CONTRACTIVE MULTIPLIERS FROM HARDY SPACE TO WEIGHTED HARDY SPACE

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Abstract. It is shown how any contractive multiplier from the Hardy space to a weighted Hardy space \( H^2_\beta \) can be factored as a fixed factor composed with the classical Schur multiplier (contractive multiplier between Hardy spaces). The result is applied to get results on interpolation for a Hardy-to-weighted-Hardy contractive multiplier class.

1. Introduction

Given a sequence \( \beta = \{\beta_j\}_{j \geq 0} \) of positive numbers such that \( \beta_0 = 1 \) and \( \liminf \beta_j \geq 1 \), the weighted Hardy space \( H^2_\beta \) is defined as the set of all functions analytic on the open unit disk \( \mathbb{D} \) and with finite norm \( \|f\|_{H^2_\beta} \) given by

\[
\|f\|_{H^2_\beta}^2 = \sum_{j=0}^{\infty} \beta_j |f_j|^2 \quad \text{if} \quad f(z) = \sum_{j=0}^{\infty} f_j z^j.
\]

Polynomials are dense in \( H^2_\beta \) and the monomials \( \{z^k\}_{k \geq 0} \) form an orthogonal set uniquely defining the weight sequence \( \beta \) by \( \beta_j = \|z^j\|^2 \) for \( j \geq 0 \). The space \( H^2_\beta \) can be alternatively characterized as the reproducing kernel Hilbert space with reproducing kernel

\[
k_\beta(z, \overline{\zeta}) = \sum_{j=0}^{\infty} \frac{z^j \overline{\zeta}^j}{\beta_j}.
\]

For a Hilbert (coefficient) space \( \mathcal{Y} \), we denote by \( H^2_\beta(\mathcal{Y}) \) the reproducing kernel Hilbert space with reproducing kernel \( k_\beta(z, \overline{\zeta})I_\mathcal{Y} \) which can be characterized explicitly as follows:

\[
H^2_\beta(\mathcal{Y}) = \left\{ f(z) = \sum_{k \geq 0} f_k z^k : \|f\|_{H^2_\beta(\mathcal{Y})}^2 := \sum_{k \geq 0} \beta_k \cdot \|f_k\|_{\mathcal{Y}}^2 < \infty \right\}.
\]

We will write \( 1 \) rather than \( \beta \) if \( \beta_j = 1 \) for all \( j \geq 0 \). Observe that \( H^2_1(\mathcal{Y}) \) is the classical vector Hardy space \( H^2(\mathcal{Y}) \) of the unit disk (with reproducing kernel \( k_1(z, \overline{\zeta}) = (1 - z\overline{\zeta})^{-1} \cdot I_\mathcal{Y} \)); the choice \( \beta_j = \frac{j!(n-1)!}{(j+n-1)!} \) yield the standard weighted Bergman space \( A^2_n(\mathcal{Y}) \) \((n \geq 1)\) and in particular, the classical Bergman space \( A^2_2(\mathcal{Y}) \) in case \( n = 2 \). A general reference for such spaces and the associated weighted shift operators is the article of Shields [11].

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For $\mathcal{U}$ and $\mathcal{V}$ any pair of Hilbert spaces, we use the notation $\mathcal{L}(\mathcal{U}, \mathcal{V})$ to denote the space of bounded, linear operators from $\mathcal{U}$ to $\mathcal{V}$, shortening $\mathcal{L}(\mathcal{X}, \mathcal{X})$ to $\mathcal{L}(\mathcal{X})$.

**Definition 1.1.** An $\mathcal{L}(\mathcal{U}, \mathcal{V})$-valued function $S$ defined on $\mathbb{D}$ is called a **contractive multiplier** from $H^2_{\alpha}(\mathcal{U})$ to $H^2_{\beta}(\mathcal{V})$, denoted as $S \in \mathcal{S}_{\alpha \to \beta}(\mathcal{U}, \mathcal{V})$, if the multiplication operator $M_S: f \to Sf$ is a contraction from $H^2_{\alpha}(\mathcal{U})$ to $H^2_{\beta}(\mathcal{V})$.

The latter means that the operator $I - M_SM_S^*$ (with $M_S$ considered as acting from the Hardy space $H^2_{\alpha}(\mathcal{U})$ into $H^2_{\beta}(\mathcal{V})$) is positive semidefinite; this in turn is equivalent to the kernel

$$K_S(z, \zeta) := k_\beta(z, \zeta)I_\mathcal{V} - S(z)S(\zeta)^* \cdot k_\alpha(z, \zeta)$$

being positive on $\mathbb{D} \times \mathbb{D}$, denoted symbolically as $K_S(z, \zeta) \succeq 0$. In case $\beta = \alpha = 1$, the set of contractive multipliers from $H^2(\mathcal{U})$ to $H^2(\mathcal{V})$ is the classical Schur class $\mathcal{S}(\mathcal{U}, \mathcal{V})$ of functions analytic on $\mathbb{D}$ whose values are contractive operators from $\mathcal{U}$ to $\mathcal{V}$. Our main focus here will be on the intermediate case where $\alpha = 1$, and we will write $\mathcal{S}_\beta(\mathcal{U}, \mathcal{V})$ (rather than $\mathcal{S}_{1 \to \beta}(\mathcal{U}, \mathcal{V})$) for the set of all contractive multipliers from $H^2(\mathcal{U})$ to $H^2_{\beta}(\mathcal{V})$.

In Section 2 below, for the case where the sequence $\beta$ is non-increasing, we show that elements of $\mathcal{S}_\beta(\mathcal{U}, \mathcal{V})$ can be factored as $S(z) = \Psi_\beta(z)S(z)$ where $\Psi_\beta$ is a fixed factor, and where $S$ is in a classical Schur class (see Theorem 2.1 below). This enables us to obtain a complete solution of a general (left-tangential with operator argument) interpolation problem in the class $\mathcal{S}_\beta(\mathcal{U}, \mathcal{V})$ by reducing the problem to a well understood interpolation problem for the function $S$ in the classical Schur class (see Theorems 3.6 and 3.8 below). In Section 3 we give an example to illustrate how the results do not generalize to the more general Schur class $\mathcal{S}_{\alpha \to \beta}$ in case $\alpha \neq 1$.

