Ramsey-Type Results for Path Covers and Path Partitions

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Abstract

A family $\mathcal{P}$ of subgraphs of $G$ is called a path cover (resp. a path partition) of $G$ if $\bigcup_{P \in \mathcal{P}} V(P) = V(G)$ (resp. $\bigcup_{P \in \mathcal{P}} V(P) = V(G)$) and every element of $\mathcal{P}$ is a path. The minimum cardinality of a path cover (resp. a path partition) of $G$ is denoted by $pc(G)$ (resp. $pp(G)$). In this paper, we characterize the forbidden subgraph conditions assuring us that $pc(G)$ (or $pp(G)$) is bounded by a constant. Our main results introduce a new Ramsey-type problem.

Mathematics Subject Classifications: 05C38, 05C55

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. For terms and symbols not defined in this paper, we refer the reader to [2].

Let $G$ be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. For a vertex $x \in V(G)$, let $N_G(x)$ denote the neighborhood of $x$ in $G$; thus $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. For a subset $X$ of $V(G)$, let $N_G(X) = \cup_{x \in X} N_G(x)$.
There exists a constant \( A_1 \) (resp. \( G - X \)) denote the subgraph of \( G \) induced by \( X \) (resp. \( V(G) \) \( X \)). Let \( \alpha(G) \) denote the independence number of \( G \), i.e., the maximum cardinality of an independent set of \( G \). Let \( K_n, P_n, C_n \) and \( K_{1,n} \) denote the complete graph of order \( n \), the path of order \( n \), the cycle of order \( n \) and the star of order \( n + 1 \), respectively. For two positive integers \( n_1 \) and \( n_2 \), the Ramsey number \( R(n_1, n_2) \) is the minimum positive integer \( R \) such that any graph of order at least \( R \) contains a clique of cardinality \( n_1 \) or an independent set of cardinality \( n_2 \).

For two graphs \( G \) and \( H \), \( G \) is said to be \( H \)-free if \( G \) contains no induced copy of \( H \).

For a family \( \mathcal{H} \) of graphs, a graph \( G \) is said to be \( \mathcal{H} \)-free if \( G \) is \( H \)-free for every \( H \in \mathcal{H} \). In this context, the members of \( \mathcal{H} \) are called forbidden subgraphs. For two families \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) of graphs, we write \( \mathcal{H}_1 \leq \mathcal{H}_2 \) if for every \( H_2 \in \mathcal{H}_2 \), there exists \( H_1 \in \mathcal{H}_1 \) such that \( H_1 \) is an induced subgraph of \( H_2 \). The relation “\( \leq \)" between two families of forbidden subgraphs was introduced in [7]. Note that if \( \mathcal{H}_1 \leq \mathcal{H}_2 \), then every \( \mathcal{H}_1 \)-free graph is also \( \mathcal{H}_2 \)-free.

Let \( A \) be a family of graphs. A family \( \mathcal{P} \) of subgraphs of \( G \) is called an \( A \)-cover of \( G \) if \( \bigcup_{P \in \mathcal{P}} V(P) = V(G) \) and each element of \( \mathcal{P} \) is isomorphic to a graph belonging to \( A \). Note that some elements of an \( A \)-cover of \( G \) might have common vertices. An \( A \)-cover \( \mathcal{P} \) of \( G \) is called an \( A \)-partition of \( G \) if the elements of \( \mathcal{P} \) are pairwise vertex-disjoint. A \( \{P_i : i \geq 1\} \)-cover (resp. a \( \{P_i : i \geq 1\} \)-partition) of \( G \) is called a path cover (resp. a path partition) of \( G \). Since \( \{G[\{x\}] : x \in V(G)\} \) is a path partition of \( G \) (and so a path cover of \( G \)), the minimum cardinality of a path cover (or a path partition) of any graph is well-defined. The value \( \min\{|\mathcal{P}| : \mathcal{P} \text{ is a path cover of } G\} \) (resp. \( \min\{|\mathcal{P}| : \mathcal{P} \text{ is a path partition of } G\} \), denoted by \( pc(G) \) (resp. \( pp(G) \)), is called the path cover number (resp. the path partition number) of \( G \). It is trivial that \( pc(G) \leq pp(G) \). Since a graph \( G \) has a Hamiltonian path if and only if \( pp(G) = 1 \), the decision problem for the path partition number is a natural generalization of the Hamiltonian path problem. In fact, it has been widely studied in, for example, [12–14,16,17]. Throughout this paper, we implicitly use the following fact.

**Fact 1.** Let \( G \) be a graph, and let \( \{X_1, X_2, \ldots, X_m\} \) be a partition of \( V(G) \). Then \( pc(G) \leq \sum_{1 \leq i \leq m} pc(G[X_i]) \) and \( pp(G) \leq \sum_{1 \leq i \leq m} pp(G[X_i]). \)

In this paper, we focus on the following conditions concerning a family \( \mathcal{H} \) of forbidden subgraphs:

(A1) There exists a constant \( c_1 = c_1(\mathcal{H}) \) such that \( pc(G) \leq c_1 \) for every connected \( \mathcal{H} \)-free graph \( G \).

(A2) There exists a constant \( c_2 = c_2(\mathcal{H}) \) such that \( pp(G) \leq c_2 \) for every connected \( \mathcal{H} \)-free graph \( G \).

Our main aim is to characterize the finite families \( \mathcal{H} \) of connected graphs satisfying (A1) or (A2).

Let \( m \) and \( n \) be two positive integers. We define five graphs which will be used as forbidden subgraphs in our main result (see Figure 1).
• Let $K^*_m$ denote the graph with $V(K^*_m) = \{x_i, y_i : 1 \leq i \leq m\}$ and $E(K^*_m) = \{x_i x_j : 1 \leq i < j \leq m\} \cup \{x_i y_i : 1 \leq i \leq m\}$.

• Let $A := \{x_1, x_2\} \cup \{y_i : 1 \leq i \leq m\} \cup \{z_i : 1 \leq i \leq n\}$. We define four graphs as follows:
  - Let $F^{(1)}_{m,n}$ denote the graph on $A$ such that $E(F^{(1)}_{m,n}) = \{x_1 x_2, x_1 y_1, x_1 z_1\} \cup \{y_i y_{i+1} : 1 \leq i \leq m - 1\} \cup \{z_i z_{i+1} : 1 \leq i \leq n - 1\}$.
  - Let $F^{(2)}_{m,n}$ be the graph obtained from $F^{(1)}_{m,n}$ by adding the edge $y_1 z_1$.
  - Let $F^{(3)}_{m,n}$ denote the graph on $A$ such that $E(F^{(3)}_{m,n}) = \{x_1 y_1, x_1 z_1, x_2 y_1, x_2 z_1\} \cup \{y_i y_{i+1} : 1 \leq i \leq m - 1\} \cup \{z_i z_{i+1} : 1 \leq i \leq n - 1\}$.
  - Let $F^{(4)}_{m,n}$ be the graph obtained from $F^{(3)}_{m,n}$ by adding the edge $y_1 z_1$.

