1. Introduction. Let $X$ be a smooth projective variety over $\mathbb{C}$ of dimension $n$ equipped with an ample line bundle $L$ and a subscheme $Z \subset X$ of length $d$. Serre duality provides a natural isomorphism of vector spaces (for each $i = 0, \ldots, n$):

\begin{equation}
(\ast) \quad \text{Ext}^i(L \otimes I_Z, \mathcal{O}_X) \cong H^{n-i}(X, K_X \otimes L \otimes I_Z)^\vee
\end{equation}

Thaddeus pairs and Reider’s theorem concern the cases $i = 1$ and $n = 1, 2$. In these cases one associates a rank two coherent sheaf $E_\epsilon$ to each extension class $\epsilon \in \text{Ext}^1(L \otimes I_Z, \mathcal{O}_X)$ via the short exact sequence:

\begin{equation}
(\ast\ast) \quad \epsilon : 0 \to \mathcal{O}_X \to E_\epsilon \to L \otimes I_Z \to 0
\end{equation}

and the Mumford stability (or instability) of $E_\epsilon$ allows one to distinguish among extension classes. The ultimate aim of this paper is to show how a new notion of Bridgeland stability can similarly be used to distinguish among higher extension classes, leading to a natural higher-dimensional generalization of Thaddeus pairs as well as the setup for a higher-dimensional Reider’s theorem.

Reider’s theorem gives numerical conditions on an ample line bundle $L$ on a surface $S$ that guarantee the vanishing of the vector spaces $H^1(S, K_S \otimes L \otimes I_Z)$ which in turn implies the base-point-freeness (the $d = 1$ case) and very ampleness (the $d = 2$ case) of the adjoint line bundle $K_S \otimes L$.

In the first part of this note we will revisit Reider’s Theorem in the context of Bridgeland stability conditions. Reider’s approach, following Mumford, uses the Bogomolov inequality for Mumford-stable coherent sheaves on a surface to argue (under suitable numerical conditions on $L$) that no exact sequence $(\ast\ast)$ can produce a Mumford stable sheaf $E_\epsilon$, and then uses the Hodge Index Theorem to argue that the only exact sequences $(\ast\ast)$ that produce non-stable sheaves must split. Thus one concludes that $\text{Ext}^1(L \otimes I_Z, \mathcal{O}_S) = 0$ and $H^1(S, K_S \otimes L \otimes I_Z) = 0$, as desired.

Here, we will regard an extension class in $(\ast\ast)$ as a morphism to the \textit{shift} of $\mathcal{O}_S$:

\begin{equation}
\epsilon : L \otimes I_Z \to \mathcal{O}_S[1]
\end{equation}

in one of a family of \textit{tilts} $\mathcal{A}_s$ ($0 < s < 1$) of the abelian category of coherent sheaves on $X$ within the bounded derived category $D(X)$ of complexes of coherent sheaves on $X$. Reider’s argument for a surface $S$ is essentially equivalent to ruling out non-trivial extensions by determining that:

- $\epsilon$ is neither injective nor surjective and
- if neither injective nor surjective, then $\epsilon = 0$ (using Hodge Index).

The second author was partially supported by NSF grant DMS 0501000.
This way of looking at Reider’s argument allows for some minor improvements, but more importantly leads to the notion of Bridgeland stability conditions, which are stability conditions, not on coherent sheaves, but rather on objects of \( \mathcal{A}_s \).

In [AB], it was shown that the Bogomolov Inequality and Hodge Index Theorem imply the existence of such stability conditions on arbitrary smooth projective surfaces \( S \) (generalizing Bridgeland’s stability conditions for \( K \)-trivial surfaces [Bri08]). Using these stability conditions, we investigate the stability of objects of the form \( L \otimes \mathcal{I}_Z \) and \( \mathcal{O}_S[1] \) with a view toward reinterpreting the vanishing:

\[
\text{Hom}(L \otimes \mathcal{I}_Z, \mathcal{O}_S[1]) = 0
\]
as a consequence of an inequality \( \mu(L \otimes \mathcal{I}_Z) > \mu(\mathcal{O}_S[1]) \) of Bridgeland slopes. Since this is evidently a stronger condition than just the vanishing of the Hom, it is unsurprising that it should require stronger numerical conditions. This reasoning easily generalizes to the case where \( \mathcal{O}_S \) is replaced by \( \mathcal{I}_W \), the derived dual of the ideal sheaf of a finite length subscheme \( W \subset S \).

The Bridgeland stability of the objects \( L \otimes \mathcal{I}_Z \) and \( \mathcal{I}_W \) is central to a new generalization of Thaddeus pairs from curves to surfaces. A Thaddeus pair on a curve \( C \) is an extension of the form:

\[
\epsilon : 0 \rightarrow \mathcal{I}_W \rightarrow E_{\epsilon} \rightarrow L \otimes \mathcal{I}_Z \rightarrow 0
\]

where \( L \) is a line bundle and \( Z, W \subset C \) are effective divisors. Normally we would write this:

\[
\epsilon : 0 \rightarrow \mathcal{O}_C(W) \rightarrow E_{\epsilon} \rightarrow L(-Z) \rightarrow 0
\]
since finite length subschemes of a curve are effective Cartier divisors. The generic such extension determines a Mumford-stable vector bundle \( E_{\epsilon} \) on \( C \) whenever:

\[
\text{deg}(L(-Z)) > \text{deg}(\mathcal{O}_C(W)) \quad \text{(and \( C \neq \mathbb{P}^1 \))}
\]
or, equivalently, whenever the Mumford slope of \( L(-Z) \) exceeds that of \( \mathcal{O}_C(W) \) (both line bundles are trivially Mumford-stable). Moreover, the Mumford-unstable vector bundles arising in this way are easily described in terms of the secant varieties to the image of \( C \) under the natural linear series map:

\[
\phi : C \rightarrow \mathbb{P}(H^0(C, K_C \otimes L(-Z - W))^\vee) \cong \mathbb{P}(\text{Ext}^1(L(-Z), \mathcal{O}_C(W)))
\]
since an unstable vector bundle \( E_{\epsilon} \) can only be destabilized by a sub-line bundle \( L(-Z') \subset L(-Z) \) that lifts to a sub-bundle of \( E_{\epsilon} \):

\[
\begin{array}{c}
L(-Z') \\
\downarrow \quad \\
E_{\epsilon} \\
\downarrow \quad \\
L(-Z) \\
0
\end{array}
\]

In the second part of this paper, we note that Thaddeus pairs naturally generalize to surfaces as extensions of the form:

\[
\epsilon : 0 \rightarrow \mathcal{I}_W[1] \rightarrow E_{\epsilon}^\bullet \rightarrow L \otimes \mathcal{I}_Z \rightarrow 0
\]
in the categories \( \mathcal{A}_s \) under appropriate Bridgeland stability conditions for which both \( L \otimes \mathcal{I}_Z \) and \( \mathcal{I}_W[1] \) are Bridgeland stable and their Bridgeland slopes satisfy \( \mu(L \otimes \mathcal{I}_Z) > \mu(\mathcal{I}_W[1]) \). Note that \( E_{\epsilon}^\bullet \) is not ever a coherent sheaf.

This is a very satisfying generalization of Thaddeus pairs since:

\[
\text{Ext}^1_{\mathcal{A}_s}(L \otimes \mathcal{I}_Z, \mathcal{I}_W[1]) \cong H^0(S, K_S \otimes L \otimes \mathcal{I}_Z \otimes \mathcal{I}_W)^\vee
\]
by Serre duality. In this case, however, there are subobjects:

\[ K \subset L \otimes I_Z \]

not of the form \( L \otimes I_Z \) that may destabilize \( E_\bullet \), as in (†). These subobjects are necessarily coherent sheaves, but may be of higher rank than one, and therefore not subsheaves of \( L \otimes I_Z \) in the usual sense. This leads to a much richer geometry for the locus of “unstable” extensions than in the curve case.

