1. Introduction

The theoretical study of the dynamics of a population leads to complex models due to the inherent complexity of the phenomenon through the interactions and the state dimension. In our study we will present a model describing the dynamics of an epidemic of HIV/AIDS in four municipalities in the city of Tamanrasset (Southern Algeria): this choice is due to several factors:

- Geo strategic position
- Migration flows (mixing national domestic, foreign-
  national)
- 1200km border between Niger and Mali for the city of Tamanrasset where the specific program for Tamanrasset in the south of Algeria.

2. Epidemiological Model

We will present in this section a mathematical model, considering the different epidemiological and demographic processes and in the end we will introduce an equilibrium model.

2.1 Model Description

Let ‘S’ means likely classified into three groups rated «S_i».

Such as

- $S_1$: ANC (prenatal visit).
- $S_2$: MSM (men who have sex with men)
- $S_3$: ANC (prenatal visit).

$\Delta S_i$: Migrants in each group $S_i$.

And each “i” is made up of “j” subgroups with $j = 1, \ldots, 6$.

For $j = 1$: “age” classified $k = 1, \ldots, 7$.

For $j = 2$: “NI: education” ranked $k = 1, \ldots, 6$.

For $j = 3$: MS: marital status “ranked $k = 1, \ldots, 4$.

For $j = 4$: “Sex” classified $k = 1, \ldots, 2$.

For $j = 5$: “Socio-professional class” in classified $k = 1, \ldots, 2$.

Or $k_j$: number of classes in each subgroup.

The model is presented in the following diagram:

This leads us to write the following system of differential equations:

\[
\begin{align*}
S_1' & = \Delta S_1 - \mu_1 S_1 \gamma_1 - F(S,I)S_1, \\
S_2' & = \Delta S_2 - \mu_2 S_2 \gamma_2 - F(S,I)S_2, \\
S_3' & = \Delta S_3 - \mu_3 S_3 \gamma_3 - F(S,I)S_3, \\
I_1' & = F(S,I)S_1 \gamma_1 - (\mu_1 + \delta) I_1, \\
I_2' & = F(S,I)S_2 \gamma_2 - (\mu_2 + \delta) I_2, \\
I_3' & = F(S,I)S_3 \gamma_3 - (\mu_3 + \delta) I_3, \\
R & = \delta (I_1 + I_2 + I_3) - \mu R + \Delta R, \\
\end{align*}
\]

Where $F(S,I)$ the force of infection defined by :

\[
F(S,I) = \alpha S + \alpha I + \alpha
\]
With  \[ S = \sum_{i=1}^{3} S_i; I = \sum_{j=1}^{3} I_j; \] and \( \alpha \): Parameter of infection outside.

Note:

All parameters in this system are positive because they represent positive quantities with a biological significance. Let the vector \( w = (S_{1k}, S_{2k}, S_{3k}, I_{1k}, I_{2k}, I_{3k}, R) \) under the initial condition \( w(0) \) states that variable is the size of the population, it is assumed that the model structure ensures that the state variables remain non-negative over time.

2.2 Time Scale

In this epidemic, epidemiological parameters are faster than the demographic parameters, which we introduce a lover has time parameter (Scaling parameter) \( \varepsilon \) as:

\[ \alpha_s = \frac{\alpha_s}{\varepsilon}; \alpha_t = \frac{\alpha_t}{\varepsilon}; \delta = \frac{\delta}{\varepsilon} \quad \text{with } 0 < \varepsilon < < 1. \]

Considering a population size, \( N_m \) it ensures that the parameters \( N_m \alpha_s, N_m \alpha_t, \) are the same order as the parameters of the model. The parameter \( \alpha \) is very small compared to other epidemiological parameters and its order is less than the order of population parameters. Finally, \( \delta \) it ensures that the smallest order. In this case the previous system has two time scales. It is reasonable to consider that people spend more time in class than that of likely infected during this time the with drawl process is spreading faster than infection, which implies that \( \delta \) has the greatest order that \( N_m \alpha_s, N_m \alpha_t, \) and the remaining parameters in the model. In this case we introduce a second parameter of time \( \varepsilon' \) as:

\[ \varepsilon' = \frac{\delta}{\varepsilon} \quad \text{with } 0 < \varepsilon' < < \varepsilon. \]

And in this case we can obtain three time scales.

Note:

All these assumptions about magnetizer parameters will be verified by their numerical values that provide a time scale that will be distinguished by numerical simulations.

2.3 Equilibrium

The epidemiological model has two equilibria; a trivial equilibrium, corresponding to the extinction of the population (free equilibrium) this will be the point stable endemic. Our system has a single endemic, that is because all parameters are positive.

