THE ZERO STABILITY FOR THE ONE-ROW COLORED $\mathfrak{sl}_3$-JONES POLYNOMIAL

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ABSTRACT. The stability of coefficients of colored ($\mathfrak{sl}_2$-) Jones polynomials $\{J_{\mathfrak{sl}_2}^{K|n}(q)\}_n$ was discovered by Dasbach and Lin. This stability is now called the zero-stability of $J_{\mathfrak{sl}_2}^{K|n}(q)$. Armond showed zero stability for a $B$-adequate link by using the linear skein theory based on the Kauffman bracket. In this paper, we prove the zero stability of one-row colored $\mathfrak{sl}_3$-Jones polynomials $\{J_{\mathfrak{sl}_3}^{K|n}(q)\}_n$ for $B$-adequate links $L$ with anti-parallel twist regions by using the linear skein theory based on Kuperberg’s $\mathfrak{sl}_3$-webs. It implies the existence of many $q$-series obtained from a quantum invariant associated with $\mathfrak{sl}_3$.

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1. INTRODUCTION

the colored $\mathfrak{g}$-Jones polynomial of a knot $K$ is a quantum invariant obtained from an irreducible representation of a simple Lie algebra $\mathfrak{g}$. In this paper, we will discuss some stability of coefficients of the one-row colored $\mathfrak{sl}_3$-Jones polynomials $\{J_{\mathfrak{sl}_3}^{K|n}(q)\}_n \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ which is a quantum invariant of $K$ associated with irreducible representations of $\mathfrak{sl}_3$ corresponding to the one-row Yang diagram $(n)$. This kind of stability for the colored ($\mathfrak{sl}_2$-) Jones polynomials was discovered by Dasbach and Lin [DL06, DL07]. They showed that some leading coefficients, concerning the degree of $q$, of $\{J_{\mathfrak{sl}_2}^{K|n}(q)\}_n$ are independent of the colorings $n$ (where $n+1$ is the dimension of an irreducible representation) if a knot $K$ is alternating. They also conjectured that the first $n$ coefficients of $J_{\mathfrak{sl}_2}^{K|n}(q)$ are constant for all $N$ greater than $n$ if $K$ is alternating. Armond [Arm13] proved this conjecture for a larger class of links called adequate links, which contain alternating links. Independently, Garoufalidis and Lê [GL15] proved more general stability, called $k$-stability, for alternating links where $k$ is a non-negative integer. in the sense of Garoufalidis and Lê, the
stability proved by [Arm13] corresponds to the zero stability. The \( k \)-stability also ensures the existence of the \( q \)-series called the \( k \)-limit, which is closely related to quantum modular forms. The \( 0 \)-limit is also known as the tail of \( K \).

**Definition 1.1.** Let \( J_{K,n}^{sl_2}(q) := \pm q^d J_{K,n}^{sl_2}(q) = a_0 + \sum_{i=1}^{\infty} a_i q^i \) be a normalization of the colored Jones polynomial \( J_{K,n}^{sl_2}(q) \) of a knot \( K \) where the sign is determined so that \( a_0 \) becomes positive. The tail of \( K \) is a \( q \)-series \( \Phi_K(q) \in \mathbb{Z}[\![q]\!] \) satisfying

\[
\Phi_K(q) - J_{K,n}^{sl_2}(q) \in q^{n+1} \mathbb{Z}[\![q]\!],
\]

for any positive integer \( n \).

Note that the integrality theorem for the colored Jones polynomial proved in [Lê00] claims that the coefficients of \( J_{K,n}(q) \) become integral, therefore its tail \( \Phi_K(q) \) belongs to \( \mathbb{Z}[\![q]\!] \). Armond and Dasbach [AD17] showed that the tail of an adequate knot is determined by its reduced \( B \)-graph.\(^1\) A similar result was also obtained by Garoufalidis–Norin [GNV16]. They stated that the first three coefficients of \( \Phi_K(q) \) of an alternating link \( K \) are described in terms of its reduced Tait graph. From these results, we can see that the tail is not useful in distinguishing links. However, tails of knots and links give us interesting \( q \)-series related to quantum modular forms. For example, Garoufalidis–Lê [GL15] showed that tails of alternating links are described as a generalization of Nahm sums. In particular, the tail of \((2,m)\)-torus link is the (false) theta series. In [AD11, Haj16, Yua18], the Andrews-Gordon type identity for the (false) theta series was derived from two explicit formulas for the tail of a \((2,m)\)-torus link. Explicit formulas for tails of other knots and links have been studied in [GL15, EH17, KO16, BO17], and for quantum spin networks in [Haj16].

Our goal is to develop a study of the stability and tails for \( J_{K,n}^{sl_2}(q) \) to quantum invariants \( J_{K,n}^{g}(q) \) associated with a higher rank simple Lie algebra \( g \). Many problems arise when we consider higher rank cases. For example, we have to choose a sequence of irreducible representations to consider the stability because the colored \( g \)-Jones polynomial of a knot is parametrized by dominant weights. Moreover, the explicit computation of the colored \( g \)-Jones polynomials of a given knot is much more difficult than in the \( sl_2 \) case.

The aim of this paper is to show zero stability of the one-row colored \( sl_3 \)-Jones polynomial \( \{ J_{K,n}^{sl_3}(q) \}_n \) of a \( B \)-adequate link \( K \) with anti-parallel twist regions. The one-row coloring \( n \) for \( K \) means that all components of \( K \) are colored by the irreducible representation of the highest weight \( n \varpi_1 \) (or we write it as \((n,0)\)) where \( \{ \varpi_i \}_{i=1,2} \) correspond to the fundamental weights of \( sl_3 \). There are some studies on the explicit computation of the colored \( sl_3 \) Jones polynomial, for the trefoil knot in [Law03], for the \((2,2m+1)\) - and \((4,5)\)-torus knots with general coloring in [GMV13, GV17], and for 2-bridge links with one-row coloring in [Yua17], for pretzel links with one-row coloring in [Kaw22]. These explicit formulae give tails of the colored \( sl_3 \)-Jones polynomial of some links in [GV17, Yua18, Yua21]. For the \( \lambda \)-colored \( g \)-Jones polynomial of the \((a,b)\)-torus knot when \( g \) is a simple Lie algebra of rank 2, Garoufalidis and Vuong [GV17] proved the \( k \)-stability for any \( k \). They used the formula of Rosso and Jones in [RJ93] to prove it. In this paper, we prove the zero stability of the one-row colored \( sl_3 \)-Jones polynomial for any anti-parallel \( B \)-adequate link using

\(^1\)They consider \( A \)-graphs. However, this corresponds to \( B \)-graphs in our convention. That is, our \( q \) is \( q^{-1} \) in [AD17].
the linear skein theory for $\mathfrak{sl}_2$ developed by Kuperberg [Kup96]. Our proof is inspired by the work of Armond [Arm13] using the Kauffman bracket.

**Theorem 1** (Zero stability for the one-row colored $\mathfrak{sl}_3$-Jones polynomial, see Theorem 4.16). For any anti-parallel $B$-adequate link $L$, there exists $\Phi_{L}^{\mathfrak{sl}_3}(q)$ in $\mathbb{Z}[\![q]\!]$ such that

$$\Phi_{L}^{\mathfrak{sl}_3}(q) - \hat{J}_{L,n}^{\mathfrak{sl}_3}(q) \in q^{n+1}\mathbb{Z}[\![q]\!] .$$

An anti-parallel $B$-adequate link is an oriented link whose representative is a $B$-adequate link diagram with only anti-parallel twist regions; see details in Definition 4.1.

This result and proof are an extension of the work on the zero stability of colored $\mathfrak{sl}_2$-Jones polynomials in [Arm13] to $\mathfrak{sl}_3$. We will discuss the zero stability for general $B$-adequate link in the forthcoming paper.

This paper is organized as follows. In Section 2, we introduce the $\mathfrak{sl}_3$ version of the linear skein theory and review properties of $\mathfrak{sl}_3$-webs and $\mathfrak{sl}_3$-clasps. In Section 3, we discuss the lower bound of the minimum degree of a clasped $\mathfrak{sl}_3$-web. In Section 4, we prove the zero stability of the one-row colored $\mathfrak{sl}_3$-Jones polynomials by calculating clasped $\mathfrak{sl}_3$-webs. In Appendix A, we prove some new formulas for the clasped $\mathfrak{sl}_3$-web used in this paper.

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## 2. $\mathfrak{sl}_3$-WEBS AND $\mathfrak{sl}_3$-CLASPS

We mainly treat a space of $\mathfrak{sl}_3$-webs which is a linear combination of oriented planar trivalent graphs with coefficients in $\mathcal{R} = \mathbb{Z}(\![\hat{\pi}]\!)$ . Let us introduce some useful symbols for elements in $\mathcal{R}$. We set

- a **quantum integer** by $[n] := \frac{q^n - q^{-n}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ for any non-negative integer $n \in \mathbb{Z}_{\geq 0}$, and

- a **quantum binomial coefficient** by $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$ for $0 \leq k \leq n$, and $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $k > n$ where $[n]! := [n][n-1]\cdots[1]$.

Let us define $\mathfrak{sl}_3$-web spaces based on [Kup96]. We consider a surface $\Sigma$ equipped with signed marked points $(P, s)$ where $P \subset \partial \Sigma$ is a finite set and $s : P \to \{+,-\}$ a map.

A **tangled trivalent graph** on $\Sigma$ is an immersion of a directed graph $G$ into $\Sigma$ satisfying:

1. the valency of a vertex of $G$ is 1 or 3,
2. all crossing points are transversal double points of two edges with under/over-passing information,
3. the set of univalent vertices of $G$ coincides with $P$,
4. a neighborhood of a vertex in $\Sigma$ is one of the followings:

```
  \begin{tikzpicture}
    \draw (0,0) -- (0.5,0.5) -- (1,0) -- (0.5,-0.5) -- (0,0);
  \end{tikzpicture}  \quad
  \begin{tikzpicture}
    \draw (0,0) -- (0.5,0.5) -- (1,0) -- (0.5,-0.5) -- (0,0);
  \end{tikzpicture}  \quad
  \begin{tikzpicture}
    \draw (0,0) -- (0.5,0.5) -- (1,0) -- (0.5,-0.5) -- (0,0);
  \end{tikzpicture}  \quad
  \begin{tikzpicture}
    \draw (0,0) -- (0.5,0.5) -- (1,0) -- (0.5,-0.5) -- (0,0);
  \end{tikzpicture}
```
A tangled trivalent graph is **flat** if it has no crossings. An **elliptic face** of a flat trivalent graph $G$ is a 0-gon (*i.e.*, a disk), 2- or 4-gon in the set of connected components of $\Sigma \setminus G$ which does not touch the boundary of $\Sigma$.

