CAUSALITY PROPERTIES OF TOPOLOGICALLY NONTRIVIAL
SPACE-TIME MODELS

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Received 10 September 1997

Some problems of the space-time causal structure are discussed using models with traversable wormholes. For this purpose the conditions of traversable wormhole matching with the exterior space-time are considered in detail and a mixed boundary problem for the Einstein equations is formulated and analyzed. The influence of these matching conditions on the space-time properties and causal structure is analyzed. These conditions have a non-dynamical nature and cannot be determined by any physical process. So, the causality violation cannot be a result of dynamical evolution of some initial hypersurface. It is also shown that the same conditions which determine the wormhole joining with the outer space provide the self-consistency of solutions and the absence of paradoxes in the case of causality violation.

1. Introduction

Topologically nontrivial space-time models are described by finite or countable sets of maps, which are joined with each other in some order. The maps joining conditions induce constraints for the field variables. The meaning of these constraints is twofold: they are part of the definition of geometrical objects on the manifold and must be considered as additional boundary conditions for the field equations.

In the present paper the influence of these boundary conditions on the causal properties of space-time will be considered for models with traversable wormholes. For this purpose the topological structure of space-time models with traversable wormholes will be considered and the mixed boundary problem for Einstein equation will be formulated. Then the applicability of the relativity principle in the exterior space-time will be discussed for the particular case of wormhole joining. After that some estimations for causality violation will be obtained and a simple example of a spherical wormhole will be considered. A discussion of the “paradoxes” of a time travel and the so-called “self-consistency conditions” concludes the paper.

2. The manifold structure for traversable wormholes

The simplest space-time model with a traversable wormhole consists of two parts: the interior space-time with the topology $M^4_{\text{int}} = T_{\text{int}} \times M^3_{\text{int}}$, and the exterior space-time $M^4_{\text{ext}} = T_{\text{ext}} \times M^3_{\text{ext}}$, where $T_{\text{int}}$ and $T_{\text{ext}}$ are the interior and exterior timelike axes, $M^3_{\text{int}}$ and $M^3_{\text{ext}}$ are the interior and exterior spaces, and the interior space of the wormhole $M^3_{\text{int}}$ has a topological structure of the direct product $M^3_{\text{int}} = I \times M^2$ of the interval $I = (-L_1, L_2)$ and a compact orientable 2-dimensional manifold $M^2$ and is often called
a wormhole handle. The sum $L = L_1 + L_2$ is called
the coordinate length of the wormhole handle. The
boundary of the wormhole handle $M^3_{int}$ is a disjoint
sum of two manifolds $M^2_1$ and $M^2_2$, which are often
called the “left” and “right” mouths of the wormhole
and are homeomorphic in the simplest models. In the
general case $M^2_1$ and $M^2_2$ may be arbitrary compact
2-manifolds and $M^3_{int}$ is an interpolating manifold.

Let $t \in T_{ext}$ and $\tau \in T_{int}$ be the exterior and in-
terior time coordinates. Then, for fixed $\tau$ the wormhole
connects the points of the external spacelike hypersur-
face $(t_1(\tau), M^3_{ext1})$ with those of the external spacelike
hypersurface $(t_2(\tau), M^3_{ext2})$, where $t_1 \neq t_2$ in the
general case.

To discuss the causal structure of wormhole-type
models, consider the simplest case $M^2 = M^2_1 = M^2_2$.
Let $\{\tau, \xi^1, \xi^2, \xi^3\}$ be the local coordinates in the
wormhole interior, such that $-\infty < \tau < \infty$ is an interior
time-like coordinate, $\xi^1$ is the coordinate along the
line $l$ which connects the left and right mouths of the
wormhole, $-(\sigma_1 + L_1) < \xi^1 < L_2 + \sigma_2$, where $\sigma_1$,
$\sigma_2$, $L_1$ and $L_2$ are some positive constants, the values
$\xi^1 = -L_1$ and $\xi^1 = L_2$ correspond to the left and
right mouths of the wormhole, respectively, the regions
$-(\sigma_1 + L_1) < \xi^1 < L_1$ and $L < \xi^1 < L + \sigma_2$ correspond
to the wormhole intersection with the exterior space-
time, $\xi^2$ and $\xi^3$ are the coordinates on the wormhole
motions.

