Complete particle-pair annihilation as a dynamical signature of the spectral singularity

G. R. Li, X. Z. Zhang, and Z. Song
School of Physics, Nankai University, Tianjin 300071, China

Motivated by the physical relevance of a spectral singularity of interacting many-particle system, we explore the dynamics of two bosons as well as fermions in one-dimensional system with imaginary delta interaction strength. Based on the exact solution, it shows that the two-particle collision leads to amplitude-reduction of the wave function. For fermion pair, the amplitude-reduction depends on the spin configuration of two particles. In both cases, the residual amplitude can vanish when the relative group velocity of two single-particle Gaussian wave packets with equal width reaches the magnitude of the interaction strength, exhibiting complete particle-pair annihilation at the spectral singularity.

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I. INTRODUCTION

Non-Hermitian operator has been introduced phenomenologically as an effective Hamiltonian to fit experimental data in various fields of physics [1–5]. In spite of the important role played non-Hermitian operator in different branches of physics, it has not been paid due attention by the physics community until the discovery of non-Hermitian Hamiltonians with parity-time symmetry, which have a real spectrum [6]. It has boosted the research on the complex extension of quantum mechanics on a fundamental level [7–17]. Non-Hermitian Hamiltonian can possess peculiar feature that has no Hermitian counterpart. A typical one is the spectral singularity (or exceptional point for finite system), which is a mathematic concept. It has gained a lot of attention recently [18–26], motivated by the possible physical relevance of this concept since the pioneer work of Mostafazadeh [27].

The majority of previous works focus on the non-Hermitian system arising from the complex potential, mean-field nonlinearity [20, 28–37] as well as imaginary hopping integral [38]. In this paper, we investigate the physical relevance of the spectral singularities for non-Hermitian interacting many-particle system. The non-Hermiticity arises from the imaginary interaction strength. For two-particle case, the exact solution shows that there exist a series of spectral singularities, forming a spectrum of singularity associated with the central momentum of the two particles. We consider dynamics of two bosons as well as fermions in one-dimensional system with imaginary delta interaction strength. It shows that the two-particle collision leads to amplitude-reduction of the wave function. For fermion pair, the amplitude-reduction depends on the spin configuration of two particles. Remarkably, in both cases, the residual amplitude can vanish only when the relative group velocity of two single-particle Gaussian wave packets with equal width reaches the magnitude of the interaction strength. This phenomenon of complete particle-pair annihilation is the direct result of the spectral singularity. We also discuss the complete annihilations of a singlet fermion pair and a maximally two-mode entangled boson pair based on the second quantization formalism.

This paper is organized as follows. In Section II we present the model Hamiltonian and exact solution. In Section III we construct the local boson pair initial state as initial state which is allowed to calculate the time evolution. Based on this, we reveal the connection between the phenomenon of complete pair annihilation and the spectral singularity. In Section IV we extend our study a singlet fermion pair and a maximally two-mode entangled boson pair based on the second quantization formalism. Finally, we give a summary in Section V.

II. HAMILTONIAN AND SOLUTIONS

We start with an one-dimensional two-distinguishable particle system with imaginary delta interaction. The solution can be used to construct the eigenstates of two-fermion and boson systems. The Hamiltonian has the form

*Electronic address: songtc@nankai.edu.cn
\[ H_{2p} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - i2\gamma\delta(x_1 - x_2) \]  

(1)

where \( \gamma > 0 \) and we use dimensionless units \( \hbar = m = 1 \) for simplicity.

Introducing new variables \( R \) and \( r \), where

\[
R = \frac{(x_1 + x_2)}{2}, \quad r = x_1 - x_2,
\]

(2)

we obtain the following Hamiltonian

\[ H_{2p} = H_R + H_r, \]

(3)

with

\[
H_R = -\frac{\partial^2}{4\partial R^2}, \quad H_r = -\frac{\partial^2}{\partial r^2} - i2\gamma\delta (r).
\]

Here \( R \) is the center-of-mass coordinate and \( r \) is the relative coordinate. The Hamiltonian is separated into a center-of-mass part and a relative part, and can be solvable exactly.

