Anomalous chiral transports and spin polarization in heavy-ion collisions

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The relativistic heavy-ion collisions create hot quark-gluon plasma as well as very strong electromagnetic and fluid vortical fields. The strong electromagnetic field and vorticity can induce intriguing macroscopic quantum phenomena such as chiral magnetic effect, chiral separation effect, chiral electric separation effect, chiral vortical effects, and spin polarization of hadrons. These phenomena provide us the experimentally feasible means to study the nontrivial topological sector of the quantum chromodynamics, the possible parity violation of strong interaction at high temperature, and the subatomic spintronics of quark-gluon plasma. These studies, both in theory and in experiment of heavy-ion collisions, are strongly entangled with other subfields of physics, such as condensed matter physics, astrophysics, and cold atomic physics, and thus form an emerging interdisciplinary research area. We will give an introduction to the above phenomena induced by electromagnetic field and vorticity and an overview about the current status of their experimental search in heavy-ion collisions. We also briefly discuss the spin hydrodynamics and the chiral and spin kinetic theories.

I. INTRODUCTION

As is well known, it is the strong interaction that binds the quarks and gluons together to form the hadrons, like the protons and neutrons. The contemporary theory of strong interaction is the quantum chromodynamics (QCD). It is a $SU(3)$ quantum gauge theory. The non-Abelian nature of QCD has important consequences such as the color confinement at low energy scale and the asymptotic freedom at high energy scale. The color confinement means that at low energy scales the color carriers, i.e. quarks and gluons, are always confined in color singlet hadrons; thus no isolated quark and gluon can be observed. However, when the energy scale grows, e.g. when the temperature or the baryon density of the hadronic matter is increased, QCD would undergo a deconfinement phase transition and quarks and gluons would be liberated from the hadrons. When the energy scale is very high, the coupling constant of QCD becomes small and the system goes into the perturbative regime of QCD; in this regime the coupling constant decreases with increasing energy scale, a phenomenon called asymptotic freedom. Reliable perturbative calculation can apply in this regime.

In reality, the conditions for the deconfinement phase transition is hard to achieve. In fact, the confinement energy scale of QCD is about $\Lambda_{\text{QCD}} \sim 200$ MeV which, in terms of temperature, is about $T_c \sim \Lambda_{\text{QCD}} \sim 10^{12}$ K. Such a high temperature may once exist in the early universe right
after the Big Bang and can so far only be realized on earth experimentally by relativistic heavy-ion collisions. The currently operating facilities of heavy-ion collisions include the Relativistic Heavy-Ion Collider (RHIC) at Brookhaven National Laboratory of United States of America and the Large Hadron Collider (LHC) at European Organization for Nuclear Research (CERN). RHIC has been in operation since 2000 and its current top colliding energy for Au + Au collisions is $\sqrt{s} = 200$ GeV. LHC has been in operation since 2010 and its current top colliding energy for Pb + Pb collisions is $\sqrt{s} = 5.02$ TeV. In these colliders, two counterpropagating beams of ions are accelerated to ultrahigh speed and then collide. The large kinematic energies of the ions are accumulated at the colliding point so that the transient energy density can be high enough to achieve the deconfinement phase transition. Such produced deconfined quark-gluon matter is usually called the quark-gluon plasma (QGP). The data collected at RHIC and LHC have shown strong evidence of the existence of QGP and also revealed a number of extraordinary properties of QGP. Here we list a few; more discussions can be found in, e.g. Ref. [1]. The QGP is considered as the “most perfect fluid” in the sense that its shear viscosity over entropy density ratio is the smallest among all the known fluids including the helium superfluid. The QGP can strongly quench the energetic jets (i.e., a particle or a collimated shower of particles of high transverse momenta), a phenomenon called jet quenching, which indicates that the energetic jets interact strongly with the constituents of QGP. The color force between two heavy quarks may be screened in QGP, similar to the usual Debye screening of the electric charges in electromagnetic (EM) plasmas, so that the heavy quarkonia, like the $J/\Psi$, are easily dissociated in QGP, leading to a suppression in the final measured yields.

In addition to the phenomena mentioned above, in recent years, people have realized that relativistic heavy-ion collisions can also generate strong EM field and fluid vorticity. More importantly, under the strong EM field and vorticity, a number of intriguing macroscopic quantum phenomena may occur. On one hand, these phenomena provide us opportunities to study the nontrivial chiral properties of the quark-gluon matter, especially those related to quantum anomaly, and the spin dynamics of QGP. On the other hand, these phenomena are closely related to other subfields of physics such as particle physics, condensed matter physics, astrophysics, and cold atomic physics, and thus give rise to a new interdisciplinary research area. Some review articles are already available, e.g. Refs. [2–9]. In the following, we will start with an introduction to the EM field and vorticity in heavy-ion collisions.

II. ELECTROMAGNETIC FIELD AND VORTICITY

Let us consider a non-central collision between two nuclei. The collision geometry is depicted in Fig. 1. The $z$ direction is along the motion of the projectile, $x$ direction is along the impact parameter $b$ (from the target to the projectile), and $y$ direction is along $\hat{z} \times \hat{x}$, the $x$-$z$ plane is the reaction plane. As the nucleus is positively charged, its motion generates an electric current which
in turn generates a magnetic field. At the moment of collision, due to the geometric symmetry, a magnetic field perpendicular to the reaction plane is produced at the collision center ($x = 0$). Let us estimate the strength of this magnetic field by using the Biot-Savart formula. For a Au + Au collision at $\sqrt{s} = 200$ GeV with $b = 10$ fm, we have

$$eB_y \approx -2Z_{\text{Au}}\gamma \frac{e^2}{4\pi} \frac{v_z}{(b/2)^2} \approx -10m_{\pi}^2 \approx -10^{19} \text{ Gauss},$$  

(1)

where $v_z = \sqrt{1 - (2m_N/\sqrt{s})^2} \approx 0.99995$ is the velocity of the nucleus in the laboratory frame with $m_N$ the nucleon mass, $\gamma = 1/\sqrt{1 - v_z^2} \approx 100$ is the Lorentz factor, and $Z_{\text{Au}} = 79$ is the proton number of Au nucleus.

This is a huge magnetic field. It is much larger than the squared masses of the electron and the light quarks ($u, d$ quarks) and thus may induce significant quantum effects in systems composed of electrons and light quarks. In fact, this is the strongest known magnetic field in the current universe, it is of several orders stronger than the surface magnetic fields of neutron stars including the magnetars ($eB \approx 10^{14} - 10^{15}$ Gauss) [10]. The result in Eq. (1) is very rough; more advanced simulations can be done using transport models like HIJING, AMPT, UrQMD, and so on [11–26]. In such simulations, one can determine the positions and momenta of each charged particle before and after the collision, and then use, e.g. the Lienard-Wiechert formula to calculate the EM fields. The possible quantum correction to the Lienard-Wiechert formula can be estimated (which was found to be insignificant) [3, 16]. Many aspects of the EM field were studied in this way, like the event-by-event fluctuations of the strength and orientation of the EM fields [13, 15, 16], the azimuthal correlation between EM field and the matter geometry [16, 17], the EM fields in different collision systems [18, 22], the influence of the charge distribution of nucleon [16, 17], and so on; see reviews [3, 4]. In Fig. 2, we show the impact parameter dependence of the EM fields computed using HIJING model for Au + Au and Pb + Pb collisions at RHIC and LHC energies, respectively. It is seen that the strength of the fields is roughly proportional to the collision energy $\sqrt{s}$ [15].

Let us consider once again a non-central collision of energy $\sqrt{s}$ and impact parameter $b$. The
system possesses an angular momentum

\[ J_y \approx -\frac{Ab\sqrt{s}}{2}, \]  

(2)

where \( A \) is the mass number of the nucleus. For RHIC Au + Au collisions at \( \sqrt{s} = 200 \) GeV and \( b = 10 \) fm, we obtain \( |J_y| \approx 10^6 \hbar \). Compared to the total spin of the produced hadrons (e.g. for a typical number of produced hadrons of 1000, the total spin would be \( \sim 10^3 \hbar \)), this is a huge angular momentum. After the collision, a part of this angular momentum is kept in the produced QGP. As the equation of state of the QGP is very soft, this part of the angular momentum does not lead to rigid rotation of the QGP but rather induces local fluid vortices. The strength of a fluid vortex is described by the vorticity. In non-relativistic hydrodynamics, the vorticity is defined by

\[ \omega = \frac{1}{2} \nabla \times v, \]  

(3)

where \( v \) is the flow velocity. From this definition, it is clear that the physical meaning of the vorticity is the local angular velocity of the fluid cell. In relativistic hydrodynamics, according to different physical contexts, different vorticities can be defined. Here we list several commonly used ones. They are the kinematic vorticity, the temperature vorticity, and the thermal vorticity. The kinematic vorticity is a natural generalization of the non-relativistic vorticity,

\[ \omega^\mu = \frac{1}{2} e^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma, \]  

(4)

where \( u^\mu = \gamma(1, v) \) is the flow four velocity. In many situations, it is more convenient to use its tensorial representation, \( \omega_{\mu\nu} = (1/2)(\partial_\nu u_\mu - \partial_\mu u_\nu) \), which is related \( \omega^\mu \) by \( \omega^\mu = -(1/2)e^{\mu\nu\rho\sigma} u_\nu \omega_{\rho\sigma} \). The temperature vorticity is defined as

\[ \omega_T^\mu = \frac{1}{2} e^{\mu\nu\rho\sigma} u_\nu \partial_\rho (Tu_\sigma), \]  

(5)
where $T$ is the temperature. The special property of the temperature vorticity is that, for ideal neutral fluid, it satisfies the Carter-Lichnerowicz equation $\omega_{\mu\nu}^{T} u^\nu = 0$, which yields two interesting consequences [27, 28]. One consequence is the relativistic Helmholtz-Kelvin theorem stating that the flow circulation, defined as $l(\tau) = \oint T u_\mu dx^\mu$, is a co-moving invariant of the fluid, $dl/d\tau = 0$. Another consequence is the conservation of $T \omega_{\mu}^{\nu}, \partial_{\mu}(T \omega_{\mu}^{\nu}) = 0$. The conserved charge $H_T = (1/2) \int d^3x T^2 \gamma^2 v \cdot \nabla \times v$ defines the relativistic fluid helicity which measures the degree of linkage of the vortex lines. The thermal vorticity, in the tensorial form, is defined as

$$\omega_{\mu\nu} = \frac{1}{2} [\partial_\nu (\beta u_\mu) - \partial_\mu (\beta u_\nu)],$$

where $\beta = 1/T$ is the inverse temperature. The importance of the thermal vorticity relies on the fact that it characterizes the global equilibrium of a rotating fluid, and determines the spin polarization of the constituent particles in the fluid at global thermal equilibrium [29, 30]. We will go into more details of the spin polarization is Sec. V.

