TWO-PARAMETER IDENTITIES FOR DIVISOR SUMS IN ALGEBRAIC NUMBER FIELDS

BRUCE C. BERNDT, MARTINO FASSINA, SUN KIM, AND ALEXANDRU ZAHARESCU

ABSTRACT. In a one-page fragment published with his lost notebook, Ramanujan stated two double series identities associated, respectively, with the famous Gauss Circle and Dirichlet Divisor problems. The identities contain an “extra” parameter, and it is possible that Ramanujan derived these identities with the intent of attacking these famous problems. Similar famous unsolved problems are connected with \( f_K(n) \), the number of integral ideals of norm \( n \) in an algebraic number field \( K \). In this paper we establish Riesz sum identities containing an “extra” parameter and involving \( f_K(n) \), or divisor functions associated with \( K \). Upper bounds for the sums as the upper index tends to infinity are also established.

1. INTRODUCTION

Let \( d(n) \) denote the number of positive divisors of the positive integer \( n \), and set
\[ D(x) := \sum_{n \leq x} ' d(n), \] (1.1)
where the prime on the summation sign indicates that if \( x \) is an integer, then only \( \frac{1}{2} d(x) \) is counted in (1.1). Dirichlet showed that
\[ D(x) = x (\log x + 2 \gamma - 1) + \frac{1}{4} + \Delta(x), \] (1.2)
where \( \gamma \) denotes Euler’s constant, and \( \Delta(x) \) is the “error term.” One of the most famous unsolved problems in analytic number theory is to find the optimal order of magnitude for \( \Delta(x) \), as \( x \to \infty \); this is the Dirichlet Divisor Problem. (The fraction \( \frac{1}{4} \) is present because it arises naturally from analytic investigations and, of course, does not affect nontrivial bounds for \( \Delta(x) \).) Dirichlet’s elementary argument also yields the first upper bound for \( \Delta(x) \), namely,
\[ \Delta(x) = O(\sqrt{x}), \quad x \to \infty. \] (1.3)
Berndt, Kim, and Zaharescu [4] provide a summary of upper bounds that have been achieved for \( \Delta(x) \), as \( x \to \infty \), in the twentieth century. A theorem of G. H. Hardy [15] implies that \( \Delta(x) \neq O(x^{1/4}) \), as \( x \to \infty \).

For over a century, those seeking increasingly better upper bounds for \( \Delta(x) \) have found that a famous formula of G. F. Voronoï [22] to be useful. In order to state his formula, we
need some definitions. Let \( J_\nu(x) \) denote the ordinary Bessel function of order \( \nu \). The Bessel function \( Y_\nu(z) \) of the second kind \([23, p. 64, Equation (1)]\) is defined by
\[
Y_\nu(z) := \frac{J_\nu(z) \cos(\nu \pi) - J_{-\nu}(z)}{\sin(\nu \pi)},
\]
and the modified Bessel function \( K_\nu(z) \) \([23, p. 78, Equation (6)]\) is defined, for \(-\pi < \arg z < \frac{1}{2} \pi\), by
\[
K_\nu(z) := \frac{\pi}{2} e^{\pi i \nu/2} J_{-\nu}(iz) - e^{-\pi i \nu/2} J_{\nu}(iz).
\]
If \( \nu \) is an integer \( n \), it is understood that we take the limits in (1.4) and (1.5) as \( \nu \to n \). Then Voronoï’s formula is given by
\[
\sum_{n \leq x'} n = x (\log x + 2\gamma - 1) + \frac{1}{4} + \sum_{n=1}^1 d(n) \left(\frac{x}{n}\right)^{1/2} I_1(4\pi \sqrt{nx}),
\]
where \( x > 0 \), and \( I_1(z) \) is defined by
\[
I_1(z) := -Y_0(z) - \frac{2}{\pi} K_0(z).
\]
In a one-page fragment published with his lost notebook \([21, p. 335]\), Ramanujan offered without proof an identity involving the same Bessel functions that appear on the right-hand side of (1.6). For \( x > 0 \) and \( 0 < \theta < 1 \),
\[
\sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor \cos(2\pi n \theta) = \frac{1}{4} - x \log(2 \sin(\pi \theta))
\]
\[
+ \frac{1}{2} \sqrt{x} \sum_{m=1}^1 \sum_{n=0}^1 \left\{ \frac{I_1(4\pi \sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} + \frac{I_1(4\pi \sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}} \right\},
\]
where \( I_1(z) \) is defined in (1.7). An elementary argument shows that
\[
D(x) = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor.
\]
Hence, if \( \theta = 0 \), the left side of (1.8) reduces to \( D(x) \). Thus, (1.8) can be considered as a two-variable analogue of (1.6). In his first letter to Hardy \([6, p. 23]\), Ramanujan expressed his interest in Dirichlet’s Divisor Problem, although it is doubtful that he would have been familiar with this attribution. Perhaps Ramanujan had derived (1.8) with the intention of applying it to the Divisor Problem.

Berndt, Kim, and Zaharescu first proved (1.8), but with the order of summation reversed and with the assumption that the double series on the right-hand side of (1.8) converges for at least one value of \( \theta \) \([2]\). A complete proof of (1.8) was later established by Berndt and Zaharescu along with J. Li \([5]\). Kim established similar theorems for weighted divisor sums and Riesz sums \([16]\). Very briefly and roughly, a Riesz sum is an extension of a sum such as (1.1), but with the summands now weighted by \((x - n)^\rho\), where \( \rho > a \), for some real number \( a \).
In this paper, we establish analogues of (1.8) for divisor sums associated with Dedekind zeta functions attached to algebraic number fields. As in [16], our analogues are in the more general setting of Riesz sums. Our motivation arises from unsolved problems for these divisor sums that are analogous to the Dirichlet Divisor Problem.

Let $K$ be an algebraic number field. For $\sigma = \text{Re}(s) > 1$, the Dedekind zeta function associated with $K$ is defined by

$$\zeta_K(s) := \sum_{I \subseteq \mathcal{O}_K} \frac{1}{(N_{K/Q}(I))^s},$$

where $\mathcal{O}_K$ is the ring of integers of $K$, the sum is over all non-zero integral ideals $I$ of $\mathcal{O}_K$, and $N_{K/Q}(I)$ denotes the norm of $I$. Furthermore, define $f_K(n)$ by

$$\zeta_K(s) =: \sum_{n=1}^{\infty} \frac{f_K(n)}{n^s}. \quad (1.9)$$

Set

$$\sum_{n \leq x} f_K(n) = \gamma_1(K)x + E_K(x),$$

where $\gamma_1(K)$ is a constant defined by (2.1) in the following section, and where $E_K(x)$ is the “error term.” E. Landau’s began this study in 1912 when he proved that [17]

$$\sum_{n \leq x} f_K(n) = \gamma_1(K)x + O(x^{(k-1)/(k+1)}),$$

where $k$ is the degree of $K$, and the implied constant in the big-O term depends upon $K$. Currently, the best results are due to W. G. Nowak [19] and B. Paul and A. Sankaranarayanan [20]. The former author proved that, for $k \geq 3$,

$$E_K(x) = O \left( x^{1-\frac{2}{k} + \frac{\epsilon}{2k} (\log x)^{\frac{3}{k}}} \right),$$

while the latter two authors proved that, for $k \geq 10$,

$$E_K(x) = O \left( x^{1-\frac{4}{k} + \epsilon} \right),$$

for every $\epsilon > 0$. Observe that

$$\frac{k - 1}{k + 1} = 1 - \frac{2}{k} + \frac{2}{k(k + 1)},$$

and, for $k \geq 10$,

$$\frac{3}{k + 6} > \frac{2}{k} > \frac{3}{2k^2} = \frac{4k - 3}{2k^2}.$$ 

Our two primary theorems below, Theorem 3.3 and Theorem 5.3, provide analogues of (1.8) involving, respectively, $f_K(n)$ and $d_{\chi_D}(n)$, where $d_{\chi_D}(n) := \sum_{k|n} \chi_D(k)$, and $D$ is a fundamental discriminant. In Theorems 6.3 and 6.4, we obtain upper bounds for the “error” terms associated with these sums. Perhaps our identities with the additional parameter $\theta$ may be of use in attacking these long-standing unsolved problems originating with Landau.
2. Preliminary Results

Define, for a Dirichlet character \( \chi \),
\[
F_K(x) := \sum_{n \leq x} f_K(n), \quad D_K(n) := \sum_{d|n} f_K(d), \quad D_{K,\chi}(n) := \sum_{d|n} f_K(d)\chi(n/d).
\]
If \( \sigma > 1 \),
\[
\zeta_K(s)\zeta(s) = \sum_{n=1}^{\infty} \frac{f_K(n)}{n^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{D_K(n)}{n^s},
\]
and
\[
\zeta_K(s)L(s, \chi) = \sum_{n=1}^{\infty} \frac{f_K(n)}{n^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{D_{K,\chi}(n)}{n^s},
\]
where \( \zeta(s) \) denotes the Riemann zeta function, and \( L(s, \chi) \) denotes the Dirichlet \( L \)-function associated with the character \( \chi \).

