A CASE OF THE RODRIGUEZ VILLEGAS CONJECTURE
WITH AN APPENDIX BY FERNANDO RODRIGUEZ VILLEGAS

TED CHINBURG, EDUARDO FRIEDMAN AND JAMES SUNDSTROM

ABSTRACT. Let \( L \) be a number field and let \( E \) be any subgroup of the units \( \mathcal{O}_L^\ast \) of \( L \). If \( \text{rank}_\mathbb{Z}(E) = 1 \), Lehmer’s conjecture predicts that the height of any non-torsion element of \( E \) is bounded below by an absolute positive constant. If \( \text{rank}_\mathbb{Z}(E) = \text{rank}_\mathbb{Z}(\mathcal{O}_L^\ast) \), Zimmert proved a lower bound on the regulator of \( E \) which grows exponentially with \( [L : \mathbb{Q}] \). Fernando Rodriguez Villegas made a conjecture in 2002 that “interpolates” between these two extremes of rank. Here we prove a high-rank case of this conjecture. Namely, it holds if \( L \) contains a subfield \( K \) for which \( [L : K] \gg [K : \mathbb{Q}] \) and \( E \) contains the kernel of the norm map from \( \mathcal{O}_L^\ast \) to \( \mathcal{O}_K^\ast \).

1. Introduction

In 2002 Fernando Rodriguez Villegas conjectured a surprising lower bound on a natural 1-norm of any non-trivial element of the \( j \)-th exterior power of the units of a number field. For \( j \) minimal, i.e., \( j = 1 \), Rodriguez Villegas’ conjecture is equivalent to Lehmer’s 1933 conjectural lower bound on the height of an algebraic number [Le] [Sm2]. For \( j \) maximal, i.e., \( j = \text{rank}_\mathbb{Z}(\mathcal{O}_L^\ast) \), it is equivalent to Zimmert’s 1981 theorem stating that the regulator of a number field grows at least exponentially with the degree of the number field [Zi].

We now state his conjecture in its strongest possible form.

RV Conjecture. (Rodriguez Villegas) There exist two absolute constants \( c_0 > 0 \) and \( c_1 > 1 \) such that for any number field \( L \) and any \( j \in \mathbb{N} \),

\[
\| \omega \|_1 \geq c_0 c_1^j \quad \left( \forall \omega \in \bigwedge^j \text{LOG}(\mathcal{O}_L^\ast) \subset \bigwedge^j \mathbb{R}^{\mathcal{A}_L}, \ \omega \neq 0 \right).
\]  

(1)

Here \( \bigwedge^j \text{LOG}(\mathcal{O}_L^\ast) \) denotes the \( j \)-th exterior power of the lattice \( \text{LOG}(\mathcal{O}_L^\ast) \subset \mathbb{R}^{\mathcal{A}_L} \), \( \mathcal{A}_L \) denotes the set of archimedean places of \( L \), and \( \text{LOG} : \mathcal{O}_L^\ast \to \mathbb{R}^{\mathcal{A}_L} \) is defined by

\[
(\text{LOG}(\gamma))_v := e_v \log |\gamma|_v, \quad e_v := \begin{cases} 1 & \text{if } v \text{ is real}, \\ 2 & \text{if } v \text{ is complex} \end{cases} \quad (\gamma \in \mathcal{O}_L^\ast, \ v \in \mathcal{A}_L),
\]

(2)

2010 Mathematics Subject Classification. 11R06, 11R27.

Key words and phrases. Lehmer’s conjecture, Mahler measure, units.

Partially supported by U.S. N.S.F. grant NSF FRG Grant DMS-1360767 (Chinburg and Sundstrom), U.S. N.S.F. SaTC Grants CNS-1513671/1701785 (Chinburg) and by Chilean FONDECYT grant 1170176 (Friedman).

1 The original 2002 write-up of this conjecture was kindly supplied to us by F. Rodriguez Villegas and appears with his permission for the first time in print here (see §7). The 2002 conjecture is somewhat weaker, but F. Rodriguez Villegas later strengthened it to the form given here.
The 1-norm on $\|\cdot\|$ is not possible for the Euclidean norm
$\|\cdot\|$. Namely, for $A_L$ having cardinality $j$, for each such $I$ we fix an ordering
$\{v_1, ..., v_j\}$ of $I$ and
$$\delta^I := \delta^{v_1} \wedge \delta^{v_2} \wedge \cdots \wedge \delta^{v_j}.$$ The 1-norm on $\bigwedge^j R^A_L$ in the RV conjecture (1) is defined with respect to this basis.
Namely, for $\omega = \sum_{I \in A_L^{|j|}} c_I \delta^I$, we let $\|\omega\|_1 := \sum_{I \in A_L^{|j|}} |c_I|$.

It is worth mentioning that Siegel [Sie] showed that the conjectural inequality (1) is not possible for the Euclidean norm $\|\omega\|_2 := \sqrt{\sum_J c_J^2}$. Indeed, if $p > 2$ is a prime, if $\varepsilon \in \mathbb{C}$ satisfies $\varepsilon^p - \varepsilon + 1 = 0$ and $L := \mathbb{Q}(\varepsilon)$, then $\|\text{LOG}(\varepsilon)\|_2 \leq \sqrt{2\log(p)/\sqrt{p}}$. Hence, the RV conjecture is necessarily for the 1-norm, at least for $j = 1$.

However, for $j$ close to the maximal value $r_L = \text{rank}_\mathbb{Z}(O_L^*)$, the 1-norm and the Euclidean norm are interchangeable for the purposes of Rodriguez Villegas’ conjecture. This is simply because on any Euclidean space $V$, we have $\sqrt{\dim(V)} \|v\|_2 \geq \|v\|_1 \geq \|v\|_2$, provided the 1-norm is taken with respect to an orthonormal basis for $V$. In this paper we will work only with the Euclidean norm and $j$ close to $r_L$.

Aside from Zimmert’s theorem on the regulator [Zi] and the known cases of Lehmer’s conjecture [Sm2], the cleanest result in favor of the RV conjecture is
$$\|\text{LOG}(\varepsilon_1) \wedge \cdots \wedge \text{LOG}(\varepsilon_j)\|_1 > 0.001 \cdot 1.4^j,$$
proved for all $j$, but only for totally real fields $L$. This follows from work of Pohst [Po] dating back to 1978. Indeed, Pohst showed for $L$ totally real that
$$\|\text{LOG}(\varepsilon)\|_2 \geq \sqrt{|L : \mathbb{Q}|} \log((1 + \sqrt{5})/2) \quad (\varepsilon \in O_L^*, \varepsilon \neq \pm 1).$$

Using estimates of Hermite’s constant, he deduced good lower bounds for the regulator of a totally real field. The same calculations show that the $j$-dimensional co-volume $\mu$ of the lattice spanned by $\text{LOG}(\varepsilon_1), ..., \text{LOG}(\varepsilon_j)$ satisfies [CF, p. 293]
$$\mu > \frac{|L : \mathbb{Q}|/j}{(j + 2)\sqrt{j}} 1.406^j \quad (1 \leq j < |L : \mathbb{Q}|).$$

Since
$$\|\text{LOG}(\varepsilon_1) \wedge \cdots \wedge \text{LOG}(\varepsilon_j)\|_1 \geq \|\text{LOG}(\varepsilon_1) \wedge \cdots \wedge \text{LOG}(\varepsilon_j)\|_2 = \mu,$$
a short numerical computation with (5) yields (4).

---

2 Although Rodriguez Villegas phrased the 1-norm in terms of the archimedean embeddings rather than places (see [7,4]), the 1-norm is unchanged as we inserted a factor of 2 at complex places in (2). However, the embedding using places gives a larger 2-norm if the field is not totally real, and so is better for our purposes.
As far as we know, the only proved cases of the RV conjecture involve “pure wedges,” i.e., \( \omega \) of the form \( \omega = \text{LOG}(\varepsilon_1) \wedge \cdots \wedge \text{LOG}(\varepsilon_j) \), where the \( \varepsilon_i \) are independent elements of \( \mathcal{O}_L^* \). If \( j = r_L \) or \( j = 1 \), every element of \( \wedge^j \) is (trivially) a pure wedge, but this also holds if \( j = r_L - 1 \) (see Lemma 22 below). In particular, if \( L \) is a totally real field of degree \( n \) over \( \mathbb{Q} \), then

\[
\|\omega\|_1 > 0.001 \cdot 1.4^{n-2},
\]

for all \( \omega \in \wedge^{n-2}\text{LOG}(\mathcal{O}_L^*) \). In general, however, the RV conjecture makes a stronger prediction than simply a lower bound on the 1-norm of pure wedges.

Another known case of the RV conjecture occurs when

\[
E = E(L/K) := \{ \varepsilon \in \mathcal{O}_L^* | \text{Norm}_{L/K}(\varepsilon) \text{ is a root of unity} \}
\]

is the group of relative units associated to an extension \( L/K \). Friedman and Skoruppa [FS] proved in 1999 that inequality (1) in the RV conjecture holds for pure wedges if \( [L : K] \geq N_0 \) for some absolute constant \( N_0 \). To prove their result, Friedman and Skoruppa defined a \( \Theta \)-type series \( \Theta_E \) associated to any subgroup \( E \subset \mathcal{O}_L^* \) of arbitrary rank and used it to produce a complicated inequality for the co-volume \( \mu(E) \) associated to the lattice \( \text{LOG}(E) \). In the case of \( E = E(L/K) \) they obtained the desired inequality using the saddle-point method to estimate the terms in the series \( \Theta_E \) as \( [L : K] \to \infty \). Although the saddle-point method in one variable is a standard tool, the difficulty in the asymptotic estimates in [FS, §5] was that the estimates needed to depend only on \( [L : K] \).

The results cited so far all pre-date the RV conjecture and essentially dealt with regulators or Lehmer’s conjecture. Inspired by the RV conjecture, Sundstrom [Su1] [Su2] dealt in his 2016 thesis with a new kind of subgroup of the units. Namely, suppose \( L \) contains two distinct real quadratic subfields \( K_1, K_2 \), and let \( E := E(L/K_1) \cap E(L/K_2) \). The series \( \Theta_E \) is still defined and yields an inequality for the co-volume \( \mu(\text{LOG}(E)) \), but to estimate the terms in the inequality Sundstrom had to apply the saddle-point method to a triple integral. Keeping all estimates uniform in this case proved considerably harder than in the one-variable case treated in [FS]. In the end, Sundstrom was able to verify the RV conjecture in this case for pure wedges. More precisely, he proved the existence of absolute constants \( N_0, c_0 > 0 \) and \( c_1 > 1 \) such that \( \mu(E) \geq c_0 c_1^j \) for \( [L : \mathbb{Q}] \geq N_0 \) and \( j = \text{rank}_\mathbb{Z}(E) = \text{rank}_\mathbb{Z}(\mathcal{O}_L^*) - 2 \).

Here we extend Sundstrom’s result, letting the \( K_i \) be arbitrary, as follows.

---

3 The inequality proved in [FS] is for the relative regulator \( \text{Reg}(L/K) \) rather than for the co-volume \( \mu \) of the relative units. This suffices since \( \mu = \text{Reg}(L/K) \prod_{v \in \mathcal{A}_K} \sqrt{\rho_v} \geq \text{Reg}(L/K) \), where \( r_v \) is the number of places of \( L \) above \( v \). The proof of this relation between the co-volume and the relative regulator mimics the determinant manipulations in the case \( K = \mathbb{Q} \) [BS, p. 115]. We note that J. Sundstrom, in the appendix to his doctoral thesis [Su1], corrected an error in Skoruppa and Friedman’s proof. Namely, in the bound on what is called \( J_1 \) in the proof of Lemma 5.5 of [FS], the real part of the error term \( \rho \) in the exponential was neglected. This did not affect the proof of their Main Theorem, but it did affect the numerical constants claimed in Theorem 4.1 and its corollaries. By improving the asymptotic estimates in [FS] and using extensive computer calculations, Sundstrom was able to prove the estimate in Theorem 4.1 of [FS], with the constants as given there. In particular, \( N_0 = 40 \). If we are willing to settle for \( N_0 = 400 \), the proof in [FS] will do after adjusting the constants to correct for the error in the proof of Lemma 5.5.
Let $K_1, \ldots, K_\ell$ be subfields of a number field $L$, let $K := K_1 \cdots K_\ell \subset L$ be the compositum of the $K_i$, let $E := \bigcap_{i=1}^{\ell} E(L/K_i) \subset \mathcal{O}_L^*$ be the subgroup of the units of $L$ whose norm to each $K_i$ is a root of unity, and let $\varepsilon_1, \ldots, \varepsilon_j$ be independent elements of $E$, where $j := \operatorname{rank}_\mathbb{Z}(E)$. Then there is an absolute constant $N_0$ such that

$$
\|\varepsilon_1 \land \cdots \land \varepsilon_j\|_1 \geq \|\varepsilon_1 \land \cdots \land \varepsilon_j\|_2 \geq 1.1^j,
$$

whenever $[L : K] \geq N_0 \cdot 2.01^{[K : \mathbb{Q}]}$.

In fact the above is an immediate corollary of our

**Main Theorem.** Suppose $E \subset \mathcal{O}_L^*$ is a subgroup of the units of the number field $L$ such that $E(L/K) \subset E$ for some subfield $K \subset L$, where $E(L/K)$ are the relative units defined in ($\mathfrak{u}$). Let $\varepsilon_1, \ldots, \varepsilon_j$ be independent elements of $E$, where $j := \operatorname{rank}_\mathbb{Z}(E)$. Then the RV conjecture ($\mathfrak{u}$) holds for $\omega := \varepsilon_1 \land \cdots \land \varepsilon_j$ and $[L : K]$ large enough compared to $[K : \mathbb{Q}]$.

More precisely, there is an absolute constant $N_0$ such that if $[L : K] \geq N_0 \cdot 2.01^{[K : \mathbb{Q}]}$, then

$$
\|\varepsilon_1 \land \cdots \land \varepsilon_j\|_1 \geq \|\varepsilon_1 \land \cdots \land \varepsilon_j\|_2 \geq 1.1^j \quad (j := \operatorname{rank}_\mathbb{Z}(E)).
$$

Our proof of the Main Theorem is again through an asymptotic analysis of the inequality for $\Theta_E$ in [FS], but there are several new features which bring the proof closer to the case of a general high-rank subgroup $E \subset \mathcal{O}_L^*$.

In both [FS] and [Su2], the uniformity of the asymptotic estimates depends on having explicit expressions for the orthogonal complement of $\log \mathcal{O}(E)$ inside $\mathbb{R}^{A_L}$, but here we have very little knowledge of $\log \mathcal{O}(E)^\perp$. As in [FS] and [Su2], we take a Mellin transform of the terms of $\Theta_E$ and invert it to express each term in $\Theta_E$ as a $k$-dimensional complex contour integral (see Lemma $\mathfrak{3}$ below). Here $k := 1 + \operatorname{rank}_\mathbb{Z}(\mathcal{O}_L^*/E)$ is the co-rank of $E \subset \mathcal{O}_L^*$, shifted by 1.

To apply the saddle-point method to our integral, we need a saddle point. In the case of [FS] one could easily write down a formula for the saddle point in terms of the logarithmic derivative of the classical $\Gamma$-function. In [Su2] the equations for the critical point were explicit enough that monotonicity arguments proved the existence of the saddle point. In our case the equations are too complicated to analyse directly. Instead, in $\mathfrak{3}$ we obtain the existence and uniqueness of the saddle point by re-interpreting it as the value of the Legendre transform of a convex function on $\mathbb{R}^k$, closely related to $\log \Gamma$.

Since (what will prove to be) the main term in our asymptotic expansion depends on the saddle point $\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^k$, of which we can only control $\sigma_1$, in $\mathfrak{4}$ we prove inequalities for the main term which depend only on $\sigma_1$. We need these inequalities to prove that the main term has the exponential growth claimed in the Main Theorem.

The results proved in $\mathfrak{2}$-$\mathfrak{4}$ are valid for any subgroup $E \subset \mathcal{O}_L^*$. In $\mathfrak{5}$ we carry out the required uniform asymptotic estimates, assuming $E(L/K) \subset E$ and $[L : K] \gg 0$ to show that the purported main term actually dominates. Finally, in $\mathfrak{6}$ we put everything together and prove the Main Theorem.
2. The $\Theta$-function

In this section we recall the series $\Theta_E(t;a)$ associated to a subgroup $E \subset \mathcal{O}_L^*$ of the units and to a fractional ideal $a$ of the number field $L$. We also recall the inequality for the co-volume of $\text{LOG}(E)$ resulting from the functional equation of $\Theta_E$. This is all quoted from [FS] §2. Our main new task here is to express the terms in the inequality as an inverse Mellin transform.

2.1. The basic inequality. Given a subgroup $E \subset \mathcal{O}_L^*$, we define $E_\mathbb{R} \subset \mathbb{R}^{A_L}$ as the group generated by all elements of the form

$$x = (x_v)_{v \in A_L} = (|\xi|^v_v)_{v \in A_L}$$

(\varepsilon \in E, \xi \in \mathbb{R}).

