AMENABILITY AND APPROXIMATION PROPERTIES FOR
PARTIAL ACTIONS AND FELL BUNDLES

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Abstract. Building on previous papers by Anantharaman-Delaroche we introduce and study the notion of AD-amenability for partial actions and Fell bundles over discrete groups. If the Fell bundle is AD-amenable, the full and reduced crossed products coincide. We prove that the cross-sectional C*-algebra of the Fell bundle is nuclear if and only if the underlying unit fibre is nuclear and the Fell bundle is AD-amenable. If a partial action is globalisable, then it is AD-amenable if and only if its globalisation is AD-amenable. Moreover, we prove that AD-amenability is invariant under (weak) equivalence of Fell bundles and show that AD-amenability is equivalent to a weak form of the approximation property introduced by Exel. For Fell bundles whose unit fibre is (Morita equivalent to) a commutative C*-algebra we prove that AD-amenability is equivalent to the approximation property.

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1. Introduction

In her seminal paper [10] Anantharaman-Delaroche introduced a notion of amenability (that we here call AD-amenability) for actions of discrete groups on C*-algebras. Her definition is based on previous papers [7, 8] where she studies amenability for group actions on W*-algebras (i.e. von Neumann algebras). More precisely, an action \( \gamma \) of a discrete group \( G \) on a W*-algebra \( N \) is said to be amenable in the sense of Anantharaman-Delaroche (or just W*AD-amenable for short) if there exists a G-equivariant conditional expectation \( P: \ell^\infty(G, N) \to N \) with respect to the diagonal \( G \)-action \( \tilde{\gamma} \) on \( \ell^\infty(G, N) = \ell^\infty(G) \otimes N \) where \( G \) acts on \( \ell^\infty(G) \) by (left) translations; the map \( P \) should be interpreted as a \( G \)-invariant mean for the action. An action \( \alpha \) of \( G \) on a C*-algebra \( \mathcal{A} \) is then said to be AD-amenable if the induced action \( \alpha'' \) on the enveloping (bidual) W*-algebra \( \mathcal{A}'' \) is W*AD-amenable.

One of the main results in [10] (namely Theorem 3.3) shows that an invariant mean \( P: \ell^\infty(G, N) \to N \) can be always approximated with respect to the pointwise weak* (i.e. ultraweak) topology by using certain nets of functions from \( \mathcal{A} \). This is indeed possible for commutative \( \mathcal{N} \) is amenable.

Then one adds undesirable actions. For instance the adjoint action \( G \) requires (and is equivalent to) the existence of a net as above with values in \( G \). One precise form of such approximation that will be specially important to us in this paper is given by a net of functions of finite support \( \{a_i : G \to \mathcal{A}\}_{i \in I} \) which is bounded when viewed as a net of the Hilbert \( \mathcal{N} \)-module \( \ell^2(G, N) \) and satisfies

\[
\langle a_i | \gamma_g(a_i) \rangle_2 = \sum_{h \in G} a_i(h) \gamma_g(a_i(g^{-1}h)) \to 1
\]

with respect to the weak* topology for all \( g \in G \). This condition indeed characterises amenability and shows, among other things, that \( \gamma \) is W*AD-amenable if and only if so is its restriction to the centre \( Z(N) \). Moreover, W*AD-amenability behaves well with respect to injectivity of W*-algebras and nuclearity of C*-algebras: if \( N \) is injective then \( \gamma \) is W*AD-amenable if and only if the W*-crossed product \( N \rtimes_\gamma G \) is injective. And similarly, if \( A \) is a nuclear C*-algebra, then the (reduced) C*-crossed product \( A \rtimes_\alpha G \) is nuclear if and only if \( \alpha \) is AD-amenable.

Notice that the AD-amenability of an action on a C*-algebra \( \mathcal{A} \) requires (and is equivalent to) the existence of a net as above with values in \( Z(A) \). While this is a huge commutative algebra in general, finding explicitly such an approximate invariant mean might be a very difficult task – if not impossible. Hoping for more concrete realisations of such approximate means one might wonder whether it is not always possible to find a net with values in \( Z(A) \) or at least in \( \mathcal{Z}M(A) \) (the central multiplier algebra). This is indeed possible for commutative \( A \) (by [10] Theorem 4.9) and hence more generally for \( A \) admitting (nondegenerate) \( G \)-equivariant *-homomorphism \( C_0(X) \to \mathcal{Z}M(A) \) for some amenable \( G \)-space \( X \). Unfortunately this is not possible in general: striking recent results by Suzuki in [33] show that every exact group admits an AD-amenable action on a unital simple nuclear C*-algebra \( \mathcal{A} \) (and one can even choose such algebra for which the crossed product is in the same class). For such an \( A \) we have \( Z(A) = \mathcal{Z}M(A) = \mathcal{C} \cdot 1 \) so that the existence of an approximate mean as above with values in \( Z(A) \) forces \( G \) to be amenable. On the other hand, dropping the commutativity completely and asking only for a net \( \{a_i\}_{i \in I} \subset \ell^2(G, N) \) satisfying (1.1) is also not a good idea because then one adds undesirable actions. For instance the adjoint action \( \gamma = Ad \lambda \) of the left regular representation on \( N = \mathcal{L}(\ell^2 G) = \mathcal{K}(\ell^2 G)'' \) has this weaker property because \( \ell^\infty(G) \hookrightarrow \mathcal{L}(\ell^2 G) \) equivariantly. But this action is AD-amenable only if \( G \) is amenable.

\[
(1.1)
\]
Fortunately there is an alternative out of this: we only need to change (1.1) slightly, requiring instead the existence of a bounded net of finitely supported functions $\{a_i\}_{i \in I} \subset \ell^2(G,N)$ satisfying
\[
\langle a_i | b\gamma_g(a_i) \rangle_2 = \sum_{h \in G} a_i(h)^* b\gamma_g(a_i(g^{-1}h)) \to b
\]
for the weak*-topology for all $g \in G$ and $b \in N$. It turns out that this is equivalent to the $W^*$-AD-amenability of $\gamma$ (and hence to the existence of a central net satisfying (1.1)). Moreover, we prove that if $A$ is a weak*-dense $G$-invariant $C^*$-subalgebra of $N$, then the above condition (hence the $W^*$-AD-amenability of $N$) is equivalent to the existence of a bounded net of finitely supported functions $\{a_i\}_{i \in I} \subset \ell^2(G,A)$ satisfying (1.2). In particular an action $\alpha$ on a $C^*$-algebra $A$ is AD-amenable if and only if there exists a bounded net $\{a_i\}_{i \in I} \subset \ell^2(G,A)$ of functions with finite supports satisfying
\[
\langle a_i | b\alpha_g(a_i) \rangle_2 = \sum_{h \in G} a_i(h)^* b\alpha_g(a_i(g^{-1}h)) \to b
\]
with respect to the weak topology on $A$ for all $g \in G$ and $b \in A$. This now brings us to a close connection with the approximation property (AP) as defined by Exel in [22] for Fell bundles over discrete groups. If $B_{\alpha} = A \times G$ is the semidirect-product Fell bundle over $G$ associated with $\alpha$, then the AP for $B_{\alpha}$ is equivalent to the existence of a bounded net of finitely supported functions $\{a_i\}_{i \in I} \subset \ell^2(G,A)$ satisfying exactly the same condition (1.3) except that the weak convergence is replaced by the convergence with respect to the norm on $A$. In particular the AP of an action (in the sense that its associated Fell bundle has the AP) always implies its AD-amenability. It seems that this simple fact has not been recognised before except for the case of actions on nuclear $C^*$-algebras, where ones uses the fact that the AP implies nuclearity of the crossed product and that this is equivalent to AD-amenability, see for instance [20 Corollary 4.5].

Since the AP makes sense for general Fell bundles (in particular for partial actions) and since it is so close to the AD-amenability of actions, it is a natural task trying to extend the notion of AD-amenability also to Fell bundles. This is one of our main goals in this paper. It is indeed not difficult to give a possible definition of AD-amenability for a general Fell bundle $B$. One can use for instance the $C^*$-algebra of kernels $\mathbb{K}(B)$ of $B$. This carries a canonical global action whose associated Fell bundle is (weakly) equivalent to $B$, see [5]. One can then say that $B$ is AD-amenable if the action on $\mathbb{K}(B)$ is AD-amenable.

Of course, in practice one does not want to go to an equivalent Fell bundle in order to check its AD-amenability. In order to have a direct description of AD-amenability for Fell bundles we proceed as in the case of ordinary actions ‘transporting’ everything to a von Neumann algebraic context. For this we introduce the notion of $W^*$-Fell bundles and prove that every Fell bundle $B$ has an enveloping $W^*$-Fell bundle $B''$; indeed, the fibres $B''_t$ of $B''$ are just the bidual Banach spaces of the original fibres $B_t$ of $B$. We then define a $W^*$-version of the approximation property of Exel: we say that a $W^*$-Fell bundle $M = (M_t)_{t \in G}$ has the $W^*$AP if there is a bounded net $\{a_i\}_{i \in I} \subset \ell^2(G,M_e)$ of finitely supported functions satisfying an approximation condition very similar (1.3); it is indeed the same condition as in the original definition of Exel in [22] except that we replace the convergence in
norm by the weak*-convergence, see our Definition 6.3 for details. We then say that a (C*-algebraic) Fell bundle $B$ has the WAP if $B''$ has the W*AP.

Our main results show that all these notions behave nicely and have the expected properties. We prove that the W*AP of $\mathcal{M}$ is equivalent to its W*AD-amenability in the sense that the canonical action on its W*-algebra of kernels $k_{w^*}(\mathcal{M})$ (a certain W*-completion of its C*-algebra of kernels) is W*AD-amenable. Moreover, we prove that the WAP of a Fell bundle $B$ is equivalent to a weak form of Exel's approximation property – the only difference, again, is with respect to the convergence: for the WAP we have weak convergence while for the AP we have norm convergence. This weak form of the AP is still enough to prove some of the main desirable properties: for instance, we prove the coincidence of full and reduced cross-sectional C*-algebras $C^*_\mathcal{B} = C^*_r(\mathcal{B})$ whenever $\mathcal{B}$ has the WAP.

The advantage of the WAP is that it corresponds exactly to nuclearity of cross-sectional C*-algebras for Fell bundles with nuclear unit fibre: we prove that $C^*_\mathcal{B}$ is nuclear if and only if $B_e$ is nuclear and $B$ has the WAP. This equivalence is unclear for the AP of Exel. Indeed, we know that the AP always implies the WAP but the converse is not clear in general. In particular it is unclear whether the nuclearity of $C^*_\mathcal{B}$ implies the AP of $\mathcal{B}$. This is an open question already raised by Exel. Our methods and results can be viewed as a partial answer to this question. The only remaining question is then to see whether the weak convergence appearing in the WAP can be always replaced by norm convergence, hence showing that the WAP and the AP are equivalent. We prove that this is indeed true for a huge class of Fell bundles, namely all Fell bundles whose unit fibres are C*-algebras Morita equivalent to a commutative C*-algebra. In particular this applies to the important case of partial actions on commutative C*-algebras and shows that a partial crossed product $C_0(X) \rtimes_r G$ is nuclear if and only if the underlying partial action has the AP.

The structure of the paper is organised as follows. In Section 2 we introduce and study the notion of partial actions of groups on W*-algebras. It seems this has not been studied before, but it will be important to us here as it opens a canonical general link between the C*- and W*-theory of partial actions. In particular we show that every partial action $\alpha$ on a C*-algebra $A$ extends to a canonical enveloping W*-partial action $\alpha''$ on $A''$. One of the main results of this section states that every W*-partial action admits an enveloping W*-global action. This in particular allows us to canonically extend Anantharaman-Delaroche’s notion of amenability to partial actions on C*- and W*-algebras. This is done in Section 3. We give some basic examples and prove that amenability in this sense behaves well with respect to taking restrictions and enveloping actions. In Section 4 we prove that AD-amenability is invariant under equivalences of partial actions, both for C*- and W*-algebras.

In Section 5 we start to extend the theory of AD-amenability to Fell bundles. We introduce the notion of W*-Fell bundles and prove that every Fell bundle admits a canonical enveloping W*-Fell bundle. We also introduce the W*-algebra of kernels in this section. This algebra always carries a global W*-action and allows us to extend AD-amenability to W*-Fell bundles and hence also to C*-algebraic Fell bundles by taking W*-envelopings.

In Section 6 we study Exel’s approximation property and translate it to the context of W*-Fell bundles. We prove that this new notion, the W*AP, gives an
alternative description of the W*-AD-amenability. We also define the WAP for C*-algebraic Fell bundles and give alternative characterisations, proving that it is equivalent to AD-amenability and to a weak form of Exel’s AP. Moreover, in the final part of Section 6 we show that the WAP behaves well with respect to (weak) equivalences of Fell bundles. In Section 7 we give some elementary properties for AD-amenability and the approximation property. In particular we prove that the WAP is enough to conclude $C^*(B) = C^r(B)$, and that nuclearity of this algebra is equivalent to the WAP in case $B$ is already nuclear. We also prove an analogous property for W*-Fell bundles using injectivity in place of nuclearity.

Finally, we add an appendix at the end of the paper where we review some basic theory of Hilbert W*-modules and W*-equivalences. This makes the paper more self-contained — with the disadvantage of making it longer — and probably easier to follow to people not used to these aspects of von Neumann algebra theory. In principle all the stuff added in our appendix is already known, at least to specialists from the W*-community. But it is probably not very common to people working with C*-algebras and it is certainly not easy to grasp all things we need from the literature.

We shall only consider discrete groups in this paper although probably many of the things we do here also extend to locally compact groups. The approximation property of Fell bundles has been extended to locally compact groups by Exel and Ng in [20]. The notions of equivalences of Fell bundles are also available to locally compact groups [5, 6]. On the other hand, the theory of amenable actions of locally compact groups on C*-algebras has not been touched yet. Anantharaman-Delaroche only defines it for discrete groups in [10] although she actually considers locally compact groups when acting on von Neumann algebras [7, 8].

2. Partial actions on von Neumann algebras

A lot is already known about partial actions of groups on C*-algebras but it seems that partial actions on W*-algebras (i.e. von Neumann algebras) have never been studied. Probably the main reason is that every W*-algebra is unital, so that every partial action in this setting automatically has an enveloping action, that is, they are always restrictions of a global action on a W*-algebra (see Proposition 2.7). This means that W*-partial actions are not as interesting as their C*-companions. However, starting with a partial action $\alpha$ on a C*-algebra $A$, its bidual $A''$ von Neumann algebra carries a natural partial action $\alpha''$ that will serve as one of our main tools in this paper. This is the reason why we develop the basic theory of partial actions on von Neumann algebras here.

We start by recalling some basic facts about partial actions and their globalisations.

**Definition 2.1.** A partial action of a group $G$ on a set $X$ is a family of functions $\sigma = \{\sigma_t : X_{t^{-1}} \to X_t\}_{t \in G}$ such that:

(i) For every $t \in G$, $X_t$ is a subset of $X$.

(ii) $X_e = X$ and $\alpha_e$ is the identity of $X$.

(iii) Given $s, t \in G$ and $x \in X_{s^{-1}}$ such that $\sigma_t(x) \in X_{s^{-1}}$, it follows that $x \in X_{(st)^{-1}}$ and $\sigma_{st}(x) = \sigma_s(\sigma_t(x))$.

An action is just a partial action $\sigma$ such that $X_t = X$ for every $t \in G$. In this case we also say that $\sigma$ is a global action.
Given two partial actions $\sigma$ and $\tau$ of $G$ on sets $X$ and $Y$ respectively, a morphism $f: \sigma \to \tau$ is a function $f: X \to Y$ such that $f(X_t) \subseteq Y_t$ and $f(\sigma_t(x)) = \tau_t(f(x))$ for all $t \in G$ and $x \in X_{t^{-1}}$. The composition of morphisms is just the composition of functions.

The restriction of $\sigma$ to a subset $Y \subseteq X$ is $\sigma|_Y := \{\sigma|_{Y_t}: Y_t \to Y_t\}$, where $Y_t := Y \cap \sigma_t(X_{t^{-1}} \cap Y)$ and $\sigma|_Y(y) = \sigma_t(y)$. It follows that $\sigma|_Y$ is a partial action of $G$ on the set $Y$ and, if $Z \subseteq Y$, then $\sigma|_Z = \sigma|_{|Z}$. When a partial action $\tau$ can be expressed as a restriction $\tau = \sigma|_Y$ of a global action $\sigma$, we say that $\tau$ is globalisable, and that $\sigma$ is a globalisation of $\tau$.

In this article we work with discrete groups exclusively, so here “group” actually means “discrete group”.

A partial action $\sigma$ of a group $G$ on a $C^*$-algebra $A$ is a partial action of $G$ on the set $A$ for which each $A_t$ is a closed two-sided ideal of $A$, and each $\sigma_t: A_{t^{-1}} \to A_t$ is an isomorphism of $C^*$-algebras. If $\beta$ is a global action of $G$ on a $C^*$-algebra $B$ and $A$ is a closed two-sided ideal of $B$, then the restriction $\alpha := \beta|_A$ is a partial action on the $C^*$-algebra. We then say that $\beta$ is a globalisation of $\alpha$, and that $\alpha$ is globalisable.

**Definition 2.2.** A $W^*$-partial action of a group $G$ on a $W^*$-algebra $M$ is a set theoretic partial action of $G$ on $M$, $\gamma := \{\gamma_t: M_{t^{-1}} \to M_t\}_{t \in G}$, where each $M_t$ is a $W^*$-ideal of $M$ (possibly $\{0\}$) and each $\gamma_t$ is a $W^*$-isomorphism. A morphism of $W^*$-partial actions is just a morphism of set theoretic partial actions which is also a morphism of $W^*$-algebras (a $w^*$-continuous morphism of *-algebras).

Here we always view $W^*$-algebras as Banach space duals, $M \cong (M_\tau)'$, endowed with the $w^*$-topology. A $W^*$-ideal is then just a *-ideal of $M$ that is closed for the $w^*$-topology. And a $W^*$-isomorphism between two $W^*$-algebras is a *-isomorphism that is $w^*$-continuous. Actually, every *-isomorphism between $W^*$-algebras is normal (preserves suprema of increasing bounded nets) and it is therefore automatically a $W^*$-isomorphism, see [12] Proposition III.2.2.2.

As in the case of actions on sets or on $C^*$-algebras, we have suitable notions of restriction and globalisation of partial actions in the category of $W^*$-algebras:

**Example 2.3** (Restriction and globalisation). Given an ordinary (global) $W^*$-action $\gamma$ of $G$ on a $W^*$-algebra $N$ and a $W^*$-ideal $M \triangleleft N$, the restriction $\gamma|_M$ is a $W^*$-partial action. When a given $W^*$-partial action $\alpha$ on a $W^*$-algebra $M$ can be written as $\alpha := \gamma|_M$, where $\gamma$ is a global $W^*$-action on a $W^*$-algebra that contains $M$ as a $W^*$-ideal, then we say that $\alpha$ is globalisable, and that $\gamma$ is a globalisation of $\alpha$. More generally, the restriction of a $W^*$-partial action to a $W^*$-ideal is again a $W^*$-partial action.

We will see below in Proposition 2.7 that, unlike the case of partial actions on $C^*$-algebras, any $W^*$-partial action has a globalisation, the so called $W^*$-enveloping action, which is essentially unique when a certain natural minimality condition is required to hold.

**Example 2.4** (Bidual partial actions). Given a $C^*$-partial action $\alpha = \{\alpha_t: A_{t^{-1}} \to A_t\}_{t \in G}$ of $G$ on a $C^*$-algebra $A$, the double dual (enveloping) $W^*$-algebra $A''$ of $A$ carries a canonical $W^*$-partial action $\alpha'' := \{\alpha''_t: A''_{t^{-1}} \to A''_t\}_{t \in G}$ which is the unique $W^*$-partial action such that $\alpha''|_A = \alpha$. Here we view the bidual algebra $A''$ as a $W^*$-ideal of $A''$ and $\alpha''_t$ as the unique $w^*$-continuous extension of $\alpha_t$. 
One of our goals is to show that every $\textsl{W}^*$-partial action is (isomorphic to) a restriction of a global $\textsl{W}^*$-action as in Example 2.3, that is, every $\textsl{W}^*$-partial action will automatically have an enveloping action. One may think that this is trivial since every von Neumann algebra has a unit, so that all the ideals of a partial action are unital (possibly zero). It is well known from the $\textsl{C}^*$-algebra theory of partial actions that in this situation the partial action has an enveloping action in the $\textsl{C}^*$-algebra category (see for instance [27]). However the following example shows that the $\textsl{C}^*$-enveloping action might be not a $\textsl{W}^*$-algebra.

**Example 2.5.** Consider the “trivial” partial action of $G$ on the $\textsl{W}^*$-algebra $M := \mathbb{C}$ in which all the ideals are zero except for $M_s := M$. This can also be viewed as the restriction of the global action of $G$ by (left) translations on the $\textsl{C}^*$-algebra $C_0(G)$ to the ideal $\mathbb{C} \cong \mathbb{C}\delta_e \subseteq C_0(G)$. Moreover, since the linear orbit of this ideal is dense in the entire algebra $C_0(G)$, this action is (up to isomorphism) the enveloping action of the original partial action on $\mathbb{C}$. But if $G$ is infinite, $C_0(G)$ is not a $\textsl{W}^*$-algebra. On the other hand, we may also view $\mathbb{C} \cong \mathbb{C}\delta_e$ as a $\textsl{W}^*$-ideal of the $\textsl{W}^*$-algebra $\ell^\infty(G)$. And since the linear orbit of this ideal is $\textsl{w}^*$-dense, this is a $\textsl{W}^*$-enveloping action in the following sense.

**Definition 2.6.** A $\textsl{W}^*$-enveloping action of a $\textsl{W}^*$-partial action $\gamma$ of a group $G$ on a $\textsl{W}^*$-algebra $M$ is a $\textsl{W}^*$-global action $\sigma$ of $G$ on a $\textsl{W}^*$-algebra $N$ together with a $\textsl{W}^*$-ideal $\tilde{M}$ of $N$ and an isomorphism of $\textsl{W}^*$-partial actions $\iota: \gamma \to \sigma|_{\tilde{M}}$, such that the linear $\sigma$-orbit of $\tilde{M}$ is $\textsl{w}^*$-dense in $N$. We summarise this situation by saying that $(N, \sigma)$ is a $\textsl{W}^*$-enveloping action of $(M, \gamma)$.

Note that a $\textsl{W}^*$-enveloping action of $(M, \gamma)$ is essentially a minimal $\textsl{W}^*$-globalisation of $\gamma$ (see Example 2.3). The reader should not confuse the $\textsl{W}^*$-enveloping actions we defined above with the enveloping actions or the Morita enveloping actions defined in [2].

**Proposition 2.7.** Every $\textsl{W}^*$-partial action $\gamma$ of $G$ on a $\textsl{W}^*$-algebra $M$ has a $\textsl{W}^*$-enveloping action that is unique up to isomorphism.

**Proof.** We define $\iota: M \to \ell^\infty(G, M)$ by $\iota(x)(t) := \gamma_{t^{-1}}(x \cdot 1_t)$, where $1_t$ denotes the unit of the $\textsl{W}^*$-algebra $M_t$ ($1_t = 0$ if $M_t = \{0\}$). This unit is a central projection of $M$ because $M_t$ is a $\textsl{W}^*$-ideal of $M$. The map $\iota$ is an injective $\textsl{w}^*$-continuous $\ast$-homomorphism whose image consists of functions $f \in \ell^\infty(G, M)$ with $f(t) = \gamma_{t^{-1}}(f(e)1_t)$. The image $\tilde{M} := \iota(M)$ is a $\textsl{w}^*$-closed $\textsl{W}^*$-subalgebra of $\ell^\infty(G, M)$ which is therefore isomorphic to $M$ via $\iota$. We now endow $\ell^\infty(G, M)$ with the $G$-action $\tau$ by left translations: $\tau_t(f)(s) := f(t^{-1}s)$. This is a $\textsl{W}^*$-global action and $\iota: \gamma \to \tau$ is a morphism. Let $N$ be the $\textsl{w}^*$-closure of the linear $\tau$-orbit of $\tilde{M}$, that is, the $\textsl{w}^*$-closure of span$\{\tau_t(f) : f \in \tilde{M}, \ t \in G\}$. Moreover, $N$ is $\tau$-invariant, so that $\tau$ restricts to a $\textsl{W}^*$-global action $\sigma$ of $G$ on $N$ and this is the desired $\textsl{W}^*$-enveloping action of $(M, \gamma)$, as we now show.

It is important to note that it follows from Definition 2.1 that $\gamma_t(1_{t^{-1}1_s}) = 1_t1tsy$ for all $s, t \in G$, because $1_{t^{-1}1_s}$ and $1_11_s$ are the units of $M_{t^{-1}} \cap M_s = M_{t^{-1}1_s}$ and $M_{t^{-1}} \cap M_s = M_{t^{-1}}M_s$, respectively. This implies $\iota: \gamma \to \sigma$ is a morphism because, for all $s, t \in G$ and $x \in M_t$:

$$\tau_t(\iota(x))(s) = \gamma_{s^{-1}t^{-1}}(x1_{t^{-1}1_s}) = \gamma_{s^{-1}}(\iota(\gamma_t(x1_{t^{-1}1_s}))) = \gamma_{s^{-1}}(\gamma_t(x1_s)) = \iota(\gamma_t(x))(s).$$
In order to prove that $\tilde{M}$ is a $W^*$-ideal of $N$ and that $\iota: \gamma \to \sigma|_{\tilde{M}}$ is an isomorphism it suffices to show that $\gamma_t(\tilde{M}) \cap \tilde{M} = \iota(M_t)$, for all $t \in G$. For all $x, y \in M$:

$$[\gamma_t(\iota(a))\iota(b)](s) = \gamma_{s-t}(a1_{t-1}s)\gamma_{s-1}(b1_s) = \gamma_{s-t}(a1_{t-1}s)1_{t-1}s-1\gamma_{s-1}(b1_s)$$

$$= \gamma_{s-t}(a1_{t-1}s)\gamma_{s-1}(1_{t-1}s1_t\gamma_{s-1}(b1_s)$$

$$= \gamma_{s-t}(a1_{t-1}s1_t\gamma_{s-1}(b1_s) = \gamma_{s-1}(\gamma_t(a1_{t-1}s)b1_{s-1})$$

$$= \iota(\gamma_t(a1_{t-1}s)b)(s).$$

We then conclude that $\gamma_t(\tilde{M}) \cap \tilde{M} = \iota(M_t)M \subseteq \iota(M_t) \subseteq \gamma_t(\tilde{M}) \cap \tilde{M}$, where the last inclusion follows from the fact that $\iota: \gamma \to \sigma$ is a morphism. At this point we know $(N, \sigma)$ is a $W^*$-enveloping action for $(M, \gamma)$.

For uniqueness, assume that $M$ is a $W^*$-ideal of a $W^*$-algebra $\tilde{N}$ carrying a $W^*$-global action $\tilde{\sigma}$ whose restriction to $M$ is $\gamma$ and such that the linear $\tilde{\sigma}$-orbit of $M$ is $w^*$-dense in $\tilde{N}$ (for simplicity, we omit the inclusion map $M \hookrightarrow \tilde{N}$ here, that is, we already assume $M \subseteq \tilde{N}$). Then we extend $\iota$ to $\tilde{\iota}: \tilde{N} \to C^\infty(G, M)$ by $\tilde{\iota}(x)(t) := \tilde{\sigma}_t^{-1}(x)1_e$. First we show that $\tilde{\iota}$ is in fact an extension of $\iota$. For $x \in M$:

$$\tilde{\iota}_e(x)(t) = \tilde{\sigma}_t^{-1}(x)1_e = \tilde{\sigma}_{t-1}(x)1_{t-1} = \tilde{\sigma}_t^{-1}(x_{1_t}) = \sigma_{t^{-1}}(x_{1_t}) = \iota(x)(t)$$

because $\tilde{\sigma}_{t^{-1}}(x_{1_t}) \in \tilde{\sigma}_{t^{-1}}(M)M = M_{1_t}$. A similar computation shows that $\tilde{\iota}$ is equivariant. Observe that $\tilde{\iota}$ is injective (hence isometric) because, if $\tilde{\iota}(x) = 0$, then $x\tilde{\sigma}_t(1_e) = 0$ for all $t \in G$. This is equivalent to $xy = 0$ for all $y$ in the linear $\tilde{\sigma}$-orbit of $M$, which is $w^*$-dense in $\tilde{N}$ by assumption. Since $\tilde{\iota}$ is an isometry and it is $w^*$-continuous in $\{x \in \tilde{N} : \|x\| \leq 1\}$, its range is $w^*$-closed (hence a $W^*$-subalgebra) and it is a $W^*$-isomorphism over its image. Finally, $\tilde{\iota}(\tilde{N})$ is the $w^*$-closure of the linear span of $\tilde{\iota}(M) = \iota(M)$. Thus $\tilde{\iota}(\tilde{N}) = N$ and $\tilde{\sigma}$ is isomorphic, as a $W^*$-partial action, to $\sigma$.

**Proposition 2.8.** Let $\gamma = \{\gamma_t: M_{t^{-1}} \to M_t\}_{t \in G}$ be a $W^*$-partial action of $G$ on a $W^*$-algebra $M$ and let $(N, \sigma)$ be its enveloping $W^*$-action. Then the restriction $\gamma|_{Z(M)} = \{\gamma_t: Z(M_{t^{-1}}) \to Z(M_t)\}$ of $\gamma$ to $Z(M)$ is a $W^*$-partial action whose enveloping $W^*$-action is the restriction of $\sigma$ to $Z(N)$.

**Proof.** First notice that $Z(M_t)$ is indeed a $W^*$-ideal of $Z(M)$. In fact, if $M_t = 1_t M$, where $1_t \in Z(M)$ is the central projection of $M$ representing the unit of $M_t$, then $Z(M_t) = 1_t Z(M)$. It is then clear that the restriction of $\gamma$ to $Z(M)$ defines a $W^*$-partial action. For the same reason, viewing $M$ as a $W^*$-ideal of $N$, $M$ is then the ideal generated by the central projection $p = 1_e$, and then $Z(M) = pZ(N)$ is the $W^*$-ideal of $Z(N)$ generated by the same projection. The restriction $\sigma|_{Z(N)}$ is clearly $\sigma|_{Z(N)}|_{Z(M)} = \sigma|_{Z(M)} = \sigma|_{M}|_{Z(M)} = \gamma|_{Z(M)}$. To see that $\sigma|_{Z(N)}$ is the enveloping action of $\gamma|_{Z(M)}$ it remains to show that the linear $\sigma$-orbit of $Z(M)$ is $w^*$-dense in $Z(N)$. For each finite subset $F \subseteq G$, we define $M_F := \sum_{t \in F} \sigma_t(M)$. This is a $W^*$-ideal of $N$ (being a finite sum of such) and the union of all these ideals is the linear $\sigma$-orbit of $M$, so it is $w^*$-dense in $N$ since $(N, \sigma)$ is the enveloping action of $(M, \gamma)$. On the other hand the linear $\sigma$-orbit of $Z(M)$ is the $w^*$-closure $P$ of the ideal $\bigcup F Z(M)_F$, where $Z(M)_F = \sum_{t \in F} \sigma_t(Z(M))$. Note that $P$ is a $W^*$-ideal of $Z(N)$. To see that $P = Z(N)$ it is enough to show that the unit of $N$ is contained in $P$. For this notice that the unit $1_F$ of $M_F$ is a (finite) linear combination of $1_t$, and this is also the unit of $Z(M)_F$. The net $(1_F)_F$ is increasing and bounded and its $w^*$-limit is the unit of $N$ because $\bigcup F M_F$ is $w^*$-dense in $N$. However this limit is also the unit of $P$. 

$\square$
Remark 2.9. If \((N, \sigma)\) is a \(W^*\)-enveloping action of \((M, \gamma)\), then \(N\) is abelian if and only if \(M\) is abelian. Indeed, clearly \(M\) is abelian if \(N\) is. For the converse observe that, by the proof of the Proposition 2.7, \(N\) is isomorphic to a \(W^*\)-subalgebra of the abelian algebra \(\ell^\infty(G, M)\).

