Abstract

As previously shown BRST singlets $|s\rangle$ in a BRST quantization of general gauge theories on inner product spaces may be represented in the form

$$|s\rangle = e^{[Q,\psi]}|\phi\rangle$$

where $|\phi\rangle$ is either a trivially BRST invariant state which only depends on the matter variables, or a solution of a Dirac quantization. $\psi$ is a corresponding fermionic gauge fixing operator. In this paper it is shown that the time evolution is determined by the singlet states of the corresponding reparametrization invariant theory. The general case when the constraints and Hamiltonians may have explicit time dependence is treated.

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1 Introduction.

In BRST quantization of general gauge theories on inner product spaces the BRST singlets, $|s\rangle$, are BRST invariant states that describe the true physical degrees of freedom and represent the BRST cohomology $(|s\rangle \in \ker Q/\im Q)$ \[1\]. By means of a generalized quartet mechanism a simple general representation of the BRST singlets for general gauge theories with finite number of degrees of freedom was obtained in \[2\]. (For Lie group theories corresponding gauge invariant states were previously obtained by means of a bigrading in \[3, 4]\.) The representation is

$$|s\rangle = e^{[Q, \psi]}|\phi\rangle$$

(1.1)

where $Q$ is the hermitian nilpotent BRST operator in BFV form \[5, 6\], $\psi$ a hermitian fermionic gauge fixing operator, and $|\phi\rangle$ BRST invariant states determined by a hermitian set of operators which are BRST doublets in involution. $|\phi\rangle$ does not belong to an inner product space although $|s\rangle$ does. Since the BRST quartets of operators may always be split into two sets of hermitian BRST doublets there are at least two dual choices for $|\phi\rangle$ and the corresponding $\psi$. For general, both irreducible and reducible, gauge theories of arbitrary rank within the BFV formulation it was found in \[2\] that there always exist solutions such that $|\phi\rangle$ are trivial BRST invariant states which only depend on the matter variables for one set of BRST doublets. For the complementary set of BRST doublets in the BRST quartets $|\phi\rangle$ must be solutions of a Dirac quantization (which not always exist). At the end of \[3\] some aspects that needed further clarifications was pointed out like e.g. the exact connection to the coBRST formulation and the freedom in the choice of gauge fixing fermions, as well as the question of how the time evolution in terms of a nontrivial Hamiltonian should be defined. The first aspects were clarified in ref.\[7\]. The second aspect is the subject of the present paper.

As in \[8\] it is natural to expect the time evolution to be given by

$$|s, t\rangle = e^{itH_0}|s\rangle$$

(1.2)

in the case when $H_0$ is a BRST invariant Hamiltonian $([Q, H_0] = 0)$ with no explicit time dependence. On the other hand one could equally well have

$$|\phi, t\rangle = e^{itH_0}|\phi\rangle, \quad |s, t\rangle = e^{[Q, \psi]}|\phi, t\rangle$$

(1.3)

which is not the same as (1.2) if $H_0$ does not commute with $[Q, \psi]$. In \[8\] eq.(1.2) was assumed together with

$$[H_0, [Q, \psi]] = 0$$

(1.4)

which makes (1.2) and (1.3) equivalent. In this case one could bring the representation (1.1) in contact with the standard path integral formulation in phase space as formulated in \[3\].

In this paper we shall determine the time evolution of states of the form (1.1) for the general case when the Hamiltonian and the gauge fixing fermions $\psi$ in (1.1) also may have explicit time dependence. This we shall do without invoking any new ingredients of the BRST quantization as formulated in \[3\]. What we shall do is to make use of a well-known trick how to make any theory reparametrization invariant. The resulting
reparametrization invariant theory has then no non-trivial Hamiltonian. Instead it has a new constraint which involves the original Hamiltonian. The method of \[3\] may therefore be directly applied to this extended formulation and at the end we may interpret the resulting BRST singlets as the time dependent singlets of the original theory by means of an appropriate gauge choice. In section 2 we define what we mean by the corresponding reparametrization invariant theory and determines its BRST charge operator. In section 3 we apply the rules of \[2\] to this extended formulation and discuss how the gauge fixing should be performed in order for the result to be interpretable as the time evolution of the original theory. In section 4 we describe how the formulation looks in the corresponding path integral formulation. In section 5 the simple example of a general regular theory is treated and in section 6 we give the conclusion.

