SOLVABILITY OF MATRIX RICCATI INEQUALITIES

KEVIN KISSI

Abstract. We consider matrix Riccati inequality arising in the theory of absolute stability, $H_\infty$ control problem, $LQ$ problem, and optimal estimation problem. In the case of sign definite frequency domain function, the solvability of Riccati inequalities is a subject of the famous Kalman-Yakubovich lemma. This paper presents necessary and sufficient conditions for solvability of Riccati inequality in the general sign indefinite case. To this end we use special representations of Hamiltonian matrices. The results are illustrated by an example.

1. Introduction and Problem statement

Consider the Riccati inequality

$$HA + A^*H + G - HB\Gamma^{-1}B^*H < 0,$$

where $A, B, G, \Gamma$ are given matrices of dimensions $n \times n, n \times m, n \times n$, and $m \times m$ respectively; $G, \Gamma$ are Hermitian matrices, $\det \Gamma \neq 0$. We are looking for necessary and sufficient conditions for existence of stabilizing and anti stabilizing solutions of the inequality (1), that is, for matrices $H_-$ and $H_+$ such that (1) holds for both of them, and matrices $A - B\Gamma^{-1}B^*H_-, -(A - B\Gamma^{-1}B^*H_+)$ are Hurwitz.

In case when matrix $\Gamma$ is sign definite, the answer to this problem is given in the famous Kalman-Yakubovich lemma [1]. This lemma, has been originally proved in [4], then extended to infinite dimensional case in [5] and formulated in the most general form in [7].

For positive definite matrices $\Gamma$ solvability of inequality (1) may be reduced to solvability of the algebraic Riccati equation (ARE)

$$HA + A^*H + G - HB\Gamma^{-1}B^*H = 0,$$

which is closely related to the existence and properties of maximal $J$-orthogonal invariant subspaces of Hamiltonian matrices (see [6], [2]). Such equations are very important in the theories of optimal control, absolute stability, game theory, $H_\infty$ control (in the last two cases the matrix $\Gamma$ is sign indefinite) [9].

A number of important results concerning solvability of Riccati inequalities were published in [8]. In this paper a new idea related to positive definiteness of matrix $M$ (see the text below) is used.

Theory of absolute stability study systems of the form

$$\frac{dx}{dt} = Ax + B\xi, \quad \sigma = C^*x,$$

$$\xi = \varphi(\sigma, t)$$

Adviser: Professor Nikita Barabanov.
in the class $N_F$ of nonlinearities $\varphi$ satisfying the following Local Quadratic Constraint (LQC):

$$F(x, \varphi(C^*x, t)) \geq 0 \quad \forall x, t,$$

where $F$ is given quadratic form. The first and most known LQC is so-called sector condition:

$$(\xi - \alpha C^*x)^* \Gamma (\beta C^*x - \xi) \geq 0,$$

where $\alpha, \beta, \Gamma > 0$ are diagonal matrices, but there are a lot of other useful constraints [1].

System (2) is called absolutely stable in the class $N_F$, if system (2) with every function $\varphi \in N_F$ is globally asymptotically stable, and this stability is uniform with respect to functions $\varphi \in N_F$.

If there exists a positive definite quadratic form $V(x) = x^*Hx$ such that $V(x(t))$ is decreasing along all non zero solutions of systems (2) with all functions $\varphi \in N_F$, then system (2) is absolutely stable in $N_F$.

Such Hermitian matrix $H$ exists if and only if the quadratic form

$$x^*H(Ax + B\xi) < 0$$

for all non zero vectors $(x^*, \xi^*)^*$, for which $F(x, \xi) \geq 0$.

According to Dines theorem such matrix $H$ exists if and only if there exists a Hermitian matrix $H$ such that

$$2x^*H(Ax + B\xi) + F(x, \xi) < 0$$

for all non zero vectors $(x^*, \xi^*)^*$.

Assume $F(x, \xi) = x^*Gx - \xi^*\Gamma\xi$ with $\Gamma > 0$. Then inequality (1) holds if and only if the following Riccati inequality holds

$$HA + A^*H + G + HB\Gamma^{-1}B^*H < 0.$$ (5)

Notice that the quadratic term of the left hand side is sign semidefinite.

