Decay of harmonic functions for discrete time Feynman-Kac operators with confining potentials

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Analysis in Tatra, 2022

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10.09.2022
○ $X$ - discrete, countably infinite state space.

○ $P : X \times X \to [0, 1]$ - a (sub-)probability kernel, i.e.
  \[
  \sum_{y \in X} P(x, y) \leq 1 \text{ for all } x \in X,
  \]

○ Equivalently, there exists a time-homogeneous Markov chain on $X$ \( \{Y_n : n \in \mathbb{N}_0\} \) such that for all $x, y \in X$

\[
\mathbb{P}(Y_{n+1} = y | Y_n = x) = P(x, y).
\]
Let $V : X \to (0, \infty)$ be a function such that $\inf_{x \in X} V(x) > 0$. Such function is called a *potential*.

We define a semigroup of operators \( \{U_n : n \in \mathbb{N}_0 \} \):

\[
U_0 f = f, \quad U_n f(x) = \mathbb{E}^x \left[ \prod_{k=0}^{n-1} \frac{1}{V(Y_k)} f(Y_n) \right], \quad n \geq 1
\]

for all admissible functions $f$. Observe that $U_n = U^n$, where

\[
U f(x) = \frac{1}{V(x)} \sum_{y \in X} P(x, y) f(y), \quad x \in X.
\]

The operator $U - I$ is called *the discrete Feynman–Kac operator*. 
Consider a discrete Schrödinger operator

\[ H f(x) = \sum_{y \in X} P(x, y)(f(x) - f(y)) + V(x)f(x), \]

where \( \inf_{x \in X}(V(x) + \sum_{y \in X} P(x, y)) > 0 \). Then we have

\[ \frac{1}{V(x) + \sum_{y \in X} P(x, y)} H f(x) = (I - U)f(x), \]

where \( U \) is defined with a \textit{shifted potential}

\[ V_*(x) = V(x) + \sum_{y \in X} P(x, y). \]

This implies \( H \) and \( (I - U) \) share many analytic properties.
Our goal is to obtain estimates for \((\mathcal{U} - I)\)-harmonic functions. We will focus on two important cases:

- Markov chains with the direct step property
- nearest-neighbour random walks
To find satisfactory estimates, we assume the following:

(A) We have $P(x, y) > 0$ for all $x, y \in X$, and there exists $C_* > 0$ such that

$$\sum_{z \in X} P(x, z)P(z, y) \leq C_* P(x, y).$$

This property is called the direct step property (DSP).

(B) For all $M > 0$, there exists a finite set $B_M \subset X$ such that $V(x) \geq M$ for $x \in B_M^c$. A potential with this property is called a confining potential.
In this section, we assume that $(X, d)$ is a metric space.

**Theorem**

Let $P(x, y)$ be a (sub-)probability kernel such that

$$P(x, y) \asymp J(d(x, y)), \quad x, y \in X,$$

(1)

for a non-increasing function $J : [0, \infty) \to (0, \infty)$ which satisfies the following doubling condition: there exists a constant $C > 0$ such that

$$J(r) \leq CJ(2r), \quad \text{for all } r > 0.$$  (2)

Then the kernel $P(x, y)$ satisfies assumption (A).
Let $P(x, y)$ be a (sub-)probability kernel such that

$$P(x, y) \asymp J(d(x, y))K(d(x, y)),$$  \hspace{1cm} x, y \in X,  \hspace{1cm} (3)$$

where $J, K : [0, \infty) \to (0, \infty)$ are non-increasing functions such that $J$ satisfies (2) and $K$ is such that

$$K(r)K(s) \leq \tilde{C}K(r + s), \hspace{1cm} r, s > 0. \hspace{1cm} (4)$$

Then the kernel $P(x, y)$ satisfies assumption (A).
DSP can also be inherited from a subordinator.

- $\{Z_n : n \geq 0\}$ - time-homogeneous Markov chain with values in $X$,
- $\{\tau_n : n \geq 0\}$ - arbitrary increasing random walk starting at 0 with values in $\mathbb{N}_0$, which is independent of $\{Z_n : n \geq 0\}$ (by saying that it is a random walk we mean that $\tau_{n+1} - \tau_n$, $n = 0, 1, 2, \ldots$ are i.i.d. random variables)

The *subordinate Markov chain* $\{Y_n : n \geq 0\}$ is then defined as

$$Y_n := Z_{\tau_n}, \quad n = 0, 1, 2, \ldots$$
Lemma
Suppose \( \{\tau_n\} \) has the DSP, that is
\[
\mathbb{P}(\tau_2 = n) \leq C_* \mathbb{P}(\tau_1 = n), \quad n = 2, 3, \ldots
\] (5)
for some constant \( C_* > 0 \). Then \( \{Y_n\} \) inherits the DSP with the same constant.

