BLOCH’S CONJECTURE, DELIGNE COHOMOLOGY
AND HIGHER CHOW GROUPS

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Introduction

Let $X$ be a smooth projective complex surface. D. Mumford [34] showed that the kernel of the Albanese map $\text{CH}_0(X)^0 \to \text{Alb}(X)$ is ‘huge’ if $p_g(X) \neq 0$ (where $p_g(X) = \dim \Gamma(X, \Omega^2_X)$). Then S. Bloch [5] conjectured that the condition $p_g(X) = 0$ should conversely imply

(a) The Albanese map $\text{CH}_0(X)^0 \to \text{Alb}(X)$ is injective.

This conjecture was proved in [9] if $X$ is not of general type, but the general case still remains open. It was suggested in [40] that condition (a) in the case $p_g(X) = 0$ would be related closely to the following conditions:

(b) $\lim_{\to} H^3_D(U, \mathbb{Q}(2)) = 0$, where $U$ runs over the nonempty open subvarieties of $X$.

(c) The cycle map $\text{CH}^2(D, 1)_\mathbb{Q} \to H^3_D(U, \mathbb{Q}(2))$ is surjective for any (sufficiently small) open subvarieties $U$ of $X$.

Here $\text{CH}^p(U, n)$ is Bloch’s higher Chow group, and $H^i_D(U, \mathbb{Q}(k))$ is $\mathbb{Q}$-Deligne cohomology. Note that condition (a) is equivalent to the injectivity of the Albanese map tensored with $\mathbb{Q}$ due to A. Roitman [37]. The equivalence of (a) and (b) has been proved by L. Barbieri-Viale and V. Srinivas [1] constructing the exact sequence (see also [24], [38]):

$$H^3_D(X, \mathbb{Z}(2)) \to \lim_{\to} H^3_D(U, \mathbb{Z}(2)) \to \text{CH}_0(X)^0 \to \text{Alb}(X).$$

By [11], condition (a) implies the decomposability of $\text{CH}^2(X, 1)_\mathbb{Q}$ (i.e. it is generated by the image of $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^*$), see also [21]. So it is conjectured that $\text{CH}^2(X, 1)_\mathbb{Q}$ is decomposable if $p_g(X) = 0$. Thus it would be interesting whether the reduced higher Abel-Jacobi map (induced by the cycle map in condition (c))

$$\text{CH}^2_{\text{ind}}(X, 1)_\mathbb{Q} \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q})(2))/\text{NS}(X)_\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}^*$$

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is injective in general, where $\text{CH}^2_{\text{ind}}(X,1)_Q$ is the quotient of $\text{CH}^2(X,1)_Q$ by the image of $\text{Pic}(X)_Q \otimes \mathbb{C}^*$, and $\text{NS}(X)_Q \otimes \mathbb{C}^*$ is identified with $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, N^1H^2(X,\mathbb{Q})(2))$. Here $N^1H^2(X,\mathbb{Q})$ is the $\mathbb{Q}$-submodule generated by algebraic cycle classes. This injectivity is related to Voisin's conjecture [44] on the countability of $\text{CH}^2_{\text{ind}}(X,1)_Q$, because the image of the reduced higher Abel-Jacobi map is countable [33]. The kernel of this map is isomorphic to

$$\text{Coker}(K_2(C(X))_Q \to \lim_{U} \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^2(U,\mathbb{Q})(2)))$$

by [35], [38], where the morphism of $K_2(C(X))_Q$ is given by $d\log \wedge d\log$ at the level of integral logarithmic forms, and the inductive limit is taken over the nonempty open subvarieties $U$ of $X$ (see also [3], 6.1). This isomorphism follows easily from the localization sequence of mixed Hodge structures together with the fact that the residue of $d\log f \wedge d\log g$ coincides with the differential of the tame symbol of $\{f, g\}$ up to sign. It holds also for open subvarieties $U$ if $H^3(U,\mathbb{Q}) = 0$. In view of condition (c) it would be interesting to study the higher Abel-Jacobi map in the smooth nonproper case, which is closely related to Griffiths' Abel-Jacobi map for the complement via (2.5), and whose image is not countable. By Beilinson [3] and Levine [31], it is described explicitly by using currents like Griffiths' Abel-Jacobi map if $X$ is smooth proper, see also [25], [26], [33], etc. In this paper, we extend this to the smooth nonproper case.

It has been observed often that constructing a nontrivial indecomposable higher cycle is not an easy task, see e.g. [2], [14], [15], [25], [33], [44], etc. If we take the composition of the cycle map with the projection to real Deligne cohomology in the smooth proper case, it is expressed by using the current defined by $\log |g_j|$, where the $g_j$ are rational functions on the irreducible components of the support of the higher cycle, and are given as a part of the definition of higher cycle. Using this, Gordon and Lewis [25] constructed an example of a nontrivial indecomposable higher cycle on a product of two elliptic curves. (Here it is enough to show that a certain integral is nonzero.)

This argument does not work when the two elliptic curves are isogeneous, and have complex multiplications. If we use the cycle map to rational Deligne cohomology, we get an integral of a 2-form on a certain topological chain whose boundary is related to the above $g_j$. This is analogous to Griffiths' Abel-Jacobi map. In order to show that the image of a higher cycle by this cycle map is not zero, we have to prove that this integral is not a $\mathbb{Q}$-linear combination of periods, and the argument is much more complicated. For example, the image of the higher cycle in [25] is zero, and it may be suspected that there are no nontrivial indecomposable higher cycles in this case, see [33]. Note that the desired property of the integrals is verified in some cases, for instance, when the cycle is the one-point compactification of $\mathbb{G}_m$, and is a fiber of a certain elliptic surface. This gives an example of an indecomposable higher cycle on a smooth affine surface which can be extended to a compactification of the surface. (It is easy to get a cycle which cannot be extended.) The argument uses double integrals which are the integrals of relative period integrals over 1-chains on the base of the elliptic surface, see (3.5). Recently, del Angel and Müller-Stach [16] have constructed another example of a higher cycle on a (proper) $K3$-surface such that the above property of the integrals is satisfied. The support of their cycle is reducible, and they use a differential equation on the parameter space, which is
satisfied by the period integrals.

Going back to Bloch's conjecture, we generalize the three conditions mentioned above to the higher dimensional case, using Deligne cohomology in a generalized sense, and prove their equivalence. The notion of Deligne cohomology was first introduced by Deligne in the case \( X \) is smooth and proper. It is a natural generalization of the first two terms of the exponential sequence (see [20]). The generalization to the open or singular case was first done by A. Beilinson [2] and H. Gillet [24], where the weight filtration was not used. Later Beilinson [3] found a more natural generalization from the viewpoint of mixed Hodge theory, which he calls absolute Hodge cohomology, and denotes by \( H^p_{\text{AH}}(X, A(k)) \), \( H^p_{\text{MS}}(X, A(k)) \), where \( A \) is a subring of \( \mathbb{R} \). In this paper we denote them by \( H^p_D(X, A(k))' \), \( H^p_D(X, A(k))'' \) respectively, see (1.1).

Let \( X \) be a connected smooth projective variety. We denote by \( \text{CH}^p_{\text{hom}}(X) \) the subgroup of \( \text{CH}^p(X) \) consisting of homologically equivalent to zero cycles. For an open subvariety \( U \) of \( X \), let \( j_{U,X} : U \to X \) denote the inclusion morphism, and \( j^*_{U,X} \) the pull-back of Deligne cohomology. For an integer \( p \), we prove in this paper the equivalence of the following conditions:

(a) Griffiths' Abel-Jacobi map \( \text{CH}^p_{\text{hom}}(X)_{\mathbb{Q}} \to J^p(X)_{\mathbb{Q}} \) is injective.

(b) \( \lim_{Y} H^{2p-1}_D(X \setminus Y, \mathbb{Q}(p))'/\text{Im}\ j^*_{X \setminus Y, X} = 0 \), where \( Y \) runs over the closed subvarieties of \( X \) with pure codimension \( p - 1 \).

(c) The cycle map \( \text{CH}^p(X \setminus Y, 1)_{\mathbb{Q}} \to H^{2p-1}_D(X \setminus Y, \mathbb{Q}(p))'/\text{Im}\ j^*_{X \setminus Y, X} \) is surjective for any (sufficiently large) closed subvarieties \( Y \) of \( X \) with pure codimension \( p - 1 \).

Note that \( H^{2p-1}_D(X \setminus Y, \mathbb{Q}(p))' = H^{2p-1}_D(X \setminus Y, \mathbb{Q}(p))'' \) if \( Y \) has codimension \( \geq p - 1 \), and \( H^{2p-1}_D(X \setminus Y, \mathbb{Q}(p)) = H^{2p-1}_D(X \setminus Y, \mathbb{Q}(p))' \) if furthermore \( p = \dim X \). (In this case, condition (a) is related with [36].) The pull-back \( j^*_{X \setminus Y, X} \) vanishes if \( H^{2p-2}(X, \mathbb{Q}) \) is generated by algebraic cycle classes and \( Y \) is sufficiently large. I am informed from the referee that the three conditions (a), (b), (c) are further equivalent to another condition which is equivalent to (c) with \( p - 1 \) replaced by \( p \), and which has been studied by Jannsen [27], 9.10.

For the proof of the equivalence of those conditions, we study the cycle map of the higher Chow group to Deligne homology in the divisor case, and prove

0.1. Theorem. For a variety \( Y \) of pure dimension \( m \), the cycle map induces isomorphisms

\[
cl : \text{CH}^1(Y) \xrightarrow{\sim} H^{2m-2}_D(Y, \mathbb{Z}(m-1))'' = H^{2m-2}_D(Y, \mathbb{Z}(m-1))'.
\]

If \( m = 1 \), this holds also for \( H^{2m-2}_D(Y, \mathbb{Z}(m-1)) \).

Indeed, using the compatibility of the cycle map with the localization sequence, this is reduced to the smooth case, because a similar isomorphism for \( \text{CH}^1(Y, 1) \) is already known, see [26], 3.1. Then the assertion is more or less well-known, see (2.7). It is also possible to describe \( \text{CH}^1_{\text{hom}}(Y) \) by using the normalization of \( Y \), see (2.10). This may be useful for explicit calculation. By the localization sequence, (0.1) immediately implies
0.2. Corollary. For an integer \( p \), we have a canonical exact sequence

\[
H_{\mathcal{D}}^{2p-1}(X, \mathbb{Z}(p)) \to \lim_{Y} H_{\mathcal{D}}^{2p-1}(X \setminus Y, \mathbb{Z}(p))'' \to \text{CH}^{p}_{\text{hom}}(X) \to J^{p}(X),
\]

where the inductive limit is taken over the closed subvarieties \( Y \) of \( X \) with pure codimension \( p-1 \). Here \( H_{\mathcal{D}}^{2p-1}(X \setminus Y, \mathbb{Z}(p))'' \) may be replaced by \( H_{\mathcal{D}}^{2p-1}(X \setminus Y, \mathbb{Z}(p))' \) in general, and by \( H_{\mathcal{D}}^{2p-1}(X \setminus Y, \mathbb{Z}(p)) \) if \( p = \dim X \).