We will finally discuss the moduli problem for families of Bridgeland stable objects with the particular invariants:

\[ [E] = [L \otimes I_Z] + [I_W^\vee [1]] = [L \otimes I_Z] - [I_W^\vee] \]

in the Grothendieck group (or cohomology ring) of \( S \), and finish by describing wall-crossing phenomena of (some of) these moduli spaces in the \( K \)-trivial case, following [AB].

This line of reasoning suggests a natural question for three-folds \( X \). Namely, might it be possible to prove a Reider theorem for \( L \) and \( Z \subset X \) by ruling out non-trivial extensions of the form:

\[ \epsilon : 0 \to O_X[1] \to E_\bullet \to L \otimes I_Z \to 0 \]

in some tilt \( \mathcal{A}_s \) of the category of coherent sheaves on \( X \) via a version of the Bogomolov Inequality and Hodge Index Theorem for objects of \( \mathcal{A}_s \) on threefolds?

We do not know versions of these results that would allow a direct application of Reider’s method of proof, but this seems a potentially fruitful direction for further research, and ought to be related to the current active search for examples of Bridgeland stability conditions on complex projective threefolds.

2. The Original Reider. Fix an ample divisor \( H \) on a smooth projective variety \( X \) over \( \mathbb{C} \) of dimension \( n \). A torsion-free coherent sheaf \( E \) on \( X \) has Mumford slope:

\[ \mu_H(E) = \frac{c_1(E) \cdot H^{n-1}}{\text{rk}(E) H^n} \]

and \( E \) is \( H \)-Mumford-stable if \( \mu_H(K) < \mu_H(E) \) for all subsheaves \( K \subset E \) with the property that \( Q = E/K \) is supported in codimension \( \leq 1 \).

Bogomolov Inequality: Suppose \( E \) is \( H \)-Mumford-stable and \( n \geq 2 \). Then:

\[ \text{ch}_2(E) \cdot H^{n-2} \leq \frac{c_2^2(E) \cdot H^{n-2}}{2 \text{rk}(E)} \]

(in case \( X = S \) is a surface, the conclusion is independent of the choice of \( H \))

Application 2.1: For \( S, L, Z \) as above, suppose \( \epsilon \in \text{Ext}^1(L \otimes I_Z, \mathcal{O}_S) \) and:

\[ \epsilon : 0 \to \mathcal{O}_S \to E \to L \otimes I_Z \to 0 \]

yields a Mumford-stable sheaf \( E \). Then \( c_2^2(L) \leq 4d \).

Proof: By the Bogomolov inequality:

\[ \text{ch}_2(E) = \frac{c_2^2(L)}{2} - d \leq \frac{c_2^2(E)}{4} = \frac{c_2^2(L)}{4} \]

Hodge Index Theorem: Let \( D \) be an arbitrary divisor on \( X \). Then:

\[ (D^2 \cdot H^{n-2-2}) (H^n) \leq (D \cdot H^{n-1})^2 \]
and equality holds if and only if $D \cdot E \cdot H^{n-2} = kE \cdot H^{n-1}$ for some $k \in \mathbb{Q}$ and all divisors $E$.

**Application 2.2:** Suppose $E$ is not $c_1(L)$-Mumford-stable in Application 2.1. Then either $\epsilon = 0$ or else there is an effective curve $C \subset S$ such that:

(a) $C \cdot c_1(L) \leq \frac{1}{2}c_1^2(L)$ and
(b) $C \cdot c_1(L) \leq C^2 + d$

and it follows that $-d < C^2 \leq d$. Moreover,

(c) $c_1^2(L) > 4d \Rightarrow C^2 < d$ and
(d) $c_1^2(L) > (d + 1)^2 \Rightarrow C^2 \leq 0$

**Proof:** By definition of (non)-stability, there is a rank-one subsheaf $K \subset E$ such that $c_1(K) \cdot c_1(L) \geq \frac{1}{2}c_1(E) \cdot c_1(L) = \frac{1}{2}c_1^2(L)$. It follows that the induced map $K \to L \otimes I_Z$ is non-zero and either $K$ splits the sequence, or else $K \subset L \otimes I_Z$ is a proper subsheaf. In the latter case, $K = L(-C) \otimes I_W$ for some effective curve $C$ and zero-dimensional $W \subset S$, and (a) now follows immediately.

The inequality (b) is seen by computing the second Chern character of $E$ in two different ways. We may assume without loss of generality that the cokernel $Q = E/K$ is also torsion-free by replacing $Q$ with its torsion-free quotient $Q'$ and $K$ with the kernel of the induced map $E \to Q'$ if necessary (this will only increase the value of $c_1(K) \cdot c_1(L)$). Then $Q$ has the form $O_S(C) \otimes I_V$ for some $V \subset S$, and in particular:

$$\operatorname{ch}_2(E) = \frac{c_1^2(L)}{2} - d = \left(\frac{c_1(E) - C}{2}\right)^2 - l(W) + \frac{C^2}{2} - l(V) \leq \frac{c_1^2(L)}{2} + C^2 - C \cdot c_1(L)$$

which gives (b).

Next, applying Hodge Index to (a) and (b) gives:

$$C^2 c_1^2(L) \leq (C \cdot c_1(L))^2 \leq \frac{1}{2}c_1^2(L) \left(C^2 + d\right)$$

from which we conclude that $C^2 \leq d$. That $C^2 > -d$ follows immediately from (b) and the fact that $L$ is ample. Finally, let $C^2 = d - k$ for $0 \leq k < d$ and apply Hodge Index to (b) to conclude that:

$$(d - k)c_1^2(L) \leq (C \cdot c_1(L))^2 \leq (2d - k)^2 \Rightarrow c_1^2(L) \leq 4d + \frac{k^2}{d - k}$$

and then (c) and (d) follow from the cases $k = 0$ and $k \leq d - 1$, respectively.

All of this gives as an immediate corollary a basic version of:

**Reider’s Theorem:** If $L$ is an ample line bundle on a smooth projective surface $S$ such that $c_1^2(L) > (d + 1)^2$ and $C \cdot c_1(L) > C^2 + d$ for all effective divisors $C$ on $S$ satisfying $C^2 \leq 0$, then “$K_S + L$ separates length $d$ subschemes of $S$,” i.e. $H^1(S, K_S \otimes L \otimes I_Z) = 0$ for all subschemes $Z \subset S$ of length $d$ (or less).

**Corollary (Fujita’s Conjecture for Surfaces):** If $L$ is an ample line bundle on a smooth projective surface $S$, then $K_S + (d + 2)L$ separates length $d$ subschemes.

**Note:** For other versions of Reider’s theorem, see e.g. [Laz91].
3. Reider Revisited. A torsion-free coherent sheaf $E$ is $H$-Mumford semi-stable (for $X$ and $H$ as in §2) if

$$
\mu_H(K) \leq \mu_H(E)
$$

for all subsheaves $K \subseteq E$ (where $\mu_H$ is the Mumford slope from §2). A Mumford $H$-semi-stable sheaf $E$ has a Jordan-Hölder filtration:

$$
F_1 \subset F_2 \subset \cdots \subset F_M = E
$$

where the $F_{i+1}/F_i$ are Mumford $H$-stable sheaves all of the same slope $\mu_H(E)$. Although the filtration is not unique, in general, the associated graded coherent sheaf $\oplus F_i = \text{Ass}_H(E)$ is uniquely determined by the semi-stable sheaf $E$ (and $H$).

The Mumford $H$-slope has the following additional crucial property:

**Harder-Narasimhan Filtration:** Every coherent sheaf $E$ on $X$ admits a uniquely determined (finite) filtration by coherent subsheaves:

$$
0 \subset E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_N = E
$$

such that

- $E_0$ is the torsion subsheaf of $E$ and
- Each $E_i/E_{i-1}$ is $H$-semi-stable of slope $\mu_i$ with $\mu_1 > \mu_2 > \cdots > \mu_N$.

Harder-Narasimhan filtrations for a fixed ample divisor class $H$ give rise to a family of “torsion pairs” in the category of coherent sheaves on $X$:

**Definition:** A pair $(\mathcal{F}, \mathcal{T})$ of full subcategories of a fixed abelian category $\mathcal{A}$ is a *torsion pair* if:

(a) For all objects $T \in \text{ob}(\mathcal{F})$ and $F \in \text{ob}(\mathcal{F})$, $\text{Hom}(T, F) = 0$.

