Limiting Distributions for Particles Near the Frontier of Spatially Inhomogeneous Branching Brownian Motions

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Abstract

Our purpose in this paper is to determine the limiting distribution and the evolution rate of particles near the frontier of branching Brownian motions. Here the branching rate is given by a Kato class measure with compact support in Euclidean space. Our investigation focuses on the two dimensional case.

Keywords

Branching process · Feynman-Kac functionals

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1 Introduction

We consider branching Brownian motions with splitting on a compact set in $\mathbb{R}^d$. The maximal displacement $L_t$ is the maximum of the Euclidean norm of particles at time $t$. Then, $L_t$ grows linearly and the growth rate is determined by $\lambda < 0$, the principal eigenvalue of the Schrödinger-type operator induced by the branching Brownian motion ([7], [13] and [23]). Therefore, the frontier of particles lies around the boundary of a ball with a linear-growing radius centered at the origin. The aim of this research is to investigate asymptotic behaviors of the population size and distribution of particles near the frontier.

We first explain the model of branching Brownian motions on $\mathbb{R}^d$. A particle starts at $x \in \mathbb{R}^d$ and moves according to the law of a standard Brownian motion $\{B_t, t \geq 0\}$. The initial particle splits into $n$-particles with probability $p_n(B_T)$ at a random time $T$. Here, $p_n(x)$ is the offspring distribution which depends on the site $x$ of the initial particle. The splitting time $T$ is determined by the branching rate measure $\mu$ as follows:

$$P_x(T > t | B_s, s \geq 0) = e^{-A_{t\mu}}. \quad (1.1)$$

Here $P_x$ is the law of the branching Brownian motion initiated from $x$ and $A_{t\mu}$ is the positive continuous additive functional (PCAF for short) which is in the Revuz correspondence with

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μ. Then, new \( n \)-particles perform independently each other and their law is the same as the initial particle. This particle system is usually denoted by a point process \( \mathcal{E}_t \):

\[
\mathcal{E}_t = \sum_{u \in Z_t} \delta_{B_t^u},
\]

(1.2)

where \( Z_t \) is a label set of particles at time \( t \), \( \delta_x \) is the Dirac measure on \( x \) and \( B_t^u \) is the position of \( u \in Z_t \).

If \( \mu \) is the Lebesgue measure, then \( T \) has the exponential distribution with parameter one and the process is a spatially-homogeneous branching Brownian motion (HBBM for short); otherwise, the process is said to be inhomogeneous or catalytic. In particular, the branching rate depends on the trajectories of particles and no branch occurs outside the support of \( \mu \).

The branching Brownian motion on \( \mathbb{R} \) has the rightmost and the leftmost particles at each time \( t \) and \( L_t \) is the maximum of these distance from the origin. McKean ([19] and [20]) proved that for a binary-HBBM, the distribution function of \( L_t - R(t) \) converges to a unique solution of the F-KPP traveling-wave equation at speed \( \sqrt{2} \) if \( R(t) \sim \sqrt{2t} \). He also remarked on the logarithmic correction of \( R(t) \). Bramson [9] revealed it and Mallein [18] extended it to \( d \geq 2 \) and gave an estimate of the tail probability \( P_x(L_t > R(t)) \).

On the other hand, Erickson [13], Bocharov and Harris [7] for \( d = 1 \), and Shiozawa [23] for \( d \geq 1 \) proved the linear growth of \( L_t \) ((2.5) below) for various inhomogeneous-BBMs. Moreover, Lalley and Sellke [16], Bocharov and Harris [8], and Nishimori and Shiozawa [21] proved that when

\[
R^*(t) := \sqrt{-\lambda/2} t + \frac{d-1}{2\sqrt{-2\lambda}} \log t,
\]

the tail distribution of \( L_t - R^*(t) \) converges to the Gumbel distribution which has the parameter mixed by the limit of a martingale ((2.4) below). Bocharov and Harris [7, Theorem 2], and Shiozawa [23, Theorem 2.8] proved the following for the number of particles outside a ball with radius \( \delta t \) centered at the origin: if \( \delta > \sqrt{-\lambda/2} \) (supercritical case), then it converges to zero as \( t \to \infty \). On the other hand, if \( 0 < \delta < \sqrt{-\lambda/2} \) (subcritical case), then it increases exponentially, almost surely. Recently in [6] Bocharov showed that in the case of BBM with a single-point catalyst at the origin \((d = 1, \mu = \beta \delta_0)\), the distribution of particles near the frontier converges to the Poisson point process with random density. For \( \delta_0 \), the PCAF \( A^{\delta_0}_t \) is \( \beta \ell_t \), where \( \ell_t \) is the local time at the origin of the Brownian motion.

In this paper, we extend the results of Bocharov [6] to a BBM with \( d \geq 1 \) so that the branching rate measure is a Kato class measure \( \mu \) with compact support in \( \mathbb{R}^d \). To do so, we develop the moment calculus of the population and the uniformly asymptotic behavior of the Feynman-Kac semigroup induced by \( e^{A_t} \) as in [21]. There is a lot of research on the frontier particles and the extremal process. Aïdekon et al. [1] and Arguin et al. [4] determined the limiting process of the extreme process of BBM as a Poisson point process. Madaule [17] also showed it for branching random walks.

For convenience, we use an annulus to explain our results. Let us denote by \( B_0(R) \) a ball of radius \( R \), centered at the origin. We write \( A_t(R) = B_0(R+r) \setminus B_0(R) \) for a \( d \)-dimensional annulus with the width \( r > 0 \). When \( t \) is large, \( A_t(R^*(t)) \) is a domain near the frontier. For \( A \subset \mathbb{R}^d \), let \( \mathcal{E}_t(A) \) denote the number of particles in \( A \) at time \( t \). We claim the following:

(i) Supercritical case: \( R(t) \) is such that \( \lim_{t \to \infty} (R(t) - R^*(t)) = \infty \). Then, \( P_x(\mathcal{E}_t(A_t(R(t))) > 0) \) converges exponentially to zero.
(ii) Subcritical case: $R(t)$ is such that $\lim_{t \to \infty} (R(t) - R^*(t)) = -\infty$. Then, $E_t(A_r(R(t)))$ is increasing exponentially in probability.

(iii) Critical case: $R(t)$ is such that $\lim_{t \to \infty} (R(t) - R^*(t)) \in (-\infty, \infty)$. For $d = 1, 2$, the distribution of $E_t(A_r(R(t)))$ converges to the Poisson one.

They would then correspond to Theorem 2.1, Theorem 2.2 and Theorem 2.3, respectively.

As mentioned above, Bocharov studied the one-dimensional BBM with a single-point catalyst. In [6], he considered $R(t) = \delta t$ and established (i) for $\delta \in (\beta/2, \beta)$, (ii) for $\delta \in (0, \beta/2)$ and (iii) for $\delta = \beta/2$.

Our proofs are similar to those introduced by Bocharov [6]. He computed the asymptotic behavior of the distribution of particles near the frontier by using the first and second moments of the number of particles. By the Many-to-One Lemma, the first order moment is represented by the Feynman-Kac functional. He computed it directly by using the joint distribution of Brownian motion and local time. We show it for more general cases and use an analytic method established in [21]. By the Many-to-Two Lemma and the crucial estimate of the Feynman-Kac semigroup, we show that the second order moment is asymptotically the same as the first order moment.

In Sect. 2, we introduce the notions of branching Brownian motions and present our results. In Sect. 3, we compute the first and second moments of the population size near the frontier by using long time asymptotic properties of Feynman-Kac semigroups. Section 4 is devoted to the proofs of our results.

Throughout this paper, the letters $c$ and $C$ (with subscript and superscript) denote finite positive constants which may vary from place to place. For positive functions $f(t)$ and $g(t)$ on $(0, \infty)$, we write $f(t) \preceq g(t), t \to \infty$ if positive constants $T$ and $c$ exist such that $f(t) \leq cg(t)$ for all $t \geq T$. We also write $f(t) \sim g(t), t \to \infty$ if $f(t)/g(t) \to 1$ as $t \to \infty$. We will omit “$t \to \infty$” for short when the meaning is clear.

2 Frameworks and Results

2.1 Notations and Some Facts

We use the same notation as in [21]. Let $(\{B_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}, \{\mathcal{F}_t\}_{t \geq 0})$ be the Brownian motion on $\mathbb{R}^d$ and $p_t(x, y)$ its transition function, where $\{\mathcal{F}_t\}$ is the minimal augmented admissible filtration. For $\alpha > 0$, the $\alpha$-resolvent density $G_\alpha(x, y)$ of the Brownian motion is given by

$$G_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y)dt, \quad x, y \in \mathbb{R}^d, \quad t > 0.$$  

**Definition 2.1**

(i) A positive Radon measure $\nu$ on $\mathbb{R}^d$ is in the Kato class ($\nu \in \mathcal{K}$ in notation) if

$$\lim_{\alpha \to \infty} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_\alpha(x, y)\nu(dy) = 0.$$  

(ii) For $\beta > 0$, a measure $\nu \in \mathcal{K}$ is $\beta$-Green tight ($\mu \in \mathcal{K}_\infty(\beta)$ in notation) if

$$\lim_{R \to \infty} \sup_{x \in \mathbb{R}^d} \int_{|y| \geq R} G_\beta(x, y)\nu(dy) = 0.$$  

When $d \geq 3$, $\nu \in \mathcal{K}$ belongs to $\mathcal{K}_\infty(0)$ if the equality above is valid for $\beta = 0$.  

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We know by [26] that $\mathcal{K}_\infty(\beta)$ is independent of $\beta > 0$. Any Kato class measure with compact support is 1-Green tight by definition.

For $\nu \in \mathcal{K}$, let $A^\nu_t$ be the positive continuous additive functional associated with $\nu$ under the Revuz correspondence (see [14, p. 401]). For a signed measure $\nu = \nu^+ - \nu^- \in \mathcal{K} - \mathcal{K}$, we define $A^\nu_t = A^{\nu^+}_t - A^{\nu^-}_t$. The Feynman-Kac semigroup $\{p^\nu_t\}_{t \geq 0}$ is defined by

$$p^\nu_t f(x) := E_x \left[ e^{A^\nu_t} f(B_t) \right], \quad f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d),$$

where $\mathcal{B}_b(\mathbb{R}^d)$ is the collection of all bounded Borel measurable functions on $\mathbb{R}^d$. By [3, Theorem 6.1 (ii)], $\{p^\nu_t\}_{t \geq 0}$ is a strongly continuous symmetric semigroup on $L^2(\mathbb{R}^d)$. The corresponding $L^2$-generator is called a Schrödinger-type operator $\mathcal{H}^\nu = -\Delta/2 - \nu$. Since $\{p^\nu_t\}_{t \geq 0}$ is extended to $L^p(\mathbb{R}^d)$ for any $p \in [1, \infty]$ by [3, Theorem 6.1 (i)], we use the same notation $\{p^\nu_t\}_{t \geq 0}$ as the extended one. By [3, Theorems 7.1], $p^\nu_t$ possesses a jointly continuous integral kernel $p^\nu_t(x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$$p^\nu_t f(x) = \int_{\mathbb{R}^d} p^\nu_t(x, y) f(y) \, dy, \quad f \in \mathcal{B}_b(\mathbb{R}^d).$$

For $\nu = \nu^+ - \nu^- \in \mathcal{K}_\infty(1) - \mathcal{K}_\infty(1)$, let us denote by $\lambda(\nu)$ the bottom of the spectrum for $\mathcal{H}^\nu$:

$$\lambda(\nu) = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \int_{\mathbb{R}^d} u^2 \, d\nu \left| u \in C^\infty_0(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \, dx = 1 \right. \right\},$$

where $C^\infty_0(\mathbb{R}^d)$ is the collection of all smooth functions on $\mathbb{R}^d$ with compact support. If $\lambda(\nu) < 0$, then $\lambda(\nu)$ is the principal eigenvalue of $\mathcal{H}^\nu$ ([25, Lemma 4.3] or [26, Theorem 2.8]) and $h$ is the corresponding eigenfunction. Then $h$ has a strictly positive, bounded and continuous version ([26, Sect. 4]). We also write $h$ for this version with $L^2$-normalization $\|h\|_{L^2(\mathbb{R}^d)} = 1$. Hence for any $x \in \mathbb{R}^d$ and $t > 0$,

$$p^\nu_t h(x) = e^{-\lambda(\nu)t} h(x).$$

We assume that both $\nu^+$ and $\nu^-$ are compactly supported in $\mathbb{R}^d$. By the proof of [26, Theorem 5.2] or [24, Appendix A.1], there exist positive constants $c_1$ and $c_2$ such that

$$c_1 e^{-\sqrt{2\lambda(\nu)}|x|} |x|^{(d-1)/2} \leq h(x) \leq c_2 e^{-\sqrt{2\lambda(\nu)}|x|} |x|^{(d-1)/2}, \quad |x| \geq 1. \quad (2.1)$$

Let $\lambda_2(\nu)$ be the second bottom of the spectrum for $\mathcal{H}^\nu$:

$$\lambda_2(\nu) := \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \int_{\mathbb{R}^d} u^2 \, d\nu \left| u \in C^\infty_0(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \, dx = 1, \int_{\mathbb{R}^d} uh \, dx = 0 \right. \right\}.$$

If $\lambda(\nu) < 0$, then $\lambda(\nu) < \lambda_2(\nu) \leq 0$ because the essential spectrum of $\mathcal{H}^\nu$ is the interval $[0, \infty)$ by [10, Theorem 3.1] or [5, Lemma 3.1].

### 2.2 Branching Brownian Motions

In this subsection, we introduce the branching Brownian motion. Let $\mu \in \mathcal{K}$ be a branching rate measure and $\{p_n(x), n \geq 1\}$ a branching mechanism, where

$$0 \leq p_n(x) \leq 1, \quad n \geq 1 \quad \text{and} \quad \sum_{n=1}^\infty p_n(x) = 1, \quad x \in \mathbb{R}^d.$$
A random time $T$ has an exponential distribution 

$$P_x(T > t \mid \mathcal{F}_\infty) = e^{-\Lambda t}, \quad t > 0.$$ 

A Brownian particle starts at $x \in \mathbb{R}^d$. After an exponential random time $T$, the particle splits into $n$-particles with probability $p_n(B_T)$. New ones are independent Brownian particles starting at $B_T$ and each one independently splits into some particles, the same as the first.

