Invariant and Group Theoretical Integrations over the $U(n)$ Group

S. Aubert and C. S. Lam
Department of Physics, McGill University
3600 University St., Montreal, QC, Canada H3A 2T8
Emails: samuel.aubert@elf.mcgill.ca, lam@physics.mcgill.ca

Abstract
In a previous article, an ‘invariant method’ to calculate monomial integrals over the $U(n)$ group was introduced. In this paper, we study the more traditional group-theoretical method, and compare its strengths and weaknesses with those of the invariant method. As a result, we are able to introduce a ‘hybrid method’ which combines the respective strengths of the other two methods. There are many examples in the paper illustrating how each of these methods works.

1 Introduction
This article deals with the calculation of integrals of the form
\[
\int (dU) \ U^*_{i_1 j_1} \cdots U^*_{i_p j_p} \ U_{k_1 l_1} \cdots U_{k_q l_q}
\]
over the $U(n)$ group, where $(dU)$ is the invariant Haar measure normalized to \( \int (dU) = 1 \), and \( U_{ij} \) is a $U(n)$ matrix element, with \( U^*_{ij} \) being its complex conjugate.

These integrals and their generating functions are useful in many areas of physics, including two-dimensional quantum gravity \[1\], QCD, matrix models, and statistical and condensed-matter problems of various sorts \[2\]. They are also needed in the parton saturation problem at small Feynman-x \[3\].

The integral depends on the indices \( I = \{i_1 \cdots i_p\}, J = \{j_1 \cdots j_p\}, K = \{k_1 \cdots k_q\}, \) and \( L = \{l_1 \cdots l_q\} \), so it will be denoted as \( \langle IJ|KL \rangle \):
\[
\langle IJ|KL \rangle = \int (dU) \ U^*_{IJ} U_{KL},
\]
where \( U^*_{IJ} = \prod_{a=1}^{p} U^*_{i_a j_a} \) and similarly for \( U_{KL} \). Since the matrix elements commute, \( U^*_{IJ} = U^*_{I^P J^P} \) where \( I^P = \{i_{P(1)} \cdots i_{P(p)}\} \) is obtained from \( I \) by a permutation \( P \in S_p \) of its \( p \) indices. Hence
\[
\langle IJ|KL \rangle = \langle I^P J^P|K^T L^T \rangle
\]
for any $P \in S_p$ and $T \in S_q$.

The integral is nonzero only when $p = q$, a number which will be referred to as the degree of the integral. Without loss of generality, it turns out that we may assume $K = I$ and $L$ to be a permutation of $J$, namely, $L = JQ$ for some $Q \in S_p$. The value of the integral depends on what the index sets $I, J$, and what the element $Q$ are, so even for a given $p$, there are many distinct cases. The best way to distinguish them is to represent each integral by a diagram in a way to be explained in the next section.

Integral (1) has been computed using a graphical technique [4]. It can also be obtained using the Itzykson-Zuber formula [5] as a generating function, or directly from group theory [6] using the Frobenius formula [7]. We shall refer to this last method as the group-theoretical method, or GTM for short. In the GTM, a general formula is available to compute (1). It involves a triple sum over an expression containing characters of the symmetric group $S_p$, as well as the dimensions of irreducible representations of $S_p$ and of the unitary group $U(n)$. One of the sums is taken over all the relevant irreducible representations, and the others are taken over the symmetry groups of the index sets $I$ and $J$. These sums could be long and tedious for a large $p$, and for most symmetry groups.

A different way to calculate (1) was introduced in a recent paper [8]. This method relies only on the unitary nature of the matrix elements, and the invariance of the Haar measure. In particular, no knowledge of group theory is necessary. The invariance of the Haar measure, as well as the off-diagonal unitarity relation, are used to derive relations between integrals of the same degree. The diagonal unitarity relation connects integrals of degree $p$ with ones of degree $p - 1$. Through a chain of these relations, the desired integral is finally related to the basic integral of degree 0, which is $\int dU = 1$. The desired integral is then solved from this chain of relations. We have called this method the invariant method, or IM for short.

The purpose of this paper is to compare the pros and cons of the GTM with the IM. In order to do so we must first study and understand better the nature of the GTM. Armed with this comparison, we will be able to design a new hybrid method which combines the strengths of these two other methods.

The IM is reviewed in Sec. 2. It is used to derive a new ‘double-fan’ relation needed in the example for the hybrid method.

The GTM is reviewed and studied in Sec. 3. It can be used to derive simple relations, but the relations derived in this way are nowhere as powerful as those derived with the IM. The group-theoretical formula can be used to calculate any integral, but generally that is tedious and has to be done integral by integral. However, for integrals whose two symmetry groups are disjoint, or one is contained in the other, systematics emerge to make the calculation simpler. Several of these ‘orderly’ integrals are studied in Sec. 3.

A comparison of the strengths and weaknesses of the two methods is to be found in Sec. 4. Armed with an understanding of their relative merits, we design a ‘hybrid method’ in Sec. 5 to take advantage of their respective strengths. This method is illustrated by the calculation of a class of ‘double-fan’ integrals. There
are also four Appendices showing details of various calculations.

2 The Invariant Method (IM)

2.1 A Brief Review

The invariant method presented in a previous paper [8] can be used to calculate the integral (1). The method exploits the unitarity of the $U(n)$ group elements,

$$\sum_{j=1}^{n} U_{ij}^* U_{lj} = \sum_{j=1}^{n} U_{ji}^* U_{jl} = \delta_{il},$$

and the invariance of the Haar measure, in the form

$$\int (dU) f(U, U^*) = \int (dU) f(U^*, U) = \int (dU) f(U^T, U^{*T})$$

$$= \int (dU) f(VU, V^*U^*) = \int (dU) f(UV, U^*V^*),$$

for any function $f$ and any $V \in U(n)$.

The unitarity relation (4) relates integrals of the same degree if $i \neq l$, and it relates integrals of degree $p$ to integrals of degree $p - 1$ if $i = l$. Other relations between integrals of the same degree can be obtained from (5), by suitable choices of $V$. Here are some of them discussed in the previous paper [8].

1. Using $V_{ij} = e^{i\phi} \delta_{ij}$, it follows that $p$ must be equal to $q$ in order to avoid the vanishing of integral (2). The number $p$ will be called the degree of the integral.

2. Using $V_{ij} = e^{i\phi} \delta_{ij}$, it follows that (2) is nonzero only when $K = I_M$ and $L = J_R$ for some $M, R \in S_p$. Using (3), and denoting $RM^{-1}$ by $Q$, only integrals of the type

$$\langle IJ | IJ_Q \rangle = \int (dU) U_{IJ}^* U_{IJ_Q}$$

are nonzero, so from now on we need to consider only integrals of this type.

An integral with $J_Q = J$ will be called a direct integral. For such integrals we may always choose $Q = e$, the identity permutation. Otherwise, the integral is an exchange integral.

Integrals are represented diagrammatically as follows. Each distinct value in the index set $I$ is represented by a dot on the left (L-dot), and each distinct value of the index set $J$ or $J_Q$ is represented by a dot on the right (R-dot). The factor $U_{ij}$ is shown as a thin solid line between the L-dot $i$ and the R-dot $j$, and the factor $U_{ij}$ is shown as a dotted line between
Figure 1: Examples of $U(n)$ integral diagrams. (a) The unique exchange integral for $p = 2$; (b) a $Z$-integral with arbitrary multiplicities $m_1, m_2, \text{ and } m_3$.

these two dots. The factor $U^{*}_{ij}U_{ij}$ is represented by a thick line, or more generally, the factor $U^{*m}_{ij}U^{n}_{ij}$ is represented by a thick line with a pair of numbers $(m,n)$ written beside it. If $m = n$, then only a single number $m$ is written. The numbers $(m,n)$ or $m$ will be known as the multiplicities of the line. See Fig. 1 for an illustration.

3. With $V$ chosen to be a permutation matrix of $n$ objects, it follows that

$$\langle IJ | KL \rangle = \langle I'J | K'L \rangle = \langle IJ' | KL' \rangle,$$

where $I'$ is obtained from $I$ by a reassignment of the values of its indices, e.g. if $I = (334)$, then $I'$ may be $(558)$. $K'$ is obtained from $K$ by the same reassignment, and similarly for $J'$ and $L'$. As a result, there is no need to know the values of the indices of the L-dots, nor the R-dots. This is why the dots in Fig. 1 are not labelled.

4. As a consequence of the first two equalities in (5), an integral remains the same under the interchange of the solid lines with the dotted lines, or the L-dots with the R-dots.

5. Using $V = R(ab)$, the rotation matrix in the $(a,b)$ plane, a `spin-off relation’ is obtained. Consider a R-dot imbedded in an arbitrary integral $M_0 = \langle IJ | IJ_0 \rangle$, with $d$ pairs of solid-dotted lines attached to the dot. Now spin off $e$ pairs of these lines to create a new R-dot and a new integral. There are many ways to choose the $e$ pair of lines, each possibly corresponds to a different integral. Let $M_e$ be the sum of all these integrals. Then the quantities $M_0$ and $M_e$ are related by the spin-off relation

$$M_e = M_0 \binom{d}{e},$$

where $\binom{d}{e}$ is the binomial coefficient.

The relation is local in that it is independent of the the structure of the rest of the diagram. The same relation can also be used to spin off a L-dot.
Figure 2: The fan diagrams shown here can be a part of a larger diagram. In that case, there may be many more dots and many more lines in the complete diagram, provided none of the additional lines land on the R-dots shown. (a) A closed fan; (b) a partially opened fan. If all \( m_i = 1 \), then it is said to be a fully opened fan, or just an opened fan.

In what follows, we summarize two general results obtained in the previous paper \cite{8} using the IM.

2.1.1 The fan relation

The fan relation

\[
\int (dU) A |U_{ac_1}|^{2m_1} \cdots |U_{ac_t}|^{2m_t} = \frac{\left(\prod_{j=1}^t m_j!\right)}{\left(\sum_{j=1}^t m_j\right)!} \int (dU) A |U_{ac}|^{2m} \quad (8)
\]

relates integrals of the same \( m = \sum_{j=1}^t m_j \), where \( A \) is an arbitrary product of matrix elements of \( U \) and \( U^\ast \) whose column indices are different from \( c_1, c_2, \ldots, c_t \). The column index \( c \) on the right could be taken to be one of the \( c_i \)'s.

Diagrammatically, the integral on the right of (8) is shown in Fig. 2(a), and the integral on the left is shown in Fig. 2(b). The additional lines and dots corresponding to the factor \( A \) are not shown, because they do not affect the spin-off relation. We shall refer to Fig. 2(a) as a closed fan, and Fig. 2(b) as a partially opened fan. If every \( m_i = 1 \), then it will be said to be a fully opened fan, or simply an opened fan.

In particular, a closed fan integral is \( m! \) times an opened fan integral. In fact, this relation between the two types of fans immediately gives rise to the relation (8) between a closed fan and a partially opened fan. To see it, note that each branch of a partially opened fan is itself a closed fan. By opening up all of them, we get the fully opened fan integral, multiplied by a multiplicity factor \( \prod_j m_j! \) from all the branches. Thus a closed fan is \( m! / \prod_j m_j! \) times a partially opened fan, as given by (8).
2.1.2 The $Z$-integral and the fan integral

In [8], we also obtained a general formula for integrals of the type shown in Fig. 1(b), for arbitrary nonnegative integers $m_1$, $m_2$, and $m_3$. We call that the ‘$Z$-formula’ because of the shape of the graph. It is

$$Z(m_1, m_2, m_3) \equiv \int (dU) |U_{ij}|^{2m_1} |U_{il}|^{2m_2} |U_{kl}|^{2m_3}$$

$$= \frac{m_1! m_2! m_3! (n-2)!(n-1)! (n+m_1+m_3-2)!}{(n+m_1-2)!(n+m_3-2)!(n+m_1+m_2+m_3-1)!}.$$  \hfill (9)

In the special case where $m_2 = m_3 = 0$, this becomes

$$F(m) \equiv Z(m,0,0) = \frac{m!(n-1)!}{(n+m-1)!}.$$ \hfill (10)

which is the integral for Fig. 2(a) when there are no additional dots or lines around.

2.2 Double-Fan Relation

The fan relation can be generalized to a double-fan relation, connecting the closed ‘double-fan’ diagram of Fig. 3(a) with a (fully) opened double-fan diagram such as Fig. 3(b). As in Fig. 2 there may be additional dots and lines in the integral, but none of them may end up on the R-dots shown. From that relation, we can also deduce relations between a closed double-fan and a partially opened double-fan, as done in the single-fan case.

The double-fan relation is considerably more complicated than the single-fan relation, because there are many more double-fan graphs. Each R-dot of a (fully) opened (double-fan) graph such as Fig. 3(b) falls into one of four basic patterns: [A]a, [A]b, [B]a, and [B]b, shown in Fig. 4. If the solid and dotted lines end up on the same L-dot, the pattern is a [B]; otherwise it is an [A]. The subscripts $a$ and $b$ tell us which L-dot the solid line emerges from.
Figure 4: The four basic patterns for the R-dots of a fully opened double-fan graph.

Suppose there are \( \alpha_i \) number of \([A_i]\) and \( \beta_i \) number of \([B_i]\) patterns in a (fully) opened (double-fan) graph. Then there are \( m_a \) solid and \( n_a \) dotted lines emerging from the L-dot \( a \), and \( m_b \) solid and \( n_b \) dotted lines emerging from the L-dot \( b \), where

\[
m_a = \alpha_a + \beta_a, \quad n_a = \alpha_b + \beta_a, \quad m_b = \alpha_b + \beta_b, \quad n_b = \alpha_a + \beta_b.
\] (11)

The total number of R-dots in the opened graph is \( N = m_a + m_b = n_a + n_b \).

When these \( N \) R-dots are merged together, we get the closed (double-fan) graph depicted in Fig. 3(a), which will be denoted by \([([m_a n_a]([m_b n_b]))\]). If \( N = 1 \), this is just one of the four basic patterns discussed before. If \( N > 1 \), we will call it a compound pattern.

From (11), we see that if we replace \( \alpha_i \) and \( \beta_i \) by \( \alpha'_i = \alpha_i + \xi \) and \( \beta'_i = \beta_i - \xi \), with any integral \( \xi \) which keeps \( \alpha'_i \) and \( \beta'_i \) nonnegative, then we get the same closed graph by collapsing this new opened graph. Conversely, it will be shown in Appendix A that the closed graph \([([m_a n_a]([m_b n_b]))\]) can be spun off into a sum of several opened graphs, one for each \((\alpha'_i, \beta'_i)\). The double-fan relation expressing that quantitatively is

\[
([m_a n_a]([m_b n_b])) = \sum v(\alpha'_a \alpha'_b \beta'_a \beta'_b) \ [A_a]^{\alpha'_a} [B_a]^{\beta'_a} [A_b]^{\alpha'_b} [B_b]^{\beta'_b}
\]
\[
= \sum [\alpha'_a A_a + \alpha'_b A_b + \beta'_a B_a + \beta'_b B_b],
\] (12)

where

\[
v(\alpha'_a \alpha'_b \beta'_a \beta'_b) = \frac{m_a! n_a! m_b! n_b!}{\alpha'_a! \alpha'_b! \beta'_a! \beta'_b!}.
\] (13)

and the sum is over all solutions \((\alpha'_a, \alpha'_b, \beta'_a, \beta'_b)\) of (11).

