NEWTONIAN LORENTZ METRIC SPACES

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Abstract. This paper studies Newtonian Sobolev-Lorentz spaces. We prove that these spaces are Banach. We also study the global $p,q$-capacity and the $p,q$-modulus of families of rectifiable curves. Under some additional assumptions (that is, $X$ carries a doubling measure and a weak Poincaré inequality), we show that when $1 \leq q < p$ the Lipschitz functions are dense in these spaces; moreover, in the same setting we show that the $p,q$-capacity is Choquet provided that $q > 1$. We also provide a counterexample to the density result in the Euclidean setting when $1 < p \leq n$ and $q = \infty$.

1. Introduction

In this paper $(X, d)$ is a complete metric space endowed with a nontrivial Borel regular measure $\mu$. We assume that $\mu$ is finite and nonzero on nonempty bounded open sets. In particular, this implies that the measure $\mu$ is $\sigma$-finite. Further restrictions on the space $X$ and on the measure $\mu$ will be imposed later.

The Sobolev-Lorentz relative $p,q$-capacity was studied in the Euclidean setting by Costea [6] and Costea-Maz’ya [8]. The Sobolev $p$-capacity was studied by Maz’ya [24] and Heinonen-Kilpeläinen-Martio [16] in $\mathbb{R}^n$ and by Costea [7] and Kinmune-Martio [21] and [22] in metric spaces. The relative Sobolev $p$-capacity in metric spaces was introduced by J. Björn in [2] when studying the boundary continuity properties of quasiminimizers.

After recalling the definition of $p,q$-Lorentz spaces in metric spaces, we study some useful property of the $p,q$-modulus of families of curves needed to give the notion of $p,q$-weak upper gradients. Then, following the approach of Shanmugalingam in [27] and [28], we generalize the notion of Newtonian Sobolev spaces to the Lorentz setting. There are several other definitions of Sobolev-type spaces in the metric setting when $p = q$; see Hajlasz [12], Heinonen-Koskela [17], Cheeger [4], and Franchi-Hajlasz-Koskela [11]. It has been shown that under reasonable hypotheses, the majority of these definitions yields the same space; see Franchi-Hajlasz-Koskela [11] and Shanmugalingam [27].

We prove that these spaces are Banach. In order to this, we develop a theory of Sobolev $p,q$-capacity. Some of the ideas used here when proving the properties of the $p,q$-capacity follow Kinmune-Martio [21] and [22] and Costea [7]. We also use this theory to prove that, in the case $1 \leq q < p$, Lipschitz functions are dense in the Newtonian Sobolev-Lorentz space if the space $X$ carries a doubling measure $\mu$ and a weak $(1, L^{p,q})$-Poincaré inequality. Newtonian Banach-valued Sobolev-Lorentz spaces were studied by Podbrdsky in [26].

We prove that under certain restrictions (when $1 < q \leq p$ and the space $(X,d)$ carries a doubling measure $\mu$ and a certain weak Poincaré inequality) this capacity is a Choquet set function.

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We recall the standard notation and definitions to be used throughout this paper. We denote by \( B(x, r) = \{ y \in X : d(x, y) < r \} \) the open ball with center \( x \in X \) and radius \( r > 0 \), while \( \overline{B}(x, r) = \{ y \in X : d(x, y) \leq r \} \) is the closed ball with center \( x \in X \) and radius \( r > 0 \). For a positive number \( \lambda \), \( \lambda B(a, r) = B(a, \lambda r) \) and \( \lambda \overline{B}(a, r) = \overline{B}(a, \lambda r) \).

Throughout this paper, \( C \) will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. \( C(a, b, \ldots) \) is a constant that depends only on the parameters \( a, b, \ldots \). For \( E \subset X \), the boundary, the closure, and the complement of \( E \) with respect to \( X \) will be denoted by \( \partial E \), \( \overline{E} \), and \( X \setminus E \), respectively; \( \text{diam} \ E \) is the diameter of \( E \) with respect to the metric \( d \).

2. LORENTZ SPACES

Let \( f : X \to [-\infty, \infty] \) be a \( \mu \)-measurable function. We define \( \mu[f] \), the distribution function of \( f \) as follows (see Bennett-Sharpley [1, Definition II.1.1]):

\[
\mu[f](t) = \mu(\{ x \in X : |f(x)| > t \}), \quad t \geq 0.
\]

We define \( f^* \), the nonincreasing rearrangement of \( f \) by

\[
f^*(t) = \inf \{ v : \mu[f](v) \leq t \}, \quad t \geq 0.
\]

(See Bennett-Sharpley [1, Definition II.1.5].) We note that \( f \) and \( f^* \) have the same distribution function. For every positive \( \alpha \) we have

\[
(|f|^\alpha)^* = (|f^*|^\alpha)
\]

and if \( |g| \leq |f| \) \( \mu \)-almost everywhere on \( X \), then \( g^* \leq f^* \). (See [1, Proposition II.1.7].)

We also define \( f^{**} \), the maximal function of \( f^* \) by

\[
f^{**}(t) = m_{f^*}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds, \quad t > 0.
\]

(See [1, Definition II.3.1].)

Throughout the paper, we denote by \( p' \) the H"older conjugate of \( p \in [1, \infty] \).

The Lorentz space \( L^{p,q}(X, \mu) \), \( 1 < p < \infty \), \( 1 \leq q \leq \infty \), is defined as follows:

\( L^{p,q}(X, \mu) = \{ f : X \to [-\infty, \infty] : f \text{ is } \mu\text{-measurable}, ||f||_{L^{p,q}(X, \mu)} < \infty \} \),

where

\[
||f||_{L^{p,q}(X, \mu)} = ||f||_{p,q} = \begin{cases} \left( \int_0^{\infty} \left( t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{t > 0} t^{1/p} \mu[f](t)^{1/p} = \sup_{s > 0} s^{1/p} f^{**}(s), & q = \infty. \end{cases}
\]

(See Bennett-Sharpley [1, Definition IV.4.1] and Stein-Weiss [29, p. 191].)

If \( 1 \leq q \leq p \), then \( || \cdot ||_{L^{p,q}(X, \mu)} \) represents a norm, but for \( p < q \leq \infty \) it represents a quasinorm, equivalent to the norm \( || \cdot ||_{L^{[p,q]}(X, \mu)} \), where

\[
||f||_{L^{[p,q]}(X, \mu)} = ||f||_{(p,q)} = \begin{cases} \left( \int_0^{\infty} \left( t^{1/p} f^{**}(t) \right)^q \frac{dt}{t} \right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{t > 0} t^{1/p} f^{**}(t), & q = \infty. \end{cases}
\]
(See [1, Definition IV.4.4].) Namely, from [1, Lemma IV.4.5] we have that
\[ \|f\|_{L^{p,q}(X,\mu)} \leq \|f\|_{L^{(p,q)}(X,\mu)} \leq p'\|f\|_{L^{p,q}(X,\mu)} \]
for every \( q \in [1, \infty] \) and every \( \mu \)-measurable function \( f : X \to [-\infty, \infty] \).

It is known that \( (L^{p,q}(X,\mu), \|\cdot\|_{L^{p,q}(X,\mu)}) \) is a Banach space for \( 1 \leq q \leq p \), while \( (L^{p,q}(X,\mu), \|\cdot\|_{L^{(p,q)}(X,\mu)}) \) is a Banach space for \( 1 < p < \infty, 1 \leq q \leq \infty \). In addition, if the measure \( \mu \) is nonatomic, the aforementioned Banach spaces are reflexive when \( 1 < q < \infty \). (See Hunt [18, p. 259-262] and Bennett-Sharpley [1, Theorem IV.4.7 and Corollaries I.4.3 and IV.4.8].) A measure \( \mu \) is called nonatomic if for every measurable set \( A \) of positive measure there exists a measurable set \( B \subset A \) such that \( 0 < \mu(B) < \mu(A) \).

**Definition 2.1.** (See [1, Definition I.3.1].) Let \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \). Let \( Y = L^{p,q}(X,\mu) \). A function \( f \) in \( Y \) is said to have absolutely continuous norm in \( Y \) if and only if \( \|f\chi_{E_k}\|_Y \to 0 \) for every sequence \( E_k \) of \( \mu \)-measurable sets satisfying \( E_k \to \emptyset \) \( \mu \)-almost everywhere.

Let \( Y_a \) be the subspace of \( Y \) consisting of functions of absolutely continuous norm and let \( Y_b \) be the closure in \( Y \) of the set of simple functions. It is known that \( Y_a = Y_b \) whenever \( 1 \leq q \leq \infty \). (See Bennett-Sharpley [1, Theorem I.3.13].) Moreover, since \((X,\mu)\) is a \( \sigma \)-finite measure space, we have \( Y_b = Y \) whenever \( 1 \leq q < \infty \). (See Hunt [18, p. 258-259].)

We recall (see Costea [6]) that in the Euclidean setting (that is, when \( \mu = m_n \) is the \( n \)-dimensional Lebesgue measure and \( d \) is the Euclidean distance on \( \mathbb{R}^n \)) we have \( Y_a \neq Y \) for \( Y = L^{p,\infty}(X, m_n) \) whenever \( X \) is an open subset of \( \mathbb{R}^n \). Let \( X = B(0, 2) \setminus \{0\} \). As in Costea [6] we define \( u : X \to \mathbb{R}, \)
\[
(1) \quad u(x) = \begin{cases} 
|x|^{\frac{n}{p}} & \text{if } 0 < |x| < 1 \\
0 & \text{if } 1 \leq |x| \leq 2.
\end{cases}
\]

It is easy to see that \( u \in L^{p,\infty}(X, m_n) \) and moreover,
\[
\|u\chi_{B(0,\alpha)}\|_{L^{p,\infty}(X,m_n)} = \|u\|_{L^{p,\infty}(X,m_n)} = m_n(B(0, 1))^{1/p}
\]
for every \( \alpha > 0 \). This shows that \( u \) does not have absolutely continuous weak \( L^p \)-norm and therefore \( L^{p,\infty}(X, m_n) \) does not have absolutely continuous norm.

**Remark 2.2.** It is also known (see [1, Proposition IV.4.2]) that for every \( p \in (1, \infty) \) and \( 1 \leq r < s \leq \infty \) there exists a constant \( C(p, r, s) \) such that
\[
(2) \quad \|f\|_{L^{p,r}(X,\mu)} \leq C(p, r, s)\|f\|_{L^{p,s}(X,\mu)}
\]
for all measurable functions \( f \in L^{p,r}(X,\mu) \). In particular, the embedding \( L^{p,r}(X,\mu) \hookrightarrow L^{p,s}(X,\mu) \) holds.

**Remark 2.3.** Via Bennett-Sharpley [1, Proposition II.1.7 and Definition IV.4.1] it is easy to see that for every \( p \in (1, \infty) \), \( q \in [1, \infty] \) and \( 0 < \alpha \leq \min(p, q) \), we have
\[
\|f\|_{L^{p,q}(X,\mu)} = \|f^\alpha\|_{L^{p,\frac{q}{\alpha}}(X,\mu)}
\]
for every nonnegative function \( f \in L^{p,q}(X,\mu) \).
2.1. The subadditivity and superadditivity of the Lorentz quasi-norms. We recall the known results and present new results concerning the superadditivity and the subadditivity of the Lorentz \( p, q \)-quasinorm. For the convenience of the reader, we will provide proofs for the new results and for some of the known results.

The superadditivity of the Lorentz \( p, q \)-norm in the case \( 1 \leq q \leq p \) was stated in Chung-Hunt-Kurtz [5, Lemma 2.5].

**Proposition 2.4.** (See [5, Lemma 2.5].) Let \((X, \mu)\) be a measure space. Suppose that \( 1 \leq q \leq p \). Let \( \{E_i\}_{i \geq 1} \) be a collection of pairwise disjoint \( \mu \)-measurable subsets of \( X \) with \( E_0 = \bigcup_{i \geq 1} E_i \) and let \( f \in L^{p,q}(X, \mu) \). Then

\[
\sum_{i \geq 1} ||\chi_{E_i} f||_{L^{p,q}(X,\mu)}^p \leq ||\chi_{E_0} f||_{L^{p,q}(X,\mu)}^p.
\]

A similar result concerning the superadditivity was obtained in Costea-Maz’ya [8, Proposition 2.4] for the case \( 1 < p < q < \infty \) when \( X = \Omega \) was an open set in \( \mathbb{R}^n \) and \( \mu \) was an arbitrary measure. That result is valid for a general measure space \((X, \mu)\).

**Proposition 2.5.** Let \((X, \mu)\) be a measure space. Suppose that \( 1 < p < q < \infty \). Let \( \{E_i\}_{i \geq 1} \) be a collection of pairwise disjoint \( \mu \)-measurable subsets of \( X \) with \( E_0 = \bigcup_{i \geq 1} E_i \) and let \( f \in L^{p,q}(X, \mu) \). Then

\[
\sum_{i \geq 1} ||\chi_{E_i} f||_{L^{p,q}(X,\mu)}^q \leq ||\chi_{E_0} f||_{L^{p,q}(X,\mu)}^q.
\]

**Proof.** We mimic the proof of Proposition 2.4 from Costea-Maz’ya [8]. We replace \( \Omega \) with \( X \). \(\square\)

We have a similar result for the subadditivity of the Lorentz \( p, q \)-quasinorm. When \( 1 < p < q \leq \infty \) we obtain a result that generalizes Theorem 2.5 from Costea [6].

**Proposition 2.6.** Let \((X, \mu)\) be a measure space. Suppose that \( 1 < p < q \leq \infty \). Suppose \( f_i, i = 1, 2, \ldots \) is a sequence of functions in \( L^{p,q}(X, \mu) \) and let \( f_0 = \sup_{i \geq 1} |f_i| \). Then

\[
||f_0||_{L^{p,q}(X,\mu)}^p \leq \sum_{i=1}^{\infty} ||f_i||_{L^{p,q}(X,\mu)}^p.
\]

**Proof.** Without loss of generality we can assume that all the functions \( f_i, i = 1, 2, \ldots \) are nonnegative. We have to consider two cases, depending on whether \( p < q < \infty \) or \( q = \infty \).

Let \( \mu_{[f_i]} \) be the distribution function of \( f_i \) for \( i = 0, 1, 2, \ldots \). It is easy to see that

\[
\mu_{[f_0]}(s) \leq \sum_{i=1}^{\infty} \mu_{[f_i]}(s) \text{ for every } s \geq 0.
\]

Suppose that \( p < q < \infty \). We have (see Kauhanen-Koskela-Malý [20, Proposition 2.1])

\[
||f_i||_{L^{p,q}(X,\mu)}^p = \left(p \int_0^{\infty} s^{q-1} \mu_{[f_i]}(s)^{\frac{q}{p}} ds\right)^{\frac{p}{q}}
\]
for \(i = 0, 1, 2, \ldots\). From this and (3) we obtain
\[
\|f_0\|_{L^{p,q}(\Omega, \mu)}^p = \left( p \int_0^\infty s^{q-1} \mu_{[f_0]}(s) \frac{d}{s^{p}} ds \right)^\frac{p}{q} \leq \sum_{i \geq 1} \left( p \int_0^\infty s^{q-1} \mu_{[f_i]}(s) \frac{d}{s^{p}} ds \right)^\frac{p}{q} = \sum_{i \geq 1} \|f_i\|_{L^{p,q}(\Omega, \mu)}^p.
\]

Now, suppose that \(q = \infty\). From (3) we obtain
\[
s^p \mu_{[f_0]}(s) \leq \sum_{i \geq 1} (s^p \mu_{[f_i]}(s)) \text{ for every } s > 0,
\]
which implies
\[
(5) \quad s^p \mu_{[f_0]}(s) \leq \sum_{i \geq 1} \|f_i\|_{L^{p,\infty}(X, \mu)}^p \text{ for every } s > 0.
\]

By taking the supremum over all \(s > 0\) in (5), we get the desired conclusion. This finishes the proof.

We recall a few results concerning Lorentz spaces.

**Theorem 2.7.** (See [6, Theorem 2.6].) Suppose \(1 < p < q \leq \infty\) and \(\varepsilon \in (0, 1)\). Let \(f_1, f_2 \in L^{p,q}(X, \mu)\). We denote \(f_3 = f_1 + f_2\). Then \(f_3 \in L^{p,q}(X, \mu)\) and
\[
\|f_3\|_{L^{p,q}(X, \mu)}^p \leq (1 - \varepsilon)^{-p} \|f_1\|_{L^{p,q}(X, \mu)}^p + \varepsilon^{-p} \|f_2\|_{L^{p,q}(X, \mu)}^p.
\]

**Proof.** The proof of Theorem 2.6 from Costea [6] carries verbatim. We replace \(\Omega\) with \(X\).

□

Theorem 2.7 has an useful corollary.

