Evanescent modes stored in cavity resonators with backward-wave slabs

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Abstract

As was shown by N. Engheta, electromagnetic fields in two adjacent slabs bounded by two metal walls can satisfy the boundary conditions even if the distance between the two walls is much smaller than the wavelength. This is possible if one of the slabs has a negative permeability. Here we show that these subwavelength “resonators” resonate only if the permeability of at least one of the slabs is frequency dependent. Thus, there is no advantage of using these structures as frequency-selective devices. However, we have found that these systems can be in principle used as memory devices for evanescent fields, because the boundary conditions are identically satisfied for all plane evanescent waves inside the cavity. The physical meaning and practical limitations for possible realizations are discussed. The analysis is supported by FDTD simulations.

Key terms: evanescent modes, subwavelength resonator, Veselago materials, left-handed materials, double negative materials, FDTD
1 Introduction

Recently, a lot of attention has been payed to electromagnetic properties of materials with negative parameters (called Veselago media, backward-wave media, double negative materials, left-handed materials). Conceptualized by V.G. Veselago [1] and realized by R.A. Shelby et. al. as a composite with metal inclusions of specific shapes [2], these materials are very much debated because of their exotic properties and potential applications. One of the applications is perfect lens, (theoretically) capable to focus not only propagating, but also evanescent modes [3]. Contradictory opinions about realizability of this device have been published in the literature [4].

Another exciting idea is a possibility to design subwavelength resonators [5]. N. Engheta has shown that a pair of plane waves traveling in the system of two planar slabs positioned between two metal planes can satisfy the boundary conditions on the walls and on the interface between two slabs even for arbitrarily thin layers, provided that one of the slabs has negative material parameters. In paper [5], waves propagating in the direction orthogonal to the interface and the metal walls have been considered. The possibility to satisfy the boundary conditions for small distances between metal plates is based on the fact that plane waves in Veselago media are backward waves, meaning that the phase shift due to propagation in a usual slab can be compensated by a negative phase shift inside a backward-wave slab.

In this paper, we analyze the system proposed by Engheta, both analytically and numerically, for arbitrary wave solutions. The geometry of the problem is shown in Figure 1: it is a two-layer planar waveguide, and we study general field solutions in form of propagating or evanescent modes with arbitrary tangential propagation factors $k_t$. The results allow better understanding of the phenomenon predicted by N. Engheta, and lead to important conclusions regarding evanescent modes in the system. It will be shown that evanescent excitations can be “stored” in the resonator volume after the external field has been removed.

![Figure 1: Planar two-layer waveguide. One of the slabs can be a Veselago medium.](image)

2 Eigenvalue equations

Let us assume that the field between the two metal walls with an arbitrary distribution in a plane orthogonal to $\mathbf{n}$ is expanded into a Fourier integral or series in that plane. The field harmonics are plane waves traveling in the transverse plane with two-dimensional
wave vectors $\mathbf{k}_t$. The corresponding propagation factors along $\mathbf{n}$ we denote by

$$\beta_{1,2} = \sqrt{k_{1,2}^2 - k_t^2} \quad (1)$$

In the Fourier domain, the exact boundary condition on the free interface of a slab backed by an ideally conducting surface reads (e.g., [6]):

$$\mathbf{E}_{t+} = j\omega\mu_{1,2} \frac{\tan \beta_{1,2} d_{1,2}}{\beta_{1,2}} \overrightarrow{A}_{1,2} \cdot \mathbf{n} \times \mathbf{H}_{t+} \quad (2)$$

where

$$\overrightarrow{A}_{1,2} = \overrightarrow{I}_t - \frac{\mathbf{k}_t \mathbf{k}_t}{k_{1,2}^2} = \frac{\beta_{1,2}^2 \mathbf{k}_t \mathbf{k}_t}{k_{1,2}^2} + \frac{\mathbf{n} \times \mathbf{k}_t \mathbf{n} \times \mathbf{k}_t}{k_t^2} \quad (3)$$

Because the tangential fields $\mathbf{E}_{t+}$ and $\mathbf{n} \times \mathbf{H}_{t+}$ are continuous on the interface between the two slabs, we can write

$$\left(j\omega\mu_1 \frac{\tan \beta_1 d_1}{\beta_1} \overrightarrow{A}_1 + j\omega\mu_2 \frac{\tan \beta_2 d_2}{\beta_2} \overrightarrow{A}_2\right) \cdot \mathbf{n} \times \mathbf{H}_{t+} = 0 \quad (4)$$

