Formalizing Anisotropic Inflation in Modified Gravity

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Motivated by the fact that the pre-inflationary era may evolve in an exotic way, in this work we formalize anisotropic evolution in the context of modified gravity, focusing on pre-inflationary and near the vicinity of the inflationary epochs. We specialize on specific metrics like Bianchi and Taub and we formalize the inflationary theory in vacuum $F(R)$ gravity, in $F(R)$ gravity with an extra scalar field and in Gauss-Bonnet gravity. We discuss the qualitative effects of the anisotropies on the evolution of the Universe and also we consider several specific solutions, like the de Sitter solution, in both the isotropic and anisotropic contexts. Furthermore, several exotic modified gravity cosmological solutions, like the ones which contain finite time singularities, are also discussed in brief.

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I. INTRODUCTION

The Standard Cosmological Model provides a consistent general picture of the Universe, at least after the inflationary era, and it is entirely based on the standard Einstein-Hilbert gravity, on the Standard model of particles and on the adoption of a Friedmann-Robertson-Walker spacetime metric to describe the geometry of the Universe. However, it is a well-known fact that, at very primordial instants, at a pre-inflationary stage, the physical description of the Universe can be more complicated than a simple isotropic model and additional effects from modified and quantum gravity can also become important. What is a well accepted fact in cosmology is that the Universe when the presumed inflationary era commenced, was four dimensional and it was basically described by classical physics. The inflationary era itself [1–4] can either be described by a single scalar field, in a purely general relativistic (GR) framework. However, in the case that inflation is controlled by the inflaton, many shortcomings of the theory exist, such as the way that the inflaton couples to the Standard Model particles is arbitrary, and so on. In conjunction with the shortcomings of the scalar field description of the inflationary era, the dark energy era described by a scalar field can also be problematic, mainly due to the fact that observations [5] allow the dark energy equation of state parameters to take values that may describe slightly a phantom evolution and specifically $\omega_{DE} = -0.957 \pm 0.080$ and hence the minimum observationally allowed value of the dark energy equation of state parameter is $\omega_{DE} = -1.037$. Although the Planck data allow an exact de Sitter (cosmological constant) and quintessential dark energy era, it also allows a phantom dark energy era. This perspective is problematic in the context of simple GR, since a phantom evolution can only be realized using a phantom scalar. The latter are in general non appealing descriptions. Thus GR descriptions with scalar fields have several conceptual issues that cannot be appropriately resolved easily.

Modified gravity [6–10] on the other hand provide a consistent theoretical framework in the context of which, both inflation and late-time acceleration can be described in a unified way, see the pioneer work [11] for this topic and also Refs. [12–24] for later developments on this topic. In fact, in the context of modified gravity, the dark energy era can turn into phantom, and at the same time be compatible with the Planck data, without relying to phantom scalars, see Ref. [25] for a several examples of this sort.

Now about the pre-inflationary era, this is a mystery for scientists, and for the moment it is experimentally inaccessible to us in a direct way. It is however possible that the pre-inflationary epoch, described by some version of M-theory, may leave its imprints on the low-energy effective inflationary theory, and this can be done in several ways, for example some higher order curvature terms, Einstein-Gauss-Bonnet terms or even via some exotic evolution scenarios, like pre-inflationary bounces [26], other effects [27] or even via effects of anisotropic evolution. Thus we can have both UV-corrections on the inflationary era or even anisotropy could be generated during the pre-inflation era,

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leading to anisotropic inflation. In this paper we shall consider the effects of the second perspective.

In this paper we shall adopt the last line of research and we shall formalize anisotropic inflation \[28-46\] in the context of modified gravity. The most natural generalization of the isotropic FRW geometry is provided by the Bianchi metrics classification. In particular, the Bianchi type VIII and IX cosmological model yield chaotic dynamics in ordinary GR contexts, which constitutes a prototype for the asymptotic behavior to the singularity of the generic inhomogeneous Universe \[45\]. We aim to describe the features of the Universe adopting the Bianchi IX cosmology, which is the most general geometry allowed by the homogeneity constraint. The relevance of the dynamics of Bianchi models consists in the role these geometries could have played in a very primordial Universe, before the inflationary phase. For a quantum discussion on the Bianchi IX model see \[48-49\]. The model has been extensively studied in modified Gauss-Bonnet and \[F(R)\] gravity.

The paper is thematically organized in three distinct sections, each of which is devoted to studying anisotropic inflation in distinct modified gravities. Specifically, in section II we present the formalism of anisotropic inflation in the context of vacuum \[F(R)\] gravity. In section III we present the \[F(R)\] gravity with an extra scalar field anisotropic inflation. In section IV we consider the anisotropic Gauss-Bonnet inflation and in section we formalize anisotropic Gauss-Bonnet inflation in Taub spacetimes. Finally the conclusions follow in the end of the article.

II. ANISOTROPIC INFLATION IN THE CASE OF \([F(R)]\) GRAVITY

In this section we shall formalize the anisotropic inflation in the context of vacuum Jordan frame \([F(R)]\) gravity. Firstly let us fix the metric, so we assume the following homogeneous and anisotropic metric,

\[
ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} e^{2\beta_i(t)} \left(dx^i\right)^2 .
\]

Let the average of \(\beta_i(t)\) be \(\bar{\beta}(t)\), \(\bar{\beta}(t) \equiv \frac{1}{3} \sum_{i=1}^{3} \beta_i(t)\). By redfining \(a(t)\) and \(\beta_i(t)\) as \(a(t) \rightarrow a(t) + \bar{\beta}(t)\) and \(\beta_i(t) \rightarrow \beta_i(t) - \bar{\beta}(t)\), we obtain

\[
0 = \sum_{i=1}^{3} \beta^i .
\]

and therefore

\[
0 = \sum_{i=1}^{3} \dot{\beta}^i .
\]

In the following, we assume Eqs. \[2\] and \[3\].

Then, the connections are given by,

\[
\Gamma^i_{ij} = a^2 e^{2\beta_i} \left(H + \dot{\beta}^i\right) \delta_{ij} , \quad \Gamma^i_{ij} = \Gamma^i_{ji} = \left(H + \dot{\beta}^i\right) \delta^i_j , \quad \text{other components} = 0 .
\]

Since the energy-momentum tensor is given by,

\[
T_{00} = -\rho , \quad T_{ij} = p g_{ij} ,
\]

the conservation law is given by,

\[
0 = \nabla^\mu T_{\mu t} = -\dot{\rho} - \sum_{i,j=1}^{3} a^{-2} e^{-2\beta_i} \delta_{ij} a^2 e^{2\beta_i} \left(H + \dot{\beta}^i\right) \delta_{ij} \rho - a^{-2} e^{-2\beta_i} \delta_{ij} \left(H + \dot{\beta}^i\right) \delta^k_j a^2 e^{2\beta_i} \delta_{kj} p
\]

\[
= - \left\{ \rho + 3H (\rho + p) \right\} .
\]

Because we define,

\[
R_{\mu \nu} = -\Gamma^\rho_{\mu \rho,\nu} + \Gamma^\rho_{\mu \nu,\rho} - \Gamma^\eta_{\mu \rho} \Gamma^\rho_{\nu \eta} + \Gamma^\eta_{\mu \nu} \Gamma^\rho_{\rho \eta} ,
\]

we find that the components of the Ricci tensor are,

\[
R_{tt} = -3H - \sum_{i=1}^{3} \left(H + \dot{\beta}^i\right)^2 = -3H - 3H^2 - \sum_{i=1}^{3} \left(\dot{\beta}^i\right)^2 ,
\]
where we have used $0 = \sum_{i=1}^{3} \beta^i = \frac{d}{dt} \sum_{i=1}^{3} \beta^i$. The field equations for $F(R)$ gravity are given by,

$$G_{\mu\nu}^F = \frac{1}{2} g_{\mu\nu} F - R_{\mu\nu} F - g_{\mu\nu} \Box F_R + \nabla_{\mu} \nabla_{\nu} F_R = -\kappa^2 T_{\mu\nu},$$  