In Fig. 3, we present the numerical results of the non-relativistic vorticity and the relativistic kinematic vorticity in Au + Au collisions at $\sqrt{s} = 200$ GeV based on HIJING simulation [28]. The results are averaged over the reaction region and over $10^5$ events; see more details in Ref [28]. As seen from Fig. 3, the vorticities grow with $b$ at $b < 2R_A$ ($R_A$ the radius of the nucleus) simply because the total angular momentum of the system increases and decrease at $b \geq 2R_A$ because of the shrinking of the reaction region. The numerical results show that the vorticity can be huge (with peak value $|\langle \omega_y \rangle| \sim 10 \text{ MeV} \sim 10^{21} \text{ s}^{-1}$). This is the strongest vorticity we have ever known. For this reason, sometimes we call QGP the “most vortical fluid” [31]. In Fig. 4, we show the numerical results for the time evolution of the thermal vorticity in Au + Au collisions for $\sqrt{s} = 19.6, 62.4$, and 200 GeV obtained using AMPT model [32]. It is quite natural that the vorticity decays in time due to the fire-ball expansion. However, it is surprising that the vorticity decreases when $\sqrt{s}$ increases; this is a relativistic effect which we will discuss later. The numerical simulations for the vorticities can also be found in, e.g. Refs. [27, 28, 32–38].
FIG. 4. The time evolution of the thermal vorticity in Au + Au collisions for several different collision energies. Figure is from Ref. [32].

III. CHIRAL ANOMALY AND TRANSPORT PHENOMENA

What are the consequences of the strong EM fields and vorticity in heavy-ion collisions? During the past decade or so, there have been many discussions about this question and a lot of interesting effects have been studied. Among them, perhaps the most intriguing ones are the quantum phenomena that are closely related to the spin dynamics of quarks. For massless fermions, such phenomena are also deeply related to chiral anomaly of QCD and QED, and can be called anomalous chiral transports (ACTs). For the massive case, the spin polarization of hyperons by vorticity is a remarkable example. Of course, in general, both the ACTs and spin polarization could occur for both massless and massive particles; but they are most manifestable for the massless and massive particles, respectively. In this section, we will focus on ACTs. The noticeable examples of ACTs are the chiral magnetic effect (CME), chiral vortical effects (CVEs), chiral separation effect (CSE), chiral electric separation effect (CESE), etc. We here give a pedagogical discussion about the underlying mechanisms of the ACTs [39, 40].

Consider a massless Dirac fermion of charge $e > 0$ in a strong constant magnetic field along $z$ direction. This is the usual Landau problem in quantum mechanics. The energy spectrum can be obtained by solving the Dirac equation and the result is presented as Landau levels,

$$\mathcal{E}_n^2 = p_z^2 + 2neB, \quad n = 0, 1, 2, \cdots, \quad (7)$$

where $n$ labels the Landau levels. The lowest Landau level (LLL), corresponding to $n = 0$, is special; see Fig. 5 (left). First, the LLL is gapless while all the higher Landau levels are gapped by $\sqrt{2neB}$. Thus for large $eB$, we only need to consider the LLL. Second, the spin of LLL is fully polarized, namely, the LLL is non-degenerate in spin. All the states of the LLL are of spin up. In
FIG. 5. (Left) Lowest Landau level in a strong magnetic field. (Right) The electric field induces spectral flow and results in the chiral anomaly.

In a many-body picture, this means that the LLL fermions are all of spin up. Third, the dynamics of LLL fermion is 1+1 dimensional because the transverse motion is frozen and $E_{\pi=0}$ is independent of $B$. We define the chirality for each LLL fermion according to its momentum direction relative to its spin direction. If $p_z$ is parallel to the spin, we call it a right-handed (RH) fermion; if $p_z$ is opposite to its spin, we call it a left-handed (LH) fermion. In this situation, the numbers of RH and LH fermions are conserved separately, i.e. \( \partial_\mu J^\mu_{R/L} = 0 \) with \( J^\mu_{R/L} = (1/2)\bar{\psi}\gamma_\mu(1 \pm \gamma_5)\psi \), or equivalently, \( \partial_\mu J^\mu_{V/A} = 0 \), where the vector and axial currents are defined as \( J^\mu_{V/A} = J^\mu_R \pm J^\mu_L \).

Now suppose an electric field is imposed in the same direction as the magnetic field; see Fig. 5 (right). Near the level crossing node, \( p_z = 0 \), the downward moving particles can be easily flipped by the electric field to move upward and thus some LH fermions are tuned to RH fermions. This is a typical spectral flow phenomenon. Therefore, although the total number of RH and LH fermions, \( N_V = N_R + N_L \), is still conserved, the difference \( N_A = N_R - N_L \) is not. We can calculate the time derivative of \( N_A \) in the following way. Let \( p^\mu_{R/L} \) denote Fermi momenta of the RH and LH fermions. We have

\[
N_{R/L} = V \frac{p^\mu_{F/L}}{2\pi} \frac{eB}{2\pi},
\]

where \( eB/(2\pi) \) is the transverse density of state and \( V \) is the volume of the system. The electric force gives \( \dot{p}^\mu_F = \pm eE \). Thus,

\[
\frac{dN_{R/L}}{dt} = V \frac{p^\mu_{F/L}}{2\pi} \frac{eB}{2\pi} = \pm V \frac{eE}{2\pi} \frac{eB}{2\pi},
\]

or equivalently \( dN_V/dt = 0 \) and \( dN_A/dt = Ve^2EB/(2\pi^2) \). Written in differential forms, they give \( \partial_\mu J^\mu_V = 0 \) and

\[
\partial_\mu J^\mu_A = \frac{e^2}{2\pi^2}E \cdot B.
\]

This is the well-known chiral or axial anomaly [41, 42]. We note that although we obtain Eq. (10) by considering strong magnetic field so that only the LLL is occupied, the result is actually true
for arbitrary magnetic field, as the higher Landau levels are degenerate in chirality and do not contribute to Eq. (10).

With the above preparation, we now remove the electric field and calculate the RH and LH currents along the magnetic field; see Fig. 6. A current is equal to the carrier density times the velocity of the constitute particles. For massless particle, the velocity is the speed of light so that

\[ J_{R/L} = \pm n_{R/L} = \pm \frac{p_{R/L}}{2\pi} \frac{eB}{2\pi}, \]  

where the minus sign is because LH fermions move opposite to the direction of the magnetic field. We can re-write Eq. (11) as

\[ J_V = \frac{p^R_F - p^L_F}{2\pi} \frac{eB}{2\pi} = \frac{\mu_A}{2\pi^2} eB, \]  

(12)

and

\[ J_A = \frac{p^R_F + p^L_F}{2\pi} \frac{eB}{2\pi} = \frac{\mu_V}{2\pi^2} eB, \]  

(13)

where we have defined the vector and axial chemical potentials as \( \mu_{V/A} = (p^R_F \pm p^L_F)/2 \). The current (12) is the CME current [43, 44] and the current (13) is the CSE current [45, 46] which appears even when \( p^R_F = p^L_F \). The CME exhibits very special properties. First, it is a macroscopic quantum effect. Second, its occurrence requires P and CP violation in the medium. Third, the generation of the CME current is time reversal even, namely, there is no associated entropy production. Thus the CME current is a sort of superconducting current. We also emphasize that the CME conductivity is fixed by the chiral anomaly and thus is free of renormalization.

In classical physics, Larmor theorem tells us that the motion of a charged particle of mass \( m \) in a magnetic field is equivalent to the motion in a rotating frame with frequency \( eB/(2m) \). This suggests the existence of analogous effects to CME and CSE but induced by rotation or vorticity. Consider a massless particle in a rotating frame. The particle feels a Coriolis force, \( F = 2p \times \)
ω + O(ω²), where ω is the rotating frequency. We have assumed that ω is so small that we neglect the centrifugal force which is O(ω²). As the Coriolis force is very similar to the Lorentz force (replacing eB by 2pω), we can consider the “Landau level problem” in rotating frame. Let us again consider only the LLL and consider a many-body system co-rotating with the frame. Comparing to the magnetic case, the only difference is that the expression for the density is modified: \( n_{R/L} = (2\pi)^{-2} \int_0^{\mu_{R/L}^f} dp_z 2p_z \omega = (\mu_{R/L}^f)^2 \omega / (2\pi)^2 \). Now the currents read

\[ J_V = n_R - n_L = \frac{\mu_V \mu_A}{\pi^2} \omega, \quad (14) \]

\[ J_A = n_R + n_L = \frac{\mu_V^2 + \mu_A^2}{2\pi^2} \omega. \quad (15) \]

They are the vector and axial CVEs [47–50]. More rigorous consideration shows that there is an additional term, \( T^2 \omega / 6 \), in \( J_A \) which may be related to the global gravitational anomaly [51, 52].

