Chiral behavior of the $B_{d,s}^0 - \overline{B}_{d,s}^0$ mixing amplitude in the Standard Model and beyond

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**Abstract**

We compute the chiral logarithmic corrections to the $B_{d}^0 - \overline{B}_{d}^0$ and $B_{s}^0 - \overline{B}_{s}^0$ mixing amplitudes in the Standard Model and beyond. We then investigate the impact of the inclusion of the lowest-lying scalar heavy-light states to the decay constants and bag-parameters and show that this does not modify the pion chiral logarithms, but it does produce corrections which are competitive in size with the $K$- and $\eta$-meson chiral logarithms. This conclusion is highly relevant to the lattice studies since the pion chiral logarithms represent the most important effect in guiding the chiral extrapolations of the lattice data for these quantities. It is also important to stress that the pion chiral logarithmic corrections are useful in guiding those chiral extrapolations as long as $m_\pi \ll \Delta_S$, where $\Delta_S$ stands for the mass difference between the heavy-light mesons belonging to $\frac{1}{2}^+$ and $\frac{3}{2}^-$ doublets.
1 Introduction

The oscillations in the $B^0_{d,s} - \overline{B}^0_{d,s}$ systems are mediated by the flavor changing neutral currents which are forbidden at tree level of the Standard Model (SM) and therefore their detection gives access to the particle content in the corresponding loop diagrams. First experimental measurement of a large value of $\Delta m_{B_d}$ indicated that the top quark mass was very heavy [1], which was confirmed almost a decade later in the direct measurements, $m_t = 172.5 \pm 1.3 \pm 1.9$ GeV through the $p\bar{p}$-collisions [2]. Nowadays, the accurately measured $\Delta m_{B_q} = 0.509(5)(3)$ ps$^{-1}$ [3] and $\Delta m_{B_s} = 17.31(33)(7)$ ps$^{-1}$ [4], are used to constrain the shape of the Cabibbo-Kobayashi-Maskawa (CKM) unitarity triangle and thereby determine the amount of the CP-violation in the SM [5]. This goal is somewhat hampered by the theoretical uncertainties in computing the values for the two decay constants, $f_{B_{s,d}}$, and the corresponding “bag” parameters, $B_{B_{s,d}}$. These quantities can, in principle, be computed on the lattice. However, a major obstacle in the current lattice studies is that the $d$-quark cannot be reached directly but through an extrapolation of the results obtained by working with larger light quark masses down to the physical $d$-quark mass. This extrapolation induces systematic uncertainties which are hard to control as the spontaneous chiral symmetry breaking effects are expected to become increasingly pronounced as one lowers the light quark mass [7]. Heavy meson chiral perturbation theory (HMChPT) allows us to gain some control over these uncertainties because it predicts the chiral behavior of the hadronic quantities relevant to the heavy-light quark phenomenology which then can be implemented to guide the extrapolation of the lattice results. HMChPT combines the heavy quark effective theory (HQET) with the common pattern of spontaneous breaking of the chiral symmetry, $SU(3)_L \otimes SU(3)_R \to SU(3)_V$ [8].

Like in the standard ChPT, in HMChPT one computes the chiral logarithmic corrections (the so-called non-analytic terms) which are expected to be relevant to the very low energy region, i.e., $m_q \ll \Lambda_{\text{QCD}}$. While this condition is satisfied for $u$- and $d$-quarks, the situation with the $s$-quark is still unclear [9]. Also ambiguous is the size of the chiral symmetry breaking scale, $\Lambda_\chi$. Some authors consider it to be around $4\pi f_\pi \approx 1$ GeV [10], while others prefer identifying it with the mass of the first vector resonance, $m_\rho = 0.77$ GeV (see e.g. ref. [11]), and sometimes even lower [12]. In the heavy-light quark systems the situation becomes more complicated because the first orbital excitations ($j^P = 1/2^+$) are not far away from the lowest lying states ($j^P = 1/2^-$). The recent experimental evidence for the scalar $D^*_0 s$ and axial $D_1 s$ mesons indicate that this splitting is only $\Delta_{S_s} \equiv m_{D^*_0 s} - m_{D_s} = m_{D^*_1 s} - m_{D^*_2 s} = 350$ MeV [13], and somewhat larger for the non-strange states $\Delta_{S_{u,d}} = 430(30)$ MeV [14]. This and the result of the lattice QCD study in the static heavy quark limit [17] suggest that the size of this mass difference remains as such in the $b$-quark sector as well. One immediately observes that both $\Delta_{S_s}$ and $\Delta_{S_{u,d}}$ are smaller than $\Lambda_\chi$, $m_\eta$, and even $m_K$, which requires revisiting the predictions based on HMChPT and reassessing their range of validity. In this paper we investigate this issue on the specific examples of

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1Recent reviews on the current status of the lattice QCD computations of $B^0_q - \overline{B}^0_q$ mixing amplitudes can be found in ref. [6].

2We note, in passing, that the experimentally established fact, $\Delta_{S_s} < \Delta_{S_{u,d}}$, is not yet understood [15] although a recent lattice study with the domain wall quarks indicates a qualitative agreement with experiment [16].
the decay constants $f_{B_d,s}$ and the bag parameters which enter the investigation of the SM and supersymmetric (SUSY) effects in the $B_d^0 - \overline{B}_d^0$ and $B_s^0 - \overline{B}_s^0$ mixing amplitudes [18].

2 Bases of $\Delta B = 2$ operators and $B$-parameters

The SUSY contributions to the $B_d^0 - \overline{B}_d^0$ mixing amplitude, where $q$ stands for either $d$- or $s$-quark, are usually discussed in the so called SUSY basis of $\Delta B = 2$ operators [19]:

\begin{align}
O_1 &= \bar{b}^i \gamma_\mu (1 - \gamma_5) q^j \bar{b}^j \gamma^\mu (1 - \gamma_5) q^j , \\
O_2 &= \bar{b}^i (1 - \gamma_5) q^j \bar{b}^j (1 - \gamma_5) q^j , \\
O_3 &= \bar{b}^i (1 - \gamma_5) q^j \bar{b}^j (1 - \gamma_5) q^j , \\
O_4 &= \bar{b}^i (1 - \gamma_5) q^j \bar{b}^j (1 + \gamma_5) q^j , \\
O_5 &= \bar{b}^i (1 - \gamma_5) q^j \bar{b}^j (1 + \gamma_5) q^j ,
\end{align}