Here $\mathbb{R}_+ := (0, \infty)$ is the multiplicative group of the positive real numbers, $\mathcal{A}_L$ denotes the set of Archimedean places of $L$, and $| \cdot |_v$ is the (un-normalized) absolute value associated to the archimedean place $v \in \mathcal{A}_L$. Thus, for $a \in L$ we have

$$|\text{Norm}_{L/Q}(a)| = \prod_{v \in \mathcal{A}_L} |a_v|_{v}^{e_v}, \quad (e_v := 1 \text{ if } v \text{ is real, } e_v := 2 \text{ if } v \text{ is complex}).$$

(9)

Note that

$$\sum_{v \in \mathcal{A}_L} e_v = [L : \mathbb{Q}] := n,$$

(10)

and that $\varepsilon \in E$ acts on $x = (x_v)_{v \in E_\mathbb{R}}$, via $(\varepsilon \cdot x)_v := |\varepsilon|_v x_v$.

We fix a Haar measure on $E_\mathbb{R} \subset \mathbb{R}^{A_L}$ as follows. The standard Euclidean structure on $\mathbb{R}^{A_L}$, in which the $\delta^v$ in (3) form an orthonormal basis of $\mathbb{R}^{A_L}$, induces a Euclidean structure (and therefore a unique Haar measure) on any $\mathbb{R}$-subspace of $\mathbb{R}^{A_L}$. We give $E_\mathbb{R}$ the Haar measure $\mu_{E_\mathbb{R}}$ that results from pulling back the Haar measure on the $\mathbb{R}$-subspace $\text{LOG}(E_\mathbb{R})$ via the isomorphism $\text{LOG}$, and let $\mu_{E_k}(E_\mathbb{R}/E)$ be the measure of a fundamental domain for the action of $E$ on $E_\mathbb{R}$.

Following [FS] p. 120, for a fractional ideal $a \subset L$ and $t > 0$, we let

$$\Theta_E(t; a) := \frac{\mu_{E_\mathbb{R}}(E_\mathbb{R}/E)}{|E_{\text{tor}}|} + \sum_{a \in \mathcal{A}/a \neq 0} \int_{E_\mathbb{R}} e^{-c_a t \|ax\|^2} d\mu_{E_\mathbb{R}}(x),$$

where $|E_{\text{tor}}|$ is the number of roots of unity in $E$,

$$c_a := \pi \left( \sqrt{|D_L|} \text{Norm}_{L/Q}(a) \right)^{-2/n}, \quad D_L := \text{discriminant of } L, \quad n := [L : \mathbb{Q}].$$

(12)

Note that the integral in (12) depends only on the $E$-orbit of $a$, and hence is independent of the representative $a \in \mathcal{A}/a$ taken for the $E$-orbit of $a$.

Our starting point for proving lower bounds on co-volumes is the inequality [FS] Corol. p. 121, valid for any $t > 0$ and any fractional ideal $a$ of $L$.

$$\Theta_E(t; a) + \frac{2 \Theta_E(t; a)}{n} \geq 0 \quad (t > 0, \Theta'_E := \frac{d\Theta_E}{dt}).$$

(13)

Writing out the individual terms of (13), we have [FS] p. 121, eq. (2.6) the
Basic Inequality.

\[ \frac{\mu_{E_\mathbb{R}}(E_\mathbb{R}/E)}{|E_{\text{tor}}|} \geq \sum_{a \in \mathbb{R}/a \neq 0} \int_{x \in E_\mathbb{R}} \left( \frac{2t ||ax||^2}{n} - 1 \right) e^{-t ||ax||^2} d\mu_{E_\mathbb{R}}(x) \quad (t > 0). \quad (14) \]

Note that in [FS] we find \( t e_a \) instead of \( t \) in (14), but \( t > 0 \) is arbitrary there too.

2.2. Mellin transforms. Our main task in this section is to re-write the \( r \)-dimensional integral in (12) as an inverse Mellin transform. For this it will prove convenient to characterize \( E_\mathbb{R} \subset G := \mathbb{R}_{+}^{A_L} \) not through generators, but rather through generators of the orthogonal complement in \( \mathbb{R}^{A_L} \) of \( \text{Log}_G(E_\mathbb{R}) \). Here \( \text{Log}_G : G \to \mathbb{R}^{A_L} \) is the group isomorphism defined by

\[ (\text{Log}_G(g))_v := \log(g_v) \quad (v \in A_L, \ g = (g_v)_v \in G := \mathbb{R}_{+}^{A_L}). \quad (15) \]

Note that \( \text{Log}_G \) is not the traditional logarithmic embedding \( \text{LOG} \) in (2), as we do not insert a factor of \( e_v \) in (15). Instead we endow \( \mathbb{R}^{A_L} \) with a new inner product

\[ \langle \beta, \gamma \rangle := \sum_{v \in A_L} e_v \beta_v \gamma_v \quad (\beta = (\beta_v)_v, \ \gamma = (\gamma_v)_v \in \mathbb{R}_{+}^{A_L}), \quad (16) \]

where \( e_v = 1 \) or 2 as in (9). Let \( \{q_j\}_{j=1}^k = \{(q_{jv})_v\}_{j=1}^k \) be an \( \mathbb{R} \)-basis of the orthogonal complement of \( \text{Log}_G(E_\mathbb{R}) \) in \( \mathbb{R}^{A_L} \) such that

\[ q_{1v} := 1 \quad (\forall v \in A_L), \quad \sum_{v \in A_L} e_v q_{iv} q_{jv} = 0 \quad (1 \leq i \neq j \leq k := 1 + \text{rank}_\mathbb{Z}(\mathcal{O}_L^*/E)). \quad (17) \]

Thus, for \( g = (g_v)_v \in G := \mathbb{R}_{+}^{A_L} \),

\[ g \in E_\mathbb{R} \iff \sum_{v \in A_L} e_v q_{jv} \log(g_v) = 0 \quad (1 \leq j \leq k). \quad (18) \]

Let \( H := \mathbb{R}_{+}^k \). Define a homomorphism \( \delta : G \to H \) by

\[ (\delta(g))_j := \prod_{v \in A_L} g_v^{e_v q_{jv}} \quad (1 \leq j \leq k, \ g = (g_v)_v \in G := \mathbb{R}_{+}^{A_L}), \quad (19) \]

so that by (18) we have an exact sequence

\[ 1 \longrightarrow E_\mathbb{R} \longrightarrow G \overset{\delta}{\longrightarrow} H \longrightarrow 1. \quad (20) \]

Let \( \sigma : H \to G \) be a homomorphism splitting the exact sequence (20), i.e., \( \delta \circ \sigma \) is the identity map on \( H \). Such a splitting exists because \( G \) and \( H \) are real vector spaces. Let

\[ d\mu_G := \prod_{v \in A_L} \frac{dg_v}{g_v}, \quad d\mu_H := \prod_{j=1}^k \frac{dh_j}{h_j} \quad (21) \]

be the usual Haar measures on \( G := \mathbb{R}_{+}^{A_L} \) and \( H := \mathbb{R}_{+}^k \).

Recall that in order to define \( \Theta_E \) in (12) we fixed a Haar measure \( \mu_{E_\mathbb{R}} \) on \( E_\mathbb{R} \). In order to calculate Mellin transforms below, we will need to compare the Haar
measure $\mu_H \times \mu_{E_k}$ on $H \times E_{\mathbb{R}}$ with a Haar measure coming from $\mu_G$. Namely, if $\gamma: E_{\mathbb{R}} \times H \to G$ is the isomorphism defined by the splitting $\sigma$, i.e.,
\[
\gamma(x, h) := x\sigma(h),
\] (22)
then the measure $\mu_G \circ \gamma$ is a Haar measure on $E_{\mathbb{R}} \times H$. Hence
\[
c \mu_G \circ \gamma = \mu_{E_k} \times \mu_H,
\] (23)
where the positive constant $c$ is evaluated in the next lemma.

**Lemma 1.** Let $Q$ be the $|A_L| \times k$ matrix whose rows are indexed by $v \in A_L$ and whose columns are indexed by $j = 1, \ldots, k$, with entry $Q_{v,j} := q_{jv}$ in the $v$th row and the $j$th column, with $q_{jv}$ as in (17). Then $c$ in (23) is independent of the splitting $\sigma$ in (22) and is given by
\[
c = 2^{r_2} \sqrt{\det(Q^TQ)},
\] (24)
where $Q^T$ is the transpose of $Q$ and $r_2$ is the number of complex places of $L$.

**Proof.** For $x = (x_v)$ and $y = (y_v) \in \mathbb{R}^{|A_L|}$, let $x \cdot y$ be the standard dot product $x \cdot y := \sum_{v \in A_L} x_v y_v$. Recall that we defined in (16) another inner product on $\mathbb{R}^{|A_L|}$, namely $(x,y) := \sum_v e_v x_v y_v$. To relate these products, let $T: \mathbb{R}^{|A_L|} \to \mathbb{R}^{|A_L|}$ be given by $(T(x))_v := e_v x_v$. Then
\[
\langle x, y \rangle = x \cdot T(y) = T(x) \cdot y.
\] (25)

Note that $\det(T) = 2^{r_2}$.

Let $u_1, \ldots, u_r$ be an orthonormal basis of $V$ (with respect to the dot product), let $C_1 := \{ \sum_{\ell} x_{\ell} u_{\ell} | 0 \leq x_{\ell} \leq 1 \} \subset V$ be the $r$-cube spanned by the $u_{\ell}$, and let $B_1 := \text{LOG}^{-1}(C_1)$. By the definition of the measure $\mu_{E_k}$ given in the paragraph preceding (12), $\mu_{E_k}(B_1) = 1$.

We define next an analogous subset $B_2 \subset H := \mathbb{R}^k_+$ with $\mu_H(B_2) = 1$. Let $F_1, \ldots, F_k$ be the “standard” orthonormal basis of $\mathbb{R}^k_+$ as an $\mathbb{R}$-vector space; that is, $(F_j)_i = e$ if $i = j$, and $(F_j)_i = 1$ otherwise. Let $B_2 \subset \mathbb{R}^k_+$ be the $k$-cube spanned by $F_1, \ldots, F_k$, so that $\mu_H(B_2) = 1$.

Set $B := B_1 \times B_2 \subset E_{\mathbb{R}} \times H$, so that $(\mu_{E_k} \times \mu_H)(B) = 1$. Thus $c$ in (23) satisfies
\[
c^{-1} = \mu_G(\gamma(B)).
\] (26)

Now, $\gamma(x, h) := x\sigma(h)$ and $\mu_G$ is the measure on $G$ that maps by $\text{Log}_G$ to the standard Haar measure on $\mathbb{R}^{|A_L|}$ (see (15), (21) and (22)). Hence, $c^{-1} = \det(M)$, where $M$ is the $|A_L| \times |A_L|$-matrix whose first $r$ columns are the vectors $w_\ell := \text{Log}_G(\text{LOG}^{-1}(u_\ell)) \in \mathbb{R}^{|A_L|} (1 \leq \ell \leq r)$. The remaining $k$ columns of $M$ are the vectors $\text{Log}_G(\sigma(F_j)) (1 \leq j \leq k)$.

Suppose $\tilde{\sigma}$ is another splitting of (20). Then $\sigma(F_j)\tilde{\sigma}(F_j)^{-1} \in E_{\mathbb{R}}$, and therefore $\text{Log}_G(\sigma(F_j)) - \text{Log}_G(\tilde{\sigma}(F_j))$ lies in the span of the columns $w_1, \ldots, w_r$. Hence $c$ is independent of the splitting $\sigma$, as claimed in the lemma. We are therefore free to use the splitting $\sigma$ determined by
\[
(\sigma(F_j))_v := \exp(q_{jv}/d_j) \quad (v \in A_L, 1 \leq j \leq k, d_j := \langle q_j, q_j \rangle := \sum_{\rho \in A_L} e_{\rho} q_{j\rho}^2).
\]
Using (19) and the orthogonality relations (17), one checks that this is indeed a splitting of $\delta$. With this $\sigma$, the last $k$ columns of $M$ are just $\log_G(\sigma(F_j)) = d^{-1}_j q_j \in \mathbb{R}^{A_L}$. As $T \circ \log_G = \log$ and $\det(T) = 2^{-r}$ (see (23)), we have
\[ c^{-1} = |\det(M)| = 2^{-r} |\det(N)|, \]
where $N$ is the $(|A_L| \times |A_L|)$-matrix whose columns are $T$ applied to the columns of $M$, i.e., the columns of $N$ are $u_1, \ldots, u_r$, followed by $d^{-1}_1 T(q_1), \ldots, d^{-1}_k T(q_k)$.

To prove the lemma we must show that $|\det(N)|^{-1} = \sqrt{\det(Q^t Q)}$. We calculate $|\det(N)|$ as
\[ |\det(N)| = \frac{|\det(R^t N)|}{\sqrt{\det(R^t R)}}, \]
where $R$ is the $(|A_L| \times |A_L|)$-matrix whose columns are $u_1, \ldots, u_r$, followed by $q_1, \ldots, q_k$ (i.e., $Q$). Using the orthonormality of the $u_\ell$’s (with respect to the dot product), we see that $R^t R$ can be divided into four blocks, the upper left one being the $r \times r$ identity matrix $I_{r \times r}$. Below it, $R^t R$ has a $k \times r$ block with entries
\[ q_j \cdot u_\ell = q_j \cdot T(\log_G(\log^{-1}(u_\ell))) = \langle q_j, \log_G(\log^{-1}(u_\ell)) \rangle = 0, \]
where we used (25) and the definition of the $q_j$’s as a basis of the orthogonal complement of $\log_G(E_{\mathbb{R}}) \subset \mathbb{R}^{A_L}$ (with respect to $\langle \rangle$, see (18)). Since the bottom right $k \times k$ block of $R^t R$ is $Q^t Q$, we find that $R^t R = \begin{pmatrix} I_{r \times r} & 0_{r \times k} \\ 0_{k \times r} & Q^t Q \end{pmatrix}$. Thus,
\[ \det(R^t R) = \sqrt{\det(Q^t Q)}. \]
A similar calculation shows $R^t N = \begin{pmatrix} I_{r \times r} & *_{r \times k} \\ 0_{k \times r} & I_{k \times k} \end{pmatrix}$, whence $\det(R^t N) = 1$.

In order to study the $\Theta$-series (12), we need to consider integrals of the form
\[ \int_{x \in E_{\mathbb{R}}} e^{-\|gx\|^2} d\mu_{E_{\mathbb{R}}}(x) \quad \left( \|gx\|^2 := \sum_{v \in A_L} e_v g_v^2 x_v^2 \right), \]
for $g = (g_v)_v \in G := \mathbb{R}^{A_L}$. For $h = (h_1, \ldots, h_k) \in H := \mathbb{R}^k_{++}$, define $\psi$ by substituting $g = \sigma(h)$ above:
\[ \psi(h) := \int_{x \in E_{\mathbb{R}}} e^{-\|\sigma(h)x\|^2} d\mu_{E_{\mathbb{R}}}(x). \]
Note that the integral (27) depends only on $g$ modulo $E_{\mathbb{R}}$, so the function $\psi$ is independent of the choice of $\sigma$ splitting the exact sequence (20). The fact that (27) depends only on $g$ modulo $E_{\mathbb{R}}$ also shows that
\[ \int_{x \in E_{\mathbb{R}}} e^{-|gx|^2} d\mu_{E_{\mathbb{R}}}(x) = \int_{x \in E_{\mathbb{R}}} e^{-|\sigma(\delta(g))x|^2} d\mu_{E_{\mathbb{R}}}(x) = \psi(\delta(g)), \]
so we will concentrate on $\psi$, a function of only $k$ variables.

Define a linear map $S : \mathbb{C}^k \to \mathbb{C}^{A_L}$ by $S(s) = Qs$, where $Q$ is the matrix whose $j$-th column is $q_j \in \mathbb{R}^{A_L} \subset \mathbb{C}^{A_L}$, as in Lemma 11. Also define maps $S_v : \mathbb{C}^k \to \mathbb{C}$ for
each \( v \in \mathcal{A}_L \) by \( S_v(s) = (S(s))_v \). That is,
\[
S(s) = \sum_{j=1}^{k} s_j q_j, \quad S_v(s) = \sum_{j=1}^{k} q_{jv} s_j \quad (s = (s_1, \ldots, s_k)). \tag{30}
\]

Note that \( S \) is injective since the \( q_j \in \mathbb{R}^{\mathcal{A}_L} \) are linearly independent.

Our first aim is to calculate the \((k\text{-dimensional})\) Mellin transform
\[
(M\psi)(s) := \int_{H} \psi(h) h^s d\mu_H(h) := \int_{h_1=0}^{\infty} \cdots \int_{h_k=0}^{\infty} \psi(h) h_1^{s_1} \cdots h_k^{s_k} \frac{dh_1}{h_1} \cdots \frac{dh_k}{h_k}, \tag{31}
\]
where \( \text{Re}(s) := (\text{Re}(s_1), \ldots, \text{Re}(s_k)) \in \mathcal{D} \), with
\[
\mathcal{D} := \left\{ \sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^k \mid S_v(\sigma) > 0 \ \forall v \in \mathcal{A}_L \right\}. \tag{32}
\]

As \( q_{1v} := 1 \) for all \( v \in \mathcal{A}_L \) (see (17)), for \( t > 0 \) we have \((t, 0, 0, \ldots, 0) \in \mathcal{D} \). Hence \( \mathcal{D} \) is a non-empty, open, convex subset of \( \mathbb{R}^k \). We will presently prove that the Mellin transform \((M\psi)(s)\) in (31) converges if \( \text{Re}(s) \in \mathcal{D} \).