**Definition 2.1** ($\mathfrak{sl}_3$-web spaces [Kup96]). Let $\mathcal{G}(s; \Sigma)$ be the set of the boundary fixing isotopy classes of tangled trivalent graphs on $\Sigma$. The $\mathfrak{sl}_3$-**web space** $\mathcal{W}(s; \Sigma)$ is a quotient of the $\mathcal{R}$-module spanned by $\mathcal{G}(s; \Sigma)$ modulo the following $\mathfrak{sl}_3$-**skein relations**:

- $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation1.png}
\end{array}
\end{array} = q^\frac{3}{2} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation2.png}
\end{array}
\end{array} - q^{-\frac{1}{2}} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation3.png}
\end{array}
\end{array},
\end{array}$

- $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation4.png}
\end{array}
\end{array} = q^{\frac{3}{2}} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation5.png}
\end{array}
\end{array} - q^{-\frac{1}{2}} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation6.png}
\end{array}
\end{array},
\end{array}$

- $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation7.png}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation8.png}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation9.png}
\end{array}
\end{array} \text{ (the 4-gon relation)},
\end{array}$

- $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation10.png}
\end{array}
\end{array} = [2] \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation11.png}
\end{array}
\end{array} \text{ (the 2-gon relation)},
\end{array}$

- $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation12.png}
\end{array}
\end{array} = [3] \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation13.png}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{relation14.png}
\end{array}
\end{array} \text{ (the trivial loop relation)}.
\end{array}$

An $\mathfrak{sl}_3$-**web** is an element in $\mathcal{W}(s; \Sigma)$ and a **basis web** is an $\mathfrak{sl}_3$-web represented by a graph in $\mathcal{G}(s; \Sigma)$ with no elliptic faces.

The $\mathfrak{sl}_3$-skein relation realizes the **Reidemeister moves** (R1'), and (R2) – (R4):

- (R1') $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister1.png}
\end{array}
\end{array} \leftrightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister2.png}
\end{array}
\end{array},$

- (R2) $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister3.png}
\end{array}
\end{array} \leftrightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister4.png}
\end{array}
\end{array},$

- (R3) $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister5.png}
\end{array}
\end{array} \leftrightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister6.png}
\end{array}
\end{array},$

- (R4) $\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister7.png}
\end{array}
\end{array} \leftrightarrow \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister8.png}
\end{array}
\end{array},$

The above means that $\mathfrak{sl}_3$-webs on the left and right sides represent the same element in an $\mathfrak{sl}_3$-web space for any choice of the orientation of edges.

It is easy to see that any tangled trivalent graph decomposes into a sum of basis web by using the $\mathfrak{sl}_3$ skein relation. In fact, the set of basis webs consists of a basis of the $\mathfrak{sl}_3$-web space.

**Theorem 2.2** ([Kup96], [SW07]). The set of basis web on a surface $\Sigma$ with a signed marked points $s: P \to \{+, -\}$ is a basis of $\mathcal{W}(s; \Sigma)$ as a $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-module.

In some cases, one can give the set of basis webs via an argument concerning the Euler characteristic.
Example 2.3. Let $D$ be a disk with a base point $* \in \partial D$. We identify signed marked points on $\partial D \setminus \{*\}$ with a sequence of signs. Then, the following isomorphisms hold for any $\epsilon \in \{+, -, \}$.

1. $\mathcal{W}(\emptyset; D)$ of a disk $D$ with no marked points is isomorphic to a free $\mathcal{R}$-module spanned by the empty diagram $\emptyset$.
2. $\mathcal{W}(\epsilon; D) = 0$.
3. $\mathcal{W}(\epsilon\epsilon; D) = 0$.
4. $\mathcal{W}(\epsilon\epsilon; D)$ is a free $\mathcal{R}$-module spanned by an oriented simple arc.
5. $\mathcal{W}(\epsilon\epsilon\epsilon; D) = 0$.
6. $\mathcal{W}(\epsilon\epsilon\epsilon; D)$ is a free $\mathcal{R}$-module spanned by a tripod with a sink or source vertex.

In the above, $\epsilon$ means the opposite sign of $\epsilon$.

We review a diagrammatic definition of an $s\mathfrak{l}_3$-clasp introduced in [Kup96, OY97, Kim07, Yua21] and note some useful properties. The $s\mathfrak{l}_3$-clasp plays a similar role to the Jones-Wenzl projector in the Kauffman bracket skein theory.

In what follows, we will mainly treat tangled trivalent graphs or $s\mathfrak{l}_3$-webs in a rectangle $D = [0, 1] \times [0, 1]$. We assume that the set of marked points lies in the top edge $I_1 = [0, 1] \times \{1\}$ and the bottom edge $I_0 = [0, 1] \times \{0\}$, and a base point $*$ at $(0, 0)$. In this situation, the set of marked points is divided into $P^{(0)}$ and $P^{(1)}$ where $P^{(j)} \coloneqq P \cap I_j$ and we denote the assignment of signs by $s^{(j)}: P^{(j)} \rightarrow \{+, -, \}$ for $j = 0, 1$. One can identify $s^{(j)}$ with a sequence of signs on $[0, 1] \times \{j\}$ arranged from 0 to 1. We simplify a symbol $G(s^{(0)} \sqcup s^{(1)}; D)$ and $W(s^{(0)} \sqcup s^{(1)}; D)$ by $G(s^{(0)}, s^{(1)})$ and $\mathcal{T}\mathcal{L}(s^{(0)}, s^{(1)})$ respectively where $s^{(1)}$ is a sequence consists of the opposite signs of $s^{(1)}$. When we describe diagrams representing $s\mathfrak{l}_3$-webs in $D$, we omit to write the signs, the basepoint, and the boundary of $D$. The composition $\mathcal{T}\mathcal{L}(s_1, s_2) \otimes \mathcal{T}\mathcal{L}(s_0, s_1) \rightarrow \mathcal{T}\mathcal{L}(s_0, s_2)$ is defined by gluing the top side of an $s\mathfrak{l}_3$-web in $\mathcal{T}\mathcal{L}(s_0, s_1)$ and the bottom side of an $s\mathfrak{l}_3$-web in $\mathcal{T}\mathcal{L}(s_1, s_2)$.

We firstly define the $s\mathfrak{l}_3$-clasp in $\mathcal{T}\mathcal{L}(-m, -m)$.

Definition 2.4 (One-row colored $s\mathfrak{l}_3$-clasps). The $s\mathfrak{l}_3$-clasp $\text{JW}_{-m}$ described by a white box with $m \in \mathbb{Z}_{>0}$ is defined as follows.

1. $\text{JW}_{-} = \begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array} := \begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}$
2. $\text{JW}_{-m+1} = \begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array} := \begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}$

In the above, an edge labeled by $m$ represents the $m$ parallelization of the edge. $\text{JW}_{+m}$ is defined by the same diagram with oppositely directed edges.

We next introduce the $s\mathfrak{l}_3$-clasp in $\mathcal{T}\mathcal{L}(-m+n, -m+n)$. 

---

\footnote{We take the opposite sign $\bar{s}^{(1)}$ to be consistent with the composition.}
Definition 2.5 (Two-row colored $sl_3$-clasps).

$$JW_{-m+n} = \sum_{i=0}^{\min\{m,n\}} (-1)^i \binom{m}{i} \binom{n}{i} \binom{m+n+1}{i}.$$ 

One can define $JW_{+m-n}$ in the same way.

For convenience in computation of $sl_3$-web, we will introduce “stair-step” and “triangle” webs in Definitions 2.6 and 2.7 that also used in [Kim06, Kim07, Yua17, FS22]. In these definitions, the orientation of edges of $sl_3$-webs is not explicitly given because it is uniquely determined according to your choice of the orientation of an edge around the box.

Definition 2.6. For positive integers $n$ and $m$, a stair-step web $\begin{array}{c} n \cr 1 \end{array}$ is defined as

$$\begin{array}{c} n \cr 1 \end{array} = n \begin{array}{c} \vdots \cr \vdots \end{array} \quad \text{and} \quad \begin{array}{c} m \cr 1 \end{array} = \begin{array}{c} \vdots \cr \vdots \end{array} \quad \text{for } m > 1.$$ 

Definition 2.7. For a positive integer $n$, $\begin{array}{c} n \cr 1 \end{array}$ is defined by

$$\begin{array}{c} n \cr 1 \end{array} = \begin{array}{c} 1 \cr \vdots \cr 1 \end{array} \quad \text{and} \quad \begin{array}{c} n \cr 1 \end{array} = \begin{array}{c} 1 \cr \vdots \cr 1 \end{array} \quad \text{for } n > 1.$$ 

Definition 2.8 (General type of the $sl_3$-clasps). Let $s_1$ and $s_2$ be two sequences of signs which consist of $m$ pluses and $n$ minuses. We define an $sl_3$-clasp in $TL(s_1, s_2)$ by gluing stair-step webs to the top and the bottom of $JW_{-m+n}$ as follows,

$$\begin{array}{c} a \cr b \end{array}, \quad \begin{array}{c} a \cr b \end{array}.$$ 

Repeating these operations, one can arbitrarily exchange signs in sequences on the top and bottom of the disk respectively, and we obtain an $sl_3$-web in $\overline{TL}(s_1, s_2)$. We denote it by $JW_{s_2}^{s_1}$ and depict it as

$$\begin{array}{c} s_2 \cr s_1 \end{array}.$$ 

The resulting $sl_3$-clasp is independent of a choice of sequences of stair-step webs, see [Yua21]. Thus, $JW_{s_2}^{s_1}$ is uniquely determined by $s_1$ and $s_2$.

