Non-rational varieties with the Hilbert Property

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Abstract

A variety $X/k$ is said to have the Hilbert Property if $X(k)$ is not thin. We shall describe some examples of varieties, for which the Hilbert Property is a new result.

We give a criterion for determining when the Hilbert Property for a variety $X$ implies the Hilbert Property for quotients $X/G$ of the variety by an action of a finite group. In the case of linear actions of the group $G$, this gives examples of (non-rational) unirational varieties with the Hilbert Property, providing positive examples to a conjecture by Colliot-Thélène and Sansuc.

We focus then on the study of the Hilbert Property for K3 surfaces that have two elliptic fibrations, in particular on diagonal quartic surfaces, i.e., varieties of the form $ax^4 + by^4 + cz^4 + dw^4 = 0$. We then show, through an explicit application, how one may use the criterion above to provide other examples of K3 surfaces with the Hilbert Property. Since the Hilbert Property is related to an abundance of rational points, K3 surfaces should (conjecturally) represent a limiting case in dimension 2.

1 Introduction

This paper will be concerned with providing new examples of varieties with the so-called “Hilbert Property”, concerning the set of $k$-rational points $X(k)$, for an algebraic variety $X$ over a field $k$. Throughout this paper $k$ will always denote a field of characteristic 0, and by “variety” we will always mean an algebraic variety.

We start by recalling the main definitions and some basic facts regarding this property. For a treatment of the basic theory we refer to [12].

Definition 1.1. Let $X$ be a variety over a field $k$. We say that $S \subset X(k)$ is thin if it is contained in a finite union of sets of two types:

(A) $Z(k) \subset X(k)$, where $Z \subset X$ is a closed proper subvariety;

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(B) $\pi(E(k)) \subset X(k)$, where $\pi : E \to X$ is an irreducible non-trivial cover of $X$, by which we mean that $E$ is an irreducible variety and $\pi$ is a dominant generically finite morphism, with $\deg \pi_i > 1$.

**Definition 1.2.** We say that a variety $X/k$ is of Hilbert type, or has the Hilbert Property, if $X(k)$ is not thin.

In the sequel we will use the abbreviation HP to denote the Hilbert Property.

The classical theorem of Hilbert can be reformulated by saying that, if $k$ is a number field, then $\mathbb{A}_n/k$ has the Hilbert Property. If, over the field $k$, there exists a variety with the Hilbert Property, then $k$ is said to be Hilbertian. In this case it can be shown that $\mathbb{A}_n/k$ has the Hilbert Property for any $n \geq 1$.

Motivation for the study of the HP comes from the following conjecture, of which a proof would settle the Inverse Galois Problem (as noted in [3]):

**Conjecture 1.3** (Colliot-Thélène, Sansuc). Let $X/k$ be a unirational variety over a number field, then $X$ has the HP.

In the second section of this paper we recall some terminology and facts about ramification of a finite morphism, since this concept will play a fundamental role in this work. This is all standard theory, that we collect in this section to make the article more self-contained.

The last two sections are dedicated to the new results of this paper, namely Theorems 1.5 and 1.7.

The third section is concerned with providing a (partial) inverse of the following classical theorem:

**Theorem 1.4.** Let $X/k$ be a variety, and $G$ a finite group acting (generically freely) on $X$. If the quotient $X/G$ has the HP, then $G$ is realizable as a Galois group over $k$.

Namely we are going to prove the following:

**Theorem 1.5.** Let $X/k$ be a quasi-projective variety, which has the Hilbert Property over a perfect field $k$. Let $G$ be a finite group acting generically freely on $X/k$. Assume that there exist Galois field extensions $\{L_i/k\}_{i \in I}$, with $G_i = \text{Gal}(L_i/k) \cong G$, and isomorphisms $\alpha_i \in \text{Iso}(G,G_i)$ such that:

(i) For each $i$, the twist of $X$ by $\alpha_i$ has the HP;

(ii) For any finite field extension $E/k$ there exists an $i \in I$ such that $L_i/k$ and $E/k$ are linearly disjoint.

Then the quotient variety $X/G$ has the HP.
Remark 1.6. We note that the latter of the two conditions necessarily implies that the extensions $L_i/k$ are an infinite amount.

As a consequence of Theorem 1.5 we will provide some new examples of non-rational unirational varieties with the Hilbert Property, making them non-immediate examples of Conjecture 1.3.

The last section of this paper is concerned with providing new examples of K3 surfaces over $\mathbb{Q}$ with the Hilbert Property.

We remark that K3 surfaces (and, in general, Calabi-Yau varieties) represent a "limiting case" for the study of rational points, at least conjecturally. In fact, the conjectures of Vojta suggest that on algebraic varieties there should be "less" rational points as the canonical bundle gets "bigger". Hence, since for K3 surfaces the canonical bundle is trivial by definition, we expect the rational points here not to be "too much", yet their existence (and Zariski-density) is not precluded. In fact, proving the HP, we are providing some examples of abundance of rational points in such surfaces.

We will prove the following:

**Theorem 1.7.** Let $V_{a,b,c,d}/\mathbb{Q}$ be the surface defined by the equation $ax^4 + by^4 + cz^4 + dw^4 = 0$ in $\mathbb{P}_3/\mathbb{Q}$. Assume that we have a rational point $p \in V_{a,b,c,d}(\mathbb{Q})$, not lying in $\Omega$. Then the surface $V_{a,b,c,d}/\mathbb{Q}$ has the Hilbert Property.

Theorem 1.7 is a generalization of a result of Corvaja and Zannier, presented in [4], who prove that the Fermat surface $E : \{x^4 + y^4 = z^4 + w^4\} \subset \mathbb{P}_3/\mathbb{Q}$ has the Hilbert Property.

The diagonal surfaces of Theorem 1.7 have already been studied in the literature. For instance, in [6], the authors have proven that these surfaces have Zariski-dense rational points under the same hypothesis as those of Theorem 1.7.

The argument of the proof of Theorem 1.7 is inspired from the one in [4], but differs in some key points: one uses two elliptic fibrations of the variety; but we will have to use a different method to "produce" the rational points for the Hilbert Property. In fact, in [4], the authors use global sections of the elliptic fibrations to do so; whereas in our case, where these sections are not available, we will provide "enough" rational points by looking at the orbit of a starting point $P$ by the action of some endomorphisms of the variety $V_{a,b,c,d}$.

We conclude by proving that the HP holds for a quotient (which is a K3 surface) of the Fermat surface $E$ by an action of the cyclic group $C_2$. This has been suggested to be true in [4]. To prove this last result we employ both Theorems 1.5 and 1.7.

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1. Here $\Omega$ will denote a specific Zariski closed subset of $V_{a,b,c,d}$.
2 Ramification

In this section we will always work with schemes of finite type over a field \( k \) of characteristic 0.

In this work, we are going to need some facts about ramification of a finite morphism \( f : X \to Y \). In particular, we will be interested in what happens at ramification when we are restricting \( f \) to a (closed) subvariety in the codomain \( Z \subset Y \). We are going now to briefly recap the basic definitions and facts which we are going to need, omitting the proofs (for a reference see e.g. [13, Tag 0C3H]).

