Analytic functions with conic domains associated with certain generalized $q$-integral operator

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Abstract

In this paper, we define a new subclass of $k$-uniformly starlike functions of order $\gamma$, $(0 \leq \gamma < 1)$ by using certain generalized $q$-integral operator. We explore geometric interpretation of the functions in this class by connecting it with conic domains. We also investigate $q$-sufficient coefficient condition, $q$-Fekete-Szegő inequalities, $q$-Bieberbach-De Branges type coefficient estimates and radius problem for functions in this class. We conclude this paper by introducing an analogous subclass of $k$-uniformly convex functions of order $\gamma$ by using the generalized $q$-integral operator. We omit the results for this new class because they can be directly translated from the corresponding results of our main class.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

that are analytic in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$. Denote by $\mathcal{P}$ the class of functions $p$ which are analytic and have positive real part in $\mathbb{D}$ with $p(0) = 1$. Let $\Omega$ be the family of Schwarz functions $w$ which are analytic in $\mathbb{D}$ satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. If $f$ and $g$ are analytic functions in $\mathbb{D}$, then we say that $f$ is subordinate to $g$, written as $f \prec g$, if there exists a Schwarz

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function \( w \in \Omega \) such that \( f(z) = g(w(z)) \). We also note that if \( g \) is univalent in \( \mathbb{D} \), then by subordination rules we get \( f(0) = g(0) \) and \( f(\mathbb{D}) \subset g(\mathbb{D}) \).

We denote the class \( S \) of all functions in \( \mathcal{A} \) that are univalent in \( \mathbb{D} \). Also, let \( ST \) and \( CV \) denote the subclasses of \( S \) that are starlike and convex, respectively. For definitions and properties of these classes, one may refer to the survey article by first author [2].

In 1991, Goodman [9] introduced the concept of uniform convexity and uniform starlike functions in \( S \). In fact, he defined such uniform classes, denoted by \( UCV \) and \( UST \), by their geometrical properties. A function \( f \) in \( \mathcal{A} \) is said to be uniformly convex (uniformly starlike) in \( \mathbb{D} \) if \( f \) is in \( CV \) (\( ST \)) and has the property that for every circular arc \( \gamma \) contained in \( \mathbb{D} \) with center \( \xi \) also in \( \mathbb{D} \), the arc \( f(\gamma) \) is convex (starlike) with respect to \( f(\xi) \).

In 1993, Ronning [24] proved the basis for further investigation of the classes \( UCV \) and \( UST \).

**Theorem A.** ([24]) If \( f \in \mathcal{A} \), then \( f \in UCV \) if and only if

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{D}).
\]

Applying the classic Alexander Theorem found by Alexander [1] in 1915, Ronning [24] obtained a characterization of the class \( UST \).

**Theorem B.** ([24]) If a function \( f \) belongs to \( \mathcal{A} \), then \( f \) belongs to \( UST \) if and only if

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{D}).
\]

For \( k \geq 0 \), Kanas et al. [18] introduced the class of \( k \)-uniformly starlike functions denoted by \( kUST \). Such a class consists of functions \( f \in \mathcal{A} \) that satisfy the inequality

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{D}).
\]

We note that \( 1UST \equiv UST \). In 1997, Bharati et al. [8] introduced the class of \( k \)-uniformly starlike functions of order \( \gamma \), \( (0 \leq \gamma < 1) \) for functions in the class \( kUST(\gamma) \).

**Definition 1.1.** ([8]) Let \( 0 \leq \gamma < 1 \) and \( k \geq 0 \). A function \( f \in \mathcal{A} \) is said to be in \( kUST(\gamma) \), called \( k \)-uniformly starlike functions of order \( \gamma \), if \( f \) satisfies the inequality

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \gamma \quad (z \in \mathbb{D}).
\]
We note that $k\text{-UST}(0) \equiv k\text{-UST}$. The class $1\text{-UST}(\gamma)$ was investigated in [6] and [25].

We next recall some concepts and notations of $q$-calculus that we need to define a new class which connects $k\text{-UST}(\gamma)$ and a generalized integral operator defined by $q$-calculus.

Quantum calculus (or $q$-calculus) is a theory of calculus where smoothness is not required. A systematic study of $q$-differentiation and $q$-integration was initiated by Jackson [12, 13]. The $q$-derivative (or $q$-difference) operator of a function $f$, defined on a subset of $\mathbb{C}$, is defined by

$$ (D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & \text{if } z \neq 0 \\ f'(0), & \text{if } z = 0, \end{cases} $$

where $q \in (0, 1)$. Note that $\lim_{q \to 1-} (D_q f)(z) = f'(z)$ if $f$ is differentiable at $z$. Under the hypothesis of the definition of $q$-derivative, we have the following rules:

$$ D_q(a f(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z) \quad (a, b \in \mathbb{C}), $$

$$ D_q(f(z)g(z)) = f(qz)D_q g(z) + g(z)D_q f(z). \quad (1.2) $$

It easily follows that if a function $f$ is given by (1.1), then

$$ (D_q f)(z) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}, \quad D_q(zD_q f(z)) = \sum_{n=1}^{\infty} [n]_q^2 a_n z^{n-1}, $$

where

$$ [n]_q = \frac{1 - q^n}{1 - q} $$

is called $q$-number or $q$-bracket of $n$. Clearly, $\lim_{q \to 1-} [n]_q = n$. In [13], Jackson defined $q$-integral of a function $f$ as follows:

$$ \int_0^x f(t)d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(xq^n), $$

provided the series converges. It is known that $q$-gamma function is given by

$$ \Gamma_q(x + 1) = [x]_q \Gamma_q(x), \quad \Gamma_q(x + 1) = [x]_q!, \quad (1.3) $$

where $q$-factorial $[x]_q!$ is given by

$$ [x]_q! = \begin{cases} [x]_q[x - 1]_q \cdots [2]_q[1]_q, & \text{if } x \geq 1 \\ 1, & \text{if } x = 0. \end{cases} $$
Note that in the limiting case when \( q \to 1^- \), \( \Gamma_q(x) \to \Gamma(x) \); see \[11\].

