Suboptimal \( s \)-union families and \( s \)-union antichains for vector spaces*

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Abstract

Let \( V \) be an \( n \)-dimensional vector space over the finite field \( \mathbb{F}_q \), and let \( \mathcal{L}(V) = \bigcup_{0 \leq k \leq n} [V]_k \) be the set of all subspaces of \( V \). A family of subspaces \( F \subseteq \mathcal{L}(V) \) is \( s \)-union if \( \dim(F + F') \leq s \) holds for all \( F, F' \in F \). A family \( F \subseteq \mathcal{L}(V) \) is an antichain if \( F \nsubseteq F' \) holds for any two distinct \( F, F' \in F \). The optimal \( s \)-union families in \( \mathcal{L}(V) \) have been determined by Frankl and Tokushige in 2013. The upper bound of cardinalities of \( s \)-union \((s < n)\) antichains in \( \mathcal{L}(V) \) has been established by Frankl recently, while the structures of optimal ones have not been displayed. The present paper determines all suboptimal \( s \)-union families for vector spaces and then investigates \( s \)-union antichains. For \( s = n \) or \( s = 2d < n \), we determine all optimal and suboptimal \( s \)-union antichains completely. For \( s = 2d + 1 < n \), we prove that an optimal antichain is either \([V]_d\) or contained in \([V]_d \cup [V]_{d+1}\) which satisfies an equality related with shadows.

Key words \( s \)-union antichain cross-intersecting shadow vector space

1 Introduction

Let \( X \) be an \( n \)-element set and let \( \binom{X}{k} \) denote the set of all \( k \)-element subsets of \( X \). For the power set of \( X \), we use the notation \( 2^X \). We say that a family of subsets \( F \subseteq 2^X \) is \( s \)-union if \( |F \cup F'| \leq s \) holds for all \( F, F' \in F \). A family \( F \) is called \( t \)-intersecting if for all \( F, F' \in F \), we have \( |F \cap F'| \geq t \). Since \( F \subseteq 2^X \) is \( s \)-union if and only if \( \{X \setminus F : F \in F\} \) is an \((n-s)\)-intersecting family, the two concepts are essentially the same.

Let \( F \subseteq 2^X \) be \( s \)-union. An \( s \)-union family is said to be \( \textit{optimal} \) if it has the largest possible cardinality. It is obvious that \( F = 2^X \) is the optimal \( s \)-union family if \( s = n \). For \( s = n - 1 \), there are many optimal \( s \)-union families achieving the maximum cardinality \( 2^{n-1} \).\(^{4}\) For \( s \leq n - 2 \), Katona \(^{9}\) showed that \( |F| \leq \sum_{i=0}^{d} \binom{n}{i} \) if \( s = 2d \), and \( |F| \leq \sum_{i=0}^{d} \binom{n}{i} + \binom{n-1}{d} \) if \( s = 2d + 1 \). The optimal \( s \)-union families are proved to be isomorphic to the Katona families \( K(n, s) \) defined as follows. For \( s = 2d \), let

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For $s = 2d + 1$, let

$$\mathcal{K}(n, 2d + 1) = \{ F \subseteq X : |F| \leq d \} \cup \{ F \subseteq X : |F| = d + 1, y \in F \},$$

where $y$ is a fixed element of $X$. In 2017, Frankl \cite{frankl2017} investigated the suboptimal $s$-union families ($s < n$), meaning that they have the largest possible cardinality under the condition that they are not contained in any of the optimal $s$-union families. Frankl established the following theorem.

**Theorem 1.1.** (\cite{frankl2017} Theorem 2) Let $X$ be an $n$-element set and $2 \leq s < n$. Suppose that $\mathcal{F} \subseteq 2^X$ is $s$-union and $\mathcal{F} \not\subseteq \mathcal{K}(n, s)$. Then the following hold.

(i) When $s = 2d$,

$$|\mathcal{F}| \leq \sum_{i=0}^{d} \binom{n}{i} - \binom{n-d-1}{d} + 1.$$

Moreover for $s \leq n - 2$, equality holds if and only if

$$\mathcal{F} = (\{ F \subseteq X : |F| \leq d \} \setminus \{ F \in \binom{X}{d} : F \cap D = \emptyset \}) \cup \{ D \},$$

where $D$ is a fixed $(d+1)$-subset of $X$.

(ii) When $s = 2d + 1$,

$$|\mathcal{F}| \leq \sum_{i=0}^{d} \binom{n}{i} + \binom{n-1}{d} - \binom{n-d-2}{d} + 1.$$

Moreover for $s \leq n - 2$, equality holds if and only if

$$\mathcal{F} = \{ F \subseteq X : |F| \leq d \} \cup \{ F \in \binom{X}{d+1} : y \in F, F \cap D \neq \emptyset \} \cup \{ D \},$$

where $D \in \binom{X}{d+1}$, $y \in X$ are fixed with $y \notin D$ or

$$\mathcal{F} = \{ F \subseteq X : |F| \leq 2 \} \cup \{ F \in \binom{X}{3} : |F \cap D| \geq 2 \},$$

where $D$ is a fixed 3-subset of $X$ if $s = 5$.

The problems in extremal set theory have natural extensions to families of subspaces over a finite field. Throughout the paper we always let $V$ be an $n$-dimensional vector space over the finite field $\mathbb{F}_q$. Let $\binom{V}{k}$ denote the family of all $k$-dimensional subspaces of $V$. For $m \in \mathbb{R}, k \in \mathbb{Z}^+$, define the Gaussian binomial coefficient by

$$\left[ \begin{array}{c} m \\ k \end{array} \right]_q := \prod_{0 \leq i < k} \frac{q^{m-i} - 1}{q^{k-i} - 1}.$$

Obviously, the size of $\binom{V}{k}$ is $\left[ \begin{array}{c} n \\ k \end{array} \right]_q$. If $k$ and $q$ are fixed, then $\left[ \begin{array}{c} m \\ k \end{array} \right]_q$ is a continuous function of $m$ which is positive and strictly increasing when $m \geq k$. If there is no ambiguity, the subscript $q$ can be omitted. Let

$$\mathcal{L}(V) = \bigcup_{0 \leq k \leq n} \binom{V}{k}.$$
For any two subspaces $A, B \in \mathcal{L}(V)$, let $A \leq B$ denote that $A$ is a subspace of $B$ and $A + B$ denote the linear span of $A, B$. Let $\mathcal{F} \subseteq \mathcal{L}(V)$ be a family of subspaces, we say that $\mathcal{F}$ is $s$-union if $\dim(F + F') \leq s$ holds for all $F, F' \in \mathcal{F}$. A family $\mathcal{F} \subseteq \mathcal{L}(V)$ is called $t$-intersecting if for all $F, F' \in \mathcal{F}$, we have $\dim(F \cap F') \geq t$. In particular, we say that $\mathcal{F}$ is an intersecting family if $t = 1$. Endow $V$ with the usual inner product $\langle \cdot, \cdot \rangle$.

For a subspace $U$ of $V$, let $U^\perp = \{v \in V : \langle u, v \rangle = 0 \text{ for all } u \in U\}$ be the orthogonal complement of $U$. For a family $\mathcal{F} \subseteq \mathcal{L}(V)$, denote $\mathcal{F}^\perp = \{F^\perp : F \in \mathcal{F}\}$.

Since $(F + F')^\perp = F^\perp \cap F'^\perp$ holds for all $F, F' \in \mathcal{F}$, it is obvious that $\mathcal{F}$ is $s$-union if and only if $\mathcal{F}^\perp$ is $(n-s)$-intersecting.

Obviously, $\mathcal{F} = \mathcal{L}(V)$ is the optimal $s$-union family if $s = n$. For $s = n - 1$, the optimal $s$-union families in $\mathcal{L}(V)$ were determined by Blokhuis et al. [1]. For $s \leq n - 2$, we record the optimal families $\mathcal{K}[n,s]$ as follows, which were displayed in the form of $t$-intersecting families in [3]. For $s = 2d$, let $\mathcal{K}[n,2d] = \{F \leq V : \dim(F) \leq d\}$.

For $s = 2d + 1$, let $\mathcal{K}[n,2d+1] = \{F \leq V : \dim(F) \leq d\} \bigcup \{F \in \begin{bmatrix} V \\ d+1 \end{bmatrix} : E \leq F\}$, where $E$ is a fixed 1-dimensional subspace of $V$.

**Theorem 1.2.** ([3 Theorem 4]) Suppose that $\mathcal{F} \subseteq \mathcal{L}(V)$ is $s$-union, $2 \leq s < n$. Then the following hold.

(i) When $s = 2d$,

$$|\mathcal{F}| \leq \sum_{i=0}^{d} \binom{n}{i}.$$

Moreover, for $s \leq n - 2$, equality holds if and only if $\mathcal{F} = \mathcal{K}[n,2d]$.

(ii) When $s = 2d + 1$,

$$|\mathcal{F}| \leq \sum_{i=0}^{d} \binom{n}{i} + \binom{n-1}{d}.$$

Moreover, for $s \leq n - 2$, equality holds if and only if $\mathcal{F} = \mathcal{K}[n,2d+1]$.