One can prove the following useful formulas for $sl_3$-clasps by straightforward computation.
Lemma 2.9. Let $s_1, s_2,$ and $s_3$ are sequences of signs. An arc labeled by a positive integer $m$ (resp. $n$) denotes $m$ (resp. $n$) parallelization of the arc.

\[
\begin{align*}
(1) \quad \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$s_1$} -- (1,0) node[above] {$s_2$};
    \draw[thick] (1,0) node[above] {$s_3$} -- (2,0);
    \end{tikzpicture} &= \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$s_1$} -- (1,0) node[above] {$s_3$};
    \draw[thick] (1,0) node[above] {$s_2$} -- (2,0);
    \end{tikzpicture}, \quad \ldots \ldots &= 0, \quad \ldots \ldots &= 0, \\
(2) \quad \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$m$} -- (1,0) node[above] {$n$};
    \draw[thick] (1,0) node[above] {$m$} -- (2,0) node[below] {$n$};
    \end{tikzpicture} &= \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$m$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$m$} -- (2,0) node[below] {$n$};
    \end{tikzpicture}, \\
(3) \quad \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$m$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$m$} -- (2,0) node[below] {$n$};
    \end{tikzpicture} &= (-1)^{mn} q^{\frac{mn}{6}} \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$m$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$m$} -- (2,0) node[below] {$n$};
    \end{tikzpicture}, \\
(4) \quad \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$m$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$m$} -- (2,0) node[below] {$n$};
    \end{tikzpicture} &= q^{\frac{mn}{6}} \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$m$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$m$} -- (2,0) node[below] {$n$};
    \end{tikzpicture}, \\
(5) \quad \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$n$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$n$} -- (2,0) node[below] {$n$};
    \end{tikzpicture} &= q^{\frac{n^2+3n}{3}} \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$n$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$n$} -- (2,0) node[below] {$n$};
    \end{tikzpicture}, \\
(6) \quad \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$n$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$n$} -- (2,0) node[below] {$n$};
    \end{tikzpicture} &= q^{\frac{n^2+3n}{3}} \tikz[baseline=-0.5ex,inner sep=0.5ex]{
    \draw[thick] (0,0) node[above] {$n$} -- (1,0) node[below] {$n$};
    \draw[thick] (1,0) node[above] {$n$} -- (2,0) node[below] {$n$};
    \end{tikzpicture}.
\end{align*}
\]

Proof. One can prove (1)–(6) by using induction on labels and skein relations. See [Yua17, Yua21] for example. □

We give a definition of the one-row colored $\mathfrak{sl}_3$-Jones polynomial of oriented framed links via an $\mathfrak{sl}_3$-web. Firstly, we introduce a normalization of a Laurent series by shifting the $q$-degree and changing the sign.

Definition 2.10 (Minimum degree). We define the minimum degree $d_* : \mathcal{R} = \mathbb{Z}((q^{1/6})) \to \frac{1}{6}\mathbb{Z}\cup\infty$ by $d_*(f(q)) := d/6$ for a non-zero series $f(q) = \sum_{i=d}^{\infty} a_i q^{\frac{i}{6}}$ in $\mathbb{Z}((q^{1/6}))$ such that $a_d \neq 0$. For the zero polynomial, we define its minimum degree as $\infty$. We also introduce a normalization $\hat{f}(q)$ of a non-zero Laurent series $f(q)$ as

\[
\hat{f}(q) := \pm q^{-d_*(f(q))} f(q) = \pm \sum_{i=0}^{\infty} a_{i+d} q^{\frac{i}{6}} \in \mathbb{Z}[[q^{1/6}]].
\]

In the above, we choose the sign so that the constant term $\pm a_d$ becomes positive.

We note some properties for the minimum degree and useful examples.

Lemma 2.11. For any $f(q), g(q) \in \mathcal{R}$,

1. $d_*(f(q) + g(q)) \geq \min\{d_*(f(q)), d_*(g(q))\}$,
2. $d_*(f(q)g(q)) = d_*(f(q)) + d_*(g(q))$. 

The equality in (1) holds if and only if \( \delta_*(f(q)) \neq \delta_*(g(q)) \) or \( \delta_*(f(q)) = \delta_*(g(q)) = \sum_{i=d}^{\infty} a_i q^i \) and \( g(q) = \sum_{i=d}^{\infty} b_i q^i \).

**Example 2.12.** For any positive integer \( n \) and \( 1 \leq k \leq n \),
\[
\left|\delta_*(\[n\])\right| = -(n-1)/2, \quad \left|\delta_*(\[n\]^{-1})\right| = (n-1)/2,
\[
\delta_*(\left[\begin{array}{c} n \\ k \end{array}\right]) = -k(n-k)/2, \quad \delta_*(\left[\begin{array}{c} n \\ k \end{array}\right]^{-1}) = k(n-k)/2.
\]

We remark that one can confirm them by \((1-q^m)^{-1} = 1 + q^m + q^{2m} + \cdots \in \mathcal{R} = \mathbb{Z}((q^{1/2}))\) and Lemma 2.11.

**Definition 2.13.** Let \( L \) be a link diagram of a framed link whose framing is given by the blackboard framing. One can replace arcs of the link diagram with \( n \) parallelized arcs and put white boxes on the \( n \) parallelized arcs. The resulting diagram denoted by \( L^{(n)} \) represents an \( sl_3 \)-web in a disk \( D \) with no marked points.\(^3\) The **one-row colored \( sl_3 \)-Jones polynomial** \( j_{L,n}^{sl_3}(q) \) with \((n,0)\)-coloring (or \( n \) boxes) is defined by \( L^{(n)} = j_{L,n}^{sl_3}(q) \emptyset \). We also define a variation of the one-row colored \( sl_3 \)-Jones polynomial as \( j_{L,n}^{sl_3}(q) \) according to Definition 2.10.

**Remark 2.14.**
- \( sl_3 \)-webs realize the Reidemeister moves (R1')--(R4) for arcs with one-row colored clasps because clasped arcs are expressed as a linear combination of \( sl_3 \)-webs. Hence \( j_{L,n}^{sl_3}(q) \) is an invariant of framed links.
- The choice of framing of \( L \) appears as multiplication by \( \pm q^* \), see (6) in Lemma 2.9. This difference is ignored in the normalization \( \hat{j}_{L,n}^{sl_3}(q) \), thus it is an invariant of links.
- Lê showed the integrality theorem for a quantum \( g \) invariant of links in [Lê00]. It says that \( \hat{j}_{L,n}^{sl_3}(q) \) belongs to \( \mathbb{Z}[q] \).

We will discuss **zero stability** of \( j_{L,n}^{sl_3}(q) \) for a certain class of links in the following sections. Let us recall the definition of zero stability and tails of the one-row colored \( sl_3 \)-Jones polynomials.

**Definition 2.15** (One-row colored \( sl_3 \)-tail). The one-row colored \( sl_3 \)-Jones polynomial \( \{j_{L,n}^{sl_3}(q)\}_n \) of a link \( L \) is **zero stable** if there exists a formal power series \( \Phi^{sl_3}_L(q) \) in \( \mathbb{Z}[[q]] \) such that
\[
\Phi^{sl_3}_L(q) - j_{L,n}^{sl_3}(q) \in q^{n+1}\mathbb{Z}[[q]]
\]
for all \( n \geq 1 \). We call \( \Phi^{sl_3}_L(q) \) the **one-row colored \( sl_3 \)-tail** of \( L \) or simply the **\( sl_3 \)-tail** of \( L \) when \( \{j_{L,n}^{sl_3}(q)\}_n \) is zero stable.

### 3. The Minimum Degree of Clasped \( sl_3 \)-webs

We will prove the existence of the \( sl_3 \)-tail of the one-row colored \( sl_3 \)-Jones polynomial by developing \( sl_3 \) analog of Armond’s argument using the Kauffman bracket in [Arm13]. In the present section, we will discuss a lower bound of the minimum degree of a clasped \( sl_3 \)-web with no crossings in a disk.

\(^3\)Such \( sl_3 \)-web space \( \mathcal{W}(\emptyset; D) \) is spanned by the empty diagram \( \emptyset \), see Example 2.3.
First, we prepare a lemma that studies isomorphisms in Example 2.3 in detail.

**Lemma 3.1.** Let \( G \in \mathcal{G}(−+; D) \) be a connected flat trivalent graph in a disk \( D \) with two marked points such that \( G \neq 0 \). There exists a sequence composed only of 2- and 4-gon relations such that it reduces \( G \) to an oriented simple arc \( \gamma \) without increasing the number of connected components of intermediate graphs.

**Proof.** Let us assume that \( G \) has at least one elliptic face and \( G \neq 0 \). We only have to attend to the 4-gon relation because the 2-gon relation does not change the number of connected components of a graph. Let us prove the claim by induction on the number \( v(G) \) of vertices. A connected flat trivalent graph \( G \) with \( v(G) = 2 \) has to become a diagram in the left-hand side of the 2-gon relation in Definition 2.1, and \( G \) with \( v(G) = 3 \) does not exist because a trivalent vertex is a sink or source. Let \( G \) be a connected flat trivalent graph with \( v(G) \geq 4 \) and we assume that it has at least one internal 4-gon \( F \). Example 2.3 (2) claims that we cannot describe a circle in \( D \) which intersect with \( G \) at a single edge. Example 2.3 (5) claims that no circle intersects two incoming (resp. outgoing) edges and one outgoing (resp. incoming) edge of \( G \). This fact requires that four edges incident to corners of \( F \) connect to other parts of \( G \) as (i) or (ii) in below;

\[
\text{(i) } \begin{array}{c}
\begin{array}{c}
\text{subgraphs } X_1, X_2, \text{ and } X_3 \text{ of } G \text{ are connected. We apply the 4-gon relation at } F \text{ in (i);} \\
\end{array}
\end{array}
\]

where subgraphs \( X_1, X_2, \) and \( X_3 \) of \( G \) are connected. We apply the 4-gon relation at \( F \) in (i);

\[
\begin{array}{c}
\begin{array}{c}
\text{Graphs after applying the 4-gon relation are divided into the left and right parts containing } X_1 \text{ and } X_2, \text{ respectively, by cutting along a vertical line in } D. \text{ Right and left subgraphs are considered } \mathfrak{s}_3\text{-web in a disk with two marked points whose number of vertices is smaller than } v(G). \text{ The right subgraph containing } X_2 \text{ is connected due to Example 2.3 (2). Hence these subgraphs satisfy the induction hypothesis. For case (ii), a sequence of the 2- and 4-gon relations changes } X_3 \text{ into an arc without increasing the number of components because } v(X_3) < v(G). \text{ Then, one can obtain a graph consisting of } X_1 \text{ and } X_2 \text{ by applying the 2-gon relation twice. The resulting graph also satisfies the induction hypothesis.} \\
\end{array}
\end{array}
\]

\( \square \)

**Proposition 3.2.** Let \( G \) be a flat trivalent graph in a disk with no marked points, and we identify \( G \) with its value in \( R \) (see Example 2.3). Then,

\[
d_*(G) \geq \frac{v(G)}{4} - c(G),
\]

where \( v(G) \) is the number of trivalent vertices of \( G \), \( c(G) \) the number of connected components of \( G \). Moreover, the equality \( d_*(G) = c(G) \) holds when \( v(G) = 0 \).

