QUANTUM EXOTIC PDE’S

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ABSTRACT. Following the previous works on the A. Prástaro’s formulation of algebraic topology of quantum (super) PDE’s, it is proved that a canonical Heyting algebra (integral Heyting algebra) can be associated to any quantum PDE. This is directly related to the structure of its global solutions. This allows us to recognize a new inside in the concept of quantum logic for microworlds. Furthermore, the Prástaro’s geometric theory of quantum PDE’s is applied to the new category of quantum hypercomplex manifolds, related to the well-known Cayley-Dickson construction for algebras. Theorems of existence for local and global solutions are obtained for (singular) PDE’s in this new category of noncommutative manifolds. Finally the extension of the concept of exotic PDE’s, recently introduced by A.Prástaro, has been extended to quantum PDE’s. Then a smooth quantum version of the quantum (generalized) Poincaré conjecture is given too. These results extend ones for quantum (generalized) Poincaré conjecture, previously given by A. Prástaro.

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1. INTRODUCTION

This paper aims to further develop the A. Prástaro’s geometric theory of quantum PDE’s, by considering three different (even if related) subjects in this theory. The first is a way to characterize quantum PDE’s by means of suitable Heyting algebras. Nowadays these algebraic structures are considered important in order to characterize quantum logics and quantum topoi. (See, e.g., [33, 43, 44, 48].) Really we prove that to any quantum PDE can be associated a Heyting algebra, naturally arising from the algebraic topologic structure of the PDE’s and that encodes its integral bordism group. Another aspect that we shall consider is the extension of the category $\mathcal{Q}$ of quantum manifolds, or $\mathcal{Q}_S$ of quantum supermanifolds, to the ones $\mathcal{Q}_{\text{hyper}}$ of quantum hypercomplex manifolds. These generalizations are obtained by extending a quantum algebra $A$, in the sense of A. Prástaro, by means of Cayley-Dickson algebras. In this
way one obtains a new category of noncommutative manifolds, that are useful in some geometric and physical applications. In fact there are some fashioned research lines, concerning classical superstrings and classical super-2-branes, where one handles with algebras belonging just to some term in the Cayley-Dickson construction. Thus it is interesting to emphasize that the Prástaro’s geometric theory of quantum PDE’s can be directly applied also to PDE’s for such quantum hypercomplex manifolds. This allows us to encode quantum micro-worlds, by a general theory that goes beyond the classical simple description of classical extended objects, and solves also the problem of their quantization.

Let us emphasize that in some previous works we have formulated a geometric theory of quantum manifolds that are noncommutative manifolds, where the fundamental algebra is a suitable associative noncommutative topological algebra, there called quantum algebra. Extensions to quantum supermanifolds and quantum-quaternionic manifolds are also considered too. Furthermore, we have built a geometric theory of quantum PDE’s in these categories of noncommutative manifolds, that allows us to obtain theorems of existence of local and global solutions, and constructive methods to build such solutions too.[61, 62, 63, 64, 65, 66, 67, 68, 77, 78, 79, 80, 81, 83, 84].

On the other hand it is well known that the sequences
\[ \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset S, \]
where \( S \) is the sedenionic algebra, fit in the so-called Cayley-Dikson construction,
\[ \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset S \subset \cdots \subset \mathbb{A}_r \subset \cdots, \]
where \( \mathbb{A}_r \) is a Cayley algebra of dimension \( 2^r \): \( \mathbb{A}_r \cong \mathbb{R}^{2^r}, r \geq 0, \mathbb{A}_0 = \mathbb{R} \). Thus we can also consider quantum hypercomplex algebras \( \mathbb{A} \otimes \mathbb{A}_r, 1 \leq r \leq \infty \), where \( \mathbb{A} \) is a quantum algebra in the sense of Prástaro, obtaining the natural inclusions \( \mathbb{A} \otimes \mathbb{A}_r \hookrightarrow \mathbb{A} \otimes \mathbb{A}_{r+1} \). It is important to note that the Cayley algebras \( \mathbb{A}_r \) are not associative for \( r \geq 3 \), (even if they are exponential associative\(^1\)) hence also the corresponding quantum hypercomplex algebras are non-associative for \( r \geq 3 \), despite the associativity of the quantum algebra \( \mathbb{A} \). This fact introduces some particularity in the theory of PDE’s on such algebras. The purpose of this paper is just to study which new behaviours have PDE’s on the category \( \Omega_{\text{hyper}} \) of quantum hypercomplex manifolds.

Let us emphasize that a first justification to use the category of quantum (super) manifolds to formulate PDE’s that encode quantum physical phenomena, arises from the fact that quantized PDE’s can be identified just with quantum (super) PDE’s, i.e., PDE’s for such noncommutative manifolds. However, to formulate equations just in the category \( \Omega \) (or \( \Omega_S \)) allows us to go beyond the point of view of quantization of classical systems, and capture more general nonlinear phenomena in quantum worlds, that should be impossible to characterize by some quantization process. (See Refs. [77, 78, 79, 80, 81, 83, 84].) In fact the concept of quantum algebra (or quantum superalgebra) is the first important brick to put in order to build a theory on quantum physical phenomena. In other words it is necessary to extend the fundamental algebra of numbers, \( \mathbb{R} \), to a noncommutative algebra \( \mathbb{A} \), just called quantum algebra. The general request on such a type of algebra can be obtained on the ground of the mathematical logic. (See [63, 68].) In fact, we have shown that the meaning of quantization of a classical theory, encoded by a PDE \( E_k \), in the category of commutative manifolds, is a representation of the logic \( \mathcal{L}(E_k) \) of the classic theory, into a quantum logic \( \mathcal{L}_q \). More precisely, \( \mathcal{L}(E_k) \) is the Boolean algebra of subsets of the set \( \Omega(E_k)_c \) of solutions of \( E_k \): \( \mathcal{L}(E_k) \equiv \mathcal{P}(\Omega(E_k)_c) \). (The

\(^1\) i.e., \( z^{n+m} = z^n z^m, \forall z \in \mathbb{A}_r \) and \( n, m \in \mathbb{N} \).
infinite dimensional manifold $\Omega(E_k)_c$ is called also the \textit{classic limit} of the quantum situs of $E_k$. Furthermore, $\mathcal{L}_q$ is an algebra $A$ of (self-adjoint) operators on a locally convex (or Hilbert) space $\mathcal{H}$: $\mathcal{L}_q \equiv A \subset L(\mathcal{H})$. Then to quantize a PDE $E_k$, means to define a map $\mathcal{L}(E_k) \to \mathcal{L}_q$, or an homomorphism of Boolean algebras $q : \mathcal{P}(\Omega(E_k)_c) \to \mathcal{P}_r(\mathcal{H})$, where $\mathcal{P}_r(\mathcal{H})$ is a Boolean algebra of projections on $\mathcal{H}$. (For details see [63, 68].) This construction allowed us also to prove that a quantization of a classical theory, can be identified by a functor relating the category of differential equations for commutative (super)manifolds, with the category of quantum (super) PDE’s. (See Refs. [59, 77, 79, 80, 81, 83, 84].)

We can also extend a quantum algebra $A$, when the particular mathematical (or physical) problem requires it useful. Then the extended algebra does not necessitate to be associative. This is, for example, the case when the extension is made by means of some algebra in the Cayley-Dickson construction, obtaining a \textit{quantum-Cayley-Dickson construction}:

\[
\begin{array}{c}
Q_0 \hookrightarrow Q_1 \hookrightarrow Q_2 \hookrightarrow \cdots \hookrightarrow Q_r \hookrightarrow \cdots
\end{array}
\]

where $Q_r \equiv A \otimes_R A_r$, hence $Q_0 = A$, $Q_1 = A \otimes_R \mathbb{C}$ and $Q_2 = A \otimes_R \mathbb{H}$, etc.

The main of this paper is just to show that the Prástaro’s algebraic topologic theory of quantum PDE’s, formulated starting from 80s, directly applies to these non-associative quantum algebras arising in the above quantum-Cayley-Dickson construction.

Finally, the last purpose of this paper is to extend the concept of \textit{exotic PDE’s}, recently introduced by A. Prástaro, for PDE’s in the category of commutative manifolds [85, 86, 87, 88], also for the ones in the category of quantum PDE’s. In particular, we prove also smooth versions of quantum generalized Poincaré conjectures.

In the following we list the main results of this paper, assembled for sections.

2. Theorem 2.42 and Theorem 2.43 show that to any quantum hypercomplex PDE $\hat{E}_k \subset \hat{J}_k(W)$, can be associated a topological spectrum (\textit{integral spectrum}) and a Heyting algebra (\textit{integral Heyting algebra}) encoding some algebraic topologic properties of such a PDE. This allows us to give a new constructive point of view to the actual approach to consider quantum logic in field theory by means of topoi. 3. Theorem 3.10 and Theorem 3.11 extend our formal geometric theory of PDE’s from the category of quantum (super)manifolds to the ones for quantum hypercomplex manifolds. Theorem 3.19 and Theorem 3.26 characterize global solutions of PDE’s in the category $\Omega_{\text{hyper}}$, by means of suitable bordism groups. 4. Theorem 4.8 gives an algebraic topologic characterization of singular PDE’s in the category $\Omega_{\text{hyper}}$. 5. Here we extend to the category $\Omega_{\text{hyper}}$, the concept of exotic PDE’s, perviously introduced by A. Prástaro for PDE’s in the category of commutative manifolds. Theorem 5.38 and Theorem 5.39 give characterizations of global solutions for exotic PDE’s in the category $\Omega_{\text{hyper}}$, that allow to classify smooth solutions starting from quantum homotopy spheres. In particular, an integral $h$-cobordism theorem in quantum Ricci flow PDE’s is proved.

2. Spectra in Quantum PDE’s

In this section we prove that to any quantum PDE can be canonically associated a Heyting algebra directly related to its integral bordism group. This is made

\[\text{\textsuperscript{2}Compare with the meaning of quantum logic given by G. Birkoff and J. Von Neumann [7].}\]
since an algebraic topologic spectrum is recognized to characterize such an integral bordism group, and topologic spectra can be encoded in Heyting algebras. This result allows us to look to the meaning of quantum logic from a more gen eral point of view, since Heyting algebras contain Boolean algebras. In order to make this paper as self-contained as possible, let us first recall some fundam ental definitions and results about Heyting algebras. (There propositions are gene rally given without proof since they are standard. See, e.g., Refs [6, 90] for more informations about.)

**Definition 2.1.** 1) A partially ordered system \((E, \leq)\) is a non-empty set \(E\), together with a relation \(\leq\) on \(E\), such that: (a) if \(a \leq b\) and \(b \leq c\) \(\Rightarrow a \leq c\); (b) \(a \leq a\). The relation \(\leq\) is called an order relation in \(E\). The notation \(y \geq x\) is sometimes used in place of \(\leq\).

2) A totally ordered subset \(F\) of partially ordered system \((E, \leq)\) is a subset of \(E\) such that for every pair \(x, y \in F\) either \(x \leq y\) or \(y \leq x\).

3) If \(F\) is a subset of a partially ordered system \((E, \leq)\) then an element \(x\) in \(E\) is said to be an upper bound for \(F\) if every \(f \in F\) has the property \(f \leq x\). An upper bound for \(F\) is said to be a least upper bound of \(F\) (sup \(F\)) if every upper bound \(y\) of \(F\) has the property \(x \leq y\).

4) In a similar fashion, the terms lower bound and greatest lower bound (in \(F\)) may be defined.

5) An element \(x \in E\) is said to be maximal if \(x \leq y\) implies \(y \leq x\).

**Example 2.2.** Let \(X\) be a set and \(\mathcal{P}(X)\) be the set of all subsets of \(X\). The couple \((\mathcal{P}(X), \subseteq)\) is a partially ordered system where the order relation is the inclusion relation \(\subseteq\) between the sets contained in \(X\). An upper bound for a subfamily \(B \subseteq \mathcal{P}(X)\) is any set containing \(\cup B\), and \(\cup B\) is the only least upper bound of \(B\). Similarly \(\cap B\) is the only greatest lower bound of \(B\). The only maximal element of \(\mathcal{P}(X)\) is \(X\).

**Definition 2.3.** Let \((E, \leq)\) be a non-void partially ordered system. A subset \(B \subseteq E\) with the following three properties will be called admissible with respect to a function \(f : E \rightarrow E\) such that \(f(x) \geq x\); and with respect to an element \(a \in E\):

(a) \(a \in B\);
(b) \(f(B) \subseteq B\);
(c) Every least upper bound of a totally ordered subset of \(B\) is in \(B\).

**Theorem 2.4.** 1) (Hausdorff maximality theorem). Every partially ordered system contains a maximal totally ordered subsystem.

2) (Zorn’s lemma). A partially ordered system has a maximal element if every totally ordered subsystem has an upper bound.

3) (Zermelo well-ordering theorem). Every set \(E\) may be well-ordered that is a partially ordered system \((E, \leq)\) such that:

(i) \(a \leq b, b \leq a \Rightarrow a = b\);
(ii) any non-void subset of \(E\) contains a lower bound for itself.

**Example 2.5.** The set \(\mathbb{N}\) of positive integers in the usual order is a familiar example of well-ordered system.

**Definition 2.6.** A partially ordered system \((E, \leq)\) is said to be complete if:

(i) \(a \leq b\) and \(b \leq a \Rightarrow a = b\);
(ii) Every non-void subset has a least upper bound and a greatest lower bound.
Definition 2.14. A connected compact Hausdorff space is isomorphic to the Boolean ring of all open and closed subsets of a totally disconnected logical space.

Theorem 2.13

Proposition 2.7 (Tarski). If \((E, \leq)\) is a complete partially ordered system, \(f : E \rightarrow E\), and \(x \leq y\) implies \(f(x) \leq f(y)\), then \(f\) has a fixed element \(x_0\), i.e., \(f(x_0) = x_0\), and the set of all fixed elements contains its least upper bound and its greatest lower bound.

Definition 2.8. An element \(x\) in a ring \(R\) is said to be idempotent if \(x^2 = x\), and to be nilpotent if \(x^n = 0\) for some positive integer \(n\). A Boolean ring is one in which every element is idempotent.

Proposition 2.9

Example 2.10. The smallest Boolean ring with unit is \(\mathbb{Z}_2 \equiv \{0, 1\}\), i.e., the set of integers modulo 2. \(\mathbb{Z}_2\) is actually a field. Conversely every Boolean ring with unit which is also a field is necessarily isomorphic to the field \(\mathbb{Z}_2\).

Proposition 2.7

Example 2.11. Every Boolean ring is a commutative ring where the identity \(x + x = 0\) (or equivalently \(x = -x\)) holds.

Example 2.12. In a Boolean ring, any prime ideal is maximal.

Example 2.13. The smallest Boolean ring with unit is \(\mathbb{Z}_2 \equiv \{0, 1\}\), i.e., the set of integers modulo 2. \(\mathbb{Z}_2\) is actually a field. Conversely every Boolean ring with unit which is also a field is necessarily isomorphic to the field \(\mathbb{Z}_2\).

Proposition 2.14

Definition 2.15. A Boolean algebra is a lattice with unit and zero which is distributive and complemented.

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3Let us recall that \(\mathbb{Z}_n\) has nonempty set \(\text{Zero}(\mathbb{Z}_n)\) of zero divisors iff \(n\) is composite, i.e., it is of the type \(n = pq\). If \(n\) is prime \(\text{Zero}(\mathbb{Z}_n) = \{0\}\). In this last case \(\mathbb{Z}_n\) is a field.
Proposition 2.16. The concepts of Boolean algebra and Boolean ring with unit are equivalent.

Proof. In fact let \( B \) be a Boolean algebra and define multiplication and addition as: \( xy = x \land y \), \( x + y = (x \land y)' \lor (x' \land y) \). Then, it may be verified that \( B \) is a Boolean ring with 1 as unit. On the other hand, if \( B \) is a Boolean ring with unit denoted by 1, then if \( x \leq y \) is defined to mean \( x = \lambda y \) and \( x' = 1 + x \), then \( B \) is a Boolean algebra and \( x \lor y = x + y \), \( x \land y = xy \).

Definition 2.17. If \( B \) and \( C \) are Boolean algebras and \( h : B \to C \), then \( h \) is said to be a Boolean algebra homomorphism, if \( h(x \land y) = h(x) \land h(y) \), \( h(x \lor y) = h(x) \lor h(y) \), \( h(x') = h(x)' \). If \( h \) is one-to-one, it is called an isomorphism Boolean algebra. If \( h \) is an isomorphism and \( h(B) = C \), then we say that \( B \) is isomorphic to \( C \).

Example 2.18. The following are examples of Boolean algebras:
1) \( \mathbb{Z}_2 \equiv \{0, 1\} \).
2) \( \mathcal{P}(X) \), where \( \leq \) is taken to be the inclusion, and \( \land \) and \( \lor \) are taken as intersection and union respectively.

Theorem 2.19. Every Boolean algebra is isomorphic with the Boolean algebra of all open and closed subsets of a totally disconnected compact Hausdorff space.

Definition 2.20. 1) A projection in a vector space \( V \) means a linear operator \( E \in L(V) \) with \( E^2 = E \).
2) The intersection \( A \land B \) and the union \( A \lor B \) of two commutating projections \( A \) and \( B \) in \( V \) are projections:
   (a) (intersection) \( A \land B = A \circ B \);
   (b) (union) \( A \lor B = A + B - (A \circ B) \).
The ranges of the intersection and union of two commutating projections are given by:
\( (A \land B)(V) = A(V) \cap B(V) \); \( (A \lor B)(V) = A(V) \oplus B(V) = Sp(A(V), B(V)) \), where \( Sp(A(V), B(V)) \) means the closed linear manifold spanned by the sets \( A(V) \) and \( B(V) \).
3) The natural ordering \( A \leq B \) between two commutating projections \( A \) and \( B \) has the geometrical significance that \( A \leq B \) is equivalent to \( A(V) \subseteq B(V) \).
4) A Boolean algebra of projections in \( V \) is a set of projections in \( V \) which is a Boolean algebra under the operations \( A \lor B \) and \( A \land B \) which has for its zero and unit elements the operators \( 0 \) and \( id_V \equiv 1 \in L(V) \).
5) The notions of abstract \( \sigma \)-completeness, and completeness can be extended.

Boolean algebras can be considered particular cases of more general algebras (Heyting algebras) that play important roles in Algebraic Topology.

Definition 2.21. A Heyting algebra \( H \) is a bounded lattice such that for all \( a, b \in H \) there is a greatest element \( x \in H \) such that \( a \land x \leq b \). This element \( x \) is called the relative pseudo-complement of \( a \) with respect to \( b \), and denoted \( x \equiv a \to b \). The largest (resp. smallest) element in \( H \) is denoted by 1 (resp. 0).
- The pseudo-complement of \( x \in H \) is the element \( \neg x = x \to 0 \).
- A complete Heyting algebra is a Heyting algebra that is a complete lattice.\(^4\)

\(^4\)One has \( a \land \neg a = 0 \), and \( \neg a \) is the largest element having this property. For a Boolean algebra, \( \neg a \) is a true complement, but for a Heyting algebra this is not generally assured. In fact, in general \( a \lor \neg a \neq 1 \).

\(^5\)Complete Heyting algebras are also called frames, or locales, or complete Brouwerian lattices. In a frame the meet distributes over infinite joins: \( a \land \lor b_i = \lor (a \land b_i) \).
• A subalgebra of a Heyting algebra $H$ is a subset $K \subset H$, containing 0 and 1 and closed under the operations $\land$, $\lor$ and $\rightarrow$.\(^6\)

• If there exists an element $a \in H$, such that, $\neg a = a$, i.e., negation has a fixed point, then $H = \{a\}$.

**Proposition 2.22.** The following propositions are equivalent.

(i) $H$ is a Heyting algebra.

(ii) (Lattice theoretic definition-a). $H$ is a bounded lattice and the mappings $f_a : H \rightarrow H$, $f_a(x) = a \land x$, $a \in H$, are the lower adjoint of a monotone Galois connection.\(^7\)

(iii) (Lattice theoretic definition-b). $H$ is a residual lattice whose monoid operation is $\land$.\(^8\)

(iv) (Bounded lattice with an implication). Let $H$ be a bounded lattice with largest and smallest elements 1 and 0 respectively, and a binary operation $\rightarrow$. such that the following hold:

1. $a \rightarrow a = 1$.
2. $a \land (a \rightarrow b) = a \land b$.
3. $b \land (a \rightarrow b) = a \land b$.
4. $a \rightarrow (b \land c) = (a \rightarrow b) \land (a \rightarrow c)$.

(v) (Axioms of intuitionistic propositional logic). $H$ is a set with three binary operations $\rightarrow$, $\land$ and $\lor$, and two distinguished elements 0 and 1, such that the following hold for any $x, y, z \in H$.\(^9\)

1. If $x \rightarrow y = 1$ and $y \rightarrow x = 1$ then $x = y$.
2. If $1 \rightarrow y = 1$ then $y = 1$.
3. $x \rightarrow (y \rightarrow x) = 1$.
4. $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$.
5. $x \land y \rightarrow x = 1$.
6. $x \land y \rightarrow y = 1$.
7. $x \rightarrow (y \rightarrow (x \land y)) = 1$.
8. $x \rightarrow x \lor y = 1$.
9. $y \rightarrow x \lor y = 1$.
10. $(x \rightarrow z) \rightarrow ((y \rightarrow z) \rightarrow (x \lor y \rightarrow z)) = 1$.
11. $0 \rightarrow x = 1$.

**Example 2.23.** Every Boolean algebra is a Heyting algebra with $p \rightarrow q \equiv \neg p \lor q$.

**Example 2.24.** Every totally ordered set that is a bounded lattice is also a Heyting algebra with $\rightarrow$ defined in (1).

\[
(1) \quad p \rightarrow q \equiv \begin{cases} q, & \text{if } p > q \\ 1, & \text{otherwise}. \end{cases}
\]

For example the set $H \equiv \{0, \frac{1}{2}, 1\}$, with $\rightarrow$ defined in (1) is a Heyting algebra that is not a Boolean algebra. It is important to emphasize that in this Heyting algebra does not hold the law of excluded middle. In fact one has: $\frac{1}{2} \lor \neg \frac{1}{2} = \frac{1}{2} \lor (\frac{1}{2} \rightarrow 0) = \frac{1}{2} \lor 0 = \frac{1}{2}$.\(^{10}\)

\(^{10}\)Then it follows that $K$ is closed also under $\neg$, hence $K$ is necessarily a Heyting algebra under the same operations of $H$.

\(^7\)Then, the relative upper adjoints $g_a$ are given by $g_a(x) = a \rightarrow x$.

\(^8\)The monoid unit must be top element 1. Commutativity of this monoid implies that the two residuals coincide as $a \rightarrow b$.

\(^9\)The relation $\leq$ is defined by the condition $a \leq b$ when $a \rightarrow b = 1$. Furthermore, $\neg x = x \rightarrow 0$.\(^{11}\)
Example 2.25 (Topological Heyting algebra). Let \((X, \mathcal{T})\) be a topological space, where \(\mathcal{T}\) is its sets of open sets. \(\mathcal{T}\) has a natural structure of lattice that is a complete Heyting algebra with binary operations given in (2).\(^\text{10}\)

\[
\begin{cases}
A \rightarrow B \equiv (\overline{C \cup B})^c = \overline{C \setminus B} \\
A \land B = A \cap B \\
A \lor B = A \cup B
\end{cases}
\]
\(A \leq B \iff A \subseteq B.\)

Example 2.26 (Topological Heyting algebras induced from partially ordered sets). Let \((X, \leq)\) be a partially ordered set. Then the increasing sets (resp. decreasing sets) form a topology \(\mathcal{T}^+\) (resp. \(\mathcal{T}^-\)) on \(X\), hence are identified topological Heyting algebras, say \(H^+\) (resp. \(H^-\)). In such algebras infinite distributive laws hold.

Proposition 2.27 (Properties of Heyting algebra). 1) (Provable identities). To prove true a formula \(F(A_1, \ldots, A_n)\) of the intuitionistic propositional calculus, by means \(\land\), \(\lor\), \(\neg\), \(\rightarrow\) and the constants 0 and 1, is equivalent to state \(F(a_1, \ldots, a_n) = 1\) for any \(a_1, \ldots, a_n \in H\), where \(H\) is a Heyting algebra generated by \(n\) variables. 2) (Distributivity). Heyting algebras are always distributive, i.e., relations (3) hold.

\[
\begin{align*}
(a \land (b \lor c)) &= (a \lor (b \land c)) \\
(a \lor (b \land c)) &= (a \lor b) \land (a \lor c)
\end{align*}
\]

• Furthermore in complete Heyting algebras one has relation given in (4)

\[
x \land (\bigwedge Y) = \bigwedge \{x \land y : y \in Y\} \forall x \in H, Y \subseteq H.
\]

• Vice versa, any complete lattice satisfying the infinite distributive law (4) is a complete Heyting algebra, with relative pseudo-complement operation defined by \(a \rightarrow b \equiv \bigwedge \{c | a \land c \leq b\}\).