(9)

where $F_R \equiv \frac{dF(R)}{dR}$. Since $\Box F_R \equiv \frac{1}{\sqrt{-g}} \partial_{\mu} \left(g^{\mu\nu} \sqrt{-g} \partial_{\nu} F_R\right) = - \left(\frac{d^2 F_R}{dt^2} + 3H \frac{dF_R}{dt}\right)$, we find that the $tt$ component of (9) is given by,

$$- \frac{1}{2} F_R - \left(3\dot{H} + 3H^2 + \sum_{i=1}^{3} \left(\dot{\beta}^i\right)^2\right) F_R = - \left(\frac{d^2 F_R}{dt^2} + 3H \frac{dF_R}{dt}\right)$$

$$= - \frac{1}{2} F_R - \left(3\dot{H} + 3H^2 + \sum_{i=1}^{3} \left(\dot{\beta}^i\right)^2\right) F_R - 3H F_R + 3H \frac{dF_R}{dt} = -\kappa^2 \rho,$$

(10)

and $ij$ component yields the following,

$$\frac{1}{2} F_R - \left(H + 3H^2 + \beta^i + 3H \beta^i\right) F_R + \left(\frac{d^2 F_R}{dt^2} + 3H \frac{dF_R}{dt}\right) = - \left(H + \beta^i\right) \frac{dF_R}{dt}$$

$$= \frac{1}{2} F_R - \left(H + 3H^2 + \beta^i + 3H \beta^i\right) F_R + \frac{d^2 F_R}{dt^2} + \left(2H - \beta^i\right) \frac{dF_R}{dt} = -\kappa^2 \rho,$$

(11)

which gives,

$$\frac{1}{2} F_R - \left(H + 3H^2\right) F_R + \left(\frac{d^2 F_R}{dt^2} + 2H \frac{dF_R}{dt}\right) = -\kappa^2 \rho,$$

(12)

$$- \left(\dot{\beta}^i + 3H \dot{\beta}^i\right) F_R - \dot{\beta}^i \frac{dF_R}{dt} = 0.$$

(13)

The above equation, namely Eq. (12), can be obtained by summing up Eq. (11) with respect to $i$ and using (9) and also the fact that $0 = \sum_{i=1}^{3} \beta^i$. We obtain Eq. (13) by subtracting Eq. (11) and Eq. (12).

We now check the consistency between the conservation law (6) with the equations (10), (12), and (13), and we get,

$$-\kappa^2 \left\{ \dot{\rho} + 3H (\rho + p) \right\}$$

$$= - \frac{1}{2} F_R \left(6\dot{H} + 24H \dot{H} + 2 \sum_{i=1}^{3} \dot{\beta}^i \dot{\beta}^i\right) + \left(3\dot{H} + 6H \dot{H} + 2 \sum_{i=1}^{3} \dot{\beta}^i \dot{\beta}^i\right) F_R + \left(3\dot{H} + 3H^2 + \sum_{i=1}^{3} \left(\dot{\beta}^i\right)^2\right) \frac{dF_R}{dt}$$

$$- 3H \frac{dF_R}{dt} - \frac{d^2 F_R}{dt^2} + 3H \left\{ - \frac{1}{2} F_R + \left(3\dot{H} + 3H^2 + \sum_{i=1}^{3} \left(\dot{\beta}^i\right)^2\right) F_R - 3H \frac{dF_R}{dt}\right\}$$

$$+ \frac{1}{2} F_R - \left(H + 3H^2\right) F_R + \frac{d^2 F_R}{dt^2} + 2H \frac{dF_R}{dt}$$

$$= - \frac{1}{2} F_R \left(24H \dot{H} + 2 \sum_{i=1}^{3} \dot{\beta}^i \dot{\beta}^i\right) + \left(24H \dot{H} + 2 \sum_{i=1}^{3} \dot{\beta}^i \dot{\beta}^i\right) F_R + \left(3\dot{H} + 3H^2 + \sum_{i=1}^{3} \left(\dot{\beta}^i\right)^2\right) \frac{dF_R}{dt}$$

$$+ 3H \left\{ \left(3\dot{H} + 3H^2 + \sum_{i=1}^{3} \left(\dot{\beta}^i\right)^2\right) F_R - 3H \frac{dF_R}{dt} - \left(H + 3H^2\right) F_R + 2H \frac{dF_R}{dt}\right\}$$

$$= \left\{ (-12 + 6 + 9 - 3) \dot{H} \ddot{H} + (9 - 9) H^2 \right\} F_R + (3 - 9 + 6) H^2 \frac{dF_R}{dt}$$. 


\[
\sum_{i=1}^{3} \dot{\beta}^i \left\{ (-1 + 2) \dot{\beta}^i F_R + 3H \dot{\beta}^i F_R + \dot{\beta}^i \frac{dF_R}{dt} \right\} = 0.
\]

Therefore there is no contradiction in the equations which we obtained, these are consistent. Eq. (13) can be integrated with respect to \(\dot{\beta}^i\) to yield,

\[
\dot{\beta}^i = \frac{C^i a^{-3}}{F_R},
\]

where the free parameters \(C^i\)'s are constants. By integrating \(\dot{\beta}^i\) with respect to \(t\), we obtain \(\beta^i\) and also Eq. (3) indicates that,

\[
0 = \sum_{i=1}^{3} C^i.
\]

In order for the late Universe to satisfy \(a \to \infty\) and \(R \to 0\), and in order for it to become isotropic, we must require \(a^{-3} \frac{F_R}{} \to 0\).

By combining Eqs. (10) and (12) and eliminating \(\dot{\beta}^i\) by using (15), we obtain,

\[
\left( 2 \dot{H} + \frac{C^2 a^{-6}}{F_R^2} \right) F_R - H \frac{dF_R}{dt} + \frac{d^2 F_R}{dt^2} = -\kappa^2 (\rho + p),
\]

with

\[
C^2 \equiv \sum_{i=1}^{3} C^i.
\]

We may assume that the matter is composed by perfect fluids with constant EoS parameters \(w_i\),

\[
\rho + p = \sum_{i} (1 + w_i) \rho_0^i a^{-3(1+w_i)},
\]

where the parameters \(\rho_0^i\)'s are constants. If we give the time evolution of the scale factor \(a = a(t)\), and therefore \(H = H(t)\) and \(\dot{H} = \dot{H}(t)\), Eq. (17) becomes a non-linear differential equation for \(F_R\). Then by solving Eq. (17), we may find the time dependence of \(F_R\) as \(F_R = F_R(t)\). On the other hand, since the scalar curvature \(R\) is given by,

\[
R = R(t) = 12H(t)^2 + 6\dot{H}(t) + \frac{C^2 a^{-6}}{F_R^2},
\]

if we delete \(t\) by combining \(F_R = F_R(t)\) and \(R = R(t)\), \(F_R\) is given by a function of \(R\) and by integrating \(F_R(R)\) with respect to \(R\), we find the form of \(F(R)\).

