Fast Track Communications

Supersymmetric field theory with benign ghosts

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Abstract

We construct a supersymmetric (1+1)-dimensional field theory involving extra derivatives and associated ghosts: the spectrum of the Hamiltonian is not bounded from below or from above. In spite of that, there is neither classical nor quantum, collapse and unitarity is preserved.

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(Some figures may appear in colour only in the online journal)

1. Introduction

We still do not know what quantum gravity is. One of the problems is that the quantum version of Einstein gravity is not renormalizable: the expansion runs over $G \Lambda^2$ ($G$ being the Newton constant and $\Lambda$ the ultraviolet cutoff) and severe power divergences are manifest. This is probably also true for Einstein supergravity.

On the other hand, the higher-derivative gravity, involving the structures $R^2$ and $R^2_{\mu\nu}$ in the Lagrangian, has dimensionless coupling constants and is renormalizable [7]. A supersymmetric version of this theory is even asymptotically free [8]. One can then imagine a scenario where

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1 On leave of absence from ITEP, Moscow, Russia.
2 Classical and quantum gravity also suffer from other problems associated with the geometric nature of the theory and the absence of the flat universal time. These problems (first of all the problem of causality violation) are no less troubling [1]. We refer the reader to [2] for a detailed discussion of these and related issues.
3 At the moment, one can still hope that the maximal $N=8$ extended supergravity is finite. Remarkable cancellations through at least four loops were observed in [3]. These cancellations are not without a reason [4]. However, most theorists expect that the divergences would arise at the seventh loop level and higher [5, 6].
the fundamental gravity theory involves higher derivatives, with the usual gravity of our world appearing as an effective low-energy theory [9–11].

Recently, there has been a revival of interest in higher-derivative gravity (see e.g. [12–14]). Still, this scenario has mostly been considered unattractive by the scientific community because higher-derivative theories have been known to be ghost-ridden [15]. Ghosts are usually conceived as the poles with the wrong residue in the propagators. The associated states have negative norm and the common lore is that their production violates unitarity.

It is important to understand that one is able to quantize the theory so that the norm of all states remains positive. The price is that states with arbitrary negative energy exist such that the Hamiltonian does not have a ground state; the perturbative vacuum is absent. At the level of the free Hamiltonian, this is not yet a problem: different states with positive or negative energy do not interact with each other and the evolution operator is perfectly unitary. A detailed discussion of this issue for the simplest higher-derivative system, the Pais–Uhlenbeck oscillator, along with a criticism of somewhat confusing recent Bender and Mannheim papers [16], can be found in [17].

Ghosts usually still strike back for interacting nonlinear higher-derivative systems. The copious creation of positive and negative energy states brings about the collapse so that unitarity is indeed violated. For this to happen, one does not need to have higher derivatives—the simplest system where this occurs is the quantum problem of 3D motion in an attractive potential $V(r) = -\alpha / r^2$ with large enough $\alpha$. Falling on the centre in this problem means breaking of unitarity [18]. Note that the associated classical problem is also sick: there are trajectories that fall on the centre in finite time. Note also that the classical problem is sick for any positive $\alpha$, while the quantum one starts having trouble only when $\alpha \geq \hbar^2 / (8m)$. This is a typical universal situation: whenever the quantum problem is sick, so is the classical one; the inverse is not true. Quantum fluctuations can sometimes cope with the singularity and prevent the system from falling there. In other words, if we are interested in the state of health of some quantum system, we can also explore its classical brother. If the latter is OK, so is the former.

A natural nonlinear generalization of the Pais–Uhlenbeck oscillator,

$$L = \frac{1}{2} (q^2 + \Omega^2 \dot{q}^2) - \frac{\alpha}{4} q^4 - \frac{\beta}{2} q^2 \dot{q}^2,$$

was considered in [19]. We have unravelled the presence of ‘islands of stability’—in a certain range of the parameters $\alpha$, $\beta$ and, for small enough initial fluctuations $q(0), \dot{q}(0), \ddot{q}(0), q^{(3)}(0)$, the trajectories do not display any collapse, but oscillate near the perturbative vacuum $q = 0$. However, when the deviations are large enough, the trajectories go astray and hit infinity. Thus, in spite of the presence of some benign trajectories, such a model is not benign as a whole.

