SYMBOLIC ARITHMETIC AND INTEGER FACTORIZATION

SAMUEL J. LOMONACO

Abstract. In this paper, we create a systematic and automatic procedure for transforming the integer factorization problem into the problem of solving a system of Boolean equations. Surprisingly, the resulting system of Boolean equations takes on a “life of its own” and becomes a new type of integer, which we call a generic integer.

We then proceed to use the newly found algebraic structure of the ring of generic integers to create two new integer factoring algorithms, called respectively the Boolean factoring (BF) algorithm, and the multiplicative Boolean factoring (MBF) algorithm. Although these two algorithms are not competitive with current classical integer factoring algorithms, it is hoped that they will become stepping stones to creating much faster and more competitive algorithms, and perhaps be precursors of a new quantum algorithm for integer factoring.

Contents

1. Introduction 2
2. Lopsided division 3
3. Generic arithmetic: “Algebraic parallel processing” 9
4. Generic lopsided division and the Boolean factoring (BF) algorithm 14
5. Examples of the application of the BF algorithm 16
6. The multiplicative Boolean factoring (MBF) algorithm 20
7. A method for solving scarcely satisfiable Boolean equations 20
8. A topdown overview: The “big picture” 23
9. Conclusions and open questions 26
10. Appendix. Generic integer multiplication defined in terms of fundamental Boolean operations 27
References 30

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1. Introduction

This paper is the result of a research program seeking to gain a better qualitative and quantitative understanding of the computational complexity of integer factorization. By the phrase “computational complexity” is meant the **Boolean complexity**, i.e., the minimum number of fundamental Boolean operations required to factor an integer, as a function of integer size. The strategy chosen for accomplishing this research objective was to develop a systematic and automatic procedure for the decomposition of arithmetic operations into Boolean operations.

The results of this endeavor produced unanticipated results. As expected, what resulted was a conversion of the problem of integer factoring into the problem of solving a system of Boolean equations. But these resulting systems of Boolean equations unexpectedly took on a “life of their own,” and became a new type of integer in their own right. We call this new type of integer a **generic integer**, and the corresponding algebraic object \( \mathbb{G} \langle x \rangle \), the **ring of generic integers**. Such a ring is a fascinating mix of characteristic 0 and characteristic 2 algebraic structure.

Yet another surprise was that this symbolic approach to integer factoring naturally led to the larger context of the ring of **dyadic integers** \( \mathbb{Z}_2 \), and then on to the corresponding ring of **generic dyadic integers** \( \mathbb{G}_2 \langle x \rangle \). The advantage of looking at integer factorization within the larger context of the dyadics is that every odd (generic) integer dyadic has a (generic) dyadic integer inverse. This naturally led to the creation of the **(generic) lopsided division algorithm** for computing inverses of odd integers in the dyadic integers \( \mathbb{Z}_2 \) (in the generic dyadic integers \( \mathbb{G}_2 \langle x \rangle \)).

This in turn led to the creation of the **Boolean factoring (BF) algorithm**, which systematically and automatically translates the problem of factoring an integer \( N \) into the problem of solving a system of Boolean equations. These Boolean equations were obtained by using generic lopsided division to divide the integer \( N \) by carefully chosen odd generic integer \( x \). The sought system of Boolean equations is simply the generic remainder resulting from this division.

Next it is noted that the generic inverse \( x^{-1} \) of \( x \) need only be computed once, and then used over and over again. This immediately leads to the creation of a second factoring algorithm, the **multiplicative Boolean factoring (MBF) algorithm**, which simply computes the generic product

\[
N \cdot x^{-1},
\]

to produce the system of Boolean equations to be solved.

A general framework (i.e., scarce satisfiability) for solving the system of Boolean equations produced by the BF and MBF algorithms is then discussed. This framework was later used by Gamal Abdali to create a LISP implementation of the BF algorithm. Sumeet Bagde then extended these methods by using binary decision diagrams (BDDs)\(^2\) to create a Mathematica program that also implemented the BF algorithm.
Both the LISP and Mathematica programs were used to factor a large number of integers. The runtime statistics indicated that the BF algorithm runs in exponential time, and hence, is not competitive with the best classical factoring algorithms. For an algebraic basis as to why this is the case, we refer the reader to the topdown overview given toward the end of this paper.

Open questions and future possible research directions are discussed in the conclusion of this paper. Connections with the satisfiability problem are also discussed.

2. Lopsided division

Let $\mathbb{Z}_{(2)}$ denote the ring of dyadic integers, and let $\mathbb{Z}$ denote its subring of all rational integers (i.e., its subring of all standard integers.)

Given below are examples of the dyadic expansion of some rational integers. Please note that the dyadic expansion of a non-negative rational integer is the conventional radix 2 expansion. The dyadic expansion of a negative rational integer is an “infinite 2’s complement” of the radix 2 expansion of its absolute value.

\[
\begin{align*}
5 &= \ldots 00101 \\
4 &= \ldots 00100 \\
3 &= \ldots 00011 \\
2 &= \ldots 00010 \\
1 &= \ldots 00001 \\
0 &= \ldots 00000 \\
-1 &= \ldots 11111 \\
-2 &= \ldots 11110 \\
-3 &= \ldots 11101 \\
-4 &= \ldots 11100 \\
-5 &= \ldots 11011 \\
\end{align*}
\]

Every odd dyadic integer is a unit in the ring of dyadic integers $\mathbb{Z}_{(2)}$, i.e., given any odd dyadic integer $a$, there exists a unique dyadic integer $a^{-1}$ such that

\[a \cdot a^{-1} = 1\]

Thus,
Proposition 1. Let \( b \) be an odd dyadic integer, and let \( a \) be an arbitrary dyadic integer. Then \( a \) divided by \( b \), written \( a/b \), is also a dyadic integer.

Corollary 1. Let \( b \) be an odd rational integer, and let \( a \) be an arbitrary rational integer. Then \( a \) divided by \( b \), written \( a/b \), is a well defined dyadic integer. Moreover, the dyadic integer \( a/b \) is a rational integer if and only if \( b \) is an exact divisor of \( a \) in the ring of rational integers \( \mathbb{Z} \).

Definition 1. Let \( \ldots, a^{(2)}, a^{(1)}, a^{(0)} \) be a sequence of dyadic integers, and let \( a^{(i)}_j \) denote the \( j \)-th bit of the dyadic expansion of \( a^{(i)} \). Then the sequence \( \ldots, a^{(2)}, a^{(1)}, a^{(0)} \) is said to be convergent provided for each \( j \geq 0 \) there exists a non-negative integer \( n = n(j) \) such that \( a^{(i)}_j = a^{(n(j))}_j \) for \( i \geq n(j) \). Otherwise, the sequence is said to be divergent. If the above sequence is convergent, its limit, written \( \lim_{i \to \infty} a^{(i)} \), is said to exist, and is defined as the dyadic integer with dyadic expansion given by \( \left( \lim_{i \to \infty} a^{(i)} \right)_j = a^{(n(j))}_j \).

Remark 1. The above limit is the standard limit in the valuation topology.

Definition 2. Let \( a \) be a dyadic integer with dyadic expansion \( \ldots, a_2, a_1, a_0 \). Then let \( S : \mathbb{Z}(2) \to \mathbb{Z}(2) \) denote a left shift by 1 bit, i.e., \( Sa = \ldots, a_2, a_1, a_0, 0 \).
Remark 2. Hence, \( Sa \) is the same as a multiplied by the dyadic
\[
2 = \ldots 0 0 0 1 0
\]

Definition 3. Let \( a, b, \) and \( c \) be dyadic integers with dyadic expansions
\[
\begin{align*}
\ldots, a_2, a_1, a_0 \\
\ldots, b_2, b_1, b_0 \\
\ldots, c_2, c_1, c_0
\end{align*}
\]
respectively. Define the first bitwise symmetric function of \( a, b, c \), written
\[
\text{Bitwise}_1(a, b, c)
\]
as the dyadic with \( i \)-th bit given by
\[
\sigma_1(a_i, b_i, c_i) = a_i + b_i + c_i
\]
where “\( + \)” denotes the exclusive “or” binary operation. Define the second bitwise symmetric function of \( a, b, c \), written
\[
\text{Bitwise}_2(a, b, c)
\]
as the dyadic with \( i \)-th bit given by
\[
\sigma_2(a_i, b_i, c_i) = (a_i \circ b_i) + (b_i \circ c_i) + (c_i \circ a_i)
\]
where “\( \circ \)” again denotes exclusive “or” and “\( \circ \)” denotes logical “and”.