There has been much interest of late in so-called Bergman inner functions, i.e., functions which map the coefficient space $\mathcal{U}$ isometrically onto a shift-invariant subspace $\mathcal{L}$ for the shift operator $S_\beta: f(z) \mapsto zf(z)$ on $H^2_{\beta}(\mathcal{U})$ (see [4, 7, 8]), especially for the case where $\beta_j = \frac{\beta_{j+1}-1}{\beta_j+1}$ yields one of the standard weighted Bergman spaces. It is known that Bergman inner functions are contractive multipliers from the Hardy space $H^2$ to the Bergman space $H^2_{\beta}$. Hence there is a potential for the results of this paper to apply to Bergman inner functions as well.

2. The Hardy-to-weighted Hardy contractive multiplier class

Let us assume now that the weighted sequence $\beta = \{\beta_j\}_{j \geq 0}$ is non-increasing and introduce the positive sequence $\gamma = \{\gamma_j\}_{j \geq 0}$ by

$$\gamma_0 = 1, \quad \gamma_j = (\beta_j\beta_{j-1}-\beta_{j-1})^{-1} = \frac{\beta_j\beta_{j-1}}{\beta_j-\beta_j} \quad (j \geq 1).$$

(2.1)

Since the sequence $\gamma$ is positive, the kernel

$$k_\beta(z, \zeta) := (1 - z\bar{\zeta}) \cdot k_\beta(z, \zeta) = \sum_{j=0}^\infty \gamma_j^{-1}z^j\bar{\zeta}^j = k_\gamma(z, \zeta)$$

(2.2)
It is readily seen from (2.2) that
\[ \Psi_\beta(z) = \{y_j\}_{j \geq 0} \mapsto y_0 + \sum_{j=1}^{\infty} \sqrt{\beta_j^{-1} - \beta_{j-1}^{-1}} \cdot y_j z^j = \sum_{j=0}^{\infty} \frac{y_j}{\sqrt{\gamma_j}} z^j. \] (2.3)

The multiplication operator \( M_{\Psi_\beta} \) is an isometry from \( \ell_2(Y) \) onto the weighted Hardy space \( H^2_\beta(Y) \) (in fact, \( \Psi_\beta \) is a weighted \( \mathcal{Z} \)-transform). It is convenient to represent \( \Psi_\beta(z) \) and the elements \( y = \{y_j\}_{j \geq 0} \) in \( \ell_2(Y) \) in the matrix form

\[ \Psi_\beta(z) = \begin{bmatrix} I_Y & \frac{z}{\sqrt{\gamma_1}} & \frac{z^2}{\sqrt{\gamma_2}} & I_Y & \cdots \end{bmatrix}, \quad y = \begin{bmatrix} y_0 & y_1 & \cdots \end{bmatrix} \in \ell_2(Y). \]

It is readily seen from (2.2) that
\[ \Psi_\beta(z)\Psi_\beta(\zeta)^* = k_\beta(z, \zeta) \cdot I_Y = (1 - z\zeta) \cdot k_\beta(z, \zeta) \cdot I_Y. \] (2.4)

**Theorem 2.1.** Let the weight sequence \( \beta \) be non-increasing. The function \( S \) is in the class \( \mathcal{S}_\beta(\mathcal{U}, Y) \) if and only if there is an \( S \) in the Schur class \( \mathcal{S}(\mathcal{U}, \ell_2(Y)) \) so that
\[ S(z) = \Psi_\beta(z)S(z). \] (2.5)

**Proof.** Suppose first that \( S \) has the form (2.5) with \( S \) in \( \mathcal{S}(\mathcal{U}, \ell_2(Y)) \). Then we compute, making use of (2.4), that
\[ k_\beta(z, \zeta) \cdot I_Y - \frac{S(z)S(\zeta)^*}{1 - z\zeta} = k_\beta(z, \zeta) \cdot I_Y - \frac{\Psi_\beta(z)S(z)S(\zeta)^*\Psi_\beta(\zeta)^*}{(1 - z\zeta)} \]
\[ = \Psi_\beta(z) \left[ I - S(z)S(\zeta)^* \right] \frac{1}{1 - z\zeta} \Psi_\beta(\zeta)^* \geq 0, \]
and it follows that \( S \in \mathcal{S}_\beta(\mathcal{U}, Y) \) by the criterion that the kernel in (1.3) be positive.

Conversely, suppose that \( S \) is in the class \( \mathcal{S}_\beta(\mathcal{U}, Y) \). It then follows that the kernel (1.3) is positive. From (2.4) we see that
\[ K_S(z, \zeta) = \frac{\Psi_\beta(z)\Psi_\beta(\zeta)^* - S(z)S(\zeta)^*}{1 - z\zeta}, \]
and hence the right-hand side is a positive kernel. It then follows from the theorem of Leech [9, p. 107] that there is an \( S \) in the Schur class \( \mathcal{S}(\mathcal{U}, \ell_2(Y)) \) so that \( S = \Psi_\beta S \), i.e., (2.5) holds. \( \square \)

The representation formula (2.5) makes it possible to reduce certain questions concerning the generalized Schur class \( \mathcal{S}_{1-\beta} \) to well-understood questions concerning the classical Schur class \( \mathcal{S} \). In the next section we demonstrate how this principle can be applied in the context of interpolation.

## 3. Multiplier interpolation problems

In this section we study a Nevanlinna-Pick type interpolation problem in the class \( \mathcal{S}_\beta(\mathcal{U}, Y) \). To formulate the problem we need several definitions.