Also,
\[
\sum_{n \leq x} F_K \left( \frac{x}{n} \right) \chi(n) = \sum_{n \leq x} \sum_{m \leq x/n} f_K(m)\chi(n) = \sum_{mn \leq x} f_K(m)\chi(n)
= \sum_{n \leq x} \sum_{d|n} f_K(d)\chi(n/d) = \sum_{n \leq x} D_{K,\chi}(n).
\]

Next, let \( \Delta_K \) denote the discriminant of \( K \), and let \( r_1 \) and \( r_2 \) denote the number of real embeddings and the number of conjugate pairs of complex embeddings of \( K \), respectively. The Dedekind zeta function \( \zeta_K(s) \) has a simple pole at \( s = 1 \) with residue \[18, p. 467\]
\[
\gamma_{-1}(K) := \frac{2^{r_1+r_2}\pi^{r_2} \text{Reg}_K h_K}{w_K \sqrt{\left| \Delta_K \right|}},
\]
where \( h_K \) is the class number of \( K \), \( \text{Reg}_K \) is the regulator of \( K \), and \( w_K \) denotes the number of roots of unity in \( K \). The Laurent expansion of \( \zeta_K(s) \) around \( s = 1 \) is given by
\[
\zeta_K(s) = \frac{\gamma_{-1}(K)}{s-1} + \sum_{n=0}^{\infty} \gamma_n(K)(s-1)^n.
\]

The function
\[
\Lambda_K(s) := \left| \Delta_K \right|^{s/2} \left( \pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\right)^{r_1} \left(2(2\pi)^{-s}\Gamma(s)\right)^{r_2} \zeta_K(s)
\]
satisfies the functional equation \[18, p. 466\]
\[
\Lambda_K(s) = \Lambda_K(1-s).
\]
We also recall that the functional equation of the Riemann zeta function \( \zeta(s) \) is given by \[9, p. 59\]
\[
\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1}{2}(1-s)\right)\zeta(1-s).
\]
If \( \chi \) is an even, non-principal, primitive character of modulus \( q \), then the Dirichlet \( L \)-function \( L(s, \chi) \) satisfies the functional equation \([9], p. 69\)
\[
\left( \frac{\pi}{q} \right)^{-s/2} \Gamma \left( \frac{1}{2} s \right) L(s, \chi) = \frac{G(\chi)}{\sqrt{q}} \left( \frac{\pi}{q} \right)^{-(1-s)/2} \Gamma \left( \frac{1}{2} (1 - s) \right) L(1 - s, \overline{\chi}),
\]
(2.6)
where \( G(\chi) \) denotes the Gauss sum
\[
G(\chi) := \sum_{h=1}^{q-1} \chi(h) e^{2\pi i h/q}.
\]

**Definition 2.1.** The Meijer \( G \)-function is defined by \([12], p. 374\)
\[
G^{m,n}_{p,q} \left( x \middle| a_1, a_2, \ldots, a_p ; b_1, b_2, \ldots, b_q \right) := \frac{1}{2\pi i} \int_L \prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)x^s ds,
\]
(2.8)
where \( L \) is a path from \(-i\infty\) to \(+i\infty\) separating the poles of \( \Gamma(b_1 - s) \cdots \Gamma(b_m - s) \) from those of \( \Gamma(1 - a_1 + s) \cdots \Gamma(1 - a_n + s) \). When the arguments \( a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q \) are omitted, we write \( G^{m,n}_{p,q} \) or \( G^{m,n}_{p,q}(x) \).

3. **The Case: \( r_2 = 0 \)**

We consider the case \( r_2 = 0 \). Let \( \chi \) be a non-principal, even, primitive character of modulus \( q \). By (2.3), (2.4), and (2.6), the functional equation of \( \zeta_K(2s)L(2s, \chi) \) is given by
\[
\pi^{-s(1+1)}q^{s}|\Delta_K|^s \Gamma(s)^{r_1+1} \zeta_K(2s)L(2s, \chi) = \frac{G(\chi)}{\sqrt{q}} \pi^{-\left( \frac{1}{2} - s \right)(r_1+1)} q^{\frac{1}{2} - s} |\Delta_K|^\frac{1}{2} - s \Gamma \left( \frac{1}{2} - s \right) \zeta_K(1 - 2s)L(1 - 2s, \overline{\chi}).
\]
(3.1)

Below and in the sequel it is tacitly assumed that if \( \rho = 0 \), only one-half of the last term in a finite sum is counted.

**Theorem 3.1.** Let \( q \) be a positive integer, and let \( \chi \) be a non-principal even primitive character modulo \( q \). Then, for \( x > 0 \) and \( \rho > \frac{1}{2} r_1 - 1 \),
\[
\frac{1}{\Gamma(\rho+1)} \sum_{n \leq x} D_{K,\chi}(n)(x^2 - n^2)^\rho = \frac{G(\chi)|\Delta_K|^{1/2} x^{2\rho}}{\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} \frac{D_{K,\chi}(n)}{n} G^{0,r_1+1}_{2(r_1+1),0} \left( \frac{q^2|\Delta_K|^2}{\pi^{2(r_1+1)}(nx)^2} \right)^{\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},\rho+1,0,\ldots,0} - \frac{G(\chi)\sqrt{\pi} \gamma_{1-K}(x)^{1+2\rho}}{2q \Gamma(\rho+\frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log (2 \sin(\pi n/q)),
\]
(3.2)
where \( G^{0,r_1+1}_{2(r_1+1),0} \) is defined in Definition 2.7 and where there are \( r_1 \) \{0\}’s and \( r_1 + 1 \) \{\frac{1}{2}\}’s in \( G^{0,r_1+1}_{2(r_1+1),0} \).
Using (3.1), we apply Theorems 2 and 4 in [1, pp. 351, 356] with, in the notation of [1],

$$m = r_1 + 1, \quad r = \frac{1}{2}, \quad \lambda_n = \mu_n = \frac{\pi r_1 + n}{q|\Delta K|}, \quad a_n = D_{K,\chi}(n), \quad b_n = \frac{G(\chi)}{\sqrt{q}}D_{K,\chi}(n).$$

Then, for $x > 0$ and $\rho > \frac{1}{2}r_1 - 1$,

$$\frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} D_{K,\chi}(n)(x - \lambda_n)^\rho = \frac{G(\chi)}{2\pi i \sqrt{q}} \sum_{n=1}^{\infty} D_{K,\chi}(n) \frac{x}{\mu_n}^{1/2 + \rho} K_{1/2 + \rho}(2^{r_1 + 1} \sqrt{\mu_n x_1; -\frac{1}{2}; r_1 + 1}) + Q_\rho(x). \quad (3.3)$$

Here,

$$Q_\rho(x) = \frac{1}{2\pi i} \int_{C_\rho} q^s |\Delta K|^s \frac{\Gamma(s) \zeta_K(2s)L(2s, \chi)x^{s+\rho}}{\pi^{s(r_1 + 1)} \Gamma(s + \rho + 1)} ds,$$

where $C_\rho$ is a positively oriented, closed curve encircling the poles of the integrand,

$$K_{1/2 + \rho}(x; -\frac{1}{2}; r_1 + 1) = \int_0^1 u_r^\rho J_{-1/2}^-(u_r) du_r \int_0^1 u_{r-1}^\rho J_{-1/2}^-(u_{r-1}) du_{r-1} \cdots \int_0^1 u_1^\rho J_{-1/2}^-(u_1) J_{1/2 + \rho}^-(x/u_1 u_2 \cdots u_{r_1}) du_1 \quad (3.4)$$

[1] p. 348, Definition 4], and $J_{\nu}(z)$ denotes the ordinary Bessel function of order $\nu$.

We first calculate $Q_\rho(x)$. The left-hand side of (2.6) is an entire function, and so $L(0, \chi) = 0$. Therefore, the only pole of the integrand is at $s = \frac{1}{2}$, arising from the simple pole of $\zeta_K(2s)$. Thus, using (2.1) and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have

$$Q_\rho(x) = \frac{\sqrt{q|\Delta K| \gamma_{-1}(K)x^{1/2 + \rho}}}{2\pi^{r_1/2} \Gamma(\rho + \frac{3}{2})} L(1, \chi).$$

Using an evaluation for $L(1, \chi)$ in [10] p. 182, Equation (3.5)), we obtain

$$Q_\rho(x) = -\frac{G(\chi)}{2\pi^{r_1/2} \sqrt{q} \Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log |1 - \xi^n_q| \log |1 - \xi_q^n|$$

$$= -\frac{G(\chi)}{2\pi^{r_1/2} \sqrt{q} \Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log(2 \sin(\pi n/q)), \quad (3.5)$$

where we used the identity

$$\log |1 - \xi^n_q| = \log |\xi^{-n/2}_q - \xi^{n/2}_q| = \log(2 \sin(\pi n/q)).$$

Lastly, we employ the identity

$$K_{1/2 + \rho}(x; -\frac{1}{2}; r_1 + 1) = 2^{-\rho+(r_1+1)/2}x^{\rho-1/2}G_{2(r_1+1),0}^{0,1} \left(\frac{4^{r_1+1}}{x^2} \left| \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \rho + 1, 0, \ldots, 0 \right. \right), \quad (3.6)$$

where there are $r_1 \{0\}$’s and $(r_1 + 1) \{\frac{1}{2}\}$’s, which we prove in Theorem[7,1]
Now, we replace $x$ by $\frac{x^{r_1+1}}{q|\Delta_K|}$ in (3.3). Then, by (3.5) and (3.6), we have

$$\frac{1}{\Gamma(\rho + 1)} \sum_{n \leq x} D_{K, \chi}(n)(x^2 - n^2)^\rho$$

$$= \frac{G(\chi)(q|\Delta_K|)^\rho}{(2^{r_1}r_{r_1+1})^\rho} \sqrt{q} \sum_{n=1}^{\infty} D_{K, \chi}(n) \left( \frac{x}{n} \right) \frac{1}{\Gamma(\rho + 1)} K^{\frac{r+\rho}{2}} \left( \frac{(2\pi)^{r_1+1}n}{q|\Delta_K|}; -\frac{1}{2}; r_1 + 1 \right)$$

$$- \frac{G(\chi) \sqrt{\pi} \gamma_{-1}(K) x^{1+2\rho}}{2q \Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log (2 \sin(\pi n/q))$$

$$\left( \frac{\Delta_K}{\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} D_{K, \chi}(n) \right) G_{0, r_1+1}^{0, 0} \left( \frac{q^2|\Delta_K|^2}{\pi^{2(r_1+1)}(nx^2)} \right) \left[ \frac{1\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \rho + 1, 0, \ldots, 0 \right]$$

$$- \frac{G(\chi) \sqrt{\pi} \gamma_{-1}(K) x^{1+2\rho}}{2q \Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log (2 \sin(\pi n/q)).$$

This completes the proof. □

The following theorem is an analogue of Voronoï’s formula (1.6). Note that if $\rho = 0$ and $r_1 = 1$, Theorem 3.2 reduces to (1.6). See also (8.1) below.