Definition 2.1. We say that the map \( f : X \to Y \) is unramified (resp. étale) in \( x \in X \), if its differential \( df_x : T_xX \to T_{f(x)}Y \) is injective (resp. an isomorphism). Otherwise we say that \( f \) is ramified at \( x \).

The set of points where \( f \) is unramified has a closed subscheme structure in \( X \). We give hence the following definition:

Definition 2.2. Let \( f : X \to Y \) be a finite morphism of schemes. The ramification locus \( R_f \) is the reduced closed subscheme of \( X \), whose points are the points where \( f \) is unramified.

Remark 2.3. We could define the ramification locus differently to keep track of the multiplicities (as in [13, Tag 0C3H]). One way to do this is, for instance, to say that \( R_f \) is the closed subscheme of \( X \) defined by the 0-th Fitting ideal of the sheaf of relative differentials \( \Omega_{X/Y} \). This gives rise to the same reduced structure as the one we defined above, but has better functorial properties. Since, for our purposes, the study of the reduced structure of \( R_f \) is sufficient, we content ourselves of Definition 2.2.

Proposition 2.4. Let \( f : X \to Y \) be a finite morphism, and let \( j : Z \hookrightarrow Y \) be an embedding. Let \( X_Z = X \times_Y Z = f^{-1}(Z) \), let \( j_X : X_Z \hookrightarrow X \) denote the projection and let \( f_Z : X_Z \to Z \) denote the base change morphism. We have then that \( R_{f_Z} = (j_X^{-1}R_f)_{\text{red}} \).

Although Proposition 2.4 is a standard result, we give here a self-contained proof for convenience of the reader.

Proof of Proposition 2.4. Since both \( R_{f_Z} \) and \( (j_X^{-1}R_f)_{\text{red}} \) are reduced by definition, it suffices to show that the set of closed points of these two schemes are the same. The inclusion \( R_{f_Z} \subset (j_X^{-1}R_f)_{\text{red}} \) is immediate from the definition, hence it is enough to prove that all closed points in \( j_X^{-1}R_f \) are contained in \( R_{f_Z} \).

Let \( p' \in j_X^{-1}R_f \subset X_Z \) be a closed point, and let \( p = j_X(p') \) and \( p'' = f_Z(p') \). Say \( p' \cong p \cong \text{Spec } K \), where \( K \) is an algebraic extension of the base field \( k \). We
denote then by \( v_p \) the scheme \( \text{Spec} \, K[\epsilon] \), where \( K[\epsilon] := K[t]/(t^2) \) (this is just a point with a tangent vector). By definition of ramification we know that there is an embedding \( \iota : v_p \hookrightarrow X \), such that the reduced structure of the image of \( \iota \) is the point \( p \), and the differential of the composition \( f \circ \iota : v_p \rightarrow Y \) is 0. Hence we have the following commutative diagram:

\[
\begin{array}{ccc}
  v_p & \xrightarrow{O_{\rho''}} & Z \\
  \downarrow{j} & & \downarrow{j} \\
  X & \xrightarrow{f} & Y
\end{array}
\]

where \( O_{\rho''} \) is the map sending the reduced point of \( v_p \) to \( p'' \), and whose differential is 0. By definition of fibered product, there exists then a morphism \( \iota' : v_p \hookrightarrow X_Z \) that makes the following diagram commute:

\[
\begin{array}{ccc}
  v_p & \xrightarrow{O_{\rho''}} & Z \\
  \downarrow{j} & & \downarrow{j} \\
  X & \xrightarrow{f} & Y
\end{array}
\]

We have now exhibited a non-zero tangent vector \( v'_{p'} \) to \( p' \) in \( X_Z \) (i.e. the tangent vector of \( \iota'(v_p) \)), such that \( df'_{p'}[v'_{p'}] = 0 \), and hence \( p' \in R_{f_z} \), as we wanted to prove.

We recall now the definition of diramation of a finite morphism, which will be used heavily in section 4. In particular, Corollary 2.7 is a technical tool that will play a fundamental role.

**Definition 2.5.** Let \( f : X \rightarrow Y \) be a finite morphism of schemes. The diramation (or branch) locus of \( f \) is the closed reduced subscheme \( D_f = f(R_f) \), the image of \( R_f \) under \( f \).

**Remark 2.6.** Keeping the notation of Definition 2.5, we recall that, when \( Y \) is smooth and \( X \) is normal, the diramination locus of \( f \) is of pure codimension 1, by Zariski’s Purity Theorem. In this case, it makes sense to talk about the diramation divisor.

**Corollary 2.7.** Let \( f : X \rightarrow Y \) be a finite morphism of schemes. Let \( Z \subset Y \) be a subscheme, and consider the base change map \( f_Z : X_Z \rightarrow Z \) (here \( X_Z \) denotes the scheme-theoretic fiber \( f^{-1}(Z) \)). Then the diramation locus \( D_{f_Z} \) of \( f_Z \) is \( D_f \cap Z \).

**Proof.** This follows immediately from Proposition 2.4.
3 Descending the Hilbert Property

3.1 Passing HP to quotient varieties

Before coming to the proof of Theorem 1.5, we briefly fix the notation and recall two basic facts (in Proposition 3.1) about twists of variety.

Let $X/k$ be a quasi-projective variety over a perfect field $k$. Let $G$ be a finite group acting on $X$, and let $L/k$ be a finite Galois extension of fields with $\text{Gal}(L/k) \cong G$. Let $\alpha \in \text{Hom}(\text{Gal}(L/k), G)$, we then denote by $X_\alpha$ the twist of $X$ by $\alpha$.

**Proposition 3.1.** Keeping the above notation, we have that:

- There exists an isomorphism $\varphi_\alpha : X_\alpha \times_k L \cong X \times_k L$.
- The isomorphism $\varphi_\alpha$ is such that:

$$\varphi_\alpha(X_\alpha(k)) = \{ x \in X(L) \mid x^\sigma = g_\sigma x \ \forall \sigma \in \text{Gal}(L/k) \}.$$ 

**Proof.** See [11].

We are going to need the following:

**Proposition 3.2.** Keeping the above notation, assume that the action of $G$ on $X$ is free, and that $\alpha$ is an isomorphism. Consider a proper subgroup $H \subset G$, and the quotient $X/H$. We denote by $\pi_H : X \to X/H$ the quotient map. We identify, with abuse of notation, the rational points of $X_\alpha$ with $L$-points of $X$ through the embedding $X_\alpha(k) \subset X(L)$ described in Proposition 3.1. We have then that the field of definition of each point $P \in \pi_H(X_\alpha(k))$ is $L^N$.

**Proof.** Let $\tilde{P} \in X_\alpha(k)$, and $P = \pi_H(\tilde{P})$. Certainly the field of definition of $P$ is contained in $L$, since $X_\alpha(k) \subset X(L)$. Hence the field of definition of $P$ is $L^N$, where $N$ is the stabilizer of $P$ through the Galois action of $\text{Gal}(L/k)$. We claim that $N = H$. In fact, let $\sigma \in \text{Gal}(L/k)$, then:

$$P^\sigma = (\pi_H(\tilde{P}))^\sigma = \pi_H((\tilde{P})^\sigma) = \pi_H(g_\sigma(\tilde{P})).$$

Hence we have that $P = P^\sigma$ if and only if $g_\sigma(\tilde{P}) \in H \cdot \tilde{P}$. Since we assumed the action of $G$ to be free, we are done.