2 Extended formulation

A general gauge theory with finite number of degrees of freedom may be given in terms of the phase space Lagrangian \(i = 1, \ldots, n; \quad \alpha = 1, \ldots, m < n\)

\[L_0(t) = p_i q^i - H - v^\alpha \Phi_\alpha\] (2.1)

where \(v^\alpha\) are Lagrange multipliers and \(\Phi_\alpha\) first class constraints. In distinction to the treatment in ref. \[3\] \(H\) and \(\Phi_\alpha\) are here allowed to depend on time \(t\) explicitly. (Such gauge theories have been treated in \[9, 10\].) We shall now make use of the fact that the gauge theory (2.1) may be embedded in a larger gauge theory described by the Lagrangian

\[L(\tau) = p_i q^i + \pi i - v(\pi + H + v^\alpha \Phi_\alpha)\] (2.2)

where we have promoted time \(t\) to a dynamical variable \(t(\tau)\) with \(\pi\) as its conjugate momentum. (In eq. (2.2) dots represent differentiation with respect to the parameter \(\tau\)!) This is a well known trick how to make a theory reparametrization invariant (see e.g. the books \[10, 11\]). Equivalence between (2.1) and (2.2) requires \(v = dt/d\tau\). We shall require \(t\) and \(\tau\) to be in a one-to-one correspondence which demands the condition \(v > 0\) or \(v < 0\) on the Lagrange multiplier \(v\). (A natural way to satisfy this condition is to replace \(v\) by \(e^\omega\) where \(\omega\) is unrestricted.) The Lagrangian (2.2) gives rise to the constraints

\[\pi + H = 0, \quad \Phi_\alpha = 0\] (2.3)

which we require to be of first class so that \(v\) and \(v^\alpha\) in (2.2) are independent variables. Since \(H\) and \(\Phi_\alpha\) do not depend on \(\pi\), the Poisson algebra of the constraints (2.3) becomes

\[\{\Phi_\alpha, \Phi_\beta\} = c^\gamma_{\alpha\beta} \Phi_\gamma, \quad \{\Phi_\alpha, \pi + H\} = c^\beta_{\alpha\beta} \Phi_\beta\] (2.4)

where the last condition corresponds to Dirac’s consistency conditions for the original theory.

We turn now to the BRST quantization of theses models. Following the BFV prescription for the nilpotent BRST charge (see \[3\]) we find the following expression for the model (2.2)

\[Q = Q_0 + C(\pi + H_0) + \bar{\mathcal{P}} \pi v\] (2.5)
where $\pi_v$ is the conjugate momentum to $v$, $C$ a fermionic ghost variable and $\bar{P}$ conjugate momentum to the corresponding fermionic antighost $\bar{\tilde{C}}$. Their fundamental nonzero (graded) commutators are

$$[C, P] = 1, \quad [\bar{C}, \bar{P}] = 1 \quad [t, \pi] = i, \quad [v, \pi_v] = i$$  \hfill (2.6)$$

$Q_0$ and $H_0$ are the BFV-BRST charge and BFV Hamiltonian for the model (2.1). $Q_0$ and $H_0$ are independent of $\pi, v, \pi_v, C, \bar{P}, \bar{\tilde{C}}$. Since $Q^2 = 0$ they satisfy (cf. [9])

$$Q_0^2 = 0, \quad [Q_0, \pi + H_0] = 0 \quad (2.7)$$

For instance, in the irreducible case their forms are

$$Q_0 = C^\alpha \Phi_\alpha + P^\alpha \pi_\alpha + \text{terms depending on $P_\alpha$}$$

$$H_0 = H + \text{terms depending on $C^\alpha$ and $P_\alpha$} \quad (2.8)$$

where we have introduced the ghosts and antighosts $C^\alpha, \bar{C}_\alpha$ and their conjugate momenta $P_\alpha, \bar{P}_\alpha$. $\pi_\alpha$ are the conjugate momenta to the Lagrange multipliers $v^\alpha$. All variables are hermitian and their fundamental nonzero (graded) commutators are (we assume here for simplicity that $C^\alpha, \bar{C}_\alpha, P_\alpha, \bar{P}_\alpha$ are fermionic and $\pi_\alpha, v^\alpha, \Phi_\alpha$ bosonic)

$$[q^i, p_j] = i\delta^i_j, \quad [v^\alpha, \pi_\beta] = i\delta^\alpha_\beta, \quad [C^\alpha, P_\beta] = \delta^\alpha_\beta, \quad [\bar{C}_\alpha, \bar{P}_\beta] = \delta^\alpha_\beta \quad (2.9)$$