where $i$ and $j$ are the color indices. Although the operators in the above bases are written with both parity even and parity odd parts, only the parity even ones survive in the matrix elements. In SM, only $O_1$ (left-left) operator is relevant in describing the $B_d^0 - \overline{B}_d^0$ mixing amplitude. The matrix elements of the above operators are conventionally parameterized in terms of bag-parameters, $\bar{B}_{1-5}$, as a measure of the discrepancy with respect to the estimate obtained by using the vacuum saturation approximation (VSA),

$$
\frac{\langle \overline{B}_q^0 | O_{1-5}(\nu) | B_q^0 \rangle}{\langle \overline{B}_q^0 | O_{1-5}(\nu) | B_q^0 \rangle_{\text{VSA}}} = \bar{B}_{1-5}(\nu),
$$

where $\nu$ is the renormalisation scale of the logarithmically divergent operators, $O_i$, at which the separation between the long-distance (matrix elements) and short-distance (Wilson coefficients) physics is made. We remind the reader that

\begin{align}
\langle \overline{B}_q^0 | O_1 | B_q^0 \rangle_{\text{VSA}} &= 2 \left(1 + \frac{1}{3}\right) \langle \overline{B}_q^0 | A_\mu | 0 \rangle \langle 0 | A^\mu | B_q^0 \rangle , \\
\langle \overline{B}_q^0 | O_2 | B_q^0 \rangle_{\text{VSA}} &= -2 \left(1 - \frac{1}{6}\right) \left| \langle 0 | P | B_q^0 \rangle \right|^2 , \\
\langle \overline{B}_q^0 | O_3 | B_q^0 \rangle_{\text{VSA}} &= \left(1 - \frac{2}{3}\right) \left| \langle 0 | P | B_q^0 \rangle \right|^2 , \\
\langle \overline{B}_q^0 | O_4 | B_q^0 \rangle_{\text{VSA}} &= \frac{1}{3} \langle \overline{B}_q^0 | A_\mu | 0 \rangle \langle 0 | A^\mu | B_q^0 \rangle + 2 \left| \langle 0 | P | B_q^0 \rangle \right|^2 , \\
\langle \overline{B}_q^0 | O_5 | B_q^0 \rangle_{\text{VSA}} &= \langle \overline{B}_q^0 | A_\mu | 0 \rangle \langle 0 | A^\mu | B_q^0 \rangle + \frac{2}{3} \left| \langle 0 | P | B_q^0 \rangle \right|^2 ,
\end{align}

with $A_\mu = \bar{b} \gamma_\mu \gamma_5 q$ and $P = \bar{b} \gamma_5 q$ being the axial current and the pseudoscalar density, respectively. In HQET, in which we will be working from now on, the field $\bar{b}$ is replaced by the static one, $h^\dagger$, which satisfies $h^\dagger \gamma_0 = h^\dagger$. This equation and the fact that the amplitude is invariant under the Fierz transformation in Dirac indices, eliminate the operator $O_3$ from
further discussion, i.e., \( \langle \bar{B}_0^0|\tilde{O}_3 + \tilde{O}_2 + \frac{1}{2}\tilde{O}_1|B_0^0\rangle = 0 \), where the tilde is used to stress that the operators are now being considered in the static limit of HQET (|\( \vec{v} \)| = 0). Furthermore, in the same limit

\[
\lim_{m_b \to \infty} \frac{\langle 0|A_\mu|B_0^0(p)\rangle_{\text{QCD}}}{\sqrt{2m_B}} = \lim_{m_b \to \infty} \frac{\langle 0|P|B_0^0(p)\rangle_{\text{QCD}}}{\sqrt{2m_B}} = \langle 0|\tilde{A}_0|B_0^0(v)\rangle_{\text{HQET}} = i\hat{f}_q ,
\]

where \( \hat{f}_q \) is the decay constant of the static 1/2\(^-\) heavy-light meson, and the HQET states are normalized as

\[
\langle B_0^0|\tilde{O}_1(v)|B_0^0\rangle = \delta(v - v'),
\]

so that we finally have

\[
\langle \bar{B}_q^0|\tilde{O}_1(\nu)|B_q^0\rangle = \frac{8}{3}\hat{f}_q(\nu)^2\tilde{B}_{1q}(\nu) ,
\]

\[
\langle \bar{B}_q^0|\tilde{O}_2(\nu)|B_q^0\rangle = -\frac{5}{3}\hat{f}_q(\nu)^2\tilde{B}_{2q}(\nu) ,
\]

\[
\langle \bar{B}_q^0|\tilde{O}_4(\nu)|B_q^0\rangle = \frac{7}{3}\hat{f}_q(\nu)^2\tilde{B}_{4q}(\nu) ,
\]

\[
\langle \bar{B}_q^0|\tilde{O}_5(\nu)|B_q^0\rangle = \frac{5}{3}\hat{f}_q(\nu)^2\tilde{B}_{5q}(\nu) .
\]

One of the reasons why lattice QCD is the best currently available method for computing these matrix elements is the fact that it enables a control over the \( \nu \)-dependence by verifying the corresponding renormalisation group equations, which is essential for the cancellation against the \( \nu \)-dependence in the corresponding perturbatively computed Wilson coefficients [20]. From now on we will assume that the UV divergences are being taken care of and the scale \( \nu \) will be implicit.

## 3 Chiral logarithmic corrections

In this section we use HMChPT to describe the low energy behavior of the matrix elements [3]. Before entering the details, we notice that the operators \( \tilde{O}_4 \) and \( \tilde{O}_5 \) differ only in the color indices, i.e., by a gluon exchange, which is a local effect that cannot influence the long distance behavior described by ChPT. In other words, from the point of view of ChPT, the entire difference of the chiral behavior of the bag parameters \( \tilde{B}_{1q} \) and \( \tilde{B}_{5q} \) is encoded in the local counter-terms, whereas their chiral logarithmic behavior is the same. Similar observation has been made for the operators entering the SUSY analysis of the \( \bar{K}^0-K^0 \) mixing amplitude, as well as for the electromagnetic penguin operators in \( K \to \pi\pi \) decay [21]. Thus, in the static heavy quark limit (\( m_Q \to \infty \)), we are left with the first three
operators in eq. (5) which, in their bosonised version, can be written as [22]

\[ \tilde{O}_1 = \sum_x \beta_1 x \text{Tr} \left[ (\xi H^Q)_q \gamma_\mu (1 - \gamma_5) X \right] \text{Tr} \left[ (\xi H^Q)_q \gamma^\mu (1 - \gamma_5) X \right] + \text{c.t.}, \]

\[ \tilde{O}_2 = \sum_x \beta_2 x \text{Tr} \left[ (\xi H^Q)_q (1 - \gamma_5) X \right] \text{Tr} \left[ (\xi H^Q)_q (1 - \gamma_5) X \right] + \text{c.t.}, \]

\[ \tilde{O}_4 = \sum_x \beta_4 x \text{Tr} \left[ (\xi H^Q)_q (1 - \gamma_5) X \right] \text{Tr} \left[ (\xi H^Q)_q (1 + \gamma_5) X \right] + \text{c.t.}, \]

\[ + \tilde{\beta}_4 x \text{Tr} \left[ (\xi H^Q)_q (1 - \gamma_5) X \right] \text{Tr} \left[ (\xi H^Q)_q (1 + \gamma_5) X \right] + \text{c.t.}, \]  

where \( X \in \{1, \gamma_5, \gamma_{\mu}, \bar{\gamma}_{\mu}, \gamma_{\mu} \gamma_5, \sigma_{\mu \nu} \}. \)