**Proof.** We first prove \( d_*(G) = -c(G) \) when \( v(G) = 0 \). If \( G \) has no trivalent vertices, then it consists only of loop components. By using an innermost argument and the trivial loop relation in Definition 2.1, it is easy to see that \( G = [3]^c(G) \). We obtain \( d_*(G) = c(G)d_*([3]) = -c(G) \) because \([3] = q + 1 + q^{-1}\).
Let us consider when $G = \bigcup_{i=1}^{c(G)} G_i$ is a non-trivial flat trivalent graph with $v(G) > 0$ where $G_i$’s are connected components of $G$. Choose a point $p_i$ on the outermost edge of $G_i$ and a small interval $I_{p_i}$ for each $i = 1, 2, \ldots, c(G)$. Then, one can take disks $\{D_i\}_{i=1}^{c(G)}$ with two marked points for all $i = 1, 2, \ldots, c(G)$ such that $G_i \setminus \text{int}(I_{p_i}) \subset D_i$, $\partial I_{p_i}$ are identical with its marked points, and $D_i \cap D_j = \emptyset$ for $i \neq j$. Apply Lemma 3.1 to $G_i \cap D_i$ for all $i \in \{1, 2, \ldots, c(G)\}$ and we obtain a disjoint union $\Gamma := \bigsqcup_{i=1}^{c(G)} \gamma_i$ of simple loops from $G$. Each component $\gamma_i$ is obtained from $G_i$ by a sequence of 2- and 4-gon relations preserving the number of connected components. Let $G = G' + G''$ be a 4-gon relation appearing in the above sequence, and we can assume that $G' \neq 0$ and $d_s(G') \leq d_s(G'')$ without loss of generality. Then,

- $d_s(G) \geq \min\{d_s(G'), d_s(G'')\} = d_s(G')$,
- $v(G) = v(G') + 4$, and
- $c(G) = c(G')$.

Thus,

$$d_s(G) + \frac{1}{4}v(G) + c(G) \geq d_s(G') + \frac{1}{4}v(G') + c(G') + 1.$$

Suppose instead that $G'$ is obtained by a 2-gon relation, that is, $G = [2]G'$. Then,

- $d_s(G) = d_s(G') - \frac{1}{2}$,
- $v(G) = v(G') + 2$, and
- $c(G) = c(G')$.

Thus,

$$d_s(G) + \frac{1}{4}v(G) + c(G) = d_s(G') + \frac{1}{4}v(G') + c(G').$$

As mentioned above, we can choose a reduction sequence from $G$ to $\Gamma$ so that flat trivalent graphs in this sequence satisfy the above inequality for the minimum degree. We remark that $d_s(\Gamma) = -c(\Gamma)$ because $v(\Gamma) = 0$. Hence, $G$ and $\Gamma$ should satisfy

$$d_s(G) + \frac{1}{4}v(G) + c(G) \geq d_s(\Gamma) + \frac{1}{4}v(\Gamma) + c(\Gamma) = 0.$$

\[\square\]

Next, we give a lower bound of the minimum degree of a flat trivalent graph with $s_{13}$-clasps. Let us consider the minimum degree of coefficients appearing in expansion formulas of $s_{13}$-clasps.

**Lemma 3.3** (The single clasp expansion formula [Kim07, Proposition 3.1]). For any positive integer $m$,

$$\text{JW}_{-m} = \frac{m}{q^m} = \sum_{j=0}^{m-1} f_j^{(m)}(q),$$

where $f_j^{(m)}(q) := (-1)^j \frac{(m-j)}{[m]}$.

One can obtain the following lemma from the single clasp expansion formula and induction on $m$. 
Lemma 3.4. The one-row colored $\mathfrak{sl}_3$-clasp has an expansion

$$JW_{-m} = \sum_M f_M(q) M$$

with $d_*(f_M(q)) = v(M)/4$ where the sum runs over finitely many flat trivalent graphs $M$, and $v(M)$ is the number of trivalent vertices in $M$. We remark that $M$ may contain $4$-gons or $2$-gons.

Proof. It is obvious that the claim is true for $m = 1, 2$. We prove it by induction on $m$. A flat trivalent graph in the right-hand side of the single clasp expansion of $JW_{-m}$ has a stair-step web with $2j$ vertices and $JW_{-m-1}$. We know that $d_*(f_j(m)(q)) = j/2$ by Example 2.12. Lemma 3.3 and the induction hypothesis derive an expansion:

$$JW_{-m} = \sum_{j=0}^{m-1} \sum_M f_j(m)(q) f_M(q)$$

where $d_*(f_M(q)) = v(M)/4$. The flat trivalent graph in the right-hand side has $2j + v(M)$ vertices, and $d_*(f_j(m)(q) f_M(q)) = d_*(f_j(m)(q)) + d_*(f_M(q)) = (2j + v(M))/4$ by Lemma 2.11. This expansion satisfies the condition of our claim. \hfill \Box

Remark 3.5. The proof of Lemma 3.4 can be used to show that $M$ constructed by composing $I_j$s where

$$I_j := \left\{ \begin{array}{c} \bullet \quad \bullet \quad \cdots \quad \bullet \\ \quad \bullet \quad \cdots \quad \bullet \\ \end{array} \right\} \in \mathcal{TC}(\cdots, \cdots)^j$$

for $j = 1, 2, \ldots, m - 1$. Labels $j$ and $j + 1$ in the bottom mean the $j$-th and $(j + 1)$-th marked points, respectively.

Lemma 3.6. The two-row colored $\mathfrak{sl}_3$-clasp has the following expansion:

$$JW_{-m+n} = \sum_{t=0}^{\min(m,n)} \sum_{M_1,M_2,M_3,M_4} f_{(m,n,t)}(M_1, M_2, M_3, M_4; q) \uparrow M_4 M_3 M_1 M_2 \downarrow (m,n,t)$$

with $d_*(f_{(m,n,t)}(M_1, M_2, M_3, M_4; q)) = \frac{t(t+1)}{2} + \sum_{i=1}^4 \frac{1}{4} v(M_i)$ where

$$\uparrow M_4 M_3 M_1 M_2 \downarrow (m,n,t) := \left\{ \begin{array}{c} \bullet \quad \bullet \quad \cdots \quad \bullet \\ \cdots \quad \bullet \\ \end{array} \right\}$$

Proof. Apply Lemma 3.4 to each one-row colored $\mathfrak{sl}_3$-clasp in the right-hand side of Definition 2.5. Then, we obtain an expansion as in the statement such that

$$f_{(m,n,t)}(M_1, M_2, M_3, M_4; q) := (-1)^t \left[ \frac{m}{t} \right] \left[ \frac{n}{t} \right] \left[ \frac{m+n+1}{t} \right] f_{M_1}(q) f_{M_2}(q) f_{M_3}(q) f_{M_4}(q).$$
One can calculate the minimum degree as follows:
\[
d_*(f_{(m,n,t)}(M_1,M_2,M_3,M_4;q)) = d_* \left( (-1)^t \frac{m}{m+n+t+1} \right) + \sum_{i=1}^{4} d_*(f_{M_i}(q))
\]
\[
= \frac{t(t+1)}{2} + \sum_{i=1}^{4} v(M_i). 
\]

We introduce a notation of a planar algebra specializing in our situation because it is useful for writing \( \mathfrak{sl}_3 \)-webs in the form of an equation. Let \( D \) be a disk, and \( D_i (i = 1, 2, \ldots, k) \) specified disjoint \( k \) rectangles in \( D \setminus \partial D \). \( D_i \) is homeomorphic to \([0,1] \times [0,1]\) and it has a base point at \((0,0)\) and marked points \( P_i = P_i^{(0)} \sqcup P_i^{(1)} \). The set \( P_i^{(j)} \) of marked points lies in the edge of \( D_i \) corresponding to \([0,1] \times \{j\}\). We have a \( k \)-holed disk \( D(k) := D \setminus \cup_{i=1}^{k} \text{int}(D_i) \) with marked points \( \bigcup_{i=1}^{k} P_i \). A small disk \( D_i \) share \( P_i \) with \( D(k) \) for \( i = 1, 2, \ldots, k \), see Figure 3.1. Let a sequence of signs \( s_i^{(0)} \) (resp. \( s_i^{(1)} \)) be an assignment of signs to a set of marked points \( P_i^{(0)} \) (resp. \( P_i^{(1)} \)) of \( D(k) \) for each \( i = 1, 2, \ldots, k \). Then, we consider the following \( \mathfrak{sl}_3 \)-web spaces:

- \( \mathcal{W}(\bigotimes_{i=1}^{k} s_i; D(k)) \) where \( s_i = s_i^{(0)} \cup s_i^{(1)} \),
- \( \mathcal{T}_L(s_i^{(0)}, s_i^{(1)}) = \mathcal{W}(s_i^{(0)} \cup s_i^{(1)}; D_i) \) for \( i = 1, 2, \ldots, k \).

As I mentioned above, \( D(k) \) and \( D_i \) share the set of marked points \( P_i = P_i^{(0)} \sqcup P_i^{(1)} \). Then, the sequence of sign \( s_i^{(j)} \) of \( P_i^{(j)} \) in \( D(k) \) consistent with \( s_i^{(j)} \) of \( P_i^{(j)} \) in \( D_i \) for \( j = 1, 2 \). For example, an edge terminating at \( p \in P_i \) in \( D(k) \) can be composed with an edge starting from \( p \) in \( D_i \). Thus, the sign of \( p \) in \( D(k) \) and \( D_i \) are different. For a tangled trivalent graph \( G \in \mathcal{G}(\bigotimes_{i=1}^{k} s_i; D(k)) \), a linear map

\[
G: \bigotimes_{i=1}^{k} \mathcal{T}_L(s_i^{(0)}, s_i^{(1)}) \to \mathcal{W}(\emptyset, D) \cong \mathbb{Z}((q^{1/3}))
\]

is induced by a map \( D(k) \sqcup \bigcup_{i=1}^{k} \bigcup_{j=1}^{k} (D_1 \sqcup D_2 \sqcup \cdots \sqcup D_k) \to D \). This map composes \( \mathfrak{sl}_3 \)-webs in \( D_i \) \((i = 1, 2, \ldots, k)\) with \( G \) in \( D(k) \).

In this paper, we only consider a segregated sign sequence \( s_i^{(0)} = \varepsilon_{m_i} \varepsilon_{n_i} \) and \( s_i^{(1)} = s_i^{(0)} = \varepsilon_{m_i} \varepsilon_{n_i} \) where \( \varepsilon \) is \(+\) or \(-\) and \( m_i, n_i \in \mathbb{Z}_{\geq 0} \) satisfy \( m_i + n_i = \#P_i^{(0)} = \#P_i^{(1)} \). The identity web denoted by \( 1_{s_i^{(1)}} \) in \( \mathcal{T}_L(s_i^{(0)}, s_i^{(1)}) \) is \((m_i + n_i)\) parallel strands in \( D_i \). The identity web \( 1_{s_i^{(1)}} \) and the \( \mathfrak{sl}_3 \)-clasp \( J W_{s_i^{(1)}} \) in \( D_i \) are simply denoted by \( 1_{D_i} \) and \( J W_{D_i} \), respectively.