Excluding the optimal families provided in Theorem 1.2, we will consider the suboptimal $s$-union families in $\mathcal{L}(V)$ and establish a vector space version of Theorem 1.1. Let us define the families $\mathcal{T}[n,s]$. For $s = 2d$, define

$$\mathcal{T}[n,2d] = (\{F \leq V : \dim(F) \leq d\} \setminus \{F \subseteq \begin{bmatrix} V \\ d \end{bmatrix} : \dim(F \cap U) = 0\}) \bigcup \{U\},$$

where $U$ is a fixed $(d+1)$-dimensional subspace of $V$. 
For $s = 2d + 1$, define
\[
\mathcal{T}[n, 2d + 1] = \{F \leq V : \dim(F) \leq d\} \cup \{F \in \left[\frac{V}{d + 1}\right] : E \leq F, \dim(F \cap U) \geq 1\} \cup \left[\frac{E + U}{d + 1}\right],
\]
where $E \in \left[\frac{V}{1}\right]$ and $U \in \left[\frac{V}{d + 1}\right]$ are fixed subspaces of $V$ with $E \not\subseteq U$.

For $s = 5$, define also
\[
\mathcal{J}[n, 5] = \{F \leq V : \dim(F) \leq 2\} \cup \{F \in \left[\frac{V}{3}\right] : \dim(F \cap D) \geq 2\},
\]
where $D$ is a fixed 3-dimensional subspace of $V$.

A main result of this paper in the next theorem shows that $\mathcal{T}[n, 2d]$, $\mathcal{T}[n, 2d + 1]$ and $\mathcal{J}[n, 5]$ (for $s = 5$) are suboptimal $s$-union families.

**Theorem 1.3.** Let $2 \leq s < n$. Suppose that $\mathcal{F} \subseteq \mathcal{L}(V)$ is $s$-union and $\mathcal{F} \not\subseteq \mathcal{K}[n, s]$. Then the following hold.

(i) If $s = 2d$, then
\[
|\mathcal{F}| \leq \sum_{i=0}^{d} \left[\frac{n}{i}\right] - q^{d(d+1)} \left[\frac{n-d-1}{d}\right] + 1.
\]
Moreover for $s \leq n - 2$, equality holds if and only if $\mathcal{F} = \mathcal{T}[n, 2d]$.

(ii) If $s = 2d + 1$, either $q \geq 3$ and $n \geq 2d + 3$, or $q = 2$ and $n \geq 2d + 4$, then
\[
|\mathcal{F}| \leq \sum_{i=0}^{d} \left[\frac{n}{i}\right] + \left[\frac{n-1}{d}\right] - q^{d(d+1)} \left[\frac{n-d-2}{d}\right] + q^{d+1}.
\]
Moreover for $s \leq n - 2$, equality holds if and only if $\mathcal{F} = \mathcal{T}[n, 2d + 1]$ or alternatively $\mathcal{F} = \mathcal{J}[n, 5]$ if $s = 5$.

A family $\mathcal{F} \subseteq \mathcal{L}(V)$ is an antichain if $F \not\leq F'$ holds for any two distinct $F, F' \in \mathcal{F}$. In 2021, Frankl \cite{6} obtained the upper bound of the cardinalities of $s$-union antichains for vector spaces.

**Theorem 1.4.** (\cite{6} Theorem 3.5) If $\mathcal{F} \subseteq \mathcal{L}(V)$ is an $s$-union antichain with $2 \leq s < n$, then
\[
|\mathcal{F}| \leq \left[\frac{n}{\left\lfloor\frac{s}{2}\right\rfloor}\right].
\]

The second main objective of the paper is to determine the structures of all optimal $s$-union antichains and then the suboptimal ones. Let us define the families $\mathcal{A}[n, s]$, $\mathcal{B}[n, s]$. For $2 \leq s \leq n$, define
\[
\mathcal{A}[n, s] = \left(\left[\frac{V}{\left\lfloor\frac{s}{2}\right\rfloor}\right] \setminus \{F \in \left[\frac{V}{\left\lfloor\frac{s}{2}\right\rfloor}\right] : U \leq F\}\right) \cup \{U\},
\]
where $U$ is a fixed $\left(\left\lfloor\frac{s}{2}\right\rfloor\right)$-dimensional subspace of $V$; define
\[
\mathcal{B}[n, s] = \left(\left[\frac{V}{\left\lfloor\frac{s}{2}\right\rfloor}\right] \setminus \{F \in \left[\frac{V}{\left\lfloor\frac{s}{2}\right\rfloor}\right] : F \leq W\}\right) \cup \{W\},
\]
where $W$ is a fixed $\left(\left\lceil\frac{s}{2}\right\rceil\right)$-dimensional subspace of $V$.

It is obvious that an $s$-union antichain is just an antichain if $s = n$. We determine the structures of all optimal and suboptimal antichains in this paper.
Theorem 1.5. Let $F \subseteq \mathcal{L}(V)$ be an antichain and $n > 1$. Then the following hold.

(i) $|F| \leq \left\lceil \frac{n}{2} \right\rceil$. Moreover, equality holds if and only if $F = \left[ \frac{V}{1} \right]$ or $F = \left[ \frac{V}{1} \right]$.

(ii) If $F \not\subseteq \left[ \frac{V}{1} \right]$ and $F \not\subseteq \left[ \frac{V}{1} \right]$, then

$$|F| \leq \left\lceil \frac{n}{2} \right\rceil - q\left\lceil \frac{n}{2} \right\rceil.$$

Moreover, equality holds if and only if $F = \mathcal{A}[n,n]$ or $\mathcal{B}[n,n]$.

For $0 \leq u \leq n$, let us define the $u$-shadow of $\mathcal{H} \subseteq \mathcal{L}(V)$ by

$$\Delta_u(\mathcal{H}) = \{G \in \left[ \frac{V}{u} \right] : G \leq H \text{ for some } H \in \mathcal{H}\}.$$

In particular, the $(u-1)$-shadow of the family $\mathcal{H} \subseteq \left[ \frac{V}{u} \right]$ is denoted by $\Delta(\mathcal{H})$ for convenience.

For $s$-union antichains with $s < n$, we establish another main theorem as follows.

Theorem 1.6. Let $F \subseteq \mathcal{L}(V)$ be an $s$-union antichain with $s < n$. Then the following hold.

(i) $|F| \leq \left\lceil \frac{n}{s} \right\rceil$. Moreover, equality holds if and only if either (a) or (b) holds.

(a) $F = \left[ \frac{V}{1} \right]$;

(b) $F = F_d \cup F_{d+1}$ for $s = 2d + 1$, where $F_{d+1} \subseteq \left[ \frac{V}{d+1} \right], F_d = \left[ \frac{V}{d} \right] \setminus \Delta(F_{d+1})$ and $|\Delta(F_{d+1})| = |F_{d+1}|$.

(ii) Suppose $s = 2d$ and $F \not\subseteq \left[ \frac{V}{d} \right]$. Then (a) or (b) holds.

(a) If $d = 1$, then

$$|F| \leq \left\lceil \frac{n}{1} \right\rceil - q.$$

Moreover, equality holds if and only if $F = \mathcal{B}[n,2]$.

(b) If $d \geq 2$, then

$$|F| \leq \left( \frac{n}{d} \right) - q\left( \frac{n-d}{1} \right).$$

Moreover, equality holds if and only if $F = \mathcal{A}[n,2d]$.

The main objective of the paper is to prove Theorems 1.3, 1.5 and 1.6. For Theorem 1.6, we first consider the case $d = 1$, $s = 2 < n$. It is obvious that if $d = 1$, the optimal 2-union antichain is $\left[ \frac{V}{1} \right]$. Let $F$ be a suboptimal 2-union antichain. We can easily find that if $|F| > 1$ then any $i$-dimensional subspace with $i = 0$ or $i \geq 3$ does not belong to $F$ and that $|F \cap \left[ \frac{V}{2} \right]| \leq 1$. Thus $F = \mathcal{B}[n,2]$ and $|F| = \left( \frac{n}{1} \right) - \left( \frac{n}{1} \right) + 1 = \left( \frac{n}{1} \right) - q$. 


2 Preliminaries

In this section, we recall a number of basic theorems and establish several new lemmas in the vector spaces, which are essential for our proofs. Firstly, we introduce the celebrated Erdős-Ko-Rado theorem and Hilton-Milner theorem for vector spaces.

**Theorem 2.1.** ([7, Theorem 1], [10, Theorem 3]) Let \(1 \leq t \leq k\). Suppose \(H \subseteq \binom{V}{k}\) is a \(t\)-intersecting family. Then we have

\[
|H| \leq \begin{cases} 
\binom{n-t}{k-t}, & \text{if } n \geq 2k, \\
\binom{2k-t}{k}, & \text{if } 2k - t < n \leq 2k.
\end{cases}
\]

Moreover, equality holds if and only if one of the following holds:

(i) If \(n > 2k\), then \(H = \{H \in \binom{V}{k} \colon T \leq H\} \text{ for some } T \in \binom{V}{t}\).

(ii) If \(2k - t < n < 2k\), then \(H = \binom{Y}{k} \text{ for some } Y \in \binom{V}{2k-t}\).

(iii) If \(n = 2k\), then \(H = \{H \in \binom{V}{k} \colon T \leq H\} \text{ for some } T \in \binom{V}{t} \text{ or } H = \binom{Y}{k} \text{ for some } Y \in \binom{V}{2k-t}\).