Proof
Since \( \mathbb{P}(\tau_2 = 1) = 0 \), for any \( x, y \in X \) we have
\[
\mathbb{P}(Y_2 = y \mid Y_0 = x) = \sum_{k=1}^{\infty} \mathbb{P}(Z_k = y \mid Z_0 = x) \mathbb{P}(\tau_2 = k)
\]
\[
\leq C_* \sum_{k=1}^{\infty} \mathbb{P}(Z_k = y \mid Z_0 = x) \mathbb{P}(\tau_1 = k)
\]
\[
= C_* \mathbb{P}(Y_1 = y \mid Y_0 = x),
\]
as desired.
Sufficient condition for assumption (A)

**Theorem**

If \( \{Z_n : n \geq 0\} \) is irreducible, \( \{\tau_n : n \geq 0\} \) is such that (5) holds and there exists \( n_0 \in \mathbb{N} \) such that

\[
P(\tau_1 = n) > 0, \quad n \geq n_0,
\]

then the subordinate chain \( \{Y_n : n \geq 0\} \) satisfies assumption (A).
For every finite set $B \subset X$ we define
\[
K_B := \inf \left\{ \frac{P(x, y)}{P(x, z)} : x \in X; y, z \in B \right\},
\]
\[
\overline{K}_B := \sup \left\{ \frac{P(x, y)}{P(x, z)} : x \in X; y, z \in B \right\}.
\]

It is easy to check that $0 < K_B \leq \overline{K}_B < \infty$. We fix a finite set $B_0 \subset X$ such that
\[
C_1 := \sup \left\{ \frac{1}{V(x)} : x \in B_0^c \right\} < 1 \wedge \frac{1}{C_*}.
\]

The existence of such a set is secured by assumption (B). Note that $B_0$ depends on $V$ and $P$. 
Theorem

**Under assumptions (A) and (B), there exists a constant** $C_2 > 0$ **such that for any finite set** $B \subset X$ **with** $B \supseteq B_0$, **and for any non-negative bounded function** $f$ **which is subharmonic in** $B^c$ **we have**

$$f(x) \leq C_2 \frac{1}{V(x)} \sum_{y \in B} P(x, y) f(y), \quad x \in B^c.$$ 

**In particular,**

$$f(x) \leq C_2 \overline{K}_B \frac{P(x, x_0)}{V(x)} \sum_{y \in B} f(y), \quad x \in B^c, \; x_0 \in B.$$ 

**The constant** $C_2$ **depends neither on** $f$, $V$, **nor on the set** $B$. 

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For any $D \subset X$, any non-negative function $f$ which is superharmonic in $D$, and for any finite set $B \subset X$ we have

$$f(x) \geq \frac{1}{V(x)} \sum_{y \in B} P(x, y) f(y), \quad x \in D.$$ 

In particular, under assumption (A),

$$f(x) \geq K_B \frac{P(x, x_0)}{V(x)} \sum_{y \in B} f(y), \quad x \in D, \ x_0 \in B.$$
Theorem (Cygan, Kaleta, MŚ)

Under assumptions (A) and (B), for any finite set $B \subset X$ with $B \supseteq B_0$, for any set $D \subset X$, and for any non-negative, non-zero and bounded function $f$ which is harmonic in $D$ and such that $f(x) = 0$ for $x \in D^c \cap B^c$ we have

$$K_B \leq \frac{f(x)}{P(x,x_0)} \frac{\sum_{y \in B} f(y)}{V(x)} \leq C_2 \overline{K}_B, \quad x \in D \cap B^c, \quad x_0 \in B,$$

where $C_2$ is independent of $f$, $V$ and $B$.

In particular, the uniform Boundary Harnack Inequality at infinity holds: if $f$ and $g$ are two such non-zero harmonic functions, then

$$\left( \frac{K_B}{C_2 \overline{K}_B} \right)^2 \leq \frac{f(x)g(y)}{g(x)f(y)} \leq \left( \frac{C_2 \overline{K}_B}{K_B} \right)^2, \quad x, y \in D \cap B^c.$$
First we impose a graph structure on $X$.

- Graph $G = (X, E)$ is defined by specifying a set of edges $E \subset \{\{x, y\} : x, y \in X\}$. Two vertices $x, y \in X$ are connected by an edge (are neighbours) in $G$ iff $\{x, y\} \in E$ (we have $\{x, y\} = \{y, x\}$). Notation: $x \sim y$.

- $G$ is said to be connected, if every two different vertices $x$ and $y$ are connected by a path in $G$.

