Some Remarks on Real-Time Turing Machines  
(Preliminary Report)

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Abstract

The power of real-time Turing machines using sublinear space is investigated. In contrast to a claim appearing in the literature, such machines can accept non-regular languages, even if working in deterministic mode. While maintaining a standard binary counter appears to be impossible in real-time, we present a guess and check approach that yields a binary representation of the input length. Based on this technique, we show that unary encodings of languages accepted in exponential time can be recognized by nondeterministic real-time Turing machines.

1 Introduction

From one of the earliest papers on computational complexity [SHL65], it is known that there are lower space bounds for accepting non-regular languages. In the case of deterministic two-way machines space proportional to log log n is required, while for one-way machines the bound is log n. An interesting claim of a stronger gap for real-time machines appears in [Bru10], namely that even nondeterministic machines of this kind require linear space for accepting non-regular sets. In contrast to this claim we show here that deterministic real-time Turing machines can accept non-regular languages in sublinear space. While it seems impossible to count in a standard way with a real-time machine, we develop a counting technique for nondeterministic real-time Turing machines such that they have access to a binary representation of the input length before having consumed more than half of their input. We use this technique for a general result about single-letter languages acceptable by nondeterministic real-time Turing machines. A consequence of this result is that the primes in unary can be accepted by nondeterministic real-time Turing machines disproving a conjecture appearing in the literature.

2 Discussion of Bruda’s Model

The definition of Turing machines given in [Bru10] deviates in several ways from the definition of classical papers like [BG70]. There is a single halt state.
h (in comparison to a set of such states in [BG70]) and no transition from this state is possible. More significant is the fact that the definition of acceptance in [Bru10] is independent of the contents of input- and work-tapes. A consequence of this aspect of its formal definition is that a Turing machine working in real-time accepts any extension of a shorter accepted input, since an accepting configuration is reached on the longer input string as well. In particular, every single-letter language accepted by machines of this kind is either empty or co-finite and thus regular. While this does not contradict Bruda’s claim that languages accepted by real-time Turing machines in sublinear space are regular, we will now present a language accepted in real-time and $O(\sqrt{n})$ space. Let the set of states of the real-time Turing machine $M_s$ with one input-tape and one work-tape be $K = \{q_0, q_1, q_2, q_3\}$ with $q_0$ the initial state. The alphabet is $\Sigma = \{\#, a, 0, 1\}$ (where $\#$ is the blank) and the transition function mapping $K \times \Sigma^2$ to $(K \cup \{h\}) \times \{R, L, N\}^2 \times \Sigma^2$ is given by

$$
\begin{align*}
\delta(q_0, a, \#) &= (q_1, R, R, a, 0) \\
\delta(q_0, a, 0) &= (q_0, R, R, a, 0) \\
\delta(q_1, a, \#) &= (q_2, R, L, a, 0) \\
\delta(q_1, b, \#) &= (h, R, L, b, 0) \\
\delta(q_2, a, 0) &= (q_2, R, L, a, 0) \\
\delta(q_2, a, \#) &= (q_3, R, L, a, 0) \\
\delta(q_3, a, \#) &= (q_0, R, R, a, 0) \\
\delta(q_3, b, \#) &= (h, R, L, b, 0)
\end{align*}
$$

The recognition of perfect squares is based on the well-known sum of the first $k$ odd numbers:

$$
\sum_{i=0}^{k-1} 2i + 1 = k^2
$$

Machine $M_s$ marks successively segments of the tape having odd lengths. For a prefix $a^n b$ with $n = k^2$ it will sweep over a segment with $2k - 1$ cells while reading $a^n$ and mark an additional cell for the $b$. For $n$ strictly between perfect squares no additional space is required during the sweep. Therefore, the space usage of $M_s$ is bounded by $2\sqrt{n}$ and thus sublinear. Next we argue that

$$
S = \{a^n b x \mid n \text{ is a perfect square}, x \in \{a, b\}^*\}
$$

is not regular. The right quotient $S/b\{a, b\}^* = \{a^n \mid n \text{ is a perfect square}\}$ is a well-known non-regular language (Exercise 4.1.2(a) in [HU79]). This quotient would be regular if $S$ was, as follows from the closure of the regular languages under quotient (Theorem 9.13 from [HU69]).

3 A Nondeterministic Counting Technique

In the present section we adopt the standard definition of acceptance by final state and empty storage.

**Lemma 1** There is a nondeterministic real-time Turing machine $M_c$ (counter) with the following properties:
1. Mₖ writes down a guess of $n$ in binary on a designated work tape before having read $n/2$ symbols of an input of length $n$.

