Superconductivity in Graphene Induced by the Rotated Layer

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Abstract

Recent discoveries in graphene bilayers revealed that when one of the layers is rotated, superconductivity emerges. We provide an explanation for this phenomenon. We find that due to the layer rotations, the spinors are modified in such way that a repulsive interaction, becomes attractive in certain directions. This result is obtained following a sequence of steps: when layer 2 is rotated by an angle $\theta$, this rotation is equivalent to a rotation of an angle $-\theta$ of the linear momentum. Due to the discreet lattice, in layer 1, the Fourier transform conserves the linear momentum modulo the hexagonal reciprocal lattice vector. In layer 2, due to the rotation, the linear momentum is conserved modulo the Moire reciprocal lattice vector. Periodicity is achieved at the magical angles obtained from the condition of commensuration of the two lattices. We find that the rotations transform the spinors around the nodal points, such that a repulsive interaction becomes attractive, giving rise to superconductivity.
I. INTRODUCTION

A commensurate triangular Moire pattern is formed when a top-layer graphene is rotated with respect to the bottom layer at certain angles [1–3]. A model with a large amount of atoms in a commensurable unit cell was considered in order to explain the formation of the flat bands that might lead to superconductivity. The model consists of interlayer interaction of \( \pi \) orbitals. The fit of the tight binding model was reproduced using a Density Functional Theory (DFT) calculation of stacked bilayers. Further progress has been achieved by [3–5] who computed the spectrum considering Coulombic interactions and phonon-mediated superconductivity. Moire insulators have been viewed as a surface for a Symetric Protected Topological phases [9] and by proximity to Mott insulators [6]. The proximity to Wigner crystallization and Mott insulator has also been considered [7]. The effect of Van Hove singularity was used by [8] in analogy with high \( T_c \) superconductivity calculation were double logarithmic singularity was the cause of Superconductivity for bar-repulsive interactions. The uniform rotation generates flat bands. When the rotation angle generates a periodic Moire structure commensurate with the honeycomb lattice, and the unit cell contains a large number of atoms, the Brillouine Zone (BZ) becomes small, the fermi velocity vanishes and a flat band appears [15, 16]. For simplicity, we consider the rotation of layer 2 in such a way that the site occupied by atom \( B' \) is situated directly opposite from atom \( A \) in layer 1. A commensurate structure is obtained if atom \( B' \) is moved by the rotation to a position formerly occupied by an atom of the same kind [17]. Following [17, 18], we determine the condition for the angles \( \theta(n) \) of a commensurate rotation. Our goal is to investigate the uniform rotation which can shine light on the mechanism which is responsible for the attraction and for causing superconductivity.

To achieve this goal, we need to compute the effect of rotation on the spinors. We will use the tight binding model for bilayer graphene [10, 12] and take into consideration the discreetness of the lattice [11]. A two-dimensional honeycomb array of carbon atoms forming a hexagonal lattice can be viewed as a superposition of two triangular sublattices, \( A \) and \( B \). The generators of lattice \( A \) are vectors \( \vec{a}_1 \) and \( \vec{a}_2 \). We have three vectors \( \delta_i \) connecting any site from lattice \( A \) to nearest neighbor sites belonging to \( B \) (layer 1). Layer \( i = 1 \) is not rotated and the sum over the position of atom \( A \) and \( B \) give rise to a summation over the reciprocal lattice vectors \( \sum_{n,m} e^{i\vec{p} \cdot \vec{R}_{n,m}} = N \delta_{k_x G_x^{(1)} + G_x^{(2)} k_y G_y^{(1)} + G_y^{(2)}} \) where, \( \vec{R}_{n,m} = n\vec{a}_1 + m\vec{a}_2 \) with integers \( n \) and \( m \).
Layer 2 is uniformly rotated. In real space the uniform rotation of the coordinates by an angle \( \theta \) is equivalent to a rotation \(-\theta\) of the momentum space \( e^{i\vec{p}[\theta]} \vec{R}_{n,m} = e^{i\vec{\theta}} \vec{R}_{n,m} \) where \( \vec{R}_{n,m} \) is the rotated vector. Performing the discrete summation in layer 2, we obtain \( \sum_{n,m} e^{i\vec{p}[\theta]} \vec{R}_{n,m} = N \delta_{k_x,g_x[\theta]} \delta_{k_y,g_y[\theta]} \) where \( g_x[\theta] = (G_x^{(1)} + G_x^{(2)}) \text{Cos}[\theta] \) and \( g_y[\theta] = (G_x^{(1)} + G_x^{(2)}) \text{Sin}[\theta] \) are the reciprocal lattice vectors which at the magic angle become the new reciprocal lattice vector which emerges in the following way: layer 2 is rotated about a site occupied by atom \( B' \) directly opposite of an atom \( A \) (layer 1). A commensurate structure is obtained if a \( B' \) atom is moved by rotation to a position formerly occupied by an atom \( B' \) of the same kind. The Moire pattern is periodic and a translation from the center to the position of \( B' \) is a translation symmetry given by, \( \text{Cos}[\theta_i] = \frac{3\pi^2 + 3\pi + 0.5}{3\pi^2 + 3\pi + 1} \), \( i = 0,1,2 \) vectors. A superlattice with basis vectors \( \vec{t}_1 = i\vec{a}_1 + (i + 1)\vec{a}_2; \vec{t}_2 = -(i + 1)\vec{a}_1 + (2i + 1)\vec{a}_2 \) is formed [18]. The Moire reciprocal lattice vector is given by \( g_x[\theta] = g_x[\theta_1] \) and \( g_y[\theta] = g_y[\theta_i] \) \( (\text{Cos}[\theta_i] = \frac{3\pi^2 + 3\pi + 0.5}{3\pi^2 + 3\pi + 1}) \). Using the tunneling between the layers at magic angles we obtain flat bands.

We linearize the bilayer Hamiltonian with respect to the nodal position and obtain a Dirac representation. For layer 2, the nodal position depends on the rotated angles \( \vec{K}_2 = \frac{4\pi}{3a} \left[ \text{Sin}[\theta], \frac{1}{\sqrt{3}} \text{Cos}[\theta] \right] \) and \( \vec{K'}_2 = -\frac{4\pi}{3a} \left[ \text{Sin}[\theta], \frac{1}{\sqrt{3}} \text{Cos}[\theta] \right] \).

As a result, the spinor will depend on the rotated angles. For certain angles and for certain valley components, the repulsive interaction becomes attractive:

\[
L_{\text{int.}} = |U[\theta]| \int \text{dy} \left[ \hat{C}_{2,R,\uparrow}^\dagger (p_x = 0, y) \hat{C}_{2,R,\downarrow}^\dagger (p_x = 0, y) \hat{C}_{2,L,\downarrow}^\dagger (p_x = 0, y) \hat{C}_{2,L,\uparrow}^\dagger (p_x = 0, y) + \hat{C}_{2,L,\uparrow}^\dagger (p_x = 0, y) \hat{C}_{2,L,\downarrow}^\dagger (p_x = 0, y) \hat{C}_{2,R,\downarrow}^\dagger (p_x = 0, y) \hat{C}_{2,R,\uparrow}^\dagger (p_x = 0, y) \right],
\]

(1)

where \( p_x = 0 \) corresponds to the nodal component \( K_{2,x} \). As a result, the superconductor is one dimensional with periodicity in the transversal direction.

The band satisfy \( E(p) = \epsilon(p) - \mu < \Lambda \) (cut-off). In our case the flat band, could be of the order of the chemical potential , and the condition \( E(p) = \epsilon(p) - \mu < \Lambda \) is not obeyed and superconducting is not achieved.

The outline of this paper is: in chapters II and III we consider the model in the real space representation. In chapter IV we linearize the model around the nodal points obtaining a Dirac representation for the two valleys. We show that at the magic angles, the low energy bands are flat and the spinors transform the repulsive interaction to an attractive interactions.
in certain directions. In chapter V, we include the spin degrees of freedom and double the number components of spinor.

III- The real space approach

In order to investigate the effect of the rotation in real space, we introduce the spinors \( \Psi_i = [a_i, b_i] \), where \( a_i \) and \( b_i \) represent the two honeycomb lattices and \( i = 1, 2 \) is the index of the two layers. For layer 1, we have a two dimensional honeycomb array of Carbon atoms forming a hexagonal lattice which can be viewed as a superposition of two triangular sublattices , A and B. The generators of lattice A are vectors \( \vec{a}_1 \) and \( \vec{a}_2 \). We have three vectors \( \delta_r \) connecting any site from lattice A to nearest neighbor sites belonging to B. We have the representation:

\[
a_1(\vec{R}) = \frac{1}{\sqrt{L^2}} \sum_{\vec{k}} a_1(\vec{k}) e^{i\vec{k} \cdot \vec{R}}
\]

\[
b_1(\vec{R} + \delta_r) = \frac{1}{\sqrt{L^2}} \sum_{\vec{k}} b_1(\vec{k}) e^{i\vec{k} \cdot (\vec{R} + \delta_r)} ; r = 1, 2, 3
\]

(2)

where \( a_1(\vec{R}) = a_1(\vec{R} + L) \) and \( b_1(\vec{R}) = b_1(\vec{R} + L) \) obey the Born-Von Karman condition \[11\] \( \vec{k} = \frac{2\pi}{L} \vec{n} \).