We are now ready to define an algorithm called lopsided division.

Theorem 1. Let \( b \) be an odd dyadic integer, and let \( a \) be an arbitrary dyadic integer, with dyadic expansions
\[
\begin{align*}
\ldots, b_2, b_1, b_0 \\
\ldots, a_2, a_1, a_0
\end{align*}
\]
respectively (where \( b_0 = 1 \)). Let
\[
\begin{align*}
c^{(0)} &= a \\
\text{borrows}^{(0)} &= 0
\end{align*}
\]
and
\[
\begin{align*}
c^{(i+1)} &= \text{Bitwise}_1 \left( c^{(i)}, \text{borrows}^{(i)}, c_i^{(i)} \circ S^i b \right) \\
\text{borrows}^{(i+1)} &= S \cdot \text{Bitwise}_2 \left( c^{(i)*}, \text{borrows}^{(i)}, c_i^{(i)} \circ S^i b \right)
\end{align*}
\]
where \( c^{(i)*} \) denotes the complement of \( c^{(i)} \), i.e.,
\[
c^{(i)*} = 1 + c^{(i)} \quad \text{for } j \geq 0
\]
where ‘\(\lor\)’ denotes exclusive “or”, and where \(c_i^{(i)} \odot S^i b\) denotes the bitwise logical “and”, i.e.,

\[
(c_i^{(i)} \odot S^i b)_j = c_i^{(i)} \odot (S^i b)_j
\]

Then the sequences \(c^{(i)}\) and \(\text{borrows}^{(i)}\) are convergent, converging respectively to:

\[
\begin{align*}
\lim_{i \to \infty} c^{(i)} &= a/b \\
\lim_{i \to \infty} \text{borrows}^{(i)} &= 0
\end{align*}
\]

This algorithm for computing \(a/b\) is called \textit{lopsided division}.

\[\text{Remark 3.} \text{ Please note that this is an algorithm in the sense that}
\]

\[
(a/b)_j = c_j^{(i)}
\]

\[\text{Definition 4. Let } a \text{ be a positive integer. The } \text{length} \text{ of } a, \text{ written}
\]

\[
\text{\text{\texttt{lgth}}}(a),
\]

\[
\text{is defined as}
\]

\[
\text{\text{\texttt{lgth}}}(a) = j + 1
\]

where \(j\) is the largest non-negative integer such that \(a_j = 1\).

\[\text{Corollary 2. Let } b \text{ be a positive odd rational integer, and let } a \text{ be an arbitrary positive rational integer. Let}
\]

\[
\Gamma = 1 + \text{\text{\texttt{lgth}}}(a) - \text{\text{\texttt{lgth}}}(b)
\]

Then \(b\) is an exact divisor of \(a\) if and only if

\[c^{(\Gamma)} = \text{\text{\texttt{borrows}}^{(\Gamma)}}.\]

If \(b\) is an exact divisor of \(a\), then the radix 2 expansion of \(a/b\) is

\[
a/b = c_{\Gamma-1}^{(\Gamma-1)} c_{\Gamma-2}^{(\Gamma-2)} \cdots c_1^{(1)} c_0^{(0)}.
\]

\[\text{Example 1. Lopsided division of } a = 209 \text{ by } b = 19. \text{ The radix 2 representations of 209 and 19 are respectively:}
\]

\[
\begin{align*}
\{ & a = 11010001 \\
& b = 10011
\end{align*}
\]
The lopsided division of $209$ by $19$ is given in the tableau below:

\[
\begin{array}{cccccc}
& c_3^{(3)} & c_2^{(2)} & c_1^{(1)} & c_0^{(0)} \\
\hline
a &=& a/b & = & 10011 &=& b \\
\hline & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & & & & & & \\
\end{array}
\]

Please note that
\[
1 + \text{lgth}(209) - \text{lgth}(19) = 1 + 8 - 5 = 4
\]
and
\[
c^{(4)} = \text{borrows}^{(4)}
\]

Hence, $19$ is an exact divisor of $209$, and
\[
209/19 = 11 \text{ (base 10)} = 10011 \text{ (base 2)}.
\]

**Example 2.** Lopsided division of $209$ by $21$.

\[
\begin{array}{cccccc}
& c_3^{(3)} & c_2^{(2)} & c_1^{(1)} & c_0^{(0)} \\
\hline
a &=& a/b & = & 10101 &=& b \\
\hline & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & & & & & & \\
\end{array}
\]

Please note that
\[
1 + \text{lgth}(209) - \text{lgth}(21) = 1 + 8 - 5 = 4
\]
and
\[
\ne^{(4)} = \text{borrows}^{(4)}
\]
Hence, \( \frac{a}{b} = \frac{209}{21} \) is not a rational integer, and \( b = 21 \) is not an exact divisor of \( a = 209 \). So

\[
1101 \text{(base 2)} \neq \left\lfloor \frac{209}{21} \right\rfloor
\]

**Example 3.** Lopsided division of 209 by 17.

Please note that

\[
1 + \text{lgth}(209) - \text{lgth}(17) = 1 + 8 - 5 = 4
\]

and

\[
\text{c}(4) \neq \text{borrows}(4)
\]

Hence, 17 is not an exact divisor of 209.
Example 4. Lopsided division of 513 by 27.

\[
\begin{array}{ccccccc}
\text{c}(5) & c(4) & c(3) & c(2) & c(1) & c(0) & a/b \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & & \\
1 & 1 & 0 & 1 & 0 & & \\
1 & 1 & 0 & 1 & 1 & & \\
0 & 1 & 1 & 0 & 0 & & \\
1 & 1 & 1 & & & & \\
0 & 0 & 0 & 0 & 0 & & \\
1 & 1 & 1 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & & \\
1 & & & & & & \\
0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & & \\
\end{array}
\]

Please note that

\[
1 + \text{lgth}(513) - \text{lgth}(27) = 1 + 10 - 5 = 6
\]

and

\[
c(6) = \text{borrows}(6)
\]

Hence, 27 is an exact divisor of 513. Moreover

\[
513/27 = 19 = 10011 \text{ (base 2)}
\]

3. Generic arithmetic: “Algebraic parallel processing”

We now lift the algebraic operations on the ring of rational integers \(\mathbb{Z}\) and on the ring of dyadics integers \(\mathbb{Z}_2\) to the generic level by creating respectively the ring \(\mathbb{G}(x)\) of generic rational integers and the ring \(\mathbb{G}_2(x)\) of generic dyadic integers. It will then be observed that each arithmetic operation in these generic rings, \(\mathbb{G}(x)\) and \(\mathbb{G}_2(x)\), is equivalent to performing many simultaneous arithmetic operations in the corresponding respective rings of rational integers \(\mathbb{Z}\) and dyadic integers \(\mathbb{Z}_2\). This is what is meant by the phrase “algebraic parallel processing.” (Please refer to section 7 for a topdown overview.)
Definition 5. A Boolean ring $\mathbb{B}$ (with addition denoted by “$\pm$” and with multiplication denoted by “$\circ$”) is a ring with multiplicative identity, denoted by “1”, such that each element $a$ of $\mathbb{B}$ is an idempotent, i.e., such that

$$a^2 = a.$$ 

It follows that $\mathbb{B}$ is a commutative ring, and that each element $a$ of $\mathbb{B}$ is its own additive inverse, i.e.,

$$a + a = 0,$$

where “0” denotes the additive identity of $\mathbb{B}$. The additive operation will often be referred to as exclusive “or” and the multiplicative operation “$\circ$” will often be referred to as logical “and”. The complement of an element $a$ of $\mathbb{B}$, written $a^*$, is defined as

$$a^* = 1 + a.$$ 

Definition 6. Let $\nu$ be an arbitrary but fixed non-negative integer. Let $x_{\nu-1}, \ldots, x_2, x_1, x_0$ denote a finite sequence of $\nu$ distinct symbols, let $x$ denote the set of these symbols, and let

$$\mathbb{B} \langle x \rangle$$

denote the free Boolean ring on the symbols in $x$. The elements of $x$ are called the free basis elements of $\mathbb{B} < x >$ and $x$ is called the free basis of $\mathbb{B} < x >$.

