On the Mixed Steklov–Neumann and Steklov-type Biharmonic Problems in Unbounded Domains

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Abstract. We study the properties of generalized solutions in unbounded domains and the asymptotic behavior of solutions of elliptic boundary value problems at infinity. Moreover, we study the unique solvability of the mixed Steklov–Neumann and Steklov-type biharmonic problems in unbounded domains with a compact and non-compact boundaries (in particular, the exterior of a compact set, half-space, the domain with conical points) under the assumption that generalized solutions of these problems has a bounded Dirichlet (energy) integral with weight $|x|^a$. Depending on the value of the parameter $a$, we obtained uniqueness (non-uniqueness) theorems of these problems or present exact formulas for the dimension of the space of solutions.

Keywords: biharmonic operator; mixed Steklov–Neumann; Steklov-type problem; Dirichlet integral; weighted spaces

MSC: 35J35; 35J40; 31B30

1. Introduction

Let $\Omega$ be an unbounded domain in $\mathbb{R}^n$, $n \geq 2$, $x = (x_1, \ldots, x_n)$, and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

In the domain $\Omega$ we consider the following mixed problems for the biharmonic equation

$$\Delta^2 u = 0$$

with the Steklov–Neumann boundary conditions

$$u|_{\Gamma_1} = \left(\Delta u + \tau \frac{\partial u}{\partial \nu}\right)|_{\Gamma_1} = 0, \quad \Delta u|_{\Gamma_2} = \frac{\partial \Delta u}{\partial \nu}|_{\Gamma_2} = 0,$$

and the Steklov-type boundary conditions

$$\frac{\partial u}{\partial \nu}|_{\partial \Omega} = \left(\frac{\partial \Delta u}{\partial \nu} + \tau u\right)|_{\partial \Omega} = 0,$$

where $\Gamma_1 \cup \Gamma_2 = \partial \Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\text{mes}_{n-1} \Gamma_1 \neq 0$, $\nu = (\nu_1, \ldots, \nu_n)$ is the outer unit normal vector to $\partial \Omega$, $\tau \in C(\partial \Omega)$, $\tau \geq 0$, $\tau \neq 0$, and $\tau > 0$ on a set of positive $(n-1)$-dimensional measure on $\partial \Omega$, $\overline{\Omega} = \Omega \cup \partial \Omega$ is the closure of $\Omega$.

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Elliptic problems with parameters in the boundary conditions have been called Steklov or Steklov-type problems since their first appearance in [32]. For the biharmonic operator, these conditions were first considered in [2], [13] and [30], whose authors the isoperimetric properties of the first eigenvalue were studied.

Note that standard elliptic regularity results are available in [6]. The monograph covers higher order linear and nonlinear elliptic boundary value problems, mainly with the biharmonic or polyharmonic operator as leading principal part. The underlying models and, in particular, the role of different boundary conditions are explained in detail. As for linear problems, after a brief summary of the existence theory and $L^p$ and Schauder estimates, the focus is on positivity. The required kernel estimates are also presented in detail.

In [5] and [6], the spectral and positivity preserving properties for the inverse of the biharmonic operator under Steklov and Navier boundary conditions are studied. These are connected with the first Steklov eigenvalue. It is shown that the positivity preserving property is quite sensitive to the parameter involved in the boundary condition. Moreover, positivity of the Steklov boundary value problem is linked with positivity under boundary conditions of Dirichlet and Navier type.

In [4], the boundary value problems for the biharmonic equation and the Stokes system are studied in a half space, and, using the Schwartz reflection principle in weighted $L^q$-space, the uniqueness of solutions of the Stokes system or the biharmonic equation is proved.

As is well known, if $\Omega$ is an unbounded domain, one should additionally characterize the behavior of the solution at infinity. As a rule, to this end, one usually poses either the condition that the Dirichlet (energy) integral is finite or a condition on the character of vanishing of the modulus of the solution as $|x| \to \infty$. Such conditions at infinity are natural and were studied by several authors (e.g., [10]–[12]).