Our first main result is the following, which is proved in Section 2.

**Theorem 2.** Let $\mathcal{H}$ be a finite family of connected graphs. Then the following hold:

(i) The family $\mathcal{H}$ satisfies (A1) if and only if $\mathcal{H} \leq \{K_{1,n}, K^*_n, F^{(1)}_{n,n}, F^{(2)}_{n,n}\}$ for an integer $n \geq 2$.

(ii) The family $\mathcal{H}$ satisfies (A2) if and only if $\mathcal{H} \leq \{K_{1,n}, K^*_n, F^{(1)}_{n,n}, F^{(2)}_{n,n}, F^{(3)}_{n,n}, F^{(4)}_{n,n}\}$ for an integer $n \geq 2$.

Our motivation derives from two different lines of research. The first one is forbidden subgraph conditions for the existence of a Hamiltonian path. Now we focus on the condition that
(H1) every connected $\mathcal{H}$-free graph (of sufficiently large order) has a Hamiltonian path for a family $\mathcal{H}$ of connected graphs. Duffus et al. [3] proved $\mathcal{H} = \{K_{1,3}, K^*_3\}$ satisfies (H1), and Faudree and Gould [5] showed that if a family $\mathcal{H}$ satisfying (H1) consists of two connected graphs, then $\mathcal{H} \subseteq \{K_{1,3}, K^*_3\}$. Thereafter a series by Gould and Harris [9–11] characterized the families $\mathcal{H}$ of connected graphs with $|\mathcal{H}| = 3$ satisfying (H1). Since a graph has a Hamiltonian path if and only if its path cover number (or its path partition number) is exactly one, it is natural to study the forbidden subgraph conditions assuring us that the path cover/partition number is bounded by a constant as a next step. Our main result gives a complete solution for the problem in a sense.

Our second motivation is an analysis of gap between minimum $A$-covers and minimum $A$-partitions. A path cover/partition, which is the main topic in this paper, is just one of examples of $A$-cover/partition problems, and there also exist many other cover/partition problems. One of representative other examples is the case where $A$ is the family of all stars, where we regard $K_1$ as one of stars. If we define the star cover number and the star partition number in the same way as $pc(G)$ and $pp(G)$, we can easily verify that the values are always equivalent. (Indeed, the star cover number also equals to the domination number, which is one of classical invariants in graph theory. The forbidden subgraph conditions assuring us that the domination number is bounded by a constant were characterized in [8].) On the other hand, as it is evident from Theorem 2, there is a gap between the path cover number and the path partition number. By Theorem 2, we discover that $F_{n,n}^{(3)}$ and $F_{n,n}^{(4)}$ play an important role for essential structures giving such a gap.

We also obtain an analogy of Theorem 2 considering a cycle cover/partition problem. A $\{K_1, K_2, C_i : i \geq 3\}$-cover (resp. a $\{K_1, K_2, C_i : i \geq 3\}$-partition) of $G$ is called a cycle cover (resp. a cycle partition) of $G$. The value $\min\{|P| : P \text{ is a cycle cover of } G\}$ (resp. $\min\{|P| : P \text{ is a cycle partition of } G\}$), denoted by $cc(G)$ (resp. $cp(G)$), is called the cycle cover number (resp. the cycle partition number) of $G$. One might feel that it is strange to admit the existence of $K_1$ and $K_2$ in cycle covers/partitions. However, if we exclude $K_1$ and $K_2$ from the definition of cycle covers/partitions, then the minimum cardinality of a $\{C_i : i \geq 3\}$-cover cannot be defined for some graphs, for example, trees and graphs having a vertex of degree one. Thus the cycle covers/partitions defined above are sometimes considered instead of $\{C_i : i \geq 3\}$-covers/partitions (see, for example, [4,6]), and we focus on it in this paper. In Section 3, as the second result, we characterize the families $\mathcal{H}$ of forbidden subgraphs satisfying one of the following:

(A’1) There exists a constant $c_1 = c_1(\mathcal{H})$ such that $cc(G) \leq c_1$ for every connected $\mathcal{H}$-free graph $G$.

(A’2) There exists a constant $c_2 = c_2(\mathcal{H})$ such that $cp(G) \leq c_2$ for every connected $\mathcal{H}$-free graph $G$.

Theorem 3. Let $\mathcal{H}$ be a family of connected graphs. Then the following are equivalent.

(i) The family $\mathcal{H}$ satisfies (A’1).
(ii) The family $\mathcal{H}$ satisfies (A')2.

(iii) For an integer $n \geq 2$, $\mathcal{H} \subseteq \{K_{1,n}, K_n^*, P_n\}$.

We conclude this section by defining a new Ramsey-type concept concerning the path cover/partition number. Let $\mathcal{H}$ be a family of graphs. The path cover Ramsey number $R^{pc}(\mathcal{H})$ (resp. the path partition Ramsey number $R^{pp}(\mathcal{H})$) is the minimum positive integer $R$ such that any connected graph $G$ with $pc(G) \geq R$ (resp. $pp(G) \geq R$) contains an induced copy of an element of $\mathcal{H}$, where $R^{pc}(\mathcal{H}) = \infty$ (resp. $R^{pp}(\mathcal{H}) = \infty$) if such an integer does not exist. Then it follows from Theorem 2 that the following hold:

(P1) For a finite family $\mathcal{H}$ of connected graphs, $R^{pc}(\mathcal{H})$ is a finite number if and only if $\mathcal{H} \subseteq \{K_{1,n}, K_n^*, F^{(1)}_{n,n}, F^{(2)}_{n,n}\}$ for an integer $n \geq 2$.

(P2) For a finite family $\mathcal{H}$ of connected graphs, $R^{pp}(\mathcal{H})$ is a finite number if and only if $\mathcal{H} \subseteq \{K_{1,n}, K_n^*, F^{(1)}_{n,n}, F^{(2)}_{n,n}, F^{(3)}_{n,n}, F^{(4)}_{n,n}\}$ for an integer $n \geq 2$.

Note that $R^{pc}(\mathcal{H}) = 2$ if and only if $R^{pp}(\mathcal{H}) = 2$. As we mentioned above, it is known that $R^{pc}(\{K_{1,3}, K_3^*\}) = 2$ and the study of triples $\{H_1, H_2, H_3\}$ of connected graphs with $R^{pc}(\{H_1, H_2, H_3\}) = 2$ is completed. Since the $K_{1,3}$-freeness tends to give an important structure to many Hamiltonian properties, one might be interested in a relationship between such new Ramsey-type values and $K_{1,3}$-freeness. Here we focus on the values $R^{pc}(\mathcal{H})$ and $R^{pp}(\mathcal{H})$ for the case where $\mathcal{H}$ contains $K_{1,3}$. Note that for positive integers $m$ and $n$ with $m + n \geq 3$, all of $F^{(1)}_{m,n}$, $F^{(3)}_{m,n}$ and $F^{(4)}_{m,n}$ contain $K_{1,3}$ as an induced copy. Thus if $K_{1,3} \in \mathcal{H}$, then

$$R^{pc}(\mathcal{H}) = R^{pc}(\mathcal{H} \setminus \{F^{(1)}_{m,n}, F^{(3)}_{m,n}, F^{(4)}_{m,n} : m \geq 1, n \geq 1, m + n \geq 3\})$$

and

$$R^{pp}(\mathcal{H}) = R^{pp}(\mathcal{H} \setminus \{F^{(1)}_{m,n}, F^{(3)}_{m,n}, F^{(4)}_{m,n} : m \geq 1, n \geq 1, m + n \geq 3\}).$$

Considering (P1) and (P2), we leave the following open problem which will be a next interesting target on this concept for readers.

