EXPONENTIAL TIME COMPLEXITY OF WEIGHTED COUNTING OF INDEPENDENT SETS

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ABSTRACT. We consider weighted counting of independent sets using a rational weight $x$: Given a graph with $n$ vertices, count its independent sets such that each set of size $k$ contributes $x^k$. This is equivalent to computation of the partition function of the lattice gas with hard-core self-repulsion and hard-core pair interaction. We show the following conditional lower bounds: If counting the satisfying assignments of a 3-CNF formula in $n$ variables ($\#3$SAT) needs time $2^{\Omega(n)}$ (i.e., there is a $c > 0$ such that no algorithm can solve $\#3$SAT in time $2^{cn}$), counting the independent sets of size $n/3$ of an $n$-vertex graph needs time $2^{\Omega(n)}$ and weighted counting of independent sets needs time $2^{\Omega(n/\log^3 n)}$ for all rational weights $x \neq 0$.

We have two technical ingredients: The first is a reduction from 3SAT to independent sets that preserves the number of solutions and increases the instance size only by a constant factor. Second, we devise a combination of vertex cloning and path addition. This graph transformation allows us to adapt a recent technique by Dell, Husfeldt, and Wahlén which enables interpolation by a family of reductions, each of which increases the instance size only polylogarithmically.

1. INTRODUCTION

Finding independent sets with respect to certain restrictions is a fundamental problem in theoretical computer science. Perhaps the most studied version is the maximum independent set problem: Given a graph, find an independent set of maximum size. This problem is closely related to the clique and vertex cover problems. The decision versions of these are among the 21 problems considered by Karp in 1972 [17], and they are used as examples in virtually every exposition of the theory of NP-completeness [10, Section 3.1], [20, Section 9.3], [3, Section 34.5]. Exact algorithms for the independent set problem have been studied since the 70s of the last century [27, 16, 21] and there is still active research [9, 18, 3].

Besides finding a maximum independent set, algorithms that count the number of independent sets have also been developed [6]. If the counting process is done in a weighted manner (as in 1 below), we arrive at a problem from statistical physics: computation of the partition function of the lattice gas with hard-core self-repulsion and hard-core pair interaction [24]. In graph theoretic language, this

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1 A subset $A$ of the vertices of a graph is independent iff no two vertices in $A$ are joined by an edge of the graph.

Full version of a contribution to IPEC 2010. This work has been done while the author was a research assistant at Saarland University, Germany.
is the following problem: Given a graph $G = (V, E)$ and a weight $x \in \mathbb{Q}$, compute
\begin{equation}
I(G; x) = \sum_{A \subseteq V} x^{|A|}.
\end{equation}

$I(G; x)$ is also known as the independent set polynomial of $G$ \cite{12, 11}. Luby and Vigoda mention that “equivalent models arise in the Operations Research community when considering properties of stochastic loss systems which model communication networks” \cite{19}. Evaluation of $I(G; x)$ has received a considerable amount of attention, mainly concerning approximability if $x$ belongs to a certain range depending on $\Delta$, the maximum degree of $G$ \cite{19, 8, 29, 30}.

In this paper, we give evidence that exact evaluation of $I(G; x)$ needs almost exponential time (Theorem 1.2). We do this by reductions from the following problem:

Name: $\#d$-SAT
Input: Boolean formula $\varphi$ in $d$-CNF with $m$ clauses in $n$ variables
Output: Number of satisfying assignments for $\varphi$

All lower bounds of this work are based on the following assumption, which is a counting version of the exponential time hypothesis (ETH) \cite{14, 7}:

$\#\text{ETH}$ (Dell, Husfeldt, Wahlén 2010). There is a constant $c > 0$ such that no deterministic algorithm can compute $\#3$-SAT in time $\exp(c \cdot n)$.

Our first result concerns the following problem:

Name: $\#\frac{1}{3}$-IS
Input: Graph with $n$ vertices
Output: Number of independent sets of size exactly $n/3$ in $G$

\begin{theorem}
$\#\frac{1}{3}$-IS requires time $\exp(\Omega(n))$ unless $\#\text{ETH}$ fails.
\end{theorem}

Theorem 1.1 gives an important insight for the development of exact algorithms counting independent sets: Let us consider algorithms that count independent sets of a particular kind. (For example: algorithms that count the independent sets of maximum size. Another example: algorithms that simply count all independent sets). Using only slight modifications, many of the actual algorithms that have been suggested for these problems can be turned into algorithms that solve $\#\frac{1}{3}$-IS. Theorem 1.1 tells us that there is some $c > 1$ such that every such algorithm has worst-case running time $\geq c^n$—unless $\#\text{ETH}$ fails. In other words: There is a universal $c^n$ barrier for counting independent sets that can only be broken 1) if substantial progress on counting SAT is made or 2) by approaches that are custom-tailored to the actual version of the independent set problem such that they can not be used to solve $\#\frac{1}{3}$-IS.

The proof of Theorem 1.1 is different from the standard constructions that reduce the decision version of 3SAT to the decision version of maximum independent set \cite{17, 20, Theorem 9.4}. This is due to the fact that these constructions do not preserve the number of solutions. Furthermore, the arguments for counting problems that have been applied in $\#P$-hardness proofs also fail in our context, as they increase the instance size by more than a constant factor\footnote{For instance, Valiant’s step from perfect matchings to prime implicants \cite{28} includes transforming a $\Theta(n)$ vertex graph into a $\Theta(n^2)$ vertex graph.} and thus do not preserve subexponential time.

2 For instance, Valiant’s step from perfect matchings to prime implicants \cite{28} includes transforming a $\Theta(n)$ vertex graph into a $\Theta(n^2)$ vertex graph.
Theorem 1.1 is proved in Section 2 using, with hindsight, simple reduction from \#3-SAT. But for the reasons given in the last paragraph, it is important to work this out precisely. In this way, we close a non-trivial gap to a result that is very important as it concerns a fundamental problem.