The \( q \)-beta function has the \( q \)-integral representation, which is a \( q \)-analogue of Euler’s formula (see \[13\]):

\[
B_q(t, s) = \int_0^1 x^{t-1}(1 - qx)^{s-1} d_q x, \quad (0 < q < 1; t, s > 0).
\]  

(1.4)

Jackson \[11\] also showed that the \( q \)-beta function defined by the formula

\[
B_q(t, s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t + s)},
\]  

(1.5)

tends to \( B(t, s) \) as \( q \to 1^- \). Also, the Gauss \( q \)-binomial coefficients are given by

\[
\left( \begin{array}{c} n \\ k \end{array} \right)_q := \frac{[n]_q!}{[k]_q![n-k]!}.
\]  

(1.6)

For more details, one may refer to \[10, 15\].

In recent years, quantum calculus approach has led to a great development in geometric function theory. Ahuja and Çetinkaya \[3\] recently wrote a survey on the use of quantum calculus approach in mathematical sciences and its role in geometric function theory. One may also refer to the recent paper by Ahuja \textit{et al.} \[4\].

Motivated by Jung \textit{et al.} \[14\], Mahmood \textit{et al.} \[21\] introduced the generalized \( q \)-integral operator \( \chi_{\alpha, \beta, q} f : \mathcal{A} \to \mathcal{A} \) defined by

\[
\chi_{\alpha, \beta, q} f(z) = \left( \begin{array}{c} \alpha + \beta \\ \beta \end{array} \right)_q \left( \frac{[\alpha]_q}{\beta} \right) \int_0^z \left( 1 - \frac{qt}{z} \right)^{\alpha-1} q^{-1} f(t) d_q t,
\]  

(1.7)

where \( \alpha > 0, \beta > -1, q \in (0, 1) \). Using (1.3), (1.4), (1.5), and (1.6), they observed that

\[
\chi_{\alpha, \beta, q} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} a_n z^n.
\]  

(1.8)

For special values of the parameters, the generalized integral operator (1.8) gives the following known integral operators as special cases:

(i) For \( q \to 1^- \), the operator \( \chi_{\beta, q}^\alpha f \) reduces to the integral operator \( \chi_{\beta}^\alpha f \) defined in \[14\] by

\[
\chi_{\beta}^\alpha f(z) = \left( \begin{array}{c} \alpha + \beta \\ \beta \end{array} \right) \frac{\alpha}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} (1 - t)^{\beta-1} f(t) dt = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta + n)}{\Gamma(\alpha + \beta + n)} \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + 1)} a_n z^n.
\]  

(1.9)
(ii) For $\alpha = 1$, the operator $\chi_{\beta,q}^{\alpha} f$ yields $q$-Bernardi integral operator $J_{\beta,q} f$ defined in [22] by

$$J_{\beta,q} f = \frac{[1 + \beta]_q}{z^\beta} \int_0^z t^{\beta-1} f(t) d_q t = \sum_{n=1}^{\infty} \frac{[1 + \beta]_q}{[n + \beta]_q} a_n z^n. \quad (1.10)$$

(iii) For $\alpha = 1, q \to 1^-$, the operator $\chi_{\beta,q}^{\alpha} f$ gives Bernardi integral operator $J_{\beta} f$ defined in [7] by

$$J_{\beta} f(z) = \frac{1 + \beta}{z^\beta} \int_0^z t^{\beta-1} f(t) dt = \sum_{n=1}^{\infty} \frac{1 + \beta}{n + \beta} a_n z^n. \quad (1.11)$$

(iv) For $\alpha = 1, \beta = 0, q \to 1^-$, the operator $\chi_{\beta,q}^{\alpha} f$ reduces to the Alexander integral operator $J_{0} f$ given in [26] by

$$J_{0} f(z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n. \quad (1.12)$$

Making use of $q$-integral operator, $\chi_{\beta,q}^{\alpha} f$ and $k$-uniformly starlike function of order $\gamma$ given in Definition 1.1, we define the the new class $k$-$J\!U\!S\!T(q; \alpha, \beta, \gamma)$.

**Definition 1.2.** Let $0 \leq \gamma < 1$, $q \in (0, 1)$, $k \geq 0$, $\alpha > 0, \beta > -1$. A function $f \in A$ is in the class $k$-$J\!U\!S\!T(q; \alpha, \beta, \gamma)$ if and only if

$$\text{Re} \left( \frac{z D_q(\chi_{\beta,q}^{\alpha} f(z))}{\chi_{\beta,q}^{\alpha} f(z)} \right) > k \left| \frac{z D_q(\chi_{\beta,q}^{\alpha} f(z))}{\chi_{\beta,q}^{\alpha} f(z)} - 1 \right| + \gamma \quad (z \in \mathbb{D}) \quad (1.13)$$

where $\chi_{\beta,q}^{\alpha} f(z)$ is given by (1.8).

