THE EFFECT OF MASS SEGREGATION ON GRAVITATIONAL WAVE SOURCES NEAR MASSIVE BLACK HOLES

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Received 2006 March 13; accepted 2006 May 30; published 2006 June 29

ABSTRACT

Gravitational waves (GWs) from the inspiral of compact remnants (CRs) into massive black holes (MBHs) will be observable to cosmological distances. While a CR spirals in, two-body scattering by field stars may cause it to fall into the central MBH before reaching a short-period orbit that would give an observable signal. As a result, only CRs very near (\(\sim 0.01\) pc) the MBH can spiral in successfully. In a multimass stellar population, the heaviest objects sink to the center, where they are more likely to slowly spiral into the MBH without being swallowed prematurely. We study how mass segregation modifies the stellar distribution and the rate of GW events. We find that the inspiral rate per galaxy is 30 \(\text{Gyr}^{-1}\) for white dwarfs, 6 \(\text{Gyr}^{-1}\) for neutron stars, and 250 \(\text{Gyr}^{-1}\) for 10 \(M_{\odot}\) stellar black holes (SBHs). The high rate for SBHs is due to their extremely steep density profile, \(n_{\text{SB}}(r) \propto r^{-\frac{7}{2}}\). The GW detection rate will be dominated by SBHs.

Subject headings: black hole physics — Galaxy: center — gravitational waves — stellar dynamics

Online material: color figures

1. INTRODUCTION

Massive black holes (MBHs) with masses \(M_{\bullet} \lesssim 5 \times 10^6 M_{\odot}\) have Schwarzschild radii \(r_S = 2GM_{\bullet}/c^2\), such that a test mass orbiting at a few \(r_S\) emits gravitational waves (GWs) with frequencies \(10^{-4} \text{Hz} \lesssim \nu \lesssim 1 \text{Hz}\), detectable by the planned space-based Laser Interferometer Space Antenna (LISA). Main-sequence (MS) stars with mass \(M_*\) and radius \(R_*\) will be disrupted at the tidal radius \(r_t = (M_*/M_{\bullet})^{1/3}R_* > r_S\) and are therefore unlikely to be sources of observable GWs (our own Galactic center may be an exception; Freitag 2003). Compact remnants (CRs) such as white dwarfs (WDs), neutron stars (NSs), and stellar black holes (SBHs) have tidal radii \(r_t < r_S\) and can emit GWs that are observable to cosmological distances. The inspiral of a CR into an MBH (“extreme mass ratio inspiral” [EMRI]) is among the main targets of LISA.

The event rate of EMRIs has been estimated by numerous authors (Hils & Bender 1995; Sigurdsson & Rees 1997; Miralda-Escudé & Gülü 2004; Freitag 2005; Ivanov 2002; Freitag 2003; Alexander & Hopman 2003; Hopman & Alexander 2005, 2006) but remains rather uncertain, in part because of the slow nature of the inspiral process, which occurs on many dynamical times. This makes the inspiraling star very susceptible to scattering by other stars, which can change the orbital parameters. The formalism for inspiral rates is similar to that for the prompt consumption of stars (Bahcall & Wolf 1977, hereafter BW77; Lightman & Shapiro 1977; Frank & Rees 1976; Cohn & Kulsrud 1978; Syer & Ulmer 1999; Magorrian & Tremaine 1999), but there are some important differences because the process is much slower.

The picture can be understood as follows: Let \(t_\text{i}\) be the relaxation time of a star with negative energy \(E\) (hereafter “energy”); \(E > 0\) for bound stars and specific angular momentum \(J\) (hereafter “angular momentum”). The relaxation time is the timescale for a change of order of energy \(E\), or a change in angular momentum of order \(J(E)\), the circular angular momentum. The change in \(J\) of a star per orbital period \(P\) is \(\Delta J = (P/t_\text{i})^{\frac{1}{2}}I_j\). The timescale for a change of order \(J\) is \(t_\text{i} = (J/J_c)^2t_\text{i}\). In particular, the timescale for a change in \(J\) by the order of the loss cone, determined by the angular momentum of the last stable orbit \(J_{\text{LSO}} = 4GM_{\bullet}/c^2\), is \(t_{\text{ls}} = (J_{\text{LSO}}/I_j)^2t_\text{i}\), which for highly eccentric orbits has a very strong angular momentum dependence, \(t_{\text{ls}}(J) \propto J^2\). If \(t_{\text{ls}} \ll t_\text{i}(E, J > J_{\text{LSO}})\), the angular momentum will be modified even if the star has \(J \gtrsim J_{\text{LSO}}\). As a result, it is very likely that the star will be scattered into the loss cone (or away from it, to an orbit where energy dissipation is very weak). Such CRs will eventually be consumed by the MBH and add to its mass, but they will not be observable as GW emitters (GW bursts in our own Galactic center may form an exception; Rubbo et al. 2006).

