Abelian Varieties, RCFTs, Attractors, and Hitchin Functional in Two Dimensions

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We consider a generating function for the number of conformal blocks in rational conformal field theories with an even central charge \( c \) on a genus \( g \) Riemann surface. It defines an entropy functional on the moduli space of conformal field theories and is captured by the gauged WZW model whose target space is an abelian variety. We study a special coupling of this theory to two-dimensional gravity. When \( c = 2g \), the coupling is non-trivial due to the gravitational instantons, and the action of the theory can be interpreted as a two-dimensional analog of the Hitchin functional for Calabi-Yau manifolds. This gives rise to the effective action on the moduli space of Riemann surfaces, whose critical points are attractive and correspond to Jacobian varieties admitting complex multiplication. The theory that we describe can be viewed as a dimensional reduction of topological M-theory.

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1. Introduction

The purpose of this paper is to look at a two-dimensional toy model of the topological M-theory \cite{1,2} in order to gain some insights on its plausible quantum description. The analysis of this model supports the idea that quantum partition function of the topological M-theory is given by a generalized index theorem for the moduli space. In particular, this implies that the OSV conjecture \cite{3} should be viewed as a higher dimensional analog of the E. Verlinde's formula for the number of conformal blocks in a two-dimensional conformal field theory. Below we briefly sketch some relatively old ideas that provide a motivation for this picture.\footnote{This introductory section is inspired by the talks of R. Dijkgraaf \cite{4} and A. Gerasimov \cite{5}.}
1.1. Universal Partition Function and Universal Index Theorem

It is well known that many generic features of topological theories can be nicely described using the category theory. Roughly speaking, the translation between the category theory and quantum mechanics language goes as follows (for more details and references see, e.g. [6]). One starts with associating a wave-function $|\Psi_0(M)\rangle$ to a $d$-dimensional manifold $M$:

$$
|\Psi_0(M)\rangle.
$$

(1.1)

A natural generalization is assigning some additional structures $E$ (bundles, sheaves, gerbes, etc.) to $M$: $|\Psi_0(M)\rangle \rightarrow |\Psi_E(M)\rangle$. In the language of physics, this is equivalent to putting some branes and/or fluxes on $M$. The morphisms in the category of $(d+1)$-dimensional manifolds with extra structures are bordisms $E \rightarrow F$:

$$
\langle \Psi_E(M)|\Psi_F(M)\rangle,
$$

(1.2)

interpreted as quantum mechanical propagators between the states $E$ and $F$. The composition law of two bordisms given by "gluing" two boundaries is a basic feature of the functional integral:

$$
\langle \Psi_E|\Psi_G\rangle = \int D F \langle \Psi_E|\Psi_F\rangle \langle \Psi_F|\Psi_G\rangle.
$$

(1.3)

These pictures, of course, mimic the well-known operations with the world-sheets in string theory. The universal partition function $Z_{M \times S^1}$ is assigned to the manifold of the form $M \times S^1$:

$$
\text{Tr}_E \langle \Psi_E(M)|\Psi_E(M)\rangle.
$$

(1.4)

Roughly speaking, it counts the number of topological (massless) degrees of freedom, or (super) dimension of the corresponding Hilbert space. Relation (1.4) is a manifestation of the equivalence between the Lagrangian and Hamiltonian formulation of the path integral. In the framework of geometric quantization, the Hilbert space is given by the cohomology
groups of the moduli space $\mathcal{M}_E$ of $E$-structures on $M$, with coefficients in the (prequantum) line bundle $\mathcal{L}$. Therefore, the universal partition function is associated with the corresponding index: $Z_{M \times S^1} = \text{Ind}_E$, where

$$\text{Ind}_E = \sum_n (-1)^n \dim H^n(\mathcal{M}_E, \mathcal{L}).$$

(1.5)

In many interesting cases higher cohomology groups vanish, and the partition function computes the dimension of the Hilbert space: $\dim \text{Hilb}_{\mathcal{M}_E} = \dim H^0(\mathcal{M}_E, \mathcal{L})$. On the other hand, the partition function can also be computed via the universal index theorem:

$$\text{Ind}_E = \int_{\mathcal{M}_E} \text{ch}(\mathcal{L}) \text{Td}(T\mathcal{M}_E),$$

(1.6)

where the integral over the moduli space arises after localization in the functional integral.

Moreover, since one can think of the wave-function (1.1) as of a partition function itself: $Z_M = |\Psi(M)\rangle$, the definitions (1.2)-(1.4) imply the quadratic relation of the form

$$Z_{M \times S^1} \sim Z_M Z_M^*.$$  

(1.7)

If $M$ is a (generalized) complex manifold, this can be true even at the level of the Lagrangian for the local (massive) degrees of freedom. Indeed, in the functional integral formalism we are dealing with the generalized Laplacian operator $\Delta_E = \mathcal{D}_E^\dagger \mathcal{D}_E$ constructed from the generalized Dirac operator $\mathcal{D}_E$. The square factor for the local degrees of freedom arises from the Quillen theorem:

$$\det' \Delta_E = e^{-\mathcal{A}(\mathcal{D}_E)} |\det' \mathcal{D}_E|^2.$$  

(1.8)

Here $\mathcal{A}(\mathcal{D}_E)$ is the holomorphic anomaly: $\partial \overline{\partial} \mathcal{A}(\mathcal{D}_E) \neq 0$. It is natural to assign this anomaly to the integration measure over the moduli space, and then interpret the deviation from the quadratic relation (1.7) as a quantum correction.

Below we list some examples that illustrate these phenomena, which sometimes is referred to as the bulk/boundary correspondence (for more examples, see, e.g., [1,2,7,8]).

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2 In a more general setup $\dim H^\bullet$ is substituted by $\text{Tr}_{H^\bullet} \mathcal{D}_E$.

3 This formula is very schematic, and its exact form depends on the details of the problem. For example, twisting by $K^{1/2}$ will result in appearance of $\widehat{A}$ instead of the Todd class.
Table 1: Examples of the bulk/boundary correspondence.

| Dimension | Correspondence                          | Index Theorem                           |
|-----------|-----------------------------------------|-----------------------------------------|
| 5\(C + 1\) | M-theory/Type IIA                        | \(Z_M \sim Z_{\text{IIA}}^L Z_{\text{IIA}}^R\) |
| 4\(C + 1\) | ?/8d Donaldson-like theory               | ?                                       |
| 3\(C + 1\) | \(G_2/\text{CY}_3\) in topological M-theory | \(Z_{\text{BH}} \sim |Z_{\text{top}}|^2\) |
| 2\(C + 1\) | 5d SYM/Donaldson-Witten theory           | \(Z_{\text{SYM}} \sim Z_{\text{DW}} Z_{\text{DW}}^*\) |
| 1\(C + 1\) | Chern-Simons theory/CFT                  | \(Z_{\text{CS}} \sim |Z_{\text{CFT}}|^2\) |
| 0\(C + 1\) | 2d quantum gravity/2d topological gravity | \(Z_{\text{qg}} \sim Z_{\text{tg}}^2\) |

Relation of the type (1.7) is known in the context of the matrix models (0\(C + 1\) dimensions) as a manifestation of the correspondence between the quantum gravity and topological gravity in two dimensions (see, e.g. [9,10]). Another form of this relation is \(\tau = \sqrt{Z_{\text{qg}}}\), where \(\tau\) is a tau-function of the KdV hierarchy [11]. The index formula (1.6) represents computation of the Euler characteristic of \(M_{g,h}\) via the Penner matrix model.

In 1\(C + 1\) dimensions (1.7) is the famous relation between the Chern-Simons theory and two-dimensional conformal field theory [12]. The index theorem in this case gives the E. Verlinde’s formula for the number of conformal blocks [13]. It will be the subject of primary interest of this paper.

In 2\(C + 1\) dimensions (1.6) and (1.7) express computation of the Gromov-Witten invariants from the counting of the BPS states in the five dimensional supersymmetric gauge theory [14,15].

In 3\(C + 1\) dimensions relation (1.7) is known as the OSV conjecture [3]. The exact formulation of the index theorem (1.6) in this case is not known, and the question about the non-pertubative (quantum) corrections to (1.7) is very important for clarifying the relation between the topological strings and the black holes entropy. It is expected that the answer can be given in the framework of the topological M-theory [1] (which was called Z-theory in [2], see also [16,17,18,19,20,21,22,23] for a discussion on the related issues).
Not much is known about the $4\mathcal{C} + 1$ dimensional example, apart from its relation to the Donaldson-like theory in eight dimensions \cite{2}.

The M-theory/Type IIA relation (topological version of which is the $5\mathcal{C} + 1$ dimensional example) was a source of tremendous progress in string theory over the last decade. Needless to say, there are many subtle details involved in this correspondence (see, e.g. \cite{24}).

The $5\mathcal{C} + 1$ dimensions is not the end of the story, it probably continues to higher dimensions (F-theory, etc.). Also, it is worth mentioning that there are many signs for the (hidden) integrability in these theories, which is intimately related to the free fermion representation. It allows for a tau-function interpretation of the partition function and is responsible for the appearance of the integrable hierarchies.

Finally, let us note that the theories in different dimensions from Table 1 are connected (apart from the obvious dimensional reduction) via the generalized transgression and descent equations (at least at the classical level), which allow for going one complex dimension up or down. From the geometry-categoric viewpoint this is related to the sequence of $d$-manifolds serving as a boundaries for $d + 1$-manifolds:

$$0 \rightarrow M_1 \rightarrow \ldots \rightarrow M_d \rightarrow M_{d+1} \rightarrow \ldots$$

By analogy with \cite{2}, one can call the theory unifying these theories in different dimensions the $Z$-theory.

1.2. The Entropic Principle and Quantum Mechanics on the Moduli Space

So far we discussed theories describing topological invariants of some structures living on a fixed $d$-manifold $M$. It is interesting to ask how these invariants, captured by the universal partition function (1.4), change if we vary $M$ within its topological class. For example, we can talk about transport on the moduli space of genus $g$ Riemann surfaces $\mathcal{M}_g$ or even fantasize about the moduli space of "all" Calabi-Yau threefolds $\mathcal{M}_{CY_3}$.

This question typically arise in quantum gravity, where one is interested in comparing different possible universes (vacuum states) $M$ and choosing "preferred" ones. The original suggestion of Hartle and Hawking \cite{25} is to weight each vacuum by the probability of creation from nothing (see also recent discussion in \cite{26}). This gives some measure on the landscape of vacua. In \cite{27}, this proposal was interpreted in the context of string compactification with fluxes on $\text{AdS}_2 \times \mathbb{S}^2 \times M$, where $M$ is a Calabi-Yau threefold, using the OSV conjecture \cite{3}. The weight, associated to a given $M$, is the norm of the Hartle-Hawking wave-function, which is related to the entropy $S_{p,q}$ of the dual black hole, obtained
by wrapping a $D3$ brane with magnetic and electric charges $(p, q)$ on $M$. This is called the entropic principle \[27,28\]:

$$
\langle \Psi_{p,q}(M) | \Psi_{p,q}(M) \rangle \sim \exp(S_{p,q}) \quad (1.9)
$$

The complex moduli of $M$ are fixed by the charges $(p, q)$ via the attractor mechanism \[29,30,31,32\].

The entropic principle implies that one can define corresponding quantum mechanical problem on the moduli space $M_M$ by summing over all Calabi-Yau manifolds $M$ with the weight (1.9). This path integral can be used, for example, for computing correlation functions of the gravitation fluctuations around the points on the moduli space that correspond to ”preferred” string compactifications (see \[33\] for some steps in this direction). This approach can help us to shed some light on the fundamental physical problems, such as quantum cosmology and string landscape.

The entropic principle in general can be formulated by saying that the entropy function is the Euler characteristic of the moduli space, associated with the problem. It is expected that critical points of the entropy function on the moduli space correspond to special manifolds with extra (arithmetic) structures, such as complex multiplication, Lie algebra lattices, etc. There is also a hidden integrality involved coming from the quantization of (at least partially) compact moduli space. As a result, we expect appearance of the nice modular functions and automorphic forms at the critical points of the entropy function. This will be very clear in a simple two-dimensional example considered in the paper.

It was noted in \[1,2,16\] that some new geometric functionals introduced by Hitchin \[34,35\] might be useful for formulation of this problem in the context of topological strings. There are several reasons why the approach based on the Hitchin functional is attractive. Since it is a diffeomorphism-invariant functional depending only on the cohomology classes of some differential forms, it is a proper candidate for the description of topological degrees of freedom. It can also be used for incorporating the generalized geometry moduli. Moreover, at the classical level it reproduces the black hole entropy. We suggest that in order to use the Hitchin functional for a quantum mechanical description of the moduli space one has to study variation of the cohomology classes that are usually fixed. The space of cohomologies has a natural symplectic structure, defined by the cup product, and therefore is easy to quantize. Moreover, the mapping class group $U = \text{Diff}/\text{Diff}_0$ acts naturally on the space of cohomologies, which might be useful for the non-perturbative description of the theory.
In this paper we take a modest step in this direction by applying this idea to the two-dimensional toy model, that has many interesting features which are expected to survive in higher dimensions. The advantage of taking digression to the two dimensions is that in this case (almost) everything becomes solvable. Our goal is to find a two-dimensional sibling of the Hitchin functional, formulate an analog of the entropic principle in $1+1$ dimensions, and describe corresponding quantum theory. It turns out that this way we find a unified description of all two-dimensional topologies. Moreover, study of the two-dimensional model leads to a natural generalization of the six-dimensional Hitchin functional, which may be useful for understanding of the topological M-theory at quantum level [36].

**Organization of the Paper**

The organization of this paper is as follows: In section 2 we review the Hitchin construction for Calabi-Yau threefolds and formulate the problem of describing the moduli space of Riemann surfaces in terms of the cohomology classes of 1-forms, in the spirit of Hitchin. In section 3 we construct a two-dimensional analog of the Hitchin functional and comment on quantization of the corresponding theory. In section 4 we show that this functional can be related to the gauged WZW model with a target space an abelian variety. In section 5 we describe corresponding quantum theory and interpret its partition function (which as a generating function for the number of conformal blocks in $c=2g$ RCFTs) as an entropy functional on the moduli space of complex structures. The non-perturbative coupling to two-dimensional gravity generates an effective potential on the moduli space, critical points of which are attractive and correspond to Jacobian varieties admitting complex multiplication. We end in section 6 with conclusions and discussion on the possible directions for future research.

**2. The Hitchin Construction**

The problem of characterizing a complex manifold in terms of the data associated with closed 1-forms on it goes back to Calabi [37]. In the context of Riemann surfaces, the non-trivial information encoded in a closed 1-forms reveals itself in the ergodicity and integrability of the associated Hamiltonian systems, which have been extensively studied since 1980s by the Novikov school (see, *e.g.* [38] and the references therein). Kontsevich and Zorich observed an interesting relation between these systems and $c=1$ topological strings [39]. A new twist to the story became possible after Hitchin [34,35] discovered some diffeomorphism-invariant functionals on stable $p$-forms.
The critical points of these topological functionals yield special geometric structures. For example, the Hitchin functional on 3-forms defines a complex structure and holomorphic 3-form in 6 dimensions, and $G_2$ holonomy metrics in 7 dimensions. Hitchin’s construction provides an explicit realization of the idea that geometrical structures on a manifold can be described via the cohomology class of a closed form on this manifold. In this approach, geometric structures arise as solutions to the equations obtained by extremizing canonical topological action.

In this section we review Hitchin’s approach to parameterizing complex structures on a Calabi-Yau threefold, and formulate the problem of describing the moduli space of genus $g$ Riemann surfaces in a similar manner.