Another property that is preserved by taking enveloping actions is injectivity in the sense that if \(N\) is the \(W^*\)-enveloping action of \(M\), then \(M\) is injective if and only if \(N\) is injective. Indeed, since injectivity passes to ideals, the reverse direction is clear. For the converse one uses that injectivity passes to (finite) sums and directed unions of ideals and the description of \(N\) as the \(w^*\)-Closure of \(\cup_F MF\) as in the proof of the previous proposition.

3. Amenity of partial actions

First let us recall the notion of amenability for (global) actions of groups on \(C^*\)-algebras and \(W^*\)-algebras introduced by Anantharaman-Delaroche, see [7,8,10].

Definition 3.1. A (global) action of a group \(G\) on a \(W^*\)-algebra \(M\) is Anantharaman-Delaroche amenable (or just \(W^*\)AD-amenable for short) if there exists a linear positive contractive and \(G\)-equivariant map \(P: \ell^\infty(G, M) \to M\) whose composition with the canonical embedding (by constant functions) \(M \hookrightarrow \ell^\infty(G, M)\) is the identity map \(M \to M\). Here \(\ell^\infty(G, M)\) is endowed with the diagonal \(G\)-action: \(\gamma_t(f)(r) = \gamma_t(f(t^{-1}r))\), where \(\gamma\) denotes the \(G\)-action on \(M\).

An action \(\alpha\) of \(G\) on a \(C^*\)-algebra \(A\) is \(AD\)-amenable if the corresponding double dual \(W^*\)-action \(\alpha^{**}\) on \(A^{**}\) is \(W^*\)AD-amenable.

Let us recall some basic examples of amenable actions.

Example 3.2. The translation \(G\)-action on itself, viewed as a \(G\)-action on the \(C^*\)-algebra \(C_0(G)\), is always \(AD\)-amenable (this action is even proper). By definition, this means that the translation action on \(C_0(G)^{**} \cong \ell^\infty(G)\) is always \(W^*AD\)-amenable as a \(W^*\)-action. However the translation action on \(\ell^\infty(G)\) is \(AD\)-amenable as a \(C^*\)-action if and only if \(G\) is exact, see [14] Theorem 5.1.7.

More generally, if \(M\) is a \(G\)-\(W^*\)-algebra, the \(G\)-\(W^*\)-algebra \(\ell^\infty(G, M)\) endowed with the diagonal \(G\)-action is always \(W^*AD\)-amenable because we have a canonical \(G\)-equivariant unital embedding \(\ell^\infty(G) \hookrightarrow Z\ell^\infty(G, M) = \ell^\infty(G, Z(M))\) (see [8, Corollary 3.8]). As before, here \(\ell^\infty(G)\) carries the translation \(G\)-action.

Before we proceed, let us highlight some of the most important characterisations of \(AD\)-amenability obtained by Anantharaman-Delaroche in her papers [7,8,10].

Theorem 3.3 (Anantharaman-Delaroche). The following are equivalent for a global action \(\gamma\) of a group \(G\) on a \(W^*\)-algebra \(M\):

(i) \(\gamma\) is \(W^*\)AD-amenable;

(ii) the restriction of \(\gamma\) to the center \(Z(M)\) is \(W^*\)AD-amenable, that is, there is a \(G\)-equivariant norm-one projection \(\ell^\infty(G, Z(M)) \to Z(M)\);

(iii) there is a net \(\langle a_i \rangle_{i \in I} : G \to Z(M)\) of finitely supported functions with \(\langle a_i \mid a_i \rangle_{2} := \sum_{g \in G} a_i(g)\overline{a_i(g)} \leq 1\) for all \(i\) and \(\langle a_i \mid \gamma_g(a_i) \rangle_{2} \to 1\) ultraweakly for all \(g \in G\).

Moreover, if \(M\) is injective as a \(W^*\)-algebra, then the above are also equivalent to

(iv) the \(W^*\)-crossed product \(M \rtimes G\) is an injective \(W^*\)-algebra.
If $\alpha$ is an AD-amenable action of $G$ on a $C^*$-algebra $A$, then the full and reduced $C^*$-crossed products coincide, that is, $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$. And if $A$ is nuclear, then $\alpha$ is AD-amenable if and only if $A \rtimes_{\alpha,r} G$ is a nuclear $C^*$-algebra.

Let us also remark that for an action on a commutative $C^*$-algebra $A = C_0(X)$, its AD-amenability is equivalent to amenability of the associated transformation groupoid $X \rtimes G$ in the sense of Anantharaman-Delaroche and Renault, see [11]. Moreover, the AD-amenability in this case is equivalent to the existence of a net $\{a_i: G \to Z(A) = A\}_{i \in I}$ with the same properties as in (iii) above, except that the ultraweak convergence in (iii) can be strengthened to the convergence with respect to the strict topology on $A \subseteq M(A)$ (the multiplier algebra), see [10, Théoréme 4.9].

This cannot be expected – and indeed it is not true – for noncommutative algebras because simple unital $C^*$-algebras can carry AD-amenable actions of non-amenable groups, see Remark 3.6.

We are now ready to introduce the notion of amenability for partial actions on $C^*$-algebras and $W^*$-algebras:

**Definition 3.4.** We say that a partial action of a group $G$ on a $W^*$-algebra $M$ is $W^*$AD-amenable if its enveloping $W^*$-action (provided by Proposition 2.7) is $W^*$AD-amenable.

We say that a partial action of $G$ on a $C^*$-algebra $A$ is AD-amenable if the induced $W^*$-partial action on $A''$ is $W^*$AD-amenable.

Of course, a global action is AD-amenable if and only if it is AD-amenable as a partial action. Before we give some proper examples of amenable partial actions, we observe the following general fact:

**Proposition 3.5.** A $W^*$-partial action $(M, \gamma)$ is $W^*$AD-amenable if and only if its restriction to the center $Z(M)$ is $W^*$AD-amenable.

**Proof.** This follows directly from the definition, Proposition 2.8 and Theorem 3.3. \qed

**Remark 3.6.** The above result does not hold for partial actions on $C^*$-algebras, not even for global actions. Indeed, the results of Suzuki in [33] show that every exact group admits an AD-amenable action on a unital simple (and nuclear) $C^*$-algebra. Such an algebra has trivial center and a trivial global action can only be AD-amenable if the group is amenable.

**Example 3.7.** The “trivial” partial action of $G$ on $A = C$ appearing in Example 2.5 is AD-amenable, both in $C^*$- and $W^*$-sense. This is because $A = A''$ has as its enveloping $W^*$-action the global translation $G$-action on $\ell^\infty(G)$ as explained in Example 2.5 and this $W^*$-action is $W^*$AD-amenable. In the same way, we can consider the partial action on a $W^*$-algebra $M$ as in Example 2.5 with all domain ideals $M_g = 0$ except for $M_e = M$. This partial action is always $W^*$AD-amenable because its enveloping $W^*$-action is the translation action on $\ell^\infty(G,M)$, which is $W^*$-amenable by Example 3.2. For the same reason, any $C^*$-algebra $A$ endowed with the “trivial” partial action (in which all the domain ideals are $A_g = 0$ except for $A_e = A$) is always AD-amenable because then $A''$ carries the “trivial” partial $G$-action which is $W^*$AD-amenable.

**Example 3.8.** More generally, the following holds: take an amenable subgroup $H \subseteq G$ acting (globally) on a $C^*$-algebra $A$ (or on a $W^*$-algebra $M$) and “extend” this
to a partial $G$-action on $A$ “by zero” in the sense that $A_h = A$ for $h \in H$, $A_g = 0$ for $g \in G \setminus H$ and $\alpha_g : A_{g^{-1}} \to A_g$ acts via the original $H$-action for $g \in H$ and by zero otherwise. This partial action (which is global only if $H = G$) is always AD-amenable. Indeed, the canonical action of $G$ on $\ell^\infty(G/H)$, $\nu_t(f)(sH) = f(t^{-1}sH)$, plays an important role here. This $W^*$-action is $W^*$AD-amenable if (and only if) $H$ is amenable. Indeed, $\ell^\infty(G/H) = C_0(G/H)^\pi$ and the crossed product $C_0(G/H) \rtimes_t G$ is Morita equivalent to $C^*_\tau(H)$ by Green’s imprimitivity theorem.

To see that the partial action of $G$ defined above is AD-amenable, it is enough to consider the von Neumann algebraic situation. For this, let us write $\gamma$ for the action of $H$ on a $W^*$-algebra $M$ and name $\bar{\gamma}$ its extension to $G$.

To prove amenability of this partial $G$-action, we give an explicit description of its $W^*$-enveloping action. Consider the $W^*$-subalgebra of $\ell^\infty(G, M)$

$$N := \{ f \in \ell^\infty(G, M) : f(s) = \gamma_{s^{-1}}t(f(t)) \text{ if } sH = tH \}.$$  

This subalgebra is invariant under the action $\tau$ of $G$ on $\ell^\infty(G, M)$ given by $\tau_t(f)(s) = f(t^{-1}s)$. We name $\delta$ the restriction of $\tau$ to $N$. In order to view $M$ as a $W^*$-ideal of $N$ in such a way that $\delta$ is the $W^*$-globalisation of $\bar{\gamma}$, we consider the map $\iota : M \to N$ given by

$$\iota(a)(s) = \begin{cases} \gamma_{s^{-1}}(a) & \text{if } s \in H \\ 0 & \text{if } s \notin H. \end{cases}$$

Note that in case $M = C$, we have $N = \ell^\infty(G/H)$ and $\tau = \nu$. In any case, we may view $\ell^\infty(G/H)$ as a unital $\delta$-invariant subalgebra of $Z(N)$ by considering the inclusion $\kappa : \ell^\infty(G/H) \to N$, $\kappa(f)(t) = f(tH)$. Moreover, the restriction of $\delta$ to $\ell^\infty(G/H)$ is $\nu$, from which it follows that $\delta$ is $W^*$AD-amenable, hence so is $\bar{\gamma}$.

**Example 3.9.** If $(M, \gamma)$ is an AD-amenable partial $W^*$-action of $G$ and $H \subseteq G$ is any subgroup, then the restriction of the partial $G$-action on $M$ to $H$, namely, $\gamma|_H = \{ \gamma_h : M_{h^{-1}} \to M_h \}_{h \in H}$ is also AD-amenable. A similar assertion holds for $C^*$-partial actions. Indeed, it is clearly enough to check this for $W^*$-partial actions. This is known to hold for global $W^*$-actions (it follows trivially from (iii) in Theorem 3.3). Now for a general $W^*$-partial action $(M, \gamma)$ of $G$, take its globalisation $W^*$-action $(N, \sigma)$. For simplicity we identify $M$ as a $W^*$-ideal of $N$. Let $H \cdot M = \sum_{t \in H} \sigma_t(M)$ be the $H$-linear orbit of $M$ in $N$. Notice that this is an ideal of $N$ and its $w^*$-closure $N_H := \overline{H \cdot M}^{w^*}$ is an $H$-invariant $W^*$-ideal of $N$ which can be viewed as an $H$-globalisation of $\gamma|_H$. If $(M, \gamma)$ is $W^*$AD-amenable, then by definition $(N, \sigma)$ is $W^*$AD-amenable, and then so is $(N, \sigma|_H)$ and hence also every $H$-invariant $W^*$-ideal, like $(N_H, \sigma|_H)$. Therefore $(M, \gamma|_H)$ is $W^*$AD-amenable.

Next we look at restrictions of partial actions to ideals and prove that amenability behaves nicely also in this direction.

**Proposition 3.10.** The restriction of a $W^*$AD-amenable $W^*$-partial action of a group to a $W^*$-ideal is again $W^*$AD-amenable. Moreover, the analogous statement holds for AD-amenity on $C^*$-algebras, that is, AD-amenity is also preserved by restrictions.

**Proof.** First we deal with $W^*$-partial actions. Let $\gamma$ be a $W^*$AD-amenable $W^*$-partial action of the group $G$ on $M$ and let $J$ be a $W^*$-ideal of $M$. We know that $M$ can be viewed as a $W^*$-ideal of a $W^*$-algebra $N$ carrying a $W^*$AD-amenable
$W^*$-global action $\sigma$ of $G$ with $\sigma|_M = \gamma$. Then $J$ is a $W^*$-ideal of $N$ and the $w^*$-closure of $\sum_{t \in G} \sigma_t(J)$, denoted by $[J]$, is a $\sigma$-invariant $W^*$-ideal of $N$. Moreover, $\sigma|_{[J]}$ is the $W^*$-enveloping action of $\gamma|_J$ because $\sigma|_{[J]}|_J = \sigma|_J = \sigma|_M|_J = \gamma|_J$ and it is also $W^*$AD-amenable because it is a restriction of a global $W^*$AD-amenable $W^*$-action to a $G$-invariant $W^*$-ideal.

Now let $\beta$ be an AD-amenable $C^*$-partial action of $G$ on $B$ and let $A$ be a $C^*$-ideal of $B$. Then we may view $A''$ as the $w^*$-closure of $A$ in $B''$. Note that $(\beta|_A)'''$ is the unique $W^*$-partial action of $G$ on $A''$ extending $\beta|_A$. But $\beta''|_{A''}$ is a $W^*$-partial action such that $\beta''|_{A''}|_A = \beta''|_A = \beta''|_B|_A = \beta|_A$. Thus $\beta''|_{A''} = (\beta|_A)'''$. By the previous paragraph, $\beta|_A$ is AD-amenable if $\beta$ is.

\begin{proof}
We view $A''$ as the $w^*$-closure of $A$ in $B''$, thus $A''$ is a $W^*$-ideal of $B''$. In the proof of Proposition 3.10 we showed that $\beta''|_{A''} = \alpha''$. Thus, to show that $\beta''$ is a $W^*$-enveloping action of $\alpha''$, it suffices to prove that $B''$ is the $w^*$-closure of $J_0 := \sum_{t \in G} \beta''_t(A''_0)$; let us write $J$ for this closure. Note that $\sum_{t \in G} \beta_t(A) \subseteq J_0 \subseteq J$ and, taking norm closure, this implies $B \subseteq J$. Now taking $w^*$-closure we get $J = B''$. The rest of the proof follows directly from the definition of AD-amenability for partial actions. \hfill \Box
\end{proof}

4. Morita equivalence of partial actions

Many $C^*$-partial actions do not admit a $C^*$-enveloping action, but every $C^*$-partial action has a Morita enveloping action, as defined in [2], which is unique up to Morita equivalence of actions. It is therefore important to see how amenability behaves in terms of Morita equivalences.

Equivalences of partial actions are defined in [2]. We shortly recall the definition: two partial actions $\alpha = \{\alpha_t : A_t \to A_t\}$ and $\beta = \{\beta_t : B_t \to B_t\}$ of $G$ on $C^*$-algebras $A$ and $B$ are equivalent if there exist an equivalence Hilbert $A$-$B$-bimodule $X$ carrying a (set theoretic) partial action $\gamma = \{\gamma_t : X_t \to X_t\}$ of $G$ by linear maps on $X$ such that $X_t \subseteq X$ are Hilbert $A$-$B$-submodules implementing an equivalence between the domain ideals $A_t$ and $B_t$, that is, the images of $X_t$ by the left and right inner products are contained and generate $A_t$ and $B_t$ as $C^*$-algebras, and the usual compatibility between the actions holds: $\langle \gamma_t(x)|\gamma_t(y) \rangle_B = \beta_t(\langle x|y \rangle_B)$ and $\langle \gamma_t(x)|\gamma_t(y) \rangle_B = \alpha_t(\langle x|y \rangle_B)$ for all $x,y \in X_{t-1}$. Using $W^*$-equivalebcoequivalences (see the Appendix) one defines equivalences between $W^*$-partial actions in a similar way.

Both notions of equivalences can be conveniently described in terms of linking algebras: given an equivalence bimodule $X$ as above, one considers its linking $C^*$-algebra $L$. This carries a partial action of $G$ where the domain ideal $L_t$ is (isomorphic to) the linking algebra of $X_t$. If $X$ is an equivalence between $W^*$-partial actions, then $L$ is a $W^*$-algebra carrying a $W^*$-action encoding the $W^*$-equivalence.

\begin{proposition}
Let $\mu$ and $\nu$ be $W^*$-Morita equivalent $W^*$-partial actions of a group $G$ on the algebras $M$ and $N$, respectively. Then the restrictions $\sigma := \mu|_{Z(M)}$ and $\tau := \nu|_{Z(N)}$ are isomorphic (as $W^*$-partial actions).
\end{proposition}
AD-amenability of Fell bundles

5. AD-amenability of Fell bundles

One of our goals in this paper is to extend Anantharaman-Delaroche’s notion of amenability to Fell bundles over discrete groups. For this we need some preparation because, as the case of partial actions already indicates, the definition of AD-amenability requires going to the $W^*$-setting.

Proof. Let $X$ be a $W^*$-Morita equivalence bimodule between $M$ and $N$ equipped with a $W^*$-partial action $\gamma$ of $G$ inducing $\mu$ and $\nu$. More precisely:

- $\gamma = \{(X_t)_{t \in G}, \{\gamma_t \}_{t \in G}\}$ is a set theoretic partial action on $W^*$-ideals by linear isometries.
- For every $t \in G$, $M_t(N_t, \text{respectively})$ is the $w^*$-closure of the space spanned by $M(X_t, X_t)$ ($(X_t, X_t)_N$, respectively).
- For every $t \in G$ and $x,y \in X_{t^{-1}}$, $M(\gamma_t(x), \gamma_t(y)) = \mu_t(M(x,y))$ and $\langle \gamma_t(x), \gamma_t(y) \rangle_N = \nu_t((x,y)_N)$

The conditions above imply that, for all $t \in G$, $x \in X_{t^{-1}}$, and $a \in M_{t^{-1}}$, $\gamma_t(ax) = \mu_t(a)\gamma_t(x)$. The same holds for $\gamma$ and $\nu$.

Proposition 4.1 yields a unique $W^*$-isomorphism $\pi : Z(N) \to Z(M)$ such that $ax = \pi(a)x$ for all $a \in Z(N)$ and $x \in X$. To finish the proof we check that the isomorphism $\pi : Z(N) \to Z(M)$ above intertwines the partial actions $\sigma$ and $\tau$. First of all, if $p_t$ and $q_t$ are the units of $M_t$ and $N_t$ (respectively), then for all $x \in X$:

$$p_t x = (p_t x) q_t = p_t (x q_t) = x q_t.$$  

Hence $\pi(q_t) = p_t$ and $\pi(Z(N)_t) = Z(M)_t$.

Now fix $t \in G$ and $a \in Z(N)_{t^{-1}}$. For every $x \in X$ we have

$$\pi(\pi_t(a)x) = x \pi_t(a) = x q_t \pi_t(a) = \gamma_t(\gamma_{t^{-1}}(x q_t)a)$$

$$= \gamma_t(\pi(a)\gamma_{t^{-1}}(p_t x)) = \mu_t(\pi(a))p_t x = \mu_t(\pi(a)) x.$$  

This implies $\pi(\mu_t(a)) = \nu_t(\pi(a))$ and the proof is complete.  

As a consequence we derive the following important result:

Proposition 4.2. AD-amenability is preserved by Morita equivalence of partial actions, both in $C^*$- and $W^*$-contexts.

Proof. A $C^*$-equivalence $A-B$-bimodule $X$ induces a $W^*$-equivalence $A''-B''$-bimodule $X''$, so it is enough to deal with the $W^*$-case. But this follows directly as a combination of Propositions 4.1 and 3.5.

Remark 4.3. The above result applies in particular to global actions and shows that AD-amenability is invariant under Morita equivalence of group actions. We believe that this is known for specialists but we could not find a reference.

Corollary 4.4. A $C^*$-partial action is AD-amenable if and only if one (hence all) of its Morita enveloping actions is AD-amenable.

Proof. Let $\alpha$ be a $C^*$-partial action of a group and let $\beta$ be one of its Morita enveloping actions. This means that $\alpha$ is Morita equivalent to a restriction $\gamma$ of $\beta$ and $\beta$ is the $C^*$-enveloping action of $\gamma$. By Propositions 4.2 and 3.11 $\alpha$ is AD-amenable if and only if $\gamma$ is AD-amenable if and only if $\beta$ is AD-amenable.
5.1. W*-enveloping Fell bundles. Given a Fell bundle $\mathcal{B}$ over a group $G$, we want to turn the bundle of biduals $\mathcal{B}'' := \{B''_r\}_{r \in G}$ into a Fell bundle in such a way that $\mathcal{B}$ becomes a Fell subbundle of $\mathcal{B}''$ and there is a certain continuity of the operations with respect to the $w^*$-topology. This requires to equip every fibre $B''_r$ with a Hilbert bimodule structure over $B''_e$ (or a ternary $W^*$-ring structure) extending that of $B_r$.

The machinery described here is not new: biduals of Hilbert modules are known to be Hilbert $W^*$-modules (see for instance [13], biduals of ternary $W^*$-rings are described in [35], and biduals of Fell bundles over inverse semigroups are already described in [18 Section 3], although there only saturated Fell bundles are considered. For the convenience of the reader and to make this article as self-contained as possible, we provide the complete constructions here.

**Lemma 5.1.** Let $\mathcal{B}$ be a Fell bundle over a group $G$. For every nondegenerate representation $\pi: \mathcal{B}_e \to \mathcal{L}(H)$ there exists a nondegenerate representation $\psi: \mathcal{B} \to \mathcal{L}(K)$ such that $\pi$ is a sub-representation of $\psi|_{\mathcal{B}_e}$.

**Proof.** Define $H_t := B_t \otimes_a H$, where we view $B_t$ as a Hilbert $B_e$-module. We want to construct, for every $s, t \in G$ and $b \in B_s$, a linear bounded map $\psi_{b,t}: B_t \otimes_a H \to B_{st} \otimes_a H$ such that $\psi_{b,t}(a \otimes h) = ba \otimes h$. For this it suffices to show that, given $\sum_{j=1}^n a_j \otimes h_j \in B_t \otimes_a H$, we have $\|\sum_{j=1}^n a_j \otimes h_j\| \leq \|\sum_{j=1}^n a_j \otimes h_j\|^2$. Viewing $B_t$ as a left $B_e$-Hilbert module we get

\[
\|\sum_{j=1}^n a_j \otimes h_j\|^2 = \sum_{j=1}^n \|h_j, a_j^* ba_jh_j\|^2 = \sum_{j=1}^n \|b^* b\|^{1/2} a_j \otimes h_j\|^2 \leq \|b\|^2 \|a_j \otimes h_j\|.
\]

The computations above show that $\psi_{b,t}$ is well defined and $\|\psi_{b,t}\| \leq \|b\|$.

Define $K$ as the $\ell_2$-direct sum $\bigoplus_{t \in G} H_t$. We claim that there exists a representation $\psi: \mathcal{B} \to \mathcal{L}(K)$ such that, for all $s, t \in G$, $b \in B_s$ and $f \in K$, $\psi(b)(f)(t) = \psi_{b,s^{-1}t}(f(s^{-1}t))$. First we show that $\psi$ is well defined. Take $f \in K$, $b \in B_s$ and a finite set $\lambda \subseteq G$. Then $\sum_{t \in \lambda} \|\psi_{b,s^{-1}t}(f(s^{-1}t))\|^2 \leq \|b\|^2 \sum_{t \in \lambda} \|f(s^{-1}t)\|^2 \leq \|b\|^2 \|f\|^2$. This shows that $t \mapsto \psi_{b,s^{-1}t}(f(s^{-1}t))$ is square summable in norm.

The facts that $\psi(ab) = \psi(a)\psi(b)$ and $\psi(b^*) = \psi(b^*)$ follow from the facts that $\psi_{b_t,s^*} = \psi_{b^*,st}$ and $\psi_{a,at}\psi_{b,t} = \psi_{ab,t}$ for all $s, t \in G$, $a \in \mathcal{B}$ and $b \in B_t$. These last identities are straightforward to prove and are left to the reader.

The natural identification of $H$ with $B_e \otimes_a H$ provides an inclusion $\iota: H \to K$. Under this inclusion we see that $\psi(B_e)H = H$ and, for all $a, b \in B_e$ and $h \in H$:

\[
\psi(b)(\iota(\pi(a)h)) = ba \otimes h = \iota(\pi(b)\pi(a)h).
\]

This clearly implies that $\pi$ is a sub-representation of $\psi$. Finally, $\psi$ is nondegenerate because for every approximate unit $(e_j)_{j \in J}$ of $B_e$ and $a \otimes h \in H_t$ we have $\lim_j \psi(e_j)a \otimes h = e_ja \otimes h = a \otimes h$.

**Theorem 5.2.** Let $\mathcal{B} = \{B_t\}_{t \in G}$ be a Fell bundle. Then the bundle of biduals $\mathcal{B}'' := \{B''_r\}_{r \in G}$ has a unique Fell bundle structure extending that of $\mathcal{B}$ and such that, for every $r, s \in G$ and $a \in B_r$, the functions $B''_a \to B''_{rs}, b \mapsto ab$, and $B''_a \to B''_s, b \mapsto b^*$, are $w^*$-continuous.
Proof. Let \( \psi : \mathcal{B} \to \mathcal{L}(H) \) be a nondegenerate representation such that \( \psi|_{\mathcal{B}_e} \) contains the universal representation of \( \mathcal{B}_e \) as a sub-representation. Then we view the bidual \( \mathcal{B}'' \) as the weak closure (or the bicommutant) of \( \psi|_{\mathcal{B}_e} \subseteq \mathcal{L}(H) \). For every \( t \in G \) we view the bidual \( \mathcal{B}''_t \) of the ideal \( \mathcal{I}_t := \mathcal{B}_t \mathcal{B}_t^* \subseteq \mathcal{B}_e \) as the weak closure of \( \psi|_{\mathcal{I}_t} \).

We claim that there exists a fibre-wise linear and \( W^* \)-continuous extension \( \psi'' : \mathcal{B}'' \to \mathcal{L}(H) \). For this we view the fibre \( \mathcal{B}_t \) as a \( \mathcal{I}_t - \mathcal{I}_{t-1} \)-equivalence bimodule with the operations inherited from \( \mathcal{B} \). Let \( 1_t \) be the unit of \( \mathcal{I}''_t \) and \( H_t := 1_t \mathcal{H}_t \). Note that \( \mathcal{B}_t \mathcal{B}_t^* \mathcal{B}_t \mathcal{B}_t^* \) is the unique element of \( \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \mathcal{I}''_t \mathcal{H}_t \). By Corollary \ref{cor:psi_unique}, \( \psi|_{\mathcal{B}_t} \) has a unique \( W^* \)-continuous extension \( \psi''|_{\mathcal{B}_t} : \mathcal{B}_t'' \to \mathcal{L}(H) \), which is faithful. The map \( \psi'' \) is then given by \( \psi''|_{\mathcal{B}_t} := \psi''|_{\mathcal{B}_t} \) for every \( t \in G \).

Notice that \( \psi'' \) is faithful on each fiber. Indeed, viewing \( \psi''|_{\mathcal{B}_t} \) as a map with image in \( \mathcal{L}(\mathcal{H}_t, \mathcal{H}_t) \), it follows that \( (\psi''|_{\mathcal{B}_t})^\gamma = (\psi|_{\mathcal{I}_t})^\gamma \) is faithful because \( (\psi|_{\mathcal{I}_t})^\gamma \) is faithful. Hence, by Proposition \ref{prop:faithful}, \( \psi''|_{\mathcal{B}_t} \) is faithful.

Using \( W^* \)-density arguments we get
\[
\psi''(\mathcal{B}_t') \psi''(\mathcal{B}_t') \subseteq \psi''(\mathcal{B}_t'') \quad \text{and} \quad \psi''(\mathcal{B}_t')^* = \psi''(\mathcal{B}_t'').
\]
Thus we may define the multiplication in such a way that, for \( x \in \mathcal{B}_t'' \) and \( y \in \mathcal{B}_t'' \), \( xy \) is the unique element of \( \mathcal{B}_t'' \) such that \( \psi''(xy) = \psi''(x) \psi''(y) \). The involution is determined by the condition \( \psi''(x^*) = \psi''(x)^* \) and the norm is \( \|x\| := \|\psi''(x)\|_2 \). These operations clearly extend those of \( \mathcal{B} \) and \( \mathcal{B}'' \) is a \( W^* \)-bundle. Finally, \( (\mathcal{B}_t')^* \mathcal{B}_t'' \) is a \( W^* \)-ideal because \( \psi''((\mathcal{B}_t')^* \mathcal{B}_t'') = \mathcal{I}''_{t-1} \).

\[ \square \]

Remark 5.3. Let \( \alpha \) be a partial action of \( G \) on a \( C^* \)-algebra \( A \). Then \( (\mathcal{B}_\alpha)'' \) is canonically \( W^* \)-isomorphic to \( \mathcal{B}_\alpha'' \).

Inspired by the previous result we introduce the following definition, which is the natural \( W^* \)-analogue of Fell bundles – also called \( C^* \)-algebraic bundles in \cite{20}.

**Definition 5.4.** A \( W^* \)-Fell bundle (or \( W^* \)-algebraic bundle) over the group \( G \) is a Fell bundle \( \mathcal{M} = \{M_t\}_{t \in G} \) such that each \( M_t \) is isometrically isomorphic to the dual of a Banach space and, for every \( s, t \in G \) and \( a \in M_s \), the functions \( M_t \to M_t, b \mapsto b^a \), and \( M_t \to M_{st}, b \mapsto ab \), are \( W^* \)-continuous.

By \cite{25} the predual of each fibre \( M_t \) is unique because \( M_t \) is a \( M_t \)-\( M_t \)-Hilbert bimodule (not necessarily full) with respect to the canonical operations coming from the product and involution of \( \mathcal{M} \).

5.2. Central partial actions of \( W^* \)-Fell bundles. Take a \( W^* \)-Fell bundle \( \mathcal{M} \) over a group \( G \). For every \( t \in G \) we define \( I_t \) as the \( W^* \)-algebra generated by \( M_t M_t^* \) in \( M_e \). Note that \( I_t \) is in fact a \( W^* \)-ideal of \( M_e \). The fiber \( M_t \) has a natural \( W^* \)-equivalence \( I_t - I_{t-1} \)-bimodule structure with the multiplication of \( \mathcal{M} \) defining the left and right actions and the inner products \( i_t(x, y) := xy^* + (x, y)_{t-1} := x^*y \). Then Proposition \ref{prop:partial_action} provides an isomorphism \( \sigma_t : Z(I_{t-1}) \to Z(I_t) \). We claim that \( \sigma := \{\sigma_t : Z(I_{t-1}) \to Z(I_t)\}_{t \in G} \) is a \( W^* \)-partial action of \( G \) on \( Z(M_e) \). To prove this it suffices to show that \( \sigma \) is a set theoretic partial action. To simplify the notation we write \( Z_t \) instead of \( Z(I_t) \).
It is clear that $I_e = M_e$. Moreover, $\sigma_e$ is the isomorphism corresponding to the $W^*$-algebra $M_e$ viewed as the identity $W^*$-equivalence $M_e-M_e$-bimodule, hence $\sigma_e$ is the identity of $Z_e$.

Let $s = \text{span}^w M_1 p_{t-1} p_s M^*_1$.

$$\leq \text{span}^w M_1 M^*_1 M^*_s M^*_t \subseteq \text{span}^w M_1 M^*_1 M^*_s M^*_t M^*_1 \subseteq I_t \cap I_s$$

Thus $\sigma_t(p_{t-1} p_s p_{t-1} p_s M^*_1) = \text{span}^w M_1 M^*_1 M^*_s M^*_t p_t$.

$$\text{span}^w M_1 M^*_1 M^*_s M^*_t \subseteq \text{span}^w M_1 M^*_1 M^*_s M^*_t M^*_1 \subseteq I_t \cap I_s$$

Now take $z \in Z_{t-1} \cap Z_{t-1}$. We already know that $\sigma_z(x) \in Z_{t-1} \cap Z_{s-1}$ and $\sigma_x(z) \in Z_{t-1} \cap Z_s$. Also $\sigma_{st}(x) \in Z_{st} \cap Z_s$. We can write $p_s$ as a $w^*$-limit of the form $p_s = \lim_i \sum_{j=1}^{n_i} u_{i,j} v^*_{i,j}$ with $u_{i,j}, v_{i,j} \in M_s$. Then, for all $z \in M_{st}$:

$$\sigma_s(\sigma_t(x)) p_s z = \sigma_s(\sigma_t(x)) \lim_i \sum_{j=1}^{n_i} u_{i,j} v^*_{i,j} z = \lim_i \sum_{j=1}^{n_i} \sigma_s(\sigma_t(x)) u_{i,j} v^*_{i,j} z$$

$$= \lim_i \sum_{j=1}^{n_i} u_{i,j} \sigma_s(\sigma_t(x)) v^*_{i,j} z = \lim_i \sum_{j=1}^{n_i} u_{i,j} v^*_{i,j} x = p_s x = \sigma_{st}(x) p_s z = \sigma_{st}(z) x.$$

This implies $\sigma_{st}(x) = \sigma_s(\sigma_t(x))$.