We now consider the case without matter, so \(\rho + p = 0\). Then Eq. (17) takes the following form,

\[
2 \dot{H} F_R - H \frac{dF_R}{dt} + \frac{d^2 F_R}{dt^2} = -\frac{C^2 a^{-6}}{F_R}.
\]

During the inflationary era, that is, the era after the first horizon crossing and much earlier than the reheating, the Hubble rate \(H\) is almost constant, \(H = H_0\), which indicates that we may neglect the first term in the left hand side of Eq. (21) and \(a = e^{H_0 t}\). Then, we may write Eq. (21) in the following form,

\[
-H_0 \frac{dF_R}{dt} + \frac{d^2 F_R}{dt^2} = e^{H_0 t} \frac{d}{dt} \left( e^{-H_0 t} \frac{dF_R}{dt} \right) = -\frac{C^2 e^{-6H_0 t}}{F_R}.
\]

In the late epoch of the inflationary era, but much before than the reheating, the right hand side of (22) should be small and can be treated as a perturbation. The leading behavior can be obtained when the right hand side of Eq. (22) vanishes, and we find that,

\[
F_R = F_1 + F_2 e^{H_0 t},
\]
where $F_1$ and $F_2$ are constants, which may be determined by the initial condition. When $F_2 = 0$ by an initial condition, the solution of Eq. (22) including the leading correction is given by,

$$F_R = F_1 - \frac{C^2 e^{-6H_0 t}}{42H_0^2 F_1}.$$  \hspace{1cm} (24)

On the other hand, when $F_2 \neq 0$, the second term in Eq. (23) becomes dominant in the late-era and, we find,

$$F_R = F_1 + F_2 e^{H_0 t} - \frac{C^2 e^{-7H_0 t}}{56H_0^2 F_2}.$$  \hspace{1cm} (25)

In the case that $F_2 = 0$ or $F_2 \neq 0$, we find that $\frac{dF}{dR}$ decreases very rapidly and the Universe becomes isotropic. Eq. (20) also indicates that,

$$R \sim \begin{cases} 12H_0^2 + \frac{C^2 e^{-6H_0 t}}{F_1} & \text{when } F_2 = 0 \\ 12H_0^2 + \frac{C^2 e^{-8H_0 t}}{F_2} & \text{when } F_2 \neq 0 \end{cases},$$ \hspace{1cm} (26)

and consequently we find that,

$$F_R \sim \begin{cases} F_1 + \frac{F_2}{42H_0} \left( R - 12H_0^2 \right) & \text{when } F_2 = 0 \\ F_1 + F_2 \left\{ \frac{F_2^2(R - 12H_0^2)}{C^2} \right\}^{-\frac{2}{7}} - \frac{C^2}{56H_0^2 F_2} \left\{ \frac{F_2^2(R - 12H_0^2)}{C^2} \right\}^{-\frac{8}{7}} & \text{when } F_2 \neq 0 \end{cases},$$ \hspace{1cm} (27)

By integrating $F_R$ with respect to $R$, we find the form of $F$ as follows,

$$F(R) \sim \begin{cases} F_0 + F_1 R - \frac{F_1}{42H_0} \left( \frac{R}{2} - 12H_0^2 R \right) & \text{when } F_2 = 0 \\ F_0 + F_1 R + \frac{8C^2}{42H_0^2} \left\{ \frac{F_2^2(R - 12H_0^2)}{C^2} \right\}^{-\frac{2}{7}} - \frac{8C^2}{56H_0^2 F_2} \left\{ \frac{F_2^2(R - 12H_0^2)}{C^2} \right\}^{-\frac{8}{7}} & \text{when } F_2 \neq 0 \end{cases} \left( \frac{R}{2} - 12H_0^2 R \right),$$ \hspace{1cm} (28)

with a constant $F_0$ parameter. On the other hand, by using Eqs. (11) or (12) with $\rho = \rho_0 = 0$, we can determine $F(R)$. In order to check the consistency and to determine the constant $F_0$, we consider (11),

$$F = 2 \left( 3H + 3H^2 + \sum_{i=1}^{3} \left( \beta^i \right)^2 \right) F_R - 6H \frac{dF_R}{dt},$$ \hspace{1cm} (29)

and using Eqs. (24) and (27), we find,

$$F \sim \begin{cases} \left( 6H_0^2 + \frac{2C^2 e^{-6H_0 t}}{F_1} \right) F_1 - \frac{\left( C^2 e^{-6H_0 t} \right)}{42H_0^2 F_1} - \frac{8C^2 e^{-6H_0 t}}{56H_0^2 F_2} & \text{when } F_2 = 0 \\ \left( 6H_0^2 + \frac{2C^2 e^{-6H_0 t}}{F_2} \right) F_1 + \frac{\left( C^2 e^{-7H_0 t} \right)}{56H_0^2 F_2} - 6H_0^2 \left( F_2 H_0 e^{H_0 t} + \frac{C^2 e^{-7H_0 t}}{8H_0^2 F_2} \right) & \text{when } F_2 \neq 0 \end{cases},$$ \hspace{1cm} (30)

By using Eq. (26), we find that Eq. (30) is consistent with Eq. (28) if and only if the following condition holds true,

$$F_0 = \begin{cases} -\frac{54}{7} H_0^2 F_1 & \text{when } F_2 = 0 \\ -\frac{6}{7} H_0^2 F_1 & \text{when } F_2 \neq 0 \end{cases},$$ \hspace{1cm} (31)
It is well-known that the $F(R)$ gravity can be rewritten as the scalar-tensor form by the scale transformation of the metric,

$$g_{\mu\nu} \rightarrow e^\varphi g_{\mu\nu}, \quad \varphi = -\frac{1}{\sqrt{3}} \ln F_R(A),$$

which gives the so-called Einstein frame action as follows,

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right),$$

$$V(\varphi) = e^\varphi g \left( e^{-\varphi} \right) - e^{\frac{2\varphi}{3}} F \left( g \left( e^{-\varphi} \right) \right) = \frac{A}{F_R(A)} - \frac{F(A)}{F_R(A)^2}. \quad (33)$$

Here $A$ corresponds to the scalar curvature in the original Jordan frame and $g \left( e^{-\varphi} \right)$ is obtained by solving the equation $\varphi = -\frac{1}{\sqrt{3}} \ln F_R(A)$ with respect to $A = g \left( e^{-\varphi} \right)$.

For the scalar-tensor theory with the potential $V(\varphi)$ for the scalar field $\varphi$, which may be identified with the inflaton field, the slow-roll parameters $\epsilon$ and $\eta$ are defined by follows,

$$\epsilon = \frac{1}{2} \left( \frac{V'(\varphi)}{V(\varphi)} \right)^2, \quad \eta = \frac{V''(\varphi)}{V(\varphi)}. \quad (34)$$

The above expressions could be used even in the anisotropic universe. By using the slow-roll indexes $\epsilon$ and $\eta$, we can express the observational indices $n_s$ and $r$ as follows,

$$n_s - 1 = -6\epsilon + 2\eta, \quad r = 16\epsilon. \quad (35)$$

In the case that we can neglect the contributions from the matter, the slow-roll parameters $\epsilon$ and $\eta$ and the observational indices $n_s$ and $r$ do not change in the frame [56]. Therefore we work in the Einstein frame.

Because

$$\frac{d}{d\varphi} = \frac{dA}{d\varphi} \frac{d}{dA} = -\sqrt{3} \frac{F_R(A)}{F_R(A)} \frac{d}{dA},$$

we find

$$V'(\varphi) = \sqrt{3} \left( \frac{A}{F_R(A)} - \frac{2F(A)}{F_R(A)^2} \right), \quad V''(\varphi) = 3 \left( \frac{1}{F_R(A)} + \frac{A}{F_R(A)} - \frac{4F(A)}{F_R(A)^2} \right), \quad (37)$$

and therefore, by using [54], we obtain [57],

$$\epsilon = \frac{3}{2} \left( \frac{AF_R(A) - 2F(A)}{AF_R(A) - F(A)} \right)^2, \quad \eta = \frac{3 \left( \frac{F_R(A)^2 + AF_R(R)F_R(A) - 4F_R(R)F(A)}{F_R(A) \left( AF_R(A) - F(A) \right)} \right)}{F_R(R) \left( AF_R(A) - F(A) \right)}. \quad (38)$$