**Problem 4.** For positive integers $p$, $q$ and $r$ with $p \geq 3$ and $q + r \geq 4$ and for a family $\mathcal{H}$ of graphs with $\mathcal{H} \subseteq \{K_{1,3}, K_p^*, F^{(2)}_{q,r}\}$, determine the value $R^{pc}(\mathcal{H})$ and $R^{pp}(\mathcal{H})$.

## 2 Proof of Theorem 2

### 2.1 The “if” parts of Theorem 2

In this subsection, we prove the following theorem, which implies that the “if” parts of Theorem 2 hold.

**Theorem 5.** Let $n \geq 2$ be an integer. Then the following hold:

- (i) There exists a constant $c_1 = c_1(n)$ depending on $n$ only such that $pc(G) \leq c_1$ for every connected $\{K_{1,n}, K_n^*, F^{(1)}_{n,n}, F^{(2)}_{n,n}\}$-free graph $G$. 

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(ii) There exists a constant $c_2 = c_2(n)$ depending on $n$ only such that $\text{pp}(G) \leq c_2$ for every connected $\{K_{1,n}, K_{n}^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}, F_{n,n}^{(3)}\}$-free graph $G$.

The following lemma was proved in [15].

**Lemma 6** (Pósa [15]). For a graph $G$, $\text{cp}(G) \leq \alpha(G)$.

Let $\mathcal{C}$ be a cycle partition of a graph $G$. For each cycle $C$ belonging to $\mathcal{C}$, fix an edge $e_C \in E(C)$. Then $\mathcal{F} = \{C - e_C : C \in \mathcal{C}, |V(C)| \geq 3\} \cup \{C : C \in \mathcal{C}, |V(C)| \leq 2\}$ is a path partition of $G$. This implies that $\text{pp}(G) \leq \text{cp}(G)$ for all graphs $G$. Hence the following lemma is obtained by Lemma 6.

**Lemma 7.** For a graph $G$, $\text{pc}(G) \leq \text{pp}(G) \leq \alpha(G)$.

**Lemma 8.** Let $n \geq 2$ and $\alpha \geq 1$ be integers. Let $G$ be a $\{K_{1,n}, K_n^*\}$-free graph, and let $X$ be a subset of $V(G)$ with $\alpha(G[X]) \leq \alpha$. Then $\alpha(G[V(G) \cup X]) \leq (n - 1)R(n, \alpha + 1) - 1$.

**Proof.** By way of contradiction, we suppose that there exists a subset $Y$ of $N_G(X)$ such that $Y$ is an independent set of $G$ and $|Y| = (n - 1)R(n, \alpha + 1)$. Take a subset $X_0$ of $X$ with $Y \subseteq N_G(X_0)$ so that $|X_0|$ is as small as possible. If $|X_0| \leq R(n, \alpha + 1) - 1$, then $|Y| \geq \frac{(n - 1)R(n, \alpha + 1)}{R(n, \alpha + 1) - 1} > n - 1$, and hence there exists a vertex $x_0 \in X_0$ with $|N_G(x_0) \cap Y| \geq n$, which contradicts the $K_{1,n}$-freeness of $G$. Thus $|X_0| \geq R(n, \alpha + 1)$. Since $\alpha(G[X_0]) \leq \alpha(G[X]) \leq \alpha$, this implies that there exists a subset $X_1$ of $X_0$ such that $X_1$ is a clique of $G$ and $|X_1| = n$. By the minimality of $X_0$, $(N_G(x) \cap Y) \setminus N_G(X_0 \setminus \{x\}) \neq \emptyset$ for every $x \in X_0$. For each $x \in X_0$, let $y_x \in (N_G(x) \cap Y) \setminus N_G(X_0 \setminus \{x\})$. Then $X_1 \cup \{y_x : x \in X_1\}$ induces a copy of $K_n^*$ in $G$, which contradicts the $K_n^*$-freeness of $G$.

In the remainder of this subsection, we fix an integer $n \geq 2$ and a connected $\{K_{1,n}, K_{n}^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}\}$-free graph $G$. Set $n_0 = \max\{\lceil \frac{n^2-n-2}{2} \rceil, n\}$. Take a longest induced path $P$ of $G$, and write $P = u_1 u_2 \cdots u_m$. Let $X_0 = \{u_i : 1 \leq i \leq n_0 \text{ or } m - n_0 + 1 \leq i \leq m\}$ and $Y = N_G(V(P) \setminus X_0) \setminus (X_0 \cup N_G(X_0))$. Note that if $|V(P)| < 2n_0$, then $X_0 = V(P)$ and $Y = \emptyset$. We further remark that $N_G(y) \cap V(P) \subseteq \{u_i : n_0 + 1 \leq i \leq m - n_0\}$ for every $y \in Y$ (and in the remainder of this subsection, we frequently use the fact without mentioning). For each $i$ with $n_0 + 1 \leq i \leq m - n_0$, let $Y_i = \{y \in Y : \min\{j : n_0 + 1 \leq j \leq m - n_0, y u_j \in E(G)\} = i\}$. In other words, $Y_i$ is the set of vertices $y \in Y$ whose neighbor in $P$ with the smallest index is $u_i$. Now we recursively define the sets $X_i$ ($i \geq 1$) as follows: Let $X_1 = N_G(X_0 \setminus V(P))$, and for $i$ with $i \geq 2$, let $X_i = N_G(X_{i-1}) \setminus (V(P)) \cup \cup (\bigcup_{1 \leq j \leq i-1} X_j)$ (see Figure 2). Then $X_1 \cap Y = \emptyset$ and $X_1 \cup Y = N_G(V(P))$.

**Lemma 9.** We have $X_{2n_0} = \emptyset$.

**Proof.** Suppose that $X_{2n_0} \neq \emptyset$. Let $x_{2n_0} \in X_{2n_0}$. Then we can recursively take a vertex $x_{2n_0-i} \in N_G(x_{2n_0-i+1}) \cap X_{2n_0-i}$ for $i$ with $1 \leq i \leq 2n_0$. Note that $x_0 = u_k$ for some $k$ with $1 \leq k \leq n_0$ or $m - n_0 + 1 \leq k \leq m$. By symmetry, we may assume that $1 \leq k \leq n_0$. Under this condition, we choose $k$ so that $k$ is as large as possible. Since $x_0 x_1 \cdots x_{2n_0}$ is
an induced path of $G$ having $2n_0 + 1$ vertices, it follows from the maximality of $P$ that $|V(P)| \geq 2n_0 + 1$. In particular, $V(P) \setminus X_0 \neq \emptyset$.