2.1. Stable forms in Six Dimensions

Let us describe the Hitchin construction, using a Calabi-Yau threefold $M$ as an example. We present below a Polyakov-like version of the Hitchin functional $[2]$ (see also $[1,16]$), although originally it was written in a Nambu-Goto-like form $[34,35]$. The reason why we need the Polyakov-like version is that it is quadratic in fields, and therefore is more suitable for quantization. We will also extend the construction of $[2]$ in order to incorporate the generalized geometric structures $[40]$.

Let us introduce a (stable) closed poliform $\rho$, which is a formal sum of the odd differential forms $\rho = \rho^{(1)} + \rho^{(3)} + \rho^{(5)}$ on a compact oriented six-dimensional manifold $M$. If we fix the cohomology class $[\rho]$ of this poliform, it defines a generalized Calabi-Yau structure on $M$ as follows. Consider the functional

$$S = -\frac{\pi}{2} \int_M \left( \sigma(\rho) \wedge J^\varsigma \Gamma_{\varsigma \upsilon} \rho + \sqrt{-1} \lambda \text{tr}(J^2 + \text{Id}) \right),$$

(2.1)

where $\sigma(\rho^{(k)}) = (-)^{[k/2]} \rho^{(k)}$ transforms the standard wedge pairing between the differential forms into the Mukai pairing $[11]$, the 6-form $\lambda$ serves as a Lagrange multiplier, $\varsigma, \upsilon = 1, \ldots, 12$ are indices in $TM \oplus T^*M$, the matrix $\Gamma_{\varsigma \upsilon} = [\Gamma_{\varsigma}, \Gamma_{\upsilon}]$ is defined by the gamma-matrices $\Gamma_{\varsigma}$ of Clifford$(6,6)$, and the tensor field $J \in \text{End}(TM \oplus T^*M)$. After solving the equations of motion and using the constraint imposed by $\lambda$, this field becomes a generalized almost complex structure on $M$: $J^2 = -\text{Id}$. Hitchin $[10]$ proved that this almost complex structure is integrable and can be used to reduce the structure group of $TM \oplus T^*M$ to $SU(3,3)$. This endows $M$ with a generalized Calabi-Yau structure.
It is perhaps more illuminating to see how this construction gives rise to the ordinary Calabi-Yau structure, when \( \rho \) is a stable closed 3-form: \( \rho = \rho^{(3)} \). The Polyakov-like version of the Hitchin functional has the form

\[
S = -\frac{\pi}{2} \int_M \left( \rho \wedge \nu_K \rho + \sqrt{-1} \lambda \text{tr}(K^2 + \text{Id}) \right).
\]

(2.2)

Here \( K \in \text{End}T_{\mathbb{R}}M \) is a traceless vector valued 1-form. We denote it as \( K \) in order to distinguish it from the generalized complex structure \( J \). Also, \( \rho = \rho^{(3)} \) is a closed 3-form in a fixed de Rham cohomology class. It can be decomposed as \( \rho = [\rho] + d\beta \), where \( [\rho] \in H^3(M, \mathbb{R}) \) and \( \beta \in \Lambda^2 T^*M \). The equations of motion, obtained by varying \( \beta \) in the functional (2.2), accompanied by the closeness condition for \( \rho \), take the form

\[
d\rho = 0, \quad d\nu_K \rho = 0.
\]

(2.3)

The Lagrange multiplier \( \lambda \) imposes the constraint \( \text{tr}K^2(\rho) = -6 \) on the solution of the equation of motion for the field \( K \), which in some local coordinates on \( M \) can be written as \( K^b_a \sim \epsilon^{ba_1a_2a_3a_4a_5} \rho_{a_1a_2a_3} \rho_{a_4a_5} \). This allows to identify \( K(\rho) \) as an almost complex structure: \( K^2(\rho) = -\text{Id} \). Moreover, it can be shown [34], that this almost complex structure is integrable. Therefore, solutions of (2.3), parameterized by the cohomology class \([\rho]\), define a unique holomorphic 3-form on the Calabi-Yau manifold \( M \), according to

\[
\Omega = \rho + \sqrt{-1} \nu_{K(\rho)} \rho.
\]

(2.4)

We can use the periods of (2.4) to introduce local coordinates on the complex moduli space of Calabi-Yau. Then, a holomorphic 3-form \( \Omega \), viewed as a function of the cohomology class \([\rho]\), gives a map between an open set in \( H^3(M, \mathbb{R}) \) and a local Calabi-Yau moduli space [34]. We will call it the Hitchin map. After integrating out the field \( K \) we arrive at the original Hitchin functional [34], written in the Nambu-Goto-like form:

\[
S = -\frac{\pi}{2} \int_M \rho \wedge \ast_{\rho} \rho.
\]

(2.5)

Here \( \ast_{\rho} \) denotes the Hodge star-operator for the Ricci-flat Kähler metric on \( M \), compatible with the complex structure \( K(\rho) \). Finally, the value of the Hitchin functional (2.2) calculated at the critical point (2.3) can also be written in terms of the holomorphic 3-form (2.4) as follows:

\[
S = -i\frac{\pi}{4} \int_M \Omega \wedge \overline{\Omega}.
\]

(2.6)

\footnote{The coefficient \(-\pi/2\) in front of the integral can be fixed after comparing the Hitchin action with the black hole entropy functional (see, e.g., [1,28]). It is tempting to speculate that this normalization factor can also be determined from the topological considerations, similarly to the way the coefficient \(-1/8\pi\) in front of the WZW functional is fixed.}
2.2. Riemann Surfaces and Cohomologies of 1-forms

We want to find an analog of the Hitchin construction for the two-dimensional surfaces. It is natural to expect that the role that was played by the closed 3-forms in 3\text{C} dimensions, in \text{1C} dimensions will be played by the closed 1-forms. Therefore, we want to construct a functional, depending on closed 1-forms in a fixed cohomology class, critical points of which will determine the complex structure on a genus \(g\) Riemann surface \(\Sigma_g\). In fact, a two-dimensional version of the Hitchin functional was already discussed in [2,16]. There, it was pointed out that it is very similar to Polyakov’s formulation of the bosonic string as a sigma model coupled to the two-dimensional gravity.

However, before discussing the explicit form of this functional, we want to explain why a naive carry-over of the Hitchin idea from six to two dimensions will not work. First, the very existence of the Hitchin map is based on the fact that in the case of Calabi-Yau threefold \(M\) the dimension of the intermediate cohomology space \(\dim H^3(M, \mathbb{R})\) coincide with the dimension of the moduli space of calibrated Calabi-Yau manifolds\(^5\), which is equal to \(2 + 2h^{2,1}\). In the case of a genus \(g\) Riemann surface \(\dim H^1(\Sigma_g, \mathbb{R}) = 2g\), but dimension of the moduli space \(\mathcal{M}_g\) for \(g > 1\) is \(\dim \mathcal{M}_g = 6g - 6\). Therefore, the cohomology class of a closed 1-form on \(\Sigma_g\) does not contain enough data to describe the moduli space. This is, of course, not surprising, as it is well known that natural parameterization of the moduli space \(\mathcal{M}_g\) is given in terms of the Beltrami differentials \(\mu\), which are dual to the holomorphic quadratic differentials \(\chi \in H^0(\Sigma_g, \Omega^{\otimes 2})\). In particular, \(\dim H^0(\Sigma_g, \Omega^{\otimes 2}) = 6g - 6\), as it should be. One could then try to use \(H^0(\Sigma_g, \Omega^{\otimes 2})\) instead of \(H^1(\Sigma_g, \mathbb{R})\), but if we go by this route, we will lose the ”background independence” on the complex structure on \(\Sigma_g\), which is a nice feature of the Hitchin construction. The relevant set-up in this case seems to be provided by the theory of beta-gamma systems [42,43]. However, it turns out that it is hard to write down an analog of the Hitchin functional for \((\mu, \chi)\) system with decoupled conformal factor.

The only exception, when the dimension of the moduli space coincides with the dimension of the first cohomology space, is the elliptic curve \(\Sigma_1\), which is in fact a direct one-dimensional analog of the Calabi-Yau threefold. In this case, \(\dim H^1(\Sigma_1, \mathbb{R}) = 2 = \dim \mathcal{M}_1\)

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\(^5\) The calibrated Calabi-Yau manifold is a pair: \((M, \Omega)\), where \(M\) Calabi-Yau threefold and \(\Omega\) is a fixed non-vanishing holomorphic 3-form on \(M\). Hitchin construction naturally gives calibrated Calabi-Yau manifolds.
and therefore we might expect that one closed 1-form can play the role of \( \rho \) in two dimensions. However, as it was noted in [2], one needs at least two closed 1-forms in order to write down a two-dimensional analog of the Hitchin functional. In a certain sense, it is a lower dimensional artefact, as there just don’t happen to be enough indices to write down a non-zero expression.

Clearly, some modification of the Hitchin construction is needed in the two-dimensional case. We suggest the following extension that preserves the spirit of the original construction. First, we will use complex cohomologies instead of the real ones:

\[
H^1(\Sigma_g, \mathbb{R}) \rightarrow H^1(\Sigma_g, \mathbb{C}).
\]  

(2.7)

Second, for a genus \( g \) surface we will consider \( g \) closed 1-forms:

\[
H^1(\Sigma_g, \mathbb{C}) \rightarrow (H^1(\Sigma_g, \mathbb{C}))^{\otimes g}.
\]

(2.8)

As we will see, this will allow us to define close analog of the Hitchin functional.

3. Construction of the Lagrangian

The case of interest for us in this section is a complex valued 1-forms on a two-dimensional compact surface \( \Sigma_g \) of genus \( g \). We will not assume that \( \Sigma_g \) is endowed with any additional structures, such as a metric or complex structure. Instead, in the spirit of Hitchin, we would like to construct a functional, critical points of which will define a complex structure on \( \Sigma_g \), making it a Riemann surface. In order to keep the presentation self-contained and to fix the notations, we start with a brief review of the basics of Riemann surfaces. Then we proceed to the construction of the functional on the space of closed 1-forms, the critical points of which in a fixed cohomology class are harmonic 1-forms. The complex structure on \( \Sigma_g \) will arise from the cohomology classes of these 1-forms. We will also briefly discuss the quantization of the corresponding theory.

3.1. Mathematical Background on Riemann Surfaces

We summarize below some basic facts from the theory of compact Riemann surfaces [44,45]. Let \( \Sigma_g \) be a topological surface with \( g \) handles, that is a compact connected oriented differentiable manifold of real dimension 2. The number of handles \( g \) is the genus
of $\Sigma_g$. Topologically, $\Sigma_g$ is completely specified by the Euler number $\chi(\Sigma_g) = 2 - 2g$. In particular, the dimensions of the homology groups are

$$\dim H_0(\Sigma_g) = 1, \quad \dim H_1(\Sigma_g) = 2g, \quad \dim H_2(\Sigma_g) = 1. \quad (3.1)$$

On $\Sigma_g$ one can choose the canonical symplectic basis of 1-cycles $\{A_I, B_I\}$, $I = 1, \ldots, g$ for $H_1(\Sigma_g)$, with the intersection numbers

$$\#(A_I, A_J) = 0, \quad \#(A_I, B_J) = \delta_{IJ}, \quad \#(B_I, B_J) = 0. \quad (3.2)$$

This basis, however, is not unique. The ambiguity is controlled by the Siegel modular group $\Gamma_g = \text{Sp}(2g, \mathbb{Z})$ preserving symplectic pairing (3.2):

$$\begin{pmatrix} B \\ A \end{pmatrix} \rightarrow \begin{pmatrix} B' \\ A' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g \quad (3.3)$$

Once we have chosen a homology basis $\{A_I, B_I\}$, or “marking” for $\Sigma_g$, we can cut surface along $2g$ curves homologous to the canonical basis and get a $4g$-sided polygon with appropriate boundary identifications. This representation of $\Sigma_g$ in terms of the polygon is very helpful in deriving some important identities. For example, one can show that for any closed 1-forms $\eta$ and $\theta$ on $\Sigma_g$

$$\int_{\Sigma_g} \eta \wedge \theta = \sum_{I=1}^g \left( \oint_{A_I} \eta \oint_{B_I} \theta - \oint_{A_I} \theta \oint_{B_I} \eta \right), \quad (3.4)$$

which is called the Riemann bilinear identity. The scalar product of two closed 1-forms $\eta$ and $\theta$ on $\Sigma_g$ is given by the Petersson inner product:

$$\langle \eta, \theta \rangle = \frac{i}{2} \int_{\Sigma_g} \eta \wedge \overline{\theta}. \quad (3.5)$$

As follows from the Riemann bilinear identity, this scalar product depends only on the cohomology class of the closed forms: $\langle \eta, \theta \rangle = \langle [\eta], [\theta] \rangle$. The canonical symplectic form

$$\mathcal{S}(\eta, \theta) = \int_{\Sigma_g} \eta \wedge \theta \quad (3.6)$$

for closed 1-forms also depends only on the cohomology class: $\mathcal{S}(\eta, \theta) = \mathcal{S}([\eta], [\theta])$. 

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Let us introduce the basis \( \{\alpha^I, \beta^I\} \), \( I = 1, \ldots, g \) for \( H^1(\Sigma_g, \mathbb{R}) \), which is dual to the canonical homology basis (3.2):

\[
\oint_{A_I} \alpha^J = \delta^{IJ}, \quad \oint_{B_I} \alpha^J = 0 \\
\oint_{B_I} \beta^J = \delta^{IJ}, \quad \oint_{A_I} \beta^J = 0.
\] (3.7)

The ambiguity of this basis is controlled by an exact 1-forms on \( \Sigma_g \). Therefore, we can think of \( \{\alpha^I, \beta^I\} \) as of some fixed representatives in the de Rham cohomology class. A natural way to fix this ambiguity is to pick some Riemann metric \( h \) on \( \Sigma_g \) and require \( \{\alpha^I, \beta^I\} \) to be harmonic:

\[
d\ast_h \alpha^I = 0, \quad d\ast_h \beta^I = 0
\] (3.8)

where \( \ast_h \) is a Hodge \( \ast \)-operator defined by \( h \). This choice provides a canonical basis for \( H^1(\Sigma_g, \mathbb{R}) \), associated with the metric \( h \). We will always use Euclidean signature on \( \Sigma_g \).

Topological surface \( \Sigma_g \) endowed with a complex structure is called a Riemann surface. Let us recall that an almost complex structure on \( \Sigma_g \) is a section \( J \) of a vector bundle \( \text{End}(T\Sigma_g) \) such that \( J^2 = -1 \). Here \( T\Sigma_g \) is a real tangent bundle of \( \Sigma_g \). If we pick some (real) local coordinates \( \{x^a\} \), \( a = 1, 2 \) on \( \Sigma_g \), then \( J \) can be represented by a real tensor field which components \( J^a_b \) obey

\[
J^a_b J^b_c = -\delta^a_c
\] (3.9)

Here and in what follows, a sum over the repeating indices is always assumed. We reserve the indices \( \{a, b, c, \ldots\} \) that range from 1 to 2, for the world-sheet (Riemann surface), and indices \( \{I, J, K, \ldots\} \) that range from 1 to \( g \), for the complex coordinates on the target (first cohomology) space. The indices \( \{i, j, k, \ldots\} \) label real coordinates on the target space and range from 1 to \( 2g \). We do not distinguish between the upper and lower indices. In particular, we do not use any metric to contract it. We will also sometimes omit indices and use matrix notations in the target space for shortness.

