Decoherence and information encoding in quantum reference frames

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Abstract: Reference frames are of special importance in physics. They are usually considered to be idealized entities. However, in most situations, e.g. in laboratories, physical processes are described within reference frames constituted by physical systems. As new technological developments make it possible to demonstrate quantum properties of complex objects an interesting conceptual problem arises: Could one use states of quantum systems to define reference frames? Recently such a framework has been introduced in [F. Giacomini, E. Castro-Ruiz, and . Brukner, Nat Commun 10, 494 (2019)]. One of its consequences is the fact that quantum correlations depend on a physical state of an observers reference frame. The aim of this work is to examine the dynamical aspect of this phenomena and show that the same is true for correlations established during an evolution of a composite systems. Therefore, decoherence process is also relative: For some observers the reduced evolution of subsystems is unitary, whereas for others it is not. I also discuss implications of this results for modern developments of decoherence theory: Quantum Darwinism and spectrum broadcast structures.

I. INTRODUCTION

Description of physical phenomena is usually made within a reference frame. As a result, definition of certain physical quantities, such as states of physical systems, is relative and may change with a different choice of a reference frame. In most considerations reference frames are regarded as some idealized entities, whereas in practice, e.g. in laboratories, they are constituted by physical objects that are in states with well defined properties such as position or momentum. However, with the rapid development of the quantum technologies field new tests of the quantum superposition principle are made with ever heavier and more complex objects. Therefore, it is both important and interesting to consider extensions of reference frames to the quantum domain and a substantiation of this framework has been introduced in [F. Giacomini, E. Castro-Ruiz, and . Brukner, Nat Commun 10, 494 (2019)]. Here I follow a recent approach to that problem introduced in [7] as it allows to associate reference frames with quantum states. As a consequence, a novel feature of this framework is that, in contrary to previous studies, it allows to consider reference frames with genuinely quantum features, e.g. being constituted by a state in a superposition of distinct classical states. The main conclusion of [7] is that correlations between subsystems of a composite quantum system are relative to a quantum state of the observers reference frame. For instance, for one observer a system is in an entangled state with some other system, whereas for the other the state of the same system is product and hence uncorrelated.

Properties of quantum states of composite systems are often used to explain emergence of classical behavior out of quantum laws. Let us consider a distinguished physical system prepared in a pure state that interacts with some degrees of freedom - the environment. Usually, due to the interaction, correlations are established between the system and the environment. As a result, as the system evolves, its state loses purity and some of its quantum features, such as coherences, may be suppressed. This is the essence of decoherence theory: Initial quantum properties of the system become non-local features of the joint system-environment state. Moreover, recent developments of decoherence theory, i.e. quantum Darwinism [8] and spectrum broadcast structures [9, 10], put emphasis on the fact that during decoherence information about the system is encoded in the environment. Redundant information encoding allows to explain how information about decohered pointer states becomes locally available to many independent observers, what is an important feature of our everyday experience.

From the above discussion it is clear that dynamically established quantum correlations are essential to explain emergence of certain classical features out of quantum theory. On the other hand, if one adopts the framework of quantum reference frames [7], then quantum correlations are relative to a quantum state of a reference frame. Therefore, one may ask whether decoherence and information encoding are also relative phenomena? Here I explore this question by investigating reduced dynamics of composite systems as seen from reference frames associated to different quantum states. An example of a transformation between two reference frames is shown, for which in one reference frame the reduced subsystem dynamics is unitary whereas in the second it is not. In one reference frame a subsystem undergoes decoherence and information about its pointer states in deposited in the other subsystem, whereas this not the case in the second reference frame.

The paper is structured as follows: In Section [11] I...
briefly outline the quantum reference frame framework of \[7\]. In Sec. \[III\] the main results of the paper are presented: It is shown that a generalized Galilean transform between quantum reference frames leads to different description of reduced subsystems dynamics in the reference frames. Implication of this result for advanced forms of decoherence theory, quantum Darwinism and spectrum broadcast structures, are discussed in Sec. \[IV\]. Sec. \[V\] concludes the paper, and an additional example is presented in the Appendix \[A\].