We use capital letters with tilde for sets in $\mathbb{R} \times S^{d-1}$ and capital letters without tilde for sets in $\mathbb{R}^d$ throughout the whole text. Particularly, for fixed $r_1 < r_2$, we set 

$$\tilde{F}_0 = [r_1, r_2] \times \Theta, \quad \tilde{F}_t = [R(t) + r_1, R(t) + r_2] \times \Theta \subset \mathbb{R} \times S^{d-1} \quad (2.2)$$

and we write $F_t(\subset \mathbb{R}^d)$ which is identified with $\tilde{F}_t$. Let $Z_t$ be the set of all particles and $B^u \in \mathbb{R}^d$ the position of $u \in Z_t$ at time $t$. Alternatively to the point process $E_t = \sum_{u \in Z_t} \delta_{B^u}$ mentioned in (1.2), the branching Brownian motion may be realized by the point process 

$$\tilde{E}_t := \sum_{u \in Z_t} \delta_{\left|B^u\right| \left|B^u\right|}.$$ 

That is, the branching process $((\tilde{E}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d}, (\mathcal{G}_t)_{t \geq 0})$ is a point process on $\mathbb{R} \times S^{d-1}$. Here, $\{\mathcal{G}_t\}_{t \geq 0}$ is the natural filtration of the branching process and $S^{d-1} (S^0 = \{-1, 1\})$ is a unit sphere. We denote by $P_x$ the law of the branching process starting at $x \in \mathbb{R}^d$, instead of $\tilde{F}_t$ and $\tilde{E}_t$. Then, 

$$\tilde{E}_t(\tilde{F}_t) = E_t(F_t). \quad (2.3)$$

We set 

$$E_t(f) = \sum_{u \in Z_t} f(B^u), \quad f \in \mathcal{B}_+(\mathbb{R}^d).$$

When $f$ is the indicator function of set $A$, $E_t(\mathbb{1}_A) = E_t(A)$. Assume that $\nu$ is a Kato class measure with compact support in $\mathbb{R}^d$ and $\lambda := \lambda(\nu) < 0$. Let $h$ be the eigenfunction of $\mathcal{H}^\nu$ corresponding to $\lambda$ and 

$$M_t := e^{\lambda t} E_t(h), \quad t \geq 0. \quad (2.4)$$

By the same argument as in [22, Lemma 3.4], we see that $M_t$ is a square integrable non-negative $\mathbb{P}_x$-martingale. Therefore, the limit $M_\infty := \lim_{t \to \infty} M_t \in [0, \infty)$ exists $\mathbb{P}_x$-a.s. and $\mathbb{P}_x(M_\infty > 0) > 0$. In particular, $\mathbb{P}_x(M_\infty > 0) = 1$ for $d = 1, 2$ by [23, Remark 2.11].

Let 

$$L_t = \max_{u \in Z_t} |B^u|.$$ 

By [23, Corollary 2.9], 

$$\lim_{t \to \infty} \frac{L_t}{t} = \sqrt{-\lambda} \cdot \frac{2}{2}, \quad \mathbb{P}_x(\cdot \mid M_\infty > 0) \text{-a.s.} \quad (2.5)$$

For $d = 1, 2$, (2.5) holds $\mathbb{P}_x$-a.s.
Let us recall the Many-to-One and Many-to-Two Lemmas. Let
\[ Q(x) = \sum_{n=1}^{\infty} np_n(x), \quad R(x) = \sum_{n=2}^{\infty} n(n-1)p_n(x). \]
For a measure \( \mu \), \( Q\mu \) and \( R\mu \) denote the measure \( Q(x)\mu(dx) \) and \( R(x)\mu(dx) \), respectively.

**Lemma 2.1** ([22, Lemma 3.3] and [21]) Let \( \mu \in \mathcal{K} \).
(i) If the measure \( Q\mu \) also belongs to Kato class, then for any \( f \in \mathcal{B}_b(\mathbb{R}^d) \),
\[ E_x [\mathcal{E}_t(f)] = E_x \left[ e^{A_t(Q-1)\mu} f(B_t) \right]. \] (2.6)
(ii) If the measure \( R\mu \) also belongs to Kato class, then for any \( f, g \in \mathcal{B}_b(\mathbb{R}^d) \),
\[ E_x [\mathcal{E}_t(f)\mathcal{E}_t(g)] = E_x \left[ e^{A_t(Q-1)\mu} f(B_t) g(B_t) \right] \\
+ E_x \left[ \int_0^t e^{A_s(Q-1)\mu} \mathcal{E}_s \left( f - \mathbb{E}_t - s f \right) \right] \mathcal{E}_s \left( g - \mathbb{E}_t - s g \right) dA_s \mu. \] (2.7)

Our results are obtained by the asymptotic behaviors of the first- and second-moments of \( \mathcal{E}_t(\mathcal{F}_t) \). By (2.3), both asymptotic behaviors reduce to ones of the Feynman-Kac functionals as on the right hand side of (2.6) and (2.7). We show these in Sect. 3.

**2.3 Results**

We will make the following assumptions:

**Assumption 2.1**
(i) \( \mu \) is a Kato class measure with compact support in \( \mathbb{R}^d \).
(ii) \( R\mu \in \mathcal{K} \).
(iii) \( \lambda := \lambda((Q-1)\mu) < 0 \).

We now introduce two measures \( \pi^S \) and \( \pi^R \):

\[ \pi^S(\Theta) = \frac{\sqrt{-2\lambda}}{2\pi} \left( \int_{\mathbb{R}^d} e^{-2\lambda(z,\theta)} d\Theta h(z)\mu(dz) \right), \quad \Theta \subset S^{d-1}, \] (2.8)
where \( d\Theta \) is the surface measure on \( S^{d-1} \) (particularly, \( d\Theta = \delta_{-1} + \delta_1 \) on \( S^0 = \{-1, 1\} \));

\[ \pi^R(A) := \sqrt{-2\lambda} \int_A e^{-\sqrt{-2\lambda}r} dr, \quad A \subset \mathbb{R}. \] (2.9)

Then, we set \( \pi = \pi^R \otimes \pi^S \). From [21, Remark 3.2], we see that
\[ \int_{\mathcal{F}_t} h(y) dy \sim \pi(\mathcal{F}_0) e^{-\sqrt{-2\lambda}R(t)} R(t)^{(d-1)/2}. \] (2.10)
When \( d = 1 \) and \( \mu = 3\delta_0 \), we can explicitly calculate \( \pi \) (see Example 2.1).
Theorem 2.1 (supercritical growth of $R(t)$) Let Assumption 2.1 hold and let $R(t)$ be such that $\lim_{t \to \infty} (R(t) - R^*(t)) = \infty$ and $R(t) \gtrsim \delta t$ for some $\delta < \frac{\sqrt{2\lambda}}{2}$. Then,

$$\lim_{t \to \infty} e^{\sqrt{2\lambda}(R(t) - R^*)} (t / R(t))^{(d-1)/2} P_x (\tilde{E}_i(\tilde{F}) > 0) = \pi(\tilde{F}_0) h(x), \quad \text{for all } x \in \mathbb{R}^d.$$  

By the Paley-Zygmund inequality (4.6 below) and the moment calculations of $\mathcal{E}_i(F_t)$, we show that $P_x (\tilde{E}_i(\tilde{F}) > 0) \sim E_x [\tilde{E}_i(\tilde{F})] \sim \pi(\tilde{F}_0) h(x) e^{-\sqrt{2\lambda}(R(t) - R^*)(R(t)/t)^{(d-1)/2}}, \quad t \to \infty.$

We next show the convergence in probability of the normalization of $\tilde{E}_i(\tilde{F})$.

Theorem 2.2 (subcritical growth of $R(t)$) Let Assumption 2.1 hold and suppose $d = 1, 2$. If $R(t)$ is such that $\lim_{t \to \infty} (R(t) - R^*(t)) = -\infty$ and $R(t) \lesssim \delta t$ for some $\delta > 0$. Then, for any $x \in \mathbb{R}^d$,

$$e^{\sqrt{2\lambda}(R(t) - R^*)} (t / R(t))^{(d-1)/2} \tilde{E}_i(\tilde{F}) \to \pi(\tilde{F}_0) M_\infty, \quad t \to \infty \quad \text{in probability } P_x.$$  

We remark on the case of $R(t) = R^*(t) + \gamma(t)$. When $\gamma(t) \to \infty$, we see from the proof of Theorem 2.1 that

$$\lim_{t \to \infty} e^{\sqrt{2\lambda}(R(t) - R^*)} (t / R(t))^{(d-1)/4} P_x (\tilde{E}_i(\tilde{F}) > 0) = \left(\frac{-\lambda}{2}\right)^{(d-1)/4} \pi(\tilde{F}_0) h(x), \quad \text{for all } x \in \mathbb{R}^d. \quad (2.11)$$

This allows for $\gamma(t) = \sqrt{2\lambda} \log t$. When $\gamma(t) \to -\infty$ and $\gamma(t) = o(\log t)$, we see also from the proof of Theorem 2.2 that

$$e^{\sqrt{2\lambda}(R(t) - R^*)} \tilde{E}_i(\tilde{F}) \to \left(\frac{-\lambda}{2}\right)^{(d-1)/4} \pi(\tilde{F}_0) M_\infty, \quad t \to \infty \quad \text{in probability } P_x. \quad (2.12)$$

We finally show that, when $d = 1, 2$, the number of particles near the frontier converges in distribution to the Poisson-like distribution.

Theorem 2.3 (critical growth of $R(t)$) Let Assumption 2.1 hold and suppose $d = 1, 2$. If $R(t)$ is such that $\lim_{t \to \infty} (R(t) - R^*(t)) = \gamma$ for some $\gamma \in (-\infty, \infty)$, then the branching Brownian motion $\tilde{E}_i$ translated by $R(t)$:

$$\mathcal{E}^*_i = \sum_{u \in Z_t} \delta (|B^*_u - R(t), \frac{P^*_u}{|B^*_u|}|), \quad t \geq 0$$

weakly converges to the Poisson Point process on $\mathbb{R} \times S^{d-1}$ with the random intensity

$$\left(\frac{-\lambda}{2}\right)^{(d-1)/4} e^{-\sqrt{2\lambda}\gamma} \pi(\cdot) M_\infty.$$  

In particular, for $n \in \mathbb{N}$, $\tilde{F}_0^1, \ldots, \tilde{F}_0^n$ mutually disjoint relatively compact sets in $\mathbb{R} \times S^{d-1}$ and $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$,

$$\lim_{t \to \infty} P_x \left(\bigcap_{i=1}^n \left\{\mathcal{E}^*_i(\tilde{F}_0^i) = k_i\right\}\right) = E_x \left[\exp \left\{\left(-\frac{\lambda}{2}\right)^{(d-1)/4} e^{-\sqrt{2\lambda}\gamma} M_\infty \sum_{i=1}^n \pi(\tilde{F}_0^i)\right\}\right].$$
We consider the one-dimensional binary-BBM. Here \( p_i(x) \equiv 1 \) and \( Q(x) = R(x) \equiv 2 \), then the corresponding Schrödinger-type operator is \( -\Delta/2 - \mu \), for the branching rate measure \( \mu \).

We assume that the branching rate measure is \( \beta \delta_0 \) and \( \beta > 0 \), which is the basic model used by [6]. The binary-BBM splits only at the origin. Since \( \lambda = -\beta^2/2 \), Assumption 2.1 holds. The corresponding \( L^2 \)-normalized eigenfunction is \( h(x) = \sqrt{\beta} e^{-\beta|x|} \). Hence, \( R(t) = \sqrt{-\lambda/2t} = \frac{\beta}{2} t \) and the martingale \( M_t \) is given by

\[
e^{\lambda t} \mathcal{E}_t(h) = \sqrt{\beta} e^{-\beta^2 t} \sum_{a \in \mathbb{Z}_t} e^{-\beta|b_a|},
\]

and then \( E_x[M_t] = h(x) = \sqrt{\beta} e^{-\beta|x|} \). In this case, we can compute the right side of (2.10) for \( \widetilde{F}_t = [R(t), R(t) + r] \times S^0 \) and \( F_t \):

\[
\int_{F_t} h(y)dy = 2\sqrt{\beta} \int_{R(t)}^{R(t)+r} e^{-\beta y}dy = \frac{2}{\sqrt{\beta}} \left( 1 - e^{-\beta r} \right) e^{-\beta R(t)},
\]

where

\[
\pi(\widetilde{F}_0) = \pi^R([0, r]) \cdot \pi^S((-1, 1)) = \left( 1 - e^{-\beta r} \right) \cdot \frac{2}{\sqrt{\beta}}.
\]

Theorems 2.1–2.3 state the following:

(a) Supercritical growth of \( R(t) \) (as in Theorem 2.1), \( \lim_{t \to \infty} (R(t) - \frac{\beta}{2} t) = \infty \). Since \( e^{\lambda t + \sqrt{\lambda} R(t)} = e^\beta (R(t) - \frac{\beta}{2} t) \to \infty \) as \( t \to \infty \), \( \mathbb{P}_x(\mathcal{E}_t(\widetilde{F}_t) > 0) \) exponentially converges to zero by Theorem 2.1.

As a special case, \( R(t) = \beta t/2 + (\gamma/\beta) \log t \) is included. By (2.11),

\[
\lim_{t \to \infty} t^\gamma \mathbb{P}_x(\mathcal{E}_t(\widetilde{F}_t) > 0) = \pi(\widetilde{F}_0)h(x) = 2 \left( 1 - e^{-\beta t} \right) e^{-\sqrt{\beta}|x|},
\]

and thus the distribution has a heavy-tail.

(b) Subcritical growth of \( R(t) \) (as in Theorem 2.2), \( \lim_{t \to \infty} (R(t) - \frac{\beta}{2} t) = -\infty \). By Theorem 2.2, \( \lim_{t \to \infty} e^{-\beta^2 t/2 + \beta R(t)} \mathcal{E}_t(\widetilde{F}_t) \to \pi(\widetilde{F}_0)M_\infty \), \( t \to \infty \) in probability.

(c) Critical growth of \( R(t) \) (as in Theorem 2.3), \( \lim_{t \to \infty} (R(t) - \frac{\beta}{2} t) = \gamma \in (-\infty, \infty) \). By Theorem 2.3, \( \mathcal{E}_t(\widetilde{F}_0) \) weakly converges to the Poisson distribution with the random intensity \( e^{-\beta \gamma} \pi(\widetilde{F}_0)M_\infty \).

These results were proved in [6].

