High confidence estimates of the mean of heavy-tailed real random variables

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Abstract: We present new estimators of the mean of a real valued random variable, based on PAC-Bayesian iterative truncation. We analyze the non-asymptotic minimax properties of the deviations of estimators for distributions having either a bounded variance or a bounded kurtosis. It turns out that these minimax deviations are of the same order as the deviations of the empirical mean estimator of a Gaussian distribution. Nevertheless, the empirical mean itself performs poorly at high confidence levels for the worst distribution with a given variance or kurtosis (which turns out to be heavy tailed). To obtain (nearly) minimax deviations in these broad class of distributions, it is necessary to use some more robust estimator, and we describe an iterated truncation scheme whose deviations are close to minimax. In order to calibrate the truncation and obtain explicit confidence intervals, it is necessary to dispose of a prior bound either on the variance or the kurtosis. When a prior bound on the kurtosis is available, we obtain as a by-product a new variance estimator with good large deviation properties. When no prior bound is available, it is still possible to use Lepski’s approach to adapt to the unknown variance, although it is no more possible to obtain observable confidence intervals.

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Introduction

This paper is devoted to the estimation of the mean of a real random variable from an independent identically distributed sample. We will emphasize the following issues:

• obtaining non asymptotic confidence intervals;

• getting high confidence levels;

• proving nearly minimax bounds in the class of distributions with a bounded variance and in the class of distributions with a bounded kurtosis.
To achieve these goals, we combine two kinds of tools: truncated estimates and PAC-Bayesian theorems ([9], [8], [10], [5], [2], [1]).

The general conclusion is that the empirical mean estimate behaves poorly at high confidence levels and that the worst case is reached for heavy tailed distributions, as the proofs of the lower bounds show.

This is the bad news. The good news is that, using iterated truncation schemes, it is possible to recover confidence intervals whose widths are close to the (optimal) width of the confidence interval of the empirical mean of a Gaussian distribution, even at very high confidence levels. From a technical point of view, it is possible to build an estimator with an exponential tail even when the sample distribution has only a finite variance. This came out to us as a surprise while working on the more elaborate topic of regression estimation [3], and gave us the spur to work out the estimation of the mean in details, this simpler case lending itself to tighter computations.

The weakest hypothesis we will consider is the existence of a finite variance. While it is possible to adapt a truncation scheme when the variance is unknown, using Lepski’s approach, some more information is required to compute an observable confidence interval. We study two situations: the case when the variance or some upper bound is known and the case when the kurtosis or some upper bound is known. In order to assess the quality of the results, we prove corresponding lower bounds for the best estimator of the worst distribution (following thus the minimax approach), and for the empirical mean estimate of the worst distribution, to assess the improvement brought by the PAC-Bayesian truncation scheme. We plot the numerical values of these upper and lower bounds for typical finite sample sizes to show the gap between them.

Let us end this introduction with a few words in favour of high confidence levels. One reason to seek them is when the estimated quantity is critical from a safety or economical point of view. We will not elaborate on this. Another setting where high confidence levels are required is when lots of estimates are to be computed and compared in some statistical learning scenario. Let us imagine, for instance, that some parameter $\theta \in \Theta$ is to be tuned in order to optimize the answer of some loss function $f_\theta$ to some random input $X$. Let us consider a split sample scheme where two i.i.d. samples $X_1, \ldots, X_s \overset{\text{def}}{=} X_s^s$ and $X_{s+1}, \ldots, X_{s+n} \overset{\text{def}}{=} X_{s+1}^{s+n}$ are used, one to build some estimators $\hat{\theta}_k(X_s^s)$ of $\arg\min_{\theta \in \Theta} E[f_\theta(X)]$ in subsets $\Theta_k$, $k = 1, \ldots, K$ of $\Theta$, and the other to test those estimators and keep hopefully the best. This is a very common model selection situation. One can think for instance of the choice of a basis to expand some regression function. If $K$ is large, estimates of $E[f_{\hat{\theta}_k(X_s^s)}(X_{s+1})]$ will be required for a lot of values of $k$. In order to keep safe from over-fitting, very high confidence levels will be required if the resulting confidence level is to be computed through a union bound (because no
special structure of the problem can be used to do better). Namely, a confidence level of $1 - \epsilon$ on the final result of the optimization on the test sample will require a confidence level of $1 - \epsilon/K$ for each mean estimate on the test sample. Even if $\epsilon$ is not very small (like, say, $5/100$), $\epsilon/K$ may be very small. For instance, if 10 parameters are to be selected among a set of 100, this gives $K = \binom{100}{10} \approx 1.7 \cdot 10^{13}$. In practice some heuristic scheme will be used to compute only a limited number of estimators $\hat{\theta}_k$, like adding parameters one at a time, choosing at each step the one with the best estimated performance increase (in our example, this requires to compute 1000 estimators instead of $\binom{100}{10}$). Nonetheless, asserting the quality of the resulting choice requires a union bound on the whole set of possible outcomes of the data driven heuristic, and therefore calls for very high confidence levels for each estimate of the mean performance $E[f_{\hat{\theta}_k(X)}(X_{s+1})]$ on the test set.

Our study has several reasons to recommend itself as addressing the question of robust statistics: we prove distribution free bounds, truncation operates mainly on outliers and the lower bounds show that the worst behaviour of the empirical mean is achieved on heavy tailed distributions. Anyhow, our point of view is quite different from the classical setting of robust statistics, as epitomised by Peter Huber [6]. Indeed, our framework is not perturbative, — we do not assume the sample to be drawn from a mixture of known and unknown distributions —, and is not asymptotic either, since it is based on finite sample exponential inequalities for suitable auxiliary variables. Moreover Huber’s approach is a minimax study of the variance of estimators, whereas we analyze the minimax properties of their deviations. From a more practical point of view, the fact that the empirical mean is unstable is well known, and any statistical package provides tools to deal with outliers. It is interesting though that it shows in the equations, even when no sample contamination is assumed, by simply considering a minimax setting on a broad set of distributions including heavy tailed ones, and looking at the deviations of estimators, rather than focussing on their variance.

1. SOME TRUNCATED MEAN ESTIMATE

Let $(Y_i)_{i=1}^n$ be an i.i.d. sample drawn from some unknown probability distribution $P$ on the real line $\mathbb{R}$ equipped with the Borel $\sigma$-algebra $\mathcal{B}$. Let $Y$ be independent from $(Y_i)_{i=1}^n$ with the same marginal distribution $P$. Let $m$ be the mean of $Y$ and let $\nu$ be its variance:

$$E(Y) = m \quad \text{and} \quad E[(Y - m)^2] = \nu.$$ 

Starting from some initial guess $\theta_0$ about the value of the mean, prior to any
Some truncated mean estimate

Observation, let us consider the thresholded estimator

\[ \hat{\theta}_\alpha(\theta_0) = \theta_0 + \frac{1}{n\alpha} \sum_{i=1}^{n} T[\alpha(Y_i - \theta_0)], \]  

(1.1)

where the threshold function \( T \) is defined as

\[ T(x) = \frac{1}{2} \log \left( \frac{1 + x + \frac{x^2}{2}}{1 - x + \frac{x^2}{2}} \right). \]

Proposition 1.1 Assume that \( v \leq v_0 \) and \( |\theta_0 - m| \leq \delta_0 \), where \( v_0 \) and \( \delta_0 \) are known prior bounds. With probability at least \( 1 - 2\epsilon \),

\[ |\hat{\theta}_\alpha(\theta_0) - m| \leq \frac{\alpha v_0}{2} + \frac{\log(\epsilon^{-1})}{n\alpha} + \frac{\alpha^2 \delta_0}{2} \left( 1 + \alpha \delta_0 \right) \left( \frac{\delta_0^2}{3} + v_0 \right). \]

Choosing \( \alpha = \sqrt{\frac{2\log(\epsilon^{-1})}{nv_0}} \), we get, with probability \( 1 - 2\epsilon \),

\[ |\hat{\theta}_\alpha(\theta_0) - m| \leq \sqrt{\frac{2v_0 \log(\epsilon^{-1})}{n}} + \frac{\log(\epsilon^{-1}) \delta_0}{3nv_0} \left( \delta_0^2 + 3v_0 \right) \left( 1 + \delta_0 \sqrt{\frac{2\log(\epsilon^{-1})}{nv_0}} \right). \]
Choosing $\alpha = \sqrt{\frac{2}{nv_0}}$ independently of $\epsilon$ we get with probability at least $1 - 2\epsilon$,

$$\left| \hat{\theta}_\alpha(\theta_0) - m \right| \leq \left[ 1 + \log(\epsilon^{-1}) \right] \sqrt{\frac{v_0}{2n} + \frac{\delta_0}{3nv_0} (\delta_0^2 + 3v_0) \left( 1 + \delta_0 \sqrt{\frac{2}{nv_0}} \right)}.$$ 

**PROPOSITION 1.2** Assume that $v \leq v_0$ and $|m - \theta_0| \leq \delta_0$, where $v_0$ and $\delta_0$ are known prior bounds. With probability at least $1 - 2\epsilon$,

$$\left| \hat{\theta}_\alpha(\theta_0) - m \right| \leq \frac{\alpha(v_0 + \delta_0^2)}{2} + \frac{\log(\epsilon^{-1})}{n\alpha}.$$ 

Choosing $\alpha = \sqrt{\frac{2 \log(\epsilon^{-1})}{n(v_0 + \delta_0^2)}}$, we get

$$\left| \hat{\theta}_\alpha(\theta_0) - m \right| \leq \sqrt{\frac{2(v_0 + \delta_0^2) \log(\epsilon^{-1})}{n}}.$$ 

Let us remark that the estimates proved here are valid for any confidence level $1 - 2\epsilon$. In particular, when $\hat{\theta}_\alpha(\theta_0)$ is independent of $\epsilon$, it has a subexponential tail distribution, even in the case when $\mathbb{P}$ has not. The proofs are gathered in the last section of the paper.

2. **Iterated Mean Estimates**

The width of the confidence intervals proved in the previous section depends heavily on the value of the prior bound $\delta_0$. On the other hand, they lend themselves naturally to an iterated scheme. Here we will iterate Proposition 1.2 (page 5), where the dependence of the bound on $\delta_0$ is the best.

**PROPOSITION 2.1** Let us assume that $v \leq v_0$ and $|m - \theta_0| \leq \delta_0$, where $v_0$, $\theta_0$ and $\delta_0$ are known prior to observing the sample. Let $U_i$, $i = 2, \ldots, k$ be uniform real random variables in the interval $(-1, 1)$, independent of each other and of everything else. Let us define

$$\delta_1 = \sqrt{\frac{2(v_0 + \delta_0^2) \log(\epsilon_1^{-1})}{n}},$$
$$\alpha_1 = \sqrt{\frac{2 \log(\epsilon_1^{-1})}{n(v_0 + \delta_0^2)}},$$
$$\tilde{\theta}_1 = \hat{\theta}_{\alpha_1}(\theta_0).$$
Iterated Mean Estimates

\[ \gamma_i = \log(1 + x_i^{-1}) , \]
\[ \delta_i = \sqrt{\frac{2[v_0 + (1 + x_i)^2\delta_{i-1}^2][\log(\epsilon_i^{-1}) + \gamma_i]}{n}} , \]
\[ \alpha_i = \sqrt{\frac{2[\log(\epsilon_i^{-1}) + \gamma_i]}{n[v_0 + (1 + x_i)^2\delta_{i-1}^2]}} , \]
\[ \tilde{\theta}_i = \tilde{\theta}_{\alpha_i}(\tilde{\theta}_{i-1} + x_i\delta_{i-1}U_i) , \]

With probability at least \( 1 - 2\sum_{i=1}^{k} \epsilon_i \),

\[ |m - \tilde{\theta}_k| \leq \delta_k . \]

Let us see how it behaves, choosing \( x_i = 1/10 \) and \( \epsilon_1 = \cdots = \epsilon_{k-1} = (\epsilon - \epsilon_k)/(k-1) = \epsilon/10 \). The following two plots of \( \delta_k \) against \( \epsilon \) show that this iterated estimate permit very large values of the prior bound \( \delta_0 \), without any substantial loss of accuracy (for a suitable number of iterations).

\( n = 1000, v_0 = 1, \delta_0 = 5 \), starting from 0.5, confidence level \( = 1 - 2\epsilon \)

\[ \epsilon, \text{ starting from 0.5, confidence level } = 1 - 2\epsilon \]
3. WITH NO PRIOR KNOWLEDGE OF THE MEAN

In the case when the prior bound $\delta_0$ is not available, we can modify the iterated scheme of the previous section, using the empirical mean estimator as a first step.

**Proposition 3.1** Let us assume that $v \leq v_0$, where $v_0$ is a known prior bound. Let $U_i, i = 2, \ldots, k$ be uniform real random variables in the interval $(-1, +1)$, independent of each other and of everything else. Let us define

$$
\delta_1 = \sqrt{\frac{v_0}{2n\epsilon_1}},
$$

$$
\tilde{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i,
$$

$$
\vdots
$$

$$
\gamma_i = \log(1 + x_i^{-1}),
$$

$$
\delta_i = \sqrt{\frac{2[v_0 + (1 + x_i)^2\delta_{i-1}^2][\log(\epsilon_i^{-1}) + \gamma_i]}{n}},
$$

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With no prior knowledge of the mean

\[ \alpha_i = \sqrt{\frac{2[\log(\epsilon_i^{-1}) + \gamma_i]}{n[v_0 + (1 + x_i)^2\delta_i^2]}}, \]

\[ \tilde{\theta}_i = \tilde{\theta}_{\alpha_{i-1}}(\tilde{\theta}_{i-1} + x_i\delta_{i-1}U_i), \]

With probability at least \( 1 - 2\sum_{i=1}^{k} \epsilon_i \),

\[ |m - \tilde{\theta}_k| \leq \delta_k. \]

So in this iterated scheme, we start from the empirical mean estimator and improve it gradually. We will show later that the confidence interval used here for the empirical mean is close to optimal in the worst case. What the next plot shows therefore, is that the iterated estimate brings a huge improvement for high confidence levels, allowing to stay close to the deviations of the empirical mean of a Gaussian distribution for confidence levels virtually as high as wished: this iterative truncation scheme behaves almost as the empirical mean estimate of a Gaussian distribution would behave, for any distribution with a known finite variance, and beats the empirical mean in the worst case for confidence levels starting from around 94% for a sample of size 1000.

