Hidden Symmetries of Higher-Dimensional Black Hole Spacetimes

Valeri P. Frolov

Theoretical Physics Institute, University of Alberta,
Edmonton, Alberta, Canada T6G 2G7

The paper contains a brief review of recent results on hidden symmetries in higher dimensional black hole spacetimes. We show how the existence of a principal CKY tensor (that is a closed non-degenerate conformal Killing-Yano 2-form) allows one to generate a ‘tower’ of Killing-Yano and Killing tensors responsible for hidden symmetries. These symmetries imply complete integrability of geodesic equations and the complete separation of variables in the Hamilton-Jacobi, Klein-Gordon and Dirac equations in the general Kerr-NUT-(A)dS metrics.

§1. Introduction

The concept of symmetries is one of the most powerful tools of modern theoretical physics. Noether’s theorem relates continuous symmetries to conservation laws. The most fundamental of them are connected with the symmetries of the background spacetime. In a curved spacetime continuous isometries are generated by Killing vector fields. For each of the Killing vector fields there exists a conserved quantity. For example, for a geodesic particle motion this conserved quantity is a projection of the particle momentum on the Killing vector. Besides these symmetries, which are ‘naturally’ connected with the spacetime isometries, there exist hidden symmetries, generated by either symmetric or antisymmetric tensor fields. These objects are connected with conserved quantities which are higher than first order in momentum.

A well known example is the famous Kerr spacetime. It was demonstrated by Carter\cite{1} that the Hamilton-Jacobi equation for a particle motion in the Kerr metric allows a separation of variables and the geodesic equations are completely integrable. The natural conservation laws connected with Kerr metric symmetries, which is stationary and axisymmetric, are not sufficient to explain this ‘miracle’. Really, spacetime symmetries are ‘responsible’ for two integrals of motion, the energy and the azimuthal component of the angular momentum. This together with the conservation of $p^2$ gives only 3 integrals of motion. Carter\cite{1} constructed the fourth required integral of motion, which is quadratic in momentum and is connected with the Killing tensor $K_{ab}\cite{2}$. Penrose and Floyd\cite{3} showed that this Killing tensor is a ‘square’ of an antisymmetric Killing-Yano tensor.\cite{4}

In many aspects a Killing-Yano tensor is more fundamental than a Killing tensor. Namely, its ‘square’ is always Killing tensor, but the opposite is not generally true (see, e.g., 5)). In 4-dimensional spacetime, as it was shown by Collinson,\cite{6} if a vacuum solution of the Einstein equations allows a non-degenerate Killing-Yano tensor it is of the type D. All the vacuum type D solutions were obtained by Kinnersley,\cite{7} Demianski and Francaviglia\cite{8} showed that in the absence of the acceleration these solutions admit Killing and Killing-Yano tensors. It should be also mentioned
that if a spacetime admits a non-degenerate Killing-Yano tensor it always has at least one Killing vector.\textsuperscript{9}

Recently a lot of interest focuses on higher dimensional black hole solutions. In the widely discussed models with large extra dimensions it is assumed that one or more additional spatial dimensions are present. In such models one expects mini black hole production in the high energy collisions of particles. Mini black holes can serve as a probe of the extra dimensions. At the same time their interaction with the brane, representing our physical world, can give the information about the brane properties. If a black hole is much smaller than the size of extra dimension and the brane tension can be neglected, its metric is an asymptotically flat or (A)dS solution of higher dimensional Einstein equations. The most general known black hole solution, which besides the mass and rotation parameters, admits also NUT parameters and the cosmological constant was obtained.\textsuperscript{10} Recently a remarkable progress in study the properties of such black holes was achieved. Namely, it was demonstrated that in many respects these metrics are similar to the 4-dimensional Kerr-NUT-(A)dS metric. They possess wide enough symmetry to make it possible the complete integrability of the geodesic motion equations and the complete separation of variables in the Hamilton-Jacobi, Klein-Gordan and Dirac equations. Main technical tool for obtaining these results is connected with the existence of hidden symmetries. This paper contains a brief review of these results.