According to the Newlander-Nirenberg theorem \( J \), is an integrable complex structure if it is covariantly constant:

\[
\nabla_a J^b_c = 0.
\] (3.10)

In fact, any almost complex structure on a topological surface is integrable, and therefore below we will just call it a complex structure. In particular, we will be interested in a
complex structure compatible with the metric $h$. In local coordinates the metric has the form

$$h = h_{ab} dx^a \otimes dx^b,$$  \hspace{1cm} (3.11)

and the corresponding complex structure is given by

$$J(h)^a_b = \sqrt{\det |h_{df}|} \epsilon_{bc} h^{ca},$$  \hspace{1cm} (3.12)

where $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$. It is straightforward to check that this complex structure indeed obeys (3.9)-(3.10). Notice that complex structure (3.12) depends only on the conformal class of the metric, since it is invariant under the conformal transformations:

$$h \rightarrow e^{\phi} h, \quad J(h) \rightarrow J(h).$$  \hspace{1cm} (3.13)

The complex coordinates $z, \overline{z}$ on $\Sigma_g$ associated with (3.12) are determined from the solution of the Beltrami equation

$$J^a_b \frac{\partial z}{\partial x^a} = i \frac{\partial \overline{z}}{\partial x^b}.$$  \hspace{1cm} (3.14)

Given a marking for $\Sigma_g$, there is a unique basis of holomorphic abelian differentials of the first kind $\omega^I \in H^0(\Sigma_g, \Omega)$, normalized as follows

$$\oint_{A_I} \omega^J = \delta^{IJ}.$$  \hspace{1cm} (3.15)

Here $\omega^I = \omega^I_z dz$. Holomorphic 1-differentials span $-i$ eigenspace of the Hodge $*$-operator for the metric compatible with the complex structure:

$$*\omega = -i \omega,$$

$$*\overline{\omega} = +i \overline{\omega}.$$  \hspace{1cm} (3.16)

The period matrix of $\Sigma_g$ is defined by

$$\tau^{IJ} = \oint_{B_I} \omega^J.$$  \hspace{1cm} (3.17)

If we apply the Riemann identity (3.4) to the trivial 2-form $\omega^I \wedge \omega^J = 0$, we find that the period matrix is symmetric:

$$\tau^{IJ} = \tau^{JI}.$$  \hspace{1cm} (3.18)
The imaginary part of the period matrix can be represented as follows:

$$\text{Im} \tau^{IJ} = \frac{i}{2} \int_{\Sigma_g} \omega^I \wedge \overline{\omega}^J. \quad (3.19)$$

If we use the fact that the norm (3.15) of the non-zero holomorphic differentials of the form $$\nu = \nu_I \omega^I$$ is positive: $$\langle \nu, \nu \rangle > 0$$, we find that the period matrix has a positive definite imaginary part:

$$\text{Im} \tau > 0. \quad (3.20)$$

Now we can express the holomorphic abelian differentials (3.15)-(3.17) via the canonical cohomology basis (3.7) of harmonic 1-forms (3.8) on $$\Sigma_g$$ as follows

$$\omega = \alpha + \tau \beta, \quad (3.21)$$

where we used the matrix notations. Under the modular transformations (3.3) the period matrix transforms as

$$\tau \rightarrow \tau' = (a \tau + b)(c \tau + d)^{-1}, \quad (3.22)$$

while the basis of abelian differentials transforms as

$$\omega \rightarrow \omega' = (\tau^T c + d^T)^{-1} \omega. \quad (3.23)$$

The space of a complex $$g \times g$$ matrices obeying (3.18), (3.20) is the Siegel upper half-space $$\mathcal{H}_g$$. We will call it the Siegel space, for short. Torelli’s theorem states that a complex structure of $$\Sigma_g$$ is uniquely defined by the period matrix up to a diffeomorphism. Moreover, to each complex structure there corresponds a unique point in the fundamental domain of the modular group

$$\mathcal{A}_g = \mathcal{H}_g / \Gamma_g. \quad (3.24)$$

Unfortunately, for higher genus surfaces the converse is not true (Schottky’s problem). This is easy to see, since for $$g > 3$$ the dimension of (3.24) $$\text{dim}_C \mathcal{A}_g = \frac{g(g+1)}{2}$$ is bigger than the dimension of the complex structures moduli space $$\text{dim}_C \mathcal{M}_g = 3g - 3$$. 

15
3.2. The Canonical Metric

There is a canonical Kähler metric on a Riemann surface, the so-called Bergmann metric. Sometimes it is also called the Arakelov metric in literature. It can be written in terms of the abelian differentials (3.15) as

$$ h^B_{z\bar{z}} = (\text{Im} \tau)^{-1}_{I,J} \omega^I \bar{\omega}^J. $$

(3.25)

This metric has a nonpositive curvature. If $g \geq 2$, the curvature vanishes at most in a finite number of points, and by an appropriate conformal transformation (3.25) can be brought into a metric of constant negative curvature (see, e.g. [46]). The Kähler form corresponding to the Bergmann metric is given by

$$ \omega^B = \frac{i}{2} (\text{Im} \tau)^{-1}_{I,J} \omega^I \wedge \bar{\omega}^J. $$

(3.26)

It is easy to see that the volume of the Riemann surface in this metric is independent of the complex structure and is equal to the genus:

$$ \int_{\Sigma_g} \omega^B = g. $$

(3.27)

The special role of the Bergmann metric will become clear if we consider the period map $z \to \xi^I$ from the Riemann surface $\Sigma_g$ into its Jacobian variety $\text{Jac}(\Sigma_g) = \mathbb{C}^n/(\mathbb{Z}^n \oplus \tau \mathbb{Z}^n)$:

$$ \xi^I = \int_{z_0}^z \omega^I. $$

(3.28)

Here $z_0$ is some fixed point on $\Sigma_g$, the exact choice of which is usually not important. Jacobian variety, being a flat complex torus, is endowed with a canonical metric, which is induced from the Euclidian metric on $\mathbb{C}^n$. The Bergmann metric (3.23) is nothing but a pull-back of this canonical metric from $\text{Jac}(\Sigma_g)$ to $\Sigma_g$ under the period map (3.28).

The metric (3.25) does not depend on the choice of a basis in a space of holomorphic differentials $H^0(\Sigma_g, \Omega)$. In particular, it is invariant under the modular transformations (3.22)-(3.23). If we consider $\omega^I$ as a set of $g$ closed 1-forms on $\Sigma_g$ in a fixed cohomology class, parameterized by the period matrix $\tau \in \mathbb{A}_g$, then (3.25) combined with (3.12) gives an explicit realization of the Torelli’s theorem, by providing the map $\mathbb{A}_g \to \mathcal{M}_g$.

\footnote{In the orthonormal basis $\{\omega^I_{o} = \omega^I_{oz} dz : \langle \omega^I_{o}, \omega^J_{o} \rangle = \delta^{IJ}\}$, the metric takes the canonical form: $h_{z\bar{z}} = \sum_{I=1}^{g} |\omega^I_{oz}|^2$.}
This should be viewed as a two-dimensional analog of the Hitchin map from the cohomology space of the stable forms on a compact six-dimensional manifolds to the moduli space of the calibrated Calabi-Yau threefolds.

Indeed, let us recall that the space of all metrics on a genus $g$ surface $\Sigma_g$ is factorized as follows

$$\text{Met}(\Sigma_g) = \mathcal{M}_g \times \text{Diff}(\Sigma_g) \times \text{Conf}(\Sigma_g). \quad (3.29)$$

Once we fixed the cohomology class of $\omega^I$, we are not allowed to do the conformal transformations, since this will spoil the closeness of $\omega^I$. Therefore, expression (3.29) provides a unique representative among the conformal structures on $\Sigma_g$. This takes care of the $\text{Conf}(\Sigma_g)$ factor in (3.29). Moreover, since diffeomorphisms do not change the cohomology class, the Bergmann metric (3.25) is invariant under the action of the $\text{Diff}(\Sigma_g)$ group, and we end up on the moduli space of genus $g$ Riemann surfaces $\mathcal{M}_g$.

### 3.3. An analog of the Hitchin Functional in Two Dimensions

As we discussed earlier, the problem in defining an analog of the Hitchin functional in two dimensions is that the cohomology class of only one 1-form is not enough to parameterize the moduli space of complex structures. However, if we take $g$ closed 1-forms on a genus $g$ surface, this can be done. In fact, this will give us even more degrees of freedom than we need ($g^2$ complex parameters instead of $3g - 3$), but it is a minimal set of data that we can start with, because of the Schottky problem. The functional that we will use is a direct generalization of [4,10]. The fields of the theory are

- $\zeta^I$: $g$ closed complex valued 1-forms, $d\zeta^I = 0$
- $\mathcal{K}$: real traceless vector valued 1-form, $\mathcal{K} \in \text{End}(T_{\mathbb{R}} \Sigma_g)$
- $\lambda$: imaginary 2-form

The Lagrangian has the form

$$L = \frac{k\pi}{4} \langle \zeta^I, \zeta^J \rangle^{-1} \int_{\Sigma_g} \left( \zeta^I \wedge i_\mathcal{K} \overline{\zeta}^J + \overline{\zeta}^J \wedge i_\mathcal{K} \zeta^I \right) - i \frac{k\pi}{4} \int_{\Sigma_g} \lambda \text{tr}(\mathcal{K}^2 + \text{Id}), \quad (3.30)$$

where $k$ is a coupling constant, 2-form $\lambda$ serves as a Lagrange multiplier, and $\text{Id}$ is a unit $2 \times 2$ matrix. Hermitian $g \times g$ matrix $\langle \zeta^I, \zeta^J \rangle^{-1}$ is an inverse of the scalar product (3.5) for 1-forms. We assume that cohomology classes of 1-forms $[\zeta^I]$ are linear independent. In order to discuss classical equations of motion for the action (3.30) and their solutions it is useful to write it is explicitly in components:

$$L = \frac{k\pi}{4} \int_{\Sigma_g} \left( \langle \zeta^J, \zeta^J \rangle^{-1} \left( \zeta_a^I \overline{\zeta}^J_c + \zeta_c^I \overline{\zeta}^J_a \right) \mathcal{K}^c_{ab} - \frac{i}{2} \lambda(x) \epsilon_{ab} \left( \mathcal{K}^d_{cd} \mathcal{K}^c_{da} + 2 \right) \right) dx^a \wedge dx^b, \quad (3.31)$$
where $\zeta^I = \zeta_a^I dx^a$, $K = K^a_b \frac{\partial}{\partial x^a} \otimes dx^b$, $\lambda = \frac{1}{2} \lambda(x) \epsilon_{ab} dx^a \wedge dx^b$, and $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$. The equations of motion for $K$ give

$$K^a_b = \frac{i}{2\lambda(x)} \langle \zeta^I, \zeta^J \rangle^{-1} (\zeta^I_b \zeta^J_c + \zeta^I_c \zeta^J_b) \epsilon^{ca}, \quad (3.32)$$

where $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = -1$, such that $\epsilon^{ac} \epsilon_{cb} = \delta^a_b$. Variation with respect to $\lambda$ imposes the constraint

$$K^a_b K^b_a = -2, \quad (3.33)$$

which is solved by setting

$$\lambda(x) = \pm i \sqrt{\det \|h_{ab}\|}, \quad (3.34)$$

where, by definition, $\det \|h_{ab}\| = \frac{1}{2} h_{ab} h_{cd} \epsilon^{ac} \epsilon^{bd}$, and $h_{ab}$ is the metric induced on $\Sigma$:

$$h_{ab} = \frac{1}{2} \langle \zeta^I, \zeta^J \rangle^{-1} (\zeta^I_a \zeta^J_b + \zeta^I_b \zeta^J_a). \quad (3.35)$$

We choose the “+” sign in (3.34) by requiring positivity of the Lagrangian (3.31) after solving for $K$ and $\lambda$:

$$L = \frac{k\pi}{2} \int_{\Sigma} \sqrt{\det \|h_{ab}\|}. \quad (3.36)$$

The corresponding solution

$$K^a_b = \frac{1}{\sqrt{\det \|h_{ab}\|}} h_{bc} \epsilon^{ca} = -\sqrt{\det \|h_{df}\| \epsilon_{bc} h^{ca}} \quad (3.37)$$

represents the action of the complex structure, compatible with the metric (3.35), on the cotangent bundle $T^*_\Sigma$:

$$K = J^{-1} = -J. \quad (3.38)$$

This should be compared with (3.12). Let us introduce the notation $\ast_\zeta$ for a Hodge star operator, defined by the metric (3.35). For example, the Hodge dual of a 1-form $\theta = \theta_a dx^a$ is given by

$$\ast_\zeta \theta = \theta_a \sqrt{\det \|h_{df}\|} h^{ab} \epsilon_{bc} dx^c. \quad (3.39)$$

Notice that it is possible for $\lambda$ to vanish at some points, if the determinant of the induced metric becomes zero. In this case, expression (3.32) is not well defined, and potentially is singular. Later we will see, that in order to give well-defined complex structure, the cohomologies $[\zeta^I]$ should lie in the Jacobian locus in the Siegel upper half-space.
This Hodge \(*\)-operator acts on 1-forms exactly as the field \( \mathcal{K} \):

\[
\ast \zeta = \mathcal{K} \zeta.
\]  

(3.40)

Therefore, we can rewrite the action (3.30) in yet another form:

\[
L = \frac{k \pi}{4} \langle \zeta^I, \zeta^J \rangle^{-1} \int_{\Sigma_g} \zeta^I \wedge \ast \zeta^J + \text{h.c.}
\]  

(3.41)

This expression is similar to the Hitchin functional (2.5) and relation between (3.30) and (3.41) is very much like relation between the Polyakov and Nambu-Goto actions in string theory, as was noted in [2,16].

Following the idea of Hitchin, we should restrict this functional to the closed forms on \( \Sigma_g \) in a given de Rham cohomology class, and look for the critical points. In order to parameterize variations of \( \zeta^I \) in a fixed cohomology class \( [\zeta^I] \in H^1(\Sigma_g, \mathbb{C}) \), we decompose it as

\[
\zeta^I = [\zeta^I] + d\xi^I,
\]  

(3.42)

where \( \xi^I \) is a proper function \( \Sigma_g \to \mathbb{C}^g \). By varying \( \xi^I \) in (3.30), we get

\[
d \ast \zeta^I = 0.
\]  

(3.43)

Thus, the critical points of the functional (3.41) correspond to the harmonic forms on \( \Sigma_g \). The complex dimension of the space of harmonic 1-forms on \( \Sigma_g \) is equal to \( g \). Since initial conditions (3.42) are parameterized by \( g \) linear independent vectors \( [\zeta^I] \), solution to (3.43) will give us a basis in the space of harmonic 1-forms. We can parameterize cohomology classes \( [\zeta^I] \) using their periods over the \( A \) and \( B \)-cycles:

\[
[\zeta^I] = A^{IJ} \alpha^J + B^{IJ} \beta^J,
\]  

(3.44)

where \( A^{IJ} \) and \( B^{IJ} \) are \( g \times g \) complex matrices. We impose some natural restrictions on the form of these matrices. First, since the action (3.30) is invariant under the linear transformations

\[
\zeta^I \to M^{IJ} \zeta^J, \quad M^{IJ} \in \text{GL}(g, \mathbb{C}),
\]  

(3.45)

we can always set \( A^{IJ} = \delta^{IJ} \) by using this transformation with \( M = A^{-1} \). Then, (3.44) becomes

\[
[\zeta^I] = \alpha^I + \Pi^{IJ} \beta^J,
\]  

(3.46)
where \( \Pi = A^{-1}B \). The fact that all cohomology classes \([\zeta^I]\) are linear independent means that

\[
\text{rank } \Pi = g. \tag{3.47}
\]

The second restriction comes from the fact that the matrix of the scalar products of 1-forms

\[
\langle \zeta^I, \zeta^J \rangle = \frac{i}{2} (\Pi^\dagger - \Pi)^{IJ} \tag{3.48}
\]

should be invertible and positive definite for the theory, based on the action \((3.30)\), to be well-defined. Moreover, it is natural to require that cohomology classes \([\zeta^I]\) do not intersect

\[
\int_{\Sigma_g} \zeta^I \wedge \zeta^J = \Pi^{IJ} - \Pi^{JI} = 0. \tag{3.49}
\]

Let us recall that this intersection number is essentially the canonical symplectic form \(S(\zeta^I, \zeta^J)\) on \(H^1(\Sigma_g, \mathbb{C})\), defined in \((3.4)\). Therefore, from the perspective of future quantization of the cohomology space, it is necessary to require that the points \([\zeta^I]\) in the configuration space commute. This requirement is similar to considering only commuting set of the periods in quantum mechanics of the self-dual form (see, e.g., \([17]\)).

