The Biker-Hiker Problem

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Abstract

There are \( n \) travellers who have \( k \) bicycles and they wish to complete a journey in the shortest possible time. We investigate optimal solutions of this problem where each traveller cycles for \( \frac{k}{n} \) of the journey. Each solution is represented by an \( n \times n \) binary matrix \( M \) with \( k \) non-zero entries in each row and column. We determine when such a matrix gives an optimal solution. This yields an algorithm deciding the question of optimality of complexity \( O(n^2 \log n) \). We introduce three symmetries of matrices that preserve optimality, allowing identification of minimal non-optimal members of this class. An adjustment to optimal solutions that eliminates unnecessary handovers of cycles is established, which maintains all other features of the solution. We identify two mutually transpose solution types, the first uniquely minimises the number of handovers, while the second keeps the number of separate cohorts to three while bounding their overall separation, in the case \( 2k \leq n \), to under \( \frac{2}{n} \) of the journey.

1 The problem: not enough bicycles

There are \( n \) friends who have \( k \) bicycles between them and the group needs to reach its destination as soon as possible. How should they go about doing this? An early allusion to this problem is in the novel *The Great House* by Cynthia Harnett [2]. Here a pair of 17th century travelling companions with only one horse between them adopt the ‘ride and tie’ method for their journey from Henley-on-Thames to London.

Assumptions Every person walks and cycles at the same speed as all the others, and cycling is faster than walking. We assume that the time required to swap from one form of locomotion to the other is negligible. For brevity, individual travellers will sometimes be referred to as ‘he’ while a set of travellers will be referred to as ‘they’.

Solution Suppose that we devise a scheme, we shall call it an *optimal* scheme, in which each traveller cycles for \( \frac{k}{n} \) of the length of the journey and never stops moving forward at any stage. Each will then have cycled and walked the same distance as each of their companions and so all \( n \) friends will arrive at their destination simultaneously. We claim that, if it exists, such a scheme is truly optimal in that it delivers the entire group to its destination in the least possible time, and that any non-optimal scheme is inferior in this respect.
It is convenient to consider the length of the journey to be \( n \) units, (although we will consider divisors other than \( n \)). To see that a optimal scheme is best, note that the maximum (net) forward progress by bicycle of any scheme is \( kn \). It follows that if one member of the group of \( n \) travellers cycled more than \( k \) units, then some other member must cycle less than \( k \) units. This latter traveller would then take longer than others who have cycled \( k \) units (or more). Hence any approach that involved any member cycling forward a total distance other than \( k \) units would take longer to deliver the entire group to their destination as opposed to an approach that adopted an optimal scheme.

That cycling is faster than walking makes the problem more interesting, a fact that is highlighted by considering the Backpack-hiker problem. Here there are \( k \) heavy backpacks to be transported to the finish and any traveller carrying a backpack walks more slowly than one that is unencumbered. The change in relative speeds makes this problem much simpler and less interesting as for any value of \( k \) \((1 \leq k \leq n)\) the minimum time for the group to complete the journey is the length of the journey divided by the speed of a backpack walker.

In Section 2 we list the properties of optimal schemes more formally through a discretised representation of the Biker-hiker problem based on square binary matrices. In Section 3 we characterise those matrices that correspond to optimal solutions and show that we may decide the question of optimality for a given matrix with an algorithm that involves \( O(n^2 \log n) \) comparisons of partial sums of the rows of the matrix. We identify three symmetries of these optimal schemes, which leads to the discovery of minimal schemes that assign \( k \) cycled stages to each traveller and \( k \) cyclists to each stage but are nonetheless not optimal. In Sections 4 and 5 we identify and investigate a certain mutually transpose pair of optimal matrices for arbitrary values of the parameters \( n \) and \( k \). Section 6 looks at certain facets of these special schemes.

### 2 k-uniform solutions

It will be convenient to allocate a measure of \( n \) units for the total length of the road the travellers will take, which we may take to be either linear or a circuit. Along the length of the journey we imagine there to be \( n + 1 \) equally spaced staging posts \( P_0, P_1, \ldots, P_n \), with \( P_0 \) and \( P_n \) marking the beginning and end of the trip respectively, so that the distance between successive signposts is 1 unit.

We assign numbered symbols to each of the \( n \) travellers as we shall call them, \( t_1, t_2, \ldots, t_n \).

**Definition 2.1** (a) The problem of delivering the \( n \) travellers equipped with \( k \) bicycles \((0 \leq k \leq n)\) to their common destination in a way that minimizes the time of the last arrival will be called the \((n,k)\)-problem.

(b) The leg of the journey from \( P_{j-1} \) to \( P_j \) is called stage \( j \) and is denoted by \( s_j \) \((1 \leq j \leq n)\).

(c) An \( n \)-scheme \( S \) is one in which each traveller \( t_i \) is directed to travel each stage \( s_j \) \((1 \leq j \leq n)\) either on foot, or by bicycle.
(d) The incidence matrix $M = M(S)$ of an $n$-scheme $S$ is the $n \times n$ binary matrix $M = (m_{i,j})$ $(1 \leq i, j \leq n)$ where $m_{i,j} = 0$ or $m_{i,j} = 1$ according as traveller $t_i$ is directed to walk or cycle respectively stage $s_j$ from $P_{j-1}$ to $P_j$. We shall write $R_i$ and $C_j$ for the $i$th row and $j$th column of $M$ respectively.

(e) The scheme $S = S(M)$ of an $n \times n$ binary matrix $M = (m_{i,j})$ is that in which traveller $t_i$ travels $s_j$ on foot or by bicycle according as $m_{i,j} = 0$ or $m_{i,j} = 1$ $(1 \leq i, j \leq n)$. Note that $S(M(S)) = S$ and $M(S(M)) = M$.

**Definition 2.2** An $n \times n$ binary matrix $M = (m_{i,j})$ is $k$-uniform if each row and each column contains exactly $k$ entries equal to $1$.

**Proposition 2.3** A scheme $S$ is optimal if and only if

(i) $M(S)$ is $k$-uniform and

(ii) whenever a set of travellers $C$ arrives at a post $P_j$, the number of cycles at $P_j$ is at least as great as the number of $t_i \in C$ such that $m_{i,j} + 1 = 1$.

**Proof** If $S$ is optimal then each traveller $t_i$ rides exactly $k$ stages so that $R_i$ has exactly $k$ entries which equal $1$. There are then $nk$ entries of $M$ equal to $1$. If it were not the case that each column had exactly $k$ non-zero entries, then some column would contain more than $k$ $1$’s, which is impossible as no cycle may travel twice through the same stage. Therefore $M$ is $k$-uniform. As for Condition (ii), if it were violated then some traveller would have to stop at some stage to wait for a bicycle to arrive for their use. The time taken for their journey would then exceed the optimal time unless he cycled more than $k$ stages, in which case some other traveller would cycle fewer than $k$ stages, and the overall time for the group to complete the journey would exceed the optimal time. Hence if $S$ is optimal, both Conditions (i) and (ii) must be met.

Conversely any scheme $S$ represented by a $k$-uniform matrix $M$ has exactly $k$ entries of $1$ in each row so that each traveller is scheduled to ride $k$ stages.

**Definition 2.4** We call a square $k$-uniform binary matrix $M$ optimal if $S(M)$ is optimal.

### 3 Optimal matrices and their symmetries

#### Assignment mappings

We will now introduce assignment mappings $\phi_j$ for the each stage $s_j$ $(1 \leq j \leq n - 1)$ of a scheme $S$. Suppose $m_{i,j} = 1$, meaning that $t_i$ cycles $s_j$. Then $\phi_j(i) = p$ conveys the information that $t_p$ will cycle $s_{j+1}$ on the cycle left behind at $P_j$ by $t_i$.

**Definition 3.1** Let $S$ denote an $n \times n$ scheme with matrix $M = M(S) = (m_{i,j})$. A one-to-one partial mapping $\phi_j$ $(1 \leq j \leq n - 1)$ is an assignment mapping for $S$ if
\[ \text{dom} \phi_j = \{ i : m_{i,j} = 1 \}, \text{ran} \phi = \{ i : m_{i,j+1} = 1 \}. \]

The main result of this section characterises optimal schemes in terms of the existence of a collection of assignment mappings that satisfy two constraints. The first is the optional constraint that allows a rider to stay on the same bike if he is required to ride two successive stages. The second constraint ensures that \( S \) is in accord with Proposition 2.3.

**Theorem 3.2** Let \( M \) be an \( n \times n \) binary matrix. Then \( S(M) \) is optimal if and only if \( M \) is \( k \)-uniform for some \( k \) \((0 \leq k \leq n)\) and for each \( j, (1 \leq j \leq n-1) \) there exist assignment mappings \( \phi_j \) such that

\[ \phi_j(i) = i \iff (m_{i,j} = m_{i,j+1} = 1) \quad \text{(1)} \]

\[ \sum_{i'=1}^j m_{i',1} \leq \sum_{l=1}^j m_{i,l}, \text{ where } i' \text{ denotes } \phi_j(i). \quad \text{(2)} \]

**Proof** Suppose that \( M \) is \( k \)-uniform and satisfies Conditions (1) and (2). Suppose inductively that the scheme \( S(M) \) has not failed up to stage \( s_j \), which holds when \( j = 1 \) as \( C_1 \) has \( k \) entries that equal 1, and so travellers assigned to cycle \( s_1 \) may do so.

Next consider stage \( s_{j+1} \) from \( P_j \) to \( P_{j+1} \). For each \( i' \) such that \( m_{i',j+1} = 1 \) there exists a unique \( i \) such that \( m_{i,j} = 1 \) and \( \phi_j(i) = i' \). By the inductive hypothesis, \( t_i \) has arrived at \( P_j \) by cycle without stalling. Condition (2) is then exactly the requirement that ensures that this has occurred no later than the arrival of \( t_{i'} \) at \( P_j \). Hence \( S \) may continue with \( t_{i'} \) riding \( s_{j+1} \) on the cycle that \( t_i \) has ridden on \( s_j \). Therefore \( s_{j+1} \) may be completed without stalling, and the induction continues. The process will therefore end with \( S(M) \) being fully executed without stalling, and so \( S(M) \) is indeed optimal.

Conversely, suppose that \( S(M) \) is optimal. Then at stage \( s_{j+1} \) \((j \geq 0)\), for each \( i' \) such that \( m_{i',j+1} = 1 \), it is possible for \( t_{i'} \) to ride \( s_{j+1} \) on a cycle that has been left at \( P_j \) by some traveller \( t_i \). It follows that Condition (2) is then met. This correspondence defines a partial one-to-one mapping:

\[ \phi_j^{-1} : \{ i : m_{i,j+1} = 1 \} \rightarrow \{ i : m_{i,j} = 1 \}. \]

By uniformity, \( \phi_j^{-1} \) is also surjective and so the partial one-to-one mapping \( \phi_j \) is an assignment mapping which satisfies Condition (2). We now show that \( \phi_j \) may be modified so that it also satisfies Condition (1). The forward direction of the implication in (1) follows from the definition of an assignment map, but the reverse implication does not follow from the optimality of \( S(M) \).