\section{Killing-Yano and Killing tensors}

Let us consider a $D$-dimensional spacetime with a metric $g$. In order to cover both cases of odd and even dimensions we write the spacetime dimension $D$ in the form $D = 2n + \varepsilon$, where $\varepsilon = 0$ ($\varepsilon = 1$) for the even (odd) dimensional case. One says that the spacetime has a symmetry generated by the vector field $\xi^a$ if this vector obeys the \textit{Killing equation}

$$\xi_{(a;b)} = 0.$$ (2.1)

For a geodesic motion of a particle in such a curved spacetime the quantity $p^a \xi_a$, where $p^a$ is its momentum, remains constant along the particle trajectory. Similarly, for a null geodesic, $p^a \xi_a$ is conserved provided $\xi^a$ is a \textit{conformal Killing vector} obeying the equation

$$\xi_{(a;b)} = \tilde{\xi} g_{ab} \, , \quad \tilde{\xi} = D^{-1} \xi^b_{,b}.$$ (2.2)

There exist natural generalizations of this symmetry which result in the conservation laws for objects higher than of the first power in the momentum $p$. In such a case one says that the spacetime possesses a \textit{hidden symmetry}.

A \textit{symmetric} tensor $K_{a_1 a_2 \ldots a_p} = K_{(a_1 a_2 \ldots a_p)}$ is called a conformal Killing tensor if it obeys the equation

$$K_{(a_1 a_2 \ldots a_p;b)} = g_{b(a_1} \tilde{K}_{a_2 \ldots a_p)}.$$ (2.3)

As in the case of a conformal Killing vector, the tensor $\tilde{K}$ is determined by tracing the both sides of the equation (2.3). If $\tilde{K}$ vanishes, the tensor $K$ is called a \textit{Killing tensor}. In a presence of the Killing tensor $K$ the conserved quantity for a geodesic
motion is \( K_{a_1a_2...a_p}p^{a_1}p^{a_2}...p^{a_p} \). For null geodesics this quantity is conserved not only for the Killing tensor, but also for a conformal Killing tensor.

An antisymmetric generalization of the Killing vector is known as a Killing-Yano tensor. An antisymmetric tensor \( h_{a_1a_2...a_p} = h[a_1a_2...a_p] \) is called a conformal Killing-Yano tensor (or, briefly, CKY tensor) if it obeys the following equation

\[
\nabla(a_1h_{a_2}a_3...a_{p+1}) = g_{a_1a_2}h_{a_3...a_{p+1}} - (p - 1)g[a_3(a_1h_{a_2})...a_{p+1}].
\]

(2.4)

By tracing the both sides of this equation one obtains the following expression for \( \tilde{h} \)

\[
\tilde{h}_{a_2a_3...a_p} = \frac{1}{D - p + 1} \nabla^{a_1}h_{a_1a_2...a_p}.
\]

(2.5)

In the case when \( \tilde{h} = 0 \) one has a Killing-Yano tensor, or, briefly, KY tensor. If \( f_{a_1a_2...a_p} \) is a Killing-Yano tensor than \( K_{ab} = f_{aa}...f_{a_p} \) is a Killing tensor.

Let \( h_{ab} \) be a CKY tensor then the vector

\[
\xi^{(0)}{}^a = \frac{1}{D - 1} \nabla_bh^{ab}
\]

obeys the following equation

\[
\xi^{(0)}{}_{(a;b)} = -\frac{1}{D - 2} R_{c(a}h_{b)}{}^c.
\]

(2.7)

Thus, in an Einstein space, that is when \( R_{ab} = \Lambda g_{ab} \), \( \xi^{(0)} \) is the Killing vector.

