Fractional-Power-Law Level-Statistics due to Dynamical Tunneling

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For systems with a mixed phase space we demonstrate that dynamical tunneling universally leads to a fractional power law of the level-spacing distribution $P(s)$ over a wide range of small spacings $s$. Going beyond Berry-Robnik statistics, we take into account that dynamical tunneling rates between the regular and the chaotic region vary over many orders of magnitude. This results in a prediction of $P(s)$ which excellently describes the spectral data of the standard map. Moreover, we show that the power-law exponent is proportional to the effective Planck constant $\hbar_{\text{eff}}$.

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Spectra of quantum systems whose classical dynamics are either regular or chaotic usually show universal statistical properties. This fascinating relation between classical motion and quantum spectra is demonstrated in Ref. [1], where it was argued that the spectra of generic regular systems show Poissonian statistics. In contrast, spectral correlations of classically chaotic systems can be described by random matrix theory [2,3]. A justification of this conjecture was given in terms of periodic orbit theory [4,5]. The nearest-neighbor level-spacing distribution $P(s)$ is of central importance to the study of universal spectral properties [6,7]. Results from these studies are of broad interest for applications in, e.g., solid state physics [8,9], mesoscopic physics [10], cold atom physics [11], and atomic as well as acoustic physics [12].

The spacing distribution of generic Hamiltonian systems has been the subject of an active debate over the last decades [14–22]. These systems show a mixed phase space, where disjoint regions of either regular or chaotic motion coexist [see Fig. 1]. Assuming statistically independent subspectra corresponding to regular and chaotic regions in phase space, Berry and Robnik computed the level-spacing distribution of mixed systems [14]. In contrast to the predicted level-clustering behavior, $P(s) > 0$ at $s = 0$, numerically a fractional power-law distribution

$$P(s) \propto s^{\beta}$$

for small spacings $s$ with exponent $\beta \in [0, 1]$ was observed [15,16]. Qualitatively this behavior may be described by the Brody distribution [23,24], as recently discussed in Refs. [10,11]. Yet, this approach involves a free fitting parameter which is not related to any physical property of the system.

Dynamical tunneling [17,25–30] plays an important role for the level-spacing distribution, as it weakly couples regular and chaotic states and thus enlarges small spacings between the corresponding levels. In Refs. [19–22] a phenomenological coupling strength between regular and chaotic states was introduced, while in Refs. [17,18] a fit-free prediction of the level-spacing distribution was given. However, these results do not explain the numerically observed power-law distribution, Eq. (1).

In this Letter we show that a fractional power-law distribution universally arises over a wide range of small spacings because tunneling rates from different regular states range over many orders of magnitude. We give an analytical prediction of the level-spacing distribution, which is in excellent agreement with numerical data of a modified standard map [see Fig. 1(a) and a designed kicked
system [see Fig. 2]. Moreover, we demonstrate that the power-law exponent $\beta$ scales like $\beta \propto h_{\text{eff}}$. For the smallest spacings below the power-law regime our result recovers the well known linear level repulsion [3].

As model systems we study kicked systems described by the Hamiltonian $H(q, p) = T(p) + V(q) \sum_{n \in \mathbb{Z}} \delta(t - n)$ [32]. By a stroboscopic view at integer times one gets an area preserving map on the two-torus. The relative areas of the regular and chaotic regions in phase space are denoted by $\rho_r$ and $\rho_c = 1 - \rho_r$. The map is quantum mechanically given by a unitary operator $U$ on a Hilbert space of dimension $N$ with effective Planck constant $h_{\text{eff}} = 1/N$. The semiclassical limit is approached as $h_{\text{eff}} \to 0$. Solving the eigenvalue equation $U |n\rangle = e^{i\phi_n} |n\rangle$ yields $N$ eigenphases $\phi_n$ with eigenvectors $|n\rangle$. According to the semiclassical eigenfunction hypothesis [33–35] one expects $N_r \approx \rho_r N$ regular states as well as $N_c \approx \rho_c N$ chaotic states. The unfolded spacings are $s_n = (\phi_{n+1} - \phi_n)/2\pi$ for phases $\phi_n$ which are ordered by increasing size.