3 As before the index “\( q \)” denotes the light quark flavor, and “c.t.” stands for the local counter-terms. To relate \( \beta_i \)'s to the bag parameters in eq. (5) we should recall that the field \( H_q \) is built up from the pseudoscalar \((P)\) and the vector \((P^*)\) meson fields as

\[ H^Q_q(v) = \frac{1 + \frac{\ell}{2}}{2} \left[ P^Q_{\mu}(v) \gamma^\mu - P^Q(v) \gamma_5 \right]_q, \]

\[ H^Q_q(v) = \left[ P^Q_{\mu}(v) \gamma^\mu - P^Q(v) \gamma_5 \right] \frac{1 - \frac{\ell}{2}}{2}, \]  

thus exhibiting the heavy quark spin symmetry for the lowest lying \( J^P = 1/2^- \) states. After evaluating the traces in eq. (6) we obtain

\[ \tilde{B}_1 = \frac{3}{2 f^2} \tilde{\beta}_1, \quad \tilde{B}_2 = \frac{12}{5 f^2} \tilde{\beta}_2, \quad \tilde{B}_4 = \frac{12}{7 f^2} \tilde{\beta}_4, \quad \tilde{B}_5 = \frac{12}{5 f^2} \tilde{\beta}_4, \]  

where

\[ \tilde{\beta}_1 = \beta_1 + \beta_1 \gamma_5 - 4(\beta_1 \gamma_\nu + \beta_1 \gamma_{\nu} \gamma_5) - 12 \beta_1 \sigma_{\mu \nu}, \]

\[ \tilde{\beta}_2 = -\beta_2 - \beta_2 \gamma_5 + \beta_2 \gamma_\nu + \beta_2 \gamma_{\nu} \gamma_5, \]

\[ \tilde{\beta}_4 = \beta_4 - \beta_4 \gamma_5 - \beta_4 \gamma_\nu + \beta_4 - \beta_4 \gamma_{\nu} - \beta_4 \gamma_5 + \beta_4 - \beta_4 \gamma_{\nu} + \beta_4 \gamma_{\nu} \gamma_5. \]

4 Contraction of Lorentz indices and HQET parity conservation requires the same \( X \) to appear in both traces of a summation term. Any insertions of \( \ell \) can be absorbed via \( \ell H = H \), while any nonfactorisable contribution with a single trace over Dirac matrices can be reduced to this form by using the \( 4 \times 4 \) matrix identity

\[ 4 \text{Tr}(AB) = \text{Tr}(A) \text{Tr}(B) + \text{Tr}(\gamma_5 A) \text{Tr}(\gamma_5 B) + \text{Tr}(A \gamma_\mu) \text{Tr}(\gamma^\mu B) \]

\[ + \text{Tr}(A \gamma_{\mu} \gamma_5) \text{Tr}(\gamma_5 \gamma^\mu B) + 1/2 \text{Tr}(A \sigma_{\mu \nu}) \text{Tr}(\sigma^{\mu \nu} B). \]
We will use the well known form of the HMChPT lagrangian \[8\]

\[
\mathcal{L} = \mathcal{L}_{\text{light}} + \mathcal{L}^{\frac{1}{2}} + \mathcal{L}_{\text{ct.}},
\]

\[
\mathcal{L}_{\text{light}} = \frac{f^2}{8} \text{tr} \left[ \left( \partial_{\mu} \Sigma \right)^\dagger \left( \partial^\mu \Sigma \right) + \Sigma^\dagger \chi + \chi^\dagger \Sigma \right],
\]

\[
\mathcal{L}^{\frac{1}{2}} = i \text{Tr} \left[ H_b v \cdot D_{ba} \overline{H}_a \right] + g \text{Tr} \left[ H_b \gamma_{\mu} \gamma_5 A^a_{ba} \overline{H}_a \right],
\]

\[
\mathcal{L}_{\text{ct}} = k_1 \text{Tr} \left[ \overline{H}_a H_b \left( \xi \mathcal{M} \xi + \xi^\dagger \mathcal{M} \xi^\dagger \right)_{ba} \right] + k_2 \text{Tr} \left[ \overline{H}_a H_a \left( \xi \mathcal{M} \xi + \xi^\dagger \mathcal{M} \xi^\dagger \right)_{bb} \right],
\]

where \( \overline{H}_a(v) = \gamma_0 H_a^\dagger(v) \gamma_0 \), \( g \) is the coupling of the pseudo-Goldstone boson to the pair of heavy-light mesons, and

\[
D^\mu_{ba} H_b = \partial^\mu H_a - H_b \frac{1}{2} \left[ \xi \partial^\mu \xi + \xi \partial_\mu \xi^\dagger \right]_{ba}, \quad A^{ab\mu} = \frac{i}{2} \left[ \xi \partial_\mu \xi - \xi \partial_\mu \xi^\dagger \right]_{ab}, \quad \mathcal{M} = \text{diag}(m_u, m_d, m_s),
\]

\[
\xi = \sqrt{\Sigma}, \quad \Sigma = \exp \left( 2i \frac{\phi}{f} \right), \quad \mathcal{M} = \text{diag}(m_u, m_d, m_s),
\]

\[
\phi = \begin{pmatrix}
\frac{1}{\sqrt{2}} n^0 + \frac{1}{\sqrt{6}} \eta \\
\frac{1}{\sqrt{2}} n^- + \frac{1}{\sqrt{6}} \eta \\
\frac{1}{\sqrt{2}} n^+ + \frac{1}{\sqrt{6}} \eta
\end{pmatrix}
\]

