UNIFORM CONVERGENCE TO EQUILIBRIUM FOR COUPLED PARABOLIC PDES AND LINEAR EVOLUTION EQUATIONS

ALEXANDER DOBRICK AND JOCHEN GLÜCK

Abstract. Consider two $L^p$-solutions $u_1$ and $u_2$ of the heat equation over a bounded domain with Neumann boundary conditions; then both $u_1(t)$ and $u_2(t)$ converge, uniformly with respect to the initial value, to an equilibrium state as $t \to \infty$. However, if we couple the equations for $u_1$ and $u_2$ by a matrix-valued potential, the long-time behaviour of the solutions becomes much more involved and, for instance, periodic solutions may occur. In this article, we prove that the solutions still converge to an equilibrium if the matrix-valued potential satisfies an appropriate geometric condition. Our proof relies on a general convergence theorem for operator semigroups on $L^p$-spaces.

We also study a similar equation in the space of bounded continuous functions over the whole space $\mathbb{R}^d$. In this case, the solution semigroup lacks strong continuity. This motivates us to show in the second part of the paper that our semigroup convergence theorem holds in fact without any time regularity assumptions on the semigroup. The proof of this result is based on a variation of the concept of the semigroup at infinity recently introduced by M. Haase and the second-named author.

1. Introduction

A system of coupled heat equations. On a bounded domain $\Omega \subseteq \mathbb{R}^d$, consider the system of coupled heat equations

$$\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} + V \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

subject to Neumann boundary conditions and appropriate initial conditions; here, $V : \Omega \to \mathbb{R}^{2 \times 2}$ is a matrix-valued bounded measurable function. Let us consider solutions $u_{1}(t), u_{2}(t) \in L^{p}(\Omega)$ ($1 \leq p \leq \infty$) and discuss their long-term behaviour as $t \to \infty$. Of course the potential $V$ can cause the solutions to tend to $\infty$ or to $0$ for large times. But even if this does not happen, $V$ can have very strange effects on the solutions; this is illustrated by the following simple example.

Example 1.1. Let $p = 2$ and endow the space $L^2(\Omega)^2$ with the norm $\| \cdot \|_2$ given by

$$\|(v_1, v_2)\|_2^2 := \|v_1\|_{L^2(\Omega)}^2 + \|v_2\|_{L^2(\Omega)}^2.$$

Moreover, let $V(x) = V := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for all $x \in \Omega$. Then it is easy to see that the solutions to (*) satisfy an energy estimate which implies that their norm is nonincreasing (or, in a more operator theoretic language, that the operator $\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} +$...
$V$ with Neumann boundary conditions is dissipative on $L^2(\Omega)$). However, if $\mathbb{1}$ denotes the constant function on $\Omega$ with value 1, then

$$e^{tV} \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix}$$

is a solution of ($\ast$) which is periodic and, thus, does not converge as $t \to \infty$.

**Spectral considerations.** From a spectral theoretic point of view, the essence of the above example is quite simple: the matrix $V$ has two eigenvalues on the imaginary axis (namely $i$ and $-i$), so the matrix exponential function $e^{tV}$ rotates both components of the fixed vector $\begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix}^T$ of the vector-valued Laplace operator $\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$. One might thus suspect that the solutions to ($\ast$) converge to an equilibrium if the spectrum of $V$ is contained in the left half plane and intersects the imaginary axis only in 0. Under appropriate assumptions this is indeed true and can even be shown for non-constant $V$ by the following kind of reasoning:

If $A$ is a dissipative self-adjoint operator on a Hilbert-space and $C$ is a bounded dissipative (but not necessarily self-adjoint) operator on the same space with no eigenvalues in $i\mathbb{R} \setminus \{0\}$, then $A + C$ does not have any eigenvalues in $i\mathbb{R} \setminus \{0\}$, either. This is a consequence of the spectral theorem for self-adjoint operators; see Proposition 2.9 for details. By applying this observation to the operator $\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} + V$, we obtain criteria for the solutions of ($\ast$) to converge as time tends to infinity; see Theorem 2.11 and Corollary 2.12. However, this approach has two serious limitations:

(i) It only works since the Laplace operator is self-adjoint; if we replace the Laplace operators in the matrix $\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} + V$ with more general elliptic operators with non-symmetric coefficients, an analysis based on the spectral theorem breaks down and we can no longer use Proposition 2.9.

(ii) The approach requires the potential $V$ to be $\ell^2$-dissipative. If $V$ is instead, for instance, $\ell^1$-dissipative, one still can show that $\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} + V$ generates a bounded semigroup on an appropriate $L^1$-space and thus, a fortiori, on the entire $L^p$-scale by an ultra-contractivity argument; however, convergence of this semigroup does no longer follow from arguments relying on the spectral theorem.

Consequently, if $\Delta$ is replaced with non-symmetric elliptic operators or $V$ is no longer $\ell^2$-dissipative, another approach is needed to obtain convergence of the coupled heat equation. Such an approach is described next.

**Convergence of contractive semigroups on $L^p$ and on spaces of continuous functions.** The matrix $V$ in Example 1.1 has two properties that are essential for the example: (i) the matrix exponentials $e^{tV}$ ($t \geq 0$) leave the $\ell^2$-unit ball in $\mathbb{R}^2$ invariant (which causes the solutions of ($\ast$) to be bounded) and (ii) the spectrum of $V$ contains non-zero numbers on the imaginary axis. Now it is interesting to observe that if $p \neq 2$ and if $e^{tV}$ leaves the $p$-unit ball instead of the $2$-unit ball in $\mathbb{R}^2$ invariant, then $V$ cannot have non-zero eigenvalues on $i\mathbb{R}$. This follows easily from geometric considerations, but an entire theory about the long-term behaviour of contractive operator semigroups on $L^p$-spaces for $p \neq 2$ can be built on this basic idea. This was done in [15] and [14, Part I], based on earlier work of Lyubich [21].

In this article, we employ the results of [15] to prove an operator norm convergence theorem for contractive $C_0$-semigroups on $L^p$-space ($p \neq 2$), and we apply
this theorem to study coupled heat equations of the type (\(\ast\)), where the Laplace operators are replaced with general non-symmetric elliptic operators.

We call a family \((T_s)_{s \in [0, \infty)}\) of bounded linear operators on a Banach space an \textit{operator semigroup} if \(T_0\) is the identity operator and \(T_{s+t} = T_s T_t\) for all \(s, t \in [0, \infty)\); in general, we require no regularity with respect to the time parameter \(s\). Our analysis of the coupled heat equation with non-symmetric coefficients (in Subsection 2.5) is based on the following theorem, which is one of the theoretical main results of our paper.

\begin{theorem}
Let \((T_s)_{s \in [0, \infty)}\) be an operator semigroup on \(L^p(\Omega, \mu; \mathbb{C})\), where \((\Omega, \mu)\) is an arbitrary measure space and \(p \in [1, \infty] \setminus \{2\}\). If each \(T_s\) maps real-valued functions to real-valued functions and has norm \(\|T_s\| \leq 1\), then the following assertions are equivalent:

(i) \(T_s\) converges with respect to the operator norm to a finite rank projection as \(s \to \infty\).
(ii) There exists a time \(s_0 \in [0, \infty)\) such that \(T_{s_0}\) is quasi-compact.
\end{theorem}

Here, an operator \(T\) is called \textit{quasi-compact} if some power of \(T\) is strictly closer than 1 (with respect to the operator norm) to a compact operator. We will prove this theorem in Section 6 as a consequence of the more general Corollary 6.6.

In order to apply Theorem 1.2 to systems of equations of the type (\(\ast\)), we have to ensure that the state space \(L^p(\Omega)^2\) is isometrically isomorphic to a (scalar-valued) \(L^p\)-space. To this end, we endow it with the norm \(\| \cdot \|_p\) given by \(\|(u_1, u_2)\|_p^p = \|u_1\|_p^p + \|u_2\|_p^p\) which renders \(L^p(\Omega)^2\) isometrically isomorphic to the \(L^p\)-space over the disjoint union \(\Omega \cup \Omega\). Of course, this is equivalent to considering \(L^p(\Omega)^2\) as a \textit{vector-valued} space \(L^p(\Omega; \mathbb{R}^2)\), where \(\mathbb{R}^2\) is now endowed with the \(p\)-norm rather than with the Euclidean norm. This already suggests that, in order to apply Theorem 1.2 we need the matrix \(V(x)\) to be \(p\)-\textit{dissipative} for each \(x \in \Omega\) as to ensure that the solution semigroup of (\(\ast\)) is contractive. A detailed application of Theorem 1.2 to coupled system of heat equations is presented in Section 2.

A few more remarks on Theorem 1.2 in order: We point out that the theorem is geometric in essence: it is based on the fact that if \(F\) is a two-dimensional subspace of an \(L^p\)-space (\(p \neq 2\)), then \(F\) is either not isometric to a Hilbert space, or \(F\) is not the range of a contractive projection; see [15] Section 3.1) for details about spaces with this property. Moreover, spaces of continuous functions have the same property (see [14] Examples 1.2.7), so Theorem 1.2 also remains true on such spaces. Let us state this explicitly for later reference.

\begin{theorem}
Let \(E = C_b(\ell; \mathbb{C})\) or \(E = C_0(\ell; \mathbb{C})\) for a locally compact Hausdorff space \(\ell\), or more generally, let \(E\) be a complex Banach lattice which is an AM-space. Let \((T_s)_{s \in [0, \infty)}\) be an operator semigroup on \(E\). If each \(T_s\) maps real-valued functions to real-valued functions and has norm \(\|T_s\| \leq 1\), then the following assertions are equivalent:

(i) \(T_s\) converges with respect to the operator norm to a finite rank projection as \(s \to \infty\).
(ii) There exists a time \(s_0 \in [0, \infty)\) such that \(T_{s_0}\) is quasi-compact.
\end{theorem}

Here, \(C_b(\ell; \mathbb{C})\) denotes the space of all bounded and continuous complex-valued functions on \(\ell\), and \(C_0(\ell; \mathbb{C}) \subseteq C_b(\ell; \mathbb{C})\) describes the space of all continuous complex-valued functions on \(\ell\) that vanish at infinity. For a definition of when a real Banach lattice is called an AM-space we refer for instance to [26] Section II.7; a complex Banach lattice is called an AM-space if its real part is an AM-space, and if \(E\) is a complex AM-space in Theorem 1.3, then the condition “each operator \(T_s\) maps real-valued functions to real-valued functions” has of course to be understood in the more abstract sense that each operator \(T_s\) leaves the real part of \(E\) invariant.
Theorem 1.3 is also a consequence of a more general result in Corollary 6.6, so we again refer to Section 6 for the proof.

It might not come as a surprise that the geometric condition \( \|T_s\| \leq 1 \) in Theorems 1.2 and 1.3 can be replaced with another condition: if the semigroup is merely bounded (instead of contractive), but leaves invariant the positive cone of the underlying space, then the equivalence of Theorem 1.2 (respectively, of Theorem 1.3) remains valid (even if \( p = 2 \)). This was proved by Lotz [20, Theorem 4] in the setting of general Banach lattices and fits into a long list of convergence results for positive semigroups. We will see below (Theorem 6.2) that Lotz’ result (and in fact, a generalization of it) can also be derived by our methods.

**Semigroups without time regularity.** If the operator \( T_{s_0} \) is even compact and \((T_s)_{s \in [0, \infty)} \) is a \( C_0 \)-semigroup, the assertion of Theorem 1.2 has already been proved in [15, Corollary 3.8] (and almost the same argument can be used to also cover the case where \( T_{s_0} \) is only quasi-compact, see [14, Corollary 2.2.5(b)]). The major progress made in Theorem 1.2 compared to [15, Corollary 3.8] is that we do no longer need any time regularity assumption on the semigroup. This severely restricts the toolbox available for the proof (for instance, we cannot use generators, resolvents or Laplace transform techniques) but we are rewarded by two advantages of our general approach:

- **Applicability:** Semigroups that occur in concrete applications always satisfy some kind of time regularity (otherwise the solution of the corresponding evolution equation would not be related to the initial value in any topological sense). However, the types of time regularity that we encounter can be quite disparate and vary, for instance, from strong continuity on \([0, \infty) \) over strong continuity on \((0, \infty) \) (see e.g. [3, 4] for two applications) to such concepts as bi-continuity [18] and continuity on norming dual pairs [19]. Instead of developing a separate convergence theory for each of these cases, we prefer the all-in approach to consider semigroups without any time regularity assumptions. For instance, our analysis of coupled heat equations on \( L^p \) of a bounded domain (see Subsection 2.5) requires only the theory of \( C_0 \)-semigroups for \( p \in [1, \infty) \); for the case \( p = \infty \), though, we have to deal with semigroups that are no longer continuous. Similarly, our study of coupled heat equations on the whole space \( \mathbb{R}^d \) and with unbounded coefficients requires convergence results for semigroups which are not \( C_0 \).

- **Theoretic considerations:** Given any convergence theorem for a certain class of semigroups \((T_t)_{t \in [0, \infty)}\) one might wonder what is so special about the time domain \([0, \infty)\) that it enforces convergence under seemingly weak conditions; for instance, it is very easy to see that Theorem 1.2 does no longer remain true for time-discrete semigroups \((T^n)_{n \in \mathbb{N}_0}\).

If convergence theorems remains true without any time regularity, this demonstrates that not the topological properties of the time domain \([0, \infty)\) but rather its algebraic properties are essential to obtain semigroup convergence in many situations. This has already been a central aspect in the papers [12, 16] which focus on strong convergence, and it will also appear in Sections 5 and 6 of this paper.

Motivated by the above theoretic considerations, and by the results from [12] and [16], we will also derive a version of Theorem 1.2 for representation of more general commutative semigroups than just \(([0, \infty), +)\); see Corollary 6.3 for details.

**Organisation of the article.** In Section 2 we consider coupled systems of parabolic equations on a bounded domain in \( \mathbb{R}^d \) and give sufficient criteria for them to converge as time tends to infinity. In Section 3 we will then turn to equations on the
whole space \( \mathbb{R}^d \), with possibly unbounded coefficients. In Sections 3 and 4 we develop a general theory for the long-term behaviour (with respect to the operator norm) of bounded representations of commutative semigroups; Theorems 6.2 and 6.3 follow, along with several similar results, from this general theory.

**Notation and Terminology.** All Banach spaces in this paper can be either real or complex, unless otherwise specified. To clarify whether the element of certain function spaces are assumed to be real- or complex-valued we use notation such as \( L^p(\Omega, \mu; \mathbb{R}) \) and \( L^p(\Omega, \mu; \mathbb{C}) \), etc.

Let \( E, F \) be Banach spaces (over the same scalar field). We endow the space \( \mathcal{L}(E; F) \) of bounded linear operators from \( E \) to \( F \) with the operator norm topology throughout; moreover, we use the abbreviation \( \mathcal{L}(E) := \mathcal{L}(E; E) \). The dual Banach space of \( E \) will be denoted by \( E' \). If the underlying scalar field is complex, the spectrum of a linear operator \( A : E \supseteq D(A) \to E \) will be denoted by \( \sigma(A) \); for \( \lambda \in \mathbb{C} \setminus \sigma(A) \), the resolvent of \( A \) at \( \lambda \) is denoted by \( \mathcal{R}(\lambda, A) := (\lambda - A)^{-1} \). If the underlying scalar field of \( E \) is real, the spectrum of an operator \( A \) is defined as the spectrum of the canonical extension of \( A \) to any complexification of \( E \).

Basic terminology for semigroup representations is introduced at the beginning of Section 4.

## 2. Coupled parabolic equations on bounded domains

### 2.1. Setting

Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^d \) be a bounded domain which has the extension property in the sense that every Sobolev function in \( H^1(\Omega; \mathbb{C}) \) is the restriction of a Sobolev function in \( H^1(\mathbb{R}^d; \mathbb{C}) \). This is the case, e.g., if \( \Omega \) has Lipschitz boundary (see Section 7.3.6).

We fix an integer \( N \geq 1 \) (which will denote the number of coupled equations on \( \Omega \)) and almost all \( x \in \Omega \), the uniform ellipticity condition

\[
\text{Re}(\xi^T A_k(x) \xi) \geq \nu \| \xi \|_2^2
\]

holds for all \( \xi \in \mathbb{C}^d \). We will study the long-term behaviour of the solutions to the coupled parabolic equation that is formally given by

\[
\frac{d}{dt} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} \text{div}(A_1 \nabla u_1) \\ \vdots \\ \text{div}(A_N \nabla u_N) \end{pmatrix} + V \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \quad (\text{CP}_{\text{bounded}})
\]

and subject to Neumann boundary conditions. Due to the weak regularity assumptions on the coefficients and on the boundary of \( \Omega \), we use form methods to give precise meaning to the elliptic operators \( u \mapsto \text{div}(A_k \nabla u) \); for each \( k \in \{1, \ldots, N\} \) we define a bilinear form

\[
a_k : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}, \quad a_k(u, v) = \int_{\Omega} \nabla u^T A_k \nabla v \, dx.
\]

This form induces a linear operator \( -A_k : L^2(\Omega) \supseteq H^1(\Omega) \to L^2(\Omega), \) and \( A_k \) is interpreted as a realization of the differential operator \( u \mapsto \text{div}(A_k \nabla u) \) with Neumann boundary conditions. For an general overview of form methods in the context of heat equations we refer the reader to [25].

Each operator \( A_k \) generates a positive (in the sense of Banach lattices) and contractive \( C_0 \)-semigroup \( (e^{tA_k})_{t \in [0, \infty)} \) on \( L^2(\Omega) \). Moreover, this semigroup and its dual have the constant function \( 1 \) as a fixed point, so it follows from interpolation theory that it also induces a positive and contractive \( C_0 \)-semigroup on \( L^p(\Omega) \) for
each $p \in [1, \infty)$; we denote the corresponding generator by $A_{k,p}$ (and thus, $A_{k,2} = A_k$).

