Majority-vote dynamics on multiplex networks with two layers

Jeehye Choi and K-I Goh
Department of Physics, Korea University, Seoul 02841, Republic of Korea
E-mail: kgoh@korea.ac.kr

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Abstract

Majority-vote model is a much-studied model for social opinion dynamics of two competing opinions. With the recent appreciation that our social network comprises a variety of different ‘layers’ forming a multiplex network, a natural question arises on how such multiplex interactions affect the social opinion dynamics and consensus formation. Here, the majority-vote processes will be studied on multiplex networks with two layers to understand the effect of multiplexity on opinion dynamics. We will discuss how global consensus is reached by different types of voters: AND- and OR-rule voters on multiplex-network and voters on single-network system. The AND-model reaches the largest consensus below the critical noise parameter $q_c$. It needs, however, much longer time to reach consensus than other models. In the vicinity of the transition point, the consensus collapses abruptly. The OR-model attains smaller level of consensus than the AND-rule but reaches the consensus more quickly. Its consensus transition is continuous. The numerical simulation results are supported qualitatively by analytical calculations based on the approximate master equation.

1. Introduction

From the network study of cascading failures on interdependent systems [1] to networks of networks [2], multiplex network [3, 4] has been a general theme to describe real complex systems recently. Multiplex system is not a simply aggregated or combined system of many single networks but a functionally integrated system of them. Multiplex systems have shown to exhibit exotic phase transitions exhibiting, e.g. discontinuity. Percolations [5–9], spreading of epidemics and information [10, 11], and many other standard models of interdisciplinary studies [12–14] have been studied on multiplex networks [1]. Statistical-mechanics models such as ferromagnetic spin models have also been studied on multiplex and connected network setting [15, 16].

Social network is a prime example multiplex networks where more than one layers of social interactions play roles towards the societal function. Multiplex network framework is therefore essential to better understanding of structure and dynamics on social networks. There has been a vast body of studies using statistical physics models such as the Ising model and the voter model as a tool for addressing and solving problems in social dynamics [17, 18].

Among them, the voter model is the simplest and much studied model. This model gives binary states to a node and each time a randomly chosen node updates its opinion into that of its randomly chosen neighbor. The voter model-type processes were studied in the multilayer network perspective [19, 20], in which individuals tend to maintain the same opinion in different layers through interlayer links. It was found that there could be coexistence phase of two states.

A well-known variant of the voter model is the majority-vote model [21, 22], in which a node updates its state towards the local majority one. Majority-rule has been argued to play important role in social processes [23]. In this regard, here we study the problem of social consensus dynamics focused on the effect of network multiplexity by using the majority-vote model as a model of social consensus dynamics. Majority-vote model is a simple toy model of social consensus dynamics that could be mapped onto a non-equilibrium spin dynamics [22]. It has been studied steadily in the complex networks literature [24–26], yet the effect of network multiplexity remains to be fully understood, which is the main aim of this study. A recent paper [27] studied the
majority-vote model on multiplex networks, in which a different majority-vote rule than our models was considered, reporting continuous consensus transitions.

This paper is organized as follows. We give a succinct introduction to the majority-vote model in section 2 and introduce the multiplex majority-vote processes studied in this work in section 3. The consensus transitions of the multiplex models are studied on random regular networks in section 4 and on Erdős–Rényi networks in section 5 and by the approximate master equation and pair approximation in section 6. Summary and discussion follow in section 7.

2. Majority-vote process

Many human social behaviors can be modeled by a sequence of decision processes among two alternative choices (to purchase A or B, to select A or B, and to vote for A or B, etc), which can conveniently be modeled as a binary spin state $\sigma = \{-1, +1\}$. The majority-vote model posits that a node (voter) tends to follow the majority state of its neighbors. More precisely, the system’s state $\{\sigma_1, \ldots, \sigma_N\}$ ($N$ denotes the number of nodes) evolves by the following dynamic rules at each step, following [22]:

(i) A site is chosen randomly and this site looks at its nearest neighbor sites’ states.

(ii) If there is a majority-vote state taken by more than half of its neighbors, the site takes the majority-vote state with probability $1 - q$, while with the rest probability $q$ it takes the opposite (minority) state.

(iii) In case of no majority, it takes any of the binary states with equal probability.

