Black Hole Radiation in the Brane World and Recoil Effect

Valeri Frolov$^1$ and Dejan Stojković$^1$

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$^1$ Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada T6G 2J1

Abstract

A black hole attached to a brane in a higher dimensional space emitting quanta into the bulk may leave the brane as a result of a recoil. We study this effect. We consider black holes which have a size much smaller than the characteristic size of extra dimensions. Such a black hole can be effectively described as a massive particle with internal degrees of freedom. We consider an interaction of such particles with a scalar massless field and prove that for a special choice of the coupling constant describing the transition of the particle to a state with smaller mass the probability of massless quanta emission takes the form identical to the probability of the black hole emission. Using this model we calculate the probability for a black hole to leave the brane and study its properties. The discussed recoil effect implies that black holes which might be created in interaction of high energy particles in colliders the thermal emission of the formed black hole could be terminated and the energy non-conservation can be observed in the brane experiments.

E-mails: frolov@phys.ualberta.ca, dstojkov@phys.ualberta.ca
1 Introduction

Recently, there has been much interest in the idea that our $(3 + 1)$-dimensional universe is only a sub-manifold on which the standard model fields are confined inside a higher dimensional space (for a review see [1]). The original ADD (Arkani-Hamed, Dimopoulos and Dvali) idea [2] implements extra space as a multi-dimensional compact manifold, so that our universe is a direct product of an ordinary $(3 + 1)$-dimensional FRW (Freedman, Robertson and Walker) universe and an extra space. This construction was primarily motivated by attractive particle physics feature — namely a solution to the hierarchy problem (large difference between the Planck scale, $M_{Pl} \sim 10^{16}$ TeV and the electroweak scale, $M_{EW} \sim 1$ TeV). By allowing only geometrical degrees of freedom to propagate in extra dimensions and making the volume of the extra space large, we can lower a fundamental quantum gravity scale, $M_*$, down to the electroweak scale ($\sim$ TeV). The size of extra dimensional manifold is then limited from above only by short distance gravity experiments (current experiments do not probe any deviations from a four-dimensional Newton’s gravity law on distances smaller than 0.2mm). Thus, for different numbers of extra dimensions compactified on a flat manifold (for an alternative way of compactification see [3, 4]) the compactification radius can vary from the fundamental length scale $M_*^{-1}$ to the macroscopic dimensions of order 0.2mm.

The other option, exercised in [5], uses a non-factorizable geometry with a single extra dimension which can be large or even infinite. If we are interested in solving the hierarchy problem, we use the so-called RSI (Randal, Sundrum) scenario in which we make extra dimension compact by introducing two branes (one with positive and one with negative tension) with a piece of anti-de Sitter space between them. If we put all the standard model fields on the negative tension brane, due to exponential scaling properties of masses in this background, we can solve the hierarchy problem by setting the distance between the two branes only one or two orders of magnitude larger than the anti-de Sitter radius.

Alternatively, we can put all the standard model fields on the positive tension brane and make the extra dimension infinite by moving the negative tension brane to infinity (so called RSII scenario). It is still possible to recover four-dimensional gravity on the positive tension brane at large distances due to a non-trivial warp factor of the anti-de Sitter radius which sets most of the physical volume of the extra space in a narrow region along the brane. This set-up does not say anything about the hierarchy problem, but opens a possibility for new phenomena arising at the energy scales above TeV.

A common feature of all theories with large extra dimension is that a lot of new interesting phenomena can be expected at the energy scale not much above the energy currently available in accelerators. Probably the most interesting and intriguing is the
possibility of production of mini black holes in future collider and cosmic rays experiments. Preliminary calculations [6, 7] indicate that the probability for creation of a mini black hole in near future hadron colliders such as the LHC (Large Hadron Collider) is so high that they can be called “black hole factories”.

It is straightforward to estimate the total geometrical cross section for a production of an \((N+1)\)-dimensional black hole [6, 7]. Consider two particles (partons in the case of the LHC) moving in opposite direction with the center of mass energy \(\sqrt{s}\). If the impact parameter is less than the Schwarzschild radius of a \((N+1)\)-dimensional black hole

\[
R_S = \frac{1}{\sqrt{\pi M_*}} \left[ \frac{M}{M_*} \left( \frac{8\Gamma(N/2)}{N-1} \right) \right]^{\frac{1}{N-2}}
\]  

than a black hole with a mass \(M = \sqrt{s}\) forms. Thus, the total geometrical cross section is

\[
\sigma(M) \approx \pi R_S^2 = \frac{1}{M_*^2} \left[ \frac{M}{M_*} \left( \frac{8\Gamma(N/2)}{N-1} \right) \right]^{\frac{2}{N-2}}.
\]  

Since at the LHC partons carry only a part of the total center of mass energy in a \(pp\) (proton-proton) collision, the total production cross section is estimated as

\[
\frac{d\sigma(pp \rightarrow BH + X)}{dM} = \frac{dL}{dM} \sigma(ab \rightarrow BH)|_{s=M^2}.
\]  

Here, the luminosity \(\frac{dL}{dM}\) is defined as the sum over all the initial parton types

\[
\frac{dL}{dM} = \frac{2M}{s} \sum_{a,b} \int_{M_{min}^2/s}^1 \frac{dx_a}{x_a} f_a(x_a) f_b\left( \frac{M^2_{min}}{s x_a} \right),
\]  

where \(f_i(x_i)\) are the parton distribution functions and \(M_{min}\) is a minimal mass for which this formula is applicable \((M_{min} \sim M_*)\). Numerical estimates for the total production cross section at the LHC give for example the number of \(10^7\) black holes per year if \(M_* = 1\)TeV with the peak luminosity of \(30\)\(fb^{-1}/year\).

The other potential source of mini black holes are ultra high energy cosmic rays which have been observed to interact in the Earth’s atmosphere with center of mass energy

\[\text{We should mention that there is still some debate in literature whether or not the total probability should be suppressed by an exponential pre-factor from semi-classical point of view [8, 9]. For the effect we describe in this paper, this factor does not play any significant role.}\]

\[\text{For high energy scattering of two particles the formation of a rotating black hole is much more probable that the formation of a non-rotating black hole. One may expect that mainly extremely rotating mini black holes are to be formed in such scattering. To simplify calculations of cross section of mini black hole production, effects connected with rotation of the black hole are usually neglected. The same simplification is usually made when quantum decay of mini black holes is discussed. In the present paper we also make such an assumption.}\]
of over 100 TeV. In particular, cosmic neutrinos could produce black hole deep in the atmosphere, which after a rapid decay initiate quasi-horizontal showers far above the standard model rate. Such events could be observed in the Auger Observatory. The neutrino-nucleon scattering cross section is calculated similarly as the $pp$ one:

$$
\sigma(\nu N \rightarrow BH) = \sum_i \int_{M_{min}^2/s}^1 dx \hat{\sigma}_i(xs) f_i(x, Q), \quad (1.5)
$$

where $s = 2m_N E_\nu$ ($m_N$ is the nucleon mass and $E_\nu$ is the neutrino energy), the sum goes over all partons in the nucleon, $f_i$ are parton distribution functions and $Q$ is momentum transfer. Calculations indicate that hundreds of such black holes events may be observed at the Auger Observatory \[10\] before the LHC starts operating.

