The ill-posedness for the rotation Camassa-Holm equation in Besov space $B^1_{\infty,1}(\mathbb{R})$

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Abstract

In this paper, we present a construction of $u_0 \in B^1_{\infty,1}$ and get the local ill-posedness for the rotation Camassa-Holm equation modelling the equatorial water waves with the weak Coriolis effect by proving the norm inflation.

Mathematics Subject Classification: 35Q35, 35B30, 35G25

Keywords: The rotation Camassa-Holm equation, Besov spaces, Continuous dependence, Non-uniform dependence.

1 Introduction

In this paper, we consider the Cauchy problem for the following rotation Camassa Holm (R-CH) equation in the equatorial water waves with the weak Coriolis effect [6]

$$\begin{align*}
&u_t - \beta \mu u_{xxt} + cu_x + 3\alpha \varepsilon uu_x - \beta_0 \mu u_{xxx} + \omega_1 \varepsilon^2 u_x^2 + \omega_2 \varepsilon^3 u_x^3 = \alpha \beta \varepsilon \mu (2u_x u_{xx} + uu_{xxx}), \\
&u(0, x) = u_0(x),
\end{align*}$$

where $\varepsilon$ is the amplitude parameter, $\mu$ is the shallowness, $\Omega$ characterizes the constant rotational speed of the Earth, we define the other coefficients of [1,2]

$$\begin{align*}
c &= \sqrt{1 + \Omega^2} - \Omega, & \alpha &= \frac{c^2}{1 + c^2}, & \beta_0 &= \frac{c(c^4 + 6c^2 - 1)}{6(c^2 + 1)^2}, & \beta &= \frac{3c^4 + 8c^2 - 1}{6(c^2 + 1)^2}, \\
\omega_1 &= -\frac{3c(c^2 - 1)(c^2 - 2)}{2(c^2 + 1)^3}, & \omega_2 &= \frac{(c^2 - 1)^2(c^2 - 2)(8c^2 - 1)}{2(c^2 + 1)^5}.
\end{align*}$$

Applying the scaling

$$x \rightarrow x - c_0 t, \quad u \rightarrow u - \gamma, \quad t \rightarrow t$$

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where \( c_0 = \frac{\beta}{\alpha} - \gamma \) and \( \gamma \) is the real root of \( c - \frac{\beta}{\alpha} - 2\gamma + \frac{\alpha^2}{\beta} \gamma^2 - \frac{2\beta}{\alpha} \gamma^3 = 0 \), one can rewrite (1.1) as
\[
 u_t + uu_x = -\partial_x(1 - \partial_x)^{-1}\left(\frac{1}{2}u_x^2 + c_1u^2 + c_2u^3 + c_3u^4\right) =: G(u), \tag{1.3}
\]
where
\[
 c_1 = 1 + \frac{3\gamma^2\omega_2}{2\alpha^3} - \frac{\omega_1\gamma}{\alpha^2}, \quad c_2 = \frac{\omega_1}{3\alpha^2} - \frac{\omega_2\gamma}{\alpha^3}, \quad c_3 = \frac{0.2\omega_2}{4\alpha^3}.
\]

The study of nonlinear equatorial geophysical waves is of great current interest. The Earth’s rotation affects the atmosphere-ocean flow near the Equator in such a way that waves propagate practically along the Equator \[11\]. Following the idea of \[21\] and \[12\], Chen et al. \[6\] derived the R-CH equation \( \text{(1.1)} \) with the influence of the Coriolis effect, and justified that the R-GN equations tend to associated solution of the R-CH equations in the Camassa-Holm regime \( \mu \ll 1 \), \( \epsilon = O(\sqrt{\mu}) \).

For \( \Omega = 0 \), then the Eq. \( \text{(1.3)} \) becomes the classical Camassa-Holm (CH) equation \[3,12\]
\[
 u_t - uu_{xx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}.
\tag{1.4}
\]
The CH equation is completely integrable \[10\] and has a bi-Hamiltonian structure \[16\]. That means that the system can be transformed into a linear flow at constant speed in suitable action-angle variables (in the sense of infinite-dimensional Hamiltonian systems), for a large class of initial data.