We also obtain the expressions of the observational indices $n_s$ and $r$ in [55],

$$n_s - 1 = -9 \left( \frac{AF_R(A) - 2F(A)}{AF_R(A) - F(A)} \right)^2 + \frac{6 \left( F_R(A)^2 + AF_R(R)F_R(A) - 4F_R(R)F(A) \right)}{F_R(R) \left( AF_R(A) - F(A) \right)},$$

$$r = 24 \left( \frac{AF_R(A) - 2F(A)}{AF_R(A) - F(A)} \right)^2. \quad (39)$$

As an example and just for simplicity by choosing $F_2 = 0$, we consider the model [28] with [31],

$$F(R) = -\frac{54}{7} H_0^2 F_1 + \frac{9}{7} F_1 R - \frac{F_1}{84H_0^2} R^2, \quad (40)$$

where $R$ behaves as $R \sim 12H_0^2 + \frac{C e^{-6H_0 t}}{F_1}$ in [26]. Then in the late time $t \rightarrow \infty$, we find

$$F(R) \sim 6H_0^2 F_1 + \frac{C e^{-6H_0 t}}{F_1}, \quad F_R(R) \sim F_1 - \frac{C e^{-6H_0 t}}{42H_0^2 F_1}, \quad F_{RR}(R) = -\frac{F_1}{42H_0^2}, \quad (41)$$

and therefore

$$n_s - 1 \sim 9, \quad r \sim \frac{54 C e^{-12H_0 t}}{49 F_1^4}. \quad (42)$$

The obtained value of $n_s - 1$ might not be realistic and we may need to consider more realistic model.
III. $F(R)$ GRAVITY WITH AN EXTRA SCALAR FIELD AND ANISOTROPIC INFLATION

Now let us consider anisotropic inflation in the case of $F(R)$ gravity with an extra scalar field, in which case we consider the following gravitational action,

$$ S_φ = \frac{1}{2\kappa^2} \int \! d^4x \sqrt{-g} \left[ F(R) - \frac{ω(φ)}{2} \partial_ρ φ \partial^ρ φ - V(φ) \right], \tag{43} $$

the variation of which with respect to the scalar field $φ$, yields,

$$ 0 = \nabla^α (ω(φ) \partial_α φ) - \frac{ω'(φ)}{2} \partial_ρ φ \partial^ρ φ - V'(φ). \tag{44} $$

Assuming that $φ = t$, Eqs. 10, 12, and 13 can be rewritten as follows,

$$ \frac{1}{2} ω + V - \frac{1}{2} F + \left( 3 \dot{H} + 3H^2 + \sum_{i=1}^{3} \left( \dot{β}^i \right)^2 \right) F_R - 3H \frac{dF_R}{dt} = -κ^2 ρ, \tag{45} $$

$$ \frac{1}{2} ω - V + \frac{1}{2} F - \left( \ddot{H} + 3H^2 \right) F_R + \frac{d^2F_R}{dt^2} + 2H \frac{dF_R}{dt} = -κ^2 p, \tag{46} $$

$$ - \left( \dot{β}^i + 3H \dot{β}^i \right) F_R - \dot{β}^i \frac{dF_R}{dt} = 0. \tag{47} $$

We should note that Eq. 47 is identical to Eq. 13, which could indicate that the isometry is mainly controlled by the functional form of $F(R)$. Eqs. 13 and 16 can be combined and can be rewritten as follows,

$$ ω = - \left( 2 \dot{H} + \sum_{i=1}^{3} \left( \dot{β}^i \right)^2 \right) F_R + H \frac{dF_R}{dt} - \frac{d^2F_R}{dt^2} - κ^2 (ρ + p), \tag{48} $$

$$ V = \frac{1}{2} F - \left( 2 \dot{H} + 3H^2 - \frac{1}{2} \sum_{i=1}^{3} \left( \dot{β}^i \right)^2 \right) F_R + \frac{5}{2} H \frac{dF_R}{dt} + \frac{1}{2} \frac{d^2F_R}{dt^2} + \frac{κ^2}{2} (−ρ + p). \tag{49} $$

As in Eq. 19, we may assume that the matter is composed by perfect fluids with constant EoS parameters $w_i$, which yields the scale factor $a$-dependence of $ρ$ and $p$, $ρ = ρ(a)$ and $p = p(a)$. Then if we consider an arbitrary expansion of the Universe, with an arbitrary scale factor $a = a(t)$ the time-dependence of the right hand sides of Eqs. 41 and 42, can be obtained and therefore we find the time-dependence of the kinetic term function $ω$ and of the scalar potential $V$, namely $ω(t)$ and $V(t)$. By replacing $t$ in $ω(t)$ and $V(t)$ with the scalar field $φ$, we find the $φ$-dependence of $ω$ and $V$, $ω(φ) = ω(t = φ)$ and $V(φ) = V(t = φ)$ and we can specify the model realizing the given scale factor $a = a(t)$.

As a special case we consider the Einstein gravity where $F(R) = R$. In the case of the Einstein gravity, Eqs. 17, 51, and 52 have the following forms,

$$ \dot{β}^i + 3H \dot{β}^i = 0, \tag{50} $$

$$ ω = 2 \dot{H} + \sum_{i=1}^{3} \left( \dot{β}^i \right)^2 - κ^2 (ρ + p), \tag{51} $$

$$ V = \dot{H} + 3H^2 + \frac{1}{2} \sum_{i=1}^{3} \left( \dot{β}^i \right)^2 + \frac{κ^2}{2} (−ρ + p). \tag{52} $$

Eq. 50 can be integrated to give,

$$ β^i = \hat{C}^i a^{-3}, \tag{53} $$

where $\hat{C}^i$'s are constants of the integration. As in Eq. 19, Eq. 53 indicates that,

$$ 0 = \sum_{i=1}^{3} \hat{C}^i. \tag{54} $$
From Eq. (53), we can see that the anisotropy disappears in the late Universe where the scale factor \(a\) is large, even in the scalar-Einstein theory. By using Eq. (53), Eqs. (51) and (52) can be rewritten as follows,

\[
\omega = 2H + \ddot{C}^2a^{-6} - \kappa^2 (\rho + p)
\]

(55)

\[
V = \dot{H} + 3H^2 + \ddot{C}^2a^{-6} + \frac{\kappa^2}{2} (-\rho + p)
\]

(56)

where in this case as in Eq. (19), we define \(\dot{C}^2\) as follows,

\[
\dot{C}^2 \equiv \sum_{i=1}^{6} (\dot{\mathcal{C}}^i)^2.
\]

(57)

Let us now consider an example, having to do with the inflationary epoch. During inflation, the contribution from the perfect matter fluids could be neglected, and therefore we put \(\rho = p = 0\). We may assume,

\[
a(t) = \frac{1}{e^{-H_0(t-t_0)} + \left(\frac{t_0}{t}\right)^n},
\]

(58)

where \(H_0\), \(t_0\), and \(n\) are constants and we assume that \(H_0\) and \(n\) are positive. When \(t \ll t_0\), \(a(t)\) behaves as an exponential function \(a \sim e^{H_0t}\), which corresponds to the de Sitter expansion and therefore the era \(t \ll t_0\) can be regarded to be the inflationary era. On the other hand, when \(t \gg t_0\), the scale factor \(a(t)\) behaves as a power function of \(t\), \(a(t) \sim \left(\frac{t}{t_0}\right)^n\). Therefore, the era \(t \gg t_0\) can be identified with the post-inflationary era. For the scale factor \(a\) in (58), by using Eq. (54), we find the behavior of \(\beta^i(t)\) as follows,

\[
\beta^i = \ddot{C}^i \left( e^{-H_0(t-t_0)} + \left(\frac{t_0}{t}\right)^n \right)^3,
\]

