Singularity Formation in the Inviscid Burgers Equation

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Abstract: We provide a lower bound for the blow up time of the $H^2$ norm of the entropy solutions of the inviscid Burgers equation in terms of the $H^2$ norm of the initial datum. This shows an interesting symmetry of the Burgers equation: the invariance of the space $H^2$ under the action of such nonlinear equation. The argument is based on a priori estimates of energy and stability type for the (viscous) Burgers equation.

Keywords: existence; uniqueness; stability; Burgers equation; Cauchy problem

MSC: 35G25; 35K55

1. Introduction

Consider the Cauchy problem for the inviscid Burgers equation:

$$\begin{cases}
\partial_t u + u \partial_x u = 0, & 0 < t < T, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases} \tag{1}$$

and on the initial datum assume

$$u_0 \in H^2(\mathbb{R}), \ u_0 \neq 0. \tag{2}$$

Following the classical Kružkov approach [1], we use the following definition of solution.

Definition 1. A function $u : [0, T) \times \mathbb{R} \to \mathbb{R}$ is an entropy solution of (1) if

$$u \in L^\infty_{loc}((0, T) \times \mathbb{R})$$

and for every constant $c \in \mathbb{R}$ and every nonnegative test function $\varphi \in C^\infty((-\infty, T) \times \mathbb{R})$ with compact support

$$\int_0^T \int_{\mathbb{R}} \left( |u - c| \varphi_t + \text{sign}(u - c) \frac{u^2 - c^2}{2} \varphi_x \right) dt dx + \int_{\mathbb{R}} |u_0(x) - c| \varphi(0, x) dx \geq 0.$$

We know that, independently of the regularity assumptions on the initial datum, such as (2), the entropy solutions are unique and may develop a discontinuity in finite time (see [2–4]). On the other hand, the existence of smooth solutions of (1) is guaranteed for a short time by the Cauchy-Kovalevskaya Theorem [5]. Several papers on the development of singularities of nonlinear hyperbolic equations are available in the literature on this topic starting from the classical ones [6–8] and arriving at the more modern ones like [9–11] where tools of complex analysis are used and [12] where the time derivative is replaced by the Caputo one.

A very detailed blow-up analysis on the solutions of (1) can be carried using the characteristic lines of the equation for $\partial_x u$ that is
\[ \partial_{x}^{2} u + u \partial_{t}^{2} u + (\partial_{x} u)^{2} = 0. \]

Indeed, in ([2] Example 1.4), choosing
\[ u_0(x) = \frac{1}{1 + x^2}, \]
the author shows that the characteristic lines cross at time \( T = 8/\sqrt{27} \) causing the creation of a shock.

A more refined tool that can be used for the analysis of the geometric structure and the large-time behavior of the solution of (1) is the Hopf-Cole [13–15] transformation that turns the (inviscid) Burgers equation into the linear heat equation. It provides a simple geometric construction of the solution, and allows to conclude that

- the initial information \( u_0(x_0) \) propagates along the characteristic line \( x = u_0(x_0)t + x_0; \)
- the initial information that reaches a given location \((t, x)\) depends on the global minimum of the function
\[ F(t, x, y) = \frac{(x - y)^2}{2t} + \int_0^y u_0(\xi) d\xi; \]
- a shock wave in (1) is experienced when the minimum of \( F \) is attained in more than one location.

Here, we provide a lower bound for the maximal time \( T \) of existence of an \( H^2 \) solution of (1) in terms of the \( H^2 \) norm of the initial datum. This result shows also a symmetry of the Burgers equations: the invariance of the space \( H^2 \) under the action of that nonlinear equation in the time interval \([0, T]\). In addition, we also prove that as soon as the solution stays in \( H^2 \) it is also \( L^2 \) stable. It is interesting to note that, while the \( L^1 \) distance between entropy solutions of (1) is non-increasing, that is not the case for the \( L^2 \) distance. Our arguments are based on energy and stability estimates on the (viscous) Burgers equation.