(11)

with \( f \approx 130 \text{ MeV} \), and \( \chi = 2B_0 \mathcal{M} \). In the above formulae \( H_a \) refers to either \( H^Q_a \) or \( H^Q_{\bar{d}} \), defined in eq. (7). Note also that we distinguish between the trace over Dirac (“Tr”) and flavor (“tr”) indices. In the chiral power counting the lagrangian \( \mathcal{L}_{\text{light}} \) in eq. (10) is of \( \mathcal{O}(p^2) \) while the rest of \( \mathcal{L} \) is of \( \mathcal{O}(p^4) \). To get the chiral logarithmic corrections to \( \tilde{B}_{iq} \)-parameters, we should subtract twice the chiral corrections to the decay constant \( \hat{f}_q \) from the chiral corrections to the four-quark operators \[10\]. The former is obtained from the study of the bosonised left-handed weak current

\[
(V - A)^\mu_q = \frac{i \alpha}{2} \left\{ \text{Tr} \left[ (\xi \overline{H})_q \gamma^\mu (1 - \gamma_5) \right] (1 + \varkappa_2 \text{tr} \mathcal{M}) + \varkappa_1 \text{Tr} \left[ (\xi \mathcal{M} \overline{H})_q \gamma^\mu (1 - \gamma_5) \right] \right\},
\]

(12)

where \( \alpha \) is the tree level decay constant in the chiral expansion, and \( \varkappa_{1,2} \) are the counter-term coefficients. Together with the strong coupling \( g \), these parameters are not predicted within HMChPT. Instead, they are expected to be fixed by matching the HMChPT expressions with the results of lattice QCD for a given quantity (see reviews in ref. \[23\]). The notation used above is the same as in ref. \[24\]. The chiral logarithmic corrections to the decay constant come from the diagrams shown in fig. \[1\]

\[
\hat{f}_d = \alpha \left[ 1 - \frac{1}{(4\pi f)^2} \left( \frac{3}{4} m_\pi^2 \log \frac{m_\pi^2}{\mu^2} + \frac{1}{2} m_K^2 \log \frac{m_K^2}{\mu^2} + \frac{1}{12} m_\eta^2 \log \frac{m_\eta^2}{\mu^2} \right) \\
+ \varkappa_1(\mu) m_d + \varkappa_2(\mu)(m_u + m_d + m_s) + \frac{1}{2} \delta Z_d \right],
\]

\[
\hat{f}_s = \alpha \left[ 1 - \frac{1}{(4\pi f)^2} \left( m_K^2 \log \frac{m_K^2}{\mu^2} + \frac{1}{3} m_\eta^2 \log \frac{m_\eta^2}{\mu^2} \right) \\
+ \varkappa_1(\mu) m_s + \varkappa_2(\mu)(m_u + m_d + m_s) + \frac{1}{2} \delta Z_s \right],
\]

(13)
Figure 1: The diagrams which give non-vanishing chiral logarithmic corrections to the pseudoscalar heavy-light meson decay constant. The double line indicates the heavy-light meson and the dashed one the pseudo-Goldstone boson propagator. The square stands for the weak current vertex. The full dot is proportional to the coupling $g$.

Figure 2: The diagrams relevant to the chiral corrections to the SM bag parameter $\tilde{B}_{1q}$. In the text we refer to the left one as “sunset”, and to the right one as “tadpole”. Only the tadpole diagram gives a non-vanishing contribution to the bag parameters $\tilde{B}_{2,4q}$.

where it should be stressed that we work in the exact isospin limit ($m_u = m_d$) so that the index $d$ means either $u$- or $d$-quark. Only explicit in the above expressions is the term arising from the tadpole diagram (right in fig. 1), whereas $Z_{d,s}$, the heavy meson field renormalization factors, come from the self energy diagram (left in fig. 1) and they read

$$Z_d = 1 - \frac{3g^2}{(4\pi f)^2} \left( \frac{3}{2} \frac{m_\pi^2 \log m_\pi^2}{\mu^2} + \frac{m_K^2 \log m_K^2}{\mu^2} + \frac{1}{6} \frac{m_\eta^2 \log m_\eta^2}{\mu^2} \right) + k_1(\mu) m_d + k_2(\mu) (m_u + m_d + m_s),$$

$$Z_s = 1 - \frac{3g^2}{(4\pi f)^2} \left( 2 \frac{m_K^2}{\mu^2} \log \frac{m_K^2}{\mu^2} + \frac{2}{3} \frac{m_\eta^2}{\mu^2} \log \frac{m_\eta^2}{\mu^2} \right) + k_1(\mu) m_s + k_2(\mu) (m_u + m_d + m_s).$$

In both eqs. (13) and (14) the $\mu$ dependence in the logarithm cancels against the one in the local counter-terms.

With these ingredients in hands it is now easy to deduce that the only diagrams which contribute to the SM bag parameter, $\tilde{B}_{1q}$, are the two shown in fig. 2. They arise from the two terms in $\tilde{O}_1 = 4\tilde{\beta}_1[(\xi \bar{P}_n^{\mu} q)(\xi \bar{P}^{\nu} q) + (\xi \bar{P}) q (\xi \bar{P}) q]$ and yield

“sunset” : $4\tilde{\beta}_1 \frac{3g^2}{(4\pi f)^2} \sum_i (t_{iq}^i)^2 m_i^2 \log \frac{m_i^2}{\mu^2}$,

“tadpole” : $-4\tilde{\beta}_1 \frac{1}{(4\pi f)^2} \sum_i (t_{iq}^i)^2 m_i^2 \log \frac{m_i^2}{\mu^2},$ (15)

respectively, where $t^i$ are the SU(3) generators and $m_i$ masses of the pseudo-Goldstone
bosons. The SM bag parameters now read

\[
\bar{B}_{1d} = \bar{B}_{1d}^{\text{Tree}} \left[ 1 - \frac{1 - 3g^2}{(4\pi f)^2} \left( \frac{1}{2} m_{\pi}^2 \log \frac{m_{\pi}^2}{\mu^2} + \frac{1}{6} m_{\eta}^2 \log \frac{m_{\eta}^2}{\mu^2} \right) + b_1(\mu)m_d + b'_1(\mu)(m_u + m_d + m_s) \right],
\]

\[
\bar{B}_{1s} = \bar{B}_{1s}^{\text{Tree}} \left[ 1 - \frac{1 - 3g^2}{(4\pi f)^2} \left( \frac{1}{2} m_{\pi}^2 \log \frac{m_{\pi}^2}{\mu^2} + \frac{1}{6} m_{\eta}^2 \log \frac{m_{\eta}^2}{\mu^2} \right) + b_1(\mu)m_s + b'_1(\mu)(m_u + m_d + m_s) \right],
\]

where we also wrote the counter-term contributions and, for short, we wrote \(\bar{B}_{1d}^{\text{Tree}} = 3\beta_1/2\alpha^2\). The above results agree with the ones presented in refs. [25, 26], in which the pion loop contribution was left out, and with the ones recently presented in ref. [27].

