WEAK FORM OF EQUIDISTRIBUTION THEOREM
FOR HARMONIC MEASURES OF FOLIATIONS
BY HYPERBOLIC SURFACES

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Abstract. We show that the equidistribution theorem of C. Bonatti and
X. Gómez-Mont for a special kind of foliations by hyperbolic surfaces does
not hold in general, and we seek a weaker form valid for general foliations by
hyperbolic surfaces.

1. Introduction

Let $M$ be a smooth closed manifold, and let $\mathcal{F}$ be a smooth foliation by hyperbolic
surfaces, i.e. a 2-dimensional foliation equipped with a smooth leafwise metric $g_P$
of constant curvature $-1$. Let $v_P$ be the leafwise Poincaré volume form, and for a
point $z \in M$ and $\rho > 0$, let $B_\rho(z)$ be the leafwise $\rho$-disk centered at $z$. When $B_\rho(z)$
is an embedded disk in $M$, let $\beta_\rho(z)$ be the probability measure of $M$ supported on
$B_\rho(z)$ defined by
$$
\beta_\rho(z) = \frac{1}{\int_{B_\rho(z)} v_P} v_P|_{B_\rho(z)}.
$$
When $B_\rho(z)$ is not embedded, define $\beta_\rho(z)$ using the universal cover of the leaf.

In [BGM], Christian Bonatti and Xavier Gómez-Mont have shown the following theorem.

Theorem 1.1. Let $\Sigma$ be a closed oriented hyperbolic surface, and let $\Phi: \pi_1(\Sigma) \to
\text{PSL}(2, \mathbb{C})$ be a nonelementary representation. Endow leaves of the associated fo-
liated $\mathbb{P}^1$ bundle $(N, \mathcal{G})$ with a hyperbolic metric lifted from $\Sigma$. Then there exists
a probability measure $\mu$ on $N$ such that for any sequences $z_n \in N$ and $\rho_n \to \infty$,
$\beta_{\rho_n}(z_n)$ converges weakly to $\mu$.

The measure $\mu$ turns out to be the unique harmonic measure of the foliation $\mathcal{G}$
in the sense of [G]. See Section 4 for more detail. Thus one may ask the following
question.

Question 1.2. For $(M, \mathcal{F})$ as above, if $\beta_{\rho_n}(z_n)$ converges weakly to a measure $\mu$
as $\rho_n \to \infty$, is it true that $\mu$ is a harmonic measure of $\mathcal{F}$?

In Section 2, we shall answer this question in the negative, and in Section 3, we
propose a measure $\mu_{\rho, \rho'}(z)$ modified for the positive answer. In Section 4, we raise

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a further question and give a new example of foliations for which the conclusion of
the theorem of Bonatti and Gómez-Mont holds.

2. A COUNTEREXAMPLE

Let $Solv_3$ be the 3-dimensional unimodular solvable nonnilpotent Lie group. The
multiplication of $Solv_3 = \{(x, q, t)\}$ is given by

$$(x, q, t)(x', q', t') = (e^t x' + x, e^{-t} q' + q, t + t').$$

It has a semidirect product structure:

$$1 \to \mathbb{R}^2 \to Solv_3 \to \mathbb{R} \to 1.$$ 

Any lattice $\Gamma$ of $Solv_3$ is a semidirect product

$$1 \to \mathbb{Z}^2 \to \Gamma \to \mathbb{Z} \to 1$$

such that $\Gamma \cap \mathbb{R}^2 = \mathbb{Z}^2$. The multiplication is given by

$$(n, m, \ell)(n', m', \ell') = ((n', m')A^\ell + (n, m), \ell + \ell')$$

for some hyperbolic matrix $A \in SL(2, \mathbb{Z})$. The quotient manifold $M = \Gamma \setminus Solv_3$ is
a $T^2$ bundle over $S^1$ with monodromy $A$:

$$T^2 = \mathbb{Z}^2 \setminus \mathbb{R}^2 \to M = \Gamma \setminus Solv_3 \to S^1 = \mathbb{Z} \setminus \mathbb{R}.$$ 