2. Mₖ enters a special state at the end of its computation if and only if the guessed value was correct.

3. Mₖ uses $O(\log n)$ space in every computation.

Proof. Machine $Mₖ$ executes several processes in parallel making use of multiple tapes and the cross product of states of Turing machines implementing these processes. The work tapes will be referenced by the following names:

- **CURRENT**: Tape contents of a single-tape Turing machine $M_d$ defined below representing the number of input symbols processed by $Mₖ$.
- **FINAL**: Guess of the configuration $M_d$ reaches when $Mₖ$ has read its entire input.
- **WORK**: Copy of the guessed configuration.
- **LENGTH**: Stores input length computed from guessed configuration on WORK.
- **DIFF**: Unary counter keeping track of the number of differences between CURRENT and FINAL.

One process using tape CURRENT is the simulation of the deterministic single-tape machine $M_d$ (no input tape) that maintains a binary counter:

$$
\begin{align*}
\delta(q₀, \#) &= (q₁, L, \#) \\
\delta(q₀, 0) &= (q₀, R, 0) \\
\delta(q₀, 1) &= (q₀, R, 1) \\
\delta(q₁, \#) &= (q₀, R, 1) \\
\delta(q₁, 0) &= (q₀, R, 1) \\
\delta(q₁, 1) &= (q₁, L, 0)
\end{align*}
$$

State $q₁$ propagates a carry and $q₀$ moves the head back to the least significant digit. The simulation is carried out on tape CURRENT by $Mₖ$. Notice that $M_d$ has no halt state, since we are only interested in its configuration when the input of $Mₖ$ has been read completely. This configuration includes the head position and the internal state of $M_d$ and we extend the alphabet $\{\#, 0, 1\}$ of $M_d$ with symbols $\{\#, 0, 1\} \times \{0, 1\}$, where the second component represents state and head position of $M_d$. Whenever $M_d$ writes a digit on a new tape cell (except for the two most significant ones), $Mₖ$ guesses a symbol $\alpha$ from the extended alphabet of $M_d$ and writes $\alpha$ onto tapes FINAL and WORK, on which the tape heads move in parallel with $M_d$’s head. The unary counter DIFF is increased when $\alpha$ differs from the corresponding symbol on CURRENT. On tapes CURRENT, FINAL and WORK identical head movements are carried out and each time the scanned symbols become equal or different by the simulation of $M_d$, the unary counter DIFF is adjusted accordingly.

If the two most significant digits are touched by $M_d$ (which it has to guess), a new phase of processing starts. While FINAL is treated in the same way
as in the first phase, $M_d$ remembers a guess of the most significant digit it
its finite control and writes this digit onto WORK (it cannot be written onto
FINAL because $M_c$ necessarily moves its head to the right after the carry has
been propagated and FINAL still executes identical head movements). Next
$M_c$ continues the simulation of $M_d$ starting from the configuration on WORK
until its head is located on the blank to the right of the least significant digit on
WORK. The number of steps executed is counted in binary on tape LENGTH.
Now the number on LENGTH is subtracted from the number on WORK and all
proper prefixes of the binary number on WORK are added to the number on tape
LENGTH. Finally this number is doubled (which simply means a concatenation
of 0).

We claim that LENGTH stores the binary encoding of the number of st-
eps that $M_d$ executes until reaching the configuration stored on FINAL. For con-
figurations with $M_d$’s head to the right of the least significant digit this is easily
checked, since a prefix increases by one for every carry propagation. This in-
volves two crossings of the right border of the prefix. We subtracted the number
of steps until such a configuration is reached, which adjusts the count.

In parallel the simulation of $M_d$ continues and $M_c$ enters the designated
state if DIFF stores 0.

The configuration stored on CURRENT includes a binary counter that in-
creases at most once per input symbol. This shows an $O(\log n)$ space bound for
CURRENT and the other tapes that depend on CURRENT. \qed

For a binary word $w \in \{0,1\}^*$ define a padded string $\text{pad}(w)$ as follows:

\begin{align*}
\text{pad}(1) &= a \\
\text{pad}(w'0) &= \text{pad}(w')^2 \\
\text{pad}(w'1) &= \text{pad}(w')^2 a
\end{align*}

We generalize padding to a language $L$ by $\text{pad}(L) = \{\text{pad}(w) | w \in L\}$.

**Theorem 1** Let $L \subseteq \{0,1\}^*$. If $L \in \text{NTIME}(2^n)$ then $\text{pad}(L)$ is accepted by
a nondeterministic real-time Turing machine.

Proof. By assumption there is a nondeterministic Turing machine $M$
accepting $L$ in time $2^n$. Given an input of the form $a^k$, a simulator first uses the technique
from Lemma 1 for guessing $k$ in binary. Then it's starts a simulation of $M$
on this binary string with a speed up in order to compensate the time needed for
guessing $k$. If $M$ accepts and the guess was correct, the input $a^k$ is accepted. \qed

We can conclude from the previous theorem that the conjecture from [BG70]
that

\{a^p | p \text{ is prime}\}

cannot be accepted in real-time is wrong. By Pratt’s result [Pra75] the primes
in binary notation are in NP and therefore clearly accepted in time $2^n$.

**References**

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