\( n = 1000, v_0 = 1, \) starting from the empirical mean

\( \epsilon, \) starting from 0.5, confidence level = \( 1 - 2\epsilon \)
4. Last Step Improvement

In this section, we introduce a more elaborate estimate to perform the last step of the iteration. The result of the previous steps will be described as $\tilde{\theta}_1$, assumed to be some mean estimator satisfying with probability at least $1 - 2\epsilon_1$

$$|m - \tilde{\theta}_1| \leq \delta_1.$$  \hfill (4.1)

Let us consider, for any $\theta_0 \in \mathbb{R}$, the Gaussian distribution on the real line with variance $(n\beta\alpha^2)^{-1}$ and mean $\theta_0$, where $\alpha$ and $\beta$ are positive parameters to be chosen later:

$$\rho_{\theta_0}(d\theta) = \sqrt{\frac{n\beta\alpha^2}{2\pi}} \exp\left[-\frac{n\beta\alpha^2}{2}(\theta - \theta_0)^2\right] d\theta.$$

This will serve to define some truncated mean estimate

$$M_\alpha(\theta_0) = \frac{1}{n\alpha} \sum_{i=1}^{n} \log \left\{ \int \rho_{\theta_0}(d\theta) \left[ 1 + \alpha(Y_i - \theta_0) + \frac{\alpha^2}{2}(Y_i - \theta_0)^2 \right] \right\}$$

$$= \frac{1}{n\alpha} \sum_{i=1}^{n} \log \left\{ 1 + \alpha(Y_i - \theta_0) + \frac{\alpha^2}{2}(Y_i - \theta_0)^2 + \frac{1}{2n\beta} \right\}. \hfill (4.2)$$

**Proposition 4.1** With probability at least $1 - \epsilon$, for any $\theta_0 \in \mathbb{R}$,

$$-n\alpha M_\alpha(\theta_0) \leq n\alpha (\theta_0 - m) + \frac{\alpha^2}{2} [(\theta_0 - m)^2 + v] + \frac{1}{2\beta} + \frac{n\beta\alpha^2}{2}(\theta_0 - m)^2 - \log(\epsilon).$$

Let us insist on the fact that this result holds with probability $1 - 2\epsilon$ uniformly with respect to $\theta_0$, which may therefore be a random variable — such as $\tilde{\theta}_1$ — if required.

**Proposition 4.2** For any $\theta_0 \in \mathbb{R}$, with probability at least $1 - \epsilon$,

$$n\alpha M_\alpha(\theta_0) \leq -n\alpha (\theta_0 - m) + \frac{n\alpha^2}{2} [(\theta_0 - m)^2 + v] + \frac{1}{2\beta} - \log(\epsilon).$$

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**Proposition 4.3** Let $v \leq v_0$, where $v_0$ is some known prior bound. Let $\tilde{\theta}_1$ be some estimator satisfying equation (4.1) with probability at least $1 - 2\epsilon_1$. Let us consider the estimator

$$\hat{\theta}_\alpha = \inf \left\{ \theta \geq \tilde{\theta}_1 - \delta_1 : M_\alpha(\theta) \leq 0 \right\}. \quad (4.3)$$

Let us define the ancillary function

$$\varphi(x) = \begin{cases} \frac{2x}{1 + \sqrt{1 - 4x}} \leq \frac{x}{1 - 2x}, & x \leq 1/4, \\ +\infty, & \text{otherwise}. \end{cases} \quad (4.4)$$

For any real positive constants $\epsilon_2$ and $\alpha$ such that

$$4n\alpha\delta_1 \leq \left[ na^2v + \beta^{-1} + 2\log(e^{-1}) \right]$$

$$\times \varphi \left( \frac{(1 + \beta) \left[ na^2v + \beta^{-1} + 2\log(e^{-1}) \right]}{4n} \right)^{-1},$$

which is the case at least when

$$\epsilon_2 \geq \exp \left\{ -n \left[ \frac{1}{1 + \beta} - \alpha\delta_1 - \frac{(na^2v + \beta^{-1})}{2n} \right] \right\}, \quad (4.5)$$

with probability at least $1 - 2\epsilon_1 - 2\epsilon_2$.

$$|m - \hat{\theta}_\alpha| \leq \frac{2}{1 + \beta} \varphi \left( \frac{(1 + \beta) \left[ \beta^{-1} + 2\log(e^{-1}) \right]}{2n} \right)^{-1},$$

Considering $\alpha = \left( \frac{\beta^{-1} - 2\log(e^{-1})}{nv_0} \right)^{1/2}$, we deduce that as soon as

$$2\delta_1 \leq \sqrt{\left[ \beta^{-1} + 2\log(e^{-1}) \right] v_0} \varphi \left( \frac{(1 + \beta) \left[ \beta^{-1} + 2\log(e^{-1}) \right]}{2n} \right)^{-1}, \quad (4.6)$$

which is the case at least as soon as

$$\epsilon_2 \geq \exp \left( \frac{1}{2\beta} - \frac{nv_0}{2(1 + \beta)^2\delta_1^2 + 8(1 + \beta)v_0} \right),$$

with probability at least $1 - 2\epsilon_1 - 2\epsilon_2$.

$$|m - \hat{\theta}_\alpha| \leq \frac{2}{1 + \beta} \left( \frac{nv_0}{\beta^{-1} - 2\log(e^{-1})} \right)^{1/2} \varphi \left( \frac{(1 + \beta) \left[ \beta^{-1} - 2\log(e^{-1}) \right]}{2n} \right).$$
In the following plot, we took the same parameters as in the previous section, and substituted only the last step. It shows some improvement, especially for moderate confidence levels, but requires more involved computations.

\[ n = 1000, v_0 = 1, \text{ first step = empirical mean, improved last step} \]

(\[ \epsilon, \text{ starting from } 0.5, \text{ confidence level } = 1 - 2\epsilon \])

This is what we obtain when we decrease \( n \) to 300.

\[ n = 300, v_0 = 1, \text{ first step = empirical mean, improved last step} \]

\[ \epsilon, \text{ starting from } 0.5, \text{ confidence level } = 1 - 2\epsilon \]
When the sample size is thus decreased, the last step improvement works up to \( \epsilon \simeq 10^{-8} \), after which the iterated estimate without last step improvement takes the lead, as shown on the previous plot.

5. MEAN ESTIMATE FROM A KURTOSIS PRIOR BOUND

Situations where the variance is unknown are likely to happen. It is possible to deal with them while making assumptions on the kurtosis.

More precisely, let us introduce some uniform kurtosis coefficient, that we define as

\[
\kappa = \sup_{\theta \in \mathbb{R}} \frac{E[(Y - \theta)^4]}{E[(Y - \theta)^2]^2}.
\]

Its relation to the classical centered kurtosis \( \kappa = \frac{E[(Y - m)^4]}{E[(Y - m)^2]^2} \) is given by the following lemma.

**Lemma 5.1** The two kurtosis coefficients defined above satisfy the inequalities

\[
\kappa \leq c \leq \frac{1}{9} \left( \kappa^{1/2} + 2 \left( \kappa + 3 \right)^{1/2} \right)^2 \leq \kappa + 2.
\]

On the other hand, if \( \kappa_P \) and \( c_P \) are the kurtosis and uniform kurtosis of the probability measure \( P \)

\[
\sup_P c_P - \kappa_P = 2,
\]

where the supremum is taken over all probability measures on the real line, proving that the previous bound is tight in the worst case.

Anyhow, in the favourable case when the skewness is null, meaning that \( E[(Y - m)^3] = 0 \), the two coefficients are equal whenever \( \kappa \geq 3 \), and more precisely

\[
c = \begin{cases} 
\kappa + \frac{(3 - \kappa)^2}{5 - \kappa}, & 1 \leq \kappa \leq 3, \\
\kappa, & \kappa \geq 3.
\end{cases}
\]

Let us consider for any \( \theta_0 \in \mathbb{R} \) and \( \delta \in (0, 1) \) the estimator of the mean \( \hat{\theta}_{\alpha}(\theta_0) \) already considered in previous sections and defined by equation (1.1, page 4). Let us also consider the increasing mapping \( \alpha \in \mathbb{R}_+ \mapsto Q_{\theta, \delta}(\alpha) \) defined as

\[
Q_{\theta, \delta}(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \log \left\{ 1 + \alpha(Y_i - \theta)^2 + \delta + \frac{1}{2} \left( \alpha(Y_i - \theta)^2 - \delta \right)^2 \right\}.
\]
We will use the ancillary function \( h(a, y) = \frac{4y}{(1 + y)\left\{1 + \sqrt{1 - \frac{4ay^2}{(1 + y)^2}}\right\}} \).

The next proposition is concerned with random confidence intervals, whose lengths are defined with the help of some estimator of the variance. More precisely, we are going to iterate a process where we successively estimate \( v + (m - \theta)^2 \) and \( m \).

**Proposition 5.2** Let us choose positive real constants \( x_i \), and confidence levels \( \epsilon_i \), \( i = 1, \ldots, 2k \). Let \( U_i \), \( i = 2, \ldots, 2k \) be uniform real random variables in the interval \((-1, +1)\), independent of each other and of everything else. Let us start with some prior guess \( \hat{\theta}_1 \) for \( m \) and let us define by induction the sequence of values

\[
\tilde{\theta}_1 = \theta_1,
\]

\[
\delta_1 = \sqrt{\frac{2\log(\epsilon_1)}{(c - 1)n}},
\]

\[
\zeta_1 = -\frac{1}{2} \log \left\{1 - h \left[\frac{c}{c - 1}, (c - 1)\delta_1\right]\right\},
\]

\[
\tilde{\gamma}_1 = \frac{\delta_1 \exp(-\zeta_1)}{Q_{\hat{\theta}_1, \delta_1} \left[-(c - 1)\delta_1^2\right]},
\]

\[
\tilde{\gamma}_2 = \tilde{\gamma}_1 \exp(x_2\zeta_1 U_2),
\]

\[
\gamma_2 = \log(1 + x_2^{-1}),
\]

\[
\alpha_2 = \exp \left[\frac{(1 + x_2)\zeta_1}{2}\right] \sqrt{\frac{2[\log(\epsilon_2^{-1}) + \gamma_2]}{nq_2}},
\]

\[
\zeta_2 = \exp \left[\frac{(1 + x_2)\zeta_1}{2}\right] \sqrt{\frac{2q_2[\log(\epsilon_2^{-1}) + \gamma_2]}{n}},
\]

\[
\tilde{\theta}_2 = \tilde{\theta}_{\alpha_2}(\theta_1),
\]

\[
\vdots
\]

\[
\gamma_{2i-1} = \gamma_{2i-2} + \log(1 + x_{2i-1}^{-1}),
\]

\[
\delta_{2i-1} = \sqrt{\frac{2[\log(\epsilon_{2i-1}^{-1}) + \gamma_{2i-1}]}{(c - 1)n}},
\]

\[
\tilde{\theta}_{2i-1} = \tilde{\theta}_{2i-2} + \zeta_{2i-2} x_{2i-1} U_{2i-1},
\]

\[
\zeta_{2i-1} = -\frac{1}{2} \log \left\{1 - h \left[\frac{c}{c - 1}, (c - 1)\delta_{2i-1}\right]\right\},
\]

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Let us assume that \( \gamma_{i} \), \( \delta_{2i-1} \) and \( \zeta_{2i-1} \) are non random and known prior to observing the sample.

Let us remark that \( \gamma_{i} = \sum_{j=2}^{i} \log(1 + x_{j}^{-1}) \), \( \delta_{2i-1} \) and \( \zeta_{2i-1} \), \( i = 1 \ldots k \) are non random and known prior to observing the sample.

With probability at least \( 1 - 2 \sum_{i=1}^{2k} \epsilon_{i} \), for any \( i = 1 \ldots k \),

\[
| m - \tilde{\theta}_{2i} | \leq \zeta_{2i}, \quad | \log(\tilde{q}_{2i-1}) - \log [ v + (m - \tilde{\theta}_{2i-1})^{2} ] | \leq \zeta_{2i-1}.
\]

As a consequence, on the same event of probability at least \( 1 - 2 \sum_{i=1}^{2k} \epsilon_{i} \), for any \( i = 1 \ldots k \),

\[
(m - \tilde{\theta}_{2i-1})^{2} \leq (1 + x_{2i-1})^{2} \zeta_{2i-2},
\]

\[
\tilde{q}_{2i-1} \leq \left[ v + (1 + x_{2i-1})^{2} \zeta_{2i-2}^{2} \right] \exp(\zeta_{2i-1}),
\]

\[
\zeta_{2i} \leq \exp \left[ (1 + x_{2i})\zeta_{2i-1} \right] \sqrt{\frac{2 [ v + (1 + x_{2i-1})^{2} \zeta_{2i-2}^{2} ] \log(e_{2i}^{-1}) + \gamma_{2i}}{n}},
\]

\[
\zeta_{2} \leq \exp \left[ (1 + x_{2})\zeta_{1} \right] \sqrt{\frac{2 [ v + (m - \theta_{1})^{2} ] \log(e_{1}^{-1}) + \gamma_{2}}{n}},
\]

\[
\exp(-\zeta_{2i-1})\tilde{q}_{2i-1} - (1 + x_{2i-1})^{2} \zeta_{2i-2}^{2} \leq v \leq \exp(\zeta_{2i-1})\tilde{q}_{2i-1}.
\]

These equations allow to compute by induction a (non observable) deterministic bound for \( \zeta_{2k} \), which is itself a random observable confidence interval half width for the estimate of the mean given by \( \tilde{\theta}_{2k} \). The last equation (better used with \( i = 1 \)}
k) shows that we get as a by-product an estimate of the variance with observable as well as theoretical confidence bounds.

In the sequel, we will give lower and upper bounds for the worst case behaviour of the empirical mean depending on the variance and the kurtosis. Note that here we do better, since we also estimate the variance and assume only a known prior bound on the kurtosis. Obtaining a similar observable confidence interval for the empirical mean would require to estimate the variance under a kurtosis bound, which is not something straightforward, as will also be discussed a little later. In the following plot, we chose a sample of size 2000, a size where things start to behave nicely under these assumptions.

Although the lower and upper deviation bounds shown for the empirical mean estimator do not correspond to observable confidence intervals, we see that for confidence levels higher than \(1 - 2 \cdot 10^{-8}\), the observable confidence interval of our estimator outperforms the deviations of the empirical mean, up to confidence levels as high as \(1 - 2 \cdot 10^{-14}\). We also plotted the upper estimate for the standard deviation (assumed to be equal to one). We took \(x_i = 0.5, i < 2k, x_{2k} = 0.1\) and assumed that \(|m - \theta_1| \leq 10\sqrt{v}\). We kept the kurtosis to 3, the kurtosis of the Gaussian distribution.

This is now what happens when we increase \(c\) to 6 (taking this time \(x_i \equiv 0.1\)).
When we increase the sample size to $n = 5000$, this makes things easier:

$n = 5000, v = 1$ (unknown), $c = 3$ (known)
This is the influence of the kurtosis on the bounds for a sample of size \( n = 5000 \):

\[ n = 5000, \; v = 1 \; \text{(unknown)}, \; \text{influence of } c \]

6. ADAPTING TO AN UNKNOWN VARIANCE WHEN THE KURTOSIS IS UNKNOWN OR EVEN INFINITE

In this section, we will point out that Lepski’s renowned adaptation method can be put to good use when nothing is known, neither the variance (still assumed to be finite) nor the kurtosis (not even assumed to be finite !). Of course, under so uncertain, (but unfortunately so frequent) circumstances, it is not possible to provide any observable confidence level. Nevertheless, it is still possible to adapt to the variance and to give deviation bounds depending on the unknown variance. Here, a clear distinction should be made between adapting to the variance and estimating the variance: estimating the variance at any predictable rate is impossible in this context where we do not assume any higher moment to be bounded.