Using the case of \( k = 0 \) in Theorem 2.3, we have the limiting distribution for the maximal displacement \( \mathcal{L}_t \). Let \( R(t) = \beta t/2 + \gamma \) for any \( \gamma \in (-\infty, \infty) \), where \( R^*(t) = \beta t/2 \). We now consider \( \widetilde{F}_t = [R(t), \infty) \times \{-1, 1\} \) and \( \widetilde{F}_0 = [0, \infty) \times \{-1, 1\} \). Then \( \pi(\widetilde{F}_0) = 2/\sqrt{\beta} \) and

\[
\mathbb{P}_x \left( L_t - \frac{\beta}{2} t \geq \gamma \right) = \mathbb{P}_x(\mathcal{E}_t(\widetilde{F}_t) > 0) = \mathbb{P}_x(\mathcal{E}_t^*(\widetilde{F}_0) > 0)
\]
\( = 1 - P_x (E_x^c (\bar{F}_0) = 0) \rightarrow 1 - E_x \left[ \exp \left( -\frac{2}{\sqrt{\beta}} e^{-\beta y} M_\infty \right) \right]. \)

Namely, the limiting distribution of \( L_t - \frac{\beta}{2} t \) is the Gumbel one. This result was shown by [8, Theorem 1.2] for \( d = 1 \).

**Example 2.2** We assume that the branching rate measure is \( \mu = \alpha_1 \delta_{y_1} + \alpha_2 \delta_{y_2} \), where \( \alpha_1, \alpha_2 \neq 0 \) and \( y_1 < y_2 \). Then the BBM on \( \mathbb{R} \) has two-point catalysts on \( \{ y_1, y_2 \} \). This model does not fall under the category of [6]. In [6], to give the asymptotic properties as in Theorems 2.1–2.3, Bocharov used the explicit formula of the joint probability distribution of the Brownian motion and the local time at the origin, which controls a particle motion and its splitting time. However, there is no explicit formula to control this two-point catalysts model. We extended his argument to contain these general models.

For simplicity, we set \( p_2(y_1) = p_2(y_2) = 1 \). According to [2, Theorem 2.1.3, pp. 142–145], the negative eigenvalue \( \lambda \) is the solution of

\[(\alpha_1 - 2\sqrt{-\lambda})(\alpha_2 - 2\sqrt{-\lambda}) = \alpha_1 \alpha_2 e^{2\sqrt{-\lambda(y_2 - y_1)}}.\]

Thus Assumption 2.1 is fulfilled and Theorems 2.1–2.3 hold (see also [24, Example 3.10 (ii)]).

**Example 2.3** We consider the \( d \)-dimensional \( (d \geq 2) \) binary-BBM splitting on surface

\[ S_{\rho}^{d-1} = \{ x \in \mathbb{R}^d : |x| = \rho \}. \]

The BBM has the Schrödinger-type operator \(-\Delta/2 - \beta \theta_{\rho}\), where \( \theta_{\rho} \) is the surface measure on \( S_{\rho}^{d-1} \). We see the condition \( \lambda := \lambda(\beta \theta_{\rho}) < 0 \) holds for some \( \beta, \rho > 0 \), from [23, Example 2.14] and references therein. Thus, Theorems 2.1–2.3 hold for the appropriate \( R(t) \).

We here consider \( d = 2 \) and \( F_t \) as the annulus centered at the origin with width \( r \):

\[ F_t := A_r(R(t)) = B_0(R(t) + r) \setminus B_0(R(t)) \subset \mathbb{R}^d, \]

and \( \bar{F}_0 = [0, r] \times S^1 \). In this case,

\[ \pi(\bar{F}_0) = \pi_R([0, r]) \pi^S(S^1) = \left(1 - e^{-\sqrt{-2\lambda}r}\right) \cdot \frac{\beta \sqrt{-2\lambda}}{2 \pi} \int_{S^1} \left( \int_{S^1} e^{-\sqrt{2\lambda} (\theta, z)} d \theta \right) h(z) \theta_{\rho}(dz). \]

It is worth pointing out that \( R^*(t) \) has the logarithmic correction. That is, \( R^*(t) = \sqrt{-\lambda/2} t + \frac{\gamma}{\sqrt{-2\lambda}} \log t \), when \( d = 2 \). We especially consider \( R(t) = R^*(t) + \gamma(t) \) below.

(a) Supercritical growth of \( R(t) \) (as in Theorem 2.1), \( \lim_{t \to \infty} (R(t) - R^*(t)) = \infty \). Taking \( \gamma > 0 \) and \( \gamma(t) = \frac{\gamma}{\sqrt{-2\lambda}} \log(t \lor 1) \), we see from (2.11) that

\[ \lim_{t \to \infty} t^\gamma P_x (E_x^c (\bar{F}_t) > 0) = \left( \frac{-\lambda}{2} \right)^{1/4} \pi(\bar{F}_0) h(x). \]

(b) Subcritical growth of \( R(t) \) (as in Theorem 2.2), \( \lim_{t \to \infty} (R(t) - R^*(t)) = -\infty \). Taking \( \gamma(t) = -\sqrt{-2\lambda} \log log t \), we see from (2.12) that

\[ (\log t)^{-1} E_x (\bar{F}_t) \to \left( \frac{-\lambda}{2} \right)^{1/4} \pi(\bar{F}_0) M_\infty, \quad \text{in probability } P_x. \]
(c) Critical growth of $R(t)$ (as in Theorem 2.3), \( \lim_{t \to \infty} (R(t) - R^*(t)) = \gamma \in (-\infty, \infty) \). By Theorem 2.3, \( \mathcal{E}_{t}^*(\tilde{F}_0) \) asymptotically follows the Poisson distribution with the random intensity
\[
\left( \frac{-\lambda}{2} \right)^{1/4} e^{-\sqrt{-2\lambda\gamma} \pi(\tilde{F}_0) M_\infty}.
\]

3 Estimates of the First and Second Moments

In this section, we calculate the moments of \( \mathcal{E}_t(F_t) \). Using Lemma 2.1 and the Feynman-Kac semigroup \( p_t^\nu \), we obtain the moments of \( \mathcal{E}_t(F_t) \). To estimate \( p_t^\nu \), we introduce a kernel \( q_t^\nu \):
\[
q_t^\nu(x,y) := p_t^\nu(x,y) - p_t(x,y) - e^{-\lambda t} h(x) h(y), \quad x, y \in \mathbb{R}^d, \quad t \geq 0.
\]

In the next section, we will consider the asymptotic behaviors of \( \tilde{E}_t(\tilde{F}_t) \), which are derived from the asymptotic behavior of \( \mathcal{E}_t(F_t) \) via (2.3). Sets \( F_t, \tilde{F}_t \) and \( \tilde{F}_0 \) are determined by the relation (2.2). Since \( \tilde{F}_0 \) is relatively compact, there exist constants \( r_1, r_2 \) and \( \Theta \subset S^{d-1} \) such that \( \tilde{F}_0 \subset [r_1, r_2] \times \Theta \). We can assume that \( r_1 = 0 \) without loss of generality.

We write \( \gamma(t) = R(t) - R^*(t) \) and
\[
\delta := \lim_{t \to \infty} \frac{R(t)}{t} \in \left( 0, \sqrt{-2\lambda} \right).
\]

In the following, we always assume that \( s(t), b(t) \) and \( x(t) \) such that
- for some \( \alpha \in (0, 1 - \delta/\sqrt{-2\lambda}) \), \( 0 \leq s(t) < \alpha t \) for any \( t \geq 0 \) and \( s(t) = o(t) \) as \( t \to \infty \);
- \( b(t) \geq 0 \) for any \( t \geq 0 \) and \( b(t) = o(t) \) as \( t \to \infty \);
- \( x(t) : [0, \infty) \to \mathbb{R}^d \) satisfies \( |x(t)| \leq b(t) \) for all large \( t > 0 \).

We consider the increasing speed of \( s(t) \) for only large \( t \). The condition of \( \alpha \) is not essential. Indeed, we set \( s(t) \), for example,
\[
s(t) = \begin{cases} 0, & t \in [0, T_0], \\ \log t, & t > T_0. \end{cases}
\]

Then, for any \( \alpha > 0 \), there exists \( T_0 \geq 1 \) such that \( s(t) \leq \alpha t \), for all \( t \geq 0 \). We impose this restriction on \( \alpha \) to clear the proof of Lemma 3.4.

3.1 The First Moment

The main claim is provided by Proposition 3.1. We concisely explain the outline of the proof of Proposition 3.1 and show details in Lemma 3.1–3.6 and Proposition 3.2.

**Proposition 3.1** Let \( \mu \) be a Kato class measure with compact support in \( \mathbb{R}^d \) and \( \lambda := \lambda((Q - 1)\mu) < 0 \). Then there exist \( T > 0 \) and \( \theta_{\pm}(t) \) such that \( \theta_{\pm}(t) \to 1 \) as \( t \to \infty \) and
\[
\theta_{-}(t) \leq \frac{\mathbf{E}_{x(t)}[\mathcal{E}_{t-s(t)}(F_t)]}{\pi(\tilde{F}_0) h(x(t)) e^{-\lambda s(t)} e^{-\sqrt{-2\lambda R(t)(d-1)/2}}} \leq \theta_{+}(t),
\]
for all \( t \geq T \).
Here, we can rewrite
\[ e^{-\lambda t} - \sqrt{-2\lambda R(t)} = e^{\sqrt{-2\lambda(R^*(t) - R(t))} t - (d-1)/2}. \]

**Proof** This follows by the same method as in [21, (3.14)]. By Lemma 2.1 (i) and (3.1),
\[
E_{x(t)} \left[ E_{t-s(t)}(F_t) \right] = E_{x(t)} \left[ e^{A(Q_{t-s(t)})} \ ; \ B_{t-s(t)} \in F_t \right] = P_{x(t)} \left( B_{t-s(t)} \in F_t \right) + e^{-\lambda(t-s(t))} h(x(t)) \int_{F_t} h(y) dy + \int_{F_t} q_{t-s(t)}^{(Q_{t-s(t)})}(x(t), y) dy
\]
\[
=: (I) + (II) + (III). \tag{3.2}
\]
In Lemma 3.1 (i) and Proposition 3.2 (ii) below, we will show that there exist
\[ C > 0 \text{ and } T > 0 \] such that for all \( t \geq T \),
\[
(I), \ |(III)| \leq h(x(t)) e^{-Ct} e^{-\lambda(t-s(t))} - \sqrt{-2\lambda R(t)}. \tag{3.3}
\]
Then we have by (2.10),
\[
\frac{(I) + |(III)|}{(II)} \sim \frac{(I) + |(III)|}{\pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t))} - \sqrt{-2\lambda R(t)}},
\]
and we see from (3.3) that the right-hand side is bounded above by \( C' e^{-Ct} \). Therefore,
\[
\frac{E_{x(t)} \left[ E_{t-s(t)}(F_t) \right]}{\pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t))} - \sqrt{-2\lambda R(t)} R(t)} \leq \frac{(I) + |(III)|}{\pi(\tilde{F}_0) e^{-\sqrt{-2\lambda R(t)}} R(t)^{(d-1)/2}} \left( 1 + \frac{(I) + |(III)|}{(II)} \right) =: \theta_{+}(t),
\]
where \( \theta_{+}(t) \to 1 \) as \( t \to \infty \) by (2.10) and (3.3).

Similarly to the above,
\[
\left| \frac{(I) - |(III)|}{(II)} \right| \lesssim C' e^{-Ct} \to 0, \quad t \to \infty,
\]
and
\[
\frac{E_{x(t)} \left[ E_{t-s(t)}(F_t) \right]}{\pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t))} - \sqrt{-2\lambda R(t)} R(t)} \geq \frac{\int_{F_t} h(y) dy}{\pi(\tilde{F}_0) e^{-\sqrt{-2\lambda R(t)}} R(t)^{(d-1)/2}} \left( 1 + \frac{(I) - |(III)|}{(II)} \right) =: \theta_{-}(t),
\]
where \( \theta_{-}(t) \to 1 \) as \( t \to \infty \).

We now give an upper estimate of (3.3). By Lemma 3.1 (i) and
\[
P_{x(t)} \left( B_{t-s(t)} \in F_t \right) \leq P_{x(t)} \left( \left| B_{t-s(t)} \right| \geq R(t) \right),
\]

\[ \square \]
we have (3.3) for (I).

In the rest of this subsection, we consider a signed measure $\nu = \nu^+ - \nu^-$, generally. Let $\lambda = \lambda(\nu)$. The proofs are based on [21, Sects. 3.2 and 3.3].

**Lemma 3.1** Let $\nu^+$ and $\nu^-$ be Kato class measures such that $\lambda < 0$ and the support of $\nu$ is contained in $B_0(M)$ for some $M > 0$. Then the following three assertions hold.