The double fan becomes a single fan if the L-dot \( b \) is not connected, namely, if \( m_b = n_b = 0 \) and \( m_a = n_a \equiv m \). In that case (12) becomes

\[
[m B_a] = m! [B_a]^m,
\]
which is just (8) (when all \( m_i = 1 \)) in another notation.
3 Group-Theoretical Method (GTM)

3.1 A Brief Review

Using group theory, the integral (6) can be turned into a multiple sum. In the notation used in Appendix A of [8], the formula is

\[
\langle IJ | IJ_Q \rangle = \sum_{R \in G_I} \sum_{S \in G_J} \sum_f \frac{d_f^2}{(pl)^2 d_f} \chi_f(SQR),
\]

where \( p \) is the degree of the integral. The symbols \( G_I \) and \( G_J \) represent the symmetry groups of the row and column index sets. More precisely,

\[
G_X = \{ P \in S_p | P(X) = X \},
\]

with \( X \) being either \( I \) or \( J \). The irreducible representations of the symmetric and unitary groups are both labelled by a sequence \( f = (f_1, f_2, \ldots, f_p) \), with \( f_1 \geq f_2 \geq \cdots \geq f_p \geq 0 \). \( \chi_f(P) \) is the character of \( P \in S_p \) in the irreducible representation with signature \( f \). The dimension of the irreducible representation \( f \) is given by \( d_f = \chi_f(e) \) for \( S_p \), and by \( \tilde{d}_f \) for \( U(n) \).

A formula for \( \tilde{d}_f \) is:

\[
\tilde{d}(f_1, \ldots, f_n) = \frac{D(f_1 + (n - 1), f_2 + (n - 2), \ldots, f_n)}{D(n - 1, n - 2, \ldots, 0)},
\]

where \( D(x_1, \ldots, x_n) \) is the Vandermonde determinant given by \( \prod_{i < k} (x_i - x_k) \).

Since \( \chi_f(g) \) depends only on the class \( c_g \) that \( g \) belongs to, we may write it as \( \chi_f(c_g) \). With this notation, eq. (14) can be re-written as

\[
\langle IJ | IJ_Q \rangle = \sum_c N[c] \xi[c],
\]

where the sum is taken over all classes \( c \) of \( S_p \),

\[
N[c] = \sum_{R \in G_I} \sum_{S \in G_J} \delta(SQR \in c)
\]

is the number of elements of the type \( SQR \) in the class \( c \), and

\[
\xi[c] = \sum_f \frac{d_f^2}{(pl)^2 d_f} \chi_f(c).
\]

It is straightforward but generally very tedious to compute \( N[c] \), because we need to calculate the product \( QTR \) for every \( T \in G_{I_Q} \), every \( R \in G_I \), and

\[
N[c] = \sum_{R \in G_I} \sum_{T \in G_{I_Q}} \delta(QTR \in c).
\]
determine what class $c$ the product belongs to. Then we have to count up all the products that are in a given class $c$ to get $N[c]$. However, the task becomes considerably more manageable if either $G_I$ and $G_J$ are disjoint, or if one is contained in the other. We shall refer to integrals with those properties as orderly. Further simplification occurs for direct integrals, because in that case $Q$ can always be chosen to be the identity $e$, so the triple product is reduced to a double product $TR$.

The calculation of $\xi[c]$ in (18) is simpler than the calculation of $N[c]$, but still we know of no closed form of it valid for every class $c$ and every symmetric group $S_p$. The best we can do is to compute them case by case. Results are given in Sec. 3.3. Each $\xi[c]$ is actually an orderly integral with $G_I = G_J = e$, to be referred to as a primitive integral.

Other integrals can be computed in terms of the primitive integrals, if $N[c]$ is known. We shall discuss two orderly integrals for which $N[c]$ can easily be obtained. In Sec. 3.4, we discuss the stack integrals, which are direct integrals with $G_I = G_J$. In Sec. 3.5, we discuss the fully opened double-fan integrals of the type $[A_a]^{\alpha}[A_b]^{\alpha}$.

Relations between orderly integrals may be obtained without knowing the explicit values of $\xi[c]$, if their $N[c]$'s are related in a simple way. This is the case for the single-fan relation, and the double-fan relation with $n_a = m_b = 0$ and $m_a = n_b = m$. They will be discussed in Sec. 3.2. However, general double-fan integrals are not orderly, so we cannot obtain the general double-fan relation by the group-theoretical method, at least not in the present way. We will also show that the closed (single-)fan integral can also be computed without explicitly knowing what $\xi[c]$ are. This is one of the very few cases where integrals can be obtained group-theoretically without explicitly knowing $\xi[c]$.

That leaves the non-orderly integrals, for which each term of the summand in (19) has to be calculated separately to get $N[c]$. The first non-orderly integral occurs in degree $p = 3$. In Sec. 3.6, we shall show how to calculate some of the $p = 3$ and $p = 4$ non-orderly integrals.

### 3.2 Single-Fan and Simple Double-Fan Relations

The single-fan integrals are orderly. The index sets for the closed fan Fig. 2(a) are

$$
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & \cdots & m & \cdots \\
a & \cdots & a & \cdots \\
c & \cdots & c \\
c & \cdots & c
\end{pmatrix}.
$$

(20)

The first row gives the index labels, and the next three rows give the values of the indices in the sets $I, J,$ and $J_Q$ respectively. Different letters are understood to correspond to different values. Additional dots and lines may be present in the graph, as long as none of the lines end up in the R-dots shown. These additional lines and dots are not drawn because they do not affect the fan relation in any way. Similarly, they are not shown in the index sets in (20) other than the

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ellipses in the first two rows, which remind us that there may be more lines connected to the L-dots. Such ellipses are absent in the last two rows because no additional lines are allowed to be connected to the R-dots shown.

Similarly, the index sets for the fully opened fan, Fig. 2(b) with all \( m_i = 1 \), are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & \cdots & m & \cdots \\
a & \cdots & a & \cdots \\
c_1 & \cdots & c_m \\
c_1 & \cdots & c_m
\end{pmatrix}.
\]

Using \( S_m \) to denote the symmetric group for the permutation of the first \( m \) labels, the symmetric groups for Fig. 2(a) can be read off from (21) to be \( G_I \supset S_m \) and \( G_J = G_{JQ} = S_m \). Similarly, the symmetric groups for Fig. 2(b) can be read off from (21) to be \( G_I \supset S_m \) and \( G_J = G_{JQ} = e \). We may choose \( Q \) and \( N \) in both cases. Then for Fig. 2(a), \( T G_I = G_I \) for every \( T \in G_{JQ} \), hence \( N[c] = m! \sum_{R \in G_I} \delta(R \in c) \). But the last sum is simply the \( N[c] \) for Fig. 2(b) and Fig. 2(a). Hence it follows from (10) that the fan relation (with all \( m_i = 1 \)) is true.

Next, let us derive the double-fan relation (12) and (13) for the case \( n_a = m_b = 0 \) and \( m_a = n_b \). The solution of (11) is now unique. It gives \( \alpha_a = m_a = n_b \equiv m \), and \( \alpha_b = \beta_a = \beta_b = 0 \). The double-fan relation (12) then becomes

\[
[m A_a] = m! [A_a]^m.
\]

The closed double-fan is shown in Fig. 3(a). Its index sets are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & \cdots & m & \cdots & n & \cdots & n + m & \cdots \\
a & \cdots & a & \cdots & b & \cdots & b & \cdots \\
c & \cdots & c & \cdots & c & \cdots & c & \cdots
\end{pmatrix}.
\]

The opened double-fan is shown in Fig. 3(b). Its index sets are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & \cdots & m & \cdots & n & \cdots & n + m & \cdots \\
a & \cdots & a & \cdots & b & \cdots & b & \cdots \\
c_1 & \cdots & c_m & \cdots & c_m & \cdots & c_m & \cdots
\end{pmatrix}.
\]

We may choose \( Q = (1, n)(2, n + 1) \cdots (m, n + m) \) in both cases. For (23), \( G_I \supset S_m \otimes S'_m \), \( G_I = S_m \), and \( G_{JQ} = S'_m \), where \( S_m \) is the permutation group of the first \( m \) labels, and \( S'_m \) is the permutation group for the labels \( n, n + 1, \ldots, n + m \). For (24), \( G_I \supset S_m \otimes S'_m \), but \( G_J = G_{JQ} = e \).

For Fig. 3(a), \( T G_I = G_I \) for every \( T \in G_{JQ} \). Hence \( N[c] = m! \sum_{T \in G_I} \delta(QR \in c) \). But the last sum is simply the \( N[c] \) of Fig. 3(b). In this way (22) is proven by the GTM.

The fan integral (10) can also be obtained from the GTM. It is given by Fig. 2(a) without extra dots and lines, or Fig. 2(b) without the ellipses at the end.
Then $G_I = S_m$, and $N[c] = m! \sum_R \delta(R \in c)$. From (16) and (18), we get

$$\langle IJ | JQ \rangle \equiv F(m) = m! \sum_{\mathcal{G}_I} \sum_f \frac{d_f^2}{(m!)^2} \chi_f(R)$$

$$= \frac{1}{m!} \sum_f \frac{d_f^2}{d_f} R \in S_m \sum \chi_f(R) \chi^*_f(R) = \frac{d_f^2}{d_f}$$

$$= \frac{m!(n-1)!}{(m+n-1)!}.$$  

In getting from (25) to (26), the character $\chi^*_f(R) = 1$ of the totally symmetric representation $(m)$ of the permutation group has been inserted, and the orthogonality relation of the characters has been used. To get to (27), $d_f = \chi_f(e) = 1$ as well as $d_f = (m+n-1)!/(n-1)!m!$ (see (15)) have been used.

The result in (27) agrees with the result (10). It is one of the very few cases where the value of the integrals can be obtained group-theoretically without knowing the values of the individual $\xi[c]$’s.

### 3.3 Primitive Integrals

Integrals in which both symmetry groups $G_I$ and $G_J$ consist only of the identity $e$ will be called primitive. This happens when all the indices $i_a$ in the set $I$ assume distinct values, and all the indices $j_b$ in the set $J$ are also different. The corresponding diagrams have $p$ dots each on both columns, and precisely one solid and one dotted lines connecting to each of the dots. The primitive diagrams for $p \leq 3$ are shown in Fig. 3, and the ones for $p = 4, 5$ are contained in Appendix C.

Since $G_I = G_J = e$, it follows from (17) that $N[c] = \delta(Q \in c)$, where $Q$ can be any element of $S_p$. The primitive integrals (16) are simply $\xi[c]$, one for
Figure 6: Primitive diagrams for (a) \( p = 1 \), (b) \( p = 2 \), and (c) \( p = 3 \). The identity element is everywhere denoted by \( e \).
3.4 Stack Integrals

The stack diagrams (see Fig. 7) are direct integrals made up of disconnected lines of arbitrary multiplicities. As such, \( Q = e \), and \( J \) differs from \( I \) only by relabelling. Using item 3 of Sec. 2.2, we may assume \( J = I \). Hence stack integrals are integrals of the form \( \langle II | II \rangle \).

Let \( p_1, p_2, \ldots, p_t \) be the multiplicities of the disconnected lines in a stack diagram. Then \( G_I = G_J \equiv G = S_{p_1} \otimes S_{p_2} \otimes \cdots \otimes S_{p_t} \), and \( N[c] \) is nonzero only when the class \( c \) is a direct product of the classes \( c_i \) of the groups \( S_{p_i} \). In that case,

\[
N[c] = \prod_{i=1}^{t} p_i! n_i(c_i),
\]

where \( n_i(c_i) \) is the number of elements of \( S_{p_i} \) in the class \( c_i \). In other words,

\[
n_i(c_i) = \frac{p_i!}{\prod_{j=1}^{\alpha_i} j^{\alpha_i} \alpha_i!},
\]

where the class \( c_i \) consists of \( \alpha_j \) cycles of length \( j \). Denoting the stack integral \( \langle II | II \rangle \) by \( \Xi(p_1, p_2, \ldots, p_t) \), we get

\[
\Xi(p_1, p_2, \ldots, p_t) = \sum_{c_1, c_2, \ldots} \left( \prod_{i=1}^{t} p_i! n_i(c_i) \right) \xi(c_1 \otimes c_2 \otimes \cdots \otimes c_t).
\]

All stack diagrams can be obtained by making the assignment \( f_i \rightarrow p_i \) from each representation. In this way, we expect a same number of stack diagrams as of primitive diagrams, or classes. Using the \( \xi \) expressions obtained in the preceding subsection, the stack integrals for \( p \leq 3 \) can be computed to yield the expressions in Table 2.
Table 2: Algebraic expressions for the stack diagrams of $p = 1, 2,$ and $3$.

| $\Xi(p_1, \ldots, p_p)$ | $\Xi(1) = \frac{1}{n}$ | $\Xi(3) = \frac{3!}{n(n+1)(n+2)}$ |
|--------------------------|------------------------|----------------------------------|
| $\Xi(2) = \frac{2}{n(n+1)}$ | $\Xi(2, 1) = \frac{2}{(n-1)n(n+2)}$ |
| $\Xi(1, 1) = \frac{1}{n^2-1}$ | $\Xi(1, 1, 1) = \frac{n^2-2}{n(n-1)(n-2)}$ |

3.5 Special Double-Fan Integrals

The index sets for the fully opened double-fan integrals $[A_a]^\alpha[A_b]^\alpha$ (Fig. 3(b) with $N = 2\alpha$) are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} = \begin{pmatrix}
1 & \cdots & a & a+1 & \cdots & 2\alpha \\
b & \cdots & b & a & \cdots & a \\
c_1 & \cdots & c_\alpha & c_{\alpha+1} & \cdots & c_{2\alpha} \\
c_{\alpha+1} & \cdots & c_{2\alpha} & c_1 & \cdots & c_\alpha
\end{pmatrix}.
\tag{31}
\]

Hence both $G_I$ and $G_{J_Q}$ consist only of the identity $e$. As for $G_I$, it is given by $S_\alpha \otimes S_\alpha$, where the permutation groups $S_\alpha$ act respectively on the $b$ and $a$ indices in $I$. The element $Q$ maps $J_Q$ to $J$, i.e. $Q = (1, \alpha+1)(2, \alpha+2)\cdots(\alpha, 2\alpha)$.