(59)

which becomes small in the late Universe and therefore the universe becomes isometric since the anisotropies get smoothed away. For the scale factor \(a\) in Eq. (58) the Hubble rate is,

\[
H = \frac{H_0 e^{-H_0(t-t_0)} + \frac{n}{t_0} \left(\frac{t_0}{t}\right)^{n+1}}{e^{-H_0(t-t_0)} + \left(\frac{t_0}{t}\right)^n},
\]

\[
\dot{H} = \frac{-H_0^2 e^{-H_0(t-t_0)} - \frac{n(n+1)}{t_0^2} \left(\frac{t_0}{t}\right)^{n+2} + \left(\frac{H_0 e^{-H_0(t-t_0)} + \frac{n}{t_0} \left(\frac{t_0}{t}\right)^{n+1}}{e^{-H_0(t-t_0)} + \left(\frac{t_0}{t}\right)^n}\right)^2}{e^{-H_0(t-t_0)} + \left(\frac{t_0}{t}\right)^n}
\]

\[
= \frac{2H_0 e^{-H_0(t-t_0)} \left(\frac{t_0}{t}\right)^{n+1} - \frac{n(n+1)}{t_0^2} \left(\frac{t_0}{t}\right)^{n+2} + \frac{n^2}{t_0^2} \left(\frac{t_0}{t}\right)^{2n+2}}{e^{-H_0(t-t_0)} + \left(\frac{t_0}{t}\right)^n},
\]

(60)

thus Eq. (61) gives,

\[
\omega(\phi) = \frac{4nH_0 e^{-H_0(\phi-t_0)} \left(\frac{t_0}{\phi}\right)^{n+1} - 2n(n+1) \left(\frac{t_0}{\phi}\right)^{n+2} + 2n^2 \left(\frac{t_0}{\phi}\right)^{2n+2}}{e^{-H_0(\phi-t_0)} + \left(\frac{t_0}{\phi}\right)^n} + \dot{C}^2 \left( e^{-H_0(\phi-t_0)} + \left(\frac{t_0}{\phi}\right)^n \right)^6,
\]

(61)

\[
V(\phi) = \frac{3H_0^2 e^{-2H_0(\phi-t_0)} + 8H_0 e^{-H_0(\phi-t_0)} \left(\frac{t_0}{\phi}\right)^{n+1} - \frac{n(n+1)}{t_0^2} \left(\frac{t_0}{\phi}\right)^{n+2} + \frac{n^2}{t_0^2} \left(\frac{t_0}{\phi}\right)^{2n+2}}{e^{-H_0(\phi-t_0)} + \left(\frac{t_0}{\phi}\right)^n},
\]

(62)

which specify the model.

In the late time \(t = \phi \to +\infty\), the expressions in (61) and (62) are reduced as

\[
\omega(\phi) \sim -\frac{2n(n+1)}{t_0^2} \left(\frac{t_0}{\phi}\right)^{-n+2}, \quad V(\phi) \sim -\frac{n(n+1)}{t_0^2} \left(\frac{t_0}{\phi}\right)^{-n+2}
\]

(63)
Because $\omega(\phi)$ is negative if $\phi$ is positive, the scalar field becomes a ghost, and therefore the model could be regarded as a kind of effective theory.  \(^1\) We now normalize the scalar field by

\[
\varphi \equiv \int d\phi \sqrt{-\omega(\phi)} \sim 2\sqrt{\frac{n+1}{n}} \left(\frac{\phi}{t_0}\right)^{\frac{2}{n}} = 2\sqrt{\frac{n+1}{n}} \left(\frac{t}{t_0}\right)^{\frac{2}{n}},
\]

and we find $V(\varphi) \propto \varphi^{2-\frac{4}{n}}$. Then if we define the slow-roll parameters by (34) and the observational indices by (35), we find

\[
\epsilon \sim \frac{1}{2\varphi^2} \left(2 - \frac{4}{n}\right)^2, \quad \eta \sim \frac{1}{\varphi^2} \left(2 - \frac{4}{n}\right) \left(1 - \frac{4}{n}\right),
\]

\[
n_s - 1 \sim -\frac{1}{\varphi^2} \left(2 - \frac{4}{n}\right) \left(4 - \frac{4}{n}\right), \quad r \sim \frac{8}{\varphi^2} \left(2 - \frac{4}{n}\right)^2. \tag{65}
\]

By using (64), we find

\[
\epsilon \sim \left(2 - \frac{4}{n}\right)^2 \frac{n}{8(n+1)} \left(\frac{t}{t_0}\right)^{-n}, \quad \eta \sim \left(2 - \frac{4}{n}\right) \left(1 - \frac{4}{n}\right) \frac{n}{4(n+1)} \frac{1}{(n+1)} \left(\frac{t}{t_0}\right)^{-n},
\]

\[
n_s - 1 \sim -\left(2 - \frac{4}{n}\right) \left(4 - \frac{4}{n}\right) \frac{n}{4(n+1)} \left(\frac{t}{t_0}\right)^{-n}, \quad r \sim \left(2 - \frac{4}{n}\right)^2 \frac{2n}{(n+1)} \left(\frac{t}{t_0}\right)^{-n}. \tag{66}
\]

**IV. ANISOTROPIC INFLATION IN MODIFIED GAUSS-BONNET GRAVITY**

In this section, we shall study anisotropic inflation in the context of another mainstream modified gravity, namely for modified Gauss-Bonnet gravity (MGB), for Bianchi IX cosmology. We start with the action of the MGB-gravity and cast it into a convenient formulation using auxiliary scalar degrees of freedom, and then specialize it to the Bianchi IX cosmology. We will also consider two additional cosmological models, related with the Bianchi IX geometry: the Bianchi I and the Taub Cosmology. The aim of this section is to derive the Lagrangian and the equation of motion for these models.

Let us consider a class of modified gravity theories where an arbitrary function of the topological Gauss-Bonnet term is added to the Lagrangian of GR.

The action of the modified Gauss-Bonnet gravity \(^6\) is the following:

\[
S = \int d^4x \sqrt{-g} \left(\frac{1}{2k^2} R + f(G)\right), \tag{67}
\]

where $G$ is the Gauss-Bonnet invariant, defined as:

\[
G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \tag{68}
\]

The action (67) can be rewritten in a convenient form by introducing two auxiliary fields $A$ and $B$, as follows,

\[
S = \int d^4x \sqrt{-g} \left(\frac{1}{2k^2} R + B(G - A) + f(A)\right). \tag{69}
\]

Upon varying the action with respect to the auxiliary scalar degrees of freedom $B$ we obtain $A = G$, and by substituting this into Eq. (69), the action of Eq. (67) is recovered. On the other hand, by varying the action with respect to $A$ we get $B = f'(A)$. Hence,

\[
S = \int d^4x \sqrt{-g} \left(\frac{1}{2k^2} Rf'(A)G - Af'(A) + f(A)\right). \tag{70}
\]

\(^1\) In order to avoid this problem, we may consider only the case that $n$ is an odd integer and $\phi$ is negative, and furthermore, we identify $\phi = -t$ instead of $\phi = t$. 
Renaming the variable \( A = \phi \) and \( f'(A) = \xi(\phi), V(\phi) = A f'(A) + f(A) \), and so the action \([70]\) becomes,

\[
S = \int d^4 x \sqrt{-g} \left( \frac{1}{2k^2} R - \xi(\phi) G - V(\phi) \right). \tag{71}
\]

We want to describe the features of the Universe adopting the Bianchi IX cosmology, which is the most general geometry allowed by the homogeneity constraint. The relevance of the dynamics of Bianchi models consists in the role these geometries could have played in a very primordial Universe, before the inflationary phase. For a quantum discussion on the Bianchi IX model see \([48, 49]\). The model has been extensively studied in GR by C. Misner \([50, 51]\), but few analysis were made in modified Gauss-Bonnet gravity.

