Casimir friction: relative motion more generally

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Abstract
This paper extends our recent study on Casimir friction forces for dielectric plates moving parallel to each other (Høye and Brevik 2014 Eur. Phys. J. D 68 61), to a case where the plates are no longer restricted to rectilinear motion. Part of the mathematical formalism thereby becomes more cumbersome, but reduces in the end to the form that we expected to be the natural one in advance. As an example, we calculate the Casimir torque on a planar disc rotating with constant angular velocity around its vertical symmetry axis next to another plate.

Keywords: Casimir friction, Casimir effect, general Casimir friction

1. Introduction
In this work we will continue our study of the Casimir friction force between two dielectric plates (half-spaces) that move longitudinally with respect to each other with a small separation d. See figure 1.

This is a topic that has attracted considerable interest in the recent past. In our recent paper [1] a problem of this type was analyzed: we calculated the Casimir friction for constant velocity, at zero temperature as well as at finite temperature, requiring vanishing initial and final velocities in order to obtain a closed loop motion, meaning a return to the starting position. The friction force was found via the dissipated energy by which the net contribution from the slow velocity part could be neglected and thus did not require further specification. Moreover, we assumed the simple situation with a constant velocity v between the times −τ and τ. In addition we assumed very low initial and final velocities in the opposite directions in order to be able to return to the starting position.

In the present work we want to extend our results to a situation where the finite velocity contributing to dissipation is not restricted to be constant, but may be slowly varying, and not necessarily restricted to rectilinear motion. In that way, circular motion with a constant speed can be considered too. A nice feature of the latter kind of motion, besides the constant speed, is its return to the initial position as required by the energy dissipation method.

On physical grounds it is reasonable to expect that with slowly varying velocity the total dissipated energy will be the sum of contributions from the various velocities. This provided the constant velocity case considered in [1] leading to a correct result. However, this extension of the problem is non-trivial. The reason is that somehow contributions from different velocities have to be separated from each other while in the reference the contributions from the very slow initial and final velocities could be neglected anyway. As will be seen in section 2 to facilitate this separation of contributions, we find it necessary to split the integrand of equation (2.20) below into two terms. These terms are subdivided in different time intervals as given by equation (2.21).

For the more general motion considered here, it actually transpires that essentially all the derivations and results of [1] remain unchanged. The exception is the integral containing the specified motion, which becomes more cumbersome and requires a detailed and more accurate treatment to handle nonzero contributions for different velocities.

Some papers dealing with Casimir friction—most of them quite new—are listed in [2–12, 13–25]. Here one will also find studies for the case where one single particle is travelling close to a dielectric surface. In principle, the theory of systems of this kind can be obtained from the full theory of interacting dielectric planes, in the limit of large dilution for one of the planes.

It should be noted that the formalism we use to obtain the friction force has been developed by us in previous works starting with a pair of polarizable particles as a basis and
computing the response via the Kubo formalism [26–33]. The methods we use are quite different to the approaches used by others, referred to above. The bases for the use of the Kubo formula are developments and results obtained in the statistical mechanics of polar and polarizable fluids. This formalism was extended, further extended by the authors to evaluate Casimir forces [37] and to evaluate Casimir friction [27, 38]. Then it transpired that time dependent interactions like the radiating dipole interaction could also be included in the statistical mechanical treatment where imaginary time is the fourth dimension. The advantage of this formalism is that the electromagnetic field can be disregarded (or eliminated). Instead it is replaced by dipolar interactions between pairs of polarizable particles. Another advantage is the possibility to consider media on a microscopic level where particles are separated by a minimum distance due to molecular hard cores. In this way it is possible to evaluate (approximately) the finite Casimir energy in the bulk of a simple fluid model [39]. With the statistical mechanical approach the Casimir forces may be given an alternative physical interpretation; they are induced molecular attractions due to fluctuating dipole moments.