Here $q$ is called the noise parameter and takes a value in the range $0 \leq q < 1/2$. Nonzero $q$ allows fluctuations around local consensus, thus playing a role of ‘temperature’ in the consensus formation dynamics. This decision rule can be reformulated in terms of the ‘spin flip’ probability $w_i$ that the site $i$ with the current spin state $\sigma_i$ will flip its spin state at the step, which can be expressed as

$$w_i(\sigma) = \frac{1}{2} \left[ 1 - (1 - 2q) \sigma_i \text{sgn} \left( \sum_{j \in \partial} \sigma_j \right) \right], \quad (1)$$

where $\text{sgn}(x)$ is signum function [defined as $\text{sgn}(x) = 1$ for $x > 0$, $-1$ for $x < 0$, and $0$ for $x = 0$] and the summation runs over the nearest neighbors of $i$ denoted as $\partial$.

3. Multiplex majority-vote processes: OR- and AND-models

Our aim in this paper is to investigate the effects of multiplexity to consensus formation dynamics using the majority-vote model. To this end, we generalize the majority-vote model to voters on multiplex networks. Let us suppose that a person interacts via two social network layers such as the family layer and the fellow workers layer. In general the ‘local’ majority opinion within each of the two layers may or may not be the same. Facing such a multiplex social environment, the person has to make decision on which majority she would follow. The basic rationale of multiplex network approach is that the decision rule is formulated layerwise. To implement this rationale concretely, we define two multiplex decision rules, the so-called OR- and AND-models, for the majority-vote processes in multiplex networks.

3.1. OR-model

The OR-voters takes the layerwise majority opinions disjunctively, that is, they tend to follow the majority of any one of the layers. Specifically, we define the following dynamic rules at each step for the OR-model:

(i) A site is chosen randomly and this site looks at its nearest neighbor sites’ states in the randomly-chosen layer.

(ii) If there is a majority state in the chosen layer, the site takes the majority-vote state with probability $1 - q$, while with the rest probability $q$ it takes the opposite (minority) state.

(iii) In case of no majority state in the chosen layer, it takes any of the binary states with equal probability.

The OR-voters behave like the usual majority-voters except that they randomly switch their opinion-consulting layer at each step.
The spin flip probability of the OR-model can be expressed as

\[ w_i^{\text{OR}}(\{\sigma\}) = \frac{1}{\mathcal{L}} \sum_\alpha \frac{1}{2} \left[ 1 - (1 - 2q)\sigma_i \text{sgn} \left( \sum_{j \in \mathcal{N}_\alpha} \sigma_j \right) \right] \]

\[ = \frac{1}{2} \left[ 1 - (1 - 2q)\sigma_i \frac{1}{\mathcal{L}} \sum_\alpha \text{sgn} \left( \sum_{j \in \mathcal{N}_\alpha} \sigma_j \right) \right], \quad (2) \]

where \( \mathcal{L} \) is the total number of layers; \( \alpha \) denotes the network layer index; \( \mathcal{N}_\alpha \) denotes the set of nearest neighbors of \( i \) in the \( \alpha \)-layer. From equation (2), the OR-voters may also be interpreted as the voters following the average majority opinions among all the layers.

### 3.2. AND-model

We suppose that the AND-voters takes the layerwise majority opinions conjunctively, that is, they tend to follow the common majority of all the layers. Specifically, we define the following dynamic rules at each step for the AND-model:

(i) A site is chosen randomly and this site looks at its nearest neighbor sites’ states separately in each layer.

(ii) If there is a common majority state among all the layers, the site takes this common majority-vote state with probability \( 1 - q \), while with the rest probability \( q \) it takes the opposite (minority) state.

(iii) In case of no common majority, it remains at its current state.

Note the third step, where we made additional modification to the majority-vote model by assuming that the AND-voters do not care to update their state when there is no common majority over all layers instead of flipping its state randomly. This is a quite strict way of applying the conjunctive rule in the decision process, which we take deliberately to discern the effect of conjunctive multiplexity to the greatest. The effect of this specific rule will further be discussed by comparing other multiplex scenarios of majority-vote processes in section 4.2.