The differential cross-section and the total probability for a black hole production in colliders and by ultra high energy cosmic rays are very well studied in literature \[6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\].

After the black hole is formed (either at LHC or the Auger Observatory), it decays by emitting Hawking radiation with temperature

$$
T_H = M_* \left[ \frac{M_*}{M} \left( \frac{N - 1}{8\Gamma(\frac{N}{2})} \right) \right]^{\frac{1}{N - 2}} \frac{N - 2}{4\sqrt{\pi}}. \quad (1.6)
$$

After its formation a black hole emits particles. The number of particles emitted is determined by the entropy of a black hole

$$
S = \frac{(N - 2)M}{(N - 1)T_H}. \quad (1.7)
$$

For example, if $M_* = 1$ TeV and $N + 1 = 10$, a 5 TeV and 10 TeV black holes will emit of order 30 and 50 quanta respectively.

Thermal Hawking radiation consists of two parts: (1) particles propagating along the brane, and (2) bulk radiation. The bulk radiation includes bulk gravitons. Usually the bulk radiation is neglected since the total number of species which are living on the brane is quite large ($\sim 60$, see e.g. \[3, 11\]). It should be noted that when the number of extra dimensions is greater than 1 this argument may not work. Really, the number of degrees of freedom of gravitons in the $(N + 1)$-dimensional space-time is $\mathcal{N} = (N + 1)(N - 2)/2$. For example, for $N + 1 = 7$ (3 extra dimensions) $\mathcal{N} = 14$. One may expect that if a black hole is non-rotating, emission of particles with non-zero spin (e.g. gravitons) is suppressed with respect to emission of scalar quanta as it happens in (3 + 1)-dimensional space-time (\[20\], see also Section 10.5 \[23\] and references therein). For extremely rotating black hole the emission of gravitons may be a dominating effect. For example, (3 + 1)-dimensional
numerical calculations done by Don Page \cite{20} (see also \cite{23}) show that the probability of emission of a graviton by an extremely rotating black hole is about 100 times higher than the probability of emission of a photon or neutrino. Since mini black holes created in the high energy scattering are expected to have high angular momentum their bulk radiation may be comparable with (or even dominate) the radiation along the brane.

But even for small number of extra dimensions the role of bulk graviton emission might be important. As a result of the emission of the graviton into the bulk space, the black hole recoil can move the black hole out of the brane. It should be emphasized that even if the probability of the recoil effect is not high it is of virtual importance. The reason is that after the black hole leaves the brane, it cannot emit brane-confined particles anymore. Black hole radiation would be terminated and an observer located on the brane would register the virtual energy non-conservation.

The aim of the present paper is to study this effect.

Don Page was first who observed that a recoil due to Hawking radiation can be very significant for small primordial black holes \cite{24}. For example, a (3 + 1)-dimensional black hole of mass $10^{15} \text{g}$ will have a recoil larger than its Schwarzschild radius after it emits $10^{-13}$ of its mass. We focus our attention on the recoil effect in the framework of theories with large extra dimensions. Since the problem in its complete scope is very complicated we make some simplifying assumptions. First, we assume that the compactification radius of extra dimensions in ADD scenario or anti-de Sitter radius in RS scenario is much larger than the Schwarzschild radius of the black hole, so that we can effectively describe the black hole by a higher dimensional Schwarzschild solution.

Such a black hole can be effectively described as a massive particle with internal degrees of freedom. We characterize these different internal states $I$ by the value of black hole mass $M_I$. Emission of quanta of a bulk field $\varphi$ by the black hole changes its mass and hence provides a transition $I \rightarrow J$ to the lower energy state $J$. To simplify calculations we assume that $\varphi$ is a bulk massless field or a set of such fields, if one wants to include effects connected with the number of degrees of freedom of the bulk gravitons.

In Section 2 we demonstrate that for a special choice of the coupling constant describing the transition of the particle to a state with smaller mass the probability of massless quanta emission takes the form identical to the probability of the black hole emission. To describe the motion of the center of mass of the black hole we use $(N + 1)$-dimensional wave functions.

In such a field-theoretical description of a black hole it is possible to take into account an effect of interaction of the black hole with a brane\footnote{Note that interaction of a black hole with a thin and thick test branes has been studied numerically}. We propose a simple model for
this in Section 3.

In Section 4 we analyze the black hole recoil effects for models with one extra dimension. The generalization to the case of higher number of extra dimensions is straightforward. Namely we calculate the probability of the black hole emission for two cases when the black hole after emission remains on the brane and when it leaves the brane as a result of the recoil.

Section 5 contains discussion of possible consequences of the recoil effect.

2 Field-Theoretical Model of an Evaporating Black Hole

2.1 Probability of emission of a scalar particle with internal degrees of freedom

The center-of-mass motion of black hole of mass $M$ is described by a scalar wave function $\Phi$ with an action

$$W = -\frac{1}{2} \int d^Dx \left[ (\nabla \Phi)^2 + M^2 \Phi^2 \right],$$

and obeying the equation

$$\Box \Phi - M^2 \Phi = 0.$$  (2.2)

Here $D$ is the number of space-time dimensions.

We use the following mode decomposition for the quantum field $\hat{\Phi}$

$$\hat{\Phi}(X^A) = \int \frac{d^N P}{\sqrt{2 \omega_P}} \frac{1}{(2\pi)^{N/2}} \left[ e^{-i\omega_P t + iP_X \hat{A}(P)} + e^{i\omega_P t - iP_X \hat{A}^\dagger(P)} \right].$$  (2.3)

$N \equiv D - 1 = 3 + n$ is the total number of spatial dimensions, $n$ being the number of extra dimensions. The bulk energy is $\omega_P = \sqrt{P^2 + M^2}$.