The Hamiltonian for layer 1 is given by \[12, 13\].

\[
H_1 = -\sum_{\vec{R}} \sum_{r=1,2,3} \left[ \gamma_0 a_1^\dagger(\vec{R}) b_1(\vec{R} + \delta_r) + H.C. \right]
\]

\[
= -\frac{1}{N_1N_2} \sum_{\vec{R}} \sum_{r=1,2,3} \sum_k \sum_p \left[ \gamma_0 a_1^\dagger(\vec{k}) b_1(\vec{p}) e^{-i(\vec{k} - \vec{p}) \cdot \vec{R}} e^{i\vec{p} \cdot \delta_r} + H.C. \right]
\]

\[
= -\sum_{\vec{k}} \sum_{\vec{p}} \left[ b_1(\vec{k}) a_1(\vec{p}) \phi_1(\vec{p}) \delta_{k_x,p_x+G_x^{(1)}+G_x^{(2)},k_y,p_y+G_y^{(1)}+G_y^{(2)}} + H.C. \right]
\]

(3)

Using the periodicity of the reciprocal lattice \( a_1(p_x,p_y) = a_1(p_x + G_x^{(1)} + G_x^{(2)}, p_y + G_y^{(1)} + G_y^{(2)}) \), and \( b_1(p_x,p_y) = b_1(p_x + G_x^{(1)} + G_x^{(2)}, p_y + G_y^{(1)} + G_y^{(2)}) \) we obtain:

\[
H_1 = \sum_{\vec{p} \in B.Z.} \left[ a_1^\dagger(\vec{p}) b_1(\vec{p}) \phi_1(\vec{p}) + H.C. \right]
\]

(4)

where the hexagonal reciprocal lattice vectors are \( \vec{G}^{(1)} = \frac{2\pi}{3a} \left[ 1, \sqrt{3} \right] \) and \( \vec{G}^{(2)} = \frac{2\pi}{3a} \left[ 1, -\sqrt{3} \right] \), with the two Bravais unit cell vectors \( \vec{a}_{(1)} = \frac{a}{2} \left[ 3, \sqrt{3} \right] \), \( \vec{a}_{(2)} = \frac{a}{2} \left[ 3, -\sqrt{3} \right] \).
The discrete sum over the integers \(n\) and \(m\), \(\tilde{R}_{n,m} = n\tilde{a}(1) + m\tilde{a}(2)\) determines the position of the lattice atom \(A\)\[10\]. Atom \(B\) is given by the relative vectors \(\delta_r\), \(r=1,2,3\), with respect to atom \(A\) at position \(\tilde{R}_{n,m}\). The sum over the vectors \(\delta_r\) determines the function \(\phi_1(\vec{p})\).

\[
\phi_1(\vec{p}) = -\gamma_0 \sum_{r=1,2,3} e^{-i\vec{p} \cdot \delta_r} = e^{-ip_x} \left[ 1 + 2e^{i\frac{3\pi}{2}p_y} \cos \left( \frac{\sqrt{3}}{2}p_y \right) \right]
\]  

(5)

The location of the two nodes in layer \(i = 1\) is given by \(\vec{R}_1 = [0, \frac{4\pi}{3\sqrt{3}}]\) and \(\vec{R}'_1 = [0, -\frac{4\pi}{3\sqrt{3}}]\) which obey \(\phi_1(\vec{p} = \vec{R}_1) = 0\) and \(\phi_1(p = \vec{R}'_1) = 0\). Layer 2 is rotated with respect to layer 1. In layer 1 the atoms are \(A_1\) and \(B_1\) while in layer 2 the atoms are \(A' = A_2\) and \(B' = B_2\). In a stacked \(A,B\) bilayer, \(A'\) and \(B'\) have the same horizontal position as atoms \(A_1\) and \(B_1\).

The rotation of layer 2 occurs in such a way that the site occupied by atom \(B'\) is located directly opposite an atom \(A\) (layer 1). A commensurate structure is obtained if a \(B'\) atom is moved by rotation to a position formerly occupied by an atom \(B'\) of the same kind. The \textit{Moire} pattern is periodic and a translation from the center to the position of \(B'\) is a translation symmetry.

\[
\cos[\theta_i] = \frac{3\beta^2 + 3\alpha + 0.5}{3\beta^2 + 3\alpha + 1}, \text{where the vector } i \text{ is } i=0,1,2...
\]

The superlattice bases are \(\vec{t}_1 = i\tilde{a} + (i + 1)\tilde{a}_2; \vec{t}_2 = -(i + 1)\tilde{a} + (2i + 1)\tilde{a}_2;\)

In addition, in layer 2, the vector \(\vec{R}_{n,m}\) is replaced by the rotated vector \(\vec{R}'_{n,m}; \vec{R}'_{n,m} = \vec{R}(\theta)_{n,m} = \left[R_x, R_y\right]_{n,m} = \left[R_x \cos[\theta] - R_y \sin[\theta], R_x \sin[\theta] + R_y \cos[\theta]\right]_{n,m}\) and \(\vec{\delta}_r\) is replaced by, \(\vec{\delta}'_r(\theta) = \left[\delta'_{r,x}, \delta'_{r,y}\right] = \left[\delta_{r,x} \cos[\theta] - \delta_{r,y} \sin[\theta], \delta_{r,x} \sin[\theta] + \delta_{r,y} \cos[\theta]\right]\)

The Hamiltonian for the rotated layer 2 is given by:

\[
H_2 = -\sum_{\vec{R}} \sum_{r=1,2,3} \gamma_0 \left[ b^\dagger_2(\vec{R}') a_2(\vec{R}' - \vec{\delta}_r) + H.C. \right]
\]

\[
= -\frac{1}{N_1 N_2} \sum_{\vec{R}} \sum_{r=1,2,3} \sum_{\vec{k}} \sum_{\vec{p}} \left[ \gamma_0 b^\dagger_2(\vec{k}) a_2(\vec{p}) e^{-i(\vec{k} - \vec{p}) \bullet \vec{R}' - i\vec{p} \cdot \vec{\delta}_r} + H.C. \right]
\]

\[
= -\frac{1}{N_1 N_2} \sum_{\vec{R}} \sum_{r=1,2,3} \sum_{\vec{k}} \sum_{\vec{p}} \left[ \gamma_0 b^\dagger_2(\vec{k}) a_2(\vec{p}) e^{-i(\vec{k} - \vec{p}) + i\vec{p} \cdot \vec{\delta}_r} + H.C. \right]
\]

(6)

When we rotate the coordinates of layer 2 by an angle \(\theta\), the momentum is rotated by angle \(-\theta, \vec{k}' = \left[k_x \cos[\theta] + k_y \sin[\theta], k_y \cos[\theta] - k_x \sin[\theta]\right]\)

\[
H_2 = -\sum_{\vec{k}} \sum_{\vec{p}} \left[ a_2(\vec{k}) b_2(\vec{p}) \phi_2(\vec{p}) \delta_{k_x,p_x + g_{x}[\theta]} \delta_{k_y,p_y + g_{y}[\theta]} + H.C. \right]
\]

\[
g_x[\theta] = (G_x^{(1)} + G_x^{(2)}) \cos[\theta]; \quad g_y[\theta] = (G_x^{(1)} + G_x^{(2)}) \sin[\theta]
\]

(7)
$g_x[\theta]$ and $g_y[\theta]$ are the new BZ with two lattice vectors $L_1, L_2$.

At special angles $g_x[\theta = \theta_{magic}] = g_x^{Moire}$, $g_y[\theta = \theta_{magic}] = g_y^{Moire}$ given by $\cos[\theta(n)] = \frac{3n^2 + 3n + 1}{3n^2 + 3n + 1}$, we obtain the Moire commensurate rotations such that $g_x[\theta_{magic} = \theta(n)]L_1 = 2\pi, g_y[\theta_{magic} = \theta(n)]L_2 = 2\pi$.