It was shown that there exist the global strong solutions \[2,3,13\], blow-up strong solutions \[2,3,13,14\], global weak solutions \[3\], global conservative solutions and dissipative solutions \[16,17\]. For local well-posedness and ill-posedness for the CH equation, the local well-posedness to CH equation in \( H^s(\mathbb{R}) \) for \( s > \frac{3}{2} \) was proved by Li and Olver \[24\]. The ill-posed in \( H^s(\mathbb{R}) \) for \( s < \frac{3}{2} \) to \( \text{(1.4)} \) was established by Byers \[3\].

Danchin et al. extended the well-posed space to Besov spaces, and presented that the Cauchy problem \( \text{(1.4)} \) is well-posed in Besov spaces \( B_{p,r}^s(\mathbb{R}) \) with \( s > \max\left(\frac{3}{2}, 1 + \frac{1}{p}\right) \), \( r < +\infty \) or \( s = 1 + \frac{1}{p} \), \( p \in [1,2] \), \( r = 1 \) \( \left(1 + \frac{1}{p} \geq \frac{3}{2}\right) \) \( \text[14,15,22] \). The non-continuity of the CH equation in \( B_{p,\infty}^s(\mathbb{R}) \) with \( s > 2 + \max\left(\frac{3}{2}, 1 + \frac{1}{p}\right) \) was demonstrated by constructing a initial data \( u_0 \) such that corresponding solution to the CH equation that starts from \( u_0 \) does not converge back to \( u_0 \) in the norm of \( B_{p,\infty}^s(\mathbb{R}) \) as time goes to zero \[23\]. The ill-posedness for the Camassa-Holm type equations in Besov spaces \( B_{p,r}^{1+\frac{1}{p}}(\mathbb{R}) \) with \( p \in [1, +\infty] \), \( r \in (1, +\infty) \) was established by Guo et al. \[20\], which implies \( B_{p,1}^{1+\frac{1}{p}}(\mathbb{R}) \) is the critical Besov space for the CH equation.

Recently, by the compactness argument and Lagrangian coordinate transformation, Ye et al. \[26\], proved the local well-posedness for the Cauchy problem of CH equation in critial Besov spaces \( B_{p,r}^{1+1/p}(\mathbb{R}) \) with \( p \in [1, +\infty) \). More recently, Guo et al. \[18\] constructed an special initial data \( u_0 \in B_{1,1}^1(\mathbb{R}) \) but \( u_{0x}^2 \notin B_{1,1}^0(\mathbb{R}) \) to get the norm inflation and hence the ill-posedness for the Cauchy problem of CH equation in \( B_{1,1}^1(\mathbb{R}) \).

Due to the influence of Earth’s deflection force caused by the rotation of the Earth, there are three or even four nonlinear terms in the R-CH model, which is different from the CH equation and has an important influence on the fluid movement. So this model has attracted some attention and got some results. The wave-breaking phenomena and persistence properties for \( \text{(1.3)} \) was studied by Zhu, Liu and Ming \[27\].
Theorem 1.2. Our main result in the following: \[G\]

Moreover, Guo and Tu obtained the well-posedness and the non-uniform dependence about initial data for \[
\text{(1.3)} \]
in supercritical and critical Besov spaces \[\text{(1.4)}\].

First, we recall some recent results for the R-CH equation in critical Besov spaces.

**Theorem 1.1** (See [19]). Let \(u_0 \in B^{1+\frac{1}{p}}_{p,1}(\mathbb{R})\) with \(1 \leq p < \infty\). Then there exists a time \(T > 0\) such that the R-CH equation with the initial data \(u_0\) is locally well-posed in the sense of Hadamard.

