Abstract—In this paper, we investigate the performance of a practical aggregated LiFi-WiFi system with the discrete constellation inputs from a practical view. We derive the achievable rate expressions of the aggregated LiFi-WiFi system for the first time. Then, we study the rate maximization problem via optimizing the constellation distribution and power allocation jointly. Specifically, a multilevel mercy-filling power allocation scheme is proposed by exploiting the relationship between the mutual information and minimum mean-square error (MMSE) of discrete inputs. Meanwhile, an inexact gradient descent method is proposed for obtaining the optimal probability distributions. To strike a balance between the computational complexity and the transmission performance, we further develop a framework that maximizes the lower bound of the achievable rate where the optimal power allocation can be obtained in closed forms and the constellation distributions problem can be solved efficiently by Frank-Wolfe method. Extensive numerical results show that the optimized strategies are able to provide significant gains over the state-of-the-art schemes in terms of the achievable rate.

Index Terms—Aggregated LiFi-WiFi system, power allocation, probabilities allocation, discrete constellation inputs.

I. INTRODUCTION
A. Motivation and Contributions

The increasing number of Internet of Things (IoT) devices exerts tremendous bandwidth burden continuously on the wireless networks. According to Ericsson’ report [1], more than 80% of the wireless data are generated in indoor environments. Visible light communication (VLC) or light fidelity (LiFi), with a vast license-free visible band in 400-790 THz, can support both high speed data transmission and illumination simultaneously. LiFi exploits the off-the-shelf light emitting diodes (LEDs) and photodiodes (PDs) as transceivers, which can be integrated into IoT devices. Although LiFi serves as a competitive candidate for the next generation wireless solution, its vulnerability to the blockage and small its signal coverage still impose many challenges on various indoor applications. Therefore, a more practical solution is to aggregate the LiFi and WiFi systems and exploit their unique advantages under certain conditions.

In this paper, we consider an aggregated LiFi-WiFi system from a practical communication perspective. First of all, we derive the achievable rate expression of the system with the discrete constellation input signals, rather than the Gaussian inputs adopted in most of the existing works. Then, we further investigate the optimal input distribution and power allocation for the considered system. Our results provide a relatively practical design framework for the aggregated LiFi-WiFi communication system. Specifically, the main contributions of this work are given as follows:

• Generally, the inputs of the practical communications systems follow a finite-set discrete distribution rather than Gaussian distributions. To obtain the performance description of the aggregated LiFi-WiFi system with an arbitrary discrete distribution, we derive the achievable rate expressions of LiFi links and WiFi links, respectively. Comparing with the existing rate expressions with equiprobable discrete constellation points, the derived results are more general and practical. Given that such rate expression is not in closed-form, we further derive both the lower and the upper bounds. All these results can be used as the performance metric for the considered system.

• We jointly optimize the discrete constellation input distribution and the power allocation to maximize the derived achievable rate. To handle this non-convex problem, we propose a multi-level mercy-filling method to obtain the suboptimal power allocation scheme, which exploits the relationship between the mutual information and the minimum mean square error (MMSE). The optimal probability distributions of the discrete constellation are calculated by the inexact gradient descent method.

• To reduce the computation complexity in the previous design problem, we further adopt the derived lower bound as the performance metric. Specifically, we jointly optimize the discrete constellation input distribution and the power allocation to maximize the derived lower bound. To overcome the difficulty of the nonconvex problem, we iteratively optimize the power allocation and discrete probability distribution, where the optimal power allocation scheme is derived with closed expression, and the discrete probability distribution sub-problems is optimized via the proposed Frank-Wolfe method.
B. Related Works and Organization

There are two schemes for the LiFi-WiFi transmission: hard-switching (hybrid) [2]–[12] and aggregating [10]–[15]. The former realizes the transmission via LiFi link or WiFi link, while the later uses both links simultaneously. In general, the former scheme usually results in a lower spectral efficiency and may cause frequent link switches. The LiFi-WiFi aggregated systems, on the contrary, can increase the system data rate and provide reliable communication.

The LiFi-WiFi aggregated system receives great attentions in recent years. For example, by leveraging the bonding technique in the Linux operating system, the authors in [10] prove that the aggregated system outperforms the conventional WiFi. Based on the cross-layer analysis of the physical and datalink layers, link selection approaches were proposed in [11] to maximize the average data arrival rate and minimize the non-asymptotic bounds on data buffering delay. In [12], the coverage probability and rate were analyzed for the aggregated RF/VLC networks. It has been shown that selecting the correct intensity of OBSs (optical BSs) plays a crucial role, and the performance of the aggregated scheme outperforms all other schemes (RF-only, VLC-only, and opportunistic RF/VLC). In [13], the access point selection strategies were optimized via the multi-armed bandit scheme. In [14], both subchannel allocation and power control scheme were developed for improving the energy efficiency of the aggregated VLC/RF network.

In most of existing works, e.g. [2]–[15], the achievable rate of the considered system is derived based on the assumption that the input signal follows the Gaussian distribution. However, Gaussian signals may not be an accurate representation of the inputs in practical communication systems. The inputs of the practical RF-only system are always generated based on the discrete constellations, such as pulse amplitude modulation (PAM), quadrature amplitude modulation (QAM), and phase shift keying (PSK), etc., rather than the Gaussian codebook. Thus, those resource allocation strategies based on the Gaussian assumption would lead to serious performance loss. It has been shown that the mutual information maximization strategies can improve performance of the practical communications systems [16]. Meanwhile, it is proved that the optimal inputs of the VLC-only system follow a finite-set discrete distribution [17]. Although some existing works studied the achievable rate based on the discrete constellation points with equal probability [16], [18]–[22], those results cannot be directly extended to the LiFi-WiFi systems with arbitrary discrete inputs. Therefore, it is necessary to investigate the optimal transmission scheme with the discrete inputs for the aggregated LiFi-WiFi system.

The rest of this paper is organized as follows. We provide the model of the aggregated LiFi-WiFi system in Section II. The optimal power allocation and optimal probability distributions schemes of the aggregated LiFi-WiFi system are presented in Section III. In Section IV, we provide the solutions for the lower bound as the throughput metric. The simulation results are presented for the lower bound as the throughput metric. Section VI concludes the paper.

Notations: $\cdot^T$, $(\cdot)^*$, $\|\cdot\|$, and $\text{Tr}(\cdot)$ represent the transpose, conjugate, Frobenius norm, and trace of a matrix respectively. The Hadamard product of $\mathbf{A}$ and $\mathbf{B}$ is denoted as $\mathbf{A} \odot \mathbf{B}$. $\mathcal{M} \triangleq \{1, 2, ..., M\}$ and $\mathcal{N} \triangleq \{1, 2, ..., N\}$.

II. System model

As illustrated in Fig. 1 we consider the downlink transmission of an aggregated LiFi-WiFi system, where the transmitter is equipped with single LED and single WiFi antenna, and the receiver is equipped with single PD and single RF antenna. The transmitter simultaneously transmits information via both the LiFi link and the WiFi link, where the bandwidths of LiFi link and WiFi link are $B_1 \text{Hz}$ and $B_2 \text{Hz}$, respectively. Let $\mathbf{x} \triangleq [x_1, x_2]^T$ denote the transmitted signal vector, where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{C}$ denote the independently transmitted signals of the LiFi link and WiFi link, respectively.

In a practical LiFi-WiFi communication system, the transmitted signals are distributed in discrete constellation. Suppose that the LiFi link signal is sent via $M$-pulse-amplitude modulation ($M$-PAM) and the WiFi link signal is sent via $N$-quadrature amplitude modulation ($N$-QAM). To be more specific, the signal $x_1$ is taken from a non-negative real discrete constellation set $\Omega_1$ with cardinality $M$, which is given as

$$
\Omega_1 \triangleq \left\{ x_1 \mid \begin{array}{ll}
\frac{\Pr (x_1 = x_{1,k}) = p_{1,k}}{0 \leq x_{1,k} \leq A_k} \\
\sum_{k=1}^{M} p_{1,k} = 1 \\
\sum_{k=1}^{M} p_{1,k} x_{1,k}^2 \leq \bar{\mu} \\
\sum_{k=1}^{M} p_{1,k} \leq P_{e,1} \\
x_{1,k} \in \mathbb{R}, k = 1, ..., M 
\end{array}\right\},
$$

where $x_{1,k}$ denotes the constellation point, $p_{1,k}$ represents probability that $x_1$ equals $x_{1,k}$, and parameters $A$, $\bar{\mu}$, and $P_{e,1}$ denotes the peak optical power, maximum average optical power, and maximum average electric power of $x_1$, respectively. On the other hand, the WiFi signal $x_2$ is taken from a complex discrete constellation set $\Omega_2$ with cardinality $N$,
which is given as

\[ \Omega_2 = \left\{ x_1 : \begin{aligned} \Pr (x_2 = x_{2,l}) = p_{2,l}, \\ \sum_{l=1}^{N} p_{2,l} = 1, \\ \sum_{l=1}^{N} p_{2,l} |x_{2,l}|^2 \leq P_{e,2}, \\ x_{2,l} \in \mathbb{C}, l = 1, \ldots, N, \end{aligned} \right\} \]

(2)

where \( x_{2,l} \) denotes the constellation point, \( p_{2,l} \) denotes the probability that the constellation \( x_{2,l} \) is chosen, \( P_{e,2} \) denotes the maximum average electric power of \( x_2 \), respectively.

Let \( q_1 \in \mathbb{R} \) and \( q_2 \in \mathbb{C} \) denote the power amplification factors for \( x_1 \) and \( x_2 \), respectively. Here, \( q_1 \) and \( q_2 \) need to satisfy the average electrical power constraint, i.e.,

\[ \eta_1 \varepsilon_1 q_1^2 + \eta_2 \varepsilon_2 |q_2|^2 \leq P_T, \]

(3)

where \( \eta_1 \) and \( \eta_2 \) denote the efficiency of the power amplifier of the LiFi link and WiFi link, respectively; \( \varepsilon_1 = \sum_{k=1}^{M} p_{1,k} x_{1,k}^2 \); \( \varepsilon_2 = \sum_{l=1}^{N} p_{2,l} |x_{2,l}|^2 \); and \( P_T \) denotes the total electrical power threshold. Moreover, for human eye safety considerations, the power control over the LiFi signal also needs to meet the average optical power and peak optical power requirement: \( \mathbb{E} \{ q_1 x_1 \} = q_1 \mu \leq P_o \), and \( q_1 A \leq P_{ins} \), where \( \mu = \sum_{k=1}^{M} p_{1,k} x_{1,k}^2 \); \( P_o \) and \( P_{ins} \) denote the maximum average optical power threshold and the instantaneous optical power threshold, respectively.