The main result of this paper is the following theorem:

**Theorem 1.** Assume (2) and
\[ T < \frac{1}{3} \log(A_0), \quad \text{or} \quad \|\partial_{x} u_0\|_{H^1(\mathbb{R})}^2 < \frac{1}{e^3 - 1}, \quad (3) \]
where
\[ A_0 = \frac{\|\partial_{x} u_0\|_{H^1(\mathbb{R})}^2 + 1}{\|\partial_{x} u_0\|_{H^1(\mathbb{R})}^2}. \]

There, the unique entropy solution \( u \) of (1) in the sense of Definition 1 satisfies
\[ u \in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^2(\mathbb{R})) \cap W^{1,\infty}((0, T) \times \mathbb{R}), \]
\[ \partial_{t} \partial_{x} u \in L^2(\mathbb{R}), \quad \text{for every} \ 0 \leq t \leq T. \quad (4) \]

Moreover, if \( u_1 \) and \( u_2 \) are the two entropy solutions of (1) obtained in correspondence of the initial data \( u_{1,0} \) and \( u_{2,0} \) satisfying (3), the following stability estimate holds
\[ \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq e^{Ct} \|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})}^2, \quad (5) \]
for every \( 0 \leq t < T \), and for some positive constant \( C \).
Our argument is based on a priori estimates on the smooth solution \( u_\varepsilon \) of the (viscous) Burgers equation of energy and stability type (see \([16–18]\)):

\[
\begin{aligned}
\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon &= \varepsilon \partial_x^2 u_\varepsilon, & 0 < t < T, & \quad x \in \mathbb{R}, \\
\varepsilon \partial_x u_\varepsilon(0, x) &= \varepsilon u_\varepsilon(0, x), & x \in \mathbb{R},
\end{aligned}
\]

(6)

where \( 0 < \varepsilon < 1 \) and \( u_{\varepsilon,0} \) is an analytic approximation of \( u_0 \), such that

\[
\|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \leq \|u_0\|_{H^2(\mathbb{R})}, \quad \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}, \quad \varepsilon \|\partial_x^3 u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 \leq C_0,
\]

(7)

where \( C_0 \) an a positive constant independent of \( \varepsilon \). Indeed we know that \([1,19]\)

\[
u_\varepsilon \to u \quad \text{a.e. and in every } L^p_{loc}((0,\infty) \times \mathbb{R}), \ 1 \leq p < \infty \text{ ad } \varepsilon \to 0.
\]

(8)

Thanks to the uniqueness of the entropy solution \( u \) of (1) the limit is taken along all the family \( \varepsilon \to 0 \) and not only merely along a subsequence.

We obtain the regularity stated in (4), proving several energy estimates of \( u_\varepsilon \) and using (4). Those are also the key tool for the proof of the stability estimate (5).

The proof of Theorem 1 is given in the next section.

2. Proof of Theorem 1

In what follows, we denote with \( C \) all the positive constants independent on \( \varepsilon \) and \( t \).

The key tool in our argument is a precise analysis on the blow-up of the \( H^1 \) norm of the solution \( u_\varepsilon \) of (6). For the sake of readability we state and prove an \( L^\infty \) and \( L^2 \) estimates on \( u_\varepsilon \), that do not need assumption (3) (see \([1]\)).

**Lemma 1.** Assume that (7) holds. For each \( 0 < \varepsilon < 1 \), we have that

\[
\|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})},
\]

(9)

for every \( 0 \leq t \leq T \).

**Proof.** Let \( 0 \leq t \leq T \). Observe that the constant maps \( \|u_0\|_{L^\infty(\mathbb{R})} \) and \(-\|u_0\|_{L^\infty(\mathbb{R})} \) solve the equation in (6). Therefore, is consequence of the comparison principle for parabolic equation and of the fact that (see (7))

\[
-\|u_0\|_{L^\infty(\mathbb{R})} \leq u_{0,\varepsilon} \leq \|u_0\|_{L^\infty(\mathbb{R})},
\]

that is (9). \( \Box \)

Directly multiplying (6) by \( u_\varepsilon \) we get the following \( L^2 \) estimate.

**Lemma 2.** Assume that (7) holds. We have that

\[
\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq \|u_0\|_{L^2(\mathbb{R})}^2,
\]

(10)

for every \( 0 \leq t \leq T \).