In the present note, this condition is the boundedness of the weighted Dirichlet integral:

$$D_\alpha(u, \Omega) \equiv \int_\Omega |x|^\alpha \sum_{|\alpha|=2} |\partial^\alpha u|^2 dx < \infty, \quad \alpha \in \mathbb{R}.$$

In various classes of unbounded domains with finite weighted Dirichlet (energy) integral, one of the author [14]–[27] studied uniqueness (non-uniqueness) problem and found the dimensions of the spaces of solutions of boundary value problems for the elasticity system and the biharmonic (polyharmonic) equation.

By developing an approach based on the use of Hardy type inequalities [8], [10]–[12], in the present note, we obtain a uniqueness (non-uniqueness) criterion for a solution of the mixed Dirichlet–Neumann and Dirichlet–Navier problems for the biharmonic equation.

**Notation:** $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions in $\Omega$ with compact support in $\Omega$.

We denote by $H^m(\Omega, \Gamma)$, $\Gamma \subset \overline{\Omega}$, the Sobolev space of functions in $\Omega$ obtained by the completion of $C^\infty(\overline{\Omega})$ vanishing in a neighborhood of $\Gamma$ with respect to the norm

$$||u; H^m(\Omega, \Gamma)|| = \left( \int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha u|^2 dx \right)^{1/2}, \quad m = 1, 2,$$

where $\partial^\alpha \equiv \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $\alpha_i \geq 0$ are integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$; if $\Gamma = \emptyset$, we denote $H^m(\Omega)$ by $H^m(\Omega)$.

$H^m_0(\Omega)$ is the space obtained by the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u; H^m(\Omega)||$;
Lemma 2.2. Let $u$ be a solution of the homogeneous biharmonic equation (1) in Definition 2.1. Let $\Omega = \mathbb{R}^n \setminus G$ for which $\Omega \cap B_0(R) \neq \emptyset$.

As is known \cite{31}, the fundamental solution of the biharmonic equation has the form

$$\Gamma(x) = \begin{cases} C|x|^{4-n}, & \text{if } 4-n < 0 \text{ or } n \text{ is odd}, \\ C|x|^{4-n} \ln |x|, & \text{if } 4-n \geq 0 \text{ and } n \text{ is even}. \end{cases}$$

Remark 2.3. As is known \cite{31}, the fundamental solution $\Gamma(x)$ of the biharmonic equation has the form

$$\Gamma(x) = \\begin{cases} C|x|^{4-n}, & \text{if } 4-n < 0 \text{ or } n \text{ is odd}, \\ C|x|^{4-n} \ln |x|, & \text{if } 4-n \geq 0 \text{ and } n \text{ is even}. \end{cases}$$

Proof of Lemma 2.2 Consider the function $v(x) = \theta_N(x)u(x)$, where $\theta_N(x) = \theta(|x|/N), \theta \in C^\infty(\mathbb{R}^n)$, $0 \leq \theta \leq 1$, $\theta(s) = 0$ for $s \leq 1$, $\theta(s) = 1$ for $s \geq 2$, while $N \gg 1$ and $G \subset \{x : |x| < N\}$.

We extend $v$ to $\mathbb{R}^n$ by setting $v = 0$ on $G = \mathbb{R}^n \setminus \Omega$.

Then the function $v$ belongs to $C^\infty(\mathbb{R}^n)$ and satisfies the equation

$$\Delta^2 v = f,$$

where $f \in C_0^\infty(\mathbb{R}^n)$ and $\text{supp} \ f \subset \{x : |x| < 2N\}$. It is easy to see that $D_a(v, \mathbb{R}^n) < \infty$.