**Definition 3.7.** Set \( s_i^{(0)} = s_i^{(1)} = \varepsilon_{m_i} \varepsilon_{n_i} \) for all \( i = 1, 2, \ldots, k \). \( G \in \mathcal{G}(\bigotimes_{i=1}^{k} s_i; D(k)) \) is **adequate** if

- \( G \) is a disjoint union of oriented simple arcs, and
- For every \( j = 1, 2, \ldots, k \), any pair of strands in \( 1_{D_j} \) belongs to different connected components of the graph \( G(\otimes_{i=1}^{k} 1_{D_i}) \) which composed of oriented simple loops.

See Example 3.8. We also call the clasped \( \mathfrak{sl}_3 \)-web \( G(\otimes_{i=1}^{k} J W_{D_i}) \) is **adequate** when \( G \) is adequate.
Figure 3.1. It is a k-holed disk $D(k)$ with $k = 3$. A shaded rectangle labeled by $i$ is $D_i$. Marked points with sign $+$ (resp. $-$) are described as black (resp. white) dots. In this case, $s_i^{(0)} = \bar{s}_i^{(1)} = ++$, $s_i^{(0)} = \bar{s}_i^{(1)} = --$, and $s_i^{(0)} = \bar{s}_i^{(1)} = -++$. A tangled trivalent graph $G$ defines $\mathcal{TL}(++,++) \otimes \mathcal{TL}(+-,+-) \rightarrow \mathbb{R}$.

Example 3.8. The left $sl_3$-web is adequate, and the right is not adequate because of the red arc.

Proposition 3.9. Let $D(k) = D \setminus \sqcup_{i=1}^{k} \text{int}(D_i)$ be a k-holed disk with signed marked point $s_i^{(0)} = \bar{s}_i^{(1)} = e^{m_i}e^{n_i}$, on the $i$-th boundary component for $i = 1, 2, \ldots, k$. For any flat trivalent graph $G$ in $G(\sqcup_{i=1}^{k} s_i; D(k))$,

$$d_*(G(\otimes_{i=1}^{k} \mathcal{JW} D_i)) \geq -\nu(G) - c \left(G(\otimes_{i=1}^{k} \mathbb{1}_{D_i})\right).$$

 Particularly, $d_*(G(\otimes_{i=1}^{k} \mathcal{JW} D_i)) \geq d_*(G(\otimes_{i=1}^{k} \mathbb{1}_{D_i}))$ holds when $G$ has no trivalent vertices due to Proposition 3.2. Moreover, the equality $d_*(G(\otimes_{i=1}^{k} \mathcal{JW} D_i)) = d_*(G(\otimes_{i=1}^{k} \mathbb{1}_{D_i}))$ holds when $G$ is adequate.

Proof. Lemma 3.6 expands all $sl_3$-clasps in $D_i (i = 1, 2, \ldots, k)$ as below.

$$G(\otimes_{i=1}^{k} \mathcal{JW} D_i) = \min\{m_1, n_1\} \cdots \min\{m_k, n_k\} \sum_{t_1=0}^{\min\{m_1, n_1\}} \cdots \sum_{t_k=0}^{\min\{m_k, n_k\}} \sum_{M^{(1)}} \cdots \sum_{M^{(k)}} \left[ \prod_{i=1}^{k} f_{t_i}(M^{(i)}; q) \right] G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow t_i),$$

where

- $\sum_{M^{(i)}}$ means a summation over $M_1^{(i)}, M_2^{(i)}, M_3^{(i)}, M_4^{(i)}$,
- $\uparrow M^{(i)} \downarrow t_i$ is the $sl_3$-web $\uparrow M^{(i)}_{M_2^{(i)}} M_3^{(i)} M_4^{(i)} \downarrow (m_i, n_i, t_i)$ defined in Lemma 3.6, and
• \( f_{t_i} (M^{(i)}; q) := f_{(m_i, n_i, t_i)}(M_1^{(i)}, M_2^{(i)}, M_3^{(i)}, M_4^{(i)}; q) \) with \( d_*(f_{t_i} (M^{(i)}; q)) = \frac{t_i(t_i+1)}{2} + \sum_{j=1}^{4} \frac{v(M_j^{(i)})}{4} \).

We remark that \( v(G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i})) = \sum_{i=1}^{k} \sum_{j=1}^{4} v(M_j^{(i)}) + v(G) \) by definition.

\[
(3.1) \quad d_* \left( \left( \prod_{i=1}^{k} f_{t_i} (M^{(i)}; q) \right) G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i}) \right) \\
= \sum_{i=1}^{k} \frac{t_i(t_i+1)}{2} + \sum_{i=1}^{k} \sum_{j=1}^{4} \frac{v(M_j^{(i)})}{4} + d_* \left( G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i}) \right) \cdot
\]

Proposition 3.2 gives a lower bound of \( d_* \left( G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i}) \right) \) using the number of vertices and connected components;

\[
d_* \left( G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i}) \right) \geq - \frac{1}{4} \left( \sum_{i=1}^{k} \sum_{j=1}^{4} v(M_j^{(i)}) + v(G) \right) - c(G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i})).
\]

Let \( G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i}) \) be an \( \mathfrak{s}3 \)-web such that \( G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i}) \cap D(k) = G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i}) \cap D_i = \uparrow \mathbb{1}^{(i)} \downarrow_{t_i} \), where \( \uparrow \mathbb{1}^{(i)} \downarrow_{t_i} := \begin{array}{ccc} m_i-t_i & t_i & n_i-t_i \end{array} \). In other words, \( G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i}) \) is obtained by replacing all \( M_j^{(i)} \) in \( G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i}) \) with identity webs.

Remark 3.5 says that \( G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i}) \) is obtained from \( G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i}) \) by a sequence of \textbf{zip cobordisms} which replace parallel strands with \( I_j \)'s. Then, \( c(G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i})) \leq c(G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i})) \) holds because a zip cobordism reduces the number of connected components. Combining these inequalities with eq. (3.1), we obtain

\[
(3.2) \quad d_* \left( \left( \prod_{i=1}^{k} f_{t_i} (M^{(i)}; q) \right) G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i}) \right) \geq \sum_{i=1}^{k} \frac{t_i(t_i+1)}{2} - \frac{v(G)}{4} - c(G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i})).
\]

We remark that one can make a similar argument when JW \( D_i \)'s have one-row colored \( \mathfrak{s}3 \)-clasps by using Lemma 3.4. If the \( i \)-th disk \( D_i \) has a one-row colored \( \mathfrak{s}3 \)-clasp, then we read \( \uparrow M^{(i)} \downarrow_{t_i} \) as \( M^{(i)}, f_{t_i} (M^{(i)}; q) \) as \( f_{M^{(i)}}(q) \), and replace \( \sum_{j=1}^{4} v(M_j^{(i)})/4 \) with \( v(M^{(i)})/4 \).

Finally, we observe how the right-hand side of eq. (3.2) changes by a single orientable saddle cobordism, which transforms \( t_i \) to \( t_i + 1 \). One can see that the single orientable saddle cobordism changes \( c(G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i})) \) by \( \pm 1 \) by considering the orientation of strands in \( \uparrow \mathbb{1}^{(i)} \downarrow_{t_i} \). Hence, the right-hand side of eq. (3.2) is a monotonically increasing function on \( 0 \leq t_i \leq \min\{m_i, n_i\} \). Consequently,

\[
d_* \left( \left( \prod_{i=1}^{k} f_{t_i} (M^{(i)}; q) \right) G(\otimes_{i=1}^{k} \uparrow M^{(i)} \downarrow_{t_i}) \right) \geq - \frac{v(G)}{4} - c(G(\otimes_{i=1}^{k} \uparrow \mathbb{1}^{(i)} \downarrow_{t_i})).
\]
4. Zero Stability of the One-Row Colored $\mathfrak{sl}_3$-Jones Polynomial

We show the zero stability of the one-row colored $\mathfrak{sl}_3$-tail for a certain class of $B$-adequate oriented links.

**Definition 4.1.** Let $D$ be a disk equipped with twist regions $\sqcup_{i=1}^k D_i$ in $\text{int}(D)$ such that all $D_i$s are isomorphic to $[0,1] \times [0,1]$ with a base point at $(0,0)$ and an assignment $l: \{D_i\}_{i=1}^k \to 2\mathbb{Z}_{>0}$. An **anti-parallel $B$-adequate link** is an oriented link represented by an oriented link diagram $L$ in $D$ satisfying the following condition:

- $L \cap D(k)$ is an adequate graph $G$ where $D(k) := D \setminus \sqcup_{i=1}^k \text{int}(D_i)$, and
- $L \cap D_i$ is a twist region $R_{l_i}$ with negative $l_i := l(D_i)$ half twists of antiparallel strands for each $i$, see Figure 4.1.

**Example 4.2.** The diagram below represents an anti-parallel $B$-adequate link.
Example 4.3 (Plumbed-like links). Let $X$ be a planar embedded graph equipped with a weight $l: E(X) \to 2\mathbb{Z}_{\geq 0}$ for the edge set $E(X)$. Then, we obtain an anti-parallel $B$-adequate link diagram from $X$ by replacing all vertices with positively oriented circles and then adding a twist $R_l(e)$ between two circles connected by an edge $e \in E(X)$.

Before we prove the zero stability for anti-parallel $B$-adequate links, let us introduce some symbols for values of special $\mathfrak{sl}_3$-webs and coefficients. We can describe the one-row colored $\mathfrak{sl}_3$-Jones polynomial by using these values.