Another example of problems where the Riccati inequalities arise is the problem of $H_\infty$ control.

Consider system

$$\frac{dx}{dt} = Ax + B_w w + B_u u,$$ (6)

where matrices $A, B_w, B_u$ are constant, $x$ is state vector, $w$ is exogenous input (noise for example), and $u$ is control.

Consider a quadratic function

$$F(x, w, u) = x^*Gx + u^*\Gamma_u u - w^*\Gamma_w w,$$
where constant Hermitian matrices \( G, \Gamma_u > 0 \) and \( \Gamma_w > 0 \) are given. The problem consists of finding a controller \( u = hx \) such that the closed loop matrix \( A + Bu_h \) is Hurwitz, and for all non zero functions \( w \in L_2(0, \infty) \) along solutions \( x \) with trivial initial value \( x(0) = 0 \) we have

\[
\int_0^\infty F(x(t), w(t), u(t)) dt < 0.
\]

This problem may be reduced to solvability of the following Riccati inequality

\[
HA + A^*H + G + HB_u \Gamma_u^{-1} B_w^* H - HB_u \Gamma_u^{-1} B_u^* H < 0.
\]

Notice that matrix of quadratic form in this inequality in many cases is sign indefinite.

Now consider a relation between the Riccati inequalities and the Kalman-Yakubovich lemma.

Without loss of generality we assume that pair \((A, B)\) is controllable and matrix \( A \) has no pure imaginary eigenvalues.

If matrix \( \Gamma \) is negative definite, then inequality (1) may be represented as linear matrix inequality (LMI):

\[
\begin{pmatrix}
HA + A^*H + G & HB_u \\
B_u^* H & \Gamma
\end{pmatrix} < 0.
\]

It may be solved via well-known technique.

Solvability of inequality (1) is also a subject of the famous Kalman-Yakubovich lemma [1]. According to this lemma, inequality (1) has a solution, if and only if, the following frequency domain inequality holds:

\[
\pi(i\omega) < 0, \quad (7)
\]

for all \( \omega \in [-\infty, \infty] \), where

\[
\pi(\lambda) = \Gamma + B^*(\lambda I + A^*)^{-1} G(A - \lambda I)^{-1} B,
\]

or, which is the same,

\[
\det \pi(i\omega) \neq 0 \quad (8)
\]

for all \( \omega \in [-\infty, \infty] \).

But if matrix \( \Gamma \) is not sign definite, the inequality (8) proved to be no longer necessary for solvability of inequality (1). In this paper we present the desired necessary and sufficient conditions, which may be considered as generalization of Kalman-Yakubovich lemma to the case of sign-indefinite quadratic forms.

2. Hamiltonian matrices

In this section we present a new necessary condition for solvability of inequality (1), which will be proved later on to be also sufficient.

In the sequel we shall use the following matrices:
\[ R = \begin{pmatrix} A & -B \Gamma^{-1}B^* \\ -G & -A^* \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \]

Matrix \( JR \) is clearly Hermitian, therefore, matrix \( R \) is \((J-)\)Hamiltonian \[2\].

The set of eigenvalues of Hamiltonian matrix is symmetric with respect to the imaginary axis. Indeed, if \( \det(R - \lambda I) = 0 \), then
\[
0 = \det(R - \lambda I) \det(J) = \det(JR - \lambda J) = \det(R^*J^* - \lambda J) = \det(-R^* - \lambda I) \det(J),
\]
Therefore
\[
0 = \det(R^* + \lambda I) = \det(R - (-\bar{\lambda} I)),
\]
and \(-\bar{\lambda}\) is an eigenvalue of \( R \).

It has been shown \[6\] that the structure of Jordan blocks corresponding to eigenvalues \( \lambda \) and \(-\bar{\lambda}\) of matrix \( R \) coincide.