\section{II. QUANTUM REFERENCE FRAMES}

Here we briefly review the main features of quantum reference frames formalism from Ref. \[7\]. In the simplest, one-dimensional, scenario there are three quantum systems A, B, C, and one of them, say the system C, is chosen to be the quantum reference frame (see also Fig. \[1\]). Form the point of view of reference frame constituted by the state of C the quantum state of A and B is \(\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B\). Evolution of this state is generated by a Hamiltonian \(\hat{H}^{(C)}\). We would like to see how this state and its dynamics is perceived from the perspective of subsystem A. In \[7\] it was shown that a transformation between quantum reference frames of C and A is given by an appropriately chosen unitary transformation \(\hat{S} : \mathcal{H}_A^{(C)} \otimes \mathcal{H}_B^{(C)} \rightarrow \mathcal{H}_A^{(A)} \otimes \mathcal{H}_B^{(A)}\). The general structure of such a transformation is

\[\hat{S} = e^{-\frac{i}{\hbar} \hat{H}_{AC} t} \hat{P}_{AC}^{(C)} \Pi_n e^{\frac{i}{\hbar} \hat{f}_{AC}^{(n)}(t) \hat{O}_{AB}^n} e^{\frac{i}{\hbar} \hat{H}_A t},\]

where \(\hat{P}_{AC} : \mathcal{H}_A^{(C)} \rightarrow \mathcal{H}_A^{(A)}\) is a parity swap operator exchanging Hilbert spaces A with C, and \(\Pi_n e^{\frac{i}{\hbar} \hat{f}_{AC}^{(n)}(t) \hat{O}_{AB}^n}\) can be seen as a quantum generalization of a transformation between classical reference frames, whose realization may in general require \(n\) terms. For example a "coherent" translation \(\hat{U}_X = e^{\frac{i}{\hbar} \hat{x} \hat{p} \hat{\mu}}\) is a generalization of a quantum state transformation between two classical reference frames translated by a distance \(X_0\): \(\hat{U}_{X_0} = e^{\frac{i}{\hbar} \hat{x} \hat{p} \hat{\mu}}\). Quantum state of subsystems AB transforms as

\[\rho_{BC}^{(A)} = \hat{S} \rho_{AB}^{(C)} \hat{S}^\dagger \in \mathcal{H}_B^{(A)} \otimes \mathcal{H}_C^{(A)},\]

and its dynamics is governed by the standard von-Neumann equation

\[i\hbar \frac{d\rho_{BC}^{(A)}}{dt} = \left[\hat{H}_{BC}^{(A)}, \rho_{BC}^{(A)}\right],\]

with the transformed Hamiltonian

\[\hat{H}_{BC}^{(A)} = \hat{S} \hat{H}^{(C)} \hat{S}^\dagger + i\hbar \frac{d\hat{S}}{dt} \hat{S}^\dagger.\]

As shown in Ref. \[7\] in this framework correlations of quantum states become relative to the state of a reference frame. For example it may happen that, from the point of the reference frame associated to the state of subsystem C, subsystem B is in an entangled state with subsystem A, whereas in reference frame of A the state of subsystem B is pure and hence uncorrelated with subsystem C. Our aim here is to investigate how dynamics of subsystems is perceived in different reference frames. More precisely, we are interested in correlations that are established during subsystems evolution. Consider a situation, in which in A reference frame subsystem B becomes entangled with subsystem C, and this entanglement grows with time, whereas in C reference frame subsystems A and B remain product during the evolution. Then in A reference frame the reduced evolution of subsystem B is non-unitary, B decoheres, and information about B could in principle be encoded in subsystem C. On the other hand in reference frame of C this is not the case. In the next Section it is shown that generalization of the Galilean transform provides such an example.