The coupled parabolic equation $(\text{CP}_{\text{bounded}})$ can now precisely be stated as the abstract Cauchy problem

$$
\frac{d}{dt} u = B_p u + Vu
$$
on the Banach space $L^p(\Omega; \mathbb{C}^N)$, where $p \in [1, \infty)$ and

$$B_p = \left( \begin{array}{ccc}
A_{1,p} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{N,p}
\end{array} \right).$$

In the following, we endow $L^p(\Omega; \mathbb{C}^N)$, $p \in [1, \infty]$, with the norm $\| \cdot \|_p$ given by

$$\| u \|_p^p = \int_{\Omega} \| u(x) \|_p^p \, dx = \sum_{k=1}^{N} \| u_k \|_{p,(\Omega)}^p \quad \text{for } p \in [1, \infty),$$

and

$$\| u \|_{\infty} = \max \{ \| u_k \|_{\infty} : k \in \{1, \ldots, N\} \}$$

for $u = (u_1, \ldots, u_N)^T \in L^p(\Omega; \mathbb{C}^N)$. This has the following simple but important consequence.

**Remark 2.1.** Let $p \in [1, \infty]$. The norm defined in $(2.2)$ is of course equivalent to the norm that we would obtain by endowing $\mathbb{C}^N$ with the Euclidean norm and then endowing $L^p(\Omega; \mathbb{C}^N)$ with the vector-valued $p$-norm. However, the main advantage of the norm $\| \cdot \|_p$ defined in $(2.2)$ is that it renders $L^p(\Omega; \mathbb{C}^N)$ isometrically lattice isomorphic to the $L^p$-space of scalar-valued functions over $N$ disjoint copies of $\Omega$ – i.e. we can treat $L^p(\Omega; \mathbb{C}^N)$ as a scalar-valued $L^p$-space.

### 2.2. The coupled heat semigroup on the $L^p$-scale.

In what follows, we will use the symbol $V$ both to denote the function $V : \Omega \to \mathbb{R}^{N \times N}$ introduced in the previous subsection and the operator $L^p(\Omega; \mathbb{C}^N) \to L^p(\Omega; \mathbb{C}^N)$ given by multiplication with this function (for any $p \in [1, \infty]$).

Since $V$ is a bounded operator, it follows from standard perturbation theory that $B_p + V$ generates a $C_0$-semigroup $(e^{t(B_p+V)})_{t \in [0, \infty)}$ on $L^p(\Omega; \mathbb{C}^N)$ for each $p \in [1, \infty)$. In this subsection we briefly discuss how those semigroups act on the $L^p$-scale. We will see, by means of an ultracontractivity argument, that most of the properties which are relevant for us do not depend on the choice of $p$. The arguments in this subsection are fairly standard, but there are a few subtleties – in particular since we also want to consider the semigroup on $L^\infty(\Omega; \mathbb{C}^N)$ – so we prefer to state all the relevant properties in detail.

The semigroups $(e^{t(B_p+V)})_{t \in [0, \infty)}$ are consistent on the $L^p$-scale. This follows from a perturbation argument (for instance, by means of Trotter’s product formula or, if one prefers, by the Dyson–Phillips series expansion) since the semigroups generated by $B_p$ are consistent.

Moreover, the semigroups $(e^{t(B_p+V)})_{t \in [0, \infty)}$ leave $L^\infty(\Omega; \mathbb{C}^N)$ invariant as the following proposition shows. For a proper reading of the proposition, note that the realizations of the multiplication operator $V$ as a bounded operator on $L^p(\Omega; \mathbb{C}^N)$ are consistent for $p \in [1, \infty]$; moreover, the exponential operators $e^{tV}$ are, for every $t \in [0, \infty)$, also consistent on the $L^p(\Omega; \mathbb{C}^N)$-scale; in other words, for $1 \leq p \leq q \leq \infty$, it does not matter whether we consider the exponential $e^{tV}$ on $L^p(\Omega; \mathbb{C}^N)$ first and then restrict it to $L^q(\Omega; \mathbb{C}^N)$ or whether we consider it on $L^q(\Omega; \mathbb{C}^N)$ in the first place.
Proposition 2.2. There exists a number $\omega \in \mathbb{R}$ such that $\|e^{tV}\|_{\infty \to \infty} \leq e^{\omega t}$ for all $t \in [0, \infty)$. For such $\omega$, and for each $t \in [0, \infty)$, the operator $e^{(c_2 + V)}$ on $L^p(\Omega; C^N)$ (for any $p \in [1, \infty)$) leaves $L^\infty(\Omega; C^N)$ invariant and satisfies $\|e^{(c_2 + V)}\|_{\infty \to \infty} \leq e^{\omega t}$.

Proof. The existence of $\omega$ follows from the fact that $\|e^{tV}\|_{\infty \to \infty} \leq e^{\|V\|_{\infty \to \infty}}$ for all $t \in [0, \infty)$.

Now, fix such an $\omega$ and $t \in [0, \infty)$. By Trotter’s product formula (see for instance [7 Corollary III. V.8]) we have $e^{i(tB_2 + V)}f = \lim_{n \to \infty}(e^{\frac{t}{n}B_2}e^{\frac{t}{n}V})nf$ with respect to the $L^p$-norm for each $f \in L^p$. The semigroup generated by $B_2$ is $L^\infty$-contractive; thus, if $f$ is an element of the unit ball of $L^\infty$, then $(e^{\frac{t}{n}B_2}e^{\frac{t}{n}V})nf$ is an element of $e^{\omega t}$ times this unit ball for each $n \in \mathbb{N}$ and hence, so is the limit as $n \to \infty$ (since the $L^\infty$-unit ball is closed in $L^p$).

Of course, the restriction of the operator $e^{i(tB_2 + V)}$ to $L^\infty(\Omega; C^N)$ is the same operator for all $p \in [1, \infty)$ (since our semigroups act consistently on the $L^p$-scale). From now on we denote, be abuse of notation, the restriction of $e^{i(tB_2 + V)}$ to $L^\infty(\Omega; C^N)$ by $e^{i(tB_2 + V)}$. Note that we use this purely as a notation; we do not define an operator $B_2$, nor do we make any assertions about such an operator. Note that $(e^{i(tB_2 + V)})_{t \in [0, \infty]}$ is, of course, an operator semigroup, but it is certainly not a $C_0$-semigroup, in general. However, it follows from Proposition 2.3 below that this semigroup is strongly continuous (and in fact, even continuous with respect to the operator norm) on the open time interval $(0, \infty)$.

Next we show that the semigroup operators $e^{i(tB_2 + V)}$ do not only leave $L^\infty(\Omega; C^N)$ invariant, but that they even map $L^p(\Omega; C^N)$ boundedly into $L^\infty(\Omega; C^N)$.

Proposition 2.3. Let $p \in [1, \infty]$.

(a) For each $t \in (0, \infty)$ the operator $e^{i(tB_2 + V)}$ maps $L^p(\Omega; C^N)$ boundedly into $L^\infty(\Omega; C^N)$, i.e. it is a bounded operator from $L^p(\Omega; C^N)$ to $L^\infty(\Omega; C^N)$.

(b) For each $t \in (0, \infty)$ the operator $e^{i(tB_2 + V)}$ is compact on $L^p(\Omega; C^N)$.

Proof. We consider the generator $B_2 + V$ on $L^2(\Omega; C^N)$. The operator $-B_2$ is associated with a form $b$ that is defined on $H^1(\Omega; C^N) \simeq (H^1(\Omega; C))^N$ and given by

$$b(u, v) = \sum_{k=1}^N a_k(u_k, v_k) \quad \text{for all } u, v \in H^1(\Omega; C^N);$$

hence, the operator $-(B_2 + V)$ is associated with the form $c$ on the same domain given by $c(u, v) = b(u, v) - \int_\Omega (Vu)^2 dx$.

As $\Omega$ has the extension property, $H^1(\Omega; C)$ embeds continuously into $L^q(\Omega; C)$ for some $q > 2$, and hence, the form domain $H^1(\Omega; C^N)$ embeds continuously into $L^q(\Omega; C^N)$. It thus follows from an ultracontractivity argument (see the theorem in [2 Section 7.3.2] for details) that, for each $p \in [1, \infty)$, $e^{i(tB_2 + V)}$ maps $L^p(\Omega; C^N)$ into $L^\infty(\Omega; C^N)$. Boundedness of this mapping is also a consequence of the same ultracontractivity argument (alternatively, it also follows from the closed graph theorem).

(b) For $p \in [1, \infty)$ this follows from (a) by Dunford–Pettis theorem, see for instance [13 Theorem 7.1] for details. For $p = \infty$, we use that $e^{i(tB_\infty + V)}$ factors as

$$L^\infty(\Omega; C^N) \xrightarrow{id} L^2(\Omega; C^N) \xrightarrow{e^{i(tB_2 + V)}} L^2(\Omega; C^N) \xrightarrow{e^{i(tB_2 + V)}} L^\infty(\Omega; C^N),$$

and so the assertion follows from the case $p = 2$. \qed
As two consequences of the above proposition, we obtain that boundedness and operator norm convergence of the semigroup does not depend on the choice of \( p \).

**Corollary 2.4.** The following assertions are equivalent:

(i) There exists \( p \in [1, \infty] \) such that the semigroup \( (e^{t(B_q+V)})_{t \in [0, \infty)} \) is bounded on \( L^p \).

(ii) For every \( p \in [1, \infty] \) the semigroup \( (e^{t(B_q+V)})_{t \in [0, \infty)} \) is bounded on \( L^p \).

Here, we used the abbreviation \( L^p := L^p(\Omega; \mathbb{C}^N) \).

**Proof.** Obviously, (ii) implies (i), so assume conversely that (i) holds and consider some \( q \in [1, \infty] \) with \( q \neq p \). For each \( t \geq 2 \), the operator \( e^{t(B_q+V)} \) factors as

\[
L^q \xrightarrow{e^{B_q}} L^\infty \xrightarrow{id} L^p \xrightarrow{e^{t-2}(B_q+V)} L^p \xrightarrow{e^{B_q}} L^\infty \xrightarrow{id} L^q.
\]

Therefore, \( \sup_{t \in [2, \infty]} \|e^{t(B_q+V)}\| < \infty \). On the other hand, we also observe that

\[
\sup_{t \in [0, 2]} \|e^{t(B_q+V)}\| \leq \infty;
\]

for \( q \in [1, \infty) \) this follows from the \( C_0 \)-property, and for \( q = \infty \) this follows from Proposition 2.2. This proves the assertion.

**Corollary 2.5.** The following assertions are equivalent:

(i) There exists \( p \in [1, \infty] \) such that \( e^{t(B_q+V)} \) converges with respect to the operator norm on \( L^p \) as \( t \to \infty \).

(ii) For every \( p \in [1, \infty] \) the operator \( e^{t(B_q+V)} \) converges with respect to the operator norm on \( L^p \) as \( t \to \infty \).

(iii) For every \( p \in [1, \infty] \) the operator \( e^{t(B_q+V)} \) converges with respect to the operator norm in \( \mathcal{L}(L^p; L^\infty) \) as \( t \to \infty \).

Here, we used again the abbreviation \( L^r := L^r(\Omega; \mathbb{C}^N) \) for \( r \in [1, \infty] \).

**Proof.** Obviously, (iii) implies (ii) and (ii) implies (i). Now assume that (i) holds; in order to show (iii), consider any number \( q \in [1, \infty] \). For \( t \geq 2 \) the operator \( e^{t(B_q+V)} : L^q \to L^\infty \) factors as

\[
L^q \xrightarrow{e^{B_q}} L^\infty \xrightarrow{id} L^p \xrightarrow{e^{t-2}(B_q+V)} L^p \xrightarrow{e^{B_q}} L^\infty.
\]

This proves that \( e^{t(B_q+V)} \) converges in \( \mathcal{L}(L^q; L^\infty) \) (with respect to the operator norm) as \( t \to \infty \).

**Definition 2.6.** We say that the solutions to the coupled Cauchy problem (2.1) converge uniformly as \( t \to \infty \) if one, and thus all, of the equivalent assertions of Corollary 2.5 are satisfied.

The purpose of the Subsections 2.4 and 2.5 below is to provide sufficient criteria for the solutions to (2.1) to converge uniformly, in the sense of Definition 2.6 as time tends to infinity.
2.3. **Intermezzo: Dissipativity.** In view of Corollary 2.4 boundedness of the solution semigroup to (2.1) for some \( p \) implies boundedness on the entire \( L^p \)-scale. This easiest way to obtain boundedness for some \( p \in [1, \infty) \) is to assume that the multiplication operator \( V \) is dissipative on \( L^p(\Omega; \mathbb{C}^N) \). In this subsection this is discussed in a bit more detail; for a general treatment of dissipative operators we refer for instance to [7, Section II.3.b].

**Proposition 2.7.** Let \( p \in [1, \infty) \). If \( V \) is dissipative on \( L^p(\Omega; \mathbb{C}^N) \), then the semigroup \((e^{t(B_p + V)})_{t \in [0, \infty)} \) is contractive on \( L^p(\Omega; \mathbb{C}^N) \).

**Proof.** First assume that \( p \in [1, \infty) \). Then \( B_p \) generates a contractive \( C_0 \)-semigroup on \( L^p(\Omega; \mathbb{C}^N) \), and it follows from the characterisation of dissipativity in [7, Proposition II.3.23] that \( B_p + V \) is dissipative, too.

Now let \( p = \infty \). Then \( \|e^{tV}\|_{\infty \to \infty} \leq 1 \) for each \( t \in [0, \infty) \), so the assertion follows from Proposition 2.2. \( \square \)

Proposition 2.7 indicates that we should be interested in dissipativity of the multiplication operator \( V \) on \( L^p(\Omega; \mathbb{C}^N) \); this property can be characterised in terms of the matrices \( V(x) \):

**Proposition 2.8.** For each \( p \in [1, \infty) \) the following assertions are equivalent:

(i) The multiplication operator \( V \) on \( L^p(\Omega; \mathbb{C}^N) \) is dissipative.

(ii) For almost all \( x \in \Omega \), the matrix \( V(x) \) is dissipative with respect to the \( \ell^p \)-norm on \( \mathbb{C}^N \).

(iii) For almost all \( x \in \Omega \), the matrix \( V(x) \) is dissipative with respect to the \( \ell^p \)-norm on \( \mathbb{R}^N \).

**Proof.** The equivalence of (i) and (ii) is an immediate consequence of our choice of the norm on \( L^p(\Omega; \mathbb{C}^N) \) (see formula (2.2)), and the implication from (ii) to (iii) is obvious.

To show that (iii) implies (ii) we note that, for every matrix \( M \in \mathbb{R}^{N \times N} \), its operator norm induced by the \( p \)-norm on \( \mathbb{R}^N \) coincides with its operator norm induced by the \( p \)-norm on \( \mathbb{C}^N \). Indeed, for \( p \in [1, \infty) \) this can e.g. be found in [9, Proposition 2.1.1], and for \( p = \infty \) this follows from the fact that \( \|\xi\|_{\infty} = \sup_{\|\xi\|_{\infty} = 1} \|\xi\|_{\infty} = \sup_{\|\xi\|_{\infty} = 1} \|\xi\| \leq \|\xi\|_{\infty} \) for each \( \xi \in \mathbb{C}^N \). Hence, if \( e^{tV(x)} \) is contractive on \( \mathbb{R}^N \), then it is also contractive on \( \mathbb{C}^N \); so (iii) indeed implies (ii). \( \square \)

In view of Proposition 2.8 it is worthwhile to note that \( p \)-dissipativity of matrices in \( \mathbb{R}^{N \times N} \) can be characterised quite explicitly; for the convenience of the reader, we recall this in detail in Proposition A.1 in the Appendix.

2.4. **Convergence in the symmetric case.** In this subsection we characterise the convergence of the solutions to the coupled heat equation (2.1) in the case that the coefficients of the elliptic operators are symmetric and the matrices \( V(x) \) are dissipative with respect to the \( \ell^2 \)-norm on \( \mathbb{R}^N \).

We start with a general observation on Hilbert spaces. Recall that an operator \( A : H \ni u \in \text{dom} \ A \rightarrow H \) on a complex Hilbert space \( H \) is dissipative if and only if \( \text{Re} \langle Au, u \rangle \leq 0 \) for all \( u \in \text{dom} \ A \).

**Proposition 2.9.** Let \( A : H \ni u \in \text{dom} \ A \rightarrow H \) be a self-adjoint, dissipative operator and let \( C \) be a bounded, dissipative operator on a complex Hilbert space \( H \). If

\[
(A + C)u = i\beta u
\]

for some \( i\beta \in \mathbb{R} \) and \( u \in \text{dom} \ A \), then \( Au = 0 \) and \( Cu = i\beta u \). In particular, we always have

\[
\sigma_{\text{pnt}}(A + C) \cap i\mathbb{R} \subseteq \sigma_{\text{pnt}}(C) \cap i\mathbb{R}.
\]
Proof. Let \(i\beta \in \sigma(A)\) be as in the statement of the proposition. Then 
\[
\langle (A + C)u, u \rangle = i\beta \|u\|^2
\]
and thus
\[
0 = \text{Re}\langle (A + C)u, u \rangle = \langle Au, u \rangle + \text{Re}\langle Cu, u \rangle.
\]
As \(A\) and \(C\) are dissipative on \(H\), it follows that both terms \(\langle Au, u \rangle\) and \(\text{Re}\langle Cu, u \rangle\) are not larger than 0 and thus, in fact, equal to 0. Since \(A\) is self-adjoint and \(\sigma(A) \subseteq (-\infty, 0]\), the equality \(\langle Au, u \rangle = 0\) implies, due to the spectral theorem, that \(Au = 0\). Consequently, \(Cu = i\beta u\).