§3. Killing-Yano equations in terms of differential forms

The CKY tensors are forms and operation with them are greatly simplified if one uses the “language” of differential forms. We just remind some of the relations we use in the present paper. If \( \alpha_p \) and \( \beta_q \) are \( p \) - and \( q \)-forms, respectively, the external derivative \( (d) \) of their external product \( (\wedge) \) obeys a relation

\[
d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p\alpha_p \wedge d\beta_q.
\]

(3.1)

A Hodge dual \(*\alpha_p \) of the \( p \)-form \( \alpha_p \) is \((D - p)\)-form defined as

\[
*\alpha_p \leftrightarrow (*\alpha)a_1...a_{D-p} = \frac{1}{p!}\alpha^{b_1...b_p}e_{b_1...b_p}a_1...a_{D-p},
\]

(3.2)

where \( e_{a_1...a_D} \) is a totally anti-symmetric tensor. The exterior co-derivative \( \delta \) is defined as follows

\[
\delta\alpha_p = (-1)^p\epsilon_p * d * \alpha_p, \quad \epsilon_p = (-1)^p(D - p) \frac{\det(g)}{|\det(g)|}.
\]

(3.3)

One also has \(* * \alpha_p = \epsilon_p \alpha_p \).

If \( \{ e_a \} \) is a basis of vectors, then dual basis of 1-forms \( \omega^a \) is defined by the relations \( \omega^a(e_b) = \delta^a_b \). We denote \( \eta_{ab} = g(e_a, e_b) \) and by \( \eta^{ab} \) the inverse matrix.
Then the operations with the indices enumerating the basic vectors and forms are performed by using these matrices. In particular, \( e^a = \eta^{ab} e_b \), and so on. We denote a covariant derivative along the vector \( e_a \) by \( \nabla_a = \nabla_{e_a} \). One has

\[
d = \omega^a \land \nabla_a, \quad \delta = -e^a \land \nabla_a.
\] (3.4)

In the tensor notations the ‘hook’ operator applied to a \( p \)-form \( \alpha_p \) corresponds to a contraction

\[
X \hook \alpha_p \not\equiv X^{a_1} \alpha_{a_1 a_2 \ldots a_p}.
\] (3.5)

For a given vector \( X \) one defines \( X^{\flat} \) as a corresponding 1-form with the components

\[
(X^{\flat})^a = g_{ab} X^b.
\]

In particular, one has \( \eta^{ab} (e^b)^{\flat} = \omega^a \). We refer to 13), 14) where these and many other useful relations can be found.

The definition (2.4) of the CKY tensor \( h \) (which is a \( p \)-form) is equivalent to the following equation (see e.g. 14))

\[
\nabla_X h = \frac{1}{p+1} X \hook d h - \frac{1}{D-p+1} X^b \land \delta h.
\] (3.6)

If \( \delta h = 0 \) this is an equation for the Killing-Yano tensor.

Using a relation

\[
X \hook * \omega = *(\omega \land X^b)
\] (3.7)

it is easy to show that (3.6) implies

\[
\nabla_X (*h) = \frac{1}{p_* + 1} X \hook d(*h) - \frac{1}{D - p_* + 1} X^b \land \delta(*h), \quad p_* = D - p.
\] (3.8)

It means that a Hodge dual \( *\) of a CKY tensor \( h \) is again a CKY tensor. Moreover, if the CKY is closed, \( dh = 0 \), then its dual \((D - p)\)-form \( f = *h \) is a Killing-Yano tensor.

The following result\(^{15}\) plays an important role in the construction of the hidden symmetry objects in higher dimensional spacetimes.

**Proposition.** If \( h^{(1)} \) and \( h^{(2)} \) are two closed CKY tensors their external product \( h = h^{(1)} \land h^{(2)} \) is also a closed CKY tensor.

To prove this proposition we first notice that Eq. (3.1) implies that the form \( h \) is closed. Suppose now that a \( p \)-form \( \alpha_p \) obeys a relation

\[
\nabla_X \alpha_p = X^b \land \gamma,
\] (3.9)

then

\[
\gamma = -\frac{1}{D-p+1} \delta \alpha_p.
\] (3.10)

Really

\[
\delta \alpha_p = -e^a \land \nabla_a \alpha_p = -e^a \land (\omega_a \land \gamma)
\]

\[
= -(e^a \land \omega_a) \land \gamma + \omega_a \land (e^a \land \gamma) = -(D - p + 1) \gamma.
\] (3.11)

We used the relations

\[
e^a \land \omega_a = D, \quad \omega_a \land (e^a \land \gamma) = (p - 1) \gamma.
\] (3.12)
The relation (3.10) follows from (3.11).