**Proof.** Let \( 0 \leq t \leq T \). Multiplying (6) by \( 2u_\varepsilon \), an integration on \( \mathbb{R} \) gives

\[
\frac{d}{dt}\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 2\int_{\mathbb{R}} u_\varepsilon \partial_x u \, dx
\]

\[
= -2\int_{\mathbb{R}} \varepsilon \partial_x^2 u \, dx + 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \, dx = -2\varepsilon\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]
Therefore, we have

\[ \frac{d}{dt} \| u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2\varepsilon \| \partial_x u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} = 0. \]

Integrating on \((0, t)\), thanks to (7), we have (10).

In order to prove the invariance of the space \( H^2 \), we need to prove an a priori estimate on the \( H^2 \)-norm of \( u_\varepsilon \) on a time interval independent on \( \varepsilon \). As a first step in that direction we prove the following estimate that holds for any \( t \geq 0 \) and does not requires (3).

**Lemma 3.** Assume that (7) holds. We have that

\[ \frac{\| \partial_x u_\varepsilon(t, \cdot) \|^2_{H^1(\mathbb{R})}}{\| \partial_x u_\varepsilon(t, \cdot) \|^2_{H^1(\mathbb{R})} + 1} \leq \frac{\| \partial_x u_0 \|^2_{H^1(\mathbb{R})} e^{3t}}{\| \partial_x u_0 \|^2_{H^1(\mathbb{R})} + 1}, \tag{11} \]

for every \( t \geq 0 \).

**Proof.** Multiplying (6) by \(-2\partial_x^2 u_\varepsilon + 2\partial_x^4 u_\varepsilon\), an integration on \( \mathbb{R} \) gives

\[
\begin{align*}
\frac{d}{dt} \left( \| \partial_x u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
= -2 \int_\mathbb{R} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx + 2 \int_\mathbb{R} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon dx \\
= 2 \int_\mathbb{R} \partial_x u_\varepsilon \partial_x \partial_x^2 u_\varepsilon dx - 2 \int_\mathbb{R} \partial_x u_\varepsilon \partial_x \partial_x^4 u_\varepsilon dx - 2\varepsilon \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_\mathbb{R} \partial_x^2 u_\varepsilon \partial_x^2 u_\varepsilon dx \\
= \int_\mathbb{R} (\partial_x u_\varepsilon)^3 dx - 5 \int_\mathbb{R} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx - 2\varepsilon \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} - 2\varepsilon \| \partial_x^4 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})}.
\end{align*}
\]

Therefore, we have that

\[
\begin{align*}
\frac{d}{dt} \left( \| \partial_x u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
+ 2\varepsilon \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2\varepsilon \| \partial_x^4 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
= - \int_\mathbb{R} (\partial_x u_\varepsilon)^3 dx - 5 \int_\mathbb{R} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx \\
\leq \| \partial_x u_\varepsilon(t, \cdot) \|_{L^\infty(\mathbb{R})} \| \partial_x u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} \| \partial_x^2 u_\varepsilon(t, \cdot) \|_{L^2(\mathbb{R})} + 5 \| \partial_x u_\varepsilon(t, \cdot) \|_{L^\infty(\mathbb{R})} \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
\leq 6 \| \partial_x u_\varepsilon(t, \cdot) \|_{L^\infty(\mathbb{R})} \left( \| \partial_x u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} \right).
\end{align*}
\]

Define the following function

\[ X_\varepsilon(t) := \| \partial_x u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} = \| \partial_x u_\varepsilon(t, \cdot) \|^2_{H^1(\mathbb{R})}. \tag{13} \]

Therefore, by (12),

\[
\frac{dX_\varepsilon(t)}{dt} + 2\varepsilon \| \partial_x^2 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2\varepsilon \| \partial_x^4 u_\varepsilon(t, \cdot) \|^2_{L^2(\mathbb{R})} \leq 6 \| \partial_x u_\varepsilon(t, \cdot) \|_{L^\infty(\mathbb{R})} X_\varepsilon(t). \tag{14}
\]
Due to the Young inequality,
\[ 6\|\partial_x u_e(t, \cdot)\|_{L^\infty(R)}^2 X_e(t) \leq 3\|\partial_x u_e(t, \cdot)\|_{L^\infty(R)}^2 + 3X_e^2(t). \]

