A numerical method of computing Hadamard finite-part integrals with an integral power singularity at the endpoint on a half infinite interval

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Abstract

In this paper, we propose a numerical method of computing Hadamard finite-part integrals with an integral power singularity at the endpoint on a half infinite interval, that is, a finite value assigned to a divergent integral with an integral power singularity at the endpoint on a half infinite interval. In the proposed method, we express a desired finite-part integral using a complex integral, and we obtain the integral by evaluating the complex integral by the DE formula. Theoretical error estimate and some numerical examples show the effectiveness of the proposed method.

1 Introduction

The integral

$$\int_0^\infty x^{-1} f(x) dx,$$

where $f(x)$ is an analytic function on the half infinite interval $[0, +\infty)$ such that $f(0) \neq 0$ and $f(x) = O(x^{-\alpha})$ with $\alpha > 0$, is divergent. However, we can assign a finite value to this divergent integral. In fact, we consider the integral

$$\int_\epsilon^\infty x^{-1} f(x) dx$$

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with $\epsilon > 0$, and, by integrating by part, we have

$$\int_{\epsilon}^{\infty} \frac{1}{x} f(x) \, dx = \int_{\epsilon}^{\infty} (\log x)' f(x) \, dx$$

$$= \left[ \log x f(x) \right]_{\epsilon}^{\infty} - \int_{\epsilon}^{\infty} \log x f'(x) \, dx$$

$$= -f(\epsilon) \log \epsilon - \int_{\epsilon}^{\infty} \log x f'(x) \, dx$$

$$= -f(0) \log \epsilon - \int_{0}^{\infty} \log x f'(x) \, dx + O(1) \quad (\text{as } \epsilon \downarrow 0).$$

Then, the limit

$$\lim_{\epsilon \downarrow 0} \left\{ \int_{\epsilon}^{\infty} \frac{1}{x} f(x) \, dx + f(0) \log \epsilon \right\}$$

exists and is finite. We call this limit the Hadamard finite-part (f.p.) integral and denote it by

$$\text{f.p.} \int_{0}^{\infty} \frac{1}{x} f(x) \, dx.$$ In general, we can define the f.p. integral

$$f^{(n)}[f] = \text{f.p.} \int_{0}^{\infty} x^{-n} f(x) \, dx \quad (n = 1, 2, \ldots), \quad (1)$$

where $f(x)$ is an analytic function such that $f(0) \neq 0$ and $f(x) = O(x^{n-\alpha-1})(x \to +\infty)$ with $\alpha > 0$ [3].

In this paper, we propose a numerical method of computing f.p. integrals (1). In the proposed method, we express the f.p. integral (1) using a complex integral, and we obtain the f.p. integral by evaluating the complex integral by the DE formula [11]. Theoretical error estimate and some numerical examples show the exponential convergence of the proposed formula as the number of sampling points increases.

Previous works related to this paper are as follows. The author and Hi-rayama proposed a numerical method of computing ordinary integrals related to hyperfunction theory [7], a theory of generalized functions based on complex function theory. The author proposed numerical methods for computing Hadamard finite-part integrals with a singularity at an endpoint on a finite interval [4-5]. In these methods, we express a desired integral using a complex integral, we obtain the integral by evaluating the complex integral by conventional numerical integration formulas. For Cauchy principal-value integrals or Hadamard finite-part integrals on a finite interval with a singularity in the interior of the integral interval

$$\text{f.p.} \int_{0}^{1} \frac{f(x)}{(x-\lambda)^n} \, dx \quad (0 < \lambda < 1, \ n = 1, 2, \ldots), \quad (2)$$

many methods were proposed. Elliot and Paget proposed Gauss-type numerical integration formulas for (2) [3-4]. Bialecki proposed Sinc numerical integration formulas for (2) [1-2], where the trapezoidal formula is used together with variable transform technique as in the DE formula [11]. Ogata and et al. improved them and proposed a DE-type numerical integration formula for (2) [5].
The remainder of this paper is structured as follows. In Section 2, we define the f.p. integral (1) and propose a numerical method of computing it. In addition, we show theoretical error estimate of the proposed method. In Section 3, we show some numerical example which show the effectiveness of the proposed method. In Section 4, we give a summary of this paper.