A completely benign higher-derivative supersymmetric quantum mechanical model was constructed in [20]. Its action has the form

$$S = \int dt dx d\theta d\bar{\theta} \left[ \frac{i}{2} D\Phi \frac{d}{dt} D\Phi + V(\Phi) \right],$$

where

$$\Phi = \phi + \theta \psi + \psi \bar{\theta} + D\theta \bar{\theta}$$

is a real $(0+1)$-dimensional superfield. The Lagrangian in (2) has an extra time derivative compared to the well-known Witten’s SQM Lagrangian [21]. As a result, the former auxiliary field $D$ becomes dynamical. The bosonic part of the component Lagrangian reads

$$L_B = \dot{\phi} D + DV(\phi).$$
The canonical bosonic Hamiltonian now involves two pairs of dynamic variables: \((\phi, p)\) and \((D, P)\),

\[
H_B = pP - DV'(\phi).
\]

It is not positive definite, and its spectrum does not have a bottom. Still, the dynamics of the system are completely benign. In fact, it involves two integrals of motion (dimensionally reduced versions of the expressions (12) and (13) below), and is thus integrable. For the simplest nontrivial superpotential,

\[
V(\Phi) = -\frac{\omega^2 \Phi^2}{2} - \frac{\lambda \Phi^4}{4},
\]

the solutions are expressed in terms of Jacobi elliptic functions (see equations (15), (16) below). Also the wave functions of the quantum states (with arbitrary high and arbitrary low energies) were found explicitly. The spectrum is continuous with eigenvalues lying in two intervals \([-\infty, -\omega] \cup [\omega, +\infty]\) plus the eigenvalue \(E = 0\). The existence of zero energy eigenstates (infinitely many of them) annihilated by the action of the supercharges may be interpreted as the absence of spontaneous supersymmetry breaking. On the other hand, zero energy states are no longer ground states...

Thus, benign quantum-mechanical systems with ghosts do exist. However, no benign ghost-ridden field theory models are known to date. We will present an example of such a model below.

2. A benign 2D higher-derivative model

The model is a straightforward two-dimensional (2D) generalization of (2). We take the real superfield (3) and assume that its components depend not only on \(t\), but also on \(x\). The superfield action is

\[
S = \int dt \, dx \, d\bar{\theta} \, d\theta \left[ -2i \Phi \partial_\Phi \Phi \Phi + V(\Phi) \right],
\]

where \(\partial_{\pm} = (\partial_t \pm \partial_x) / 2\) and

\[
\mathcal{D} = \frac{\partial}{\partial \theta} + i \theta \partial_{-}, \quad \bar{\mathcal{D}} = \frac{\partial}{\partial \bar{\theta}} - i \bar{\theta} \partial_{+}
\]

are the 2D supersymmetry covariant derivatives. Note that the first term in (7) can also be written as \(-2i \Phi \partial_\Phi \Phi \Phi \). The integrals over \(d\bar{\theta} \, d\theta\) of these two expressions coincide.

Expressing (7) in the components, we obtain

\[
L = \partial_\mu \phi \partial_\mu \Phi \Phi + \partial_\mu \bar{\psi} \partial_\mu \psi + DV'(\phi) + V''(\phi) \bar{\psi} \psi.
\]

Its dimensional reduction gives the Lagrangian studied in [20]. The bosonic part of (9) is

\[
L_B = \partial_\mu \phi \partial_\mu \Phi \Phi + DV'(\phi).
\]

When it is benign, the whole system is benign.

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4 One could, of course, get rid of the derivative \(D\) by adding a total derivative to the Lagrangian (4). In this case \(D\) becomes a Lagrange multiplier. But then the second derivative \(\phi\) would appear in the Lagrangian so that the number of degrees of freedom stays the same.

5 It is not only (2). An example of a stable nonlinear system with ghosts suggested back in 2003 [22] was discussed in details in recent [23]. Other examples were found in [24].

6 See [25] for a good pedagogical description of the 2D superfield formalism.

7 The latter is not so relevant for us in this communication, where our primary goal is to construct an example of benign theory with ghosts. The Lagrangian (10) is already such an example.
Let us study the classical dynamics of (10) with the superpotential (6). The equations of motion are

\[
\Box \phi + \omega^2 \phi + \lambda \phi^3 = 0 \tag{11} \\
\Box D + D(\omega^2 + 3\lambda \phi^2) = 0.
\]

We see that \( \phi(x, t) \) satisfies a nonlinear wave equation. The solutions to this equation cannot grow—the positive definite integral of motion

\[
N = \int dx \left[ \frac{1}{2} (\phi'^2 + (\phi')^2) + \frac{\omega^2 \phi^2}{2} + \frac{\lambda \phi^4}{4} \right] \tag{12}
\]

does not allow this. The equation for \( D(x, t) \) is more tricky. \( D(x, t) \) can grow with time, but this growth is at worst linear and does not lead to collapse. Besides (12), the system has also the usual energy integral of motion,

\[
E = \int dx \left[ \phi D + \phi' D' + D\phi (\omega^2 + \lambda \phi^2) \right], \tag{13}
\]

which can be both positive and negative.