\section{III. A CASE STUDY: GENERALIZED GALILEAN TRANSFORM}

The unitary operator \(\hat{S}_B\) that provides transformation between two reference frames related by the generalization of the Galilean transform is

\[\hat{S}_B = e^{-\frac{i}{\hbar} \frac{\hat{p}_B^2}{2m_B} t} \hat{P}_{AC}^{(v)} e^{\frac{i}{\hbar} \hat{p}_A^v \hat{\mu}_A} e^{\frac{i}{\hbar} \frac{\hat{p}_A^2}{2m_A} t},\]

where \(\hat{G}_B = \hat{\rho}_B t - m_B \hat{x}_B\), and and \(\hat{P}_{AC}^{(v)}\) maps velocity of A to the opposite velocity of C: \(\hat{P}_{AC}^{(v)} = \hat{P}_{AC} \exp \left(\frac{i}{\hbar} \frac{\mu_A}{m_A} (\hat{x}_A \hat{\rho}_A + \hat{\rho}_A \hat{x}_A)\right)\). Let us assume that in the reference frame of observer systems C and B are free particles. Their Hamiltonian is thus

\[\hat{H}_{AB}^{(C)} = \frac{\hat{p}_A^2}{2m_A} + \frac{\hat{p}_B^2}{2m_B},\]

and we assume that the particles are initially uncorrelated \(\Psi_0^{(C)}\)

\[\Psi_0^{(C)} = |\psi_0\rangle_A \otimes |\psi_0\rangle_B\]

According to the
transformation law Eq. (2), the evolution of the initial state in reference frame of A is 
\[ \hat{S}_b e^{-\frac{i}{\hbar} H_{BC}^{(A)} t} |\Psi_0^{(C)}\rangle \rangle_{AB}. \] 
The state after this transformation is 
\[ |\psi^{(A)}(t)\rangle_{BC} = e^{-\frac{i}{\hbar} H_{BC}^{(A)} t} e^{-\frac{i}{\hbar} \frac{\pi C}{m_c} G_B} |\Psi_0^{(A)}\rangle_{BC}. \] 
For simplicity we assume that \( m_A = m_C \). The initial state in the transformed frame is still a product one
\[ |\Psi_0^{(A)}\rangle_{BC} = \int \int d\pi_B d\pi_C \psi_0^{(B)} (\pi_B) |\pi_B\rangle_B \otimes \phi_0(-\pi_C) |\pi_C\rangle_C, \]
where the minus in \( \phi_0(-\pi_C) \) indicates that subsystem C evolves with opposite velocity to that of A. To investigate dynamics of \( |\Psi^{(A)}(t)\rangle_{BC} \) it is convenient to rewrite the Galilean operator using the Baker - Campbell - Hausdorff formula
\[ e^{-\frac{i}{\hbar} \frac{\pi C}{m_c} G_B} = \int d\pi_C e^{-\frac{i}{\hbar} \frac{\pi C}{m_c} G_B} \otimes |\pi_C\rangle_C = \int d\pi_C e^{-\frac{i}{\hbar} \frac{\pi C}{\sqrt{2}m_c} \pi x \cdot \pi C} e^{-\frac{i}{\hbar} \frac{m_B}{m_c} \pi C \phi_0} \otimes |\pi_C\rangle_C. \]
The form of Eq. (9) suggests that the evolution of subsystem B is controlled by the momentum of subsystem C. One can also show that evolution of subsystem C is controlled by subsystem B.