The idea of Lepski’s method is powerful and simple: consider a sequence of confidence intervals obtained by assuming a variance bound \( v_0 \) to take a range of possible values and pick up as an estimator the middle of the smallest interval intersecting all the larger ones. For this to be legitimate, we need all the confidence regions for which the variance bound is valid to hold together, which is performed using a union bound.
Let us describe this idea more precisely. Let \( \hat{\theta}(v_0) \) be some estimator of the mean depending on some assumed variance bound \( v_0 \), as the ones described in the beginning of this paper. Let \( \delta(v_0, \epsilon) \in \mathbb{R}_+ \cup \{+\infty\} \) be the corresponding confidence bound: namely let us assume that with probability at least \( 1 - 2\epsilon \),

\[
|m - \hat{\theta}(v_0)| \leq \delta(v_0, \epsilon).
\]

Presumably, except for distributions with bounded support, \( \delta(v_0, 0) = +\infty \).

Let \( \nu \in \mathcal{M}_1^+(\mathbb{R}_+) \) be some coding atomic sub-probability measure on the positive real line, which will serve to take a union bound on a (countable) set of possible values of \( v_0 \).

We can choose for instance for \( \nu \) the following coding distribution: expressing \( v_0 \) by comparison with some reference value

\[
v_0 = V^{2e} \sum_{k=0}^d c_k 2^{-k}, \quad s \in \mathbb{Z}, d \in \mathbb{N}, (c_k)_{k=0}^d \in \{0, 1\}^{d+1}, c_0 = c_d = 1,
\]

we set \( \nu(v_0) = \left[ (|s| + 2)(|s| + 3)(d + 1)(d + 2)2^{d-1}\right]^{-1} \), and otherwise we set \( \nu(v_0) = 0 \). It is easy to see that this defines a subprobability distribution on \( \mathbb{R}_+ \) (supported by dyadic numbers scaled by the factor \( V \)). It is clear that, as far as possible, the reference value \( V \) should be chosen as close as possible to the true variance \( v \).

Let us consider for any \( v_0 \) such that \( \delta(v_0, \epsilon \nu(v_0)) < +\infty \) the confidence interval

\[
I(v_0) = \hat{\theta}(v_0) + \delta[v_0, \epsilon \nu(v_0)] \times (-1, 1).
\]

Let us put \( I(v_0) = \mathbb{R} \) when \( \delta(v_0, \epsilon \nu(v_0)) = +\infty \).

Let us consider the non-decreasing family of closed intervals

\[
J(v_1) = \bigcap \left\{ I(v_0) : v_0 \geq v_1 \right\}, \quad v_1 \in \mathbb{R}_+.
\]

A union bound shows immediately that with probability at least \( 1 - 2\epsilon \), \( m \in J(v) \), implying as a consequence that \( J(v) \neq \emptyset \).

**Proposition 6.1** Since \( v_1 \mapsto J(v_1) \) is a non decreasing family of closed intervals, the intersection

\[
\bigcap \left\{ J(v_1) : v_1 \in \mathbb{R}_+, J(v_1) \neq \emptyset \right\}
\]

is a non empty closed interval, and we can therefore pick up an adaptive estimator \( \hat{\theta} \) belonging to it, choosing for instance the middle of this interval.
With probability at least \(1 - 2\epsilon\), \(m \in J(v)\), which implies that \(J(v) \neq \emptyset\), and therefore that \(\tilde{\theta} \in J(v)\).

Thus with probability at least \(1 - 2\epsilon\)

\[
|m - \tilde{\theta}| \leq |J(v)| \leq 2 \inf_{v_0 > v} \delta(v_0, \epsilon v_0).
\]

If the confidence bound \(\delta(v_0, \epsilon)\) is homogeneous, in the sense that

\[
\delta(v_0, \epsilon) = \delta(1, \epsilon) \sqrt{v_0},
\]

as it is the case in Proposition 3.7 (page 7) and Proposition 4.3 (page 10) when used in conjunction with Proposition 3.1 then with probability at least \(1 - 2\epsilon\),

\[
|m - \tilde{\theta}| \leq 2 \inf_{v_0 > v} \delta(1, \epsilon v_0) \sqrt{v_0}.
\]

Since usually \(\epsilon \mapsto \delta(1, \epsilon)\) is quite flat in the high confidence region, as shown on previous plots, we see that, in the high confidence region we are mostly interested in in this paper, the order of magnitude of the adaptive confidence bound is not much more than twice the value \(\delta(v, \epsilon)\) of the confidence bound we would have obtained for the estimator \(\hat{\theta}(v)\) which we could have used had we known the exact value of the variance beforehand.

### 7. Worst Case Empirical Mean Deviations for a Given Kurtosis Value

In the previous sections, we studied truncation techniques suited to various prior hypotheses on the sample distribution. It is interesting to compare them to the performance of the empirical mean estimator. This section is devoted to upper bounds, whereas the next will study corresponding lower bounds.

When the variance is known and nothing else, it is easy to see, using Chebyshev’s inequality for the second moment that the empirical mean

\[
M = \frac{1}{n} \sum_{i=1}^{n} Y_i
\]

is such that

\[
P\left(\left|\frac{M - m}{\sqrt{\frac{v}{2\epsilon n}}}\right| \geq 2\epsilon \right) \leq 2\epsilon.
\]

The behaviour of the empirical mean for a given kurtosis is not so straightforward. The following bound uses a truncation argument, allowing to study separately the behaviour of large and rare values. It is to our knowledge a new result. We will show later in this paper that its leading term is essentially tight (up to the \((3/2)^{1/4}\) multiplicative constant due to the union bound argument).
Proposition 7.1 For any probability distribution whose kurtosis is not greater than $\kappa$, the empirical mean $M$ is such that with probability at least $1 - 2\epsilon$,

$$\frac{|M - m|}{\sqrt{v}} \leq \frac{2 \log(\frac{3}{2} \epsilon^{-1}) \sqrt{\kappa}}{5n} + \frac{2 \log(\frac{3}{2} \epsilon^{-1})}{n} + \left( \frac{3\kappa}{2\epsilon n^3} \right)^{1/4} \left( 1 + \frac{3^5(n - 1) \log(\frac{3}{2} \epsilon^{-1})^2 \kappa}{2500n^2} + \frac{12 \sqrt{2} \log(\frac{3}{2} \epsilon^{-1})^{3/2} \sqrt{\kappa}}{25n^{3/2}} \right)^{1/4}.$$ 

Let us also stress here the fact that estimating the variance under a kurtosis bound, using the empirical estimator

$$M_2 = \frac{1}{n} \sum_{i=1}^{n} Y_i^2$$

of the moment of order two is likely to be unsuccessful at high confidence levels. Indeed, computing the quadratic mean

$$E \left\{ [M_2 - E(Y^2)]^2 \right\} = \frac{E(Y^4) - E(Y^2)^2}{n} \leq \frac{(c - 1)}{n} E(Y^2)^2,$$

we can only conclude, using Chebyshev’s inequality, that with probability at least $1 - 2\epsilon$

$$E(Y^2) \leq \frac{M_2}{1 - \sqrt{\frac{c-1}{2me}}},$$

a bound which breaks down at level of confidence $\epsilon = \frac{c - 1}{2n}$, and which we do not suspect to be substantially improvable in the worst case. In contrast to this, Proposition 5.2 (page 13) provides a variance estimator at high confidence levels.

8. LOWER BOUNDS

8.1. LOWER BOUND FOR GAUSSIAN DISTRIBUTIONS. This lower bound is well known. We recall it here for the sake of completeness.

The empirical mean cannot be improved in the Gaussian case in the following precise sense.

Proposition 8.1 For any estimator of the mean $\hat{\theta} : \mathbb{R}^n \to \mathbb{R}$, any variance value $v > 0$, and any deviation level $\eta > 0$, there is some Gaussian measure

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\( \mathcal{N}(m, v) \) (with variance \( v \) and mean \( m \)) such that the i.i.d. sample of length \( n \) drawn from this distribution is such that

\[
P(\hat{\theta} \geq m + \eta) \geq P(M \geq m + \eta) \quad \text{or} \quad P(\hat{\theta} \leq m - \eta) \geq P(M \leq m - \eta),
\]

where \( M = \frac{1}{n} \sum_{i=1}^{n} Y_i \) is the empirical mean.

This means that any distribution free symmetric confidence interval based on the (supposedly known) value of the variance has to include the confidence interval for the empirical mean of a Gaussian distribution, whose length is exactly known and equal to the properly scaled quantile of the Gaussian measure.

Let us state this more precisely. With the notations of the previous proposition

\[
P(M \geq m + \eta) = P(M \leq m - \eta) = G \left[ \left( \sqrt{\frac{n \eta}{v} } , +\infty \right) \right] = 1 - F \left( \sqrt{\frac{n \eta}{v} } \right),
\]

where \( G \) is the standard normal measure and \( F \) its distribution function.

The upper bounds proved in this paper can be decomposed into

\[
P(\hat{\theta} \geq m + \eta) \leq \epsilon \quad \text{and} \quad P(\hat{\theta} \leq m - \eta) \leq \epsilon,
\]

although we preferred for simplicity to state them in the slightly weaker form

\[
P(|\hat{\theta} - m| \geq \eta) \leq 2\epsilon.
\]

As the Gaussian shift model made of Gaussian distributions with a given variance and varying means, is included in all the models we consider to state bounds, we necessarily should have according to the previous proposition

\[
\epsilon \geq 1 - F \left( \sqrt{\frac{n \eta}{v} } \right),
\]

which can be also written as

\[
\eta \geq \sqrt{\frac{v}{n} F^{-1}(1 - \epsilon)}.
\]

Therefore some visualisation of the quality of our bounds can be obtained by plotting \( \epsilon \rightarrow \eta \) against \( \epsilon \rightarrow \sqrt{\frac{v}{n} F^{-1}(1 - \epsilon)} \), as we did in the previous sections.
8.2. **Worst performance of the empirical mean for a given variance.** Another way to measure the quality of the bound is to compare it to the empirical mean outside from the Gaussian shift model, where we have seen that the deviations of the empirical mean are minimax at any confidence level. This is done in the following proposition.

**Proposition 8.2** For any value of the variance $v$, any deviation level $\eta > 0$, there is some distribution with variance $v$ and mean 0 such that the i.i.d. sample of size $n$ drawn from it satisfies

$$P(M \geq \eta) = P(M \leq -\eta) \geq v \left( 1 - \frac{v}{n^2 \eta^2} \right)^{n-1}.$$  

Thus, as soon as $\epsilon \leq (2e)^{-1}$, with probability at least $2\epsilon$,

$$|M - m| \geq \sqrt{\frac{v}{2n\epsilon}} \left( 1 - \frac{2\epsilon e}{n} \right)^{\frac{n-1}{2}}.$$  

Let us remark that this bound is pretty tight, as shown in the next plot, since, according to equation (7.1, page 19) with probability at least $1 - 2\epsilon$,

$$|M - m| \leq \sqrt{\frac{v}{2n\epsilon}}.$$  

\[ n = 1000, v_0 = 1 \]

\[ \epsilon, \text{ starting from } 0.5, \text{ confidence level } = 1 - 2\epsilon \]
8.3. **Worst Performance of the Empirical Mean for a Given Kurtosis.**

**Proposition 8.3** For any \( c \geq 1 + 1/n \), and any \( \epsilon \leq (4\epsilon)^{-1} \), there is a probability measure on the real line, with uniform kurtosis equal to \( c \) and unit variance, such that with probability at least \( 2\epsilon \),

\[
|M - m| \geq \left( \frac{c - 1}{4n^3\epsilon} \right)^{1/4} \left( 1 - \frac{4\epsilon\epsilon}{n} \right)^{(n-1)/4}.
\]

Let us plot this lower bound as well as the corresponding upper bound given by Proposition 7.1 (page 20), for a sample of size \( n = 2000 \) and a kurtosis \( c = 6 \). The space between the two curves is of moderate size, showing that we got the order of magnitude right in these bounds.

9. **Proofs**

9.1. **Proof of Proposition 1.1 (Page 4).** Let us start with some bounds for the map \( x \mapsto \log\left(1 + x + \frac{x^2}{2}\right) \).

**Lemma 9.1** The map \( x \mapsto \log\left(1 + x + \frac{x^2}{2}\right) \) satisfies for any \( x \in \mathbb{R} \),

\[
-\frac{x^4}{38} \leq \log\left(1 + x + \frac{x^2}{2}\right) - x + \frac{x^3}{6} \leq \frac{x^4}{6}.
\]
PROOFS

PROOF. Let us consider for some positive real parameter \( a \) the function

\[
f(x) = \log(1 + x + \frac{x^2}{2}) - x + \frac{x^3}{6} - \frac{ax^4}{8}.
\]

We can study its sign through its derivative

\[
f'(x) = \frac{1 + x}{1 + x + \frac{x^2}{2}} - 1 + \frac{x^2}{2} - \frac{ax^3}{2} = \frac{x^2(1 + \frac{x^2}{2})}{2(1 + x + \frac{x^2}{2})} - \frac{ax^3}{2}
\]

\[
= \frac{x^3(1 + \frac{x}{2} - a - ax - \frac{ax^2}{2})}{2(1 + x + \frac{x^2}{2})} = -\frac{x^3[(a - 1) + (a - \frac{1}{2})x + \frac{ax^2}{2}]}{2(1 + x + \frac{x^2}{2})}.
\]

When \((a - \frac{1}{2})^2 - 2a(a - 1) \leq 0\), \(f'(x)\) has the same sign as \(-x\), showing that \(\sup_{\mathbb{R}} f = 0\), since \(f(0) = 0\). This condition can also be written as \(a^2 - a - \frac{1}{4} \geq 0\), and is fulfilled when \(a = \frac{1 + \sqrt{2}}{2}\). Thus

\[
\log(1 + x + \frac{x^2}{2}) - x + \frac{x^3}{6} \leq \frac{(1 + \sqrt{2})x^4}{16} \leq \frac{x^4}{6}, \quad x \in \mathbb{R}.
\]

Let us proceed to the lower bound now. Consider the same computations as above, but with a negative parameter \(a\). In this case, under the same discriminant condition, \(f'(x)\) has the same sign as \(x\), showing that \(\inf_{\mathbb{R}} f = 0\). For the lower bound, we can thus take \(a = \frac{1 - \sqrt{2}}{2}\), proving that

\[
-\frac{x^4}{38} \leq -\frac{(\sqrt{2} - 1)x^4}{16} \leq \log(1 + x + \frac{x^2}{2}) - x + \frac{x^3}{6}, \quad x \in \mathbb{R}.
\]