**Definition 5.5.** Let $\mathcal{M}$ be a $W^*$-Fell bundle over a group $G$. The central partial action of $\mathcal{M}$ is the $W^*$-partial action $\sigma$ of $G$ on $Z(M_e)$ constructed above.

**Example 5.6.** Let $\gamma = \{\{M_t\}_{t \in G}, \{\gamma_t\}_{t \in G}\}$ be a $W^*$-partial action of a group $G$ on a $W^*$-algebra $M$. If $\mathcal{M}$ is the semidirect product bundle of $\gamma$, which is a $W^*$-Fell bundle, then the central partial action of $\mathcal{M}$ is the restriction of $\gamma$ to $Z(M)$.

To prove the claim above note that the $W^*$-ideals of $I_t = M \delta_e$ generated by $(M \delta_e)(M \delta_e)^* = \gamma_t(\gamma_{t-1} M \delta_e) \delta_e = M \delta_e$ are just $M_i$ seen as a subalgebra of $M = M \delta_e$. Then the domains of $\sigma$ and $\gamma|_{Z(M)}$ agree. If $x \in Z(M_{t-1})$ and $y \in M_t$, then

$$\gamma_t(x) \delta_e y \delta_i = \gamma_t(x) y \delta_i = y \gamma_t(x) \delta_i = \gamma_t(y \gamma_t(x) \delta_i) = \gamma_t(y) \delta_i \delta_i = \sigma_t(x) \delta_i y \delta_i$$

and this implies $\gamma_t(x) = \sigma_t(x)$ (because we identify $x \in M_e$ with $x \delta_e \in M \delta_e$).

### 5.3. Cross-sectional $W^*$-algebras of $W^*$-Fell bundles

To a $W^*$-Fell bundle $\mathcal{M}$ one can naturally assign a cross-sectional $W^*$-algebra $W^*_c(\mathcal{M})$ as follows: the usual Hilbert $M_e$-module $\ell^2(M)$ is not suitable here because it might not be a $W^*$-module, that is, it is possibly not self-dual. We look at its self-dual completion that can be concretely described as follows. Let $\ell^2_w(\mathcal{M})$ be the space of sections $\xi: G \to \mathcal{M}$ for which the net of finite sums $\sum_{x \in F} \xi(t)^* x(t)$ (for $G \subseteq F$ finite) is bounded; since this net is increasing and consists of positive elements, it $w^*$-converges to some element $\langle \xi | \xi \rangle_{M_e} := \sum_{x \in G} \xi(t)^* x(t) \in M_e$. The space $\ell^2_w(\mathcal{M})$ is then a right $W^*$-Hilbert $M_e$-module when endowed with right $M_e$-action $(\xi \cdot b)(t) := \xi(t) \cdot b$ and inner product $\langle \xi | \eta \rangle_{M_e} := \sum_{x \in G} \xi(t)^* \eta(t)$, the limit of this sum being with respect to the $w^*$-topology, for all $\xi, \eta \in \ell^2_w(\mathcal{M})$. 
Next we define the left regular representation of \( \mathcal{M} \). This is done as in the \( C^* \)-case, except that we now act on \( \ell^2_{\omega,*}(\mathcal{M}) \). More precisely, for each \( t \in G \) we define the map \( \Lambda_t: M_t \to \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M})) \) by \( \Lambda_t(a)\xi(s) := a \cdot \xi(ts) \) (the multiplication performed in \( \mathcal{M} \)) for all \( s \in G \), \( a \in M_t \) and \( \xi \in \ell^2_{\omega,*}(\mathcal{M}) \). As in the \( C^* \)-setting, a routine argument shows that \( \Lambda_t(a) \) is a well-defined adjointable operator with \( \Lambda_t(a)^* = \Lambda_{t^{-1}}(a^*) \) and that \( \Lambda = (\Lambda_t)_{t \in G} \) is a representation of \( \mathcal{M} \). Note that \( \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M})) \) is a \( W^* \)-algebra, see for example [31] Proposition 3.10.

**Definition 5.7.** The cross-sectional \( W^* \)-algebra of \( \mathcal{M} \) is the \( W^* \)-subalgebra \( W^*_r(\mathcal{M}) \) of \( \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M})) \) generated by the image of its regular representation \( \Lambda \).

The linear span of the image of \( \Lambda \) is already a \( * \)-subalgebra, so that \( W^*_r(\mathcal{M}) \) is just the \( w^* \)-closure of that subalgebra. We also observe that the cross-sectional \( C^* \)-algebra \( C^*_r(\mathcal{M}) \) embeds as a \( w^* \)-dense \( C^* \)-subalgebra of \( W^*_r(\mathcal{M}) \). Moreover, since \( \ell^2_{\omega,*}(\mathcal{M}) \) is the self-dual completion of \( \ell^2(\mathcal{M}) \), every adjointable operator on \( \ell^2(\mathcal{M}) \) extends to an adjointable operator on \( \ell^2_{\omega,*}(\mathcal{M}) \) and this gives a \( C^* \)-embedding \( \mathcal{L}(\ell^2(\mathcal{M})) \hookrightarrow \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M})) \) that restricts to the embedding \( C^*_r(\mathcal{M}) \hookrightarrow W^*_r(\mathcal{M}) \).

The (reduced) \( W^* \)-algebra of a \( W^* \)-Fell bundle \( \mathcal{M} \) is exactly the \( W^* \)-counterpart of the reduced \( C^* \)-algebra as defined by Exel and Ng in [26].

**Proposition 5.8.** Let \( \mathcal{M} \) be a \( W^* \)-Fell bundle over a group \( G \) and \( \pi: \mathrm{M}_{\epsilon} \to \mathcal{L}(H) \) be a weak*-continuous representation. Then the map

\[
\Lambda_\pi : \mathcal{M} \to \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M}) \otimes_\pi H), \ \ b \mapsto \Lambda(b) \otimes \mathrm{id}
\]

is a representation which is \( w^* \)-continuous on each fiber. The integrated form \( \tilde{\Lambda}_\pi \) factors through a representation of \( C^*_r(\mathcal{M}) \) that can be extended to a \( w^* \)-continuous representation \( \tilde{\Lambda}_{\omega,*} \) of \( W^*_r(\mathcal{M}) \) in a unique way. Moreover, \( \tilde{\Lambda}_{\omega,*} \) is unital and \( \tilde{\Lambda}_{\omega,*} (W^*_r(\mathcal{M})) = \tilde{\Lambda}_{\omega,*} (\mathcal{M})'' \) (the bicommutant). If \( \pi \) is injective then so is \( \tilde{\Lambda}_{\omega,*} \).

**Proof.** Consider the map \( \rho : \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M})) \to \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M}) \otimes_\pi H), \ \rho(R) = R \otimes \mathrm{id}, \) of Lemma \[A.2\]. Then \( \Lambda_{\epsilon} := \rho \circ \Lambda : \mathcal{M} \to \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M}) \otimes_\pi H) \) is clearly a representation that, when restricted to the closed unit ball of a fiber, is \( w^* \)-continuous by Lemma \[A.2\]. Hence \( \Lambda_{\epsilon} \) is a representation which is \( w^* \)-continuous on each fiber. In case \( \pi \) is faithful then so is \( \rho \) and hence \( \Lambda_{\epsilon} \) is faithful if \( \pi \) is.

Note that \( \ell^2_{\omega,*}(\mathcal{M}) \otimes_\pi H = \ell^2_\omega(\mathcal{M}) \otimes_\pi H. \) Thus we may very well think of \( \Lambda_{\epsilon} : \mathcal{M} \to \mathcal{L}(\ell^2_\omega(\mathcal{M}) \otimes_\pi H) \) as the composition of the \( C^* \)-regular representation \( \mathcal{M} \to \mathcal{L}(\ell^2(\mathcal{M})) \) with \( C^*_r(\mathcal{M}) \subset \mathcal{L}(\ell^2(\mathcal{M})) \to \mathcal{L}(\ell^2_\omega(\mathcal{M}) \otimes_\pi H), \ \mathcal{T} \mapsto \mathcal{T} \otimes \mathrm{id}. \) This clearly implies that \( \Lambda_{\epsilon} \) factors through \( C^*_r(\mathcal{M}) \).

The restriction \( \rho|_{\mathcal{M}_{\epsilon}} : W^*_r(\mathcal{M}) \to \mathcal{L}(\ell^2_{\omega,*}(\mathcal{M}) \otimes_\pi H) \) is a \( w^* \)-continuous representation that clearly extends the integrated form of \( \Lambda_{\epsilon} \). Hence this integrated form can be extended to \( W^*_r(\mathcal{M}) \) (as \( \rho|_{\mathcal{M}_{\epsilon}} \)) and the extension is faithful if \( \pi \) is. The rest of the proof follows by the Bicommutant Theorem. \[\square\]

**Theorem 5.9** (c.f. [26] Corollary 2.15]). Assume that \( \mathcal{M} \) is a \( W^* \)-Fell bundle over \( G \) and write \( \lambda \) for the left regular representation of \( G \) by unitary operators on \( \ell^2(G) \). Let \( T : \mathcal{M} \to \mathcal{L}(H) \) be a nondegenerate representation which is weak*-continuous on each fiber and let \( \mu_{\lambda,T} : \mathcal{M} \to \mathcal{L}(\ell^2(G,H)) \) be the representation such that \( \mu_{\lambda,T}(b) = \lambda_t \otimes T_b \) for every \( b \in B_t \) and \( t \in G \). Then the integrated form of \( \mu_{\lambda,T} \), denoted \( \tilde{\mu}_{\lambda,T} \), factors through a representation of \( C^*_r(\mathcal{M}) \) that has a unique \( w^* \)-continuous extension to a representation \( \tilde{\mu}_{\lambda,T}^{\omega,*} \) of \( W^*_r(\mathcal{M}) \). Moreover, if \( T|_{B_e} \) is faithful then so is \( \tilde{\mu}_{\lambda,T}^{\omega,*} \).
Proof. It was shown in [26] that \( \tilde{\mu}_{\Lambda,T} \) factors through a representation of \( C^*_r(\mathcal{M}) \). To extend \( \tilde{\mu}_{\Lambda,T} \) to \( W^*_r(\mathcal{M}) \) take \( R \in W^*_r(\mathcal{M}) \). Then, by [19], there exists a bounded net \((f_j)_{j \in J} \subset C^*_r(\mathcal{M}) \) such that \( R = w^* \lim_j \tilde{\lambda}_n^\ast(f_j) \).

Every closed ball of \( \mathcal{L}(\ell_2(G,H)) \) is compact in the weak* topology, thus there exists \( S \in \mathcal{L}(\ell_2(G,H)) \) and a subnet \((f_j)_{j \in J} \subset \mathcal{L}(\ell_2(G,H)) \) such that \( S = w^* \lim_j \tilde{\mu}_{\Lambda,T}(f_j) \). In order to prove that \( S = w^* \lim_j \tilde{\mu}_{\Lambda,T}(f_j) \) it suffices to show the existence of a set \( X \subset \ell_2(G,H) \) spanning a dense subset of \( \ell_2(G,H) \) and such that, for every \( x, y \in X \), \( (\langle x, \tilde{\mu}_{\Lambda,T}(f_j)y \rangle)_{j \in J} \) is a convergent net.

Using the notation of [26, Proposition 2.13] we define

\[
X := \bigcup_{r \in G} \{ (\rho_r \otimes 1) \circ Vu : u \in \ell_2(\mathcal{M}) \otimes_T H, h \in H \}.
\]

Recall that \( \rho : G \to \ell_2(G) \) is the right regular representation and that \( \rho_r \otimes 1 \) lies in the commutant of \( \tilde{\mu}_{\Lambda,T}(C^*_r(\mathcal{M})) \). Recall also that \( V : \ell_2(\mathcal{M}) \otimes_T H \to \ell_2(G,H) \) is an isometry such that \( V(z \otimes h)(t) = T_{z(t)} h \). Take \( x = \rho_r \otimes 1 \circ Vu \in X \) and \( y = \rho_r \otimes 1 \circ Vu \in X \). Then, by [26, Proposition 2.6],

\[
\lim_j \langle x, \tilde{\mu}_{\Lambda,T}(f_j)y \rangle = \lim_j \langle Vu, (\rho_{r^{-1}} \otimes 1) \tilde{\mu}_{\Lambda,T}(f_j)(\rho_s \otimes 1)Vv \rangle
= \lim_j \langle Vu, (\rho_{r^{-1}s} \otimes 1)\tilde{\lambda}_n^\ast(f_j \otimes 1)v \rangle
= \langle Vu, (\rho_{r^{-1}s} \otimes 1)\tilde{\lambda}_n^\ast(V(\rho_{r^{-1}} \otimes 1)v) \rangle.
\]

This not only shows that \( (\tilde{\mu}_{\Lambda,T}(f_j))_{j \in J} \) converges in the weak (and weak*) topology, but also that its limit is completely determined by \( R = w^* \lim_j \tilde{\lambda}_n^\ast(f_j) \). Of course, we define \( \tilde{\mu}_{\Lambda,T}^w(R) := S \).

Define \( V^r := (\rho_r \otimes 1) \circ V \). Then \( \tilde{\mu}_{\Lambda,T}^w : W^*_r(\mathcal{M}) \to \mathcal{L}(\ell_2(G,H)) \) is uniquely determined by the condition

\[
\langle V^r u, \tilde{\mu}_{\Lambda,T}^w(R)V^s v \rangle = \langle V^{s^{-1}r} u, (V \otimes 1)v \rangle, \quad \forall u, v \in \ell_2(\mathcal{M}), \quad r, s \in G.
\]

This condition immediately implies that \( \tilde{\mu}_{\Lambda,T}^w \) is linear and \( w^* \)-continuous in any closed ball. Moreover, it is also straightforward to prove that \( \tilde{\mu}_{\Lambda,T}^w \) preserves the involution. To show that \( \tilde{\mu}_{\Lambda,T}^w \) is multiplicative take \( R, S \in W^*_r(\mathcal{M}) \) and bounded nets \((f_j)_{j \in J} \subset C^*_r(\mathcal{M}) \) weak* converging to \( R \) and \( S \), respectively. Then, using that multiplication is separately weakly continuous, we deduce

\[
\tilde{\mu}_{\Lambda,T}^w(RS) = \lim_j \tilde{\mu}_{\Lambda,T}^w(f_jS) = \lim_j \tilde{\mu}_{\Lambda,T}^w(f_j) \tilde{\mu}_{\Lambda,T}^w(S) = \mu_{\Lambda,T}^w(R) \tilde{\mu}_{\Lambda,T}^w(S) = \tilde{\mu}_{\Lambda,T}^w(RS).
\]

Assume \( T|_{\mathcal{N}_e} \) is faithful and \( \tilde{\mu}_{\Lambda,T}^w(R) = 0 \). Then (5.10) implies (with \( r = s = e \)) that \( \langle u, (R \otimes 1)v \rangle = 0 \) for all \( u, v \in \ell_2^0(\mathcal{M}) \otimes_T \mathcal{N}_e H \). Since \( T|_{\mathcal{N}_e} \) is faithful, we have \( R = 0 \). \( \Box \)

**Definition 5.11.** Let \( \mathcal{M} \) be a \( W^* \)-Fell bundle over the discrete group \( G \). We say that the subset \( \mathcal{N} \subset \mathcal{M} \) is a \( W^* \)-Fell subbundle if it is a Fell subbundle and the \( w^* \) topology of \( \mathcal{N}_e \) is the restriction of the \( w^* \) topology of \( M_e \).

Since each fiber of \( \mathcal{N} \) is a \( W^* \)-equivalence bimodule between \( W^* \)-ideals of \( \mathcal{N}_e \), the definition above actually implies that the \( w^* \) topology of each fiber \( \mathcal{N}_e \) is the restriction of the \( w^* \) topology of \( B_1 \).
Proposition 5.12. Let \( \mathcal{N} \subset \mathcal{M} \) be a \( W^* \)-Fell subbundle. If we view \( C_r^*(\mathcal{N}) \) as a \( C^* \)-subalgebra of \( C_r^*(\mathcal{M}) \subset W^*_r(\mathcal{M}) \) as in \cite[Proposition 3.2]{2}, then \( W^*_r(\mathcal{N}) \) is isomorphic to the \( w^* \)-closure of \( C_r^*(\mathcal{N}) \) in \( W^*_r(\mathcal{M}) \).

Proof. Our proof is a slight modification of that of \cite[Proposition 3.2]{2}. By Proposition 5.3 there exists a representation \( \lambda : \mathcal{M} \to \mathcal{L}(\mathcal{H}) \) with \( \lambda|_M \) faithful and \( w^* \)-continuous on each fiber. Define \( H_0 := T_{1_N} H \), where \( 1_N \) is the unit of \( N \), and the restriction map \( R : \mathcal{N} \to \mathcal{L}(\mathcal{H}_0) \) by \( R_a := T_{a|H_0} \). Then \( R \) is a representation \( w^* \)-continuous on each fiber and with \( R|_{\mathcal{N}_e} \) faithful.

In terms of the decomposition \( \ell_2(G,H) = \ell_2(G,H_0) \oplus \ell_2(G,H_0)^{\perp} \), we have

\[
\mu_{\lambda,T}(a) = \begin{pmatrix} \mu_{\lambda,R}(a) & 0 \\ 0 & 0 \end{pmatrix}, \forall a \in \mathcal{N}.
\]

We get the desired result by considering the integrated forms of \( \mu_{\lambda,T} \) and \( \mu_{\lambda,R} \) and the respective \( w^* \)-continuous extensions to \( W^*_r(\mathcal{M}) \) and \( W^*_r(\mathcal{N}) \), respectively. \( \square \)

Remark 5.14. If we add to the hypotheses of the last theorem the condition that \( \mathcal{N} \) is hereditary in \( \mathcal{M} \) (that is \( \mathcal{N},\mathcal{M} \mathcal{N} \subset \mathcal{N} \) then \( W^*_r(\mathcal{N}) \) is hereditary in \( W^*_r(\mathcal{M}) \). Indeed, it follows from separate \( w^* \)-continuity of the product and the fact that \( C_r^*(\mathcal{N}) \) is hereditary in \( C_r^*(\mathcal{M}) \).

5.4. The \( W^* \)-algebra of kernels. Let \( \mathcal{M} \) be a \( W^* \)-Fell bundle over a group \( G \).

A kernel of \( \mathcal{M} \) is a function \( k : G \times G \to \mathcal{M} \) such that \( k(r,s) \in M_{rs^{-1}} \). As usual we denote by \( \mathbb{k}(\mathcal{M}) \) the \( C^* \)-algebra of kernels of \( \mathcal{M} \) and \( \mathbb{k}_c(\mathcal{M}) \) the kernels of compact support, see \cite{2} for more details. Recall that there exists a canonical action of \( G \) on \( \mathbb{k}(\mathcal{M}) \), given by \( \beta_k(r,s) = k(rt,st) \). We are going to define a \( W^* \)-version of \( \mathbb{k}(\mathcal{M}) \) and also of \( \beta, 0 \).

Consider the canonical representation \( \pi : \mathbb{k}(\mathcal{M}) \to \mathcal{L}(\ell^2(\mathcal{M})) \) given by \( \pi(k)f(s) = \sum_{s \in G} k(s,t)f(t) \) for every \( k \in \mathbb{k}(\mathcal{M}) \) and \( f \in \ell^2(\mathcal{M}) \) with finite support. This representation has been already considered in \cite{2}. Using the canonical embedding \( \mathcal{L}(\ell^2(\mathcal{M})) \hookrightarrow \mathcal{L}(\ell^2_w(\mathcal{M})) \), we may view \( \pi \) as a representation \( \pi : \mathbb{k}(\mathcal{M}) \to \mathcal{L}(\ell^2_w(\mathcal{M})) \).

With the canonical action of \( G \) on \( k(\mathcal{M}) \) we construct the *-homomorphism \( \pi^\beta : k(\mathcal{M}) \to \ell^\infty(G,\mathcal{L}(\ell^2_w(\mathcal{M}))) \) defined by \( \pi^\beta(f)x = \pi(f^{-1}(x)) \). Note that \( \pi^\beta \) is equivariant with respect to the translation \( W^* \)-action \( \gamma \) on \( \ell^\infty(G,\mathcal{L}(\ell^2_w(\mathcal{M}))) \).

Recall that we may view the algebra of (generalised) compact operators \( \mathcal{K}(\mathcal{M}) := \mathcal{K}(\ell^2(\mathcal{M})) \) as an ideal of \( \mathbb{k}(\mathcal{M}) \) and \( \beta \) is the enveloping action of \( \beta|_{\mathcal{K}(\mathcal{M})} \). In the \( C^* \)-case we know that \( \pi : \mathbb{k}(\mathcal{M}) \to \mathcal{L}(\ell^2(\mathcal{M})) \) is the identity when restricted to \( \mathcal{K}(\mathcal{M}) \). In particular \( \pi : \mathbb{k}(\mathcal{M}) \to \mathcal{L}(\ell^2_w(\mathcal{M})) \) is injective on \( \mathcal{K}(\mathcal{M}) \).

Now we can prove that \( \pi^\beta \) is injective. Indeed, \( \pi^\beta(f) = 0 \) implies that \( \pi(\beta_k(f)x) = 0 \) for every \( t \in G \) and \( x \in \mathcal{K}(\mathcal{M}) \) and since \( \pi \) is faithful on \( \mathcal{K}(\mathcal{M}) \), this implies \( f\beta_k(x) = 0 \) for every \( t \in G \) and \( x \in \mathcal{K}(\mathcal{M}) \), and this is equivalent to \( f = 0 \) because the linear \( G \)-orbit of \( \mathcal{K}(\mathcal{M}) \) is dense in \( \mathbb{k}(\mathcal{M}) \).

Let \( \mathbb{k}_w(\mathcal{M}) \) and \( \mathbb{k}_w^*(\mathcal{M}) \) be the \( w^* \)-closures of \( \pi^\beta(\mathbb{k}(\mathcal{M})) \) and \( \pi^\beta(\mathcal{K}(\mathcal{M})) \), respectively. Then clearly \( \mathbb{k}_w^*(\mathcal{M}) \) is a \( W^* \)-ideal of \( \mathbb{k}_w(\mathcal{M}) \) and \( \beta^w := \gamma|_{\mathbb{k}_w^*(\mathcal{M})} \) is the \( W^* \)-enveloping action of \( \beta^w|_{\mathbb{k}_w^*(\mathcal{M})} = \gamma|_{\mathbb{k}_w^*(\mathcal{M})} \).

Our construction implies that \( \beta^w \) is a quotient of \( \beta^w \). This quotient is such that we can faithfully view \( \beta \) as a restriction of \( \beta^w \). Notice that \( \mathcal{K}(\ell^2_w(\mathcal{M})) \) is \( w^* \)-dense in \( N := \mathcal{L}(\ell^2_w(\mathcal{M})) \) (this follows, for instance, from \cite[Lemma 8.5.23]{13}). We
claim that $K_w^\ast(\mathcal{M})$ is canonically isomorphic to $N$. Indeed, the evaluation at $e \in G$, $ev_e : \ell^\infty(G,N) \to N$, is a surjective $w^\ast$-continuous $^\ast$-homomorphism. Moreover, $ev_e$ is injective when restricted to $K_w^\ast(\mathcal{M})$ because $ev_e \circ \pi^\beta|_{K_w^\ast(\mathcal{M})}$ is just $\pi|_{K_w^\ast(\mathcal{M})}$. Thus $ev_e|_{K_w^\ast(\mathcal{M})}$ is an isomorphism between $K_w^\ast(\mathcal{M})$ and $N = \mathcal{L}(\ell^2_w(\mathcal{M}))$.

**Definition 5.15.** The $W^\ast$-algebra $b_w^\ast(\mathcal{M})$ constructed above will be called the $W^\ast$-algebra of kernels of $\mathcal{M}$. It will be always endowed with the canonical $W^\ast$-action $\beta^w$ of $G$ defined above.

**Definition 5.16.** We say that two $W^\ast$-Fell bundles are weakly $W^\ast$-equivalent if their canonical actions on their $W^\ast$-algebras of kernels are $W^\ast$-Morita equivalent.

**Remark 5.17.** $W^\ast$-equivalence of $W^\ast$-Fell bundles is an equivalence relation because, as in the $C^\ast$-case, we have inner tensor products of $W^\ast$-equivalence bimodules.

**Theorem 5.18.** Let $\mathcal{M}$ be a $W^\ast$-Fell bundle over a group $G$. Then the $W^\ast$-enveloping action of the central partial action $\sigma$ of $\mathcal{M}$ is the restriction of $\beta^w$ to the centre of $b_w(\mathcal{M})$.

**Proof.** By Proposition 2.3 $\beta^w|_{Z(b_w^\ast(\mathcal{M}))}$ is the $W^\ast$-enveloping action of $\tau := \beta^w|_{Z(K_w^\ast(\mathcal{M}))}$. Hence all we need is to show that $\tau$ is isomorphic to $\sigma$.

The module $\ell^2_w(\mathcal{M})$ is a $W^\ast$-equivalence bimodule between $K_w^\ast(\mathcal{M})$ and $M_e$, hence it induces a $W^\ast$-isomorphism $\mu : Z(M_e) \to Z(K_w^\ast(\mathcal{M}))$ which we claim is an isomorphism of $W^\ast$-partial actions between $\sigma$ and $\tau$.

To simplify our notation we write $ZM$ and $Z\mathcal{M}$ instead of $Z(M_e)$ and $Z(K_w^\ast(\mathcal{M}))$, respectively. Consequently, the domains of $\sigma$ and $\tau$ will be denoted $ZM_t$ and $Z\mathcal{M}_t$ for $t \in G$.

We must show that $\mu(ZM_t) = ZM_t$ or, equivalently, that $\ell^2_w(\mathcal{M})$ induces the ideal $I_t = \overline{\text{span}}^{w^\ast} M_t^\ast$ to $J_t := K_w^\ast(\mathcal{M}) \cap \beta^w(\mathcal{M})$. From the proof of [5] Theorem 3.5 we know that $\ell^2(\mathcal{M}) \subseteq \ell^2_w(\mathcal{M})$ induces $\overline{\text{span}}^{w^\ast} M_t^\ast$ to $K(M) \cap \beta_t(K(M))$. By taking $w^\ast$-closures in $M_e$ and $\ell^\infty(G,\mathcal{L}(\ell^2_w(\mathcal{M})))$, respectively, we get the desired induction.

The composition $\mu \circ \sigma_t$ equals the composition $\mu|_t \circ \sigma_t$, where $\mu|_t$ represents the restriction and co-restriction of $\mu$ to $ZM_t$ (in the domain) and $Z\mathcal{M}_t$ (in the co-domain). But $\mu|_t$ is the isomorphism corresponding to the bimodule $X_t := J_t \ell^2_w(\mathcal{M}) I_t = \ell^2_w(\mathcal{M}) I_t = J_t \ell^2_w(\mathcal{M})$.

Hence we may view $\mu \circ \sigma_t$ as the isomorphism corresponding to the bimodule $X_t \otimes^{\ell^2_w(\mathcal{M})} M_t$. In the same way we may view $\tau \circ \mu$ as the isomorphism corresponding to the bimodule $J_t \delta_t \otimes^{\ell^2_w(\mathcal{M})} X_{t-1}$, where $J_t \delta_t$ is the fiber over $t$ of the semidirect product bundle of $\beta^w|_{K_w^\ast(\mathcal{M})}$, $B^w$, and $J_{t-1} \delta_e$ is the ideal $J_{t-1}$ seen as an ideal of the unit fiber of that bundle. Once again we will make use of the $C^\ast$-version of all these constructions.

The semidirect product bundle of $\beta|_{K(M)}$ will be denoted $\mathcal{B}$, and we will think of it as a Fell subbundle of $\mathcal{B}^w$. The fibre over $t$ of $\mathcal{B}$ is $K(M)_t \delta_t$ and $K(M)_t \delta_t (K(M)_t \delta_t)^\ast = K(M)_t \delta_t \subseteq K(M) \delta_e$.

Define $I_t^{\parallel}$ and $J_t^{\parallel}$ as the $C^\ast$-algebras generated by $M_t^\ast$ and $(K(M)_t \delta_t (K(M)_t \delta_t)^\ast)$ in $M_e$ and $K(M) \delta_e$, respectively. If we set $X_t^{\parallel} := J_t^{\parallel} \ell^2(\mathcal{M}) J_t^{\parallel} = \ell^2(\mathcal{M}) I_t^{\parallel} = J_t^{\parallel} \ell^2(\mathcal{M})$, we find that $\mu|_t$ is the isomorphism corresponding to the bimodule $X_t^{\parallel} \otimes^{\ell^2(\mathcal{M})} M_t$.
then $X^\|_{t} \otimes J^\|_{i=1} M_t$ and $\mathcal{K}(\mathcal{M}) \delta_{t} \otimes J^\|_{i=1} X_{t-1}$ are isomorphic as $C^*$-trings. To prove this claim consider the canonical $L^2$-bundle of $\mathcal{M}$, $\mathcal{LM} = \{L_t\}_{t \in G}$, which establishes a strong equivalence between $\mathcal{B}$ and $\mathcal{M}$. Then $X^\|_{t}$ is exactly $J^\|_{i=1} L_\nu = L_\nu J^\|_{i=1} = J^\|_{i=1} L_\nu J^\|_{i=1}$, and we have canonical injective maps
\[
\nu_1: X^\|_{t} \otimes J^\|_{i=1} M_t \to L_t, \ x \otimes y \mapsto xy, \\
\nu_2: \mathcal{K}(\mathcal{M}) \delta_{t} \otimes J^\|_{i=1} X_{t-1} \to L_t, \ T \otimes x \mapsto Tx,
\]
where the actions used are the actions of $\mathcal{B}$ and inner products of $\mathcal{LM}$. It can be directly shown that $\nu_1$ and $\nu_2$ are $L_\nu M_t \subseteq L_t$ and $\mathcal{K}(\mathcal{M}) \delta_{t} L_\nu \subseteq L_t$, respectively, because $M_t = J^\|_{i=1} M_t$ and $\mathcal{K}(\mathcal{M}) \delta_{t} = \mathcal{K}(\mathcal{M}) J^\|_{i=1}$ (due to Cohen-Hewitt Theorem we do not need closed linear spans here). Recalling the definition of strong equivalence and understanding the identities on $I$ and $\nu_1, \nu_2$ are the identities (on $J_i$ and $I_i$, respectively). Now the isomorphism in (5.19) and Corollary A.10 imply $\nu_1 \circ \nu_2$ to an isomorphism
\[
\nu_2^{-1} \circ \nu_1: X^\|_{t} \otimes J^\|_{i=1} M_t \to \mathcal{K}(\mathcal{M}) \delta_{t} \otimes J^\|_{i=1} X_{t-1}
\]
is an isomorphism of ternary $C^*$-trings with $(\nu_2^{-1} \circ \nu_1)^r$ and $(\nu_2^{-1} \circ \nu_1)^l$ being the identities on $I^\|_{i=1}$ and $J^\|_{i=1}$, respectively.

The question is now if we can extend $\nu_2^{-1} \circ \nu_1$ to an isomorphism
\[
(5.19)
\]
In fact we can follow the same line of reasoning we used when constructing the inner $W^*$-tensor product (see the construction preceding Definition B.3). The idea is to represent the $W^*$-equivalence modules of (5.19) using the same representation of $I_t$ and then to translate the continuity into continuity of inner products. At that point everything will follow immediately because $(\nu_2^{-1} \circ \nu_1)^r$ is the identity operator.

After constructing the isomorphism $\nu_2^{-1} \circ \nu_1$ as a $w^*$-extension of $\nu_2^{-1} \circ \nu_1$ it follows directly that $\nu_2^{-1} \circ \nu_1$ and $\nu_2^{-1} \circ \nu_1$ are the identities (on $J_i$ and $I_i$, respectively). Now the isomorphism in (5.19) and Corollary A.10 imply $\mu \circ \sigma$ is $\tau \circ \mu$.