(i) There exist $C > 0$ and $T \geq 0$ such that for all $t \geq T$,
\[ P_x(t) \left( \left| B_{t-s(t)} \right| \geq R(t) \right) \leq h(x(t)) e^{-C t} e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t)}. \]  

(ii) There exist $C > 0$ and $T \geq 1$ such that for all $x \in B_0(M)$, $t \geq T$ and $w \in [0, t-1]$,
\[ P_x \left( \left| B_{t-w} \right| \geq R(t) \right) \leq C e^{-\lambda(t-w) - \sqrt{-2\lambda} R(t)} t^{(d-2)/2}. \]

(iii) There exist $C > 0$ and $T \geq 0$ such that for all $x \in B_0(M)$, $t \geq T$ and $w \in [0, 1]$,
\[ P_x \left( \left| B_{w} \right| \geq R(t) \right) \leq C e^{-(R(t) - M)^2/2} t^{d-2}. \]

**Proof** (i) We have by the spatial homogeneity and scaling property of Brownian motion that, for any $x \in B_0(M)$,
\[ P_x \left( \left| B_{t-s(t)} \right| \geq R(t) \right) \leq P_0 \left( \left| B_{1} \right| \geq \frac{R(t) - M}{\sqrt{t-s(t)}} \right). \]

Hence if $|x(t)| \leq b(t)$, then
\[ P_x(t) \left( \left| B_{t-s(t)} \right| \geq R(t) \right) \leq P_0 \left( \left| B_{1} \right| \geq \frac{R(t) - b(t)}{\sqrt{t-s(t)}} \right). \]

We set $R_b(t) := R(t) - b(t)$. Since
\[ \int_R e^{-u^2/2} u^{d-1} du \sim e^{-R^2/2} R^{d-2}, \quad R \to \infty \]  

and $R_b(t)/\sqrt{t-s(t)} \to \infty$ as $t \to \infty$, there exists $c > 0$ such that for any large $t$,
\[ P_0 \left( \left| B_{1} \right| \geq \frac{R_b(t)}{\sqrt{t-s(t)}} \right) = (2\pi)^{-d/2} \omega_d \int_{R_b(t)/\sqrt{t-s(t)}}^\infty e^{-u^2/2} u^{d-1} du \leq c \exp \left\{ -\frac{1}{2} \left( \frac{R_b(t)}{\sqrt{t-s(t)}} \right)^2 \right\} \left( \frac{R_b(t)}{\sqrt{t-s(t)}} \right)^{d-2}, \]

where $\omega_d$ is the area of the unit ball in $\mathbb{R}^d$. By (2.1) and $|x(t)| \leq b(t)$ for large $t$,
\[ 1 = \frac{h(x(t))}{h(x(t))} \leq h(x(t)) e^{\sqrt{-2\lambda} |x(t)|} |x(t)|^{(d-1)/2} \leq ch(x(t)) e^{\sqrt{-2\lambda} b(t)} b(t)^{(d-1)/2}. \]

Thus,
\[ P_0 \left( \left| B_{1} \right| \geq \frac{R_b(t)}{\sqrt{t-s(t)}} \right) \leq h(x(t)) \exp \left\{ \log c - \frac{1}{2} \left( \frac{R_b(t)}{\sqrt{t-s(t)}} \right)^2 \right\}. \]
\[(d - 2) \log \frac{R_b(t)}{\sqrt{t - s(t)}} + \sqrt{-2\lambda b(t)} + \frac{d - 1}{2} \log b(t) \\
= h(x(t)) \exp \left\{ -\frac{1}{2} \left( \frac{R_b(t)}{\sqrt{t - s(t)}} \right)^2 + o(t) \right\}.
\]

Then
\[-\frac{1}{2} \left( \frac{R_b(t)}{\sqrt{t - s(t)}} \right)^2 = \lambda(t - s(t)) \left\{ 1 - \frac{R_b(t)}{\sqrt{-2\lambda(t - s(t))}} \right\}^2 + \sqrt{-2\lambda b(t)} + \sqrt{-2\lambda R(t)}
- \lambda(t - s(t))
\]

and
\[
\lim_{t \to \infty} \frac{1}{t} \left[ \lambda(t - s(t)) \left\{ 1 - \frac{R_b(t)}{\sqrt{-2\lambda(t - s(t))}} \right\}^2 + \sqrt{-2\lambda b(t)} + o(t) \right] = \lambda \left( 1 - \frac{\delta}{\sqrt{-2\lambda}} \right)^2
< 0.
\]

Hence we can choose \(C > 0\) and large \(T > 0\) such that for any \(t \geq T\),
\[-\frac{1}{2} \left( \frac{R_b(t)}{\sqrt{t - s(t)}} \right)^2 + o(t) \leq -Ct - \sqrt{-2\lambda R(t)} - \lambda(t - s(t)).\]

This proves (i).

(ii) For any \(x \in B_0(M)\) and \(w \in [0, t - 1]\),
\[
P_x (|B_{t - w}| \geq R(t)) \leq P_0 \left( |B_1| \geq \frac{R(t) - M}{\sqrt{t - w}} \right) = (2\pi)^{-d/2} \omega_d \int_{R_{M(t)}/\sqrt{t-w}}^{\infty} e^{-r^2/2} r^{d-1} dr,
\]
where \(R_M(t) := R(t) - M\). Then for any \(w \in [0, t - 1]\),
\[
\frac{R_M(t)}{\sqrt{t - w}} \geq \frac{R_M(t)}{\sqrt{t}} \to \infty, \quad t \to \infty.
\]

Therefore, in the same way as (i), for any large \(t\),
\[
\int_{R_{M(t)}/\sqrt{t-w}}^{\infty} e^{-r^2/2} r^{d-1} dr \leq c \exp \left\{ -\frac{1}{2} \left( \frac{R_M(t)}{\sqrt{t - w}} \right)^2 \right\} \left( \frac{R_M(t)}{\sqrt{t - w}} \right)^{d-2}.
\]

Then
\[-\frac{1}{2} \left( \frac{R_M(t)}{\sqrt{t - w}} \right)^2 = -\left\{ \sqrt{-\lambda(t - w)} - \frac{R_M(t)}{\sqrt{2(t - w)}} \right\}^2 + \sqrt{-2\lambda R_M(t)} - \lambda(t - w)
\leq -\sqrt{-2\lambda R_M(t)} - \lambda(t - w),
\]
and \(R_M(t)/\sqrt{t - w})^{d-2} \leq c t^{(d-2)/2} \). Thus, there is some large \(T > 0\) such that for any \(t \geq T\) and \(w \in [0, t - 1]\),
\[
P_x (|B_{t - w}| \geq R(t)) \leq C e^{-\lambda(t - w) - \sqrt{-2\lambda R(t)} t^{(d-2)/2}}.
\]
(iii) For any \( x \in \overline{B_0(M)} \) and \( w \in [0, 1] \),

\[
P_x (|B_w| \geq R(t)) \leq P_0 (|B_1| \geq R(t) - M) = (2\pi)^{-d/2} \omega_d \int_{R_M(t)}^{\infty} e^{-r^2/2} r^{d-1} \, dr.
\]

There exists \( T > 0 \) such that for any \( t \geq T \),

\[
\int_{R_M(t)}^{\infty} e^{-r^2/2} r^{d-1} \, dr \leq c_1 e^{-R^2(t)/2} R^{d-2}(t) \leq c_2 e^{-(R(t)-M)^2/2} t^{d-2}
\]

and the proof is complete. \( \square \)

We estimate (III). By the same argument as [21, (3.19)],

\[
\int_{F_t} q^v_t(x, y) \, dy = \int_0^1 \left( \int_{\mathbb{R}^d} p^v_s(x, z) P_z(B_{t-s} \in F_t) v(\,dz) \right) \, ds + \int_1^t \left\{ \int_{\mathbb{R}^d} \left( p^v_s(x, z) - e^{-\lambda s} h(x) h(z) \right) P_z(B_{t-s} \in F_t) v(\,dz) \right\} \, ds - h(x) e^{-\lambda t} \int_1^t e^{\lambda s} \left( \int_{\mathbb{R}^d} h(z) P_z(B_s \in F_t) v(\,dz) \right) \, ds =: K_1(x, t, R(t)) + K_2(x, t, R(t)) - K_3(x, t, R(t)).
\]

We write \( K_1(x, t, R(t)) \) instead of \( K_1(x, t, F_t) \), because \( F_t \) depends only on \( R(t) \). The task is now to estimate the following:

\[
|\text{(III)}| \leq |K_1(x(t), t-s(t), R(t))| + |K_2(x(t), t-s(t), R(t))| + |K_3(x(t), t-s(t), R(t))|.
\]

We divide the estimate into a sequence of lemmas for each \( K_i \).

**Lemma 3.2** Under the same setting as in Lemma 3.1, the following three assertions hold.

(i) ([21, Lemma 3.5]) There exists \( C > 0 \) such that for all \( x \in \mathbb{R}^d, t \geq 1 \) and \( R > M \),

\[
\left| \int_0^1 \left( \int_{\mathbb{R}^d} p^v_s(x, z) P_z (|B_{t-s}| \geq R) \, v(\,dz) \right) \, ds \right| \leq C h(x) P_0 (|B_1| > R - M).
\]

(ii) There exist \( C > 0 \) and \( T > 1 \) such that for all \( x \in \mathbb{R}^d, t \geq T \) and \( w \in [0, t-1] \),

\[
|K_1(x, t-w, R(t))| \leq C h(x) e^{-\lambda(t-w)-\sqrt{-2\lambda R(t)} (d-2)/2}.
\]

(iii) There exist \( C > 0 \) and \( T > 0 \) such that for all \( t \geq T \),

\[
|K_1(x(t), t-s(t), R(t))| \leq h(x(t)) e^{-Ct} e^{-\lambda(t-s(t)) - \sqrt{-2\lambda R(t)}}.
\]

For a signed measure \( v = v^+ - v^- \), we write \( |v| = v^+ + v^- \).

**Proof** (ii) By Lemma 3.1 (ii), there exists \( T > 1 \) such that for any \( t \geq T, R(t) > M \) and

\[
P_0 (|B_{t-w}| \geq R(t) - M) \leq C e^{-\lambda(t-w) - \sqrt{-2\lambda R(t)} (d-2)/2}, \quad w \in [0, t-1].
\]
Thus, we see from (i) that
\[
|K_1(x, t - w, R(t))| \leq \left| \int_0^1 \int_{\mathbb{R}^d} p^y_s(x, z) P_z (|B_{t-w-s}| \geq R(t)) \nu(dz) \right| \\
\leq c_1 h(x) P_0 (|B_t| \geq R(t) - M) \\
\leq c_2 h(x) e^{-\lambda(t-w)-\sqrt{\lambda R(t)} t \frac{d(d-2)}{2}}.
\]

(iii) follows by (3.4) and the same argument of (ii).

For \(c > 0\), we set
\[
I_c(t, R) = \\
\begin{cases} \\
e^{-\sqrt{\lambda R} R(t-1/2)}, & \lambda_2 < 0, \\
e^{-\sqrt{\lambda R} R(t-1/2) \wedge t P_0 (|B_t| > R - M)}, & \lambda_2 = 0. \\
\end{cases}
\]

In [21, Lemma 3.6], the following lemma was shown.

**Lemma 3.3** Under the same setting as in Lemma 3.1, the following three assertions hold.

(i) For any \(c > 0\) with \(c \geq -\lambda_2\), there exists \(C > 0\) such that for all \(t \geq 1\) and \(R > 2M\),
\[
\int_1^t \left( \int_{\mathbb{R}^d} e^{-\sqrt{\lambda R} R} P_z (|B_t| > R) |\nu|(dz) \right) \, ds \leq C e^{-\sqrt{\lambda R} R(t-1/2)}.
\]

(ii) If \(\lambda_2 = 0\), then there exists \(C > 0\) such that for all \(t \geq 1\) and \(R > M\),
\[
\int_1^t \left( \int_{\mathbb{R}^d} P_z (|B_t| > R) |\nu|(dz) \right) \, ds \leq C t P_0 (|B_t| > R - M).
\]

(iii) For any \(c > 0\) with \(c \geq -\lambda_2\), there exists \(C > 0\) such that for all \(x \in \mathbb{R}^d\), \(t \geq 1\) and \(R > M\),
\[
\int_1^t \left\{ \int_{\mathbb{R}^d} \left| p^y_s(x, z) - e^{-\sqrt{\lambda R} R} h(x) h(z) \right| P_z (|B_t| > R) |\nu|(dz) \right\} \, ds \leq C I_c(t, R).
\]

Lemma 3.3 yields upper estimates of \(K_2\).

**Lemma 3.4** Under the same setting as in Lemma 3.1, the following assertions hold.

(i) For any \(c > 0\) with \(c \geq -\lambda_2\), there exists \(C > 0\) such that for any \(x \in \mathbb{R}^d\), \(t \geq 1\) and \(R > M\),
\[
|K_2(x, t, R)| \leq C I_c(t, R). \tag{3.6}
\]

In particular, there exists \(T \geq 1\) such that for any \(x \in \mathbb{R}^d\), \(t \geq T\) and \(w \in [0, t - 1]\),
\[
|K_2(x, t - w, R(t))| \leq C e^{-\lambda(t-w)-\sqrt{\lambda R(t)} t \frac{d(d-2)}{2}}.
\]

(ii) There exist \(C > 0\) and \(T \geq 1\) such that for any \(t \geq T\),
\[
|K_2(x(t), t - s(t), R(t))| \leq h(x(t)) e^{-C t e^{-\lambda(t-s(t))-\sqrt{\lambda R(t)}}}.
\]
**Proof**  (i) (3.6) follows by Lemma 3.3 (iii). Substituting \( c = -\lambda \) in (3.6), we have

\[
|K_2(x, t - w, R(t))| \leq C I_{-\lambda}(t - w, R(t)) \leq C e^{-\lambda(t-w) - \sqrt{-2\lambda} R(t)} R(t)^{(d-1)/2} \\
\leq C' e^{-\lambda(t-w) - \sqrt{-2\lambda} R(t)} R(t)^{(d-1)/2},
\]

for any large \( t \).

(ii) Let \( \alpha \in (0, 1 - \delta/\sqrt{-2\lambda}) \), which is the constant at the beginning of Sect. 3. Fix a constant \( c_0 \) with

\[
\left\{ (-\lambda_2) \lor \left( \frac{\sqrt{2\delta}}{1 - \alpha} - \sqrt{-\lambda} \right)^2 \right\} < c_0 < -\lambda.
\]

Since \( s(t) < \alpha t \), \( R(t) > M \) and \( t - s(t) \geq 1 \) for any large \( t \), (3.6) becomes

\[
|K_2(x(t), t - s(t), R(t))| \leq c_1 I_{c_0}(t - s(t), R(t)).
\]

By (2.1),

\[
c_1 I_{c_0}(t - s(t), R(t)) \leq c_2 h(x(t)) e^{\sqrt{-2\lambda} |x(t)|} |x(t)|^{(d-1)/2} I_{c_0}(t - s(t), R(t)).
\]

Since

\[
c_2 e^{\sqrt{-2\lambda} |x(t)|} |x(t)|^{(d-1)/2} \leq c_2 e^{\sqrt{-2\lambda} b(t)} b(t)^{(d-1)/2} = e^{o(t)},
\]

it suffices to show that there exists \( c > 0 \) such that for any large \( t \),

\[
e^{o(t)} I_{c_0}(t - s(t), R(t)) \leq e^{-C t} e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t)}.
\]

Note that

\[
e^{o(t)} I_{c_0}(t - s(t), R(t)) \leq e^{o(t)} e^{c_0(t-s(t)) - \sqrt{2c_0} R(t)(d-1)/2} = e^{c_0(t-s(t)) - \sqrt{2c_0} R(t) + o(t)}.
\]

Then,

\[
c_0(t - s(t)) - \sqrt{2c_0} R(t) + o(t)
\]

\[
= \left\{ (c_0 + \lambda) \left( 1 - \frac{s(t)}{t} \right) - \frac{(\sqrt{2c_0} - \sqrt{-2\lambda}) R(t)}{t} + o(1) \right\} t - \lambda(t - s(t)) - \sqrt{-2\lambda} R(t)
\]

and

\[
\lim_{t \to \infty} \left\{ (c_0 + \lambda) \left( 1 - \frac{s(t)}{t} \right) - \frac{(\sqrt{2c_0} - \sqrt{-2\lambda}) R(t)}{t} + o(1) \right\}
\]

\[
= c_0 + \lambda - (\sqrt{c_0} - \sqrt{-\lambda}) \sqrt{2\delta} < (\sqrt{c_0} - \sqrt{-\lambda}) \frac{\sqrt{2\delta} \alpha}{1 - \alpha} < 0.
\]