**Proof of Theorem 1.5.** Let $\pi : X \longrightarrow X/G$ be the projection to the quotient. If necessary, we can restrict $X$ to a $G$-invariant open subset, and assume that the action of $G$ on $X$ is free, in particular the quotient map $\pi$ will be assumed to be finite and étale. We denote by $k(X)$ and $k(X)^G$, the fields of functions of $X$ and $X/G$, respectively.
3 Descending the Hilbert Property

Suppose that $X/G$ does not have the HP. There exist then covers $\varphi_j : E_j \to X/G$, $j \in J$, where $|J|$ is finite and $\deg \varphi_j > 1$ for each $j$, such that $X(k) \setminus \bigcup_{j \in I} \varphi_j(E_j(k))$ is not Zariski-dense. We can assume, without loss of generality (see [12]), that the $E_j$ are geometrically irreducible. By restricting again $X$ to an open $G$-invariant subvariety, we may assume that:

$$X(k) \subset \bigcup_{j \in J} \varphi_j(E_j(k)).$$  \hfill (3.1)

Note that, if we restrict even further $X$ to an open $G$-invariant subvariety, the assumption (3.1) remains valid. We are going to make use of this several times.

For $j \in J$, we say that $E_j$ is good if $\overline{k}(X)$ and $\overline{k}(E_j)$ are linearly disjoint over $\overline{k}(X)^G$, and bad otherwise. We say that $j \in J$ is good or bad, if the corresponding $E_j$ is respectively good or bad.

We call $k(E_j)$ the field of functions of $E_j$. We denote by $F_j$ the fibered products $E_j \times_{X/G} X$, and call $\pi_j : F_j \to X$ the projection on the second factor.

If $E_j$ is good, the tensor product $\overline{k}(X) \otimes_{k(X)^G} \overline{k}(E_j)$ is a field. This implies that $F_j$ is geometrically irreducible.

If $E_j$ is bad, the field extensions $\overline{k}(X)/\overline{k}(X)^G$ and $\overline{k}(E_j)/\overline{k}(X)^G$ have a common subextension, say $\mathcal{L}_j/\overline{k}(X)^G$, with $[\mathcal{L}_j : \overline{k}(X)^G] > 1$. We call $\iota_{X,j} : \mathcal{L}_j \hookrightarrow \overline{k}(X)$ and $\iota_{E_j} : \mathcal{L}_j \hookrightarrow \overline{k}(E_j)$, the two associated field embeddings. By classical Galois theory we have that $\iota_{X,j}(\mathcal{L}_j)$ is of the form $\overline{k}(X)^{H_j}$, where $H_j$ is a proper subgroup of $G$. We assume, with abuse of notation, that $\mathcal{L}_j = \overline{k}(X)^{H_j}$, i.e. $\iota_{X,j}$ is an inclusion. The field embeddings $\iota_{X,j}, \iota_{E_j}$ will correspond to dominant rational $\overline{k}$-maps $\alpha_{X,j} : X \dashrightarrow X/H_j$, $\alpha_{E_j} : E_j \dashrightarrow X/H_j$. We will assume that $\alpha_{X,j}$ and the $\alpha_{E_j}$’s are morphisms (and not just rational maps), again defined over $\overline{k}$. Note that this assumption is not restrictive, as it may be achieved by restricting $X$ to an open $G$-invariant subvariety, if necessary. Of course, by construction, $\alpha_{X,j}$ is just the projection to the quotient.

We call now $F$ a common field of definition, finite over $k$, for all the $\alpha_{E_j}$, for all the bad $j$’s.

We choose now $i \in I$ such that the field $L_i$, as defined in the hypothesis of this theorem, is coprime with $F$. We call $\pi_i$ the morphism $X_{\alpha_i} \to X/G$.

Let us now prove that $\pi_i(X_{\alpha_i}(k))$ is not contained in $\bigcup_{\text{good } j} \varphi_j(E_j(k))$. We denote by $F'_j$, for any good $j$, the fibered product $E_j \times_{X/G} X_{\alpha_i}$. We denote by $\pi'_j : F'_j \to X_{\alpha_i}$ the projection on the second factor. We note that $F'_j \times_k L \cong F_j \times_k L$, and hence the $F'_j$’s are geometrically irreducible and $\pi'_j : F'_j \to X_{\alpha_i}$ has degree equal to the degree of $\varphi_j$, which is $> 1$.

We have that:

$$\pi_i^{-1} \left( \bigcup_{\text{good } j} \varphi_j(E_j(k)) \right) = \bigcup_{\text{good } j} \pi_i^{-1}(\varphi_j(E_j(k))) = \bigcup_{\text{good } j} \pi'_j(F'_j(k)).$$
Hence, since the $F_j'$s are geometrically irreducible and $\deg \pi_j' > 1$, and $X_{\alpha_i}$ has the HP, there exists $Q \in X_{\alpha_i}(k) \setminus \pi_i^{-1}(\bigcup_{j \in \text{good}} \varphi_j(E_j(k)))$. Hence $\pi_i(Q) \in \pi_i(X_{\alpha_i}(k))$, and $\pi_i(Q) \notin \bigcup_{j \in \text{good}} \varphi_j(E_j(k))$, and therefore $\pi_i(X_{\alpha_i}(k))$ is not contained in $\bigcup_{j \in \text{good}} \varphi_j(E_j(k))$, as we wanted to prove.

We claim now that:

$$\pi_i(X_{\alpha_i}(k)) \cap \bigcup_{j \in \text{bad}} \varphi_j(E_j(k)) = \emptyset. \quad (A)$$

For any bad $j$, we denote by $\xi_j : X/H_j \to X/G$ the natural projection.

In order to prove (A), since we have that

$$\pi_i(X_{\alpha_i}(k)) \cap \bigcup_{j \in \text{bad}} \varphi_j(E_j(k)) = \bigcup_{j \in \text{bad}} \xi_j(\alpha_{X,j}(X_{\alpha_i}(k)) \cap \alpha_{E_j}(E_j(k))),$$

it is enough to prove that:

$$\alpha_{X,j}(X_{\alpha_i}(k)) \cap \alpha_{E_j}(E_j(k)) = \emptyset, \quad (B)$$

for each bad $j$.

To prove (B), we look at the field of definition of a point $P \in \alpha_{X,j}(X_{\alpha_i}(k)) \cap \alpha_{E_j}(E_j(k))$, where $j$ is bad. Since $j$ is bad and $P$ is contained in $\alpha_{E_j}(E_j(k))$, we have that the field of definition of $P$ is contained in $F$. On the other hand, we have that $P \in \alpha_{X,j}(X_{\alpha_i}(k))$. Hence, by Proposition 3.2, we have that the field of definition of $P$ is a subextension of $L_i/k$, which, since $H_j \neq G$, is nontrivial. By our choice of $i \in I$, $L_i$ and $F$ are linearly disjoint, therefore this is impossible. This concludes the proof of (A).

We are now done. In fact, we proved that $\pi_i(X_{\alpha_i}(k)) \subset X/G(k)$ is not contained in $\bigcup_{j \in \text{good}} \varphi_j(E_j(k))$, which, together with (A), implies that $\pi_i(X_{\alpha_i}(k)) \subset X/G(k)$ is not contained in $\bigcup_{j \in \text{bad}} \varphi_j(E_j(k))$, as we wanted to prove.