In order to study the influence of dynamical tunneling on spectral statistics we consider the standard map [36] which is the paradigmatic kicked system. We use parameter values for which it has one large regular island. To obtain significant statistics for small spacings we consider an ensemble of modified standard maps, where the regular island is preserved, while the chaotic dynamics is varied. Thus, the regular levels and the tunneling rates remain essentially unchanged, while the chaotic levels are strongly varied. This is realized by varying the kicking potential $V(q)$ in the chaotic part [31]. Moreover, since partial barriers in the chaotic region may affect spectral statistics beyond dynamical tunneling [20], we remove their influence by the above choice of the kicking potential. Numerically, we focus on the $h_{\text{eff}}$-regime with few regular levels (e.g. $N_r = 6$), as semiclassically ($N_r \to \infty$) the influence of tunneling on spectral statistics becomes less pronounced. We find a power-law distribution for $P(s)$ at small $s$ [see Fig. 1]. This behavior is also observed for another ensemble of kicked systems [37] [see Fig. 2].

We model the spectral statistics of systems with a mixed phase space by considering a random matrix Hamiltonian $H$ which contains regular levels $\varepsilon_r$ and chaotic levels $\varepsilon_c$ on its diagonal. These levels are coupled by off-diagonal elements $v_{m,j}$, which account for tunneling contributions between the $m$th regular and the $j$th chaotic state. $H$ is scaled such that the mean level spacing is unity and can be chosen real and symmetric for time-reversal invariant systems. The regular levels $\varepsilon_r$ are semiclassically determined by the torus structure of the regular island [32]. Since $N_r$ is small, they do not show the Poissonian behavior assumed in Refs. [18, 19, 22]. The chaotic levels $\varepsilon_c$ behave like eigenphases of a random matrix from the circular orthogonal ensemble [3]. Here we assume that there are no additional phase-space structures within the chaotic region. The coupling matrix elements $v_{m,j}$ are modeled by independent Gaussian random variables with zero mean. The standard deviation $v_m$ of $v_{m,j}$ does not depend on the chaotic state $j$ but is specific for each regular state $m$. In particular $v_m$ is smaller for states $m$ which quantize closer to the center of the regular island, $v_0 < \ldots < v_{N_c-1}$. The typical coupling $v_m$ is related to the tunneling rate $\gamma_m$ of the $m$th regular state by

$$v_m = \frac{N}{2\pi} \sqrt{\frac{2m}{N_c}},$$

which follows from the dimensionless form of Fermi’s golden rule in kicked systems [38]. Hence, we model the probability density $P(v)$ of all couplings by

$$P(v) = \frac{1}{N_r} \sum_{m=0}^{N_c-1} \frac{1}{\sqrt{2\pi v_m}} e^{-v^2/v_m^2}.$$ 

The tunneling rates $\gamma_m$ are parameters of the random matrix model which can either be determined numerically or analytically, e.g. using the fictitious integrable system approach [28, 30]. Since the tunneling rates $\gamma_m$ vary over many orders of magnitude, the typical couplings $v_m$ embrace a wide range on a logarithmic scale [see the triangles in Fig. 1]. Hence, in contrast to previous studies [18, 19, 22] $P(v)$ is not Gaussian but strongly peaked around small couplings.

In the spirit of the semiclassical eigenfunction hypothesis [32–34] we partition the level-spacing distribution into three distinct contributions

$$P(s) = p_{r-r}(s) + p_{c-c}(s) + p_{r-c}(s).$$

Here, $p_{r-r}(s)$ describes the fraction of $r$-$r$ spacings formed by two regular levels, $p_{c-c}(s)$ the fraction of $c$-$c$ spacings formed by two chaotic levels, and $p_{r-c}(s)$ the fraction of $r$-$c$ spacings formed by one regular and one chaotic level in the superposed spectrum [14, 18, 22].

We evaluate the three contributions by making the following assumptions: (i) The spacings of the chaotic subspectrum of $H$ can be approximated by the Wigner distribution $P_{\text{c}}(s) = \pi \delta_{\rho_{c}^2}/2e^{-\pi(s-\rho_{c})^2/4}$ with mean spacing $1/\rho_c$ [4–8]; (ii) consecutive regular levels are separated on scales larger than the mean level spacing. This is generically the case if there are less regular than chaotic states ($\rho_r < \rho_c$) and $h_{\text{eff}}$ is not much smaller than the regular region $\rho_r$ ($h_{\text{eff}} \lesssim \rho_r$). Then one has just few regular levels, $N_r \approx \rho_r/h_{\text{eff}}$, which are semiclassically determined by the torus structure of the regular island. The interval between such consecutive regular levels then typically contains chaotic levels.