We have of course the usual ghost grading such that $C, \bar{P}, C^\alpha, \bar{C}_\alpha, Q, Q_0$ have ghost number one, $\bar{\tilde{C}}, P, C_\alpha, P_\alpha$ have ghost number minus one, while $(q^i, p_j) (v^\alpha, \pi_\beta) (t, \pi) (v^\alpha, \pi_\beta)$ and $H_0$ have ghost number zero. The last conditions imply that the ghost number operator

$$N = C^\alpha P_\alpha - \bar{C}_\alpha \bar{P}_\alpha + \bar{\tilde{C}} \bar{\tilde{P}} - \bar{\tilde{C}} \bar{\tilde{P}} \quad (2.10)$$

is conserved, i.e. $[N, \pi + H_0] = [N, H_0] = 0$.

### 3 Time evolution as BRST singlets of the extended formulation

It is now straight-forward to apply the method of ref. [2] to the above BRST theory of the extended formulation (2.2) of the general gauge theory (2.1). This implies that the BRST singlet states may be chosen to have the form (1.1). For the irreducible case (2.8) we have the two expressions

$$|s\rangle_1 = e^{[Q, \psi_1]}|\phi_1\rangle, \quad |s\rangle_2 = e^{[Q, \psi_2]}|\phi_2\rangle$$  \hfill (3.1)$$

where $|\phi\rangle_{1,2}$ satisfy

$$\chi^\alpha |\phi\rangle_1 = C^\alpha |\phi\rangle_1 = \bar{C}_\alpha |\phi\rangle_1 = \pi_\alpha |\phi\rangle_1 = 0 \quad (3.2)$$

$$\chi(t) |\phi\rangle_1 = C |\phi\rangle_1 = \bar{C} |\phi\rangle_1 = \pi_v |\phi\rangle_1 = 0 \quad (3.3)$$

$$\Lambda^\alpha (v^\beta) |\phi\rangle_2 = \bar{P}^\alpha |\phi\rangle_2 = P_\alpha |\phi\rangle_2 = [Q, P_\alpha] |\phi\rangle_2 = 0 \quad (3.4)$$
\[ \Lambda(v)|\phi\rangle_2 = \bar{\mathcal{P}}|\phi\rangle_2 = \mathcal{P}|\phi\rangle_2 = (\pi + H_0)|\phi\rangle_2 = 0 \] (3.5)

where the hermitian operators \( \chi^\alpha, \chi(t), \Lambda^\alpha(v^\beta) \), and \( \Lambda(v) \) are gauge fixing operators to \([Q, \mathcal{P}_\alpha], \pi + H_0, \pi_\alpha, \text{and } \pi_t \) respectively. \( \chi(t), \Lambda^\alpha(v^\beta) \), and \( \Lambda(v) \) should therefore be such that the first conditions in (3.3)-(3.5) fix \( t, v^\alpha \), and \( v \) uniquely. In particular they must satisfy \( \partial_v \Lambda \neq 0, \partial_t \chi \neq 0 \) and \( \det \Lambda^\beta \neq 0 \) where \( \partial_\alpha \equiv \partial/\partial v^\alpha \). (The most natural choice is of course \( \chi(t) = t, \Lambda^\alpha(v^\beta) = v^\alpha, \text{and } \Lambda(v) = v \).) The gauge fixing fermions \( \psi_{1,2} \) in (3.1) may then always be chosen to be \( 2 \)

\[ \psi_1 = \psi_{01} + \mathcal{P}\Lambda(v), \quad \psi_2 = \psi_{02} + \bar{C}\chi(t) \] (3.6)

where

\[ \psi_{01} = \mathcal{P}_\alpha \Lambda^\alpha(v^\beta), \quad \psi_{02} = \bar{C}_\alpha \chi^\alpha \] (3.7)