It follows from (14) that
\[
\frac{dX_e(t)}{dt} + 2\|\partial_x^2 u_e(t, \cdot)\|_{L^2(R)}^2 + 2\|\partial_x^3 u_e(t, \cdot)\|_{L^2(R)}^2 \leq 3\|\partial_x u_e(t, \cdot)\|_{L^\infty(R)}^2 + 3X_e^2(t). \quad (15)
\]

Thanks to the Hölder inequality,
\[
(\partial_x u_e(t, x))^2 \leq 2 \int_{-\infty}^\infty |\partial_x u_e| |\partial_x^2 u_e| dy \leq 2 \int_{R} |\partial_x u_e| |\partial_x^2 u_e| \, dx 
\leq 2\|\partial_x u_e(t, \cdot)\|_{L^2(R)} \|\partial_x^2 u_e(t, \cdot)\|_{L^2(R)}.
\]

Hence,
\[
\|\partial_x u_e(t, \cdot)\|_{L^\infty(R)}^2 \leq 2\|\partial_x u_e(t, \cdot)\|_{L^2(R)} \|\partial_x^2 u_e(t, \cdot)\|_{L^2(R)}.
\]

Due to (13) and the Young inequality,
\[
\|\partial_x u_e(t, \cdot)\|_{L^\infty(R)}^2 \leq \|\partial_x u_e(t, \cdot)\|_{L^2(R)}^2 + \|\partial_x^2 u_e(t, \cdot)\|_{L^2(R)}^2 = X_e(t). \quad (16)
\]

Consequently, by (15),
\[
\frac{dX_e(t)}{dt} + 2\|\partial_x^2 u_e(t, \cdot)\|_{L^2(R)}^2 + 2\|\partial_x^3 u_e(t, \cdot)\|_{L^2(R)}^2 \leq 3X_e(t)(X_e(t) + 1).
\]

Then,
\[
\frac{dX_e(t)}{dt} \leq 3X_e(t)(X_e(t) + 1), \quad (17)
\]

by (13),
\[
\frac{1}{X_e(t)(X_e(t) + 1)} \frac{dX_e(t)}{dt} \leq 3.
\]

Integrating on \((0, t)\), we get
\[
\log \left( \frac{X_e(t)}{X_e(t) + 1} \right) - \log \left( \frac{X_{0,e}}{X_{0,e} + 1} \right) \leq 3t.
\]

Hence, thanks to (13),
\[
\log \left( \frac{\|\partial_x u_e(t, \cdot)\|_{H^1(R)}^2}{\|\partial_x u_e(t, \cdot)\|_{H^1(R)}^2 + 1} \right) \leq \log \left( \frac{\|\partial_x u_{0,e}\|_{H^1(R)}^2}{\|\partial_x u_{0,e}\|_{H^1(R)}^2 + 1} \right) + 3t,
\]

which gives
\[
\frac{\|\partial_x u_e(t, \cdot)\|_{H^1(R)}^2}{\|\partial_x u_e(t, \cdot)\|_{H^1(R)}^2 + 1} \leq \frac{\|\partial_x u_{0,e}\|_{H^1(R)}^2 e^{3t}}{\|\partial_x u_{0,e}\|_{H^1(R)}^2 + 1}.
\quad (18)
\]

Using (7) in (18) we gain (11). \(\Box\)

Using the assumption (3) on the time \(T\), we are able to deduce for the previous general estimate the boundedness of the family \(\{u_e\}_e\) in the spaces \(L^\infty(0, T; H^2(R))\) and \(L^\infty(0, T; W^{1,\infty}(R))\). This is the core of our argument because it allows us to find that the entropy solution \(u\) of (1) lives in the same spaces.
Lemma 4. Assume that (3) and (7) hold. We have that
\[
\frac{-1}{\|\partial_x u_\epsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2} \leq \frac{-1}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2} + \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2} + 1 (e^{3\epsilon} - 1),
\]
(19)
\[
\|\partial_x u_\epsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3\epsilon T}}{1 - \|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2 e^{3\epsilon T} - 1},
\]
(20)
for every $0 \leq t \leq T$. In particular, we have
\[
\|\partial_x u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C,
\]
(21)
\[
\|\partial_x^2 u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C,
\]
(22)
\[
\|\partial_x u_\epsilon\|_{L^\infty(0,T) \times \mathbb{R}} \leq C,
\]
(23)
for every $0 \leq t \leq T$ and some constant $C > 0$, that depends on $u_0$ and $T$ but not on $\epsilon$.