The spin flip probability of the AND-model can be expressed as

\[ w_i^{\text{AND}}(\{\sigma\}) = \frac{1}{\mathcal{L}} \left[ 1 - (1 - 2q)\sigma_i \text{sgn} \left( \sum_{j \in \mathcal{N}_\alpha} \sigma_j \right) \right] \times \delta \left( \mathcal{L} - \left| \sum_\alpha \text{sgn} \left( \sum_{j \in \mathcal{N}_\alpha} \sigma_j \right) \right| \right), \quad (3) \]

where \( \delta(x) \) denotes the delta function and \( \alpha_0 \) can be any of the layers \( \alpha = 1, \ldots, \mathcal{L} \).

### 3.3. Numerical simulations

We perform Monte Carlo simulations of the multiplex models on two-layer multiplex networks. Nodes are assigned their initial states at random. Each step, a node, say \( i \), is chosen at random and the spin-flip probability \( w_i \) is calculated. The node flips its spin state \( \sigma_i \) stochastically with probability \( w_i \). \( N \) consecutive steps defines one Monte Carlo time. Simulation continues for a time long enough to reach the stationary state. Typically, our simulations run for the Monte Carlo time up to \( t = 2^{16} \) (OR- and AND-model with odd degree) and \( t = 2^{20} \) (AND-model with even degree) on networks with sizes up to \( N \leq 2 \times 10^5 \).

Our main quantity of interest is the consensus level or the ‘magnetization’ \( M \) of the system. We first define the instantaneous magnetization \( m_t \) of the spin configuration \( \{\sigma(t)\} \) at time \( t \) as

\[ m_t = \left| \frac{1}{N} \sum_{i=1}^N \sigma_i(t) \right|, \quad (4) \]

where \( N \) is the total number of nodes. Magnetization \( M \) is given by its average as

\[ M = \langle m_t \rangle, \quad (5) \]

where \( \langle \cdots \rangle \) denotes the temporal average calculated in the stationary state for the interval \( (t_{\text{final}}/2, t_{\text{final}}) \) where \( t_{\text{final}} \) is the final time of the simulation; and \( \langle \cdots \rangle \) is the ensemble average over different simulation runs and network configurations. \( M \) takes the value 1 if all the nodes are in the same spin state, that is, in complete consensus, and 0 if the nodes are split into two equal-size groups of opposite opinions without a clear majority opinion. In that sense, \( M \) may be called colloquially the consensus level of the system.

In this study we examined the two cases of duplex networks: one with layers of random regular networks and the other with layers of Erdős–Rényi networks.
with the order parameter exponent for different network sizes.

Figure 1. The magnetization $M$ on $k$-RR duplexes with degrees $k$ from 3 to 10 for the OR-model (a) and the AND-model (b). Both the symbols and connecting lines are the Monte Carlo simulation results. Note that for the OR-model, the data for $k = 7, 8$ and $k = 9, 10$, respectively, almost overlap. (c), (d) Alternative plots for the results of OR- and AND-model displayed simultaneously for a comparison on 3-RR and 4-RR duplexes, respectively. (e) Phase diagram of the OR- and the AND-model on $k$-RR duplexes. Solid boundary lines are for continuous phase transitions and dashed lines are for discontinuous phase transitions.

4. Consensus transition on duplexes with random regular network layers

We first consider the multiplex models on two-layer (duplex) networks with each layer being random regular network of the same degree $k$ (referred hereafter to as $k$-RR duplexes for short). Figure 1 displays the main results of this study. OR-model exhibits a continuous transition (figure 1(a)) whereas the AND-model displays more abrupt, and even discontinuous for even $k$, change of consensus level at the transition (figure 1(b)). We highlight two notable features: first, for a given ensemble of networks (that is, given $k$), AND-model tends not only to yield higher consensus level (larger $M$) than OR-model, but also to hold the consensus up against higher level of noise (larger $q_c$, with the exception for $k = 4$) (figures 1(c), (d)). Secondly, distinct behavior is observed for random regular duplexes with even degrees where the consensus transition for AND-mode turns into a discontinuous one (figure 1(b)), in marked contrast to the continuous transitions in other cases. To summarize those results, the phase diagram shows different consensus levels and phase transitions between OR and AND model (figure 1(e)). In the following, we present further results and discussions to elaborate on these main findings.