Therefore, instead of dealing with all non-degenerate matrices \(\Pi \in \text{GL}(g, \mathbb{C})\), we can concentrate only on the matrices that obey

\[
\Pi^T = \Pi, \quad \text{Im}\Pi > 0. \tag{3.50}
\]

In other words, we parameterize cohomology classes \([\zeta^I]\) by the points on the Siegel upper half-space \(\mathcal{H}_g\). In fact, \(\mathcal{H}_g\) is the smallest linear space where we can embed Jacobian variety \(\text{Jac}(\Sigma_g)\) without knowing its detailed description, which is unavailable for \(g > 4\) because of the Schottky problem. There is a natural action of the symplectic group on \(\mathcal{H}_g\).

We will denote this ”target” modular group as \(\text{Sp}(2g, \mathbb{Z})_t\), in order to distinguish it from the ”world-sheet” modular group \(\text{Sp}(2g, \mathbb{Z})_{ws}\) acting on the cover of the moduli space of Riemann surfaces.

Given the solution to \((3.43)\), the complex structure on \(\Sigma_g\) is uniquely determined by the corresponding cohomology class via \((3.35)-(3.32)\), very much in the spirit of Hitchin.

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8 This condition can be imposed, for example, by adding a term of the form \(iA_{IJ} \int_{\Sigma_g} \zeta^I \wedge \zeta^J\) to the action \((3.30)\) and integrating out antisymmetric matrix \(A_{IJ}\). This term is purely topological (it is not coupled to \(K\) and depends only on the cohomology classes), so it does not affect the ordinary Hitchin story.
The map $\zeta^I \to *\zeta\zeta^I$ globally defines a decomposition of $\zeta^I$ on components of type $(1,0)$ and $(0,1)$ with respect to this complex structure. For example, the $(1,0)$ component of the solution to (3.43)

$$\zeta^{I,(1,0)} = \zeta^I + i *\zeta \zeta^I,$$

being a harmonic, must be a linear combination of the abelian differentials (3.15). This observation allows us to express the period matrix as a function of the cohomology classes:

$$\tau^{IJ} = \sum_K \left( \oint_{A_K} \zeta^{I,(1,0)} \right)^{-1} \oint_{B_J} \zeta^K_{(1,0)}.$$

In practice, however, we will have to solve the equation (3.43) in order to compute corresponding period matrix via (3.52). This should be as hard to do as to solve the Schottky problem. Furthermore, the complex structure (3.32), that we will get, will in general be different from the background complex structure on the abelian variety $T(\Pi)$, that we use to parameterize the cohomologies. Only if we start from a point on the Siegel space that corresponds to the Jacobian variety $T(\Pi) = \text{Jac}(\Sigma_g(\tau))$, the critical point of the functional (3.30) will give us the same complex structure on the world-sheet as on the target space. In this case harmonic maps (3.43) are promoted to the holomorphic maps, and (3.52) gives $\tau = \Pi$. The metric (3.35) then is the Bergmann metric (3.25). This will happen on a very rare occasion, since Jacobian locus has measure zero in the Siegel space. However, in general there is no obstruction for the map $\mathcal{H}_g \to \mathcal{M}_g$ defined by (3.52), since all almost complex structures on $\Sigma_g$ are integrable.

Formally, this is the end of the ordinary Hitchin story in two dimensions. However, a new interesting direction for study emerges if we allow the cohomology classes $[\zeta^I]$ to vary. In this case we will be dealing with the effective quantum mechanics of $g$ points on the Siegel space $\mathcal{H}_g$ defined by the functional (3.30).

Let us discuss the dependence of this functional on the "massless" degrees of freedom encoded in $\Pi$ and $K$. We choose some complex structure on $\Sigma_g$, which is equivalent to fixing the corresponding value of the field $K$. Modulo diffeomorphisms, it is defined by the corresponding period matrix $\tau$. This is equivalent to choosing a set of the abelian

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9 To be precise, we can also use any point that can be obtained from this one by the action of the modular group $\text{Sp}(2g, \mathbb{Z})_t$.

10 We assume that some marking for $\Sigma_g$ is fixed, and discuss the modular group $\text{Sp}(2g, \mathbb{Z})_w$ issues later.
differentials of the first kind (3.15). Then, we can express the cohomology class (3.46) as follows:

$$[ζ] = (Π − τ) \frac{1}{τ − τ} Ω − (Π − τ) \frac{1}{τ − τ} ω.$$ (3.53)

The background dependence on the complex structure on $Σ_g$ is encoded in the period matrix $τ$. Let $∗$ be the Hodge star-operator compatible with this complex structure: $\iota_K → ∗$. Using the identity

$$ζ^I ∧ ∗ζ^J = iζ^I ∧ ζ^J − \frac{i}{2} (ζ^I − i ∗ ζ^I) ∧ (ζ^J + i ∗ ζ^J)$$ (3.54)

and assuming that the classical equations of motion (3.43) are satisfied, we get the following expression for the functional (3.30):

$$L(Π, τ) = kgπ + \frac{kπ}{4} \text{Tr} \frac{1}{\text{Im}Π} (Π − τ) \frac{1}{\text{Im}τ} (Π − τ).$$ (3.55)

It is clear that this expression has a maximum at the point $Π = τ$ on the Siegel plane. Moreover, it is straightforward to check, using the symmetry of the matrices $Π$ and $τ$, and the positivity of $\text{Im}τ$, that

$$Π = τ$$ (3.56)

is the only solution\(^{11}\) of the corresponding equation of motion

$$\frac{∂L(Π, τ)}{∂Π} = 0.$$ (3.57)

Therefore, if we allow the cohomology classes in the theory with Lagrangian (3.31) to fluctuate, we find the following picture. For the generic period matrix $Π$, parameterizing the cohomology classes, solution to the equation of motion (3.43) for the ”massive” degrees of freedom (scalars $ξ^I$ in (3.42)) give harmonic maps $ζ : Σ_g → H_g$. Further extremization with respect to $Π$ picks up only the holomorphic maps that correspond to the Jacobian variety $\text{Jac}(Σ_g(τ)) \in H_g$ of the Riemann surface $Σ_g(τ)$ with the period matrix $τ$.

Another important feature of the expression (3.55) is that it is invariant under the diagonal subgroup of the group $\text{Sp}(2g, \mathbb{Z})_t × \text{Sp}(2g, \mathbb{Z})_{ws}$, which acts as follows

$$Π → Π’ = (aΠ + b)(cΠ + d)^{-1}$$
$$τ → τ’ = (aτ + b)(cτ + d)^{-1}.$$ (3.58)

\(^{11}\) There is also a nonphysical solution $Π = τ$, that does not lie on the Siegel upper-space, since $\text{Im}Π < 0$ in this case.
This can be easily checked using the basic relations for the Sp(2g, \mathbb{Z}) matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \):

\[
\begin{align*}
ad^T - bc^T &= a^T d - c^T b = \mathbb{I}_{g \times g} \\
ab^T - ba^T &= cd^T - dc^T = 0 \\
a^T c - c^T a &= b^T d - d^T b = 0.
\end{align*}
\] (3.59)

This, in particular, implies that after integrating \( e^{-L(\Pi, \tau)} \) over the Siegel space \( \mathcal{H}_g \) we will get a modular invariant function of \( \tau \). As we will see shortly, this gives an interesting topological quantum mechanical toy model on \( \mathcal{M}_g \). However, this toy model can hardly be interpreted from the entropic principle perspective. Therefore, further refinement of the functional (3.30) will be needed.

3.4. Towards the Quantum Theory

Consider the following partition function defined by the functional (3.30):

\[
Z_g(k, \tau) = \int \frac{D\Pi D\xi}{\det \text{Im}\Pi} e^{-L},
\] (3.60)

where the canonical modular invariant measure is used. It is assumed that we have fixed the value of the field \( K \), corresponding to the complex structure on a Riemann surface \( \Sigma_g(\tau) \) with the period matrix \( \tau \). After performing the Gaussian integral over \( \xi \) and using (3.55), we find

\[
Z_g(k, \tau) = e^{-k g \pi} \int D\Pi \exp \left( -\frac{k \pi}{4} \text{Tr} \frac{1}{\text{Im}\Pi} (\Pi - \tau) \frac{1}{\text{Im}\tau} (\Pi - \tau) \right).
\] (3.61)

The canonical modular invariant measure on \( \mathcal{H}_g \) is

\[
D\Pi = (\det \text{Im}\Pi)^{-g-1} \prod_{I \leq J} |d\Pi_{IJ}|^2
\] (3.62)

Since we know that the exponent in (3.61) has only one minimum (3.56), for large \( k \) we can study the perturbative expansion of the matrix integral (3.61) near \( \Pi = \tau \). It is convenient to describe the fluctuations by introducing the matrix \( H \) as follows

\[
\Pi = \tau - \text{Im}\tau H.
\] (3.63)

Then (3.61) becomes

\[
Z_g(k, \tau) = e^{-k g \pi} \int D\Pi \exp \left( -\frac{k \pi}{4} \text{Tr} \frac{H \Pi}{1 - \text{Im}H} \right).
\] (3.64)
As we discussed earlier, because of the invariance of the exponent \(3.55\) in \(3.61\) under the diagonal modular action \(3.58\), the partition function \(Z_g(k, \tau)\) is a modular invariant function of \(\tau\). Therefore, it descends to a function on the moduli space \(\mathcal{M}_g\) of genus \(g\) Riemann surfaces. Notice that expression \(3.64\) does not depend on \(\tau\) at all. Therefore, the modular invariant function that we will get is actually a constant \(12\).

The integral in \(3.64\), as a function of \(k\), can be expressed in terms of the 1-matrix model. Let us split the matrix \(H\) into its real and imaginary parts

\[
H = H_1 + iH_2.
\]

Then we can rewrite \(3.64\) as

\[
Z_g(k, \tau) = e^{-kg\pi} \int \mathcal{D}H_2 \int \mathcal{D}H_1 \exp \left( -\frac{k\pi}{4} \text{Tr} \left( H_1^2 + H_2^2 \right) \frac{1}{1 - H_2^2} \right),
\]

where we integrate over the symmetric matrices, and the matrix measures are defined in accordance with \(3.62\)

\[
\mathcal{D}H_1 = \prod_{I \leq J}^{g} dH_1^{IJ}, \quad \mathcal{D}H_2 = (\det H_2)^{-g-1} \prod_{I \leq J}^{g} dH_2^{IJ} \tag{3.67}
\]

The integral over \(H_1\) is Gaussian, and gives \((\frac{2}{k\pi})^{\frac{g(g+1)}{2}} \det(1 - H_2)^{g+1}\), up to a numerical constant. Then, \(3.66\) becomes

\[
Z_g(k, \tau) = (\frac{2}{k\pi})^{\frac{g(g+1)}{2}} e^{-kg\pi} \int \prod_{I \leq J}^{g} dH_2^{IJ} \left( \det \frac{1 - H_2}{H_2^2} \right)^{\frac{g+1}{2}} e^{-\frac{k\pi}{4} \text{Tr} \frac{H_2^2}{H_2}}, \tag{3.68}
\]

It is unclear, however, whether this expression has any interesting interpretation. In principle, we could use an alternative definition of the partition function, where the canonical measure is multiplied by some function of \(\Pi\), instead of \(3.60\). Morally, this is equivalent to adding corresponding topological terms (depending only on the cohomology classes \([\zeta]\), without coupling to \(\mathcal{K}\)) to the action \(3.30\). This will bring us into the realm of the matrix models. However, it looks like just a digression to \(0+1\) theory, and we are looking for the links with the higher dimensional theories.

\[12\] Here we ignored possible contribution from the boundary terms. On the boundary of the moduli space, when \(\det \text{Im} \Pi = 0\) and \(\det \text{Im} \tau = 0\), the integral \(3.61\) needs to be carefully regularised. This could result in a non-trivial \(\tau\)-dependence, but such effects are beyond the scope of this paper.
Thus, we suggest a natural generalization of the theory, that comes from the following observation. If we concentrate only on the massive modes in the functional (3.30), described by the free non-compact fields $\xi$, it looks very much like the action for the string propagating on a complex torus $\mathcal{T}_g^\mathbb{C} = \mathbb{C}^g/\mathbb{Z}^g \oplus \Pi \mathbb{Z}^g$ with the period matrix $\Pi$. Indeed, as we discussed earlier, the metric $(\text{Im} \Pi)^{-1}$ is a canonical metric induced on the torus from the flat Euclidian metric on $\mathbb{C}^g$. Therefore, from the stringy point of view the non-compact scalars $\xi : \Sigma_g \to \mathbb{C}^g$ can be promoted to the maps $\phi : \Sigma_g \to \mathcal{T}_g^\mathbb{C}$ with non-trivial winding numbers. Then, the cohomology class $[\zeta]$ in the combination $\zeta = [\zeta] + d\xi$ can be interpreted as a background abelian gauge field on the torus: $[\zeta] \to A$. Therefore, we want to substitute

$$\zeta \to d\phi + A$$

(3.69)

in the functional (3.30) and study resulting quantum theory. This modification will give us a new insight on the gauged WZW model for abelian varieties, coupled to the complex structure on $\Sigma_g$ in the specific way (3.30).

4. Gauged WZW Model for Abelian Varieties and the Hitchin Functional

In this section we argue that in order to use quantum theory based on the Hitchin functional for computing topological invariants, one has to incorporate stringy effects into it. In particular, the target space has to be compactified, and in accordance with that one has to consider topologically non-trivial maps $\Sigma_g \to \mathcal{T}_g^\mathbb{C}$, instead of $\Sigma_g \to \mathbb{C}^g$. Here $\mathcal{T}_g^\mathbb{C}$ is a complex $g$-torus, viewed as a principally polarized abelian variety. Moreover, the translation group of the target space has to be gauged, so that a two-dimensional stringy version of the Hitchin functional becomes the gauged WZW model with an abelian gauge group. It is well known that the partition function of this model, representing the number of conformal blocks in corresponding toroidal CFT, is independent of the complex structure on $\Sigma_g$. However, as we will see, in the Hitchin extension of the model the coupling to the two-dimensional gravity appears non-perturbatively via the instanton effects.