Let us now turn attention to the reduced state of subsystem B. In C reference frame the state of subsystem B is initially pure and remains pure during the evolution. On the other hand, Eq. (10) indicates that this is not the case in A reference frame as one finds
\[ \hat{\rho}_B(t) = Tr C \left[ |\psi^{(A)}(t)\rangle \langle \psi^{(A)}(t)|_{BC} \right] = \int \int d\pi_B d\pi_B' \Gamma_{\pi_B,\pi_B'}(t) \psi_0(\pi_B) \psi_0'(\pi_B') |\pi_B\rangle_B \langle \pi_B' |_B, \]
where (assuming that \( m_A = m_C \))
\[ \Gamma_{\pi_B,\pi_B'}(t) \equiv \langle \phi_{\pi_B}(t) | \phi_{\pi_B'}(t) \rangle = \int d\pi_C e^{-\frac{i}{\hbar} \frac{\pi C}{\sqrt{2}m_c} \tau_{\Delta_{\pi_B}}} |\phi_0(-\pi_C)\rangle^2. \]
For simplicity we choose \( \phi_0(\pi_C) \) to be a Gaussian \( \phi_0(\pi_C) = \left( \Delta_{\pi_C}^2 \pi \right)^{-1/4} e^{-\frac{(\pi - \pi_0)^2}{2\Delta_{\pi_C}^2}} \), then decay of coherences is a Gaussian one
\[ \Gamma_{\pi_B,\pi_B'}(t) = e^{-\left( \frac{\tau_{\Delta_{\pi_B}}}{\Delta_{\pi_B}} \right)^2} e^{-\frac{m_B}{m_c} \tau_{\Delta_{\pi_B}}}, \]
where
\[ \tau_{\Delta_{\pi_B}} = \frac{2\hbar m_c}{(\pi_B - \pi_B') \Delta_{\pi_B}}. \]
The above result implies that in the A reference frame the state of B subsystem will decohere in momentum.

The above analysis shows that, as time passes, in the A reference frame subsystems B and C become entangled, and this in turn leads to decoherence of the state of subsystem B (for pure states the only source of decoherence is entanglement). One finds that, in momentum basis, off-diagonal elements of subsystem B density matrix are suppressed. As have been shown in recent developments of decoherence theory, usually the subsystem causing decoherence (in our case the subsystem C) carries some information about decohering system. In order to check whether this is the case we investigate properties of the reduced state of subsystem C, which is
\[ \hat{\rho}_C(t) = \int \pi_B |\psi_0(\pi_B)|^2 |\phi_{\pi_B}(t)\rangle \langle \phi_{\pi_B}(t)|_C. \]
IV. ADVANCED FORMS OF DECOHERENCE IN QUANTUM REFERENCE FRAMES

In the previous section it was demonstrated that if the physical states of reference frames are incorporated into description of the considered physical problem as in Ref. [4], then decoherence and encoding of information into the system causing decoherence are relative phenomena that depend on the state of the observers reference frame.

Let us now discuss implications of the above findings for recently studied advanced forms of decoherence: Quantum Darwinism and spectrum broadcast structures (SBS) [8, 9], which recently have become a vivid theoretical (see e.g. [16 (22)]) and experimental (23,25) area of research. Those frameworks aim to explain some aspects of quantum-to-classical transition problem. Decoherence accounts for the fact that states of complex (macroscopic) objects are rarely found in some state being a superposition of distinct classical states. In addition to that quantum Darwinism and SBS explain how it is possible for multiple observers to independently retrieve the same information about the state of decohering system, and do not disturb it nor results of measurements of other observers. These are properties of classical states. A multipartite quantum state describing the system and some available to observers fragment of the environment can exhibit these properties since, usually, the environment causing decoherence consists of numerous degrees of freedom, and each of them carries some information about decohereing systems hence this information is redundantly encoded in the environment (it may also be the case that a single degree of freedom is not enough to retrieve information about the system but a group of them is, such groups are referred to as macrofractions [9]). Quantum Darwinism as well as SBS provide tools in order to check whether in a considered situation quantum state will exhibit such classical features. In general both approaches are not equivalent as conditions of SBS are stricter than those of quantum Darwinism. A multipartite state that admits SBS form fulfills also conditions of quantum Darwinism, whereas the converse statement is not true [26]. Therefore here we focus on SBS and the conclusions reached here hold also for quantum Darwinism. The state admits spectrum broadcast structure form if it can be written as

$$\hat{\sigma}_{SBS} = \sum_i p_i |i\rangle \otimes \hat{\rho}^i_1 \ldots \otimes \hat{\rho}^i_N$$

(19)