### 3.2 Quotients by a linear action

Theorem 1.7 goes in the opposite direction with respect to the Inverse Galois Problem. It can be used to obtain varieties with the Hilbert Property, knowing that some group can be realized "enough" times as a Galois group over the field $k$. To make this explicit, we give the following definition:

**Definition 3.3.** We say that a finite group $G$ is strongly realizable as a Galois group over $k$, if, for every finite field extension $L/k$, there exists a Galois extension $E/k$, with Galois group $G$, such that $E/k$ and $L/k$ are linearly disjoint.
Theorem 1.4 can actually be strengthened to obtain the following, whose proof is presented in [12]:

**Proposition 3.4.** Let $X/k$ be a variety, assume that $X/G$ has the HP, then $G$ is strongly realizable as a Galois group over $k$.

We give now a corollary to Theorem 1.5. From this Corollary we will obtain some new examples of varieties with the Hilbert Property.

**Corollary 3.5.** Let $G$ be a finite group acting linearly on $\mathbb{A}^n$ for some $n \geq 1$, then $\mathbb{A}^n/G$ has the HP, if and only if $G$ is strongly realizable as a Galois group over $k$.

*Proof.* The "only if" part is a direct consequence of Theorem 3.4.

The "if" part is a consequence of Theorem 1.5, and the fact that twists of $\mathbb{A}^n/k$ by a linear group action are always trivial, by Hilbert’s Theorem 90.

The interesting part of this result is the "if" part, and although it does not directly help in finding answers to the Inverse Galois Problem, it may be considered nonetheless of independent interest. For instance, this result gives us the Hilbert Property of (some) unirational varieties, proving therefore some particular cases of Conjecture 1.3. We would now like to provide some explicit examples of applications of Theorem 1.5 through Corollary 3.5. In order for these examples to be of interest, we are going to give examples where $\mathbb{A}^n/G$ is not rational.

**Proposition 3.6.** Let $G$ be a finite solvable group, then it is strongly realizable over $\mathbb{Q}$. Hence, $\mathbb{A}^n/G$ has the Hilbert Property.

*Proof.* Let $L/\mathbb{Q}$ be a finite field extension. Denote by $S \subset \mathbb{P}$ the set of primes where $L/\mathbb{Q}$ ramifies. By [9, 597] we know that we can find a $G$-Galois extension $E$ of $\mathbb{Q}$ with no ramification in $S$ (in fact we can choose that all the primes in $S$ split completely in $E$). It is then well known that $E$ and $L$ are linearly disjoint.

*Remark 3.7.* For abelian groups $G$, Proposition 3.6 follows from the well known fact that these groups are regularly realizable over $\mathbb{Q}$, and Proposition 3.4.

Proposition 3.6 provides examples of non-rational (unirational) varieties with the Hilbert Property:

**Example 3.8.** By taking $G = \mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/47\mathbb{Z}$, Lenstra [5] and, respectively, Swan [14] proved that $\mathbb{A}^n/G$ is never rational over $\mathbb{Q}$ (although it is over $\mathbb{C}$) for any faithful linear action of $G$, and has the Hilbert Property by Corollary 3.5.

However, we should mention that this is not a new result, as for varieties of the form $\mathbb{A}^n/G$, where $G$ is an abelian group acting linearly on $\mathbb{A}^n$, even a stronger property has already been proven (namely, weak weak approximation [7]).
Example 3.9. Saltman [10] has given an example of a linear action of a $p$-group $G_p$ of order $p^9$ (where $p$ may be chosen to be any prime), such that the quotient $\mathbb{A}^n/G_p$ is never rational (even over $\overline{\mathbb{Q}}$), for any linear faithful action of $G_p$. This example has been later generalized by Bogomolov in [2] to other examples of nilpotent groups.

Again, for each of these groups $G$, the quotient $\mathbb{A}^n/G$ is not rational (for any faithful linear action of $G$), and has the Hilbert Property by Proposition 3.6.

Hilbert Property has not been proven before in general for varieties of the form $\mathbb{A}^n/G$, where $G$ is solvable, although some partial progress has been made regarding weak weak approximation (see e.g. [8])

4 K3 surfaces with the Hilbert Property

In this section, we are going to provide new examples of K3 surfaces with the Hilbert Property.

We recall that an example of K3 surface with the Hilbert Property has already been given in [4]. Namely, the following surface, also known in the literature as the Fermat surface, has the HP:

$$E := \{ [x : y : z : w] \in \mathbb{P}_3 : x^4 + y^4 = z^4 + w^4 \} / \mathbb{Q}.$$ 

In the next subsection we are going to prove Theorem 1.7, which provides a family of examples of such surfaces with the HP.

The last subsection is devoted to another example of a surface $E'$, for which the HP has been suggested to be true by Corvaja and Zannier in [4].

We explain here the construction of this surface and the idea of the proof of its HP, although we will postpone the details to the end of this section. We feel that this proof is an interesting application of the other results given in this work.

Consider the following automorphism of the Fermat surface $E$:

$$\sigma : E \longrightarrow E, \quad \sigma([x : y : z : w]) = [x : -y : -z : w].$$

By construction, $\sigma^2 = 1$. Therefore we have an induced action of the group $C_2 = \mathbb{Z}/2\mathbb{Z}$ on $E$. The surface $E'$ is then defined as (a smoothening of) the quotient $E/C_2$. This is a K3 surface (see [4]).

To prove that $E'$ has the HP we are going to prove that enough, as in Theorem 1.5, $C_2$-twists of $E$ have the HP. Each of the $C_2$-twists of $E$ may be described as follows:

Let $\mathbb{Q} \subset \mathbb{Q}(\sqrt{d})$ be a quadratic extension, and let $\alpha_d$ be the unique element of $\text{Iso}(C_2, \text{Gal}(\mathbb{Q}(\sqrt{d}/\mathbb{Q})))$, the twist $E_d$ of $E$ by $\alpha_d$ is then:

$$E_d = E_{\alpha_d} = \{ [x : y : z : w] \in \mathbb{P}_3 \mid x^4 + d^2y^4 = d^2z^4 + w^4 \}.$$
We will prove the following, using mainly Theorem 4.7, from which the HP of \( E' \) follows:

**Theorem 4.1.** Given any finite field extension \( E/\mathbb{Q} \), there exists \( a \in \mathbb{Q}^* \), which is not a perfect square, and such that \( \mathbb{Q}(\sqrt{a}) \) and \( E \) are linearly disjoint over \( \mathbb{Q} \) and the surface \( E_d/\mathbb{Q} \) has the HP.

### 4.1 Hilbert Property for certain K3 surfaces with two elliptic fibrations

In this section our field of definition will always be assumed to be a number field \( K \).

We will denote by \( V_{a,b,c,d}/K \), where \( a, b, c, d \in K^* \), the following surface embedded in \( \mathbb{P}_3/K \):

\[
ax^4 + by^4 + cz^4 + dw^4 = 0.
\]

We will always assume that \( abcd \neq 0 \) is a square in \( K^* \). This class of varieties has been widely studied, e.g. in [6]. Obviously we have that \( V_{a,b,c,d} \cong V_{1,1,-1,-1} \) over \( \mathbb{C} \), when \( a, b, c, d \in K^* \).