(b) Each $A \in \text{ob}(\mathcal{A})$ fits into a (unique) extension $0 \to T \to A \to F \to 0$ for some (unique up to isomorphism) objects $T \in \text{ob}(\mathcal{T})$ and $F \in \text{ob}(\mathcal{F})$.

**Application 3.1:** For each real number $s$, let $\mathcal{T}_s$ and $\mathcal{F}_s$ be full subcategories of the category $\mathcal{A}$ of coherent sheaves on $X$ that are closed under extensions and which are generated by, respectively:

$$
\mathcal{F}_s \supset \{ \text{torsion-free } H\text{-stable sheaves of } H\text{-slope } \mu \leq s \}
$$

$$
\mathcal{T}_s \supset \{ \text{torsion-free } H\text{-stable sheaves of } H\text{-slope } \mu > s \} \cup \{ \text{torsion sheaves} \}
$$

Then $(\mathcal{F}_s, \mathcal{T}_s)$ is a torsion pair of $\mathcal{A}$.

**Proof:** Part (a) of the definition follows from the fact that $\text{Hom}(T, F) = 0$ if $T, F$ are $H$-stable and $\mu_H(T) > \mu_H(F)$, together with the fact that $\text{Hom}(T, F) = 0$ if $T$ is torsion and $F$ is torsion-free.

A coherent sheaf $E$ is either torsion (hence in $\mathcal{T}_s$ for all $s$) or else let $E(s) := E_i$ be the largest subsheaf in the Harder-Narasimhan of $E$ with the property that $\mu(E_i/E_{i-1}) > s$. Then $0 \to E(s) \to E \to E/E(s) \to 0$ is the desired short exact sequence for (b) of the definition.

**Theorem (Happel-Reiten-Smalø) [HRS96]:** Given a torsion pair $(\mathcal{T}, \mathcal{F})$, then there is a $t$-structure on the bounded derived category $\mathcal{D}(\mathcal{A})$ defined by:

$$
\text{ob}(\mathcal{D}^{<0}) = \{ E^\bullet \in \text{ob}(\mathcal{D}) | H^i(E^\bullet) \in \mathcal{F}, \text{H}^i(E^\bullet) = 0 \text{ for } i < -1 \}
$$

$$
\text{ob}(\mathcal{D}^{>0}) = \{ E^\bullet \in \text{ob}(\mathcal{D}) | H^0(E^\bullet) \in \mathcal{T}, \text{H}^i(E^\bullet) = 0 \text{ for } i > 0 \}
$$
In particular, the heart of the $t$-structure:

$$A_{(F,T)} := \{ E^\bullet \mid H^{-1}(E^\bullet) \in F, \text{Ext}^0(E^\bullet) \in T, H^1(E^\bullet) = 0 \text{ otherwise} \}$$

is an abelian category (referred to as the “tilt” of $A$ with respect to $(F, T)$).

**Notation:** We will let $A_s$ denote the tilt with respect to $(F_s, T_s)$ (and $H$).

In practical terms, the category $A_s$ consists of:

- Extensions of torsion and $H$-stable sheaves $T$ of slope $> s$
- Extensions of shifts $F[1]$ of $H$-stable sheaves $F$ of slope $\leq s$
- Extensions of a sheaf $T$ by a shifted sheaf $F[1]$

Extensions in $A_s$ of coherent sheaves $T_1, T_2$ in $T$ or shifts of coherent sheaves $F_1[1], F_2[1]$ in $F$ are given by extension classes in $\text{Ext}_{A_s}^1(T_1, T_2)$ or $\text{Ext}_{A_s}^1(F_1, F_2)$, which are first extension classes in the category of coherent sheaves.

However, an extension of a coherent sheaf $T$ by a shift $F[1]$ in $A_s$ is quite different. It is given by an element of $\text{Ext}_{A_s}^1(T, F[1])$ by definition, but:

$$\text{Ext}_{A_s}^1(T, F[1]) = \text{Ext}_{A_s}^2(T, F)$$

and this observation will allow us to associate objects of $A_s$ to certain “higher” extension classes of coherent sheaves in $A$ just as coherent sheaves are associated to first extension classes of coherent sheaves.

First, though, recall that the **rank** is an integer-valued linear function:

$$r : K(D) \rightarrow \mathbb{Z}$$

on the Grothendieck group of the derived category of coherent sheaves, with the property that $r(E) \geq 0$ for all coherent sheaves $E$.

We may define an analogous rank function for each $s \in \mathbb{R}$ (and $H$):

$$r_s : K(D) \rightarrow \mathbb{R}; \quad r_s(E) = c_1(E) \cdot H^{n-1} - s \cdot r(E)H^n$$

which has the property that $r_s(E^\bullet) \geq 0$ for all objects $E^\bullet$ of $A_s$ and $r_s(T) > 0$ for all coherent sheaves in $T_s$ supported in codimension $\leq 1$. This rank is evidently rational-valued if $s \in \mathbb{Q}$.

Now consider the objects $\mathcal{O}_X[1]$ and $L \otimes \mathcal{I}_Z$ of $A_s$ for $0 \leq s < 1$ ($H = c_1(L)$) where $Z \subset X$ is any closed subscheme supported in codimension $\geq 2$.

**Sub-objects of $\mathcal{O}_X[1]$;**

An exact sequence $0 \rightarrow K^\bullet \rightarrow \mathcal{O}_X[1] \rightarrow Q^\bullet \rightarrow 0$ of objects of $A_s$ (for any $s \geq 0$) induces a long exact sequence of cohomology sheaves:

$$0 \rightarrow H^{-1}(K^\bullet) \rightarrow \mathcal{O}_X \rightarrow H^{-1}(Q^\bullet) \xrightarrow{\delta} H^0(K^\bullet) \rightarrow 0$$

Since $H^{-1}(Q^\bullet)$ is torsion-free and $\delta$ is not a (non-zero) isomorphism, then either:

(i) $H^{-1}(K^\bullet) = \mathcal{O}_X$ and $Q^\bullet = 0$, $\delta = 0$ and $K^\bullet = \mathcal{O}_X[1]$, or else

(ii) $H^{-1}(K^\bullet) = 0$, and $K = K^\bullet$ is a coherent sheaf with no torsion subsheaf supported in codimension two. The quotient $Q^\bullet = Q[1]$ is the shift of a torsion-free sheaf satisfying:

$$0 \leq r_s(Q[1]) = -c_1(Q) \cdot H^{n-1} + s \cdot r(Q)H^n < r_s(\mathcal{O}_X[1]) = sH^n$$
hence in particular:

\[ s \left(1 - \frac{1}{r(Q)}\right) < \mu_H(Q) \leq s \]

Moreover, if \( E \) is a stable coherent sheaf appearing in the associate graded of a semi-stable coherent sheaf in the Harder-Narasimhan filtration of \( Q \), then the same inequality holds for \( \mu_H(E) \) (because the \( r_s \) rank is additive).

**Sub-objects of \( L \otimes I_Z \):**

An exact sequence: \( 0 \to K' \to L \otimes I_Z \to Q' \to 0 \) induces a long exact sequence of cohomology sheaves:

\[ 0 \to H^{-1}(Q') \to H^0(K') \to L \otimes I_Z \to H^0(Q') \to 0 \]

from which it follows that \( K' := K' \) is a torsion-free coherent sheaf, and either:

1. \((i') r(K') = 1\), so that \( K' = L \otimes I_Z \); and \( Q' = L \otimes (I_Z/I_Z') \), or else:
2. \((ii') r(K') > 1\) and \( H^{-1}(Q') \neq 0 \).

In either case, we have the inequality:

\[ s < \mu_H(K') \leq s + \frac{(1 - s)}{r(K')} \]

and the same inequality when \( K' \) is replaced by any \( E' \) appearing in the associated graded of a semi-stable coherent sheaf in the Harder-Narasimhan filtration of \( K' \).

