Approximately Strongly Regular Graphs
Ferdinand Ihringer
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Abstract
We give variants of the Krein bound and the absolute bound for graphs with a spectrum similar to that of a strongly regular graph. In particular, we investigate what we call approximately strongly regular graphs.

We apply our results to extremal problems. Among other things, we show the following:

(1) Caps in $\text{PG}(n, q)$ for which the number of secants on exterior points does not vary too much, have size at most $O(q^{\frac{3}{4}n})$ (as $q \to \infty$ or as $n \to \infty$).

(2) Optimally pseudorandom $K_m$-free graphs of order $v$ and degree $k$ for which the induced subgraph on the common neighborhood of a clique of size $i \leq m - 3$ is similar to a strongly regular graph, have $k = O(v^{\frac{1}{3} - \frac{1}{3m - 2i - 5}})$.

1 Introduction
We investigate graphs and families of graphs which asymptotically behave like strongly regular graphs (SRGs). In particular, we generalize existence conditions. Our interest stems from the fact that for some extremal problems such as the cap set problem or optimally pseudorandom clique-free graphs (see §5) it is natural to look for constructions which behave very similarly to strongly regular graphs.

All graphs in this document are finite and simple. Let us repeat some basic facts about strongly regular graphs and bounds on their parameters: Our notation for strongly regular graphs is standard, cf. [8, 9]. A strongly regular graph $\Gamma$ with parameters $(v, k, \lambda, \mu)$ is a $k$-regular graph (not complete, not edgeless) of order $v$ such that two distinct adjacent vertices have precisely $\lambda$ common neighbors, while two distinct nonadjacent vertices have precisely $\mu$ common neighbors. One of the parameters depends on the others: For a fixed vertex $a$, counting the pairs $(b, c)$ with $a \sim b \sim c \not\sim a$ in two ways shows that $(v - k - 1)\mu = k(k - \lambda - 1)$.

Call an eigenvalue of the adjacency matrix of a regular graph restricted if it has an eigenvector orthogonal to the all-ones vector. Then, alternatively, a strongly regular graph can be defined as a $k$-regular graph whose adjacency matrix $A$ has exactly two restricted eigenvalues $r \geq 0$ and $s < 0$. Denote the multiplicity of $r$ by $f$ and the multiplicity of $s$ by $g$. We have the identities

$$\lambda - \mu = r + s, \quad k - \mu = -rs.$$
Explicit formulas for $f$ and $g$ can be found using $1 + f + g = v$ and $k + fr + gs = \text{tr}(A) = 0$.

As a toy example for this introduction, we consider the parameter set $v = (1 + o(1))\lambda^{11}$, $k = (1 + o(1))\lambda^{10}$, and $\mu = (1 + o(1))\lambda^9$ (as $\lambda \to \infty$). See \[\text{11}\] for a discussion of big-$O$ (and similar) notation.

The Krein bound\[\text{11}\] and the absolute bound provide asymptotic conditions on the parameters $(v, k, \lambda, \mu)$ of a strongly regular graph.

**Theorem 1.1** (Krein Bound for SRGs, \[9\] p. 26]). The eigenvalues $k \geq r \geq 0 > s$ of a strongly regular graph satisfy

$$1 + \frac{s^3}{k^2} - \frac{(s + 1)^3}{(v - k - 1)^2} \geq 0,$$

$$1 + \frac{r^3}{k^2} - \frac{(r + 1)^3}{(v - k - 1)^2} \geq 0.$$

In the toy example above, $s = (-1 + o(1))\lambda^3$, so Theorem \[\text{11}\] implies $1 + (-1 + o(1))\lambda^7 - (1 + o(1))\lambda^5 \geq 0$ which is impossible.

The absolute bound for strongly regular graphs is a corollary of the well-known result by Delsarte, Goethals, and Seidel that a family of $v$-regular graphs are examples for

\[\text{Theorem 1.2}} \text{ (Absolute Bound for SRGs, \[9\] Prop. 1.3.14}). \text{ The multiplicities } f, g \text{ of a primitive strongly regular graph satisfy } v \leq \frac{1}{2}f(f + 3) \text{ and } v \leq \frac{1}{2}g(g + 3).

In the toy example above, $g = (1 + o(1))\lambda^3$, so Theorem \[\text{12}\] implies $(1 + o(1))\lambda^{11} \leq \frac{1}{2}\lambda^6$ which is impossible.

In the first part of this document, we generalize Theorem \[\text{11}\].

**Proposition 1.3** (Krein Bound, Variant for Regular Graphs). Let $\Gamma$ be a $k$-regular graph of order $v$ with adjacency matrix $A$. Let $r$ denote the second largest and $s$ the smallest eigenvalue of $A$. Then

$$(s + r^2)v + 2(k - r)(r - s) \geq 0,$$

$$(r + s^2)v + 2(k - s)(s - r) \geq 0.$$

An $1$-walk-regular graph is a graph in which the number of walks of length $\ell$ between vertices $a$ and $b$ with $a = b$ or $a, b$ adjacent only depends on $\ell$ and $a = b$, not the choice of $a$ and $b$, cf. \[\text{13}\]. Arc-transitive graphs and strongly regular graphs are examples for $1$-walk-regular graphs. In \[\text{12}\] we will provide a variant of Proposition \[\text{13}\] for a special type of $1$-walk-regular graphs which is significantly stronger.

There is a poor man’s version of the absolute bound which only shows $v \leq f^2$ and $v \leq g^2$. We give a variant of this poor man’s result.

**Proposition 1.4** (Absolute Bound, Variant). Consider a $k$-regular graph of order $v$ with adjacency matrix $A$. Let $r, s$ be real numbers with $k > r \geq 0 > s$. Suppose that $A$ has at least $f_1$ and at most $f_2$ restricted eigenvalues in $[r, k]$, all eigenvalues of $A$ are at least $s - \varepsilon$ for some $\varepsilon > 0$, and at least $v - f_2$ eigenvalues of $A$ are in $[s - \varepsilon, s]$. If $s^2 + s > \varepsilon$, then $v \leq f_2(f_2 + 1) - f_1$.

\[\text{1}^{\text{Named somewhat indirectly after Mark Grigorievich Krein, cf.}} \text{[9] p. 26}].
In the second part of this document, we consider what we call approximately strongly regular graphs. For two adjacent vertices $a$ and $b$ of a graph $\Gamma$, let $\lambda_{ab}$ denote the number of common neighbors of $a$ and $b$ in $\Gamma$. Similarly, for two distinct nonadjacent vertices $a$ and $b$ of a graph $\Gamma$, let $\mu_{ab}$ denote the number of common neighbors of $a$ and $b$ in $\Gamma$. Let $\Lambda$ (respectively, $M$) denote the set of all pairs of adjacent (respectively, distinct nonadjacent) vertices in $\Gamma$.

We call a $k$-regular graph (not complete, not edgeless) $\Gamma$ of order $v$ an approximately strongly regular graph with parameters $(v, k, \lambda, \mu, \sigma)$, where $\sigma \geq 0$, if

$$E(\lambda_{ab}) := \frac{1}{|\Lambda|} \sum_{(a,b) \in \Lambda} \lambda_{ab} = \lambda$$

and

$$\text{Var}(\lambda_{ab}) := \frac{1}{|\Lambda|} \sum_{(a,b) \in \Lambda} (\lambda_{ab} - \lambda)^2 \leq \sigma^2,$$

and

$$E(\mu_{ab}) := \frac{1}{|M|} \sum_{(a,b) \in M} \mu_{ab} = \mu$$

and

$$\text{Var}(\mu_{ab}) := \frac{1}{|M|} \sum_{(a,b) \in M} (\mu_{ab} - \mu)^2 \leq \sigma^2.$$

Strongly regular graphs are precisely the approximately strongly regular graphs with $\sigma = 0$. The complement of an approximately strongly regular graph with parameters $(v, k, \lambda, \mu, \sigma)$ is an approximately strongly regular graph with parameters $(v, v - k - 1, v - 2k + \mu, v - 2k + \lambda; \sigma)$. Counting triples $(a, b, c)$ with $a \sim b \sim c \not\sim a$ shows $\sum_{(a, b) \in M} \mu_{bc} = \sum_{(a, b) \in M} (k - \lambda_{ab} - 1)$. Hence, $(v - k - 1)\mu = (k - \lambda - 1)$ also holds for approximately strongly regular graphs.

In our toy example with $v = (1 + o(1))\lambda^{11}$, $k = (1 + o(1))\lambda^{10}$, and $\mu = (1 + o(1))\lambda^9$, Proposition 1.3 rules out the existence of approximately strongly regular graphs with $\sigma = o(\lambda^{1.5})$. Under slightly stronger conditions, see Proposition 2.1, we also obtain $\sigma = o(\lambda^8)$. If our toy example contains a coclique of size $(1 + o(1))q^9$, then we will also rule out $\sigma = o(\lambda^{7.5})$.

In the third part of this document, we apply our results to the cap set problem and to optimally pseudorandom clique-free graphs.

For instance, if there exists a cap of size $(1 + o(1))q^9$ in the projective space $\text{PG}(10, q)$, then a standard construction yields a approximately strongly regular graph with the same parameters of our toy example (where $\lambda = q - 2$).

## 2 Bounds

Denote the all-ones vector by $j$, the all-ones matrix by $J$, and the identity matrix by $I$. We denote the Hadamard product of two matrices by $\circ$.

### 2.1 Krein Bounds

The following proof is based on Remark (i) on page 50 in [8].

**Proof of Proposition 1.3.** Consider the matrices $E_1$ and $E_2$ defined by

$$E_1 = \frac{1}{s - r} \left( A - sI - \frac{k-r}{v}J \right), \quad E_2 = \frac{1}{s - r} \left( A - rI - \frac{k-r}{v}J \right).$$

The spectrum of $E_2$ is in $[0, 1]$ as $(s - r)E_2$ has only eigenvalues in $[s, r]$. Write $E_2 \circ E_2$ as a linear combination of the matrices $A, I, J$. Then the coefficients of $I$ and $A$ are

$$\frac{1}{(s - r)^2} \left( t^2 + 2r \frac{k-r}{v} \right) \text{ for } I, \quad \frac{1}{(s - r)^2} \left( 1 - 2 \frac{k-r}{v} \right) \text{ for } A.$$

Now we write $E_2 \circ E_2$ as a linear combination of the matrices $E_1, E_2, J$, that is we replace $A$ and $I$ by $E_1$ and $E_2$. We obtain the coefficients

$$\frac{r + t^2}{(s - r)^2} \text{ for } E_1, \quad t := \frac{(s + r^2)v + 2(k - r)(r - s)}{v(s - r)^2} \text{ for } E_2.$$
Let $\chi$ be an eigenvector of $A$ with $A\chi = s\chi$. Then $E_1\chi = 0$. Hence, $(E_2 \circ E_1)\chi = tE_2\chi = t\chi$. Hence, $\chi$ is an eigenvector of $E_2 \circ E_2$ which shows that $t \geq 0$. Hence, using our expression for $\beta$ and $u_i - r < u_i$, we obtain the first inequality.