**Corollary 2.8.** (See [6, Corollary 2.7].) Suppose \(1 < p < \infty\) and \(1 \leq q \leq \infty\). Let \(f_k\) be a sequence of functions in \(L^{p,q}(X, \mu)\) converging to \(f\) with respect to the \(p, q\)-quasinorm and pointwise \(\mu\)-almost everywhere in \(X\). Then
\[
\lim_{k \to \infty} \|f_k\|_{L^{p,q}(X, \mu)} = \|f\|_{L^{p,q}(X, \mu)}.
\]

**Proof.** The proof of Corollary 2.7 from Costea [6] carries verbatim. We replace \(\Omega\) with \(X\).

□

### 3. \(p, q\)-Modulus of the Path Family

In this section, we establish some results about \(p, q\)-modulus of families of curves. Here \((X, d, \mu)\) is a metric measure space. We say that a curve \(\gamma\) in \(X\) is rectifiable if it has finite length. Whenever \(\gamma\) is rectifiable, we use the arc length parametrization \(\gamma : [0, \ell(\gamma)] \to X\), where \(\ell(\gamma)\) is the length of the curve \(\gamma\).

Let \(\Gamma_{\text{rect}}\) denote the family of all nonconstant rectifiable curves in \(X\). It may well be that \(\Gamma_{\text{rect}} = \emptyset\), but we will be interested in metric spaces for which \(\Gamma_{\text{rect}}\) is sufficiently large.
Definition 3.1. For $\Gamma \subset \Gamma_{\text{rect}}$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_{\gamma} \rho \geq 1 \text{ for every } \gamma \in \Gamma.$$ 

Now for each $1 < p < \infty$ and $1 \leq q \leq \infty$ we define

$$\operatorname{Mod}_{p,q}(\Gamma) = \inf_{\rho \in F(\Gamma)} ||\rho||_{L^{p,q}(X,\mu)}^p.$$ 

The number $\operatorname{Mod}_{p,q}(\Gamma)$ is called the $p,q$-modulus of the family $\Gamma$.

3.1. Basic properties of the $p,q$-modulus. Usually, a modulus is a monotone and subadditive set function. The following theorem will show, among other things, that this is true in the case of the $p,q$-modulus.

Theorem 3.2. Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. The set function $\Gamma \mapsto \operatorname{Mod}_{p,q}(\Gamma)$, $\Gamma \subset \Gamma_{\text{rect}}$, enjoys the following properties:

(i) $\operatorname{Mod}_{p,q}(\emptyset) = 0$.

(ii) If $\Gamma_1 \subset \Gamma_2$, then $\operatorname{Mod}_{p,q}(\Gamma_1) \leq \operatorname{Mod}_{p,q}(\Gamma_2)$.

(iii) Suppose that $1 \leq q < p$. Then

$$\operatorname{Mod}_{p,q}(\bigcup_{i=1}^{\infty} \Gamma_i)^{q/p} \leq \sum_{i=1}^{\infty} \operatorname{Mod}_{p,q}(\Gamma_i)^{q/p}.$$ 

(iv) Suppose $p < q \leq \infty$. Then

$$\operatorname{Mod}_{p,q}(\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} \operatorname{Mod}_{p,q}(\Gamma_i).$$

Proof. (i) $\operatorname{Mod}_{p,q}(\emptyset) = 0$ because $\rho \equiv 0 \in F(\emptyset)$.

(ii) If $\Gamma_1 \subset \Gamma_2$, then $F(\Gamma_2) \subset F(\Gamma_1)$ and hence $\operatorname{Mod}_{p,q}(\Gamma_1) \leq \operatorname{Mod}_{p,q}(\Gamma_2)$.

(iii) Suppose that $1 \leq q < p$. The case $p = q$ corresponds to the $p$-modulus and the claim certainly holds in that case. (See for instance Hajlasz [13, Theorem 5.2 (3)].) So we can look at the case $1 \leq q < p$.

We can assume without loss of generality that

$$\sum_{i=1}^{\infty} \operatorname{Mod}_{p,q}(\Gamma_i)^{q/p} < \infty.$$

Let $\varepsilon > 0$ be fixed. Take $\rho_i \in F(\Gamma_i)$ such that

$$||\rho_i||_{L^{p,q}(X,\mu)}^q < \operatorname{Mod}_{p,q}(\Gamma_i)^{q/p} + \varepsilon 2^{-i}.$$ 

Let $\rho := (\sum_{i=1}^{\infty} \rho_i^q)^{1/q}$. We notice via Bennett-Sharpley [1, Proposition II.1.7 and Definition IV.4.1] and Remark 2.3 applied with $\alpha = q$ that

$$\rho_i^q \in L^\frac{q}{q-1}(X,\mu) \text{ and } ||\rho_i||_{L^\frac{q}{q-1}(X,\mu)}^q = ||\rho_i||_{L^{p,q}(X,\mu)}^q.$$ 

for every $i = 1, 2, \ldots$. By using (6) and Remark 2.3 together with the definition of $\rho$ and the fact that $|| \cdot ||_{L^\frac{q}{q-1}(X,\mu)}$ is a norm when $1 \leq q \leq p$, it follows that $\rho \in F(\Gamma)$ and

$$\operatorname{Mod}_{p,q}(\Gamma_i)^{q/p} \leq ||\rho||_{L^{p,q}(X,\mu)}^q \leq \sum_{i=1}^{\infty} ||\rho_i||_{L^{p,q}(X,\mu)}^q \leq \sum_{i=1}^{\infty} \operatorname{Mod}_{p,q}(\Gamma_i)^{q/p} + 2\varepsilon.$$ 

Letting $\varepsilon \to 0$, we complete the proof when $1 \leq q \leq p$. 


(iv) Suppose now that \( p < q \leq \infty \). We can assume without loss of generality that
\[
\sum_{i=1}^{\infty} \text{Mod}_{p,q}(\Gamma_i) < \infty.
\]
Let \( \varepsilon > 0 \) be fixed. Take \( \rho_i \in F(\Gamma_i) \) such that
\[
||\rho_i||_{L^p,q(X,\mu)}^p \leq \text{Mod}_{p,q}(\Gamma_i) + \varepsilon 2^{-i}.
\]
Let \( \rho := \sup_{i \geq 1} \rho_i \). Then \( \rho \in F(\Gamma) \). Moreover, from Proposition 2.6 it follows that
\[
\rho \in L^p,q(X,\mu) \quad \text{and} \quad \text{Mod}_{p,q}(\Gamma) \leq \sum_{i=1}^{\infty} ||\rho_i||_{L^p,q(X,\mu)}^p < \sum_{i=1}^{\infty} \text{Mod}_{p,q}(\Gamma_i) + 2\varepsilon.
\]
Letting \( \varepsilon \to 0 \), we complete the proof when \( p < q \leq \infty \).
\[\square\]

So we proved that the modulus is a monotone function. Also, the shorter the curves, the larger the modulus. More precisely, we have:

**Lemma 3.3.** Let \( \Gamma_1, \Gamma_2 \subset \Gamma_{\text{rect}} \). If each curve \( \gamma \in \Gamma_1 \) contains a subcurve that belongs to \( \Gamma_2 \), then \( \text{Mod}_{p,q}(\Gamma_1) \leq \text{Mod}_{p,q}(\Gamma_2) \).

**Proof.** \( F(\Gamma_2) \leq F(\Gamma_1) \). \[\square\]

The following theorem provides an useful characterization of path families that have \( p,q \)-modulus zero.

**Theorem 3.4.** Let \( \Gamma \subset \Gamma_{\text{rect}} \). Then \( \text{Mod}_{p,q}(\Gamma) = 0 \) if and only if there exists a Borel measurable function \( 0 \leq \rho \in L^{p,q}(X,\mu) \) such that \( f_{\gamma} \rho = \infty \) for every \( \gamma \in \Gamma \).

**Proof.** Sufficiency. We notice that \( \rho/n \in F(\Gamma) \) for every \( n \) and hence
\[
\text{Mod}_{p,q}(\Gamma) \leq \lim_{n \to \infty} ||\rho/n||_{L^{p,q}(X,\mu)}^p = 0.
\]

Necessity. There exists \( \rho_i \in F(\Gamma) \) such that \( ||\rho_i||_{L^{p,q}(X,\mu)}^p < 2^{-i} \) and \( f_{\gamma} \rho_i \geq 1 \) for every \( \gamma \in \Gamma \). Then \( \rho := \sum_{i=1}^{\infty} \rho_i \) has the required properties.
\[\square\]

**Corollary 3.5.** Suppose \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \) are given. If \( 0 \leq g \in L^{p,q}(X,\mu) \) is Borel measurable, then \( f_{\gamma} g < \infty \) for \( p,q \)-almost every \( \gamma \in \Gamma_{\text{rect}} \).

The following theorem is also important.

**Theorem 3.6.** Let \( u_k : X \to \overline{\mathbb{R}} = [-\infty, \infty] \) be a sequence of Borel functions which converge to a Borel function \( u : X \to \overline{\mathbb{R}} \) in \( L^{p,q}(X,\mu) \). Then there is a subsequence \( (u_{k_j})_j \) such that
\[
\int_{\gamma} |u_{k_j} - u| \to 0 \quad \text{as} \quad j \to \infty,
\]
for \( p,q \)-almost every curve \( \gamma \in \Gamma_{\text{rect}} \).

**Proof.** We follow Hajlasz [13]. We take a subsequence \( (u_{k_j})_j \) such that
\[
||u_{k_j} - u||_{L^{p,q}(X,\mu)} < 2^{-2j}.
\]
Set \( g_j = |u_k - u| \), and let \( \Gamma \subset \Gamma_{\text{rect}} \) be the family of curves such that

\[
\limsup_{j \to \infty} \int_{\gamma} g_j > 0.
\]

We want to show that \( \text{Mod}_{p,q}(\Gamma) = 0 \). Denote by \( \Gamma_j \) the family of curves in \( \Gamma_{\text{rect}} \) for which \( \int_{\gamma} g_j > 2^{-j} \). Then \( 2^j g_j \in F(\Gamma_j) \) and hence \( \text{Mod}_{p,q}(\Gamma_j) < 2^{-pj} \) as a consequence of (7). We notice that

\[
\Gamma \subset \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \Gamma_j.
\]

Thus

\[
\text{Mod}_{p,q}(\Gamma) \leq \sum_{j=i}^{\infty} \text{Mod}_{p,q}(\Gamma_j) \leq \sum_{j=i}^{\infty} 2^{-j} = 2^{1-i}
\]

for every integer \( i \geq 1 \), which implies \( \text{Mod}_{p,q}(\Gamma) = 0 \). \( \square \)

3.2. Upper gradient.

**Definition 3.7.** Let \( u : X \to [-\infty, \infty] \) be a Borel function. We say that a Borel function \( g : X \to [0, \infty] \) is an upper gradient of \( u \) if for every rectifiable curve \( \gamma \) parametrized by arc length parametrization we have

\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma} g
\]

whenever both \( u(\gamma(0)) \) and \( u(\gamma(\ell(\gamma))) \) are finite and \( \int_{\gamma} g = \infty \) otherwise. We say that \( g \) is a \( p,q \)-weak upper gradient of \( u \) if (8) holds on \( p,q \)-almost every curve \( \gamma \in \Gamma_{\text{rect}} \).

The weak upper gradients were introduced in the case \( p = q \) by Heinonen-Koskela in [17]. See also Heinonen [15] and Shanmugalingam [27] and [28].

If \( g \) is an upper gradient of \( u \) and \( \tilde{g} = g, \mu \)-almost everywhere, is another nonnegative Borel function, then it might happen that \( \tilde{g} \) is not an upper gradient of \( u \). However, we have the following result.

**Lemma 3.8.** If \( g \) is a \( p,q \)-weak upper gradient of \( u \) and \( \tilde{g} \) is another nonnegative Borel function such that \( \tilde{g} = g, \mu \)-almost everywhere, then \( \tilde{g} \) is a \( p,q \)-weak upper gradient of \( u \).

**Proof.** Let \( \Gamma_1 \subset \Gamma_{\text{rect}} \) be the family of all nonconstant rectifiable curves \( \gamma : [0, \ell(\gamma)] \to X \) for which \( \int_{\gamma} |g - \tilde{g}| > 0 \). The constant sequence \( g_n = |g - \tilde{g}| \) converges to 0 in \( L^{p,q}(X,\mu) \), so from Theorem 3.6 it follows that \( \text{Mod}_{p,q}(\Gamma_1) = 0 \) and \( \int_{\gamma} |g - \tilde{g}| = 0 \) for every nonconstant rectifiable curve \( \gamma : [0, \ell(\gamma)] \to X \) that is not in \( \Gamma_1 \).

Let \( \Gamma_2 \subset \Gamma_{\text{rect}} \) be the family of all nonconstant rectifiable curves \( \gamma : [0, \ell(\gamma)] \to X \) for which the inequality

\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma} g
\]

is not satisfied. Then \( \text{Mod}_{p,q}(\Gamma_2) = 0 \). Thus \( \text{Mod}_{p,q}(\Gamma_1 \cup \Gamma_2) = 0 \). For every \( \gamma \in \Gamma_{\text{rect}} \) not in \( \Gamma_1 \cup \Gamma_2 \) we have

\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma} g = \int_{\gamma} \tilde{g}.
\]

This finishes the proof. \( \square \)
The next result shows that $p, q$-weak upper gradients can be nicely approximated by upper gradients. The case $p = q$ was proved by Koskela-MacManus [23].

**Lemma 3.9.** If $g$ is a $p, q$-weak upper gradient of $u$ which is finite $\mu$-almost everywhere, then for every $\varepsilon > 0$ there exists an upper gradient $g_\varepsilon$ of $u$ such that

$$g_\varepsilon \geq g \text{ everywhere on } X \text{ and } ||g_\varepsilon - g||_{L^{p,q}(X, \mu)} \leq \varepsilon.$$

**Proof.** Let $\Gamma \subseteq \Gamma_{\text{rect}}$ be the family of all nonconstant rectifiable curves $\gamma : [0, \ell(\gamma)] \to X$ for which the inequality

$$|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_\gamma g$$

is not satisfied. Then $\text{Mod}_{p,q}(\Gamma) = 0$ and hence, from Theorem 3.4 it follows that there exists $0 \leq \rho \in L^{p,q}(X, \mu)$ such that $\int_\gamma \rho = \infty$ for every $\gamma \in \Gamma$. Take $g_\varepsilon = g + \varepsilon \rho/||\rho||_{L^{p,q}(X, \mu)}$. Then $g_\varepsilon$ is a nonnegative Borel function and

$$|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_\gamma g_\varepsilon$$

for every curve $\gamma \in \Gamma_{\text{rect}}$. This finishes the proof. □

If $A$ is a subset of $X$ let $\Gamma_A$ be the family of all curves in $\Gamma_{\text{rect}}$ that intersect $A$ and let $\Gamma^+_A$ be the family of all curves in $\Gamma_{\text{rect}}$ such that the Hausdorff one-dimensional measure $\mathcal{H}_1(|\gamma| \cap A)$ is positive. Here and throughout the paper $|\gamma|$ is the image of the curve $\gamma$.

The following lemma will be useful later in this paper.

**Lemma 3.10.** Let $u_i : X \to \mathbb{R}, i \geq 1$, be a sequence of Borel functions such that $g \in L^{p,q}(X)$ is a $p, q$-weak upper gradient for every $u_i, i \geq 1$. We define $u(x) = \lim_{i \to \infty} u_i(x)$ and $E = \{x \in X : |u(x)| = \infty\}$. Suppose that $\mu(E) = 0$ and that $u$ is well-defined outside $E$. Then $g$ is a $p, q$-weak upper gradient for $u$.

**Proof.** For every $i \geq 1$ we define $\Gamma_{1,i}$ to be the set of all curves $\gamma \in \Gamma_{\text{rect}}$ for which

$$|u_i(\gamma(0)) - u_i(\gamma(\ell(\gamma)))| \leq \int_\gamma g$$

is not satisfied. Then $\text{Mod}_{p,q}(\Gamma_{1,i}) = 0$ and hence $\text{Mod}_{p,q}(\Gamma_1) = 0$, where $\Gamma_1 = \bigcup_{i=1}^\infty \Gamma_{1,i}$.

Let $\Gamma_0$ be the collection of all paths $\gamma \in \Gamma_{\text{rect}}$ such that $\int_\gamma g = \infty$. Then we have via Theorem 3.4 that $\text{Mod}_{p,q}(\Gamma_0) = 0$ since $g \in L^{p,q}(X, \mu)$.