However, for \(p = \infty\), the local well-posedness or ill-posedness for the Cauchy problem \[
\text{(1.3)} \]
of the R-CH equation have not been solved yet. The main difficulty is that the space \(B^0_{\infty,1}(\mathbb{R})\) is not a Banach algebra, one can’t obtain an a priori estimate of the force term \(G(u) = -\partial_x (1 - \partial_{xx})^{-1} (\frac{1}{2}u^2 + c_1u^2 + c_2u^3 + c_3u^4)\) in \(B^1_{\infty,1}\) (we will focus on \(-\partial_x (1 - \partial_{xx})^{-1}(\frac{u^2}{2})\) in below since the other terms are lower order terms). Note that for any \(u_0 \in B^1_{\infty,1}(\mathbb{R})\), the following formula holds:

\[
E_0 := -\partial_x (1 - \partial_{xx})^{-1}(\frac{u^2}{2}) \in B^1_{\infty,1}(\mathbb{R}) \iff u^2_0 \in B^0_{\infty,1}(\mathbb{R}), \quad -\partial_{xx}(1 - \partial_{xx})^{-1} = \text{Id} - (1 - \partial_{xx})^{-1}.
\]

So we can conclude that the R-CH equation is ill-posed in \(C_T(B^1_{\infty,1}(\mathbb{R}))\) by proving the norm inflation. See our main result in the following:

**Theorem 1.2.** For any \(N \in \mathbb{N}^+\) large enough, there exists a \(u_0 \in C^\infty(\mathbb{R})\) such that the following hold:

1. \(\|u_0\|_{B^1_{\infty,1}} \leq CN^{-\frac{1}{4}}\);
2. There is a unique solution \(u \in C_T(C^\infty(\mathbb{R}))\) to the Cauchy problem \[
\text{(1.3)} \]
   with a time \(T \leq \frac{2}{N^2}\);
3. There exists a time \(t_0 \in [0,T]\) such that \(\|u(t_0)\|_{B^1_{\infty,1}} \geq \ln N\).

Our paper unfolds as follows. In the second section, we introduce some preliminaries which will be used in this sequel. In the third section, we establish the local ill-posedness for \[
\text{(1.3)} \]
in \(B^1_{\infty,1}\).

## 2 Preliminaries

In this section, we first introduce some properties of the Littlewood-Paley theory in [1].

Let \(\chi : \mathbb{R} \to [0,1]\) be a radical, smooth, and even function which is supported in \(B = \{x : |x| \leq \frac{3}{4}\}\). Let \(\varphi : \mathbb{R} \to [0,1]\) be a radical, smooth, function which is supported in \(C = \{x : \frac{3}{4} \leq |x| \leq \frac{8}{3}\}\).

Denote \(\mathcal{F}\) and \(\mathcal{F}^{-1}\) by the Fourier transform and the Fourier inverse transform respectively as follows:

\[
\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_\mathbb{R} e^{-ix\xi}u(x)dx;
\]

\[
u(x) = (\mathcal{F}^{-1}\hat{u})(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{ix\xi}\hat{u}(\xi)d\xi.
\]

For any \(u \in \mathcal{S}'(\mathbb{R}^d)\) and all \(j \in \mathbb{Z}\), define \(\Delta_j u = 0\) for \(j \leq -2\); \(\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u)\); \(\Delta_j u = \mathcal{F}^{-1}(\varphi (2^{-j}\cdot) \mathcal{F} u)\) for \(j \geq 0\); and \(S_j u = \sum_{j' < j} \Delta_{j'} u\).
Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. We define the nonhomogeneous Besov space $B_{p,r}^s(\mathbb{R}^d)$

$$B_{p,r}^s = B_{p,r}^s(\mathbb{R}^d) = \left\{ u \in S'(\mathbb{R}^d) : \|u\|_{B_{p,r}^s} = \|\langle 2^ju \rangle_{L^p} \|_{L^{r,2}} < \infty \right\}.$$  

Then, we recall some properties about the Besov spaces.