As for the bag parameters \(\bar{B}_{2q}\) and \(\bar{B}_{4q}\) we obtain

\[
\bar{B}_{2,4d} = \bar{B}_{2,4d}^{\text{Tree}} \left[ 1 + \frac{3g^2}{(4\pi f)^2} \left( \frac{1}{2} m_{\pi}^2 \log \frac{m_{\pi}^2}{\mu^2} + \frac{1}{6} m_{\eta}^2 \log \frac{m_{\eta}^2}{\mu^2} \right) + b_{2,4}(\mu)m_d + b'_{2,4}(\mu)(m_u + m_d + m_s) \right],
\]

\[
\bar{B}_{2,4s} = \bar{B}_{2,4s}^{\text{Tree}} \left[ 1 + \frac{2g^2/3}{(4\pi f)^2} m_{\pi}^2 \log \frac{m_{\pi}^2}{\mu^2} + b_{2,4}(\mu)m_s + b'_{2,4}(\mu)(m_u + m_d + m_s) \right],
\]

where \(\bar{B}_{2}^{\text{Tree}} = 12\beta_2/4\alpha^2\), \(\bar{B}_{4}^{\text{Tree}} = 12\beta_2/7\alpha^2\ Y = (\beta_{2,4}^*/\beta_{2,4})\), with \(\beta_2^* = \beta_{2\gamma} + \beta_{2\gamma_\nu\gamma_5} + 4\beta_{2\sigma_{\nu\rho}}\), and \(\beta_4^* = \beta_{4\gamma_\nu\gamma_5} + \beta_{4\gamma_\nu} - \beta_{4\gamma_\mu\gamma_5}\). We checked that our results agree with those presented in ref. [27] where also the partially quenched theory has been considered. In our paper we refer only to the full (unquenched) theory.

## 4 Impact of the 1/2^+-mesons

In this section we examine the impact of the heavy-light mesons belonging to the 1/2^+ doublet when propagating in the loops onto the chiral logarithmic corrections derived in the previous section. We first extend the lagrangian by adding to eq. (10) the following terms [5]:

\[
\mathcal{L}_{\beta} = -\text{Tr} \left[ S_b(\bar{v}i\gamma_5D_b + \Delta_5)\bar{S}_a \right] + \bar{g}\text{Tr} \left[ S_b\gamma_\mu\gamma_5 A_{ba}^\mu \bar{S}_a \right],
\]

\[
\mathcal{L}_{\text{mix}} = h\text{Tr} \left[ S_b\gamma_\mu\gamma_5 A_{ba}^\mu \bar{T}_a \right] + \text{h.c.},
\]

\[
\mathcal{L}_{\text{ct}} = k_1\text{Tr} \left[ \bar{T}_a S_b (\xi M\xi + \xi^\dagger M^\dagger)_{ba} \right] + k_2\text{Tr} \left[ S_a S_b (\xi M\xi + \xi^\dagger M^\dagger)_{ba} \right] + k_3\text{Tr} \left[ \bar{T}_a S_b (\xi M\xi + \xi^\dagger M^\dagger)_{ba} \right] + \text{h.c.},
\]

where the fields of the scalar \(P_0\) and the axial \(P_{1\mu}\) mesons are organised in a superfield

\[
S_4(v) = \left[ \begin{array}{c} P_{1\mu}\gamma_5 + P_{1\mu}(v) \end{array} \right]_q, \quad \bar{S}_4(v) = \gamma_0 S_4(v)\gamma_0.
\]
\( \tilde{g} \) is the coupling of the \( P \)-wave Goldstone boson to the pair of 1/2\(^+\) heavy-light mesons, and \( h \) is the coupling of the \( S \)-wave Goldstone boson to the heavy-light mesons, one of which belongs to 1/2\(^-\) and the other to 1/2\(^+\) doublet. Before including the 1/2\(^+\) doublet we were free to set \( \Delta = 0 \) because all the chiral loop divergences are cancelled by \( \mathcal{O}(m_q) \) counter-terms in the static heavy quark limit. Once the 1/2\(^+\) doublet is included, the mass difference between the 1/2\(^+\) and 1/2\(^-\) states (\( \Delta S \approx 400 \text{ MeV} \)) must be included in the lagrangian, but since it does not vanish in the chiral nor in the heavy quark limit it is of \( \mathcal{O}(p^0) \) in the chiral power counting (see also ref. [28] where, in addition to the static heavy quark limit, the chiral power counting is discussed also when the 1/m\(_Q\)-corrections are included).

### 4.1 Decay constants

Beside the lagrangian, the 1/2\(^+\) mesons contribution should also be added to the left vector current (12), which now reads

\[
(V - A)_q^\mu = \frac{i\alpha}{2} \text{Tr} \left[(\xi \bar{H})_q \gamma^\mu(1 - \gamma_5)\right] + \frac{i\alpha^+}{2} \text{Tr} \left[(\xi \bar{S})_q \gamma^\mu(1 - \gamma_5)\right] + \frac{i\alpha}{2} \varepsilon_1 \text{Tr} \left[(\xi \bar{M}H)_q \gamma^\mu(1 - \gamma_5)\right] + \frac{i\alpha^+}{2} \tilde{\varepsilon}_1 \text{Tr} \left[(\xi \bar{M}S)_q \gamma^\mu(1 - \gamma_5)\right] + \frac{i\alpha}{2} \varepsilon_2 \text{Tr} \left[(\xi \bar{H})_q \gamma^\mu(1 - \gamma_5)\right] \text{tr}\mathcal{M} + \frac{i\alpha^+}{2} \tilde{\varepsilon}_2 \text{Tr} \left[(\xi \bar{S})_q \gamma^\mu(1 - \gamma_5)\right] \text{tr}\mathcal{M},
\]

where \( \alpha^+ \) is the coupling of one of the 1/2\(^+\) mesons to the weak left current, and \( \tilde{\varepsilon}_{1,2} \) are the coefficients of two new counter-terms. From the recent lattice results reported in ref. [29] we extract, \( \alpha^+ / \alpha = 1.1(2) \). In other words, at least in the static heavy quark mass limit, the weak current coupling of 1/2\(^+\) mesons is not suppressed with respect to the 1/2\(^-\) ones. Since we focus on the pseudoscalar meson decay constant, it should be clear that only the scalar meson from the 1/2\(^+\) doublet can propagate in the loop. The diagrams that give non-vanishing contributions are shown in fig. 3 and the corresponding expressions now read