**Corollary 3.2:** The alternatives for a non-zero homorphism:

\[ f \in \text{Hom}_{\mathcal{A}_s}(L \otimes I_Z, \mathcal{O}_X[1]) = \text{Ext}^1_{\mathcal{A}}(L \otimes I_Z, \mathcal{O}_X) \] for some fixed \( 0 < s < 1 \) are as follows:

1. \( f \) is injective, with quotient \( Q' = Q[1] \):
   \[ 0 \to L \otimes I_Z \xrightarrow{f} \mathcal{O}_X[1] \to Q[1] \to 0 \]
   which in particular implies that \( 1/2 = \mu_H(Q) \leq s \) and, more generally, that each stable \( E \) in the Harder-Narasimhan filtration of \( Q \) has Mumford-slope \( \mu_H(E) \leq s \).
   
   (b) \( f \) is surjective, with kernel \( (K')' = K' \):
   \[ 0 \to K' \to L \otimes I_Z \xrightarrow{f} \mathcal{O}_X[1] \to 0 \]
   which in particular implies that \( 1/2 = \mu_H(K') > s \) and, more generally, that each stable \( E' \) in the Harder-Narasimhan filtration of \( K' \) has Mumford-slope \( \mu_H(E') > s \).
   
2. \( f \) is neither injective nor surjective, inducing a long exact sequence:
   \[ 0 \to L(-D) \otimes I_W \to L \otimes I_Z \xrightarrow{f} \mathcal{O}_X[1] \to (\mathcal{O}_X(D) \otimes I_V)[1] \to 0 \]
   for some effective divisor \( D \) satisfying \( D \cdot H^{n-1} \leq sH^n \) and \( D \cdot H^{n-1} < (1 - s)H^n \), as well as subschemes \( V, W \subset X \) supported in codim \( \geq 2 \).

**Proof:** Immediate from the considerations above.

**Example:** At \( s = 1/2 \), we nearly get the same dichotomy as in §2. Here:

\[ f \] is injective in \( \mathcal{A}_{1/2} \iff Q \) is \( H \)-semistable

so the injectivity (or not) of \( f \) is equivalent to the semi-stability (or not) of the coherent sheaf \( E \) expressed as the corresponding (ordinary) extension.
Next, recall that the degree is an integer-valued linear function:
\[ d : K(D) \rightarrow \mathbb{Z}; \quad d(E) = c_1(E) \cdot H^{n-1} \]
(depending upon \( H \)) with the property that for all coherent sheaves \( E \):
\[ r(E) = 0 \Rightarrow (d(E) \geq 0 \text{ and } d(E) = 0 \iff E \text{ is supported in codim } \geq 2) \]

There is an analogous two-parameter family of degree functions \((s, t) \in \mathbb{R}, t > 0\):
\[ d_{(s, t)} : K(D) \rightarrow \mathbb{R}; \quad d_{(s, t)}(E) = \text{ch}_2(E) \cdot H^{n-2} - sc_1(E) \cdot H^{n-1} + \left( \frac{s^2 - t^2}{2} \right) r(E)H^n \]
(i.e. a ray of degree functions for each rank \( r_s \)). Suppose \( E^* \) is an object of \( A_s \) and
\[ r_s(E^*) = c_1(E) \cdot H^{n-1} - s \cdot r(E)H^n = 0 \]
Then \( E^* \) fits into a (unique) exact sequence (in \( A_s \)):
\[ 0 \rightarrow F[1] \rightarrow E^* \rightarrow T \rightarrow 0 \]
where \( T \) is a torsion sheaf supported in codimension \( \geq 2 \), and \( F \) is an \( H \)-semistable torsion-free sheaf with \( \mu_H(F) = s \).

**Proposition 3.3:** Suppose \( r_s(E^*) = 0 \) for an object \( E^* \) of \( A_s \). Then for all \( t > 0 \),
\[ d_{(s, t)}(E^*) \geq 0 \text{ and } d_{(s, t)}(E^*) = 0 \iff E^* \text{ is a sheaf, supported in codim } \geq 3 \]

**Proof:** Because \( d_{(s, t)} \) is linear, it suffices to prove the Proposition for torsion sheaves \( T \) supported in codimension \( \geq 2 \) and for shifts \( F[1] \) of \( H \)-stable torsion-free sheaves of slope \( s \). In the former case:
\[ d_{(s, t)}(T) = \text{ch}_2(T) \cdot H^{n-2} \geq 0 \text{ with equality } \iff T \text{ is supported in codim } \geq 3 \]

In the latter case:
\[ d_{(s, t)}(F[1]) = -\text{ch}_2(F) \cdot H^{n-2} + sc_1(F) \cdot H^{n-1} - \left( \frac{s^2 - t^2}{2} \right) r(F)H^n \]
and \( \mu_H(F) = s \Rightarrow (c_1(F) - sr(F)H) \cdot H^{n-1} = 0 \Rightarrow (c_1(F) - sr(F)H)^2H^{n-2} \leq 0 \)
by the Hodge Index Theorem. It follows from the Bogomolov inequality that:
\[ d_{(s, t)}(F[1]) \geq -\left( \frac{c_1^2(F)}{2r(F)} \right) \cdot H^{n-2} + sc_1(F) \cdot H^{n-1} - \left( \frac{s^2}{2} \right) r(F)H^n + \left( \frac{t^2}{2} \right) r(F)H^n \]
\[ = -\left( \frac{1}{2r(F)} \right) (c_1(F) - sr(F)H)^2 \cdot H^{n-2} + \left( \frac{t^2}{2} \right) r(F)H^n > 0 \square \]

**Corollary 3.4:** If \( X = S \) is a surface, then the complex linear function:
\[ Z_{s+it} := -(d_{(s, t)} + itr_s) : K(D) \rightarrow \mathbb{C}; \quad s \in \mathbb{R}, t > 0, i^2 = -1 \]
has the property that \( Z_{s+it}(E^*) \neq 0 \) for all objects \( E^* \neq 0 \) of \( \text{ob}(A_s) \), and:
\[ 0 < \arg(Z_{s+it}(E^*)) \leq 1 \text{ (where } \arg(re^{i\rho}) = \rho) \]
i.e. \( Z_{s+it} \) takes values in the (extended) upper half plane.

In higher dimensions, the Corollary holds modulo coherent sheaves supported in codimension \( \geq 3 \), just as the ordinary \( H \)-degree and rank lead to the same conclusion modulo torsion sheaves supported in codimension \( \geq 2 \).

**Remark:** The “central charge” \( Z_{s+it} \) has the form:
\[ Z_{s+it}(E) = -d_{(s, t)}(E) + itr_s(E) = -\int_S e^{-(s+it)H} \text{ch}(E)H^{n-2} \]
which is a much more compact (and important) formulation.
Corollary 3.5: Each “slope” function:

$$\mu := \mu_{s+it} = \frac{d(s,t)}{tr_s} = -\frac{\text{Re}(Z_{s+it})}{\text{Im}(Z_{s+it})}$$

has the usual properties of a slope function on the objects of $$A_s$$. That is, given an exact sequence of objects of $$A_s$$:

$$0 \rightarrow K^* \rightarrow E^* \rightarrow Q^* \rightarrow 0$$

then $$\mu(K^*) < \mu(E^*) \iff \mu(E^*) < \mu(Q^*)$$ and $$\mu(K^*) = \mu(E^*) \iff \mu(E^*) = \mu(Q^*)$$

Also, when we make the usual:

**Definition:** $$E^*$$ is $$\mu$$-stable if $$\mu(K^*) < \mu(E^*)$$ whenever $$K^* \subset E^*$$ and the quotient has nonzero central charge (i.e. is not a torsion sheaf supported in codim $$\geq 3$$).

Then $$\text{Hom}(E^*, F^*) = 0$$ whenever $$E^*, F^*$$ are $$\mu$$-stable and $$\mu(E^*) > \mu(F^*)$$.

