Coexistence of spanning clusters in directed percolation

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(March 24, 2022)

The probability distribution for the number of top to bottom spanning clusters in Directed percolation in two and three dimensions appears to be universal and is of the form $P(n) \sim \exp(-\alpha n^\gamma)$. We argue that $\alpha$ is a new critical quantity vanishing at the upper critical dimension. The probability distribution of the individual masses of the spanning clusters is found to have a Pearson distribution with a lower cutoff. Various properties of the clusters are reported.

PACS 05.70.Jk,64.60.Ak,64.60.-i,05.50.+q

In a large variety of equilibrium or nonequilibrium critical systems spatial or space-time anisotropy plays a crucial role. Many such athermal cases exhibit a continuous transition in the geometric properties like the sizes or shapes of clusters, and the critical behavior is best described as belonging to the universality class of directed percolation (DP) [1]. Examples abound [2] as fluid flow through porous medium in an external field, threshold dynamics in random systems, forest-fire or epidemic models, reaction diffusion systems, self organized criticality, resistance(or insulator)-diode network, damage spreading, Reggeon field theory etc. Naturally, the detailed properties right at the percolation threshold $p_c$ are influenced by the nature of the spanning clusters. But, then, how many spanning clusters does one expect at the threshold?

Let us consider a DP problem on a square or a cubic lattice where each lattice site is occupied independently with a probability $p$. We want percolation that involves a top to bottom spanning (TB-spanning) cluster. Above the percolation threshold, there is only one infinite cluster occupying a finite density of sites of the lattice. However, at the threshold, a spanning cluster is a zero-density self-affine fractal (requiring direction dependent length rescaling). The lattice can then in principle support many such TB-spanning clusters.

Although renormalization group arguments have led to a coherent understanding of long distance (and time) properties of DP, especially the universal exponents, detailed questions like the number of spanning clusters at the critical point still remains unanswered. The known exponents do tell us that there is a problem in high enough dimensions due to violations of the hyperscaling relation and this can be resolved [3] if there is an infinite number of spanning clusters at $p = p_c$. This argument is not applicable for dimensions below the upper critical dimension ($d_u = 5$) and therefore the question stays whether there will be a finite number of spanning clusters or a finite density or just one.

In a number of recent studies [4–11], it has been found that there exists more than one TB-spanning cluster at the percolation threshold for $d < 6$ for ordinary (i.e., undirected or isotropic) percolation (OP) [12]. For two dimensions, it has been shown analytically that there exists more than one TB-spanning cluster at the percolation threshold and the probability $P(n)$ of the existence of $n$ such clusters follows a behavior like $\exp(-\alpha n^\gamma)$ with $\gamma = 2$ for large $n$ [4]. There is some controversy about the behavior of $P(n)$ in higher dimensions, though recent results indicate that $\gamma$ is independent of dimension and remains equal to 2 while $\alpha$ decreases with dimension [9]. The individual masses of the spanning clusters have been shown to follow the scaling behavior [3] of the unique spanning cluster in all dimensions 2 to 5. It has been conjectured that in two dimensions, the ratio of the masses of the largest and the second largest spanning cluster is $\alpha$ decreasing with dimension [9].

In sharp contrast to ordinary percolation [14], very few exact or rigorous results are available (to the best of our knowledge) for DP (except mean field results). In order to develop an analytical understanding, attempts have recently been made to bridge the gap by studying crossover from OP to DP [13]. Hence the necessity to know better the difference or similarity between OP and DP in features that are universal but yet to be resolved in the RG framework.

Here we have studied coexisting TB-spanning clusters by simulating DP on square and cubic lattices with helical boundary conditions in the transverse directions. The clusters are identified by using the Hoshen Kopelman algorithm, respecting the directedness of the paths. For any finite lattice, we have first estimated the percolation threshold using the method described in [6]. Estimates of $p_c$ in the thermodynamic limit are quite well-established for DP; however, we have worked with our estimated $p_c$ whenever finite size effects are strong, especially in three dimensions and rectangular geometries in two dimensions. The shape of the lattice (i.e., when the size of the lattice is not equal in different dimensions) itself may also play an important role in determining, e.g., the spanning probabilities etc. Even in OP, it was shown [10] that the TB-spanning probabilities show scaling behav-
ior with respect to the aspect ratio. However, the percolation threshold is not affected except showing stronger finite size effects. With this in mind, and angular dependence being special to DP, several variations have been introduced in two dimensions, e.g., two different directions of propagation have been considered and also the aspect ratio has been varied. In three dimensions, we only consider the DP to be propagating along the diagonal.