In a general form the interior metric of a traversable
wormhole may be written in the coordinates $\{\tau, \xi^1,$
$\xi^2, \xi^3\}$ as
\[
ds^2_{int} = a^2(\tau, \xi) \tau \delta \tau^2 - 2b_i(\tau, \xi) \tau \delta \xi^i - \gamma_{ij}(\tau, \xi) \delta \xi^i \delta \xi_j
\]
where $\xi = \{\xi^1, \xi^2, \xi^3\}$, $\gamma_{ij}(\tau, \xi)$ denotes the metric of
the interior 3-space $\tau =$ const., $a^2(\tau, \xi) > 0$ because of
the assumption of the wormhole traversability and, in
general, $b_i(\tau, \xi) \neq 0$.

Without loss of generality it may be supposed that
in the exterior space-time both wormhole mouths are
covered by the same map $\{t, x^1, x^2, x^3\}$, where $-\infty <
t < \infty$ is an exterior time and $\{x^1, x^2, x^3\}$ are the
coordinates on the exterior space section $t =$ const.
For simplicity it will be supposed, in addition, that the
exerior space-time coordinates are synchronous, so that the exterior space-time metric has the form
\[
ds^2_{ext} = dt^2 - \gamma_{ij} dx^i dx^j
\]
where $\gamma_{ij}$ is a metric of the exterior 3-space $t =$ const.

Let us suppose, in addition, that the exterior co-
ordinates are comoving the left mouth of the wormhole,
so that for $-(\sigma_1 + L_1) < \xi^1 < -L_1$ we have
\[
t_{left} = \tau, \quad x^i_{left} = x^i(\xi^1, \xi^2, \xi^3)
\]
while for the right mouth, i.e. for $L_2 < \xi^1 < L_2 + \sigma_2$,
we have in general
\[
t_{right} = t_r(\tau), \quad x^i_{right} = x^i(\tau, \xi^1, \xi^2, \xi^3).
\]
These equations determine matching of the interior
space-time of the wormhole with the exterior space-
time.

3. Mixed boundary problem for
wormhole models

Eqs. (3)-(4) of wormhole joining with the outer space,
which determine the manifold structure, induce the fol-
lowing boundary conditions for the interior and exter-
ior metrics:
\[
a^2(\tau, \xi) = \begin{cases}
1 & \text{for } -(\sigma_1 + L_1) < \xi^1 < -L_1, \\
\alpha^2(1 - v_i v^i) & \text{for } L_2 < \xi^1 < L_2 + \sigma_2;
\end{cases}
\]
\[
\beta_i = \begin{cases}
0 & \text{for } -(\sigma_1 + L_1) < \xi^1 < -L_1, \\
\gamma_{kl} \frac{\partial x^k}{\partial \tau} \frac{\partial x^l}{\partial \xi^i} & \text{for } L_2 < \xi^1 < L_2 + \sigma_2;
\end{cases}
\]
and
\[
\tilde{\gamma}_{ij} = \begin{cases}
\gamma_{kl} \frac{\partial x^k}{\partial \tau} \frac{\partial x^l}{\partial \xi^i} & \text{for } -(\sigma_1 + L_1) < \xi^1 < -L_1, \\
\gamma_{kl} \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^l}{\partial \xi^j} & \text{for } L_2 < \xi^1 < L_2 + \sigma_2;
\end{cases}
\]
where $x^k = x^k(\xi^1)$ near the left mouth ($\xi^1 \rightarrow -(\sigma_1 +
L_1)$) and $x^k = x^k(\tau, \xi^1)$ near the right mouth ($\xi^1 \rightarrow$ $L_2$),
and
\[
\alpha = \frac{dt_r}{d\tau}, \quad v_i = \frac{1}{\alpha} \gamma_{kl} \frac{\partial x^k}{\partial \tau} \frac{\partial x^l}{\partial \xi^i}, \quad \beta_i = \gamma_{kl} \frac{\partial x^k}{\partial \tau} \frac{\partial x^l}{\partial \xi^j}.
\]

The traversability condition gives the following
restrictions on the functions $t_r(\tau)$ and $x^i(\tau, \xi)$:
\[
a^2(1 - v_i v^i) > 0,
\]
and hence
\[
a^2 > \varepsilon > 0, \quad 0 \leq v_i v^i < 1 - \varepsilon_1
\]
where $\varepsilon, \varepsilon_1 = \text{const} > 0$, i.e. $t_r(\tau)$ must be a monoton-
ous function of $\tau$ without stationary points.