4.1. Nature of consensus transitions

In order to discern the nature of consensus transitions of the AND-model in different cases, we examine the behaviors in the vicinity of the transitions more closely. To this end, we calculate the Binder’s 4th-order cumulant $U$ given by

$$U = 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2},$$

for different network sizes $N$. For 3-RR duplex, $U(N)$ for different network size $N$ crosses at $q_c = 0.186$ (1) (figure 2(a)), which is a signature of continuous transition at $q_c = q_c$. On the contrary, for 4-RR duplex, $U(N)$ does not show clear crossing but the curves tend to converge to a step-function change at $q_{\text{step}} = 0.152(1)$ (figure 2(b)), which is a typical feature of the discontinuous transition with $q_c = q_{\text{step}}$.

The critical behaviors at the continuous transition of 3-RR duplex are further examined by the finite-size scaling analyses of $U$ and $M$, using the standard finite-size scaling ansatz

$$U(N) = \tilde{U}[(q - q_c)N^{1/\nu}],$$

$$M(N) = N^{-\beta/\nu} \tilde{M}[(q - q_c)N^{1/\nu}],$$

with the order parameter exponent $\beta$ and the correlation volume exponent $\nu$. Despite the apparently more abrupt change of $M$ near the transition point of the AND-model with odd-$k$ than those of the OR-model (figure 1), the finite-size scaling analyses on 3-RR duplex (figures 2(c), (d)) suggest that the critical behaviors of
the AND-model on (odd k)-RR duplexes are consistent with the mean-field exponents $\beta = 1/2$ and $\nu = 2$ for the consensus transitions of the majority-vote model in the single-layer networks [25]. We also checked that the OR-model shows the mean-field behaviors as expected from its spin-flip probability, equation (2).

The discontinuous consensus transitions observed for the AND-model on (even k)-RR networks deserve closer examinations as well. To this end, we calculated the probability distribution of instantaneous magnetization at random moment, $P(m_t)$, in the long time limit, which are shown in figure 3 for cases of $k = 7$ and $k = 8$, representative of continuous and discontinuous transition, respectively. The calculations are done for the networks of relatively small size $N = 12500$ to observe transient behaviors clearly. The statistics were collected for time interval from $t = 2^{19}$ to $t = 2^{20}$ Monte Carlo times and averaged over 50 different networks. For 7-RR networks, the distribution exhibits a single peak, whose location varies continuously from $m_{\text{peak}} = 0$ for $q > q_c$ through $m_{\text{peak}} > 0$ for $q < q_c$ (figures 3(a)–(c)). In contrast, for 8-RR networks, the distribution

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**Figure 2.** Binder’s forth-order cumulant $U$ on $k$-RR duplexes with degree $k = 3$ (a) and $k = 4$ (b). Simulation results with network sizes ranging $N = 6250$ to $2 \times 10^5$, averaged over 50 network configurations, are used for data analysis. The transition point is estimated as $q_c = 0.1861 \pm 0.0001$ ($k = 3$) and $q_c = 0.152 \pm 0.001$ ($k = 4$), respectively. Finite-size scaling plots of $U$ and $M$ for 3-RR duplexes based on equations (7) and (8) are displayed in (c) and (d), using the mean-field critical exponents $1/\nu = 1/2$ and $\beta = 1/2$.

**Figure 3.** Distributions of instantaneous magnetization $P(m_t)$ (a)–(c), (e)–(g) and typical transient time trajectory of $m_t$ (d), (h) of the AND-model on RR duplexes with $k = 7$ (a)–(d) and $k = 8$ (e)–(h). Simulations are performed with network size $N = 12500$ and the distributions are obtained over the Monte Carlo time interval $[2^{19}, 2^{20}]$ and over 50 network configurations. Noise parameters used are $q = \{0.307, 0.308, 0.309\}$ for $k = 7$ and $q = \{0.305, 0.306, 0.307\}$ for $k = 8$. Newly, the AND-model on (odd k)-RR duplexes are consistent with the mean-field exponents $\beta = 1/2$ and $\nu = 2$ for the consensus transitions of the majority-vote model in the single-layer networks [25]. We also checked that the OR-model shows the mean-field behaviors as expected from its spin-flip probability, equation (2).
develops double peaks as the system approaches to the transition point, across which the location of the higher dominant peak changes discontinuously (figures 3(c)–(g)). Also shown are the typical time series of $m_i$ at the transition point $q_c$ (figures 3(d), (h)). The time series fluctuates around $m = 0$ for a 7-RR network (figure 3(d)). On the other hand, it displays transient behavior manifesting stochastic switchings between $m = 0$ and $m = \pm m_{\text{peak}} \approx \pm 0.3$ (figure 3(h)), resulting in the double-peak distribution in figure 3(f). These observations support the conclusion that the consensus transition of the AND-model on (even $k$)-RR networks is genuinely discontinuous.