Since $\mu(E) = 0$, it follows that $\text{Mod}_{p,q}(\Gamma_E^+) = 0$. Indeed, $\infty \cdot \chi_E \in F(\Gamma_E^+)$ and $||\infty \cdot \chi_E||_{L^{p,q}(X, \mu)} = 0$. Therefore $\text{Mod}_{p,q}(\Gamma_0 \cup \Gamma_E^+ \cup \Gamma_1) = 0$.

For any path $\gamma$ in the family $\Gamma_{\text{rect}} \setminus (\Gamma_0 \cup \Gamma_E^+ \cup \Gamma_1)$, by the fact that the path is not in $\Gamma_E^+$, there exists a point $y$ in $|\gamma|$ such that $y$ is not in $E$, that is $y \in |\gamma|$ and $|u(y)| < \infty$. For any point $x \in |\gamma|$, we have (since $\gamma$ is not in $\Gamma_{1,i}$)

$$|u_i(x)| - |u_i(y)| \leq |u_i(x) - u_i(y)| \leq \int_\gamma g < \infty.$$

Therefore

$$|u_i(x)| \leq |u_i(y)| + \int_\gamma g.$$ 

Taking limits on both sides and using the facts that $|u(y)| < \infty$ and that $\gamma$ is not in $\Gamma_0 \cup \Gamma_1$, we see that

$$\lim_{i \to \infty} |u_i(x)| \leq \lim_{i \to \infty} |u_i(y)| + \int_\gamma g = |u(y)| + \int_\gamma g < \infty.$$
and therefore $x$ is not in $E$. Thus $\Gamma_E \subset \Gamma_0 \cup \Gamma_E^+ \cup \Gamma_1$ and $\text{Mod}_{p,q}(\Gamma_E) = 0$.

Next, let $\gamma$ be a path in $\Gamma_{\text{rect}} \setminus (\Gamma_0 \cup \Gamma_E^+ \cup \Gamma_1)$. The above argument showed that $|\gamma|$ does not intersect $E$. If we denote by $x$ and $y$ the endpoints of $\gamma$, we have

$$|u(x) - u(y)| = |\lim_{i \to \infty} u_i(x) - \lim_{i \to \infty} u_i(y)| = \lim_{i \to \infty} |u_i(x) - u_i(y)| \leq \int_{\gamma} g.$$

Therefore $g$ is a $p,q$-weak upper gradient for $u$ as well. 

\[\square\]

The following proposition shows how the upper gradients behave under a change of variable.

**Proposition 3.11.** Let $F : \mathbb{R} \to \mathbb{R}$ be $C^1$ and let $u : X \to \mathbb{R}$ be a Borel function. If $g \in L^{p,q}(X,\mu)$ is a $p,q$-weak upper gradient for $u$, then $|F'(u)|g$ is a $p,q$-weak upper gradient for $F \circ u$.

**Proof.** Let $\Gamma_0$ to be the set of all curves $\gamma \in \Gamma_{\text{rect}}$ for which

$$|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma} g$$

is not satisfied. Then $\text{Mod}_{p,q}(\Gamma_0) = 0$. Let $\Gamma_1 \subset \Gamma_{\text{rect}}$ be the collection of all curves having a subcurve in $\Gamma_0$. Then $F(\Gamma_0) \subset F(\Gamma_1)$ and hence $\text{Mod}_{p,q}(\Gamma_1) \leq \text{Mod}_{p,q}(\Gamma_0) = 0$.

Let $\Gamma_2$ be the set of curves $\gamma \in \Gamma_{\text{rect}}$ for which $\int_{\gamma} g = \infty$. Then we have via Theorem 3.4 that $\text{Mod}_{p,q}(\Gamma_2) = 0$ since $g \in L^{p,q}(X,\mu)$. Thus $\text{Mod}_{p,q}(\Gamma_1 \cup \Gamma_2) = 0$.

The claim will follow immediately after we show that

$$|(F \circ u)(\gamma(0)) - (F \circ u)(\gamma(\ell(\gamma)))| \leq \int_0^{\ell(\gamma)} (|F'(u(\gamma(s)))| + \varepsilon)g(\gamma(s)) \, ds.$$  \hspace{1cm} (9)

for all curves $\gamma \in \Gamma_{\text{rect}} \setminus (\Gamma_1 \cup \Gamma_2)$ and for every $\varepsilon > 0$.

So fix $\varepsilon > 0$ and choose a curve $\gamma \in \Gamma_{\text{rect}} \setminus (\Gamma_1 \cup \Gamma_2)$. Let $\ell = \ell(\gamma)$. We notice immediately that $u \circ \gamma$ is uniformly continuous on $[0,\ell]$ and $F'$ is uniformly continuous on the compact interval $I := (u \circ \gamma)([0,\ell])$. Let $\delta, \delta_1 > 0$ be chosen such that

$$|F'(u \circ \gamma)(t) - F'(u \circ \gamma)(s)| + |(u \circ \gamma)(t) - (u \circ \gamma)(s)| < \delta_1$$

for all $t, s \in [0,\ell]$ with $|t - s| < \delta$ and such that

$$|F'(u) - F'(v)| < \varepsilon \text{ for all } u, v \in I \text{ with } |u - v| < \delta_1.$$  

Fix an integer $n > 1/\delta$ and put $\ell_i := (i\ell)/n, i = 0, \ldots, n-1$. For every $i = 0, \ldots, n-1$ we have

$$|(F \circ u \circ \gamma)(\ell_{i+1}) - (F \circ u \circ \gamma)(\ell_i)| = |F'(t_{i,i+1})||u \circ \gamma)(\ell_{i+1}) - (u \circ \gamma)(\ell_i)| \leq |F'(t_{i,i+1})|\int_{\ell_i}^{\ell_{i+1}} g(\gamma(s)) \, ds,$$

where $t_{i,i+1} \in I_{i,i+1} := (u \circ \gamma)((\ell_i, \ell_{i+1}))$. From the choice of $\delta$ it follows that

$$|(F \circ u \circ \gamma)(\ell_{i+1}) - (F \circ u \circ \gamma)(\ell_i)| \leq \int_{\ell_i}^{\ell_{i+1}} (|F'(u(\gamma(s)))| + \varepsilon)g(\gamma(s)) \, ds,$$

for every $i = 0, \ldots, n-1$. If we sum over $i$ we obtain easily (9). This finishes the proof. \[\square\]

As a direct consequence of Proposition 3.11, we have the following corollaries.
Corollary 3.12. Let \( r \in (1, \infty) \) be fixed. Suppose \( u : X \to \mathbb{R} \) is a bounded nonnegative Borel function. If \( g \in L^{p,q}(X, \mu) \) is a p,q-weak upper gradient of \( u \), then \( ru^{-1}g \) is a \( p,q \)-weak upper gradient for \( u \).

Proof. Let \( M > 0 \) be such that \( 0 \leq u(x) < M \) for all \( x \in X \). We apply Proposition 3.11 to any \( C^1 \) function \( F : \mathbb{R} \to \mathbb{R} \) satisfying \( F(t) = t^r, 0 \leq t \leq M \).

\( \square \)

Corollary 3.13. Let \( r \in (0,1) \) be fixed. Suppose that \( u : X \to \mathbb{R} \) is a nonnegative function that has a \( p,q \)-weak upper gradient \( g \in L^{p,q}(X, \mu) \). Then \( r(u + \varepsilon)^{-1}g \) is a \( p,q \)-weak upper gradient for \( u + \varepsilon \) for all \( \varepsilon > 0 \).

Proof. Fix \( \varepsilon > 0 \). We apply Proposition 3.11 to any \( C^1 \) function \( F : \mathbb{R} \to \mathbb{R} \) satisfying \( F(t) = t^r, \varepsilon \leq t < \infty \).

\( \square \)

Corollary 3.14. Suppose \( 1 \leq q \leq p < \infty \). Let \( u_1, u_2 \) be two nonnegative bounded real-valued functions defined on \( X \). Suppose \( g_i \in L^{p,q}(X, \mu), i = 1, 2 \) are \( p,q \)-weak upper gradients for \( u_i, i = 1, 2 \). Then \( L^{p,q}(X, \mu) \ni g := (g_1^q + g_2^q)^{1/q} \) is a \( p,q \)-weak upper gradient for \( u := (u_1^q + u_2^q)^{1/q} \).

Proof. The claim is obvious when \( q = 1 \), so we assume without loss of generality that \( 1 < q \leq p \). We prove first that \( g \in L^{p,q}(X, \mu) \). Indeed, via Remark 2.3 it is enough to show that \( g^q \in L^{q,1}(X, \mu) \). But \( g^q = g_1^q + g_2^q \) and \( g_i^q \in L^{q,1}(X, \mu) \) since \( g_i \in L^{p,q}(X, \mu) \). (See Remark 2.3.) This, the fact that \( \| \cdot \|_{L^{q,1}(X,\mu)} \) is a norm whenever \( 1 < q \leq p \), and another appeal to Remark 2.3 yield \( g \in L^{p,q}(X, \mu) \) with

\[
\|g\|_{L^{p,q}(X,\mu)} = \|g^q\|_{L^{q,1}(X,\mu)} \leq \|g_1^q\|_{L^{q,1}(X,\mu)} + \|g_2^q\|_{L^{q,1}(X,\mu)} = \|g_1\|_{L^{p,q}(X,\mu)} + \|g_2\|_{L^{p,q}(X,\mu)}.
\]

For \( i = 1, 2 \) let \( \Gamma_{i,1} \) be the family of nonconstant rectifiable curves in \( \Gamma_{\text{rect}} \) for which

\[
|u_i(x) - u_i(x)| \leq \int_\gamma g_i
\]

is not satisfied. Then \( \text{Mod}_{p,q}(\Gamma_{i,1}) = 0 \) since \( g_i \) is a \( p,q \)-weak upper gradient for \( u_i, i = 1, 2 \).

Let \( \Gamma_{i,2} \) be the family of nonconstant rectifiable curves in \( \Gamma_{\text{rect}} \) for which \( f, g_i = \infty \). Then for \( i = 1, 2 \) we have \( \text{Mod}_{p,q}(\Gamma_{i,2}) = 0 \) via Theorem 3.4 because by hypothesis \( g_i \in L^{p,q}(X, \mu), i = 1, 2 \). Let \( \Gamma_0 = \Gamma_{1,1} \cup \Gamma_{1,2} \cup \Gamma_{2,1} \cup \Gamma_{2,2} \). Then \( \text{Mod}_{p,q}(\Gamma_0) = 0 \).

Fix \( \varepsilon > 0 \). By applying Corollary 3.12 with \( r = q \) to the functions \( u_i \) for \( i = 1, 2 \), we see that \( L^{p,q}(X, \mu) \ni q\varepsilon^{-1}g_i \) is a \( p,q \)-weak upper gradient of \( (u_i + \varepsilon)^q \) for \( i = 1, 2 \). Thus via Hölder’s inequality it follows that \( G_\varepsilon \) is a \( p,q \)-weak upper gradient for \( U_\varepsilon \), where

\[
G_\varepsilon := q((u_1 + \varepsilon)^q + (u_2 + \varepsilon)^q)^{1/(q-1)}(g_1^q + g_2^q)^{1/q} \text{ and } U_\varepsilon := (u_1 + \varepsilon)^q + (u_2 + \varepsilon)^q.
\]

We notice that \( G_\varepsilon \in L^{p,q}(X, \mu) \). Indeed, \( G_\varepsilon = qU_\varepsilon^{(q-1)/q}g, \) with \( U_\varepsilon \) nonnegative a bounded and \( g \in L^{p,q}(X, \mu) \), so \( G_\varepsilon \in L^{p,q}(X, \mu) \).

Now we apply Corollary 3.13 with \( r = 1/q \), \( u = U_\varepsilon \) and \( g = G_\varepsilon \) to obtain that \( u_\varepsilon := U_\varepsilon^{1/q} \) has \( 1/qU_\varepsilon^{(1-q)/q}G_\varepsilon = g \) as a \( p,q \)-weak upper gradient that belongs to \( L^{p,q}(X, \mu) \). In fact, by looking at the proof of Proposition 3.11, we see that

\[
|u_\varepsilon(x) - u_\varepsilon(y)| \leq \int g
\]
for every curve \( \gamma \in \Gamma_{\text{rect}} \) that is not in \( \Gamma_0 \). Letting \( \varepsilon \to 0 \), we obtain the desired conclusion. This finishes the proof of the corollary.

\[ \Box \]

**Lemma 3.15.** If \( u_i, i = 1, 2 \) are nonnegative real-valued Borel functions in \( L^{p,q}(X,\mu) \) with corresponding \( p,q \)-weak upper gradients \( g_i \in L^{p,q}(X,\mu) \), then \( g := \max(g_1, g_2) \in L^{p,q}(X,\mu) \) is a \( p,q \)-weak upper gradient for \( u := \max(u_1, u_2) \in L^{p,q}(X,\mu) \).

**Proof.** It is easy to see that \( u, g \in L^{p,q}(X,\mu) \). For \( i = 1, 2 \) let \( \Gamma_{0,i} \subset \Gamma_{\text{rect}} \) be the family of nonconstant rectifiable curves \( \gamma \) for which \( \int_{\gamma} g_i = \infty \). Then we have via Theorem 3.4 that \( \text{Mod}_{p,q}(\Gamma_{0,i}) = 0 \) because \( g_i \in L^{p,q}(X,\mu) \). Thus \( \text{Mod}_{p,q}(\Gamma_0) = 0 \), where \( \Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2} \).

For \( i = 1, 2 \) let \( \Gamma_{i,1} \subset \Gamma_{\text{rect}} \) be the family of curves \( \gamma \in \Gamma_{\text{rect}} \setminus \Gamma_0 \) for which

\[ |u_i(\gamma(0)) - u_i(\gamma(\ell(\gamma)))| \leq \int_{\gamma} g_i \]

is not satisfied. Then \( \text{Mod}_{p,q}(\Gamma_{1,i}) = 0 \) since \( g_i \) is a \( p,q \)-weak upper gradient for \( u_i \), \( i = 1, 2 \). Thus \( \text{Mod}_{p,q}(\Gamma_1) = 0 \), where \( \Gamma_1 = \Gamma_{1,1} \cup \Gamma_{1,2} \).

It is easy to see that

\[ |u(x) - u(y)| \leq \max(\{|u_1(x) - u_1(y)|, |u_2(x) - u_2(y)|\}). \tag{10} \]

On every curve \( \gamma \in \Gamma_{\text{rect}} \setminus (\Gamma_0 \cup \Gamma_1) \) we have

\[ |u_i(x) - u_i(y)| \leq \int_{\gamma} g_i \leq \int_{\gamma} g. \]

This and (10) show that

\[ |u(x) - u(y)| \leq \int_{\gamma} g \]

on every curve \( \gamma \in \Gamma_{\text{rect}} \setminus (\Gamma_0 \cup \Gamma_1) \). This finishes the proof.

\[ \Box \]

**Lemma 3.16.** Suppose \( g \in L^{p,q}(X,\mu) \) is a \( p,q \)-weak upper gradient for \( 0 \leq u \in L^{p,q}(X,\mu) \). Let \( \lambda \geq 0 \) be fixed. Then \( u_\lambda := \min(\lambda, u) \in L^{p,q}(X,\mu) \) and \( g \) is a \( p,q \)-weak upper gradient for \( u_\lambda \).

**Proof.** Obviously \( 0 \leq u_\lambda \leq u \) on \( X \), so it follows via Bennett-Sharpley [1, Proposition I.1.7] and Kauhanen-Koskela-Malý [20, Proposition 2.1] that \( u_\lambda \in L^{p,q}(X,\mu) \) with \( \|u_\lambda\|_{L^{p,q}(X,\mu)} \leq \|u\|_{L^{p,q}(X,\mu)} \). The second claim follows immediately since \( |u_\lambda(x) - u_\lambda(y)| \leq |u(x) - u(y)| \) for every \( x, y \in X \).

\[ \Box \]

4. **Newtonian \( L^{p,q} \) spaces**

We denote by \( \hat{N}^{1,L^{p,q}}(X,\mu) \) the space of all Borel functions \( u \in L^{p,q}(X,\mu) \) that have a \( p,q \)-weak upper gradient \( g \in L^{p,q}(X,\mu) \). We note that \( \hat{N}^{1,L^{p,q}}(X,\mu) \) is a vector space, since if \( \alpha, \beta \in \mathbb{R} \) and \( u_1, u_2 \in \hat{N}^{1,L^{p,q}}(X,\mu) \) with respective \( p,q \)-weak upper gradients \( g_1, g_2 \in L^{p,q}(X,\mu) \), then \( \alpha g_1 + \beta g_2 \) is a \( p,q \)-weak upper gradient of \( \alpha u_1 + \beta u_2 \).