Lemma 4.4.

$$
\Delta^{(n)}(j) = \frac{[n-j+1][2n-2j+2]}{2}, \quad \Theta^{(n)}(j) = \frac{[2n-j+2]}{[n]^{2j}} \Delta^{(n)}(j),
$$

$$
\gamma^{(n)}(j) = (-1)^{n-j} q^{-\frac{1}{2}} n^2 q^{-\frac{1}{2} j^2 + (n+1) j},
$$

where

$$
\Delta^{(n)}(j) = \begin{array}{c}
\includegraphics[width=1in]{diagram1.png}
\end{array}, \quad \Theta^{(n)}(j) = \begin{array}{c}
\includegraphics[width=1.5in]{diagram2.png}
\end{array}, \quad \gamma^{(n)}(j) = \begin{array}{c}
\includegraphics[width=1.5in]{diagram3.png}
\end{array}.
$$

**Proof.** It is well-known that the value of a closure of $JW_{-m+n}$ is obtained by quantum dimension $\frac{[m+1][n+1][m+n+2]}{2}$, and it is just $\Delta_j^{(n)}$. One can compute the value $\gamma$ by using Lemma 2.9 (3), (4), and (6). $\Theta^{(n)}(j)$ was computed in [Yua18]. \qed

Lemma 4.5.

$$
d_*\left(\Delta^{(n)}(j)\right) = d_*\left(\Delta^{(n)}(j - 1)\right) + 2,
$$

$$
d_*\left(\Theta^{(n)}(j)\right) = d_*\left(\Theta^{(n)}(j - 1)\right) + 1,
$$

$$
d_*\left(\gamma^{(n)}(j)\right) = d_*\left(\gamma^{(n)}(j - 1)\right) + \left(n - j + \frac{3}{2}\right).
$$

**Proof.** From Lemma 2.11 and Example 2.12, one can compute the minimum degrees of $\Delta^{(n)}(j)$ and $\Theta^{(n)}(j)$ as $d_*(\Delta^{(n)}(j)) = -2n + 2j$ and $d_*(\Theta^{(n)}(j)) = -2n + j$. \qed

An $n$-parallelization $L_n$ of the anti-parallel $B$-adequate link diagram $L$ defines an adequate graph $G_n := L_n \cap D(k) \in \mathcal{G}(\bigcup_{i=1}^k s_i; D(k))$ where $s_i^{(0)} = \tilde{s}_i^{(1)} = \epsilon_i^m \in \mathcal{M}_{L_n}$ for some $\epsilon_i \in \{\pm\}$ and an $n$-parallelization $(R_{l_i})_n := L_n \cap D_i$ of $l_i$ half twists for any $i$. Then, we define a clasped $\mathfrak{sl}_3$-web $R_{l_i}^{(n)}$ in $D_i$ by inserting one-row colored $\mathfrak{sl}_3$-clasps for each $n$ parallelized strands of $(R_{l_i})_{n}$. The one-row colored $\mathfrak{sl}_3$-Jones polynomial $J_{L,n}^{(n)}(q)$ of the anti-parallel $B$-adequate link $L$ is obtained by replacing $(R_{l_i})_n$ of $L_n$ with $R_{l_i}^{(n)}$. Using a linear map defined by $G_n$, this replacement is described as $G_n(\otimes_{i=1}^k (R_{l_i}^{(n)}))$. 
Lemma 4.6. For an anti-parallel $B$-adequate link diagram $L$ with twist regions $D_i$

$$L^{(n)} = G_n(\otimes_{i=1}^{k} (R_{i}^{(n)})) = \sum_{t_1, t_2, \ldots, t_k = 0}^{n} \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) G_n(\otimes_{i=1}^{k} M(t_i; n)),$$

where

$$\Gamma^{(n)}(t_i; l_i) = \gamma^{(n)}(t_i)^{l_i} \frac{\Delta^{(n)}(t_i)}{\Theta^{(n)}(t_i)}.$$

and

$$M(t_i; n) = \text{or}.$$

Proof. Apply the formula

$$\frac{n}{\Delta^{(n)}(t)} = \sum_{t=0}^{n} M(t; n)$$

shown in [Yua18] to all twist regions, and resolve twists by definition of $\gamma^{(n)}(j)$ in Lemma 4.4. We obtain the desired formula. \hfill \Box

Lemma 4.7. $d_*(\Gamma^{(n)}(t_i; l_i)) = d_*(\Gamma^{(n)}(t_i - 1; l_i)) + l_i (n - t_i + \frac{3}{2}) + 1.$

Proof. By Lemma 2.11 and Lemma 4.5,

$$d_*(\Gamma^{(n)}(t_i; l_i)) = l_i d_*(\gamma^{(n)}(t_i)) + d_*(\Delta^{(n)}(t_i)) - d_*(\Theta^{(n)}(t_i))$$

$$= l_i d_*(\gamma^{(n)}(t_i - 1)) + d_*(\Delta^{(n)}(t_i - 1)) - d_*(\Theta^{(n)}(t_i - 1))$$

$$+ l_i \left(n - t_i + \frac{3}{2}\right) + 1$$

$$= d_*(\Gamma^{(n)}(t_i - 1; l_i)) + l_i \left(n - t_i + \frac{3}{2}\right) + 1$$

\hfill \Box

Proposition 4.8. Let $L$ be an anti-parallel $B$-adequate link diagram with twist regions $\sqcup_{i=1}^{k} D_i$, $L: \{D_i\}^{k}_{i=1} \to 2\mathbb{Z}_{>0}$, and $G = L \cap D(k)$. Then,

$$G_n(\otimes_{i=1}^{k} (R_{i}^{(n)})) - \prod_{i=1}^{k} \gamma^{(n)}(0)^{l_i} G_n(\otimes_{i=1}^{k} JW_{D_i}) \in q^{2(n+2)} + d_*(L^{(n)} \mathbb{Z}[q]).$$

Proof. The proof is located below Lemma 4.12. \hfill \Box
To prove Proposition 4.8, we prepare several lemmas. First of all, let us introduce an operation $S_j$ corresponding to the single orientable saddle cobordism at $D_j$ for $i = 1, 2, \ldots, k$. More precisely, $S_j$ acts on the set $\{\otimes_{i=1}^k \uparrow l_{ti} | 0 \leq t_i \leq n \}$ of an $\mathfrak{sl}_3$-webs in $\sqcup_{i=1}^k D_i$ as follows: $S_j$ replaces $\uparrow l_{ti}$ with $\uparrow l_{ti-1}$ and acts by identity on $\uparrow l_{t_0}$ in $D_j$ or elements in $D_i$ with $i \neq j$. We remark that $\otimes_{i=1}^k \uparrow l_{ti}$ change to $\otimes_{i=1}^k \uparrow r_{ti}$ by a composition of orientable saddle cobordisms. For instance, $S_1^5 \cdots S_2^4 S_1^3$ realize this deformation.

**Lemma 4.9.** For an adequate graph $G$ of the anti-parallel $B$-adequate link diagram $L$ in Proposition 4.8 and any fixed tuple $(t_1, \ldots, t_k) \in \{0, \ldots, n\}^k$,

$$d_* \left( G_n(\otimes_{i=1}^k M(t_i; n)) \right) \geq d_* \left( G_n(\otimes_{i=1}^k \uparrow l_{ti}) \right).$$

**Proof.** A clasped $\mathfrak{sl}_3$-web $M(t_i; n)$ has five $\mathfrak{sl}_3$-clasps in $D_i$. For all $i = 1, 2, \ldots, k$, one can choose five small disks $D_{(i,1)}, \ldots, D_{(i,5)}$ in $D_i$ so that each small disk surrounds a single $\mathfrak{sl}_3$-clasp. The intersection of $L$ and $D \setminus \sqcup_{(i,j)} D_{(i,j)} | 1 \leq i \leq k, 1 \leq j \leq 5 \}$ become a graph $G'$ with no trivalent vertices. Then, $G_n(\otimes_{i=1}^k M(t_i; n)) = G_n(\otimes_{i=1}^k \uparrow l_{ti})$. We remark that $\otimes_{i=1}^k \uparrow l_{ti}$ change to $\otimes_{i=1}^k \uparrow r_{ti}$ by a composition of orientable saddle cobordisms. For instance, $S_1^5 \cdots S_2^4 S_1^3$ realize this deformation.

**Lemma 4.10.** For an adequate graph $G$ of the anti-parallel $B$-adequate link diagram $L$ in Proposition 4.8 and any fixed tuple $(t_1, \ldots, t_k) \in \{0, \ldots, n\}^k$ with $0 < t_j \leq n$,

$$d_* \left( \Gamma(n)(t_j; l_j)G_n(\otimes_{i=1}^k \uparrow l_{ti}) \right) > d_* \left( \Gamma(n)(t_j-1; l_j)G_n(S_j(\otimes_{i=1}^k \uparrow l_{ti})) \right).$$

**Proof.** We firstly note that $d_*(G_n(\otimes_{i=1}^k \uparrow l_{ti})) = -c(G_n(S_j(\otimes_{i=1}^k \uparrow l_{ti})))$. Hence $d_*(G_n(\otimes_{i=1}^k \uparrow l_{ti})) = -c(G_n(\otimes_{i=1}^k \uparrow l_{ti}))$ by Proposition 3.2. The orientable saddle cobordism $S_j$ changes the number of connected components by $\pm 1$ or 0. Hence $d_*(G_n(\otimes_{i=1}^k \uparrow l_{ti})) \geq d_*(G_n(S_j(\otimes_{i=1}^k \uparrow l_{ti}))) - 1$. This inequality and Lemma 4.7 conclude

$$d_* \left( \Gamma(n)(t_j; l_j)G_n(\otimes_{i=1}^k \uparrow l_{ti}) \right) = d_* \left( \Gamma(n)(t_j; l_j) \right) + d_* \left( G_n(S_j(\otimes_{i=1}^k \uparrow l_{ti})) \right) - 1$$

$$= d_* \left( \Gamma(n)(t_j-1; l_j) \right) + d_* \left( G_n(S_j(\otimes_{i=1}^k \uparrow l_{ti})) \right) + l_i(n - t_i + \frac{3}{2})$$

One can easily see that $l_i(n - t_i + \frac{3}{2}) \geq 3$ because $l_i \in \mathbb{Z}_{>0}$ and $0 < t_j \leq n$. □
Lemma 4.11. For an adequate graph $G$ of the anti-parallel $B$-adequate link diagram $L$ in Proposition 4.8 and any $0 < j \leq k$,
\[
d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(\delta_{ij}; l_i) G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow \delta_{ij}) \right) - d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) G_n(\otimes_{i=1}^{k} \uparrow D_i) \right) \geq 2n + 3,
\]
where $\delta_{ij}$ is the Kronecker delta function.