Proof. We begin by observing that, arguing as in Lemma 3, we have (17). Therefore, thanks to (11), (13) and (17),
\[
\frac{dX_\epsilon(t)}{dt} \leq \frac{3}{6} X_\epsilon(t)(X_\epsilon(t) + 1)
\]
\[
= \frac{3X_\epsilon(t)}{X_\epsilon(t) + 1} (X_\epsilon(t) + 1)^2 \leq \frac{3\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3\epsilon}}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2} (X_\epsilon(t) + 1)^2.
\]
Therefore,
\[
\frac{1}{(X_\epsilon(t) + 1)^2} \frac{dX_\epsilon(t)}{dt} \leq \frac{3\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3\epsilon}}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2}.
\]
Integrating on $(0, t)$, we have that
\[
\frac{-1}{X_\epsilon(t) + 1} + \frac{1}{X_\epsilon(t) + 1} \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2} (e^{3\epsilon} - 1).
\]
Thanks to (13), we get
\[
\frac{-1}{\|\partial_x u_\epsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2} \leq \frac{-1}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2} + \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2} + 1 (e^{3\epsilon} - 1),
\]
which gives (19).

We conclude by proving (20). Thanks to (19), we have
\[
\frac{-1}{\|\partial_x u_\epsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{-1}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2 + 1} + \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2} + 1 (e^{3\epsilon} - 1).
\]
Consequently, we obtain that
\[
\frac{\left[1 - \|\partial_x u_\epsilon\|_{H^1(\mathbb{R})}^2 (e^{3\epsilon} - 1)\right]}{\|\partial_x u_\epsilon\|_{H^1(\mathbb{R})}^2} \leq \frac{1}{\|\partial_x u_\epsilon,0\|_{H^1(\mathbb{R})}^2 + 1},
\]
(24)
Multiplying (24) by $\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2$, we have that
\[
\frac{1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)}{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 + 1} \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 \leq \|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2.
\]
This gives (20).

Proof. Let 0
\[
\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3T}.
\]
It follows from (25) and (26) that
\[
\frac{1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)}{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 + 1} \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 \leq \|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2.
\]
that is
\[
1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1) \leq \|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2.
\]
Thanks to (3), we have that
\[
1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1) > 0.
\]
Therefore, by (27) and (28), we get
\[
\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3T}}{1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)},
\]
which gives (20).

Finally, (21)–(23) follows from (13), (16) and (20), respectively.

In order to prove the $L^2$-regularity of $\partial_t u$ we need the following energy estimate on the $H^1$ norm of $u_\varepsilon$.

**Lemma 5.** Assume that (3) and (7) hold. We have that
\[
\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x^2 u_\varepsilon(s, \cdot)\|_{H^1(\mathbb{R})}^2 \, ds \leq C,
\]
for every $0 \leq t \leq T$ and some constant $C > 0$, that depends on $u_0$ and $T$ but not on $\varepsilon$.

**Proof.** Let $0 \leq t \leq T$, where $T$ is defined in (3). Arguing as in Lemma 3, we (12). Therefore, by (12), (21)–(23), we get
\[
\frac{d}{dt} \left( \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right)
\]
\[
+ 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]
\[
\leq 6(\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \leq C.
\]
Integrating on $(0, t)$, by (7), we have
\[
\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]
\[ +2\epsilon \int_0^t \left\| \partial_s^2 u_e(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + 2\epsilon \int_0^t \left\| \partial_t^2 u_e(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \]
\[ \leq \|u_0\|_{H^2(\mathbb{R})} + Ct \leq C, \]

which gives (29). \(\square\)

We complete the proof of the \(H^1\) regularity of \(u\) proving the following bound \(L^2\) on \(\partial_t u_e\).

**Lemma 6.** Assume that (3) and (7) hold. We have that
\[ \left\| \partial_t u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C, \]
for every \(0 \leq t \leq T\) and some constant \(C > 0\), that depends on \(u_0\) and \(T\) but not on \(\epsilon\).