In the system (1), there is no extinction of the population, and is not a zero balance. If we consider \( \alpha = 0 \) (no infection outside), the equilibrium is given by:

\[ w^* = (S_{1k}, S_{2k}, S_{3k}, I_{1k}, I_{2k}, I_{3k}, R) \]

And one can easily find \( w^* \)

3. The Order of Scale Model

Using different time scale parameters, we can apply the “theory of singular perturbations” to approximate the original system by another system of lower dimension. Whereas a standard singular perturbation

\[ x = f(t,x,z,\varepsilon); x \in \mathbb{R}^n; x(t_0) = \xi(\varepsilon). \]

\[ \varepsilon z = g(t,x,z,\varepsilon); z \in \mathbb{R}^m; z(t_0) = \phi(\varepsilon) \quad (a) \]

The reduced system is \( x = f(t, x, h(t, x), 0) \) (b)

Taking \( y = zh(t, x) \), we defined the boundary layer system:

\[ \frac{dy}{d\varepsilon} = g(t, x, y + h(t, x), 0) \quad (c) \]

The singular perturbation theory gives the following result (Khalil 1996)

**Theorem 1:** Approximation for \( \varepsilon \) small enough

Suppose that the conditions are met for all \( (t, x, z, h(t, x), \varepsilon) \in [0, \varepsilon_0], \varepsilon_0 < 0 \)...

- The functions \( f, g \) and their respective partial derivatives \( (x, z) \) are continuous and bounded.
- The function \( h(t, x) \) and the Jacobian \( \frac{\partial g(t, x, z, \varepsilon)}{\partial z} \) has a partial derivative bounded.
- The Jacobian \( \frac{\partial f(t, x, h(t, x), 0)}{\partial x} \) is bounded.
- Initial conditions \( \xi(\varepsilon) \) and \( \phi(\varepsilon) \) are regular features \( \varepsilon \).
- The origin of the reduced system (b) is exponentially stable.
- Origin of the boundary layer system (c) exponentially stable uniformly in \( t, x \).

Then:
∃ positifs constants $\mu_1$, $\mu_2$, and $\varepsilon^*$ as
\[ \forall \|x(0)\| < \mu_1, \|y(0) - h(t_0 - \xi(0))\| < \mu_1 \] and
\[ 0 < \varepsilon << \varepsilon^* \] we obtain:

The singular perturbation in (a) has a unique solution $x(t, \varepsilon)$ $z(t, \varepsilon)$ defined $\forall t \geq t_0$; and $x(t,\varepsilon)$-$x'(t)=\theta(\varepsilon)$
\[ z(t,\varepsilon)-h(t, x^*(t))=\theta'(t/\varepsilon) =\theta(\varepsilon). \]
Uniformly for all $t$, $\in\left[t_0, \infty\right[$, where $x'(t)$ and $y'(t)$ are solution of (b).
In addition to all $t \geq t_0$ et $\varepsilon^* \leq \varepsilon^*$ we have:
\[ z(t,\varepsilon)-h(t, x^*(t)) = \theta(\varepsilon) \] converges uniformly for $t$, $\in\left[t_0, \infty\right[$.

Theorem 2: Stability for $\varepsilon$ small enough.
- the equation $g(t, x, z, 0)=0$ has an isolated root $z=h(t, x)$
- such that $h(t, 0) = 0$
- functions $f$, $g$ and $h$ and their second order partial derivatives are bounded for
\[ z-h(t,x) \in B_\rho. \]
- the origin of (b) is exponentially stable.
- the origin of (c) exponentially stable uniformly in $(t, x)$.
- the origin of the initial system(a) exponentially stable.

3.1 Two Time Scales

Either a change of variable:
\[ A_1 = S_1 + I_1 + R. \]
\[ A_2 = S_2 + I_2. \]
\[ A_3 = S_3 + I_3. \]

This change allows us to identify slow and fast variables, and each fast variable is expressed in terms of slow variables, to keep our system and we can minimize it. The fast variables arrive faster at quasi-equilibrium, so our system can be written as follows:
\[ A_1 = \Delta_1 - \mu_1 A_1 - \delta (I_2 + I_3) - \mu R + \Delta_R \]
\[ A_2 = \Delta_2 - \mu_2 A_2 - \delta I_2 \]
\[ A_3 = \Delta_3 - \mu_3 A_3 - \delta I_3 \]
\[ R = \Delta + \delta (I_1 + I_2 + I_3) - \mu R \]

Where $\varepsilon F(S, I, I') = \varepsilon S + \varepsilon I + \varepsilon I$.
Assume that $X=(A_1, A_2, A_3, R)$ and $Z=(I_1, I_2, I_3)$.
This system takes the form of an autonomous model singular perturbation.
\[ X = f(t, x, \varepsilon) ; \quad (a) \]
\[ \varepsilon Z = g(x, z, \varepsilon) \quad (b) \]