\[
Z_q = 1 + \frac{\lambda}{(4\pi f)^2} \left[3g^2 \lim_{x \to 0} \frac{d}{dx} [x J_1(m_i^2, x)] - h^2 \left( J_1(m_i^2, \Delta S) + J_2(m_i^2, \Delta S) \right) + \Delta S \frac{d}{d\Delta S}\left(J_1(m_i^2, \Delta S) + J_2(m_i^2, \Delta S)\right) \right],
\]

\[
\hat{f}_q = \alpha \left[1 + \frac{\lambda}{2(4\pi f)^2} \left[3g^2 \lim_{x \to 0} \frac{d}{dx} [x J_1(m_i^2, x)] - I_1(m_i^2) - h^2 \left( J_1(m_i^2, \Delta S) + J_2(m_i^2, \Delta S) \right) + \Delta S \frac{d}{d\Delta S}\left(J_1(m_i^2, \Delta S) + J_2(m_i^2, \Delta S)\right) \right] - 2h \frac{\alpha^+}{\alpha} \left(I_1(m_i^2) + I_2(m_i^2)\right) \right],
\]

where the summation over “\( i \)” is implicit, and we omit the counter-term contributions to make the expressions simpler. We stress that in all our formulae, the terms of \( \mathcal{O}(p^0) \) and higher in the chiral power counting are neglected. The integrals \( I_{1,2} \) and \( J_{1,2} \) are the same as the ones used in ref. [24] and can be found in the appendix of that paper. The terms proportional to \( h^2 \) comes from the inclusion of the left diagram shown in fig. 3 while the
Figure 3: In addition to the diagrams shown in fig. 1, these two diagrams contribute the loop corrections to the pseudoscalar meson decay constant after the 1/2+ mesons are included in HMChPT. The full dot in these graphs is proportional to the coupling $h$.

last term in the decay constant (the one proportional to $h$) comes from the right graph in fig. 3. Obviously such terms were absent before including the scalar mesons. Notice also that

$$\lim_{x \to 0} \frac{d}{dx}[xJ_1(m_\pi^2, x)] = -m_\pi^2 \log \frac{m_\pi^2}{\mu^2}. \quad (22)$$

When $\Delta_S > m_i$, which in our case is true for the pion mass, we can expand around $m_\pi^2 = 0$ and obtain

$$J_1(m_\pi^2, \Delta_S) + J_2(m_\pi^2, \Delta_S) =$$

$$I_1(m_\pi^2) + I_2(m_\pi^2, \Delta_S) \to 2\Delta_S^2 (1 - \log \frac{4\Delta_S^2}{\mu^2}) + m_\pi^2 (1 + \log \frac{4\Delta_S^2}{\mu^2}) + \ldots,$$

$$-\Delta_S \frac{d}{d\Delta_S} [J_1(m_\pi^2, \Delta_S) + J_2(m_\pi^2, \Delta_S)] \to 4\Delta_S^2 \log \frac{4\Delta_S^2}{\mu^2} - 2m_\pi^2 + \ldots, \quad (23)$$

where the dots stand for higher powers in $m_\pi^2$. In other words, the presence of the nearby 1/2+ state does not affect the pionic logarithmic behavior of the decay constant. It does, however, affect the kaon and $\eta$-meson loops because those states are heavier than $\Delta_S$ ($m_\pi < \Delta_S \lesssim m_K < m_\eta$) and the coefficients of their logarithms, although still predictions of this approach, cease to be numerically relevant because those logarithms are competitive in size with the terms proportional to $\Delta_S^2 \log(4\Delta_S^2/\mu^2)$, as indicated in eq. (23). Stated equivalently, the relevant chiral logarithmic corrections are those coming from the $SU(2)_L \otimes SU(2)_R \to SU(2)_V$ theory, and the pseudoscalar decay constant reads

$$\hat{f}_q = \alpha \left[ 1 - \frac{1 + 3g^2}{2(4\pi f)^2} \frac{3}{2} m_\pi^2 \log \frac{m_\pi^2}{\mu^2} + c_f(\mu)m_\pi^2 \right], \quad (24)$$

where $c_f(\mu)$ stands for the combination of the counter-term coefficients considered in the previous section. 5 At this point we also note that we checked that the chiral logarithms in the scalar heavy-light meson decay constant, which has recently been computed on the

5More specifically, $2B_0c_f(\mu) + \frac{3h^2}{4(4\pi f)^2} [3 + \log(4\Delta_S^2/\mu^2)] + \frac{3h\alpha^+}{2(4\pi f)^2} [1 + \log(4\Delta_S^2/\mu^2)] = \frac{1}{2} k_1(\mu) + \frac{1}{2} k_2'(\mu) + k_2(\mu) + k_1(\mu) + k_2(\mu) + k_2(\mu) + k_2(\mu) + k_2(\mu)$, where we use the Gell-Mann–Oakes–Renner formula, $m_\pi^2 = 2B_0m_d$. The exact isospin symmetry ($m_u = m_d$) is assumed throughout this work.
lattice in ref. [29], are the same as for the pseudoscalar meson, with the coupling $g$ being replaced by $\tilde{g}$, i.e.,

$$ f_q^+ = \alpha^+ \left[ 1 - \frac{1 + 3\tilde{g}^2}{2(4\pi f)^2} \frac{3m^2_{\pi}}{2m^2_{\pi} \log \frac{m^2_{\pi}}{\mu^2} + c^+_f(\mu)m^2_{\pi}} \right].$$

(25)

Since $\tilde{g}^2/g^2 \approx 1/9$ [30], the deviation from the linear behavior in $m^2_{\pi}$ is less pronounced for $\hat{f}_q^+$ than it is for $f_q$. Finally, it should be emphasized that the counter-term coefficients relevant to the $SU(2)_V$-theory, namely $c^+_f(\mu)$ in eqs. (24,25), are not the same as those in $SU(3)_V$.