We also considered RR-duplexes with different layer mean degrees (figure 4(a)). Of particular interest is the case where one layer is of odd degree and the other is of even degree. We found that for the cases of $(k_1, k_2) = (3, 4)$ and $(k_1, k_2) = (3, 6)$ the consensus transition remains discontinuous. This suggests further the important role of even-degree nodes in establishing the discontinuous transition in the AND-model.

### 4.2. Impact of multiplexity

To highlight the impact of multiplexity, in figure 4(b) we show comparisons between the multiplex models and single-layer majority-vote processes. Along with the multiplex OR- and AND-model on $k$-RR duplexes, shown are the results of majority-vote model on single-layer RR network with degree $k$ (denoted single) and $2k$ (denoted double), for the case of $k = 6$. Also shown is the multiplex majority-vote model studied in [27] (denoted KGS).

We note two features: first, the consensus level of OR-model is equal to neither of the single and double cases but lies in the intermediate level between them. Second, for the AND-model, the transition point is between those of single and double and the consensus level below $q_c$ is even larger, albeit slightly, than that of double. These results illustrate primary multiplex effects: in understanding the majority-rule-driven consensus dynamics on multiplex social networks, ignoring the presence of other social interaction layer (single) will underestimate both the consensus level ($M$-value) and the tolerance of it against the noise ($q_c$-value); on the other hand, ignoring the multiplexity of interactions and treating them as simple aggregate (double) will generally overestimate the consensus level and its tolerance to noise, but also with the flip-side possibility that even higher consensus could be attained when the conjunctive multiplexity is sufficiently strong as in our AND-model.

The discontinuous transition displayed by the AND-model on (even $k$)-RR duplexes is reminiscent of discontinuous or hybrid transitions in other multiplex models such as the mutual percolation [5] and the multiplex threshold cascade dynamics [28]. We have assumed for our AND-model that a node would not change its state when it does not have a common majority state across all the layers. This creates an ‘inertia’-like effect [26] against the local field produced by the neighboring nodes as well as the stochastic noise parametrized by the $q$ factor. The specific way we formulated the AND-model is to maximize this effect. The KGS model studied in [27], which can be thought of as a variant of AND-model, is also found to display discontinuous transition in 6-RR duplexes, however, both the consensus level and the critical noise parameter are much suppressed compared to the AND-model introduced in this paper (figure 4(b)). It suggests that the random flipping rule of KGS model in case of no common majority has detrimental effect in attaining global consensus. Note that [27] reported continuous consensus transitions, which we speculate to be due to the large mean degrees of layers considered therein.
4.3. Time to consensus

We observed that the AND-model could attain larger consensus level than the OR-model as well as the single-layer counterparts below $q_f$. In achieving such largest consensus level, the inertia-like rule (zero flip-probability for no common majority) is thought to play an important role, by inducing the ratchet-like effect to drive the system towards increasing consensus against the fluctuating local fields and noise. A node without the common majority would ‘wait’, without changing its state, until the common majority is formed in its neighborhood. This could have the effect of slowing down the dynamics towards the consensus. To examine the timescale of consensus formation quantitatively, we define the consensus time $t_C$ of each simulation run by

$$t_C = \inf \{ t : |m_t - \bar{m}| < \sigma_m \},$$

where $\bar{m}$ and $\sigma_m$, respectively, are as in equation (5) the time average and standard deviation of $m_t$ at the final time interval ($t_{\text{final}}/2$, $t_{\text{final}}$), where $t_{\text{final}}$ is the total simulation time (typically we use $t_{\text{final}} = 2^{20}$). $t_C$ gives the timescale after which the system can be thought to reach the stationary state.