are the natural gauge fixing fermions for \( Q_0 \). For this choice we find

\[ [Q, \psi_1] = [Q_0, \psi_{01}] + \bar{C} [\pi + H_0, \psi_{01}] + \Lambda(v)(\pi + H_0) - i\Lambda'(v)\bar{P}\mathcal{P} \]

\[ [Q, \psi_2] = [Q_0, \psi_{02}] + \bar{C} [\pi + H_0, \psi_{02}] + \pi_v \chi(t) - i\dot{\chi}(t)\bar{C}\mathcal{C} \] (3.8)

where \( \Lambda'(v) \equiv \partial_v \Lambda(v) \) and \( \dot{\chi}(t) \equiv \partial_t \chi(t) \). The resulting BRST singlets (3.1) do then represent the general time dependent BRST singlets of the original BRST theory of (2.1). In particular one may notice that \( |\phi\rangle_2 \) always satisfy the Schrödinger equation due to the last condition in (3.3). In the general reducible or irreducible case (3.4) and (3.7) are replaced by the more general expressions given in ref. [2]. (The last condition in (3.4) may be replaced by \([Q_0, \mathcal{P}_\alpha]|\phi\rangle_2 = 0 \) since \( H_0 \) only depends on \( C^\alpha \) in terms like \( t_\alpha^2 C^\alpha \mathcal{P}_\beta \). Such reductions are always valid.)

The time evolution of \( |s\rangle_1 \) and \( |s\rangle_2 \) in (3.1) are determined by the following subset of the conditions in (3.3)-(3.5)

\[ \chi(t)|\phi\rangle_1 = \Lambda(v)|\phi\rangle_2 = 0 \] (3.9)

\[ \pi_v|\phi\rangle_1 = (\pi + H_0)|\phi\rangle_2 = 0 \] (3.10)

The conditions (3.9) in (3.1) imply

\[ \chi(t - i\Lambda(v))|s\rangle_1 = \Lambda(v - i\chi(t))|s\rangle_2 = 0 \] (3.11)

which relate \( t \) and \( v \). (Notice that \( \chi(t) \) and \( \Lambda(v) \) should be monotonic functions.) The conditions (3.10) in (3.1) yield then generalized Schrödinger equations for \( |s\rangle_1 \) and \( |s\rangle_2 \) when combined with (3.11). They are in general rather involved and therefore difficult to express in terms of the original theory. However, if \( \chi(t) \) and \( \Lambda(v) \) are chosen to be linear functions in their arguments, \textit{i.e.}

\[ \Lambda(v) = \alpha (v - v_0), \quad \chi(t) = \beta (t - t_0) \] (3.12)

where \( \alpha, \beta, v_0, t_0 \) are real constants satisfying \( \alpha \neq 0, \beta \neq 0 \), and if \( \psi_{01}, \psi_{02} \) are chosen to be conserved,

\[ [\psi_{01}, \pi + H_0] = 0, \quad [\psi_{02}, \pi + H_0] = 0, \] (3.13)
then it is straightforward to show that (3.10) implies

\[
(\pi - i(\Lambda')^{-1}\pi_0 + H_0)|s\rangle_1 = 0 \\
(\pi + i\pi_0 \chi + H_0)|s\rangle_2 = 0
\]

which when combined with (3.11) reduce to the ordinary Schrödinger equations. Notice that (3.13) implies

\[
[\pi + H_0, |Q_0, \psi_{01(02)}]\rangle = 0
\]

which is the natural generalization of (1.14). The time evolution seems to be effectively that of (3.14) even if only (3.12) and (3.15) are required, since the modifications in (3.14) will only affect the ghost part of \(|s\rangle\) in such a way that they will not contribute to the inner products. The condition (3.15) also allows \(e^{[Q_0, \psi_0]}\) to be factored out from \(e^{[Q, \psi]}\). Since the gauge fixing conditions must satisfy Dirac’s consistency conditions the original gauge fixing fermions should be conserved in some weak sense. Whether or not (3.15) is the weakest possible conditions remains to determine.