Let us write \( \phi \) for \( \phi_j \) and, as before, abbreviate \( \phi_j(i) \) to \( i' \). Suppose then that \( m_{i,j} = m_{i,j+1} = 1 \) but \( i \neq i' \). We consider the sequence \( I = i, \phi(i), \phi^2(i), \cdots \). If \( I \) is a cycle, so that for some positive integer \( p \), \( \phi^p(i) = i \), then it follows that \( m_{i,j} = 1 = m_{\phi(t),j} \) for all \( t = \phi^k(i) \) \((k \geq 0)\). In this case we may modify \( \phi \) (while
retaining the same symbol \( \phi \) for the mapping) such that \( \phi(t) = t \) for all \( t = \phi^k(i) \), in accord with Condition (1). Moreover, applying Condition (2) repeatedly yields a cycle of inequalities that begins and ends with the same sum, and so are in fact equalities, indicating that all the travellers \( t_i, t_{\phi(i)}, \ldots, t_{\phi^e(i)} = t_i \) arrive at \( P_j \) simultaneously. The original assignment mapping \( \phi \) instructed this set of travellers to exchange bicycles in accord with the cycle \( I \). The modified mapping simply allows each traveller to remain on the bike he is currently riding.

Alternatively, the sequence \( I \) does not generate a cycle. Then by definition of \( \phi \) there exists a sequence of maximal length:

\[
i_{-r}, i_{-r+1}, \cdots i_0 = i, i_1 = \phi(i), i_2, \cdots, i_{s-1}, i_s
\]

such that \( \phi(i_p) = i_{p+1}, (-r \leq p \leq s-1), (r, s \geq 1) \).

In (3), \( m_{i_{-r},j+1} = 0 = m_{i_s,j} \) and \( m_{t,j} = m_{t,j+1} = 1 \) for all \( -r + 1 \leq t \leq s - 1 \). We now modify \( \phi \) by putting

\[
\phi(t) = t \forall -r + 1 \leq t \leq s - 1
\]

\[
\phi(i_{-r}) = i_s,
\]

for then Condition (2) holds trivially for \( i = t \) as in (4), and (2) also holds for (5) for \( i = i_{-r} \), \( i' = i_s \) as applying Condition (2) repeatedly for \( \phi \) we have:

\[
\sum_{l=1}^{j} m_{i_{-r}, l} \geq \sum_{l=1}^{j} m_{i_{-r+1}, l} \geq \cdots \geq \sum_{l=1}^{j} m_{i_{s-1}, l} \geq \sum_{l=1}^{j} m_{i_s, l},
\]

which, in the notation of Theorem 3.2, provides the required inequality concerning \( i_{-r} \) and \( i_s = \phi_j(i_{-r}) = i'_{-r} \):

\[
\sum_{l=1}^{j} m_{i_{-r}, l} \leq \sum_{l=1}^{j} m_{i_{-r+1}, l}.
\]

We modify \( \phi \) for each such \( i \), which is possible as the sequences as in (3) that arise are pairwise disjoint as \( \phi \) is one-to-one, \( i_r \) is not in the range of \( \phi \), and \( i_s \) is not in the domain of \( \phi \). Modifying \( \phi \) as necessary for each \( i \) such that \( m_{i,j} = m_{i,j+1} = 1 \) ensures that the partial one-to-one mapping \( \phi \) satisfies both Conditions (1) and (2), thereby completing the proof.

**Definitions 3.3** Let \( M \) be a \( k \)-uniform matrix.

(i) For any \( j (1 \leq j \leq n - 1) \) we shall call an assignment mapping \( \phi_j \) optimal if \( \phi_j \) satisfies Conditions (1) and (2) of Theorem 3.2.

(ii) For any \( j (1 \leq j \leq n - 1) \) consider the partition of \( X_n = \{1, 2, \cdots, n\} \) induced by \( M \) into the following four (possibly empty) disjoint subsets:

\[
X_{1,1} = \{ i : m_{i,j} = m_{i,j+1} = 1 \}, \ X_{1,0} = \{ i : m_{i,j} = 1, m_{i,j+1} = 0 \},
\]

\[
X_{0,1} = \{ i : m_{i,j} = 0, m_{i,j+1} = 1 \}, \ X_{0,0} = \{ i : m_{i,j} = m_{i,j+1} = 0 \}.
\]
When necessary, we write $X_{i,j}^t$ etc. to indicate that the set refers to column $C_j$.

An assignment mapping $\phi_j$ then satisfies the conditions that:

$$\text{dom} \phi_j = X_{1,1} \cup X_{1,0}, \quad \text{ran} \phi_j = X_{1,1} \cup X_{0,1}$$

(7)

with $\phi_j$ acting identically on $X_{1,1}$ if $\phi_j$ is optimal.

(iii) We shall denote the $i$th row sum up to column $C_j$ by $S_{i,j}$:

$$S_{i,j} = \sum_{t=1}^{j} m_{i,t} \ (1 \leq i, j \leq n).$$

(8)

Suppress the second subscript $j$ by writing $\bar{S}_i$ for $S_{i,j}$, and form ordered sets, written in ascending order as:

$$\overline{X}_{1,0} = \{(i_1, S_{i_1}), \ldots, (i_p, S_{i_p}), S_{i_1} \leq \cdots \leq S_{i_p}, i_t \in X_{1,0}, (1 \leq t \leq p)\}.$$  
(9)

$$\overline{X}_{0,1} = \{(j_1, S_{j_1}), \ldots, (j_p, S_{j_p}), S_{j_1} \leq \cdots \leq S_{j_p}, j_t \in X_{0,1}, (1 \leq t \leq p)\}.$$  
(10)

To make each order unique, in the case of ties, we order by subscript value, so if $S_{i_1} = S_{i_2}$ then $(i_1, S_{i_1}) < (i_2, S_{i_2})$ for $\overline{X}_{1,0}$ if $i_1 < i_2$, and similarly for $\overline{X}_{0,1}$. We now meld these two lists to define a total order on $Y = \overline{X}_{1,0} \cup \overline{X}_{0,1}$. The order $(Y, \leq)$ is equal to the order defined in (9) and (10) when restricted to $\overline{X}_{1,0}$ and to $\overline{X}_{0,1}$ respectively. For $(i, S_i) \in \overline{X}_{1,0}$ and $(j, S_j) \in \overline{X}_{0,1}$ we define $(i, S_i) < (j, S_j)$ if $|S_i| \leq |S_j|$ and $(i, S_i) > (j, S_j)$ if $|S_i| > |S_j|$. In this way $\leq$ is indeed a linear order on $Y$ as transitivity is readily checked by cases.

**Definition 3.4**  
(i) The reverse order, $(Y, \geq)$ of the linear order $(Y, \leq)$ is the canonical order of $Y$.

Let $A = \{a, b\}$ be a two-letter alphabet.

(ii) The canonical word $w = a_1 a_2 \cdots a_{2p} \in A^{2p}$ ($a_r \in A, 1 \leq p \leq k$) is defined by $a_r = a$ or $a_r = b$ according as the $r$th entry in the canonical order belongs to $\overline{X}_{1,0}$ or to $\overline{X}_{0,1}$.

(iii) For any word $w \in A^m$ ($m \geq 0$) we write $|w|_c$ for the number of instances of $c \in A$ in $w$. The length of $w$, denoted by $|w|$, is then $|w| = |w|_a + |w|_b$.

(iv) If $w \in A^m$ ($m \geq 0$) has a factorization $w = uv$ we call $u$ a prefix and $v$ a suffix of $w$.

(v) A word $w \in A^{2m}$ ($m \geq 0$) such that $|w|_a = |w|_b$ is called a Dyck word if for every prefix $u$ of $w$, $|u|_a \geq |u|_b$.

(vi) For $w \in A^m$ ($m \geq 0$), the dual reverse word $\overline{w}$ is formed by taking the reverse word $w^R$ of $w$ and interchanging all instances of the letters $a$ and $b$.

**Remark 3.5**  
The set of all words of any length that satisfy the conditions of (v) is called the Dyck language. This is the language of well-formed parentheses in that replacing $a$ and $b$ by the left and right brackets '{' and '}' respectively, a Dyck word corresponds to a string of brackets that represents a meaningful bracketing of some binary operation. For further information, see [3].

**Proposition 3.6**  
(i) The dual reverse word $\overline{w}$ of a Dyck word $w$ is also a Dyck word.
(ii) There exists an optimal assignment mapping \( \phi_j \) \((1 \leq j \leq n-1)\) if and only if the canonical word \( w = w_j \) is a Dyck word.

**Proof** (i) Let \( \overline{w} = uv \), whence \( w = \overline{w} = \overline{u} \overline{v} \). Since \( w \) is a Dyck word, \(|\overline{v}|_{a} \geq |\overline{v}|_{b} \), whence \(|\overline{u}|_{a} \leq |\overline{u}|_{b} \), and so \(|u|_{a} \geq |u|_{b} \). Hence \( \overline{w} \) is a Dyck word.

(ii) Suppose that \( \phi = \phi_j \) is an optimal assignment mapping. The action of this mapping induces a bijection from letters \( a_s = a \) in the canonical word \( w \) to letters \( a_t = b \) in \( w \), which acts, by Condition (2) of Theorem 3.2, so that \( a_s \) lies to the left of \( a_t \) in \( w \). It follows that for any initial prefix \( u = w = uv \), we must have \(|u|_{a} \geq |u|_{b} \), for if \(|u|_{a} < |u|_{b} \), there would be some instance of \( b \) in \( u \) that was not in the range of the induced mapping, contradicting that \( \phi_j \) is one-to-one. Hence \( w \) is a Dyck word.

Conversely, given that \( w \) is a Dyck word, we map \( i \in X_{1,0} \) to \( i' \in X_{0,1} \) whereby if \( i \) corresponds to the \( r \)th instance of \( a \) in \( w \), then \( i' \) corresponds to the \( r \)th position of \( b \) in \( w \). By the given condition, the \( r \)th \( a \) in \( w \) lies to the left of the \( r \)th \( b \) in \( w \), whence \(|S_i| \geq |S_{i'}| \). The map \( \phi \) thereby defined satisfies Condition (2) of Theorem 3.2. Extending \( \phi \) to act identically on \( X_{1,1} \) then produces a required optimal assignment map. \( \square \)

**Theorem 3.7** Algorithm to decide optimality of a \( k \)-uniform matrix \( M \).

For the columns \( C_j \) \((1 \leq j \leq n-1)\) of \( M \):

1. Calculate the partial sums \( S_{i,j} \) \((i \in X_{1,0} \cup X_{0,1})\);
2. Rank the \( 2p \) \((0 \leq p \leq k)\) partial sums from Step 1 in descending order, with members of \( X_{1,0} \) taking precedence over members of \( X_{0,1} \) in the case of a tie, as per Definition 3.3(iii);
3. Form the canonical word \( w = w_j = a_1 \cdots a_{2p} \) where \( a_r = a \) or \( b \) according as the \( r \)th member of this ranking lies in \( X_{1,0} \) or \( X_{0,1} \).
4. \( M \) is optimal if and only if \( w_j \) is a Dyck word for all \( 1 \leq j \leq n-1 \).