In the following calculation of \((M\psi)(s)\) the reader should initially consider only real \( s_j \), so that the integrand is positive. At the end of the calculation it will become clear that the integral converges for \( s \) in the open subset of \( \mathbb{C}^k \) where \( \text{Re}(s) \in \mathcal{D} \).

\[
(M\psi)(s) = 2^{r_2} \sqrt{\det(Q^T Q)} \int_{g \in G} \delta(g)^s \exp(-\|g\|^2) \, d\mu_G(g)
= 2^{r_2} \sqrt{\det(Q^T Q)} \int_{g \in G} \exp(-\|g\|^2) \prod_{j=1}^{k} \delta(g)_{j}^{s_j} \, d\mu_G(g)
= 2^{r_2} \sqrt{\det(Q^T Q)} \int_{g \in G} \exp\left(-\sum_{v \in \mathcal{A}_L} e_v g_v^2 \prod_{v \in \mathcal{A}_L} g_v^{e_v q_{jv}} \prod_{j=1}^{k} g_{\sum_{j=1}^{k} q_{jv}} \prod_{v \in \mathcal{A}_L} \frac{dg_v}{g_v}\right)
\exp\left(-e_v g_v^2 \frac{dg_v}{g_v}\right) \frac{\sqrt{\det(Q^T Q)}}{2^{r_1} \prod_{v \in \mathcal{A}_L} \frac{\Gamma(e_v S_v(s)/2)}{e_v S_v(s)/2}}, \tag{33}
\]
where \( r_1 \) is the number of real places of \( L \).

**Lemma 2.** For any \( \sigma \in \mathcal{D} \) (see \([32]\)), the Mellin inversion formula holds:

\[
\psi(h) = \frac{1}{(2\pi i)^k} \int_{I_\sigma} (M\psi)(s) h^{-s} \, ds \quad (h \in \mathbb{R}_+^k),
\]

where \( s = (s_1, \ldots, s_k) \) and \( I_\sigma \subset \mathbb{C}^k \) is the product of the \( k \) vertical lines \( \text{Re}(s_j) = \sigma_j \), taken from \( \sigma_j - i\infty \) to \( \sigma_j + i\infty \).

**Proof.** The calculation \([33]\) shows that the Mellin transform \((M\psi)(s)\) is defined for \( s \in I_\sigma \). Thus Mellin inversion will work provided that \( \int_{I_\sigma} |(M\psi)(s)| h^{-s} \, ds < \infty \). Since \( |h^{-s}| \) and \( |e_v^{e_v S_v(s)/2}| \) are constant on \( I_\sigma \), we turn to the factors \( |\Gamma(e_v S_v(s)/2)| \) in \([33]\). Write \( s = \sigma + iT, \ T \in \mathbb{R}^k \). In a strip \( 0 < C_1 \leq \text{Re}(z) \leq C_2 \), we have

\[
|\Gamma(z)| < C_3 \exp(-|\text{Im}(z)|) \tag{33}
\]

Since \( \text{Re}(e_v S_v(s)) = e_v S_v(\sigma) > 0 \) for \( s \in I_\sigma \),

\[
\prod_{v \in \mathcal{A}_L} |\Gamma(e_v S_v(s)/2)| < C_4 \exp\left(-\sum_{v \in \mathcal{A}_L} e_v |S_v(T)|/2\right) \leq C_4 \exp\left(-\|S(T)\|_1/2\right),
\]

where \( \| (m_v) \|_1 := \sum_{v \in \mathcal{A}_L} |m_v| \) is the 1-norm on \( \mathbb{R}^k \), and \( S \) is the linear function from \([30]\). Since \( S \) is injective, there exists \( C_5 > 0 \) such that

\[
\|S(T)\|_1 \geq C_5 \|T\|_1 := C_5 \sum_{j=1}^k |T_j|.
\]

Thus \((M\psi)(s)h^{-s}\) is integrable over \( I_\sigma \) and Mellin inversion \([34]\) holds. \(\square\)

Let

\[
\Gamma_v(z) := \begin{cases} 
\Gamma(z) & \text{if } v \text{ is real}, \\
\Gamma(z)\Gamma(z + \frac{1}{2}) & \text{if } v \text{ is complex},
\end{cases}
\]

and

\[
\alpha(s) := \sum_{v \in \mathcal{A}_L} \log \Gamma_v(S_v(s)) = \sum_{v \in \mathcal{A}_L} \log \Gamma_v\left(\sum_{j=1}^k q_{jv}s_j\right). \tag{35}
\]

We take the branch of \( \log \Gamma_v(z) \) which is real when \( z \) is real and positive.

**Lemma 3.** Let \( y = (y_1, \ldots, y_k) \in \mathbb{R}^k \) and \( \chi = \chi(y) := (e^{y_1/2}, \ldots, e^{y_k/2}) \in H := \mathbb{R}_+^k \). Then

\[
\psi(\chi) = \frac{\sqrt{\det(Q)\overline{Q}}}{2\pi(2\sqrt{\pi})^{r_2}2\pi i^k} \int_{s \in I_\sigma} \exp(\alpha(s) - \sum_{j=1}^k y_j s_j) \, ds \quad \text{(for any } \sigma \in \mathcal{D}),
\]

with \( \psi \) as in \([28]\), \( \alpha \) as in \([36]\), \( Q \) as in Lemma\([1]\), \( I_\sigma \) as in Lemma\([2]\) and \( r_1 \) (resp. \( r_2 \)) being the number of real (resp. complex) places of \( L \).

**Proof.** If \( v \) is complex, so \( e_v = 2 \), the duplication formula gives

\[
\frac{\Gamma(e_v S_v(s))}{e_v^{e_v S_v(s)}} = \frac{\Gamma(2S_v(s))}{2^{2S_v(s)}} = \frac{\Gamma(S_v(s))\Gamma\left(\frac{1}{2} + S_v(s)\right)}{2\sqrt{\pi}} = \frac{\Gamma_v(S_v(s))}{2\sqrt{\pi}}.
\]

\footnote{In fact, \( |\Gamma(z)| < C_\varepsilon \exp(-\pi|\varepsilon|\text{Im}(z))/2 \) holds for any \( \varepsilon > 0 \) \([AAR]\) Cor. 1.4.4.}
If $v$ is real, so $e_v = 1$, then
\[
\frac{\Gamma(e_v S_v(s))}{e_v S_v(s)} = \Gamma(S_v(s)) = \Gamma_v(S_v(s)).
\]

From (33) and Mellin inversion (34) we get
\[
\psi(\chi) = \frac{1}{(2\pi i)^k} \int_{s \in \mathbb{I}_{2}^2} \chi^{-s} \cdot (M\psi)(s) \, ds
\]
\[
= \sqrt{\det(Q^{-1}Q)} \frac{2^{\pi_1(2\pi i)k}}{2^{\pi_1(2\pi i)k}} \sum_{s \in \mathbb{I}_{2}^2} \prod_{j=1}^{k} \chi_j^{-s_{j}} \prod_{v \in \mathbb{A}_L} \frac{\Gamma(e_v S_v(s))}{e_v S_v(s)}/2 \, ds
\]
\[
= \sqrt{\det(Q^{-1}Q)} \frac{2^{\pi_1(2\sqrt{\pi}r^2\pi i)^k}}{2^{\pi_1(2\sqrt{\pi}r^2\pi i)^k}} \sum_{s \in \mathbb{I}_{2}^2} \prod_{j=1}^{k} \chi_j^{-2s_{j}} \prod_{v \in \mathbb{A}_L} \Gamma(S_v(s)) \, ds
\]
\[
= \sqrt{\det(Q^{-1}Q)} \frac{2^{\pi_1(2\sqrt{\pi}r^2\pi i)^k}}{2^{\pi_1(2\sqrt{\pi}r^2\pi i)^k}} \sum_{s \in \mathbb{I}_{2}^2} \exp(-\sum_{j=1}^{k} y_j s_{j}) \prod_{v \in \mathbb{A}_L} \Gamma_v(S_v(s)) \, ds \quad \square
\]

Now we apply the lemma to the Basic Inequality (14).

**Corollary 4.** For $t > 0$ and $a \in L^*$, define $y = y_{a,t} \in \mathbb{R}^k$ by
\[
y_j = (y_{a,t})_j := \begin{cases} 
\log(t) + \frac{2}{n} \log |\text{Norm}_{L/Q}(a)| & \text{if } j = 1, \\
\frac{2}{n} \sum_{v \in \mathbb{A}_L} e_v q_{jv} \log|a|_v & \text{if } 2 \leq j \leq k.
\end{cases}
\]

Then, with $L := \sqrt{\det(Q^{-1}Q)} / (2^{\pi_1(2\sqrt{\pi}r^2\pi i)^k})$, for any $\sigma \in \mathcal{D}$ we have
\[
\int_{x \in \mathbb{E}_k} e^{-t\|ax\|^2} \, d\mu_{E_k}(x) = \frac{L}{t^k} \int_{s \in \mathbb{I}_{2}^2} \exp \left( \alpha(s) - n \sum_{j=1}^{k} y_j s_{j} \right) \, ds,
\]
\[
\frac{2t}{n} \int_{x \in \mathbb{E}_k} \|ax\|^2 e^{-t\|ax\|^2} \, d\mu_{E_k}(x) = \frac{2L}{t^k} \int_{s \in \mathbb{I}_{2}^2} \exp \left( \alpha(s) - n \sum_{j=1}^{k} y_j s_{j} \right) \, ds.
\]

**Proof.** Define $r = r_{a,t} \in G := \mathbb{A}_L^\infty$ by $r_v := t^{1/2}|a|_v$. In view of (29) and Lemma 3, (39) will follow from $(\delta(r))_j = e^{ny_j/2}$. Indeed, by (19),
\[
(\delta(r))_j := \prod_{v \in \mathbb{A}_L} (t^{1/2}|a|_v)^{e_v q_{jv}} = t^{\frac{1}{2} \sum_{v \in \mathbb{A}_L} e_v q_{jv}} \prod_{v \in \mathbb{A}_L} |a|_v^{e_v q_{jv}}.
\]

If $j = 1$, then by (17) we have $q_{jv} = 1$ for all $v \in \mathbb{A}_K$. Using (9) and (10) we find
\[
(\delta(r))_1 = t^{n/2} |\text{Norm}_{L/Q}(a)| = e^{ny_j/2}.
\]

If $j > 1$, then $\sum_v e_v q_{jv} = 0$ (see (17)), so
\[
(\delta(r))_j = \prod_{v \in \mathbb{A}_L} |a|_v^{e_v q_{jv}} = e^{ny_j/2},
\]
as claimed. To prove (40), apply $-\frac{2t}{n} \frac{d}{dt}$ to (39), noting that $\frac{dy_j}{dt} = 0$ for $j \geq 2$. \quad \square
3. Existence and uniqueness of the critical point

We shall show that for every \( y \in \mathbb{R}^k \) there is a unique \( \sigma = \sigma(y) \in \mathcal{D} \) (see (32)) which is a critical point of \( F_y : \mathcal{D} \to \mathbb{R} \), defined as

\[
F_y(\sigma) := \alpha(\sigma) - \sum_{j=1}^{k} y_j \sigma_j = \alpha(\sigma) - y \cdot \sigma,
\]

with \( \alpha \) as in (36). The map taking \( y \in \mathbb{R}^k \) to the critical point \( \sigma(y) \in \mathcal{D} \) is closely related to the Legendre transform of \( \alpha : \mathcal{D} \to \mathbb{R} \), but we will develop the theory from scratch as ours is an easy case of the general theory of the Legendre transform [HUL, §E] [Sim, §1 and §5].

Lemma 5. Let \( \alpha : \mathcal{D} \to \mathbb{R} \) be as in (36). Then \( \alpha \) is steep [Sim, p. 30], i.e.,

\[
\lim_{\|\sigma\| \to \infty} \frac{\alpha(\sigma)}{\|\sigma\|} = +\infty,
\]

where the limit is taken over \( \sigma \in \mathcal{D} \) as its Euclidean norm \( \|\sigma\| \) tends to infinity.

Proof. Recall that the linear map \( S \) in (30) is injective. Hence there exists \( C > 0 \) such that, for all \( \sigma \in \mathcal{D} \),

\[
\max_{v \in A_L} \{|S_v(\sigma)|\} =: \|S(\sigma)\|_\infty \geq C\|\sigma\|.
\]

For any \( \sigma \in \mathcal{D} \), there is a \( v_0 = v_0(\sigma) \in A_L \) such that \( S_{v_0}(\sigma) = \max_{v \in A_L} \{S_v(\sigma)\} \). The previous inequality says that

\[
S_{v_0}(\sigma) \geq C\|\sigma\|. \tag{42}
\]

The known behavior of \( \Gamma(z) \) for \( z > 0 \) shows that there is a \( \kappa < 0 \) such that

\[
\log \Gamma_v (z) > \kappa \quad (\Gamma_v \text{ as in (35)}), \tag{43}
\]

for all \( z > 0 \) and all \( v \in A_L \) (\( \kappa = -1/5 \) will do). Also, Stirling’s formula shows that

\[
\log \Gamma_v (z) > \frac{z \log z}{2} \tag{44}
\]

for \( z \gg 0 \). It follows from (43), (42), and (44) that when \( \|\sigma\| \) is large,

\[
\alpha(\sigma) := \sum_{v \in A_L} \log \Gamma_v (S_v(\sigma)) > n\kappa + \log \Gamma_{v_0} (S_{v_0}(\sigma)) > n\kappa + \frac{1}{2}C\|\sigma\| \log(C\|\sigma\|),
\]

and the lemma follows. \( \square \)

The next lemma amounts to the fact that the gradient \( \nabla f \) of a steep and differentiable strictly convex function \( f \) is a bijection. However, in our case the domain \( \mathcal{D} \neq \mathbb{R}^k \), which means that we would need to check the boundary behavior of \( \alpha \) before citing results from convex analysis. We prefer not to quote and instead adapt the usual proof [Sim, §1] [HUL, §E] to our nicely behaved function \( \alpha \).

Lemma 6. For any \( y \in \mathbb{R}^k \) there is a unique \( \sigma = \sigma(y) \in \mathcal{D} \) such that \( y = \nabla_\alpha(\sigma) \).
Proof. For any \( y \in \mathbb{R}^k \), let \( F_y : D \to \mathbb{R} \), \( F_y(\tau) := \alpha(\tau) - y \cdot \tau \), and let
\[
\alpha^+(y) := \inf_{\tau \in D} \{ F_y(\tau) \},
\]
which we will now prove to be finite, i.e., \( \alpha^+(y) \neq -\infty \). Let \( \tau^{(i)} \) be a sequence in \( D \) such that \( F_y(\tau^{(i)}) \) converges to \( \alpha^+(y) \). By \( (43) \), \( \alpha(\tau^{(i)}) \) is bounded below, so it suffices to check that the sequence \( \tau^{(i)} \) is bounded. By Lemma 5 \( \alpha(\tau) > (\|y\| + 1) \|\tau\| \) for \( \tau \in D \) with \( \|\tau\| \) sufficiently large. For such \( \tau \),
\[
F_y(\tau) > (\|y\| + 1) \|\tau\| - \|y\| \|\tau\| = \|\tau\|,
\]
which shows that \( \tau^{(i)} \) is bounded.

We now prove that the infimum defining \( \alpha^+(y) \) is assumed at a point in the open set \( D \subset \mathbb{R}^k \). Passing to a subsequence of the bounded sequence \( \tau^{(i)} \), we may assume that the \( \tau^{(i)} \in D \) converge to a point \( \sigma \) in the closure of \( D \) in \( \mathbb{R}^k \). Recall from (32) that \( D \) is the (non-empty) open set consisting of \( \tau \in \mathbb{R}^k \) such that \( S_v(\tau) > 0 \) for all \( v \in A_L \). If \( \sigma \notin D \), then \( S_v(\sigma) = 0 \) for some \( v \in A_L \). Since \( \log \Gamma_v(S_v(\tau^{(i)})) \to +\infty \) as \( S_v(\tau^{(i)}) \to 0^+ \), and the remaining summands in the definition of \( \alpha \) remain bounded from below (as does \( y \cdot \tau^{(i)} \)), we conclude that \( \sigma \in D \). Since \( \sigma \) is an interior minimum of the smooth function \( F_y \), we have \( \nabla F_y(\sigma) = 0 \). By (41), \( y = \nabla \alpha(\sigma) \), as claimed.

To prove the uniqueness of \( \sigma \), it suffices to prove that \( F_y \) is a strictly convex function on \( D \). The strict convexity of \( F_y \) follows from the strict convexity of \( \log \Gamma(z) \) for \( z > 0 \). Indeed,
\[
F_y(t\tau + (1-t)\tilde{\tau}) = -(t\tau + (1-t)\tilde{\tau}) \cdot y + \alpha(t\tau + (1-t)\tilde{\tau})
= -(t\tau + (1-t)\tilde{\tau}) \cdot y + \sum_{v \in A_L} t \log \Gamma_v(S_v(t\tau + (1-t)\tilde{\tau}))
\leq -(t\tau + (1-t)\tilde{\tau}) \cdot y + \sum_{v \in A_L} t \log \Gamma_v(S_v(\tau)) + (1-t) \log \Gamma_v(S_v(\tilde{\tau}))
= tF_y(\tau) + (1-t)F_y(\tilde{\tau}),
\]
with strict inequality holding for \( t \in (0,1) \) unless \( S_v(\tau) = S_v(\tilde{\tau}) \) for all \( v \in A_L \). But this is impossible because \( S \) in (30) is injective.