In what follows, we shall first look at the geometric interpretation of (1.14). Since

$$\frac{z D_q(\chi_{\beta,q}^{\alpha} f(z))}{\chi_{\beta,q}^{\alpha} f(z)} = 1 + ([2]_q - 1) \psi_2 a_2 z + \left( [[3]_q - 1) \psi_3 a_3 - ([2]_q - 1) \psi_2^2 a_2^2 \right) z^2 + \cdots \quad (1.14)$$

where

$$\psi_n = \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} \quad (n \geq 2).$$
It follows that for \( z = 0 \), we have

\[
\frac{zD_q(\chi^\alpha_{\beta,q}f(z))}{\chi^\alpha_{\beta,q}f(z)} = 1.
\]

Thus \( p(z) = zD_q(\chi^\alpha_{\beta,q}f(z))/\chi^\alpha_{\beta,q}f(z) \), where \( p \) belongs to the class \( P \). Therefore (1.13) is equivalent to

\[
\text{Re } p(z) > k|p(z) - 1| + \gamma \quad (z \in \mathbb{D}).
\]

Thus, \( p(z) \) takes the values in the conic domain \( \Omega_{k,\gamma} \) defined by

\[
\Omega_{k,\gamma} = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2 + \gamma}, \ 0 \leq \gamma < 1, k \geq 0 \right\}.
\]

Note that \( 1 \in \Omega_{k,\gamma} \) and \( \partial \Omega_{k,\gamma} \) is a curve defined by

\[
\partial \Omega_{k,\gamma} = \left\{ u + iv : (u-\gamma)^2 = k^2(u-1)^2 + k^2v^2, \ 0 \leq \gamma < 1, k \geq 0 \right\}.
\]

Elementary computations show that \( \partial \Omega_{k,\gamma} \) represents a conic section symmetric about the real axis. Hence \( \Omega_{k,\gamma} \) is an elliptic domain for \( k > 1 \), parabolic domain for \( k = 1 \), hyperbolic domain for \( 0 < k < 1 \) and a right half plane \( u > \gamma \) for \( k = 0 \).

Denote by \( P(p_{k,\gamma}) \) the family of functions \( p \) such that \( p \in P \) and \( p \prec p_{k,\gamma} \) in \( \mathbb{D} \), where \( p_{k,\gamma} \) maps \( \mathbb{D} \) conformally onto the domain \( \Omega_{k,\gamma} \).

Kanas et al. [17, 18] found that the function \( p_{k,\gamma}(z) \) plays a role of extremal function of the class \( P(p_{k,\gamma}) \) and is given by

\[
p_{k,\gamma}(z) = \begin{cases} 
\frac{1+(1-2\gamma)z}{1-z}, & (k = 0) \\
\frac{1-\gamma}{1-k^2} \cos \left\{ \frac{2}{\pi} (\cos^{-1} k) i \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{k^2-\gamma}{1-k^2}, & (0 < k < 1) \\
1 + \frac{2(1-\gamma)}{\pi^2} \left( \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & (k = 1) \\
\frac{1-\gamma}{k^2-1} \sin \left\{ \frac{\pi}{2K(t)} \int_0^{u(z)/\sqrt{1-x^2}} \frac{dx}{\sqrt{1-x^2(1-t+x^2)}} \right\} + \frac{k^2-\gamma}{k^2-1}, & (k > 1),
\end{cases}
\]  

(1.15)

where \( u(z) = (z - \sqrt{t})/(1 - \sqrt{t})z, t \in (0, 1) \) and \( t \) is chosen such that \( k = \cosh \frac{\pi K'(t)}{4K(t)} \) and \( K(t) \) is Legendre’s complete elliptic integral of the first kind and \( K'(t) \) is complementary integral of \( K(t) \). Furthermore, since \( p_{k,\gamma}(\mathbb{D}) = \Omega_{k,\gamma} \) and \( p_{k,\gamma}(\mathbb{D}) \) is convex univalent in \( \mathbb{D} \) (see [17]), it follows that (1.13) is equivalent to

\[
\frac{zD_q(\chi^\alpha_{\beta,q}f(z))}{\chi^\alpha_{\beta,q}f(z)} < p_{k,\gamma}(z).
\]  

(1.16)
Remark 1.3. For special values of parameters $q$, $\alpha$, $\beta$, $\gamma$ and $k$, we get the following new classes as special cases of Definition 1.2; for example:

1. If $q \to 1^{-}$ and $\alpha = 1$, we get $k$-$\text{JUST}(\beta, \gamma) := \lim(k$-$\text{JUST}(q; 1, \beta, \gamma))$ with Bernardi operator (1.11)

2. If $q \to 1^{-}$, $\alpha = 1$, $\beta = 0$ and $k = 0$, we get $\text{JUST}(\gamma) := \lim(0$-$\text{JUST}(q; 1, 0, \gamma))$ with Alexander operator (1.12)

In this paper, we shall investigate the class $k$-$\text{JUST}(q; \alpha, \beta, \gamma)$. In particular, we obtain $q$-sufficient coefficient condition, $q$-Fekete-Szegö inequalities, $q$-Bieberbach-De Branges type coefficient estimates and solve radius problem for the functions in this class. In the concluding section, we introduce another new class $k$-$\text{JUSTY}(q; \alpha, \beta, \gamma)$ and omit the results for this class, because analogous results can be directly translated from the corresponding results found in Section 2 for the class $k$-$\text{JUST}(q; \alpha, \beta, \gamma)$.

Unless otherwise stated, we assume in the reminder of the article that $0 \leq \gamma < 1$, $q \in (0, 1)$, $k \geq 0$, $\alpha > 0$, $\beta > -1$ and $z \in \mathbb{D}$.

2 Main Results

We first obtain $q$-sufficient coefficient condition for the functions belonging to the class $k$-$\text{JUST}(q; \alpha, \beta, \gamma)$.

Theorem 2.1. If a function $f$ defined by (1.1) satisfies the inequality

$$
\sum_{n=2}^{\infty} ([n]_q(k + 1) - (k + \gamma)) \frac{\Gamma_q(\alpha + \beta + n)}{\Gamma_q(\alpha + \beta + n)} |a_n| \leq 1 - \gamma,
$$

then $f$ belongs to $k$-$\text{JUST}(q; \alpha, \beta, \gamma)$. The result is sharp.