In Fig. 2, we see that, in addition to the strong magnetic field, heavy-ion collisions create also strong electric field due to the fluctuation of the proton distribution. In geometrically asymmetric collisions like Cu + Au collision, there can also exist strong electric field pointing from the Au nucleus to the Cu nucleus with strength comparable to the magnetic field [18, 53, 54]. The electric field can also lead to anomalous transport, i.e., the CESE [55]; see also derivation in holographic models [56, 57] and discussion in Weyl semimetal [58]. The CESE is not directly related to the chiral anomaly and its appearance requires both P and C violation. The CESE represents an axial current along the direction of the electric field. Its expression for two flavor QCD up to leading-log accuracy is given by [59]

\[ J_A \approx 14.5163 \text{Tr}(Q_e Q_A) \frac{\mu_V \mu_A}{T^2} \frac{eT}{g^4 \ln(1/g)} E, \quad (16) \]

where \( Q_e \) and \( Q_A \) are charge matrix and axial matrix in flavor space, \( g \) is the strong coupling constant. Of course, in addition to the CESE, the electric field induces also the Ohm current \( J_V = \sigma E \) where \( \sigma \) is the electric conductivity which, for QGP, is actually very large meaning that the QGP is a good conducting matter [60].

There emerge interesting collective modes from the coupled evolution of the axial and vector charges via CME and CSE, or vector CVE and axial CVE, or CESE and the usual Ohm’s law. For example, the continuity equations for vector and axial charges can be written in terms of RH and LH charges

\[ \partial_t f_{R/L}^0 + \nabla \cdot J_{R/L} = 0. \quad (17) \]

Substituting the CME and CSE expressions and considering small fluctuations in \( J_{R/L}^0 \) and \( \mu_{R,L} \), we obtain

\[ \partial_t \delta J_R^0 + \frac{e^2}{4\pi^2 \chi} B \cdot \nabla \delta J_R^0 = 0, \quad (18) \]

\[ \partial_t \delta J_L^0 - \frac{e^2}{4\pi^2 \chi} B \cdot \nabla \delta J_L^0 = 0, \quad (19) \]
TABLE I. Table of anomalous chiral transports

|       | $eE$       | $eB$       | $\omega$          |
|-------|-----------|-----------|-------------------|
| $J_V$ | $\sigma$  | $\frac{\mu_A}{2\pi^2}$ | $\frac{\mu_V\mu_A}{\pi^2}$ |
| $J_A$ | $\propto \frac{\mu_V\mu_A}{T^2}\sigma$ | $\frac{\mu_V}{2\pi^2}$ | $\frac{T^2}{6} + \frac{\mu_V^2 + \mu_A^2}{2\pi^2}$ |

Collective mode chiral electric wave chiral magnetic wave chiral vortical wave

where $\chi = \partial f^0_{R}/\partial \mu_R \approx \partial f^0_{L}/\partial \mu_L$ is the number susceptibility, and we keep only linear terms in fluctuations. These two equations express two collective, gapless, wave modes which are called the chiral magnetic waves (CMWs) [61]. Similarly, if we consider the CESE and the Ohm’s law, we can find new collective modes, the chiral electric waves and axial or vector density waves [55]; if we consider the vector and axial CVEs, we can find chiral vortical waves (CVWs) [62] described by $\partial_t \delta J^0_{R/L} \pm v_{CVW} \partial_z \delta J^0_{R/L} = 0$ with $v_{CVW} = \mu_{V0}/(2\pi^2\chi)$ the propagating velocity of the CVWs. Note that, different from the CMWs, the occurrence of the CVWs needs background vector density (characterized by $\mu_{V0}$). Finally, we summarize the ACTs (and the usual Ohm’s law) in Table I.

IV. ACTS IN HEAVY-ION COLLISIONS

The ACTs have attracted remarkable attention in many subfields of physics including nuclear physics, particle physics, astrophysics, condensed matter physics, atomic physics, and quantum optics. For heavy-ion collisions, in particular, the ACTs provide valuable means to detect the possible P and CP violation of QCD at high temperature. It is a well-known experimental fact that the strong interaction respects P and CP invariance in vacuum to very high precision, although QCD itself permits the existence of P and CP violating $\theta$ term. This lacks a natural explanation and is one of the main puzzles in contemporary physics. It was proposed that in the high-temperature environment created by heavy-ion collisions, metastable domains leading to P and CP violation could be produced through, e.g. sphaleron induced transition between gauge field vacua of different topological winding numbers [63–65]. In these domains, the interaction between gluons and quarks (via triangle anomaly) can induce chirality imbalance in quarks which can be characterized by the parameter $\mu_A$. Thus, the EM fields or vorticity exerting to these domains cause the CME, CVE, CESE, and so on. Therefore, the detection of ACTs is highly demanded in heavy-ion collisions.
A. Experimental search of CME

Because the magnetic field is roughly perpendicular to the reaction plane, the CME would drive a current that finally leads to a charge separation with respect to the reaction plane. However, the production of $\mu_A$ has strong spatial fluctuation (among the metastable P-violating domains) and event-by-event fluctuation so that the event-averaged CME-induced charge separation vanishes. What can be observed is the fluctuation of the charge separation. This can be done by designing appropriate hadronic observables. One commonly used observable is the $\gamma$-correlation introduced by Voloshin [66],

$$\gamma_{\alpha\beta} \equiv \langle \cos(\phi_\alpha + \phi_\beta - 2\Psi_{RP}) \rangle,$$

where $\alpha, \beta = \pm$ denote the charge signs, $\phi_\alpha$ and $\phi_\beta$ are the corresponding azimuthal angles, $\Psi_{RP}$ is the reaction plane angle, and $\langle \cdots \rangle$ is event average. It is easy to see that a charge separation with respect to the reaction plane results in positive $\gamma_{+ -}$ and $\gamma_{- +}$ (denoted as $\gamma_{OS}$) and negative $\gamma_{++}$ and $\gamma_{--}$ (denoted as $\gamma_{SS}$). In real experiments, one additional reference hadron (of arbitrary charge) is needed to determine $\Psi_{RP}$, so practically Eq. (20) is a three-particle correlation.

The correlation $\gamma_{\alpha\beta}$ was first measured by STAR Collaboration at RHIC for Au + Au collisions at $\sqrt{s} = 200$ GeV [67, 68]; see Fig. 7. The same quantity was also measured by ALICE Collaboration at LHC for Pb + Pb collisions at $\sqrt{s} = 2.76$ TeV [69, 70], by CMS Collaboration at LHC for Pb+Pb collisions at $\sqrt{s} = 5.02$ TeV [71, 72], and by STAR Collaboration for Au + Au collisions at different beam energies down to $\sqrt{s} = 19.6$ GeV [73]. For mid-central collisions, these measurements show positive $\gamma_{OS}$ and negative $\gamma_{SS}$ with features consistent with the expectation of CME. However, there are non-CME background effects in the $\gamma$-correlation, noticeably the transverse momentum conservation (TMC) and local charge conservation (LCC). Before we have a convincing way to subtract these backgrounds, we cannot claim the observation of the CME. The TMC induces a back-to-back correlation to $\gamma_{\alpha\beta}$ [74, 75] which can be subtracted by making a difference $\Delta \gamma \equiv \gamma_{OS} - \gamma_{SS}$ as the TMC is charge blind. The LCC is more difficult to subtract [76, 77] which gives a finite contribution to $\Delta \gamma$, $\Delta \gamma^{LCC} \propto M v_2 / N$, where $M$ is the number of hadrons in a local neutral cell, $N$ is the multiplicity, and $v_2$ is the elliptic flow.

So now the main challenge for the experiments is how to disentangle the elliptic-flow driven background effects and the magnetic-field driven CME signal. One important experimental progress is the measurement of the $\gamma$-correlation in small systems like p(d) + A collisions. In p(d) + A collisions, although the magnetic field could be big, its orientation is not correlated to the participant plane (or $v_2$ plane); thus, the magnetic field is not expected to drive a strong $\gamma$-correlation measured with respect to the $v_2$ plane. Therefore, the p(d) + A collisions can serve as a baseline for the background contributions. The recent results from CMS [71, 72] and STAR [78] Collaborations showed that the $\gamma$-correlation in p(d) + A collisions is comparable to or even larger than that in A + A collisions at the same energy and multiplicity. This suggests that the
γ-correlation contains a large portion of background contribution for peripheral A + A collisions; see more discussions in Refs. [7, 71, 72, 78].

Another important experimental progress was made in 2018 at RHIC, that is the isobar collision. In this experimental program, two different sets of collisions are operated, one is for $^{96}_{44}$Ru + $^{96}_{44}$Ru and the other is for $^{96}_{40}$Zr + $^{96}_{40}$Zr [22, 23, 79–84]. It is expected that these two collisions with same beam energy and same centrality will produce roughly equal elliptic flow but a 10% difference in magnetic fields. If $\Delta \gamma$ contains contribution from CME, we should see a difference in $\Delta \gamma$ between Ru + Ru and Zr + Zr collisions. To quantify the sensitivity of the isobar collisions, let us define the relative difference of the eccentricity $R_{\epsilon^2} = 2(\epsilon_{2}^{\text{Ru }+ \text{Ru}} - \epsilon_{2}^{\text{Zr }+ \text{Zr}})/(\epsilon_{2}^{\text{Ru }+ \text{Ru}} + \epsilon_{2}^{\text{Zr }+ \text{Zr}})$ (Note that, usually $v_2$ is proportional to $\epsilon_2$); similarly, we can define $R_{B_{\text{sq}}}$ to quantify the relative difference in the projected magnetic field squared $B_{\text{sq}} \equiv \langle (eB/m_i^2)^2 \cos[2(\Psi_B - \Psi_{\text{RP}})] \rangle$ (with $\Psi_B$ the azimuthal angle of the magnetic field) [16, 17] and $R_S$ to quantify the relative difference in the corrected γ-correlation $S = N_{\text{part}} \Delta \gamma$ ($N_{\text{part}} \propto N$ the participant number, used to compensate for the dilution effect). Since $R_{\epsilon^2}, R_{B_{\text{sq}}}, R_S$ are small, we can take a linear approximation to link them, $R_S = (1 - bg)R_{B_{\text{sq}}} + bgR_{\epsilon^2}$. The quantities $R_{\epsilon^2}$ and $R_{B_{\text{sq}}}$ can be easily obtained from theoretical simulation, then we obtain $R_S$ as a function of the background level bg through this relation. In Fig. 8, we show the numerical results for $R_{\epsilon^2}$ and $R_S$ for $bg = 2/3$ with 400 million events for each collision type [22]. In this situation, the significance level of the discovery of the CME signal reaches 5σ for centrality region 20 – 60%. In the 2018 experiment, the total number of collision events is 6.3 billion [85] and a 5σ significance level of the discovery of CME can be reached even for $bg \approx 88\%$ or 3σ significance level for $bg \approx 93\%$ in centrality region 20 – 60% ¹. Currently,

¹ We thank G. Wang for discussion.
the STAR Collaboration is making the blind analysis of the isobar data and we are really looking forward to their results.