**Proposition 2.1** (See [1]). Let $s \in \mathbb{R}$, $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$.

1. $B_{p,r}^s$ is a Banach space, and is continuously embedded in $S'$.
2. If $r < \infty$, then $\lim_{j \to \infty} \|S_j u - u\|_{B_{p,r}^s} = 0$. If $p, r < \infty$, then $C_0^\infty$ is dense in $B_{p,r}^s$.
3. If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-(\frac{1}{p_1} - \frac{1}{p_2})}$. If $s_1 < s_2$, then the embedding $B_{p_2,r_2}^{s_2} \hookrightarrow B_{p_1,r_1}^{s_1}$ is locally compact.
4. $B_{p,r}^s \hookrightarrow L^\infty \iff s > \frac{d}{p}$ or $s = \frac{d}{p}$, $r = 1$.
5. Fatou property: if $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence in $B_{p,r}^s$, then an element $u \in B_{p,r}^s$ and a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ exist such that

$$\lim_{k \to \infty} u_{n_k} = u \text{ in } S' \quad \text{and} \quad \|u\|_{B_{p,r}^s} \leq C \liminf_{k \to \infty} \|u_{n_k}\|_{B_{p,r}^s}.$$  

6. Let $m \in \mathbb{R}$ and $f$ be a $S^m$-multiplier (i.e. $f$ is a smooth function and satisfies that $\forall \alpha \in \mathbb{N}^d$, $\exists C = C(\alpha)$ such that $|\partial^\alpha f(\xi)| \leq C(1 + |\xi|)^{m-|\alpha|}$, $\forall \xi \in \mathbb{R}^d$). Then the operator $f(D) = F^{-1}(f \hat{f})$ is continuous from $B_{p,r}^s$ to $B_{p,r}^{s-m}$.

The 1-D Moser-type estimates are provided as follows.

**Lemma 2.2** (See [1]). For any $s > 0$ and any $1 \leq p, r \leq +\infty$, the space $L^\infty \cap B_{p,r}^s$ is an algebra, and a constant $C = C(s)$ exists such that

$$\|uv\|_{B_{p,r}^s} \leq C(\|u\|_{L^\infty}\|v\|_{B_{p,r}^s} + \|u\|_{B_{p,r}^s}\|v\|_{L^\infty}).$$  

We also introduce a space $B_{\infty,\infty,1}^s$ with the norm $\|f\|_{B_{\infty,\infty,1}^s} = \sup_j 2^j \|\Delta_j f\|_{L^\infty}.$

**Lemma 2.3** (See [18]). For any $f \in B_{\infty,1}(\mathbb{R}) \cap B_{\infty,\infty,1}^3(\mathbb{R})$, we have

$$\|f\|_{B_{\infty,\infty,1}^s} \leq C\|f\|_{B_{\infty,1}^s} \|f\|_{B_{\infty,\infty,1}^s}.$$  

**Lemma 2.4** (See [18]). Define $R_j = \Delta_j(f g_x) - f \Delta_j g_x$. Then we have

$$\sup_j \|R_j\|_{L^\infty} \leq C\|f\|_{B_{\infty,1}^s} \|g\|_{B_{\infty,\infty,1}^s}.$$  

$$\sum_j \|R_j\|_{L^\infty} \leq C\|f\|_{B_{\infty,1}^s} \|g\|_{B_{\infty,1}^s} \cap B_{\infty,\infty,1}^s.$$  

Here is the useful Gronwall lemma.
Lemma 2.5. Let \( q(t), a(t) \in C^1([0, T]), q(t), a(t) > 0 \). Let \( b(t) \) is a continuous function on \([0, T]\). Suppose that, for all \( t \in [0, T] \),
\[
\frac{1}{2} \frac{d}{dt} q^2(t) \leq a(t)q(t) + b(t)q^2(t).
\]
Then for any time \( t \in [0, T] \), we have
\[
q(t) \leq q(0) \exp \int_0^t b(\tau)d\tau + \int_0^t a(\tau) \exp \left( \int_\tau^t b(t')dt' \right) d\tau.
\]

Lemma 2.6 (See [1]). Let \( 1 \leq p \leq \infty, 1 \leq r \leq \infty \) and \( \theta > - \min(\frac{1}{p}, \frac{1}{p'}) \). Suppose \( f_0 \in B^{\theta}_{p,r} \), \( g \in L^1(0,T; B^\theta_{p,r}) \), and \( v \in L^p(0,T; B^{-\theta}_{\infty,\infty}) \) for some \( \rho > 1 \) and \( M > 0 \), and
\[
\begin{align*}
\partial_x v & \in L^1(0,T; B^{1}_{p,\infty} \cap L^\infty), & \text{if } \theta < 1 + \frac{1}{p}, \\
\partial_x v & \in L^1(0,T; B^{\theta}_{p,r}), & \text{if } \theta = 1 + \frac{1}{p}, \ r > 1, \\
\partial_x v & \in L^1(0,T; B^{\theta-1}_{p,r}), & \text{if } \theta > 1 + \frac{1}{p} \ (\text{or } \theta = 1 + \frac{1}{p}, \ r = 1).
\end{align*}
\]