The eigenfunctions of the center-of-mass motion \( H_R \) are simply plane waves, while the Hamiltonian \( H_r \) is equivalent to that of a single-particle in an imaginary delta-potential, which has been exactly solved in the Ref. [39]. Then the eigenfunctions of the original Hamiltonian can be obtained and expressed as

\[
\psi_+ (K, k, x_1, x_2) = e^{iK(x_1 + x_2)/2} \left\{ \cos [k(x_1 - x_2)] - \frac{i\gamma}{k} \sin [k(x_1 - x_2)] \text{sign} (x_1 - x_2) \right\},
\]

(5)

in symmetrical form, and

\[
\psi_- (K, k, x_1, x_2) = e^{iK(x_1 + x_2)/2} \sin [k(x_1 - x_2)],
\]

(6)

in antisymmetrical form. The corresponding energy is

\[ E(K, k) = K^2/4 + k^2, \]

(7)

with the central and relative momenta \( K, k \in (-\infty, \infty) \). The symmetrical wavefunction \( \psi (K, k, x_1, x_2) \) is the spatial part wavefunction for two bosons or two fermions in singlet pair, while the antisymmetrical wavefunction \( \varphi (K, k, x_1, x_2) \) only for two triplet fermions.

Before starting the investigation on dynamics of two-particle collision, we would like to point that there exist spectral singularities in the present Hamiltonian. It arises from the same mechanism as that in the single-particle systems [39, 40].

We can see that the eigen functions with even parity and momentum \( k = -\gamma \) can be expressed in the form

\[
\psi_{ss} (K) \equiv \psi(K, -\gamma, x_1, x_2) = e^{iK(x_1 + x_2)/2} e^{-i\gamma|x_1 - x_2|},
\]

(8)

with energy

\[ E_{ss} (K) = K^2/4 + \gamma^2. \]

(9)

We note that function \( \psi_{ss} (K) \) satisfies

\[
\lim_{x_1 - x_2 \to \pm \infty} \left[ \frac{\partial \psi_{ss} (K)}{\partial (x_1 - x_2)} \pm i\gamma \psi_{ss} (K) \right] = 0,
\]

(10)

which accords with the definition of the spectral singularity in Ref. [20]. It shows that there exist a series of spectral singularities associated with energy \( E_{ss} (K) \) for \( K \in (-\infty, \infty) \), which constructs a spectrum of spectral singularities. We will demonstrate in the following section that such a singularity spectrum leads to a peculiar dynamical behavior of two local boson pair or equivalently, singlet fermion pair.
III. DYNAMICAL SIGNATURE

A. Construction of initial state

The emergence of the spectral singularity induces a mathematical obstruction for the calculation of the time evolution of a given initial state, since it spoils the completeness of the eigen functions and prevents the eigenmode expansion. Nevertheless, the completeness of the eigen functions is not necessary for the time evolution of a state with a set of given coefficients of expansion. It does not cause any difficulty in deriving the time evolution of an initial state with arbitrary combination of the eigen functions. Namely, any linear combination of function set \{\psi (K,k,x_1,x_2)\} or \{\varphi (K,k,x_1,x_2)\} can be an initial state, and the time evolution of it can be obtained simply by adding the factor e^{-iE(K,k)t}.

In order to investigate the dynamical consequence of the singularity spectrum, we consider the time evolution of the initial state of the form

\[
\Psi (x_1,x_2,0) = \frac{1}{\sqrt{\Lambda}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(K)g(k)\psi (K,k,x_1,x_2) dK dk, \tag{11}
\]

where \(\Lambda\) is the normalization factor, which will be given in the following and

\[
G(K) = \exp \left[ -\frac{1}{2\alpha^2}(K-K_0)^2 - iKR_0 \right], \tag{12}
\]

\[
g(k) = \exp \left[ -\frac{1}{2\beta^2}(k-k_0)^2 - ikr_0 \right]. \tag{13}
\]

Here \(\alpha, \beta, r_0, k_0 > 0\) and \(K_0\) is arbitrary real number. We explicitly have

\[
\Psi (x_1,x_2,0) = \frac{\pi \alpha \beta}{2\sqrt{\Lambda k_0}} [(k_0 + \gamma) \exp \theta_+ + (k_0 - \gamma) \exp \theta_-] \tag{14}
\]

where

\[
\theta_\pm = -\frac{\alpha^2}{2} \left( \frac{x_1 + x_2}{2} - R_0 \right)^2 - \frac{\beta^2}{2} (|x_1 - x_2| + r_0)^2 \]
\[
+ i \left( K_0 \frac{x_1 + x_2}{2} \mp k_0 |x_1 - x_2| - k_0 r_0 + K_0 R_0 \right). \tag{15}
\]