For the second inequality, consider $E_1 \circ E_1$ instead of $E_2 \circ E_2$. \hfill \square

For an eigenspace $U_i$ of a real symmetric matrix $A$ with eigenvalue $u_i$, let $F_i$ be the orthogonal projection onto $U_i$, so $F_i$ is idempotent and $AF_i = u_i F_i$. We will use repeatedly without further notice that the eigenspaces of $A$ are pairwise orthogonal, so $F_i F_j = 0$ if $u_i \neq u_j$. For the remainder of this subsection, consider the case that $\Gamma$ is 1-walk-regular. Recall that $(A^{\ell})_{ab}$ is the number of walks from $a$ to $b$ of length $\ell$. It follows from $A^\ell = \sum_{i} u_i^{\ell} F_i$ that $(F_i)_{aa}$ is constant for all vertices $a$, and that $(F_i)_{ab}$ is constant for all $a, b$ adjacent, cf. [11] Theorem 3.1.

Let $m_i$ denote the multiplicity of $F_i$. From $m_i = \text{tr}(F_i)$ we obtain that $(F_i)_{aa} = \frac{m_i}{v}$. We have $u_i \cdot \frac{m_i}{v} = (u_i F_i)_{aa} = (AF_i)_{aa} = k(F_i)_{ab}$ for $a, b$ adjacent, so

$$(F_i)_{ab} = \frac{m_i u_i}{vk}.$$ 

Hence, for a 1-walk-regular graph we can control $I \circ F_i$ and $A \circ F_i$ as we could control $I \circ J$ and $A \circ J$ in the proof of Proposition 1.3.

For a matrix $M$, let $\rho(M)$ denote its spectral radius. Now we are ready to give an example for how one can increase regularity conditions in Krein-type bounds to obtain better nonexistence results.

**Proposition 2.1** (Krein Bound, Variant for 1-Walk-Regular Graphs). Let $\Gamma$ be a $k$-regular 1-walk-regular graph of order $v$ with adjacency matrix $A$. Let $s$ be the smallest eigenvalue of $A$. For some $r \geq 0$, let $I$ denote the set of indices of eigenvalues $u_i$ of $A$ in the interval $(r, k]$. Put $K_1 = \frac{2}{v} \sum_{i \in I} u_i^2$, $K_2 = \frac{2}{v} \sum_{i \in I} u_i$, $L = \rho(\sum_{i,j \in I} |u_i u_j| F_i \circ F_j)$. Then

$$(1 - K_1)s + r^2 + rK_2 + L \geq 0.$$ 

**Proof.** Consider the matrices $E_1$ and $E_2$ defined by

$$(E_1)_{ij} = \begin{cases} \frac{1}{r - s} & (A - sI - \sum_{i \in I} (u_i - s) F_i)_{ij} \\ 0 & \text{otherwise} \end{cases},$$

$$(E_2)_{ij} = \begin{cases} \frac{1}{s - r} & (A - rI - \sum_{i \in I} (u_i - r) F_i)_{ij} \\ 0 & \text{otherwise} \end{cases}.$$ 

The spectrum of $E_2$ is in $[0, 1]$ as $(s - r)E_2$ has only eigenvalues in $[s, r]$. Write

$$(E_2 \circ E_2) = \sum_{i,j \in I} \frac{(u_i - r)(u_j - r)}{(s - r)^2} F_i \circ F_j$$

as a linear combination of the matrices $I$ and $A$. Put $\alpha_i = (F_i)_{aa}$ and $\beta_i = (F_i)_{ab}$ for adjacent vertices $a, b$. Then the coefficients of $I$ and $A$ are

$$\frac{1}{(s - r)^2} \left( r^2 + 2r \sum_{i \in I} (u_i - r) \alpha_i \right)$$

for $I$,

$$\frac{1}{(s - r)^2} \left( 1 - 2 \sum_{i \in I} (u_i - r) \beta_i \right)$$

for $A$. 

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Put $\alpha = 2r\sum_i (u_i - r)\alpha_i$ and $\beta = 2\sum_i (u_i - r)\beta_i$. Put
\[ \tilde{E}_1 = \frac{1}{r-s} (A - sI), \quad \tilde{E}_2 = \frac{1}{s-r} (A - rI). \]

Now we write $E_2 \circ E_2$ as a linear combination of the matrices $\tilde{E}_1, \tilde{E}_2$, that is we replace $I$ and $A$ by $\tilde{E}_1$ and $\tilde{E}_2$. We obtain the coefficients
\[ \frac{(1-\beta)r + r^2 + \alpha}{(s-r)^2} \text{ for } \tilde{E}_1, \quad \frac{(1-\beta)s + r^2 + \alpha}{(s-r)^2} =: t \text{ for } \tilde{E}_2. \]

Let $\chi$ be an eigenvector of $A$ with $A\chi = s\chi$ and $\|\chi\| = 1$. Then $\tilde{E}_1\chi = 0$. Hence,
\[ (E_2 \circ E_2)\chi = t\tilde{E}_2\chi + \frac{1}{(s-r)^2} \sum_{i,j \in I} |u_i u_j| \cdot (F_i \circ F_j)\chi. \]

We have $t\tilde{E}_2\chi = t\chi$ and
\[ \left\| \frac{1}{(s-r)^2} \sum_{i,j \in I} |u_i u_j| \cdot (F_i \circ F_j)\chi \right\| \leq \frac{L}{(s-r)^2}. \]

The matrix $E_2 \circ E_2$ is a principle submatrix of $E_2 \otimes E_2$, so the eigenvalues of $E_2 \circ E_2$ interlace those of $E_2 \otimes E_2$, thus they are in $[0,1]$. In particular, $E_2 \circ E_2$ is positive semidefinite. Hence, $t + \frac{L}{(s-r)^2} \geq 0$. Hence, using our expression for $\beta$ and $u_i - r < u_i$, we obtain the assertion.

We still lack control over $J \circ F_i$ and, more generally, $F_i \circ F_j$. For this, we need one last concept: We say that an eigenvalue $u_i$ of a 1-walk-regular is $h$-flat if for $a,c$ nonadjacent, we have
\[ |(F_i)_{ac}| \leq h \frac{m_i m_j |u_i|}{v^k}. \]

Recall that the spectral radius $\rho(M)$ of a matrix $M$ is at most its 1-norm $\|M\|_1$. Hence,
\[ \rho(J \circ F_i) \leq \frac{m_i}{v} + k \cdot \frac{m_i |u_i|}{v^k} + \frac{m_i |u_i|}{v^k} \cdot (v - k - 1) \cdot h \frac{m_i |u_i|}{v^k}, \quad \text{and, more generally}, \]
\[ \rho(F_i \circ F_j) \leq \frac{m_i m_j}{v^2} + k \cdot \frac{m_i m_j |u_i u_j|}{v^{2k}} + \frac{m_i m_j |u_i u_j|}{v^{2k}} \cdot (v - k - 1) \cdot h \frac{2 m_i m_j |u_i u_j|}{v^{2k}}. \]

For a set $I$, we need a convenient bound on $\sum_{i,j \in I} \rho(u_i u_j F_i \circ F_j)$. Put $M =$
\[ \sum_{i \in I} m_i. \] As an example, suppose that the \( u_i \) are 1-flat for \( i \in I \). Then

\[ \rho \left( \sum_{i,j \in I} |u_iu_j| F_i \circ F_j \right) \leq \sum_{i,j \in I} |u_iu_j| \rho(F_i \circ F_j) \]

\[ \leq \sum_{i,j \in I} \left( \frac{u_i^2 u_j^2}{v^2} + \frac{|u_iu_j| \cdot m_im_j}{v^2} \right) \]

\[ \leq \frac{1}{v k^2} \sum_{i \in I} u_i^2 m_i \sum_{j \in I} u_j^2 m_j + \sum_{i,j \in I} |u_iu_j| \frac{m_i m_j}{v^2} \]

\[ \leq \frac{1}{v k^2} \left( \sum_{i \in I} m_i u_i^2 \right)^2 + \sum_{i \in I} 2 u_i^2 \frac{m_i}{v} \]

\[ \leq \frac{1}{v k^2} \left( \sum_{i \in I} m_i u_i^2 \right)^2 + \frac{2}{v} \left( \sum_{i \in I} m_i u_i^2 \right). \] (1)

Assuming that the matrices are 1-flat might be very generous for cases where \( v \) is much larger than \( k \). As \( F_i^2 = F_i \), we have

\[ \frac{m_i^2}{v^2} + k \frac{m_i^2 u_i^2}{v^2} + \sum_{a \neq c} (F_i)_{ac}^2 = (F_i)_{aa}^2 + \sum_{b \neq c} (F_i)_{ab}^2 + \sum_{a \neq c} (F_i)_{ac}^2 = (F_i)_{aa} = \frac{m_i}{k_i}. \]

Hence,

\[ E((F_i)_{ac}^2) = \frac{m_i v_i - \frac{n_i^2}{v^2} - k \frac{m_i^2 s^2}{v^2}}{v - k - 1}. \] (2)

Thus, if \( k \) is small compared to \( v \), then we expect \( (F_i)_{ac} \) for \( a, c \) nonadjacent to be small compared to \( (F_i)_{ab} \) for \( a, b \) adjacent. Hence, the condition of being 1-flat in (1) is not particularly strong when \( k \) is small compared to \( v \). In particular, later we will consider applications with \( k = o(v) \).

### 2.2 Absolute Bounds

The following is based on the proof of Theorem 2.3.3 in [8]. We use eigenvalue interlacing, cf. [22]. More precisely, if \( A \) is a real symmetric matrix of order \( v \) with eigenvalues \( u_1 \geq \ldots \geq u_v \) and \( B \) is a principal submatrix of \( A \) of order \( w \) with eigenvalues \( \nu_1 \geq \ldots \geq \nu_w \), then \( u_i \geq \nu_i \geq u_{i-w+v} \).