Definition 2.28. Two complements elements \(x, y \in H\) of a Heyting algebra \(H\) are characterized by the following conditions: \(x \land y = 0\) and \(x \lor y = 1\). If \(x\) admits a complement, we say that it is complemented.

Proposition 2.29 (Properties of complement elements in Heyting algebra). 1) If there exists a complement \(y\) of \(x \in H\), it is unique and one has \(y = \neg x\). 2) If \(x \in H\) is complemented, then so is \(\neg x\), and both are to each other complement.\(^\text{11}\) 3) In any Heyting algebra, 0 and 1 are complements to each other.

Definition 2.30. A regular element \(x\) of a Heyting algebra \(H\) is characterized by either of the following equivalent conditions.\(^\text{12}\)

(i) \(x = \neg \neg x\).
(ii) \(x = \neg y\) for some \(y \in H\).

Proposition 2.31 (Properties of regular elements in Heyting algebra). 1) Any complemented element of a Heyting algebra is regular. (The converse is not true in general.) Therefore, 0 and 1 are regular elements. 2) For any Heyting algebra \(H\) the following conditions are equivalent.

\(^{10}\)Complete Heyting algebras are seen in categorical topology as generalized topological spaces.

\(^{11}\)Even if \(x\) is not complemented, \(\neg x\) can be complemented, with complement different from \(x\).

\(^{12}\)The equivalence of conditions (i) and (ii) follows from the fact that \(\neg \neg x = \neg x\).
(i) $H$ is a Boolean algebra.
(ii) Every $x \in H$ is regular.
(iii) Every $x \in H$ is complemented.
3) Any Heyting algebra, $H$, contains two Boolean algebras $H_{\text{reg}}$ and $H_{\text{comp}}$ made respectively by the regular and complemented elements. In both Boolean algebras the operations $\land$, $\lnot$ and $\rightarrow$ are the same than in $H$. For $H_{\text{comp}}$, also the operation $\lor$ is the same than in $H$, so that this Boolean algebra is a sub-algebra of $H$. Instead, for $H_{\text{reg}}$ we get the following different "or" operation: $x \lor_{\text{reg}} y = \lnot(\lnot x \land \lnot y)$.
4) (The De Morgan laws in a Heyting algebra). In any Heyting algebra one has:
\[
\begin{cases}
\text{(regular De Morgan law): } & \lnot(x \lor y) = \lnot x \land \lnot y \\
\text{(weak De Morgan law): } & \lnot(x \land y) = \lnot\lnot(\lnot x \lor \lnot y).
\end{cases}
\]  
\forall_{x,y\in G}.
5) The following propositions are equivalent for all Heyting algebras $H$.
1. $H$ satisfies both De Morgan laws. 
2. $\lnot(x \land y) = \lnot x \lor \lnot y, \forall x, y \in H$.
3. $\lnot(x \land y) = \lnot x \lor \lnot y$, for all regular $x, y \in H$.
4. $\lnot(x \lor y) = \lnot x \land \lnot y, \forall x, y \in H$.
5. $\lnot(x \lor y) = x \land y$, for all regular $x, y \in H$.
6. $\lnot(\lnot x \land \lnot y) = x \lor y$, for all regular $x, y \in H$.
7. $\lnot x \lor \lnot x = 1, \forall x \in H$.

**Definition 2.32.** A morphism $f : H_1 \to H_2$ between Heyting algebras is characterized by the following equivalent relations.
(i) $f(0) = 0$.
(ii) $f(1) = 1$.
(iii) $f(x \land y) = f(x) \land f(y)$.
(iv) $f(x \lor y) = f(x) \lor f(y)$.
(v) $f(x \rightarrow y) = f(x) \rightarrow f(y)$.
(vi) $f(\lnot x) = \lnot f(x)$.

**Proposition 2.33 (Properties of Heyting morphisms).** 1. Let $H_1$ and $H_2$ be structure with operations $\rightarrow$, $\land$, $\lor$ (and possibly $\lnot$) and constants 0 and 1. Let $f : H_1 \to H_2$ be a surjective mapping satisfying properties (i)–(vi) in above definition. Then if $H_1$ is a Heyting algebra so is $H_2$ too.
2. Heyting algebras form a category $\mathcal{H}_{\text{Heyting}}$.
3. Let $K \subseteq H$ be a subalgebra of a Heyting algebra $H$. Then the inclusion $i : K \rightarrow H$ is a morphism.
4. One has a canonical morphism $H \to H_{\text{reg}}$, given by $x \mapsto \lnot\lnot x$.

**Definition 2.34.** A filter on a Heyting algebra $H$ is a subset $F \subseteq H$ such that the following conditions hold.
(i) $1 \in F$.
(ii) $x, y \in F$ then $x \land y \in F$.
(iii) $x \in F$, $y \in H$, $x \leq y$ then $y \in F$.

---

\(^{13}\)Complete Heyting algebras form three different categories having all the same objects, but having different morphism. In Tab. 1 are resumed these categories. The category $\mathfrak{Top}$ of topologic spaces admits a representation in the category $\mathfrak{Locales}$ of locales. However, many important theorems in point-set topology require axiom of choice, that has not an analogue in $\mathfrak{Locales}$. Therefore, not all propositions in $\mathfrak{Top}$ can be translated in $\mathfrak{Locales}$.

\(^{14}\)Note that the composition $H \rightarrow H_{\text{reg}} \rightarrow H$ is not in general a morphism, since the join operation of $H_{\text{reg}}$ can be different from that of $H$. 

Table 1. Complete Heyting algebras categories.

| Symbol | Definition | Note |
|--------|------------|------|
| Øheyting | \(\text{Ob}(\text{Øheyting}) = \{\text{complete Heyting algebras}\}\) | |
| \(\text{Hom}(\text{Øheyting}) = \{\text{homomorphism of complete Heyting algebras}\}\) | | |
| \(\text{Frames}\) | \(\text{Ob}(\text{Frames}) = \{\text{lattices } L, \text{ where every subset } \{a_i\} \subseteq L \text{ has a supremum } \bigvee a_i\}\) | |
| \(\text{Hom}(\text{Frames}) = \{\text{lattices homomorphisms respecting arbitrary suprema}\}\) | | |
| \(\text{Locales}\) | \(\text{Locales} = \{\text{Frames}\}^{\text{op}}\) | There exists a natural functor \(\text{Top} \to \text{Locales}\) |

\(\text{Top}\) denotes the category of topological spaces. The axiom of choice is required for many important theorems in point-set topology, but it does not exist for locales.

**Proposition 2.35** (Properties of filter on Heyting algebra). 1) The intersection of filters on a Heyting algebra \(H\) is again a filter.

2) We call filter generated by a subset \(S \subseteq H\) of a Heyting algebra \(H\), the smallest filter \(F\) on \(H\) containing \(S\). If \(S = \emptyset\) then \(F = \{1\}\). If \(S \neq \emptyset\) then \(F = \{x \in H \mid y_1 \wedge y_2 \wedge \cdots \wedge y_n \leq x \in S\}\).

3) If \(F\) is a filter on a Heyting algebra \(H\), there is a Heyting algebra \(H/F\), called quotient of \(H\) by \(F\), such that \(p_F : H \to H/F\) is a morphism. More precisely \(H/F = H/\sim\), where \(\sim\) is the equivalence relation \((6)\) in \(H\) induced by \(F\).

\[
(6)\quad x \sim y \iff \begin{cases} x \to y \in F \\ y \to x \in F. \end{cases}
\]

4) (Universal property). Let \(S \subseteq H\) be a subset of a Heyting algebra \(H\), and let \(F\) the corresponding filter generated by \(S\). Given any morphism \(f : H \to H'\) of Heyting algebras, such that \(f(y) = 1, \forall y \in S\), there exists a unique morphism \(f' : H/F \to H'\), such that the diagram \((7)\) is commutative.

\[
(7)\quad \\
H \xrightarrow{f} H' \\
\downarrow p_F \downarrow \Rightarrow \\
H/F \xrightarrow{f'}
\]

5) The kernel of a morphism \(f : H_1 \to H_2\) of Heyting algebras is the filter \(\ker f \equiv f^{-1}(\{1\}) \subseteq H_1\). The morphism \(f' : H_1/(\ker f) \to H_2\), for universal property, is an isomorphism: \(H_1/(\ker f) \cong f(H_1) \preceq H_2\).

**Theorem 2.36** (Heyting algebra of propositional formulas in \(n\) variables up to intuitionistic equivalence). To the set of propositional formulas in the variables \(A_1, \ldots, A_n\), one can canonically associate a Heyting algebra. (Similar properties hold also for any set of variables \(\{A_i\}_{i \in I}\) that are conditioned to some theory \(T\).)

**Proof.** Let us introduce in the set \(L\) of propositional formulas in the variables \(A_1, \ldots, A_n\), a preorder \(\leq\) defined by \(F \leq G\) if \(G\) is an (intuitionistic) local consequence of \(F\). This preorder induces an equivalence relation \(\sim\) in \(L\): \(F \sim G \iff F \leq G, G \leq F\). Then \(H_0 \equiv L/\sim\) is a Heyting algebra. Furthermore, the preorder on \(L\) induces on \(H_0\) an order relation \(\leq\).

**Theorem 2.37** (Spectra in Algebraic Topology and Heyting algebras). Spectra in algebraic topology identify Heyting algebras.
Proof. We say that the spectrum $X$ is acyclic with respect to certain theory $E$ if $E \wedge X$ is contractible: $E \wedge X \simeq pt$. Two spectra are Bousfield equivalent if they have the same acyclic spectra:

$$E \sim F \iff \forall X : E \wedge X \simeq pt \iff F \wedge X \simeq pt.$$ 

Let us denote by $< E >$ the Bousfield class of the spectrum $E$. One has the following lemmas.

**Lemma 2.38** ([55]). Bousfield classes form a set $B$ of cardinality at most $\beth_2 > c$.\(^{15}\)

**Lemma 2.39** ([36]). One can define partial ordering in the set of Bousfield classes by

$$< E > \triangleright < F > \iff \forall X E \wedge X \simeq pt \Rightarrow F \wedge X \simeq pt.$$ 

The set of Bousfield classes form a complete lattice. The join is given by the wedge $\vee$. The smallest element is $< pt >$ and the largest element is $< E(S^0) >$. The meet $\wedge$ is given by the join of all lower bounds.\(^{16}\) Let us denote by $DL \subset B$ the subset of the Bousfield lattice such that $< E > \wedge < E > \equiv < E \wedge E > = < E >$. $DL$ is a distributive lattice, (distributive Bousfield lattice), and it is just a complete Heyting algebra.\(^{17}\)

The inclusion $DL \hookrightarrow B$ preserves joins but does not preserve meets. Furthermore, there is a retraction $r : B \to DL$, defined by

$$r < X > = \bigvee \{ < Y > \in DL \mid < Y > \leq < X > \}.$$ 

$r$ can be considered the right adjoint of the functor $i$, by considering any partially ordered set a category with a unique map from $x$ to $y$ iff $x \leq y$.\(^{18}\) $r$ preserves smash product: $r(< X > \wedge < Y >) = r < X > \wedge r < Y >$.

After the above two lemma the proof of the theorem is done. \(\Box\)

**Example 2.40.** All ring spectra and all finite spectra are in $DL$.

**Proposition 2.41.** The set of $< E > \in DL$ such that the pseudocomplement $< E > \not\triangleright < pt >$\(^{19}\) is really the complement is a Boolean algebra $BA \subset DL$. Furthermore into $BA$ is contained a Boolean algebra $FBA$ isomorphic to the Boolean algebra of finite and co-finite subsets of $\mathbb{N}$; this is just the subalgebra of all finite $p$-local spectra.

**Theorem 2.42** (Integral spectrum of quantum PDEs). 1) Let $\hat{E}_k \subset J_k^s(W)$ be a PDE in the category $\Omega$ of quantum manifolds [66, 68, 77, 78, 79, 80, 81]. Then there is a spectrum $\{ \Xi_a \}$ (singular integral spectrum of quantum PDEs), such that $\Omega_{p,s}^{\hat{E}_k} = \lim_{r \to \infty} \pi_{p+r}(\hat{E}_k^+ \wedge \Xi_r)$, $\Omega_{p,s}^E_k = \lim_{r \to \infty} [S^r \hat{E}_k^+ : \Xi_{p+r}]$, $p \in \{ 0, 1, \ldots, n-1 \}$.

\(^{15}\)The cardinality of all subsets $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$ and it is greater than the cardinality $c$ of the continuum, i.e., that of $\mathbb{R}$ (Recall that $\beth_1 = 2^\aleph_0 - 1$, and $\beth_0 = \aleph_0$.)

\(^{16}\)But, this meet does not distribute over infinite joins and the smash does. So it is interesting to consider a structure where the meet coincides with the smash.

\(^{17}\)This is endowed with three operations: wedge, $\vee$ and $\Rightarrow$, where $a \Rightarrow b$ is the greatest $x$ such that $a \wedge x \leq b$. Therefore $DL$ is a frame. (See Definition 2.21 and [10].)

\(^{18}\)A functor between complete lattices preserves colimits iff it preserves arbitrary joins. For maps $f : A \to B$, $g : B \to A$, between partially ordered sets $A$ and $B$, $g$ is right adjoint to $f$ iff $f(x) \leq y$ is equivalent to $x \leq g(y)$.

\(^{19}\)That is the greatest $< F >$ such that $< E > \wedge < F > \leq pt$, i.e., $< E \wedge F > \wedge X \simeq pt \Rightarrow pt \wedge X \simeq pt$. 

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**Quotient Exotic PDE’s**

11
Remark 2.45. Direct extensions of Theorem 2.42 and Theorem 2.43 for PDE’s, category of quantum hypercomplex manifolds. Following our previous works devoted to quantum PDE’s, we consider, now, the product preserves the norm. In fact, for quantum PDE’s, considering PDE’s in quantum hypercomplex number.\[Q\]

Definition 3.1. Let in the following, introduce the new definitions related to quantum PDE’s. In fact we have the following.

Theorem 2.43 (Heyting algebra of a quantum PDE). Let \(\hat{E}_k \subset J^k(W)\) be a PDE in the category \(\Omega\) of quantum manifolds. Then \(\hat{E}_k\) identifies a Heyting algebra, \(H(\hat{E}_k)\), that we call spectral Heyting algebra of \(\hat{E}_k\).

Proof. After Theorem 2.37 and Theorem 2.42 we can conclude that to the spectrum \(\Xi_\ast\) on can associate a Heyting algebra that is just \(H(\hat{E}_k)\).\]

Remark 2.44. Compare this new result with reinterpretations of quantum theory by using toposi. (See, e.g. Refs. [5, 43, 33]. Recall that subobjects of any object in a topos form a Heyting algebra. Furthermore, it is possible identify a topos for any non-abelian \(C^\ast\)-algebra \(B\), as the topos of covariant functors over the category \(C\) of abelian subalgebras of \(B\). \(C\) induces an internal abelian \(C^\ast\)-algebra \(C\) in this topos of functors. The Gel’fand spectrum of \(C\) is the spectral presheaf.\]

Remark 2.45. Direct extensions of Theorem 2.42 and Theorem 2.43 for PDE’s, in the category \(\Omega_S\) of quantum super manifolds, hold too.

3. Quantum hypercomplex PDE’s

Following our previous works devoted to quantum PDE’s, we consider, now, the category of quantum hypercomplex manifolds \(\Omega_{\text{hyper}}\) and we recaste our theory of quantum PDE’s in \(\Omega_{\text{hyper}}\) emphasizing the geometric new mathematical structures and the characterizations that these generate. Then theorems for existence of local and global solutions are obtained for such quantum PDE’s that extend analogous previous results for quantum PDE’s.

For definitions and fundamental results on quantum manifolds see [77, 78, 79, 80, 81, 83, 84]. Let in the following, introduce the new definitions related to quantum hypercomplex manifolds.

Definition 3.1. A quantum hypercomplex \(r\)-algebra, \(0 \leq r \in \mathbb{N}\), is the extension \(Q_r \equiv B \otimes_\mathbb{R} A_r\), where \(B\) is a quantum algebra in the sense of [77, 78, 79, 80, 81, 83, 84] and \(A_r\), is an \(\mathbb{R}\)-algebra in the Cayley-Dikson construction. (We assume that \(B\) is a quantum \(\mathbb{K}\)-algebra, with \(\mathbb{K} \equiv \mathbb{R}\) or \(\mathbb{K} \equiv \mathbb{C}\).) We call any \(q \in Q_r\) a quantum hypercomplex number.\[21\]

\[20\]For general informations on topos theory see, e.g., the following [5, 44, 47, 48, 99].

\[21\]Let us recall that the Cayley-Dikson algebra \(A_r \equiv \mathbb{R}^{2^r}\) is an \(\mathbb{R}\)-algebra structure on the \(\mathbb{R}\)-vector space \(\mathbb{R}^{2^r} = \mathbb{R}^{2^r-1} \times \mathbb{R}^{2^r-1}\), given inductively by the formula \((a_1, a_2)(b_1, a_2) = (a_1b_1 - b_2a_2, b_1a_1 + a_2b_1)\), where \(a = (a_1, a_2) \in \mathbb{R}^{2^r} = \mathbb{R}^{2^r-1} \times \mathbb{R}^{2^r-1}\), with \(A_0 = \mathbb{R}\) and \(A_1 = \mathbb{C}\), \(A_2 = \mathbb{H}\), \(A_3 = \mathbb{O}\). The algebras \(A_r\), \(0 \leq r \leq 3\) are called the classical Cayley-Dickson algebras. For classical Cayley-Dickson algebras holds the Hurewicz’s theorem: \(||ab|| = ||a|| ||b||\), \(\forall a, b \in A_r\), \(r = 0, 1, 2, 3\), where \(||\|\|\) denotes the euclidean norm in \(\mathbb{R}^{2^r}\). For \(r \geq 4\) does not necessitate the product preserves the norm. In fact, for \(r \geq 4\) one has Cayley-Dickson algebras \(A_r\) with nonempty set \(\mathbb{Z}_{\text{ev}}(A_r)\), of zero divisors.
Proposition 3.2 (Properties of quantum hypercomplex algebras). 1) \( Q_r \) is a quantum vector space of dimension \( 2^r \) with respect to the quantum algebra \( B \). Therefore, \( Q_r \) is a metrizable, complete, Hausdorff, locally convex topological \( K \)-vector space.

2) \( Q_r \) has a natural structure of \( K \)-algebra. Furthermore one has the following properties:

(i) \( Q_r \) is also a ring with unit \( e \);
(ii) \( e : K \to Z(Q_r) \subset Q_r \) is a ring homomorphism, where \( Z \equiv Z(Q_r) \) is the centre of \( Q_r \);
(iii) \( c : Q_r \to K \) is a \( K \)-linear morphism, with \( c(e) = 1 \), \( e \) = unit of \( Q_r \). For any \( a \in Q_r \) we call \( a_c = c(a) \in K \) the classic limit of \( a \);
(iv) \( Q_r \) is a non-associative \( K \)-algebra, for \( r \geq 3 \). We say that \( Q_r = B \otimes_R K_r \) is \( m \)-associative if \( K_r \) is \( m \)-associative, i.e., there exists an \( m \)-dimensional subspace \( V \subset K_r \) such that \( (y,z)z = y.(x,z) \), for all \( y, z \in K_r \) and \( x \in S \). This means that one has also \((a \otimes y),(b \otimes x),(c \otimes z) = (a \otimes y),(b \otimes x),(c \otimes z)\), \( \forall a, b \in B \).
(v) \( Q_r \) is an alternative \( K \)-algebra, i.e., \( a^2b = a(ab) \) and \( a^2b = a(ab) \), \( \forall a, b \in Q_3 \).
(vi) One has the following implications for the quantum hypercomplex algebras \( Q_r \):

- Associative \( \Rightarrow \) Alternative \( \Rightarrow \) Flexible, but the backwards implications are not true. The canonical product in \( Q_r \) is nonassociative for \( r \geq 3 \), since \( K_r \) is a nonassociative algebra for \( r \geq 3 \).

3) One has the short exact sequences reported in (8).

\[
\begin{align*}
\begin{array}{c}
0 & \to & B & \overset{a}{\to} & Q_r & \overset{b}{\to} & Q_r/B & \to & 0 \\
0 & \to & K_r & \overset{c}{\to} & Q_r & \overset{d}{\to} & Q_r/K_r & \to & 0 \\
\end{array}
\end{align*}
\]

This means that any quantum hypercomplex algebra \( Q_r \) is an extension of a quantum algebra \( B \) and a Cayley-Dickson algebra \( K_r \), where both can be considered subalgebras of \( Q_r \), and one has the canonical isomorphisms reported in (9).

\[
Q_r \cong (B \otimes_R 1).(e_B \otimes K_r) \cong (e_B \otimes K_r).(B \otimes_R 1).
\]

4) The nucleus \( \text{Nucleus}(Q_r) \) of \( Q_r \), i.e., the set of elements in \( Q_r \) which associates with every pair of elements \( a, b \in Q_r \), is an associative subalgebra of \( Q_r \), containing the center \( Z(Q_r) \) that is a commutative associative subalgebra of \( Q_r \).

Proof. 1) In fact for any basis \( \{e_p\}_{1 \leq p \leq 2^r} \), one has an \( R \)-isomorphism \( K_r \cong R^{2^r} \). This induces an isomorphism \( Q_r \cong A^{2^r} \). This means that \( \{1 \otimes e_p\}_{1 \leq p \leq 2^r} \), \( 1 \in A \), is a \( A \)-basis for \( Q_r \), hence any vector \( q \in Q_r \), i.e., any quantum hypercomplex \( r \)-number, admits the linear representation given in (10).

\[
q = \sum_{1 \leq p \leq 2^r} a^p(1 \otimes e_p), \; a^p \in A.
\]

In fact, any \( q \in Q_r \) can be written in the form \( q = \sum_{i \in I} a^i \otimes b_i \), with \( a^i \in B \) and \( b_i \in K_r \). Representing each \( b_i \) in a basis \( \{e_p\}_{1 \leq p \leq 2^r} \) of \( K_r \), we get:
2) The product in $\mathcal{Q}_r$ is given by (11).

\[
\begin{aligned}
q &= \sum_{i \in I} a^i \otimes b_i \\
&= \sum_{i \in I} a^i \otimes (\sum_{1 \leq p \leq 2r} b^p_i e_p), \quad b^p_i \in \mathbb{R}, \quad e_p \in \Lambda_r \\
&= \sum_{i \in I, 1 \leq p \leq 2r} a^i b^p_i \otimes e_p \\
&= \sum_{1 \leq p \leq 2r} c^p \otimes (1 \otimes e_p), \quad c^p \equiv \sum_{i \in I} a^i b^p_i \in B, \quad 1 \otimes e_p \in \mathcal{Q}_r.
\end{aligned}
\]

By using the linear representations of $a, b \in \mathcal{Q}_r$, with respect to a $A$-basis $\{1 \otimes e_p\}_{1 \leq p \leq 2r}$, we get the expression (12).\footnote{The $2^m$ numbers $\gamma_{pq}^k \equiv (e_pe_q)^k \in \mathbb{R}$, are called multiplication constants of $\mathcal{Q}_r$.}

\[
\begin{aligned}
\{a, b\} &= \left(\sum_{1 \leq p \leq 2r} a^p (1 \otimes e_p) \right) \left(\sum_{1 \leq q \leq 2r} b^q (1 \otimes e_q) \right), \quad a^p, b^q \in B \\
&= \sum_{1 \leq p, q \leq 2r} a^p b^q (1 \otimes e_pe_q) \\
&= \sum_{1 \leq k \leq 2r} c^k (1 \otimes e_k) \\
c^k &\equiv \sum_{1 \leq p, q \leq 2r} a^p b^q (e_pe_q)^k \in B, \quad (e_pe_q)^k \in \mathbb{R} \\
epeq &\equiv \sum_{1 \leq k \leq 2r} (e_pe_q)^k e_k \in \Lambda_r.
\end{aligned}
\]

Thus this product is $\mathbb{R}$-bilinear and unital with unit $e \in \mathcal{Q}_r$ given by $e = e_B \otimes 1$, where $e_B$ is the unity in $B$ and 1 is the unity in $\Lambda_r$. Then, the properties listed above, follow directly from analogous ones for the Cayley-Dickson algebras $\Lambda_r$.