Theorem 3.2. For $x > 0$ and $\rho > \frac{1}{2}r_1 - 1$,

$$\frac{1}{\Gamma(\rho + 1)} \sum_{n \leq x} D_{K}(n)(x^2 - n^2)^\rho$$

$$= \frac{|\Delta_K|^{1/2}x^{2\rho}}{\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} D_{K}(n) \left( \frac{\Delta_K}{\pi^{2(r_1+1)}(nx^2)} \right) \left[ \frac{1\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \rho + 1, 0, \ldots, 0 \right]$$

$$+ \frac{\sqrt{\pi} x^{1+2\rho}}{2 \Gamma(\rho + \frac{3}{2})} \left\{ \gamma_0(K) + \gamma_{-1}(K) \gamma + \frac{\gamma_{-1}(K) \Gamma'(\frac{1}{2})}{2 \sqrt{\pi}} \right\}$$

$$- \frac{1}{2} \gamma_{-1}(K) \psi(\rho + \frac{3}{2}) + \gamma_{-1}(K) \log x \right\}, \quad (3.7)$$

where $\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)}$. $\gamma$ denotes Euler’s constant, and $\gamma_{-1}(K)$ and $\gamma_0(K)$ are defined in (2.1) and (2.2), respectively. Also, if $r_1 = 1$, then we have the following additional term on the right-hand side.

$$\frac{x^{2\rho}}{4 \Gamma(\rho + 1)}.$$

Proof. From (2.3), (2.4), and (2.5), we see that $\zeta_K(2s)\zeta(2s)$ satisfies the functional equation

$$\pi^{s(r_1+1)}|\Delta_K|^s \Gamma(s)^{r_1+1} \zeta_K(2s)\zeta(2s)$$

$$= \pi^{\left(\frac{s}{2}\right)(r_1+1)}|\Delta_K|^\frac{s}{2} \Gamma\left(\frac{1}{2} - s\right)^{r_1+1} \zeta_K(1 - 2s)\zeta(1 - 2s). \quad (3.8)$$
Similarly, we apply Theorems 2 and 4 in [1] with

\[ m = r_1 + 1, \quad r = \frac{1}{2}, \quad \lambda_n = \mu_n = \frac{\pi^{r_1+1}n^2}{|\Delta_K|}, \quad a_n = b_n = D_K(n). \]

Then, we have, for \( x > 0 \) and \( \rho > \frac{1}{2}r_1 - 1, \)
\[
\frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} D_K(n)(x - \lambda_n)^\rho
\]
\[ = 2^{-r_1\rho} \sum_{n=1}^{\infty} D_K(n) \left( \frac{x}{\mu_n} \right)^{\frac{1}{2} + \frac{\rho}{2}} K_{\frac{1}{2} + \rho} \left( 2^{r_1+1} \sqrt{\mu_n x}; -\frac{1}{2}; r_1 + 1 \right) + Q_{\rho}(x), \quad (3.9) \]

where

\[ Q_{\rho}(x) = \frac{1}{2\pi i} \int_{C_{\rho}} \frac{|\Delta_K|^s \Gamma(s) \zeta_K(2s) \zeta(2s)x^{s+\rho}}{\pi^{s(r_1+1)} \Gamma(s + \rho + 1)} \, ds, \]

and where \( C_{\rho} \) is a positively oriented, closed curve encircling the poles of the integrand. Note that each of \( \zeta_K(2s) \) and \( \zeta(2s) \) has a simple pole at \( s = \frac{1}{2}. \) If \( r_1 = 1, \) then the integrand also has a simple pole at \( s = 0 \) arising from \( \Gamma(s). \) From the following Laurent expansions

\[ \zeta(2s) = \frac{1}{s - \frac{1}{2}} + \gamma + \sum_{n=1}^{\infty} \frac{(-1)^n \gamma_n}{n!} 2^n (s - \frac{1}{2})^n, \quad (3.10) \]
\[ \zeta_K(2s) = \frac{1}{s - \frac{1}{2}} + \sum_{n=0}^{\infty} \gamma_n(K) 2^n (s - \frac{1}{2})^n, \quad (3.11) \]
\[ \Gamma(s) = \sqrt{\pi} + \Gamma'(\frac{1}{2}) (s - \frac{1}{2}) + \cdots, \quad (3.12) \]
\[ \frac{1}{\Gamma(s + \rho + 1)} = \frac{1}{\Gamma(\rho + \frac{3}{2})} \frac{\psi(\rho + \frac{3}{2})}{\Gamma(\rho + \frac{3}{2})} (s - \frac{1}{2}) + \cdots, \quad (3.13) \]

we derive

\[ Q_{\rho}(x) = \frac{\sqrt{|\Delta_K|} x^{\rho + \frac{1}{2}}}{2\Gamma(\rho + \frac{3}{2}) \pi^{\rho/2}} \left\{ \gamma_0(K) + \gamma_1(K) \gamma + \frac{\gamma_1(K) \Gamma'(\frac{1}{2})}{2\sqrt{\pi}} - \frac{1}{2} \gamma_1(K) \psi(\rho + \frac{3}{2}) + \frac{1}{2} \gamma_1(K) \log \left( \frac{|\Delta_K|^x}{\pi^{r_1+1}} \right) \right\}, \quad (3.14) \]

where if \( r_1 = 1, \) \( Q_{\rho}(x) \) has an additional term

\[ \frac{x^\rho}{4\Gamma(\rho + 1)}. \]

Now, replacing \( x \) by \( \frac{\pi^{r_1+1}x^2}{|\Delta_K|} \) in (3.9) and using (3.14) and (3.6), we obtain for \( r_1 \geq 2, \)
\[
\frac{1}{\Gamma(\rho + 1)} \sum_{n \leq x} D_K(n) (x^2 - n^2)^\rho
\]
Proof. It suffices to prove (3.16) for $\theta = h/q$, where $q$ is prime and $0 < h < q$. Using the identity [3] Lemma 2.5
\[
\cos \left( \frac{2\pi h a}{q} \right) = \frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(a) \chi(h) G(\chi),
\]
where $f_K(n)$ is defined in (1.9).
where $\phi(q)$ is the Euler’s $\phi$-function and $G(\chi)$ denotes the Gauss sum in (2.7), we deduce that

$$
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r \mid n} f_K \left( \frac{n}{r} \right) \cos \left( \frac{2\pi rh}{q} \right)
= \sum_{n \leq x} (x^2 - n^2)^\rho \left\{ \sum_{\substack{q \mid n \atop q \mid r}} f_K \left( \frac{n}{r} \right) + \sum_{\substack{r \mid n \atop q \nmid r}} f_K \left( \frac{n}{r} \right) \cos \left( \frac{2\pi rh}{q} \right) \right\}
= \sum_{n \leq x} (x^2 - n^2)^\rho \left\{ \sum_{\substack{q \mid n \atop q \mid r}} f_K \left( \frac{n}{r} \right) + \frac{1}{\phi(q)} \sum_{\substack{r \mid n \atop q \nmid r}} f_K \left( \frac{n}{r} \right) \sum_{\chi \text{ even}} \chi(h) \chi(r) G(\chi) \right\}
= \sum_{n \leq x/q} (x^2 - q^2 n^2)^\rho \sum_{d \mid n} f_K(d) + \frac{1}{\phi(q)} \sum_{\substack{r \mid n \atop q \nmid r}} \chi(h) G(\chi) \sum_{n \leq x} (x^2 - n^2)^\rho \sum_{\chi \text{ even}} \chi(r) f_K \left( \frac{n}{r} \right)
= q^{2\rho} \sum_{n \leq x/q} \left( \left( \frac{x}{q} \right)^2 - n^2 \right)^\rho D_K(n) + \frac{1}{\phi(q)} \sum_{\substack{r \mid n \atop q \nmid r}} \chi(h) G(\chi) \sum_{n \leq x} (x^2 - n^2)^\rho f_K \left( \frac{n}{r} \right) \chi(r)
= q^{2\rho} \sum_{n \leq x/q} \left( \left( \frac{x}{q} \right)^2 - n^2 \right)^\rho D_K(n) - \frac{1}{\phi(q)} \sum_{n \leq x} (x^2 - n^2)^\rho D_{K,\chi_0}(n)
+ \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \text{ even}} \chi(h) G(\chi) \sum_{n \leq x} (x^2 - n^2)^\rho D_{K,\chi}(n),
\quad (3.17)
$$

where $\chi_0$ denotes the principal character modulo $q$.

Note that

$$
\sum_{n \leq x} (x^2 - n^2)^\rho D_{K,\chi_0}(n)
= \sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r \mid n} f_K \left( \frac{n}{r} \right) \chi(r) \chi_0(r)
= \sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r \mid n \atop q \nmid r} f_K \left( \frac{n}{r} \right)
= \sum_{n \leq x} (x^2 - n^2)^\rho \left\{ \sum_{r \mid n \atop q \nmid r} f_K \left( \frac{n}{r} \right) - \sum_{\substack{q \mid n \atop q \nmid r}} f_K \left( \frac{n}{r} \right) \right\}
= \sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r \mid n} f_K \left( \frac{n}{r} \right) - \sum_{n \leq x/q} (x^2 - q^2 n^2)^\rho \sum_{d \mid n} f_K \left( \frac{n}{d} \right)
= \sum_{n \leq x} (x^2 - n^2)^\rho D_K(n) - q^{2\rho} \sum_{n \leq x/q} \left( \left( \frac{x}{q} \right)^2 - n^2 \right)^\rho D_K(n).
\quad (3.18)
$$
Thus, using (3.2), (3.7), and (3.18) in (3.17), we have