Our main results can be summarized as follows:

(1) We measure $P(n)$ the probability for $n$ TB-spanning clusters. From the absence of any significant finite size effect, we conclude that there is a nonzero probability of a finite number of such clusters at $p = p_c$ both in two and three dimensions.

(2) The probability distribution $P(n) \sim \exp(-\alpha n^2)$ and the functional form is perhaps universal.

(3) We conjecture that $\alpha$ is a weakly universal quantity that vanishes above the upper critical dimension and $\alpha = O(c)$ for $\epsilon = d_c - d \to 0$. This $\alpha$ is a new characteristic for percolation.

(4) The fractal dimension of the spanning clusters is the same as expected for the unique cluster at $p_c$.

(5) The probability distribution of the masses (from all clusters) has a universal asymmetric Pearson distribution (Type III) in terms of a scaled mass variable. See Eq. 1.

(6) The mass ratio of the two largest clusters also has a universal probability distribution.

(7) The ratio of the average masses, universal for OP, may have an angular dependence.

In two dimensions, we consider lattices of dimensions $L_x \times L_y$. With $L_x = L_y$ (square lattices), we check that for both directions, there is a non-vanishing probability for the existence of more than one spanning cluster. In the case of the diagonal DP, the number of spanning clusters $n \leq 2$ always, and the normalized probability $P(2) = P(2)/P(1)$ here is found to be not significantly different from that of OP, where the occurrence of coexisting spanning clusters is quite well established. This is shown in Fig. 1 and to be noted is the absence of any significant finite size effect.

For the DP parallel to the $y$ axis, a larger number of spanning clusters is obtained, giving us an opportunity to study the behavior of $P(n)$. Since in OP, $P(n) \sim \exp(-\alpha n^2)$ we check whether this behavior is still true for DP. We plot, in Fig. 2, $P(n)$ against $n$ and $n^2$ and the latter gives a better straight line, at least for the range of $n$ in the present study. Fig. 2 shows a plot of the finite log-derivative defined as $\gamma_n = \ln(z)/\ln(n/{\ln(n)}+1)$ where $z = \ln(P(n+1))/\ln(P(n))$ for two cases with $n \geq 5$ for both $d = 2$ and 3. This plot of $\gamma_n$ shown against $1/n$, though not very systematic, is indicative of an extrapolated value of 2 for $n \to \infty$ and definitely rules out a small value like $\gamma = 1$. That we obtain a larger number of clusters, compared to OP, may be due to a lesser value of $\alpha$ in DP. Its significance is discussed later.

For rectangular geometries, i.e., when $L_x \neq L_y$, we find that for the diagonal DP, $P(n > 1)$ vanishes for $\rho = L_y/L_x > 1$. The plot $P(n > 1)$ against $\rho$ in Fig. 3 for different sets of $L_x, L_y$ shows that it is only dependent on the aspect ratio $\rho$, a result which was also obtained in OP. A larger number of spanning clusters is obtained even for the diagonal DP when $\rho < 1$. No such crossover is observed for the DP parallel to the spanning direction in the sense $P(n > 1)$ is always non-zero even for $\rho$ as high as 8. These results tell us that such a crossover behavior is dependent on the direction of the DP, apparently vanishing when it is parallel to the spanning direction. In fact, the crossover in the diagonal (and not in the parallel) DP is possibly because of the relatively larger spread of the clusters in the $x$ direction. The study of the probability distribution $P(n)$ is possible for the diagonal DP with $\rho < 1$ as we get around 5-6 spanning clusters, as well as for the parallel DP for any aspect ratio. We again get results compatible to the value $\gamma = 2$ for each individual case.