Remark 4. Thus, $\mathbb{B} \langle \emptyset \rangle$ denotes the free Boolean ring on the empty free basis. Hence, $\mathbb{B} \langle \emptyset \rangle$ may be identified with the field of two elements $\mathbb{F}_2$.

A generic dyadic integer $e$ is an infinite sequence

$$e = \ldots, e_2, e_1, e_0$$

of elements $e_i$ of $\mathbb{B} \langle x \rangle$. If there exists an integer $k$ such that

$$e_i = e_k$$

for all $i \geq k$, then $e$ is called a generic rational integer.

Let

$$\mathbb{G}(2) = \mathbb{G}(2) \langle x \rangle = \mathbb{G}(2) \langle x_{\nu-1}, \ldots, x_2, x_1, x_0 \rangle$$

and

$$\mathbb{G} = \mathbb{G} \langle x \rangle = \mathbb{G} \langle x_{\nu-1}, \ldots, x_2, x_1, x_0 \rangle$$

denote respectively the set of all generic dyadic integers and the set of all generic rational integers.

The generic integers

$$\begin{cases} 0 = \ldots, 0, 0, 0, 0 \\ 1 = \ldots, 0, 0, 0, 1 \end{cases}$$

will be called zero and one, respectively.
Remark 5. Please note that the dyadic integers and the rational integers lie in the set of generic dyadic integers and the set of generic rational integers, respectively.

Definition 7. A \textit{generic integer} \( e \) such that
\[ e_i = 1 \]
for almost all \( i \) will be said to be \textit{negative}. A generic integer \( e \) not equal 0 such that
\[ e_i = 0 \]
for almost all \( i \) will be said to be \textit{positive}. The generic integer \( e \) will be said to be \textit{non-negative} if \( e \) is either 0 or positive.

Remark 6. Please note that there are generic rational integers which are neither positive nor negative nor non-negative.

Let \( a \) and \( b \) be generic dyadic integers. The \textit{component-wise exclusive “or”} of \( a \) and \( b \), written \( a \lor b \), is defined as:
\[ a \lor b = \ldots, a_2 \lor b_2, a_1 \lor b_1, a_0 \lor b_0 \]
where \( a_i \lor b_i \) denotes the exclusive “or” of the \( i \)-th components of \( a \) and \( b \).

The \textit{component-wise logical “and”} of \( a \) and \( b \), written \( a \land b \), is defined as:
\[ a \land b = \ldots, a_2 \land b_2, a_1 \land b_1, a_0 \land b_0 \]
where \( a_i \land b_i \) denotes the logical “and” of the \( i \)-th components of \( a \) and \( b \).

The \textit{component-wise complement} of \( a \), written \( a^\ast \), is defined as:
\[ a^\ast = \ldots, a^\ast_2, a^\ast_1, a^\ast_0 \]
where \( a^\ast_i \) denotes the complement of the \( i \)-th component of \( a \).

Let \( \alpha \) be an element of the free Boolean ring \( \mathbb{B} \langle x \rangle \). Then the \textit{scalar product} of \( \alpha \) and \( a \), written \( \alpha \cdot a \), is defined as
\[ \alpha \cdot a = \ldots, \alpha \cdot a_2, \alpha \cdot a_1, \alpha \cdot a_0, \]
where \( \alpha \cdot a_i \) denotes the logical “and” of \( \alpha \) and the \( i \)-th component of \( a \).

The \textit{unit left shift} of \( a \), written \( Sa \), is defined as:
\[ Sa = \ldots, a_2, a_1, a_0, 0. \]

Let
\[ \ldots, e^{(2)}, e^{(1)}, e^{(0)} \]
be an infinite sequence of elements of \( \mathbb{G}^{(2)} \langle x \rangle \). If for every \( j \geq 0 \), there exists a non-negative integer \( n(j) \) such that
\[ e_j^{(i)} = e_j^{(n(j))} \quad \text{for } i \geq n(j) \]
then the sequence is said to be **convergent**. Otherwise, it is said to be **divergent**. If the above sequence is convergent, its **limit**, written

\[ \lim_{i \to \infty} e^{(i)} \]

is said to **exist**, and is defined as the generic dyadic integer

\[ \lim_{i \to \infty} e^{(i)} = \ldots, e_2^{(n(2))}, e_1^{(n(1))}, e_0^{(n(0))} \]

**Definition 8.** An **instantiation** is a mapping

\[ \Phi : x \rightarrow B \langle \rangle , \]

where \( B \langle \rangle \) denotes the free Boolean ring on the empty set of symbols. Hence, \( B \langle \rangle \) may be identified with the field of two elements \( F_2 \). Since \( B \langle x \rangle \) is free on \( x \), every instantiation uniquely and naturally extends to a Boolean ring epimorphism

\[ \Phi : B \langle x \rangle \rightarrow B \langle \rangle \]

which again is called an **instantiation**. Moreover, each instantiation uniquely extends to epimorphisms:

\[
\begin{align*}
\Phi : G \langle x \rangle &\rightarrow \mathbb{Z} \\
\Phi : G_2 \langle x \rangle &\rightarrow \mathbb{Z}_2
\end{align*}
\]

which are also called **instantiations**.

The following will be helpful in proving theorems:

**The Principle of Instantiation.**

**a1):** Let \( a \) and \( b \) be elements of \( B \langle x \rangle \). If for every instantiation \( \Phi \),

\[ \Phi(a) = \Phi(b) , \]

then \( a = b \).

**a2):** Let \( \Phi \) and \( \Omega \) be instantiations. If for every element \( a \) of \( B < x > \),

\[ \Phi(a) = \Omega(a) , \]

then \( \Phi = \Omega \).

**b1):** Let \( a \) and \( b \) be generic dyadic integers. If for every instantiation \( \Phi \),

\[ \Phi(a) = \Phi(b) , \]

then \( a = b \).

**b2):** Let \( \Phi \) and \( \Omega \) be instantiations. If for every generic dyadic integer \( a \),

\[ \Phi(a) = \Omega(a) \]

then \( \Phi = \Omega \).
Finally, $G(x)$ and $G_{(2)}(x)$ can now be made into commutative rings by defining two binary operations, **addition** "+" and **multiplication** ".", as follows:

**Definition 9** (of addition "+"). Let $a$ and $b$ be generic dyadic integers. Let

\[
\begin{align*}
    c^{(0)} & = a \\
    carries^{(0)} & = b
\end{align*}
\]

and let

\[
\begin{align*}
    c^{(i+1)} & = c^{(i)} + \text{carries}^{(i)} \\
    \text{carries}^{(i+1)} & = S(c^{(i)} \circ \text{carries}^{(i)})
\end{align*}
\]

Then the sequences $c^{(i)}$ and carries$^{(i)}$ are convergent. The generic dyadic integer $a + b$ is defined as:

\[
a + b = \lim_{i \to \infty} c^{(i)}
\]

It can also be shown that

\[
\lim_{i \to \infty} \text{carries}^{(i)} = 0
\]

Moreover, if $a$ and $b$ are generic rational integers, then $a + b$ is also a generic rational integer.