The above reduction to Schrödinger equations may be made clearer if we turn to the wave function representations of \(|s\rangle_{1,2}\). Inserting (3.8) into (3.1) using (3.13) we have formally

\[
\Psi_1(t, \nu, \mathcal{P}, \omega) = \langle t, \nu, \mathcal{P}, \omega|s\rangle_1 = e^{-i\mathcal{P}(\omega)} e^{\Lambda(\nu)(-i\partial t + H_0)} e^{[Q_0, \psi_0]} \phi_1(t, \omega) \\
\Psi_2(t, \nu, \mathcal{C}, \mathcal{C}', \omega) = \langle t, \nu, \mathcal{C}, \mathcal{C}', \omega|s\rangle_2 = e^{-i\mathcal{P}(\omega)} e^{\Lambda(\nu)(-i\partial t + H_0)} e^{[Q_0, \psi_0]} \phi_2(t, \omega)
\]

where \(\omega\) is a collective coordinate for \(q^i, \nu^\alpha, C^\alpha, \bar{C}_\alpha\) which are the variables of the original BRST theory. In (3.16) we have used the fact that the wave function representation of \(|\phi\rangle_{1,2}\) may be written as

\[
\langle t, \nu, \mathcal{P}, \omega|\phi\rangle_1 = \delta(\chi(t))\phi_1(t, \omega), \quad \langle t, \nu, \mathcal{C}, \mathcal{C}', \omega|\phi\rangle_2 = \delta(\Lambda(\nu))\phi_2(t, \omega)
\]

where \(\phi_1(t, \omega)\) and \(\phi_2(t, \omega)\) satisfy the conditions (3.2) and (3.4) respectively. Eq. (3.17) are explicit solutions of (3.3) and (3.5) except for the last condition in (3.5) which requires \(\phi_2(t, \omega)\) to satisfy the Schrödinger equation \(i\partial_t \phi_2 = H_0 \phi_2\).

For the natural gauge choice (3.12) the wave functions in (3.16) become

\[
\Psi_1(t, \nu, \mathcal{P}, \omega) = e^{-i\mathcal{P}(\omega)} e^{-i\alpha_0 \mathcal{P}} \delta(t - t_0 - i\alpha(\nu - v_0)) \Phi_1(t, \omega) \\
\Psi_2(t, \nu, \mathcal{C}, \mathcal{C}', \omega) = e^{-i\mathcal{C}(\omega)} e^{-i\beta_0 \mathcal{C}} \delta(\beta(t - t_0) + i(\nu - v_0)) \Phi_2(t, \omega)
\]

where both \(\Phi_1(t, \omega)\) and \(\Phi_2(t, \omega)\) satisfy the Schrödinger equation (notice that \(e^{[Q_0, \psi_0]}\) and \(e^{[Q, \psi]}\) are conserved)

\[
i\partial_t \Phi_{1,2}(t, \omega) = H_0 \Phi_{1,2}(t, \omega)
\]

\(\Phi_{1,2}(t, \omega)\) may be identified with the time dependent BRST singlets of the original BRST theory.

So far the expressions (3.18) for the singlets are only formal. One should notice that \(q^i, \nu^\alpha, C^\alpha, \bar{C}_\alpha\) in the wave functions (3.16) and (3.18) are the eigenvalues of the corresponding hermitian operators. The question that remains to answer is which eigenvalues
are real and which are imaginary. In [12] we gave a general rule which says that the existence of solutions of the form (3.19) requires bosonic unphysical (gauge) variables and corresponding Lagrange multipliers to be quantized in an opposite manner. This implies that in (3.16) and (3.18) \( t \) and \( v \) must be quantized with real and imaginary eigenvalues respectively, or vice versa. This follows also since the argument of the delta functions in (3.16) and (3.18) must be real. The most natural choice is to let \( t \) have real eigenvalues and to let \( v \) be quantized with imaginary eigenvalues \( iu, u \) real. \( v \) in e.g. (3.18) must then be replaced by \( iu \) after which (3.18) become true solutions if \( v_0 = 0 \). By means of the properties

\[
\begin{align*}
|iu\rangle &= iu|iu\rangle, \quad \langle -iu| = (|iu\rangle)^\dagger \\
\langle iu'|iu\rangle &= \delta(u' - u), \quad \int du| -iu\rangle\langle -iu| = \int du|iu\rangle\langle iu| = 1
\end{align*}
\]  

