermat- or Mersenne prime.

Other interesting questions concerning torsion units of the integral group ring of $G = \text{PSL}(2, p^f)$ were mentioned at the end of [HHK09] and are still open today: If the order of $u \in V(ZG)$ is divisible by $p$, is $u$ of order $p^2$? Are units of order $p$ rationally conjugate to elements of $G$? Are there non-abelian $p$-subgroups in $V(ZG)$?

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