**Theorem 2.2.** ([1, Theorem 1.4]) Suppose \(k \geq 2\), and either \(q \geq 3\) and \(n \geq 2k+1\), or \(q = 2\) and \(n \geq 2k+2\). Let \(H \subseteq \binom{V}{k}\) be an intersecting family with \(\dim(\bigcap_{H \in H} H) = 0\). Then

\[
|H| \leq \binom{n-1}{k-1} - q^{k(k-1)} \binom{n-k-1}{k-1} + q^k.
\]

Moreover, equality holds if and only if

\[
H = \{H \in \binom{V}{k} \colon E \leq H, \dim(H \cap U) \geq 1\} \bigcup \binom{E + U}{k},
\]

where \(E, U\) are fixed 1-dimensional, \(k\)-dimensional subspaces of \(V\) (respectively) with \(E \not\subseteq U\) or for \(k = 3\),

\[
H = \{F \in \binom{V}{3} \colon \dim(F \cap D) \geq 2\},
\]

where \(D\) is a fixed 3-dimensional subspace of \(V\).

Shadow is an important notion in extremal set theory. We will make use of the following two vector space version of theorems on shadows.

**Theorem 2.3.** ([8, Theorem 3]) Let \(1 \leq t \leq k \leq n\) and let \(H \subseteq \binom{V}{k}\) be \(t\)-intersecting. Then for \(k - t \leq u \leq k\), we have

\[
\frac{|\triangle_u(H)|}{|H|} \geq \binom{2k-t}{k-t} \binom{u}{k-t}.
\]

Note that for \(k - t < u < k\), the RHS of (4) is strictly greater than 1.
Theorem 2.4. (3 Theorem 1.4) Let \( H \subseteq \binom{V}{k} \) and let \( m \geq k \) be the real number which satisfies \( |H| = \binom{m}{k} \). Then

\[
|\Delta(H)| \geq \binom{m}{k - 1}.
\]

Moreover, equality holds if and only if \( H = \binom{M}{k} \) for some \( M \in \binom{V}{m} \), \( m \in \mathbb{Z}^+ \).

The following result is an analog of [5, Proposition 1] for finite sets.

Lemma 2.5. Suppose \( F \subseteq \mathcal{L}(V) \) is \( s \)-union and \( 2 \leq s < n \). Let \( F_i = F \cap \binom{V}{i} \) for \( 0 \leq i \leq \lfloor \frac{n}{2} \rfloor \). Then

\[
|F_i| + |F_{s+1-i}| \leq \binom{n}{i}.
\]  \hspace{1cm} (5)

Moreover, for \( s \leq n - 2 \), equality holds if and only if \( F_{s+1-i} = \emptyset \).

Proof. When \( F_{s+1-i} = \emptyset \), the theorem holds trivially. We suppose \( F_{s+1-i} \neq \emptyset \) in the following. For a fixed \( i \leq \lfloor \frac{n}{2} \rfloor \), define the family

\[
\mathcal{H} = \{ F^\perp : F \in F_{s+1-i} \} \subseteq \binom{V}{n + i - s - 1}.
\]

We claim that \( \Delta_i(\mathcal{H}) \cap \mathcal{F}_i = \emptyset \). Suppose \( F \in \Delta_i(\mathcal{H}) \cap \mathcal{F}_i \). Then there exists \( H \in \mathcal{H} \) such that \( F \subseteq H, H^\perp \in F_{s+1-i} \). So we have \( \dim(F \cap H^\perp) = 0 \), i.e., \( \dim(F + H^\perp) = i + (s + 1 - i) = s + 1 \), a contradiction to the \( s \)-union property of \( F \). By the claim, we have

\[
|\Delta_i(\mathcal{H})| + |\mathcal{F}_i| \leq \binom{n}{i}.
\]  \hspace{1cm} (6)

Since \( F_{s+1-i} \subseteq F \) is \( s \)-union, \( \mathcal{H} \) is an \((n - s)\)-intersecting family. In Theorem 2.3 setting \( u = i, t = n - s, k = n + i - s - 1 \), yields the following inequality for \( s \leq n - 1 \):

\[
|\Delta_i(\mathcal{H})| \geq |\mathcal{H}|.
\]

When \( s \leq n - 2 \), since \( F_{s+1-i} \neq \emptyset \), i.e., \( \mathcal{H} \neq \emptyset \), we have \( |\Delta_i(\mathcal{H})| > |\mathcal{H}| \) by (4). Hence, we have \( |\mathcal{H}| + |\mathcal{F}_i| \leq \binom{n}{i} \) by (5). It is clear that \( |\mathcal{H}| = |F_{s+1-i}| \). Therefore, the desired result follows. \( \square \)

We introduce a counting formula for vector spaces, which is further interpreted by applying the \( q \)-analog of inclusion-exclusion principle in [2].

Lemma 2.6. Let \( Z \) be an \( m \)-dimensional subspace of the \( n \)-dimensional vector space \( V \) over \( \mathbb{F}_q \). For a positive integer \( l \) with \( m + l \leq n \), let \( x \) denote the number of \( l \)-dimensional subspaces \( W \) of \( V \) such that \( \dim(Z \cap W) = 0 \). Then the following hold.

(i) \( x = q^{lm} \binom{n-m}{l} = \sum_{0 \leq t \leq \min(m, l)} (-1)^t q^{\binom{l-t}{2}} \binom{m}{t} \binom{n-t}{l-t} \).

(ii) \( x \geq \binom{n}{l} - \binom{m}{1} \binom{n-1}{l-1} \).
Proof. (i) This is the result of Propositions 2.2 and 2.3 in [2].

(ii) Let $a = \min\{m, l\}$, if $a = 1$, then the equality in (ii) holds by (i). If $a \geq 2$, we have

$$x = \binom{n}{l} - \binom{m}{l} \binom{n-1}{l-1} + \sum_{t=2}^{a} (-1)^t q^{\binom{t-1}{2}} \binom{m}{t} \binom{n-t}{l-t}.$$  

It suffices to prove that $q^{\binom{t-1}{2}} \binom{m}{t} \binom{n-t}{l-t} \geq q^{\binom{t+1}{2}} \binom{m}{t+1} \binom{n-t-1}{l-t-1}$, where $a \geq t \geq 2$. Since $n \geq m + l \geq 2a$, we have

$$\frac{q^{n-t-1}}{q^{t-1}} \cdot \frac{q^{t+1}}{q^{n-t}} > \frac{(q^{n-t-1})(q^{t+1}-1)}{q^{n-t}+2},$$

which implies the desired result. 

Two families $A$ and $B$ in $\mathcal{L}(V)$ are said to be cross-Sperner if there exist no $A \in A$ and $B \in B$ with $A \subseteq B$ or $B \subseteq A$. Wang and Zhang [11] obtained the upper bound of sizes of a pair of cross-Sperner families of finite vector spaces.

**Theorem 2.7.** ([11] Theorem 1.5) Let $a, b, t$ be positive integers with $a < b < n$. If $A \subseteq \binom{V}{a}$ and $B \subseteq \binom{V}{b}$ are cross-Sperner, then

$$|A| + |B| \leq \max\{\binom{n}{b} - \binom{n-a}{b-a} + 1, \binom{n}{a} - \binom{b}{a} + 1\}.$$  

Moreover equality holds if and only if one of the following holds:

(i) $\binom{n}{a} \leq \binom{n}{b}$ and $A = \{A\}$ for some $A \subseteq \binom{V}{a}$ and $B = \{B \in \binom{V}{b} : A \nsubseteq B\}$;

(ii) $\binom{n}{a} \geq \binom{n}{b}$ and $B = \{B\}$ for some $B \in \binom{V}{b}$ and $A = \{A \in \binom{V}{a} : A \nsubseteq B\}$.

We say that two families $A$ and $B$ in $\mathcal{L}(V)$ are cross-$t$-intersecting if $\dim(A \cap B) \geq t$ for all $A \in A, B \in B$, where $t \geq 1$. In particular, $A$ and $B$ are said to be cross-intersecting if $t = 1$. Wang and Zhang [11] also obtained the upper bound of sizes of cross-$t$-intersecting families.

**Theorem 2.8.** ([11] Theorem 1.4) Let $n \geq 4, a, b, t$ be positive integers with $a, b \geq 2, t < \min\{a, b\}, a+b < n+t$, and $\binom{n}{a} \leq \binom{n}{b}$. If $A \subseteq \binom{V}{a}$ and $B \subseteq \binom{V}{b}$ are cross-$t$-intersecting, then

$$|A| + |B| \leq \binom{n}{b} - \sum_{i=0}^{t-1} q^{(a-i)(b-i)} \binom{a}{i} \binom{n-a}{b-i} + 1.$$  

Moreover equality holds if and only if one of the following holds:

(i) $A = \{A\}$ for some $A \subseteq \binom{V}{a}$ and $B = \{B \in \binom{V}{b} : \dim(A \cap B) \geq t\}$;
Proof. If \(A = \{b\} \) and \(B = \{r\} \) for some \( b \in \left[ V \right] \) and \( A = \{a \in \left[ V \right] : \dim(A \cap B) \geq t \} \).

We will sharpen the upper bound of \((7)\) in a special case in the following lemma, which will be very useful in the proofs of Theorems 1.3 and 1.6.