![Graph showing relative difference in eccentricity and corrected γ-correlation for background level bg = 2/3 in isobar collisions with 400 million events for each collision type.](image)

FIG. 8. The relative difference in eccentricity, $R_{e_2}$, and corrected $γ$-correlation, $R_S$, for background level bg = 2/3 in isobar collisions with 400 million events for each collision type. Figure is from Ref. [22].

Recently, there were also other methods proposed for the purpose of disentangling the CME signal and the backgrounds. They include the pair invariant mass dependence of $γ$-correlation [86, 87], the comparative measurement of the $γ$-correlation with respect to reaction and participant planes [87, 88], the signed balance functions [89], and the charge-sensitive in-event correlations [90]; the detailed discussion can be found in the cited papers.

B. Experimental search of other ACTs

The chiral magnetic wave can transport both the vector and axial charges and can lead to an electric quadrupole in the QGP with more positive charges on the tips of the fireball and more negative charges in the equator of the fireball [91, 92]. Therefore, hydrodynamic expansion of the fireball drives a larger $v_2$ for negative charges (say, $π^-$) than the positive charges (say, $π^+$) [92–96]. The difference $Δv_2 = v_2(π^-) - v_2(π^+)$ is proportional to the net charge asymmetry $A_{ch} = (N_+ - N_-)/(N_+ + N_-)$ (this is because the CSE is proportional to $μ_V$). Such a charge dependence of $v_2$ was measured by STAR Collaboration [97] at RHIC and by ALICE Collaboration [98] and CMS Collaboration [99] at LHC. The data shows indeed an elliptic-flow difference $Δv_2$ linear in $A_{ch}$ with a positive slope whose centrality dependence is consistent with the expectation of the CMW. But we should emphasize that, like the measurement of the $γ$-correlation, there are non-CMW background effects contributing to $Δv_2$ [15, 100–106]. Before we can successfully subtract the background effects, we cannot make a conclusive claim about the experimental results for the CMW search.

In heavy-ion collisions, the transverse space-averaged vorticity at mid-rapidity region is roughly perpendicular to the reaction plane, therefore, similar to the CME case, the vector CVE
induces a baryon number separation with respect to the reaction plane. We can use a correlation similar to the $\gamma$-correlation for CME to detect the vector CVE induced baryon number separation, i.e., $\eta_{a\beta} = \langle \cos(\phi_a + \phi_\beta - 2\Psi_{RP}) \rangle$, where $\alpha, \beta = \pm$ denote baryons or anti-baryons and $\phi_{a,\beta}$ is the corresponding azimuthal angle. But, similar to the situation for the CME search, it would be challenging to subtract the possible background contributions, like the transverse momentum conservation and local baryon number conservation in $\eta$-correlation. The implication of the CVW in heavy-ion collisions is that it could induce a baryon quadrupole in the QGP in such a way that more baryons are distributed on the tips of the fireball and more anti-baryons are distributed in the equator of the fireball. After the collective expansion of the fireball, the baryons (say, $\Lambda$) would have smaller $v_2$ than the anti-baryons (say, $\bar{\Lambda}$) with the difference proportional to the net baryon asymmetry $A^{\Lambda}_{\pm} = (N_\Lambda - N_{\bar{\Lambda}}) / (N_\Lambda + N_{\bar{\Lambda}})$; see Fig. 9 for a theoretical simulation of $v_2(\bar{\Lambda}) - v_2(\Lambda)$ versus $p_t$ [62]. As the produced $\Lambda$ and $\bar{\Lambda}$ are much rarer than $\pi^{\pm}$, the detection of this difference is statistically more challenging than $v_2(\pi^-) - v_2(\pi^+)$. We expect that the phase II of the RHIC beam energy scan program would provide new possibility for the search of CVE and CVW [107].

![Graph](image)

**FIG. 9.** The splitting of $v_2$ between $\Lambda$ and $\bar{\Lambda}$ induced by the chiral vortical wave. Figure is from Ref. [62].

The non-central Cu + Au collisions may be used to test the CESE as they generate a persistent electric field orientating from the Au to Cu nuclei [18]. As illustrated in Fig. 10, the CESE induces an axial charge separation along the impact parameter direction (e.g., RH chirality on the near-Cu side and LH chirality on the near-Au side), the CME in turn induces a charge separating pattern as shown in the last step (which is superposed by an Ohm-current induced in-plane charge separation). A possible observable for this special quadrupolar pattern of charge distribution can be the charge dependence of the event planes, namely, a finite $\Delta\Psi = \langle |\Psi^+_2 - \Psi^-_2| \rangle$ increasing with the centrality where $\Psi^\pm_2$ is the event plane reconstructed from positively/negatively charged hadrons [108]. Another possible observable is the $\zeta$-correlation [108], $\zeta_{a\beta} = \langle \cos[2(\phi_a + \phi_\beta - 2\Psi_{RP})]\rangle$. But we should note that as the CESE is proportional to $\mu_v \mu_A / T^2$ which is small for typical heavy-ion collisions, the test of CESE requires a large number of collision events.
V. SPIN POLARIZATION IN HEAVY-ION COLLISIONS

A remarkable effect of vorticity is that it could polarize the spin of the constituent particles [109–112]. It is simply due to the quantum mechanical spin-orbit coupling. The motion of the fluid cell with finite vorticity generates an orbital angular momentum which can be transferred to the spin degree of freedom of the particles that constitute the fluid. If the system reaches thermal equilibrium, we can use the statistical mechanics to make an estimation about the spin polarization. The density operator is

$$\rho = Z^{-1} \exp \left[ -\beta (\hat{H} - \hat{S} \cdot \omega) \right]$$

with \(\omega\) the non-relativistic vorticity, \(\hat{H}\) the spin-unpolarized Hamiltonian, \(\hat{S}\) the spin operator (the orbital-angular-momentum part is absorbed into \(\beta \hat{H}\) term), and \(Z\) the partition function. The spin polarization is given by

$$P = \frac{\text{Tr} [\hat{S} \rho]}{s}$$

where \(s\) is spin quantum number. For fermions of spin1/2, we have \(\hat{S} = \sigma/2\) with \(\sigma\) the Pauli matrices, and thus

$$P = \frac{\omega}{2T} + o(\omega/T).$$

More rigorous derivation shows that, for spin-1/2 fermions, the spin four-vector reads [30, 113–115]

$$S^{\mu}(x, p) = -\frac{1}{8m} (1 - n_F) e^{\mu\nu\rho\sigma} p_\nu \rho\sigma(x) + O(\omega^2), \quad (21)$$

where \(n_F(p_0)\) with \(p_0 = \sqrt{p^2 + m^2}\) is the Fermi-Dirac distribution function, \(\omega_{\rho\sigma}(x)\) is the thermal vorticity. For \(\Lambda\) and \(\bar{\Lambda}\) hyperons, \(s = 1/2\), and we approximately have \(1 - n_F \approx 1\) as they are heavy. In the rest frame of the particle, \(S^{*\mu} = (0, S^*)\), where \(S^*\) can be obtained by using Lorentz transformation,

$$S^* = S - \frac{p \cdot S}{p_0(p_0 + m)} p. \quad (22)$$

Hence, we obtain the polarization vector in the rest frame of the particle as

$$P^* = \frac{S^*}{s}. \quad (23)$$

In the following, without of confusion, we will simply denote the polarization vector in the rest frame by \(P\) as well.

Before turning to the discussion of the experimental measurements and the numerical computations, let us explain the relation and the distinction between the spin polarization of hyperons and the ACTs in heavy-ion collisions. The ACTs are closely related to the chiral anomaly of QCD.
and/or QED which is of fundamental importance in modern physics. The detection of the ACTs would also serve as a strong evidence for the chiral symmetry restoration in the hot QGP. On the other hand, the underlying mechanism of the spin polarization is not related to chiral anomaly but to the quantum mechanical spin-orbit coupling. Importantly, the spin polarization measurements provide a new probe to the QGP, namely, the spin probe, which is complementary to the usual probes using, e.g. the charges. The ACTs and the spin polarization of hyperons are also closely related to each other. First, they all represent the responses of the hot medium to the external vortical field or electromagnetic field (in fact, as we will see in the following, the spin polarizations of $\Lambda$ and $\bar{\Lambda}$ are not identical which probably reflects the response to the magnetic field). Second, as we discussed in Sec. III, using QED as an example, the chiral anomaly is also understood as a type of spin polarization: the spin is fully polarized in the LLL which is responsible for the chiral anomaly. Therefore, the ACTs and the spin polarization of hyperons give us different angles to look at how the spin degree of free in the medium responses to the vortical and electromagnetic fields and that is why we discuss them together in this article.