Proof. To show that $f \in k$-$\text{JUST}(q; \alpha, \beta, \gamma)$, it suffices to prove that

$$
k |zD_q(\chi_{\alpha, \beta}^q f(z))| \chi_{\beta}^q f(z) - 1 | - \Re \left( \frac{zD_q(\chi_{\alpha, \beta}^q f(z))}{\chi_{\beta}^q f(z)} - 1 \right) \leq 1 - \gamma.
$$

We note that

$$
|\frac{zD_q(\chi_{\alpha, \beta}^q f(z))}{\chi_{\beta}^q f(z)} - 1| = \left| \frac{zD_q(\chi_{\alpha, \beta}^q f(z)) - \chi_{\beta}^q f(z)}{\chi_{\beta}^q f(z)} \right|
$$
\[
\begin{align*}
\frac{\sum_{n=2}^{\infty} ([n]q - 1) \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} a_n z^n}{z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} a_n z^n}
\leq \frac{\sum_{n=1}^{\infty} ([n]q - 1) \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} \left| a_n \right|}{1 - \sum_{n=1}^{\infty} \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} \left| a_n \right|}.
\end{align*}
\]

(2.2)

In view of (2.1), it follows that
\[
1 - \sum_{n=1}^{\infty} \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} \left| a_n \right| > 0.
\]

Using (2.2), we have
\[
\begin{align*}
k \left| \frac{zDq(\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} - 1 \right| &\leq k \left| \frac{zDq(\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} - 1 \right| \quad \text{Re} \left( \frac{zDq(\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} - 1 \right) \\
&\leq (k + 1) \left| \frac{zDq(\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} - 1 \right| + \left| \frac{zDq(\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} - 1 \right| \\
&\leq (k + 1) \left\{ \frac{\sum_{n=2}^{\infty} ([n]q - 1) \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} \left| a_n \right|}{1 - \sum_{n=1}^{\infty} \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)} \left| a_n \right|} \right\} \\
&\leq 1 - \gamma
\end{align*}
\]

which proves (2.1).

For sharpness, consider the function \( f_n : \mathbb{D} \rightarrow \mathbb{C} \) defined by
\[
f_n(z) = z - \frac{(1 - \gamma) \Gamma_q(\alpha + \beta + n)}{([n]q(k + 1) - (k + \gamma)) \Gamma_q(\beta + n)} \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\alpha + \beta + 1)} z^n.
\]

Since
\[
\text{Re} \left( \frac{zDq(\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} \right) = \text{Re} \left( \frac{[n]q(k + 1) - (k + \gamma) - (1 - \gamma)[n]q z^{n-1}}{[n]q(k + 1) - (k + \gamma) - (1 - \gamma)z^{n-1}} \right) > \frac{k + \gamma}{k + 1}
\]
and
\[
k \left| \frac{zDq(\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} - 1 \right| = k \left| \frac{(1 - \gamma)(1 - [n]q) z^{n-1}}{[n]q(k + 1) - (k + \gamma) - (1 - \gamma)z^{n-1}} \right| < \frac{k(1 - \gamma)}{k + 1},
\]

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it follows that \( f_n \in k\text{-JUST}(q; \alpha, \beta, \gamma) \). Also, it is easy to shows that the equality holds in (2.1) for the function \( f_n \). Thus the result is sharp.

**Corollary 2.2.** If \( f(z) = z + a_n z^n \) and
\[
|a_n| \leq \frac{(1 - \gamma) \Gamma_q(\alpha + \beta + n)}{([n]_q(k + 1) - (k + \gamma)) \Gamma_q(\beta + n) \Gamma_q(\alpha + \beta + 1)}, \quad (n \geq 2)
\]
then \( f \in k\text{-JUST}(q; \alpha, \beta, \gamma) \).

Using Remark 1.3.1 and Remark 1.3.2, Theorem 2.1 gives the following new results.

**Corollary 2.3.** If a function \( f \) defined by (1.1) is in the class \( k\text{-JUST}(\beta, \gamma) \), then
\[
\sum_{n=2}^{\infty} \left( n(k+1)-(k+\gamma) \right) \frac{1+\beta}{n+\beta} |a_n| \leq 1 - \gamma.
\]

**Corollary 2.4.** If a function \( f \) defined by (1.1) is in the class \( \text{JUST}(\gamma) \), then
\[
\sum_{n=2}^{\infty} \left( n-\gamma \right) \frac{1}{n} |a_n| \leq 1 - \gamma.
\]

In order to determine \( q \)-Fekete-Szegö inequalities for the functions in the class \( k\text{-JUST}(q; \alpha, \beta, \gamma) \), we need next three lemmas.

**Lemma 2.5.** (2.17) Let \( k \geq 0 \) be fixed and \( p_{k, \gamma} \) defined by (1.15). If
\[
p_{k, \gamma}(z) = 1 + P_1 z + P_2 z^2 + \ldots,
\]
then
\[
P_1(z) = \begin{cases} 
\frac{8(1-\gamma)(\arccos k)^2}{\pi^2(1-k^2)}, & 0 \leq k < 1 \\
\frac{8(1-\gamma)}{\pi^2}, & k = 1 \\
\frac{\pi^2(2-\gamma)}{4\sqrt{\pi(1+t)K^2(t)(k^2-1)}}, & k > 1,
\end{cases} \quad (2.3)
\]

and
\[
P_2(z) = \begin{cases} 
\frac{(A^2+2)}{3} P_1, & 0 \leq k < 1 \\
\frac{3}{2} P_1, & k = 1 \\
\frac{4K^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{K^2(t)(1+t)}} P_1, & k > 1,
\end{cases} \quad (2.4)
\]

where \( A = \frac{2}{\pi} (\cos^{-1} k) \) and \( t \in (0,1) \) are chosen such that \( k = \cosh(\pi K'(t)/4K(t)) \) and \( K(t) \) is Legendre’s complete elliptic integral of the first kind and \( K'(t) \) is complementary integral of \( K(t) \).
Lemma 2.6. ([19] Lemma 3, p.254) If \( p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \) is in class \( \mathcal{P} \) and \( \eta \) is a complex number, then
\[
|c_2 - \eta c_1^2| \leq 2 \max\{1, |2\eta - 1|\}.
\]
The result is sharp for the functions \( p(z) = (1 + z^2)/(1 - z^2) \) and \( p(z) = (1 + z)/(1 - z) \).