Assume matrix \( R \) has no eigenvalues on the imaginary axis. Then there exist \( n \times n \)-matrices \( \Lambda, X_1, \Psi_1, X_2, \Psi_2 \) such that all eigenvalues of matrix \( \Lambda \) have negative real parts, and
\[
R \begin{pmatrix} X_1 & X_2 \\ \Psi_1 & \Psi_2 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ \Psi_1 & \Psi_2 \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda^* \end{pmatrix}. \tag{9}
\]

Hence, we have the following equalities:
\[
AX_1 - B \Gamma^{-1}B^* \Psi_1 = X_1 \Lambda \\
-GX_1 - A^* \Psi_1 = \Psi_1 \Lambda.
\]

Assume matrix \( X_1 \) is nonsingular. Multiply the first equation by \( \Psi_1 X_1^{-1} \) from the left, by \( X_1^{-1} \) from the right. Multiply the second equation by \( X_1^{-1} \) from the right. Add the equations. Then with notation \( H = \Psi_1 X_1^{-1} \) we have
\[
HA + A^*H + G - H B \Gamma^{-1}B^*H = 0. \tag{10}
\]

Hence, \( H \) is a solution of the Riccati equation \((10)\).

Why \( H \) is Hermitian? Denote \( Z = \text{col}(X_1, \Psi_1) \). Then \( RZ = Z \Lambda \) and
\[
Z^* JZ \Lambda = Z^* JRZ = Z^* (JR)^* Z = (RZ)^* J^* Z = (-\Lambda^*) Z^* JZ.
\]
But matrices \( \Lambda \) and \(-\Lambda^* \) have no common eigenvalues. Therefore
\[
Z^* JZ = 0.
\]

Recalling the definition of \( Z \) we get
\[
-X_1^* \Psi_2 + \Psi_1^* X_1 = 0,
\]
which it turn is equivalent to \((\Psi_1 X_1^{-1})^* = \Psi_1 X_1^{-1} \), or \( H^* = H \).
Notice that \( A - B \Gamma^{-1} B^* H = X_1 \Lambda X_1^{-1} \). Therefore matrix \( A - B \Gamma^{-1} B^* H \) is Hurwitz, and \( H \) is a stabilizing solution of the Riccati equation (10).

If we consider matrix \( \text{col}(X_2, \Psi_2) \) instead of \( Z \), we arrive to a solution \( H \) such that matrix \(- (A - B \Gamma^{-1} B^* H)\) is Hurwitz. Then \( H \) is an anti-stabilizing solution of the Riccati equation (10).

Inverse, if \( H \) is a solution of the Riccati inequality (9), then for some positive definite matrix \( \Delta G \), matrix \( H \) is a solution of the Riccati equation

\[
HA + A^* H + G + \Delta G - H B \Gamma^{-1} B^* H = 0.
\]

(11)

Assume \( H \) is a stabilizing solution of equation (11). Then for some representation (9) of matrix \( R_{\text{new}} = (A - V_1 V_2^* - B \Gamma^{-1} B^* + V_1 V_1^* - G - V_2 V_2^* - A^* + V_2 V_1^*) \)

we have \( A - B \Gamma^{-1} B^* H = X_1 \Lambda X_1^{-1} \), and therefore matrix \( \Lambda \) is Hurwitz. In particular, it means that matrix \( R_{\text{new}} \) has no pure imaginary eigenvalues.

Thus, the problem of finding necessary and sufficient conditions for existence of stabilizing and anti-stabilizing solutions of Riccati inequality (1) is reduced to a problem of existence of a positive definite matrix \( \Delta G \) such that matrix \( R_{\text{new}} \) has no pure imaginary eigenvalues, and in the representation (9) matrices \( X_1 \) and \( X_2 \) are nonsingular.

3. Special Transformation

Assume \( V_1, V_2 \) are \( n \times m \)-matrices, and \( V = \text{col}(V_1, V_2) \). Denote by \( R(V) \) the Hamiltonian matrix

\[
R + V(JV)^* = \begin{pmatrix}
A - V_1 V_2^* & -B \Gamma^{-1} B^* + V_1 V_1^* \\
-G - V_2 V_2^* & -A^* + V_2 V_1^*
\end{pmatrix}.
\]

The corresponding Riccati equation has a form

\[
H(A - V_1 V_2^*) + (A - V_1 V_2^*)^* H + G + V_2 V_2^* - H B \Gamma^{-1} B^* H + HV_1 V_1^* H = 0.
\]

(12)

For a solution \( H \) of this equation we have

\[
HA + A^* H + G - H B \Gamma^{-1} B^* H = -(V_2 - HV_1)(V_2 - HV_1)^* \leq 0.
\]

If the right hand side is strictly negative, then \( H \) is a solution of inequality (1).