\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r|n} f_K \left( \frac{n}{r} \right) \cos \left( \frac{2\pi rh}{q} \right)
\]

\[
= \frac{q^{2\rho+1}}{\phi(q)} \sum_{n \leq x/q} \left( \left( \frac{x}{q} \right)^2 - n^2 \right)^\rho D_K(n) - \frac{1}{\phi(q)} \sum_{n \leq x} (x^2 - n^2)^\rho D_K(n) 
\]

\[
+ \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \chi(h)G(\chi) \sum_{n \leq x} (x^2 - n^2)^\rho D_K(\chi)(n) 
\]

\[
= \frac{q^{2\rho+1}\Gamma(\rho + 1)}{\phi(q)} \left\{ \frac{|\Delta_K|^{1/2}x^{2\rho}}{\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} \frac{D_K(n)}{n} \frac{x^{-r_1+1}}{(2r_1+1,0)} \left( \frac{|\Delta_K|^2 q^2}{\pi^{2(r_1+1)}(nx)^2} \right) 
\]

\[
+ \frac{\sqrt{\pi x^{1+2\rho}\gamma_1(K)}}{2q^{\rho+1}\Gamma(\rho + \frac{3}{2})} \left( \frac{1}{\psi_0(K)} + \frac{1}{2\sqrt{\pi}} - \frac{1}{2} \psi_0(K) + \frac{1}{2} \log (2\pi n^2) \right) \right\} 
\]

\[
- \frac{\Gamma(\rho + 1)}{\phi(q)} \left\{ \frac{|\Delta_K|^{1/2}x^{2\rho}}{\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} \frac{D_K(n)}{n} \frac{G^{0,r_1+1}}{(2r_1+1,0)} \left( \frac{|\Delta_K|^2 q^2}{\pi^{2(r_1+1)}(nx)^2} \right) 
\]

\[
+ \frac{\sqrt{\pi x^{1+2\rho}\gamma_1(K)}}{2\Gamma(\rho + \frac{3}{2})} \left( \frac{1}{\psi_0(K)} + \frac{1}{2\sqrt{\pi}} - \frac{1}{2} \psi_0(K) + \frac{1}{2} \log x \right) \right\} 
\]

\[
+ \frac{\Gamma(\rho + 1)}{\phi(q)} \sum_{\chi \neq \chi_0} \chi(h)G(\chi) 
\]

\[
\times \left\{ \frac{G(\chi)|\Delta_K|^{1/2}x^{2\rho}}{\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} \frac{D_K(\chi)(n)}{n} \frac{G^{0,r_1+1}}{(2r_1+1,0)} \left( \frac{|\Delta_K|^2 q^2}{\pi^{2(r_1+1)}(nx)^2} \right) \right. 
\]

\[
- G(\chi)\sqrt{\pi} \gamma_1(K)x^{1+2\rho} \sum_{n=1}^{q-1} T(n) \log (2\sin(\pi n/q)) \right\} 
\]

\[
= \frac{\Gamma(\rho + 1)|\Delta_K|^{1/2}x^{2\rho}q}{\phi(q)\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} \frac{D_K(n)}{n} \frac{G^{0,r_1+1}}{(2r_1+1,0)} \left( \frac{|\Delta_K|^2 q^2}{\pi^{2(r_1+1)}(nx)^2} \right) 
\]

\[
- \frac{\Gamma(\rho + 1)|\Delta_K|^{1/2}x^{2\rho}}{\phi(q)\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} \frac{D_K(n)}{n} \frac{G^{0,r_1+1}}{(2r_1+1,0)} \left( \frac{|\Delta_K|^2 q^2}{\pi^{2(r_1+1)}(nx)^2} \right) 
\]

\[
- \frac{\Gamma(\rho + 1)\sqrt{\pi} \gamma_1(K)x^{1+2\rho}}{2\phi(q)\Gamma(\rho + \frac{3}{2})} \log q 
\]

\[
+ \frac{\Gamma(\rho + 1)q}{\phi(q)} \sum_{\chi \neq \chi_0} \chi(h) 
\]

\[
\times \left\{ \frac{|\Delta_K|^{1/2}x^{2\rho}}{\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} \frac{D_K(\chi)(n)}{n} \frac{G^{0,r_1+1}}{(2r_1+1,0)} \left( \frac{|\Delta_K|^2 q^2}{\pi^{2(r_1+1)}(nx)^2} \right) \right. 
\]

\[
- G(\chi)\sqrt{\pi} \gamma_1(K)x^{1+2\rho} \sum_{n=1}^{q-1} T(n) \log (2\sin(\pi n/q)) \right\} 
\]
\[-\frac{\sqrt{\pi} \gamma_{-1}(K) x^{1+2\rho}}{2q \Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log (2 \sin (\pi n/q)) \] 
\[-\frac{\Gamma(\rho + 1) q |\Delta_K|^{1/2} x^{2\rho}}{\pi^{(r_1+1)/2}} \sum_{n=1}^{\infty} \frac{D_{K,\bar{\chi}}(n)}{n} G^{0, r_1+1}_{2(r_1+1), 0} \left( \frac{q^2 |\Delta_K|^2}{\pi^{2(r_1+1)} (nx)^2} \right) 
+ \frac{\Gamma(\rho + 1) \sqrt{\pi} \gamma_{-1}(K) x^{1+2\rho}}{2\Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \log (2 \sin (\pi n/q)). \tag{3.20} \]

Employing the identity,

\[D_{K,\bar{\chi}}(n) = D_K(n) - D_K\left(\frac{n}{q}\right),\]

the formula \[14\] p. 41, Formula 1.392, no. 1, 

\[\prod_{n=1}^{q-1} \sin \left(\frac{\pi n}{q}\right) = \frac{q}{2^{q-1}}, \tag{3.21} \]

and \[2\] equation (3.7)]

\[\sum_{\chi \text{ even}} \chi(a) \bar{\chi}(h) = \begin{cases} \frac{\phi(q)}{2} & \text{if, } h = \pm a \pmod{q}, \\ 0, & \text{otherwise,} \end{cases} \tag{3.22} \]

we find that

\[\sum_{n \leq x} \sum_{r|n} f_K\left(\frac{n}{r}\right) \cos \left(\frac{2\pi rh}{q}\right) \]
\[-= \frac{\Gamma(\rho + 1) q |\Delta_K|^{1/2} x^{2\rho}}{\phi(q) \pi^{(r_1+1)/2}} \sum_{\chi \text{ even}} \chi(h) \sum_{n=1}^{\infty} \frac{D_{K,\bar{\chi}}(n)}{n} G^{0, r_1+1}_{2(r_1+1), 0} \left( \frac{q^2 |\Delta_K|^2}{\pi^{2(r_1+1)} (nx)^2} \right) 
- \frac{\Gamma(\rho + 1) \sqrt{\pi} \gamma_{-1}(K) x^{1+2\rho}}{2\phi(q) \Gamma(\rho + \frac{3}{2})} \log q \]
\[-= \frac{\Gamma(\rho + 1) q |\Delta_K|^{1/2} x^{2\rho}}{\phi(q) \pi^{(r_1+1)/2}} \sum_{\chi \text{ even}} \chi(h) \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} f_K\left(\frac{n}{d}\right) \bar{\chi}(d) G^{0, r_1+1}_{2(r_1+1), 0} \left( \frac{q^2 |\Delta_K|^2}{\pi^{2(r_1+1)} (nx)^2} \right) 
- \frac{\Gamma(\rho + 1) \sqrt{\pi} \gamma_{-1}(K) x^{1+2\rho}}{2\phi(q) \Gamma(\rho + \frac{3}{2})} \left\{ \sum_{n=1}^{q-1} \log (2 \sin (\pi n/q)) \sum_{\chi \text{ even}} \chi(h) \bar{\chi}(n) \right\} \]
\[-= \frac{\Gamma(\rho + 1) q |\Delta_K|^{1/2} x^{2\rho}}{\phi(q) \pi^{(r_1+1)/2}} \sum_{m,d=1}^{\infty} \frac{f_K(m)}{md} \sum_{\chi \text{ even}} \chi(h) \bar{\chi}(d) G^{0, r_1+1}_{2(r_1+1), 0} \left( \frac{q^2 |\Delta_K|^2}{\pi^{2(r_1+1)} (mdx)^2} \right) \]
\[-\frac{\Gamma(\rho+1) \sqrt{\pi \gamma_{-1}(K)} x^{1+2\rho} \log (2 \sin h/q)}{2 \Gamma(\rho+\frac{3}{2})} \]

\[= \frac{\Gamma(\rho+1)|\Delta_K|^{1/2} x^{2\rho}}{2\pi^{(r_1+1)/2}} \sum_{m=1}^\infty \frac{f_K(m)}{m} \]

\[
\times \sum_{n=0}^\infty \left\{ \frac{G^{0,r_1+1}_{0,0}\left( \frac{|\Delta_K|^2}{(m(m+n+h/q)x)^2} \right)}{n+h/q} + \frac{G^{0,r_1+1}_{0,0}\left( \frac{|\Delta_K|^2}{(m(m+n-1+h/q)x)^2} \right)}{n+1-h/q} \right\} \]

\[-\frac{\Gamma(\rho+1) \sqrt{\pi \gamma_{-1}(K)} x^{1+2\rho} \log (2 \sin h/q)}{2 \Gamma(\rho+\frac{3}{2})} . \]

This completes the proof. \(\square\)

4. DIRICHLET’S THEOREM

We state a classical theorem of Dirichlet [11].

**Theorem 4.1.** Let \(\left( \frac{\cdot}{m} \right)\) denote the Kronecker symbol, where \(m\) is any positive integer [9, p. 39]. Let \(n\) be a positive integer coprime with the discriminant \(D\) of a positive-definite primitive integral binary quadratic form. Let \(R_D(n)\) denote the number of all representations of \(n\) by a representative set of positive-definite primitive integral binary quadratic forms of discriminant \(D\). Then

\[R_D(n) = w_D \sum_{k|n} \left( \frac{D}{k} \right), \quad (4.1)\]

where

\[w_D = \begin{cases} 
2, & \text{if } D < -4, \\
4, & \text{if } D = -4, \\
6, & \text{if } D = -3. 
\end{cases} \quad (4.2)\]

The following is Theorem 2.2.15 in [8].

**Theorem 4.2.** If \(D\) is a fundamental discriminant, the Kronecker symbol \(\left( \frac{D}{n} \right)\) defines a real primitive character modulo \(|D|\). Conversely, if \(\chi\) is a real primitive character modulo \(m\), then \(D = \chi(-1)m\) is a fundamental discriminant and \(\chi(n) = \left( \frac{D}{n} \right)\).