**Proof (of the Corollary):** Simple arithmetic.

**Example:** In dimension $$n \geq 2$$:

$$\mu_{s+it}(O_X[1]) = \frac{t^2 - s^2}{2st}$$ and $$\mu_{s+it}(L \otimes I_Z) = \frac{(1 - s)^2 - t^2 - \frac{2t}{H}}{2t(1 - s)}$$

where $$d = [Z] \cap H^{n-2}$$ is the (codimension two) degree of the subscheme $$Z \subset X$$. Thus $$\mu_{s+it}(L \otimes I_Z) > \mu_{s+it}(O_X[1])$$ if and only if:

$$t^2 + \left(s - \left(\frac{1}{2} - \frac{d}{H^n}\right)^2\right) < \left(\frac{1}{2} - \frac{d}{H^n}\right)^2$$ and $$t > 0$$.

This describes a nonempty subset (interior of a semicircle) of $$\mathbb{R}^2$$ if $$H^n > 2d$$.

**Proposition 3.6:** For all smooth projective varieties $$X$$ of dimension $$\geq 2$$ (and $$L$$)

(a) $$O_X[1]$$ is a $$\mu_{s+it}$$-stable object of $$A_s$$ for all $$s \geq 0$$ and $$t > 0$$.

(b) $$L$$ is a $$\mu_{s+it}$$-stable object of $$A_s$$ for all $$s < 1$$ and $$t > 0$$.

**Proof:** (a) Suppose $$0 \neq K^* \subset O_X[1]$$, and let $$E$$ be an $$H$$-stable torsion-free sheaf in the associated graded of $$Q$$, where $$Q[1]$$ is the quotient object. Recall that $$0 < \mu_H(E) \leq s$$. The Proposition follows once we show $$\mu_{s+it}(O_X[1]) < \mu_{s+it}(E[1])$$ for all $$E$$ with these properties. We compute:

$$\mu_{s+it}(E[1]) = \frac{-2ch_2(E)H^{n-2} + 2sc_1(E)H^{n-1} - (s^2 - t^2)r(E)H^n}{2t(-c_1(E)H^{n-1} + sr(E)H^n)}$$

and we conclude (using the computation of $$\mu_{s+it}(O_X[1])$$ above) that:

$$\mu_{s+it}(O_X[1]) > \mu_{s+it}(E[1]) \iff (s^2 + t^2)c_1(E)H^{n-1} > (2s)ch_2(E)H^{n-2}$$

But by the Bogomolov Inequality:

$$(2s)ch_2(E)H^{n-2} \leq s(c_1^2(E)H^{n-2})/r(E)$$

and by the Hodge Index Theorem and the inequality $$c_1(E) \cdot H^{n-1} \leq sr(E)H^n$$:

$$sc_1^2(E)H^{n-2}/r(E) \leq s^2c_1(E)H^{n-1}$$

The desired inequality follows from the fact that $$t > 0$$ and $$c_1(E) \cdot H^{n-1} > 0$$.

The proof of (b) proceeds similarly. Suppose $$0 \neq (K')^* \subset L$$ in $$A_s$$, and let $$E'$$ be an $$H$$-stable coherent sheaf in the Harder-Narasimhan filtration of $$K'$$. 
Then $s < \mu_H(E') < 1$, and we need to prove that $\mu_{s+it}(E') < \mu_{s+it}(L)$. This follows as in (a) from the Bogomolov Inequality and Hodge Index Theorem. □.

**Corollary 3.7:** (Special case of Kodaira vanishing):

$$H^{n-1}(X, K_X + L) = 0 \text{ for all } n > 1$$

**Proof:** Within the semicircle $\{(s, t) \mid t^2 + (s - \frac{1}{2})^2 < \frac{1}{4} \text{ and } t > 0\}$ the inequality $\mu_{s+it}(L) > \mu_{s+it}(\mathcal{O}_X[1])$ holds. But $L$ and $\mathcal{O}_X[1]$ are always $\mu_{s+it}$-stable, hence:

$$0 = \text{Hom}_A(L, \mathcal{O}_X[1]) \cong \text{Ext}^1_{\mathcal{O}_X}(L, \mathcal{O}_X) \cong H^{n-1}(X, K_X + L)^{\vee}$$

□

**Remark:** The Bogomolov Inequality and Hodge Index Theorem are trivially true in dimension one. However, the computation of $\mu_{s+it}(L)$ is different in dimension one, and indeed in that case the inequality $\mu_{s+it}(L) > \mu_{s+it}(\mathcal{O}_X[1])$ never holds (which is good, since the corollary is false in dimension one)!

Restrict attention to $X = S$ a surface for the rest of this section, and consider:

$$\mathcal{I}_W'[1]$$

the shifted derived dual of the ideal sheaf of a subscheme $W \subset S$ of length $d$. Since:

$$H^{-1}(\mathcal{I}_W'[1]) = \mathcal{O}_S \text{ and } H^0(\mathcal{I}_W'[1])$$

it follows that $\mathcal{I}_W'[1]$ is in $A_s$ for all $s \geq 0$.

Every quotient object $\mathcal{I}_W'[1] \to Q^\bullet$ satisfies:

- $H^0(Q^\bullet)$ is supported in codimension two (on the scheme $W$, in fact).
- Let $Q = H^{-1}(Q^\bullet)$ (a torsion-free sheaf). Then every $H$-stable term $E$ in the Harder-Narasimhan filtration of $Q$ satisfies:

$$0 \leq r_s(E[1]) = -c_1 H + rsH^2 < r_s(\mathcal{I}_W'[1]) = sH^2 \iff (r-1)sH^2 < c_1 H \leq rsH^2$$

where $r = r(E)$ and $c_1 = c_1(E)$ (because the $r_s$ rank of the kernel object is positive).

**Proposition 3.8:** For subschemes $Z, W \subset S$ of the same length $d$ (and $H = c_1(L)$):

(a) If $H^2 > 8d$, then $\mu_{s+it}(L \otimes \mathcal{I}_Z) > \mu_{s+it}(\mathcal{I}_W'[1])$ for all $(s, t)$ in the semicircle:

$$C(d, H^2) := \left\{(s, t) \mid t^2 + \left(s - \frac{1}{2}\right)^2 < \frac{1}{4} - \frac{2d}{H^2} \text{ and } t > 0\right\}$$

centered at the point $(1/2, 0)$ (and the semicircle is nonempty!).

(b) If $H^2 > 8d$ and $\mathcal{I}_W'[1]$ or $L \otimes \mathcal{I}_Z$ is not stable at $(s, t) = (\frac{1}{2}, \sqrt{\frac{1}{4} - \frac{2d}{H^2}})$, then there is a divisor $D$ on $S$ and an integer $r > 0$ such that:

$$\frac{r - 1}{2} H^2 < D \cdot H \leq \frac{r}{2} H^2, \text{ and } \frac{D}{r} \cdot H < \frac{D^2}{r^2} + 2d$$

**Proof:** Part (a) is immediate from:

$$\mu_{s+it}(L \otimes \mathcal{I}_Z) = \frac{(1 - s)^2 - t^2 - \frac{2d}{H^2}}{2(1 - s)t} \text{ and } \mu_{s+it}(\mathcal{I}_W'[1]) = \frac{t^2 - s^2 + \frac{2d}{H^2}}{2st}$$

We prove part (b) for $\mathcal{I}_W'[1]$ (the proof for $L \otimes \mathcal{I}_Z$ is analogous).
Let $\mathcal{I}_W^\vee[1] \to Q^*$ be a surjective map in the category $\mathcal{A}_s$ and let $Q = H^{-1}(Q^*)$. Since $H^0(Q^*)$ is torsion, supported on $W$, it follows that $\mu_{s+t+i}(Q[1]) \leq \mu_{s+t+i}(Q^*)$ with equality if and only if $H^0(Q^*) = 0$.

Thus if $\mathcal{I}_W^\vee[1]$ is not $\mu_{s+t+i}$-stable, then $\mu_{s+t+i}(\mathcal{I}_W^\vee[1]) \geq \mu_{s+t+i}(Q[1])$ for some torsion-free sheaf $Q$ satisfying $(r - 1)sH^2 < c_1(Q) \cdot H \leq rsH^2$, and moreover, the same set of inequalities hold for (at least) one of the stable torsion-free sheaves $E$ appearing in the Harder-Narasimhan filtration of $Q$. We let $D = c_1(E)$ and $r = \text{rk}(E)$. Then $\mu_{s+t+i}(\mathcal{I}_W^\vee[1]) \geq \mu_{s+t+i}(E[1])$ if and only if:

$$(i^2 + s^2)(D \cdot H) \leq (2s)\text{ch}_2(E) + \frac{2d}{H^2}(rsH^2 - D \cdot H)$$

and by the Bogomolov inequality, $(2s)\text{ch}_2(E) \leq s\frac{D^2}{H^2}$. Setting $(s, t) = \left(\frac{1}{2}, \sqrt{\frac{1}{4} - \frac{2d}{H^2}}\right)$, we obtain the desired inequalities.