Using the periodicity in the BZ with respect to the Moire reciprocal lattice $a_2(p_x, p_y) = a_2(p_x + g_x[\theta, p_y + g_y[\theta])$, $b_2(p_x, p_y) = b_2(p_x + g_x[\theta], p_y + g_y[\theta])$, we find that at the magic angle $[\theta_{magic}]$ it is commensurate with the hexagonal lattice $a_2(p_x, p_y) = a_2(p_x + g_x[\theta = 0], p_y + g_y[\theta = 0])$, $b_2(p_x, p_y) = b_2(p_x + g_x[\theta = 0], p_y + g_y[\theta = 0])$. As a result of the periodicity and commensuration we obtain:

$$H_2 = \sum_{\vec{p} \in MoireB.Z} \left[ b_2(\vec{p})a_2(\vec{p})\varphi_2(\vec{p}) + H.C. \right]$$ (8)

We replace $\vec{p} \cdot \vec{δ}_r(\theta)$ with $\vec{p}[-\theta] \cdot \vec{δ}_r$ and compute $\varphi_2(\vec{p})$:

$$\varphi_2(\vec{p}) = \phi_1(\vec{p}[-\theta]) = \gamma_0 \sum_{r=1,2,3} e^{-ip_r\delta_r} = \gamma_0 \sum_{r=1,2,3} e^{-ip[-\theta]r\delta_r}$$

$$\varphi_2(\vec{p}) = \gamma_0 e^{-ip_x\cos[\theta]}e^{-ip_y\sin[\theta]} \left[ 1 + 2e^{\frac{2\pi}{3}\psi_x\cos[\theta]}e^{\frac{2\pi}{3}\psi_y\sin[\theta]} \cos \left( \frac{\sqrt{3}}{2}(p_x\cos[\theta] - p_y\sin[\theta]) \right) \right]$$ (9)

The Dirac points in layer 2 are at momentum $\vec{K}_2 = \frac{4\pi}{3a} \left[ \sin[\theta], \frac{1}{\sqrt{3}} \cos[\theta] \right]$ and $\vec{K}_2' = -\frac{4\pi}{3a} \left[ \sin[\theta], \frac{1}{\sqrt{3}} \cos[\theta] \right]$.

The tunneling Hamiltonian between layer 1 and 2 is given by [17]:

$$H_{1,3} = -\frac{\gamma_3}{\gamma_0} \sum_{r=1,2,3} \sum_{\vec{R}} \left[ b_2^\dagger(\vec{R})a_1(\vec{R}) + a_2^\dagger(\vec{R})b_1(\vec{R} + \vec{δ}_r) \right] =$$

$$-\frac{1}{N_1N_2} \frac{\gamma_3}{\gamma_0} \sum_{r=1,2,3} \sum_{\vec{R}} \sum_{\vec{k}} \sum_{\vec{p}} \left[ b_2^\dagger(\vec{k})a_1(\vec{p})e^{-i\vec{k}[-\theta] - \vec{p}}e^{-i\vec{k}[\theta] - \vec{p}} \delta_{\vec{k}, \vec{p}}e^{-i\vec{k}[\theta] - \vec{p}}\vec{R} e^{ip_r\delta_r} + a_2(\vec{k})b_1(\vec{p})e^{-i\vec{k}[\theta] - \vec{p}}\vec{R} e^{ip_r\delta_r} \right]$$

$$-\frac{1}{N_1N_2} \frac{\gamma_3}{\gamma_0} \sum_{r=1,2,3} \sum_{\vec{R}} \sum_{\vec{k}} \sum_{\vec{p}} \left[ b_2^\dagger(\vec{k})a_1(\vec{p})e^{-i\vec{k}[-\theta] - \vec{p}}\vec{R} e^{ip_r\delta_r} \varphi_2^*\vec{p}(\vec{k}) + a_2(\vec{k})b_1(\vec{p})e^{-i\vec{k}[\theta] - \vec{p}}\vec{R} \phi_1(\vec{p}) \right] =$$

$$-\frac{\gamma_3}{\gamma_0} \sum_{\vec{k}} \sum_{\vec{p}} \left[ b_2^\dagger(\vec{k})a_1(\vec{p})\delta_{\vec{k}, \vec{p}} e^{-i\vec{k}[\theta] - \vec{p}}\vec{R} + a_2(\vec{k})b_1(\vec{p})e^{-i\vec{k}[\theta] - \vec{p}}\vec{R} \phi_1(\vec{p}) \right]$$

$$+ a_2(\vec{k})b_1(\vec{p})\delta_{\vec{k}, \vec{p}} e^{-i\vec{k}[\theta] - \vec{p}}\vec{R} + a_2(\vec{k})b_1(\vec{p})e^{-i\vec{k}[\theta] - \vec{p}}\vec{R} \phi_1(\vec{p}),$$

where $\vec{p}[-\theta] = [p_x\cos[\theta] + p_y\sin[\theta], p_y\cos[\theta] - p_x\sin[\theta]]$.
For small angle rotations, we replace $H_{\perp,3}$ with the Moire BZ:

$$H_{\perp,3} = -\frac{\gamma_3}{\gamma_0} \sum_{\vec{p} \in \text{Moire BZ}} \left[ b_2^\dagger(\vec{p})a_1(\vec{p})\phi_1(\vec{p}) + a_2^\dagger(\vec{p})b_1(\vec{p})\phi_1^*(\vec{p}) \right]$$  \hspace{1cm} (11)

Here, the tunneling coupling constant is given by $\gamma_0 = 2.8eV, \gamma_3 = 0.3eV$ \[10\].

**IV-Computation of the eigenvalues**

We will compute the eigenvalues for small angles which correspond to a Moire commensurate lattice. The Hamiltonian $H_1$ and $H_2$ are diagonalized using the following representation: for layer $i = 1$ the eigenvalues are $\pm |\varphi_1(\vec{p})|$, with the two eigenvectors $u_1(\vec{p}) = \frac{1}{\sqrt{2}} \left[ 1, -e^{-i\alpha_1(\vec{p})} \right]^T = \frac{1}{\sqrt{2}} \left[ 1, -\frac{\phi_1(\vec{p})}{\phi_1(\vec{p})} \right]^T$ and for negative eigenvalues the eigenvector is $v_1(\vec{p}) = \frac{1}{\sqrt{2}} \left[ 1, e^{-i\alpha_1(\vec{p})} \right]^T$

For layer $i = 2$ we have the eigenvalue $\pm |\varphi_2(\vec{p})|$. The eigenvector for layer $i = 2$ is $u_2(\vec{p}) = \frac{1}{\sqrt{2}} \left[ 1, -e^{-i\alpha_2(\vec{p})} \right]^T = \frac{1}{\sqrt{2}} \left[ 1, -\frac{\varphi_2(\vec{p})}{|\varphi_2(\vec{p})|} \right]^T$ for positive eigenvalues, while for the negative eigenvalues $v_2(\vec{p}) = \frac{1}{\sqrt{2}} \left[ 1, e^{-i\alpha_2(\vec{p})} \right]^T$

$$\Psi_2(\vec{p}) = C_2(\vec{p})u_2(\vec{p}) + D_2^\dagger(\vec{p})v_2(\vec{p})$$
$$H_2 = \sum_{\vec{p}} C^\dagger_2(\vec{p})C_2(\vec{p})|\varphi_2(\vec{p})| + D^\dagger_2(\vec{p})D_2(\vec{p})|\varphi_2(\vec{p})|$$
$$\Psi_1(\vec{p}) = C_1(\vec{p})u_1(\vec{p}) + D_1^\dagger(\vec{p})v_1(\vec{p})$$
$$H_1 = \sum_{\vec{p}} C^\dagger_1(\vec{p})C_1(\vec{p})\phi_1(\vec{p})| + D^\dagger_1(\vec{p})D_1(\vec{p})\phi_1(\vec{p})|$$

\hspace{1cm} (12)

$C^\dagger_2(\vec{p}),C_2(\vec{p}),C^\dagger_1(\vec{p}),C_1(\vec{p})$ are the particle operators and $D^\dagger_2(\vec{p}),D_2(\vec{p}),D^\dagger_1(\vec{p}),D_1(\vec{p})$ are the anti-particles operators.

**We neglect the anti-particles and rewrite the Hamiltonian in terms of the particle operators only.** We consider the small angles such that the Moire lattice is commensurate with the hexagonal lattice.

$$H_1 = \sum_{\vec{p}} C^\dagger_1(\vec{p})C_1(\vec{p})\phi_1(\vec{p})|$$
$$H_2 = \sum_{\vec{p}} C^\dagger_2(\vec{p})C_2(\vec{p})\varphi_2(\vec{p})|$$

\hspace{1cm} (13)
\[
H_{\perp,3} = -\frac{\gamma_3}{\gamma_0} \sum_{\vec{p} \in \text{MoireB.Z}} C_2^\dagger(\vec{p})C_1(\vec{p})|\phi_1(\vec{p})|\cos[\alpha_2(\vec{p}) + \alpha_1(\vec{p})] + H.C. \tag{14}
\]

The lowest eigenvalue is given by:
\[
E^{-}(\vec{p}) = -\mu + \frac{1}{2} \left( |\phi_1(\vec{p})| + |\phi_2(\vec{p})| - \sqrt{(|\phi_1(\vec{p})| - |\phi_2(\vec{p})|)^2 + 4\left(\frac{\gamma_3}{\gamma_0} |\phi_1(\vec{p})|\cos[\alpha_2(\vec{p}) + \alpha_1(\vec{p})]\right)^2} \right) \tag{15}
\]

We include the chemical potential \( \mu \) and observe that the band is quasi-flat.