Figure 5 shows the results regarding the consensus time. The mean consensus time $\langle t_C \rangle$ is plotted in figure 5(a) and typical time courses of $m_t$ for different models in figure 5(b). These results show that the AND-model takes considerably longer time to reach the stationary consensus state, often several orders of magnitude longer than other models do. As a result, one might expect that the AND-model requires much longer time to arrive the final consensus state than other models (figure 5). Furthermore, the evolution of consensus level in time is step-like rather than gradual (in logarithmic timescale). Consequently, if one were concerned with the consensus level attainable by a fixed amount of time, one would arrive at a different conclusion than that for the stationary-state answer in the previous sections. For example, as illustrated in figure 5(b), for the 4-RR duplex with timescale of $t \approx 2^7$, the OR-model can achieve larger consensus ($t > t_C \approx 2^7$) than the AND-model, because the AND-model requires much longer time ($t < t_C \approx 2^{12}$) to reach its full consensus level and remains nearly zero consensus state at the time of interest $t \approx 2^7$.

The above analyses on time to consensus demonstrate that indeed the slowing-down effect in the AND-model can be prominent. Such effect gets more pronounced for weaker noise as $(t_C)$ increases considerably as $q$ decreases for the AND-model whereas it saturates for other models as seen in figure 5(a). It also highlights the importance of consideration of timescales in consensus formation dynamics.

5. Majority-vote processes on duplexes with Erdős–Rényi layers

The analyses on the RR duplexes in the previous section have unveiled important role of the parity of node degrees in the network on multiplex majority-vote processes, especially the AND-model. In reality, nodes with different degrees form a network. As a next step, we consider the AND- and OR models on duplexes of Erdős–Rényi network layers of common mean degree $\langle k \rangle$ (hereafter referred to as ER duplexes, for short). In ER duplexes some nodes may have no links ($k = 0$) in one or all layers. The nodes with no links in both layers do not participate to the dynamics. For the nodes with neighbors in only one layer, we assume that they work as single-layer network voters, that is, they update their state following equation (1).

Results of numerical simulations of the AND- and OR-model on ER duplexes are summarized in figure 6. The AND-model displays discontinuous transition with nonzero jump of consensus level at the transition point (figure 6(a)), whereas the transition is continuous for OR-model (figure 6(b)). The discontinuity in AND-model on ER duplexes is weaker than that on (even $k$)-RR duplexes. For example, for ER duplexes with $\langle k \rangle = 6$, we observe the bimodal-peak distribution of magnetization near the transition point and the multistable
transient switching dynamics, both indications of discontinuous transition, albeit with less pronounced peaks and noisier switchings (figures (c)–(e)). This is likely due to the competition between the even-degree nodes that cause discontinuity and the odd-degree nodes that do not. The discontinuity gets lessened as the mean degree \( \langle k \rangle \) decreases and it becomes barely identifiable for \( \langle k \rangle \lesssim 3 \). It is not fully clear whether the discontinuity disappears completely or stays at very small nonzero value in this case. More detailed analysis is called for to answer it conclusively.

Overall, the qualitative behaviors of the multiplex majority-vote processes on ER duplexes can be understood from their behaviors on RR duplexes. These include the nature of transitions (discontinuous for AND- and continuous for OR-model) and the strength of consensus (larger \( M \) and larger \( q \), of AND- than OR-model in most mean degrees). Quantitative questions remain as to both the accurate consensus level and the precise nature of transition for the AND-model on small mean degrees, which we leave for future study.

6. Approximate master equations and pair approximation

Numerical solutions for the multiplex majority-vote processes, the OR- and AND-model, on multiplex networks are obtained by using the approximate master equation formalism [29]. The master equation is set up by following [29] with modifications to account for the multiplexity. To start with, the binary opinion \( \alpha \) is the degree in the \( \alpha \)-layer and \( m_\alpha \) is the number of con-neighbors in \( \alpha \)-layer. The multiplex network is then characterized by the joint degree distribution \( q_{\alpha \beta} \). Distributions of instantaneous magnetization \( M_t \) and typical transient time trajectory of \( M_t \) of the AND-model in the case of \( \langle k \rangle = 6 \) at \( q = 0.279, 0.280, 0.281 \), displaying bimodality at \( q = 0.280 \). Simulations are done with networks of size \( N = 5 \times 10^4 \) and the distributions are obtained with 50 network configurations.