Let \( g \triangleq [g_1, g_2]^T \) denotes the channel vector, where \( g_1 \) and \( g_2 \) are the channel parameters of the LiFi link and WiFi link, respectively. Assume that the channel parameters \( g_1 \) and \( g_2 \) are quasi-static in the paper. Specifically, \( g_1 \) is given by [22]

\[ g_1 = \frac{(m+1) A_1}{2 \pi d_1^2} \cos^m (\phi) \cos \varphi \alpha_g \alpha_c (\varphi), \]

(4)

where \( m \) is the order of the Lambertian emission; \( A_1 \) is the detector area of PD receiver; \( d_1 \) is the distance between the LED and PD; \( \phi \) and \( \varphi \) are the radiance and incidence angle of the LiFi link; \( \alpha_g \) denotes the gain of the optical filter; \( \alpha_c (\varphi) \) denotes the gain of optical concentrator, which is given as

\[ \alpha_c (\varphi) = \left\{ \begin{array}{ll} \frac{n^2}{\sin^2 (\Psi_c)}, & 0 \leq \varphi \leq \Psi_c, \\ 0, & \text{otherwise}, \end{array} \right. \]

(5)

where \( n \) is the refractive index, \( \Psi_c \) represents the field-of-view (FoV) of the LiFi receiver.

As to the WiFi link, \( g_2 \) is modeled by [8]

\[ g_2 = \left\{ \begin{array}{ll} g_2 10^{-\left( L_F (d_2) + L_o \right)/20}, & d_2 \leq d_B, \\ g_2 10^{-\left( L_F (d_2) + 35 \log_{10} \left( \frac{d_2 + 2}{4} \right) \right)/20}, & d_2 > d_B, \end{array} \right. \]

(6)

where \( g_r = \sqrt{\frac{K}{N + 1}} e^{j \psi} \) denotes the small-scale fading gain, \( a \sim \mathcal{CN} (0, 1) \); \( \psi \) is the angle of arrival/departure of the WiFi link; \( K \) denotes the Ricean \( K \)-factor; \( d_2 \) denotes the distance between user and the RF antenna; \( d_B \) is the breakpoint distance; \( L_o \) denotes the Ricean \( K \)-factor; \( d_2 \) denotes the small-scale fading, \( \sigma = 3 \text{dB} \) with \( d_2 \leq d_B \), \( \sigma = 5 \text{dB} \) with other conditions; \( L_F (d_2) \) is the free space loss at the central carrier frequency \( f_c \) as follows:

\[ L_F (d_2) = 20 \log_{10} (d_2) + 20 \log_{10} (f_c) - 147.5. \]

(7)

Let \( y_1 \) and \( y_2 \) denote the received signals from the LiFi link and WiFi link, respectively, which can be written in the vector form as

\[ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} g_1 q_1 x_1 \\ g_2 q_2 x_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \]

(8)

where \( z_1 \sim \mathcal{N} (0, \sigma_1^2) \) is the received real Gaussian noise from the LiFi link, and \( z_2 \sim \mathcal{CN} (0, \sigma_2^2) \) denotes the received complex Gaussian noise from the WiFi link. Then, the signal-to-noise ratio (SNR) of the LiFi link and WiFi link can be defined as

\[ \frac{g_1^2 \varepsilon_1^2}{B_1 \sigma_1^2}, \quad \frac{g_2^2 |q_2|^2 \varepsilon_2}{B_2 \sigma_2^2}. \]

(9)

Note that, for the aggregated LiFi-WiFi system with the finite-alphabet inputs, the achievable rate is still unknown. To address this issue, we define the achievable rate \( R_{\text{LiFi-WiFi}} \) as

\[ R_{\text{LiFi-WiFi}} = I (x_1, x_2; y_1, y_2) = R_{\text{LiFi}} + R_{\text{WiFi}}, \]

(10)

where \( R_{\text{LiFi}} \triangleq I (x_1; y_1) \) and \( R_{\text{WiFi}} \triangleq I (x_2; y_2) \) denote the achievable rates of the LiFi link and WiFi link, respectively.

**Proof:** According to the mutual information, the upper bound of \( R_{\text{LiFi-WiFi}} \) is given as

\[ R_{\text{LiFi-WiFi}} = I (x_1, x_2; y_1, y_2) \]

\[ \leq I (x_1; y_1) + I (x_2; y_2) \]

\[ \leq R_{\text{LiFi}} + R_{\text{WiFi}}, \]

(11)

where the equality (11c) holds if and only if \( x_1 \) and \( x_2 \) are independent. Thus, when \( x_1 \) and \( x_2 \) are independent, the upper bound (11d) is achievable, i.e., the upper bound (11d) is the channel capacity of the aggregated LiFi-WiFi system. In other words, for the considered aggregated LiFi-WiFi system, the optimal inputs of \( x_1 \) and \( x_2 \) are independent, and the achievable rate is \( R_{\text{LiFi-WiFi}} = I (x_1, x_2; y_1, y_2) = R_{\text{LiFi}} + R_{\text{WiFi}}. \)

**Lemma 1:** With the finite-alphabet inputs and the given bandwidths \( B_1 \) and \( B_2 \), the achievable rates of the aggregated LiFi-WiFi system \( R_{\text{LiFi}} \) and \( R_{\text{WiFi}} \) are respectively given as

\[ R_{\text{LiFi}} = \frac{B_1}{\ln 2} - 2 B_1 \sum_{k=1}^{M} p_{1,k} E_2 \left\{ \log_2 \frac{M}{\rho_{m,m} \exp (\Lambda_{k,m})} \right\}, \]

(12a)

\[ R_{\text{WiFi}} = \frac{B_2}{\ln 2} - 2 B_2 \sum_{l=1}^{N} p_{2,l} E_{z_2} \left\{ \log_2 \frac{N}{\rho_{n,n} \exp (\Gamma_{l,n})} \right\}, \]

(12b)

where \( \Lambda_{k,m} \triangleq \frac{(q_1 q_2 (x_{1,k-1} - x_{1,m}) + \sqrt{M} z_1)^2}{2 \pi d_1^2}, \) and \( \Gamma_{l,n} \triangleq \frac{(q_1 q_2 (x_{2,l-1} - x_{2,n}) + \sqrt{M} z_2)^2}{2 \pi d_1^2} \).
Due to the property of the complex multiplication. Thus, the \( q_1 \) and (13g) denote the probability distribution constraint.

Based on the derived expression of \( R_{\text{LiFi-WiFi}} \), the following question is to optimize the signal distributions and power allocation schemes for the two links to obtain the maximal achievable rate of the aggregated system. The optimization problem is formulated as

\[
\begin{align*}
\max_{q_1,q_2,\{p_{1,k}\},\{p_{2,l}\}} & \quad R_{\text{LiFi-WiFi}} \\
\text{s.t.} & \quad \eta_1 P_{e,1} q_1^2 + \eta_2 P_{e,2} |q_2|^2 \leq P_T, \quad (13b) \\
& \quad q_1 \leq \min \left( P_e / \bar{\mu}, P_{\text{ins}} / A \right), \quad (13c) \\
& \quad \sum_{k=1}^M p_{1,k} x_{1,k} \leq \bar{\mu}, \sum_{k=1}^M p_{1,k} x_{1,k}^2 \leq P_{e,1}, \quad (13d) \\
& \quad \sum_{k=1}^M p_{1,k} = 1, p_{1,k} \geq 0, \forall k \in M, \quad (13e) \\
& \quad \sum_{l=1}^N p_{2,l} |x_{2,l}|^2 \leq P_{e,2}, \quad (13f) \\
& \quad \sum_{l=1}^N p_{2,l} = 1, p_{2,l} \geq 0, \forall l \in N, \quad (13g)
\end{align*}
\]

where (13b) denotes the total average electrical power of two links; (13c) and (13d) denote the maximal optical power constraint, average optical power constraint, and average power constraint on the LiFi link, respectively; (13e) denotes the average electrical power constraint on the WiFi link; (13f) and (13g) denote the probability distribution constraint.

For problem (13), the optimal phase of \( q_2 \) is the same as that of \( q_2 \), which can be proved by the counter-evidence method due to the property of the complex multiplication. Thus, the optimal \( q_2 \) can be written as

\[
q_2 = \sqrt{q_2^*}, \quad (14)
\]

where \( \hat{q}_2 \triangleq q_2^* \). Furthermore, by substituting (14) into (10) and \( \hat{q}_1 \triangleq q_1^* \), the achievable rate \( R_{\text{LiFi-WiFi}} \) can be reformed as (15).

By defining \( \hat{x}_1 \triangleq [x_{1,1}, \ldots, x_{1,M}]^T \) and \( \hat{p}_1 \triangleq [p_{1,1}, \ldots, p_{1,M}]^T \), constraints (13d) and (13g) can be reformulated as

\[
\begin{align*}
\Upsilon_1 \triangleq \left\{ \hat{p}_1 1_M^T \hat{p}_1 = 1, \hat{p}_1 \succeq 0, (\hat{x}_1 \circ \hat{x}_1) \hat{p}_1 \leq P_{e,1} \right\}, \quad p_1 \in \Upsilon_1, \quad (16)
\end{align*}
\]

Similarly, by defining \( \hat{x}_2 \triangleq [x_{2,1} \ldots x_{2,N}]^T \) and \( \hat{p}_2 \triangleq [p_{2,1} \ldots p_{2,N}]^T \), constraints (13e) and (13g) can be reformulated as

\[
\begin{align*}
\Upsilon_2 \triangleq \left\{ \hat{p}_2 1_N^T \hat{p}_2 = 1, \hat{p}_2 \succeq 0, (\hat{x}_2 \circ \hat{x}_2) \hat{p}_2 \leq P_{e,2} \right\}, \quad p_2 \in \Upsilon_2, \quad (17)
\end{align*}
\]

Next, we introduce auxiliary variables, i.e.,

\[
\begin{align*}
w & \triangleq \left[ \log_2 p_1^T \bar{w}_1, \ldots, \log_2 p_M^T \bar{w}_M \right]^T, \quad (18a) \\
\bar{w}_k & \triangleq [\bar{w}_{k,1}, \ldots, \bar{w}_{k,M}]^T, \quad (18b) \\
\bar{w}_{k,m} & \triangleq \exp \left( -\frac{(q_1 \sqrt{q_1} (x_{1,k} - x_{1,m}) + \sqrt{B_1} z_1)^2}{2B_1 \sigma^2_1} \right), \quad (18c) \\
r & \triangleq [\log_2 p_2^T \bar{r}_1, \ldots, \log_2 p_N^T \bar{r}_N]^T, \quad (18d) \\
\bar{r}_l & \triangleq [\bar{r}_{l,1}, \ldots, \bar{r}_{l,N}]^T, \quad (18e) \\
\bar{r}_{l,n} & \triangleq \exp \left( -\frac{|q_2| \sqrt{q_2} (x_{2,l} - x_{2,n}) + \sqrt{B_2} z_2|^2}{2B_2 \sigma^2_2} \right), \quad (18f)
\end{align*}
\]

where \( k \) and \( m \) in \( \mathcal{M} \), \( l \) and \( n \) in \( \mathcal{N} \). Then, we rewrite the achievable rate \( R_{\text{LiFi-WiFi}} \) as

\[
R_{\text{LiFi-WiFi}} = -\frac{B_1 + B_2}{\ln 2} - 2B_1 \mathbb{E}_{z_1} \left\{ p_1^T w \right\} - 2B_2 \mathbb{E}_{z_2} \left\{ p_2^T r \right\}. \quad (19)
\]

