A generalization of the Tutte polynomials

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Abstract

In this paper, we introduce the concept of the Tutte polynomials of genus \( g \) and discuss some of its properties. We note that the Tutte polynomials of genus one are well-known Tutte polynomials. The Tutte polynomials are matroid invariants, and we claim that the Tutte polynomials of genus \( g \) are also matroid invariants. The main result of this paper and the forthcoming paper declares that the Tutte polynomials of genus \( g \) are complete matroid invariants.

Key Words and Phrases. matroid, Tutte polynomial.

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1 Introduction

This is an announcement paper.

Let \( E \) be a set. A matroid \( M \) on \( E = E(M) \) is a pair \((E, \mathcal{I})\), where \( \mathcal{I} \) is a non-empty family of subsets of \( E \) with the properties

\[
\begin{align*}
(i) & \quad \text{if } I \in \mathcal{I} \text{ and } J \subset I, \text{ then } J \in \mathcal{I}; \\
(ii) & \quad \text{if } I_1, I_2 \in \mathcal{I} \text{ and } |I_1| < |I_2|, \\
& \quad \text{then there exists } e \in I_2 \setminus I_1 \\
& \quad \text{such that } I_1 \cup \{e\} \in \mathcal{I}.
\end{align*}
\]

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Each element of the set $I$ is called an independent set. A matroid $(E, \mathcal{I})$ is isomorphic to another matroid $(E', \mathcal{I}')$ if there is a bijection $\varphi$ from $E$ to $E'$ such that $\varphi(I) \in \mathcal{I}'$ holds for each member $I \in \mathcal{I}$, and $\varphi^{-1}(I') \in \mathcal{I}$ holds for each member $I' \in \mathcal{I}'$.

It follows from the second axiom that all maximal independent sets in a matroid $M$ take the same cardinality, called the rank of $M$. These maximal independent sets $\mathcal{B}(M)$ are called the bases of $M$. The rank $\rho(A)$ of an arbitrary subset $A$ of $E$ is the cardinality of the largest independent set contained in $A$.

We provide examples below.

**Example 1.1.** Let $A$ be a $k \times n$ matrix over a finite field $\mathbb{F}_q$. This offers a matroid $M$ on the set

$$E = \{ z \in \mathbb{Z} | 1 \leq z \leq n \}$$

in which a set $I$ is independent if and only if the family of columns of $A$ whose indices belong to $I$ is linearly independent. Such a matroid is called a vector matroid.

**Example 1.2.** Let

$$E = \{ z \in \mathbb{Z} | 1 \leq z \leq n \}$$

and $r$ a natural number. We define a matroid on $E$ by taking every $r$-element subset of $E$ to be a basis. This is known as the uniform matroid $U_{r,n}$.

The classification of the matroids is one of the most important problems in the theory of matroids. One tool to classify the matroids is the Tutte polynomials. Let $M$ be a matroid on the set $E$ having a rank function $\rho$. The Tutte polynomial of $M$ is defined as follows \cite{2, 3, 4}:

$$T(M) := T(M; x, y) := \sum_{A \subseteq E} (x - 1)^{\rho(E) - \rho(A)} (y - 1)^{|A| - \rho(A)}.$$  

For example, the Tutte polynomial of the uniform matroid $U_{r,n}$ is

$$T(U_{r,n}; x, y) = \sum_{i=0}^{r} \binom{n}{i} (x - 1)^{r-i} + \sum_{i=r+1}^{n} \binom{n}{i} (y - 1)^{i-r}.$$  

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It is easy to demonstrate that \( T(M; x, y) \) is a matroid invariant. Two matroids are called \( T \)-equivalent if their Tutte polynomials are equivalent.