Solution for the eigenwaves is now very easy because dyadics $\overrightarrow{A}_{1,2}$ are diagonal with the same set of eigenvectors: $\mathbf{k}_t$ and $\mathbf{n} \times \mathbf{k}_t$. Writing the two-dimensional vector $\mathbf{n} \times \mathbf{H}_{t+}$ in this basis:

$$\mathbf{n} \times \mathbf{H}_{t+} = a \mathbf{k}_t / |k_t| + b \mathbf{n} \times \mathbf{k}_t / |k_t| \quad (5)$$

and substituting into (4), we arrive to equations for the propagation constant $k_t$. If $a \neq 0$ and $b = 0$, vector $\mathbf{H}_{t+}$ is directed along $\mathbf{n} \times \mathbf{k}_t$, that is, orthogonal to the propagation direction. This gives the TM mode solution. The eigenvalue equation in this case is

$$\frac{\beta_1}{\epsilon_1} \tan \beta_1 d_1 + \frac{\beta_2}{\epsilon_2} \tan \beta_2 d_2 = 0 \quad (6)$$

For the other mode, when $b \neq 0$ and $a = 0$, the magnetic field vector is along $\mathbf{k}_t$ (TE mode), and we get

$$\frac{\mu_1}{\beta_1} \tan \beta_1 d_1 + \frac{\mu_2}{\beta_2} \tan \beta_2 d_2 = 0 \quad (7)$$

Note that the square root branch defining the normal components of the propagation factors (1) can be chosen arbitrarily, which is natural for the system with standing waves.

Let us now assume that the thicknesses of the two layers are the same ($d_1 = d_2 = d$), and the material parameters differ by sign: $\mu_2 = -\mu_1 = -\mu$, $\epsilon_2 = -\epsilon_1 = -\epsilon$. In this particular case the eigenvalue equations (6) and (7) are satisfied identically for all $\mathbf{k}_t$! This result can be perhaps better understood deriving the matrix relation between tangential fields on the two metal boundaries. The exact solution for tangential fields on the two sides of an isotropic slab can be conveniently written in matrix form as (e.g. [6]):

$$\left( \begin{array}{c} \mathbf{E}_{t+} \\ \mathbf{n} \times \mathbf{H}_{t+} \end{array} \right) = \left( \begin{array}{cc} \overline{\tau}_{11} & \overline{\tau}_{12} \\ \overline{\tau}_{21} & \overline{\tau}_{22} \end{array} \right) \cdot \left( \begin{array}{c} \mathbf{E}_{t-} \\ \mathbf{n} \times \mathbf{H}_{t-} \end{array} \right) \quad (8)$$

The dyadic coefficients in this matrix read

$$\overline{\tau}_{11} = \overline{\tau}_{22} = \cos(\beta d)\overrightarrow{I}_t \quad (9)$$
\( a_{12} = \frac{j\omega \mu}{\beta} \sin(\beta d)A, \quad a_{21} = \frac{j\omega \epsilon}{\beta} \sin(\beta d)C \) \hspace{1cm} (10)

where \( \overline{I}_t \) is the transverse unit dyadic, \( \overline{A} \) is given by (3), and
\[
\overline{C} = \overline{I}_t - \frac{n \times k_t n \times k_t}{k^2} = \frac{k_t k_t}{k^2} + \frac{\beta^2 n \times k_t n \times k_t}{k^2} \hspace{1cm} (11)
\]

The total transmission matrix for the system of two slabs is the product of matrices (8) for individual slabs. For the slab with negative parameters we have
\[
\begin{pmatrix}
\mathbf{E}_{t+} \\
\mathbf{n} \times \mathbf{H}_{t+}
\end{pmatrix} = \begin{pmatrix}
a_{11} & -a_{12} \\
-a_{21} & a_{22}
\end{pmatrix} \cdot \begin{pmatrix}
\mathbf{E}_{t-} \\
\mathbf{n} \times \mathbf{H}_{t-}
\end{pmatrix} \hspace{1cm} (12)
\]