Thus, if \(D\) is a fundamental discriminant, we can rewrite \(R_D(n)\) as

\[R_D(n) = w_D \sum_{k|n} \chi_D(k) = w_D d_\chi(n), \]

where

\[d_\chi(n) := \sum_{k|n} \chi(k). \]

Note that, for \(\sigma > 1\),

\[\zeta(s)L(s, \chi_D) = \sum_{n=1}^\infty \frac{1}{n^s} \sum_{n=1}^\infty \frac{\chi_D(n)}{n^s} = \sum_{n=1}^\infty \frac{d_\chi_D(n)}{n^s}. \]
Define
\[ D_{D,\chi}(n) := \sum_{k|n} d_{\chi_D}(k)\chi(n/k), \quad \text{and} \quad D_D(n) := \sum_{k|n} d_{\chi_D}(k). \]
Also,
\[ D_{\overline{\chi},\chi}(n) := \sum_{k|n} d_{\overline{\chi}_D}(k)\overline{\chi}(n/k), \quad \text{and} \quad D_{\overline{\chi}}(n) := \sum_{k|n} d_{\overline{\chi}_D}(k). \]
Then,
\[ \zeta(s)L(s,\chi_D)L(s,\chi) = \sum_{n=1}^{\infty} \frac{d_{\chi_D}(n)}{n^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{D_{D,\chi}(n)}{n^s}, \]
and
\[ \zeta^2(s)L(s,\chi_D) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n=1}^{\infty} \frac{d_{\chi_D}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{D_D(n)}{n^s}. \]

5. THE CASE: $\chi_D$ EVEN

**Theorem 5.1.** Let $\chi_D$ be a nonprincipal even primitive character modulo $|D|$, where $D$ is a fundamental discriminant. Let $q$ be a positive integer, and let $\chi$ be a nonprincipal even primitive character modulo $q$. Then, for $x > 0$ and $\rho > 0$,
\[
\frac{1}{\Gamma(\rho + 1)} \sum_{n \leq x} D_{D,\chi}(n)(x^2 - n^2)\rho
= \frac{G(\chi)G(\chi_D)}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{D_{\overline{\chi},\chi}(n)}{n}G_{0,3}(\frac{D^2 q^2}{\pi^6(nx)^2})
- \frac{G(\chi)\sqrt{|D|L(1,\chi_D)}}{2q\Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \overline{\chi}(n) \log (2 \sin(\pi n/q)). \quad (5.1)
\]

**Proof.** By (2.5) and (2.6), we see that $\zeta(2s)L(2s,\chi_D)L(2s,\chi)$ satisfies the functional equation
\[
\pi^{-3s}(|D|q)^s\Gamma^3(s)\zeta(2s)L(2s,\chi_D)L(2s,\chi)
= \frac{G(\chi)G(\chi_D)}{\sqrt{|D|q}} \pi^{-3(1/2-s)}(|D|q)^{(1/2-s)} \Gamma^3(1/2-s)\zeta(1 - 2s)L(1 - 2s,\chi_D)L(1 - 2s,\chi).
\]
We apply Theorems 2 and 4 in [1] with
\[ m = 3, \quad r = \frac{1}{2}, \quad \lambda_n = \mu_n = \frac{n^2 \pi^{3}}{|D|q}, \quad a_n = D_{D,\chi}(n) \quad \text{and} \quad b_n = \frac{G(\chi_D)G(\chi)}{\sqrt{|D|q}} D_{\overline{\chi},\chi}(n). \]
Then, we obtain, for $x > 0$ and $\rho > 0$,
\[
\frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} D_{D,\chi}(n)(x - \lambda_n)^\rho
\]
where
\[ Q_\rho(x) = \frac{1}{2\pi i} \int_{C_\rho} (|D|q)^s \frac{\Gamma(s)\zeta(2s)L(2s,\chi_D)L(2s,\chi)x^{s+\rho}}{\pi^{3s}\Gamma(s+\rho+1)} ds, \]
where \( C_\rho \) is a positively oriented closed curve encircling the poles of the integrand.

Theorem 5.2. Let \( \chi_D \) be a nonprincipal even primitive character modulo \( |D| \), where \( D \) is a fundamental discriminant. For \( x > 0 \) and \( \rho > 0 \),
\[
\frac{1}{\Gamma(\rho+1)} \sum_{n \leq x} \mathcal{D}_{D,\chi}(n)(x^2 - n^2)^\rho
= \frac{G(\chi_D)x^{2\rho}}{\pi^{3/2}} \sum_{n=1}^\infty \mathcal{D}_{D,\chi}(n) \left( \frac{D^2q^2}{\pi^6(nx)^2} \right) G_{6,0}^0 \left( \frac{D^2q^2}{\pi^6(nx)^2} \right)
+ \frac{\sqrt{\pi}x^{1+2\rho}L(1,\chi_D)}{4\Gamma(\rho+\frac{3}{2})} \left\{ \frac{2L'(1,\chi_D)}{L(1,\chi_D)} + \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} - \psi(\rho+\frac{3}{2}) + 2 \log x + 4\gamma \right\},
\]
where \( \psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} \).

Proof. Similarly, from (2.5) and (2.6), it is easy to see that \( \zeta^2(2s)L(2s,\chi_D) \) satisfies the functional equation
\[
\pi^{-3s}|D|^s \Gamma^3(s)\zeta^2(2s)L(2s,\chi_D) = \pi^{-3(\frac{1}{2}-s)}|D|^{\frac{1}{2}-s} \Gamma^3(s)\zeta^2(1-2s)L(1-2s,\chi_D).\]
Using Theorems 2 and 4 in [1] with
\[ m = 3, \quad r = \frac{1}{2}, \quad \lambda_n = \mu_n = \frac{n^2 \pi^3}{|D|}, \quad a_n = D_D(n), \quad \text{and} \quad b_n = \frac{G(\chi_D)}{\sqrt{|D|}} D_D(n), \]
we find that, for \( x > 0 \) and \( \rho > 0 \),
\[ \frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} D_D(n)(x - \lambda_n)^\rho \]
\[ = \frac{G(\chi_D)}{2^{2\rho} \sqrt{|D|}} \sum_{n=1}^{\infty} D_D(n) \left( \frac{x}{\mu_n} \right)^{\frac{1}{2} + \frac{\rho}{2}} K_{\frac{1}{2} + \rho} (2^3 \sqrt{\mu_n x}; -\frac{1}{2}; 3) + Q_\rho(x), \quad (5.5) \]
where
\[ Q_\rho(x) = \frac{1}{2\pi i} \int_{C_\rho} \frac{|D|^s \Gamma(s) \zeta^2(2s) L(2s, \chi_D)x^{s+\rho}}{\pi^{3s} \Gamma(s + \rho + 1)} \, ds. \]
where \( C_\rho \) is a closed curve containing all of the integrand’s singularities in its interior. Note that \( Q_\rho(x) \) has only a double pole at \( s = \frac{1}{2} \).

From (3.10), we have
\[ \zeta^2(2s) = \frac{1}{(s - \frac{1}{2})^2} + \frac{\gamma}{s - \frac{1}{2}} + \cdots. \quad (5.6) \]

Also, we consider
\[ \left( \frac{|D|^x}{\pi^3} \right)^s = \left( \frac{|D|^x}{\pi^3} \right)^{1/2} + \left( \frac{|D|^x}{\pi^3} \right)^{1/2} \log \left( \frac{|D|^x}{\pi^3} \right) (s - \frac{1}{2}) + \cdots, \quad (5.7) \]
\[ L(2s, \chi_D) = L(1, \chi_D) + 2L'(1, \chi_D)(s - \frac{1}{2}) + \cdots. \quad (5.8) \]

Using the Laurent expansions (3.12), (3.13), (5.6), (5.7) and (5.8), we can evaluate \( Q_\rho(x) \) as follows:
\[ Q_\rho(x) = \frac{L(1, \chi_D) \sqrt{|D|^x} x^{\rho + \frac{1}{2}}}{4\Gamma(\rho + \frac{3}{2})} \left( \frac{2L'(1, \chi_D)}{L(1, \chi_D)} + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} - \psi(\rho + \frac{3}{2}) + \log \left( \frac{|D|^x}{\pi^3} \right) + 4\gamma \right). \quad (5.9) \]

We replace \( x \) by \( \frac{n^2 \pi^2}{|D|} \) in (5.5) and use (5.9) and (3.6) to deduce that
\[ \frac{1}{\Gamma(\rho + 1)} \sum_{n \leq x} D_D(n)(x^2 - n^2)^\rho \]
\[ = \frac{G(\chi_D)x^{2\rho}}{\pi^{3/2}} \sum_{n=1}^{\infty} D_D(n) G_{6,0}^0 \left( \frac{D^2}{\pi^6 (nx)^2} \right) \]
\[ + \frac{\sqrt{\pi} x^{1+2\rho} L(1, \chi_D)}{4\Gamma(\rho + \frac{3}{2})} \left\{ \frac{2L'(1, \chi_D)}{L(1, \chi_D)} + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} - \psi(\rho + \frac{3}{2}) + 2 \log x + 4\gamma \right\}. \]

This completes our proof. \( \square \)
Theorem 5.3. Let \( \chi_D \) be a nonprincipal even primitive character modulo \( |D| \), where \( D \) is a fundamental discriminant. For \( x > 0, 0 < \theta < 1 \) and \( \rho > 0 \),

\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r|n} d_{\chi_D}(\frac{n}{r}) \cos(2\pi r \theta) = \frac{\Gamma(\rho + 1) G(\chi_D) x^{2\rho}}{2\pi^{3/2}} \sum_{m=1}^{\infty} \frac{d_{\chi_D}(m)}{m} \sum_{n=0}^{\infty} \left\{ \frac{G_{0.3}^0 \left( \frac{D^2}{\pi^6 (n+\theta x)^2} \right)}{n + \theta} + \frac{G_{0.3}^0 \left( \frac{D^2}{\pi^6 (n+1-\theta x)^2} \right)}{n + 1 - \theta} \right\} - \frac{\Gamma(\rho + 1) \sqrt{\pi} L(1, \chi_D) x^{1+2\rho}}{2\Gamma(\rho + \frac{3}{2})} \log (2 \sin(\pi \theta)). \tag{5.10}
\]

Proof. Similarly to the proof of \((5.16)\), we prove \((5.10)\) for \( \theta = h/r \), where \( q \) is prime and \( 0 < h < q \). With the same argument, we can derive the following identity, analogous to \((3.19)\):

\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r|n} d_{\chi_D}(\frac{n}{r}) \cos \left( \frac{2\pi r h}{q} \right) = \frac{q^{2\rho+1}}{\phi(q)} \sum_{n \leq x/q} \left( \left( \frac{n}{q} \right)^2 - n^2 \right)^\rho \mathcal{D}_D(n) - \frac{1}{\phi(q)} \sum_{n \leq x} (x^2 - n^2)^\rho \mathcal{D}_D(n) + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \text{ even}} \chi(h) G(\chi) \sum_{n \leq x} (x^2 - n^2)^\rho \mathcal{D}_{D,\chi}(n). \tag{5.11}
\]