Later we consider states when there is one particle in given space volume, say $N$-cube of size $L$. Such states are more easily described by using not continuous, but discrete levels. The procedure is well known. By using periodic boundary conditions, instead of waves along the $x$-axis $\exp(iPx)/\sqrt{2\pi}$ one has $\exp(i2\pi n/L)/\sqrt{L}$. In means that the following modification

$$\frac{1}{(2\pi)^{N/2}} \int d^N P \rightarrow \frac{1}{L^{N/2}} \sum_{\{n_1, \ldots, n_L\}}$$

is to be applied to (2.3).

in [25, 26] and nucleation of black holes in the presence of the thick domain wall in [21, 22].
The operators of creation and annihilation for continuous representation (2.3) obey the standard commutation relation

$$\left[ \hat{A}(P), \hat{A}^\dagger(P') \right] = \delta^N(P - P') .$$

One particle states $\left| P \right\rangle = \hat{A}^\dagger(P) |0\rangle$ obey the completeness condition

$$\int \! d^N P |P\rangle \langle P| = \hat{I} ,$$

where $\hat{I}$ is a unit operator.

In a similar manner we write the mode decomposition for the bulk massless scalar field

$$\hat{\varphi}(X^A) = \int \! \frac{d^N K}{\sqrt{2\omega}} \frac{1}{(2\pi)^{N/2}} \left[ e^{-i\omega t + iK X} \hat{a}(K) + e^{i\omega t - iK X} \hat{a}^\dagger(K) \right] .$$

Here $\omega = K = |K|$ is the bulk energy.

We choose the interaction action in the following form

$$W_{\text{int}} = \sum_{I \neq J} \lambda_{IJ} \int \! dX^D \hat{\Phi}_I(X) \hat{\Phi}_J(X) \hat{\varphi}(X) .$$

The amplitude of probability $A_{JK,I}$ of the particle (“black hole”) transition from the initial state $I$ to the final state $J$ with emission of a massless quantum $K$ is

$$A_{JK,I} = i \langle P_J, K | W_{\text{int}} | P_I \rangle = i \lambda_{IJ} \left( \frac{2^{3/2}}{(2\pi)^{N/2 - 1}} \right) (\omega_{P_I} \omega_{P_J} \omega)^{-1/2} \delta^N(P_I - P_J - K) \delta(\omega_{P_I} - \omega_{P_J} - \omega) .$$

We assume that initially black hole is at rest, so that $P_I = 0$ and $\omega_{P_J} = M_I$. The probability for the black hole to emit a quantum with energy $\omega$ per unit time is

$$w(\omega) = \frac{(2\pi)^N}{\Delta t V_N} \sum J \int \! d^N P_J \int \! d\mathbf{n}_K \omega^{N-1} |A_{JK,I}|^2 .$$

Here $V_N$ is the space volume and $\Delta t$ is the total time duration. The factor $\frac{(2\pi)^N}{V_N}$ arises because we consider the initial state when there is only one particle in the volume $V_N$ and hence are to use the discrete normalization for $\text{in}$-states. Since we average over final states no additional factors will appear. As usual the factor $\Delta t V_N$ is canceled by $\delta^{N+1}(0)$ term in $|A_{JK,I}|^2$ so that $w(\omega)$ is finite. We also denoted $\mathbf{n}_K = \mathbf{K}/K$ so that $\int d\mathbf{n}_K$ is the averaging over direction of $\mathbf{K}$ which in our case results in the additional factor

$$\Omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} .$$
equal to the volume of \((N-1)\)-dimensional unit sphere \(S^{N-1}\).

Performing integrations we get

\[
w_I(\omega) = \alpha_N F_I(\omega), \quad \alpha_N = \frac{\Omega_{N-1}}{4(2\pi)^{N-1}},
\]

\[
F_I(\omega) = \frac{\omega^{N-2}}{2M_I^2} \sum J \lambda^2_{IJ} \frac{\delta(M_I - \sqrt{M_I^2 + \omega^2 - \omega})}{\sqrt{M_I^2 + \omega^2}}.
\]

Using relation \(\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|\) where \(f(x_0) = 0\), we can rewrite (2.12) in the form

\[
F_I(\omega) = \frac{\omega^{N-2}}{M_I} \sum J \lambda^2_{IJ} \delta(M_I^2 - (M_I^2 - 2M_I \omega)).
\]

We assume that

\[
M_I^2 - M_J^2 = \epsilon(I - J).
\]

In the limit \(\epsilon \to 0\) one has continuous spectrum for black hole mass. We assume that \(\epsilon \ll T(M)/M\), where \(T(M)\) is the temperature of a black hole of mass \(M\). In this case the discreteness of the levels practically does not affect the “black hole” radiation. Since the mass \(M\) is the only parameter which specify the properties of a black hole we have

\[
\lambda_{IJ} = \sqrt{\epsilon} \Lambda(M_I^2, M_J^2).
\]

We include the factor \(\sqrt{\epsilon}\) into this relation to provide the correct limit to the continuous mass spectrum case.

Changing the summation over the discrete levels \(J\) by integration over \(M_J^2\) one has

\[
F_I(\omega) = \frac{\omega^{N-2}}{M_I} \int dM_J^2 \Lambda^2(M_I^2, M_J^2) \delta(M_J^2 - (M_I^2 - 2M_I \omega)) = \frac{\omega^{N-2}}{M_I} \Lambda^2(M_I^2, M_I^2 - 2M_I \omega).
\]

To summarize, the probability of emission of a massless particle of energy \(\omega\) per unit time by our “back hole” of mass \(M\) is

\[
p(\omega|M) = \frac{\alpha_N \omega^{N-2}}{M} \Lambda^2(M^2, M^2 - 2M \omega).
\]

### 2.2 Black hole radiation

We demonstrate now that for a special choice of the function \(\Lambda\) the probability rate (2.18) coincides with the probability of emission of scalar massless quanta by a black hole of mass \(M\). For this purpose let us consider \((N+1)\)-dimensional non-rotating black hole. Its metric is

\[
ds^2 = -A dt^2 + \frac{dr^2}{A} + r^2 d\Omega^2_{N-1},
\]
The $R_0$ is the length parameter defining the position of the horizon which is related to the mass $M$ of the black hole as

$$M = \frac{(N - 1)\Omega_{N-1}}{16\pi G_*} R_0^{N-2}.$$  \hfill (2.21)

where $G_* = 1/M_*^{N-1}$ is a fundamental gravitational constant determined by a fundamental energy (mass) scale $M_*$. The surface gravity $\kappa$ is

$$\kappa = \frac{1}{2} A' R_0 = \frac{N - 2}{2R_0}.$$  \hfill (2.22)

Thus the Hawking temperature of the black hole is

$$T_H = \frac{\kappa}{2\pi} = \frac{N - 2}{4\pi R_0}.$$  \hfill (2.23)

Consider a scalar massless field $\varphi$ obeying the equation

$$\Box \varphi = 0.$$  \hfill (2.24)

Using spherical harmonics $Y_{lm}$ which are eigenfunctions of the Laplace operator on a unit sphere $S^{N-1}$

$$\Delta_{N-1} Y_{lm} = -l(l + N - 2) Y_{lm}$$  \hfill (2.25)

one can write the mode decomposition of $\varphi$

$$\varphi \sim \sum_l \sum_m \frac{\varphi_{lm}(t,r)}{r^{l/2}} Y_{lm}(\Omega).$$  \hfill (2.26)