**Corollary 5.20.** For a $W^*$-Fell bundle $\mathcal{M}$ the following are equivalent:

(i) The canonical action $\beta^w$ on $k_{w^*}(\mathcal{M})$ is $W^*$-AD-amenable.

(ii) The restriction of $\beta^w$ to $Z(\mathcal{K}_{w^*}(\mathcal{M}))$ is $W^*$-AD-amenable.

(iii) The central partial action of $\mathcal{M}$ is $W^*$-AD-amenable.

**Proof.** Follows at once from the definition of $W^*$-AD-amenability of partial actions, Theorem 5.18 and Theorem 5.3. □

**Definition 5.21.** A $W^*$-Fell bundle is said to be $W^*$-AD-amenable if the equivalent conditions of the corollary above are satisfied. A Fell bundle $\mathcal{B}$ is $AD$-amenable if the enveloping $W^*$-Fell bundle $B^\nu$ is $W^*$-AD-amenable.

**Remark 5.22.** $W^*$-AD-amenability is preserved by weak equivalence of $W^*$-Fell bundles.
Remark 5.23. Proposition [5.3] and Example [5.4] imply that a $W^*$-partial action is $W^*$-AD-amenable if and only if its semidirect product bundle (which is a $W^*$-Fell bundle) is $W^*$-AD-amenable. Hence the same conclusion holds for $C^*$-partial actions and AD-amenity.

Theorem 5.24. Let $\mathcal{B}$ be a Fell bundle over a group and let $\mathcal{B}''$ be the enveloping $W^*$-Fell bundle of $\mathcal{B}$. Then the canonical action $\beta''$ on $\mathcal{K}_w'(\mathcal{B}'')$ and the bidual of the canonical action $\beta$ on $\mathcal{K}_w(\mathcal{A})$, $\beta''$, are isomorphic as $W^*$-actions. In particular, $\mathcal{K}_w'(\mathcal{B}'')$ is isomorphic to $\mathcal{K}(\mathcal{B})''$.

Proof. We first show that $\ell^2(\mathcal{B}'')$ is Morita equivalent through a partial action $(\mathcal{A}'')^\gamma$ on $\mathcal{K}_w'(\mathcal{B}'')$ as $W^*$-Hilbert $M_v$-modules. This is the crucial point if we follow the original construction of $\mathcal{K}_w'(\mathcal{B}'')$ at the beginning of Section 5.4.

Let $\rho: B_e \to L(H)$ be the universal representation; we extend it to the bidual and view it as a faithful $W^*$-representation $\rho'': B''_e \to L(H)$. We may then view $\ell^2(\mathcal{B}'')$ as a wot-closed subspace of $L(H, \ell^2(\mathcal{B}'') \otimes_{\rho''} H)$. But $\ell^2(\mathcal{B}'') \otimes_{\rho''} H = \ell^2(\mathcal{B}) \otimes_{\rho''} H = K$ and we have a faithful representation $U: \ell^2(\mathcal{B}) \to L(H, K)$ such that $U(x)h = x \otimes h$. Moreover, $\ell^2(\mathcal{B}'')$ is the wot-closure of $U(\ell^2(\mathcal{B}))$, i.e. $\ell^2(\mathcal{B}'')$.

Looking at the linking algebra $L$ of $\ell^2(\mathcal{B})$, we may view $\mathcal{K}_w'(\mathcal{G}, \mathcal{B}'')$ as the $W^*$-completion of $\mathcal{K}(\mathcal{B})''$ in $L''$. But this closure is also equal to $\mathcal{K}(\mathcal{B})''$. Now we have

$$\beta''|_{\mathcal{K}(\mathcal{B})''} = \beta''|_{\mathcal{K}_w(\mathcal{B})'} = \beta'|_{\mathcal{K}_w(\mathcal{B})'}|_{\mathcal{K}(\mathcal{B})''} = \beta''|_{\mathcal{K}_w(\mathcal{B})'}|_{\mathcal{K}(\mathcal{B})''} = (\beta''|_{\mathcal{K}(\mathcal{B})''})|_{\mathcal{K}(\mathcal{B})'}.$$  

Hence we have two $W^*$-partial actions on $\mathcal{K}(\mathcal{B})''$, namely $\beta''|_{\mathcal{K}(\mathcal{B})'}$ and $\beta''|_{\mathcal{K}_w(\mathcal{B})'}$, which are the unique $W^*$-actions extending the $C^*$-partial action $\beta|_{\mathcal{K}(\mathcal{B})'}$. Therefore $\beta''|_{\mathcal{K}(\mathcal{B})'} = \beta''|_{\mathcal{K}_w(\mathcal{B})'}$. But $\beta''$ and $\beta''$ are both $W^*$-enveloping actions of $\beta''|_{\mathcal{K}(\mathcal{B})'}$, then uniqueness of $W^*$-enveloping actions implies that $\beta''$ is isomorphic to $\beta''$. □

Corollary 5.25. If two Fell bundles $\mathcal{A}$ and $\mathcal{B}$ over a group are weakly equivalent then their enveloping $W^*$-Fell bundles $\mathcal{A}''$ and $\mathcal{B}''$ are weakly $W^*$-equivalent. In particular, AD-amenity of Fell bundles is preserved by weak equivalence of Fell bundles.

Proof. The canonical partial actions on $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{B})$, $\alpha$ and $\beta$ respectively, are Morita equivalent through a partial action $\gamma$ on a $\mathcal{K}(\mathcal{A})\otimes_{\mathcal{K}(\mathcal{B})}$-equivalence bimodule $X$. If $L$ is the linking partial action of $X$ and $\nu$ the linking partial action of $\gamma$, then the $w^*$-closure of $X$ in $L''$ and the restriction $\gamma'' := \nu''|_{X''}$ provide a $W^*$-equivalence between $\alpha''$ and $\beta''$. The rest follows from Theorem [5.24] and Definitions [5.16] and [5.21]. □

Corollary 5.26. A Fell bundle $\mathcal{B}$ is AD-amenable if and only if the canonical action on $\mathcal{K}(\mathcal{B})$, $\beta$, is AD-amenable.

Proof. Just recall that $\mathcal{B}$ is weakly equivalent to $\mathcal{B}_\beta$ [5.6] and use the Corollary above. □

5.5. The dual coaction: another picture for the $W^*$-algebra of kernels. In this section we want to show that the sectional $W^*$-algebra $W^*_x(\mathcal{M})$ of a $W^*$-Fell bundle $\mathcal{M}$ over $G$ carries a canonical $G$-coaction and identify the crossed product by this coaction with the $W^*$-algebra of kernels $\mathcal{K}_w(\mathcal{M})$.  

Recall that a coaction of $G$ on a $W^*$-algebra $N$ is a faithful unital $W^*$-homomorphism $\delta: N \to N \otimes W^*_r(G)$ satisfying $(\delta \otimes \text{id})\delta = (\text{id} \otimes \delta_G)\delta$, were $\otimes$ denotes the (spatial) tensor product of $W^*$-algebras. Given such a coaction, the $W^*$-crossed product is defined as the $W^*$-subalgebra of $N \bar{\otimes} \mathcal{L}(\ell^2(G))$ generated by $\delta(N)$ and $1 \otimes \ell^\infty(G)$, where, as usual, $\ell^\infty(G)$ is represented as a $W^*$-subalgebra of $\mathcal{L}(\ell^2(G))$ via multiplication operators. We omit this representation here for simplicity, that is, we view $\ell^\infty(G)$ as a subalgebra of $\mathcal{L}(\ell^2(G))$. It turns out that

$$N \bar{\otimes} G = \text{span}^{\ast\ast} \{ \delta(n)(1 \otimes f) : n \in N, f \in \ell^\infty(G) \}.$$  

Representing $N$ on a Hilbert space or, more generally, on a self-dual Hilbert module $H$, the $W^*$-crossed product $N \bar{\otimes} G$ gets represented as a $W^*$-subalgebra of $\mathcal{L}(H) \bar{\otimes} \mathcal{L}(\ell^2(G)) = \mathcal{L}(H \otimes \ell^2(G))$. This crossed product carries a canonical $G$-action, the so called dual action $\bar{\delta}$. It is given on a generator $\delta(n)(1 \otimes f)$ by $\bar{\delta}(\delta(n)(1 \otimes f)) = \delta(n)(1 \otimes \tau_l(f))$, where $\tau_l(f)(s) := f(st)$ denotes the right translation $G$-action on $\ell^\infty(G)$. This can also be described as $\bar{\delta}(x) = (1 \otimes \rho_l)x(1 \otimes \rho_{l}^{-1})$, where $\rho: G \to \mathcal{L}(\ell^2(G))$ denotes the right regular representation of $G$.

Now, returning to the case of a $W^*$-Fell bundle $\mathcal{M}$, we want to define a coaction $\delta_\mathcal{M}: W^*_r(\mathcal{M}) \to W^*_r(\mathcal{M}) \bar{\otimes} W^*_r(G)$ that acts on generators $\Lambda_t(a) \in M_t$ with $a \in M_t$ by the formula

$$\delta_\mathcal{M}(\Lambda_t(a)) = \Lambda_t(a) \otimes \lambda_t. \tag{5.27}$$

This is therefore an extension of the usual dual coaction on $C^*_r(\mathcal{M}) \subseteq W^*_r(\mathcal{M})$. Here $W^*_r(G)$ denotes the group $W^*$-algebra of $G$, that is, the $W^*$-subalgebra of $\mathcal{L}(\ell^2(G))$ generated by the left regular representation $\lambda: G \to \mathcal{L}(\ell^2(G))$.

To prove that $\delta_\mathcal{M}$ exists, we proceed as in the $C^*$-algebra situation (see [24 Section 8] or [26]): Let $\mathcal{M} \times G$ be the pullback of $\mathcal{M}$ along the first coordinate projection $G \times G \to G$. This is a $W^*$-Fell bundle over $G \times G$ whose $W^*$-algebra is canonically isomorphic to $W^*_r(\mathcal{M} \times G) = W^*_r(\mathcal{M}) \bar{\otimes} W^*_r(G)$, in particular we have a canonical $W^*$-embedding

$$W^*_r(\mathcal{M}) \bar{\otimes} W^*_r(G) \subseteq \mathcal{L}(\ell^2_u(\mathcal{M} \times G)).$$

Now we define a unitary operator $V$ on the Hilbert $W^*$-module $\ell^2_u(\mathcal{M} \times G)$ by the formula

$$V \zeta(s, t) := \zeta(s, s^{-1}t) \quad \text{for all } \zeta \in \ell^2_u(\mathcal{M} \times G), \ s, t \in G.$$  

Straightforward computations show that this is indeed a unitary operator with adjoint $V^* \zeta(s, t) = \zeta(s, stt)$. Now we define a $w^*$-continuous injective unital homomorphism $\delta_\mathcal{M}: \mathcal{L}(\ell^2_u(\mathcal{M})) \to \mathcal{L}(\ell^2_u(\mathcal{M} \times G))$ by

$$\delta_\mathcal{M}(a) := V(a \otimes 1)V^*, \quad a \in W^*_r(\mathcal{M}).$$

It is easy to see that $(5.27)$ is satisfied. Moreover, since the elements $\Lambda_t(a)$ with $a \in M_t$ generate $W^*_r(\mathcal{M})$ as a $W^*$-algebra, the above formula restricts to an injective $w^*$-continuous unital homomorphism

$$\delta_\mathcal{M}: W^*_r(\mathcal{M}) \to W^*_r(\mathcal{M}) \bar{\otimes} W^*_r(G).$$

This is indeed a coaction, that is, the coassociativity identity $(\delta_\mathcal{M} \otimes \text{id}) \circ \delta_\mathcal{M} = (\text{id} \otimes \delta_G) \circ \delta_\mathcal{M}$ holds, where $\delta_G: W^*_r(G) \to W^*_r(G) \bar{\otimes} W^*_r(G)$ denotes the comultiplication of $W^*_r(G)$ (which, incidentally, is the coaction $\delta_M$ for the trivial one-dimensional Fell bundle $\mathcal{M} = C \times G$).
Remark 5.28. There is a canonical normal conditional expectation $E: W^*_r(M) \to M_e$ given on generators by $E(\Lambda(a)) = \delta_{e,c}(a)$ for all $a \in M_t$. This can be proved as in the $C^*$-case, or it can be deduced from the existence of the dual coaction $\delta_M$ above as follows: Consider the canonical tracial state $\tau: W^*_r(G) \to \mathbb{C}$ given by $\tau(x) = \langle \delta_e | x \delta_e \rangle$. Then $E = (id \otimes \tau) \circ \delta_M$ is the desired conditional expectation.

**Proposition 5.29.** For a $W^*$-Fell bundle $M$, we have a canonical isomorphism

$$W^*_r(M)\rtimes_{\delta_M} G \cong \mathbb{K}_{w^*}(M),$$

that identifies a generator $\delta_M(a)(1 \otimes f) \in W^*_r(M)\rtimes_{\delta_M} G$ with the kernel $k_{a,f}(s,t) := a(st^{-1})f(t)$ for $a \in C_c(M)$ and $f \in \ell^\infty(G)$. This isomorphism is $G$-equivariant with respect to the dual $G$-action on $W^*_r(M)\rtimes_{\delta_M} G$ and the canonical $G$-action on $\mathbb{K}_{w^*}(M)$.

**Proof.** Let $N := W^*_r(M)$ and $\delta := \delta_M$. We show how to turn $\ell^2_{\mathbb{K}}(M)$ into a $W^*$-Hilbert $N\rtimes_{\delta} G$-$M_e$-bimodule.

Consider the map $\iota: C_c(M) \to N \otimes_{\text{alg}} C_c(G) \subseteq N \otimes \ell^2(G)$ defined by $\iota(\xi) = \delta(\xi)(1 \otimes \delta_e) = \sum_{g \in G} \Lambda(g, r) \otimes \delta_e$. Here and throughout this proof $(\delta_g)_{g \in G}$ will also denote the standard right normal isomorphism of $\ell^2(G)$ -- apologies for the overuse of the symbol $\delta$ here! Let $X$ be the $w^*$-closure of the image of $\iota$ in $N \otimes \ell^2(G)$. Notice that with respect to the $N$-valued inner product on $N \otimes \ell^2(G)$ we have

$$\langle \iota(\xi) | \iota(\eta) \rangle_N = \sum_{s,t \in G} \langle \Lambda(\xi(s) \otimes \delta_e) | \Lambda(\eta(t)) \otimes \delta_e \rangle = \langle \Lambda(\xi) | \Lambda(\eta) \rangle_{M_e},$$

for all $\xi, \eta \in C_c(M)$, where $\langle \xi | \eta \rangle_{M_e}$ denotes the $M_e$-valued inner product on $C_c(M) \subseteq \ell^2_{\mathbb{K}}(M)$. Since $\Lambda$ is a $W^*$-embedding $M_e \hookrightarrow N$, it follows that the image of the $N$-valued inner product on $X$ takes values in $\Lambda(M_e) = M_e$ so that $X$ can be viewed as a right $W^*$-Hilbert $M_e$-module and $\iota$ extends to an isomorphism $\ell^2_{\mathbb{K}}(M) \cong X$ of $W^*$-Hilbert $M_e$-modules. The advantage of this picture is that $X$ is also canonically a left $W^*$-Hilbert $N\rtimes_{\delta} G$-module, where the left inner product is defined by $\iota(\xi) | \iota(\eta) \rangle_N := \delta(\xi)(1 \otimes \chi_e)\delta(\eta^*) \in I$ for $\xi, \eta \in C_c(M)$. The image of this inner product generates a $W^*$-ideal of $I$ of $N \rtimes_{\delta} G$, namely the $W^*$-ideal generated by the projection $p := \chi_e$. It follows that $I \cong \mathcal{L}(\ell^2_{\mathbb{K}}(M))$; this isomorphism identifies $\delta(\xi)(1 \otimes \chi_e)\delta(\eta^*)$ with $\theta_{\xi,\eta} = \langle \xi | \eta \rangle \in \mathcal{K}(\ell^2_{\mathbb{K}}(M)) \subseteq \mathcal{K}(\ell^2_{\mathbb{K}}(M)))$, and it is determined by this formula and the fact that it is $w^*$-continuous.

Next, considering the dual $G$-action $\tilde{\delta}$ on $Q := N \rtimes_{\delta} G$, we notice that the linear $G$-orbit of $I$ is $w^*$-dense. This is because $\tilde{\delta}_{t^{-1}}(\chi_e) = \chi_t$, so that $\tilde{\delta}_{t^{-1}}(I)$ is the $\tilde{\delta}$-ideal of $N \rtimes_{\delta} G$ generated by the projection $p_t = \chi_t$, and these projections generate $\ell^\infty(G)$ as a $W^*$-algebra. Therefore $\tilde{\delta}$ can be viewed as the $W^*$-enveloping action of its restriction $\tilde{\delta}|_I$. On the other hand, the $G$-action on the $W^*$-algebra of kernels $k_{w^*}(M)$ is also enveloping for a partial action on $\mathcal{L}(\ell^2_{\mathbb{K}}(M))$. By uniqueness of enveloping $W^*$-actions (Proposition 2.7), to see that $k_{w^*}(M) \cong Q$, it is enough to show that the restriction of $\tilde{\delta}$ to $I$ coincides with the partial action on $\mathcal{L}(\ell^2_{\mathbb{K}}(M))$ obtained as restriction of the $G$-action $\beta_{w^*}$ on $k_{w^*}(M)$. But by definition, $\beta_{w^*}$ is the unique $w^*$-continuous extension of the $G$-action $\beta$ on the $C^*$-algebra of kernels $k(M)$ given by $\beta_k(k)(s,t) = k(st, tr)$ for a kernel $k \in k_c(M)$. An elementary compact operator $\theta_{\xi, \eta} \in \mathcal{K}(\ell^2(M)) \subseteq \mathcal{K}(\ell^2_{\mathbb{K}}(M))$ identifies with the kernel function $k_{\xi, \eta}(s,t) := \langle \xi(s) | \eta(t) \rangle_{\mathbb{K}}$. And by [2] Proposition 8.1] we have a $C^*$-isomorphism $k(M) \cong B := C^*_r(M) \times_{\delta} G$ that is $G$-equivariant for the dual $G$-action $\tilde{\delta}$ on $B$ and $\beta$ on $k(M)$. Here $\delta$ also denotes the dual coaction of $G$ on $C^*_r(M)$; this is
a restriction of the dual coaction on \( N = \ell^2(G) \), denoted by the same symbol \( \delta \). The isomorphism \( \kappa(\mathcal{M}) \cong B \) is given as in the statement (see the proof of Proposition 8.1 in [2]). The \( C^* \)-algebra of compact operators \( K(\ell^2(\mathcal{M})) \) identifies, as above, with the \( C^* \)-ideal \( J \) of \( B \) generated by \( p = \chi_e \). This is \( w^* \)-dense in \( I \). Since the partial \( G \)-action on \( J \) we get from viewing it as an ideal of \( \kappa(\mathcal{M}) \) coincides with the partial action coming from the dual action on \( B \), the same has to be true for the \( w^* \)-closures, that is, via the isomorphism \( \mathcal{L}(\ell^2_\kappa(\mathcal{M})) \cong I \) the partial action on \( I \) we get by restriction of \( \hat{\delta} \) is the partial action we get from \( \kappa_{w^*}(\mathcal{M}) \) by restricting it to the \( w^* \)-ideal \( \mathcal{L}(\ell^2_\kappa(\mathcal{M})) \).

**Corollary 5.30.** For every \( W^* \)-Fell bundle \( \mathcal{M} \) we have a canonical isomorphism
\[
\kappa_{w^*}(\mathcal{M}) \rtimes_{\beta_{w^*}} G \cong \mathcal{L}(\ell^2(\mathcal{M}) \overline{\otimes} \mathcal{L}(\ell^2 G)).
\]

**Proof.** This follows from Proposition 5.29 and general duality theory for crossed products by \( W^* \)-coactions, see [30]. □

We recall from [5] that given a Fell subbundle \( \mathcal{A} \) of \( \mathcal{B} \) we can identify \( \kappa(\mathcal{A}) \) with the norm closure of \( \kappa_e(\mathcal{A}) \) in \( \kappa(\mathcal{B}) \). This inclusion has a \( W^* \)-counterpart.

**Corollary 5.31.** If \( \mathcal{N} \) is a \( W^* \)-Fell subbundle of \( \mathcal{M} \) and we view \( \kappa(\mathcal{N}) \) as a \( C^* \)-subalgebra of \( \kappa(\mathcal{M}) \subset \kappa_{w^*}(\mathcal{M}) \), then \( \kappa_{w^*}(\mathcal{N}) \) is isomorphic to the \( w^* \)-closure of \( \kappa(\mathcal{N}) \) in \( \kappa_{w^*}(\mathcal{M}) \).

**Proof.** The inclusion \( \kappa_{w^*}(\mathcal{N}) \subset \kappa_{w^*}(\mathcal{M}) \) is just the inclusion \( W^*_*(\mathcal{N}) \rtimes_{\delta_{\mathcal{N}}} G \subset W^*_*(\mathcal{M}) \rtimes_{\delta_{\mathcal{M}}} G \) provided by Proposition 5.29. □

6. Exel’s approximation property and AD-amenability

The main goal of this section is to compare the notion of amenability in the sense of Anantharaman-Delaroche with the approximation property introduced by Exel in [22]. We start by recalling Exel’s approximation property:

**Definition 6.1.** A Fell bundle \( \mathcal{B} = \{ B_t \}_{t \in G} \) has the approximation property (AP) if there exists a net \( \{ a_i \}_{i \in I} \) of functions \( a_i : G \to B_e \) with finite support such that

(i) \( \sup_{t \in I} \| \sum_{r \in G} a_i(r)^* a_i(r) \| < \infty \).

(ii) For every \( t \in G \) and \( b \in B_t \), \( \lim_i \| b - \sum_{r \in G} a_i(tr)^* b a_i(r) \| = 0 \).

A partial action \( \alpha \) on a \( C^* \)-algebra has the AP if the semidirect product bundle \( \mathcal{B}_\alpha \) has the AP.

**Remark 6.2.** Notice that (1) above means that \( \{ a_i \}_{i \in I} \) is a bounded net when viewed as a net in the Hilbert \( B_e \)-module \( \ell^2(G, B_e) \). Indeed, the original definition of the AP in [22] uses such nets and Proposition 4.5 in [22] says that both definitions are equivalent (the difference being whether the supports of the functions are required to be finite or not).

Condition (2) can also be weakened: it is enough to check the norm convergence in (2) for \( b \) in total subsets of \( B_t \), that is, for \( b \) in a subset \( B_t^0 \) spanning a norm-dense subset of \( B_t \) for each \( t \in G \).

As a way of combining Exel’s approximation property and amenability in the sense of Anantharaman-Delaroche ([7][8][10]), we introduce the following:

**Definition 6.3.** A \( W^* \)-Fell bundle \( \mathcal{M} = \{ M_t \}_{t \in G} \) has the \( W^* \)-approximation property (\( W^* \)AP) if there exists a net of functions \( \{ a_i : G \to M_e \}_{i \in I} \) with finite support such that
WAP if and only if there exists a net \(\gamma\) and only if there exists a net \(\alpha\), we say that a \(\gamma\) has the WAP if the associated W*-Fell bundle \(\mathcal{B}_\gamma\) has the WAP.

A \((C^*-algebraic)\) Fell bundle \(\mathcal{B}\) has the WAP if its W*-enveloping Fell bundle \(\mathcal{B}''\) has the WAP and a \((C^*-partial action)\) \(\alpha\) has the WAP if \(\alpha''\) has the WAP.

The reader should read WAP as “weak approximation property”, the reason for this will be clear after Theorem 6.4. We recommend to read the statement of that theorem at this point to get a feeling of what we want to do next.

Remark 5.3 implies that a \(\gamma\)-partial action \(\alpha\) has the WAP if and only if \(\mathcal{B}_\alpha\) has the WAP. We shall prove in what follows that AD-amenability and the WAP are equivalent notions, first for global actions and later also for general Fell bundles. This is not trivial, even for global actions, because the AD-amenability of a global action requires the existence of a certain net that takes central values (see Theorem 6.3) while for the WAP this is not explicitly necessary (Definition 6.3).

Let \(\gamma\) be a (global) action of \(G\) on the W*-algebra \(N\). As usual, we write \(\tilde{\gamma}\) for the action of \(G\) on \(\ell^\infty(G,N)\) given by \(\tilde{\gamma}(f)(r) = \gamma_t(f(t^{-1}r))\) and view \(N\) as the subalgebra of constant functions in \(\ell^\infty(G,N)\). Abusing the notation we also use the same notation for the \(G\)-action on functions \(f \in \ell^2(G,N)\). The following result gives an explicit characterisation of the WAP for global actions.

**Proposition 6.4.** Let \(\gamma\) be a global action of \(G\) on a W*-algebra \(N\). Then \(\gamma\) has the WAP if and only if there exists a net \(\{a_i\}_{i \in I}\) of finitely supported functions \(a_i : G \to N\) such that \(\{a_i\}_{i \in I}\) is bounded in \(\ell^2(G,N)\) and \(\{\langle a_i, b\gamma_t(a_i) \rangle\}_{i \in I}\) \(w^*\)-converges to \(b\) for all \(b \in N\) and \(t \in G\).

**Proof.** We view the fibre of \(\mathcal{B}_\gamma\) at \(t\) as \(N\delta_t\) and denote its elements by \(x\delta_t\). With this notation \(x\delta_t\delta_s = x\delta_{ts}\) and \(x\delta_t^* = \delta_{t^{-1}}(x^*)\delta_{t^{-1}}\). Viewing \(x \in N\) as \(x\delta_e\), we can then think of a function \(a : G \to N\) as a function from \(G\) to the unit fibre \(N\delta_e \equiv N\). If \(a : G \to N\) has finite support, then for every \(t \in G\) and \(b \in N\) we have:

\[
\sum_{r \in G} (a(r)\delta_e)^*a(r)\delta_e = \sum_{r \in G} a(r)^*a(r) = \sum_{r \in N}(a(r)^*a(r))\delta_t = \sum_{r \in G} a(r)^*b\gamma_t(a(r))\delta_t = \sum_{r \in G} a(r)^*b\gamma_t(a(t^{-1}r))\delta_t = \langle a, b\gamma_t(a) \rangle\delta_t.
\]

The proof follows directly from the computations above and from the fact that under the identification \(N \to N\delta_t, x \mapsto x\delta_t\), the \(w^*\)-topology of \(N\delta_t\) is just the \(w^*\)-topology of \(N\).

In order to show that the W*-AD-amenability is equivalent to WAP for W*-Fell bundles we shall need the following result:

**Lemma 6.5.** Let \(\gamma\) be a W*-global action of \(G\) on \(N\). Then \(\gamma\) is AD-amenable if and only if \(\mathcal{B}_\gamma\) has the WAP for all \(\gamma\)-invariant \(w^*\)-dense \(*\)-subalgebra \(A \subset N\) and a bounded net \(\{a_i\}_{i \in I} \subset \ell^2(G,N)\) of functions with finite support such that for all \(b \in A\) and \(t \in G\), \(\{\langle a_i, b\gamma_t(a_i) \rangle\}_{i \in I}\) \(w^*\)-converges to \(b\).
Proof. The direct implication follows from Theorem 3.3 (with $A = N$). For the converse we view $N$ as a concrete (unital) von Neumann algebra of operators on some Hilbert space $H$, $N \subseteq \mathcal{L}(H)$, and take a *-subalgebra $A \subseteq N$ and a net $\{a_i\}_{i \in I}$ as in the statement. For $t \in G$ and $i \in I$ we define $\varphi^i_t : N \to N$ by $\varphi^i_t(f) := (a_i | b\gamma_t(a_i))$. Then $\{\varphi^i_t\}_{i \in I}$ is a net of uniformly bounded linear maps (with uniform bound $c := \sup_i \|a_i\|_2 < \infty$). By assumption $\varphi^i_t(b) \to b$ in the weak*-topology for every $b \in A$ and $t \in G$. A standard argument shows that the same happens for all $b$ in the norm closure of $A$ which is then a (w*-dense) C*-subalgebra of $N$. Hence we may assume, without loss of generality, that $A$ is already a C*-algebra. In particular we may assume that $A$ is closed by continuous functional calculus and that $\Lambda := \{x \in A : 0 \leq x \leq 1\}$ is an approximate unit for $A$ and thus w*-converges to $1_N$ (this follows from the assumption that $A$ is w*-dense in $N$).

For each $(i, \lambda) \in I \times \Lambda$ we define

$$P_{i, \lambda} : \ell^\infty(G, N) \to N, \quad P_{i, \lambda}(f) = \langle \lambda^{1/2}a_i, f\lambda^{1/2}a_i \rangle,$$

where the product $f\lambda^{1/2}a_i$ represents the diagonal action of $f\lambda^{1/2} \in \ell^\infty(G, N)$ on $\ell^2(G, N)$. Each $P_{i, \lambda}$ is a completely positive linear map with norm $\|P_{i, \lambda}\| = \|\lambda^{1/2}a_i\|_2 \leq c := \sup_i \|a_i\|_2 < \infty$.

Let $K$ be the Hilbert space $\ell^2(\Lambda, H)$ and define, for each $i \in I$, the function

$$P_i : \ell^\infty(G, N) \to \mathcal{L}(K), \quad P_i(f)g|_\lambda = P_{i, \lambda}(f)(g|_\lambda).$$

If we view $K = \ell^2(\Lambda, H)$ as the direct sum of $\Lambda$-copies of $H$, then $P_i(f)$ is the “diagonal” operator formed by the family $(P_{i, \lambda})_\lambda$. Thus $P_i$ is completely positive and $\|P_i\| \leq c$ for all $i \in I$.

The set of completely positive maps $Q : \ell^\infty(G, N) \to \mathcal{L}(K)$ with $\|Q\| \leq c$ is compact with respect to the topology of pointwise w*-convergence, thus there exists a completely positive map $P : \ell^\infty(G, N) \to \mathcal{L}(K)$ and a subnet $\{P_{i_j}\}_{j \in J}$ such that $P(f) = \lim_j P_{i_j}(f)$ in the w*-topology for every $f \in \ell^\infty(G, N)$. By passing to a subnet we may therefore assume that $\{P_{i_j}\}_{j \in J}$ converges to $P$ for the pointwise w*-topology.

As a consequence of the last paragraph we get that, for each $f \in \ell^\infty(G, N)$ and $\lambda \in \Lambda$, $\{P_{i, \lambda}(f)\}_{i \in I}$ w*-converges to some $P_{\lambda}(f)$. In fact, the map $P_{\lambda} : \ell^\infty(G, N) \to N, f \mapsto P_{\lambda}(f)$, is completely positive and $\|P_{\lambda}\| \leq c$ for all $\lambda \in \Lambda$.

For each $\lambda \in \Lambda$ we define $Q_{\lambda} : \ell^\infty(G, Z(N)) \to N \subseteq \mathcal{L}(H)$ as the restriction of $P_{\lambda}$. We claim that $\{Q_{\lambda}\}_{\lambda \in \Lambda}$ converges w*-pointwise. Indeed, it suffices to prove that for each positive $f \in \ell^\infty(G, Z(N))$ the net $\{Q_{\lambda}(f)\}_{\lambda \in \Lambda}$ is increasing. Take $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$. Then, for every $h \in H$ and $i \in I$:

$$\langle h, P_{i, \lambda}(f) h \rangle = \sum_{r \in G} \langle a_i(r)h, \lambda^{1/2}f(r)\lambda^{1/2}a_i(r)h \rangle$$

$$= \sum_{r \in G} \langle a_i(r)h, f^{1/2}(r)\lambda f^{1/2}(r)a_i(r)h \rangle$$

$$\leq \sum_{r \in G} \langle a_i(r)h, f^{1/2}(r)\mu f^{1/2}(r)a_i(r)h \rangle = \langle h, P_{i, \mu}(f) h \rangle.$$

Taking limit in $i$ we get $\langle h, Q_{\lambda}(f) h \rangle \leq \langle h, Q_{\mu}(f) h \rangle$ and it follows $Q_{\lambda}(f) \leq Q_{\mu}(f)$. Let $Q : \ell^\infty(G, Z(N)) \to N$ be the pointwise w*-limit of $\{Q_{\lambda}\}_{\lambda \in \Lambda}$.