The condition \(t_{\text{i}}(E, J > J_{\text{LSO}}) < t_{\text{ls}}(E)\) translates into a minimal energy or maximal semimajor axis \(a_{\text{GW}}\) that a CR must have in order to spiral in and become a LISA source (“successful inspiral”); Hopman & Alexander (2005, hereafter HA05) estimate that for an MBH of \(M_{\bullet} = 3 \times 10^6 M_{\odot}\), \(a_{\text{GW}} \approx 0.01\) pc. Nearly all CRs with \(a > a_{\text{GW}}\) are promptly captured or deflected without giving an observable signal, while nearly all stars with \(a < a_{\text{GW}}\) do spiral in successfully.

The fact that the distribution of CRs near MBHs is crucial to the observational outcome implies that mass segregation is likely to play a very important role for EMRIs (see Freitag et al. 2006 for a detailed treatment of mass segregation near MBHs). Mass segregation drives the heaviest objects to the center, so their concentration within increases, and the lightest stars to larger radii, so that they are relatively rare within \(a_{\text{GW}}\). The importance of mass segregation for inspiral processes was demonstrated in N-body simulations (Baumgardt et al. 2004, 2005) of the tidal capture of MS stars (Alexander & Morris 2003; Hopman et al. 2004). Baumgardt et al. (2005) studied tidal capture by an \(\sim 10^6 M_{\odot}\) black hole in a young stellar cluster with MS masses up to \(\sim 100 M_{\odot}\). In spite of the fact that massive stars are scarce, captured stars typically had masses \(^{1}M_{\bullet} \sim 20 M_{\odot}\).

In this Letter we study the implications of mass segregation on the EMRI rate.

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2. Model

Our model is based on the analysis of Bahcall & Wolf (1976) and BW77. Here we briefly recapitulate the main assumptions and discuss our treatment of GW capture. A detailed discussion of our model can be found in HA05 and Hopman & Alexander (2006).

2.1. Dynamics

The MBH dominates the dynamics of stars within its radius of influence, \( r_g = GM_*/\sigma_*^2 \), where \( \sigma_* \) is the velocity dispersion of a typical star of mass \( M_* \) (assumed of solar type), which we will use to scale our expressions. Orbits are assumed to be Keplerian within \( r_g \). Each species with mass \( M \) is described by a distribution function (DF) in energy space \( f_M(E) \). We define a dimensionless time \( \tau = t/t_h \) in terms of the relaxation time at the radius of influence

\[
T_h = \frac{3(2\pi^2/M_*)^{3/2}}{32\pi^2G^2M_*^2n_* \ln \Lambda},
\]

where \( n_* \) is the number density at \( r_g \), for the typical star \( M_* \), \( \beta = M/\sigma_*^2 \), and \( \Lambda = M_*/M_* \). The dimensionless energy \( x = (M/M_*)(E/E_0) \) and the dimensionless DF \( g_M(x) = [(2\pi\beta M_*)^{3/2}n_*^{-1}]f_M(E) \), the Fokker-Planck equation in energy space is (BW77, eq. [26])

\[
\frac{\partial g_M(x, \tau)}{\partial \tau} = -x^{5/2} \frac{\partial}{\partial x} Q_M(x) - R_M(x).
\]

The spatial number density of stars is related to the DF by

\[
n_M(r) = 2\pi^{-1/2}n_* \int_{-\infty}^{r_g} dx g_M(x)(r_g/|r - x|)^{1/2}.
\]

We fit our numerical results by power laws \( n_M(r) \propto r^{-n_M} \).

In equation (2), \( Q_M(x) \) is the (dimensionless) rate at which stars flow to energies larger than \( x \),

\[
Q_M(x) = \sum_M \frac{M}{M_*} \int_0^{x_h} dx' \left[ \frac{g_M(x')}{\partial x'} - M' \frac{\partial g_M(x')}{\partial x} \right].
\]

The dimensional stellar current is related to \( Q_M \) by \( I_M(E, t) = I_0Q_M(x, t) \), where

\[
I_0 = \frac{8\pi^2}{3\sqrt{2}} r_g n_* (GM_*)^2 \ln \Lambda_n \sigma_*^{-3}
\]

(Bahcall & Wolf 1976; Hopman & Alexander 2006).

The last term in equation (2) represents losses of stars due to loss-cone effects (both prompt infall and inspiral) in \( J \)-space. The sink term in the diffusive régime for the loss cone is

\[
R_M(x) = \frac{g_M(x)}{\tau(x) \ln \left[ J(x)/M_0 \right]},
\]

where \( J(x)/M_0 = 1/(4\sqrt{2}\epsilon/\epsilon_a) - x^{-1/2} \). The full loss-cone régime, \( x \lesssim 10 \), does not contribute to the GW event rate (Alexander & Hopman 2003; HA05; Hopman & Alexander 2006). In our calculations we neglect the sink term in the full loss-cone régime by setting \( R_M \rightarrow 0 \) for \( x < 10 \).