**Definition 4.1.** If \( u \) is a function in \( \hat{N}^{1,L^{p,q}}(X,\mu) \), let

\[ \|u\|_{\hat{N}^{1,L^{p,q}}} := \begin{cases} \left( \|u\|_{L^{p,q}(X,\mu)}^p + \inf_g \|g\|_{L^{p,q}(X,\mu)}^p \right)^{1/p}, & 1 \leq q \leq p, \\ \left( \|u\|_{L^{p,q}(X,\mu)}^q + \inf_g \|g\|_{L^{p,q}(X,\mu)}^p \right)^{1/p}, & p < q \leq \infty, \end{cases} \]
where the infimum is taken over all \( p, q \)-integrable \( p, q \)-weak upper gradients of \( u \).

Similarly, let

\[
\|u\|_{\widetilde{N}^{1,L,p,q}} := \left\{ \left( \|u\|_{L^{p,q}(X,\mu)}^{q} + \inf_{g} \|g\|_{L^{p,q}(X,\mu)}^{q} \right)^{1/q} \right\}, \quad 1 \leq q \leq p,
\]

\[
\left( \|u\|_{L^{p,q}(X,\mu)}^{p} + \inf_{g} \|g\|_{L^{p,q}(X,\mu)}^{p} \right)^{1/p}, \quad p < q \leq \infty,
\]

where the infimum is taken over all \( p, q \)-integrable \( p, q \)-weak upper gradients of \( u \).

If \( u, v \) are functions in \( \widetilde{N}^{1,L,p,q}(X,\mu) \), let \( u \sim v \) if \( \|u - v\|_{\widetilde{N}^{1,L,p,q}} = 0 \). It is easy to see that \( \sim \) is an equivalence relation that partitions \( \widetilde{N}^{1,L,p,q}(X,\mu) \) into equivalence classes. We define the space \( N^{1,L,p,q}(X,\mu) \) as the quotient \( \widetilde{N}^{1,L,p,q}(X,\mu) / \sim \) and

\[
\|u\|_{N^{1,L,p,q}} = \|u\|_{\widetilde{N}^{1,L,p,q}} \quad \text{and} \quad \|u\|_{N^{1,L,p,q}(u)} = \|u\|_{\widetilde{N}^{1,L,p,q}((u)}
\]

**Remark 4.2.** Via Lemma 3.9 and Corollary 2.8, it is easy to see that the infima in Definition 4.1 could as well be taken over all \( p, q \)-integrable upper gradients of \( u \). We also notice (see the discussion before Definition 2.1) that \( \|\,\cdot\|_{N^{1,L,p,q}} \) is a norm whenever \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \), while \( \|\,\cdot\|_{N^{1,L,p,q}} \) is a norm when \( 1 \leq q \leq p < \infty \). Moreover (see the discussion before Definition 2.1)

\[
\|u\|_{N^{1,L,p,q}} \leq \|u\|_{N^{1,L,p,q}(u)} \leq p^{|u|}_{N^{1,L,p,q}}
\]

for every \( 1 < p < \infty \), \( 1 \leq q \leq \infty \) and \( u \in N^{1,L,p,q}(X,\mu) \).

**Definition 4.3.** Let \( u : X \to [-\infty, \infty] \) be a given function. We say that

(i) \( u \) is absolutely continuous along a rectifiable curve \( \gamma \) if \( u \circ \gamma \) is absolutely continuous on \([0, \ell(\gamma)]\).

(ii) \( u \) is absolutely continuous on \( p, q \)-almost every curve (has \( ACC_{p,q} \) property) if for \( p, q \)-almost every \( \gamma \in \Gamma_{\text{rect}} \), \( u \circ \gamma \) is absolutely continuous.

**Proposition 4.4.** If \( u \) is a function in \( \widetilde{N}^{1,L,p,q}(X,\mu) \), then \( u \) is \( ACC_{p,q} \).

**Proof.** We follow Shanmugalingam [27]. By the definition of \( \widetilde{N}^{1,L,p,q}(X,\mu) \), \( u \) has a \( p, q \)-weak upper gradient \( g \in L^{p,q}(X,\mu) \). Let \( \Gamma_{0} \) be the collection of all curves in \( \Gamma_{\text{rect}} \) for which

\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma))))| \leq \int_{\gamma} g
\]

is not satisfied. Then by the definition of \( p, q \)-weak upper gradients, \( \text{Mod}_{p,q}(\Gamma_{0}) = 0 \). Let \( \Gamma_{1} \) be the collection of all curves in \( \Gamma_{\text{rect}} \) that have a subcurve in \( \Gamma_{0} \). Then \( \text{Mod}_{p,q}(\Gamma_{1}) \leq \text{Mod}_{p,q}(\Gamma_{0}) = 0 \).

Let \( \Gamma_{2} \) be the collection of all curves in \( \Gamma_{\text{rect}} \) such that \( \int_{\gamma} g = \infty \). Then \( \text{Mod}_{p,q}(\Gamma_{2}) = 0 \) because \( g \in L^{p,q}(X,\mu) \). Hence \( \text{Mod}_{p,q}(\Gamma_{1} \cup \Gamma_{2}) = 0 \). If \( \gamma \) is a curve in \( \Gamma_{\text{rect}} \setminus (\Gamma_{1} \cup \Gamma_{2}) \), then \( \gamma \) has no subcurves in \( \Gamma_{0} \), and hence

\[
|u(\gamma(\beta)) - u(\gamma(\alpha))| \leq \int_{\alpha}^{\beta} g(\gamma(t)) \, dt, \quad \text{provided} \quad [\alpha, \beta] \subset [0, \ell(\gamma)].
\]

This implies the absolute continuity of \( u \circ \gamma \) as a consequence of the absolute continuity of the integral. Therefore \( u \) is absolutely continuous on every curve \( \gamma \) in \( \Gamma_{\text{rect}} \setminus (\Gamma_{1} \cup \Gamma_{2}) \). \( \square \)

**Lemma 4.5.** Suppose \( u \) is a function in \( \widetilde{N}^{1,L,p,q}(X,\mu) \) such that \( \|u\|_{L^{p,q}(X,\mu)} = 0 \). Then the family

\[
\Gamma = \{ \gamma \in \Gamma_{\text{rect}} : u(x) \neq 0 \text{ for some } x \in |\gamma| \}
\]

has \( p,q \)-modulus.
Proof. We follow Shanmugalingam [27]. Since \(|u|_{L^{p,q}(X,\mu)} = 0\), the set \(E = \{x \in X : u(x) \neq 0\}\) has measure zero. With the notation introduced earlier, we have
\[
\Gamma = \Gamma_E = \Gamma_E^{+} \cup (\Gamma_E \setminus \Gamma_E^{+}).
\]
We can disregard the family \(\Gamma_E^{+}\), since
\[
\text{Mod}_{p,q}(\Gamma_E^{+}) \leq ||\infty \cdot \chi_E||^p_{L^{p,q}(X,\mu)} = 0,
\]
where \(\chi_E\) is the characteristic function of the set \(E\). The curves \(\gamma\) in \(\Gamma_E \setminus \Gamma_E^{+}\) intersect \(E\) only on a set of linear measure zero, and hence with respect to the linear measure almost everywhere on \(\gamma\) the function \(u\) is equal to zero. Since \(\gamma\) also intersects \(E\), it follows that \(u\) is not absolutely continuous on \(\gamma\). By Proposition 4.4, we have \(\text{Mod}_{p,q}(\Gamma_E \setminus \Gamma_E^{+}) = 0\), yielding \(\text{Mod}_{p,q}(\Gamma) = 0\). This finishes the proof.

\[\square\]

**Lemma 4.6.** Let \(F\) be a closed subset of \(X\). Suppose that \(u : X \to [-\infty, \infty]\) is a Borel ACC\(p,q\) function that is constant \(\mu\)-almost everywhere on \(F\). If \(g \in L^{p,q}(X,\mu)\) is a \(p,q\)-weak upper gradient of \(u\), then \(g\chi_{X \setminus F}\) is a \(p,q\)-weak upper gradient of \(u\).

Proof. We can assume without loss of generality that \(u = 0\) \(\mu\)-almost everywhere on \(F\). Let \(E = \{x \in F : u(x) \neq 0\}\). Then by assumption \(\mu(E) = 0\). Hence \(\text{Mod}_{p,q}(\Gamma_E^{+}) = 0\) because \(\infty \cdot \chi_E \in F(\Gamma_E^{+})\).

Let \(\Gamma_0 \subset \Gamma_{\text{rect}}\) be the family of curves on which \(u\) is not absolutely continuous or on which
\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma} g
\]
is not satisfied. Then \(\text{Mod}_{p,q}(\Gamma_0) = 0\). Let \(\Gamma_1 \subset \Gamma_{\text{rect}}\) be the family of curves that have a subcurve in \(\Gamma_0\). Then \(F(\Gamma_0) \subset F(\Gamma_1)\) and thus \(\text{Mod}_{p,q}(\Gamma_1) \leq \text{Mod}_{p,q}(\Gamma_0) = 0\).

Let \(\Gamma_2 \subset \Gamma_{\text{rect}}\) be the family of curves on which \(\int_{\gamma} g = \infty\). Then via Theorem 3.4 we have \(\text{Mod}_{p,q}(\Gamma_2) = 0\) because \(g \in L^{p,q}(X,\mu)\).

Let \(\gamma : [0, \ell(\gamma)] \to X\) be a curve in \(\Gamma_{\text{rect}} \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_E^{+})\) connecting \(x\) and \(y\). We show that
\[
|u(x) - u(y)| \leq \int_{\gamma} g\chi_{X \setminus F}
\]
for every such curve \(\gamma\).

The cases \(|\gamma| \subset F \setminus E\) and \(|\gamma| \subset (X \setminus F) \cup E\) are trivial. So is the case when both \(x\) and \(y\) are in \(F \setminus E\). Let \(K := (u \circ \gamma)^{-1}(\{0\})\). Then \(K\) is a compact subset of \([0, \ell(\gamma)]\) because \(u \circ \gamma\) is continuous on \([0, \ell(\gamma)]\). Hence \(K\) contains its lower bound \(c\) and its upper bound \(d\). Let \(x_1 = \gamma(c)\) and \(y_1 = \gamma(d)\).

Suppose that both \(x\) and \(y\) are in \((X \setminus F) \cup E\). Then we see that \([c, d] \subset (0, \ell(\gamma))\) and \(\gamma([0, c] \cup [d, \ell(\gamma)]) \subset (X \setminus F) \cup E\).

Moreover, since \(\gamma\) is not in \(\Gamma_1\) and \(u(x_1) = u(y_1)\), we have
\[
|u(x) - u(y)| \leq |u(x) - u(x_1)| + |u(y_1) - u(y)| \leq \int_{\gamma([0, c])} g + \int_{\gamma([d, \ell(\gamma)])} g \leq \int_{\gamma} g\chi_{X \setminus F}
\]
because the subcurves \(\gamma|_{[0, c]}\) and \(\gamma|_{[d, \ell(\gamma)]}\) intersect \(E\) on a set of Hausdorff 1-measure zero.

Suppose now by symmetry that \(x \in (X \setminus F) \cup E\) and \(y \in F \setminus E\). This means in terms of our notation that \(c > 0\) and \(d = \ell(\gamma)\). We notice that \(\gamma([0, c]) \subset (X \setminus F) \cup E\) and
Lemma 4.7. Assume that \( u \in N^{1,L^{p,q}}(X,\mu) \), and that \( g, h \in L^{p,q}(X,\mu) \) are \( p, q \)-weak upper gradients of \( u \). If \( F \subset X \) is a closed set, then
\[
\rho = g\chi_F + h\chi_{X \setminus F}
\]
is a \( p, q \)-weak upper gradient of \( u \) as well.

Proof. We follow Hajłasz [13]. Let \( \Gamma_1 \subset \Gamma_{\text{rect}} \) be the family of curves on which \( f_i(g + h) = \infty \). Then via Theorem 3.4 it follows that \( \text{Mod}_{p,q}(\Gamma_1) = 0 \) because \( g + h \in L^{p,q}(X,\mu) \).

Let \( \Gamma_2 \subset \Gamma_{\text{rect}} \) be the family of curves on which \( u \) is not absolutely continuous. Then via Proposition 4.4 we see that \( \text{Mod}_{p,q}(\Gamma_2) = 0 \).

Let \( \Gamma'_3 \subset \Gamma_{\text{rect}} \) be the family of curves on which
\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \min\left( \int_{\gamma} g, \int_{\gamma} h \right)
\]
is not satisfied. Let \( \Gamma_3 \subset \Gamma_{\text{rect}} \) be the family of curves which contain subcurves belonging to \( \Gamma'_3 \). Since \( F(\Gamma'_3) \subset F(\Gamma_3) \), we have \( \text{Mod}_{p,q}(\Gamma_3) \leq \text{Mod}_{p,q}(\Gamma'_3) = 0 \). Now it remains to show that
\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma} \rho
\]
for all \( \gamma \in \Gamma_{\text{rect}} \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3) \). If \( |\gamma| \subset F \) or \( |\gamma| \subset X \setminus F \), then the inequality is obvious. Thus we can assume that the image \( |\gamma| \) has a nonempty intersection both with \( F \) and with \( X \setminus F \).

The set \( \gamma^{-1}(X \setminus F) \) is open and hence it consists of a countable (or finite) number of open and disjoint intervals. Assume without loss of generality that there are countably many such intervals. Denote these intervals by \( ((t_i, s_i))_{i=1}^\infty \). Let \( \gamma_i = |t_i, s_i| \). We have
\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq |u(\gamma(0)) - u(\gamma(t_i))| + |u(\gamma(t_i)) - u(\gamma(s_i))| + |u(\gamma(s_i)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma \setminus \gamma_1} g + \int_{\gamma_1} h,
\]
where \( \gamma \setminus \gamma_1 \) denotes the two curves obtained from \( \gamma \) by removing the interior part \( \gamma_1 \), that is the curves \( \gamma|_{[0,t_i]} \) and \( \gamma|_{[s_i, b]} \). Similarly we can remove a larger number of subcurves of \( \gamma \). This yields
\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma \setminus \cup_{i=1}^n \gamma_i} g + \int_{\cup_{i=1}^n \gamma_i} h
\]
for each positive integer \( n \). By applying Lebesgue dominated convergence theorem to the curve integral on \( \gamma_i \), we obtain
\[
|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_{\gamma} g\chi_F + \int_{\gamma} h\chi_{X \setminus F} = \int_{\gamma} \rho.
\]
Next we show that when $1 < p < \infty$ and $1 \leq q < \infty$, every function $u \in N^{1,L^{p,q}}(X, \mu)$ has a ‘smallest’ $p,q$-weak upper gradient. For the case $p = q$ see Kallunki-Shanmugalingam [19] and Shanmugalingam [28].

**Theorem 4.8.** Suppose that $1 < p < \infty$ and $1 \leq q < \infty$. For every $u \in N^{1,L^{p,q}}(X, \mu)$, there exists the least $p,q$-weak upper gradient $g_u \in L^{p,q}(X, \mu)$ of $u$. It is smallest in the sense that if $g \in L^{p,q}(X, \mu)$ is another $p,q$-weak upper gradient of $u$, then $g \geq g_u$ $\mu$-almost everywhere.

**Proof.** We follow Hajlasz [13]. Let $m = \inf_g \|g\|_{L^{p,q}(X, \mu)}$, where the infimum is taken over the set of all $p,q$-weak upper gradients of $u$. It suffices to show that there exists a $p,q$-weak upper gradient $g_u$ of $u$ such that $\|g_u\|_{L^{p,q}(X, \mu)} = m$. Indeed, if we suppose that $g \in L^{p,q}(X, \mu)$ is another $p,q$-weak upper gradient of $u$ such that the set $\{g < g_u\}$ has positive measure, then by the inner regularity of the measure $\mu$ there exists a closed set $F \subset \{g < g_u\}$ such that $\mu(F) > 0$. Via Lemma 4.7 it follows that the function $\rho := g\chi_F + g_u\chi_{F^c}$ is a $p,q$-weak upper gradient. Via Kauhanen-Koskela-Malý [20, Proposition 2.1] that would give $\|\rho\|_{L^{p,q}(X, \mu)} < \|g_u\|_{L^{p,q}(X, \mu)} = m$, in contradiction with the minimality of $\|g_u\|_{L^{p,q}(X, \mu)}$.