In order to obtain a better description of the bands, we will expand the model around the Dirac Cones and we will observe the Dirac dispersion.

**IV-The continuum model**

In order to see how the interactions are affected by the rotations we will use a continuum model. The continuum model will show the Dirac dispersion around the Dirac cones. For each layer we introduce a linear model around the position of the two Dirac nodes. For layer 1, we replace \( \vec{p} = \vec{K}'_1 + \vec{q} \) for the left valley and \( \vec{p} = \vec{K}_1 + \vec{q} \) for the right valley. The two valley are represented by the Pauli matrix \( \vec{\tau} \).

\[
\sum_{\vec{p} \in \text{B.Z}} . \left[ a_1^\dagger(\vec{p})b_1(\vec{p})\phi_1(\vec{p}) + H.C. \right] = \sum_{\vec{q}} \left[ a_1^\dagger(\vec{p} = \vec{K}'_1 + \vec{q})b_1(\vec{p} = \vec{K}'_1 + \vec{q})\phi_1(\vec{p} = \vec{K}'_1 + \vec{q}) + a_1^\dagger(\vec{p} = \vec{K}_1 + \vec{q})b_1(\vec{p} = \vec{K}_1 + \vec{q})\phi_1(\vec{p} = \vec{K}_1 + \vec{q}) + H.C. \right] = \sum_{\vec{q}} \left[ a_{1,L}^\dagger(\vec{q})b_{1,L}(\vec{q})\Phi_{1,L}(\vec{q}) + a_{1,R}^\dagger(\vec{q})b_{1,R}(\vec{q})\Phi_{1,R}(\vec{q}) + H.C. \right] = \sum_{\vec{q}} \left[ v_F\Phi_{1,L}^\dagger(\vec{q}) (\tau_1q_y - \tau_2q_x)\Phi_{1,L}(\vec{q}) + v_F\Phi_{1,R}^\dagger(\vec{q}) ( - \tau_1q_y - \tau_2q_x)\Phi_{1,R}(\vec{q}) \right] \tag{16}\]

where
\[
\Phi_{1,L}(\vec{q}) = \begin{bmatrix} a_1(\vec{p} = \vec{K}'_1 + \vec{q}), b_1(\vec{p} = \vec{K}'_1 + \vec{q}) \end{bmatrix}^T = \begin{bmatrix} a_{1,L}(\vec{q}), b_{1,L}(\vec{q}) \end{bmatrix}^T
\]
\[
\Phi_{1,R}(\vec{q}) = \begin{bmatrix} a_1(\vec{p} = \vec{K}_1 + \vec{q}), b_1(\vec{p} = \vec{K}_1 + \vec{q}) \end{bmatrix}^T = \begin{bmatrix} a_{1,R}(\vec{q}), b_{1,R}(\vec{q}) \end{bmatrix}^T
\]
\[
\Psi_1(\vec{r}) = \Phi_{1,L}(\vec{r})e^{i\vec{K}'_1 \cdot \vec{r}} + \Phi_{1,R}(\vec{r})e^{i\vec{K}_1 \cdot \vec{r}}
\]
\[
\Psi_1(\vec{p}) = \Phi_1(\vec{p})\mu[-p_x] + \Psi_1(\vec{p})\mu[p_x] = \Phi_{1,L}(\vec{q} + \vec{K}'_1)\mu[-q_x - K'_{x,1}] + \Phi_{1,R}(\vec{q} + \vec{K}_1)\mu[q_x + K_{x,1}],
\]
(17)

where \(\mu[q_x + K_{x,1}]\) is the step function which obeys \(\mu[-q_x - K'_{x,1}] + \mu[q_x + K_{x,1}] = 1\). From equation (15) we obtain the eigen spinors and represent \(\Phi_{1,L}(\vec{q})\), \(\Phi_{1,R}(\vec{q})\) in terms of the valley operators \(C_{1,L}(\vec{q})\), \(C_{1,R}(\vec{q})\)

\[
\Phi_{1,L}(\vec{q}) = C_{1,L}(\vec{q})U_{1,L}(\vec{q})
\]
\[
\Phi_{1,R}(\vec{q}) = C_{1,R}(\vec{q})U_{1,R}(\vec{q})
\]
\[
U_{1,L}(\vec{q}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1, -ie^{\alpha(\vec{q})} \end{bmatrix}^T, \quad U_{1,R}(\vec{q}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1, ie^{-i\alpha(\vec{q})} \end{bmatrix}^T, \quad \alpha(\vec{q}) = ArcTan\left(\frac{q_y}{q_x}\right)
\]
\[
\Psi_1(\vec{r}) = \sum_{\vec{q}} \left[ C_{1,L}(\vec{q})U_{1,L}(\vec{q})e^{i(\vec{q} + \vec{K}'_1 \cdot \vec{r})} + C_{1,R}(\vec{q})U_{1,R}(\vec{q})e^{i(\vec{q} + \vec{K}_1 \cdot \vec{r})} \right]
\]
\[
= \sum_{\vec{p}} \left[ C_{1,L}(\vec{p} - \vec{K}'_1)U_{1,L}(\vec{p} - \vec{K}'_1)e^{i(\vec{p} - \vec{K}'_1 \cdot \vec{r})}\mu[-p_x] + C_{1,R}(\vec{p})U_{1,R}(\vec{p} - \vec{K}_1)e^{i(\vec{p} - \vec{K}_1 \cdot \vec{r})}\mu[p_x] \right],
\]
\[
\tilde{C}_{1,L}(\vec{p}) = C_{1,L}(\vec{p} - \vec{K}'_1), \quad \tilde{C}_{1,R}(\vec{p}) = C_{1,R}(\vec{p} - \vec{K}_1)
\]
(18)

For layer \(i = 2\) with the rotated nodes at momentum \(\vec{K}'_2 = \frac{4\pi}{3a}\left[Sin[\theta], \frac{1}{\sqrt{3}}Cos[\theta]\right]\) and \(\vec{K}_2 = -\frac{4\pi}{3a}\left[Sin[\theta], \frac{1}{\sqrt{3}}Cos[\theta]\right]\), eq.(8) gives the linearized form:

\[
H_2 = \sum_{\vec{p} \in \text{Moiré B.Z}} \left[ b_2(\vec{p})a_2(\vec{p})\varphi_2(\vec{p}) + H.C. \right]
\]
\[
= \sum_{\vec{q}} \left[ b_2(\vec{p} = \vec{q} + \vec{K}_2)a_2(\vec{p} = \vec{q} + \vec{K}_2)\varphi_2(\vec{p} = \vec{q} + \vec{K}_2) + b_2(\vec{p} = \vec{q} + \vec{K}'_2)a_2(\vec{p} = \vec{q} + \vec{K}'_2)\varphi_2(\vec{p} = \vec{q} + \vec{K}'_2) + H.C. \right] =
\]
\[
- \sum_{\vec{q}} \left[ a_{2,L}(\vec{q})b_{2,L}(\vec{q})\Phi_{2,L}(\vec{q}) + a_{2,R}(\vec{q})b_{2,R}(\vec{q})\Phi_{2,R}(\vec{q}) + H.C. \right]
\]
\[
= \sum_{\vec{q}} \left[ v_F\Phi_{2,L}(\vec{q})\left( \tau_1(q_yCos[\theta] - q_xSin[\theta]) + \tau_2(q_ySin[\theta] + q_xCos[\theta]) \right)\Phi_{2,L}(\vec{q}) + v_F\Phi_{2,R}(\vec{q})\left( \tau_1(q_yCos[\theta] - q_xSin[\theta]) - \tau_2(q_xCos[\theta] + q_ySin[\theta]) \right)\Phi_{2,R}(\vec{q}) \right]
\]
(19)

The field in layer \(i = 2\) has the representation:

\[
\Psi_2(\vec{r}) = \Phi_{2,L}(\vec{r})e^{i\vec{K}'_2 \cdot \vec{r}} + \Phi_{2,R}(\vec{r})e^{i\vec{K}_2 \cdot \vec{r}}
\]
(20)
with the representation: \( \Phi_{2,L}(\vec{q}), \Phi_{2,R}(\vec{q}) \) in terms of the operators \( C_{2,L}(\vec{q}), C_{2,R}(\vec{q}) \)

\[
\Phi_{2,L}(\vec{q}) = C_{2,L}(\vec{q})U_{2,L}(\vec{q}), \quad \Phi_{2,R}(\vec{q}) = C_{2,R}(\vec{q})U_{2,R}(\vec{q})
\]

\[
U_{2,L} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1, ie^{i\theta}e^{-i\alpha(\vec{q})} \end{bmatrix}^T, \quad U_{2,R} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1, -ie^{-i\theta}e^{i\alpha(\vec{q})} \end{bmatrix}^T, \quad \alpha(\vec{q}) = ArcTan\left(\frac{q_y}{q_x}\right)
\]

\[
\Psi_2(\vec{r}) = \sum_{\vec{q}} \left[ C_{2,L}(\vec{q})U_{2,L}(\vec{q})e^{i(\vec{q}\cdot\vec{r})}, R \right] + C_{2,R}(\vec{q})U_{2,R}(\vec{q})e^{i(\vec{q}\cdot\vec{r})}\right]
\]

\[
= \sum_{\vec{p}} \left[ \hat{C}_{2,L}(\vec{p})U_{2,L}(\vec{p} - \vec{K}', \vec{p})e^{i\vec{p}\cdot\vec{r}} + \hat{C}_{2,R}(\vec{p})U_{2,R}(\vec{p} - \vec{K})e^{i\vec{p}\cdot\vec{r}} \right]
\]