\(\square\)

We will also need the following property of the truncated exponential function

\[
\frac{1}{2} \leq 1 + x + \frac{x^2}{2} \simeq \exp(x):
\]

\[
-\log\left(1 - x + \frac{x^2}{2}\right) = \log\left(\frac{1 + x + \frac{x^2}{2}}{1 + x + \frac{x^2}{4}}\right) \leq \log\left(1 + x + \frac{x^2}{2}\right), \quad x \in \mathbb{R}, \quad (9.1)
\]

so that

\[
-\log\left(1 - x + \frac{x^2}{2}\right) \overset{\text{def}}{=} T_-(x) \leq T(x) \overset{\text{def}}{=} \log\left(1 + x + \frac{x^2}{2}\right).
\]

Accordingly

\[
\tilde{\theta}_\alpha(\theta_0) \leq \theta_0 + \frac{1}{n\alpha} \sum_{i=1}^{n} T_+ [\alpha (Y_i - \theta_0)].
\]
We can then compute the exponential moment

\[
\mathbb{E}\left\{ \exp\left[ \sum_{i=1}^{n} T_i \left( \alpha(Y_i - \theta) \right) \right] \right\} = \prod_{i=1}^{n} \mathbb{E}\left( 1 + \alpha(Y_i - \theta) + \frac{\alpha^2}{2} (Y_i - \theta)^2 \right) = \exp\left[ n \log \left( 1 + \alpha(m - \theta) + \frac{\alpha^2}{2} [v + (m - \theta)^2] \right) \right].
\]

From the exponential Chebyshev inequality \( \mathbb{P}(X \geq \eta) \leq \exp(-\eta) \), considering \( \epsilon = \exp(-\eta) \), we deduce that with probability at least \( 1 - \epsilon \),

\[
\hat{\theta}_\alpha(\theta_0) \leq m + \frac{\alpha v}{2} + \frac{\log(\epsilon^{-1})}{n\alpha} \leq m + \frac{\alpha v}{2} + \frac{\log(\epsilon^{-1})}{n\alpha}.
\]

Let us now remark that for any \( x \in \mathbb{R} \), any \( y \in \mathbb{R}_+ \), according to Lemma 9.1 (page 23),

\[
\log \left( 1 + x + \frac{x^2}{2} + y \right) = \log \left( 1 + x + \frac{x^2}{2} \right) + \log \left( 1 + \frac{y}{1 + x + \frac{x^2}{2}} \right) \leq x - \frac{x^3}{6} + \frac{x^4}{6} + \frac{y}{1 + \frac{x^2}{4}} \left( 1 - x + \frac{x^2}{2} \right) \leq x - \frac{x^3}{6} + \frac{x^4}{6} + y - xy + \frac{x^2 y}{2}.
\]

Thus

\[
\hat{\theta}_\alpha(\theta_0) \leq m + \frac{\alpha v}{2} + \frac{\log(\epsilon^{-1})}{n\alpha} - \frac{\alpha^2 (m - \theta_0)^3}{6} + \frac{\alpha^3 (m - \theta_0)^4}{6} - \frac{\alpha^2 v}{2} \left( m - \theta_0 - \frac{\alpha(m - \theta_0)^2}{2} \right).
\]

In the same way, considering \( \theta_0 - Y_i \) instead of \( Y_i - \theta_0 \) and using the symmetry of \( T(x) \), we get with probability at least \( 1 - \epsilon \)
\[ \hat{\theta}_\alpha(\theta_0) \geq m - \frac{\alpha v}{2} - \frac{\log(\epsilon^{-1})}{n\alpha} - \frac{\alpha^2(m - \theta_0)^3}{6} - \frac{\alpha^3(m - \theta_0)^4}{6} - \frac{\alpha^2 v}{2} \left(m - \theta_0 + \frac{\alpha(m - \theta_0)^2}{2}\right). \]

Therefore with probability at least 1 − 2\(\epsilon\),

\[ |\hat{\theta}_\alpha(\theta_0) - m| \leq \frac{\alpha v}{2} + \frac{\log(\epsilon^{-1})}{n\alpha} + \frac{\alpha^2|m - \theta_0|^3}{6}(1 + \alpha|m - \theta_0|) + \frac{\alpha^2|m - \theta_0|v}{2} \left(1 + \frac{\alpha|m - \theta_0|}{2}\right) \leq \frac{\alpha v}{2} + \frac{\log(\epsilon^{-1})}{n\alpha} + \frac{\alpha^2|m - \theta_0|}{2} \left(1 + \frac{\alpha|m - \theta_0|}{2}\right) \left(\frac{(m - \theta_0)^2}{3} + v\right). \]

Specifically, when \(\alpha = \sqrt{\frac{2\log(\epsilon^{-1})}{nv_0}}\), with \(v_0 \geq v\),

\[ |\hat{\theta}_\alpha(\theta_0) - m| \leq \sqrt{\frac{2v_0\log(\epsilon^{-1})}{n}} + \frac{\log(\epsilon^{-1})|m - \theta_0|}{3nv_0} \left[(m - \theta_0)^2 + 3v_0\right] \left[1 + |m - \theta_0| \sqrt{\frac{2\log(\epsilon^{-1})}{nv_0}} \right]. \]

If we prefer to keep the estimator \(\hat{\theta}_\alpha(\theta_0)\) independent from the confidence level 1 − \(\epsilon\), we can choose \(\alpha = \sqrt{\frac{2}{nv_0}}\) and obtain for this value and with probability at least 1 − 2\(\epsilon\),

\[ |\hat{\theta}_\alpha(\theta_0) - m| \leq \left[1 + \log(\epsilon^{-1})\right] \sqrt{\frac{v_0}{2n}} + \frac{|m - \theta_0|}{3nv_0} \left[(m - \theta_0)^2 + 3v_0\right] \left[1 + |m - \theta_0| \sqrt{\frac{2}{nv_0}} \right]. \]

9.2. **Proof of Proposition 1.2 (Page 5).** If \(|m - \theta_0|\) is already small, or if you are aiming at an iterative scheme, you can be content with the inequality

\[ |\hat{\theta}_\alpha(\theta_0) - m| \leq \frac{\alpha}{2} \left[v_0 + (m - \theta_0)^2\right] + \frac{\log(\epsilon^{-1})}{n\alpha}, \]

which holds with probability at least 1 − 2\(\epsilon\). This is a consequence of Equation 9.2, page 25 and the coarse inequality \(\log(1 + x) \leq x\).
9.3. PROOF OF PROPOSITION 2.1 (PAGE 5). We are going here to iterate the use of Proposition 1.2 (page 5). Applying it once, to start with, we get that with probability at least $1 - 2\epsilon_1$

$$|\tilde{\theta}_1 - m| \leq \delta_1.$$ 

Let

$$\tilde{\theta}_2 = \begin{cases} 
\tilde{\theta}_1 + \delta_1 x_2 U_2 & \text{when } |\tilde{\theta}_1 - m| \leq \delta_1, \\
 m + \delta_1 x_2 U_2, & \text{otherwise.}
\end{cases}$$

We are going to use some PAC-Bayesian theorem to overcome the fact that the sequence of estimators $\tilde{\theta}_i$ is computed on the same sample.

Let us consider the prior distribution $\pi_1$ defined as the uniform probability measure on the interval $m + (1 + x_2)\delta_1 \times (-1, +1)$. Let $\rho_1$ be the conditional distribution of $\tilde{\theta}_2$ knowing the sample. From the definition of $\tilde{\theta}_2$, we see that for any value of the sample $(Y_i)_{i=1}^n$, the support of $\rho_1$ is included in the support of $\pi_1$, and therefore that $\rho_1$ is absolutely continuous with respect to $\pi_1$, with density

$$\frac{d\rho_1}{d\pi_1} = 1 + x_2^{-1}, \quad \rho_1 \text{ almost surely.}$$

Let us define the family of random variables

$$X(\theta) = \sum_{i=1}^n T_+ [\alpha_2(Y_i - \theta)] - n \log \left(1 + \alpha_2(m - \theta) + \frac{\alpha_2^2}{2} [v + (m - \theta)^2] \right).$$

Integrating with respect to $\rho_1$, and using Fubini’s theorem we get

$$\mathbb{E} \left\{ \int \rho_1(d\theta) \exp \left[ X(\theta) - \log(1 + x_2^{-1}) \right] \right\}$$

$$= \mathbb{E} \left\{ \int \rho_1(d\theta) \mathbb{I} \left( \frac{d\rho_1}{d\pi_1}(\theta) > 0 \right) \exp \left[ X(\theta) - \log \left( \frac{d\rho_1}{d\pi_1}(\theta) \right) \right] \right\}$$

$$= \mathbb{E} \left\{ \int \pi_1(d\theta) \mathbb{I} \left( \frac{d\rho_1}{d\pi_1}(\theta) > 0 \right) \exp [X(\theta)] \right\}$$

$$\leq \mathbb{E} \left\{ \int \pi_1(d\theta) \exp [X(\theta)] \right\} = \int \pi_1(\theta) \mathbb{E} \left\{ \exp [X(\theta)] \right\} = 1.$$ 

We can now use the fact that $\mathbb{P}_{\rho_1}$ is the joint distribution of the sample and of $\tilde{\theta}_2$ and Chebyshev’s exponential inequality, to prove with probability at least $1 - \epsilon_2$ that

$$X(\tilde{\theta}_2) \leq \log(\epsilon_2^{-1}) + \log \left(1 + x_2^{-1} \right).$$

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As \[ \hat{\theta}_{\alpha_2}(\bar{\theta}_2) \leq \bar{\theta}_2 + \frac{X(\bar{\theta}_2)}{n\alpha_2} + \frac{1}{\alpha_2} \log \left( 1 + \alpha_2(m - \bar{\theta}_2) + \frac{\alpha_2^2}{2} [v + (m - \bar{\theta}_2)^2] \right) \]
\[ \leq m + \frac{\alpha_2}{2} [v + (m - \bar{\theta}_2)^2] + \frac{X(\bar{\theta}_2)}{n\alpha_2}, \]
we deduce that with probability at least \( 1 - \epsilon_2 \),
\[ \hat{\theta}_{\alpha_2}(\bar{\theta}_2) - m \leq \frac{\alpha_2}{2} [v + (m - \bar{\theta}_2)^2] + \frac{\log(\epsilon_2^{-1}) + \log(1 + x_2^{-1})}{n\alpha_2} \leq \delta_2. \]

We can prove in the same way that with probability at least \( 1 - \epsilon_2 \),
\[ m - \hat{\theta}_{\alpha_2}(\bar{\theta}_2) \leq \delta_2. \]

We deduce that with probability at least \( 1 - 2\epsilon_2 \),
\[ |m - \hat{\theta}_{\alpha_2}(\bar{\theta}_2)| \leq \delta_2. \]

Moreover, we see from the definition of \( \bar{\theta}_2 \) that with probability at least \( 1 - 2\epsilon_1 \),
\[ \bar{\theta}_2 = \hat{\theta}_{\alpha_2}(\bar{\theta}_2), \]
therefore with probability at least \( 1 - 2(\epsilon_1 + \epsilon_2) \),
\[ |m - \tilde{\theta}_2| \leq \delta_2. \]

The induction carries on in the same way. Assuming that with probability at least \( 1 - 2 \sum_{i=1}^{k-1} \epsilon_i, |m - \tilde{\theta}_{k-1}| \leq \delta_{k-1} \), we deduce that with probability at least \( 1 - 2 \sum_{i=1}^{k} \epsilon_i, |m - \tilde{\theta}_k| \leq \delta_k \).

9.4. PROOF OF PROPOSITION 3.1 (PAGE 7). The proof is the same as the previous one, except for the first step, which is a consequence of the Chebyshev inequality, applied to the second moment of the empirical mean:
\[ P\left( |\tilde{\theta}_1 - m| \geq \delta_1 \right) \leq \frac{E[(\tilde{\theta}_1 - m)^2]}{\delta_1^2} \leq 2\epsilon_1. \]

9.5. PROOF OF PROPOSITION 4.1 (PAGE 9). Jensen’s inequality for convex functions can serve to pull the integration with respect to \( \rho_{\theta_0} \) out of the logarithm. Using moreover Equation (9.1 page 24), we get the following chain of inequalities:
\[-n\alpha M_\alpha(\theta_0) \leq -\int \rho_{\theta_0}(d\theta) \sum_{i=1}^{n} \log \left[ 1 - \alpha (\theta - Y_i) + \frac{\alpha^2}{2} (\theta - Y_i)^2 \right] \]
\[\leq \int \rho_{\theta_0}(d\theta) \sum_{i=1}^{n} \log \left[ 1 + \alpha (\theta - Y_i) + \frac{\alpha^2}{2} (\theta - Y_i)^2 \right].\]

To proceed, let us consider the empirical process

\[W(\theta) = \sum_{i=1}^{n} \log \left[ 1 + \alpha (\theta - Y_i) + \frac{\alpha^2}{2} (\theta - Y_i)^2 \right].\]

It satisfies

\[E \{ \exp [W(\theta)] \} = E \left\{ \prod_{i=1}^{n} \left[ 1 + \alpha (\theta - Y_i) + \frac{\alpha^2}{2} (\theta - Y_i)^2 \right] \right\} \]
\[= \left\{ 1 + \alpha (\theta - m) + \frac{\alpha^2}{2} \left[ (\theta - m)^2 + E[(Y - m)^2] \right] \right\}^{n} \]
\[\leq \left\{ 1 + \alpha (\theta - m) + \frac{\alpha^2}{2} \left[ (\theta - m)^2 + v \right] \right\}^{n}.\]

Thus if we put

\[w(\theta) = n \log \left\{ 1 + \alpha (\theta - m) + \frac{\alpha^2}{2} \left[ (\theta - m)^2 + v \right] \right\}\]

we see that

\[E \{ \exp [W(\theta) - w(\theta)] \} \leq 1,\]

(with equality when \(E[(Y - m)^2] = v\)). We can then follow the usual PAC-Bayesian route, choosing as reference measure \(\rho_m\). This consists in the inequalities

\[E \left\{ \exp \left[ \sup_{\theta_0 \in \mathcal{R}} \int \rho_{\theta_0}(d\theta) \left[ W(\theta) - w(\theta) \right] - \mathcal{K}(\rho_{\theta_0}, \rho_m) \right] \right\} \]
\[\leq E \left\{ \int \rho_m(d\theta) \exp \left[ W(\theta) - w(\theta) \right] \right\} \]
\[= \int \rho_m(d\theta) E \left\{ \exp \left[ W(\theta) - w(\theta) \right] \right\} = 1,\]

where we have used Fubini’s theorem and the convex inequality.
\[ \sup_{\rho \in \mathcal{M}_1^+} \int \rho(d\theta) \left[ W(\theta) - w(\theta) \right] - \mathcal{K}(\rho, \rho_m) = \log \left\{ \int \rho_m(d\theta) \exp \left[ W(\theta) - w(\theta) \right] \right\}. \]

(See [4, page 159] for a proof.)