Based on the above definitions, problem (13) can be equivalent to a compact from as

\[
\begin{align*}
\min_{\hat{q}_1,\hat{q}_2,\hat{p}_1,\hat{p}_2} & \quad 2B_1 \mathbb{E}_{z_1} \left\{ p_1^T w \right\} + 2B_2 \mathbb{E}_{z_2} \left\{ p_2^T r \right\} \\
\text{s.t.} & \quad \eta_1 P_{e,1} \hat{q}_1 + \eta_2 P_{e,2} \hat{q}_2 \leq P_T, \quad (20b) \\
& \quad \hat{q}_1 \leq \tau^2, \quad (20c) \\
& \quad \hat{p}_1 \in \Upsilon_1, \hat{p}_2 \in \Upsilon_2, \quad (20d)
\end{align*}
\]

where \( \tau \triangleq \min \left( P_e / \bar{\mu}, P_{\text{ins}} / A \right) \). Note that in problem (20), the power allocation variables \( \hat{q}_1 \) and \( \hat{q}_2 \) are only contained in constraints (20b) and (20c), while the distribution variables \( \hat{p}_1 \) and \( \hat{p}_2 \) are only contained in the constraint (20d). Therefore, the optimization problem (18) can be decomposed into two sub-problems solved alternately until the objective function converges: power allocation sub-problem 1: optimizing \( \hat{q}_1 \)
and \( \hat{q}_2 \) with given \( p_1 \) and \( p_2 \), and probability distribution sub-problem 2: optimizing \( p_1 \) and \( p_2 \) with given \( \hat{q}_1 \) and \( \hat{q}_2 \). Next, we will present the solutions to these two sub-problems.

### A. Power Allocation Sub-Problem

With given \( p_1 \), \( p_2 \), problem (20) can be simplified to an optimal power allocation problem as

\[
\min_{\hat{q}_1, \hat{q}_2} \quad h(\hat{q}_1, \hat{q}_2) \\
s.t. \quad \eta_1 P_{e,1}\hat{q}_1 + \eta_2 P_{e,2}\hat{q}_2 \leq P_T, \\
\hat{q}_1 \leq \tau^2, \\
\hat{q}_2 \leq \tau^2,
\]

where \( h(\hat{q}_1, \hat{q}_2) \) is defined as in (18) and (21a). Note that

\[
\frac{(q_2 \sqrt{(x_{i,2} - x_{i,1})} + \sqrt{\sigma_1^2})^2}{B_1^2 \sigma_2^2} = \frac{1}{B_1^2 \sigma_1^2}
\]

are convex over \( \hat{q}_1 \) and \( \hat{q}_2 \), respectively. Furthermore, since \( \log \sum_i e^{f_i(x)} \) is convex as long as \( f_i(x) \) is convex.(24), problem (21) is convex over \( \hat{q}_1 \) and \( \hat{q}_2 \). To obtain the optimal power allocation \( \hat{q}_1 \) and \( \hat{q}_2 \), we write the Lagrangian function of problem (1) as

\[
\mathcal{L}(\hat{q}_1, \hat{q}_2, \gamma, \nu) = h(\hat{q}_1, \hat{q}_2) + \gamma (\eta_1 P_{e,1}\hat{q}_1 + \eta_2 P_{e,2}\hat{q}_2 - P_T) + \nu (\hat{q}_1 - \tau^2),
\]

where \( \gamma \geq 0 \) and \( \nu \geq 0 \) are the Lagrangian multipliers associated with constraints (21b) and (21c), respectively. Then, the Karush-Kuhn-Tucker (KKT) conditions of problem (21) are

\[
\begin{align*}
\frac{\partial h}{\partial \hat{q}_1} + \gamma \eta_1 P_{e,1} + \nu &= 0, \\
\frac{\partial h}{\partial \hat{q}_2} + \gamma \eta_2 P_{e,2} &= 0, \\
\gamma (\eta_1 P_{e,1}\hat{q}_1 + \eta_2 P_{e,2}\hat{q}_2 - P_T) &= 0, \quad \gamma \geq 0, \\
\nu (\hat{q}_1 - \tau^2) &= 0, \quad \nu \geq 0.
\end{align*}
\]

Furthermore, based on the definition of \( h(\hat{q}_1, \hat{q}_2) \), we have

\[
\begin{align*}
\frac{\partial h}{\partial \hat{q}_1} &= -\frac{g_1^2 \xi_1}{g_1^2} \frac{\partial I(x_1; y_1)}{\partial \text{SNR}_1}, \\
\frac{\partial h}{\partial \hat{q}_2} &= -\frac{g_2^2 \xi_2}{g_2^2} \frac{\partial I(x_2; y_2)}{\partial \text{SNR}_2}.
\end{align*}
\]

Moreover, according to relationship between the mutual information and the MMSE [22], we have

\[
\begin{align*}
\frac{\partial I(x_1; y_1)}{\partial \text{SNR}_1} &= \frac{\text{MMSE}_1(\text{SNR}_1)}{2}, \\
\frac{\partial I(x_2; y_2)}{\partial \text{SNR}_2} &= \text{MMSE}_2(\text{SNR}_2),
\end{align*}
\]

where \( \text{MMSE}_1(\text{SNR}_1) \equiv \mathbb{E} \{ |x_i - \hat{x}_i| \} \) denotes the MMSE between \( x_i \) and \( \hat{x}_i, i = 1, 2 \), and \( \hat{x}_i \) is the conditional mean of the MMSE estimate of \( x_i \), i.e., \( \tilde{x}_i \equiv \mathbb{E} \{ x_i | y_1 = g_1 q_i x_i + z_i \} \) and \( \hat{x}_i \equiv \mathbb{E} \{ x_i | y_2 = g_2^* q_2 x_i + z_2 \} \). Note that the calculation of the above MMSE involves non-holonomic function, i.e., it is non-trivial to compute the exact MMSE for an arbitrary input distribution. Meanwhile, it is an opening problem to find the lower and upper bounds of the MMSE for the discrete inputs

\( \mathcal{X} \). Thus, most of existing works adopt numerical methods such as Monte Carlo integral to obtain the results, which may lead to sub-optimality and high complexity. To overcome this issue, we approximate the MMSEs by a scaled linear MMSE upper bound [23], i.e.,

\[
\begin{align*}
\text{MMSE}_1(\text{SNR}_1) &\approx \text{LMMSE}_{e_1}(\text{SNR}_1), \\
\text{MMSE}_2(\text{SNR}_2) &\approx \text{LMMSE}_{e_2}(\text{SNR}_2),
\end{align*}
\]

where \( \text{LMMSE}_{e_i}(x) = \frac{x}{1 + x} \). Although the approximation based on LMMSE, like a continuous Gaussian MMSE, would lead to the optimality loss, it can provide a fast acceptable closed-form method. Then, [23] can be reformulated as

\[
\begin{align*}
\frac{\partial I(x_1; y_1)}{\partial \text{SNR}_1} &= \frac{1}{2} \text{LMMSE}_{e_1}(\text{SNR}_1), \\
\frac{\partial I(x_2; y_2)}{\partial \text{SNR}_2} &= \text{LMMSE}_{e_2}(\text{SNR}_2) .
\end{align*}
\]

With (21) and (27), we arrive at

\[
\begin{align*}
\frac{\partial h}{\partial \hat{q}_1} &= -\frac{g_1^2 \xi_1}{2g_1^2} \text{LMMSE}_{e_1} \left( \frac{g_1^2 q_1 \xi_1}{B_1 \sigma_1^2} \right), \\
\frac{\partial h}{\partial \hat{q}_2} &= -\frac{g_2^2 \xi_2}{2g_2^2} \text{LMMSE}_{e_2} \left( \frac{g_2^2 q_2 \xi_2}{B_2 \sigma_2^2} \right).
\end{align*}
\]

Substituting equation (28) into equation (23), we have

\[
\frac{g_1^2 \xi_1}{2g_1^2} \text{LMMSE}_{e_1} \left( \frac{g_1^2 q_1 \xi_1}{B_1 \sigma_1^2} \right) - \gamma \eta_1 P_{e,1} - \nu = 0, \\
\frac{g_2^2 \xi_2}{2g_2^2} \text{LMMSE}_{e_2} \left( \frac{g_2^2 q_2 \xi_2}{B_2 \sigma_2^2} \right) - \gamma \eta_2 P_{e,2} = 0.
\]

Then, \( \hat{q}_1 \) and \( \hat{q}_2 \) can be derived through solving the equation (29) as

\[
\begin{align*}
\hat{q}_1 &= \frac{B_1 \sigma_1^2}{g_1^2 \xi_1} \text{LMMSE}_{e_1}^{-1} \left( \frac{2B_1 \sigma_1^2 (\gamma \eta_1 P_{e,1} + \nu)}{g_1^2 \xi_1} \right), \\
\hat{q}_2 &= \frac{B_2 \sigma_2^2}{g_2^2 \xi_2} \text{LMMSE}_{e_2}^{-1} \left( \frac{2B_2 \sigma_2^2 (\gamma \eta_2 P_{e,2})}{g_2^2 \xi_2} \right).
\end{align*}
\]

where \( \text{LMMSE}_{e_i}^{-1}(x) = \frac{1}{x} - \frac{1}{x^2} \). The Lagrangian multipliers \( \gamma \) and \( \nu \) can be solved by the Water-filling (WF) Method. The detailed algorithm is given in Algorithm 1.