We can now use Theorem 1 of \cite{5} since it is based on Lemma 2 of \cite{5}, which imposes no constraint on the sign of $\sigma$. Hence, the expansion

$$v(x) = P(x) + \sum_{\beta_0 < |\alpha| \leq \beta} \partial^{\alpha} \Gamma(x) C_\alpha + v_\beta(x),$$

holds for each $a$, where $P(x)$ is a polynomial of order $\text{ord} P(x) < m_0 = \max\{2, 2-n/2-a/2\}$, $\beta_0 = 2-n/2+a/2, C_\alpha = \text{const}$ and

$$|\partial^{\alpha} v_\beta(x)| \leq C_{\gamma \beta}|x|^{3-n-\beta-|\gamma|}, \quad C_{\gamma \beta} = \text{const}.$$

Therefore, by the definition of $v$, we obtain (4). The proof of Lemma 2.2 is complete.
Definition 2.4. By a solution of the mixed Steklov-Neumann problem (1), (2), we mean a function \( u \in H^0_0(\Omega) \cap H^1_0(\Omega), u = 0 \) on \( \Gamma_1 \), such that, for every function \( \varphi \in C^\infty_0(\mathbb{R}^n) \), \( \varphi = 0 \) on \( \Gamma_1 \), the following identity holds:

\[
\int \Delta u \Delta \varphi \, dx + \int_{\Gamma_1} \tau \frac{\partial u}{\partial \nu} \frac{\partial \varphi}{\partial \nu} \, ds = 0.
\] (5)

Theorem 2.5. The mixed Steklov-Neumann problem (1),(2) with the condition \( D(u, \Omega) < \infty \) has \( n + 1 \) linearly independent solutions.

Proof. For any nonzero vector \( A \) in \( \mathbb{R}^n \), we construct a generalized solution \( u_A \) of the biharmonic equation (1) with the boundary conditions

\[
u \big|_{\Gamma_1} = (Ax) \big|_{\Gamma_1}, \quad \left( \Delta u_A + \tau \frac{\partial u_A}{\partial \nu} \right) \big|_{\Gamma_1} = \tau \frac{\partial (Ax)}{\partial \nu} \big|_{\Gamma_1}, \quad \Delta u_A \big|_{\Gamma_2} = \frac{\partial \Delta u_A}{\partial \nu} \big|_{\Gamma_2} = 0
\] (6)

and the condition

\[
\chi(u_A, \Omega) \equiv \begin{cases} 
\int_\Omega \left( \frac{|u_A|^2}{|x|^4} + \frac{|\nabla u_A|^2}{|x|^2} + |\nabla \nabla u_A|^2 \right) dx < \infty & \text{for } n > 4, \\
\int_\Omega \left( \frac{|u_A|^2}{|x|^4 \ln |x|^2} + \frac{|\nabla u_A|^2}{|x|^2 \ln |x|^2} + |\nabla \nabla u_A|^2 \right) dx < \infty & \text{for } 2 \leq n \leq 4,
\end{cases}
\] (7)

for \( A, x \in \mathbb{R}^n \), where \( Ax \) denotes the standard scalar product of \( A \) and \( x \).

Such a solution of problem (1), (6) can be constructed by the variational method [31], minimizing the functional

\[
\Phi(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx
\]

in the class of admissible functions \( \{ v : v \in H^2(\Omega), v(x) \big|_{\Gamma_1} = (Ax) \big|_{\Gamma_1}, \left( \Delta v + \tau \frac{\partial v}{\partial \nu} \right) \big|_{\Gamma_1} = \tau \frac{\partial (Ax)}{\partial \nu} \big|_{\Gamma_1}, \Delta v \big|_{\Gamma_2} = \frac{\partial \Delta v}{\partial \nu} \big|_{\Gamma_2} = 0, \, v \text{ is compactly supported in } \Omega \} \).

The validity of condition (7) as a consequence of the Hardy inequality follows from the results in [10]–[12].