Then the problem (2.1) has a unique solution \( f \) in
- the space \( C([0,T]; B^{\theta}_{p,r}) \), if \( r < \infty \),
- the space \( (\bigcap_{\rho < \theta} C([0,T]; B^\rho_{p,\infty})) \cap C_0([0,T]; B^{\theta}_{p,\infty}) \), if \( r = \infty \).

Lemma 2.7 (See [1, 22]). Let \( 1 \leq p, r \leq \infty \) and \( \theta > - \min(\frac{1}{p}, \frac{1}{p'}) \). There exists a constant \( C \) such that for all solutions \( f \in L^\infty(0,T; B^{\theta}_{p,r}) \) of (2.1) with initial data \( f_0 \) in \( B^{\theta}_{p,r} \) and \( g \) in \( L^1(0,T; B^{\theta}_{p,r}) \), we have, for a.e. \( t \in [0,T] \),
\[
\|f(t)\|_{B^\theta_{p,r}} \leq \|f_0\|_{B^\theta_{p,r}} + \int_0^t \|g(\tau)\|_{B^\theta_{p,r}} d\tau' + \int_0^t V'(t') \|f(t')\|_{B^\theta_{p,r}} d\tau'
\]
or
\[
\|f(t)\|_{B^\theta_{p,r}} \leq e^{CV(t)} \left( \|f_0\|_{B^\theta_{p,r}} + \int_0^t e^{-CV(t')} \|g(t')\|_{B^\theta_{p,r}} d\tau' \right)
\]
with
\[
V'(t) = \begin{cases} 
\|\partial_x v(t)\|_{B^{\rho}_{p,\infty} \cap L^\infty} & \text{if } \theta < 1 + \frac{1}{p}, \\
\|\partial_x v(t)\|_{B^{\theta}_{p,r}} & \text{if } \theta = 1 + \frac{1}{p}, \ r > 1, \\
\|\partial_x v(t)\|_{B^{\theta-1}_{p,r}} & \text{if } \theta > 1 + \frac{1}{p} \ (\text{or } \theta = 1 + \frac{1}{p}, \ r = 1).
\end{cases}
\]
If \( \theta > 0 \), then there exists a constant \( C = C(p,r,\theta) \) such that the following statement holds
\[
\|f(t)\|_{B^\theta_{p,r}} \leq \|f_0\|_{B^\theta_{p,r}} + \int_0^t \|g(\tau)\|_{B^\theta_{p,r}} d\tau' + C \int_0^t \left( \|f(\tau)\|_{B^\theta_{p,r}} \|\partial_x v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{B^{\theta-1}_{p,r}} \|\partial_x f(\tau)\|_{L^\infty} \right) d\tau.
\]
In particular, if \( f = av + b, a, b \in \mathbb{R} \), then for all \( \theta > 0 \), \( V'(t) = \|\partial_x v(t)\|_{L^\infty} \).
3 The ill-posedness

In this section, we present the local ill-posedness for the Cauchy problem of (1.3) in Besov space $B_{\infty,1}^1$.

We first present some crucial lemmas that we will use later.