In the next section dealing with rectilinear motion where the velocity may vary, the key point will be how to correctly handle the $\dot{Q}$ integral (equation (2.20) below) and the product with its complex conjugate. This is necessary in order to obtain the total dissipated energy. Under the present general circumstances there will be contributions to the dissipation from various velocities that can not be neglected. In [1] the simplifying situation with only one velocity was regarded as the contribution from the slow initial, and return motions can be neglected anyway. After carrying out this more careful analysis, we consider two-dimensional motion in the horizontal plane in section 3. Finally, in section 4, we consider as an example, a rotating planar disc above a resting plate and evaluate the Casimir torque.

As before, we find the friction force to be proportional to $v^3$ at $T = 0$, assuming $v$ is small, while it is proportional to $v$ at finite $T$, assuming $v$ is small.

2. Rectilinear motion

As in [1] we consider a two-plate setup in which the lower plate (2) is at rest, while the upper plate (1) executes motion in a closed loop meaning that it finally slides back to its initial position. The friction force is evaluated via the dissipation of energy, the latter point being advantageous since one avoids the problem of separating a reversible part of the inter-particle force from the total force.

Let us outline some essentials of the theory given in [1] keeping out details of the evaluation that can be found there. For simplicity the numeral I will be used below to designate the equations of [1]. So consider a quantum mechanical harmonic two-oscillator system whose Hamiltonian $H$ is perturbed by a time-dependent term written in general form as $-AF(t)$. Here $A$ is a time independent operator and $F(t)$ is a classical function that depends upon time. For simplicity consider for the moment a pair of one-dimensional oscillators for which we can write

$$-AF(t) = -\psi(r(t))s_1s_2$$ \hspace{1cm} (2.1)

where $r$ is the separation between the pair of oscillators, $\psi(r)$ is the coupling strength, and $s_1, s_2$ are the vibrational coordinates of the oscillators. The instantaneous force between the oscillators is $B = -\nabla\psi(r)$. Its thermal average is, according to the Kubo formula given by equation (I3),

$$\langle B(t) \rangle = \int_{-\infty}^{t} \phi_{BA}(t - t') F(t') \, dt',$$ \hspace{1cm} (2.2)

where, as just mentioned, the numeral I refers to the equations in [1]. With

$$A = s_1s_2 \quad \text{and} \quad F(t) = -\psi(r(t))$$ \hspace{1cm} (2.3)

the response function is

$$\phi_{BA}(t) = \frac{1}{i\hbar} \text{Tr} \left\{ \rho[A, B(t)] \right\}.$$ \hspace{1cm} (2.4)

Here $\rho$ is the density matrix and $B(t) = e^{iHt/\hbar}Be^{-iHt/\hbar}$ is the Heisenberg operator. Furthermore we can write the response function as

$$\phi_{BA}(t) = \nabla\psi \phi(t),$$ \hspace{1cm} (2.5)

where

$$\phi(t) = \frac{1}{i\hbar} \text{Tr} \left\{ \rho[s_1s_2, s_1(t)s_2(t)] \right\} .$$ \hspace{1cm} (2.6)

The $\phi(t)$ depends upon the temperature and the polarizabilities, $\alpha_1$ and $\alpha_2$, and the eigenfrequencies, $\omega_1$ and $\omega_2$, of the two oscillators as given by equations (I18)–(I22).

$$\phi(t) = C_- \sin(\omega_-t) + C_+ \sin(\omega_+ t),$$ \hspace{1cm} (2.7)

$$C_- = \frac{H}{\hbar} \sinh \left( \frac{1}{2} \beta \hbar \omega_- \right), \quad H = \frac{\hbar^2 \omega_1 \omega_2 \alpha_1 \alpha_2}{4 \sinh(\frac{1}{2} \beta \hbar \omega_1) \sinh(\frac{1}{2} \beta \hbar \omega_2)}$$ \hspace{1cm} (2.8)

with $\omega_\pm = |\omega_1 \pm \omega_2|$ ($\phi(t) = 0$ for $t < 0$) and $\beta = 1/(k_B T)$ where $T$ is temperature and $k_B$ is the Boltzmann’s constant. The relative position between the two oscillators can be written as

$$r = r_0 + vq(t).$$ \hspace{1cm} (2.9)