**Definition 10** (of multiplication). Let $a$ and $b$ be generic dyadic integers. The **product** of $a$ and $b$, written $a \cdot b$, is defined as:

\[
a \cdot b = \sum_{i=0}^{\infty} b_i \circ (S^i a) = \sum_{i=0}^{\infty} a_i \circ (S^i b)
\]

where "$\sum_{i=0}^{\infty}$" denotes "$\lim_{j \to \infty} \sum_{i=0}^{j}$", and "$\sum$" denotes a sum using the operation "+" defined above.

**Remark 7.** Generic integer multiplication "." was defined above in terms of the secondary operation of generic integer addition "+". Please refer to the appendix for a definition of generic integer multiplication "." in terms of more fundamental Boolean operations.

Generic division "/" will be defined in the next section.
4. Generic lopsided division and the Boolean factoring (BF) algorithm

One of the objectives of this paper is to lift the algebraic structure (i.e., the fundamental binary operations) of the rational integers \( \mathbb{Z} \) and the dyadic integers \( \mathbb{Z}_2 \) to the generic level. In the previous section, this was accomplished for all of the fundamental binary operations but for the exception of division "/". In this section, we complete this part of our research program by lifting the lopsided division defined in section III to the generic level.

An immediate consequence of achieving his objective will be the creation of the Boolean factoring (BF) algorithm, which transforms the problem of integer factoring into the problem of solving a system of Boolean equations. This system of Boolean equations is nothing more than the generic remainder arising from generic lopsided division algorithm.

**Definition 11.** Let \( u, v, w \) be generic integers. The **first component-wise symmetric function** of \( u, v, w \), written

\[
\text{Component}_Wise_1(u, v, w)
\]

is the generic integer whose \( i \)-th component is the first symmetric function of the \( i \)-th components of \( u, v, w \), i.e., whose \( i \)-th component is

\[
\sigma_1(u, v, w) = u_i + v_i + w_i
\]

The **second component-wise symmetric function** of \( u, v, w \), written

\[
\text{Component}_Wise_2(u, v, w)
\]

is the generic integer whose \( i \)-th component is the second symmetric function of the \( i \)-th components of \( u, v, w \), i.e., whose \( i \)-th component is

\[
\sigma_2(u_i, v_i, w_i) = (u_i \diamond v_i) + (v_i \diamond w_i) + (w_i \diamond u_i)
\]

**Definition 12.** Let \( u \) and \( v \) be two generic rational integers. Let

\[
u \equiv v
\]

denote the following element of the free Boolean ring \( \mathbb{B} < x > \)

\[
(u \equiv v) = \prod_{i=0}^{\infty} (1 + u_i + v_i).
\]

**Remark 8.** Please note that, since \( u \) and \( v \) are generic integers, almost all terms in the above product are 1.
In the theorem below, \( \lfloor \rfloor \) and \( \lceil \rceil \) denote respectively the \textbf{floor} and \textbf{ceiling} functions:

\[
\lfloor \rfloor : \mathbb{R} \rightarrow \mathbb{Z} \quad \text{and} \quad \lceil \rceil : \mathbb{R} \rightarrow \mathbb{Z}
\]

\[
\lfloor u \rfloor = \max \{ k \in \mathbb{Z} : k \leq u \} \quad \text{and} \quad \lceil u \rceil = \min \{ k \in \mathbb{Z} : k \geq u \}
\]

where \( \mathbb{R} \) denotes the set of real numbers.

**Theorem 2** (Main). \( (\text{The Boolean Factoring Algorithm.}) \) Let \( N \) be a fixed positive rational integer, and let

\[
N = \ldots , 0, 0, N_{\alpha-1}, N_{\alpha-2}, \ldots , N_1, N_0
\]

denote its radix 2 representation. Let

\[
\beta = \lfloor (1 + \alpha) / 2 \rfloor
\]

and let \( x \) denote the positive odd generic rational integer

\[
x = \ldots , 0, 0, x_{\beta-1}, x_{\beta-2}, \ldots , x_2, x_1, 1
\]

in \( \mathbb{G} < x > \). (Hence, \( x_i = 0 \) for \( i \geq \beta \).) Let

\[
\begin{cases}
c(0) = N \\
borrow(0) = 0
\end{cases}
\]

and

\[
\begin{cases}
c(i+1) = \text{Component}_\text{Wise}_1 \left( \left( c(i), \text{borrows}(i), c(i) \circ S^{i} x \right) \right) \\
\text{borrows}(i+1) = \text{Component}_\text{Wise}_2 \left( \left( c(i)^{+}, \text{borrows}(i), c(i) \circ S^{i} x \right) \right)
\end{cases}
\]

where \( c(i)^{+} \) denotes the component-wise complement of \( c(i) \) and \( S \) denotes the unit left shift operator defined in section II of this paper. Let

\[
\Gamma = \lceil (1 + \alpha) / 2 \rceil
\]

Finally, let \( e_k \) denote the following element of \( \mathbb{B} < x > \)

\[
e_k = \prod_{i=0}^{\infty} \left( c_i^{(k)} + \text{borrows}_i^{(k)} + 1 \right) = \left( c^{(k)} \equiv \text{borrows}^{(k)} \right)
\]

for \( k \geq 0 \). Then \( a \) has an odd rational integral factor of length \( \beta - j \) \((0 \leq j \leq \beta - 1)\) if and only if there exists an instantiation \( \Phi \) such that

\[
\begin{cases}
\Phi(x_{\beta-p}) = 0 \quad \text{for} \ 1 \leq p \leq j \\
\Phi(x_{\beta-(j+1)}) = 1 \\
\Phi(e_{\Gamma+1}) = 1
\end{cases}
\]

Moreover, if there exists such an instantiation \( \Phi \), then \( N \) is the product of the following two positive rational integers

\[
\Phi(x)
\]
and

\[ N/\Phi(x) = \Phi \left( \ldots, 0, c^{(\Gamma+j-1)}_1, c^{(\Gamma+j-2)}_1, \ldots, c^{(1)}_1, c^{(0)}_0 \right) \]

**Summary 1.** Thus, given an arbitrary positive integer \( N \), the BF algorithm produces a system of Boolean equations, namely

\[ c^{(\Gamma+1)} = \text{borrows}^{(\Gamma+1)}, \]

which we have expressed as the equality of two generic integers. Solving the above system of Boolean equations, is equivalent to finding a satisfying set of values for the Boolean variables \( x_1, x_2, \ldots, x_{\beta-1} \) (i.e., an instantiation \( \Phi \)) for the following single Boolean function

\[ e^{\Gamma+1} = \prod_{i=0}^{\infty} \left( c^{(\Gamma+1)}_i + \text{borrows}^{(\Gamma+1)}_i + 1 \right), \]

i.e., finding a solution \( \Phi \) such that

\[ \Phi(e^{\Gamma+1}) = 1. \]

Each such satisfying set of values (i.e., each instantiation \( \Phi \)) produces a rational integer divisor \( \Phi(x) \) of the rational integer \( N \), i.e.,

\[ \Phi(x)/N. \]

5. **Examples of the application of the BF algorithm**

We now give a number of examples of integer factorization using the BF algorithm.