Therefore, there exists \( C > 0 \) such that for any large \( t \),

\[
(c_0 + \lambda) \left( 1 - \frac{s(t)}{t} \right) - \frac{(\sqrt{2c_0} - \sqrt{-2\lambda}) R(t)}{t} + o(1) < -C
\]

and (3.7) is proved. \( \square \)
Let us set
\[ J(t, R) = e^{-\lambda t - \sqrt{2\lambda R}} R^{(d-1)/2} \int_{(\sqrt{2\lambda R}-R)/\sqrt{2\lambda}}^{\infty} e^{-v^2} \, dv. \]

**Lemma 3.5** ([21, Lemma 3.7]) Under the same setting as in Lemma 3.1, there exists \( C > 0 \) such that for any \( x \in \mathbb{R}^d \), \( t \geq 1 \) and \( R > 2M \),
\[
e^{-\lambda t} h(x) \int_{t-1}^{\infty} e^{\lambda s} \left( \int_{\mathbb{R}^d} h(z) P_z (|B_z| > R) |v|(dz) \right) ds
\leq C h(x) \left( P_0 (|B_t| > R - M) + J(t, R) \right).
\]

**Lemma 3.6** Under the same setting as in Lemma 3.1, the following assertions hold.

(i) There exists \( C > 0 \) such that for any \( x \in \mathbb{R}^d \), \( t \geq 1 \) and \( R > 2M \),
\[
|K_3(x, t, R)| \leq C h(x) \left( P_0 (|B_t| > R - M) + J(t, R) \right).
\]

In particular, there exist \( C' > 0 \) and \( T \geq 1 \) such that for any \( x \in \mathbb{R}^d \), \( t \geq T \) and \( w \in [0, t-1] \),
\[
|K_3(x, t-w, R(t))| \leq C' h(x) e^{-\lambda(t-w)-\sqrt{2\lambda} R(t)} (t^{(d-2)/2} \lor t^{(d-1)/2}).
\]

(ii) There exist \( C > 0 \) and \( T > 0 \) such that for any \( t \geq T \),
\[
|K_3(x(t), t-s(t), R(t))| \leq h(x(t)) e^{-\lambda(t-s(t))-\sqrt{2\lambda} R(t)}.\]

**Proof** (i) For \( t \geq 1 \) and \( R > 2M \). By Lemma 3.5, for any \( x \in \mathbb{R}^d \),
\[
|K_3(x, t, R)| \leq e^{-\lambda t} h(x) \int_{t-1}^{\infty} e^{\lambda s} \left( \int_{\mathbb{R}^d} h(z) P_z (|B_z| > R) |v|(dz) \right) ds
\leq C h(x) \left( P_0 (|B_t| > R - M) + J(t, R) \right).
\]

In particular, taking any large \( t \) with \( R(t) > 2M \), we see that for any \( w \in [0, t-1] \),
\[
|K_3(x, t-w, R(t))| \leq C h(x) \left( P_0 (|B_t-w| > R(t) - M) + J(t-w, R(t)) \right).
\]

Lemma 3.1 (ii) yields
\[
P_0 (|B_{t-w}| > R(t) - M) \leq c e^{-\lambda(t-w)-\sqrt{2\lambda} R(t)} t^{(d-2)/2}.
\]

Then,
\[
J(t-w, R(t)) \leq e^{-\lambda(t-w)-\sqrt{2\lambda} R(t)} R(t)^{(d-1)/2} \leq c' e^{-\lambda(t-w)-\sqrt{2\lambda} R(t)} t^{(d-1)/2}.
\]

It follows that
\[
|K_3(x, t-w, R(t))| \leq c'' h(x) e^{-\lambda(t-w)-\sqrt{2\lambda} R(t)} (t^{(d-2)/2} \lor t^{(d-1)/2}).
\]

(ii) Note that \( t-s(t) \geq 1 \) and \( R(t) > 2M \) for any large \( t \). By (i),
\[
|K_3(x(t), t-s(t), R(t))| \leq c_1 h(x(t)) \left( P_0 (|B_{t-s(t)}| > R(t) - M) + J(t-s(t), R(t)) \right).
\]
Let us show that the right-hand side is bounded above by
\[ h(x(t))e^{-Ct} e^{-\lambda t - s(t) - \sqrt{-2\lambda} R(t)}. \] (3.8)

Lemma 3.1 (i) implies that for large \( t \),
\[ c_1 P_0 (|B_{t-s(t)}| > R(t) - M) \leq c_1 \|h\|_\infty e^{-c_2 t} e^{-\lambda t - s(t) - \sqrt{-2\lambda} R(t) - M} \leq e^{-c_3 t} e^{-\lambda t - s(t) - \sqrt{-2\lambda} R(t)}, \] (3.9)
where \( c_1 \|h\|_\infty e^{-c_2 t} e^{-\sqrt{-2\lambda} M} \leq e^{-c_3 t} \). On the other hand,
\[ c_1 J(t-s(t), R(t)) = c_1 e^{-\lambda t - s(t) - \sqrt{-2\lambda} R(t)} \left( d \frac{1}{2} \right) \int_0^\infty e^{-\gamma^2} d\gamma. \] (3.10)

By (3.5), \( \int_0^\infty e^{-\gamma^2} d\gamma \sim e^{-L^2 / (2L)} \) as \( L \to \infty \). Since for any \( \delta \in (0, \sqrt{-2\lambda}) \),
\[ L(t) := \sqrt{\frac{-2\lambda(t-s(t)) - R(t)}{2(t-s(t))}} = \sqrt{\frac{-2\lambda - \delta + o(1)}{2 + o(1)}} \sqrt{t} \to \infty, \quad t \to \infty, \]
there exists some \( c > 0 \) such that \( L^2(t) \geq ct \) for all sufficiently large \( t \). We thus have for some \( C > 0 \),
\[ c_1 R(t)^{d-1/2} \int_{L(t)}^\infty e^{-\gamma^2} d\gamma \leq c' R(t)^{d-1/2} \frac{e^{-L^2(t)}}{L(t)} \leq e^{-Ct}. \]

Therefore we have by (3.10)
\[ c_1 J(t-s(t), R(t)) \leq e^{-Ct} e^{-\lambda t - s(t) - \sqrt{-2\lambda} R(t)}. \] (3.11)

Combining (3.9) and (3.11), we obtain (3.8). \( \square \)

From Lemmas 3.2, 3.4 and 3.6, we have the following proposition. In particular, when \( \nu = (Q - 1)\mu \), this gives an upper estimate of (III) in the proof of Proposition 3.1.

**Proposition 3.2** Under the same setting as in Lemma 3.1, the following assertions hold.

(i) There exist \( C > 0 \) and \( T > 0 \) such that for all \( x \in \mathbb{R}^d, t \geq T \) and \( w \in [0, t - 1] \),
\[ \left| \int_{F_t} q_{t-w}^\nu (x, y) dy \right| \leq C e^{-\lambda (t-w) - \sqrt{-2\lambda} R(t)} \left( t^{(d-2)/2} \sqrt{t^{(d-1)/2}} \right). \]

(ii) There exist \( C > 0 \) and \( T > 0 \) such that for all \( t \geq T \),
\[ \left| \int_{F_t} q_{t-s(t)}^\nu (x(t), y) dy \right| \leq h(x(t)) e^{-Ct} e^{-\lambda t - s(t) - \sqrt{-2\lambda} R(t)}. \]

**3.2 The Second Moment**

From now on, \( \mu \) is a branching rate measure, that is, \( \mu \) is a positive measure and satisfies Assumption 2.1. We set
\[ C_d(t) = (d - 1) \log(t \vee 1), \quad d \geq 1. \]
Lemma 3.7 Under the same setting as in Proposition 3.1, there exists $T \geq 1$ such that for any $t \geq T$,
\[
\mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right] \leq \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \right] \leq \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right] + h(x(t)) e^{-2\lambda(t-s(t)) - 2\sqrt{2\lambda R(t)} + C_d(t)}.
\] (3.12)

Proof Since $\mathcal{E}_{t-s(t)}(F_t)$ is a non-negative integer, the first inequality of (3.12) holds.

Let us denote by $\sigma_M$ the hitting time of some particles to $B_0(M)$. Because $\mathcal{E}_{t-s(t)}(F_t) \in \{0, 1\}$ on the event $\{t - s(t) < \sigma_M\}$,
\[
\mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \right] = \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2; t - s(t) \geq \sigma_M \right] + \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t); t - s(t) < \sigma_M \right].
\] (3.13)

It is sufficient to show that the first term of the right-hand side in (3.13) is bounded above by
\[
\mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t); t - s(t) \geq \sigma_M \right] + C h(x(t)) e^{-2\lambda(t-s(t)) - 2\sqrt{2\lambda R(t)} + C_d(t)},
\] (3.14)

which gives the second inequality in (3.12). Lemma 3.8 below shows that for large $t, u \leq t - s(t)$, $\tau \in [0, t - s(t) - u]$ and $x \in B_0(M)$,
\[
\mathbb{E}_x \left[ \int_0^{t-s(t)-u} e^{A(t-s(t)) \mu} \mathbb{E}_{B_t} \left[ \mathcal{E}_{t-s(t)-u-t}(F_t)^2 \right] dA_{t} \right] \leq C h(x) e^{-2\lambda(t-s(t)-u) - 2\sqrt{2\lambda R(t)} + C_d(t)}.
\] (3.15)

Before the initial particle hits $B_0(M)$, the branching Brownian motion has no branch. Hence by the strong Markov property, Lemma 2.1 (ii) and (3.15),
\[
\mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2; t - s(t) \geq \sigma_M \right] = \mathbb{E}_{x(t)} \left[ \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \left| G_{t-s(t)} \right. \right] ; t - s(t) \geq \sigma_M \right]
\]
\[
= \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t); t - s(t) \geq \sigma_M \right] + \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t); t - s(t) \geq \sigma_M \right]
\]
\[
= \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t); t - s(t) \geq \sigma_M \right]
\]
\[
\leq \mathbb{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t); t - s(t) \geq \sigma_M \right]
\]
\[
+ C e^{-2\lambda(t-s(t)) - 2\sqrt{2\lambda R(t)} + C_d(t)} \mathbb{E}_{x(t)} \left[ h(B_{\sigma_M}) e^{2\lambda \sigma_M}; t - s(t) \geq \sigma_M \right].
\]

Since $e^{\lambda t} p_t h = h$, $e^{\lambda t} + A^\gamma \mu h(B_t)$ is a $P_x$-martingale. Then the optional stopping theorem yields that
\[
\mathbb{E}_{x(t)} \left[ h(B_{\sigma_M}) e^{2\lambda \sigma_M}; t - s(t) \geq \sigma_M \right] \leq \mathbb{E}_{x(t)} \left[ h(B_{\sigma_M}) e^{\lambda \lambda \sigma_M + A^\gamma \mu}; t - s(t) \geq \sigma_M \right]
\]
\[
\leq \mathbb{E}_{x(t)} \left[ h(B_{\sigma_M \wedge (t-s(t))}) e^{\lambda \lambda \sigma_M \wedge (t-s(t)) + A^\gamma \mu \wedge (t-s(t))}; t - s(t) \geq \sigma_M \right] = h(x(t)).
\]

Therefore we have (3.14).

Lemma 3.8 There exist $C > 0$ and $T > 0$ such that for any $x \in B_0(M)$, $t \geq T$ and $u \in [0, t - s(t)]$,
\[
\mathbb{E}_x \left[ \int_0^{t-s(t)-u} e^{A(t-s(t)) \mu} \mathbb{E}_{B_t} \left[ \mathcal{E}_{t-s(t)-u-t}(F_t)^2 \right] dA_{t} \right] \leq C h(x) e^{-2\lambda(t-s(t)-u) - 2\sqrt{2\lambda R(t)} + C_d(t)}.
\]
According to Lemma 3.1 (i) and (iii), we can take $t$ so large enough that

$$P_x \left( |B_{t-s(t)}| \geq R(t) \right) \leq e^{-Ct} e^{-\lambda(t-s(t)) - \sqrt{2\pi R(t)}}$$

and

$$P_0(|B_1| \geq R(t) - M) \leq ce^{-(R(t) - M)^2/2}t^{d-2}.$$  \hspace{1cm} (3.16)

We first assume that $w_1 := t - s(t) - u \leq 1$. For any $s \in [0, w_1]$,

$$E_x \left[ \mathcal{E}_{w_1-s} (F_t) \right] = E_x \left[ e^{\lambda(Q^{-1})u} ; B_{w_1-s} \in F_t \right] = \int_{F_t} p_{w_1-s}^{(Q^{-1})u}(x, y) dy$$

$$= \int_{F_t} \left( p_{w_1-s}(x, y) + p_{w_1-s}^{(Q^{-1})u}(x, y) - p_{w_1-s}(x, y) \right) dy$$

$$= P_x \left( B_{w_1-s} \in F_t \right) + \int_{F_t} \left( p_{w_1-s}^{(Q^{-1})u}(x, y) - p_{w_1-s}(x, y) \right) dy$$

$$= P_x \left( B_{w_1-s} \in F_t \right) + \int_{F_t} \int_{0}^{w_1-s} \int_{\mathbb{R}^d} p_v^{(Q^{-1})u}(x, z) p_{w_1-s-v}(z, y) \mu(dz) dy d\pi_v^{(Q^{-1})u}(x, z) \mu(dz) dy.$$  \hspace{1cm} (3.17)