Eqs. (3-4) must be considered as additional
boundary conditions for the components of the metric
tensor. Indeed, in the regions $-(\sigma_1 + L_1) < \xi^1 < -L_1$
and $L_2 < \xi^1 < L_2 + \sigma_2$, where the interior space-
time intersects with the exterior one, Eqs. (3-4) and
the induced equations (5-8) have the form of coordinate
transformations and have no effect on the energy-
momentum tensor. Therefore these equations are inde-
pendent of the field equations and have a nondynamical
nature.

To complete the formulation of the mixed boundary
problem for traversable wormhole models it is neces-
sary to fix the field equations and initial conditions for
internal and external spaces.
For simplicity, assume that the space-time metric must satisfy the standard Einstein equations

$$G_0^0 = \kappa T_0^0, \quad G_i^0 = \kappa T_i^0$$

and

$$G_j^i = \kappa T_j^i$$

(10)

where $G_0^0$ is the Einstein tensor, $\kappa$ is the Einstein gravitational constant and $T_0^0$ is the energy-momentum tensor of matter and non-gravitational fields. Eqs. (10) are the constraint equations and Eqs. (11) are dynamical. These equations must be supplemented by matter and non-gravitational field equations, not to be considered here.

The usual initial conditions for the Einstein equations must be also specified for both interior and exterior space-times, namely: the components of the interior metric tensor and their first partial derivatives with respect to the interior time coordinate $\tau$, i.e.

$$a^2(\tau_0, \xi), \quad \beta_i(\tau_0, \xi), \quad \gamma_{ij}(\tau_0, \xi), \quad \partial a^2(\tau_0, \xi)/\partial \tau, \quad \beta_{i,\tau}(\tau_0, \xi), \quad \gamma_{ij,\tau}(\tau_0, \xi),$$

(12)

with

$$a^2(\tau_0, \xi) > 0,$$

and the components of the exterior metric and their first partial derivatives with respect to the exterior time coordinate $t$, i.e.

$$\gamma_{ij}(t_0, x), \quad \gamma_{ij,t}(t_0, x).$$

(13)

These quantities must satisfy the standard constraint equations (10), the boundary conditions (1) - (2) near the left mouth $(-\sigma_1 + L_1) < \xi^1 < -L_1$ and the corresponding conditions for derivatives, which take the form

$$\frac{\partial a^2(\tau_0, \xi)}{\partial \tau} = 0, \quad \frac{\partial \beta_i(\tau_0, \xi)}{\partial \tau} = 0$$

(14)

and

$$\gamma_{ij,\tau}(\tau_0, \xi) = \gamma_{kl,\tau}(t_0, x)x^k_i x^l_j.$$ 

(15)

So the mixed boundary problem for the wormhole models may be formulated as follows:

**Statement.** Any space-time model with a traversable wormhole, whose interior and exterior metrics have the forms (1) and (2), respectively, is subject of the mixed boundary problem for the Einstein equations (10) - (11), which is formed by (i) the manifold structure equations (1) - (2), (ii) the boundary conditions (1) - (2) with the traversability constraints (9), (iii) the interior and exterior initial conditions (12), (13) which must satisfy the boundary conditions (1) - (2) and (14) - (15) near the left mouth.

An analogous mixed boundary problem may be formulated for other types of space-time models with non-trivial topology, in particular, for space-time models with a cosmic string which were used in 3 for time machine construction.

4. The relativity principle and the twin paradox for a traversable wormhole

In this section the particular case of wormhole joining with the exterior space-time (1) - (2) will be considered. Namely, it will be assumed that $t_{\text{left}} = t_{\text{right}} = \tau$, i.e. $t = \tau$ is a global time coordinate and both mouths are placed on the same external space-like hypersurface. Let the wormhole mouths be also placed along the $z = x^1$ axis. For simplicity we assume, in addition, that the external space-time is flat.