This expresses the kernel of Griffiths’ Abel-Jacobi map in terms of Deligne cohomology (in the generalized sense) using inductive limit, and generalizes the result of [1] mentioned above. Note that \( \text{CH}^p(X) \) is the inductive limit of \( \text{CH}^1(Y) \) with \( Y \) as above.

By (0.2) the equivalence of (a) and (b) is clear. Since the cycle map is compatible with the localization sequence, (0.1) implies also the equivalence of (a) and (c). (A similar argument has been used for the equivalence with Jannsen’s condition, see [27], 9. 10.) As for the equivalence of (b) and (c), it follows also from an isomorphism for higher cycles similar to (0.1), see (2.9).

In Sect. 1, we review some elementary facts from the theories of Deligne cohomology and mixed Hodge Modules which are needed in this paper. In Sect. 2, we recall the definition of Bloch’s higher Chow groups and the cycle map, and prove (0.1–2) together with the equivalence of (b) and (c). In Sect. 3 we give an explicit description of the cycle map for higher cycles, and construct an example of an indecomposable higher cycle.

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In this paper a variety means a separated scheme of finite type over \( \mathbb{C} \). All sheaves are considered on the associated analytic spaces, and \( H^j(X^{\text{an}}, \mathbb{Q}) \) is denoted by \( H^j(X, \mathbb{Q}) \).

1. Deligne cohomology and mixed Hodge Modules

1.1. Deligne cohomology (see [2, 3, 18, 20, 22, 24, 26], etc.) Let \( X \) be a smooth variety, and \( \overline{X} \) a smooth compactification of \( X \) such that \( D := \overline{X} \setminus X \) is a divisor with normal crossings. Let \( j : X \to \overline{X} \) denote the inclusion morphism. Let \( A \) be \( \mathbb{Z} \) or \( \mathbb{Q} \) for simplicity in this paper. Then \( A \)-Deligne cohomology \( H^i_{\mathcal{D}}(X, A(k)) \) is defined to be the \( i \)-th hypercohomology group of

\[
C^\bullet_{\mathcal{X}(D)}(k) := C(\mathcal{R}j_*A_X(k) \oplus \sigma_{\geq k}\Omega^\bullet_{\mathcal{X}}(\log D) \to \mathcal{R}j_*\Omega^\bullet_{\mathcal{X}})[-1],
\]

where \( A_X(k) = (2\pi i)^k A_X \subset \mathbb{C}_X \).

For a complex variety \( X \) in general, let \( K = (K_A, (K_Q, W), (K_C, F, W)) \) be the complex of graded-polarizable mixed \( A \)-Hodge structures corresponding by [3, 3.11] to the mixed Hodge complex calculating the cohomology of \( X \) which is defined by using a simplicial resolution of a compactification of \( X \) as in [17]. Let \( \text{MHS}(A)^p \) (resp. \( \text{MHS}(A) \)) denote the abelian category of graded-polarizable (resp. not necessarily graded-polarizable) mixed \( A \)-Hodge structures. Then we define \( A \)-Deligne cohomology in the generalized sense by

\[
H^i_{\mathcal{D}}(X, A(k)) = H^i(C(K_A(k) \oplus F^kK_C \to K_C)[-1]),
\]

\[
H^i_{\mathcal{D}}(X, A(k))' = \mathcal{R}\text{Hom}_{\text{MHS}(A)}(A, K(k)[i]),
\]

\[
H^i_{\mathcal{D}}(X, A(k))'' = \mathcal{R}\text{Hom}_{\text{MHS}(A)^p}(A, K(k)[i]).
\]
We have also natural morphisms
\[ p \text{ polarizable mixed } X, A \\]

Then we define
\[ H^i_D(X, A(k))'' \rightarrow H^i_D(X, A(k))' \rightarrow H^i_D(X, A(k)). \]

Similarly, let \( K' = (K'_A, (K'_Q, W), (K'_C, F, W)) \) be the dual of the complex of graded-polarizable mixed \( A \)-Hodge structures corresponding by [3, 3.11] to the mixed Hodge complex calculating the cohomology with compact support of \( X \) which is defined by using a simplicial resolution of a compactification of \( X \) together with that of the divisor at infinity. Then we define \( A \)-Deligne homology in the generalized sense by

\[ H^i_D(X, A(k)) = H^{-i}(C(K'_A(-k) \oplus F^{-k}K'_C \rightarrow K'_C)[-1]), \]
\[ H^i_D(X, A(k))' = \text{RHom}_{\text{MHS}(A)}(A, K'(-k)[-i]), \]
\[ H^i_D(X, A(k))'' = \text{RHom}_{\text{MHS}(A)^p}(A, K'(-k)[-i]). \]

We have also natural morphisms

\[ H^i_D(X, A(k))'' \rightarrow H^i_D(X, A(k))' \rightarrow H^i_D(X, A(k)). \]

If \( X \) is smooth of pure dimension \( n \), then \( K = K'(n)[2n] \) so that

\[ H^i_D(X, A(k)) = H^{2n-i}(X, A(n-k)), \]

and similarly for \( H^i_D(X, A(k))' \), etc.

For a mixed \( A \)-Hodge structure \( H = (H_A, (H_Q, W), (H_C, F, W)) \) and an integer \( k \), we define

\[ J(H(k)) = H_C/(H_A(k)_{\text{free}} + F^k H_C), \]
\[ J'(H(k)) = W_{2k} H_C/((W_{2k} H_A)(k)_{\text{free}} + F^k H_C), \]
\[ J''(H(k)) = W_{2k-1} H_C/((W_{2k} H_A)(k)_{\text{free}} + F^k H_C) \cap W_{2k-1} H_C), \]

where \( H_A(k)_{\text{free}} = H_A(k)/H_A(k)_{\text{tor}} \). We define also

\[ F^k W_{2k} H_A(k) = \text{Ker}(H_A(k) \rightarrow H_C/F^k W_{2k} H_C). \]

(Similarly for \( F^k H_A(k) \).) Then we have short exact sequences

\[ 0 \rightarrow J(H^{i-1}(X, A)(k)) \rightarrow H^i_D(X, A(k)) \rightarrow F^k H^i(X, A)(k) \rightarrow 0, \]
\[ 0 \rightarrow J(H^{i-1}(X, A)(k))' \rightarrow H^i_D(X, A(k))' \rightarrow F^k W_{2k} H^i(X, A)(k) \rightarrow 0, \]
\[ 0 \rightarrow J(H^{i-1}(X, A)(k))'' \rightarrow H^i_D(X, A(k))'' \rightarrow F^k W_{2k} H^i(X, A)(k) \rightarrow 0, \]

because \( J'(H(k)) = \text{Ext}^1_{\text{MHS}(A)}(A, H(k)), J''(H(k)) = \text{Ext}^1_{\text{MHS}(A)^p}(A, H(k)) \) by [12] and the semisimplicity of polarizable Hodge structures. We have also

\[ 0 \rightarrow J(H^{BM}_{i+1}(X, A)(-k))'' \rightarrow H^i_D(X, A(k))'' \rightarrow F^{-k} W_{-2k} H^i_{BM}(X, A)(-k) \rightarrow 0, \]
etc. Here \( H^\text{BM}(X, A) \) denotes Borel-Moore homology.

It is known that \( H^D_i(X, A(k)) \) and \( H^P_i(X, A(k)) \) (together with Deligne local cohomology) satisfy the axioms of Bloch-Ogus \([10]\), see \([2], [24], [26]\), etc. In particular, we have a canonical long exact sequence

\[
(1.1.3) \quad \rightarrow H^D_i(Y, A(k)) \rightarrow H^P_i(X, A(k)) \rightarrow H^P_i(U, A(k)) \rightarrow H^P_{i-1}(Y, A(k)) \rightarrow
\]

for a closed subvariety \( Y \) of \( X \) and \( U = X \setminus Y \). (This is functorial for \( Y, U \).) Similar assertions hold for \( H^P_i(X, A(k))' \), \( H^P_i(X, A(k))'' \). (The assertion for \( H^P_i(X, \mathbb{Q}(k))'' \) follows also from \([42]\) using (1.4) below.)

**Remark.** Let \( K = (K_Z, (K_Q, W), (K_C; F, W); K'_Q, (K'_C, W); \alpha_1, \alpha_2, \alpha_3, \alpha_4) \) be a polarizable mixed Hodge complex in the sense of \([3, 3.9]\), where

\[
\alpha_1: K_Z \otimes \mathbb{Z} Q \rightarrow K'_Q, \quad \alpha_2: K_Q \rightarrow K'_Q,
\]

\[
\alpha_3: (K_Q, W) \otimes \mathbb{C} \rightarrow (K'_C, W), \quad \alpha_4: (K_C, W) \rightarrow (K'_C, W)
\]

are (filtered) quasi-isomorphisms, and \((\text{Gr}^W_i K_Q, \text{Gr}^W_i (K_C, F))\) with the isomorphism in the derived category \( \text{Gr}^W_i K_Q \otimes \mathbb{Q} \mathbb{C} = \text{Gr}^W_i K_C \) induced by \( \alpha_3, \alpha_4 \) is a polarizable Hodge complex of weight \( i \) in the sense of \([17]\), i.e. \( \text{Gr}^W_i (K_C, F) \) is strict and \( H^j(\text{Gr}^W_i K_Q, \text{Gr}^W_i (K_C, F)) \) is a polarizable Hodge structure of weight \( i + j \).

Let \( \text{Dec} W \) be as in loc. cit. By definition, we have a canonical surjection

\[
(\text{Dec} W)_0 K^j_Q \rightarrow H^j \text{Gr}^W_{-j} K_Q,
\]

(and similarly for \( K_C, K'_C \)). For a \( \mathbb{Q} \)-Hodge structure \( H = (H_Q, (H_C, F)) \) of weight 0, let

\[
H^{(0)} = \text{Hom}_{\text{MHS}}(\mathbb{Q}, H),
\]

which is identified with a subgroup of \( H_Q, H_C \). We define the subcomplex \((\text{Dec} W)_0^{(0)} K_Q\) of \((\text{Dec} W)_0 K_Q\) so that \((\text{Dec} W)_0^{(0)} K^j_Q\) is the inverse image of \((H^j \text{Gr}^W_{-j} K)^{(0)}\) by the above morphism (and similarly for \((\text{Dec} W)_0^{(0)} K_C\), \((\text{Dec} W)_0^{(0)} K'_C\)).