Here (3.17) follows from [21, Lemma 3.1 (i)]. The first and second terms of (3.17) are bounded above, respectively, by $P_0(|B_1| \geq R(t) - M)$ and

$$\int_{F_t} \int_{0}^{w_1-s} \int_{\mathbb{R}^d} p_v^{(Q^{-1})u}(x, z) p_{w_1-s-v}(z, y) \mu(dz) dy d\pi_v^{(Q^{-1})u}(x, z) \mu(dz) dy$$

$$\leq \int_{0}^{w_1-s} \int_{\mathbb{R}^d} p_v^{(Q^{-1})u}(x, z) P_v(B_{w_1-s-v} \in C(R(t))) |\mu||(dz) dy$$

$$\leq \int_{0}^{w_1-s} \int_{\mathbb{R}^d} p_v^{(Q^{-1})u}(x, z) P_0(|B_{w_1-s-v}| \geq R(t) - M) |\mu||(dz) dy$$

$$\leq P_0(|B_1| \geq R(t) - M) \int_{0}^{w_1-s} \int_{\mathbb{R}^d} p_v^{(Q^{-1})u}(x, z) |\mu||(dz) dy$$

$$\leq c_3 P_0(|B_1| \geq R(t) - M).$$

Thus, we see from (3.16) and (3.17) that for any $x \in \overline{B_0(M)}$,

$$E_x \left[ \mathcal{E}_{w_1-s} (F_t) \right] \leq c_1 P_0(|B_1| \geq R(t) - M) \leq c_2 e^{-(R(t) - M)^2/2}t^{d-2}. \hspace{1cm} (3.18)$$

By [12, Proposition 3.8],

$$\sup_{x \in \mathbb{R}^d} E_x \left[ \left( A_t^{R_u} \right)^2 \right] \leq \sup_{x \in \mathbb{R}^d} E_x \left[ e^{A_t^{R_u}} \right] \leq e^{c_1+c_2 t}, \quad \text{for all } t > 0,$$

and then for any large $t$,

$$E_x \left[ \int_{0}^{t-s(t)-u} e^{A_t^{(Q^{-1})u}} dA_s^{R_u} \right] \leq E_x \left[ e^{A_t^{(Q^{-1})u}} \int_{0}^{t-s(t)-u} dA_s^{R_u} \right] \leq E_x \left[ e^{A_t^{(Q^{-1})u}} A_t^{R_u} \right]$$

$$\leq E_x \left[ e^{2A_t^{(Q^{-1})u}} \right]^{1/2} E_x \left[ \left( A_t^{R_u} \right)^2 \right]^{1/2} \leq e^{c_3 t}. \hspace{1cm} (3.19)$$
These imply that
\[ E_x \left[ \int_0^{t-s(t)-u} e^{A_s^{(Q-1)\mu}} E_{B_x} \left[ \mathcal{E}_{t-s(t)-u-s} (F_t) \right]^2 dA_s^{R\mu} \right] \]
\[ \leq c_1 e^{-(R(t)-M)^2} t^{2(d-2)} E_x \left[ \int_0^{t-s(t)-u} e^{A_s^{(Q-1)\mu}} dA_s^{R\mu} \right] \]
\[ \leq c_2 e^{c_3 t-(R(t)-M)^2} t^{2(d-2)} = e^{-(R(t)-M)^2+o(t^2)}. \]  
(3.20)

Next, suppose that \( t-s(t)-u > 1 \). We set
\[ E_x \left[ \int_0^{t-s(t)-u} e^{A_s^{(Q-1)\mu}} E_{B_x} \left[ \mathcal{E}_{t-s(t)-u-s} (F_t) \right]^2 dA_s^{R\mu} \right] \]
\[ = E_x \left[ \int_0^{t-s(t)-u-1} \cdots \right] + E_x \left[ \int_{t-s(t)-u-1}^{t-s(t)-u} \cdots \right] = I_1 + I_2, \]
and \( w_2 = w_2(s) = s(t) + u + s \), then \( 0 \leq w_2 \leq t - 1 \) for any \( s \in [0, t - s(t) - u - 1] \). Now (3.2) gives
\[ E_x \left[ \mathcal{E}_{t-w_2} (F_t) \right] = E_x \left[ e^{A_{(Q-w_2)}^{(Q-1)\mu}} ; B_{t-w_2} \in F_t \right] \]
\[ = P_x \left( B_{t-w_2} \in F_t \right) + e^{-\lambda(t-w_2)} h(x) \int_{F_t} h(y) d\mu + \int_{F_t} q_{(Q-w_2)}^{(Q-1)\mu} (x, y) dy. \]
Owing to Lemma 3.1 (ii), (2.10) and Proposition 3.2 (i), respectively, we have
\[ P_x \left( B_{t-w_2} \in F_t \right) \leq C e^{-\lambda(t-w_2)-\sqrt{2\lambda R(t)}} t^{(d-2)/2}, \]
\[ e^{-\lambda(t-w_2)} h(x) \int_{F_t} h(y) d\mu \leq C e^{-\lambda(t-w_2)-\sqrt{2\lambda R(t)}} R(t)^{(d-1)/2} \leq C' e^{-\lambda(t-w_2)-\sqrt{2\lambda R(t)}} t^{(d-1)/2} \]
and
\[ \left| \int_{F_t} q_{(Q-w_2)}^{(Q-1)\mu} (x, y) dy \right| \leq C e^{-\lambda(t-w_2)-\sqrt{2\lambda R(t)}} \left( t^{(d-2)/2} \vee t^{(d-1)/2} \right), \]
for any large \( t, w_2 \in [0, t - 1] \) and \( x \in \overline{B_2(M)} \). Hence
\[ E_x \left[ \mathcal{E}_{t-w_2} (F_t) \right]^2 \leq C h^2(x) e^{-2\lambda(t-w_2)-2\sqrt{2\lambda R(t)}+C_d(t)}, \]
which implies
\[ I_1 \leq E_x \left[ \int_0^{t-s(t)-u-1} e^{A_s^{(Q-1)\mu}} E_{B_x} \left[ \mathcal{E}_{t-w_2(s)} (F_t) \right]^2 dA_s^{R\mu} \right] \]
\[ \leq C \|h\|_{\infty}^2 e^{-2\lambda(t-s(t)-u)-2\sqrt{2\lambda R(t)}+C_d(t)} E_x \left[ \int_0^{t-s(t)-u-1} e^{A_s^{(Q-1)\mu}+2\lambda s} dA_s^{R\mu} \right]. \]
By the same argument as [11, Proposition 3.3 (i)], we also have
\[ E_x \left[ \int_0^{t-s(t)-u-1} e^{A_s^{(Q-1)\mu}+2\lambda s} dA_s^{R\mu} \right] \leq \sup_{x \in \mathbb{R}^d} E_x \left[ \int_0^\infty e^{A_s^{(Q-1)\mu}+2\lambda s} dA_s^{R\mu} \right] < \infty. \]
Therefore,

\[ I_1 \leq C e^{-2\lambda (t-s(t)-u)-2\sqrt{-\lambda} R(t)+C_d(t)}. \]  

(3.21)

We set \( w_3 = t - s(t) - u \), where \( 0 \leq w_3 - s \leq 1 \) for any \( s \in [w_3 - 1, w_3] \). In the same way as (3.18), we have for any large \( t \),

\[ \mathbb{E}_x \left[ \mathcal{E}_{w_3-s}(F_t) \right] \leq C e^{-(R(t)-M)^2/2} t^{-d-2}, \]

and by the same as (3.19),

\[ \mathbb{E}_x \left[ \int_{w_3-1}^{w_3} e^{A_s^{(Q-1)\mu}} dA_{s}^{R\mu} \right] \leq e^{ct} \]

Thus,

\[ I_2 = \mathbb{E}_x \left[ \int_{w_3-1}^{w_3} e^{A_s^{(Q-1)\mu}} \mathbb{E}_x \left[ \mathcal{E}_{w_3-s}(F_t) \right]^2 dA_{s}^{R\mu} \right] \leq C e^{-(R(t)-M)^2} e^{2(2-d)\lambda^2} \mathbb{E}_x \left[ e^{A_s^{(Q-1)\mu}} dA_{s}^{R\mu} \right]. \]

Combining this with (3.21), we obtain

\[ \mathbb{E}_x \left[ \int_{0}^{t-s(t)-u} e^{A_s^{(Q-1)\mu}} \mathbb{E}_x \left[ \mathcal{E}_{t-s(t)-u-s}(F_t) \right]^2 dA_{s}^{R\mu} \right] \leq C e^{-2\lambda(t-s(t))-2\sqrt{-\lambda} R(t)+C_d(t)}. \]  

(3.22)

Consequently, (3.20) and (3.22) yield that

\[ \mathbb{E}_x \left[ \int_{0}^{t-s(t)-u} e^{A_s^{(Q-1)\mu}} \mathbb{E}_x \left[ \mathcal{E}_{t-s(t)-u-s}(F_t) \right]^2 dA_{s}^{R\mu} \right] \leq C e^{-2\lambda(t-s(t))-2\sqrt{-\lambda} R(t)+C_d(t)} \]

\[ \leq C'h(x(t)) e^{-2\lambda(t-s(t))-2\sqrt{-\lambda} R(t)+C_d(t)}, \]

for any large \( t, u \in [0, t-s(t)] \) and \( x \in \bar{B}_0(M) \).

Set \( \tilde{F}_0 \) is determined by \( \tilde{F}_t \), according to (2.2). As before, \( \delta = \lim_{t \to \infty} R(t)/t \) and \( \gamma(t) = R(t) - R^*(t) \).

**Proposition 3.3** We assume the same setting as in Proposition 3.1.

(i) If \( \delta \in (0, \sqrt{-\lambda}/2) \), then there exist \( T \geq 1 \) and \( \Theta(t) \) with \( \Theta(t) \to 1 \) as \( t \to \infty \), such that for any \( t \geq T \),

\[ \mathbb{E}_x(t) \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \right] \leq \Theta(t) h(x(t)) e^{-2\lambda(t-s(t))-2\sqrt{-\lambda} R(t)+C_d(t)}. \]  

(3.23)

(ii) Let \( \delta \in [\sqrt{-\lambda}/2, \sqrt{-2\lambda}) \). When \( \delta = \sqrt{-\lambda}/2 \), we choose \( s(t) \) and \( \gamma(t) \) such that

\[ \lambda s(t) - \sqrt{-2\lambda} \gamma(t) \to -\infty, \quad t \to \infty. \]  

(3.24)

Then,

\[ \mathbb{E}_x(t) \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \right] \sim \pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t)} R(t)^{(d-1)/2}, \quad t \to \infty. \]
Proof} (i) If $\delta \in (0, \sqrt{-\lambda/2})$, then $-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) > 0$, for large $t$. Hence, Proposition 3.1 and Lemma 3.7 show that for any large $t$,

$$
E_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \right] \leq \pi(\tilde{F}_0) \theta_+ (t) h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t)} R(t)^{(d-1)/2} + h(x(t)) e^{-2\lambda(t-s(t)) - 2\sqrt{-2\lambda} R(t) + C_d(t)} 
$$

$$
= \left( \frac{\pi(\tilde{F}_0) \theta_+ (t) R(t)^{(d-1)/2}}{e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) + C_d(t) + 1}} \right) h(x(t)) e^{-2\lambda(t-s(t)) - 2\sqrt{-2\lambda} R(t) + C_d(t)} 
$$

$$
= \Theta(t) h(x(t)) e^{-2\lambda(t-s(t)) - 2\sqrt{-2\lambda} R(t) + C_d(t)}. 
$$

(ii) Let $\delta \in [\sqrt{-\lambda/2}, \sqrt{-2\lambda})$. By the same way as (3.25),

$$
E_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \right] \leq 1 + \frac{e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) + C_d(t)}}{\pi(\tilde{F}_0) \theta_+ (t) R(t)^{(d-1)/2}} \pi(\tilde{F}_0) \theta_+ (t) 
$$

$$
\times h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}} 
$$

$$
=: \pi(\tilde{F}_0) \Theta'(t) h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}}. 
$$

If we make the assumption that

$$
e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}} = e^{\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}} \rightarrow 0, \quad (3.26)
$$

then $\Theta'(t) \rightarrow 1$. By Proposition 3.1 and Lemma 3.7, we also have the corresponding lower estimate. Thus,

$$
E_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \right] \sim E_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t)^2 \right] \sim \pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}}, 
$$

as $t \rightarrow \infty$. \hfill \Box

Remark 3.1 In the critical case of $\lim_{t \to \infty}(R(t) - R^*(t)) \in (-\infty, \infty)$, i.e., $\delta = \sqrt{-\lambda/2}$, we need (3.24) such that $\Theta'(t) \rightarrow 1$. For example, if $s(t) = 0$, then $\gamma(t) = \log(t \vee 1)$; if $\gamma(t) \equiv \gamma$, then $s(t) = \alpha \log(t \vee 1)$, $0 < \alpha < 1$; if $s(t) \rightarrow \infty$ and $\gamma(t) \rightarrow -\infty$, then $\gamma(t) = o(s(t))$.

4 Proofs of the Main Results

Our proofs of Theorems 2.1–2.3 follow the same approach as [6]. We note that $\gamma(t) = R(t) - R^*(t)$, $\delta = \lim_{t \to \infty} R(t)/t \in (0, \sqrt{-2\lambda})$ and $C_d(t) = (d-1) \log(t \vee 1)$ for $d \geq 1$.

Proposition 4.1 Let $d \geq 1$ and $\delta \in (\sqrt{-\lambda/2}, \sqrt{-2\lambda})$. Then, there exist $\theta_i(t)$ ($i = 1, 2, 3, 4$) with $\theta_i(t) \rightarrow 1$ as $t \rightarrow \infty$ and $T > 0$ such that for all $t \geq T$,

$$
P_{x(t)} \left( E_{t-s(t)}(F_t) = 0 \right) \geq 1 - \pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}} \theta_1(t), \quad (4.1)
$$

$$
P_{x(t)} \left( E_{t-s(t)}(F_t) = 0 \right) \leq 1 - \pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}} \theta_2(t), \quad (4.2)
$$

$$
P_{x(t)} \left( E_{t-s(t)}(F_t) = 1 \right) \geq \pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}} \theta_3(t), \quad (4.3)
$$

$$
P_{x(t)} \left( E_{t-s(t)}(F_t) = 1 \right) \leq \pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t)) - \sqrt{-2\lambda} R(t) R(t)^{(d-1)/2}} \theta_4(t), \quad (4.4)
$$
\[ \mathbf{P}_{x(t)} \left( \mathcal{E}_{t-s(t)}(F_t) > 1 \right) \leq h(x(t)) e^{-2\lambda(t-s(t))-2\sqrt{-2\lambda} R(t)+C_d(t)} . \] (4.5)

When \( \delta = \sqrt{-\lambda/2} \), we assume that \( s(t) \) and \( \gamma(t) \) satisfy (3.24) and \( \gamma(t) = o(\log t) \) as \( t \to \infty \). Then (4.1)–(4.5) hold for \( R(t) = R^*(t) + \gamma(t) \).