However, it is not necessary to check the first two nor the last two assignment mappings for optimality by virtue of part (ii) of our next result.

**Lemma 3.8** (i) For a given \( j \) \((1 \leq j \leq n-1)\), all assignment mappings \( \phi_j \) are optimal if and only if the canonical word \( w_j = a^p b^p \), \((p = |X_{1,0}|)\).

(ii) An assignment mapping \( \phi_j \) is optimal if \( j \in \{1, 2, n - 2, n - 1\} \) or if \( k \in \{1, 2, n - 2, n - 1\} \).

**Proof** (i) Every \( \phi_j \) is optimal if and only if \( S_{i_{1,j}} \geq S_{i_{2,j}} \) for all \( i_1 \in X_{1,0} \) and \( i_2 \in X_{0,1} \), which in turn is equivalent to \( w_j = a^p b^p \), where \( p = |X_{1,0}| \).

(ii) Let \( i_1 \in X_{1,0} \) and \( i_2 \in X_{0,1} \). For \( \phi_1 \) and \( \phi_2 \) we have \( S_{i_{1,j}} \geq 1 \) and \( S_{i_{2,j}} \leq 1 \) \((j = 1, 2)\) whence it follows that \( w_j = a^p b^p \). For \( \phi_{n-2} \) or \( \phi_{n-1} \) we have \( S_{i_{1,j}} \geq k - 1 \) while \( S_{i_{2,j}} \leq k - 1 \), \((j = n - 2, n - 1)\) and again \( w_j = a^p b^p \). The claim now follows from part (i).

Similarly if \( k \leq 2 \) then \( S_{i_{1,j}} \geq 1 \) and \( S_{i_{2,j}} \leq 1 \), while if \( k \geq n - 2 \) then \( S_{i_{1,j}} \geq j - 1 \) and \( S_{i_{2,j}} \leq j - 1 \) and again the result follows. \( \square \)

**Corollary 3.9** (i) An \( n \times n \) uniform matrix \( M \) is optimal if \( n \leq 5 \).

(ii) For any non-optimal \( k \)-uniform matrix \( M \), \( 3 \leq k \leq n - 3 \).
(iii) Optimality of a \( k \)-uniform matrix \( M \) is preserved under the exchange of columns \( C_1 \) and \( C_2 \), and under the exchange of columns \( C_{n-1} \) and \( C_n \).

**Proof** (i) For \( n \leq 5 \), for any scheme there are at most \( 5 - 1 = 4 \) assignment mappings which are among the four mappings listed in Lemma 3.8(ii), and so all are optimal.

(ii) This follows from Lemma 3.8(ii).

(iii) Indeed we may replace \( C_1 \) and \( C_2 \) by any pair of binary columns that retains \( k \)-uniformity of \( M \), for then the transformed matrix retains its status with respect to optimality by Lemma 3.8(ii). These correspond to exchanging adjacent instances of 0 and 1 in the two columns in opposite pairs. In particular, since complete exchange of \( C_1 \) and \( C_2 \) retains \( k \)-uniformity, the result follows, as it does likewise for the exchange of the final column pair. \( \square \)

**Proposition 3.10** Let \( S = S(M) \) be an \((n, k)\)-uniform scheme with a given set of assignment mappings \( \phi_j \) \((1 \leq j \leq n - 1)\). If all travellers complete \( c \) cycled stages of \( S \) without the scheme failing, (that is, without any traveller being stalled) then the scheme, with this set of assignment mappings, will not fail before some traveller is due to ride their \((c + 3)\)rd cycled stage.

In particular, \( S \) will not fail prior to some traveller being due to ride their 3rd stage, and if all travellers complete \( k - 2 \) stages without \( S \) failing, then \( S \) is an optimal scheme, which is realised by the given set of assignment mappings.

**Proof** Suppose all travellers have completed \( c \) cycled stages without failure in \( S \). Suppose a walking traveller \( t_j \) arrives at a staging post \( P_j \) \((1 \leq j \leq n - 1)\), where \( s_j \) represents cycle stage number \( c + 1 \) or \( c + 2 \) for that traveller. Let \( i = \phi_j^{-1}(i') \). Then \( S_{i', j} = c \) in the first case, and \( S_{i', j} = c + 1 \) in the second. If \( t_j \) stalls at \( P_j \) then it follows that \( S_{i', j} < c \). However, since \( m_{i', j} = 1 \), it follows that \( t_j \) has not yet completed \( c \) cycled stages when the stall occurs, contrary to hypothesis. Therefore if all travellers complete \( c \) cycle stages without the scheme failing, then the scheme will not fail prior to some traveller attempting to cycle a stage for the \((c + 3)\)rd occasion. The final statement simply draws attention to the special cases where \( c = 0 \), and where \( c = k - 2 \). \( \square \)

**Examples 3.11** It follows from Corollary 3.9 that the smallest dimension \( n \) that might admit a non-optimal matrix \( M \) is \( n = 6 \). In this case, the inequality of Corollary 3.9(ii) becomes \( 3 \leq k \leq 6 - 3 \), so that \( k = 3 \). Consider the simple scheme \( S(M_1) \), where \( M_1 \) is given below. This scheme is clearly optimal: travellers \( t_1, t_2, t_3 \) ride the first three stages and then leave their bikes to be collected later by \( t_4, t_5, t_6 \) who then ride together to the finish. The assignment mappings all act identically except for \( \phi_3 \), which may be taken as any bijection such that \( \phi_3(\{1, 2, 3\}) = \{4, 5, 6\} \). However, if we swap columns \( C_3 \) and \( C_4 \) in \( M_1 \), we have the array \( M_2 \). By Lemma 3.8, the only canonical word of \( M_2 \) that may fail to be a Dyck word is \( w_3 \). However for \( j = 3 \) we have \( X_{1,0} = \{4, 5, 6\} \) and \( X_{0,1} = \{1, 2, 3\} \). For any \( t_1 \in X_{1,0} \) and \( t_2 \in X_{0,1} \) we have \( S_{t_1, t_2} = 1 < 2 = S_{t_2, t_3} \) and so \( w_3 = b^3a^3 \), which is not a Dyck word. Therefore \( M_2 \) is not optimal. Indeed this example shows that the class of optimal matrices is not closed under permutation of columns, nor under the taking of transpositions.
The question of whether an \( n \times n \) binary matrix \( M \) is optimal may be decided by an algorithm of complexity \( O(n^2 \log n) \).

**Proof.** 1. By inspecting rows and columns of \( M \), decide whether \( M \) is uniform, an operation of order \( O(n^2) \).

If \( M \) is uniform, we may decide optimality of \( M \) by carrying out the following procedure for each \( j \) with \( 1 \leq j \leq n - 1 \).

2. Compute \( S_{t,j+1} \) from \( S_{t,j} \) for all \( 1 \leq i \leq n - 1 \), which consists of \( n \) additions. This allows identification of the sets \( X_{i,0}^j, X_{i,0}^j, X_{i,1}^j \) and \( X_{i,1}^j \).

3. Form the two sets \( X_{1,0}^j \) and \( X_{0,1}^j \) and sort in descending order, a process which has time complexity \( O(n \ln n) \), as this is the least possible for any comparison algorithm [1], from which may be read the canonical word, \( w_j \).

4. At most \( O(n) \) comparisons determine whether or not \( w_j \) is a Dyck word.

For each \( j \), the total complexity of steps 2, 3, and 4 is \( O(n) + O(n \ln n) + O(n) = O(n \ln n) \). These steps are carried out \( n - 1 \) times, (strictly speaking, by Lemma 3.8(ii), at most \( n - 5 \) applications are needed), which, including Step 1, yields an overall complexity of \( O(n^2) + O(n^2 \log n) = O(n^2 \log n) \). \( \square \)

**Definition 3.13** Let \( M = (m_{i,j}) \) be an \( n \times n \) \( k \)-uniform binary matrix. Let \( S_n \) denote the symmetric group on \( X_n \). The \( n \times n \) \( k \)-uniform matrices \( M_\pi = (p_{i,j}) \ (\pi \in S_n), M_r = (r_{i,j}), \) and \( \overline{M} = (d_{i,j}) \) are defined by:

(i) \( p_{i,j} = m_{\pi(i),j} \),
(ii) \( r_{i,j} = m_{i,n-j+1} \),
(iii) \( d_{i,j} = (m_{i,j} + 1) \pmod{2} \).

We may denote \( d_{i,j} \) by \( \overline{m}_{i,j} \).

**Theorem 3.14** Suppose that \( S(M) \) is an optimal scheme. Then so are the schemes (i) \( S(M_\pi) \), (ii) \( S(M_r) \), and (iii) \( S(\overline{M}) \).

**Lemma 3.15** Let \( M \) be an \( n \times n \) \( k \)-uniform matrix. Then

(i) The \( j \)th canonical word of \( M_\pi \) (\( \pi \in S_n \)) is \( w_j \), the \( j \)th canonical word of \( M \) (\( 1 \leq j \leq n \)).

(ii) The \( j \)th canonical word of \( \overline{M} \) is \( \overline{w}_j \).

(iii) The \( j \)th canonical word of \( M_r \) is \( \overline{w}_{n-j} \).

**Proof** (i) The canonical words \( w_j \) (\( 0 \leq j \leq n - 1 \)) of \( M \) are defined by \( (Y, \leq) \) based on the partial orders as in (9) and (10). Replacing \( M \) by \( M_\pi \), results in replacing each of the symbols \( i_t, j_s \) by \( \pi^{-1}(i_t), \pi^{-1}(j_s) \) in the sets (9) and (10). Since the value of \( w_j \) is independent of the naming of these symbols, each canonical word \( w_j \) is unaltered.

(ii) Write \( \overline{S}_{i,j} \) for a typical partial sum of \( \overline{M} \). Since for any matrix position \((i, j)\), \( \overline{S}_{i,j} = j - S_{i,j} \) the list of inequalities in (9) and (10), apart from tied sums,
is reversed when passing from $M$ to $\overline{M}$. Moreover, $i_1 \in X_{1,0}, i_2 \in X_{0,1}$ for $M$ if and only if $i_1 \in X_{0,1}, i_2 \in X_{1,0}$ for $\overline{M}$. It follows from this pair of observations that the $j$th canonical word of $M$ is $w_j$, the dual reverse canonical word of $w_j$.