**Corollary 3.9:** (a) If $L = \mathcal{O}_S(H)$ is ample on $S$ and satisfies $H^2 > (2d + 1)^2$ and:

$$H^1(S, K_S \otimes L \otimes \mathcal{I}_W \otimes \mathcal{I}_W) \neq 0$$

for a pair $Z, W \subset S$ of length $d$ subschemes then there is a divisor $D$ on $S$ satisfying $D^2 \leq 0$ and $0 < D \cdot H \leq D^2 + 2d$.

(b) (Fujita-type result) If $L$ is an arbitrary ample line bundle on $S$, then

$$H^1(S, K_S \otimes L \otimes \mathcal{I}_W \otimes \mathcal{I}_W) = 0$$

for all subschemes $Z, W \subset S$ of length $d$ (or less).

**Proof:** Part (b) immediately follows from (a). Since:

$$H^1(S, K_S \otimes L \otimes \mathcal{I}_W \otimes \mathcal{I}_W) \cong \text{Hom}_{\mathcal{A}_s}(L \otimes \mathcal{I}_W, \mathcal{I}_W^\vee[1])^\vee$$

and $(2d + 1)^2 \geq 8d + 1$ for all $d \geq 1$, the non-vanishing of $H^1$ implies that either $L \otimes \mathcal{I}_W$ or $\mathcal{I}_W^\vee[1]$ must not be stable at $\left(\frac{1}{2}, \sqrt{\frac{1}{4} - \frac{2d}{H^2}}\right)$, and from Proposition 3.8 (b) there is a divisor $D$ and integer $r \geq 1$ such that the $\mathbb{Q}$-divisor $C = D/r$ satisfies:

$$(1 - \frac{1}{r}) \frac{H^2}{2} < C \cdot H \leq \frac{H^2}{2} \quad \text{and} \quad C \cdot H \leq C^2 + 2d$$

(similar to Application 2.2). The result now follows as in Application 2.2 once we prove that $C^2 \geq 1$ whenever $r > 1$. To this end, note:

(i) $r \geq 3 \Rightarrow C^2 + 2d \geq C \cdot H > \frac{H^2}{3} > \frac{2d + 1}{3} \Rightarrow C^2 > \frac{2d + 1}{3} \geq 1$.

(ii) $r = 2 \Rightarrow C^2 + 2d \geq C \cdot H > \frac{H^2}{4} > 2d + \frac{1}{4} \Rightarrow \frac{H^2}{2} \geq 2d + \frac{1}{4}, C \cdot H \geq 2d + 1$, and $C^2 \geq 1$, since $C$ is of the form $D/2$ for an “honest” divisor $D$.

Thus either $C^2 \leq 0$, in which case $r = 1$ and $C = D$ is an “honest” divisor, or else $C^2 \geq 1$. Furthermore, by the Hodge index theorem:

$$C^2H^2 \leq (C \cdot H)^2 \leq \frac{H^2}{2} (C^2 + 2d) \Rightarrow C^2 \leq 2d$$

and if $C^2 = \kappa$ for $1 \leq \kappa \leq 2d$, then $\kappa^2H^2 \leq (C \cdot H)^2 \leq (\kappa + 2d)^2 \Rightarrow H^2 \leq (1 + \frac{2d}{\kappa})^2$.

This is a decreasing function, giving us $H^2 \leq (2d + 1)^2$, contradicting $H^2 > (2d + 1)^2$.

---

1The authors thank Valery Alexeev for pointing out the embarrassing omission of this step in the original version of the paper.
Remark: This variation resembles other variations of Reider’s theorem, e.g. [Lan99], though the authors do not see how to directly obtain this result from the others.

In a special case, Proposition 3.8 can be made even stronger, as noted in [AB].

**Proposition 3.10:** If Pic($S$) = $\mathbb{Z}$, generated by $c_1(L) = H$, then the two objects $L \otimes I_Z$ and $I_W \otimes [1]$ are $\mu_{(\frac{1}{2}, \mu)}$-stable for all $t > 0$ and any degree of $Z$ (and $W$).

**Proof:** Again we do this for $I_W \otimes [1]$, the proof for $L \otimes I_Z$ being analogous. Consider again the condition on every subbundle $E \subset Q$, where $Q = H^{-1}(Q^*)$, and $Q^*$ is a quotient object of $I_W \otimes [1]$:

$$ (r(E) - 1)(\frac{1}{2})H^2 < c_1(E) \cdot H \leq r(E)(\frac{1}{2})H^2 $$

Since $c_1(E) = kH$ is an integer multiple of $H$, by assumption, it follows immediately that $Q$ is itself of even rank and $H$-stable, satisfying $c_1(Q) = (r(Q)/2)H$. But in that case, $Q[1]$ has “Bridgeland rank” $r_+^b(Q[1]) = 0$, hence has maximal phase (infinite slope), and thus cannot destabilize $I_W \otimes [1]$.

**Remark:** This argument is highly sensitive to setting $s = \frac{1}{2}$, and indeed the conclusion is not true when $s \neq \frac{1}{2}$.

**Corollary 3.11:** If Pic($S$) = $\mathbb{Z}H$ and $H^2 > 8d$, then $H^1(S, K_S \otimes L \otimes I_Z \otimes I_W) = 0$ for pairs of subschemes $Z, W \subset S$ of length $d$.

### 4. Thaddeus Pairs Revisited.

Let $S$ be a surface with ample line bundle $L$ and Pic($S$) = $\mathbb{Z}H$ with $H = c_1(L)$. Consider the objects of $\mathcal{A}_s$ ($0 < s < 1$) appearing as extensions:

$$ \epsilon : 0 \to \mathcal{O}_S[1] \to E^*_\epsilon \to L \to 0 $$

parametrized by:

$$ \epsilon \in \text{Ext}^1_{\mathcal{A}_s}(L, \mathcal{O}_S[1]) = \text{Ext}^2_{\mathcal{O}_S}(L, \mathcal{O}_S) \cong H^0(S, K_S \otimes L)^\vee $$

As we saw in Proposition 3.6 and the preceding calculation, $\mathcal{O}_S[1]$ and $L$ are both $\mu_{s+it}$ stable for all $(s, t)$. Moreover, $\mu_{s+it}(\mathcal{O}_S[1]) < \mu_{s+it}(L)$ inside the semicircle:

$$ C := \left\{ (s, t) \mid t^2 + \left( s - \frac{1}{2} \right)^2 < \frac{1}{4} \text{ and } t > 0 \right\} $$

**Remark:** Here and earlier, we are using the notion of stability a little bit loosely. The correct definition, given by Bridgeland [Bri08] requires the existence of finite-length Harder-Narasimhan filtrations for all objects of $\mathcal{A}_s$. This is straightforward to prove when $(s, t)$ are both rational numbers (following Bridgeland), but much more subtle in the irrational case. For the purposes of this paper, the rational values will suffice.

We investigate the dependence of the $\mu_{\frac{1}{2}+it}$-stability of $E^*_\epsilon$ upon the extension class $\epsilon$ for $\frac{1}{2} + it$ inside the semicircle $S$. If $E^*_\epsilon$ is $\mu_{\frac{1}{2}+it}$-unstable, destabilized by $K^* \subset E^*_\epsilon$, then:

(i) $K^* = H^0(K^*) =: K$ is a coherent sheaf with $\mu_{\frac{1}{2}+it}(K) > 0$.

(ii) $K$ is $H$-stable of odd rank $r$ and $c_1(K) = ((r + 1)/2)H$.