From the exponential Chebyshev inequality \( P[X \geq \eta] \leq \mathbb{E}[\exp(X - \eta)] \), it follows that with probability at least \( 1 - \epsilon \), for any \( \theta_0 \in \mathbb{R} \),

\[ \int \rho_{\theta_0}(d\theta) W(\theta) \leq \int \rho_{\theta_0}(d\theta) w(\theta) + \mathcal{K}(\rho_{\theta_0}, \rho_m) - \log(\epsilon). \]

We can then remark that \( \mathcal{K}(\rho_{\theta_0}, \rho_m) = \frac{n^2\alpha}{2}\beta (\theta_0 - m)^2 \) and that

\[ \int \rho_{\theta_0}(d\theta) w(\theta) \leq n \int \rho_{\theta_0}(d\theta) \left\{ \alpha(\theta - m) + \frac{\alpha^2}{2}[(\theta - m)^2 + v] \right\} \]

\[ = n\alpha(\theta_0 - m) + \frac{n\alpha^2}{2}[(\theta_0 - m)^2 + v] + \frac{1}{2\beta}, \]

to conclude that with probability at least \( 1 - \epsilon \), for any \( \theta_0 \in \mathbb{R} \),

\[ \int \rho_{\theta_0}(d\theta) W(\theta) \leq n\alpha(\theta_0 - m) + \frac{n\alpha^2}{2}[(\theta_0 - m)^2 + v] + \frac{1}{2\beta} + \frac{n\beta}{2} - \log(\epsilon). \]

As we have already established that \( -n\alpha M_\alpha(\theta_0) \leq \int \rho_{\theta_0}(d\theta) W(\theta) \), this completes the proof of the proposition.

9.6. PROOF OF PROPOSITION 4.2 (PAGE 9). It is straightforward to realize that

\[ \mathbb{E}\left\{ \exp[n\alpha M_\alpha(\theta_0)] \right\} \leq \left\{ 1 - \alpha(\theta_0 - m) + \frac{\alpha^2}{2}(\theta_0 - m)^2 + v + \frac{1}{2n\beta} \right\}^n. \]

The result then follows as in the previous proof from the exponential Chebyshev inequality.

9.7. PROOF OF PROPOSITION 4.3 (PAGE 10). We will need the following elementary lemma.

**Lemma 9.2** For any positive real constants \( a \) and \( c \) such that \( 4ac \leq 1 \),

\[ \{ x \in \mathbb{R} : x > ax^2 + c \} \]
\[
= \left( \frac{2c}{1 + \sqrt{1 - 4ac}} \right) \left( \frac{1 + \sqrt{1 - 4ac}}{2a} \right) \supset \left( \frac{c}{1 - 2ac}, \frac{1 - 2ac}{a} \right).
\]

Using Proposition 4.1 (page 9) and the previous lemma, we see that with probability 
\[1 - \epsilon_2\]
\[
m \leq \tilde{\theta}_1 + \frac{2c}{(1 + \beta)\alpha} \varphi \left( \frac{(1 + \beta) \left[ n\alpha^2 v + \beta^{-1} - 2 \log(\epsilon_2) \right]}{4n} \right)
\]

or \[m \geq \tilde{\theta}_1 + \frac{2c}{(1 + \beta)\alpha} \left( 1 - \frac{(1 + \beta) \left[ n\alpha^2 v + \beta^{-1} - 2 \log(\epsilon_2) \right]}{2n} \right) \varphi \left( \frac{(1 + \beta) \left[ n\alpha^2 v + \beta^{-1} + 2 \log(\epsilon_2^{-1}) \right]}{4n} \right)^{-1} \]

\[\geq \tilde{\theta}_1 + \frac{2c}{(1 + \beta)\alpha} \left( 1 - \frac{(1 + \beta) \left[ n\alpha^2 v + \beta^{-1} - 2 \log(\epsilon_2) \right]}{2n} \right) \varphi \left( \frac{(1 + \beta) \left[ n\alpha^2 v + \beta^{-1} + 2 \log(\epsilon_2^{-1}) \right]}{4n} \right)^{-1} \]

Let us make sure that the second condition cannot be fulfilled when 
\[|\tilde{\theta}_1 - m| \leq \delta_1\]

assuming that \[\epsilon_2 > \exp\left\{ -n \left[ \frac{1}{1 + \beta} - \alpha \delta_1 - \left( \frac{n\alpha^2 v + \beta^{-1}}{2n} \right) \right] \right\},\]

or more accurately that

\[4n\alpha \delta_1 < \left[ n\alpha^2 v + \beta^{-1} + 2 \log(\epsilon_2^{-1}) \right] \times \varphi \left( \frac{(1 + \beta) \left[ n\alpha^2 v + \beta^{-1} + 2 \log(\epsilon_2^{-1}) \right]}{4n} \right)^{-1} \]

In this case, with probability at least \[1 - \epsilon_2\], either \[|\tilde{\theta}_1 - m| > \delta_1\] or

\[m \leq \hat{\theta}_1 + \frac{2c}{(1 + \beta)\alpha} \varphi \left( \frac{(1 + \beta) \left[ n\alpha^2 v + \beta^{-1} - 2 \log(\epsilon_2) \right]}{4n} \right) \]

On the other hand, let us consider

\[\theta_0 = m + \frac{2c}{\alpha} \varphi \left( \frac{n\alpha^2 v + \beta^{-1} - 2 \log(\epsilon_2)}{4n} \right).\]
From Proposition 4.2 (page 9), with probability at least \(1 - \epsilon\), \(M(\theta_0) \leq 0\), and therefore
\[
\hat{\theta}_\alpha \leq \theta_0 \leq m + \frac{2}{(1 + \beta)\alpha} \varphi \left( \frac{(1 + \beta)[n\alpha^2v + \beta^{-1} - 2\log(\epsilon)]}{4n} \right).
\]
Thus with probability at least \(1 - 2\epsilon\), either \(|\tilde{\theta}_1 - m| > \delta_1\) or
\[
|\hat{\theta}_\alpha - m| \leq \frac{2}{(1 + \beta)\alpha} \varphi \left( \frac{(1 + \beta)[n\alpha^2v + \beta^{-1} - 2\log(\epsilon)]}{4n} \right).
\]
This proves the first part of the proposition. The consequences drawn from special choices of \(\alpha\) are obvious, except for the last condition which may require some verification: when \(\alpha = \left(\frac{\beta^{-1} - 2\log(\epsilon)}{nv}\right)^{1/2}\), putting \(\gamma = \beta^{-1} - 2\log(\epsilon)\) condition 4.5 page 10 becomes
\[
\delta_1 \leq \frac{1}{(1 + \beta)\gamma} \left( \frac{nv}{\gamma} \right)^{1/2} \left( 1 - \frac{(1 + \beta)\gamma}{n} \right).\]
This can also be written as
\[
\frac{(1 + \beta)}{n} \gamma + \frac{(1 + \beta)\delta_1}{\sqrt{nv}} \sqrt{\gamma} - 1 \leq 0,
\]
which is a second order inequality in \(\sqrt{\gamma}\). Considering that \(\sqrt{\gamma} \geq 0\), its solution is
\[
\sqrt{\gamma} \leq \frac{2}{(1 + \beta)\delta_1} + \sqrt{\frac{(1 + \beta)^2\delta_1^2}{nv} + \frac{4(1 + \beta)}{n}}.
\]
To simplify formulas, we can remark that this inequality is satisfied when
\[
\sqrt{\gamma} \leq \left( \frac{(1 + \beta)^2\delta_1^2}{nv} + \frac{4(1 + \beta)}{n} \right)^{-1/2},
\]
that is when
\[
\epsilon_2 \geq \exp \left( \frac{1}{2\beta} - \frac{nv}{2(1 + \beta)^2\delta_1^2 + 8(1 + \beta)v} \right).
\]

9.8. PROOF OF LEMMA 5.1 (PAGE 12). Using the fact that the \(L_2\) norm of a sum is less than the sum of the norms, and the definition of the kurtosis, we get
\[ E(Y^4) = E\left\{ (Y - m)^2 + m(2Y - m) \right\}^2 \]

\[ \leq \left\{ E[(Y - m)^4]^{1/2} + |m|E[(2Y - m)^2]^{1/2} \right\}^2 \]

\[ \leq \left\{ \kappa^{1/2}E[(Y - m)^2] + |m|E\left\{ [2(Y - m) + m]^2 \right\}^{1/2} \right\}^2 \]

\[ = \left\{ \kappa^{1/2}(E(Y^2) - m^2) + |m|(4E(Y^2) - 3m^2)^{1/2} \right\}^2. \]

Introducing \( y = \frac{m^2}{E(Y^2)} \), this gives

\[ \frac{E(Y^4)}{E(Y^2)^2} \leq \left( \kappa^{1/2}(1 - y) + y^{1/2}(4 - 3y)^{1/2} \right)^2. \]

Let us consider the function \( f : (0, 1) \mapsto \mathbb{R} \) defined as

\[ f(y) = \kappa^{1/2}(1 - y) + y^{1/2}(4 - 3y)^{1/2}. \]

It reaches its maximum at point \( x \) satisfying \( f'(x) = 0 \), that is

\[ -\kappa^{1/2} + \frac{1}{2}x^{-1/2}(4 - 3x)^{1/2} - \frac{3}{2}x^{1/2}(4 - 3x)^{-1/2} = 0. \]

Therefore \( x \) satisfies \( \kappa x(4 - 3x) = (2 - 3x)^2 \) or \( 3x^2 - 4x + \frac{4}{\kappa + 3} = 0 \). Thus

\[ x = \frac{2}{3} \left( 1 - \sqrt{\frac{\kappa}{\kappa + 3}} \right), \]

and

\[ \sup_{y \in (0, 1)} f(y) = \kappa^{1/2} \left( \frac{1}{3} + \frac{2}{3} \sqrt{\frac{\kappa}{\kappa + 3}} \right) + 2 \sqrt{\frac{1}{\kappa + 3}} = \frac{2}{3} \sqrt{\kappa + 3} + \frac{1}{3} \sqrt{\kappa}. \]

This proves that

\[ \frac{E(Y^4)}{E(Y^2)^2} \leq \frac{1}{9} \left( \sqrt{\kappa} + 2\sqrt{\kappa + 3} \right)^2 = \frac{\kappa}{9} \left( 5 + \frac{12}{\kappa} + 4 \sqrt{1 + \frac{3}{\kappa}} \right) \]

\[ \leq \frac{\kappa}{9} \left( 5 + \frac{12}{\kappa} + 4 + \frac{6}{\kappa} \right) = \kappa + 2. \]

The same is of course true for \( Y - \theta \) for any shift \( \theta \), as a mere change of notations shows, proving the first assertion of the lemma.

Consider now for the lower bound the Bernoulli distribution with parameter \( p \).

In this case

\[ E[(Y - m)^2] = p(1 - p)^2 + (1 - p)p^2 = p(1 - p), \]

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\[
E[(Y - m)^4] = p(1 - p)^4 + (1 - p)p^4 = p(1 - p)(1 - 3p + 3p^2),
\]

thus
\[
\kappa = \frac{1 - 3p + 3p^2}{p(1 - p)} = p^{-1} - 2 + \frac{p}{1 - p}.
\]

Moreover \(\frac{E(Y^4)}{E(Y^2)^2} = p^{-1} \leq c\), and thus
\[
c - \kappa \geq 2 - \frac{p}{1 - p}.
\]

While \(p\) tends to zero, this proves that \(\sup_{P \in \mathcal{M}_1^+(\mathbb{R})} c_P - \kappa_P \geq 2\), and therefore, due to the already proved upper bound, that \(\sup_{P \in \mathcal{M}_1^+(\mathbb{R})} c_P - \kappa_P = 2\).

When the skewness is null, that is when \(E[(Y - m)^3] = 0\), we can write, assuming without loss of generality that \(m = 0\),
\[
E[(Y + \theta)^4] = E(Y^4) + 6\theta^2E(Y^2) + \theta^4
= \kappa E(Y^2)^2 + 6\theta^2E(Y^2) + \theta^4
= \kappa \left\{ E[(Y + \theta)^2] - \theta^2 \right\}^2 + 6\theta^2 \left\{ E[(Y + \theta)^2] - \theta^2 \right\} + \theta^4.
\]

Thus, introducing \(y = \frac{\theta^2}{E[(Y + \theta)^2]}\), we see that
\[
c = \sup_{\theta \in \mathbb{R}} \frac{E[(Y + \theta)^4]}{E[(Y + \theta)^2]^2} = \sup_{y \in (0,1)} \kappa(1 - y)^2 + 6y(1 - y) + y^2
= \sup_{y \in (0,1)} \kappa - 2(\kappa - 3)y + (\kappa - 5)y^2
= \begin{cases} 
\kappa + \frac{(3 - \kappa)^2}{(5 - \kappa)}, & 1 \leq \kappa \leq 3, \\
\kappa, & \kappa \geq 3.
\end{cases}
\]

9.9. **Proof of Proposition 5.2 (Page 13)**. We are going to build a non observable variant of the construction made in the proposition, for which the conclusions of the proposition are always fulfilled because we enforced them.