The variety \( V_{a,b,c,d} \) possesses a natural morphism \((\tau : [x : y : z : w] \mapsto [x^2 : y^2 : z^2 : u^2])\) to the variety:

\[
Q_{a,b,c,d} : ax^2 + by^2 + cz^2 + dw^2 = 0,
\]

which, being a conic in \( \mathbb{P}_3 \), is rational as soon as there is at least one rational point on it. Under the hypothesis that there exists a rational point \( P \in Q_{a,b,c,d} \) and that \( abcd \neq 0 \) is a square, the two usual rulings of \( Q_{a,b,c,d} \) are defined over \( K \) (for a proof of this fact, see [6]).

We will briefly recall some natural structures that these surfaces have (for a more detailed exposition, see e.g. [6])

**Definition 4.2.** Let \( a, b, c, d \in K^* \) be such that \( abcd \) is a square, and let \( V = V_{a,b,c,d} \) be a diagonal quartic surface as defined above. Assume that the associated quartic \( Q = Q_{a,b,c,d} \) has at least a rational point in it. Fix a rational point \( R \in Q \), and decompose the intersection of \( Q \) with the plane in \( \mathbb{P}_3 \) tangent to \( Q \) at \( R \) into two lines \( l_1, l_2 \). Let \( f_1, f_2 : Q \to \mathbb{P}_1 \) be two rulings on \( Q \) such that \( l_i \) is a fiber of \( f_i \). For \( i = 1, 2 \), set \( \pi_i = f_i \circ \tau : V \to \mathbb{P}_1 \).

When considering purely geometric questions we may assume that \( V_{a,b,c,d} \cong V_{1,1,-1,-1} \), and the fibrations are:

\[
\pi_1 = \frac{x^2 - z^2}{w^2 - y^2}, \quad \text{and} \quad \pi_2 = \frac{x^2 - z^2}{w^2 + y^2}.
\]
One can easily check that the $\pi_i$’s are elliptic fibrations.
We define rational automorphisms $e_1, e_2$ as follows:

**Definition 4.3.** The morphism $e_1$ sends any point $P$ to the unique second intersection point between the fiber of $\pi_1$ through $P$ and the tangent at $P$ to the fiber of $\pi_2$ through $P$. The morphism $e_2$ is defined by interchanging the roles of $\pi_1, \pi_2$.

On $V_{1,1,1,1}$ (and therefore on $V_{a,b,c,d}$) we have exactly 48 lines, defined over $\mathbb{C}$, 16 of which are defined by equations: $x = i^{k+1/2}y, z = i^{j+1/2}w$, where $k = 1, \ldots, 4$, $j = 1, \ldots, 4$. The other 32 are defined analogously by permutation of the coordinates.

We denote by $\Omega$ the union of these lines and the points with at least one zero coordinate.

The main result in this section will be Theorem 1.7, in its proof we are going to use techniques from both [4] and [6].

We are going to prove the following, and then deduce Theorem 1.7 from it:

**Theorem 4.4.** Let $E/K$ be a smooth algebraically simply connected surface with two elliptic fibrations over $\mathbb{P}^1/K$, denoted by $\pi_1, \pi_2$. We denote the fibers of these fibrations by $E_{j,x} = \pi_1^{-1}(x)$, where $x \in \mathbb{P}^1(K)$. Assume that:

(a) There exists a finite set of points $Z \subset \mathbb{P}^1(\bar{K})$, such that, for $x \in \mathbb{P}^1(K) \setminus Z$, if $E_{2,x}$ has infinitely many $K$-rational points, all but finitely many $P_i \in E_{2,x}$ lie on a curve $E_{1,y}$ with infinitely many $K$-rational points;

(b) Denote by $E_{1,\lambda_0}$ the generic fiber of $\pi_1 : E \to \mathbb{P}^1/K$, and $K(\lambda)$ the field of rational functions of the codomain of $\pi_1$. Take $\pi_{2,0} : E_{1,\lambda_0} \to \mathbb{P}^1_{K(\lambda)}$ to be the restriction of $\pi_2$ to $E_{1,\lambda_0}$. Then all the diramation points of $\pi_{2,0}$ (viewed as points in $\mathbb{P}^1_{K(\lambda)}$) are non-constant in $\lambda$, and the same holds inverting $\pi_1$ and $\pi_2$;

(c) There are two non-thin subsets $N_1, N_2$ of $\mathbb{P}^1(K)$, such that, for $j = 1, 2$, $E_{j,x}$ has infinitely many $K$-rational points for each $x \in N_j$.

We have then that the surface $E/K$ has the Hilbert Property.

### 4.2 Proof of Theorem 4.4

Before coming to the proof of Theorem 4.4, we recall the following (we refer to [4] for a proof):

**Lemma 4.5.** Let $G$ be an finitely generated abelian group of positive rank. Assume that $G \setminus \bigcup_{u \in U}(h_u + H_u)$ is finite, where $U$ is finite, $h, u \in G$ and $H_u$ is a subgroup of $G$. Then $\bigcup_{u \in U}(h_u + H_u) = G$. 12
Proof of Theorem 4.4. Suppose that the surface $E/K$ does not have the HP. Then there exist covers $\varphi_i : Y_i \to E$, $i \in I$, $|I|$ finite, each of degree $d_i > 1$, such that $E(K) \subset \bigcup_i \varphi_i(Y_i(K)) \cup D(K)$, where $D$ is a proper closed subvariety of $E$. We may assume that the $Y_i$ are geometrically irreducible (see [12, 20]). Moreover, we may assume the morphisms $\varphi_i$ to be finite maps, and the varieties $Y_i$ to be normal.

Since the surface $E$ is algebraically simply connected, we know that each of the $\varphi_i$ will have a nontrivial ramification divisor. Call $R_i \subset E$ the diramation locus for $\varphi_i$ (which will be necessarily of codimension 1). For $j = 1, 2$, we divide now the covers in three kinds (this division will depend on $j$):

(a) we say that the cover $\varphi_i : Y_i \to E$ is almost $j$-good if $\pi_j \circ \varphi_i$ has a geometrically irreducible generic fiber, and $R_i$ is contained in a finite number of fibers of $\pi_j$;

(b) we say that the cover $\varphi_i : Y_i \to E$ is $j$-good if $\pi_j \circ \varphi_i$ has a geometrically irreducible generic fiber, and the morphism $\pi_j |_{R_i} : R_i \to \mathbb{P}_1$ is surjective;

(c) we say that the cover $\varphi_i : Y_i \to E$ is $j$-bad if the morphism $\pi_j \circ \varphi_i$ has a geometrically reducible generic fiber.

We will say, with abuse of notation, that $i \in I$ is $j$-bad or (almost) $j$-good if $\varphi_i$ is respectively $j$-bad or (almost) $j$-good.

When $\varphi_i$ is $j$-bad, we have the following factorization of $\pi_j \circ \varphi_i$ (see [1, Lemma 3.1]):

$$\pi_j \circ \varphi_i : Y_i \xrightarrow{\varphi_i} C_{i,j} \xrightarrow{r_{i,j}} \mathbb{P}_1,$$

where the $C_{i,j}$ are geometrically irreducible projective curves /$K$, and the $r_{i,j}$ are finite morphisms, with $\deg r_{i,j} > 1$.