The second step in the proof of the Proposition is to show that if \( \alpha_p \) and \( \beta_q \) are two closed CKY tensors then

\[
\nabla_X (\alpha_p \land \beta_q) = X^b \land \gamma_p+q-1.
\]

Really, one has

\[
\nabla_X (\alpha_p \land \beta_q) = \nabla_X \alpha_p \land \beta_q + \alpha_p \land \nabla_X \beta_q
\]

\[
= -\frac{1}{D-p+1} (X^b \land \delta \alpha_p) \land \beta_q - \frac{1}{D-q+1} \alpha_p \land (X^b \land \delta \beta_q) = X^b \land \gamma_p+q-1,
\]

\[
\gamma_p+q-1 = -\frac{1}{D-p+1} \delta \alpha_p \land \beta_q - \frac{(-1)^p}{D-q+1} \alpha_p \land \delta \beta_q.
\]

Combining the relations (3.9)–(3.14) one arrives at the result given in the Proposition.

§4. Principal CKY tensors and ‘towers’ of Killing and Killing-Yano tensors

Let us consider now a special case which is important for applications. Namely we assume that a spacetime allows a 2-form \( h \) which is a closed conformal Killing-Yano tensor. Such a 2-form can be written, at least locally, as \( h = db \). We also assume that the 2-form \( h \) is non-degenerate, that is its rank is 2. We call such an object a principal CKY tensor.

It is easy to show that \( S_{ab} = h_{ac} h^c_b \) is a symmetric tensor, and its eigen-values \( x^2 \),

\[
S^a_{\; b} v^b = -x^2 v^a,
\]

are real and non-negative. Using a modified Gram-Schmidt procedure it is possible to show that there exists such an orthonormal basis in which the operator \( h^a_{\; b} \) has the following form

\[
h = \text{diag}(0, \ldots, 0, A_1, \ldots, A_p),
\]

where \( A_i \) are matrices of the form

\[
A_i = \begin{pmatrix} 0 & -x_i I_i \\ x_i I_i & 0 \end{pmatrix}
\]

and \( I_i \), are unit matrices. Such a basis is known as the Darboux basis (see e.g. 16)). Its elements are unit eigen-vectors of the problem (4.1).

For a non-degenerate 2-form \( h \) the number of zeros in the Darboux decomposition (4.2) coincides with \( \varepsilon \). If all the eigen-values \( x \) in (4.1) are different (we denote them \( x_\mu, \mu = 1, \ldots, n \)), the matrices \( A_i \) are 2-dimensional. Denote the vectors of the Darboux basis by \( e_\mu \) and \( e_{\bar{\mu}} \equiv e_{n+\mu} \), where \( \mu = 1, \ldots, n \). In the odd dimensional spacetime we also have an additional basic vector \( e_0 \) (an eigen-vector of (4.1) with \( x = 0 \)). Orthonormal vectors \( e_\mu \) and \( e_{\bar{\mu}} \) span a 2-dimensional plane of eigen-vectors of (4.1) with the same eigen-value \( x_\mu \). We denote by \( \omega^{\mu} \) and \( \omega^{\bar{\mu}} \equiv \omega^{n+\mu} \) (and \( \omega^0 \) if
the dual basis of 1-forms. The metric $g_{ab}$ and the principal CKY tensor $h$ in this basis take the form

$$g_{ab} = \sum_{\mu=1}^{n} (\omega^\mu_a \omega^\mu_b + \omega^\overline{\mu}_a \omega^\overline{\mu}_b) + \epsilon \omega^0_a \omega^0_b,$$

(4.4)

$$h = \sum_{\mu=1}^{n} x_\mu \omega^\mu \land \omega^\overline{\mu}.$$

(4.5)

According to the Proposition of the previous section, the principal CKY tensor generates a set (‘tower’) of new closed CKY tensors

$$h^{(j)} = h \land \ldots \land h \text{ total of } j \text{ factors}.$$  

(4.6)

$h^{(j)}$ is a $2j$ form. In particular for $j = 1$ $h^{(1)} = h$. Since $h$ is non-degenerate, one has a set of $n$ non-vanishing closed CKY tensors. In the even dimensional spacetime $h^{(n)}$ is proportional to the totally antisymmetric tensor.