(3.20)

of indefinite metric states, the solutions (3.18) become

\[
\begin{align*}
\Psi_1(t, iu, \mathcal{P}, \mathcal{P}, \omega) &= e^{-i\alpha P_\omega} \delta(t - t_0 + \alpha u) \Phi_1(t, \omega) \\
\Psi_2(t, iu, \mathcal{C}, \mathcal{C}, \omega) &= e^{-i\beta C_\omega} \delta(\beta(t - t_0) - u) \Phi_2(t, \omega)
\end{align*}
\]  

(3.21)

and we find

\[
\begin{align*}
1\langle s|s \rangle_1 &= \int d\omega dt dud\mathcal{P} d\mathcal{P} \Psi^*_1(t, -iu, \mathcal{P}, \mathcal{P}, \omega^*) \Psi_1(t, iu, \mathcal{P}, \mathcal{P}, \omega) = \\
&\quad = \int d\omega dt dud\mathcal{P} d\mathcal{P} e^{-2i\alpha P_\omega} \delta(t - t_0 - \alpha u)\delta(t - t_0 + \alpha u) \Phi^*_1(t, \omega^*) \Phi_1(t, \omega) = \\
&\quad = \int d\omega \Phi^*_1(t_0, \omega^*) \Phi_1(t_0, \omega) > 0
\end{align*}
\]  

(3.22)

\[
\begin{align*}
2\langle s|s \rangle_2 &= \int d\omega dt dud\mathcal{C} d\mathcal{C} \Psi^*_2(t, -iu, \mathcal{C}, \mathcal{C}, \omega) \Psi_2(t, iu, \mathcal{C}, \mathcal{C}, \omega) = \\
&\quad = \int d\omega dt dud\mathcal{C} d\mathcal{C} e^{-2i\beta C_\omega} \delta(\beta(t - t_0) + u)\delta(\beta(t - t_0) - u) \Phi^*_2(t, \omega^*) \Phi_2(t, \omega) = \\
&\quad = \int d\omega \Phi^*_2(t_0, \omega^*) \Phi_2(t_0, \omega) > 0
\end{align*}
\]  

(3.23)

This is the expected result since the choice \( \chi(t) = \beta(t - t_0) \) in (3.3) and in \( \psi_2 \) in (3.4) should be interpreted as a gauge choice in which we fix \( t \) to be \( t_0 \). The right-hand sides can of course be further reduced so that all imaginary bosonic coordinates in \( \omega \) are eliminated and we end up with a regular theory with square integrable wave functions \( \Phi \).

The two different gauge fixing \( \chi(t) = \beta t \) and \( \chi(t) = \beta(t - t_0) \) should be connected by a gauge transformation

\[
|s, t_0 \rangle = |s, 0 \rangle + Q|\chi\rangle
\]  

(3.24)

where \( |\chi\rangle \) is restricted by the condition that both \( |s, t_0 \rangle \) and \( |s, 0 \rangle \) are inner product states. In fact, we have

\[
|s, t_0 \rangle = e^{-i(\pi + H_0)t_0} |s, 0 \rangle
\]  

(3.25)

which agrees with (3.24) since \( \pi + H_0 = [Q, \mathcal{P}] \). Thus,

\[
\langle s, t_0|s, t_0 \rangle = \langle s, 0|s, 0 \rangle
\]  

(3.26)
From (3.22) and (3.23) this requires in turn the time evolution of the original BRST invariant theory to be unitary. Thus, the different gauge choices are connected by a unitary gauge transformation exactly as far as the Schrödinger equation (3.19) allows us to connect all time instances in a unitary fashion. Another consequence of (3.24) and (3.25) is that
\[ \langle s,t|s,t_0\rangle = \langle s,t_0|s,0\rangle = \langle s,0|s,0\rangle \] (3.27)

An explicit calculation using (3.22) yields also
\[ \langle s,t_0|s,0\rangle = \int d\omega \Phi^*(t_0/2,\omega^*) \Phi(t_0/2,\omega) \] (3.28)
in agreement with (3.27) since (3.26) is valid for arbitrary \( t_0 \).