With $\dot{F}(t) = -(v \nabla \psi) \dot{q}(t)$, we can write the dissipated energy for fixed $r_0$ as

$$\Delta E(r_0) = -\int_{-\infty}^{\infty} vq(t) \langle B(t) \rangle \, dt = -\int_{-\infty}^{\infty} \int_{-\infty}^{t} F(t') \phi(t - t') \times F(t') \, dt' \, dt.$$ \hspace{1cm} (2.10)
For two half-planes with surfaces located at \( z = 0 \) and \( z = d \) one can integrate for low densities to obtain the total energy dissipation per unit surface as

\[
\Delta E = \rho_1 \rho_2 \int_{z_1>0, z_2<0} \Delta E(r_0) \, dx_1 \, dy_1 \, dz_1 \, dz_2,
\]

where \( \rho_1, \rho_2 \) are the uniform number densities. We write this as

\[
\Delta E = \rho_1 \rho_2 \int_{t>t'} L(t, t') \phi(t - t') \, dt \, dt',
\]

and find after some calculation by use of the Fourier transform methods that \( L(t, t') \) takes the form of equation (112)

\[
L(t, t') = - \int \hat{F}(t) \hat{F}(t') \, dx_1 \, dy_1 \, dz_1 \, dz_2 = - \frac{1}{(2\pi)^2} \times \int_{z_1>0, z_2<0} \hat{\psi}(z_0, k_\perp) \hat{\psi}(z_0, -k_\perp) A(t, t') \, dk_\perp \, dz_1 \, dz_2.
\]

where

\[
A(t, t') = -ik_x v(t) e^{-ik_x x_0(t) - q(t) t'}
\]

with \( dk_\perp = dk_x dk_y \), and \( z_0 = z_1 - z_2 \).

The \( A(t, t') \) is to be integrated together with the \( \phi(t - t') \) of equation (2.7). With use of the condition of return \( q(\infty) = q(-\infty) = 0 \) one finds that it can be rewritten as equation (117)

\[
A(t, t') = \frac{1}{2} \sum_{n=\pm 1} \hat{Q}(t, n\omega_v) \hat{Q}(t', -n\omega_v),
\]

where

\[
\hat{Q}(t, \omega) = e^{-i\omega \cdot \hat{v}(t)} - 1, \quad \text{with} \quad \hat{v}(t) = k_\perp v(t).
\]

Then the dissipated energy becomes expression (123) which is

\[
\Delta E = \frac{\rho_1 \rho_2}{(2\pi)^3} \int_{z_1>0, z_2<0} \hat{\psi}(z_0, k_\perp) \hat{\psi}(z_0, -k_\perp) J(\omega_v) \, dk_\perp \, dz_1 \, dz_2
\]

\[
J(\omega_v) = \int_{t>t'} A(t, t') \phi(t - t') \, dt \, dt' = C_\pm I(\omega_v) + C_+ I(\omega_v)
\]

The \( C_\pm \) are the coefficients given by equation (2.8). For higher densities straightforward summation (or integration) of particle pairs is no longer valid due to dipolar interactions within each half-plane. This, however, is taken into account by replacing the polarizability \( \alpha = \alpha(\omega) \) with the corresponding dielectric constant \( \varepsilon \). The replacement is \( 2\pi \rho \alpha \rightarrow (\varepsilon - 1)/(\varepsilon + 1) \) as given by equation (150). This extension to arbitrary densities was shown in section 4 of [33].