Each $2j$-form $h^{(j)}$ determines a $(D - 2j)$-form of the Killing-Yano tensors

$$f^{(j)} = \ast h^{(j)}.$$  

(4.7)

In its turn, these tensors determine the Killing tensors $K^{(j)}$

$$K^{(j)}_{ab} = \frac{1}{(D - 2j - 1)! (j)!^2} f^{(j)}_{ac1 \ldots c_{D-2j-1}} f^{(j)}_{b} c_{1 \ldots c_{D-2j-1}}.$$  

(4.8)

A choice of the coefficient in the definition (4.8) is a matter of convenience. It is convenient to include the metric $g$, which is a trivial Killing tensor, as an element $K^{(0)}$ of the ‘tower’ of the Killing tensors. The total number of elements of the ‘extended tower’ is $n$. 

**§5. Hidden symmetries of the higher dimensional Kerr-NUT-(A)dS spacetimes**

The most general known higher dimensional solution describing rotating black holes with NUT parameters in an asymptotically (Anti) deSitter spacetime (Kerr-NUT-(A)dS metric) was obtained in 10). This metric has the form (4.4) where

$$\omega^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad \omega^\overline{\mu} = \sqrt{Q_\mu} \sum_{k=0}^{n-1} A^{(k)}_\mu d\psi_k, \quad \omega^0 = (c/A^{(n)})^{1/2} \sum_{j=0}^{n} A^{(j)} d\psi_j.$$  

(5.1)

Here

$$Q_\mu = X_\mu/U_\mu, \quad U_\mu \equiv \prod_{\nu \neq \mu} (x_\nu^2 - x_\mu^2).$$  

(5.2)

Footnote 1: For example, in 5D spacetime where $n = 2$, this ‘tower’ contains only one non-trivial Killing tensor. For the 5D rotating black hole solution this Killing tensor was first found in 17),18) by using the Carter’s method of separation of variables in the Hamilton-Jacobi equation.
and the coefficients $A^{(k)}_\mu$ and $A^{(j)}$ are polynomial functions of coordinates $x_\nu$ determined by the following expansions (their explicit form can be found in 10))

$$
\prod_{\nu=1}^{n}(1 + tx_\nu^2) = \sum_{j=0}^{n} t^j A^{(j)}_\nu, \quad (1 + tx_\nu^2)^{-1} \prod_{\nu=1}^{n}(1 + tx_\nu^2) = \sum_{k=0}^{n-1} t^k A^{(k)}_\nu. \quad (5.3)
$$

The coefficients $X_\mu$ are functions of one variable, $x_\mu$, which for the Kerr-NUT-(A)dS metric are

$$
X_\mu = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} - \frac{\varepsilon c}{x_\mu^2}. \quad (5.4)
$$

This metric is of the Petrov type D.\(^{19}\) The total number of constants which enter the solution is $2n + 1$: $\varepsilon$ constants $c$, $n + 1 - \varepsilon$ constants $c_k$ and $n$ constants $b_\mu$. The form of the metric is invariant under a 1-parameter scaling coordinate transformations, thus a total number of independent parameters is $D - \varepsilon$. These parameters are related to the cosmological constant, mass, angular momenta, and NUT parameters. One of these parameters can be used to define a scale, while the other $D - 1 - \varepsilon$ parameters can be made dimensionless.\(^{10}\) This solution may be considered as a higher dimensional generalization of 4-dimensional Kerr-NUT-(A)dS solution obtained by Carter.\(^{20}\) Moreover, the coordinates used in the metric (4.4), (5.1)–(5.4) are higher dimensional analogue of the Debever-Carter coordinates.\(^{21)}, \, 22)\)