**Lemma 2.9.** Suppose \( A \subseteq \left[ V \right] \) and \( B \subseteq \left[ V \right] \) are cross-intersecting families. Further suppose \( B \) is 2-intersecting. Then for \( n \geq 2k + 1 \),

\[
|A| + |B| \leq \left[ \frac{n}{k} \right] - q^{k(k+1)} \left[ \frac{n-k-1}{k} \right] + 1. \tag{8}
\]

Moreover for \( n \geq 2k + 2 \), equality holds if and only if \( B = \{b\} \) for some \( b \in \left[ V \right] \) and \( A = \{a \in \left[ V \right] : \dim(A \cap B) \geq 1 \} \).

**Proof.** If \( k = 1 \), since \( B \subseteq \left[ \frac{V}{2} \right] \) is 2-intersecting, then \( |B| = 1 \) and \((8)\) holds by Lemma 2.6 (i). We always let \( k \geq 2 \) in the following proof.

First we consider the case \( n = 2k + 1 \). Since \( \left[ \frac{n}{k} \right] = \left[ \frac{n}{k+1} \right] \), we can set \( a = k + 1 \), \( b = k, t = 1 \) in Theorem 2.8. Then \((8)\) holds by \((7)\).

Next we let \( n \geq 2k + 2 \). Since \( B \subseteq \left[ \frac{V}{k+1} \right] \) is 2-intersecting and \( n \geq 2k + 2 \), then by Theorem 2.11 we have

\[
1 \leq |B| \leq \left[ \frac{n-2}{k-1} \right]. \tag{9}
\]

For any \( C \subseteq \left[ \frac{V}{k+1} \right] \), define \( \Gamma(C) = \{T \in \left[ \frac{V}{k} \right] : \dim(T \cap C) = 0 \text{ for some } C \in C\} \).

Since \( \dim(A \cap B) \geq 1 \) for all \( A \in A \) and \( B \in B \), we have \( A \cap B = \emptyset \). Hence,

\[
|A| + |\Gamma(B)| \leq \left[ \frac{n}{k} \right]. \tag{10}
\]

Case a: Suppose \( B = \{b\} \subseteq \left[ \frac{V}{k+1} \right] \). Then \( \Gamma(B) = \{T \in \left[ \frac{V}{k} \right] : \dim(T \cap B) = 0\} \).

Hence we have \( |\Gamma(B)| = q^{k(k+1)} \left[ \frac{n-k-1}{k} \right] \) by Lemma 2.6 (i), which implies \((8)\) by using \((10)\). Moreover, the equality in \((8)\) holds if and only if the equality in \((11)\) holds, that is, \( A = \{a \in \left[ V \right] : \dim(A \cap B) \geq 1\} \).

Case b: Suppose \( |B| \geq 2 \). Let \( B = \{B_1, B_2, \ldots, B_r\} \subseteq \left[ \frac{V}{k+1} \right] \),

where \( r \geq 2 \). We claim that

\[
|\Gamma(B)| \geq q^{k(k+1)} \left[ \frac{n-k-1}{k} \right] + q^{(k-1)(k+1)} \left[ \frac{n-k-2}{k-1} \right]. \tag{11}
\]

Since \( \{B_1, B_2\} \subseteq B \), then \( \Gamma(\{B_1, B_2\}) \subseteq \Gamma(B) \) by the definition of \( \Gamma(B) \). Let \( \Gamma_1 = \{T \in \left[ \frac{V}{k} \right] : \dim(T \cap B_1) = 0\} \), \( \Gamma_2 = \{T \in \left[ \frac{V}{k} \right] : \dim(T \cap B_2) = 0, \dim(T \cap B_1) > 0\} \).

It is clear that \( \Gamma_1 \cup \Gamma_2 \subseteq \Gamma(\{B_1, B_2\}) \) and \( \Gamma_1 \cap \Gamma_2 = \emptyset \). Then \( |\Gamma(B)| \geq |\Gamma_1| + |\Gamma_2| \). By Lemma 2.6 (i), we have

\[
|\Gamma_1| = q^{k(k+1)} \left[ \frac{n-k-1}{k} \right]. \tag{12}
\]
Obviously, there exists $E \in \left[ B_1 \right]$, $E \not\subseteq B_2$ such that $B_2 \leq W \in \left[ V \right]$, where $V = E \oplus W$ (namely $V = E + W$ and $E \cap W = \{0\}$). Define a subfamily of $\Gamma_2$ by $\Gamma_3 = \{ T \in \left[ V \right] : \dim(T \cap B_2) = 0, E \leq T \}$, which has a one-to-one correspondence to $\Gamma_4 = \{ R \in \left[ W \right] : \dim(R \cap B_2) = 0 \}$. There is a natural injective map $\varphi : \Gamma_3 \to \Gamma_4$ by $T \mapsto T \cap W$. For any $R \in \Gamma_4$, since $R + B_2 \leq W, E \not\subseteq W$, we have $E + R \in \Gamma_3$. So $\varphi$ is also surjective. By Lemma 2.6 (i), we have

$$|\Gamma_2| \geq |\Gamma_3| = |\Gamma_4| = q^{(k-1)(k+1)} \left[ \frac{n - k - 2}{k - 1} \right].$$

Together with (12), we complete the proof of (11). Combining (9), (10), and (11), we have

$$|A| + |B| \leq \left[ \frac{n}{k} \right] - q^{k+1} \left[ \frac{n-k-1}{k-1} \right] - q^{(k-1)(k+1)} \left[ \frac{n-k-2}{k-1} \right] + \left[ \frac{n-2}{k-1} \right]. \quad (13)$$

Hence, we only need to show that

$$\left[ \frac{n - 2}{k - 1} \right] < 1 + q^{(k-1)(k+1)} \left[ \frac{n - k - 2}{k - 1} \right] \quad (14)$$

to prove the final conclusion. If $k = 2$, (14) clearly holds. If $k \geq 3$, by Lemma 2.6 (ii), we have the inequality:

$$q^{(k-1)(k+1)} \left[ \frac{n - k - 2}{k - 1} \right] > \left[ \frac{n - 1}{k - 1} \right] - \left[ \frac{k + 1}{k - 1} \right] \left[ \frac{n - 2}{k - 2} \right].$$

Since $q^{n-k} \leq \frac{q^{n-1}}{q^{k-1}} \leq q^{n-k+1}$ and $n \geq 2k + 2$, we have

$$\left[ \frac{n-1}{k-1} \right] - \left[ \frac{k+1}{k-2} \right] \left[ \frac{n-2}{k-1} \right] + 1 - \left[ \frac{n-2}{k-1} \right] > \left[ \frac{n-1}{k-1} \right] - \left[ \frac{k+1}{k-2} \right] \left[ \frac{n-2}{k-2} \right] - \left[ \frac{n-2}{k-1} \right]
= q^{n-k-1} \left[ \frac{n-2}{k-2} \right] - \frac{q^{k+1}-1}{q-1} \left[ \frac{n-2}{k-2} \right] - \frac{q^{n-k-1}}{q^{k-1}} \left[ \frac{n-2}{k-2} \right]
\geq (q^{n-k} - q^{k+1} - q^{2k-2}) \left[ \frac{n-2}{k-2} \right]
\geq (2q^{n-k-1} - q^{k+1} - q^{n-2k+2}) \left[ \frac{n-2}{k-2} \right]
\geq 0,$$

meaning that (14) holds as well. So we obtain that $|A| + |B| < \left[ \frac{n}{k} \right] - q^{k+1} \left[ \frac{n-k-1}{k-1} \right] + 1$ by (13) and (14).

Reviewing the whole proof, we get that the families $A$ and $B$ attaining the equality in (3) are just those stated in the lemma. \hfill $\square$

### 3 Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. Let $\mathcal{F} \subseteq \mathcal{L}(V)$ be an s-union family of maximum size with $\mathcal{F} \not\subseteq \mathcal{K}[n, s]$. We will calculate the cardinality

$$|\mathcal{F}| = \sum_{0 \leq i \leq s} |\mathcal{F}_i|, \text{ where } \mathcal{F}_i = \mathcal{F} \cap \left[ \frac{V}{i} \right]. \quad (15)$$
We consider the singularity of $s$.

(1) Let $s = 2d + 1$. It is easily seen that $\mathcal{F}_i = \emptyset$ for $i \geq 2d + 2$. Adding up (5) for $0 \leq i \leq d$, we have

$$\sum_{i=0}^{d} (|\mathcal{F}_i| + |\mathcal{F}_{2d+2-i}|) \leq \sum_{i=0}^{d} \binom{n}{i}. \quad (16)$$

Since $\mathcal{F}_{d+1} \subseteq \mathcal{F}$ is $(2d + 1)$-union, we have that $\mathcal{F}_{d+1}$ is an intersecting family. Since for $G \subseteq F \in \mathcal{F}$, the family $\mathcal{F} \cup \{G\}$ is also $s$-union. So we can assume that $G \subseteq F$ and $F \in \mathcal{F}$ implies $G \in \mathcal{F}$. Then we distinguish two cases.