By some calculation one finds equation (126)

\[
I(\omega) = \frac{\omega}{4} \sum_{n=\pm 1} \hat{Q}(-\omega, n\omega_v) \hat{Q}(\omega, -n\omega_v).
\]

So far, the formalism works out similar to the previous case of [1]. The new element in our analysis is to calculate the integral (128) of [1]. This integral is

\[
\hat{Q}(\omega, -\omega_v) = \int_{-\infty}^{t_0} e^{i\omega t} dt \int_{-\infty}^{t_0} e^{i\omega t} \, dt' = 2 \hat{e}^{i\omega t_0} \frac{\sin(\omega(t - t'))}{\omega}
\]

With \( q(t) \) consisting of only a few linear parts in \( t \), the integral (2.20) is easily evaluated. But to obtain the appropriate form of the result was less trivial as the product of \( \hat{Q} \) and its complex conjugate in equation (126) or equation (2.19) should produce the \( \delta \)-functions of equation (129). But the corresponding \( \delta \)-functions for the slow initial and return motions were not considered as the dissipation should vanish anyway for these parts. For the present situation with varying velocity all finite velocities will contribute and thus cannot be neglected. So to obtain the desired result in this more general situation, the difference between the two terms of the integral must be taken in a proper way.

If the velocity \( v_x = v(t) \) varies slowly the \( q(t) \) can be considered piecewise linear in \( t \) such that explicit integrations can be performed. However, the additional problem is that expression (2.20) should be multiplied with its complex conjugate as mentioned above, by which cross-terms will appear. The problem is to get rid of these cross-terms. As will be seen below, this is possible by separating the integrand into two parts that are subdivided in a different manner in intervals.

Then consider a time interval from \( t_1 \) to \( t_2 \) of length \( 2\tau = t_2 - t_1 \). These times are chosen as limits for part of the first term of the integral of equation (2.20). The corresponding interval for the second term of the integral is chosen from \( t'_1 \) to \( t'_2 \) such that

\[
\omega t_1' = \omega t_1 - \omega_v q(t_1) \quad \omega t_2' = \omega t_2 - \omega_v q(t_2)
\]

With this subdivision of the two terms of the integrand the full integral will be covered properly by such intervals when the motion that starts at time \( t_1 \) ends at the same position at time \( t_2 \), i.e. the condition \( q(t_1) = q(t_2) = 0 \) is fulfilled.

Relation (2.21) can now be expanded around the middle of the time intervals. So to linear order with \( t_1 = t_0 - \tau, t_2 = t_0 + \tau, t_1' = t_0' - \tau', t_2' = t_0' + \tau' \) condition (2.21) becomes

\[
\omega(t_0' - \tau') = \omega(t_0 - \tau) - \omega_v(q(t_0) - q(t_0)\tau)
\]

\[
\omega(t_0' + \tau') = \omega(t_0 + \tau) - \omega_v(q(t_0) + q(t_0)\tau)
\]

from which follows

\[
\omega t_1' = (\omega - \omega_v \hat{q}(t_0))\tau \quad \text{and} \quad \omega t_2' = \omega t_0 - \omega_v q(t_0).
\]

(2.23)

For the chosen interval, one now gets the integrals (with \( x = t' - t_0' \) and then \( x = t - t_0 \))

\[
S_1 = \int_{t_1'}^{t_2'} \frac{e^{-i\omega t'}}{\omega} \, dt' = \hat{e}^{i\omega t_0'} \int_{t_1}^{t_2} \frac{e^{-i\omega x}}{\omega} \, dx = 2 \hat{e}^{i\omega t_0'} \frac{\sin(\omega(t - t'))}{\omega}
\]

(2.24)
of such phase factors should vanish by the further integrations varying. So from this argument we find that cross-terms with the ones in equation (2.28) will add such that equation (I29) is 

\[ \frac{\omega I(\omega)}{\Delta_1 I(\omega)} = 2e^{i\omega_0 q} \sin \left( \frac{(\omega - \omega_0 q) \tau}{\omega - \omega_0 q} \right). \]  

(2.25)

Here the relations of equation (2.23) are utilized and \( \dot{q}(t_0) = \dot{q} \) is used as a simplification. From this the contribution to integral (2.20) becomes 

\[ \Delta Q(\omega, -\omega v) = S_2 - S_1 = 2e^{i\omega_0 q} \sin \left( \frac{(\omega - \omega_0 q) \tau}{\omega - \omega_0 q} \right) \]  

(2.26)