It is easy to see that, unlike the Minkowski space-time, in the model under consideration the exterior time $t$ will be the global time coordinate only in the restricted class of inertial reference frames of the outer space-time. As a result, the relativity principle cannot be applied to the motion of the wormhole mouths in the outer space. To see that, compare the observer motion in the outer space with respect to the wormhole mouths with the mouths motion with respect to the observer.

In the first case the outer and inner synchronizations of events coincide in the outer space frame of reference where the wormhole mouths are at rest and the observer moves. In the frame comoving the observer this coincidence is violated.

In the second case the interior and exterior synchronizations coincide in the outer space frame of reference where the observer is at rest. In the frame comoving with one of the wormhole mouths this coincidence is violated.

The above considerations may be easily generalized to accelerated motion of the wormhole mouths, e.g. to the “twin paradox” motion. According to the conclusion of 2, 3, “in the wormhole case the twin paradox is a true paradox involving causality violation”. This conclusion is based on the additional implicit supposition that in the interior wormhole metric the proper times of the mouths coincide with each other. However, it is not necessary because the junction conditions (1) - (2) are independent. In particular, we may consider the case when both mouths move as described above in the $t - z$ plane of the outer space and $t_1 = t_2 = \tau$. In this case $\tau$ is a global time coordinate in the whole space-time that defines the absolute synchronization of events near the wormhole mouths. For this reason the time delay of the right mouth relative to the left one is absolute and independent of the
space path along which the comparison of clock readings is realized.

To show that it is indeed the case, recall that the time delay of the right mouth relative to the left one is determined from the equations of the world lines of the comoving observers

\[ ds_L^2 = d\tau^2, \quad ds_R^2 = (1 - V_R^2)d\tau^2 \]

where \( V_R = dz_R/d\tau \) is the velocity of the right mouth in the outer space. Eqs. (16) have the same form for both the inner and outer spaces and are a special case of the equality (9) determining the asymptotic form for the component \( g_{00} \) of the interior wormhole metric. Hence the proper gravitational field of the right mouth induces the same time delay as the right mouth motion in the outer space.

We have considered only an accelerated motion of the right mouth along the straight line. More general motion may be considered in a similar manner. Hence we can make the general conclusion that, contrary to the statements of the papers [3-5], accelerated motion of the mouths of a wormhole does not lead to its transformation into a time machine and to closed time-like curve (CTC) creation. So, the statements about unavoidable or “absurdly easy” wormhole transformation into a time machine [3-5] are wrong. This conclusion conforms with the well-known theorems about the space-time causal structure and the Cauchy problem [6-8].

5. Causality violation in traversable wormhole models

The above equations make it possible to obtain some estimates for causality violation in traversable wormhole models. Without loss of generality one may assume that the mouth sizes are much smaller than the distance between them in the outer space (the approximation of thin mouths) and the \( x^1 \) axis connects the centres of the wormhole mouths. Let, moreover, \( t_r(\tau_1) > \tau_1 \) for some \( \tau_1 > \tau_0 \). Consider a light signal sent at the moment \( t_r(\tau_1) \) from the right mouth to the left one through the wormhole and then returning to the right mouth through the outer space. It is clear that the time delay between the moments of signal sending and receiving is equal to

\[ \Delta t = \delta t_1 + \delta t_2 - \delta t_3 \]

where \( \delta t_1 \) and \( \delta t_2 \) are the signal passing times through the wormhole and the exterior space, respectively, and \( \delta t_3 = t(\tau_1) - \tau_1 \). The causality violation appears if \( \Delta t \leq 0 \). An estimate of the times \( \delta t_1 \) and \( \delta t_2 \) may be obtained from Eqs. (10) and (11). Namely, let

\[ a_0^2 = \min_{-L_1 \leq \xi^1 \leq L_2} a^2(\tau, \xi), \quad b = \max_{-L_1 \leq \xi^1 \leq L_2} |b_1(\tau, \xi)|, \]

\[ N = \max_{-L_1 \leq \xi^1 \leq L_2} \tilde{\gamma}_{ij}(\tau, \xi), \]

and

\[ R = x^1(\tau_1, L_2), \quad C_{\text{ext}} = \max_{0 \leq x^1 \leq R} \gamma_{11}(t, x), \]

\[ C_{\text{int}} = \frac{b + \sqrt{b^2 + N}}{a_0^2}, \]

then

\[ \delta t_1 \leq C_{\text{int}} L, \quad \delta t_2 \leq C_{\text{ext}} R, \]

where \( L = L_1 + L_2 \), and thus

\[ \Delta t \leq C_{\text{int}} L + C_{\text{ext}} R - |t_r(\tau_1) - \tau_1|. \]