Now, for any arbitrary number \( e \neq 0 \), we construct a generalized solution \( u_e \) of equation (1) with the boundary conditions

\[
u \big|_{\Gamma_1} = e, \quad \left( \Delta u_e + \tau \frac{\partial u_e}{\partial \nu} \right) \big|_{\Gamma_1} = 0, \quad \Delta u_e \big|_{\Gamma_2} = \frac{\partial \Delta u_e}{\partial \nu} \big|_{\Gamma_2} = 0
\] (8)

and the condition

\[
\chi(u_e, \Omega) \equiv \begin{cases} 
\int_\Omega \left( \frac{|u_e|^2}{|x|^4} + \frac{|\nabla u_e|^2}{|x|^2} + |\nabla \nabla u_e|^2 \right) dx < \infty & \text{for } n > 4, \\
\int_\Omega \left( \frac{|u_e|^2}{|x|^4 \ln |x|^2} + \frac{|\nabla u_e|^2}{|x|^2 \ln |x|^2} + |\nabla \nabla u_e|^2 \right) dx < \infty & \text{for } 2 \leq n \leq 4.
\end{cases}
\] (9)
The solution of problem (1), (8) also is constructed by the variational method with the minimization of the corresponding functional in the class of admissible functions \( \{v : v \in H^2(\Omega), v|_{\Gamma_1} = e, (\Delta v + \tau \frac{\partial v}{\partial n})|_{\Gamma_1} = 0, \Delta v|_{\Gamma_2} = \frac{\partial \Delta v}{\partial n}|_{\Gamma_2} = 0, v \) is compactly supported in \( \bar{\Omega} \rangle \).

The condition (9) as a consequence of the Hardy inequality follows from the results in [10]-[12].

Consider the function \( v(x) = (u_A(x) - Ax) - (u_e - e) \).

Obviously, \( v \) is a solution of problem (1), (2):

\[
\Delta^2 v = 0, \quad x \in \Omega, \quad v|_{\Gamma_1} = (\Delta v + \tau \frac{\partial v}{\partial n})|_{\Gamma_1} = 0, \quad \Delta v|_{\Gamma_2} = \frac{\partial \Delta v}{\partial n}|_{\Gamma_2} = 0.
\]

One can easily see that \( v \neq 0 \) and \( D(v, \Omega) < \infty \).

To each nonzero vector \( A = (A_0, A_1, \ldots, A_n) \) in \( \mathbb{R}^{n+1} \), there corresponds a nonzero solution \( v_A = (v_{A_0}, v_{A_1}, \ldots, v_{A_n}) \) of problem (1), (2) with the condition \( D(v_A, \Omega) < \infty \), and moreover,

\[
v_A(x) = u_A(x) - u_e - Ax + e.
\]

Let \( A_0, A_1, \ldots, A_n \) be a basis in \( \mathbb{R}^{n+1} \). Let us prove that the corresponding solutions \( v_{A_0}, v_{A_1}, \ldots, v_{A_n} \) are linearly independent. Let

\[
\sum_{i=0}^{n} C_i v_{A_i} = 0, \quad C_i = \text{const}.
\]

Set \( W(x) \equiv \sum_{i=1}^{n} C_i A_i x - C_0 e \). We have

\[
W(x) = \sum_{i=1}^{n} C_i u_{A_i}(x) - C_0 u_e,
\]

\[
\int_{\Omega} |x|^{-2} |\nabla W|^2 \, dx < \infty, \quad n > 4,
\]

\[
\int_{\Omega} ||x| \ln |x||^{-2} |\nabla W|^2 \, dx < \infty, \quad 2 \leq n \leq 4.
\]

Let us show that

\[
W(x) \equiv \sum_{i=1}^{n} C_i A_i x - C_0 e \equiv 0.
\]

Let \( T = \sum_{i=0}^{n} C_i A_i = (t_0, \ldots, t_n) \), where \( A_0 = -e \). Then

\[
\int_{\Omega} |x|^{-2} |\nabla W|^2 \, dx = \int_{\Omega} |x|^{-2}(t_1^2 + \cdots + t_n^2) \, dx = \infty, \quad n > 4,
\]

\[
\int_{\Omega} ||x| \ln |x||^{-2} |\nabla W|^2 \, dx = \int_{\Omega} ||x| \ln |x||^{-2}(t_1^2 + \cdots + t_n^2) \, dx = \infty, \quad 2 \leq n \leq 4,
\]

if \( T \neq 0 \).