2 Hadamard finite-part integrals and a numerical method

The f.p. integral (1) is defined by

\[
I_n(f) = \lim_{\epsilon \to 0} \{ \int_{\epsilon}^{\infty} x^{-n} f(x) dx - \sum_{k=0}^{n-2} \frac{\epsilon^{k+1-n}}{k!(n-1-k)} f^{(k)}(\epsilon) + \frac{\log \epsilon}{(n-1)!} f^{(n-1)}(\epsilon) \} \quad (n = 1, 2, \ldots),
\]

where \( f(x) \) is an analytic function on \([0, +\infty)\) such that \( f(0) \neq 0 \) and \( f(x) = O(x^{n-1-\alpha}) \) as \( x \to +\infty \) with \( \alpha > 0 \), and the second term on the right-hand side is zero if \( n = 1 \). We can show that it is well-defined as follows. In fact, for \( \epsilon > 0 \), we can show by integrating by part

\[
\int_{\epsilon}^{\infty} x^{-n} f(x) dx = \frac{\epsilon^{1-n}}{n-1} f(\epsilon) + \frac{1}{n-1} \int_{\epsilon}^{\infty} x^{1-n} f'(x) dx
\]

\[
= \frac{\epsilon^{1-n}}{n-1} f(\epsilon) + \frac{\epsilon^{2-n}}{(n-1)(n-2)} f'(\epsilon) + \frac{1}{(n-1)(n-2)} \int_{\epsilon}^{\infty} x^{2-n} f''(x) dx
\]

\[
= \ldots
\]

\[
= \sum_{k=0}^{n-2} \frac{\epsilon^{k+1-n}}{(n-1)(n-2) \cdots (n-1-k)} f^{(k)}(\epsilon) - \frac{\log \epsilon}{(n-1)!} f^{(n-1)}(\epsilon)
\]

\[
+ \frac{1}{(n-1)!} \int_{\epsilon}^{\infty} \log x f^{(n)}(x) dx
\]

\[
= \sum_{k=0}^{n-2} \frac{\epsilon^{k+1-n}}{(n-1)(n-2) \cdots (n-1-k)} \left\{ \sum_{l=0}^{\infty} \frac{f^{(k+l)}(0)}{l!} \epsilon^l \right\}
\]

\[
- \frac{\log \epsilon}{(n-1)!} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \epsilon^k + \frac{1}{(n-1)!} \int_{\epsilon}^{\infty} \log x f^{(n)}(x) dx
\]

\[
= \sum_{l=0}^{n-2} \left\{ \sum_{k=0}^{l} \frac{1}{(l-k)!} \frac{f^{(k)}(0)}{((n-1)(n-2) \cdots (n-1-k))} \epsilon^{l+1-n} f^{(l)}(0) \right\}
\]

\[
- \frac{\log \epsilon}{(n-1)!} f^{(n-1)}(0) + O(1) \quad (as \ \epsilon \downarrow 0),
\]
and

\[ \ast = \frac{1}{l!(n-1)} + \frac{1}{(l-1)!(n-1)(n-2)} + \ldots + \frac{1}{l!(n-1)(n-2)\ldots(n-l+1)(n-l)} + \frac{1}{(n-1)(n-2)\ldots(n-l+1)(n-l)(n-l-1)} \]

Then, we have

\[ \int_{\epsilon}^{\infty} x^{-n} f(x) dx = \sum_{l=0}^{n-2} \frac{\epsilon^{l+1-n}}{l!(n-l-1)} f^{(l)}(0) - \frac{\log \epsilon}{(n-1)!} f^{(n-1)}(0) + O(1) \quad (\text{as } \epsilon \downarrow 0). \]

Therefore, the limit in (3) exists and is finite.

Theorem 1 We suppose that \( f(z) \) is analytic in a complex domain \( D \), which contains the half infinite interval \([0, +\infty)\) in its interior. Then, the f.p. integral (3) is expressed as

\[ I^{(n)} = \frac{1}{2\pi i} \oint_{C} z^{-n} f(z) \log(-z) dz, \]  

where \( C \) is a complex integral path such that it encircles \([0, +\infty)\) in the positive sense and is contained in \( D \).