Now we return to the setting of the Subsection \(2.7\). Suppose, additionally to the assumptions made there, that the functions \(A_1, \ldots, A_N : \Omega \to \mathbb{R}^{d \times d}\) have values in the symmetric matrices a.e. on \(\Omega\). Then the associated forms \(a_k\) are symmetric, and hence, the operators \(A_k\) are self-adjoint on \(L^2(\Omega)\). Further, suppose that \(V : \Omega \to \mathbb{R}^{N \times N}\) is \(L^2\)-dissipative. In this case, we obtain the following spectral result.

Proposition 2.10. Assume that, for almost all \(x \in \Omega\), the matrices \(A_1(x), \ldots, A_N(x)\) are symmetric and the matrix \(V(x)\) is dissipative with respect to the \(\ell^2\)-norm on \(\mathbb{R}^N\). For each \(i\beta \in i\mathbb{R}\) the following two assertions are equivalent:

(i) \(i\beta \in \sigma_{\text{pnt}}(B_2 + V)\).

(ii) There exists a measurable subset \(\tilde{\Omega} \subseteq \Omega\) which has the same measure as \(\Omega\) such that

\[
\bigcap_{x \in \tilde{\Omega}} \ker(i\beta - V(x)) \neq \{0\}.
\]

In this case, each component function of every eigenvector \(u \in \ker(i\beta - (B_2 + V))\) is constant on \(\Omega\) (more precisely: it has a constant representative).

Proof. (i) \(\Rightarrow\) (ii): Let \(u\) be an eigenvector of \(B_2 + V\) to the eigenvalue \(i\beta\). Then Proposition 2.9 shows that \(B_2u = 0\), i.e. \(A_ku_k = 0\) for each \(k = 1, \ldots, N\). Since the semigroups \((e_tA_k)\) are all positive and irreducible on \(L^2(\Omega)\) and their generators \(A_k\) have compact resolvent, it follows that \(\dim(\ker A_k) = 1\), i.e., \(u_k\) is constant for all \(k \in \{1, \ldots, N\}\). Hence, there is a non-zero vector \(z \in \mathbb{C}^N\) such that \(u(x) = z\) for almost all \(x \in \tilde{\Omega}\) and thus,

\[
V(x)z = V(x)u(x) = i\beta u(x) = i\beta z
\]

for almost all \(x \in \Omega\). Therefore, there exists a measurable set \(\tilde{\Omega} \subseteq \Omega\) of full measure such that \(z \in \bigcap_{x \in \tilde{\Omega}} \ker(i\beta - V(x))\).

(ii) \(\Rightarrow\) (i): Let \(0 \neq z \in \bigcap_{x \in \tilde{\Omega}} \ker(i\beta - V(x)) \subseteq \mathbb{C}^N\) and consider the constant function \(u(x) = z\). Then \(u \in D(B_2 + V)\) and \((B_2 + V)u = i\beta u\), and thus \(i\beta \in \sigma_{\text{pnt}}(B_2 + V)\).

Proposition 2.10 characterises, in terms of the matrices \(V(x)\), whether \(B_2 + V\) has a non-zero imaginary eigenvalue. This yields the following characterisation of uniform convergence for the solutions to the coupled heat equation (2.1).

Theorem 2.11. Assume that, for almost all \(x \in \Omega\), the matrices \(A_1(x), \ldots, A_N(x)\) are symmetric and the matrix \(V(x)\) is dissipative with respect to the \(\ell^2\)-norm on \(\mathbb{R}^N\). Then the following assertions are equivalent:

(i) The solutions to the coupled heat equation (2.1) converge uniformly (in the sense of Definition 2.6) as \(t \to \infty\).

(ii) For every \(i\beta \in i\mathbb{R} \setminus \{0\}\) and every measurable subset \(\tilde{\Omega} \subseteq \Omega\) with the same measure as \(\Omega\) we have

\[
\bigcap_{x \in \tilde{\Omega}} \ker(i\beta - V(x)) = \{0\}.
\]
Proof. According to Proposition 2.10, we have to show that the solutions to (2.1) converge uniformly as \( t \to \infty \).

According to Proposition 2.10, assertion (ii) of the theorem is equivalent to uniform convergence of the solutions to (2.1).

If \( B_2 + V \) has an eigenvalue \( i\beta \in \mathbb{R} \setminus \{0\} \) with eigenvector \( u \), then \( e^{(B_2+V)t}u \) does not converge as \( t \to \infty \), so (i) fails. Assume now conversely that \( B_2 + V \) does not have any eigenvalues on the imaginary axis, except for possibly 0.

According to Proposition 2.13(b), the semigroup \( \left\{ e^{(B_2+V)t} \right\}_{t \in [0,\infty)} \) is immediately compact. It is also contractive as \( B_2 + V \) is dissipative. Therefore, it follows from [12, Corollary V.3.2] that \( e^{(B_2+V)t} \) converges with respect to the operator norm on \( L^2(\Omega;\mathbb{C}^N) \) as \( t \to \infty \); thus, the solutions to (2.1) converge uniformly in the sense of Definition 2.6 as \( t \to \infty \).

Let us state the following special case of Theorem 2.11 explicitly.

**Corollary 2.12.** Assume that, for almost all \( x \in \Omega \), the matrices \( A_1(x), \ldots, A_N(x) \) are symmetric and the matrix \( V(x) \) is dissipative with respect to the \( \ell^2 \)-norm on \( \mathbb{R}^N \). If, for almost all \( x \in \Omega \),

\[
\sigma_{\text{pnt}}(V(x)) \cap i\mathbb{R} \subseteq \{0\},
\]

then the solutions to the coupled heat equation (2.1) converge uniformly as \( t \to \infty \).

It is instructive to compare Corollary 2.12 to Example 1.1 and to the subsequent comments in the Subsection “Spectral considerations” of the Introduction.

### 2.5. Convergence in the non-symmetric case

A drawback of the techniques employed in the preceding section is that they rely heavily on the Hilbert space structure of the function space and the self-adjointness of the involved elliptic operators. This prevents us from considering (i) non-symmetric elliptic operators and (ii) more general potentials \( V(x) \) that might be dissipative with respect to the \( \ell^p \)-norm for some \( p \neq 2 \).

Therefore, in this section we will not suppose that the coefficient functions \( A_1, \ldots, A_N : \Omega \to \mathbb{R}^{d \times d} \) are symmetric. However, we will assume that, for almost all \( x \in \Omega \), \( V(x) \) is dissipative with respect to the \( \ell^p \)-norm on \( \mathbb{R}^N \) for some \( p \in [1,\infty) \), \( p \neq 2 \). This assumption is stronger than assuming \( \sigma_{\text{pnt}}(V(x)) \cap i\mathbb{R} \subseteq \{0\} \) for almost all \( x \in \Omega \); this follows from the following proposition which is a special case of [13, Theorem 3.7].

**Proposition 2.13.** Let \( p \in [1,\infty) \setminus \{2\} \). If a matrix in \( \mathbb{R}^{N \times N} \) is dissipative with respect to the \( \ell^p \)-norm on \( \mathbb{R}^N \), then its spectrum intersects the imaginary axis at most in \( \{0\} \).

Note that the assertion of Proposition 2.13 fails in the case \( p = 2 \): consider once again the matrix

\[
V := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

from the introduction. This matrix satisfies \( \langle Vz, z \rangle = 0 \) for each \( z \in \mathbb{R}^2 \), i.e., is dissipative with respect to the \( \ell^2 \)-norm on \( \mathbb{R}^2 \), but \( \sigma_{\text{pnt}}(V) \cap i\mathbb{R} = \{-i, i\} \). Moreover, we stress that it is essential in Proposition 2.13 that the matrices \( V(x) \) have only real entries.

**Theorem 2.14.** Let \( p \in [1,\infty) \setminus \{2\} \) and assume that, for almost all \( x \in \Omega \), the matrix \( V(x) \) is dissipative with respect to the \( \ell^p \)-norm on \( \mathbb{R}^N \). Then the solutions to the coupled heat equation (2.1) converge uniformly (in the sense of Definition 2.6) as \( t \to \infty \).
Proof. By Propositions 2.7 and 2.8 the semigroup \((e^{(B_s + V)})_{t \in [0, \infty)}\) is contractive on \(L^p(\Omega; \mathbb{C}^N)\), and clearly, it leaves \(L^p(\Omega; \mathbb{R}^N)\) invariant. Since the operators \(e^{(B_s + V)}\) are compact for \(t \in (0, \infty)\) and since \(L^p(\Omega; \mathbb{C}^N)\) is isometrically lattice isomorphic to a scalar-valued \(L^p\)-space, the assertion follows from Theorem 1.2. \(\square\)

Observe that the value of \(p\) enters Theorem 2.14 only as an assumption on the matrix potential \(V\). The convergence of the coupled heat semigroup takes place on the entire \(L^p\)-scale as shown in Corollary 2.5.

We point out, for \(p \in [1, \infty)\), the above proof only needs a version of Theorem 1.2 for \(C_0\)-semigroups; such a version can be found in [15 Corollary 3.8] (for eventually compact semigroups) or in [14 Corollary 2.2.5(b)] (for the case where \(T_{s_0}\) is, as in Theorem 1.2 quasi-compact). However, for \(p = \infty\) we do note have a \(C_0\)-semigroup, so we need a convergence result for more general semigroups – which can be found in Theorem 1.2. We also point out that we consider the case \(p = \infty\) to be quite important (rather than just an interesting side note) since the assumption that \(V(x)\) be dissipative with respect to the \(L^p\)-norm is easiest to check if \(p\) is either 1 or \(\infty\), see Proposition A.1.

Let us illustrate our above results by providing a simple example where the results from Subsection 2.5 can be applied, but the results from the Subsection 2.4 cannot:

**Example 2.15.** For an arbitrary bounded domain \(\Omega \subseteq \mathbb{R}^d\) with the extension property, consider the \(C^2\)-valued evolution equation

\[
\begin{pmatrix}
\dot{u}_1 \\
\dot{u}_2
\end{pmatrix} = \begin{pmatrix}
\Delta u_1 \\
\Delta u_2
\end{pmatrix} + \begin{pmatrix}
-v(x) & -v(x) \\
-2v(x) & -2v(x)
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
\]

with Neumann boundary conditions on \(\Omega\), where \(v(x) : \Omega \to (0, \infty)\) is a bounded and measurable function. It follows from Proposition A.1 in the appendix that the matrices

\[
V(x) = \begin{pmatrix}
-v(x) & -v(x) \\
-2v(x) & -2v(x)
\end{pmatrix},
\]

with spectrum \([-3v(x), 0)\] are \(L^\infty\)-dissipative; however, they are not \(L^2\)-dissipative since their symmetric parts have eigenvalues \(\frac{v(x)}{2}(-3 - \sqrt{11})\) and \(\frac{v(x)}{2}(-3 + \sqrt{11})\).

Hence, we cannot apply Corollary 2.72 but Theorem 2.14 tells us that the solutions to our equation converge uniformly as \(t \to \infty\). Moreover, we note that there are solutions that do not converge to 0 since the vector \((-1, 1)^T\) is an equilibrium of the equation.

Finally, we remark that the matrix semigroup generated by \(V(x)\) is not positive for any \(x \in \Omega\), so we cannot expect the solution semigroup of (2.3) to be positive - and hence we cannot apply convergence theorems for positive semigroups such as [20, Theorem 4].

Subsections 2.4 and 2.5 give a rather complete description of the long-term behaviour of the coupled heat equation (2.1) in case that the potential is dissipative with respect to an \(L^p\)-norm – with the exception of the following question for the case \(p = 2\):

**Open Problem 2.16.** Does the non-trivial implication “(ii) \(\Rightarrow\) (i)” in Theorem 2.11 remain true if the coefficient matrices \(A_1, \ldots, A_N\) are not assumed to be symmetric?

Of course, much more remains to be done in case that the potential \(V\) is not dissipative, since the methods presented in this paper do not work at all in this case. In fact, it is not even clear to the authors how to check boundedness of the solutions to (2.1) if \(V\) is not dissipative with respect to any \(L^p\)-norm on \(\mathbb{R}^N\).
3. Coupled parabolic equations on the whole space

After dealing with coupled parabolic equations on bounded domains in the \( L^p \)-setting in the former section, we now turn our attention to a related type of partial differential equations on the entire space \( \mathbb{R}^d \), but with possibly unbounded coefficients. Of course, the unboundedness of the coefficients forces us to impose other conditions on the equation in order to have well-posedness. Throughout the section we mainly rely on the results of [5], and as in this paper, we work on the space of bounded continuous functions over \( \mathbb{R}^d \).

3.1. Setting. We fix an integer \( N \geq 1 \) (which again will denote the numbers of coupled equations) as well as functions \( A : \mathbb{R}^d \to \mathbb{R}^{d \times d} \), \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( V : \mathbb{R}^d \to \mathbb{R}^{N \times N} \) and assume that the following conditions are satisfied:

1. For all \( x \in \mathbb{R}^d \) the matrix \( A(x) \) is symmetric and there exists a continuous function \( \nu : \mathbb{R}^d \to (0, \infty) \) such that the ellipticity condition
   \[ \xi^T A(x) \xi \geq \nu(x) \| \xi \|^2 \]
   holds for all \( x \in \mathbb{R}^d \) and all \( \xi \in \mathbb{R}^d \).
2. There exists \( \alpha \in (0, 1) \) such that the functions \( A \), \( b \) and \( V \) are locally \( \alpha \)-Hölder continuous on \( \mathbb{R}^d \).
3. The function \( V \) is bounded.
4. There exists a twice continuously differentiable function \( \varphi : \mathbb{R}^d \to (0, \infty) \) such that
   \[ \varphi(x) \to \infty \quad \text{as} \quad \|x\|_2 \to \infty \]
   and a number \( \lambda_0 > 0 \) such that the estimate
   \[ \lambda_0 \varphi - \sum_{i,j=1}^d A_{ij} \partial_{ij} \varphi - \sum_{j=1}^d b_j \partial_j \varphi \geq 0 \]
   holds on \( \mathbb{R}^d \).

Those are essentially the assumptions from [5, Hypotheses 2.1], with two exceptions:

- Instead of boundedness of \( V \) a weaker condition is used there (see [5, Hypotheses 2.1(iii) and Remark 2.2]). The reason why we assume boundedness of \( V \) is explain after Corollary 3.2.
- At first glance, the inequality in [5, Hypotheses 2.2(iv)] looks slightly different form the inequality that we use in (4). However, since we assume \( V \) to be bounded, both inequalities are actually equivalent in our setting (if one changes \( \lambda_0 \) appropriately).

We point out that both \( A \) and \( b \) are allowed to be unbounded and that \( A(x) \) need not be bounded away from 0 as \( \|x\|_2 \to \infty \). We are interested in the (possibly degenerate) parabolic equation

\[
\dot{u} = (\mathcal{B} + V)(u)
\]

on the space \( C_b(\mathbb{R}^d; \mathbb{C}^N) \) of bounded continuous function on \( \mathbb{R}^d \) with values in \( \mathbb{C}^N \), where the operator \( \mathcal{B} \) is given by

\[
\mathcal{B}u := \begin{pmatrix}
\left( \sum_{i,j=1}^d A_{ij} \partial_{ij} + \sum_{j=1}^d b_j \partial_j \right) u_1 \\
\vdots \\
\left( \sum_{i,j=1}^d A_{ij} \partial_{ij} + \sum_{j=1}^d b_j \partial_j \right) u_N
\end{pmatrix}
\]

for all \( u \) in the domain

\[
D(\mathcal{B}) := \{ u \in C_b(\mathbb{R}^d; \mathbb{C}^N) \cap \bigcap_{1 \leq p < \infty} W^{2,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^N) : \text{the expression in (3.2) is in } C_b(\mathbb{R}^d; \mathbb{C}^N) \}.
\]
Note that (3.1) differs from the parabolic problem (2.1) that we considered in Section 2 with respect to the following points: (i) we now consider the equation on the whole space $\mathbb{R}^d$ and on the space of bounded continuous functions; (ii) the elliptic operator is now the same in each component of the equation, and the coefficient matrices $A(x)$ are throughout assumed to be symmetric; (iii) the elliptic operator is now in non-divergence form; (iv) we now also allow for a drift term (which is also the same in each component of the equation); (v) we now assume more regularity of the coefficients, but we allow for both degeneracy and unboundedness of $A$ and $b$ at $\infty$.

The above setting allows us to employ the results from [5]. In particular, we will need the subsequent proposition. Similarly as in (2.2) we endow $C_b(\mathbb{R}^d; \mathbb{C}^N)$ with the norm $\| \cdot \|_\infty$ given by

$$\| u \|_\infty = \max\{ \| u_k \|_\infty : k \in \{1, \ldots, N\} \}$$

for all $u$ in this space (we note in passing that in [5], the space $C_b(\mathbb{R}^d; \mathbb{C}^N)$ is equipped with the norm $\| u \| = \sum_{k=1}^N \| u_k \|_\infty$ – which is equivalent to the norm introduced above, but which does not render $C_b(\mathbb{R}^d; \mathbb{R}^N)$ an AM-space).

**Proposition 3.1.** The operators $\mathcal{B}$ and $\mathcal{B} + V$ (with $D(\mathcal{B} + V) := D(\mathcal{B})$) on $C_b(\mathbb{R}^d; \mathbb{C}^N)$ are closed and all sufficiently large real numbers belong to the resolvent sets of both $\mathcal{B}$ and $\mathcal{B} + V$.