(21)

The rotated layer \( i = 2 \), \( \Psi_2(\vec{r}') \) is represented in terms of the rotated Dirac nodes, \( \vec{K}'_2 = -\frac{2\pi}{\sqrt{3}} \left[ \sin[\theta], \frac{1}{\sqrt{3}} \cos[\theta] \right], \vec{K}_2 = \frac{2\pi}{\sqrt{3}} \left[ -\sin[\theta], \frac{1}{\sqrt{3}} \cos[\theta] \right].
\]

The eigenvalues for the two layers around the two Dirac points for particles with respect to the momentum \( \vec{q} \) are: \( \epsilon_{1,L}(\vec{q}) = \epsilon_{1,R}(\vec{q}) = v_F\sqrt{(q_x)^2 + (q_y)^2} \) and \( \epsilon_{2,L}(\vec{q}) = \epsilon_{2,R}(\vec{q}) = v_F\sqrt{(q_x)^2 + (q_y)^2}. \)

For the tunneling Hamiltonian, we obtain:

\[
H_{1,3} = -\gamma_3 \sum_{\vec{R}} \sum_{\tau=1,2,3} \left[ b_{\tau}^\dagger(\vec{R}' - \vec{\delta}_\tau)a_{\tau}(\vec{R}) + a_{\tau}^\dagger(\vec{R}')b_{\tau}(\vec{R} + \vec{\delta}_\tau) + H.C. \right]
\]

\[
= -\gamma_3 \sum_{\vec{R}} \sum_{\vec{\delta}_\tau} \left[ (\Phi_{2}(\vec{R}' - \vec{\delta}_\tau)_{\tau}(\Phi_{1}(\vec{R}))_\tau + (\Phi_{2}(\vec{R}')_\tau(\Phi_{1}(\vec{R} + \vec{\delta}_\tau))_{\tau} + H.C. \right]
\]

(22)

where

\[
\Phi_2(\vec{R}' - \vec{\delta}_\tau) = \sum_{\vec{p}} \left[ \hat{C}_{2,L}(\vec{p})U_{2,L}(\vec{p} - \vec{K}', \vec{p})e^{i\vec{p}\cdot\vec{r}} + \hat{C}_{2,R}(\vec{p})U_{2,R}(\vec{p} - \vec{K}_2)e^{i\vec{p}\cdot\vec{r}} \right]
\]

\[
\Phi_1(\vec{R}) = \sum_{\vec{p}} \left[ \hat{C}_{1,L}(\vec{p})U_{1,L}(\vec{p} - \vec{K}'_1)e^{i\vec{p}\cdot\vec{R}} + \hat{C}_{1,R}(\vec{p})U_{1,R}(\vec{p} - \vec{K}_1)e^{i\vec{p}\cdot\vec{R}} \right]
\]

(23)

\[
H_{1,3} = -\gamma_3 \sum_{\vec{p}} \sum_{\vec{R}} \sum_{\vec{\delta}_\tau} \left[ \hat{C}_{2,L}(\vec{p})U_{2,L}(\vec{p} - \vec{K}', \vec{p})e^{i\vec{p}\cdot\vec{r}} \right] \sum_{\tau=1,2,3} e^{-i\vec{p}\cdot\vec{\delta}_\tau} +
\]

\[
\hat{C}_{2,R}(\vec{p})U_{2,R}(\vec{p} - \vec{K}_2)e^{i\vec{p}\cdot\vec{r}} \right] \sum_{\tau=1,2,3} e^{-i\vec{p}\cdot\vec{\delta}_\tau} \]

\[
\left( \hat{C}_{1,L}(\vec{p})U_{1,L}(\vec{p} - \vec{K}'_1)e^{i\vec{p}\cdot\vec{R}} + \hat{C}_{2,R}(\vec{p})U_{2,R}(\vec{p} - \vec{K}_2)e^{i\vec{p}\cdot\vec{R}} \right) + H.C. \]

(24)
Using the periodicity with respect to the *Moire* reciprocal lattice which at magical angles is commensurate with the honeycomb lattice, at small angles we obtain in the *Moire* BZ the representation:

\[
H_{1,3} \approx -\frac{\gamma_3}{\gamma_0} \sum_{\vec{p} \in \text{MoireB.Z}} \left[ (\tilde{C}_{1,L}(\vec{p})U_{1,L}^{(2)}(\vec{p} - \vec{K}_1)\phi_1(\vec{p}) + \tilde{C}_{2,R}(\vec{p})U_{2,R}^{(2)}(\vec{p} - \vec{K}_2)\phi_1(\vec{p}))^\dagger \right] \\
\left( \tilde{C}_{1,L}(\vec{p})U_{1,L}^{(1)}(\vec{p} - \vec{K}_1) + \tilde{C}_{2,R}(\vec{p})U_{2,R}^{(1)}(\vec{p} - \vec{K}_2) \right) + \\
\left( \tilde{C}_{2,L}(\vec{p})U_{2,L}^{(1)}(\vec{p} - \vec{K}_2) + \tilde{C}_{2,R}(\vec{p})U_{2,R}^{(1)}(\vec{p} - \vec{K}_2) \right)^\dagger \left( \tilde{C}_{1,L}(\vec{p})U_{1,L}^{(2)}(\vec{p} - \vec{K}_1)\phi_1(\vec{p}) + \tilde{C}_{2,R}(\vec{p})U_{2,R}^{(2)}(\vec{p} - \vec{K}_2)\phi_1(\vec{p}) \right) + H.C. \right] 
\]

(25)

\[
H = H_L + H_R + H_{L,R} \\
H_L = \sum_{\vec{p}} \left[ (\tilde{C}_{1,L}(\vec{p})\tilde{C}(\vec{p})\epsilon_{1,L}(\vec{p} - \vec{K}_1) + \tilde{C}_{2,L}(\vec{p})\tilde{C}(\vec{p})\epsilon_{2,L}(\vec{p} - \vec{K}_2)) + \\
g_{1L,2L}\tilde{C}_{1,L}(\vec{p})\tilde{C}_{2,L}(\vec{p}) + H.C. \right] \\
H_R = \left[ \tilde{C}_{1,R}(\vec{p})\tilde{C}(\vec{p})\epsilon_{1,R}(\vec{p} - \vec{K}_1) + \tilde{C}_{2,R}(\vec{p})\tilde{C}(\vec{p})\epsilon_{2,R}(\vec{p} - \vec{K}_2) + \\
g_{1R,2R}\tilde{C}_{1,R}(\vec{p})\tilde{C}_{2,R}(\vec{p}) + H.C. \right] \\
H_{L,R} = \left[ g_{1L,2R}\tilde{C}_{1,L}(\vec{p})\tilde{C}_{2,R}(\vec{p}) + g_{1R,2L}\tilde{C}_{1,R}(\vec{p})\tilde{C}_{2,L}(\vec{p}) + H.C. \right] 
\]

(26)

where the effective tunneling coefficients are given by:

\[
g_{2L,1L} = -\frac{\gamma_3}{\gamma_0} \left[ U_{2,L}^{(2)}(\vec{p} - \vec{K}_2)U(1)_{1,L}(\vec{p} - \vec{K}_1)\phi_1^*(\vec{p}) + \tilde{U}_{2,L}^{(1)}(\vec{p} - \vec{K}_2)U(2)_{1,L}(\vec{p} - \vec{K}_1)\phi_1(\vec{p}) \right] \\
g_{2R,1R} = -\frac{\gamma_3}{\gamma_0} \left[ U_{2,R}^{(2)}(\vec{p} - \vec{K}_2)U(1)_{1,R}(\vec{p} - \vec{K}_1)\phi_1^*(\vec{p}) + \tilde{U}_{2,R}^{(1)}(\vec{p} - \vec{K}_2)U(2)_{1,R}(\vec{p} - \vec{K}_1)\phi_1(\vec{p}) \right] \\
g_{2R,1L} = -\frac{\gamma_3}{\gamma_0} \left[ U_{2,R}^{(2)}(\vec{p} - \vec{K}_2)U(1)_{1,L}(\vec{p} - \vec{K}_1)\phi_1^*(\vec{p}) + \tilde{U}_{2,R}^{(1)}(\vec{p} - \vec{K}_2)U(2)_{1,L}(\vec{p} - \vec{K}_1)\phi_1(\vec{p}) \right] \\
g_{2L,1R} = -\frac{\gamma_3}{\gamma_0} \left[ \tilde{U}_{2,L}^{(2)}(\vec{p} - \vec{K}_2)U(1)_{1,R}(\vec{p} - \vec{K}_1)\phi_1^*(\vec{p}) + \tilde{U}_{2,L}^{(1)}(\vec{p} - \vec{K}_2)U(2)_{1,R}(\vec{p} - \vec{K}_1)\phi_1(\vec{p}) \right] 
\]