Remember that the sequences \(\delta_i\), \(\gamma_i\) and \(\zeta_{2i-1}\) take non random values, and let us define
\[
\bar{\gamma}_1 = \theta_1,
\]
\[
\bar{\gamma}_1 = \begin{cases} 
\frac{\delta_1 \exp(-\zeta_1)}{Q_{\bar{\gamma}_1, \delta_1}(-c - 1)^2}, & \text{when } \bar{\gamma}_1 \text{ thus defined satisfies} \\
v + (m - \bar{\gamma}_1)^2, & \text{otherwise},
\end{cases}
\]

\text{when } \bar{\gamma}_1 \text{ thus defined satisfies}
\[
| \log(\bar{\gamma}_1) - \log[v + (m - \bar{\gamma}_1)^2] | \leq \zeta_1.
\]

\text{otherwise},

\text{otherwise},
\( q_2 = q_1 \exp(x_2 \zeta_1 U_2), \)

\( \bar{\alpha}_2 = \sqrt{2 \frac{\log(\epsilon_2^{-1}) + \gamma_2}{n \bar{q}_2}}, \)

\( \bar{\zeta}_2 = \exp \left[ \frac{(1 + x_2)\zeta_1}{2} \right] \sqrt{2 \frac{\log(\epsilon_2^{-1}) + \gamma_2}{n}}, \)

\( \bar{\theta}_2 = \begin{cases} \hat{\theta}_{\bar{\pi}_2}(\bar{\theta}_1), & \text{when } \bar{\theta}_2 \text{ thus defined satisfies } m - \bar{\theta}_2 \leq \bar{\zeta}_2, \\ m, & \text{otherwise,} \end{cases} \)

\( \vdots \)

\( \bar{q}_{2i-1} = \bar{q}_{2i-2} + \bar{z}_{2i-2} x_{2i-1} U_{2i-1}, \)

\( \bar{q}_{2i} = \begin{cases} \delta_{2i-1} \exp(-\zeta_{2i-1}) \frac{Q_{\bar{\theta}_{2i-1}, \delta_{2i-1}}}{-(c-1)\delta_{2i-1}}, & \text{when } \bar{q}_{2i-1} \text{ thus defined satisfies } \\
\left| \log(\bar{q}_{2i-1}) - \log[\nu + (m - \bar{\theta}_{2i-1})^2] \right| \leq \zeta_{2i-1}, \\
\nu + (m - \bar{\theta}_{2i-1})^2, & \text{otherwise,} \end{cases} \)

\( \bar{\bar{\alpha}}_{2i} = \exp \left[ \frac{(1 + x_{2i})\zeta_{2i-1}}{2} \right] \sqrt{2 \frac{\log(\epsilon_2^{-1}) + \gamma_{2i}}{n \bar{q}_{2i}}}, \)

\( \bar{\bar{\zeta}}_{2i} = \exp \left[ \frac{(1 + x_{2i})\zeta_{2i-1}}{2} \right] \sqrt{2 \frac{\log(\epsilon_2^{-1}) + \gamma_{2i}}{n}}, \)

\( \bar{\bar{\theta}}_{2i} = \begin{cases} \hat{\theta}_{\bar{\pi}_2}(\bar{\theta}_{2i-1}), & \text{when } \bar{\theta}_{2i} \text{ thus defined satisfies } m - \bar{\theta}_{2i} \leq \bar{\zeta}_{2i}, \\ m, & \text{otherwise,} \end{cases} \)

\( \vdots \)

By construction, these modified quantities are such that for any \( i = 1, \ldots, k, \)

\( |m - \bar{\theta}_{2i}| \leq \zeta_{2i}, \)

\( \left| \log(\bar{q}_{2i-1}) - \log[\nu + (m - \bar{\theta}_{2i-1})^2] \right| \leq \zeta_{2i-1}. \)

Let us defined “the modified sequence”

\( S_{2j-1} = \{ (\bar{\theta}_{2i-1})_{i=2}^j; \left[ \log(\bar{q}_{2i}) \right]_{i=1}^{j-1} \} = S_{2j-2} \cup \{ \bar{\theta}_{2K-1} \}, \)

\( S_{2j} = \{ (\bar{\theta}_{2i-1})_{i=2}^j; \left[ \log(\bar{q}_{2i}) \right]_{i=1}^j \} = S_{2j-1} \cup \{ \log(\bar{q}_{2j}) \}. \)
The first step of the proof will be to prove by induction on $j$ the following lemma.

**Lemma 9.3** There exists some prior distribution $\pi_j$ on the modified sequence $S_j$ (that is some non random probability measure on $\mathbb{R}^{j-1}$) such that the joint conditional distribution $\rho_j$ of the modified sequence $S_j$ knowing the sample $(\bar{Y}_i)_{i=1}^n$ is such that $\log \left( \frac{d\rho_j}{d\pi_j} \right) \leq \gamma_j$.

**Proof.** Indeed, assuming that this is true for $2j - 2$, we build $\pi_{2j-1}$ and $\pi_{2j}$ from $\pi_{2j-2}$ by deciding that $\pi_{2j-2}$ is the marginal of $\pi_{2j-1}$ on $S_{2j-2}$ and that $\pi_{2j-1}$ is the marginal of $\pi_{2j}$ on $S_{2j-1}$. We complete the definition of $\pi_{2j-1}$ by defining the conditional distribution of $\tilde{\theta}_{2j-1}$ knowing $S_{2j-2}$ under $\pi_{2j-1}$ as the uniform probability distribution on the interval

$$m + (1 + x_{2j-1})\zeta_{2j-2} \times (-1, +1).$$

Similarly we complete the definition of $\pi_{2j}$ by defining the conditional distribution of $\log(\tilde{\theta}_{2j})$ knowing $S_{2j-2} \cup \{\tilde{\theta}_{2j-1}\}$ as the uniform probability distribution on the interval

$$\log \left[ v + (m - \tilde{\theta}_{2j-1})^2 \right] + (1 + x_{2j})\zeta_{2j-1} \times (-1, +1).$$

As the conditional distribution of $\tilde{\theta}_{2j-1}$ and $\log(\tilde{\theta}_{2j})$ knowing $S_{2j-2}$ and the sample $(Y_i)_{i=1}^n$ is the product of the uniform probability measure on the interval

$$\tilde{\theta}_{2j-2} + \zeta_{2j-2} x_{2j-1} \times (-1, +1)$$

and the uniform probability measure on the interval

$$\log(\tilde{\theta}_{2j-1}) + \zeta_{2j-1} x_{2j} \times (-1, +1),$$

it is readily seen that $\frac{d\rho_{2j-1}(\cdot|S_{2j-2})}{d\pi_{2j-1}(\cdot|S_{2j-2})} = 1 + x_{2j-1}^{-1}$ on the support of $\rho_{2j-1}(\cdot|S_{2j-2})$.

Using the induction hypothesis we deduce that

$$\frac{d\rho_{2j-1}}{d\pi_{2j-1}} = \frac{d\rho_{2j-2}}{d\pi_{2j-2}} \times \frac{d\rho_{2j-1}(\cdot|S_{2j-2})}{d\pi_{2j-1}(\cdot|S_{2j-2})} \leq \exp(\gamma_{2j-2})(1 + x_{2j-1}^{-1}) = \exp(\gamma_{2j-1}).$$

We deduce in the same way that

$$\frac{d\rho_{2j}}{d\pi_{2j}} = \frac{d\rho_{2j-1}}{d\pi_{2j-1}} \times \frac{d\rho_{2j}(\cdot|S_{2j-1})}{d\pi_{2j}(\cdot|S_{2j-1})} \leq \exp(\gamma_{2j}).$$

Moreover the first step is easy to prove, taking for $\pi_1$ the uniform probability measure on the interval

$$\log \left[ v + (m - \theta_1)^2 \right] + \zeta_1 x_2 \times (-1, +1).$$

This achieves to prove the lemma by induction. □ Let us now proceed with a second lemma.
Lemma 9.4 With probability at least $1 - 2\epsilon_{2i-1}$,

$$\bar{q}_{2i-1} = \frac{\delta_{2i-1} \exp(-\zeta_{2i-1})}{Q_{\theta_{2i-1}, \delta_{2i-1}}^{-1}[-(c - 1)\delta_{2i-1}^2]}.$$  

Proof. Let us remark that

$$\mathbb{E}\left\{ \exp\left[nQ_{\theta_{2i-1}}(\alpha) \right]\right\} \leq \exp\left\{ n\alpha[v + (\theta - m)^2] - \delta_{2i-1} \right\} + \frac{n}{2}\mathbb{E}\left\{ \alpha(Y_i - \theta)^2 - \alpha[v + (\theta - m)^2] \right\}^2 + \frac{n}{2}\left\{ \alpha[v + (\theta - m)^2] - \delta_{2i-1} \right\}^2 \leq \exp[n\alpha(\theta)],$$

where

$$g(\theta) = \alpha[v + (\theta - m)^2] - \delta_{2i-1} + \frac{(c - 1)\alpha^2}{2} [v + (\theta - m)^2]^2 + \frac{1}{2}\left\{ \alpha[v + (\theta - m)^2] - \delta_{2i-1} \right\}^2.$$  

Integrating the previous exponential moment with respect to $\rho_{2i-1}$, and taking expectations with respect to the distribution of the sample $(Y_i)_{i=1}^n$, we get, for any measurable mapping $S_{2i-1} \mapsto \alpha(S_{2i-1})$ to be chosen afterward,

$$\mathbb{E}\left\{ \int \rho_{2i-1}(dS_{2i-1}) \exp\left[nQ_{\theta_{2i-1}, \delta_{2i-1}}(\alpha) - ng(\bar{q}_{2i-1}) - \gamma_{2i-1} \right]\right\} \leq \mathbb{E}\left\{ \int \rho_{2i-1}(dS_{2i-1}) \mathbb{I} \left( \frac{d\rho_{2i-1}}{d\pi_{2i-1}} > 0 \right) \right\} \exp\left[nQ_{\theta_{2i-1}, \delta_{2i-1}}(\alpha) - ng(\bar{q}_{2i-1}) - \log\left( \frac{d\rho_{2i-1}}{d\pi_{2i-1}} \right) \right] \leq \mathbb{E}\left\{ \int \pi_{2i-1}(dS_{2i-1}) \exp\left[nQ_{\theta_{2i-1}, \delta_{2i-1}}(\alpha) - ng(\bar{q}_{2i-1}) \right]\right\} \leq 1.$$

(Let us remember that $\bar{q}_{2i-1}$ is the last component of $S_{2i-1}$. We made some dependences explicit, but not all of them, and more specifically the dependence of $\alpha$ here has been kept hidden.)

Using Chebyshev’s exponential inequality, we deduce from this moment inequality that, for any measurable mapping $\bar{q}_{2i-1} \mapsto \alpha(\bar{q}_{2i-1})$, (that is for any choice of $\alpha$ which may depend on the value of $\bar{q}_{2i-1}$), with probability at least $1 - \epsilon_{2i-1},$

$$Q_{\theta_{2i-1}, \delta_{2i-1}}[\alpha(\bar{q}_{2i-1})] \leq g(\bar{q}_{2i-1}) + \frac{\gamma_{2i-1} + \log(\epsilon_{2i-1})}{n}.$$
It is useful at this point to realize that the mapping \( \alpha \mapsto Q_{\theta, \delta}(\alpha) \) is increasing for any \( \theta \in \mathbb{R} \) and \( \delta \in (0, 1) \), as its derivative shows

\[
Q'_{\theta, \delta}(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \frac{(1 - \delta)(Y_i - \theta)^2 + \alpha(Y_i - \theta)^4}{1 + \alpha(Y_i - \theta)^2 + \frac{1}{2}[(Y_i - \theta)^2 - \delta]^2} \geq 0.
\]

In order to choose \( \alpha \) (which is allowed to depend on \( \bar{\theta}_{2i-1} \)), let us introduce temporarily \( y = \alpha \left[ v + (\bar{\theta}_{2i-1} - m)^2 \right] \) - \( \delta_{2i-1} \). We can rephrase what we just proved saying that with probability at least \( 1 - \epsilon_{2i-1} \),

\[
\frac{y + \delta_{2i-1}}{v + (\bar{\theta}_{2i-1} - m)^2} \leq Q_{\bar{\theta}_{2i-1}, \delta_{2i-1}}^{-1} \left[ y + \frac{(c - 1)(y + \delta_{2i-1})^2}{2} + \frac{y^2}{2} + \frac{(c - 1)\delta_{2i-1}^2}{2} \right]
= Q_{\bar{\theta}_{2i-1}, \delta_{2i-1}}^{-1} \left[ \frac{cy^2}{2} + [(c - 1)\delta_{2i-1} + 1]y + (c - 1)\delta_{2i-1}^2 \right].
\]

Let us choose \( y \) such that the argument of \( Q_{\bar{\theta}_{2i-1}, \delta_{2i-1}}^{-1} \) in this inequality is equal to \(- (c - 1)\delta_{2i-1}^2 \). This requires that \( y \) should satisfy

\[
\frac{cy^2}{2} + [(c - 1)\delta_{2i-1} + 1]y + 2(c - 1)\delta_{2i-1}^2 = 0.
\]

This has (negative) real roots when

\[
\delta_{2i-1} \leq \frac{1}{2\sqrt{c(c - 1) - (c - 1)}},
\]

and it is elementary to check that the largest of these negative roots is

\[
y = -\delta_{2i-1} h\left[ \frac{c}{c - 1}, (c - 1)\delta_{2i-1} \right] = -[1 - \exp(-2\zeta_{2i-1})] \delta_{2i-1}.
\]

Thus with probability at least \( 1 - \epsilon_{2i-1} \),

\[
\frac{\exp(-2\zeta_{2i-1})\delta_{2i-1}}{v + (\bar{\theta}_{2i-1} - m)^2} \leq Q_{\bar{\theta}_{2i-1}, \delta_{2i-1}}^{-1} \left[ -(c - 1)\delta_{2i-1}^2 \right]. \quad (9.3)
\]

To get the reverse inequality, we may notice, due to Equation \(9.1\) page \(24\) that

\[
-Q_{\theta, \delta}(\alpha) \leq \frac{1}{n} \sum_{i=1}^{n} \log \left\{ 1 - \alpha(Y_i - \theta)^2 + \delta + \frac{1}{2} \left[ \alpha(Y_i - \theta)^2 - \delta \right]^2 \right\}.
\]

Consequently

\[
E\left\{ \exp \left[ -nQ_{\theta, \delta}(\alpha) \right] \right\} \leq \exp \left[ n\bar{y}(\theta) \right],
\]

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This can be rewritten as

\[
\overline{g}(\theta) = \exp\left\{ n\delta - n\alpha [v + (\theta - m)^2] + \frac{(c - 1)\alpha^2}{2} [v + (\theta - m)^2] + \frac{1}{2} \left\{ \alpha [v + (\theta - m)^2] - \delta \right\}^2 \right\}.
\]

Thus, integrating with respect to \( \pi_{2i-1} \) as previously, we deduce that with probability at least 1 - \( \epsilon_{2i-1} \),

\[
-\overline{g}(\theta_{2i-1}) - \frac{\gamma_{2i-1} + \log(\epsilon_{2i-1}^2)}{n} = -\overline{g}(\theta_{2i-1}) - \frac{(c - 1)}{2}\delta_{2i-1}^2 \leq Q_{\pi_{2i-1},\delta_{2i-1}}(\alpha),
\]

where the choice of \( \alpha \) may depend on \( \theta_{2i-1} \). Choosing then \( \alpha = \frac{\delta_{2i-1}}{v + (\theta_{2i-1} - m)^2} \), we get that with probability at least 1 - \( \epsilon_{2i-1} \),

\[
Q_{\pi_{2i-1},\delta_{2i-1}}^{-1} \left[ -(c - 1)\delta_{2i-1}^2 \right] \leq \frac{\delta_{2i-1}}{v + (\theta_{2i-1} - m)^2}.
\] (9.4)

Taking the union bound of inequalities (9.3 page 38) and (9.4 page 39), we see that with probability at least 1 - 2\( \epsilon_{2i-1} \),

\[
\frac{\exp(-2\zeta_{2i-1})\delta_{2i-1}}{v + (\theta_{2i-1} - m)^2} \leq Q_{\pi_{2i-1},\delta_{2i-1}}^{-1} \left[ -(c - 1)\delta_{2i-1}^2 \right] \leq \frac{\delta_{2i-1}}{v + (\theta_{2i-1} - m)^2}.
\]

This can be rewritten as

\[
\left| \log [v + (\theta_{2i-1} - m)^2] - \log \left[ Q_{\pi_{2i-1},\delta_{2i-1}}^{-1} \left[ -(c - 1)\delta_{2i-1}^2 \right] \right] \right| \leq \zeta_{2i-1}.
\]

Therefore, coming back to the definition of \( \overline{q}_{2i-1} \), we see that, with probability at least 1 - 2\( \epsilon_{2i-1} \),

\[
\overline{q}_{2i-1} = \frac{\delta_{2i-1} \exp(-\zeta_{2i-1})}{Q_{\pi_{2i-1},\delta_{2i-1}}^{-1} \left[ -(c - 1)\delta_{2i-1}^2 \right]}.
\]

\[ \square \]

**Lemma 9.5** With probability at least 1 - 2\( \epsilon_{2i} \), \( \overline{q}_{2i} = \hat{\theta}_{\pi_{2i}}(\theta_{2i-1}) \).