Here $m$ denotes a collective index which enumerates the states with given angular momentum $l$. Modes $\varphi_{lm}(t,r)$ obey 2D wave equations

$$(2\Box \varphi_{lm} - V_l) \varphi_{lm} = 0,$$  \hfill (2.27)

with a potential barrier

$$V_l = \frac{l(l + N - 2)}{r^2} + \frac{N - 1}{2} \left[ \frac{N - 3}{2} \frac{A}{r^2} + \frac{A'}{r} \right].$$  \hfill (2.28)

The operator $^2\Box$ is the box-operator in the 2D metric of $(t,r)$ sector.
Let \( r_\ast \) be the tortoise like radial coordinate

\[
r_\ast = \int \frac{dr}{A} ,
\]

and \( \varphi_{lm}(t, r) = \exp(-i\omega t) \varphi_{\omega,lm}(r) \) be a monochromatic wave solution of (2.27) then \( \varphi_{\omega,lm}(r) \) obeys the equation

\[
\left[ \frac{d^2}{dr_\ast^2} + [\omega^2 - AV_l] \right] \varphi_{\omega,lm} = 0 .
\]

This equation can be rewritten in the dimensionless form by using variables \( x = r/R_0 \), \( x_\ast = r_\ast/R_0 \), \( \vartheta = R_0\omega \) as follows

\[
\left[ \frac{d^2}{dx_\ast^2} + [\vartheta^2 - W_l(x)] \right] \varphi_{\omega,lm} = 0 ,
\]

where

\[
W_l(x) = (1 - \frac{1}{x^{N-2}}) \left[ A + \frac{(N-1)^2}{2} \frac{1}{x^{N}} \right] .
\]

\[
A = l(l + N - 2) + \frac{(N-1)(N-3)}{4} .
\]

Denote by \( T_l(\vartheta) \) the amplitude probability to penetrate the potential barrier \( W_l \) for a mode with frequency \( \omega \). Then the greybody factor is defined as

\[
\Gamma(\vartheta) = \sum_{l=0}^{\infty} \Gamma_l(\vartheta) , \quad \Gamma_l(\vartheta) = \sum_m |T_l(\vartheta)|^2 .
\]

The probability of emission of the massless field quanta of energy \( \omega \) by the black hole is (see e.g. [23], section 10.4.4)

\[
P(\omega|M) = \frac{1}{2\pi} \sum_{l=0}^{\infty} \sum_m \frac{(e^{\vartheta\beta} - 1) |T_l(\vartheta)|^2}{(e^{\vartheta\beta} - 1 + |T_l(\vartheta)|^2)^2} .
\]

Here \( \beta \) is the dimensionless inverse temperature

\[
\beta = (TR_0)^{-1} = 4\pi/(N - 2) .
\]

\( s \)-modes (that is modes with \( L = 0 \)) give the main contribution to the Hawking radiation. For this reason we shall keep only \( L = 0 \) contribution in (2.35). Moreover, numerical calculations show that \( |T_0(\vartheta)|^2 \ll e^{\vartheta\beta} - 1 \). For \( N = 4, 5, 6 \) the functions

\[
F(\vartheta) = \frac{\Gamma_0(\vartheta)}{e^{\vartheta\beta} - 1} .
\]
Figure 1: Greybody factor $\Gamma$ as a function of $\varpi = R_0 \omega$ for $N = 4$ (a1), $N = 5$ (a2), and $N = 6$ (a3). The curves representing $\Gamma$ in the vicinity of $\omega = 0$ are well approximated by the curves shown at the same figures. At large values of $\omega$, $\Gamma \to 1$. The figures b1, b2, and b3 show the function $F(\varpi)$ for $N = 4$, $N = 5$, and $N = 6$ respectively. Similar plots for the low energy approximation are also shown.
are shown at Fig. 1.

Under these conditions one has

\[ P(\omega|M) = \frac{1}{2\pi} F(\omega) . \]

For small frequencies one has (see e.g. [27, 28] and references therein)

\[ \Gamma_{ul}(\omega) = \frac{\pi \omega^{N-2}}{2^{N-3} \Gamma^2(N/2)} . \]

Plots presented at Fig. 1 demonstrate that the probability function \( F(\varphi) \) can be reasonably well approximated if one use \( \Gamma_{ul}(\omega) \) instead of the exact value of the grey body factor. In what follows we shall use this low frequency approximation.

Comparing (2.18) with (2.38), in the low frequency approximation, one can conclude that if we want a decaying massive particle \( M \) to emit massless quanta with the same probability as a evaporating black hole one must choose

\[ \Lambda^2(M^2, M^2 - 2M\omega) = \frac{\gamma_N M R_0^{N-2}}{e^{\beta\omega} - 1} , \]

where

\[ \gamma_N = \frac{(2\pi)^{N-1}}{\pi^{N/2} 2^{N-3} \Gamma(N/2)} . \]

Denote

\[ \xi = M^2, \quad \zeta = M^2 - 2M\omega , \]

then we can write

\[ \Lambda^2(\xi, \zeta) = \frac{\gamma_N f(\xi)}{\exp \left[ \frac{\beta R_0(\xi)(\xi - \zeta)}{2\sqrt{\xi}} \right] - 1} , \]

where

\[ f(\xi) = M R_0^{N-2} , \quad R_0(\xi) = \left[ \frac{16\pi G_*}{(N-1)\Omega_{N-1}} \right]^{1/(N-2)} \xi^{1/(2(N-2))} . \]

In the next section we shall use the expression for \( \Lambda^2(\xi, \zeta) \) for the case \( N = 4 \). Calculations give

\[ \Lambda^2(\xi, \zeta) = \frac{2^5}{3} \frac{G_* \xi}{\exp \left[ b \sqrt{G_* (\xi - \zeta)/\xi^{1/4}} \right] - 1} , \]

where \( b = \sqrt{8\pi}/3 \).
3 Interaction of a black hole with a brane

Suppose now that there exists a brane representing our physical world embedded in higher dimensional universe. For simplicity we assume that the brane has a co-dimension 1, i.e. there exist only one extra dimension. In different models it is usual to consider either compactified extra dimensions (see e.g [3]) or to assume that the space out of the brane is anti-de Sitter one [5]). We assume here that the size $R_0$ of a $(N+1)$-dimensional black hole is much smaller than the compactification scale or the anti-de Sitter curvature radius. In this case one can neglect the action of the environment onto the metric of the black hole. Certainly, this is true only for the gravitational field near the black hole. At far distances, the gravitational field of the black hole is modified.