Before describing the topological extension of Hitchin functional in two dimensions, we will recall some general aspects of the gauged Wess-Zumino-(Novikov)-Witten model. This theory was extensively studied in the literature, see e.g. [48,49,50,51,52,53], therefore below we just summarize basic features of the model, following [51,52,53]. We will also use some facts about the abelian Chern-Simons theories with the gauge group $U(1)^d$, which has been discussed recently in great details in [56,57]. We will be particularly interested in the case $d = 2g$, and focus on viewing gauge group as a complex algebraic variety.
4.1. Review of the Gauged WZW Model

Let $G$ be a compact Lie group. The group $G$ acts on itself by left and right multiplication, which is convenient to view as the action of $G_L \times G_R$. For any subgroup $H_L \times H_R \subseteq G_L \times G_R$ consider a principal $H_L \times H_R$ bundle $X$ over Riemann surface $\Sigma_g$, with connection $(A_L, A_R)$. Let $J$ be a complex structure on $\Sigma_g$. As we discussed earlier, it determines the action of the Hodge $*$-operator, corresponding to the Riemann metric compatible with this complex structure, on 1-forms. Consider the functional:

$$I(A_L, A_R; g) = -\frac{1}{8\pi} \int_{\Sigma_g} \text{Tr}(g^{-1} d_A g \wedge * g^{-1} d_A g) - i \Gamma(g) +$$

$$+ \frac{i}{4\pi} \int_{\Sigma_g} \text{Tr}(A_L \wedge d g^{-1} + A_R \wedge g^{-1} d g + A_R \wedge g^{-1} A_L g),$$

(4.1)

where $d_A$ is the gauge-covariant extension of the exterior derivative:

$$d_A g = dg + A_L g - g A_R,$$

(4.2)

and $\Gamma(g)$ is the topological WZNW term:

$$\Gamma(g) = \frac{1}{12\pi} \int_{B: \partial B = \Sigma_g} \text{Tr}(g^{-1} d g)^{\wedge 3}.$$  

(4.3)

Here Tr is an invariant quadratic form on the Lie algebra $\text{Lie} G$ of the group $G$, normalized so that $\Gamma(g)$ is well-defined modulo $2\pi \mathbb{Z}$. The field $g$ is promoted from the map $g : \Sigma_g \to G$ to a section of the bundle $X \times_{H_L \times H_R} G$, where $G$ is understood as a trivial principal $G$ bundle over $\Sigma_g$. We are interested in the non-anomalous gauging, which is only possible if for all $t, t' \in \text{Lie}(H_L \times H_R)$

$$\text{Tr}_L t t' - \text{Tr}_R t t' = 0$$

(4.4)

where $\text{Tr}_L$ and $\text{Tr}_R$ are traces on $\text{Lie} H_L$ and $\text{Lie} H_R$. The standard choice for a non-abelian group is $H_L = G_L$ and $H_R = G_R$, with diagonal action $g \to h^{-1} g h$. This gives the $G/G$ gauged WZW model.

Consider the following propagator

$$\langle \Psi_{A_L}(\Sigma_g) | \Psi_{A_R}(\Sigma_g) \rangle = \int Dg e^{-k I(A_L, A_R; g)}.$$

(4.5)

\footnote{not to be confused with the genus $g$ of $\Sigma_g$.}
This should be compared to (1.2). In order to simplify the notations, we will often write $\Psi_\mathcal{A}$ instead of $\Psi_\mathcal{A}(\Sigma_g)$, when the dependence on the complex structure of $\Sigma_g$ is not essential. We will use the notation $\Psi_\mathcal{A}(\Sigma_g(\tau))$ if we want to stress dependence on the complex structure, parameterized by the period matrix $\tau$ of $\Sigma_g$.

By performing the change of variables $g \to g^{-1}$ in the functional integral (4.5), we find that the propagator has the necessary property

$$
\langle \Psi_\mathcal{A}_L | \Psi_\mathcal{A}_R \rangle = \langle \Psi_\mathcal{A}_R | \Psi_\mathcal{A}_L \rangle.
$$

Furthermore, using the Gaussian integration and the Polyakov-Wiegmann formula

$$
I(0, 0, gh) = I(0, 0, g) + I(0, 0, h) - \frac{1}{4\pi} \int_{\Sigma_g} \text{Tr} g^{-1} dg \wedge dh h^{-1},
$$

it is easy to check that propagator (4.5) satisfies the "gluing" condition (1.3). It is also straightforward to obtain the relation

$$
I(A^h_L, A^\tilde{h}_R; h^{-1}g\tilde{h}) = I(A_L, A_R; g) - i\Phi(A_L; h) + i\Phi(A_R; \tilde{h}),
$$

where the gauge transformed connection is

$$
A^h_L = h^{-1} A_L + h^{-1} dh,
\quad A^\tilde{h}_R = \tilde{h}^{-1} A_R + \tilde{h}^{-1} d\tilde{h},
$$

and the cocycles

$$
\Phi(A; h) = \frac{1}{4\pi} \int_{\Sigma_g} \text{Tr} A \wedge dh h^{-1} - \Gamma(h)
$$

are independent of the complex structure (metric) on $\Sigma_g$, and satisfy

$$
\Phi(A; hh') = \Phi(A^h; h') + \Phi(A; h).
$$

Infinitesimal form of these global gauge transformations, combined with a direct variation over $A^L_z$ in the functional integral (4.3), leads to the following set of the equations for the propagator

$$
\frac{D}{DA^L_z} \langle \Psi_\mathcal{A}_L | \Psi_\mathcal{A}_R \rangle = 0
$$

$$
\left( \frac{D}{DA^L_a} + \frac{i k}{4\pi} \epsilon^{ab} F^L_{ab} \right) \langle \Psi_\mathcal{A}_L | \Psi_\mathcal{A}_R \rangle = 0.
$$
where we introduced a connection \( \frac{D}{DA_L} \) on the line bundle \( L^k \) over the space of \( H_L \)-valued connections

\[
\frac{D}{DA_L} = \frac{\delta}{\delta A_L} + \frac{k}{4\pi} A_L \bar{z} \tag{4.13}
\]

and covariant derivatives on the principal \( H_L \) bundle over \( \Sigma_g \)

\[
D^L_a = \partial_a + [A_L, .], \tag{4.14}
\]

with the curvature form

\[
\mathcal{F}^L = [D^L, D^L] = dA_L + A_L \wedge A_L. \tag{4.15}
\]

Connections (4.13) obey canonical commutation relation

\[
\left[ \frac{D}{DA_L(z)}, \frac{D}{DA_L(w)} \right] = + \frac{k}{2\pi} \delta(z, w) \tag{4.16}
\]

The propagator (4.13) also satisfies a set of conjugate equations that describe its dependence on \( A_R \). These equations are obtained from (4.12)-(4.16) by a change of the indices and signs, according to

\[
L \leftrightarrow R, \quad +k \leftrightarrow -k. \tag{4.17}
\]

Geometrically, it means that the propagator \( \langle \Psi_{A_L} | \Psi_{A_R} \rangle \) is an (equivariant) holomorphic section of the line bundle \( \mathcal{L} = L^k \otimes L^{-k} \).

The quantum field theory with Lagrangian \( L = kI(A, A; g) \), \( k \in \mathbb{Z}_+ \) is conformal and gauge invariant and is called the \( G/G \) gauged WZW model for the non-abelian group \( G \) at level \( k \):

\[
Z_k(G/G; \Sigma_g) = \int \frac{DgDA}{\text{vol}(\text{Gauge})} e^{-kI(A, A; g)}. \tag{4.18}
\]

It is a two-dimensional sigma model with target space \( G \) gauged by a non-anomalous subgroup diag(\( G_R \times G_R \)). The partition function of the \( G/G \) gauged WZW model can be also written as

\[
Z_k(G/G; \Sigma_g) = \text{Tr}_A \langle \Psi_A(\Sigma_g) | \Psi_A(\Sigma_g) \rangle = \int \frac{DA_LDA_R}{\text{vol}^2(\text{Gauge})} \left| \langle \Psi_{A_L} | \Psi_{A_R} \rangle \right|^2, \tag{4.19}
\]

where we used (4.16). This should be compared to (4.14), with the identification \( Z_k(G/G; \Sigma_g) = Z_{\Sigma_g \times S^1} \). After performing the Gaussian integration, applying the Polyakov-Wiegmann formula and relation (4.8), we indeed get (4.18).
The gauged WZW functional (4.1) allows one to connect three-dimensional Chern-Simons theory and its dual two-dimensional rational conformal field theory in a simple and effective way. The partition function of the WZW model is

\[ Z_k(G; \Sigma_g) = \int Dg e^{-kI(0,0;g)}. \] (4.20)

The holomorphic factorization of the WZW model into the conformal blocks can be explained by observing [51] that (4.20) can also be written as

\[ Z_k(G; \Sigma_g) = \langle \psi_0(\Sigma_g) | \psi_0(\Sigma_g) \rangle = | \langle \psi_0(\Sigma_g) \rangle |^2 = \int \frac{DA}{\text{vol(Gauge)}} | \langle \psi_0 | \psi_A \rangle |^2. \] (4.21)

The WZW model is a rational conformal field theory if it is constructed from a finite number of conformal blocks [14]. In this case the conformal blocks of the WZW model are in one-to-one correspondence with the states in a Hilbert space [15], obtained from canonical quantization of the Chern-Simons theory on \( \Sigma_g \times \mathbb{R} \) [12,58]. A geometrical interpretation of this Hilbert space (achieved in the framework of the geometrical quantization [59]) is that it is a space \( V_{g,k}(G) \) of (equivariant) holomorphic sections of \( k \)-th power of the determinant line bundle \( \mathcal{L} \) over the moduli space of (semistable) holomorphic \( G_{\mathbb{C}} \)-connections on \( \Sigma_g \), which by the Narasimhan-Seshadri theorem is the same as the moduli space \( \mathcal{M}_G \) of flat connections on the principal \( G \)-bundle over \( \Sigma_g \). Thus, the Hilbert space is given exactly by the holomorphic sections that satisfy (4.12).

Let \( s_\gamma(A; \tau), \gamma = 1, \ldots, \text{dim} H^0(\mathcal{M}_G, \mathcal{L}^k) \) be an orthonormal basis in the space \( H^0(\mathcal{M}_G, \mathcal{L}^k) \) of holomorphic sections. Then we can write the propagator in (4.21) as

\[ \langle \psi_0(\Sigma_g(\tau)) | \psi_A(\Sigma_g(\tau)) \rangle = \sum_{\gamma=1}^{\text{dim} H^0(\mathcal{M}_G, \mathcal{L}^k)} \frac{F_\gamma(\tau)}{|F_\gamma(\tau)|^2} s_\gamma(A; \tau). \] (4.22)

The coefficients \( F_\gamma(\tau) \) in (4.22) are the conformal blocks of the WZW model. Of course, the dimensions of the space \( V_{g,k}(G) \) of conformal blocks and the space \( H^0(\mathcal{M}_G, \mathcal{L}^k) \) of holomorphic sections coincide. After plugging (4.22) into (4.21) and using the orthonormality of the basis \( s_\gamma(A; \tau) \), we obtain

\[ Z_k(G; \Sigma_g(\tau)) = \sum_{\gamma=1}^{\text{dim} V_{g,k}} |F_\gamma(\tau)|^2. \] (4.23)

---

14 There are many definitions of RCFT, but this one is the most convenient for our purposes.

15 For simplicity we are not considering marked points on \( \Sigma_g \).
The propagator (4.3) can now be written as
\[
\langle \Psi_{\mathcal{A}_L} (\Sigma_g (\tau)) | \Psi_{\mathcal{A}_R} (\Sigma_g (\tau)) \rangle = \dim V_{g,k} \sum_{\gamma=1}^{\dim V_{g,k}} s_\gamma (\mathcal{A}_L; \tau) s_\gamma (\mathcal{A}_R; \tau),
\] (4.24)
which is a unique solution to the equations (4.12) (and conjugate equations (4.17)) obeying the "gluing" condition (1.3). After plugging this into (4.19) we find
\[
Z_k (G/G; \Sigma_g) = \dim V_{g,k} (G) = \dim H^0 (\mathcal{M}_G, \mathcal{L}^k).
\] (4.25)
Therefore, the partition function of the gauged WZW model computes the dimension of the Chern-Simons Hilbert space \( V_{g,k} (G) \), which coincides with the number of conformal blocks in the corresponding RCFT. This can be viewed as an example of the universal index theorem (1.5) for the universal partition function (1.4) of \( \Sigma_g \times S^1 \), which in this case is equal to \( Z_k (G/G; \Sigma_g) \). The higher cohomology groups vanish since we are dealing with the integrable representations of RCFT.

Another way to explain (4.25) is to observe \([53,54]\) that the propagator (4.5) is exactly the free propagator of the Chern-Simons theory multiplied by the projector on the gauge invariant subspace, enforcing the Gauss law. In other words, equation (4.18) is equivalent to \( Z_k (G/G; \Sigma_g) = \text{Tr} \, 1 \), which yields (4.25).

From the CFT algebra viewpoint, the number of conformal blocks \( \dim V_{g,k} (G) \) is given by the E. Verlinde’s formula \([13]\). For example, when \( G = SU(2) \),
\[
\dim V_{g,k} (SU(2)) = \left( \frac{k + 2}{2} \right)^{g-1} \sum_{j=0}^{k} \sin^2 \frac{2g(j+1)}{k+2} \pi.
\] (4.26)
The gauged WZW model provides a constructive method of computing the dimension of the Verlinde algebra via the localization of the functional integral \([52,53]\).

4.2. Abelian Case

We are particularly interested in the case when \( G \) is an abelian group: \( G \sim U(1)^{2g} \). Moreover, we want to view it as an algebraic complex variety with a fixed complex structure. Therefore, we will describe \( G \) as a \( g \)-dimensional complex torus \( \mathcal{T} \), which is a principally polarized abelian variety: \( \mathcal{T} = \mathbb{C}^g / \Lambda, \Lambda = \mathbb{Z}^g \oplus \Pi \mathbb{Z}^g \). Sometimes we will use the notation \( \mathcal{T}(\Pi) \) to show the explicit dependence of \( \mathcal{T} \) on the defining period matrix \( \Pi \).
Let us first describe an analog of the functional (4.1) for the case of abelian group $\mathcal{T} \sim U(1)^{2g}$. In complex coordinates, it has the form:

$$I(A_L, A_R; \phi) = \frac{\pi G_{IJ}}{2} \int_{\Sigma_g} dA^I \wedge \star d\overline{A}^J + i(A^I_L + A^I_R) \wedge d\overline{\phi}^J + i(\overline{A}^I_L + \overline{A}^I_R) \wedge d\phi^J - i \frac{\pi G_{IJ}}{2} \int_{\Sigma_g} (A^I_L \wedge \overline{A}^J_R + \overline{A}^I_L \wedge A^J_R) - i \Gamma(\phi),$$

(4.27)

where $(A_L, A_R)$ are connections on a principal bundle $\mathcal{T}_L \times \mathcal{T}_R$ over $\Sigma_g$, and the scalar fields $\phi \sim \phi + \mathbb{Z} + \Pi \mathbb{Z}$ describe the maps $\Sigma_g \to \mathcal{T}$, which, after coupling to the gauge fields, are promoted to the corresponding sections, with the covariant derivative defined as

$$d_{\mathcal{A}}\phi^J = d\phi^J + A^I_L - A^I_R.$$

(4.28)

The role of the trace operator $\text{Tr}$ in (4.1) now is played by the matrix $G_{IJ} = (\frac{1}{\text{Im} \Pi})_{IJ}$, that defines a canonical metric on $\mathcal{T}$:

$$G(\phi, \phi) = \frac{1}{2} G_{IJ}(\phi^I \otimes \overline{\phi}^J + \overline{\phi}^I \otimes \phi^J).$$

(4.29)

The analog of the topological WZNW term is

$$\Gamma(\phi) = \pi G_{IJ} \int_{\Sigma_g} d\phi^I \wedge d\overline{\phi}^J,$$

(4.30)

which obey the corresponding Polyakov-Wiegmann formula

$$\Gamma(\phi + \psi) = \Gamma(\phi) + \Gamma(\psi) + \pi G_{IJ} \int_{\Sigma_g} d\phi^I \wedge d\overline{\psi}^J - d\overline{\phi}^I \wedge d\psi^J.$$  

(4.31)

Under the small gauge transformations

$$A^\psi_L = A_L + d\psi, \quad A^\psi_R = A_R + d\bar{\psi},$$ 

(4.32)

We should be careful with the expression (4.1) in the case of $U(1)^{2g}$ abelian group. Naively, it looks like we can take $2g$ copies of the WZNW term (4.3) for $U(1)$ group, but this expression vanishes for abelian group element $g = e^{i\phi}$. The analog of this term in the abelian group case is $\Gamma(\phi) = \pi G_{IJ} \int_{\Sigma_g} d\phi^I \wedge d\overline{\phi}^J$, which can be interpreted a $B$-field. It is crucial for a global identification with the corresponding three-dimensional Chern-Simons theory.
the change of the functional (4.27) depends on $A_{L,R}$ but not on $\phi$ or the complex structure of $\Sigma_g$:

$$I(A^\psi_L, A^{\bar{\psi}}_R; \phi + \bar{\psi} - \psi) - I(A_L, A_R; \phi) = i \frac{G_{IJ}}{2} \int_{\Sigma_g} A^I_L \wedge d\bar{\psi}^J + \bar{A}^I_R \wedge d\psi^J - A^I_L \wedge d\bar{\psi}^J - \bar{A}^I_R \wedge d\psi^J.$$

(4.33)

This should be compared to (4.8). There are certain restrictions [58, 60] on the possible choice of the period matrix $\Pi$ of the torus $T(\Pi)$. First, in order for the functional (4.27) to be a well defined modulo $2\pi i \mathbb{Z}$, the lattice $\Lambda = \mathbb{Z} \oplus \Pi \mathbb{Z}$ has to be integral. Second, modular invariance requires $\Lambda$ to be even lattice. Therefore,

$$\mathbb{Z}^g \oplus \Pi \mathbb{Z}^g \in \Gamma_{2g}^\mathbb{Z}.$$

(4.34)

where $\Gamma_{2g}^\mathbb{Z}$ denotes the moduli space of even integral $2g$-dimensional lattices. The dual conformal field theory in this case is rational.