3) In fact the mappings $a$ and $e$ in (8) are the canonical inclusions $b \mapsto b \otimes 1$ and $a \mapsto e_B \otimes a$ respectively, for $b \in B$ and $a \in \Lambda_r$. Furthermore, the mappings $b$ and $d$ in (8) are the canonical projections.

4) These properties directly follow from definitions. Let us emphasize only that $Z(\mathcal{Q}_r) = \{a \in N_{\text{unital}}(\mathcal{Q}_r) | ab = ba, \forall b \in \mathcal{Q}_r\}$. \hfill \square

**Definition 3.3.** A quantum hypercomplex vector space of dimension $(m_0, \ldots, m_s) \in \mathbb{N}^s$, built on the quantum hypercomplex algebra $A \equiv \mathcal{Q}_0 \times \cdots \times \mathcal{Q}_s$, is a locally convex topological $\mathbb{K}$-vector space $E$ isomorphic to $\mathcal{Q}_0^{m_0} \times \cdots \times \mathcal{Q}_s^{m_s}$.

**Definition 3.4.** A quantum hypercomplex manifold of dimension $(m_0, \ldots, m_s)$ over a quantum algebra $A \equiv \mathcal{Q}_0 \times \cdots \times \mathcal{Q}_s$ of class $Q^k_w$, $0 \leq k \leq \infty, \omega$, is a locally convex manifold $M$ modelled on $E$ and with a $Q^k_w$-atlas of local coordinate mappings, i.e., the transition functions $f : U \subset E \rightarrow U' \subset E$ define a pseudogroup of local $Q^k_w$-homeomorphisms on $E$, where $Q^k_w$ means $C^k_w$, i.e., weak differentiability [38, 63, 66, 68], and derivatives $Z$-linaires, with $Z \equiv Z(A)$ the centre of $A$. So for each open coordinate set $U \subset M$ we have a set of $m_0 + \cdots + m_s$ coordinate functions $x^A : U \rightarrow A$, (quantum hypercomplex coordinates).
Definition 3.5. The tangent space $T_pM$ at $p \in M$, where $M$ is a quantum hypercomplex manifold of dimension $(m_0, \ldots, m_s)$, over a quantum hypercomplex algebra $A \equiv Q_0 \times \cdots \times Q_s$, of class $Q^k_{\mu^s}$, $k > 0$, is the vector space of the equivalence classes $v \equiv [f]$ of $C^1_{\mu}(0)$ curves $f: I \to M$, $I \equiv$ open neighborhood of $0 \in \mathbb{R}$, $f(0) = p$; two curves $f, f'$ are equivalent if for each (equivalently, for some) coordinate system $\mu$ around $p$ the functions $\mu \circ f, \mu \circ f'$ have the same derivative at $0 \in \mathbb{R}$.

Remark 3.6. Then, derived tangent spaces associated to a quantum hypercomplex manifold $M$ can be naturally defined similarly to what made for quantum manifolds. (For details see [63, 66, 68, 77, 78, 80, 81, 83, 84].)

Definition 3.7. We say that a quantum hypercomplex manifold of dimension $(m_0, \ldots, m_s)$ is classic regular if it admits a projection $c: M \to M_{c}$ on an $n$-dimensional manifold $M_{c}$. We will call $M_{c}$ the classic limit of $M$ and in order to emphasize this structure we say that the dimension of $M$ is $(n, m_0, \ldots, m_s)$.

Definition 3.8. The category $\mathfrak{Q}_{\text{hyper}}$ of quantum hypercomplex manifolds, is made by $\text{Ob}(\mathfrak{Q}_{\text{hyper}})$ that contains all quantum hypercomplex manifolds, and morphisms $\text{Hom}(\mathfrak{Q}_{\text{hyper}})$ are mappings of class $Q^f_{\mu}$ between quantum hypercomplex manifolds. In order that $\text{Hom}_{\mathfrak{Q}_{\text{hyper}}}(M, N)$ should be non empty, it is necessary that whether $M$, (resp. $N$), is modeled on the quantum hypercomplex algebra $A$, (resp. $A'$), and $A'$ should be also a $Z(A)$-module, where $Z \equiv Z(A)$ is the centre of $A$.

Remark 3.9. Let $\pi: W \to M$ be a fiber bundle in the category $\mathfrak{Q}_{\text{hyper}}$, such that $M$ is of dimension $m$ on $A$ and $W$ of dimension $(m, s)$ on $B \equiv A \times E$, where $E$ is also a $Z(A)$-module. Then we can define the $k$-order jet-derivative space for sections of $\pi$, $J^k\pi^k(W)$ as an object in $\mathfrak{Q}_{\text{hyper}}$, similarly to what made for jet-derivative spaces in the category $\mathfrak{Q}$. So in the following we will formally resume the definition of PDE’s in the category $\mathfrak{Q}_{\text{hyper}}$, by adapting the language for analogous geometric structures just previously considered in the category $\mathfrak{Q}$. (See Refs. [63, 66, 68, 77, 78, 79, 80, 81, 83, 84].)

A quantum PDE (QPDE) of order $k$ on the fibre bundle $\pi: W \to M$, defined in the category of quantum hypercomplex manifolds, is a subset $E_k \subset JD^k(W)$ of the jet-quantum derivative space $JD^k(W)$ over $M$. We can formally extend the geometric theory for quantum PDEs, previously considered in [63, 66], to PDE’s in the category $\mathfrak{Q}_{\text{hyper}}$, since the intrinsic formulation therein is not influenced by the non-associativity of the underlying quantum hypercomplex algebra. In the following we shall emphasize some important definitions and results about a QPDE $E_k$ is quantum regular if the $r$-quantum prolongations $E_{k+r} \equiv JD^r(E_k) \cap JD^{k+r}(W)$ are subbundles of $\pi_{k+r,k+r-1}: JD^{k+r}(W) \to JD^{k+r-1}(W)$, $\forall r \geq 0$. Furthermore, we say that $E_k$ is formally quantum integrable if $E_k$ is quantum regular and if the mappings $E_{k+r+1} \to E_{k+r}, \forall r \geq 0$, and $\pi_{k,0}: E_k \to W$ are surjective. The quantum symbol $g_k+r$ of $E_{k+r}$ is a family of $Z \equiv Z(A)$-modules over $E_k$ characterized by means of the following short exact sequence of $Z$-modules: $0 \to \pi^+_{k+r,k+r} \to \psi T E_{k+r} \to \pi^+_{k+r,k+r-1} \psi T E_{k+r-1}$. Then one has the following

---

23In the following we shall consider fiber bundles of this type.
24This is, instead, important in the local writing and meaning of PDE’s.
complex of $\mathbb{Z}$-modules over $\hat{E}_k$ ($\delta$-quantum complex):

$$
\begin{align*}
0 & \xrightarrow{\delta} \hat{g}_m \xrightarrow{\delta} \text{Hom}_Z(TM; \hat{g}_{m-1}) \xrightarrow{\delta} \text{Hom}_Z(\hat{A}_0^M; \hat{g}_{m-2}) \xrightarrow{\delta} \cdots \\
& \xrightarrow{\delta} \text{Hom}_Z(\hat{A}_0^{m-k}M; \hat{g}_k) \xrightarrow{\delta} \delta(\text{Hom}_Z(\hat{A}_0^{m-k}M; \hat{g}_k)) \xrightarrow{\delta} 0
\end{align*}
$$

where $\hat{A}_0^M$ is the skewsymmetric subbundle of $\hat{T}_0^1M \equiv TM \otimes \cdots \otimes Z TM$. We call $\text{Spencer quantum cohomology}$ of $\hat{E}_k$ the homology of such complex. We denote by $\{H_q^{m-j}\}_{q \in \hat{E}_k}$ the homology at $(\text{Hom}_Z(\hat{A}_0^M; \hat{g}_{m-j}))_q$. We say that $\hat{E}_k$ is $r$-quantumacyclic if $H_q^{m-j} = 0$, $m \geq k$, $0 \leq j \leq r$, $\forall q \in \hat{E}_k$. We say that $\hat{E}_k$ is quantum involutive if $H_q^{m-j} = 0$, $m \geq k$, $j \geq 0$. We say that $\hat{E}_k$ is $\delta$-regular if there exists an integer $\kappa_0 \geq \kappa$, such that $\hat{g}_{\kappa_0}$ is quantum involutive or 2-quantumacyclic.

**Theorem 3.10** ($\delta$-Poincaré lemma for quantum PDE’s in the category $\Omega_{\text{hyper}}$). Let $\hat{E}_k \subset J\hat{D}^k(W)$ be a quantum regular QPDE. If $Z$ is a Noetherian $\mathbb{K}$-algebra, then $\hat{E}_k$ is a $\delta$-regular QPDE.

**Proof.** The proof can be copied by the analogous theorem formulated in the category of $\Omega$ [63, 66]. In fact that proof is given there in intrinsic way, thus it does not depend on the particular coordinates representation. Really, from the formal point of view the difference between a quantum manifold $M$ of dimension $m$ over a quantum algebra $B$, and a quantum hypercomplex manifold $N$ of dimension $m$ over a quantum hypercomplex algebra $\mathcal{Q}_r = B \otimes \mathbb{R} \mathbb{A}_r$, is that the quantum coordinates $x^A$ on $M$ take values in the associative algebra $B$ and the quantum coordinates $y^A$ on $N$ take values in the algebra $\mathcal{Q}_r$, that is not associative for $r \geq 3$. But the eventual non-associativity does not influence the geometric intrinsic formal properties of (nonlinear) PDE’s built in the category $\Omega_{\text{hyper}}$. (For complementary informations on nonassociative algebras see, e.g., [91].) \qed

**Theorem 3.11** (Criterion of formal quantum integrability in the category $\Omega_{\text{hyper}}$). Let $\hat{E}_k \subset J\hat{D}^k(W)$ be a quantum regular, $\delta$-regular QPDE. Then if $\hat{g}_{k+r+1}$ is a bundle of $\mathbb{Z}$-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$ is surjective for $0 \leq r \leq m$, then $\hat{E}_k$ is formally quantum-integrable.

**Proof.** The proof can be copied by the analogous theorem formulated in the category of $\Omega$ [63, 66]. (Considerations similar to the proof in Theorem 3.10 can be made here too.) \qed

An initial condition for QPDE $\hat{E}_k \subset J\hat{D}^k(W)$ is a point $q \in \hat{E}_k$. A solution of $\hat{E}_k$ passing for the initial condition $q$ is a $m$-dimensional quantum hypercomplex manifold $N \subset \hat{E}_k$ such that $q \in N$ and such that $N$ can be represented in a neighborhood of any of its points $q' \in N$, except for a nowhere dense subset $\Sigma(N) \subset N$ of dimension $\leq m - 1$, as image of the $k$-derivative $D^k$s of some $Q_\text{reg}$-section $s$ of $\pi : W \to M$. We call $\Sigma(N)$ the set of singular points (of Thom-Bordman type) of $N$. If $\Sigma(N) \neq \emptyset$ we say that $N$ is a regular solution of $\hat{E}_k \subset J\hat{D}^k(W)$. Furthermore, let us denote by $J^k_m(W)$ the $k$-jet of $m$-dimensional quantum manifolds (over $A$) contained into $W$. One has the natural embeddings $\hat{E}_k \subset J\hat{D}^k(W) \subset J^k_m(W)$. Then, with respect to the embedding $\hat{E}_k \subset J^k_m(W)$ we can consider solutions of $\hat{E}_k$ as $m$-dimensional (over $A$) quantum hypercomplex manifolds $V \subset \hat{E}_k$ such that $V$ can be represented in the neighborhood of any of its points $q' \in V$, except for a nowhere dense subset $\Sigma(V) \subset V$, of dimension $\leq m - 1$, as $N^{(k)}$, where $N^{(k)}$ is
the $k$-quantum prolongation of a $m$-dimensional (over $A$) quantum hypercomplex manifold $N \subset W$. In the case that $\Sigma(V) = \emptyset$, we say that $V$ is a regular solution of $\hat{E}_k \subset J^k_m(W)$. Of course, solutions $V$ of $\hat{E}_k \subset J^k_m(W)$, even if regular ones, are not, in general diffeomorphic to their projections $\pi_k(V) \subset M$, hence are not representable by means of sections of $\pi : W \to M$.

Therefore, from Theorem 3.10 and Theorem 3.11 we are able to obtain existence theorems of local solutions. Now, in order to study the structure of global solutions it is necessary to consider the integral bordism groups of QPDEs. In [63, 66] we extended to QPDEs our previous results on the determination of integral bordism groups of PDEs [64, 65, 66]. Let us denote by $\Omega^{\hat{E}_k}_p$, $0 \leq p \leq m - 1$, the integral bordism groups of a QPDE $\hat{E}_k \subset J^k_m(W)$ for closed integral quantum hypercomplex submanifolds of dimension $p$, over a quantum hypercomplex algebra $A$, of $\hat{E}_k$. The structure of smooth global solutions of $\hat{E}_k$ are described by the integral bordism group $\Omega^{\hat{E}_k}_{\infty - 1}$ corresponding to the $\infty$-quantum prolongation $\hat{E}_\infty$ of $\hat{E}_k$. Beside the groups $\Omega^{\hat{E}_k}_p$, $0 \leq p \leq m - 1$, we can also introduce the integral singular $p$-bordism groups $\hat{B}\Omega^{\hat{E}_k}_p$, $0 \leq p \leq m - 1$, where $B$ is a quantum hypercomplex algebra. Then one can prove that $\hat{B}\Omega^{\hat{E}_k}_p \cong \Omega^{\hat{E}_k}_p \otimes_k B$, where $\Omega^{\hat{E}_k}_p$ are the integral singular bordism groups for $B = \mathbb{K}$. Furthermore, the equivalence classes in the groups $\hat{B}\Omega^{\hat{E}_k}_p$ are characterized by means of suitable characteristic numbers (belonging to $B$), similarly to what happens for PDEs in the category of commutative manifolds and quantum (super) manifolds. (For details see [63, 66, 68, 77, 78, 79, 80, 81, 83, 84].)

Example 3.12 (Quantum quaternionic manifolds). Since the category of $\Omega_\tau$, for $0 \leq r \leq 1$, coincides with the category of quantum manifolds over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, that we have just explicitly considered in some our previous works, let us first concentrate our attention on the category $\Omega_\tau$, i.e., with the category of quantum quaternionic manifolds. Really, also an algebraic topology of PDE’s in this category has been considered by us in some previous works (see [68] and references therein). However, may be useful to recall here these results in some details.

Let us first recall some algebraic definitions and results on quaternionic and Cayley algebra [9]. Let $R$ be a commutative ring. Let $\alpha, \beta \in R$, $(e_1,e_2)$ the canonical basis of the $R$-module $R^2$. We say quadratic algebra of type $(\alpha, \beta)$ over $R$ the $R$-module $R^2$ endowed with the structure of algebra defined by means of the following multiplication:

$$e_1^2 = e_1, e_1e_2 = e_2e_1 = e_2, e_2^2 = \alpha e_1 + \beta e_2.$$  

Any $R$-algebra $E$, isomorphic to a quadratic algebra is called a quadratic algebra too. (Any $R$-algebra $E$ that admits a basis of two elements (one belonging the identity) is a quadratic algebra.) Then the basis is called a basis of type $(\alpha, \beta)$. A quadratic algebra $E$ is associative and commutative. Let $E$ be a quadratic $R$-algebra, $e$ its unit. Let $u \in E$ and $T(u)$ the trace of the endomorphism $x \mapsto ux$ of the free $R$-module $E$. Then the application $s : E \to E$, $s(u) = T(u)e - u$, is an endomorphism of the $R$-algebra $E$ and one has $s^2(u) = u$, $\forall u \in E$. A Cayley algebra on $R$ is a couple $(E,s)$, where $E$ is a $R$-algebra, with unit $e \in E$, and $s$ is a skewendomorphism of $E$ such that: (a) $u + \bar{u} \in Re$, (b) $u\bar{u} \in Re$, with $\bar{u} \equiv s(u)$, $\forall u \in E$. $s$ is called conjugation of the Cayley algebra $E$ and $s(u) \equiv \bar{u}$ is the conjugated of $u$. From the

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25See also Refs. [69, 70, 71, 72, 73, 74, 75, 76].
condition (a) it follows that \( u \bar{u} = \bar{u}u \). One defines Cayley trace and Cayley norm respectively the following maps: \( T : E \to \mathbb{R} \), \( u \mapsto T(u) = u + \bar{u} \); \( N : E \to \mathbb{R} \), \( u \mapsto N(u) = u\bar{u} \). One has the following properties:

1. \( \bar{e} = e \);
2. \( s(u + s(u)) = u + s(u) \Rightarrow s(u) + s^2(u) = u + s(u) \Rightarrow s^2(u) = u \Rightarrow s^2 = \text{id}_E \);
3. \( T(\bar{u}) = T(u) \);
4. \( N(\bar{u}) = N(u) \);
5. \( (u - \bar{u})(u - \bar{u}) = 0 \Rightarrow u^2 - T(u)u + N(u) = 0 \);
6. Let \( E \) be a \( R \)-algebra and let \( s, s' \) be skewendomorphisms of \( E \) such that \((E, s)\) and \((E, s')\) are Cayley algebras. If \( E \) admits a basis containing \( e \), one has \( s = s' \);
7. \( u + v = \bar{u} + \bar{v} \); \( \bar{uv} = \alpha \bar{u} \bar{v} \); \( \bar{uv} = \bar{v} \bar{u} \); \( \forall \alpha \in \mathbb{R}, \forall u, v \in E \);
8. \( T(e) = 2e \); \( N(e) = e \);
9. \( T(uv) = T(vu) \);
10. \( T(uv) = T(\bar{u}\bar{v}) = N(u + v) - N(u) - N(v) = T(u)T(v) - T(uv) \);
11. \( N(\alpha u) = \alpha^2 N(u) \);
12. \( (T(u))^2 - T(u^2) = 2N(u) \);
13. \( T \) is a linear form on \( E \) and \( N \) is a quadratic form on \( E \).

• (Cayley extension of a Cayley algebra \((E, s)\) defined by an element \( \gamma \in R \). Let \((E, s)\) be a Cayley algebra and let \( \gamma \in R \). Let \( F \) be the \( R \)-algebra with underlying module \( E \times E \) and with multiplication \((x, y)(x', y') = (xx' + \gamma y'y, yx' + y'y') \). Then \((e, 0)\) is the unit of \( F \) and \( E \times \{0\} \) is a subalgebra of \( F \) isomorphic to \( E \) that can be identified with \( E \). Let \( t \) be the permutation of \( F \) defined by \( t(x, y) = (\bar{x}, -y) \), \( \forall x, y \in E \). Then the couple \((F, t)\) is a Cayley algebra over \( R \). Let \( j = (0, e) \). So we can write \((x, y) = (x, 0)(e, 0) + (0, y)(0, e) = xe + yj \). One has \( yj = j\bar{y}, x(yj) = (xy)j - (xj)y = (xy)j - (xj)y = j\bar{x}e, j^2 = e \). Furthermore, one has \( T_F(xe + yj) = T(x), N_F(xe + yj) = N(x) - \gamma N(y) \). \( F \) is associative iff \( E \) is associative and commutative.

As a particular case one has: If \( E = R \) (hence \( s = \text{id}_R \)), the Cayley extension of \((R, \id_R)\) by an element \( \gamma \in R \) is a quadratic \( R \)-algebra with basis \((e, j)\) with \( j^2 = \gamma e \). Another particular case is the following. Let \( E \) be a quadratic algebra of type \((\alpha, \beta)\) such that the underlying module is \( \mathbb{R}^2 \) with multiplication rule given by means of \((13)\) for the canonical basis. Let the conjugation \( s \) be the conjugation in \( E \). Then for any \( \gamma \in R \), the Cayley extension \( F \) of \((E, s)\) by means of \( \gamma \) is called quaternionic algebra of type \((\alpha, \beta, \gamma)\). (This is an associative algebra.) The underlying module is \( \mathbb{R}^4 \). Let us denote by \((0, i, j, k)\) the canonical basis of \( \mathbb{R}^4 \). Then the corresponding multiplication rule is given in Tab. 2. (In the same table it are also reported the trace and norm formulas.)
An $R$-algebra isomorphic to a quaternionic algebra is called a quaternionic algebra of type $(\alpha, \gamma, \beta)$, if it has a basis with multiplication table given in Tab. 2.

If $0 = \alpha$ we say that the quaternionic algebra is of type $(\alpha, \beta, \gamma)$. The corresponding multiplication table is given in Tab. 3.

In particular if $R = \mathbb{K} = \mathbb{R}$, $\alpha = \gamma = -1$, $\beta = 0$, $F$ is called the Hamiltonian quaternionic algebra and is denoted by $\mathbb{H}$. In this case $N(u) \neq 0$, hence $u$ admits an inverse $u^{-1} = N(u)^{-1}u$ in $\mathbb{H}$, therefore $\mathbb{H}$ is a noncommutative corpr. Any finite $\mathbb{R}$-algebra that is also a corpr (noncommutative) is isomorphic to $\mathbb{H}$. Any quaternion $q \in \mathbb{H}$ can be represented by $q = \rho e + \xi i + \eta j + \zeta k$, where $i, j, k$ are linearly independent symbols that satisfy the following multiplication rules: $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$, $i^2 = j^2 = k^2 = -1$. One has the following $R$-algebras homomorphism: $A : \mathbb{H} \to M(2; \mathbb{C})$, $q \mapsto \left( \begin{array}{cc} a + bi & c + di \\ -c + di & a - bi \end{array} \right)$, where $i$ is the imaginary unity of $\mathbb{C}$. The matrices $\sigma_x := -iA(k), \sigma_y := -iA(j), \sigma_z := -iA(i)$, where $A(i) = \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), A(j) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), A(k) = \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)$, are called Pauli matrices and satisfy $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$, $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$. The set $\mathbb{H}_1 := N^{-1}(1)$ of quaternions of norm 1 is isomorphic to the group $SU(2): \mathbb{H}_1 \cong SU(2)$. The $n$-dimensional quaternionic space $\mathbb{H}^n$ has a canonical basis $\{e_k\}_{1 \leq k \leq n}$, $e_k \in \mathbb{H}$, and any $v \in \mathbb{H}^n$ can be represented in the form $v = \sum_{1 \leq k \leq n} q^k e_k, q^k \in \mathbb{H}$, $(q^k \equiv \text{quaternionic components})$. As any quaternionic number $q$ admits the following representation $q = x + yj + yk = x + yj + yk$, with $x = \rho e + \xi i, y = \eta j + \zeta k$, where $x$ and $y$ can be considered complex numbers, then one has the following isomorphism $\mathbb{H}^n \cong \mathbb{C}^{2n}$, $(q^k) \mapsto (x^k, y^k)$, where $\mathbb{C}^{2n}$ has the following basis $(e_1, \ldots, e_n, je_1, \ldots, je_n)$. We write $\dim \mathbb{H}^n = n, \dim \mathbb{H}^2 = 2n$. By using different quaternionic bases in $\mathbb{H}^n$ one has that the quaternionic components of any vector $v \in \mathbb{H}^n$ transform by means of the following rule $q^k = \sum_{1 \leq i \leq n} q^i \lambda^k_i, (\lambda^k_i) \in \mathbb{GL}(n, \mathbb{H})$. Furthermore, the corresponding complex components transform in the following way:

$\{x^h = x^i a_i^h - y^i b_i^h, y^h = x^i b_i^h + y^i a_i^h\}, \lambda^h_i = a_i^h + b_i^h j$.