Simple calculation\(^1\) shows that in this particular case
\[
\begin{pmatrix}
\overline{a}_{11} & \overline{a}_{12} \\
\overline{a}_{21} & \overline{a}_{22}
\end{pmatrix} \cdot \begin{pmatrix}
\overline{a}_{11} & -\overline{a}_{12} \\
-\overline{a}_{21} & \overline{a}_{22}
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \overline{I}_t \hspace{1cm} (13)
\]

identically for any \( k_t \) and arbitrary thickness \( d \) of the layers. This fact means that the boundary conditions at the metal walls are satisfied not only for normally propagating waves, but also for obliquely traveling waves, and, most important, for all evanescent modes.

### 3 Discussion

Let us first discuss the possible use of this system as a subwavelength resonator, as proposed by N. Engheta \[5\]. If the propagation constant along the slabs \( k_t = 0 \), both equations for TM and TE modes reduce to
\[
\frac{\mu_1}{k_1} \tan k_1 d_1 + \frac{\mu_2}{k_2} \tan k_2 d_2 = 0 \hspace{1cm} (14)
\]

that is the resonance condition for standing waves in the dual-layer system between two metal plates. If the thicknesses of both layers are small compared with the wavelength, one can simplify this equation replacing tangent functions by the first terms of their Taylor expansions:
\[
\mu_1 d_1 + \mu_2 d_2 = 0 \hspace{1cm} (15)
\]

From here it is obvious that if both permeabilities are positive (or both negative), no resonance is possible in thin layers: the thickness should be of the order of the wavelength (half-wavelength resonance). However, as was noticed by N. Engheta \[5\], if one of the permeabilities is negative, condition (15) can be satisfied even for very thin layers. This appears to open a possibility to realize very compact resonant cavities.

However, let us think what kind of resonator we get this way? Resonance as such means that a certain circuit function (reflection coefficient, input impedance . . . ) sharply changes with the frequency. In the conventional case of positive media parameters, the

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\(^1\)Calculating the product, note that \( \overline{A} \cdot \overline{C} = \frac{\beta^2}{k^2} \overline{I}_t \).
two-layer cavity is a resonator because the tangent functions in (14) quickly vary with respect to the frequency near the resonance. Now look at relation (15): there is no explicit dependence on the frequency at all! So, if the material parameters are assumed to be approximately frequency independent over a certain frequency range, the system does not resonate, although the “resonance condition” (15) is satisfied. Indeed, equation (15) is satisfied for all frequencies in this range, thus this system is not frequency selective. What actually happens is that the inductive reactive part of the input impedance of a usual layer \((+j\omega\mu_1d_1)\) is compensated by the negative inductive impedance of the other layer \((-j\omega|\mu_2|d_2)\). What is left is only resistance, defined by losses in the materials of the slab. Of course, in realistic situations material parameters are frequency dependent, but since the frequency variations in the response functions are determined by the permeability only and do not depend on the frequency explicitly, the resonant phenomena in subwavelength resonators suggested by Engheta are determined only by the resonant properties of the permeability function of the material layers. In this sense phenomena in thin resonant layers resemble resonance in reflection from a thin ferrite layer on a metal plane or from a ferrite sphere near a microstrip line or in a closed waveguide.

However, the most interesting case is when the waves between the planes are evanescent, meaning that \(k_2^2 > |k_{1,2}|\). In this case the waves actually decay in the direction orthogonal to the interface, because the corresponding propagation factors \(\beta_{1,2} = \sqrt{k_{1,2}^2 - k_t^2}\) are imaginary. Suppose that the volume between the two metal screens is excited by a source with a certain fast variation of the current or field in space. Under the above assumptions regarding the media properties, all evanescent modes of this source satisfy the boundary conditions. Conceptually, this is an ideal memory device, because after the source has been removed, the field distribution near the interface of the two slabs will be preserved (until losses will consume the field energy), and the field distribution will correspond to the field distribution of the source.