**Proof of Proposition 1.4.** Consider \( M := A - sI - \frac{k-a}{v} J \). Then \( M \) has at least \( v - f_2 \) eigenvalues in \([-\varepsilon, 0]\), and between \( f_1 \) and \( f_2 \) eigenvalues at least \( r - s \). Hence, \( M \otimes M \) has at most \( f_2^2 \) eigenvalues at least \((r-s)^2\), while all its other eigenvalues are at most \( \varepsilon^2 \). Furthermore,

\[ M \circ M = (1 - \frac{k-a}{v})A + (s^2 + \frac{k-a}{v})I + \left( \frac{k-a}{v} \right)^2 J. \]

Hence, \( M \circ M \) has one eigenvalue \((k-k \frac{k-a}{v}) + (s^2 + s \frac{k-a}{v}) + \left( \frac{k-a}{v} \right)^2 \) \( v \geq s^2 + s \) with eigenvector \( j \), at least \( f_1 \) eigenvalues at least \((r-k \frac{k-a}{v}) + (s^2 + s \frac{k-a}{v}) \geq s^2 + s \) and at least \( v - f_2 \) eigenvalues at least \((s - s \frac{k-a}{v}) + (s^2 + s \frac{k-a}{v}) = s^2 + s \). The matrix \( M \circ M \) is a principal submatrix of \( M \otimes M \), so the eigenvalues of \( M \circ M \) interlace those of \( M \otimes M \). Hence, \( M \circ M \) has at most \( f_2^2 \) eigenvalues greater than \( \varepsilon^2 < s^2 + s \). We obtain that \( v - f_2 + f_1 \leq f_2^2 \).
2.3 Cvetković Bound or Inertia Bound

The following bound will prove useful for some of our applications. For a graph \( \Gamma \) of order \( v \), let \( M \) be a matrix with \( M_{ac} = 0 \) if \( a, c \) nonadjacent. Let \( n^+(M) \) denote the number of positive eigenvalues of \( M \) and let \( n^-(M) \) denote the number of negative eigenvalues of \( M \). Then a coclique (independent set, stable set) of \( \Gamma \) has size at most

\[
\min(v - n^+(M), v - n^-(M)).
\]

This bound is known as *Cvetković bound* or *inertia bound*, cf. [9, p. 13]. Here we always use the adjacency matrix for \( M \).

3 Approximately Strongly Regular Graphs

Consider the adjacency matrix \( A \) of an approximately strongly regular graph \( \Gamma \) with parameters \((v, k, \lambda, \mu; \sigma)\). We can write

\[
A^2 = kI + \lambda A + \mu (J - I - A) + E,
\]

where

\[
(E)_{ab} = \lambda_{ab} - \lambda \text{ when } a, b \text{ are adjacent, } (E)_{ab} = \mu_{ab} - \mu \text{ when } a, b \text{ are distinct nonadjacent, and } (E)_{ab} = 0 \text{ when } a = b.
\]

Let \( \chi \) be an eigenvector of \( A \) orthogonal to the all-ones vector \( j \) with eigenvalue \( u \). Then

\[
u u^2 \chi = A^2 \chi = (k - \mu)(\chi) + \lambda(\chi) + \mu(\chi) + \nu(\chi).
\]

Hence, \( \chi \) is an eigenvector of \( E \) with some eigenvalue \( \nu \). By solving for \( u \), we find that

\[
u = \frac{1}{2} \left( (\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu + \nu)} \right).
\]

We say that \( u \) has *positive form* if

\[
u = \frac{1}{2} \left( (\lambda - \mu) + \sqrt{(\lambda - \mu)^2 + 4(k - \mu + \nu)} \right),
\]

and that \( u \) has *negative form* if

\[
u = \frac{1}{2} \left( (\lambda - \mu) - \sqrt{(\lambda - \mu)^2 + 4(k - \mu + \nu)} \right).
\]

Let \( u_1, u_2, \ldots, u_v \) denote the eigenvalues of \( A \). For \( u_i \) an eigenvalue of \( A \), let \( \nu_i \) denote the corresponding eigenvalue of \( E \).

The next result shows that if \( \sigma \) and \( v \) are sufficiently small, then there are few (if any) large \( \nu_i \).

**Lemma 3.1.** The eigenvalues \( \nu_1, \ldots, \nu_v \) of \( E \) satisfy \( \sum \nu_i^2 \leq v(v - 1)\sigma^2 \).

**Proof.** We have

\[
u_i^2 = \text{tr}(E^2) = \sum_{a \sim b} (\lambda_{ab} - \lambda)^2 + \sum_{a \neq b} (\mu_{ab} - \mu)^2 \leq v(v - 1)\sigma^2.
\]

Call a graph \( \Gamma \) *edge-regular* if \( \text{Var}(\lambda_{ab}) = 0 \) and *coedge-regular* if \( \text{Var}(\mu_{ab}) = 0 \). Clearly, Lemma 3.1 can be improved to \( \sum \nu_i^2 \leq v(k\sigma^2 \text{ for edge-regular graphs and to } \sum \nu_i^2 \leq v(v - k - 1)\sigma^2 \text{ for coedge-regular graphs.}

In the introduction, we define strongly regular graphs in two ways, combinatorially and spectrally. Lemma 3.1 shows that that for small \( \sigma \), the restricted
eigenvalues of $\Gamma$ are concentrated at two values. Let us also show the reverse, namely if $\Gamma$ restricted eigenvalues are concentrated around two values, then $\Gamma$ is approximately strongly regular.

Call a $k$-regular (not complete, not edgeless) graph $\Gamma$ of order $v$ spectrally approximately strongly regular with parameters $(v, k, r, s; \sigma)$ if

$$(A - rI)(A - sI) = \mu J + \tilde{E},$$

for some constant $\mu$, where the eigenvalues $\tilde{\nu}$ of $\tilde{E}$ are also eigenvalues of $A$ and satisfy $\sum \tilde{\nu}^2 \leq v(v - 1)\sigma^2$. Furthermore,

$$A = (r + s - \mu)A + (\mu - rs)I + \mu J + \tilde{E}.$$

We see that $\Gamma$ is spectrally approximately strongly regular with parameters $(v, k, r, s; \sigma)$ if and only if $\Gamma$ is approximately strongly regular with parameters $(v, k, r + s - k - rs, k + rs; \sigma)$.

### 3.1 Big-O Notation

We use the symbols $O, \Omega, \Theta, o, \omega$ in the following way:

- $f(x) = O(g(x))$ (as $x \to a$) if and only if $\limsup_{x \to a} \frac{f(x)}{g(x)} < \infty$,
- $f(x) = \Omega(g(x))$ (as $x \to a$) if and only if $\limsup_{x \to a} \frac{f(x)}{g(x)} > 0$,
- $f(x) = \Theta(g(x))$ (as $x \to a$) if and only if $0 < \limsup_{x \to a} \frac{|f(x)|}{|g(x)|} < \infty$,
- $f(x) = o(g(x))$ (as $x \to a$) if and only if $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$,
- $f(x) = \omega(g(x))$ (as $x \to a$) if and only if $\lim_{x \to a} \frac{|f(x)|}{|g(x)|} = \infty$.

Usually, we have $a = \infty$. If there are several variables involved, then we specify the relevant one. We also use the big-$O$ notation in minor order terms. For instance, we can write $x^2 + x + 100 = x^2 + O(x)$ (as $x \to \infty$) as $x + 100 = O(x)$. For us the sign of $f(x)$ is often important, so for convenience, we aim to use big-$O$ notation with $f(x) > 0$. For instance, we write $-s = O(\mu)$ even though $s = O(\mu)$ is equally correct.

If we talk about a $k$-regular graph $\Gamma$ of order $v$ with $k = O(g(v))$ for some function $g$, then we mean that we consider an infinite family of graphs $(\Gamma_n)$, where $\Gamma_n$ is of order $v_n$ and $k_n$-regular with $k_n = O(g(v_n))$ as $n \to \infty$. In particular, if we say that $\Gamma$ is an approximately strongly regular graph or a family of approximately strongly regular graphs with parameters $(v, k, \lambda, \mu; \sigma)$ and $k = o(\mu - \lambda)$, then there is an infinite family of approximately strongly regular graphs $(\Gamma_n)$ with parameters $(v_n, k_n, \lambda_n, \mu_n; \sigma_n)$ such that $k_n = o(|\mu_n - \lambda_n|\frac{\sigma_n}{\mu_n})$ as $n \to \infty$.

We also use big-$O$ notation for the eigenvalues $\nu_i$ and $\nu_i$: Assume that $|\nu_1| \geq |\nu_2| \geq \cdots \geq |\nu_v|$. If we write $\nu_i = O(g(n))$, then there is a function $h(n)$ such that $\nu_{h(n)} = O(g(n))$. For instance, we might assume that $\nu_i = O(\mu_n)$ and show some property (P) for $\nu_i$. By Lemma 5.1, $\sum \nu_i^2 \leq v^2 \sigma^2$, so the number of $\nu_i$ with $\nu_i = o(\mu)$ is at most $o(\frac{\mu}{\sigma})$. Thus, (P) holds for $\nu_{h(n)}$ with $h(n) = \Omega(\frac{\mu_n \sigma_n}{\mu_n})$. 

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3.2 Asymptotic Bounds on Eigenvalues

Lemma 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(y) = o(y^2)$ (as $y \to \infty$). Then

$$\sqrt{y^2 + f(y)} - y = (\frac{1}{2} + o(1)) \frac{f(y)}{y}.$$

Proof. Write $\sqrt{y^2 + f(y)} - y = y(\sqrt{1 + \frac{f(y)}{y^2}} - 1)$. The Taylor expansion of $\sqrt{1 + x}$ at 1 shows that (as $x \to 0$)

$$\sqrt{1 + x} = 1 + \frac{1}{2}x + O(x^2).$$

The following gives us approximate versions of the equations $\mu - \lambda = r + s$ and $k - \mu = -rs$ for strongly regular graphs.

Lemma 3.3. For a family of approximately strongly regular graphs with parameters $(v, k, \lambda, \mu; \sigma)$, consider an eigenvalue $u_i$ with associated eigenvalue $\nu_i$.