---

**Algorithm 1** Water-filling (WF) Method for Power Allocation Sub-Problem (21).

1. Given \( \zeta \geq 0, \gamma \in [0, \gamma] \);
2. **Initialization**: \( \gamma_{\text{min}} = 0, \gamma_{\text{max}} = \gamma \);
3. **repeat**
4. Set \( \gamma \leftarrow (\gamma_{\text{min}} + \gamma_{\text{max}}) / 2 \);
5. Obtain \( \hat{q}_2 \) by (30b);
6. Find the minimum \( \nu \) satisfying the constraint \( \hat{q}_1 \leq \tau^2 \);
7. Obtain \( \hat{q}_1 \) by (30a);
8. If \( \eta_1 P_{e,1}\hat{q}_1 + \eta_2 P_{e,2}\hat{q}_2 \leq P_T \), set \( \gamma_{\text{max}} \leftarrow \gamma \); otherwise, \( \gamma_{\text{min}} \leftarrow \gamma \);
9. **until** \( |\nu_{\text{max}} - \nu_{\text{min}}| \leq \zeta \);
10. **return** \( \hat{q}_1 \) and \( \hat{q}_2 \).
B. Probability Distribution Sub-Problem

When \( \hat{q}_1 \) and \( \hat{q}_2 \) are given, problem (20) can be reformulated as

\[
\begin{align*}
\min_{p_1, p_2} & \quad 2B_1E_{z_1}\{p_1^T w\} + B_2E_{z_2}\{p_2^T r\} \\
\text{s.t.} & \quad p_1 \in \Upsilon_1, \quad p_2 \in \Upsilon_2,
\end{align*}
\]

(31a)

(31b)

which is a convex optimization problem with two variables \( p_1 \) and \( p_2 \). However, there is no analytical expression of the objective function, which prevents us from calculating the optimal probability distributions.

To overcome this bottleneck, we adopt the inexact gradient descent method\cite{17} to calculate the optimal probability distribution. Besides, the Blahut-Arimoto algorithm\cite{26, 28} can also solve this problem. Let \( \phi_1(p_1) \triangleq 2B_1E_{z_1}\{p_1^T w\} \) and \( \nabla \phi_1(p_1) \) denote its gradient as

\[
\nabla \phi_1(p_1) = 2B_1E_{z_1}\{w + Wp_1\} = 2B_1 \int_{-\infty}^{\infty} f_{z_1}(z_1)(w + Wp_1)dz_1,
\]

(32)

where \( W \triangleq [W_{i,j}] \), \( W_{i,j} \triangleq \frac{w_i^T e_j}{w_i^T p_1 \ln 2} \), \( e_j \) denotes the unit vector in which the \( i \)-th element is 1 and the other elements are zeros, and \( f_{z_1}(z_1) \triangleq \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp \left( -\frac{z_1^2}{2\sigma_1^2} \right) \) denotes the probability density function of \( z_2 \).

Since neither \( \phi_j(p_j) \) nor \( \nabla \phi_j(p_j) \) has a closed-form expression, we perform the truncation of the integration from infinity to an finite interval. More specifically, let \( [-\tau_1, \tau_1] \) and \( [-\tau_2, \tau_2] \) denote the integration intervals of \( \phi_1(p_1) \) and \( \nabla \phi_1(p_1) \). Then, let \( \hat{\phi}_1(p_1) \) and \( \hat{\nabla} \phi_1(p_1) \) denote the approximation of \( \phi_1(p_1) \) and \( \nabla \phi_1(p_1) \) respectively, which are given by

\[
\begin{align*}
\hat{\phi}_1(p_1) & \triangleq 2B_1 \int_{-\tau_1}^{\tau_1} f_{z_1}(z_1)p_1^T w dz_1, \\
\hat{\nabla} \phi_1(p_1) & \triangleq 2B_1 \int_{-\tau_2}^{\tau_2} f_{z_1}(z_1)(w + Wp_1)dz_1,
\end{align*}
\]

(33a)

(33b)

where \( \tau_1 > 0 \) and \( \tau_2 > 0 \).

With the approximated objective function and its gradient, i.e., \( \hat{\phi}_1(p_1) \) and \( \hat{\nabla} \phi_1(p_1) \), we adopt the gradient projections method to solve problem (31). Specifically, let \( p_1^{[i]} \) denote the \( n \)-th iteration feasible point. With inexact gradient \( \hat{\nabla} \phi_1(p_1) \), the gradient projection iteration between \( p_1^{[i]} \) and \( p_1^{[i+1]} \) is given by

\[
p_1^{[i+1]} = \text{Proj}_{\Upsilon_1}(p_1^{[i]} - \alpha_1^{[i]} \hat{\nabla} \phi_1(p_1^{[i]})),
\]

(34)

where \( \alpha_1^{[i]} \) denotes the \( i \)-th iteration step size, \( \text{Proj}_{\Upsilon_1}(\hat{x}) \) denotes the projection of \( x \) onto \( \Upsilon_1 \):

\[
\text{Proj}_{\Upsilon_1}(x) = \begin{cases} 
  x, & \text{if } x \in \Upsilon_1, \\
  \arg \min_{\hat{x} \in \Upsilon_1} \|x - \hat{x}\|^2, & \text{otherwise}.
\end{cases}
\]

(35)

The details of the inexact gradient descent method are listed in Algorithm 2, and the distribution of LiFi link \( p_1 \) can be obtained. The optimal distribution of WiFi link \( p_2 \) can be obtained analogously. Thus, we omit the detailed derivations for brevity. The optimal probability distribution \( p_1, p_2 \) can be achieved by the distribution matching (DM)\cite{29, 31}.

In summary, the achievable rate maximization problem (13) can be solved by Algorithm 3, therefore the maximal achievable rate \( R_{\text{LiFi-WiFi}} \) can be achieved.

Algorithm 2 Inexact Gradient Descent Method for Probability Distribution Sub-Problem (31)

1: \textbf{Initialization:} Given \( \delta \geq 0 \), set \( i = 1 \), and choose \( p_1^{[i]} \in \Upsilon_1 \).

2: \textbf{repeat}

3: \quad \quad i \leftarrow i + 1.

4: \quad \quad Update \( \hat{\phi}_1(p_1^{[i-1]}) \) and \( \hat{\nabla} \phi_1(p_1^{[i-1]}) \) by (33).

5: \quad \quad Compute the step size \( \alpha_1^{[i-1]} \) by Armijo rule (24).

6: \quad \quad Update \( p_1^{[i]} = \text{Proj}_{\Upsilon_1}(p_1^{[i-1]} - \alpha_1^{[i-1]} \hat{\nabla} \phi_1(p_1^{[i-1]})) \).

7: \quad \quad until \( \|p_1^{[i]} - p_1^{[i-1]}\| \leq \delta \).

8: \quad \quad return \( p_1 = p_1^{[i]} \).

Algorithm 3 Optimal discrete constellation inputs of problem (20)

1: \textbf{Initialization:} Given \( \xi \geq 0 \), set \( k = 1 \), and choose \( p_1^{[k]} \in \Upsilon_1 \), \( p_2^{[k]} \in \Upsilon_2 \).

2: \textbf{repeat}

3: \quad \quad Update \( \hat{q}_1^{[k]} \) and \( \hat{q}_2^{[k]} \) by Algorithm 1.

4: \quad \quad Obtain \( R_{\text{LiFi-WiFi}}^{[k]} \) by substituting \( \hat{q}_1^{[k]} \), \( \hat{q}_2^{[k]} \), \( p_1^{[k]} \) and \( p_2^{[k]} \) into formulation (19).

5: \quad \quad \quad \textbf{repeat}

6: \quad \quad \quad \quad k \leftarrow k + 1.

7: \quad \quad \quad \quad Update \( \hat{q}_1^{[k]} \) and \( \hat{q}_2^{[k]} \) by Algorithm 1 with given \( p_1^{[k]} \) and \( p_2^{[k]} \).

8: \quad \quad \quad \quad Obtain \( R_{\text{LiFi-WiFi}}^{[k]} \) by substituting \( \hat{q}_1^{[k]} \), \( \hat{q}_2^{[k]} \), \( p_1^{[k]} \) and \( p_2^{[k]} \) into formulation (19).

9: \quad \quad \quad \quad until \( \|R_{\text{LiFi-WiFi}}^{[k]} - R_{\text{LiFi-WiFi}}^{[k-1]}\| \leq \xi \).

10: \quad \quad \quad \quad return \( p_1 = p_1^{[k]} \), \( p_2 = p_2^{[k]} \), \( \hat{q}_1 = \hat{q}_1^{[k]} \) and \( \hat{q}_2 = \hat{q}_2^{[k]} \).

IV. OPTIMAL DISCRETE CONSTELLATION INPUT DISTRIBUTIONS BASED ON LOWER BOUNDS AND UPPER BOUNDS

Recall that, neither the achievable rate\cite{12a} nor \( \hat{R}_{\text{LiFi-WiFi}} \) is in a closed-form expression, and thus the calculation of \( \hat{R}_{\text{LiFi-WiFi}} \) is computationally inefficient. To reduce the computational complexity, we may replace the objective function with an explicit expression. Thus, we turn to the capacity bound of LiFi link and WiFi link.

Lemma 2: With the finite-alphabet inputs, the closed-form upper bound \( R_{\text{LiFi}}^U \) and lower bound \( R_{\text{LiFi}}^L \) of the LiFi link achievable rate are respectively given as

\[
R_{\text{LiFi}}^U = -2B_1 \sum_{k=1}^{M} p_{1,k} \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left( 2\hat{L}_{k,m} \right), \quad (36a)
\]
where \( \hat{\kappa}_m \triangleq \frac{(x_k - x_{1,m})^2}{4B_1 \sigma_q^2} \).

**Proof:** Please find the proof in Appendix [13].

Furthermore, we develop the closed-form upper bound and lower bound of the achievable rate for WiFi link \( R_\text{WiFi} \).

**Lemma 3:** The upper bound \( R^U_\text{WiFi} \) and lower bound \( R^L_\text{WiFi} \) of the achievable rate for the WiFi link are respectively given as

\[
R^U_\text{WiFi} = -B_2 \sum_{l=1}^{N} p_{2,l} \log_2 \left( \sum_{n=1}^{N} p_{2,n} \exp \left( \hat{\Gamma}_l, n \right) \right),
\]

\[
R^L_\text{WiFi} = B_2 - \frac{B_2}{\ln 2} - B_2 \sum_{l=1}^{N} p_{2,l} \log_2 \left( \sum_{n=1}^{N} p_{2,n} \exp \left( \hat{\Gamma}_l, n \right) \right),
\]

where \( \hat{\Gamma}_l, n = \frac{(x_l - x_{2,n})^2}{2B_2 \sigma_q^2} \).