A pair \((E, T)\) consisting of operators \( T \in \mathcal{L}(\mathcal{X}) \) and \( E \in \mathcal{L}(\mathcal{X}, Y) \) is called an output pair. An output pair \((E, T)\) is called \( \beta \)-output-stable if the associated
\( \beta \)-observability operator

\[
O_{\beta,E,T} : x \mapsto E \sum_{j=0}^{\infty} \left( \beta_j^{-1} ET^j x \right) z^j
\]

maps \( \mathcal{X} \) into \( H_\beta^2(\mathcal{Y}) \) and is bounded. If \((E,T)\) is \( \beta \)-output stable, then the \( \beta \)-observability gramian

\[
G_{\beta,E,T} := (O_{\beta,E,T})^* O_{\beta,E,T}
\]

is bounded on \( \mathcal{X} \) and can be represented via the series

\[
G_{\beta,E,T} = \sum_{k=0}^{\infty} \beta_j^{-1} \cdot T^k E^* ET^k
\]

converging in the strong operator topology. Observe that in case \( \beta = 1 \), the observability operator (3.1) amounts to the well-known observability operator

\[
O_{1,E,T} : x \mapsto E(I - zT)^{-1} x
\]

and the 1-output stability means that \( O_{1,E,T} \) is bounded as an operator from \( \mathcal{X} \) to \( H^2(\mathcal{Y}) \).

For a \( \beta \)-output stable pair \((E,T)\), we define the tangential functional calculus \( f \mapsto (E^* f)^\wedge L(T^*) \) on \( H_\beta^2(\mathcal{Y}) \) by

\[
(E^* f)^\wedge L(T^*) = \sum_{j=0}^{\infty} T^{*j} E^* f_j \quad \text{if} \quad f(z) = \sum_{j=0}^{\infty} f_j z^j \in H_\beta^2(\mathcal{Y}).
\]

The computation

\[
\left\langle \sum_{j=0}^{\infty} T^{*j} E^* f_j, x \right\rangle \mathcal{X} = \sum_{j=0}^{\infty} \left\langle f_j, ET^j x \right\rangle \mathcal{Y} = \sum_{j=0}^{\infty} \beta_j \cdot \left( f_j, \beta_j^{-1} ET^j x \right) \mathcal{Y} = \left\langle f, O_{\beta,E,T} x \right\rangle_{H_\beta^2(\mathcal{Y})}
\]

shows that the \( \beta \)-output stability of \((E,T)\) is exactly what is needed to verify that the infinite series in the definition of \( (E^* f)^\wedge L(T^*) \) converges in the weak topology on \( \mathcal{X} \). The same computation shows that tangential evaluation with operator argument amounts to the adjoint of \( O_{\beta,E,T} \):

\[
(E^* f)^\wedge L(T^*) = O_{\beta,E,T}^* f \quad \text{for} \quad f \in H_\beta^2(\mathcal{Y}).
\]

The evaluation map extends to multipliers \( S \in S_\beta(\mathcal{U}, \mathcal{Y}) \) by

\[
(E^* S)^\wedge L(T^*) = O_{\beta,E,T}^* M_S |_{\mathcal{U}}.
\]

The objective of this section is to study the interpolation problem \( \text{IP} \) whose data set consists of three operators

\[
T \in \mathcal{L}(\mathcal{X}), \quad E \in \mathcal{L}(\mathcal{X}, \mathcal{Y}), \quad N \in \mathcal{L}(\mathcal{X}, \mathcal{U})
\]

such that the pair \((E,T)\) is \( \beta \)-output stable and the pair \((N,T)\) is 1-output stable.

\( \text{IP} \): Given operators \( (3.6) \), find all contractive multipliers \( S \in S_\beta(\mathcal{U}, \mathcal{Y}) \) such that

\[
(E^* S)^\wedge L(T^*) := O_{\beta,E,T}^* M_S |_{\mathcal{U}} = N^*.
\]
Example 3.1. By way of motivation of problem IP, we note that if we take the data set \((T, E, N)\) of the form

\[
T = \begin{bmatrix}
z_1 I_Y & 0 \\
. & . \\
0 & z_k I_Y
\end{bmatrix}, \quad E = [I_Y \ldots I_Y], \quad N = [V_1^* \ldots V_k^*]
\]

for some \(z_1, \ldots, z_k \in \mathbb{D}\) and \(V_1, \ldots, V_k \in \mathcal{L}(U, Y)\), then it follows from (3.3) that

\[
(E^* S)^\wedge T^* (T^*) = \begin{bmatrix}
S(z_1) \\
. \\
S(z_k)
\end{bmatrix},
\]

so that condition (3.7) transcribes to Nevanlinna-Pick interpolation conditions

\[
S(z_i) = V_i \quad \text{for} \quad i = 1, \ldots, k.
\] (3.8)

The stability assumption for the pair \((E, T)\) is needed to define the expression on the left side of (3.7). We next show that the stability assumption for the pair \((N, T)\) is not restrictive.

Proposition 3.2. Let us assume that the pair \((E, T)\) is \(\beta\)-output stable and that there is a function \(S \in S^*_{\beta}(U, Y)\) satisfying condition (3.7). Then the pair \((N, T)\) is 1-output stable and the following equality holds:

\[
O_{\beta, E, T}^* M_S = O_{1, N, T}^* : H^2(U) \to X.
\] (3.9)

Furthermore, the observability gramian

\[
G_{1, N, T} := O_{1, N, T}^* O_{1, N, T} = \sum_{j=0}^{\infty} T^* j N^* N T^j
\] (3.10)

satisfies the Stein identity

\[
G_{1, N, T} - T^* G_{1, N, T} T = N^* N.
\] (3.11)

Proof. Let \(S\) be a function in \(S_{\beta}(U, Y)\) subject to (3.7). Then for a function \(h(z) = \sum_{j=0}^{\infty} h_j z^j \in H^2(U)\), we have \((M_S h)(z) = \sum_{j=0}^{k} \left( \sum_{j=0}^{k} S_j h_{\ell-j} \right) z^\ell\). Hence, as a consequence of (3.3) we have

\[
O_{\beta, E, T}^* M_S h = (E^* (Sh))^\wedge T^* (T^*) = \sum_{\ell=0}^{\infty} T^* \ell E^* \left( \sum_{j=0}^{\ell} S_j h_{\ell-j} \right),
\]

where the latter series converges weakly since the pair \((E, T)\) is \(\beta\)-output stable and since \(Sh \in H^2_{\beta}(Y)\). If we regularize the series by replacing \(S_j\) by \(r^j S_j\) and replacing \(h_i\) by \(r^i h_i\), we even get that the double series in (3.12), after taking the inner product against a fixed vector \(x \in X\), converges absolutely. We may then rearrange the series to have the form

\[
(O_{\beta, E, T}^* M_S, h, x) = \sum_{j,k=0}^{\infty} \langle r^{j+k}(T^*)^{j+k} E^* S_j h_k, x \rangle.
\]
We may then invoke Abel’s theorem to take the limit as $r \uparrow 1$ (justified by the facts that $(E, T)$ is $\beta$-output stable and that $Sh \in H_2^\beta(Y)$—see [10, page 175]) to get

$$O_{\beta,E,T}^* M_S h = (E^* Sh)^{\wedge L}(T^*) = \sum_{j,k=0}^{\infty} (T^*)^{j+k} E^* S_j h_k. \quad (3.12)$$