Thus it remains to prove the existence of a $p,q$-weak upper gradient $g_u$ such that $\|g_u\|_{L^{p,q}(X, \mu)} = m$. Let $(g_i)_{i=1}^\infty$ be a sequence of $p,q$-weak upper gradients of $u$ such that $\|g_i\|_{L^{p,q}(X, \mu)} < m + 2^{-i}$. We will show that it is possible to modify the sequence $(g_i)$ in such a way that we will obtain a new sequence of $p,q$-weak upper gradients $(\rho_i)$ of $u$ satisfying

$$\|\rho_i\|_{L^{p,q}(X, \mu)} < m + 2^{1-i}, \quad \rho_1 \geq \rho_2 \geq \rho_3 \geq \ldots \mu\text{-almost everywhere.}$$

The sequence $(\rho_i)_{i=1}^\infty$ will be defined by induction. We set $\rho_1 = g_1$. Suppose the $p,q$-weak upper gradients $\rho_1, \rho_2, \ldots, \rho_i$ have already been chosen. We will now define $\rho_{i+1}$. Since $\rho_i \in L^{p,q}(X, \mu)$, the measure $\mu$ is inner regular and the $(p,q)$-norm has the absolute continuity property whenever $1 < p < \infty$ and $1 \leq q < \infty$ (see the discussion after Definition 2.1), there exists a closed set $F \subset \{g_{i+1} < \rho_i\}$ such that

$$\|\rho_i\chi_{\{g_{i+1} < \rho_i\}}\|_{L^{p,q}(X, \mu)} < 2^{-i-1}.$$ 

Now we set $\rho_{i+1} = g_{i+1}\chi_F + \rho_i\chi_{F^c}$. Then

$$\rho_{i+1} \leq \rho_i \text{ and } \rho_{i+1} \leq g_{i+1}\chi_{F \cup \{g_{i+1} \geq \rho_i\}} + \rho_i\chi_{\{g_{i+1} < \rho_i\}}\chi_F .$$

Suppose first that $1 \leq q \leq p$. Since $\|\cdot\|_{L^{p,q}(X, \mu)}$ is a norm, we see that

$$\|\rho_{i+1}\|_{L^{p,q}(X, \mu)} \leq \|g_{i+1}\chi_{F \cup \{g_{i+1} \geq \rho_i\}}\|_{L^{p,q}(X, \mu)} + \|\rho_i\chi_{\{g_{i+1} < \rho_i\}}\chi_F\|_{L^{p,q}(X, \mu)} < m + 2^{-i-1} + 2^{-i-1} = m + 2^{-i} .$$

Suppose now that $p < q < \infty$. Then we have via Proposition 2.6

$$\|\rho_{i+1}\|_{L^{p,q}(X, \mu)} \leq \|g_{i+1}\chi_{F \cup \{g_{i+1} \geq \rho_i\}}\|_{L^{p,q}(X, \mu)}^p + \|\rho_i\chi_{\{g_{i+1} < \rho_i\}}\chi_F\|_{L^{p,q}(X, \mu)}^p < (m + 2^{i-1})^p + 2^{-p(i+1)} < (m + 2^{-i})^p .$$

Thus, no matter what $q \in [1, \infty)$ is, we showed that $m \leq \|\rho_{i+1}\|_{L^{p,q}(X, \mu)} < m + 2^{-i} . \quad \Box$

The sequence of $p,q$-weak upper gradients $(\rho_i)_{i=1}^\infty$ converges pointwise to a function $\rho$. The absolute continuity of the $(p,q)$-norm (see Bennett-Sharpley [1, Proposition I.3.6] and the discussion after Definition 2.1) yields

$$\lim_{i \to \infty} \|\rho_i - \rho\|_{L^{p,q}(X, \mu)} = 0.$$ 

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Obviously $||\rho||_{L^{p,q}(X,\mu)} = m$. The proof will be finished as soon as we show that $\rho$ is a $p, q$-weak upper gradient for $u$.

By taking a subsequence if necessary, we can assume that $||\rho_i - \rho||_{L^{p,q}(X,\mu)} \leq 2^{-2i}$ for every $i \geq 1$.

Let $\Gamma_1 \subset \Gamma_{\text{rect}}$ be the family of curves on which $f_\gamma(\rho + \rho_i) = \infty$ for some $i \geq 1$. Then via Theorem 3.4 and the subadditivity of $\text{Mod}_{p,q}$ follows via Theorem 3.6 that $\text{Mod}_{p, q}$ is not satisfied. Then $\text{Mod}_{p,q}(\Gamma_1) = 0$ since $\rho + \rho_i \in L^{p,q}(X,\mu)$ for every $i \geq 1$.

For any integer $i \geq 1$ let $\Gamma_{2,i} \subset \Gamma_{\text{rect}}$ be the family of curves for which

$$|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_\gamma \rho_i$$

is not satisfied. Then $\text{Mod}_{p,q}(\Gamma_{2,i}) = 0$ because $\rho_i$ is a $p, q$-weak upper gradient for $u$.

Let $\Gamma_2 = \bigcup_{i=1}^\infty \Gamma_{2,i}$.

Let $\Gamma_3 \subset \Gamma_{\text{rect}}$ be the family of curves for which $\limsup_{i \to \infty} \int_\gamma |\rho_i - \rho| > 0$. Then it follows via Theorem 3.6 that $\text{Mod}_{p,q}(\Gamma_3) = 0$.

Let $\gamma$ be a curve in $\Gamma_{\text{rect}} \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$. On any such curve we have (since $\gamma$ is not in $\Gamma_{2,i}$)

$$|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \int_\gamma \rho_i$$

for every $i \geq 1$.

By letting $i \to \infty$, we obtain (since $\gamma$ is not in $\Gamma_1 \cup \Gamma_3$)

$$|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \leq \lim_{i \to \infty} \int_\gamma \rho_i = \int_\gamma \rho < \infty.$$

This finishes the proof of the theorem.

\[\square\]

5. Sobolev $p, q$-capacity

In this section, we establish a general theory of the Sobolev-Lorentz $p, q$-capacity in metric measure spaces. If $(X, d, \mu)$ is a metric measure space, then the Sobolev $p, q$-capacity of a set $E \subset X$ is

$$\text{Cap}_{p,q}(E) = \inf\{||u||_{N^1, L^{p,q}}^p : u \in \mathcal{A}(E)\},$$

where

$$\mathcal{A}(E) = \{u \in N^1, L^{p,q}(X, \mu) : u \geq 1 \text{ on } E\}.$$

We call $\mathcal{A}(E)$ the set of admissible functions for $E$. If $\mathcal{A}(E) = \emptyset$, then $\text{Cap}_{p,q}(E) = \infty$.

\textbf{Remark 5.1.} It is easy to see that we can consider only admissible functions $u$ for which $0 \leq u \leq 1$. Indeed, for $u \in \mathcal{A}(E)$, let $v := \min(u_+, 1)$, where $u_+ = \max(u, 0)$. We notice that $|v(x) - v(y)| \leq |u(x) - u(y)|$ for every $x, y$ in $X$, which implies that every $p, q$-weak upper gradient for $u$ is also a $p, q$-weak upper gradient for $v$. This implies that $v \in \mathcal{A}(E)$ and $||v||_{N^1, L^{p,q}} \leq ||u||_{N^1, L^{p,q}}$.

\textbf{5.1. Basic properties of the Sobolev $p, q$-capacity.} A capacity is a monotone, subadditive set function. The following theorem expresses, among other things, that this is true for the Sobolev $p, q$-capacity.

\textbf{Theorem 5.2.} Suppose that $1 < p < \infty$ and $1 \leq q \leq \infty$. Suppose that $(X, d, \mu)$ is a complete metric measure space. The set function $E \mapsto \text{Cap}_{p,q}(E)$, $E \subset X$, enjoys the following properties:

(i) If $E_1 \subset E_2$, then $\text{Cap}_{p,q}(E_1) \leq \text{Cap}_{p,q}(E_2)$.
(ii) Suppose that $\mu$ is nonatomic. Suppose that $1 < q \leq p$. If $E_1 \subset E_2 \subset \ldots \subset E = \bigcup_{i=1}^{\infty} E_i \subset X$, then
\[ \text{Cap}_{p,q}(E) = \lim_{i \to \infty} \text{Cap}_{p,q}(E_i). \]
(iii) Suppose that $p < q \leq \infty$. If $E = \bigcup_{i=1}^{\infty} E_i \subset X$, then
\[ \text{Cap}_{p,q}(E) \leq \sum_{i=1}^{\infty} \text{Cap}_{p,q}(E_i). \]
(iv) Suppose that $1 \leq q \leq p$. If $E = \bigcup_{i=1}^{\infty} E_i \subset X$, then
\[ \text{Cap}_{p,q}(E)^{q/p} \leq \sum_{i=1}^{\infty} \text{Cap}_{p,q}(E_i)^{q/p}. \]

Proof. Property (i) is an immediate consequence of the definition.

(ii) Monotonicity yields
\[ L := \lim_{i \to \infty} \text{Cap}_{p,q}(E_i) \leq \text{Cap}_{p,q}(E). \]
To prove the opposite inequality, we may assume without loss of generality that $L < \infty$. The reflexivity of $L^{p,q}(X, \mu)$ (guaranteed by the nonatomicity of $\mu$ whenever $1 < q \leq p < \infty$) will be used here in order to prove the opposite inequality.

Let $\varepsilon > 0$ be fixed. For every $i = 1, 2, \ldots$ we choose $u_i \in A(E_i)$, $0 \leq u_i \leq 1$ and a corresponding upper gradient $g_i$ such that
\[ \|u_i\|_{N^{1,L^{p,q}}(X, \mu)} \leq \text{Cap}_{p,q}(E_i)^{q/p} + \varepsilon \leq L^{q/p} + \varepsilon. \]
We notice that $u_i$ is a bounded sequence in $N^{1,L^{p,q}}(X, \mu)$. Hence there exists a subsequence, which we denote again by $u_i$ and functions $u, g \in L^{p,q}(X, \mu)$ such that $u_i \to u$ weakly in $L^{p,q}(X, \mu)$ and $g_i \to g$ weakly in $L^{p,q}(X, \mu)$. It is easy to see that
\[ u \geq 0 \mu\text{-almost everywhere and } g \geq 0 \mu\text{-almost everywhere.} \]
Indeed, since $u_i$ converges weakly to $u$ in $L^{p,q}(X, \mu)$ which is the dual of $L^{p',q'}(X, \mu)$ (see Hunt [18, p. 262]), we have
\[ \lim_{i \to \infty} \int_X u_i(x) \varphi(x) \, d\mu(x) = \int_X u(x) \varphi(x) \, d\mu(x) \]
for all $\varphi \in L^{p',q'}(X, \mu)$. For nonnegative functions $\varphi \in L^{p',q'}(X, \mu)$ this yields
\[ 0 \leq \lim_{i \to \infty} \int_X u_i(x) \varphi(x) \, d\mu(x) = \int_X u(x) \varphi(x) \, d\mu(x), \]
which easily implies $u \geq 0 \mu\text{-almost everywhere on } X$. Similarly we have $g \geq 0 \mu\text{-almost everywhere on } X$.

From the weak-* lower semicontinuity of the $p, q$-norm (see Bennett-Sharpley [1, Proposition II.4.2, Definition IV.4.1 and Theorem IV.4.3] and Hunt [18, p. 262]), it follows that
\[ \|u\|_{L^{p,q}(X, \mu)} \leq \liminf_{i \to \infty} \|u_i\|_{L^{p,q}(X, \mu)} \text{ and } \|g\|_{L^{p,q}(X, \mu)} \leq \liminf_{i \to \infty} \|g_i\|_{L^{p,q}(X, \mu)}. \]
Using Mazur’s lemma simultaneously for $u_i$ and $g_i$, we obtain sequences $v_i$ with correspondent upper gradients $\overline{g}_i$ such that $v_i \in A(E_i)$, $v_i \to u$ in $L^{p,q}(X, \mu)$ and $\mu$-almost everywhere and $\overline{g}_i \to g$ in $L^{p,q}(X, \mu)$ and $\mu$-almost everywhere. These sequences can be found in the following way. Let $i_0$ be fixed. Since every subsequence of $(u_i, g_i)$ converges to $(u, g)$ weakly in the reflexive space $L^{p,q}(X, \mu) \times L^{p,q}(X, \mu)$, we may use the Mazur lemma (see Yosida [30, p. 120]) for the subsequence $(u_i, g_i), i \geq i_0$.  

We obtain finite convex combinations $v_{i_0}$ and $\tilde{g}_{i_0}$ of the functions $u_i$ and $g_i$, $i \geq i_0$ as close as we want in $L^{p,q}(X, \mu)$ to $u$ and $g$ respectively. For every $i = i_0, i_0 + 1, \ldots$ we see that $u_i = 1$ in $E_i \supseteq E_{i_0}$. The intersection of finitely many supersets of $E_{i_0}$ contains $E_{i_0}$. Therefore, $v_{i_0}$ equals 1 on $E_{i_0}$. It is easy to see that $\tilde{g}_{i_0}$ is an upper gradient for $v_{i_0}$.

Passing to subsequences if necessary, we may assume that $v_i$ converges to $u$ pointwise $\mu$-almost everywhere, that $\tilde{g}_i$ converges to $g$ pointwise $\mu$-almost everywhere and that for every $i = 1, 2, \ldots$ we have

$$\|v_{i+1} - v_i\|_{L^{p,q}(X,\mu)} + \|\tilde{g}_{i+1} - \tilde{g}_i\|_{L^{p,q}(X,\mu)} \leq 2^{-i}. \quad \text{(13)}$$

Since $v_i$ converges to $u$ in $L^{p,q}(X, \mu)$ and pointwise $\mu$-almost everywhere on $X$ while $\tilde{g}_i$ converges to $g$ in $L^{p,q}(X, \mu)$ and pointwise $\mu$-almost everywhere on $X$ it follows via Corollary 2.8 that

$$\lim_{i \to \infty} \|v_i\|_{L^{p,q}(X,\mu)} = \|u\|_{L^{p,q}(X,\mu)} \quad \text{and} \quad \lim_{i \to \infty} \|\tilde{g}_i\|_{L^{p,q}(X,\mu)} = \|g\|_{L^{p,q}(X,\mu)}. \quad \text{(14)}$$

This, (11) and (12) yield

$$\|u\|_{L^{p,q}(X,\mu)}^p + \|g\|_{L^{p,q}(X,\mu)}^p = \lim_{i \to \infty} \|v_i\|_{N^{1,p,q}(X,\mu)}^q \leq L^{q/p} + \varepsilon. \quad \text{(15)}$$

For $j = 1, 2, \ldots$ we set

$$w_j = \sup_{i \geq j} v_i \quad \text{and} \quad \tilde{g}_j = \sup_{i \geq j} \tilde{g}_i.$$ 

It is easy to see that $w_j = 1$ on $E$. We claim that $\tilde{g}_j$ is a $p,q$-weak upper gradient for $w_j$. Indeed, for every $k > j$, let

$$w_{j,k} = \sup_{k \geq j} v_i.$$ 

Via Lemma 3.15 and finite induction, it follows easily that $\tilde{g}_j$ is a $p,q$-weak upper gradient for every $w_{j,k}$ whenever $k > j$. It is easy to see that $w_j = \lim_{k \to \infty} w_{j,k}$ pointwise in $X$. This and Lemma 3.10 imply that $\tilde{g}_j$ is indeed a $p,q$-weak upper gradient for $w_j$.