Let us define the propagator as

$$\langle \Psi_{A_L} | \Psi_{A_R} \rangle = \int D\phi e^{-kI(A_L, A_R; \phi)}.$$

(4.35)

Notice that interactions in (4.27) are such that $\phi$ is coupled only to $A_L^L$, via the term $d\phi^I \wedge (i - *) \bar{A}^I_L$, and to $A_R^L$, via the term $d\phi^I \wedge (i + *) \bar{A}^I_R$. Moreover, ”left” and ”right” gauge fields interact only via the coupling $A^I_{Rz} \wedge \bar{A}^I_{Lz}$, and its complex conjugate. This observation, combined with the Ward identity, that follows from (4.33), leads to the following set of equations [7], which the propagator obeys:

$$\frac{D}{D A^I_{Lz}} \langle \Psi_{A_L} | \Psi_{A_R} \rangle = 0$$

$$\frac{D}{D A^I_{Rz}} \langle \Psi_{A_L} | \Psi_{A_R} \rangle = 0$$

$$\left( \partial_a \frac{D}{D A^I_{La}} + i k \pi G_{IJ} \epsilon^{ab} \partial_a A^J_{Lb} \right) \langle \Psi_{A_L} | \Psi_{A_R} \rangle = 0$$

$$\left( \partial_a \frac{D}{D A^I_{Ra}} - i k \pi G_{IJ} \epsilon^{ab} \partial_a A^J_{Rb} \right) \langle \Psi_{A_L} | \Psi_{A_R} \rangle = 0$$

(4.36)

\[\text{We treat } A_{L,R} \text{ and } \bar{A}_{L,R} \text{ as independent variables, and there is also a corresponding set of equations with } A_{L,R} \rightarrow \bar{A}_{L,R}.\]
Here we introduced a connection
\[
\frac{D}{D A^I_{Lz}} = \frac{\delta}{\delta A^I_{Lz}} + k \pi G_{IJ} A^J_{Lz}, \quad \frac{D}{D A^I_{L\tau}} = \frac{\delta}{\delta A^I_{L\tau}} - k \pi G_{IJ} A^J_{L\tau},
\]
\[
\frac{D}{D A^I_{Rz}} = \frac{\delta}{\delta A^I_{Rz}} - k \pi G_{IJ} A^J_{Rz}, \quad \frac{D}{D A^I_{R\tau}} = \frac{\delta}{\delta A^I_{R\tau}} + k \pi G_{IJ} A^J_{R\tau},
\]
(4.37)
on the line bundle $\mathcal{L}^k \times \mathcal{L}^{-k}$ over the space $A$ of $T_L \times T_R$-valued connections on $\Sigma_g$.

The geometrical interpretation of the equations (4.36) is very simple. We pick a standard complex structure on the space $A$ of connections induced from the complex structure on $\Sigma_g$. In this complex structure, $A^I_{Lz}$ and $A^I_{Rz}$ are holomorphic, and $A^I_{L\tau}$ and $A^I_{R\tau}$ are antiholomorphic. Then the propagator $\langle \Psi_{A_L} | \Psi_{A_R} \rangle$ is a holomorphic section of the line bundle $\mathcal{L} = \mathcal{L}^k \otimes \mathcal{L}^{-k}$, equivariant with respect to the action of the abelian group $T_L \times T_R$.

It is well-known (see, e.g., [59,61]) that the basis in the corresponding space $H^0(M_T, \mathcal{L}^k)$ of the gauge invariant holomorphic sections of $\mathcal{L}^k$ is provided by the level $k$ Narain-Siegel theta-functions $\Theta_\gamma(A; \tau|\Lambda, k)$, associated with the lattice $\Lambda$, that defines the torus $T = \mathbb{C}^g/\Lambda$. We will not need an explicit expression for $\Theta_\gamma(A, \tau|\Lambda, k)$ (it can be found, for example, in [56,57]). What is important for us is that the linear independent Narain-Siegel theta-functions are labelled by the index $\gamma \in (\Lambda^*/k\Lambda)^{\otimes g}$, where $\Lambda^*$ is the dual lattice. From the viewpoint of the three-dimensional abelian Chern-Simons theory, these theta-functions are exactly the wave-functions: $\Psi_\gamma(A; \tau) \sim \Theta_\gamma(A; \tau|\Lambda, k)$. Therefore, the dimension of the corresponding Hilbert space is
\[
\dim \text{Hilb}_{CS}(\Lambda, k) = |\Lambda^*/k\Lambda|^{g}. \quad (4.38)
\]

We can repeat the steps that we did in the non-abelian case, and connect abelian Chern-Simons theory and its dual CFT via the functional (4.27) and the propagator (4.35). The property of the functional (4.27)
\[
I(A_L, A_R; \phi) = I(A_R, A_L; -\phi)
\]
(4.39)
guarantees that the propagator (4.35) is hermitian:
\[
\langle \Psi_{A_L} | \Psi_{A_R} \rangle = \langle \Psi_{A_R} | \Psi_{A_L} \rangle,
\]
(4.40)
since we can always change the variables $\phi \to -\phi$ in the functional integral (4.40). Moreover, it is straightforward to show that the propagator obeys the gluing condition (1.3)

$$\langle \Psi_{A_L} | \Psi_{A_R} \rangle = \int \frac{DA}{\text{vol(Gauge)}} \langle \Psi_{A_L} | \Psi_A \rangle \langle \Psi_A | \Psi_{A_R} \rangle,$$

(4.41)

by performing the Gaussian integral over $A$, using the Polyakov-Wiegmann formula, and the fact that $\int D\phi = \text{vol(Gauge)}$. This allows us to write down the following expression for the propagator in terms of the Narain-Siegel theta-functions:

$$\langle \Psi_{A_L} | \Psi_{A_R} \rangle = \sum_{\gamma \in (\Lambda^*/k\Lambda)^g} \Theta_\gamma(A_L; \tau|\Lambda, k) \Theta_{\gamma}(A_R; \tau|\Lambda, k)$$

(4.42)

The partition function of the gauged WZW model for abelian group $\mathcal{T}$ at level $k$ is defined as

$$Z_k(\mathcal{T}/\mathcal{T}; \Sigma_g) = \int \frac{D\phi DA}{\text{vol(Gauge)}} e^{-kI(A, A; \phi)} = \text{Tr}_A \langle \Psi_A | \Psi_A \rangle$$

(4.43)

Using (4.42) and orthonormality of the Narain-Siegel theta-functions

$$\int \frac{DA}{\text{vol(Gauge)}} \Theta_\gamma(A; \tau|\Lambda, k) \Theta_{\gamma'}(A; \tau|\Lambda, k) = \delta_{\gamma\gamma'},$$

(4.44)

it is easy to see that the partition function (4.43) indeed computes the dimension of the Chern-Simons theory Hilbert space

$$Z_k(\mathcal{T}/\mathcal{T}; \Sigma_g) = |\Lambda^*/k\Lambda|^g.$$

(4.45)

### 4.3. Hitchin Extension of the Abelian GWZW Model

Now we are ready to discuss the Hitchin extension of the gauged WZW functional (4.27) for abelian group. We want to introduce non-trivial coupling to the complex structure on $\Sigma_g$ by using the operator $\imath_K$ instead of the Hodge $*$-operator, and adding the term $i\lambda \text{tr}(K^2 + \mathbb{1})$ to the action. This leads to the following functional

$$I(A_L, A_R; \phi|\lambda, K) = \frac{G_{IJ} \pi}{4} \int_{\Sigma_g} dA^I \phi^J + \imath_K d_{A^I} \bar{\phi}^J + d_{A^I} \bar{\phi}^J \wedge \imath_K dA^I - 4\sqrt{-1} d\phi^I \wedge d\bar{\phi}^I +$$

$$+ \frac{\sqrt{-1} \pi}{2} G_{IJ} \int_{\Sigma_g} (A^I_L + A^I_R) \wedge d\bar{\phi}^J + (\bar{A}_L^I + \bar{A}_R^I) \wedge d\phi^J -$$

$$- \frac{\sqrt{-1} \pi}{2} G_{IJ} \int_{\Sigma_g} (A^I_L \wedge \bar{A}_R^I + A^I_R \wedge \bar{A}_L^I) - \frac{\sqrt{-1} \pi}{4} \int_{\Sigma_g} \lambda \text{tr}(K^2 + \mathbb{1}).$$

(4.46)
The Hitchin extension of the propagator (4.40) formally is given by
\[
\langle \Psi(A_L|\lambda, \mathcal{K}) | \Psi(A_R|\lambda, \mathcal{K}) \rangle = \int D\phi e^{-kI(A_L, A_R; \phi|\lambda, \mathcal{K})}.\] (4.47)

However, this expression can be interpreted as a propagator only for the ”on-shell” values of the field \(\mathcal{K}\), such that
\[
\mathcal{K}^2 = -\text{Id}.\] (4.48)

In this case \(\mathcal{K}\) defines a complex structure on \(\Sigma_g\), and we can glue together two propagators defined in the same complex structure, according to the gluing rule (4.41). Moreover, if we define a ”partition function” as
\[
Z_k(T|\lambda, \mathcal{K}) = \int \frac{D\phi D\mathcal{A}}{\text{vol(Gauge)}} e^{-kI(A, A; \phi|\lambda, \mathcal{K})},\] (4.49)

then formally we can write
\[
\int D\lambda \ Z_k(T|\lambda, \mathcal{K}) = |\Lambda^*/k\Lambda|^g \delta(\text{tr}\mathcal{K}^2 + 2).\] (4.50)

The meaning of this expression is that perturbatively the Hitchin extension (4.46) is equivalent to the ordinary gauged WZW model (4.27). There is no non-trivial \(\mathcal{K}\) dependence in (4.50), and after performing the integration over \(D\mathcal{K}\) we will just get some multiplicative constant, depending on \(g\). This should not be surprising. After all, the gauged WZW model computes the number of conformal blocks (the dimension of the corresponding Hilbert space), and this number does not depend on the choice of the complex structure on \(\Sigma_g\), which is controlled by the field \(\mathcal{K}\).

However, the very new feature of the Hitchin extension is that dependence on \(\mathcal{K}\) can be restored non-perturbatively. Indeed, if the action \(kI(A, A; \phi|\lambda, \mathcal{K})\) has non-trivial critical point, we have to do expansion around this point in the functional integral. In this case, the answer will depend on the value \(\mathcal{K}_{\text{min}}\) of the complex structure tensor at the minimal point of the action.

4.4. Attractor Points and Complex Multiplication

Therefore, we have to study the critical points of the functional
\[
I(A, A; \phi|\lambda, \mathcal{K}) = \frac{G_{I,J} \pi}{4} \int_{\Sigma_g} (d\phi^I \wedge \iota_\mathcal{K} d\phi^J + d\phi^I \wedge \iota_\mathcal{K} d\phi^J - 4\sqrt{-1}d\phi^I \wedge d\phi^J) + \\
+ \sqrt{-1} \pi G_{I,J} \int_{\Sigma_g} (A^I \wedge d\phi^J + \bar{A}^I \wedge d\phi^J) - \frac{\sqrt{-1} \pi}{4} \int_{\Sigma_g} \lambda \text{tr} (\mathcal{K}^2 + \text{Id}).\] (4.51)
Let us recall that in the functional integral (4.49) we integrate over the exact parts $\varphi^I$ of the fields $d\phi^I = [d\phi^I] + d\varphi^I$ and sum over non-trivial maps $[d\phi^I] \in H^1(\Sigma_g, \Lambda)$. After dividing by the gauge transformations, we need to integrate only over the space of gauge inequivalent flat gauge fields $A^I \in H^1(\Sigma_g, \mathbb{C}^g)/H^1(\Sigma_g, \Lambda)$. Therefore, in the functional (4.51) the exact part of $d\phi$ couples only to the term $i_K d\bar{\varphi}$. By varying $\varphi$, we get a classical equation of motion, analogous to (3.43):

$$d i_K d\phi^I = 0.$$ (4.52)

After solving the constraint $\text{tr} K^2 = -2$, imposed by the Lagrange multiplier $\lambda$, the equations of motion for $K$ give

$$K^a_b = \frac{G_{IJ}}{2\sqrt{\det|h|}} (d\phi^I_b d\bar{\phi}^J_c + d\bar{\phi}^I_c d\phi^J_b) e^c a,$$ (4.53)

where $h$ is the metric induced on $\Sigma_g$. If we recall that $G_{IJ} = \left(\frac{1}{\text{Im}\Pi}\right)_{IJ}$, this metric takes the form

$$h_{ab} = \left(\frac{1}{2\text{Im}\Pi}\right)_{IJ} (d\phi^I_{a} d\bar{\phi}^J_{b} + d\bar{\phi}^I_{b} d\phi^J_{a}).$$ (4.54)

Expressions (4.53) and (4.54) should be compared with (3.32) and (3.35).

For generic choice of the matrix $\Pi \in \mathcal{H}_g$ and cohomology vectors $[d\phi^I] \in H^1(\Sigma_g, \Lambda)$ expression (4.53) for the complex structure can be singular at some points on $\Sigma_g$. Those are the points where the determinant of the metric (4.54) vanishes. However, it is easy to find a family of non-singular solutions (4.53). Let us compare (4.54) with the expression (3.25) for the canonical Bergmann metric on the Riemann surface $\Sigma_g(\tau)$

$$h^B_{ab} = \left(\frac{1}{2\text{Im}\tau}\right)_{IJ} (\omega^I_a \omega^J_b + \bar{\omega}^I_b \omega^J_a).$$ (4.55)

The complex structure on $\Sigma_g(\tau)$ is defined by the period matrix $\tau$, and is such that the differentials $\omega^I$ are holomorphic. If we set

$$d\phi^I = \omega^I,$$ (4.56)

18 For example, the metric $h_{zz} = |\omega^1_z|^2$ vanishes at zeroes of the abelian differential $\omega^1$. Strictly speaking, even in this singular case it is possible to define complex structure globally on $\Sigma_g$ via appropriate conformal transformation and analytical continuation, but the resulting complex structure will not be given by (4.53).
and choose the torus $\mathcal{T}$, for which
\[ \Pi = \tau, \] (4.57)
then the metric (4.54) coincides with the Bergmann metric: $h = h^B$, and therefore the complex structure defined by $\mathcal{K}$ coincides with the complex structure defined by $\tau$.