(27)

In order to address the question of flat bands, we will solve the model under an approximations which neglects the higher order couplings.
The effective tunneling Hamiltonian which includes the coupling between the two valleys is:

\[ E_{L}^{(-)} = \frac{1}{2} \left[ \epsilon_{1, L}(\vec{p} - \vec{K}_1) + \epsilon_{2, L}(\vec{p} - \vec{K}_2) - \sqrt{\left( \epsilon_{1, L}(\vec{p} - \vec{K}_1) - \epsilon_{2, L}(\vec{p} - \vec{K}_2) \right)^2 + 4|g_{1L2L}|^2} \right] \]

\[ E_{L}^{(+)} = \frac{1}{2} \left[ \epsilon_{1, L}(\vec{p} - \vec{K}_1) + \epsilon_{2, L}(\vec{p} - \vec{K}_2) + \sqrt{\left( \epsilon_{1, L}(\vec{p} - \vec{K}_1) - \epsilon_{2, L}(\vec{p} - \vec{K}_2) \right)^2 + 4|g_{2L1L}|^2} \right] \]

We find the two operators, \( C_{L}^{(-)} \) and \( C_{L}^{(+)} \) using the approximation, \( E_{L}^{(+)} \gg E_{L}^{(-)} \). The approximations is based on projecting out the states \( E_{L}^{(+)} \) using the constraint \( C_{L}^{(+)} \approx 0 \) (which is justified when \( E_{L}^{(+)} \gg E_{L}^{(-)} \)). Thus we obtain:

\[ \tilde{C}_{2L}(\vec{p}) = Z_{2L}C_{L}^{(-)}(\vec{p}), \quad Z_{2L} = \frac{-\sqrt{(E_{L}^{(-)} - \epsilon_{1, L}(\vec{p} - \vec{K}_1))^2 + |g_{1L2L}|^2}}{(E_{L}^{(-)} - E_{L}^{(+))}} \]

\[ \tilde{C}_{1L}(\vec{p}) = Z_{1L}C_{L}^{(-)}(\vec{p}), \quad Z_{1L} = -\frac{E_{L}^{(-)} - \epsilon_{1, L}(\vec{p} - \vec{K}_1)}{E_{L}^{(+)} - E_{L}^{(-)}} \cdot \frac{-\sqrt{(E_{L}^{(-)} - \epsilon_{1, L}(\vec{p} - \vec{K}_1))^2 + |g_{1L2L}|^2}}{g_{1L2L}} \]

We diagonalize the \( H_{R} \) Hamiltonian using the approximation based on projecting out the state \( E_{L}^{(+)} \):

\[ E_{R}^{(-)} = \frac{1}{2} \left[ \epsilon_{1, R}(\vec{p} - \vec{K}_1) + \epsilon_{2, R}(\vec{p} - \vec{K}_2) - \sqrt{\left( \epsilon_{1, R}(\vec{p} - \vec{K}_1) - \epsilon_{2, R}(\vec{p} - \vec{K}_2) \right)^2 + 4|g_{1R2R}|^2} \right] \]

\[ E_{R}^{(+)} = \frac{1}{2} \left[ \epsilon_{1, R}(\vec{p} - \vec{K}_1) + \epsilon_{2, R}(\vec{p} - \vec{K}_2) + \sqrt{\left( \epsilon_{1, R}(\vec{p} - \vec{K}_1) - \epsilon_{2, R}(\vec{p} - \vec{K}_2) \right)^2 + 4|g_{1R2R}|^2} \right] \]

Performing the projection in the right valley gives:

\[ \tilde{C}_{2R}(\vec{p}) = Z_{2R}C_{R}^{(-)}(\vec{p}), \quad Z_{2R} = \frac{-\sqrt{(E_{R}^{(-)} - \epsilon_{1, R}(\vec{p} - \vec{K}_1))^2 + |g_{1R2R}|^2}}{(E_{R}^{(-)} - E_{R}^{(+))}} \]

\[ \tilde{C}_{1R}(\vec{p}) = Z_{1R}C_{R}^{(-)}(\vec{p}), \quad Z_{1R} = -\frac{E_{R}^{(-)} - \epsilon_{1, R}(\vec{p} - \vec{K}_1)}{E_{R}^{(+)} - E_{R}^{(-)}} \cdot \frac{-\sqrt{(E_{R}^{(-)} - \epsilon_{1, R}(\vec{p} - \vec{K}_1))^2 + |g_{1R2R}|^2}}{g_{1R2R}} \]

The effective tunneling Hamiltonian which includes the coupling between the two valleys is:

\[ H_{eff.} = \sum \tilde{C}_{\vec{p}} \]
where the spinors are given by:

\[ C_L^{(-\hat{\theta})}(\vec{p})C_L^{(-)}(\vec{p})E_L^{(-)}(\vec{p}) + C_R^{(-\hat{\theta})}(\vec{p})C_R^{(-)}(\vec{p})E_R^{(-)}(\vec{p}) + G \cdot C_L^{(-\hat{\theta})}(\vec{p})C_R^{(-)}(\vec{p}) + G^* \cdot C_R^{(-\hat{\theta})}(\vec{p})C_R^{(-)}(\vec{p}) \]

\[ G = g_{1L,2R}Z_{1L}Z_{2R} + (g_{1L,2R})^* Z_{2L}Z_{1R} \]

The lowest eigenvalue of the effective Hamiltonian will give the band \( E_0 \):

\[ E_0 \frac{1}{2} \left[ E_L^{(-)}(\vec{p}) + E_R^{(-)}(\vec{p}) - \sqrt{(E_L^{(-)}(\vec{p}) - E_R^{(-)}(\vec{p}))^2 + 4|G|^2} \right] \]  

(33)

The lowest energy band \( E_0 \) is flat \( E_{\text{flat}} = E_0 \) and justifies the name when \( \theta \approx \theta(n)_{\text{magic}} \)

V-Superconductivity induced by the rotated layer \( i = 2 \)

Here we will use the spinor representation to demonstrate that the rotation by angle \( \theta \) affects the electron-electron interactions. For simplicity we consider a repulsive Hubbard interaction. Due to the spinor rotation the Hubbard interaction becomes attractive in the \( y \) direction and periodic in the \( x \) direction. Effectively, this is described as a set of one dimensional superconducting wires separated by metallic regions. This result is additive to the attractive interactions mediated by the phonons.

The interaction in layer \( i = 2 \) is controlled by two fields, \( \Psi_2(\vec{R}) \) and \( \Psi_2(\vec{R} + \delta \vec{r}) \):

\[ \Psi_2(\vec{R}) = \sum_{\vec{p}} \left[ \hat{C}_{2,L}(\vec{p})U_{2,L}(\vec{p} - \vec{K}_2) e^{i\vec{p} \cdot \vec{R}} + \hat{C}_{2,R}(\vec{p})U_{2,R}(\vec{p} - \vec{K}_2) e^{i\vec{p} \cdot \vec{R}} \right] \]

\[ = \sum_{0<p_x<\Lambda} \sum_{-\Lambda<p_y<\Lambda} \left[ \hat{C}_{2,L}(-\vec{p})U_{2,L}(\vec{p} - \vec{K}_2) e^{i(-p_x[-\theta] \cdot R_x + p_y[-\theta] \cdot R_y)} + \hat{C}_{2,R}(\vec{p})U_{2,R}(\vec{p} - \vec{K}_2) e^{i\vec{p} \cdot \vec{R}} \right] \]

\[ \Psi_2(\vec{R} + \delta \vec{r}) \approx \sum_{0<p_x<\Lambda} \sum_{-\Lambda<p_y<\Lambda} \left[ \hat{C}_{2,L}(-\vec{p})U_{2,L}(\vec{p} - \vec{K}_2) e^{i(-p_x[-\theta] \cdot R_x + p_y[-\theta] \cdot R_y) + i(-p_x[-\theta] \cdot R_x + p_y[-\theta] \cdot R_y) + i(-p_x[-\theta] \cdot \delta_{r_x} + p_y[-\theta] \cdot \delta_{r_y})} \right] \]

\[ + \hat{C}_{2,R}(\vec{p})U_{2,R}(\vec{p} - \vec{K}_2) e^{i\vec{p} \cdot \vec{R} + \vec{R} \cdot \delta \vec{r}} \]  

\[ \Psi_2(\vec{R}) \approx \sum_{0<p_x<\Lambda} \sum_{-\Lambda<p_y<\Lambda} \left[ \hat{C}_{2,L}(\vec{p})U_{2,L}(\vec{p} - \vec{K}_2) e^{i(-p_x[-\theta] \cdot R_x + p_y[-\theta] \cdot R_y) + i(-p_x[-\theta] \cdot \delta_{r_x} + p_y[-\theta] \cdot \delta_{r_y})} \right] \]

\[ \hat{C}_{2,R}(\vec{p})U_{2,R}(\vec{p} - \vec{K}_2) e^{i\vec{p} \cdot \vec{R} + \vec{R} \cdot \delta \vec{r}} \]  

(34)

where the spinors are given by: \( U_{2,L}(\vec{p} - \vec{K}_2) \approx U_{2,L}(\vec{K}_2) = \frac{1}{\sqrt{2}} \left[ 1, ie^{i\theta} e^{-i\vec{K}_2} \right] \), \( U_{2,R}(\vec{p} - \vec{K}_2) \approx U_{2,R}(\vec{K}_2) = \frac{1}{\sqrt{2}} \left[ 1, ie^{i\theta} e^{-i\vec{K}_2} \right] \).