Lemma 2.7. ([20] Lemma 1, p.162) If \( p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \) is in class \( \mathcal{P} \) and \( \eta \) is a real number, then
\[
|c_2 - \eta c_1^2| \leq \begin{cases} 
-4\eta + 2 & \text{if } \eta \leq 0, \\
2 & \text{if } 0 \leq \eta \leq 1, \\
4\eta - 2 & \text{if } \eta \geq 1.
\end{cases}
\]
When \( \eta < 0 \) and \( \eta > 1 \), equality holds if and only if \( p(z) = (1 + z)/(1 - z) \) or one of its rotations. If \( 0 < \eta < 1 \), then equality holds if and only if \( p(z) = (1 + z^2)/(1 - z^2) \) or one of its rotations. If \( \eta = 0 \), equality holds if and only if
\[
p(z) = \frac{1 + \lambda}{2} \left( \frac{1 + z}{1 - z} \right) + \frac{1 - \lambda}{2} \left( \frac{1 - z}{1 + z} \right), \quad 0 \leq \lambda \leq 1
\]
or one of its rotations. If \( \eta = 1 \), equality holds if and only if \( p(z) \) is the reciprocal of one of the functions such that the equality holds in the case \( \eta = 0 \).

Theorem 2.8. Let \( k \geq 0 \) and \( f \in k-\text{JUST}(q; \alpha, \beta, \gamma) \), where \( f \) is of the form \( (11) \). Then, for a complex number \( \eta \), \( q \)-Fekete-Szegő inequality is given by
\[
|a_3 - \eta a_2^2| \leq \frac{P_1}{2([3]_q - 1)\psi_3} \max\{1, |2\nu - 1|\},
\]
where
\[
\nu = 1 - \frac{P_2}{2P_1} - \frac{P_1}{2([2]_q - 1)} + \eta \frac{P_1([3]_q - 1)\psi_3}{2([2]_q - 1)\psi_2^2},
\]
and where \( P_1 \) and \( P_2 \) are given by Lemma 2.5.

Proof. If \( f \in k-\text{JUST}(q; \alpha, \beta, \gamma) \), then there is a Schwarz function \( w \), analytic in \( \mathbb{D} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that
\[
\frac{zD_q(\lambda_{\beta,q}^\alpha f(z))}{\lambda_{\beta,q}^\alpha f(z)} = p_{k,\gamma}(w(z)).
\]
Define the function \( p \) by
\[
p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots \quad (z \in \mathbb{D}).
\]

Since \( p \in \mathcal{P} \) is a function with \( p(0) = 1 \) and \( \text{Re}(p(z)) > 0 \), we get
\[
p_{k,\gamma}(w(z)) = p_{k,\gamma}(p(z) - 1) + 1
\]
\[
= p_{k,\gamma}\left(\frac{c_1}{2} z + \frac{1}{2} \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \frac{1}{2} \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \ldots\right)
\]
\[
= 1 + \frac{P_1 c_1}{2} z + \left(\frac{P_1}{2} \left(c_2 - \frac{c_1^2}{2}\right) + \frac{P_2 c_1^2}{4}\right) z^2 + \ldots \quad (2.6)
\]

Comparing the coefficients of (2.6) and (1.14), we get
\[
a_2 = \frac{P_1 c_1}{2([2]_q - 1) \psi_2}
\]
and
\[
a_3 = \frac{P_1}{2([3]_q - 1) \psi_3} \left\{ c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2} \left(\frac{P_2}{P_1} + \frac{P_1}{[2]_q - 1}\right) \right\}.
\]

For any complex number \( \eta \), we have
\[
a_3 - \eta a_2^2 = \frac{P_1}{2([3]_q - 1) \psi_3} \left\{ c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2} \left(\frac{P_2}{P_1} + \frac{P_1}{[2]_q - 1}\right) \right\} - \eta \frac{P_1^2 c_1^2}{4([2]_q - 1)^2 \psi_2^2}.
\]

Equation (2.7) can be written as:
\[
a_3 - \eta a_2^2 = \frac{P_1}{2([3]_q - 1) \psi_3} \left\{ c_2 - \nu c_1^2 \right\}, \quad (2.8)
\]
where \( \nu \) is defined by (2.5). Applying Lemma 2.6, proof is completed. The result is sharp for a function \( f \) given by
\[
\frac{z D_q (\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} = p_{k,\gamma}(z) \quad \text{or} \quad \frac{z D_q (\chi_{\beta,q}^\alpha f(z))}{\chi_{\beta,q}^\alpha f(z)} = p_{k,\gamma}(z^2).
\]

In view of Remark 1.3.1 and Remark 1.3.2, we get the following new results as special cases of Theorem 2.8.
Corollary 2.9. If a function $f$ defined by (1.1) is in the class $\text{JUST}(\beta, \gamma)$, then

$$|a_3 - \eta a_2^2| \leq \frac{P_1(3 + \beta)}{4(1 + \beta)} \max \left\{ 1, \left| -\frac{P_2}{P_1} - P_1 + \eta \frac{2P_1(2 + \beta)^2}{(3 + \beta)(1 + \beta)} \right| \right\}.$$ 

Corollary 2.10. If a function $f$ defined by (1.1) is in the class $\text{JUST}(\gamma)$, then

$$|a_3 - \eta a_2^2| \leq \frac{3P_1}{4} \max \left\{ 1, \left| -\frac{P_2}{P_1} - P_1 + \eta \frac{8P_1}{3} \right| \right\}.$$ 