Proof of Theorem 1 From Cauchy’s integral theorem, we have

\[ \frac{1}{2\pi i} \oint_{C} z^{-n} f(z) \log(-z) dz = \frac{1}{2\pi i} \left( \oint_{\Gamma_1^+} + \oint_{\Gamma_2} + \oint_{\Gamma_3^-} \right) z^{-n} f(z) \log(-z) dz, \]
for $\epsilon > 0$, where $\Gamma_{\epsilon}^{(+)}$ and $C_{\epsilon}$ are complex integral paths respectively defined by

$$
\Gamma_{\epsilon}^{(+)} = \{ x \in \mathbb{R} + i0 \mid +\infty > x \geq \epsilon \},
$$

$$
\Gamma_{\epsilon}^{(-)} = \{ x \in \mathbb{R} - i0 \mid \epsilon \leq x < +\infty \},
$$

$$
C_{\epsilon} = \{ e^{i\theta} \mid 0 \leq \theta \leq 2\pi \}
$$

(see Figure 1), and the complex logarithmic function $\log z$ is the principal value, that is, the branch such that it takes a real value on the positive real axis. As

![Figure 1: The complex integral paths $\Gamma_{\epsilon}^{(\pm)}$ and $C_{\epsilon}$.](image)

...
where we exchanged the order of the integral and the infinite sum since the infinite series is uniformly convergent on \(0 \leq \theta \leq 2\pi\). Since
\[
\int_{0}^{2\pi} e^{i(k-n+1)\theta} d\theta = 2\pi \delta_{k,n-1},
\]
\[
\int_{0}^{2\pi} (\theta - \pi) e^{i(k-n+1)\theta} d\theta = \begin{cases} \frac{-2\pi}{i(n-1-k)} & (0 \leq k \leq n-2) \\ 0 & (k = n-1) \end{cases},
\]
we have
\[
[5] = \log \epsilon \frac{f^{(n-1)}(0)}{(n-1)!} - \sum_{k=0}^{n-2} \frac{\epsilon^{k-n+1}}{k!(n-1-k)} f^{(k)}(0) + O(\epsilon) \quad (\epsilon \downarrow 0).
\]

Summarizing the above calculations, we have
\[
\frac{1}{2\pi i} \oint_{C} z^{-n} f(z) \log(-z) dz = \int_{1}^{\epsilon} x^{-n} f(x) dx - \sum_{k=0}^{n-2} \frac{\epsilon^{k-n+1}}{k!(n-1-k)} f^{(k)}(0)
\]
\[
+ \log \epsilon \frac{f^{(n-1)}(0)}{(n-1)!} \approx \epsilon^{n-1} f^{(n-1)}(0) + O(\epsilon) \quad (\epsilon \downarrow 0),
\]
and, taking the limit \(\epsilon \downarrow 0\), we obtain [4].

We obtain the f.p. integral by evaluating the complex integral on the right-hand side of [4] by a conventional numerical integration formula such as the DE formula [11], that is,
\[
\int_{-\infty}^{\infty} g(u) du = \int_{-\infty}^{\infty} g(\psi_{DE}(v)) \psi'_{DE}(v) dv \simeq h \sum_{k=-N_{-}}^{N_{+}} g(\psi_{DE}(kh)) \psi'_{DE}(kh),
\]
where \(h > 0\) is the mesh of the trapezoidal formula, \(u = \psi_{DE}(v)\) is the DE transform
\[
\psi_{DE}(v) = \begin{cases} \sinh(\sinh v) & \left( g(u) = O(|u|^{-\alpha-1}) \quad \text{as} \quad u \to \pm \infty, \alpha > 0 \right) \\ \sinh v & \left( g(u) = O(\exp(-c|u|)) \quad \text{as} \quad u \to \pm \infty, c > 0 \right), \end{cases}
\]
and \(N_{\pm}\) is a positive integer such that the transformed integrand \(g(\psi_{DE}(kh))\psi'_{DE}(kh)\) is sufficiently small at \(k = -N_{-}, N_{+}\). We can take \(N_{\pm}\) small since \(g(\psi_{DE}(v))\psi'_{DE}(v)\) decays double exponentially as \(v \to \pm \infty\). Then, we have the approximation formula
\[
f^{(n)}[f] \simeq I^{(n)}_{h,N_{+},N_{-}}[f]
\]
\[
= \frac{h}{2\pi i} \sum_{k=-N_{-}}^{N_{+}} \varphi(\psi_{DE}(kh))^{-n} f(\varphi(\psi_{DE}(kh))) \log(-\varphi(\psi_{DE}(kh))) \times \varphi'(\psi_{DE}(kh))\psi'_{DE}(kh), \quad (6)
\]
where \(z = \varphi(u), -\infty < u < +\infty\) is a parameterization of the complex integral path \(C\).