Consequently, \( T = \sum_{i=0}^{n} C_i A_i = 0 \), and since the vectors \( A_0, A_1, \ldots, A_n \) are linearly independent, we obtain \( C_i = 0 \), \( i = 0, 1, \ldots, n \).

Thus, the Steklov–Neumann problem (1), (2) with the condition \( D(u, \Omega) < \infty \) has at least \( n + 1 \) linearly independent solutions.
Let us prove that each solution $u$ of problem (1), (2) with the condition $D(u, \Omega) < \infty$ can be represented as a linear combination of the functions $v_{A_0}, v_{A_1}, \ldots, v_{A_n}$, i.e.

$$u = \sum_{i=0}^{n} C_i v_{A_i}, \quad C_i = \text{const}.$$ 

Since $A_0, A_1, \ldots, A_n$ is a basis in $\mathbb{R}^{n+1}$, it follows that there exists constants $C_0, C_1, \ldots, C_n$ such that

$$A = \sum_{i=0}^{n} C_i A_i.$$ 

We set

$$u_0 = u - \sum_{i=0}^{n} C_i v_{A_i}.$$ 

Obviously, the function $u_0$ is a solution of problem (1), (2), and $D(u_0, \Omega) < \infty$, $\chi(u_0, \Omega) < \infty$.

Let us show that $u_0 \equiv 0$, $x \in \Omega$. To this end, we substitute the function $\varphi(x) = u_0(x) \theta_N(x)$ into the integral identity (5) for the function $u_0$, where $\theta_N(x) = \theta(|x|/N)$, $\theta \in C^\infty(\mathbb{R})$, $0 \leq \theta \leq 1$, $\theta(s) = 0$ for $s \geq 2$ and $\theta(s) = 1$ for $s \leq 1$; then we obtain

$$\int_{\Omega} (\Delta u_0)^2 \theta_N(x) \, dx + \int_{\Gamma_1} \tau \left| \frac{\partial u_0}{\partial \nu} \right|^2 \theta_N(x) \, ds = -J_1(u_0) - J_2(u_0),$$

where

$$J_1(u_0) = 2 \int_{\Omega} \Delta u_0 \nabla u_0 \nabla \theta_N(x) \, dx, \quad J_2(u_0) = \int_{\Omega} u_0 \Delta u_0 \Delta \theta_N(x) \, dx.$$ 

By applying the Cauchy–Schwarz inequality and by taking into account the conditions $D(u_0, \Omega) < \infty$ and $\chi(u_0, \Omega) < \infty$, one can easily show that $J_1(u_0) \to 0$ and $J_2(u_0) \to 0$ as $N \to \infty$. Consequently, by passing to the limit as $N \to \infty$ in (10), we obtain

$$\int_{\Omega} (\Delta u_0)^2 \theta_N(x) \, dx + \int_{\Gamma_1} \tau \left| \frac{\partial u_0}{\partial \nu} \right|^2 \theta_N(x) \, ds \to 0.$$ 

Using the integral identity

$$\int_{\Omega} (\Delta u_0)^2 \, dx + \int_{\Gamma_1} \tau \left| \frac{\partial u_0}{\partial \nu} \right|^2 \, ds = 0,$$

we find that if $u_0$ is a solution of the homogeneous problem (1), (2), then $\Delta u_0 = 0$.