On the other hand, due to (3.7),

$$O_{1,N,T}^* h = (N^* h)^{\wedge L}(T^*) = \sum_{k=0}^{\infty} T^{*k} N^* h_k = \sum_{k=0}^{\infty} T^{*k} O_{\beta,E,T}^* Sh_k$$

$$= \sum_{k=0}^{\infty} T^{*k} \left( \sum_{j=0}^{\infty} T^{*j} E^* S_j \right) h_k = \sum_{j,k=0}^{\infty} (T^*)^{j+k} E^* S_j h_k$$

where all the series converge weakly, since that in (3.12) does. Since $h$ was picked arbitrarily in $H_2^2(U)$, the last equality and (3.12) imply (3.9). Therefore, the operator $O_{1,N,T}^* : H_2^2(U) \to X$ is bounded and hence the pair $(N, T)$ is $1$-output stable. Therefore, the series in (3.10) converges strongly and (as is well known) satisfies the Stein identity (3.11).

We shall have need of the auxiliary observation operator described in the following lemma.

**Lemma 3.3.** Let us assume that the weight sequence $\beta$ is non-increasing and that the pair $(E, T)$ is $\beta$-output stable. Then the operator

$$\tilde{O}_{\beta,E,T} : x \mapsto \left\{ \frac{1}{\sqrt{\gamma_j}} ET^j x \right\}_{j \geq 0} \quad (3.13)$$

(where $\gamma_j$’s are defined from $\beta$ as in (2.1)) maps $X$ into $\ell^2(Y)$. Furthermore, the pair $(\tilde{O}_{\beta,E,T}, T)$ is $1$-output stable and the following relations hold:

$$\tilde{O}_{\beta,E,T} \tilde{O}_{\beta,E,T} = G_{\beta,E,T} - T^* G_{\beta,E,T} T \quad \text{and} \quad G_{1,\tilde{O}_{\beta,E,T}} = G_{\beta,E,T}. \quad (3.14)$$

**Proof.** Making use of the power series representation (3.12) for $G_{\beta,E,T}$ and the formulas (2.1) for $\gamma_j$, we get

$$G_{\beta,E,T} - T^* G_{\beta,E,T} T = \sum_{k=0}^{\infty} \beta_{k-1} T^{*k} E^* E T^k - \sum_{k=1}^{\infty} \beta_{k-1} T^{*k} E^* E T^k$$

$$= E^* E + \sum_{k=1}^{\infty} \left( \beta_{k-1} - \beta_{k-1}^{-1} \right) T^{*k} E^* E T^k$$

$$= \sum_{k=0}^{\infty} \gamma_k^{-1} T^{*k} E^* E T^k = G_{\gamma,E,T} \quad (3.15)$$

from which we conclude that the series on the right side of (3.15) converges strongly. Then we see from (3.13) that

$$\| \tilde{O}_{\beta,E,T} x \|_{\ell^2(Y)}^2 = \langle G_{\gamma,E,T} x, x \rangle_X < \infty,$$
We conclude that $\tilde{O}_{\beta,E,T} x$ belongs to $\ell^2(\mathcal{Y})$ for any $x \in \mathcal{X}$ and that $\tilde{O}_{\beta,E,T} \tilde{O}_{\beta,E,T} = \mathcal{G}_{\gamma,E,T}$. Substituting this last relation into (3.15) gives the first relation in (3.14).

Finally, from (2.1) we see that $\sum_{k=0}^j \gamma^{-1}_{k} = \beta^{-1}_j$ for all $j \geq 0$. We therefore have

$$G_{1,\tilde{O}_{\beta,E,T}} := \sum_{j=0}^\infty T^{*k} \tilde{O}_{\beta,E,T} \tilde{O}_{\beta,E,T} T^k = \sum_{j=0}^\infty T^{*k} G_{\gamma,E,T} T^k$$

$$= \sum_{j=0}^\infty \sum_{k=0}^j \gamma^{-1}_{k} \cdot T^{*(k+j)} E^* E T^{k+j} = \sum_{j=0}^\infty \sum_{k=0}^j \gamma^{-1}_{k} \cdot T^{*j} E^* E T^{j}$$

$$= \sum_{j=0}^\infty \beta^{-1}_j \cdot T^{*j} E^* E T^{j} = G_{\beta,E,T},$$

and the second equality in (3.14) follows. This in turn implies in particular that the pair $(\tilde{O}_{\beta,E,T}, T)$ is 1-output stable. \qed

We next show how the auxiliary observation operator constructed in Lemma 3.3 can be used to reduce the problem IP to a well understood problem for a classical Schur-class function.