Thus a sufficient condition for causality violation may be written in the form

\[ |t(\tau_1) - \tau_1| \geq C_{\text{int}} L + C_{\text{ext}} R. \]

It is necessary to note that this estimate is very rough, so the causality violation may occur even if the inequality (10) is not satisfied.

Analogously, if

\[ |t_r(\tau_1) - \tau_1| < C_{\text{int}} L + C_{\text{ext}} R, \]

then

\[ C_{\text{int}} = \frac{b_m + \sqrt{b_m^2 + N_m}}{a_1^2}, \quad C_{\text{ext}} = \min_{0 \leq x^1 \leq R} \gamma_{11}(t, x), \]

and

\[ a_1^2 = \max_{-L_1 \leq \xi^1 \leq L_2} a^2(\tau, \xi), \quad b_m = \min_{-L_1 \leq \xi^1 \leq L_2} |b_1(\tau, \xi)|, \]

\[ N_m = \min_{-L_1 \leq \xi^1 \leq L_2} \tilde{\gamma}_{ij}(\tau, \xi) \]

for all \( \tau \in (-\infty, \infty) \), then there are no CTCs in the model considered.

Similar estimates may be also obtained for a wormhole with finite mouth sizes.

Thus the main parameters which determine the causal structure of wormhole-type models with a given function \( t_r(\tau) \) are the interior “coordinate length” \( L \) of the wormhole handle, the exterior “coordinate distance” \( R \) between its mouths and the factors \( C_{\text{int}} \) and \( C_{\text{ext}} \). It follows from the above consideration that the parameters \( L \) and \( R \) are subject to boundary conditions for space-time models with traversable wormhole. These parameters are independent of each other, of the field equations and of the function \( t_r(\tau) \). So, using the appropriate choice of the parameters \( L \) and \( R \) (the conditions (3) and (4)) both causal and non-causal space-time models with traversable wormholes may be obtained for the same \( t_r(\tau) \neq \tau \). Of course, for given boundary conditions (3)-(4) with \( t_r(\tau) \neq \tau \) the causality violation depends on the factors \( C_{\text{int}} \) and \( C_{\text{ext}} \) determined by the field equations. On the other hand, if \( t_r(\tau) \equiv \tau \), then a causality violation is impossible in the model under consideration, contrary to
the statement of \([3]\) about an unavoidable wormhole transformation into a time machine.

This confirms our earlier statements about a non-dynamical nature of CTC's and the impossibility of dynamical wormhole transformation into a time machine \([4, 11]\).

### 5.1. A spherical wormhole in Minkowski space-time

To demonstrate that the causality violation is not directly related to any physical processes, consider the special case of a traversable spherical wormhole with immovable mouths, joint to flat Minkowskian exterior space-time.

The exterior flat region of such model is described by the Cartesian coordinates \(\{t, x, y, z\}\), which vary from \(\infty\) to \(\infty\), and the metric

\[
\text{ds}^2 = \text{dt}^2 - \text{dx}^2 - \text{dy}^2 - \text{dz}^2,
\]

while the interior region is described by the coordinates \(\{\tau, l, \theta, \phi\}\), \(-\infty < \tau < \infty\), \(-\sigma_1 + L_1 < l < L_2 + \sigma_2\), and \((\theta, \phi)\) are polar coordinates on the 2-sphere \(S^2\). If the wormhole mouths are placed on the \(x\) axis with the centres at \(x_{\text{left}} = 0\) and \(x_{\text{right}} = R = \text{const}\), then the matching conditions \([3]-[4]\) read

\[
t_{\text{left}} = \tau, \ x_{\text{left}} = l \cos(\phi) \cos(\theta), \ y_{\text{left}} = l \cos(\phi) \sin(\theta), \ z_{\text{left}} = l \sin(\phi)
\]

and

\[
t_{\text{right}} = t_\tau(\tau), \ x_{\text{right}} = R + l \cos(\phi) \cos(\theta), \ y_{\text{right}} = l \cos(\phi) \sin(\theta), \ z_{\text{right}} = l \sin(\phi).
\]