- $G$ is said to be of finite geometry, if every vertex has finitely many neighbours.
Throughout this section we assume that
(C) $G$ is a connected graph of finite geometry.

This allows us to impose a \textit{geodesic metric $d$} on $G$, where $d(x, y)$ is the length of the shortest path between $x$ and $y$.

We consider a (sub-)probability kernel $P$ such that

$$P(x, y) > 0 \iff x \sim y. \quad (7)$$

We restrict our attention to potentials with the following property:
(D) There exist $x_0 \in X$ and an increasing profile function $W : \mathbb{N}_0 \to (0, \infty)$ such that $V(x) = W(d(x_0, x))$, for any $x \in X$.

Additionally, we use $B_r(x_0)$ to denote an open ball (with respect to $d$) with radius $r$ and center $x_0$. 
Let assumptions (C) and (D) hold with a fixed $x_0 \in X$ and a profile function $W$. Let $\mathcal{U} - I$ be the Feynman–Kac operator corresponding to the kernel $P(x, y)$ satisfying (7). Then for any $r \in \mathbb{N}$ and for any non-negative and bounded function $f$ which is $(\mathcal{U} - I)$-subharmonic in $B_r(x_0)^c$ we have

$$f(x) \leq \|f\|_{\infty} \prod_{i=r}^{d(x,x_0)} \frac{1}{W(i)}, \quad x \in B_r(x_0)^c.$$
To obtain the lower bound for \((U - I)\)-superharmonic functions we consider connected and geodesically convex subsets of \(X\).

The set \(D \subset X\) is called \textit{geodesically convex} in a graph \(G = (X, E)\) if \(D\) contains each vertex on any geodesic path connecting vertices in \(D\).

We also need an additional regularity assumption on the kernel \(P(x, y)\):

\[
M := \inf \{P(x, y) : x, y \in X, \ x \sim y\} > 0. \tag{8}
\]
Theorem

Let assumptions (C) and (D) hold with some \( x_0 \in X \) and a profile function \( W \). Let \( \mathcal{U} - I \) be the Feynman–Kac operator corresponding to the kernel \( P(x, y) \) satisfying (7) and (8). Then, for any connected geodesically convex set \( D \subset X \), for any non-negative function \( f \) which is \( (\mathcal{U} - I) \)-superharmonic in \( D \), for any \( x \in D \), and for any \( x_r \in D \) which lies on the geodesic path connecting \( x \) with \( x_0 \) and is such that \( d(x_r, x_0) = r < d(x, x_0) \), we have

\[
f(x) \geq f(x_r) \prod_{i=r+1}^{d(x, x_0)} \frac{M}{W(i)}.
\]

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Consider a kernel $b : X \times X \to [0, \infty)$ such that

(i) $b(x, y) = b(y, x)$, for every $x, y \in X$;

(ii) $\sum_{y \in X} b(x, y) > 0$, for every $x \in X$, and
$\sup_{x \in X} \sum_{y \in X} b(x, y) < \infty$.

Let $m : X \to (0, \infty)$ be a (strictly positive) measure on $X$. We additionally consider a function $V : X \to \mathbb{R}$ such that $\inf_{x \in X} V(x) > -\infty$. The graph Laplacian $H$ is defined by

$$Hf(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)) + V(x)f(x),$$

for all functions $f \in F := \{f : X \to \mathbb{R} : \sum_{y} b(x, y)|f(y)| < \infty, \text{ for every } x \in X\}$.

The triple $(X, b, V)$ can be seen as a weighted graph over $X$ (two points $x, y \in X$ form an edge if and only if $b(x, y) > 0$).
We set

\[ b(x) = \sum_{y \in X} b(x, y), \quad b^* := \sup_{x \in X} b(x), \quad P(x, y) = \frac{b(x, y)}{b^*}, \]

(9)

and

\[ \tilde{V}(x) = \begin{cases} \frac{m(x)V(x) + b(x)}{b^*}, & x \in A^c, \\ 1, & x \in A, \end{cases} \]

(10)

where \( A = \{ x \in X : m(x)V(x) + b(x) \leq b^* \} \).