When a black hole is close to the brane or intersect the latter, its gravitational field is modified. Unfortunately till now there is no exact solutions for a black hole on the $(3+1)$-brane (see however [29] where such a solution was found for $(2+1)$-brane). In what follows we study the recoil effect caused by the emission of massless field quanta in the bulk space. As earlier, we neglect the effects connected with the spin and consider emission of the scalar massless field. Moreover we use a simplified model to take into account the effects connected with the interaction of the black hole with the brane. Namely, we neglect the effects of the brane gravitational field on the Hawking radiation, and use one parameter, similar to a chemical potential, to describe the interaction of the black hole with the brane. Below we describe this model.

As earlier we consider the black hole as a point particle with internal degrees of freedom like in the section 2.1 but now we rewrite its action (2.1) as follows

$$W = -\frac{1}{2} \int d^D x \left[ (\nabla \Phi)^2 + U \Phi^2 \right],$$

$$(3.1)$$

$$U = M^2 - 2\mu \delta(y).$$

$$(3.2)$$

The field $\Phi$ obeys the equation

$$\Box \Phi - U \Phi = 0.$$  

$$(3.3)$$

To construct the mode decomposition of the field operator $\hat{\Phi}$ we first consider the following eigenvalue problem

$$\left[ \frac{d^2}{dy^2} + 2\mu \delta(y) \right] \Phi = \lambda \varphi.$$  

$$(3.4)$$

Denote a solution of this equation in the interval of negative and positive values of $y$ as $\Phi^-(y)$ and $\Phi^+(y)$, respectively. Then the junction conditions at $y = 0$ are

$$\Phi^+(0) = \Phi^-(0), \quad \Phi'_+(0) - \Phi'_-(0) = -2\mu \Phi^+(0).$$

$$(3.5)$$
It is easy to see that there is only one level with positive \( \lambda = \mu^2 \). A wave function of the corresponding bound state is

\[
\Phi^{(0)}(y) = \sqrt{\mu} e^{-\mu |y|} .
\] (3.6)

We choose the coefficient so that \( \Phi_0(y) \) obeys the following normalization condition

\[
\int_{-\infty}^{\infty} dy |\Phi_0(y)|^2 = 1.
\] (3.7)

This normalization condition is in agreement with the normalization of fields in the previous section.

For negative \( \lambda \) the spectrum is continuous. We denote \( \lambda = -p^2_\perp \). Solving the scattering problem for the \( \delta \)-like potential one obtains the following set of solutions

\[
\Phi^{(+)}_{p\perp}(y) = \begin{cases} 
  e^{ip_\perp y} - \frac{\mu}{\mu + ip_\perp} e^{-ip_\perp y}, & y < 0, \\
  \frac{ip_\perp}{\mu + ip_\perp} e^{ip_\perp y}, & y > 0,
\end{cases}
\] (3.8)

\[
\Phi^{(-)}_{p\perp}(y) = \begin{cases} 
  \frac{ip_\perp}{\mu + ip_\perp} e^{-ip_\perp y}, & y < 0, \\
  e^{-ip_\perp y} - \frac{\mu}{\mu + ip_\perp} e^{ip_\perp y}, & y > 0.
\end{cases}
\] (3.9)

obeying the normalization conditions

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dy \Phi^{(+)}_{p\perp}(y) \Phi^{(+)*}_{p\perp}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \Phi^{(-)}_{p\perp}(y) \Phi^{(-)*}_{p\perp}(y) = \delta(p_\perp - p'_\perp).
\] (3.10)

The other similar integrals vanish.

The complete set of modes consists of two types of solutions. The first type are solutions describing a black hole attached to the brane which can freely propagate only along it. These solutions are

\[
\Phi^{(0)}_{p\parallel}(X) = \frac{e^{-i\tilde{\omega}t}}{\sqrt{2\tilde{\omega}(2\pi)^{(N-1)/2}}} \Phi_0(y) e^{ip_{\parallel} X_{\parallel}},
\] (3.11)

where

\[
\tilde{\omega}^2 = \tilde{M}^2 + p^2_{\parallel} , \quad \tilde{M}^2 = M^2 - \mu^2.
\] (3.12)

The second type of modes are bulk modes of the form

\[
\Phi^{(\pm)}_p(X) = \frac{e^{-i\omega t}}{\sqrt{2\omega(2\pi)^{N/2}}} \Phi^{(\pm)}_{p\perp}(y) e^{ip_{\parallel} X_{\parallel}},
\] (3.13)

\[
\omega^2 = M^2 + p^2_{\parallel} + p^2_{\perp}.
\] (3.14)
Here and later we use the following notations. \( P_i^A (A = 1, ..., N) \) is total bulk momentum which has components \( p_i^\parallel (i = 1, 2, 3) \) along the brane, and \( p_a^\perp (a = 1, ..., n) \) in the bulk direction. We also denote \( x^i \) coordinated along the brane, and \( y^a \) bulk coordinates. Thus we have

\[
P X = p^\parallel x + p^\perp y.
\] (3.15)

The field operator decomposition takes the form

\[
\hat{\Phi}(X) = \int d^{N-1}p^\parallel \left[ \Phi^{(0)}(X) \hat{B}(p^\parallel) + \Phi^{(0)}(X) \hat{B}^\dagger(p^\parallel) \right] + \sum_{\pm} \int d^{N}P \left[ \Phi^{(\pm)}(X) \hat{A}_{\pm}(P) + \Phi^{(\pm)}(X) \hat{A}_{\pm}^\dagger(P) \right].
\] (3.16)

For the scalar massless field we shall use the decomposition (2.7). This means that we neglect possible interaction of scalar quanta with the brane.

It should be emphasized that in a general case the constant \( \mu \) may depend on \( M \). The form of this dependence can be calculated when one obtains a solution describing a black-hole-brane system. An important example of a 3-dimensional black hole attached to 2-brane was studied in [29]. Unfortunately at the moment no generalization of these solutions to higher dimensions are known. We shall discuss different options of the choice of \( \mu(M) \) in the Section 5.