Let \((W_0 H^j K_Q)^{(0)}\) be the inverse image of \((\text{Gr}^W_0 H^j K_Q)^{(0)}\) by the projection of \( W_0 H^j K_Q \) to \( \text{Gr}^W_0 H^j K_Q \). Since \( d_1 \) of the weight spectral sequence is a morphism of Hodge structures, we can show the canonical quasi-isomorphism

\[
\tau_{\leq j}(\text{Dec} W)_0^{(0)} K_Q/\tau_{< j}(\text{Dec} W)_0^{(0)} K_Q \rightarrow (W_0 H^j K_Q)^{(0)},
\]

and similarly for \( K_C, K'_C \).

We define a complex \( \Gamma(D''_H K) \) to be the single complex associated with

\[
K_Z \oplus (\text{Dec} W)_0^{(0)} K_Q \oplus F^0(\text{Dec} W)_0^{(0)} K_C \rightarrow K'_Q \oplus (\text{Dec} W)_0^{(0)} K'_C,
\]

where \( \phi \) is induced by \((\alpha_1 - \alpha_2) \oplus (\alpha_3 - \alpha_4)\), and the degree of the source of \( \phi \) is zero. Then, by an argument similar to \([3]\), we can show the isomorphism

\[
(1.1.4) \quad \text{Hom}_{\text{D''}}(\mathbb{Z}, K) = H^0 \Gamma(D''_H K),
\]
where $\mathcal{D}''$ denotes the category of polarizable mixed Hodge complexes in the sense of [3, 3.9]. So we can define $H_D^i(X, \mathbb{Z}(k))''$ taking a mixed Hodge complex which calculates the cohomology of $X$ as in [17]. Note that (1.1.4) implies the equivalence of categories

$$D^b_{\text{MHS}}(\mathbb{Z})'' \sim \mathcal{D}''$$

in Lemma 3.1 of [3], and that it is easy to show the exact sequence

$$0 \to \text{Ext}^1_{\text{MHS}(\mathbb{Z})'}(\mathbb{Z}, H^{-1}K) \to \text{Hom}_{\mathcal{D}''}(\mathbb{Z}, K) \to \text{Hom}_{\text{MHS}(\mathbb{Z})}(\mathbb{Z}, H^0K) \to 0,$$

using the truncation $\tau$.

We can similarly define $\Gamma(D_H'K), \Gamma(D_HK)$ to be the single complex associated with

$$K_Z \oplus (\text{Dec} W)_0K_Q \oplus F^0(\text{Dec} W)_0K_C \to K'_Q \oplus (\text{Dec} W)_0K'_C, \quad K_Z \oplus K_Q \oplus F^0K_C \to K'_Q \oplus K'_C,$$

respectively. They can be defined also for a mixed Hodge complex $K$ in the sense of [3, 3.2] (where $\text{Dec} W$ is replaced by $W$). Using these, we can also define $H_D^i(X, A(k))'$, $H_D^i(X, A(k))$. Note that $\Gamma(D_HK)$ is canonically isomorphic to

$$K_Z \oplus F^0K_C \to K'_C,$$

if there is a canonical morphism $\alpha' : K_Z \to K_Q$ such that $\alpha_1 = \alpha_2 \circ \alpha'$.

1.2. Lemma. The canonical morphism $H_D^i(X, A(k))' \to H_D^i(X, A(k))$ is an isomorphism if

$$H^{i-1}(X, A) \text{ and } H^i(X, A) \text{ have weights } \leq 2k,$$

and $H_D^i(X, A(k))''' \to H_D^i(X, A(k))'$ is an isomorphism if

$$\text{Gr}^{W}_{2k}H^{i-1}(X, \mathbb{Q}) \text{ is isomorphic to a direct sum of } \mathbb{Q}(-k).$$

We have the corresponding assertion for Deligne homology where $H^{i-1}(X, A)$, $H^i(X, A)$ and $k$ are replaced respectively by $H^{BM}_{i+1}(X, A)$, $H^i_{BM}(X, A)$ and $-k$.

Proof. This is clear by (1.1.1).

Remark. Condition (1.2.2) is satisfied for $H^{2p-2}(X \setminus Y, \mathbb{Q})$ with $i = 2p-1$ and $k = p$, if $X$ is smooth and $Y$ is a closed subvariety of codimension $\geq p-1$. Indeed, $\text{Gr}^{W}_{2p}H^{2p-1}_Y(X, \mathbb{Q})$ is a direct sum of $\mathbb{Q}(-p)$. A similar assertion holds also for $\text{Gr}^{W}_{2m-2}H^{2m-1}_{BM}(Y, \mathbb{Q})$ if $Y$ is of pure dimension $m$.

1.3. Mixed Hodge Modules (see [39]). For a variety $X$ we denote by $\text{MHM}(X)$ the abelian category of mixed $\mathbb{Q}$-Hodge Modules on $X$, and $D^b_{\text{MHM}}(X)$ its derived category consisting of bounded complexes of mixed $\mathbb{Q}$-Hodge Modules. There is a natural functor $\text{rat} : D^b_{\text{MHM}}(X) \to D^b_{\text{c}}(X, \mathbb{Q})$ assigning the underlying $\mathbb{Q}$-complexes where $D^b_{\text{c}}(X, \mathbb{Q})$
denotes the full subcategory of $D^b_c(\mathcal{X}^\text{an}, \mathbb{Q})$ consisting of $\mathbb{Q}$-complexes whose cohomology sheaves are algebraically constructible. We denote by $H^i : D^b\text{MHM}(X) \to \text{MHM}(X)$ the usual cohomology functor.

For morphisms $f$ of algebraic varieties we have canonically defined functors $f_*, f_!, f^*, f^!$ between the derived categories of mixed $\mathbb{Q}$-Hodge Modules. They are compatible with the corresponding functors of $\mathbb{Q}$-complexes via the functor rat. For a closed embedding $i : X \to Y$, the direct image $i_*$ will be omitted sometimes in order to simplify the notation, because

$$i_* : D^b\text{MHM}(X) \to D^b\text{MHM}(Y)$$

is fully faithful.

If $X = \text{Spec} \mathbb{C}$ we have naturally an equivalence of categories

$$\text{MHM}(\text{Spec} \mathbb{C}) = \text{MHS}(\mathbb{Q})^p.$$  

Here the right-hand side is as in (1.1). So $\text{MHM}(\text{Spec} \mathbb{C})$ will be identified with $\text{MHS}(\mathbb{Q})^p$.

We denote by $\mathbb{Q}(j)$ the mixed Hodge structure of type $(-j, -j)$ whose underlying $\mathbb{Q}$-vector space is $(2\pi)^j \mathbb{Q} \subset \mathbb{C}$, see [17]. For a variety $X$ with structure morphism $a_X : X \to \text{Spec} \mathbb{C}$, we define

$$\mathbb{Q}^H_X(j) = a_X^* \mathbb{Q}(j), \quad \mathbb{D}^H_X(j) = a_X^! \mathbb{Q}(j),$$

so that $\mathbb{D}^H_X(j)$ is the dual of $\mathbb{Q}^H_X(-j)$. We will write $\mathbb{Q}^H_X$ for $\mathbb{Q}^H_X(0)$, and similarly for $\mathbb{D}^H_X$.

If $X$ is smooth of pure dimension $n$, we have a canonical isomorphism

$$\mathbb{D}^H_X = \mathbb{Q}^H_X(n)[2n].$$

1.4. Proposition. With the notation of (1.1) and (1.3) we have canonical isomorphisms

$$H^i_P(X, \mathbb{Q}(k))'' = \text{Ext}^i(\mathbb{Q}, (a_X)_* \mathbb{Q}^H_X(k)),$$

$$H^i_P(X, \mathbb{Q}(k))'' = \text{Ext}^{-i}(\mathbb{Q}, (a_X)_* \mathbb{D}^H_X(-k)).$$

Proof. In the case $A = \mathbb{Q}$, we have canonical isomorphisms $K = (a_X)_* \mathbb{Q}^H_X$, $K' = (a_X)_* \mathbb{D}^H_X$ by [43].

1.5. Remark. Let $X$ be a reduced variety of pure dimension $n$, and $X_i$ be the irreducible components of $X$. Let $\text{Rat}(X)^* = \prod \text{Rat}(X_i)^*$ with $\text{Rat}(X_i)$ the rational function field of $X_i$. Then by [22, 2.12], [26, 3.1] we have a canonical isomorphism

$$H^i_{2n-1}(X, \mathbb{Z}(n-1)) = \{g \in \text{Rat}(X)^* : \text{div} g = 0\},$$

where $\text{div} g = \sum \text{div} g_i$ if $g = (g_i)$ with $g_i \in \text{Rat}(X_i)^*$. (Here (1.2.1–2) are satisfied.)

If $X$ is smooth, this is due to [22, 2.12]. In this case, the left-hand side of (1.5.1) is isomorphic to $\text{Ext}^1(\mathbb{Z}_X, \mathbb{Z}_X(1))$ (where $\text{Ext}^1$ is taken in the category of admissible variation...
of mixed Hodge structures), and the assertion is related with the theory of 1-motives [17], and is more or less well-known. Indeed, if $X$ is a point, the assertion is verified by calculating the period of the mixed Hodge structure on $H^1(\mathbb{A}^1 \setminus \{0\}, \{1\} \cup \{x\})$ for $x \in \mathbb{C} \setminus \{0, 1\}$, i.e., by using the integral of $dt/t$ on the relative cycle connecting $\{1\}$ and $\{x\}$, where $t$ is the coordinate of $\mathbb{A}^1$. The general case is reduced to the smooth case using a long exact sequence, see [26, 3.1].

2. Higher Chow groups and cycle maps

2.1. Higher Chow groups ([6]). Let $\Delta^n = \text{Spec}(\mathbb{C}[t_0, \ldots, t_n]/(\sum t_i - 1))$. For a subset $I$ of $\{0, \ldots, n\}$, let $\Delta^n_I = \{t_i = 0 (i \in I)\} \subset \Delta^n$. It is naturally isomorphic to $\Delta^m$ with $m = n - |I|$ (fixing the order of the coordinates), and is called a face of $\Delta^n$. For $0 \leq i \leq n$, we have inclusions $\iota_i : \Delta^{n-1} \to \Delta^n$ such that its image is $\Delta^n_I$.

Let $X$ be an equidimensional variety. Then $X \times \Delta^n_I$ is also called a face of $X \times \Delta^n$. Following Bloch, we define $z^p(X, n)$ to be the free abelian group with generators the irreducible closed subvarieties of $X \times \Delta^n$ of codimension $p$, intersecting all the faces of $X \times \Delta^n$ properly. We have face maps

$$\partial_i : z^p(X, n) \to z^p(X, n - 1),$$

induced by $\iota_i$. Let $\partial = \sum (-1)^i \partial_i$. Then $\partial^2 = 0$, and $\text{CH}^p(X, n)$ is defined to be $\text{Ker} \partial/\text{Im} \partial$ which is a subquotient of $z^p(X, n)$. By [6] it is isomorphic to

$$\frac{\bigcap_{0 \leq i \leq n} \text{Ker}(\partial_i : z^p(X, n) \to z^p(X, n - 1))}{\partial_{n+1}(\bigcap_{0 \leq i \leq n} \text{Ker}(\partial_i : z^p(X, n + 1) \to z^p(X, n)))}$$

(2.1.1)

Indeed, let $z^p(X, \bullet)'$ be the subcomplex of $z^p(X, \bullet)$ defined by

$$z^p(X, n)' = \bigcap_{0 \leq i < n} \text{Ker}(\partial_i : z^p(X, n) \to z^p(X, n - 1)).$$

Then

$$z^p(X, \bullet)' \to z^p(X, \bullet)$$

(2.1.2)

is a quasi-isomorphism. (For this, we can consider first the subcomplex defined by $\text{Ker} \partial_0$, using a homotopy given by the zeroth degeneracy, and then proceed inductively.)