Theorem 2.11. Let $k \geq 0$ and $f \in \text{JUST}(q; \alpha, \beta, \gamma)$ where $f$ is of the form (1.1). Then, for a real number $\eta$, we have

$$|a_3 - \eta a_2^2| \leq \frac{1}{([3]_q - 1)\psi_3} \times \begin{cases} P_2 + \frac{P_2^2}{[2]_q - 1} - \eta \frac{P_2([3]_q - 1)\psi_3}{([2]_q - 1)^2\psi_2^2}, & \text{if } \eta \leq \sigma_1 \\ P_1, & \text{if } \sigma_1 \leq \eta \leq \sigma_2 \\ -P_2 - \frac{P_2^2}{[2]_q - 1} + \eta \frac{P_2([3]_q - 1)\psi_3}{([2]_q - 1)^2\psi_2^2}, & \text{if } \eta \geq \sigma_2 \end{cases},$$

where

$$\psi_n = \frac{\Gamma_q(\beta + n)}{\Gamma_q(\alpha + \beta + n)} \frac{\Gamma_q(\alpha + \beta + 1)}{\Gamma_q(\beta + 1)}, \quad (n = 2, 3),$$

$$\sigma_1 = \frac{([2]_q - 1)^2\psi_2^2}{([3]_q - 1)\psi_3} \left( \frac{P_2}{P_1^2} + \frac{1}{[2]_q - 1} - \frac{1}{P_1} \right),$$

$$\sigma_2 = \frac{([2]_q - 1)^2\psi_2^2}{([3]_q - 1)\psi_3} \left( \frac{P_2}{P_1^2} + \frac{1}{[2]_q - 1} + \frac{1}{P_1} \right),$$

and $P_1$ and $P_2$ are given by Lemma 2.5.

Proof. Using (2.5), (2.8) and Lemma 2.7 we get the proof. The bounds are sharp as can be seen by defining the following functions for $n \geq 2$ and $0 \leq \lambda \leq 1$.

$$\frac{zD_q(\chi_{\beta,q}^n F_n(z))}{\chi_{\beta,q}^n F_n(z)} = p_{k,\gamma}(z^{n-1}), \quad F_n(0) = F_n'(0) = 1 = 0,$$

$$\frac{zD_q(\chi_{\beta,q}^n G\lambda(z))}{\chi_{\beta,q}^n G\lambda(z)} = p_{k,\gamma} \left( \frac{z(z + \lambda)}{1 + \lambda z} \right), \quad G\lambda(0) = G'\lambda(0) = 1 = 0,$$

$$\frac{zD_q(\chi_{\beta,q}^n H\lambda(z))}{\chi_{\beta,q}^n H\lambda(z)} = p_{k,\gamma} \left( \frac{z(z + \lambda)}{1 + \lambda z} \right), \quad H\lambda(0) = H'\lambda(0) = 1 = 0.$$
When $\eta < \psi_1$ or $\eta > \psi_2$, equality holds if and only if $f$ is $F_2$ or one of its rotations. When $\psi_1 < \eta < \psi_2$, equality holds if and only if $f$ is $F_3$ or one of its rotations. If $\eta = \psi_1$, equality holds if and only if $f$ is $G_\lambda$ or one of its rotations and if $\eta = \psi_2$, equality holds if and only if $f$ is $H_\lambda$ or one of its rotations.

For investigating $q$-Bieberbach-De Branges inequalities, we need the following result called Rogogonki's Theorem.

**Lemma 2.12.** ([23, Theorem 2.3, p.70]) Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be subordinate to $p_{k,\gamma}(z) = 1 + \sum_{n=1}^{\infty} P_n z^n$ in $D$. If $p_{k,\gamma}$ is univalent in $D$ and $p_{k,\gamma}(D)$ is convex, then

$$|c_n| \leq P_1, \quad (n \geq 1).$$

**Theorem 2.13.** If a function $f$ of the form (1.1) belongs to the class $k$-J UST $(q; \alpha, \beta, \gamma)$, then

$$|a_2| \leq \frac{P_1}{q\psi_2} \quad \text{and} \quad |a_3| \leq \frac{qP_2 + P_1^2}{q^2(1 + q)\psi_3}.$$  

These results are sharp for the function given by (2.13).

**Proof.** Let $p(z) = \frac{zD_q(x^{\alpha}\beta f(z))}{x^{\alpha}\beta f(z)}$. Using the relation (1.8) for $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, we have

$$(\lbrack n \rbrack_q - 1)\psi_n a_n = \sum_{k=1}^{n-1} \psi_k a_k c_{n-k}, \quad a_1 = 1.$$  

Comparing the coefficients for $n = 2$ and $n = 3$, we get

$$a_2 = \frac{c_1}{([2]_q - 1)\psi_2} \quad \text{and} \quad a_3 = \frac{c_2 + c_1 \psi_2 a_2}{([3]_q - 1)\psi_3}. \tag{2.9}$$

It is obvious that

$$|a_2| = \frac{|c_1|}{([2]_q - 1)\psi_2} \leq \frac{P_1}{q\psi_2}, \quad \text{(2.10)}$$

where $|c_1| \leq P_1$.

Now, Lemma 2.12 (2.9) together with inequality $|c_1^2| + |c_2| \leq |P_1|^2 + |P_2|$ (see [16]) yield

$$|a_3| = \frac{q|c_2| + c_2^2}{q^2(1 + q)\psi_3} \leq \frac{q(|c_2| + |c_1|^2)(1 - q)|c_1^2|}{q^2(1 + q)\psi_3} \leq \frac{q|P_2| + |P_1|^2 + (1 - q)|P_1^2|}{q^2(1 + q)\psi_3} \leq \frac{qP_2 + P_1^2}{q^2(1 + q)\psi_3}. \tag{2.9}$$
In our next result, we state and prove a $q$-Bieberbach-De Branges inequality.