**Proof.** Let \(0 \leq t \leq T\). We begin by observing that, by (6),
\[ (\partial_t u_e)^2 = \left( -u_e \partial_x u_e + \epsilon \partial_x^2 u_e \right)^2 = u_e^2 \left( \partial_x u_e \right)^2 - 2\epsilon u_e \partial_x u_e \partial_x^2 u_e + \epsilon^2 \left( \partial_x^2 u_e \right)^2. \] (31)

Since
\[ -2\epsilon \int_{\mathbb{R}} u_e \partial_x u_e \partial_x^2 u_e dx = \epsilon \int_{\mathbb{R}} (\partial_x u_e)^3 dx, \]
integrating (31) on \(\mathbb{R}\), we have
\[ \left\| \partial_t u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u_e^2 (\partial_x u_e)^2 dx + \epsilon \int_{\mathbb{R}} (\partial_x u_e)^3 dx + \epsilon^2 \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \] (32)

Since \(0 < \epsilon < 1\), due to (9), (21) and (22),
\[ \int_{\mathbb{R}} u_e^2 (\partial_x u_e)^2 dx \leq \|u_0\|_{L^\infty((0,T) \times \mathbb{R})}^2 \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C, \]
\[ \epsilon \int_{\mathbb{R}} (\partial_x u_e)^3 dx \leq \epsilon \left\| \partial_x u_e(t, \cdot) \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C, \]
\[ \epsilon^2 \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C. \]

Hence, by (32),
\[ \left\| \partial_t u_e(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C, \]
which gives (30). \(\square\)

In order to prove the \(L^2\)-regularity of \(\partial_x^2 u\), we need the following energy estimate on the \(H^1\) norm of \(u_e\).

**Lemma 7.** Assume that (3) and (7) hold. We have that
\[ \epsilon \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\epsilon^2 \int_0^t \left\| \partial_t^2 u_e(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C, \]
for every \(0 \leq t \leq T\) and some constant \(C > 0\), that depends on \(u_0\) and \(T\) but not on \(\epsilon\).

**Proof.** Let \(0 \leq t \leq T\). Multiplying (6), by \(-2\epsilon \partial_x^2 u_e\), we have that
\[ -2\epsilon \int_{\mathbb{R}} \partial_x^2 u_e \partial_t u_e dx = \frac{d}{dt} \left\| \partial_x^2 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ = 2\epsilon \int_{\mathbb{R}} u_e \partial_x u_e \partial_x^2 u_e dx + 2\epsilon^2 \int_{\mathbb{R}} \partial_x^2 u_e \partial_x^2 u_e dx \]
\[ = -2\epsilon \int_{\mathbb{R}} (\partial_x u_e)^2 \partial_x^2 u_e dx - 2\epsilon \int_{\mathbb{R}} u_e \partial_x^2 u_e \partial_x^2 u_e dx - 2\epsilon^2 \int_{\mathbb{R}} \partial_x^2 u_e \partial_x^2 u_e dx \]
Lemma 8. Assume that (3) and (7) hold. We have that

\[ \| \partial_t \partial_x u_e (t, \cdot) \|_{L^2(\mathbb{R})} \leq C, \]

\[ \| \partial_t u_e \|_{L^{\infty}(0,T) \times \mathbb{R}} \leq C, \]

for every \( 0 \leq t \leq T \) and some constant \( C > 0 \), that depends on \( u_0 \) and \( T \) but not on \( \varepsilon \).

**Proof.** Let \( 0 \leq t \leq T \), where \( T \) is defined in (3). Multiplying (6) by \(-2 \partial_t \partial_x^2 u_e \), we have that

\[ -2 \int_\mathbb{R} \partial_t \partial_x^2 u_e \partial_t u_e dx = 2 \int_\mathbb{R} u_e \partial_x u_e \partial_t \partial_x^2 u_e dx - 2\varepsilon \int_\mathbb{R} \partial_x^2 u_e \partial_t \partial_x u_e dx. \]  

(37)