This unique property wholly depends on the assumption that in one of the slab the material parameters are negative. The slab thickness is irrelevant, and the same effect exists also on a single interface between two half spaces filled by the same materials as the two slabs in the “resonator”. Indeed, the “memory” effect for evanescent modes with arbitrary propagation constants is due to the fact that the interface supports surface modes with arbitrary propagation constants, at the frequency where \(\mu_2 = -\mu_1\) and \(\varepsilon_2 = -\varepsilon_1\). Actually, the same phenomenon is the core effect that makes the planar slab of a Veselago medium act as a perfect lens [3]. In that device, there are two such interfaces supporting surface waves with arbitrary wavenumbers. Resonant excitation of these modes leads to amplification of evanescent waves crossing the slab.

The main assumption has been that the Veselago material is an effective magnetodielectric medium, and the main challenge in realizing any device using the principle explained here is to design such a material with as small spatial period as possible. Higher-order evanescent modes vary extremely fast in space, and as soon as the spatial period of the exciting field becomes comparable with the spatial period of the artificial material with negative parameters, spatial dispersion effects degrade the properties of the device. Clearly, there are other limitations related to absorption present in any realistic medium and to the final size of the slabs in the transverse direction.
4 FDTD simulated behavior

Figure 2: Evanescent field distribution near an interface with a backward-wave material region. The source (over the interface) excites an eigenmode of the interface. Cell numbers shown near the grid edges.

We have simulated fields near an interface between free space and a backward-wave medium slab. Negative permittivity and permeability are assumed to follow the Lorentz dispersion model:

$$
\varepsilon(\omega) = \varepsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j \Gamma \omega}\right), \quad \mu(\omega) = \mu_0 \left(1 + \frac{\omega_m^2}{\omega_0^2 - \omega^2 + j \Gamma \omega}\right)
$$

(16)

This model corresponds to a realization of Veselago materials as mixtures of conductive spirals or omega particles, as discussed in [7]. Used discrete FDTD model of the material is based on the constitutive relation discretized after one integration. This approach, described in detail in [8], leads to much better accuracy and, what is also important, to considerably better stability than the conventional direct discretization without integration. In the simulations, the loss factors have been set to zero, and the other parameters chosen so that at the central frequency of the source spectrum both relative parameters equal $-1$.

To visualize the behavior of evanescent modes, the interface has been excited by an array of line sources with the period smaller than the free space wavelength. Each line source is out-of-phase with its two nearest sources. In the numerical example, the source spectrum is concentrated near 0.4377 GHz, corresponding to 68.5 cm wavelength. The distance between the line sources is 9 cm (6 cells, cell size is 1.5 cm). Thus, the incident field exponentially decays in the direction orthogonal to the source plane and forms a standing wave with a small period in the source plane. The distance from the source to
the interface is equal to the array period (9 cm). The time dependence of the source is

\[ E_{\text{inc}}(t) = e^{-\left(\frac{t-t_0}{t_1}\right)^4} \sin(\omega_0 t) \]  

where \( t_0 = 200\Delta t \), \( t_1 = 100\Delta t \), and \( \Delta t \) is the time step.

Computed spatial distributions of the field amplitude are shown in Figures 2 and 3. The time dependence of the field in the source plane and at the plane of the interface is plotted in Figure 4. The simulated results confirm the theoretical predictions. First, the source excites oscillations near the interface between free space and the backward-wave medium, because the interface supports modes with arbitrary propagation constants along the interface plane. Next, the source is switched off, but the oscillations near the boundary stay there and get distorted very slowly (due to medium dispersion), since the field satisfies the Maxwell equations and the boundary conditions at the interface. In this numerical example, electric walls are not present, and it is obvious from the results that they are not relevant for the memory effect for evanescent modes: evanescent fields are concentrated only near the source and the media interface.

### 5 Conclusion

Detailed analysis of field solutions in thin subwavelength cavity resonators using materials with negative permittivity and permeability reveals that these cavities can in principle support evanescent fields concentrated near the interface of the two slabs. Numerical simulations show that when the evanescent field source is removed, resonant field excited near the interface continues to oscillate for a long time, as expected from the analytical
Figure 4: Time dependence of the field of the source and on the interface. It is seen that the field at the interface continues oscillations with the same frequency as that of the source after the source has been switched off.

Numerical experiments also indicate that the system is rather sensitive to the material parameter values: slight deviations from the resonant values reduce the excited field amplitude. The analysis of the cavity as a resonator for waves traveling along \( n \) shows that this system can be used as a frequency selective device only due to resonant frequency dependence of the material parameters of one or both materials slabs.

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