**Proof:** Please see the proof in Appendix [C].

Then, let \( R^L_{\text{LiFi- WiFi}} \) and \( R^U_{\text{LiFi- WiFi}} \) respectively denote the lower bound and upper bound of \( R_{\text{LiFi- WiFi}} \), which are given by

\[
R^L_{\text{LiFi- WiFi}} \triangleq R^L_\text{LiFi} + R^L_\text{WiFi},
\]

\[
R^U_{\text{LiFi- WiFi}} \triangleq R^U_\text{LiFi} + R^U_\text{WiFi}.
\]

The lower bound \( R^L_{\text{LiFi- WiFi}} \) and \( R^L_\text{LiFi} \) are not tight, and might be negative [32]. When \( P_T \to 0 \) and \( P_T \to +\infty \), there is a constant gap, i.e., \( B_1 \left( \frac{1}{\ln 2} - 1 \right) \), between the \( R^L_{\text{LiFi}} \) and \( R^L_{\text{LiFi- WiFi}} \). There also is a constant gap, i.e., \( \left( \frac{1}{\ln 2} - 1 \right) (B_1 + B_2) \), between the \( R^L_{\text{LiFi- WiFi}} \) and \( R^L_{\text{WiFi}} \). Thus, when \( P_T \to 0 \) and \( P_T \to +\infty \), there is a constant gap, i.e., \( \left( \frac{1}{\ln 2} - 1 \right) (B_1 + B_2) \), between the \( R^L_{\text{LiFi- WiFi}} \) and \( R^L_{\text{WiFi}} \). If the constant gap is directly added to \( R^L_{\text{LiFi- WiFi}} \), there might be intersections between \( R_{\text{LiFi- WiFi}} \) and \( R^L_{\text{LiFi- WiFi}} \) as similar as Fig. 1-3 in [32].

Based on the lower bound of the achievable rate \( R^L_{\text{LiFi- WiFi}} \), we aim to obtain the optimal distribution of the signal magnitudes and power allocation jointly, which can be formulated as

\[
\max_{q_1, q_2, \{p_1, k \}, \{p_2, k \}} R^L_{\text{LiFi- WiFi}} \quad (39)
\]

s.t. \( \{13b\}, \{13c\}, \{13e\}, \{13d\}, \{13m\}, \{13g\} \).

Furthermore, by defining \( \hat{q}_1 \triangleq q_1^2 \) and \( \hat{q}_2 \triangleq |q_2|^2 \), the lower bound of the achievable rate \( R^L_{\text{LiFi- WiFi}} \) can be rewritten as

\[
R^L_{\text{LiFi- WiFi}} = \left( 1 - \frac{1}{\ln 2} \right) (B_1 + B_2 - 2B_1 \mathbf{p}_1^T \mathbf{u}(\hat{q}_1) - B_2 \mathbf{p}_2^T \mathbf{v}(\hat{q}_2)),
\]

where

\[
\mathbf{u}(\hat{q}_1) \triangleq [\log_2 b_1^T (\hat{q}_1) \mathbf{p}_1, \ldots, \log_2 a_M^T (\hat{q}_1) \mathbf{p}_1]^T,
\]

\[
\mathbf{a}_k (\hat{q}_1) \triangleq [a_{k,1} (\hat{q}_1), \ldots, a_{k,M} (\hat{q}_1)]^T,
\]

\[
\mathbf{v}(\hat{q}_2) \triangleq \left[ \log_2 b_1^T (\hat{q}_2) \mathbf{p}_2, \ldots, \log_2 b_N^T (\hat{q}_2) \mathbf{p}_2 \right]^T,
\]

\[
b_1 (\hat{q}_2) \triangleq \left[ b_{1,1} (\hat{q}_2), \ldots, b_{1,N}(\hat{q}_2) \right]^T,
\]

\[
b_{l,n} (\hat{q}_2) \triangleq \exp \left( \frac{|x_{2,l} - x_{2,n}|^2}{2B_2 \sigma_q^2} \right), \forall l, n \in N.
\]

Then, problem [39] can be formulated as

\[
\min_{\hat{q}_1, \hat{q}_2, \mathbf{p}_1, \mathbf{p}_2} 2B_1 \mathbf{p}_1^T \mathbf{u}(\hat{q}_1) + B_2 \mathbf{p}_2^T \mathbf{v}(\hat{q}_2)
\]

s.t. \( \eta_1 P_{e,1} \hat{q}_1 + \eta_2 P_{e,2} \hat{q}_2 \leq P_T, \)

\[
\hat{q}_1 \leq \tau^2,
\]

\[
\hat{q}_1 \geq 0, \hat{q}_2 \geq 0.
\]

Since \( \log_2 x \) is a concave function, at least one of constraint \( 13b \) and constraint \( 43c \) is active for the optimal power allocation, i.e., \( \eta_1 P_{e,1} \hat{q}_1 + \eta_2 P_{e,2} \hat{q}_2 = P_T \) or \( \hat{q}_1 = \tau^2 \). When \( \hat{q}_1 = \tau^2 \), constraint \( 13b \) is also active. Therefore, constraint \( 43c \) is always active, i.e., \( \eta_1 P_{e,1} \hat{q}_1 + \eta_2 P_{e,2} \hat{q}_2 = P_T \). In the following, we will discuss the optimal power allocation of problem [42] based on whether constraint \( 43c \) is active or not.

1) \( \eta_1 \tau^2 > P_T \): In this case, constraint \( 43c \) is inactive, i.e., \( \hat{q}_1 < \tau^2 \). Problem [43b] can be reformulated as

\[
\min_{\hat{q}_1, \hat{q}_2} 2B_1 \mathbf{p}_1^T \mathbf{u}(\hat{q}_1) + B_2 \mathbf{p}_2^T \mathbf{v}(\hat{q}_2)
\]

s.t. \( \eta_1 P_{e,1} \hat{q}_1 + \eta_2 P_{e,2} \hat{q}_2 \leq P_T, \)

\[
\hat{q}_1 \geq 0, \hat{q}_2 \geq 0.
\]

Substituting \( \hat{q}_2 = \frac{P_T - \eta_1 P_{e,1} \hat{q}_1}{\eta_2 P_{e,2}} \) into problem [44a], the objective function [44a] is reformulated as

\[
\Phi (\hat{q}_1) \triangleq 2B_1 \mathbf{p}_1^T \mathbf{u}(\hat{q}_1) + B_2 \mathbf{p}_2^T \mathbf{v}(\hat{q}_1),
\]

where \( \mathbf{v}(\hat{q}_1) \triangleq \left[ \log_2 b_1^T (\hat{q}_1) \mathbf{p}_1, \ldots, \log_2 b_N^T (\hat{q}_1) \mathbf{p}_1 \right]^T \),

\[
\mathbf{b}_1 (\hat{q}_1) \triangleq \left[ b_{1,1} (\hat{q}_1), \ldots, b_{1,N}(\hat{q}_1) \right]^T,
\]

\[
\mathbf{b}_{l,n} (\hat{q}_1) \triangleq \left[ b_{l,1} (\hat{q}_1), \ldots, b_{l,N}(\hat{q}_1) \right]^T, \forall l, n \in N,
\]
\[
\exp\left(-\frac{||x_{2,t} - x_{2,n}||^2}{2\sigma_{\xi}^2 \sigma_{\eta}^2}\right), \forall l, n \in \mathcal{N}.
\]

Thus, problem (43) can be reformulated as
\[
\begin{align*}
\min_{\tilde{q}_1} & \quad \Phi(\tilde{q}_1) \\
\text{s.t.} & \quad 0 \leq \tilde{q}_1 \leq \frac{P_T}{\eta_1 \xi_1},
\end{align*}
\] (46a)

Let \( \tilde{q}_1^{\text{opt}} \) denote the optimal solution of problem (46), and \( \Phi^{\text{opt}} \) denote the maximum rate of the aggregated LiFi - WiFi system. Then, we can obtain a stationary point \( \tilde{q}_1^{\text{sta}} \) by setting \( \frac{\partial \Phi(\tilde{q}_1^{\text{sta}})}{\partial \tilde{q}_1} = 0 \) so that the optimal solution \( \tilde{q}_1^{\text{sta}} \) satisfies the following equation
\[
2B_1 p_1^T \tilde{a}(\tilde{q}_1) = B_2 p_2^T \tilde{b}(\tilde{q}_1),
\] (47)

where
\[
\tilde{a}(\tilde{q}_1) = [\tilde{a}_1(\tilde{q}_1), \ldots, \tilde{a}_M(\tilde{q}_1)]^T,
\]
\[
\tilde{a}_k(\tilde{q}_1) = c_k^T \odot a_k(\tilde{q}_1) p_1,
\] (48a)

\[
\tilde{b}(\tilde{q}_1) = [\tilde{b}_1(\tilde{q}_1), \ldots, \tilde{b}_N(\tilde{q}_1)]^T,
\]
\[
\tilde{b}_i(\tilde{q}_1) = \frac{d_i^T \odot b_i^T(\tilde{q}_1) p_2}{d_i^T(\tilde{q}_1) p_2},
\] (48d)

\[
d_i = \frac{\eta_1 \xi_1 ||2||^2}{2B_2 \sigma_2^2 \eta_2 \xi_2} \begin{bmatrix} ||x_{2,t} - x_{2,1}||^2, \ldots, ||x_{2,t} - x_{2,N}||^2 \end{bmatrix}^T, \forall l \in \mathcal{N}.
\] (48f)

Let \( \Phi^{\text{opt},1} \) and \( \tilde{q}_1^{\text{opt},1} \) respectively denote the minimal objective function and the optimal power allocation \( \tilde{q}_1 \) of problem (43). Thus, if \( 0 \leq \tilde{q}_1 \leq P_T \), we have
\[
\tilde{q}_1^{\text{opt},1} = \arg \min_{\tilde{q}_1} \left\{ \Phi(\tilde{q}_1^{\text{sta}}), \Phi(0), \Phi \left( \frac{P_T}{\eta_1 \xi_1} \right) \right\},
\] (49b)

Otherwise, we have
\[
\begin{align*}
\Phi^{\text{opt},1} & = \min \left\{ \Phi(0), \Phi \left( \frac{P_T}{\eta_1 \xi_1} \right) \right\}, \\
\tilde{q}_1^{\text{opt},1} & = \arg \min_{\tilde{q}_1} \left\{ \Phi(0), \Phi \left( \frac{P_T}{\eta_1 \xi_1} \right) \right\}.
\end{align*}
\] (50a)

2) \( \eta_1 P_{e,1} \tau^2 \leq P_T \): When \( \eta_1 P_{e,1} \tau^2 \leq P_T \), constraint (43c) could be either active or inactive, i.e., \( \hat{q}_1 = \tau^2 \) or \( \hat{q}_1 < \tau^2 \). When constraint (43c) is inactive, the minimal objective function of problem (43) is the same as the case \( \eta_1 P_{e,1} \tau^2 < P_T \). Otherwise, the optimal allocated power of problem (43) are \( \hat{q}_1 = \tau^2 \), and \( \hat{q}_2 = \frac{P_T - \eta_1 P_{e,1} \tau^2}{\eta_2 \xi_2} \). Here, the corresponding minimal objective function of problem (43) is \( \Phi(\tau^2) \). Let \( \Phi^{\text{opt},2} \) and \( \hat{q}_1^{\text{opt},2} \) respectively denote the minimal objective function and the optimal power allocation \( \hat{q}_1 \) of problem (43), which are given by
\[
\Phi^{\text{opt},2} = \min \left\{ \Phi^{\text{opt},1}, \Phi(\tau^2) \right\},
\]
(51a)

\[
\hat{q}_1^{\text{opt},2} = \arg \min_{\hat{q}_1} \left\{ \Phi^{\text{opt},1}, \Phi(\tau^2) \right\}.
\] (51b)

By solving the power allocation sub-problem, we can obtain the optimal power of LiFi and WiFi links \( \hat{q}_1 \) and \( \hat{q}_2 \).