If $N_G(x_1) \cap (V(P) \setminus X_0) = \emptyset$, then $x_{2n_0}x_{2n_0-1} \cdots x_1u_ku_{k+1} \cdots u_{m-n_0}$ is an induced path of $G$ having $2n_0 + m - n_0 - k + 1 \geq m + 1$ vertices, which contradicts the maximality of $P$. Thus $N_G(x_1) \cap (V(P) \setminus X_0) \neq \emptyset$.

Now we consider an operation recursively defining integers $j_1, j_2, \ldots$ with $1 \leq j_p \leq m$ ($p \geq 1$) and $j_1 < j_2 < \cdots$ as follows (see Figure 3): Let $j_1 = \min\{j : 1 \leq j \leq m, x_1u_j \in E(G)\}$. For $p \geq 2$, we assume that the integer $j_{p-1}$ has defined. If $\{j : j_{p-1} + 2 \leq j \leq m, x_1u_j \in E(G)\} \neq \emptyset$, we let $j_p = \min\{j : j_{p-1} + 2 \leq j \leq m, x_1u_j \in E(G)\}$; otherwise, we finish the operation. Let $S = \{u_{j_p} : p \geq 1\}$, and set $s = |S|$. Let $j^* = \max\{j : 1 \leq j \leq m, x_1u_j \in E(G)\}$. Note that $j^* \in \{j_s, j_s + 1\}$. Since $j_p \geq j_{p-1} + 2$, $S$ is an independent set of $G$. Since $G$ is $K_{1,n}$-free and $\{x_1, x_2\} \cup S$ induces a copy of $K_{1,s+1}$ in $G$, we have $s + 1 \leq n - 1$.

For the moment, suppose that $s = 1$. Since $N_G(x_1) \cap \{u_j : 1 \leq j \leq n_0\} \neq \emptyset$ and $N_G(x_1) \cap (V(P) \setminus X_0) \neq \emptyset$, this forces $N_G(x_1) = \{u_{n_0}, u_{n_0+1}\}$. Then

$$\{x_1, x_2, u_{n_0}, u_{n_0-1}, \ldots, u_{n_0-n_0+1}, u_{n_0+1}, u_{n_0+2}, \ldots, u_{n_0+n}\}$$

induces a copy of $F_{n,n}^{(2)}$ in $G$, which is a contradiction. Thus $s \geq 2$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Path $P$ and sets $X_i$ and $Y$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{An example for $s = 5$.}
\end{figure}
Let $Q_1 = u_1 u_2 \cdots u_j$ and $Q_{s+1} = u_j u_{j+1} \cdots u_m$ be subpaths of $P$. For $p$ with $2 \leq p \leq s$, let $Q_p = u_{j_p-1+2} u_{j_p-1+3} \cdots u_{j_p}$ be a subpath of $P$. Then $V(P) \setminus (\bigcup_{1 \leq p \leq s+1} V(Q_p)) = \{u_{j_p+1} : 1 \leq p \leq s-1\}$, and hence

$$2n_0 + 1 \leq |V(P)|$$

$$= \left| V(P) \setminus \left( \bigcup_{1 \leq p \leq s+1} V(Q_p) \right) \right| + \bigcup_{1 \leq p \leq s+1} V(Q_p)$$

$$\leq (s-1) + \sum_{1 \leq p \leq s+1} |V(Q_p)|$$

$$= \sum_{1 \leq p \leq s+1} (|V(Q_p)| + 1) - 2.$$ 

This implies that $\sum_{1 \leq p \leq s+1} (|V(Q_p)| + 1) \geq 2n_0 + 3 \geq 2^{\frac{n^2-n-2}{2}} + 3 \geq n^2 - n + 1$. If $|V(Q_p)| \leq n-1$ for all $p$ with $1 \leq p \leq s+1$, then $n^2 - n + 1 \leq \sum_{1 \leq p \leq s+1} (|V(Q_p)| + 1) \leq (s+1)n \leq (n-1)n$, which is a contradiction. Thus $|V(Q_q)| \geq n$ for some $q$ with $1 \leq q \leq s+1$.

Note that $|N_G(x_1) \cap V(Q_q)| = 1$. Write $N_G(x_1) \cap V(Q_q) = \{u_j\}$. If $q \neq s+1$, then $j \in \{j_p : 1 \leq p \leq s\}$; otherwise, $j = j^*$ (in $\{j_s, j_{s+1}\}$). Since $u_j$ is an endvertex of $Q_q$, there exists a subpath $Q$ of $Q$ such that $u_j$ is an endvertex of $Q$ and $|V(Q)| = n$. Since $|S| \geq 2$, we can take a vertex $v \in S$ as follows: If $q \neq s+1$, let $v \in S \setminus \{u_j\}$; otherwise (i.e., $j = j^*$), let $v = u_{j_1}$. Then by the definition of $Q_p$ ($1 \leq p \leq s+1$), $N_G(v) \cap V(Q_q) = \emptyset$. Since $2n_0 \geq 2n \geq n+1$, the vertices $x_i$ with $2 \leq i \leq n+1$ have been defined, and hence this implies that $\{x_1, v, x_2, x_3, \ldots, x_{n+1}\} \cup V(Q)$ induces a copy of $F_{n,n}^{(1)}$ in $G$, which is a contradiction.

**Lemma 10.** Let $i$ be an integer with $n_0 + 1 \leq i \leq m-n_0$, and let $y \in Y_i$. Then the following hold:

(i) If $yu_{i+1} \notin E(G)$, then $N_G(y) \cap V(P) = \{u_i, u_{i+2}\}$.

(ii) We have $Y_{m-n_0} = \emptyset$.

(iii) If $G$ is $F_{n,n}^{(3)}$-free, then $yu_{i+1} \in E(G)$.

**Proof.** (i) Suppose that $yu_{i+1} \notin E(G)$ and $N_G(y) \cap V(P) \neq \{u_i, u_{i+2}\}$. Let $k = \max \{ j : n_0 + 1 \leq j \leq m - n_0, yu_j \in E(G) \}$. If $k = i$ (i.e., $N_G(y) \cap V(P) = \{u_i\}$), then

$$\{u_i, y, u_{i-1}, u_{i-2}, \ldots, u_{i-n}, u_{i+1}, u_{i+2}, \ldots, u_{i+n}\}$$

induces a copy of $F_{n,n}^{(1)}$ in $G$, which is a contradiction. Since $yu_{i+1} \notin E(G)$ and $N_G(y) \cap V(P) \neq \{u_i, u_{i+2}\}$, this forces $k \geq i+3$. Then

$$\{u_i, u_{i+1}, u_{i-1}, u_{i-2}, \ldots, u_{i-n}, y, u_k, u_{k+1}, \ldots, u_{k+n-2}\}$$

induces a copy of $F_{n,n}^{(1)}$ in $G$, which is a contradiction.
(ii) By (i), if there exists a vertex \( y \in Y_{m-n_0} \), then it follows that \( yu_{m-n_0+1} \in E(G) \) or \( N_G(y) \cap V(P) = \{ u_{m-n_0}, u_{m-n_0+2} \} \), and in particular, \( N_G(y) \cap X_0 \neq \emptyset \), which contradicts the definition of \( Y \). Thus we have \( Y_{m-n_0} = \emptyset \).