The fully opened integral can be computed using (16), with $N[c]$ given by (17) or (19). Thus,

\[
N[c] = \sum_{R \in G_I} \sum_{T \in G_{J_Q}} \delta(QTR \in c)
= \sum_{R \in G_I} \delta(QR \in c) = \sum_{Q'} \delta(Q' \in c),
\tag{32}
\]

where the last sum is over every permutation $Q'$ that sends all $b$ indices in $J$ to the positions labelled from $\alpha + 1$ to $2\alpha$, and similarly all $a$ indices to the positions labelled from $1$ to $\alpha$. As a consequence, the allowed cycles of $Q'$ must be of even length, and they can be specified by a sequence of nonnegative integers $(k) \equiv (k_1k_2\cdots k_\alpha)$, $k_i$ being the number of cycles of length $2i$. The number of $Q'$ with the class structure $(k)$ that is related to $c$ is given by

\[
N[c] = \frac{(\alpha!)^2}{\prod_{i=1}^n i^{k_i} \cdot k_i!}.
\tag{33}
\]

In order to see how this is arrived at, consider an example where $k_1 = 2$, $k_2 = 2$, and all other $k_i$ values are zero. Then $Q'$ is of the form $(ba)(ba)(baba)$, where the $b$ and $a$ letters should take the distinct index labels in $(1, \ldots, \alpha)$ and $(\alpha+1, \ldots, 2\alpha)$ respectively. Another $Q'$ with the same cycle structure can thus
\[ a_1 - 1_n \left( n_2 - 1 \right) \]
\[ 2_2 \left( n_2 - 1 \right) \]
\[ n_2 \left( n + 2 \right) \left( n + 3 \right) \]
\[ 3 - 6 \left( n - 1 \right) n_2 \left( n + 1 \right) \]
\[ 2 \left( n + 2 \right) \left( n + 3 \right) \left( n + 4 \right) \left( n + 5 \right) \]

Table 3: Values of the monomial integrals \([A_a]_a [A_b]_a\) for \(\alpha = 1, 2, 3\).

\[
\begin{array}{|c|c|c|}
\hline
\alpha & [A_a]_a [A_b]_a \\
\hline
1 & \frac{-1}{n (n^2 - 1)} \\
2 & \frac{2}{(n^2 - 1)n_2 (n + 2)(n + 3)} \\
3 & \frac{-6}{(n - 1)n_2 (n + 1)n_2 (n + 3)(n + 4)(n + 5)} \\
\hline
\end{array}
\]

Figure 8: Non-orderly integrals of \(p = 3\).

be obtained by permuting individually all the \(a\) and \(b\) labels. This accounts for the numerator in (33). However, such permutations do not necessarily give distinct \(Q'\) elements. The cyclic nature of a cycle tells us that each cycle of length \(2i\) will appear \(i\) times; this accounts for the \(i^k_i\) factor in the denominator. Moreover, no new \(Q'\) is obtained if we permute cycles of the same length; that accounts for the other factor \(k_i!\) in the denominator.

We may now return to (10) to calculate the integral \([A_a]_a [A_b]_a\) in terms of the primitive integrals \(\xi[c]\). The result for the first few \(\alpha\) values are listed in Table 3.

### 3.6 Non-Orderly Integrals

All integrals with degree \(p < 3\) are orderly. The non-orderly integrals of \(p = 3\) are shown in Fig. 8 and those related to them by the fan relation (8). The calculation of \(N[c]\) and the integral for each of them is discussed below. The integrals will be labelled by their figure, \(e.g.,\) integral \(I(8)\).

The index sets for Fig. 8 (a) are

\[
\begin{pmatrix}
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 3 \\
b & b & a \\
d & c & c \\
d & c & c
\end{pmatrix}.
\]

They give rise to the symmetry groups \(G_I = \{e, (12)\}\) and \(G_J = G_{J_Q} = \{e, (23)\}\). Moreover, the element \(Q\) can be taken to be the identity element. In order to
obtain the coefficients $N[e]$ of equation (16), we need to compute $QTR$ for all $T \in \mathcal{G}_J$ and $R \in \mathcal{G}_I$. That triple product is $Q \mathcal{G}_J \mathcal{G}_I = \{e, (12), (23), (132)\}$. As a result,

$$I(8a) = Z(1,1,1) = \xi[e] + 2\xi[(12)(3)] + \xi[(123)] = \frac{1}{(n^2-1)(n+2)}. \quad (35)$$

In the same way, the index sets of Fig. 8(b) are

$$\begin{pmatrix} \text{label} \\ I \\ J \\ J_Q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ b & a & a \\ e & d & c \\ d & e & c \end{pmatrix}, \quad (36)$$

and hence the symmetry groups are $\mathcal{G}_J = \{e\}$ and $\mathcal{G}_I = \{e, (23)\}$, and the exchange element is $Q = (12)$. We thus obtain $Q \mathcal{G}_J \mathcal{G}_I = \{(12), (123)\}$, from which

$$I(8b) = \xi[(12)(3)] + \xi[(123)] = \frac{-1}{(n^2-1)n(n+2)}. \quad (37)$$

follows.

Finally, for Fig. 8(c), the index sets are

$$\begin{pmatrix} \text{label} \\ I \\ J \\ J_Q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ b & b & b & a \\ d & c & c & c \\ c & d & c & c \end{pmatrix}, \quad (38)$$

The relevant symmetry groups are $\mathcal{G}_J = \{e, (13)\}$ and $\mathcal{G}_I = \{e, (23)\}$. With $Q = (12)$, the set $Q \mathcal{G}_J \mathcal{G}_I$ is $\{(12), (13), (123), (132)\}$, and formula (16) gives:

$$I(8c) = 2(\xi[(12)(3)] + \xi[(123)]) = \frac{-2}{(n^2-1)n(n+2)}. \quad (39)$$

The calculation of $Q \mathcal{G}_J \mathcal{G}_I$ is not that cumbersome for $p = 3$, but it gets worse pretty quickly as $p$ increases. For example, let us look at some examples of $p = 4$.

Let us first calculate $Z(2,1,1)$ of Fig. 11(b), whose index sets are

$$\begin{pmatrix} \text{label} \\ I \\ J \\ J_Q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ b & b & b & a \\ d & d & c & c \\ d & d & c & c \end{pmatrix}. \quad (40)$$

Then $Q = e$, $\mathcal{G}_J = \{e, (12), (34), (12)(34)\}$ and $\mathcal{G}_I = \{e, (12), (13), (23), (123), (132)\}$. Thus $Q \mathcal{G}_J \mathcal{G}_I$ is $\{(e, (12), (13), (23), (123), (132), (12), (132), (123), (13), (34), (12)(34), (143), (243), (1243), (1432), (12)(34), (34), (1432), (1243), (243), (143))\}$, hence

$$Z(2,1,1) = 2\xi[e] + 8\xi[(12)(3)(4)] + 8\xi[(123)(4)] + 2\xi[(12)(34)] + 4\xi[(1234)] = \frac{2}{(n-1)n(n+2)(n+3)}. \quad (41)$$
Our last example is the $\Sigma$-integral, shown in Fig. 9. Its index sets are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_{Q}
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 \\
b & b & a & a \\
e & d & d & c \\
e & d & d & c
\end{pmatrix}.
\]

Hence $Q = e$, $G_{J_{Q}} = \{e, (23)\}$ and $G_{J} = \{e, (12), (34), (12)(34)\}$. Multiplying such elements accordingly, the set $\{e, (12), (34), (12)(34), (23), (132), (234), (1342)\}$ is obtained for $QG_{J_{Q}}G_{J}$. Hence

\[
\Sigma = \xi[e] + 3\xi[(12)(3)(4)] + 2\xi[(123)(4)] + \xi[(12)(34)] + \xi[(1234)]
= \frac{n + 1}{(n - 1)n^2(n + 2)(n + 3)}.
\]

4 Comparison of the IM and the GTM

We have discussed the computation of $U(n)$ integrals in two ways: the IM in Sec. 2, and the GTM in Sec. 3. Each of these two methods has its own merits, and drawbacks, and in a way they complement each other. The purpose of this section is to compare their relative strong and weak points.

The IM is based solely on the unitarity condition and the invariance of the Haar Measure. The method is simple because there is no need to know group theory. The conditions relate integrals of the same degree, and also integrals of degree $p$ to integrals of degree $p - 1$. Through these relations, specific integrals such as the fan integrals and the $Z$-integrals can be obtained, and general relation such as the single-fan relation and the double-fan relation can be worked out.

The GTM has the advantage of being general, in the sense that all integrals can be computed using the formula or . The price to pay is that we have to know the characters of the irreducible representations of the appropriate symmetric group, and a triple sum has to be carried out, which can prove to be very tedious for integrals of high degrees. Furthermore, unlike the IM, relations between integrals are hard to come by, so one must calculate the integrals one by one. There are however certain class of integrals, the orderly integrals, for which relations can be developed, and the quantity $N[e]$ in can be relatively easily computed. Then we merely have to know the primitive integrals $\xi[e]$ in to get the value of the orderly integral on hand. The stack integrals.
The sum is performed on the index indicated by an arrow. Using the fan relation \( \mathcal{S} \), the unitary sum can be written as: 
\[
\frac{n-3}{4} + 2 = Z(2,0,2) + 2 \Sigma = \frac{1}{2} Z(2,0,1).
\]

Figure 10: Unitarity sum relation involving the \( \Sigma \) diagram (second from the left).

and the special opened double-fan integrals \( \mathcal{S}_\Omega \) are examples of this kind. The primitive integrals \( \xi[c] \) themselves must be calculated using \( \mathcal{I} \).

To summarize, the IM gives a huge number of relations but it is not easy to obtain the value of any specific integral. The GTM allows us to calculate any specific integral, albeit rather tedious at times, but it is difficult to obtain relations between integrals. In the next section, we shall discuss a hybrid method which makes use of the advantages of both methods. We shall use the general GTM formula to calculate a specific set of integrals, and then use the IM relations to obtain all the other integrals.

In the rest of this section, we shall enlarge these general remarks about the IM and GTM, by using specific examples presented in the last two sections as concrete illustrations.

The single-fan relation \( \mathcal{S} \) can be obtained by both the IM and the GTM. However the double-fan relation \( \mathcal{I}_\Omega \) in its general form can be obtained only by the IM, because most of the integrals involved are not orderly, making it hard to derive relations using the GTM. Nevertheless, in special cases involving only orderly integrals, \( \mathcal{I}_\Omega \), the GTM can also be used to derive the relation.

The \( Z \)-formula \( \mathcal{I}_\Omega \) is obtained using the IM, by a series of relations connecting it down to \( \int dU = 1 \). Since the \( Z \)-integrals are not orderly, it is hard to compute them using the GTM except at low degrees. The calculation of those by the GTM is shown in equations \( \mathcal{I}_\Omega \) and \( \mathcal{I}_\Omega \).

However, since the values of the integrals in the IM are obtained only through relations, it may be relatively complicated to calculate just one specific integral. This is where the GTM is superior, because of the general formula \( \mathcal{I}_\Omega \) valid for any one integral. For example, it is easy to obtain the \( \Sigma \) integral \( \mathcal{I}_\Omega \), assuming of course the \( \xi[c] \)'s to be already known. We can also obtain it using the IM, as we shall show below, but that involves a few steps because we must get it from relations. To see how that is done, look at Fig. 10 which is the unitarity relation applied to the dot of the first diagram indicated by an arrow. The first and third diagrams can be related to \( Z(2,0,2) \) by using the fan relation \( \mathcal{S} \), and similarly the diagram on the right can be related to \( Z(2,0,1) \). Using the \( Z \)-formula \( \mathcal{I}_\Omega \), we then obtain

\[
\Sigma = \frac{1}{4} \left[ Z(2,0,1) - \left( \frac{n-3}{2} + 1 \right) Z(2,0,2) \right].
\]
\[
\frac{n + 1}{(n - 1)n^2(n + 2)(n + 3)}.
\]
the same as the result obtained by the GTM.

5 Hybrid Method

Having understood the relative merits of the GTM and the IM, it is possible to combine their strengths into a more efficient hybrid calculational scheme.

The strategy is to start with one or more integrals that can be computed by the GTM with relative ease. Generally speaking, such integrals are ordered. Once they are obtained, the many relations of the IM can be used to calculate other integrals from them.

To illustrate this strategy, we will consider how the hybrid method can be used to calculate all double-fan integrals.

By a double-fan integral, we mean any integral with two L-dots and any number of R-dots. Fig. 3(a) shows a closed (double-fan) integral (with the understanding that there are no extra dots or lines than those shown), and Fig. 3(b) shows a fully opened (double-fan) integral. We may also have partially opened (double-fan) integrals, in which every branch, namely, every R-dot with its connecting lines, can be regarded as a closed integral. See Fig. 11 for an example of a partially opened integral.

As in Sec. 2.2, a fully opened integral is denoted by
\[
[A_a]^{\alpha_a}[B_a]^{\beta_a}
\]
and its corresponding closed integral is denoted by $\alpha_a A_a + \beta_a B_a$. For a partially opened integral, we will denote it as a product of the closed integrals of each branch. See Fig. 11 for examples.

Using (12) and (13), all double-fan integrals can be expressed as sums of fully opened integrals. Integrals of the form $[A_a]^{\alpha_a}[B_b]^{\beta_b}$ are given by Table 3, but we still have to know how to calculate a fully opened integral when $\beta_i \neq 0$. As shown in Appendix D, the IM allows us to relate them to those with $\beta_i = 0$, by using the following formula

\[
[A_a]^{\alpha_a}[B_a]^{\beta_a} = \sum_{e=0}^{\min(\beta_a, \beta_b)} \left\{ (-1)^e e! \left( \begin{array}{c} \beta_a \\ e \end{array} \right) \left( \begin{array}{c} \beta_b \\ e \end{array} \right) (n + 2\alpha - 1 + 2e) \right. \cdot \left( \begin{array}{c} n + 2\alpha - 2 + e \\ e \end{array} \right) \cdot \left( \begin{array}{c} n + 2\alpha - 1 + 2e \\ e \end{array} \right) \cdot \left( \begin{array}{c} n + 2\alpha + \beta_a - 1 + e \\ e \end{array} \right) \cdot \left( \begin{array}{c} n + 2\alpha + \beta_b - 1 + e \\ e \end{array} \right) \right\}.
\]

We close this section by showing how to use (12) and (13) to calculate the integrals in Fig. 11.

5.1 Fig. 11(a)

There are two equivalent forms for this diagram. One is

\[
[A_a + 2A_b][A_a] = (2[A_a][A_a]^2) [A_a] = 2[A_a]^2[A_b]^2
\]

(46)
Figure 11: Partially opened double-fan integrals. (a) There are two equivalent forms for this graph: $[A_a + 2A_b][A_a]$ and $[A_b + B_a + B_b][A_a]$. (b) There are four equivalent forms for this graph: $[A_a + A_b + B_a][A_a + A_b]$, $[2B_a + B_b][A_a + A_b]$, $[A_a + A_b + B_a][B_a + B_b]$, $[2B_a + B_b][B_a + B_b]$.

and the other is

$$[A_b + B_a + B_b][A_a] = 4 ([A_b][B_a][B_b]) [A_a] = 4[A_a][B_a][B_b],$$

where equations (12) and (13) have been used. The integral $I_{11a}$ is obtained by adding up (46) and (47).