According to equation (2.19) this should be multiplied with its complex conjugate to obtain the following contribution

\[ \Delta I(\omega) = \sum_{n=\pm 1} \frac{(\omega n q)^2}{\omega} \left( \frac{\sin(\omega - \omega_0 q) \tau}{\omega - \omega_0 q} \right)^2 \]  

(2.27)

For large \( \tau (\rightarrow \infty) \) \( \delta \)-functions are obtained with the amplitude determined by the integral \( \int_{-\infty}^{\infty} (\sin x/x)^2 \, dx = \pi \). Thus for large \( \tau \)

\[ \Delta I(\omega) = \pi \frac{(\omega n q)^2}{\omega} \left[ \delta(\omega - \omega_0 q) + \delta(\omega + \omega_0 q) \right]. \]  

(2.28)

With \( \dot{q} = 1 \) this is equation (I29).

Likewise there will be similar contributions from the other time intervals of the motion. When adding these contributions to equation (2.26) they will form cross-terms when multiplied together. However, products of terms for different time intervals with midpoints \( t_0 \) and \( t_0' \) will have a phase factor \( e^{i\omega(t_0 - t_0')} \). This phase factor will vary rapidly as the function of \( \omega \) since \( |t_0' - t_0| \) can be chosen as large when \( q(t) \) is slowly varying. So from this argument we find that cross-terms with such phase factors should vanish by the further integrations of \( \omega \) and \( \omega_0 \). With the lack of cross-terms, contributions like the ones in equation (2.28) will add such that equation (I29) is modified into (\( dt_0 \geq 2 \tau \))

\[ \begin{align*}
I(\omega) &= \int \Delta I(\omega) \, dt_0 \\
&= \int \pi \frac{(\omega n q)^2}{\omega} \left[ \delta(\omega - \omega_0 q) + \delta(\omega + \omega_0 q) \right] \, dt_0.
\end{align*} \]  

(2.29)

So altogether with varying velocity the various velocities give independent and additive contributions to the dissipation. With two eigenfrequencies one only has \( \omega = \omega_0 \pm \omega_2 \). In the general situation bands of eigenfrequencies and integrations of \( I(\omega) \) are performed as in [1] starting with equation (I33) to obtain the resulting dissipation.

3. Motion in the plane

The results in the previous section are for rectilinear motion. However, it can be extended to more general motion in a straightforward way. Without relative rotation the motion is such that the term \( v_\omega q(t) \) of equation (2.9) is replaced by

\[ x = v_\omega q_\theta(t), \quad y = y(t) = v_\gamma q_\gamma(t). \]  

(3.1)

But integral (2.20) can be kept now where

\[ \omega_0 q(t) \to k_\omega x + k_\gamma y. \]  

(3.2)

As before, the velocity is expected to vary slowly to be considered approximately constant within a long time interval \( 2\tau \). Expanding around its midpoint \( t = t_0 \) we have with \( u = t - t_0 \) (with \( q_{\ell_1} = q_{\ell_2}(t_0) \) etc)

\[ x = v_\omega q_\theta + v_\omega q_\theta u, \quad y = v_\gamma q_\gamma + v_\gamma q_\gamma u, \quad \omega_0 q \to k_\omega v_\omega q_\theta + k_\gamma v_\gamma q_\gamma + \omega_0 q_\gamma u \]  

(3.3)

where now \( \omega_0 = kv \) with

\[ v_\omega = v \cos \varphi_\omega = q, \quad v_\gamma = v \sin \varphi_\gamma = \dot{q}, \quad \dot{q}^2 = \dot{q}_\omega^2 + \dot{q}_y^2, \quad \omega_0 = kv \cos(\varphi_\omega - \varphi_\gamma), \quad k_\omega = k \cos \varphi_\gamma, \quad k_\gamma = k \sin \varphi_\gamma. \]  

(3.4)

Integral (2.20) can now be performed as before, and for its two terms, condition (2.21) will be modified to

\[ \omega_0' = \omega_0 - (k_\omega v_\omega q_\theta(t_1) + k_\gamma v_\gamma q_\gamma(t_1)) \]  

(3.5)

Likewise expansion (2.22) can be used and conditions (2.23) are still valid with the minor replacement \( \omega_0 q(t_0) \to k_\omega v_\omega q_\theta(t_0) + k_\gamma v_\gamma q_\gamma(t_0) \). With this, all remaining results (2.24)–(2.29) are still valid.