**Theorem 2.14.** If $f \in k\text{-JUST}(q; \alpha, \beta, \gamma)$, then

$$|a_n| \leq \frac{P_1}{([n]_q - 1)\psi_n} \prod_{k=1}^{n-2} \left(1 + \frac{P_1}{([k+1]_q - 1)}\right), \quad (n \geq 3) \quad (2.11)$$

where $P_1$ is given by (2.3) and

$$\psi_n = \frac{\Gamma_q(\beta+n)}{\Gamma_q(n+\alpha+\beta+1)} \frac{\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\beta+1)}, \quad (n \geq 3). \quad (2.12)$$

**Proof.** In view of Definition 1.2, we can write

$$zD_q(\chi_{q,\alpha,\beta,\gamma}f(z)) = p(z) \prec p_{k,\gamma}$$

where $p \in \mathcal{P}$ is analytic in $\mathbb{D}$. Since $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $\chi_{q,\alpha,\beta,\gamma}f$ given by (1.8), we have

$$z + \sum_{n=2}^{\infty} [n]_q \psi_n a_n z^n = \left(z + \sum_{n=2}^{\infty} \psi_n a_n z^n\right) \left(1 + \sum_{n=1}^{\infty} c_n z^n\right),$$

where $\psi_n$ are given by (2.12).

Comparing the coefficients of $z^n$ on both sides, we observe

$$[n]_q \psi_n a_n = a_n + c_1 \psi_{n-1} a_{n-1} + c_2 \psi_{n-2} a_{n-2} + ... + c_{n-2} \psi_2 a_2 + c_{n-1}$$

for all integer $n \geq 3$. Taking absolute value on both sides and applying Lemma 2.12, we have

$$|a_n| \leq \frac{P_1}{([n]_q - 1)\psi_n} \{1 + \psi_2 |a_2| + ... + \psi_{n-2} |a_{n-2}| + \psi_{n-1} |a_{n-1}|\}.$$

We will prove (2.11) by using mathematical induction. For $n = 2$, the result follows by (2.10). Let us assume that (2.11) is true for $n \leq m$, that is

$$|a_m| \leq \frac{P_1}{([m]_q - 1)\psi_m} \{1 + \psi_2 |a_2| + ... + \psi_{m-1} |a_{m-1}|\} \leq \frac{P_1}{([m]_q - 1)\psi_m} \prod_{k=1}^{m-2} \left(1 + \frac{P_1}{([k+1]_q - 1)}\right),$$

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Consider
\[
|a_{m+1}| \leq \frac{P_1}{([m+1]q-1)\psi_{m+1}} \left\{ 1 + \psi_2|a_2| + \ldots + \psi_m|a_m| \right\}
\]
\[
\leq \frac{P_1}{([m+1]q-1)\psi_{m+1}} \left\{ 1 + \frac{P_1}{([2]q-1)} + \frac{P_1}{([3]q-1)} \left( 1 + \frac{P_1}{([2]q-1)} \right) + \ldots + \frac{P_1}{([m]q-1)} \prod_{k=1}^{m-2} \left( 1 + \frac{P_1}{([k+1]q-1)} \right) \right\}
\]
\[
= \frac{P_1}{([m+1]q-1)\psi_{m+1}} \prod_{k=1}^{m-1} \left( 1 + \frac{P_1}{([k+1]q-1)} \right).
\]

Thus (2.11) is true for \( n = m + 1 \). Consequently, mathematical induction shows that (2.11) holds for \( n, n \geq 2 \). This completes the proof. The result is sharp for a function \( f \) given by
\[
z D_q(\chi_{\alpha,\beta,q}^\alpha f(z)) = p_{k,\gamma}(z).
\]

(2.13)

For different values of the parameters, Theorem 2.14 gives several new results. In particular in view of Remark 1.3.1 and Remark 1.3.2. Theorem 2.14 gives the following results.

**Corollary 2.15.** If a function \( f \) defined by (1.1) is in the class \( k-JUST(\beta,\gamma) \), then
\[
|a_n| \leq \frac{P_1(\beta + n)}{(n-1)(\beta + 1)} \prod_{k=1}^{n-2} \left( 1 + \frac{P_1}{k} \right), \quad (n \geq 3)
\]

**Corollary 2.16.** If a function \( f \) defined by (1.1) is in the class \( JUST(\gamma) \), then
\[
|a_n| \leq \frac{nP_1}{(n-1)} \prod_{k=1}^{n-2} \left( 1 + \frac{P_1}{k} \right), \quad (n \geq 3)
\]

We now conclude this section by exploring \( q \)-radius for the functions in the class \( k-JUST(q;\alpha,\beta,\gamma) \).

**Theorem 2.17.** If \( f \in k-JUST(q;\alpha,\beta,\gamma) \), then \( f(\mathbb{D}) \) contains an open disc of radius
\[
r = \frac{q\psi_2}{2q\psi_2 + P_1},
\]
where \( P_1 \) is given by (2.3).
Proof. Let \( w_0 \neq 0 \) be a complex number such that \( f(z) \neq w_0 \) for \( z \in \mathbb{D} \). Then
\[
f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left( a_2 + \frac{1}{w_0} \right) z^2 + \ldots.
\]
Since \( f_1 \) is univalent in \( \mathbb{D} \), it follows that
\[
\left| a_2 + \frac{1}{w_0} \right| \leq 2.
\]
By using (2.10), we get
\[
\left| \frac{1}{w_0} \right| \leq 2 + \frac{P_1}{q \psi_2} = \frac{2q \psi_2 + P_1}{q \psi_2}.
\]
Consequently, we obtain
\[
|w_0| \geq \frac{q \psi_2}{2q \psi_2 + P_1}.
\]

3 Concluding Remarks

Using the well-known formula (1.2) and replacing \( \chi_{\beta,q}^\alpha f(z) \) in (1.13) by \( z D_q(\chi_{\beta,q}^\alpha f(z)) \), we obtain a new subclass \( k-\mathcal{JUCV}(q; \alpha, \beta, \gamma) \) of \( k \)-uniformly convex functions of order \( \gamma \) associated with the generalized \( q \)-integral operator given by (1.7).