2.2. Functoriality. Let $f : X \to Y$ be a proper morphism of varieties, and put $r = \dim X - \dim Y$. Then we have the pushforward functor

$$f_* : \text{CH}^p(X, n) \to \text{CH}^{p-r}(Y, n).$$

In fact, for a face map $\iota : \Delta^m \to \Delta^n$, Bloch showed the commutative diagram

$$
\begin{array}{ccc}
z^p(X, n) & \xrightarrow{\iota^*} & z^p(X, m) \\
\downarrow f_* & & \downarrow f_* \\
z^{p-r}(X, n) & \xrightarrow{\iota^*} & z^{p-r}(X, m)
\end{array}
$$
As for the pull-back, we have $f^*: \text{CH}^p(Y, n) \to \text{CH}^p(X, n)$ if $f$ is flat. In the case $X, Y$ are quasi-projective and smooth, we have $f^*: \text{CH}^p(Y, n)_Q \to \text{CH}^p(X, n)_Q$ by [32]. Here we have a quasi-isomorphic subcomplex $z^p_f(Y, \bullet)_Q$ of $z^p(Y, \bullet)_Q$ on which the pull-back $f^*$ is naturally defined.

2.3. Cycle map. Let $X$ be an equidimensional variety. By [2], [7], [18], etc., we have a cycle map

\[(2.3.1)\quad cl: \text{CH}^p(X, n) \to H^{2d+n}_\mathcal{D}(X, Q(d))''\]

where $d = \dim X - p$. The target becomes $H^{2p-n}_\mathcal{D}(X, Q(p))$ by if $X$ is smooth. Using mixed Hodge Modules [41], the cycle map (2.3.1) is defined as follows.

Let $S^{n-1} = \bigcap \Delta^n_i \subset \Delta^n, U = \Delta^n \setminus S^{n-1}$ with the inclusion morphisms $i: S^{n-1} \to \Delta^n, j: U \to \Delta^n$. Then

\[(2.3.2)\quad (a_{\Delta^n})_* j_! Q^H_U = Q^H_{\text{pt}}[-n],\]

where $a_{\Delta^n}: \Delta^n \to \text{pt} := \text{Spec} \mathbb{C}$ is the structure morphism. Let $\zeta = \sum_k n_k[Z_k] \in \bigcap_{0 \leq i \leq n} \text{Ker} \partial_i \subset z^p(X, n)$ (see (2.1)), where $Z_k$ are irreducible closed subvarieties of $X \times \Delta^n$. Let $d' = \dim Z_k = d + n$. Put $Z = \bigcup_k Z_k$. Then the coefficients $n_k$ of $Z_k$ induces a morphism

\[(2.3.3)\quad Q^H_Z \to \bigoplus_k \text{IC}_{Z_k} Q^H[-d'] \to \mathbb{D}^H_Z(-d')[-2d'] \to \mathbb{D}^H_{X \times \Delta^n}(-d')[-2d'],\]

where $\text{IC}_{Z_k} Q^H$ denotes the mixed Hodge Module whose underlying perverse sheaf is the intersection complex $\text{IC}_{Z_k} Q$ [4]. Let $\pi: X \times \Delta^n \to X$ be the first projection, and let $j$ denote also $\text{id} \times j: X \times U \to X \times \Delta^n$ (and the same for $i$). Then

$$\pi_* j_! \mathbb{D}^H_{X \times U} = \mathbb{D}^H_X(n)[n]$$

by (1.3.4) and (2.3.2). So it is enough to show that (2.3.3) is uniquely lifted to

$$Q^H_Z \to j_! \mathbb{D}^H_{X \times U}(-d')[-2d'],$$

i.e., the composition of (2.3.3) with

$$\mathbb{D}^H_{X \times \Delta^n}(-d')[-2d'] \to i_* i^* \mathbb{D}^H_{X \times \Delta^n}(-d')[-2d']$$

is zero and $\text{Hom}(Q^H_Z, i_* i^* \mathbb{D}^H_{X \times \Delta^n}(-d')[-2d' - 1]) = 0$. But they follow from the condition on proper intersection together with $\zeta \in \bigcap_{0 \leq i \leq n} \text{Ker} \partial_i$. For the well-definedness of the cycle map, it is enough to show its invariance under a deformation of cycle parametrized by $\mathbb{A}^1$ (using a blow-up of $\Delta^n$).

Remark. The cycle map for $n = 0$ is defined with integral coefficients by the composition of

\[(2.3.4)\quad Z \to H^D_{2d}(Z, Z(d)) \to H^D_{2d}(X, Z(d))\]
for \( \zeta = \sum_k n_k [Z_k] \in CH_d(X) \), where \( Z = \cup_k Z_k \), see [2], [24], etc. If \( X \) is smooth proper, this coincides with Deligne’s cycle map, which is defined by the composition of

\[
Z \to H_{2p}^2(X, \mathbb{Z}(p)) \to K(p)[2p]
\]

where \( K \) is as in (1.1). It induces Griffiths’ Abel-Jacobi map

(2.3.5) \[ CH^p_{\text{hom}}(X) \to J^p(X) := \text{Ext}^1_{\text{MHS}}(\mathbb{Z}, H^{2p-1}(X, \mathbb{Z}(p))), \]

see [20] (and also [39, (4.5.20)]), etc. Here \( J^p(X) \) is Griffiths’ intermediate Jacobian by [12]. (This can be defined even if \( X \) is not proper.) If \( p = \dim X \), (2.3.5) is the Albanese map.

2.4. Compatibility. The cycle map (2.3.1) is compatible with \( f_* \) for a proper morphism \( f \), and also with \( f^* \) for a morphism of smooth quasi-projective varieties \( f : X \to Y \) in the case of rational coefficients. Indeed, this is reduced to the case of the usual Chow groups by (2.2) and the construction of the cycle map (2.3), and follows from [42].

2.5. Proposition. Let \( X \) be a quasi-projective variety, and \( Y \) a closed subvariety. Assume \( X, Y \) are equidimensional. Let \( r = \text{codim}_X Y \), and \( d = \dim X - p \). Then the cycle map induces a morphism of long exact sequences

\[
\begin{array}{cccc}
CH^p(X \setminus Y, n + 1) & \longrightarrow & CH^{p-r}(Y, n) & \longrightarrow & CH^p(X, n) \\
\downarrow & & \downarrow & & \downarrow \\
H^{2d+n+1}_{2d+n+1}(X \setminus Y, Q(d))'' & \longrightarrow & H^{2d+n}_{2d+n}(Y, Q(d))'' & \longrightarrow & H^{2d+n}_{2d+n}(X, Q(d))'' \\
& & & &
\end{array}
\]

where the first exact sequence is the localization sequence [8] (choosing the sign appropriately), and the second comes from (1.1.3).

Proof. The assertion is clear except for the commutativity of the left part of the diagram. Let \( U' = \Delta^{n+1} \setminus \cup_{0 \leq i \leq n} \Delta_{n}^{n+1} \{i\} \), and identify \( \Delta_{n+1}^{n+1} \cap U' \) with \( U^n := \Delta^n \setminus S^n \). Let \( j : X \setminus Y \to X, j' : U' \to \Delta^{n+1}, j^n : U^n \to \Delta^n \) denote the inclusion morphisms so that we have distinguished triangles

\[
\begin{array}{cccc}
\to \mathbb{D}_Y^H & \to \mathbb{D}_X^H & \to j_* \mathbb{D}_{X \setminus Y}^H & \to, \\
& \to j_1^{n+1} \mathbb{D}_{U_{n+1}}^H & \to j'_1 \mathbb{D}_{U'}^H & \to j_1^n \mathbb{D}_{U_n}^H(1)[2] & \to,
\end{array}
\]

where the direct images by closed embeddings are omitted to simplify the notation, see (1.3.1).

Let \( \zeta \in CH^p(X \setminus Y, n + 1) \). By (2.1.2) and [8], it is represented by \( \zeta \in z^p(X \setminus Y, n + 1)' \) which is extended to \( \zeta' \in z^p(X, n + 1)' \) so that its restriction to \( X \times \Delta^n \) is \( \overline{\zeta} \in z^p(Y, n)' \). Then the image of \( \zeta \) by the morphism of the localization sequence is \( \overline{\zeta} \). By definition \( \zeta' \) gives

\[
\xi \in \text{Hom}(\mathbb{Q}_Z^H, j'_1 \mathbb{D}_{X \times U'}^H(-d' - 1)[-2d' - 2])
\]
such that its restriction to \((X \setminus Y) \times \Delta^n\) vanishes. (Here \(d' = \dim X-p+n\), and \(j'\) denotes also \(id \times j'\).) So it induces

\[
\xi' \in \text{Hom}(\mathbb{Q}^H_2, j^nD_{Y, U}^H ((-d')[-2d']),
\xi'' \in \text{Hom}(\mathbb{Q}^H_2, j^nD_{(X \setminus Y) \times U}^H ((-d'-1)[-2d'-2]),
\]

using the external product of the above two distinguished triangles. We see that \(\xi'\) coincides with the image of \(\xi\) by the cycle map, and the second distinguished triangle induces an isomorphism \((a_{\Delta^n})_*j^nD_{U}^H (1)[1] \rightarrow (a_{\Delta^n+1})_*j^{n+1}D_{U_{n+1}}^H\). So the assertion is reduced to the next lemma. (Here we represent the middle terms of the distinguished triangles by the mapping cone of the morphism of the other terms so that we get short exact sequences as below.)

2.6. Lemma. Let \(\{K^{i,j,k}\}\) be a square diagram of short exact sequences of complexes of an abelian category, i.e. \(K^{i,j,k} = 0\) for \(|i| > 1\) or \(|j| > 1\), and \(K^{i+1,j,k} \rightarrow K^{i,j,k} \rightarrow K^{i+1,j,k}\) is exact (and the same for the index \(j\)). Let \(\xi \in H^k(K^{0,0,*})\) such that its image in \(H^k(K^{1,1,*})\) vanishes. Let \(\xi' \in H^k(K^{1,-1,*})\), \(\xi'' \in H^k(K^{1,-1,*})\) such that the images of \(\xi\), \(\xi'\) in \(H^k(K^{0,1,*})\) coincide and the images of \(\xi\), \(\xi''\) in \(H^k(K^{1,0,*})\) coincide. Then the images of \(\xi'\), \(\xi''\) in \(H^{k+1}(K^{1,-1,*})\) coincide up to sign.