If $\mu > \lambda$, $k = o(|\lambda - \mu|)$, and $|\nu_i| = o(|\lambda - \mu|)$, then the following holds:

(i) If $u_i$ has positive form, then $u_i = (1 + o(1)) \frac{k - \mu + \nu_i}{\mu - \lambda}$.

(ii) If $u_i$ has negative form, then $u_i = -(1 + o(1))(\mu - \lambda)$.

If $\mu < \lambda$, $k = o(|\lambda - \mu|)$, and $|\nu_i| = o(|\lambda - \mu|)$, then the following holds:

(iii) If $u_i$ has positive form, then $u_i = (1 + o(1))(\lambda - \mu)$.

(iv) If $u_i$ has negative form, then $u_i = -(1 + o(1)) \frac{k - \mu + \nu_i}{\mu - \lambda}$.

If $k = \Omega(|\lambda - \mu|^2)$ and $|\nu_i| = O(k)$, then the following holds:

(v) We have $u_i = \Theta(\sqrt{k})$.

If $|\nu_i| = \Omega(|\lambda - \mu|^2)$ and $|\nu_i| = \Omega(k)$, then the following holds:

(vi) We have $u_i = \Theta(\sqrt{|\nu_i|})$.

Proof. We only show (i) and (ii) as the other cases are similar. Using Lemma 3.2 with $y = \mu - \lambda$ and $\mu = o(k)$, we find that if $u_i$ has positive form, then

$$u_i = \frac{1}{2} \left( \lambda - \mu + (1 + o(1)) \sqrt{(\lambda - \mu)^2 - 4(k - \mu + \nu_i)} \right)$$

$$= (1 + o(1)) \frac{k - \mu + \nu_i}{\mu - \lambda}.$$

If $u_i$ has negative form, then

$$u_i = \frac{1}{2} \left( \lambda - \mu - (1 + o(1)) \sqrt{(\lambda - \mu)^2 - 4(k - \mu + \nu_i)} \right)$$

$$= -(1 + o(1))(\mu - \lambda).$$

3.3 Krein Bounds

The expected size of the common neighborhood of two distinct vertices is $(1 + o(1)) \frac{k^2}{v}$. Hence, if $k = o(v)$, then $\mu = (1 + o(1)) \frac{k}{v}$, so $\mu = o(k)$. 

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Proposition 3.4 (Krein Bound, Approximately Strongly Regular Graphs). Consider a family of approximately strongly regular graphs with $\mu > \lambda$, $k = o(v)$, and $k = o((\mu - \lambda)^2)$. Then $\sigma \geq (1 + o(1))(\mu - \lambda)^2 v^{-1}$.

Proof. Suppose to the contrary that $\sigma \leq (D + o(1))(\mu - \lambda)^2 v^{-1}$ for some constant $D < 1$. By Lemma 3.3 (ii), if $u_i$ is an eigenvalue of $A$, then

$$u_i = (1 + o(1)) \frac{k - u}{\mu - \lambda} + D\sqrt{\mu - \lambda} \leq (D + o(1))\sqrt{\mu - \lambda} =: r.$$

By Lemma 3.3 (ii), if $u_i$ has negative form, then

$$u_i = -(1 + o(1))(\mu - \lambda) =: s.$$

Hence, $r^2 = -(D^2 + o(1))s$. By Proposition 1.3,

$$0 \leq (s + r^2)v + 2(k - r)(r - s) = (1 - D^2 + o(1))sv - 2sk = (1 - D^2 + o(1))sv < 0.$$

This is a contradiction, so $\sigma \geq (1 + o(1))(\mu - \lambda)^2 v^{-1}$.

We call a 1-walk-regular graph positive-$\alpha$-flat if all its positive eigenvalues are $\alpha$-flat. Recall from the discussion at the end of §2.1 that for $k = o(v)$ it is natural to assume that a 1-walk-regular graph is positive-1-flat.

Proposition 3.5 (Krein Bound, Positive-1-Flat 1-Walk-Regular Approximately Strongly Regular Graphs). Consider a family of positive-1-flat 1-walk-regular approximately strongly regular graphs with $\mu > \lambda$, $k = o(v)$, and $k = o((\mu - \lambda)^2)$. Then $\sigma \geq (1 + o(1))(\mu - \lambda)^2 \cdot v^{-1} k^{\frac{3}{2}}$.

Proof. Let $K_1$, $K_2$, and $L$ be as in Proposition 2.1 where we use the estimate from Equation $1$ for $L$.

Our plan is as follows: Choose $r = \Theta\left(\frac{\sqrt{\mu}}{\mu - \lambda}\right)$. We suppose that $\sigma \leq (D + o(1))(\mu - \lambda)^2 \cdot v^{-1} k^{\frac{3}{2}}$ for some constant $D < 1$, so smaller than claimed. From this we show that we have an eigenvalue $s$ of size $-(1 + o(1))(\mu - \lambda)$, so that

$$0 \leq (1 - K_1)s + rK_2 + L$$

yields a contradiction if $K_1 = o(1)$, $rK_2 = o(-s)$, and $L + o(1) < (D^4 + o(1))(-s)$.

We have $B = \sum u_i^2 = \text{tr}(A) = vk = (1 + o(1))vk$. Note that in this sum we can ignore eigenvalues $u_i$ with $|u_i| = O\left(\frac{k - u}{\mu - \lambda}\right)$ as these contribute at most $O\left(\frac{vk^2}{(\mu - \lambda)^2}\right) = o(vk)$ to it (here we use $k - \mu = o((\mu - \lambda)^2)$). An eigenvalue $u_i$ of positive form with associated eigenvalue $v_i$ of $E$ with $v_i = O(k)$ satisfies, by Lemma 3.3 (i),

$$u_i = (1 + o(1))\frac{k - u + u_0}{\mu - \lambda} = O\left(\frac{k - u}{\mu - \lambda}\right).$$

Hence, an eigenvalue $u_i$ of positive form either has a significant contribution from $E$, that is $v_i = \omega(k)$, or does not contribute to $\sum u_i^2$. By Lemma 3.3 we find

$$B := \sum u_i^2 \leq \frac{\sum u_i^2}{(\mu - \lambda)^2} \leq (D^2 + o(1))(\mu - \lambda)^2 v^{\frac{3}{2}} k. \quad (3)$$
We find, using Equation (3),

\[ K_1 = \frac{1}{\nu^2} B \leq (2D^2 + o(1))(\mu - \lambda)^{\frac{3}{2}} v^{-\frac{3}{2}} = o(1), \text{ and} \]
\[ L = \frac{1}{\nu^2} B^2 + \frac{2}{v} B \leq (D^4 + o(1))(\mu - \lambda) = (D^4 + o(1))(-s). \]

The Cauchy-Schwarz inequality applied to Equation (3) shows that \((\sum u_i)^2 \leq (D^2 + o(1))(\mu - \lambda)^{\frac{3}{2}} v^{-\frac{3}{2}} k\). Hence,

\[ rK_2 \leq (D + o(1))(\mu - \lambda)^{-\frac{3}{2}} v^{-\frac{3}{2}} k^{\frac{3}{2}} = o(-s). \]

It remains to show that there exists an eigenvalue \(s\) as asserted. Only considering \(u_i\) with \(\nu_i = \omega(k)\), so we are in one of the cases (i) or (vi) of Lemma 3.3, we see as in Equation (3) that

\[ \sum (\mu - \lambda)^2 u_i^2 \leq (D^2 + o(1))(\mu - \lambda)^{\frac{3}{2}} v^{\frac{3}{2}} k. \]

Hence, the contribution of \(E\) to the sum \(\sum u_i^2\) is bounded by \((D^2 + o(1))(\mu - \lambda)^{\frac{3}{2}} v^{\frac{3}{2}} k\). Hence, eigenvalues without a significant contribution from \(E\) account for at least \((1 + o(1)) v k\) of \(\sum u_i^2 = (1 + o(1)) v k\). We saw earlier that such \(u_i\) have negative form. Hence, by Lemma 3.3(ii), \(u_i = -(1 + o(1))(\mu - \lambda)\) for some \(u_i\).

In light of (2), it might be more reasonable (at least if one is an optimist) to assume that the graph \(O(\frac{k}{v})\)-flat. Then in the same notation as the previous proof and the argument from (1), we have

\[ L \leq \frac{1 + o(1)}{v^2 k} \left( \sum_{i \in I} m_i u_i^2 \right)^2. \]

Let us also state the absolute bound for this case.

**Proposition 3.6** (Krein Bound, Positive-\(O(\frac{k}{v})\)-Flat 1-Walk-Regular Approximately Strongly Regular Graphs). Consider a family of positive-\(O(\frac{k}{v})\)-flat 1-walk-regular approximately strongly regular graphs with \(\mu > \lambda\), \(k = o(v)\), and \(k = o((\mu - \lambda)^{\frac{3}{2}})\). Then \(\sigma = \Theta((\mu - \lambda)^{\frac{3}{2}} \cdot v^{-\frac{3}{2}} k^{\frac{3}{2}})\).

### 3.4 An Absolute Bound

**Proposition 3.7** (Absolute Bound, Approximately Strongly Regular Graphs). Consider a family of approximately strongly regular graphs such that \(\lambda > \mu\) and \(\sqrt{\nu} \cdot k = o((\lambda - \mu)^2)\). Then \(\sigma \geq \frac{1}{2} + o(1) \cdot \frac{k}{\sqrt{\nu}}\).

**Proof.** Our plan for applying Proposition 1.1 is as follows: We can ignore \(f_1\) as it is a minor order term. We suppose that \(\sigma \leq (D + o(1)) \frac{k}{\nu}\) for some constant \(D < \frac{1}{\nu}\). Put \(r = (1 + o(1))(\lambda - \mu)\), \(s = -(1 - D + o(1)) \frac{k - \mu}{\lambda - \mu}\), and \(\varepsilon = (2D + o(1)) \frac{k - \mu}{\lambda - \mu}\). As \(D < \frac{1}{\nu}\), we have that \(s^2 + s < \varepsilon^2\).