We use the identities \((5.1)\) and \((5.4)\) in \((5.11)\) to deduce that

\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r|n} d_{\chi_D}(\frac{n}{r}) \cos \left( \frac{2\pi r h}{q} \right) = \frac{\Gamma(\rho + 1) q G(\chi_D) x^{2\rho}}{\phi(q) \pi^{3/2}} \sum_{n=1}^{\infty} \frac{\mathcal{D}_{\pi}(n)}{n} G_{0.3}^0 \left( \frac{(D q)^2}{\pi^6 (n x)^2} \right) + \frac{\Gamma(\rho + 1) \sqrt{\pi} x^{1+2\rho} L(1, \chi_D)}{4\phi(q) \Gamma(\rho + \frac{1}{2})} \left\{ \frac{2L'(1, \chi_D)}{L(1, \chi_D)} + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} - \psi(\rho + \frac{3}{2}) + 2 \log \left( \frac{4}{q} \right) + 4\gamma \right\} + \frac{\Gamma(\rho + 1) G(\chi_D) x^{2\rho}}{4\phi(q) \Gamma(\rho + \frac{1}{2})} \sum_{n=1}^{\infty} \frac{\mathcal{D}_D(n)}{n} G_{0.3}^0 \left( \frac{D^2}{\pi^6 (n x)^2} \right) + \frac{\Gamma(\rho + 1) \sqrt{\pi} x^{1+2\rho} L(1, \chi_D)}{4\phi(q) \Gamma(\rho + \frac{1}{2})} \left\{ \frac{2L'(1, \chi_D)}{L(1, \chi_D)} + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} - \psi(\rho + \frac{3}{2}) + 2 \log x + 4\gamma \right\} + \frac{\Gamma(\rho + 1)}{\phi(q) \chi \neq \chi_0 \text{ even}} \sum_{\chi \neq \chi_0 \text{ even}} \chi(h) G(\chi) \left\{ \frac{G(\chi) G(\chi_D) x^{2\rho}}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{\mathcal{D}_{\pi}(n)}{n} G_{0.3}^0 \left( \frac{D^2 q^2}{\pi^6 (n x)^2} \right) - \frac{G(\chi) \sqrt{\pi} L(1, \chi_D) x^{1+2\rho}}{2q \Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log (2 \sin(\pi n/q)) \right\}.
\]
This completes the proof of (5.10). □

Using (3.21) and the easy identity

\[ \frac{\sqrt{\pi} L(1, \chi_D) x^{1+2\rho}}{2\Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log (2 \sin(\pi n/q)) \]

we can rewrite (5.12) as

\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r \mid n} d_{\chi_D}(\frac{n}{r}) \cos (\frac{2\pi rh}{q})
\]

\[
= \frac{\Gamma(\rho + 1)}{\phi(q)\pi^{3/2}} \sum_{\chi \text{ even}} \chi(h) \left\{ \frac{qG(\chi_D)x^{2\rho}}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{D\chi(n)}{n} G_{6,0}^{0,3} \left( \frac{D^2q^2}{\pi^6(nx)^2} \right) \right. \\
- \frac{\sqrt{\pi} L(1, \chi_D) x^{1+2\rho}}{2\Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \chi(n) \log (2 \sin(\pi n/q)) \left. \right\}
\]

\[
= \frac{\Gamma(\rho + 1)}{\phi(q)\pi^{3/2}} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k \mid n} d_{\chi_D}(\frac{n}{k}) \sum_{\chi \text{ even}} \chi(h) \overline{\chi}(k) G_{6,0}^{0,3} \left( \frac{D^2q^2}{\pi^6(nx)^2} \right) \\
- \frac{\Gamma(\rho + 1)\sqrt{\pi} L(1, \chi_D) x^{1+2\rho}}{2\phi(q)\Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \log (2 \sin(\pi n/q)) \sum_{\chi \text{ even}} \chi(h) \overline{\chi}(n)
\]

\[
= \frac{\Gamma(\rho + 1)G(\chi_D)x^{2\rho}}{2\pi^{3/2}} \sum_{m=1}^{\infty} d_{\chi_D}(m) \sum_{m=0}^{\infty} \left\{ G_{6,0}^{0,3} \left( \frac{D^2}{\pi^6(mnhx)^2} \right) \frac{D^2}{n + h/q} + G_{6,0}^{0,3} \left( \frac{D^2}{\pi^6(mn+1-h/q)x^2} \right) \frac{D^2}{n + 1 - h/q} \right\} \\
- \frac{\Gamma(\rho + 1)\sqrt{\pi} L(1, \chi_D) x^{1+2\rho}}{2\Gamma(\rho + \frac{3}{2})} \log (2 \sin(\pi h/q)).
\]

This completes the proof of (5.10). □

Using (4.1), we derive the following corollary.

**Corollary 5.4.** Let \( x > 0, 0 < \theta < 1, \) and \( \rho > 0. \) If \( D \) is a fundamental discriminant and \( \chi_D \) is a nonprincipal even primitive character, then

\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r \mid n} R_D \left( \frac{n}{r} \right) \cos(2\pi r \theta)
\]
\[
\frac{\Gamma(\rho + 1)G(\chi_D)x^{2\rho}}{2\pi^{3/2}} \sum_{m=1}^{\infty} \frac{R_D(m) \sum_{n=0}^{\infty} \left\{ G_{6,0}^0 \left( \frac{D^2}{\pi^2(m+\theta)x^2} \right) \right\}}{n + \theta} + \frac{G_{6,0}^0 \left( \frac{D^2}{\pi^2(m+1-\theta)x^2} \right)}{n + 1 - \theta} \\
- \frac{\Gamma(\rho + 1)w_D\sqrt{\pi}L(1, \chi_D)x^{1+2\rho}}{2\Gamma(\rho + \frac{3}{2})} \log (2\sin(\pi\theta)).
\]

6. Big-O Results

We employ a theorem of Chandrasekharan and Narasimhan [7]. We first provide the general setting [7, p. 95–98].

**Definition 6.1.** Let \(a(n)\) and \(b(n)\) be two sequences of complex numbers, where not all terms are equal to 0 in either sequence. Let \(\lambda_n\) and \(\mu_n\) be two sequences of positive numbers, strictly increasing to \(\infty\). Let \(\delta > 0\). Throughout, \(s = \sigma + it\), where \(\sigma\) and \(t\) are both real. For \(N \geq 1\), let

\[
\Delta(s) := \prod_{n=1}^{N} \Gamma(\alpha_n s + \beta_n),
\]

where, for \(1 \leq n \leq N\), \(\beta_n\) is complex and \(\alpha_n > 0\). Assume that

\[
A := \sum_{n=1}^{N} \alpha_n \geq 1.
\]

Let

\[
\varphi(s) := \sum_{n=1}^{\infty} \frac{a(n)}{\lambda_n^s} \quad \text{and} \quad \psi(s) := \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s}
\]

converge absolutely in some half-plane, and suppose they satisfy the functional equation

\[
\Delta(s)\varphi(s) = \Delta(\delta - s)\psi(\delta - s).
\]

Furthermore, assume that there exists in the \(s\)-plane a domain \(\mathcal{D}\), which is the exterior of a compact set \(S\), in which there exists an analytic function \(\chi\) with the properties

\[
\lim_{|t| \to \infty} \chi(s) = 0,
\]

uniformly in every interval \(-\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty\), and

\[
\chi(s) = \Delta(s)\varphi(s), \quad \sigma > \alpha,
\]

\[
\chi(s) = \Delta(\delta - s)\psi(\delta - s), \quad \sigma < \beta,
\]

where \(\alpha\) and \(\beta\) are particular constants.

For \(\rho \geq 0\), let

\[
A_\rho(x) := \frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} a(n)(x - \lambda_n)^\rho,
\]

where the prime \(\prime\) indicates that if \(x = \lambda_n\) and \(\rho = 0\), the last term is to be multiplied by \(\frac{1}{2}\). Furthermore, let

\[
Q_\rho(x) := \frac{1}{2\pi i} \int_{C_\rho} \frac{\Gamma(s)\varphi(s)}{\Gamma(s + \rho + 1)} x^{s+\rho} ds,
\]
where \( C_\rho \) is a closed curve enclosing all of the singularities of the integrand to the right of \( \sigma = -\rho - 1 - k \), where \( k \) is chosen such that \( k > |\delta/2 - 1/(4A)| \), and all of the singularities of \( \varphi(s) \) lie in \( \sigma > -k \).

**Theorem 6.2.** [7, p. 98–99, Theorem 3.2] Suppose that the functional equation
\[
\Delta(s)\varphi(s) = \Delta(\delta - s)\psi(\delta - s)
\]
is satisfied. If \( \rho \geq 2A\beta - A\delta - \frac{1}{2} \), where \( \beta \) is such that \( \sum_{n=1}^{\infty} |b_n| \mu_n^{-\beta} < \infty \), then
\[
A_\rho(x) - Q_\rho(x) = O(x^\theta),
\]
where
\[
\theta = \frac{A\delta + \rho(2A - 1) - 1/2}{2A}.
\]

**Theorem 6.3.** Let \( q \) be prime and \( 0 < h < q \). For \( \rho > r_1/2 \),
\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r | n} f_K \left( \frac{n}{r} \right) \cos \left( \frac{2\pi rh}{q} \right)
= -\frac{\sqrt{\pi}\Gamma(\rho + 1)}{2\Gamma(\rho + \frac{3}{2})} \log (2\sin(\pi h/q))x^{1 + 2\rho} + O \left( x^{2\rho - \frac{3}{2} - \frac{2\rho + 1}{2(r_1 + 1)}} \right).
\]

**Proof.** For the convenience of readers, we recall (3.19):
\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r | n} f_K \left( \frac{n}{r} \right) \cos \left( \frac{2\pi rh}{q} \right)
= \frac{q^{\rho + 1}}{\phi(q)} \sum_{n \leq x/q} \left( \frac{x}{q} \right)^2 - n^2 \right)^\rho D_K(n) - \frac{1}{\phi(q)} \sum_{n \leq x} (x^2 - n^2)^\rho D_K(n)
+ \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \text{ even}} \chi(h) G(\chi) \sum_{n \leq x} (x^2 - n^2)^\rho D_{K,\chi}(n).
\]