Furthermore, from the identity

\[
\alpha^2 (x_1 + x_2 + A)^2 + 4\beta^2 (x_1 - x_2 + B)^2 = \alpha^2 \left( (x_1 + A)^2 + (x_2 + A)^2 - A^2 \right) \]
\[
+ 4\beta^2 \left( (x_1 + B)^2 + (x_2 - B)^2 - B^2 \right) + (\alpha^2 - 4\beta^2) x_1 x_2 \tag{16}
\]

we can see that the cross term \(x_1 x_2\) vanishes if we take \(\alpha = 2\beta\). The initial state can be written as a separable form

\[
\Psi (x_1,x_2,0) = \frac{\pi \beta^2}{\sqrt{\Lambda k_0}} \{(k_0 + \gamma) [\varphi_+(x_1) \varphi_-(x_2) u(x_2 - x_1) + \varphi_+(x_2) \varphi_-(x_1) u(x_1 - x_2)] \}
\]
\[
+ (k_0 - \gamma) \left[ \varphi_+(x_2) \varphi_-(x_1) u(x_2 - x_1) + \varphi_+(x_1) \varphi_-(x_2) u(x_1 - x_2) \right]\} , \tag{17}
\]

where \(u(x)\) is Heaviside step function and

\[
\varphi_{\pm}(x) = \exp \left[ -\beta^2 \left( x \mp \frac{r_0}{2} \right)^2 + i \frac{1}{2} (K_0 \pm 2k_0) x \right]. \tag{18}
\]

In this case, \(\Lambda\) can be obtained as

\[
\Lambda = \frac{4\pi^3 \beta^2 (k_0 - \gamma)^2}{k_0^2}. \tag{19}
\]
Under the condition $\beta r \gg 1$, we can see that the perfect pair annihilation in the case of (a) and imperfect pair annihilation in the cases of (b) and (c) when the initial center-of-mass coordinate $R_0 = 0$ and dropped an overall phase $k_0 r_0$. We note that functions $\varphi_+ (x)$ and $\varphi_- (x)$ represent Gaussian functions with centers at $r_0 / 2$ and $-r_0 / 2$, respectively. Obviously, the probability contributions of $\varphi_+ (x_2) \varphi_- (x_1) u (x_2 - x_1)$ and $\varphi_+ (x_1) \varphi_- (x_2) u (x_1 - x_2)$ are negligible under the condition $\beta r_0 \gg 1$. We then yield

$$\Psi (x_1, x_2, 0) \approx \beta \sqrt{\frac{1}{2\pi}} \left[ \varphi_+ (x_1) \varphi_- (x_2) + \varphi_+ (x_2) \varphi_- (x_1) \right],$$

which represents two-boson wavepacket state with the same width, group velocity $K_0 / 2 \pm k_0$, and location $\mp r_0 / 2$. Here the renormalization factor has been readily calculated by Gaussian integral. So far we have construct an expected initial state without using the biothogonal basis set. The dynamics of two separated boson wavepackets can be described by the time evolution as that in the conventional quantum mechanics.

**B. Annihilating collision**

It is presumable that before the bosons start to overlap they move as free particles with the center moving in their the group velocities $K_0 / 2 \pm k_0$ and the width spreading as function of time $\sqrt{(4\beta^2 t^2 + 1) / \beta^2}$. We concern the dynamic behavior after the collision. To this end, we calculate the time evolution of the given initial state, which can be expressed as

$$\Psi (x_1, x_2, t) = \frac{1}{\sqrt{\Omega}} \int_{-\infty}^{\infty} G (K) g (k) \psi (K, k, x_1, x_2) e^{-i(K^2/4+k^2)t} dK dk. \tag{20}$$

By the similar procedure as above, we find that the evolved wave function can always be written in the separated form

$$\Psi (x_1, x_2, t) = \Phi (R, t) \phi (r, t), \tag{21}$$

where

$$\Phi (R, t) = \sqrt{\frac{4\beta^2}{\pi (1 + 4\beta^2 t^2)}} \exp \left[ -\frac{2\beta^2 (R - K_0 / 2)^2}{1 + 4\beta^2 t^2} + i \frac{(16\beta^4 R^2 + 4RK_0 - tK_0^2)}{4 + 16\beta^2 t^2} \right], \tag{22}$$

and

$$\phi (r, t) = \frac{1}{\sqrt{\Omega}} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2\beta^2} (k - k_0)^2 - ikr_0 \right] \left[ \cos (kr) - \frac{i\gamma}{k} \sin (k \left| r \right|) \right] \exp (-ik^2 t) dk. \tag{23}$$
where the normalization factor
\[ \Omega = \frac{\pi^{3/2} \beta (k_0 - \gamma)^2}{k_0^2}. \]  