4. Special case

Assume matrix \( R \) has no pure imaginary eigenvalues. Then for sufficiently small positive number \( \epsilon \) matrix \( R(\epsilon I) \) also has no pure imaginary eigenvalues.

Consider a representation (9) of matrix \( R(\epsilon I) \). For every positive number \( \delta \) there exists a Hamiltonian matrix
\[
\tilde{R} = \begin{pmatrix}
\tilde{A} & -\tilde{B}\tilde{\Gamma}^{-1}\tilde{B}^* + \epsilon I \\
-\tilde{G} - \epsilon I & -\tilde{A}^*
\end{pmatrix}
\]

such that \( \|R(\epsilon I) - \tilde{R}\| < \delta \), and in the representation (1) of matrix \( \tilde{R} \) matrices \( X_1, X_2 \) are nonsingular. Denote by \( H \) the stabilizing solution of corresponding Riccati equation. Then

\[
HA + A^*H + G - H\tilde{B}\tilde{\Gamma}^{-1}\tilde{B}^*H = \\
= H(A - \tilde{A}) + (A - \tilde{A})^*H + (G - \tilde{G} + \epsilon I) - \\
- (H\tilde{B}\tilde{\Gamma}^{-1}\tilde{B}^*H - H\tilde{\Gamma}^{-1}\tilde{B}^*H) + H\epsilon H.
\]

For sufficiently small number \( \delta \) the right hand side of the equality is negative. Therefore, \( H \) is a solution to the Riccati inequality (1). Since matrix \( \tilde{A} - \tilde{B}\tilde{\Gamma}^{-1}\tilde{B}^*H \) is Hurwitz, for sufficiently small number \( \delta \) the matrix \( A - B\Gamma^{-1}B^*H \) is Hurwitz, and \( H \) is a stabilizing solution of inequality (1).

The same conclusion is true for the anti stabilizing solution of inequality (1).

Thus, if matrix \( R \) has no pure imaginary eigenvalues, then the Riccati inequality (1) has both stabilizing and anti stabilizing solutions.

5. General case

Now consider general case: matrix \( R \) may have pure imaginary eigenvalues. Our next goal is to figure out how eigenvalues of matrix \( R + V(JV)^* \) depend on \( V \).

We need the following definitions.

**Definition 5.1.** Let \( P \) be a subspace of \( \mathbb{C}^{2n} \), and \( T \) be a matrix such that set of columns of \( P \) is a basis of \( P \). Denote by \( n_-(P) \), \( n_0(P) \), and \( n_+(P) \) the numbers of positive, zero, and negative eigenvalues of matrix \( T^*iJ^T \) respectively.

Obviously, these numbers do not depend on the choice of basis \( T \) of \( P \).

Denote by \( J_1, \ldots, J_m \) all Jordan blocks of matrix \( R \) with pure imaginary eigenvalues \( i\omega_1, \ldots, i\omega_m \) respectively:

\[
J_j = \begin{pmatrix}
i\omega_j & 1 & 0 & \ldots & 0 \\
0 & i\omega_j & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & i\omega_j
\end{pmatrix}.
\]

We assume that the Jordan blocks are arrange in such an order that \( \omega_1 \leq \omega_2 \leq \ldots \leq \omega_m \). Let \( P_1, \ldots, P_m \) be R-invariant subspaces associated to Jordan blocks \( J_1, \ldots, J_m \) of dimensions \( n_1, \ldots, n_m \) respectively. It is known that for every \( j = 1, \ldots, m \) the value \( n_+(J_j) - n_-(J_j) \) is equal to \(-1, 0, 1\), and \( n_0(J_j) = 0 \). We use classification given by M. Krein.