The basis \{|$i\rangle\$\} constitutes the so-called pointer states of the central system to which it decoheres, \(p_i\) are initial pointer probabilities, \(k\) enumerates the environments, and \(\hat{\rho}^k_i\) are some states of the observed parts of the environment which have mutually orthogonal supports for different pointer index \(i\) so that they are perfectly distinguishable with the help of a single measurement.
Let us consider a trivial extension of the model considered in the previous Section to the case, in which subsystems A and C consist of two degrees of freedom and the transformation [5] concerns the joint momentum \( \hat{p}_{A_1 A_2} = \hat{p}_{A_1} \otimes \hat{I} + \hat{I} \otimes \hat{p}_{A_2} \), and similarly for subsystem C. Then if in the A reference frame one part of subsystem C is not observed the joint state will be

\[
\hat{\rho}_{BC_1}(t) = \text{Tr}_{C_2} \left[ \left| \psi^{(A)}(t) \right\rangle \left\langle \psi^{(A)}(t) \right|_{BC_1 C_2} \right]
\]

\[
\int \int d\pi_B d\pi_B' \Gamma_{\pi_B, \pi_B'}(t) \psi_0(\pi_B) \psi_0^\dagger(\pi_B') \left| \pi_B \right\rangle \left\langle \pi_B' \right| \otimes \left| \phi_{\pi_B}(t) \right\rangle \left\langle \phi_{\pi_B}(t) \right|_{C_1},
\]

where \( \Gamma_{\pi_B, \pi_B'}(t) \) is given by Eq. [13]. For the same choice of the initial state for subsystems \( C_1 C_2 \) as in the previous Section one can see that the structure of the above state, after decoherence time, will be in a good approximation a SBS one (for discussion on SBS in continuous variables setting see e.g. [13 27 25]). On the other hand, in C reference frame there will be no decoherence or SBS formation even if one subsystem of A is traced out. Therefore, formation of SBS is relative to the choice of reference frame and SBS and quantum Darwinism frameworks, in order to correctly capture the mentioned aspects of quantum-to-classical transition, need to assume existence of classical reference frames between different observers. As has been discussed in the previous Section, in the limit of large masses and small coherences of the quantum states that define quantum reference frames one recovers classical reference frames. Therefore, for observers using complex objects to constitute their reference frames this requirement is usually satisfied.

V. CONCLUSIONS

In this paper I investigated reduced dynamics of subsystems as seen form different quantum reference frames, using the framework introduced in [7]. Using the generalized Galiliean transform I demonstrated that both decoherence and information encoding about decohering system in the environment are relative to a quantum state constituting the reference frame. I discussed implications of this results for quantum Darwinism and SBS, arguing that, to correctly capture some features of quantum-to-classical transition, those frameworks need to assume existence of classical reference frames.

Note added- While this work was being completed a preprint appeared [29], in which a similar conclusion is reached, namely that SBS are relative to quantum reference frames. However, there the static case is studied, i.e. the transformation between reference frames does not involve momentum. Therefore, the results reported here are complementary to those of Ref. [29].

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Appendix A: Derivation of decoherence factor for a cat state of subsystem B

Here we present a detailed derivation of the decoherence factor in the case of the generalized Galilean transform between reference frames, for a more general choice of subsystems B state.

We choose the initial state to be a product one between reference frames, for a more general choice of subsystems B state. In this way one the state can be rewritten is a form similar to that of Eq. (10) in the main text

\[ |\Psi_0(A)\rangle = \hat{P}_{AC} |\Psi_0(C)\rangle_A = \int d\pi_B d\pi_C \psi_0(\pi_B) |\pi_B\rangle_B \otimes \phi_0(-\pi_C) |\pi_C\rangle_C, \]  