The main result of our paper is based on Theorem 1.1:

**Theorem 1.2.** Let $x \in \mathbb{Q}$, $x \neq 0$. On input $G = (V, E)$, $n = |V|$, evaluating the independent set polynomial at $x$, i.e. computing
\[
\sum_{A \subseteq V, \text{ independent set}} x^{|A|},
\]
requires time $\exp(\Omega(n/\log^3 n))$ unless \#ETH fails.

This shows that we cannot expect that the partition function of the lattice gas with hard-core self-repulsion and hard-core pair interaction can be computed much faster than in exponential time.

Let us state an important consequence of Theorem 1.2, the case $x = 1$.

**Corollary 1.3.** Every algorithm that, given a graph $G$ with $n$ vertices, counts the independent sets of $G$ has worst-case running time $\exp(\Omega(n/\log^3 n))$ unless \#ETH fails.

Referring to the discussion after Theorem 1.1, this gives a conditional lower bound for our second example (i.e. counting all independent sets of a graph). The bound of Corollary 1.3 is not as strong as the bound of Theorem 1.1 but holds for every algorithm, not only for algorithms that can be modified to solve \#3-IS.

**Techniques and Relation to Previous Work.** Theorem 1.1 is proved in two steps: First, we reduce from \#3-SAT to \#X3SAT. \#X3SAT is a version of SAT that counts the assignments that satisfy exactly one literal per clause. From \#X3SAT we can reduce to independent sets using a modified version of a standard reduction from SAT to independent sets [20, Theorem 9.4]. We also use the fact that the exponential time hypothesis with number of variables as a parameter is equivalent to the hypothesis with number of clauses as parameter. Impagliazzo, Paturi, and Zane proved this for the decision version [14]. We use the following version for counting problems:

**Theorem 1.4 ([7, Theorem 1]).** For all $d \geq 3$, \#ETH holds if and only if \#d-SAT requires time $\exp(\Omega(m))$.

Our main result (Theorem 1.2) is inspired by recent work of Dell, Husfeldt, and Wahlén on the Tutte polynomial [7, Theorem 3(ii)]. These authors use Sokal’s formula for the Tutte polynomial of generalized Theta graphs [26]. In Section 3 we devise and analyze $S$-clones, a new graph transformation that can be used with the independent set polynomial in a similar way as generalized Theta graphs with the Tutte polynomial. $S$-clones are a combination of vertex cloning (used under this name for the interlace polynomial [2], but generally used in different situations for a long time [23, Theorem 1, Reduction 3.], [13, 22, Lemma A.3]) and addition of paths. Having introduced $S$-clones, we are able to transfer the construction of Dell et al. quite directly to the independent set polynomial. The technical details are more involved than in the previous work on the Tutte polynomial, but the general idea is the same: Use the graph transformation ($S$-clones) to evaluate
the graph polynomial (independent set polynomial) at different points, and use the result for interpolation. An important property of the construction is that the graph transformation increases the size of the graph only polylogarithmically. More details on this can be found at the beginning of Section 4.

Before we start with the detailed exposition, let us mention that the reductions we devise for the independent set polynomial can be used with the interlace polynomial \[15\] as well \[15\].

2. Reduction from counting SAT to counting independent sets

We give the details of a reduction from SAT to independent sets which increases the instance size only by a constant factor and preserves the number of solutions. This yields the conditional lower bound for counting independent sets of size \(n/3\) (Theorem \[1.1\]).

Name: \#X3SAT

Input: Boolean formula \(\varphi = C_1 \land \ldots \land C_m\) where each clause \(C_i\) is a disjunction of two or three literals over variables \(x_1, \ldots, x_n\).

Output: Number of assignments for \(\varphi\) such that in every clause exactly one literal is satisfied.

By a polynomial time reduction from a counting problem \(A\) to a counting problem \(B\) we mean a polynomial time algorithm that maps an input instance \(x\) for \(A\) to an input instance \(y\) for \(B\) such that the number of solutions for \(x\) equals the number of solutions for \(y\).

**Lemma 2.1.** There is a polynomial time reduction from \#3-SAT to \#X3SAT that maps formulas with \(m\) clauses to formulas with \(O(m)\) clauses.

**Proof.** Schaefer \[23, Lemma 3.5\] gives the following construction. For a clause \(C = (a \lor b \lor c)\), define \(F = (a \lor u_1 \lor u_2)(b \lor u_3 \lor u_5)(u_1 \lor u_2 \lor u_4)(u_3 \lor u_4 \lor u_6)(c \lor u_5)\).

It is not hard to check that every assignment that satisfies \(C\) in the usual sense corresponds to exactly one assignment that satisfies \(F\) in the sense of \#X3SAT.

Using this construction, we reduce \#3-SAT to \#X3SAT. If an instance \(\varphi\) for \#3-SAT is given, we construct an instance \(\varphi'\) for \#X3SAT by applying the above construction for every clause in \(\varphi\), each time using “fresh” variables \(u_1, \ldots, u_6\). \(\square\)

**Lemma 2.2.** There is a polynomial time reduction from \#X3SAT to \#\(\frac{1}{3}\)-IS that maps formulas with \(m\) clauses to graphs with \(3m\) vertices.

**Proof.** We reduce from \#X3SAT. Let \(\varphi = C_1 \land \ldots \land C_m\) be an instance for \#X3SAT, i.e. a 3-CNF formula with \(m\) clauses in \(n\) variables. We assume that every variable appears in \(\varphi\), otherwise a factor of \(2^r\) is introduced in the following reduction, where \(r\) is the number of variables that do not appear in \(\varphi\). Furthermore, we assume that no literal appears twice in a clause and that, if a literal \(\ell\) appears in a clause, its negation \(\neg \ell\) does not appear in the same clause. The construction in Lemma \[2.1\] complies with these assumptions. Therefore, we do not lose generality.