The contribution of zeroth order $r$-$c$ spacings $\tilde{s} = |\varepsilon_r - \varepsilon_c|$ to the level-spacing distribution, neglecting couplings between regular and chaotic states, is given by

$$p_{r-c}^{(0)}(\tilde{s}) = 2\rho_r \rho_c \exp \left( -\pi(\tilde{s}\rho_c)^2/4 \right).$$

Here $\rho_c$ is the probability to have a chaotic level $\varepsilon_c$ in the distance $\tilde{s}$ from the regular level $\varepsilon_r$, $\int_{-\infty}^{\infty} P_{\text{c}}(s) ds = \rho_c$. 

The probability density $P(v)$ of all couplings is given by

$$P(v) = \frac{1}{N_r} \sum_{m=0}^{N_c-1} \frac{1}{\sqrt{2\pi v_m}} e^{-v^2/v_m^2}.$$
\[
\exp \left(-\pi (\delta \rho) \right)^2/4 \right) \text{ is the probability to have no further chaotic level between } \varepsilon_r \text{ and } \varepsilon_c, \text{ and } 2\rho, \text{ is the probability of a zeroth order r-c spacing to contribute to } P(s) \text{ [14].}
\]

Dynamical tunneling leads to enlarged r-c spacings, which can be modeled by the \(2 \times 2\) submatrices of \(H\)
\[
\begin{pmatrix}
\varepsilon_r & v \\
v & \varepsilon_c
\end{pmatrix}.
\]

This relies on degenerate perturbation theory \([18, 19]\) and is applicable because typically both r-r and c-c spacings are large compared to the couplings \((v_m \ll 1/p_{r,c})\). From Eq. (6) we calculate the tunneling improved r-c spacings
\[
s = \sqrt{s^2 + 4v^2} \text{ such that }
p_{r-c}(s) = \int_0^\infty ds \bar{P}(s) (s) \delta \left(s - \sqrt{s^2 + 4v^2}\right).
\]

This expression reflects that r-c spacings result from all possible zeroth order r-c spacings \(\bar{s}\) and all possible couplings \(v\), which are described by \(\bar{P}(s)\).

In order to compute the integrals in Eq. 7 we introduce polar coordinates \((s', \varphi)\) with \(s' = s \cos \varphi\) and \(2v = s \sin \varphi\) \([13]\), such that
\[
p_{r-c}(s) = \int_0^{\pi/2} d\varphi \int_0^\infty \bar{P}(s \cos \varphi) P_{r-c}(s \sin \varphi).
\]

Calculating the remaining integral gives
\[
p_{r-c}(s) = \frac{p_{r-c}(0)}{N} \sum_{m=0}^{N-1} \bar{v}_m X \left(\frac{s}{2v_m}\right)
\]
with \(X(x) = \sqrt{\pi/2 e^{-x^2/4} I_0(x^2/4)}\), where \(I_0\) is the zeroth order modified Bessel function of the first kind and
\[
\bar{v}_m = v_m / \left(1 - 2\rho^2 v_m^2\right)^{1/2}.
\]

The contribution of r-r spacings to the level-spacing distribution is insignificant, due to assumption (ii)
\[
p_{r-r}(s) = 0.
\]

The contribution of c-c spacings to \(P(s)\) is
\[
p_{c-c}(s) = P_c(s) \left[1 - p_r s\right],
\]
where the first factor is the probability of finding a c-c spacing of size \(s\) in the chaotic subspectrum of \(H\). The second factor describes the probability of having no regular level within this c-c spacing, which is valid in the regime where \(s\) is smaller than all r-r spacings.

Combining Eqs. (8) and (10) in Eq. 11 gives our prediction of the level-spacing distribution in the presence of dynamical tunneling. Figures 1 and 2 show that this result is in excellent agreement with spectral data of our example systems for \(s \lesssim 1\). The small deviations for \(s \gtrsim 1\) can be attributed to approximation (i).