Substituting the theoretically calculated thermal vorticity shown in Fig. 4 into Eqs. (21)-(23), one can obtain the $y$ component of the spin polarization. It reflects the global angular momentum of the collision system and is called the global spin polarization; see Fig. 11 for global spin polarization of $\Lambda$ and $\bar{\Lambda}$ (in short, “$\Lambda$ polarization”) [32]. Also shown is the experimental data from STAR Collaboration [31, 116]. We find that the theoretical results fit the data very well. We note that similar calculations were done by using either transport models or hydrodynamic models and good matches with the experimental data were seen in all those calculations [32, 117–124]. From Fig. 11, we can see two special features of the global $\Lambda$ polarization. One is that the global
Λ polarization (as well as the vorticity) is smaller for larger $\sqrt{s}$. This contradicts our intuition, since the total angular momentum of the system should be larger for larger $\sqrt{s}$. This is a relativistic effect: with increasing collision energy, the created hot matter at mid-rapidity behaves more and more boost invariant along the beam direction, thus support less and less vorticity at mid-rapidity \([28, 33]\). But for very-low-energy collision, the system may be non-relativistic and in this case, the initial vorticity (which well reflects the angular momentum retained in the mid-rapidity region) would increase with $\sqrt{s}$ \([38]\). The other one is that, albeit with a big error bar, the experimental data show that Λ spin polarization is smaller than ¯Λ spin polarization. Some possibilities for such a difference are discussed recently \([122, 125–127]\). As the Zeeman coupling between the magnetic field and spin depends on the magnetic moment of the particle, ¯Λ which has a positive magnetic moment is more easily polarized than Λ which has a negative magnetic moment.

Recently, the STAR collaboration published the measurement of the differential spin polarization, namely, the dependence of the Λ polarization on the kinematic variables like the azimuthal angle and the transverse momentum \([116, 128]\). In describing the differential spin polarization, the theoretical calculations so far are not satisfactory. In particular, the calculations based on hydrodynamic and transport models show that $P_y(\phi)$ (φ: azimuthal angle) at mid-rapidity increases when $\phi$ grows from 0 to $\pi/2$, but the experimental data is the opposite; see Fig. 12 \([32]\). Similarly, for non-central collisions, a non-zero longitudinal Λ polarization $P_z(\phi)$ is observed in experiments (which vanishes when integrated over all the angles $\phi$) which shows a $\phi$ dependence also qualitatively opposite to the theoretical calculations of the thermal vorticity \([32, 129, 130]\); see Fig. 13. Expressed in formula,

$$\frac{dP_y}{d\phi} = P_{y,z} + 2f_{2y,z} \sin[2(\phi - \Psi_{RP})] + 2g_{2y,z} \cos[2(\phi - \Psi_{RP})] + \cdots , \quad (24)$$

the second-order harmonic coefficient $f_{2y}$ (and $g_{2y}$) has opposite sign in current theoretical calculations and in experimental data, i.e., $f_{2y}^{\text{ther}} < 0, g_{2y}^{\text{ther}} < 0$ while $f_{2y}^{\text{exp}} > 0, g_{2y}^{\text{exp}} > 0$. This is a big puzzle. We will call it the “spin sign problem”. In order to resolve the spin sign problem, some important issues should be carefully (re)-examined. (1) About 80% of the measured Λ and ¯Λ are from decays of other higher-lying hadrons. During such decays, it is possible (e.g., in $\Sigma^0 \to \Lambda + \gamma$) that the spin-polarization direction of the daughter Λ is flipped compared with the parent particle. The recent studies showed that such decay contributions, though suppress $\sim 10\%$ of the primary Λ polarization, are not enough to resolve the spin sign problem \([131, 132]\). (2) The possible initial local spin polarization or initial flow profile that can lead to finite local vorticity have not been encoded in hydrodynamic and transport models. It is a important future task to perform a numerical test of such possible initial conditions. (3) The formula (21) are derived under the assumption that both momentum and spin degrees of freedom are at global equilibrium. This is a strong assumption which may not be the realistic case in heavy-ion collisions. Away from global equilibrium, spin polarization is no longer enslaved to thermal vorticity and should be treated as an independent dynamical variable. It is very desirable to develop new theoreti-
cal framework that is beyond the global equilibrium assumption. Such frameworks, in both the hydrodynamic setup and the kinetic setup, have much progressed recently. We will discuss the hydrodynamic and kinetic frameworks with spin as dynamical variable in the following sections.

(4) It is important to understand the polarization dynamics. Recent studies are Refs. [133–137].

(5) Other issues that may influence the $\Lambda$ polarization should also be explored, e.g., the hadronic mean-fields [122], the chiral-anomaly induced effects [138, 139], the other possible spin chemical potentials [140–142], and the gluonic contribution. It is also helpful to test complementary observables for measuring the vorticity, e.g., the $\phi$- and $K^\ast_0$-spin alignment [143], the CVEs and CVW, and the recently-proposed vorticity-dependent hadron yields [144].

![Image](206x327 to 404x527)

FIG. 12. The $\Lambda$ polarization along $y$ direction as a function of the azimuthal angle at mid-rapidity. Figure is from Ref. [32].

To end this section, we explain how the special pattern of the thermal vorticity shown in Fig. 13 emerges. Although we discussed in Sec. II that the global angular momentum of the collision system is the cause of the vorticity, it is not the only cause. There are many other sources for vorticity. One important source is the inhomogeneous expansion of the fireball. Since in the non-central collisions the fireball is almond shaped, the gradient of pressure would more strongly drive the fireball expand along the reaction plane (that is why we observe positive elliptic flow $v_2$). In such an expansion, it is easy to imagine that a vortical structure with four vortices in four quadrants would appear. Of course, the temperature is also inhomogeneous and its gradient also contributes to the thermal vorticity which together with the gradient of the velocity field gives the pattern shown in Fig. 13.
VI. SPIN HYDRODYNAMICS

In order to solve the spin sign problem, many attempts have been made. But so far no satisfactory solution is found. From the theoretical point of view, a key step forward should be to develop new theoretical frameworks to describe the spin polarization beyond the global equilibrium assumption. One promising framework is the hydrodynamics, which is very successful in describing the bulk evolution of the fireball in heavy-ion collisions, with the dynamical spin degree of freedom encoded. Such a framework is the spin hydrodynamics in which the spin polarization density (or equivalently the spin chemical potential) is treated on the same foot as the temperature $T$ and flow velocity $u^\mu$ [145–149].

In the so-called first order hydrodynamics, the energy-momentum tensor and the spin current tensor are given by

\[
T^{\mu \nu} = e u^\mu u^\nu - P \Delta^{\mu \nu} + \sigma^{\mu \nu}_\eta + \sigma^{\mu \nu}_\zeta + 2q^{[\mu} u^{\nu]} + \phi^{\mu \nu},
\]

\[
\Sigma^{\mu \alpha \beta} = u^\mu S^{\alpha \beta},
\]

where we have chosen the Landau-Lifshitz frame, $e$ is the energy density, $P$ is the pressure, $\sigma^{\mu \nu}_\eta$, $\sigma^{\mu \nu}_\zeta$ are shear and bulk viscous tensors, $q^\mu$ and $\phi^{\mu \nu} = \phi^{[\mu \nu]}$ are related to the spin degree of freedom and represent the strength of the torque on the temporal and spacial components of the spin current tensor. The constitutive relations are [148]

\[
\sigma^{\mu \nu}_\eta = 2 \eta \partial^\mu (u^\nu),
\]

\[
\sigma^{\mu \nu}_\zeta = \zeta \theta \Delta^{\mu \nu},
\]

\[
q^\mu = \lambda (Du^\mu + \beta \partial^\mu T - 4 \Omega^{\mu \nu} u_\nu),
\]

\[
\phi^{\mu \nu} = 2 \gamma (\partial^\mu (u_\perp^\nu) + 2 \Omega^{\mu \nu}_\perp),
\]
where $X^{[\alpha\beta]} = (X^{\alpha\beta} - X^{\beta\alpha})/2$ is anti-symmetrization in indices $\alpha, \beta$, $X^{(\alpha\beta)} = (X^{\alpha\beta} + X^{\beta\alpha})/2 - X^\mu_\mu \Delta^{\alpha\beta}/3$ is traceless symmetrization in indices $\alpha, \beta, \theta = \partial_\mu u^\mu$ is the expansion rate, $\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ is the spatial projection, $D = u \cdot \partial$ is the co-moving time derivative, $\partial_\perp^\mu = \Delta_{\mu\nu} \partial_\nu$ is the spatial derivative, $\Omega^{\mu\nu}$ is called the spin chemical potential, $\Omega^{\mu\nu}_\perp = \Delta_{\mu\rho} \Delta_{\nu\sigma} \Omega^{\rho\sigma}$. Here, $\eta, \zeta, \lambda, \gamma$ are the transport coefficients which must be semi-positive. We call them the shear viscosity, the bulk viscosity, the boost heat conductivity, and the rotational viscosity [148]. A recent attempt for the calculation of the spin-related transport coefficient is given in Ref. [150]. The hydrodynamic equations are

$$\partial_\mu T^{\mu\nu} = 0,$$

$$\partial_\mu \Sigma^{\mu,\alpha\beta} = 4q^{[\beta} \mu] 2 + \phi^{\beta\alpha}.$$

To make the above equation close, we also need the equation of state which links $e, P, S^{\alpha\beta}$. In practical use, the above first-order theory has severe problem. It has non-physical modes at ultraviolet region which violates the relativistic causality and leads to numerical instability. The origin of this problem stems from the constitutive relations Eqs. (26)-(29) which represent simple proportionality between the responses of the fluid (i.e., the LHSs) and the corresponding forces (i.e., the RHSs). To overcome this shortage of the first-order hydrodynamics, the simplest way is to amend Eqs. (26)-(29) to the Israel-Stewart form,

$$\tau_\eta (D\sigma^{\mu\nu}_\eta)_\perp + \sigma^{\mu\nu}_\eta = 2\eta \partial^\mu (u^\nu),$$

$$\tau_\zeta (D\sigma^{\mu\nu}_\zeta)_\perp + \sigma^{\mu\nu}_\zeta = \zeta \theta \Delta^{\mu\nu},$$

$$\tau_\lambda (Dq^{\mu\nu})_\perp + q^{\mu\nu} = \lambda (Du^{\mu\nu} + \beta \partial_\perp^\mu T - 4\Omega^{\mu\nu}_\perp u_\nu),$$

$$\tau_\gamma (D\phi^{\mu\nu})_\perp + \phi^{\mu\nu} = 2\gamma (\partial_\perp^\mu u^\nu + 2\Omega^{\mu\nu}_\perp),$$

where $(\cdots)_\perp$ means taking the components transverse to $u^\mu$ (e.g., $(D\sigma^{\mu\nu}_\eta)_\perp = \Delta^{\mu\nu}_\rho \Delta^\nu_\sigma D\sigma^{\rho\sigma}_\eta$). In these equations $\sigma^{\mu\nu}_\eta, \sigma^{\mu\nu}_\zeta, q^{\mu\nu}, \phi^{\mu\nu}$ are treated as dynamical variables as well so we also need additional initial conditions for them in practical use. They relax to the constitutive relations Eqs. (26)-(29) after a time scale much larger than the relaxation times, $\tau_\eta, \tau_\zeta, \tau_\lambda, \tau_\gamma$. In such a way, we obtain a set of closed, numerically stable, hydrodynamic equations. The next step would be the development of a numerical application to heavy-ion collisions and, hopefully, it will provide us valuable insights into the spin sign problem.