Then one has a group-homomorphism $\mathbb{GL}(n, \mathbb{H}) \hookrightarrow \mathbb{GL}(2n, \mathbb{C})$, such that if $\Lambda = A + Bj \in \mathbb{GL}(n, \mathbb{H})$, then $c(\Lambda) = \left( \begin{array}{cc} A & B \\ -B & A \end{array} \right)$. On $\mathbb{H}^n$ there is a canonical quadratic form $|v|^2 = \sum_{1 \leq k \leq n} |q^k|^2 = \sum_{1 \leq k \leq n} q^k q^k = \sum_{1 \leq k \leq n} (|x^k|^2 + |y^k|^2) \in \mathbb{R}$, where $v = q^k e_k, q^k \in \mathbb{H}$, $q^k = x^k + y^k j, x^k, y^k \in \mathbb{C}$. So such a quadratic form coincides with the ordinary norm of the vector space $\mathbb{C}^{2n}$. Furthermore one has on $\mathbb{H}^n$ the following form $< v_1, v_2 >_H = \sum_{1 \leq k \leq n} q^k e_k \bar{q}^k e_k \in \mathbb{H}, v_i = \sum_{1 \leq k \leq n} q^k e_k, i = 1, 2$. The quaternionic transformations of $\mathbb{H}^n$, that conserve above form a group $\mathbb{Sp}(n) \subset \mathbb{GL}(n, \mathbb{H})$. As we can write $< v_1, v_2 >_H$ in the following way:

| $i$ | $j$ | $k$ | trace and norm formulas |
|-----|-----|-----|-------------------------|
| $i$ | $\alpha e$ | $\kappa$ | $\alpha j$ | $Tv(u) = 2\rho$ |
| $j$ | $-\kappa$ | $\gamma e$ | $-\gamma i$ | $N_p(u) = \rho^2 - \alpha \xi^2 - \gamma \eta^2 + \alpha \zeta^2$ |
| $k$ | $-\alpha j$ | $\gamma i$ | $-\alpha \gamma e$ | |

$u = \rho e + \xi i + \eta j + \zeta k, \rho, \xi, \eta, \zeta \in \mathbb{K}; \bar{u} = \rho e - \xi i - \eta j - \zeta k$.
Let $B$ algebra $C$ over $K$ from $M_c$ we see that $\Lambda$ is a quantum manifold $\bar{\Omega}$

Example 3.13 (Quantum quaternionic Möbius strip). Let us denote $I \equiv \{ -\pi, \pi \} \subset \mathbb{R}, N \equiv I \times \mathbb{H}$. Let us introduce the following equivalence relation in $N$: $(x, y) \sim (x, y)$ if $x \neq -\pi, \pi$, $(-\pi, -y) \sim (\pi, y)$. Then $N/\sim \equiv M$ is called noncommutative quaternionic Möbius strip. One has a natural projection $p : M \to S^1$, given by $p([x, y]) = x \in S^1$ if $x \neq -\pi, \pi$, and $p([\pi, y]) = * \in S^1$, where $*$ is the point of $S^1 \equiv I/\{ -\pi, \pi \}$, corresponding to $\{ -\pi, \pi \}$. One can recover $M$ with two open sets:

$$\{ \Omega_1 \equiv p^{-1}(U_1), \quad U_1 \equiv -\pi, \pi; \quad \Omega_2 \equiv p^{-1}(U_2), \quad U_2 \equiv S^1 \setminus \{ \} \}.$$  

We put quaternionic coordinates on $\Omega_i$, $i = 1, 2$, in the following way. On $\Omega_1$, 

$$\{ x^1, x^2, y \mid x^1 \in \mathbb{R}, x^2 \in \mathbb{H}, y \in \mathbb{H} \}.$$  

On $\Omega_2$, if $x \neq -\pi, \pi$, $\bar{x}^1[x, y] = -\pi + x \in \mathbb{R}$, $\bar{x}^1[-x, y] = \pi - x \in \mathbb{R}$, $\bar{x}^2[x, y] = y \in \mathbb{H}$; $\bar{x}^1[-\pi, -y] = \bar{x}^1[\pi, y] = 0$, $\bar{x}^2[-\pi, -y] = \bar{x}^2[\pi, y] = |y| \in c^{-1}(\mathbb{R}^+) \subset \mathbb{H}$, with $c = \frac{1}{2}T$. The change of coordinates is given by:

$$\forall q \in \Omega_1 \cap \Omega_2, \quad (p(q) \in S^1 \setminus \{ *, 0 \} \equiv U_+ \cup U_-), \quad \begin{cases} \bar{x}^1|_{U_-} = \pi - x^1 \in \mathbb{R} \\ \bar{x}^1|_{U_+} = -\pi + x^1 \in \mathbb{R} \\ \bar{x}^2 = x^2 \in \mathbb{H} \end{cases}.$$  

The structure group, i.e. the group of the Jacobian matrix, is isomorphic to $\mathbb{Z}_2$. In fact one has:

$$\left\{ (\partial x_j, \bar{x}^k) = \begin{pmatrix} \partial x_1, \bar{x}^1 \\ \partial x_2, \bar{x}^1 \end{pmatrix} \begin{pmatrix} \partial x_1, \bar{x}^2 \\ \partial x_2, \bar{x}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset GL(2; \mathbb{H}).$$  

Therefore, $M$ is a quantum quaternionic manifold modelled on $\mathbb{R} \times \mathbb{H} \subset \mathbb{H}^2$, hence $\dim_{\mathbb{H}} M = 2$. Furthermore one has the canonical projection $M \to S^1$, therefore $M$

---

$^{26}$ $Sp(1) \cong SU(2) \subset U(2)$. So all the transformations contained in $c(Sp(1))$ are unimodular.

$^{27}$ As a particular case we can take $B = K$. In this case $C = \mathbb{H}$ and we call such quantum $K$-quaternionic manifolds simply quantum quaternionic manifolds.
is a regular quantum manifold of dimension $(1 \downarrow 2)$. Finally remark that as $M$ is not covered by a global chart, it is a non-trivial example of quantum quaternionic manifold. Of course this can be also seen by means of homological arguments. In fact one has $H_1(M; \mathbb{R}) \cong H_1(M_C; \mathbb{R}) \cong H_1(S^1; \mathbb{R}) \cong \mathbb{R}$. Therefore, $M$ is not homotopy equivalent to $\mathbb{R}^5$, as $H_1(\mathbb{R}^5; \mathbb{R}) = 0$.

**Example 3.14** (Quaternionic manifolds [96, 97]). The category $C_{\mathbb{H}}$ of quaternionic manifolds is a subcategory of $C_{\mathbb{H}^\mathbb{R}}$, where the morphisms are quaternionic affine maps. Therefore any of such morphisms $f \in \text{Hom}_{C_{\mathbb{H}}}(M, N)$, where $\dim M = 4m$, $\dim N = 4n$, are locally represented by formulas like the following: $f^k = A_i^k q^i + r^k$, $A_i^k, q^i, r^k \in \mathbb{H}$, $1 \leq k \leq n, 1 \leq j \leq m$. $A_i^k$ identify $m \times n$ matrices with entries in $\mathbb{H}$, or equivalently, real matrices of the form

$$(A_i^k) = \left( \begin{array}{ccc} \hat{A}_1^1 & \ldots & \hat{A}_1^m \\ \vdots & \ddots & \vdots \\ \hat{A}_n^1 & \ldots & \hat{A}_n^m \end{array} \right), \quad \hat{A}_i^j = \left( \begin{array}{cccc} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & b & a \end{array} \right), \quad a, b, c, d \in \mathbb{R}.$$

The set of such matrices is denoted by $M(n, m; \mathbb{H})$. The structure group of a 4-dimensional quaternionic manifold is $\text{GL}(n; \mathbb{H})$, (that is the subset of $M(n, m; \mathbb{H})$ of invertible matrices). Therefore, quaternionic manifolds are quantum quaternionic manifolds where the local maps $f : U \subset \mathbb{H}^n \to \hat{U} \subset \mathbb{H}^n$, change of coordinates, are $\mathbb{H}$-linear. Hence $Df(p) \in \mathbb{H}^n$, $\forall p \in U$. In fact, one has the following commutative diagram:

$$
\begin{array}{ccc}
\text{Hom}_{\mathbb{H}}(\mathbb{H}^n; \mathbb{H}^n) & \longrightarrow & \text{Hom}_{\mathbb{R}}(\mathbb{H}^n; \mathbb{H}^n) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathbb{H}}(\mathbb{H}; \mathbb{H})^{n^2} & \longrightarrow & \text{Hom}_{\mathbb{R}}(\mathbb{H}; \mathbb{H})^{n^2} \\
\downarrow & & \downarrow \\
\mathbb{H}^{n^2} & \sim & \mathbb{R}^{4n^2} \\
\downarrow & & \downarrow \\
\mathbb{R}^{4n^2} & \longrightarrow & \mathbb{R}^{4n^2}
\end{array}
$$

On the other hand the tangent space $T_p M$ has a natural structure of $\mathbb{H}$-module iff $M$ is an affine manifold. (As in this case the action of $\mathbb{H}$ on $T_p M \cong \mathbb{H}^n$ does not depend on the coordinates used to obtain the identification of $T_p M$ with $\mathbb{H}^n$.) Hence the category $C_{\mathbb{H}^\mathbb{R}}$ is the subcategory of $C_{\mathbb{H}}$ of affine quantum quaternionic manifolds. A trivial example of quaternionic manifold is $\mathbb{R}^{4n} \cong \mathbb{H}^n$. If $\{x^i, y^i, u^i, v^i\}_{1 \leq i \leq n}$ are real coordinates on $\mathbb{R}^{4n}$, then the almost quaternionic structure given by

$$
\left\{ \begin{array}{llll}
J(\partial x_i) = \partial y_i, & J(\partial y_i) = -\partial x_i, & J(\partial u_i) = -\partial v_i, & J(\partial v_i) = \partial u_i \\
K(\partial x_i) = \partial u_i, & K(\partial y_i) = \partial v_i, & K(\partial u_i) = -\partial x_i, & K(\partial v_i) = -\partial y_i
\end{array} \right. 
$$
is called the standard right quaternionic structure on $\mathbb{R}^{4n}$.\footnote{An almost complex structure on a $C^\infty$ manifold $M$ is a fiberwise endomorphism $J$ of the tangent bundle $TM$ such that $J^2 = -1$. A complex analytic map between almost complex manifolds $(X, J_1)$ and $(Y, J_2)$ is a $C^\infty$ map $\phi : X \to Y$, such that $T(\phi) \circ J_1 = J_2 \circ T(\phi)$. An almost quaternionic structure on a $C^\infty$ manifold $M$ is a pair of two almost complex structures $J$ and $K$ such that $JK + KJ = 0$. A quaternionic map $\phi$ between two almost quaternionic manifolds $(X, J_1, K_1)$ and $(Y, J_2, K_2)$ is a map $\phi : X \to Y$ that is complex analytic from $(X, J_1)$ to $(Y, J_2)$ and from $(X, K_1)$ to $(Y, K_2)$. A quaternionic manifold is a $C^\infty$ manifold $M$ endowed with an atlas $\{ \phi_i : U_i \to \mathbb{R}^{4n} \}$, for some $n$, such that $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a quaternionic function with respect to the standard structure on $\mathbb{R}^{4n}$. (See also [96, 97].)}

A non trivial example of quaternionic manifold is $\mathbb{R}^{4n}$ with the standard quaternionic structure quotiented by a discrete translation group that gives a torus.

**Example 3.15 (Almost quaternionic manifolds).** The category $\widetilde{C}_H$ of almost quaternionic manifolds is a subcategory of $C^*_H \equiv \mathbb{R}$, where the structure group of a 4-dimensional quantum quaternionic manifold is $GL(n; \mathbb{H})Sp(1) \subset GL(4n; \mathbb{R})$. The category $\widetilde{C}_H$ properly contains $C_\mathbb{H}$. An example of almost quaternionic manifold, that is not contained into $C_\mathbb{H}$, is the quaternionic projective space $\mathbb{H}P^1$. This cannot be a quaternionic manifold, since it does not admit a structure of complex manifold.

**Remark 3.16.** As the centre $Z(C)$ of $C \equiv B \otimes_\mathbb{K} \mathbb{H}$ is isomorphic to $Z(B)$, that is the centre of $B$, we get that, whether $Z(B)$ is Noetherian, one can apply above Theorem 1.1 and Theorem 1.2 for QPDEs, in order to state the formal quantum-integrability for quantum $B$-quaternionic PDEs. Note that in such a way we obtain as solutions submanifolds that have natural structures of quantum $B$-quaternionic manifolds. Then applying our theorems on the integral bordism groups for quantum PDEs, we can also calculate theorem of existence of global solutions for quantum $B$-quaternionic PDEs.

**Example 3.17 (Quantum $B$-quaternionic heat equation).** Let us consider the fiber bundle $\pi : W \equiv C^3 \to C^2 \equiv M$ with coordinates $(t, x, u) \mapsto (t, x)$. The quantum $B$-quaternionic heat equation is the following QPDE: $(\hat{H}E)_C \subset \hat{JD}^2(W) \subset \hat{J}_2^2(W)$:

$$u_{xx} - u_t = 0.$$  

This is a formally (quantum)integrable QPDE. Hence, for $(\hat{H}E)_C$ we have the existence of local solutions for any initial condition. This means that in the neighborhood of any point $q \in (\hat{H}E)_C$ we can built an integral quantum $B$-quaternionic manifold of dimension 2 over $C$, $V \subset (\hat{H}E)_C$, such that $V \equiv \pi_2(V) \subset M$, where $\pi_2$ is the canonical projection $\pi_2 : \hat{JD}^2(W) \to M$. Then by using a Theorem 5.6 given in [66] we have that the first integral bordism groups of $(\hat{H}E)_C$ is: $\Omega_1^{(\hat{H}E)_C} \cong H_1(W; \mathbb{K}) \otimes_\mathbb{K} C \cong 0$. Hence we get that any admissible closed integral 1-dimensional quantum $B$-quaternionic manifold, $N \subset (\hat{H}E)_C$, is the boundary of an integral 2-dimensional quantum $B$-quaternionic manifold $V$, $\partial V \subset N$, $V \subset (\hat{H}E)_C$, such that $V$ is diffeomorphic to its projection into $W$ by means of the canonical projection $\pi_{2,0} : \hat{J}_2^2(W) \to W$.

**Example 3.18 (Quantum quaternionic heat equation).** As a particular case of above equation one can take $B \equiv \mathbb{R}$. Then one has:

$$\begin{cases} 
\pi : W \equiv \mathbb{H}^3 \to \mathbb{H}^2 \equiv M : (t, x, u) \mapsto (t, x) \\
(\hat{H}E)_H \subset \hat{JD}^2(W) \subset \hat{J}_2^2(W) : u_{xx} - u_t = 0 
\end{cases}, \quad \Omega_1^{(\hat{H}E)_H} \cong H_1(W; \mathbb{R}) \otimes_\mathbb{R} \mathbb{H} \cong 0.$$

\[28\]
We can see that the set $\text{Sol}((\hat{HE})_\mathbb{H})$ of solutions of $(\hat{HE})_\mathbb{H}$ contains also quaternionic manifolds, i.e., affine quantum quaternionic solutions. For example a torus $29 \ X \subset \mathbb{H}^2 \cong M$ can be embedded into $(\hat{HE})_\mathbb{H}$ by means of the second holonomic prolongation of the zero section $u \equiv 0 : M \to W$. In fact, $(\hat{HE})_\mathbb{H}$ is a linear equation. Therefore, $X(2) \equiv D^2 u(X)$ is a 1-dimensional smooth closed compact admissible integral manifold contained into $(\hat{HE})_\mathbb{H}$, that is the boundary of a 2-dimensional integral admissible manifold contained into $(\hat{HE})_\mathbb{H}$ too. This last is also a quaternionic manifold. Moreover, all the regular solutions of $(\hat{HE})_\mathbb{H} \subset JD^2(W)$ are quaternionic manifolds, as they are diffeomorphic to $\mathbb{H}^2$. However, no all the regular solutions of $(\hat{HE})_\mathbb{H} \subset J^2(W)$ are necessarily quaternionic manifolds too.

We are ready now to state the following theorem.

**Theorem 3.19.** Let $B$ be a quantum algebra such that its centre $Z(B)$ is a Noetherian $\mathbb{R}$-algebra. Let $\hat{E}_k \subset JD^k(W)$ be a quantum regular QPDE in the category $C^B_\mathbb{H}$, where $\pi : W \to M$ is a fibre bundle with $\dim \mathcal{C} = m$, $C \equiv B \otimes \mathbb{R} \mathbb{H}$. If $\hat{g}_{k+r+1}$ is a bundle of $Z(C)$-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$ is surjective for $0 \leq r \leq m$, then $\hat{E}_k$ is formally quantumintegrable. In such a case, and further assuming that $W$ is $p$-connected, $p \in \{0, \ldots, m-1\}$, then the integral bordism groups of $\hat{E}_k \subset J^k_m(W)$ are given by:

$$\Omega^\hat{E}_k_p \cong H_p(W; \mathbb{R}) \otimes \mathbb{R} C, \quad 0 \leq p \leq m - 1.$$  

All the regular solutions of $\hat{E}_k \subset J^k_m(W)$ are quantum $B$-quaternionic submanifolds of $\hat{E}_k$ of dimension $m$, over $C$, identified with $m$-dimensional quantum $B$-quaternionic submanifolds of $W$.

**Proof.** It follows directly from above definitions and remarks by specializing Theorem 1.1, Theorem 1.2 and our results in [66], about integral bordism groups in QPDEs, to the category $C^B_\mathbb{H}$. □

**Corollary 3.20.** Let $\hat{E}_k \subset JD^k(W)$ be a quantum regular QPDE in the category $C^B_\mathbb{H}$, (resp. $\tilde{C}_\mathbb{H}$), where $\pi : W \to M$ is a fibre bundle with $\dim \mathcal{C} = m$. If $\hat{g}_{k+r+1}$ is a bundle of $\mathbb{R}$-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \to \hat{E}_{k+r}$ is surjective for $0 \leq r \leq m$, then $\hat{E}_k$ is formally quantumintegrable. In such a case, further assuming that $W$ is $p$-connected, $p \in \{0, \ldots, m-1\}$, then the integral bordism groups of $\hat{E}_k \subset J^k_m(W)$ are given by:

$$\Omega^\hat{E}_k_p \cong H_p(W; \mathbb{R}) \otimes \mathbb{R} \mathbb{H}, \quad 0 \leq p \leq m - 1.$$  

All the regular solutions of $\hat{E}_k \subset J^k_m(W)$ are quantum quaternionic, (resp. almost quaternionic), submanifolds of $\hat{E}_k$ of dimension $m$, identified with $m$-dimensional quantum quaternionic, (resp. almost quaternionic), submanifolds of $W$.

Before to pass to the following example on the heat PDE over octonions, let us first recall some useful definitions and results about alternative algebras.\(^{30}\)

\(^{29}\)Recall [96, 97] that if $(X, J, K)$ is a quaternionic manifold, then $X$ with the complex structure $aJ + bK + c(JK)$, $a, b, c \in \mathbb{R}$, is an affine complex manifold, hence has zero rational Pontryagin classes. Furthermore, if $X$ is compact has zero index and Euler characteristic. Moreover, if $\dim X = 1$ and, for some $a, b, c$, $X$ is Kähler, then it is a torus.

\(^{30}\)The algebra $\mathcal{O}$ of octonions is just a distinguished example of alternative algebra.
Definition 3.21. The associator of an algebra $A$ is a trilinear map $[, , ] : A \times A \times A \to A$, such that $[a, b, c] \equiv (ab)c - a(bc)$. When $[a, b, c] = 0$ we say that the three elements associate.

An $n$-multilinear mapping $f : A \times \cdots \times A \to A$ is alternating if it vanishes whenever two of its arguments are equal.

An alternative algebra is one where the associator is alternative.

Proposition 3.22 (Properties of alternative algebras). 1) Given an algebra $A$ the following propositions are equivalent.
(i) $A$ is an alternative algebra.
(ii) The following propositions hold:
(a) (left alternative identity): $[a, a, b] = 0$, $b \in A$.
(b) (Right alternative identity): $[a, b, b] = 0$, $b \in A$.
(c) (Flexibility identity): $a(ba) = (ba)a$, $b \in A$.

2) An alternating associator is always totally skew-symmetric, i.e., $[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}] = \epsilon(\sigma)[a_1, a_2, a_3]$, where $\epsilon(\sigma)$ is the signature of the permutation $\sigma \in S_3$ and $S_3$ is the group of permutations of three objects.\footnote{31}

3) (Artin’s theorem) In an alternative algebra $A$ the subalgebras generated by any two elements are associative, and vice versa.

4) Let $A$ be an alternative algebra, then the subalgebra generated by three elements $a, b, c \in A$, such that $[a, b, c] = 0$, is associative.

5) Alternative algebras are power-associative, that is, the subalgebra, generated by a simple element is associative.\footnote{32}

6) (Moufang identities) In any alternative algebra hold the following identities:
(a) $a(x(ay)) = (axa)y$.
(b) $((xa)y)a = x(aya)$.
(c) $(ax)(ya) = a(xy)a$.

7) If $A$ is a unital alternative algebra, multiplicative inverse are unique whenever they exist. Furthermore one has the identities reported in (15).

\begin{equation}
\begin{cases}
 b = a^{-1}(ab) \\
 [a^{-1}, a, b] = 0 \\
 (ab)^{-1} = b^{-1}a^{-1}
\end{cases}
\end{equation}

if $a$ and $b$ are invertible.

Example 3.23 (Quantum octonionic heat equation). The octonions form a normed division $\mathbb{R}$-algebra that is non-associative, non-commutative, power-associative and alternative. In Tab. 4 it is reported the multiplication table with respect to a basis $\{e_\alpha\}_{0 \leq \alpha \leq 7}$, with $e_0 = 1 \in \mathbb{R}$, (unit octonions), i.e., any $q \in \mathbb{O}$ has the following linear representation $q = \sum_{0 \leq \alpha \leq 7} x^\alpha e_\alpha$, $x^\alpha \in \mathbb{R}$.\footnote{33}

An example of quantum octonionic manifold is the quantum octonionic Möbius strip, that can be obtained similarly to the quantum quaternionic manifold, just considered in Example 3.13. In fact it is enough to copy the construction given in

\footnote{31}{\text{The converse holds too when the characteristic of the base field $K$ of $A$ is not 2, (e.g. $K = \mathbb{R}$).}}

\footnote{32}{\text{The converse need not hold: the sedenions are power-associative, but not alternative.}}

\footnote{33}{\text{There are 480 isomorphic octonionic algebras generated by different multiplication tables. The group $G_2 = Aut(\mathbb{O})$ is a simply connected, compact real Lie group, dim$_\mathbb{R}G_2 = 4$. One has $\mathbb{O} \cong H \times H$, with multiplication $(a, b)(c, d) = (ac - db, da + bc)$. The corresponding representation of the basis is the following: $e_0 = (1, 0)$, $e_1 = (i, 0)$, $e_2 = (j, 0)$, $e_3 = (k, 0)$, $e_4 = (0, 1)$, $e_5 = (0, i)$, $e_6 = (0, j)$, $e_7 = (0, k)$.}}
Table 4. Multiplication table, trace and norm formulas for octonionic algebra $O \cong \mathbb{R}^8$ with basis $(e_\alpha)_{0 \leq \alpha \leq 7}$.