So the simplest interior metric which satisfies the boundary conditions \([3]-[4]\) has the form

\[
\text{ds}^2 = a^2(\tau, l)d\tau^2 - dl^2 - r^2(l)(d\theta^2 + \sin^2(\theta)d\phi^2),
\]

with \(a(\tau, l) = 1\) for \(l < L_1\) and \(a(\tau, l) = dt/\tau\) for \(l > L_2\) and \(r(l) = l\) for \(l < -L_1\) or \(l > L_2\). In the special case when \(a = a(l)\) this metric coincides with the static metric considered in \([3]-[12]\).

A direct calculation gives the following values of non-zero components of the Einstein tensor in the wormhole interior:

\[
G_{0}^0 = -\frac{2\rho'' + r'' - 1}{r^2}, \quad G_{1}^1 = -\frac{ar'^2 + 2ar' + a}{ar^2}, \quad G_{2}^2 = G_{3}^3 = -\left(\frac{r'''}{r} + \frac{a'}{a} \frac{r'}{r} + \frac{a''}{a}\right)
\]

where a prime ('') denotes \(\partial/\partial l\).

It is easy to see that the Einstein tensor \(G_{\alpha}^\alpha\) and hence the energy-momentum tensor for the interior space in this model in the non-static case \((t_{\text{left}} = \tau, \ t_{\text{right}} = t_{\tau}(\tau) \neq \tau)\) have the same structure and properties as in the static case \((a(\tau, l) = a(l), \ t_{\text{left}} = t_{\text{right}} = \tau)\) which were considered in \([12]\). In particular, the matter in the wormhole interior must have the same "exotic" properties as in the static case. Further, the Einstein equations do not restrict the dependence \(a(\tau, l)\) on the interior time \(\tau\). Taking into consideration that \(t_{\tau}(\tau)\) is an arbitrary monotonic function \((dt_{\tau}(\tau)/dr \neq 0)\), one may conclude that the Einstein equations (and hence the physical processes in the space-time) have no effect on the causal structure of the model.

Evidently the same result may be obtained for any model with an arbitrary static exterior space-time, in particular, for the so-called ring-hole model, which was considered recently in \([13]\).

### 6. The "paradoxes" of time machine and "self-consistency conditions"

Causality violation is associated traditionally with different paradoxes which are usually formulated in the following way \([14]\): somebody, after passing through a time machine, kills his parents, which makes impossible his time travel. It is clear, that the "paradox" appears because the observer is considered as an object which moves along its world line, while in the presence of closed time-like curves such a consideration is incorrect and the whole world line which represents the object must be considered.

As an example, consider the motion of a self-interacting test particle of mass \(m\) in a background with a wormhole "time machine". The exterior space-time is supposed to be Minkowskian and the sizes of the wormhole mouths are negligibly small (the point-like mouths approximation) and at rest in some reference frame. For definiteness, we shall assume that the wormhole mouths in the exterior space-time have the coordinates \((t, \vec{r}_A)\) and \((t+a, \vec{r}_B)\), where \(a = a(t) > 0\), and \(a(t) \geq |\vec{r}_B - \vec{r}_A|\) for some time interval \(t_1 \leq t \leq t_2\), where \(t_1\) and \(t_2\) are some constants. So, the region \(t_1 \leq t \leq t_2 + a(t_2)\) of the exterior space-time contains the paths of the closed time-like or null curves which violate causality (we use geometric units where \(c = 1\)).

As in \([14]\), the following particle motion will be considered. The particle starts at a time \(t_0\) in the position \(\vec{r}_i\), enters the mouth (B) of the wormhole at a time \(T + a(T)\) (the position \(\vec{r}_B\)), where \(T > t_1\), goes out of the other mouth (A) at an earlier time \(\tilde{T}\) (position \(\vec{r}_A\)) and finally ends its trajectory at a time \(t_f\) in the position \(\vec{r}_f\). The path length of the wormhole handle is assumed to be infinitely short, so the motion through the wormhole in the proper time of the particle is almost simultaneous. According to an external observer, the particle traversing the time machine...
travels back in time by the amount $\Delta t = -a(\overline{t})$ where $a(\overline{t}) \geq |\overrightarrow{r}_0 - \overrightarrow{r}_A|$ by assumption. For simplicity, the motion with the only self-intersection of the particle world line at the point with coordinates $(t_0, \overrightarrow{r}_0)$ will be considered here.

