MOTION OF GRAIN BOUNDARIES WITH DYNAMIC LATTICE MISORIENTATIONS AND WITH TRIPLE JUNCTIONS DRAG

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Abstract. Most technologically useful materials are polycrystalline microstructures composed of a myriad of small monocrystalline grains separated by grain boundaries. The energetics and connectivities of grain boundaries play a crucial role in defining the main characteristics of materials across a wide range of scales. In this work, we propose a model for the evolution of the grain boundary network with dynamic boundary conditions at the triple junctions, triple junctions drag, and with dynamic lattice misorientations. Using the energetic variational approach, we derive system of geometric differential equations to describe motion of such grain boundaries. Next, we relax curvature effect of the grain boundaries to isolate the effect of the dynamics of lattice misorientations and triple junctions drag, and we establish local well-posedness result for the considered model.

1. Introduction

Most technologically useful materials are polycrystalline microstructures composed of a myriad of small monocrystalline grains separated by grain boundaries. The energetics and connectivities of grain boundaries play a crucial role in defining the main characteristics of materials across a wide range of scales. More recent mesoscale experiments and simulations provide large amounts of information about both geometric features and crystallography of the grain boundary network in material microstructures.

For the time being, we will focus on a planar grain boundary network. A classical model, due to Mullins and Herring [17, 27, 28], for the evolution of grain boundaries in polycrystalline materials is based on the motion by mean curvature as the local evolution law. Under the assumption that the total grain boundary energy depends only on the surface tension of the grain boundaries, the motion by mean curvature is consistent with the dissipation principle for the total grain boundary energy. In addition, to have a well-posed model of the evolution of the grain boundary network, one has to impose a separate condition at the triple junctions where three grain boundaries meet [19]. Note, that at equilibrium state, the energy is minimized, which implies that a force balance, known as the Herring Condition, holds at the triple junctions. Herring condition is the natural boundary condition for the system at the equilibrium. However, during the evolution of the grain boundaries, the normal velocity of the boundary is proportional to a driving force. Therefore, unlike the equilibrium state, there is no natural boundary condition for an evolutionary system, and one must be stated. A standard choice is the Herring condition [8, 9, 19, 18], and reference therein. There are several mathematical studies about the motion by mean curvature of grain boundaries with the Herring condition at the triple junctions, see for example [19, 22, 23, 24, 25, 26, 8, 4, 5, 2, 21, 6, 1]. There are some computational studies too [30, 31, 14, 13, 12, 2].

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A basic assumption in the theory and simulations of the grain growth is the motion of the grain boundaries themselves and not the motion of the triple junctions. However, recent experimental studies indicate that the motion of triple junctions together with anisotropy of the grain boundary network can have an important effect on the grain growth [6]. In this work, to investigate the evolution of the anisotropic network of grain boundaries, we propose a new model that assumes that interfacial/grain boundary energy density is a function of dynamic lattice misorientations. Moreover, we impose a dynamic boundary condition at the triple junctions, a triple junctions drag. The proposed model can be viewed as a multiscale model containing the local and long-range interactions of the lattice misorientations and the interactions of the triple junctions of the grain boundaries. Using the energetic variational approach, we derive the system of geometric differential equations to describe the motion of such grain boundaries. Next, we relax the curvature effect of the grain boundaries to isolate the effect of the dynamics of lattice misorientations and triple junctions drag, and we establish local well-posedness result for the considered model.

The paper is organized as follows. In Section 2, we derive a new model for the grain boundaries. In Sections 3-6, we show local well-posedness of the proposed model under assumption of a single triple junction. Finally, in Section 7, we extend the obtained results for a system with a single triple junction to the grain boundary network with multiple junctions.

2. Derivation of the model

First, we obtain our model for the evolution of the grain boundaries using energy dissipation principle for the system. Note, while critical events (such as, disappearance of the grains and/or grain boundaries during coarsening of the system) pose a great challenge on the modeling, simulation and analysis, see Fig. 1, here we start with a system of one triple junction to obtain a consistent model, see Fig. 2. Thus, we start the derivation by considering the system of three curves, that meet at a single point – a triple junction \( a(t) \), see Fig. 2:

\[
\Gamma_t^{(j)} : \xi^{(j)}(s, t), \quad 0 \leq s \leq 1, \quad t > 0, \quad j = 1, 2, 3.
\]