The function \( \alpha^+ \) in (45) is a concave function of \( y \in \mathbb{R}^k \), being the infimum over \( \tau \in D \) of the set of concave (in fact, affine) functions \( \leftarrow y \cdot \tau + \alpha(\tau) \). The convex function \(-\alpha^+ \) is known as the Legendre transform of \( \alpha \).

4. Inequalities at the critical point

To take advantage of the inequality (14), we will later need to drop all terms in (14) corresponding to algebraic integers \( a \neq 1 \). For this we will need some control of the first coordinate \( \sigma_1(y) \) of the function \( \sigma \) in Lemma 6. In this subsection we...
take advantage of the concavity of \( \Psi := \Gamma' / \Gamma \) to find a lower bound for \( \sigma_1(y) \). Then we use the convexity of \( \log \Gamma \) to find a lower bound for \( \alpha(\sigma(y)) \). Let

\[
\Psi_v(z) := \Psi(z) + \frac{1}{2}, (v \in A_L).
\]

This definition ensures that \( \Psi_v(z) \) is a concave function of \( z \) for \( z > 0 \). We also note that \( \Psi_v: (0, \infty) \to \mathbb{R} \) has an inverse function \( \Psi_v^{-1}: \mathbb{R} \to (0, \infty) \) since \( \Psi(z) \) is strictly increasing when \( z > 0 \), tends to \(-\infty\) as \( z \to 0^+ \), and tends to \(+\infty\) as \( z \to +\infty \).

Writing out the \( \ell \)-th coordinate of the equation \( y = \nabla \alpha(\sigma) \) in Lemma \( \ref{lem:gradient} \) we get

\[
y_\ell = \sum_{v \in A_L} \Psi_v(S_v(\sigma))q_v
\]

which for \( \ell = 1 \) simplifies to

\[
y_1 = \sum_{v \in A_L} \Psi_v(S_v(\sigma)).
\]

\[\text{Lemma 7.}\] Let \( L \) be a number field of degree \( n \), with \( r_2 \) complex places. For \( y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k \), let \( \sigma_1(y) \) be the first coordinate of the function \( \sigma(y) \) defined in Lemma \( \ref{lem:gradient} \). Then

\[
\sigma_1(y_1, y_2, \ldots, y_k) \geq \Psi^{-1} \left( \frac{y_1}{n} \right) - \frac{r_2}{2n}.
\]

\[\text{Proof.}\] We prove (50) using the concavity of \( \Psi \). Namely, from (49),

\[
y_1 = \sum_{v \in A_L} \Psi_v(S_v(\sigma)) = \sum_{v \in A_L} \Psi(S_v(\sigma)) + \sum_{v \text{ complex}} \Psi(\frac{1}{2} + S_v(\sigma))
\]

\[
\leq n\Psi \left( \frac{1}{n} \left( \sum_{v \in A_L} S_v(\sigma) + \sum_{v \text{ complex}} \left( \frac{1}{2} + S_v(\sigma) \right) \right) \right)
\]

\[
= n\Psi \left( \frac{1}{n} \sum_{v \in A_L} e_v S_v(\sigma) + \frac{r_2}{2n} \right) = n\Psi \left( \sigma_1 + \frac{r_2}{2n} \right),
\]

where the last step uses

\[
\frac{1}{n} \sum_{v \in A_L} e_v S_v(\sigma) = \sigma_1 \quad (\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbb{C}^k),
\]

which follows from (17) since

\[
\sum_{v \in A_L} e_v S_v(\sigma) = \sum_{v \in A_L} \sum_{j=1}^k e_v q_j \sigma_j = \sum_{j=1}^k \sigma_j \sum_{v \in A_L} e_v q_j = \sigma_1 \sum_{v \in A_L} e_v = \sigma_1 n.
\]

Inequality (50) now follows, since \( \Psi^{-1} \) is an increasing function.

Our next result is a similar inequality for \( \alpha(\sigma) \).

\[\text{Lemma 8.}\] With notation as in Lemma \( \ref{lem:asymptotic} \) we have

\[
\alpha(\sigma) \geq n \log \Gamma(\sigma_1 + \frac{r_2}{2n}) \quad (\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathcal{D}).
\]
Proof. We compute directly from the definition (36) of $\alpha$, using the convexity of $z \mapsto \log \Gamma(z)$ for $z > 0$ and (51):

\[
\alpha(\sigma) = \sum_{v \in A_L} \log \Gamma(S_v(\sigma)) + \sum_{v \text{ complex}} \log \Gamma(S_v(\sigma) + \frac{1}{2}) \\
\geq n \log \Gamma\left(\frac{1}{n} \left(\sum_{v \in A_L} S_v(\sigma) + \sum_{v \text{ complex}} \left(\frac{1}{2} + S_v(\sigma)\right)\right)\right) \\
= n \log \Gamma\left(\frac{1}{n} \sum_{v \in A_L} e_v S_v(\sigma) + \frac{r_2}{2n}\right) = n \log \Gamma(\sigma_1 + \frac{r_2}{2n}).
\]

We now prove a lower bound for $S_v(\sigma)$ in terms of $\sigma_1$ and $y_1$.

Lemma 9. Let $u \in A_L$, $y \in \mathbb{R}^k$, and let $\sigma := \sigma(yu) \in D$ be as in Lemma 6. Assume that $y_1 \geq t_0$ for some $t_0 \in \mathbb{R}$, and $n := [L : \mathbb{Q}] \geq 2$. Then $S_u(\sigma) \geq 2/5$ or

\[
S_u(\sigma) \geq \frac{1}{(n-1)\Psi\left(\frac{\sigma_1}{n-1} + \frac{1}{2}\right) - nt_0} \geq \frac{1}{(n-1)\log(2\sigma_1 + \frac{1}{2} - nt_0)} > 0. \tag{53}
\]

Proof. We shall show below that both denominators in (53) are positive if $S_u(\sigma) < 2/5$, as we may assume. Replacing $y$ with $ny$ in (49), we have

\[
ny_1 = \sum_{v \in A_L} \Psi(S_v(\sigma)) + \sum_{v \in A_L \text{ complex}} \Psi\left(\frac{1}{2} + S_v(\sigma)\right).
\]

Since $-\Psi$ is a monotone decreasing convex function on $(0, \infty)$, we find

\[
\Psi(S_u(\sigma)) = ny_1 - \sum_{v \in A_L \text{ complex}} \Psi(S_v(\sigma)) - \sum_{v \in A_L \text{ complex}} \Psi\left(\frac{1}{2} + S_v(\sigma)\right) \\
\geq ny_1 - (n-1)\Psi\left(\frac{1}{n-1} \left(\sum_{v \in A_L \text{ complex}} S_v(\sigma) + \sum_{v \in A_L \text{ complex}} \frac{1}{2} + S_v(\sigma)\right)\right) \\
= ny_1 - (n-1)\Psi\left(\frac{1}{n-1} \left(-S_u(\sigma) + \sum_{v \in A_L \text{ complex}} e_v S_v(\sigma) + \sum_{v \in A_L \text{ complex}} \frac{1}{2}\right)\right) \\
= ny_1 - (n-1)\Psi\left(-\frac{S_u(\sigma)}{n-1} + \frac{\sigma_1}{n-1} + \frac{r_2}{2(n-1)}\right) \quad \text{(see (51))} \\
\geq ny_1 - (n-1)\Psi\left(\frac{\sigma_1}{n-1} + \frac{r_2}{2(n-1)}\right) \geq ny_1 - (n-1)\Psi\left(\frac{\sigma_1}{n-1} + \frac{1}{2}\right).
\]

From $x\Gamma(x) = \Gamma(x+1)$ and the fact that $\Psi(x) < 0$ for $x < 1.461$,

\[
\Psi(x) = -\frac{1}{x} + \Psi(1+x) < -\frac{1}{x} \quad \text{for} \quad x < 0.461.
\]

Hence, as we are assuming $S_u(\sigma) < 2/5$,

\[
-\frac{1}{S_u(\sigma)} > \Psi(S_u(\sigma)) \geq ny_1 - (n-1)\Psi\left(\frac{\sigma_1}{n-1} + \frac{1}{2}\right) \geq nt_0 - (n-1)\Psi\left(\frac{\sigma_1}{n-1} + \frac{1}{2}\right).
\]

Since $S_u(\sigma) > 0$, the right-hand side above is negative. Hence the left-most inequality in (53) is proved.
Next recall [Ni, §71, eq. (11)],
\[
\log(x) - \Psi(x) = \frac{1}{2x} + 2 \int_0^\infty \frac{t}{(t^2 + x^2)(e^{2\pi t} - 1)} \, dt > 0 \quad (x > 0).
\]
Whence \(\Psi(x) < \log(x)\) for \(x > 0\), and so
\[
\Psi\left(\frac{n\sigma_1}{n-1} + \frac{1}{2}\right) < \log\left(\frac{n\sigma_1}{n-1} + \frac{1}{2}\right) \leq \log(2\sigma_1 + \frac{1}{2}).
\]
Now the second inequality in (53) follows as before. \(\square\)

5. Asymptotics

With a view to applying Corollary 4 and the Basic Inequality (14), in this section we will estimate integrals of the type
\[
\frac{1}{2k} \int_{s \in I_a} e^{\alpha(s) - ny \cdot s} \, ds = \int_{T \in \mathbb{R}^k} e^{\alpha(\sigma + iT) - n y \cdot (\sigma + iT)} \, dT =: \int_{\mathbb{R}^k} G(T) \, dT, \tag{54}
\]
where \(n := [L : \mathbb{Q}], y = (y_1, \ldots, y_k) \in \mathbb{R}^k, \sigma := \sigma(ny) \in D \subset \mathbb{R}^k\) as in Lemma 6 and \(y \cdot s := \sum_{j=1}^k y_j s_j\). We will let \(H(T)\) be a Gaussian approximating \(G(T)\) (see (63) below) in a bounded neighborhood \(\Delta \subset \mathbb{R}^k\) of \(T = 0\) (see (85)). As usual with the saddle point method, we decompose the integral (54) into four pieces
\[
\int_{\mathbb{R}^k} G(T) \, dT = \int_{\mathbb{R}^k} H(T) \, dT + \int_{\mathbb{R}^k - \Delta} G(T) \, dT - \int_{\mathbb{R}^k - \Delta} H(T) \, dT + \int_{\Delta} (G(T) - H(T)) \, dT =: I_1 + I_2 - I_3 + I_4. \tag{55}
\]
The term \(I_1\) (i.e., \(\int_{\mathbb{R}^k} H\)) is readily computed and gives (as we will prove in this section) the main term in (55). Thus, we shall prove that the terms \(I_2, I_3\) and \(I_4\) are \(o(I_1)\) as \([L : K] \to \infty\), uniformly in \(y \in \mathbb{R}^k\).

From now on we always (and usually tacitly) assume that the relative units \(E(L/K) \subset E \subset \mathcal{O}_L^*\) for some subfield \(K \subset L\) (see (17)). Define \(\Log : L^* \to \mathbb{R}^{A_L}\) by
\[
(\Log(a))_v := \log(|a|_v) \quad (a \in L^*, \ v \in A_L).
\]
Note that the complex places do not carry a factor of 2. Instead we use this factor in the inner product (16) on \(\mathbb{R}^{A_L}\) defined by \(\langle \beta, \gamma \rangle := \sum_{v \in A_L} e_v \beta_v \gamma_v\). The usefulness of assuming \(E(L/K) \subset E\) lies in the following.

**Lemma 10.** Suppose \(E(L/K) \subset E \subset \mathcal{O}_L^*\) and \(q = (q_v)_{v \in A_L} \in \Log(E)^\perp\) lies in the orthogonal complement of \(\Log(E)\) inside \(\mathbb{R}^{A_L}\) with respect to the above inner product. Then \(q_v = q_{v'}\) whenever \(v\) and \(v'\) lie above the same place of \(K\) and
\[
1 \leq k := \dim_{\mathbb{R}} (\Log(E)^\perp) \leq |A_K| \leq [K : Q]. \tag{56}
\]
**Proof.** The lemma will follow from the fact that \(\Log(E)^\perp\) is contained in the \(\mathbb{R}\)-span of \(\Log(K^*)\) in \(\mathbb{R}^{A_L}\). Clearly \(\Log(E)^\perp \subset \Log(E(L/K))^\perp\), so it suffices to prove that span \(\Log(K^*) = \Log(E(L/K))^\perp\). This follows from span \(\Log(K^*) \subset \Log(E(L/K))^\perp\) and \(\dim(\text{span } \Log(K^*)) = \dim(\Log(E(L/K))^\perp) = |A_K|\). \(\square\)
Recall that in (17) we fixed a basis $q_1, \ldots, q_k$ of $\text{Log}(E)^\perp$ such that $q_{1v} = 1$ for all $v \in A_L$ and $\langle q_1, q_j \rangle = 0$ for $2 \leq j \leq k$. In view of Lemma 10, we will write $q_{jv} := q_{jv}$ for any $v \in A_L$ extending $w \in A_K$.

For a place $w \in A_K$, let $r_{1w}$ and $r_{2w}$ denote respectively the number of real and complex places of $L$ extending $w$, and let (cf. [FS, p. 134])

$$m_w := r_{1w} + 2r_{2w}, \quad \kappa_w := \frac{r_{1w} + r_{2w}}{m_w}, \quad \alpha_w(z) := \kappa \log \Gamma(z) + (1 - \kappa) \log \Gamma(z + \frac{1}{2}). \quad (57)$$

Note that $m_w = e_w[L : K] = [L : K]$ or $2[L : K]$, and that $\frac{1}{2} \leq \kappa_w \leq 1$.

Lemma 10 implies that $S_v$ defined in (30) satisfies

$$S_v(s) := \sum_{j=1}^{k} q_{jv}s_j = \sum_{j=1}^{k} q_{jw}s_j =: S_w(s) \quad (s \in \mathbb{C}^k), \quad (58)$$

where $v \in A_L$ is any place extending $w \in A_K$. We therefore rewrite $\alpha$ in (36) as

$$\alpha(s) := \sum_{v \in A_L} \log \Gamma_v(S_v(s)) = \sum_{w \in A_K} \sum_{v | w} \log \Gamma_v(S_w(s)) = \sum_{w \in A_K} m_w \alpha_{\kappa_w}(S_w(s)), \quad (59)$$

where we write $v | w$ if $v$ extends $w$, and $\alpha_{\kappa_w}$ was defined in (57).

For each $w \in A_K$ and $\sigma \in D$ (see (32)), define $\rho_w : \mathbb{R}^k \to \mathbb{C}$ by

$$\rho_w(T) := \alpha_{\kappa_w}(S_w(\sigma + iT)) - \alpha_{\kappa_w}(S_w(\sigma)) - i\alpha'_{\kappa_w}(S_w(\sigma))S_w(T) + \frac{1}{2} \alpha''_{\kappa_w}(S_w(\sigma))(S_w(T))^2, \quad (60)$$

i.e., $\rho_w$ is the error in the degree-2 Taylor approximation of $T \mapsto \alpha_{\kappa_w}(S_w(\sigma + iT))$ at $T = 0$. We shall henceforth take any $y \in \mathbb{R}^k$ and let $\sigma := \sigma(ny)$ be the corresponding saddle point in Lemma 6. Thus $\nabla \alpha(\sigma) = ny$. Using this and (59), we find

$$\sum_{j=1}^{k} ny_jT_j = \sum_{j=1}^{k} T_j \sum_{w \in A_K} m_w \alpha'_{\kappa_w}(S_w(\sigma))q_{jw} = \sum_{w \in A_K} m_w \alpha'_{\kappa_w}(S_w(\sigma))S_w(T). \quad (61)$$

It follows from (59), (61) that

$$\alpha(\sigma + iT) - ny \cdot (\sigma + iT) = \alpha(\sigma) - ny \cdot \sigma - \frac{1}{2} \sum_{w \in A_K} m_w \alpha''_{\kappa_w}(S_w(\sigma))S_w(T)^2 + \rho(T), \quad (62)$$

$$\rho(T) := \sum_{w \in A_K} m_w \rho_w(T).$$

The linear terms in $T$ have disappeared as $\sigma$ is a critical point of $s \mapsto \alpha(s) - ny \cdot s$.