Case a: Suppose there exists $G \in \mathcal{F}$ with $\dim(G) \geq d + 2$. Then $\binom{G}{d+1} \subseteq \mathcal{F}_{d+1}$, which implies that $\dim(\bigcap_{F \in \mathcal{F}_{d+1}}) = 0$. By Theorem 2.2 we have

$$|\mathcal{F}_{d+1}| \leq \binom{n-1}{d} - q^{d(d+1)} \binom{n-d-2}{d} + q^{d+1}. \quad (17)$$

Moreover, there exists $\mathcal{F}_{2d+2-i} \neq \emptyset$ for $0 \leq i \leq d$ in this case. Then (5) is a strict inequality for some $i$, which implies the inequality in (16) is strict as well. Hence, by (15)-(17), we have

$$|\mathcal{F}| = \sum_{i=0}^{d} (|\mathcal{F}_i| + |\mathcal{F}_{2d+2-i}|) + |\mathcal{F}_{d+1}| \leq \sum_{i=0}^{d} \binom{n}{i} + \binom{n-1}{d} - q^{d(d+1)} \binom{n-d-2}{d} + q^{d+1}. \quad (18)$$

Case b: Suppose that $\mathcal{F}_i = \emptyset$ for all $i \geq d + 2$. Now (5) holds trivially, that is (16) holds trivially as well. We have $\dim(\bigcap_{F \in \mathcal{F}_{d+1}}) = 0$, because otherwise $\mathcal{F} \subseteq \mathcal{K}[n,2d+1]$. We use Theorem 2.2 again. The upper bound in (2) can be obtained by (16) and (17).

Moreover, if the equality in (2) holds, then $\mathcal{F}_i = \left\{\begin{array}{l}V \\ V \end{array}\right\} \forall 0 \leq i \leq d$ and the equality in (17) holds as well. So applying Theorem 2.2 we have

$$\mathcal{F}_{d+1} = \{F \in \binom{V}{d+1} : E \subseteq F, \dim(F \cap U) \geq 1\} \cup \binom{E+U}{d+1},$$

where $E,U$ are fixed 1-dimensional, $(d+1)$-dimensional subspaces of $V$ with $E \nsubseteq U$. If $d+1 = 3$, the equality is also attained by taking $\mathcal{F}_3 = \{F \in \binom{V}{3} : \dim(F \cap D) \geq 2\}$, where $D$ is a fixed 3-dimensional subspace of $V$.

(2) Let $s = 2d$. For $0 \leq i < d$, we use (5) as well. Then

$$\sum_{0 \leq i < d} (|\mathcal{F}_i| + |\mathcal{F}_{2d+1-i}|) \leq \sum_{0 \leq i < d} \binom{n}{i}. \quad (18)$$

When $i = d$, we have a stronger result. For convenience, we set $\mathcal{A} = \mathcal{F}_d$, $\mathcal{B} = \mathcal{F}_{d+1}$ in the following. Since $\mathcal{F} \nsubseteq \mathcal{K}[n,2d]$, there exists $G \in \mathcal{F}$ with $\dim(G) \geq d + 1$. Hence $\binom{G}{d+1} \subseteq \mathcal{B} \neq \emptyset$.

Since $\mathcal{B} \subseteq \mathcal{F}$ is $2d$-union, then for all $B,B' \in \mathcal{B}$, we have

$$\dim(B \cap B') = \dim B + \dim B' - \dim(B + B') \geq 2d + 2 - 2d = 2,$$

i.e., $\mathcal{B}$ is a 2-intersecting family. Similarly, we have that $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting. Then we apply Lemma 2.3 for $k = d$ to obtain
\[ |A| + |B| \leq \binom{n}{d} - q^{d+1} \binom{n - d - 1}{d} + 1. \]  

(19)

Since for \( i \geq 2d + 1 \), \( F_i = \emptyset \), we can add up (18) along with (19) to obtain (1) by (15). Moreover, if the equality in (11) holds, then the equalities in (18) and (19) hold as well. Then by Lemmas 2.5 and 2.9, we have

\[ F = \{ F \leq V : \dim(F) \leq d \} \setminus \{ F \subseteq \binom{V}{d} : \dim(F \cap U) = 0 \} \cup \{U\}. \]

where \( U \) is a fixed \((d + 1)\)-dimensional subspace of \( V \).

\[ \square \]

4 Proofs of Theorems 1.5 and 1.6

In this section, we will prove Theorems 1.5 and 1.6. The main approach adopts a series of replacement in an \( s \)-union antichain by shadows or shades. First, we will define the concept of shade and disclose a new relationship between a family of \( k \)-dimensional subspaces and its shadows or shades.

For a family \( \mathcal{H} \subseteq \binom{V}{k} \), we define its shade by

\[ \triangledown(\mathcal{H}) = \{ G \in \binom{V}{k+1} : H \leq G \text{ for some } H \in \mathcal{H} \}. \]

Lemma 4.1. Suppose that \( \mathcal{H} \subseteq \binom{V}{k} \), \( n \geq 3 \). Then the following hold.

(i) If \( k \geq \left\lceil \frac{n}{2} \right\rceil + 1 \), then \( |\nabla(\mathcal{H})| - |\mathcal{H}| \geq q \left\lceil \frac{k-1}{1} \right\rceil \). Moreover, equality holds if and only if \( \mathcal{H} = \{U\} \), where \( U \) is a fixed \( k \)-dimensional subspace of \( V \).

(ii) If \( k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \), then \( |\nabla(\mathcal{H})| - |\mathcal{H}| \geq q \left\lceil \frac{n-k-1}{1} \right\rceil \). Moreover, equality holds if and only if \( \mathcal{H} = \{U\} \), where \( U \) is a fixed \( k \)-dimensional subspace of \( V \).

Proof. (i) Let \( |\mathcal{H}| = \left\lceil \frac{x}{k} \right\rceil \), where \( k \leq x \leq n \leq 2k - 2 \). Then by Theorem 2.4, \( |\triangle(\mathcal{H})| \geq \left\lceil \frac{x}{k-1} \right\rceil \). So we have

\[ |\triangle(\mathcal{H})| - |\mathcal{H}| \geq \left\lceil \frac{x}{k-1} \right\rceil - \left\lceil \frac{x}{k} \right\rceil = \frac{q^k - q^{x-k+1}}{q^{k-1} - 1} \left\lceil \frac{x}{k-1} \right\rceil = \frac{q^k - q^{x-k+1}}{q^{k-1} - 1} \prod_{i=0}^{k-2} \frac{q y^{q^{i-1}} - 1}{q^{k-1} - 1}. \]

Let \( f(x) \) be the RHS of the above inequality. By setting \( y = q^x \), we can rewrite \( f(x) \) as a polynomial \( g(y) \) of degree \( k \), namely

\[ g(y) = \frac{q^k y - q^{x-k+1}}{q^{k-1} - 1} \prod_{i=0}^{k-2} \frac{y q^{i-1} - 1}{q^{k-1} - 1}. \]

Because the polynomial \( g(y) \) has \( k \) simple roots \( 1, q, \ldots, q^{k-2}, q^{2k-1} \), \( f(x) \) has \( k \) simple roots \( 0, 1, \ldots, k - 2, 2k - 1 \). It is clear that \( f'(x) \) has a simple root in each interval between these roots. Since \( f(x) < 0 \) if \( x > 2k - 1 \), then in \([k-2, 2k-1]\), \( f(x) \) is increasing up to some value and then decreasing. Hence for \( k \leq x \leq n \leq 2k - 2 \),

\[ f(x) \geq \min\{ f(k), f(2k-2) \}. \]
Clearly, we have
\[
f(2k - 2) - f(k) = \left[ \frac{2k-2}{k-1} \right] - \left[ \frac{2k-2}{k} \right] - \left[ \frac{k}{k-1} \right] + 1
\]
\[
= q^k - q^{k-1} \left[ \frac{2k-2}{k-2} \right] - q \left[ \frac{k-1}{1} \right]
\]
\[
> \left[ \frac{2k-2}{k-2} \right] - q^k
\]
\[
> 0.
\]

Therefore,
\[
f(x) \geq f(k) = q \left[ \frac{k-1}{1} \right].
\]

The equality holds if and only if \(x = k\), that is \(|\mathcal{H}| = 1\). Equivalently, \(\mathcal{H} = \{U\}\), where \(U\) is a fixed \(k\)-dimensional subspace of \(V\).

(ii) We claim that \(|\bigtriangledown(\mathcal{H})| = |\bigtriangleup(\mathcal{H})|\). For any \(G \in \bigtriangledown(\mathcal{H})\), there exists \(H \in \mathcal{H}\) such that \(H \leq G\). Then \(G^\perp \leq H^\perp\), where \(\dim(G^\perp) = n - k - 1\), \(\dim(H^\perp) = n - k\), and \(H^\perp \in \mathcal{H}^\perp\). Thus \(G^\perp \in \bigtriangleup(\mathcal{H}^\perp)\). This gives an injective map \(\varphi: \bigtriangledown(\mathcal{H}) \rightarrow \bigtriangleup(\mathcal{H}^\perp)\) by \(G \mapsto G^\perp\), which is obviously surjective.

Let \(k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1\). Then \(n - k \geq \left\lceil \frac{n}{2} \right\rceil + 1\). We can obtain the following inequality by the above claim and the result of (i):
\[
|\bigtriangledown(\mathcal{H})| - |\mathcal{H}| = |\bigtriangleup(\mathcal{H}^\perp)| - |\mathcal{H}^\perp| \geq q \left[ \frac{n - k - 1}{1} \right].
\]

Moreover, the equality holds if and only if \(\mathcal{H}^\perp = \{U^\perp\}\), that is \(\mathcal{H} = \{U\}\), where \(U\) is a fixed \(k\)-dimensional subspace of \(V\).