(iii) Suppose that \( G \) is \( F_{n,n}^{(3)} \)-free and \( yu_{i+1} \notin E(G) \). Then it follows from (i) that \( N_G(y) \cap V(P) = \{ u_i, u_{i+2} \} \), and hence

\[
\{ y, u_{i+1}, u_i, u_{i-1}, \ldots, u_{i-n+1}, u_{i+2}, u_{i+3}, \ldots, u_{i+n+1} \}
\]

induces a copy of \( F_{n,n}^{(3)} \) in \( G \), which is a contradiction. \( \square \)

**Lemma 11.** We have \( V(G) = V(P) \cup N_G(V(P)) \cup (\bigcup_{2 \leq i \leq 2n_0-1} X_i) \).

**Proof.** Suppose that \( V(G) \neq V(P) \cup N_G(V(P)) \cup (\bigcup_{2 \leq i \leq 2n_0-1} X_i) \). Since \( G \) is connected, there exists a vertex \( z \in V(G) \setminus (V(P) \cup N_G(V(P)) \cup (\bigcup_{2 \leq i \leq 2n_0-1} X_i)) \) adjacent to a vertex \( y \in V(P) \cup N_G(V(P)) \cup (\bigcup_{2 \leq i \leq 2n_0-1} X_i) \) in \( G \). By Lemma 9 and the definition of \( X_i \) and \( Y \), this implies that \( y \in Y \). Let \( i \) be the integer such that \( y \in Y_i \). Then by Lemma 10(ii), \( n_0 + 1 \leq i \leq m - n_0 - 1 \). Let \( k = \max \{ j : n_0 + 1 \leq j \leq m - n_0, yu_j \in E(G) \} \). By Lemma 10(i), \( k \geq i + 1 \). If \( k = i + 1 \), then

\[
\{ y, z, u_i, u_{i-1}, \ldots, u_{i-n+1}, u_{i+1}, u_{i+2}, \ldots, u_{i+n} \}
\]

induces a copy of \( F_{n,n}^{(2)} \) in \( G \); if \( k \geq i + 2 \), then

\[
\{ y, z, u_i, u_{i-1}, \ldots, u_{i-n+1}, u_k, u_{k+1}, \ldots, u_{k+n-1} \}
\]

induces a copy of \( F_{n,n}^{(1)} \) in \( G \). In either case, we obtain a contradiction. \( \square \)

Now we recursively define the values \( \alpha_i \) (\( i \geq 0 \)) as follows: Let \( \alpha_0 = 2\lceil \frac{m_0}{2} \rceil \), and for \( i \) with \( i \geq 1 \), let \( \alpha_i = (n-1)R(n, \alpha_{i-1} + 1) - 1 \).

**Lemma 12.** For an integer \( i \) with \( i \geq 0 \), \( \alpha(G[X_i]) \leq \alpha_i \).

**Proof.** We proceed by induction on \( i \). If \( |V(P)| \leq 2n_0 \), then \( G[X_0] \) equals to \( P \), and hence \( \alpha(G[X_0]) = \alpha(P) = \lceil \frac{|V(P)|}{2} \rceil \leq n_0 \leq \alpha_0 \); if \( |V(P)| \geq 2n_0 + 1 \), then \( G[X_0] \) consists of two components each of which is a path of order \( n_0 \), and hence \( \alpha(G[X_0]) = 2\lceil \frac{m_0}{2} \rceil = \alpha_0 \). In either case, we have \( \alpha(G[X_0]) \leq \alpha_0 \). Thus we may assume that \( i \geq 1 \), and suppose that \( \alpha(G[X_{i-1}]) \leq \alpha_{i-1} \). Since \( X_i \subseteq N_G(X_{i-1}) \), it follows from Lemma 8 that \( \alpha(G[X_i]) \leq \alpha(G[N_G(X_{i-1})]) \leq (n-1)R(n, \alpha_{i-1} + 1) - 1 = \alpha_i \), as desired. \( \square \)

Note that the value \( \sum_{1 \leq i \leq 2n_0-1} \alpha_i \) is a constant depending on \( n \) only. Thus, considering Lemmas 7, 11 and 12, it suffices to show that

- \( \text{pc}(G[V(P) \cup Y]) \) is bounded by a constant depending on \( n \) only, and
- if \( G \) is \( \{ F_{n,n}^{(3)}, F_{n,n}^{(4)} \} \)-free, then \( \text{pp}(G[V(P) \cup Y]) \) is bounded by a constant depending on \( n \) only.

Hence the following lemma completes the proof of Theorem 5.
Lemma 13.  

(i) We have $\text{pc}(G[V(P) \cup Y]) \leq \max\{3n - 6, 1\}$.  

(ii) If $G$ is $\{F_{n,n}^{(3)}, F_{n,n}^{(4)}\}$-free, then there exists a Hamiltonian path of $G[V(P) \cup Y]$, i.e., $\text{pp}(G[V(P) \cup Y]) = 1$.

Proof. If $Y = \emptyset$, then $P$ is a Hamiltonian path of $G[V(P) \cup Y]$, and hence $\text{pc}(G[V(P) \cup Y]) = \text{pp}(G[V(P) \cup Y]) = 1$. Thus we may assume that $Y \neq \emptyset$. By Lemma 10(ii), $Y_{m-n_0} = \emptyset$.

We first prove (i). Fix an integer $i$ with $n_0 + 1 \leq i \leq m - n_0 - 1$. Let $Y_{i,1} = \{y \in Y_i : yu_{i+1} \in E(G)\}$ and $Y_{i,2} = Y_i \setminus Y_{i,1}$. Then by Lemma 10(i), $N_G(y) \cap V(P) = \{u_i, u_{i+2}\}$ for all $y \in Y_{i,2}$. Let $j \in \{1, 2\}$. If there exists an independent set $U \subseteq Y_{i,j}$ of $G$ with $|U| = n - 1$, then $\{u_{i-1}, u_i\} \cup U$ induces a copy of $K_{1,n}$ in $G$, which is a contradiction. Thus $\alpha(G[Y_{i,j}]) \leq n - 2$. Since $Y \neq \emptyset$, i.e., $Y_{p,q} \neq \emptyset$ for some $p$ and $q$ with $n_0 + 1 \leq p \leq m - n_0 - 1$ and $q \in \{1, 2\}$, this implies that $n \geq 3$. By Lemma 7, there exists a path partition $\mathcal{P}_{i,j} = \{Q_{i,j}^{(1)}, Q_{i,j}^{(2)}, \ldots, Q_{i,j}^{(s_{i,j})}\} \subseteq G[Y_{i,j}]$ with $s_{i,j} \leq n - 2$, where $\mathcal{P}_{i,j} = \emptyset$ and $s_{i,j} = 0$ if $Y_{i,j} = \emptyset$. For an integer $t$ with $1 \leq t \leq n - 2$, if $t \leq s_{i,j}$, let $R_{i,j}^{(t)}$ be the path $u_{i}vQ_{i,j}^{(t)}wu_{i+j}$, where $\{v, w\}$ is the set of endvertices of $Q_{i,j}^{(t)}$; otherwise, let $R_{i,j}^{(t)}$ be the path between $u_i$ and $u_{i+j}$ on $P$ (i.e., $R_{i,j}^{(t)} = u_{i}u_{i+1}$ and $R_{i,j}^{(t)} = u_{i}u_{i+1}u_{i+2}$). We define the value $\xi_2$ (resp. $\xi_3$) with $\xi_2 = m - n_0$ or $\xi_2 = m - n_0 - 1$ (resp. $\xi_3 = m - n_0 - 1$ or $\xi_3 = m - n_0$) according as $m$ is odd or even. Let