For each clause \(C_i = \ell_{i,1} \lor \ell_{i,2} \lor \ell_{i,3}\) of \(\varphi\), we construct a triangle \(T_i\) whose vertices \(v_{i,1}, v_{i,2}, v_{i,3}\) are labeled by \(\ell(v_{i,j}) = \ell_{i,j}\), \(1 \leq j \leq 3\), the literals of \(C_i\). In this way, we obtain the vertex set \(V = \{v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq 3\}\) for the \#\(\frac{1}{3}\)-IS instance \(G\). Besides the triangle edges, we add the following edges to \(G\): For each pair \(\{u, v\}\) of vertices, where \(\ell(u) = \ell(v)\) or \(\ell(u) = \neg \ell(v)\), let \(u_2, u_3\) be the other two vertices in \(u\)’s triangle and \(v_2, v_3\) be the other two vertices in \(v\)’s triangle. If \(\ell(u) = \ell(v)\), we
connect $u$ to $v_2$ and $v_3$, and we connect $v$ to $u_2$ and $u_3$. If $\ell(u) = \neg \ell(v)$, we connect $u$ and $v$, and we connect every vertex of $\{u_2, u_3\}$ to every vertex of $\{v_2, v_3\}$.

It is not difficult to argue that the number of satisfying assignments for $\varphi$ (i.e. assignments such that in each clause exactly one literal evaluates to true) equals the number of independent sets $A$ of $G$ with $|A| = m$. □

Proof of Theorem 1.1. Follows from Theorem 1.4, Lemma 2.1, and Lemma 2.2. □

3. $S$-clones and the Independent Set Polynomial

In this section, we analyze the effect of the following graph transformation on the independent set polynomial.

Definition 3.1. Let $S$ be a finite multiset of nonnegative integers and $G = (V, E)$ be a graph. We define the $S$-clone $G_S = (V_S, E_S)$ of $G$ as follows:

- For every vertex $a \in V$, there are $|S|$ vertices $a(|S|) := \{a_1, \ldots, a_{|S|}\}$ in $V_S$.
- For every edge $uv \in E$, there are edges in $E_S$ that connect every edge in $u(|S|)$ to every edge in $v(|S|)$.
- Let $S = \{s_1, \ldots, s_t\}$. For every vertex $a \in V$, we add a path of length $s_i$ to $a_i$, the $i$th clone of $a$. Formally: For every $i$, $1 \leq i \leq |S|$, and every $a \in V$ there are vertices $a_{i,1}, \ldots, a_{i,s_i}$ in $V_S$ and edges $a_i a_{i,1}, a_{i,1} a_{i,2}, \ldots, a_{i,s_i-1} a_{i,s_i}$ in $E_S$.
- There are no other vertices and no other edges in $G_S$ but the ones defined by the preceding conditions.

The effect of $S$-cloning on a single vertex is illustrated in Figure 1. The purpose of $S$-clones is that $I(G_S; x)$ can be expressed in terms of $I(G; x(S))$, where $x(S)$ is some number derived from $x$ and $S$. For technical reasons, we restrict ourselves to $x$ that fulfill the following condition:

Definition 3.2. Let $x \in \mathbb{R}$. We say that $x$ is nondegenerate for path reduction if $x > \frac{1}{4}$ and $x \neq 0$. Otherwise, we say that $x$ is degenerate for path reduction.

Definition 3.3. Let $x \in \mathbb{R}$ be nondegenerate for path reduction. Then we define $\lambda_1$ and $\lambda_2$ to be the two roots of

$$\lambda^2 - \lambda - x,$$

i.e.

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + x}.$$

The following condition ensures that (11) is well-defined (cf. (10)).

Definition 3.4. Let $x \in \mathbb{R}$ be nondegenerate for path reduction. We say that a set $S$ of nonnegative integers is compatible with $x$ if $\lambda_1^{s+2} \neq \lambda_2^{s+2}$ for all $s \in S$. 

\[\text{Figure 1. Effect of a } \{0, 2, 3\}-\text{clone on a single vertex.}\]
Now we can state the effect of $S$-cloning on the independent set polynomial:

**Theorem 3.5.** Let $G = (V, E)$ be a graph, $x$ be nondegenerate for path reduction, and $S$ be a finite multiset of nonnegative integers that is compatible with $x$. Then we have

$$I(G_S; x) = (\prod_{s \in S} C_s)^{|V|} I(G; x(S)),$$

where

$$x(S) + 1 = \prod_{s \in S} \left(1 + \frac{B_s}{C_s}\right) \quad \text{with}$$

$$B_k = \frac{x}{\lambda_2 - \lambda_1} \cdot (-\lambda_1^{k+1} + \lambda_2^{k+1}),$$

$$C_k = \frac{1}{\lambda_2 - \lambda_1} \cdot (-\lambda_1^{k+2} + \lambda_2^{k+2}),$$

and $\lambda_1, \lambda_2$ as in Definition 3.3.

The rest of this section is devoted to the proof of Theorem 3.5.

3.1. **Notation.** We use a multivariate version of the independent set polynomial. This means that every vertex has its own variable $x$. Formally, we define a vertex-indexed variable $x$ to be a set of independent variables $x_a$ such that, if $G = (V, E)$ is a graph, $x$ contains $\{x_a \mid a \in V\}$. If $x$ is a vertex-indexed variable and $A$ is a subset of the vertices of $G$, we define

$$x_A := \prod_{a \in A} x_a.$$  

The multivariate independent set polynomial \[24\] is defined as

$$I(G; x) = \sum_{A \subseteq V \text{ independent}} x_A.$$  

We have $I(G; x) = I(G; x)[x_a := x \mid a \in V]$, i.e. the single-variable independent set polynomial is obtained from the multivariate version by substituting every vertex-indexed variable $x_a$ by one and the same ordinary variable $x$.

We will use the following operation on graphs: Given a graph $G$ and a vertex $b$ of $G$, $G - b$ denotes the graph that is obtained from $G$ by removing $b$ and all edges incident to $b$.

3.2. **Proof of Theorem 3.5** Let us first analyze the effect of a single leaf on the independent set polynomial.