**Lemma 3.4.** Let $\beta$ be a non-increasing weight sequence and let $(E, T)$ be a $\beta$-output stable pair. Suppose that $S \in \mathcal{S}_{\beta}(\mathcal{U}, \mathcal{Y})$ is presented in the form $S = \Psi \beta S$ with $S \in \mathcal{S}(\mathcal{U}, \ell_2(\mathcal{Y}))$ as in Lemma 2.1. Then

$$(E^* S)^\wedge (T^*) = (E^* (\Psi \beta S))^\wedge (T^*) = \left((\tilde{O}_{\beta,E,T})^* S\right)^\wedge (T^*). \quad (3.16)$$

**Proof.** Write out $S(z) : \mathcal{U} \to \ell_2(\mathcal{Y})$ as a column

$$S(z) = \begin{bmatrix} S_1(z) \\ S_2(z) \\ \vdots \end{bmatrix} \text{ where } S_j(z) = \sum_{k=0}^\infty S_{j,k} z^k \text{ with } S_{j,k} \in \mathcal{L}(\mathcal{U}, \mathcal{Y}).$$

Then $S(z) = \Psi \beta(z) S(z)$ is given explicitly as

$$\Psi \beta(z) S(z) = \sum_{j=0}^\infty \gamma^{-\frac{j}{2}}_{j} S_j(z) z^j = \sum_{j=0}^\infty \sum_{k=0}^j \gamma^{-\frac{j}{2}}_{j} S_{j,k} z^{j+k} = \sum_{k=0}^\infty \sum_{j=0}^\ell \gamma^{-\frac{j}{2}}_{k} S_{k,\ell-k} z^\ell$$

where the rearrangement of the infinite series can be justified much as in the proof of Proposition 3.2. We conclude that

$$(E^* (\Psi \beta S))^\wedge (T^*) = \sum_{\ell=0}^\infty \sum_{k=0}^\ell \gamma^{-\frac{j}{2}}_{k} T^{*\ell} E^* S_{k,\ell-k}. \quad (3.17)$$

On the other hand,

$$\left((\tilde{O}_{\beta,E,T})^* S \right)^\wedge (T^*) = \sum_{j=0}^\infty \gamma^{-\frac{j}{2}}_{j} T^{*j} E^* S_{j,j} = \sum_{j=0}^\infty \sum_{k=0}^j \gamma^{-\frac{j}{2}}_{j} T^{*j} E^* S_{j,k} z^k$$

and hence

$$\left((\tilde{O}_{\beta,E,T})^* S\right)^\wedge (T^*) = \sum_{\ell=0}^\infty \sum_{k=0}^\ell \gamma^{-\frac{j}{2}}_{k} T^{*\ell} E^* S_{k,\ell-k}.$$
Comparison of the latter equality and (3.17) now gives (3.16).

The following consequence of Lemma 3.4 is immediate.

**Corollary 3.5.** A function \( S \) belongs to the class \( S_\beta(U, Y) \) and satisfies interpolation condition (3.7) if and only if it is of the form (2.5) with a Schur-class function \( S \in S(U, \ell_2(Y)) \) subject to interpolation condition
\[
((\tilde{O}_{\beta,E,T})^* S)^\vee L(T^*) = N^*.
\] (3.18)

Applying the known theory for the left-tangential interpolation problem with operator argument for the classical Schur class now leads to the following result.

**Theorem 3.6.** The problem \( IP \) with the data set (3.6) and the associated observability gramians \( G_{\beta,E,T} \) and \( G_{1,N,T} \) given in (3.2), (3.10) has a solution if and only if the associated Pick matrix
\[
P := G_{\beta,E,T} - G_{1,N,T}
\] (3.19)
is positive semidefinite.

**Proof.** By Corollary 3.5 solutions \( S \) of \( IP \) exist if and only if the problem (3.18) has a solution \( S \) in the Schur class \( S(U, \ell_2(Y)) \). Since the pairs \((\tilde{O}_{\beta,E,T}, T)\) and \((N,T)\) are both 1-output stable, by the general theory of left-tangential operator-argument Schur-class interpolation (see e.g. Theorem 4.4 in [1]), we know that the latter holds if and only if the associated Pick matrix is positive semidefinite:
\[
P := \sum_{j=0}^{\infty} T^j [(\tilde{O}_{\beta,E,T})^* \tilde{O}_{\beta,E,T} - N^* N] T^j \geq 0.
\] (3.20)

It is readily seen from (3.2) and (3.10) that \( P = G_{\beta,E,T} - G_{1,N,T} \) is as in (3.19).

**Example 2.1 continued:** In the case of the Nevanlinna-Pick problem from Example 3.1, we have \( G_{\beta,E,T} = [k_\beta(z_i, \bar{z}_j) I]_{i,j=1}^{k} \) and \( G_{1,N,T} = [k_1(z_i, \bar{z}_j) V_i V_j^*]_{i,j=1}^{k} \) and we conclude from Theorem 3.6 that there is a contractive multiplier \( S \in S_\beta(U, Y) \) satisfying the interpolation conditions (3.8) if and only if the following block-operator is positive semidefinite:
\[
P = \left[ k_\beta(z_i, \bar{z}_j) I - \frac{V_i V_j^*}{1 - z_i \bar{z}_j} \right]_{i,j=1}^{k} \geq 0.
\]

**Remark 3.7.** Let \( S_\beta \) denote the shift operator on \( H^2_\beta(Y) \) defined as \( S_\beta : f(z) \to z f(z) \). It follows that \( \|S_\beta\| = \sup_{j \geq 0} \frac{\beta_{j+1}}{\beta_j} \) and thus \( S_\beta \) is a contraction if and only if the weight sequence \( \beta \) is non-increasing. Reproducing kernel calculations show that its adjoint \( S_\beta^* \) is given by
\[
S_\beta^* f = \sum_{k=0}^{\infty} \frac{\beta_{k+1}}{\beta_k} \cdot f_{k+1} z^k \quad \text{if} \quad f(z) = \sum_{k=0}^{\infty} f_k z^k.
\] (3.21)