By Lemma 3.1 any \(\nu_i\) satisfies \(|\nu_i| \leq v \sigma \leq (D + o(1)) k\). If an eigenvalue \(u_i\) has positive form, then, by Lemma 3.3(iii),

\[ u_i = (1 + o(1))(\lambda - \mu) = r. \]
If an eigenvalue \( u_i \) has negative form, then, by Lemma 3.3 (iv),
\[
s - \varepsilon = -(1 + o(1)) \frac{k - \mu + Dk}{\lambda - \mu} \leq u_i \leq -(1 + o(1)) \frac{k - \mu - Dk}{\lambda - \mu} = s.
\]

To apply Proposition 1.4, it remains to determine \( f_2 \), that is we need to bound the number of restricted eigenvalues in \([r, k]\). We already saw that all such \( u_i \) are of positive form. Using \( \sum u_i^2 = \text{tr}(A^2) = vk \), we see that there are at most \((1 + o(1))vk(\lambda - \mu)^2\) eigenvalues \( u_i \) of positive form in this interval. Using \( \sqrt{v} \cdot k = o((\lambda - \mu)^2) \), we see \( \frac{vk}{(\lambda - \mu)^2} = o(\sqrt{v}) \).

For instance, for \( v = (1 + o(1))q^{11}, k = (1 + o(1))q^9, \lambda = (1 + o(1))q^8, \) and \( \mu = (1 + o(1))q^7 \), Proposition 3.7 shows \( \sigma \geq (\frac{1}{3} + o(1))q^{-2} \).

### 3.5 Cocliques

**Proposition 3.8 (Inertia Bound for ARSGs).** Consider a family of approximately strongly regular graphs \( \Gamma \) such that \( k = o(v), k = o(|\lambda - \mu|^2) \). Then a coclique of \( \Gamma \) has at most size
\[
(1 + o(1)) \left( \frac{vk}{(\mu - \lambda)^2} + \frac{v^2\sigma^2}{k^2} \right).
\]

**Proof.** Suppose without loss of generality that \( \mu > \lambda \). We want to apply the inertia bound from [2.3] with the adjacency matrix, so we count negative eigenvalues. We need to count the number \( g_1 \) of eigenvalues \( u_i \) with \( \nu_i = \Omega(k) \), as these can be negative by Lemma 3.3 (i) and (vi), and we need to count the number \( g_2 \) of eigenvalues of negative form with \( \nu_i = O(k) \).

For \( g_1 \), Lemma 3.1 shows \( g_1 k^2 \leq v^2\sigma^2 \). For \( g_2 \), note that all eigenvalues considered have size \((1 + o(1))(\mu - \lambda)\), so \((1 + o(1))g_2(\mu - \lambda)^2 \leq \sum u_i^2 = \text{tr}(A^2) = vk \) yields the claim.

### 4 Examples

Clearly, strongly regular graphs provide plenty of examples for approximately strongly regular graphs with \( \sigma = 0 \). Let us present examples with small, but nonzero \( \sigma \).

#### 4.1 Very Small Examples

We list, using the classification of small regular graphs [26], the number of connected graphs with smallest \( \sigma \) for given \( v \) and \( k \). The last column contains a common name or a structure description of the automorphism group.
4.2 Some Examples from Literature

Various examples for small $\sigma$ occur in the literature. Here we list some.

(i) In [28] Radziszowski and Xiaodong describe an approximately strongly regular graph with parameters $(127, 42, 11, \mu; \sigma)$ where $\mu \in [14, 16]$ and $\sigma \leq 1$.

(ii) In [7] Bollobás and Thomason construct an approximately strongly regular graph with parameters $(2^r, 2^r - 1 - 1, \lambda, \mu; 1)$ where $\lambda \in [2^r - 2 - 2, 2^r - 2 - 1]$ and $\mu \in [2^r - 2 - 1, 2^r]$.

(iii) In [29, Theorem 4] Shi, Dong, Petersen, and Johansson show that certain graphs related to quantum networks are edge-regular approximately strongly regular graphs with parameter $(v, k, n - 2, \mu; \sigma)$ where $\mu \in [0, 1]$ and $\sigma \leq 1$.

(iv) An edge-regular graph with parameters $(v, k, \lambda)$ is quasi-strongly regular graph with parameters $(v, k, \lambda; \mu_1, \mu_2)$ if $\mu_{ab} \in \{\mu_1, \mu_2\}$ for all nonadjacent vertices $a, b$. For $\mu_1 < \mu_2$, it is approximately strongly regular with parameters $(v, k, \lambda, \mu; \sqrt{\mu_2 - \mu_1})$ where $\mu \in [\mu_1, \mu_2]$, cf. [18].

(v) A $k$-regular graph of order $v$ is a Deza graph with parameters $(v, k, \beta, \alpha)$, where $\alpha \leq \beta$, if $\lambda_{ab}, \mu_{ab} \in \{\alpha, \beta\}$ for all vertices $a, b$. It is approximately strongly regular with parameters $(v, k, \lambda, \mu; \sqrt{\beta - \alpha})$ where $\lambda, \mu \in [\alpha, \beta]$, cf. [19].

(vi) A random $k$-regular graph of order $v$, cf. [6 §2.4], is an approximately strongly regular graph with parameters (roughly) $(v, k, \frac{k^2}{v}, \frac{k^2}{v}; \sqrt{k})$. 

| $v$ | $k$ | $\lambda$ | $\mu$ | $\sigma$ | $nr$ | remarks |
|-----|-----|-----------|-------|---------|------|---------|
| 8   | 3   | 0         | 1.5   | 0.5     | 1    | $D_8$   |
| 10  | 3   | 0         | 1     | 0       | 1    | Petersen graph, $NO_{h,5}^{-}$ |
| 12  | 3   | 0         | 0.75  | $\sim$ 0.43 | 2    | $D_8, D_9$ |
| 14  | 3   | 0         | 0.8   | $\sim$ 0.49 | 9    |         |
| 16  | 3   | 0.625     | 0.34375 | $\sim$ 0.48 | 2    | $D_6, D_9$ |
| 18  | 3   | 0.675     | 0.3571428 | $\sim$ 0.47 | 2    | $D_6, S_3 \rtimes C_2$ |
| 20  | 3   | 0.3       | 0.31875 | $\sim$ 0.47 | 5993 |         |
| 22  | 3   | 0.275     | 0.287  | $\sim$ 0.45 | 86977|         |
| 9   | 4   | 1         | 2     | 0       | 1    | Paley(9)|
| 10  | 4   | 0.75      | 1.8   | $\sim$ 0.43 | 1    | $D_5$   |
| 11  | 4   | 1.09      | 1.27  | $\sim$ 0.44 | 1    | $C_2 \times S_3$ |
| 12  | 4   | 1         | 1.142857 | 0.41    | 1    | $C_2 \times D_4$ |
| 13  | 4   | 0.692307  | 1.153846 | $\sim$ 0.46 | 1    | $D_8$   |
| 14  | 4   | 0.32142857 | 1.190476 | $\sim$ 0.47 | 2    | id, $C_2^2$ |
| 15  | 4   | 0.1       | 1.16  | $\sim$ 0.37 | 1    | $D_6$   |
| 16  | 4   | 0         | 1.09  | 0.36    | 1    | $C_4 \times C_2$ |
| 12  | 5   | 0.7       | 2.75  | $\sim$ 0.46 | 1    | $S_3^2$ |
| 14  | 5   | 1.02857144 | 1.857142 | $\sim$ 0.45 | 1    | $C_2 \times D_4$ |
| 13  | 6   | 2         | 3     | 0       | 1    | Paley(13)|


Let \( \Gamma \) be the orthogonal graph of \( d \)-spaces in \( \mathbb{F}_q^n \), that is the graph with the \( d \)-spaces of \( \mathbb{F}_q^n \) as vertices, two adjacent if they meet nontrivially, cf. \([9]\) §1.2.4. Let the number of \( b \)-spaces in \( \mathbb{F}_q^n \) by \([\binom{n}{b}]_q \). It is easy to see that for \( n \gg 2d \) and \( q \to \infty \), \( \Gamma \) is approximately strongly regular with parameters \((\binom{n}{d}, \binom{n}{d}, q^{2d} \binom{n-d}{d}, \lambda, \mu; 1)\), where \( \lambda = \Theta(\binom{n-d}{d}) \) and \( \mu = \Theta(q^{2(d-1)} \binom{n-d}{d-1}) \).

Consider a 2-(\( V, K, \Lambda \)) design \( D \), cf. \([9]\) §6.2. Let \( \Gamma \) be the graph with the blocks of \( D \) as vertices, two adjacent if they intersect. If \( \Gamma \) is regular, then \( \Gamma \) is approximately regular with parameters \((v, k, \lambda, \mu; \sigma)\) where \( v \sim \frac{V^2}{K^2} \), \( k \sim \Lambda \), \( \lambda \sim \frac{\Lambda}{2V} \), \( \mu \sim \frac{\Lambda^2}{2K^2} \), and \( \sigma \to 0 \) if \( K, \Lambda \) are constant and \( V \to \infty \). Due to the seminal work by Keevash \([24]\), these exist if \( V \) is sufficiently large and some divisibility conditions are satisfied, but we do not know if we can guarantee that each block is disjoint to the same number of blocks, that is the regularity of \( \Gamma \).

It is well-known that if one has equality in the relative (or special) bound for equiangular lines in \( \mathbb{R}^d \), then one obtains a strongly regular graph from the Gram matrix of the set of equiangular lines, cf. \([9]\) §8.14. Similarly, if one is close, then the graph from the Gram matrix has its spectrum concentrated at two values. This is used often, for instance recently in \([20]\) \([21]\). Thus, if the graph is regular, then it is approximately strongly regular with small \( \sigma \). We did not estimate \( \sigma \).

### 4.3 Orthogonality Graphs

Our application in \([5.2]\) is partially motivated by the construction described here.

For \( q \) an odd prime power, let \( V \) be the \( n \)-dimensional vector spaces over \( \mathbb{F}_q \), the finite field with \( q \) elements. As \( q \) is odd, \( \mathbb{F}_q \) contains \( \frac{\sqrt{q}-1}{2} \) (nonzero) squares and \( \frac{\sqrt{q}+1}{2} \) nonsquares. Put \( \gamma = 1 \) if \( q \equiv 1 \mod 4 \) and \( \gamma = -1 \) if \( q \equiv 3 \mod 4 \).

A *quadratic form* over \( \mathbb{F}_q \) is a map \( Q : V \to \mathbb{F}_q \) such that \( Q(\alpha v) = \alpha^2 Q(x) \) for all \( \alpha \in \mathbb{F}_q \) and \( x \in V \) and the function \( B(x, y) := Q(x + y) - Q(x) - Q(y) \) is bilinear. We can find an \((n \times n)\)-matrix \( M \) such that \( Q(x) = x^T M x \). We say that \( Q \) is *nondegenerate* if \( \det(Q) := \det(M) \neq 0 \). From now on we assume that \( Q \) is nondegenerate. For \( x \in V \) nonzero, call \( \langle x \rangle \) a *point*. Call a point \( \langle x \rangle \) *singular* when \( Q(x) = 0 \). Cf. \( \S 2 \) and \( \S 3 \) in \([9]\) and \( \S 1 \) in \([30]\).