In this case $X$ and $Z$ are respectively the slow and fast variables. The system (b) has a single non-negative quasi-equilibrium when $\varepsilon \to 0$.
\[ Z^* = (I_1^*, I_2^*, I_3^*) = h(X)=(0,0,0) \]

Substituting this quasi-equilibrium in the slow system(a) we obtain the reduced linear system
\[ A_1 = \Delta_1 - \mu_1 A_1 - \mu R + \Delta_R \]
\[ A_2 = \Delta_2 - \mu_2 A_2 \]
\[ A_3 = \Delta_3 - \mu_3 A_3 \]
\[ R = \Delta_R - \mu R \]

Which is a linear system easier to study its stability.
And the equilibrium point is not trivial(0,0,0,0) but
a non-zero point. Under certain conditions there is an equilibrium state non-negative \( X^* = (A_1^*, A_2^*, A_3^*, R^*) \) with
\[
A_1^* = \frac{\Delta_1}{\mu_1}; A_2^* = \frac{\Delta_2}{\mu_2}; A_3^* = \frac{\Delta_3}{\mu_3}; R^* = \frac{\Delta_R}{\mu}
\]

4. Conclusion

- Theorem 1) the model gives a good approximation of the initial system with \( Z = Z^* = h(X) \)
- Theorem 2) gives an endemic of the initial system and globally-stable.

And after some matrix computations we can easily find the Jacobian of this matrix.

Idea 2

\( R_{\text{inconnu}} \): AIDS unknown

\( R_{\text{connu}} \): with AIDS known to the system

The system is written as follows:
\[
\begin{align*}
\dot{S}_1 &= \Delta_2 - \mu_1 S_1 R_1 - F(S, I) S_1 R_1; \\
\dot{S}_2 &= \Delta_3 - \mu_2 S_2 R_2 - F(S, I) S_2 R_2; \\
\dot{S}_3 &= \Delta_3 - \mu_3 S_3 R_3 - F(S, I) S_3 R_3; \\
\dot{I}_1 &= F(S, I) S_1 R_1 - (\mu + \delta) I_1 R_1; \\
\dot{I}_2 &= F(S, I) S_2 R_2 - (\mu + \delta) I_2 R_2; \\
\dot{I}_3 &= F(S, I) S_3 R_3 - (\mu + \delta) I_3 R_3; \\
\dot{R}_{\text{inc}} &= \delta (I_3 R_3 + I_2 R_2 + I_1 R_1) - (\mu + \beta) R_{\text{inc}} + \Delta_{R_{\text{inc}}}; \\
\dot{R}_{\text{con}} &= \beta R_{\text{inc}} + \Delta_{R_{\text{con}}} - \mu R_{\text{con}}.
\end{align*}
\]

The slow variables are: \( S_{\text{inc}}; R_{\text{inconnu}} \) and

The fast variables are: \( I_{\text{inc}}; R_{\text{con}} \)

We made the following change of variables:
\[
\begin{align*}
A_1 &= S_1 + I_1 + R_{\text{inc}}; \\
A_2 &= S_2 + I_2; \\
A_3 &= S_3 + I_3.
\end{align*}
\]

And write the system as before, replacing the values of \( A_1, A_2, A_3, R_{\text{inc}}, \varepsilon I_1, \varepsilon I_2, \varepsilon I_3, \varepsilon R; \)

When \( \varepsilon \to 0 \), we found \( I_1 = I_2 = I_3 = 0 \).

\( X = (A_1, A_2, A_3, R_{\text{inc}}) \) is variable and the system can be written:
\[
\begin{align*}
A_1 &= \Delta_1 + \Delta_R - \mu R_{\text{inc}} - \mu_1 (A_1 - R_{\text{inc}}); \\
A_2 &= \Delta_2 - \mu_2 A_2; \\
A_3 &= \Delta_3 - \mu_3 A_3; \\
R_{\text{inc}} &= \Delta_{R_{\text{inc}}} - \mu R_{\text{inc}} - \beta R_{\text{inc}}.
\end{align*}
\]

\( X^* = (A_1^*, A_2^*, A_3^*, R_{\text{inc}}^*) \) with
\[
R_{\text{inc}}^* = \frac{\Delta_{R_{\text{inc}}}}{\mu + \beta}; A_1^* = \frac{\Delta_1}{\mu_1}; A_2^* = \frac{\Delta_2}{\mu_2};
\]
\[
A_3^* = \frac{\Delta_3 (\mu + \beta) - \Delta_{R_{\text{inc}}}}{\mu_3 (\mu + \beta)}
\]

And we can easily find the Jacobian matrix of the \( X^* \).

5. References

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