Denote by $G = \{q = 0\}$ the subgroup of $Solv_3$, isomorphic to the 2-dimensional
solvable nonabelian Lie group, and let $\tilde{F}$ be the orbit foliation of the right $G$-action.
Notice that the leaf passing through $(x_0, q_0, t_0)$ is just $L_{q_0} = \{q = q_0\}$. The left
action of the lattice $\Gamma$ commutes with the right $G$-action, and therefore $\tilde{F}$ descends
to a foliation $\mathcal{F}$ on $M$. Now

$$g = e^{-2t}dx^2 + e^{2t}dq^2 + dt^2$$

is a left invariant metric on $Solv_3$. The restriction of $g$ to each leaf $L_q$ of $\tilde{F}$ is
written as

$$g_P = e^{-2t}dx^2 + dt^2.$$ 

If we change the variable by $y = e^t$, then we get

$$g_P = (dx^2 + dy^2)/y^2,$$

the Poincaré metric on the half plane $\mathbb{H}$. That is, we have an identification

$$Solv_3 \ni (x, q, t) \leftrightarrow (x + e^t i, q) \in \mathbb{H} \times \mathbb{R},$$

where the right $G$-action leaves each leaf $L_q = \mathbb{H} \times \{q\}$ invariant. The action of the
one-parameter subgroup $\{Y^t = (0, 0, t)\}$ of $G$ on each leaf $L_q \cong \mathbb{H}$ is given by

$$Y^t(x + yi) = x + e^t yi,$$

and the one-parameter subgroup $\{S^s = (s, 0, 0)\}$ is given by

$$S^s(x + yi) = ys + x + yi.$$ 

They satisfy

$$Y^t \circ S^s = S^{se^{-t}} \circ Y^t.$$ 

See Figure 1. On the other hand, the left $Solv_3$-action (in particular $\Gamma$-action)
leaves the product structure invariant, and the action on the first factor $\mathbb{H}$ is given by

$$(x, q, t) \cdot z = e^t z + x.$$
It is not only $g_P$-isometric but also leaves the point $\infty$ on $\partial \mathbb{H}$ invariant. That is, each leaf of the foliation $\mathcal{F}$ of the quotient manifold $M$ admits a pointed hyperbolic structure.

The flow $\{S^s\}$ leaves the coordinate $y$, whence the old coordinate $t$, invariant. Thus it leaves fibers of the fibration $T^2 \to M \to S^1$ invariant and is a linear flow on it parallel to an eigenvector of the matrix $A$.

Now $m = dx \wedge dq \wedge dt$ is a biinvariant Haar measure of $\text{Solv}_3$. By a criterion in [G], $m$ is a harmonic measure of $\mathcal{F}$, since the function $y$ is a harmonic function on $\mathbb{H}$. Moreover by a general theorem of Bertrand Deroin and Victor Kleptsyn [DK], it is the unique harmonic measure. The rest of this section is devoted to the proof of the following theorem.

**Theorem 2.1.** There exist $z_n \in M$ such that $\beta_n(z_n)$ converges to $\mu \neq m$.

We consider an infinite cyclic covering $\hat{M}$ of $M$ and the lift $\hat{\mathcal{F}}$ of the foliation $\mathcal{F}$. Precisely,

$$\hat{M} = \mathbb{Z}^2 \setminus \text{Solv}_3 = T^2 \times \mathbb{R},$$

where $\mathbb{Z}^2$ is the normal subgroup of the lattice $\Gamma$. Let us denote by $\mathcal{P}(\hat{M})$ the space of the Radon probability measures of $\hat{M}$, endowed with the pointwise convergence topology on the space $C_0(\hat{M})$ of continuous functions on $\hat{M}$ with compact support.

Every leaf of $\hat{\mathcal{F}}$ is pointedly isometric to $\mathbb{H}$. Choose one leaf and identify it with $\mathbb{H}$. For $\rho > 0$, let $z_\rho = e^{\rho i} \in \mathbb{H} \subset \hat{M}$. Notice that the hyperbolic distance of $z_\rho$ to the horocycle $\{y = 1\}$ is $\rho$. We shall show that the probability measure $\beta_\rho(z_\rho) \in \mathcal{P}(\hat{M})$ converges to a measure $\hat{\mu} \in \mathcal{P}(\hat{M})$ as $\rho \to \infty$. The boundary $\partial B_\rho(z_\rho)$ of the disk $B_\rho(z_\rho)$ is tangent to $\{y = 1\}$ and satisfies the equation:

$$x^2 + (y - \frac{R + 1}{2})^2 = \frac{1}{4}(R - 1)^2,$$

where $R = e^{2\rho}$.