The line element of a generic Bianchi model \([52]\) is,

\[
ds^2 = -N^2(t) dt^2 + a^2(t) (\omega^i)^2 + b^2(t) (\omega^m)^2 + c^2(t) (\omega^n)^2. \tag{72}
\]

where the functions \( \omega^i \) are the 1-forms that define the geometry of the model. For the Bianchi I and Bianchi IX cosmologies the 1-forms are:

\[
\begin{align*}
\omega^i &= dx, \quad \omega^j = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\
\omega^m &= dy, \quad \omega^m = \sin \psi d\theta - \cos \psi \sin \theta d\phi, \\
\omega^n &= dz, \quad \omega^n = d\psi + \cos \theta d\phi.
\end{align*} \tag{73}
\]

We can calculate the Gauss-Bonnet invariant, the Ricci Scalar and the \( R_{\mu\nu}R^{\mu\nu} \) for the Bianchi IX cosmology using the following relationships,

\[
R^0_0 = -\frac{1}{2} \frac{\partial}{\partial t} \chi^a_\alpha \chi^a_\beta, \quad R^3_3 = -P^3_3 - \frac{1}{2} \frac{\partial}{\partial t} (\sqrt{\gamma} \chi^a_\alpha). \tag{74}
\]

where \( P^3_3 \) is the 3-dimensional Ricci scalar and \( \gamma \) is the determinant of the spatial metric. The Riemann tensor will be computed using the relationship for the Riemann tensor in the tetrad formalism,

\[
R_{abcd} = \gamma_{abc,d} - \gamma_{ab,c} + \gamma_{abf} \left( \gamma^f_{cd} - \gamma^f_{dc} + \gamma_{af \epsilon} \gamma^f_{\epsilon c} - \gamma_{af d} \gamma^f_{\epsilon c} \right), \tag{75}
\]

where \( g_{ik} = \eta_{ab}e^a_i e^b_k \) and the \( \gamma_{abc} = e^{(a)}_i e^{(b)}_j e^{(c)}_k \) are the Ricci rotation coefficient. We can introduce their linear combinations \( \lambda_{abc} = \gamma_{abc} - \gamma_{acb} \). For the Bianchi IX cosmology we can express them in terms of the structure constants and the time derivative of the spatial metric as follows,

\[
\lambda^a_{bc} = C^a_{bc}, \quad \lambda^0_{ab} = \chi_{ab}. \tag{76}
\]

The non-vanishing components of the Riemann tensor are,

\[
\begin{align*}
R_{0101} &= R_{0202} = R_{0303}, \\
R_{1212} &= R_{1313} = R_{2323}, \\
R_{0123} &= R_{0231} = R_{0312} \tag{77}
\end{align*}
\]

and their permutations allowed by the Bianchi identities. We can now compute the Gauss-Bonnet invariant, which has the following form,

\[
G = 8 \left\{ \frac{\dot{a}}{a} \left[ \frac{\dot{b}}{b} \frac{\dot{c}}{c} - \frac{1}{2} U_1 \right] + \frac{\dot{b}}{b} \left[ \frac{\dot{a}}{a} \frac{\dot{c}}{c} - \frac{1}{2} U_2 \right] + \frac{\dot{c}}{c} \left[ \frac{\dot{a}}{a} \frac{\dot{b}}{b} - \frac{1}{2} U_3 \right] + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 \frac{\partial U_1}{\partial \log a} + \frac{1}{2} \left( \frac{\dot{b}}{b} \right)^2 \frac{\partial U_2}{\partial \log b} + \frac{1}{2} \left( \frac{\dot{c}}{c} \right)^2 \frac{\partial U_3}{\partial \log c} + \frac{1}{2} \frac{\partial U_1}{\partial \log b} - \frac{\partial U_2}{\partial \log c} \right\}, \tag{78}
\]

where \( U_1, U_2 \) and \( U_3 \) are functions defined for the Bianchi I model,

\[
U_1 = U_2 = U_3 = 0, \tag{79}
\]

and for the Bianchi IX model:

\[
U_1 = \frac{-3a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 - 2b^2c^2}{2a^2b^2c^2},
\]
\[ U_2 = \frac{a^4 - 3b^4 + c^4 + 2a^2b^2 - 2a^2c^2 + 2b^2c^2}{2a^2b^2c^2}, \]
\[ U_3 = \frac{a^4 + b^4 - 3c^4 - 2a^2b^2 + 2a^2c^2 + 2b^2c^2}{2a^2b^2c^2}. \]

The Lagrangian of the Bianchi IX model assumes the following form,
\[
\mathcal{L}_{IX} = -abcV(\phi) + 2abc + 2\dot{a}\dot{b} + 2\dot{a}\dot{c} + \frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}{2abc} + 8\xi'(\phi)\dot{\phi}(\dot{a}\dot{b}c - \frac{3a^2\dot{a}}{4b} + \frac{c^3\dot{a}}{2c} + \frac{cb}{2b} + \frac{b^3\dot{a}}{2a^2c} - \frac{bc\dot{a}}{2a^2} + \frac{a^3b}{2a^2} + \frac{ab}{2c} + \frac{ac\dot{b}}{2b^2} + \frac{c^2b}{4a^2} + \frac{\dot{c}b}{2a} - \frac{3b^2\dot{b}}{4bc} + \frac{a^3c}{2bc} - \frac{a^2\dot{c}}{2a} + \frac{ab\dot{c}}{2c} + \frac{b^3c}{2a^2} - \frac{3c^2\dot{c}}{4ab} ).
\]

The equations of motion can be found using the field equations for the theory at hand, and we eventually get,
\[
0 = \frac{b^3}{2a^2c} + \frac{bc}{a^2} - \frac{c^3}{2a^2b} + \frac{3a^2}{2bc} - 2\dot{c}b - 2\dot{b}c - 2\dot{bc} - \frac{b}{c} \frac{b}{c} - bcV(\phi) + \dot{\phi} + \xi'(\phi) \left( \frac{6b^2\dot{b}}{a^2c} - \frac{6b^2\dot{b}}{a^2} - 8\dot{bc} - 8\dot{bc} \right) + \dot{\phi}^2\xi''(\phi) \left( \frac{-2b^3}{a^2c} - \frac{4bc}{a^2} - \frac{2c^3}{ab^2} + \frac{6a^2}{bc} - \frac{4b}{c} - \frac{4c}{b} \right) + \ddot{\phi}^8(\phi) + \left( \frac{-2b^3}{a^2c} + \frac{4bc}{a^2} - \frac{2c^3}{ab^2} - \frac{6a^2}{bc} - \frac{8bc}{c} - \frac{8bc}{c} \right) ;
\]

where the first two are dynamical equations for the metric \( a(t) \) and \( b(t) \) and the last one is the equation for the scalar field \( \phi(t) \).

### A. Theory Reconstruction in Taub Cosmology and Specific Cosmological Solutions

We can use the equation of motion (88) to find the potentials \( \xi(\phi) \) and \( V(\phi) \) which lead to interesting cosmological solutions. The reconstruction of the corresponding theory is a well known procedure in the flat FRW geometry [54, 55] and here we try to extend these results to the Taub cosmological model. For the sake of simplicity, we will derive the potentials \( V(\phi) \) and \( \xi'(\phi) = \partial \xi(\phi)/\partial \phi \), because only the derivative of this potential appears in the action (51).
1. Anisotropic de Sitter Solution

An interesting solution of the Einstein field equations is the de Sitter one, which describes an empty Universe with a cosmological constant. The line element of this solution is:

$$ds^2 = dt^2 - a_0^2 \cosh(H_0 t) d\Omega_3,$$

where we are dealing with a non-stationary metric in a synchronous reference frame. We can identify $a(t) = a_0 \cosh(H_0 t)$ and bring back this line element to the FRW one with $K = 1$. This form of the metric represents a de Sitter solution in an isotropic closed Universe. The de Sitter model describes a singularity free Universe, which collapses to a finite minimum of the volume and then re-expands.