Moreover, there exist operator semigroups $(S_t)_{t \in [0, \infty)}$ and $(T_t)_{t \in [0, \infty)}$ on $C_b(\mathbb{R}^d; \mathbb{C}^N)$ with the following properties:

(a) For each $f \in C_b(\mathbb{R}^d; \mathbb{C}^N)$, each $x \in \mathbb{R}^d$ and all sufficiently large real numbers $\lambda$ the functions

$$0 < t \mapsto e^{-\lambda t}S_t f(x) \in \mathbb{C}^N$$

are continuous and in $L^1((0, \infty); \mathbb{C}^N)$, and their integrals equal $\mathcal{R}(\lambda, \mathcal{B}) f(x)$ and $\mathcal{R}(\lambda, \mathcal{B} + V) f(x)$, respectively.

(b) Both semigroups leave the set of functions with values in $\mathbb{R}^N$ invariant. Moreover, the semigroup $(S_t)_{t \in [0, \infty)}$ is contractive.

**Proof.** The assertions about $\mathcal{B}$ and $\mathcal{B} + V$, as well as the existence of both semigroups and property (a) follow from [5] Section 3; to see that we can really use the domain $D(\mathcal{B})$ as domain of the operator $\mathcal{B} + V$ we need the assumption that $V$ is bounded. Moreover, we note that the authors of [5] actually work with the real Banach space $C_b(\mathbb{R}^d; \mathbb{R}^N)$; but from this one can easily derive the same results for the complex case, as well as the fact that both semigroups leave $C_b(\mathbb{R}^d; \mathbb{R}^N)$ invariant – which is the first part of (b).

Since $\mathcal{B}$ acts separately in every component, so does the semigroup $(S_t)_{t \in [0, \infty)}$; hence, contractivity of $(S_t)_{t \in [0, \infty)}$ follows from contractivity in the scalar case, which can for instance be found in [5] Proposition 2.3(i)].

The semigroup $(T_t)_{t \in [0, \infty)}$ describes the solutions to our parabolic equation (3.1); see [5] Section 3. We note that, in our setting where the matrix potential $V$ is bounded, one could – alternatively to the approach from [5] – employ the theory of bi-continuous semigroups to study the perturbed operator $\mathcal{B} + V$; see [22] beginning of Section 5 and [8] Theorem 3.5. (There are also results about unbounded perturbations of bi-continuous semigroups such as in [11 Corollary 4.2], but we do not know whether such results can be applied under the assumptions of [5] Hypotheses 2.1]).
We point out that the semigroup \((S_t)_{t \in [0, \infty)}\) is positive, but the semigroup \((T_t)_{t \in [0, \infty)}\) is not positive, in general. Moreover, we cannot expect those semigroups to be strongly continuous, in general (see for instance the discussion at the beginning of [5 Subsection 3.1]).

If we assume \(\ell^\infty\)-dissipativity of the matrices \(V(x)\), then we also obtain that the semigroup \((T_t)_{t \in [0, \infty)}\) is contractive:

**Corollary 3.2.** Assume that, for each \(x \in \mathbb{R}^d\), the matrix \(V(x)\) is dissipative with respect to the \(\ell^\infty\)-norm on \(\mathbb{R}^N\). Then the semigroup \((T_t)_{t \in [0, \infty)}\) is contractive, too.

We note that the real matrix \(V(x)\) is dissipative with respect to the \(\ell^\infty\)-norm on \(\mathbb{R}^N\) if and only if it is dissipative with respect to the \(\ell^\infty\)-norm on \(\mathbb{C}^N\); this follows from the same arguments as in the proof of Proposition 2.8.

**Proof of Corollary 3.2.** For each \(f \in C_b(\mathbb{R}^d; \mathbb{C}^N)\), each \(x \in \mathbb{R}^d\) and each \(\lambda > 0\) the mapping \((0, \infty) \ni t \mapsto e^{-\lambda S_t f(x)} \in \mathbb{C}^N\) is continuous and in \(L^1((0, \infty); \mathbb{C}^N)\), and its integral equals \(\mathcal{R}(\lambda, B)f(x)\); this follows from Proposition 3.1 and from the identity theorem for analytic functions. As \((S_t)_{t \in [0, \infty)}\) is contractive, so is the operator \(\mathcal{A}\mathcal{R}(\lambda, B)\) for each \(\lambda > 0\), and thus it follows that \(B\) is dissipative.

The matrix-valued multiplication operator \(V\) is dissipative by assumption, and since it is a bounded operator, it is thus even strictly dissipative. Consequently, the operator \(B + V\) is dissipative, too. It now follows from Post’s inversion formula for the Laplace transform (for \(C^N\)-valued functions) and, again, from Proposition 3.1 that \((T_t)_{t \in [0, \infty)}\) is contractive.

The proof of Corollary 3.2 is the reason why we assumed \(V\) to be bounded; we needed the boundedness on two occasions in the proof: (i) in order to derive strict dissipativity of \(V\) from mere dissipativity, and (ii) in order for \(B + V\) to have the same domain as \(B\). The authors do not know whether Corollary 3.2 remains true for unbounded \(V\) which satisfies, besides dissipativity, only the assumptions of [5 Hypotheses 2.1].

### 3.2. A convergence result.

After the preparations of the preceding subsection, we now arrive at the following convergence result for the solutions to (3.1): if the matrices \(V(x)\) are \(\ell^\infty\)-dissipative and if the operator semigroup \((T_t)_{t \in [0, \infty)}\) is immediately compact, then the solutions to (3.1) converge uniformly (for initial values in the unit ball) as time tends to infinity.

**Theorem 3.3.** Assume that, for each \(x \in \mathbb{R}^d\), the matrix \(V(x)\) is dissipative with respect to the \(\ell^\infty\)-norm on \(\mathbb{R}^N\). If the operators \(T_t\) are compact for \(t > 0\), then \(T_t\) converges with respect to the operator norm to a finite-rank projection as \(t \to \infty\).

**Proof.** This is a consequence of Corollary 3.2 and Theorem 1.3 since \(C_b(\mathbb{R}^d; \mathbb{C}^N)\) is isometrically isomorphic to the space \(C_b(L; \mathbb{C})\), where \(L\) is composed of \(N\) disjoint copies of \(\mathbb{R}^d\) (and since the isomorphism can be chosen to map the real part of \(C_b(\mathbb{R}^d; \mathbb{C}^N)\) onto the real part of \(C_b(L; \mathbb{C})\)).

Of course, we do not really need to assume that all operators \(T_t\) (for \(t > 0\)) are compact in order to apply Theorem 1.3; it would suffice to assume that at least one operator \(T_{t_0}\) is quasi-compact. However, the property that all \(T_t\) are compact is quite a reasonable assumption in this setting since there are several sufficient criteria for this property available; we refer to [5 Subsection 3.2] for such conditions and refrain from stating them explicitly here. However, let us illustrate the above result by the following simple concrete example, where the matrix potential is the same as in Example 2.15 above, and where the differential operator is a special case of the one considered in [5 Section 4].
Example 3.4. Consider the $C^2$-valued evolution equation

$$\begin{align*}
\begin{pmatrix}
\dot{u}_1 \\
\dot{u}_2
\end{pmatrix} &= \begin{pmatrix}
\Delta u_1 \\
\Delta u_2
\end{pmatrix} - \begin{pmatrix}
(1 + \|x\|_2^\beta) x^T \nabla u_1 \\
(1 + \|x\|_2^\beta) x^T \nabla u_2
\end{pmatrix} + \begin{pmatrix}
-v(x) & -v(x) \\
-2v(x) & -2v(x)
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
\end{align*}$$

on $\mathbb{R}^d$, where $\beta > 0$ is a fixed real number and where $v : \mathbb{R}^d \to (0, \infty)$ is bounded and locally $\alpha$-Hölder continuous with $\alpha \in (0, 1)$.

Examples of this type (in fact, of a more general type) are considered in [5, Section 4], where it is shown that this equation fits into the setting of the present section and that the solution semigroup of (3.3) is immediately compact on $C_b(\mathbb{R}^d; C^2)$ [5, Theorem 4.2] (but note that the parameter $\alpha$ is used with another meaning there).

Since, according to Proposition A.1(c), $V(x)$ is dissipative with respect to the $\ell^\infty$-norm on $\mathbb{R}^d$ for each $x \in \mathbb{R}^d$, it follows from Theorem 3.3 that the solution semigroup of (3.3) converges with respect to the operator norm on $C_b(\mathbb{R}^d; C^2)$ as $t \to \infty$.

4. Semigroup representations and the semigroup at infinity

In the remaining part of the paper, we develop a general framework to analyse whether an operator semigroup converges with respect to the operator norm as time tends to infinity. The most important situation that occurs in applications (for instance, in Sections 2 and 3) is that the semigroup contains a quasi-compact operator, and this situation will also be one of our main interests (though not our only interest). In the case of $C_0$-semigroups, a rather complete description of the long-term behaviour in the case of quasi-compactness can be found in [7, Section V.3]. However, as has become apparent in the preceding sections, the case of $C_0$-semigroups is not always sufficient and, as explained in the introduction, we do not wish to develop a separate convergence theory for each different type of time regularity that might occur in applications. Thus, we stick to the other extreme and develop a single theory that does not assume any time regularity at all.

Given this goal, it is also consequent to leave the restricted setting of semigroups of the type $(T_s)_{s \in [0, \infty)}$, and to consider operator representations of arbitrary commutative semigroups $(S, +)$ instead. This allows us to also treat the time-discrete case $(T^n)_{n \in \mathbb{N}_0}$ and, for instance, the case of multi-parameter semigroups within our one theory. Moreover, it allows for some interesting theoretical observations in the spirit of [12] and [16].

Our approach is based on the famous Jacobs–de Leeuw–Glicksberg (JdLG) theory which applies abstract results about (semi-)topological semigroups to the more concrete situation of operator semigroups, and we combine this with the construction of a semigroup at infinity which is inspired by [16]. In this context, we find it also worthwhile mentioning that there exist other quite abstract approaches to general operator semigroups, too, that do not rely on JdLG theory (see for instance [10]); however, we will mainly stick to JdLG theory in this paper.

4.1. Setting. Throughout the rest of this paper, let $(S, +)$ be a commutative semigroup with neutral element 0 (i.e., in a more algebraic language, $(S, +)$ is a monoid). We define a reflexive and transitive relation (i.e. a pre-order) $\leq$ on $S$ by setting

$$s \leq t \quad \text{if and only if} \quad \text{there exists } r \in S \text{ such that } t = s + r$$

for $s, t \in S$.

A representation of $S$ on a Banach space $E$ is any mapping $T : S \to \mathcal{L}(E)$ that satisfies

$$T(0) = \text{id}_E \quad \text{and} \quad T(s + t) = T(s)T(t) \quad \text{for all } t, s \in S.$$
In the following we will often use the index notation $T_s$ instead of $T(s)$ and call $(T_s)_{s \in S}$ an operator semigroup on $E$.

Let $(T_s)_{s \in S}$ be an operator semigroup on $E$, and let $\mathbb{K}$ be the underlying scalar field of $E$. A function $\lambda : S \to \mathbb{K}$ is called an eigenvalue of $(T_s)_{s \in S}$ if there exists a non-zero vector $x \in E$ such that

$$T_s x = \lambda_s x$$

for all $s \in S$;

in this case, the vector $x$ is called a corresponding eigenvector. Note that an eigenvalue $\lambda = (\lambda_s)_{s \in S}$ is always a representation of $(S, +)$ on the space $\mathbb{K}$. Moreover, we call an eigenvalue $\lambda = (\lambda_s)_{s \in S}$ unimodular if $|\lambda_s| = 1$ for all $s \in S$.

An operator semigroup $(T_s)_{s \in S}$ is called bounded if $\sup_{s \in S} \|T_s\| < \infty$. Note that, as $S$ is a directed set, every operator semigroup $(T_s)_{s \in S}$ becomes a net, and hence it makes sense to talk about convergence of $(T_s)_{s \in S}$. At this point we recall that, throughout the article, we always endow the operator space $\mathcal{L}(E)$ with the operator norm, i.e. for us, convergence always means convergence with respect to the operator norm. In the case of a bounded operator semigroup one has the following simple characterization of convergence to the zero operator.

**Proposition 4.1.** Let $(T_s)_{s \in S}$ be a bounded representation of $(S, +)$ on a Banach space $E$. The following assertions are equivalent:

(i) $\lim_{s \in S} T_s = 0$.

(ii) There exists $s_0 \in S$ such that $\|T_{s_0}\| < 1$.

(iii) $0$ is contained in the closure of the set $\{T_s : s \in S\}$.

**Proof.** (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Let $s_0 \in S$ such that $\|T_{s_0}\| < 1$. Let $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\|T_{s_0}\|^n < \varepsilon$. Hence,

$$\|T_{ns_0}\| \leq \|T_{s_0}\|^n < \varepsilon.$$

Therefore, $0 \in \{T_s : s \in S\}$.

(iii) $\Rightarrow$ (i): Let $\varepsilon > 0$. Then there exists $s_0 \in S$ such that $\|T_{s_0}\| \leq \varepsilon$. Thus,

$$\|T_t\| \leq \varepsilon M$$

for all $t \in s_0 + S$,

where $M := \sup_{s \in S} \|T_s\|$. So it follows that $\lim_{s \in S} T_s = 0$. \qed

4.2. The semigroup at infinity. In [16, Section 2] the concept of the semigroup at infinity with respect to the strong operator topology was used to study strong convergence of operator semigroups. In reminiscence of this concept we define the semigroup at infinity now with respect to the operator norm topology.

**Definition 4.2.** Let $(T_s)_{s \in S}$ be a representation of $(S, +)$ on a Banach space $E$. We call the set

$$\mathcal{T}_\infty := \bigcap_{r \in S} \{T_s : s \geq r\}$$

the semigroup at infinity associated with $(T_s)_{s \in S}$ with respect to the operator norm. Since we restrict ourselves to the operator norm topology throughout the paper and since we only consider a single operator semigroup, we will sometimes just call $\mathcal{T}_\infty$ the semigroup at infinity.

Note that the semigroup at infinity consists of all cluster points (with respect to the operator norm) of the net $(T_s)_{s \in S}$.

If the semigroup at infinity, $\mathcal{T}_\infty$, is non-empty and compact, then we can apply the Jacobs–de Leeuw–Glicksberg theory to the topological semigroup $\mathcal{T}_\infty$. This yields a smallest non-empty closed ideal $\mathcal{I}$ in $\mathcal{T}_\infty$ (where ideal means that $T \mathcal{I} \subseteq \mathcal{I}$ for all $T \in \mathcal{T}_\infty$), and the ideal $\mathcal{I}$ – the so-called Sushkevich kernel of $\mathcal{T}_\infty$ – is a
compact topological group with respect to operator multiplication. For details we refer for instance to [10] Section 16.1 or to [11] Theorem V.2.3.

We denote the neutral element in \( \mathcal{T} \) by \( P_\infty \) – it is a projection in \( \mathcal{L}(E) \) which we call the projection at infinity; we denote the range of \( P_\infty \) by \( E_\infty \).

The following theorem demonstrates why the semigroup at infinity is important for the analysis of the long term behaviour of a semigroup representation. It is very close in spirit to a similar theorem for the strong operator topology that can be found in [16] Theorem 2.2.

**Theorem 4.3.** Let \( (T_s)_{s \in S} \) be a bounded representation of \( (S,+) \) on a Banach space \( E \) and assume that the semigroup at infinity, \( \mathcal{T}^\infty_\infty \), is non-empty and compact. Set \( \mathcal{T} := \{ T_s : s \in S \} \subseteq \mathcal{L}(E) \). Then the following assertions hold:

(a) The projection at infinity, \( P_\infty \), commutes with all operators in \( \mathcal{T} \), and \( T P_\infty = P_\infty T \).

(b) The semigroup at infinity, \( \mathcal{T}^\infty_\infty \), is a group with respect to operator multiplication with neutral element \( P_\infty \). Moreover, we have

\[
\mathcal{T}|_{E_\infty} = \mathcal{T}^\infty_\infty|_{E_\infty} = \{ T_s | s \in S \}_{|E_\infty} \mathcal{L}(E_\infty)
\]

and this set is a compact subgroup of the bijective operators in \( \mathcal{L}(E_\infty) \). Finally, \( \mathcal{T}^\infty_\infty \) and \( \mathcal{T}^\infty_\infty|_{E_\infty} \) are isomorphic (in the category of compact topological groups) via the mapping \( R \mapsto R|_{E_\infty} \).

(c) We have \( \lim_{s \in S} T_s | ker P_\infty = 0 \) with respect to the operator norm on \( \mathcal{L}(ker P_\infty) \).

(d) For every vector \( x \in E \) the following assertions are equivalent:

(i) \( P_\infty x = 0 \).

(ii) \( 0 \) is contained in the weak closure of the orbit \( \{ T_s x : s \in S \} \).

(iii) The net \( (T_s x)_{s \in S} \) norm converges to \( 0 \) (in \( E \)).

(iv) We have \( Rx = 0 \) for each \( R \in \mathcal{T}^\infty_\infty \).

(v) We have \( Rx = 0 \) for at least one \( R \in \mathcal{T}^\infty_\infty \).

(e) If the underlying scalar field of \( E \) is complex, then

\[
E_\infty = \overline{\{ x \in E : \forall s \in S \exists \lambda_s \in \mathbb{T} \text{ with } T_s x = \lambda_s x \}}.
\]

Note that the first part of assertion (a) implies that every operator in \( \mathcal{T} \) – and thus in particular every operator \( T_s \) – leaves \( E_\infty \) and ker \( P_\infty \) invariant.