2.2. Boundary Conditions and Model Parameters

Let the number of stars accreted to the MBH before giving an observable GW signal be \( N_M(x) \), and let the number of those that spiral in successfully and do give a signal be \( N(x) \).

The steady state result for \( \tau(x) \) is used to determine the probability for inspiral \( S_M(x) = N_M(x)/[N_M(x) + N(x)] \) by Monte Carlo simulations (HA05) as follows. At every orbit, a star of initial energy \( E \) and initially large \( J \) makes a step in \( J \) of order \( \Delta J = [P(x)/I_0(x)]^{1/2}J \), with random sign because of scattering and loses energy \( dE_{GW} = (85\pi/24576)M(M_0)M_*^{-2}(J/L_{SO}) \) to GWs (Peters 1964). This is repeated many times, and the outcome is recorded. The total rate of successful inspirals for species \( M \) is then given by \( \Gamma_M = I_0 \int_0^\infty dx S_M(x)x^{-3/2}R_M(x) \). It is convenient to express the capture rate in terms of the semimajor axis \( a = r_g/2x \) of the stars,

\[
\Gamma_M(<a) = \frac{2\pi}{I_0 a^{3/2}} \int_0^a da a^{1/2}S_M(a)R_M(a).
\]

4 The Galactic MBH obeys the \( M_\bullet-M_\ast \) relation (Ferrarese & Merritt 2000; Gebhardt et al. 2000; Tremaine et al. 2002), so that these values may also be representative of other MBHs.
TABLE 1

| Star | $M$ $(M_\odot)$ | $C_u$ | $\alpha_{\rm BH}$ | $N_d(<0.01)$ pc | $N_d(<0.1)$ pc | $a_{\rm GW}$ (mpc) | $\Gamma_\nu$ (Gyr$^{-1}$) |
|------|----------------|-------|-------------------|----------------|----------------|----------------|----------------|
| MS   | 1              | 1.0   | 1.4               | $10^4$         | $3 \times 10^0$ | $\ldots$      | $\ldots$      |
| WD   | 0.6            | 0.1   | 1.4               | 80             | $2.7 \times 10^4$ | 4              | 30            |
| NS   | 1.4            | 0.01  | 1.5               | 12             | 374            | 5              | 6             |
| SBH  | 10             | $10^{-1}$ | 2.0               | 150            | $1.8 \times 10^3$ | 13             | 250           |

3. RESULTS

We integrated equation (2) until a steady state is obtained, after time $t \approx 1$. In Figure 1 we show the resulting densities for the different species. The DF of the SBHs is much steeper than that of the other types ($\alpha_{\rm BH} = 2.0$), and at $r \approx 0.01$ pc the number density of SBHs becomes comparable to that of the WDs. MS stars dominate everywhere by number, although we did not take into account stellar collisions [Freitag & Benz 2002, 2005], which could deplete the MSs close to the MBH. SBHs also determine the functional behavior of $t_r \propto r^p$, where $p \approx 3/2 \approx 0.5$.

Throughout most of the cusp $\alpha_{\rm BH} = 2$, but near $x_p$, the DF flattens, as required for the integrals in equation (4) to converge at high energies (BW77). Large slopes at intermediate energies are allowed by these equations, and they arise when a population of massive objects with a low number density exists, as is the case in our model. At low energies the massive objects sink effectively to the center by dynamical friction. At high energies the massive objects dominate the dynamics, decouple from the lighter objects, and form an $\alpha = 7/4$ "minicusp." This process is reminiscent of the Spitzer instability in globular clusters, where SBHs decouple from the other stars [Spitzer & Hart 1971; Khalisi et al. 2006].

In Figure 2 we show the cumulative rates of successful inspiral (eq. [8]) for all CRs as a function of distance from the MBH. We summarize some results in Table 1, where we also give the enclosed number of stars $N_d(<a)$ within $a$.

4. SUMMARY AND DISCUSSION

In our model, SBHs dominate the EMRI rate, in spite of their small number density at $r_c$. The combination of a very steep cusp ($\alpha_{\rm BH} = 2$) and a larger $a_{\rm GW}$ due to their larger mass leads to a GW inspiral rate per galaxy $\Gamma_{\rm BH} \geq \Gamma_{\rm WD}, \Gamma_{\rm NS}$. We also note that the amplitude of the GWs is proportional to the stellar mass, so that the distance at which these objects can be observed is $\sim 10$ times larger than that for WDs and NSs. Thus, SBHs will dominate the cosmic detection rate.