**VII. CHIRAL AND SPIN KINETIC THEORIES**

In addition to hydrodynamics, kinetic theory is another commonly used method to study many-body systems in and out of equilibrium. Let us start with a short review of the classical kinetic theory.
A. Classical kinetic theory

Classically, kinetic theory is built based on a single particle distribution function which is a scalar function defined in the phase space. The physical meaning of the single particle distribution which we denote \( f(t, x, p) \) is the number of particles with a specific space location \( x \) and momentum \( p \) at the time \( t \). The kinetic equation determines the time evolution of \( f(t, x, p) \) and was first proposed by Boltzmann in the following form

\[
(\partial_t + u \cdot \partial_x + F \cdot \partial_p) f(t, x, p) = C(t, x, p),
\]

(36)

where \( u \equiv p/m \) is the single particle velocity with the particle mass \( m \), \( F \) is the external force and \( C(t, x, p) \) is the collision term which is a functional of \( f \). The LHS of the above equation is the evolution of \( f \) due to streaming in the phase space in the existence of the external force field, namely, the particle at the phase space point \((x, p)\) is moving with the velocity \( \dot{x} = u \) and the momentum-space velocity \( \dot{p} = F \) at time \( t \) that leads to the change of the distribution function \( f(t, x, p) \). The RHS denotes the collision effects among particles which can change the momentum (and possibly also the location) of the particle under study.

Considering the special relativity, we can generalize Eq. (36) into the relativistic kinetic equation [151]. We adopt the Minkowski metric \( \eta_{\mu\nu} = \text{diag}\{1, -1, -1, -1\} \) and the convention \( c = e = k_B = 1 \), and define the eight-dimensional phase space coordinates as \((x, p)\), where \( x = x^{\mu} = (t, x) \) and \( p = p^{\mu} = (p^0, p) \) with \( p^0 \) the energy coordinate. Particles satisfy the following on-shell condition \( p^0 = \sqrt{p^2 - m^2} \). Defining the distribution function \( f(x, p) \) in the eight-dimensional phase space, we write the relativistic kinetic equation in the form

\[
u^{\mu} \partial_{\mu} f(x, p) + F^\mu \partial_0 f(x, p) = C(x, p)
\]

(37)

where \( u^{\mu} \equiv p^{\mu}/p^0 \) is the single particle four velocity and \( F^{\mu} = (F^0, F) \) is the four external force. The external force is called mechanical if it satisfies the condition \( F^{\mu} = \dot{p}^{\mu} \), which leads to the condition \( p^{\mu} F_{\mu} = 0 \) according to the on-shell condition. Further, we obtain \( F^0 = F \cdot p/p^0 \). In the following discussion, we always assume \( F^{\mu} \) to be mechanical. Substituting the solution of \( F^0 \) into Eq. (37) and using the chain rule

\[
\frac{\partial p^0}{\partial p} \frac{\partial}{\partial p^0} + \frac{\partial}{\partial p} \rightarrow \frac{\partial}{\partial p}
\]

(38)

we reproduce the form of the LHS of Eq. (36) in the non-relativistic kinetic representation.

The relations between physical quantities and the distribution function are readily obtained. The most elementary quantity is the particle density \( n(t, x) \), which is expressed as \( n(x, t) \equiv \int \frac{d^3p}{(2\pi)^3} f(x, p) \). The particle three current is defined as \( j(x, t) \equiv \int \frac{d^4p}{(2\pi)^4} u f(x, p) \). Combining the particle density and the three current, we obtain the four current as \( j^{\mu}(t, x) \equiv (n, j) \). In the relativistic kinetic theory, the covariant four current can be written concisely as follows

\[
j^{\mu}(x) = \int \frac{d^4p}{(2\pi)^3} \delta(p^0 - \sqrt{p^2 - m^2}) u^{\mu} f(x, p)
\]

(39)
where the delta function makes sure that the particles are on-shell. Next, we consider the energy-momentum tensor. Classically, the energy-momentum tensor can be explained as the covariant current of the four momentum and thus reads

\[ T^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^3} \delta(p^0 - \sqrt{p^2 - m^2}) u^\mu p^\nu f(x, p). \] (40)

The energy-momentum tensor is symmetric because the four velocity is proportional to the momentum \( u^\mu = p^\mu / p^0 \). The entropy density is defined as

\[ s = -\int \frac{d^3p}{(2\pi)^3} f(x, p) \left[ \ln f(x, p) - 1 \right]. \]}

Similarly, we define the covariant entropy current

\[ s^\mu = -\int \frac{d^4p}{(2\pi)^3} \delta(p^0 - \sqrt{p^2 - m^2}) u^\mu f(x, p) \left[ \ln f(x, p) - 1 \right]. \] (41)

The entropy current satisfies the second law of thermodynamics (the Boltzmann H-theorem)

\[ \partial_\mu s^\mu \geq 0 \] where the equality holds in the global equilibrium state.

### B. Winger function in non-relativistic physics

When quantum mechanics is in action, the above kinetic theory needs to be modified. The quantum kinetic theory can be built based on the Winger function method [152]. Winger function is the quantum correspondence of the classical distribution function which was first proposed by Winger in 1932. In quantum mechanics, the properties of a particle is described by the wave function \( \psi(t, x) \). The dynamics of a non-relativistic particle is governed by the Schrödinger equation

\[ i\partial_t \psi = -\frac{\partial^2}{2m} \psi + V \psi. \] (42)

where \( V = V(t, x) \) is the external potential. After the second quantization, we define the Winger function as

\[ W(t, x, p) = \int d^3y e^{ip\cdot y} \langle \psi^*_+ \psi_- \rangle \] (43)

where \( \psi^*_+ \equiv \psi^*(x + \frac{y}{2}, t), \psi_- \equiv \psi(x - \frac{y}{2}, t) \) and \( \langle \cdots \rangle \) means the ensemble average. Note that the Winger function is real.

The dynamics of the Winger function is derived from the Schrödinger equation (42). Define \( x_\pm \equiv x \pm \frac{y}{2} \). We obtain

\[ \left( \partial_t + \frac{1}{m} p \cdot \partial_x \right) W(t, x, p) = i \int d^3y e^{ip\cdot y} \langle [V(x_+, t) - V(x_-, t)] \psi^*_+ \psi_- \rangle, \] (44)

where we have used the integration by part. Next, we suppose the gradient of the potential \( V \) is small so that we can make a gradient expansion. At the first order in \( \partial_x \), we have \( V(x_+, t) - V(x_-, t) = y \cdot \partial_x V(x, t) \) and thus Eq. (44) reduces to

\[ \partial_t W + \frac{p}{m} \cdot \partial_x W - \partial_x V \cdot \partial_p W = 0. \] (45)
We thus identify the Winger function as the single particle distribution function \( f(t, x, p) = W(t, x, p) \) and identify the external force \( F = -\partial_x V(x, t) \). Thus Eq. (45) is reduced into the classical kinetic equation (36) without the collision term. In order to obtain the collision term, we need to start with an interacting theory rather than the Schrödinger equation. The Winger function method is particularly useful in performing the semiclassical approach to the quantum kinetic theory of spinful particles. So let us discuss the quantum kinetic theory about spin-\( \frac{1}{2} \) particles.

C. Kinetic theory for Spin-\( \frac{1}{2} \) fermions

With the above warmup preparation, we now consider the Dirac fermions. We will not only introduce the Wigner function for the spinor field [153] but also review the derivation of the kinetic theory available in curved spacetime and external EM field for Dirac fermions [115, 154–157]. In quantum field theory in Minkowski spacetime, the spin-\( \frac{1}{2} \) particle is described by the Dirac field \( \psi(x) \), which is in general a four-component spinor field. We have to establish a local flat frame to introduce the spinor into curved spacetime. This is naturally done by using the vierbein field \( e^a_\mu \) which can be considered as the coordinate transformation between the general coordinate manifold at a given point. The local inner product of the momentum space and the spacetime manifold constitutes the phase space, which is the cotangent bundle [158].