If \( n = 2m + 1 \) is odd, then there is only one choice for \( Q \) up to isomorphism.

The set of nonsingular points \( \langle x \rangle \) splits into two parts of sizes \( \frac{1}{2}q^n(q^m + \varepsilon) \) for \( \varepsilon \in \{-1, 1\} \). Here \( \varepsilon \) depends on \( Q(x) \) being a (nonzero) square or a nonsquare. Let \( NO_{\varepsilon,q}^+ \) denote the graph with one of the parts as vertices, two vertices \( x, y \) adjacent when they are orthogonal, that is \( B(x, y) = 0 \). We identify \( \varepsilon = 1 \) with + and \( \varepsilon = -1 \) with −, so we write \( NO_{+q}^+ \) and \( NO_{-q}^+ \). In \([9]\) these graphs are called \( NO_{\varepsilon,q}^+(q) \) for \( q \in \{3, 5\} \).

The automorphism group of \( NO_{\varepsilon,q}^+ \) acts transitively on cliques of a given size as the corresponding orthogonal group acts transitively on tuples of pairwise orthogonal points of the same type. In particular, \( NO_{\varepsilon,q}^+ \) is edge-regular with parameters

\[
v = \frac{1}{2}q^m(q^m + \varepsilon), \quad k = \frac{1}{2}q^{m-1}(q^m - \varepsilon), \quad \lambda = \frac{1}{2}q^{m-1}(q^{m-1} + \gamma \varepsilon).
\]
For $q = 3, 5$, the graph $NO^\perp_{n,q}$ is strongly regular with $\mu = \frac{1}{2}q^{m-1}(q^{m-1} - \varepsilon)$. Standard counting for quadratic forms shows that
\[
\frac{1}{2}q^{m-1}(q^{m-1} - 1) \leq \mu_{xy} \leq \frac{1}{2}q^{m-1}(q^{m-1} + 1).
\]
Hence, for fixed $n$ and $q \to \infty$, the graph $NO^\perp_{n,q}$ is approximately strongly regular with $\sigma \leq (1 + o(1))q^{m-1}$.

If $n = 2m$ is even, then there are two choices for $Q$ up to isomorphism, depending on if $Q$ is of elliptic (put $\varepsilon = -1$) or hyperbolic type (put $\varepsilon = 1$). We can distinguish them by the number of singular points which is $\frac{(q^{m-1}+\varepsilon)(q^{m-1})}{q-1}$. We can also distinguish them by $\det(Q)$ being a (nonzero) square or a nonsquare. The nonsingular points $(x)$ split into two orbits of equal size $\frac{1}{2}q^{m-1}(q^m - \varepsilon)$ each, depending on $Q(x)$ being a square or a nonsquare. Let $NO^{\perp}_{n,q}$ denote the graph with one of the parts as vertices, two vertices adjacent when orthogonal, that is $B(x, y) = 0$. In [9] these graphs are called $NO^\perp_n(q)$ for $q = 3$.

As for $n$ odd, the automorphism group of $NO^\perp_{n,q}$ acts transitively on cliques. In particular, it is edge-regular with parameters
\[
v = \frac{1}{2}q^{m-1}(q^m - \varepsilon), \quad k = \frac{1}{2}q^{m-1}(q^{m-1} - \gamma\varepsilon), \quad \lambda = \frac{1}{2}q^{m-2}(q^{m-1} + \gamma\varepsilon).
\]
For $q = 3$, the graph $NO^{\perp}_{n,q}$ is strongly regular with $\mu = \frac{1}{2}q^{m-1}(q^{m-2} - \varepsilon)$. Standard counting for quadratic forms shows that
\[
\frac{1}{2}q^{m-1}(q^{m-2} - 1) \leq \mu_{xy} \leq \frac{1}{2}q^{m-1}(q^{m-2} + 1).
\]
Hence, for fixed $n$ and $q \to \infty$, the graph $NO^\perp_{n,q}$ is approximately strongly regular with $\sigma \leq (1 + o(1))q^{m-1}$.

There is the following tower of graphs (see [9] p. 89) for the case $q = 3$:

Let $NO^{\perp}_{n,q}(x)$ denote the induced subgraph on the neighborhood of $x$ in $NO^{\perp}_{n,q}$. We find that $NO^{\perp}_{2m+1,q}(x)$ is isomorphic to $NO^{\perp}_{2m,q}$, and that $NO^{\perp}_{2m,q}(x)$ is isomorphic to $NO^{\perp}_{2m-1,q}$. The graph $NO^{\perp}_{2,q}$ is edgeless if $\gamma = -1$, otherwise it is the union of $\frac{q+1}{2}$ pairwise disjoint edges. The graph $NO^{\perp}_{2,q}$ is edgeless if $\gamma = 1$, otherwise it is the union of $\frac{q+1}{2}$ pairwise disjoint edges. By induction, we find that the clique number of $NO^{\perp}_{n,q}$ for $n = 2m + 1$ or $n = 2m$ is $n - 1$ if $\gamma\varepsilon = (-1)^m$ and $n$ if $\gamma\varepsilon = -(-1)^m$.

5 Some Applications

5.1 Large Caps

Let $n \geq 2$ and let $q$ be a prime power. Consider a set of points $\mathcal{C}$ in $PG(n, q)$, the $n$-dimensional projective space over $\mathbb{F}_q$. We use that the number of points in $PG(n, q)$ is $\frac{q^{n+1} - 1}{q - 1}$. If no three points in $\mathcal{C}$ are collinear, then $\mathcal{C}$ is called a cap.

For the regime of $q = 3$ and $n \to \infty$, the cap set problem recently gained much prominence due to the breakthrough result by Ellenberg and Gijswijt, see [10]. Here we consider the regimes where $n$ is fixed and $q \to \infty$ as well as where
lies on a constant number of secants. Then Lemma 5.1. We can say the following about this case.

Proof. We only prove the first part. Let us calculate the negative eigenvalue $t = 2n$. Size this only changes bounds by constant factor. We always assume that $q \geq 3$ as $q = 2$ is trivial. As the calculations for $n$ fixed require slightly more care than those for $q$ fixed (but are essentially identical), we will only include those. It is easy to see that a cap has size at most $(1 + o(1))q^{n-1}$ for $n$ fixed and $q \to \infty$. The largest known constructions for caps have size $\Theta(q^{\frac{3}{2}n})$. This is tight for $n = 2, 3$. See [13, 14] for constructions of caps for large $n$ or large $q$.

It is well-known that caps define graphs in various ways. For a cap $C$ of $\text{PG}(n, q)$, define an associated graph $\Gamma$ as follows: Consider $\mathbb{F}^n_q$ with $\text{PG}(n, q)$ as hyperplane at infinity. Take the vectors of $\mathbb{F}^{n+1}_q$ as vertices, two distinct $a, b \in \mathbb{F}^{n+1}_q$ adjacent if $(a, b)$ meets $\text{PG}(n, q)$ in a point of $C$. Put $t = |C|$. It is well-known and easy to verify that this defines an edge-regular graph with $(v, k, \lambda) = (q^{n+1}, t(q - 1), q - 2)$. An exterior point of $C$ is a point of $\text{PG}(n, q)$ not in $C$ and a secant of $C$ is a line of $\text{PG}(n, q)$ which meets $C$ in precisely two points.

If each exterior point lies on precisely the same number $h$ of secants, then we obtain a strongly regular graph with $\mu = \frac{(t-1)(q-1)^2}{q^{n+1}t(q-1)}$. See §8.7.1(vi) in [9]. We can say the following about this case.

**Lemma 5.1.** Let $C$ be a cap of size $t$ in $\text{PG}(n, q)$ such that each exterior point lies on a constant number of secants. Then

$$
t \leq (1 + o(1))q^{\frac{3}{2}n-\frac{3}{4}} \quad \text{(as $q \to \infty$),}
$$

$$
t = O(q^{s^2}) \quad \text{(as $n \to \infty$).}
$$

**Proof.** We only prove the first part. Let us calculate the negative eigenvalue $s < 0$ of the associated graph $\Gamma$. We find $s = -(1 + o(1))\mu$. One of the Krein conditions, Theorem I.1 requires

$$0 \leq 1 + \frac{s^3}{k^2} - \frac{(s + 1)^3}{(v - k - 1)^2} = 1 - (1 + o(1))\frac{(t^{2}q^{-n-1})^3}{(tq)^2} = 1 - (1 + o(1))\frac{t^4}{q^{3n-1}}.$$

Hence, $t \leq (1 + o(1))q^{\frac{3}{2}n-\frac{3}{4}}$.

We can obtain the same bound using the inertia bound (estimates for $q \to \infty$ only): We find that the multiplicity $g$ of $s$ is $(1 + o(1))k/s^2 = (1 + o(1))q^{3n}/t^4$. As $C$ has a clique of size $t$, we find $t^4 \leq (1 + o(1))q^{3n}$. Lastly, the absolute bound shows $t \leq (1 + o(1))q^{\frac{3}{2}n+\frac{3}{4}}$.

How much can we weaken the condition on the exterior points and secants? From now on let $h$ be the expected number of secants through an exterior point $p$ of $\text{PG}(n, q)$ and let $h_p$ the actual number of secants through $p$.