Now we derive the fractional power-law of \(P(s)\), Eq. 11, in the tunneling regime. Here \(s\) is in between the smallest typical coupling \(v_0\) and the largest typical coupling \(v_{N-1}\). In this regime r-r spacings do not contribute [see Eq. (9)]. Furthermore, the repulsion between chaotic levels [see Eq. (10)] ensures that c-c spacings only have significant probability for larger spacings and are insignificant deep in the tunneling regime \((s \ll 1)\). Here, Eq. 5 reduces to
\[
p_{r-c}(s) \approx 2\rho \rho_c v_m X \left(\frac{s}{2v_m}\right).
\]

This explicitly shows that the fractional power-law arises from typical couplings which range over many orders of magnitude.

We now evaluate the scaling of the exponent \(\beta\) under variations of \(h_{\text{eff}}\). With \(N_r = \rho_r/h_{\text{eff}}, \rho_{N_r-1} = 1\), and the rough estimate \(v_0 \sim \exp(-C\rho_r/h_{\text{eff}})\) \([28, 39]\), we get
\[
\beta \propto h_{\text{eff}}.
\]
This type of scaling behavior, phenomenologically derived in Ref. 16, is confirmed in Fig. 2 \((h_{\text{eff}} = 1/80, 1/120, \text{and } 1/160)\), with the almost constant ratio \(\beta/h_{\text{eff}} = 26, 24, \text{and } 21\) for the three cases of \(h_{\text{eff}}\). This result demonstrates the inapplicability of the Brody distribution [23, 24] for mixed systems, as it fails to simultaneously describe the \(h_{\text{eff}}\)-dependent power-law exponent for small spacings and the fixed Berry-Robnik-type distribution of large spacings beyond the tunneling regime.

Let us conclude by considering the level-spacing distribution in the semiclassical limit \(h_{\text{eff}} \to 0\). In this limit small r-r spacings appear such that assumption (ii) is violated. A generalized derivation shows that the r-c contribution to \(P(s)\) then still follows a power-law with exponent \(\beta \to 0\), as in Eq. 13. At the same time the r-r contribution approaches Poisson statistics. In combination one thus recovers Berry-Robnik statistics in the semiclassical limit.

To summarize, we have demonstrated how the wide range of dynamical tunneling rates universally leads to a power-law distribution of \(P(s)\) at small spacings. We expect that this is also the fundamental mechanism which explains spacing-statistics in mixed systems with more complicated phase-space structures.

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[31] We consider the standard map on the torus with unit length, using \(T(p) = p^2/2\) and \(V(q) = \kappa \cos (2\pi q)/(2\pi)^2\) for \(q \in [\tilde{q}, 1 - \tilde{q}]\). For \(q \in [0, \tilde{q}]\) and \(q \in [1 - \tilde{q}, 1]\) we use the modified potential \(V(q) = c(q^2 - \tilde{q}^2)/2 + \kappa \cos (2\pi \tilde{q}q)/(2\pi)^2\) and \(V(q) = c((q - 1)^2 - (\tilde{q} - 1)^2)/2 + \kappa \cos (2\pi \tilde{q}q)/(2\pi)^2\), respectively. We choose \(\kappa = 3.0 \text{ and } \tilde{q} = 0.275\) such that the regular region (of size \(r_\ell \approx 0.12\)) of the standard map is preserved. Taking \(10^6\) equidistant values \(c \in [10, 1000]\) enables an ensemble average.
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[37] Similar to Ref. [30] we design a kicked system by defining the functions \(t'(p)\) and \(v'(q)\) with \(t'(p) = -p\) for \(p \in [-1/4, 1/4]\) and \(t'(p) = p\) for \(p \in (1/4, 1/2)\) as well as \(v'(q) = c(2q + 1)\) for \((q \in [-1/2, -1/4]\), \(v'(q) = -2v + 4Rq^2\) for \(q \in (-1/4, 1/4)\), and \(v'(q) = (c + 3R/2)(2q - 1)\) for \(q \in [1/4, 1/2]\). Smoothing the periodically extended functions with a Gaussian, \(G(z) = \exp (-z^2/2\sigma^2) / \sqrt{2\pi}\sigma^2\), yields analytic functions \(T'(p) = \int dz G(z)t'(p + z)\) and \(V'(q) = \int dz G(z)v'(q + z)\). We choose \(r = 0.26\), \(R = 0.4\), and \(\varepsilon = 0.001\), leading to \(r_\ell \approx 0.07\). Taking \(10^6\) equidistant values \(c \in [5, 1000]\) enables an ensemble average.
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