Proof. The adequacy of $G_n$ says that $c(G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow \delta_{ij})) - c(G_n(\otimes_{i=1}^{k} \uparrow D_i)) = -1$. We know $d_* (\Gamma^{(n)}(1; l_j)) - d_* (\Gamma^{(n)}(0; l_j)) = l_j(n + \frac{1}{2}) + 1$ by Lemma 4.7, $d_* (G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow \delta_{ij})) = -c(G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow \delta_{ij}))$ and $d_* (G_n(\otimes_{i=1}^{k} \uparrow D_i)) = -c(G_n(\otimes_{i=1}^{k} \uparrow D_i))$ by Proposition 3.2. Hence,
\[
d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(\delta_{ij}; l_i) G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow \delta_{ij}) \right) - d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) G_n(\otimes_{i=1}^{k} \uparrow D_i) \right)
= d_* (\Gamma^{(n)}(1; l_j)) - d_* (\Gamma^{(n)}(0; l_j)) + d_* (G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow \delta_{ij})) - d_* (G_n(\otimes_{i=1}^{k} \uparrow D_i))
= l_j(n + \frac{1}{2}) + 1 - c(G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow \delta_{ij})) + c(G_n(\otimes_{i=1}^{k} \uparrow D_i))
= l_j(n + \frac{1}{2}) + 2 \geq 2n + 3.
\]

The last inequality holds because $l_j$ is a positive even integer. \hfill \□

Lemma 4.12. For an adequate graph $G$ of the anti-parallel $B$-adequate link diagram $L$ in Proposition 4.8,
\[
d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) G_n(\otimes_{i=1}^{k} \uparrow \uparrow \downarrow D_i) \right) = d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) G_n(\otimes_{i=1}^{k} \uparrow \uparrow \downarrow D_i) \right)
\]

Proof. This assertion comes from the adequacy of $G$ and Proposition 3.9. \hfill \□

Proof of Proposition 4.8. By Lemma 2.11 and Lemma 4.9, we obtain
\[
d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) G_n(\otimes_{i=1}^{k} \uparrow \downarrow M(t_i; n)) \right) = \left( \sum_{i=1}^{k} d_* (\Gamma^{(n)}(t_i; l_i)) \right) + d_* (G_n(\otimes_{i=1}^{k} \uparrow \uparrow M(t_i; n)))
\geq \left( \sum_{i=1}^{k} d_* (\Gamma^{(n)}(t_i; l_i)) \right) + d_* (G_n(\otimes_{i=1}^{k} \uparrow \uparrow \downarrow D_i))
= d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) G_n(\otimes_{i=1}^{k} \uparrow \uparrow \downarrow D_i) \right).
\]

Choose a sequence of orientable saddle cobordisms that changes $G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow t_i)$ to $G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow 0) = G_n(\otimes_{i=1}^{k} \uparrow D_i)$, and apply Lemma 4.10 to $d_* \left( \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) G_n(\otimes_{i=1}^{k} \uparrow \uparrow^{(i)} \downarrow t_i) \right)$ along the sequence until just before the last step. We can apply Lemma 4.11 to the last orientable saddle
cobordism given by $S_j$. This operation gives

$$d_*\left( \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{d_{t_i}} \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{d_{t_i}} \right) > d_*\left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{D_{t_i}} \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{D_{t_i}} \right) + 2n + 3.$$

The above two inequalities and Lemma 4.12 conclude the following:

(4.1)

$$d_*\left( \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{M(t_i; n)} \right) > d_*\left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{D_{t_i}} \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{D_{t_i}} \right) + 2n + 3$$

= $d_*\left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{\mathbb{JW}_{D_{t_i}}} \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{\mathbb{JW}_{D_{t_i}}} \right) + 2n + 3$

for any $(t_1, t_2, \ldots, t_k) \neq (0, 0, \ldots, 0)$. Finally, we will compare $d_*\left( \bigotimes_{i=1}^{k} \gamma^{(n)}(0)_{l_i} \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{M(0; n)} \right)$ to $d_*\left( \bigotimes_{i=1}^{k} \Gamma^{(n)}(0; l_i) \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{D_{t_i}} \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{D_{t_i}} \right)$ by using the expansion in Lemma 4.6 and eq. (4.1). By Lemma 4.6,

$$G_n(\bigotimes_{i=1}^{k} \gamma^{(n)}(0)_{l_i}) = \left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) \right) G_n(\bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{M(0; n)}) + \sum_{(t_1, t_2, \ldots, t_k) \neq (0, 0, \ldots, 0)} \left( \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) \right) G_n(\bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{M(t_i; n)})$$

= $\left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) \right) G_n(\bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{\mathbb{JW}_{D_{t_i}}}) + \sum_{(t_1, t_2, \ldots, t_k) \neq (0, 0, \ldots, 0)} \left( \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) \right) G_n(\bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{M(t_i; n)})$.

By Lemma 2.11 and eq. (4.1), one can obtain

$$d_*\left( G_n(\bigotimes_{i=1}^{k} \gamma^{(n)}(0)_{l_i}) \bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{\mathbb{JW}_{D_{t_i}}} \right) - \left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) \right) G_n(\bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{\mathbb{JW}_{D_{t_i}}})$$

= $\min_{(t_1, t_2, \ldots, t_k) \neq (0, 0, \ldots, 0)} \left\{ d_*\left( \left( \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i) \right) G_n(\bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{M(t_i; n)}) \right) \right\}$

= $d_*\left( \left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) \right) G_n(\bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{\mathbb{JW}_{D_{t_i}}}) \right) + 2n + 3$

We remark that

$$d_*\left( G_n(\bigotimes_{i=1}^{k} \gamma^{(n)}(0)_{l_i}) \right) = d_*\left( \prod_{i=1}^{k} \Gamma^{(n)}(0; l_i) \right) G_n(\bigotimes_{i=1}^{k} \mathbb{1}^{(i)}_{\mathbb{JW}_{D_{t_i}}})$$
by eq. (4.1) and \( \Gamma^{(n)}(0; l_i) = \prod_{i=1}^{k} \gamma^{(n)}(0)^{l_i} \) by definition.

**Definition 4.13.** For \( f(q) \) and \( g(q) \) in \( \mathbb{Z}[[q]] \), we define \( f(q) \equiv_n g(q) \) if \( d_s\left( \hat{f}(q) - \hat{g}(q) \right) \geq n+1 \).

**Proposition 4.14.** Let \( JW_{D_i} \) represent \( M(0; n) \) in \( D_i \) defined in Lemma 4.6. If \( G \) is adequate, then

\[
G_{n+1}(\otimes_{i=1}^{k} JW_{D_i}) \equiv_{n+1} G_n(\otimes_{i=1}^{k} JW_{D_i}).
\]

**Proof.** We prove it below Lemma 4.15.

We will prove this proposition similar strategy to [Arm13]. Let us explain it in our situation. Choose a one-row colored \( sl_3 \)-clasp with \( n+1 \) strands in \( G_{n+1}(\otimes_{i=1}^{k} JW_{D_i}) \). It corresponds to \( n+1 \) parallel circles in \( G(\otimes_{i=1}^{k} 1D_i) \). First, we move the \( n+1 \) parallel strands to the left side of the two-row colored \( sl_3 \)-clasps at the center of \( D_i \). The “left side” is determined by the orientation of \( n+1 \) parallel strands at each \( sl_3 \)-clasp, see the following picture:

In the above, the chosen \( n+1 \) parallel strands are expressed as a red arc labeled by \( n+1 \). We remark that this deformation of \( sl_3 \)-webs does not change the coefficients. We assume that the chosen \( n+1 \) parallel strands pass through \( m \) two-row colored \( sl_3 \)-clasps \( JW_{D_1}, JW_{D_2}, \ldots, JW_{D_m} \) in this order by replacing labels of twist regions if necessary. We denote the initial \( sl_3 \)-web by \( G^{(n+1)}(0) \), and a clasped \( sl_3 \)-web obtained by unclasping the leftmost strand of the \( n+1 \) strands from \( JW_{D_1}, \ldots, JW_{D_{j-1}} \), and \( JW_{D_j} \) in \( G^{(n+1)}(0) \) by \( G^{(n+1)}(j) \) for \( j = 1, 2, \ldots, m-1 \). If one could unclasp the leftmost strand from \( JW_{D_1}, \ldots, JW_{D_{m-1}} \), then the \( sl_3 \)-web becomes \( G^{(n+1)}(m-1) \). One can shrink the unclasped strand in \( G^{(n+1)}(m-1) \) to \( JW_{D_m} \) as follows.
Lemma 4.15.

Proof of Proposition 4.14. Let us do the unclasping operation that we explained. Choose \( n + 1 \) parallel circles passing the left side of \( \mathfrak{sl}_3 \)-clasps \( JW_{D_1}, \ldots, JW_{D_m} \). We apply Lemma 4.15 to the \( j \)-th \( \mathfrak{sl}_3 \)-clasp \( JW_{D_j} \) in \( G^{(n+1)}(j - 1) \) for \( j = 1, 2, \ldots, m - 2 \). Then, we obtain

\[
G^{(n+1)}(j - 1) = G^{(n+1)}(j) + (-1)^{n+1} \frac{[k_j]}{n + k_j + 2} H^{(n+1)}(j),
\]

where \( H^{(n+1)}(j) \) is a clasped \( \mathfrak{sl}_3 \)-web corresponding to the second term in Lemma 4.15. We use \( k_j \) above although \( k_j = n + 1 \) in this situation because it is useful for later discussion. We compare \( d_*(G^{(n+1)}(j)) \) to \( d_*(H^{(n+1)}(j)) \). Let \( \tilde{G}^{(n+1)}(j) \) and \( \tilde{H}^{(n+1)}(j) \) denote a flat trivalent graph obtained by replacing all \( \mathfrak{sl}_3 \)-clasps in \( G^{(n+1)}(j) \) and \( H^{(n+1)}(j) \) with identity webs, respectively. Proposition 3.9 says that the lower bound of the minimum degree \( d_*(H^{(n+1)}(j)) \) is given by the number of vertices and connected components of \( \tilde{H} \). By tracing strands of \( H^{(n+1)}(j) \) as in Figure 4.2, one can see that \( c(\tilde{G}^{(n+1)}(j - 1)) - c(\tilde{H}^{(n+1)}(j)) = n + 1 \) and \( v(\tilde{H}^{(n+1)}(j)) = 2n \). By Proposition 3.9,

\[
d_*(H^{(n+1)}(j)) \geq -\frac{1}{4} v(\tilde{H}^{(n+1)}(j)) - c(\tilde{H}^{(n+1)}(j))
= -\frac{1}{4} (2n) - \left( c(\tilde{G}^{(n+1)}(j)) - (n + 1) \right)
= \frac{n + 2}{2} - c(\tilde{G}^{(n+1)}(j))
\]
Thus we obtain $G(\tilde{H}^{(n+1)}(j)) = 0$ claim $d_*(G^{(n+1)}(j)) = -c(G^{(n+1)}(j))$. Moreover $d_*(G^{(n+1)}(j)) = d_*(\tilde{G}^{(n+1)}(j))$ holds by adequacy of $G^{n+1}(j)$ and Proposition 3.9. Using Lemma 2.11 and Example 2.12, the above facts lead to the following inequality.

$$d_*\left((-1)^{n+1} \frac{[k]}{[n+k_j+2]} H^{(n+1)}(j)\right) \geq \frac{n+2}{2} + \left(\frac{n+2}{2} + d_*\left(G^{(n+1)}(j)\right)\right)$$

$$= (n+2) + d_*\left(G^{(n+1)}(j)\right).$$

It also holds that $d_*(G^{(n+1)}(j-1)) = d_*(G^{(n+1)}(j))$ due to Proposition 3.2 and Proposition 3.9. In fact, the adequacy of these clasped $\mathfrak{sl}_3$-webs and $G^{(n+1)}(j-1) = \tilde{G}^{(n+1)}(j)$ imply

$$d_*\left(G^{(n+1)}(j-1)\right) = d_*\left(G^{(n+1)}(j)\right) = -c(\tilde{G}^{(n+1)}(j-1)) = -c(\tilde{G}^{(n+1)}(j)) = d_*\left(G^{(n+1)}(j)\right).$$

Thus we obtain $G^{(n+1)}(j) - G^{(n+1)}(j-1) = (-1)^{n+1} \frac{[k]}{[n+k_j+2]} H^{(n+1)}(j) \in q^{(n+2)+d_*G^{(n+1)}} \mathbb{Z}[\mathbb{q}^{\frac{1}{2}}]$ where $d_*(G^{(n+1)}) := d_*(G^{(n+1)}(j-1)) = d_*(G^{(n+1)}(j))$.