4.2 Bag parameters

In this subsection we show that the situation with the bag parameters is similar to the one with decay constant, namely the pion loop chiral logarithms remain unchanged when the nearby scalar meson is included in HMChPT. To that end, besides eq. (18), we should include the contributions of $1/2^-$-mesons to the operators (6). Generically the operators $\tilde{O}_{1,2,4}$ now become

$$ \tilde{O}_1 = \sum_X \beta_{1X} \text{Tr} \left[ (\xi H^Q)^\gamma (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi H^Q)^\gamma (1 - \gamma_5)X \right] $$

$$ + \beta'_1 \left\{ \text{Tr} \left[ (\xi H^Q)^\gamma (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi S^Q)^\gamma (1 - \gamma_5)X \right] + \text{h.c.} \right\} $$

$$ + \beta''_1 \text{Tr} \left[ (\xi S^Q)^\gamma (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi S^Q)^\gamma (1 - \gamma_5)X \right],$$

(26)

where $\beta_{1X}$ are the couplings of the operator $\tilde{O}_1$ to both $1/2^-$ and $1/2^+$ mesons, while $\beta''_1$ come from the coupling to the $1/2^+$ mesons only. Similarly, the operators $\tilde{O}_{2,4}$ now read:

$$ \tilde{O}_2 = \sum_X \beta_{2X} \text{Tr} \left[ (\xi H^Q)^\gamma (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi H^Q)^\gamma (1 - \gamma_5)X \right] $$

$$ + \beta'_2 \left\{ \text{Tr} \left[ (\xi H^Q)^\gamma (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi S^Q)^\gamma (1 - \gamma_5)X \right] + \text{h.c.} \right\} $$

$$ + \beta''_2 \text{Tr} \left[ (\xi S^Q)^\gamma (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi S^Q)^\gamma (1 - \gamma_5)X \right],$$

(27)
Note that we indicate when either the vector or scalar meson can propagate in the loop by Figure 4: All diagrams which enter in the calculation of the chiral corrections to the operators \( \tilde{O}_{1,2,4} \). Note that we indicate when either the vector or scalar meson can propagate in the loop by eq. (9). In addition, in eqs. (26,27,28), \( \gamma^5 \) and \( \gamma^5P \) have forms analogous to the ones written in eq. (7). The corresponding tree and 1-loop chiral diagrams are shown in fig. 4. Since the couplings of the four-quark operators to the scalar meson are proportional to \( \beta_i^\prime \) and of the pseudoscalar decay constant

\[
\tilde{O}_4 = \sum_X \beta_{4X} \text{Tr} \left[ (\xi \overline{T}^Q)_q (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi^\dagger \overline{H}^Q)_q (1 + \gamma_5)X \right] \\
+ \bar{\beta}_{4X} \text{Tr} \left[ (\xi H^Q)_q (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi^\dagger \overline{T}^Q)_q (1 + \gamma_5)X \right] \\
+ \beta_{4X}' \left\{ \text{Tr} \left[ (\xi \overline{T}^Q)_q (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi^\dagger S^Q)_q (1 + \gamma_5)X \right] + \text{h.c.} \right\} \\
+ \bar{\beta}_{4X}' \left\{ \text{Tr} \left[ (\xi H^Q)_q (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi^\dagger S^Q)_q (1 + \gamma_5)X \right] + \text{h.c.} \right\} \\
+ \beta_{4X}'' \text{Tr} \left[ (\xi S^Q)_q (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi^\dagger \overline{T}^Q)_q (1 + \gamma_5)X \right] \\
+ \bar{\beta}_{4X}'' \text{Tr} \left[ (\xi S^Q)_q (1 - \gamma_5)X \right] \text{Tr} \left[ (\xi^\dagger \overline{S}^Q)_q (1 + \gamma_5)X \right].
\]  

(28)

After evaluating the traces in eqs. (26), and keeping in mind that the external states are the pseudoscalar mesons, we have

\[
\tilde{O}_1 = 4\tilde{\beta}_1 \left[ (\xi \overline{T}^{Q* \mu})_q (\xi P^Q_\mu)_q + (\xi \overline{T}^{Q* \mu})_q (\xi P^Q_\mu)_q \right] \\
+ 4\tilde{\beta}_1' \left[ (\xi \overline{T}^{Q^*})_q (\xi P^Q_0)_q + (\xi \overline{T}^{Q^*})_q (\xi P^Q_0)_q \right] + 4\tilde{\beta}_1'' \left[ (\xi P^Q_0)_q (\xi P^Q_0)_q \right],
\]  

(29)

where \( \tilde{\beta}_1^{(\prime,\prime)} \) have forms analogous to the ones written in eq. (9). In addition, in eqs. (26,27,28), the fields \( S^Q_q \) and \( S^Q_q \) are defined in a way similar to eq. (7). The corresponding tree and 1-loop chiral diagrams are shown in fig. 4. Since the couplings of the four-quark operators to the scalar meson are proportional to \( \beta_i^\prime \) and of the pseudoscalar decay constant
to $\alpha^+$, the cancellation between the chiral loop corrections in the operators $\tilde{O}_i$ and in the decay constant is not automatic. For that reason, instead of writing the chiral logarithmic corrections to the bag-parameter, we will write them for the full operator, namely

$$
\tilde{B}_{1q}\hat{f}_q^2 = \frac{3}{2}\beta_1 \left\{ 1 - \frac{i\gamma_5\gamma^\dagger}{2(4\pi f)^2} \left[ -6g^2 \lim_{x \to 0} \frac{d}{dx} [xJ_1(m_i^2, x)] + 2I_1(m_i^2) \right] 
+ 2h^2 \left( J_1(m_i^2, \Delta_S) + J_2(m_i^2, \Delta_S) + \Delta_S \frac{d}{d\Delta_S} \left( J_1(m_i^2, \Delta_S) + J_2(m_i^2, \Delta_S) \right) \right) 
+ 4h^2 \frac{\beta_1}{\beta_1} \left( I_1(m_i^2) + I_2(m_i^2, \Delta_S) \right) - \frac{i\gamma_5\gamma^\dagger}{2(4\pi f)^2} \left[ 6g^2 \lim_{x \to 0} \frac{d}{dx} [xJ_1(m_i^2, x)] + 2I_1(m_i^2) \right] 
+ 4h^2 \frac{\beta_1}{\beta_1} \left( I_1(m_i^2) + I_2(m_i^2, \Delta_S) \right) + h^2 \frac{\beta_1}{\beta_1} \sum_{k=1,2; s=\pm 1} J_k(m_i^2, s\Delta_S) \right\},
$$

where the ellipses stand for the terms of higher order in $m$. The similar formulae for $\tilde{B}_{2q}\hat{f}_q^2$ are lengthy and we will not write them explicitly. For the point that we want to make in this section it is enough to consider eq. (30) because in the expressions for $\tilde{B}_{2q}\hat{f}_q^2$ the loop functions $I_{1,2}$ and $J_{1,2}$ occur in the same form as in eq. (30).