(The proof is straightforward. For a similar assertion, where the triangle is slightly shifted, a proof is given in [28], p. 268.)

2.7. Proof of (0.1). It is enough to show the assertion for \(H^2_{2m-2}(Y, \mathbb{Z}(m-1))''\) by (1.2). We apply (2.5) to \(Y\) and a divisor \(Z\) on \(Y\) containing \(\text{Sing} Y\). Let \(U = Y \setminus Z\). The assertion follows from [42, I, (3.4)] if \(Y\) is smooth (i.e. if \(U = Y\)). Note that we have the surjectivity of the cycle map \(\text{CH}^1(U) \rightarrow H^2_{2m-2}(U, \mathbb{Z}(m-1))''\) in loc. cit, because \(H^1(X, \mathbb{Z})\) is torsion-free. By [22, 2.12], we have a similar isomorphism

\[
(2.7.1) \quad \text{CH}^1(U, 1) \rightarrow H^2_{2m-1}(U, \mathbb{Z}(m-1))''
\]

So the general case is reduced to the smooth case by using the cycle map of the localization sequence

\[
\text{CH}^1(U, 1) \rightarrow \text{CH}^0(Z) \rightarrow \text{CH}^1(Y) \rightarrow \text{CH}^1(U) \rightarrow 0
\]

to the corresponding exact sequence of Deligne homology.

2.8. Remarks. (i) In general, the isomorphism \(\text{CH}^1(Y) = H^2_{2m-2}(Y, \mathbb{Z}(m-1))\) does not hold (even for a smooth \(Y\)), see [42, I, (3.5)]. (In Remark (i) of loc. cit. the assumption of the second statement should be replaced by the condition that \(H^2(X, \mathbb{Q}) \cap F^1H^2(X, \mathbb{C})\) is not contained in \(W_2H^2(X, \mathbb{Q})\).)

(ii) It is well known (see e.g. [33]) that any element \(\xi\) of \(\text{CH}^1(X, 1)\) can be represented by \(\sum_j (Z_j, g_j)\) where \(Z_j\) are irreducible (and reduced) subvarieties of \(X\) with pure codimension \(p-1\) and \(g_j\) are rational functions on \(Z_j\) such that \(\sum_j \text{div} g_j = 0\). (Here we use an automorphism of \(\mathbb{P}^1\) sending \(0, 1, \infty\) to \(0, \infty, 1\) respectively.) The relation is given by the tame symbol.
(iii) If we assume \( g_j = \text{const} \) in Remark (ii) above, we get a natural morphism

\[
\text{CH}^{p-1}(X) \otimes \mathbb{C}^* \to \text{CH}^p(X, 1).
\]

Its image is denoted by \( \text{CH}^{p-1}_{\text{dec}}(X, 1) \), and is called the subgroup of decomposable higher cycles. We put

\[
\text{CH}^{p-1}_{\text{ind}}(X, 1)_\mathbb{Q} = \text{CH}^{p-1}(X, 1)_\mathbb{Q}/\text{CH}^{p-1}_{\text{dec}}(X, 1)_\mathbb{Q}.
\]

The image of \( \text{CH}^{p-1}_{\text{dec}}(X, 1)_\mathbb{Q} \) by the higher Abel-Jacobi map (see (3.4.3) below) is contained in

\[
J(N^{p-1}H^{2p-2}(X, \mathbb{Q}))(p) \subset \text{Hdg}^{p-1}(X)_\mathbb{Q} \otimes \mathbb{C}^* ,
\]

where \( N^{p-1}H^{2p-2}(X, \mathbb{Q}) \) is the \( \mathbb{Q} \)-submodule generated by algebraic cycle classes, and \( \text{Hdg}^{p-1}(X)_\mathbb{Q} \) is the group of Hodge cycles with rational coefficients, see [25], [33], etc.

2.9. Proof of the equivalence of (a), (b) and (c). The equivalence of (b) and (c) in the introduction follows from (2.5) together with an isomorphism for higher cycles similar to (0.1), which we apply to \( X \) and a closed subvariety of pure codimension \( p - 1 \). The equivalence of (a) and (c) follows from (2.5) and (0.1).

Remark. Consider the condition (c) with \( p - 1 \) replaced by \( p \). Then it is equivalent to the condition in [27], 9.10, and the equivalence of this condition and condition (a) was shown by using a compatibility (9.8 in loc. cit.) which is similar to (2.5).

2.10. Relation with the normalization. We can express the subgroup \( \text{CH}^1_{\text{hom}}(Y) \) of \( \text{CH}^1(Y) \) consisting of Borel-Moore homologically equivalent to zero cycles by using the normalization of \( Y \). This may be useful for explicit calculation.

Let \( Y \) be a connected variety of pure dimension \( m \) with \( Y_k \) (\( 1 \leq k \leq r \)) the irreducible components of \( Y \). Let \( \tilde{Y} \) be the disjoint union of the normalizations \( \tilde{Y}_k \) of \( Y_k \) with \( \pi : \tilde{Y} \to Y \) the natural morphism. Let \( D = \{ y \in Y : |\pi^{-1}(y)| > 1 \} \). We assume \( \tilde{Y} \) is smooth, \( D \) is a smooth closed subvariety of \( Y \) with pure codimension one, and \( \pi_* \mathbb{Z}_{\tilde{Y}} |_D \) is a local system. (We may assume these because \( \text{CH}^1(Y) \) and \( H^P_{2m-2}(Y, \mathbb{Z}(m-1)) \) do not change by deleting a closed subvariety of codimension \( > 1 \).)

Let \( \tilde{D} = \pi^{-1}(D) \), and \( \tilde{D}_i (i \in I), D_j (j \in J) \) be connected components of \( \tilde{D}, D \). Put

\[
I_j = \{ i \in I : \tilde{D}_i \subset \pi^{-1}(D_j) \}, \quad I(k) = \{ i \in I : \tilde{D}_i \subset \tilde{Y}_k \}.
\]

We define \( \mathcal{E}_j = \ker(\text{Tr} : \oplus_{i \in I_j} \pi_* \mathbb{Z}_{\tilde{D}_i} \to \mathbb{Z}_{D_j}) \), \( E_j = H^0(D_j, \mathcal{E}_j) \), and \( E = \oplus_j E_j \). Let \( d_i \) be the degree of \( \tilde{D}_i \) over \( \pi(\tilde{D}_i) \). Then \( E_j \) is naturally identified with

\[
\{ a_i \in \mathbb{Z} (i \in I_j) : \sum_{i \in I_j} d_i a_i = 0 \}.
\]

Let \( E' = \oplus_{j \in J} H^1(D_j, \mathcal{E}_j) \). (This may have torsion which is related to the cokernel of \( \text{CH}^1(\tilde{Y}) \to \text{CH}^1(Y) \).) Then we have an exact sequence

\[
0 \to H^{BM}_{2m-1}(\tilde{Y}, \mathbb{Z}) \to H^{BM}_{2m-1}(Y, \mathbb{Z}) \to E(m-1)
\]

\[
\to H^{BM}_{2m-2}(\tilde{Y}, \mathbb{Z}) \to H^{BM}_{2m-2}(Y, \mathbb{Z}) \to E'(m-1),
\]
where \( \gamma \) is defined by \( (a_i) \to \sum_i a_i cl([\tilde{D}_i]) \). Here \( cl([\tilde{D}_i]) \) denotes the cycle class.

We define \( E^0 = \text{Ker} \gamma \subset E \) so that we get

\[
0 \to H^1(\tilde{Y}, \mathbb{Z})(1) \to H_{2m-1}^{BM}(Y, \mathbb{Z})(1-m) \to E^0 \to 0,
\]

The associated extension class is denoted by \( e \in \text{Ext}^1_{\text{MHS}}(E^0, H^1(\tilde{Y}, \mathbb{Z})(1)) \). The cycle map induces an isomorphism of exact sequences

\[
\begin{array}{cccc}
E^0 & \longrightarrow & \text{CH}^1_{\text{hom}}(\tilde{Y}) & \longrightarrow & \text{CH}^1_{\text{hom}}(Y) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
E^0 & \longrightarrow & J^1(\tilde{Y}) & \longrightarrow & J^1(Y)^{BM} & \longrightarrow & 0,
\end{array}
\]

where \( J^1(Y)^{BM} := \text{Ext}^1_{\text{MHS}}(\mathbb{Z}, H_{2m-1}^{BM}(Y, \mathbb{Z})(1-m)) \) and \( J^1(\tilde{Y}) \) is as in (2.3.5) (and is a quotient of the Jacobian of a smooth compactification of \( \tilde{Y} \)).

Indeed, let \( \text{CH}^1(Y)' = \text{Im}(\text{CH}^1(\tilde{Y}) \to \text{CH}^1(Y)) \), \( \text{CH}^1_{\text{hom}}(Y)' = \text{CH}^1(Y)' \cap \text{CH}^1_{\text{hom}}(Y) \). Then, for the exactness of the first row, it is sufficient to show

\[
\text{CH}^1_{\text{hom}}(Y)' = \text{CH}^1_{\text{hom}}(Y).
\]

This is reduced to the case where the cycle is supported on \( \text{Sing} Y \), and follows from the localization sequence for Borel-Moore homology. The second row is induced by (2.10.2), and we can show that for \( u \in \text{Hom}_{\text{MHS}}(\mathbb{Z}, E^0) \),

\[
e \circ u \in \text{Ext}^1_{\text{MHS}}(\mathbb{Z}, H^1(\tilde{Y}, \mathbb{Z})(1))
\]

collides with the image of \( \sum_i a_i [\tilde{D}_i] \) by the Abel-Jacobi map, where \( (a_i) = u(1) \in E^0 \).