We apply Theorem 6.2 to estimate the sums
\[
\sum_{n \leq x} (x^2 - n^2)^\rho D_K(n) \quad \text{and} \quad \sum_{n \leq x} (x^2 - n^2)^\rho D_{K,\chi}(n).
\]

We begin with the first sum in (6.6). From (3.8), (6.1) and (6.2), we have \( A = r_1 + 1 \) and \( \delta = \frac{1}{2} \). So, by (6.4),
\[
\theta = \rho + \frac{1}{4} - \frac{2\rho + 1}{4(r_1 + 1)}.
\]

Also, note that \( \beta = \frac{1}{2} + \epsilon \), where \( \epsilon > 0 \). Thus, by (6.3), we see that, for \( \rho > r_1/2 \),
\[
\frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} (x - \lambda_n)^\rho D_K(n) = Q_\rho(x) + O(x^\theta),
\]
where \( Q_\rho(x) \) is given in (3.14). Replacing \( x \) by \( \frac{x^{1+2\rho}}{|\Delta_K|} \), as in (3.15), we have
\[
\frac{1}{\Gamma(\rho + 1)} \sum_{n \leq x} (x^2 - n^2)^\rho D_K(n) = \frac{\sqrt{\pi}x^{1+2\rho}}{2\Gamma(\rho + \frac{3}{2})} \left\{ \gamma_0(K) + \gamma_1(K) + \gamma_{-1}(K) \Gamma(\frac{1}{2}) \right\}.
\]
- \frac{1}{2} \gamma_1(K) \psi(\rho + \frac{3}{2}) + \gamma_1(K) \log x \right) + O(x^{2\theta}).
\end{equation}

(6.8)

We estimate the second sum in (6.6) in a similar fashion. From (3.1), (6.1), (6.2) and (6.4), we see that the values of \(A, \delta \) and \(\theta \) are the same as above. Applying (6.3) yields for \(\rho > r_1/2\),

\[
\frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} (x - \lambda_n)^\rho D_{K, \lambda}(n) = Q_\rho(x) + O(x^\theta),
\]

where \(Q_\rho(x)\) is evaluated in (3.5). We replace \(x\) by \(\frac{\pi r_1 + 1}{q |\Delta_K|} x \) to obtain

\[
\frac{1}{\Gamma(\rho + 1)} \sum_{n \leq x} (x^2 - n^2)^\rho D_{K, \lambda}(n)
= - \frac{G(\chi) \sqrt{\pi} \gamma_1(K) x^{1+2\rho}}{2q \Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \overline{\chi}(n) \log (2 \sin(\pi n/q)) + O(x^{2\theta}).
\]  

(6.9)

Now, substituting (6.8) and (6.9) into (6.5), we deduce that

\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r|n} f_K \left( \frac{n}{r} \right) \cos \left( \frac{2\pi rh}{q} \right)
= - \frac{\sqrt{\pi} \Gamma(\rho + 1) \gamma_1(K) \log q}{2 \phi(q) \Gamma(\rho + \frac{3}{2})} x^{1+2\rho} + O(x^{2\theta})
- \frac{\sqrt{\pi} \Gamma(\rho + 1) \gamma_1(K)}{2 \phi(q) \Gamma(\rho + \frac{3}{2})} x^{1+2\rho} \sum_{\chi \neq \chi_0 \text{ even}} \frac{1}{\chi} \sum_{n=1}^{q-1} \overline{\chi}(n) \log (2 \sin(\pi n/q))
- \frac{\sqrt{\pi} \Gamma(\rho + 1) \gamma_1(K)}{2 \phi(q) \Gamma(\rho + \frac{3}{2})} x^{1+2\rho} \sum_{\chi \text{ even}} \frac{1}{\chi} \sum_{n=1}^{q-1} \overline{\chi}(n) \log (2 \sin(\pi n/q)) + O(x^{2\theta})
- \frac{\sqrt{\pi} \Gamma(\rho + 1) \gamma_1(K)}{2 \Gamma(\rho + \frac{3}{2})} x^{1+2\rho} \log (2 \sin(\pi h/q)) + O(x^{2\theta}),
\]  

(6.10)

where we employed (3.21) and (3.22). Using (6.7) in (6.10), we complete our proof. \(\square\)

**Theorem 6.4.** Let \(q\) be prime and \(0 < h < q\). For \(\rho > 1\),

\[
\sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r|n} d_{\chi_D} \left( \frac{n}{r} \right) \cos \left( \frac{2\pi rh}{q} \right)
= - \frac{\sqrt{\pi} \Gamma(\rho + 1) L(1, \chi_D)}{2 \Gamma(\rho + \frac{3}{2})} \log (2 \sin(\pi h/q)) x^{1+2\rho} + O(x^{(5\rho+1)/3}).
\]
Proof. For convenience, we record (5.11):
\[
\sum_{n \leq x} \left( x^2 - n^2 \right)^\rho \sum_{r | n} d_{\chi_D} \left( \frac{n}{r} \right) \cos \left( \frac{2\pi r h}{q} \right) = \frac{q^{2\rho+1}}{\phi(q)} \sum_{n \leq x/q} \left( \left( \frac{x}{q} \right)^2 - n^2 \right)^\rho D_D(n) - \frac{1}{\phi(q)} \sum_{n \leq x} (x^2 - n^2)^\rho D_D(n) + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \text{ even}} \chi(h)G(\chi) \sum_{n \leq x} (x^2 - n^2)^\rho D_{D,\chi}(n). \tag{6.11}
\]

Next, apply Theorem 6.2. Observe that \( A = 3, \delta = \frac{1}{2}, \beta = \frac{1}{2} + \epsilon, \) for \( \epsilon > 0, \) and \( \rho > 1. \) Referring to Theorem 5.2 and its proof, we replace \( x \) by \( \frac{\pi x^2}{|\Delta_R|}. \) Thus, we can derive, for \( \rho > 1, \)
\[
\frac{1}{\Gamma(\rho+1)} \sum_{n \leq x} D_D(n)(x^2 - n^2)^\rho = \frac{\sqrt{\pi}L(1, \chi_D)}{4\Gamma(\rho + \frac{3}{2})} x^{1+2\rho} \left\{ \frac{2L'(1, \chi_D)}{L(1, \chi_D)} + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} - \psi(\rho + \frac{3}{2}) + 2 \log x + 4\gamma \right\} + O(x^{(5\rho+1)/3}), \tag{6.12}
\]
where we recall that \( \psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}. \)

Next, refer to Theorem 5.1 and its proof, in particular, (5.3), in which we replaced \( x \) by \( \frac{\pi x^2}{|\Delta_R|}. \) Thus, for \( \rho > 1, \)
\[
\frac{1}{\Gamma(\rho+1)} \sum_{n \leq x} D_{D,\chi}(n)(x^2 - n^2)^\rho = -\frac{G(\chi)\sqrt{\pi}L(1, \chi_D)}{2q\Gamma(\rho + \frac{3}{2})} x^{1+2\rho} \sum_{n=1}^{q-1} \chi(n) \log (2 \sin (\pi n/q)) + O(x^{(5\rho+1)/3}). \tag{6.13}
\]
Now, we substitute (6.12) and (6.13) into (6.11) to find that
\[
\frac{1}{\Gamma(\rho+1)} \sum_{n \leq x} (x^2 - n^2)^\rho \sum_{r | n} d_{\chi_D} \left( \frac{n}{r} \right) \cos \left( \frac{2\pi r h}{q} \right) = -\frac{\sqrt{\pi}L(1, \chi_D) \log q}{2\phi(q)\Gamma(\rho + \frac{3}{2})} x^{1+2\rho} + O(x^{(5\rho+1)/3})
- \frac{\sqrt{\pi}L(1, \chi_D) x^{1+2\rho}}{2\phi(q)\Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \log (2 \sin (\pi n/q)) \sum_{\chi \neq \chi_0 \text{ even}} \chi(h) \overline{\chi}(n)
- \frac{\sqrt{\pi}L(1, \chi_D) x^{1+2\rho}}{2\phi(q)\Gamma(\rho + \frac{3}{2})} \sum_{n=1}^{q-1} \log (2 \sin (\pi n/q)) \sum_{\chi \text{ even}} \chi(h) \overline{\chi}(n) + O(x^{(5\rho+1)/3})
- \frac{\sqrt{\pi}L(1, \chi_D) x^{1+2\rho}}{2\Gamma(\rho + \frac{3}{2})} \log (2 \sin (\pi h/q)) + O(x^{(5\rho+1)/3}),
\]
where we employed (3.21) and (3.22). This completes the proof of the theorem. □

7. The Meijer $G$-Function

Recall the definition of the Meijer $G$-function in Definition 2.1. Also, recall that to complete the proof of Theorem 3.1, we need to prove the following theorem.

**Theorem 7.1.** If $K_{\rho+1/2}(x; -\frac{1}{2}; r_1 + 1)$ is defined by (3.6), then

$$K_{\rho+1/2}(x; -\frac{1}{2}; r_1 + 1) = 2^{-\rho+(r_1+1)/2}x^{\rho-1/2}G^{0, r_1+1}_{2(r_1+1), 0} \left( \frac{4^{r_1+1}}{\sqrt{\pi}} \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \rho + 1, 0, \ldots, 0 \end{array} \right. \right),$$

where there are $r_1 \{0\}$'s and $(r_1 + 1) \{\frac{1}{2}\}$'s.