Using (15), we can express all fully opened integrals in the form $[A_a]^\alpha[A_b]^\alpha$. Applying to the present case, we get

$$[A_a][A_b][B_a][B_b] = \frac{1}{(n + 2)^2}[A_a][A_b] - \frac{1}{(n + 2)}[A_a]^2[A_b]^2.$$

Using Table 3, we finally obtain

$$I_{11a} = 2[A_a]^2[A_b]^2 + 4[A_a][A_b][B_a][B_b]$$

$$= \frac{2n}{(n + 2)}[A_a]^2[A_b]^2 + \frac{4}{(n + 2)^2}[A_a][A_b]$$

$$= -4 \frac{(n^2 - 1)n(n + 2)(n + 3)}{n(n + 1)(n + 2)(n + 3)}.$$

5.2 Fig. 11(b)

As shown in Fig. 11(b), $I_{11b}$ has four equivalent forms. For one branch, the factors are

$$[A_a + A_b + B_a] = 4 [A_a][A_b][B_a],$$

$$[2B_a + B_b] = 2 [B_a]^2[B_b];$$

and for the other branch, they are

$$[A_a + A_b] = [A_a][A_b],$$

$$[B_a + B_b] = [B_a][B_b].$$
Hence

\[ I(11) = 4[A_a]^2[A_b]^2[B_a] + 6[A_a][A_b][B_a]^2[B_b] + 2[B_a]^3[B_b]^2. \quad (51) \]

We will now express each of the three monomial integrals in (51) in terms of \([A_a]^\alpha[A_b]^\beta\). First, with respect to (45), \([A_a]^2[A_b]^2[B_a]\) is characterized by \(\alpha = 2, \beta_a = 1, \) and \(\beta_b = 0\). The vanishing of \(\beta_b\) causes (45) to consist of the single term:

\[ [A_a]^2[A_b]^2[B_a] = \frac{1}{(n + 4)}[A_a]^2[A_b]^2. \quad (52) \]

Second, \([A_a][A_b][B_a]^2[B_b]\) has \(\alpha = 1, \beta_a = 2, \) and \(\beta_b = 1\), and the sum in (45) gives:

\[ [A_a][A_b][B_a]^2[B_b] = \frac{1}{(n + 2)^2(n + 3)}[A_a][A_b] - \frac{2}{(n + 2)(n + 4)}[A_a]^2[A_b]^2. \quad (53) \]

Finally, \([B_a]^3[B_b]^2\), having \(\alpha = 0, \beta_a = 3, \) and \(\beta_b = 2\), can be expressed as

\[ [B_a]^3[B_b]^2 = \frac{1}{n^2(n + 1)^2(n + 2)} - \frac{6}{n(n + 2)^2(n + 3)}[A_a][A_b] + \frac{6}{(n + 1)(n + 2)(n + 4)}[A_a]^2[A_b]^2. \quad (54) \]

from equation (55). Using the fan relation, notice that \([B_a]^3[B_b]^2\) can also be reduced to \(\frac{1}{4^{10}} Z(3, 0, 2)\).

The expressions of \([A_a][A_b]\) and \([A_a]^2[A_b]^2\) in terms of \(n\) have already been determined in Example 1. The final answer is obtained by inserting (52)–(54) into (51). The result is:}

\[ [A_a + A_b + B_a][A_a + A_b] = \frac{2}{n^2(n + 1)^2(n + 2)} + \frac{6(n - 2)}{n(n + 2)^2(n + 3)}[A_a][A_b] + \frac{4(n^2 + 2)}{(n + 1)(n + 2)(n + 4)}[A_a]^2[A_b]^2 \]

\[ = \frac{2(n^2 + 2n + 4)}{(n^2 - 1)n^2(n + 2)(n + 3)(n + 4)}, \]

which can be verified using the plain group theoretical formula (14).

6 Conclusion

In this article, we have pursued the goal of finding an efficient method to calculate the monomial integral (1) or (2). We find that the IM discussed in Sec. 2 is superior for deriving relations between integrals, but the GTM is able to give a formula to calculate any integral. The GTM formula involves a triple sum whose computation is often tedious and prone to mistakes. The sums simplify for orderly integrals, in which the invariant groups \(G_I\) and \(G_J\) are either disjoint, or one is contained in the other. For non-orderly integrals, the hybrid
method is probably the most efficient. It uses the IM to relate them to some orderly integrals that can be calculated by the GTM with relative ease.

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Appendices

A Derivation of Equations (12) and (13)

To prove (12) and (13), we use the rotation technique discussed in item 5 of Section 2.1 and equation (7), to spin off from the R-dot of Fig. 3(a) a new R-dot attached to a pair of solid-dotted lines. Depending on whether the basic pattern of this new R-dot is \([A_a]\), \([A_b]\), \([B_a]\), or \([B_b]\), we get the graphs shown in Fig. 12(a), 12(b), 12(c), and 12(d) respectively.

Repeating this spin-off operation over and over again on the R-dot still containing a compound pattern, eventually we come to a graph where every R-dot is given by a basic pattern. The resulting integrals are given in (12), corresponding to the decompositions of the compound pattern into a sum of all possible fully opened integrals obtained by spinning off.
The coefficient \( v(\alpha'_a \alpha'_b \beta'_a \beta'_b) \) of these basic integrals is derived from a combination of three factors:

1. Each time that we spin off a basic pattern from a compound pattern with \( d \) pairs of solid-dotted lines, there is a factor \( 1/d \) arising from eq. 7, by taking \( e = 1 \). Since we start from a compound pattern with \( N \) pair of lines, by the time we come to a fully-opened pattern we have accumulated a factor \( 1/N! \).

2. The \( N \) R-dots in the final pattern that is fully opened can be spun off in a different sequential order. According to (7), they must be summed over. This gives rise to a factor \( N!/\alpha'_a \alpha'_b \beta'_a \beta'_b! \).

3. At any time when we spin off a basic pattern, we can choose its single pair of solid-dotted lines in all possible ways. Eq. 7 says that we must sum over all these possibilities. The multiplicity factor is given by the number of permutations of these lines that lead back to the same basic integral. It is a factor of \( m_a! m_b! n_a! n_b! \).

Assembling these three factors, we get

\[
v(\alpha'_a \alpha'_b \beta'_a \beta'_b) = \frac{m_a! m_b! n_a! n_b!}{\alpha'_a \alpha'_b \beta'_a \beta'_b!}, \quad (55)
\]

which is identical to (13).

## B Character Tables

The character tables for \( p = 2, 3 \) are given here in the form used by M. Hamermesh in [9]. The rows are labelled by the partitions that define the representations, and the columns are labelled by the cycle structures that define the classes. The number of elements in each class, \( n(c) \), is written above the classes. The table for \( p = 1 \) is trivial, and it consists of the sole value 1.

### B.1 \( p = 2 \)

| Part. \ Class | 1 | 1 |
|--------------|---|---|
| (1)          | 1 |   |
| (2)          | 1 |   |
| (1^2)        | 1 | -1|

### B.2 \( p = 3 \)

| Part. \ Class | 1 | 3 | 2 |
|---------------|---|---|---|
| (1^3)         | 1 |   |   |
| (1,2)         | 2 | 0 | -1|
| (1,3)         | 1 | -1| 1 |
C Primitive Diagrams for $p = 4$ and $p = 5$

Using (14) and the character tables for $S_4$ and $S_5$, the algebraic expressions for the primitive diagrams of Fig. 13 and Fig. 14 can be obtained, and they are given in Table 4.

![Figure 13: The $p = 4$ primitive diagrams.](image1)

![Figure 14: The $p = 5$ primitive diagrams.](image2)
\[ \xi(c_Q) \]

| \( Q \) | \( p = 4 \) | \( p = 5 \) |
|---|---|---|
| \( (12)(3)(4) \) | \( \frac{n^4 - 8n^2 + 6}{n(n^2-1)(n^2-4)(n^2-9)} \) | \( \frac{n^4 - 20n^2 + 78}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)} \) |
| \( (123)(4) \) | \( \frac{-1}{n(n^2-1)(n^2-4)(n^2-9)} \) | \( \frac{-1}{n^2(n^2-4)(n^2-9)(n^2-16)} \) |
| \( (12)(34) \) | \( \frac{2n^2 - 3}{n^2(n^2-1)(n^2-4)(n^2-9)} \) | \( \frac{2}{n^2-2} \) |
| \( (1234) \) | \( \frac{n^2 + 6}{n^2(n^2-1)(n^2-4)(n^2-9)} \) | \( \frac{n^2 - 2}{n(n^2-4)(n^2-9)(n^2-16)} \) |
| \( (123)(45) \) | \( \frac{-5}{n^2(n^2-1)(n^2-4)(n^2-9)(n^2-16)} \) | \( \frac{-5n^2 + 24}{n^2(n^2-4)(n^2-9)(n^2-16)} \) |
| \( (12345) \) | \( \frac{-2(n^2 + 12)}{n^2(n^2-1)(n^2-4)(n^2-9)(n^2-16)} \) | \( \frac{-2(2n^2 + 12)}{n^2(n^2-4)(n^2-9)(n^2-16)} \) |

Table 4: Algebraic expressions for the primitive diagrams of fourth and fifth degrees. In the \( p = 5 \) case, the elements from row two to row five should be written with the additional \( (5) \) one-cycle.

## D Derivation of Equation (45)

We would like to express the general fully-opened integral \([A_a]^\alpha[A_b]^\beta[A_b]^{\beta_a}\) in terms of the special ones of the form \([A_a]^\alpha[A_b]^\beta\). The idea is to apply a unitarity sum on the \([B_a]\) or \([B_b]\) basic patterns to get rid of them. To get the final result we also need to apply the fan relation \(8\) or the double-fan relation \(12\) and \(13\). Our approach is to first determine how can \([A_a]^\alpha[A_b]^{\beta_a}\) be reduced to fully opened integrals involving only the \([A_a]\) and \([A_b]\) patterns. With such an information at hand, we will then try to reduce the more general \([A_a]^\alpha[A_b]^\beta[B_a]^{\beta_a}\) integrals into the \([A_a]^\alpha[A_b]^\beta[B_a]^{\beta_a}\) integrals.

Let us apply a unitarity sum on one of the \([B_a]\) patterns in \([A_a]^\alpha[A_b]^\beta[B_a]^{\beta_a}\):

\[
(n - (2\alpha + \beta_a - 1)) [A_a]^\alpha[A_b]^\beta[B_a]^{\beta_a} + (\beta_a - 1) [A_a]^\alpha[A_b]^\beta[B_a]^{\beta_a - 2}[2B_a] + \alpha \left( [A_a]^\alpha[A_b]^{\alpha - 1}[A_a + B_a][B_a]^{\beta_a - 1} + [A_a]^{\alpha - 1}[A_a + B_a][A_b]^\alpha[B_a]^{\beta_a - 1} \right)
\]

\[= [A_a]^\alpha[A_b]^\alpha[B_a]^{\beta_a - 1} \quad (56)\]

Using \(8\), \(2[B_a]\), in the second term above, can be rewritten as \(2[B_a]^2\). Furthermore, equation \(13\) tells us that \([A_a + B_a]\) and \([A_a + B_a]\) can respectively be rewritten as \(2[A_a][B_a]\) and \(2[A_a][B_a]\), and the term in parentheses above simplifies to \(4[A_a]^\alpha[A_b]^{\alpha - 1}[B_a]^{\beta_a}\). As a result, relation \(15\) reduces to:

\[
[A_a]^\alpha[A_b]^{\alpha - 1}[B_a]^{\beta_a} = \frac{1}{n + 2\alpha + \beta_a - 1} [A_a]^\alpha[A_b]^{\alpha - 1}[B_a]^{\beta_a - 1}. \quad (57)
\]

Using the relation \(57\) recursively on its right-side, until no \([B_a]\) remains, we
and the first step of the work is completed.

Assuming that \( \beta_b \leq \beta_a \), let us perform a unitarity sum on a \([B_b]\) pattern in \([A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}[B_b]^{\beta_b}\):

\[
\sum_{\beta_b=0}^{\beta_a} \frac{1}{(n+2\alpha+\beta_a-1)} \cdot \frac{1}{(n+2\alpha+\beta_a-2)} \cdot \ldots \cdot \frac{1}{(n+2\alpha)} [A_a]^{\alpha}[A_b]^{\alpha} [B_a]^{\beta_a} [B_b]^{\beta_b} = \frac{(n+2\alpha-1)!}{(n+2\alpha+\beta_a-1)!} [A_a]^{\alpha}[A_b]^{\alpha},
\]

Relation (8), or formula (13), again permit to make some simplifications, i.e.

\[
\begin{align*}
\frac{1}{(n+2\alpha+\beta_a-1)} [A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}\beta_b &= \frac{(n+2\alpha-1)!}{(n+2\alpha+\beta_a-1)!} [A_a]^{\alpha}[A_b]^{\alpha} [B_a]^{\beta_a} [B_b]^{\beta_b}
\end{align*}
\]

Relating (8), or formula (13), again permit to make some simplifications, i.e.

\[
[A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}[B_b]^{\beta_b} = \frac{1}{(n+2\alpha+\beta_b-1)} \left\{ [A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a} [B_b]^{\beta_b} \right\}
\]

results. The above can be solved to give

\[
\begin{align*}
[A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}[B_b]^{\beta_b} &= \sum_{\beta_b=0}^{\beta_a} \left\{ (-1)^{n+2\alpha-2+\beta_a} \cdot \frac{1}{(n+2\alpha+\beta_b-1)!} \right\} \cdot [A_a]^{\alpha+\delta_a}[A_b]^{\alpha+\delta_b}[B_a]^{\beta_a}[B_b]^{\beta_b}
\end{align*}
\]

which upon substitution of (58) yields the desired equation (45).

References

[1] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, Phys. Rep. 254, 1 (1995); S.L. Shatashvili, Comm. Math. Phys. 154, 421 (1993); A. Morozov, Mod. Phys. Lett. A7, 3503 (1992).
[2] A. Mueller-Groeling, and H.A. Weidenmueller, Phys. Rep. 299, 189 (1998) cond-mat/9707301.

[3] G. Mahlon, C.S. Lam, and W. Zhu, Phys. Rev. D66, 074005 (2002).

[4] M. Creutz, J. Math. Phys. 19, 2043 (1978).

[5] C. Itzykson and J.-B. Zuber, J. Math. Phys. 21, 411 (1980).

[6] S. Samuel, J. Math. Phys. 21, 2695 (1980); I. Bars, J. Math. Phys. 21 (1980) 2678; I. Bars, Phys. Scripta 23, 983 (1981); A. Morozov Mod. Phys. Lett. A7, 3503 (1992); A.B. Balantekin, Phys. Rev. D62, 085017 (2000); B. Collins, math-ph/0205010; B. Schlittgen and T. Wettig, J. Phys. A36, 3195 (2003); P. Zinn-Justin and J.-B. Zuber, J. Phys. A36, 3173 (2003).

[7] H. Weyl, ‘The Classical Groups’, (Princeton University Press, 1966).

[8] S. Aubert and C.S. Lam, J. Math. Phys. 44, 6112 (2003). math-ph/0307012.