Since

\[ -2 \int_\mathbb{R} \partial_t \partial_x^2 u_e \partial_t u_e dx = 2 \| \partial_t \partial_x u_e (t, \cdot) \|_{L^2(\mathbb{R})}^2, \]

\[ 2 \int_\mathbb{R} u_e \partial_x u_e \partial_t \partial_x^2 u_e = -2 \int_\mathbb{R} (\partial_x u_e)^2 \partial_t \partial_x u_e dx - 2 \int_\mathbb{R} u_e \partial_x^2 u_e \partial_t \partial_x u_e dx, \]

\[ 2\varepsilon \int_\mathbb{R} \partial_x^2 u_e \partial_t \partial_x^2 u_e dx = -2\varepsilon \int_\mathbb{R} \partial_x^2 u_e \partial_t \partial_x u_e dx, \]

an integration of (37) on \( \mathbb{R} \) gives

\[ 2 \| \partial_t \partial_x u_e (t, \cdot) \|_{L^2(\mathbb{R})}^2 = -2 \int_\mathbb{R} (\partial_x u_e)^2 \partial_t \partial_x u_e dx - 2 \int_\mathbb{R} u_e \partial_x^2 u_e \partial_t \partial_x u_e dx \]

\[ - 2\varepsilon \int_\mathbb{R} \partial_x^2 u_e \partial_t \partial_x u_e dx. \]  

(38)
Since \(0 < \varepsilon < 1\), thanks to (9), (21)–(23), (33) and the Young inequality,
\[
2 \int_{\mathbb{R}} (\partial_x u_t)^2 |\partial_1 \partial_x u_e| \, dx 
\leq 2C \int_{\mathbb{R}} |\partial_x u_e| |\partial_1 \partial_x u_e| \, dx 
\leq 2 \left[ \max_{t \in (0, T)} \left\| \partial_1 u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \left\| \partial_2 \partial_x u_e(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right],
\]
which gives (35).

We are ready for the proof of Theorem 1.

Proof of Theorem 1. The convergence stated in (8) and the bounds proved in the previous lemmas gives the regularity on \(u\) claimed in (4).

We prove (5). Let \(u_1\) and \(u_2\) be two solutions of (1), which verify (4), that is
\[
\begin{align*}
&\partial_t u_1 + \partial_x u_1 = 0, \quad 0 < t < T, \; x \in \mathbb{R}, \\
&u_1(0, x) = u_{1,0}(x), \quad x \in \mathbb{R}, \\
&\partial_t u_2 + \partial_x u_2 = 0, \quad 0 < t < T, \; x \in \mathbb{R}, \\
&u_2(0, x) = u_{2,0}(x), \quad x \in \mathbb{R}.
\end{align*}
\]

Then, the function
\[
\omega = u_1 - u_2
\]
is the solution of the following Cauchy problem:
\[
\begin{align*}
&\partial_t \omega + \partial_x \partial_x \omega + \partial_x u_2 \omega = 0, \quad 0 < t < T, \; x \in \mathbb{R}, \\
&\omega(0, x) = u_{1,0}(x) - u_{2,0}(x), \quad x \in \mathbb{R}.
\end{align*}
\]
Multiplying (40) by $2\omega$, an integration on $\mathbb{R}$ gives
\[
\frac{d}{dt} \| \partial_t \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 = -2 \int_{\mathbb{R}} u_1 \omega \partial_t \omega dx - 2 \int_{\mathbb{R}} \partial_x u_1 \omega^2 dx - 2 \int_{\mathbb{R}} \partial_x u_2 \omega^2 dx.
\]
(41)

Since $u_1, u_2 \in H^2(\mathbb{R})$ for every $0 \leq t \leq T$
\[
\| \partial_x u_1 \|_{L^\infty((0,T) \times \mathbb{R})} \| \partial_x u_2 \|_{L^\infty((0,T) \times \mathbb{R})} \leq C.
\]
(42)

Therefore, by (41) and (42),
\[
\frac{d}{dt} \| \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C \| \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]

The Gronwall Lemma and (39) give (5).

3. Discussion

The goal of this paper is to investigate the blow-up of the $H^2$ norm of the solution of the (inviscid) Burgers equation. The interest for this result is twofold. First of all, we prove an easy to verify relation between the $H^2$ norm of the initial datum and the blow-up time for the $H^2$-norm of the solution. Moreover, we show a symmetry of the (inviscid) Burgers equation that consists in the invariance of the $H^2$ space under the action of that nonlinear equation. Finally, we show that as soon as the solution of the (inviscid) Burgers equation stays in $H^2$ is is $L^2$ stable with respect to the initial datum. The arguments are based on energy and stability estimates on the the solution of the (viscous) Burgers equation.

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