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $-1$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $-e_7$ | $e_6$ |
| $e_2$ | $-e_3$ | $-1$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $-e_4$ | $-e_1$ | $-1$ | $e_7$ | $-e_6$ | $e_5$ | $-e_4$ |
| $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $-1$ | $e_1$ | $e_2$ | $e_3$ |
| $e_5$ | $-e_6$ | $-e_7$ | $-e_1$ | $-1$ | $-e_4$ | $e_2$ | $e_1$ |
| $e_6$ | $-e_7$ | $-e_1$ | $-e_4$ | $-e_2$ | $e_1$ | $-1$ | $-e_4$ |
| $e_7$ | $-e_1$ | $-e_2$ | $-e_3$ | $-e_4$ | $-e_5$ | $-e_6$ | $-1$ |

$e_ie_j = -\delta_{ij}e_0 + e_{ijk}e_k, \ 1 \leq i, j, k \leq 7.$

$e_ie_j = -e_j e_i. (e_i e_j) e_k = -e_i (e_j e_k).$

$x = \sum_{0 \leq \alpha \leq 7} x^\alpha e_\alpha = x^0 + \sum_{1 \leq k \leq 7} x^k e_k.$

$\bar{x} = x^0 - \sum_{1 \leq k \leq 7} x^k e_k.$

$\mathbb{R}(x) \equiv \frac{1}{2}(x + \bar{x}); \ \mathbb{Q}(x) \equiv \frac{1}{2}(x - \bar{x}).$

Norm: $N(x) \equiv \|x\| = \sqrt{x\bar{x}}.$

Square root: $\|x\|^2 = \bar{x}x = \sum_{0 \leq \alpha \leq 7} (x^\alpha)^2 > 0.$

Inverse: $x^{-1} = \bar{x}/\|x\|^2; \ x^{-1}x = x^{-1} = 1.$

Example 3.13, after substituting $\mathbb{H}$, with $O$, to get a quantum octonion manifold. Similarly we can obtain the quantum octonionic heat equation as given in (16).

$$\left\{ \begin{array}{ll}
\pi : W \equiv O^3 \rightarrow O^2 \equiv M : \ (t, x, u) \mapsto (t, x) \\
\Omega_{\hat{(HE)}_O} \subset JD^2(W) \subset J^2_2(W) : \ u_{xx} - u_1 = 0
\end{array} \right\} \\ \hat{(HE)}_O \cong H_4(W; \mathbb{R}) \otimes_{\mathbb{R}} O \cong 0.
$$

We can see that the set $\text{Sol}(\hat{(HE)}_O)$ of solutions of $\hat{(HE)}_O$ contains also octonionic manifolds, i.e., affine quantum octonionic solutions. $X \subset O^2 \equiv M$ can be embedded into $\hat{(HE)}_O$ by means of the second holonomic prolongation of the zero section $u \equiv 0 : M \rightarrow W$. In fact, $(HE)_O$ is a linear equation. Therefore, $X^{(2)} \equiv D^2u(X)$ is a 1-dimensional smooth closed compact admissible integral manifold contained into $(\hat{HE})_O$, that is the boundary of a 2-dimensional integral admissible manifold contained into $(\hat{HE})_O$ too. This last is also a octonionic manifold. Moreover, all the regular solutions of $(\hat{HE})_O \subset JD^2(W)$ are octonionic manifolds, as they are diffeomorphic to $O^2$. However, no all the regular solutions of $(HE)_O \subset J^2_2(W)$ are necessarily octonionic manifolds too.

We are ready now to state one of main results of this paper.

Theorem 3.24. Let $B$ be a quantum algebra such that its centre $Z(B)$ is a Noetherian $\mathbb{R}$-algebra. Let $\hat{E}_k \subset JD^k(W)$ be a quantum regular QPDE in the category $C^B_\mathbb{O}$, where $\pi : W \rightarrow M$ is a fibre bundle with $\dim_{\mathbb{C}} M = m$, $C \equiv B \otimes_{\mathbb{R}} \mathbb{O}$. If $\hat{y}_{k+r+1}$ is a bundle of $Z(C)$-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \rightarrow \hat{E}_{k+r}$ is surjective for $0 \leq r \leq m$, then $\hat{E}_k$ is formally quantumintegrable. In such a case, and further assuming that $W$ is $p$-connected, $p \in \{0, \ldots, m - 1\}$, then the integral bordism groups of $\hat{E}_k \subset J^m_k(W)$ are given in (17).
Table 5. Multiplication table, trace and norm formulas for sedenionic algebra $S \cong \mathbb{R}^{16}$ with basis $\{e_\alpha\}_{0 \leq \alpha \leq 15}$.

| $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|
| $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_{14}$ | $e_{15}$ |

S = $A_4$ in the Cayley-Dickson construction.

All the regular solutions of $\hat{E}_k \subset \hat{j}_m(W)$ are quantum B-octonionic submanifolds of $\hat{E}_k$ of dimension $m$, over $C$, identified with $m$-dimensional quantum B-octonionic submanifolds of $W$.

Example 3.25 (Quantum sedenionic heat equation). Sedenionic algebra, $S$, is a 16-dimensional non-commutative and non-associative $\mathbb{R}$-algebra, that is power-associative (but not even alternative), where any $q \in S$ has the following linear representation $q = \sum_{0 \leq \alpha \leq 15} x^\alpha e_\alpha$, $x^\alpha \in \mathbb{R}$, where the basis $\{e_\alpha\}_{0 \leq \alpha \leq 15}$, (unit sedenions), has $e_0 = 1 \in \mathbb{R}$, with multiplication table reported in Tab. 5. $S$ is not a division algebra since there are zero divisors, i.e., there are non-zero $a, b \in S$ such that their product is zero: $ab = 0$.

Similarly to the two above examples, we can define and characterize quantum sedenionic manifolds, and quantum sedenionic PDE’s. In particular we get a natural extension, in the category of quantum sedenionic manifolds, of Theorem 3.24. In fact, we get the following theorem.

Theorem 3.26. Let $B$ be a quantum algebra such that its centre $Z(B)$ is a Noetherian $\mathbb{R}$-algebra. Let $\hat{E}_k \subset J^k(W)$ be a quantum regular QPDE in the category $C_B$, where $\pi : W \rightarrow M$ is a fibre bundle with $\dim C M = m$, $C \equiv B \otimes_{\mathbb{R}} S$. If $\hat{j}_{k+r+1}$ is a bundle of $Z(C)$-modules over $\hat{E}_k$, and $\hat{E}_{k+r+1} \rightarrow \hat{E}_{k+r}$ is surjective for $0 \leq r \leq m$, then $\hat{E}_k$ is formally quantum integrable. In such a case, and further assuming that $W$ is $p$-connected, $p \in \{0, \ldots, m - 1\}$, then the integral bordism groups of $\hat{E}_k \subset J^m(W)$ are given in (18).

For example $(e_3 + e_{10})(e_6 - e_{15}) = 0$. Note that all algebras $A_r$ in the Cayley-Dickson construction contain zero divisors, when $r \geq 4$. Recall that all the Cayley-Dickson algebras $A_r$, with $0 \leq r \leq 2$, are associative, division algebras, hence have no zero divisors. [A finite-dimensional unital associative algebra (over a field) is a division algebra if and only if it has no zero divisors.]

Let us emphasize that a quantum algebra $A$ is not, in general, a division algebra. Therefore, in general, a quantum hypercomplex algebra $Q_r$, $r \geq 0$, is not a division algebra.
(18) \[ \hat{\Omega}_p^k \cong H_p(W; \mathbb{R}) \otimes_{\mathbb{R}} C, \quad 0 \leq p \leq m - 1. \]

All the regular solutions of \( \hat{\hat{E}}_k \subset \hat{\hat{J}}^k_m(W) \) are quantum \( B \)-sedenionic submanifolds of \( \hat{\hat{E}}_k \) of dimension \( m \), over \( C \), identified with \( m \)-dimensional quantum \( B \)-sedenionic submanifolds of \( W \).

4. Quantum hypercomplex singular (super) PDE’s

In this section we shall consider singular quantum super PDE’s extending our previous theory of singular PDE’s, i.e., by considering singular quantum (super) PDE’s as singular quantum sub-(super)manifolds of jet-derivative spaces in the category \( \Omega \) or \( \Omega_S \) or \( \Omega_{hyper} \). In fact, our previous formal theory of quantum (super) PDE’s works well on quantum smooth or quantum analytic submanifolds, since these regularity conditions are necessary to develop such a theory. However, in many mathematical problems and physical applications, it is necessary to work with less regular structures, so it is useful to formulate a general geometric theory for such more general quantum PDE’s in the category \( \Omega \) or \( \Omega_S \) or \( \Omega_{hyper} \). Therefore, we shall assume that quantum singular super PDE’s are subsets of jet-derivative spaces where are presents regular subsets, but also other ones where the conditions of regularity are not satisfied. So the crucial point to investigate is to obtain criteria that allow us to find existence theorems for solutions crossing "singular points" and study their stability properties. In some previous works we have considered quantum singular PDE’s in the categories \( \Omega \) and \( \Omega_S \). Here we shall specialize on quantum singular PDE’s in the category \( \Omega_{S-hyper} \) of quantum hypercomplex supermanifolds. There the fundamental non-commutative algebras considered are of the type \( \hat{\hat{Q}}_r \equiv \hat{\hat{B}} \otimes_{\mathbb{R}} \mathbb{A}_r \), where \( B \) is a quantum superalgebra in the sense of A. Prástaro, and \( \mathbb{A}_r \) is a Cayley-Dickson algebra, as just considered in the previous sections.

The main result of this section is Theorem 4.8 that relates singular integral bordism groups of singular quantum PDE’s to global solutions passing through singular points. Some example are explicitly considered.

Let us, now, first begin with a generalization of algebraic formulation of quantum super PDE’s, starting with the following definitions. (See also Refs.[78, 82, 83].)

**Definition 4.1.** The general category of quantum hypercomplex superdifferential equations, \( \Omega_{S-hyper}^G \), is defined by the following: 1) \( \hat{\hat{B}} \in Ob(\Omega_{S-hyper}^G) \) iff \( \hat{\hat{B}} \) is a filtered quantum hypercomplex superalgebra \( \hat{\hat{B}} \equiv \{ \hat{\hat{B}}_i \}, \hat{\hat{B}}_i \subset \hat{\hat{B}}_{i+1} \), such that in the differential calculus in the category \( \Omega_{S-hyper}^G(\hat{\hat{B}}) \) over \( \hat{\hat{B}} \) is defined a natural operation \( C \) that satisfies \( C\hat{\hat{\Omega}}^1 \wedge \hat{\hat{\Omega}}^* = C\hat{\hat{\Omega}}^* \), where \( \hat{\hat{\Omega}}^i \equiv \hat{\hat{B}} \wedge \cdots \wedge \hat{\hat{B}} \) are the representative objects of the functor \( \hat{\hat{D}}_i \) in the category \( \Omega_{S-hyper}^G(\hat{\hat{B}}) \) over \( \hat{\hat{B}} \), where \( \hat{\hat{D}}_i \equiv \hat{\hat{D}} \cdots \hat{\hat{D}}_i \), being \( \hat{\hat{D}}_i \) the \( \hat{\hat{B}} \)-module of all quantum superdifferentiations of algebra \( \hat{\hat{B}} \) with values in module \( P \). Furthermore, \( \hat{\hat{\Omega}}^* \equiv \bigoplus_{i \geq 0} \hat{\hat{\Omega}}^i \), \( \hat{\hat{\Omega}}^0 \equiv A \). 2) \( f \in Hom(\Omega_{S-hyper}^G) \) iff \( f \) is a homomorphism of filtered quantum hypercomplex superalgebras preserving operation \( C \).

\[35\] See Refs.[68, 82, 83]. See also [3] where some interesting applications are considered.
Table 6. Some quantum hypercomplex singular PDE’s defined by differential polynomials.

| Name                          | Singular PDE                                                                 |
|-------------------------------|-----------------------------------------------------------------------------|
| PDE with node and triple point| $p_1 = (u_1^2)^4 + (u_2^1)^4 - (u_2^2)^3 = 0$                              |
| $\mathcal{R}_i \subset JD(E)$ | $p_2 = (u_1^2)^6 + (u_2^1)^6 - u_3 u_4 = 0$                                |
| PDE with cusp and tacnode     | $q_1 = (u_1^2)^6 + (u_2^1)^4 - (u_2^2)^2 = 0$                              |
| such that $\mathcal{S}_1 \subset JD(E)$ | such that $q_2 = (u_1^2)^4 + (u_2^1)^3(u_2^2 + u_3) - (u_2^1)^2(u_2^2)^3 = 0$. |
| PDE with conical double point, | $r_1 = (u_1^2)^4 - (u_2^1)^4 = 0$                                          |
| double line and pinch point   | $r_2 = (u_2^1)^2 - (u_2^2)^2 = 0$                                          |
| $\mathcal{T}_1 \subset JD(F)$ | $r_3 = (u_1^3)^3 + (u_2^1)^3 + (u_2^2)^2 = 0$                              |

Remark 4.2. In practice we shall take $\hat{\mathcal{B}} \equiv \{ \hat{\mathcal{B}}_i \equiv Q^\infty_{\mathcal{B}}(M_i; A) \}$, where $M_i$ is a quantum hypercomplex supermanifold and $A$ is a quantum hypercomplex superalgebra. Then, we have a canonical inclusion: $j_i : M_i \to Sp(\hat{\mathcal{B}}_i), x \mapsto j_i(x) \equiv e_x \equiv$ evaluation map at $x \in M_i$. To the inclusion $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ corresponds the quantum smooth map $M_{i+1} \to M_i$. So we set $M_\infty = \lim_{\leftarrow} M_i$. One has $\overline{M}_\infty = Sp(\hat{\mathcal{B}}_\infty)$. However, as $M_\infty$ contains all the "nice" points of $Sp(\hat{\mathcal{B}}_\infty)$, we shall use the space $M_\infty$ to denote an object of the category of quantum hypercomplex superdifferential equations.

Definition 4.3. The category of quantum hypercomplex superdifferential equations $\mathfrak{Q}^{\mathfrak{E}}$-hyper is defined by the Frobenius full quantum superdistribution $\tilde{C}(X) \subset \tilde{T}X \equiv Hom_{\mathbb{Z}}(A; TX)$, which is locally the same as $\tilde{E}_\infty$, i.e., the Cartan quantum superdistribution of $E_\infty$ for some quantum hypercomplex super PDE $\tilde{E}_k \subset JD^k(W)$. We set: $sdim X \equiv dim \tilde{C}(X) = (m + n|m + n)$, i.e., the Cartan quantum superdimension of $X \in Ob(\mathfrak{g}^{\mathfrak{E}}$-hyper). $f \in Hom(\mathfrak{g}^{\mathfrak{E}}$-hyper) iff it is a quantum supersmooth map $f : X \to Y$, where $X, Y \in Ob(\mathfrak{g}^{\mathfrak{E}}$-hyper), such that conserves the corresponding Frobenius full superdistributions: $\tilde{T}(f) : \tilde{C}(X) \to \tilde{C}(Y), f \in Hom_{\mathfrak{g}^{\mathfrak{E}}$-hyper}(X, Y), $sdim X = (m+n|m+n)$, $sdim Y = (m'+n'|m'+n')$, $srank f = (r|s) = dim(T(f)_x(\tilde{C}(X)_x)), x \in X$. Then the fibers $f^{-1}(y), y \in im(f) \subset Y$, are $(m+n-r|m+n-s)$-quantum superdimensional objects of $\mathfrak{g}^{\mathfrak{E}}$-hyper. Isomorphisms of $\mathfrak{g}^{\mathfrak{E}}$-hyper: quantum supermorphisms with fibres consisting of separate points. Covering maps of $\mathfrak{g}^{\mathfrak{E}}$-hyper: quantum supermorphisms with zero-quantum superdimensional fibres.

Example 4.4 (Some quantum hypercomplex singular PDE’s). In Tab. 6 we report some quantum singular PDE’s having some algebraic singularities. For the first two equations these are quantum singular PDE’s of first order defined on the quantum fiber bundle $\pi : E \to M$, with $E \equiv A^4, M \equiv A^3$, where $A$ is a quantum algebra. Then $JD(E) \cong B_1^{1,4} = A^3 \times X^4$. Furthermore, for the third equation one has the quantum fiber bundle $\tilde{\pi} : F \to M$, with $F \equiv A^5, M \equiv A^2$, and $JD(F) \cong B_1^{5,6} = A^5 \times \tilde{A}^6$. We follow our usual notation introduced in some previous works on the
same subject. In particular for a given quantum (super)algebra $A$, we put

$$
\begin{cases}
\tilde{T}_0^r(H) \equiv H \otimes \cdots \otimes H, & r \geq 0 \\
\tilde{\mathcal{A}} = \text{Hom}_Z(\tilde{T}_0^r(A); A), & r \geq 0 \\
\tilde{\mathcal{A}} = \text{Hom}_Z(\tilde{T}_0^0(A); A) \equiv \text{Hom}_Z(A; A) = \tilde{\mathcal{A}}
\end{cases}
$$

(19)

with $Z$ the centre of $A$ and $H$ any $Z$-module. Furthermore, we denote also by $\tilde{S}_0(H)$ and $\tilde{\Lambda}_0(H)$ the corresponding symmetric and skew-symmetric submodules of $\tilde{T}_0(H)$. To the ideals $a \equiv < p_1, p_2 > \supseteq \mathfrak{B}_1$, $b \equiv < q_1, q_2 > \supseteq \mathfrak{B}_2$, and $c \equiv < r_1, r_2, r_3 > \supseteq \mathfrak{P}_1$, where $\mathfrak{B}_1 \equiv Q_w^\infty(\tilde{J} D(E), B_2)$, with $B_2 \equiv A \times \tilde{A} \times \tilde{A}$, and $\mathfrak{P}_1 \equiv Q_w^\infty(\tilde{J} D(F), B_1)$, one associates the corresponding algebraic sets $\tilde{R}_1 = \{ q \in B^{4,4}_1 | f(q) = 0, \forall f \in a \} \subset B^{4,4}_1$, $\tilde{S}_1 = \{ q \in B^{4,4}_1 | f(q) = 0, \forall f \in b \} \subset B^{4,4}_1$ and $\tilde{T}_1 = \{ q \in B^{5,6}_1 | f(q) = 0, \forall f \in c \} \subset B^{5,6}_1$.

Let us consider in some details, for example, the first equation in Table 5. There the node and the triple point refer to the singular points in the planes $(u_x^1, u_y^1)$ and $(u_x^2, u_y^2)$, respectively, with respect to the $\mathbb{R}$-restriction. However, the equation $\tilde{R}_1$ has a set $\Sigma(\tilde{R}_1) \subset \tilde{R}_1$ of singular points that is larger than one reported in (20).

(20) $\Sigma(\tilde{R}_1)_0 = \{ q_0 = (x, y, u_x^1, u_y^2, 0, 0, 0, 0) \} \equiv A^4 \subset \Sigma(\tilde{R}_1)_1 \subset \tilde{R}_1$

In fact the jacobian $(j(F))_{ij}, i = 1, 2, j = 1, \cdots, 8$, with $(F_i) \equiv (p_1, p_2) : \tilde{J} D(E) \rightarrow B_2$, is given by the following matrix with entries in the quantum algebra $B_2$:

(21) 

$$(j(F))_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2u_x^1[2(u_x^1)^2 - 1] & 0 & 0 & 4(u_y^2)^3 \\ 0 & 0 & 0 & 0 & 0 & 6(u_y^1)^5 - u_x^2 & 0 & 6(u_x^2)^5 - u_y^1 \end{pmatrix}.$$  

Since, in general, the quantum hypercomplex algebra $A$ has non-empty the set $Z_{zero}(A)$ of zero-divisors,\(^{36}\) in order $\tilde{Y}_1 \equiv \tilde{R}_1 \setminus \Sigma(\tilde{R}_1)$ should represent $\tilde{R}_1$ without singular points, i.e., in order to apply the implicit quantum function theorem (see Theorem 1.38 in [68]), it is enough to take the points $q \in \tilde{R}_1$, where there are $2 \times 2$ minors in (21) with invertible determinant. In other words, we shall add the condition that at least one of determinants $\det_i, 1 \leq i \leq 4$, reported in (22) should be invertible elements of the quantum algebra $\tilde{A} \equiv \text{Hom}_Z(\tilde{T}_0^0(A); A)$, for suitable $s \in \mathbb{N}$.\(^{37}\)  

\(^{36}\)Let us emphasize that $Z_{zero}(A) = \bigcup_{i} p_i$, where $p_i$ is an associated prime ideal of $A$, i.e., a prime ideal that is the annihilator of some element of $A$. (Let us recall that an ideal $p$ of $A$ is called prime if $p \subsetneq A$ and whenever two ideals $a, b \subset A$ are such that $ab \subset p$, then at least one of $a$ and $b$ is contained in $A$.) The $\mathbb{R}$-algebra $A$ is called tame if $Z_{zero}(A) = \bigoplus_{1 \leq i \leq r < \infty} S_i$, where $S_i$ is a subspace of $A$. Note that even if $Z_{zero}(A) = \emptyset$, $Z_{zero}(A^2) \neq \emptyset$, since $(0, a)(b, 0) = (0, 0) = 0 \in A^2$.

\(^{37}\)See the implicit quantum function theorem. (Theorem 1.38 in [68]..)
\( (22) \)
\[
\begin{align*}
\text{det}_1 & \equiv \text{det} \left( \begin{array}{cc}
2u_x^1[2(u_x^1)^2 - 1] & 0 \\
0 & 6(u_y^1)^2 - u_x^2
\end{array} \right) = 2u_x^1[2(u_x^1)^2 - 1][6(u_y^1)^2 - u_x^2] \in \mathbb{A}^3 \\
\text{det}_2 & \equiv \text{det} \left( \begin{array}{cc}
2u_x^1[2(u_x^1)^2 - 1] & 0 \\
0 & 6(u_x^2)^5 - u_y^1
\end{array} \right) = 2u_x^1[2(u_x^1)^2 - 1][6(u_x^2)^5 - u_y^1] \in \mathbb{A}^3 \\
\text{det}_3 & \equiv \text{det} \left( \begin{array}{cc}
0 & 4(u_y^1)^3 \\
6(u_y^1)^5 - u_x^2 & 0
\end{array} \right) = -[6(u_y^1)^5 - u_x^2]^2 [4(u_y^1)^3] \in \mathbb{A}^2 \\
\text{det}_4 & \equiv \text{det} \left( \begin{array}{cc}
0 & 4(u_y^1)^3 \\
6(u_x^2)^5 - u_y^1 & 0
\end{array} \right) = -[6(u_x^2)^5 - u_y^1]^2 [4(u_y^1)^3] \in \mathbb{A}^2.
\end{align*}
\]

Therefore, we must identify the sets \( \hat{X}_1 \equiv \{ q \in \hat{J}D(E) | \text{det}_i(q) \in \hat{A}(G(A)) \}, \)
\( 1 \leq i \leq 4, \) with \( s(1) = s(2) = 3 \) and \( s(3) = s(4) = 2. \) Here \( G(A) \) denotes
the abelian group of unities in \( A \) and \( \hat{A}(G(A)) \equiv \text{Hom}_{\mathbb{Z}(A)}(T_0^{s(i)}(A); G(A)) \subset \hat{A}. \)

Then taking into account that \( \hat{A}(G(A)) \) is an open set in the quantum hypercomplex
algebra \( \hat{A} \), \( 38 \) we get that \( \hat{X}_1 \equiv \bigcup_{1 \leq i \leq 4} \text{det}_i^{-1}(\hat{A}(G(A))) = \bigcup_{1 \leq i \leq 4} \hat{X}_1 \subset \hat{J}D(E) \)
is an open submanifold of \( \hat{J}D(E). \) We call \( \hat{Y}_1 \equiv \hat{R}_1 \cap \hat{X}_1 \) the regularized quantum
PDE corresponding to \( \hat{X}_1. \) Since \( \hat{X}_1 \) is an open submanifold of \( \hat{J}D(E) \), it follows
that \( \hat{Y}_1 \) is a quantum submanifold of \( \hat{J}D(E) \) of dimension as reported in formulas
\( (25). \) Let us define the following subsets of \( \hat{R}_1: \)
\[
\begin{align*}
\hat{Y}_1 & \equiv \hat{R}_1 \setminus \Sigma(\hat{R}_1) \subset \hat{R}_1 \\
2\hat{R}_1 & \equiv \left\{ q \in \hat{R}_1 | u_y^1(q) = 0, u_x^2(q) = 0 \right\} \subset \hat{R}_1 \\
3\hat{R}_1 & \equiv \left\{ q \in \hat{R}_1 | u_x^1(q) = 0, u_y^2(q) = 0 \right\} \subset \hat{R}_1.
\end{align*}
\]

One has \( 2\hat{R}_1 \cap 3\hat{R}_1 \neq \emptyset, \hat{Y}_1 \cap 2\hat{R}_1 \neq \emptyset, \hat{Y}_1 \cap 3\hat{R}_1 \neq \emptyset. \) Furthermore the set of
singular points \( \Sigma(2\hat{R}_1) \) (resp. \( \Sigma(3\hat{R}_1) \)) of \( 2\hat{R}_1 \) (resp. \( 3\hat{R}_1 \)) is contained in \( \Sigma(\hat{R}_1) \) and
contains \( \Sigma(2\hat{R}_1)_0 \equiv 2\hat{R}_1 \cap \Sigma(\hat{R}_1)_0 \equiv A^4 \) (resp. \( \Sigma(3\hat{R}_1)_0 \equiv 3\hat{R}_1 \cap \Sigma(\hat{R}_1)_0 \equiv A^4 \)).