For fixed $y \in \mathbb{R}^k$ and $\sigma := \sigma(ny) \in D$, define the following functions of $T \in \mathbb{R}^k$:

$$\mathcal{H}(T) := e^{\alpha(\sigma) - ny \cdot \sigma - \frac{1}{2}\mathcal{H}(T)}, \quad (63)$$

$$H(T) := \sum_{w \in A_K} m_w \alpha''_{\kappa_w}(S_w(\sigma))S_w(T)^2, \quad (64)$$

$$\mathcal{G}(T) := e^{\alpha(\sigma+iT) - ny \cdot (\sigma+iT)} = e^{\rho(T)}\mathcal{H}(T). \quad (65)$$

Although $\mathcal{H}, H, \mathcal{G}$ and $\rho$ depend on $y \in \mathbb{R}^k$, we do not include $y$ in our notation.
5.1. **The main term.** In Lemma 11 we defined the $|A_L| \times k$ matrix $Q$ of rank $k$ whose coefficients are $Q_{v,j} := q_{jv}$. We will write $Q$ for the $|A_K| \times k$ matrix with entries $Q_{w,j} := q_{jw}$ and rank $k$. Recall that we write $q_{jw} := q_{jv}$ for any $v \in A_L$ extending $w \in A_K$. Let $A_K^{[k]}$ be the set of $k$-element subsets of $A_K$. For $\eta \in A_K^{[k]}$, let $Q_\eta$ be the $k \times k$ submatrix of $Q$ whose rows are indexed by the elements of $\eta$. In the computation of $\psi(\chi)$ in Lemma 3 the term $\det(Q^TQ)$ appears. Using the smaller matrix $Q$ we have

$$\det(Q^TQ) = \det(Q^TQ) \prod_{w \in A_K} (r_{1,w} + r_{2,w}) \quad \text{for } (r_{1,w}, r_{2,w} \text{ as in (57)}), \quad (66)$$

as follows from

$$(Q^TQ)_{i,j} = \sum_{w \in A_L} q_{iw}q_{jw} = \sum_{w \in A_K} q_{iw}q_{jw} \sum_{v | w} 1 = \sum_{w \in A_K} q_{iw}q_{jw}(r_{1,w} + r_{2,w}).$$

Next we calculate some integrals such as $I_1$ in (55), and its derivatives.

**Lemma 11.** Let $Q$ and $Q_\eta$ be as above, where $\eta \in A_K^{[k]}$, let $(b_w)_{w \in A_K} \in \mathbb{R}_+^{A_K}$, and define

$$D_\eta := \det^2(Q_\eta) \prod_{w \in \eta} b_w, \quad D := \sum_{\eta \in A_K^{[k]}} D_\eta. \quad (67)$$

Then, with $S_w$ as in (58),

$$\int_{T \in \mathbb{R}^k} \exp \left( -\frac{1}{2} \sum_{w \in A_K} b_w S_w(T)^2 \right) dT = (2\pi)^{k/2} D^{-1/2}. \quad (68)$$

Furthermore, for any $w_0 \in A_K$ we have

$$\int_{\mathbb{R}^k} S_{w_0}(T)^4 \exp \left( -\frac{1}{2} \sum_{w \in A_K} b_w S_w(T)^2 \right) dT = 3(2\pi)^{k/2} D^{-5/2} b_{w_0}^{-2} \left( \sum_{\eta \supset w_0} D_\eta \right)^2 \leq 3(2\pi)^{k/2} D^{-1/2} b_{w_0}^{-2}$$

and

$$\int_{\mathbb{R}^k} S_{w_0}(T)^6 \exp \left( -\frac{1}{2} \sum_{w \in A_K} b_w S_w(T)^2 \right) dT = 15(2\pi)^{k/2} D^{-7/2} b_{w_0}^{-3} \left( \sum_{\eta \supset w_0} D_\eta \right)^3 \leq 15(2\pi)^{k/2} D^{-1/2} b_{w_0}^{-3}.$$

**Proof.** Let $P = (P_{w,j})$ be the $|A_K| \times k$ matrix with entries $P_{w,j} := \sqrt{b_w} q_{jw}$ ($w \in A_K$, $1 \leq j \leq k$). Then for $T = (T_1, ..., T_k) \in \mathbb{R}^k$, considered as a $k \times 1$ matrix, $PT \in \mathbb{R}_+^{A_K}$ satisfies $(PT)_w = \sqrt{b_w} S_w(T)$. Hence

$$\sum_{w \in A_K} b_w S_w(T)^2 = (PT)^\top PT = T^\top (P^\top P) T = T^\top HT \quad (H := P^\top P).$$

The $k \times k$ matrix $H$ is clearly positive semi-definite. The Cauchy-Binet formula gives $\det(H) = D$, with $D$ as in (67). But $D > 0$ as $D_\eta > 0$ for at least one $\eta \in A_K^{[k]}$.

6 The Cauchy-Binet formula computes $\det(AB)$, where $A$ is a $k \times \ell$ and $B$ is $\ell \times k$, in terms of the $k \times k$ minors of $A$ and $B$. 
since $Q$ has rank $k$. Hence $H$ is positive definite, and so the integral in (68) is the well-known Gaussian integral attached to a positive definite quadratic form $H$ in $k$ variables, as claimed in (68).

The other equalities in Lemma 11 are obtained by differentiating (68) with respect to $b_{w_0}$ repeatedly. Indeed, noting that the partial derivative $\frac{\partial}{\partial b_{w_0}} = b^{-1}_{w_0} \sum_{\eta \supseteq w_0} D_\eta$ is independent of $b_{w_0}$, i.e., $\frac{\partial^2}{\partial b_{w_0}^2} = 0$, we have

$$-\frac{1}{2} \int_{\mathbb{R}^k} S_{w_0}(T)^2 \exp\left(-\frac{1}{2} \sum_{w \in A_K} b_w S_w(T)^2\right) \, dT = -\frac{1}{2} \left(2\pi\right)^{k/2} \sum_{\eta \supseteq w_0} D_\eta,$$

$$\frac{1}{4} \int_{\mathbb{R}^k} S_{w_0}(T)^4 \exp\left(-\frac{1}{2} \sum_{w \in A_K} b_w S_w(T)^2\right) \, dT = \frac{3}{4} \left(2\pi\right)^{k/2} \sum_{\eta \supseteq w_0} D_\eta^2,$$

$$\frac{1}{8} \int_{\mathbb{R}^k} S_{w_0}(T)^6 \exp\left(-\frac{1}{2} \sum_{w \in A_K} b_w S_w(T)^2\right) \, dT = \frac{15}{8} \left(2\pi\right)^{k/2} \sum_{\eta \supseteq w_0} D_\eta^3,$$

proving the equalities. The inequalities follow from $\sum_{\eta \supseteq w_0} D_\eta \leq 0$, as $D_\eta \geq 0$. □

As $\alpha''(t) > 0$ for $t > 0$, we can now evaluate $I_1$.

**Corollary 12.** With notation as in (63), for $y \in \mathbb{R}^k$ we have

$$I_1 = I_1(ny) := \int_{\mathbb{R}^k} \mathcal{H}(T) \, dT = \frac{(2\pi)^{k/2} e^{\alpha(\sigma)} - \eta - \sigma}{\sqrt{\det(H(\sigma))}},$$

where $\sigma := \sigma(ny) \in D$ as in Lemma 10 and

$$\det(H(\sigma)) = \sum_{\eta \in A_K} \det^2(Q_\eta) \prod_{w \in \eta} m_w \alpha''(S_w(\sigma)).$$

### 5.2. The small terms.

We begin by quoting some one-variable estimates.

**Lemma 13.** If $m \geq 1000$, $\kappa \in [\frac{1}{2}, 1]$, and $r > 0$, then

$$\int_{-\infty}^{\infty} |e^{m\alpha_\kappa(r+it)}| \, dt < 1.0021 \frac{\sqrt{2\pi} e^{m\alpha_\kappa(r)}}{ma''_\kappa(r)},$$

$$\int_{-\infty}^{\infty} |te^{m\alpha_\kappa(r+it)}| \, dt < 0.83 \frac{\sqrt{2\pi} e^{m\alpha_\kappa(r)}}{ma''_\kappa(r)}.$$  \hspace{1cm} (70) \hspace{1cm} (71)

**Proof.** The estimate (70) is proved in [Su2, Lemma 4.4]. We now prove (71). From [Su2, Lemma 4.11] we have

$$\int_{\frac{r}{\sqrt{2\pi} \alpha''_\kappa}}^{\frac{r}{\sqrt{2\pi} \alpha''_\kappa}} |te^{m\alpha_\kappa(r+it)}| \, dt < \frac{72 e^{m\alpha_\kappa(r)}}{35 m a''_\kappa(r)},$$

while from [FS, Lemma 5.3] we have

$$\int_{|t| > \frac{r}{3\sqrt{2}}} |te^{m\alpha_\kappa(r+it)}| \, dt < \frac{2r^2 e^{m\alpha_\kappa(r)}}{m(\kappa - \frac{2}{m})(1 + \frac{1}{72})m^{(r)}(1 + \frac{1}{18})(m\kappa - 2)/2},$$

(72)
where \( \lfloor r \rfloor \) is the floor of \( r \). Since \( 0 < r^2 \alpha''(r) < 1 + r \) [FS, p. 141], we have
\[
\frac{r^2}{(1 + \frac{1}{72})m\kappa[r]/2} \leq \frac{1}{\alpha''(r)} (1 + \frac{1}{72})m\kappa[r]/2 \leq \frac{2}{\alpha''(r)}.
\]
Indeed, for \( 0 < r < 1 \) the last inequality is obvious, while for \( r \geq 1 \) a much better inequality follows from \( m\kappa \geq 500 \). Hence
\[
\int_{|t| > \frac{r}{4\sqrt{2}}} |te^{m\alpha''(r+it)}| \, dt < \frac{1}{m\alpha''(r)} (\frac{72}{16} - \frac{2}{1000})(1 + \frac{1}{18})(500-2)/2 < \frac{0.00002e^{m\alpha''(r)}}{m\alpha''(r)}.
\]
Combining this with (72) we obtain (71).

We will need the following inequality, proved by elementary calculus.
\[
x^{5/2}e^{-x} \leq \left( \frac{5}{2e} \right)^{5/2} < 0.8112 \quad (x \geq 0).
\]

**Lemma 14.** Suppose \( m \geq 1000 \), \( \frac{1}{2} \leq \kappa \leq 1 \), \( 0 < D \leq m^{1/3}\kappa \), and let
\[
\delta := \frac{D}{m^{1/3}\sqrt{\alpha''(r)}}.
\]
Then, for any \( r > 0 \),
\[
\int_{|t| > \delta} |e^{m\alpha''(r+it)}| \, dt < \left( \frac{10^{-76} + 41.43}{D^6} \right) \frac{\sqrt{2\pi}e^{m\alpha''(r)}}{m\alpha''(r)},
\]
and
\[
\int_{|t| > \delta} e^{-\frac{1}{2}m\alpha''(r)t^2} \, dt < \frac{3.67 \sqrt{2\pi}}{mD^6} \frac{\sqrt{m\alpha''(r)}}{m\alpha''(r)}.
\]

**Proof.** Inequality (77) follows from
\[
\int_{|t| > \delta} e^{-\frac{1}{2}m\alpha''(r)t^2} \, dt \leq \frac{2e^{-m^{1/3}D^2/2}}{m^{2/3}D\sqrt{\alpha''(r)}} = \frac{\sqrt{2\pi}}{m\alpha''(r)} \frac{8(m^{1/3}D^2/2)^{5/2}e^{-m^{1/3}D^2/2}}{m\sqrt{\pi}D^6} < \frac{3.67 \sqrt{2\pi}}{m\alpha''(r)} \frac{\sqrt{m\alpha''(r)}}{mD^6},
\]
where the first inequality is from [FS, p. 139] and the last one uses (74) with \( x := m^{1/3}D^2/2 \). To prove (76) we use [Su2, Lemma 4.5],
\[
\int_{|t| > \delta} |e^{m\alpha''(r+it)}| \, dt < 10^{-76} + \frac{23/2m^{5/6}\exp(-m^{1/3}D^2/4)}{\sqrt{\pi}D} < 10^{-76} + \frac{41.43}{D^6},
\]
where the second inequality again follows from (74).

Next we deal with the second order remainder term in the Taylor expansion about \( a \) of log \( \Gamma(a + ib) \), taking \( a = S_w(\sigma) \) and \( b = S_w(T) \).
Lemma 15. For \( w \in \mathcal{A}_K, \sigma \in \mathcal{D} \) (see \footnote{12}), \( T \in \mathbb{R}^k \) and \( \rho_w \) as in \footnote{60}, we have
\[
|\text{Im}(\rho_w(T))| \leq -\frac{\alpha^{(3)}(S_w(\sigma))}{3!} |S_w(T)|^3 \leq \frac{\sqrt{2}}{3} \alpha''(S_w(\sigma))^3/2 |S_w(T)|^3, \quad (78)
\]
\[
|\text{Re}(\rho_w(T))| \leq \frac{\alpha^{(4)}(S_w(\sigma))}{4!} S_w(T)^4 \leq \frac{1}{2} \alpha''(S_w(\sigma))^2 S_w(T)^4, \quad (79)
\]
\[
\text{Im}(\rho_w(\text{−}T)) = -\text{Im}(\rho_w(T)), \quad \text{Re}(\rho_w(\text{−}T)) = \text{Re}(\rho_w(T)), \quad (80)
\]
if \( |S_w(T)| \leq S_w(\sigma) \), then \( 0 \leq \text{Re}(\rho_w(T)) \leq \frac{\alpha''(S_w(\sigma))}{4} S_w(T)^2. \quad (81)
\]

**Proof.** The first inequalities in \footnote{78} and \footnote{79} are proved in \footnote{Su2, Lemma 4.7}, as is also \footnote{81}. The second inequalities in \footnote{78} and \footnote{79} follow from \footnote{FS, Lemma 5.2} and \( \kappa_w \geq \frac{1}{2} \). The identities in \footnote{80} follow from \footnote{60} and \( \log \Gamma(z) = \log \Gamma(z) \). \( \square \)

Lemma 16. \footnote{FS (5.11)]} If \( u, v \in \mathbb{R} \) with \( 0 \leq u \leq R \), then
\[
|\text{Re}(e^{u+iv} - 1)| \leq \frac{u^2}{2} + u \frac{e^R - 1}{R}.
\]

We first estimate the easier “outer” terms, \( I_2 \) and \( I_3 \) in \footnote{65}, i.e., where the region of integration is \( \mathbb{R}^k - \Delta \). For \( y \in \mathbb{R}^k \), let \( \eta_0(y) = \eta_0(y) \in \mathcal{A}_K^{[k]} \) correspond to a maximal summand in \footnote{69}, so
\[
\det^2(Q_{\eta}) \prod_{w \in \eta} m_w \alpha''_{\kappa_w}(S_w(\sigma)) \leq \det^2(Q_{\eta_0}) \prod_{w \in \eta_0} m_w \alpha''_{\kappa_w}(S_w(\sigma)) \quad (\forall \eta \in \mathcal{A}_K^{[k]}). \quad (82)
\]
Thus,
\[
\det(H(\sigma)) \leq |\mathcal{A}_K^{[k]})| \det^2(Q_{\eta_0}) \prod_{w \in \eta_0} m_w \alpha''_{\kappa_w}(S_w(\sigma)),
\]
and so
\[
\frac{1}{|\det(Q_{\eta_0})| \prod_{w \in \eta_0} \sqrt{m_w \alpha''_{\kappa_w}(S_w(\sigma))}} \leq \sqrt{|\mathcal{A}_K^{[k]}|} \sqrt{\det(H(\sigma))}, \quad (83)
\]
For \( y \in \mathbb{R}^k, \ w \in \eta_0(y) \) and \( D > 0 \), let \( (\text{cf. \footnote{75}}) \)
\[
\delta_w := \frac{D}{m_w^{1/3} \sqrt{\alpha''_{\kappa_w}(S_w(\sigma))}}. \quad (84)
\]
Define the neighborhood \( \Delta \subset \mathbb{R}^k \) of \( T = 0 \in \mathbb{R}^k \) as
\[
\Delta = \Delta(y) := \{ T \in \mathbb{R}^k | S_w(T) < \delta_w \ (\forall w \in \eta_0) \}. \quad (85)
\]
The next lemma shows that \( I_2 \) and \( I_3 \) are small compared to \( I_1 \) in Corollary \footnote{12}.