See Figure 2. Putting $y = e^t$, we get

$$x = \pm \sqrt{R(e^t - 1)} + e^t - e^{2t}.$$
Now since $v_P = y^{-2}dx \wedge dy = e^{-t}dx \wedge dt$, the area $A(R, t, \Delta t)$ of the set $B_\rho(z_\rho) \cap (T^2 \times [t, t + \Delta t])$ is given by

$$A(R, t, \Delta t) = 2 \int_t^{t+\Delta t} \sqrt{R(e^{-t} - e^{-2t}) + e^{-t} - 1} \, dt.$$ 

On the other hand, the area $A(R)$ of $B_\rho(z_\rho)$ is given by

$$A(R) = \pi(e^\rho + e^{-\rho} - 2) = \pi(R^{1/2} + R^{-1/2} - 2).$$

Now we have

$$\beta_\rho(z_\rho)(T^2 \times [t, t + \Delta t]) = \frac{A(R, t, \Delta t)}{A(R)} = \frac{2}{\pi} \int_t^{t+\Delta t} \frac{\sqrt{R(e^{-t} - e^{-2t}) + e^{-t} - 1}}{R^{1/2} + R^{-1/2} - 2} \, dt.$$ 

Therefore the limit measure $\hat{\mu}$ as $\rho \to \infty$ should satisfy

$$\hat{\mu}(T^2 \times [t, t + \Delta t]) = \frac{2}{\pi} \int_t^{t+\Delta t} \sqrt{e^{-t} - e^{-2t}} \, dt.$$ 

On the other hand, the portion of the measure $\beta_\rho(z_\rho)$ supported on $T^2 \times [t, t + \Delta t]$ $(t > 0)$ becomes more and more invariant by the flow $S^s$ as $\rho \to \infty$, while $S^s$ is a linear flow of irrational slope on $T^2 \times \{t\}$ and $dx \wedge dq$ is the unique measure invariant by $S^s$. Therefore one concludes that

$$\beta_\rho(z_\rho) \to \hat{\mu} = \hat{\Phi} \, dx \wedge dq \wedge dt \text{ as } \rho \to \infty,$$

where

$$\hat{\Phi}(t) = \begin{cases} \frac{2}{\pi} \sqrt{e^{-t} - e^{-2t}} & \text{if } t \geq 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

The actual proof needs the evaluation on a function from $C_0(\hat{M})$, which is routine and omitted. But $\beta_\rho(z_\rho) \to \hat{\mu}$ does not guarantee that $\hat{\mu}$ is a probability measure, since the constant 1 does not belong to $C_0(\hat{M})$ and some part of $\beta_\rho(z_\rho)$ may escape to $\infty$. This, however, is assured by the following concrete computation:

$$\int_0^\infty \sqrt{e^{-t} - e^{-2t}} \, dt = \pi/2.$$ 

Also this implies a stronger fact that $\beta_\rho(z_\rho) \to \hat{\mu}$ pointwise on any bounded continuous function. The function $\hat{\Phi}$ takes the maximum value at $t = \log 2$. See Figure 3. Returning to the compact manifold $M$, the previous observation shows that the limit measure $\mu$ of $\beta_\rho(z_\rho)$ is the projected image of $\hat{\mu}$ and is written as $\mu = (\Phi \circ \rho) \, m$, where
where \( p : M \to S^1 \) is the bundle projection and \( \Phi \) is a continuous function on \( S^1 \) given by
\[
\Phi(t) = \sum_{k \in \mathbb{Z}} \hat{\Phi}(t + k).
\]
But \( \Phi \) is not a constant function since \( \Phi(0) < \Phi(\log 2) \), showing that \( \mu \neq m \), as is required.

The author is grateful to Hiroki Kodama for showing him this simple proof.