Moreover,

$$w_j \leq v_j + \sum_{i=j}^{\infty} |v_{i+1} - v_i| \quad \text{and} \quad \tilde{g}_j \leq \tilde{g}_j + \sum_{i=j}^{k-1} |\tilde{g}_{i+1} - \tilde{g}_i|.$$ 

Thus

$$\|w_j\|_{L^{p,q}(X,\mu)} \leq \|v_j\|_{L^{p,q}(X,\mu)} + \sum_{i=j}^{\infty} \|v_{i+1} - v_i\|_{L^{p,q}(X,\mu)} \leq \|v_j\|_{L^{p,q}(X,\mu)} + 2^{-j+1} \quad \text{and}$$

$$\|\tilde{g}_j\|_{L^{p,q}(X,\mu)} \leq \|\tilde{g}_j\|_{L^{p,q}(X,\mu)} + \sum_{i=j}^{\infty} \|\tilde{g}_{i+1} - \tilde{g}_i\|_{L^{p,q}(X,\mu)} \leq \|\tilde{g}_j\|_{L^{p,q}(X,\mu)} + 2^{-j+1},$$

which implies that $w_j, \tilde{g}_j \in L^{p,q}(X, \mu)$. Thus $w_j \in A(E)$ with $p,q$-weak upper gradient $\tilde{g}_j$. We notice that $0 \leq g = \inf_{j \geq 1} \tilde{g}_j$ $\mu$-almost everywhere on $X$ and $0 \leq u = \inf_{j \geq 1} w_j$ $\mu$-almost everywhere on $X$. Since $w_i$ and $\tilde{g}_i$ are in $L^{p,q}(X, \mu)$, the absolute continuity of the $p,q$-norm (see Bennett-Sharpley [1, Proposition I.3.6] and the discussion after Definition 2.1) yields

$$\lim_{j \to \infty} \|w_j - u\|_{L^{p,q}(X,\mu)} = 0 \quad \text{and} \quad \lim_{j \to \infty} \|\tilde{g}_j - g\|_{L^{p,q}(X,\mu)} = 0. \quad \text{(17)}$$

By using (15), (17), and Corollary 2.8 we see that

$$\text{Cap}_{p,q}(E)^{q/p} \leq \lim_{j \to \infty} \|w_j\|_{N^{1,p,q}(X,\mu)}^q = \|u\|_{L^{p,q}(X,\mu)}^q + \|g\|_{L^{p,q}(X,\mu)}^q \leq L^{q/p} + \varepsilon.$$
By letting $\varepsilon \to 0$, we get the converse inequality so (ii) is proved.

(iii) We can assume without loss of generality that
\[ \sum_{i=1}^{\infty} \text{Cap}_{p,q}(E_i)^{q/p} < \infty. \]

For $i = 1, 2, \ldots$ let $u_i \in A(E_i)$ with upper gradient $g_i$ such that
\[ 0 \leq u_i \leq 1 \text{ and } ||u_i||_{p,1,L^p,q}^q < \text{Cap}_{p,q}(E_i)^{q/p} + \varepsilon 2^{-i}. \]

Let $u := (\sum_{i=1}^{\infty} u_i^q)^{1/q}$ and $g := (\sum_{i=1}^{\infty} g_i^q)^{1/q}$. We notice that $u \geq 1$ on $E$. By repeating the argument from the proof of Theorem 3.2 (iii), we see that $u, g \in L^{p,q}(X, \mu)$ and
\[ ||u||_{L^{p,q}(X,\mu)}^p + ||g||_{L^{p,q}(X,\mu)}^p \leq \sum_{i=1}^{\infty} \left( ||u_i||_{L^{p,q}(X,\mu)}^p + ||g_i||_{L^{p,q}(X,\mu)}^p \right) \leq 2\varepsilon + \sum_{i=1}^{\infty} \text{Cap}_{p,q}(E_i)^{q/p}. \]

We are done with the case $1 \leq q \leq p$ as soon as we show that $u \in A(E)$ and that $g$ is a $p, q$-weak upper gradient for $u$. It follows easily via Corollary 3.14 and finite induction that $g$ is a $p, q$-weak upper gradient for $\tilde{u}_n := (\sum_{i=1}^{n} u_i^q)^{1/q}$ for every $n \geq 1$. Since $u(x) = \lim_{i \to \infty} \tilde{u}_i(x) < \infty$ on $X \setminus F$, where $F = \{ x \in X : u(x) = \infty \}$ it follows from Lemma 3.10 combined with the fact that $u \in L^{p,q}(X, \mu)$ that $g$ is in fact a $p, q$-weak upper gradient for $u$. This finishes the proof for the case $1 \leq q \leq p$.

(iv) We can assume without loss of generality that
\[ \sum_{i=1}^{\infty} \text{Cap}_{p,q}(E_i) < \infty. \]

For $i = 1, 2, \ldots$ let $u_i \in A(E_i)$ with upper gradients $g_i$ such that
\[ 0 \leq u_i \leq 1 \text{ and } ||u_i||_{p,1,L^p,q}^p < \text{Cap}_{p,q}(E_i) + \varepsilon 2^{-i}. \]

Let $u := \sup_{i \geq 1} u_i$ and $g := \sup_{i \geq 1} g_i$. We notice that $u = 1$ on $E$. Moreover, via Proposition 2.6 it follows that $u, g \in L^{p,q}(X, \mu)$ with
\[ ||u||_{L^{p,q}(X,\mu)}^p + ||g||_{L^{p,q}(X,\mu)}^p \leq \sum_{i=1}^{\infty} \left( ||u_i||_{L^{p,q}(X,\mu)}^p + ||g_i||_{L^{p,q}(X,\mu)}^p \right) \leq 2\varepsilon + \sum_{i=1}^{\infty} \text{Cap}_{p,q}(E_i). \]

We are done with the case $p < q \leq \infty$ as soon as we show that $u \in A(E)$ and that $g$ is a $p, q$-weak upper gradient for $u$. Via Lemma 3.15 and finite induction, it follows that $g$ is a $p, q$-weak upper gradient for $\tilde{u}_n := \max_{1 \leq i \leq n} u_i$ for every $n \geq 1$. Since $u(x) = \lim_{i \to \infty} \tilde{u}_i(x)$ pointwise on $X$, it follows via Lemma 3.10 that $g$ is in fact a $p, q$-weak upper gradient for $u$. This finishes the proof for the case $p < q \leq \infty$.

\[ \square \]

Remark 5.3. We make a few remarks.

(i) Suppose $\mu$ is nonatomic and $1 < q < \infty$. By mimicking the proof of Theorem 5.2 (ii) and working with the $(p, q)$-norm and the $(p, q)$-capacity, we can also show that
\[ \lim_{i \to \infty} \text{Cap}_{p,q}(E_i) = \text{Cap}_{p,q}(E) \]
whenever $E_1 \subset E_2 \subset \ldots \subset E = \bigcup_{i=1}^{\infty} E_i \subset X$.

(ii) Moreover, if $\text{Cap}_{p,q}$ is an outer capacity then it follows immediately that
\[ \lim_{i \to \infty} \text{Cap}_{p,q}(K_i) = \text{Cap}_{p,q}(K) \]
whenever \((K_i)_{i=1}^\infty\) is a decreasing sequence of compact sets whose intersection set is \(K\). We say that \(\text{Cap}_{p,q}\) is an outer capacity if for every \(E \subset X\) we have
\[
\text{Cap}_{p,q}(E) = \inf \{ \text{Cap}_{p,q}(U) : E \subset U \subset X, U \text{ open} \}.
\]

(iii) Any outer capacity satisfying properties (i) and (iii) of Theorem 5.2 is called a Choquet capacity. (See Appendix II in Doob [9].)

We recall that if \(A \subset X\), then \(\Gamma_A\) is the family of curves in \(\Gamma_{\text{rect}}\) that intersect \(A\) and \(\Gamma^+_A\) is the family of all curves in \(\Gamma_{\text{rect}}\) such that the Hausdorff one-dimensional measure \(H_1(\gamma \cap A)\) is positive. The following lemma will be useful later in this paper.

**Lemma 5.4.** If \(F \subset X\) is such that \(\text{Cap}_{p,q}(F) = 0\), then \(\text{Mod}_{p,q}(\Gamma_F) = 0\).

**Proof.** We follow Shanmugalingam [27]. We can assume without loss of generality that \(q \neq p\). Since \(\text{Cap}_{p,q}(F) = 0\), for each positive integer \(i\) there exists a function \(v_i \in \mathcal{A}(F)\) such that \(0 \leq v_i \leq 1\) and such that \(\|v_i\|_{N_1(L^{p,q})} \leq 2^{-i}\). Let \(u_n := \sum_{i=1}^{n} v_i\). Then \(u_n \in N^{1,L^{(p,q)}}(X, \mu)\) for each \(n\), \(u_n(x)\) is increasing for each \(x \in X\), and for every \(m > n\) we have
\[
\|u_n - u_m\|_{N_1,L^{(p,q)}} \leq \sum_{i=m+1}^{n} \|v_i\|_{N_1,L^{(p,q)}} \leq 2^{-m} \to 0, \text{ as } m \to \infty.
\]

Therefore the sequence \(\{u_n\}_{n=1}^{\infty}\) is a Cauchy sequence in \(N^{1,L^{(p,q)}}(X, \mu)\).

Since \(\{u_n\}_{n=1}^{\infty}\) Cauchy in \(N^{1,L^{(p,q)}}(X, \mu)\), it follows that it is Cauchy in \(L^{p,q}(X, \mu)\). Hence by passing to a subsequence if necessary, there is a function \(\tilde{u} \in L^{p,q}(X, \mu)\) to which the subsequence converges both pointwise \(\mu\)-almost everywhere and in the \(L^{(p,q)}\) norm. By choosing a further subsequence, again denoted by \(\{u_i\}_{i=1}^{\infty}\) for simplicity, we can assume that
\[
\|u_i - \tilde{u}\|_{L^{(p,q)}(X, \mu)} + \|g_{i,i+1}\|_{L^{(p,q)}(X, \mu)} \leq 2^{-2i},
\]
where \(g_{i,j}\) is an upper gradient of \(u_i - u_j\) for \(i < j\). If \(g_1\) is an upper gradient of \(u_1\), then \(u_2 = u_1 + (u_2 - u_1)\) has an upper gradient \(g_2 = g_1 + g_{12}\). In general,
\[
u_i = u_1 + \sum_{k=1}^{i-1} (u_{k+1} - u_k)
\]
has an upper gradient
\[
g_i = g_1 + \sum_{k=1}^{i-1} g_{k,k+1}
\]
for every \(i \geq 2\). For \(j < i\) we have
\[
\|g_i - g_j\|_{L^{(p,q)}(X, \mu)} \leq \sum_{k=j}^{i-1} \|g_{k,k+1}\|_{L^{(p,q)}(X, \mu)} \leq \sum_{k=j}^{i-1} 2^{-2k} \leq 2^{1-2j} \to 0 \text{ as } j \to \infty.
\]

Therefore \(\{g_i\}_{i=1}^{\infty}\) is also a Cauchy sequence in \(L^{(p,q)}(X, \mu)\), and hence converges in the \(L^{(p,q)}\) norm to a nonnegative Borel function \(g\). Moreover, we have
\[
\|g_j - g\|_{L^{(p,q)}(X, \mu)} \leq 2^{1-2j}
\]
for every \(j \geq 1\).

We define \(u\) by \(u(x) = \lim_{i \to \infty} u_i(x)\). Since \(u_i \to \tilde{u}\) \(\mu\)-almost everywhere, it follows that \(u = \tilde{u}\) \(\mu\)-almost everywhere and thus \(u \in L^{p,q}(X, \mu)\). Let
\[
E = \{ x \in X : \lim_{i \to \infty} u_i(x) = \infty \}.
\]
The function \( u \) is well defined outside of \( E \). In order for the function \( u \) to be in the space \( N^{1,L^{p,q}}(X, \mu) \), the function \( u \) has to be defined on almost all paths by Proposition 4.4. To this end it is shown that the \( p,q \)-modulus of the family \( \Gamma_E \) is zero. Let \( \Gamma_1 \) be the collection of all paths from \( \Gamma_{\text{rect}} \) such that \( \int_\gamma g = \infty \). Then we have via Theorem 3.4 that \( \text{Mod}_{p,q}(\Gamma_1) = 0 \) since \( g \in L^{p,q}(X, \mu) \).

Let \( \Gamma_2 \) be the family of all curves from \( \Gamma_{\text{rect}} \) such that \( \limsup_{j \to \infty} \int_\gamma |g_j - g| > 0 \). Since \( ||g_j - g||_{L^{p,q}(X, \mu)} \leq 2^{1 - 2j} \) for all \( j \geq 1 \), it follows via Theorem 3.6 that \( \text{Mod}_{p,q}(\Gamma_2) = 0 \).

Since \( u \in L^{p,q}(X, \mu) \) and \( E = \{ x \in X : u(x) = \infty \} \), it follows that \( \mu(E) = 0 \) and thus \( \text{Mod}_{p,q}(\Gamma_2) = 0 \). Therefore \( \text{Mod}_{p,q}(\Gamma_1 \cup \Gamma_2 \cup \Gamma_E^c) = 0 \). For any path \( \gamma \) in the family \( \Gamma_{\text{rect}} \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_E^c) \), by the fact that \( \gamma \) is not in \( \Gamma_E^c \), there exists a point in \( |\gamma| \setminus E \). For any point \( x \) in \( \gamma \), since \( g_i \) is an upper gradient of \( u_i \), it follow that

\[
u_i(x) - u_i(y) \leq |u_i(x) - u_i(y)| \leq \int_\gamma g_i.
\]

Therefore

\[
u_i(x) \leq u_i(y) + \int_\gamma g_i.
\]

Taking limits on both sides and using the fact that \( \gamma \) is not in \( \Gamma_1 \cup \Gamma_2 \), it follows that

\[
\lim_{i \to \infty} \nu_i(x) \leq \lim_{i \to \infty} u_i(y) + \int_\gamma q = u(y) + \int_\gamma g < \infty,
\]

and therefore \( x \) is not in \( E \). Thus \( \Gamma_E \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_E^c \) and \( \text{Mod}_{p,q}(\Gamma_E) = 0 \). Therefore \( g \) is a \( p,q \)-weak upper gradient of \( u \), and hence \( u \in N^{1,L^{p,q}}(X, \mu) \). For each \( x \) not in \( E \) we can write \( u(x) = \lim_{i \to \infty} u_i(x) < \infty \). If \( F \setminus E \) is nonempty, then

\[
u F \setminus E \geq u_n |_{F \setminus E} = \sum_{i=1}^{n} v_i |_{F \setminus E} = n
\]

for arbitrarily large \( n \), yielding that \( \nu F \setminus E = \infty \). But this impossible, since \( x \) is not in the set \( E \). Therefore \( F \subset E \), and hence \( \Gamma_F \subset \Gamma_E \). This finishes the proof of the lemma.

Next we prove that \( (N^{1,L^{p,q}}(X, \mu), || \cdot ||_{N^{1,L^{p,q}}}) \) is a Banach space.

**Theorem 5.5.** Suppose \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \). Then \( (N^{1,L^{p,q}}(X, \mu), || \cdot ||_{N^{1,L^{p,q}}}) \) is a Banach space.

**Proof.** We follow Shanmugalingam [27]. We can assume without loss of generality that \( q \neq p \). Let \( \{u_i\}_{i=1}^{\infty} \) be a Cauchy sequence in \( N^{1,L^{p,q}}(X, \mu) \). To show that this sequence is convergent in \( N^{1,L^{p,q}}(X, \mu) \), it suffices to show that some subsequence is convergent in \( N^{1,L^{p,q}}(X, \mu) \). Passing to a further subsequence if necessary, it can be assumed that

\[
||u_{i+1} - u_i||_{L^{p,q}(X, \mu)} + ||g_{i,j+1}||_{L^{p,q}(X, \mu)} \leq 2^{-2i},
\]

where \( g_{i,j} \) is an upper gradient of \( u_i - u_j \) for \( i < j \). Let

\[
E_j = \{ x \in X : |u_{j+1}(x) - u_j(x)| \geq 2^{-j} \}.
\]

Then \( 2^j|u_{j+1} - u_j| \in A(E_j) \) and hence

\[
\text{Cap}_{p,q}(E_j)^{1/p} \leq 2^j||u_{j+1} - u_j||_{N^{1,L^{p,q}}} \leq 2^{-j}.
\]

Let \( F_j = \cup_{k=j}^{\infty} E_k \). Then

\[
\text{Cap}_{p,q}(E_j)^{1/p} \leq \sum_{k=j}^{\infty} \text{Cap}_{p,q}(E_k)^{1/p} \leq 2^{1-j}.
\]
Let \( F = \cap_{j=1}^{\infty} F_j \). We notice that \( \text{Cap}_{p,q}(F) = 0 \). If \( x \) is a point in \( X \setminus F \), there exists \( j \geq 1 \) such that \( x \) is not in \( F_j = \cup_{k=j}^{\infty} E_k \). Hence for all \( k \geq j \), \( x \) is not in \( E_k \). Thus \( |u_{k+j}(x) - u_k(x)| \leq 2^{-k} \) for all \( k \geq j \). Therefore whenever \( l \geq k \geq j \) we have that
\[
|u_k(x) - u_l(x)| \leq 2^{1-k}.
\]
Thus the sequence \( \{u_k(x)\}_{k=1}^{\infty} \) is Cauchy for every \( x \in X \setminus F \). For every \( x \in X \setminus F \), let 
\[
u(x) = \lim_{i \to \infty} u_i(x).
\]
Therefore for each \( x \) in \( X \setminus F \),
\[
(18)
\nu(x) = u_k(x) + \sum_{n=k}^{\infty} (u_{n+1}(x) - u_n(x)).
\]
Noting by Lemma 5.4 that \( \text{Mod}_{p,q}(\gamma) = 0 \) and that for each path \( \gamma \) in \( \Gamma_{\text{rect}} \setminus \Gamma_F \) equation (18) holds pointwise on \( |\gamma| \), we conclude that \( \sum_{n=k}^{\infty} g_{n,n+1} \) is a \( p, q \)-weak upper gradient of \( u - u_k \). Therefore
\[
\|u - u_k\|_{N^{1,L^{p,q}}(X,\mu)} \leq \|u - u_k\|_{L^{p,q}(X,\mu)} + \sum_{n=k}^{\infty} \|g_{n,n+1}\|_{L^{p,q}(X,\mu)}
\]
\[
\leq \|u - u_k\|_{L^{p,q}(X,\mu)} + \sum_{n=k}^{\infty} 2^{-2n}
\]
\[
\leq \|u - u_k\|_{L^{p,q}(X,\mu)} + 2^{1-2k} \to 0 \text{ as } k \to \infty.
\]
Therefore the subsequence converges in the norm of \( N^{1,L^{p,q}}(X,\mu) \) to \( u \). This completes the proof of the theorem.

\[\Box\]

6. Density of Lipschitz functions in \( N^{1,L^{p,q}}(X,\mu) \)

6.1. Poincaré inequality. Now we define the weak \((1, L^{p,q})\)-Poincaré inequality. Podbrdsky in [26] introduced a stronger Poincaré inequality in the case of Banach-valued Newtonian Lorentz spaces.