In order to parameterize general non-singular complex structure solutions (4.53) we proceed as follows. Suppose that some $\mathcal{K}$, given by (4.53), provides a globally well-defined complex structure on $\Sigma_g$. All complex structures on $\Sigma_g$ are parameterized by the period matrices. Therefore, there is the period matrix $\tau$ that defines the same complex structure on $\Sigma_g$ as $\mathcal{K}$. Then $\mathcal{K}$ must be equal to the corresponding Bergmann complex structure: $\mathcal{K} = \mathcal{K}^B(\tau)$, which is a canonical complex structure compatible with the metric $h^B$ (4.54).

This gives
\[ \sqrt{\det h^B} \left( \frac{1}{\text{Im}\Pi} \right)_{IJ} (d\phi_b^I d\bar{\phi}_c^J + d\bar{\phi}_c^I d\phi_b^J) = \sqrt{\det h} \left( \frac{1}{\text{Im} \tau} \right)_{IJ} (\omega_b^I \bar{\omega}_c^J + \bar{\omega}_c^J \omega_b^I). \] (4.58)
Since the 1-forms $d\phi$ are harmonic, we can express them in terms of the abelian differentials:
\[ d\phi = M\omega + N\overline{\omega}, \] (4.59)
where $M$ and $N$ are certain $g \times g$ complex matrices representing non-trivial mappings $\Sigma_g(\tau) \to \mathcal{T}(\Pi)$. If $A$ and $B$ are the period matrices of the 1-forms:
\[ A^{IJ} = \oint_{A_J} d\phi^I, \quad B^{IJ} = \oint_{B_J} d\phi^I, \] (4.60)
then
\[ M = (B - A\tau) \frac{1}{\tau - \overline{\tau}}, \quad N = -(B - A\tau) \frac{1}{\tau - \overline{\tau}}. \] (4.61)
We stress that (4.59) is an exact expression for the 1-forms $d\phi$, that solves classical equations of motion, as opposed to (3.53), that captures only the cohomology class. Once the cohomology class $[d\phi]$ of the 1-forms is fixed, the exact part $d\phi - [d\phi]$ is uniquely determined by (4.52), which states that $d\phi$ is a linear combination of the harmonic representatives.

Combining the $\omega_b^I \omega_c^J$ terms in (4.58), we find
\[ M^T \frac{1}{\text{Im} \Pi} N = 0, \] (4.62)
which means that either $N = 0$ or $M = 0$, since $\text{Im} \Pi$ is non-degenerate. The terms of the form $\omega_b^I \overline{\omega}_c^J$ give
\[ \text{Tr} \left( \omega_b^T M^T \frac{1}{\text{Im} \Pi} M \overline{\omega}_c \right) + \text{Tr} \left( \omega_b^T N^T \frac{1}{\text{Im} \Pi} N \omega_c \right) = \sqrt{\frac{\det h^B}{\det h}} \text{Tr} \left( \omega_b^T \frac{1}{\text{Im} \tau} \overline{\omega}_c \right). \] (4.63)
Thus, the only way to satisfy (4.62)-(4.63) is to set \( N = 0 \), and
\[
M^T \frac{1}{\text{Im}\Pi} M = \frac{1}{\text{Im}\tau}.
\] (4.64)

This equation implies that \( \det M \neq 0 \), since the matrices \( \text{Im}\Pi \) and \( \text{Im}\tau \) are not degenerate. Moreover, in this case we also have \( h = h^B \). From (4.61) we see, that the condition \( N = 0 \) is equivalent to
\[
B = A\tau,
\] (4.65)
so that
\[
M = A.
\] (4.66)

The columns of the matrix \( A \) (4.61) are the vectors of the lattice \( \Lambda = \mathbb{Z}^g \oplus \Pi\mathbb{Z}^g \). We can write it as \( A = P_Z + \Pi Q_Z \), where \( P_Z \) and \( Q_Z \) are integral \( g \times g \) matrices. Therefore, the complex structures \( K_* \) corresponding to the critical points of the functional (4.51) can be parameterized by the period matrices \( \tau \), obeying
\[
\frac{1}{\text{Im}\tau} = \left( P_Z^T + Q_Z^T \Pi \right) \frac{1}{\text{Im}\Pi} \left( P_Z + \Pi Q_Z \right).
\] (4.67)

This equation puts additional constraint on the period matrix, which according to (4.63) can be written as \( \tau = A^{-1} B \). Since the columns of the matrix \( B \) are also the vectors of the lattice \( \Lambda \), we can write it as \( B = P'_Z + \Pi Q'_Z \), where \( P'_Z \) and \( Q'_Z \) are integral \( g \times g \) matrices. Then (4.65) takes the form
\[
\tau = \frac{1}{P'_Z + \Pi Q'_Z} \left( P'_Z + \Pi Q'_Z \right).
\] (4.68)

Equations (4.67)-(4.68) can be interpreted as a two-dimensional analog of the attractor equations [29,30,31,32]. In the \( 3 \)-dimensional case attractor equations define complex structure of the Calabi-Yau threefold in terms of the integral cohomology class, given by the magnetic and electric charges of the associated black hole. In \( 1 \)-dimensional case equations (4.67)-(4.68) define the complex structure of the Riemann surface \( \Sigma_g(\tau) \) in terms of the integral matrices \( P_Z, Q_Z, P'_Z, \) and \( Q'_Z \).

The critical points (4.53) minimize the value of the functional (4.51), viewed as a function on the moduli space of complex structures. Indeed, the second variation of the functional at the critical point is
\[
\frac{\delta^2 I(A, A; \phi|\lambda, \mathcal{K})}{\delta \mathcal{K}^2} \bigg|_* = -\frac{i\pi}{2} \lambda_* = \frac{\pi}{2} \sqrt{\det\|h^B\|} > 0.
\] (4.69)
If we perform functional integration over $\mathcal{D}\mathcal{K}$ with the weight $e^{-kI(A,A;\phi|\lambda,K)}$, the main contribution will come from these critical points. Therefore, from the point of view of the corresponding quantum mechanical problem on the moduli space of complex structures, these points are attractive. We will denote a set of these points on the moduli space of genus $g$ Riemann surfaces as $\text{Attr}_g$.

For the particular choice $P_Z = Q'_Z = \mathbb{I}$, and $Q_Z = P'_Z = 0$ the attractor equations (4.67)-(4.68) reduce to (4.56)-(4.57). This allows us to generate all solutions to (4.67)-(4.68) from (4.56)-(4.57) by an appropriate symplectic transformation. Indeed, a compatibility of (4.67) and (4.68), combined with the symmetry requirement $\tau^T = \tau$ imposes certain restrictions on the possible choice of the integer matrices. After some algebra one finds that these restrictions are equivalent to relations (3.59) for the symplectic group, with the identification: $a = Q'_TZ$, $b = P'_TZ$, $c = Q_TZ$, $d = P_TZ$. Therefore,

$$\begin{pmatrix} Q'_TZ & P'_TZ \\ Q_TZ & P_TZ \end{pmatrix} \in \text{Sp}(2g,\mathbb{Z}), \quad (4.70)$$

and all solutions to (4.67)-(4.68) for a given $\Pi$ correspond to the same Riemann surface, with a different choices of the symplectic basis. To summarize, we find that the critical points of the functional (4.51) on the moduli space of complex structures $\mathcal{M}_g$ are given by the intersection of the Jacobian locus $\text{Jac}(\Sigma_g) \subset \mathcal{H}_g$ with a set $\Gamma_{2g}^2Z$ of abelian varieties generated by the even integer $2g$-dimensional lattices:

$$\text{Attr}_g = \text{Jac}(\Sigma_g) \cap \Gamma_{2g}^2Z. \quad (4.71)$$

There is another interesting property of the critical points defined by (1.65) and (1.67): the corresponding Riemann surface $\Sigma_g(\tau)$ admits a non-trivial endomorphism, known as the complex multiplication (CM). The notion of the complex multiplication appears in the study of black hole attractors and rational conformal field theories (see, e.g., [32,62] for more details and references). In particular, it was shown in [28], that the critical attractor points of the Calabi-Yau holomorphic volume functional (2.6) (which is morally the higher-dimensional analog of the functional (4.51)) lead to the abelian varieties (associated with the coupling constant matrix) admitting complex multiplication.

In order to illustrate the CM-property of the critical points (4.53), we will use a simple criterion [62] that says that an abelian variety defined by the period matrix $\tau$ admits complex multiplication, if $\tau$ obeys a second order matrix equation

$$\tau n\tau + \tau m - n'\tau - m' = 0, \quad (4.72)$$
for some integer $g \times g$ matrices $m, n, m', n'$, with rank$(n) = g$. It is straightforward to show that any solution to the attractor equations (4.67)-(4.68) also obeys the CM-type equation (1.72). After using (4.61), the equation (4.64) takes the form

$$ (B^T - \tau A^T) \frac{1}{\text{Im} \Pi} (B - A\tau) = 4\text{Im} \tau. \quad (4.73) $$

By substituting $\tau = \tau - 2i\text{Im} \tau$ into the real part of (4.73), we find

$$ \tau \text{Re} \left( A^T \frac{1}{\text{Im} \Pi} A \right) \tau - \tau \text{Re} \left( A^T \frac{1}{\text{Im} \Pi} B \right) - \text{Re} \left( B^T \frac{1}{\text{Im} \Pi} A \right) \tau + \text{Re} \left( B^T \frac{1}{\text{Im} \Pi} B \right) = 0, \quad (4.74) $$

where we used (4.67) and the attractor equation (1.67) in the form $A^T \frac{1}{\text{Im} \Pi} A = \frac{1}{\text{Im} \tau}$. Let us now recall that $\Lambda = \mathbb{Z}^g \oplus \Pi \mathbb{Z}^g$ is an even integral lattice. This guarantees that the corresponding three-dimensional abelian Chern-Simons theory is well-defined, and the associated two-dimensional conformal field theory is rational. Therefore, for any two vectors $a, b \in \Lambda$ a scalar product, defined as

$$ (a, b) = \text{Re} \left( a^I (\text{Im} \Pi)^{-1} j_B^j \right) \quad (4.75) $$

is an integer, and the norm of any vector of the lattice $\Lambda$ is an even number:

$$ (a, b) \in \mathbb{Z}, \ a \neq b; \quad (a, a) \in 2\mathbb{Z}. \quad (4.76) $$

We can write the period matrices (4.60) in terms of the lattice vectors, as $A = (a_1, \ldots, a_g)$, and $B = (b_1, \ldots, b_g)$. Then all elements of the matrices

$$ n = \text{Re} \left( A^T \frac{1}{\text{Im} \Pi} A \right), \quad m = -\text{Re} \left( A^T \frac{1}{\text{Im} \Pi} B \right), $$

$$ n' = \text{Re} \left( B^T \frac{1}{\text{Im} \Pi} A \right), \quad m' = -\text{Re} \left( B^T \frac{1}{\text{Im} \Pi} B \right), \quad (4.77) $$

according to (4.76) are integral, and thus equation (4.74) is indeed of the CM-type (1.72). This fact should not be surprising. As was proven recently in [63], the complex multiplication on abelian variety is equivalent to the existence of the rational Kähler metric. This is, of course, true in our case, since we consider abelian varieties generated by the even integer lattices. The fact that the associated CFT in this case is rational, fits nicely with the observations of Gukov and Vafa [52].
5. Quantization and the Partition Function

In this section, we define a generating function for the dimension of the space of conformal blocks in a family of toroidal $c = 2g$ RCFTs on a genus $g$ Riemann surface. We use Hitchin construction to introduce coupling to two-dimensional gravity. The universal index theorem in the context of the Chern-Simons/CFT correspondence is a computation of the number of conformal blocks via the gauged WZW model. After coupling to two-dimensional gravity it gives, according to the entropic principle, the effective entropy functional on the moduli space of complex structures. The functional is peaked at the attractor points. We will be interested in the fluctuation of the complex geometry around the gravitational instanton solution corresponding to these points. It gives some version of the two-dimensional Kodaira-Spencer theory of gravity.

5.1. Generating Function for the Number of Conformal Blocks and Attractors

We learned that the Hitchin extension $I(A, A; \phi|\lambda, K)$ of the abelian gauged WZW model gives rise to effective potential on the moduli space of the complex structures, whose critical points correspond to Jacobians of Riemann surfaces admitting complex multiplication. In order to describe all such points we have to sum over all even integer lattices \( \Lambda(\Pi) = \mathbb{Z}^g \oplus \Pi \mathbb{Z}^g : \Lambda(\Pi) \in \Gamma_{2g}^2 \mathbb{Z}^2 \). This discrete sum is basically a sum over the moduli space of the toroidal rational two-dimensional conformal field theories:

\[
Z_{g,k}(\Theta|\lambda, K) = \sum_{\Pi: \Lambda(\Pi) \in \Gamma_{2g}^2 \mathbb{Z}} e^{i \text{Tr} \Theta \Pi} \int \frac{\mathcal{D} \phi \mathcal{D} A}{\text{vol}(\text{Gauge})} e^{-k I(A, A; \phi|\lambda, K)}. \tag{5.1}
\]

We perform a sum with the weight factor \( e^{i \text{Tr} \Theta \Pi} \), where \( \Theta \) is an auxiliary symmetric matrix, so that \( Z_k(\Theta|\lambda, K) \) can be interpreted as a generating function capturing all the relevant information about the theory. In principle, we can go one step further and sum over the theories at different levels \( k \) as well:

\[
Z_g(q, \Theta|\lambda, K) = \sum_{k=1}^{\infty} q^k Z_{g,k}(\Theta|\lambda, K). \tag{5.2}
\]

\[\text{To be more precise, the moduli space of a } 2g\text{-dimensional torus is } \frac{\text{SO}(2g, 2g)}{\text{SO}(2g) \times \text{SO}(2g) \times \text{SO}(2g, 2g, \mathbb{Z})}. \]

We are interested in the subspace of the complex algebraic tori \( \frac{\text{Sp}(2g)}{U(g) \times \text{Sp}(2g, \mathbb{Z})} \) in this moduli space, and moreover, consider only the tori generated by the even integral lattices.
If we compute (5.2) at any classical value $K^*_2 = -\text{Id}$, the functional $I(A, A; \phi|\lambda, K_*)$ describes the ordinary gauged WZW model, and therefore (5.2) becomes a generating function\footnote{The simplest example of such generating function corresponds to the $U(1)_k$ theory, describing the free boson at $k$ times the self-dual radius. The holomorphic wave-functions of the dual Chern-Simons theory are the level $k$ Jacobi theta-functions. The dimension of the corresponding Hilbert space is $k^g$. Therefore, on this case $Z_g(q) = \sum_{k=1}^{\infty} q^k k^g = Li_{-g}(q)$.} for the number of conformal blocks in $c = 2g$ RCFTs

$$Z_k(q, \Theta|\lambda, K_*) = \sum_k \sum_{\Pi: \Lambda(\Pi) \in \Gamma_{2g}^2} q^k e^{i\text{Tr} \Theta \Pi} |\Lambda^*(\Pi)/k \Lambda(\Pi)|^g. \quad (5.3)$$

Let us discuss the quantum aspects of the theory on the moduli space of the complex structures that arises after averaging the generating function (5.3) over the fluctuations of the fields $K$ and $\lambda$, according to

$$Z_{g,k}(\Theta) = \int \frac{D\mathcal{K} D\lambda}{\text{vol}(\text{Diff}(\Sigma_g))} Z_{g,k}(\Theta|\lambda, K). \quad (5.4)$$