Throughout the remainder of the section, let \(\mathcal{F} \subseteq \mathcal{L}(V)\) be a given \(s\)-union antichain and let \(d = \left\lfloor \frac{s}{2} \right\rfloor\). For any family \(\mathcal{G} \subseteq \mathcal{L}(V)\), define
\[
l(\mathcal{G}) = \max\{\dim(G) : G \in \mathcal{G}\},
\]
\[
m(\mathcal{G}) = \min\{\dim(G) : G \in \mathcal{G}\}.
\]

For short \(l(\mathcal{F})\), \(m(\mathcal{F})\) are briefly denoted by \(l\) and \(m\) respectively. Denote \(\mathcal{F}_i = \mathcal{F} \cap \left[ \begin{array}{c} V \\ i \end{array} \right] \) for \(m \leq i \leq l\).

**Lemma 4.2.** If \(l \leq d\), then the following hold.

(i) \(|\mathcal{F}| \leq \left\lceil \frac{n}{d} \right\rceil\). Moreover, equality holds if and only if \(\mathcal{F} = \left[ \begin{array}{c} V \\ d \end{array} \right] \).

(ii) If \(\mathcal{F} \nsubseteq \left[ \begin{array}{c} V \\ d \end{array} \right]\), then \(|\mathcal{F}| \leq \left\lceil \frac{n}{d} \right\rceil - q \left\lceil \frac{n-d}{1} \right\rceil\). Moreover, equality holds if and only if \(\mathcal{F} = \left( \left[ \begin{array}{c} V \\ d \end{array} \right] \setminus \{F \in \left[ \begin{array}{c} V \\ d \end{array} \right] : U \leq F\} \right) \cup \{U\}\), where \(U\) is a fixed \((d-1)\)-dimensional subspace of \(V\).
Proof. (1) Suppose \( m = l < d \). Then \(|\mathcal{F}| \leq \left[ \frac{n}{d} \right] \leq \left[ \frac{n}{d-1} \right]\). If \( d = 1 \), \(|\mathcal{F}| = 1 = \left[ \frac{n}{1} \right] - q \left[ \frac{n-1}{1} \right]\). If \( d \geq 2 \), we have

\[
\left[ \frac{n}{d} \right] - q \left[ \frac{n-d}{1} \right] - \left[ \frac{n-d}{d-1} \right] = \frac{q}{q-1} \left[ \frac{n-d+1}{1} \right] - q \left[ \frac{n-d}{1} \right]
\]

\[
\left[ \frac{n}{d} \right] - q \left[ \frac{n-d}{1} \right] > \left[ \frac{n-1}{d-1} \right] - q \left[ \frac{n-d}{1} \right]
\]

\[
\left[ \frac{n}{d} \right] - q \left[ \frac{n-d}{1} \right] > \frac{q^{n-1}}{q^{d-1}} - q^{n-d+1}
\]

\[
> 0.
\]

Hence,

\[
|\mathcal{F}| \leq \left[ \frac{n}{d-1} \right] < \left[ \frac{n}{d} \right] - q \left[ \frac{n-d}{1} \right].
\]

(2) Suppose \( m = l = d \). Then \(|\mathcal{F}| \leq \left[ \frac{n}{d} \right]\), and equality holds if and only if \( \mathcal{F} = \left[ \frac{V}{d} \right] \).

(3) Suppose \( m < l \leq d \). Let \( \mathcal{H} = \mathcal{F}_m \), \( \mathcal{G}_1 = (\mathcal{F} \setminus \mathcal{H}) \cup \bigvee (\mathcal{H}) \). It is clear that \( \mathcal{G}_1 \) is also an \( s \)-union antichain in this case and \((\mathcal{F} \setminus \mathcal{H}) \cap \bigvee (\mathcal{H}) = \emptyset\). Then by Lemma 4.1 we have

\[
|\mathcal{G}_1| = |\mathcal{F}| - |\mathcal{H}| + |\bigvee (\mathcal{H})| \geq |\mathcal{F}| + q \left[ \frac{n-m-1}{1} \right].
\]

If \( m(\mathcal{G}_1) = m + 1 < l \), then let \( \mathcal{H}_1 = \mathcal{G}_1 \cap \left[ \frac{V}{m(\mathcal{G}_1)} \right] \) and \( \mathcal{G}_2 = (\mathcal{G}_1 \setminus \mathcal{H}_1) \cup \bigvee (\mathcal{H}_1) \). After this we obtain an \( s \)-union antichain \( \mathcal{G}_2 \) for which by Lemma 4.1

\[
|\mathcal{G}_2| = |\mathcal{G}_1| - |\mathcal{H}_1| + |\bigvee (\mathcal{H}_1)|
\]

\[
\geq |\mathcal{G}_1| + q \left[ \frac{n-m(\mathcal{G}_1)-1}{1} \right]
\]

\[
\geq |\mathcal{F}| + q \left[ \frac{n-m-1}{1} \right] + q \left[ \frac{n-m-2}{1} \right].
\]

Repeat doing like this until we raise the minimum dimension of the spaces in \( \mathcal{F} \) to \( l \), and we obtain an \( s \)-union antichain \( \mathcal{G}_{l-m} \) satisfying

\[
|\mathcal{G}_{l-m}| \geq |\mathcal{F}| + q \left[ \frac{n-m-1}{1} \right] + q \left[ \frac{n-m-2}{1} \right] + \cdots + q \left[ \frac{n-l}{1} \right].
\]

Since \( \mathcal{G}_{l-m} \leq \left[ \frac{V}{l} \right], l \leq d \), then \( |\mathcal{G}_{l-m}| \leq \left[ \frac{n}{d} \right] \). Since \( q \left[ \frac{n-i}{1} \right] \leq q \left[ \frac{n-i}{1} \right] \) for \( m + 1 \leq i \leq l \) and \( m < l \leq d \), we have

\[
|\mathcal{F}| \leq \left[ \frac{n}{d} \right] - q \left[ \frac{n-m-1}{1} \right] - q \left[ \frac{n-m-2}{1} \right] - \cdots - q \left[ \frac{n-l}{1} \right]
\]

\[
\leq \left[ \frac{n}{d} \right] - q \left[ \frac{n-m-1}{1} \right]
\]

\[
\leq \left[ \frac{n}{d} \right] - q \left[ \frac{n-l}{1} \right]
\]

\[
\leq \left[ \frac{n}{d} \right] - q \left[ \frac{n-d}{1} \right].
\]

(21)
Moreover, equality holds if and only if \( m + 1 = l = d \), \( \mathcal{G}_1 = \binom{V}{d} \) and the equality in (20) holds. That is \( \mathcal{H} = \{U\} \), where \( U \) is a fixed \((d - 1)\)-dimensional subspace of \( V \) by Lemma 4.4.1. Thus

\[
\mathcal{F} = (\mathcal{G}_1 \setminus \vee(\mathcal{H})) \cup \mathcal{H} = (\binom{V}{d} \setminus \{F \in \binom{V}{d}: U \leq F\}) \cup \{U\}.
\]

Combining (1)-(3) yields part (i) of the lemma. Note that if \( \mathcal{F} \not\subseteq \binom{V}{d} \) then \( m = l < d \) or \( m < l \leq d \) and the arguments in (1) and (3) apply. This proves part (ii).

**Lemma 4.3.** If \( l \geq d + 1 \), then the following hold.

(i) If \( m < d < l \), then \(|\mathcal{F}| \leq \binom{n}{d} - q \binom{n-d}{1} \) and equality holds only if \( s \) is odd.

(ii) If \( m = d < l \), then

\[
|\mathcal{F}| \leq \begin{cases} 
\binom{n}{d} - q^{d+1} \binom{n-d-1}{d} + 1, & \text{if } s = 2d < n, \\
\binom{n}{d}, & \text{if } s = 2d + 1 \leq n.
\end{cases}
\]

(iii) If \( d < m \leq l \), then

\[
|\mathcal{F}| \leq \begin{cases} 
\binom{s}{d+1}, & \text{if } n \leq 2m, \\
\binom{n+s-2d-2}{s-d-1}, & \text{if } n > 2m.
\end{cases}
\]

**Proof.** Let \( \mathcal{D} = \mathcal{F}_l, \mathcal{F}^1 = (\mathcal{F} \setminus \mathcal{D}) \cup \Delta(\mathcal{D}) \). It is obvious that \( \mathcal{F}^1 \) is also an \( s \)-union antichain and \((\mathcal{F} \setminus \mathcal{D}) \cap \Delta(\mathcal{D}) = \emptyset \). Since \( \mathcal{F} \) is an \( s \)-union antichain and \( l \geq d + 1 \), then for any \( D, D' \in \mathcal{D} \), we have

\[
\dim(D \cap D') = 2l - \dim(D + D') \geq 2l - s \geq 1.
\]

By Theorem 2.3 we have \(|\Delta(\mathcal{D})| \geq |\mathcal{D}|\). Then

\[
|\mathcal{F}^1| = |\mathcal{F}| - |\mathcal{D}| + |\Delta(\mathcal{D})| \geq |\mathcal{F}|.
\]

Note that the inequality is strict if either \( s = 2d \) or \( s = 2d + 1 \) and \( l \geq d + 2 \). If \( l(\mathcal{F}^1) = l - 1 \geq \max\{d+1, m+1\} \), then let \( \mathcal{D}_1 = \mathcal{F}^1 \cap \binom{V}{l(\mathcal{F}^1)}, \mathcal{F}^2 = (\mathcal{F}_1 \setminus \mathcal{D}_1) \cup \Delta(\mathcal{D}_1) \). After this we obtain an \( s \)-union antichain \( \mathcal{F}^2 \) for which by Theorem 2.3

\[
|\mathcal{F}^2| = |\mathcal{F}^1| - |\mathcal{D}_1| + |\Delta(\mathcal{D}_1)| \geq |\mathcal{F}^1| \geq |\mathcal{F}|.
\]

(i) Suppose \( m < d < l \). Repeat the above process until we decrease the maximum dimension of the spaces in \( \mathcal{F} \) to \( d \).