**Definition 5.2.** We say that the block \( J_j \) contains \( n_+(J_j) \) eigenvalues \( i\omega_j \) of the first type, and \( n_-(J_j) \) eigenvalues \( i\omega_j \) of the second type.

For every subspace \( P_j \) there exist a number \( \beta_j \in \{-1, 1\} \), and a matrix \( S_j \) such that the columns of \( S_j \) span \( P_j \), \( RS_j = S_jJ_j \).
\[ S_j^*JS_j = \epsilon_j \begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & \ldots & (-1)^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m_j} & 0 & \ldots & 0 & 0 \end{pmatrix}, \]

and

(a) if the size \( n_j \) of \( J_j \) is even, then \( n_+(J_j) - n_-(J_j) = 0 \) and \( \epsilon_j = (-1)^{n_j/2}\beta_j \);
(b) if the size \( n_j \) of \( J_j \) is odd, then \( n_+(J_j) - n_-(J_j) = \beta_j \) and \( \epsilon_j = (-1)^{(m_j-1)/2}i\beta_j \).

Moreover, matrices \( S_j \) for distinct \( j \) are \( J \)-orthogonal: \( S_{j_1}^*JS_{j_2} = 0 \) if \( j_1 \neq j_2 \).

The value \( \beta_j \) is called index of \( J_j \), and we call invariant space \( P_j \) neutral, of the first type, or of the second type, \( n_+(J_j) - n_-(J_j) \) is equal to respectively zero, one or negative one.

Denote by \( S_+ (S_-) \) a matrix whose columns span the \( R \)-invariant subspace associated to eigenvalues with positive (respectively, negative) real parts. Then \( S_+ \) and \( S_- \) are \( J \)-orthogonal to \( S_j \) for every \( j = 1, \ldots, m \). Besides, all columns of matrices \( S_-, S_+, S_1, \ldots, S_m \) present a basis of \( \mathbb{C}^{2n} \).

Fix \( j \in \{1, \ldots, m\} \) and consider a matrix \( V \) such that \( JV \) is orthogonal to \( S_- , S_+ , \) and all matrices \( S_k \) with \( k \neq j \). Then

\[
\det(R + V(JV)^* - \lambda I) = \det(R - \lambda I)(1 + (JV)^*(R - \lambda I)^{-1}V) = \\
= \det(R - \lambda I)(1 + (JV)^*S_j(J_j - \lambda I)^{-1}(S_j^*JS_j)^{-1}S_j^*JV).
\]

Denote \( P_j = S_j^*JS_j/\epsilon_j \). Consider the matrix of quadratic form in the last brackets. We have \( P_j^{-1} = (-1)^{m_j} - P_j \), and

\[
(J_j - \lambda I)^{-1} = \begin{pmatrix} \frac{1}{i\omega_j - \lambda} & \frac{-1}{(i\omega_j - \lambda)^2} & \ldots & \frac{(-1)^{m_j - 1}}{(i\omega_j - \lambda)^m_j} \\ 0 & \frac{1}{i\omega_j - \lambda} & \ldots & \frac{(-1)^{m_j - 1}}{(i\omega_j - \lambda)^m_j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{i\omega_j - \lambda} \end{pmatrix}.
\]

Therefore

\[
(J_j - \lambda I)^{-1}(\epsilon_j P_j)^{-1} = \epsilon_j^{-1} \begin{pmatrix} \frac{(-1)^{m_j}}{i\omega_j - \lambda} & \frac{(-1)^{m_j}}{(i\omega_j - \lambda)^2} & \ldots & \frac{(-1)^{m_j}}{(i\omega_j - \lambda)^m_j} \\ \frac{(i\omega_j - \lambda)^{m_j - 1}}{i\omega_j - \lambda} & \frac{(i\omega_j - \lambda)^{m_j - 1}}{(i\omega_j - \lambda)^2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{i\omega_j - \lambda} & 0 & \ldots & 0 \end{pmatrix}.
\]