These curves satisfy the following conditions at the triple junction and at the end points of the curves,

\[
a(t) := \xi^{(1)}(0, t) = \xi^{(2)}(0, t) = \xi^{(3)}(0, t), \quad \text{and} \quad \xi^{(j)}(1, t) = x^{(j)}, \quad j = 1, 2, 3.
\]

Here, we assume that curves \( \Gamma_t^{(j)}, j = 1, 2, 3 \) are sufficiently smooth functions of parameter \( s \) (not necessarily the arc length) and time \( t \). Also, for now we assume that endpoints of the curves \( x^{(j)} \in \mathbb{R}^2 \) are fixed points, see Fig. 2. We define a tangent vector \( b^{(j)} = \xi_s^{(j)} \) and a normal vector \( n^{(j)} = Rb^{(j)} \) (not necessarily the unit vectors) to each curve, where \( R \) is the rotation matrix through \( \pi/2 \). We denote \( \Gamma_t := \Gamma_t^{(1)} \cup \Gamma_t^{(2)} \cup \Gamma_t^{(3)} \). We also consider below a standard euclidean vector norm denoted \(| \cdot |\).

Now, for \( j = 1, 2, 3 \), let \( \alpha^{(j)} = \alpha^{(j)}(t) \) be the lattice orientation of the grain which is enclosed between grain boundaries \( \Gamma_t^{(j)} \) and \( \Gamma_t^{(j+1)} \), and we set that \( \Gamma_t^{(4)} = \Gamma_t^{(1)} \) for the simplicity of the notation. In this work, we make an assumption that lattice orientations are functions of time \( t \) (we assume that during grain growth, grains can change their lattice orientations due to rotation), but independent of the parameter \( s \). Next, we define, the surface energy density or interfacial grain boundary energy of \( \Gamma_t^{(j)} \) as

\[
\sigma = \sigma(n^{(j)}, \alpha^{(j)} - \alpha^{(j-1)}) = \sigma(n^{(j)}, \Delta \alpha^{(j)}) \geq 0,
\]
where we denote $\Delta \alpha^{(j)} := \alpha^{(j-1)} - \alpha^{(j)}$ to be misorientation angle across the grain boundary (a common boundary for two neighboring grains with orientations $\alpha^{(j-1)}$ and $\alpha^{(j)}$), and we set for convenience $\alpha^{(0)} := \alpha^{(3)}$, see Fig. 2. Therefore, the total grain boundary energy of the system $\Gamma_t$ can be obtained as

$$ (2.1) \quad E(t) = \sum_{j=1}^{3} \int_{\Gamma_t} \sigma(\mathbf{n}^{(j)}, \Delta \alpha^{(j)}) \, d\mathcal{H}^1 = \sum_{j=1}^{3} \int_{0}^{1} \sigma(\mathbf{n}^{(j)}, \Delta \alpha^{(j)}) |b^{(j)}| \, ds, $$

where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure, (see Fig. 2). Next, assume that $\sigma$ is taken to be positively homogeneous of degree 0 in $\mathbf{n}^{(j)}$. For simplicity of notations, we denote $\sigma^{(j)} := \sigma(\mathbf{n}^{(j)}, \Delta \alpha^{(j)})$.

Let us now define grain boundary motion that will result in the dissipation of the total grain boundary energy (2.1). Denote by $\hat{\mathbf{n}}^{(j)}$ the normalization operator of vectors, e.g. $\hat{\mathbf{n}}^{(j)} = \frac{\mathbf{n}^{(j)}}{|\mathbf{n}^{(j)}|}$. Then, we can compute the rate of change in energy at time $t$ due to grain boundary motion as
follows:

\[
\frac{d}{dt} E(t) = \sum_{j=1}^{3} \left( \int_0^1 \nabla_n \sigma(j) \cdot \frac{d\mathbf{n}}{dt} |\mathbf{b}(j)| \, ds + \int_0^1 \sigma(j) \frac{\mathbf{b}(j)}{|\mathbf{b}(j)|} \cdot \frac{d\mathbf{b}(j)}{dt} \, ds \right.
\]

\[
+ \int_0^1 \sigma(j) \frac{d(\Delta \alpha(j))}{dt} |\mathbf{b}(j)| \, ds \bigg) \right)
\]

(2.2)

\[
= \sum_{j=1}^{3} \left( \int_0^1 \left( |\mathbf{b}(j)| R \nabla_n \sigma(j) + \sigma(j) \hat{\mathbf{n}}(j) \right) \cdot \frac{d\mathbf{b}(j)}{dt} \, ds \right.
\]

\[
+ \int_0^1 \sigma(j) \frac{d(\Delta \alpha(j))}{dt} |\mathbf{b}(j)| \, ds \bigg) \right).
\]