If \( T \in L(X) \) is strongly stable and the pair \((E,T)\) is \( \beta \)-output stable, then the range space \( \text{Ran} \tilde{O}_{\beta,E,T} \) with lifted norm is an \( S_\beta \)-invariant (closed) subspace of...
$H_2^2(\mathcal{Y})$. Indeed, making use of power series expansion (3.1) and of (3.21) we get

$$S_\beta^* O_{\beta,E,T} x = S_\beta^* \sum_{k=0}^{\infty} \beta_k^{-1}(ET^k)x z^k = \sum_{k=0}^{\infty} \beta_k^{-1}(ET^{k+1})x z^k = O_{\beta,E,T} Tx$$

from which the desired invariance follows. For a strongly stable $T \in \mathcal{L}(\mathcal{X})$ and operators $E \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ and $N \in \mathcal{L}(\mathcal{X},\mathcal{U})$ so that the pairs $(E,T)$ and $(N,T)$ are respectively, $\beta$-stable and 1-stable, define two subspaces

$$\mathcal{M}_1 = \text{Ran} O_{\beta,E,T} \subset H_2^2(\mathcal{Y}) \quad \text{and} \quad \mathcal{M}_2 = \text{Ran} O_{1,N,T} \subset H_2^2(\mathcal{U})$$

which are invariant under $S_\beta^*$ and $S_1^*$ respectively. Let us define the operator $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ by

$$\Phi: O_{\beta,E,T} \to O_{1,N,T}x \quad \text{for all} \quad x \in \mathcal{X}.\quad (3.22)$$

From the formulation (3.6) of the interpolation condition (3.7), it is clear that a necessary condition for the problem IP to have a solution is that $\|\Phi\| \leq 1$, or equivalently, that

$$P := G_{\beta,E,T} - G_{1,N,T} \geq 0.\quad (3.23)$$

Furthermore, the computation (where $y = O_{\beta,E,T}x \in \mathcal{M}_1$)

$$\Phi S_\beta^* y = \Phi S_\beta^* O_{\beta,E,T}x = \Phi O_{\beta,E,T} Tx = O_{1,N,T}x = S_1^* O_{1,N,T}x = S_1^* \Phi O_{\beta,E,T}x = S_1^* \Phi y$$

shows that $\Phi$ intertwines $S_\beta^*$ and $S_1^*$. Since $S_\beta$ is a contraction and $S_1$ is an isometry, it follows from the Treil-Volberg commutant lifting result [12] that $\Phi$ can be extended to an operator $R : H_2^2(\mathcal{Y}) \to H^2$ such that $\|R\| = \|\Phi\| \leq 1$ and $RS_\beta^* = S_1^* \Phi$. Its adjoint $R^*$ necessarily is the operator of multiplication by a function $S \in S_\beta(\mathcal{U},\mathcal{Y})$. In this way we see that the condition (3.23) is necessary and sufficient for the existence of solutions of the problem IP, i.e., we arrive at an alternative proof of Theorem 3.6. One of the contributions of the present paper is to obtain an explicit description of the set of all solutions of the problem IP.

It is known how to parametrize solutions of a left-tangential operator-argument interpolation problem for Schur-class functions; the description is more transparent in the case where $P (= P)$ is invertible. Let $J$ be the operator given by

$$J = \begin{bmatrix} I_{\ell_2(\mathcal{Y})} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix} \quad \text{and let} \quad \Theta(z) = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix}.\quad (3.24)$$

be an $\mathcal{L}(\ell_2(\mathcal{Y}) \oplus \mathcal{U})$-valued function such that for all $z, \zeta \in \mathbb{D},$

$$\frac{J - \Theta(z)J \Theta(\zeta)^*}{1 - z\zeta} = \begin{bmatrix} \tilde{O}_{\beta,E,T} & N \end{bmatrix} (I - zT)^{-1}P^{-1}(I - zT^*)^{-1} \begin{bmatrix} \tilde{O}_{\beta,E,T}^* & N^* \end{bmatrix}.\quad (3.25)$$

The function $\Theta$ is determined by equality (3.25) uniquely up to a constant $J$-unitary factor on the right. One possible choice of $\Theta$ satisfying (3.25) is

$$\Theta(z) = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + z \begin{bmatrix} \tilde{O}_{\beta,E,T} & N \end{bmatrix} (I - zT)^{-1}R$$

where the operator

$$\begin{bmatrix} R \\ D_1 \\ D_2 \end{bmatrix}: \ell_2(\mathcal{Y}) \oplus \mathcal{U} \to \begin{bmatrix} \mathcal{X} \\ \ell_2(\mathcal{Y}) \end{bmatrix}$$
Proof. Since the pair \((E, T)\) is \(\beta\)-output stable, it follows from Lemma 3.3 that the pair \((\tau_{\beta,E,T}, T)\) is 1-output stable. By the theory for left-tangential operator-argument interpolation in the Schur class (see e.g. [1]), it is known that in case the operator \((3.20)\) (which is the same as \(P\)) is strictly positive definite, then all solutions \(S\) of problem \(\text{IP}\) are parametrized by the formula

\[
S = \Psi_{\beta}(AE + B)(CE + D)^{-1}.
\]  

(3.28)

where \(E\) is a free parameter from the Schur class \(\mathcal{S}(U, \ell_2(\mathcal{Y}))\).

Proof. Since the pair \((E, T)\) is \(\beta\)-output stable, it follows from Lemma 3.3 that the pair \((\tau_{\beta,E,T}, T)\) is 1-output stable. By the theory for left-tangential operator-argument interpolation in the Schur class (see e.g. [1]), it is known that in case the operator \((3.20)\) (which is the same as \(P\)) is strictly positive definite, then all solutions \(S\) of problem \(\text{IP}\) are parametrized by the formula

\[
S = \Psi_{\beta}(AE + B)(CE + D)^{-1}.
\]  

(3.28)

where \(E\) is a free parameter from the Schur class \(\mathcal{S}(U, \ell_2(\mathcal{Y}))\). The Theorem now follows as a consequence of Lemma 3.3.