We can write:
\[
\hat{R}_1 = \hat{Y}_1 \cup [\Sigma(\hat{R}_1) \cap 2\hat{R}_1] \times [\Sigma(\hat{R}_1) \cap 3\hat{R}_1] \subset \hat{J}D(E),
\]
where \( \hat{Y}_2 \equiv 2\hat{R}_1 \setminus \Sigma(2\hat{R}_1) \) (resp. \( \hat{Y}_3 \equiv 3\hat{R}_1 \setminus \Sigma(3\hat{R}_1) \)). Then we can see that
\( \hat{Y}_1 \) is a formally quantum integrable and completely quantum integrable quantum
PDE of first order. (For the theory of formal integrability of quantum PDE’s, see
Refs. [72, 76, 85, 86, 87].) In fact \( \hat{Y}_1 \) and its prolongations \( (\hat{Y}_1)_{+r} \subset \hat{J}D^{r+1}(E), \)
are subbundles of \( \hat{J}D^{r+1}(E) \to \hat{J}D^r(E), r \geq 0. \) One can also see that the canonical
maps \( \pi_{r+1,r} : (\hat{Y}_1)_{+r} \to (\hat{Y}_1)_{+r-1}, \) are surjective mappings. For example, for

\[38\] This is a direct consequence of Lemma 3.32 in [84]. It is useful to emphasize that the abelian
group \( G(A) \) of a quantum (hypercomplex) algebra \( A \), has no zero divisors, \( Zer(A) = \emptyset, \) hence
it is a division algebra. (Left or right divisors can never be units.) Furthermore, any \( b \in G(A), \)
with \( b \neq 1, \) cannot be idempotent, since an idempotent element must be a zero divisor: \( b^2 = b \Rightarrow b(b - 1) = 0. \)
$r = 1$, one has the following isomorphisms:

$$
\begin{align*}
\text{(25)} \quad & \dim_{B_1} \mathcal{J}D(E) = (4, 4) \\
& \hat{Y}_1 \cong A^4 \times \hat{A}^2 \Rightarrow \dim_{B_1} \hat{Y}_1 = (4, 2) \\
& \mathcal{J}D^2(E) \cong A^4 \times \hat{A}^4 \times (A)^8 \Rightarrow \dim_{B_2} \mathcal{J}D^2(E) = (4, 4, 8) \\
& \langle \hat{Y}_1 \rangle_{+1} \cong A^4 \times \hat{A}^2 \times (A)^4 \Rightarrow \dim_{B_2} \langle \hat{Y}_1 \rangle_{+1} = (4, 2, 4) \\
& \textrm{Hom}_Z(\bar{S}_q^2(T_pM);\nu T_qE) \cong (A)^8 \Rightarrow \dim_{B_2} \textrm{Hom}_Z(\bar{S}_q^2(T_pM);\nu T_qE) = (0, 0, 8) \\
& \langle (\hat{g}_1)_{+1} \rangle_{q \in \hat{Y}_1} \cong (A)^4 \Rightarrow \dim_{B_2} \langle (\hat{g}_1)_{+1} \rangle_{q \in \hat{Y}_1} = (0, 0, 4) \\
& \left[ \dim_{B_2} \langle \hat{Y}_1 \rangle_{+1} \right] = \left[ \dim_{B_2} \hat{Y}_1 \right] + \left[ \dim_{B_2} \langle (\hat{g}_1)_{+1} \rangle_{q \in \hat{Y}_1} \right].
\end{align*}
$$

Therefore, $\langle \hat{Y}_1 \rangle_{+1} \rightarrow \langle \hat{Y}_1 \rangle$, is surjective, and by iterating this process, we get that also the mappings $\langle \hat{Y}_1 \rangle_{+r} \rightarrow \langle \hat{Y}_1 \rangle_{+(r-1)}$, $r \geq 0$, are surjective. We put $\langle \hat{Y}_1 \rangle_{+(r-1)} \equiv E$. Thus $\hat{Y}_1$ is a quantum regular quantum PDE, and under the hypothesis that $A$ has a Noetherian centre, it follows that $\hat{Y}$ is quantum $\delta$—regular too. Then, from Theorem 3.10 and Theorem 3.11, it follows that $\hat{Y}_1$ is formally quantum integrable. Since it is quantum analytic, it is completely quantum integrable too.

**Definition 4.5.** We define quantum extended crystal hypercomplex singular super PDE, a quantum extended crystal hypercomplex singular super PDE $\hat{E}_k \subset J_{m|n}(W)$ that splits in irreducible components $\hat{A}_i$, i.e., $\hat{E}_k = \bigcup_{i} \hat{A}_i$, where each $\hat{A}_i$ is a quantum extended crystal hypercomplex singular super PDE. Similarly we define quantum extended 0-crystal hypercomplex singular PDE, (resp. quantum 0-crystal hypercomplex singular PDE), a quantum extended crystal hypercomplex singular PDE where each component $\hat{A}_i$ is a quantum extended 0-crystal hypercomplex PDE, (resp. quantum 0-crystal hypercomplex PDE).

**Definition 4.6.** (Algebraic singular solutions of quantum singular hypercomplex super PDE’s). Let $\hat{E}_k \subset J_{m|n}(W)$ be a quantum singular super PDE, that splits in irreducible components $\hat{A}_i$, i.e., $\hat{E}_k = \bigcup_{i} \hat{A}_i$. Then, we say that $\hat{E}_k$ admits an algebraic singular solution $V \subset \hat{E}_k$, if $V \cap \hat{A}_r \equiv V_r$ is a solution (in the usual sense) in $\hat{A}_r$ for at least two different components, say $\hat{A}_i, \hat{A}_j, i \neq j$, and such that one of the following conditions are satisfied: (a) $(ij)\hat{E}_k \equiv \hat{A}_i \cap \hat{A}_j \neq \emptyset$; (b) $(ij)\hat{E}_k \equiv \hat{A}_i \cup \hat{A}_j$ is a connected set, and $(ij)\hat{E}_k = \emptyset$. Then we say that the algebraic singular solution $V$ is in the case (a), weak, singular or smooth, if it is so with respect to the equation $(ij)\hat{E}_k$. In the case (b), we can distinguish the following situations: (weak solution): There is a discontinuity in $V$, passing from $V_i$ to $V_j$; (singular solution): there is not discontinuity in $V$, but the corresponding tangent spaces $TV_i$ and $TV_j$ do not belong to a same $n$-dimensional Cartan sub-distribution of $J_{m|n}(W)$, or alternatively $TV_i$ and $TV_j$ belong to a same $(m|n)$-dimensional Cartan sub-distribution of $J_{m|n}(W)$, but the kernel of the canonical projection $(\pi_{k,0})_* : T^*J_{m|n}(W) \rightarrow TW$, restricted to $V$ is larger than zero; (smooth solution): there is not discontinuity in $V$ and
the tangent spaces $TV_i$ and $TV_j$ belong to a same $(m|n)$-dimensional Cartan subdistribution of $J^k_{m|n}(W)$ that projects diffeomorphically on $W$ via the canonical projection $(\pi_{k,0})_* : T^k_{m|n}(W) \to TW$. Then we say that a solution passing through a critical zone bifurcate.\footnote{Note that the bifurcation does not necessarily imply that the tangent planes in the points of $V_{ij} \subset V$ to the components $V_i$ and $V_j$, should be different.}

**Definition 4.7.** (Integral bordism for quantum singular hypercomplex super PDE’s). Let $E_k \subset J^k_{m|n}(W)$ be a quantum super PDE on the fiber bundle $\pi : W \to M$, $\dim_B W = (m|n,r|s)$, $\dim_A M = m|n$, $B = A \times E$, $E$ a quantum hypercomplex superalgebra that is also a $Z$-module, with $Z = Z(A)$ the centre of $A$. Let $N_1, N_2 \subset E_k \subset J^k_{m|n}(W)$ be two $(m - |n - 1|)-$dimensional, (with respect to $A$), admissible closed integral quantum hypercomplex supermanifolds. We say that $N_1$ algebraic integral bords with $N_2$, if $N_1$ and $N_2$ belong to two different irreducible components, say $N_1 \subset A_i$, $N_2 \subset A_j$, $i \neq j$, such that there exists an algebraic singular solution $V \subset E_k$ with $\partial V = N_1 \varnothing N_2$.

In the integral bordism group $\Omega^E_k$ of a quantum singular hypercomplex super PDE $E_k \subset J^k_{m|n}(W)$, we call algebraic class a class $[N] \in \Omega^E_k$, (resp. $[N] \in \Omega^E_k$, resp. $[N] \in \Omega^E_k$, with $N \subset A_j$, such that there exists a closed $(m - |n - 1|)-$dimensional, (with respect to $A$), admissible integral quantum hypercomplex supermanifolds $X \subset A_i \subset E_k$, algebraic integral bording with $N$, i.e., there exists a smooth (resp. singular, resp. weak) algebraic singular solution $V \subset E_k$, with $\partial V = N \varnothing X$.

**Theorem 4.8** (Singal integral bordism group of quantum hypercomplex singular super PDE). Let $E_k \equiv \bigcup_i A_i \subset J^k_{m|n}(W)$ be a quantum singular super PDE. Then under suitable conditions, algebraic singular solutions integrability conditions, we can find (smooth) algebraic singular solutions bording assigned admissible closed smooth $(m - |n - 1|)-$dimensional, (with respect to $A$), integral quantum hypercomplex supermanifolds $N_0$ and $N_1$ contained in some component $A_i$ and $A_j$, $i \neq j$.

**Proof.** In fact, we have the following lemmas.

**Lemma 4.9.** Let $E_k \equiv \bigcup_i A_i \subset J^k_{m|n}(W)$ be a quantum singular super PDE with $E_k \equiv A_i \cap \hat{A}_j \neq \varnothing$. Let us assume that $\hat{A}_i \subset J^k_{m|n}(W)$, $\hat{A}_j \subset J^k_{m|n}(W)$ and $A_i \subset J^k_{m|n}(W)$ be formally integrable and completely integrable quantum hypercomplex super PDE’s with nontrivial symbols. Then, one has the following isomorphisms:

$$\begin{align*}
\Omega_{m-1|n-1,w} \cong & \quad \Omega_{m-1|n-1,w} \\
\cong & \quad \Omega_{m-1|n-1,s} \\
\cong & \quad \Omega_{m-1|n-1,k} \\
\cong & \quad \Omega_{m-1|n-1,s} \\
\cong & \quad \Omega_{m-1|n-1,w} \\
\cong & \quad \Omega_{m-1|n-1,k} \\
\cong & \quad \Omega_{m-1|n-1,s}.
\end{align*}$$

So we can find a weak or singular algebraic singular solution $V \subset E_k$ such that $\partial V = N_0 \varnothing N_1$, $N_0 \subset A_i$, $N_1 \subset A_j$, iff $N_1 \in [N_0]$.\footnote{Note that the bifurcation does not necessarily imply that the tangent planes in the points of $V_{ij} \subset V$ to the components $V_i$ and $V_j$, should be different.}
Proof. In fact, under the previous hypotheses one has that we can apply Theorem 2.1 in [75] to each component \( \hat{A}_i \), \( \hat{A}_j \) and \( (ij) \hat{E}_k \) to state that all their weak and singular integral bordism groups of dimension \((m-1)n-1\) are isomorphic to \( H_{m-1|n-1}(W;A) \). \( \square \)

Lemma 4.10. Let \( \hat{E}_k = \bigcup_i \hat{A}_i \) be a quantum 0-crystal hypercomplex singular PDE. Let \( (ij) \hat{E}_k \equiv \hat{A}_i \cup \hat{A}_j \) be connected, and \( (ij) \hat{E}_k \equiv \hat{A}_i \cap \hat{A}_j \neq \emptyset \). Then \( \Omega^{(ij)}_{m-1|n-1, s} = 0 \). \( \square \)

Proof. In fact, let \( Y \subset (ij) \hat{E}_k \) be an admissible closed \((m-1)n-1\)-dimensional closed integral quantum hypercomplex supermanifold, then there exists a smooth solution \( V_i \subset A_i \) such that \( \partial V_i = N_0 \partial Y \) and a solution \( V_j \subset A_j \) such that \( \partial V_j = Y \partial N_1 \). Then, \( V = V_i \cup_Y V_j \) is an algebraic singular solution of \( \hat{E}_k \). This solution is singular in general. \( \square \)

After above lemmas the proof of the theorem can be considered done besides the algebraic singular solutions integrability conditions. \( \square \)

Example 4.11 (Quantum sedenionic d’Alembert equation). Above examples about heat equations in the categories of quantum hypercomplex manifolds are linear equations, however the general theory works too also for non-linear equations. For example, we can consider the quantum sedenionic d’Alembert equation are reported in (27).

\[
\left\{ \pi : W \equiv S^3 \to S^2 \equiv M; \hspace{1cm} (x, y, u) \mapsto (x, y) \hspace{1cm} (d^*A)_{\bar{S}} \subset J\bar{D}^2(W) \subset J_d^2(W) : \hspace{0.5cm} F \equiv uu_{xy} - u_x u_y = 0 \right\}
\]

In this case, we can apply the same machinery, for the quantum smooth submanifold of \( \hat{X}_2 \subset J\bar{D}^2(W) \), where \( \hat{X}_2 \) is the open quantum hypercomplex submanifold of \( J\bar{D}^2(W) \), identified with the condition \( u \neq 0 \). In fat, there the jacobian

\[ \begin{pmatrix} 0 & 0 & u_{xy} & -u_y & -u_x & 0 & 0 \end{pmatrix} \]

has rank 1. Here \( (\xi^j) = (x, y; u, u_x, u_y, u_{xx}, u_{xy}) \) are quantum hypercomplex coordinates on \( J\bar{D}^2(W) \). Thus, in the case, in order to regularize \( (d^*A)_{\bar{S}} \), it is enough to consider \( (d^*A)_{\bar{S}} \cap \hat{X}_2 \), \( \hat{X}_2 \equiv u^{-1}(0) \), with \( u : J\bar{D}^2(W) \to A \). (Compare with singular quantum hypercomplex PDE’s considered in Example 4.4.)

5. Quantum hypercomplex exotic PDE’s

In this section we extend to PDE’s in the category \( \Omega_{\text{hyper}} \), a previous definition, and some results, about ”exotic PDE’s” given in the category of commutative manifolds [85, 86, 87, 88]. This allows us the opportunity to discuss about a classification of smooth solutions of PDE’s in the category \( \Omega_{\text{hyper}} \) and to prove a smooth generalized version of the quantum generalized Poincaré conjecture, (previously proved in [80] in the category \( \Omega_S \) of quantum supermanifolds). There it is proved that a closed quantum supermanifold \( M \), of dimension \((m|n)\), with respect to a quantum superalgebra \( A \), homotopy equivalent to the quantum \((m|n)\)-dimensional supersphere, \( S^m|n \), is homeomorphic (but not necessarily diffeomorphic) to \( S^m|n \), if \( M \) is classic regular, with classic limit a \( m \)-dimensional manifold \( M_C \), identified by means of a

\( ^{40} \)But, in general, one has \( \Omega^{(ij)}_{m-1|n-1} \neq 0 \).
fiber structure $\bar{\pi}_C : M \to M_C$, such that the homotopy equivalence is realized by means of a couple of mappings $(f, f_C)$ such that the diagram (28) is commutative.

$$\begin{array}{ccc}
M & \xrightarrow{f} & \hat{S}^m[n] \\
\downarrow \bar{\pi}_C & & \downarrow \pi_C \\
M_C & \xrightarrow{f_C} & S^m \\
\end{array}$$

Thus in this section we will assume quantum manifolds with classic regular quantum manifold structures $\bar{\pi}_C : M \to M_C$, $\dim_A M = \dim_B M_C = n$, that are compact, closed and homotopy equivalent to the quantum $n$-sphere $\hat{S}^n$, (with respect to the same quantum (hypercomplex) algebra $A$), are homeomorphic to $\hat{S}^n$ too.

**Definition 5.1** (Quantum homotopy $n$-sphere). We call quantum homotopy $n$-sphere (with respect to a quantum algebra $A$) a smooth, compact, closed $n$-dimensional quantum manifold $\hat{\Sigma}^n$, that is homeomorphic to $\hat{S}^n$, with classic regular structure $\pi_C : M \to M_C$, where $M, M_C$ are homotopy $n$-spheres, and such that the homotopy equivalence between $M$ and $\hat{S}^n$ is realized by a commutative diagram (28).

**Remark 5.2.** Let $\hat{\Sigma}^n_1$ and $\hat{\Sigma}^n_2$ be two quantum diffeomorphic, quantum homotopy $n$-spheres: $\hat{\Sigma}^n_1 \cong \hat{\Sigma}^n_2$. Then the corresponding classic limits $\hat{\Sigma}^n_{1,C}$ and $\hat{\Sigma}^n_{2,C}$ are diffeomorphic too: $\hat{\Sigma}^n_{1,C} \cong \hat{\Sigma}^n_{2,C}$. This remark is the natural consequence of the fact that quantum diffeomorphisms here considered respect the fiber bundle structures of quantum homotopy $n$-spheres with respect their classic limits: $\pi_C : \hat{\Sigma}^n \to \Sigma^n$. Therefore quantum diffeomorphisms between quantum homotopy $n$-spheres are characterized by a couple $(f, f_C) : (\hat{\Sigma}^n_1, \hat{\Sigma}^n_{1,C}) \to (\hat{\Sigma}^n_2, \hat{\Sigma}^n_{2,C})$ of mappings related by the commutative diagram in (29).

$$\begin{array}{ccc}
\hat{\Sigma}^n_1 & \xrightarrow{f} & \hat{\Sigma}^n_2 \\
\downarrow \pi_{1,C} & & \downarrow \pi_{2,C} \\
\hat{\Sigma}^n_{1,C} & \xrightarrow{f_C} & \hat{\Sigma}^n_{2,C} \\
\end{array}$$

There $f$ is a quantum diffeomorphism between quantum manifolds and $f_C$ is a diffeomorphism between manifolds. Note that such diffeomorphisms of quantum homotopy $n$-spheres allow to recognize that $\hat{\Sigma}^n_1$ has also $\hat{\Sigma}^n_{2,C}$ as classic limit, other than $\hat{\Sigma}^n_{1,C}$. (See commutative diagram in (30).)

$$\begin{array}{ccc}
\hat{\Sigma}^n_1 & \xrightarrow{f} & \hat{\Sigma}^n_2 \\
\downarrow \pi_{1,C} & & \downarrow \pi_{2,C} \\
\hat{\Sigma}^n_{1,C} & \xrightarrow{f_C} & \hat{\Sigma}^n_{2,C} \\
\end{array}$$

This clarifies that the classic limit of a quantum homotopy $n$-sphere is unique up to diffeomorphisms.
Let us also emphasize that (co)homology properties of quantum homotopy $n$-spheres are related to the ones of $n$-spheres, since we here consider classic regular objects only.

**Lemma 5.3.** In (31) are reported the cohomology spaces for quantum homotopy $n$-spheres.

\[ H^p(\Sigma^n; \mathbb{Z}) \cong H^p(\hat{S}^n; \mathbb{Z}) \cong H^p(S^n; \mathbb{Z}) = \begin{cases} 0 & p \neq 0, n \\ \mathbb{Z} & p = 0, n. \end{cases} \]

**Proof.** Let us first calculate the homology groups in integer coefficients $\mathbb{Z}$, of quantum $n$-spheres. In (32) are reported the homology spaces for quantum $n \neq 0$-spheres.

\[ H_p(\hat{S}^n; \mathbb{Z}) \cong H_p(S^n; \mathbb{Z}) = \begin{cases} 0 & p \neq 0, n \\ \mathbb{Z} & p = 0, n. \end{cases} \]

Furthermore, for $n = 0$ we get

\[ H_p(\hat{S}^0; \mathbb{Z}) \cong H_p(S^0; \mathbb{Z}) = \begin{cases} 0 & p \neq 0 \\ \mathbb{Z} \oplus \mathbb{Z} & p = 0. \end{cases} \]

Above formulas can be obtained by the reduced Mayer-Vietoris sequence applied to the triad $(\hat{S}^n, \hat{D}^n_+, \hat{D}^n_0)$ since we can write $\hat{S}^n = \hat{D}^n_+ \cup \hat{D}^n_-$, where $\hat{D}^n_+$ and $\hat{D}^n_-$ are respectively the north quantum $n$-disk and south quantum $n$-disk that cover $\hat{S}^n$. Taking into account that $\hat{D}^n_+ \cap \hat{D}^n_- = \hat{S}^{n-1}$, we get the long exact sequence (33).

\[
\begin{array}{ccccccccc}
\cdots & \tilde{H}_p(\hat{S}^{n-1}; \mathbb{Z}) & \longrightarrow & \tilde{H}_p(\hat{D}^n_+; \mathbb{Z}) \oplus \tilde{H}_p(\hat{D}^n_-; \mathbb{Z}) & \longrightarrow & \tilde{H}_p(\hat{S}^n; \mathbb{Z}) & \longrightarrow & \\
& \tilde{H}_{p-1}(\hat{S}^n; \mathbb{Z}) & \longrightarrow & \tilde{H}_{p-1}((\hat{D}^n_+)_D; \mathbb{Z}) \oplus \tilde{H}_{p-1}((\hat{D}^n_-)_D; \mathbb{Z}) & \longrightarrow & \tilde{H}_{p-1}(\hat{S}^{n-1}; \mathbb{Z}) & & \\
& \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & & \\
& \tilde{H}_0(\hat{S}^{n-1}; \mathbb{Z}) & \longrightarrow & \tilde{H}_0(\hat{D}^n_+; \mathbb{Z}) \oplus \tilde{H}_0(\hat{D}^n_-; \mathbb{Z}) & \longrightarrow & \tilde{H}_0(\hat{S}^n; \mathbb{Z}) & \longrightarrow & 0.
\end{array}
\]

Taking into account that $\tilde{H}_0(\hat{D}^n_+; \mathbb{Z}) = 0$, we get $\tilde{H}_p(\hat{S}^n; \mathbb{Z}) \cong \tilde{H}_{p-1}(\hat{S}^{n-1}; \mathbb{Z}) = 0$. Therefore, we get

\[ \tilde{H}_p(\hat{S}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = n \\ 0 & \text{if } p \neq n \end{cases} \Rightarrow \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z} & \text{if } p = 0, n \\ 0 & \text{if } p \neq 0, n. \end{cases} \]

Therefore we get formulas (32). To conclude the proof we shall consider that $H^p(\hat{S}^n; \mathbb{Z}) \cong Hom_\mathbb{Z}(H_p(S^n; \mathbb{Z}); \mathbb{Z})$. Furthermore, quantum homotopy $n$-spheres have same (co)homology of quantum spheres since are homotopy equivalent to these last ones.

□
Lemma 5.4. The (quantum) Euler characteristic numbers for quantum homotopy \( n \)-spheres are reported in (35).

\[
\hat{\chi}(\Sigma^n) = \hat{\chi}(\hat{S}^n) = \chi(S^n) = (-1)^0 \beta_0 + (-1)^n \beta_n = 1 + (-1)^n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}
\]

Proof. We have considered that \( \hat{S}^n \) admits the following quantum-cell decomposition: \( \hat{S}^n = \hat{e}^n \cup \hat{e}^0 \), where \( \hat{e}^n = \hat{D}^n \) is a \( n \)-dimensional quantum cell, with respect to the quantum algebra \( A \), and \( \hat{e}^0 = \hat{D}^0 \) is the 0-dimensional quantum cell with respect to \( A \). Therefore we can consider the quantum homological Euler characteristic \( \hat{\chi}(\hat{S}^n) \) of \( \hat{S}^n \), given by formulas (36).

\[
\begin{align*}
\hat{\chi}(\hat{S}^n) &= (-1)^0 \dim_A H_0(\hat{S}^n; A) + (-1)^n \dim_A H_n(\hat{S}^n; A) \\
&= (-1)^0 \dim_A A + (-1)^n \dim_A A \\
&= 1 + (-1)^n \\
&= \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}
\end{align*}
\]

So the homological quantum Euler characteristic of the quantum \( n \)-sphere is the same of the homological Euler characteristic of the usual \( n \)-sphere. Furthermore, since quantum homotopy \( n \)-spheres are homotopy equivalent to quantum \( n \)-spheres, it follows that the quantum Euler characteristic of a quantum homotopy \( n \)-sphere is equal to the one of \( \hat{S}^n \). \( \square \)

Theorem 5.5. Let \( \hat{\Theta}_n \) be the set of equivalence classes of quantum diffeomorphic quantum homotopy \( n \)-spheres over a quantum (hypercomplex) algebra \( A \) (and with Noetherian centre \( Z(A) \)). In \( \hat{\Theta}_n \) it is defined an additive commutative and associative composition map such that \( [\hat{S}^n] \) is the zero of the composition. Then one has the exact commutative diagram reported in (38).