[9] M. Hamermesh, “Group Theory and its Application to Physical Problems”, (Dover, 1962).
Invariant and Group Theoretical Integrations over the $U(n)$ Group

S. Aubert and C. S. Lam
Department of Physics, McGill University
3600 University St., Montreal, QC, Canada H3A 2T8
Emails: samuel.aubert@elf.mcgill.ca, lam@physics.mcgill.ca

Abstract
In a previous article, an ‘invariant method’ to calculate monomial integrals over the $U(n)$ group was introduced. In this paper, we study the more traditional group-theoretical method, and compare its strengths and weaknesses with those of the invariant method. As a result, we are able to introduce a ‘hybrid method’ which combines the respective strengths of the other two methods. There are many examples in the paper illustrating how each of these methods works.

1 Introduction
This article deals with the calculation of integrals of the form
\[
\int (dU) U_{i_1j_1}^* \cdots U_{i_pj_p}^* U_{k_1l_1} \cdots U_{k_ql_q}
\]
over the $U(n)$ group, where $(dU)$ is the invariant Haar measure normalized to $\int (dU) = 1$, and $U_{ij}$ is a $U(n)$ matrix element, with $U_{ij}^*$ being its complex conjugate.

These integrals and their generating functions are useful in many areas of physics, including two-dimensional quantum gravity [1], QCD, matrix models, and statistical and condensed-matter problems of various sorts [2]. They are also needed in the parton saturation problem at small Feynman-$x$ [3].

The integral (1) depends on the indices $I = \{i_1 \cdots i_p\}, J = \{j_1 \cdots j_p\}, K = \{k_1 \cdots k_q\}$, and $L = \{l_1 \cdots l_q\}$, so it will be denoted as $\langle IJ|KL \rangle$:
\[
\langle IJ|KL \rangle = \int (dU) U_{i_j}^* U_{k_l}
\]
where $U_{i_j} = \prod_{a=1}^p U_{i_a,j_a}^*$, and similarly for $U_{k_l}$. Since the matrix elements commute, $U_{i_j}^* = U_{i_p,j_p}^*$, where $I_P = \{i_P(1) \cdots i_P(p)\}$ is obtained from $I$ by a permutation $P \in S_p$ of its $p$ indices. Hence
\[
\langle IJ|KL \rangle = \langle I_P J_P|K_T L_T \rangle
\]
for any $P \in S_p$ and $T \in S_q$.

The integral is nonzero only when $p = q$, a number which will be referred to as the degree of the integral. Without loss of generality, it turns out that we may assume $K = I$ and $L$ to be a permutation of $J$, namely, $L = J_Q$ for some $Q \in S_p$. The value of the integral depends on what the index sets $I, J$, and what the element $Q$ are, so even for a given $p$, there are many distinct cases. The best way to distinguish them is to represent each integral by a diagram in a way to be explained in the next section.

Integral (1) has been computed using a graphical technique [4]. It can also be obtained using the Itzykson-Zuber formula [5] as a generating function, or directly from group theory [6] using the Frobenius formula [7]. We shall refer to this last method as the group-theoretical method, or GTM for short. In the GTM, a general formula is available to compute (1). It involves a triple sum over an expression containing characters of the symmetric group $S_p$, as well as the dimensions of irreducible representations of $S_p$ and of the unitary group $U(n)$. One of the sums is taken over all the relevant irreducible representations, and the others are taken over the symmetry groups of the index sets $I$ and $J$. These sums could be long and tedious for a large $p$, and for most symmetry groups.

A different way to calculate (1) was introduced in a recent paper [8]. This method relies only on the unitary nature of the matrix elements, and the invariance of the Haar measure. In particular, no knowledge of group theory is necessary. The invariance of the Haar measure, as well as the off-diagonal unitarity relation, are used to derive relations between integrals of the same degree. The diagonal unitarity relation connects integrals of degree $p$ with ones of degree $p - 1$. Through a chain of these relations, the desired integral is finally related to the basic integral of degree 0, which is $\int dU = 1$. The desired integral is then solved from this chain of relations. We have called this method the invariant method, or IM for short.

The purpose of this paper is to compare the pros and cons of the GTM with the IM. In order to do so we must first study and understand better the nature of the GTM. Armed with this comparison, we will be able to design a new hybrid method which combines the strengths of these two other methods.

The IM is reviewed in Sec. 2. It is used to derive a new ‘double-fan’ relation needed in the example for the hybrid method.

The GTM is reviewed and studied in Sec. 3. It can be used to derive simple relations, but the relations derived in this way are nowhere as powerful as those derived with the IM. The group-theoretical formula can be used to calculate any integral, but generally that is tedious and has to be done integral by integral. However, for integrals whose two symmetry groups are disjoint, or one is contained in the other, systematics emerge to make the calculation simpler. Several of these ‘orderly’ integrals are studied in Sec. 3.

A comparison of the strengths and weaknesses of the two methods is to be found in Sec. 4. Armed with an understanding of their relative merits, we design a ‘hybrid method’ in Sec. 5 to take advantage of their respective strengths. This method is illustrated by the calculation of a class of ‘double-fan’ integrals. There
are also four Appendices showing details of various calculations.

2 The Invariant Method (IM)

2.1 A Brief Review

The invariant method presented in a previous paper [8] can be used to calculate the integral (1). The method exploits the unitarity of the $U(n)$ group elements,

$$\sum_{j=1}^{n} U_{ij}^* U_{lj} = \sum_{j=1}^{n} U_{ji}^* U_{jl} = \delta_{il}, \quad (4)$$

and the invariance of the Haar measure, in the form

$$\int (dU) f(U, U^*) = \int (dU) f(U^*, U) = \int (dU) f(U^T, U^{*T})$$

$$= \int (dU) f(VU, U^*V^*) = \int (dU) f(UV, U^*V^*), \quad (5)$$

for any function $f$ and any $V \in U(n)$.

The unitarity relation (4) relates integrals of the same degree if $i \neq l$, and it relates integrals of degree $p$ to integrals of degree $p-1$ if $i = l$. Other relations between integrals of the same degree can be obtained from (5), by suitable choices of $V$. Here are some of them discussed in the previous paper [8].

1. Using $V_{ij} = e^{i\phi} \delta_{ij}$, it follows that $p$ must be equal to $q$ in order to avoid the vanishing of integral (2). The number $p$ will be called the degree of the integral.

2. Using $V_{ij} = e^{i\phi_j} \delta_{ij}$, it follows that (2) is nonzero only when $K = I_M$ and $L = J_R$ for some $M, R \in S_p$. Using (3), and denoting $RM^{-1}$ by $Q$, only integrals of the type

$$\langle IJ|IJ_Q \rangle = \int (dU) U_{iJ}^* U_{1J_Q} \quad (6)$$

are nonzero, so from now on we need to consider only integrals of this type.

An integral with $J_Q = J$ will be called a direct integral. For such integrals we may always choose $Q = e$, the identity permutation. Otherwise, the integral is an exchange integral.

Integrals are represented diagrammatically as follows. Each distinct value in the index set $I$ is represented by a dot on the left (L-dot), and each distinct value of the index set $J$ or $J_Q$ is represented by a dot on the right (R-dot). The factor $U_{ij}^*$ is shown as a thin solid line between the L-dot $i$ and the R-dot $j$, and the factor $U_{ij}$ is shown as a dotted line between
these two dots. The factor $U_{ij}^* U_{ij}$ is represented by a thick line, or more
generally, the factor $U_{ij}^{*m} U_{ij}^n$ is represented by a thick line with a pair of
numbers $(m, n)$ written beside it. If $m = n$, then only a single number $m$
is written. The numbers $(m, n)$ or $m$ will be known as the *multiplicities*
of the line. See Fig. 1 for an illustration.

3. With $V$ chosen to be a permutation matrix of $n$ objects, it follows that
$$\langle I J | K L \rangle = \langle I' J' | K' L' \rangle = \langle I J' | K L' \rangle,$$
where $I'$ is obtained from $I$ by a reassignment of the values of its indices,
e.g. if $I = (334)$, then $I'$ may be $(558)$. $K'$ is obtained from $K$ by the
same reassignment, and similarly for $J'$ and $L'$. As a result, there is no
need to know the values of the indices of the L-dots, nor the R-dots. This
is why the dots in Fig. 1 are not labelled.

4. As a consequence of the first two equalities in (5), an integral remains the
same under the interchange of the solid lines with the dotted lines, or the
L-dots with the R-dots.

5. Using $V = R(ab)$, the rotation matrix in the $(a, b)$ plane, a ‘spin-off re-
lation’ is obtained. Consider a R-dot imbedded in an arbitrary integral
$M_0 = \langle I J | I J \rangle$, with $d$ pairs of solid-dotted lines attached to the dot. Now
spin off $e$ pairs of these lines to create a new R-dot and a new integral.
There are many ways to choose the $e$ pair of lines, each possibly corre-
sponds to a different integral. Let $M_e$ be the sum of all these integrals.
Then the quantities $M_0$ and $M_e$ are related by the spin-off relation
$$M_e = M_0 \binom{d}{e},$$
where $\binom{d}{e}$ is the binomial coefficient.

The relation is local in that it is independent of the the structure of the
rest of the diagram. The same relation can also be used to spin off a L-dot.
In what follows, we summarize two general results obtained in the previous paper [8] using the IM.

2.1.1 The fan relation

The fan relation

$$\int (dU) A |U_{ac1}|^{2m_1} \cdots |U_{ac_t}|^{2m_t} = \frac{(\prod_{j=1}^{t} m_j!)}{(\sum_{j=1}^{t} m_j)!} \int (dU) A |U_{ac}|^{2m}$$  \hspace{1cm} (8)

relates integrals of the same $m = \sum_{j=1}^{t} m_j$, where $A$ is an arbitrary product of matrix elements of $U$ and $U^*$ whose column indices are different from $c_1, c_2, \ldots, c_t$. The column index $c$ on the right could be taken to be one of the $c_i$'s.

Diagrammatically, the integral on the right of (8) is shown in Fig. 2(a), and the integral on the left is shown in Fig. 2(b). The additional lines and dots corresponding to the factor $A$ are not shown, because they do not affect the spin-off relation. We shall refer to Fig. 2(a) as a closed fan, and Fig. 2(b) as a partially opened fan. If every $m_i = 1$, then it will be said to be a fully opened fan, or simply an opened fan.

In particular, a closed fan integral is $m!$ times an opened fan integral. In fact, this relation between the two types of fans immediately gives rise to the relation (8) between a closed fan and a partially opened fan. To see it, note that each branch of a partially opened fan is itself a closed fan. By opening up all of them, we get the fully opened fan integral, multiplied by a multiplicity factor $\prod_{j} m_j!$ from all the branches. Thus a closed fan is $m!/ \prod_{j} m_j!$ times a partially opened fan, as given by (8).
2.1.2 The $Z$-integral and the fan integral

In [8], we also obtained a general formula for integrals of the type shown in Fig. 1(b), for arbitrary nonnegative integers $m_1$, $m_2$, and $m_3$. We call that the ‘Z-formula’ because of the shape of the graph. It is

$$Z(m_1, m_2, m_3) \equiv \int (dU) |U_{ij}|^{2m_1} |U_{il}|^{2m_2} |U_{kl}|^{2m_3}$$

$$= \frac{m_1! m_2! m_3! (n-2)! (n-1)! (n + m_1 + m_3 - 2)!}{(n + m_3 - 2)! (n + m_1 + m_3 - 1)!}.$$  (9)

In the special case where $m_2 = m_3 = 0$, this becomes

$$F(m) \equiv Z(m, 0, 0) = \frac{m! (n-1)!}{(n+m-1)!},$$  (10)

which is the integral for Fig. 2(a) when there are no additional dots or lines around.

2.2 Double-Fan Relation

The fan relation can be generalized to a double-fan relation, connecting the closed ‘double-fan’ diagram of Fig. 3(a) with a (fully) opened double-fan diagram such as Fig. 3(b). As in Fig. 2, there may be additional dots and lines in the integral, but none of them may end up on the R-dots shown. From that relation, we can also deduce relations between a closed double-fan and a partially opened double-fan, as done in the single-fan case.

The double-fan relation is considerably more complicated than the single-fan relation (8), because there are many more double-fan graphs. Each R-dot of a (fully) opened (double-fan) graph such as Fig. 3(b) falls into one of four basic patterns: $[A_a]$, $[A_b]$, $[B_a]$, and $[B_b]$, shown in Fig. 4. If the solid and dotted lines end up on the same L-dot, the pattern is a $[B]$; otherwise it is an $[A]$. The subscripts $a$ and $b$ tell us which L-dot the solid line emerges from.
Figure 4: The four basic patterns for the R-dots of a fully opened double-fan graph.

Suppose there are \( \alpha_i \) number of \([A_i]\) and \( \beta_i \) number of \([B_i]\) patterns in a (fully) opened (double-fan) graph. Then there are \( m_a \) solid and \( n_a \) dotted lines emerging from the L-dot \( a \), and \( m_b \) solid and \( n_b \) dotted lines emerging from the L-dot \( b \), where

\[
m_a = \alpha_a + \beta_a, \quad n_a = \alpha_b + \beta_a, \\
m_b = \alpha_b + \beta_b, \quad n_b = \alpha_a + \beta_b.
\]  

The total number of R-dots in the opened graph is \( N = m_a + m_b = n_a + n_b \).

When these \( N \) R-dots are merged together, we get the closed (double-fan) graph depicted in Fig. 3(a), which will be denoted by \([m_a n_a](m_b n_b)\). If \( N = 1 \), this is just one of the four basic patterns discussed before. If \( N > 1 \), we will call it a compound pattern.

From (11), we see that if we replace \( \alpha_i \) and \( \beta_i \) by \( \alpha_i' = \alpha_i + \xi \) and \( \beta_i' = \beta_i - \xi \), with any integral \( \xi \) which keeps \( \alpha_i' \) and \( \beta_i' \) nonnegative, then we get the same closed graph by collapsing this new opened graph. Conversely, it will be shown in Appendix A that the closed graph \([m_a n_a](m_b n_b)\] can be spinned off into a sum of several opened graphs, one for each \((\alpha_i', \beta_i')\). The double-fan relation expressing that quantitatively is

\[
[(m_a n_a)(m_b n_b)] = \sum v(\alpha', \alpha', \beta', \beta') [A_a]^{\alpha'_a}[A_b]^{\alpha'_b}[B_a]^{\beta'_a}[B_b]^{\beta'_b} = \sum [\alpha'_a A_a + \alpha'_b A_b + \beta'_a B_a + \beta'_b B_b],
\]  

where

\[
v(\alpha', \alpha', \beta', \beta') = \frac{m_a! n_a! m_b! n_b!}{\alpha'_a! \alpha'_b! \beta'_a! \beta'_b!},
\]  

and the sum is over all solutions \((\alpha_i', \beta_i')\) of (11).