**Lemma 3.6.** Let $G = (V, E)$ be a graph and $a \neq b$ be two vertices such that $a$ is the only neighbor of $b$. Then, as a polynomial equation, we have

$$I(G, x_a, x_b) = (1 + x_b)I(G - b, x_a/(1 + x_b)),$$

where $I(G, y, z)$ denotes $I(G; x)$ with $x_a = y$ and $x_b = z$ and $I(G - b, z)$ denotes $I(G - b; y)$ with $y_a = z$ and $y_v = x_v$ for all $v \in V \setminus \{a, b\}$. ($G - b$ is defined on Page 2.)
Proof. Let $V' = V \setminus \{a, b\}$ and $i(A) = 1$ if $A \subseteq V$ is a independent set in $G$ and $i(A) = 0$ otherwise. We have

$$I(G, x_a, x_b) = \sum_{A \subseteq V'} x_A \left( i(A) + x_a i(A \cup \{a\}) + x_b i(A \cup \{b\}) \right)$$

$$= \sum_{A \subseteq V'} x_A \left( i(A) + x_a i(A \cup \{a\}) + x_b i(A) \right)$$

$$= \sum_{A \subseteq V'} x_A \left( i(A)(1 + x_b) + x_a i(A \cup \{a\}) \right),$$

from which the claim follows. \qed

In other words, Lemma 3.6 states that a single leaf $b$ and its neighbor $a$ can be “contracted” by incorporating the weight of $b$ into $a$. In a very similar way, two vertices with the same set of neighbors can be contracted:

**Lemma 3.7.** Let $G = (V, E)$ be a graph and $a, b \in V$ two vertices that have the same set of neighbors. Then

$$I(G; x) = I(G - b; y),$$

where $y_v = x_v$ for all $v \in V \setminus \{a, b\}$ and $y_a + 1 = (1 + x_a)(1 + x_b)$.

**Proof.** Similar to the proof of Lemma 3.6. \qed

Consider the following special case of a $S$-clone.

**Definition 3.8.** Let $S$ be the multiset that consists of $k$ times the number 0 and $G = (V, E)$ be a graph. Then we write $k^G$ to denote $G_S$. We call $k^G$ the $k$-clone of $G$.

Applying Lemma 3.7 repeatedly yields the following statement. Observations of this kind have been used for a long time [28, 15, 22, 2].

**Theorem 3.9.** Let $G = (V, E)$ be a graph. We have the polynomial identity

$$I(k^G; x) = I(G; (1 + x)^k - 1).$$

Let us now use Lemma 3.6 to derive a formula that describes how a path, attached to one vertex, influences the independent set polynomial. Basically, we derive an explicit formula from recursive application of (8) (cf. the proof of the formula for the interlace polynomial of a path by Arratia et al. [14 Proposition 14]).

**Theorem 3.10.** Let $G = (V, E)$ be a graph and $a_0 \in V$ a vertex. For a positive integer $k$, let $\tau_k G$ denote the graph $G$ with a path of length $k$ added at $a_0$, i.e. $\tau_k G = (V \cup \{a_1, \ldots, a_k\}, E \cup \{a_0a_1, a_1a_2, \ldots, a_{k-1}a_k\})$ with $a_1, \ldots, a_k$ being new vertices. Let $x$ be a vertex labeling of $\tau_k G$ with variables. Then the following polynomial equation holds:

$$I(\tau_k G; x) = I(G; B_k/C_k),$$

where $y_v = x_v$ for all $v \in V \setminus \{a_0, \ldots, a_k\}$, $y_{a_0} = B_k/C_k$, and $B_0 = x_k$, $C_0 = 1$ and, for $0 \leq i < k$,

$$(10) \quad \frac{B_{i+1}}{C_{i+1}} = M(x_{k-i-1}) \left( \frac{B_i}{C_i} \right),$$
\[(12) \quad M(x) = \begin{pmatrix} 0 & x \\ 1 & 1 \end{pmatrix}.\]

Let now \(x \in \mathbb{R}\) be nondegenerate for path reduction, \(\lambda_1, \lambda_2\) be as in Definition 3.3, \(x_a = x\) for all \(a \in \{a_0, \ldots, a_k\}\) and \(\lambda_1^{k+2} \neq \lambda_2^{k+2}\). Then (10) holds with \(B_k\) and \(C_k\) defined as in (5) and (6).

**Proof.** Let us write \(I(\tau_k G; x_0, \ldots, x_k)\) for \(I(\tau_k G; \mathbf{x})\) where \(x_{a_j} = x_j, 0 \leq j \leq k\). Let us argue that we defined the \(B_i, C_i\) such that, for \(0 \leq i \leq k\),
\[(13) \quad I(\tau_k G; x_0, \ldots, x_k) = C_i I \left( \tau_{k-i} G; x_0, \ldots, x_{k-i-1}, \frac{B_i}{C_i} \right).\]

This is trivial for \(i = 0\). As we have
\[1 + \frac{B_i}{C_i} (1 + x_{k-i-1} u^2) = \frac{C_i + B_i (1 + x_{k-i-1} u^2)}{C_i} = \frac{C_{i+1}}{C_i},\]
we see that (13) holds for all \(1 \leq i \leq k\): Use Lemma 3.6 in the following inductive step:
\[I(\tau_k G; x_0, \ldots, x_k) = C_i I(\tau_{k-i} G; x_0, \ldots, x_{k-i-1}, \frac{B_i}{C_i})\]
\[= C_i \frac{C_{i+1}}{C_i} I \left( \tau_{k-i-1} G; x_0, \ldots, x_{k-i-2}, \frac{x_{k-i-1} C_i}{C_{i+1}} \right)\]
\[= C_{i+1} I \left( \tau_{k-i-1} G; x_0, \ldots, x_{k-i-2}, \frac{B_{i+1}}{C_{i+1}} \right).\]

Thus, (10) holds as a polynomial equality.