However, there might be a problem remaining as the velocity changes direction by which the angle \( \varphi_\gamma \) of equation (3.4) will vary slowly with time. But this will not influence the remaining integration with respect to \( k \) when following the derivations in [1] since only the relative angle between \( k \) and \( v \) will occur anyway.

Altogether, we have found that the result for the energy dissipation and friction obtained in [1] is valid for more general motion. This is the situation for plates that move relative to each other in a closed circle with only one constant speed \( \dot{q} = \text{const} \).

4. Rotating planar disc

The results obtained in section 3 will be valid for more general motion where the plates can also rotate with respect to each other. Such a situation will be pure rotation around a centre at constant angular velocity. See figure 1, where the upper plate (radius \( R \)) now rotates with angular velocity \( \Omega \) around the vertical axis \( z \). The lower plate is at rest, and is of infinite extent, as before.

The argument is that a rotating plate can be subdivided in small areas whose linear dimension is large compared to the separation from the plate at rest. Each area can thus be regarded as a macroscopic plate that moves around. This latter small area will also perform a rotation. But since its linear size is much smaller than that of the whole plate, this rotation contributes to negligible differences between velocities within each small area by which they can be considered equal. Thus, for each of them the results of section 3 are valid. This is at least obvious for a low dielectric constant in which case the resulting friction force is the sum of the contributions for each separate particle.

For a rotating plate it is of interest to have the torque acting due to friction. For two metal plates of the same material at

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temperature \( T = 0 \) the friction force per unit area in [1] was by its equation (156) found to be

\[
F_p = -C_p v^3, \quad C_p = \frac{15\pi^2}{64\alpha^2} D^3 h^3, \quad D = \frac{h v}{\rho (\pi h \omega_p)^2}
\]  
(4.1)

with dielectric function \( \varepsilon = 1 + \alpha^2 / (\xi (\xi + v)) \) where \( \xi = i \omega_c \). (Here only small frequencies \( \omega < \omega_c \) in the corresponding frequency distribution were needed.) The \( \rho \) is the particle density of free electrons, \( \omega_p \) is the corresponding plasma frequency, and \( d \) is the separation between the plates.

Likewise at finite temperature \( T \), the corresponding friction force was found by equation (159) in the reference to be

\[
F = -C v, \quad C = \frac{\pi^4}{4 \beta^2} D^3 h^3
\]  
(4.2)

with \( \beta = 1/(k_B T) \) where \( k_B \) is the Boltzmann’s constant. Here \( d / (\beta h v) \gg 1 \) is assumed which holds unless \( T \) is very small or \( v \) is very large.

To obtain results (4.1) and (4.2) the frequency distribution for the dielectric function given below (4.1) for both metal half-planes is needed. It is found via the imaginary part of this function and is given by equation (151) (for small \( m \))

\[
m^2 \alpha_1 (m^2) = D m, \quad m = h \omega_c.
\]  
(4.3)

The \( \alpha_1 (m^2) \) with \( m = m_1 \) and \( m = m_2 \) respectively replaces the product of polarizabilities \( \alpha_1 \) and \( \alpha_2 \) in the \( C \) given by equation (2.8). With this replacement the \( J(\omega_c) \) and thus the \( \delta \)-functions of \( I(\omega_c) \) are integrated with the volume element \( d(m_1^2) d(m_2^2) \). Then for \( T = 0 \) only the \( C_+ \) term contributes while for finite \( T \) only the \( C_- \) term contributes as \( \omega_c \) inside the \( \delta \)-functions and can then be neglected. This results in equations (164) and (152) respectively for \( J(\omega_c) \). Then its \( \omega = k v \) dependence is averaged over directions. Furthermore the electrostatic dipolar interaction has to be inserted in equation (2.17). This is obtained from the Coulomb interaction \( \psi = \psi(r) \) by which the corresponding dipolar interaction \( \psi_{ij} \) \( (i, j = x, y, z) \) is given by equation (136) as

\[
\psi_{ij} = -\frac{\hat{a}^2}{\hat{A} x \hat{A} y} \psi, \quad \psi = \frac{1}{r}.
\]  
(4.4)