**Definition 6.1.** The space \((X, d, \mu)\) is said to support a weak \((1, L^{p,q})\)-Poincaré inequality if there exist constants \( C > 0 \) and \( \sigma \geq 1 \) such that for all balls \( B \) with radius \( r \), all \( \mu \)-measurable functions \( u \) on \( X \) and all upper gradients \( g \) of \( u \) we have
\[
(19) \quad \frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C r^{\sigma} \frac{1}{\mu(\sigma B)^{1/p}} \|g\chi_{\sigma B}\|_{L^{p,q}(X,\mu)}.
\]
Here
\[
u_B = \frac{1}{\mu(B)} \int_B u(x) \, d\mu(x)
\]
whenever \( u \) is a locally \( \mu \)-integrable function on \( X \).

In the above definition we can equivalently assume via Lemma 3.9 and Corollary 2.8 that \( g \) is a \( p, q \)-weak upper gradient of \( u \). When \( p = q \) we have the weak \((1, p)\)-Poincaré inequality. For more about the Poincaré inequality in the case \( p = q \) see Hajlasz-Koskela [14] and Heinonen-Koskela [17].
A measure $\mu$ is said to be **doubling** if there exists a constant $C \geq 1$ such that

$$\mu(2B) \leq C \mu(B)$$

for every ball $B = B(x, r)$ in $X$. A metric measure space $(X, d, \mu)$ is called **doubling** if the measure $\mu$ is doubling. Under the assumption that the measure $\mu$ is doubling, it is known that $(X, d, \mu)$ is proper (that is, closed bounded subsets of $X$ are compact) if and only if it is complete.

Now we prove that if $1 \leq q \leq p$, the measure $\mu$ is doubling, and the space $(X, d, \mu)$ carries a weak $(1, L^{p,q})$-Poincaré inequality, the Lipschitz functions are dense in $N^1, L^{p,q}(X, \mu)$.

In order to prove this we need a few definitions and lemmas.

**Definition 6.2.** Suppose $(X, d)$ is a metric space that carries a doubling measure $\mu$. For $1 < p < \infty$ and $1 \leq q \leq \infty$ we define the noncentered maximal function operator by

$$M_{p,q}u(x) = \sup_{B \ni x} \frac{|u_B|_{L^{p,q}(X, \mu)}}{\mu(B)^{1/p}},$$

where $u \in L^{p,q}(X, \mu)$.

**Lemma 6.3.** Suppose $(X, d)$ is a metric space that carries a doubling measure $\mu$. If $1 \leq q \leq p$, then $M_{p,q}$ maps $L^{p,q}(X, \mu)$ to $L^{p,\infty}(X, \mu)$ boundedly and moreover,

$$\lim_{\lambda \to \infty} \lambda^p \mu(\{x \in X : M_{p,q}u(x) > \lambda\}) = 0.$$

**Proof.** We can assume without loss of generality that $1 \leq q < p$. For every $R > 0$ let $M_{p,q}^R$ be the restricted maximal function operator defined on $L^{p,q}(X, \mu)$ by

$$M_{p,q}^Ru(x) = \sup_{B \ni x, \text{diam}(B) \leq R} \frac{|u_B|_{L^{p,q}(X, \mu)}}{\mu(B)^{1/p}}.$$

Denote $G_\lambda = \{x \in X : M_{p,q}u(x) > \lambda\}$ and $G_\lambda^R = \{x \in X : M_{p,q}^R u(x) > \lambda\}$. It is easy to see that $G_\lambda^{R_1} \subset G_\lambda^{R_2}$ if $0 < R_1 < R_2 < \infty$ and $G_\lambda^R \to G_\lambda$ as $R \to \infty$.

Fix $R > 0$. For every $x \in G_\lambda^R$ and some $r > 0$, there exists a ball $B(y_x, r_x)$ with diameter at most $R$ such that $x \in B(y_x, r_x)$ and such that

$$||u \chi_{B(y_x, r_x)}||_{L^{p,q}(X, \mu)} > \lambda^p \mu(B(y_x, r_x)).$$

We notice that $B(y_x, r_x) \subset G_\lambda^R$. The set $G_\lambda^R$ is covered by such balls and by Theorem 1.2 in Heinonen [15] it follows that there exists a countable disjoint subcollection $\{B(x_i, r_i)\}_{i=1}^\infty$ such that the collection $\{B(x_i, 5r_i)\}_{i=1}^\infty$ covers $G_\lambda^R$. Hence

$$\mu(G_\lambda^R) \leq \sum_{i=1}^\infty \mu(B(x_i, 5r_i)) \leq C \left( \sum_{i=1}^\infty \mu(B(x_i, r_i)) \right) \leq \frac{C \lambda^p}{\lambda^p} \sum_{i=1}^\infty ||u \chi_{B(x_i, r_i)}||_{L^{p,q}(X, \mu)}^p \leq \frac{C \lambda^p}{\lambda^p} ||u \chi_{G_\lambda^R}||_{L^{p,q}(X, \mu)}^p.$$

The last inequality in the sequence was obtained by applying Proposition 2.4. (See also Chung-Hunt-Kurtz [5, Lemma 2.5].)

Thus

$$\mu(G_\lambda^R) \leq \frac{C \lambda^p}{\lambda^p} ||u \chi_{G_\lambda^R}||_{L^{p,q}(X, \mu)}^p \leq \frac{C \lambda^p}{\lambda^p} ||u \chi_{G_\lambda}||_{L^{p,q}(X, \mu)}^p.$$
for every $R > 0$. Since $G_\lambda = \bigcup_{R > 0} G_\lambda^R$, we obtain (by taking the limit as $R \to \infty$)

$$
\mu(G_\lambda) \leq \frac{C}{N Industries and others. The absolute continuity of the $p, q$-norm (see the discussion after Definition 2.1), the $p, q$-integrability of $u$ and the fact that $G_\lambda \to \emptyset$ $\mu$-almost everywhere as $\lambda \to \infty$ yield the desired conclusion.

**Question 6.4.** Is Lemma 6.3 true when $p < q < \infty$?

The following proposition is necessary in the sequel.

**Proposition 6.5.** Suppose $1 < p < \infty$ and $1 \leq q < \infty$. If $u$ is a nonnegative function in $N^{1,p,q}(X, \mu)$, then the sequence of functions $u_k = \min(u, k)$, $k \in \mathbb{N}$, converges in the norm of $N^{1,p,q}(X, \mu)$ to $u$ as $k \to \infty$.

**Proof.** We notice (see Lemma 3.16) that $u_k \in L^{p,q}(X, \mu)$. That lemma also yields easily $u_k \in N^{1,p,q}(X, \mu)$ and moreover $\|u_k\|_{N^{1,p,q}} \leq \|u\|_{N^{1,p,q}}$ for all $k \geq 1$.

Let $E_k = \{ x \in X : u(x) > k \}$. Since $\mu$ is a Borel regular measure, there exists an open set $O_k$ such that $E_k \subset O_k$ and $\mu(O_k) \leq \mu(E_k) + 2^{-k}$. In fact the sequence $(O_k)_{k=1}^\infty$ can be chosen such that $O_{k+1} \subset O_k$ for all $k \geq 1$. Since $\mu(E_k) \leq \frac{C(p,q)}{k^p} \|u\|_{L^{p,q}(X, \mu)}^p$, it follows that

$$
\mu(O_k) \leq \mu(E_k) + 2^{-k} \leq \frac{C(p,q)}{k^p} \|u\|_{L^{p,q}(X, \mu)}^p + 2^{-k}.
$$

Thus $\lim_{k \to \infty} \mu(O_k) = 0$. We notice that $u = u_k$ on $X \setminus O_k$. Thus $2g\chi_{O_k}$ is a $p, q$-weak upper gradient of $u - u_k$ whenever $g$ is an upper gradient for $u$ and $u_k$. See Lemma 4.6. The fact that $O_k \to \emptyset$ $\mu$-almost everywhere and the absolute continuity of the $(p, q)$-norm yield

$$
\limsup_{k \to \infty} \|u - u_k\|_{N^{1,p,q}} \leq 2 \limsup_{k \to \infty} \left( \|u\|_{N^{1,p,q}} + \|g\chi_{O_k}\|_{L^{p,q}(X, \mu)} \right) = 0.
$$

**Counterexample 6.6.** For $q = \infty$ Proposition 6.5 is not true. Indeed, let $n \geq 2$ be an integer and let $1 < p \leq n$ be fixed. Let $X = B(0, 1) \setminus \{0\} \subset \mathbb{R}^n$, endowed with the Euclidean metric and the Lebesgue measure.

Suppose first that $1 < p < n$. Let $u_p$ and $g_p$ be defined on $X$ by

$$u_p(x) = |x|^{1 - \frac{2}{p}} - 1 \quad \text{and} \quad g_p(x) = \left( \frac{n}{p} - 1 \right) |x|^{- \frac{2}{p}}.$$ 

It is easy to see that $u_p, g_p \in L^{p,\infty}(X, m_n)$. Moreover, (see for instance Hajlasz [13, Proposition 6.4]) $g_p$ is the minimal upper gradient for $u_p$. Thus $u_p \in N^{1,p,\infty}(X, m_n)$. For every integer $k \geq 1$ we define $u_{p,k}$ and $g_{p,k}$ on $X$ by

$$u_{p,k}(x) = \begin{cases} k & \text{if } 0 < |x| \leq (k + 1)^{\frac{p}{p-n}}, \\ |x|^{1 - \frac{2}{p}} - 1 & \text{if } (k + 1)^{\frac{p}{p-n}} < |x| < 1
\end{cases}$$

and

$$g_{p,k}(x) = \begin{cases} \left( \frac{n}{p} - 1 \right) |x|^{- \frac{2}{p}} & \text{if } 0 < |x| < (k + 1)^{\frac{p}{p-n}} \\ 0 & \text{if } (k + 1)^{\frac{p}{p-n}} \leq |x| < 1.
\end{cases}$$

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We notice that \( u_{p,k} \in N^{1,L^{p,\infty}}(X, m_n) \) for all \( k \geq 1 \). Moreover, via [13, Proposition 6.4] and Lemma 4.6 we see that \( g_{p,k} \) is the minimal upper gradient for \( u_p - u_{p,k} \) for every \( k \geq 1 \). Since \( g_{p,k} \searrow 0 \) on \( X \) as \( k \to \infty \) and \( \|g_{p,k}\|_{L^{p,\infty}(X, m_n)} = \|g_p\|_{L^{p,\infty}(X, m_n)} = C(n, p) > 0 \) for all \( k \geq 1 \), it follows that \( u_{p,k} \) does not converge to \( u_p \) in \( N^{1,L^{p,\infty}}(X, m_n) \) as \( k \to \infty \).

Suppose now that \( p = n \). Let \( u_n \) and \( g_n \) be defined on \( X \) by

\[
    u_n(x) = \ln \frac{1}{|x|} \quad \text{and} \quad g_n(x) = \frac{1}{|x|}.
\]

It is easy to see that \( u_n, g_n \in L^{p,\infty}(X, m_n) \). Moreover, (see for instance Hajlasz [13, Proposition 6.4]) \( g_n \) is the minimal upper gradient for \( u_n \). Thus \( u_n \in N^{1,L^{n,\infty}}(X, m_n) \).

For every integer \( k \geq 1 \) we define \( u_{n,k} \) and \( g_{n,k} \) on \( X \) by

\[
    u_{n,k}(x) = \begin{cases} 
        k & \text{if } 0 < |x| \leq e^{-k}, \\
        \ln \frac{1}{|x|} & \text{if } e^{-k} < |x| < 1 
    \end{cases}
\]

and

\[
    g_{n,k}(x) = \begin{cases} 
        \frac{1}{|x|} & \text{if } 0 < |x| < e^{-k} \\
        0 & \text{if } e^{-k} \leq |x| < 1.
    \end{cases}
\]

We notice that \( u_{n,k} \in N^{1,L^{n,\infty}}(X, m_n) \) for all \( k \geq 1 \). Moreover, via [13, Proposition 6.4] and Lemma 4.6 we see that \( g_{n,k} \) is the minimal upper gradient for \( u_n - u_{n,k} \) for every \( k \geq 1 \). Since \( g_{n,k} \searrow 0 \) on \( X \) as \( k \to \infty \) and \( \|g_{n,k}\|_{L^{p,\infty}(X, m_n)} = \|g_n\|_{L^{n,\infty}(X, m_n)} = C(n) > 0 \) for all \( k \geq 1 \), it follows that \( u_{n,k} \) does not converge to \( u_n \) in \( N^{1,L^{n,\infty}}(X, m_n) \) as \( k \to \infty \).

The following lemma will be used in the paper.

**Lemma 6.7.** Let \( f_1 \in N^{1,L^{p,q}}(X, \mu) \) be a bounded Borel function with \( p, q \)-weak upper gradient \( g_1 \in L^{p,q}(X, \mu) \) and let \( f_2 \) be a bounded Borel function with \( p, q \)-weak upper gradient \( g_2 \in L^{p,q}(X, \mu) \). Then \( f_3 := f_1 f_2 \in N^{1,L^{p,q}}(X, \mu) \) and \( g_3 := |f_1| g_2 + |f_2| g_1 \) is a \( p, q \)-weak upper gradient of \( f_3 \).

**Proof.** It is easy to see that \( f_3 \) and \( g_3 \) are in \( L^{p,q}(X, \mu) \). Let \( \Gamma_0 \subset \Gamma_{\text{rect}} \) be the family of curves on which \( f_1(g_1 + g_2) = \infty \). Then it follows via Theorem 3.4 that \( \text{Mod}_{p,q}(\Gamma_0) = 0 \) because \( g_1 + g_2 \in L^{p,q}(X, \mu) \).

Let \( \Gamma_{i,\ell} \subset \Gamma_{\text{rect}}, i = 1, 2 \) be the family of curves for which

\[
    |f_i(\ell(\gamma))| - f_i(\ell(\gamma)))| \leq \int_{\gamma} g_i
\]

is not satisfied. Then \( \text{Mod}_{1,\ell} = 0, i = 1, 2 \). Let \( \Gamma_1 \subset \Gamma_{\text{rect}} \) be the family of curves that have a subcurve in \( \Gamma_{1,1} \cup \Gamma_{1,2} \). Then \( F(\Gamma_{1,1} \cup \Gamma_{1,2}) \subset F(\Gamma_1) \) and thus \( \text{Mod}_{p,q}(\Gamma_1) \leq \text{Mod}_{p,q}(\Gamma_{1,1} \cup \Gamma_{1,2}) = 0 \). We notice immediately that \( \text{Mod}_{p,q}(\Gamma_0 \cup \Gamma_1) = 0 \).