Next, consider a polar angle \( \theta(j) \) and set \( \hat{\mathbf{n}}(j) = (\cos \theta(j), \sin \theta(j)) \). Since \( \sigma(j) \) is positively homogeneous of degree 0 in \( \mathbf{n}(j) \), we have

\[
\nabla_n \sigma \cdot \mathbf{n} = 0, \quad R \nabla_n \sigma = (\nabla_n \sigma \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}, \quad \sigma(j) \hat{\mathbf{n}}(j) = |\mathbf{b}(j)| R \nabla_n \sigma(j),
\]

and, thus, we define the vector \( \mathbf{T}(j) \) known as the line tension or capillary stress vector,

\[
\mathbf{T}(j) := \sigma(j) \hat{\mathbf{n}}(j) + \sigma(j) \hat{\mathbf{b}}(j) = |\mathbf{b}(j)| R \nabla_n \sigma(j) + \sigma(j) \hat{\mathbf{b}}(j).
\]

Now, using the change of variable

\[
\frac{d\mathbf{b}(j)}{dt} = \frac{d}{ds} \frac{\mathbf{b}(j)}{ds} \cdot \frac{d\xi(j)}{dt},
\]

we can rewrite (2.2) as:

\[
\frac{d}{dt} E(t) = \sum_{j=1}^{3} \left( \int_0^1 \mathbf{T}(j) \cdot \frac{d\xi(j)}{ds} \, ds + \int_0^1 \sigma(j) \frac{d(\Delta \alpha(j))}{dt} |\mathbf{b}(j)| \, ds \right)
\]

(2.3)

\[
= -\sum_{j=1}^{3} \int_0^1 \mathbf{T}_s(j) \cdot \frac{d\xi(j)}{ds} \, ds + \sum_{j=1}^{3} \int_0^1 \sigma(j) \frac{d(\Delta \alpha(j))}{dt} |\mathbf{b}(j)| \, ds
\]

\[
- \sum_{j=1}^{3} \mathbf{T}(j)(0, t) \cdot \frac{d\mathbf{a}}{dt}(t).
\]

For the reader’s convenience, we will recall below the following property for a divergence of the capillary stress vector \( \mathbf{T}(j) \).

**Lemma 2.1.** Let \( \kappa(j) \) is the curvature of \( \Gamma_i(j) \). Then

\[
\mathbf{T}_s(j) = |\mathbf{b}(j)| (\sigma_{\theta \theta}(j) + \sigma(j) \kappa(j)) \hat{\mathbf{n}}(j).
\]

**Proof.** From the Frenet-Serret formula for the non-arc length parameter,

\[
\hat{\mathbf{b}}_s(j) = |\mathbf{b}(j)| \kappa(j) \hat{\mathbf{n}}(j), \quad \hat{\mathbf{n}}_s(j) = -|\mathbf{b}(j)| \kappa(j) \hat{\mathbf{b}}(j).
\]
Thus we obtain,

\[
T_s^{(j)} = \left( \nabla_n \sigma^{(j)} \cdot n_s^{(j)} \right) \hat{n}^{(j)} + \sigma^{(j)} \hat{b}_s^{(j)} + \left( \nabla_n \sigma^{(j)} \cdot n_s^{(j)} \right) \hat{b}_s^{(j)} + \sigma^{(j)} \hat{b}_s^{(j)}
\]

\[
= \left( R \nabla_n \sigma^{(j)} \cdot b_s^{(j)} + |b^{(j)}| \sigma^{(j)} \kappa^{(j)} \right) \hat{n}^{(j)} + \left( -|b^{(j)}| \sigma^{(j)} \kappa^{(j)} + R \nabla_n \sigma^{(j)} \cdot b_s^{(j)} \right) \hat{b}_s^{(j)}.
\]

(2.6)

Since \(\sigma^{(j)}\) and \(\sigma^{(j)}_{\theta}\) are positively homogeneous of degree 0 in \(n^{(j)}\), we have,

\[
\sigma^{(j)} \hat{n}^{(j)} = |b^{(j)}| R \nabla_n \sigma^{(j)}, \quad \sigma^{(j)}_{\theta} \hat{n}^{(j)} = |b^{(j)}| R \nabla_n \sigma^{(j)}_{\theta}.
\]

(2.7)