**Lemma 3.1.** Let $u \in L^\infty([0,T];C^{0,1})$ solve (1.3) with initial data $u_0 \in C^{0,1}$. There exist a constant $C$ such that for all $t \in [0,T]$, we have

$$\|u_t\|_{L^\infty} + \|u\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^3 \leq \left(\|u_{0t}\|_{L^\infty} + \|u_0\|_{L^\infty} + \|u_0\|_{L^\infty}^2 + \|u_0\|_{L^\infty}^3\right)e^{C \int_0^t \|\partial_x u\|_{L^\infty} \, dt}.$$  

Thanks to Lemma (3.1), we can easily obtain the local existence and uniqueness of the solution $u$ to the R-CH equation with the initial data $u_0 \in C^{0,1}$ and a lifespan $T \approx \frac{\|u_0\|_{C^{0,1}}}{\|u_0\|_{C^{0,1}} + \|u_0\|_{C^{0,1}}^3} + \|u_0\|_{C^{0,1}}^2$ such that

$$\|u\|_{L^\infty(C^{0,1})} \leq C \left(\|u_0\|_{C^{0,1}} + \|u_0\|_{C^{0,1}}^2 + \|u_0\|_{C^{0,1}}^3\right).$$

**The proof of Theorem 1.2.** Choose

$$u_0(x) = -(1 - \partial_{xx})^{-1} \partial_x \left[\cos 2^{N+5}x \cdot (1 + N^{-\frac{1}{2}} S_N h(x))\right] N^{-\frac{1}{2}}$$

where $S_N f = \sum_{-1 \leq j < N} \Delta_j f$ and $h(x) = 1_{x \geq 0}(x)$. Let $u$ be a solution to the R-CH equation with the initial data $u_0$ defined as (3.1). Similar to [18], we can deduce

$$\|u_0\|_{B_{\infty,1}^1} \approx \|u_0\|_{L^\infty} \|u_{0x}\|_{B_{\infty,1}^0} \leq C N^{-\frac{1}{2}} N^{-\frac{1}{2}}, \quad \|u_0\|_{B_{\infty,1}^1} \leq CN^{-\frac{1}{2}}, \quad \|u_{0x}\|_{B_{\infty,1}^0} \geq CN^{-\frac{1}{2}}.$$  

Set

$$\frac{d}{dt} y(t, \xi) = u(t, y(t, \xi)), \quad y_0(\xi) = \xi.$$  

The R-CH equation has a solution $u(t, x)$ with the initial data $u_0$ in $C^{0,1}(\mathbb{R})$ such that

$$\|u(t)\|_{C^{0,1}} \leq C \left(\|u_0\|_{C^{0,1}} + \|u_0\|_{C^{0,1}}^2 + \|u_0\|_{C^{0,1}}^3\right) \leq CN^{-\frac{1}{2}}, \quad \forall t \in [0, T_0]$$

where $T_0 < T$, $\|u_0\|_{C^{0,1}} \leq C \|u_0\|_{B_{\infty,1}^1}$, and $C$ is a constant independent of $N$.

Therefore, according to (3.3) and (3.4), we can find a $T_1 > 0$ sufficiently small such that $\frac{1}{2} \leq y_N(t) \leq 2$ for any $t \in [0, \min\{T_0, T_1\}]$. Let $\bar{T} = \frac{3}{N^2} \leq \min\{T_0, T_1\}$ for $N > 10$ large enough. To prove the norm inflation, it suffices to prove there exists a time $t_0 \in \left(0, \frac{2}{N^2}\right)$ such that $\|u_{x}(t_0)\|_{B_{\infty,1}^0} \geq \ln N$ for $N > 10$ large enough. Let us assume the opposite. Namely, we suppose that

$$\sup_{t \in [0, \frac{2}{N^2}]} \|u_{x}(t)\|_{B_{\infty,1}^0} < \ln N.$$  

Applying $\Delta_j$ and the Lagrange coordinates to Eq. (1.4), and then integrating with respect to $t$, we get

$$(\Delta_j u \circ y = \Delta_j u_0 + \int_0^{y} (\Delta_j \bar{R}_j \circ y + (\Delta_j \bar{R}_j) \circ y + \Delta_j \bar{E} \circ y - \Delta_j \bar{E}_0) \, ds + t \Delta_j \bar{E}_0)$$

(3.6)
where
\[ R_j = \Delta_j(ux_x) - u\Delta_ju_x, \]
\[ R_L = -(1 - \partial_{xx})^{-1}\partial_x(c_1u^2 + c_2u^3 + c_3u^4), \]
\[ E(t, x) = -(1 - \partial_{xx})^{-1}\partial_x\left(\frac{u^2}{2}\right). \]

Let \( T = \frac{1}{N^2} < \hat{T} \) (Indeed we can extend \( T \) to \( \hat{T} \) by using the method of continuity).