Figure 6. (a), (b) Consensus level \( M \) of OR- (a) and AND-model (b) on ER duplexes with layer mean degree \( z = 3, 4, 5, 6, 7, 8 \). Both the symbols and the connecting lines are the Monte Carlo simulation results. (c)–(f) Distributions of instantaneous magnetization \( P(m_t) \) and typical transient time trajectory of \( m_t \) for the AND-model on small mean degrees, which we leave for future study.
The first line of the right-hand side of equations (10), (11) accounts for the changes of $p_{k,m}(t)$ and $c_{k,m}(t)$ due to the update of the node’s own state, whereas the second and third lines describe the changes due to state updates of the node’s neighbors. These terms can be expressed in terms of the transition probabilities $F_{k,m}$ and $R_{k,m}$. $F_{k,m}$ denotes the probability that a pro-node with k-degree and m-con-neighbors switches into a con-node. Similarly, the reverse transition rate $R_{k,m}$ is the probability that a con-node with k-degree and m–con-neighbors turns into a pro-node. These transition probabilities should encode the layerwise majority-vote rule of the specific model. They are given in the two multiplex models as follows: for OR-model, they are given by

\begin{equation}
F_{k,m}^{\text{OR}} = \frac{1}{\mathcal{L}} \left[ A_{k,m} q + B_{k,m} (1 - q) + \frac{1}{2} C_{k,m} \right],
\end{equation}

\begin{equation}
R_{k,m}^{\text{OR}} = \frac{1}{\mathcal{L}} \left[ A_{k,m} (1 - q) + B_{k,m} q + \frac{1}{2} C_{k,m} \right],
\end{equation}

whereas for the AND-model they are given by

\begin{equation}
F_{k,m}^{\text{AND}} = \frac{1}{\mathcal{L}} \left[ A_{k,m} q \delta(a - \mathcal{L}) + B_{k,m} (1 - q) \delta(b - \mathcal{L}) \right] ,
\end{equation}

\begin{equation}
R_{k,m}^{\text{AND}} = \frac{1}{\mathcal{L}} \left[ A_{k,m} (1 - q) \delta(a - \mathcal{L}) + B_{k,m} q \delta(b - \mathcal{L}) \right].
\end{equation}

Here the factors $A_{k,m}$ and $B_{k,m}$ denote the number of layers where pro and con are the majority, respectively, whereas $C_{k,m}$ is the number of layers with no majority; that is, $A_{k,m} = \left\{ \alpha | m_\alpha > \frac{k}{2} \right\}$ and $C_{k,m} = \left\{ \alpha | m_\alpha = \frac{k}{2} \right\}$.

In the latter part of equations (10), (11) $m_\alpha^+$ is used as a shorthand notation to denote the con-number vector which differs from $m$ only in the $\alpha$-layer by a single unit, ±1, that is, $m_\alpha^+ \equiv (\ldots, m_\alpha \pm 1, \ldots)$. The factors $\beta$s and $\gamma$s denote the transition probabilities as follows: $\beta^p$ is the probability that an edge from a pro-node changes from pro–pro to pro–con due to the state-flipping of the pro-neighbor; $\gamma^p$ is the probability that an edge from a pro-node changes from pro–con into pro–pro due to the state-flipping of the con-neighbor. Likewise, $\beta^c$ is the probability that an edge from a con-node changes from con–pro to con–con due to the state-flipping of the pro-neighbor; $\gamma^c$ is the probability that an edge from a con-node changes from con–con into con–pro due to the state-flipping of the con-neighbor. The transition probabilities $\beta$s and $\gamma$s are time-dependent. To make the equations amenable to solution, mean-field-type approximation is employed, which replaces these terms by the average transition probabilities. Under the pair approximation, they are given by

\begin{equation}
\beta^p = \frac{\sum_k \sum_\alpha \sum_m \xi (k_\alpha - m_\alpha) F_{k,m} p_{k,m}}{\sum_k \sum_\alpha \sum_m \xi (k_\alpha - m_\alpha) p_{k,m}},
\end{equation}

\begin{equation}
\gamma^p = \frac{\sum_k \sum_\alpha \sum_m \xi (k_\alpha - m_\alpha) R_{k,m} c_{k,m}}{\sum_k \sum_\alpha \sum_m \xi (k_\alpha - m_\alpha) c_{k,m}},
\end{equation}

\begin{equation}
\beta^c = \frac{\sum_k \sum_\alpha \sum_m m_\alpha F_{k,m} p_{k,m}}{\sum_k \sum_\alpha \sum_m m_\alpha p_{k,m}},
\end{equation}

\begin{equation}
\gamma^c = \frac{\sum_k \sum_\alpha \sum_m m_\alpha R_{k,m} c_{k,m}}{\sum_k \sum_\alpha \sum_m m_\alpha c_{k,m}}.
\end{equation}