Using the orthogonal relation \(b^{(j)} \cdot \hat{n}^{(j)} = 0\) and the Frenet-Serret formula (2.5), we obtain,

\[
b_s^{(j)} \cdot \hat{n}^{(j)} = -b^{(j)} \cdot \hat{n}^{(j)} = |b^{(j)}|^2 \kappa^{(j)}.
\]

(2.8)

Plugging (2.7) and (2.8) into (2.6), we derive (2.4).

\[\square\]

Next, to ensure that the entire system of grain boundaries is dissipative, i.e.

\[
\frac{d}{dt} E(t) \leq 0,
\]

we impose Mullins theory (curvature driven growth) \[28, 29\] as the local evolution law stating that the normal velocity \(v_n^{(j)}\) of a grain boundary of \(\Gamma_i^{(j)}\) (the rate of growth of area adjacent to the boundary \(\Gamma_i^{(j)}\)), is proportional to the line force \(T_s^{(j)}\) (to the work done through deforming the curve), through the factor of the mobility \(\mu^{(j)} > 0\) :

\[
v_n^{(j)} \hat{n}^{(j)} = \mu^{(j)} \frac{1}{|b^{(j)}|} T_s^{(j)} = \mu^{(j)} (\sigma^{(j)}_{\theta} + \sigma^{(j)} \kappa^{(j)}) \hat{n}^{(j)} \quad \text{on} \quad \Gamma_i^{(j)}, \quad j = 1, 2, 3.
\]

(2.9)

Note, that using variation of the energy \(E\) with respect to the curve \(\xi^{(j)}\), namely,

\[
v_n^{(j)} \hat{n} = -\mu^{(j)} \frac{\delta E}{\delta \xi^{(j)}}.
\]

one can derive the following relation for the line force \(T_s^{(j)}\) \[19\],

\[
\mu^{(j)} \frac{1}{|b^{(j)}|} T_s^{(j)} = \mu^{(j)} (\sigma^{(j)}_{\theta} + \sigma^{(j)} \kappa^{(j)}) \hat{n}^{(j)} \quad \text{on} \quad \Gamma_i^{(j)}, \quad j = 1, 2, 3.
\]

(2.10)

Since \(v_n^{(j)} = \frac{d\xi^{(j)}}{dt} \cdot \hat{n}^{(j)}\), we obtain that,

\[
T_s^{(j)} \cdot \frac{d\xi^{(j)}}{dt} = \mu^{(j)} |v_n^{(j)}|^2 |b^{(j)}| \geq 0,
\]

(2.11)

and, thus, the first term on the right-hand side of (2.3) is non-positive. Next, we consider the second term on the right-hand side of (2.3) which depends on the derivative of lattice misorientation, we have that (since \(\alpha^{(j)}\) is independent of \(s\)),

\[
\sum_{j=1}^{3} \int_{0}^{1} \sigma^{(j)}_{\alpha} \frac{d(\Delta \alpha^{(j)})}{dt} |b^{(j)}| ds = \sum_{j=1}^{3} \left( \int_{0}^{1} \left( \sigma^{(j+1)}_{\alpha} |b^{(j+1)}| - \sigma^{(j)}_{\alpha} |b^{(j)}| \right) ds \right) \frac{d\alpha^{(j)}}{dt},
\]

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where we used that $\sigma^{(4)} = \sigma^{(1)}$. To ensure, $\frac{d}{dt}E(t) \leq 0$ in (2.3), we make an assumption that for a constant $\gamma > 0$, we have the following relation for the rate of change of the lattice orientations,

\begin{equation}
\frac{d\alpha^{(j)}}{dt} = -\gamma \left( \int_0^1 \left( \sigma_0^{(j)}(\sigma^{(j+1)} - \sigma_0^{(j)}) \right) ds \right), \quad j = 1, 2, 3
\end{equation}

since the relation (2.12) results in the condition,

\begin{equation}
\sum_{j=1}^3 \int_0^1 \sigma_0^{(j)} d(\Delta\alpha^{(j)}) |b^{(j)}| ds = -\frac{1}{\gamma} \sum_{j=1}^3 \left( \frac{d\alpha^{(j)}}{dt} \right)^2 \leq 0
\end{equation}

on the second term in the right-hand side of (2.3). Note, that the proposed relation (2.12) can also be derived using variation of the energy $E$ with respect to lattice orientation $\alpha^{(j)}$, namely,

\begin{equation}
\frac{d\alpha^{(j)}}{dt} = -\gamma \frac{\delta E}{\delta \alpha^{(j)}}.
\end{equation}