4 Path integral formulations

The representation (1.1) of BRST singlets, i.e. \( |s\rangle = e^{[Q,\psi]}|\phi\rangle \), constitutes a missing link between operator and path integral quantization of general gauge theories. The quantization rules in [12] may easily be understood when the inner products of (1.1) is written as path integrals [8]. The inner products of the extended singlets (3.1) are of the form
\[ \langle s'|s\rangle = \langle \phi|e^{[Q,\psi]}|\phi\rangle = \int dR'dR \phi^*(R')\phi(R)\langle R'|e^{[Q,\psi]}|R'\rangle \] (4.1)
where \( R \) and \( R' \) represent all the involved coordinates (some may have imaginary eigenvalues). \( \gamma \) is a real constant. Since this expression is independent of the value of \( \gamma \) as long
as $\gamma \neq 0$, we may make the identification $\gamma = \tau - \tau'$. This allows us to interpret the inner product (4.1) as the transition amplitude

$$\langle s, \tau' | s, \tau \rangle = \int dR' dR \phi^*(R') \phi(R) \langle R', \tau' | R^*, \tau \rangle \tag{4.2}$$

where

$$\langle R', \tau' | R^*, \tau \rangle = \langle R' | e^{i(\tau - \tau') [Q, \psi]} | R^* \rangle = \int DR'D\pi e^{i \int_{\tau}^{\tau'} (i\pi \dot{Q} - \{Q, \psi\})} \tag{4.3}$$

where $P$ are conjugate momenta to $R$. The last path integral formula is obtained through the time slice formula (see [8]). The first equality in (4.3) requires $\{Q, \psi\}$ to be interpreted as a conserved Hamiltonian, which is possible since $\{Q, \psi\}$ has no explicit $\tau$ dependence. Notice that the conditions (3.2)-(3.5) act as boundary conditions in (4.2). (Actually (4.2) is independent of the precise choice of the first conditions in (3.2)-(3.5) since they are gauge fixing conditions.) $\{Q, \psi\}$ represents the 'classical' counterpart (Weyl symbol) of (3.8). As shown in [8], $\{Q, \psi\}$ should be real provided all Lagrange multipliers as well as all the ghosts (or antighosts) are chosen to have imaginary eigenvalues. This is also true here for imaginary $v$ if $\Lambda(v) = \alpha v$ in agreement with the results of the previous section. Thus, (4.3) is a good path integral formula for $\tau, t$ and $C (\bar{C})$ real, while all the ghosts (or antighosts) are chosen to have imaginary eigenvalues. This is also true here for imaginary $v$ if $\Lambda(v) = \alpha v$ in agreement with the results of the previous section.

All the above path integrals may be reduced to the appropriate path integrals of the original BRST invariant theory (see [8])

$$\langle s, t' | s, t \rangle = \int d\omega' d\omega \phi^*(\omega'^*) \phi(\omega) \langle \omega', t' | \omega^*, t \rangle \tag{4.4}$$

where $\phi(\omega)$ only satisfies (3.2) or (3.4) and where

$$\langle \omega', t' | \omega^*, t \rangle = \int D\omega D\pi_\omega e^{i \int_t^{t'} (i\pi_\omega \dot{\omega} - H_0 - \{Q_0, \psi_0\})} \tag{4.5}$$

where $\omega$ denotes all coordinates of the original BRST invariant theory and $\pi_\omega$ their conjugate momenta. This reduction may be performed by an integration over $t$, $\pi$, $v$, $\pi_v$, $C$, $P$, $\bar{C}$, $\bar{P}$ in (3.3) for all the above cases provided $\Lambda(v)$ and $\chi(t)$ have the linear form (3.12). In particular one must choose $\Lambda(v) = \alpha v$ or $\chi(t) = \beta t$ when $v$ or $t$ are imaginary in agreement with the results of the previous section.