We observe that the spinor depends on the a rotated angle \( \theta \).
The Hubbard interaction in layer \( i = 2 \) is:

\[
H_U = \sum_{\vec{R}} Un_\uparrow(\vec{R})n_\downarrow(\vec{R}) + \sum_{\vec{R}} \sum_{r=1,2,3} Un_\uparrow(\vec{R} + \delta_r) n_\downarrow(\vec{R} + \delta_r)
\]

(35)

We will use the summation over the twisted vector \( \vec{\delta}_r \) introduced in equation (8) which relates the rotated momentum to the Moire reciprocal lattice (see Eq.(6)):

\[
\sum_{r=1,2,3} e^{-i(\vec{p}_1[-\theta] + \vec{p}_2[-\theta] + \vec{p}_3[-\theta] + \vec{p}_4[-\theta]) \cdot \vec{\delta}} \delta_\theta \vec{p}_1[-\theta] + \vec{p}_2[-\theta] + \vec{p}_3[-\theta] + \vec{p}_4[-\theta)] = \vec{g} \cdot \vec{R}, g_z \theta, g_y \theta
\]

\[
e^{-i(g_z \theta + g_y \theta)} \left[ 1 + 2e^{i(\frac{1}{2} g_z \theta + \frac{1}{2} g_y \theta)} \cos \left( \frac{\sqrt{3}}{2} (g_z \theta - g_y \theta) \right) \right]
\]

(36)

Next, in the Hubbard interaction we substitute the fields \( \Psi_{2,\sigma}(\vec{R}') \) and \( \Psi_{2,\sigma}(\vec{R}' + \delta_r) \) given in Eq.(33) including the spin dependence. We notice that the intervalleys depend on the rotated angles. If we use a long range Coulomb potential we can use the same strategy as used for the Hubbard model, by identifying directions where we have an attractive potential. Those terms are given by \( \tilde{C}_{2,L,\uparrow}^\dagger(\vec{p}_1) \tilde{C}_{2,L,\downarrow}^\dagger(\vec{p}_2) \tilde{C}_{2,R,\uparrow}^\dagger(\vec{p}_3) \tilde{C}_{2,R,\downarrow}^\dagger(\vec{p}_4) \) and \( \tilde{C}_{2,R,\uparrow}^\dagger(\vec{p}_1) \tilde{C}_{2,R,\downarrow}^\dagger(\vec{p}_2) \tilde{C}_{2,L,\uparrow}^\dagger(\vec{p}_3) \tilde{C}_{2,L,\downarrow}^\dagger(\vec{p}_4) \). We introduce the short notation:

\[
\begin{align*}
\sum_{p_x} = & \sum_{0 < p_{2x} < \Lambda} \sum_{0 < p_{3x} < \Lambda} \sum_{0 < p_{4x} < \Lambda} \\
\sum_{p_y} = & \sum_{-\Lambda < p_{2y} < \Lambda} \sum_{-\Lambda < p_{3y} < \Lambda} \sum_{-\Lambda < p_{4y} < \Lambda}
\end{align*}
\]

(37)

Using the periodicity with respect to the Moire reciprocal lattice \( \tilde{C}_{2,L}(\vec{p} + \vec{g}[\theta]) = \tilde{C}_{2,L}(\vec{p}) \), \( \tilde{C}_{2,R}(\vec{p} + \vec{g}[\theta]) = \tilde{C}_{2,R}(\vec{p}) \) we obtain the following form of the Hubbard interaction:

\[
H_U = \sum_{p_x} \sum_{p_y} \left[ \frac{U}{4} \left( e^{-i4\theta} + e^{-i(4\theta + g_z \theta + g_y \theta)} + 2e^{-i(4\theta - \frac{1}{2} g_z \theta - \frac{1}{2} g_y \theta)} \cos \left( \frac{\sqrt{3}}{2} (g_z \theta - g_y \theta) \right) \right) \right.
\]

\[
\tilde{C}_{2,R,\uparrow}^\dagger(-p_{2x} - p_{3x} - p_{4x}, -p_{2y} - p_{3y} - p_{4y}) \tilde{C}_{2,L,\downarrow}^\dagger(p_{2x}, p_{2y}) \tilde{C}_{2,L,\downarrow}^\dagger(p_{2x} - p_{3x}, p_{3y}) \tilde{C}_{2,L,\uparrow}^\dagger(-p_{4x}, p_{4y})
\]

\[
+ \frac{U}{4} \left( e^{i4\theta} + e^{i(4\theta + g_z \theta + g_y \theta)} + 2e^{i(4\theta - \frac{1}{2} g_z \theta - \frac{1}{2} g_y \theta)} \cos \left( \frac{\sqrt{3}}{2} (g_z \theta - g_y \theta) \right) \right)
\]

\[
\tilde{C}_{2,L,\uparrow}^\dagger(p_{2x} + p_{3x} + p_{4x}, -p_{2y} - p_{3y} - p_{4y}) \tilde{C}_{2,L,\downarrow}^\dagger(-p_{2x}, p_{2y}) \tilde{C}_{2,R,\downarrow}^\dagger(p_{3x}, p_{3y}) \tilde{C}_{2,R,\uparrow}^\dagger(p_{4x}, p_{4y})
\]

(38)
At certain angles, the imaginary part of the effective Hubbard potential vanishes and the real part is negative:

\[
\text{Im.} \left[ \frac{U}{4} \left( e^{-i4\theta} + e^{-i(4\theta+g_x[\theta]+g_y[\theta])} + 2e^{-i(4\theta-\frac{1}{2}g_x[\theta]-\frac{1}{2}g_y[\theta])} \cos \left( \frac{\sqrt{3}}{2} (g_x[\theta] - g_y[\theta]) \right) \right) \right] = 0
\]

\[
U[\bar{\theta}] = \frac{U}{2} \left( \cos[4\theta] + \cos[4\theta + g_x[\theta] + g_y[\theta]] + 2\cos[(4\theta - \frac{1}{2}g_x[\theta] - \frac{1}{2}g_y[\theta]) \cos \left( \frac{\sqrt{3}}{2} (g_x[\theta] - g_y[\theta]) \right) \right)
\]

(39)

We observe in Figure 1 that the interaction term \(U[\bar{\theta}]\) is attractive for certain angles \(\theta = s\).

\[
H_U \approx \sum_{p_y} \sum_{p_x} \left[ -|U[\bar{\theta}]| \left( \tilde{C}^\dagger_{2,R,\uparrow}(p_{2x} - p_{3x} - p_{4x}, -p_{2y} - p_{3y} - p_{4y}) \tilde{C}^\dagger_{2,L,\uparrow}(p_{2x}, p_{2y}) \tilde{C}_{2,L,\downarrow}(p_{3x}, p_{3y}) \tilde{C}_{2,R,\downarrow}(p_{4x}, p_{4y}) + \tilde{C}^\dagger_{2,L,\uparrow}(p_{2x} + p_{3x} + p_{4x}, -p_{2y} - p_{3y} - p_{4y}) \tilde{C}^\dagger_{2,L,\downarrow}(p_{2x} + p_{3x} + p_{4x}, -p_{2y} - p_{3y} - p_{4y}) \tilde{C}_{2,R,\uparrow}(p_{4x}, p_{4y}) \right) \right]
\]

(40)

This allows us to write a one dimensional pairing Hamiltonian at a fixed linear momentum \(p_x = 0\) (this corresponds to \(\tilde{K}'_{2,x} = \tilde{K}_{2,x}\)):

\[
H_U = -|U[\bar{\theta}]| \int dy \left[ \tilde{C}^\dagger_{2,R,\uparrow}(p_x = 0, y) \tilde{C}^\dagger_{2,R,\downarrow}(p_x = 0, y) \tilde{C}_{2,L,\downarrow}(p_x = 0, y) \tilde{C}_{2,L,\uparrow}(p_x = 0, y) + \tilde{C}^\dagger_{2,L,\uparrow}(p_x = 0, y) \tilde{C}^\dagger_{2,L,\downarrow}(p_x = 0, y) \tilde{C}_{2,R,\downarrow}(p_x = 0, y) \tilde{C}_{2,R,\uparrow}(p_x = 0, y) \right]
\]