Finally, as a part of $\frac{d}{dt}E(t) \leq 0$ condition in (2.3), we also assume the dynamic boundary conditions for the triple junctions, namely, for a constant $\eta > 0$,

\begin{equation}
\frac{da^{(j)}}{dt}(t) = \eta \sum_{j=1}^3 T^{(j)}(0, t), \quad t > 0.
\end{equation}

This assumption implies that the last term in (2.3) satisfies,

\begin{equation}
- \sum_{j=1}^3 T^{(j)}(0, t) \cdot \frac{da^{(j)}}{dt}(t) = -\frac{1}{\eta} \left| \frac{da^{(j)}}{dt}(t) \right|^2 \leq 0.
\end{equation}

Therefore, we obtain from (2.11), (2.13), and (2.15), that the entire system of grain boundaries $\Gamma^{(j)}$ is dissipative, namely,

\begin{equation}
\frac{d}{dt}E(t) = -\sum_{j=1}^3 \int_{\Gamma^{(j)}} \frac{1}{\mu^{(j)}} |v_n^{(j)}|^2 d\mathcal{H}^1 - \frac{1}{\eta} \left| \frac{da^{(j)}}{dt}(t) \right|^2 - \frac{1}{\gamma} \sum_{j=1}^3 \left( \frac{d\alpha^{(j)}}{dt} \right)^2 \leq 0.
\end{equation}

Finally, we combine assumptions (2.9), (2.12), and (2.14) to obtain the following system of geometric evolution differential equations to describe motion of grain boundaries $\Gamma^{(j)}, j = 1, 2, 3$ together with a motion of the triple junction $a(t)$:

\begin{equation}
\begin{cases}
v_n^{(j)} = \mu^{(j)} (\sigma_{\theta}^{(j)} + \sigma^{(j)}), & \text{on } \Gamma^{(j)}, \quad t > 0, \quad j = 1, 2, 3, \\
\frac{d\alpha^{(j)}}{dt} = -\gamma \left( \int_0^1 \left( \sigma_0^{(j+1)} |b^{(j+1)}| - \sigma_0^{(j)} |b^{(j)}| \right) ds \right), \quad j = 1, 2, 3, \\
\frac{da^{(j)}}{dt}(t) = \eta \sum_{j=1}^3 T^{(k)}(0, t) = \eta \sum_{j=1}^3 (\sigma_{\theta}^{(k)} \hat{n}^{(k)} + \sigma^{(k)} \hat{b}^{(k)})(0), \quad t > 0, \\
\Gamma^{(j)} : \xi^{(j)}(s, t), \quad 0 \leq s \leq 1, \quad t > 0, \quad j = 1, 2, 3, \\
a(t) = \xi^{(1)}(0, t) = \xi^{(2)}(0, t) = \xi^{(3)}(0, t), \quad \text{and} \quad \xi^{(j)}(1, t) = x^{(j)}, \quad j = 1, 2, 3.
\end{cases}
\end{equation}

Remark 2.2. The entire system (2.17) satisfies energy dissipation principle (2.16). However, it is important to note, that there are three independent relaxation time scales in the system (2.17), namely, $\mu^{(j)}, \gamma$ and $\eta$ (length, misorientation and position of the triple junction). Classical approach is to let $\gamma \to \infty$ and $\eta \to \infty$. 

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In this work, we let $\mu^{(j)} \to \infty$, and set $\gamma = \eta = 1$ to study the effect of the dynamics of lattice orientations $\alpha^{(j)}(t)$, $j = 1, 2, 3$ together with the effect of the dynamics of a triple junction $a(t)$ on a grain boundary motion. Then, in this limit, $\Gamma^{(j)}$ becomes a line segment from the triple junction $a(t)$ to the boundary point $x^{(j)}$. Hence, we have

\[
\begin{cases}
\xi^{(j)}(s, t) = a(t) + s b^{(j)}(t), & 0 \leq s \leq 1, \quad t > 0, \quad j = 1, 2, 3, \\
a(t) + b^{(j)}(t) = x^{(j)}, & j = 1, 2, 3.
\end{cases}
\]

Further, for simplicity of the calculations (we anticipate that similar results will hold true for the surface energy given by the convex function), we set the surface energy to be a quadratic function of lattice misorientation,

\begin{equation}
\sigma(n^{(j)}, \Delta \alpha^{(j)}) = 1 + \frac{1}{2} (\Delta \alpha^{(j)})^2 = 1 + \frac{1}{2} (\alpha^{(j-1)} - \alpha^{(j)})^2.
\end{equation}