B. Probability Distribution Sub-problem

Next, we solve the probability distribution sub-problem: optimizing the LiFi and the WiFi links probability distribution, i.e., \( p_1 \) and \( p_2 \), with given \( \hat{q}_1 \) and \( \hat{q}_2 \). Then, the maximization problem of the aggregated LiFi-WiFi system (43) can be formulated as
\[
\begin{align*}
\min_{p_1, p_2} & \quad 2B_1 p_1^T u(p_1) + B_2 p_2^T v(p_2) \\
\text{s.t.} & \quad p_1 \in \mathcal{T}_1, \quad p_2 \in \mathcal{T}_2,
\end{align*}
\] (52a)

where \( u(p_1) = [\log_2 a_1^T(\hat{q}_1) p_1, \ldots, \log_2 a_M^T(\hat{q}_1) p_1]_T \), and \( v(p_2) = [\log_2 b_1^T(\hat{q}_2) p_2, \ldots, \log_2 b_N^T(\hat{q}_2) p_2]_T \). Then, problem (52) can be divided into two independent subproblems as
\[
\begin{align*}
\min_{p_1} & \quad f_1(p_1) \\
\text{s.t.} & \quad p_1 \in \mathcal{T}_1,
\end{align*}
\] (53a)

and
\[
\begin{align*}
\min_{p_2} & \quad f_2(p_2) \\
\text{s.t.} & \quad p_2 \in \mathcal{T}_2.
\end{align*}
\] (54a)

where \( f_1(p_1) \triangleq 2B_1 p_1^T u(p_1) \), and \( f_2(p_2) \triangleq B_2 p_2^T v(p_2) \). We can adopt the Frank-Wolfe method (53) to solve problem (53) and (54). Specifically, let \( p_1[i] \) denote a feasible point at the \( i \)th iteration. The first-order Taylor expansion of \( f_1(p_1) \) is given by
\[
\begin{align*}
f_1(p_1) \approx f_1(p_1[i]) + \nabla_{p_1} f_1^T \left( p_1[i] \right) (p_1 - p_1[i]),
\end{align*}
\] (55)

where \( \nabla_{p_1} f_1^T \left( p_1[i] \right) \) denotes the gradient of the objective function \( f_1(p_1) \). Since \( p_1[i] \) is obtained, problem (53) is equivalent to
\[
\begin{align*}
\min_{p_1} & \quad \nabla_{p_1} f_1^T \left( p_1[i] \right) p_1 \\
\text{s.t.} & \quad p_1 \in \mathcal{T}_1.
\end{align*}
\] (56a)

Then, the \( i + 1 \)th iteration point \( p_1[i+1] \) is updated by
\[
p_1[i+1] = p_1[i] + \lambda_1[i] d_1[i],
\] (57)

where \( \lambda_1[i] \) denotes the stepsize of the \( i \)th iteration, and \( d_1[i] \) denotes the feasible descending direction of the \( i \)th iteration. The details of the Frank-Wolfe method are summarized in Algorithm 4, which outputs the probability distribution of the LiFi link, i.e., \( p_1 \). Similar as problem (53), we can obtain the optimal probability distribution of the WiFi link, i.e., \( p_2 \), by solving problem (54) through Frank-Wolfe method. We omit the detailed derivations for brevity. The optimal probability distribution \( p_1, p_2 \) can be achieved by the distribution matching (DM) (29) - (41).
In summary, the achievable rate maximization of the aggregated LiFi-WiFi system can be solved by Algorithm 5. The optimal power allocation of LiFi and WiFi links \( \hat{q}_1 \) and \( \hat{q}_2 \), the probability distribution of LiFi and WiFi links \( p_1 \) and \( p_2 \), and the maximal lower bound of the achievable rate \( R_{L_{\text{LiFi-WiFi}}}^L \) can be obtained by Algorithm 3. Note that, based on the upper bound of the achievable rate \( R_{L_{\text{LiFi-WiFi}}}^U \), we can also find the optimal distribution of the signal magnitudes and power allocation to maximize the achievable rate of the aggregated LiFi-WiFi system, which is similar to the lower bound case.

Algorithm 4 Frank-Wolfe Method for the Probability Distribution Sub-Problem (52).

1: Initialization: Given \( \xi \geq 0 \), set \( i = 1 \), and choose \( p_1^{[1]} \in \mathcal{Y}_1 \).
2: repeat
3: \( i \leftarrow i + 1 \).
4: Obtain \( \tilde{p}_1^{[i-1]} \) by solving problem (56).
5: Construct a feasible descending direction \( d_1^{[i-1]} = p_1^{[i-1]} - \tilde{p}_1^{[i-1]} \).
6: Use the bisection search to obtain the optimal \( \lambda_1^{[i-1]} = \arg \min_{\lambda_1} f_1 (p_1^{[i-1]} + \lambda d_1^{[i-1]}) \).
7: Update \( p_1^{[i]} = p_1^{[i-1]} + \lambda_1^{[i-1]} d_1^{[i-1]} \).
8: until \( \left| \nabla_p f_1 (p_1^{[i-1]}) d_1^{[i-1]} \right| \leq \xi \).
9: return \( p_1 = p_1^{[+1]} \).

Algorithm 5 Optimal discrete constellation input distributions based on lower bounds.

1: Initialization: Given \( \xi \geq 0 \), set \( k = 1 \), and choose \( p_1^{[1]} \in \mathcal{Y}_1 \), \( p_2^{[1]} \in \mathcal{Y}_2 \).
2: Update \( \hat{q}_1^{[1]} \) and \( \hat{q}_2^{[1]} \) by solving problem (43).
3: Obtain \( R_{L_{\text{LiFi-WiFi}}}^L \) by substituting \( \hat{q}_1^{[1]} \), \( \hat{q}_2^{[1]} \), \( p_1^{[1]} \) and \( p_2^{[1]} \) into formulation (40).
4: repeat.
5: \( k \leftarrow k + 1 \).
6: Update \( \hat{q}_1^{[k]} \) and \( \hat{q}_2^{[k]} \) by solving problem (55).
7: Update \( p_1^{[k]} \) and \( p_2^{[k]} \) by Frank-Wolfe method (in Algorithm 4).
8: Obtain \( R_{L_{\text{LiFi-WiFi}}}^{L_{\text{LiFi}}} \) by substituting \( \hat{q}_1^{[k]} \), \( \hat{q}_2^{[k]} \), \( p_1^{[k]} \) and \( p_2^{[k]} \) into formulation (40).
9: until \( \left| R_{L_{\text{LiFi-WiFi}}}^{L_{\text{LiFi}}} - R_{L_{\text{LiFi-WiFi}}}^{L_{\text{LiFi}}[k-1]} \right| \leq \xi \).
10: return \( p_1 = p_1^{[k]} \), \( p_2 = p_2^{[k]} \), \( \hat{q}_1 = \hat{q}_1^{[k]} \) and \( \hat{q}_2 = \hat{q}_2^{[k]} \).

Table I: Basic Parameters

| LiFi Link | WiFi Link |
|-----------|-----------|
| Parameters | Value | Parameters | Value |
| \( \sigma_1 \) | \( 10^{-24} \) A²/Hz | \( \sigma_2 \) | \( -57 \) dBm/MHz |
| FoV \( \Psi_c \) | \( 90^\circ \) | \( d_B \) | \( 5 \) m |
| \( A_{PD} \) | \( 1 \) cm² | \( d_2 \) | \( 4 \) m |
| \( I_{th} \) | \( 8 \) A | \( f_c \) | \( 2.4 \) GHz |
| \( \theta_{1/2} \) | \( 60^\circ \) | \( \psi \) | \( 45^\circ \) |
| \( B_1 \) | \( 40 \) MHz | \( B_2 \) | \( 20 \) MHz |

V. Simulation Results and Discussions

In this section, we illustrate the performance of the proposed schemes for the considered aggregated LiFi-WiFi system. Moreover, the simulation results also demonstrate the impact of key parameters on both the achievable rate and energy efficiency of the aggregated LiFi-WiFi system, such as total power threshold, bandwidths, etc. In the simulations, we consider the locations of the LED and PD as \( (0, 0, 5.7) \) m and \( (0, 0, 1.7) \) m, respectively. More detailed system parameters of the LiFi and WiFi links are given in Table I.

A. Performance of the Aggregated LiFi-WiFi System based on \( R_{L_{\text{LiFi-WiFi}}} \)

In the following, we evaluate the performance of the proposed transmission schemes in Fig. 2 and Fig. 3.

Fig. 2(a) illustrates the optimal probability distribution of input \( \{x_{1,k}, p_{1,k}\} \) versus SNR \( \gamma_1 \) of the LiFi link, where \( K_1 = 8 \). It shows that in the low SNR region, the optimal input positions include two discrete points with an equal probability. While in the high SNR region, the optimal input positions have more than two discrete points. As the increase of SNR, the optimal probability distribution is closer to the equiprobable distribution. Fig. 2(c) illustrates the achievable rate of LiFi link \( R_{L_{\text{LiFi}}} \) versus SNR. We can observe that the achievable rate of LiFi link \( R_{L_{\text{LiFi}}} \) with the proposed method is higher than that of the equiprobable distribution. Moreover, with the increase of the SNR, the gap between the proposed method and the equiprobable distribution shrinks.

Fig. 2(b) shows the optimal probability distribution of input \( \{x_{2,k}, p_{2,k}\} \) with SNR \( \gamma_2 = 4 \) dB of the WiFi link, where \( K_2 = 16 \). It shows that for SNR \( \gamma_2 = 4 \) dB, the optimal input positions include sixteen discrete points, and the optimal probability distribution is equiprobable. Fig. 2(d) illustrates the achievable rate of WiFi link \( R_{L_{\text{WiFi}}} \) versus SNR. We can observe that the achievable rate of WiFi link \( R_{L_{\text{WiFi}}} \) with the proposed method is higher than that obtained by the equiprobable distribution. Moreover, with the increase of the SNR, the gap between the proposed method and the equiprobable distribution decreases.

Fig. 3(a) shows the achievable rate \( R_{L_{\text{LiFi-WiFi}}} \) and link transmit power of the aggregated LiFi-WiFi system versus total power threshold \( P_T \), respectively. We observe that the achievable rates \( R_{L_{\text{LiFi-WiFi}}} \) with both the proposed method and equiprobable distribution increase as total power threshold \( P_T \) increases.