Fix \( \varepsilon > 0 \). By using the argument from Lemma 1.7 in Cheeger [4], we see that

\[
    |f_3(\ell(\gamma))| - f_3(\ell(\gamma)))| \leq \int_{0}^{\ell(\gamma)} (|f_1(\gamma(s))| + \varepsilon) g_2(\gamma(s)) + (|f_2(\gamma(s))| + \varepsilon) g_1(\gamma(s)) ds
\]

for every curve \( \gamma \) in \( \Gamma_{\text{rect}} \setminus (\Gamma_0 \cup \Gamma_1) \). Letting \( \varepsilon \to 0 \) we obtain the desired claim.

\[\square\]
Fix $x_0 \in X$. For each integer $j > 1$ we consider the function
\[
\eta_j(x) = \begin{cases} 
1 & \text{if } d(x_0, x) \leq j - 1, \\
 j - d(x_0, x) & \text{if } j - 1 < d(x_0, x) \leq j, \\
0 & \text{if } d(x_0, x) > j.
\end{cases}
\]

**Lemma 6.8.** Suppose $1 \leq q < \infty$. Let $u$ be a bounded function in $N^{1,p,q}(X, \mu)$. Then the function $v_j = u\eta_j$ is also in $N^{1,p,q}(X, \mu)$. Furthermore, the sequence $v_j$ converges to $u$ in $N^{1,p,q}(X, \mu)$.

**Proof.** We can assume without loss of generality that $u \geq 0$. Let $g \in L^{p,q}(X, \mu)$ be an upper gradient for $u$. It is easy to see by invoking Lemma 4.6 that $h_j := \chi_{B(x_0,j)\setminus B(x_0,j-1)}$ is a $p, q$-weak upper gradient for $\eta_j$ and for $1 - \eta_j$. By using Lemma 6.7, we see that $v_j \in N^{1,p,q}(X, \mu)$ and that $g_j := uh_j + g\eta_j$ is a $p, q$-weak upper gradient for $v_j$. By using Lemma 6.7 we notice that $h_j := uh_j + g(1 - \eta_j)$ is a $p, q$-weak upper gradient for $u - v_j$. We have in fact
\[
0 \leq u - v_j \leq u\chi_{X\setminus B(x_0,j-1)} \text{ and } h_j \leq (u + g)\chi_{X\setminus B(x_0,j-1)}.
\]
for every $j > 1$. The absolute continuity of the $(p, q)$-norm when $1 \leq q < \infty$ (see the discussion after Definition 2.1) together with the $p, q$-integrability of $u, g$ and (20) yield the desired claim.

Now we prove the density of the Lipschitz functions in $N^{1,p,q}(X, \mu)$ when $1 \leq q < p$. The case $q = p$ was proved by Shanmugalingam. (See [27] and [28].)

**Theorem 6.9.** Let $1 \leq q \leq p < \infty$. Suppose that $(X, d, \mu)$ is a doubling metric measure space that carries a weak $(1, L^{p,q})$-Poincaré inequality. Then the Lipschitz functions are dense in $N^{1,p,q}(X, \mu)$.

**Proof.** We can consider only the case $1 \leq q < p$ because the case $q = p$ was proved by Shanmugalingam in [27] and [28]. We can assume without loss of generality that $u$ is nonnegative. Moreover, via Lemmas 6.5 and 6.7 we can assume without loss of generality that $u$ is bounded and has compact support in $X$. Choose $M > 0$ such that $0 \leq u \leq M$ on $X$. Let $g \in L^{p,q}(X, \mu)$ be a $p, q$-weak upper gradient for $u$. Let $\sigma \geq 1$ be the constant from the weak $(1, L^{p,q})$-Poincaré inequality.

Let $G_\lambda := \{x \in X : M_{p,q}g(x) > \lambda\}$. If $x$ is a point in the closed set $X \setminus G_\lambda$, then for all $r > 0$ one has that
\[
\frac{1}{\mu(B(x, r))} \int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C r \|g\chi_{B(x, \sigma r)}\|_{L^{p,q}(X, \mu)} \leq C r M_{p,q}g(x) \leq C \lambda r.
\]
Hence for $s \in [r/2, r]$ one has that
\[
|u_{B(x, s)} - u_{B(x, r)}| \leq \frac{1}{\mu(B(x, s))} \int_{B(x, s)} |u - u_{B(x, r)}| \, d\mu \leq \frac{1}{\mu(B(x, s))} \cdot \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C \lambda r.
\]
whenever $x$ is in $X \setminus G_\lambda$. For a fixed $s \in (0, r/2)$ there exists an integer $k \geq 1$ such that $2^{-kr} \leq 2s < 2^{-k+1}r$. Then
\[
|u_{B(x,s)} - u_{B(x,r)}| \leq |u_{B(x,s)} - u_{B(x,2^{-k}r)}| + \sum_{i=0}^{k-1} |u_{B(x,2^{-i}r)} - u_{B(x,2^{-i-1}r)}| \leq C\lambda \left( \sum_{i=0}^{k} 2^{-i}r \right) \leq C\lambda r.
\]
For any sequence $r_i \searrow 0$ we notice that $(u_{B(x,r_i)})_{i=1}^\infty$ is a Cauchy sequence for every point $x$ in $X \setminus G_\lambda$. Thus on $X \setminus G_\lambda$ we can define the function
\[
u_\lambda(x) := \lim_{r \rightarrow 0} u_{B(x,r)}.
\]
We notice that $u_\lambda(x) = u(x)$ for every Lebesgue point $x$ in $X \setminus G_\lambda$.

For every $x, y$ in $X \setminus G_\lambda$ we consider the chain of balls $\{B_i\}_{i=-\infty}^\infty$, where
\[
B_i = B(x, 2^{1+i}d(x,y)), i \leq 0 \text{ and } B_i = B(y, 2^{-i-1}d(x,y)), i > 0.
\]
For every two such points $x$ and $y$ we have that they are Lebesgue points for $u_\lambda$ by construction and hence
\[
|u_\lambda(x) - u_\lambda(y)| \leq \sum_{i=-\infty}^\infty |u_{B_i} - u_{B_{i+1}}| \leq C\lambda d(x,y),
\]
where $C$ depends only on the data on $X$. Thus $u_\lambda$ is $C\lambda$-Lipschitz on $X \setminus G_\lambda$. By construction it follows that $0 \leq u_\lambda \leq M$ on $X \setminus G_\lambda$. Extend $u_\lambda$ as a $C\lambda$-Lipschitz function on $X$ (see McShane [25]) and denote the extension by $v_\lambda$. Then $v_\lambda \geq 0$ on $X$ since $u_\lambda \geq 0$ on $X \setminus G_\lambda$. Let $w_\lambda := \min(v_\lambda, M)$. We notice that $w_\lambda$ is a nonnegative $C\lambda$-Lipschitz function on $X$ since $v_\lambda$ is. Moreover, $w_\lambda = v_\lambda = u_\lambda$ on $X \setminus G_\lambda$ whenever $\lambda > M$.

Since $u = w_\lambda \mu$-almost everywhere on $X \setminus G_\lambda$ whenever $\lambda > M$ we have
\[
\|u - w_\lambda\|_{L^p,\mu} = \|(u - w_\lambda)\chi_{G_\lambda}\|_{L^p,\mu} \leq \|u\chi_{G_\lambda}\|_{L^p,\mu} + C(p, q)\lambda \mu(G_\lambda)^{1/p}
\]
whenever $\lambda > M$. The absolute continuity of the $p, q$-norm when $1 \leq q \leq p$ together with Lemma 6.3 imply that
\[
limit_{\lambda \rightarrow \infty} \|u - w_\lambda\|_{L^p,\mu} = 0.
\]

Since $u - w_\lambda = 0 \mu$-almost everywhere on the closed set $G_\lambda$, it follows via Lemma 4.6 that $(C\lambda + g)\chi_{G_\lambda}$ is a $p, q$-weak upper gradient for $u - w_\lambda$. By using the absolute continuity of the $p, q$-norm when $1 \leq q \leq p$ together with Lemma 6.3, we see that
\[
limit_{\lambda \rightarrow \infty} \|(C\lambda + g)\chi_{G_\lambda}\|_{L^p,\mu} = 0.
\]
This finishes the proof of the theorem.

Theorem 6.9 yields the following result.

**Proposition 6.10.** Let $1 \leq q \leq p < \infty$. Suppose that $(X, d, \mu)$ satisfies the hypotheses of Theorem 6.9. Then $\text{Cap}_{p, q}$ is an outer capacity.

In order to prove Proposition 6.10 we need to state and prove the following proposition, thus generalizing Proposition 1.4 from Björn-Björn-Shanmugalingam [3].

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Proposition 6.11. (See [3, Proposition 1.4]) Let $1 < p < \infty$ and $1 \leq q < \infty$. Suppose that $(X, d, \mu)$ is a proper metric measure space. Let $E \subset X$ be such that $\text{Cap}_{p,q}(E) = 0$. Then for every $\varepsilon > 0$ there exists an open set $U \supset E$ with $\text{Cap}_{p,q}(U) < \varepsilon$.

Proof. We adjust the proof of Proposition 1.4 in Björn-Björn-Shanmugalingam [3] to the Lorentz setting with some modifications. It is enough to consider the case when $q \neq p$. Due to the countable subadditivity of $\text{Cap}_{p,q}(\cdot)^{1/p}$ we can assume without loss of generality that $E$ is bounded. Moreover, we can also assume that $E$ is Borel. Since $\text{Cap}_{p,q}(E) = 0$, we have $\chi_E \in N^{1,L^{p,q}}(X, \mu)$ and $\|\chi_E\|_{N^{1,L^{p,q}}} = 0$. Let $\varepsilon \in (0, 1)$ be arbitrary. Via Lemma 3.9 and Corollary 2.8, there exists $g \in L^{p,q}(X, \mu)$ such that $g$ is an upper gradient for $\chi_E$ and $\|g\|_{L^{p,q}(X, \mu)} < \varepsilon$. By adapting the proof of the Vitali-Carathéodory theorem to the Lorentz setting (see Folland [10, Proposition 7.14]) we can find a lower semicontinuous function $\rho \in L^{p,q}(X, \mu)$ such that $\rho \geq g$ and $\|\rho - g\|_{L^{p,q}(X, \mu)} < \varepsilon$. Since $\text{Cap}_{p,q}(E) = 0$, it follows immediately that $\mu(E) = 0$. By using the outer regularity of the measure $\mu$ and the absolute continuity of the $(p, q)$-norm, there exists a bounded open set $V \supset E$ such that

$$\|\chi_V\|_{L^{p,q}(X, \mu)} + \|\rho + 1\|\chi_V\|_{L^{p,q}(X, \mu)} < \frac{\varepsilon}{2}.$$ 

Let

$$u(x) = \min\left\{1, \inf_{\gamma} \int_{\gamma} (\rho + 1)\right\},$$ 

where the infimum is taken over all the rectifiable (including constant) curves connecting $x$ to the closed set $X \setminus V$. We notice immediately that $0 \leq u \leq 1$ on $X$ and $u = 0$ on $X \setminus V$. By Björn-Björn-Shanmugalingam [3, Lemma 3.3] it follows that $u$ is lower semicontinuous on $X$ and thus the set $U = \{x \in X : u(x) > \frac{1}{2}\}$ is open. We notice that for $x \in E$ and every curve connecting $x$ to some $y \in X \setminus V$, we have

$$\int_{\gamma} (\rho + 1) \geq \int_{\gamma} \rho \geq \chi_E(x) - \chi_E(y) = 1.$$ 

Thus $u = 1$ on $E$ and $E \subset U \subset V$. From [3, Lemmas 3.1 and 3.2] it follows that $(\rho + 1)\chi_V$ is an upper gradient of $u$. Since $0 \leq u \leq \chi_V$ and $u$ is lower semicontinuous, it follows that $u \in N^{1,L^{p,q}}(X, \mu)$. Moreover, $2u \in \mathcal{A}(U)$ and thus

$$\text{Cap}_{p,q}(U)^{1/p} \leq 2\|u\|_{N^{1,L^{p,q}}} \leq 2\|u\|_{L^{p,q}(X, \mu)} + \|\rho + 1\|\chi_V\|_{L^{p,q}(X, \mu)}$$

$$\leq 2\|\chi_V\|_{L^{p,q}(X, \mu)} + \|\rho + 1\|\chi_V\|_{L^{p,q}(X, \mu)} < \varepsilon.$$ 

This finishes the proof of Proposition 6.11.

Now we prove Proposition 6.10.

Proof. We start the proof of Proposition 6.10 by showing that every function $u$ in $N^{1,L^{p,q}}(X, \mu)$ is continuous outside open sets of arbitrarily small $p, q$-capacity. (Such a function is called $p, q$-quasicontinuous.) Indeed, let $u$ be a function in $N^{1,L^{p,q}}(X, \mu)$. From Theorem 6.9 there exists a sequence $\{u_j\}_{j=1}^{\infty}$ of Lipschitz functions on $X$ such that

$$\|u_j - u\|_{N^{1,L^{p,q}}} < 2^{-2j}$$

for every integer $j \geq 1$.

For every integer $j \geq 1$ let

$$E_j = \{x \in X : |u_{j+1}(x) - u_j(x)| > 2^{-j}\}.$$
Then all the sets $E_i$ are open because the all functions $u_j$ are Lipschitz. By letting $F = \cap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} E_k$ and applying the argument from Theorem 5.5 to the sequence $\{u_k\}_{k=1}^{\infty}$ which is Cauchy in $N^{1,L^{p,q}}(X,\mu)$, we see that $\operatorname{Cap}_{p,q}(F) = 0$ and the sequence $\{u_k\}$ converges in $N^{1,L^{p,q}}(X,\mu)$ to a function $\bar{u}$ whose restriction on $X \setminus F$ is continuous. Thus $||u - \bar{u}||_{N^{1,L^{p,q}}} = 0$ and hence if we let $E = \{x \in X : u(x) \neq \bar{u}(x)\}$, we have $\operatorname{Cap}_{p,q}(E) = 0$. Therefore $\operatorname{Cap}_{p,q}(E \cup F) = 0$ and hence, via Proposition 6.11 we have that $\forall \in \bar{u}$ outside open supersets of $E \cup F$ of arbitrarily small $p,q$-capacity. This shows that $\forall$ is quasicontinuous.

Now we fix $E \subseteq X$ and we show that

$$\operatorname{Cap}_{p,q}(E) = \inf \{\operatorname{Cap}_{p,q}(U), E \subseteq U \subseteq X, U \text{ open} \}.$$  

For a fixed $\varepsilon \in (0,1)$ we choose $u \in \mathcal{A}(E)$ such that $0 \leq u \leq 1$ on $X$ and such that

$$||u||_{N^{1,L^{p,q}}} < \operatorname{Cap}_{p,q}(E)^{1/p} + \varepsilon.$$  

We have that $u$ is $p,q$-quasicontinuous and hence there is an open set $V$ such that $\operatorname{Cap}_{p,q}(V)^{1/p} < \varepsilon$ and such that $u|_{X \setminus V}$ is continuous. Thus there exists an open set $U$ such that $U \setminus V = \{x \in X : u(x) > 1 - \varepsilon\} \setminus V \supset E \setminus V$. We see that $U \cup V = (U \setminus V) \cup V$ is an open set containing $E \cup V = (E \setminus V) \cup V$. Therefore

$$\operatorname{Cap}_{p,q}(E)^{1/p} \leq \operatorname{Cap}_{p,q}(U \cup V)^{1/p} \leq \operatorname{Cap}_{p,q}(U \setminus V)^{1/p} + \operatorname{Cap}_{p,q}(V)^{1/p} \leq \frac{1}{1 - \varepsilon} \left(||u||_{N^{1,L^{p,q}}} + \operatorname{Cap}_{p,q}(V)^{1/p}\right) \leq \frac{1}{1 - \varepsilon} \left(\operatorname{Cap}_{p,q}(E)^{1/p} + \varepsilon\right) + \varepsilon.$$  

Letting $\varepsilon \to 0$ finishes the proof of Proposition 6.10.

Theorems 5.2 and 6.9 together with Proposition 6.10 and Remark 5.3 yield immediately the following capacitability result. (See also Appendix II in Doob [9].)

**Theorem 6.12.** Let $1 < q \leq p < \infty$. Suppose that $(X,d,\mu)$ satisfies the hypotheses of Theorem 6.9. The set function $E \mapsto \operatorname{Cap}_{p,q}(E)$ is a Choquet capacity. In particular, all Borel subsets (in fact, all analytic subsets) $E$ of $X$ are capacitable, that is

$$\operatorname{Cap}_{p,q}(E) = \sup \{\operatorname{Cap}_{p,q}(K) : K \subseteq E, K \text{ compact} \}$$  

whenever $E \subseteq X$ is Borel (or analytic).

**Remark 6.13.** Counterexample 6.6 gives also a counterexample to the density result for $N^{1,L^{p,\infty}}$ in the Euclidean setting for $1 < p \leq n$ and $q = \infty$.

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