3. Weak form of equidistribution

Let \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) be the disk model of the Poincaré plane. For \( 0 < R < 1 \), denote \( \mathbb{D}(R) = \{ |z| < R \} \) and let \( \rho \) be the Poincaré distance from 0 to the circle \( \partial \mathbb{D}(R) \), i.e.
\[
\rho = \frac{1}{2} \log \frac{1 + R}{1 - R}.
\]
For \( R < R' < 1 \), let \( \rho' \) be the Poincaré distance between \( \partial \mathbb{D}(R) \) and \( \partial \mathbb{D}(R') \). Define a function \( \psi_{\rho, \rho'} : \mathbb{D} \to [0, \infty) \) by
\[
\psi_{\rho, \rho'}(z) = \begin{cases} 
1 & \text{if } |z| \leq R, \\
\frac{R'-|z|}{R'-R} & \text{if } R \leq |z| \leq R', \\
0 & \text{if } R' \leq |z| < 1.
\end{cases}
\]
The function \( \psi_{\rho, \rho'} \) is determined by \( \rho \) and \( \rho' \). See Figure 4. Define a probability measure \( \mu_{\rho, \rho'} \) on \( \mathbb{D} \) by
\[
\mu_{\rho, \rho'} = \frac{1}{\int_{\mathbb{D}} \psi_{\rho, \rho'} v_{\rho'}} \psi_{\rho, \rho'} v_{\rho'},
\]
where \( v_{\rho} \) denotes the Poincaré volume form.
Let \((M, \mathcal{F})\) be as in Section 1. For any \(x \in M\), let \(L_x\) be the leaf through \(x\) with the universal cover \(\tilde{L}_x\) identified with \(\mathbb{D}\). Define a map \(j_x : \mathbb{D} \to M\) as the composite

\[ j_x : \mathbb{D} \cong \tilde{L}_x \to L_x \subset M \]

such that \(j_x(0) = x\). Define \(\mu_{\rho, \rho'}(x) \in \mathcal{P}(M)\) by \(\mu_{\rho, \rho'}(x) = (j_x)_* \mu_{\rho, \rho'}\). The main result of this section is the following.

**Theorem 3.1.** If \(\mu_{\rho_n, \rho'_n}(x_n)\) converges for some sequences \(x_n \in M, \rho_n \to \infty\) and \(\rho'_n \to \infty\), then the limit is a harmonic measure for \(\mathcal{F}\).

To show this, we approximate \(\psi_{\rho, \rho'}\) by another function \(\varphi_{\rho, \rho'}\) which is a combination of harmonic functions. Let \(A = 1/\log R'/R\). Define a function \(\varphi_{\rho, \rho'} : \mathbb{D} \to [0, \infty)\) by

\[
\varphi_{\rho, \rho'}(z) = \begin{cases} 
1 & \text{if } |z| \leq R, \\
A \log \frac{R'}{|z|} & \text{if } R \leq |z| \leq R', \\
0 & \text{if } R' \leq |z| < 1.
\end{cases}
\]

Define a probability measure \(\nu_{\rho, \rho'}\) on \(\mathbb{D}\) by

\[
\nu_{\rho, \rho'} = \frac{1}{\int_{\mathbb{D}} \varphi_{\rho, \rho'} v_P} \varphi_{\rho, \rho'} v_P,
\]

and define \(\nu_{\rho, \rho'}(x) = (j_x)_* \nu_{\rho, \rho'} \in \mathcal{P}(M)\) just as before. Theorem 3.1 reduces to the following two propositions. Denote by \(\| \cdot \|\) the norm of \(\mathcal{P}(M) \subset \mathcal{C}(M)'\) dual to the sup norm \(\| \cdot \|_\infty\) of the Banach space \(\mathcal{C}(M)\) of the continuous functions of \(M\).

**Proposition 3.2.** We have \(\|\mu_{\rho_n, \rho'_n}(x_n) - \nu_{\rho_n, \rho'_n}(x_n)\| \to 0\) as \(\rho_n, \rho'_n \to \infty\).

**Proposition 3.3.** If \(\nu_{\rho_n, \rho'_n}(x_n)\) converges for some sequences \(x_n \in M\) and \(\rho_n, \rho'_n \to \infty\), then the limit is a harmonic measure for \(\mathcal{F}\).

We shall first show Proposition 3.3. For a \(C^2\) function \(f : M \to \mathbb{R}\), we denote by \(\Delta_P f\) the leafwise Laplacian with respect to the leafwise Poincaré metric. What we have to prove is that

\[
\int_M \Delta_P f \nu_{\rho_n, \rho'_n}(x_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \(j_{x_n}\) is a local isometry onto the leaf, we have \(\Delta_P f \circ j_{x_n} = \Delta_P (f \circ j_{x_n})\). Rewriting \(f \circ j_{x_n}\) as \(f\), this follows from the following proposition about \(\mathbb{D}\).

**Proposition 3.4.** For any nonzero bounded \(C^2\) function \(f : \mathbb{D} \to \mathbb{R}\), we have

\[
\frac{\int_{\mathbb{D}} \varphi_{\rho, \rho'} \Delta_P f v_P \cdot \|f\|_\infty^{-1}}{\int_{\mathbb{D}} \varphi_{\rho, \rho'} v_P} \to 0
\]

as \(\rho, \rho' \to 0\) uniformly on \(f\).