(41)

Following [19] we use the Lagrangian representation and perform a saddle point computation:

\[
L = L_0 + L_{\text{int.}}
\]

\[
L_{\text{int.}} = |U[\bar{\theta}]| \int dy \left[ \tilde{C}^\dagger_{2,R,\uparrow}(p_x = 0, y) \tilde{C}^\dagger_{2,R,\downarrow}(p_x = 0, y) \tilde{C}_{2,L,\downarrow}(p_x = 0, y) \tilde{C}_{2,L,\uparrow}(p_x = 0, y) + \tilde{C}^\dagger_{2,L,\uparrow}(p_x = 0, y) \tilde{C}^\dagger_{2,L,\downarrow}(p_x = 0, y) \tilde{C}_{2,R,\downarrow}(p_x = 0, y) \tilde{C}_{2,R,\uparrow}(p_x = 0, y) \right]
\]

(42)

We use the Hubbard Stratonovici fields \(\Delta, \Delta^*, D\) and \(D^*\). This is done by replacing:
\[\hat{C}_{2,R,\uparrow}(p_x = 0, y)\hat{C}_{2,R,\downarrow}(p_x = 0, y)\hat{C}_{2,L,\downarrow}(p_x = 0, y)\hat{C}_{2,L,\uparrow}(p_x = 0, y) = \frac{1}{4}(\hat{C}_{2,R,\uparrow}(p_x = 0, y)\hat{C}_{2,R,\downarrow}(p_x = 0, y)\hat{C}_{2,L,\downarrow}(p_x = 0, y)\hat{C}_{2,L,\uparrow}(p_x = 0, y))^2\]

and further decoupling the two body interaction by a Gaussian integration \[19\].

The variation with respect to the fields \(\Delta, \Delta^*\), \(D\) and \(D^*\) gives the equations:

\[
\begin{align*}
\Delta^*(p_x = 0, y; t) &= 4\langle \hat{C}_{2,L,\uparrow}(p_x = 0, y; t)\hat{C}_{2,L,\downarrow}(p_x = 0, y; t)\rangle, \\
\Delta(p_x = 0, y; t) &= 4\langle \hat{C}_{2,R,\uparrow}(p_x = 0, y; t)\hat{C}_{2,R,\downarrow}(p_x = 0, y; t)\rangle, \\
D(p_x = 0, y; t) &= 4\langle \hat{C}_{2,L,\uparrow}(p_x = 0, y; t)\hat{C}_{2,L,\downarrow}(p_x = 0, y; t)\rangle, \\
D^*(p_x = 0, y; t) &= 4\langle \hat{C}_{2,R,\uparrow}(p_x = 0, y; t)\hat{C}_{2,L,\downarrow}(p_x = 0, y; t)\rangle.
\end{align*}
\]

The variation with respect to the fields \(\Delta, \Delta^*, D\) and \(D^*\) gives the equations:

\[
\begin{align*}
\Delta^*(p_x = 0, y; t) &= 4\langle \hat{C}_{2,L,\uparrow}(p_x = 0, y; t)\hat{C}_{2,L,\downarrow}(p_x = 0, y; t)\rangle, \\
\Delta(p_x = 0, y; t) &= 4\langle \hat{C}_{2,R,\uparrow}(p_x = 0, y; t)\hat{C}_{2,R,\downarrow}(p_x = 0, y; t)\rangle, \\
D(p_x = 0, y; t) &= 4\langle \hat{C}_{2,L,\uparrow}(p_x = 0, y; t)\hat{C}_{2,L,\downarrow}(p_x = 0, y; t)\rangle, \\
D^*(p_x = 0, y; t) &= 4\langle \hat{C}_{2,R,\uparrow}(p_x = 0, y; t)\hat{C}_{2,L,\downarrow}(p_x = 0, y; t)\rangle.
\end{align*}
\]

From these equations we obtain:

\[
\langle \hat{C}_{2,L,\uparrow}(p_x = 0, y; t)\hat{C}_{2,L,\downarrow}(p_x = 0, y; t)\rangle = \left(\langle \hat{C}_{2,R,\uparrow}(p_x = 0, y; t)\hat{C}_{2,R,\downarrow}(p_x = 0, y; t)\rangle\right)^*.
\]

This allows us to reduce the four fields to only two: \(D = \Delta = \chi(p_x = 0, y), \Delta^* = D^* = \chi(p_x = 0, y)^*\). Using this saddle point, we simplify the interaction term:

\[
\begin{align*}
\int dt L_{\text{int.}}(t) &= \int dt \int dy \left[ -\frac{\chi(p_x = 0, y; t)\chi^*(p_x = 0, y; t)}{2|U[\theta]|} \\
+ \left(\hat{C}_{2,L,\uparrow}(p_x = 0, y; t)\hat{C}_{2,L,\downarrow}(p_x = 0, y; t) + \hat{C}_{2,R,\uparrow}(p_x = 0, y; t)\hat{C}_{2,R,\downarrow}(p_x = 0, y; t)\right)\chi(p_x = 0, y; t) + \\
\chi^*(p_x = 0, y; t)\left(\hat{C}_{2,R,\downarrow}(p_x = 0, y; t)\hat{C}_{2,R,\uparrow}(p_x = 0, y; t) + \hat{C}_{2,L,\downarrow}(p_x = 0, y; t)\hat{C}_{2,L,\uparrow}(p_x = 0, y; t)\right)\right]
\end{align*}
\]

This gives a one-dimensional superconducting order parameter periodic in \(x\) with periodicity \(K_{1,x}\).

\[
\chi(p_x = 0, y; t) = \cos[K_{1,x}x]\chi_0(T)
\]
To compute the critical temperature $T = T_c$, we perform a variation with respect to $\chi$.

Performing the computation in the Euclidean form, we obtain:

$$L = -\frac{|\chi(p_x = 0)|^2}{2|U[\theta]|} - \frac{T}{L} \sum \sum_p \log[(i\omega_n)^4 - E^4 - |\chi|^4];$$

$$\log[(i\omega_n)^4 - E^4 - |\chi|^4] = \log\left((i\omega_n - \sqrt{E^2 + |\chi|^2})\left((i\omega_n + \sqrt{E^2 + |\chi|^2})\left((i\omega_n - i\sqrt{E^2 + |\chi|^2})\right)\right)\right).$$

We obtain the equation:

$$\frac{1}{2|U[\theta]|} = \frac{1}{L} \sum_p \left[ \frac{1}{E(p)} \tanh\left(\frac{E(p)}{2T}\right) + \frac{\sin\left[\frac{E(p)}{T}\right]}{1 + \cos\left[\frac{E(p)}{T}\right]} \right]$$

(48)

with $E(\vec{p}) = \epsilon(\vec{p}) - \mu < \Lambda$

The electronic dispersion is not known. The flat band allows us to replace $E(p) = \epsilon(\vec{p})_{max} - \mu \approx 0$ and superconductivity is destroyed. The critical temperature for limit $\mu \approx 0$ gives $T_c \approx \frac{L}{\epsilon(\vec{p})_{max}} |U[\theta]|$.

VI-Conclusions

To conclude, we have shown that in graphene lower rotation induces an attractive interaction for some directions and certain valley components. This results in a one-dimensional superconducting parameter periodic in $x$ with periodicity $K_{1,x}$. We want to mention that alternative explanations have been proposed. These explanations are complementary to the model proposed here and can give rise to two-dimensional superconductivity.

These results have been obtained under the following conditions:

a) A superlattice with basis vectors $\vec{t}_1 = i\vec{a}_1 + (i + 1)\vec{a}_2; \vec{t}_2 = -(i + 1)\vec{a}_1 + (2i + 1)\vec{a}_2$ is formed at magic angles $\cos[\theta_i] = \frac{3i^2 + 3i + 0.5}{3i^2 + 3i + 1}$, $i=0,1,2,$...

b) A uniform rotation in real space by an angle $\theta$ is equivalent to a rotation $-\theta$ of the momentum space.

c) The discrete summation with respect to the lattice points relates the momentum to the reciprocal lattice vectors $(G^{(1)}_x + G^{(2)}_x)\cos[\theta]$ and $(G^{(1)}_x + G^{(2)}_x)\sin[\theta]$ which at the magic angles become the Moire reciprocal lattice.
FIG. 1: The possible attractive potential, $U[\theta] = \frac{U}{2} \left( \cos[4\theta] + \cos[4\theta + g_x[\theta] + g_y[\theta]] + 2\cos[(4\theta - \frac{1}{2}g_x[\theta] - \frac{1}{2}g_y[\theta])\sqrt{3}(g_y[\theta] - g_x[\theta])] \right)$.

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