Let us now consider \(x\) as a real number, \(x > -1/4\). Matrix \(M(x)\) in (12) can be diagonalized as \(M(x) = SDS^{-1}\) with
\[S = \begin{pmatrix} x & x \\ \lambda_1 & \lambda_2 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad S^{-1} = \frac{1}{x(\lambda_2 - \lambda_1)} \begin{pmatrix} \lambda_2 & -x \\ -\lambda_1 & x \end{pmatrix},\]
where \(\lambda_{1,2}\) as in (4). Now we substitute variable \(x_v\) by real number \(x\) for all \(v \in \{a_0, \ldots, a_k\}\) and \(M(x)\) by \(SDS^{-1}\). This yields the statement of the theorem. \(\square\)

Now we see that Theorem 3.5 can be proved by repeated application of Lemma 3.7 and Theorem 3.10.

### 4. Interpolation via \(S\)-clones

In this section, we give a reduction from evaluation of the independent set polynomial at a fixed point \(x \in \mathbb{Q} \setminus \{0\}\) to computation of the coefficients of the independent set polynomial. Thus, given a graph \(G\) with \(n\) vertices, we would like to interpolate \(I(G; X)\), where \(X\) is a variable. The degree of this polynomial is at most \(n\), thus it is sufficient to know \(I(G; x_i)\) for \(n + 1\) different values \(x_i\). Our approach is to modify \(G\) in \(n + 1\) different ways to obtain \(n + 1\) different graphs \(G_0, \ldots, G_n\). Then we evaluate \(I(G_0; x), I(G_1; x), \ldots, I(G_n; x)\). We will prove that \(I(G_i; x) = p_i I(G; x_i)\) for \(n + 1\) easy to compute \(x_i\) and \(p_i\), where \(x_i \neq x_j\) for all \(i \neq j\). This will enable us to interpolate \(I(G; X)\).

If the modified graphs \(G_i\) are \(c\) times larger than \(G\), we lose a factor of \(c\) in the reduction, i.e. a \(2^n\) running time lower bound for evaluating the graph polynomial at \(x\) implies only a \(2^{n/c}\) lower bound for evaluation at the interpolated points.
Thus, we can not afford simple cloning (i.e. constructing $\kappa^2 G$, $\kappa^3 G$, ... to use Theorem 3.9): To get enough points for interpolation, we would have to evaluate the graph polynomials on graphs of sizes $2n, 3n, \ldots, n^2$. To overcome this problem, we transfer a technique of Dell, Husfeldt, and Wahlén [7], which they developed for the Tutte polynomial to establish a similar reduction: We clone every vertex $O(\log n)$ times and use $n$ different ways to add paths of different (but at most $O(\log n)$) length at the different clones. Eventually, this will lead to the following result:

**Theorem 4.1.** Let $x_0 \in \mathbb{Q}$ such that $x_0$ is nondegenerate for path reduction and the independent set polynomial $I$ of every $n$-vertex graph $G$ can be evaluated at $x_0$ in time $2^{O(n/\log^3 n)}$.

Then, for every $n$-vertex graph $G$, the $X$-coefficients of the independent set polynomial $I(G; X)$ can be computed in time $2^{O(n)}$. In particular, the independent set polynomial $I(G; x_1)$ can be evaluated in this time for every $x_1 \in \mathbb{Q}$.

Using this theorem, we can prove our main result.

**Proof of Theorem 4.1.** For $x > -1/4$, the corollary follows from Theorem 4.1 and Theorem 3.1.

Let us now consider $x < -2$. Then we have $|1 + x| > 1$, which implies $(1 + x)^2 - 1 > 0$. On input graph $G = (V, E)$, we have $I(\kappa^2 G; x) = I(G; (1 + x)^2 - 1)$ by Theorem 3.9. Graph $\kappa^2 G$ has $2|V|$ vertices. This establishes a reduction from $I(-; (1 + x)^2 - 1)$ to $I(-; x)$, where the instance size increases only by a constant factor. As $(1 + x)^2 - 1 > 0$, we have reduced from an evaluation point where we have already proved the (conditional) lower bound of the lemma. Thus, the same bound, which is immune to constant factors in the input size, holds for $I(-; x)$.

Let us consider $x \in (-2, 0) \setminus \{-1\}$. We have $|x + 1| < 1$ and $|x + 1| \neq 0$. In a similar way as we just used a 2-clone, we can use the comb reduction [2] Section 3.2]. Let $k$ be a positive even integer such that $k > \frac{\log(-2x)}{\log(|x|+1)}$. Then we have $y := \frac{x}{(1+x)^2} < -2$. On input graph $G = (V, E)$, we can construct $G_k$ as in the comb identity for the interlace polynomial [2] Theorem 3.5, and we have $I(G_k; x) = (1 + x)^k I(G; y)$. As $k$ does not depend on $n = |V|$, $|V(G_k)| = O(|V|)$. Thus, we have reduced from $y < -2$, an evaluation point where we have already proved the lower bound, to evaluation at $x$.

To handle $x = -2$ and $x = -1$, add cycles [2] Theorem 3.7 and Proposition 3.8.

The rest of this section is devoted to the proof of Theorem 4.1 which is quite technical. The general idea is similar to Dell et al. [7] Lemma 4, Theorem 3(ii)].

**Definition 4.2.** Let $S$ be a set of numbers. Then we define $\|S\| = \sum_{s \in S} s$.

**Remark 4.3.** The $S$-clone $G_S$ of a graph $G = (V, E)$ has $|V|(\|S\| + |S|)$ vertices.

**Lemma 4.4.** Assume that $x \in \mathbb{R}$ is nondegenerate for path reduction. Then there are sets $S_0, S_1, \ldots, S_n$ of positive integers, constructible in time $\text{poly}(n)$, such that

1. $x(S_i) \neq x(S_j)$ for all $i \neq j$ and
2. $|S_0| \in O(\log^3 n)$ and $|S_i| \in O(\log n)$ for all $i, 0 \leq i \leq n$.