**Example 5.** Factoring 21 with the Boolean factoring algorithm. The radix 2 representation of 21 is:

\[ 1 \ 0 \ 1 \ 0 \ 1 \]

Thus,

\[
\begin{array}{ccccc}
\alpha &=& 5, & \beta &=& 3, & \Gamma &=& 3, \\
\end{array}
\]

\[
\begin{array}{cccc}
& & & c^{(1)}_2 \ c^{(0)}_1 \ c^{(0)}_0 \\
\hline
a & = & \begin{array}{cccc}
1 & 0 & 1 & 0 & 1 \\
1 + x_2 & x_1 & 1 \\
1 + x_2 & x_1 & 0 \\
0 & x_1 & 1 + x_2 & x_1 & x_1 & x_1 \\
x_1 x_2 & x_1 & x_1 & 1 + x_2 & 0 \\
0 & x_1 & x_1 & x_1 & 1 + x_2 \\
1 + x_1 x_2 & 0 & 0 \\
x_1 + x_1 x_2
\end{array}
\end{array}
\]

Hence,

\[ e_3 = 1 + x_1 x_2 + x_1 + x_1 x_2 + 1 = x_1 \]
and
\[
\begin{align*}
\Phi(x_2) &= 1 \\
\Phi(x_1) &= 1
\end{align*}
\]
is a solution. Thus, 21 is a product of the rational integers:
\[
\begin{align*}
\Phi(x_2, x_1, 1) &= 111 \text{ (base } 2) = 7 \text{ (base } 10) \\
\Phi(1 + x_2, x_1, 1) &= 011 \text{ (base } 2) = 3 \text{ (base } 10)
\end{align*}
\]

Example 6. Factoring 77 with the Boolean factoring algorithm.
\[
\alpha = 7, \ \beta = 4, \ \Gamma = 4
\]

| \(c_3^{(3)}\) | \(c_2^{(2)}\) | \(c_1^{(1)}\) | \(c_0^{(0)}\) | \(a/x\) |
|----------|----------|----------|----------|----------|
| 1        | 0        | 0        | 1        | \(x_3, x_2, x_1, 1 = x\) |
| \(x_3\)  | \(x_2\)  | \(x_1\)  | 1        | \(c_0^{(0)} \circ S^0 x\) |
| 0        | \(x_1\)  | \(x_1x_3\) | \(x_1x_2\) | \(c_1^{(1)} \circ S^1 x\) |
|          |          | \(x_3\)  | \(1 + x_3\) | \(c_2^{(2)} \circ S^2 x\) |
|          |          | \(x_1x_3\) | \(x_1x_2\) | \(c_3^{(3)} \circ S^3 x\) |
|          |          |          | \(1 + x_2\) | \(c_4^{(4)}\) |
|          |          |          | 0        | \(\text{borrows}^{(3)}\) |
|          |          |          | 0        | \(\text{borrows}^{(4)}\) |

Hence,
\[
e_4 = [(1 + x_3 + x_2x_3 + x_1x_2x_3) + (x_2 + x_2x_3) + 1] \circ [(x_2 + x_3 + x_2x_3) + (x_1 + x_1x_3) + 1]
\]
\[
= (x_2 + x_3 + x_1x_2x_3) \circ (x_1 + x_2 + x_3 + 1)
\]
\[
= x_1(x_2 + x_3)
\]
Therefore, find an instantiation \(\Phi\) such that
\[
\Phi(x_3) = 1 \text{ and } \Phi(x_1(x_2 + x_3)) = 1
\]

An algorithm for finding the solutions to Boolean equations of the above scarcely satisfiable kind can be found in Section IV of this paper. The solution \(\Phi\) to these
Boolean equations is:

\[ x_1 \mapsto 1 \]

\[ \Phi : x_2 \mapsto 0 \]

\[ x_3 \mapsto 1 \]

Thus, 77 is the product of the following positive rational integers:

\[ \Phi (x_3, x_2, x_1, 1) = 1011 \text{ (base 2)} = 11 \text{ (base 10)} \]

and

\[ \Phi (1 + x_3, 1 + x_2, x_1, 1) = 0111 \text{ (base 2)} = 7 \text{ (base 10)} \]

**Example 7.** Factoring 95 with the Boolean factoring algorithm.

\[ \alpha = 7, \beta = 4, \Gamma = 4 \]

\[
\begin{array}{cccc|ccc}
\hline
& c_3^{(3)} & c_2^{(2)} & c_1^{(1)} & c_0^{(0)} \\
1 & 1 + x_1 & 1 + x_2 & 1 + x_1 & 1 \\
+ x_2 + x_3 & 0 & 0 & 0 & 0 \\
1 + x_3 & 1 + x_2 & 0 & 0 & 0 \\
x_3 + x_1 x_3 & x_2 + x_1 x_2 & 0 & 0 & 0 \\
& 1 + x_3 + x_1 x_3 & + x_3 + x_1 x_2 & + x_2 + x_3 & + x_2 + x_3 \\
x_3 + x_2 x_3 & x_2 x_3 + x_1 x_2 x_3 & x_1 x_3 + x_1 x_2 x_3 & x_1 x_2 + x_1 x_3 & 1 + x_1 \\
& 1 + x_3 + x_1 x_3 & + x_2 x_3 + x_1 x_2 x_3 & + x_2 + x_3 & + x_2 + x_3 \\
x_1 x_3 + x_2 x_3 & x_1 x_2 + x_2 x_3 & x_1 x_2 + x_1 x_3 & 1 + x_1 & 0 \\
& 1 + x_3 & 1 + x_1 x_2 & 0 & 0 \\
+ x_1 x_3 & + x_2 x_3 + x_1 x_2 x_3 & + x_1 x_3 + x_2 x_3 & 0 & 0 \\
x_1 x_2 + x_2 x_3 & x_1 x_3 + x_1 x_2 x_3 & 0 & 0 & 0 \\
\hline
\end{array}
\]

The leftmost expression in \( \text{borrows}^{(4)} \), i.e., \( x_1 x_3 + x_1 x_2 x_3 \), is not shown in the above tableau because there is no room.

Hence,

\[ e_4 = (0 + x_1 x_3 + x_1 x_2 x_3 + 1) \cdot (1 + x_3 + x_1 x_3 + x_1 x_2 + x_2 x_3 + 1) \]

\[ \cdot (x_3 + x_1 x_2 + x_2 x_3 + x_1 x_2 x_3 + x_1 x_3 + x_1 x_2 x_3 + 1) \]

\[ \cdot (1 + x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 + 0 + 1) \cdot 1 \cdot 1 \]
Therefore,  
\[ e_4 = (x_1 x_3 + x_1 x_2 x_3 + 1) \diamond (x_3 + x_1 x_3 + x_1 x_2 + x_2 x_3) \]
\[ \diamond (x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 + 1) \diamond (x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3) \]

Hence,  
\[ e_4 = 0 \]

Thus, there is no factor of 95 of length 4.

So we set \( x_3 = 0 \), and continue.

\[
\begin{array}{cccccc}
0 & 0 & 1 & 1 & 1 & 1 \\
0 & x_2 & x_1 & 1 & 1 & 1 \\
0 & 0 & x_2 + x_1 x_2 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 x_2 & x_1 x_2 & 1 + x_1 & 1 + x_2 & 0 \\
0 & x_1 x_2 & 1 + x_1 x_2 & 0 & 0 & 0 \\
0 & x_2 + x_1 x_2 & x_1 + x_1 x_2 & 1 + x_1 x_2 & 0 & 0 \\
0 & x_2 + x_1 x_2 & 1 + x_1 x_2 & 0 & 0 & 0 \\
0 & 1 + x_2 & x_1 & 0 & 0 & 0 \\
0 & x_1 + x_1 x_2 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Therefore,  
\[ e_5 = (1 + x_2 + x_1 + x_1 x_2 + 1) \diamond (x_1 + 0 + 1) \diamond 1 \diamond 1 \diamond \cdots \diamond 1 \]
\[ = (x_1 + x_2 + x_1 x_2) \diamond (1 + x_1) = x_2 (1 + x_1) \]

Thus, a solution \( \Phi \) is:
\[
\begin{align*}
x_1 & \mapsto 0 \\
\Phi : x_2 & \mapsto 1 \\
x_3 & \mapsto 0
\end{align*}
\]

Hence, 95 is the product of:
\[
\Phi (x_3, x_2, x_1, 1) = 0101 \text{ (base 2)} = 5 \text{ (base 10)}
\]
The multiplicative Boolean factoring (MBF) algorithm

The BF algorithm defined in the previous section is based on generic lopsided division. But there is no need to perform generic lopsided division each time one factors an integer. One need only pre-compute the generic inverse $x^{-1}$ of a judiciously chosen odd generic integer $x$, and then use the pre-computed inverse $x^{-1}$ over and over again to factor arbitrarily chosen integers.