Lemma 17. Suppose \( m := [L : K] \geq 1000, \ 0 < D < m^{1/3}/\sqrt{2}, \) and \( y \in \mathbb{R}^k \). Then, with \( \Delta \) as in \footnote{85}, \( \sigma := \sigma(\eta_0) \in \mathcal{D} \) as in Lemma \footnote{4} \( \mathcal{H} \) and \( \mathcal{G} \) as in \footnote{63} and \footnote{65},
we have

\[ |I_2| = \left| \int_{\mathbb{R}^k - \Delta} \mathcal{G}(T) \, dT \right| \leq \frac{1.0021^k - 1 \left( 10^{-76} + \frac{41.43}{D^u} \right) k \sqrt{|A_K^{|k|}}}{m} I_1, \tag{86} \]

\[ |I_3| = \left| \int_{\mathbb{R}^k - \Delta} \mathcal{H}(T) \, dT \right| \leq \frac{3.67k \sqrt{|A_K^{|k|}}}{m D^u} I_1. \tag{87} \]

**Proof.** We first prove (86). Note that \( \Gamma(z) = \int_0^\infty x^z e^{-x} \frac{dx}{x} \) implies

\[ |\Gamma(z)| \leq \Gamma(\text{Re}(z)) \quad (\text{Re}(z) > 0). \tag{88} \]

Using this, (85) and (59) we have,

\[ \int_{\mathbb{R}^k - \Delta} |\mathcal{G}(T)| \, dT \leq e^{-ny} \prod_{w \in A_K^{|k|}} e^{m_w \alpha_w(S_w(\sigma))} \int_{\mathbb{R}^k - \Delta} \left| \prod_{w \in \eta_0} e^{m_w \alpha_w(S_w(\sigma+iT))} \right| \, dT. \]

Let \( B \subset \mathbb{R}^{\eta_0} \) denote the \( k \)-dimensional box

\[ B = B(y) := \{ \tilde{T} \in \mathbb{R}^{\eta_0} \mid |\tilde{T}_w| \leq \delta_w \quad (\forall w \in \eta_0) \}, \tag{89} \]

and let \( B^c := \mathbb{R}^{\eta_0} - B \) denote its complement. Making the change of variables \( \tilde{T}_w := S_w(T) \) for \( w \in \eta_0 \), we have

\[ \int_{\mathbb{R}^k - \Delta} \left| \prod_{w \in \eta_0} e^{m_w \alpha_w(S_w(\sigma+iT))} \right| \, dT = \frac{1}{|\det(Q_{\eta_0})|} \int_{\tilde{T} \in B^c} \left| \prod_{w \in \eta_0} e^{m_w \alpha_w(S_w(\sigma+i\tilde{T}_w))} \right| \, d\tilde{T}. \]

The latter integral is easy to bound using Lemmas 13 and 14. We integrate over \( k \) (overlapping) regions, each of which has \( k-1 \) of the \( \tilde{T}_w \) range over all of \( \mathbb{R} \), and the remaining \( \tilde{T}_{w_0} \) over \( |\tilde{T}_{w_0}| > \delta_{w_0} \). Since \( m_w \geq m := [L : K] \), we conclude that

\[ \int_{\mathbb{R}^k - \Delta} |\mathcal{G}(T)| \, dT \leq \frac{k(2\pi)^{k/2} 1.0021^k - 1 \left( 10^{-76} + \frac{41.43}{D^u} \right) e^{\alpha(\sigma) - ny\sigma}}{m |\det(Q_{\eta_0})| \prod_{w \in \eta_0} \sqrt{m_w \alpha''_w(S_w(\sigma))}}. \]

Now inequality (83) and Corollary 12 prove (86).

Next we prove (87). Changing variables as before, we have

\[ |I_3| = e^{\alpha(\sigma) - ny\sigma} \int_{\mathbb{R}^k - \Delta} \exp \left( -\frac{1}{2} \sum_{w \in A_K} m_w \alpha''_w(S_w(\sigma)) S_w(T)^2 \right) \, dT \]

\[ \leq e^{\alpha(\sigma) - ny\sigma} \int_{\mathbb{R}^k - \Delta} \exp \left( -\frac{1}{2} \sum_{w \in \eta_0} m_w \alpha''_w(S_w(\sigma)) S_w(T)^2 \right) \, dT \]

\[ = e^{\alpha(\sigma) - ny\sigma} \left| \frac{1}{|\det(Q_{\eta_0})|} \int_{B^c} \exp \left( -\frac{1}{2} \sum_{w \in \eta_0} m_w \alpha''_w(S_w(\sigma)) \tilde{T}_w^2 \right) \, d\tilde{T} \right|. \]
Once again, we bound \( \int_{B^c} \) using \( k \) overlapping regions, one for each \( w_0 \in \eta_0 \). The integral over the region given by all \( \tilde{T} \in \mathbb{R}^\eta \) such that \( |\tilde{T}_{w_0}| > \delta_{w_0} \) is bounded by

\[
\int_{|\tilde{T}_{w_0}| > \delta_{w_0}} e^{-\frac{1}{2}m_{w_0}\alpha_{w_0}''(S_{w_0}(\sigma))\tilde{T}_{w_0}^2} d\tilde{T}_{w_0} \prod_{w \in \eta_0 \atop w \neq w_0} \int_{-\infty}^{\infty} e^{-\frac{1}{2}m_{w}\alpha_{w}''(S_{w}(\sigma))\tilde{T}_{w}^2} d\tilde{T}_{w}.
\]

We can use (77) to bound the first integral, and the remaining integrals are explicitly known. Hence, summing over the \( k \) regions,

\[
|I_3| \leq \frac{(2\pi)^{k/2}e^{\alpha(\sigma) - ny_\sigma}}{|\det(Q_{\eta_0})|} \frac{3.67k}{mD^K} \prod_{w \in \eta_0} \frac{1}{\sqrt{m_{w}\alpha_{w}''(S_{w}(\sigma))}}.
\]

We again conclude using (83).

For the “inner” integral \( I_4 = \int_{\Delta} (G - H) \) in (55), we can only expect estimates of the kind \( O(I_1/m) \), whereas \( I_2 \) and \( I_3 \) are essentially \( O(I_1 \exp(-m^{1/3})) \). This allowed us to use simple estimates for the contribution of places \( w \notin \eta_0 \). However, to estimate \( I_4 \) we shall need the following geometric result.

**Lemma 18.** Let \( M = (m_{i,j}) \) be an \( N \times k \) matrix of rank \( k \), and let \( a_i > 0 \) (\( 1 \leq i \leq N \)). Define linear maps \( P_i : \mathbb{R}^k \to \mathbb{R} \) by \( P_i(T) := \sum_{j=1}^{k} m_{i,j} T_j \), where \( T = (T_1, \ldots, T_k) \). For any \( k \)-element subset \( \eta = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, N\} \), let \( M_\eta \) denote the \( k \times k \) submatrix of \( M \) given by \( (M_\eta)_{i,j} = m_{i,j} \). Define \( E_\eta := |\det(M_\eta)| \prod_{i \in \eta} a_i \), and let \( \eta_0 \) maximize \( E_\eta \). Then

\[
a_i |P_i(T)| \leq \sum_{j \in \eta_0} a_j |P_j(T)| \quad \quad \quad (1 \leq i \leq N, \ T \in \mathbb{R}^k).
\]

**Proof.** Replacing \( m_{i,j} \) with \( a_i m_{i,j} \), we may assume \( a_i = 1 \). Hence \( \eta_0 \) simply maximizes \( |\det(M_\eta)| \). Fix \( i \in \{1, 2, \ldots, N\} \), and define \( \lambda_j \in \mathbb{R} \) for \( j \in \eta_0 \) by \( P_i = \sum_{j \in \eta_0} \lambda_j P_j \). For \( j \in \eta_0 \), let \( M_j \) denote \( M_\eta \) replaced by the \( i \)th row. Then, by Cramer’s rule, \( |\lambda_j \det(M_\eta)| = |\det(M_j)| \leq |\det(M_\eta)| \), so \( |\lambda_j| \leq 1 \). Hence

\[
|P_i(T)| = \left| \sum_{j \in \eta_0} \lambda_j P_j(T) \right| \leq \sum_{j \in \eta_0} |P_j(T)|.
\]

**Lemma 19.** For \( y \in \mathbb{R}^k \) and \( D > 0 \) we have

\[
|I_4| = \left| \int_{\Delta} (G(T) - H(T)) \right| dT \leq \frac{|A_K| (\frac{\pi}{2} |A_K| + \frac{3}{2} Z)}{m} I_1,
\]

with notation as in (55), \( m := [L : K] \) and \( Z := (e^{A_K |k^4D^4m^{-1/3}|} - 1) / |A_K| |k^4D^4m^{-1/3}| \).

**Proof.** Lemma 18 applied to the matrix \( Q \) and \( A_w := \sqrt{m_w \alpha_w''(S_w(\sigma))} \), shows

\[
\sqrt{m_w \alpha_w''(S_w(\sigma))} |S_w(T)| \leq \sum_{w_0 \in \eta_0} \sqrt{m_{w_0} \alpha_{w_0}''(S_{w_0}(\sigma))} |S_{w_0}(T)| \quad \quad \quad (91)
\]
for $w \in A_K$, $T \in \mathbb{R}^k$ and $\eta_0$ as in (82). Since $x \mapsto x^4$ is convex, we have,

$$m_w^2 \alpha''_{\kappa_w}(S_w(\sigma))^2 S_w(T)^4 \leq \left( \sum_{w_0 \in \eta_0} \left( \frac{\sqrt{m_w^2 \alpha''_{\kappa_{w_0}}(S_{w_0}(\sigma)) |S_{w_0}(T)|}}{S_{w_0}(T)} \right)^4 \right)^k$$

$$\leq k^3 \sum_{w_0 \in \eta_0} m_w^2 \alpha''_{\kappa_{w_0}}(S_{w_0}(\sigma))^2 S_{w_0}(T)^4.$$  

For $T \in \Delta$ and $w_0 \in \eta_0$, by (84) and (85) we have

$$m_w \alpha''_{\kappa_{w_0}}(S_{w_0}(\sigma))^2 S_{w_0}(T)^4 \leq m_w \alpha''_{\kappa_{w_0}}(S_{w_0}(\sigma))^2 \delta_{w_0}^4 = D^4 m_w^{-1/3}.$$  

Hence,

$$m_w \alpha''_{\kappa_{w_0}}(S_{w}(\sigma))^2 S_{w}(T)^4 \leq k^3 \sum_{w_0 \in \eta_0} \frac{m_{w_0}^2 D^4}{m_w m^{1/3}} \leq k^3 \sum_{w_0 \in \eta_0} \frac{2D^4}{m^{1/3} m^{1/3}} = \frac{2k^4 D^4}{m^{1/3}}.$$  

Combining this with Lemma 15 we conclude that for $T \in \Delta$,

$$|\operatorname{Re}(\rho(T))| = \left| \sum_{w \in A_K} m_w \operatorname{Re}(\rho_w(T)) \right| \leq \sum_{w \in A_K} k^4 D^4 m_w^{-1/3} = |A_K| k^4 D^4 m^{-1/3}.$$  

Lemmas 15 and 16 now show that for $T \in \Delta$,

$$|\operatorname{Re}(e^{\rho(T)} - 1)| \leq \frac{|\operatorname{Im}(\rho(T))|^2}{2} + \operatorname{Re}(\rho(T)) \left[ \frac{\sqrt{2}}{3} \sum_{w \in A_K} m_w \alpha''_{\kappa_{w_0}}(S_w(\sigma))^3/2 |S_w(T)|^3 \right] + \frac{Z}{2} \sum_{w \in A_K} m_w \alpha''_{\kappa_{w_0}}(S_w(\sigma))^2 S_w(T)^4$$

$$\leq \frac{|A_K|}{9} \sum_{w \in A_K} m_w^2 \alpha''_{\kappa_{w_0}}(S_w(\sigma))^3 S_w(T)^6 + \frac{Z}{2} \sum_{w \in A_K} m_w \alpha''_{\kappa_{w_0}}(S_w(\sigma))^2 S_w(T)^4, \quad (92)$$

where in the last step we used the convexity of $x \mapsto x^2$.

By Lemma 15 $\operatorname{Im}(e^{\rho(T)})$ is odd, while $\operatorname{Re}(e^{\rho(T)})$ is even in $T$. Furthermore, $\mathcal{H}(T)$ is a real and even function of $T$, and $\Delta$ is mapped to itself by $T \mapsto -T$. Hence, using (65) and (92),

$$|\int_\Delta (\mathcal{G}(T) - \mathcal{H}(T)) \, dT| = |\int_\Delta (e^{\rho(T)} - 1) \mathcal{H}(T) \, dT| = \left| \int_\Delta \operatorname{Re}(e^{\rho(T)} - 1) \mathcal{H}(T) \, dT \right|$$

$$\leq \sum_{w \in A_K} \int_\mathbb{R}^k \left( \frac{|A_K|}{9} m_w^2 \alpha''_{\kappa_{w_0}}(S_w(\sigma))^3 S_w(T)^6 + \frac{Z}{2} m_w \alpha''_{\kappa_{w_0}}(S_w(\sigma))^2 S_w(T)^4 \right) \mathcal{H}(T) \, dT.$$  

Using Lemma 11 and Corollary 12 we find

$$\left| \int_\Delta (\mathcal{G}(T) - \mathcal{H}(T)) \, dT \right| \leq \left( \sum_{w \in A_K} \frac{5}{9} |A_K| + \frac{3}{2} Z \right) \left( \frac{2\pi)^{k/2} e^{\alpha(\sigma) - \eta_0 \sigma}}{\sqrt{\det(H(\sigma))}} \right)$$

$$\leq \frac{|A_K|}{m} \left( \frac{5}{9} |A_K| + \frac{3}{2} Z \right) I_1. \quad \square$$  

Our next estimate will let us deal with the term $\int_{E_a} \|ax\|^2 e^{-t \|ax\|^2} \, d\mu(x)$ in the Basic Inequality (14) and (40).
Lemma 20. For \( y \in \mathbb{R}^k \) and \( m := [L : K] \geq 1000 \) we have

\[
\int_{T \in \mathbb{R}^k} \left| T_1 e^{\alpha(y + iT)} \right| dT \leq \frac{1.66 \cdot 1.0021^{k-1} k \sqrt{|A_K|}}{\sqrt{m}} \sigma_1 I_1, \tag{93}
\]

with \( I_1 \) as in (55), \( \alpha \) as in (59) and \( \sigma = (\sigma_1, \ldots, \sigma_k) := \sigma(y) \) as in Lemma 6.

Proof. By (51), for \( T \in \mathbb{R}^k \) we have

\[
nT_1 = \sum_{v \in \mathcal{A}_L} e_v S_v(T) = \sum_{w \in \mathcal{A}_K} \sum_{v | w} e_v S_w(T) = \sum_{w \in \mathcal{A}_K} m_w S_w(T). \tag{94}
\]

Hence we will need to bound integrals of the kind \( \int_{\mathbb{R}^k} |S_w(T) e^{\alpha(y + iT)}| dT \).

Let \( \eta_0 \) be as in (82) and let \( w_0 \in \eta_0 \). Then, using (83) and changing variables as in the proof of Lemma 17,

\[
\int_{\mathbb{R}^k} |S_{w_0}(T) e^{\alpha(y + iT)} - \alpha(\sigma)| dT \leq \int_{\mathbb{R}^k} |S_{w_0}(T) \prod_{w \in \eta_0} \prod_{w \notin \eta_0} e^{m_w \alpha_w (S_w(y + iT) - m_w \alpha_w (S_w(\sigma)))} dT
\]

\[
= \frac{1}{\det(Q_{\eta_0})} \int_{-\infty}^{\infty} \prod_{w \in \eta_0} \prod_{w \notin \eta_0} e^{m_w \alpha_w (S_w(y + iT) - m_w \alpha_w (S_w(\sigma)))} dT
\]

Using Lemma 13 and (83) we obtain,

\[
\int_{\mathbb{R}^k} |S_{w_0}(T) e^{\alpha(y + iT)} - \alpha(\sigma)| dT \leq \frac{1}{\det(Q_{\eta_0})} \prod_{w \in \eta_0} \prod_{w \notin \eta_0} \frac{0.83 \sqrt{2\pi}}{\sqrt{m_w \alpha''_{w_0} (S_{w_0}(\sigma))}} \sqrt{m \alpha''_{w_0} (S_{w_0}(\sigma))} \sqrt{\det(H(\sigma))}
\]

By inequality (91),

\[
\sum_{w \in \mathcal{A}_K} m_w |S_w(T)| = \sum_{w \in \mathcal{A}_K} \sqrt{\alpha''_{w_0} (S_w(\sigma))} \sqrt{m_w \alpha''_{w_0} (S_w(\sigma))} |S_w(T)|
\]

\[
\leq \sum_{w \in \mathcal{A}_K} \sqrt{\alpha''_{w_0} (S_w(\sigma))} \sum_{w_0 \in \eta_0} \sqrt{m_w \alpha''_{w_0} (S_w(\sigma))} |S_{w_0}(T)|
\]

\[
\leq 2 \sum_{w \in \mathcal{A}_K} m_w \sum_{w_0 \in \eta_0} \sqrt{\alpha''_{w_0} (S_{w_0}(\sigma))} |S_{w_0}(T)|,
\]
where the last inequality uses \( m_{w_0} \leq 2m_w \) and \( x^2 \alpha''_{\kappa_w}(x) > \kappa_w \geq 1/2 \) for \( x > 0 \) [FS (5.7)]. Hence, by (94),

\[
\sum_{w \in A_K} m_w |S_w(T)| \leq 2n\sigma_1 \sum_{w_0 \in \eta_0} \sqrt{\alpha''_{\kappa_{w_0}}(S_{w_0}(\sigma))} |S_{w_0}(T)|.
\]

It follows that

\[
\int_{T \in \mathbb{R}^k} \left| T_1 e^{\alpha(\sigma + iT) - ny(\sigma + iT)} \right| dT = \frac{e^{-ny\sigma}}{n} \int_{\mathbb{R}^k} \left| \left( \sum_{w \in A_K} m_w S_w(T) \right) e^{\alpha(\sigma + iT)} \right| dT
\]

\[
\leq \frac{e^{\alpha(\sigma) - ny\sigma}}{n} \cdot 2n\sigma_1 \sum_{w_0 \in \eta_0} \sqrt{\alpha''_{\kappa_{w_0}}(S_{w_0}(\sigma))} \int_{\mathbb{R}^k} |S_{w_0}(T) e^{\alpha(\sigma + iT) - \alpha(\sigma)} | dT
\]

\[
\leq 2\sigma_1 \sum_{w_0 \in \eta_0} \frac{0.83 \cdot 1.0021^{k-1} \sqrt{|A^K|} (2\pi)^{k/2} e^{\alpha(\sigma) - ny\sigma}}{\sqrt{\det(H(\sigma))}} \sigma_1 I_1,
\]

where the last equality uses Corollary 12.

\[\square\]

6. Proof of the Main Theorem

The next lemma will allow us to ensure that each integral in the Basic Inequality (14) is positive. As in [15, we always assume that \( E(L/K) \subset E \subset O_L \).