Fig. 3(b) illustrates the achievable rate of aggregated LiFi-WiFi system \( R_{L_{\text{LiFi-WiFi}}} \) versus instant optical power threshold \( P_{\text{ins}} \). We observe that the achievable rates \( R_{L_{\text{LiFi-WiFi}}} \) with the proposed method increase as the instant optical power threshold \( P_{\text{ins}} \) first increases, and remains constant. Moreover, as instant optical power threshold \( P_{\text{ins}} \) increases, the transmit power of LiFi link \( \hat{q}_1 \) first increases, and then...
Fig. 2. (a) The optimal probability distribution of input $\{x_{1,k}, p_{1,k}\}$ versus SNR$_1$ of LiFi link; (b) The optimal probability distribution of input $\{x_{2,l}, p_{2,l}\}$ with SNR$_2=4$dB of WiFi link. (c) Achievable rate of LiFi link $R_{LiFi}$ versus SNR$_1$; (d) Achievable rate of WiFi link $R_{WiFi}$ versus SNR$_2$.

keeps as a constant. The transmit power of WiFi link $\hat{q}_2$ first decreases, and then keeps as a constant. This is due to $\tau \leq \min\left(\frac{P_o}{\bar{\mu}}, \frac{P_{\text{ins}}}{A}\right)$, and we assume that $P_o = 0.8P_{\text{ins}}$, and $\bar{\mu} = 0.5A$. For a lower $P_{\text{ins}}$, $\hat{q}_1$ is constrained by $\tau$, while for a high $P_{\text{ins}}$, $\hat{q}_1$ is constrained by the total electrical power threshold $P_T$.

**B. Optimal Discrete Constellation Input Distributions Based on $R_{LiFi-WiFi}^L$ (38a)**

In the following, the performance of the proposed aggregated LiFi-WiFi system based on the lower bound of achievable rate is investigated in Fig. 4 (a) and (b).

Fig. 4 (a) demonstrates the lower bound of achievable rate $R_{LiFi-WiFi}^L$ and link transmit power of aggregated LiFi-WiFi system versus total power threshold $P_T$, respectively. We observe that the lower bound of achievable rates $R_{LiFi-WiFi}^L$ with both the proposed method and equiprobable distribution increase as total power threshold $P_T$ increases. Moreover, as total power threshold $P_T$ increases, the transmit power of the LiFi link, i.e., $\hat{q}_1$, first increases, and remains as a
constant and the transmit power of the WiFi link, i.e., $\hat{q}_2$, increases. The reason is that $\hat{q}_1$ is also limited by the optical power constraint.

Fig. 4 (b) describes the lower bound of achievable rates of aggregated LiFi-WiFi system $R_{L_{\text{LiFi}}-\text{WiFi}}^L$ versus instant optical power threshold $P_{\text{ins}}$, and we assume that $P_o = 0.8 P_{\text{ins}}$, and $\mu = 0.5 A$. We observe that the lower bound of achievable rates $R_{L_{\text{LiFi}}-\text{WiFi}}^L$ with the proposed method increases as instant optical power threshold $P_{\text{ins}}$ first increases, and remains as a constant. Moreover, as instant optical power threshold $P_{\text{ins}}$ increases, the transmit power of the LiFi link, i.e., $q_1$, first increases, and then keeps as a constant. The transmit power of WiFi link $\hat{q}_2$ first decreases, and then keeps as a constant. This is due to $\tau \triangleq \min (P_o / \mu, P_{\text{ins}} / A)$. For a lower $P_{\text{ins}}, q_1$ is constrained by $\tau$, while for high $P_{\text{ins}}, \hat{q}_1$ is constrained by the total electrical power threshold $P_T$.

In the following, we investigate the comparison between the achievable rate $R_{L_{\text{LiFi}}-\text{WiFi}}$ (10), it’s lower bound $R_{L_{\text{LiFi}}-\text{WiFi}}^L$ (38a) and the upper bound $R_{L_{\text{LiFi}}-\text{WiFi}}^U$ (38b) of the proposed aggregated LiFi-WiFi system in Fig. 5 (a) and (b), Fig. 6 (a) and (b).

Fig. 5 (a) shows the achievable rate $R_{L_{\text{LiFi}}-\text{WiFi}}$ (10), it’s lower bound $R_{L_{\text{LiFi}}-\text{WiFi}}^L$ (38a) and the upper bound $R_{L_{\text{LiFi}}-\text{WiFi}}^U$ (38b) of the aggregated LiFi-WiFi system versus total power threshold $P_T$ with the optimal solutions of problem (13), respectively. And Fig. 5 (b) illustrates the achievable rate $R_{L_{\text{LiFi}}-\text{WiFi}}$ (10), it’s lower bound $R_{L_{\text{LiFi}}-\text{WiFi}}^L$ (38a) and the upper bound $R_{L_{\text{LiFi}}-\text{WiFi}}^U$ (38b) of the aggregated LiFi-WiFi system versus total power threshold $P_T$ with the optimal solutions of problem (39), respectively. Both Fig. 5 (a) and (b) show that for a low $P_T$, the gap between $R_{L_{\text{LiFi}}-\text{WiFi}}$ and $R_{L_{\text{LiFi}}-\text{WiFi}}^L$ is larger than the one of between $R_{L_{\text{LiFi}}-\text{WiFi}}$ and $R_{L_{\text{LiFi}}-\text{WiFi}}^U$, and for a high $P_T$, the gap between $R_{L_{\text{LiFi}}-\text{WiFi}}$ and $R_{L_{\text{LiFi}}-\text{WiFi}}^L$ is lower than the one of between $R_{L_{\text{LiFi}}-\text{WiFi}}$ and $R_{L_{\text{LiFi}}-\text{WiFi}}^U$. Moreover, the $R_{L_{\text{LiFi}}-\text{WiFi}}$, $R_{L_{\text{LiFi}}-\text{WiFi}}^L$ and $R_{L_{\text{LiFi}}-\text{WiFi}}^U$...
of Algorithm 3 and Algorithm 5, respectively. This is because that solutions of Algorithm 3 are suboptimal.

![Graph](image)

**Table II**

| Time(s) | Scheme | Algorithm 3 | Algorithm 5 |
|---------|--------|-------------|-------------|
| M = N   |        |             |             |
| 4       | 5.47   | 0.2466      |
| 8       | 30.16  | 4.60        |
| 16      | 79.01  | 7.68        |
| 32      | 311.44 | 29.75       |
| 64      | 4217.26| 264.95      |

**VI. Conclusion**

In this paper, we investigated the aggregated LiFi-WiFi system, which simultaneously used both the LiFi link and WiFi link to transmit information. First of all, with both LiFi link and WiFi link bandwidths consideration, we derived the achievable rate expression of the system with the discrete constellation input signals. Then, we further investigated the optimal input distribution and power allocation for the considered system. Moreover, we derived the upper and lower bounds of the achievable rate expression of the system with the discrete constellation input signals, and we investigated the optimal input distribution and power allocation for the considered system. At last, the effects of critical parameters of the aggregated LiFi-WiFi system on the maximum achievable rate, such as total threshold, optical power, and bandwidth, etc., are numerically analyzed. In the future, we would further investigate both imperfect CSI and interference management for multi-user LiFi-WiFi networks.

**Appendix A**

**Proof of the Formulation Lemma 1**

The achievable rates of the LiFi link $R_{\text{LiFi}}$ are given as follows

$$R_{\text{LiFi}} = h(y_1) - h(y_1|x_1)$$  \hspace{1cm} (58a)

$$= -\int_{-\infty}^{\infty} \sum_{k=1}^{L} p_{1,k} \exp \left(-\frac{x_1^2}{2\sigma_1^2}\right) \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left(\Lambda_{k,m}\right) \, dz_1$$

$$- \frac{1}{2} \log_2 2 \pi \sigma_1^2$$  \hspace{1cm} (58b)

$$= -\frac{1}{2} \log_2 2 \pi \sigma_1^2 - \sum_{k=1}^{L} \sum_{m=1}^{M} p_{1,k} E_{z_1} \left\{ \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left(\Lambda_{k,m}\right) \right\},$$  \hspace{1cm} (58c)

where $\Lambda_{k,m} \triangleq (g_1 y_1 (x_{1,k} - x_{1,m}) + z_1^2) / 2\sigma_1^2$. For the given modulation order, the average executing time of the two algorithms increases as the modulation order increases, and the CPU time of Algorithm 5 is significantly less than that of Algorithm 3.
the noise is $\frac{n^2}{2}$ watts/Hz, which leads to the noise power $\sigma_n^2 B_1$. For the time interval $[0, T]$, there are $2BT$ noise samples, and the variance of each sample is $\sigma_n^2 B_2 T = \frac{\sigma_n^2}{2}$. Moreover, if the power of signal is $P_1$, the energy of signal per sample is $\frac{P_1}{2B_1 T} = \frac{\sigma_n^2}{2}$. Therefore, we obtain the achievable rate expression of the LiFi link with the bandwidth of $B_1$ as given in (123).

Then, the achievable rates of the WiFi link and $R_{\text{WiFi}}$ are given as follows

$$R_{\text{WiFi}} = -\sum_{i=1}^{N} \frac{p_{2,l} \exp \left( -\frac{|z_i|^2}{\sigma^2_2} \right)}{\pi \sigma^2_2} \log_2 \left( \frac{\sum_{n=1}^{N} p_{2,n} \exp (\Gamma_{l,n})}{\pi \sigma^2_2} \right) \quad \text{(59a)}$$

$$= -\frac{1}{\ln 2} - \sum_{i=1}^{N} \frac{p_{2,l} E_{z_2} \left\{ \log_2 \left( \sum_{n=1}^{N} p_{2,n} \exp (\Gamma_{l,n}) \right) \right\}}{\sigma^2_2} \quad \text{(59b)}$$

where $\Gamma_{l,n} = \frac{|z_i|^2}{\sigma^2_2} - p_{2,l}^z + \frac{p_{2,l}^2}{2B_2}$. Suppose that the bandwidth of the WiFi link is $B_2$ (Hz). Then, both the input and output can be represented by complex samples taken by $\frac{1}{B_2}$ seconds apart. Note that since the noise is independent in the L and Q components, each use of the complex channel can be thought of as two independent uses of a real AWGN channel. If the power of signal is $P_2$, the noise variance and the power constraint per real symbol are $\frac{\sigma_n^2}{2}$ and $\frac{p_{2,l}^2}{2B_2^2}$, respectively. Thus, we have the achievable rate expression of the WiFi link with the bandwidth of $B_1$ as given in (123).

Hence, the achievable rates of the aggregated LiFi-WiFi system $R_{\text{LiFi-WiFi}}$ is given in (14).