\[
R_1^{(t)} = u_1u_2u_3 \cdots u_{n_0-1}u_{n_0+1}R_{n_0+1,1}^{(t)}u_{n_0+2}R_{n_0+2,1}^{(t)}u_{n_0+3} \cdots u_{m-n_0-1}u_{m-n_0}u_{m-n_0+1} \cdots u_m,
\]

\[
R_2^{(t)} = u_1u_2u_3 \cdots u_{n_0+1}R_{n_0+1,1}^{(t)}u_{n_0+2}R_{n_0+2,1}^{(t)}u_{n_0+3} \cdots u_{m-n_0}u_{m-n_0+1} \cdots u_m,
\]

\[
R_3^{(t)} = u_1u_2u_3 \cdots u_{n_0+2}R_{n_0+2,1}^{(t)}u_{n_0+3} \cdots u_{m-n_0}u_{m-n_0+1} \cdots u_m,
\]

Then we easily verify that $\{R_a^{(t)} : a \in \{1, 2, 3\}, 1 \leq t \leq n - 2\}$ is a path cover of $G[V(P) \cup Y]$ having cardinality at most $3(n - 2)$, which proves (i).

Next we prove (ii). Suppose that $G$ is $\{F_{n,n}^{(3)}, F_{n,n}^{(4)}\}$-free. We start with the following claim.

Claim 14. For an integer $i$ with $n_0 + 1 \leq i \leq m - n_0 - 1$, $\{u_i, u_{i+1}\} \cup Y_i$ is a clique of $G$.

Proof. Suppose that there exist two vertices $y, y' \in \{u_i, u_{i+1}\} \cup Y_i$ with $yy' \notin E(G)$. By the definition of $Y_i$ and Lemma 10(iii), every vertex in $Y_i$ is adjacent to both $u_i$ and $u_{i+1}$ in $G$. Thus $y, y' \in Y_i$. Recall that $N_G(Y) \cap V(P) \subseteq \{u_j : n_0 + 1 \leq j \leq m - n_0\}$. Let $k = \max\{j : n_0 + 1 \leq j \leq m - n_0, N_G(u_j) \cap \{y, y'\} \neq \emptyset\}$. We may assume that $yu_k \in E(G)$. Note that $k \geq i + 1$. If $k = i + 1$, then

\[
\{y, y', u_i, u_{i-1}, \ldots, u_{i-n+1}, u_{i+1}, u_{i+2}, \ldots, u_{i+n}\}
\]

induces a copy of $F_{n,n}^{(4)}$ in $G$, which is a contradiction. Thus $k \geq i + 2$. If $y'u_k \in E(G)$, then

\[
\{y, y', u_i, u_{i-1}, \ldots, u_{i-n+1}, u_k, u_{k+1}, \ldots, u_{k+n-1}\}
\]
induces a copy of $F_{n,n}^{(3)}$ in $G$; if $y'u_k \notin E(G)$, then
$$\{u_i, y', u_{i-1}, u_{i-2}, \ldots, u_{i-n}, y, u_k, u_{k+1}, \ldots, u_{k+n-2}\}$$
induces a copy of $F_{n,n}^{(1)}$ in $G$. In either case, we obtain a contradiction. \(\square\)

For an integer $i$ with $n_0 + 1 \leq i \leq m - n_0 - 1$, it follows from Claim 14 that there exists a Hamiltonian path $R_i$ of $G'[\{u_i, u_{i+1}\} \cup Y_i]$ with the endvertices $u_i$ and $u_{i+1}$. Then
$$u_1u_2\cdots u_{n_0+1}R_{n_0+1}u_{n_0+2}R_{n_0+2}u_{n_0+3}\cdots u_{m-n_0-1}u_{m-n_0}u_{m-n_0+1}\cdots u_m$$
is a Hamiltonian path of $G[V(P) \cup \left(\bigcup_{n_0+1 \leq i \leq m-n_0-1} Y_i\right)] = G[V(P) \cup Y]$, as desired. \(\square\)

2.2 The “only if” parts of Theorem 2

Let $s \geq 2$ and $t \geq 3$ be integers, and let $Q_i = u_1^{(i)}u_2^{(i)}\cdots u_i^{(i)}$ ($1 \leq i \leq s$) be $s$ pairwise vertex-disjoint paths. We define four graphs.

- Let $H_{s,t}^{(1)}$ be the graph obtained from the union of the paths $Q_1, \ldots, Q_s$ by adding $2(s-1)$ vertices $v_i, w_i$ ($1 \leq i \leq s - 1$) and $3(s-1)$ edges $v_iw_i, v_iu_i^{(i)}, v_iu_{i+1}^{(i)}$ ($1 \leq i \leq s - 1$).

- Let $H_{s,t}^{(2)}$ be the graph obtained from $H_{s,t}^{(1)}$ by adding $s-1$ edges $u_i^{(i)}u_{i+1}^{(i)}$ ($1 \leq i \leq s - 1$).

- Let $H_{s,t}^{(3)}$ be the graph obtained from the union of the paths $Q_1, \ldots, Q_s$ by adding $2(s-1)$ vertices $v_i, w_i$ ($1 \leq i \leq s - 1$) and $4(s-1)$ edges $v_iu_i^{(i)}, v_iu_{i+1}^{(i)}, w_iu_i^{(i)}, w_iu_{i+1}^{(i)}$ ($1 \leq i \leq s - 1$).

- Let $H_{s,t}^{(4)}$ be the graph obtained from $H_{s,t}^{(3)}$ by adding $s-1$ edges $u_i^{(i)}u_{i+1}^{(i)}$ ($1 \leq i \leq s - 1$).