Another interesting behavior is observed for the DP propagating along the diagonal direction when we consider $\rho < 1$. Here $p_c$ decreases as $L_x$ is made larger, keeping $L_y$ constant. With $L_x/L_y = m$ an integer, one may visualize the lattice as comprising of $m$ square cells placed adjacent to each other. We see that though the value of $p_c$ is higher in each individual cell, it is lesser for the entire system. This could happen if the clusters in the individual cells do not TB-span but join with those in the neighboring cells to span the larger lattice. If this picture is correct then the width of the spanning cluster along the $x$ direction should increase with $L_x$. We have verified that this is true.

The mass of the spanning cluster in a DP is expected to behave as $\langle M_{i,n} \rangle \sim L_{\|}^{D_{\|}}$ where $M_{i,n}$ is the mass of the $i$th largest spanning cluster with $n$ spanning clusters present. $L_{\|}$ is the length parallel to the preferred direction, $D_{\|} = -\beta/\nu_{\|} + 1 + (d-1)/b$, $b = \nu_{\|}/\nu_{\perp}$, $(\nu_{\perp})$ is the correlation length exponents parallel (perpendicular) to the direction of the DP, and $\beta$ is the DP order parameter exponent $\beta/\nu_{\|}$. $D_{\|} = 1.47$ in two dimensions with the values of $\beta$, $\nu_{\|}$ and $\nu_{\perp}$ as given in [13]. In the $L \times L$ lattice with the direction of propagation along the diagonal, it is expected that $M_{i,n} \sim L_{\|}^{D_{\|}}$ as the spanning is considered from top to bottom. This is because the length scale in the direction of propagation is $\sim L_{\|}$ and the width of the cluster is $\sim L_{\perp}$ and here $L_{\|} \sim L$. We indeed find that $D_{\|} \sim 1.5$ (comparable to 1.47) here. See Fig. 3. For the DP parallel to the spanning direction, $L_{\|} \sim L$ obviously, and again the above scaling behavior is observed. Only exception is the diagonal DP case with $\rho > 1$ where the mass exponent is found to take a new value $\sim 1.8$. A value close to 2 suggests a nearly compact
structure which could happen from strong boundary effect in this particular geometry. The data of Fig. 3 show that $(M_{i,n})$ is independent of the number $n$ of clusters present, signifying that the largest cluster is uncorrelated to the remainder of the lattice.

The ratios of the masses of the individual spanning clusters should be constants as the scaling behaviors are identical. Defining $r_{i,j}^{(n)} = (M_{i,n})/(M_{j,n})$, we find that $r_{1,2}^{(2)}$ is $1.36 \pm 0.03$ for the diagonal DP and $1.31 \pm 0.02$ for the parallel DP for $\rho = 1$. The variations of these ratios are lesser when the aspect ratio is changed but the direction of the DP remains same. For comparison, we add that $r_{1,2}^{(2)} \simeq 1.4$ for OP [4].

In three dimensional symmetric lattices ($L^3$), we find a significantly different larger number of spanning clusters compared to OP with the same number of random initial configurations (106). However, when we plot $P(n)$ against $n$ and $n^2$, again the latter gives a better straight line. The extrapolated value of $\gamma_n$ is also possibly close to 2 (see Fig 3). The fractal dimension of the cluster from the known values is around 1.67 and the log-log plot of $M_{i,j}$ against $L$ is compatible with this value (Fig 4). We obtain several mass ratios, for example, $r_{1,2}^{(2)} = 1.52 \pm 0.02$, $r_{1,2}^{(3)} = 1.34 \pm 0.03$, and $r_{1,3}^{(3)} = 1.77 \pm 0.05$. As $n$ becomes larger, the fluctuations also increase and the values of the ratios for $n \geq 4$ may be less reliable. It may be mentioned here that $r_{1,2}^{(2)} \simeq 1.8$ for OP. Though our results are not inconsistent with the expectation of angular dependence in DP [4], still it is perhaps premature to comment quantitatively on the universality of these ratios from the present data. However, one can safely conclude that at least for three dimensions, these ratios are not same for OP and DP.