---

41In general, quantum characteristic numbers for quantum manifolds with quantum algebras \( A \), are \( A \)-valued characteristic forms. (For details see [63, 66, 68, 77, 78, 80, 81].) However, taking into account the canonical ring homomorphism \( \epsilon : \mathbb{R} \to Z(A) \subset A \), we can consider also above characteristic numbers for quantum homotopy \( n \)-spheres as belonging to the corresponding quantum algebra \( A \).

42Quantum diffeomorphisms are meant in the sense specified in Remark 5.2.

43For the definition of the groups \( \Theta_n \), see [86]. Let us emphasize here that \( \Theta_n \) is a finite abelian group (Kervaire-Milnor). These are particular cases of finitely generated abelian groups \( G \) that admit a finite direct sum decomposition like in (37).

\[
G \cong \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r} \oplus \mathbb{Z}^r.
\]

The partial sum \( T \equiv \oplus_{1 \leq i \leq r} \mathbb{Z}_{p_i} \) is called torsion subgroup of \( G \). It is a finite group and consists of all elements of \( G \) of finite order. The quotient \( G/T \cong \mathbb{Z}^r \) is called free part of \( G \). The number \( r \) of summands \( \mathbb{Z} \) in \( G/T \) is called the rank (or Betti number) of \( G \). It does not depend on the particular direct sum decomposition (37). In fact, rank(\( G \)) is the maximal number of linearly independent elements in \( G \). The numbers \( p_j \) (torsion coefficients in \( G \)) that occur in (37) are not unique. However, they can be chosen as powers of prime numbers \( p_j = q_j^{\rho_j} \), \( q_j \) prime, \( \rho_j > 0 \), and then they are unique (independent of the decomposition) up to permutation coefficients. Two finitely generated abelian groups are isomorphic iif they have the same rank and the same system of torsion coefficients. \( \Theta_n \) as a finite abelian group has rank \( r = 0 \). Let us emphasize that the fundamental theorem of finite abelian group states that any such a group \( H \) can be written in a direct product of cyclic groups: \( H = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r} \), such that \( (p_1, \ldots, p_r) \) are powers of primes, or \( p_i \) divides \( p_{i+1} \). For example we get the isomorphism: \( \mathbb{Z}_{15} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5 \), but
where $\Theta_n$ is the set of equivalence classes for diffeomorphic homotopy $n$-spheres and $j_C$ is the canonical mapping $j_C : [\Sigma^n] \to [\Sigma_C^n]$. One has the canonical isomorphisms:

$$(39) \quad Z \bigotimes \hat{\Theta}_n \cong Z \hat{\Theta}_n, \text{ as right } \hat{\Theta}_n\text{-modules.}$$

**Proof.** After above Remark 5.2 we can state that the mapping $j_C$ is surjective. In other words we can write

$$\hat{\Theta}_n = \bigcup_{[\Sigma_2^n] \in \Theta_n} (\hat{\Theta}_n)_{[\Sigma_2^n]}.$$ 

The fiber $(\hat{\Theta}_n)_{[\Sigma_2^n]}$ is given by all classes $[\Sigma^n]$ such their classic limits are diffeomorphic, hence belong to the same class in $\Theta_n$. Furthermore one has $\ker(j_C) = \hat{\Theta}_n \subset \hat{\Theta}_n$. Therefore, we can state that $\hat{\Theta}_n$ is an extension of $\Theta_n$ by $\hat{\Theta}_n$. Such extensions are classified by $H^2(\Theta_n; \hat{\Theta}_n)$.

### Table 7. Homology of finite cyclic group $\mathbb{Z}_i$ of order $i$.

| $r$   | $H_r(\mathbb{Z}_i; \mathbb{Z})$ |
|-------|----------------------------------|
| 0     | $\mathbb{Z}$                     |
| $r$ odd | $\mathbb{Z}_r$                  |
| $r > 0$ even | 0                               |

The composition map in $\hat{\Theta}_n$ is defined by *quantum fibered connected sum*, i.e., a connected sum on quantum manifolds that respects the connected sum on their corresponding classic limits. More precisely let $M \to M_C$ and $N \to N_C$ be connected $n$-dimensional regular quantum manifolds. We define quantum fibered connected sum of $M$ and $N$ the classic regular $n$-dimensional quantum manifold $M \sharp N \to M_C \sharp N_C$, where

$$\begin{cases} 
M \sharp N &= (M \setminus \hat{D}^n) \cup (\hat{S}^{n-1} \times \hat{D}^1) \cup (N \setminus \hat{D}^n) \\
M_C \sharp N_C &= (M_C \setminus D^n) \cup (S^{n-1} \times D^1) \cup (N_C \setminus D^n).
\end{cases}$$

$\mathbb{Z}_8 \not\cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \not\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. (In other words $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$ iff $p$ and $q$ are coprime.) Let us emphasize that after above representation of finite abelian groups, we can for abuse of notation denote $\dim_2 H = \text{order}(H)$. (A notation for the order of a group $H$ is also $|H|$.) This is justified since the integral group ring $\mathbb{Q}H$ of $H$ is just a vector space of dimension equal to $|H|$. (See Tab. 4 in [86].)

In Tab. 7 are reported useful formulas to explicitly calculate these groups.
Then the additive composition law is $+ : \hat{\Theta}^n \times \hat{\Theta}^n \to \hat{\Theta}^n$, $[M] + [N] = [M \times N]$. $[\hat{S}^n]$ is the zero of this addition. In fact, since $\hat{S}^n \setminus \hat{D}^n \cup_{\hat{S}^{n-1}} (\hat{S}^{n-1} \times \hat{D}^1) \cong \hat{D}^n$, we get

$$\hat{M} \hat{n} \cong M \setminus \hat{D}^n \cup_{\hat{S}^{n-1}} (\hat{S}^{n-1} \times \hat{D}^1) \cong M.$$ 

Analogous calculus for $M_C$ completes the proof.

In the following remark we will consider some examples and further results to better understand some relations between quantum homotopy spheres and their classic limits.

**Example 5.6 (Quantum homotopy 7-sphere).** Let us calculate the extension classes

$$(41) \quad 0 \to \hat{\Theta}_7 \xleftarrow{\text{Ext}} \hat{\Theta}_7 \xrightarrow{\text{Hom}} \Theta_7 \to 0$$

These are given by $H^2(\Theta_7; \hat{\Theta}_7) \cong H^2(\hat{Z}_28; \hat{\Theta}_7)$. We get

$$(42) \quad H^2(\hat{Z}_28; \hat{\Theta}_7) = \text{Hom}_Z(H_2(\hat{Z}_28; \hat{Z}); \hat{\Theta}_7) = \text{Hom}_Z(0; \hat{\Theta}_7) \bigoplus \text{Ext}_Z(H_1(\hat{Z}_28; \hat{Z}); \hat{\Theta}_7).$$

We shall prove that $\text{Ext}_Z(H_1(\hat{Z}_28; \hat{Z}); \hat{\Theta}_7) = \hat{\Theta}_7/28 \cdot \hat{\Theta}_7$. Let us look in some detail to this $\hat{Z}$-module. By using the projective resolution of $\hat{Z}_28$ given in (43),

$$(43) \quad 0 \to \hat{Z} \xrightarrow{\mu = 28} \hat{Z} \xrightarrow{\epsilon} \hat{Z}_28 \to 0$$

we get the exact sequence (44).

$$(44) \quad 0 \to \text{Hom}_Z(\hat{Z}_28; \hat{\Theta}_7) \xrightarrow{\epsilon_*} \text{Hom}_Z(\hat{Z}; \hat{\Theta}_7) \xrightarrow{\mu_*} \text{Hom}_Z(\hat{Z}; \hat{\Theta}_7) \xrightarrow{\text{Hom}} \hat{\Theta}_7$$

Therefore we get

$$\text{Ext}_Z(\hat{Z}_28; \hat{\Theta}_7) = \hat{\Theta}_7/\text{im} (\mu_*).$$

In order to see what is $\text{im} (\mu_*)$ let us consider that $\mu_*$ is defined by the commutative diagram in (45).

$$(45) \quad \begin{array}{c}
0 \\
\text{Hom}_Z(\hat{Z}_28; \hat{\Theta}_7) \xrightarrow{\epsilon_*} \text{Hom}_Z(\hat{Z}; \hat{\Theta}_7) \xrightarrow{\mu_*} \text{Hom}_Z(\hat{Z}; \hat{\Theta}_7) \xrightarrow{\text{Hom}} \hat{\Theta}_7 \\
\text{Hom}_Z(\hat{Z}_28; \hat{\Theta}_7) \xrightarrow{\epsilon_*} \hat{\Theta}_7 \xrightarrow{\mu_*} \hat{\Theta}_7
\end{array}$$

Since $\alpha$ is identified by means of the image of $1 \in \hat{Z}$, i.e., $\alpha(1) \in \hat{\Theta}_7$, similarly also $\mu_*(\alpha)$ is determined by image of $1 \in \hat{Z}$, i.e., $\mu_*(\alpha)(1) = \alpha(\mu(1)) = \alpha(28) = 28 \cdot \alpha(1) \in \hat{\Theta}_7$. Therefore $\text{im} (\mu_*) = 28 \cdot \hat{\Theta}_7$. Thus we get $\text{Ext}_Z(\hat{Z}_28; \hat{\Theta}_7) = \hat{\Theta}_7/28 \cdot \hat{\Theta}_7$.

The particular structure of this module, depends on the particular quantum algebra considered. For example, take $A = \mathbb{C}$. Since $S^7 \to S^7$, is just the fiber bundle $S^{14} \to S^7$, we can easily see that $\hat{\Theta}_7 = \Theta_14 = \hat{Z}_2$. In this case the extensions (41)

\[45\] We have used the fact that $H_2(\hat{Z}_28; \hat{Z}) = 0$.\]
are classified by $\text{Ext}_2(\mathbb{Z}_{2^8}, \mathbb{Z}_2) = \mathbb{Z}_2/(2^8\cdot \mathbb{Z}_2) = \mathbb{Z}_2/(14\cdot(2\cdot \mathbb{Z}_2)) = \mathbb{Z}_2/(14\cdot0) = \mathbb{Z}_2$.\textsuperscript{46} Therefore we get that all the extensions in (41) are in correspondence one-to-one with $\Theta_{14}$. Let us explicate in some more details this result. Since $S^7 \cong S^{14}$ and $\Theta_{14} = \mathbb{Z}_2$, it follows that other $S^{14}$ there is another class of homotopy 14-spheres, non-diffeomorphic to $S^{14}$. Let us denote one of these spheres by $\Sigma^{14}$. Let us denote by $f : \Sigma^{14} \to S^{14}$ the homotopy map existing between $S^{14}$ and $\Sigma^{14}$. Then one has also a continuous fibration $\pi = \pi_C \circ f : \Sigma^{14} \to S^{14} \to S^7$. Since $C^s(\Sigma^{14}, S^7)$ is dense in $C^s(\Sigma^{14}, S^7)$, for $0 \leq s < r$, we can approximate $\pi$ by a smooth mapping, yet denoted $\pi$. Therefore we can consider the identification $\hat{\Theta}_7 = \mathbb{Z}_2$. Furthermore, since there exists homeomorphisms $\phi : S^7 \to \Sigma^7$, where $\Sigma^7$ are homotopy 7-spheres belonging to different classes in $\Theta_7$, we get also continuous fibered structures $\Sigma^{14} \to \Sigma^7$, that again can be approximated by smooth mappings. In this way we can have the identification $\hat{\Theta}_7/\hat{\Theta}_7 \cong \Theta_7 \cong \mathbb{Z}_{2^8}$, and all the possible diffeomorphic classes of quantum homotopy 7-spheres, with respect to the quantum algebra $A = C$ are $\Theta_7 = \mathbb{Z}_2 \times_\alpha \mathbb{Z}_{2^8}$, where $\alpha$ are homomorphisms $\alpha : \mathbb{Z}_{2^8} \to \text{Aut}(\mathbb{Z}_2)$. On the other hand, one has $\text{Aut}(\mathbb{Z}_2) = \mathbb{Z}_2^* = \{1\}$, where $\mathbb{Z}_2^*$ is the group of units of $\mathbb{Z}_2$, that coincide with all numbers $0 \leq p < 2$ coprime to 2. Therefore the quantum homotopy 7-spheres diffeomorphic classes are $2 \times 28 = 56$, i.e., the order of $\hat{\Theta}_7 = \mathbb{Z}_2 \oplus \mathbb{Z}_{2^8}$.

**Example 5.7 (Quantum homotopy n-spheres for the limit case $A = \mathbb{R}$).** In the limit case where the quantum algebra is $A = \mathbb{R}$, then the classic limit of a quantum homotopy $n$-sphere $\hat{\Sigma}^n$ is just $\Sigma^n = \hat{\Sigma}^n$, hence $\pi_C = \text{id}_{\Sigma^n}$. Furthermore $\Theta_n = \Theta_n$ and $\hat{\Theta}_n = 0 = [S^n] \in \Theta_n$. In particular if $n = \{1, 2, 3, 4, 5, 6\}$, we get $\Theta_n = \Theta_n = \hat{\Theta}_n = 0$. (For the smooth case $n = 4$ see [88].)

**Theorem 5.8 (Homotopy groups of quantum n-sphere).** Quantum homotopy $n$-spheres cannot have, in general, the same homotopy groups of $n$-spheres:\textsuperscript{47}

\begin{equation}
\pi_k(\hat{\Sigma}^n) \cong \pi_k(\hat{S}^n) \neq \pi_k(S^n).
\end{equation}

Furthermore, $S^n$ can be identified with a contractible subspace, yet denoted $\hat{S}^n$, of $\hat{\Sigma}^n$. There exists a mapping $\hat{S}^n \to S^n$, but this is not a retraction, and the inclusion $S^n \to \hat{S}^n$, cannot be a homotopy equivalence.\textsuperscript{48}

**Proof.** Since must necessarily be $\pi_k(\hat{\Sigma}^n) \cong \pi_k(\hat{S}^n)$, $k \geq 0$, it is enough prove theorem for $\hat{S}^n$. We shall first recall some useful definitions and results of Algebraic Topology, here codified as lemmas.

**Definition 5.9.** A pair $(X, A)$ has the homotopy extension property if a homotopy $f_t : A \to Y$, $t \in I$, can be extended to homotopy $f_t : X \to Y$ such that $f_0 : X \to Y$ is a given map.

**Lemma 5.10.** If $(X, A)$ is a CW pair, then it has the homotopy extension property.

**Lemma 5.11.** If the pair $(X, A)$ satisfies the homotopy extension property and $A$ is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.

\textsuperscript{46}$\mathbb{Z}_2$ is a field of characteristic 2, hence $2 \cdot \mathbb{Z}_2 = 0$.

\textsuperscript{47}In other words, quantum homotopy n-spheres are not homotopy equivalent to the n-sphere.

\textsuperscript{48}For any convenience, in Tab. 8 are reported some homotopy groups for $S^n$. 

Table 8. Homotopy groups of $n$-sphere.

| $k$ | $\pi_k(S^n)$ |
|-----|--------------|
| $k < n$ | 0 |
| $n$ | $\mathbb{Z}$ |

Examples for $k > n$.

$\pi_k(S^0) = 0$, $k \geq 0$.

$\pi_k(S^1) = 0$, $k > 1$.

$\pi_3(S^2) = \mathbb{Z}$, $\pi_4(S^2) = \pi_5(S^3) = \mathbb{Z}_2$, $\pi_6(S^2) = \mathbb{Z}_{12}$, $\pi_7(S^2) = \mathbb{Z}_2$.

$\pi_4(S^3) = \pi_5(S^3) = \mathbb{Z}_2$, $\pi_6(S^3) = \mathbb{Z}_{12}$, $\pi_7(S^3) = \mathbb{Z}_2$.

$\pi_5(S^4) = \pi_6(S^4) = \mathbb{Z}_2$, $\pi_7(S^4) = \mathbb{Z} \times \mathbb{Z}_{12}$.

$\pi_6(S^5) = \pi_7(S^5) = \mathbb{Z}_2$, $\pi_8(S^5) = \mathbb{Z}_{24}$.

$\pi_7(S^6) = \pi_8(S^6) = \mathbb{Z}_2$.

$\pi_8(S^7) = \mathbb{Z}_2$.

Let us consider that we can represent $S^n$ into $\hat{S}^n$ by a continuous mapping $s : S^n \to \hat{S}^n$, defined by means of the commutative diagram in (47).

$$
\begin{array}{ccc}
\hat{S}^n & \xrightarrow{\pi_C} & A^n \cup \{\infty\} \\
\downarrow{s|} & & \downarrow{s|_{\equiv (\epsilon^n, id_{\infty})}} \\
S^n & \xrightarrow{\epsilon^n : \mathbb{R}^n \rightarrow A^n} & \mathbb{R}^n \cup \{\infty\}
\end{array}
$$

where $\epsilon^n : \mathbb{R}^n \rightarrow A^n$ is induced by the canonical ring homomorphism $\epsilon : \mathbb{R} \rightarrow A$. $s$ is a section of $\pi : \pi \circ s = id_{S^n}$. Let us yet denote by $S^n$ the image of $s$. So we can consider the canonical couple $(\hat{S}^n, S^n)$ as a CW pair, hence it has the homotopy extension property. $S^n$ is not a contractible subcomplex of $\hat{S}^n$, so in general the quotient map $\hat{q} : \hat{S}^n \to \hat{S}^n / S^n$ is not a homotopy equivalence. We have the following lemma.

**Lemma 5.12.** The couple $(S^n, \infty)$ can be deformed into $(\hat{S}^n, \infty)$ to the base point $\{\infty\}$.

**Proof.** In fact, let $p \in \hat{S}^n \setminus S^n$. Then the inclusion $i : S^n \to \hat{S}^n$ is nullhomotopic since $\hat{S}^n \setminus \{\infty\} \approx A^n$ (homeomorphism). \hfill \Box

Since $S^n$ is contractible into $\hat{S}^n$, to the point $\infty \in \hat{S}^n$, the quotient map $\hat{q} : \hat{S}^n \to \hat{S}^n / S^n$ can be deformed into quotient mapping $\hat{q}_I$ over deformed quotient spaces $X_t \equiv \hat{S}^n / S^n_t$, with $S^n_t \equiv f_t(S^n) \subset \hat{S}^n$, for some homotopy $f : I \times S^n \to \hat{S}^n$, such that $X_0 = \hat{S}^n / S^n$, $X_1 = \hat{S}^n$ and $\hat{q}_t = id_{\hat{S}^n}$. (See diagram (48).)

$$
\begin{array}{ccc}
\hat{S}^n & \xrightarrow{\hat{q}_0 = \hat{q}} & \hat{S}^n / S^n \equiv X_0 \\
\downarrow{\hat{q}_I} & & \downarrow{\hat{q}_I} \\
\hat{S}^n / S^n_t \equiv X_t & & \hat{S}^n / \{\infty\} \equiv \hat{S}^n \equiv X_1
\end{array}
$$
Let us emphasize that we have a natural continuous mapping \( i \) commutative diagram (47). The inclusion \( n \) the surjection between the quantum A map Lemma 5.15. equivalence.

(49) \[ 0 \rightarrow \pi_k(S^n, \infty) \xrightarrow{i_*} \pi_k(\hat{S}^n, \infty) \rightarrow \pi_k(\hat{S}^n, S^n, \infty) \rightarrow \infty \]

hence we should have the splitting given in (50). (For details on relations between homotopy groups and retractions see, e.g. [68].)

(50) \[ \pi_k(\hat{S}^n, \infty) \cong \text{im } (i_*) \bigoplus \ker(r_*) \cong \pi_k(S^n, \infty) \bigoplus \ker(r_*) \]

But this cannot work. In fact, in the case \( A = C \), we should have the commutative diagram (51) with exact horizontal lines.

(51) \[ 0 \rightarrow \pi_1(S^1, \infty) \xrightarrow{i_*} \pi_1(\hat{S}^1, \infty) \rightarrow \pi_1(\hat{S}^1, S^1, \infty) \rightarrow \infty \]

This should imply that \( \pi_1(S^1, \infty) = 0 \), instead that \( \mathbb{Z} \), hence the bottom horizontal line in (51) cannot be an exact sequence, hence \( \pi_C : \hat{S}^1 \cong S^1 \rightarrow S^1 \) cannot be a retraction !

**Corollary 5.16.** Quantum homotopy spheres cannot be homotopy equivalent to \( S^n \), except in the case that the quantum algebra \( A \) reduces to \( \mathbb{R} \).

Quantum homotopy groups for quantum supermanifolds are introduced in [80]. In Tab. 9 are resumed their definitions and properties.

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49 Rally \( S^n \) is contractible in \( \hat{S}^n \), but is not a contractible sub-complex of \( \hat{S}^n \). This clarifies the meaning of Lemma 5.11. For example, in the case \( A = C \), one has that \( S^1/S^1 \) is not homotopy equivalent to \( S^2 \). In fact \( \pi_2(S^1) = Z \) and \( \pi_2(S^2/S^1) \cong \pi_2(S^2 \vee S^2) \cong H_2(S^2 \vee S^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \).

50 It is enough to consider the counterexample when \( A = C \) and \( \hat{S}^1 = C \cup \{\infty\} = \mathbb{R}^2 \cup \{\infty\} = S^2 \). Then \( S^1 \) cannot be homotopy equivalent to \( \hat{S}^1 = S^2 \), since \( \pi_1(S^1) = \mathbb{Z} \) and \( \pi_1(S^2) = 0 \).
Lemma 5.21

As a by-product we get the isomorphism \( \hat{\pi}_n(\mathcal{D}^n) \rightarrow \hat{\pi}_n(\mathcal{D}^n) \) for \( X \) connected, by substituting cells with quantum cells. For example we can prove for examples that \( \hat{\pi}_n(\mathcal{D}^n) \rightarrow \hat{\pi}_n(\mathcal{D}^n) \) is an isomorphism for \( k < m \) and \( n \geq 2 \).

\[ p : (X, x_0) \rightarrow (X, x_0) \text{ covering in } \Omega_{\text{hyper}} : p_* : \hat{\pi}_n(X, x_0) \cong \hat{\pi}_n(X, x_0), n \geq 2. \]

(1) Whenever \( X \) has a contractible universal cover \( \hat{\pi}_n(X, x_0) = 0, n \geq 2. \)

(2) If \( A \rightarrow S^1 \), \( \hat{\pi}_n(S^1) = 0, n \geq 2. \)

(3) For \( k = 0, n \geq 2. \)

\[ A^n \rightarrow \hat{T}^n = \overline{S^1 \times S^1} \hat{\pi}_k(T^n) = 0, k \geq 1. \]

Table 9. Quantum homotopy groups: definitions and some properties.

| Definition | Sum-law |
|-----------|---------|
| \( \hat{\pi}_n(X, x_0) = ([S^n, s_0], (X, x_0)] \) | \( f + g : \hat{\pi}_n(\mathcal{D}^n) \rightarrow \hat{\pi}_n(\mathcal{D}^n) \) |
| \( \hat{\pi}_n(X, A, x_0) = ([\hat{D}^n, s_0], (X, A, x_0)] \) | \( f + g : \hat{\pi}_n(\mathcal{D}^n) \rightarrow \hat{\pi}_n(\mathcal{D}^n) \) |

Some Properties.

If \( X \) is path connected one can simply write \( \hat{\pi}_n(X), x_0 \) since it does not depend on \( x_0 \).

If \( A \) is path connected one can simply write \( \hat{\pi}_n(A, x_0), \) since it does not depend on \( x_0 \).

\( \hat{\pi}_n(A, x_0) = \hat{\pi}_n(X, x_0, x_0) \) is abelian for \( n \geq 2. \)

\( \hat{\pi}_n(\Pi_n(X_0)) = \Pi_n(\hat{\pi}_n(X_0)) \) for \( (X_0, x_0) \) path-connected.

\[ p : (X, x_0) \rightarrow (X, x_0) \text{ covering in } \Omega_{\text{hyper}} : p_* : \hat{\pi}_n(X, x_0) \cong \hat{\pi}_n(X, x_0), n \geq 2. \]

(1) Whenever \( X \) has a contractible universal cover \( \hat{\pi}_n(X, x_0) = 0, n \geq 2. \)

(2) If \( A \rightarrow S^1 \), \( \hat{\pi}_n(S^1) = 0, n \geq 2. \)

(3) For \( k = 0, n \geq 2. \)

\[ A^n \rightarrow \hat{T}^n = \overline{S^1 \times S^1} \hat{\pi}_k(T^n) = 0, k \geq 1. \]

Theorem 5.17 (Quantum homotopy groups of quantum \( n \)-sphere). Quantum homotopy \( n \)-spheres have quantum homotopy groups isomorphic to homotopy groups of \( n \)-spheres:

\[ \hat{\pi}_k(S^n) = \hat{\pi}_k(S^n) \cong \pi_k(S^n). \]

Proof. In fact, we can prove for examples that \( \hat{\pi}_k(S^n) \cong 0, \) for \( k < n, \) and \( \hat{\pi}_n(S^n) \cong \mathbb{Z}. \) For this it is enough to reproduce the analogous proofs for the commutative spheres, by substituting cells with quantum cells. For example we can have the following quantum versions of analogous propositions for commutative CW complexes.