(iii) Denote the partial sums of $M_r$ by $S_{t,j}^r$. Then $S_{t,j}^r + S_{k,n-j} = k$ (1 \leq j \leq n, taking $S_{k,0} = 0$). Moreover $i_1 \in X_{1,0}, i_2 \in X_{0,1}$ for $M$ if and only if $i_1 \in X_{0,1}, i_2 \in X_{1,0}$ for $M_r$. Now

$$S_{t_1,j}^r \leq S_{t_2,j}^r \iff k - S_{t_1,n-j} \leq k - S_{t_2,n-j} \iff S_{t_2,n-j} \leq S_{t_1,n-j}.$$ 

This pair of observations imply that the $j$th canonical word of $M_r$ is $\overline{w_{n-j}}$. □

**Proof of Theorem 3.14.** Since $M$ is optimal, by Theorem 3.7 all canonical words $w_j$ of $M$ are Dyck words. By Lemma 3.15, the corresponding canonical words of $M_r, \overline{M},$ and $M^r$ are respectively $w_j, \overline{w}_j, \text{ and } \overline{w}_{n-j}$. Since the reverse dual word of a Dyck word is a Dyck word (Proposition 3.6(i)) it follows, again by Theorem 3.7, that each of $M_r, \overline{M},$ and $M^r$ is optimal. □

**Definition 3.16** Define the complementary assignment function $\overline{\phi}_j$ of an assignment function $\phi_j$ by putting

$$\text{dom } \overline{\phi}_j = X_{0,0} \cup X_{1,0}, \text{ ran } \overline{\phi}_j = X_{0,0} \cup X_{1,0} \quad (11)$$

with $\overline{\phi}_j(i) = i$ if $i \in X_{0,0}$ and $\overline{\phi}_j(i) = \phi_j^{-1}(i)$ if $i \in X_{1,0}$.

**Remark 3.17** We may prove Theorem 3.14 directly by identifying optimal assignment mappings $\psi_j$ for the matrix of the transformed scheme in terms of given optimal assignment mappings $\phi_j$ of $M(S)$. In case (iii) for instance, put $\psi_j = \overline{\phi}_j (1 \leq j \leq n - 1)$, as per Definition 3.16. For $\overline{M}$ we have

$$\text{dom } \psi_j = \{i : m_{i,j} = 0\} = \{i : d_{i,j} = 1\},$$

$$\text{ran } \psi_j = \{i : m_{i,j+1} = 0\} = \{i : d_{i,j+1} = 1\},$$

whence it follows that the $\psi_j$ qualify as assignment mappings for $S(\overline{M})$. Moreover, by definition, $\psi_j(i) = i$ if and only if $d_{i,j} = d_{i,j+1} = 1$, and so Condition (1) is satisfied. For $i \in X_{0,0}$ we have $\psi_j(i) = i$ and so in this case the inequality of Condition (2) becomes an equality, and is thus satisfied. Otherwise $i \in X_{1,0}$.

Then we have

$$\sum_{i=1}^{j} d_{\psi_j(i),l} = j - \sum_{i=1}^{j} m_{\psi_j(i),l} = j - \sum_{i=1}^{j} m_{\overline{\phi}_j(i),l} = j - \sum_{i=1}^{j} m_{\phi_j^{-1}(i),l} \leq j - \sum_{i=1}^{j} m_{i,l} = \sum_{i=1}^{j} d_{i,l},$$

where the inequality comes from Condition (2) applied to the $\phi_j$, thereby verifying Condition (2) for the $\psi_j$. For parts (i) and (ii) the corresponding assignment mappings are given respectively by $\psi_j = \pi^{-1}\phi_j\pi$, and $\psi_j = \phi_{n-j}^{-1}$, $(1 \leq j \leq n - 1)$.
Removing unnecessary handovers from an optimal scheme

Optimal schemes may have unnecessary cycle handovers, which can be removed, resulting in a scheme that is still optimal and displays the same character as the original. Suppose that $S = S(M)$ is an optimal $(n, k)$-scheme and for some $j$ we have $i_1 \in X_{1,0}, i_2 \in X_{0,1}$ and $S_{i_1, j} = S_{i_2, j}$. Then $t_{i_1}$ and $t_{i_2}$ arrive at $P_j$ simultaneously, the former by bike and the latter on foot, whereupon $t_{i_2}$ takes one of the bikes parked at $P_j$ and goes on to cycle $s_{j+1}$. However, one cycle handover could be avoided if the pair of travellers swapped labels at this point, with $t_{i_1}$ taking on the mantle of $t_{i_2}$ and vice-versa. In other words $t_{i_1}$ would complete the journey as instructed by the final part of $R_{i_2}$ from $m_{i_1, j+1}$ onwards and similarly $t_{i_2}$ would follow $R_{i_1}$ from $m_{i_1, j+1}$ onwards, allowing $t_{i_1}$ to remain on his bike for $s_{j+1}$.

This does not alter any column sums, and nor does it alter rows sums as the initial portions are equal: $S_{i_1, j} = S_{i_2, j}$, and hence so are the latter portions, as together they each sum to $k$. Applying this procedure repeatedly will lead to a more efficient scheme that will appear to be identical, meaning that if both schemes were to run simultaneously, at any given moment the set of positions of walking travellers and the set of positions of cycling travellers for the two schemes are identical. We shall call such a scheme reduced, with it being free of excess handovers. In summary we have the following theorem.

**Theorem 3.18** Given any optimal scheme $S = S(M)$ for the $(n, k)$-problem we may construct an optimal scheme $S(M')$ that is free of unnecessary handovers by repetition of the rule that if for some $j$ we have $i_1 \in X_{1,0}, i_2 \in X_{0,1}$ and $S_{i_1, j} = S_{i_2, j}$ we replace $R_{i_1}$ and $R_{i_2}$ in $M$ by

$$R_{i_1}' = (m_{i_1, 1}, \ldots, m_{i_1, j}, m_{i_2, j+1}, \ldots, m_{i_2, n}),$$
$$R_{i_2}' = (m_{i_2, 1}, \ldots, m_{i_2, j}, m_{i_1, j+1}, \ldots, m_{i_1, n}).$$

(12)

**Remark 3.19** Removal of unnecessary handovers yields a stronger form of Condition (2) of Theorem 3.2 in which all the associated inequalities for which $i' \neq i$ are strict, for all collections of optimal assignment mappings. However, this process does alter the scheme, whereas imposing Condition (1) merely chooses a special type of set of assignment maps for a given scheme.

Conversely, if $S(M)$ is optimal and every set of optimal assignment mappings yields strict inequalities in Condition (2), it follows that $S(M)$ has no unnecessary handover. However an optimal scheme may have some collection of assignment mappings for which the non-trivial inequalities in Condition (2) are all strict, yet the scheme still not be reduced. Such a collection of assignment mappings has the added feature that each traveller will find a parked cycle waiting for him whenever he is due to pick one up.

Simple comparison arguments like those in the proof of Theorem 3.14 give the following result.

**Proposition 3.20** For any optimal matrix $M$, the number $h = h(M)$ of excess handovers is the same for the optimal schemes $M_\pi$, $M_\tau$ and $M$.  

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4 Solution to the Biker-hiker Problem

We now provide a particular solution type to the general Biker-hiker problem. Because of the cyclic nature of our solutions, it will be convenient in this section to label the travellers as \( t_0, t_1, \ldots, t_{n-1} \) and the entries of an \( n \times n \) matrix \( M \) as \( m_{i,j} \) (\( 0 \leq i, j \leq n - 1 \)), and stages are labelled \( s_0, s_1, \ldots, s_{n-1} \) also.

**Definition 4.1 The Cyclic Scheme** We define the cyclic \((n, k)\)-scheme \( S = S_{n,k} \) with matrix \( M(S) = M_{n,k} \) by assigning the cycling quota for \( t_i \) to consist of the \( k \) cyclically successive stages, which run from \( P_{ik} \) to \( P_{(i+1)k} \), where arithmetic is conducted modulo \( n \). The matrix \( M_{n,k} \) of the \( n \times n \) cyclic scheme will be called the cyclic \((n, k)\)-matrix.

Since we are working modulo \( n \), we identify \( P_0 \) and \( P_n \), thereby making the journey a circuit. However, the following analysis holds whether the journey is linear or circular in nature.

**Theorem 4.2** The \((n, k)\)-cyclic scheme \( S_{n,k} \) is optimal.

**Proof** By construction, \( M = M_{n,k} \) is row \( k \)-uniform. The entry \( m_{i,j} = 1 \) if and only if \( j \) belongs the cyclic sequence \( ik, ik + 1, \ldots, ik + k - 1 \) which is equivalent to the statement that \( ki \mod n \) lies in the cyclic interval \( I_j = (j - k + 1, j - k + 2, \ldots, j) \). Therefore the number of \( 1 \)'s in \( C_j \) is the number of solutions to the congruences \( kx \equiv a \mod n \), \( a \in I_j \). Such a congruence has no solution if \( d = \gcd(n, k) \) is not a divisor of \( a \), otherwise there are \( d \) solutions. Since \( d | n \), it follows that the number of \( a \) such that \( d | a \) is the number of multiples of \( d \) in \( I_j \) when \( I_j \) is regarded as an interval of \( k \) consecutive integers, which is \( k \frac{d}{n} \), and so that there are exactly \( d \cdot \frac{k}{n} = k \) non-zero entries in each column of \( M \).

(Indeed every column of \( M \) represents the same cyclic sequence: see Prop. 4.9.)

To prove optimality of the matrix \( M \) of a cyclic scheme we appeal to Proposition 3.10, which says that a uniform scheme will not stall prior to some traveller attempting to mount a bicycle for the third time. Since no-one mounts a bike more than twice in a cyclic scheme, it follows that there is no stalling and the scheme is optimal. □

For \( M = (m_{i,j}) \), a square matrix, \( M_r \), the matrix that results from reversing the rows of \( M \) is described by permuting the columns of \( M \) by \( C_j \leftrightarrow C_{n-j-1} \). Similarly we now define \( M_c \) by reversing the columns of \( M \), which is effected by the row permutation whereby \( R_i \leftrightarrow R_{n-i-1} \). Of course both these permutations are respectively involutions on the set of columns and the set of rows of \( M \).

Writing \( M_{rc} \) for \((M_r)c\), and similarly defining \( M_{cr}, M_{r^2} \) and so on, we see that \( M_{rc} = M_{cr} = (m_{n-1-i,n-1-j}) \).

**Proposition 4.3** (i) For the cyclic \((n, k)\)-matrix \( M \\
(i) M_c = M_r, \ (ii) M_{cr} = M_{rc} = M, \ (iii) (MT)_r = (M_r)^T, \ (MT)_c = (M_c)^T \)

(iv) \((MT)_{rc} = M^T\).