The covariant Wigner operator under the U(1) gauge, local Lorentz transformation and diffeomorphism is defined as [157]

\[
\hat{W}(x, p) = \int \sqrt{-g(x)} d^4y e^{-ip\cdot y/\hbar} \hat{\rho}(x, y),
\]

(47)
with \( \hat{\rho}(x,y) \equiv \hat{\psi}(x,y/2) \otimes \hat{\psi}(x,-y/2) \) and \( \hat{\psi}(x,y) \equiv e^{yD}\hat{\psi}(x) \) where \( D_\mu \) also contains the \( U(1) \) gauge field when acting on a charged spinor: \( D_\mu \hat{\psi}(x,y) = (\nabla_\mu - \Gamma^\lambda_\mu \gamma_\lambda \partial^y + i A_\mu / \hbar) \hat{\psi}(x,y) \). The Wigner function is defined by replacing the operator \( \hat{\rho}(x,y) \) with the ensemble average \( \rho(x,y) \equiv \langle \hat{\rho}(x,y) \rangle \) in Eq. (47). The dynamics of the Wigner function with the full quantum corrections is derived with the help of the Dirac equation (46), which can be solved by the expansion method respecting to \( \hbar \) with the power counting scheme \( p_\mu = O(1) \) and \( y^\mu \sim i \hbar \partial_\mu = O(\hbar) \) [157]. Up to \( O(\hbar^2) \), the dynamic equation reads [157]

\[
\left[ \gamma^\mu \left( \Pi_\mu + \frac{i\hbar}{2} \Delta_\mu \right) - m \right] W = \frac{i\hbar^2}{32} \gamma^\mu \left( R_\mu v_\alpha p_\beta + \frac{i\hbar}{6} \partial_\mu \cdot \nabla R_\mu v_\alpha \right) \left[ \partial^\nu W, \sigma^{\alpha\beta} \right],
\]

with

\[
\Pi_\mu = p_\mu - \frac{\hbar^2}{12} (\nabla_\rho F_{\mu\nu}) \partial^\rho \partial^\nu + \frac{\hbar^2}{24} R^\rho_{\sigma\mu\nu} \partial^\rho \partial^\nu p_\rho + \frac{\hbar^2}{4} R_{\mu\nu} \partial^\nu,
\]

\[
\Delta_\mu = \nabla_\mu + ( - F_{\mu\lambda} + \Gamma^\nu_\mu p_\nu ) \partial^\lambda - \frac{\hbar^2}{12} (\nabla_\rho R_{\mu\nu}) \partial^\rho \partial^\nu
\]

\[
- \frac{\hbar^2}{24} (\nabla_\lambda R^\rho_{\sigma\mu\nu}) \partial^\rho \partial^\nu \partial^\lambda p_\rho + \frac{\hbar^2}{8} R^\rho_{\sigma\mu\nu} \partial^\rho \partial^\nu \partial^\rho p_\rho
\]

\[
+ \frac{\hbar^2}{24} (\nabla_\lambda \nabla_\beta F_{\mu\nu} + 2 R^\rho_{\alpha\mu\nu} F_{\beta\rho}) \partial^\rho \partial^\nu \partial^\rho p_\rho,
\]

where \( R_{\mu\nu} = R^\rho_{\mu\nu\rho} \) is the Ricci tensor. We find the spacetime curvature comes at \( O(\hbar^2) \) at least. The Wigner function for Dirac field is a \( 4 \times 4 \) matrix, which is different from the scalar case we discussed in the previous subsection. Thus, the relation between the Wigner function and the semiclassical distribution function is less obvious in the spinor case. Equation (48) holds 16 scalar equations if we separate its matrix components, which can be decomposed into hermitian and antihermitian parts further.

Thus, we decompose the Wigner function on the basis of the Clifford algebra: \( W = \frac{1}{4} [\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu}] \), where \( \gamma^5 = (-i/4!) \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \) and all the Clifford coefficients are real. Further, Eq. (48) can be decomposed into dynamic equations for the Clifford coefficients. The Clifford coefficients are not independent. We choose the independent variables as \( \mathcal{V}^\mu \) and \( \mathcal{A}^\mu \). The physical meanings of \( \mathcal{V}^\mu \) and \( \mathcal{A}^\mu \) are the vector current density and the axial current density in the phase space, the latter is also related to the canonical spin current density in phase space. Therefore, we have the vector current, the axial current, and the canonical spin current given by \( J^\mu \equiv \langle \bar{\psi} \gamma^\mu \psi \rangle = \int_p \mathcal{V}^\mu, J^5_\mu \equiv \langle \bar{\psi} \gamma^5 \gamma^\mu \psi \rangle = \int_p \mathcal{A}^\mu \), and \( \mathcal{S}^{\lambda,\mu\nu} \equiv \langle \frac{i}{4} \bar{\psi} \{ \sigma^{\mu\nu}, \gamma^\lambda \} \psi \rangle = -\frac{\hbar}{2} \int_p e^{\lambda\mu\nu} A_\nu \) with \( \int_p \equiv \int_{(2\pi)^4} \frac{d^4p}{\sqrt{-g(x)}} \). In the limit \( \hbar \to 0 \), the vector \( \mathcal{V}_\mu \) is proportional to the momentum \( p_\mu \), which is in accordance with Eq. (39). However, the axial vector \( \mathcal{A}_\mu \) has different forms in the massless case and the massive case due to the fact that spin is parallel (or anti-parallel) to the momentum for a massless particle and is perpendicular to
the momentum for a massive particle. Although spin is not an independent variable in the mass-
less case, it induces a Berry curvature, which leads to nontrivial topological effect and results in
the chiral kinetic theory. While in the massive case, spin becomes an independent variable which
induces two new degree of freedom (the orientation of the spin vector). The complete set of ki-
netic equations in the massive case is thus composed of four equations. We call such theory the
spin kinetic theory. Let us discuss the chiral kinetic theory and the spin kinetic theory separately
in the following.

1. Chiral kinetic theory

For massless fermions, in the classical limit, not only $V_\mu$ but also $A_\mu$ is parallel to the momen-
tum, and up to $O(\hbar)$, they read

$$
(V, A)^\mu = 4\pi \left\{ [p^\mu (f, f_5) + \hbar \Sigma^\mu_\nu \Delta_\nu (f_5, f)] \delta(p^2) 
+ \hbar \tilde{F}^{\mu\nu} p_\nu (f_5, f) \delta'(p^2) \right\},
$$

(50)

where $f = f(x, p)$ and $f_5 = f_5(x, p)$ are two scalar coefficients, $\Sigma^\mu_\nu = \frac{1}{2p^\mu p^\nu} e^{\mu\nu\rho\sigma} n_\rho n_\sigma$ is the spin
tensor for chiral fermions with $n^\mu$ a unit timelike frame vector, the delta function $\delta(p^2)$ imposes
the mass-shell condition at classical limit. Comparing the vector current and the axial current
for massless fermions with Eq. (39), we find that the two scalar functions $f$ and $f_5$ represent the
semiclassical vector distribution function and axial distribution function. The second term in
Eq. (50) is called the side-jump term which ensures the total angular momentum to be conserved
during collisions of two massless fermions [159], and the last term comes from the interaction
between the spin and the external EM field.

Define the right-hand and left-hand distribution functions as $f_{R/L} = \frac{1}{2} (f \pm f_5)$. The kinetic
equations for $f_R$ and $f_L$ are derived as [157]

$$
0 = \delta(p^2 \mp \hbar F_{\alpha\beta} \Sigma^{\alpha\beta}_n) \left[ p_\mu \Delta^\mu f_{R/L} 
\mp \frac{\hbar}{p \cdot n} \tilde{F}_{\mu\nu} n^\mu \Delta^\nu f_{R/L} \pm \hbar \Delta^\mu \left( \Sigma^\mu_\nu \Delta^\nu f_{R/L} \right) \right],
$$

(51)

where the mass-shell condition is corrected by the interaction between spin and the external EM
field at $O(\hbar)$. The flat spacetime version of the above chiral kinetic equation has been under
intensive investigations recently [159–174], which can be written in the following form (for right
hand particles only) after integrating over $p_0$:

$$
0 = \left\{ \left( 1 - \frac{\hbar (B \cdot p)}{2|p|^3} \right) \partial_t + \left( v - \frac{\hbar}{2|p|^3} [(E - \nabla \epsilon_p) \times p] 
- \frac{\hbar B}{2|p|^2} \cdot \nabla + \left( \frac{E - \nabla \epsilon_p}{|p|} \right) + v \times B 
- \frac{\hbar}{2|p|^3} (E - \nabla \epsilon_p) \cdot \frac{B}{|p|} \right) \nabla \right\} f_R,
$$

(52)
where we have chosen \( n^\mu = (1, 0, 0, 0) \), \( \epsilon_p \equiv p_0 = |p| - \frac{hB \cdot p}{2|p|^2} \) is the particle energy and \( v = \frac{\partial \epsilon_p}{\partial p} \) is the effective velocity. We find that there is a phase space correction factor \((1 - \frac{hB \cdot b}{2})\), where \( b = \frac{p}{2|p|^2} \) is Berry curvature. The dispersion relation is also corrected by the Berry curvature at \( O(h) \). The three components current for the right hand particles take the form

\[
J_R = \int \frac{d^3 p}{(2\pi)^3} \left( \mathbf{v} - \frac{hB}{2|p|^2} - \frac{h}{2|p|^3} \mathbf{E} \times \mathbf{p} + \frac{h}{2|p|^3} \epsilon_p \mathbf{v} \times \nabla \right) f_R.
\] (53)

Similarly, the kinetic equation and the current for left hand particles can be readily derived.

The kinetic theory in curved spacetime can be used to study the rotating frame. We consider the frame is rotating with the angular velocity \( \Omega \) in the inertial frame, and choose the frame vector \( n^\mu = (1, x \times \Omega) \). The kinetic equation reads [157]

\[
\left[ (1 + 2h \Omega \cdot \mathbf{b}) \frac{\partial}{\partial t} + \left\{ \mathbf{v} + 2h|p|(\mathbf{v} \cdot \mathbf{b}) \Omega \right\} \cdot \nabla_x \right. \\
\left. + 2|p|(\mathbf{v} \times \Omega) \cdot \frac{\partial}{\partial p} \right] f_R = 0,
\] (54)

where \( \mathbf{v} = \partial \epsilon_p / \partial p \) and \( \epsilon_p = |p| - \frac{h}{2} \mathbf{b} \cdot \Omega \). We find the correspondence between the rotation velocity and the magnetic field in \( \tilde{\epsilon}_p \) (hence in \( \tilde{\mathbf{v}}_p \)) is \( |p| \Omega \leftrightarrow \mathbf{B} \) while in other place is \( 2|p| \Omega \leftrightarrow \mathbf{B} \).