Straightforward algebra shows that
\[ \phi (r, t) = (k_0 + \gamma) \Theta_+ + (k_0 - \gamma) \Theta_- \]  

where
\[ \Theta_\pm = \frac{\sqrt{3} |k_0 - \gamma|^{-1}}{2\pi^{1/2} (1 + 2\beta^2 t)} \exp \left\{ -\beta^2 \left[ |r| |r + (r_0 - 2k_0 t)| \right] \frac{1}{2 (4\beta^4 t^2 + 1)} + i\Delta_\pm \right\}, \]
\[ \Delta_\pm = \frac{\beta^4 (|r| \pm r_0)^2 t - 2k_0^2 t \mp 2k_0 (|r| \pm r_0)}{2 (4\beta^4 t^2 + 1)}. \]

In the case of \( \beta^4 t^2 \gg 1, k_0 t \gg r_0 \) the probability distribution is
\[ |\phi (r, t)|^2 \approx \frac{\pi (k_0 + \gamma)^2}{4\Omega^2 k_0^2 t} \exp \left\{ -\left[ (r_0 + k_0 t)^2 \right] \frac{1}{4\beta^2 t^2} \right\} + \frac{\pi (k_0^2 - \gamma^2)}{2\Omega^2 k_0^2 t} e^{-k_0^2/\beta^2} \exp \left\{ -\frac{r^2}{4\beta^2 t^2} \right\}, \]
which leads the total probability under the case \( k_0/\beta \gg 1 \)
\[ \int_{-\infty}^{\infty} |\phi (r, t)|^2 \, dr \approx \frac{(k_0 + \gamma)^2}{(k_0 - \gamma)^2}. \]

We can see that, after the collision the residual probability becomes a constant and vanishes when \( k_0 = -\gamma \). It shows that when the relative group velocity of two single-particle Gaussian wave packets with equal width reaches the magnitude of the interaction strength, the dynamics exhibits complete particle-pair annihilation.

In order to demonstrate such dynamic behavior and verify our approximate result, the numerical method is employed to simulate the time evolution process for several typical situations. The profiles of \( |\varphi (r, t)|^2 \) are plotted in Fig. 1. We would like to point that the complete annihilation depends on the relative group velocity, which is the consequence of singularity spectrum. This enhances the probability of the pair annihilation for a cloud of bosons, which may provide an detection method of the spectral singularity in experiment.

**IV. SECOND QUANTIZATION REPRESENTATION**

In this section, we will investigate the two-particle collision process from another point of view and give a more extended example. By employing the second quantization representation, the initial state in Eq. (19) can be expressed as the form \( a_1^\dagger a_2^\dagger |0\rangle \), where \( a_i^\dagger \) (\( i = 1, 2 \)) is the creation operator for a boson in single-particle state with the wavefunction
\[ \varphi_+ (x) = (x| a_1^\dagger |0\rangle, \varphi_- (x) = (x| a_2^\dagger |0\rangle, \]
and \( |0\rangle \) denotes the vacuum state of the particle operator. Similarly, if we consider a fermion pair, the initial state in Eq. (19) can be written as
\[ \frac{1}{\sqrt{2}} \left( c_{1,\uparrow}^\dagger c_{2,\downarrow}^\dagger - c_{1,\downarrow}^\dagger c_{2,\uparrow}^\dagger \right) |0\rangle, \]
where \( c_{i,\sigma}^\dagger (i = 1, 2; \sigma = \uparrow, \downarrow) \) is the creation operator for a fermion in single-particle state with the wavefunction
\[ \varphi_+ (x) \chi_\sigma = (x| c_{1,\sigma}^\dagger |0\rangle, \varphi_- (x) \chi_\sigma = (x| c_{2,\sigma}^\dagger |0\rangle. \]
FIG. 2: (Color online) The profiles of $|\varphi(r, t)|^2$ of a maximally two-mode entangled boson pair are plotted for different values of $k_0$ and $\alpha = 2\beta$: (a) $\gamma = 10.0, k_0 = 10.0, \alpha = 2\beta = 1.0$; (b) $\gamma = 10.0, k_0 = 10.0, \alpha = 2\beta = 3.0$; (c) $\gamma = 4.0, k_0 = 10.0, \alpha = 2\beta = 1.0$.

One can see that the perfect pair annihilation in the case of (a) and imperfect pair annihilation in the cases of (b) and (c) when the width of the initial wavepackets becomes narrower, and the relative group velocity deviates from $\gamma$, respectively.