**Proof.** We start from the exponential moment inequality

\[
\mathbb{E} \left\{ \exp \left[ n\alpha \hat{\theta}_\alpha(\theta) \right] \right\} \leq \exp \left[ n\alpha g(\theta) \right], \text{ where } g(\theta) = m + \frac{\alpha}{2} [v + (m - \theta)^2].
\]

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Integrating with respect to $\pi_{2i}$, we get, choosing the parameter $\alpha$ to be $\overline{\alpha}_{2i}$, depending on $S_{2i}$ through $\overline{\eta}_{2i}$,

$$
E\left\{ \int \rho_{2i}(dS_{2i}) \exp \left[ n\overline{\alpha}_{2i} \left( \hat{\theta}_{\pi_{2i}}(\overline{\theta}_{2i-1}) - g(\overline{\theta}_{2i-1}) \right) - \gamma_{2i} \right] \right\}
\leq E\left\{ \int \rho_{2i}(dS_{2i}) I \left( \frac{d\rho_{2i}}{d\pi_{2i}} > 0 \right) \right\}
\times \exp \left[ n\overline{\alpha}_{2i} \left( \hat{\theta}_{\pi_{2i}}(\overline{\theta}_{2i-1}) - g(\overline{\theta}_{2i-1}) \right) - \log \left( \frac{d\rho_{2i}}{d\pi_{2i}} \right) \right] \}
\leq E\left\{ \int \pi_{2i}(dS_{2i}) \exp \left[ n\overline{\alpha}_{2i} \left( \hat{\theta}_{\pi_{2i}}(\overline{\theta}_{2i-1}) - g(\overline{\theta}_{2i-1}) \right) \right] \right\}
= \int \pi_{2i}(dS_{2i}) E\left\{ \exp \left[ n\overline{\alpha}_{2i} \left( \hat{\theta}_{\pi_{2i}}(\overline{\theta}_{2i-1}) - g(\overline{\theta}_{2i-1}) \right) \right] \right\} \leq 1.

Thus, according to Chebyshev’s inequality, with probability at least $1 - \epsilon_{2i}$,

$$
\hat{\theta}_{\pi_{2i}}(\overline{\theta}_{2i-1}) \leq m + \frac{\overline{\alpha}_{2i}}{2} \left[ v + (m - \overline{\theta}_{2i-1})^2 \right] + \frac{\gamma_{2i} + \log(\epsilon_{2i}^{-1})}{n\overline{\alpha}_{2i}}
\leq m + \exp \left[ \frac{(1 + x_{2i})\zeta_{2i}}{2} \right] \sqrt{\frac{2\overline{\eta}_{2i} \left( \gamma_{2i} + \log(\epsilon_{2i}^{-1}) \right)}{n}} = m + \zeta_{2i}.
$$

In the same way, considering $\overline{\theta}_{2i-1} - Y_i$ instead of $Y_i - \overline{\theta}_{2i-1}$, we can prove with probability at least $1 - \epsilon_{2i}$ that

$$m \leq \hat{\theta}_{\pi_{2i}}(\overline{\theta}_{2i-1}) + \zeta_{2i}.
$$

A union bound argument then proves that with probability at least $1 - 2\epsilon_{2i}$,

$$|m - \hat{\theta}_{\pi_{2i}}(\overline{\theta}_{2i-1})| \leq \zeta_{2i}.
$$

Coming back to the definition of $\overline{\theta}_{2i}$, we see that it means that with probability at least $1 - 2\epsilon_{2i}$, $\overline{\theta}_{2i} = \hat{\theta}_{\pi_{2i}}(\overline{\theta}_{2i-1})$. $\square$

We can now take a union bound of Lemma 9.4 (page 37) and Lemma 9.5 (page 39), for $i = 1, \ldots, k$, to see that with probability at least $1 - 2 \sum_{i=1}^{2k} \epsilon_i$, the constructions of $\overline{\eta}_i$ and $\overline{\theta}_i$ coincide with the definitions of $\overline{\eta}_i$ and $\overline{\theta}_i$, and therefore that $\overline{\theta}_i = \overline{\theta}_i$ and $\overline{\eta}_i = \overline{\eta}_i$, $i = 1, \ldots, 2k$. Consequently, with probability at least $1 - 2 \sum_{j=1}^{2k} \epsilon_j$, for any $i = 1, \ldots, k$,

$$|m - \overline{\theta}_{2i}| \leq \zeta_{2i},
|\log \left[ v + (m - \overline{\theta}_{2i-1})^2 \right] - \log(\overline{\theta}_{2i-1})| \leq \zeta_{2i-1},
$$

proving Proposition 5.2 (page 13).
Let us consider the function
\[ g(x) = x - \frac{1}{2} \log \left( \frac{1 + x + \frac{x^2}{2}}{1 - x + \frac{x^2}{2}} \right), \quad x \in \mathbb{R}. \]

**Plot of** \( x \mapsto g(x) \) (with a zoom near the origin)

**Lemma 9.6** The function \( g \) is bounded by
\[ |g(x)| \leq \min \left\{ \frac{|x|^3}{5}, \frac{3x^2}{10}, |x| \right\}, \quad x \in \mathbb{R}. \]

The derivative of \( g \) is
\[ g'(x) = \frac{x^2}{4} \left( \frac{1}{1 + x + \frac{x^2}{2}} + \frac{1}{1 - x + \frac{x^2}{2}} \right) = \frac{x^2(2 + x^2)}{4 + x^4} \geq 0, \quad x \in \mathbb{R}, \]
showing that \( g \) has the same sign as \( x \). The fact that \( |g(x)| \leq |x| \) is then clear from the sign of \( \frac{1}{2} \log \left( \frac{1 + x + \frac{x^2}{2}}{1 - x + \frac{x^2}{2}} \right) \), which is the same as the sign of \( x \).

Let us prove now that \( |g(x)| \leq \frac{|x|^3}{5} \). It is clearly enough to prove it for \( x > 0 \), because \( |g| \) is symmetric. Let us consider \( h(x) = g(x) - \frac{x^3}{5} \) and let us compute
\[ h'(x) = \frac{x^2}{4 + x^4} \left( -\frac{2}{5} + x^2 - \frac{3}{5} x^4 \right) = -\frac{x^2(x^2 - 1)(3x^2 - 2)}{5(4 + x^4)}. \]
From the sign of $h'$, we see that $h$ has a unique local maximum on the positive real line at point $x = 1$. Moreover $h(1) = \frac{4}{5} - \frac{1}{2} \log(5) < 0$ (it is close to $-0.005$, as can be checked numerically). Thus $h(x) \leq 0$ for $x \in \mathbb{R}_+$, implying that $|g(x)| \leq \frac{|x|^3}{5}$ on the whole real line, as announced.

To prove $|g(x)| \leq \frac{3x^2}{10}$, we consider $h^2(x) = g(x) - \frac{3x^2}{10}$. A small computation shows that

$$h^2_2(x) = -\frac{x(x-1)(x-2)(3x^2 + 4x + 6)}{5(4 + x^4)}.$$

Thus it has a unique local maximum on the positive real line at point $2$. Moreover $h^2(2) = \frac{4}{5} - \frac{1}{2} \log(5) < 0$, showing that $|g|$ is upper-bounded by $\frac{3x^2}{10}$ on the positive real line, and therefore on the whole real line because it is symmetric.

Let us remark now that with probability at least $1 - \epsilon$,

$$n\alpha(M - m) - \sum_{i=1}^{n} g[\alpha(Y_i - m)] \leq \sum_{i=1}^{n} \log \left[ 1 + \alpha(Y_i - m) + \frac{\alpha^2}{2} (Y_i - m)^2 \right] \leq \frac{n\alpha^2}{2} v + \log(\epsilon^{-1}).$$

The first of these two inequalities comes from the fact that

$$\frac{1}{2} \log \left( \frac{1 + x + \frac{x^2}{2}}{1 - x + \frac{x^2}{2}} \right) \leq \log(1 + x + \frac{x^2}{2}), \quad x \in \mathbb{R}.$$

In the same way, with probability at least $1 - \epsilon$,

$$n\alpha(M - m) - \sum_{i=1}^{n} g[\alpha(Y_i - m)] \geq -\sum_{i=1}^{n} \log \left[ 1 - \alpha(Y_i - m) + \frac{\alpha^2}{2} (Y_i - m)^2 \right] \geq -\frac{n\alpha^2}{2} v - \log(\epsilon^{-1}).$$

Let us now deal with $\sum_{i=1}^{n} g[\alpha(Y_i - m)]$. We need some compact notations to manipulate this. Let $G_i = g[\alpha(Y_i - m)]$ and $G = g[\alpha(W - m)]$. Let us remark that

$$|G| \leq \min \left\{ \frac{\alpha^3}{5} |Y - m|^3, \frac{3\alpha^2}{10} (Y - m)^2, \alpha |Y - m| \right\}.$$
Moreover, using the fact that \( \min\{a, b\} \leq a^{2/3}b^{1/3} \), we see that
\[
|G| \leq \left( \frac{3}{10} \right)^{1/3} \alpha^{4/3} |Y - m|^{4/3}.
\]

With probability at least \( 1 - \epsilon \),
\[
\left| \frac{1}{n} \sum_{i=1}^{n} [G_i - \mathbb{E}(G)] \right| \leq \epsilon^{-1/4} \mathbb{E} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} [G_i - \mathbb{E}(G)] \right)^{4} \right\}^{1/4}
\]
\[
= (\epsilon n^3)^{-1/4} \left\{ 3(n - 1) \left[ \mathbb{E}(G^2) - \mathbb{E}(G)^2 \right]^{2} + \mathbb{E} \left\{ \left[ G - \mathbb{E}(G) \right]^4 \right\} \right\}^{1/4}
\]
\[
= (\epsilon n^3)^{-1/4} \left\{ 3(n - 1) \left[ \mathbb{E}(G^2)^2 - 2\mathbb{E}(G^2)\mathbb{E}(G)^2 + \mathbb{E}(G)^4 \right] 
+ \mathbb{E}(G^4) - 4\mathbb{E}(G^3)\mathbb{E}(G) + 6\mathbb{E}(G^2)^2 \mathbb{E}(G) - 3\mathbb{E}(G)^4 \right\}^{1/4}
\]
\[
= (\epsilon n^3)^{-1/4} \left\{ 3(n - 1)\mathbb{E}(G^2)^2 - 6(n - 2)\mathbb{E}(G^2)\mathbb{E}(G)^2 
+ 3(n - 2)\mathbb{E}(G)^4 + \mathbb{E}(G^4) - 4\mathbb{E}(G^3)\mathbb{E}(G) \right\}^{1/4}
\]
\[
\leq (\epsilon n^3)^{-1/4} \left\{ 3(n - 1)\mathbb{E}(G^2)^2 + \mathbb{E}(G^4) + 4\mathbb{E}(|G|^3)\mathbb{E}(|G|) \right\}^{1/4}
\]
\[
\leq (\epsilon n^3)^{-1/4} \left\{ 3(n - 1) \left[ \frac{9}{100} \alpha^4 \mathbb{E}[(Y - m)^4] \right]^{2} + \alpha^4 \mathbb{E}[(Y - m)^4] 
+ \frac{6}{25} \alpha^7 \mathbb{E}[(Y - m)^4] \mathbb{E}[|Y - m|^3] \right\}^{1/4}.
\]

Let us now use the fact that \( \mathbb{E}[(Y - m)^4] \leq \kappa v^2 \) to deduce that
\[
\mathbb{E}[(Y - m)^3] \leq \sqrt{\mathbb{E}[(Y - m)^2] \mathbb{E}[(Y - m)^4]} \leq \sqrt{\kappa v^3}.
\]

Let us set \( \alpha = \sqrt{\frac{2\log(\epsilon^{-1})}{nv}} \), for this value of \( \alpha \)
\[ \left| \frac{1}{n} \sum_{i=1}^{n} [G_i - \mathbb{E}(G)] \right| \leq \left( \epsilon n^3 \right)^{-1/4} \left\{ \frac{3^5}{10^4} (n-1) \kappa^2 \alpha^8 v^4 + \kappa \alpha^4 v^2 + \frac{6}{25} \kappa^{3/2} \alpha^7 v^{7/2} \right\}^{1/4} \]

\[ \leq \left( \epsilon n^3 \right)^{-1/4} \alpha \sqrt{v} \left\{ \kappa + \frac{3^5 (n-1) \log(\epsilon^{-1})^2}{2500 n^2} \kappa^2 + \frac{12 \sqrt{2} \log(\epsilon^{-1})^{3/2}}{25 n^{3/2}} \kappa^{3/2} \right\}^{1/4} \]

Let us remark also that

\[ |\mathbb{E}(G)| \leq \frac{\alpha^3}{5} \mathbb{E}(|Y - m|^3) \leq \frac{2 \alpha \log(\epsilon^{-1}) \sqrt{\kappa v}}{5n}. \]

Putting all this together, we see that with probability at least \( 1 - 3\epsilon \),

\[ \frac{|M - m|}{\sqrt{v}} \leq \sqrt{\frac{2 \log(\epsilon^{-1})}{n}} + \frac{2 \log(\epsilon^{-1}) \sqrt{\kappa}}{5n} \]

\[ + \left( \frac{\kappa}{\epsilon n^3} \right)^{1/4} \left( 1 + \frac{3^5 (n-1) \log(\epsilon^{-1})^2 \kappa}{2500 n^2} \kappa^2 + \frac{12 \sqrt{2} \log(\epsilon^{-1})^{3/2} \sqrt{\kappa}}{25 n^{3/2}} \right)^{1/4}. \]

The result stated in Proposition 7.1 (page 20) is then obtained by replacing \( \epsilon \) with \( \frac{2}{3} \epsilon \), to get an event with confidence level \( 1 - 2\epsilon \) as elsewhere in this paper.