We further assume that the operator \( \mathcal{U} \) is defined with a sub-probability kernel \( P(x, y) \) and the potential \( \tilde{V}(x) \) defined at (9) and (10).
For every $f \in F$ and $x \in A^c$ we have

$$Hf(x) = - \left( V(x) + \frac{b(x)}{m(x)} \right) (U - I)f(x).$$

In particular, if $D \subset A^c$ and $f \in F$, then

$$Hf(x) \geq 0, \quad x \in D \iff (U - I)f(x) \leq 0, \quad x \in D.$$
Corollary (DSP case, upper bound)

Suppose that \( b(x, y) \), \( m(x) \) and \( V(x) \) are as above. Assume that \( V \) satisfies (B), \( \inf_{x \in X} m(x) > 0 \) and that

\[
    b(x, y) > 0, \quad \sup_{x, y \in X} \sum_{z \in X} \frac{b(x, z)b(z, y)}{b(x, y)} < \infty. \quad (11)
\]

Let \( D \subset X \) and let \( f \) be a bounded solution to the equation \( Hf(x) = 0 \), \( x \in D \). Then there exists a finite set \( B_0 \subset X \) (independent of \( m, D \) and \( f \)) with \( B_0 \supseteq A \) such that for any finite set \( B \subset X \) with \( B \supseteq B_0 \) there exists a constant \( C > 0 \) (independent of \( V, m, D \) and \( f \)) such that

\[
    |f(x)| \leq C \frac{b(x, x_0)}{m(x)V(x) + b(x)} \sum_{y \in B} |f(y)|, \quad x \in D \cap B^c, \ x_0 \in B,
\]

whenever \( f(x) = 0 \) for \( x \in D^c \cap B^c \).
Corollary (DSP case, lower bound)

If, in addition, $f$ is non-negative, then for any finite set $B \subseteq X$ with $B \supseteq B_0$ there exists a constant $\tilde{C} > 0$ (independent of $V$, $m$, $D$ and $f$) such that

$$f(x) \geq \tilde{C} \frac{b(x, x_0)}{m(x) V(x) + b(x)} \sum_{y \in B} f(y), \quad x \in D \cap B^c, \quad x_0 \in B.$$ 

**Proof.** Observe that when $V$ is a confining potential, then $\tilde{V}$ is confining as well. Realize that $f$ is $(\mathcal{U} - I)$-harmonic in $D \cap A^c$. To justify the upper bound, it is enough to observe that $|f|$ is $(\mathcal{U} - I)$-subharmonic in $B^c$ and apply respective theorem. The corresponding lower bound is obtained directly from the proposition in the DSP case.
Suppose we are given a positive measure $\mu$ on $X$ such that

(i) $\sup_{y \in X} \frac{\sum_{x \in X} \mu(x)P(x, y)}{\mu(y)} < \infty$, (ii) $\sup_{x,y \in X} \frac{P(x, y)}{\mu(y)} < \infty$.

Under condition (i), the operator $U$ is bounded in $\ell^p(X, \mu)$, for any $1 \leq p < \infty$. Condition (ii) implies that the operator $U : \ell^p(X, \mu) \to \ell^\infty(X, \mu)$ is bounded for every $1 \leq p < \infty$.

**Lemma**

*Under assumption (B), the operator $U$ is compact in $\ell^2(X, \mu)$.***
We deduce that the spectrum of the operator $\mathcal{U}$ (excluding zero) consists solely of eigenvalues. Moreover, by Jentzsch theorem, the spectral radius of $\mathcal{U}$ is an eigenvalue, which we denote by $\lambda_0 > 0$, and the corresponding eigenfunction $\psi_0$ is strictly positive on $X$.

Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ be an eigenvalue of the operator $\mathcal{U}$ and let $\psi \in \ell^2(X, \mu)$ be the corresponding eigenfunction, i.e. $\mathcal{U}\psi = \lambda \psi$. We then have $|\lambda| |\psi| = |\mathcal{U}\psi| \leq \mathcal{U}|\psi|$, which implies $|\psi| \leq \mathcal{U}^\lambda |\psi|$, where

$$\mathcal{U}^\lambda f(x) = \frac{1}{V_\lambda(x)} \sum_{y \in X} P(x, y) |\psi(y)|,$$

with $V_\lambda := |\lambda| V$.

In particular, $(\mathcal{U}^\lambda - I)|\psi|(x) \geq 0$, $x \in X$, i.e. the non-negative function $\varphi := |\psi|$ is $(\mathcal{U}^\lambda - I)$-subharmonic in $X$. We show similarly that the positive function $\psi_0$ is $(\mathcal{U}^\lambda - I)$-harmonic.
After this preparation we can apply our results to obtain an upper bound for $|\psi|$ outside of a finite set in the DSP and the nearest-neighbour case, respectively.

We can also find the matching lower bound for the positive eigenfunction $\psi_0$ in these two cases.
Endre Csáki
A discrete Feynman–Kac formula
Journal of Statistical Planning and Inference 34 (1993) 63-73

Wojciech Cygan, Kamil Kaleta, Mateusz Śliwiński
Decay of harmonic functions for discrete time Feynman–Kac operators with confining potentials
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Wojciech Cygan, Kamil Kaleta, René Schilling, Mateusz Śliwiński
Kernel estimates for discrete Feynman–Kac operators
in preparation
Thank you!