Remark 3.9. If \(\Theta\) is taken in the form \((3.27)\) and the free parameter function \(E \in \mathcal{S}(U, \ell_2(\mathcal{Y}))\) is taken to have the form \(E(z) = \begin{bmatrix} E_0(z) \\ E_1(z) \\ \vdots \end{bmatrix}\) where each \(E_i\) is in the Schur class \(\mathcal{S}(U, \mathcal{Y})\) subject to \(\sum_{j=0}^{\infty} E_j(z)^*E_j(z) \leq I_U\) for \(z \in \mathbb{D}\), then the parametrization formula \((3.28)\) can be written more explicitly as

\[
S(z) = \left(\sum_{j=0}^{\infty} \frac{z^j E_j(z)}{\sqrt{\gamma_j}} + (z - \mu)E_k \Psi_{\beta}(z, T)P^{-1}(\mu I - T^*)^{-1}R \Psi_{\beta}(z)\right)
\]

\[
\times (1 + (z - \mu)N(I - zT)^{-1}P^{-1}(\mu I - T^*)^{-1}R \Psi_{\beta}(z))^{-1}.
\]  

(3.29)
Example 3.10. For a fixed integer \( n \geq 1 \) and where we have set for short
\[
R_E(z) := \sum_{j=0}^{\infty} \frac{1}{\sqrt{\gamma_j}} T^{n+j} E^* E_j(z) - N^*.
\]
To derive (3.24), it is enough to observe that
\[
\Psi_{\beta}(z) E(z) = \sum_{j=0}^{\infty} \gamma_j z^j E_j(z), \quad \overline{\Psi}_{\beta,E,T}(z) = \sum_{j=0}^{\infty} \frac{1}{\sqrt{\gamma_j}} T^{n+j} E^* E_j(z),
\]
and that on account of (2.3), (3.13), and (2.2),
\[
\Psi_{\beta}(z) \overline{\Psi}_{\beta,E,T}(z) = \sum_{j=0}^{\infty} \gamma_j^{-1} E T^j z^j = E k_{\gamma}(z, T) = E k_\beta(z, T) \cdot (I - zT),
\]
so that
\[
\Psi_{\beta}(z) \overline{\Psi}_{\beta,E,T}(z)(I - zT)^{-1} = E k_\beta(z, T).
\]
Substituting (3.27) into (3.28) and taking into account the latter expressions we arrive at (3.29).

If we choose \( E_0 \) to be an arbitrary function in \( S(U, V) \) and \( E_j \equiv 0 \) for \( j \geq 1 \), we get a family of solutions to the problem \( \text{IP} \) given by the formula
\[
S(z) = (E_0(z) + (z - \mu) E k_{\beta}(z, T) P^{-1}(\mu I - T^*)^{-1} (E^* E_0(z) - N^*))
\times (1 + (z - \mu) N(I - zT)^{-1} P^{-1}(\mu I - T^*)^{-1} (E^* E_0(z) - N^*))^{-1}.
\]

We now illustrate Theorem 3.8 and Remark 3.9 by a simple example.

**Example 3.10.** For a fixed integer \( n \geq 1 \), let \( \beta = \left\{ \frac{j(n-1)!}{(n+j-1)!} \right\}_{j \geq 0} \) and let us write \( k_n \) (rather than \( k_\beta \)) for the associated kernel (1.11). We thus have
\[
k_n(z, \zeta) = \sum_{j=0}^{\infty} \left( \frac{n+j-1}{j} \right) \gamma_j z^j \zeta^{n-j} = \frac{1}{(1 - \zeta)^n}
\]
and the associated reproducing kernel Hilbert space \( H^2_\beta \) coincides with the standard weighted Bergman space \( A^2_n \). Let us find a contractive multiplier \( S \) from \( H^2 \) to \( A^2_n \) satisfying a single interpolation condition:
\[
S(3/4) = 4/3, \quad (3.30)
\]
We have \( T = 3/4, E = 1, N = 4/3 \) and consequently,
\[
P_n := G_{n,E,T} - G_{1,N,T} = \left( \frac{16}{T} \right)^n - \frac{256}{63} > 0 \quad \text{for every} \quad n \geq 2.
\]
Therefore, the problem (3.30) has solutions for every \( n \geq 2 \), which can be described in terms of the linear fractional transformation (3.28) with free parameter
\[
E(z) = \{ E_k(z) \}_{k \geq 0} \quad \text{such that} \quad \sum_{k=0}^{\infty} |E(z)|^2 \leq 1 \quad \text{for all} \quad z \in \mathbb{D}.
\]
The numbers \( \gamma_j \) defined in (2.1) take the form
\[
\gamma_j = \frac{1}{\binom{n+j-1}{j} - \binom{n+j-2}{j-1}} = \frac{1}{\binom{n+j-2}{j}}
\]
and the kernel $\tilde{k}_0 = \tilde{k}_n$ given by (2.2) becomes simply

$$\tilde{k}_n(z, \zeta) = \tilde{k}_n(z, \zeta) = (1 - z\zeta)^{-(n-1)} = k_{n-1}(z, \zeta).$$

We can choose $\mu = 1$ in formula (3.29) to get

$$S(z) = \frac{16(z - 1)}{3(1 - \frac{3}{4}z)P_n} \sum_{k=0}^{\infty} \left[ \binom{n+k-2}{k} \cdot \frac{(3/4)^k}{n!} \cdot \frac{16(z - 1)}{3(1 - \frac{3}{4}z)^n} P_n^{-1} \right].$$

To get a particular solution $S$ of the problem (3.30) we may let $E = 0$ to arrive at

$$S(z) = \frac{7^n (1 - z)}{3 \cdot 16^{n-1} + 4 \cdot 7^{n-1} - (3/2) \cdot 16^{n-1} + (3/4) \cdot 7^{n-1}}.$$

**Remark 3.11.** In case the Pick operator (3.19) is positive semidefinite but not invertible, the solution set of the problem IP can be still parametrized by a Redheffer-type formula

$$S = \Psi_{\beta} \cdot \left( \tilde{D} + \tilde{C} E (I - \tilde{A} E)^{-1} \tilde{B} \right)$$

(3.31)

where $E$ and $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ are Schur-class functions with the coefficient spaces depending on the degeneracy of the Pick operator $P$. Parametrization (3.31) follows from the factorization result in Lemma 2.1 and known results on the degenerate interpolation problem (5.10) for Schur-class functions (see [5, 6]).

4. **Contractive Multipliers between $H^2_\alpha$ and $H^2_\beta$ in Case $\alpha \neq 1$**

It is natural to consider contractive multipliers between two weighted Hardy spaces $H^2_\alpha (U)$ and $H^2_\beta (Y)$ for given non-increasing weight sequences $\alpha$ and $\beta$ (let us denote this class by $\mathcal{S}_{\alpha \to \beta}(U, Y)$). The analog of Theorem 2.1 is the following.