For even \( m_j = 2r_j \) we have \( \epsilon_j = (-1)^{r_j}\beta_j \), and

\[
(J_j - \lambda I)^{-1}(\epsilon_j P_j)^{-1} = \beta(-1)^r \begin{pmatrix} \frac{1}{(i\omega_j - \lambda)^{2r_j}} & \frac{1}{(i\omega_j - \lambda)^{2r_j - 1}} & \ldots & 0 \\ \frac{(i\omega_j - \lambda)^{2r_j - 1}}{i\omega_j - \lambda} & \frac{(i\omega_j - \lambda)^{2r_j - 1}}{(i\omega_j - \lambda)^2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{i\omega_j - \lambda} & 0 & \ldots & 0 \end{pmatrix}.
\]

Therefore for \( \lambda = i\omega \) we have
If we choose vector \( V \) such that \( S_k^*(JV) = 0 \) for all \( k \neq j \), and \( S_j^*(JV) = (\delta, 0, \ldots, 0)^* \) (that is, \( v \) is parallel to the last column of matrix \( S_j^* \)), then

\[
1 + (JV)^*S_j(J - \lambda I)^{-1}(S^*_jJS_j)^{-1}S^*_jJV = 1 + \delta^2 \beta_j \frac{1}{(\omega_j - \omega)^{2r_j}}.
\]

Hence, for sufficiently small \( \delta \) and positive \( \beta_j \) all pure imaginary eigenvalues of matrix \( R \) corresponding to block \( J_j \) leave imaginary axis and become pairs of eigenvalues with non zero real parts symmetric with respect to imaginary axis.

If \( \beta_j = -1 \) then for sufficiently small \( \delta \) exactly two eigenvalues remain on the imaginary axis, and the rest \( m_j - 2 \) eigenvalues leave this axis. Among two remaining eigenvalues the bigger is of the first type, and the smaller is of the second type. With \( \delta \) increasing the first eigenvalue goes up the imaginary axis, and the second eigenvalue goes down.

Now consider the blocks with odd dimension: \( m_j = 2r_j + 1 \). Then \( \epsilon_j = (-1)^{r_j} i \beta_j \) and

\[
(J_j - \lambda I)^{-1}(\epsilon_j P_j)^{-1} = \beta(-1)^{r_j} \begin{pmatrix}
\frac{i}{(\omega_j - \lambda)^{r_j+1}} & \frac{i}{(\omega_j - \lambda)^{r_j}} & \cdots & \frac{i}{i(\omega_j - \lambda)} \\
\frac{i}{(\omega_j - \lambda)^{r_j+1}} & \frac{i}{(\omega_j - \lambda)^{r_j}} & \cdots & 0 \\
\frac{i}{(\omega_j - \lambda)} & \cdots & \cdots & \cdots \\
\frac{i}{i(\omega_j - \lambda)} & 0 & \cdots & 0
\end{pmatrix}.
\]

For \( \lambda = i\omega \) we get

\[
(J_j - i\omega I)^{-1}(\epsilon_j P_j)^{-1} = \beta \begin{pmatrix}
\frac{1}{(\omega_j - \omega)^{r_j+1}} & \frac{i}{(\omega_j - \omega)^{r_j}} & \cdots & \frac{i(-1)^{r_j+1}}{i(\omega_j - \omega)} \\
\frac{1}{(\omega_j - \omega)^{r_j+1}} & \frac{i}{(\omega_j - \omega)^{r_j}} & \cdots & 0 \\
\frac{i}{(\omega_j - \omega)} & \cdots & \cdots & \cdots \\
\frac{i(-1)^{r_j+1}}{i(\omega_j - \omega)} & 0 & \cdots & 0
\end{pmatrix}.
\]

Again, if \( V \) is such that \( S_k^*(JV) = 0 \) for all \( k \neq j \), and \( S_j^*(JV) = (\delta, 0, \ldots, 0)^* \) (that is, \( v \) is parallel to the last column of matrix \( S_j^* \)), then

\[
1 + (JV)^*S_j(J - \lambda I)^{-1}(S^*_jJS_j)^{-1}S^*_jJV = 1 + \delta^2 \beta_j \frac{1}{(\omega_j - \omega)^{2r_j+1}}.
\]

Therefore, for small \( \delta \) all eigenvalues of matrix \( R \) corresponding to block \( J_j \) but one leave imaginary axis. If \( \beta = 1 \), then the eigenvalue which remains on imaginary axis is of the first type, and it goes up the imaginary axis with increasing positive \( \delta \). If \( \beta = -1 \), then this eigenvalue is of the second type and it goes down imaginary axis with increasing positive \( \delta \).