**Estimate of the numerator.** First notice that \(\Delta_P f v_P = \Delta_E f v_E\) where \(E\) stands for Euclidian. (Both are equal to \(dJ^* df\), where \(J\) is the almost complex structure.) We need the following Green-Riesz formula. See [D], Chap. I, p.30.

**Theorem 3.5.** Let \(\Omega\) be a smoothly bounded compact domain in \(\mathbb{R}^n\), and let \(\vec{n}_E\) be the outward unit normal vector at \(\partial \Omega\). Denote by \(\sigma_E\) the Euclidian area measure on \(\partial \Omega\). Then for any \(C^2\) function \(\varphi\) and \(f\) defined on \(\mathbb{R}^n\), we have

\[
\int_{\Omega} (\varphi \Delta_E f - f \Delta_E \varphi) v_E = \int_{\partial \Omega} (\varphi \frac{\partial f}{\partial \vec{n}_E} - f \frac{\partial \varphi}{\partial \vec{n}_E}) \sigma_E.
\]
Let us apply this formula to $\varphi_{\rho,\rho'}$, $f$ in Proposition 3.4 and the domains $D(R)$, $D(R') \setminus D(R)$. Remark that
\[ \Delta_E \log \frac{R'}{|z|} = 0 \quad \text{and} \quad \frac{\partial}{\partial n_E} (\log \frac{R'}{|z|}) = - \frac{1}{|z|}. \]
Computation shows that
\[ \int_D \varphi_{\rho,\rho'} \Delta_P f \cdot v_P = A \left( \frac{1}{R'} \int_{\partial D(R')} f \sigma_E - \frac{1}{R} \int_{\partial D(R)} f \sigma_E \right). \]
This implies that
\[ |\int_D \varphi_{\rho,\rho'} \Delta_P f \cdot v_P| \|f\|_\infty^{-1} \leq 4\pi A. \]
**Estimate of the denominator.** We use the following notation.

**Notation 3.6.** For $0 < R < R' < 1$ and positive valued functions $F(R, R')$ and $G(R, R')$, we write $F \sim G$ if $F/G \to 1$ as $R \to 1$.

**Lemma 3.7.** (1) If $F_1 \sim G_1$ and $F_2 \sim G_2$, then $F_1 + F_2 \sim G_1 + G_2$.

(2) If $F \sim G$, then we have
\[ \int_R^{R'} F(r, R') \, dr \sim \int_R^{R'} G(r, R') \, dr. \]

(3) We have $\log(R'/R) \sim R' - R$.

Now since
\[ v_P = \frac{4dx \wedge dy}{(1 - |z|^2)^2} = \frac{4dr \wedge d\theta}{(1 - r^2)^2} \]
in the polar coordinates, we have
\[ \frac{1}{8\pi} \int_D \varphi_{\rho,\rho'} v_P = \frac{1}{0^R} \frac{r \, dr}{(1 - r^2)^2} + A \frac{\int_R^{R'} \log R' - r}{r} \frac{r \, dr}{(1 - r^2)^2}, \]
where
\[ \text{the first term} = \frac{R^2}{2(1-R)(1+R)} \sim \frac{1}{4(1-R)} = \frac{A \log(R'/R)}{4(1-R)} \sim \frac{A(R' - R)}{4(1-R)}, \]
and by Lemma 3.7 (2)
\[ \text{the second term} \sim A \int_R^{R'} \frac{(R' - r) \, dr}{4(1-r)^2} = - \frac{A (R' - R)}{4(1-R)} + A \frac{1}{4} \log \frac{1-R}{1-R'}, \]
Since both terms are positive, we get from Lemma 3.7 (1),
\[ \frac{1}{8\pi} \int_D \varphi_{\rho,\rho'} v_P \sim \frac{A}{4} \log \frac{1-R}{1-R'} \sim \frac{A \rho'}{2}, \]
where the last $\sim$ holds when $\rho'$ is bounded from below and follows from the formula
\[ \rho' = \frac{1}{2} \left( \log \frac{1 + R'}{1 - R'} - \log \frac{1 + R}{1 - R} \right). \]