## 4 Radiation of a black hole attached to the brane and the recoil effect

### 4.1 Matrix elements and probability for leaving the brane

As earlier the amplitude of probability \( A_{JK,I} \) of the particle (“black hole”) transition from the initial state \( I \) to the final state \( J \) with emission of a massless quantum \( K \) is

\[
A_{JK,I}^{off} = i \langle P_J, K | W_{\text{int}} | P_I \rangle,
\] (4.1)

where \( W_{\text{int}} \) is given by (2.8). We choose as the initial state \( |I\rangle \) a state of the black hole at rest at the brane

\[
|I\rangle = \hat{B}^\dagger(p^\parallel = 0) |0\rangle.
\] (4.2)

We are interested in those final states of a black hole when as a result of the recoil it leaves the brane and moves in the bulk space. Thus we choose

\[
|J, K\rangle = \hat{A}^\dagger_{\pm}(P) \hat{a}^\dagger(K)|0\rangle.
\] (4.3)
Simple calculations give

\[ A_{jk,I\pm}^{off} = i\lambda_{jJ} \frac{2^{-3/2}}{(2\pi)^{(N-1)/2}} (\tilde{M}_I \omega_{pj} \omega)^{-1/2} \delta^{N-1}(p_{\|j} + k_{||}) \delta(\tilde{M}_I - \omega_{pj} - \omega_k) C_\pm, \]

where

\[ C_\pm = \int dy \Phi^{(0)}(y) e^{ik_1 y} \phi^{(\pm)}_{p,l,1}(y). \]

Calculating the integral in (4.5) we get

\[ C_+ = \frac{\sqrt{\mu}}{\mu + ip_{\|l}} \left[ \frac{\mu + ip_{\|l}}{\mu + ip_{\|l} + ik_1} - \frac{\mu}{\mu - ip_{\|l} + ik_1} + \frac{ip_{\|l}}{\mu - ip_{\|l} - ik_1} \right], \]

\[ C_- = \frac{\sqrt{\mu}}{\mu + ip_{\|l}} \left[ \frac{\mu + ip_{\|l}}{\mu + ip_{\|l} - ik_1} - \frac{\mu}{\mu - ip_{\|l} - ik_1} + \frac{ip_{\|l}}{\mu - ip_{\|l} + ik_1} \right]. \]

The probability per unit time of the black hole emission which results in the recoil taking the black hole away from the brane is

\[ w_{off} = \frac{(2\pi)^{N-1}}{\Delta t V_{N-1}} \sum_j \int d^N p_j \int dK |A_{jk,I\pm}^{off}|^2. \]

By taking the integral over \( p_{\|} \) one gets

\[ w_{off} = \frac{\Omega_{N-2}}{8 (2\pi)^N \tilde{M}_I} \sum_j |\lambda_{jJ}|^2 \int_0^\infty dp_1 \int_{-\infty}^\infty dk_1 \int_0^\infty dk || k_{N-2}^{||} \delta(\tilde{M}_I - \omega_p - \omega_k) |C|^2, \]

Here

\[ \omega_p = \sqrt{M_3^2 + k_1^2 + p_1^2}, \quad \omega_k = \sqrt{k_{N-2}^{||} + k_1^2}, \]

\[ |C|^2 = |C_+|^2 + |C_-|^2 = \frac{32 \mu^3 k_1^2 p_1^2 (\mu^2 + k_1^2 + p_1^2)}{(\mu^2 + k_1^2 + 2k_1 p_1 + p_1^2)^2 (\mu^2 + k_1^2 - 2k_1 p_1 + p_1^2)^2 (\mu^2 + p_1^2)}. \]

We use the relation (see Section 2.1)

\[ \frac{1}{2} \sum_j |\lambda_{jJ}|^2 \frac{\delta(\tilde{M}_I - \omega_p - \omega_k)}{\omega_p} = \Lambda^2(\tilde{M}_I^2, m^2), \]

where

\[ m^2 = \tilde{M}_I^2 - 2\tilde{M}_I \sqrt{k_1^2 + k_\perp^2} - p_1^2. \]

We omit the subscript “I”, i.e. \( \tilde{M}_I \equiv \tilde{M} \).

Rewrite now (4.3) as follows

\[ w_{off} = \frac{\Omega_{N-2}}{4 (2\pi)^N \tilde{M}} \int_0^\infty dp_1 \int_{-\infty}^\infty dk_1 \int_0^\infty dk || k_{N-2}^{||} \frac{1}{\omega_k} \Lambda^2(\tilde{M}^2, m^2) |C|^2. \]
For $N = 4$ (that is for the $(3+1)$-brane in 5-dimensional space-time) after using (2.45) we have

$$w_{off} = \frac{2}{3\pi^3} G_* \tilde{M} \int_0^{\infty} dp_\perp \int_{-\infty}^{\infty} dk_\perp \int_0^{\infty} \frac{dk_\parallel}{\omega_k} |C|^2 Q, \quad (4.15)$$

$$Q = \frac{1}{\exp \left[ b \sqrt{G_* M} (2 \tilde{M} \sqrt{k_\parallel^2 + k_\perp^2 + k_\perp^2 - p_\perp^2}) \right] - 1}, \quad (4.16)$$

where $b = \sqrt{8\pi/3}$.

One can expect that the main contribution to $w_{off}$ is given by low frequency bulk modes since the higher frequency ones are suppressed by the thermal factor. According to this, we neglect $k_\parallel^2$ and $p_\perp^2$ terms in the exponent with respect to $\tilde{M} k$ term and approximate $Q$ by the expression

$$Q_0(k) = \frac{1}{\exp \left[ (2b \sqrt{G_* M} k) \right] - 1}, \quad (4.17)$$

where $k = \sqrt{k_\parallel^2 + k_\perp^2}$. After this simplification the integral over $p_\perp$ in (1.13) can be easily taken

$$\int_0^{\infty} dp_\perp |C|^2 = \frac{2\pi k_\perp^2 (8\mu^2 + k_\parallel^2)}{(k_\parallel^2 + 4\mu^2)^2}. \quad (4.18)$$

We substitute this expression into (1.13) and using the fact that the integrand is an even function of $k_\parallel$, extend the region of integration over this variable to $(-\infty, \infty)$. Then passing to the polar coordinates $k_{\parallel} = k \sin \phi$, $k_{\perp} = k \cos \phi$ we get

$$w_{off} = \frac{2}{3\pi^2} \tilde{M} \int_0^{\infty} dk k^4 Q_0(k) Z(k), \quad (4.19)$$

where

$$Z(k) = \int_0^{2\pi} d\phi \frac{\sin^2 \phi \cos^2 \phi (k^2 \cos^2 \phi + 8\mu^2)}{(k^2 \cos^2 \phi + 4\mu^2)^2}. \quad (4.20)$$

The integral (4.20) can be taken exactly

$$Z(k) = \frac{\pi (\sqrt{k^2 + 4\mu^2} - 2\mu)}{k^2 \sqrt{k^2 + 4\mu^2}}. \quad (4.21)$$

Thus we obtain the following representation for $w_{off}$

$$w_{off} = \frac{2}{3\pi} G_* \tilde{M} \int_0^{\infty} dk k^2 \frac{(\sqrt{k^2 + 4\mu^2} - 2\mu)}{\sqrt{k^2 + 4\mu^2}} \frac{1}{\exp(a k) - 1}, \quad (4.22)$$

where $a = 2b \sqrt{G_* M} = 4 \sqrt{2\pi/3} \sqrt{G_* M}$. 