Let us prove that the image of $Q$ is contained in $Z(N)$. It suffices to show that for $f \in \ell^\infty(G, Z(N))^+ \subseteq \Lambda$ and a self-adjoint $b \in A$, $Q(f) b$ is self-adjoint. Let $\{P_{i_j}\}_{j \in J}$ be a pointwise w*-convergent subnet of $\{P_{\lambda}\}_{\lambda \in \Lambda}$, clearly, both $f b$ and $P(fb)$ are
We know that the net \( Q(f)b = P(fb) \). Fix \( h, k \in H \). Using the inner product of \( \ell^2_w(G, N) \otimes_N H \) in the following computations, we deduce

\[
|h, (Q(f)b - P(fb))k| = \lim_i |(\lambda_j^{1/2}a_i \otimes h, f(\lambda_j^{1/2}a_i \otimes bk - b\lambda_j^{1/2}a_i \otimes k))| \\
\leq \lim_i \sqrt{\|h\| \|f\| \|\lambda_j^{1/2}a_i \otimes bk - b\lambda_j^{1/2}a_i \otimes k\|}.
\]

The double limit above is zero because \( \lim_i \|\lambda_j^{1/2}a_i \otimes bk - b\lambda_j^{1/2}a_i \otimes k\| = 0 \), which implies that

\[
\lim_i \langle h, (Q(f)b - P(fb))k \rangle = 0.
\]

Hence the proof will be complete after we show (6.6). We claim that

\[
Q(\gamma_t(f)) = \gamma_t(Q(\gamma_{t-1}(\lambda))(f)).
\]

Since \( \{\gamma_t(\lambda)\}_{t \in \Lambda} \) is a subset of \( \{\gamma\}_{t \in \Lambda} \), if we take the \( w^* \)-limit in (6.6), we obtain

\[
\lim_{\lambda} Q(\gamma_t(f)) = \gamma_t(\lim_{\lambda} Q(\gamma_{t-1}(\lambda))(f)) = \gamma_t(Q(f)).
\]

Hence the proof will be complete after we show (6.6).

Fix \( f \in \ell^2_w(G, Z(N)) \), \( \lambda \in \Lambda \) and \( t \in G \). In the \( w^* \)-topology:

\[
Q(\gamma_t(f)) = \lim_{\lambda} (\lambda^{1/2}a_i, \gamma_t(f)(\lambda^{1/2}a_i)) = \lim_{\lambda} (\gamma_t(\gamma_{t-1}(a_i), \gamma_{t-1}(\lambda)(f)\gamma_{t-1}(a_i)))
\]

We know that the set \( \{\gamma_{t-1}(a_i), \gamma_{t-1}(\lambda)(f)\gamma_{t-1}(a_i)\}_{i \in I} \) \( w^* \)-converges. We only need to prove it \( w^* \)-converges to \( Q(\gamma_{t-1}(\lambda)(f)) \). To avoid the annoying inverse \( t^{-1} \), we change \( t \) by \( t^{-1} \).

Note that \( \langle \gamma_t(a_i), \gamma_t(\lambda)(f)\gamma_{t-1}(a_i) \rangle = \langle f^{1/2}\gamma_t(\lambda^{1/2}a_i), f^{1/2}\gamma_t(\lambda^{1/2}a_i) \rangle \) is self-adjoint, and so it is \( Q(\gamma_t(\lambda))(f) \). Thus, it suffices to show that, for all \( h \in H \),

\[
\lim_i \langle h, \gamma_t(a_i), \gamma_t(\lambda)(f)\gamma_{t-1}(a_i)h \rangle = \langle h, Q(\gamma_t(\lambda))(f)h \rangle.
\]

In any Hilbert space we have \( \|x\|^2 - \|y\|^2 \leq (\|x\| + \|y\|)\|x - y\| \). In particular, we use this inequality in \( \ell^2_w(G, N) \otimes_N H \):

\[
\lim_i \langle h, \gamma_t(a_i), \gamma_t(\lambda)(f)\gamma_{t-1}(a_i)h \rangle - \langle h, Q(\gamma_t(\lambda))(f)h \rangle = \\
\leq \lim_i \|f^{1/2}\gamma_t(\lambda^{1/2}a_i) \otimes h\|^2 - \|f^{1/2}\gamma_t(\lambda^{1/2}a_i) \otimes h\|^2 \\
\leq \lim_i 2\|f^{1/2}\|a_i\| \|h\|\|f^{1/2}\gamma_t(\lambda^{1/2}a_i) - \gamma_t(\lambda^{1/2}a_i) \otimes h\| \\
\leq \lim_i 2\|f\|\|a_i\| \|h\|\|\gamma_t(\lambda^{1/2})(\gamma_t(a_i) - a_i) \otimes h\|.
\]
Moreover, \(|\{\|a_i\|_2\}_{i \in I}\) is bounded and the limit of \(\{\|\gamma_t(\lambda^{1/2})(\tilde{\gamma}_t(a_i) - a_i) \otimes \delta_t\|_2\}_{i \in I}\) is
\[
\lim_i \langle h, \gamma_t(\langle a_i, \lambda a_i \rangle) h \rangle + \langle h, \langle a_i, \gamma_t(\lambda) a_i \rangle h \rangle - \\
- \lim_i \langle h, \langle \tilde{\gamma}_t(a_i), \gamma_t(\lambda) a_i \rangle h \rangle + \langle h, \langle a_i, \gamma_t(\lambda) \tilde{\gamma}_t(a_i) \rangle h \rangle
\]
\[
= \langle h, \gamma_t(\lambda) h \rangle + \langle h, \gamma_t(\lambda) h \rangle - \langle \gamma_t(\lambda) h \rangle - \langle \gamma_t(\lambda) h \rangle = 0.
\]
This implies \([6.6]\) and the proof is complete. □

The next remark will be extremely useful to show that \(W^*\)-amenability of the W*-AP.

**Remark 6.7.** For a Fell bundle \(B = \{B_t\}_{t \in G}\) and an ordered set \(F = \{t_1, \ldots, t_n\} \subset G\), the algebra \(\mathcal{M}(B)\) formed by the \(n \times n\) matrices \(M = (M_{i,j})_{i,j=1}^n\) such that \(M_{i,j} \in B_{t_i t_j^{-1}}\), is a C*-algebra with usual matrix involution and multiplication [6. Lemma 2.8]. The C*-norm is equivalent to \(\|M\|_\infty = \max_{i,j} \|M_{i,j}\|\) and each \(B_{t_i t_j^{-1}}\) may be isometrically identified with a subspace of \(\mathcal{M}(B)\). Moreover, \(\mathcal{M}(B)' = \mathcal{M}(B''')\).

**Theorem 6.8.** A W*-Fell bundle is \(W^*\)-amenable if and only if it has the W*-AP.

**Proof.** Assume that the W*-Fell bundle \(\mathcal{M}\) over the group \(G\) is \(W^*\)-amenable. By Corollary 5.20, the central partial action \(\gamma\) on \(Z := Z(M_e)\) is \(W^*\)-amenable. Let \(\delta\) be the \(W^*\)-enveloping action of \(\gamma\), acting on the commutative W*-algebra \(Y\).

We know that \(Z\) is a W*-ideal of \(Y\) and that \(\delta\) is \(W^*\)-amenable.

Let \(\{\xi_i\}_{i \in I} \subset \ell^2(G, Y)\) be a net for \(\gamma\) as in Theorem 3.3 and let \(p \in Y\) be the unit of \(Z\). We define \(a_i := \xi_i p\) and claim that \(\{a_i\}_{i \in I} \subset \ell^2(G, Z)\) is a net as in the definition of the W*-AP.

First of all note that \(\{a_i\}_{i \in I}\) is bounded because \(\langle a_i, a_i \rangle = p(\xi_i, \xi_i)\), for all \(i \in I\). If \(p_t\) is the unit of \(Z_t = Z \cap \delta_t(Z)\), then for every \(t \in G\) and \(x \in M_t\) we have (by the definition of the central partial action):
\[
\lim_i \sum_{r \in G} a_i(tr)^* x a_i(r) = \lim_i \sum_{r \in G} a_i(tr)^* x p_{t^{-1}_i} a_i(r) = \lim_i \sum_{r \in G} a_i(tr)^* \gamma_t(p_{t^{-1}_i} a_i(r)) x
\]
\[
= \lim_i \sum_{r \in G} p_{\xi_i(tr)}^* p_t \delta_t(p) \delta_t(\xi_i(r)) x = \lim_i p_t(\xi_i, \delta_t(\xi_i)) x
\]
\[
= p_t x = x,
\]
where the limits are taken in the w*-topology. This shows that \(\mathcal{M}\) has the W*-AP.

Now assume that \(\mathcal{M}\) has the W*-AP. We will show that the canonical W*-action \(\beta_{\mathcal{M}}\) on \(k_{w^*}(\mathcal{M})\) is \(W^*\)-amenable using Lemma 6.3 and Theorem 3.3. We set \(\gamma := \beta_{\mathcal{M}}, N := k_{w^*}(\mathcal{M})\) and \(A := k_{c}(\mathcal{M})\). Recall that \(N\) is a W*-completions of \(k(\mathcal{M})\) and that \(A\) is norm dense in \(k(\mathcal{M})\). Hence \(A\) is \(w^*\)-dense in \(N\). Moreover, \(A\) is \(\gamma\) invariant because \(A\) is invariant under the canonical action on \(k(\mathcal{M})\).

We claim that \(\mathcal{M}(\mathcal{M})\) is a W*-subalgebra of \(N\). This is important because in such a case the convergence in the topology of \(\mathcal{M}(\mathcal{M})\) relative to the \(w^*\)-topology of \(N\) is just entrywise \(w^*\)-convergence on the matrix algebra \(\mathcal{M}(\mathcal{M})\).

Recall that we defined \(N = k_{w^*}(\mathcal{M})\) as the \(w^*\)-closure of the image of the map \(\pi^3 : k(\mathcal{M}) \to \ell^\infty(G, \mathcal{L}(\ell^2_{w^*}(\mathcal{M})))\) (see section 5.4). Thus it suffices to prove that the image of \(p : \mathcal{M}(\mathcal{M}) \to \mathcal{L}(\ell^2_{w^*}(\mathcal{M})), \rho(k) f(r) = \sum_{s \in G} k(r, s) f(s)\), is a W*-subalgebra. Here we think of the matrix \(k\) as a kernel of compact support. For all
$f, g \in \ell^2_w(M)$ the map $\mathcal{M}_F(M) \to M_c, k \mapsto (f, \rho(k)g)$ is $w^*$-continuous. Hence the
closed unit ball $\rho(\mathcal{M}_F(M))$ is $w^*$-compact by Lemma \[A.2\] so we conclude that the
image of $\rho$ is a $W^*$-subalgebra.

Take a net of functions $\{a_j\}_{j \in J}$ as in the definition of $W^*$AP for $M$. Let $F$ be
the set of finite subsets of $G$ and consider $\Xi := F \times J$ as a directed set with the order
$(U, j) \leq (V, i) \iff U \subseteq V$ and $j \leq i$. For each $\xi = (U, j) \in \Xi$ let $a_\xi : G \to \mathcal{M}_U(M)$
be such that for every $r \in G$, $a_\xi(r)$ is the diagonal matrix with all the entries in the
diagonal equal to $a_j(r)$. Note $\|a_\xi, a_j\| = \|a_j, a_j\|$. Observe also that

\[\gamma_t(\mathcal{M}_U(M)) = \mathcal{M}_{U^{-1}}(M).\]

Fix $t \in G$ and $k \in A$. Take a finite set $U_0 \subseteq G$ such that $\text{supp}(k) \subseteq U_0 \times U_0$. If
$\xi = (U, i) \in \Xi$ is such that $U \supseteq U_0 \cup U_0t$, that is, $U_0 \subseteq U \cap U^{-1}$, then

\[\langle a_\xi, k\gamma_t(a_\xi) \rangle = \sum_{r \in G} a_\xi(tr)^*k\gamma_t(a_\xi(r)),\]

and $a_\xi(tr)^*k\gamma_t(a_\xi(r)) \in \mathcal{M}_{\text{supp}(k)}(M)$. Moreover, considering the left and right
entrywise action of $M$, on $\mathcal{M}_{\text{supp}(k)}(M)$, $a_\xi(tr)^*k\gamma_t(a_\xi(r)) = a_j(tr)^*ka_j(r)$. It is then
clear that $\lim_{\xi} (b_\xi, k\gamma_t(b_\xi)) = (b_\xi, k\gamma_{t^{-1}}(b_\xi)) = b$ $w^*$-entrywise and hence $w^*$
in $N$. \[\square\]

Remark 6.9. In the proof above we incidentally showed that the net $\{a_i\}_{i \in I}$ in the
Definition of the $W^*$AP can be taken in the unit ball of $\ell^2(G, Z(M))$, without
altering the definition.

Corollary 6.10. A $W^*$-partial (resp. $C^*$-partial) action has the $W^*$AP (resp.
WAP) if and only if it is $W^*$AD-amenable (resp. $AD$-amenable).

Proof. Follows at once from Theorem above and Remark \[5.23\] \[\square\]

Our next goal is to give alternative characterisations of the WAP for Fell bundles.

Theorem 6.11. For every Fell bundle $B$ over a group $G$ the following are equivalent:

(i) $B$ is $AD$-amenable.
(ii) $B$ has the WAP.
(iii) $B''$ is the $W^*$AD-amenable.
(iv) $B''$ has the $W^*$AP.
(v) There exists a bounded net $\{a_i\}_{i \in I} \subset \ell^2(G, Z(B''))$ of functions with finite
support such that, for every $t \in G$ and $b \in B_1$, $\lim_i \sum_{r \in G} a_i(tr)^*b_{a_i(r)} = b$ in $B''$ with respect to the $w^*$-topology.
(vi) There exists a bounded net $\{a_i\}_{i \in I} \subset \ell^2(G, B_\epsilon)$ of functions with finite support
such that, for every $t \in G$ and $b \in B_1$, $\lim_i \sum_{r \in G} a_i(tr)^*b_{a_i(r)} = b$ in the weak
topology of $B_1$.

Proof. The equivalences between \[i\] and \[iii\] and between \[iii\] and \[iv\], follow
directly from the definitions of AD and $W^*$AD amenability, and of WAP and $W^*$AP
properties. We know from the previous theorem that $B''$ is $W^*$AD-amenable if and
only if it has the $W^*$AP, hence \[iii\] and \[iv\] are equivalent. By Remark \[6.9\] \[iv\]
and \[v\] are equivalent. To prove that \[vi\] implies \[iv\] we can proceed exactly as in the proof of the converse in Theorem \[6.8\], noticing that convergence in the
weak topology of $\mathcal{M}_F(B)$ is entrywise convergence in the weak topology and, also,
$w^*$-convergence in $\mathcal{M}_F(B'') = \mathcal{M}_F(B'')$.

We now prove that \[iv\] implies \[vi\]. First we indicate how to approximate elements of $\ell^2(G, B''_\epsilon)$ by elements of $\ell^2(G, B_\epsilon)$ in a certain particular way. We start by
representing \( \ell^2(G, B''_e) \) and \( \ell^2(G, B_e) \) faithfully. Let \( \pi : B'' \to \mathcal{L}(H) \) be a nondegenerate \(*\)-representation, fiber-wise faithful and \( w^* \)-continuous (we constructed one such representation in the proof of Theorem 5.2). Define \( \rho := \pi|_{B''_e} : B''_e \to \mathcal{L}(H) \) and note that we have canonical identifications
\[
K := \ell^2(G, H) = \ell^2(G, B_e) \otimes \rho H = \ell^2(G, B''_e) \otimes \rho H.
\]
The map \( \hat{\pi} : \ell^2(G, B''_e) \to \mathcal{L}(H, K), \hat{\pi}(f)h r = \pi(f(r))h \) is a faithful representation of the \( C^* \)-ternary ring \( \ell^2(G, B''_e) \). Then we have a canonical nondegenerate representation \( \hat{\pi}^L : K(\ell^2(G, B''_e)) \to \mathcal{L}(K) \) such that \( \hat{\pi}^L(T) \pi(f) = \pi(T f) \), and thus we get a nondegenerate representation of the linking algebra \( L \) of \( \ell^2(G, B''_e) \):
\[
\hat{\pi}^L : L \to \mathcal{L}(K \oplus H) \quad \hat{\pi}^L \left( \begin{array}{cc} T & f \\ g & S \end{array} \right) = \left( \begin{array}{cc} \hat{\pi}^L(T) & \hat{\pi}^L(f) \\ \hat{\pi}^L(g) & \hat{\pi}^L(S) \end{array} \right),
\]
where \( \hat{\pi}^L : B''_e \to \mathcal{L}(H) \) is just \( \rho \).

Fix an element \( c \in C_c(G, B''_e) \). Using a net in \( B_e \) to approximate \( c(t) \) (for each \( t \) in the finite support of \( c \)) with respect to the \( w^* \)-topology, we can construct a net \( \{c_j \}_{j \in I} \subset C_c(G, B_e) \) such that \( \text{supp}(c_j) \subset \text{supp}(c) \) and \( c_j(t) \to c(t) \) in the \( w^* \)-topology for every \( t \in G \). This construction implies that \( \{\hat{\pi}(c_j)\}_{j \in I} \) weak∗-converges to \( \hat{\pi}(c) \) because, for all \( h \in H \) and \( k \in K \):
\[
\lim_j (\hat{\pi}(c_j)h, k) = \lim_j \sum_t (\pi(c_j(r))h, k(r)) = (\hat{\pi}(c)h, k).
\]

It follows from the previous comments that \( \hat{\pi}(c) \in \overline{\pi(C_c(G, B_e))}_{w^*} \). Now, according to [30] Theorem 4.8 and [19] Part I Ch. 3, the unit ball of \( \hat{\pi}^L(L) \) is \( \hat{\pi}^L \)-strongly dense in the unit ball of \( \hat{\pi}^L(L)'' \), and this bicommutant is the weak∗-closure of \( \hat{\pi}^L(L) \). Hence there exists a net \( \{T_j \}_{j \in J} \subseteq L \) in the closed ball of radius \( \|\hat{\pi}(c)\| = \|c\| \) such that \( \{\hat{\pi}^L(T_j a_j s_j)\}_{j \in J} \) converges to \( (0 \ 0 \hat{\pi}(c) \ 0) \) \( \hat{\pi}^L \)-strongly. Then, in the strong operator topology:
\[
\lim_j \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \hat{\pi}(a_j) = \lim_j \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \hat{\pi}^L \left( \begin{array}{cc} T_j & a_j \\ b_j & s_j \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \left( \begin{array}{cc} \hat{\pi}(c) \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \hat{\pi}(c) = \left( \begin{array}{c} \hat{\pi}(c) \ 0 \end{array} \right).
\]

Now we arrange the supports of the \( a_j \)'s to be contained in \( \text{supp}(c) \). Let \( P \in \mathcal{L}(K) = \mathcal{L}(\ell^2(G, H)) \) be the multiplication by the indicator function of \( \text{supp}(c) \). Then, in the strong operator topology: \( \lim_j P \hat{\pi}(a_j) = P \hat{\pi}(c) = \hat{\pi}(c) \) and \( P \hat{\pi}(a_j) = \pi(a_j|_{\text{supp}(c)}) \). Thus we are allowed to assume \( \text{supp}(a_j) \subseteq \text{supp}(c) \) for all \( j \in J \). We must retain the following facts about the net \( \{a_j\}_{j \in J} \subseteq \ell^2(G, B_e) \):

- \( \text{supp}(a_j) \subseteq \text{supp}(c) \) for all \( j \in J \).
- \( \|a_j\| \leq \|c\| \) for all \( j \in J \), with the norm of \( \ell^2(G, B_e) \).
- \( \{\hat{\pi}(a_j)\}_{j \in J} \) converges strongly to \( \hat{\pi}(c) \).

We claim that these conditions imply, for every \( t \in G \), \( b \in B_t \) and \( \varphi \in B'_t \), that
\[
\lim_j \varphi \left( \sum_{r \in G} a_j(tr)^*b r a_j(r) \right) = \varphi \left( \sum_{r \in G} c(tr)^*b c(r) \right).
\]
In other words, we claim that the net \( \{\sum_{r \in G} a_j(tr)^*b r a_j(r)\}_{k \in J} \) weakly converges to \( \sum_{r \in G} c(tr)^*b c(r) \) in \( B_t \). Indeed, since \( \pi|_{B''_e} \) is an isomorphism over its image, and a homeomorphism considering in \( B''_e \) and in \( \mathcal{L}(H) \) the \( w^* \)-topology and the ultraweak topology \( \sigma_{w^*} \) respectively, it is enough to prove that \( \pi(\sum_{r \in G} a_j(tr)^*b r a_j(r)) \xrightarrow{\sigma_{w^*}} \pi(\sum_{r \in G} c(tr)^*b c(r)) \).
Let \( U : G \to \mathcal{L}(K) = \mathcal{L}(\ell^2(G, H)) \) be the unitary representation given by \( U(f)(r) = f(t^{-1}r) \), and \( \pi^G : \mathcal{B}' \to \mathcal{L}(K) \) be the \( \ell^2 \)-direct sum of \( G \) copies of \( \pi \), that is, \( \pi^G(b)f(r) := \pi(b)f(r) \). Note that \( \{ \sum_{r \in G} a_j(tr)^*ba_j(r) \}_{j \in J} \) is bounded because, for all \( u, v \in H \),

\[
(6.13) \quad \langle u, \pi \left( \sum_{r \in G} a_j(tr)^*ba_j(r) \right) v \rangle = \langle U_t^* \hat{\pi}(a_j)u, \pi^G(b) \hat{\pi}(a_j)v \rangle.
\]

Since the ultraweak topology coincides with the weak operator topology on bounded sets, to prove \((6.12)\) it is enough to show that \( \{ \pi(\sum_{r \in G} a_j(tr)^*ba_j(r)) \}_{j \in J} \) converges to \( \pi(\sum_{r \in G} c(tr)^*bc(r)) \) in the wot topology. But our construction of \( \{a_j\}_{j \in J} \) and \((6.13)\) implies

\[
\lim_j \langle u, \pi \left( \sum_{r \in G} a_j(tr)^*ba_j(r) \right) v \rangle = \lim_j \langle U_t^* \hat{\pi}(a_j)u, \pi^G(b) \hat{\pi}(a_j)v \rangle = \langle U_t^* \hat{\pi}(c)u, \pi^G(b) \hat{\pi}(c)v \rangle = \langle u, \pi \left( \sum_{r \in G} c(tr)^*bc(r) \right) v \rangle.
\]

Therefore \((6.12)\) holds (note that \((6.13)\) does not imply \((6.12)\) if we only know that \( \pi(a_j) \) wot \( \pi(c) \)).

Now assume that \( \mathcal{B}' \) has the W*AP and take a net \( \{c_i\}_{i \in I} \) as in the definition of the W*AP for \( \mathcal{B}' \), with all the \( c_i \)'s with compact support. Set \( M := \sup_{i \in I} \| \sum_{t \in G} c_i(t)^*c_i(t) \| \) and let \( F \) and \( F' \) be the families of finite subsets of \( \mathcal{B} \) and \( \mathcal{B}' \) respectively, where \( B_{t,1} \) is the closed unit ball of \( B_t \). On \( \Lambda := (0,1) \times F \times F' \) we consider the canonical order \( (\varepsilon, U, V) \leq (\delta, Y, Z) \iff \delta \leq \varepsilon, U \subseteq Y \) and \( V \subseteq Z \). For every \( \lambda = (\varepsilon, U, V) \in \Lambda \) there exists \( i_0 \in I \) such that, for every \( t \in G, b \in B_t \cap U \) and \( \varphi \in B_t' \cap V \),

\[
\left| \varphi \left( b - \sum_{r \in G} c_{i_0}(tr)^*bc_{i_0}(r) \right) \right| < \varepsilon / 2.
\]

Our approximation procedure ensures the existence of \( a_\lambda \in \ell^2(G, B_c) \) such that: \( \text{supp}(a_\lambda) \subseteq \text{supp}(c_i) \), \( \|a_\lambda\|^2 \leq \|c_i\|^2 \leq M \) and

\[
\left| \varphi \left( b - \sum_{r \in G} a_\lambda(tr)^*ba_\lambda(r) \right) \right| < \varepsilon,
\]

for every \( t \in G, b \in B_t \cap U \) and \( \varphi \in B_t' \cap V \). It is then clear that \( \{a_\lambda\}_{\lambda \in \Lambda} \) is a net satisfying \((vi)\).

\begin{remark}
By the proof above and Remark \((6.9)\) we could replace the condition “bounded net” in the the last Theorem by “net in the closed unit ball” without changing the conclusions.
\end{remark}

\begin{corollary}
A Fell bundle \( \mathcal{B} \) has the WAP if and only if the canonical action on its \( C^* \)-algebra of kernels \( \mathcal{K}(\mathcal{B}) \) is AD-amenable.
\end{corollary}

\begin{proof}
Follows from Theorem \((6.11)\) and Corollary \((5.26)\).
\end{proof}

\begin{corollary}
The AP implies the WAP.
\end{corollary}

\begin{proof}
The AP clearly implies condition \((vi)\) of Theorem \((6.11)\).
\end{proof}
Notice that by Example 3.7, every group partially acts AD-amenable on $C$, so that the above situation does happen for every group. For global actions the situation is different: no non-amenable group can act globally AD-amenable on a finite dimensional non-zero C*-algebra.

**Remark 6.17.** We do not know if the WAP implies (and hence is equivalent to) the AP in general. We will show in Section 8 that this is true at least in the case of Fell bundles whose unit fibre is (Morita equivalent to) a commutative C*-algebra.

6.1. **Invariance under equivalences.** We have shown that AD-amenability of Fell bundles is equivalent to the WAP and both are preserved by the weak equivalence of Fell bundles. But, is the AP preserved by weak equivalence of Fell bundles?

Every Fell bundle $B$ is weakly equivalent to the semidirect product bundle of an action $\alpha$ on a C*-algebra, see [5]. Moreover, $\alpha$ is unique up to Morita equivalence of actions and the equivalence class is that of the canonical action on the C*-algebra of kernels of $B$, see [2].

In order to show that the AP is preserved by weak equivalences we decompose such equivalences into “elementary” pieces. By [5], every weak equivalence (represented by $\sim$) between the Fell bundles $A$ and $B$ (over $G$) can be decomposed as a chain of equivalences

$$(6.18) \quad A \approx B_\alpha \sim B_\gamma \approx B_\sigma \sim B_\beta \approx B,$$

where $\approx$ represents strong equivalence and

- $\alpha$ and $\beta$ are partial actions of $G$ on C*-algebras.
- $\gamma$ ($\sigma$) is the canonical action of $G$ on the C*-algebra of kernels of $A$ ($B$, respectively) and it is also the C*-enveloping action of $\alpha$ ($\beta$, respectively).

The advantage of this decomposition is that we have changed a weak equivalence for some strong equivalences and a very specific type of weak equivalence: that of C*-enveloping actions.

**Lemma 6.19.** The AP is preserved by strong equivalence of Fell bundles.

**Proof.** Suppose $A$ and $B$ are Fell bundles over $G$, $A$ has the AP and $A'$ is an $A$-$B$-weak equivalence bundle. Let $\{a_i\}_{i \in I}$ be a set of functions for $A$ as in the definition of the AP.

Let $F$ be the collection of finite subsets of $A'$ and consider in $\Lambda := F \times (0, +\infty)$ the order $(U, \varepsilon) \leq (V, \delta) \Leftrightarrow U \subseteq V$ and $\delta \leq \varepsilon$. We will construct a net of functions $\{b_\lambda\}_{\lambda \in \Lambda}$, $b_\lambda: G \to B_\varepsilon$, with finite supports such that

- $\sup_{\lambda \in \Lambda} \|\sum_{r \in G} b_\lambda(r)^* b_\lambda(r)\| \leq \sup_{r \in I} \|\sum_{r \in G} a_i(r)^* a_i(r)\| < \infty$
- For every $\lambda = (U, V, \varepsilon) \in \Lambda$ and $u, v \in U$, if $\langle u, v \rangle_B \in B_\varepsilon$ then
  $$\left| \sum_{r \in G} b_\lambda(tr)^* \langle u, v \rangle_B b_\lambda(r) - \langle u, v \rangle_B \right| < \varepsilon.$$

This will clearly suffice to complete the proof because, for every $t \in G$,

$$\sup_{\lambda \in \Lambda} \{\langle u, v \rangle_B: u \in X_r, v \in X_{rt}, r \in G\} = B_\varepsilon.$$

Fix $\lambda = (U, \varepsilon) \in \Lambda$. Take a positive $c \in B_\varepsilon$ such that $\|c\| < 1$ and $\|c(u, v)_{Bc} - \langle u, v \rangle_B\| < \varepsilon$ for all $u, v \in U$. By [6] we can assume $c = \sum_{j=1}^n \langle x_j, x_j \rangle_B$ for some $x_1, \ldots, x_n \in X_e$. Define, for every $i \in I$, $b_i: G \to B_\varepsilon$ as $b_i(r) := \sum_{j=1}^n \langle x_j, a_i(r) x_j \rangle_B$.

The function $b_\lambda$ will be one of the $b_i$’s, that we will indicate how to choose.
We claim that \( \| \sum_{r \in G} b_r(r) \| \leq \| \sum_{r \in G} a_i(r) a_i(r) \| \). Let \( F = \{ t_1, \ldots, t_n \} \) be such that \( x_j \in X_{t_j} \ (j = 1, \ldots, n) \). Define \( \mathcal{M} := \mathcal{M}_F(A) \) as in Remark 6.24. The direct sum \( E := X_{t_1} \oplus \cdots \oplus X_{t_n} \) is an \( \mathcal{M} \)-\( \mathcal{B}_c \)-Hilbert bimodule (not full in general). If we think of the elements of \( E \) as column matrices, the action of \( \mathcal{M} \) is given by matrix multiplication and the left inner product is \( \langle \xi, \eta \rangle = \langle A(\xi, \eta) \rangle^* \). If \( x \) is the column vector \((x_1, \ldots, x_n)^t \in E \), then \( \|x\|^2 = \|e\| < 1 \) and

\[
\left\| \sum_{r \in G} b_r(r) \right\| \leq \left\| \sum_{r \in G} a_i(r) a_i(r) \right\| \leq \left\| \sum_{r \in G} a_i(r) a_i(r) \right\|.
\]

Now take \( u, v \in U \) and let \( t \in G \) be such that \( \langle u, v \rangle_B \in B_t \). For all \( i \in I \) we have

\[
\sum_{r \in G} b_i(tr)^* \langle u, v \rangle_B b_i(r) = \sum_{r \in G} \sum_{j,k=1}^n \langle x_j, a_i(tr)x_j \rangle_B^* \langle u, v \rangle_B \langle x_k, a_i(r)x_k \rangle_B
\]

\[
= \sum_{r \in G} \sum_{j,k=1}^n \langle u, a_i(tr)x_j \rangle_B \langle v, x_k, a_i(r)x_k \rangle_B
\]

\[
= \sum_{r \in G} \sum_{j,k=1}^n \langle A(u, x_j) a_i(tr)x_j, A(v, x_k) a_i(r)x_k \rangle_B
\]

\[
= \sum_{j,k=1}^n \langle \sum_{r \in G} a_i(t^{-1}r)^*A(x_k, v)A(u, x_j) a_i(r)x_j, x_k \rangle_B.
\]

Note that \( A(x_k, v)A(u, x_j) = A(x_k^*u, v, x_j) \in A_{t^{-1}} \). Then, taking limit in \( i \),

\[
\lim_i \sum_{r \in G} b_i(tr)^* \langle u, v \rangle_B b_i(r) = \sum_{j,k=1}^n \langle A(x_k, v)A(u, x_j) x_j, x_k \rangle_B
\]

\[
= \sum_{j,k=1}^n \langle A(x_k, v)A(u, x_j) x_j, x_k \rangle_B
\]

\[
= \sum_{j,k=1}^n \langle u(x_j, x_j), v(x_k, x_k) \rangle_B = c(u, v)sc.
\]

We then can choose \( i_0 \in I \) such that \( \| \sum_{r \in G} b_{i_0}(tr)^* \langle u, v \rangle_B b_{i_0}(r) - \langle u, v \rangle_B \| < \varepsilon \) for all \( u, v \in U \). Thus we take \( b_\lambda := b_{i_0} \).

**Lemma 6.20.** Let \( B \) be a Fell bundle. If \( B \) has the AP and \( \beta \) is the canonical action on the \( C^* \)-algebra of kernels of \( B \), then \( \beta \) has the AP.