It is instructive to compare the EMRI rates that we obtain here to those obtained by HA05, where mass segregation was not explicitly included. For SBHs, $\Gamma_{\rm BH}$ is larger by a factor of $\approx 50$. Part of the difference is our assumption of a larger total number of SBHs within $r_c$ (by a factor of $\approx 6$; we normalized the SBH number fraction at $r_c$ to be $C_{\rm BH} = 10^{-3}$, while HA05 assumed that the enclosed fraction of SBHs is $10^{-5}$). More importantly, the steeper cusp leads to a higher capture rate (by a factor of $\approx 9$; see eq. [32] of HA05, who assumed $\alpha_{\rm BH} = 1.75$). The BH cusp is much steeper than any of the cases studied by BW77, and in particular it is steeper than the cusp of a single mass population, $\alpha = 7/4$ (Bahcall & Wolf 1976).

The rates of $\Gamma_{\rm WD}$ and $\Gamma_{\rm NS}$ are also somewhat larger than those found by HA05. Here the difference originates mainly in the behavior of $t_r$: For WDs and NSs, $t_r$ was assumed to be constant by HA05, as appropriate for a single mass population with $\alpha = 3/2$. However, the interaction between SBHs and the other CRs leads to a decrease in $t_r$ toward the MBH (Fig. 1). Using the analytical expressions in HA05, it can be shown that $n_{\rm BH}(r) \propto r^{-3/2}$, and $t_r \propto r^p$, the successful inspiral rate is enhanced by $(d_r/r_c)^{-3p(3-2p)} \sim 10$ (for $p = 0.5$) relative to the $t_r = \infty$ case, where $d_r = ([85/3072]GM(M/M_\odot) t)^{1/3}$ (see HA05, eq. [29]).

To probe the sensitivity of our results to the assumed SBH mass, we also performed a calculation for $M_{\rm BH} = 3 M_\odot$, in which case we found $\Gamma_{\rm WD}, \Gamma_{\rm NS}, \Gamma_{\rm BH} = (37, 21, 51)$ Gyr$^{-1}$, and for $M_{\rm BH} = 20 M_\odot$, in which we found $\Gamma_{\rm WD}, \Gamma_{\rm NS}, \Gamma_{\rm BH} = (36, 5, 463)$ Gyr$^{-1}$. Clearly, the more massive the SBH,
the higher their inspiral rate, because $t_s$ decreases, and $\Delta E_{GW}$ increases (the slope remains approximately the same, $\alpha_{BH} \approx 2$). When $M_{BH} = 3 M_\odot$, $N_{NS}$ becomes larger, due to the steeper NS cusp ($\alpha_{NS} \approx 1.7$), so that more NSs are within $a_{GW}$. We also find $\alpha_{WD} \approx 1.5$ and $\alpha_{GW} \approx 2$ for this case.

The EMRI rates that we found here are promising for the LISA detection rate (Barack & Cutler 2004a; Gair et al. 2004), in spite of the fact that more sources also imply a stronger background noise (Barack & Cutler 2004b). We neglected here the effect of resonant relaxation (RR; Rauch & Tremaine 1996; Rauch & Ingalls 1998), which can increase the EMRI rate by up to an order of magnitude (Hopman & Alexander 2006). A multimass analysis of RR has yet to be performed. In addition to direct capture of CRs, EMRI can occur following the formation of SBHs in accretion disks (Levin 2003), binary disruptions (Miller et al. 2005), and tidal capture followed by a supernova explosion of the captured star (Hopman & Portegies Zwart 2005). These other mechanisms lead to low-eccentricity signals, whereas direct capture leads to high eccentricities (HA05).

Our estimate of the number fraction of SBHs at $r_h$ is somewhat uncertain, in part because we neglected dynamical evolution for unbound stars (fixed by the boundary conditions). We note that our estimate $N_{BH}(r_h) \sim 1.6 \times 10^4$ is consistent with the calculations of Miralda-Escudé & Gould (2000), who found $N_{BH}(r_h) \sim 2.5 \times 10^4$. Our Galactic center contains an MBH of mass comparable to the MBH mass considered here (Ghez et al. 2005; Eisenhauer et al. 2005). Observational effects of a cluster of SBHs near the Galactic MBH include microlensing (Chaname et al. 2001), X-ray emission (Pessah & Melia 2003), capture of massive stars by an exchange interaction (Alexander & Livio 2004), and deviations from Keplerian motion of luminous stars (Mouawad et al. 2005). Such effects could in principle be used to constrain the predicted densities.

T. A. is supported by ISF grant 295/02-1, Minerva grant 8563, and a New Faculty grant by Sir H. Djangoly, CBÉ, of London, UK.

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