**Lemma 5.2.** Let $C$ be a cap of size $t$ in $\text{PG}(n, q)$ with an associated approximately strongly regular graph $\Gamma$ with parameters $(v, k, \lambda, \mu; \sigma)$. Then

$$\text{Var}(h_p) = (\frac{1}{q} + o(1))\text{Var}(\mu_{ab}) \quad \text{(as $q \to \infty$),}
$$

$$\text{Var}(h_p) = \Theta(\text{Var}(\mu_{ab})) \quad \text{(as $n \to \infty$).}
$$
Proof. We only show the assertion for \( q \to \infty \). Let \( M \) be as in the introduction, that is all pairs of nonadjacent vertices, and let \( D \) denote the set of exterior points of \( C \). Note that \( |M| = q^{n+1}(q^{n+1} - t(q-1)) \) and that \( |C| \leq (1 + o(1))q^{n-1} \) implies that \( |D| = (1 + o(1))q^n \). If for two distinct nonadjacent vertices \( a, b \) the line \( \langle a, b \rangle \) meets \( D \) in \( p \), then \( 2h_p \) is the number of common neighbors of \( a \) and \( b \). Hence,

\[
\text{Var}(\mu_{ab}) = \frac{1}{M} \sum_{a \neq b} (\mu_{ab} - \mu)^2 = \frac{1}{M} \sum_{p \in D} \sum_{a \neq b, (a, b) \in \text{PG}(n, q) = p} (\mu_{ab} - \mu)^2
\]

\[
= \frac{1}{M} \sum_{p \in D} 4 \cdot q^{n+1}(q - 2) \cdot (h_p - h)^2 = (4 + o(1))q^{-n} \sum_{p \in D} (h_p - h)^2
\]

\[
= (4 + o(1)) \frac{1}{|D|} \sum_{p \in D} (h_p - h)^2 = (4 + o(1))\text{Var}(h_p). \tag{$\square$}
\]

**Proposition 5.3.** For \( n \geq 4 \), let \( C \) be a cap of size \( t \geq (2 + o(1))q^{\frac{3n}{4}} \) in \( \text{PG}(n, q) \) and let \( D \) denote its exterior points.

(i) If \( \text{Var}(h_p) \leq \left( \frac{1}{2} + o(1) \right)\sigma^2 \), then \( \sigma \geq (1 + o(1))t^\frac{2}{3}q^{-n} \) (as \( q \to \infty \)).

(ii) If \( \text{Var}(h_p) = \Theta(\sigma^2) \), then \( \sigma = \Omega(t^\frac{2}{3}q^{-n}) \) (as \( n \to \infty \)).

If the associated graph \( \Gamma \) is also positive-1-flat 1-walk-regular, then we have the following.

(iii) If \( \text{Var}(h_p) \leq \left( \frac{1}{2} + o(1) \right)\sigma^2 \), then \( \sigma \geq (1 + o(1))tq^{-2n+1} \) (as \( q \to \infty \)).

(iv) If \( \text{Var}(h_p) = \Theta(\sigma^2) \), then \( \sigma = \Omega(tq^{-2n}) \) (as \( n \to \infty \)).

**Proof.** Recall that from \( t = \Omega(q^{\frac{3}{4}}) \) and \( n \geq 4 \), we obtain \( k = (q-1)t = o(\mu^\frac{3}{2}) \) and \( \lambda = q - 2 = o(\mu) \) (as \( \mu = (1 + o(1))t^2q^{1-n} \)).

For the first part we apply the inertia bound from Proposition 5.8. We already saw in the discussion on strongly regular graphs that we require \( t \geq (1 + o(1))q^{\frac{3}{4}} \) for the first summand. What remains of Proposition 5.8 is

\[
t \leq (1 + o(1)) \frac{\nu^2 \sigma^2}{k^2} = (1 + o(1)) \frac{q^{2n+2} \sigma^2}{q^2 t^2}.
\]

Rearranging for \( \sigma \) yields \( \sigma \geq (1 + o(1))t^\frac{2}{3}q^{-n} \).

For (iii) and (iv), use Proposition 3.3. We find

\[
\sigma \geq (1 + o(1))(\mu - \lambda)^\frac{3}{2} \cdot e^{-\frac{3}{2}k^\frac{3}{2}} t = t^3 q^{-2n+1}. \tag{$\square$}
\]

One can also use Proposition 3.4. but the resulting bounds on \( \sigma \) are slightly worse than what is stated in Proposition 5.8.

**Proposition 5.3** implies the following.

**Corollary 5.4.** For \( n \geq 4 \), let \( C \) be a cap of size \( t \) in \( \text{PG}(n, q) \). If \( \text{Var}(h_p) = o(q^{\frac{3n}{4}}) \), then \( t = O(q^{\frac{3n}{4}}) \). If the associated graph is positive-1-flat 1-walk-regular, then already \( \text{Var}(h_p) = o(q^{\frac{3n}{4}}) \) implies \( t = O(q^{\frac{3n}{4}}) \).

The above holds for \( q \to \infty \) as well as \( n \to \infty \).
For \( t = \Theta(q^{n-1}) \) (as \( n \to \infty \)), we find \( \sigma = \Omega(q^{n-2}) \). Using Proposition 3.6 instead of Proposition 5.5 only yields a marginal improvement. For instance, for and \( t = \Theta(q^{n-1}) \) (as \( n \to \infty \)), we find \( \sigma = \Omega(q^{-2+1/25}) \).

If we assume that the setwise stabilizer of \( C \) (a subgroup of \( \Gamma L(n, q) \)) acts transitively on \( C \), then \( \Gamma \) is 1-walk-regular. Maybe it is feasible to use Corollary 5.4 to show a bound of the form \( O(q^n) \) for some \( C < 1 \) under some symmetry conditions.

For \( q = 3 \), Edel constructed caps of size \( O(2.21^n) \) [13] and there is an upper bound of \( o(2.76^n) \) by Ellenberg and Gijswijt [10]. For the special cases of Corollary 5.4 we find an upper bound of \( o(2.28^n) \); for Lemma 5.4 we also find \( o(2.28^n) \). In general it is known that \( t \leq (1 - O(q^{-1}))q^{-1} \) (as \( q \to \infty \)), cf. Table 4.4(ii) in [23].

**Explicit Bounds**

We can also find explicit bounds. In the following we will demonstrate this with some crude estimates.

Suppose that we are looking for a cap of size \( t \). We have a coclique of size \( t \) in the associated graph, so we need at least \( t \) nonpositive eigenvalues. Let \( f \) be the number of positive eigenvalues, and let \( g = g_1 + g_2 \) be the number of negative eigenvalues, \( g_1 \) of negative form (so they are at most \( -\frac{1}{2}(\lambda - \mu) \)), \( g_2 \) of positive form. Then we require (using standard trace arguments as before)

\[
g_1 + g_2 \geq t, \quad k^2 + g_1 \cdot \frac{1}{2}(\lambda - \mu)^2 \leq vk, \quad g_2(k - \mu)^2 \leq v(v - k - 1)\sigma^2.
\]

For instance, by [23], a cap in \( PG(10, 3) \) has size at most 10937. The largest known cap in \( PG(10, 3) \) has size 2744 [15]. Put \( t = 10937 \). We find \((v, k, \lambda, \mu) = (3^{11}, 2 \cdot 10937, 1, \frac{29901728}{10937})\). Then \( g_1 \leq 5731 \), so \( g_2 \geq 10937 - 5731 = 5206 \). Hence, \( 5206 \cdot (k - \mu)^2 \leq v(v - k - 1)\sigma^2 \). We obtain that \( \sigma \geq 8.84 \).

### 5.2 Optimally Pseudorandom Clique-free Graphs

A \( k \)-regular graph \( \Gamma \) of order \( v \) is called optimally pseudorandom if the second largest eigenvalue in absolute value of its adjacency matrix is in \( O(\sqrt{k}) \), cf. [25].

**Proposition 5.5** (Alon and Krivelevich, [3]). Let \( \Gamma \) be a \( K_m \)-free \( k \)-regular graph of order \( v \) with smallest eigenvalue \( s \) such that \( -s = O(\sqrt{k}) \). Then

\[
k = O(v^{1 - \frac{1}{m-1}}).
\]

This bound is tight for \( m = 3 \) due to a construction by Alon [2]. Alon and Krivelevich gave an example with \( k = \Theta(v^{1 - \frac{1}{m-1}}) \) [3]. The author noticed that there is a well-known construction with \( k = \Theta(v^{1 - \frac{1}{m-1}}) \) [11]. These are the graphs \( NO_{m,q}^{c,\perp} \) with clique number \( m - 1 \).

The Ramsey number \( R(m, n) \) is the largest number such that there exists a graph on \( R(m, n) - 1 \) vertices without a clique of size \( m \) or a coclique of size \( n \). Ajtai, Komlós, and Szemerédi [11] and Bohman and Keevash [5] proved

\[
\Omega \left( \frac{n^{m-1}}{(\log n)^{m-2}} \right) = R(m, n) = O \left( \frac{n^{m-1}}{(\log n)^{m-2}} \right) \quad \text{(as } n \to \infty).\]

Recently, Mubayi and Verstraëte showed in [27] that if the upper bound in Proposition 5.5 is tight for some \( m \), then \( R(m, n) = \Omega \left( \frac{n^{m-1}}{(\log n)^{m-2}} \right) \), nearly
matching the upper bound. Their result also implies that if one finds a construction with \( k = \Omega(v^{1 - \frac{1}{m+1}}) \) for some \( \varepsilon > 0 \), then
\[
R(m,n) = \Omega \left( \frac{n^{m+1}}{(\log n)^{m+1}} \right)
\]
(as \( n \to \infty \)),
which would improve the lower bound on \( R(n,m) \). Our technique here cannot show anything better than \( k = O(v^{1 - \frac{1}{m+1}}) \).

For the remainder of the section consider an optimally pseudorandom \( K_m \)-free \( k \)-regular graph \( \Gamma \) of order \( v \), smallest eigenvalue \( \sigma \), and second largest eigenvalue \( r \), where \( m \geq 3 \). Let \( Y \) be a clique of size \( i \) of \( \Gamma \). Let \( \Gamma(Y) \) be the induced subgraph on the common neighborhood of \( Y \). Let us define the following properties for \( \Gamma_i := \Gamma(Y) \).

(P1) The graph \( \Gamma_i \) has \( v_i \) vertices and \( \frac{1}{2}v_ik_i \) edges, where
\[
k_i \geq (1 + o(1))k \left( \frac{v_i}{v} \right)^i, \quad (1 + o(1)) \frac{v_i}{v} \leq \frac{v_i}{v_i} = o(1).
\]
(P2) The graph \( \Gamma_i \) is approximately strongly regular with parameters \((v_i, k_i, \lambda_i, \mu_i; \sigma_i)\) and its smallest eigenvalue \( \sigma_i \) satisfies
\[
-s_i = O(\mu_i - \lambda_i).
\]

Clearly, the graphs \( NO_{m,q}^{\perp} \) with clique number \( m - 1 \) satisfy (P1). Furthermore, as mentioned in [14,3] the automorphism group of \( NO_{m,q}^{\perp} \) acts transitively on cliques of a given size. Hence, \( \Gamma_i \) is regular. The graphs \( NO_{m,q}^{\perp} \) with clique number \( m - 1 \) often have property (P2). We will see that property (P2) follows from (P1) for \( i = m - 3 \) when \( \Gamma_{m-3} \) is regular as \( \Gamma_{m-3} \) is triangle-free, so \( \lambda_{m-3} = 0 \). More generally, some \( \Gamma_i \) must have property (P2) as \( \lambda_i \leq (D + o(1)) \frac{v_i}{v} \) has to occur for some \( D < 1 \) for some \( i \).