Therefore, we have

$$\Delta u_0 = 0, \quad x \in \Omega,$$

$$u_0|_{\Gamma_1} = (\Delta u_0 + \tau \frac{\partial u_0}{\partial \nu})|_{\Gamma_1} = 0, \quad \Delta u_0|_{\Gamma_2} = \frac{\partial \Delta u_0}{\partial \nu}|_{\Gamma_2} = 0.$$ 

Hence, it follows [7, Ch.2] that $u_0 \equiv 0$ in $\Omega$. The relation

$$\int_{\partial \Omega} \tau \left| \frac{\partial u_0}{\partial \nu} \right|^2 \, ds = 0$$

implies that $u_0 \equiv 0$ on a set of a positive measure on $\partial \Omega$. The proof of the theorem is complete.
Theorem 2.6. The mixed Steklov–Neumann problem (1), (2) with the condition \( D_u(u, \Omega) < \infty \) has:

(i) the trivial solution for \( n - 2 \leq a < \infty, n > 4 \);
(ii) \( n \) linearly independent solutions for \( n - 4 \leq a < n - 2, n > 4 \);
(iii) \( n + 1 \) linearly independent solutions for \( -n \leq a < -n - 4, n > 4 \);
(iv) \( k(r, n) \) linearly independent solutions for \( -2r + 2 - n \leq a < -2r + 4 - n, r > 1, n > 4 \),

where

\[
k(r, n) = \binom{r + n}{n} - \binom{r + n - 4}{n}.
\]

The proof of Theorem 2.6 is based on Lemma 2.2 about the asymptotic expansion of the solution of the biharmonic equation and the Hardy type inequalities for unbounded domains [10]–[12]. In case (iv), we need to determine the number of linearly independent solutions of the biharmonic equation (1), the degree of which not exceed the fixed number.

It is well known that the dimension of the space of all polynomials in \( \mathbb{R}^n \) of degree \( \leq r \) is equal \( \binom{r + n}{n} \) [28]. Then the dimension of the space of all biharmonic polynomials in \( \mathbb{R}^n \) of degree \( \leq r \) is equal to

\[
\binom{r + n}{n} - \binom{r + n - 4}{n},
\]

since the biharmonic equation is the vanishing of some polynomial of degree \( r - 4 \) in \( \mathbb{R}^n \). If we denote by \( k(r, n) \) the number of linearly independent polynomial solutions of equation (1) whose degree do not exceed \( r \) and by \( l(r, n) \) the number of linearly independent homogeneous polynomials of degree \( r \), that are solutions of equation (1), then

\[
k(r, n) = \sum_{s=0}^{r} l(s, n),
\]

where

\[
l(s, n) = \binom{s + n - 1}{n - 1} - \binom{s + n - 5}{n - 1}, \quad s > 0.
\]

Further, we prove that the mixed Steklov–Neumann problem (1), (2) with the condition \( D_u(u, \Omega) < \infty \) for \( -2r + 2 - n \leq a < -2r + 4 - n \) has equally \( k(r, n) \) of linearly independent solutions.

3. Steklov-type Biharmonic Problem in a Half-Space

Let \( x = (x', x_n) \), where \( x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \); then \( \mathbb{R}^n_+ = \{ x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 1 \} \) is an "open half-space" with the boundary \( \partial \mathbb{R}^n_+ := \{ x = (x', x_n) \in \mathbb{R}^n : x_n = 1 \} \equiv \mathbb{R}^{n-1} \). Let now \( \Omega \equiv \mathbb{R}^n_+ \) with the boundary \( \partial \Omega \equiv \partial \mathbb{R}^n_+ \).