In this subsection we assume de Sitter solutions for the expansion factors $a(t), b(t)$ and find the family of potentials $V(\phi), \xi'(\phi)$ which lead to these cosmological solutions. The scalar field has a power-law dependence on time and the cosmological model is still anisotropic, thus,

$$a(t) = a_0 \cosh H_0 t, \quad b(t) = b_0 \cosh H_b t,$$

and the scalar field is,

$$\phi(t) = t^k.$$

We can integrate the equations of motion and obtain the following potentials,

$$V(\phi) = -\frac{\text{sech}^4(\phi_a)}{8a_0^4} [3a_0^4 H_0^2 \cosh(4\phi_a) + 4a_0^2 + 4(2a_0^4 H_0^2 + a_0^2) \cosh(2\phi_a) + -3b_0^2 (\cosh(2\phi_a) + 1) - 3b_0^2 5a_0^4 H_0^2] +$$

$$+ c_1 \exp \left( f_1(\phi) - \frac{f_2(\phi)}{f_3(\phi)} + \frac{f_4(\phi)}{f_5(\phi)} \right)^2 + c_2 \exp \left( f_1(\phi) + \frac{f_2(\phi)}{f_3(\phi)} + \frac{f_4(\phi)}{f_5(\phi)} \right)^2,$$

$$\xi'(\phi) = \frac{a_0^3 \cosh^4(\phi_a) \cosh(\phi_a)}{f_5(\phi)} V(\phi),$$

with $\phi_a = H_0 \phi^{1/k}$. The explicit form of the auxiliary functions $f_1, f_2, f_3, f_4$ and $f_5$ is given in Appendix A.

2. Isotropic de Sitter Solution

Now let us seek for potentials which lead to an isotropic de Sitter solution for a closed Universe,

$$a(t) = b(t) = r_0 \cosh (H_0 t),$$

Let us consider a simple power-law dependence of the scalar field on time:

$$\phi(t) = t^k,$$

then we can integrate the equations of motion to find the potentials,

$$V(\phi) = -3H_0^2 e^{-\phi} \tan^{-1} \left( e^\phi \right) + 3H_0^2 e^\phi \tan^{-1} \left( e^\phi \right) - \frac{3}{4r_0^2} + \frac{3e^{-\phi} \tan^{-1} \left( e^\phi \right)}{4r_0^2} + \frac{e^{3\phi}}{2H_0 \left( e^{2\phi} + 1 \right)} +$$

$$+ e^{-2\phi} \text{sech} \left( \phi \right) - \frac{3e^\phi \tan^{-1} \left( e^\phi \right)}{4r_0^2},$$

$$\xi'(\phi) = \frac{\tan^{-1} \left( e^\phi \right) \cosh^3 \left( \phi \right)}{4H_0 f(\phi)} - \frac{H_0 r_0^3 \sinh \left( \phi \right) \cosh \left( \phi \right)}{2f(\phi)} - \frac{H_0 r_0^2 \tan^{-1} \left( e^\phi \right) \cosh^3 \left( \phi \right)}{f(\phi)} + \frac{\sinh \left( \phi \right) \cosh \left( \phi \right)}{8H_0 f(\phi)} +$$
\[
\frac{r_0^2 e^{-2 \bar{\phi}} \sinh (\bar{\phi})}{24 H_0^2 f (\bar{\phi})} + \frac{r_0^2 e^{-2 \bar{\phi}} \cosh (\bar{\phi})}{3 H_0 f (\bar{\phi})} - \frac{r_0^2 e^{-2 \bar{\phi}} \cosh (\bar{\phi})}{12 H_0^2 f (\bar{\phi})} + \frac{r_0^2 e^{-2 \bar{\phi}} \sinh (\bar{\phi})}{6 H_0 f (\bar{\phi})},
\]

\[f (\phi) = \frac{4 H_0^2 k r_0^2 \sinh^2 (\bar{\phi})}{(\phi^{1/k})^{1-k}}, \quad \bar{\phi} = H_0 \phi^{1/k}.\]

We can perturb the solutions (91) and (91) adding a small perturbation to the expansion factor and to the scalar field, as follows,

\[a(t) = r_0 \cosh (H_0 t) + \delta a_1 (t), \quad \phi (t) = t^k + \delta \phi_1 (t),\]

with \(\delta\) is a small parameter. We can study the evolution of the perturbation using the potentials (92) and the equations of motion (85). Since \(a(t) = b(t)\), the first two equations in (85) take the same form and the equations of motions have the following form,

\[0 = -4 \delta H_0 r_0 a'_1 (t) \sinh (\bar{t}) - 4 \delta H_0^2 r_0 a_1 (t) \cosh (\bar{t}) - 4 \delta r_0 a_1 (t) V (t) \cosh (\bar{t}) - 2 r_0^2 V (t) \cosh^2 (\bar{t}) + (95)\]

\[-3 H_0^2 r_0^2 \cosh (2 \bar{t}) - H_0^2 r_0^2 - \frac{1}{2} - 4 \delta r_0 a''_1 (t) \cosh (\bar{t}) - 16 H_0^3 k r_0^2 k^{k-1} \sinh (2 \bar{t}) - 4 k^2 t^{k-2} + 4 k^k - 2 \bar{t} \{4 t^{-1} \{8 \delta H_0 k r_0 a'_1 (t) \sinh (\bar{t}) + 4 H_0^2 k r_0^2 \sinh^2 (\bar{t}) (4 k^2 t^k + k^k + k^k) + \}

32 \xi'' (t) \delta H_0 \{ - k r_0 a''_1 (t) \cosh (\bar{t}) - k^2 r_0^2 a_1' (t) \cosh (\bar{t}) + k r_0 a^{k-2} a'_1 (t) \cosh (\bar{t}) \} + \]

\[-32 H_0^2 k r_0 a^{k-1}_1 (t) \cosh (\bar{t}) - 16 H_0^3 k r_0^3 \sinh^2 (\bar{t}) (2 \bar{t}) - 4 \delta \phi''_1 (t) - 16 H_0^3 k r_0^2 \phi'_1 (t) \cosh (\bar{t}) - 16 H_0^2 k r_0^2 k^{k-2} \sinh^2 (\bar{t}) + 16 H_0^2 k r_0^2 t^{k-2} \sinh^2 (\bar{t}) + (96)\]

where we considered only the terms linear in \(\delta\) and \(\bar{t} = H_0 t\). We can solve the second equation in (96) with respect to \(\phi'_1 (t)\) and substitute it into the first equation to obtain a third order differential equation for \(a_1 (t)\). The resulting equation is,

\[F_3 a'''_1 (t) + F_2 a''_1 (t) + F_1 a'_1 (t) + F_0 a_1 (t) = 0\]

with,

\[F_0 = \frac{2 \delta r_0 e^{-i} \left(2 H_0^3 \left(e^{2i} + 1 \right)^3 + 4 H_0 e^{3i} + e^{3i}\right)}{H_0 \left(e^{2i} + 1 \right)^2}.
\]

\[F_1 = \delta H_0 r_0 e^{-2i} \left(e^i \left(e^{2i} - 1 \right) + 3 \left(e^{2i} + 1 \right)^2 \tan^{-1} \left(e^i \right) \right) - \frac{3 \delta e^{-2i} \left(e^i \left(e^{2i} - 1 \right) + \left(e^{2i} + 1 \right)^2 \tan^{-1} \left(e^i \right) \right) + }{4 H_0 r_0}
\]

\[-\frac{\delta r_0 e^{2i} \left(e^{2i} + 3 \right)}{2 H_0^2 \left(e^{2i} + 1 \right)^3} - \frac{2 \delta r_0 \left(e^{2i} - 3 \right)}{H_0 \left(e^{2i} + 1 \right)}.\]

\[F_2 = \frac{\delta e^{-2i} \left(e^{2i} + 1 \right) \left(e^i + \left(e^{2i} - 1 \right) \tan^{-1} \left(e^i \right) \right) - 2 \delta r_0 e^{-2i} \left(e^{2i} + 1 \right) \left(2 e^i + \left(e^{2i} - 1 \right) \tan^{-1} \left(e^i \right) \right) + }{2 H_0^2 r_0}
\]

\[-\frac{\delta r_0 e^{2i} \left(2 e^{2i} + 3 e^{4i} + 3 \right)}{3 H_0^2 \left(e^{2i} + 1 \right)^2} - \frac{4 \delta r_0 e^{-2i} \left(2 e^{2i} + 3 e^{4i} + 1 \right)}{3 H_0^2 \left(e^{2i} + 1 \right)^2}.
\]
\[
F_3 = \frac{\delta e^{-2f} \left( e^f \left( e^{2f} - 1 \right) + (e^{2f} + 1)^2 \tan^{-1} \left( e^f \right) \right)}{4H_0^3 r_0} - \frac{\delta r_0 e^{-2f} \left( e^{2f} + 3 \right)}{6H_0^4 (e^{2f} + 1)} + \frac{2\delta r_0 e^{-2f} \left( 3e^{2f} + 1 \right)}{3H_0^3 (e^{2f} + 1)}.
\]

This equation can be solved numerically with different parameters \((H_0, r_0)\) and initial conditions \(a_1(0), a'_1(0)\) and \(a''_1(0)\).