The double fan becomes a single fan if the L-dot \( b \) is not connected, namely, if \( m_b = n_b = 0 \) and \( m_a = n_a \equiv m \). In that case (12) becomes

\[
[m B_a] = m! [B_a]^m,
\]  

which is just (8) (when all \( m_i = 1 \)) in another notation.
3 Group-Theoretical Method (GTM)

3.1 A Brief Review

Using group theory, the integral (6) can be turned into a multiple sum [5, 6]. In the notation used in Appendix A of [8], the formula is

\[
\langle IJ | IJ_Q \rangle = \sum_{R \in G_I} \sum_{S \in G_J} \sum_f \frac{d_f^2}{(pl)^2} d_f \chi_f(SQR),
\]  

where \( p \) is the degree of the integral. The symbols \( G_I \) and \( G_J \) represent the symmetry groups of the row and column index sets. More precisely, \( G_X = \{ P \in S_p | P(X) = X \} \), with \( X \) being either \( I \) or \( J \). The irreducible representations of the symmetric and unitary groups are both labelled by a sequence \( f = (f_1, f_2, \ldots, f_p) \), with \( f_1 \geq f_2 \geq \cdots \geq f_p \geq 0 \). \( \chi_f(P) \) is the character of \( P \in S_p \) in the irreducible representation with signature \( f \). The dimension of the irreducible representation \( f \) is given by \( d_f = \chi_f(e) \) for \( S_p \), and by \( \tilde{d}_f \) for \( U(n) \).

A formula for \( \tilde{d}_f \) is:

\[
\tilde{d}(f_1, \ldots, f_n) = \frac{D(f_1 + (n-1), f_2 + (n-2), \ldots, f_n)}{D(n-1, n-2, \ldots, 0)},
\]  

where \( D(x_1, \ldots, x_n) \) is the Vandermonde determinant given by \( \prod_{i<k} (x_i - x_k) \).

Since \( \chi_f(g) \) depends only on the class \( c_g \) that \( g \) belongs to, we may write it as \( \chi_f(c_g) \). With this notation, eq. (14) can be re-written as

\[
\langle IJ | IJ_Q \rangle = \sum_c N[c] \xi[c],
\]  

where the sum is taken over all classes \( c \) of \( S_p \),

\[
N[c] = \sum_{R \in G_I} \sum_{S \in G_J} \delta(SQR \in c)
\]  

is the number of elements of the type \( SQR \) in the class \( c \), and

\[
\xi[c] = \sum_f \frac{d_f^2}{(pl)^2} d_f \chi_f(c).
\]  

It is not difficult to see that \( Q \) is not unique, because \( Q' = SQ = QT \) for any \( S \in G_J \) and any \( T \in G_J \). That equation can also be written as

\[
N[c] = \sum_{R \in G_I} \sum_{T \in G_{JQ}} \delta(QTR \in c).
\]  

It is straightforward but generally very tedious to compute \( N[c] \), because we need to calculate the product \( QTR \) for every \( T \in G_{JQ} \), every \( R \in G_I \), and
determine what class \( c \) the product belongs to. Then we have to count up all the products that are in a given class \( c \) to get \( N[c] \). However, the task becomes considerably more manageable if either \( \mathcal{G}_I \) and \( \mathcal{G}_{J_Q} \) are disjoint, or if one is contained in the other. We shall refer to integrals with those properties as *orderly*. Further simplification occurs for direct integrals, because in that case \( Q \) can always be chosen to be the identity \( e \), so the triple product is reduced to a double product \( TR \).

The calculation of \( \xi[c] \) in (18) is simpler than the calculation of \( N[c] \), but still we know of no closed form of it valid for every class \( c \) and every symmetric group \( S_p \). The best we can do is to compute them case by case. Results are given in Sec. 3.3. Each \( \xi[c] \) is actually an orderly integral with \( \mathcal{G}_I = \mathcal{G}_J = e \), to be referred to as a *primitive integral*.

Other integrals can be computed in terms of the primitive integrals, if \( N[c] \) is known. We shall discuss two orderly integrals for which \( N[c] \) can easily be obtained. In Sec. 3.4, we discuss the *stack integrals*, which are direct integrals with \( \mathcal{G}_I = \mathcal{G}_J \). In Sec. 3.5, we discuss the fully opened double-fan integrals of the type \([A_a]^m[A_b]^n\)...

Relations between orderly integrals may be obtained without knowing the explicit values of \( \xi[c] \), if their \( N[c] \)'s are related in a simple way. This is the case for the single-fan relation, and the double-fan relation with \( n_a = m_b = 0 \) and \( m_a = n_b = m \). They will be discussed in Sec. 3.2. However, general double-fan integrals are not orderly, so we cannot obtain the general double-fan relation by the group-theoretical method, at least not in the present way. We will also show that the closed (single-)fan integral can also be computed without explicitly knowing what \( \xi[c] \) are. This is one of the very few cases where integrals can be obtained group-theoretically without explicitly knowing \( \xi[c] \).

That leaves the non-orderly integrals, for which each term of the summand in (19) has to be calculated separately to get \( N[c] \). The first non-orderly integral occurs in degree \( p = 3 \). In Sec. 3.6, we shall show how to calculate some of the \( p = 3 \) and \( p = 4 \) non-orderly integrals.

### 3.2 Single-Fan and Simple Double-Fan Relations

The single-fan integrals are orderly. The index sets for the closed fan Fig. 2(a) are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix}
= 
\begin{pmatrix}
1 & \cdots & m & \cdots \\
a & \cdots & a & \cdots \\
c & \cdots & c \\
c & \cdots & c
\end{pmatrix}.
\]

(20)

The first row gives the index labels, and the next three rows give the values of the indices in the sets \( I, J, \) and \( J_Q \) respectively. Different letters are understood to correspond to different values. Additional dots and lines may be present in the graph, as long as none of the lines end up in the R-dots shown. These additional lines and dots are not drawn because they do not affect the fan relation in any way. Similarly, they are not shown in the index sets in (20) other than the
ellipses in the first two rows, which remind us that there may be more lines connected to the L-dots. Such ellipses are absent in the last two rows because no additional lines are allowed to be connected to the R-dots shown.

Similarly, the index sets for the fully opened fan, Fig. 2(b) with all \( m_i = 1 \), are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & \cdots & m & \cdots \\
a & \cdots & a & \cdots \\
c_1 & \cdots & c_m \\
c_1 & \cdots & c_m
\end{pmatrix}.
\tag{21}
\]

Using \( S_m \) to denote the symmetric group for the permutation of the first \( m \) labels, the symmetric groups for Fig. 2(a) can be read off from (20) to be \( G_I \supset S_m \) and \( G_J = G_{J_Q} = S_m \). Similarly, the symmetric groups for Fig. 2(b) can be read off from (21) to be \( G_I \supset S_m \) and \( G_J = G_{J_Q} = e \). We may choose \( Q = e \) in both cases. Then for Fig. 2(a), \( T \hat{G}_I = \hat{G}_I \) for every \( T \in G_{J_Q} \), hence \( N[c] = m! \sum_{R \in \hat{G}_I} \delta(R \in c) \). But the last sum is simply the \( N[c] \) for Fig. 2(b) and (21). Hence it follows from (16) that the fan relation (with all \( m_i = 1 \)) is true.

Next, let us derive the double-fan relation (12) and (13) for the case \( n_a = m_b = 0 \) and \( n_a = n_b \). The solution of (11) is now unique. It gives \( \alpha_a = m_a = n_b = n \), \( \alpha_b = \beta_a = \beta_b = 0 \). The double-fan relation (12) then becomes

\[
[m A_a] = m! [A_a]^m.
\tag{22}
\]

The closed double-fan is shown in Fig. 5(a). Its index sets are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & \cdots & m & \cdots & n & \cdots & n+m & \cdots \\
a & \cdots & a & \cdots & b & \cdots & b & \cdots \\
c & \cdots & c & \cdots & c & \cdots & c & \cdots
\end{pmatrix}.
\tag{23}
\]

The opened double-fan is shown in Fig. 5(b). Its index sets are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & \cdots & m & \cdots & n & \cdots & n+m & \cdots \\
a & \cdots & a & \cdots & b & \cdots & b & \cdots \\
c_1 & \cdots & c_m & \cdots & c & \cdots & c & \cdots
\end{pmatrix}.
\tag{24}
\]

We may choose \( Q = (1, n)(2, n+1) \cdots (m, n+m) \) in both cases. For (23), \( G_I \supset S_m \otimes S'_m, G_J = S_m, \) and \( G_{J_Q} = S'_m, \) where \( S_m \) is the permutation group of the first \( m \) labels, and \( S'_m \) is the permutation group for the labels \( n, n+1, \ldots, n+m \). For (24), \( G_I \supset S_m \otimes S'_m, \) but \( G_J = G_{J_Q} = e \).

For Fig. 5(a), \( T \hat{G}_I = \hat{G}_I \) for every \( T \in G_{J_Q} \). Hence \( N[c] = m! \sum_{T \in \hat{G}_I} \delta(QR \in c) \). But the last sum is simply the \( N[c] \) of Fig. 5(b). In this way (22) is proven by the GTM.

The fan integral (10) can also be obtained from the GTM. It is given by Fig. 2(a) without extra dots and lines, or (20) without the ellipses at the end.
Figure 5: Double-fan diagrams with \( n_a = m_b = 0 \), and \( m_a = n_b = m \). (a) A closed fan; (b) a fully opened fan.

Then \( \mathcal{G}_I = S_m \), and \( N[c] = m! \sum_R \delta(R \in c) \). From (16) and (18), we get

\[
\langle IJ | IJ_Q \rangle \equiv F(m) = m! \sum_{R \in \mathcal{G}_I} \sum_f \frac{d_f^2}{(m!)^2} d_f \chi_f(R) \tag{25}
\]

\[
= \frac{1}{m!} \sum_f d_f^2 \sum_{R \in S_m} \chi_f(R) \chi^*_f(R) = \frac{d^2_f(m)}{d_f(m)} \tag{26}
\]

\[
= \frac{m!(n-1)!}{(m+n-1)!}. \tag{27}
\]

In getting from (25) to (26), the character \( \chi^*_f(m)(R) = 1 \) of the totally symmetric representation \( (m) \) of the permutation group has been inserted, and the orthogonality relation of the characters has been used. To get to (27), \( d_f(m) = \chi_f(m)(e) = 1 \) as well as \( d_f(m) = (m+n-1)!/(n-1)!m! \) (see (15)) have been used.

The result in (27) agrees with the result (10). It is one of the very few cases where the value of the integrals can be obtained group-theoretically without knowing the values of the individual \( \xi[c] \)'s.

### 3.3 Primitive Integrals

Integrals in which both symmetry groups \( \mathcal{G}_I \) and \( \mathcal{G}_J \) consist only of the identity \( e \) will be called primitive. This happens when all the indices \( i_a \) in the set \( I \) assume distinct values, and all the indices \( j_b \) in the set \( J \) are also different. The corresponding diagrams have \( p \) dots each on both columns, and precisely one solid and one dotted lines connecting to each of the dots. The primitive diagrams for \( p \leq 3 \) are shown in Fig. 6, and the ones for \( p = 4, 5 \) are contained in Appendix C.

Since \( \mathcal{G}_I = \mathcal{G}_J = e \), it follows from (17) that \( N[c] = \delta(Q \in c) \), where \( Q \) can be any element of \( S_p \). The primitive integrals (16) are simply \( \xi[c] \), one for...
each class $c$ of $S_p$. We may therefore use an element of each cycle structure to label the primitive integrals, as is done in Fig. 6. Diagrammatically, the cycle structure is translated into the loop structure of its diagram, as can be seen in Fig. 6. Using (18) along with (15) and the character tables found in Appendix B (note that $d_f = \chi_f(e)$), the primitive integrals for $p \leq 3$ can be easily computed, and the results are displayed in Table 1. The results for $p = 4, 5$ can also be found in Table 4 of Appendix C.

| $\xi[e]$ | $\xi[(12)]$ | $\xi[(123)]$ |
|----------|-------------|--------------|
| $\xi[e]$ | $\xi[(12)(3)]$ | $\xi[(123)]$ |

Figure 6: Primitive diagrams for (a) $p = 1$, (b) $p = 2$, and (c) $p = 3$. The identity element is everywhere denoted by $e$.

Table 1: Algebraic expressions for the primitive diagrams of $p = 1, 2, 3$.
3.4 Stack Integrals

The stack diagrams (see Fig. 7) are direct integrals made up of disconnected lines of arbitrary multiplicities. As such, $Q = e$, and $J$ differs from $I$ only by relabelling. Using item 3 of Sec. 2.2, we may assume $J = I$. Hence stack integrals are integrals of the form $\langle II|II \rangle$.

Let $p_1, p_2, \ldots, p_t$ be the multiplicities of the disconnected lines in a stack diagram. Then $G_J = G_I \equiv G = S_{p_1} \otimes S_{p_2} \otimes \cdots \otimes S_{p_t}$, and $N[c]$ is nonzero only when the class $c$ is a direct product of the classes $c_i$ of the groups $S_{p_i}$. In that case,

$$N[c] = \prod_{i=1}^{t} p_i! n_i(c_i),$$

where $n_i(c_i)$ is the number of elements of $S_{p_i}$ in the class $c_i$. In other words,

$$n_i(c_i) = \frac{p_i!}{\prod_{j=1}^{p_i} j^{\alpha_j} \alpha_j!},$$

where the class $c_i$ consists of $\alpha_j$ cycles of length $j$. Denoting the stack integral $\langle II|II \rangle$ by $\Xi(p_1, p_2, \ldots, p_t)$, we get

$$\Xi(p_1, p_2, \ldots, p_t) = \sum_{c_1, c_2, \ldots} \left( \prod_{i=1}^{t} p_i! n_i(c_i) \right) \xi(c_1 \otimes c_2 \otimes \cdots \otimes c_t).$$

All stack diagrams can be obtained by making the assignment $f_i \rightarrow p_i$ from each representation. In this way, we expect a same number of stack diagrams as of primitive diagrams, or classes. Using the $\xi$ expressions obtained in the preceding subsection, the stack integrals for $p \leq 3$ can be computed to yield the expressions in Table 2.

![Figure 7: Arbitrary stack diagram, $\Xi(p_1, p_2, \ldots, p_t)$, of degree $p = \sum_{i=1}^{t} p_i$.](image)
\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\(\Xi(1) = \frac{1}{n}\) & \(\Xi(3) = \frac{3!}{n(n+1)(n+2)}\) \\
\hline
\(\Xi(2) = \frac{2}{n(n+1)}\) & \(\Xi(2,1) = \frac{2}{(n-1)n(n+2)}\) \\
\hline
\(\Xi(1,1) = \frac{1}{n^2-1}\) & \(\Xi(1,1,1) = \frac{n^2-2}{n(n-1)(n+4)}\) \\
\hline
\end{tabular}
\caption{Algebraic expressions for the stack diagrams of \(p = 1, 2,\) and \(3.\)}
\end{table}

3.5 Special Double-Fan Integrals

The index sets for the fully opened double-fan integrals \([A_a]^\alpha[A_b]^{\alpha}\) (Fig. 3(b) with \(N = 2\alpha\)) are

\[
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix}
= \begin{pmatrix}
1 & \cdots & \alpha & \alpha + 1 & \cdots & 2\alpha \\
b & \cdots & b & a & \cdots & a \\
c_1 & \cdots & c_\alpha & c_{\alpha+1} & \cdots & c_{2\alpha} \\
c_{\alpha+1} & \cdots & c_{2\alpha} & c_1 & \cdots & c_\alpha
\end{pmatrix}.
\] (31)

Hence both \(G_J\) and \(G_{J_Q}\) consist only of the identity \(e\). As for \(G_I\), it is given by \(S_\alpha \otimes S_\alpha\), where the permutation groups \(S_\alpha\) act respectively on the \(b\) and \(a\) indices in \(I\). The element \(Q\) maps \(J_Q\) to \( J\), i.e. \(Q = (1, \alpha+1)(2, \alpha+2) \cdots (\alpha, 2\alpha)\).