**APPENDIX B**

**DERIVATION OF THE FORMULATION LEMMA 2**

Since $\log_2 \left( \sum \exp (x) \right)$ is a convex function with respect to $x$, we can obtain the upper bound of $R_{\text{LiFi}}$ by Jensen’s inequality.

$$R_{\text{LiFi}} \leq -\frac{1}{\ln 2} - \sum_{k=1}^{M} p_{1,k} \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left\{ \mathbb{E}_{z_2} \left\{ \bar{X}_{k,m} \right\} \right\} \quad \text{(60a)}$$

$$= -\sum_{k=1}^{M} p_{1,k} \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left( 2B_1 \bar{X}_{k,m} - \frac{E_{z_2}(z_1)}{2\sigma_1^2} \right) \quad \text{(60b)}$$

$$= -\frac{1}{\ln 2} - \sum_{k=1}^{M} p_{1,k} \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left( 2B_1 \bar{X}_{k,m} \right) \quad \text{(60c)}$$

The upper bound of the LiFi link $R_{\text{LiFi}}$ with the bandwidth of $B_1$ is given by

$$R_{\text{LiFi}} \leq -2B_1 \sum_{k=1}^{M} p_{1,k} \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left( 2B_1 \bar{X}_{k,m} \right). \quad \text{(61)}$$

Since $\log_2 (x)$ is a concave function, we can obtain the lower bound of $R_{\text{LiFi}}$ by Jensen’s inequality as follows

$$R_{\text{LiFi}} \geq -\frac{1}{2 \ln 2} - \sum_{k=1}^{M} p_{1,k} \log_2 \sum_{m=1}^{M} p_{1,m} \mathbb{E}_{z_2} \left\{ \exp \left( \bar{X}_{k,m} \right) \right\} \quad \text{(62a)}$$

$$= -\frac{1}{2 \ln 2} + \frac{1}{2} \sum_{k=1}^{M} p_{1,k} \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left( B_1 \bar{X}_{k,m} \right). \quad \text{(62b)}$$

The lower bound of the LiFi link $R_{\text{LiFi}}$ with the bandwidth of $B_2$ is given by

$$R_{\text{LiFi}} \geq B_1 - \frac{B_1}{\ln 2} - 2B_1 \sum_{k=1}^{M} p_{1,k} \log_2 \sum_{m=1}^{M} p_{1,m} \exp \left( \bar{X}_{k,m} \right). \quad \text{(63)}$$

**APPENDIX C**

**DERIVATION OF THE FORMULATION LEMMA 3**

The upper bound of $R_{\text{WiFi}}$ is given as

$$R_{\text{WiFi}} \leq -\frac{1}{\ln 2} - \sum_{l=1}^{N} p_{2,l} \log_2 \sum_{n=1}^{N} p_{2,n} \exp \left\{ \mathbb{E}_{z_2} \left\{ \bar{X}_{l,n} \right\} \right\} \quad \text{(64a)}$$

$$= -\sum_{l=1}^{N} p_{2,l} \log_2 \sum_{n=1}^{N} p_{2,n} \exp \left( 2B_2 \bar{X}_{l,n} - \frac{|z_i|^2}{\sigma_2^2} \right) \quad \text{(64b)}$$

$$= -\sum_{l=1}^{N} p_{2,l} \log_2 \sum_{n=1}^{N} p_{2,n} \exp \left( 2B_2 \bar{X}_{l,n} \right). \quad \text{(64c)}$$

Then, the upper bound of the Wi-Fi link $R_{\text{WiFi}}$ with the bandwidth of $B_2$ is given by

$$R_{\text{WiFi}} \leq -B_2 \sum_{l=1}^{N} p_{2,l} \log_2 \sum_{n=1}^{N} p_{2,n} \exp \left( 2B_2 \bar{X}_{l,n} \right). \quad \text{(65)}$$

Since $\log_2 (x)$ is a concave function, the lower bound of $R_{\text{WiFi}}$ can be obtained by Jensen’s inequality as well

$$R_{\text{WiFi}} \geq -\frac{1}{\ln 2} - \sum_{l=1}^{N} p_{2,l} \log_2 \sum_{n=1}^{N} p_{2,n} \frac{1}{\sigma_2^2} \int_{-\infty}^{\infty} \exp \left( -\left( c_R + z_{2,R} \right)^2 + \frac{z_{2,R}^2}{\sigma_2^2} \right) \left( c_l + z_{2,l} \right)^2 + \frac{z_{2,l}^2}{\sigma_2^2} \right) dz_{2,R} dz_{2,l} \quad \text{(66a)}$$

$$= -\frac{1}{\ln 2} + 1 - \sum_{l=1}^{N} p_{2,l} \log_2 \sum_{n=1}^{N} p_{2,n} \exp \left( 2B_2 \bar{X}_{l,n} \right), \quad \text{(66b)}$$

where $z_{2,R} \triangleq \text{Re} (z_{2}), z_{2,l} \triangleq \text{Im} (z_{2}), c_R \triangleq \text{Re} (q_{2}^z q_2 (x_{2,l} - x_{2,n}))$ and $c_l \triangleq \text{Im} (q_{2}^z q_2 (x_{2,l} - x_{2,n}))$; (66a) is true due to $\mathbb{E}_{z_2} \{ f (z_2) \} = \int p_{zz} f (z_2) dz_2$, and $z_2$ follows the complex Gaussian distribution.
Finally, the lower bound of the WiFi link $R_{\text{WiFi}}$ with the bandwidth of $B_2$ is given by

$$R_{\text{WiFi}} \geq B_2 - \frac{B_2}{\ln 2} - B_2 \sum_{l=1}^{N} p_{2,l} \log_2 \left( \sum_{n=1}^{N} p_{2,n} \exp \left( \hat{f}_{l,n} \right) \right).$$  \hspace{1cm} (67)

**REFERENCES**

[1] Ericsson, “Mobility report: On the pulse of the networked society,” 2015.

[2] Y. Wang, X. Wu, and H. Haas, “Load balancing game with shadowing effect for indoor hybrid LiFi/RF networks,” *IEEE Trans. Wireless Commun.*, vol. 16, no. 4, pp. 2366–2378, Apr. 2017.

[3] Y. Wang, D. A. Basnayaka, X. Wu, and H. Haas, “Optimization of load balancing in hybrid LiFi/RF networks,” *IEEE Trans. Commun.*, vol. 65, no. 4, pp. 1708–1720, Apr. 2017.

[4] V. K. Papanikolau, P. D. Diamantoulakis, P. C. Sofotasios, S. Muhaidat, and G. K. Karagiannidis, “On optimal resource allocation for hybrid VLC/RF networks with common backhaul,” *IEEE Trans. on Cogn. Commun. Netw.*, vol. 6, no. 1, pp. 352–365, Mar. 2020.

[5] X. Li, R. Zhang, and L. Hanzo, “Cooperative load balancing in hybrid visible light communications and WiFi,” *IEEE Trans. Commun.*, vol. 63, no. 4, pp. 1319–1329, Apr. 2015.

[6] F. Jin, R. Zhang, and L. Hanzo, “Resource allocation under delay-guarantee constraints for heterogeneous visible-light and RF femtocell,” *IEEE Trans. Wireless Commun.*, vol. 14, no. 2, pp. 1020–1034, Feb. 2015.

[7] Y. Wang and H. Haas, “Dynamic load balancing with handover in hybrid Li-Fi and Wi-Fi networks,” *J. Lightw. Technol.*, vol. 33, no. 22, pp. 4671–4682, Nov. 2015.

[8] X. Wu, M. Safari, and H. Haas, “Access point selection for hybrid Li-Fi and Wi-Fi networks,” *IEEE Trans. Commun.*, vol. 65, no. 12, pp. 5375–5385, Dec. 2017.

[9] D. A. Basnayaka and H. Haas, “Design and analysis of a hybrid radio frequency and visible light communication system,” *IEEE Trans. Commun.*, vol. 65, no. 10, pp. 4334–4347, Oct. 2017.

[10] S. Shao, A. Khreishah, M. Ayyash, M. B. Rahaim, H. Elgala, V. Jungnickel, D. Schulz, T. D. C. Little, J. Hilt, and R. Freund, “Design and analysis of a visible-light-communication enhanced WiFi system,” *IEEE/OSA J. Opt. Commun. Networking*, vol. 7, no. 10, pp. 960–973, Oct. 2015.

[11] M. Hammouda, S. Akin, A. M. Vegni, H. Haas, and J. Peissig, “Link selection in hybrid RF/VLC systems under statistical queuing constraints,” *IEEE Trans. Wireless Commun.*, Apr. 2018.

[12] H. Tabassum and E. Hosain, “Coverage and rate analysis for co-existing RF/VLC downlink cellular networks,” *IEEE Trans. Wireless Commun.*, vol. 17, no. 4, pp. 2588–2601, 2018.

[13] J. Wang, C. Jiang, H. Zhang, X. Zhang, V. C. M. Leung, and L. Hanzo, “Learning-aided network association for hybrid indoor LiFi-WiFi systems,” *IEEE Trans. Veh. Technol.*, vol. 67, no. 4, pp. 3561–3574, Apr. 2018.

[14] H. Zhang, N. Liu, K. Long, J. Cheng, V. C. M. Leung, and L. Hanzo, “Energy efficient subchannel and power allocation for software-defined heterogeneous VLC and RF networks,” *IEEE J. Sel. Areas Commun.*, vol. 36, no. 3, pp. 658–670, Mar. 2018.

[15] S. Ma, F. Zhang, H. Li, F. Zhou, M. S. Alouini, and S. Li, “Aggregated VLC-RF systems: Achievable rates, optimal power allocation, and energy efficiency,” *IEEE Trans. Wireless Commun.*, vol. 19, no. 11, pp. 7265–7278, Nov. 2020.

[16] C. Xiao, Y. R. Zheng, and Z. Ding, “Globally optimal linear precoders for finite alphabet signals over complex vector gaussian channels,” *IEEE Trans. Signal Process.*, vol. 59, no. 7, pp. 3301–3314, Jul. 2011.

[17] S. Ma, R. Yang, Y. He, S. Lu, F. Zhou, N. Al-Dhahir, and S. Li, “Achieving channel capacity of visible light communication,” *IEEE J. Sel. Areas Commun.*, vol. 35, no. 1, pp. 1552–1601, Jun. 2016.

[18] T. M. Cover and J. A. Thomas, *Elements of Information Theory, 2nd ed.*, New York:Wiley, 2006.

[19] A. E. Gamal and Y. H. Kim, *Network Information Theory*, Cambridge:Cambridge Univ. Press, 2011.

[20] D. N. C. Tse and P. Viswanath, *Fundamentals of Wireless Communications*, Cambridge:Cambridge Univ. Press, 2005.

[21] W. Zeng, C. Xiao, and J. Lu, “A low-complexity design of linear precoding for mimo channels with finite-alphabet inputs,” *IEEE Wireless Commun. Lett.*, vol. 1, no. 1, pp. 38–41, Feb. 2012.