**Proof.** We use the notation from Theorem 3.10 and assume $\lambda_1 > \lambda_2$. 
As $\frac{|\lambda_1|}{|\lambda_2|} \to \infty$ for $k \to \infty$, there is a positive integer $s_0$ such that

$$\left(\frac{\lambda_1}{\lambda_2}\right)^s \notin \left\{ \left(\frac{\lambda_2}{\lambda_1}\right)^2 \frac{\lambda_2(x + \lambda_2)}{\lambda_1(x + \lambda_1)} \right\} \forall s \geq s_0.$$

Thus, for every $i, 0 \leq i \leq n$, the following set fulfills the precondition on $S$ and $T$ in Lemma 4.5

$$S_i = \{ s_0 + \Delta(2j + b_1(i)) \mid 0 \leq j \leq \lfloor \log n \rfloor \},$$

where $\Delta$ is a positive integer defined later, $\Delta \in \Theta(\log n)$, and $[b_1(i), \ldots, b_1(i), b_0(i)]$ is the binary representation of $i$. Note that this construction is very similar to Dell et al. [7, Lemma 4]. It is important that the elements in these sets have distance at least $\Delta$ from each other. The sets are $\text{poly}(n)$ time constructible as $s_0$ does not depend on $n$. We have $|S_i| \leq (1 + \log n)(s_0 + (1 + 2 \log n)\Delta)$ and obviously $|S_i| \in O(\log n)$ for all $i$. Thus, the second statement of the lemma holds.

To prove the first statement, we use Lemma 4.5. Let $1 \leq i < j \leq n$ and $S = S_i \setminus S_j, T = S_j \setminus S_i$. Let $s_1$ be the smallest number in $S \cup T$ and $A_1 = (S \cup T) \setminus \{s_1\}$. For $f$ as in Lemma 4.5 let us prove that $|f(A_1)| > \sum_{A \subseteq S \cup T \mid f(A)}$, which yields the statement of Lemma 4.4.

Assume without loss of generality that $s_1 \in S$. As $x$ is nondegenerate, $C_1 := \min\{1, |\lambda_1|, |\lambda_2|, |x + \lambda_1|, |x|, |\lambda_1 - \lambda_2|\}$ is a nonzero constant. As $|S| = |T|$, 

$$D(S, T, A_1) = \lambda_1|T|^2(x + \lambda_1)|S|^{-1}(x + \lambda_2) - \lambda_1|S|^{-2}\lambda_2(x + \lambda_1)|T|$$

$$= \lambda_1|S|^{-1}(x + \lambda_1)|S|^{-1}(\lambda_1(x + \lambda_2) - \lambda_2(x + \lambda_1))$$

$$= \lambda_1|S|^{-1}(x + \lambda_1)|S|^{-1}x(\lambda_1 - \lambda_2),$$

and we have

$$|f(A_1)| \geq |\lambda_1|^{||S \cup T|| - s_1} |\lambda_2|^{s_1} C_1^{7|S|}. \tag{14}$$

If $A = \emptyset$ or $A = S \cup T$, we have $D(A) = 0$, which implies $f(A) = 0$. For every $A \subseteq S \cup T, A \neq \emptyset, A \neq S \cup T, A \neq A_1$, we have $|A| \leq ||S \cup T|| - s_1 - \Delta$. Thus,

$$|f(A)| \leq |\lambda_1|^{||S \cup T|| - s_1 - \Delta} |\lambda_2|^{s_1 + \Delta} C_2^{7|S|}, \tag{15}$$

where $C_2 = 2 \max\{1, |\lambda_1|, |\lambda_2|, |x + \lambda_1|, |x + \lambda_2|\}$. There are less than $2^{\lfloor \log n \rfloor + 1} \leq 2n^2$ such $A$. Combining this with (14) and (15), it follows that we have proved the lemma if we ensure

$$\frac{|\lambda_1|}{|\lambda_2|} > \left(\frac{C_2}{C_1}\right)^{7|S|} 2n^2.$$

This holds if

$$\Delta > 7 \left((\log n + 1) \log \frac{C_2}{C_1} + 2 \log n + 1 \right)/ \log \frac{\lambda_1}{\lambda_2}.$$

As $C_1$, $C_2$, $\lambda_1$, $\lambda_2$ do not depend on $n$, we can choose $\Delta \in \Theta(\log n)$.
Lemma 4.5. Let $S$ and $T$ be two sets of positive integers. Let also $x \in \mathbb{R}$ be nondegenerate for path reduction and, for all $s \in S \cup T$,

\begin{equation}
(\frac{\lambda_1}{\lambda_2})^{s+2} \neq 1 \quad \text{and} \quad (\frac{\lambda_1}{\lambda_2})^{s+1} \neq \frac{x+\lambda_2}{x+\lambda_1},
\end{equation}

where $\lambda_1, \lambda_2$ are defined as in Theorem 3.10. Then we have $x(S) = x(T)$ iff

$$
\sum_{A \subseteq S \triangle T} f(A) = 0,
$$

where

$$
f(A) = \lambda_1^{\|A\|} \lambda_2^{\|(S \triangle T) \setminus A\|} (-\lambda_1)^{\|A\|} \lambda_2^{\|(S \triangle T) \setminus A\|} \cdot D(S \setminus T, T \setminus S, A),
$$

$$
D(S, T, A) = c(S, T, A \cap S, A \cap T) - c(T, S, A \cap T, A \cap S),
$$

$$
c(S, T, S_0, T_0) = \lambda_1^{\|T_0\|} \lambda_2^{\|T \setminus T_0\|} (x+\lambda_1)^{\|S_0\|} (x+\lambda_2)^{\|S_0\|}.
$$

Proof. Let $\tilde{S} = S \setminus T$ and $\tilde{T} = T \setminus S$. We have $x(S) = x(T)$ iff $x(S) + 1 = x(T) + 1$. Condition (17) ensures $1 + \frac{\lambda_2}{\lambda_1} \neq 0$ for all $s \in S \cup T$. Thus, $x(S \cap T) + 1 \neq 0$, and $x(S) = x(T)$ iff $x(\tilde{S}) + 1 = x(\tilde{T}) + 1$. This is equivalent to $Y(\tilde{S}, \tilde{T}) = Y(T, S)$, where $Y(S, T) = \prod_{s \in S} (C_s + B_s) \prod_{t \in T} C_t$. For sets of integers $M \subseteq N$, let us define

$$
B(N, M) = \lambda_1^{\|M\|} (-\lambda_1)^{\|M\|} \lambda_2^{\|N \setminus M\|} \lambda_2^{\|N \setminus M\|}
$$

and

$$
C(N, M) = \lambda_1^{\|M\|} \lambda_2^{2\|M\|} (-1)^{\|M\|} \lambda_2^{\|N \setminus M\|} \lambda_2^{\|N \setminus M\|}.
$$