The fully opened integral can be computed using (16), with \(N[c]\) given by (17) or (19). Thus,

\[
N[c] = \sum_{R \in G_I} \sum_{T \in G_{J_Q}} \delta(QTR \in c)
= \sum_{R \in G_I} \delta(QR \in c) = \sum_{Q'} \delta(Q' \in c),
\] (32)

where the last sum is over every permutation \(Q'\) that sends all \(b\) indices in (31) to the positions labelled from \(\alpha + 1\) to \(2\alpha\), and similarly all \(a\) indices to the positions labelled from \(1\) to \(\alpha\). As a consequence, the allowed cycles of \(Q'\) must be of even length, and they can be specified by a sequence of nonnegative integers \((k) \equiv (k_1k_2 \cdots k_\alpha)\), \(k_i\) being the number of cycles of length \(2i\). The number of \(Q'\) with the class structure \((k)\) that is related to \(c\) is given by

\[
N[c] = \frac{(\alpha!)^2}{\prod_{i=1}^{\alpha} k_i! k_{i+1}!}.
\] (33)

In order to see how this is arrived at, consider an example where \(k_1 = 2\), \(k_2 = 2\), and all other \(k_i\) values are zero. Then \(Q'\) is of the form \((ba)(ba)(baba)(baba)\), where the \(b\) and \(a\) letters should take the distinct index labels in \((1, \ldots, \alpha)\) and \((\alpha + 1, \ldots, 2\alpha)\) respectively. Another \(Q'\) with the same cycle structure can thus
Table 3: Values of the monomial integrals \([A_a]^{\alpha} [A_b]^{\alpha}\) for \(\alpha = 1, 2, 3\).

| \(\alpha\) | \([A_a]^{\alpha} [A_b]^{\alpha}\) |
|-----------|-------------------------------|
| 1         | \(-\frac{1}{n(n^2-1)}\)     |
| 2         | \(-\frac{2}{(n^2-1)n(n+2)(\alpha+3)}\) |
| 3         | \(-\frac{6}{(n-1)n(n+1)^2(n+2)(\alpha+3)(\alpha+4)(\alpha+5)}\) |

Figure 8: Non-orderly integrals of \(p = 3\).

be obtained by permuting individually all the \(a\) and \(b\) labels. This accounts for the numerator in (33). However, such permutations do not necessarily give distinct \(Q'\) elements. The cyclic nature of a cycle tells us that each cycle of length \(2i\) will appear \(i\) times; this accounts for the \(i^k_i\) factor in the denominator. Moreover, no new \(Q'\) is obtained if we permute cycles of the same length; that accounts for the other factor \(k_i!\) in the denominator.

We may now return to (16) to calculate the integral \([A_a]^{\alpha} [A_b]^{\alpha}\) in terms of the primitive integrals \(\xi[c]\). The result for the first few \(\alpha\) values are listed in Table 3.

### 3.6 Non-Orderly Integrals

All integrals with degree \(p < 3\) are orderly. The non-orderly integrals of \(p = 3\) are shown in Fig. 8, and those related to them by the fan relation (8). The calculation of \(N[c]\) and the integral for each of them is discussed below. The integrals will be labelled by their figure, e.g., integral \(I(8a)\).

The index sets for Fig. 8(a) are

\[
\begin{pmatrix}
label \\
I \ \\
J \ \\
J_Q
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 \\
b & b & a \\
d & c & c \\
d & c & c
\end{pmatrix}.
\]

They give rise to the symmetry groups \(G_I = \{e, (12)\}\) and \(G_J = G_{J_Q} = \{e, (23)\}\). Moreover, the element \(Q\) can be taken to be the identity element. In order to
obtain the coefficients $N[e]$ of equation (16), we need to compute $QTR$ for all $T \in \mathcal{G}_J$ and $R \in \mathcal{G}_I$. That triple product is $Q\mathcal{G}_J\mathcal{G}_I = \{e, (12), (23), (132)\}$. As a result,

$$I(8a) = Z(1, 1, 1) = \xi[e] + 2\xi[(12)(3)] + \xi[(123)] = \frac{1}{(n^2 - 1)(n + 2)}. \quad (35)$$

In the same way, the index sets of Fig. 8(b) are

$$\begin{pmatrix} \text{label} \\ I \\ J \\ J_Q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ b & a & a \\ e & d & c \\ d & e & c \end{pmatrix}, \quad (36)$$

and hence the symmetry groups are $\mathcal{G}_J = \{e\}$ and $\mathcal{G}_I = \{e, (23)\}$, and the exchange element is $Q = (12)$. We thus obtain $Q\mathcal{G}_J\mathcal{G}_I = \{(12), (123)\}$, from which

$$I(8b) = \xi[(12)(3)] + \xi[(123)] = \frac{-1}{(n^2 - 1)n(n + 2)}. \quad (37)$$

follows.

Finally, for Fig. 8(c), the index sets are

$$\begin{pmatrix} \text{label} \\ I \\ J \\ J_Q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ b & b & a \\ d & c & c \\ c & d & c \end{pmatrix}. \quad (38)$$

The relevant symmetry groups are $\mathcal{G}_J = \{e, (13)\}$ and $\mathcal{G}_I = \{e, (23)\}$. With $Q = (12)$, the set $Q\mathcal{G}_J\mathcal{G}_I$ is $\{(12), (13), (123), (132)\}$, and formula (16) gives:

$$I(8c) = 2(\xi[(12)(3)] + \xi[(123)]) = -\frac{2}{(n^2 - 1)n(n + 2)}. \quad (39)$$

The calculation of $Q\mathcal{G}_J\mathcal{G}_I$ is not that cumbersome for $p = 3$, but it gets worse pretty quickly as $p$ increases. For example, let us look at some examples of $p = 4$.

Let us first calculate $Z(2, 1, 1)$ of Fig. 1(b), whose index sets are

$$\begin{pmatrix} \text{label} \\ I \\ J \\ J_Q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ b & b & b & a \\ d & d & c & c \\ d & d & c & c \end{pmatrix}. \quad (40)$$

Then $Q = e$, $\mathcal{G}_J = \{e, (12), (34), (12)(34)\}$ and $\mathcal{G}_I = \{e, (12), (13), (23), (123), (132)\}$. Thus $Q\mathcal{G}_J\mathcal{G}_I = \{e, (12), (13), (23), (123), (132), (23), (13), (34), (12)(34), (143), (243), (1243), (1432), (1324), (1432), (1432), (243), (143)\}$, hence

$$Z(2, 1, 1) = 2\xi[e] + 8\xi[(12)(3)(4)] + 8\xi[(123)(4)] + 2\xi[(12)(34)] + 4\xi[(1234)] = \frac{2}{(n - 1)n(n + 2)(n + 3)}. \quad (41)$$
Our last example is the $\Sigma$-integral, shown in Fig. 9. Its index sets are

$$
\begin{pmatrix}
\text{label} \\
I \\
J \\
J_Q
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 3 & 4 \\
b & b & a & a \\
e & d & d & c \\
e & d & d & c
\end{pmatrix}.
$$

(42)

Hence $Q = e$, $G_{JQ} = \{e, (23)\}$ and $G_I = \{e, (12), (34), (12)(34)\}$. Multiplying such elements accordingly, the set $\{e, (12), (34), (12)(34), (23), (132), (234), (1342)\}$ is obtained for $QG_{JQ}G_I$. Hence

$$
\Sigma = \xi[e] + 3\xi[(12)(3)(4)] + 2\xi[(123)(4)] + \xi[(12)(34)] + \xi[(1234)]
$$

$$
= \frac{n + 1}{(n - 1)n^2(n + 2)(n + 3)}.
$$

(43)

4 Comparison of the IM and the GTM

We have discussed the computation of $U(n)$ integrals (2) in two ways: the IM in Sec. 2, and the GTM in Sec. 3. Each of these two methods has its own merits, and drawbacks, and in a way they complement each other. The purpose of this section is to compare their relative strong and weak points.

The IM is based solely on the unitarity condition (4), and the invariance of the Haar Measure (5). The method is simple because there is no need to know group theory. The conditions relate integrals of the same degree, and also integrals of degree $p$ to integrals of degree $p - 1$. Through these relations, specific integrals such as the fan integrals (10) and the $Z$-integrals (9) can be obtained, and general relation such as the single-fan relation (8) and the double-fan relation (12) can be worked out.

The GTM has the advantage of being general, in the sense that all integrals can be computed using the formula (14) or (16). The price to pay is that we have to know the characters of the irreducible representations of the appropriate symmetric group, and a triple sum has to be carried out, which can prove to be very tedious for integrals of high degrees. Furthermore, unlike the IM, relations between integrals are hard to come by, so one must calculate the integrals one by one. There are however certain class of integrals, the orderly integrals, for which relations can be developed, and the quantity $N[e]$ in (16) can be relatively easily computed. Then we merely have to know the primitive integrals $\xi[e]$ in (16) to get the value of the orderly integral on hand. The stack integrals (30)
Figure 10: Unitarity sum relation involving the $\Sigma$ diagram (second from the left). The sum is performed on the index indicated by an arrow. Using the fan relation (8), the unitary sum can be written as: $\left(\frac{n-3}{2} + 1\right)Z(2, 0, 2) + 2\Sigma = \frac{1}{2}Z(2, 0, 1)$.

and the special opened double-fan integrals (33) are examples of this kind. The primitive integrals $\xi[c]$ themselves must be calculated using (18).

To summarize, the IM gives a huge number of relations but it is not easy to obtain the value of any specific integral. The GTM allows us to calculate any specific integral, albeit rather tedious at times, but it is difficult to obtain relations between integrals. In the next section, we shall discuss a hybrid method which makes use of the advantages of both methods. We shall use the general GTM formula to calculate a specific set of integrals, and then use the IM relations to obtain all the other integrals.

In the rest of this section, we shall enlarge these general remarks about the IM and GTM, by using specific examples presented in the last two sections as concrete illustrations.

The single-fan relation (8) can be obtained by both the IM and the GTM. However the double-fan relation (12) in its general form can be obtained only by the IM, because most of the integrals involved are not orderly, making it hard to derive relations using the GTM. Nevertheless, in special cases involving only orderly integrals, (22), the GTM can also be used to derive the relation.

The $Z$-formula (9) is obtained using the IM, by a series of relations connecting it down to $\int dU = 1$. Since the $Z$-integrals are not orderly, it is hard to compute them using the GTM except at low degrees. The calculation of those by the GTM is shown in equations (35) and (41).

However, since the values of the integrals in the IM are obtained only through relations, it may be relatively complicated to calculate just one specific integral. This is where the GTM is superior, because of the general formula (14) valid for any one integral. For example, it is easy to obtain the $\Sigma$ integral (43), assuming of course the $\xi[c]$’s to be already known. We can also obtain it using the IM, as we shall show below, but that involves a few steps because we must get it from relations. To see how that is done, look at Fig. 10, which is the unitarity relation applied to the dot of the first diagram indicated by an arrow. The first and third diagrams can be related to $Z(2, 0, 2)$ by using the fan relation (8), and similarly the diagram on the right can be related to $Z(2, 0, 1)$. Using the $Z$-formula (9), we then obtain

$$\Sigma = \frac{1}{4} \left[ Z(2, 0, 1) - \left(\frac{n-3}{2} + 1\right)Z(2, 0, 2) \right]$$
the same as the result (43) obtained by the GTM.

5 Hybrid Method

Having understood the relative merits of the GTM and the IM, it is possible to combine their strengths into a more efficient hybrid calculational scheme. The strategy is to start with one or more integrals that can be computed by the GTM with relative ease. Generally speaking, such integrals are ordered. Once they are obtained, the many relations of the IM can be used to calculate other integrals from them.

To illustrate this strategy, we will consider how the hybrid method can be used to calculate all double-fan integrals.

By a double-fan integral, we mean any integral with two L-dots and any number of R-dots. Fig. 3(a) shows a closed (double-fan) integral (with the understanding that there are no extra dots or lines than those shown), and Fig. 3(b) shows a fully opened (double-fan) integral. We may also have partially opened (double-fan) integrals, in which every branch, namely, every R-dot with its connecting lines, can be regarded as a closed integral. See Fig. 11 for an example of a partially opened integral.

As in Sec. 2.2, a fully opened integral is denoted by \[ A_a^\alpha A_b^\alpha + A_a^\beta A_b^\beta \] and its corresponding closed integral is denoted by \[ A_a^{\alpha+\epsilon} A_b^{\alpha+\epsilon} \].

Using (12) and (13), all double-fan integrals can be expressed as sums of fully opened integrals. Integrals of the form \[ A_a^\alpha A_b^\alpha \] are given by (33) and Table 3, but we still have to know how to calculate a fully opened integral when \( \beta_i \neq 0 \). As shown in Appendix D, the IM allows us to relate them to those with \( \beta_i = 0 \), by using the following formula

\[
\sum_{e=0}^{\min(\beta_a, \beta_b)} \left\{ (-1)^e e! \left( \begin{array}{c} \beta_a \\ e \end{array} \right) \left( \begin{array}{c} \beta_b \\ e \end{array} \right) (n + 2\alpha - 1 + 2e) \cdot \right. \\
\left. (n + 2\alpha - 2 + e)! (n + 2\alpha - 1 + 2e)! \right\} \]

We close this section by showing how to use (12) and (13) to calculate the integrals in Fig. 11.

5.1 Fig. 11(a)

There are two equivalent forms for this diagram. One is

\[
[A_a + 2A_b][A_a] = (2[A_a][A_b])^2 \]
Figure 11: Partially opened double-fan integrals. (a) There are two equivalent forms for this graph: \([A_a + 2A_b][A_a]\) and \([A_a + B_a + B_b][A_a]\). (b) There are four equivalent forms for this graph: \([A_a + A_b + B_a][A_a + A_b]\), \([2B_a + B_b][A_a + A_b]\), \([A_a + A_b + B_a][B_a + B_b]\), \([2B_a + B_b][B_a + B_b]\).

and the other is

\[ [A_a + B_a + B_b][A_a] = 4 ([A_b][B_a][B_b]) [A_a] = 4[A_a][A_b][B_a][B_b], \]

(47)

where equations (12) and (13) have been used. The integral \(I(11a)\) is obtained by adding up (46) and (47).

Using (45), we can express all fully opened integrals in the form \([A_a]^\alpha[A_b]^\alpha\).