Let $x$ denote the following particular generic integer

$$x = \ldots, x_3, x_2, x_1, 1$$

and let $x^{-1}$ be the corresponding generic inverse, which can be computed with generic lopsided division.

Let $x$ be pre-computed. Then for each chosen positive integer $N$ to be factored, one can find the appropriate system of Boolean equations to be solved to factor $N$ simply by computing the generic product

$$N \cdot x^{-1}.$$  

The resulting algorithm is called the **Multiplicative Boolean factoring (MBF) algorithm**.

We leave the remaining details to the reader.

7. A method for solving scarcely satisfiable Boolean equations

In this section, we outline a general framework for solving the system of Boolean equations produced by the BF and MBF algorithms. This framework was later used by Gamal Abdali to create a LISP implementation of the BF algorithm. Sumee Bagde then extended these methods by using binary decision diagrams (BDDs) to create a Mathematica program that also implemented the BF algorithm.

Both the LISP and Mathematica programs were used to factor many integers. The runtime statistics clearly indicated that the BF algorithm runs in exponential time, and hence, is not competitive with the best classical factoring algorithms. For an algebraic proof as to why this is the case, we refer the reader to the topdown overview given in the next section of this paper.

The main theorem, found in section IV, reduces the task of factoring a fixed positive rational integer $N$ to the task of finding a solution to a Boolean equation of the form

$$e = 1$$

where $e \in \mathbb{B} < x >$. Each solution of this equation corresponds to a divisor of $N$. 

and

$$\begin{align*}
\Phi (1 + x_1 x_2, 1 + x_1 + x_2, 1 + x_2, 1 + x_1, 1) &= 10011 \text{ (base 2)} = 19 \text{ (base 10)}
\end{align*}$$
On the other hand, we are interested in factoring large integers which only have a small number of divisors. It follows that the corresponding equation
\[ e = 1 \]
has only a small number of solutions. We now use this idea to develop a method for solving Boolean equations that each have only a small number of solutions.

**Definition 13.** Let \( e \in B \langle x \rangle \). A **solution** of the Boolean equation \( e = 1 \) is an instantiation \( \Phi \) such that \( \Phi(e) = 1 \).

**Definition 14.** Let less than, written “\(<\)”, denote the linear ordering on the free basis elements
\[ x = \{ \ldots , x_2 , x_1 , x_0 \} \]
defined by
\[ x_i < x_j \text{ if } i < j \]
A **term** in \( B \langle x \rangle \) is a finite product of distinct free basis elements which appear in the product from left-to-right in ascending order according to the relation “\(<\)”. The element 1 of \( B \langle x \rangle \) is represented as the term which is the empty product of free basis elements. Let “\(<\)” also denote the lexicographic linear ordering induced on the set of terms by the linear ordering “\(<\)” on \( x \). (Please note that 1 is the smallest term.)
A **canonical expression** is a sum of distinct terms which appear in the sum from left-to-right in ascending order according to the linear ordering “\(<\)” . (Please note that 0 is represented by the empty sum of terms.)

**Observation.** Every element of \( B \langle x \rangle \) is uniquely representable as a canonical expression.

Finally, we are in a position to define what is meant by a Boolean equation having only a small number of solutions.

**Definition 15.** Let \( e \in B \langle x \rangle \). The Boolean equation \( e = 1 \) is said to be **scarcely satisfiable** if the number of its solutions \( \Phi \) is a non-zero number which is less than the number of distinct free basis elements \( x_i \) appearing in the canonical expression for \( e \).
Next, we observe that the solutions of
\[ e = 1 \]
are in 1-1 correspondence with minterms in the minterm expansion of \( e \). It follows that a scarcely satisfiable Boolean equation
\[ e = 1 \]
is one in which there are only a small number of minterms appearing in the minterm expansion of \( e \). Thus, we have the following proposition.

**Proposition 2.** Let
\[ e = 1 \]
be a scarcely satisfiable Boolean equation. Then for all but a small number of free basis elements \( x_i \) appearing in the canonical expression for \( e \) either
\[ x_i e = e \]
or
\[ x_i^* e = e \]

**Remark 9.** Please note that
1): \( x_i e = 0 \iff x_i^* e = e \iff \Phi(x_i) = 0 \) for all solutions \( \Phi \) of \( e = 1 \).
2): \( x_i^* e = 0 \iff x_i e = e \iff \Phi(x_i) = 1 \) for all solutions \( \Phi \) of \( e = 1 \).

This leads to the following:

**Algorithm for finding a solution \( \Phi \) to a scarcely satisfiable Boolean equation**
\[ e = 1 \]

**Step 1:** For each free basis element \( x_i \) not appearing in the canonical expression for \( e \), set \( \Phi(x_i) \) arbitrarily equal to 0 or 1.

**Step 2:** For each free basis element \( x_i \) such that
\[ x_i^* e = 0 \]
set
\[ \Phi(x_i) = 1 \]

**Step 3:** For each free basis element \( x_i \) such that
\[ x_i e = 0 \]
set
\[ \Phi(x_i) = 0 \]
Step 4: Let $\mu$ denote the number of free basis elements not determined in Steps 1 through 4. Exhaustively try each of the possible $2^\mu$ assignments of $\Phi$ for these basis elements. Since $e$ is scarcely satisfiable, the number of possibilities $2^\mu$ is small.

8. A topdown overview: The "big picture"

After the careful microscopic analysis of arithmetic complexity given in the previous sections of this paper, the "big picture" emerges. We now step back, and take a discerning macroscopic look at what has been found.

We begin by defining two rings $B^{\oplus}$ and $Z^{\oplus}$, the former of characteristic 2, the latter of characteristic 0.

Construction of the first ring:

Let $B\langle \underline{x} \rangle$ be the free Boolean ring on the set

$$\underline{x} = \{x_0, x_1, \ldots, x_{n-1}\}$$

of $n$ symbols. Since the ring $B\langle \underline{x} \rangle$ is both semisimple and a principle ideal domain, it decomposes into the direct sum

$$B\langle \underline{x} \rangle = 2^{n-1} \bigoplus_{\alpha=0}^{2^{n-1}} (m_\alpha) \cong 2^{n-1} \bigoplus_{\alpha=0}^{2^{n-1}} \mathbb{F}_2$$

of principal, minimal ideals, where the ideal generators $m_\alpha$, called minterms, form a complete set of orthogonal idempotents. Consequently, each element $e \in B\langle \underline{x} \rangle$ can be uniquely written in the form

$$e = \sum_{\alpha=0}^{2^{n-1}} c_\alpha m_\alpha,$$

where each coefficient $c_\alpha$ is an element of the finite field of two elements $\mathbb{F}_2 = \{0, 1\}$.

We now define the ring $B^{\oplus}$ of concurrent Boolean functions to be the Boolean ring formed by the ring direct sum

$$B^{\oplus} = \bigoplus_{j=0}^{\infty} B\langle \underline{x} \rangle$$

with diagonal multiplication.

Construction of the second ring:

Let $Z$ be the ring of rational integers. We define the ring $Z^{\oplus}$ of concurrent integers as the direct sum

$$Z^{\oplus} = Z^{\oplus 2^n} = \bigoplus_{\alpha=0}^{2^{n-1}} Z$$
with multiplication defined diagonally. Thus each element of $\mathbb{Z}^\oplus$ can be uniquely written in the form
\[
2^{n-1} \bigoplus_{\alpha=0}^{2^n-1} b_\alpha 2^j,
\]
where $b_\alpha \in \mathbb{Z}$, and where
\[
b_\alpha = \sum_{j=0}^{\infty} b_{j\alpha} 2^j
\]
is the binary expansion of the integer $b_\alpha$.