**Remark 4.1** By the identification (2.3), (4.1)–(4.5) hold for \( \mathcal{E}_t^0(\tilde{F}_t) \) and \( \mathcal{E}_t^* (F_0) \)

Before we prove Proposition 4.1, we note the following: suppose that \( Z \) is an \( \mathbb{N} \cup \{0\} \)-valued random variable on some probability space \( (\Omega, \mathcal{F}, P) \). Then

\[ P(Z > 1) \leq E[Z^2] - E[Z] \quad \text{and} \quad \frac{E[Z]^2}{E[Z^2]} \leq P(Z > 0) \leq E[Z] . \] (4.6)

**Proof** We first prove (4.5). Lemma 3.7 and (4.6) give that for any large \( t \),

\[ \mathbf{P}_{x(t)} \left( \mathcal{E}_{t-s(t)}(F_t) > 1 \right) \leq \mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right]^2 - \mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right] \]

\[ \leq h(x(t)) e^{-2\lambda(t-s(t))-2\sqrt{-2\lambda} R(t)+C_d(t)} . \]

We next prove (4.1). On account of (4.6), we have

\[ \mathbf{P}_{x(t)} \left( \mathcal{E}_{t-s(t)}(F_t) = 0 \right) = 1 - \mathbf{P}_{x(t)} \left( \mathcal{E}_{t-s(t)}(F_t) > 0 \right) \geq 1 - \mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right] . \]

Then it follows by Proposition 3.1 that for any large \( t \),

\[ \mathbf{P}_{x(t)} \left( \mathcal{E}_{t-s(t)}(F_t) = 0 \right) \geq 1 - \theta_+(t) \pi(\tilde{F}_0) h(x(t)) e^{-\lambda(t-s(t))-\sqrt{-2\lambda} R(t) R(t)(d-1)/2} . \]

We next prove (4.2). By (4.6),

\[ \mathbf{P}_{x(t)} \left( \mathcal{E}_{t-s(t)}(F_t) = 0 \right) = 1 - \mathbf{P}_{x(t)} \left( \mathcal{E}_{t-s(t)}(F_t) > 0 \right) \leq 1 - \frac{\mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right]^2}{\mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right]^2} . \]

According to Proposition 3.1 and Lemma 3.7, we can choose \( \theta_-(t) \) and \( \theta'_+(t) \) both converging to one as \( t \to \infty \), and for any large \( t \),

\[ \frac{\mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right]^2}{\mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right]^2} \geq \frac{\mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right]^2}{\mathbf{E}_{x(t)} \left[ \mathcal{E}_{t-s(t)}(F_t) \right] + Ch(x(t)) e^{-2\lambda(t-s(t))-2\sqrt{-2\lambda} R(t)+C_d(t)}} \]

\[ \geq \frac{\left( \pi(\tilde{F}_0) \theta'_+(t) h(x(t)) e^{-\lambda(t-s(t))-\sqrt{-2\lambda} R(t) R(t)(d-1)/2} \right)^2}{\pi(\tilde{F}_0) \theta'_+(t) h(x(t)) e^{-\lambda(t-s(t))-\sqrt{-2\lambda} R(t) R(t)(d-1)/2} + Ch(x(t)) e^{-2\lambda(t-s(t))-2\sqrt{-2\lambda} R(t)+C_d(t)}} \]

\[ =: \pi(\tilde{F}_0) \theta'_+(t) h(x(t)) e^{-\lambda(t-s(t))-\sqrt{-2\lambda} R(t) R(t)(d-1)/2} , \]

where by (3.26),

\[ \theta_2(t) = \frac{\theta'_+(t)}{\theta'_+(t) + C\pi(\tilde{F}_0)^{-1} e^{-\lambda(t-s(t))-\sqrt{-2\lambda} R(t)+C_d(t)} R(t)^{(d-1)/2}} \to 1, \quad t \to \infty . \] (4.7)

Therefore,

\[ \mathbf{P}_{x(t)} \left( \mathcal{E}_{t-s(t)}(F_t) = 0 \right) \leq 1 - \pi(\tilde{F}_0) \theta'_+(t) h(x(t)) e^{-\lambda(t-s)-\sqrt{-2\lambda} R(t) R(t)(d-1)/2} . \]
Since $\theta_2(t) \to 1$ as $t \to \infty$, we have (4.2).

We finally prove (4.3) and (4.4). By (4.1), (4.2) and (4.5),

\[
P_x(t) \left( \mathcal{E}_{t-s(t)}(F_t) = 1 \right) = 1 - P_x(t) \left( \mathcal{E}_{t-s(t)}(F_t) > 1 \right) - P_x(t) \left( \mathcal{E}_{t-s(t)}(F_t) = 0 \right) \geq 1 - h(x(t)) e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2})
\]

\[
= \left( \theta_2(t) - \frac{e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2})}{\pi(F_0(t) R(t)^{(d-1)/2})} \right) \pi(F_0(t) h(x(t)) e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2})
\]

\[
= \theta_3(t) \pi(F_0(t) h(x(t)) e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2}) = \theta_3(t) \pi(F_0(t) h(x(t)) e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2})
\]

Similarly to (4.7), $\theta_3(t) \to 1$ as $t \to \infty$. By (4.6) and Proposition 3.1, there exists $\theta_4(t)$ such that $\theta_4(t) \to 1$ as $t \to \infty$ and

\[
P_x(t) \left( \mathcal{E}_{t-s(t)}(F_t) = 1 \right) \leq P_x(t) \left( \mathcal{E}_{t-s(t)}(F_t) > 0 \right) \leq P_x(t) \left( \mathcal{E}_{t-s(t)}(F_t) \right) \leq \pi(F_0(t) h(x(t)) e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2}) \theta_4(t).
\]

□

The rest of this section will be devoted to the proofs of our main theorems.

4.1 Proof of Theorem 2.1

Proof As mentioned in Remark 4.1, we can use Proposition 4.1 for $\tilde{\mathcal{E}}_i(\tilde{F}_t)$. In Proposition 4.1, we choose $x(t) \equiv x$ and $s(t) \equiv 0$. We note that if $s(t) \equiv 0$ and $\gamma(t) = R(t) - R^*(t) \to \infty$, then the condition (3.24) is fulfilled. By (4.1) and (4.2), there exist $T > 0$ and $\theta_i(t)$, $i = 1, 2$ such that for any $t \geq T$,

\[
\pi(F_0(t) h(x(t)) e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2}) \leq P_x(\tilde{\mathcal{E}}_i(\tilde{F}_t) > 0)
\]

\[
\leq \pi(F_0(t) h(x(t)) e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2}) \theta_4(t).
\]

We can rewrite

\[
e^{-\lambda t} (1 - \frac{1}{2} \sqrt{-\lambda} R(t) \theta_2(t) R(t)^{(d-1)/2}) = e^{\delta (-\frac{1}{2} \sqrt{-\lambda} R(t) - R(t)^{(d-1)/2})},
\]

and we have the conclusion. □

4.2 Proof of Theorem 2.2

We see from the Markov property and variance formula that

\[
E_x \left[ (\tilde{\mathcal{E}}_i(\tilde{F}_t) - E_x[\tilde{\mathcal{E}}_i(\tilde{F}_t) | G_{t(s(t))}]^2 \right] \leq \sum_{u \in Z(s(t))} E_{B^u_{s(t)}} \left[ (\tilde{\mathcal{E}}_{t-s(t)}(\tilde{F}_t))^2 \right].
\]

We assume that $\gamma(t) = R(t) - R^*(t) \to -\infty$ and $R(t) \sim \delta t$ for some $\delta \in (0, \sqrt{-\lambda}/2]$, which are divided into two cases: $\delta \in (0, \sqrt{-\lambda}/2)$; $\delta = \sqrt{-\lambda}/2$ and $\gamma(t) \to -\infty$ with $\gamma(t) = o(\log t)$. 

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Proof of Theorem 2.2 Since $M_t \to M_\infty$, $\mathbb{P}_x$-a.s. and

$$
\left| e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \bar{E}(\bar{F}_t) - \pi(\bar{F}_0) M_\infty \right| \\
\leq e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \left| \bar{E}(\bar{F}_t) - \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] \right| + e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] - \pi(\bar{F}_0) M_{s(t)} + \pi(\bar{F}_0) \left| M_{s(t)} - M_\infty \right|,
$$

we show the followings:

$$
e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \left| \bar{E}(\bar{F}_t) - \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] \right| \to 0, \quad \text{in probability } \mathbb{P}_x,
$$

(4.9)

$$
e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] - \pi(\bar{F}_0) M_{s(t)} \to 0, \quad \mathbb{P}_x\text{-a.s. (4.10)}
$$

We show that firstly, (4.9) holds for $\delta \in (0, \sqrt{-\lambda/2}]$; secondly, (4.10) does for $\delta \in (0, \sqrt{-2\lambda})$. Consequently, we have the desired conclusion in Theorem 2.2.

We first prove (4.9). Fix $\varepsilon > 0$ and set $b(t) = (\sqrt{-\lambda/2} + \varepsilon) s(t)$, where $s(t) \to \infty$. For any $\varepsilon > 0$,

$$
\begin{align*}
\mathbb{P}_x \left( e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \left| \bar{E}(\bar{F}_t) - \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] \right| > \varepsilon \right) \\
= \mathbb{P}_x \left( e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \left| \bar{E}(\bar{F}_t) - \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] \right| > \varepsilon, \; L_{s(t)} \leq b(t) \right) \\
+ \mathbb{P}_x \left( e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \left| \bar{E}(\bar{F}_t) - \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] \right| > \varepsilon, \; L_{s(t)} > b(t) \right).
\end{align*}
$$

(4.11)

Since $L_t / t \to \sqrt{-\lambda/2}$, $\mathbb{P}_x$-a.s. by (2.5), the last term of (4.11) converges to zero. We thus show that the middle term of (4.11) converges to zero, for $\delta \in (0, \sqrt{-\lambda/2}]$. Let $\delta \in (0, \sqrt{-2\lambda})$. Then the Chebyshev inequality and (4.8) yield that

$$
\begin{align*}
\mathbb{P}_x \left( e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \left| \bar{E}(\bar{F}_t) - \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] \right| > \varepsilon, \; L_{s(t)} \leq b(t) \right) \\
= \mathbb{E}_x \left[ \mathbb{P}_x \left( e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)/2} \left| \bar{E}(\bar{F}_t) - \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] \right| > \varepsilon \mid G_{s(t)} \right) \right]; \; L_{s(t)} \leq b(t)
\end{align*}
$$

$$
\leq \frac{e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)}}{\varepsilon^2} \mathbb{E}_x \left[ \left( \bar{E}(\bar{F}_t) - \mathbb{E}_x \left[ \bar{E}(\bar{F}_t) \mid G_{s(t)} \right] \right)^2 \mid G_{s(t)} \right]; \; L_{s(t)} \leq b(t)
$$

$$
\leq \frac{e^{-\sqrt{2\lambda} (R(t) - R^*) (t / R(t)) (d-1)}}{\varepsilon^2} \mathbb{E}_x \left[ \sum_{u \in Z_{s(t)}} \mathbb{E}_{b_{s(t)}} \left[ \left( \bar{E}_{s(t)} (\bar{F}_t) \right)^2 \right]; \; L_{s(t)} \leq b(t) \right].
$$

(4.12)

We note that $|\mathbb{B}_{s(t)}^u| \leq b(t)$ for any $u \in Z_{s(t)}$ on the event $\{L_{s(t)} \leq b(t)\}$. According to Proposition 3.3 (i), we can take non-random $T > 0$ so large that for all $t > T$, the second moment in (4.12) is bounded by (3.23) uniformly on the event $\{L_{s(t)} \leq b(t)\}$. That is, for
\[ \delta \in (0, \sqrt{-\lambda/2}), \]

\[ (4.12) \leq \frac{\Theta(t) e^{\lambda s(t)} (t/R(t))^{d-1}}{\varepsilon^2} E_x \left[ e^{\lambda s(t)} \sum_{u \in Z_{s(t)}} h(B_{s(t)}^u) : L_{s(t)} \leq b(t) \right] \]

\[ \leq \frac{\Theta(t) e^{\lambda s(t)} (t/R(t))^{d-1}}{\varepsilon^2} E_x \left[ M_{s(t)} \right] = \frac{\Theta(t) e^{\lambda s(t)} (t/R(t))^{d-1}}{\varepsilon^2} h(x) \to 0, \quad t \to \infty. \]

We now consider the critical case of \( \gamma(t) = R(t) - R^* (t) \to -\infty \) and \( \gamma(t) = o(\log t), \quad t \to \infty. \) We take \( s(t) = O(\log t), \) which satisfies both (3.24) and the condition as mentioned at the beginning of Sect. 3. By the same argument as (4.12) and Proposition 3.3 (ii),

\[ \mathbb{P}_x \left( e^{\sqrt{\lambda x} (R(t)-R^*(t))} (t/R(t))^{(d-1)/2} \left| \tilde{\xi}_t (\tilde{F}_t) - \mathbb{E}_x \left[ \tilde{\xi}_t (\tilde{F}_t) : \mathcal{G}_{s(t)} \right] \right| > \varepsilon, \quad L_{s(t)} \leq b(t) \right) \]

\[ \leq \frac{e^{2\sqrt{\lambda x} (R(t)-R^*(t))} (t/R(t))^{d-1}}{\varepsilon^2} E_x \left[ \sum_{u \in Z_{s(t)}} \mathbb{E}_{s(t)}^u \left[ \tilde{\xi}_{t-s(t)} (\tilde{F}_t)^2 \right] : L_{s(t)} \leq b(t) \right] \]

\[ \leq \frac{Ce^{\sqrt{\lambda x} (R(t)-R^*(t))} (t/R(t))^{(d-1)/2}}{\varepsilon^2} E_x \left[ e^{\lambda s(t)} \sum_{u \in Z_{s(t)}} h(B_{s(t)}^u) : L_{s(t)} \leq b(t) \right] \]

\[ \leq \frac{C}{\varepsilon^2} h(x) e^{\sqrt{\lambda x} \gamma(t)} (t/R(t))^{(d-1)/2} \to 0. \]

Thus, the middle term of (4.11) also converges to zero, for \( \delta = \sqrt{-\lambda/2}. \)

We next prove (4.10). By abuse of notation, we use

\[ A(t) = e^{\sqrt{\lambda x} (R(t)-R^*(t))} (t/R(t))^{(d-1)/2} \mathbb{E}_x \left[ \tilde{\xi}_t (\tilde{F}_t) : \mathcal{G}_{s(t)} \right]. \]