Definition 3.1. A function \( u \) is a solution of the Steklov-type biharmonic problem (1),(3) if \( u \in H^2_{\text{loc}}(\Omega), \partial u/\partial \nu = 0 \) on \( \partial \Omega \), such that, for every function \( \varphi \in C_0^\infty(\mathbb{R}^n), \partial \varphi/\partial \nu = 0 \) on \( \partial \Omega \), the following integral identity holds

\[
\int_\Omega \Delta u \Delta \varphi \, dx - \int_{\partial \Omega} \tau u \varphi \, ds = 0.
\]

Lemma 3.2. Let \( u \) be a solution of equation (1) in \( \Omega \) satisfying the boundary conditions (3) and the inequality \( |u(x)| \leq C (1 + |x|^k) \) for all \( x \in \Omega \), where \( C \) is a positive constant and \( k \geq 0 \) is an integer. Then \( u \) is a polynomial of degree at most \( k \), i.e., \( u(x) = P(x) \) and \( \text{ord} P(x) \leq k \).
Proof. This lemma is an analogue of Liouville’s theorem for systems of equations. It is valid for general Douglis–Nirenberg elliptic systems with constant coefficients. For $\Omega \equiv \mathbb{R}^n$ it was proved in [3].

The proof for $\Omega \equiv \mathbb{R}^n_+$ is similar. Namely, in the theory of elliptic systems, the following Bernstein inequality is established:

$$\max_{|x| \leq 1/2, |\alpha| = l} |\partial^\alpha u(x)| \leq C_1 \max_{|x| \leq 1} |u|, \quad l \geq 1, \quad (11)$$

where the constant $C_1$ is independent of $u$.

Let $u$ be a solution of a homogeneous elliptic system with constant coefficients in the half-ball $|x| \leq 1, x_n > 0$, and for $x_n = 0$ suppose that $u$ satisfies the Shapiro–Lopatinskii zero boundary condition [1] (for more details, see [29]). Putting $x = \lambda y$ in (11) and using the homogeneity of the system and the boundary conditions, we get

$$\max_{|x| \leq 1/2, |\alpha| = l} |\partial^\alpha u(x)| \leq \lambda^{-1} \max_{|x| \leq \lambda} |u|. \quad \text{(12)}$$

Hence, the hypotheses of Lemma 3.2 imply

$$\max_{|x| \leq 1, |\alpha| = l} |\partial^\alpha u(x)| \leq C_2 \lambda^{k-l}. \quad \text{(13)}$$

Taking $l > k$ and letting $\lambda$ tend to $\infty$, we see that $\partial^\alpha u(x) = 0$ for $|x| \leq 1$. Hence, $u$ is a polynomial of degree at most $l - 1$ for $|x| \leq 1$. Since the solution of an elliptic system is analytic, it follows that $u$ is a polynomial in $\mathbb{R}^n_+$. The condition $|u(x)| \leq C(1 + |x|^k)$ implies that the degree of this polynomial is at most $k$. The proof of Lemma 3.2 is complete. \qed

**Theorem 3.3.** The Steklov-type biharmonic problem (1),(3) with the condition $D(u, \Omega) < \infty$ has $n$ linearly independent solutions.

**Theorem 3.4.** The Steklov-type biharmonic problem (1),(3) with the condition $D_{\alpha}(u, \Omega) < \infty$ has:

(i) $n$ linearly independent solution for $-n \leq a < \infty$;

(ii) $k(r, n)$ linearly independent solutions for $-2r - n + 2 \leq a < -2r - n + 4$, $r \geq 2$, where

$$k(r, n) = \left(\frac{r+n}{n}\right) - \left(\frac{r+n-4}{n}\right) - \left(\frac{r+n-2}{n-1}\right) - \left(\frac{r+n-4}{n-1}\right).$$

The proof of Theorem 3.4 is based on Lemma 3.2, and in the case of (ii), as in Theorem 2.6, it is necessary to determine the number of linearly independent solutions of a biharmonic equation, the degree of which do not exceed the fixed number. Further, the condition for the vanishing on the hyperplane $x_n = 1$ of the polynomial and its derivatives of $x_n$ yields the equality to zero of certain polynomials of $(n - 1)$ variables of degree $(r - j)$, $j = 1, 3$. Hence, the dimension of the space of solutions of the Steklov-type biharmonic problem is $k(r, n)$.

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