It follows from the two estimates that
\[ \lim_{R \to 1} \frac{1}{\rho'} \frac{1}{\int_D \varphi_{\rho,\rho'} v_P} \leq \lim_{R \to 1} \frac{1}{\rho'} = 0, \]
and the convergence is uniform on $f$. This shows Propositions 3.3 and 3.6.
Finally Proposition 3.2 follows from the estimate
\[ \int_D \psi_{\rho,\rho'} v_P \sim \int_D \varphi_{\rho,\rho'} v_P, \]
since \( \psi_{\rho,\rho'} \geq \varphi_{\rho,\rho'} \). We have already shown that
\[ \int_D \varphi_{\rho,\rho'} v_P \sim 4 \pi A \rho'. \]
Analogous (and easier) computation using \( A(R' - R) \sim 1 \) show that
\[ \int_D \psi_{\rho,\rho'} v_P \sim 4 \pi A \rho'. \]

4. Further Question and Example

It seems that the counterexample in Section 2 is rather special. There might be more foliations which satisfy the conclusion of the theorem of Bonatti and Gómez-Mont. To consider this problem, let us recall their proof, which consists of two steps. In the first step, they consider general \((M, F, g_P)\) as in the beginning of Section 1. Let \( p : \hat{M} \to M \) be the unit tangent bundle of the foliation \( F \). The space \( \hat{M} \) admits a leafwise geodesic flow \( \{g_t\} \) and the leafwise stable horocycle flow \( \{h_s\} \), which satisfy
\[ g_t \circ h_s \circ g^{-t} = h^{s^{-t}}. \]
Therefore the two flows form a locally free action of the Lie group \( B \), the 2-dimensional nonabelian Lie group. Given a leafwise submersed \( \rho \)-disk \( B_{\rho}(z) \) of \( F \) (see Section 1), they considered the lift \( \sigma : B_{\rho}(z) \setminus \{z\} \to \hat{M} \) by the radial unit vector fields, and showed that the limit \( \lim_{\rho_n \to \infty} \sigma_{\rho_n}^* \beta_{\rho_n}(z_n) \) is \( h^s \)-invariant, if it exists. This part is true for any \((M, F, g_P)\).

On the other hand, Yuri Bakhtin and Matilde Martínez [BM] showed that the map \( p^* : \mathcal{P}(\hat{M}) \to \mathcal{P}(M) \) between the space of the probability measures gives a bijection from the subset of the \( B \)-invariant measures on \( \hat{M} \) to the subset of the harmonic measures on \( M \).

Now assume that the horocycle flow \( \{h^s\} \) is uniquely ergodic. Then the unique invariant measure \( \mu \) is also \( g^t \)-invariant by (4.1), and thus \( p_* \mu \) is a unique harmonic measure of \((M, F, g)\). In the second step, Bonatti and Goméz-Mont showed the unique ergodicity of the horocycle flow \( \{h^s\} \) for foliations in Theorem 1.1. It is plausible to expect that there are more foliations with this property. We shall raise one example.

Example 4.1. Let \( G \) be an arbitrary connected unimodular Lie group, and let \( \Gamma \subset PSL(2, \mathbb{R}) \times G \) be a cocompact lattice such that \( p_2(\Gamma) \) is dense in \( G \), where \( p_2 \) is the projection onto the second factor. Then the manifold \( M = \Gamma \setminus (\mathbb{H} \times G) \) admits a horizontal foliation \( F = \Gamma \setminus \{\mathbb{H} \times \{g\}\} \) by hyperbolic surfaces.

The unit tangent bundle \( \hat{M} \) of the above foliation \( F \) is identified with \( \Gamma \setminus (PSL(2, \mathbb{R}) \times G) \). According to Marina Ratner [R], any ergodic probability measure \( \hat{\mu} \) invariant by the leafwise stable horocycle flow is algebraic in the following sense. For any \( x \) in the support of \( \hat{\mu} \), there is a closed subgroup \( H \subset PSL(2, \mathbb{R}) \times G \) such that the closure of the horocycle orbit of \( x \) is \( x \cdot H \) and that \( \hat{\mu} = x_* m \), where \( m \) is the normalized Haar measure of \((g^{-1} \Gamma g \cap H) \setminus H \) and \( x = \Gamma g \).
On the other hand, it is shown by Fernando Alcalde Cuesta and Françoise Dal’bo [ACD] that the leafwise stable horocycle flow is minimal. Therefore $\hat{\mu}$ is the Haar measure of $\Gamma \backslash (\text{PSL}(2, \mathbb{R}) \times G)$ and is unique. In conclusion the foliation in Example 4.1 is equidistributed, i.e. satisfies the conclusion of Theorem 1.1.

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