**Lemma 15.** We have $pc(H_{s,t}^{(1)}) = pc(H_{s,t}^{(2)}) = \left\lceil \frac{s+1}{2} \right\rceil$.

**Proof.** Note that $u_1^{(1)}, u_i^{(i)}, w_i$ ($1 \leq i \leq s - 1$) have degree one in $H_{s,t}^{(2)}$. Since a path contains at most two vertices of degree at most one, $pc(G) \geq \left\lceil \frac{l}{2} \right\rceil$ for every graph $G$ where $l$ is the number of the vertices of $G$ having degree one. In particular, we have
$$pc(H_{s,t}^{(2)}) \geq \left\lceil \frac{s+1}{2} \right\rceil.$$  (1)

If $s$ is odd, let
$$P = \left\{ H_{s,t}^{(1)} - \{w_j : 1 \leq j \leq s - 1\}, \ w_{2i-1}v_{2i-1}u_{2i}^{(1)}Q_2u_{2i}^{(i)}v_{2i}w_{2i} : 1 \leq i \leq \frac{s-1}{2} \right\};$$
if $s$ is even, let
$$P = \left\{ H_{s,t}^{(1)} - \{w_j : 1 \leq j \leq s - 1\}, \ H_{s,t}^{(1)}[\{w_{s-1}\}], \ w_{2i-1}v_{2i-1}u_{2i}^{(1)}Q_2u_{2i}^{(i)}v_{2i}w_{2i} : 1 \leq i \leq \frac{s-2}{2} \right\}.$$
Then we verify that $P$ is a path cover of $H_{s,t}^{(1)}$ with $|P| = \left\lceil \frac{s+1}{2} \right\rceil$. Furthermore, since $H_{s,t}^{(1)}$ is a spanning subgraph of $H_{s,t}^{(2)}$, a path cover of $H_{s,t}^{(1)}$ is also a path cover of $H_{s,t}^{(2)}$, and hence $pc(H_{s,t}^{(2)}) \leq pc(H_{s,t}^{(1)}) \leq \left\lceil \frac{s+1}{2} \right\rceil$. This together with (1) leads to the desired conclusion. \(\square\)
Lemma 16. We have $\text{pp}(H_{s,t}^{(3)}) = \text{pp}(H_{s,t}^{(4)}) = s$.

Proof. We first prove that

$$\text{pp}(H_{s,t}^{(4)}) \geq s. \quad (2)$$

Let $P$ be a path partition of $H_{s,t}^{(4)}$. It suffices to show that $|P| \geq s$. For each $i$ with $1 \leq i \leq s$, let $R_i$ be the unique element of $P$ containing $u_i^{(2)}$. We remark that $R_i$ might equal to $R_j$ for some $1 \leq i < j \leq s$. Let $I = \{i : 1 \leq i \leq s - 1, R_i = R_{i+1}\}$, and write $I = \{i_1, i_2, \ldots, i_h\}$ with $i_1 < i_2 < \ldots < i_h$ where $h = 0$ if $I = \emptyset$. For integers $i$ and $i'$ with $1 \leq i < i' \leq s$, any paths of $H_{s,t}^{(4)}$ joining $u_i^{(2)}$ and $u_{i'}^{(2)}$ contain every vertex in $\{u_j^{(2)} : i < j < i'\}$. This implies that if $R_i = R_{i'}$ with $1 \leq i < i' \leq s$, then $i' - i + 1$ paths $R_j (i \leq j \leq i')$ are equal. In particular, we have $|\{R_i : 1 \leq i \leq s\}| = s - h$.

Fix an integer $l$ with $1 \leq l \leq h$. Then for every path $R$ of $H_{s,t}^{(4)}$ joining $u_i^{(2)}$ and $u_{i+1}^{(2)}$, we easily verify that

- $\{u_i^{(1)}, u_{i+1}^{(1)}\} \subseteq V(R)$, and
- $v_i \notin V(R)$ or $w_i \notin V(R)$.

Since $v_i, w_i \notin E(H_{s,t}^{(4)})$, this implies that there exists an element $R'_{i_l}$ of $P$ such that either $V(R'_{i_l}) = \{v_i\}$ or $V(R'_{i_l}) = \{w_i\}$. Therefore

$$|P| \geq |\{R_i : 1 \leq i \leq s\} \cup \{R'_{i_l} : 1 \leq j \leq h\}|$$

$$= |\{R_i : 1 \leq i \leq s\}| + |\{R'_{i_l} : 1 \leq j \leq h\}|$$

$$= (s - h) + h$$

$$= s,$$

which proves (2).

Since

$$P' = \{H_{s,t}^{(3)} - \{w_j : 1 \leq j \leq s - 1\}, H_{s,t}^{(3)}[\{w_i\}] : 1 \leq i \leq s - 1\}$$

is a path partition of $H_{s,t}^{(3)}$ with $|P'| = s$. Furthermore, since $H_{s,t}^{(3)}$ is a spanning subgraph of $H_{s,t}^{(4)}$, a path partition of $H_{s,t}^{(3)}$ is also a path partition of $H_{s,t}^{(4)}$, and hence $\text{pp}(H_{s,t}^{(4)}) \leq \text{pp}(H_{s,t}^{(3)}) \leq s$. This together with (2) leads to the desired conclusion. \qed

Now we prove the following proposition, which gives the “only if” parts of Theorem 2.

Proposition 17. Let $\mathcal{K}$ be a finite family of connected graphs.

(i) If $\mathcal{K}$ satisfies (A1), then $\mathcal{K} \subseteq \{K_{1,n}, K_n, F_{n,n}^{(1)}, F_{n,n}^{(2)}\}$ for an integer $n \geq 2$.

(ii) If $\mathcal{K}$ satisfies (A2), then $\mathcal{K} \subseteq \{K_{1,n}, K_n, F_{n,n}^{(1)}, F_{n,n}^{(2)}, F_{n,n}^{(3)}, F_{n,n}^{(4)}\}$ for an integer $n \geq 2$. 

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Proof. Since \( \mathcal{H} \) is a finite family, the value \( p = \max\{|V(H)| : H \in \mathcal{H}\} \) is well-defined. If \( p \leq 2 \), then the desired conclusions trivially hold. Thus we may assume that \( p \geq 3 \).

We first suppose that \( \mathcal{H} \) satisfies (A1), and show that (i) holds. There exists a constant \( c_1 = c_1(\mathcal{H}) \) such that \( pc(G) \leq c_1 \) for every connected \( \mathcal{H}\)-free graph \( G \). Since \( pc(K_{1,2c_1+1}) = c_1 + 1 \) and \( pc(K_{2c_1+1}^*) = c_1 + 1 \), neither \( K_{1,2c_1+1} \) nor \( K_{2c_1+1}^* \) is \( \mathcal{H} \)-free. This implies that

\[
\mathcal{H} \subseteq \{K_{1,2c_1+1}, K_{2c_1+1}^*\}. 
\] (3)

For each \( i \in \{1, 2\} \), it follows from Lemma 15 that \( pc(H^{(i)}_{2c_1,p}) = \left\lceil \frac{2c_1+1}{2} \right\rceil = c_1 + 1 \), and hence \( H^{(i)}_{2c_1,p} \) is not \( \mathcal{H} \)-free, i.e., \( H^{(i)}_{2c_1,p} \) contains an induced subgraph \( A_i \) isomorphic to an element of \( \mathcal{H} \). Since \( |V(A_i)| \leq p \), we have

- \( |\{j : 1 \leq j \leq 2c_1, V(A_i) \cap V(Q_j) \neq \emptyset\}| \leq 2 \), and
- \( |\{j : 1 \leq j \leq 2c_1 - 1, V(A_i) \cap \{v_j, w_j\} \neq \emptyset\}| \leq 1 \).