Applying to the present case, we get

\[ [A_a][A_b][B_a][B_b] = \frac{1}{(n + 2)^2}[A_a][A_b] - \frac{1}{(n + 2)}[A_a]^2[A_b]^2. \]

(48)

Using Table 3, we finally obtain

\[
I(11a) = 2[A_a]^2[A_b]^2 + 4[A_a][A_b][B_a][B_b] \\
= \frac{2n}{(n + 2)}[A_a]^2[A_b]^2 + \frac{4}{(n + 2)^2}[A_a][A_b] \\
= \frac{-4}{(n^2 - 1)n(n + 2)(n + 3)}.
\]

5.2 Fig. 11(b)

As shown in Fig. 11(b), \(I(11b)\) has four equivalent forms. For one branch, the factors are

\[
[A_a + A_b + B_a] = 4[A_a][A_b][B_a], \\
[2B_a + B_b] = 2[B_a]^2[B_b]; \]

(49)

and for the other branch, they are

\[
[A_a + A_b] = [A_a][A_b], \\
[B_a + B_b] = [B_a][B_b].
\]

(50)
Hence

$$I(11b) = 4 [A_a]^2 [A_b]^2 [B_a] + 6 [A_a] [A_b] [B_a]^2 [B_b] + 2 [B_a]^3 [B_b]^2. \quad (51)$$

We will now express each of the three monomial integrals in (51) in terms of $[A_a]^n [A_b]^n$. First, with respect to (45), $[A_a]^2 [A_b]^2 [B_a]$ is characterized by $\alpha = 2$, $\beta_a = 1$, and $\beta_b = 0$. The vanishing of $\beta_b$ causes (45) to consist of the single term:

$$[A_a]^2 [A_b]^2 [B_a] = \frac{1}{(n+4)} [A_a]^2 [A_b]^2. \quad (52)$$

Second, $[A_a] [A_b] [B_a]^2 [B_b]$ has $\alpha = 1$, $\beta_a = 2$, $\beta_b = 1$, and the sum in (45) gives:

$$[A_a] [A_b] [B_a]^2 [B_b] = \frac{1}{(n+2)^2 (n+3)} [A_a] [A_b] - \frac{2}{(n+2)(n+4)} [A_a]^2 [A_b]^2. \quad (53)$$

Finally, $[B_a]^3 [B_b]^2$, having $\alpha = 0$, $\beta_a = 3$, $\beta_b = 2$, can be expressed as

$$[B_a]^3 [B_b]^2 = \frac{1}{n^2 (n+1)^2 (n+2)} - \frac{6}{n(n+2)^2 (n+3)} [A_a] [A_b] + \frac{6}{(n+1)(n+2)(n+4)} [A_a]^2 [A_b]^2 \quad (54)$$

from equation (45). Using the fan relation, notice that $[B_a]^3 [B_b]^2$ can also be reduced to $\frac{1}{12} \frac{T_3}{Z(3, 0, 2)}$.

The expressions of $[A_a] [A_b]$ and $[A_a]^2 [A_b]^2$ in terms of $n$ have already been determined in Example 1. The final answer is obtained by inserting (52)–(54) into (51). The result is:

$$[A_a + A_b + B_a] [A_a + A_b] = \frac{2}{n^2 (n+1)^2 (n+2)} + \frac{6(n-2)}{n(n+2)^2 (n+3)} [A_a] [A_b] + \frac{4(n+2)}{(n+1)(n+2)(n+4)} [A_a]^2 [A_b]^2$$

$$= \frac{2(n^2 + 2n + 4)}{(n^2 - 1)n^2 (n+2)(n+3)(n+4)},$$

which can be verified using the plain group theoretical formula (14).

6 Conclusion

In this article, we have pursued the goal of finding an efficient method to calculate the monomial integral (1) or (2). We find that the IM discussed in Sec. 2 is superior for deriving relations between integrals, but the GTM is able to give a formula to calculate any integral. The GTM formula involves a triple sum whose computation is often tedious and prone to mistakes. The sums simplify for orderly integrals, in which the invariant groups $G_I$ and $G_{J_Q}$ are either disjoint, or one is contained in the other. For non-orderly integrals, the hybrid
method is probably the most efficient. It uses the IM to relate them to some orderly integrals that can be calculated by the GTM with relative ease.

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Appendices

A Derivation of Equations (12) and (13)

To prove (12) and (13), we use the rotation technique discussed in item 5 of Section 2.1, and equation (7), to spin off from the R-dot of Fig. 3(a) a new R-dot attached to a pair of solid-dotted lines. Depending on whether the basic pattern of this new R-dot is \([A_a]\), \([A_b]\), \([B_a]\), or \([B_b]\), we get the graphs shown in Fig. 12(a), 12(b), 12(c), and 12(d) respectively.

Repeating this spin-off operation over and over again on the R-dot still containing a compound pattern, eventually we come to a graph where every R-dot is given by a basic pattern. The resulting integrals are given in (12), corresponding to the decompositions of the compound pattern into a sum of all possible fully opened integrals obtained by spinning off.
The coefficient \( v(\alpha'_a \alpha'_b \beta'_a \beta'_b) \) of these basic integrals is derived from a combination of three factors:

1. Each time that we spin off a basic pattern from a compound pattern with \( d \) pairs of solid-dotted lines, there is a factor \( 1/d \) arising from eq. (7), by taking \( e = 1 \). Since we start from a compound pattern with \( N \) pair of lines, by the time we come to a fully-opened pattern we have accumulated a factor \( 1/N! \).

2. The \( N \) R-dots in the final pattern that is fully opened can be spinned off in a different sequential order. According to (7), they must be summed over. This gives rise to a factor \( N!/\alpha'_a \alpha'_b \beta'_a \beta'_b! \).

3. At any time when we spin off a basic pattern, we can choose its single pair of solid-dotted lines in all possible ways. Eq. (7) says that we must sum over all these possibilities. The multiplicity factor is given by the number of permutations of these lines that lead back to the same basic integral. It is a factor of \( m_a! m_b! n_a! n_b! \).

Assembling these three factors, we get

\[
v(\alpha'_a \alpha'_b \beta'_a \beta'_b) = \frac{m_a! m_b! n_a! n_b!}{\alpha'_a \alpha'_b \beta'_a \beta'_b!},
\]

which is identical to (13).

**B Character Tables**

The character tables for \( p = 2, 3 \) are given here in the form used by M. Hamermesh in [9]. The rows are labelled by the partitions that define the representations, and the columns are labelled by the cycle structures that define the classes. The number of elements in each class, \( n(c) \), is written above the classes. The table for \( p = 1 \) is trivial, and it consists of the sole value 1.

**B.1 \( p = 2 \)**

| Part. \( \backslash \) Class | 1 \( (1^2) \) | 1 \( (2) \) |
|-----------------------------|---------------|---------------|
| (2)                         | 1             | 1             |
| (1\(^2\))                   | 1             | -1            |

**B.2 \( p = 3 \)**

| Part. \( \backslash \) Class | 1 \( (1^3) \) | 3 \( (1, 2) \) | 2 \( (3) \) |
|-----------------------------|---------------|---------------|---------------|
| (3)                         | 1             | 1             | 1             |
| (2, 1)                      | 2             | 0             | -1            |
| (1\(^3\))                   | 1             | -1            | 1             |
C  Primitive Diagrams for $p = 4$ and $p = 5$

Using (14) and the character tables for $S_4$ and $S_5$ [9], the algebraic expressions for the primitive diagrams of Fig. 13 and Fig. 14 can be obtained, and they are given in Table 4.

Figure 13: The $p = 4$ primitive diagrams.

Figure 14: The $p = 5$ primitive diagrams.
\[\xi(c_Q)\]

| \(Q\) | \(p = 4\) | \(p = 5\) |
|-------|--------|--------|
| \(e\) | \(-\frac{n^4 - 8n^2 + 1}{n^2(n^2 - 1)(n^2 - 4)(n^2 - 9)}\) | \(-\frac{n^6 - 20n^2 + 78}{n(n^2 - 1)(n^2 - 4)(n^2 - 9)(n^2 - 16)}\) |
| (12)(3)(4) | \(-\frac{1}{n(n^2 - 1)(n^2 - 9)}\) | \(-\frac{1}{n(n^2 - 1)(n^2 - 9)(n^2 - 16)}\) |
| (123)(4) | \(-\frac{2n^2 - 3}{n^2(n^2 - 1)(n^2 - 4)(n^2 - 9)}\) | \(-\frac{n^2}{n(n^2 - 1)(n^2 - 4)(n^2 - 9)(n^2 - 16)}\) |
| (12)(34) | \(-\frac{n^4 + 6}{n^2(n^2 - 1)(n^2 - 4)(n^2 - 9)}\) | \(-\frac{5n^2 + 24}{n^2(n^2 - 1)(n^2 - 4)(n^2 - 9)(n^2 - 16)}\) |
| (123)(45) | \(-\frac{-5}{n^2(n^2 - 1)(n^2 - 4)(n^2 - 9)}\) | \(-\frac{-2(n^2 + 12)}{n(n^2 - 1)(n^2 - 4)(n^2 - 9)(n^2 - 16)}\) |
| (12345) | \(-\frac{n^4 - 1}{n^2(n^2 - 1)(n^2 - 4)(n^2 - 9)}\) | \(-\frac{n^5 - 1}{n(n^2 - 1)(n^2 - 4)(n^2 - 9)(n^2 - 16)}\) |

Table 4: Algebraic expressions for the primitive diagrams of fourth and fifth degrees. In the \(p = 5\) case, the elements from row two to row five should be written with the additional (5) one-cycle.

\section*{D Derivation of Equation (45)}

We would like to express the general fully-opened integral \([A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}[B_b]^{\beta_b}\) in terms of the special ones of the form \([A_a]^{\alpha}[A_b]^{\alpha'}\). The idea is to apply a unitarity sum on the \([B_a]\) or \([B_b]\) basic patterns to get rid of them. To get the final result we also need to apply the fan relation (8) or the double-fan relation (12) and (13). Our approach is to first determine how can \([A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}\) be reduced to fully opened integrals involving only the \([A_a]\) and \([A_b]\) patterns.

With such an information at hand, we will then try to reduce the more general \([A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}[B_b]^{\beta_b}\) integrals into the \([A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}\) integrals.

Let us apply a unitarity sum on one of the \([B_a]\) patterns in \([A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}\):

\[
\begin{align*}
(n - (2\alpha + \beta_a - 1)) [A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a} + (\beta_a - 1) [A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a - 1}2B_a + \alpha ([A_a]^{\alpha}[A_b]^{\alpha - 1}[A_a + B_a][B_a]^{\beta_a - 1} + [A_a]^{\alpha - 1}[A_a + B_a][A_b]^{\alpha}[B_a]^{\beta_a - 1}) \\
= [A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a - 1}
\end{align*}
\]

(56)

Using (8), \([2B_a]\), in the second term above, can be rewritten as \(2[B_a]^2\). Furthermore, equation (13) tells us that \([A_a + B_a]\) and \([A_a + B_a]\) can respectively be rewritten as \(2[A_a][B_a]\) and \(2[A_a][B_a]\), and the term in parentheses above simplifies to \(4[A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a}\). As a result, relation (56) reduces to:

\[
[A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a} = \frac{1}{n + 2\alpha + \beta_a - 1} [A_a]^{\alpha}[A_b]^{\alpha}[B_a]^{\beta_a - 1}.
\]

(57)

Using the relation (57) recursively on its right-side, until no \([B_a]\) remains, we
obtain:

\[
[A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a = \frac{1}{(n + 2\alpha + \beta_a - 1)(n + 2\alpha + \beta_a - 2) \cdots (n + 2\alpha)} [A_a]^\alpha [A_b]^\alpha \cdot \frac{1}{(n + 2\alpha - 1)!} [A_a]^\alpha [A_b]^\alpha , \tag{58}
\]

and the first step of the work is completed.

Assuming that \( \beta_b \leq \beta_a \), let us perform a unitarity sum on a \([B_b]\) pattern in \([A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a [B_b]^\beta_b\):

\[
(n - (2\alpha + \beta_a + \beta_b - 1)) [A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a [B_b]^\beta_b + (\beta_b - 1) [A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a [B_b]^\beta_b - 2[2B_b] + \beta_a [A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a - 1[2B_a] + B_b[B_b][B_b]^{\beta_b - 1} + \alpha (A_a)^\alpha [A_b]^\alpha - 1[A_b + B_b][B_b][B_b]^\beta_b - 1 + [A_a]^\alpha - 1[A_a + B_b][A_b]^\alpha [B_a]^\beta_a [B_b]^\beta_b - 1 = [A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a [B_b]^\beta_b - 1. \tag{59}
\]

Relation (8), or formula (13), again permit to make some simplifications, i.e. \(2B_b = 2[B_b]^2\), \([B_a + B_b] = [A_a][A_b + B_b][B_b] \), \([A_a + B_b] = 2[A_a][B_b] \), \([A_a + B_b] = 2[A_a][B_b] \). By making the proper substitutions in (59), the recursion equation,

\[
[A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a [B_b]^\beta_b = \frac{1}{(n + 2\alpha + \beta_b - 1)} \{[A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a [B_b]^{\beta_b - 1} - \beta_a [A_a]^\alpha + 1[A_b]^\alpha + 1[B_a]^{\beta_a - 1}[B_b]^{\beta_b - 1}\},
\]

results. The above can be solved to give

\[
[A_a]^\alpha [A_b]^\alpha [B_a]^\beta_a [B_b]^\beta_b = \sum_{e=0}^{\beta_b} (-1)^e \binom{\beta_a}{e} \binom{\beta_b}{e} (n + 2\alpha - 1 + 2e) \cdot \frac{(n + 2\alpha - 2 + e)!}{(n + 2\alpha + \beta_b - 1 + e)!} [A_a]^\alpha + e [A_b]^\alpha + e [B_a]^{\beta_b - e},
\]

which upon substitution of (58) yields the desired equation (45).

References

[1] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, Phys. Rep. 254 (1995) 1.

[2] A. Mueller-Groeling, and H.A. Weidenmueller, Phys. Rep. 299 (1998) 189 cond-mat/9707301.
[3] G. Mahlon, C.S. Lam, and W. Zhu, Phys. Rev. D 66 (2002) 074005.

[4] M. Creutz, J. Math. Phys. 19 (1978) 2043.

[5] C. Itzykson and J.-B. Zuber, J. Math. Phys. 21 (1980) 411.

[6] S. Samuel, J. Math. Phys. 21 (1980) 2695; I. Bars, J. Math. Phys. 21 (1980) 2678; I. Bars, Phys. Scripta 23 (1981) 983; A. Morozov Mod. Phys. Lett. A 7 (1992) 3503; A.B. Balantekin, Phys. Rev. D 62 (2000) 085017; B. Collins, math-ph/0205010; B. Schlittgen and T. Wettig, J. Phys. A 36 (2003) 3195; P. Zinn-Justin and J.-B. Zuber, J. Phys. A 36 (2003) 3173.

[7] H. Weyl, ‘The Classical Groups’, (Princeton University Press, 1966).

[8] S. Aubert and C.S. Lam, J. Math. Phys. 44 (2003) 6112 [arXiv:math-ph/0307012].

[9] M. Hamermesh, “Group Theory and its Application to Physical Problems”, (Dover, 1962).