17
Finally, we introduce dimensionless variables
\[ \chi = a k, \quad \nu = 2 \mu a = 8 \sqrt{\frac{2\pi}{3}} \mu \sqrt{G_* M}, \]  
(4.23)
and write \( w_{\text{off}} \) in the form
\[ w_{\text{off}}(\nu) = \frac{\mu}{8\pi^2 \nu} \int_0^\infty d\chi \frac{\chi^2 (\sqrt{\chi^2 + \nu^2} - \nu)}{\sqrt{\chi^2 + \nu^2}} \frac{1}{\exp(\chi) - 1}. \]  
(4.24)

4.2 Matrix elements and probability of remaining on the brane

Calculation of the probability that the black hole remains on the brane is analogous to the calculation given in the previous section. The amplitude of probability \( A_{JK,I} \) for the particle ("black hole") to stay on the brane after emitting a massless quantum \( K \) is:
\[ A_{JK,I}^{\text{on}} = i \langle P_J, K | W_{\text{int}} | P_I \rangle, \]  
(4.25)
where \( W_{\text{int}} \) is again given by (2.8). We choose as the initial state \( |I\rangle \) a state of the black hole at rest at the brane
\[ |I\rangle = \hat{B}^\dagger(p_\parallel = 0) |0\rangle. \]  
(4.26)
We are interested in those final states of a black hole when as a result of the recoil it remains on the brane, i.e. does not move to the bulk space. Thus we choose
\[ |J, K\rangle = \hat{B}^\dagger(p_\parallel \neq 0) \hat{a}^\dagger(K) |0\rangle. \]  
(4.27)

Repeating calculations from previous section we get:
\[ A_{JK,I}^{\text{on}} = i \lambda_{IJ} \frac{2^{-3/2}}{(2\pi)^{N/2-1}} (\bar{M}_I \omega_{p_j} \omega)^{-1/2} \delta^{N-1}(p_\parallel + k_\parallel) \delta(\bar{M}_I - \omega_{p_j} - \omega_k) D(k_\perp), \]  
(4.28)
with
\[ D(k_\perp) = \int dy \Phi^{(0)}(y) e^{ik_\perp y} \Phi^{(0)}(y). \]  
(4.29)
Simple integration gives
\[ D(k_\perp) = \frac{4\mu^2}{4\mu^2 + k_\perp^2}. \]  
(4.30)
The probability per unit time of the black hole emission which does not take the black hole away from the brane is
\[ w_{\text{on}} = \frac{(2\pi)^{N-1}}{\Delta t V_{N-1}} \sum_J \int d^N P_J \int dK |A_{JK,I}^{\text{on}}|^2. \]  
(4.31)
Integrating over \( p_\parallel \) we have
\[
\omega_p = \sqrt{M_f^2 + k_\parallel^2}, \quad \omega_k = \sqrt{k_\parallel^2 + k_\perp^2}, \quad p_\perp = 0.
\]

As before, we use the relation
\[
\frac{1}{2} \sum J \left| \lambda_{IJ} \right|^2 \frac{\delta(\tilde{M} - \omega_p - \omega_k)}{\omega_p} = \Lambda^2(\tilde{M}^2, m^2),
\]

where
\[
m^2 = \tilde{M}_I^2 - 2\tilde{M}_I\sqrt{k_\parallel^2 + k_\perp^2} + k_\perp^2.
\]

We omit the subscript "I", i.e. \( \tilde{M}_I \equiv \tilde{M} \).

Now, (4.31) becomes
\[
w_{on} = \frac{\Omega_{N-2}}{8(2\pi)^{N-1} M_I} \sum J \left| \lambda_{IJ} \right|^2 \int_{-\infty}^{\infty} dk_\perp \int_{0}^{\infty} dk_\parallel \frac{k_\parallel^{N-2}}{\omega_k} \frac{\delta(\tilde{M} - \omega_p - \omega_k)}{\omega_p} |D(k_\perp)|^2.
\]

For \( N = 4 \) (that is for the (3+1)-brane in 5-dimensional space-time), using (2.45) we have
\[
w_{on} = \frac{4}{3\sqrt{2}\pi^2} G_* \tilde{M} \int_{-\infty}^{\infty} dk_\perp \int_{0}^{\infty} dk_\parallel \frac{k_\parallel^2}{\omega_k} \Lambda^2(\tilde{M}^2, m^2) |D(k_\perp)|^2 Q,
\]
\[
Q = \frac{1}{\exp \left[ b\sqrt{G_*\tilde{M}}(2\tilde{M}\sqrt{k_\parallel^2 + k_\perp^2} + k_\perp^2) \right] - 1},
\]
where \( b = \sqrt{8\pi/3} \).

Neglecting \( k_\perp^2 \) term in the exponent with respect to \( \tilde{M}k \) term we approximate \( Q \) by the expression
\[
Q_0(k) = \frac{1}{\exp \left[ (2b\sqrt{G_*\tilde{M}k}) \right] - 1}.
\]
where \( k = \sqrt{k_\parallel^2 + k_\perp^2} \). Passing to the polar coordinates \( k_\parallel = k \sin \phi, k_\perp = k \cos \phi \) we get
\[
w_{on} = \frac{2}{3\pi^2} \tilde{M} \int_{0}^{\infty} dk k^2 Q_0(k) Z(k),
\]
where
\[
Z(k) = \int_{0}^{2\pi} d\phi \frac{16\mu^4 \sin^2 \phi}{(k^2 \cos^2 \phi + 4\mu^2)^2}.
\]
The integral (4.41) can be taken exactly

\[ Z(k) = \frac{2\pi\mu}{\sqrt{k^2 + 4\mu^2}}. \] (4.42)

We obtain the final representation for \( w_{on} \) as

\[ w_{on} = \frac{4}{3\pi} G_s \tilde{M} \int_0^\infty dk \frac{k^2 \mu}{\sqrt{k^2 + 4\mu^2}} \frac{1}{\exp(a k) - 1}. \] (4.43)

where \( a = 2b\sqrt{G_s \tilde{M}} = 4\sqrt{\frac{2\pi}{3}} \sqrt{G_s \tilde{M}} \). Introducing dimensionless variables

\[ \chi = ak, \quad \nu = 2\mu a = 8\sqrt{\frac{2\pi}{3}} \sqrt{G_s \tilde{M}}, \] (4.44)

we write \( w_{on} \) in the form

\[ w_{on}(\nu) = \frac{8\mu}{\pi} \int_0^\infty d\chi \frac{\chi^2}{\sqrt{\chi^2 + \nu^2}} \frac{1}{\exp(\chi) - 1}. \] (4.45)

It is easy to see that the sum

\[ w = w_{off}(\nu) + w_{on}(\nu) \] (4.46)

does not depend on \( \nu \). In fact \( w \) coincides with the total probability of emission of a massless field quantum by the 5-dimensional black hole:

\[ w = \int_0^\infty d\omega P(\omega|M), \] (4.47)

where \( P(\omega|M) \) is given by (2.38). The plots of the functions \( w_{off}(\nu)/w \) and \( w_{on}(\nu)/w \) are shown at Fig. 2. Analysis shows that, for large \( \nu \) (or \( \mu \)), the probability \( w_{off} \) falls off as \( 1/\nu \), while for small \( \nu \) (or \( \mu \)) the (normalized) probability \( w_{off} \) goes to 1.