Using this notation, it is

$$
Y(S, T) = \prod_{s \in S} (B_s + C_s) \prod_{t \in T} C_t
$$

$$
= \sum_{S_0 \subseteq S, S_0 \cap T_0 \neq \emptyset} \prod_{s \in S_0} B_s \prod_{s \not\in S \setminus S_0} C_s \prod_{t \in T} C_t
$$

$$
= (\lambda_2 - 1)^{-|S|-|T|} \sum_{S_0 \subseteq S} \sum_{S_1 \subseteq S_0} \sum_{S_2 \subseteq S \setminus S_0} B(S_0, S_1) C(S \setminus S_0, S_2)
$$

$$
\sum_{T_0 \subseteq T} C(T, T_0).
$$

We want to collect the terms $\lambda_1^{\|M\|}$ and $\lambda_2^{\|N \setminus M\|}$ in one place. Thus, we change the order in which $S$ is split into subsets $S_0, S_1, S_2$ (cf. Figure 2) such that we first choose $S_1 := S_1 \cup S_2 \subseteq S$, then $S_1 \subseteq S_1$ (which implies $S_2 = S_1 \setminus S_1$), and finally $S_0$ as $S_1 \subseteq S_0 \subseteq S \setminus S_2$. Now we can write

$$
Y(S, T) = (\lambda_2 - 1)^{-|S|-|T|} \sum_{S_1 \subseteq S} \sum_{S_0 \subseteq S \setminus S_1 \setminus T_0} \lambda_1^{\|S_1\| + |S_0\|} \lambda_2^{\|S \setminus S_1\| + |T \setminus T_0|}
$$

$$
\left((-\lambda_1)^{|S_1| + |S_0|} \lambda_2^{|S \setminus S_1| + |T \setminus T_0|}
\right) c(S, T, S_1, T_0),
$$

where

$$
c(S, T, S_1, T_0) = \lambda_1^{\|T_0\|} \lambda_2^{\|T \setminus T_0\|} \sum_{S_1 \cup S_2 = S_1} \sum_{S_1 \subseteq S_0 \subseteq S \setminus S_2} x^{\|S_0\|} \lambda_1^{\|S_0\|} \lambda_2^{\|S \setminus S_0\|}.
$$
Note that (13) as symmetric in \(S\) and \(T\), except for the term \(c(S, T, S_{12}, T_0)\). Let us analyze this non-symmetrical term. We write \(S_0 = S_1 \cup \tilde{S}_0\).

\[
c(S, T, S_{12}, T_0) = \lambda_1^{|T_0|} \lambda_2^{|T \setminus T_0|} \sum_{S_1 \cup S_2 = S_{12}} \lambda_1^{|S_2|} x^{|S_0|} \sum_{\tilde{S}_0 \subseteq S \setminus S_{12}} x^{|\tilde{S}_0|} \lambda_2^{|(S \setminus S_{12}) \setminus S_0|}
\]

\[
= \lambda_1^{|T_0|} \lambda_2^{|T \setminus T_0|} \sum_{S_1 \cup S_2 = S_{12}} \lambda_1^{|S_2|} x^{|S_0|} (x + \lambda_2)^{|S \setminus S_{12}|}
\]

\[
= \lambda_1^{|T_0|} \lambda_2^{|T \setminus T_0|} (x + \lambda_1)^{|S_{12}|} (x + \lambda_2)^{|S \setminus S_{12}|}.
\]

This implies the statement of the lemma.

\[\square\]

**Proof of Theorem 4.1.** On input \(G = (V, E)\) with \(|V| = n\), do the following. Construct \(G_{S_0}, G_{S_1}, \ldots, G_{S_n}\) with \(S_i\) from Lemma 4.4. Every \(G_{S_i}\) can be constructed in time polynomial in \(|G_{S_i}|\), which is \(\text{poly}(n)\) by Remark 4.3 and by condition 2 of Lemma 4.4. Thus, the whole construction can be performed in time \(\text{poly}(n)\).

Again by condition 2 of Lemma 4.4, there is some \(c' > 1\) such that all \(G_{S_i}\) have \(\leq c' n \log^3 n\) vertices. Evaluate \(I(G_{S_0}; x)\), \(I(G_{S_1}; x)\), \ldots, \(I(G_{S_n}; x)\). By the assumption of the theorem, one such evaluation can be performed in time

\[
2^c \frac{c' n \log^3 n}{(\log (c' n \log^3 n))^c} = 2^c \frac{c' n \log^3 n}{(\log c' + \log n + 3 \log \log n)^c} \leq 2^c \frac{c' n \log^3 n}{(\log n)^c} = 2^c c' n
\]

for every \(c > 0\).

Using Theorem 3.5, we can compute \(I(G; x(S_0))\), \(I(G; x(S_1))\), \ldots, \(I(G; x(S_n))\) from the already computed \(I(G_{S_i}; x)\) in time \(\text{poly}(n)\).

By condition 1 of Lemma 4.3 the \(n + 1\) values \(x(S_i)\) are pairwise distinct. As \(I(G; X)\) is a polynomial of degree at most \(n\) in \(X\), this enables us to interpolate \(I(G; X)\). The overall time needed is \(\text{poly}(n) 2^{c' n} \leq 2^{(c' + \varepsilon) n}\) for every \(\varepsilon > 0\).

\[\square\]

5. **Open Problems**

The most important open problem is to find a reduction that does not lose the factor \(\Theta(\log^3 n)\) in the exponent of the running time.