**The Identification:**

We now have two rings, $\mathbb{B}^\oplus$ of characteristic 2, and $\mathbb{Z}^\oplus$ of characteristic 0. Our next step is to identify these two rings as sets via the bijection defined by

\[
\mathbb{Z}^\oplus \xrightarrow{\Upsilon} \mathbb{B}^\oplus
\]

\[
\bigoplus_{\alpha=0}^{2^n-1} \sum_{j=0}^{\infty} b_{j\alpha} 2^j \xrightarrow{\Upsilon^{-1}} \bigoplus_{\alpha=0}^{2^n-1} \sum_{j=0}^{\infty} b_{j\alpha} m_\alpha,
\]

where $b_{j\alpha}$ on the left is an integer 0 or 1 in $\mathbb{Z}$, and on the right an element 0 or 1 of the finite field of two elements $\mathbb{F}_2$. The result of this identification is called the **ring of generic integers**, and denoted by $G \langle x \rangle$. This object $G \langle x \rangle$ is a set with two distinct ring structure, i.e., a bi-ring $G \langle x \rangle$, of characteristic 0 and of characteristic 2.

**Remark 10.** Please take care to note that both $\Upsilon$ and its inverse $\Upsilon^{-1}$ are bijections of sets, and not ring isomorphisms.

In like manner, the bi-ring of generic dyadic integers $G_{(2)} \langle x \rangle$ can be defined as follows:

Let $\mathbb{Z}_{(2)}$ denote the ring of dyadic integers. We define the (characteristic 0) **ring $\mathbb{Z}_{(2)}^\times$ of concurrent dyadic integers** as the direct product

\[
\mathbb{Z}_{(2)}^\times = \bigtimes_{\alpha=0}^{2^n-1} \mathbb{Z}_{(2)}
\]
with diagonal multiplication.

In turn, the (characteristic 2) **dyadic ring $\mathbb{B}_{(2)}^\times$ of concurrent Boolean functions** is defined as the direct product

\[
\mathbb{B}_{(2)}^\times = \bigtimes_{j=0}^{\infty} \mathbb{B} \langle x \rangle
\]
with diagonal multiplication.
The definition of the bijection \( \Upsilon_{(2)} \), i.e.,

\[
\begin{array}{ccc}
\mathbb{Z}^\times_{(2)} & \xrightarrow{\Upsilon_{(2)}} & \mathbb{B}^\times_{(2)} \\
\Upsilon_{(2)}^{-1} & \xleftarrow{} & 
\end{array}
\]

is similar to that of \( \Upsilon \), and is left to the reader.

Finally, the bi-ring of generic dyadic integers \( G_{(2)} \langle x \rangle, +, \cdot \) is defined by using the above bijection \( \Upsilon_{(2)} \) to identify \( \mathbb{B}^\times_{(2)} \) and \( \mathbb{Z}^\times_{(2)} \) as sets.

One immediate consequence of the above "big picture" is that our use of the phrase "algebraic parallel processing" is not unwarranted. For each arithmetic operation in \( G_{(2)} \langle x \rangle, +, \cdot \) (in \( G_{(2)} \langle x \rangle, +, \cdot \)) is the same as \( 2^n - 1 \) arithmetic operations in \( G_{(2)} \langle x \rangle, +, \cdot \).

It also follows that we have achieved one of the research objectives mentioned in the introduction, namely, the development of a procedure for decomposing arithmetic operations into fundamental Boolean operations. In other words, we have developed a procedure for decomposing the arithmetic operations of the ring \( G_{(2)} \langle x \rangle, +, \cdot \) (of the ring \( G_{(2)} \langle x \rangle, +, \cdot \)) into the elementary operations of the ring \( G_{(2)} \langle x \rangle, +, \cdot \) (of the ring \( G_{(2)} \langle x \rangle, +, \cdot \)).

In closing this section, we give below a summary of the Boolean decompositions of the elementary arithmetic operations for addition "\( + \)"; negation "\( - \)"; subtraction "\( - \)"; lopsided division "\( / \)". The Boolean decomposition of multiplication "\( \cdot \)" can be found in the appendix. Please note that all of the operations \( +, - / \) can be viewed as fixed points of either the function \( A \) or the function \( D \). The definitions of the functions \( A \) and \( D \) can be found below.

**Proposition 3.** Let \( A \) and \( P \) be the functions defined by

\[
A : G_{(2)} \langle x \rangle \times G_{(2)} \langle x \rangle \to G_{(2)} \langle x \rangle \times G_{(2)} \langle x \rangle \quad \text{and} \quad P : G_{(2)} \langle x \rangle \times G_{(2)} \langle x \rangle \to G_{(2)} \langle x \rangle .
\]

Then

**Addition "\( + \)"**

\[
a + b = P \lim_{k \to \infty} A^k (a, b) ,
\]

and hence, \( (a + b, 0) \) is a fixed point of \( A \).

**Negation "\( - \)"**

\[
-a = P \lim_{k \to \infty} A^k (a^*, 1) ,
\]

and hence, \( (-a, 0) \) is a fixed point of the function \( A \).

**Subtraction "\( - \)"**

\[
a - b = P \lim_{k \to \infty} A^k (a + b^* + 1, S (a \circ b^* + a_0 + b_0)) ,
\]

and hence, \( (a - b, 0) \) is a fixed point of the function \( A \).
Let $\sigma_1$ and $\sigma_2$ are the component-wise symmetric functions defined in section IV. Let $D$ and $P'$ be the functions defined by
\[
D : \mathbb{G}_{(2)}^{(u,v,w,\ell)} \times \mathbb{Z} \longrightarrow \mathbb{G}_{(2)}^{(u,v,w,\ell)}, \\
P' : \mathbb{G}_{(2)}^{(u,v,w,\ell)} \times \mathbb{Z} \longrightarrow \mathbb{G}_{(2)}^{(u,v,w,\ell)}.
\]

Then

Lopsided Division "/"

\[
a/b = P' \lim_{\ell \to \infty} D^k (a, 0, b, 0),
\]
and hence, $(a/b, 0, 0, \infty)$ is a fixed point of $D$, where $a$ and $b$ are odd positive generic dyadic integers.

**Question:** Is it possible to remove the characteristic $0$ counter $\ell \mapsto \ell + 1$ from the above lopsided division algorithm?

9. Conclusions and open questions

The BF and MBF algorithms described in this paper are far from competitive with current classical factoring algorithms. It is hoped that the two Boolean factoring algorithms and the generic framework described within this paper will become natural stepping stones for creating faster and more competitive algorithms, and perhaps lead to a new highly competitive quantum integer factorization algorithm.

Before closing this section, a word should be said about the natural question of what is the relationship between the satisfiability problem SAT and the BF and MBF algorithms. SAT is NP-complete. Is the task of solving the class of systems of Boolean equations produced by the BF and MBF algorithms (when reduced to a decision problem) NP-complete? Is it $\#P$-hard? The answers to these questions are not known at this time.

Certainly the BF and MBF algorithms culminate in satisfiability problems. But not all satisfiability problems are NP-complete. Not all are $\#P$ hard. For example, 2-SAT is not NP-complete, but 3-SAT is.

It should be noted that the execution of every algorithm on a digital computer is ultimately reduced by a compiler/assembler to the execution of a Boolean algorithm. This was one of the primary motivating factors for writing this paper. Certainly, it is clear that not every Boolean algorithm executed on a digital computer corresponds to an NP-complete or a $\#P$ hard problem.
10. Appendix. Generic integer multiplication defined in terms of fundamental Boolean operations

In section II, generic integer multiplication “·” was defined in terms of the secondary operation of generic integer addition “+”. Sketched below is a definition of generic integer multiplication “·” in terms of more fundamental Boolean operations.