Here, the average $\langle \cdots \rangle$ is taken over the joint degree distribution $P_k$. The summations $\sum_k$ and $\sum_\alpha^c$ are shorthand notations for $\sum_{m_\alpha=0}^{k_{\alpha}} \sum_{m_\alpha=0}^{k_{\alpha}} \cdots \sum_{m_\alpha=0}^{k_{\alpha}} \cdots \sum_{m_\alpha=0}^{k_{\alpha}}$, respectively.

The theoretical solution is obtained from the equations (10) and (11) by iterating them until stationary values of $p_{k,m}$ and $c_{k,m}$ are obtained. One can calculate the fraction of cons with k-degree as $\rho_k = 1 - \sum_m p_{k,m}$; from which, finally, the consensus level $M$ is obtained as

\begin{equation}
M = \left| 1 - 2 \sum_k \rho_k P_k \right|,
\end{equation}

the results of which are plotted in figure 7.

The results of the approximate theoretical solution (figure 7) corroborate the conclusions for the nature of transitions and the critical behaviors drawn from numerical simulations. The analytic results agree well with the
numerical simulation results in the OR-model and in the AND-model for large $k$. For the AND-model for small $k$, the deviations from numerics are noticeable, but even in this case the approximate analytic calculation predict correctly the qualitative features, such as the nature of transitions and the increase of $q_c$ with $k$.

7. Summary and discussion

In this paper, we have studied majority-vote processes on multiplex networks, by introducing two toy models, the OR- and AND-model, implementing the layerwise majority-rule in disjunctive and conjunctive manner, respectively. The results illustrate the impact of multiplexity. Both the multiplex majority-vote processes are found to behave differently from those on the isolated single-layer network and on the network with a simply-aggregated layer, strengthening the premise that the multiplex dynamic processes cannot be reduced to a single-layer one [11–13]. The AND-model can generally attain larger consensus level than OR-model but it may take considerably longer time to achieve such a higher consensus, making the available timescale of the process as an important factor as the dynamic rule itself. We also found that the AND-model can be implemented to induce discontinuous consensus transition on multiplex networks dominated by even-degree nodes, suggesting that the conjunctive layerwise decision rule can be a natural way for the inertia-like factor in the majority-vote processes to be implemented to drive discontinuous phase transitions in multiplex social networks.

Our study adds to the continuing effort of discerning general effects of multiplexity on dynamic processes on multiplex networks, compared to conventional single-layer ones. In this respect, the majority-vote processes considered in this paper has particular feature that nodes are described by a unique binary state across all the layers and the state evolution does not possess the so-called permanently active property [30]. Our results show that discontinuous transition can occur, under right conditions, in this class of dynamic processes. Several questions remain from the perspective of both the current models and the multiplex dynamic processes in general. For the current models, issues of immediate interest would include, but not limited to, identifying the weakest condition for the discontinuity in the conjunctive rule of AND-type model and improving the analytic approximation method towards better solutions for the AND-model. Extensions of the present study to more realistic network structures as well as more realistic and detailed layerwise majority-rules would also be worthwhile. From a broader theoretical perspective, still missing is a generic theoretical framework of multiplex processes, both equilibrium and nonequilibrium, from which one can understand at the fundamental level the governing physics underlying the emergent characteristics such as the discontinuity and/or hysteresis in multiplex systems. We hope this work could contribute to invigorating ongoing endeavors in all these diverse directions.

Figure 7. Numerical solutions obtained by the approximate master equations with pair approximation. Displayed are the results for the OR-model (a), (c) and the AND-model (b), (d) on $k$-RR duplexes (a), (b) and on ER duplexes with layer mean degree $\langle k \rangle$ (c), (d), respectively. Solid lines are the numerical solutions and the symbols are corresponding Monte Carlo simulation results.
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