We call now $S'_j$ the set $\bigcup_{j\text{-bad}} C_{i,j}(K) \subset \mathbb{P}_1(K)$, which is, by construction, thin.

We consider now the following subset of $\mathbb{P}_1(K)$:

$$S''_j := \{x \in \mathbb{P}_1(K) : E_{j,x} \subset D\} \cup \bigcup_{(\text{almost}) \text{ j-good } i} \{x \in \mathbb{P}_1(K) : (\pi_j \circ \varphi_i)^{-1}(x) \text{ is not irreducible and smooth}\} \cup \{x \in \mathbb{P}_1(K) : E_{j,x} \text{ is not irreducible and smooth}\}.$$

Notice that $\{x \in \mathbb{P}_1(K) : E_{j,x} \subset D\}$ is obviously finite. The set

$$\{x \in \mathbb{P}_1(K) : (\pi_j \circ \varphi_i)^{-1}(x) \text{ is not irreducible}\}$$
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is finite as well, as, for (almost) $j$-good $i \in I$, the generic fiber of $\pi_j \circ \varphi_i$ is geometrically irreducible. The set

$$\{x \in \mathbb{P}_1(K) : (\pi_j \circ \varphi_i)^{-1}(x) \text{ is not smooth}\}$$

is finite as well, in fact this follows from generic smoothness applied to the morphism $\pi_j \circ \varphi_i \circ \nu_i : \tilde{Y}_i \to \mathbb{P}_1$, where $\nu_i : \tilde{Y}_i \to Y_i$ is a desingularization of $Y_i$, and the fact that $Y_i$ is normal.

One can show analogously that the set

$$\{x \in \mathbb{P}_1(K) : E_{j,x} \text{ is not irreducible and smooth}\}$$

is finite as well.

Hence $S''_j$ is finite.

We call $S_j = S'_j \cup S''_j$, and $V_j = \pi_j^{-1}(S_j)$. We notice that $S_j$ is still thin. We claim that then the following holds:

$$\text{If } x \in \mathbb{P}_1(K) \setminus S_j \text{ and the Mordell-Weil rank of } E_{j,x} \text{ is positive, then } \pi_j^{-1}(x)(K) \subset \bigcup_{\text{almost } j-\text{good } i} \varphi_i(Y_i(K)). \quad (\star)$$

We focus now on the proof of $(\star)$, keeping the already defined notation.

We have that $\pi_j^{-1}(x)(K) \subset \bigcup_{i \in I} \varphi_i(Y_i(K)) \cup D(K)$. Since $x \notin S'_j$, we have by (A) that $\pi_j^{-1}(x)(K) \cap (\bigcup_{j-\text{bad } i} \varphi_i(Y_i(K))) = \emptyset$. Hence:

$$\pi_j^{-1}(x)(K) \subset \bigcup_{\text{almost } j-\text{good } i} \varphi_i((\pi_j \circ \varphi_i)^{-1}(x)(K))$$

$$\cup \bigcup_{j-\text{good } i} \varphi_i((\pi_j \circ \varphi_i)^{-1}(x)(K)) \cup (D \cap \pi_j^{-1}(x))(K).$$

Since $x \notin S''_j$, we have that $(D \cap \pi_j^{-1}(x))(K)$ is finite. For $j$-good $i$, we denote by $D_{i,x}$ the curve $(\pi_j \circ \varphi_i)^{-1}(x)$ (actually, $D_{i,x}$ depends on $j$ too, but since there is no risk of misinterpretation, we avoid using this index here to lighten the notation). Since $x \notin S''_j$, we have that $D_{i,x}$ is an irreducible curve, and $R_i$ intersects properly $E_{j,x}$, hence the morphism $D_{i,x} \xrightarrow{\varphi_i \mid_{D_{i,x}}} E_{j,x}$ is ramified by Corollary 2.7.

Hence, by Riemann-Hurwitz formula applied to $\varphi_i \mid_{D_{i,x}}$, the smooth curve $D_{i,x}(K)$ has genus $> 1$. Therefore, by Falting’s theorem, $D_{i,x}(K)$ is finite. We have deduced that:

$$\pi_j^{-1}(x)(K) \subset \bigcup_{\text{almost } j-\text{good } i} \varphi_i((\pi_j \circ \varphi_i)^{-1}(x)(K)) \cup A_0, \quad (B)$$

where $A_0$ is a finite set.
We notice now that, for almost $j$-good $i \in I$, $D_{i,x}$ is a curve of genus 1. In fact, the morphism:
\[ \varphi_{i|D_{i,x}} : D_{i,x} \to E_{j,x}, \]
is, since $R_i$ does not intersect $E_{j,x}$, unramified (again, by Corollary 2.7), hence, by Riemann-Hurwitz formula applied to $\varphi_{i|D_{i,x}}$, $D_{i,x}$ is a curve of genus 1. Giving now $E_{j,x}$ the structure of an elliptic curve, by choosing $O \in E_{j,x}(K)$, we have that $\varphi_{i|D_{i,x}}$ is a composition of a translation and an isogeny, hence $\phi_{i}(D_{i,x}(K)) \subset C_x(K)$ is a finite index group coset. Hence, (*) follows from (B) and Lemma 4.5.

We remark that in course of proving (⋆), we proved that, if $x \not\in S_j$, than only finitely many of the rational points in $E_{j,x}$ lift to the $j$-good covers. We are going to use this again at the end of this proof.

We will now focus our attention on the fibers of the second fibration, $\pi_2$.

Since $S_1$ is thin, we know that there are some smooth projective curves $X_1, \ldots, X_r$, and finite morphisms $\alpha_k : X_k \to \mathbb{P}_1$, of deg $\alpha_k > 1$, such that $S_1 \subset \bigcup_{k=1}^r X_k(K)$.

We call now $R_S$ the union of the diramation sets of the $\alpha_k$.

We denote by $T_2$ the following set:
\[ T_2 := \{ x \in \mathbb{P}_1(K) : \pi_1|_{E_{2,x}} : E_{2,x} \to \mathbb{P}_1 \text{ ramifies over } R_S \}. \]

By hypothesis (b), we know that $T_2$ is finite. We denote now by $\tilde{S}_2$ the union $T_2 \cup S_2 \cup Z$. By construction, $\tilde{S}_2$ is thin. We choose now, using hypothesis (c), a $y \in \mathbb{P}_1(K) \setminus \tilde{S}_2$, such that the Mordell-Weil rank of $E_{2,y}$ is positive.

We claim now that only a finite number of the $K$-rational points in $E_{2,y}(K)$ are in $V_1$.

To prove this we consider the following fibered products, indexed by $k = 1, \ldots, r$ and $y \in \mathbb{P}_1(K) \setminus \tilde{S}_2$:
\[ F_{k,y} := X_k \times_{\alpha_k, \pi_1} E_{2,y}. \]

We denote by $f_{k,y} : F_{k,y} \to E_{2,y}$ the projection on the second factor.