This implies that \( A_i \) is an induced copy of \( F^{(i)}_{p,p} \), and hence

\[
\mathcal{H} \subseteq \{F^{(1)}_{p,p}, F^{(2)}_{p,p}\}. 
\] (4)

Let \( n = \max\{2c_1 + 1, p\} \). Then by (3) and (4), \( \mathcal{H} \subseteq \{K_{1,n}, K_{n}^*, F^{(1)}_{m,n}, F^{(2)}_{n,n}\} \), which proves (i).

Next we suppose that \( \mathcal{H} \) satisfies (A2), and show that (ii) holds. There exists a constant \( c_2 = c_2(\mathcal{H}) \) such that \( pp(G) \leq c_2 \) for every connected \( \mathcal{H}\)-free graph \( G \). Since \( pp(G) \geq pc(G) \) for all graphs \( G \), \( \mathcal{H} \) also satisfies (A1). Hence by (i), there exists an integer \( m \geq 2 \) such that

\[
\mathcal{H} \subseteq \{K_{1,m}, K_{m}^*, F^{(1)}_{m,m}, F^{(2)}_{m,m}\}. 
\] (5)

For each \( i \in \{3, 4\} \), it follows from Lemma 16 that \( pp(H^{(i)}_{c_2+1,p}) = c_2 + 1 \), and hence \( H^{(i)}_{c_2+1,p} \) is not \( \mathcal{H} \)-free, i.e., \( H^{(i)}_{c_2+1,p} \) contains an induced subgraph \( B_i \) isomorphic to an element of \( \mathcal{H} \). Since \( |V(B_i)| \leq p \), we have

- \( |\{j : 1 \leq j \leq c_2, V(B_i) \cap V(Q_j) \neq \emptyset\}| \leq 2 \), and
- \( |\{j : 1 \leq j \leq c_2, V(B_i) \cap \{v_j, w_j\} \neq \emptyset\}| \leq 1 \).

This implies that \( B_i \) is an induced copy of \( F^{(i)}_{p,p} \), and hence

\[
\mathcal{H} \subseteq \{F^{(3)}_{p,p}, F^{(4)}_{p,p}\}. 
\] (6)

Let \( n' = \max\{m, p\} \). Then by (5) and (6), \( \mathcal{H} \subseteq \{K_{1,n'}, K_{n'}^*, F^{(1)}_{n',n'}, F^{(2)}_{n',n'}, F^{(3)}_{n',n'}, F^{(4)}_{n',n'}\} \), which proves (ii). \( \square \)
3 Proof of Theorem 3

In this section, we prove Theorem 3. The following lemma was implicitly proved in [1]. (To keep the paper self-contained, we give its proof.)

**Lemma 18** (Choi et al. [1]). Let $n \geq 2$ be an integer. There exists a constant $c = c(n)$ depending on $n$ only such that $\alpha(G) \leq c$ for every connected $\{K_{1,n}, K^*_n, P_n\}$-free graph $G$.

**Proof.** Let $x$ be a vertex of $G$, and for an integer $i$ with $i \geq 0$, let $X_i$ be the set of vertices $y$ of $G$ such that the distance between $x$ and $y$ in $G$ is exactly $i$. Note that $X_0 = \{x\}$ and $X_1 = N_G(x)$. Since $G$ is $P_n$-free, $X_i = \emptyset$ for all $i \geq n - 1$. Since $G$ is connected, this implies that

$$V(G) = \bigcup_{0 \leq i \leq n-2} X_i. \quad (7)$$

We recursively define the values $\alpha_i \ (i \geq 0)$ as follows: Let $\alpha_0 = 1$, and for $i$ with $i \geq 1$, let $\alpha_i = (n - 1)R(n, \alpha_{i-1} + 1) - 1$.

We prove that

$$\alpha(G[X_i]) \leq \alpha_i \text{ for an integer } i \text{ with } 0 \leq i \leq n - 2. \quad (8)$$

We proceed by induction on $i$. Since $\alpha(G[X_0]) = 1 = \alpha_0$, we may assume that $i \geq 1$ and $\alpha(G[X_{i-1}]) \leq \alpha_{i-1}$. Since $X_i \subseteq N_G(X_{i-1})$, it follows from Lemma 8 that $\alpha(G[X_i]) \leq \alpha(G[N_G(X_{i-1})]) \leq (n - 1)R(n, \alpha_{i-1} + 1) - 1 = \alpha_i$, as desired.

By (7) and (8), we have $\alpha(G) \leq \sum_{0 \leq i \leq n-2} \alpha(G[X_i]) \leq \sum_{0 \leq i \leq n-2} \alpha_i$. Since the value $\sum_{0 \leq i \leq n-2} \alpha_i$ is a constant depending on $n$ only, we obtain the desired conclusion. \qed

**Proof of Theorem 3.** By the definition of cycle cover and cycle partition, “(ii) $\implies$ (i)” clearly holds.

We show that “(iii) $\implies$ (ii)” holds. Let $n \geq 2$ be an integer, and let $c = c(n)$ be the constant as in Lemma 18. It suffices to show that there exists a constant $c_1 = c_1(n)$ depending on $n$ only such that $cp(G) \leq c_1$ for every connected $\{K_{1,n}, K^*_n, P_n\}$-free graph $G$. By the definition of $c(n)$, we have $\alpha(G) \leq c$. This together with Lemma 6 leads to $cp(G) \leq \alpha(G) \leq c$. Since $c$ is a constant depending on $n$ only, we obtain the desired conclusion.

Finally, we show that “(i) $\implies$ (iii)” holds, which completes the proof of Theorem 3. Suppose that a family $\mathcal{H}$ of connected graphs satisfies (A’1). Then there exists a constant $c_1 = c_1(\mathcal{H})$ such that $cc(G) \leq c_1$ for every connected $\mathcal{H}$-free graph $G$. Since $cc(K_{1,c_1+1}) = c_1 + 1$, $cc(K^*_{c_1+1}) = c_1 + 1$ and $cc(P_{2c_1+1}) = \lceil \frac{2c_1+1}{2} \rceil = c_1 + 1$, none of $K_{1,c_1+1}$, $K^*_{c_1+1}$ and $P_{2c_1+1}$ is $\mathcal{H}$-free. This implies that $\mathcal{H} \subseteq \{K_{1,c_1+1}, K^*_{c_1+1}, P_{2c_1+1}\}$, and hence $\mathcal{H} \subseteq \{K_{1,2c_1+1}, K^*_{2c_1+1}, P_{2c_1+1}\}$, which leads (iii). \qed
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