The measure for the vector-valued 1-form $K$ can be defined as follows. Let us first notice that for any 1-form $\theta$ on $\Sigma_g$

$$\theta \wedge i_{\mathcal{K} - \text{tr} \mathcal{K}} \theta = \theta \wedge i_{\mathcal{K}} \theta - (\text{tr} \mathcal{K}) \theta \wedge i_{\text{Id}} \theta = \theta \wedge i_{\mathcal{K}} \theta, \quad (5.5)$$

since $\theta \wedge i_{\text{Id}} \theta = \theta \wedge \theta = 0$. Therefore, $\text{tr} \mathcal{K}$ does not couple to the scalars $\phi$ in the action (4.51). This is the reason why we can integrate only over the traceless tensor fields $\text{tr} \mathcal{K} = 0$. In this case the measure on the space of the fields $\mathcal{K}$ is induced from the following metric:

$$\|\delta \mathcal{K}\|^2 = \int_{\Sigma_g} d^2 x \left(\text{tr} \mathcal{K}^2\right)^{-\frac{g}{2}} \text{tr} \left(i_{\mathcal{K}} \delta \mathcal{K}\right)^2. \quad (5.6)$$

In order to motivate this choice of the metric, we note that on-shell, $\mathcal{K}$ is linearly related to the Riemann metric $h$ on $\Sigma_g$: $\mathcal{K}^a_b \sim h_{bc} \epsilon^{ca}$. Traceless vector-valued 1-form $\mathcal{K}^a_b$ contains 3 local degrees of freedom, the same amount as the symmetric metric tensor $h_{ab}$. However, $\mathcal{K}$ and $h$ scale differently under the conformal transformations. This can be taken care of by introducing a conformal factor $\sigma$, such that

$$\mathcal{K}^a_b = \frac{h_{bc}}{\sqrt{\text{det} h}} \epsilon^\sigma \epsilon^{ca}. \quad (5.7)$$
Then it is easy to see that the metric (5.6) for the variations of \( K \) that does not involve change of \( \text{tr}K^2 \), coincides with the standard metric \([64]\) on the space of Riemann metrics

\[
\|\delta h\|^2 = \int_{\Sigma_g} d^2x \sqrt{\text{det} h} h^{ac} h^{bd} \delta h_{ab} \delta h_{cd},
\]  

(5.8)

for the variations of \( h \) that does not involve conformal transformations. In order to parameterize general variations, we follow the standard procedure \([65]\), and introduce complex coordinates on \( \Sigma_g \), in terms of which the metric takes the conformal form \( h = h_{z \bar{z}} dz \otimes d\bar{z} \). The group \( \text{Diff}(\Sigma_g) \) is generated by the coordinate transformations \( z \rightarrow z + \varepsilon(z, \bar{z}) \). Then the metric (5.6) takes the form

\[
\|\delta K\|^2 = \int_{\Sigma_g} d^2x e^\sigma ((\delta \sigma)^2 + \partial \overline{\sigma} \varepsilon) + \sum_{i, j = 1}^{3g - g} \delta m_i (N_2^{-1})^{ij} \delta \overline{m}_j
\]  

(5.9)

where \( m_i \) are coordinates on the moduli space of Riemann surfaces \( M_g \), and \( N_2 \) is the matrix of scalar products of the quadratic holomorphic differentials on \( \Sigma_g \). Therefore, the measure in the functional integral (5.4) is given by

\[
\mathcal{D}K = \text{vol}(\text{Diff}(\Sigma_g)) \frac{\det' \Delta_{-1}}{\det N_2} d\sigma dm
\]  

(5.10)

where \( \Delta_j \) denotes the Laplace-Beltrami operator acting on the space of the holomorphic \( j \)-differentials, \( N_j \) is the matrix of scalar products of holomorphic \( j \)-differentials, and the volume form on the moduli space is \( dm = \prod_{i=1}^{3g - g} dm_i \wedge d\overline{m}_i \). We see that \( \sigma \) plays the role of the Liouville field (the conformal factor of the metric). In particular, we can compute the \( \sigma \)-dependence of the determinant in (5.10) using the standard formula

\[
\frac{\det' \Delta_j}{\det N_j \det N_{1-j}} = \left| \det \overline{\sigma}_j \right|^2 e^{-\frac{c_j}{24\pi} S_L[\sigma]},
\]  

(5.11)

where \( c_j = 6j^2 - 6j + 1 \) and \( S_L[\sigma] \) is the Liouville action. However, because of the choice of the parameterization (5.7), the conformal field \( \sigma \) enters the Hitchin extension of the gauged WZW model (4.51) in a special way. The relevant terms of the functional (4.51) have the form:

\[
\mathcal{G}_{IJ} \pi \int_{\Sigma_g} d^2x e^\sigma \partial \phi^I \overline{\partial} \overline{\phi}^J + \frac{i\pi}{2} \int_{\Sigma_g} \chi (e^\sigma - 1)
\]  

(5.12)
The additional factor $e^\sigma$ makes this theory at the quantum level very different from Polyakov’s bosonic string. Let us recall that the quantum theory (5.4) is defined as an expansion around the attractor point

$$\mathcal{K} = \mathcal{K}_* + \delta \mathcal{K}, \quad \lambda = \lambda_* + \delta \lambda. \quad (5.13)$$

This means that we should expand (5.12) around $\sigma = 0$. If we formally do this expansion, in perturbation series we will encounter terms of the form

$$\sum_{n>0} \frac{1}{n!} \int_{\Sigma_g} d^2 x \sigma^n \langle \partial \phi^I \overline{\partial \phi^J} \rangle. \quad (5.14)$$

These terms are singular, since $\langle \phi(z) \overline{\phi(w)} \rangle \sim \log |z - w|$, and we are taking the limit $z \to w$, $\sigma \to 0$. Therefore, for this theory to make sense, (5.14) has to be regularized in some way.

However, in the classical (weak coupling) limit $k \to \infty$ we can ignore this regularization ambiguity. If we neglect possible contributions from the boundary of the moduli space, in this limit the main contribution to (5.4) comes from the attractor points (4.71):

$$Z_{g,k}(\Theta) \bigg|_{k \to \infty} = \sum_{\Pi \in \text{Attr}_g} e^{i \text{Tr} \Pi} \left( |\Lambda^*(\Pi)/k \Lambda(\Pi)|^g + \ldots \right). \quad (5.15)$$

From the viewpoint of the entropic principle (1.9), it means that the wave-function (1.1) on the moduli space of the complex structures $\mathcal{M}_g$ is peaked at the attractor points (1.14).

There is one physically natural way to resolve the regularization ambiguity in (5.12). We would like to think about the corresponding theory as of a $1_C$-dimensional analog of the Kodaira-Spencer theory of gravity [66]. In the $3_C$-dimensional case, the target space KS action [66] also suffers from the regularization ambiguities. However, there the topological string $B$-model provides a natural regularization. Unfortunately, the higher genus topological string amplitudes vanish if the target manifold has dimension different from the critical dimension $\hat{c} = 3$, so we can not view $1_C$-dimensional analog of KS theory as a topological strings on $\Sigma_g$. Instead, we can define it by requiring that a generating function (5.4) should be identified with the corresponding computation in the dimensional reduction of the Kodaira-Spencer theory of gravity [30] from six to two dimensions.
5.2. Dimensional Reduction of the Topological M-Theory

Let us discuss the relation between the two-dimensional Hitchin model studied above, and the dimensional reduction of the topological M-theory\(^{21}\). At the moment, there is no consistent quantum definition of the topological M-theory \([1]\). However, many ingredients of the theory can be identified at the classical level. In particular, a seven-dimensional topological action

\[
S_7 = \frac{1}{2\pi} \int_{M_7} H \wedge dH, \tag{5.16}
\]

which is a \(U(1)\) Chern-Simons theory for 3-form \(H\) plays an important role in interpreting the topological string partition function as a wave-function (see, e.g., \([1,2,16,67]\) and references therein).

On the other hand, it is well-known \([68]\), that we can get a \(1+1\)-dimensional abelian Chern-Simons theory from (5.16) via dimensional reduction on the manifold of the form \(M_7 = M_4 \times \Sigma_g \times \mathbb{R}\). Using the ansatz \(H = \sum \alpha_i A^i\), where \(\alpha_i\) are integral harmonic 2-forms on \(M_4\), we obtain:

\[
\frac{1}{2\pi} \int_{M_4 \times \Sigma_g \times \mathbb{R}} H \wedge dH \rightarrow \frac{K_{ij}}{2\pi} \int_{\Sigma_g \times \mathbb{R}} A^i \wedge dA^j. \tag{5.17}
\]

Here \(K_{ij}\) is an intersection form for harmonic 2-forms on \(M_4\). If we use the spin manifold, this form is an even integral, and therefore the dual conformal field theory is rational. In this paper, we studied a special case of such compactifications, with the form \(K_{ij}\) defining an abelian variety. In general case, \(K_{ij}\) is an integral form, and if \(b_+ \neq b_-\), we get lattices of various signatures. It would be interesting to understand how these lattices can be embedded in our framework, given that the relevant abelian (spin) Chern-Simons theories has been recently classified \([57]\).

6. Conclusions and Further Directions

In this paper we studied Hitchin-like functionals in two dimensions. They lead to topological theories of a special kind: the metric is not required for constructing the theory. Instead, it arises dynamically from the topological data, characterized by particular choice of the cohomologies. We considered the cases of non-compact and compact cohomologies.

\(^{21}\) We thank C. Vafa and E. Witten for raising the question about the relation between Hitchin functionals in different dimensions.
In both cases the theory generates a map between the cohomologies $H^1(\Sigma_g, \mathcal{C}) \otimes g$ of genus $g$ Riemann surface $\Sigma_g$ and moduli space $\mathcal{M}_g$ of the complex structures on $\Sigma_g$, in the spirit of the original Hitchin construction \cite{34,35}. The Hitchin parameterization of the moduli space in terms of the cohomologies has several useful features. The fact that we can use simplicial complexes for the description of cohomologies is a natural source of the modular group appearance. Although explicit calculations may involve a choice of a symplectic basis, the action is modular invariant and therefore provides a laboratory for generating modular invariant objects. Moreover, the symplectic structure on the cohomology space allows one to perform canonical quantization of the moduli space via the Hitchin map.

![Fig. 1: Transport on the moduli space and the Hitchin map.](image)

The geometric picture that arises in this approach is shown on Fig. 1. The cohomology space in question is parameterized by the Seigel upper half-space $\mathcal{H}_g$. It can also be viewed as a space of a complex $g$-dimensional principally polarized abelian varieties. The Hitchin map classically is just the Torelli map between $\mathcal{M}_g$ and Jacobian locus $\text{Jac}(\Sigma_g) \in \mathcal{H}_g$. In the spirit of the Kodaira-Spencer theory \cite{66}, we can start at some ”background” point $\tau$ on the moduli space $\mathcal{M}_g$ and study resulting quantum mechanical problem on the Seigel upper half-space. The corresponding wave-function is then peaked at $\Pi = \tau$, and classical trajectories $\Pi \rightarrow \Pi'$ on $\mathcal{H}_g$ are obtained from trajectories $\tau \rightarrow \tau'$ on $\mathcal{M}_g$ via the Torelli map.
In the dual approach, we start at some point $\Pi \in \mathcal{H}_g$ and study resulting quantum mechanical problem on the moduli space of Riemann surfaces. We find that in this case it is convenient to integrate over the part of the cohomology space given by the complex torus $\mathcal{T}(\Pi) = \mathbb{C}^g/\mathbb{Z}^g \oplus \Pi\mathbb{Z}^g$. Then the effective theory on $\mathcal{M}_g$ can be interpreted as the abelian gauged WZW model coupled to the two-dimensional gravity in a special way. Furthermore, the choice of the classical starting points $\Pi$ is then restricted to those that correspond to the even integral lattices. The classical solutions of the gauged WZW model correspond to the harmonic maps $\Sigma_g \to \mathcal{T}$. After coupling to the two-dimensional gravity, variation with respect to the complex structure implies that these maps are holomorphic with respect to both complex structures on $\Sigma_g(\tau)$ and $\mathcal{T}(\Pi)$. This is only possible if $\mathcal{T}$ is equivalent to the Jacobian of some Riemann surface, up to the modular transformation. In this special case the wave function in the corresponding quantum mechanical problem on $\mathcal{M}_g$ is peaked at the attractor point (4.71) $\tau = \Pi_*$. Otherwise, the wave function is extremized at the boundary of the moduli space $\partial \mathcal{M}_g$.

The probability/entropy function that we get by squaring the wave function has special value at the attractor point: it is equal to the dimension of the Hilbert space in the associated three-dimensional Chern-Simons theory with the abelian group $\mathcal{T}_\tau$. Therefore, the Hitchin construction allows us to effectively organize the moduli space of $c = 2g$ RCFTs by introducing canonical index/entropy function that weights different points on the moduli space $\mathcal{M}_g$ according to the number of conformal blocks in corresponding RCFT.

It is widely believed that there is a vast landscape of consistent theories of quantum gravity, that can be realized in string theory. It was recently suggested [39] that this landscape is surrounded by the huge area of consistent looking effective theories, that cannot be completed to a full theory, called the swampland. On the abelian varieties side, an analog of the string landscape is the Jacobian locus in the Siegel upper half-space, and the ”swampland” is a vast area of non-geometric points in Siegel space, which do not correspond to any Riemann surface. By extremizing the Hitchin functional, we land on a special set of points in the Jacobian locus, corresponding to the surfaces admitting complex multiplication. On the string theory side, similar phenomena occur [32] if the complex moduli of the compactification manifolds are fixed by the attractor mechanism [29,30,31]. Moreover, in both situations we have an entropy/index weight function assigned to those points on the moduli space. This gives us an interesting analogy between the moduli space of string compactifications and the moduli space of abelian varieties. Very schematically, it is shown on Fig. 2. It would be interesting to develop this analogy further.
In particular, it is worth mentioning that there is a direct analog of the non-geometric locus \( \mathcal{H}_g \backslash \text{Jac}(\Sigma_g) \) in Siegel space for a Calabi-Yau threefold \( M \) described in terms of the cohomologies \( H^3(M, \mathbb{R}) \). The Hitchin theorem \([34]\) states that the critical points of the functional (2.2) on a fixed cohomology class \([\rho] \in H^3(M, \mathbb{R})\) define a complex structure on a Calabi-Yau three-fold only if there is a stable solution \( \rho_* : \text{tr}K^2(\rho_*) < 0 \) everywhere\(^{22}\).

From the viewpoint of the attractor equations, the boundary of the stability region in \( H^3(M, \mathbb{R}) \) corresponds to the black hole solutions with classically vanishing entropy. There is no classical solutions outside of the stability region, but it can be probed in quantum theory. Apart form the obvious physical importance, to describe and classify the stability regions in \( H^3(M, \mathbb{R}) \) is a challenging mathematical question. To the best of our knowledge, the answer to this question is not known even in the simple case of one-parametric Calabi-Yau threefolds. This can be thought of as the Schottky problem for Calabi-Yau threefolds.

In this paper, we only considered non-degenerate Riemann surfaces and concentrated on the massless degrees of freedom. The next natural step is to incorporate punctures and holes into the story, and study contributions from the boundary of the moduli space. This way, one could control not only the fluctuations of the geometry, but also the change of topology. By concentrating on the local degrees of freedom at the punctures it should be

\[^{22}\text{The importance of this condition was stressed to us by C. Vafa.}\]
possible to find a connection with the Kodaira-Spencer theory on a local Calabi-Yau geometries, following [70]. Another interesting direction for further study is incorporating supersymmetry and considering more general curved target spaces, for example, non-abelian groups and $\beta\gamma$-systems.

It is likely that the analysis of the Hitchin functionals performed in the paper can be extended from two to six dimensions. The insight that we get from studying the two-dimensional toy model (1.46) is that the six-dimensional Hitchin functional (2.2) should be viewed as an analog of the gauged WZW model for the seven-dimensional Chern-Simons theory. Then the OSV conjecture [3] will have an interpretation in terms of the corresponding index theorem.

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