**Proof.** Take a net of functions \( \{ b_j \}_{j \in J} \subset \ell^2(G, B_c) \) as in the definition of the AP and construct a net \( \{ b_k \}_{k \in \mathbb{K}} \subset \ell^2(G, k(B)) \) exactly as in the proof of Theorem 6.8.
This time we can ensure that, for every \( k \in k_c(\mathcal{B}) \), \( \{(b_\lambda, k_\lambda b_\lambda)\}_{\lambda \in \Lambda} \) converges (entrywise) in norm to \( k \). The rest follows from Remark 6.2. □

**Lemma 6.21.** Let \( \alpha \) be a partial action of a group \( G \) on a \( C^* \)-algebra \( A \). If a Morita enveloping action of \( \alpha \) has the AP then \( \alpha \) has the AP.

**Proof.** Suppose \( \beta \) is a Morita enveloping action of \( \alpha \). By definition \( \alpha \) is Morita equivalent to a restriction of \( \beta \), but Lemma 6.19 implies the AP is preserved under Morita equivalence of actions. Thus we may assume \( \beta \) is an enveloping action of \( \alpha \). We assume \( \beta \) is an action of \( G \) on \( B \) and recall that \( A_t = \beta_t(A) \cap A \) and \( \alpha_t(a) = \beta_t(a) \). Moreover, \( \mathcal{B}_\alpha \) is a Fell subbundle of \( \mathcal{B}_\beta \). We think of \( A \) and \( B \) as the unit fibres of these bundles.

Let \( \{b_i\}_{i \in I} \) be a net of functions, as in the definition of the AP for the bundle \( \mathcal{B}_\beta \). Take a positive \( c \in A \) with \( \|c\| < 1 \) and any \( a_\delta \in \mathcal{B}_\alpha \). By \( cb_i \) we mean the function \( r \mapsto cb_i(r) \). Note that \( cb_i \) is an \( A \) valued function. Making the following computations in \( \mathcal{B}_\beta \) we deduce

\[
(6.22) \quad \lim_i \sum_{r \in G} (cb_i)(tr)^* (a_\delta)(cb_i)(r) = \lim_i \sum_{r \in G} b_i(tr)^* ca_\beta_i(c) b_i(r) \delta_t = ca_\beta_i(c).
\]

Since \( a \in A_t \), if we replace \( c \) by an approximate unit of \( A \) and take limit then \( ca_\beta_i(c) \) converges to \( a \). Imitating the ideas we used to prove Lemma 6.19 to show that \( \mathcal{B}_\alpha \) has the AP, we can construct a net of functions (for \( \mathcal{B}_\alpha \)) indexed over \( I \times F \times (0, +\infty) \), where \( F \) is the family of finite subsets of \( \mathcal{B}_\alpha \). We leave this task to the reader. □

Now we use Lemmas 6.19 and 6.21 to prove our next Theorem, which in turn implies those lemmas.

**Theorem 6.23.** The AP is preserved by the weak equivalences of Fell bundles.

**Proof.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be weakly equivalent Fell bundles and consider the equivalences in \((6.18)\). Recall that \( \gamma \) and \( \sigma \) are the canonical actions on the algebras of kernels of \( \mathcal{A} \) and \( \mathcal{B} \), respectively. If \( \mathcal{A} \) has the AP then, by Lemmas 6.20 and 6.19 \( \mathcal{B} \) has the AP. Now Lemmas 6.21 and 6.20 imply that \( \mathcal{B} \) has the AP. □

**Remark 6.24.** We know that Theorem 6.23 (and hence also Corollary 6.25) Lemmas 6.21, 6.20 and 6.19 hold if we replace the AP by the WAP because, by Theorem 6.11 and Corollary 5.25, the WAP is equivalent to AD-amenable and both are preserved by weak equivalence of Fell bundles.

The converse of Lemma 6.24 is also true.

**Corollary 6.25.** Let \( \alpha \) be a partial action of \( G \) on a \( C^* \)-algebra \( A \). Then \( \alpha \) has the AP if and only if one (hence all) Morita enveloping action of \( \alpha \) has the AP. In particular the double dual (global) action of \( G \) on \( A \rtimes_\alpha G \rtimes_\alpha G \) has the AP if and only if \( \alpha \) has the AP.

**Proof.** This follow from Theorem 6.23 and the fact that \( \mathcal{B}_\alpha \) is weakly equivalent to the semidirect product bundle of each Morita enveloping action of \( \alpha \). Moreover, the double dual action on \( A \rtimes_\alpha G \rtimes_\alpha G \) is a Morita enveloping action of \( \alpha \), which is isomorphic to the canonical action on the \( C^* \)-algebras of kernels under the isomorphism \( k(B_\alpha) \cong C^*(B_\alpha) \rtimes_{\delta_\beta} G \cong A \rtimes_\alpha G \rtimes_\alpha G \) by \([2] \) Proposition 8.1, where \( \delta_\beta \) denotes the dual coaction on \( C^*(\mathcal{B}_\alpha) \) as in Section 5.3. □
Corollary 6.26. If a partial action \( \alpha \) of \( G \) on \( A \) admits an enveloping global action \( \beta \) of \( G \) on \( B \), then \( \alpha \) has the AP if and only if \( \beta \) has the AP.

Remark 6.27. By Remark 6.24 the AP and the WAP are equivalent if and only if they are equivalent for actions on C*-algebras. Thus the AP and the WAP agree if and only if every AD-amenable action has the AP. This is an important question that will be left open. A positive answer would solve another important question raised by Exel: if the reduced cross-sectional C*-algebra \( C^*_r(B) \) of a Fell bundle (over a discrete group) is nuclear, does it follow that \( B \) has the AP? By our Proposition 6.28 below this would follow if we know that the AD-amenability implies the AP.

Corollary 6.28. Let \( B \) be a Fell bundle and \( \beta \) the canonical action on the C*-algebra of kernels of \( B \). Then \( B \) has the AP if and only if \( \beta \) has the AP.

Proof. Recall that \( B \) is weakly equivalent to \( B_\beta \), as we discussed at the beginning of Section 6.1. The conclusion now follows from Theorem 6.23. □

7. Cross-sectional C*-algebras and the WAP

After Theorem 6.11 we can think of the WAP as the Fell bundle counterpart of AD-amenability of noncommutative C*-dynamical systems. In fact many well-known results about AD-amenable actions hold for Fell bundles with the WAP.

We start with a result involving W*-Fell bundles.

Proposition 7.1. Let \( M = \{ M_t \}_{t \in G} \) be a W*-Fell bundle. Then the following assertions are equivalent.

(i) \( M_e \) injective and \( M \) has the W*AP (or, equivalently, \( M \) is W*AD-amenable);
(ii) \( \mathcal{L}(\mathbb{C}_w(M)) \) is injective and its canonical W*-action \( \beta^w \) has the W*AP (or is W*AD-amenable);
(iii) \( \mathcal{L}(\mathbb{C}_w(M)) \rtimes_{\beta^w} G \) is injective;
(iv) \( W^*_e(M) \) is injective.

Proof. First notice that \( M_e \) is injective if and only if the W*-algebra of kernels \( \mathcal{L}(\mathbb{C}_w(M)) \) is injective. Indeed, \( M_e \) is W*-Morita equivalent to \( \mathcal{L}(\ell^2_w(M)) \) (via the W*-equivalence bimodule \( \ell^2_w(M) \)); it follows that \( M_e \) is injective if and only if \( \mathcal{L}(\ell^2_w(M)) \) is injective. But \( \mathcal{L}(\mathbb{C}_w(M)) \) carries a W*-action that is enveloping for a partial W*-action on \( \mathcal{L}(\ell^2_w(M)) \). The claim now follows from Remark 2.29. Also observe that \( M_e \) is injective if \( W^*_e(M) \) is injective because we have a canonical (normal) conditional expectation \( W^*_e(M) \rightarrow M_e \) (Remark 5.28).

The discussion above implies that (1) is equivalent to (2). Since (2) involves a W*-action, \( (2) \Leftrightarrow (3) \) by Theorem 3.4. Finally, \( (3) \Leftrightarrow (4) \) by Corollary 5.30. □

Proposition 7.2. Let \( B \) be a Fell bundle and let \( \pi : C^*(B) \rightarrow C^*_r(B) \) be the canonical map between the full and reduced cross-sectional C*-algebras [20, 22]. If \( B \) has the WAP then \( \pi \) is an isomorphism.

Proof. Since \( B \) has the WAP, the canonical action \( \beta \) on the C*-algebra of kernels of \( B \) is AD-amenable (Corollary 6.15). Hence the full and reduced cross-sectional algebras \( C^*(B_\beta) \) and \( C^*_r(B_\beta) \) (i.e. the full and reduced \( \beta \)-crossed product) are canonically isomorphic. This implies that \( \pi \) is an isomorphism, see [5, 6]. □
Proposition 7.3 (c.f. [10] Théorème 4.5). Let $\mathcal{B}$ be a Fell bundle over a group $G$ with $B_\circ$ nuclear. Then the following are equivalent:

(i) $C^*(\mathcal{B})$ is nuclear.
(ii) $C^*_\rho(\mathcal{B})$ is nuclear.
(iii) $\mathcal{B}$ has the WAP.

Proof. Let $\mathcal{K}(\mathcal{B})$ be the $C^*$-algebra of kernels and $\beta$ the canonical action of $G$ on $\mathcal{K}(\mathcal{B})$. By the proof of [10] Theorem 6.3, $\mathcal{K}(\mathcal{B})$ is nuclear. Moreover, by Corollary [6,10] and [10] Théorème 4.5, (3) is equivalent to any of the following:

(1') $\mathcal{K}(\mathcal{B}) \rtimes_{\beta} G := C^*(\mathcal{B}_\beta)$ is nuclear.
(2') $\mathcal{K}(\mathcal{B}) \rtimes_{r,\beta} G := C^*_\rho(\mathcal{B}_\beta)$ is nuclear.
(3') $\beta$ is AD-amenable.

Since nuclearity is preserved by Morita equivalence of $C^*$-algebras, by [5] and [5] we know that $(n)$ is equivalent to $(n')$, for $n = 1, 2, 3$. □

When specialised to partial actions the above proposition takes the following form:

Corollary 7.4. Let $\alpha$ be a partial action of the group $G$ on a nuclear $C^*$-algebra $A$. Then the following are equivalent:

(i) The full crossed product $A \rtimes_\alpha G$ is nuclear.
(ii) The reduced crossed product $A \rtimes_{\alpha, r} G$ is nuclear.
(iii) $\alpha$ is AD-amenable.

Proof. Follows directly from the last Proposition and Corollary [5,10]. □

The last two results are examples of a general way of extending known results from $C^*$-actions to Fell bundles. The trick is to use the weak equivalence of Fell bundles and the canonical action on the $C^*$-algebra of kernels. We use this very same idea to treat exactness of cross sectional $C^*$-algebras, but first we introduce the spatial tensor product of a Fell bundle (over a discrete group) and a $C^*$-algebra. This construction is a special case of the tensor products of Fell bundles developed in [11]. We recall the basic facts here for the convenience of the reader.

Take a Fell bundle $\mathcal{B}$ and a $C^*$-algebra $C$. Let $L_t$ be the linking algebra of $B_t$ and define $B_t \otimes C$ as the closure of the algebraic tensor product $B_t \otimes C$ in $L_t \otimes C$. We claim that $\mathcal{B} \otimes C := \{B_t \otimes C\}_{t \in G}$ is a Fell bundle with a multiplication and involution such that $(a \otimes x)(b \otimes y) = ab \otimes xy$ and $(a \otimes x)^* = a^* \otimes x^*$.

For future purposes, and to prove $\mathcal{B} \otimes C$ is actually a Fell bundle, it is convenient to indicate how to construct this bundle using a representation of $\mathcal{B}$. Let $T: \mathcal{B} \to \mathcal{L}(H)$ be a nondegenerate $^*$-representation (in the sense of [21]) with $T|_{B_t}$ faithful and take a nondegenerate and faithful representation $\pi: C \to \mathcal{L}(K)$ (here $H$ and $K$ are Hilbert spaces). Consider, for each $t \in G$, the map $\rho_t: L_t \to \mathcal{L}(H \oplus H) = \mathcal{M}_2(\mathcal{L}(H))$ such that

$$\rho_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} T_a & T_b \\ T^*_c & T_d \end{pmatrix}, \quad a, b \in B_t, \quad c, d \in \overline{\text{span}}(B_t B_t^*),$$

Then we get the canonical (linear and injective) map $\rho_t \otimes \pi: L_t \otimes C \to \mathcal{L}(H^2 \otimes K)$.

If we restrict this map to $B_t \otimes C$ we note that $H \otimes K = (H \otimes 0) \otimes K \subset H^2 \otimes K$ is an invariant subspace and that the compression (of the restriction) to this subspace is an isometric linear representation of $B_t \otimes C$. Hence we obtain the map $T \otimes \pi: B \otimes C \to $
\( \mathcal{L}(H \otimes K) \) which is linear isometric on each fiber and \( T \circ \pi(a \otimes c) = T_a \otimes \pi(c) \).

Note that \( T \circ \pi(B_x \otimes C) \circ T \circ \pi(B_\cdot \otimes C) \subset T \circ \pi(B_x \otimes C) \) and that \( T \circ \pi(B_x \otimes C)^* = T \circ \pi(B_x \otimes C) \). Thus we are forced to define the multiplication and involution of \( B \otimes C \) in such a way that \( T \circ \pi(xy) = T \circ \pi(x) T \circ \pi(y) \) and \( T \circ \pi(x^*) = T \circ \pi(x)^* \).

**Proposition 7.5.** If \( B \) is a Fell bundle, then \( B_c \) is exact if and only if \( k(B) \) is exact.

**Proof.** In the proof of Theorem 6.8 we constructed an inclusion \( i_F(M) \subset k_* (M) \). That inclusion can be used to prove that \( k(B) \) is the direct limit of \( \{ i_F(B) \}_F \), where \( F \) runs over the finite subsets of \( G \). Hence \( k(B) \) is exact if and only if \( i_F(B) \) is exact for every finite set \( F \subseteq G \). Notice that if we take \( F = \{ e \} \) we conclude that \( B_c \) is exact if \( k(B) \) is.

Assume \( B_c \) is exact and take a finite set \( F \subset G \) and a short exact sequence (SES) of \( C^* \)-algebras \( I \subseteq A \rightarrow A/I \). For every \( t \in G \) the linking algebra of \( B_t \), \( L_t \), is an exact \( C^* \)-algebra because it is Morita equivalent to the ideal \( \text{span}(B_t B_I) \) of \( B_c \). Thus we get the SES \( L_t \otimes I \rightarrow L_t \otimes A \rightarrow L_t \otimes A/I \) and, by our construction of spatial tensor products, we obtain the following SES of Fell bundles

\[
B \otimes I \rightarrow B \otimes A \rightarrow B \otimes A/I.
\]

By entrywise computation of the maps above we get the SES

\[
(7.6) \quad \mathcal{M}_F(B \otimes I) \rightarrow \mathcal{M}_F(B \otimes A) \rightarrow \mathcal{M}_F(B \otimes A/I).
\]

Using a nondegenerate representation \( T: B \rightarrow \mathcal{L}(H) \) with \( T|_{B_c} \) faithful and representing \( I \) faithfully, \( I \subset \mathcal{L}(K) \), we can think of \( \mathcal{M}_F(B \otimes I) \) as a subalgebra of \( \mathcal{L}(H^n \otimes K) = \mathcal{L}((H \otimes K)^n) \). But we can also represent \( B \otimes I \) in \( \mathcal{L}(H \otimes K) \) and thus we get a faithful representation \( \mathcal{M}_F(B \otimes I) \subset \mathcal{L}((H \otimes K)^n) \). It turns out that, after taking faithful representations, \( \mathcal{M}_F(B \otimes I) = \mathcal{M}_F(B) \otimes I \). The same argument holds for the tensor products with \( A \) and \( A/I \). Moreover, the identifications \( \mathcal{M}_F(B \otimes Z) = \mathcal{M}_F(B) \otimes Z \) (for \( Z = I, A, A/I \)) are compatible with the maps of (7.6) and thus we get the SES

\[
\mathcal{M}_F(B) \otimes I \rightarrow \mathcal{M}_F(B) \otimes A \rightarrow \mathcal{M}_F(B) \otimes A/I.
\]

This proves \( \mathcal{M}_F(B) \) is exact. \( \square \)

**Corollary 7.7.** If \( B \) is the enveloping \( C^* \)-algebra of a partial action \( \alpha \) of a locally compact and Hausdorff group on a \( C^* \)-algebra \( A \), then \( B \) is exact if and only if \( A \) is exact.

**Proof.** We can drop the topology of the group and work with the discrete one because this does not affect the enveloping action [27]. The proof follows directly from Proposition 7.5 because \( B \) is Morita equivalent to \( k(B_\alpha) \) and \( A \) is the unit fiber of \( B_\alpha \). \( \square \)

It is known [28] Proposition 7.1] that the crossed product of an amenable group acting on an exact \( C^* \)-algebra is exact. Recently this was proved in [15] Theorem 6.1] for AD-amenable actions. For the convenience of the reader we give a direct proof here. If we go back to the ideas of Kirchberg [28] we see that the following Lemma is the key point.

**Lemma 7.8.** Let \( \alpha \) be an AD-amenable action of \( G \) on the \( C^* \)-algebra \( A \) and \( B \) a \( C^* \)-algebra endowed with the trivial \( G \)-action \( \text{id} \). Then the tensor product \( G \)-action \( \alpha \otimes \text{id} \) on \( A \otimes B \) is AD-amenable.
Theorem 7.11. Let \( B \) be a Fell bundle (over \( G \)) with the WAP and \( B_c \) is exact, then so is \( C^*_w(\mathcal{B}) = C^*(\mathcal{B}) \).

Proof. We know, by Proposition 7.5, that \( k(\mathcal{B}) \) is exact and the canonical action on \( k(\mathcal{B}) \) is AD-amenable (Corollaries 5.26 and 6.10). By Corollary 7.9 \( k(\mathcal{B}) \rtimes G = k(\mathcal{B}) \rtimes G \) is exact. Since this algebra is Morita equivalent to \( C^*_w(\mathcal{B}) = C^*(\mathcal{B}), C^*_w(\mathcal{B}) \) is exact.

In [24, Definition 21.19] Exel introduces the notion of conditional expectation for Fell bundles: if \( A \) is a Fell subbundle of \( B \), a conditional expectation from \( B \) to \( A \) is a map \( P : B \to A \) which restricts to bounded surjective idempotent linear maps \( P_g : B_g \to A_g \subseteq B_g \) such that \( P_e : B_e \to A_e \) is an ordinary conditional expectation and \( P_g(b)^* = P_g(b^*) \) and \( P_{gh}(ba) = P_g(b)a \) for all \( b \in B_g \) and \( a \in A_h, g,h \in G \).

Theorem 7.11. Let \( \mathcal{M} \) be a \( W^* \)-Fell bundle over \( G \), \( N \) a \( W^* \)-Fell subbundle of \( \mathcal{M} \) and \( P : \mathcal{M} \to \mathcal{N} \) a (not necessarily \( w^* \)-continuous) conditional expectation. Then there exists a conditional expectation \( P_k : k_{w^*}(\mathcal{M}) \to k_{w^*}(\mathcal{N}) \) which is equivariant with respect to the canonical \( W^* \)-actions. Moreover, the restriction of \( P_k \) to \( k(\mathcal{M}) \) is a conditional expectation onto \( k(\mathcal{N}) \).

Proof. By Corollary 5.31 we can think of \( k_{w^*}(\mathcal{N}) \) as a \( W^* \)-subalgebra of \( k_{w^*}(\mathcal{M}) \). The matrix algebras \( \mathfrak{M}_F(\mathcal{M}) \) (for \( F \subseteq G \) finite) of Remark 5.7 form an upward directed set of \( C^* \)-subalgebras of \( k_{w^*}(\mathcal{M}) \) with norm closure equal to \( k(\mathcal{M}) \). Moreover, \( \mathfrak{M}_F(\mathcal{M}) \) is hereditary in \( k(\mathcal{M}) \) and in the proof of Theorem 6.8 we showed that \( \mathfrak{M}_F(\mathcal{M}) \) is in fact a \( W^* \)-subalgebra of \( k_{w^*}(\mathcal{M}) \). Hence the family \( \{\mathfrak{M}_F(\mathcal{M})\}_F \) is an upward directed family of hereditary \( W^* \)-subalgebras of \( k_{w^*}(\mathcal{M}) \) whose union is \( w^* \)-dense in \( k_{w^*}(\mathcal{M}) \). We denote \( 1_F \) the unit of \( \mathfrak{M}_F(\mathcal{M}) \). Then \( 1_F \) may or may not be equal to the unit of \( \mathfrak{M}_F(\mathcal{N}) \), which we denote \( 1_F' \).

Define, for each finite subset \( F \subseteq G \), the map \( P_F : \mathfrak{M}_F(\mathcal{M}) \to \mathfrak{M}_F(\mathcal{N}) \subset k_{w^*}(\mathcal{N}) \) as the intrinsic application of \( P \). We claim that this map is a conditional expectation. Indeed, by Tomiyama’s theorem it suffices to prove it is contractive.

Let \( X_{\mathcal{F},\mathcal{M}} \) be the \( \mathfrak{M}_F(\mathcal{M}) \)-Hilbert module obtained by considering on \( \mathfrak{M}_F(\mathcal{M}) \) the inner product \( \langle X, Y \rangle_{\mathcal{F},\mathcal{M}} := \text{trace}(X^*Y) \) and the action given by matrix multiplication. Then matrix multiplication on the left gives a faithful representation...
In particular, this is the case if \( A \) is associated inclusion \( W^* \)-amenable. Then everything follows from [7, Proposition 3.8] and the Theorem.

Thus \( ||P_F(A)|| \leq ||A|| \) and \( P_F \) is contractive.

We can extend \( P_F \) to \( k_w^*(M) \) by defining \( P_F: k_w^*(M) \to k_w^*(N) \) as \( P_F(x) = P_F(F x 1_F) \). Then \( P_F \) is clearly ccp, in fact it is a conditional expectation over \( M(M) \). In this way we get a net of ccp maps \( \{ P_F \}_F \) from \( k_w^*(M) \) to \( k_w^*(N) \). Let \( P_k \) be a pointwise \( W^* \)-limit of a converging subnet \( \{ P_F \}_j \). Clearly \( P_k \) is ccp. Take \( x \in k_w^*(N) \). Since \( P_F(x) = 1_F x 1_F \), both \( \{ P_F(x) \}_F \) and \( \{ P_F(x) \}_F \) w*-converge to \( x \). Thus \( P_k(x) = x \) and \( P_k \) is a conditional expectation.

We claim that \( P_k \) is equivariant with respect to the canonical \( W^* \)-actions. Take \( t \in G \) and note that \( \beta^w_t^w(1_F) = 1_{F t^{-1}} \) and that, given \( x \in M(M) \), it follows that \( P_{F t^{-1}}(\beta^w_t^w(x)) = \beta^w_t^w(P_F(x)) \). Considering w*-limits we have

\[
P_k(\beta^w_t^w(x)) = \lim_j P_{F t}(1_{F t} \beta^w_t^w(x) 1_{F t}) = \lim_j P_{F t}(\beta^w_t^w(1_{F t} x 1_{F t}))
\]

\[
= \lim_j \beta^w_t^w(P_{F t}(1_{F t} x 1_{F t})) = \beta^w_t^w(\lim_j P_{F t}(1_{F t} x 1_{F t}))
\]

Thus \( \{ P_{F t}(1_{F t} x 1_{F t}) \}_j \) actually has a w*-limit (for every \( x \)) and it suffices to show that this limit is \( P_k(x) \).

Every element \( v \) of \( k_w^*(N) \) is completely determined by the products \( uvw \), for \( u, v \in k_c(M) \). Then it suffices to prove that \( \lim_j u P_{F t}(1_{F t} x 1_{F t}) w = u P_k(x) w \), for all \( u, w \in k_c(M) \). Fix \( u, v \in k_c(M) \) and take a finite set \( K \subset G \) such that \( K \times K \) contains both the supports of \( u \) and \( w \). Since the families \( \{ F_j \}_j \) and \( \{ F_t \}_j \) are cofinal in the finite subsets of \( G \) we have

\[
\lim_j u P_{F t}(1_{F t} x 1_{F t}) w = \lim_j u' K P_{F t}(1_{F t} x 1_{F t}) 1'_K w
\]

\[
= \lim_j u P_{F t}(1' K 1_{F t} x 1_{F t} 1'_K) w = u P_k(1' K x 1'_K) v
\]

\[
= u P_k(1' K x 1'_K) v = u 1'_K P_k(x) 1'_K v = u P_k(x) v.
\]

Finally, the last statement is clear from the computations above. In fact if \( F := \{ F \subset G : F \) is finite \}, then it is easy to see also that the \( C^* \)-limits of the direct systems \( \{ M(M) \}_{F \in F} \) and \( \{ M(F(N)) \}_{F \in F} \) are \( k(M) \) and \( k(N) \) respectively, and that \( P_k|_{k(M)} \) is the limit of the direct system \( \{ M(F(M)) \}_{F \in F} \xrightarrow{P_k} \{ M(F(N)) \}_{F \in F} \).

**Corollary 7.12.** Let \( M \) be a \( W^* \)-Fell bundle over \( G \), \( N \) a \( W^* \)-Fell subbundle of \( M \) and \( P: M \to N \) a (not necessarily \( W^* \)-continuous) conditional expectation. If \( M \) has the \( W^* \)AP then so does \( N \).

**Proof.** Recall that \( M \) has the \( W^* \)AP iff the canonical \( W^* \)-action on \( k_w^*(M) \) is \( W^* \)AD-amenable. Then everything follows from [7, Proposition 3.8] and the Theorem above. \( \square \)

**Corollary 7.13.** Let \( A \) be a Fell subbundle of \( B \). If \( B \) has the \( W^* \)AP and the associated inclusion \( A'' \hookrightarrow B'' \) admits conditional expectation, then \( A \) has the \( W^* \)AP. In particular, this is the case if \( A \) is hereditary in \( B \) (i.e. \( A_e B, A_e \subset A_r \) for all \( r \in G \)).

**Proof.** If \( A \) is hereditary in \( B \) and \( p \) is the unit of \( A''_e \subset B''_e \), then the map \( P: B'' \to A'' \), \( b \mapsto p b p \), is a conditional expectation. Then the proof follows from our last Corollary. \( \square \)
Corollary 7.14. Let \( \mathcal{A} \) be a Fell subbundle of a Fell bundle \( \mathcal{B} \) and suppose there exists a conditional expectation \( P: \mathcal{B} \to \mathcal{A} \). If \( \mathcal{B} \) has the WAP, then so does \( \mathcal{A} \).

Proof. It suffices to construct a conditional expectation \( P''': \mathcal{B}'' \to \mathcal{A}''' \). Take a fiber \( B_t \) and consider it’s linking algebra as

\[
L(B_t) = \begin{pmatrix}
I_t & B_t \\
B_{t^{-1}} & I_{t^{-1}}
\end{pmatrix},
\]

where \( I_t \) is (the closed linear span of) \( B_t B_t^* \) in \( B_e \). Then we can form a conditional expectation \( L(P): L(B_t) \to L(A_t) \) by entrywise computation of \( P \). Using Stinespring’s factorization theorem we can extend \( L(P)_t \) \( \tau \)-continuously to a conditional expectation \( L(P)_t''': L(B_t)'' \to L(A_t)''' \). If we now restrict this last map to \( B_t'' \) we get a map \( P''': B_t'' \to A_t''' \). We leave to the reader the verification of the fact that \( P''' := \{P'_t\}_{t \in G} : \mathcal{B}'' \to \mathcal{A}''' \) is a conditional expectation.

\[\square\]

8. Fell bundles with commutative unit fibre

This section is dedicated to the study of amenability for Fell bundles with commutative unit fibre. Let \( \mathcal{B} = (B_t)_{t \in G} \) be a Fell bundle such that \( B_e = C_0(X) \) is a commutative \( C^* \)-algebra. Such a Fell bundle canonically induces a partial action of \( G \) on \( X \) and hence also on \( C_0(X) \). This is because imprimitivity bimodules between commutative \( C^* \)-algebras yield isomorphisms between their spectra. Since each \( B_t \) can be viewed as an imprimitivity \( I_t \)-\( I_t^{-1} \)-bimodule, where \( I_t := B_t B_t^* \cong C_0(A_t) \) for some open subset \( A_t \subseteq X \), it yields an isomorphism \( \alpha_t : C_0(A_t^{-1}) \to C_0(A_t) \) or, equivalently, to a homeomorphism \( \theta_t : A_t^{-1} \to A_t \). The collection \( \alpha = (\alpha_t)_{t \in G} \) (resp. \( \theta = (\theta_t)_{t \in G} \)) is then the desired partial action of \( G \) on \( C_0(X) \) (resp. \( X \)); they are related by the equation \( \alpha_t(f) = f \circ \theta_t^{-1} \) for all \( f \in C_0(A_t^{-1}) \).

Definition 8.1. The partial action \( \theta \) of \( G \) on \( X \) or its associated partial action \( \alpha \) on \( C_0(X) = B_e \) defined above will be call the spectral partial action of \( \mathcal{B} \).

These partial actions are analogous to the central partial actions defined in Section 5.24. Moreover, they are special cases of the more general partial actions on the spectrum of \( B_e \), as in 11 for an arbitrary Fell bundle \( \mathcal{B} \) (\( B_e \) need not be abelian here). Moreover, we see from the constructions that the central partial action of \( \mathcal{B}'' \) is the double dual partial action \( \alpha'' \) of \( G \) on the \( W^* \)-algebra \( C_0(X)'' \) associated with \( \alpha \). From this we immediately derive the following result:

Corollary 8.2. A Fell bundle \( \mathcal{B} \) with commutative unit fibre is \( AD \)-amenable (or, equivalently, has the WAP) if and only if its spectral partial action is \( AD \)-amenable (or has the WAP).

Proof. By definition, \( \alpha \) is \( AD \)-amenable if and only if its double dual partial action \( \alpha'' \) is \( W^* \)-\( AD \)-amenable. Since \( \alpha'' \) is the central partial action of \( \mathcal{B}'' \), the result follows from Corollary 5.24 and the definition of \( AD \)-amenability (Definition 5.24) and the equivalence between \( AD \)-amenability and the WAP, Theorem 5.24.

\[\square\]

From the spectral partial action \( \alpha \) of \( \mathcal{B} \), we obtain another Fell bundle \( \mathcal{B}_\alpha \), the one associated to the partial action \( \alpha \). These Fell bundles are not necessarily isomorphic in general because the original Fell bundle may contain some “twist”. One form of twist is given in terms of 2-cocycles for partial actions as defined by Exel in [25]. More precisely, Exel introduces the notion of a twisted partial action. In
the commutative case, besides a partial action $\alpha$ of $G$ on $C_0(X)$, this involves certain unitary multipliers $\omega(s,t) \in \text{UM}(D_t)$. We do not need to recall the precise conditions on $\omega$ and its relation with $\alpha$. We only recall that the Fell bundle $B_{\alpha,\omega}$ associated with the twisted partial action $(\alpha,\omega)$ has fibres $B_{\alpha,\omega,t} := C_0(D_t)\delta_t \cong C_0(D_t)$ and multiplications and involutions given by:
\[ (f\delta_s) \cdot (g\delta_t) := \omega(s,t)\alpha_s(\alpha_t^{-1}(f)g)\delta_{st}, \quad (f\delta_s)^* := \omega(s^{-1},s)\alpha_{s^{-1}}(f^*)\delta_{s^{-1}} \]
for all $s,t \in G$, $f \in C_0(D_s)$ and $g \in C_0(D_t)$.