(i) Following the similar proof of Lemma 2.100 in [1], we see
\[ \sum_j 2^j||K_1||_{L^\infty} \leq \sum_j 2^j||R_j||_{L^\infty} \leq ||u_x||_{L^\infty}||u||_{B^{1}_{\infty,1}} \leq ||u||_{C^{0,1}} \cdot \ln N \leq \frac{C\ln N}{N^{1/r}}. \]

(ii) According to the Bony’s decomposition, we find
\[ \sum_j 2^j||K_2||_{L^\infty} \leq \sum_j 2^j||\Delta_j R_L||_{L^\infty} \leq C(||u||_{L^\infty} + ||u||_{L^\infty}^2 + ||u||_{L^\infty}^3)||u||_{B^{1}_{\infty,1}} \leq \frac{C \ln N}{N^{1/r}}. \]

(iii) Now we estimate \( K_3 \). Noting that \( u(t, x) \in L_T^\infty(C^{0,1}(\mathbb{R})) \) is a solution to the R-CH equation, then we have
\[ \begin{cases} \frac{d}{dt}E + u\partial_x E = F(t, x), & t \in (0, T], \\ E(0, x) = E_0(x) = -(1 - \partial_{xx})^{-1}\partial_x\left(\frac{u^2}{2}\right) \end{cases} \]
where \( F(t, x) = \frac{c_1}{3}u^3 + \frac{c_2}{4}u^4 + \frac{c_3}{6}u^5 - u(1 - \partial_{xx})^{-1}\left(\frac{u^2}{2}\right) - (1 - \partial_{xx})^{-1}\left(\frac{c_1}{3}u^3 + \frac{c_2}{4}u^4 + \frac{c_3}{6}u^5 - \frac{1}{2}uu_x - \partial_x(u_x(1 - \partial_{xx})^{-1}\left(\frac{u^2}{2}\right) + c_1u^2 + c_2u^3 + c_3u^4)\right). \)

Since \( (1 - \partial_{xx})^{-1} \) is a \( S^{-2} \) operator in nonhomogeneous Besov spaces, one can easily get
\[ \|F(t)\|_{B^{1}_{\infty,1}} \leq C(||u(t)||_{C^{0,1}}^2 + ||u(t)||_{C^{0,1}}^3)\|u(t)\|_{B^{1}_{\infty,1}} \leq CN^{-1/4}\ln N, \quad \forall t \in (0, T]. \]

Applying \( \Delta_j \) and the Lagrange coordinates to (3.7) yields
\[ (\Delta_j E) \circ y - \Delta_j E_0 = \int_0^t \hat{R}_j \circ y + (\Delta_j G) \circ yds \]
where \( \hat{R}_j = u\partial_x \Delta_j E - \Delta_j(u\partial_x E) \). By Lemmas 2.3 2.4, we discover
\[ \sum_j 2^j\|\hat{R}_j \circ y\|_{L^\infty} = \sum_j 2^j\|\hat{R}_j\|_{L^\infty} \leq C\|u\|_{B^{1}_{\infty,1}}\|E\|_{B^{1}_{\infty,1}} \leq C\ln N\|u_x\|_{B^{0}_{\infty,\infty,1}}. \]