The curvature for the general metric (5.1)–(5.3) with arbitrary functions $X_\mu(x_\mu)$ was calculated in 23). For a special choice (5.4) this metric is a solution of the higher dimensional Einstein equations

$$
R_{ab} = \Lambda g_{ab}, \quad \Lambda = (D - 1)(-1)^n c_n. \quad (5.5)
$$

It was shown in 24), 25) that this spacetime possesses a principal CKY tensor $h$ which has the form (4.5) and its potential $b$, $h = db$, is\(^{a})

$$
b = \frac{1}{2} \sum_{k=0}^{n-1} A^{(k+1)}d\psi_k. \quad (5.6)
$$

The Killing tensors associated with the principal CKY tensor for the Kerr-NUT-(A)dS spacetime can be written as follows:\(^{15})

$$
K^{(j)}_{ab} = \sum_{\mu=1}^{n} A^{(j)}_\mu (\omega^\mu_a \omega^\mu_b + \omega^\mu_b \omega^\mu_a) + \varepsilon A^{(j)}_\mu \omega^0_a \omega^0_b. \quad (5.7)
$$

We call the Killing vector $\xi^{(0)}$ generated by the principal CKY tensor $h$ (see (2.6)) a primary Killing vector. In the Kerr-NUT-(A)dS spacetime the primary Killing vector is $\xi^{(0)} = \partial_\psi_0$. Besides the primary Killing vector, this spacetime has $n - 1$ additional Killing vectors $\xi^{(j)}\(^{15})

$$
\xi^{(j)a} = K^{(j)a} b, \quad \xi^{(j)a} \partial_a = \partial_\psi_j, \quad j = 1, \ldots, n - 1. \quad (5.8)
$$

\(^{a})\) In fact, this potential generates a principal CKY tensor for a general form of the metric (4.4), (5.1)–(5.3) with arbitrary functions $X_\mu(x_\mu)$.
In odd dimensions the last Killing vector is given by the $n$-th Killing-Yano tensor $f^{(n)}$, which in the Kerr-NUT-(A)dS spacetime turns out to be $\partial \psi_n$. The total number of these Killing vectors is $n + \varepsilon$. For a geodesic motion they give $n + \varepsilon$ linear in momentum integrals of motion. The ‘extended tower’ of the Killing tensors $K^{(j)}$ $(j = 0, \ldots n - 1)$ gives $n$ additional integrals of motion, which are quadratic in the momentum. Thus the total number of conserved quantities for a geodesic motion is $2n + \varepsilon$, that is it coincides with the number of the spacetime dimensions $D$. It is possible to show that these integrals of motion are independent and in involution, so that the geodesic motion in the Kerr-NUT-(A)dS spacetime is completely integrable.\(^{26)–28)}\) Moreover, it was shown recently\(^{29)}\) that the following operators
\begin{align}
L_{(k)} &= -i\xi^{(k)a}\partial_a, \quad (k = 0, \ldots, n + \varepsilon - 1) \\
K_{(j)} &= -\frac{1}{\sqrt{|g|}}\partial_a[\sqrt{|g|}K^{(j)ab}\partial_b], \quad (j = 0, \ldots, n - 1)
\end{align}
determined by a principal CKY tensor, are mutually commutative.

It should be emphasized that the coordinates in the metric (5.1)–(5.3) have a well defined geometrical meaning. The ‘essential’ coordinates $x_\mu$ are connected with eigen-values of the principal CKY tensor $h$ (see (4.5)), while the Killing coordinates $\psi_j$ are defined by the Killing vectors generated by the principal CKY tensor. Namely this invariant definition of the coordinates in the metric (5.1)–(5.3) makes it so convenient for calculations.

The existence of a principal CKY tensor imposes non-trivial restrictions on the geometry of the spacetime. Namely, the following result was proved in 30). Let $h$ be a principal CKY tensor and $\xi^{(0)}$ be its primary Killing vector. Then if
\begin{equation}
\mathcal{L}_{\xi^{(0)}} h = 0, \tag{5.11}
\end{equation}
then the only solution of the the Einstein equations with the cosmological constant (5.5) is the Kerr-NUT-(A)dS spacetime. (Here $\mathcal{L}_u$ is a Lie derivative along the vector $u$.)