**Proof of Theorem 4.3.** (a) The first assertion is clear since \( \mathcal{T} \) is commutative. Moreover, we have \( \mathcal{T} \mathcal{T}^\infty_\infty \subseteq \mathcal{T}^\infty_\infty \subseteq \mathcal{T} \), where the second inclusion is obvious and the first inclusion follows easily from the definitions of \( \mathcal{T} \) and \( \mathcal{T}^\infty_\infty \). Therefore,

\[
\mathcal{T} P_\infty = \mathcal{T} P_\infty P_\infty \subseteq \mathcal{T}^\infty_\infty P_\infty \subseteq \mathcal{T} P_\infty.
\]

(c) Since \( P_\infty \) is trivial on ker \( P_\infty \), we have \( 0 \in \{ T_s | ker P_\infty : s \in S \} \); this is equivalent to \( \lim_{s \in S} T_s | ker P_\infty = 0 \) by Proposition 4.1.

(b) Let \( \mathcal{I} \subseteq \mathcal{T}^\infty_\infty \) denote the Sushkevich kernel of \( \mathcal{T}^\infty_\infty \), i.e. the smallest non-empty closed ideal in the semigroup \( \mathcal{T}^\infty_\infty \) (see the discussion before the theorem). We show that \( \mathcal{T}^\infty_\infty = \mathcal{I} \). To this end, let \( R \in \mathcal{T}^\infty_\infty \). Then \( R \) is a cluster point of the net \( (T_s)_{s \in S} \), so there exists a subnet \( (T_{s_j})_j \) that converges to \( R \). It follows from assertion (c), which we have already proved, that \( T_{s_j} (id_E - P_\infty) \to 0 \), so \( R (id_E - P_\infty) = 0 \) and hence, \( R = RP_\infty \). Since \( P_\infty \) is \( \mathcal{I} \) and since \( \mathcal{I} \) is an ideal in \( \mathcal{T}^\infty_\infty \) we conclude that \( R \in \mathcal{I} \). We have thus proved that \( \mathcal{T}^\infty_\infty \) is a group with respect to operator multiplication and that its neutral element is \( P_\infty \).

Next we show the equalities in the displayed formula. One has \( \mathcal{T}|_{E_\infty} = \mathcal{T}^\infty_\infty|_{E_\infty} \) by (a). As the restriction map from \( \mathcal{L}(E) \) to \( \mathcal{L}(E_\infty, E) \) is continuous, we have \( \mathcal{T}|_{E_\infty} \subseteq \{ T_s : s \in S \}_{|E_\infty} \mathcal{L}(E_\infty) \). The converse inclusion follows from \( \{ T_s : s \in S \}_{|E_\infty} P_\infty \subseteq \mathcal{T} \).
Since $\mathcal{T}_\infty^\text{on}$ is a group with neutral element $P_\infty$, it readily follows that $\mathcal{T}_\infty^\text{on}|_{E_\infty}$ is a subgroup of the invertible operators on $E_\infty$. The mapping

$$
\mathcal{T}_\infty^\text{on} \ni R \mapsto R|_{E_\infty} \in \mathcal{T}_\infty^\text{on}|_{E_\infty}
$$

is clearly a surjective and continuous group homomorphism and consequently, $\mathcal{T}_\infty^\text{on}|_{E_\infty}$ is compact. If $R|_{E_\infty} = \text{id}_{E_\infty}$ for some $R \in \mathcal{T}_\infty^\text{on}$, then $P_\infty = RP_\infty = R$, so our group homomorphism is also injective. Finally, it is also a homeomorphism by the compactness of its domain and range.

(d) Fix $x \in E$. Clearly, since $P_\infty \in \mathcal{T}_\infty^\text{on}$, (iv) implies (i) and (i) implies (v). Furthermore, (v) implies $0 \in \{T_s x \mid s \in S\}$ which is equivalent to $\lim_{s \in S} T_s x = 0$, i.e., (iii). Moreover, if (iii) holds and $\varepsilon > 0$ is fixed, then there exists $s \in S$ such that $\{T_s x \mid t \geq s\} \subseteq \varepsilon B$, where $B$ denotes the closed unit ball in $E$. Thus, $\mathcal{T}_\infty^\text{on} x \subseteq \varepsilon B$. Since $\varepsilon > 0$ was arbitrary, it follows that $\mathcal{T}_\infty^\text{on} x = \{0\}$, i.e., (iv) holds. Finally, (iii) obviously implies (ii). Conversely, suppose that (ii) holds. Then it follows that 0 is contained in the weak closure of the set $\{T_s P_\infty x \mid s \in S\}$. Moreover, it follows from (a) that the set $\{T_s P_\infty \mid s \in S\}$ is a subset of $\mathcal{T}_\infty^\text{on} P_\infty$ and thus relatively compact in $\mathcal{L}(E)$. Hence, $\{T_s P_\infty x \mid s \in S\}$ is relatively strongly compact and thus its closure must coincide with its weak closure. Thus, 0 is contained in the strong closure of $\{T_s P_\infty x \mid s \in S\}$, which implies $P_\infty x = P_\infty (P_\infty x) = 0$.

(e) Recall that, by (b), $\mathcal{G} := \{T_s \mid s \in S\}|_{E_\infty} \subseteq \mathcal{L}(E_\infty)$ is a compact group with respect to the operator norm on $\mathcal{L}(E_\infty)$. Let $\mathcal{G}^*$ denote the dual group of $\mathcal{G}$. According to [13 Corollary 15.18] we have

$$
E_\infty = \overline{\lim \{x \in E_\infty : \exists \xi \in \mathcal{G}^* \text{ with } Rx = \xi(R)x \forall R \in \mathcal{G}\}} \subseteq \lim \{x \in E_\infty : \forall s \in S \exists \lambda_s \in T \text{ with } T_s x = \lambda_s x \} \subseteq E_\infty.
$$

Now let $x \in E$ be an eigenvector associated to the unimodular eigenvalue $\lambda = (\lambda_s)_{s \in S}$. Consider $y := (I - P_\infty)x \in \ker P_\infty$. Then $T_s y \to 0$ and $T_s y = \lambda_s y$ for each $s \in S$. Since $|\lambda_s| = 1$ for all $s \in S$, this implies $y = 0$, i.e., $x \in E_\infty$. \qed

Remark 4.4. For the strong operator topology, the analogue result to Theorem 4.3 is [16 Theorem 2.2]. The assertion that the semigroup at infinity is automatically a group in case that it is non-empty and compact is not included in this reference, but it is also true in the situation there; this can be shown by exactly the same argument as in our proof of Theorem 4.3(b).

As a consequence of the above theorem, operator norm convergence of a semigroup can be characterised in terms of its semigroup at infinity. Let us state this explicitly in the following corollary.

**Corollary 4.5.** For every bounded representation $(T_s)_{s \in S}$ on a Banach space $E$ the following assertions are equivalent:

(i) $(T_s)_{s \in S}$ converges (with respect to the operator norm).

(ii) $\mathcal{T}_\infty^\text{on}$ is a singleton.

(iii) $\mathcal{T}_\infty^\text{on}$ is non-empty and compact, and acts as the identity on $E_\infty$.

(iv) $\mathcal{T}_\infty^\text{on}$ is non-empty and compact, and $(T_s)_{s \in S}$ acts as the identity on $E_\infty$.

If the equivalent conditions (i)–(iv) are satisfied, then $\lim_{s \in S} T_s$ equals $P_\infty$, the projection at infinity.

If the underlying scalar field of $E$ is complex, the above assertions (i)–(iv) are also equivalent to:

(v) $\mathcal{T}_\infty^\text{on}$ is non-empty and compact, and $\mathbb{1} := (1)_{s \in S}$ is the only unimodular eigenvalue of $(T_s)_{s \in S}$.

**Proof.** (i) $\Rightarrow$ (ii): If the net $(T_s)_{s \in S}$ converges, then its limit is the only cluster point of $(T_s)_{s \in S}$. Hence, $\mathcal{T}_\infty^\text{on}$ is a singleton.
(iii) ⇒ (ii): Assertion (ii) implies $T_{\infty}^{\cong} = \{P_{\infty}\}$ and $P_{\infty}$ acts trivially on $E_{\infty}$.

(iii) ⇒ (iv): By Theorem 4.3(a) we have $T_{E_{\infty}} = T_{E_{\infty}}^{\cong}$, so (iii) implies (iv).

(iv) ⇒ (i): By Theorem 4.3(c), assertion (iv) implies that $\lim_{s \in S} T_s = P_{\infty}$

(iv) ⇔ (v): By Theorem 4.3(c), $(T_s)_{s \in S}$ acts as the identity on $E_{\infty}$ if and only if $1 := (1)_{s \in S}$ is the only unimodular eigenvalue of $(T_s)_{s \in S}$.

**Remark 4.6.** We note once again that our results in this subsection, as well as their proofs, are quite close to similar results for the strong operator topology from [16 Subsection 2.2]. The relation between the semigroups at infinity with respect to the operator norm topology and with respect to the strong operator topology can also be formalised in the following sense.

If $(T_s)_{s \in S}$ is an operator semigroup on a Banach space $E$ we can, for each $s \in S$, define an operator $R_s$ on the Banach space $\mathcal{L}(E)$ by

$R_s : \mathcal{L}(E) \ni A \mapsto T_s A \in \mathcal{L}(E)$.

Then $(R_s)_{s \in S}$ is an operator semigroup on the Banach space $\mathcal{L}(E)$, and topological properties of $(R_s)_{s \in S}$ with respect to the strong operator topology translate into topological properties of $(T_s)_{s \in S}$ with respect to the operator norm. This observation can be used as a basis to derive the theory of the semigroup at infinity with respect to the operator norm from the corresponding theory with respect to the strong topology presented in [16].

However, in the present section we prefer to give more direct proofs in order to make our work more self-contained and to improve its accessibility for readers not familiar with [16].

In order to apply Theorem 4.3 and Corollary 4.5 we need criteria to ensure that the semigroup at infinity is non-empty and compact; in a general setting, such criteria can be found in the following proposition.

**Proposition 4.7.** For every bounded representation $(T_s)_{s \in S}$ of $(S, +)$ on a Banach space $E$, the following assertions are equivalent:

(i) The semigroup at infinity is non-empty and compact.

(ii) Every subnet of $(T_s)_{s \in S}$ has a convergent subnet.

(iii) Every universal subnet of $(T_s)_{s \in S}$ converges.

In case that $S$ contains a cofinal sequence, the above assertions (i)-(iii) are also equivalent to:

(iv) For every cofinal sequence $(s_n)_{n \in \mathbb{N}}$ in $S$, the sequence $(T_{s_n})_{n \in \mathbb{N}}$ has a convergent subsequence.

**Proof.** The equivalence (ii) ⇔ (iii) and the implication (ii) ⇒ (i) both follow from Lemma C.1.

To prove (i) ⇒ (ii), note that one has $\lim_{s \in S} (T_s(I - P_{\infty})) = 0$ by Theorem 4.3(c). Moreover, the net $(T_s P_{\infty})_{s \in S}$ is contained in the compact set $T_{\infty}^{\cong} P_{\infty}$ by Theorem 4.3(a). Thus each of it subnets has a convergent subnet. Since

$T_s = T_s P_{\infty} + T_s(I - P_{\infty})$ for all $s \in S$,

this shows that every subnet of $(T_s)_{s \in S}$ has a convergent subnet.

Finally, the remaining two implications (iii) ⇒ (iv) and (iv) ⇒ (i) are consequences of Lemma C.1.

If $(x_n)$ is a net in an arbitrary metric (or topological) space whose set of cluster points is non-empty and compact, then the set of cluster points of any subnet of $(x_n)$ might well be empty. Our next Corollary shows that, as a consequence of Proposition 4.7, the situation is different for our semigroup setting. For a proper understanding of that corollary, the following algebraic observation is important.
Remark 4.8. Let $R$ be a subsemigroup of $S$ that contains 0. Denote the pre-order on $R$ inherited from $S$ by $\leq_S$ and denote the pre-order on $R$ induced by its semigroup operation by $\leq_R$. For all $r_1, r_2 \in R$ we then have the implication

$$r_1 \leq_R r_2 \Rightarrow r_1 \leq_S r_2.$$  

(Note that $\leq_R$ and $\leq_S$ do not coincide in general, as can for instance be seen by considering the subsemigroup $\{0\} \cup [1, \infty)$ of $([0, \infty), +)$.)

Now, let $X$ be a set and for each $r \in R$, let $x_r \in X$. Let us use, within this remark, the notations $(x_r)_{r \in (R, \leq_R)}$ and $(x_r)_{r \in (R, \leq_S)}$ to distinguish the nets that we obtain be considering the different pre-orders $\leq_R$ and $\leq_S$ on $R$. Then it follows from what was said above that the net $(x_r)_{r \in (R, \leq_R)}$ is a subnet of $(x_r)_{r \in (R, \leq_S)}$.

In particular, if $R$ is cofinal in $S$ and $(x_s)_{s \in S}$ is a net in $X$, then $(x_r)_{r \in (R, \leq_R)}$ is a subnet of $(x_s)_{s \in S}$.

Corollary 4.9. Let $E$ be a Banach space. Let $R$ be a subsemigroup of $S$ that contains 0 and is cofinal in $S$ and let $(T_s)_{s \in S}$ be a bounded representation on $E$ whose associated semigroup at infinity is non-empty and compact.

Then the semigroup at infinity associated with $(T_s)_{s \in R}$ is also non-empty and compact, and the projections at infinity of $(T_s)_{s \in S}$ and $(T_s)_{s \in R}$ coincide.

Note that in the corollary the semigroup $R$ is endowed with the order inherited from its semigroup operation (denoted by $\leq_R$ in Remark 4.8). For any other order on $R$ (for instance the order inherited from $S$) we did not even define the notion semigroup at infinity.

Proof of Corollary 4.9. It follows from Remark 4.8 that $(T_s)_{s \in R}$ is a subnet of $(T_s)_{s \in S}$. In particular, every universal subnet of $(T_s)_{s \in R}$ is also a universal subnet of $(T_s)_{s \in S}$ and thus convergent by Proposition 4.7. Hence, by the same proposition the semigroup at infinity associated with $(T_s)_{s \in R}$ is non-empty and compact.

Let $P_\infty$ and $Q_\infty$ denote the projections at infinity of $(T_s)_{s \in S}$ and $(T_s)_{s \in R}$, respectively. Those two projections commute. It follows from Theorem 4.3(c) that $\lim_{s \in S} T_s \ker P_\infty = 0$ and thus, in particular, $\lim_{s \in R} T_s \ker P_\infty = 0$; Theorem 4.3(d), applied to the semigroup $(T_s)_{s \in R}$, thus implies that $Q_\infty x = 0$ for every $x \in \ker P_\infty$, i.e. $\ker P_\infty \subseteq \ker Q_\infty$.

Conversely, it also follows from Theorem 4.3(c) that $\lim_{s \in R} T_s \ker Q_\infty = 0$, so Proposition 4.4 implies that even $\lim_{s \in S} T_s \ker Q_\infty = 0$. Theorem 4.3(d), applied to the semigroup $(T_s)_{s \in S}$, thus implies that $P_\infty x = 0$ for every $x \in \ker Q_\infty$, i.e. $\ker Q_\infty \subseteq \ker P_\infty$. Therefore, we proved that the commuting projections $P_\infty$ and $Q_\infty$ have the same kernel, so they coincide. 

In order to determine the projection $P_\infty$ in concrete situations the following proposition is quite useful; it shows that $P_\infty$ is uniquely determined by some of its properties listed in Theorem 4.3.

Proposition 4.10. Let $(T_s)_{s \in S}$ be a bounded representation of $(S, +)$ on a Banach space $E$ and let $P \in \mathcal{L}(E)$ be a projection that commutes with all operators $T_s$.

Consider the following assertions:

(a) $\lim_{s \in S} T_s \ker P = 0$ (with respect to the operator norm on $\mathcal{L}(\ker P)$).

(b) The set $\{T_s | P| E, s \in S\}$ is relatively compact in $\mathcal{L}(PE)$.

(c) The net $(T_s x)_{s \in S}$ does not converge to 0 for any $x \in PE \setminus \{0\}$.

If assertions (a) and (b) are satisfied, then the semigroup at infinity, $T_\infty^S$, is non-empty and compact, and the projection at infinity satisfies $P_\infty E \subseteq PE$ and $\ker P_\infty \supseteq \ker P$.

If all assertions assertions (a)–(c) are satisfied, then we have in addition $P_\infty = P$.
Proof. First note that the representation leaves both the kernel and the range of $P$ invariant since $P$ commutes with each operator $T_s$. Now assume that (a) and (b) are satisfied and let $(T_{s_j})$ be a universal subnet of $(T_s)_{s \in S}$. By (a), $(T_{s_j} |_{\ker P})$ converges to 0 and $(T_{s_j} |_{PE})$ is convergent. Thus, the net $(T_{s_j})$ is convergent, which proves that $T_\infty$ is non-empty and compact by Proposition 4.7. It follows from assumption (a) and Theorem 4.3(d) that $\ker P_\infty \supseteq \ker P$. To show that $P_\infty E \subseteq PE$, let $x \in P_\infty E$. We have $(id_E - P)x \in \ker P \subseteq \ker P_\infty$, and since $P_\infty$ and $P$ commute, this implies that $0 = (id_E - P)x = (id_E - P)x$, so $x = Px \in PE$.