This is verified by using a natural morphism of (2.10.2) to

\[
0 \to H^1(\tilde{Y}, \mathbb{Z})(1) \to H^1(\tilde{Y} \setminus \tilde{D}, \mathbb{Z})(1) \to H^0(\tilde{D}, \mathbb{Z}).
\]

3. Higher Abel-Jacobi map

3.1. Currents. For a complex manifold \( X \) of dimension \( n \), let \( \mathcal{C}^*(X) \) denote the complex of currents on \( X \) which has the Hodge filtration \( F \) as usual. Here we normalize \( \mathcal{C}^*(X) \) so that \( \mathcal{C}^i(X) = 0 \) for \( i > 0 \) or \( i < -2n \). It has a structure of double complex such that the Hodge filtration is given by the first degree. We have a natural morphism \( \mathcal{E}^*(X)(n)[2n] \to \mathcal{C}^*(X) \), where \( \mathcal{E}^i(X) \) denotes the vector space of \( C^\infty \) \( i \)-forms on \( X \). Let \( S^*(X) \) denote the complex of locally finite \( C^\infty \) chains on \( X \) where \( S^{-j}(X) \) consists of locally finite \( j \)-chains so that \( H^j(S^*(X)) = H^{j+2n}(X, \mathbb{Z})(n) \). There is a natural morphism \( t : S^*(X) \to \mathcal{C}^*(X) \). The differential \( d \) of \( S^*(X) \) is defined in a compatible way with that of \( \mathcal{C}^*(X) \). So it differs from the usual boundary map \( \partial \) by the sign \( (-1)^{\text{deg}} \) due to the Stokes theorem. Note
that the differential of a current \(\Phi\) is defined by \((d\Phi)(\omega) + (-1)^{\deg \Phi} \Phi(d\omega) = 0\) for \(C^\infty\)
forms \(\omega\) with compact supports. For a smooth complex algebraic variety \(X\), we will denote
\(S^\bullet(X^\an), C^\bullet(X^\an)\) by \(S^\bullet(X), C^\bullet(X)\) to simplify the notation.

Let \(\overline{X}\) be a smooth proper complex algebraic variety of dimension \(n\), and \(D\) a divisor on \(\overline{X}\) with normal crossings such that each irreducible component \(D_j\) is smooth. Put \(X = \overline{X} \setminus D\). Let \(\tilde{D}^{(j)}\) be the disjoint union of the intersections of \(j\) irreducible components as in [17, II]. Then we have naturally a double complex

\[
\longrightarrow F^k C^\bullet(\tilde{D}^{(j+1)}) \longrightarrow F^k C^\bullet(\tilde{D}^{(j)}) \longrightarrow \cdots \longrightarrow F^k C^\bullet(\tilde{D}^{(0)}) \longrightarrow 0
\]

by the dual construction of [17, III] (using the push-down of currents instead of the pullback of forms), and the associated single complex will be denoted by \(F^k C^\bullet(\overline{X}(D))\).

We have the weight filtration \(W\) on \(C^\bullet(\overline{X}(D))\) such that \(\text{Gr}_W^j C^\bullet(\overline{X}(D)) = C^\bullet(\tilde{D}^{(j)})[j]\). We define similarly \(S^\bullet(\overline{X}(D))\) with the filtration \(W\) such that \(\text{Gr}_W^j S^\bullet(\overline{X}(D)) = S^\bullet(\tilde{D}^{(j)})[j]\). Then we get the polarizable mixed Hodge complex \(K(\overline{X}(D))\) defined by

\[
(S^\bullet(X), (S^\bullet(\overline{X}(D))_Q, W), (C^\bullet(\overline{X}(D)), F, W); S^\bullet(X)_Q, (C^\bullet(\overline{X}(D)), W)),
\]

which calculates the Borel-Moore homology of \(X\). By (1.1.4) we have a canonical isomorphism (see also [26], [29]):

\[
H^i_D(X, \mathbb{Z}(k)) = H^{i-2n}(\Gamma(D^\prime_H(K(\overline{X}(D)))(k-n))).
\]

3.2. Cycle class. With the above notation, let \(\zeta = \sum_j (Z_j, g_j) \in \text{CH}^p(X, 1)\) as in Remark (ii) of (2.8). Put \(d = n - p\). Let \(\gamma_j\) be the closure of the inverse image by \(g_j\) of

\[
\{z \in \mathbb{C} \mid \text{Re} z > 0, \text{Im} z = 0\} \subset \mathbb{P}^1.
\]

Using a triangulation, it is viewed as a topological chain. We give it an orientation so that \(\partial \gamma_j = \text{div} g_j\). Then \(\gamma := \sum_j \gamma_j\) is a topological cycle on \(Z := \bigcup_j Z_j\), and it belongs to \(S^{-2d-1}(X)\). Let \(\tilde{Z}_j \rightarrow Z_j\) be a resolution of singularities such that the divisor of the pullback \(\tilde{g}_j\) of \(g_j\) to \(\tilde{Z}_j\) has normal crossings. Let \(\pi_j : \tilde{Z}_j \rightarrow X\) denote its composition with the inclusion \(i_j : Z_j \rightarrow X\). Then we have the push-down of currents \((\pi_j)_* : C^\bullet(\tilde{Z}_j) \rightarrow C^\bullet(X)\).

Let \(\log_{\text{hv}} \tilde{g}_j\) denote a locally integrable function on \(\tilde{Z}_j\) which is defined by choosing a branch of \(\log \tilde{g}_j\) on \(\tilde{Z}_j \setminus \tilde{\gamma}_j\) where \(\tilde{\gamma}_j\) is the pull-back of \(\gamma_j\) to \(\tilde{Z}_j\). (Hv stands for Heaviside.) Then it is a current on \(\tilde{Z}_j\), and we can verify

\[
d(\log_{\text{hv}} \tilde{g}_j) = \tilde{g}_j^{-1} d\tilde{g}_j - (2\pi i) \lambda \tilde{\gamma}_j,
\]

\[
d(\tilde{\lambda}^{-1} d\tilde{g}_j) = (2\pi i) \lambda (\text{div} \tilde{g}_j).
\]

Note that \(\tilde{g}_j^{-1} d\tilde{g}_j\) is a form with locally integrable coefficients on \(\tilde{Z}_j\) and \(\sum_j (\pi_j)_*(\tilde{g}_j^{-1} d\tilde{g}_j)\) is a closed current. We define

\[
(i_j)_*(\tilde{g}_j^{-1} d\tilde{g}_j) = (\pi_j)_*(\tilde{g}_j^{-1} d\tilde{g}_j), \quad gdg = \sum_j (i_j)_*(\tilde{g}_j^{-1} d\tilde{g}_j) \in F^{-d}C^{-2d-1}(X),
\]

\[
(i_j)_*(\log_{\text{hv}} g_j) = (\pi_j)_*(\log_{\text{hv}} \tilde{g}_j), \quad \log_{\text{hv}} g = \sum_j (i_j)_*(\log_{\text{hv}} g_j) \in C^{-2d-2}(X),
\]
These are independent of the choice of \( \overline{Z}_j \).

Let \( \overline{Z}_j \) be the closure of \( Z_j \) in \( \overline{X} \). Then \( g_j \) is identified with a rational function \( \overline{g}_j \) on \( \overline{Z}_j \), and we can define \( \overline{g}^{-1}d\overline{g} = \sum_j (i_j)_* (\overline{g}_j^{-1}d\overline{g}_j) \), etc. similarly, where \( i_j : \overline{Z}_j \to \overline{X} \) is the inclusion morphism.

Let \( \operatorname{div} \overline{g} = \sum_j \operatorname{div} \overline{g}_j \). This is supported on \( D \), and there is a cycle \( (\operatorname{div} \overline{g})^{(1)} \) on \( \overline{D}^{(1)} \) such that its image in \( \overline{X} \) coincides with \( \operatorname{div} \overline{g} \). Taking a triangulation, \( (\operatorname{div} \overline{g})^{(1)} \) can be viewed as an element of \( S^{-2d}(\overline{D}^{(1)}) \). So we get

\[
(g^{-1}dg)^\wedge := (\overline{g}^{-1}d\overline{g}, -2\pi i \epsilon(\operatorname{div} \overline{g})^{(1)}) \in F^{-d}C^{-2d-1}(\overline{X}(D))
\]
such that \( d(g^{-1}dg)^\wedge = 0 \). Let \( \overline{\gamma} \) be the closure of \( \gamma \) in \( \overline{X} \). Since \( \partial \overline{\gamma} = \operatorname{div} \overline{g} \), we get

\[
\gamma^\wedge := (\overline{\gamma}, -(\operatorname{div} \overline{g})^{(1)}) \in S^{-2d-1}(\overline{X}(D))
\]
such that \( d\gamma^\wedge = 0 \). Then \( d(\log_{\text{Hv}} g) \in C^{-2d-1}(\overline{X}(D)) \) coincides with the sum of \(-2\pi i \epsilon \gamma^\wedge \) and \((g^{-1}dg)^\wedge \) in \( C^{-2d-1}(\overline{X}(D)) \) by \( (3.2.1) \).

**3.3. Theorem.** With the above notation, \( \mathrm{cl}(\zeta) \in H^{2p-1}_D(X, \mathbb{Z}(p))'' \) corresponds by the isomorphism \( (3.1.1) \) to

\[
(3.3.1) \quad (-2\pi i)^{-d} \gamma, -(2\pi i)^{-d} \gamma^\wedge, -(2\pi i)^{-d-1} (g^{-1}dg)^\wedge; 0, (2\pi i)^{-d-1} \log_{\text{Hv}} g)
\]
in \( S^{-2d-1}(X)(-d) \oplus S^{-2d-1}(\overline{X}(D))Q(-d) \oplus F^{-d}C^{-2d-1}(\overline{X}(D)) \oplus S^{-2d-2}(X)Q(-d) \oplus C^{-2d-2}(\overline{X}(D)) \).

**Proof.** Since the class of \( (\operatorname{div} \overline{g})^{(1)} \) in \( H^{2p-2}(\overline{D}^{(1)}, \mathbb{Q})(p - 1) \) is a Hodge cycle, we see that \( (3.3.1) \) belongs to \( H^{-2d-1}(\Gamma(D''_H(K(\overline{X}(D)))(-d))) \), see Remark after \( (1.1) \). Let \( \overline{Z} \) be the closure of \( Z \) in \( \overline{X} \), and

\[
\Sigma = \bigcup_j \mathrm{supp} \operatorname{div} \overline{g}_j \cup \text{Sing} \overline{Z}.
\]

Then the canonical morphism

\[
(3.3.2) \quad H^{2p-1}_D(X, \mathbb{Z}(p))'' \to H^{2p-1}_D(X \setminus \Sigma, \mathbb{Z}(p))''
\]
is injective by the localization sequence. So we may replace \( X \) with \( X \setminus \Sigma \), and assume that \( Z \) is smooth, and hence irreducible. Here we may assume also that the closure \( \overline{Z} \) of \( Z \) in \( \overline{X} \) is a good smooth compactification (i.e. \( \overline{Z} \setminus Z \) is a divisor with normal crossings) by taking further blowing-ups if necessary, and that every irreducible components of \( \overline{Z} \setminus Z \) is contained by only one irreducible component of \( \overline{X} \setminus X \). Then the isomorphism \( (3.1.1) \) is compatible with the push-forward by the closed embedding \( Z \to X \), and the assertion is reduced to the case \( X = Z, p = 1 \).