5 Application to the regular case

The above results are also valid when the original theory is an ordinary regular one. Consider a regular theory described by the phase space Lagrangian

$$L_0(t) = p_i \dot{q}^i - H \tag{5.1}$$
corresponding to an unconstrained Lagrangian. This is obviously a special case of \(2.1\). This regular theory may be described by an extended singular phase space Lagrangian of the form (the corresponding reparametrization invariant theory)

\[
L(\tau) = p_i \dot{q}^i + \pi \dot{t} - v(\pi + H)
\]

(5.2)

A BRST quantization of this singular model leads to the BFV-BRST charge operator

\[
Q = C(\pi + H) + \bar{P} \pi_v
\]

(5.3)

(All variables are hermitian.) The BRST singlets are

\[
|s_l\rangle = e^{[Q, \psi]|\phi|l}, \quad l = 1, 2
\]

(5.4)

where \(|\phi\rangle_{1,2}\) satisfy (3.3) and (3.5), and where the most natural choice for the gauge fixing fermions \(\psi_{1,2}\) is (cf. (3.6))

\[
\psi_1 = P \Lambda(v), \quad \psi_2 = \bar{C} \chi(t)
\]

(5.5)

Eq. (3.8) reduces here to

\[
[Q, \psi_1] = \Lambda(v)(\pi + H_0) - i\Lambda'(v)\bar{P}P
\]

\[
[Q, \psi_2] = \pi_v \chi(t) - i\dot{\chi}(t)\bar{C}C
\]

(5.6)

Conditions (3.3) and (3.5) imply now (3.11) which relates \(t\) and \(v\), the generalized Schrödinger equations

\[
(\pi - i(\Lambda')^{-1}\pi_v + H + (\Lambda')^{-1}\Lambda''\bar{P}P)|s\rangle_1 = 0
\]

\[
(\pi + i\pi_v \chi + H + \dot{\chi}\bar{C}C)|s\rangle_2 = 0
\]

(5.7)

and the ghost conditions

\[
(C - i\Lambda' \bar{P})|s\rangle_1 = (\bar{C} + i\Lambda' \bar{P})|s\rangle_1 = 0
\]

\[
(P - i\dot{\chi} \bar{C})|s\rangle_2 = (P + i\dot{\chi} \bar{C})|s\rangle_2 = 0
\]

(5.8)

For the linear choice (3.12) the equations for \(|s\rangle_1\) reduces to

\[
(t + i\alpha v)|s\rangle_1 = (C - i\alpha \bar{P})|s\rangle_1 = (\bar{C} + i\alpha P)|s\rangle_1 = (H + \pi + \frac{i}{\alpha} \pi_v)|s\rangle_1 = 0
\]

(5.9)

and for \(|s\rangle_2\) we have the same equations for \(\beta = -1/\alpha\). Thus, for this choice we may set \(|s\rangle \equiv |s\rangle_1 = |s\rangle_2\). The Schrödinger or wave function representation of \(|s\rangle\) is

\[
\Psi_s(q, t, v, C, \bar{C}) \equiv \langle q, t, v, C, \bar{C}|s\rangle
\]

(5.10)

where \(q\) represents the physical coordinates \(q^i\) in (3.1). The solutions of the equations (5.9) (i.e. (5.4)) may then be written as

\[
\Psi_s(q, t, v, C, \bar{C}) = e^{-i\mathcal{C}/\alpha \delta(t + iv)}\Phi(q, t)
\]

(5.11)

where \(\Phi(q, t)\) satisfies the Schrödinger equation

\[
(i\partial_t - H_S(t))\Phi(q, t) = 0
\]

(5.12)

where \(H_S(t)\) is the Schrödinger representation of the the Hamiltonian operator \(H(t)\). So far all equations are formal. To get the true equations we must consider the appropriate quantization rules. This proceeds exactly along the lines of section 3.
6 Conclusions

In section 2 we have shown that the BFV-BRST charge and BFV Hamiltonian for a general gauge theory, in which the constraints and Hamiltonian also may depend explicitly on time, are determined by the standard prescription of the BFV-BRST charge for the corresponding reparametrization invariant gauge theory which then has no explicit 'time' dependence. We have then investigated to what extent the extended BRST singlets may be interpreted as the time dependent BRST singlets of the original gauge theory. We have then found that this is possible at least for the sector of linear gauge choices for time and a corresponding Lagrange multiplier, and provided the gauge fixing fermions of the original theory is conserved in some weak sense. For these linear gauges the path integrals for the transition amplitudes of the extended models do also formally reduce to the appropriate transition amplitudes of the original theory. Any gauge choice for time within the linear sector yields equivalent results since there are no topological obstructions in this sector.

The main message of the present communication is, thus, that the time evolution in BRST quantization for general gauge theories is determined by the BRST quantization of the corresponding reparametrization invariant theory provided time is fixed by a linear gauge choice.

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