It is well known that there exist two inequivalent matroids, which are \( T \)-equivalent \({}^5{}\text{ p. 269}\). We provide more examples below. Let

\[
E := \{ z \in \mathbb{Z} \mid 1 \leq z \leq n \}.
\]

Let us define the subsets \( X_1, X_2, X_3 \) of \( E \) as follows:

\[
\begin{align*}
X_1 & := \{ z \in \mathbb{Z} \mid 1 \leq z \leq r \}; \\
X_2 & := \{ z \in \mathbb{Z} \mid r + 1 \leq z \leq 2r \}; \\
X_3 & := \{ z \in \mathbb{Z} \mid r \leq z \leq 2r - 1 \}.
\end{align*}
\]

Let \( R_{r,n} \) denote the matroid on \( E \) such that

\[
B(R_{r,n}) = B(U_{r,n}) \setminus \{X_1, X_2\},
\]

\( Q_{r,2n} \) denote the matroid on \( E \) such that

\[
B(Q_{r,n}) = B(U_{r,n}) \setminus \{X_1, X_3\}.
\]

Then, for \( 2r \leq n, r \geq 3 \), \( R_{r,n} \) and \( Q_{r,n} \) are matroids. Both matroids have exactly two dependent sets of size \( r \), namely \( \{X_1, X_2\} \) of \( R_{r,n} \) and \( \{X_1, X_3\} \) of \( Q_{r,n} \). Therefore, if \( R_{r,n} \) and \( Q_{r,n} \) are isomorphic, there exists \( \varphi \) such that

\[
\varphi(\{X_1, X_2\}) = \{X_1, X_3\}.
\]

This is a contradiction since \( \varphi \) is bijective, and we know that \( R_{r,n} \) and \( Q_{r,n} \) are non-isomorphic matroids.

On the other hand,

\[
T(R_{r,n}) = T(Q_{r,n}).
\]

In fact, the difference between

\[
T(U_{r,n}) - T(R_{r,n})
\]

and

\[
T(U_{r,n}) - T(Q_{r,n})
\]

are zero since \( R_{r,n} \) and \( Q_{r,n} \) are obtained from \( U_{r,n} \) after deleting the two maximal independent sets.
This gives rise to a natural question: is there a generalization of the Tutte polynomial which identifies such $T$-equivalent but inequivalent matroids? This paper aims to provide a candidate generalization that answers this. In Section 2, we provide the concept of the Tutte polynomial of genus $g$. In Section 3, we provide the main results. The details of the proofs will be presented in our forthcoming paper [1].

2 Tutte polynomials of genus $g$

We now present the concept of the Tutte polynomial of genus $g$.

\textbf{Definition 2.1.} Let $M := (E, I)$ be a matroid. Let

$$\Lambda_1 := \{z \in \mathbb{Z} \mid 1 \leq z \leq g\}$$

and let

$$\Lambda_2 := \binom{\Lambda_1}{2}.$$ 

For every element $\lambda \in \Lambda_2$, let us denote

$$A_{\cap(\lambda)} := \cap_{i \in \lambda} A_i \text{ and } A_{\cup(\lambda)} := \cup_{i \in \lambda} A_i.$$ 

Then, the genus $g$ of the Tutte polynomial $T^{(g)}(M)$ of the matroid $M$ will be defined as follows:

$$T^{(g)}(M) := T^{(g)}(M; x_{\lambda_1}, y_{\lambda_1}, x_{\cup\lambda_2}, y_{\cap\lambda_2},$$

$$x_{\cup\lambda_2}, y_{\cup\lambda_2} : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2)$$

$$:= \sum_{A_1, \ldots, A_n \subseteq E} \prod_{\lambda \in \Lambda_1} (x_{\lambda} - 1)^{\rho(M) - \rho(A_{\lambda})}$$

$$\prod_{\lambda \in \Lambda_2} (x_{\cap(\lambda)} - 1)^{\rho(M) - \rho(A_{\cap(\lambda)})}$$

$$\prod_{\lambda \in \Lambda_2} (y_{\cap(\lambda)} - 1)^{\rho(M) - \rho(A_{\cap(\lambda)})}$$

$$\prod_{\lambda \in \Lambda_2} (x_{\cup(\lambda)} - 1)^{\rho(M) - \rho(A_{\cup(\lambda)})}$$

$$\prod_{\lambda \in \Lambda_2} (y_{\cup(\lambda)} - 1)^{\rho(M) - \rho(A_{\cup(\lambda)})}.$$
It is easy to demonstrate that \( T(g)(M) \) is matroid invariant and if two matroids have the same Tutte polynomial for genus \( g \), we call them \( T(g) \)-equivalent. For example, the genus 2 for the Tutte polynomial \( T(2)(M) \) of the matroid \( M \) is as follows:

\[
T^{(2)}(M; x_1, x_2, y_1, y_2, x \cap \{12\}, y \cap \{12\}, x \cup \{12\}, y \cup \{12\}) = \sum_{A_1, A_2 \subseteq E} (x_1 - 1)^{\rho(E) - \rho(A_1)} (x_2 - 1)^{\rho(E) - \rho(A_2)} (y_1 - 1)^{|A_1| - \rho(A_1)} (y_2 - 1)^{|A_2| - \rho(A_2)} (x \cap \{1, 2\} - 1)^{\rho(E) - \rho(A_1 \cap A_2)} (y \cap \{1, 2\} - 1)^{|A_1 \cap A_2| - \rho(A_1 \cap A_2)} (x \cup \{1, 2\} - 1)^{\rho(E) - \rho(A_1 \cup A_2)} (y \cup \{1, 2\} - 1)^{|A_1 \cup A_2| - \rho(A_1 \cup A_2)}.
\]

We remark that for \( g \in \mathbb{N}_{\geq 2} \), the specialization of \( T^{(g)}(M) \) is \( T^{(g-1)}(M) \). For example,

\[
T^{(2)}(M; x_1, 2, y_1, 2, 2, 2, 2) = 2^{|E|} T^{(1)}(M; x_1, y_1).
\]

Therefore, if \( T^{(g)}(M) = T^{(g)}(M') \) then \( T^{(i)}(M) = T^{(i)}(M') \), for all \( 1 \leq i \leq g \).

### 3 Main results

The main result of the present paper is as follows:

**Theorem 3.1.** 1. The Tutte polynomial of genus \( g \) \( \{T^{(g)}\}_{g=1}^{\infty} \) is a complete invariant for matroids.

2. For \( 2r \leq n \), \( r \geq 3 \),

\[
T^{(2)}(R_{r,n}) \neq T^{(2)}(Q_{r,n}).
\]

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3. Let \[ E := \{ z \in \mathbb{Z} \mid 0 \leq z \leq 4n - 1 \} \]
with \( n \geq 3 \). Let us define the subsets \( Y_1, Y_2 \) of \( 2^E \) as follows:

\[
Y_1 := \{ \{0, 1, 2\}, \{2, 3, 4\}, \ldots, \\
\{2n - 4, 2n - 3, 2n - 2\}, \\
\{2n - 2, 2n - 1, 0\}, \\
\{2n, 2n + 1, 2n + 2\}, \\
\{2n + 2, 2n + 3, 2n + 4\}, \ldots, \\
\{4n - 4, 4n - 3, 4n - 2\}, \\
\{4n - 2, 4n - 1, 2n\} \},
\]

\[
Y_2 := \{ \{0, 1, 2\}, \{2, 3, 4\}, \ldots, \\
\{2n - 4, 2n - 3, 2n - 2\}, \\
\{2n - 2, 2n - 1, 2n\}, \\
\{2n, 2n + 1, 2n + 2\}, \\
\{2n + 2, 2n + 3, 2n + 4\}, \ldots, \\
\{4n - 4, 4n - 3, 4n - 2\}, \\
\{4n - 2, 4n - 1, 0\} \}.
\]

Let \( S_{4n} \) denote the independence system on \( E \) such that

\[
B(S_{4n}) = B(U_{3,4n}) \setminus Y_1,
\]

\( S'_{4n} \) denote the independence system on \( E \) such that

\[
B(S'_{4n}) = B(U_{3,4n}) \setminus Y_2.
\]

Then, \( S_{4n} \) and \( S'_{4n} \) are matroids. Let

\[
m_1 = \left\lfloor \frac{-1 + \sqrt{1 + 8n}}{2} \right\rfloor \text{ and } m_2 = 2\lceil \sqrt{n} \rceil.
\]

We have

\[
\begin{align*}
T^{(m_1)}(S_{4n}) &= T^{(m_1)}(S'_{4n}); \\
T^{(m_2)}(S_{4n}) &\neq T^{(m_2)}(S'_{4n}).
\end{align*}
\]

In particular, for a matroid \( M \), \( T^{(\mathcal{B}(M))}(M) \) determines \( M \). For detailed explanation, see [1].
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