By \( (1.2) \) we have isomorphisms

\[
H^1_D(X, \mathbb{Z})'' = H^1_D(X, \mathbb{Z}) = H^1(\overline{X}, C^*_X(\overline{D})(\mathbb{Z}(1)));
\]
and the cycle map is calculated by using the commutative diagram

\[
\begin{array}{ccc}
\Gamma(X, \mathbb{G}_m) & \xrightarrow{\sim} & H^1(X, C_X^\bullet(D)\mathbb{Z}(1)) \\
\downarrow & & \downarrow \\
\Gamma(X^\text{an}, \mathcal{O}_{X^\text{an}}^\star) & \xrightarrow{\sim} & H^1(X, C_X^\bullet\mathbb{Z}(1))
\end{array}
\]

(3.3.3)

where the isomorphism on the bottom row is induced by the canonical quasi-isomorphism

\[\mathcal{O}_{X^\text{an}}^\star = C(Z_{X^\text{an}}(1) \to \mathcal{O}_{X^\text{an}}).\]

Here we may replace \(H^1(X, C_X^\bullet\mathbb{Z}(1))\) with the cohomology of the single complex associated with

\[S^\bullet(X)(-d) \to C^\bullet(X)/F^{-d}C^\bullet(X).\]

Then, using (1.1.5), it is enough to show that the image of \(g \in \Gamma(X^\text{an}, \mathcal{O}_{X^\text{an}}^\star)\) in this cohomology is represented by

\[(- (2\pi i)^{-d} \gamma, (2\pi i)^{-d-1} \log_{H^0} g),\]

because the vertical morphisms of (3.3.3) are injective.

We can verify this assertion by using a Cech resolution together with the delta functions supported on faces of a triangulation of \(X^\text{an}\) compatible with \(\gamma\). (See [23] for the notion of integral current.) Indeed, let \(U = \{U_i\}_{i \in \Lambda}\) be an open covering of \(X^\text{an}\) such that \(U_i\) are simply connected. We will denote by \(C_{\mathcal{U}}^i\mathcal{F}\) the Cech complex of a sheaf \(\mathcal{F}\) associated with the covering \(\mathcal{U}\), where

\[C_{\mathcal{U}}^i\mathcal{F} = \oplus_{|I|=i+1} \Gamma(U_I, \mathcal{F}) \quad \text{with} \quad U_I = \cap_{i \in I} U_i \quad \text{for} \quad I \subset \Lambda.
\]

For \(g \in \Gamma(X^\text{an}, \mathcal{O}_{X^\text{an}}^\star)\), we have the corresponding element in the cohomology of the single complex associated with

\[C_{\mathcal{U}}^i\mathcal{Z}_{X^\text{an}}^\star(1) \to C_{\mathcal{U}}^i\mathcal{O}_{X^\text{an}}^\star
\]

(where the first term has degree one), and it is given by

\[\{(\log(g|_{U_i}) - \log(g|_{U_j}))|_{U_{i,j}}, \{\log(g|_{U_i})\}_i \in C_{\mathcal{U}}^i\mathcal{Z}_{X^\text{an}}^\star(1) \oplus C_{\mathcal{U}}^0\mathcal{O}_{X^\text{an}}^\star.
\]

We define \(C_{\mathcal{U}}^i C^j(X), C_{\mathcal{U}}^i S^j(X)\) similarly. Then

\[\{(2\pi i)^{-d-1}((\log_{H^0} g)|_{U_i} - \log(g|_{U_i}))\}_i \in C_{\mathcal{U}}^0 C^{-2d-2}(X)
\]

belongs to the image of \(C_{\mathcal{U}}^0 S^{-2d-2}(X)(-d)\). So we get the assertion, using the triple complex

\[C_{\mathcal{U}}^i S^\bullet(X)(-d) \to C_{\mathcal{U}}^i (C^\bullet(X)/F^{-d}C^\bullet(X)).\]

This completes the proof of (3.3).
3.4. Remark. The cycle map (2.3.1) induces

\[(3.4.1) \quad \text{CH}^p(X, 1) \to \text{Hom}_{\text{MHS}}(\mathbb{Z}, H^{2p-1}(X, \mathbb{Z})(p)),\]

and (3.3) implies that the image of $\zeta$ by this morphism is represented by

\[-(2\pi i)^{-d}\gamma \quad \text{and} \quad -(2\pi i)^{-d-1}\sum_j (i_j)_* (g_j^{-1} dg_j).\]

Let $\text{CH}^p_{\text{hom}}(X, 1)$ be the kernel of (3.4.1). Then, in the notation of (1.1), the cycle map (2.3.1) induces the higher Abel-Jacobi map

\[(3.4.2) \quad \text{CH}^p_{\text{hom}}(X, 1) \to J(H^{2p-2}(X, \mathbb{Z})(p)) = \text{Ext}^1_{\text{MHS}}(\mathbb{Z}, H^{2p-2}(X, \mathbb{Z})(p)).\]

Assume $H^{2p-1}(X, \mathbb{Q}) = 0$ or $X$ proper. Then $\text{CH}^p(X, 1)/\text{CH}^p_{\text{hom}}(X, 1)$ is finite because the target of (3.4.1) is torsion. So (3.4.2) induces the higher Abel-Jacobi map

\[(3.4.3) \quad \text{CH}^p(X, 1)_Q \to J(H^{2p-2}(X, \mathbb{Q})(p)).\]

By (3.3) this is expressed explicitly as follows. For $\zeta \in \text{CH}^p_{\text{hom}}(X, 1)$, there exist a $C^\infty$ chain $\Gamma$ on $X$ and $\Xi \in F^{-d} C^{-2d-1}(\overline{X}(D))$ such that

\[\partial \Gamma = \gamma, \quad d\Xi = (g^{-1} dg)^\wedge.\]

By (3.3) and (1.1.5), the image of $\zeta$ under the higher Abel-Jacobi map (3.4.3) is represented by the current

\[(3.4.4) \quad \Phi_\zeta = (2\pi i)^{-d-1}\sum_j (i_j)_* \log_{H^v}, g_j + (2\pi i) d\Gamma - \Xi|_X.\]

(Note that $d\alpha = -i\gamma$ by Stokes.) If $X$ is proper, it is enough to consider $\Phi_\zeta(\omega)$ for $C^\infty$ forms $\omega$ with compact supports which are direct sums of forms of type $(i, j)$ with $i \geq d+1$, because the dual of $H^{2p-2}(X, \mathbb{C})/FpH^{2p-2}(X, \mathbb{C})$ is $F^{d+1} H^{2d+2}(X, \mathbb{C})$. Then $\Xi$ can be neglected, and we get the higher Abel-Jacobi map in \([3], [31]\) (see also \([25], [26], \) etc.)

3.5. Example. Let $Y = \mathbb{A}^2, S = \mathbb{A}^1$, and let $f : Y \to S$ be the morphism defined by the polynomial $f = y^2 - x^3(x + 1)$. It has two singular fibers $Y_0, Y_a$ (with $a = -4/27$), which are rational curves obtained by identifying two points of $\mathbb{A}^1$. Let $\pi : \overline{S} \to S$ be a finite morphism of smooth affine curves with degree 2 which ramifies over $P_1, P_2, P_3$ and $\infty$. Let $X = Y \times_S \overline{S}$ with the projection $\pi : X \to Y$. If the $P_i$ are generic, $X$ is a smooth affine surface, and hence $H^3(X, \mathbb{Q}) = 0$. Let $Z$ be a connected component of $\pi^{-1}(Y_0)$. Let $g$ be a rational function on $Z$ whose pull-back to the normalization of $Z$ has a zero and a pole at the inverse image of $\text{Sing} \overline{Z}$. If the $P_i$ are very general, then the image of $(Z, g)$ in $J(H^2(X, \mathbb{Q})(2))$ by (3.4.3) does not belong to the image of $F^1 H^2(X, \mathbb{C})$. In particular, $(Z, g)$ is indecomposable.
Indeed, let Γ′ be the topological cycle defined by the compact connected component of \( \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \leq 0\} \). Let Γ be the connected component of the inverse image of Γ′ to \( X \) such that \( \partial \Gamma \) is contained in \( Z \). Then it is enough to show that there exists a holomorphic 2-form \( \tilde{\omega} \) on a good compactification of \( X \) such that

\[
\int_{\Gamma} \tilde{\omega} \neq \int_{\eta} \tilde{\omega}
\]

for any \( \eta \in H_2(\mathbb{X}, \mathbb{Q}) \). Note that \( H^2(\mathbb{X}, \mathbb{Q}) \) is pure of weight 2 (using the Leray spectral sequence), and is identified with the cohomology of a good compactification of \( X \) up to the Hodge cycles, because 1 and \(-1\) are not eigenvalues of the local monodromy at infinity of the direct image local system, see (3.5.2).

Let \( \omega_{\text{rel}} = -2dx \wedge dy/f^*dt = (y^{-1}dx) \) where \( t \) is the standard coordinate of \( S \). Then \( \omega_{\text{rel}} \) satisfies the Gauss hypergeometric differential equation (via the Gauss-Manin connection, see [19])

\[
t(t + 4/27)\partial^2_t \omega_{\text{rel}} + (2t + 4/27)\partial_t \omega_{\text{rel}} + (5/36)\omega_{\text{rel}} = 0.
\]

Note that this differential equation is also satisfied by the period integrals \( \int_{\gamma_t} \omega_{\text{rel}} \) for multivalued horizontal families of cycles \( \{\gamma_t\} \). By calculating the roots of the indicial equation [13] of (3.5.2) at \( 0, a \) and \( \infty \), which are respectively \( \{0,0\}, \{0,0\}, \{1/6, 5/6\} \), we see that \( \omega_{\text{rel}} \) belongs to the direct image of the relative dualizing sheaf of a good compactification of \( X \), see [30].

Let \( \alpha_i \) be the coordinate of \( P_i \) so that \( \tilde{S} \) is given by the equation

\[
s^2 = (t - \alpha_1)(t - \alpha_2)(t - \alpha_3).
\]

Then \( \omega_{\tilde{S}} \) is trivialized by \( s^{-1}dt \), and the above \( \tilde{\omega} \) is defined by \( \omega_{\text{rel}} \otimes s^{-1}dt \). Let \( \beta_1, \beta_2, \beta_3 \), be a path on \( S \) connecting respectively two of \( \{P_1, P_2, P_3\}, \{0, \infty\} \) and \( \{a, \infty\} \). Let \( \beta = \beta_1 \cup \beta_2 \cup \beta_3, \) and \( \tilde{\beta} = \pi^{-1}(\beta) \). Since the exponents of the period integrals are nonnegative by (3.5.2), and \( \int_{\eta} \tilde{\omega} = 0 \) for \( \eta \) coming from \( Y \), we see that it is sufficient to treat the case where \( \eta \) is a 1-parameter family of \( C^\infty \) 1-cycles over \( \tilde{\beta} \) (see Remark below). So we may assume that \( \int_{\eta} \tilde{\omega} \) is the sum of

\[
\int_{\beta_i} s^{-1}F_{\eta_i}(t)dt \quad \text{with} \quad F_{\eta_i}(t) = \int_{\eta_i(t)} \omega_{\text{rel}},
\]

where \( \eta_i(t) \) is a family of \( C^\infty \) 1-cycles with rational coefficients over \( \beta_i \), and \( \eta_2(t), \eta_3(t) \) belong to the image of the variation at \( 0 \) and \( a \) respectively, because \( \beta_2 \) is the limit of a loop around \( \{0, \infty\} \), and similarly for \( \beta_3 \). We may assume furthermore that \( \beta_2 \) is the real interval \([0, +\infty) \), and \( \beta_3 \) is the union of \( \beta_2 \) and the real interval \( \beta_0 := [a, 0] \) by taking the limit.