§6. Hidden symmetries and separation of variables

The massive scalar field equation
\begin{equation}
\Box \Phi - m^2 \Phi = 0, \tag{6.1}
\end{equation}
in the Kerr-NUT-(A)dS metric allows a complete separation of variables.\(^{31)}\) Namely a solution can be decompose into modes
\begin{equation}
\Phi = \prod_{\mu=1}^{n} R_\mu (x_\mu) \prod_{k=0}^{n+\varepsilon-1} e^{i\psi_k \psi_k}. \tag{6.2}
\end{equation}
Substitution of (6.2) into the equation (6.1) results in the following second order ordinary differential equations for functions $R_\mu (x_\mu)$
\begin{equation}
(X_\mu R'_\mu) + \varepsilon \frac{X_\mu}{x_\mu} R'_\mu + \left( V_\mu - \frac{W^2}{X_\mu} \right) R_\mu = 0. \tag{6.3}
\end{equation}
Here
\[ W_\mu = \sum_{k=0}^{n+\varepsilon-1} \Psi_k(-x_\mu^2)^{n-k}, \quad V_\mu = \sum_{k=0}^{n+\varepsilon-1} \kappa_k(-x_\mu^2)^{n-k}. \quad (6.4) \]

Here \( \kappa_0 = -m^2 \) and for \( \varepsilon = 1 \) we put \( \kappa_n = \psi_n^2 / c \). The parameters \( \kappa_k \) \( (k = 1, \ldots, n + \varepsilon - 1) \) are separation constants. Using (5.10) one has \( K(0) = -\Box \). Since all the operators (5.9)–(5.10) commute with one another, their common eigen-values can be used to specify the modes. It is possible to show\(^{29}\) that the eigen-vectors of these commuting operators are the modes (6.2) and one has
\[ L^{(k)} \Phi = \Psi_k \Phi, \quad K^{(j)} \Phi = \kappa_j \Phi. \quad (6.5) \]

Similarly, the Hamilton-Jacobi equation for geodesic motion
\[ \frac{\partial S}{\partial \lambda} + g^{ab} \partial_a S \partial_b S = 0, \quad (6.6) \]
in the Kerr-NUT-(A)dS spacetime allows a complete separation of variables\(^{31}\)
\[ S = m^2 \lambda + \sum_{k=0}^{n+\varepsilon-1} \psi_k \psi_k + \sum_{\mu=1}^{n} S_\mu(x_\mu). \quad (6.7) \]
The functions \( S_\mu \) obey the first order ordinary differential equations
\[ S_\mu' = V_\mu X_\mu - W_\mu^2 X_\mu^{-2}, \quad (6.8) \]
where the functions \( V_\mu \) and \( W_\mu \) are defined in (6.4).

Recently it was shown that the massive Dirac equation in the Kerr-NUT-(A)dS spacetime also allows the separation of variables\(^{32}\). It was also proved that the stationary test string equations in the Kerr-NUT-(A)dS spacetime are completely integrable.\(^{33}\)

§7. Conclusions

The Kerr-NUT-(A)dS metric is the most general known solution describing higher dimensional rotating black hole spacetimes with NUT parameters in an asymptotically (Anti) de Sitter spacetime background. It possesses, what we called, a principal CKY tensor \( h \) which determines the hidden symmetries of this spacetime. This 2-form \( h \) generates a ‘tower’ of Killing-Yano and Killing tensors, which make it possible a complete integrability of geodesic equations and separability of the Hamilton-Jacobi, Klein-Gordon and Dirac equations. Moreover, if a higher dimensional solution of the Einstein equations with the cosmological constant allows a principal CKY tensor obeying (5.11), it coincides with the Kerr-NUT-(A)dS metric. These remarkable properties of higher dimensional black hole solutions resemble the well known ‘miraculous’ properties of the Kerr spacetime. Does this analogy go further? Are all higher dimensional solutions with the principle CKY tensor of Petrov type D? Can the higher spin massless field equations be decoupled and do they allow separation of variables? These are interesting but still open problems.
Acknowledgements

The author thanks the Yukawa Institute for Theoretical Physics at Kyoto University, where this work was partially done during the scientific program on “Gravity and Cosmology 2007”, and Professor Misao Sasaki for the hospitality. He also is grateful to the Natural Sciences and Engineering Research Council of Canada and the Killam Trust for the financial support.

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