**Proof** We prove (i), from which (ii), (iii), and (iv) readily follow. For \( M = (m_{i,j}) \) we have \( M_c = (c_{i,j}) \) where \( c_{i,j} = m_{n-1-i,j} \) and \( M_r = (r_{i,j}) \), where
\( r_{i,j} = m_{i,n-1-j} \). Then we have

\[
c_{i,j} = 1 \iff m_{n-1-i,j} = 1 \iff j \equiv (n-1-i)k + a \pmod{n} \text{ for some } 0 \leq a \leq k - 1
\]

\[\iff j + ik + k \equiv a \pmod{n} \quad (13)\]

\[
r_{i,j} = 1 \iff m_{i,n-j-1} = 1 \iff n - j - 1 \equiv ik + b \pmod{n} \text{ for some } 0 \leq b \leq k - 1
\]

\[\iff j + ik + 1 \equiv -b \pmod{n} \]

\[\iff j + ik + k \equiv c \pmod{n}, \quad (14)\]

where \( c = k - 1 - b \). Now

\[0 \leq b \leq k - 1 \iff -k + 1 \leq -b \leq 0 \iff 0 \leq c \leq k - 1.\]

We now note that the conditions of (13) and (14) are the same. It follows that \( c_{i,j} = r_{i,j} \), allowing us to conclude that \( M_c = M_r \). □

**Theorem 4.4** For the \((n,k)\)-problem, \((1 \leq k \leq n - 1)\) on an \(n\)-circuit, the cyclic scheme matrix represents the unique solution, up to permutation of rows, in which each traveller mounts and dismounts a cycle only once.

**Proof** By construction \( S(M_{n,k}) \) instructs each traveller to mount and dismount a cycle exactly once on the circuit. On the other hand, a uniform scheme that has this property is the cyclic solution. To see this, take any traveller, label the traveller \( t_0 \) and label the post where \( t_0 \) mounts a cycle as \( P_0 \). Since \( t_0 \) has a single bike ride, he must pass posts that we may label, \( P \), we now note that the conditions of (13) and (14) are the same. It follows that \( c_{i,j} = r_{i,j} \), allowing us to conclude that \( M_c = M_r \). □

**Remark 4.5** The feature of one ride per circuit is preserved by any of the symmetries of Theorem 3.14. In the case of row reversal, the non-zero stages for \( t_i \) remain those between \( P_{ik} \) and \( P_{(i+1)k} \) but are now ridden in reverse. Indeed since, by Proposition 4.3, \( M_r = M_c \), we see that for the cyclic solution matrix \( M \), \( M_r \) is a special case of permutation of the rows of \( M \), and so \( M_r \) also represents an \((n,k)\)-cyclic scheme. When we pass to the binary dual we find that \( \overline{M}_{n,k} = (M_{n,n-k})_r \) and so by the previous observation it follows that \( \overline{M}_{n,k} \) indeed represents a cyclic solution to the \((n,n-k)\) problem. In detail, write \((M_{n,n-k})_r = (a_{i,j}) \) and \( M_{n,n-k} = (m_{i,j}) \) whence \( a_{i,j} = 1 \) becomes

\[m_{i,n-1-j} = 1 \iff n - 1 - j \equiv i(n-k) + a \pmod{n} \text{ for some } 0 \leq a \leq n - k - 1\]

\[\iff j + 1 + a \equiv ik \pmod{n} \quad 0 \leq a \leq n - k - 1. \quad (15)\]

For the left hand side we write \( \overline{M}_{n,k} = (b_{i,j}) \) and \( M = (m_{i,j}) \). Then \( b_{i,j} = 1 \) may be written as

\[m_{i,j} = 0 \iff j \equiv (i+1)k + b \pmod{n} \text{ for some } 0 \leq b \leq n - k - 1\]
\[ \Leftrightarrow j - (b + k) \equiv j + (n - b - k) \equiv ik \pmod{n}. \]

Now \(0 \leq n - 1 - b - k \leq n - 1 - k\). Put \(c = n - 1 - b - k\). Then

\[ j + 1 + c \equiv ik \pmod{n} \quad 0 \leq c \leq n - k - 1. \quad (16) \]

The agreement of (15) and (16) allow us to conclude that \(M_{n,k} = (M_{n,n-k})^r\) and so \(M_{n,k}\) represents a cyclic solution to the \((n, n-k)\)-problem.

**Proposition 4.6** Consider the \((n,k)\)-problem and let \(d = \gcd(n,k)\). Let \(R_i\) and \(C_j\) denote the \(i\)th row and \(j\)th column respectively of \(M\), the matrix of the cyclic solution to the \((n,k)\)-problem as defined in 4.1. Then

(i) \(R_i = R_j\) if and only if \(i \equiv j \pmod{\frac{n}{d}}\);

(ii) \(C_i = C_j\) if and only if \(dq \leq i, j \leq d(q+1)-1\) for some \(q \in \{0,1,\ldots,\frac{n}{d}-1\}\).

**Proof** (i) is trivially true if \(k = 0\) or \(k = n\). Otherwise the cyclic intervals of entries that equal \(1\) in \(R_i\) and \(R_j\) respectively are defined by the corresponding cyclic lists of staging posts: \(P_{ik}, P_{ik+1}, \ldots, P_{(i+1)k}\) and \(P_{jk}, P_{jk+1}, \ldots, P_{(j+1)k}\). These lists are identical if and only if \(ik \equiv jk \pmod{n} \Leftrightarrow i \equiv j \pmod{\frac{n}{d}}\).

(ii) We observe that the non-zero entries of each row \(R_i\) consist of two intervals: an initial interval \(I\) of \(R_i\) of length \(r\) say, and a terminal interval \(T\) of \(R_i\) of length \(k-r\) \((0 \leq r \leq k)\). We may write \(k = du\) and \(n = dv\). Then for some \(x \geq 0\) we have

\[ ik \pmod{n} = dui - dux = d(\overline{uv} - xv). \]

If non-empty, the terminal interval \(T\) begins at \(P_{ik}\) and ends at \(P_{n}\) and so has length \(|T|\) given by

\[ |T| = n - ik \pmod{n} = d(v - ui + xv). \]

It follows that \(|I| = |T|\). The length \(|I|\) of the initial interval is \(|I| = k - |T| = du - |T|\), whence \(d||I|\) also. In the case where both \(I\) and \(T\) are non-empty the (successive) zeros in \(R_i\) number \(n - |I| - |T|\), which likewise is a multiple of \(d\). Otherwise there is an initial interval of zeros of length \(ik\), which is a multiple of \(d\), from which it follows that the terminal interval of zeros has length that is too a multiple of \(d\). Therefore within any row, counting left to right by columns, the entries from one multiple of \(d\) up to but not including the next, are equal, because each maximal list of identical entries begins at a multiple of \(d\). Hence

\[ dq \leq i, j \leq d(q+1) - 1 \quad (0 \leq q \leq \frac{n}{d} - 1) \Rightarrow C_i = C_j. \quad (17) \]

In order to prove the reverse implication, we introduce the following construction. By (17), the columns of \(M\) consist of \(\frac{n}{d}\) blocks \(A_1, A_2, \ldots, A_{\frac{n}{d}}\) of contiguous columns, with each \(A_i\) consisting of \(d\) identical columns. On the other hand \(M\) is partitioned into \(\frac{n}{d}\) sets of \(d\) (non-contiguous) identical rows \(B_1, B_2, \ldots, B_{\frac{n}{d}}\). We may permute the rows of \(M\), giving a new optimal matrix \(M'\) in which the rows of \(M'\) are partitioned into \(\frac{n}{d}\) blocks \(B'_1, B'_2, \ldots, B'_{\frac{n}{d}}\) each consisting of \(d\) identical rows. The new column blocks, \(A'_1, A'_2, \ldots, A'_{\frac{n}{d}}\) that result from this row permutation each consist of \(d\) columns, and the columns...
within each block remain identical. The pairwise intersections $A'_i \cap B'_j$ partition $M'$ into $\frac{n^2}{d}$ square blocks, which are themselves $d \times d$ matrices. Each such block has identical columns and identical rows, whence it follows that all entries of any particular $A'_i \cap B'_j$ are identical. We can then form a quotient matrix, $M'_d$ by identifying each of the $A'_i \cap B'_j$ with the common value (0 or 1) of all entries in that sub-matrix. Therefore $M'_d$ is the cyclic scheme for the $(\frac{n}{d} \times \frac{n}{d})$-problem in which the travellers and the bicycles are grouped into sets of order $d$, which move together as a block throughout the scheme.

If now the reverse implication in (17) were false, it would imply that there were two identical columns in the quotient matrix $M'_d$. It is possible to prove directly by analysing the cardinality of the intersection of sets of cyclic intervals that in the case where $n$ and $d$ are coprime, no two columns are identical, which, since $(\frac{n}{d}, \frac{n}{d})$ is a pair of coprime integers, applies to $M_d$. However the desired result follows at once from the next proposition which shows that in the case of coprimality the determinant of $M$ corresponds to the number of bicycles.

**Proposition 4.7** If $n$ and $k$ are coprime then $|\det(M_{n,k})| = k$. Otherwise $M_{n,k}$ is singular.

**Proof** Let $d = \gcd(k, n)$. If $d \geq 2$ then by Proposition 4.6(i), $M_{n,k}$ has a pair of identical rows and so $\det(M) = 0$. For $d = 1$ however the rows are cyclically identical and no two are equal. It follows that the set of rows consists of all $n$ different possibilities that arise from the cyclic sequence $(1, 1, \cdots, 1, 0, \cdots, 0)$, where the initial sequence of 1’s has length $k$. By permuting the rows of $M_{n,k}$ we may obtain the circulant matrix $C_{n,k}$, where $R_i(C_{n,k})$ has for its non-zero entries $m_{i,i}, m_{i,i+1}, \cdots, m_{i,i+k-1}$, (addition modulo $n$). Hence $\det(M_{n,k}) = \pm \det(C_{n,k})$. We may therefore complete the proof by showing that $\det(C_{n,k}) = k$.