**Proof.** For the remainder of the proof, we set

$$a_j = \frac{x}{u_{j+1}u_{j+2} \cdots u_{r_1}}, \quad 1 \leq j \leq r_1 - 1, \quad a_{r_1} = x. \quad (7.1)$$

Employing (7.1), and replacing $u_1$ by $u$, we see that the inner integral is

$$f_1(\rho; a_1) := \sqrt{\frac{2}{\pi}} \int_0^{\infty} u^{\rho-1/2} \cos u J_{\rho+1/2}(a_1/u)du. \quad (7.2)$$

We use a representation for the ordinary Bessel function $J_{\nu}(x)$ in terms of a $G$-function [12, p. 380], namely,

$$x^\mu J_{\nu}(x) = 2^\mu G^{1,0}_{0, 2} \left( \frac{x^2}{4} \left| \begin{array}{c} \mu + \nu, \mu - \nu \end{array} \right. \right) \cdot (7.3)$$

Applying (7.3) with $\mu = -\rho + \frac{1}{2}, \nu = \rho + \frac{1}{2}$, and $x = a_1/u$, we find that

$$\left( \frac{a_1}{u} \right)^{-\rho+1/2} J_{\rho+1/2} \left( \frac{a_1}{u} \right) = 2^{-\rho+1/2}G^{1,0}_{0, 2} \left( \frac{a_1^2}{4u^2} \left| \begin{array}{c} 1/2, -\rho \end{array} \right. \right), \quad (7.4)$$

or

$$u^{\rho-1/2} J_{\rho+1/2} \left( \frac{a_1}{u} \right) = \left( \frac{a_1}{2} \right)^{\rho-1/2} G^{1,0}_{0, 2} \left( \frac{a_1^2}{4u^2} \left| \begin{array}{c} 1/2, -\rho \end{array} \right. \right) \quad (7.5)$$

Note that in (7.4), we employed the definition of $G$ in (2.8) to switch the roles of the parameters $\frac{1}{2}$ and $-\rho$ in (7.4) and to obtain an alternative set of parameters in (7.5). Hence, by (7.2) and (7.5),

$$f_1(\rho; a_1) = \left( \frac{a_1}{2} \right)^{\rho-1/2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos u G^{0,1}_{2, 0} \left( \frac{4u^2}{a_1^2} \left| \begin{array}{c} 1/2, \rho + 1 \end{array} \right. \right) du. \quad (7.6)$$
We now state the general cosine transform of $G$ \cite[p. 420, Formula (8)]{13}. If $a > 0$ and $c \neq 0$,

$$\int_0^\infty \cos(cx)G_{p,q}^{m,n}\left(a x^2 \left| \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \right. \right) dx = \frac{\sqrt{\pi}}{c} G_{p+2,q}^{m,n+1}\left(\frac{4a}{c^2} \left| \begin{array}{c} \frac{1}{2}, a_1, a_2, \ldots, a_p, 0 \\ b_1, b_2, \ldots, b_q \end{array} \right. \right),$$

(7.7)

provided that $p + q \leq 2(m + n); 0 \leq m + n - \frac{1}{2}p - \frac{1}{2}q; c > 0, \Re a_j \leq \frac{1}{2}, j = 1, 2, \ldots, n; \text{ and } \Re b_j \geq -\frac{1}{2}, j = 1, 2, \ldots, m$. Utilizing (7.7) in (7.6), we find that

$$f_1(\rho; a_1) = 2^{1/2} \left(\frac{a_1}{2}\right)^{p-1/2} G_{4,0}^{0,2} \left(\frac{16}{a_1^2} \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \rho + 1, 0 \\ - \end{array} \right. \right)$$

$$= 2^{-\rho+1} x^{\rho-1/2} \frac{u_j}{(u_2 \cdots u_r)} \rho - 1/2 G_{4,0}^{0,2} \left(\frac{16 u_j^2}{a_1^2} \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \rho + 1, 0 \\ - \end{array} \right. \right),$$

(7.9)

where in the last step we used (7.1).

Next, we put (7.9) in (3.6) and repeat the process. Note that with each application we use (7.1), and increase the number of $\frac{1}{2}$’s and the number of 0’s, each by 1. Hence,

$$K_{\rho+1/2}(x; -\frac{1}{2}; r_1 + 1)$$

$$= 2^{-\rho+1} x^{\rho-1/2} \int_0^\infty \sqrt{u_{j1}} J_{\rho - 1/2}(u_{j1}) du_{j1} \cdots \int_0^\infty \sqrt{u_{j2}} J_{\rho - 1/2}(u_{j2}) du_{j2}$$

$$\times G_{4,0}^{0,2} \left(\frac{4^{u_j^2}/a_1^2}{2} \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \rho + 1, 0 \\ - \end{array} \right. \right)$$

$$= 2^{-\rho+3/2} x^{\rho-1/2} \int_0^\infty \sqrt{u_{j1}} J_{\rho - 1/2}(u_{j1}) du_{j1} \cdots \int_0^\infty \sqrt{u_{j2}} J_{\rho - 1/2}(u_{j2}) du_{j2}$$

$$\times G_{6,0}^{0,3} \left(\frac{4^{u_j^2}/a_1^2}{2} \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \rho + 1, 0, 0 \\ - \end{array} \right. \right)$$

$$= \cdots$$

$$= 2^{-\rho+(r_1+1)/2} x^{\rho-1/2} G_{2(r_1+1),0}^{0,r_1+1} \left(\frac{4^{r_1+1}/x^2}{2} \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \rho + 1, 0, 0, \ldots, 0 \\ - \end{array} \right. \right),$$

where there are $r_1 \{0\}$’s and $(r_1 + 1) \{\frac{1}{2}\}$’s. In the last step, the integral evaluation is given by

$$\int_0^\infty \sqrt{u} J_{\rho - 1/2}(u) G_{2n,0}^{0,n} \left(\frac{u^2}{a_1^2} \left| \begin{array}{c} \frac{1}{2}, \rho + 1, \{0\} \\ - \end{array} \right. \right) du = \sqrt{2} G_{2n+2,0}^{0,n+1} \left(\frac{4^{r_1+1}/x^2}{2} \left| \begin{array}{c} \{\frac{1}{2}\}, \frac{1}{2}, \rho + 1, \{0\}, 0 \\ - \end{array} \right. \right),$$

where there are $r_1 \{0\}$’s and $(r_1 + 1) \{\frac{1}{2}\}$’s. This completes the proof of (7.1).

□
8. Special Case

We consider the special case of (3.4) when \( \rho = 0 \) and \( r_1 = 1 \). Recall that [23, pp. 54–55]

\[
J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z \quad \text{and} \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z.
\]

Thus, (3.4) reduces to, with \( x = y^2 \) [3, p. 74, Equation (3.5)],

\[
K_{1/2}(x; -\frac{1}{2}; 2) = \int_0^\infty J_{-1/2}(u)J_{1/2}\left(\frac{x}{u}\right) du
\]

\[
= \left(\frac{2}{\pi y}\right) \int_0^\infty \cos u \sin \left(\frac{y^2}{u}\right) du
\]

\[
= \left(\frac{2}{\pi y}\right) \left(-y \left(\frac{\pi}{2} Y_1(2y) + K_1(2y)\right)\right)
\]

\[
= -Y_1(2y) - \frac{2}{\pi} K_1(2y), \quad (8.1)
\]

where we used [3, p. 74, Equation (3.5)] (or [23, p. 184, Formula (3)]), and where \( Y_1 \) and \( K_1 \) are the Bessel functions defined in (1.4) and (1.5), respectively.

9. Acknowledgement

The authors are very grateful to Larry Glasser for guiding them to a proof of Theorem 7.1.

References

[1] B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. I, Trans. Amer. Math. Soc. 137 (1969), 345–359.
[2] B. C. Berndt, S. Kim, and A. Zaharescu, Weighted divisor sums and Bessel function series, II, Adv. Math. 229 (2012), 2055–2097.
[3] B. C. Berndt, S. Kim, and A. Zaharescu, Weighted divisor sums and Bessel function series, III, J. Reine Angew. Math. 683 (2013), 67–96.
[4] B. C. Berndt, S. Kim, and A. Zaharescu, The Circle Problem of Gauss and the Divisor Problem of Dirichlet–Still Unsolved, Amer. Math. Monthly 125 (2018), 99–114.
[5] B. C. Berndt, J. Li, and A. Zaharescu, The final problem: an identity from Ramanujan’s lost notebook, J. London Math. Soc. 100 (2019).
[6] B. C. Berndt and R. A. Rankin, Ramanujan: Letters and Commentary, American Mathematical Society, Providence, RI, 1995; jointly published by the London Mathematical Society, London, 1995; published in India by Affiliated East West, New Delhi, 1997.
[7] K. Chandrasekharan and R. Narasimhan, Functional equations with multiple gamma factors and the average order of arithmetical functions, Ann. Math. 76 (1962), 93–136.
[8] H. Cohen, Number Theory, Graduate Texts in Mathematics, vol. 239, Springer-Verlag, New York, 2007.
[9] H. Davenport, Multiplicative Number Theory, 3rd ed., Springer-Verlag, New York, 2000.
[10] C. Deninger, On the analogue of the formula of Chowla and Selberg for real quadratic fields, J. Reine Angew. Math. 351 (1984), 171–191.
[11] P. G. L. Dirichlet, Recherches sur diverses applications de l’analyse infinitésimale à théorie des nombres, J. Reine Angew. Math. 21 (1840), 1–12.
[12] A. Erdélyi, Tables of Integral Transforms, Volume 1, McGraw-Hill, New York, 1954.
[13] A. Erdélyi, Tables of Integral Transforms, Volume 2, McGraw-Hill, New York, 1954.
[14] I. S. Gradshteyn and I. M. Ryzhik, eds., Table of Integrals, Series, and Products, 5th ed., Academic Press, San Diego, 1994.
[15] G. H. Hardy, *The average order of the arithmetical functions \( P(x) \) and \( \Delta(x) \)*, Proc. London Math. Soc. (2) 15 (1916), 192–213.

[16] S. Kim, *Sums of divisors functions and Bessel function series*, J. Number Thy. 170 (2017), 142–184.

[17] E. Landau, *Über die Anzahl der gitterpunkte in gewissen Bereichen*, Nachr. Akad. Wiss. Göttingen 1912, 687–771.

[18] J. Neukirch, *Algebraic Number Theory*, Springer-Verlag, Berlin, 1999.

[19] W. G. Nowak, *On the distribution of integer ideals in algebraic number fields*, Nachr. Akad. Wiss. Göttingen 161 (1993), 59–74.

[20] B. Paul and A. Sankaranarayanan, *On the error term and zeros of the Dedekind zeta function*, J. Number Thy. 215 (2020), 98–119.

[21] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[22] G. F. Voronoi, *Sur une fonction transcendante et ses applications à la sommation de quelques séries*, *Ann. École Norm. Sup.* (3) 21 (1904), 207–267, 459–533.

[23] G. N. Watson, *Theory of Bessel Functions*, second ed., University Press, Cambridge, 1966.