Thus, we get the following result.
Theorem 5.3. There exists a nonnegative matrix $M$ such that the eigenvalues of matrix $R - tMJ$ with number $t$ increasing from zero have the following behaviour.

(a) For each Jordan block $J_j$ of matrix $R$ of odd dimension $2r_j + 1$ with index $\beta = 1$ exactly $2r_j$ eigenvalues leave imaginary axis with increasing $t$ from zero, and the rest eigenvalue goes up imaginary axis, and has the first type.

(b) For each Jordan block $J_j$ of matrix $R$ of odd dimension $2r_j + 1$ with index $\beta = -1$ exactly $2r_j$ eigenvalues leave imaginary axis with increasing $t$ from zero, and the rest eigenvalue goes down imaginary axis, and has the second type.

(c) For each Jordan block $J_j$ of matrix $R$ of even dimension $2r_j$ with index $\beta = 1$ all eigenvalues leave imaginary axis with increasing $t$ from zero.

(d) For each Jordan block $J_j$ of matrix $R$ of even dimension $2r_j$ with index $\beta = -1$ exactly $2r_j - 2$ eigenvalues leave imaginary axis with increasing $t$ from zero. One of the rest eigenvalues goes up imaginary axis and has the first type, and the other eigenvalue goes down imaginary axis and has the second type.

From the representation above we see, that if two pure imaginary eigenvalues of $R - tMJ$ having different types meet for some number $t$, then further we do not change them. More precisely, we modify matrix $M$ (by eliminating the terms corresponding to Jordan blocks with these eigenvalues) such that further these eigenvalues and corresponding eigenvectors remain constant.

We can finally get Hamiltonian matrix with all pure imaginary eigenvalues having even multiplicity only if each eigenvalue of the first type eventually meets an eigenvalue of the second type. Taking into account the fact that with increasing $t$ eigenvalues of the first type go up imaginary axis, and eigenvalues of the first type go down imaginary axis, it can happen if and only if for every $\omega$ the number of Jordan blocks of $R$ with eigenvalues $i\omega_j < \omega$ of the first type is not less than the number of Jordan blocks of $R$ with eigenvalues $i\omega_j < \omega$ of the second type plus the number of Jordan blocks of $R$ of neutral type with eigenvalue $i\omega$ and index $\beta = -1$.

Denote

$$s(\omega) = m_+ (\omega) - m_- (\omega) - m_0 (\omega),$$

where $m_+ (\omega)$ is the number of odd dimensional Jordan blocks of matrix $R$ with eigenvalues $i\omega_j$ such that $\omega_j < \omega$, $\beta = 1$, $m_- (\omega)$ is the number of odd dimensional Jordan blocks of matrix $R$ with eigenvalues $i\omega_j$ such that $\omega_j \leq \omega$, $\beta = -1$, and $m_0 (\omega)$ is the number of even dimensional Jordan blocks of matrix $R$ with eigenvalues $i\omega$ such that $\beta = -1$.

Theorem 5.4. There exists a positive definite matrix $M$ such that matrix $R - MJ$ has no pure imaginary eigenvalues if and only if for every pure imaginary eigenvalue $i\omega_j$ of matrix $R$ we have

$$s(\omega_j) \geq 0.$$  \hspace{1cm} (13)

Taking into account the result of the previous section, we get the following main result.

Theorem 5.5. The Riccati inequality (1) has a solution if and only if inequality (13) holds for all $\omega \in \mathbb{R}$. 