By a standard result on circulant matrices (see for example [4]), with $\omega$ denoting any primitive $n$th root of unity:

$$\det(C_{n,k}) = \Pi_{i=0}^{n-1}(1 + \omega^i + \omega^{2i} + \cdots + \omega^{(k-1)i}). \quad (18)$$

For $i = 0$, the bracketed term is equal to $k$. It remains to show that the product of the other terms in (18) is equal to 1. By summing each of the geometric series we see that this claim is equivalent to the equation:

$$\Pi_{i=1}^{n-1}(\omega^{ki} - 1) = \Pi_{i=1}^{k}(\omega^i - 1). \quad (19)$$

However, since $k$ and $n$ are coprime, $\omega^k$ is also a primitive $n$th root of unity, and so it follows that the products in (19) are identical up to the order of their factors, thereby completing the proof. In particular, no two columns of $M_{n,k}$ are identical, thereby also completing the proof of Proposition 4.6. □

**Remark 4.8** Note from the previous proof that for $\gcd(n, k) = 1$, $S(C_{n,k})$ is also the cyclic $(n, k)$-scheme. Moreover, the non-zero entries of $R_i(C_{n,k}^T)$ are $m_{i,i}, m_{i,i-1}, \cdots, m_{i,i-k+1}$. Hence the non-zero entries of $R_{i+k-1}(C_{n,k}^T)$ are $m_{i+k-1, i+k-1}, m_{i+k-1, i+k-2}, \cdots, m_{i+k-1, i}$, which match those of $R_i(C_{n,k})$, and so $S(C_{n,k}^T)$ is also the cyclic $(n, k)$-scheme, with $C_{n,k}^T$ obtained by rotating the
columns of \( C_{n,k} \) forward by \( k - 1 \) places. This contrasts with \( S(M_{n,k}^T) \), the subject of Section 5, which although optimal is of a different character to \( S(M_{n,k}) \).

The rows of \( M_{n,k} \) represent the same cyclic sequence. The same is true of the columns.

**Proposition 4.9** Let \( M = M_{n,k} \) be the cyclic \((n,k)\)-matrix. Then every pair or columns of \( M \) represent the same cyclic sequence.

**Proof** Let \( \gcd(n,k) = d \). For \( M_{n,k} = (m_{i,j}) \) we have, with addition modulo \( n \), that \( m_{i,j} = m_{i+1,j+k} \). Since \( \gcd(n,k) = d \), there exists a value \( r \) such that \( kr \equiv d \text{ (mod } n) \); \( r \)-fold application of the previous equation then gives \( m_{i,j} = m_{i+r,j+k} = m_{i+r,j+d} \). It follows in particular that \( C_j \) and \( C_{j+d} \) define the same cyclic sequence, with one being transformed into the other through a rotation of \( r \) positions. By Proposition 4.7(ii), the columns \( C_0, C_1, \ldots, C_{d-1} \) are identical, and so it now follows that every pair of columns of \( M_{n,k} \) define the same cyclic sequence. □

5 The transpose solution

We have noted that optimality of a uniform matrix is generally not preserved under transposition. However, the cyclic scheme is an exception to this.

**Theorem 5.1** The transpose matrix \( M = M_{n,k}^T \) of a cyclic \((n,k)\)-matrix \( M_{n,k} \) is also optimal.

We shall call \( S(M^T) \) a transpose cyclic scheme and similarly \( M^T \) is a transpose cyclic matrix. With subscripts calculated modulo \( n \), the non-zero entries of column \( C_j \) of \( M^T \) are \( m_{jk}, m_{jk+1,j}, \ldots, m_{jk+k-1,j} \) \( (0 \leq j \leq n - 1) \). The transpose cyclic matrix \( M^T \) is \( k \)-uniform, and so if \( S(M^T) \) does not stall, we have optimality. By passing to the binary dual if necessary, we may suppose that \( k \leq \frac{n}{2} \), for first note that for any binary matrix \( M = (m_{i,j}) \), we have \( \overline{M} = \overline{M^T} \) as the \((i,j)\)th entry in each of these matrices is \( \overline{m_{i,j}} \). Now let us assume that for any cyclic \((n,k)\)-matrix \( M \) with \( n \geq 2k \), the transpose matrix \( M^T \) is optimal. Suppose that \( M \) is a cyclic \((n,k)\)-matrix with \( n < 2k \) and consider \( M^T \). Then \( M^T = \overline{M^T} \), with \( \overline{M} \) a cyclic \((n,n-k)\)-matrix. Since \( n < 2k \), it follows that \( n > 2(n-k) \), and so by our assumption we have that \( \overline{M^T} \) is optimal. But \( \overline{M^T} = M^T \), whence \( M^T = M^T \) is also optimal.

We are therefore permitted to adopt the assumption that \( 2k \leq n \) in our proof that transpose cyclic matrices are optimal. For the remainder of the section we shall denote our transpose cyclic matrix by \( M \) (as opposed to \( M^T \)). For any \( t \), at least one of the entries \( m_{i,j} \) and \( m_{i,j+1} \) of \( M \) is 0, as we now show.

For any \( j \geq 0 \), there is a unique \( i (= jk \text{ mod } n) \), such that the non-zero entries of columns \( C_j \) and \( C_{j+1} \) in \( M \) have the form:

\[
(m_{i,j} = m_{i+1,j} = \cdots = m_{i+k-1,j} = 1)
\]
\[ C = S \]

the inductive hypothesis to the row sums of \( m \).

(Note that the total number of entries in (22) is indeed \( (2k+1) \).

Finally we have \( C \).

The non-zero entries of \( C \) form a cyclic block of length \( k \). This will manifest itself either as a single linear block in \( C \), or as a pair of initial and a terminal blocks. In the single block case, the initial and terminal blocks of non-zero entries are one and the same.

Lemma 5.2 Let \( (i, j) \) be the final entry of the initial block of non-zero entries of \( C \). We shall write \( i = i(j) \). Then

\[ S_{0,j} = S_{1,j} = \cdots = S_{i,j} = S_{i+1,j} + 1; \quad S_{i+1,j} = S_{i+2,j} = \cdots = S_{n-1,j}. \] (21)

Proof We proceed by induction on \( j \). For \( j = 0 \) we have \( m_{0,0} = m_{1,0} = \cdots = m_{k-1,0} = 1, \) \( m_{k,0} = \cdots = m_{n-1,0} = 0 \), in accord with (21), where \( i(0) = k - 1 \). Suppose now that (21) holds for some value of \( j \) and consider \( C_{j+1} \). Suppose first that the non-zero entries of \( C_{j+1} \) form a single linear block: \( m_{t,j+1} = m_{t+1,j+1} = \cdots = m_{t+k-1,j+1} = 1 \). If \( t = 0 \) then \( i(j) = n - 1 \) in (21) and all the row sums for \( C \) in (21) are equal. It then follows that (21) holds for \( C_{j+1} \) as in the \( j = 0 \) case. Otherwise \( t \geq 1 \) and so \( i(j) = t - 1 \). By induction:

\[ S_{0,j} = S_{1,j} = \cdots = S_{t-1,j} = S_{t,j} + 1; \quad S_{t,j} = S_{t+1,j} = \cdots = S_{n-1,j}. \]

Since \( S_{p,j} = S_{p,j+1} \) for all \( 0 \leq p \leq t - 1 \) it follows that \( S_{0,j+1} = S_{p,j+1} \) for all \( 0 \leq p \leq t - 1 \). On the other hand for \( t \leq p \leq t + k - 1 \) we have \( S_{p,j+1} = 1 + S_{p,j} = 1 + (S_{0,j} - 1) = S_{0,j} = S_{0,j+1} \). Therefore \( S_{0,j+1} = S_{p,j+1} \) for all \( 0 \leq p \leq t + k - 1 \). Finally, for the case where \( t + k \leq p \) we have \( S_{p,j+1} = S_{p,j} = S_{0,j+1} - 1 \) and so (21) is holds for the \( S_{p,j+1} (0 \leq p \leq n - 1) \).

The alternative case is where the non-zero entries of \( C_{j+1} \) break into distinct initial and terminal blocks. The two blocks then have the respective forms:

\[ m_{0,j+1} = m_{1,j+1} = \cdots = m_{i,j+1} = 1 \]

& \( m_{n-k+i+1,j+1} = m_{n-k+i+2,j+1} = \cdots = m_{n-1,j+1} = 1 \) \((0 \leq i \leq k - 2)\) (22)

(Note that the total number of entries in (22) is indeed \((i + 1) + (n - 1 - (n - k + i)) = k \).) The single linear cyclic block of non-zero entries of \( C \) ends at \( m_{n-k+i,j} = 1 \) and begins at \( m_{(n-k+i-(k-1))} = m_{n-2k+i+1,j} = 1 \). By applying the inductive hypothesis to the row sums of \( C \) we infer that:

\[ S_{0,j+1} = S_{1,j+1} = \cdots = S_{i,j+1} = S_{0,j} + 1 = \cdots = S_{k,j} + 1. \] (23)

Since \( S_{0,j} = S_{1,j} = \cdots = S_{n-k+i,j} \), it follows that

\[ S_{i+1,j+1} = \cdots = S_{n-k+i,j+1} = S_{0,j}. \] (24)

Finally we have

\[ S_{n-k+i,j} = 1 + S_{n-k+i+1,j}, \quad S_{n-k+i+1,j} = \cdots = S_{n-1,j} \]
\[ S_{n-k+t+1,j+1} = \cdots = S_{n-1,j+1} = 1 + S_{n-k+t+1,j} = S_{0,j}. \] (25)

Statements (23), (24), and (25) together give (22) as applied to \( C_{j+1} \).

Now suppose that \( n \leq 2k \). Recall from Remark 4.5 that the binary dual \( \overline{M} \) of \( M \) is the cyclic transpose matrix of the \((n, n-k)\) problem with rows reversed.

Since \( n \geq 2(n-k) \) it follows that (22) holds for the corresponding row sums of the columns of \( \overline{M} \), (denoted \( \overline{S}_{i,j} \)). Note that \( S_{i,j} + \overline{S}_{i,j} = j + 1 \). Hence for some \( i \ (0 \leq i \leq n-1) \):

\[ \overline{S}_{n-1,j} = \overline{S}_{n-2,j} = \cdots = \overline{S}_{n-i,j} = \overline{S}_{n-i-1,j} + 1; \overline{S}_{n-i-1,j} = \overline{S}_{n-i-2,j} = \cdots = \overline{S}_{0,j}, \]

\[ \Leftrightarrow S_{n-1,j} = S_{n-2,j} = \cdots = S_{n-i,j} = S_{n-i-1,j} - 1; \]

\[ S_{n-i-1,j} = S_{n-i-2,j} = \cdots = S_{0,j}, \]

\[ \Leftrightarrow S_{0,j} = \cdots = S_{n-i-1,j} = S_{n-i,j} + 1; S_{n-i,j} = \cdots = S_{n-1,j}, \]

which is in accord with (22) with \( i(j) = n - 1 - i \). This completes the proof. \( \square \)

**Proof of Theorem 5.1** As already observed, we may assume that \( n \geq 2k \), in which case it is clear from (21) that for any column \( C_j \),

\[ (i_1 \in X^j_{1,0} \land i_2 \in X^j_{0,1}) \Rightarrow S_{i_1,j} \geq S_{i_2,j}, \]

from which it follows that the canonical word \( w_j \) is the Dyck word \( w_j = a^k b^k \) \((2k \leq n \Rightarrow |X_{1,0}| = k)\). Hence every assignment mapping \( \phi_j \) is optimal, and therefore \( M \) is optimal. \( \square \)

**Proposition 5.3** If \( n \geq 2k \), then at any time point during the execution of \( S(M) \), there are at most 3 distinct positions for the travellers. Moreover the distance separating one cohort from the next is less than 1 unit.