Since the diramation of $\alpha_k$ and $\pi_1|_{E_{2,y}}$ are disjoint, we have that the curve $F_{k,y}$ is smooth and geometrically irreducible. Moreover, we know that the morphism $\pi_1|_{E_{2,y}}$ is ramified (as $\mathbb{P}_1$ is simply connected), hence the projection $f_{k,y} : F_{k,y} \to E_{2,y}$ will be ramified as well. This in turn implies, by Riemann-Hurwitz formula applied to $f_{k,y}$, that $F_{k,y}$ has genus $\geq 2$. By Falting’s theorem we have then that $F_{k,y}(K)$ is finite. In conclusion, we have:
\[
E_{2,y}(K) \cap V_1 = \pi_1|_{E_{2,y}}(S_1) \subset \pi_1|_{E_{2,y}}^{-1}(\bigcup_{k=1}^{r} X_k(K))
= \bigcup_{k=1}^{r} \pi_1|_{E_{2,y}}^{-1}(X_k(K)) = \bigcup_{k=1}^{r} f_{k,y}(F_{k,y}(K)),
\]
which is finite. Hence $E_{2,y}(K) \cap V_1$ is finite, and we proved our claim.

In particular we deduce, using $(\star)$ for $j = 1$ and hypothesis $(a)$, that all but finitely many of the points in $\pi_2^{-1}(y)(K)$ lift to the almost 1-good covers, and, since $y \notin S_2$, none of them lifts to the 2-bad covers. Therefore, they almost all lift to 1-almost good covers, that are 2 good or 2-almost good. These covers are hence necessarily 2-good, in fact their ramification will be contained in a finite union of fibers of $\pi_1$, which intersect properly the fibers of $\pi_2$.

We are now in the position to produce an absurd. In fact we have just proved that all but finitely many of the points in $\pi_1^{-1}(y)(K)$ lift to 2-good covers. Yet, we have that $y \notin \tilde{S}_2$, hence only a finite number of these points can lift to 2-good covers, as we proved before. Since we chose the Mordell-Weil rank of $E_{2,y}$ to be positive, this is absurd.

To proceed with the proof of Theorem 1.7, we need to check that the hypothesis of Theorem 4.4 hold in our case, taking as surface $E/K = V_{a,b,c,d}/\mathbb{Q}$ (where $a, b, c, d \in K^*$ are such that the hypothesis of Theorem 1.7 hold), and as fibrations $\pi_1, \pi_2$ the ones defined in Definition 4.2. We keep here the same notation as in Definition 4.2.

Condition $(b)$ of Theorem 4.4 can be easily checked. In fact, let us assume without loss of generality, that $V_{a,b,c,d} = V_{1,1,-1,-1}$ and:

$$\pi_1 = \frac{x^2 - z^2}{w^2 - y^2}, \text{ and } \pi_2 = \frac{x^2 - z^2}{w^2 + y^2}.$$ 

Let $\lambda \in \mathbb{P}^1/\mathbb{Q}$ be the generic point. We denote by $E_\lambda$ the generic fiber $\pi_1^{-1}(\lambda)$. The morphism $\pi_2 : E_\lambda \to \mathbb{P}^1$ ramifies then only if $x, y, z$ or $w = 0$. Since none of the subvarieties defined by $x, y, z$ or $w = 0$ have irreducible components that are contained in the fibers of the fibrations $\pi_1$ and $\pi_2$, we are done.

To prove that conditions $(a)$ and $(c)$ hold in our case, we will need to produce "a lot of" rational points on $V_{a,b,c,d}$. We will use the following technique, described, for instance, in [6]:

Take a rational point $P = [x_0 : y_0 : z_0 : w_0] \in V_{a,b,c,d}(K)$, such that $x_0y_0z_0w_0 \neq 0$, and $P \notin \Omega$. Denote by $C_1, C_2$ the fibers of the usual two fibrations of $V$ passing through $P$ (by our choice of $P$, $C_1$ and $C_2$ will be smooth curves of genus 1). Assume that $e_1(P) - (P)$ is not torsion, and consider the set $S \subset C_1(K)$ generated by $e_1(P)$ on $C_1$, having given the curve $C_1$ the structure of an elliptic curve by choosing $P$ to be the origin. Keeping this notation, the following has been proven in [6], which proves directly that condition $(a)$ holds:

**Lemma 4.6.** There are only finitely many points $Q \in S$ such that $(e_2(Q)) - (Q)$ has finite order in $\text{Div}_0 C_2$. 

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We anticipate now the following, which we will use to prove that condition (c) holds:

**Lemma 4.7.** Fix a number field $K$. Let $\pi_i : E_i \to \mathbb{P}_1/K$, $i \in I$, with $|I| = \infty$, be elliptic covers (i.e. finite morphisms where $E_i$ is a genus 1 curve), such that:

- For each choice $p_1, \ldots, p_n$ of points in $\mathbb{P}_1/K$, we have that for all but finitely many $i \in I$, $E_i \to \mathbb{P}_1/K$ does not ramify over the $p_i$’s;

- For each $i \in I$ there is a subset $S_i \subset E_i(K)$ of infinite cardinality.

Then $\bigcup_i \pi_i(S_i)$ is not thin in $\mathbb{P}_1/K$.

**Proof.** Suppose that $\bigcup_i \pi_i(S_i)$ were thin. Then we would have a finite number of covers $\varphi_j : C_j \to \mathbb{P}_1/K$ such that $\bigcup_i \pi_i(S_i) \setminus \bigcup_j \varphi_j(C_j(K))$ is finite. Take $p_1, \ldots, p_n \in \mathbb{P}_1/K$ to be the union of all ramification points of the $\varphi_j$’s. Take an elliptic cover $\pi_{i_0} : E_{i_0} \to \mathbb{P}_1/K$, $i_0 \in I$, with ramification disjoint from the $p_k$’s. Consider then the fibered product $D_j = E_{i_0} \times_{\mathbb{P}_1} C_j$, and denote by $\psi_j = D_j \to E_{i_0}$ the projection on the first factor. Since the diramations of $\varphi_j$ and $\pi_{i_0}$ are disjoint, $D_j$ is a smooth irreducible curve, and since $\varphi_j$ is ramified in points where $\pi_{i_0}$ is not, the morphism $\psi_j$ will be ramified as well. Therefore, by Riemann-Hurwitz formula, we have that the genus of $D_j$ is $\geq 2$. Hence, by Falting’s theorem, there are only a finite number of rational points on $D_j$. But now we have that $\pi_{i_0}^{-1}(\varphi_j(C_j(K))) = \psi_j(D_j(K))$, and this is finite. Therefore, there are infinitely many points $p$ in $E_{i_0}(K)$ such that $\pi_{i_0}(p)$ does not lie in any of the $\varphi_j(C_j(K))$, hence the absurd, and the thesis.

We complete now the proof of Theorem 1.7:

**Proof of Theorem 1.7.** We want to use Theorem 4.4 on the surface $V$, taking as fibrations $\pi_1$ and $\pi_2$ the ones defined in Definition 4.2. We just proved that hypothesis (a) and (b) hold, so the proof is concluded once we show that hypothesis (c) holds as well.