The Fourier transforms in the \( xy \)-plane are

\[
\hat{\psi}_{ij}(z_0, k_\perp) = -k_i k_j \hat{\psi}(z_0, k_\perp), \quad \hat{\psi}(z_0, k_\perp) = \frac{2\pi e^{-q|z_0|}}{q}
\]  
(4.5)

where here \( q = k_\perp, k_\perp^2 = k_i^2 + k_j^2, \) \( i k_\perp = \mp q \) (for \( z_0 \geq 0 \)). The \( \hat{\psi}_{ij} \) (with \( \sum_{ij} \)) substitutes the \( \psi \) in equation (2.17) to obtain equation (141)

\[
\hat{G}(z_0, q) = (2q^2)^2 \left( \frac{2\pi e^{-q|z_0|}}{q} \right)^2.
\]  
(4.6)

Finally the integrations of equation (2.17) are performed to obtain results (4.1) and (4.2) above with \( \rho_1 = \rho_2 = \rho \).

For the torque on a rotating plate to be finite it should have a finite radius \( R \). With this the torque due to friction for the metal plate rotating with angular speed \( \Omega \) at \( T = 0 \) will be (with \( v = \Omega r \))

\[
\tau_p = \int_0^R r F_p 2\pi r \, dr = -2\pi C_p \int_0^R \Omega^2 r^5 \, dr = -\frac{\pi}{3} C_p R^6 \Omega^3.
\]  
(4.7)

Likewise for finite temperature the torque will be

\[
\tau = \int_0^R r F 2\pi r \, dr = -2\pi C \int_0^R \Omega^3 r^3 \, dr = -\frac{\pi}{2} C R^4 \Omega.
\]  
(4.8)

As noted above in equation (2.19) these results for metal plates, with dielectric function given below in equation (4.1), are not restricted to a pairwise approximation for pairs of particles, but are valid for arbitrary densities.

Here it should be noted that the \( T > 0 \) result (4.2) (apart from a small factor \( \approx 1.2 \)) agrees with a result obtained earlier by Volokin and Persson [8] as shown in [33]. Furthermore in [1] we showed that the \( T = 0 \) result (4.1) agrees with the one obtained by Barton (except for the factor \( \zeta(5) = 1.037 \)) [18]. Except for a numerical factor 2 (or 12) it is in accordance with an earlier result by Pendry [3, 8]. In this respect, however, our results, like those mentioned, are not in agreement with the recent ones of Silveirinha [20]. There, for instance, the quantum friction force is expected to have exponential growth, but is found to be consistent with the semi-classical result of Pendry [3, 5] in the weak interaction limit. Also a velocity threshold above which quantum friction can take place was found in [20]. We can see no such threshold as the friction is present for all velocities. This reference also draws conclusions about relativistic velocities where Cherenkov radiation will appear. We, however, can not draw such conclusions about Cherenkov radiation, as we use electrostatic dipole interactions (4.4) and thus we assume non-relativistic velocities.

In a recent work a freely rotating disc or cylinder was considered [40]. This, however, is a situation quite different from the one considered in this work, with a disc or plate rotating close to another parallel plate. In addition, we limit ourselves to the electrostatic field (near field) while friction on a freely rotating cylinder or disc requires energy loss by radiation. Thus, for various reasons, our results cannot be compared to those of this recent reference.

5. Summary

We have extended our previous results for Casimir friction to a situation where the velocity may vary both in magnitude and direction. As may be expected we find that the various velocities give independent contributions to the dissipated energy. In [1] our results were compared with those of others both for temperatures \( T = 0 \) and \( T > 0 \), and an agreement with the results of [3, 8] and [18] was (mostly) obtained.
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