In order to probe the detailed mass distribution, we obtain histogram for the cluster masses when DP is parallel to $y$ axis (the TB-spanning direction). A collapse of all the probability distributions is found, for various geometries and aspect ratios, if a scaled variable $X = (M_i - \langle M_i \rangle)/S_i$ is used where $S_i^2 = \langle M_i^2 \rangle - \langle M_i \rangle^2$, and $M_i$ is the $i$th largest cluster irrespective of the value of $n$. No sensitivity to $n$ was detected. We see a definite lower cutoff for the scaled variable. The plot is shown in Fig 4 which also shows a Pearson distribution (type III). Since by construction the scaled variable $X$ has a zero mean and unit standard deviation, only parameter available for the Pearson distribution is the lower limit. The resulting curve for $-\alpha \leq x \leq \infty$, $Q(x) = \frac{\alpha}{\Gamma(\alpha^2)}[\alpha(x + \alpha)]^{x^2-1}e^{-\alpha(x+\alpha)}$, (1)

with $\alpha = 2.8$, fits remarkably the probability distribution obtained numerically.

The single variable distribution $Q(x)$ does not reflect the restriction $M_1 > M_2$. For a joint distribution, we study the probability density $R(y)$ for the mass ratio $y = M_1/M_2$ of the two largest clusters, again irrespective of $n \geq 2$. A data collapse with some finite size effect is seen, see Fig 4b. An exponential distribution $R(y) = A \exp[-A(x-1)]$ with $A = 4$ is also shown there for comparison. With this exponential distribution, the average mass ratio is expected to be $1 + A^{-1} \approx 1.25$.

Our main results have already been summarized at the beginning. We like to add a few comments here. First, the occurrence of the two simple distributions in Fig. 3 remains unexplained. The second is regarding $\alpha$ in the distribution $P(n) \sim \exp(-\alpha n^2)$. As mentioned earlier, the number of TB-spanning clusters at the threshold is infinity at the upper critical dimension ($d_c$ for DP and 6 for OP), validity for all $d < 5$ for DP also. From $P(n)$, Generalizing our result to all $d$ (as known to be the case for OP), we may take $n_c = 1/\sqrt{\alpha}$ as a measure of the number of clusters. This $n_c$, defined strictly at the critical point, has to diverge as $d \rightarrow d_c$. Therefore, $\alpha$ must decrease with dimension, vanishing at $d = d_c$. We indeed find that $\alpha$ for DP is lesser than OP (for three dimensions, where we could make a comparison), a fact compatible with $d_c(DP) < d_c(OP)$. Assuming analyticity in $\epsilon = d_c - d$, one expects $\alpha = a + b\epsilon + \ldots$, for small $\epsilon$, with geometry (and OP/DP) dependent $a$ and $b$. In fact, for OP, $a = 0.15$ and $b = 0.05$ give very good agreement with the $\alpha$ values obtained for $\epsilon = 1.2$ and 3 [3]. We conjecture that critical fluctuations are responsible for nonvanishing of $\alpha$, and therefore $\alpha$ is a new weakly universal critical quantity.

We thank D. Stauffer for a critical reading of the manuscript. PS is grateful to Saha Institute of Nuclear Physics for the use of computer facilities.

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FIG. 1. (a) The normalized probability of the existence of two spanning clusters for ordinary and directed percolation are shown for square lattices. The direction of the DP is along the diagonal and \( P(n > 2) = 0 \) for both cases. (b) The normalized spanning probabilities for \( n > 2 \) for the diagonal DP in two dimensions shown against the aspect ratio.

FIG. 2. (a) The probability distribution of spanning clusters plotted against \( n^2 \) for different DP’s. (b) Same as (a) but now plotted against \( n \). (c) The plot of \( \gamma_n \) vs \( 1/n \).

FIG. 3. The masses of different spanning clusters for DP are shown along with specific power law variations for comparison. For two dimensions, \( L = L_x \) and system sizes are enlarged by 10 for three dimensions.

FIG. 4. (a) The probability distribution of masses of individual clusters irrespective of the total number. The solid line is the Pearson distribution type III with zero mean and unit standard deviation with a lower cutoff \(-2.8\). (b) The probability distribution for the mass ratio of the two largest clusters for \( n \geq 2 \). The solid line is an exponential distribution with \( A = 4 \).