The current is

\[
J_R = \int_p \left[ \tilde{\mathbf{v}}_p + 2h|p|(\tilde{\mathbf{v}}_p \cdot \mathbf{b}_p) \Omega \right] f_R + O(\Omega^2).
\] (55)

The equilibrium state can be derived from the kinetic equation (51). We suppose the local equilibrium distribution functions depend on the linear combination of the collisional conserved quantities: the particle number, the energy and momentum, and the angular momentum. Therefore, we have \( f_{R/L}^{LE} = n_F(g_{R/L}) \) with \( g_{R/L} = p \cdot \beta + \alpha_{R/L} \pm h\Sigma_{n}^{\mu\nu} \omega_{\mu\nu} \), where the coefficients \( \beta, \alpha \)'s, \( \omega_{\mu\nu} \) depend only on \( x \), \( \beta^\mu \) is timelike and \( n_F \) is supposed to be the Fermi-Dirac distribution function. The global equilibrium condition is derived as [157]

\[
\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = \phi(x) g_{\mu\nu}, \\
\nabla_\nu \beta_\nu - 2\omega_{\mu\nu} = 0, \\
\nabla_\mu \alpha_{R/L} = F_{\mu\nu} \beta_\nu,
\] (56)

where \( \phi(x) \) is an arbitrary function which arises due to the conformal invariance in massless case. We define the four velocity of the fluid as \( U^\mu = T \beta^\mu \) with \( T \) the temperature, and the chemical potential \( \mu_{R/L} = -T \alpha_{R/L} \). Substituting the global equilibrium into Eqs. (53) and (55) and taking into account the current of left hand particles, we derive the CME and CSE

\[
\mathbf{J} = \frac{\hbar \mu}{2\pi^2} \mathbf{B}, \\
\mathbf{J}_5 = \frac{\hbar \mu}{2\pi^2} \mathbf{B},
\] (57)
where \( \mu = \frac{1}{2} (\mu_R + \mu_L) \) and \( \mu_5 = \frac{1}{2} (\mu_R - \mu_L) \), and the CVEs

\[
\mathbf{J} = \frac{\hbar}{\pi^2} \mu \Sigma, \\
\mathbf{J}^\mu_5 = \hbar \left( \frac{\mu^2 + \mu_5^2}{2\pi^2} + \frac{T^2}{6} \right) \Sigma.
\] (58)

We should note that the results for the CME and CVE currents are independent of the choice of the frame vector \( n^\mu \).

2. Spin kinetic theory

For massive fermions, the particle spin is perpendicular to its momentum up to \( O(\hbar) \). The expressions of the vector and the axial vector are as follows

\[
\mathcal{V}^\mu = 4\pi \left\{ p^\mu f \delta(p^2 - m^2) + m\hbar F^{\mu\nu} \theta_\nu \delta'(p^2 - m^2) \\
+ \frac{\hbar}{2m} \varepsilon^{\mu\nu\rho\sigma} p_\nu \Delta_\rho \left( \theta_\sigma f_A \right) \delta(p^2 - m^2) \right\},
\] (59)

\[
\mathcal{A}^\mu = 4\pi \left\{ m\theta^\mu f_A \delta(p^2 - m^2) + \hbar F^{\mu\nu} p_\nu \delta'(p^2 - m^2) \right\},
\] (60)

where \( f = f(x, p) \) and \( f_A = f_A(x, p) \) are two scalar functions and \( \theta^\mu \) is the unit spacelike spin vector which is perpendicular to momentum \( p^\mu \theta_\mu = 0 \). We define \( f_\pm \equiv \frac{1}{2} (f \pm f_A) \) which satisfy the following relation,

\[
4\pi mf_\pm \delta(p^2 - m^2) = \text{Tr} [W(x, p) P_\pm (\theta)],
\] (61)

where \( \Sigma_S^{\mu\nu} = \frac{1}{2m} \varepsilon^{\mu\nu\rho\sigma} \theta_\rho p_\sigma \) is the spin tensor for massive fermions and \( P_\pm (\theta) \equiv (1/2)(1 \pm \gamma^5 \gamma_\mu \theta^\mu) \) is the spin projection operator [175]. Thus the physical meanings of \( f_\pm \) are the semiclassical distribution functions that describe the spin-up and spin-down states with respect to \( \theta^\mu \). The kinetic equations for \( f_\pm \) are derived as [115]

\[
0 = \delta(p^2 - m^2) \left[ f_A p \cdot \Delta \theta^\mu - f_A F^{\mu\nu} \theta_\nu + \theta^\mu p \cdot \Delta f_A \\
- \frac{\hbar}{4m} \varepsilon^{\nu\rho\mu\alpha} p_\alpha \left( \nabla_\nu F_{\rho\mu} - p_\lambda R^\lambda_{\rho\mu} \right) \partial_\rho f \right].
\] (63)

The evolution equation for the spin-direction vector \( \theta^\mu \) is given by [115]

\[
0 = \delta(p^2 - m^2) \left[ f_A p \cdot \Delta \theta^\mu - f_A F^{\mu\nu} \theta_\nu + \theta^\mu p \cdot \Delta f_A \\
- \frac{\hbar}{4m} \varepsilon^{\nu\rho\mu\alpha} p_\alpha \left( \nabla_\nu F_{\rho\mu} - p_\lambda R^\lambda_{\rho\mu} \right) \partial_\rho f \right].
\] (63)
We emphasize that the third term on the right-hand side is actually $O(h)$ order. From the above kinetic equations, we can extract the Mathisson-Papapetrou-Dixon equations

\[
\frac{Dp^\mu}{D\tau} = F^\mu_\lambda \frac{p_\lambda}{m} + \frac{\hbar}{2m} \Sigma^{\alpha\beta}_s \left( \nabla^\mu F_{\alpha\beta} - p_\lambda R^\lambda_{\rho\alpha\beta} \right), \tag{64}
\]

\[
\frac{D\Sigma^{\mu\nu}_s}{D\tau} = -\frac{1}{2m} F_{\sigma}^{\left[\mu \Sigma^{v\sigma}_s \right]} + 2p^\mu \frac{dx^\nu}{d\tau}, \tag{65}
\]

where $\tau$ is the proper time along the trajectory of the particle and $dx^\mu/d\tau = p^\mu/m$. The above two equations describe the spin dynamics for a single particle in curved spacetime and external EM field.

We can derive the equilibrium state for massive fermions with the same method as the massless case. Supposing $f^{LE}_\pm = n_F(g_\pm)$ with $g_\pm = p \cdot \beta + \alpha_\pm \pm \bar{h}^\Sigma_{\mu\nu} \omega_{\mu\nu}$ and substituting it into Eq. (62), we very that the following conditions make Eq. (62) hold:

\[
\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0, \\
\nabla_\mu [\beta_\nu] - 2\omega_{\mu\nu} = 0, \\
\alpha_+ - \alpha_- = O(h), \\
\nabla_\mu \alpha_\pm = F_{\mu\nu} \beta^\nu, \tag{66}
\]

where we have used $\beta^\rho \nabla_\rho F_{\mu\nu} = F^\rho_\beta \nabla_\beta F_{\mu\nu} - F_{\mu}^\rho \nabla_\rho \beta_{\nu}$ [151]. Further, we find the following solutions of $\alpha_\pm$ and $\theta^\mu$ fulfill the spin evolution equation (63) [115]:

\[
\alpha_+ = \alpha_-, \\
\theta^\mu = -\frac{1}{2m\Gamma^2} \epsilon^{\mu\nu\rho\sigma} p_\nu \nabla_{[\beta_\sigma]} [\theta^\rho], \tag{67}
\]

where $\Gamma^2 = \frac{1}{2} \nabla_{[\beta_\nu]} \Lambda^{\mu\rho} \Lambda^{\nu\nu} \nabla_{[\beta_\sigma]}$ with $\Lambda^{\mu\nu} = g^{\mu\nu} - \frac{\bar{p} p^\nu}{m^2}$. Thus the particle spin is polarized along the thermal vorticity at global equilibrium. The spin polarization per particle in the phase space is defined by $S^\mu = A^\mu / (4\pi f)$. Substituting the global equilibrium conditions (66) and (67), we obtain

\[
S^\mu_{GE} = \frac{\hbar}{4} \epsilon^{\mu\nu\rho\sigma} p_\nu \nabla_{[\beta_\sigma]} [1 - n_F] \delta(p^2 - m^2) \\
+ \hbar \bar{F}^\mu_\nu p_\nu \delta'(p^2 - m^2), \tag{68}
\]

which after integrating $p_0$ over 0 to $\infty$ gives nothing but formula (21) for $s = 1/2$ and with contribution from the EM field added. (Note that in Eq. (21), the approximation $\sqrt{p^2 + m^2} \approx m$ is used as $m$ for, e.g., $\Lambda$ hyperon, is large.) More details on collisionless spin kinetic theory in flat space can be seen in Refs. [176–179], the discussions about the collision terms in CKT and SKT are in Refs. [150, 165, 167, 180–182].
VIII. SUMMARY

In summary, we have discussed some intriguing properties of the strong EM fields and vorticity in heavy-ion collisions. We have given heuristic introduction to the anomalous chiral transport phenomena and spin polarization in heavy-ion collisions. We briefly reviewed the recent progresses in both theory and experiments towards the understanding these novel quantum phenomena in heavy-ion collisions. The ACTs could be used to detect the nontrivial topological structure of QCD gauge sector and the possible P and CP violation of strong interaction at high-temperature environment. The spin polarization of hadrons provides us a probe to the (local) rotating properties and to the spin dynamics of the quark-gluon matter. This opens a door to a new era of subatomic spintronics.

Puzzles exist. Noticeably, the experimental observables for the ACTs, e.g. the CME, contain strong background contributions which call for more efforts and new ideas from both the theoretical and experimental sides to resolve. The experimental data for the azimuthal-angle dependence of the spin polarization show qualitatively opposite trend comparing to the theoretical calculations based thermal vorticity, which gives rise to a spin sign problem. It is promising that new theoretical frameworks with spin as independent dynamical variable may provide important insight to the spin sign problem. Now, two of such frameworks, the spin hydrodynamics and spin kinetic theory are progressing fast, and hopefully in near future the numerical simulations based on them could be achieved.

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