If \(f(x)\) is an analytic function on the real axis and \(C\) is an analytic curve, the proposed approximation [6] converges exponentially as shown in the following theorem. For the simplicity, we take \(N_{+} = N_{-} \equiv N'\).
Theorem 2 We suppose that
1. the parameterization function $\varphi(w)$ of $C$ is analytic in the strip
   \[ \mathcal{D}_d = \{ w \in \mathbb{C} \mid |\text{Im} w| < d \} \quad (d > 0) \]
such that
   \[ \varphi(\mathcal{D}_d) = \{ \varphi(w) \mid w \in \mathcal{D}_d \}, \]
is contained in $\mathbb{C} \setminus [0, +\infty],$
2. \[ \mathcal{N}(f, \varphi, \psi_{DE}, \mathcal{D}_d) \]
   \[ \equiv \lim_{\epsilon \to 0} \max_{0 \leq |x| \leq \epsilon} |\varphi(\psi_{DE}(w))^{-n}f(\psi_{DE}(w))\log(-\varphi(\psi_{DE}(w)))\psi_{DE}'(w)| < \infty, \]
   where
   \[ \mathcal{D}_d(\epsilon) = \{ w \in \mathbb{C} \mid |\text{Re} w| < 1/\epsilon, \quad |\text{Im} w| < d(1 - \epsilon) \}. \]
   and
3. there exist positive numbers $C_0$, $c_1$ and $c_2$ such that
   \[ |f(\varphi(\psi_{DE}(v)))| \leq C_0 \exp(-c_1 \exp(c_2|v|)) \quad (\forall v \in \mathbb{R}). \]

Then, we have the inequality
\[ |f^{(n)}[f] - I^{(n)}_{h,N}[f]| \leq \frac{1}{2\pi} \mathcal{N}(f, \varphi, \psi_{DE}, \mathcal{D}_d) \frac{\exp(-2\pi d/h)}{1 - \exp(-2\pi d/h)} \]
\[ + C(f, \varphi, \psi_{DE}, \mathcal{D}_d) \exp(-c_1 \exp(c_2 N' h)), \]
(7)

where $I^{(n)}_{h,N}[f] = I^{(n)}_{h,N,N}[f]$ and $C(f, \varphi, \psi_{DE}, \mathcal{D}_d)$ is a positive number depending on $f(z)$, $\varphi$, $\psi_{DE}$ and $\mathcal{D}_d$ only.

This theorem shows that the approximation formula (6) converges exponentially as the mesh $h$ decreases and the number of sampling points $2N' + 1$ increases.

Proof of Theorem 2 We have
\[ \left| f \cdot \text{p.} \int_0^{\infty} x^{-n} f(x) dx - I^{(n)}_{h,N}[f] \right| \]
\[ \leq \left| f \cdot \text{p.} \int_0^{\infty} x^{-n} f(x) dx - I^{(n)}_{h}[f] \right| \]
\[ + h \sum_{|k| > N} |\varphi(\psi_{DE}(kh))^{-n}f(\psi_{DE}(kh))\varphi'(\psi_{DE}(kh))\psi_{DE}'(kh)|, \]
(8)

where $I^{(n)}_{h}[f] = \lim_{N \to \infty} I^{(n)}_{h,N}[f]$. For the first term on the right-hand side of (8), we have
\[ \left| f \cdot \text{p.} \int_0^{\infty} x^{-n} f(x) dx - I^{(n)}_{h}[f] \right| \leq \frac{1}{2\pi} \mathcal{N}(f, \varphi, \psi_{DE}, \mathcal{D}_d) \frac{\exp(-2\pi d/h)}{1 - \exp(-2\pi d/h)} \]
by Theorem 3.2.1 in [10]. For the second term on the right-side hand, we have

\[ |\text{the second term}| \leq C_0 h \sum_{|k| > N} \exp(-c_1 \exp(c_2 kh)) \]

\[ \leq 2C_0 \int_{kh}^{\infty} \exp(-c_1 \exp(c_2 x))dx \]

\[ \leq 2C_0 \int_{kh}^{\infty} \exp(c_2 x) \exp(-c_1 \exp(c_2 x))dx \]

\[ = \frac{2C_0}{c_2} \exp(-c_1 \exp(c_2 Nh)). \]

Then, we obtain (7).