We remark that this result holds independently of the number $m - j$ of $\mathfrak{sl}_3$-clasps that the $n+1$ paralleled strands pass through, and besides, the number $k_j$ of the oppositely oriented strands adjacent to the $n+1$ paralleled strands. We repeatedly apply Lemma 4.15 to $J\mathbb{W}_{D_1}, \ldots, J\mathbb{W}_{D_{m-1}}$ and obtain

$$G^{(n+1)}(0) \equiv_{n+1} G^{(n+1)}(1) \equiv_{n+1} \cdots \equiv_{n+1} G^{(n+1)}(m-1).$$

Let $G^{(n+1)}(m)$ be a $\mathfrak{sl}_3$-web removing the small circle from $G^{(n+1)}(m-1)$, see below:
Then, we obtain
\[ G^{(n+1)}(m - 1) = \frac{[n + 2][n + k_m + 3]}{[n + 1][n + k_m + 2]} G^{(n+1)}(m). \]
by Proposition A.1. From the above equality, it is easily seen that
\[ G^{(n+1)}(1) \equiv_{n+1} G^{(n+1)}(0) \]
holds for any \( k_m \). Next, we consider the leftmost strand of the other \( n + 1 \) parallel circles. One can unclasp the leftmost strand from \( \mathfrak{s}_3 \)-clasps exactly in the same way. The label \( k_j \) in this argument might be \( n \). However, it works independently of \( k_j \) as I mentioned above. We repeatedly apply this argument until all \( n + 1 \) parallel circles passing through \( \mathfrak{s}_3 \)-clasps become \( n \) parallel strands. Consequently, we obtain \( G_{n+1}(\otimes_{i=1}^k 1_D_i) \equiv_{n+1} G_n(\otimes_{i=1}^k 1_D_i) \).

**Theorem 4.16.** Let \( L \) be an anti-parallel \( B \)-adequate link. Then,
\[ \hat{j}^{\mathfrak{s}_3}_{L,n+1}(q) - \hat{j}^{\mathfrak{s}_3}_{L,n}(q) \in q^{n+1}Z[[q]]. \]
In other words, the one-row colored \( \mathfrak{s}_3 \)-Jones polynomials \( \{ \hat{j}^{\mathfrak{s}_3}_{L,n}(q) \}_n \) of \( L \) is zero stable.

**Proof.** Let us take a link diagram \( G(\otimes_{i=1}^k R_i) \) of \( L \) with an adequate graph \( G \) and twist regions \( I: \{ D_i \}_{i=1}^k \rightarrow 2Z_{>0} \). The one-row colored \( \mathfrak{s}_3 \)-Jones polynomial \( \hat{j}^{\mathfrak{s}_3}_{L,n}(q) \) is given by the normalization in Definition 2.10 of the clasped \( \mathfrak{s}_3 \)-web \( G_n(\otimes_{i=1}^k (R_i^{(n)})) \). Lemma 4.6 and Proposition 4.8 claim that
\[ G_n(\otimes_{i=1}^k (R_i^{(n)})) \equiv_{2n+1} \prod_{i=1}^k \gamma^{(n)}(0) \gamma^{(n)}(k-n-1) \]
and Proposition 4.14 claims
\[ G_n(\otimes_{i=1}^k (R_i^{(n)})) = G_n(\otimes_{i=1}^k (R_i^{(n)})). \]
It is easy to see that \( f(q) = n \) \( g(q) \) if \( f(q) = n \) \( g(q) \) for some \( N \geq n \), and \((-1)^{k_i} q^{l_i} g(q) \) if \( f(q) = n \) \( g(q) \) for any \( k_i, l_i, l_2 \). Hence, the above equivalence relations derive
\[ G_n(\otimes_{i=1}^k (R_i^{(n)})) = G_n(\otimes_{i=1}^k (R_i^{(n)})). \]
It means \( \hat{j}^{\mathfrak{s}_3}_{L,n+1}(q) - \hat{j}^{\mathfrak{s}_3}_{L,n}(q) \in q^{n+1}Z[[q^{1/2}]]. \)
It is well-known that the closure of $\text{JW}_{-m+n}$ is given by
\[
\Delta(m,n) = \begin{array}{c}\includegraphics{circle} \end{array} = \frac{(m+1)(n+1)(m+n+2)}{2}.
\]

**Proposition A.1.**
\[
\begin{array}{c}\includegraphics{triangle} \end{array} = \frac{\Delta(m+l,n)}{\Delta(m,n)} \begin{array}{c}\includegraphics{triangle} \end{array}
\]

**Proof.** It is known that this clasped $\mathfrak{sl}_3$-web space is a one-dimensiona1 and it is spanned by $\text{JW}_{-m+n}$. Thus, we set
\[
\begin{array}{c}\includegraphics{triangle} \end{array} = C \begin{array}{c}\includegraphics{triangle} \end{array}.
\]
The closure of diagrams in the left- and right-hand sides are given by $\Delta(m+l,n)$ and $\Delta(m,n)$, respectively. Hence, $C = \Delta(m+l,n)/\Delta(m,n)$.

In order to prove Lemma 4.15, we prepare some lemmas.

**Lemma A.2** (The bubble skein expansion formula [Yua17]).
\[
\begin{array}{c}\includegraphics{triangle} \end{array} = \sum_{t = \max\{a,b\}}^{\min\{a+b,k,l\}} \begin{array}{c}\includegraphics{triangle} \end{array} \begin{array}{c}\includegraphics{triangle} \end{array} \begin{array}{c}\includegraphics{triangle} \end{array} \begin{array}{c}\includegraphics{triangle} \end{array} \begin{array}{c}\includegraphics{triangle} \end{array}
\]

**Lemma A.3** ([Kim07, Theorem 3.3]).
\[
\begin{array}{c}\includegraphics{triangle} \end{array} = \begin{array}{c}\includegraphics{triangle} \end{array} - \frac{k}{k+1} \begin{array}{c}\includegraphics{triangle} \end{array} - \frac{l}{k+1} \begin{array}{c}\includegraphics{triangle} \end{array} \begin{array}{c}\includegraphics{triangle} \end{array} \begin{array}{c}\includegraphics{triangle} \end{array} \begin{array}{c}\includegraphics{triangle} \end{array}
\]

**Lemma A.4.**
\[
\begin{array}{c}\includegraphics{triangle} \end{array} = \frac{(-1)^k}{k+1} \begin{array}{c}\includegraphics{triangle} \end{array}
\]

**Proof.** Apply Lemma 3.3 to an $\mathfrak{sl}_3$-clasp in the left above. Then, one can see that diagrams in the expansion vanish except for the last term due to the bottom $\mathfrak{sl}_3$-clasp. It becomes the right-hand side by Lemma 2.9 (2).
Lemma A.5.

\[
\begin{align*}
\frac{\k+l-1}{\k+l} & = (-1)^k \frac{[l+1]}{[k+l+1]} \quad \text{and} \\
\frac{\k+l-1}{\k+l} & = (-1)^k \frac{[l+1]}{[k+l+1]}
\end{align*}
\]

Proof. It is known that the \(\mathfrak{sl}_3\)-web space on a disk with clasped end points \(\text{JW}_{-k,l}, \text{JW}_{+k+1,l}\), and \(\text{JW}_{-k+l-1}\) is spanned by one clasped \(\mathfrak{sl}_3\)-web in the right-hand side. See, for example, [Kup96, Kim07] for details. Hence we only have to determine the coefficient \(C\) such that

\[
\frac{\k+l-1}{\k+l} = C.
\]

Attach an \(\mathfrak{sl}_3\)-web to the top of \(\mathfrak{sl}_3\)-webs in both sides. The left-hand side becomes \((-1)^k \Delta(k,l)\text{JW}_{k+l+1}\) by Lemma A.4 and Proposition A.1. The right-hand side becomes \(C[k+l+2][k+l+1]\text{JW}_{k+l+1}\) by Lemma A.2. We obtain the value \(C\) in the assertion by solving this equation. \(\square\)

Proposition A.6.

\[
\frac{\k+l}{\k+l+2} = \frac{\k+l}{\k+l+2} + (-1)^{k+1}\frac{[1]}{[k+l+2]}.
\]

Proof. Let us denote the second coefficient in the assertion by \(a_k := \frac{(-1)^{k+1}[l]}{[k+l+2]}\). We firstly attach an \(\mathfrak{sl}_3\)-clasp to the left top of diagrams in Lemma A.3. Next, we calculate the second and the third terms on the right-hand side of the resulting equation. More precisely, we will show

\[
-\frac{[k]}{[k+1]} \frac{\k+l}{\k+l+2} - \frac{[l]}{[k+1][k+l+2]} = a_k
\]

(A.1)
We remark that the second $s^3_l$-web on the left-hand side is already done in Lemma A.5, and it provides a coefficient $(-1)^k [l + 1]/[k + l + 1]$. eq. (A.1) holds if the first $s^3_l$-web provides a coefficient $a_{k-1}$ because the summation of these coefficients is calculated as follows.

$$\begin{align*}
- \frac{[k]}{[k+1]} a_{k-1} + 
\left( -\frac{[l]}{[k+1][k+l+2]} \right) 
\left( (-1)^k \frac{[l+1]}{[k+l+1]} \right) 
= (-1)^{k+1} \frac{[l]}{[k+1][k+l+1][k+l+2]} ([k][k+l+2] + [l+1]) 
= (-1)^{k+1} \frac{[l]}{[k+l+2]} = a_k.
\end{align*}$$

We used $[k+1][k+l+1] - [k][k+l+2] = [l+1]$ in the equation above. Hence, let us prove

(A.2)

\[ k + 1 \]
\[ k \]
\[ l \]
\[ \begin{array}{c}
1 \\
1
\end{array} \]
\[ k + 1 \]
\[ k \]
\[ l \]
\[ \begin{array}{c}
1 \\
1
\end{array} \]

by induction on $k$. One can easily prove eq. (3.2) at $k = 1$ by expanding the middle $s^3_l$-clasp. We assume that eq. (A.2) holds when $k = n$. It means that Proposition A.6 also holds when $k = n$ by the above argument. Thus, we apply Proposition A.6 to the middle $s^3_l$-clasp on the left-hand side of eq. (A.2) at $k = n + 1$, and one can confirm that it concludes the right-hand side of eq. (A.2). \hfill \square

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