(\text{A1})

where \( \phi_0(p_C) = (\Delta_{\gamma_0})^{-1/4} e^{-\langle p_C - \gamma_0 \rangle^2 / (2\Delta_{\gamma_0}^2)} \) is a Gaussian and \( \psi_0(p_B) = \sqrt{N}^{-1} (\Delta_{\gamma_0})^{-1/4} e^{-\langle p_B - \beta \rangle^2 / (2\Delta_{\gamma_0}^2)} + e^{-\langle m_B - \beta \rangle^2 / (2\Delta_{\gamma_0}^2)} \), is a cat state in momentum space. Using Eq. (9) one finds

\[ |\Psi(A)(t)\rangle_{BC} = e^{-\frac{i}{\hbar}\hat{H}_{BC}t} \int d\pi_B d\pi_C \psi_0 \left( \pi_B - \frac{m_B}{m_C} \pi_C \right) |\pi_B\rangle_B \otimes e^{-\frac{i}{\hbar} \pi_C\hat{e}_{BC}} e^{-\frac{i}{\hbar} \pi_C\hat{e}_{BC}} \phi_0(-\pi_C) |\pi_C\rangle_C \]  

(\text{A2})

The initial wavefunction of B is superposition of Gaussian states, therefore one can write

\[ \psi\left(\pi_B - \frac{m_B}{m_C} \pi_C\right) = \sum_{\beta, \beta'} e^{-\langle \pi_B - \beta \rangle^2 / (2\Delta_{\gamma_0}^2)} e^{\langle \pi_B - \beta' \rangle \frac{m_B}{m_C} \pi_C - \left(\frac{m_B}{m_C} \pi_C\right)^2 / 2}. \]  

(\text{A3})

In this way one the state can be rewritten is a form similar to that of Eq. (10) in the main text

\[ |\Psi(A)(t)\rangle_{BC} = e^{-\frac{i}{\hbar}\hat{H}_{BC}t} \sum_{\beta, \beta'} \int d\psi_{\beta}(\pi_B) |\pi_B\rangle_B \otimes |\phi_{\pi_B, \beta}\rangle_C \]  

(\text{A4})

where

\[ \psi_{\beta}(\pi_B) \equiv e^{-\langle \pi_B - \beta \rangle^2 / (2\Delta_{\gamma_0}^2)} \]  

(\text{A5})

\[ |\phi_{\pi_B, \beta}\rangle \equiv \int d\pi_C e^{-\frac{i}{\hbar} \pi_C\hat{e}_{BC} + \langle \pi_B - \beta \rangle \frac{m_B}{m_C} \pi_C - \left(\frac{m_B}{m_C} \pi_C\right)^2 / 2} \phi_0(-\pi_C) |\pi_C\rangle_C \]  

(\text{A6})
The reduced state of the B subsystem is thus

\[ T_{\pi C} \left[ \langle \psi^{(A)}(t) \rangle \left| \psi^{(A)}(t) \right\rangle_{\text{CB}} \right] = N^{-1} \int d\pi_B d\pi'_B \sum_{\beta \in \beta', \beta' \in \beta'} \sum_{\beta \in \beta'} \tilde{\Gamma}_{\pi_B,\pi'_B,\beta,\beta'}(t) \psi_{\beta}(\pi_B) \psi_{\beta'}(\pi'_B), \]  

(A7)

where

\[ \tilde{\Gamma}_{\pi_B,\pi'_B,\beta,\beta'}(t) = \int d\pi_C e^{-\frac{1}{\hbar} \left( \pi_B - \pi'_B \right)^2 \frac{\Delta^2_{\pi B} \Delta^2_{\pi'_B}}{\gamma_{\rho 0}^2}} \left( \frac{\pi_B + \pi'_B - \beta - \beta'}{m_{\pi C}} \right) \frac{\pi_B - \pi'_B}{m_{\pi C}} e^{-2 \left( \frac{m_{\pi B}}{\gamma_{\rho 0}} \right)^2 \pi_B^2} \left| \varphi_0 \right\rangle \left\langle -\pi_C \right| \gamma_{\rho 0}^2. \]  

(A8)

To simplify expression we will choose masses to be equal \( m_A = m_B = m_C = m \).