Consider the region of the exterior Minkowskian space-time with $\overline{t} < t < \overline{t} + a(\overline{t})$. The world line of the particle may be considered in this region as two world lines of two copies of the same particle with the positions $\overrightarrow{r}_1(t)$ and $\overrightarrow{r}_2(t)$. Both particles may be considered as independent objects which interact by means of a potential $V$ of special type. The geometry of the model imposes some additional limitations on the possible motion. Namely, the entrance of the particle into mouth B and its exit from A may be written formally as (Eq. (3) of [15])

\[ \overrightarrow{r}_1(\overline{t} + a(\overline{t})) = \overrightarrow{r}_B \quad (20) \]
\[ \overrightarrow{r}_2(\overline{t}) = \overrightarrow{r}_A \quad (21) \]

and the self-intersection of the particle world line has the form (Eq. (13) of [15])

\[ \overrightarrow{r}_1(t_0) = \overrightarrow{r}_2(t_0) = \overrightarrow{r}_0. \quad (22) \]

Carlini et al. [3] obtained an exact solution of the equations of motion, which correspond to a special case of the potential $V$, with the constraints (20)-(22). It was stated that the existence of such a solution, which minimizes the action functional, shows that the "Principle of self-consistency" is a consequence of the "Principle of minimal action" [15].

Let us analyze this statement in more detail. It is clear that Eqs. (20)-(22) are a direct consequence of the definition of the self-intersected line on the manifold. To be applied to the particle world line, the condition (20) states that the particle falls into the wormhole at some time $\overline{t} + a(\overline{t})$ independently of the previous history, and the condition (22) states that the world line $(\overrightarrow{r}_1, \overrightarrow{r}_2(\overline{t}))$ is a continuation of the world line of the same particle. Therefore, according to the constraints (20)-(22), the points $(t, \overrightarrow{r}_1(t))$ and $(t, \overrightarrow{r}_2(t))$ are points of different paths of the world line of the same particle. For the same reason the point $(t_0, \overrightarrow{r}_1(t_0)) = (t_0, \overrightarrow{r}_2(t_0)) = (t_0, \overrightarrow{r}_0)$ is a self-intersection point of the same world line. Moreover, the conditions (20)-(22) also state that the self-intersection of the particle world line does not prevent its passing through the time machine. Hence, the conditions (21)-(22) prevent the appearance of paradoxes which are usually associated with the existence of the time machine.

So, the self-consistency of the solution (the absence of some “paradoxes”) is provided not by the “Principle of minimal action”, but by the constraint equations (20)-(22) which are part of the definition of the self-intersected line in the manifold.

Of course, the constraints (20)-(22) are not independent of the equations of motion. Namely, the number of the particle entrances into the time machine, the existence and number of self-intersections of its world line, as well as the precise values of the parameters $\overline{t}$, $t_0$ and $\overrightarrow{r}_0$ for every self-intersection are determined by the equation of motion. But if the particle world line has a self-intersection, a local solution of the equations of motion near each self-intersection point must satisfy the geometrical constraints (20)-(22) or their generalization.

The above conditions (20)-(22) are pure geometrical and are not directly related to the action functional. In the quantum case there appears some additional restriction on the particle world line. Indeed, the probability amplitude $\psi = \exp\{iS/h\}$ must be a function of point, while the action $S$ is a function of the world line. Therefore, for any closed world line the action functional of a test particle must satisfy the additional constraint

\[ S = \oint Ld\sigma = 0 \mod (2\pi h) \quad (23) \]

where $\sigma$ is the canonical parameter on the world line. Eq. (23) may be called the "world line quantization condition". This condition formally coincides with the well-known Bohr-Sommerfeld quantization condition but is principally different in its nature.

Acknowledgement

This work was supported in part by the Russian Ministry of Science and the Russian Fund of Basic Research (grant N 95-02-05785-a).

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