Definition 16. Let
\[ a = \ldots, a_2, a_1, a_0 \]
be a non-negative (i.e., positive or zero) generic rational integer. Then the \( i \)-th symmetric function
\[ \sigma_i(a) \]
of \( a \) is defined as
\[ \sigma_i(a) = \sum_{j(1) < j(2) < \ldots < j(i)} a_{j(1)} \odot a_{j(2)} \odot \ldots \odot a_{j(i)} \]
where
\[ \sum_{j(1) < j(2) < \ldots < j(i)} \]
denotes a sum with respect to the operation “∔” in \( B < x > \) over the indices \( j(1), j(2), \ldots, j(i) \) subject to the condition
\[ j(1) < j(2) < \ldots < j(i) \]
This function is well-defined since
\[ a_k = 0 \]
for all but finitely many \( k \).

Definition 17. Let
\[ \ldots, a^{(2)}, a^{(1)}, a^{(0)} \]
denote an infinite sequence of generic dyadic integers such that
\[ \lim_{i \to \infty} a^{(i)} = 0 \]
Then, since the limit of the sequence is zero, the \( j \)-th components
\[ \ldots, a^{(2)}_j, a^{(1)}_j, a^{(0)}_j \]
of the above elements of the sequence form a generic rational integer. The component-wise \( i \)-symmetric function
\[ \text{Component\_Wise} \sigma_i \left( \ldots, a^{(2)}, a^{(1)}, a^{(0)} \right) \]
of the sequence
\[ \ldots, a^{(2)}, a^{(1)}, a^{(0)} \]
is defined as the generic dyadic integer whose \( j \)-th component is
\[ \sigma_i \left( \ldots, a^{(2)}_j, a^{(1)}_j, a^{(0)}_j \right) \]
Observation. Let
\[ \ldots, a^{(2)}, a^{(1)}, a^{(0)} \]
be an arbitrary sequence of generic dyadic integers, and let "\( \sum \)" denote a summation with respect to the operation "\( + \)." Then
1): \( \sum_{i=0}^{\infty} a^{(i)} \) is convergent, hence exists, if and only if \( \lim_{i \to \infty} a^{(i)} = 0 \).
2): \( \lim_{i \to \infty} S^i a^{(i)} = 0 \), and hence, \( \sum_{i=0}^{\infty} S^i a^{(i)} \) is convergent and well-defined.

Proposition 4. Let
\[ \ldots, c^{(2)}, c^{(1)}, c^{(0)} \]
be a sequence of generic dyadic integers. Define \( c^{(i,j)} \) recursively as follows:
\[
\begin{cases}
  c^{(i,0)} & = c^{(i)} \\
  c^{(i,j+1)} & = \sigma_2^i \left( c^{(0,j)}, S c^{(1,j)}, S^2 c^{(2,j)}, \ldots, S^p c^{(p,j)}, \ldots \right)
\end{cases}
\]
Then \( \lim_{j \to \infty} c^{(i,j)} \) exists and is given by
\[
\lim_{j \to \infty} c^{(i,j)} = \begin{cases}
  0 & \text{if } i > 0 \\
  \sum_{p=0}^{\infty} S^p c^{(p,q)} & \text{for all } q \geq 0 \text{ if } i = 0
\end{cases}
\]

Corollary 3. Let \( a \) and \( b \) be two generic dyadic integers. Define the generic dyadic integer
\[
c^{(i)} = b_i \odot a
\]
Then, by using the construction given in the above proposition, the generic product of \( a \) and \( b \) is given by
\[
a \cdot b = \lim_{j \to \infty} c^{(0,j)}
\]

It is amusing and insightful to note that the above Boolean decomposition is based on the repeated application of the following well known combinatorial formula which acts as a bridge between characteristic 0 and characteristic 2 algebraic structure, namely:

Theorem 3. Let \( s = s_{m-1}s_{m-2}\ldots s_2s_1s_0 \) be a binary string of length \( m \), and let \( \sigma_k (y_0, y_1, y_2, \ldots, y_{m-1}) \) be the \( k \)-th \( (0 \leq k < m) \) elementary symmetric function in the free Boolean ring \( \mathbb{B} (y_0, y_1, y_2, \ldots, y_{m-1}) \). Then the Hamming weight \( Wt (s) \) of the string \( s \) is given by the following formula
\[
Wt (s) = \sum_{j=0}^{\lfloor \log_2 m \rfloor-1} \sigma_{2^j} (s) \cdot 2^j,
\]
where the symmetric function is first evaluated in the field of two elements \( \mathbb{F}_2 \), and then interpreted as the integer 0 or 1 in \( \mathbb{Z} \). [12]
Thus, the $\sigma_{2^j}$’s are the $j$-th order carries in the multiplication algorithm. This is now made explicit in the following topdown formulation of the multiplication algorithm.

**Proposition 5.** Let $\text{Mat}_\infty (\mathbb{B} \langle \underline{x} \rangle)$ be the set of all Boolean matrices of the form

$$M = (m_{ij}) = \begin{pmatrix}
  \cdots & m_{03} & m_{02} & m_{01} & m_{00} \\
  \cdots & m_{13} & m_{12} & m_{11} & 0 \\
  \cdots & m_{23} & m_{22} & 0 & 0 \\
  \cdots & m_{33} & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},$$

where $m_{ij} \in \mathbb{B} \langle \underline{x} \rangle$ and $m_{ij} = 0$ for $i > j$. Let $\Omega$ be the map defined by

$$\Omega : \text{Mat}_\infty (\mathbb{B} \langle \underline{x} \rangle) \rightarrow \text{Mat}_\infty (\mathbb{B} \langle \underline{x} \rangle)$$

$$M = (\ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_0) \mapsto \begin{pmatrix}
  \cdots & \sigma_2 (\gamma_3) & \sigma_2 (\gamma_2) & \sigma_2 (\gamma_1) & \sigma_2 (\gamma_0) \\
  \cdots & \sigma_2 (\gamma_2) & \sigma_2 (\gamma_1) & \sigma_2 (\gamma_0) & 0 \\
  \cdots & \sigma_2 (\gamma_1) & \sigma_2 (\gamma_0) & 0 & 0 \\
  \cdots & \sigma_2 (\gamma_0) & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix} = \Omega (M)$$

where $\gamma_j$ denotes the $j$-th column of $M$, for $0 \leq j < \infty$.

Let $a$ and $b$ be positive generic dyadic integers. Then

$$a \cdot b = \lim_{k \rightarrow \infty} \Omega^k (M_0),$$

where

$$M_0 = \begin{pmatrix}
  \cdots & a_3 b_0 & a_2 b_0 & a_1 b_0 & a_0 b_0 \\
  \cdots & a_2 b_1 & a_1 b_1 & a_0 b_1 & 0 \\
  \cdots & a_1 b_2 & a_0 b_2 & 0 & 0 \\
  \cdots & a_0 b_3 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},$$

and where $P$ projects each matrix of $\text{Mat}_\infty (\mathbb{B} \langle \underline{x} \rangle)$ onto its first row.

**Remark 11.** Please note that if the columns of the matrices of $\text{Mat}_\infty (\mathbb{B} \langle \underline{x} \rangle)$ are written in reverse order, then $\text{Mat}_\infty (\mathbb{B} \langle \underline{x} \rangle)$ becomes a ring of upper triangular matrices.

**Remark 12.** Please note that each entry $\sigma_{2^j} (\gamma_j)$ is well defined because each column $\gamma_j$ is a positive generic integer.

**Remark 13.** The sub-Boolean ring of $\mathbb{B} \langle \underline{x} \rangle$ of all elementary symmetric Boolean functions is a free Boolean ring with free basis $\sigma_1, \sigma_2, \sigma_2^2, \ldots, \sigma_2^{[k \times n]}$. Hence, as vector spaces, $\dim_{\mathbb{F}_2} (\mathbb{B} \langle \sigma_2 \rangle) = [\lg \dim_{\mathbb{F}_2} (\mathbb{B} \langle \underline{x} \rangle)].$ Consequently, $\Omega^k (M_0)$ converges exponentially fast to a matrix with only two non-zero rows.
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