Thereby, we deduce that
\[ \sum_j 2^j\||\Delta_j E \circ y - \Delta_j E_0\|_{L^\infty} \leq C\int_0^t \sum_j 2^j\|\hat{R}_j \circ y\|_{L^\infty} + \sum_j 2^j\|\Delta_j F \circ y\|_{L^\infty} \leq CT \cdot (\ln N)^2 \cdot \|u_x\|_{L_T^\infty(B^{0}_{\infty,\infty,1})} + CT \cdot N^{-1/4} \cdot \ln N \]
Moreover, note that \( u_x \) solves
\[ u_{xt} + uu_{xx} = -\frac{1}{2}u_x^2 + c_1u^2 + c_2u^3 + c_3u^4 - (1 - \partial_{xx})^{-1}\left(\frac{u^2}{2} + c_1u^2 + c_2u^3 + c_3u^4\right) := -\frac{1}{2}u_x^2 + H(t, x). \]
Then, following the similar proof of Lemma 2.6 and Lemma 2.7, we deduce
\[
\|u|L^\infty_T(B^1_{\infty,1}) \leq \|u|L^\infty_T(B^0_{\infty,1} \cap B^1_{\infty,1}) \\
\leq\|u_0|B^0_{\infty,1} \cap B^1_{\infty,1} + C\int_0^T \|u^2|B^1_{\infty,1} + \|H(t)|B^0_{\infty,1} \cap B^0_{\infty,1} d\tau \\
\leq\|u_0|B^0_{\infty,1} \cap B^1_{\infty,1} + C\int_0^T \|u_x|B^0_{\infty,1} \|u|B^0_{\infty,1} + \|u^2, u^3, u^4|C^0,1 d\tau \\
\leq CN^{\frac{3}{8}} + CN^{-\frac{4}{9}} ln N\|u_x|L^\infty_T(B^0_{\infty,1}) + C \\
\leq CN^{3/8}.
\] (3.12)

Plugging (3.12) into (3.11), we discover
\[
\sum_j 2^j\|K_j|L^\infty \leq CN^{\frac{3}{8} - \frac{4}{9}} (ln N)^2 + CN^{-\frac{4}{9} - \frac{4}{9}} ln N.
\] (3.13)

Multiplying both sides of (3.6) by 2^j and performing the l^1 summation, by (i) – (ii) we gain for any t \in [0, T]
\[
\|u(t)|B^1_{\infty,1} = \sum_j 2^j\|\Delta_j u|L^\infty = \sum_j 2^j\|\Delta_j u \circ y|L^\infty \\
\geq t\|E_0|B^1_{\infty,1} - Ct(N^{-\frac{4}{9}} ln N - N^{\frac{4}{9} - \frac{4}{9}}(ln N)^2 - N^{-\frac{4}{9} - \frac{4}{9}} ln N) - \|u_0|B^1_{\infty,1} \\
\geq Ct\left(\frac{1}{4}N^{\frac{4}{9}} - N^{-\frac{4}{9}} ln N - N^{\frac{4}{9} - \frac{4}{9}}(ln N)^2 - N^{-\frac{4}{9} - \frac{4}{9}} ln N\right) - C \\
\geq \frac{1}{8}N^{\frac{4}{9}} - C.
\]
where the second inequality holds by use of (3.2). That is
\[
\|u(t)|B^1_{\infty,1} \geq \frac{1}{16}N^{\frac{4}{9} - \frac{4}{9}} - C, \quad \forall t \in \left[\frac{1}{2N^\frac{4}{9}}, \frac{1}{N^\frac{4}{9}}\right].
\]
Hence,
\[
\sup_{t \in [0, \frac{1}{N^\frac{4}{9}}]} \|u(t)|B^1_{\infty,1} \geq \frac{1}{16}N^{\frac{4}{9} - \frac{4}{9}} - C > ln N
\] (3.14)
which contradicts the hypothesis (3.5).

In conclusion, we obtain for N large enough
\[
\|u|L^\infty_T(B^1_{\infty,1}) \geq \|u_x|L^\infty_T(B^0_{\infty,1}) \geq ln N, \quad \bar{T} = \frac{2}{N^\frac{4}{9}},
\]
\[
\|u_0|B^1_{\infty,1} \leq N^{\frac{4}{9} - \frac{4}{9}},
\]
which follows that the norm inflation and hence the ill-posedness of the R-CH equation. This completes the proof of Theorem 1.2.

**Acknowledgements.** Y. Guo was supported by the GuangDong Basic and Applied Basic Research Foundation (No. 2020A1515111092) and Research Fund of Guangdong-Hong Kong-Macao Joint Laboratory for Intelligent Micro-Nano Optoelectronic Technology (No. 2020B1212030010). X. Tu was supported by National Natural Science Foundation of China (No. 11801076).
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