The main result in [25] states that every regular Fell bundle is isomorphic to one associated with a twisted partial action. The regularity of $B$ concerns the structure of the fibres $B_t$ as imprimitivity $I_t$-$I_t$-bimodules. Since the $C^*$-algebras $I_t = C_0(D_t)$ are commutative, such imprimitivity bimodules are necessarily given as $C_0$-sections of a certain (complex) line bundle $L_t$ over $D_t$. The $C_0$-section $C_0(L_t)$ of such a line bundle may be viewed as an imprimitivity $C_0(D_t)$-$C_0(D_t)$-bimodule; using the isomorphism $\alpha_t : C_0(D_t^{-1}) \xrightarrow{\sim} C_0(D_t)$ we may also view $C_0(L_t)$ as an imprimitivity $C_0(D_t)$-$C_0(D_t^{-1})$-bimodule which is then isomorphic to $B_t$. The regularity of $B_t$ is then equivalent to $L_t$ being topologically trivial as a complex line bundle. This is always the case for Fell bundles associated with twisted partial actions but it might be not the case in general, so that our original Fell bundle $B$ is not necessarily isomorphic to $B_{\alpha,\omega}$, not even as Banach bundles. However amenability does not see these differences. Indeed, again the spectral and central partial actions of all three Fell bundles $B$, $B_\alpha$ and $B_{\alpha,\omega}$ are isomorphic, so we immediately obtain the next corollary (an improvement of the previous one):

**Corollary 8.3.** A Fell bundle with commutative unit fibre $B$ is AD-amenable (or has the WAP) if and only if so is $B_\alpha$ or, equivalently, $B_{\alpha,\omega}$.

Notice that by Proposition 7.3 the equivalent conditions in the above corollary are also equivalent to nuclearity of one of the $C^*$-algebras $C^*_{\{t\}}(B)$, $C^*_{\{t\}}(B_\alpha)$ $\cong$ $C_0(X) \rtimes_{\alpha_t\{t\}} G$ or $C^*_{\{t\}}(B_{\alpha,\omega}) = C_0(X) \rtimes_{\alpha_{\omega,\{t\}}} G$. In particular we obtain the interesting consequence that $C_0(X) \rtimes_{\alpha_t\{t\}} G$ is nuclear if and only if $C_0(X) \rtimes_{\alpha_{\omega,\{t\}}} G$ is nuclear for every twisted partial action $(\alpha,\omega)$. The parenthesis around $r$ here means that we can take either the full or the reduced crossed product (or cross-sectional $C^*$-algebras) in all equivalent statements. Indeed, any other exotic crossed-product norm between the full and reduced could be used for that matter.

The above results can also be interpreted using groupoid descriptions of the associated $C^*$-algebras. To explain this, let us first recall that a partial action $\theta$ of $G$ on $X$ yields a locally compact Hausdorff étale transformation groupoid $\Gamma = X \rtimes_\theta G$ (see [M]). As a set it consists of pairs $(x,t)$ with $t \in G$ and $x \in D_t^{-1}$. The source and range maps are $s(x,t) := x$ and $r(x,t) := t \cdot x := \theta_t(x)$ and multiplication and inversion are given by
\[ (x,s) \cdot (y,t) = (y, st), \quad (x,t)^{-1} = (t \cdot x, t^{-1}) \quad \text{for } x = t \cdot y. \]

The topology is the one inherited from the product topology on $X \times G$. The domains $D_t$ give rise to subsets $\Gamma _t := D_t \times \{ t^{-1} \} \subseteq \Gamma$ that are clopen bisections of $\Gamma$ (although the domains $D_t$ are only assumed to be open in $X$). Hence $\Gamma$ decomposes as a disjoint union $\Gamma = \bigsqcup_{t \in G} \Gamma _t$ of clopen subsets. In particular the vector space $C_c(\Gamma)$ identifies canonically with the algebraic direct sum $\bigoplus_{t \in G} C_c(\Gamma _t)$, that is, functions $\zeta \in C_c(\Gamma)$ correspond bijectively to finite sets of functions $\zeta_t \in C_c(\Gamma _t)$, $t \in G$. This
identification extends to a canonical isomorphism $C^*_\alpha(\Gamma) \cong C_0(X) \rtimes_{\alpha, (r)} G$, where $\alpha$ is the partial action of $G$ on $C_0(X)$ corresponding to $\theta$.

Now, given a Fell bundle $B$ over $G$ with unit fibre $B_e = C_0(X)$ as above, let $\theta$ be its spectral action and $\Gamma$ the corresponding transformation groupoid that we call the spectral groupoid of $B$. As previously, we also identify each fibre $B_1$ with the sections $C_0(L_t)$ of a line bundle $L_t$ over $D_t$. The disjoint union $L := \sqcup_{t \in G} L_t$ can then be viewed as a line bundle over $\Gamma = \sqcup_{t \in G} \Gamma_t$. Moreover, with the Fell bundle structure inherited from $B$, $L$ is indeed a Fell line bundle over $\Gamma$; such a Fell bundle is also usually viewed as a twist over $\Gamma$. By construction we get an obvious identification $C_c(\Gamma, L) \cong C_c(B)$ that extends to an isomorphism $C^*_\alpha(\Gamma, L) \cong C^*_\beta(B)$. In other words, we have described every Fell bundle over a discrete group with commutative unit fibre in terms of a twisted groupoid. This result can be deduced from the constructions and results in [17] that describe Fell bundles over inverse semigroups with commutative fibres over idempotents (also call semi-abelian Fell bundles in [17]).

Next we relate amenability of $B$ in terms of amenability of its spectral groupoid. Amenable groupoids are defined and studied mainly in [11]. We shall use the characterisation from [13, Lemma 5.6.14] that says that an étale groupoid $\Gamma$ is amenable if and only if there is a net $(\zeta_i) \subseteq C_c(\Gamma)$ with $\|\zeta_i\|_2 \leq 1$ for all $i$ and $(\zeta_i * \zeta_i)(\gamma) \to 1$ uniformly for $\gamma$ in compact subsets of $\Gamma$. One of the main results in this direction states that $\Gamma$ is amenable if and only if $C^*_\alpha(\Gamma)$ is nuclear.

Notice that the spectral groupoids of $B$ and $B_\alpha$ (and also of $B_{\alpha, \omega}$) are the same; they are just the transformation groupoid $\Gamma = X \rtimes_\theta G$ of the spectral partial action of $B$. In particular we can reinterpret our previous results as follows:

**Corollary 8.4.** A Fell bundle with commutative unit fibre is AD-amenable if and only if its spectral groupoid is amenable.

Using the description of $B$ in terms of a twisted groupoid $(\Gamma, L)$ and that AD-amenability is equivalent to nuclearity of the corresponding $C^*$-algebras, we can also interpret the above result as the statement that $C^*_\alpha(\Gamma, L)$ is nuclear if and only if $C^*_\alpha(\Gamma)$ is nuclear in $C^*_\beta$. In other words, nuclearity of a twisted groupoid $C^*$-algebra is independent of the twist. Indeed, in this form this result is already known, see [23].

Up to this point we have only looked at AD-amenability or, equivalently, the WAP for Fell bundles with commutative unit fibre. We also want to consider the AP for such Fell bundles. We know already that the AP always implies the WAP but we do not know whether the converse holds in general. The next result aims at a partial converse:

**Theorem 8.5.** Let $B$ be a Fell bundle over $G$ with commutative unit fibre $B_e = C_0(X)$. Let $\theta$ be its spectral partial action with associated partial action $\alpha$ on $C_0(X)$, and let $\Gamma = X \rtimes_\theta G$ be its spectral groupoid. Then the following assertions are equivalent:

(i) $B$ has the WAP or is AD-amenable, that is, $C^*_\alpha(B)$ is nuclear;

(ii) $B_\alpha$ has the WAP or is AD-amenable;
We check that this net also gives the AP for $\mathcal{B}$ and we already observed that this is equivalent to amenability of $\Gamma$. This implies the desired convergence that gives the AP for $\mathcal{B}$.

Proof. We already checked the equivalences (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv). It remains to check the equivalence of these conditions with (v), (vi) and (vii). Assume that $\Gamma$ is amenable and let $\{\zeta_i\}_{i \in I}$ be a net of functions in $C_c(\Gamma)$ with $\|\zeta_i\|_2 \leq 1$ for all $i$ and $\zeta_i^* \zeta_i(\gamma) \to 1$ uniformly for $\gamma$ in compact subsets of $\Gamma$. Define $\xi_i : G \to C_0(X)$ by $\xi_i(t)(x) := \zeta_i(x, t^{-1})$ if $x \in D_t$ and 0 otherwise. In other words, we just use the canonical identification $C_c(\Gamma) \cong \oplus_{t \in G} C_c(D_t)$ to view each $\zeta_i$ as a finitely supported function $\xi_i : G \to C_c(X)$ with $\xi_i(t) \in C_c(D_t)$ for all $t \in G$. We verify that this net yields the AP for $\mathcal{B}_\alpha$. The boundedness of $\langle \xi_i \rangle$ for the $L^2$-norm implies the same boundedness for $\langle \xi_i \rangle$. It remains to check the convergence condition that gives the AP. For this it is enough to check that if $f \in C_c(D_t)$, then $\sum_{s \in G} \xi_i(ts)^* (f \delta_t) \xi_i(s)$ converges in norm to $f \delta_t$. By definition, we have

$$\sum_{s \in G} \xi_i(ts)^* (f \delta_t) \xi_i(s) = \sum_{s \in G} \xi_i(ts)^* \alpha_t(f) (\xi_t(s)) \delta_t.$$  

Computing this sum at some $x \in D_t$ we get an expression of the form:

$$\sum_{s \in G} \xi_i(x, s^{-1}t^{-1}) f(x) \xi_i(\theta_t^{-1}(x), s^{-1})$$

where the sum varies over all $s \in G$ in a finite subset (depending on the support of $\zeta_i$) satisfying $\theta_t^{-1}(x) \in D_s$ or, equivalently, $x \in D_{ts}$. The above sum can be rewritten as

$$\left( \sum_{\alpha \in \Gamma} \overline{\alpha(\gamma)} \xi_i(\alpha \gamma) \right) f(x) = (\xi_i^* \xi_i)(\gamma) f(x)$$

where $\gamma = (\theta_t^{-1}(x), t)$ and $\alpha \in \Gamma$ varies in a finite subset (depending on the support of $\zeta_i$) satisfying $s(\alpha) = r(\gamma)$; those $\alpha$ are necessarily of the form $(x, s^{-1}t^{-1})$ with $x \in D_{ts}$. If $x$ varies in the compact support $K := \text{supp}(f) \subseteq D_t$ of $f$, then $\gamma = (\theta_t^{-1}(x), t)$ varies in a compact subset of $\Gamma$ so that $(\xi_i^* \xi_i)(\gamma) \to 1$ uniformly on this compact subset. This implies the desired convergence that gives the AP for $\mathcal{B}_\alpha$. Hence (iv)$\Rightarrow$(v). Moreover, if $\omega$ is a 2-cocycle for $\alpha$, then the computation $\sum_{s \in G} \xi_i(ts)^* (f \delta_t) \xi_i(s)$ is the same because $\omega(t, e) = 1$. Therefore the same argument also yields the implication (iv)$\Rightarrow$(vi). Conversely, if $\mathcal{B}_\alpha$ or $\mathcal{B}_{\alpha, \omega}$ has the AP, then it also has the WAP and we already observed that this is equivalent to amenability of $\Gamma$. This yields the implications (v),$\text{(vi)}$,$\text{(vii)} \Rightarrow$ (iv).

It remains to check (iv)$\Rightarrow$(vii). The proof is essentially the same as before, let us give more details: let $\{\zeta_i\}_{i \in I} \subseteq C_c(\Gamma)$ be a net that gives the amenability of $\Gamma$ and let $\{\xi_i\}_{i \in I}$ be the same net as above defined from $\{\zeta_i\}_{i \in I}$ that gives the AP for $\mathcal{B}_\alpha$. We check that this net also gives the AP for $\mathcal{B}$. For this we identify $\mathcal{B}_t \cong C_0(L_t)$ for a line bundle $L_t$ as before. The structure of $C_0(L_t)$ as a Hilbert $C_0(D_t)-C_0(D_{t-1})$ is as follows: the left and right inner products are given by $\langle \xi | \eta \rangle(x) := (\eta(x))_x$ and $\langle \xi | \eta \rangle_r(x) := (\xi(\theta_t(x)) | \eta(\theta_t(x)))_x$. Here we use that $L_t$ is a Hermitian complex line bundle and $\langle \cdot | \cdot \rangle_x$ denotes the inner product on each fibre $L_{t,x}$; this inner product is assumed to be linear on the second variable and it is continuous so that the inner products on $C_0(L_t)$ are well defined. The left action of $C_0(D_t)$ and
the right action of $C_0(D_{t-1})$ on $C_0(L_t)$ are given by $(f \cdot \xi)(x) := f(x)\xi(x)$ and $(\xi \cdot g)(x) := \xi(x)g(\theta_t^{-1}(x))$ for all $x \in D_t$.

Having this, essentially the same proof as before still works: we take an element $\eta \in C_c(L_t)$ and verify that

$$
\sum_{s \in G} \xi_i(ts) \cdot \eta \cdot \xi_i(s) \to \eta
$$

with respect to the norm of $C_0(L_t) \cong B_1$. But the expression above is a section of $L_t$ and when we compute at some $x \in D_t$ we get:

$$
\sum_{s \in G} \zeta_i(x,s^{-1}t^{-1})\eta(x)\zeta_i((\theta_t^{-1}(x),s^{-1})
$$

which as before can be rewritten as

$$
\left(\sum_{\alpha \in \Gamma} \zeta_i(\alpha)\zeta_i(\alpha\gamma)\right)\eta(x) = (\zeta_i^* \cdot \zeta_i)(\gamma)\eta(x).
$$

Using that $(\zeta_i^* \cdot \zeta_i)(\gamma) \to 1$ uniformly on compacts, the desired result follows. \hfill \qed

**Corollary 8.8.** Let $\mathcal{B}$ be a Fell bundle which is weakly equivalent to a Fell bundle $\mathcal{A}$ with commutative unit fibre. Then $\mathcal{B}$ has the AP if and only if it has the WAP (i.e. is AD-amenable).

**Proof.** We showed that both the WAP and the AP are preserved by the weak equivalence of Fell bundles and our last Theorem implies $\mathcal{A}$ has the AP if and only if it has the WAP. Thus the claim follows. \hfill \qed

Given a Fell bundle $\mathcal{B}$ as in the Corollary above, there may not be a suitable candidate for the spectral grupoid, mainly because the spectrum of $B_e$ may not be Hausdorff. Consider for example the semidirect product bundle of a Morita enveloping action of a partial action on a commutative C*-algebra which does not have an enveloping action [2]. If we want to generalize Theorem 8.5, it is then reasonable to assume $B_e$ is Morita equivalent to a commutative C*-algebra.

**Theorem 8.9.** Suppose $\mathcal{B}$ is a Fell bundle over $G$, $\mathcal{A}$ is a C*-algebra and $M$ is an $A$-$B_e$-equivalence bimodule. Consider, for each $t \in G$, $B_t$ as a left $B_e$-Hilbert module and let $M \otimes B_t$ be the $B_e$-inner tensor product. Then there exists a unique right Hilbert $\mathcal{B}$-bundle structure [6 Definition 2.1] on $X := \{M \otimes B_t\}_{t \in G}$ such that, for all $x, y \in M$ and $a, b \in B$:

$$
\langle x \otimes a, y \otimes b \rangle_{\mathcal{B}} = a^*(x,y)_{B_e}b \quad \text{and} \quad (x \otimes a)b = x \otimes (ab).
$$

Let also $K(X) = \{K_t\}_{t \in G}$ be the bundle of generalized compact operators, as in [6 Theorem 3.9]. Then $X$ is a strong $K(X)$ – $\mathcal{B}$ equivalence and the unit fibre $K_e$ is isomorphic to $A$.

**Proof.** Uniqueness follows from [6] and the fact that elementary tensor products span a dense subspace of the tensor products $M \otimes B_t$, thus we only need to prove the existence claim.
First we show that the action of $B$ on $X$ is defined. Take $r,s \in G, x_1, \ldots, x_n \in M, a_1, \ldots, a_n \in B_r$ and $b \in B_s$. Then

$$\| \sum_{i=1}^{n} x_i \otimes (a_i b) \|^2 = \| \sum_{i=1}^{n} b_i^* (x_i, x_j)_{B_r} a_i b \|^2 = \| b^* (\sum_{i=1}^{n} x_i \otimes a_i) b \|^2 \leq \| b \|^2 \| \sum_{i=1}^{n} x_i \otimes a_i \|^2.$$

With the inequality above we can easily prove the existence of a bilinear map

$$(M \otimes B_r) \times B_s \to M \otimes B_r, \ (u, b) \mapsto ub,$$

such that $(x \otimes a) b = x \otimes (ab)$ and $\|ub\| \leq \|u\|\|b\|$. These maps define the action of $B$ on $X$.

We now construct the $B$-valued inner product of $X$. Take $r,s \in G, x_1, \ldots, x_n \in M, y_1, \ldots, y_n \in M, a_1, \ldots, a_n \in B_r$ and $b_1, \ldots, b_n \in B_s$. Set $u := \sum_{i=1}^{n} x_i \otimes a_i \in M \otimes B_r$ and $v := \sum_{i=1}^{n} y_i \otimes b_i \in M \otimes B_s$ and $w := \sum_{i,j=1}^{n} a_i^* (x_i, y_j)_{B_r} b_j \in B_{r^{-1}s}$. In order to prove the inner product is defined it suffices to show that

$$\| w \| \leq \| u \| \| v \|,$$

because after this inequality we can set $\langle u, v \rangle_B := w$.

Let $[u, u]_r \in K(M \otimes B_r)$ represent the generalized compact operator $z \mapsto u(u, z)$, and let $\varphi_r : A = K(M) \to K(M \otimes B_r)$ be the unique $^*$-homomorphism such that $\varphi_r(a)(z \otimes c) = (az) \otimes c$. It is straightforward to show that $[u, u]_r = \varphi_r(\sum_{i,j=1}^{n} a_i^* (x_i, y_j))$. We know $\varphi_r$ may not be injective, but it is injective when restricted to the ideal $\sum_{i} MB_i B_i^\ast, M$. Thus

$$\| w \|^2 = \| [u, u]_r \|^2 = \| \sum_{i,j=1}^{n} A(x_i a_i b_j^*, x_j) \| = \| \sum_{i,j=1}^{n} A(x_i, x_j b_j^* a_i^*) \|.$$

To prove (8.11) note that $w^* w = \sum_{i,j,k,l,p=1}^{n} b_j^* (y_j, x_i)_{B_r} a_i^* (x_i, y_k)_{B_r} b_k$. We have $a := (a_i^* a_i^*)_{i=1}^{n} \in \mathcal{M}_n(B_e)$ and, if $d := a^{1/2} \in \mathcal{M}_n(B_e)$, then $a_i a_i^* = \sum_{p=1}^{n} d_{i,p} d_{i,p}^\ast$. This implies

$$w^* w = \sum_{i,j,k,l,p=1}^{n} b_j^* (y_j, x_i)_{B_r} d_{i,p} d_{i,p}^\ast (x_i, y_k)_{B_r} b_k$$

$$= \sum_{i,j,k,l,p=1}^{n} b_j^* (A(d_{i,p} d_{i,p}^\ast, x_i) y_j, y_k)_{B_r} b_k$$

$$= \sum_{j,k=1}^{n} b_j^* \left( \sum_{i=1}^{n} A(x_i a_i^* a_i^*, x_i) \right) y_j, y_k)_{B_r} b_k.$$

Consider the direct sum of $n$ copies of $M$, $\oplus_n M$, as a $\mathcal{M}_n(A) - B_e$ equivalence bimodule with left and right inner products given by

$$\langle f_1, \ldots, f_n \rangle_{\mathcal{M}_n(A)} (g_1, \ldots, g_n) := \langle A(f_i, g_i) \rangle_{i=1}^{n},$$

$$\langle (f_1, \ldots, f_n), (g_1, \ldots, g_n) \rangle_{B_e} := \sum_{i=1}^{n} \langle f_i, g_i \rangle_{B_e}.$$
with \([u, u]_r\) via \(\varphi_r\), thus \(d \geq 0\). With these considerations and defining \(\xi := (b_1, \ldots, b_n) \in \oplus_n B_u\) and \(\eta := (y_1, \ldots, y_n) \in \oplus_n M\) we have

\[ w^*w = \langle \xi, \varphi_n(B_u) \langle \eta, \text{diag}(d)\eta \rangle \rangle_{B_u}. \]

Viewing the direct sum of adjoints \(\oplus_n M^*\) as an \(M_n(B_v) - A\) Hilbert bimodule, we deduce that \(\varphi_n(B_u) \langle \eta, \text{diag}(d)\eta \rangle \rangle \leq \|d\|\|\varphi_n(B_u)\| \|\eta\|\). By \([8,11]\), in \(B_c\) we have

\[ w^*w \leq \|d\|\|\xi, \varphi_n(B_u) \langle \eta, \text{diag}(d)\eta \rangle \rangle_{B_u} = \|u\|^2 \langle v, v \rangle_{B_u}, \]

and this clearly implies \(\|u\| \leq \|v\|\).

At this point we have shown that the inner product of \(X\) and the action of \(B\) on \(X\) are defined, the reader can now check that these operations satisfy the conditions of [6] Definition 2.1.

In the rest of the proof we use the notation of [6, Theorem 3.9]. Recall in particular that, for \(u \in M \otimes B_r\) and \(v \in M \otimes B_s\), \([u, v]\) is the adjointable operator of order \(rs^{-1}\) given by \(X \rightarrow X, w \mapsto u(v, w)_B\). The adjointable operators of order \(e\) of \(X, B_e(X)\), form a C*-algebra and

\[ K_e = \text{span}\{[u, v]: u, v \in M \otimes B_t, \ t \in G\} = \text{span}\{[x \otimes b, y \otimes c]: x \otimes b, y \otimes c \in M \otimes B_r, \ r \in G\}. \]

There exists a unique *-homomorphism \(\varphi: A \rightarrow B_e(X)\) such that \(\varphi(a)(x \otimes b) = (ax) \otimes b\). Note that \(\varphi\) is injective because we can think of the unit fiber \(M \otimes B_e\) as an \(A - B_e\) equivalence bimodule. If \(x \otimes b, y \otimes c \in M \otimes B_t\), then for all \(z \otimes d \in M \otimes B_s\) we have

\[ [x \otimes b, y \otimes c](z \otimes d) = x \otimes bc^* \langle y, z \rangle_{B_s}d = xbc^* \langle y, z \rangle_{B_s} \otimes d = \varphi(A \langle xbc^*, y \rangle) z \otimes d. \]

Since the elements \([x \otimes b, y \otimes c]\) span a dense subset of \(K_e\), we conclude that \(\varphi(A) = K_e\) is isomorphic to \(A\).

In order to prove that \(X\) is a strong equivalence we must show that

\[ \text{span} K_t K^*_t = \text{span}[M \otimes B_t, M \otimes B_t], \ \forall t \in G. \]

Recall that \(K_t\) is the closure in \(B_t(X)\) of span\{\([u, v]: u \in M \otimes B_{r1}, v \in M \otimes B_{r2}, r \in G\}\}. Fix \(t \in G\) and take \(r_1, r_2 \in G\), \(x_i \otimes a_i \in B_{r_i}, y_i \otimes b_i \in M \otimes B_{r_i}\), for \(i = 1, 2\). Then

\[ \begin{align*}
[x_1 \otimes a_1, y_1 \otimes b_1][x_2 \otimes a_2, y_2 \otimes b_2]^* (z \otimes c) &= [x_1 \otimes a_1, y_1 \otimes b_1]y_2 \otimes b_2 a_2^* \langle x_2, z \rangle_{B_s, c} = x_1 \otimes a_1 b_1^* \langle y_1, y_2 \rangle_{B_s, b_2 a_2^*} \langle x_2, z \rangle_{B_s, c} \\
&= x_1 a_1 b_1^* \langle y_1, y_2 \rangle_{B_s, b_2 a_2^*} \langle x_2, z \rangle_{B_s} \otimes c \\
&= \varphi(A \langle x_1 a_1 b_1^* \langle y_1, y_2 \rangle_{B_s}, b_2 a_2^* \langle x_2, z \rangle_{B_s} \rangle) z \otimes c \\
&= [x_1 \otimes a_1 b_1^* \langle y_1, y_2 \rangle_{B_s}, x_2 \otimes a_2 b_2^*] (z \otimes c). \end{align*} \]

This implies the inclusion \(\subseteq\) in \((8.12)\). To prove the converse take \(x \otimes a, y \otimes b \in M \otimes B_r\). We can write \(a = a_1 b_1^*\) and \(b = a_2 b_2^*\) for some \(a_1, a_2 \in B_t\) and \(b_1, b_2 \in B_e\) (by Cohen-Hewitt’s Theorem). We can also approximate \(a = a_1 b_1^*\) in norm by sums
of elements of the form $a_1 b_1^* (y_1, y_2)_B$. This allows us to approximate, in $\mathcal{B}_c(X)$, the operator $[x \otimes a, y \otimes b]$ by sums of operators of the form

$$[x_1 \otimes a_1 b_1^* (y_1, y_2)_B, x_2 \otimes a_2 b_2^*] = [x_1 \otimes a_1, y_1 \otimes b_1][x_2 \otimes a_2, y_2 \otimes b_2]^* \in K_1 K_1^*.$$ 

Thus the inclusion $\supseteq$ in (8.12) follows. 

\[\square\]

**Corollary 8.13.** Suppose $\mathcal{B}$ is a Fell bundle over $G$ and that $\mathcal{B}_c$ is Morita equivalent to a commutative C*-algebra $\mathcal{C}_0(X)$ through an equivalence bimodule $M$. Identify $X$ with the primitive ideal space of $\mathcal{B}_c$ and let $\alpha$ be the partial action defined by $\mathcal{B}$ on $\mathcal{C}_0(X)$. Let also $\Gamma$ be the groupoid associated to $\alpha$. Then the following assertions are equivalent:

1. $\mathcal{B}$ has the WAP or is AD-amenable, that is, $C^*_\alpha(\mathcal{B})$ is nuclear;
2. $\mathcal{B}_\alpha$ has the WAP or is AD-amenable;
3. $C^*_\alpha(\mathcal{B}_\alpha) = C^*_\alpha(\Gamma) = \mathcal{C}_0(X) \rtimes_{\alpha, \Gamma} G$ is nuclear;
4. $\Gamma$ is amenable;
5. $\mathcal{B}_\alpha$ has the AP;
6. for every 2-cocycle $\omega$ for $\alpha$, the corresponding Fell bundle $\mathcal{B}_\alpha \omega$ has the AP;
7. $\mathcal{B}$ has the AP.

**Proof.** Let $\mathcal{X} = \{M \otimes B_1\}_{1 \in \mathbb{G}}$ be the equivalence bundle of Theorem 8.3. Since $\mathcal{X}$ is a strong equivalence bundle and $\mathcal{C}_0(X)$ is the unit fibre of $K(\mathcal{X})$, $\alpha$ is (isomorphic to) the partial action defined by $K(\mathcal{X})$ [5]. These facts and Theorem 8.3 imply that:

- (ii) to (vi) are equivalent to: (i) $K(\mathcal{X})$ has the WAP or is AD-amenable, that is, $C^*_\alpha(K(\mathcal{X}))$ is nuclear; and to (vi') $K(\mathcal{X})$ is amenable.
- (i) $\iff$ (i').
- (vii) $\iff$ (vii'). 

\[\square\]

**Example 8.14.** In [23, Proposition 37.9] Exel provides a partial crossed product description for the C*-algebra of every directed graph $E = (s, r : E^1 \to E^0)$ with no sinks (i.e. $s^{-1}(v) \neq \emptyset$ for all $v \in E^0$). In other words, we have an isomorphism

$$C^*(E) \cong \mathcal{C}_0(X) \rtimes_{\alpha} G$$

for a certain partial action $\alpha$ of the free group $G = \mathbb{F}_n$ on $n = |E^1|$ generators (this can be infinite), and $X$ is a certain (totally disconnected) locally compact Hausdorff space. The exact description of this space and the partial action is slightly complicated in general but it simplifies under certain regularity conditions on $E$. For instance, if every vertex $v \in E^0$ is regular in the sense that $r^{-1}(v)$ is non-empty and finite, then $X$ is just the infinite path space $E^\infty$ of $E$.

Regardless of how $X$ and the partial action $\alpha$ above are defined, using that graph C*-algebras are always nuclear (a well-known fact, see [29, Proposition 2.6]), it follows from our previous theorem that $\alpha$ has the AP. Indeed, Exel gives a more direct proof of this fact in [24, Theorem 37.10].

We shall give more details about the partial action $\alpha$ and its amenability in what follows in the case of the graph $E$ that describes the Cuntz algebra $\mathcal{O}_n$, that is, the graph with one vertex and $n$ loops with $2 \leq n < \infty$. This is a special and representative case. This is a finite graph that has no sinks or sources. In this case, $X \cong \{1, \ldots, n\}^\infty$ is Cantor space and $G = \mathbb{F}_n$ is the free group on $n$ generators that we also view as the free group generated by $E^1$. The partial action $\alpha$ is defined as follows: the domains $D_g$ for $g \in \mathbb{F}_n$ are defined in terms of the cylinders $X_n = \{a\mu : \mu \in X\}$ if $g \in \mathbb{F}_n$ can be written in reduced form as
\[ g = ab^{-1} \] for \( a, b \in E^* \), the set of finite paths viewed as elements of \( \bar{F}_n \). In this case \( D_{g^{-1}} = C(X_b) \) and \( D_g = C(X_a) \) and \( \alpha_g : D_{g^{-1}} \to D_g \) is given \( \alpha_g(f) = f \circ \theta_g^{-1} \), where \( \theta_g : X_b \to X_a \) is the canonical homeomorphism sending \( b \theta \) to \( a \). If \( g \) is not of the form \( ab^{-1} \), then \( D_g \) is defined to be the zero ideal (and \( \alpha_g \) is the zero map).

The AP for \( \alpha \) means the existence of a net of finitely supported functions \( \xi_i : G \to C(X) \) that is uniformly bounded for the \( \ell^2 \)-norm and satisfying
\[
(8.15) \quad \langle \xi_i | a \tilde{\alpha}_g(\xi_i) \rangle_2 := \sum_{h \in G} \xi_i(h) \alpha_g(\alpha_g^{-1}(a \xi_i(g^{-1}h))) \to a
\]
for all \( g \in G \) and \( a \in D_g \). Notice that all the ideals \( D_g \) are unital here. If \( 1_g \) denotes its unit (so that \( D_g = A \cdot 1_g \)), then (8.15) is equivalent to
\[
\sum_{h \in \bar{F}_n} \xi_i(h) \alpha_g(1_g^{-1} \xi_i(g^{-1}h)) \to 1_g
\]
for all \( g \in G \). One explicit sequence \( \xi_i : G \to C(X) \) that gives the AP for this partial action can be defined by \( \xi_i(g) = \frac{1}{|g|} 1_g \) if \( g \in \bar{F}_n^+ \) (the positive cone of \( \bar{F}_n \)) with length \( |g| \leq i \) and \( \xi_i(g) = 0 \) otherwise. Recall that \( 1_g \) denotes the characteristic function on the cylinder set \( X_g = \{ g\mu : \mu \in X = E^\infty \} \) which makes sense because \( g \) is positive.

The fact that all domain ideals \( D_g \) are unital also implies that \( \alpha \) has an enveloping global action and we know from Corollary 3.20 that this global action also has the AP or, equivalently, it is AD-amenable. Indeed, a concrete description of the enveloping action for the partial action of \( \bar{F}_n \) on \( X \) is as follows: instead of considering only words on positive words, we also consider their inverses, that is, we consider the generators of \( \bar{F}_n \) and their inverses, and then look at all infinite reduced words on this new alphabet. This yields a new space, denoted \( \tilde{X} \) that naturally contains \( X \) as a clopen subspace. Now notice that \( \bar{F}_n \) naturally acts (globally) on \( \tilde{X} \) by (left) concatenation and the partial action on \( X \) is just the restriction of this global action. Moreover, the global action of \( \bar{F}_n \) on \( \tilde{X} \) is known to be amenable: this action can be viewed as the action on a certain boundary of \( \bar{F}_n \), and this is an amenable action, see [16] Examples 2.7(4) and [14] Proposition 5.1.8. Indeed, this is the standard way to see that \( \bar{F}_n \) is an exact group.

Appendix A. \( W^* \)-bimodules and their representations

Let \( M \) be a \( W^* \)-algebra. A \( W^* \)-Hilbert \( M \)-module is an ordinary \( C^* \)-Hilbert \( M \)-module \( X \) which is isometrically isomorphic to a dual Banach space, \( X \cong X_*' \), and such that the \( M \)-action and \( M \)-inner product are separately \( w^* \)-continuous. These are exactly the self-dual Hilbert modules; this means that every bounded \( M \)-linear map \( X \to M \) is of the form \( y \mapsto (x | y)_M \) for some (uniquely determined) element \( x \in X \). In this case the predual \( X_* \) is unique up to isomorphism.