**Proof** We begin with three useful observations.

- If \( i \leq i' \) then \( t_i \) never trails \( t_{i'} \). This follows easily from the fact that for any fixed \( j \), the \( S_{i,j} \) are monotonically decreasing in \( i \) (Lemma 5.2).

- Consider a typical column \( C_j \) of \( M \). As explained prior to Lemma 5.2, \( C_j \) consists of three blocks, each of which consists of zeros or ones. Writing 0 for a block of zeros and 1 for a block of ones, the blocks of \( C_j \) have either of the two forms 010 or 101, although in the former case the second 0-block may be empty, as may be the second 1-block in the latter case.

- Two successive columns \( C_j \) and \( C_{j+1} \) cannot both have the 101-block structure. (This is a consequence of \( n \geq 2k \), for since the combined length of the two 1-blocks is \( k \), the length of the 0-block in the 101 case is at least \( k \).)

When travelling in a common stage \( s_{j+1} \), we shall refer to \( t_i \) and \( t_{i'} \) as being *members of the block* if \( m_{i,j} \) and \( m_{i',j} \) are in the same block of \( C_j \). A set of travellers who are currently moving together will be called a *cohort*.

We now prove inductively on \( j \) that, during the period when the leading cohort is between \( P_j \) and \( P_{j+1} \), the following three conditions hold:

1. Any pair of members of the same block are in the same cohort.
2. There are at most 3 cohorts.
3. The distance between the members of two neighbouring cohorts is less than 1 unit.

Inductive verification of this trio of claims proves Proposition 5.3.

For \( j = 0 \) all three claims are clear and indeed there are only 2 cohorts. Consider \( C_j \) \( (j \geq 1) \) and suppose by way of induction that our claims hold for all lesser values of \( j \).

Take the case where \( C_j \) has a 010 block structure. Suppose first that \( C_{j-1} \) also has a 010 structure. Then, by Lemma 5.2, the members of the joint 01-block of \( C_{j-1} \) arrive together at \( P_j \) and from the structure of the transpose cyclic scheme, form the first 0-block of \( C_j \), so forming the lead cohort in \( s_{j+1} \). By induction, the lead of this cohort over the second 0-block in \( C_{j-1} \) as it becomes the first 0-block of \( C_j \) is less than 1 unit. Since the leading cohort is walking, its lead over the next cohort remains less than 1 unit as the lead cohort traverses \( s_{j+1} \). The 1-block of \( C_j \) consists of the first \( k \) entries of the members of the second 0-block of \( C_{j-1} \), whose members arrived in unison at \( P_j \). This 1-block cohort forms the second cohort, which then catches the leading cohort at \( P_{j+1} \). During this time the lead of the second cohort is less than 1 unit over the third cohort which is the remainder of the second 0-block of \( C_{j-1} \), which becomes the second 0-block of \( C_j \) upon arrival at \( P_j \). These observation taken together demonstrate that Conditions 1-3 are respected throughout the time that the lead cohort walks \( s_{j+1} \).

Next suppose that \( C_{j-1} \) has a 101 structure, in which case the members of the first 1-block of \( C_{j-1} \) arrive first at \( P_j \) and form the first 0-block of \( C_j \). By induction, this cohort is less than 1 unit ahead of the next cohort. Since the leading group of \( C_j \) is walking, its lead over the following cohort cannot increase as it traverses \( s_{j+1} \), and so remains less than 1 unit. By induction, the separation of the 0-block of \( C_{j-1} \) and the second 1-block of \( C_{j-1} \) is less than 1 unit up until the time the leading cohort of \( C_{j-1} \) reaches \( P_j \). Their separation decreases after that and the two cohorts reach \( P_j \) in unison. After that the 01-block of \( C_{j-1} \) splits into two new cohorts, the first a cohort of size \( k \) comprised of cyclists, which are the members of the 1-group of \( C_j \), with the remainder of the joint 01-block of \( C_{j-1} \) becoming the second 0-block of \( C_j \) and the third cohort, (thus maintaining Conditions 1 and 2). The second 0-block of \( C_j \) will be less than 1 unit behind the second cohort until the leading cohort completes \( s_{j+1} \). Hence Conditions 1, 2, and 3 remain valid throughout the period where the leading cohort is travelling between \( P_j \) and \( P_{j+1} \), thus continuing the induction.

Finally we examine the case where \( C_j \) has the block form 101. By the third bullet point, \( C_{j-1} \) has the block form 010. By Lemma 5.2, the members of the 01-block of \( C_{j-1} \) arrive together at \( P_j \), and by induction, the members of the second 0-block are the trailing cohort, which is less than 1 unit behind. The first 1-block of \( C_j \) is an initial segment of the joint 01-block of \( C_{j-1} \) and its members therefore proceed together as the lead cohort. By construction of the transpose cyclic scheme, the joint 01-block of \( C_{j-1} \) becomes the joint 10-block of \( C_j \), with the walking members becoming a second cohort in \( s_{j+1} \). Their distance behind the first cohort is always less than 1 unit. The second 0-block of \( C_{j-1} \) becomes the second 1-block of \( C_j \), and so its members proceed together, as the third
cohort. This cohort was also the third cohort of $C_{j-1}$ and so was less than 1 unit behind the members of the 01-block of $C_{j-1}$ (by Condition 3 and induction) when the joint block reached $P_j$. Hence the separation between the two trailing cohorts is less than 1 unit (and decreases to 0 as these cohorts traverse $s_{j+1}$). Therefore Conditions 1, 2, and 3 have been met, and so the induction continues, thereby completing the proof. □

Example 5.4 Proposition 5.3 does not hold however when $2k > n$. For example, consider the transpose matrix $M$ for the $(n, n-1)$ problem. Then the zeros consist of the non-leading diagonal running between entries $(n-1,0)$ and $(0,n-1)$. If we let the ratio of the cycling speed to walking speed become arbitrarily large, then $t_0$ will reach $P_{n-1}$ before $t_{n-1}$ has reached $P_1$, so that the separation of $t_0$ and $t_{n-1}$ approaches an upper limit of $n-1$ units.

6 Calculating features of cyclic schemes

Proposition 6.1 In respect to the $(n,k)$-cyclic solution, let $n = r + qk$, $(0 \leq r \leq k - 1)$, and let $d = \gcd(n,k)$. Let $i_0, i_1, \ldots, i_{k-1}$ be the subscripts of the $k$ travellers that ride stage $s_1$. Label the $k$ bicycles as $b_0, b_1, \ldots, b_{k-1}$, where $b_m$ is the bicycle ridden by $t_{im}$ in $s_1$. Then during the execution of the scheme:

(i) the total number of bicycle rides is $n + k - d$.

(ii) Each bicycle $b_m$ is mounted on either $\lceil \frac{n}{k} \rceil$ or $\lceil \frac{n}{k} \rceil + 1$ occasions, with the first alternative applying if and only if $r \leq c_m$, where $ki_m \equiv c_m \pmod{n}$, $1 \leq c_m \leq k$.

Proof (i) There are $k$ travellers $t_i$ that cycle $s_1$, and $t_i$ completes their quota if and only if $ki \equiv 0 \pmod{n}$. There are $d$ solutions $i$ to this congruence. Therefore $n - k + d$ travellers have a single ride while $k - d$ travellers have two separate rides, one beginning and the other ending their journey. The total number of cycle rides is therefore $n - k + d + 2(k - d) = n + k - d$.

(ii) Bicycle $b_m$ $(0 \leq m \leq k - 1)$ is mounted at $P_0$ by $t_{im}$, who dismounts at $P_{c_m}$, where $ki_m \equiv c_m \pmod{n}$ $(1 \leq c_m \leq k)$. If $c_m < r$ then

$$c_m + qk < r + qk = n.$$  \hspace{1cm} (26)

Hence $b_m$ is ridden by $1 + q + 1 = q + 2$ travellers. Since $1 \leq r$ we have $\lceil \frac{n}{k} \rceil + 1 = (q + 1) + 1 = q + 2$, as required. Otherwise $r \leq c_m$ whence

$$c_m + (q - 1)k \leq qk \leq r + qk = n.$$  \hspace{1cm} (27)

If $1 \leq r$ it follows from (27) that $b_m$ is ridden by $1 + (q - 1) + 1 = q + 1$ travellers. If $r = 0$ then $c_m = k$ and this figure is $1 + (q - 1) = q$, but in either event this number equals $\lceil \frac{n}{k} \rceil$, thus completing the proof. □

Proposition 6.2 For the $M = M^T_{n,k}$ transpose cyclic matrix and scheme:

(i) If $n \leq 2k$ then traveller $t_i$ has $k$ cycle rides; if $2k \geq n$, then $t_i$ has $n - k + 1$ rides if $n - k \leq i \leq k - 1$, and $n - k$ rides otherwise.
(ii) The total number of cycle rides is \( nk \) if \( 2k \leq n \) and is \( n(n - k - 1) + 2k \) if \( 2k \geq n \).

(iii) The number of excess handovers \( h(M) = \min(k(k-1), (n-k)(n-k-1)) \).

(iv) The number of cycle rides after the elimination of excess handovers is, in all cases, \( k(n-k+1) \).

**Proof** (i) For the case where \( 2k \leq n \), each traveller rides just one stage at a time, and so \( t_i \) has \( k \) rides. We analyse this case further. Applying Proposition 4.3(iv) to \( M_{n,k}^T \), we have that \( m_{i,j} = m_{n-1-i,n-1-j} \), and so

\[
m_{0,0} = m_{1,0} = \cdots = m_{k-2,0} = m_{k-1,0} = 1,
\]

\[
m_{n-k,n-1} = m_{n-k+1,n-1} = \cdots = m_{n-2,n-1} = m_{n-1,n-1} = 1.
\]  \hspace{1cm} (28)

\[
m_{k,0} = m_{k+1,0} = \cdots = m_{n-1,0} = 0 = m_{0,n-1} = m_{1,n-1} = \cdots = m_{n-k-1,n-1}.
\]  \hspace{1cm} (29)

It follows from (28) and (29) that for \( 0 \leq i \leq k-1 \), row \( R_i \) has \( m_{i,0} = 1 \), and that \( R_i \) has \( k \) (non-consecutive) entries equal to 1, each followed by a maximal sequence of positive length that consists of entries that equal 0. For \( n-k \leq i \leq n-1 \), the same is true for \( R_i \) but the statement applies for \( R_i \) considered in reverse order, beginning with \( m_{i,n-1} = 1 \). On the other hand, for \( k \leq i \leq n-k-1 \), \( R_i \) begins and ends with a sequence of zeros, and once again there are \( k \) entries equal to 1, but with no two consecutive entries equal to 1.