We remark here that we can reduce the number of sampling points by half if the integrand \( f(x) \) is real valued on the real axis. In fact, we have \( f(z) = \overline{f(\overline{z})} \) by the reflection principle, taking the integral path \( C \) symmetric with respect to the real axis, that is, \( \varphi(-u) = \overline{\varphi(u)} \), which leads to

\[ \varphi'(-u) = -\overline{\varphi'(u)}, \]

and taking the DE transform \( \psi_{DE}(v) \) to be an even function, we have

\[ I^{(n)}[f] \approx I_{h,N}^{(n)}[f] \]

\[ = \frac{h}{2\pi} \text{Im} \left\{ \varphi(\psi_{DE}(0))^{-n} f(\varphi(\psi_{DE}(0))) \log(-\varphi(\psi_{DE}(0))) \varphi'(\psi_{DE}(0)) \varphi'_{DE}(0) \right\} \]

\[ + \frac{h}{\pi} \text{Im} \left\{ \sum_{k=1}^{N} \varphi(\psi_{DE}(kh))^{-n} f(\varphi(\psi_{DE}(kh))) \log(-\varphi(\psi_{DE}(kh))) \right. \]

\[ \left. \times \varphi'(\psi_{DE}(kh)) \varphi'_{DE}(kh) \right\}. \]

(9)

### 3 Numerical examples

In this section, we show some numerical examples which show the effectiveness of the proposed method.

We computed the f.p. integrals

(i) \( \int_{0}^{\infty} \frac{x^{-n}}{1 + x^2} dx = \begin{cases} \frac{\pi}{2} (-1)^m & (n = 2m \text{ (even)}) \\ 0 & (n = 2m + 1 \text{ (odd)}) \end{cases} \)

(ii) \( \int_{0}^{\infty} x^{-n} e^{-x} dx = \begin{cases} -\gamma & (n = 1) \\ -1 + \gamma & (n = 2) \\ \frac{2}{n} - \frac{1}{2} \gamma & (n = 3) \\ \frac{2}{n} + \frac{1}{2} \gamma & (n = 4) \end{cases} \)

for \( n = 1, 2, 3, 4 \), where \( \gamma \) is Euler’s constant, by the formula (9). All the computations were performed using programs coded in C++ with double precision working. The complex integral path \( C \) in (10) is taken as

\[ C : z = \varphi(u) = \frac{u + 0.5i}{i\pi} \log\left( \frac{1 + i(u + 0.5i)}{1 - i(u + 0.5i)} \right), \quad +\infty > u > -\infty \]
We took the number of sampling points $N$ for given mesh $h = 2^{-1}, 2^{-2}, \ldots$ by truncating the infinite sum at the $k$-th term such that

$$\frac{h}{\pi} \times |\text{the } k\text{-th term}| < \begin{cases} 10^{-15} \times |I^{(n)}_{h,N}[f]| & \text{if } I^{(n)}_{h,N}[f] \neq 0 \\ 10^{-15} & \text{otherwise.} \end{cases}$$

Figure 2 shows the relative errors of the proposed approximation formula (9) applied to the f.p. integrals (10). These figures show the exponential convergence of the proposed formula as the number of sampling points $N$ increases.

Figure 3: The errors of the proposed approximation formula (9) applied to the f.p. integrals (10).
4 Summary

In this paper, we proposed a numerical integration formula for Hadamard finite-part integrals with an integral power singularity at the endpoint on a half-infinite interval. In the proposed method, we express the desired f.p. integral using a complex integral, and we obtain the f.p. integral by evaluating the complex integral by the DE formula. Theoretical error estimate and some numerical examples show the exponential convergence of the proposed method in the case that the integrand is an analytic function.

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