(iii) The induced map $K \to L$ is injective (in the category $\mathcal{A}_{\frac{1}{2}}$).
Thus as in the curve case, $E^*_\subset$ can only be destabilized by lifting subobjects $K \subset L$ (in the category $A^\perp_2$) of positive $\mu_{\frac{r}{t}+i\ell}$-slope to subobjects of $E^*_\subset$:

$$K$$

(i) \quad 0 \to O_S[1] \to E^*_\subset \to L \to 0

That is, the unstable objects $E^*_\subset$ correspond to extensions in the kernel of the map:

$$\text{Ext}^2(L, O_S) \to \text{Ext}^2(K, O_S)$$

for some mapping of coherent sheaves $K \to L$ with $K$ satisfying (i) and (ii).

**Proof** (of (i)-(iii)): The $d(\frac{1}{2} t, i)$-degree of $E^*_\subset$ is:

$$\text{ch}_2(E^*_\subset) - \frac{c_1 H}{2} + \frac{(1 - t^2)}{2} r H^2 = 0$$

since $\text{ch}_2(E^*_\subset) = H^2/2, c_1 = H$ and $r = 0$. Thus the slope (equivalently, the degree) of any destabilizing $K^* \subset E^*_\subset$ is positive, by definition. Moreover the “ranks”

$$r_{\frac{1}{2}}(O_S[1]) = r_{\frac{1}{2}}(L) = \frac{H^2}{2}$$

are the minimal possible (as in the curve case) without being zero, hence as in the curve case, $K^* \subset E^*_\subset$ must also have minimal rank $\frac{H^2}{r}$ (if it had the next smallest rank $H^2 = r_{\frac{1}{2}}(E^*_\subset)$, it would fail to destabilize). The presentation of $E^*_\subset$ gives $H^{-1}(E^*_\subset) = O_S$ and $H^0(E^*_\subset) = L$, hence if we let $Q^* = E^*_\subset / K^*$, then:

$$0 \to H^{-1}(K^*) \to O_S \to H^{-1}(Q^*) \to K \to L \to H^0(Q^*) \to 0$$

and, as usual, either $H^{-1}(K^*) = 0$ or $H^{-1}(K^*) = O_S$. The latter is impossible, since in that case, the rank consideration would give $K^* = O_S[1], which doesn’t destabilize for $(\frac{1}{2}, t) \in C$. Thus $K^* = K$ is a coherent sheaf. This gives (i).

Next, the condition that $r_{\frac{1}{2}}(K)$ be minimal implies that there can only be one term in the Harder-Narasimhan filtration of $K$ (i.e. $K$ is $H$-stable), and that:

$$r_{\frac{1}{2}}(K) = c_1(K)H - \frac{r(K)H^2}{2} = \frac{H^2}{2}.$$  

Since $c_1(K) = kH$ for some $k$, this gives (ii).

Finally, (iii) follows again from the minimal rank condition since any kernel of the induced map to $L$ would be a torsion-free sheaf, of positive $r_{\frac{1}{2}}$-rank.  

Suppose now that $K$ satisfies (i) and (ii). By the Bogomolov inequality:

$$d(\frac{1}{2} t, i)(K) \leq \frac{1}{2t} \left( c_1(K) - \frac{r}{2} H \right)^2 - \frac{r t^2 H^2}{2} = \frac{H^2}{2t} \left( \frac{1}{4} - r^2 t^2 \right)$$

so in particular, $t \leq \frac{1}{2r}$, or in other words, we have shown:

**Proposition 4.1:** If $t > \frac{1}{2r}$ and $\mu_{\frac{1}{2}+i\ell}(K) < 0$ for all $K \subset L$ (in $A^\perp_2$) of odd ordinary rank $\leq r$, then $E^*_\subset$ is $\mu_{\frac{1}{2}+i\ell}$-stable.
**Special Case:** Suppose $t > \frac{1}{6}$. Because $H = c_1(L)$ generates $\text{Pic}(S)$ it follows that the only rank one subobjects $K \subset L$ in $\mathcal{A}_3$ are the subsheaves $L \otimes \mathcal{I}_Z$ for $Z \subset S$ of finite length. Thus $E^*_t$ only fails to be $\mu_{\frac{1}{2}+it}$-stable if:

$$d(\mu_{\frac{1}{2}+it})(L \otimes \mathcal{I}_Z) = \frac{1}{2} \left( \frac{1}{4} - t^2 \right) H^2 - d \geq 0 \Longleftrightarrow t^2 \leq \frac{1}{4} - \frac{2d}{H^2}$$

and $\epsilon \in \ker(\text{Ext}^2(L, \mathcal{O}_S) \rightarrow \text{Ext}^2(L \otimes \mathcal{I}_Z, \mathcal{O}_S))$, so that $L \otimes \mathcal{I}_Z \subset L$ lifts to a subobject of $E^*_t$. As in the curve case, Serre duality implies that the image of such a (non-zero) extension in the projective space:

$$\mathbb{P}(H^0(S, K \otimes L)^\vee)$$

is a point of the secant $d-1$-plane spanned by $Z \subset S$ under the linear series map:

$$\phi_{K+L} : S - \rightarrow \mathbb{P}(H^0(S, K \otimes L)^\vee).$$

By Corollary 3.11, this inequality on $t$ guarantees $H^1(S, K_S \otimes L \otimes \mathcal{I}_W \otimes \mathcal{I}_Z)^\vee = 0$ for all subschemes $Z, W \subset S$ of length $d$, hence in particular, the $d-1$-secant planes spanned by $Z \subset S$ are well-defined.

Thus there are “critical points” or “walls” at $t = \sqrt{\frac{1}{4} - \frac{2d}{H^2}} > \frac{1}{6}$, i.e. $d < \frac{2H^2}{9}$ on the line $s = \frac{1}{2}$ where the objects $E^*_t$ corresponding to points of the secant variety:

$$\left( \text{Sec}^{d-1}(S) - \text{Sec}^{d-2}(S) \right) \subset \mathbb{P}(H^0(S, K \otimes L)^\vee)$$

change from $\mu$-stable to $\mu$-unstable as $t$ crosses the wall.

**Moduli.** The Chern class invariants of each $E^*_t$ are:

$$c_2 = \frac{H^2}{2}, \quad c_1 = H, \quad r = 0$$

Thus it is natural to ask for the set of all $\mu_{\frac{1}{2}+it}$-stable objects with these invariants, and further to ask whether they have (projective) moduli that are closely related (by flips or flops) as $t$ crosses over a critical point. In one case, this is clear:

**Proposition 4.2:** For $t > \frac{1}{2}$, the $\mu_{\frac{1}{2}+it}$-stable objects with Chern class invariants above are precisely the (Simpson)-stable coherent sheaves with these invariants, i.e. sheaves of pure dimension one and rank one on curves in the linear series $|H|$.

**Proof:** Suppose $E^*$ has the given invariants and is not a coherent sheaf. Then the sequence:

$$0 \rightarrow H^{-1}(E^*)[1] \rightarrow E^* \rightarrow H^0(E^*) \rightarrow 0$$

destabilizes $E^*$ for $t > \frac{1}{2}$ for the following reason. Let $E = H^{-1}(E^*)$. If $c_1(E) = kH$, then $k \leq \frac{r}{2}$ is required in order that $E[1] \in \mathcal{A}_{k+1}$. Moreover, since $r_{\frac{r}{2}}(E^*) = H^2$ and $H^0(E^*)$ has positive (ordinary) rank, hence also positive $r_{\frac{r}{2}}$-rank, it follows that $r_{\frac{r}{2}}(E[1]) = 0$ or $H^2$. But $r_{\frac{r}{2}}(E[1]) = 0$ implies $E[1]$ has maximal (infinite) slope, and then $E^*$ is unstable (for all $t$). It follows similarly that if $r_{\frac{r}{2}}(E[1]) = \frac{H^2}{2}$, then $E^*$ is unstable for all $t$ unless $E$ is $H$-stable, of rank $r = 2k+1$. In that case, by the Bogomolov inequality:

$$d(\mu_{\frac{1}{2}+it})(E[1]) = -c_2(E) + c_1(E)H - \frac{1}{2} - t^2 \right) rH^2 = \frac{H^2}{2} \left( t^2 r - \frac{1}{4r} \right)$$

and this is positive if $t > \frac{1}{3}$. \(\square\)
In fact, at \( t = \frac{1}{2} \) only \( H^{-1}(E^\bullet) = \mathcal{O}_S \) (the rank one case, moreover matching the Bogomolov bound) would fail to destabilize a non-sheaf \( E^\bullet \), and conversely, among the coherent sheaves \( T \) with these invariants, only those fitting into an exact sequence (of objects of \( \mathcal{A}_{\frac{1}{2}} \)):

\[
0 \to L \to T \to \mathcal{O}_S[1] \to 0
\]

become unstable as \( t \) crosses \( \frac{1}{2} \), and they are replaced by the “Thaddeus” extensions:

\[
0 \to \mathcal{O}_S[1] \to E^\bullet \to L \to 0
\]