6. A Numerical Example

Consider the Riccati inequality:

\[ HA + A^*H + G_1 - HB\Gamma^{-1}B^*H < 0, \]

where:

\[ \{A, B, G\} = \begin{cases} \left( \begin{array}{ccc} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{ccc} 6 & -2 & -2 \\ -2 & -3 & -2 \\ -2 & -2 & -3.9 \end{array} \right) \end{cases} \]

and

\[ \Gamma = \begin{pmatrix} -10 & 0 \\ 0 & 0.1 \end{pmatrix} \]

Now computing the eigenvalues of \( A \) yields \( \{1,1,1\} \). This it checks that matrix \( A \) has no purely imaginary eigenvalues. Notice also that matrix \( G \) and \( \Gamma \) are Hermitian, and matrix \( \Gamma \) is sign indefinite.

Associated Hamiltonian matrix of the inequality being:

\[ R = \begin{pmatrix} A & -B\Gamma^{-1}B^* \\ -G & -A^* \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0.1 & 0.1 & 0. \\ 0 & 1.1 & 0.1 & 0.1 & 0. \\ 0 & 0 & 1 & 0. & 0. & -10. \\ -6 & 2 & 2 & -1 & 0 & 0 \\ 2 & 3 & 2 & 1 & -1 & 0 \\ 2 & 2 & 3.9 & -1 & -1 & -1 \end{pmatrix} \]

Notice that eigenvalues of \( R \) are \( \pm 6.0506i, \pm 1.5866i \) and \( \pm 1.7964 \). Thus, there are simple pure imaginary eigenvalues of matrix \( R \).

The corresponding Riccati equation has no solution.

Now consider some positive definite matrix, for instance

\[ \zeta = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \]

Then

\[ G_1 = G + \zeta = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \]

Notice that the matrix \( \zeta \) is positive definite because it has the following eigenvalues \( \{8, 2, 2\} \).

The equality becomes:

\[ HA + A^*H + G_1 - HB\Gamma^{-1}B^*H = 0 \]

With all conditions satisfied, a Hermitian matrix \( H \) that satisfies (26) is equal to
RICCATI INEQUALITY

\[ H = \begin{pmatrix} 128.485 & -178.389 & -7.18338 \\ -178.389 & 259.987 & 12.4241 \\ -7.18338 & 12.4241 & 1.25879 \end{pmatrix} \] (20)

It is however straight-forward to check that this matrix \( H \) is a solution of inequality (21).

7. Conclusion

We consider the problem of solvability of Riccati inequalities with arbitrary nondegenerate quadratic form. The necessary and sufficient condition for solvability of these inequalities were obtained and expressed in terms of the associated Hamiltonian matrices. The proof of this result is based on special representations of Hamiltonian matrices. An extension of this result to matrix pencils is a subject of possible future investigations.

References

[1] Likhtarnikov AL, Barabanov NE, Leonov GA, Gelig AH, Matveev AS, Smirnova VB, Fradkov AL. Frequency domain theorem (YakubovichKalman lemma) in the control theory. Automation and Remote Control 1996; 57(10):340.

[2] R. Alam, S. Bora, M. Karow, V. Mehrmann, J. Moro, Perturbation theory for Hamiltonian matrices and the distance to bounded-realness, SIMAX, submitted for publication.

[3] Zhou K, Doyle JC. Essentials of Robust Control, Prentice Hall 1997

[4] Yakubovich VA. The solution of certain matrix inequalities in automatic control theory. Soviet Mathematics Doklady 1962; 143(6):13041307.

[5] Yakubovich VA. Frequency domain theorem in the control theory. Siberian Mathematical Journal 1973; 17(2):384420.

[6] Lancaster P, Rodman L. The Algebraic Riccati Equations. Oxford University Press: Oxford, 1995.

[7] Barabanov NE, Ortega R. On the solvability of extended Riccati equations. Transactions on IEEE Automatic Control 2004; 49(4):598602.

[8] Barabanov, Nikita E. "On the solutions of Riccati inequalities with sign indefinite quadratic terms." Decision and Control, 1996., Proceedings of the 35th IEEE Conference on. Vol. 4. IEEE, 1996.

[9] C. Scherer, The Riccati inequality and state-space H,-optimal control, Ph.D. dissertation, Univ. of Würzburg, 1990

Department of Mathematics 2750, North Dakota State University, PO BOX 6050, Fargo, ND 58105-6050, USA

E-mail address: kevin.kissi@ndsu.edu