Lemma 5.18. Let \( X \) be a quantum CW-complex admitting a decomposition in two quantum subcomplexes \( X = A \cup B, \) such that \( A \cap B = C \neq \emptyset. \) If \( (A, C) \) is \( m \)-connected and \( (B, C) \) is \( n \)-connected, \( m, n \geq 0, \) then the mappings \( \hat{\pi}_k(A, C) \rightarrow \hat{\pi}_k(X, B) \) induced by inclusion is an isomorphism for \( k < m + n, \) and a surjection for \( k = m + n. \)

Lemma 5.19 (Quantum Freudenthal suspension theorem). The quantum suspension map \( \hat{\pi}_k(S^n) \rightarrow \hat{\pi}_{k+1}(S^{n+1}) \) is an isomorphism for \( k < 2n - 1, \) and a surjection for \( k = 2n - 1. \)

As a by-product we get the isomorphism \( \hat{\pi}_n(S^n) \cong \mathbb{Z}. \)

Remark 5.20. Let us emphasize that Theorem 5.17 does not allow to state that \( S^n \) is a deformation retract of \( S^n, \) as one could conclude by a wrong application of the Whitehead’s theorem, reported in the following lemma.

Lemma 5.21 (Whitehead’s theorem). If a map \( f : X \rightarrow Y \) between connected CW complexes induces isomorphisms \( f_* : \pi_n(X) \rightarrow \pi_n(Y) \) for all \( n, \) then \( f \) is a

\[ \frac{\text{This holds also for quantum suspension } \hat{\pi}_k(X) \rightarrow \hat{\pi}_{k+1}(S^n) \text{, for an } (n-1)\text{-connected quantum CW-complex } X.} \]
homotopy equivalence. Furthermore, if \( f \) is the inclusion of a subcomplex \( f : X \hookrightarrow Y \), then \( X \) is a deformation retract of \( Y \).

In fact, in the case \( i : S^n \hookrightarrow \hat{S}^n \) we are talking about different CW structures. One for \( S^n \) is the usual one, the other, for \( \hat{S}^n \) is the quantum CW structure. In order to easily understand the difference let us refer again to the case \( A = \mathbb{C} \). Here one has \( \pi_1(S^1) = \mathbb{Z} = \pi_1(\hat{S}^1) \), but \( \hat{\pi}_1(\hat{S}^1) = [S^1, \hat{S}^1] = [S^2, S^2] = \pi_2(S^2) \). Furthermore, \( \pi_1(S^1) = |S^1, S^2| = 0 \neq \pi_1(S^1) \). Therefore, \( \hat{S}^1 \), with respect to the usual CW complex structure, has its first homotopy group zero, hence different from the first homotopy group of its classic limit \( S^1 \). In fact \( S^1 \) is not a deformation retract of \( S^2 = \hat{S}^1 \). (Therefore there is not contradiction with the Whitehead’s theorem.)

Moreover, it is useful to formulate the quantum version of the Whitehead’s theorem and some related lemmas. These can be proved by reproducing analogous proofs by substituting CW complex structure with quantum CW complex structure in quantum manifolds.

**Theorem 5.22 (Quantum Whitehead theorem).** If a map \( f : X \to Y \) between connected quantum CW complexes induces isomorphisms \( f_* : \hat{\pi}_n(X) \to \hat{\pi}_n(Y) \) for all \( n \), then \( f \) is a homotopy equivalence. Furthermore, if \( f \) is the inclusion of a quantum subcomplex \( f : X \hookrightarrow Y \), then \( X \) is a quantum deformation retract of \( Y \).

**Lemma 5.23 (Quantum compression lemma).** Let \((X, A)\) be a quantum CW pair and let \((Y, B)\) be any quantum pair with \( B \neq \emptyset \). Let us assume that for each \( n \)
\[
\hat{\pi}_n(Y, B, y_0) = 0,
\]
for all \( y_0 \in B \), and \( X \setminus A \) has quantum cells of dimension \( n \). Then, every map \( f : (X, A) \to (Y, B) \) is homotopic rel \( A \) to a map \( X \to B \).

**Lemma 5.24 (Quantum extension lemma).** Let \((X, A)\) be a quantum CW pair and let \( f : A \to Y \) be a mapping with \( Y \) a path-connected quantum manifold. Let us assume that \( \hat{\pi}_{n-1}(Y) = 0 \), for all \( n \), such that \( X \setminus A \) has quantum cells of dimension \( n \). Then, \( f \) can be extended to a map \( f : X \to Y \).

**Proof.** The proof can be done inductively. Let us assume that \( f \) has been extended over the quantum \((n-1)\)-skeleton. Then, an extension over quantum \( n \)-cells exists iff the composition of the quantum cell’s attaching map \( \hat{S}^{n-1} \to X^{n-1} \) with \( f : X^{n-1} \to Y \) is null homotopic. \( \square \)

As a by-product of above results we get also the following theorems that relate quantum homotopy groups and quantum relative homotopy groups.

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\(^{52}\)When \( n = 0 \), the condition \( \hat{\pi}_n(Y, B, y_0) = 0 \), for all \( y_0 \in B \), means that \((Y, B)\) is 0-connected. Let us emphasize that there is not difference between 0-connected and quantum 0-connected. In fact \([S^0, Y] = \hat{\pi}_0(Y) = \pi_0(Y) = [S^0, Y] \), since \( S^0 = S^0 = (\{a\}, \{b\}) \), i.e., is a set of two points. However, after Theorem 5.17, there is not difference between the notion of quantum \( p \)-connected (i.e., \( \hat{\pi}_k = 0 \), \( k \leq p \)), quantum (homotopy) \( n \)-sphere, and \( p \)-connected (i.e., \( \pi_k = 0 \), \( k \leq p \)), (homotopy) \( n \)-sphere. In other words, a quantum homotopy \( n \)-sphere is quantum \((n-1)\)-connected as well as its classic limit is \((n-1)\)-connected.
Theorem 5.25 (Quantum exact long homotopy sequence). One has the exact sequence (53).

\[
\begin{array}{ccccccccc}
\cdots & \hat{\pi}_n(A, x_0) & \xrightarrow{i_*} & \hat{\pi}_n(X, x_0) & \xrightarrow{j_*} & \hat{\pi}_n(X, A, x_0) & \xrightarrow{\partial} & \hat{\pi}_0(X, x_0) & \xrightarrow{} \cdots & \hat{\pi}_{n-1}(A, x_0)
\end{array}
\]

where \( i_* \) and \( j_* \) are induced by the inclusions \( i : (A, x_0) \hookrightarrow (X, x_0) \) and \( j : (X, x_0, x_0) \hookrightarrow (X, A, x_0) \) respectively. Furthermore, \( \partial \) comes from the following composition \( (S^{n-1}, s_0) \hookrightarrow (\hat{D}^n, \hat{S}^{n-1}, s_0) \to (X, A, x_0) \), hence \( \partial[f] = [f|\hat{S}^{n-1}] \).

Theorem 5.26 (Quantum Hurewicz theorem). The exact commutative diagram in (54) relates (quantum) homotopy groups and (quantum) homology groups for (quantum) homotopy \( n \)-spheres, \( n \geq 2 \). The morphisms \( a \) and \( b \) are isomorphisms for \( p \leq n \) and epimorphisms for \( p = n + 1 \).

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{} & \pi_p(S^n) & \xrightarrow{a} & \hat{\pi}_p(\hat{S}^n) & \xrightarrow{b} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & H_p(S^n; \mathbb{Z}) & \xrightarrow{a} & H_p(\hat{S}^n; \mathbb{Z}) & \xrightarrow{b} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]

The following propositions are also stated as direct results coming from Theorem 5.17 and analogous propositions for topologic spaces.

Proposition 5.27. The following propositions are equivalent for \( i \leq n - 1 \).
1) \( S^i \to S^n \) is homotopic to a constant map.
2) \( S^i \to S^n \) extends to a map \( D^{i+1} \to S^n \).
3) \( \hat{S}^i \to \hat{S}^n \) is homotopic to a constant map.
4) \( \hat{S}^i \to \hat{S}^n \) extends to a map \( \hat{D}^{i+1} \to \hat{S}^n \).

Proof. 1) and 2) follow from the fact that \( S^n \) is \( (n - 1) \)-connected, and 3) and 4) from the fact that \( \hat{S}^n \) is quantum \( (n - 1) \)-connected. Furthermore, let us recall the following related result of Algebraic Topology.

Lemma 5.28. The following propositions are equivalent.
(i) The space \( X \) is \( n \)-connected.
(ii) Every map \( f : S^i \to X \) is homotopic to a constant map.
(iii) Every map \( f : S^i \to X \) extends to a map \( D^{i+1} \to X \).
Example 5.29 (Quantum-complex cupola). Let us consider again \( A = \mathbb{C} \), that offers beautiful examples, (even if commutative), easy to understand. Then, from points 3) and 4) in Proposition 5.27 we get that \( \hat{S}^1 \to \hat{S}^2 \) is homotopic to a constant map and extends to a map \( \hat{D}^2 \to \hat{S}^2 \). These agree with points 1) and 2) in Proposition 5.27, as shown in the commutative diagram (55). In the quantum-complex cupola all the building is nullhomotopic to the base point \( \{\infty\} \in \hat{D}^3 \) identified with a 6-cell. By using Proposition 5.27 similar building can be obtained with any quantum algebra.

\[
\begin{array}{c}
S^0 \\
\downarrow \\
S^0 \\
\downarrow \\
S^2 \\
\downarrow \\
S^2 \\
\downarrow \\
S^2 \\
\downarrow \\
\partial D^2 \\
\downarrow \\
\partial D^2 \\
\downarrow \\
\partial D^2 \\
\downarrow \\
\partial D^2 = S^0
\end{array}
\]

Let us, now, consider quantum PDE’s with respect to quantum homotopy n-spheres.

Definition 5.30 (Quantum hypercomplex exotic PDE’s). Let \( \hat{E}_k \subset \hat{J}^k_n(W) \) be a \( k \)-order PDE on the fiber bundle \( \pi: W \to M \) in the category \( Q_{\text{hyper}} \), with \( \dim A M = n \) and \( \dim_B W = (n,m) \), where \( B = A \times E \) and \( E \) is also a \( Z(A) \)-module. We say that \( \hat{E}_k \) is a quantum exotic PDE if it admits Cauchy integral manifolds \( N \subset \hat{E}_k \), \( \dim N = n - 1 \), such that one of the following two conditions is verified.

(i) \( \Sigma^{n-2} \equiv \partial N \) is a quantum exotic sphere of dimension \( (n-2) \), i.e. \( \Sigma^{n-2} \) is homeomorphic to \( \hat{S}^{n-2} \), \( (\Sigma^{n-2} \approx \hat{S}^{n-2}) \) but not diffeomorphic to \( \hat{S}^{n-2} \), \( (\Sigma^{n-2} \not\approx \hat{S}^{n-2}) \).

(ii) \( \emptyset = \partial N \) and \( N \approx \hat{S}^{n-1} \), but \( N \not\approx \hat{S}^{n-1} \).

Definition 5.31 (Quantum hypercomplex exotic-classic PDE’s). Let \( \hat{E}_k \subset \hat{J}^k_n(W) \) be a \( k \)-order PDE as in Definition 5.30. We say that \( \hat{E}_k \) is a quantum exotic-classic PDE if it is a quantum exotic PDE, and the classic limit of the corresponding Cauchy quantum exotic manifolds are also exotic homotopy spheres.

From above results we get also the following one.

Lemma 5.32. A quantum PDE \( \hat{E}_k \subset \hat{J}^k_n(W) \), where \( n \) is such that \( \Theta_{n-1} = 0 \), cannot be a quantum exotic-classic PDE, in the sense of Definition 5.31.

---

\(^{55}\)The following Refs. [12, 34, 35, 37, 39, 40, 41, 49, 50, 51, 52, 53, 54, 55, 92, 93, 94, 95, 101, 104] are important background for differential structures and exotic spheres.
**Lemma 5.33.** For \( n \in \{1, 2, 3, 4, 5, 6\} \), one has the isomorphism reported in (56).

\[
\hat{\Theta}_n \cong \hat{\Upsilon}_n.
\]

In correspondence of such dimensions on \( n \) we cannot have quantum exotic-classic PDE’s.

**Proof.** Isomorphisms in (56), follow directly from above lemmas, and the fact that \( \Theta_n = 0 \) for \( n \in \{1, 2, 3, 4, 5, 6\} \). (See Refs. [86, 87].) \( \square \)

**Example 5.34** (The quantum hypercomplex Ricci flow equation). Let us recall that in [80] A. Prástaro proved the generalized Poincaré conjecture in the category of quantum supermanifolds for classic regular, closed compact quantum supermanifolds \( M \), of dimension \( (m|n) \), homotopy equivalent to \( S^m|n \), when the quantum superalgebra \( A \) has a Noetherian centre \( Z(A) \). More precisely one has proved that \( M \) is homeomorphic to \( S^m|n \) and its classic limit \( M_C \) is homeomorphic to \( S^m \). As a by-product it follows that the quantum Ricci flow equation is a quantum exotic PDE for quantum \( n \)-dimensional Riemannian manifolds. Furthermore, the quantum Ricci flow equation cannot be quantum exotic-classic for \( n < 7 \). (See [69, 79, 81, 82].) (For complementary informations on the Ricci flow equation see also the following Refs. [28, 29, 30, 31, 32, 56, 57].)

**Example 5.35** (The quantum hypercomplex Navier-Stokes equation). The quantum Navier-Stokes equation can be encoded on the quantum extension of the affine fiber bundle \( \pi : W \equiv M \times I \times \mathbb{R}^2 \rightarrow M \), \( (x^\alpha, \dot{x}^i, p, \theta)_{0 \leq \alpha \leq 3, 1 \leq i \leq 3} \mapsto (x^\alpha) \). (See Refs. [65, 71, 72, 74, 76, 79] for the Navier-Stokes equation in the category of commutative manifolds and [67, 68] for its quantum extension on quantum manifolds.) Therefore, Cauchy manifolds are \( 3 \)-dimensional space-like manifolds. For such dimension do not exist exotic spheres. Therefore, the Navier-Stokes equation cannot be a quantum exotic-classic PDE. Similar considerations hold for PDE’s of the classical continuum mechanics.

**Example 5.36** (The quantum hypercomplex \( n \)-d’Alembert equation). The quantum \( n \)-d’Alembert equation on \( A^n \) cannot be a quantum exotic-classic PDE for quantum \( n \)-dimensional Riemannian manifolds of dimension \( n < 7 \) in the category \( \mathcal{Q}_{\text{hyper}} \). (See Example 4.11, and [87] for exotic d’Alembert equations in the category of commutative manifolds.)

**Example 5.37** (The quantum hypercomplex Einstein equation). The quantum Einstein equation in the category \( \mathcal{Q}_{\text{hyper}} \), cannot be a quantum exotic-classic PDE for quantum \( n \)-dimensional space-times of dimension \( n < 7 \). Similar considerations hold for generalized quantum Einstein equations like quantum Einstein-Maxwell equation, quantum Einstein-Yang-Mills equation and etc, in the category \( \mathcal{Q}_{\text{hyper}} \).

**Theorem 5.38** (Integral bordism groups in quantum hypercomplex exotic PDE’s in the category \( \mathcal{Q}_{\text{hyper}} \) and stability). Let \( \hat{E}_k \subset J^\infty_k(W) \) be a quantum exotic formally integrable and completely integrable PDE on the fiber bundle \( \pi : W \rightarrow M \), in the category \( \mathcal{Q}_{\text{hyper}} \), such that \( \hat{g}_k \neq 0 \) and \( \hat{g}_{k+1} \neq 0 \). Then there exists a

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56The fiber bundle \( \pi : W \rightarrow M \) is as in Definition 5.30, hence \( \dim_A M = 0 \), \( \dim_B W = (n, m) \), with \( E \) endowed with a \( Z(A) \)-module structure too.
topologic spectrum $\Xi$, such that for the singular integral $p$-(co)boundary groups can be expressed by means of suitable homotopy groups as reported in (57).

\[
\begin{align*}
\Omega^{\hat{E}_k}_{p,s} &= \lim_{r \to \infty} \pi_{p+r}(\hat{E}_k^r \wedge \Xi_r) \\
\Omega^{\hat{E}_k}_{p,s} &= \lim_{r \to \infty} [S^r \hat{E}_k^r, \Xi_{p+r}]_{p \in \{0,1,\ldots,n-1\}}.
\end{align*}
\]

Furthermore, the singular integral bordism group for admissible smooth closed compact Cauchy manifolds, $N \subset \hat{E}_k$, is given in (58).

\[
\Omega^{\hat{E}_k}_{n-1,s} \cong H_{n-1}(W; A).
\]

In the quantum homotopy equivalence full admissibility hypothesis, i.e., by considering admissible only $(n-1)$-dimensional smooth Cauchy integral manifolds identified with quantum homotopy spheres, and assuming that the space of conservation laws is not trivial, one has $\Omega^{\hat{E}_k}_{n-1,s} = 0$. Then $\hat{E}_k$ becomes a quantum extended 0-crystal PDE. Then, there exists a global singular attractor, in the sense that all Cauchy manifolds, identified with quantum homotopy $(n-1)$-spheres, bound singular manifolds.

Furthermore, if in $W$ we can embed all the quantum homotopy $(n-1)$-spheres, and all such manifolds identify admissible smooth $(n-1)$-dimensional Cauchy manifolds of $\hat{E}_k$, then two of such Cauchy manifolds bound a smooth solution iff they are diffeomorphic and one has the following bijective mapping: $\Omega^{\hat{E}_k}_{n-1} \leftrightarrow \Theta_{n-1}$.

Moreover, if in $W$ we cannot embed all quantum homotopy $(n-1)$-spheres, but only $\hat{S}^{n-1}$, then in the quantum sphere full admissibility hypothesis, i.e., by considering admissible only quantum $(n-1)$-dimensional smooth Cauchy integral manifolds identified with $\hat{S}^{n-1}$, then $\Omega^{\hat{E}_k}_{n-1} = 0$. Therefore $\hat{E}_k$ becomes a quantum 0-crystal PDE and there exists a global smooth attractor, in the sense that two of such smooth Cauchy manifolds, identified with $\hat{S}^{n-1}$ bound quantum smooth manifolds. Instead, two Cauchy manifolds identified with quantum exotic $(n-1)$-spheres bound by means of quantum singular solutions only.

All above quantum smooth or quantum singular solutions are unstable. Quantum smooth solutions can be stabilized.

**Proof.** The relations (57) can be proved by a direct extension of analogous characterizations of integral bordism groups of PDE’s in the category of commutative manifolds. (See Theorem 4.14 in [65].) Furthermore, under the hypotheses of theorem we can apply Theorem 5.5 and Theorem 5.8 in [66] and Theorem 2.2 in [84] (or Theorem 2.3 in [83]). Thus we get directly (58). Furthermore, under the quantum homotopy equivalence full admissibility hypothesis, all admissible quantum smooth $(n-1)$-dimensional Cauchy manifolds of $\hat{E}_k$, are identified with all possible quantum homotopy $(n-1)$-spheres. Moreover, all such Cauchy manifolds have same quantum integral characteristic numbers. (The proof is similar to the one given for Ricci flow PDE’s in [80].) Therefore, all such Cauchy manifolds belong to the same singular integral bordism class, hence $\Omega^{\hat{E}_k}_{n-1,s} = 0$. Thus in such a case $\hat{E}_k$ becomes an quantum extended 0-crystal PDE. When all quantum homotopy $(n-1)$-spheres can be embedded in $W$ and so that in each smooth integral bordism class of $\Omega^{\hat{E}_k}_{n-1}$ are contained quantum homotopy $(n-1)$-spheres.\textsuperscript{57} Then, since two quantum homotopy

\textsuperscript{57}It is useful to emphasize that the possibility to embed quantum homotopy $(n-1)$-spheres in $W$, are related to the dimension of $W$. For example, in the cases where quantum algebras $A$
(n − 1)-spheres bound a quantum smooth solution of \( \hat{E}_k \) iff they are diffeomorphic, it follows that one has the bijection (but not isomorphism) \( \Omega_{n-1}^{\hat{E}_k} \cong \hat{\Theta}_{n-1} \), where \( \hat{\Theta}_{n-1} \) is the set group of equivalent classes of quantum diffeomorphic quantum homotopy \((n - 1)\)-spheres. In the quantum sphere full admissibility hypothesis we get \( \Omega_{n-1}^{\hat{E}_k} = 0 \) and \( \hat{E}_k \) becomes a quantum 0-crystal PDE.

Let us assume now, that in \( W \) we can embed only \( \hat{S}^{n-1} \) and not all quantum exotic \((n - 1)\)-spheres. Then smooth Cauchy \((n − 1)\)-manifolds identified with quantum exotic \((n - 1)\)-spheres are necessarily integral manifolds with Thom-Boardman singularities, with respect to the canonical projection \( \pi_{k,0} : \hat{E}_k \to W \). So solutions passing through such Cauchy manifolds are necessarily singular solutions. In such a case smooth solutions bord Cauchy manifolds identified with \( \hat{S}^{n-1} \), and two diffeomorphic Cauchy manifolds identified with two quantum exotic \((n - 1)\)-spheres belonging to the same class in \( \hat{\Theta}_{n-1} \), cannot bound quantum smooth solutions. Finally, if also \( \hat{S}^{n-1} \) cannot be embedded in \( W \), then there are not quantum smooth solutions bording smooth Cauchy \((n - 1)\)-manifolds in \( \hat{E}_k \), identified with \( \hat{S}^{n-1} \) or \( \hat{\Sigma}^{n-1} \) (i.e., quantum exotic \((n - 1)\)-sphere). In other words \( \Omega_{n-1}^{\hat{E}_k} \) is not defined in such a case!

We are ready to state the main result of this paper that extends to the category of quantum manifolds Theorem 4.59 in [86] and Theorem 4.7 in [88], given for homotopy spheres in the category of smooth manifolds.

**Theorem 5.39** (Integral h-cobordism in quantum hypercomplex Ricci flow PDE’s).

The quantum Ricci flow equation for quantum \( n \)-dimensional Riemannian manifolds, admits that starting from a quantum \( n \)-dimensional sphere \( \hat{S}^n \), we can dynamically arrive, into a finite time, to any quantum \( n \)-dimensional homotopy sphere \( M \). When this is realized with a smooth solution, i.e., solution with characteristic flow without singular points, then \( \hat{S}^n \cong M \). The other quantum homotopy spheres \( \hat{\Sigma}^n \), that are homeomorphic to \( \hat{S}^n \) only, are reached by means of singular solutions. For \( 1 \leq n \leq 6 \), quantum hypercomplex Ricci flow PDE’s cannot be quantum exotic-classic ones. In particular, the case \( n = 4 \), is related to the proof that the smooth Poincaré conjecture is true.

**Proof.** This is a direct consequence of Theorem 3.23 in [80] on the generalized Poincaré conjecture for quantum supermanifolds, Theorem 4.59 in [86] and Theorem 4.7 in [88] for homotopy spheres, where it is proved also the smooth Poincaré conjecture (i.e. in dimension four) for commutative manifolds. □

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and \( E \) are finite dimensional, the Whitney’s embeddings theorem assures that embeddings exist if \( \dim W \geq 2(n - 1) + 1 = 2n - 1 \). Furthermore, by considering that such embeddings should be compatible with \( n \)-dimensional submanifolds of \( W \), we should also require that \( \dim W \geq 2n + 1 \).

\(^{58}\) See Lemma 5.5. Therefore \( \Theta_{n-1}/\Gamma_{n-1} \cong \Theta_{n-1} \), where \( \Theta_{n-1} \) is the corresponding set group of equivalent classes of diffeomorphic homotopy \((n - 1)\)-spheres.
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