If \( s = 2d \), we obtain a \( 2d \)-union antichain \( \mathcal{F}^{l-d} \) satisfying

\[
|\mathcal{F}^{l-d}| > |\mathcal{F}^{l-d-1}| > \cdots > |\mathcal{F}^1| > |\mathcal{F}|.
\]

Since \( m = m(\mathcal{F}^{l-d}) < l(\mathcal{F}^{l-d}) = d < l \), then replacing \( \mathcal{F} \) with \( \mathcal{F}^{l-d} \) in (3) of the proof of Lemma 4.2 and applying (21), we have

\[
|\mathcal{F}| < |\mathcal{F}^{l-d}| \leq \binom{n}{d} - q \binom{n-d}{1}.
\]
If \( s = 2d + 1 \), we obtain a \((2d + 1)\)-union antichain \( \mathcal{F}^{l-d} \) satisfying
\[
|\mathcal{F}^{l-d}| \geq |\mathcal{F}^{l-d-1}| > \cdots > |\mathcal{F}^1| > |\mathcal{F}|.
\]
Here note that if \( l = d + 1 \), then we have \( |\mathcal{F}^{l-d}| = |\mathcal{F}^1| \geq |\mathcal{F}| \). Similarly by (21), we have
\[
|\mathcal{F}| \leq |\mathcal{F}^{l-d}| \leq \left\lfloor \frac{n}{d} \right\rfloor - q^{d(d+1)} \left\lfloor \frac{n - d - 1}{d} \right\rfloor + 1.
\]

(ii) Suppose \( m = d < l \). Similarly as above but we decrease the maximum dimension of the spaces in \( \mathcal{F} \) to \( d + 1 \) if \( l > d + 1 \). Then we obtain an \( s \)-union antichain \( \mathcal{F}^{l-d-1} \) satisfying
\[
|\mathcal{F}^{l-d-1}| > |\mathcal{F}^{l-d-2}| > \cdots > |\mathcal{F}^1| > |\mathcal{F}|.
\]
If \( l = d + 1 \), let \( \mathcal{F}^{l-d-1} = \mathcal{F} \). It is clear that \( m(\mathcal{F}^{l-d-1}) = d \), \( l(\mathcal{F}^{l-d-1}) = d + 1 \). Let
\[
\mathcal{H} = \mathcal{F}^{l-d-1} \cap \left\lfloor \frac{V}{d} \right\rfloor, \quad \mathcal{G} = \mathcal{F}^{l-d-1} \cap \left\lfloor \frac{V}{d+1} \right\rfloor.
\]
If \( s = 2d \), then the \( 2d \)-union property of \( \mathcal{F}^{l-d-1} \), we have that \( \mathcal{H} \) and \( \mathcal{G} \) are cross-intersecting families and \( \mathcal{G} \) is a 2-intersecting family. Whenever \( n \geq 2d + 1 \), by Lemma 2.9 we have
\[
|\mathcal{F}| \leq |\mathcal{F}^{l-d-1}| = |\mathcal{H}| + |\mathcal{G}| \leq \left\lfloor \frac{n}{d} \right\rfloor - q^{d(d+1)} \left\lfloor \frac{n - d - 1}{d} \right\rfloor + 1.
\]

If \( s = 2d + 1 \), then by the \((2d + 1)\)-union property of \( \mathcal{F}^{l-d-1} \), we have that \( \mathcal{G} \) is an intersecting family. By Theorem 2.3 we have
\[
|\mathcal{F}| \leq |\mathcal{F}^{l-d-1}| = |\mathcal{H}| + |\mathcal{G}| \leq \left\lfloor \frac{n}{d} \right\rfloor - \left| \triangle (\mathcal{G}) \right| + |\mathcal{G}| \leq \left\lfloor \frac{n}{d} \right\rfloor.
\] (22)

(iii) Suppose \( m > d \). As before but we decrease the maximum dimension of the spaces in \( \mathcal{F} \) to \( m \) if \( l > m \). Now we obtain an \( s \)-union antichain \( \mathcal{F}^{l-m} \subseteq \left\lfloor \frac{V}{m} \right\rfloor \) satisfying
\[
|\mathcal{F}^{l-m}| > |\mathcal{F}^{l-m-1}| > \cdots > |\mathcal{F}^1| > |\mathcal{F}|.
\]
If \( l = m \), then let \( \mathcal{F}^{l-m} = \mathcal{F} \). By the \( s \)-union property of \( \mathcal{F}^{l-m} \), for any \( F, F' \in \mathcal{F}^{l-m} \), we have \( \dim(F \cap F') = 2m - \dim(F + F') \geq 2m - s \).

If \( n \leq 2m \), then by Theorem 2.1 and noting \( m > d \), we have
\[
|\mathcal{F}^{l-m}| \leq \left\lfloor \frac{s}{m} \right\rfloor \leq \left\lfloor \frac{s}{d + 1} \right\rfloor.
\]
If \( n > 2m \), then by Theorem 2.1 and noting \( s \geq l \geq m > d \), we have
\[
|\mathcal{F}^{l-m}| \leq \left\lfloor \frac{n + s - 2m}{s - m} \right\rfloor \leq \left\lfloor \frac{n + s - 2d - 2}{s - d - 1} \right\rfloor.
\]

Proof of Theorem 1.5  Obviously, we have the two assertions in Lemma 4.2 for the case \( l \leq d \). Next, we give new upper bounds of \( |\mathcal{F}| \) for the case \( l > d \) by similar approach of Lemma 4.3. Now \( d = \left\lfloor \frac{n}{2} \right\rfloor \).
(1) Suppose \( m < d < l \). By Lemma 4.3 (i), if \( n = 2d \), then \( |\mathcal{F}| \leq \left[ \frac{n}{d} \right] - q\left[ \frac{d}{1} \right] \); if \( n = 2d + 1 \), \( |\mathcal{F}| \leq \left[ \frac{n}{d} \right] - q\left[ \frac{n-d}{1} \right] \leq \left[ \frac{n}{d} \right] - q\left[ \frac{d}{1} \right] \).

(2) Suppose \( m = d < l \). Similarly as the proof of Lemma 4.3 (ii), we obtain an antichain \( \mathcal{F}^{l-d-1} \) satisfying \( |\mathcal{F}| \leq |\mathcal{F}^{l-d-1}| \), \( m(\mathcal{F}^{l-d-1}) = d \) and \( l(\mathcal{F}^{l-d-1}) = d + 1 \). Let \( \mathcal{G} = \mathcal{F}^{l-d-1} \cap \left[ \frac{V}{d+1} \right] \).

Case a: If \( n = 2d \), let \( \mathcal{M} = (\mathcal{F}^{l-d-1} \setminus \mathcal{G}) \cup \triangle(\mathcal{G}) \). Then we obtain an antichain \( \mathcal{M} \) satisfying

\[
|\mathcal{M}| \leq \left[ \frac{n}{d} \right].
\]

By Lemma 4.1 (i), we have

\[
|\triangle(\mathcal{G})| - |\mathcal{G}| \geq q\left[ \frac{d}{1} \right].
\]

Hence,

\[
|\mathcal{F}| \leq |\mathcal{F}^{l-d-1}| = |\mathcal{M}| + |\mathcal{G}| - |\triangle(\mathcal{G})| \leq \left[ \frac{n}{d} \right] - q\left[ \frac{d}{1} \right].
\]

Moreover, the equality holds if and only if \( l = d + 1 \), \( \mathcal{M} = \left[ \frac{V}{d} \right] \), \( \mathcal{G} = \{W\} \), where \( W \) is a fixed \((d+1)\)-dimensional subspace of \( V \), that is

\[
\mathcal{F} = (\mathcal{M} \setminus \triangle(\{W\})) \cup \{W\} = B[n,n].
\]

Case b: If \( n = 2d + 1 \), let \( \mathcal{H} = \mathcal{F}^{l-d-1} \cap \left[ \frac{V}{d} \right] \). Since \( \mathcal{F}^{l-d-1} \) is an antichain, we have that \( \mathcal{H}, \mathcal{G} \) are cross-Sperner. Then by Theorem 2.7

\[
|\mathcal{F}| \leq |\mathcal{F}^{l-d-1}| = |\mathcal{H}| + |\mathcal{G}| \leq \left[ \frac{n}{d} \right] - q\left[ \frac{d}{1} \right].
\]

Moreover, equality holds if and only if \( l = d + 1 \) and either \( \mathcal{F} = A[n,n] \) or \( \mathcal{F} = B[n,n] \).