**Theorem 4.1.** Let the weight sequences $\alpha$ and $\beta$ be non-increasing and let $\Psi_{\alpha}$ and $\Psi_{\beta}$ be the associated operator-valued functions defined as in (2.2). The function $S$ is a contractive multiplier from $H^2_\alpha (U)$ to $H^2_\beta (Y)$ if and only if there is an $S$ in the Schur class $\mathcal{S}(\ell_2(U), \ell_2(Y))$ so that

$$S(z) = S_{\Psi_{\beta}}(z).$$

(4.1)

We now consider the interpolation problem IP with interpolation condition (3.7) but now with solution $S$ sought in the contractive multiplier class $\mathcal{S}_{\alpha \to \beta}(U, Y)$. One can easily see that the interpolation condition (3.7) can equivalently be expressed in the form

$$\mathcal{O}_{\beta, E, T} M_S = \mathcal{O}_{\alpha, N, T}^* : H^2_\alpha (U) \to X$$

(4.2)

where now we view $M_S$ as a multiplication operator from $H^2_\alpha (U)$ to $H^2_\beta (Y)$, i.e., the analogue of (3.9) holds. It is then easily seen that the condition

$$P := \mathcal{G}_{\beta, E, T} - \mathcal{G}_{\alpha, N, T} \geq 0.$$  

(4.3)

is a necessary condition for the existence of a solution $S$ of the interpolation condition (3.7) in the class $\mathcal{S}_{\alpha \to \beta}(U, Y)$. However, unlike the situation for the case $\alpha = 1$ where the factorization (4.1) can be used to reduce the interpolation problem to...
a solvable interpolation problem for a classical Schur-class function, the condition \((4.3)\) in general is not sufficient for the existence of \(S_{\alpha} \rightarrow g(\mathcal{U}, \mathcal{Y})\) solutions, as shown by the following example. We note also that the Treil-Volberg result \([12]\) does not cover this case since the shift \(S_\alpha\) is a nonisometric contraction, and hence not expansive.

**Example 4.2.** Let \(k_n(z, \bar{\zeta}) = (1 - z \bar{\zeta})^{-n}\) be the reproducing kernel of the standard weighted Bergman space \(A^2_n\). We want to solve the two-point interpolation problem

\[
S(\pm 1/\sqrt{2}) = \pm \sqrt{\frac{26}{15}}
\]

in the class of contractive multipliers from \(A^2_3\) to \(A^3_3\), i.e., for functions \(S\) for which the associated kernel

\[
K_S(z, \bar{\zeta}) = k_3(z, \bar{\zeta}) - k_2(z, \bar{\zeta})S(z)\overline{S(\zeta)}
\]

is positive on \(\mathbb{D} \times \mathbb{D}\). The necessary condition \((4.3)\) is satisfied:

\[
P = \begin{bmatrix}
K_S(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) & K_S(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \\
K_S(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) & K_S(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})
\end{bmatrix} = \begin{bmatrix}
16 & 1 \\
1 & 1
\end{bmatrix} \geq 0.
\]

Assume such a function exists. Then the kernel

\[
\begin{bmatrix}
K_S(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) & K_S(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) & K_S(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \\
K_S(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) & K_S(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) & K_S(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \\
K_S(z, \frac{1}{\sqrt{2}}) & K_S(z, -\frac{1}{\sqrt{2}}) & K_S(z, \frac{1}{\sqrt{2}})
\end{bmatrix}
\]

is positive and, since its \(2 \times 2\) principal submatrix is singular, we conclude that \(K_S(z, 1/\sqrt{2}) \equiv K_S(z, -1/\sqrt{2})\), i.e., that

\[
k_3(z, \frac{1}{\sqrt{2}}) - \sqrt{\frac{26}{15}}k_2(z, \frac{1}{\sqrt{2}})S(z) = k_3(z, -\frac{1}{\sqrt{2}}) + \sqrt{\frac{26}{15}}k_2(z, -\frac{1}{\sqrt{2}})S(z).
\]

Solving the latter equality for \(S\) gives

\[
S(z) = \sqrt{\frac{15}{13}} \cdot \frac{z(z^2 + 6)}{4 - z^4}.
\]

Let us show that this \(S\) is not a contractive multiplier from \(A^2_3\) to \(A^3_3\). If it were, then, since

\[
K_S(z, 1/\sqrt{2}) = \left(1 - \frac{z}{\sqrt{2}}\right)^{-3} - \left(1 - \frac{z}{\sqrt{2}}\right)^{-2} \cdot \sqrt{\frac{15}{13}} \cdot \sqrt{\frac{26}{15}} \cdot \frac{z(z^2 + 6)}{4 - z^4} = \frac{4}{4 - z^4},
\]

the kernel

\[
(z, \zeta) \mapsto \begin{bmatrix}
K_S(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) & K_S(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \\
K_S(z, \frac{1}{\sqrt{2}}) & K_S(z, \frac{1}{\sqrt{2}})
\end{bmatrix} = \begin{bmatrix}
\frac{16}{15} & -\frac{4 - z^2}{4 - z^4} \\
\frac{4 - z}{1 - z} & K_S(z, \zeta)
\end{bmatrix}
\]

would be positive semidefinite as well as the kernel

\[
\bar{K}(z, \zeta) = K_S(z, \zeta) - \frac{15}{(4 - z^4)(4 - \zeta^4)}.
\]

Therefore, the kernel

\[
\begin{bmatrix}
\bar{K}(0, 0) & \bar{K}(0, \zeta) \\
\bar{K}(z, 0) & \bar{K}(z, \zeta)
\end{bmatrix} = \begin{bmatrix}
\frac{1}{16} & \frac{1 - z^2}{4 - z^4} \\
\frac{1 - 4z^2}{1 - z} & \bar{K}(z, \zeta)
\end{bmatrix}
\]
was positive, as well as the kernel

\[ \hat{K}(z, \bar{\zeta}) = K(z, \zeta) - \frac{16(1 - 4z^4)(1 - 4\zeta^4)}{(4 - z^4)(4 - \zeta^4)} \]

which is not the case since \( \hat{K}(0.1, 0.1) = -0.93276 \).

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