Let $P$ be a rational point not in $\Omega$, whose existence is guaranteed by the hypothesis. Denote by $C_1$, $C_2$ the fibers passing through it. In [6] it is proven that, keeping our notation, there exists $i = 1$ or 2 such that $(e_i(P)) - (P)$ is not torsion (in the corresponding $\text{Div}(C_i)$). Assume, without loss of generality, that $i = 1$. Then, by Lemma 4.7, we know that we can find infinitely many points $Q_i$ on $C_1$ such that $(e_2(Q_i)) - (Q_i)$ is not torsion. This in turn generates infinitely many rational points $Q_{ij}$ on the curve $\pi_2^{-1}(\pi_1(Q_i))$. Fixing $i$, all but finitely many of the curves $\pi_1^{-1}(\pi_1(Q_{ij}))$, will be curves of genus 1 with positive Mordell-Weil rank (thanks again to Lemma 4.7). Therefore $\pi_1(Q_{ij})$ will be rational points on $\mathbb{P}_1$ such that $\pi_1^{-1}(\pi_1(Q_{ij}))$ is a curve of genus 1 with positive Mordell-Weil rank.
Hence, using Lemma 4.7 on the curves $\pi_2^{-1}(\pi_2(Q_i))$ with points $Q_{ij}$ on them, we deduce that condition (c) holds for $\pi_1$ (the ramification hypothesis follows immediately from condition (b) of Theorem 4.4, which we have already proven). By repeating the same argument interchanging the roles of $\pi_1$ and $\pi_2$ and using anyone of the $Q_i$’s as starting point, instead of $P$, we obtain that condition (c) holds for $\pi_2$ as well.

We are now ready to prove Theorem 4.1:

**Proof of Theorem 4.1.** We keep the notation used in the formulation of the theorem.

It is enough to prove it for $\mathbb{Q}$ (see [12]). We know that the twist $E_d$ has the HP as soon as it has a rational point outside $\Omega$. We will obtain such rational points by using an elliptic curve chosen ad hoc, as follows:

Take a couple of odd coprime integer numbers $n, m > 1$. Denote by $\lambda = n^4 - m^4$.

Consider the following curve of geometric genus 1:

$$E_\lambda : \lambda d^2 = t^4 - 1,$$

defined over $\mathbb{Q}$. By construction, the point $P_0 = (d, t) = \left(\frac{1}{m^2}, \frac{n}{m}\right)$ lies on $E_\lambda$. The curve $E_\lambda$ is a quadratic twist of the curve:

$$E : d^2 = t^4 - 1,$$

(Q)

with respect to the involution $\sigma : d \mapsto -d$, $t \mapsto t$, and the field $\mathbb{Q}(\sqrt{\lambda})$. The point $P_0$, in the model (Q), corresponds to the point $\left(\frac{\sqrt{\lambda}}{m^2}, \frac{n}{m}\right)$. Passing to a Weierstrass model for $E$ (we are using here the substitution $x = t^2 - d$, $y = tx$), we obtain:

$$\left(x - \frac{y^2 \lambda}{x^2}\right)^2 = d^2 = t^4 - 1 = \frac{y^4}{x^4}$$

$$2y^2 = x^3 + x.$$  

(W)

We wish now to obtain a description of the automorphism $\sigma$ through the group law on $E$, which we assume to have the structure of elliptic curve given by the Weierstrass model (W). To do so, we note that, generically, $\sigma(P)$ is the unique point on $E$ such that $t(\sigma(P)) = t(P)$, and $\sigma(P) \neq P$.

It is a straightforward verification to check now that $\sigma(P) = E_2 - P$.

Therefore, rational points on $E_\lambda$ correspond to points $P \in E(\mathbb{Q}(\sqrt{\lambda}))$ such that $P + \bar{P} = E_2$, where the bar denotes the Galois conjugation of $\mathbb{Q}(\sqrt{\lambda})/\mathbb{Q}$. Therefore odd multiples of $P_0$ on $E$ correspond to rational points on $E_\lambda$. We prove now that $P_0$ is not torsion on $E$, and to do so we will apply the Lutz-Nagell theorem to $[2]P_0$. 

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By explicit computation it can be seen that the $x$-coordinate (in the model $(W)$) of $[2]P$, where $P$ is a point in $E$, is equal to $t(P)^2 - 1/t(P)^2$, where $t(P)$ denotes the $t$-coordinate of $P$ in the model $(Q)$.

Since we have that $P + \bar{P} = E_2$, we have that $[2]P + [2]\bar{P} = O$. Therefore, by denoting by $x' = x([2]P)$, $y' = y([2]P)$ the coordinates of $[2]P$ in model $(W)$, we have that $y' = \sqrt{\lambda}y_0$, where $y_0 \in Q$. Hence we have that:

$$
\left(2\lambda\left(\frac{t^2 - 1}{t^2}\right)\right)^3 + 4\lambda^2 \left(2\lambda\left(\frac{t^2 - 1}{t^2}\right)\right) = (4\lambda y_0)^2,
$$

but $2\lambda\left(\frac{t^2 - 1}{t^2}\right) = \frac{2(n^4 - m^4)}{n^2 m^2} \notin \mathbb{Z}$. By Lutz-Nagell theorem, we know that $[2]P_0$, and hence $P_0$, is not torsion.

As noted before, this gives us now infinitely many rational solutions $(d_k, t_k)_{k \in \mathbb{N}}$ of the polynomial equation $\lambda d^2 = t^4 - 1$. Each of these points can be used to obtain a point on the surface $E_{d_k}$, namely the one with coordinates $[t_k : m : n : 1]$. By construction, for almost all $k$ this point will have nonzero coordinates and will not lie on the lines of $E_{d_k}$.

By Theorem 1.7, this proves the HP for the surfaces $E_{d_k}$.

To use Theorem 1.5, thus concluding the proof, it is enough to prove that, given a finite field extension $L/\mathbb{Q}$, there exists a $k \in \mathbb{N}$ such that $\mathbb{Q}(\sqrt{d_k})$ is linearly disjoint from $L$, and $d_k$ is not a square. Since quadratic extensions are minimal, this is equivalent to prove that we can always choose such $d_k$ such that $\mathbb{Q}(\sqrt{d_k})$ is linearly disjoint from any finite family $\mathbb{Q}(\sqrt{a_1}), \ldots, \mathbb{Q}(\sqrt{a_l})$ of quadratic extensions, and $d_k$ is not a square. This is equivalent to saying that $d_k a_j$ is not a square for $j = 0, \ldots, l$, where we set, for convenience, $a_0 := 1$.

But we have that, if $d_k a_j = s^2$ for some $s \in \mathbb{Q}$, then:

$$a_j^2 \lambda s^4 = t^4 - 1 \text{ (where } a_j \neq 0). 
\tag{C}
$$

Considering $s, t$ as variables, (C) defines an affine curve, whose projective closure $C_j$ is of genus 3 (in fact $C_j$ is a smooth curve of degree 4 in $\mathbb{P}^2$, and has therefore genus $(4 - 1) \cdot (4 - 2)/2 = 3$).

Therefore, by Falting’s theorem applied to the curves defined through (C), there are only a finite number of rational solutions $(s, t)$ of (C) for each $j = 0, \ldots, l$. Hence there are only a finite amount of $k \in \mathbb{N}$ such that $a_j d_k$ is a square for some $j = 0, \ldots, l$. Since the $d_k$’s are infinite, this concludes the proof of the theorem.

As noticed at the beginning of this section, we have the following corollary of Theorem 4.1:

**Corollary 4.8.** Keeping the notation above, the surface $E' = E/C_2$ has the HP over number fields.
Proof. This follows immediately from Theorem 1.5 and Theorem 4.1.

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