**Lemma 21.** There is an absolute constant \( N_0 \) such that if \([L : K] \geq N_0 \cdot 2.01^{[K : \mathbb{Q}]}\) and \( a \in O_L, a \neq 0 \), then for \( t := \exp(\Psi(0.51 + \frac{\alpha}{2n})) \) we have \( \sigma_1(ny_{a,t}) \geq 0.51 \) and

\[
\int_{x \in E_K} \left( \frac{2t||ax||^2}{n} - 1 \right) e^{-t||ax||^2} d\mu_E(x) > 0.01I_1(ny_{a,t}) \mathcal{L},
\]

where \( y_{a,t} \) is given by Corollary 4, \( \Psi(x) := \Gamma'(x)/\Gamma(x) \), and

\[
\mathcal{L} = \frac{\sqrt{\det(Q^T Q)}}{2r_1(2\sqrt{\pi})^{r_2r_2}k}, \quad I_1(ny) = \frac{(2\pi)^{k/2} e^{\alpha(\sigma) - ny\sigma}}{\sqrt{\det(H(\sigma))}}, \quad \sigma := \sigma(ny_{a,t}).
\]

**Proof.** We note that \( \mathcal{L} \) is as in Corollary 4 except that we used (66) to express \( \mathcal{L} \) in terms of \( Q \) rather than \( Q \). Letting \( y := y_{a,t} \), from Corollary 4 we have

\[
\int_{E_K} \left( \frac{2t||ax||^2}{n} - 1 \right) e^{-t||ax||^2} d\mu_E(x) = \int_{T \in \mathbb{R}^k} \left( 2(\sigma_1 + iT_1) - 1 \right) e^{\alpha(\sigma + iT) - ny(\sigma + iT)} dT
\]

\[
= 2\sigma_1 - 1 + \frac{2i^{1-k}}{n-k} \int_{\mathbb{R}^k} T_1 e^{\alpha(\sigma + iT) - ny(\sigma + iT)} dT.
\]

Again from Corollary 4, for \( a \in O_L, a \neq 0 \),

\[
y_1 := (y_{a,t})_1 = \log(t) + \frac{2}{n} \log |\text{Norm}_{L/\mathbb{Q}}(a)| \geq \log(t) = \Psi(0.51 + \frac{\alpha}{2n}).
\]
Applying Lemma 7 to \( ny \), since \( \Psi^{-1} \) is increasing we have,

\[
\sigma_1 = \sigma_1(ny_a, t) \geq \Psi^{-1}(y_1) - \frac{r_2}{2n} \geq \Psi^{-1}(\Psi(0.51 + \frac{r_2}{2n})) - \frac{r_2}{2n} = 0.51.
\]

(97)

Since \( k \leq |A_K| \leq [K : \mathbb{Q}] \) by (56), we have \( |A_K| = (\frac{|A_K|}{k}) \leq 2 |A_K| \leq 2^{[K : \mathbb{Q}]} \). Thus, Lemma 20 yields

\[
2 \int_{T \in \mathbb{R}^k} |T_1 e^{\alpha(\sigma+iT)-ny(\sigma+iT)}| dT \leq \frac{3.32 \cdot 1.0021^{k-1} \sqrt{|A_K|}}{m} \sigma_1 I_1(ny)
\]

\[
\leq \frac{3.32 \cdot 1.0021^{[K : \mathbb{Q}]}}{m} [K : \mathbb{Q}] 2^{[K : \mathbb{Q}]/2} \sigma_1 I_1(ny) < 0.01 \sigma_1 I_1(ny)
\]

(98)

for \( m \geq N_0 \cdot 2^{1+[K : \mathbb{Q}]} \) and some absolute \( N_0 \geq 500 \). By (55) and (54) we have

\[
\frac{1}{t^k} \int_{\mathbb{R}^k} e^{\alpha(\sigma+iT)-ny(\sigma+iT)} dT = I_1 + I_2 - I_3 + I_4,
\]

where \( I_j = I_j(ny) \). Taking \( D = 1 \) in Lemmas 17 and 19 and after possibly enlarging \( N_0 \), we obtain \( |I_2| + |I_3| + |I_4| \leq 0.01 I_1 \). Hence,

\[
\frac{1}{t^k} \int_{\mathbb{R}^k} e^{\alpha(\sigma+iT)-ny(\sigma+iT)} dT \geq 0.99 I_1,
\]

(99)

and so, since \( \sigma_1 \geq 0.51 \) by (97),

\[
2\sigma_1 - 1 + \frac{2i^{-1-k}}{i^{-k}} \int_{\mathbb{R}^k} T_1 e^{\alpha(\sigma+iT)-ny(\sigma+iT)} dT \geq 2\sigma_1 - 1 - \frac{0.01 \sigma_1}{0.99} > 1.989 \sigma_1 - 1 > 0.014.
\]

A glance at (95) shows that we are finished.

We now prove the Main Theorem in §1, which we do not repeat here. Note that

\[
\| \varepsilon_1 \land \cdots \land \varepsilon_j \|_2 = \mu_E \frac{(\mathbb{R}/E)}{E_{\text{tor}}} \geq \frac{\mu_E (\mathbb{R}/E)}{E_{\text{tor}}}.
\]

(100)

Take \( N_0 \) and \( t := \exp(\Psi(0.51 + \frac{r_2}{2n})) \) as in Lemma 21. In the Basic Inequality (14) take \( a := \mathcal{O}_L \), so that the sum there includes only nonzero \( a \in \mathcal{O}_L \). By Lemma 21 each integral in the sum is positive. Retaining only the term corresponding to \( a = 1 \in \mathcal{O}_L \) we have, again by Lemma 21

\[
\frac{\mu_E (\mathbb{R}/E)}{E_{\text{tor}}} > 0.01 \frac{2^{k/2} \sqrt{\det(Q^T Q) \prod_{w \in A_K} (r_{1,w} + r_{2,w}) (2/ \sqrt{\pi})^{r_2} e^{\alpha(\sigma)-ny-\sigma}}}{\det(H(\sigma))^{k/2} 2^n}
\]

(101)

where \( y := y_{1,t} \) and \( \sigma := \sigma(ny) \). Corollary 4 applied to \( a = 1 \) gives

\[
y = (\log(t), 0, 0, \ldots, 0) = (\Psi(0.51 + \frac{r_2}{2n}), 0, \ldots, 0).
\]

(102)

We need an upper bound for \( \det(H(\sigma)) \) in (101). In view of (69), we look for an upper bound for \( \alpha''_w(S_w(\sigma)) \). Note that

\[
\alpha''_w(x) = \kappa \Psi'(x) + (1 - \kappa) \Psi'(x + \frac{1}{2}) \leq \Psi'(x) \quad (0 \leq \kappa \leq 1, \ x > 0),
\]
since $\Psi'(x)$ is decreasing for $x > 0$. Note that $\sigma_1 \geq 0.51$ by (97) and that
\[-2 < \Psi(0.51) \leq y_1 = \Psi(0.51 + \frac{r}{2n}) \leq \Psi(0.76) < -1. \tag{103}\]

From Lemma 9 we have
\[S_w(\sigma) \geq \frac{1}{(n - 1) \log(2\sigma_1 + \frac{1}{2}) - ny_1} \geq \frac{1}{n (\log(3\sigma_1) + 2)} > \frac{1}{n \log(23\sigma_1)}. \]

Estimating the series by an integral, $\Psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^2} < \frac{1}{x} + \frac{1}{\pi^2}$, yields
\[\alpha''_w(S_w(\sigma)) < \Psi(S_w(\sigma)) < \frac{1}{S_w(\sigma)} + \frac{1}{(S_w(\sigma))^2} < 2n^2 \log^2(23\sigma_1). \]

From $\det(Q^TQ) = \sum_{\eta \in A_K^{|\eta|}} \det^2(Q_{\eta})$ (Cauchy-Binet), $r_{1,w} + r_{2,w} \geq m_w/2$ and (63),
\[\frac{2^{k/2} \sqrt{\det(Q^TQ) \prod_{w \in A_K} (r_{1,w} + r_{2,w})}}{\sqrt{\det(H(\sigma))} \pi^{k/2}} \geq \left(\frac{1}{\sqrt{2\pi n \log(23\sigma_1)}}\right)^{|K:Q|}, \tag{104}\]

where we also used $k \leq |A_K| \leq [K:Q]$.

We now bound the term $e^{\alpha(\sigma) - ny \cdot \sigma}$ in (101) from below. From (102) and (103),
\[-ny \cdot \sigma = -n\sigma_1 y_1 > n\sigma_1. \]

Using the lower bound for $\alpha(\sigma)$ in Lemma 8 we have
\[\alpha(\sigma) - ny \cdot \sigma \geq n \log \Gamma(\sigma + \frac{r}{2n}) - n\sigma_1 y_1. \tag{105}\]

We now distinguish two cases according to the size of $\sigma_1$. If $\sigma_1 \geq 4$, then $\log \Gamma(\sigma_1 + \frac{r}{2n}) \geq \log(6)$. Since $-n\sigma_1 y_1 > n\sigma_1$, after possibly increasing $N_0$, the Main Theorem follows easily from (100), (101), (104) and (105).

We now turn to the remaining case, i.e., $0.51 \leq \sigma_1 < 4$. (By Lemma 21, $\sigma_1 \geq 0.51$.) Then in (104) we can replace $\log(23\sigma_1)$ by 5. The critical points $r \in (0, \infty)$ of $r \mapsto \log \Gamma(r + \frac{r}{2n}) - ry_1$ occur where
\[\Psi(r + \frac{r}{2n}) = y_1 := \Psi(0.51 + \frac{r}{2n}). \]

But $\Psi : (0, \infty) \to \mathbb{R}$ is injective, so $r = 0.51$ is the only critical point of $r \mapsto \log \Gamma(r + \frac{r}{2n}) - ry_1$, and it is a local (therefore global) minimum. Since $\sigma_1 \geq 0.51$,
\[\alpha(\sigma) - ny \cdot \sigma \geq n \left(\log \Gamma(0.51 + \frac{r}{2n}) - 0.51 y_1\right) = n \left(\log \Gamma(0.51 + \frac{r}{2n}) - 0.51 \Psi(0.51 + \frac{r}{2n})\right). \]

Note that $0 \leq \frac{r_{1,n}}{2n} \leq \frac{1}{4}$, $\Psi(r) < -1$ for $0 < r < 0.76$, and $\Psi'(r) > 0$ for $r > 0$. Hence
\[x \mapsto \log \Gamma(0.51 + x) - 0.51 \Psi(0.51 + x) + x \log(4/\pi) \]

is decreasing for $0 \leq x \leq \frac{1}{4}$. We conclude that
\[\alpha(\sigma) - ny \cdot \sigma + r_2 \log(2/\sqrt{\pi}) - n \log(2) \geq n \left(\log \Gamma(0.76) - 0.51 \Psi(0.76) + 0.25 \log(4/\pi) - \log(2)\right) > n/10. \]

Since $e^{0.0055} > 1.1$ and $j := \text{rank}_Z(E) \leq |A_L| \leq n$, after again possibly increasing $N_0$, we can use the “spare” $\exp(0.0045n)$ to control the term in (104). \[\square\]
We note that the our proof of the Main Theorem shows that the $1.1^j$ appearing in it can be replaced by $\exp(n f(r_2/(2n)))$, where $r_2$ is the number of complex places of $L$ and

$$f(x) := \log \Gamma(0.51 + x) - 0.51\Psi(0.51 + x) + x \log(4/\pi) - \log(2).$$

In particular, if $L$ is totally real, we can replace $1.1^j$ by $2.3^n$. We can also replace $0.51$ above by $\epsilon + 1/2$ for any $\epsilon > 0$.

Finally, we prove that every element of $\bigwedge^{r_L-1} \log(\mathcal{O}_L^n)$ is represented by a pure wedge, as claimed in the Introduction.

**Lemma 22.** Suppose $M$ is a $\mathbb{Z}$-lattice in $\mathbb{R}^n$ of rank $n \geq 1$. Then every element of $w \in \bigwedge^{n-1} M$ has the form

$$w = d \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_{n-1}$$

for some integer $d$ and some basis $\{\epsilon_1, \ldots, \epsilon_n\}$ of $M$ as a $\mathbb{Z}$-module.

**Proof.** We may clearly assume $\omega \neq 0$. Define the homomorphism $\wedge_\omega : M \to \bigwedge^n M$ by $\wedge_\omega(m) := \omega \wedge m$. As $\bigwedge^n M \cong \mathbb{Z}$, $M/\ker(\wedge_\omega)$ is torsion-free and so $\ker(\wedge_\omega)$ is a direct summand of $M$ of rank $n - 1$. Let $\epsilon_1, \ldots, \epsilon_n$ be a $\mathbb{Z}$-basis of $M$ such that $\epsilon_1, \ldots, \epsilon_{n-1}$ is a $\mathbb{Z}$-basis of $\ker(\wedge_\omega)$, let $\eta := \epsilon_1 \wedge \cdots \wedge \epsilon_{n-1} \in \bigwedge^{n-1} M$, and define $d \in \mathbb{Z}$ by $\omega \wedge \epsilon_n = d\eta \wedge \epsilon_n$. Notice that $\eta \wedge \epsilon_i = 0 = \omega \wedge \epsilon_i$ for $1 \leq i \leq n - 1$.

For $m \in M$, write $m = \sum_{i=1}^n a_i \epsilon_i$ with $a_i \in \mathbb{Z}$. Then

$$\omega \wedge m = \omega \wedge \sum_{i=1}^n a_i \epsilon_i = a_n \omega \wedge \epsilon_n = a_n d\eta \wedge \epsilon_n = d\eta \wedge \sum_{i=1}^n a_i \epsilon_i = d\eta \wedge m.$$ 

As the $\wedge$-pairing of $\bigwedge^{n-1} M$ with $M$ is non-degenerate, $\omega = d\eta = d\epsilon_1 \wedge \cdots \wedge \epsilon_{n-1}$. \qed

7. **Appendix by Fernando Rodriguez Villegas (May 2002)**

Some remarks on Lehmer’s conjecture

7.1. The *logarithmic Mahler measure* of a non-zero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is defined as

$$m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| \, d\theta_1 \cdots d\theta_n$$

and its *Mahler measure* as $M(P) = e^{m(P)}$, the geometric mean of $|P|$ on the torus

$$T^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| = \ldots = |z_n| = 1\}.$$ 

When $n = 1$ Jensen’s formula gives the identity

$$M(P) = |a_0| \prod_{|\alpha_\nu| > 1} |\alpha_\nu|,$$

(107)

where $P(x) = a_0 \prod_{\nu=1}^d (x - \alpha_\nu)$, from which we clearly obtain that $M(P) \geq 1$ if $P \in \mathbb{Z}[x]$. By a theorem of Kronecker if $M(P) = 1$ for $P \in \mathbb{Z}[x]$ then $P$ is *cyclotomic*, i.e., $P$ is monic and its roots are either 0 or roots of unity.

In the early 30’s Lehmer famously asked whether there is an absolute lower bound for $M(P)$ when $P \in \mathbb{Z}[x]$ and $M(P) > 1$. The purpose of this note is to point
out a simple reformulation of this question in terms of the logarithmic embedding of units of a number field and, given this setting, to propose a natural generalization.

7.2. We start with some general observations about $m(P)$. First of all, the fact that the integral in (106) is finite for all non-zero $P$ does need a proof. Here is a sketch. Using Jensen’s formula we find, as in (107) that

$$m(P) = m(a_0) + \frac{1}{(2\pi i)^n} \sum_{\nu=1}^{d} \int_{T^{n-1}} \log^+ |\alpha_\nu(y)| \frac{dy}{y},$$  \hspace{1cm} (108)$$

where $y = (y_1, \cdots, y_{n-1})$, $dy/y = dy_1/y_1 \cdots dy_{n-1}/y_{n-1}$, $\log^+(x) = \max\{\log|x|, 0\}$, and $a_0(y), \alpha_\nu(y), d$ are the leading coefficient, roots and degree, respectively, of $P$ viewed as a polynomial in $x_n$. The $\alpha_\nu$’s are algebraic functions of $y \in \mathbb{C}^{n-1}$, continuous and piecewise smooth, except at those $y$’s where $a_0(y)$ vanishes (where some will go off to infinity).

We can apply the above procedure to any variable $x_n$ on the torus $T^n$. It is not hard to see that we may change coordinates in such a way that $a_0(y)$ is actually constant, completing the proof by induction on $n$.

This last remark can be expanded. Let $\Delta$ be the Newton polytope of $P$; i.e., the convex hull of the exponents $m \in \mathbb{Z}^n$ of monomials $x^m = x_1^{m_1} \cdots x_n^{m_n}$ such that if $P = \sum_{m \in \mathbb{Z}^n} c_m x^m,$

then $c_m \neq 0$.