(3) Suppose \( d < m \leq l \).

Case a: Let \( n = 2d \). Now we have \( n < 2m \). Then by Lemma 4.3 (iii), \(|\mathcal{F}| \leq \left[ \frac{2d}{d+1} \right] < \left[ \frac{2d}{d} \right] - q\left[ \frac{d}{1} \right] \).

Case b: Let \( n = 2d + 1 \).

If \( d < m < l \), then \(|\mathcal{F}| \leq \left[ \frac{2d+1}{m} \right] \leq \left[ \frac{2d+1}{d+1} \right] \). Moreover, equality holds if and only if \( \mathcal{F} = \left[ \frac{V}{d+1} \right] \). Further, if \( \mathcal{F} \not\subset \left[ \frac{V}{d+1} \right] \), then \(|\mathcal{F}| \leq \left[ \frac{2d+1}{d+2} \right] < \left[ \frac{2d+1}{d} \right] - q\left[ \frac{d}{1} \right] \).

If \( d < m < l \), similarly as the proof of Lemma 4.3 (iii) but we decrease the maximum dimension of the spaces in \( \mathcal{F} \) to \( m + 1 \) if \( l > m + 1 \), then we obtain an antichain \( \mathcal{F}^{l-m-1} \) fulfilling

\[
|\mathcal{F}^{l-m-1}| > \cdots > |\mathcal{F}^{1}| > |\mathcal{F}|.
\]

If \( l = m + 1 \), then let \( \mathcal{F}^{l-m-1} = \mathcal{F} \). Let \( \mathcal{G} = \mathcal{F}^{l-m-1} \cap \left[ \frac{V}{m+1} \right] \) and \( \mathcal{N} = (\mathcal{F}^{l-m-1} \setminus \mathcal{G}) \cup \triangle(\mathcal{G}) \). By Lemma 4.1 (i), we have

\[
|\triangle(\mathcal{G})| - |\mathcal{G}| \geq q\left[ \frac{m}{1} \right].
\]
Hence,

$$|F| \leq |F^{t-m-1}| = |N| + |G| - |\triangle(G)| \leq \left\lfloor \frac{2d+1}{m} \right\rfloor - q \left\lfloor \frac{m}{1} \right\rfloor < \left\lfloor \frac{2d+1}{d} \right\rfloor - q \left\lfloor \frac{d}{1} \right\rfloor.$$ 

To sum up, an optimal antichain satisfies $|F| \leq \left\lfloor \frac{n}{d} \right\rfloor$ and equality occurs in (3) Case b or Lemma 4.2 (i); a suboptimal antichain has $|F| \leq \left\lfloor \frac{n}{d} \right\rfloor - q \left\lfloor \frac{n}{1} \right\rfloor$ and equality occurs in (2) or Lemma 4.2 (ii) if $n = 2d$. This completes the proof. □

**Proof of Theorem 1.6.** We divide the proof into two parts, according to the singularity of $s$.

(1) Suppose $s = 2d$. The case of $d = 1$ is trivial and has been explained in Section 1. Hence, we only need to consider $d \geq 2$ in the following. By Lemmas 4.2 and 4.3, it suffices to show that the upper bounds provided in (ii) and (iii) of Lemma 4.3 are strictly smaller than $\left\lfloor \frac{n}{d} \right\rfloor - q \left\lfloor \frac{n-d}{1} \right\rfloor$.

Case a: It is readily checked that

$$q^{d(d+1)} \left\lfloor \frac{n-d-1}{d} \right\rfloor - 1 - q \left\lfloor \frac{n-d}{1} \right\rfloor = q^{d(d+1)} \left\lfloor \frac{n-d-1}{d} \right\rfloor - q^{n-d+1} \geq q^{d(d+1)} \left\lfloor \frac{n-d-1}{d} \right\rfloor - q^{n-d+1} \geq \frac{q^{d(d+1)}(q^{n-d-1}-1)}{q^d-1} - q^{n-d+1} \geq q^{d+2}(q^{n-d-1} - 1) - q^{n-d+1} > 0.$$ 

Hence,

$$\left\lfloor \frac{n}{d} \right\rfloor - q^{d(d+1)} \left\lfloor \frac{n-d-1}{d} \right\rfloor + 1 < \left\lfloor \frac{n}{d} \right\rfloor - q \left\lfloor \frac{n-d}{1} \right\rfloor.$$ 

Case b: Noting $n > s = 2d$, $\left\lfloor \frac{n}{d-1} \right\rfloor > q^{n-d+1} > q \left\lfloor \frac{n-d}{1} \right\rfloor$, we have

$$\left\lfloor \frac{n}{d} \right\rfloor - q \left\lfloor \frac{n-d}{1} \right\rfloor - \left\lfloor \frac{2d}{d+1} \right\rfloor = \frac{q^{n-d+1}-1}{q^d-1} \left\lfloor \frac{n}{d-1} \right\rfloor - q^{2d} - q \left\lfloor \frac{n-d}{1} \right\rfloor > 0.$$ 

Hence,

$$\left\lfloor \frac{2d}{d+1} \right\rfloor < \left\lfloor \frac{n}{d} \right\rfloor - q \left\lfloor \frac{n-d}{1} \right\rfloor.$$ 

Case c: Since $\left\lfloor \frac{a}{k} \right\rfloor = q^{a-k} \left\lfloor \frac{a-1}{k-1} \right\rfloor + \left\lfloor \frac{a-1}{k} \right\rfloor$ for $a \geq k + 1$ and note $d \geq 2$, we have
\[
\begin{align*}
\binom{n}{d} - q \binom{n-d}{1} - \binom{n-2}{d-1} &= q^{n-d} \binom{n-1}{d-1} + \binom{n-1}{d} - q \binom{n-d}{1} - \binom{n-2}{d-1} \\
&= q^{n-d} \binom{n-1}{d-1} + q^{n-d-1} \binom{n-2}{d-1} + \binom{n-2}{d} - q \binom{n-d}{1} - \binom{n-2}{d-1} \\
&> q^{n-d} \binom{n-1}{d-1} + \binom{n-2}{d} - q \binom{n-d}{1} \\
&> 0.
\end{align*}
\]

Hence,
\[
\binom{n-2}{d-1} < \binom{n}{d} - q \binom{n-d}{1}.
\]

(2) Suppose \(s = 2d + 1\). It is clear that the upper bounds provided in Lemma 4.3 (i) and (iii) are strictly smaller than \(\binom{n}{d}\); and the equality in Lemma 4.3 (ii) holds if and only if (22) holds, that is \(F = H \cup G\), where \(G \subseteq \left[\frac{V}{d+1}\right]\), \(H = \left[\frac{V}{d}\right] \triangle (G)\) and \(|\triangle (G)| = |G|\).

Then by Lemma 4.2 we complete the proof.

5 Concluding remarks

In the present paper, we determine all suboptimal \(s\)-union families for vector spaces. For \(s = n\) or \(s = 2d < n\), we determine all optimal and suboptimal \(s\)-union antichains completely. For \(s = 2d + 1 < n\), we prove that an optimal \(s\)-union antichain is either \(\left[\frac{V}{d}\right]\)

or \(F = F_d \cup F_{d+1}\), where \(F_{d+1} \subseteq \left[\frac{V}{d+1}\right]\), \(F_d = \left[\frac{V}{d}\right] \triangle (F_{d+1})\) and \(|\triangle (F_{d+1})| = |F_{d+1}|\).

It is very interesting to display all optimal \((2d+1)\)-union antichains of the latter type. Obviously, \(F = \left(\left[\frac{V}{d}\right] \setminus \left[\frac{s}{d}\right]\right) \cup \left[\frac{s}{d+1}\right]\) satisfies the above condition, where \(S\) is a fixed \((2d+1)\)-dimensional subspace of \(V\). Consulting the situation of \(s\)-union antichains in an \(n\)-element set, we conjecture that this is the unique desired structure (of the latter type).

Let \(F \subseteq \mathcal{L}(V)\) be a \((2d+1)\)-union antichain with \(F\) not contained in any optimal \((2d+1)\)-union antichain. By Lemma 4.2 if \(l \leq d\), we have

\[
|F| \leq \binom{n}{d} - q \binom{n-d}{1} < \binom{n}{d} - q \binom{d}{1}.
\]

It is obvious that the upper bounds provided in Lemma 4.3 (i) and (iii) are strictly smaller than \(\binom{n}{d} - q \binom{d}{1}\). From the proof of Lemma 4.3 (ii), we know that a suboptimal \((2d+1)\)-union antichain has the form \(F = F_d \cup F_{d+1}\), where \(F_{d+1} \subseteq \left[\frac{V}{d+1}\right]\), \(F_d = \left[\frac{V}{d}\right] \triangle (F_{d+1})\) and \(|\triangle (F_{d+1})| > |F_{d+1}|\). In view of Theorem 1.5 we make the following conjecture.

**Conjecture 5.1.** Let \(F \subseteq \mathcal{L}(V)\) be a \((2d+1)\)-union antichain with \(F\) not contained in any optimal \((2d+1)\)-union antichain and \(2d+1 < n\). Then

\[
|F| \leq \binom{n}{d} - q \binom{d}{1}.
\]

Moreover, equality holds if and only if \(F = B[n, 2d + 1]\).
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