Two integrals of motion are not enough to make the field dynamics regular, and the latter exhibits chaotic features. We are in a position to solve the equations (11) numerically. We played with different values of the parameters \( \omega, \lambda \) and with different initial conditions and never found a collapse; only, at worst, a linear growth of \( D(x, t) \) with time. Typical behaviour is displayed in figure 1 where the dispersion \( d(t) = \sqrt{\langle D^2 \rangle} \) is plotted as a function of time.
when we have chosen \( \omega = \lambda = 1 \) and the initial conditions
\[
\phi(x, t = 0) = Ce^{-x^2}, \quad \text{with} \quad C = 1, 3, 5
\]
\[
D(x, t = 0) = \cos \frac{\pi x}{L}, \quad L = 10 \quad \text{being the length of the box.} \quad (14)
\]

Due to chaoticity, \( d \) is not a smooth function of \( t \), but exhibits fluctuations. Still the linear growth trend is clearly seen. Note, however, that this growth is in a considerable extent a finite size effect. The initial conditions (14) are chosen so that the energy density (the integrand in (13)) represents a simple lump concentrated near zero. But as time proceeds, this lump starts to travel to the left and to the right and soon reaches the boundary. With larger values of \( L \), this happens later, which affects the dynamics of \( \phi(x, t) \), \( D(x, t) \), and \( d(t) \). In particular, the latter grows more slowly (see figure 2).

2.1. Possible implications for inflation

Another interesting choice for the initial conditions is the homogeneous \( \phi(x, t) \equiv \phi(t) \). The equation for \( \phi(t) \) can in this case be solved analytically [20]. It is an elliptic cosine function,
\[
\phi(t) = \phi_0 \text{cn}[\Omega t|m]
\]
with
\[
\alpha = \frac{\omega^4}{\lambda N}, \quad \Omega = [\lambda N(4 + \alpha)]^{1/4}, \quad m = \frac{1}{2} \left[ 1 - \sqrt{\frac{\alpha}{4 + \alpha}} \right],
\]
\[
\phi_0 = \left( \frac{N}{\Omega} \right)^{1/4} \sqrt{4 + \alpha} \sqrt{4 + \alpha - \sqrt{\alpha}} \quad (16)
\]
where \( N \) is the value of the integral of motion (12) and we used the Mathematica conventions.

With homogeneous \( \phi(x, t) \), we can expand \( D(x, t) \) in a Fourier series such that the second equation in (11) splits into independent equations for each Fourier component. The solutions to all these equations can also be expressed semi-analytically via certain integrals [20], or are found numerically. Either way one observes that all nonzero Fourier modes of \( D(x, t) \) stay bounded, and only the zero mode exhibits a linear growth (see figure 3). As a result, the function \( D(x, t) \) becomes more and more homogeneous.

This phenomenon may be relevant for the eventual solution of the long-standing problem of inflation initial conditions. For the inflation to take place, the scalar field which drives it...
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Figure 3. (a) The growth of the zero Fourier mode and (b) the boundedness of the mode with nonzero \( k \) for homogeneous \( \phi(x, t) \equiv \phi(t) \). The parameters \( \omega = \lambda = N = 1 \) are chosen.

should be homogeneous. At present, such homogeneity is mostly just postulated. But, maybe it appears as a result of the evolution of a higher-derivative system at the stage preceding inflation.

This speculative idea expressed first in [19] seems attractive to us. One should also mention that, as far as the toy model considered in this communication is concerned, the profiles for \( \phi(x, t) \) and \( D(x, t) \) do not become homogeneous with arbitrarily chosen initial conditions. It is only when \( \phi(x, t) \) was homogeneous at the initial moment that \( D(x, t) \) becomes homogeneous as time passes. Certainly, more studies in this direction are necessary.

3. Conclusions and outlook

In this communication, we described the first example of field theory which stays unitary despite the presence of ghosts. One can guess that there are other such models, also with number of spatial dimensions higher than 1 and probably also higher than 3. In [26], we suggested that one of such higher-dimensional higher-derivative models can play the role of the Holy Grail Theory of Everything such that our (3+1)-dimensional universe represents a brane-like classical solution of this theory embedded in a flat higher-dimensional bulk. My hopes at that time were associated with a beautiful superconformal (at the classical level) six-dimensional renormalizable SYM theory [27]. It has a nontrivial and nice looking Lagrangian and is asymptotically free. Unfortunately, beauty does not always mean efficiency. This theory involves collapsing classical trajectories\(^8\) and is not benign in the sense outlined above; one needs to search for alternatives.

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\(^8\) It is especially clearly seen in the sector of the (former auxiliary) fields \( D_{\mu=1,2,3} \) of canonical dimension 2 which enter the Lagrangian (for the \( SU(2) \) gauge group) as

\[
\mathcal{L}_D \propto \frac{1}{2} (D_{\mu}^2)^2 + \frac{1}{8} \epsilon^{abc} \epsilon_{ijkl} D^a_{\mu} D^b_{\nu} D^c_{\rho} \Delta^{ijkl}.
\]

Substituting the ansatz \( D_{\mu}^a = \Delta^{ij}_{\mu} \) in the equations of motion, we obtain the equation \( \ddot{D} - \Delta^2 = 0 \), whose solutions are obviously singular.
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