Then we see that for any \( \delta \in (0, \sqrt{-2\lambda}), A(t) \to \pi(\tilde{F}_0) M_{\infty}, \mathbb{P}_x \)-a.s. In fact, by Proposition 3.1, there exists non-random \( T' > T \) such that for any \( t > T' \), we have uniformly on \( \{ L_{s(t)} \leq b(t) \} \),

\[ A(t) = e^{\sqrt{\lambda x} (R(t)-R^*(t))} (t/R(t))^{(d-1)/2} \sum_{u \in Z_{s(t)}} \mathbb{E}_{s(t)}^u \left[ \tilde{\xi}_{t-s(t)}(\tilde{F}_t) \right] \]

\[ \leq \frac{e^{\sqrt{\lambda x} (R(t)-R^*(t))} (t/R(t))^{(d-1)/2}}{\varepsilon^2} \sum_{u \in Z_{s(t)}} \theta_+(t) \pi(\tilde{F}_0) h(B_{s(t)}^u) e^{-\lambda (t-s(t)) - \sqrt{\lambda x} R(t)} R(t)^{(d-1)/2} \]

\[ = \pi(\tilde{F}_0) \theta_+(t) M_{s(t)}. \]

We also have \( A(t) \geq \pi(\tilde{F}_0) \theta_-(t) M_{s(t)} \) so that \( A(t) \to \pi(\tilde{F}_0) M_{\infty}, \mathbb{P}_x \)-a.s. We thus have (4.10). \( \square \)

### 4.3 Proof of Theorem 2.3

Let \( d = 1, 2 \) and \( \gamma(t) = R(t) - R^*(t) \to \gamma \in (-\infty, \infty). \) If \( \gamma(t) \to \gamma, \) then we can choose \( s(t) \) which satisfies both (3.24) and the condition as mentioned at the beginning of Sect. 3. In addition, we can take such \( s(t) \) independently of \( \gamma(t), \) for example \( s(t) = \alpha \log (t \vee 1) \) for any \( \alpha \in (0, 1). \) By [15, Proposition 16.17], we only prove (4.15) and (4.16) below. The main
calculation is summed up in Lemma 4.1, which is proved by using the Markov property at \( s(t) \) and Proposition 4.1.

For a fixed \( \varepsilon > 0 \), we use \( b(t) = (\sqrt{-\lambda/2} + \varepsilon)s(t) \) as in (4.11), throughout this subsection.

**Lemma 4.1** There exists a non-random \( T \) and \( \kappa_1(t), \kappa_2(t) \) such that for all \( t \geq T \),

\[
\exp\left(-\kappa_1(t)\pi(\tilde{F}_0)M_{s(t)}e^{-\sqrt{-\lambda}\gamma(t)(R(t)/t)^{(d-1)/2}}\right) \leq \prod_{u \in Z_{s(t)}} P_{B_u^s(t)}(\tilde{E}_{t-s(t)}^u(\tilde{F}_0) = 0)
\]

\[
\leq \exp\left(-\kappa_2(t)\pi(\tilde{F}_0)M_{s(t)}e^{-\sqrt{-\lambda}\gamma(t)(R(t)/t)^{(d-1)/2}}\right)
\]

uniformly on the event \( \{L_{s(t)} \leq b(t)\} \). Here \( \kappa_1(t), \kappa_2(t) \rightarrow 1 \) as \( t \rightarrow \infty \).

By (2.5), \( \|\sum_{t \leq T}\|L_{s(t)} = \|\sum_{t \leq T\sqrt{\gamma(t)}(L_{s(t)}/s(t))} \rightarrow 1 \), \( t \rightarrow \infty \), \( \mathbf{P}_x \)-a.s. Since \( (R(t)/t)^{(d-1)/2} \rightarrow (\lambda/2)^{(d-1)/4} \) as \( t \rightarrow \infty \), we see from (4.13) that, \( \mathbf{P}_x \)-a.s.,

\[
\lim_{t \rightarrow \infty} \mathbf{P}_{x(t)}\left(L_{s(t)} \prod_{u \in Z_{s(t)}} P_{B_u^s(t)}(\tilde{E}_{t-s(t)}^u(\tilde{F}_0) = 0) \right)
\]

\[
= \begin{cases} 
\exp\left\{-\left(\frac{\lambda}{2}\right)^{(d-1)/4} e^{-\sqrt{-\lambda}\gamma(t)\pi(\tilde{F}_0)M_{s(t)}}\right\}, & \text{if } \gamma(t) \rightarrow \gamma < \infty, \\
1, & \text{if } \gamma(t) \rightarrow \infty,
\end{cases}
\]

**Proof of Lemma 4.1** As mentioned in Remark 4.1, the asymptotic behavior of \( \mathbf{P}_{x(t)}(\tilde{E}_{t-s(t)}^u(\tilde{F}_0) = 0) \) is given by Proposition 4.1. Thus, there exists a non-random \( T > 0 \) such that for any \( t > T \) and \( x(t) \) with \( |x(t)| \leq b(t) \),

\[
\mathbf{P}_{x(t)}(\tilde{E}_{t-s(t)}^u(\tilde{F}_0) = 0) \leq 1 - \pi(\tilde{F}_0)h(x(t)) e^{\lambda s(t) - \sqrt{-\lambda}\gamma(t)(R(t)/t)^{(d-1)/2}\theta_2(t)}.
\]

Since \( |B_u^s(t)| \leq b(t) \) for all \( u \in Z_{s(t)} \) on the event \( \{L_{s(t)} \leq b(t)\} \) and \( 1 - x \leq e^{-x} \) for all \( x \in \mathbb{R} \),

\[
\prod_{u \in Z_{s(t)}} P_{B_u^s(t)}(\tilde{E}_{t-s(t)}^u(\tilde{F}_0) = 0)
\]

\[
\leq \exp\left\{-\sum_{u \in Z_{s(t)}} \pi(\tilde{F}_0)h(B_u^s(t)) e^{\lambda s(t) - \sqrt{-\lambda}\gamma(t)(R(t)/t)^{(d-1)/2}\theta_2(t)}\right\}
\]

\[
= \exp\left\{-\pi(\tilde{F}_0)\theta_2(t)M_{s(t)}e^{-\sqrt{-\lambda}\gamma(t)(R(t)/t)^{(d-1)/2}}\right\}.
\]

For a fixed \( x^* \in (0, 1) \), \( \log(1-x) \geq \frac{\log(1-x^*)}{x^*} x \) for any \( x \in (0, x^*) \). Then by Proposition 4.1, for any \( t > T \),

\[
\prod_{u \in Z_{s(t)}} P_{B_u^s(t)}(\tilde{E}_{t-s(t)}^u(\tilde{F}_0) = 0)
\]

\[
\geq \prod_{u \in Z_{s(t)}} \left(1 - \pi(\tilde{F}_0)h(B_u^s(t)) e^{\lambda s(t) - \sqrt{-\lambda}\gamma(t)(R(t)/t)^{(d-1)/2}\theta_1(t)}\right)
\]
\[
\exp \left\{ \sum_{u \in \mathcal{Z}_{s(t)}} \log \left( 1 - \pi(\tilde{F}_0) h\left( B^u_{s(t)} \right) e^{\lambda s(t) - \sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2 \theta_1(t)} \right) \right\}
\geq \exp \left\{ \sum_{u \in \mathcal{Z}_{s(t)}} \frac{\log \left( 1 - \pi(\tilde{F}_0) \|h\|_\infty e^{\lambda s(t) - \sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2 \theta_1(t)} \right)}{\pi(\tilde{F}_0) \|h\|_\infty e^{\lambda s(t) - \sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2 \theta_1(t)}} \times \pi(\tilde{F}_0) h\left( B^u_{s(t)} \right) e^{\lambda s(t) - \sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2 \theta_1(t)} \right\}
= \exp \left\{ \log \left( 1 - \pi(\tilde{F}_0) \|h\|_\infty e^{\lambda s(t) - \sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2 \theta_1(t)} \right) \right\}
\times \pi(\tilde{F}_0) \theta_1(t) M_{s(t)} e^{-\sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2} \theta_1(t) \right\}.
\]

We note that \\
\[ e^{\lambda s(t) - \sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2} \sim \left( -\frac{\lambda}{2} \right)^{(d-1)/4} e^{-\sqrt{2} \lambda y(t)} \to 0, \]
by (3.24). Since \(-\frac{\log(1-x)}{x} \to 1\) as \(x \downarrow 0\), we have \\
\[ \kappa_1(t) \equiv -\frac{\log \left( 1 - \pi(\tilde{F}_0) \|h\|_\infty e^{\lambda s(t) - \sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2 \theta_1(t)} \right)}{\pi(\tilde{F}_0) \|h\|_\infty e^{\lambda s(t) - \sqrt{2} \lambda y(t)} \frac{R(t)/t}{(d-1)/2 \theta_1(t)}} \theta_1(t) \to 1, \quad t \to \infty, \]
so that (4.13) follows. \(\square\)

**Proof of Theorem 2.3** Let \(\gamma(t) = R(t) - R^*(t)\). We suppose that \(\gamma(t) \to \gamma \in (-\infty, \infty)\). By [15, Proposition 16.17], we only consider two conditions: (i) for all relatively compact sets \(\tilde{F} \subseteq \mathbb{R} \times S^{d-1}\), \\
\[ \lim_{t \to \infty} P_x \left( \mathcal{E}^*_t(\tilde{F}) = 0 \right) = E_x \left[ \exp \left\{ -\left( -\frac{\lambda}{2} \right)^{(d-1)/4} e^{-\sqrt{2} \lambda y(t)} \pi(\tilde{F}) \right\} \right] \] (4.15)
and (ii) for all compact sets \(\tilde{K} \subseteq \mathbb{R} \times S^{d-1}\), \\
\[ \limsup_{t \to \infty} E_x \left[ \mathcal{E}^*_t(\tilde{K}) \right] \leq E_x \left[ \left( -\frac{\lambda}{2} \right)^{(d-1)/4} e^{-\sqrt{2} \lambda y(t)} \pi(\tilde{K}) \right] \right\}
\[ = \left( -\frac{\lambda}{2} \right)^{(d-1)/4} e^{-\sqrt{2} \lambda y(t)} \pi(\tilde{K}) h(x). \] (4.16)

Therefore, we can conclude that \(\mathcal{E}^*_t\) converges to a Poisson point process such that the random intensity is \((-\lambda/2)^{(d-1)/4} e^{-\sqrt{2} \lambda y(t)} \pi(\cdot) \) \(M_\infty\). If we take \(\tilde{F}^1_0, \ldots, \tilde{F}^g_0\) mutually disjoint relatively compact sets in \(\mathbb{R} \times S^{d-1}\), then we have (2.13).

(i) We choose \(s(t)\) which is same as in the beginning of this subsection. Then \\
\[ P_x \left( \mathcal{E}^*_t(\tilde{F}) = 0 \right) = P_x \left( \mathcal{E}^*_t(\tilde{F}) = 0, L_{s(t)} \leq b(t) \right) + P_x \left( \mathcal{E}^*_t(\tilde{F}) = 0, L_{s(t)} > b(t) \right) \] (4.17)
and the second term converges to zero by (2.5). We here consider the limit of the first term in (4.17). By the Markov property,

\[
P_x (\mathcal{E}_t^s (\widetilde{F}) = 0, L_{s(t)} \leq b(t)) = E_x \left[ E_x \left[ \mathcal{E}_t^s (\widetilde{F}) = 0 \mid \mathcal{G}_{s(t)} \right] ; L_{s(t)} \leq b(t) \right] \\
= E_x \left[ \prod_{u \in Z_{s(t)}} P_{B_{s(t)}} (\mathcal{E}_t^u (\widetilde{F}) = 0) ; L_{s(t)} \leq b(t) \right].
\]

We see from (4.13), (4.14) and the bounded convergence theorem that,

\[
\lim_{t \to \infty} \inf P_x (\mathcal{E}_t^s (\widetilde{F}) = 0, L_{s(t)} \leq b(t)) \geq \lim_{t \to \infty} \inf E_x \left[ \exp \left( -\kappa_1(t) \pi (\widetilde{F}) M_{s(t)} e^{-\sqrt{-2} \lambda \gamma(t)} (R(t) / t)^{(d-1)/2} ; L_{s(t)} \leq b(t) \right) \right] \\
= \begin{cases} 
E_x \left[ \exp \left( -\frac{\lambda}{2} \right)^{(d-1)/4} e^{-\sqrt{-2} \lambda \gamma \pi (\widetilde{F}) M_{\infty}} \right], & \text{if } \gamma(t) \to \gamma \in (-\infty, \infty), \\
1, & \text{if } \gamma(t) \to \infty.
\end{cases}
\]

Here, the above inequality is adapted from \( P_x (L_{s(t)} > b(t)) \to 0 \) as \( t \to \infty \). Hence, when \( \gamma(t) \to \infty \), the limit of \( P_x (\mathcal{E}_t^s (\widetilde{F}) = 0, L_{s(t)} \leq b(t)) \) exists and this is equal to one. In the case of \( \gamma(t) \to \gamma \),

\[
\lim_{t \to \infty} \sup P_x (\mathcal{E}_t^s (\widetilde{F}) = 0, L_{s(t)} \leq b(t)) \\
\leq \lim_{t \to \infty} \sup E_x \left[ \exp \left( -\kappa_2(t) \pi (\widetilde{F}) M_{s(t)} e^{-\sqrt{-2} \lambda \gamma(t)} (R(t) / t)^{(d-1)/2} ; L_{s(t)} \leq b(t) \right) \right] \\
\leq E_x \left[ \exp \left( -\frac{\lambda}{2} \right)^{(d-1)/4} e^{-\sqrt{-2} \lambda \gamma \pi (\widetilde{F}) M_{\infty}} \right].
\]

Therefore, (4.15) holds.

In Proposition 3.1, we can take \( x(t) \equiv x \) and \( s(t) \equiv 0 \). As \( t \to \infty \),

\[
E_x \left[ \mathcal{E}_t^s (\widetilde{K}) \right] \sim \pi (\widetilde{K}) h(x) e^{-\lambda t - \sqrt{-2} \lambda R(t)} R(t)^{(d-1)/2} = \pi (\widetilde{K}) h(x) e^{-\sqrt{-2} \lambda \gamma(t)} (R(t) / t)^{(d-1)/2}.
\]

Therefore, (4.16) is proved. \( \square \)

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**Declarations**

**Competing Interests** The author declares no competing interests.
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