Definition 3.1. Let \( 0 \leq \gamma < 1 \), \( q \in (0,1) \), \( k \geq 0 \), \( \alpha > 0 \), \( \beta > -1 \). A function \( f \in A \) is in the class \( k-\mathcal{JUCV}(q; \alpha, \beta, \gamma) \) if and only if
\[
\text{Re} \left( 1 + q \frac{z D_q^2(\chi_{\beta,q}^\alpha f(z))}{D_q(\chi_{\beta,q}^\alpha f(z))} \right) > k \left| q \frac{z D_q^2(\chi_{\beta,q}^\alpha f(z))}{D_q(\chi_{\beta,q}^\alpha f(z))} \right| + \gamma \quad (z \in \mathbb{D})
\]
where \( \chi_{\beta,q}^\alpha f(z) \) is given by (1.7) and (1.8).

Alexander-type relationship between functions of these classes is
\[
\chi_{\beta,q}^\alpha f(z) \in k-\mathcal{JUCV}(q; \alpha, \beta, \gamma) \Leftrightarrow z D_q(\chi_{\beta,q}^\alpha f(z)) \in k-\mathcal{JUST}(q; \alpha, \beta, \gamma);
\]
that is,
\[
\chi_{\beta,q}^\alpha f(z) \in k-\mathcal{JUST}(q; \alpha, \beta, \gamma) \Leftrightarrow \int_0^z \frac{\chi_{\beta,q}^\alpha f(z)}{t} dt \in k-\mathcal{JUCV}(q; \alpha, \beta, \gamma).
\]
In view of the classical Alexander Theorem and the results for the class \( k-\mathcal{JUST}(q; \alpha, \beta, \gamma) \), it is easy to obtain the corresponding properties for the class \( k-\mathcal{JUCV}(q; \alpha, \beta, \gamma) \).

Therefore, we omit the statements and proofs of the corresponding results of the class \( k-\mathcal{JUCV}(q; \alpha, \beta, \gamma) \).
References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. Math.* **17** (1915/1916), 12-22.

[2] O. P. Ahuja, The Bieberbach conjecture and its impact on the developments in Geometric Function Theory, *Math. Chronicle*, **15** (1986), 1-28.

[3] O. P. Ahuja and A. Çetinkaya, Use of Quantum Calculus approach in Mathematical Sciences and its role in geometric function theory. *AIP Conf. Proc.* **2019**, 2095, 020001-1–020001-14.

[4] O. P. Ahuja, S. Anand and N. K. Jain, Bohr Radius Problems for Some Classes of Analytic Functions Using Quantum Calculus Approach, *Mathematics*, **8**, (2020), 623.

[5] F. M. Al-Oboudi and K. A. Al-Amoudi, On classes of analytic functions related to conic domains, *J. Math. Anal. Appl.* **339** (2008), no. 1, 655–667.

[6] R. M. Ali and V. Singh, Coefficient of parabolic starlike functions of order alpha, In: Comput.Methods Funct. Theory Ser. Approx. Decompos., 1994 (Penang), Vol. 5, World Scientific Publishing, Singapore, (1995), 23–26.

[7] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **135** (1969), 429–446.

[8] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, *Tamkang J. Math.* **28** (1997), no. 1, 17–32.

[9] A. W. Goodman, On uniformly convex functions, *Ann. Polon. Math.* **56** (1991), no. 1, 87–92.

[10] G. Gasper and M. Rahman, *Basic hypergeometric series*, second edition, Encyclopedia of Mathematics and its Applications, 96, Cambridge University Press, Cambridge, 2004.

[11] F. H. Jackson, *A generalization of the functions* $\Gamma(n)$ *and* $x^n$, *Proc. Royal Soc. London*, **74** (1904), 64-72.

[12] F. H. Jackson, On $q$-functions and a certain difference operator, *Trans. Royal Soc. Edinburgh*, **46** (1908), no. 2, 253-281.
[13] F. H. Jackson, *On q-definite integrals*, Quart. J. Pure Appl. Math. **41** (1910), 193-203.

[14] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl. **176** (1993), no. 1, 138–147.

[15] V. Kac and P. Cheung, *Quantum calculus*, Universitext, Springer-Verlag, New York, 2002.

[16] S. Kanas and A. Wiśniowska, Conic regions and $k$-uniform convexity. II, Zeszyty Nauk. Politech. Rzeszowskiej Mat. No. 22 (1998), 65–78.

[17] S. Kanas and A. Wiśniowska, Conic regions and $k$-uniform convexity, J. Comput. Appl. Math. **105** (1999), 327-336.

[18] S. Kanas and A. Wiśniowska, Conic domains and starlike functions, Rev. Roumaine Math. Pures Appl. **45** (2000), no. 4, 647–657.

[19] R. J. Libera and E. J. Złotkiewicz, Coefficient bounds for the inverse of a function with derivative in $P$, Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257.

[20] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA. 1994.

[21] S. Mahmood, N. Raza, E. S. A. Abujarad, G. Srivastava, H. M. Srivastava, S. N. Malik, Geometric properties of certain classes of analytic functions associated with $q$-integral operators, Symmetry **11** (2019), 719.

[22] K. I. Noor, S. Riaz and M. A. Noor, On $q$-Bernardi integral operator, TWMS J. Pure Appl. Math. **8** (2017), no. 1, 3–11.

[23] W. Rogosinski, On the coefficients of subordinate functions, Proc. London Math. Soc. (2) **48** (1943), 48–82.

[24] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. **118** (1993), no. 1, 189–196.

[25] F. Rønning, On starlike functions associated with parabolic regions, Ann. Univ.Mariae Curie-Sklodowska Sect. A 45 (1991), 117–122.
[26] Z. Shareef, S. Hussain and M. Darus, Convolution operators in the geometric function theory, J. Inequal. Appl. 2012, 2012:213.

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