In the evaluation of the sunset diagrams we used the standard simplification-trick

$$
\frac{1}{(vp - \Delta)(vp - \Delta')} = \frac{1}{\Delta - \Delta'} \left( \frac{1}{vp - \Delta} - \frac{1}{vp - \Delta'} \right).
$$

We now turn to the case $m_\pi \ll \Delta_S$ and study the behavior of eq. (31) around $m_\pi^2 \to 0$. In addition to the limits discussed in eq. (23), when dealing with the integrals in the last line of eq. (30) we shall proceed similarly to what has been done in ref. [31], namely we expand the integrand in $E_\pi/\Delta_S$ and write

$$
\sum_{k=1,2; s=\pm 1} J_k(m_\pi^2, s\Delta_S) = -2(4\pi)^2 v_\mu v_\nu \times i\mu^\epsilon \int \frac{d^4-p}{(2\pi)^4} \frac{p^\nu \rho^\nu}{(p^2 - m_\pi^2)[\Delta_S^2 - (vp)^2]} = -\frac{2(4\pi)^2}{\Delta_S^2} v_\mu v_\nu \left[ i\mu^\epsilon \int \frac{d^4-p}{(2\pi)^4} \frac{p^\nu \rho^\nu}{p^2 - m_\pi^2} + \mathcal{O}(1/\Delta_S^2) \right]
= -\frac{m_\pi^2}{2\Delta_S^2} (4\pi^2) I_1(m_\pi^2) + \cdots \to -\frac{m_\pi^4}{2\Delta_S^2} \log \frac{m_\pi^2}{\mu^2} + \ldots,
$$

where the ellipses stand for the terms of higher order in $m_\pi^2/\Delta_S^2$. Note, however, that even the leading term is already of higher order in the chiral expansion and thus the terms proportional to $h$ in eq. (30) do not affect the leading chiral logarithmic corrections.

On the basis of the above discussion and eqs. (22,23) we see that after expanding eq. (30) around $m_\pi^2 = 0$, the leading chiral logarithms arising from the pion loops remain unchanged even when the coupling to the scalar meson is included in the loops. On the other hand, as discussed in the previous subsection, the logarithms arising from the kaon and the $\eta$-meson are competitive in size with those arising from the coupling to the heavy-light scalar meson,
which is the consequence of the smallness of $\Delta_S$. Therefore, like for the decay constants, the relevant chiral expansion is the one derived in the $SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_V$ theory, i.e.,

$$\tilde{B}_{1q} \hat{f}_q^2 = \tilde{B}_{1q}^{\text{Tree}} \alpha^2 \left[ 1 - \frac{3g^2 + 2}{(4\pi f)^2} m^2_\pi \log \frac{m^2_\pi}{\mu^2} + c_{O_1}(\mu) m^2_\pi \right],$$

$$\tilde{B}_{2,4q} \hat{f}_q^2 = \tilde{B}_{2,4q}^{\text{Tree}} \alpha^2 \left[ 1 - \frac{3g^2(3-Y) + 3 \pm 1}{2(4\pi f)^2} m^2_\pi \log \frac{m^2_\pi}{\mu^2} + c_{O_2}(\mu) m^2_\pi \right],$$

or by using eq. (24), for the bag parameters we obtain

$$\tilde{B}_{1q} = \tilde{B}_{1q}^{\text{Tree}} \left[ 1 - \frac{1 - 3g^2}{2(4\pi f)^2} m^2_\pi \log \frac{m^2_\pi}{\mu^2} + c_{B_1}(\mu) m^2_\pi \right],$$

$$\tilde{B}_{2,4q} = \tilde{B}_{2,4q}^{\text{Tree}} \left[ 1 + \frac{3g^2Y \pm 1}{2(4\pi f)^2} m^2_\pi \log \frac{m^2_\pi}{\mu^2} + c_{B_2}(\mu) m^2_\pi \right],$$

which coincide with the pion loop contributions shown in eqs. (16) and (17), as they should.

### 5 Relevance to the analyses of the lattice QCD data

It should be stressed that the consequence of the discussion in the previous section is mainly important to the phenomenological approaches in which the sizable kaon and $\eta$-meson logarithmic corrections are taken as predictions, whereas the counter-term coefficients are fixed by matching to large $N_c$ expansion or some other model. We showed that the contributions of the nearby heavy-light scalar states are competitive in size and thus they cannot be ignored nor separated from the discussion of the kaon and/or $\eta$-meson loops.

In the extrapolation of the lattice data, instead, this is not a problem because the kaon and the $\eta$-meson loops essentially do not alter the quark mass dependence, whereas the important nonlinearity comes from the pion chiral loops. As an illustration, in fig. 5 we plot the typical chiral logarithm, $-m^2_\pi \log(m^2_\pi/\mu^2)$, as a function of $r = m_d/m_s$ which appear in the Gell-Mann–Oakes–Renner formulae,

$$m^2_\pi = 2B_0 m_s r, \quad m^2_K = 2B_0 m_s \frac{r + 1}{2}, \quad m^2_\eta = 2B_0 m_s \frac{r + 2}{3},$$

with $2B_0 m_s = 2m^2_K - m^2_\eta = 0.468 \, \text{GeV}^2$. Thus the fact that the nearby scalar heavy-light mesons do not spoil the pion logarithmic corrections to the decay constants and the bag-parameters is most welcome from the lattice practitioners’ point of view, because the formulae derived in HMCChPT can still (and should) be used to guide the chiral extrapolations of the lattice results, albeit for the pion masses lighter than $\Delta_S$. 

13
6 Conclusion

In this paper we revisited the computation of the $B^0_q - \bar{B}^0_q$ mixing amplitudes in the framework of HMChPT. Besides the SM bag parameter, we also provided the expressions for the chiral logarithmic correction to the SUSY bag parameters. More importantly, we study the impact of the near scalar mesons to the predictions derived in HMChPT in which these contributions were previously ignored. We showed that while the corrections due to the nearness of the scalar mesons are competitive in size with the kaon and η meson loop corrections, they do not alter the pion chiral logarithms. In other words the valid (pertinent) ChPT expressions for the quantities discussed in this paper are those involving pions only. This is of major importance for the chiral extrapolations of the results obtained from the QCD simulations on the lattice, because precisely the pion chiral logarithms provide the important guidance in those extrapolations as long as $m_s \ll \Delta S$. The corresponding useful formulas are given in eqs. (24,34,35). As a side-result we verified that the chiral logarithmic corrections to the scalar meson decay constant are the same as to the pseudoscalar one, modulo replacement $g \to \tilde{g}$ (c.f. eq. (25)).

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