Let \( \eta_0(t) = \Gamma \cap Y_t \) for \( t \in [a, 0] \). There is a rational number \( C \) such that \( \eta_3(t) = C\eta_0(t) \), because \( \eta_0(t) \) generates the kernel of the variation at \( a \) which coincides with the image of the variation. Let \( \eta_2(t) = \eta_2(t) + \eta_3(t) \) for \( t \in [0, +\infty[ \). Then it is enough to show

\[
(1 - C) \int_{\beta_0} s^{-1}F_{\eta_0}(t)dt \neq \int_{\beta_1} s^{-1}F_{\eta_1}(t)dt + \int_{\beta_2} s^{-1}F_{\eta_2}(t)dt
\]

(3.5.4)
as multivalued functions of the $\alpha_i$ for any families of cycles $\eta_i(t)$ over $\beta_i$ ($i = 1, 2$) as above. Assume the equality holds for some $\eta_i$ ($i = 1, 2$). Letting $\alpha_2 = \alpha_1 + \varepsilon, \alpha_3 = \alpha_1 - \varepsilon$, and taking the limit for $\varepsilon \to 0$, the first term of the right-hand side of (3.5.4) diverges, and we may assume $\eta_i(t) = 0$. If $C \neq 1$, the left-hand side diverges and the right-hand side remains bounded when the $\alpha_i$ approach $a$. So we may assume $C = 1$, i.e. the left-hand side is zero. In this case $\eta_3(t)$ and $\eta_4(t)$ are nonzero, because the images of the variations at 0 and $a$ are different. Thus the assertion is reduced to

\[(3.5.5) \quad \int_{\beta_2} s^{-1} F_{\eta_2}(t) dt \neq 0 \quad \text{if } \eta_2(t) \neq 0.\]

Here $\eta_2(t)$ is a linear combination of of the cycles $\xi_i(t)$ ($i = 1, 2$) on $Y_i$ whose images on the $x$-plane are paths between $c, \alpha, \bar{\alpha}$ respectively, where $c, \alpha, \bar{\alpha}$ are the roots of $x^2(x + 1) + t = 0$. Since $F_{\xi_i}(t)$ is a nonzero real number for $t > 0$, it is sufficient to show that $\text{Im} F_{\xi_2}(t) \neq 0$ for $t > 0$ (restricting to the case $\alpha_i < 0$). But this is verified easily.

Remark. Let $D$ be an open disk, and $U = D \setminus \{P_1, \ldots, P_n\}$. Take $P_0 \in U$, and choose a path $\gamma_i$ between $P_0$ and $P_i$ such that $\gamma_i \cap \gamma_j = \{P_0\}$. Let $f : X \to U$ be a proper smooth morphism of complex manifolds. Put $Y = f^{-1}(P_0)$. Let $\text{Var}_i$ denote the variation $T_i - \text{id}$ around $P_i$ which acts on $H_j(Y, \mathbb{Q})$ using $\gamma_i$. Then we have an isomorphism

$$\text{Coker}(H_{j+1}(Y) \to H_{j+1}(X)) = \text{Ker}(\sum_i \text{Var}_i : \oplus_i H_j(Y) \to H_j(Y)),$$

such that an element $\{\eta_i\}$ in the right-hand side corresponds to $\sum_i \Gamma_i - \Gamma_0$ in the left-hand side, where $\Gamma_i$ is a $(j + 1)$-chain on $X$ such that $\partial \Gamma_i = \text{Var}_i(\eta_i)$ and $f(\Gamma_i)$ is contained in the union of $\gamma_i$ and a sufficiently small circle around $P_i$, and $\Gamma_0$ is a $(j + 1)$-chain on $Y$ such that $\partial \Gamma_0 = \sum_i \text{Var}_i(\eta_i)$. Here $\Gamma_i$ is given by using a $C^\infty$ trivialization of the restriction of $f$ over a path in $U$.

References

[1] L. Barbieri-Viale and V. Srinivas, A reformulation of Bloch’s conjecture, C.R. Acad. Sci. Paris 321 (1995), 211–214.
[2] A. Beilinson, Higher regulators and values of $L$-functions, J. Soviet Math. 30 (1985), 2036–2070.
[3] ————, Notes on absolute Hodge cohomology, Contemporary Math. 55 (1986) 35–68.
[4] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Astérisque, vol. 100, Soc. Math. France, Paris, 1982.
[5] S. Bloch, Lectures on algebraic cycles, Duke University Mathematical series 4, Durham, 1980.
[6] ————, Algebraic cycles and higher $K$-theory, Advances in Math., 61 (1986), 267–304.
[7] ————, Algebraic cycles and the Beilinson conjectures, Contemporary Math. 58 (1) (1986), 65–79.
[8] ————, The moving lemma for higher Chow groups, J. Alg. Geom. 3 (1994), 537–568.
[9] S. Bloch, A. Kas and D. Lieberman, Zero cycles on surfaces with $p_g = 0$, Compos. Math. 33 (1976), 135–145.
[10] S. Bloch and A. Ogus, Gersten’s conjecture and the homology of schemes, Ann. Sci. Ecole Norm. Sup. 7 (1974), 181–201.
[11] S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles, Amer. J. Math. 105 (1983), 1235–1253.
[12] J. Carlson, Extensions of mixed Hodge structures, in Journées de Géométrie Algébrique d’Angers 1979, Sijthoff-Noordhoff Alphen a/d Rijn, 1980, pp. 107–128.
[13] E. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, 1955.
[14] A. Collino, Griffiths’ infinitesimal invariant and higher $K$-theory on hyperelliptic Jacobians, J. Alg. Geom. 6 (1997), 393–415.
[15] A. Collino and N. Fakhruddin, Indecomposable higher Chow cycles on Jacobians, preprint.
[16] P. del Angel and S. Müller-Stach, The transcendental part of the regulator map for $K_1$ on a mirror family of $K_3$ surfaces, preprint.
[17] P. Deligne, Théorie de Hodge I, Actes Congrés Intern. Math., 1970, vol. 1, 425–430; II, Publ. Math. IHES, 40 (1971), 5–57; III ibid., 44 (1974), 5–77.
[18] C. Deninger and A. Scholl, The Beilinson conjectures, in Proceedings Cambridge Math. Soc. (eds. Coats and Taylor) 153 (1992), 173–209.
[19] A. Dimca and M. Saito, Algebraic Gauss-Manin systems and Brieskorn modules, Am. J. Math. 123 (2001), 163–184.
[20] F. El Zein and S. Zucker, Extendability of normal functions associated to algebraic cycles, in Topics in transcendental algebraic geometry, Ann. Math. Stud., 106, Princeton Univ. Press, Princeton, N.J., 1984, pp. 269–288.
[21] H. Esnault and M. Levine, Surjectivity of cycle maps, Astérisque 218 (1993), 203–226.
[22] H. Esnault and E. Viehweg, Deligne-Beilinson cohomology, in Beilinson’s conjectures on Special Values of $L$-functions, Academic Press, Boston, 1988, pp. 43–92.
[23] H. Federer, Geometric Measure Theory, Springer, New York, 1969.
[24] H. Gillet, Deligne homology and Abel-Jacobi maps, Bull. Amer. Math. Soc. 10 (1984), 285–288.
[25] B.B. Gordon and J.D. Lewis, Indecomposable higher Chow cycles, in Proceedings of the NATO Advanced Study Institute on the arithmetic geometry of algebraic cycles (B.B. Gordon et al. eds.), Kluwer Academic, Dordrecht, 2000, pp. 193–224.
[26] U. Jannsen, Deligne homology, Hodge-$D$-conjecture, and motives, in Beilinson’s conjectures on Special Values of $L$-functions, Academic Press, Boston, 1988, pp. 305–372.
[27] , Mixed motives and algebraic $K$-theory, Lect. Notes in Math., vol. 1400, Springer, Berlin, 1990.
[28] , Letter from Jannsen to Gross on higher Abel-Jacobi maps, in Proceedings of the NATO Advanced Study Institute on The arithmetic geometry of algebraic cycles (B.B. Gordon et al. eds.), Kluwer Academic, Dordrecht, 2000, pp. 261–275.
[29] J.R. King, Log complexes of currents and functorial properties of the Abel-Jacobi map, Duke Math. J. 50 (1983), 1–53.
[30] J. Kollár, Higher direct images of dualizing sheaves, I, II, Ann. of Math. 123 (1986), 11–42; 124 (1986), 171–202.
[31] M. Levine, Localization on singular varieties, Inv. Math. 91 (1988), 423–464.
[32]———, Bloch's higher Chow groups revisited, Astérisque 226 (1994), 235–320.
[33] S. Müller-Stach, Constructing indecomposable motivic cohomology classes on algebraic surfaces, J. Alg. Geom. 6 (1997), 513–543.
[34] D. Mumford, Rational equivalence of 0-cycles on surfaces, J. Math. Kyoto Univ. 9 (1969), 195–204.
[35] C. Pedrini, Bloch's conjecture and the $K$-theory of projective surfaces, in The arithmetic and geometry of algebraic cycles, CRM Proc. Lecture Notes, 24, Amer. Math. Soc., Providence, 2000, pp. 195–213.
[36] A. Roitman, Rational equivalence of zero cycles, Math. USSR Sbornik 18 (1972), 571–588.
[37]———, The torsion in the group of zero cycles modulo rational equivalence, Ann. Math. 111 (1980), 553–569.
[38] A. Rosenschon, Indecomposable Elements of $K_1$, K-theory 16 (1999), 185–199.
[39] M. Saito, Mixed Hodge Modules, Publ. RIMS, Kyoto Univ., 26 (1990), 221–333.
[40]———, On the injectivity of cycle maps, Pub. RIMS, Kyoto Univ., 28 (1992), 99–127.
[41]———, On the formalism of mixed sheaves, RIMS-preprint 784, Aug. 1991.
[42]———, Hodge conjecture and mixed motives, I, Proc. Symp. Pure Math. 53 (1991), 283–303; II, in Lect. Notes in Math., vol. 1479, Springer, Berlin, 1991, pp. 196–215.
[43]———, Mixed Hodge complex on algebraic varieties, Math. Ann. 316 (2000), 283–331.
[44] C. Voisin, Remarks on zero-cycles of self-products of varieties, in Moduli of Vector Bundles, Lect. Notes in Pure and Applied Mathematics, vol. 179, M. Dekker, New York, 1996, pp. 265–285.

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