In other words, the moduli of Simpson-stable coherent sheaves

\[
M_S \left( 0, H, \frac{H^2}{2} \right)
\]

is known to be projective by a geometric invariant theory construction \cite{Sim94}. It is the moduli of \( \mu_{\frac{1}{2} + \nu t} \)-stable objects of \( \mathcal{A}_{\frac{1}{2}} \) for \( t > \frac{1}{2} \). The wall crossing at \( t = \frac{1}{2} \) removes:

\[
P(\text{Ext}^1(\mathcal{O}_S[1], L)) = P(H^0(S, L)) \subseteq M_S \left( 0, H, \frac{H^2}{2} \right)
\]

and replaces it with \( P(H^0(S, K_S + L)) \) in another birational model. In the case \( K_S = 0 \), the Simpson moduli spaces are holomorphic symplectic varieties, this new birational model is a Mukai flop of the moduli of stable sheaves, and the further wall crossings (up to \( t = \frac{1}{6} \), when rank three bundles appear) all replace extensions of the form:

\[
0 \to L \otimes I_Z \to (T \text{ or } E^\bullet) \to \mathcal{I}_W[1] \to 0
\]

with

\[
0 \to \mathcal{I}_W[1] \to E^\bullet \to L \otimes I_Z \to 0.
\]

This is achieved globally by Mukai flops, replacing projective bundles over the product \( \text{Hilb}^d(S) \times \text{Hilb}^d(S) \) of Hilbert schemes with their dual bundles:

\[
P(H^0(S, L \otimes \mathcal{I}_W \otimes I_Z)) \leftrightarrow P(H^0(S, L \otimes \mathcal{I}_W \otimes I_Z)^\vee)
\]

This was constructed in detail in \cite{AB}.

General questions regarding moduli of Bridgeland-stable objects remain fairly wide open, however. Toda \cite{Tod08} has shown that when \( S \) is a \( K3 \) surface, then the Bridgeland semistable objects of fixed numerical class are represented by an Artin stack of finite type. One expects the isomorphism classes of Bridgeland-stable objects, at least in special cases as above, to be represented by a proper scheme when \( (s, t) \) is not on a “wall.” However:

**Question 1:** When are the isomorphism classes of Bridgeland-stable objects of fixed numerical type represented by a (quasi)-projective scheme of finite type?

**Question 2:** Conversely, is there an example where the isomorphism classes are represented by a proper algebraic space which is not a projective scheme? (The examples produced in \cite{AB} are proper algebraic spaces. It is unknown whether they are projective.)

For each \( t < \frac{1}{2} \), we make the following provisional:
**Definition:** The space of $t$-stable Thaddeus pairs (given $S$ and ample $L$) is the proper transform of the projective space of extensions $\mathbb{P}(\text{Ext}^1(L, O_S[1]))$ under the natural rational embedding in the moduli space of (isomorphism classes of) $\mu_{2+it}$-stable objects with invariants $(0, H, H^2/2)$.

**Remark:** Note that for $t < \frac{1}{6}$, this will contain objects that have no analogue in the curve case, corresponding to destabilizing Mumford-stable torsion-free sheaves $K$ of higher odd rank $r$ and first chern class $c_1(K) = \frac{r+1}{2}H$.

**Question 3.** Can stable Thaddeus pairs, as a function of $t$ (inside the moduli of $\mu_{2+it}$-stable objects of the same numerical class) be defined as a moduli problem? If so, what are its properties? Is it projective? Smooth? What happens as $t \downarrow 0$?

5. **Reider in Dimension Three?** Let $X$ be a smooth projective three-fold, with $\text{Pic}(X) = \mathbb{Z} \cdot H$ (for simplicity) and $H = c_1(L)$ for an ample line bundle $L$. Consider:

$$
\begin{align*}
\epsilon : 0 \to O_X[1] \to E^* \to L \otimes I_Z &\to 0 \\
\end{align*}
$$

for subschemes $Z \subset X$ of finite length $d$, taken within the tilted category $\mathcal{A}_2$.

**Question 4:** Are there bounds $d_0$ and $t_0$ such that all objects $E^*$ formed in this way are $\mu_{2+it}$-unstable when $d > d_0$ and $t < t_0$? If so, does this follow from a more general Bogomolov-type codimension three inequality for the numerical invariants of $\mu$-stable objects?

As we have already discussed in the surface case, a destabilizing subobject of an unstable such $E^*$ would be exhibited by lifting:

$$
\begin{align*}
K \uparrow \downarrow \text{ (†) } 0 \to O_S[1] \to E^* \to L \otimes I_Z &\to 0 \\
\end{align*}
$$

where $K \subset L \otimes I_Z$ is a subobject in $\mathcal{A}_2$. By requiring $\text{Pic}(X) = \mathbb{Z}$, we assure that $K$ does not factor through $L(-D)$ for any effective divisor $D$. The interesting cases are therefore:

- $K = L \otimes I_C$ where $C \subset X$ is a curve, and a new-looking condition:
- $K$ is an $H$-stable torsion-free sheaf of odd rank $r$ and $c_1(K) = \frac{r+1}{2}H$.

**Question 5:** Assuming Question 4, are there examples of threefolds where the bounds of Question 4 are satisfied (hence all $E^*$ are $\mu$-unstable), but the “interesting cases” above allow for non-zero extensions?

And the last question is whether all the interesting cases can be numerically eliminated (by some form of the Hodge Index Theorem) leading to a proof of Fujita’s conjecture for threefolds.

**References**

[AB] D. Arcara, A. Bertram, Bridgeland-Stable Moduli Spaces for K-Trivial Surfaces, [arXiv:0708.2247v1 [math.AG]].

[Bri08] T. Bridgeland, Stability Conditions on K3 Surfaces, *Duke Math. J.* 141 (2008), 241–291.

[HRS96] D. Happel, I. Reiten, S. Smalø, Tilting in Abelian Categories and Quasitilted Algebras, *Mem. of the Am. Math. Soc.* (1996), vol. 120, no. 575.

[Lan99] A. Langer, A note on k-jet ampleness on surfaces, *Contemp. Math.* 241 (1999), 273–282.
[Laz97] R. Lazarsfeld, *Lectures on linear series*, Complex Algebraic Geometry (Park City, UT, 1993), IAS/Park City Math. Series, vol. 3, Amer. Math. Soc., Providence, RI, 1997, 161–219.

[Rei88] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, *Ann. Math.* 127 (1988), 309–316.

[Sim94] C. Simpson, Moduli of representations of the fundamental group of a smooth projective variety, *Inst. Hautes Etudes Sci. Publ. Math.* 80 (1994), 5–79.

[Tha94] M. Thaddeus, Stable pairs, linear systems and the Verlinde formula, *Invent. Math.* 117 (1994), no. 2, 317–353.

[Tod08] Y. Toda, Moduli stacks and invariants of semistable objects on K3 surfaces, *Advances in Math.* 217 (2008), 273–278.

Department of Mathematics, Saint Vincent College, 300 Fraser Purchase Road, Latrobe, PA 15650-2690, USA

E-mail address: daniele.arcara@email.stvincent.edu

Department of Mathematics, University of Utah, 155 S. 1400 E., Room 233, Salt Lake City, UT 84112-0090, USA

E-mail address: bertram@math.utah.edu