It is convenient to split the result of the above integration into two parts, in which one depends on \( \beta, \beta' \) and the other doesn’t. Explicitly one has:

\[ \tilde{\Gamma}_{\pi_B,\pi'_B,\beta,\beta'}(t) = \Gamma_{\pi_B,\pi'_B,\beta,\beta'}(t) \Gamma_{\pi_B,\pi'_B,\beta,\beta'}(t), \]  

(A9)

where

\[ \Gamma_{\pi_B,\pi'_B,\beta,\beta'}(t) = \Gamma_{\pi_B,\pi'_B,\beta,\beta'}(0) e^{\frac{i}{\hbar} \left( (\pi_B - \pi'_B)(\beta + \beta') \right) t}, \]  

(A10)

\[ \Gamma_{\pi_B,\pi'_B,\beta,\beta'}(0) = \sqrt{\left( \frac{\Delta^2_{\pi B}}{\gamma_{\rho 0}^2} + \frac{\Delta^2_{\pi'_B}}{\gamma_{\rho 0}^2} \right)} \left\{ -4 \left( \pi_B + \pi'_B \right) + \frac{\pi_B + \pi'_B}{\gamma_{\rho 0}^2} \right\} \frac{\pi_B - \pi'_B}{\gamma_{\rho 0}^2} e^{\frac{i}{\hbar} \left( (\pi_B - \pi'_B)(\beta + \beta') \right) t}. \]  

(A11)

\[ \Gamma_{\pi_B,\pi'_B,\beta,\beta'}(0) = e^{\frac{1}{\hbar} \left( (\pi_B - \pi'_B)(\beta + \beta') \right) t} \left\{ -4 \left( \pi_B + \pi'_B \right) + \frac{\pi_B + \pi'_B}{\gamma_{\rho 0}^2} \right\} \frac{\pi_B - \pi'_B}{\gamma_{\rho 0}^2} e^{\frac{i}{\hbar} \left( (\pi_B - \pi'_B)(\beta + \beta') \right) t}. \]  

(A12)

\[ \Gamma_{\pi_B,\pi'_B,\beta,\beta'}(0) = e^{\frac{1}{\hbar} \left( (\pi_B - \pi'_B)(\beta + \beta') \right) t} \left\{ -4 \left( \pi_B + \pi'_B \right) + \frac{\pi_B + \pi'_B}{\gamma_{\rho 0}^2} \right\} \frac{\pi_B - \pi'_B}{\gamma_{\rho 0}^2} e^{\frac{i}{\hbar} \left( (\pi_B - \pi'_B)(\beta + \beta') \right) t}. \]  

(A13)

\[ \Gamma_{\pi_B,\pi'_B,\beta,\beta'}(0) = e^{\frac{1}{\hbar} \left( (\pi_B - \pi'_B)(\beta + \beta') \right) t} \left\{ -4 \left( \pi_B + \pi'_B \right) + \frac{\pi_B + \pi'_B}{\gamma_{\rho 0}^2} \right\} \frac{\pi_B - \pi'_B}{\gamma_{\rho 0}^2} e^{\frac{i}{\hbar} \left( (\pi_B - \pi'_B)(\beta + \beta') \right) t}. \]  

(A14)

In this case there are two factors contributing to the loss of coherence. The first is that the initial state is no longer the product one. Therefore, even for \( t = 0 \), one has \( \left| \tilde{\Gamma}_{\pi_B,\pi'_B,\beta,\beta'}(0) \right| < 1 \) since the initial state is an entangled one. In addition to that entanglement between B and C subsystems grows with time as for the simpler example considered in the main text. In fact, the leading term causal dynamical suppression of B coherences is only a slight modification of that presented in the main text as

\[ \left| \tilde{\Gamma}_{\pi_B,\pi'_B,\beta,\beta'}(0) \right| \sim e^{-\frac{1}{2} \left( \frac{\pi_B}{\Delta_{\pi B}} \right)^2}, \]  

(A15)

where (cf. Eq. [15] of the main text)

\[ \Delta_{\pi B} \equiv \frac{2\hbar m_{\pi B}}{(\pi_B - \pi'_B) \Delta_{\pi B} \Delta_{\pi'_B}}. \]  

(A16)

Regarding the information content of the subsystem B, as we again deal with pure initial states, decreasing decoherence factor implies that distinguishability of states of subsystem C corresponding to different momenta \( \pi_B, \pi'_B \) increases with time.