### 4.3 Brane world with \( Z_2 \) symmetry

Here, we repeat the calculations of previous sections for the case when the forms of wave functions are restricted by additional \( Z_2 \) symmetry, like in RS scenarios. By imposing the symmetry under transformation \( y \to -y \), we restrict the wave function of the continuous spectrum (3.8) and (3.9) to a symmetric linear combination

\[ \Phi_{p_{\perp}}^{\text{sym}}(y) = \Phi_{p_{\perp}}^{(+)}(y) + \Phi_{p_{\perp}}^{(-)}(y) = \left( e^{-ip_{\perp}|y|} - \frac{\mu - ip_{\perp}}{\mu + ip_{\perp}} e^{ip_{\perp}|y|} \right). \] (4.48)

Also, the bulk part of the wave function of the emitted massless scalar particle must attain a symmetric form, i.e. \( e^{ik_{\perp}|y|} \).
First, we recalculate the probability for leaving the brane. Since main steps are the same as earlier, we give here only the final result for probability to leave the brane

\[ w^{\text{sym off}} = \frac{4}{3\pi} G_s \tilde{M} \int_0^\infty dk \left( -4\mu \sqrt{k^2 + 4\mu^2 + 8\mu^2 + k^2} \right) \frac{1}{\exp(a k) - 1}. \]  

(4.49)

where \( a = 2b\sqrt{G_s \tilde{M}} = 4\sqrt{\frac{2\pi}{3}} \sqrt{G_s \tilde{M}} \). In dimensionless variables

\[ \chi = ak, \quad \nu = 2\mu a = 8\sqrt{\frac{2\pi}{3}} \mu \sqrt{G_s \tilde{M}}, \]  

(4.50)

this is

\[ w^{\text{sym off}}(\nu) = \frac{1}{4\pi} \int_0^\infty d\chi \left( -\nu \sqrt{\chi^2 + \nu^2} + \nu^2 + \frac{1}{2} \chi^2 \right) \frac{1}{\exp(\chi) - 1}. \]  

(4.51)

Similarly, for the probability to remain on the brane we get

\[ w^{\text{sym on}} = \frac{16}{3\pi} G_s \tilde{M} \int_0^\infty dk \left( \mu \sqrt{k^2 + 4\mu^2 - 2\mu^2} \right) \frac{1}{\exp(a k) - 1}. \]  

(4.52)

where \( a = 2b\sqrt{G_s \tilde{M}} = 4\sqrt{\frac{2\pi}{3}} \sqrt{G_s \tilde{M}} \). In dimensionless variables (4.50) this is

\[ w^{\text{sym on}}(\nu) = \frac{1}{4\pi} \int_0^\infty d\chi \left( \nu \sqrt{\chi^2 + \nu^2 - \nu^2} \right) \frac{1}{\exp(\chi) - 1}. \]  

(4.53)
Once again the sum
\[ w = w_{\text{off}}(\nu) + w_{\text{on}}(\nu) \]  \hspace{1cm} (4.54)
does not depend on \( \nu \) and coincides with the total probability of emission of a massless field quantum by the 5-dimensional black hole.

5 Discussion

In literature, the process of evaporation of mini black holes in brane world models is well studied. The pattern of such a black hole decay is markedly different from any other standard model event. There are even very detailed calculations estimating the energy spectrum and ratio between emitted particles (leptons, fotons, hadrons...)\(^7\). It is also claimed that, because of very little missing energy, the determination of the mass and the temperature of the black hole may lead to a test of Hawking’s radiation. The recoil effect discussed above may change some of these predictions. Certainly the most important observable effect of a black hole recoil is a suddenly disrupted evaporation and local non-conservation of energy.

We developed a phenomenological model for description of the recoil effect. This model contains an important parameter \( \mu \) which play the role of the chemical potential. In the general case, \( \mu \) depends on the mass of a black hole \( M \) and the tension \( \sigma \) of the brane, \( \mu = \mu(M, \sigma) \). Unfortunately, to determine this dependence is not an easy task. One can expect that in the limit \( \sigma \to 0 \), that is for a test brane, \( \mu \to 0 \). In this case, it is very likely that the black hole leaves the brane as soon as emits first quanta with non-zero bulk momentum.

We can also estimate \( \mu \) for small \( \sigma \) as follows. Consider two different states. First, a brane with a black hole of radius \( R_0 \), and the second, when the same black hole is out of the brane. The second configuration has extra energy \( \Delta E \sim \sigma R_0^3 \) (for 1 extra dimension). One can identify \( \Delta E \) with \( M - \tilde{M} \) (compare with (3.11)). This gives
\[ \mu^2 \approx 2M \Delta E, \]  \hspace{1cm} (5.1)
or
\[ \mu \sim \sigma^{1/2} R_0^{5/2} \sim \sigma^{1/2} M_0^{5/4}. \]  \hspace{1cm} (5.2)

In the other limit of infinitely heavy brane, or a brane with \( Z_2 \) symmetry, the process of a black hole leaving the brane reminds to a black hole splitting into two symmetric black holes in the “mirror” space. Classically this process is forbidden in a higher dimensional space-time for the same reason as in \((3 + 1)\)-dimensional space-time in connection with non-decreasing property of the entropy. In the presence of cosmological constant, such an
effect may become possible as a tunneling process. These arguments show that in this case \( \mu \to \infty \) or is exponentially large, and the recoil effect is suppressed. This feature could help in distinguishing between the two different scenarios of extra dimensions — ADD and RS. Also, if the general conclusion that the recoil effect may be important for branes of small tension is correct, it opens an interesting possibility of using the experiments with decay of mini black holes to put restrictions on the brane tension.

The above order of magnitude estimates of the parameter \( \mu \) do not follow from the exact calculations and therefore might not be very reliable\(^4\). In any case further investigation is required, which will be the topic of eventual further publications.

We note that the results from section 2, about the field theoretical model describing a black hole as a point radiator, are quite general and are not restricted to brane world models.

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\(^4\)Our attention was drown by Don Page who pointed out that more detailed analysis shows that the estimate (5.1) is quite valid in the case of the brane of a co-dimension 3 and higher (three or more extra spatial dimensions). For co-dimensions 1 and 2 result might depend on a regime in which we extract the black hole, i.e. whether it is an adiabatic process or not.
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