Here

$$\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

are the spin part of wavefunction. We see that the initial state in Eq. (31) is singlet pair with maximal entanglement.

In coordinate space, the above wavefunction has the form

$$\Psi(x_1, x_2, 0) = \frac{2i\pi \beta^4}{\sqrt{2\Xi k_0}} \{ (k_0 + \gamma) \left[ \left( \varphi_+^{(1)}(x_1) \varphi_-(x_2) - \varphi_+(x_1) \varphi_-^{(1)}(x_2) \right) u(x_2 - x_1) \\
+ (x_1 \leftrightarrow x_2) \right] \\
- (k_0 - \gamma) \left[ \left( \varphi_-^{(1)}(x_1) \varphi_+(x_2) - \varphi_-^{(1)}(x_2) \varphi_-(x_1) \right) u(x_2 - x_1) \\
+ (x_1 \leftrightarrow x_2) \right] \},$$

which can be reduced to

$$\Psi(x_1, x_2, 0) \approx \frac{2i\pi \beta^4 (k_0 - \gamma)}{\sqrt{2\Xi k_0}} \left[ \left( \varphi_+^{(1)}(x_1) \varphi_-(x_2) - \varphi_+(x_1) \varphi_-^{(1)}(x_2) \right) + (x_1 \leftrightarrow x_2) \right],$$

under the approximation $\beta r_0 \gg 1$. Here $\Xi$ is the normalized constant, and $\varphi_\pm^{(1)}(x) = (x \mp \frac{r_0}{2}) \varphi_\pm(x)$.

By the same procedure, at time $t$ the evolved wavefunction is

$$\phi(r, t) = -(k_0 + \gamma) \Theta_+ + (k_0 - \gamma) \Theta_-,$$

with

$$\Theta_\pm = \frac{\sqrt{\pi} i \beta^3 |r| \pm (r_0 - 2k_0 t)}{2k_0 (1 + 2i \beta^2 t)^{3/2}} \exp \left\{ -\frac{\beta^2 |r| \pm (r_0 - 2k_0 t)^2}{2 (4\beta^4 t^2 + 1)} + i \Delta_\pm \right\},$$

and

$$\Delta_\pm = \frac{\beta^4 (|r| \pm r_0)^2 t - 2k_0^2 t^2 \mp 2k_0 (|r| \pm r_0)}{2 (4\beta^4 t^2 + 1)}.$$
where the normalization factor
\[ \Omega' = \frac{\pi^{3/2} \beta^3 (k_0 - \gamma)^2}{4k_0^2}. \]

In the case of \( \beta^4 t^2 \gg 1, \ k_0 t \gg r_0 \) the probability distribution is

\[
|\phi(r, t)|^2 \approx \pi \frac{(k_0 + \gamma)^2}{16\Omega' k_0^3 t^3} (|r| + 2k_0 t)^2 \exp \left\{ -\frac{(|r| + 2k_0 t)^2}{4\beta^2 t^2} \right\} \]

which leads the total probability

\[
\int_{-\infty}^{\infty} |\phi(r, t)|^2 \, dr \approx \frac{(k_0 + \gamma)^2}{(k_0 - \gamma)^2}. \quad (41)
\]

The profiles of \( |\varphi(r, t)|^2 \) are plotted in Fig. 2. We can see that the same behavior occurs in the present situation.

In order to clarify the physical picture, we still employ the second quantization representation by introducing another type of boson creation operator \( b_i^\dagger (i = 1, 2) \) with

\[
\varphi_0(x) = \langle x | b_i^\dagger | 0 \rangle, \\
\varphi_0'(x) = \langle x | b_i^\dagger | 0 \rangle.
\]

Then the initial state in Eq. (36) can be expressed as

\[
\frac{1}{\sqrt{2}} \left( a_1^\dagger b_2^\dagger + a_2^\dagger b_1^\dagger \right) |0\rangle, \quad (42)
\]

which is maximally two-mode entangled state.

V. SUMMARY AND DISCUSSION

In summary we identified a connection between spectral singularities and dynamical behavior for interacting many-particle system. We explored the collision process of two bosons as well as fermions in one-dimensional system with imaginary delta interaction strength based on the exact solution. We have showed that there is a singularity spectrum which leads to complete particle-pair annihilation when the relative group velocity is resonant to the magnitude of interaction strength. The result for this simple model implies that the complete particle-pair annihilation can only occur for two distinguishable bosons, maximally two-mode entangled boson pair and singlet fermions, which may predict the existence of its counterpart in the theory of particle physics.

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