In a similar fashion one defines left \( W^* \)-modules and \( W^* \)-bimodules (requiring both left and right inner products and actions to be separately \( w^* \)-continuous). Specially, we want to emphasise the \( W^* \)-equivalence bimodules:

**Definition A.1.** Given two \( W^* \)-algebras, \( M \) and \( N \), a \( W^* \)-equivalence bimodule is a \( W^* \)-Hilbert \( M \cdot N \)-bimodule \( X \) such that the left and right inner products span \( w^* \)-dense ideals of \( M \) and \( N \).

Here is an elementary concrete example: for Hilbert spaces \( H, K \), the space \( \mathcal{L}(H, K) \) is a \( W^* \)-equivalence \( \mathcal{L}(K) \cdot \mathcal{L}(H) \)-bimodule with respect to the obvious
operations given by composition and adjunction of operators. For example, the right inner product is given by \( \langle S | T \rangle_{\mathcal{L}(H)} := S^*T \). The predual in this case can be identified with \( \mathcal{K}(H, K) \). On bounded subsets the \( w^* \)-topology coincides with the weak topology, that is, \( T_i \to T \) with respect to the \( w^* \)-topology if and only if \( \langle u | T_i v \rangle \to \langle u | T v \rangle \) whenever \( \{T_i\}_{i \in I} \) is a norm-bounded net.

Every \( W^* \)-equivalent bimodule can be faithfully represented into some concrete bimodule of the form \( \mathcal{L}(H, K) \) as above. We explain in what follows how this can be done.

Let \( X \) be a \( W^* \)-equivalence \( M-N \)-bimodule. Using the notation of [35], we view \( X \) as a ternary \( W^* \)-ring with the ternary operation

\[
(x, y, z) := x(y, z)_N = M(x, y)z.
\]

We now indicate how to translate the fundamental results of Zettl [35] to represent \( W^* \)-equivalence bimodules on Hilbert spaces.

For example, Zettl shows that the adjointable operators of \( X_N, \mathcal{L}(X) \), form a \( W^* \)-algebra. We indicate how to represent this algebra \( W^* \)-faithfully.

**Lemma A.2.** Let \( X \) be a \( W^* \)-Hilbert right \( M \)-module. Consider a unital and faithful \( W^* \)-representation \( M \subset \mathcal{L}(H) \), and let \( K \) be the Hilbert space \( X \otimes_M H \). Then the representation \( \rho: \mathcal{L}(X) \to \mathcal{L}(K) \), such that \( \rho(T)(x \otimes h) = Tx \otimes h \), is a unital and faithful \( W^* \)-representation. Moreover, a bounded net \( \{T_i\}_{i \in I} \subset \mathcal{L}(X) \) \( w^* \)-converges to \( T \) if and only if \( \{\langle y, T_i x \rangle \}_{i \in I} \) \( w^* \)-converges to \( \langle y, Tx \rangle \), for all \( x, y \in X \).

**Proof.** Clearly \( \rho \) is an injective and unital \( * \)-homomorphism. To show that the image of \( \rho, M \), is a concrete \( W^* \)-algebra it suffices to prove that its closed unit ball \( M_1 \) is wot closed.

Take a net \( \{\rho(T_i)\}_{i \in I} \subset M_1 \) that weakly converges to \( R \in \mathcal{L}(K) \). If \( X_1 \) is the closed unit ball of \( X \) with the \( w^* \)-topology, then \( X_1 \) is compact and so it is \( Y := \Pi_{x \in X_1} (X_1 \times X_1) \). Let \( h: \mathcal{L}(X) \to Y \) be such that \( h(T)z = (Tx, T^*z) \). Then \( \{h(T_i)\}_{i \in J} \) has a converging subnet \( \{h(T_i)\}_{j \in J} \). This implies the existence of two linear maps \( U, V: X \to X \) such that \( Ux = \lim_j T_i x \) and \( Vx = \lim_j T^*_i x \), in the \( w^* \)-topology, for all \( x \in X \). Hence, for all \( x, y \in X \),

\[
\langle Ux, y \rangle = \lim_j \langle T_i x, y \rangle = \lim_j \langle x, T^*_i y \rangle = \langle x, Vy \rangle,
\]

where the limits are taken with respect to the \( w^* \)-topology. Then \( U \in \mathcal{L}(X) \) and our construction implies \( \|U\| \leq 1 \). We have \( \rho(U) = R \) because, for all \( x, y \in X \) and \( h, k \in H \),

\[
\langle x \otimes h, \rho(U)(y \otimes k) \rangle = \lim_j \langle x, T_i y \rangle k = \lim_j \langle x \otimes h, \rho(T_i)(y \otimes k) \rangle = \langle x \otimes h, R(y \otimes k) \rangle.
\]

This shows that \( M_1 \) is wot closed, hence \( M \) is a concrete \( W^* \)-algebra.

On bounded sets of \( \mathcal{L}(K) \) the wot topology (and hence the \( w^* \)-topology) is determined by the functionals \( R \mapsto \langle x \otimes h, R(y \otimes k) \rangle \). When translated to \( \mathcal{L}(X) \) this means that on bounded sets of \( \mathcal{L}(X) \) the \( w^* \)-topology is determined by the functionals \( \mathcal{L}(X) \to X, T \mapsto \langle x, T \rangle \), considering on \( X \) the \( w^* \)-topology.

**Definition A.3.** Let \( X \) be a \( W^* \)-equivalence \( M-N \)-bimodule. A representation of \( X \) is a linear and \( w^* \)-continuous map \( \pi: X \to \mathcal{L}(H, K) \), where \( H \) and \( K \) are Hilbert spaces and \( \pi(x, y)_N = \pi(x)\pi(y)^* \pi(z) \) for all \( x, y, z \in X \). We say that \( \pi \) is nondegenerate if \( K = \text{span} \pi(X)^*H \) and \( H = \text{span} \pi(X)^*K \).
Proposition A.4 (33). Every $W^*$-equivalence bimodule admits a nondegenerate faithful representation.

Proof. Every $W^*$-equivalence $M$-$N$-bimodule $X$ is a ternary $W^*$-ring, and by [33] has a faithful representation $\pi: X \to \mathcal{L}(H,K)$. Let $H_0 := \overline{\text{span}} \pi(X)^* K$ and $K_0 = \overline{\text{span}} \pi(X) H$. Clearly, $\overline{\text{span}} \pi(X) H_0 \subseteq K_0$ and $\overline{\text{span}} \pi(X)^* K_0 \subseteq H_0$. We claim that $\pi(X)(H_0^*) = 0$. If $h \in H_0^*$ and $x \in X$, then $\pi(x)^* \pi(x) h \in H_0$ and
\[ \|\pi(x) h\|^2 = \langle \pi(x) h, \pi(x) h \rangle = \langle \pi(x)^* \pi(x) h, h \rangle = 0. \]
In a similar way it can be shown that $\pi(X)^* (K_0^*) = 0$. Thus we may consider the representation $\pi_0: X \to \mathcal{L}(H_0, K_0)$ given by $\pi_0(x) h = \pi(x) h$.

It suffices to show that $\pi_0$ is nondegenerate. We show that $\overline{\text{span}} \pi_0(X) H_0 = K_0$; the proof of $\overline{\text{span}} \pi_0(X)^* K_0 = H_0$ is analogous. Take $x \in X$ and $h \in H$. We can approximate $x$ by sums of elements of the form $u(v, w)$, thus $\pi(x) h$ lies in the closed linear span of $\pi(X) \pi(X)^* \pi(X) h \subseteq \pi(X)(H_0) = \pi_0(X) H_0$. Hence $\pi(x) h \in \overline{\text{span}} \pi_0(X) H_0$.

\[ \square \]

Definition A.5. Given a representation $\pi$ of a $W^*$-equivalence bimodule, the representation $\pi_0$ constructed in the proof above is called de essential part of $\pi$.

Proposition A.6. Let $X$ be a $W^*$-equivalence $M$-$N$-bimodule and $\pi: X \to \mathcal{L}(H,K)$ a nondegenerate representation. Then there exists a unique unital and normal representation $\pi^1: M \to \mathcal{L}(K)$ such that $\pi^1(M \langle x, y \rangle) = \pi(x) \pi(y)^*$ for all $x, y \in X$. If $\pi$ is faithful then so is $\pi^1$.

Proof. Let $M_X$ be the norm closure of $\text{span}_M \langle X, X \rangle$ in $M$. By [24, Proposition 4.1] there exists a unique $^*$-homomorphism $\rho^1: M_X \to \mathcal{L}(K)$ such that $\rho^1(M \langle x, y \rangle) = \pi(x) \pi(y)^*$ for all $x, y \in X$. We claim this representation can be extended in a unique way to a normal representation $\pi^1$ of $M$.

Let $\rho^1: N_X \to \mathcal{L}(H)$ be the $^*$-homomorphism such that $\rho^1((x, y)_N) = \pi(x)^* \pi(y)$. Since $\pi$ is nondegenerate, there exists a unique unitary $U: X \otimes_{\rho^1} H \to H$ such that $U(x \otimes h) = \pi(x) h$. We also have a representation $\mu: M \to \mathcal{L}(X \otimes_{\rho^1} H) = \mathcal{L}(K)$ such that $\mu(a)(x \otimes h) = \pi(\ax) h$. We claim $\mu$ is $w^*$-continuous, to show this it suffices to prove $\mu$ is $w^*$-continuous on the closed unit ball $M_1$. Since $\mu(M_1)$ is bounded, it suffices to prove that given a net $\{a_\lambda\}_\lambda \subseteq M_1$ that $w^*$-converges to $a \in M_1$, then
\[ \lim_{\lambda} (\mu(a_\lambda)(x \otimes \pi(y)^* k), (x \otimes \pi(y)^* k)) = \langle \mu(a)(x \otimes \pi(y)^* k), (x \otimes \pi(y)^* k) \rangle, \]
for every $x, y \in X$ and $k \in K$. But
\[ \lim_{\lambda} (\mu(a_\lambda)(x \otimes \pi(y)^* k), (x \otimes \pi(y)^* k)) = \lim_{\lambda} \langle h, \pi(y(a_\lambda x), x) \pi(y)^* h \rangle = \langle h, \pi(y(\ax), x) \pi(y)^* h \rangle = \langle \mu(a)(x \otimes \pi(y)^* k), (x \otimes \pi(y)^* k) \rangle. \]

It is straightforward to show that the restriction of $\mu$ to $M_X$ is $\rho^1$, thus $\pi^1 := \mu$ is the unique $w^*$-continuous extension of $\rho^1$.

If $\pi$ is faithful and $\pi(a) = 0$ then $\pi(\ax) h = \pi^1(a) \pi(x) h = 0$ for every $x \in X$ and $h \in H$. Hence $\ax = 0$ for every $x \in X$ and this implies $a = 0$.

\[ \square \]

Definition A.7. Let $X$ be a $W^*$-equivalence $A$-$B$-bimodule and let $Y$ be a $W^*$-equivalence $M$-$N$-bimodule. A map $\pi: X \to Y$ is a $W^*$-homomorphism if it is linear, $w^*$-continuous and $\pi(x,y,z)_N = \pi(x)(\pi(y), \pi(z))_B$ for every $x, y, z \in X$. 


Proposition A.8 ([2 Proposition 4.1]). Let $X$ and $Y$ be $W^*$-equivalence $M$–$N$ and $A$-$B$-bimodules, respectively, and $\pi: X \to Y$ a $w^*$-continuous linear map such that $\pi(x(y,z)_N) = \pi(x)(\pi(y),\pi(z))_B$ for every $x,y,z \in X$. Then there exist unique $w^*$-continuous homomorphisms $\pi^i: M \to A$ and $\pi^r: N \to B$ such that $\pi^i((x,y)_N) = (\pi^i(x),\pi^i(y))_B$ and $\pi((x,y,z)) = \lambda(\pi^i(x),\pi^r(y))$ and $\pi((y,z)_N) = (\pi(x),\pi(y))_B$ for every $x,y \in X$. If $\pi$ is an isomorphism then so are $\pi^r$ and $\pi^i$.

Proof. We construct $\pi^r$, the map $\pi^i$ can be constructed considering the adjoint module of $X$. Take a nondegenerate and faithful representation $\rho: Y \to \mathcal{L}(H,K)$. In this situation $\rho^i: M \to \mathcal{L}(K)$ is a faithful unital $W^*$-representation. Then $\rho \circ \pi: X \to \mathcal{L}(H,K)$ is a representation, that may or may not be nondegenerate. In any case, the essential part $(\rho \circ \pi)_0$ is nondegenerate and we may think of $\rho \circ \pi$ as the null extension of $(\rho \circ \pi)_0$ (from $H_0$ to $H_0 \oplus (H_0^*)$).

We know $(\rho \circ \pi)^0_0: A \to \mathcal{L}(K_0)$ is a $W^*$-homomorphism. Define $(\rho \circ \pi)^r: A \to \mathcal{L}(K)$ as the null extension of $(\rho \circ \pi)^0_0$. We claim $(\rho \circ \pi)^r(A) \subseteq \rho^r(M)$. Indeed, note $(\rho \circ \pi)^r(A)$ is the $w^*$-closure of $(\rho \circ \pi)^r(A_Y)$. Considering only the C*-structure $Y$ and using the map $\pi^r: M_X \to A_Y$ of [2 Proposition 4.1], we get that $(\rho \circ \pi)^r(A_X,y)) = \rho^r(M(\pi^r(x),\pi^r(y))) \in \rho^r(M)$. Then the map $\pi^r: M \to A$ we are looking for is the unique $w^*$-continuous extension of $\pi^r: M_X \to A_Y$ and can be computed as $(\rho^r)^{-1} \circ (\rho \circ \pi)^r$. In case $\pi$ is an isomorphism $(\pi^{-1})^r$ is the inverse of $\pi^r$.

Let $X$ be a $W^*$-equivalence $M$-$N$-bimodule. The $W^*$-linking algebra of $X$ is the Banach space formed by all the matrices

$\begin{pmatrix} a & x \\ \hat{y} & b \end{pmatrix}$,

where $\hat{Y}$ is the module conjugate to $X$. To give $L$ a $W^*$-algebra structure take a faithful and nondegenerate representation $\pi: X \to \mathcal{L}(H,K)$. Then $\rho: L \to \mathcal{L}(H \oplus K) = \begin{pmatrix} \mathcal{L}(H) & \mathcal{L}(H,K) \\ \mathcal{L}(K,H) & \mathcal{L}(K,K) \end{pmatrix}$; $\rho \begin{pmatrix} a & x \\ \hat{y} & b \end{pmatrix} = \begin{pmatrix} \pi^i(a) & \pi(x) \\ \pi(y) & \pi^r(b) \end{pmatrix}$,

is a faithful representation of $^*$-algebras and $\rho$ induces a $C^*$-algebra structure on $L$. Moreover, $\rho(L)$ is a unital subalgebra closed with respect to the weak operator topology (wot) because convergence in $\mathcal{L}(H \oplus K)$ in the wot is just entrywise wot-convergence. This implies that $M,N$ and $X$ are $w^*$-closed subspaces of $L$. In particular $M$ and $N$ are hereditary $W^*$-subalgebras of $L$.

Proposition A.9. Let $X$ be a $W^*$-equivalence $M$-$N$-bimodule. Then there exists a unique $W^*$-isomorphism $\pi_X: Z(N) \to Z(M)$ such that $xa = \pi(a)x$ for all $a \in Z(M)$ and $x \in X$.

Proof. By the definition of the centre of an algebra we have

$\begin{pmatrix} a & x \\ \hat{y} & b \end{pmatrix} \in Z(L) \Leftrightarrow x = y = 0, a \in Z(M), b \in Z(N), az = zb \ \forall z \in X$.

Note $a$ and $b$ completely determine each other, thus we have an injective $w^*$-continuous $^*$-homomorphism

$\pi: Z(L) \to Z(N)$, $\pi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = b$. 

We claim $\pi$ is surjective and hence an isomorphism. Note the image of $\pi$ is a $W^*$-algebra because $\pi$ is $w^*$-continuous. Hence it suffices to show that $\text{Im}(\pi)$ contains every projection of $Z(N)$.

Let $p \in Z(N)$ be a projection and take a unital and normal representation on a Hilbert space $\rho: N \to \mathcal{L}(H)$ with $\ker(\rho) = (1 - p)N$. Consider the representation of $M$ induced by $\rho$ through $X$, $\text{Ind}(\rho)$, and let $q \in M$ be projection such that $\ker(\text{Ind}(\rho)) = (1 - q)M$. Then for all $x, y \in X$ and $h, k \in H$:

$$\langle k, \pi(y^*(qx - xp))k \rangle = \langle k, \pi(y^*qxk) - \langle k, \pi(y^*x)k \rangle = \langle \text{Ind}(\rho)(q)(y \otimes \rho k), (x \otimes \rho h) \rangle - \langle (y \otimes \rho k), (x \otimes \rho h) \rangle = 0$$

We conclude $y^*(qx - xp)p = 0$ for all $x, y \in X$ and this implies $qx = xp$ for all $x \in X$. By symmetry (or double induction) we get $qx = xp$ for all $x \in X$. Then $p = \pi \left( \begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix} \right)$ and $\pi$ is surjective.

It may look strange to consider the above isomorphism $\pi_X$ as a homomorphism $Z(N) \to Z(M)$ and not the opposite. Implicitly we have chosen this convention because we view $X$ as a “generalised” morphism from $N$ to $M$. This choice also makes it easier to see that the constructions we perform in Section 5.2 are exactly the $W^*$-counterparts of that in [1]. Another motivation for our notation is the relation between the composition and $W^*$-tensor products in Remark [3].

**Corollary A.10.** Let $X$ and $Y$ be two $W^*$-equivalence $M$-$N$-bimodules and $\rho: X \to Y$ an isomorphism such that $\pi^\rho$ and $\pi^\rho$ are the identities on $M$ and $N$, respectively. Then the isomorphisms $\pi_X$ and $\pi_Y$ of Proposition A.9 are equal.

**Proof.** Take $a \in Z(N)$. For all $x \in X$ we have $xa = \pi_x(a)x$, hence

$$\rho(x)a = \rho(x)\pi^\rho(a) = \rho(ax) = \rho(\pi_X(a)x) = \rho(\pi_X(a))\rho(x) = \pi_X(a)\rho(x).$$

Since $\rho$ is surjective we deduce that $\pi_X(a) = \pi_Y(a)$.

**Appendix B. Induction of representations and tensor products**

**Proposition B.1.** Let $X$ be an $W^*$-equivalence $M$-$N$-bimodule and $\pi: N \to \mathcal{L}(H)$ a unital $W^*$-representation. If $K := X \otimes \pi H$, then there exists a unique nondegenerate representation $\hat{\pi}: X \to \mathcal{L}(H, K)$ such that $\hat{\pi}(x)h = x \otimes h$. Moreover, if $\rho: X \to \mathcal{L}(H, K)$ is a nondegenerate representation then there exists a unique unitary $U: X \otimes_{\rho^\pi} H \to K$ such that $U(x \otimes h) = \rho(x)h$ and we have $\rho(x) = U \circ \hat{\pi}(x)$ for all $x \in X$. In particular, $\rho$ is faithful if and only if $\rho^\pi$ is faithful.

**Proof.** Regarding $\hat{\pi}$, we only have to show it is $w^*$-continuous. It suffices to show that given a bounded net $\{x_i\}_{i \in I} \subseteq X$ that $w^*$-converges to $x \in X$, we have $\lim_i \langle \hat{\pi}(x_i)h, y \otimes k \rangle = \langle \pi(x_i)y, h \otimes k \rangle$ for all $y \in X$ and $h, k \in H$. But the separate $w^*$ continuity of the inner products implies

$$\lim_i \langle \hat{\pi}(x_i)h, y \otimes k \rangle = \lim_i \langle h, \pi((x_i, y)_N)k \rangle = \langle h, \pi((x, y)_N)k \rangle = \langle \hat{\pi}(x)h, y \otimes k \rangle.$$

Now consider a nondegenerate representation $\rho$ as in the statement. There exists a unique linear isometry $U: X \otimes_{\rho^\pi} H \to K$ such that $U(x \otimes h) = \rho(x)h$ because

$$\langle \rho(x)h, \rho(y)k \rangle = \langle h, \rho^\pi((x, y)_N)k \rangle.$$

This isometry is in fact surjective because $\rho$ is nondegenerate. Then $U \circ \hat{\pi}(x)h = U(x \otimes h) = \pi(x)h$. 


We know \( \rho^r \) is faithful whenever \( \rho \) is. Assume \( \rho^r \) is faithful and \( \rho(x) = 0 \). Then, for every \( h \in H \), \( \langle h, \rho^r((x,x)N)h \rangle = ||\rho(x)h||^2 = 0 \). Thus \( \langle x,x \rangle_N = 0 \) and \( x = 0 \).

**Definition B.2.** Given a \( W^* \)-equivalence \( M-N \)-bimodule \( X \) and a unital \( W^* \)-representation \( \pi: N \to \mathcal{L}(H) \), the representation induced by \( \pi \) through \( X \), denoted \( \text{Ind}_X(\pi) \), is \( \hat{\pi}^\dagger \).

Let \( X \) and \( Y \) be \( W^* \)-equivalence \( M-N \) and \( N-P \)-bimodules, respectively. We want to construct a tensor product \( X \otimes^\ast_N Y \), with a natural \( M-P \) \( W^* \)-equivalence bimodule structure.

Take faithful and unital \( W^* \)-representation \( \pi: P \to \mathcal{L}(H) \) and define \( H_\pi := (X \otimes_N Y) \otimes_\pi H \), where \( X \otimes_N Y \) is the usual tensor product of Hilbert modules. We have a natural representation \( \hat{\pi}: X \otimes_N Y \to \mathcal{L}(H,H_\pi) \) such that \( \hat{\pi}(x \otimes y)h = x \otimes y \otimes h \). We define \( X \otimes^\ast_N Y \) as the wot-closure of \( \pi(X \otimes_N Y) \). Note

\[
\pi(P) = \text{span}^w\{\pi(u)^*\pi(v): u,v \in X \otimes_N Y\}
\]

\[
\text{Ind}_Y(\text{Ind}_X(\pi))(M) = \text{span}^w\{\pi(u)\pi(v)^*: u,v \in X \otimes_N Y\}
\]

Then we may think of \( X \otimes^\ast_N Y \) as a \( W^* \)-equivalence \( M-P \)-bimodule.

Let \( \rho: P \to \mathcal{L}(H) \) be another unital and faithful \( W^* \)-representation. We have represented \( X \otimes_N Y \) faithfully, as a ternary \( C^* \)-ring, in \( \mathcal{L}(H,H_\pi) \) and in \( \mathcal{L}(K,K_\rho) \). With some abuse of notation we denote these representations \( \hat{\pi}: X \otimes_N Y \to \mathcal{L}(H,H_\pi) \) and \( \hat{\rho} \). There exists a unique isomorphism of ternary \( C^* \)-rings \( \mu: \hat{\pi}(X \otimes_N Y) \to \hat{\rho}(X \otimes_N Y) \) such that \( \mu(\hat{\pi}(x \otimes y)) = \hat{\rho}(x \otimes y) \).

We claim that \( \mu \) is continuous on bounded sets with respect to the wot-topologies. Indeed, let \( \{u_i\}_{i \in I} \subseteq X \otimes_N Y \) be a bounded net and \( u \in X \otimes_N Y \) such that \( \{\hat{\pi}(u_i)\}_{i \in I} \) wot-converges to \( \hat{\pi}(u) \). Then, for every \( h \in H \) and \( v \in X \otimes_N Y \), we have

\[
\lim_i \langle h, \pi(\langle u_i, v \rangle_M)k \rangle = \langle \hat{\pi}(u)h, \hat{\pi}(v)k \rangle = \langle h, \pi(\langle u, v \rangle_M)k \rangle.
\]

In fact we can conclude that the wot-convergence of \( \{\hat{\pi}(u_i)\}_{i \in I} \) to \( \hat{\pi}(u) \) is equivalent to the w* convergence of \( \{\langle u_i, v \rangle_M\}_{i \in I} \) to \( \langle u, v \rangle_M \) for every \( v \in X \otimes_N Y \), which in turn is equivalent to the wot-convergence of \( \{\hat{\rho}(u_i)\}_{i \in I} \) to \( \hat{\rho}(u) \). Then \( \mu \) has a unique extension to a \( W^* \)-isomorphism \( \overline{\mu}: X \otimes^\ast_N Y \to X \otimes^\ast_N Y \).

**Definition B.3.** The \( W^* \)-tensor product \( X \otimes^\ast_N Y \) is the \( W^* \)-isomorphism class of the modules \( X \otimes^\ast_N Y \). As usual we abuse the notation and view \( X \otimes^\ast_N Y \) as any of its representatives.

**Remark B.4.** (1) Almost by construction we have, for any unital \( W^* \)-representation \( \pi: P \to \mathcal{L}(H) \), that \( \text{Ind}_{X \otimes^\ast_N Y}(\pi) \) is unitarily equivalent to \( \text{Ind}_Y(\text{Ind}_X(\pi)) \).

(2) If \( \pi_X: Z(N) \to Z(M) \) and \( \pi_Y: Z(P) \to Z(N) \) are the isomorphisms of Proposition A.9, then \( \pi_X \circ \pi_Y = \pi_{X \otimes^\ast_N Y} \).

**Appendix C. Biduals of Hilbert bimodules**

**Proposition C.1.** Let \( X \) be a Hilbert \( A-B \)-bimodule. Then there exists a unique Hilbert \( A''-B'' \)-bimodule structure on \( X'' \) extending that of \( X \) and with (left and right) inner products and actions of \( A'' \) and \( B'' \) separately w*-continuous. Moreover, if \( X \) is an equivalence \( A-B \)-bimodule, that is, if the left and right inner products on
X generate A and B as C*-algebras, then the inner products on X" generate A" and B" as W*-algebras, that is, X" is a W*- equivalence A"-B"-bimodule.

Proof. Uniqueness follows immediately because X and A and B are w*-dense in X", A" and B", respectively. Let L be the linking algebra of X. Since A and B are C*-subalgebras of L, we may view A" and B" as W*-subalgebras of L". Moreover, we also view X as a closed subspace of L and identify X" with the w*-closure of X in L". Note that AXB ⊆ X, XX* ⊆ A and X*X ⊆ B imply A"X"B" ⊆ X", X"X" ⊆ A" and X"X" ⊆ B" because the multiplication of L" is separately w*-continuous and the involution of L" is w*-continuous. The rest follows directly because the Hilbert module operations of X" are defined in terms of the W*-algebra structure of L".

The reader should note that X" is usually not an A"-B"-equivalence bimodule in the C*-sense because the images of the inner products on X" might be not linearly norm dense in A" or B" (only w*-dense).

Proposition C.2. Let A and B be C*-algebras and X an A-B-equivalence bimodule. Given a nondegenerate representation π: B → ℒ(H), write π": B" → ℒ(H) for its unique w*-continuous extension, and Ind\text{c}_X π: A → ℒ(X ⊗_π H) for the representation induced by π through X. Then (Ind\text{c}_X π)" is faithful if and only if π" is faithful.

Proof. This result is certainly well-known, but we could not find it explicitly in the literature, so we give a proof here. A quick way to prove the statement is to notice that the induction process of representations via an equivalence bimodule preserves quasi-equivalence of representations which is, in turn, determined by their central cover projections, see [32, Section 3.8]. And π" is faithful if and only if its central cover is zero.

A more elementary way to prove the result is as follows: for a ∈ A", ξ₁, ξ₂ ∈ X and v₁, v₂ ∈ H, we have

\[ \langle ξ₁ ⊗ v₁ | \text{Ind}_X^c (π)" (a)(ξ₂ ⊗ v₂) \rangle = \langle v₁ | π" (a) \langle ξ₁ | ξ₂ \rangle v₂ \rangle \]

where a · ξ₂ ∈ X" means the left action of A" on X". The above equation holds because it does for a ∈ A and all the operations involved are w*-continuous. Now, if Ind\text{c}_X π" (a) = 0, then \[ \langle ξ₁ | a · ξ₂ \rangle = 0 \] for all ξ₁, ξ₂ ∈ X from which it follows that a = 0. The converse (faithfulness of π from Ind\text{c}_X π) follows by symmetry since π can be seen as the induced representation of Ind\text{c}_X π through the dual equivalence bimodule X*.

Corollary C.3. Let X be a Hilbert A-module and π: A → ℒ(H) a nondegenerate representation. Then the representation π_X: X → ℒ(H, X ⊗_π H), π_X(x)h = x ⊗ h, has a unique w*-continuous extension π_X" : X" → ℒ(H, X ⊗_π H) to a representation of ternary W*-rings [35]. Moreover, if π" is faithful then so is π_X".

Proof. We view X, A and the algebra of generalized compact operators of X, B, as subspaces of the linking algebra L of X. Then LA = X ⊕ A and the representation Ind\text{c}_X A π of L induced by π through X ⊕ A can be seen as

\[
\text{Ind}_X^c \otimes A \pi \; : \; L \rightarrow \mathcal{L}((X \oplus A) \otimes_π H) \cong \mathcal{L}((X \otimes_π H) \oplus H),
\]

\[
\text{Ind}_X^c \otimes A \pi \left( \begin{array}{cc} T & x \\ y & a \end{array} \right) = \left( \begin{array}{cc} \text{Ind}_X^c \pi(T) & \pi_X(x) \\ \pi_X(y) & \pi(a) \end{array} \right).
\]
Since \( \text{Ind}_{X \oplus A}^L \pi \) is nondegenerate, we have a canonical extension
\[
(\text{Ind}_{X \oplus A}^L \pi)^{\prime\prime} : L'' \to \mathcal{L}(X \oplus A) \otimes \pi H).
\]
In the proof of Proposition C.1 we have identified \( X'' \) with the \( \ast \)-closure of \( X \) in \( L'' \). Thus the restriction of \( (\text{Ind}_{X \oplus A}^L \pi)^{\prime\prime} \) to \( X'' \) is a \( \ast \)-continuous extension of \( (\text{Ind}_{X \oplus A}^L \pi)|_{X} \). The image of \( (\text{Ind}_{X \oplus A}^L \pi)|_{X} \) consists entirely of operators of the form \( \begin{pmatrix} 0 & \alpha \\
\beta & 0 \end{pmatrix} \). Under the identification \( y = \begin{pmatrix} 0 & \alpha \\
\beta & 0 \end{pmatrix} \) we may view \( (\text{Ind}_{X \oplus A}^L \pi)|_{X''} \) as the unique \( \ast \)-extension of \( \pi_X \).

In case \( \pi'' \) is faithful then so is \( (\text{Ind}_{X \oplus A}^L \pi)^{\prime\prime} \), so \( \pi''_X \) is faithful because it is a restriction of a faithful map. \( \square \)

**Corollary C.4.** Let \( X \) be a Hilbert \( A \)-module. For a bounded net \( \{x_i\}_{i \in I} \subseteq X'' \) and \( x \in X'' \), the following assertions are equivalent:

(i) \( \{x_i\}_{i \in I} \subseteq X'' \) \( \ast \)-converges to \( x \).

(ii) For every \( y \in X \), \( \{\langle x_i, y \rangle A^{\ast}\}_{i \in I} \) \( \ast \)-converges to \( \langle x, y \rangle A^{\ast} \).

(iii) For every \( y \in X \), \( \{\langle x_i, y \rangle A^{\ast}\}_{i \in I} \) \( \ast \)-converges to \( \langle y, x \rangle A^{\ast} \).

**Proof.** Let \( \pi : A \to \mathcal{L}(H) \) be the universal representation and consider \( \pi''_X : X'' \to \mathcal{L}(H, X \otimes \pi H) \). On bounded sets the \( \ast \)-topology on \( X'' \) coincides with the weak operator topology if it inherits from \( \mathcal{L}(H, X \otimes \pi H) \) via \( \pi''_X \) and it is determined by the functionals of the form \( X \to \mathbb{C}, z \mapsto \langle y \otimes k, \pi''_X(z) h \rangle = \langle k, \langle y, z \rangle A^{\ast} h \rangle \) for \( k \in H \) and \( y \in X \). The proof then follows immediately. \( \square \)

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