If we now pass from \( M_{n,k}^T \) to \( \overline{M_{n,k}^T} \), the rows indexed by \( 0 \leq i \leq k-1 \) and \( n-k \leq i \leq n-1 \) each indicate \( k \) bicycle rides, while the remaining central rows each show \( k+1 \) cycle rides. By symmetry, the same conclusion applies to the matrix with rows reversed. Now by Remark 4.5 and Proposition 4.3(iii) we infer that

\[
M_{n,k}^T = (M_{n,n-k})^T_r = ((M_{n,n-k})_r)^T = ((M_{n,n-k})_r)^T = ((M_{n,n-k})_r). \quad (30)
\]

Therefore if \( n < 2k \) we infer that the rows \( R_i \) of \( M_{n,k}^T \) such that \( 0 \leq i \leq n-k-1 \) or \( k \leq i \leq n-1 \) indicate \( n-k \) bicycle rides, while those indexed by \( n-k \leq i \leq k-1 \) show \( n-k+1 \) cycle rides, as required.

(ii) If \( 2k \leq n \) then each cycle is mounted on \( n \) separate occasions, and so the total number of cycle rides is \( nk \). Otherwise it follows by (i) that the total number of cycle rides is given by:

\[
n(n-k) + (k-1 - (n-k-1)) = (n-k)(n-1) + k = n(n-k-1) + 2k.
\]

(iii) The cyclic sequence of 1’s in \( C_j \) has length \( k \) and may be written as \( T_j = (i_j, i_j + 1, \cdots, i_j + k - 1) \) with addition mod \( n \) \( (i_j = jk \mod n) \). Suppose that \( n \geq 2k \), in which case \( T_j = X_{1,0}^j \). By Lemma 5.2 it follows that \( s_j \) has no excess handovers unless for some \( t \) such that \( 0 \leq t \leq k-1 \) we have \( i_{j}+2k-1 = n+t \).

In that case \( j \leq n-2 \) and \( T_{j+1} \) consists of two linear sequences, which are \( I_1 = (0, 1, \cdots, t) \) and \( I_2 = (n-k+t+1, n-k+t+1, \cdots, n-1) \), although the latter is empty if \( t = k-1 \). (Note that \( |I_1| = t+1, |I_2| = n-1 - (n-k+t) = k-t-1 \), so that \( |I_1| + |I_2| = k \).) Observe from Lemma 5.2 that for all \( i_r \in I_1, i_r \in X_{1,1}^j \)
and $S_{i_r,j} = S_{i,j}$. Similarly for all $i_r \in I_2$, $i_r \in X_{i_0}^{j+1}$, and $i_s \in X_{i_0}^{j+1}$ for all $i_s \in (t+1, t+2, \ldots, k-1)$, an interval also of length $k - t - 1$, with $S_{i_r,j+1} = S_{i,j+1}$. There are no other unnecessary handovers in either $C_j$ or in $C_j+1$. It follows that the number of excess handovers in the pair of stages $s_j$ and $s_{j+1}$ is $|I_1| + |I_2| = k$. Conversely if $T_{j+1}$ consists of two linear sequences as above, then $T_j$ is a single linear sequence and the pair $s_j$ and $s_{j+1}$ collectively have $k$ excess handovers.

It follows that the set of excess handovers of $S(M)$ is partitioned into sets of order $k$, with one such set for every $0 \leq j \leq n-1$ such that $0 \leq jk \pmod n \leq k-1$, with one exception. In the case where $j = n - 1$ so that $j + 1 \equiv 0 \pmod n$ there are no handovers from $s_{n-1}$ to $s_0$. Let $d = \gcd(n,k)$. The number of multiples of $d$ in the interval $[0, k-1]$ is $\frac{k}{d}$. Then there are $d$ values of $j \ (1 \leq j \leq n-1)$ such that $jk \equiv td \pmod n \ (0 \leq t \leq \frac{k}{d})$. Hence the number of sets in question is $\frac{k}{d}$ = $k$. Therefore $h(M) = k^2 - k = k(k-1)$, as we subtract $k$ in recognition of no handovers occurring from $s_{n-1}$ to $s_0$.

On the other hand, if $n \leq 2k$ consider $M_{n,k}^T = (M_{n,n-k}^T)$, by (30). Since the latter matrix is also a reverse transpose matrix and $n \geq 2(n-k)$, it now follows from Proposition 3.20 that

$$h(M_{n,k}^T) = h(M_{n,n-k}^T) = h(M_{n,k}^T) = (n-k)(n-k-1).$$

Therefore $h(M) = k(k-1)$ if $n \geq 2k$ and $h(M) = (n-k)(n-k-1)$ otherwise. Combining these two cases we obtain the statement of (iii).

For part (iv), if $n \geq 2k$ we have the number of cycle rides after elimination of excess handovers is $kn - k(k-1) = k(n-k+1)$, as required. In the alternative case, by (i), the corresponding number has the same form:

$$n(n-k-1) + 2k - (n-k)(n-k-1)$$

$$= (n-k-1)(n-n+k) + 2k = k(n-k+1).$$

Example 6.3 $M_{11,7}^T$

|   | $P_0$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | $P_7$ | $P_8$ | $P_9$ | $P_{10}$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $t_0$ | 1     | 1     | 0     | 0     | 1     | 1     | 1     | 0     | 0     | 0     | 1       |
| $t_1$ | 1     | 1     | 0     | 0     | 1     | 1     | 1     | 0     | 1     | 1     | 0       |
| $t_2$ | 1     | 1     | 0     | 0     | 1     | 1     | 1     | 0     | 0     | 0     | 1       |
| $t_3$ | 1     | 1     | 0     | 0     | 1     | 1     | 1     | 0     | 0     | 1     | 1       |
| $t_4$ | 1     | 0     | 1     | 1     | 1     | 0     | 0     | 1     | 1     | 1     | 0       |
| $t_5$ | 1     | 0     | 1     | 1     | 1     | 0     | 0     | 1     | 1     | 1     | 0       |
| $t_6$ | 1     | 0     | 0     | 1     | 1     | 1     | 1     | 0     | 0     | 1     | 1       |
| $t_7$ | 0     | 1     | 1     | 1     | 0     | 0     | 1     | 1     | 1     | 0     | 1       |
| $t_8$ | 0     | 1     | 1     | 1     | 0     | 1     | 1     | 0     | 0     | 1     | 1       |
| $t_9$ | 0     | 1     | 1     | 1     | 0     | 1     | 0     | 1     | 1     | 0     | 1       |
| $t_{10}$ | 0     | 1     | 1     | 1     | 0     | 1     | 1     | 0     | 1     | 1     | 0       |

We expunge all excess handovers from $S(M_{11,7}^T)$ to yield $M'$. Since $2k \geq n$, from Proposition 6.2(iv) we find that the total ride number is $11(11 - 7 - 1) +$
2(7) = 47. The number of rides by \( t_0 \) through to \( t_{10} \) is \((4+4+4+4)+(5+5+5)+(4+4+4+4) = 47\), in accord with part (i), as \( n-k = 4 \) and \( n-k \leq i \leq k-1 \) becomes \( 4 \leq i \leq 6 \), so it is \( t_4, t_5, \) and \( t_6 \) who have the extra ride. We have \( h(M) = (11-7)(11-7-1) = 12 \). According to (iii), after elimination of excess handovers the total number of rides is \( 7(11-7+1) = 35 \), which indeed equals \( 47 - h(M) \). All travellers have three rides in \( S(M'') \) except for \( t_5 \) and \( t_9 \) who each have four.

Throughout this paper we have placed the staging posts at intervals of one unit with the journey regarded as being of length \( n \). We may however consider other partitions of the travellers’ journey. Consider a putative scheme, \( S = S_m \), based on partitions into \( m \) equal stages. Such a scheme \( S_m \) would then be represented by an \( n \times m \) binary matrix \( M = M(S_m) \).

**Theorem 6.4** For the \((n, k)\)-problem, let \( k = \gcd(n, k) \) and put \( n' = \frac{n}{k} \) and \( k' = \frac{k}{n} \). Then an optimal scheme \( S_m \) defined by an \( n \times m \) matrix exists for the \((n, k)\)-problem if and only if \( m = rn' \) for some \( r \geq 1 \), in which case each traveller cycles for \( l = rk' \) of the \( m \) stages of \( S_m \).

**Proof** As in the \( m = n \) case, for \( S_m \) to be an optimal solution, we must have each column \( C_j \) of \( M \) containing exactly \( k \) instances of 1, and each row containing a common number, \( t \) say, of 1’s. Counting the 1’s by rows, and then by columns we equate to see that \( tn = km \), whence \( m = \frac{tn}{k} = \frac{tn'}{k'} \). Since \( \gcd(n', k') = 1 \), it follows that \( k'|t \) so that \( t = rk' \) say, and \( m \) necessarily has the form \( m = rn' \), for some \( r \geq 1 \). Moreover, in any optimal scheme \( S_m \), each traveller cycles the same number, \( l \) say, of stages of \( S_m \). By optimality we then have \( \frac{k}{m} = \frac{k}{n} = \frac{k'}{n} \) so that \( l = \frac{mk'}{m} = \frac{rn'k'}{n} = rk' \). In conclusion:

\[
m = rn', \quad l = rk' \quad (r \geq 1).
\]

Conversely we now show that given that \( m \) satisfies (32), we may build a scheme \( S_m \) from copies of schemes for the \((n', k')\)-problem to yield an optimal solution for the \((n, k)\)-problem based on an \( n \times m \) binary matrix \( M \) which is \((l, k)\)-uniform, meaning that each row and each column contains exactly \( l \) and \( k \) non-zero entries respectively. To do this we take a \( d \times r \) array and at each position in the array we place an optimal \( n' \times n' \) matrix for the \((n', k')\)-problem. (There is no need for these matrices to be identical solutions.) This yields a \((dn' \times rn') = (n \times m)\) binary matrix \( M \) with \( l \) entries of 1 in each row, and \( k \) entries of 1 in each column.

The first set of \( n' \) columns represents a scheme for the initial \( \frac{1}{r} \) part of the journey. Executing this partial scheme will see \( d \) (disjoint) sets of travellers, with each set executing an optimal \((n', k')\) scheme. Since these schemes are carried out in parallel, all \( n \) travellers will complete the first \( \frac{1}{r} \) of the full journey simultaneously, as all these schemes are optimal. Each of these \( d \) sets of travellers will then repeat a similar process for the second and subsequent partial schemes, with all travellers completing each of the fractional journeys of lengths \( \frac{1}{r}, \frac{2}{r}, \ldots, \frac{i}{r}, \ldots, \frac{d}{r} = 1 \) at the same time. All \( n \) travellers complete the journey simultaneously, having cycled \( l \) stages, as required to finish in the least time. \( \square \)
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