Another interesting direction for further research are restricted classes of graphs, for example graphs of bounded maximum degree or regular graphs.

The independent set polynomial is a special case of the two-variable interlace polynomial [1]. It would be interesting to have an exponential time hardness result for this polynomial as well. In this context, the following question arises: Is the upper bound \(\exp(O(\sqrt{n}))\) [25] for evaluation of the Tutte polynomial on planar graphs sharp?

**Acknowledgments.** I would like to thank Raghavendra Rao and the anonymous referees for helpful comments.
REFERENCES

1. Richard Arratia, Béla Bollobás, and Gregory B. Sorkin, A two-variable interlace polynomial, Combinatorica 24 (2004), no. 4, 567–584.

2. Markus Bläser and Christian Hoffmann, On the complexity of the interlace polynomial, 25th International Symposium on Theoretical Aspects of Computer Science (STACS) (Susanne Albers and Pascal Weil, eds.), Dagstuhl Seminar Proceedings, vol. 08001, Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany, 2008, Updated full version: arXiv:cs.CC/0707.4565v3, pp. 97–108.

3. Nicolas Bourgeois, Bruno Escoffier, Vangelis Th. Paschos, and Johan M. M. van Rooij, A bottom-up method and fast algorithms for max independent set, SWAT (Haim Kaplan, ed.), Lecture Notes in Computer Science, vol. 6139, Springer, 2010, pp. 62–73.

4. Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, Introduction to algorithms, 2. ed. ed., MIT Press, 2001.

5. Bruno Courcelle, A multivariate interlace polynomial and its computation for graphs of bounded clique-width, The Electronic Journal of Combinatorics 15 (2008), no. 1.

6. Vilhelm Dahllof and Peter Jonsson, An algorithm for counting maximum weighted independent sets and its applications, SODA, 2002, pp. 292–298.

7. Holger Dell, Thore Husfeldt, and Martin Wahlén, Exponential time complexity of the permanent and the Tutte polynomial, ICALP (1) (Samson Abramsky, Cyril Gavoille, Claude Kirchner, Friedhelm Meyer auf der Heide, and Paul G. Spirakis, eds.), Lecture Notes in Computer Science, vol. 6198, Springer, 2010, Full paper: Electronic Colloquium on Computational Complexity TR10-078, pp. 426–437.

8. Martin E. Dyer and Catherine S. Greenhill, On Markov chains for independent sets, J. Algorithms 35 (2000), no. 1, 17–49.

9. Fedor V. Fomin, Fabrizio Grandoni, and Dieter Kratsch, A measure & conquer approach for the analysis of exact algorithms, J. ACM 56 (2009), no. 5.

10. Michael R. Garey and David S. Johnson, Computers and intractability – a guide to the theory of np-completeness, Freeman, 1979.

11. I. Gutman and F. Harary, Generalizations of the matching polynomial, Utilitas Math. 24 (1983), 97–106.

12. Cornelis Hoede and Xueliang Li, Clique polynomials and independent set polynomials of graphs, Discrete Mathematics 125 (1994), no. 1-3, 219 – 228.

13. Christian Hoffmann, Computational complexity of graph polynomials, Ph.D. thesis, Saarland University, Department of Computer Science, 2010.

14. Russell Impagliazzo, Ramamohan Paturi, and Francis Zane, Which problems have strongly exponential complexity?, J. Comput. Syst. Sci. 63 (2001), no. 4, 512–530.

15. Mark Jerrum, Leslie G. Valiant, and Vijay V. Vazirani, Random generation of combinatorial structures from a uniform distribution, Theor. Comp. Sc. 43 (1986), 169–188.

16. Tang Jian, An $O(2^{0.304n})$ algorithm for solving maximum independent set problem, IEEE Trans. Computers 35 (1986), no. 9, 847–851.

17. Richard M. Karp, Reducibility among combinatorial problems, Complexity of Computer Computations (R. E. Miller and J. W. Thatcher, eds.), Plenum Press, New York, 1972, pp. 85–103.

18. Joachim Kneis, Alexander Langer, and Peter Rossmanith, A fine-grained analysis of a simple independent set algorithm, IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2009) (Dagstuhl, Germany) (Ravi Kannan and K Narayan Kumar, eds.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 4, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2009, pp. 287–298.

19. Michael Luby and Eric Vigoda, Approximately counting up to four (extended abstract), STOC, 1997, pp. 682–687.

20. Christos H. Papadimitriou, Computational complexity, Addison Wesley Longman, 1994.

21. J. M. Robson, Algorithms for maximum independent sets, J. Algorithms 7 (1986), no. 3, 425–440.

22. Dan Roth, On the hardness of approximate reasoning, Artif. Intell. 82 (1996), no. 1-2, 273–302.

23. Thomas J. Schaefer, The complexity of satisfiability problems, STOC, ACM, 1978, pp. 216–226.
24. Alexander D. Scott and Alan D. Sokal, The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma, J. Stat. Phys. 118 (2005), 1151.
25. Kyoko Sekine, Hiroshi Imai, and Seiichiro Tani, Computing the Tutte polynomial of a graph of moderate size, ISAAC (John Staples, Peter Eades, Naoki Katoh, and Alistair Moffat, eds.), Lecture Notes in Computer Science, vol. 1004, Springer, 1995, pp. 224–233.
26. Alan D. Sokal, Chromatic roots are dense in the whole complex plane, Combinatorics, Probability & Computing 13 (2004), no. 2, 221–261.
27. Robert Endre Tarjan and Anthony E. Trojanowski, Finding a maximum independent set, SIAM J. Comput. 6 (1977), no. 3, 537–546.
28. Leslie G. Valiant, The complexity of enumeration and reliability problems, SIAM Journal on Computing 8 (1979), no. 3, 410–421.
29. Eric Vigoda, A note on the glauber dynamics for sampling independent sets, Electr. J. Comb. 8 (2001), no. 1.
30. Dror Weitz, Counting independent sets up to the tree threshold, STOC (Jon M. Kleinberg, ed.), ACM, 2006, pp. 140–149.

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