Now assume in addition that assumption (c) is satisfied. We show that the inclusion $\ker P_\infty \subseteq \ker P$ is also satisfied then. Let $x \in \ker P_\infty$. Since $P$ and $P_\infty$ commute, the projection $P$ leaves $\ker P_\infty$ invariant, i.e. we also have $Px \in \ker P_\infty$. Hence, $T_s Px \to 0$ by Theorem 4.3(d), so it follows from assumption (c) that $Px = 0$. We thus proved $\ker P = \ker P_\infty$, so the claim $P_\infty = P$ follows from the general observation that two commuting projections coincide if their kernels coincide. □

4.3. Powers of a single operator. In this subsection we consider time-discrete semigroups, i.e. semigroups of the form $(T^n)_{n \in \mathbb{N}_0}$ for a single operator $T$. Let us first note in the following lemma that, in this case, the semigroup at infinity is non-empty and compact if and only if the entire set $\{T^n : n \in \mathbb{N}_0\}$ is relatively compact in $L(E)$.

Lemma 4.11. Let $T \in L(E)$ be a power-bounded operator on a Banach space $E$. Then the semigroup at infinity, $T_\infty$, associated to the semigroup $(T^n)_{n \in \mathbb{N}_0}$ is non-empty and compact if and only if the set $\mathcal{T} = \{T^n : n \in \mathbb{N}_0\}$ is relatively compact in $L(E)$.

Proof. Let $(T^{n_k})_{k \in \mathbb{N}}$ be an arbitrary sequence in $\mathcal{T}$; we have to distinguish two cases since this sequence might not be a subsequence of $(T^n)_{n \in \mathbb{N}_0}$. In the first case, the index sequence $(n_k)_{k \in \mathbb{N}}$ is bounded; then, by the pigeon hole principle, it has a constant subsequence, so $(T^{n_k})_{k \in \mathbb{N}}$ has a constant, thus convergent, subsequence.

In the second case the index sequence $(n_k)_{k \in \mathbb{N}}$ is unbounded. Then it has a subsequence $(n_{k_j})_{j \in \mathbb{N}}$ that is cofinal in $\mathbb{N}_0$. Hence, Proposition 4.7 yields that $(T^{n_{k_j}})_{j \in \mathbb{N}}$ has a convergent subsequence, and the latter is also a subsequence of $(T^{n_k})_{k \in \mathbb{N}}$.

The converse follows directly from Proposition 4.7. □

Now we derive a spectral characterization of the compactness and non-emptiness of the semigroup at infinity associated to a single operator.

Proposition 4.12. Let $(S, +) = (\mathbb{N}_0, +)$ and let $(T_n)_{n \in \mathbb{N}_0} = (T^n)_{n \in \mathbb{N}_0}$ be a bounded representation on a complex Banach space $E$. Then the following two assertions are equivalent:

(i) The semigroup at infinity, $T_\infty$, is non-empty and compact.

(ii) All spectral values of $T$ on the unit circle are poles of the resolvent of $T$.

In this case, $P_\infty$ coincides with the spectral projection of $T$ associated with $\sigma(T) \cap \mathbb{T}$.

Proof. (i) $\Rightarrow$ (ii): Let $\lambda \in \mathbb{T}$ be a spectral value of $T$. Let $\mathcal{K}$ denote the closed convex hull of the relatively compact set

$$\mathbb{T} \cdot \{T^n : n \in \mathbb{N}_0\};$$

then $\mathcal{K}$ is compact, too. Moreover, the operator $(r\lambda - \lambda)R(r\lambda, T)$ is contained in $\mathcal{K}$ for each $r > 1$; this is a consequence of the Neumann series representation of the resolvent. Consequently, the net $((r\lambda - \lambda)R(r\lambda, T))_{r \in (1, \infty)}$ (where $(1, \infty)$ is
directed conversely to the order inherited from \( \mathbb{R} \) has a convergent subnet. This shows, according to Proposition 4.13 in the appendix, that \( \lambda \) is a pole of \( \mathcal{R}(\cdot, T) \).

(ii) \( \Rightarrow \) (i): Note that, as a consequence of (ii), \( \sigma(T) \cap \mathbb{T} \) is isolated from the rest of the spectrum of \( T \); let \( P \) denote the spectral projection associated with \( \sigma(T) \cap \mathbb{T} \). We show that \( P \) satisfies the assumptions (a)–(c) in Proposition 4.10.

The spectral radius of \( T|_{\ker P} \) is strictly less than 1, so \( T^n|_{\ker P} \to 0 \) as \( n \to \infty \); this proves assumption (a). In order to show assumptions (b) and (c), note that the set \( \sigma(T) \cap \mathbb{T} \) is finite as a consequence of (ii) and enumerate its elements (if any exist) as \( \lambda_1, \ldots, \lambda_m \).

By assumption, each \( \lambda_k \) is a pole of the resolvent of \( T \), and its pole order equals 1 since \( T \) is power bounded. Hence, \( T \) acts as \( \lambda_k \) times the identity on the range of the associated spectral projection \( P_k \). It follows that \( T \) acts on \( PE = P_1E \oplus \cdots \oplus P_nE \) as the multiplication with the tuple \( (\lambda_1, \ldots, \lambda_m) \), which readily implies that \( \{(T|_{PE})^n | n \in \mathbb{N}_0\} \) is relatively compact with respect to the operator norm and that \( T^n x \) does not converge to 0 as \( n \to \infty \) for any \( x \in PE \). Thus, all assumptions (a)–(c) of Proposition 4.10 are satisfied, which shows that \( T_0^{\infty} \) is non-empty and compact and that \( P = P_\infty \). □

4.4. Semigroups that contain a quasi-compact operator. Recall that an operator \( T \in \mathcal{L}(E) \) on a Banach space \( E \) is called quasi-compact if there exists a compact operator \( K \in \mathcal{K}(E) \) and \( n \in \mathbb{N} \) such that \( \|T^n - K\| < 1 \). It is well known that, if the underlying scalar field is complex, a quasi-compact operator \( T \) has at most finitely many spectral values on the complex unit circle, and that all those spectral values are poles of the resolvent of \( T \) with finite-rank residuum. Hence, the spectral projection associated to the part of the spectrum on the unit circle has finite rank.

Quasi-compact operators – and in particular, compact operators – appear quite often in concrete applications. This is why the following proposition, in conjunction with Theorem 4.13 and Corollary 4.14, is very useful.

Proposition 4.13. Let \( (T_s)_{s \in S} \) be a bounded semigroup on a Banach space \( E \) such that, for some \( s_0 \in S \), the operator \( T_{s_0} \) is quasi-compact. Then the semigroup at infinity associated to \( (T_s)_{s \in S} \) is non-empty and compact, and the projection at infinity has finite rank.

Proof. We may assume that the underlying scalar field of \( E \) is complex, since otherwise we can consider a complexification of \( E \). According to Proposition 4.12, the semigroup at infinity associated to \( (T_{s_0}^n)_{n \in \mathbb{N}_0} \) is non-empty and compact; let \( P \) denote projection associated to this semigroup at infinity.

Then \( P \) commutes with each operator \( T_s \), so both \( \ker P \) and \( PE \) are invariant under the action of the semigroup \( (T_s)_{s \in [0, \infty)} \). Moreover, \( (T_{s_0}|_{\ker P})^n \to 0 \) as \( n \to \infty \), so it follows from Proposition 4.11 that actually \( \lim_{n \to \infty} T_{s_0|_{\ker P}} = 0 \). Additionally, it follows from Proposition 4.12 and the quasi-compactness of \( T_{s_0} \) that \( PE \) is finite-dimensional. Since our semigroup is bounded, the set \( \{T_s|_{PE} : s \in S\} \) is thus relatively compact in \( \mathcal{L}(PE) \), so it follows from Proposition 4.10 that the semigroup at infinity associated with \( (T_s)_{s \in S} \) is non-empty and compact, and that the projection at infinity, \( P_\infty \), satisfies \( P_\infty E \subseteq PE \). Hence, \( P_\infty \) has finite rank. □

In the situation of Proposition 4.13, the projections at infinity associated with \( (T_s)_{s \in S} \) and with \( (T_{s_0}^n)_{n \in \mathbb{N}_0} \) do coincide if the subsemigroup \( \{n s_0 : n \in \mathbb{N}_0\} \) is cofinal in \( S \) (see Corollary 4.14). Without this additional assumption, the projections at infinity do not need to coincide, as the following examples show.

Examples 4.14. (a) Consider the semigroup \( S = [0, \infty) \) where the semigroup operation is given by the maximum operator \( \vee \); then the order in \( S \) coincides with
the usual order on \([0, \infty)\). Now, let \(E = \mathbb{C}^2\), let \(Q \in \mathcal{L}(\mathbb{C}^2)\) be the projection onto the first component and define
\[
T_s = \begin{cases} 
\text{id}_{\mathbb{C}^2} & \text{if } s \in [0, 1], \\
Q & \text{if } s \in (1, \infty).
\end{cases}
\]
Then \((T_s)_{s \in S}\) is a bounded representation of \(([0, \infty), \lor)\), its semigroup at infinity as non-empty and compact and its projection at infinity equals \(Q\). The operator \(T_1\) is compact, but the projection at infinity associated to \((T^n_n)_{n \in \mathbb{N}_0}\) is \(\text{id}_{\mathbb{C}^2}\).

(b) Here is also an example where the underlying semigroup is cancellative: Let \(S = [0, \infty)^2\), together with the componentwise addition \(+\). Let \(E = \mathbb{C}^2\), let \(Q \in \mathcal{L}(\mathbb{C}^2)\) denote the projection onto the first component and \(P \in \mathcal{L}(\mathbb{C}^2)\) the projection onto the second component. We define a representation \((T_{(s,t)})_{(s,t) \in [0, \infty)^2}\) by
\[
T_{(s,t)} = \begin{cases} 
\text{id}_{\mathbb{C}^2} & \text{if } (s, t) = 0, \\
Q & \text{if } s > 0 \text{ and } t = 0, \\
P & \text{if } s = 0 \text{ and } t > 0, \\
0 & \text{if } s > 0 \text{ and } t > 0.
\end{cases}
\]
Then \((T_{(s,t)})_{(s,t) \in [0, \infty)^2}\) is a bounded representation with non-empty and compact semigroup at infinity; its projection at infinity equals \(0\). The operator \(T_{(0,1)}\) is compact, but the projection at infinity associated with \((T^n_n)_{n \in \mathbb{N}_0}\) equals \(P\).

4.5. Beyond the quasi-compact case. The purpose of this subsection are various theoretical considerations which are, in our point of view, illuminating and interesting, but which are not need throughout the rest of the paper. While the situation of Proposition 4.13 is most important for applications, it is not completely satisfying from a theoretical point of view. Indeed, for every Banach space \(E\) and every commutative monoid \((S, +)\) the semigroup at infinity associated to the trivial semigroup \((\text{id}_E)_{s \in S}\) is non-empty and compact, but \(\text{id}_E\) is not quasi-compact unless \(E\) is finite-dimensional.

In the case of a time-discrete semigroup \((T^n)_{n \in \mathbb{N}_0}\) the non-quasi-compact case is still covered by Proposition 4.12 – where non-quasi-compactness of \(T\) means precisely that at least one spectral value on the unit circle has infinite-dimensional eigenspace. It would be satisfying to have a similar result for more general semigroups \((S, +)\) at hand, at least for the semigroup \(([0, \infty), +)\). However, the following example shows the things are not that simple.

Example 4.15. There exists an \(L^2\)-space and a bounded positive semigroup \(T = (T_t)_{t \in [0, \infty)}\) on it with the following properties:
(a) The spectrum of every operator \(T_t\) is finite and consists of poles of the resolvent.
(b) The semigroup at infinity, \(T^\infty\), is not compact.
Indeed, let \(U \subseteq \mathbb{C}\) denote the group of all roots of unity and consider the space \(\ell^2(U)\). Note that there exists a group homomorphism \(\varphi : \mathbb{R} \to \mathbb{Q}\) which acts as the identity on \(\mathbb{Q}\) (the existence of \(\varphi\) follows from the fact the \(\mathbb{R}\), seen as a vector space over \(\mathbb{Q}\), possesses a basis that contains the number 1). We define \(T\) by
\[
T_t f(z) = f(e^{2\pi i \varphi(t)} z)
\]
for \(t \in [0, \infty), f \in \ell^2(U)\) and \(z \in U\). Obviously, the semigroup obtained this way is bounded and positive.
For every time \(t\) there exists an integer \(n \in \mathbb{N}\) such that \(T^n_t = I\) (indeed, one simply has to choose \(n\) such that \(n\varphi(t)\) is an integer). Hence, every operator \(T_t\) is algebraic (i.e. mapped to 0 by a polynomial), so it follows that property (a) is satisfied.
On the other hand, choose a sequence \((q_n)_{n \in \mathbb{N}}\) of positive rational numbers which converges to \(\infty\) and such that \(e^{2\pi i q_n} \neq e^{2\pi i q_m}\) whenever \(n \neq m\). By applying the sequence \((T_{t_{q_n}})_{n \in \mathbb{N}}\) to any canonical unit vector in \(\ell^2(U)\) we can see that no subsequence of this sequence converges (not even strongly) as \(n \to \infty\). Hence, it follows from Proposition 4.17 that the semigroup at infinity is either empty or not compact. Since \(T^\infty\) clearly contains the identity operator, we thus conclude that \(T^\infty_{\infty}\) is not compact.

**Remark 4.16.** (a) In the situation of Example 4.15 the semigroup at infinity associated with the time discrete semigroup \((T_{nt})_{n \in \mathbb{N}}\) is, for any time \(t \in (0, \infty)\), non-empty and compact; this follows from Proposition 4.12. On the other hand, the semigroup at infinity associated with the entire semigroup \(T\) is not compact. This shows that the implication in Corollary 4.19 does not have a simple converse.

(b) It is easy to modify Example 4.15 in such a way that all orbits of the semigroup become relatively compact: just replace \(\ell^2(U)\) with \(L^2(\mathbb{T})\) in the example and construct the semigroup in the same way. Then, for each \(f \in L^2(\mathbb{T})\), the orbit \(\{T_t f : t \in [0, \infty]\}\) is a subset of the compact set \(\{f(e^{2\pi i \theta}) : \theta \in [0, 1]\}\) and thus, the orbit is relatively compact. However, we can see similarly as in Example 4.15 that the semigroup at infinity is not compact.

Example 4.15 shows that, if the semigroup at infinity associated to \((T_n^u)_{n \in \mathbb{N}_0}\) is non-empty and compact for each \(s_0 \in [0, \infty)\), we cannot automatically conclude that the semigroup at infinity associated to \((T_s)_{s \in [0, \infty)}\) is non-empty and compact. If we want this implication to be true we need an additional assumption, and this is the only time in the theoretical part of this paper where we are forced to impose a time regularity condition on our semigroup. In fact, if the semigroup is strongly continuous at a strictly positive time, we obtain the following characterisation.

**Theorem 4.17.** Let \(E\) be a Banach space and let \((T_s)_{s \in [0, \infty)}\) be a bounded semigroup on \(E\) which is strongly continuous at at least one point \(s_0 \in (0, \infty)\). The following assertions are equivalent:

(i) For each \(s \in (0, \infty)\) the semigroup at infinity associated with \((T_n^u)_{n \in \mathbb{N}_0}\) is non-empty and compact.

(ii) The semigroup at infinity associated with \((T_s)_{s \in [0, \infty)}\) is non-empty and compact.

If the underlying scalar field of \(E\) is complex, the above assertions (i) and (ii) are also equivalent to:

(iii) For each \(s \in (0, \infty)\) all spectral values of \(T_s\) on the complex unit circle are poles of the resolvent of \(T_s\).

The proof of Theorem 4.17 requires a bit of preparation. Let \(\varphi : K \to K\) be a continuous map on some compact Hausdorff space \(K\). In this case, the pair \((K; \varphi)\) is called a topological dynamical system. Further, a point \(x \in K\) is called recurrent for the system \((K; \varphi)\) if for each neighbourhood \(U \subseteq K\) of \(x\) there is \(n \in \mathbb{N}\) such that \(\varphi^n(x) \in U\). It is not hard to see that \(x \in K\) is recurrent if and only if \(x \in K\) is infinitely recurrent, that is for each neighbourhood \(U \subseteq K\) of \(x\) and each \(n_0 \in \mathbb{N}\) there is \(n \in \mathbb{N}\) with \(n \geq n_0\) such that \(\varphi^n(x) \in U\). More facts on recurrence in topological dynamical systems can for instance be found in [27] Chapter 3.2]. We now use these facts to prove the following lemma.

**Lemma 4.18.** There exists a cofinal net \((n_j)_{j \in \mathbb{N}}\) in \(\mathbb{N}\) such that the net \((\lambda^{n_j})_{j}\) converges to 1 for each \(\lambda \in \mathbb{T}\).

**Proof.** Endow \(G := \mathbb{T}^\mathbb{N}\) with the topology of pointwise convergence and with the pointwise multiplication. Then \(G\) is a compact topological group. Set \(1 := (1)_{\lambda \in \mathbb{T}}\)
and let \( \varphi : G \to G \) be given by \( \varphi(\mu) = (\lambda \mu) \lambda \in \mathbb{T} \) for each \( \mu = (\mu_\lambda) \lambda \in \mathbb{T} \). Then \( \varphi \) is continuous and the topological dynamical system \((G,\varphi)\) is a so-called group rotation. Hence, by [2] Proposition 3.12(d)] every point in \( G \) is recurrent with respect to \((G,\varphi)\) and thus, so is \( 1 \).