We define a face $\tau$ of $\Delta$ as the non-empty intersection of $\Delta$ with a half-space in $\mathbb{R}^n$. Chose a parameterization $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ of the affine subspace of smallest dimension containing $\tau$; $k$ is the dimension of the face $\tau$. Define

$$P_\tau = \sum_{m \in \mathbb{Z}^k} c_{\phi(m)} x^m,$$

a polynomial whose own Newton polytope is $\phi^{-1}(\tau)$. We call $P_\tau$ the face polynomial associated to the face $\tau$. It depends on a choice of $\phi$ but note that by changing variables in the integral $m(P_\tau)$ is actually independent of that choice.

It is not hard to see that for any facet (co-dimension 1 face) $\tau \subset \Delta$ we can choose $\phi$ and system of coordinates in $T^n$ so that, in the notation of (108), $P_\tau = a_0(y)$. By (108) and induction on $n$ we conclude [Sm1] that

$$m(P_\tau) \leq m(P), \quad \text{for all faces } \tau \subset \Delta.$$  \hspace{1cm} (109)$$

In particular,

$$m(P) \geq 0, \quad \text{for } 0 \neq P \in \mathbb{Z}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}].$$

Also, since clearly $m(PQ) = m(P) + m(Q)$, we have that

$$m(Q) \leq m(P), \quad \text{if } Q \mid P, \quad 0 \neq P, Q \in \mathbb{Z}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}].$$  \hspace{1cm} (110)$$

Though Lehmer’s conjecture is about polynomials in one variable, polynomials in more variables are also relevant due to the following result [Bo]. For any $0 \neq P \in \mathbb{Z}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$.
$\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ and $0 \neq (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we have

$$\lim_{k \to \infty} m(Q_k) = m(P) \quad \text{where} \quad Q_k(t) = P(t^a_1, \ldots, t^a_n) \quad (111)$$

That is, there are one variable polynomials $Q$ with $m(Q)$ as close to $m(P)$ as desired. (We should note that (111) is not an immediate consequence of general results about integration but requires a somewhat delicate analysis.)

7.3. Let us go back to polynomials in one variable. If we want to find polynomials $P \in \mathbb{Z}[x]$ with positive but small $m(P)$, by (109) and (110) (and Gauss’ lemma) we may as well restrict ourselves to minimal polynomials of algebraic units.

Let $F$ be a number field of degree $n$. Let $I$ be the set of embeddings $\sigma : F \longrightarrow \mathbb{C}$ and $V$ the real vector space of formal linear combinations

$$\sum_{\sigma \in I} \alpha_\sigma [\sigma], \quad \alpha_\sigma \in \mathbb{R}.$$

We have the decomposition

$$V = V^+ \oplus V^-,$$

where $V^\pm$ is the subspace of $V$ where complex conjugation acts like $\pm 1$. We let $n_\pm = \dim_{\mathbb{R}} V^\pm$ (in terms of the standard notation $n_+ = r_1 + r_2$ and $n_- = r_2$).

By Dirichlet’s theorem the image of the unit group $\mathcal{O}_F^\ast$ by the log map

$$l_1 : \mathcal{O}_F^\ast \longrightarrow \mathcal{V} \quad \epsilon \mapsto \sum_{\sigma \in I} \log |\epsilon^\sigma| [\sigma]$$

is a discrete subgroup $L_1 \subset V$ of rank $r = n^+ - 1$.

On $V$ we define the $L^1$-norm

$$\left\| \sum_{\sigma \in I} \alpha_\sigma [\sigma] \right\|_1 := \sum_{\sigma \in I} |\alpha_\sigma|$$

and we let

$$\mu_{1,1}(F) := \min_{l \in L_1 \setminus \{0\}} \|l\|_1$$

(the reason for this indexing will become clear shortly).

For any unit $\epsilon \in \mathcal{O}_F^\ast$ we have $|N_{F/Q}(\epsilon)| = 1$ hence

$$\sum_{\sigma \in I} \log |\epsilon^\sigma| = 0. \quad (112)$$

Let $P \in \mathbb{Z}[x]$ be the (monic) minimal polynomial of $\epsilon$ and

$$\|l_1(\epsilon)\|_1 = \frac{2n}{n_\epsilon} m(P).$$

This simple observation allows us to reformulate Lehmer’s conjecture as follows.

**Conjecture.** (Lehmer) There exists an absolute constant $\delta_1 > 0$ such that

$$\mu_{1,1}(F) \geq \delta_1, \quad \text{for all number fields } F \text{ with } r \geq 1. \quad (113)$$
7.4. Let $V$ be a vector space over $\mathbb{R}$ of dimension $n$ and $L \subset V$ a discrete subgroup of rank $r \geq 1$. A choice of basis $v_1, \ldots, v_n$ for $V$ determines $L^1$-norms on $\Lambda^k V$ for $k = 1, \ldots, n$ by
\[
\left\| \sum_{1 \leq j_1 < \cdots < j_k \leq n} a_{j_1, \ldots, j_k} v_{j_1} \wedge \cdots \wedge v_{j_k} \right\|_1 := \sum_{1 \leq j_1 < \cdots < j_k \leq n} |a_{j_1, \ldots, j_k}|.
\]
For each $1 \leq k \leq r$ we define (with respect to the chosen basis)
\[
\mu_k(L) := \min \|l_1 \wedge \cdots \wedge l_k\|_1,
\]
where the minimum is taken over all $l_1, \ldots, l_k \in L$ which are linearly independent over $\mathbb{R}$.

If $A$ is the $n \times k$ integral matrix whose $i$-th column consists of the coordinates of $l_i$ in the basis $v_1, \ldots, v_n$ then, as it is easily seen,
\[
\|l_1 \wedge \cdots \wedge l_k\|_1 = \sum_{A'} |\det A'|,
\]
where $A'$ runs over all $k \times k$ minors of $A$.

Returning to the number field situation of the previous section we define the invariants
\[
\mu_{1,k}(F) := \mu_k(L_1),
\]
where, as before, $L_1$ is the image of the units of $F$ under the log map.

A general version of Lehmer’s conjecture would then be

**Conjecture.** For each $k \in \mathbb{N}$ there exists an absolute constant $\delta_k > 0$ such that
\[
\mu_{1,k}(F) \geq \delta_k, \quad \text{for all number fields } F \text{ with } r \geq k.
\]

A straightforward calculation shows that the top invariant $\mu_{1,r}(F)$, with $r = n^+ - 1$ the rank of the unit group $\mathcal{O}_F^*$, equals the regulator of $F$. It is known [Zi], [Fr], [Sk] that the regulator of number fields is universally bounded below and hence the above conjecture is true for $k = r$.

In summary, we have seen (18) that Lehmer’s conjecture can be phrased in terms of the $L^1$-norm of units under the log map. The above conjecture is an attempt to quantify, in what seems to be the most natural way, the question of what is the general shape of $L_1$, the discrete group of units under the log map.

7.5. We may carry these ideas a little further still. Borel proved, generalizing Dirichlet’s result for units, that for each $j > 1$ there is a regulator map $\text{reg}_j$
\[
\xi \mapsto \sum_{\sigma \in I} \text{reg}_j(\xi^\sigma)[\sigma]
\]
whose image is a discrete subgroup $L_j$ of $V^\pm$, with $\pm = (-1)^{j-1}$, of rank $n^\pm$ and covolume related to the value of the zeta function $\zeta_F$ of $F$ at $s = j$. Here $K_{2j-1}(F)$ are the $K$ groups defined by Quillen.

We now define
\[
\mu_{j,k}(F) := \mu_k(L_j), \quad \text{for } 1 \leq k \leq n^\pm,
\]
and we may ask: what is the nature of these invariants, how do they depend on the field $F$? Does the analogue of Lehmer’s conjecture hold?

Apart from their formal analogy with Lehmer’s question, answers to such questions can be quite useful in practice as we now illustrate.

7.6. For general $j$, very little is known about the groups $K^{2j-1}(F)$ or the map $\text{reg}_j$. For $j = 2$, however, things can be made quite explicit (and of course $j = 1$ corresponds to the case of units). Indeed, up to torsion, $K_3(F)$ is isomorphic to the Bloch group $B(F)$, defined by generators and relations as follows.

For any field $F$ define

$$ A(F) := \left\{ \sum n_i[z_i] \in \mathbb{Z}[F] \mid \sum n_i(z_i \wedge (1 - z_i)) = 0 \right\}, $$

where the corresponding term in the sum is omitted if $z_i = 0, 1$ and

$$ C(F) := \left\{ [x] + [y] + \left[ \frac{1 - x}{1 - xy} \right] + [1 - xy] + \left[ \frac{1 - y}{1 - xy} \right] \mid x, y \in F, \ xy \neq 1 \right\}. $$

It is not hard to check that $C(F) \subset A(F)$. Finally, let

$$ B(F) := A(F)/C(F). $$

We recall the definition of the Bloch–Wigner dilogarithm. Starting with the usual dilogarithm

$$ \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| < 1 $$

one defines

$$ D(z) = \text{Im}(\text{Li}_2(z)) + \arg(1 - z) \log |z| $$

and checks that it extends to a real analytic function on $\mathbb{C} \setminus \{0, 1\}$, continuous on $\mathbb{C}$. See [Za] for an account of its many wonderful properties. It is obvious that

$$ D(\bar{z}) = -D(z). \quad (115) $$

The 5-term relation satisfied by $D$ guarantees that, extended by linearity to $A(F)$, it induces a well defined function on $B(\mathbb{C})$ (still denoted by $D$).

For $j = 2$ (114) can be formulated as follows

$$ l_2 : \ B(F) \longrightarrow \ V \quad \xi \mapsto \sum_{\sigma \in I} D(\xi^\sigma)[\sigma] $$

(115) makes it clear that the image $L_2$ lies in $V^-$) whose image $L_2$ is a discrete subgroup of rank $n^-$.

An a priori lower bound for $||l_2(\xi)||_1$ even for the simplest case where $L_2$ is of rank 1 (namely, for a field with only one complex embedding) would be quite useful. For example, in [BRV1] we find that an identity between the Mahler measure of certain two-variable polynomials is equivalent to the following

$$ D(7[\alpha] + [\alpha^2] - 3[\alpha^3] + [-\alpha^4]) = 0, \quad \alpha = (-3 + \sqrt{-7})/4. \quad (116) $$

This was proved by Zagier by showing that it is a consequence of series of 5-term relations. Such calculations, however, can be quite hard and at present there is
no known algorithm that is guaranteed to produce the desired result. Clearly if we knew a reasonable lower bound for the possible non-zero values of $|D(\xi)|$ for $\xi \in \mathcal{B}(\mathbb{Q}(\sqrt{-7}))$ a simple numerical verification would be enough to prove (116).

Similarly, many identities \cite{BRV2} between the Mahler measure of certain two-variable polynomials and $\zeta_F(2)$ for a corresponding number field $F$, which by Borel’s theorem are known up to an unspecified rational number, could be proved by a numerical check. For example, we can show that

$$m(x^2 - 2xy - 2x + 1 - y + y^2) = s \frac{1728^{3/2}}{2^6 \pi^7} \zeta_F(2),$$

with $s \in \mathbb{Q}^*$, where $F$ is the splitting field $x^4 - 2x^3 - 2x + 1$, of discriminant $-1728$. However, though numerically $s$ appears to be equal to 1 we cannot prove this at the moment. Again, a reasonable lower bound on $|D(\xi)|$ for non-torsion elements $\xi \in \mathcal{B}(F)$ would allow us to conclude that $s = 1$ by checking it numerically to high enough precision.

There is also some evidence that $\mu_{2,1}(F)$ might be universally bounded below, at least for fields with one complex embedding. Indeed, for such a field one can construct a hyperbolic three dimensional manifold $M$ by taking the quotient of hyperbolic space by a torsion-free subgroup of the group of units of norm 1 in a quaternion algebra over $F$ ramified at all its real places. Its associated Bloch group element $\xi(M)$, obtained from a triangulation of $M$ into ideal tetrahedra, satisfies $D(\xi(M)) = \text{vol}(M)$. On the other hand, the volume of hyperbolic 3-manifolds is known to be universally bounded below. The question becomes then, that of obtaining an upper bound for the index in $\mathcal{B}(F)$ of the subgroup generated by all such $\xi(M)$. This index is likely to be rather small; in fact, if we accept a precise form of Lichtenbaum’s conjecture, it should be essentially the order of $K_2(O_F)$, an analogue of a class group. Unfortunately, there is no known upper bound for $|K_2(O_F)|$ in terms of, say, the degree and discriminant of $F$.

Finally, to a hyperbolic 3-manifold $M$ with one cusp one may associate \cite{CCGLS} a two variable polynomial $A(x,y) \in \mathbb{Z}[x,y]$, called the $A$-polynomial of $M$. Its zero locus parameterizes deformations of the complete hyperbolic structure of $M$.

It is known that

$$m(A_r) = 0$$

for every face polynomial of $A$ and that $A$ is reciprocal, i.e. $A(1/x, 1/y) = x^a y^b A(x, y)$ for some $a, b \in \mathbb{Z}$. It is interesting that these two properties, which have a topological and $K$-theoretic origin, are, for $A$ irreducible, precisely the known necessary conditions for a polynomial in $\mathbb{Z}[x, y]$ to have to have small Mahler measure (the first, an analogue of being the minimal polynomial of an algebraic unit, because of (109); the second because $m(P)$ is known to be universally bounded below for $P$ non-reciprocal \cite{Sm1}).

Though the whole picture is still not completely clear yet one can prove \cite{BRV2} for many $M$’s identities of the form

$$2\pi m(A) = \|D(\xi(M))\|_1,$$
where $\xi(M)$ is the Bloch group element associated to $M$. This suggests a direct link between Lehmer’s conjecture and the size of the invariants $\mu_{2,1}$.

REFERENCES

[AAR] G. Andrews, R. Askey and R. Roy, Special functions, Cambridge U. Press, Cambridge (1999).
[BS] Z.I. Borevich and I.R. Shafarevich, Number Theory. Academic Press, New York (1966).
[Bo] D.W. Boyd, Speculations concerning the range of Mahler’s measure, Canad. Math. Bull. 24 (1981) 453–469.
[BRV1] D.W. Boyd and F. Rodriguez Villegas, Mahler’s measure and the dilogarithm I, Canad. J. Math. 54 (2002) 468–492.
[BRV2] D.W. Boyd and F. Rodriguez Villegas, Mahler’s measure and the dilogarithm II,
[Ca] J. Cassels, An introduction to the geometry of numbers. Springer, Berlin (1959).
[CCGLS] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen, Plane curves associated to character varieties of 3-manifolds, Invent. Math. 118 (1994) 47–84.
[CF] A. Costa and E. Friedman, Ratios of regulators in totally real extensions of number fields, J. Number Th. 37 (1991) 288–297.
[Fr] E. Friedman, Analytic formulas for the regulator of a number field, Invent. Math. 98 (1989) 599–622.
[FS] E. Friedman and N.-P. Skoruppa, Relative regulators of number fields, Invent. Math. 135 (1999) 115–144.
[HUL] J.-B. Hiriart-Urruty and C. Lemaréchal, Fundamentals of convex analysis, Springer, Berlin (2001).
[Le] D.H. Lehmer, Factorization of certain cyclotomic functions, Ann. Math. (2) 34 (1933) 461–479.
[Ni] N. Nielsen, Die Gammafunktion. Chelsea, New York (1965) (reprint of 1906 edition).
[Po] M. Polst, Eine Regulatorabschätzung, Abh. Math. Sem. Univ. Hamburg 47 (1978) 95–106.
[Sie] C. L. Siegel, Abschätzung von Einheiten, Nachr. Akad. Wiss. Göttingen (1969) 71–86.
[Sim] B. Simon, Convexity: an analytic viewpoint. Cambridge U. Press, Cambridge (2011).
[Sk] N.-P. Skoruppa, Quick lower bounds for regulators of number fields, Enseign. Math. Math. 39 (1993) 137–141.
[Sm1] C.J. Smyth, On measures of polynomials in several variables, Bull. Austral. Math. Soc. 23 (1981) 49–63.
[Sm2] C.J. Smyth, Mahler measure of one-variable polynomials: a survey, in: Conference proceedings, University of Bristol, 3–7 April 2006, LMS Lecture Note Series 352, Cambridge U. Press, Cambridge (2008) 322-349.
[Su1] J. Sundstrom, Lower bounds for generalized regulators, Thesis (Ph. D.)University of Pennsylvania (2016). 84 pp. ISBN: 978-1339-82732-2. Available at ProQuest LLC.
[Su2] J. Sundstrom, Lower bounds for generalized unit regulators, J. Théor. Nombres Bordeaux 30 (2018) 95–106.
[Za] D. Zagier, The Bloch-Wigner-Ramakrishnan polylogarithm function, Math. Ann. 286 (1990) 613–624.
[Zi] R. Zimmert, Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung, Invent. Math. 62 (1981) 131–173.
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Las Palmeras 3425, Núñoa, Santiago R.M., Chile
E-mail address: friedman@uchile.cl

The Abdus Salam International Centre for Theoretical Physics, ICTP Math Section, Strada Costiera 11, I-34151 Trieste, Italy
E-mail address: villegas@ictp.it

Department of Mathematics (038-16), Temple University, Wachman Hall, 1805 North Broad Street, Philadelphia PA 19122, USA
E-mail address: james.sundstrom@temple.edu