### B. Scalar Potentials in Bianchi I Cosmology

The Gauss-Bonnet invariant for the Bianchi I cosmology makes the potentials \(U_1, U_2\) and \(U_3\) in Eq. \((78)\) vanish, obtaining the following expression,

\[
G_I = 8 \left( \dot{a} \dot{b} \dot{c} + \dot{a} \dot{b} \dot{c} + \dot{a} \dot{b} \dot{c} \right)
\]

and so, the Lagrangian of the cosmological model becomes,

\[
\mathcal{L}_I = 2 \dot{a} \dot{b} \dot{c} + 2 \dot{b} \dot{a} \dot{c} + 2 \dot{a} \dot{b} \dot{c} + 6 \dot{a} \dot{b} \dot{c} \phi \dot{a} \dot{b} \dot{c} - abcV(\phi).
\]

Using the field equations we found the following equation of motion,

\[
-8abc\dot{\phi}'\phi''(\phi) + \dot{b}\dot{c} + \phi'(\phi) \left( -8bc\dot{\phi} - 8bc\dot{\phi} - 8bc\dot{\phi} \right) + \frac{1}{2} \left( -2\dot{c} - 4\dot{b} - 2\dot{a} \right) - bcV(\phi) = 0,
\]

\[
-8\dot{a}\dot{c}\phi'(\phi) + \dot{a}\dot{c} + \phi'(\phi) \left( -8a\dot{c}\dot{\phi} - 8a\dot{c}\dot{\phi} - 8a\dot{c}\dot{\phi} \right) + \frac{1}{2} \left( -2\dot{a}c - 4\dot{c} - 2\dot{b} \right) - acV(\phi) = 0,
\]

\[
-8\dot{a}\dot{b}\phi'(\phi) + \dot{a}\dot{b} + \phi'(\phi) \left( -8\dot{a}\dot{b}\phi - 8\dot{a}\dot{b}\phi - 8\dot{a}\dot{b}\phi \right) + \frac{1}{2} \left( -2\dot{b}a - 4\dot{a}b - 2\dot{c} \right) - abV(\phi) = 0,
\]

\[
\phi'(\phi) \left( -8\dot{a}\dot{b} - 8\dot{a}\dot{b} - 8\dot{a}\dot{b} \right) - abcV'(\phi) = 0.
\]

The simplest solution to the Einstein equation in the framework of the Bianchi classification is the Kasner one, a solution for an empty, anisotropic and flat Universe. In this case the spatial line element reduces to,

\[
dl^2 = t^{2p_1} (dx^1)^2 + t^{2p_2} (dx^2)^2 + t^{2p_3} (dx^3)^2,
\]

where \(p_1, p_2, p_3\) are the Kasner indices and they satisfy two relations,

\[
p_1 + p_2 + p_3 = 1
\]

\[
p_1^2 + p_2^2 + p_3^2 = 1
\]

and so there is only one independent parameter that describes the solution. We are looking for the form of potential which leads to Kasner solutions,

\[
a(t) = t^{p_1}, \quad b(t) = t^{p_2}, \quad c(t) = t^{p_3}.
\]

We assume a simple power-law dependence of the scalar field on time,

\[
\phi(t) = t^k,
\]

and so we can solve the equations of motion to find the following potentials,

\[
V(\phi) = -\frac{p_1 (p_2^2 + p_2 (p_3 - 1) + (p_3 - 1) p_3)}{p_1^2 + p_1 (E_1 - p_1) + 2(p_2 + p_3 - 1)} \phi^{-2/k} \frac{E_1}{p_1} + c_1 \phi^{E_1/2k} + c_2 \phi^{E_2/2k},
\]

\[
\xi'(\phi) = \frac{(\phi^{1/k})^{2-k}}{16p_1 p_2 p_3 k} \left( -4 \frac{p_2^2 + p_2 (p_3 - 1) + (p_3 - 1) p_3}{p_1^2 + p_1 (E_1 - p_1) + 2(p_2 + p_3 - 1)} \frac{i \sqrt{p_1 E_1 + p_1 E_2 + p_2 + p_3 + 1}}{p_1 E_1} \right) +
\]

\[
+ c_1 \phi^{1/2k} \left( -i \sqrt{p_1 E_1 + p_2 (p_3 - 3)} \right) + c_2 \phi^{1/2k} \left( i \sqrt{p_1 E_1 + p_2 (p_3 - 3)} \right),
\]

with \(E_1 = p_1 + p_2 + p_3 - 3\), \(E_2 = \frac{-4p_1^2 + 4p_1 (p_2 + p_3 - 3) + (p_2 + p_3 + 1)^2}{p_1 E_1}\) and \(E_3 = i \sqrt{p_1 E_1 E_2 - p_2 - p_3 - 1}\).
V. CONCLUSIONS AND FUTURE PERSPECTIVE

In this work we considered some anisotropic evolution scenarios in the context of modified gravity in several of its various forms. These scenarios can be relevant for pre-inflationary eras and may leave their imprint on the four dimensional classical effective inflationary theory. We considered several mainstream modified gravities, such as $F(R)$ gravity with an extra scalar field and in its Jordan frame vacuum form, and also Gauss-Bonnet gravities. In all the cases we considered, the field equations can be utilized as a reconstruction method and several evolutions of specific form like Taub and Bianchi Universes can be realized. Using these reconstruction techniques, we were able to present the formalism of studying anisotropic inflation in several modified gravities. This work fills in the gap in the literature between ordinary isotropic inflationary theories in modified gravity with a single scale factor and anisotropic inflation in modified gravity theories. Our aim was to present reconstruction techniques to be utilized in order to obtain phenomenological outcomes out of these theories. This is not an easy task, however with this work we made the first step in the literature towards this direction. We also answered the vital question whether cosmological solutions of elevated importance exist, like for example isotropic or anisotropic de Sitter solutions. Finally, for the Gauss-Bonnet theories, we provided expressions for the scalar potentials, in the case of a Bianchi I cosmology. Admittedly, the way towards obtaining phenomenological information for these modified gravity theories in the context of anisotropic inflation, is long and thorny, however we presented the first step in this direction, the formalization of anisotropic inflation in non-trivial modified gravity theories. Now what remains as a future perspective of this work is to quantitatively check the actual effects of the anisotropies on the effective inflationary theory starting at the first horizon crossing when the primordial modes exit the horizon for the first time. The short wavelength modes will directly be probed by future gravitational wave experiments like the LISA and DECIGO, and thus we will have a direct grasp on these modes. Therefore, if the Universe was anisotropic pre-inflationary, these evolution conditions may set and determine the initial conditions of the Universe at the pre-inflationary stages. Of course as the Universe enters the inflationary era, the anisotropies will be smoothed away, however, the anisotropies may affect directly the initial conditions of the inflationary epoch, for example the scale of inflation. These issues should be separately considered in a future work, based on the formalism we discussed in this paper.

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