Now let \( \mathcal{U} \) denote the neighbourhood filter of \( 1 \) in \( G \), ordered by converse set inclusion, and endow \( \mathcal{U} \times \mathbb{N} \) with the product order, which renders it a directed set. For each pair \((U,k) \in \mathcal{U} \times \mathbb{N}\) we can find a number \( n(U,k) \in \mathbb{N} \) such that \( n(U,k) \geq k \) and \( \varphi^n(U,k)(1) \in U \). Hence, the net \( \left( \varphi^n(U,k)(1) \right)_{(U,k) \in \mathcal{U} \times \mathbb{N}} \) converges to \( 1 \) in \( G \), which means that \( \left( \lambda^n(U,k) \right)_{(U,k) \in \mathcal{U} \times \mathbb{N}} \) converges to 1 for each \( \lambda \in \mathbb{T} \). Moreover, the net \( \left( n(U,k) \right)_{(U,k) \in \mathcal{U} \times \mathbb{N}} \) is clearly cofinal in \( \mathbb{N} \) by construction. \( \square \)

Now we can show that, if the semigroup at infinity of a time-discrete operator semigroup \((T^n)_{n \in \mathbb{N}_0}\) is non-empty and compact, then there exists a subnet \((T^n)_{n \in \mathbb{N}}\), which converges to \( P_\infty \), where \( (n_j)_j \) can be chosen independently of the operator \( T \) (and also independently of the underlying Banach space).

**Proposition 4.19.** Let \( (n_j)_j \) be a cofinal net in \( \mathbb{N} \) such that \( (\lambda^n)_j \) converges to 1 for each \( \lambda \in \mathbb{T} \). If \( E \) is a Banach space and \((T^n)_{n \in \mathbb{N}_0} \in \mathcal{L}(E) \) is a bounded semigroup on \( E \) whose semigroup at infinity is non-empty and compact, then \((T^n)_{n \in \mathbb{N}}\), converges to \( P_\infty \).

**Proof.** We may assume throughout the proof that the scalar field is complex, since otherwise we may replace \( E \) with a complexification. We know from Proposition 4.12 that \( P_\infty \) is the spectral projection of \( T \) associated with the single spectral values \( \lambda_1, \ldots, \lambda_m \). The operator \( T \) acts on the space \( E_\infty \) as the multiplication with the tuple \( (\lambda_1, \ldots, \lambda_m) \), so it follows readily that \( (T|_{E_\infty})^n_j \to 0 \). Moreover, \( E_\infty \) can be decomposed as

\[
E_\infty = P_1 E \oplus \cdots \oplus P_mE,
\]

where \( m \in \mathbb{N}_0 \), \( \sigma(T) \cap \mathbb{T} = \{\lambda_1, \ldots, \lambda_m\} \) and \( P_1, \ldots, P_m \) are the spectral projections associated with the single spectral values \( \lambda_1, \ldots, \lambda_m \). It follows that \( (T|_{E_\infty})^n_j \to 0 \). \( \square \)

**Proof of Theorem 4.17.** We may assume throughout the proof that \( E \) is a complex Banach space since we can otherwise replace \( E \) with a complexification. Assertions (i) and (iii) are equivalent by Proposition 4.12 and (ii) implies (i) by Corollary 4.14. Let us now prove that (i) implies (ii).

For each \( s \in (0,\infty) \) denote by \( P_{\infty,s} \) the corresponding projection belonging to the semigroup at infinity associated with the representation \((T_{ns})_{n \in \mathbb{N}_0}\); then \( P_{\infty,s} \) is also the spectral projection of \( T_s \) that belongs to the intersection of the spectrum with the unit circle. Let \( (n_j) \subseteq \mathbb{N} \) be a cofinal net with the property asserted in Lemma 4.18. According to Proposition 4.19 we have \( T_{ns} = T^n_{s_j} \to P_{\infty,s} \) for each \( s \in (0,\infty) \), which implies that the operator family \( (P_{\infty,s})_{s \in (0,\infty)} \) satisfies the semigroup law. This in turn implies that all the projections \( P_{\infty,s} \) coincide (see [11] Lemma 2.2); from now on, we set \( P := P_{\infty,\infty} \) for all \( s \in (0,\infty) \). Since all operators \( T_s \) commute with \( P \), our the semigroup \((T_s)_{s \in [0,\infty)} \) leaves both \( ker P \) and \( PE \) invariant. It remains to prove that \( P \) satisfies the conditions (a) and (b) of Proposition 4.10.

(a) It is an immediate consequence of Proposition 4.11 that \( (T_s|_{ker P})_{s \in [0,\infty)} \) converges to 0 as, for instance, the powers of \( T_1|_{ker P} = T_1|_{ker P_{\infty,1}} \) converge to 0.

(b) It follows from Theorem 4.3(b) that, for each \( s \in (0,\infty) \), the operator \( T_s|_{PE} \) is invertible on \( PE \). Hence, the semigroup \((T_s|_{PE})_{s \in [0,\infty)} \) extends to a group on \( PE \). Since the semigroup is strongly continuous at least one time, it thus follows
that it is strongly continuous at all times $s \in [0, \infty)$. Let $A$ denote the generator of the $C_0$-semigroup $(T_t|_{PE})_{s \in [0, \infty]}$.

Let us show that the operator $A$ has at most finitely many eigenvalues on the imaginary axis. So assume to the contrary that the set $iB := \sigma_{\text{pur}}(A) \cap i\mathbb{R}$ is infinite. Choose two time $s,t \in (0, \infty)$ such that $s/t$ is irrational. Since $e^{itB}$ consists of unimodular eigenvalues of $T_t|_{PE}$, it follows that this set is finite. Hence, there exists an infinite subset $iC$ of $iB$ whose values are all mapped to the same number by the mapping $\exp(\cdot t)$. Thus, $t(c_1 - c_2) \in 2\pi \mathbb{Z}$ for all $c_1, c_2 \in C$. Consequently, $s(c_1 - c_2) = \frac{s}{t}(c_1 - c_2) \notin 2\pi \mathbb{Z}$ for any two distinct $c_1, c_2 \in C$, which conversely implies that all the values $e^{isc}$ are distinct for $c \in C$. However, each such number is an eigenvalue of $T_s|_{PE}$; this is a contradiction since $T_s|_{PE}$ has only finitely many eigenvalues.

Let $i\beta_1, \ldots, i\beta_n$ denote the eigenvalues of $A$ on the imaginary axis (at least one such eigenvalue exists unless $PE = \{0\}$) and denote their corresponding eigenspaces by $E_1, \ldots, E_n$. We note that $PE = E_1 \oplus \cdots \oplus E_n$. To see this, choose a sufficiently small number $s_0 \in (0, \infty)$ such that all the numbers $e^{is_0\beta_1}, \ldots, e^{is_0\beta_n}$ are distinct. Then, for each $k \in \{1, \ldots, n\}$, the space $E_k$ is the eigenspace of $T_{s_0}$ for the eigenvalue $e^{is_0\beta_k}$ [7, Corollary IV.3.8(ii)]. Consequently, $E_k$ is even the spectral space of $T_{s_0}$ for the spectral value $e^{is_0\beta_k}$ since the latter number is a first order pole of the resolvent of $T_{s_0}$ (as $T_{s_0}$ is power-bounded). Moreover, $P$ is the spectral projection of $T_{s_0}$ corresponding to the part $\sigma(T_{s_0}) \cap \mathbb{T} = \{e^{is_0\beta_1}, \ldots, e^{is_0\beta_n}\}$ of the spectrum, so indeed

$$PE = \ker(e^{is_0\beta_1} - T_{s_0}) \oplus \cdots \oplus \ker(e^{is_0\beta_n} - T_{s_0}) = E_1 \oplus \cdots \oplus E_n.$$

As the semigroup $(T_t|_{PE})_{s \in [0, \infty)}$ acts on $E_k$ as the multiplication with $(e^{is\beta_k})_{s \in [0, \infty)}$, it follows that $\{T_s|_{PE} : s \in [0, \infty)\}$ is relatively compact in $\mathcal{L}(PE)$.

\section{Triviality of compact operator groups}

Loosely speaking, the major theoretical consequence of Corollary 4.3 is that, if we would like to find sufficient criteria for an operator semigroup to converge with respect to the operator norm, then we should seek for criteria which ensure that a compact operator group is trivial. This is the purpose of the present section.

\subsection{Connected groups and a spectral condition}

We start with a theorem on the triviality of connected compact groups of linear operators. The corollaries of this theorem that we list at the end of this subsection will be powerful tools in Section 6 when we finally prove various concrete convergence theorems for positive operator semigroups. It is important for us that a compact topological group $G$ is connected if and only if it is divisible in the sense that, for each $g \in G$ and each $n \in \mathbb{N}$, there exists $h \in G$ such that $h^n = g$ [24, Theorem 2].

\textbf{Theorem 5.1.} Let $E$ be a complex Banach space and let $\mathcal{G} \subseteq \mathcal{L}(E)$ be a connected and compact subgroup of the invertible linear operators on $E$. If, for each $T \in \mathcal{G}$, all spectral values of $T$ are roots of unity, then $\mathcal{G} = \{\text{id}_E\}$.

For the proof of Theorem 5.1 we need a bit of Banach algebra theory, specifically the following lemma. For the convenience of the reader, we include its simple proof.

\textbf{Lemma 5.2.} Let $A$ be a complex Banach algebra with multiplicatively neutral element $1$ and let $\mathcal{B} \subseteq A$ be a commutative subalgebra which is maximal among all commutative subalgebras of $A$. Then the following assertions hold:

(i) $\mathcal{B}$ is closed and contains $1$.

(ii) For each $b \in \mathcal{B}$ its spectrum in $A$ coincides with its spectrum in $\mathcal{B}$. 

Proof. (i) This follows immediately from the maximality of $B$.

(ii) Fix $b \in B$. Clearly, the spectrum of $b$ in $A$ is contained in the spectrum of $b$ in $B$. To show the converse inclusion, let $\lambda$ be in the resolvent set of $b$ with respect to $A$. Observe that the inverse $(\lambda - b)^{-1}$ commutes with all elements in $B$. Therefore, the linear span of the set
\[ \{ (\lambda - b)^{-n} : n \in \mathbb{N}_0 \} \cdot B \]
is a commutative subalgebra of $A$ that contains $B$ and thus coincides with $B$. Hence, $(\lambda - b)^{-1} \in B$, i.e., $\lambda$ is contained in the resolvent set of $b$ in $B$. \hfill \Box

Note that if $G \subseteq \mathcal{L}(E)$ is a compact subgroup of the invertible linear operators on a complex Banach space $E$, then $\sup_{n \in \mathbb{Z}} \|T^n\| < \infty$ for all $T \in G$, i.e., each operator in $G$ is doubly power-bounded. Now we can prove Theorem 5.1.

**Proof of Theorem 5.1.** According to [24] Theorem 2 every element of a connected compact group is contained in a connected (and closed) subgroup, so it suffices to prove the assertion for commutative $G$.

Let $B$ be a subalgebra of $\mathcal{L}(E)$ which is maximal among all commutative subalgebras of $\mathcal{L}(E)$ that contain $G$ (such a $B$ exists by Zorn’s lemma). Then $B$ is also maximal among all commutative subalgebras of $\mathcal{L}(E)$, so according to Lemma 5.2 $B$ is closed and contains $id_E$; moreover, for each $T \in B$ the spectrum $\sigma(T)$ of $T$ in $\mathcal{L}(E)$ and its spectrum in $B$ coincide. Hence, if $\Omega(B)$ denotes the character space of the Banach algebra $B$, then we have
\[ \sigma(T) = \{ \varphi(T) : \varphi \in \Omega(B) \}; \]
see e.g. [23] Theorem 1.3.4(1)]. Since each $\varphi \in \Omega(B)$ is continuous from $B$ to $\mathbb{C}$, it follows that $\varphi(G)$ is a connected subset of $\mathbb{C}$ for each character $\varphi$.

On the other hand, it follows from our spectral assumption that $\varphi(G)$ consists of roots of unity only; consequently, $\varphi(G) = \{ 1 \}$ since $\varphi(id_E)$ always contains $\varphi(id_E) = 1$. We conclude that $\sigma(T) = \{ 1 \}$ for each $T \in G$, so each such $T$ equals $id_E$ by Gelfand’s $T = id$ theorem since $T$ is doubly power-bounded (see e.g. [7] Theorem B.17). \hfill \Box

The condition that all spectral values of any $T \in G$ are roots of unity is automatically fulfilled in two important situations. The first one is that the underlying space is a Banach lattice and all operators in $T$ are positive; this is the content of the following corollary.

**Corollary 5.3.** Let $E$ be a Banach lattice and let $G \subseteq \mathcal{L}(E)$ be a connected and compact group of positive linear operators. Then $G = \{ id_E \}$.

**Proof.** We may assume that the scalar field is complex. According to Theorem 5.1 it suffices to show that the spectrum of each $T \in G$ consists of roots of unity only, so fix $T \in G$. Clearly, $\sigma(T) \subseteq \mathbb{T}$, so it follows from Proposition 4.12 that $\sigma(T)$ is finite and consists of poles of the resolvent.

It follows from infinite-dimensional Perron-Frobenius theory (see e.g. [26] Theorem V.4.4, or Theorem V.4.9 and its Corollary]) that the spectrum of $T$ is cyclic, meaning that $\lambda^n \in \sigma(T)$ for all $n \in \mathbb{Z}$ whenever $\lambda \in \sigma(T)$. By the finiteness of the spectrum, this implies that $\sigma(T)$ consists of roots of unity only. \hfill \Box

Our second corollary deals with the case of contractive operators on so-called projectively non-Hilbert spaces. This notion is taken from [15] Definition 3.1]: a real Banach space $E$ is called projectively non-Hilbert if, for no rank-2 projection $P \in \mathcal{L}(E)$, the range $PE$ is isometrically a Hilbert space. Every $L^p$-space over an arbitrary measure space is projectively non-Hilbert if $p \in [1, \infty] \setminus \{ 2 \}$, see [15] Example 3.2 and the discussion after [15] Example 3.5]. Moreover, every real Banach
lattice that is a so-called AM-space is projectively non-Hilbert [14] Example 1.2.7; this includes the space of real-valued bounded and continuous functions on any topological space.

**Corollary 5.4.** Let $E$ be a real Banach space that is projectively non-Hilbert and let $G \subseteq \mathcal{L}(E)$ be a connected and compact subgroup of the invertible bounded linear operators on $E$ and assume that each operator in $G$ is contractive. Then $G = \{ \text{id}_E \}$.

**Proof.** Let $E_C$ denote a Banach space complexification of $E$; for each $T \in G$ we denote the canonical extension of $T$ to $E_C$ by $T_C$. Then $G_C := \{ T_C : T \in G \}$ is a connected and compact subgroup of the invertible bounded linear operators on $E_C$.

Now fix $T \in G$; it suffices to prove that the spectrum of $T_C$ consists of roots of unity only. By Proposition 4.7 the semigroup at infinity associated to $(T^n_C)_{n \in \mathbb{N}_0}$ is non-empty and compact, so it follows from Proposition 4.12 that $\sigma(T_C)$ is a finite subset of the complex unit circle and consists of eigenvalues of $T_C$. Moreover, the set $\{ T^n_C : n \in \mathbb{N}_0 \}$ is relatively compact with respect to the weak operator topology, i.e. $T_C$ is weakly almost periodic. Since $E$ is projectively non-Hilbert, we can now apply [15] Theorem 3.11 to conclude that the spectrum of $T_C$ consists of roots of unity only. \qed

5.2. **Strong positivity of groups.** Another way to ensure that a group of linear operators is trivial is to ensure a certain condition of strong positivity; this works in the very general setting of ordered Banach spaces. By an ordered Banach space we mean a tuple $(E, E_+)$ where $E$ is a real Banach space and $E_+$ is a closed subset of $E$ such that $\alpha E_+ + \beta E_+ \subseteq E_+$ for all $\alpha, \beta \in [0, \infty)$ and such that $E_+ \cap (-E_+) = \{ 0 \}$; the set $E_+$ is called the positive cone in $E_+$.

Let $(E, E_+)$ be an ordered Banach space. An operator $T \in \mathcal{L}(E)$ is called positive if $TE_+ \subseteq E_+$. A functional $\varphi \in E'$ is called positive if $\langle \varphi, f \rangle \geq 0$ for all $f \in E_+$. A vector $f \in E_+$ is called an almost interior point of $E_+$ if $\langle \varphi, f \rangle > 0$ for each non-zero positive functional $\varphi \in E'$. If, for instance, $E$ is an $L^p$-space over a $\sigma$-finite measure space and $p \in [1, \infty)$, then a function $f \in E_+$ is an almost interior point if and only if $f(\omega) > 0$ for almost all $\omega \in \Omega$. For more information about almost interior points we refer to [17] Section 2). The following result is inspired by the proof of [17] Theorem 4.1.

**Theorem 5.5.** Let $(E, E_+) \neq \{ 0 \}$ be an ordered Banach space with $E_+ \neq \{ 0 \}$ and let $G \subseteq \mathcal{L}(E)$ be a norm-bounded subgroup of the invertible operators on $E$. Assume that every operator in $G$ is positive and that, for each $f \in E_+ \setminus \{ 0 \}$, there exists $T \in G$ such that $Tf$ is an almost interior point of $E_+$. Then $E$ is one-dimensional and $G = \{ \text{id}_E \}$.

**Proof.** We first show that every point in $E_+ \setminus \{ 0 \}$ is an almost interior point of $E_+$. So let $f \in E_+ \setminus \{ 0 \}$. Choose $T \in G$ such that $Tf$ is an almost interior point of $E_+$. Since $T^{-1}$ is an element of $G$, it is a positive operator on $E$, and since $T^{-1}$ is surjective it thus follows from [17] Corollary 2.22(a) that $T^{-1}$ maps almost interior points to almost interior points. Hence, $f = T^{-1} Tf$ is an almost interior point.