Let us state the expander-mixing lemma for the special case of only one set, see the proof of Proposition 1.1.6 in [9].

**Lemma 5.6** (Expander-Mixing Lemma, Variant). Let \( Y \) be a set of vertices of size \( y \) of a \( k \)-regular graph \( \Gamma \) of order \( v \) with second largest eigenvalue \( r \) and smallest eigenvalue \( s \). Then the number \( e \) of edges in the induced subgraph on \( Y \) satisfies
\[
\frac{1}{2}y \left( \frac{\omega(k-r)}{v} + s \right) \leq e \leq \frac{1}{2}y \left( \frac{\omega(k-r)}{v} + r \right).
\]

**Lemma 5.7.** Let \( 0 \leq i \leq m - 3 \).

(i) If \( k = \omega(v^{1 - \frac{1}{m+1}}) \), then there exists a \( \Gamma_i \) with property (P1).

(ii) If \( \Gamma_i \) is regular and has property (P1), then
\[
-s_i = \Omega \left( k_i \left( \frac{k_i}{v_i} \right)^{m-1-2} \right).
\]

**Proof.** First we show (i). For this, let \( k_i \) denote the average degree of \( \Gamma_i \). Clearly, the claim is true for \( i = 0 \). The condition \( k = \omega(v^{1 - \frac{1}{m+1}}) \) is equivalent to
\[
(1 + o(1))k \left( \frac{v_i}{v} \right)^i = \omega(\sqrt{v_i}). \tag{4}
\]
Suppose that the claim is true for $\Gamma_i = \Gamma(Y)$ for some clique $Y$ of size $i - 1$. Let $a$ be a vertex of $\Gamma_{i-1}$ of degree at least $k_{i-1}$. Take $\Gamma_i = \Gamma(Y \cup \{a\})$. By Lemma 5.6 applied to $\Gamma_i$, using Equation (4), the average degree $k_i$ of a vertex in $\Gamma_i$ satisfies

$$(1 + o(1))k_i \left(\frac{k_i}{v}\right)^i \leq \frac{k_{i-1}(k_i - s)}{v} + s \leq \frac{k_{i-1}(k_i - r)}{v} + r = (1 + o(1))k_{i-1} \cdot \frac{k_i}{v}.$$  

This shows property (P1) for $\Gamma_i$.

Next we show (ii). For some $j$ with $i \leq j \leq m - 3$ we require that $\lambda_j < (D + o(1))\frac{k_j}{v_j}$ for some constant $D < 1$ as $\Gamma_i$ is $K_m$-free. Similarly to the above, for the first $j$ for which this occurs, we find a $\Gamma_j$ with

$$k_j = (1 + o(1))k_i \left(\frac{k_i}{v_i}\right)^{j-i}.$$  

By Lemma 5.6 applied to the regular graph $\Gamma_i$,

$$(D + o(1))\frac{k_j^2}{v_j} \geq \lambda_j \geq (1 + o(1))k_i \left(\frac{k_i}{v_i}\right)^{j-i+1} + s_i.$$  

In the worst case is $j = m - 3$ which yields the claim. \hfill \Box

**Proposition 5.8.** Suppose that $\Gamma$ has $k = \omega(v^{1 - \frac{1}{m-1}})$. Furthermore, suppose that $\Gamma_i$ has property (P1), and that $\Gamma_i$ has property (P2) or $i = m - 3$. If $\Gamma_i$ is regular, then $\sigma_i = \Omega(k \left(\frac{k}{v}\right)^{m-2} \cdot \frac{k_i}{m - 3})$. If $\Gamma_i$ is also positive-1-flat 1-walk-regular, then $\sigma_i = \Omega(k \left(\frac{k}{v}\right)^{m-1} \cdot \frac{k_i}{m - 4}).$

**Proof.** If $\Gamma_i$ has property (P1) and is regular, then it is an approximately strongly regular graph with parameters $(v_i, k_i, \lambda_i, \mu_i; \sigma_i)$ with $\mu_i = (1 + o(1))\frac{k_i}{v_i}$ (as $k_i = o(v)$). If $\Gamma_{m-3}$ has property (P1), then $\lambda_{m-3} = 0$ and $\mu_{m-3} = (1 + o(1))\frac{k_{m-3}}{v_{m-3}}$, so $\Gamma_{m-3}$ has property (P2).

By Lemma 5.6(ii) and property (P2),

$$\mu_i - \lambda_i = \Omega(k_i \left(\frac{k_i}{v_i}\right)^{m-2}).$$  

From $k = \omega(v^{1 - \frac{1}{m-1}})$ and $k_i \geq (1 + o(1))k_i \left(\frac{k_i}{v_i}\right)^i$, we obtain that $k_i = o(|\mu_i - \lambda_i|^{\frac{1}{2}})$. By Proposition 5.4

$$\sigma_i \geq (1 + o(1))\frac{|\mu_i - \lambda_i|}{v_i} = \Omega \left(k_i \left(\frac{k_i}{v_i}\right)^{m-2} \right).$$  

This shows the general case. The case with $\Gamma_i$ positive-1-flat 1-walk-regular uses Proposition 5.5 and is otherwise similar.

For $i = m - 3$, we can also use the inertia bound.

**Proposition 5.9.** Let $m \geq 5$. Let $\Gamma_{m-3}$ be as in Proposition 5.8. If $\sigma_{m-3} = \Omega \left(k \left(\frac{k}{v}\right)^{m-2} \right)$, then $k = O(v^{1 - \frac{1}{m-1}}).$
Proof. Apply Proposition 3.8 with \(v = v_{m-3} = k_{m-2}, k = k_{m-3},\) and 

\[
\mu - \lambda = \Omega \left( k \left( \frac{k}{v} \right)^{m-2} \right).
\]

Hence, with the chosen \(\sigma\) we cannot have a coclique of size \(k_{m-3}\). \(\square\)

For \(i \leq \frac{3}{4}m - \frac{3}{2}\), we have \(k = \omega(v^{1 - \frac{1}{m-1}})\) in Proposition 5.8 so we can apply Lemma 5.7(i) and see that there exists a \(\Gamma_i\) with property (P1). Hence, the case \(i = \lfloor \frac{3}{4}m - \frac{3}{2} \rfloor\) is special.

**Corollary 5.10.** Let \(m \geq 5\). Let \(\Gamma_i\) be as in Proposition 5.8

(i) If \(i = \frac{3}{4}m - \frac{3}{2}\), and \(\sigma_i = o(k^{\frac{1}{2}} \left( \frac{k}{v} \right)^{m - \frac{1}{2}})\), then \(k = O(v^{1 - \frac{1}{m-1}})\).

(ii) If \(i = m - 3\) and \(\sigma_{m-3} = o(k^{\frac{1}{2}} \left( \frac{k}{v} \right)^{m - \frac{3}{2}})\), then \(k = O(v^{1 - \frac{1}{m-1}})\).

If \(\Gamma_i\) is also positive-1-flat 1-walk-regular, then \(\sigma_i = o(k^{\frac{1}{2}} \left( \frac{k}{v} \right)^{m - 1})\), respectively, \(\sigma_i = o(k^{\frac{1}{2}} \left( \frac{k}{v} \right)^{m - \frac{1}{2}})\) suffice in (i), respectively, (ii).

### 5.3 SRGs as Counterexamples

Glock, Janzer, and Sudakov ask in the conclusion of [17] for a family of clique-free strongly regular graphs with large \(\lambda\) to disprove several conjectures in extremal combinatorics. To our knowledge no such graph is known. Approximately strongly regular graphs with small \(\sigma\) are equally suitable for this task.

### 6 Future Work

There are countless results specific to strongly regular graphs. Generalizing them to approximately strongly regular graphs seems to be a worthwhile endeavor.

Maybe one can improve the bounds given here: our variant of the absolute bound, Proposition 5.7, is not very satisfying compared to our Krein bounds.

Bounds on approximately equiangular lines might be helpful here. This is not a completely new topic, for instance, constructions for almost equiangular lines were investigated in [10].

While anything the author tries to construct will usually satisfy the conditions of Proposition 2.1 or Proposition 3.5 (or be very close to it), the setup seems overly technical and hard to verify compared to Proposition 1.3 and Proposition 3.4. Maybe it can be simplified. There is also the question if our results – using the usual connections between caps, strongly regular graphs, and linear codes – has any interesting implications for coding theory.

One can ask several existence questions, for instance:

(i) For given \((v, k, \lambda, \mu)\), what is the smallest \(\sigma\) such that an approximately strongly regular graph with parameters \((v, k, \lambda, \mu; \sigma)\) exists?

(ii) For a given strongly regular graph with parameters \((v, k, \lambda, \mu)\), what is the smallest \(\sigma\) such that an approximately strongly regular graph with parameters \((v, k, \lambda, \mu; \sigma)\) exists which is not strongly regular?
Allen W. Herman suggested (ii) and some variants. These questions and also some examples in §4.2 suggest that a nonregular version of approximately strongly regular graphs might be interesting.

More formally, for a vertex \( a \), let \( k_a \) denote its degree. Call a graph \( \Gamma \) nonregular approximately strongly regular with parameters \((v, k, \lambda, \mu; \sigma)\) the same as for an approximately strongly regular graph, except that we no longer require \( \Gamma \) to be \( k \)-regular, just \( k = \frac{1}{v} \sum a k_a \) and \( \text{Var}(k_a) := \frac{1}{v} \sum_a (k_a - k)^2 \leq \sigma^2 \). It might be interesting to investigate how to make a nonregular approximately strongly regular graph regular without changing its parameters too much.

Our primary motivation for this document is to restrict the search space when looking for constructions for specific extremal problems. Maybe the techniques in this paper can be expanded to obtain more general bounds on caps and optimally pseudorandom clique-free graphs. At the time of writing, the author holds the weak belief that Corollary 5.4 and Corollary 5.10 state true upper bounds for the respective general cases.

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