Let us remark that the following proposition, based on the Chebyshev inequality applied directly to the fourth moment of the empirical mean does not provide the right speed when \( \epsilon \) is small and \( n \) large.

**Proposition 9.7** For any probability distribution whose kurtosis is not greater than \( \kappa \), the empirical mean \( M \) is such that with probability at least \( 1 - 2\epsilon \),

\[ |M - m| \leq \left( \frac{3(n-1) + \kappa}{2n \epsilon} \right) \frac{1}{2} \left( \frac{v}{n} \right) \]

**Proof.** Let us assume to simplify notations and without loss of generality that \( \mathbb{E}(Y) = 0 \).

\[ \mathbb{E}(M^4) = \frac{1}{n^4} \sum_{i=1}^{n} \mathbb{E}(Y_i^4) + \frac{1}{n^4} \sum_{i<j}^{n} 6 \mathbb{E}(Y_i^2) \mathbb{E}(Y_j^2) = \frac{\mathbb{E}(Y^4)}{n^3} + \frac{3(n-1) \mathbb{E}(Y^2)^2}{n^3}. \]

It implies that

\[ \mathbb{P}\left( |M - m| \geq \eta \right) \leq \frac{\mathbb{E}(M^4)}{\eta^4} \leq \frac{3(n-1) + \kappa}{n^3 \eta^4} \]

and the result is proved by considering \( 2\epsilon = \frac{3(n-1) + \kappa}{n^3 \eta^4} \).
9.11. **Proof of Proposition 8.1 (Page 20)**. Let us consider the distributions $P_1$ and $P_2$ of the sample $(Y_i)_{i=1}^n$ obtained when the marginal distributions are respectively the Gaussian measure with variance $v$ and mean $m_1 = -\eta$ and the Gaussian measure with variance $v$ and mean $m_2 = \eta$. We see that, whatever the estimator $\hat{\theta}$,

$$P_1(\hat{\theta} \geq m_1 + \eta) + P_2(\hat{\theta} \leq m_2 - \eta) = P_1(\hat{\theta} \geq 0) + P_2(\hat{\theta} \leq 0) \geq (P_1 \wedge P_2)(\hat{\theta} \geq 0) + (P_1 \wedge P_2)(\hat{\theta} \leq 0) = |P_1 \wedge P_2|,$$

where $P_1 \wedge P_2$ is the measure whose density with respect to the Lebesgue measure (or equivalently with respect to any dominating measure, such as $P_1 + P_2$) is the minimum of the densities of $P_1$ and $P_2$ and whose total variation is $|P_1 \wedge P_2|$.

Now, using the fact that the empirical mean is a sufficient statistics of the Gaussian shift model, it is easy to realize that

$$|P_1 \wedge P_2| = P_1(M \geq m_1 + \eta) + P_2(M \leq m_2 - \eta),$$

which obviously proves the proposition.

9.12. **Proof of Proposition 8.2 (Page 22)**. Let us consider the distribution with support \{-n\eta, 0, n\eta\} defined by

$$P\left(\{n\eta\}\right) = P\left(\{-n\eta\}\right) = \left[1 - P(\{0\})\right]/2 = \frac{v}{2n^2\eta^2}.$$ 

It satisfies $E(Y) = 0$, $E(Y^2) = v$ and

$$P(M \geq \eta) = P(M \leq -\eta) \geq P(M = \eta) = \frac{v}{2n\eta^2} \left(1 - \frac{v}{n^2\eta^2}\right)^{n-1}.$$

9.13. **Proof of Proposition 8.3 (Page 23)**. Let us consider for $Y$ the following distribution, with support \{-n\eta, -\xi, \xi, n\eta\}, where $\xi$ and $\eta$ are two positive real parameters, to be adjusted to obtain the desired variance and kurtosis.

$$P(Y = -n\eta) = P(Y = n\eta) = q,$$
$$P(Y = -\xi) = P(Y = \xi) = \frac{1}{2} - q.$$

In this case

$$m = 0,$$

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Let us choose $\xi$ such that $v = 1$. This is done by putting
\[
\xi^2 = \frac{1 - 2qn^2\eta^2}{1 - 2q}.
\]

The kurtosis of the distribution defined by $q$ and $\eta$, the two remaining free parameters once $\xi$ has been set as explained, is equal to
\[
\kappa = \mathbb{E}(Y^4) = \frac{(1 - 2qn^2\eta^2)^2}{1 - 2q} + 2qn^4\eta^4.
\]

It is easily seen that
\[
P(M \geq \eta) = P(M \leq -\eta) \geq nq\left(1 - 4\frac{e\epsilon}{n}\right) \frac{(1 - 2q)^n - 1}{2} = \epsilon.
\]

Indeed,
\[
P(M \geq \eta) \geq \sum_{i=1}^{n} P\left(Y_i = n\eta; Y_j \in \{-\xi, +\xi\}, j \neq i; \sum_{j,j \neq i} Y_j \geq 0\right)
\]
\[
= nP(Y_1 = n\eta)P\left(Y_j \in \{-\xi, +\xi\}, j = 2, \ldots, n; \sum_{j=2}^{n} Y_j \geq 0\right)
\]
\[
\geq \frac{nq}{2} P\left(Y_j \in \{-\xi, +\xi\}, j = 2, \ldots, n\right) = \frac{nq}{2}(1 - 2q)^n - 1.
\]

Starting from $\epsilon \leq (4e)^{-1}$, and $c \geq 1 + 1/n$, we can define a probability distribution by choosing
\[
q = \frac{2\epsilon}{n} \left(1 - \frac{4\epsilon e}{n}\right)^{-\frac{(n-1)}{2}} \leq \frac{2\epsilon e}{n} \leq \frac{1}{2n},
\]
\[
\eta = \left(\frac{c - 1}{2qn^4}\right)^{1/4} = \left(\frac{c - 1}{4\epsilon n^3}\right)^{1/4} \left(1 - \frac{4\epsilon e}{n}\right)^{(n-1)/4} \geq \frac{1}{n},
\]
whose kurtosis $\kappa$ will not be greater than $c$, since in this case
\[
\kappa = \frac{(1 - 2qn^2\eta^2)^2}{1 - 2q} + 2qn^4\eta^4 \leq 1 + 2qn^4\eta^4 \leq c,
\]
and for which
\[
P\left(|M - m| \geq \eta\right) \geq nq(1 - 2q)^n - 1 \geq nq \left(1 - \frac{4\epsilon e}{n}\right)^{(n-1)} = 2\epsilon.
\]
10. **Generalizations**

10.1. **Non identically distributed independent random variables.**

The assumption that the sample is identically distributed can be dropped. Indeed, assuming only that the random variables \((Y_i)_{i=1}^n\) are independent, meaning that their joint distribution is of the product form \(\otimes_{i=1}^n P_i\), we can still write, for \(W_i = \pm \alpha (Y_i - \theta)\) or \(W_i = \pm [\alpha (Y_i - \theta) - \delta]\),

\[
E\left\{ \exp \left[ \sum_{i=1}^n \log \left( 1 + W_i + \frac{W_i^2}{2} \right) \right] \right\} \\
= \exp \left\{ \sum_{i=1}^n \log \left[ 1 + E(W_i) + \frac{E(W_i^2)}{2} \right] \right\} \\
\leq \exp \left\{ n \log \left[ 1 + \frac{1}{n} \sum_{i=1}^n E(W_i) + \frac{1}{2n} \sum_{i=1}^n E(W_i^2) \right] \right\}.
\]

Starting from these exponential inequalities, we can reach the same conclusions as in the i.i.d. case, as long as we set

\[
m = \frac{1}{n} \sum_{i=1}^n E(Y_i),
\]

and

\[
v = \frac{1}{n} \sum_{i=1}^n E[(Y_i - m)^2].
\]

Thus here, the role that is played by the marginal sample distribution in the i.i.d. case is played by the mean marginal sample distribution \(\frac{1}{n} \sum_{i=1}^n P_i\). As moreover, the empirical mean \(M\) still satisfies

\[
E[(M - m)^2] = \frac{v}{n} - \frac{1}{n^2} \sum_{i=1}^n [E(Y_i) - m]^2 \leq \frac{v}{n},
\]

we see that Propositions 1.1 (page 4), 1.2 (page 5), 2.1 (page 5), 3.1 (page 7), 4.3 (page 10), and 5.2 (page 13) remain true, the proofs being unchanged, except for the starting inequalities mentioned above, and the kurtosis coefficients being those of the mean sample distribution \(\frac{1}{n} \sum_{i=1}^n P_i\).

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10.2. Simpler Truncating Function. We can use the simpler truncation function

\[ L(x) = \max\{ -1, \min\{ +1, x \} \}. \]

Let \( \lambda \) be the positive root of the equation

\[ -\frac{1}{\lambda} \log \left( 1 - \frac{\lambda^2}{4[\exp(\lambda) - 1 - \lambda]} \right) = 1. \]

Let us define the upper and lower bounds

\[ L_+(x) = \frac{1}{\lambda} \log \left\{ 1 + \lambda x + [\exp(\lambda) - 1 - \lambda] x^2 \right\}, \]
\[ L'_+(x) = \frac{1}{\log(2)} \log \left\{ 1 + \log(2)x + \frac{\log(2)^2}{2} x^2 + \left[ 1 - \log(2) - \frac{\log(2)^2}{2} \right] x^3 \right\}, \]
\[ L_-(x) = -L_+(-x), \]
\[ L'_-(x) = -L'_+(-x). \]

Numerically, \( 0.535 \leq \lambda \leq 0.536 \). Moreover \( \exp(\lambda) - 1 - \lambda = \frac{a\lambda^2}{2} \), with \( a = \frac{2[\exp(\lambda) - 1 - \lambda]}{\lambda^2} \approx 1.2 \).

LEMMA 10.1 They are such that

\[ L_-(x) \leq L(x) \leq L_+(x), \quad x \in \mathbb{R}, \quad (10.1) \]
\[ L'_-(x) \leq L(x) \leq L'_+(x), \quad x \in \mathbb{R}. \quad (10.2) \]

PROOF. Let us consider the function \( f(x) = \exp[\lambda L_+(x)] - \exp[\lambda L(x)] \). It is such that

\[ f(x) = 1 + \lambda x + \frac{a\lambda^2 x^2}{2} - \exp[\lambda L(x)], \]
\[ f'(x) = \begin{cases} 
\lambda + a\lambda^2 x - \lambda \exp(\lambda x), & x \in (-1, +1), \\
\lambda + a\lambda^2 x, & x \notin (-1, +1), 
\end{cases} \]
\[ f''(x) = \begin{cases} 
\lambda^2 [1 - \exp(\lambda x)], & x \in (-1, +1), \\
\lambda^2 a, & x \notin (-1, +1). 
\end{cases} \]

Since \( f'(0) = 0 \) and \( f''(x) \geq 0, -1 < x \leq \frac{\log(a)}{\lambda}, f''(x) \leq 0, \frac{\log(a)}{\lambda} \leq x < 1, \) and \( f(1) = 0 \), we see that \( f(x) \geq x, x \in (-1, +1) \). Moreover, \( f \) is quadratic.
on $\mathbb{R} \setminus (-\infty, -1)$ and reach its minimum at point $x = -\frac{1}{\alpha}$, thus it is non negative on the whole line when this minimum value is non negative, that is when
$$1 - \frac{1}{2a} - \exp(-\lambda) \geq 0,$$
which is satisfied according to the definition of $\lambda$. This proves that $L(x) \leq L_+(x), x \in \mathbb{R}$. The fact that $L_-(x) \leq L(x)$ is then a consequence of $L(x) = -L(-x)$. The proof of $L(x) \leq L'_+(x)$ is done similarly by analyzing the shape of the function $x \mapsto \exp[\log(2)L'_+(x)] - \exp[\log(2)L(x)]$.

\[\square\]
If we redefine now our truncated mean estimate as
\[ \hat{\theta}_\alpha(\theta_0) = \theta_0 + \frac{\lambda}{n\alpha} \sum_{i=1}^{n} L \left[ \frac{\alpha}{\lambda} (Y_i - \theta_0) \right], \]
and define
\[ a = \frac{2[\exp(\lambda) - 1 - \lambda]}{\lambda^2} \simeq 1.2, \]
we deduce that
\[ E\{ \exp \left[ n\alpha [\hat{\theta}_\alpha(\theta_0) - \theta_0] \right] \} \leq \exp \left\{ n \log \left[ 1 + \alpha(m - \theta_0) + \frac{a\alpha^2}{2} [v + (m - \theta_0)^2] \right] \right\}. \]

Therefore, with probability at least 1 - \epsilon,
\[ \hat{\theta}_\alpha(\theta_0) \leq \theta_0 + \frac{1}{\alpha} \log \left\{ 1 + \alpha(m - \theta_0) + \frac{a\alpha^2}{2} [v + (m - \theta_0)^2] \right\} + \frac{\log(\epsilon^{-1})}{n\alpha} \leq m + \frac{a\alpha [v + (m - \theta_0)^2]}{2} + \frac{\log(\epsilon^{-1})}{n\alpha} \]

Working out the reverse inequality in the same way gives the following variant of Proposition 1.2 (page 5).

PROPOSITION 10.2 Assume that \( v \leq v_0 \) and \( |m - \theta_0| \leq \delta_0 \), where \( v_0 \) and \( \delta_0 \) are known prior bounds. With probability at least 1 - 2\epsilon,
\[ |\hat{\theta}_\alpha(\theta_0) - m| \leq \frac{a\alpha (v_0 + \delta_0^2)}{2} + \frac{\log(\epsilon^{-1})}{n\alpha}. \]

When \( \alpha = \sqrt{\frac{2\log(\epsilon^{-1})}{a(v_0 + \delta_0^2)n}} \), we get with probability at least 1 - 2\epsilon,
\[ |\hat{\theta}_\alpha(\theta_0) - m| \leq \sqrt{\frac{2a(v_0 + \delta_0^2)\log(\epsilon^{-1})}{n}} \leq 1.1 \sqrt{\frac{2(v_0 + \delta_0^2)\log(\epsilon^{-1})}{n}}. \]

So there is a ten per cent loss of accuracy with respect to Proposition 1.2 (page 5), this is the price to pay for using a simpler truncation function. We let the reader derive by himself the equivalent of the iterated estimate of Proposition 2.1 (page 5). When there is a third moment, we can also work with Equation (10.2, page 48), instead of Equation (10.1, page 48).
11. SOME CONCLUDING REMARKS

We would like to end this paper by sharing some guess about what is going on behind the scene. The need for thresholding indicates that large values may not be reliable, and have, so to speak, a bad “signal to noise ratio”. This is somehow understandable, since values whose deviation from the mean is much larger than the standard deviation have to appear in the sample with a small and therefore hard to estimate probability whereas their large size gives them a strong impact on the empirical mean, which makes their contributions to this estimate even worse.

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