Since all vectors in $E_+ \setminus \{ 0 \}$ are almost interior points, it follows from [17] Theorem 2.10 that $E$ is one-dimensional. We can thus identify $G$ with a bounded subgroup of the multiplicative group $(0, \infty)$, so indeed $G$ does indeed consist of one element only. \qed

6. **Operator norm convergence of semigroup representations.**

In this section we finally derive convergence theorems for various classes of operator semigroups. In Subsection 6.1 we consider representation whose underlying semigroup $(S, +)$ satisfies a certain kind of divisibility condition. In Subsection 6.2...
we then deal with positive semigroups on ordered Banach spaces under an appropriate strong positivity assumption.

6.1. Convergence under divisibility conditions. We call the semigroup \((S,+)\) essentially divisible if, for each \(a \in S\) and each integer \(n \in \mathbb{N}\), there exist elements \(t_1, t_2 \in S\) such that \(nt_1 = s + nt_2\). This definition is taken from [16], where it was used as a generalisation of semigroups that generated divisible groups (which played an important role in [12]). Let us illustrate the notion of essential divisibility with a list of simple examples.

Examples 6.1. (a) The semigroup \(([0, \infty),+\)) is essentially divisible, and so is \((\mathbb{Q} \cap [0, \infty),+,\).
(b) More generally, for each \(a \geq 0\), both the semigroup \(([0] \cup [a, \infty),+,\)) and the semigroup \(([0] \cup (\mathbb{Q} \cap [a, \infty)),+,\)) are essentially divisible.
(c) The semigroup \(([0, \infty)^n,+)\) is essentially divisible for any \(n \in \mathbb{N}\).
(d) The semigroup \(([0, \infty),\max)\) is essentially divisible; here, \(\max\) denotes the binary operator which assigns the maximum to any two given elements of \([0, \infty)\).
(e) More generally, if \(L\) is a lattice with a smallest element \(i\), then \((L, \vee)\) is an essentially divisible semigroup (with neutral element \(i\)).
(f) The semigroup \(([0, \infty),+,\)) is not essentially divisible.
(g) The semigroup \((D,+)\), where \(D = \{k/2^n : k, n \in \mathbb{N}_0\}\) is the set of dyadic numbers in \([0, \infty)\), is not essentially divisible.

We now use the notion of essential divisibility to prove a convergence theorem for positive semigroups and Banach lattices and a convergence theorem for contractive semigroups on projectively non-Hilbert spaces. Let us begin with the positive case.

Theorem 6.2. Let \(E\) be a Banach lattice and let \((T_s)_{s \in S}\) be a positive and bounded representation on \(E\). If the semigroup at infinity, \(T_\infty\), is non-empty and compact and if \((S,+)\) is essentially divisible, then \((T_s)_{s \in S}\) converges with respect to the operator norm to the projection at infinity.

Proof. Note that the range \(E_\infty\) of the projection at infinity, \(P_\infty\), is again a Banach lattice since \(P_\infty\) is positive. Consider the set \(T := \{T_s : s \in S\}\). As \((S,+)\) is essentially divisible, a simple compactness argument shows that \(T|_{E_\infty} = T_\infty|_{E_\infty}\), which is a compact group according to Theorem 6.3, is divisible. Thus, [24, Theorem 2] shows that \(T|_{E_\infty}\) is connected; therefore, it is trivial by Corollary 6.3. Finally, Corollary 6.4 yields the claim.

The following corollary is due to Lotz in the special case where \(S = [0, \infty)\).

Corollary 6.3. Let \(E\) be a Banach lattice and let \((T_s)_{s \in S}\) be a positive and bounded semigroup on \(E\). If \(T_\infty\) is quasi-compact for at least one \(s_0 \in S\) and if \((S,+)\) is essentially divisible, then \((T_s)_{s \in S}\) converges with respect to the operator norm to a finite rank projection.

Proof. This is an immediate consequence of Corollary 6.3 and Theorem 6.2.

Our second corollary – which only deals with the semigroup \(([0, \infty),+)\) – has the nice theoretical feature that it covers, in contrast to Corollary 6.3, also the trivial operator semigroup that consists merely of the operator \(\text{id}_E\) – which is arguably the simplest convergent operator semigroup.

Corollary 6.4. Let \(E\) be a complex Banach lattice and let \((T_s)_{s \in [0, \infty)}\) be a positive and bounded semigroup on \(E\) which is strongly continuous at least one time \(s_0 \in (0, \infty)\). If, for each \(s \in (0, \infty)\), all spectral values of \(T_s\) on the unit circle are poles of the resolvent, then \(T_s\) converges with respect to the operator norm as \(s \to \infty\).
Proof. This is an immediate consequence of Theorems 4.17 and 6.2.

Now we deal with real Banach spaces which are projectively non-Hilbert; see the discussion before Corollary 5.4 for a definition of this property.

Theorem 6.5. Let \( E \) be a real Banach space that is projectively non-Hilbert and let \((T_s)_{s \in S}\) be a contractive semigroup on \( E \). If the semigroup at infinity, \( T_{\infty}^S \), is non-empty and compact and if \((S, +)\) is essentially divisible, then \((T_s)_{s \in S}\) converges with respect to the operator norm to the projection at infinity.

Proof. Note that the semigroup at infinity, \( P_{\infty} \), as contractive, and hence its range is itself a projectively non-Hilbert space. Now the theorem has the same proof as Theorem 6.2 except that one has to employ Corollary 5.4 instead of Corollary 5.3.

□

Again, we state the same result separately for the quasi-compact case.

Corollary 6.6. Let \( E \) be a real Banach space that is projectively non-Hilbert and let \((T_s)_{s \in S}\) be a contractive semigroup on \( E \). If \( T_{s_0} \) is quasi-compact for at least one \( s_0 \in S \) and if \((S, +)\) is essentially divisible, then \((T_s)_{s \in S}\) converges with respect to the operator norm to a finite rank projection.

Proof. This is an immediate consequence of Corollary 4.13 and Theorem 6.5.

□

As similar result as in Corollary 6.4 is, of course, also true for contractive semigroups on projectively non-Hilbert spaces, but we refrain from stating this explicitly as a corollary.

Finally, we note that Theorems 1.2 and 1.3 from the Introduction follow from Corollary 6.6:

Proof of Theorems 1.2 and 1.3. (i) \( \Rightarrow \) (ii): This implication is obvious in both theorems.

(ii) \( \Rightarrow \) (i): In both theorems, the semigroup is assumed to leave the real part of the underlying Banach lattice invariant, so it suffices to prove that the restriction of the semigroup to this real part converges to a finite rank projection. Since, in both theorems, the real part of the underlying space is projectively non-Hilbert and since the semigroup \(((0, \infty), +)\) is essentially divisible, the assertion follows from Corollary 6.6.

□

6.2. Convergence under a strict positivity condition. The following theorem is generalisation of [17, Theorem 5.3] where only the cases \( S = \mathbb{N}_0 \) and \( S = [0, \infty) \) were considered. For an ordered Banach space \((E, E^+)\) (see the beginning of Subsection 5.2) we call a function \( \varphi \in E' \) strictly positive if \( \langle \varphi, f \rangle > 0 \) for all \( f \in E^+ \setminus \{0\} \).

Theorem 6.7. Let \((E, E^+)\) be an ordered Banach space with \( E^+ \neq \{0\} \) and let \((T_s)_{s \in S}\) be a bounded semigroup on \( E \) which is positive in the sense that \( T_s E^+ \subseteq E^+ \) for each \( s \in S \). Assume moreover that \( T_{s_0} \) is quasi-compact for at least one \( s_0 \in S \) and that the following strong positivity condition holds: for each \( f \in E^+ \setminus \{0\} \) there exists \( s \in S \) such that \( T_s f \) is an almost interior point of \( E^+ \).

Then \((T_s)_{s \in S}\) converges with respect to the operator norm to a projection in \( \mathcal{L}(E) \) of rank at most 1.

Proof. According to Proposition 4.13 the semigroup at infinity, \( T_{\infty}^S \), is non-empty and compact since \((T_s)_{s \in S}\) is bounded and since \( T_{s_0} \) is quasi-compact. Let \( P_{\infty} \) denote the corresponding projection at infinity. Then \( P_{\infty} \) is a positive operator and hence, its range \( P_{\infty} E \) is also an ordered Banach space with positive cone...
The matrix $P = \{1, \ldots, N\}^N$ satisfies the assumptions of Theorem 5.5. Therefore, we recall a characterisation of dissipativity of matrices with real entries played quite an important role. In Sections 2 and 3 dissipativity of matrices with real entries played quite an important role. Therefore, we recall a characterisation of dissipativity of matrices with real entries played quite an important role.

**APPENDIX A. DISSIPATIVITY OF REAL MATRICES**

In Sections 2 and 3 dissipativity of matrices with real entries played quite an important role. Therefore, we recall a characterisation of dissipativity of matrices with respect to various $\ell^p$-norms in the following proposition.

**Proposition A.1.** Let $C = (c_{jk}) \in \mathbb{R}^{N \times N}$.

(a) The matrix $C$ is dissipative with respect to the $\ell^2$-norm on $\mathbb{R}^N$ if and only if all eigenvalues of the symmetric part $\frac{1}{2}(C + C^T)$ of $C$ are contained in $(-\infty, 0]$.  

(b) The matrix $C$ is dissipative with respect to the $\ell^1$-norm on $\mathbb{R}^N$ if and only if

$$c_{kk} \leq \sum_{j \in \{1, \ldots, N\} \setminus \{k\}} |c_{jk}|$$

for each $k \in \{1, \ldots, N\}$.

(c) The matrix $C$ is dissipative with respect to the $\ell^\infty$-norm on $\mathbb{R}^N$ if and only if

$$c_{kk} \leq \sum_{j \in \{1, \ldots, N\} \setminus \{k\}} |c_{kj}|$$

for each $k \in \{1, \ldots, N\}$.

(d) Let $p \in [1, \infty)$. The matrix $C$ is dissipative with respect to the $\ell^p$-norm on $\mathbb{R}^N$ if and only if

$$(\text{sgn} \xi \cdot |\xi|^{p-1})^T C \xi \leq 0$$

for all $\xi \in \mathbb{R}^N$; here, the vector $\text{sgn} \xi$ contains the signs of the entries of $\xi$, and its product with the vector $|\xi|^{p-1}$ is computed entrywise.

**Proof.** (a) It follows from the $C_2T C \xi = \xi^T \left[ \frac{1}{2}(C + C^T) \right] \xi$ for all $\xi \in \mathbb{R}^N$ that $C$ is dissipative with respect to the $\ell^2$-norm on $\mathbb{R}^N$ if and only if $\frac{1}{2}(C + C^T)$ is so. Since $\frac{1}{2}(C + C^T)$ is symmetric, this proves the assertion.

(d) Endow $\mathbb{R}^N$ with the $\ell^p$-norm for a fixed $p \in [1, \infty)$. For each $\xi \in \mathbb{R}^N$ of norm $\|\xi\|_p = 1$ the vector $\text{sgn} \xi \cdot |\xi|^{p-1}$, if considered as an element of the dual space of $\mathbb{R}^N$, also has norm 1 and satisfies $(\text{sgn} \xi \cdot |\xi|^{p-1})^T \xi = 1$. Since a matrix is dissipative if and only if it is strictly dissipative, this proves (d).
(b) If the estimate in (b) is satisfied, then it follows for each \( \xi \in \mathbb{R}^N \) that
\[
(sgn(\xi)^T C \xi = \sum_{j=1}^{N} \sum_{k=1}^{N} sgn(x_j) c_{jk} x_k
= \sum_{k=1}^{N} \left( c_{kk} |x_k| + \sum_{j \in \{1, \ldots, N\} \setminus \{k\}} sgn(x_j) c_{jk} x_k \right)
\leq \sum_{k=1}^{N} \left( c_{kk} |x_k| + \sum_{j \in \{1, \ldots, N\} \setminus \{k\}} |c_{jk}| |x_k| \right) \leq 0,
\]
so \( C \) is dissipative according to (d). Now assume conversely that \( C \) is dissipative.
Fix \( k \in \{1, \ldots, N\} \) and let \( e_k \in \mathbb{R}^N \) denote the \( k \)-the canonical unit vector. Moreover, we define a vector \( \xi \in \mathbb{R}^N \), which we consider as a functional on \( \mathbb{R}^N \), in the following way: we set \( \xi_k = 1 \) and \( \xi_j = sgn(c_{jk}) \) for all \( j \in \{1, \ldots, N\} \setminus \{k\} \). Then \( \|\xi\|_{\infty} = 1 \) and \( \xi^T e_k = 1 \), so the strict dissipativity of \( C \) implies that
\[
0 \geq \xi^T C e_k = \sum_{j=1}^{N} \xi_j c_{jk} = c_{kk} + \sum_{j \in \{1, \ldots, N\} \setminus \{k\}} |c_{jk}|
\]
which proves the assertion.

(c) This follows from (b) by duality. \( \square \)

In the above proof we could, of course, also first prove (c) directly and then derive (b) from (c) by duality; see for instance [13, Remark 2.1.2].

Appendix B. On poles of operator resolvents

In the following proposition we briefly recall a result about poles of the resolvent of a linear operator. This result is needed in the proof of Proposition 4.12.

**Proposition B.1.** Let \( T \) be a bounded linear operator on a complex Banach space \( E \) and let \( (\mu_j)_j \) be a net in the resolvent set of \( T \) which converges to a number \( \lambda \in \mathbb{C} \). Then the following assertions hold:

(a) \( \lambda \in \mathbb{C} \setminus \sigma(T) \) if and only if the net \( (\mu_j - \lambda)\mathcal{R}(\mu_j, T)_j \) converges to the zero operator.

(b) \( \lambda \) is a spectral value of \( T \) and a first order pole of the resolvent function \( \mathcal{R}(\cdot, T) \) if and only if the net \( (\mu_j - \lambda)\mathcal{R}(\mu_j, T)_j \) converges to a non-zero operator \( P \in \mathcal{L}(E) \).

In this case, \( P \) is the spectral projection associated with the pole \( \lambda \).

**Proof.** (a) The implication “\( \Rightarrow \)” is obvious, and the converse implication “\( \Leftarrow \)” follows from that well-known fact that, for every \( \mu \) in the resolvent set of \( T \), the norm of \( \mathcal{R}(\mu, T) \) is no less than \( 1/\text{dist}(\mu, \sigma(T)) \) (where dist denotes the distance in the complex plane).

(b) If \( \lambda \) is a spectral value of \( T \) and a first order pole of the resolvent, then the net \( (\mu_j - \lambda)\mathcal{R}(\mu_j, T)_j \) obviously converges to the spectral projection associated with \( \lambda \), and this spectral projection is non-zero.

Now assume conversely that the net \( (\mu_j - \lambda)\mathcal{R}(\mu_j, T)_j \) converges to an operator \( P \neq 0 \). It then follows from (a) that \( \lambda \) is a spectral value of \( T \); in particular, the elements of the net \( (\mu_j)_j \) are eventually distinct from \( \lambda \). Hence, it follows from the resolvent identity that
\[
(\mathcal{R}(\mu, T)P = \frac{P}{\mu - \lambda})
\]

\( (\mathcal{B.1}) \)
for each $\mu$ in the resolvent set of $T$. From this we immediately obtain $P^2 = P$, i.e. $P$ is a projection; moreover, $P$ clearly commutes with $T$, so $T$ splits over the decomposition $E = \ker P \oplus PE$.

It follows from (a) that $\lambda$ is in the resolvent set of $T|_{\ker P}$. Moreover, we conclude from (B.1) that $\lambda$ is a first order pole of the resolvent of $T|_{PE}$. Consequently, $\lambda$ is also a first order pole of the resolvent of $T$. □

Appendix C. A few facts about nets

In this appendix we recall a few facts about nets and universal nets that are needed in the main text, in particular in Proposition 4.7. Recall that a net $(x_j)_{j \in I}$ in a set $X$ is called a universal net if, for each $A \subseteq X$, the net is either eventually contained in $A$ or eventually contained in $X \setminus A$. If a subnet $(x_{j_i})$ of a net $(x_j)$ is a universal net, then we call $(x_{j_i})$ a universal subnet of $(x_j)$. It follows from Zorn’s lemma that every net has a universal subnet.

If $X$ is a topological Hausdorff space, then a subset $A \subseteq X$ is compact if and only if every universal net in $A$ converges to an element of $A$. In the following lemma we collect a few facts about metric spaces. For a proof we refer for instance to [16, Theorem B.3], where these facts are given in a slightly more general topological setting.

Lemma C.1. Let $(x_\alpha)_{\alpha \in I}$ be a net in a metric space $X$ and let

$$C := \bigcap_{\beta \in I} \{x_\alpha : \alpha \geq \beta\}$$

be its set of cluster points. Consider the following assertions.

(i) Each subnet of $(x_\alpha)_{\alpha \in I}$ has a convergent subnet.
(ii) Each universal subnet of $(x_\alpha)_{\alpha \in I}$ converges.
(iii) For each cofinal subsequence $(\alpha_n)_{n \in \mathbb{N}}$ the sequence $(x_{\alpha_n})_{n \in \mathbb{N}}$ has a cluster point.
(iv) The set $C$ is non-empty and compact.

Then (i) $\iff$ (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv). If, in addition, $I$ admits cofinal subsequences, then (iii) $\Rightarrow$ (iv) as well.

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Alexander Dobrick, *Arbeitsbereich Analysis*, Christian-Albrechts-Universität zu Kiel, Ludwig-Meyn-Str. 4, 24098 Kiel, Germany

E-mail address: dobrick@math.uni-kiel.de

Jochen Glück, *Fakultät für Informatik und Mathematik*, Universität Passau, Innstraße 33, D-94032 Passau, Germany

E-mail address: jochen.glueck@uni-passau.de