Then, it follows that $\alpha_a^{(j)} = \Delta \alpha^{(j)} = \alpha^{(j-1)} - \alpha^{(j)}$, and, hence, we deduce a simpler relation for the evolution of the lattice orientations,

\begin{equation}
\frac{d \alpha^{(j)}}{dt} = -(|b^{(j+1)}(t)| + |b^{(j)}(t)|) \alpha^{(j)} + |b^{(j+1)}(t)| \alpha^{(j+1)} + |b^{(j)}(t)| \alpha^{(j-1)}, \quad j = 1, 2, 3.
\end{equation}

Thus, the system of geometric evolution differential equations (2.17) becomes the following system of ordinary differential equations (ODE):

\begin{equation}
\begin{cases}
\frac{d \alpha^{(j)}}{dt} = -(|b^{(j+1)}(t)| + |b^{(j)}(t)|) \alpha^{(j)} + |b^{(j+1)}(t)| \alpha^{(j+1)} + |b^{(j)}(t)| \alpha^{(j-1)}, & j = 1, 2, 3, \\
\frac{d a}{dt}(t) = \sum_{j=1}^{3} \left( 1 + \frac{1}{2} (\alpha^{(j-1)}(t) - \alpha^{(j)}(t))^2 \right) \frac{b^{(j)}}{|b^{(j)}|}, & t > 0, \\
a(t) + b^{(j)}(t) = x^{(j)}, & j = 1, 2, 3.
\end{cases}
\end{equation}

Below, we continue with a study of the local well-posedness of the problem (2.20) with the initial data given by $a_0^{(1)}, a_0^{(2)}, a_0^{(3)}, a_0$.

### 3. Equilibrium

We start by rewriting the system (2.20) as

\begin{equation}
\begin{cases}
\frac{d \alpha}{dt} = -\mathcal{B}(t) \alpha, \quad t > 0, \quad \alpha(t) = \left( \alpha^{(1)}(t), \alpha^{(2)}(t), \alpha^{(3)}(t) \right)^T, \\
\mathcal{B}(t) = \begin{pmatrix}
|b^{(1)}(t)| + |b^{(2)}(t)| & -|b^{(2)}(t)| & -|b^{(1)}(t)| \\
-|b^{(2)}(t)| & |b^{(2)}(t)| + |b^{(3)}(t)| & -|b^{(3)}(t)| \\
-|b^{(1)}(t)| & -|b^{(3)}(t)| & |b^{(3)}(t)| + |b^{(1)}(t)|
\end{pmatrix}, \\
\frac{d a}{dt} = \sum_{j=1}^{3} \left( 1 + \frac{1}{2} (\alpha^{(j-1)}(t) - \alpha^{(j)}(t))^2 \right) \frac{b^{(j)}}{|b^{(j)}|}, & t > 0, \\
a(t) + b^{(j)}(t) = x^{(j)}, & t > 0, \quad j = 1, 2, 3,
\end{cases}
\end{equation}
and we study an associated equilibrium solution of the system (3.1), namely,

\[
0 = -\mathcal{B}_\infty \alpha_\infty, \quad \alpha_\infty = \left(\alpha^{(1)}_\infty, \alpha^{(2)}_\infty, \alpha^{(3)}_\infty\right),
\]

\[
\mathcal{B}_\infty = \begin{pmatrix}
|b^{(1)}_\infty| + |b^{(2)}_\infty| & -|b^{(2)}_\infty| & -|b^{(1)}_\infty| \\
-|b^{(2)}_\infty| & |b^{(2)}_\infty| + |b^{(3)}_\infty| & -|b^{(3)}_\infty| \\
-|b^{(1)}_\infty| & -|b^{(3)}_\infty| & |b^{(3)}_\infty| + |b^{(1)}_\infty|
\end{pmatrix},
\]

(3.2)

\[
a_\infty + b^{(j)}_\infty = x^{(j)}, \quad j = 1, 2, 3.
\]

To consider the equilibrium system (3.2), we define a matrix for, \(c_1, c_2, c_3 \in \mathbb{R},\)

\[
C := \begin{pmatrix}
(c_1 + c_2) & -c_2 & -c_1 \\
-c_2 & c_2 + c_3 & -c_3 \\
-c_1 & -c_3 & c_3 + c_1
\end{pmatrix}
\]

(3.3)

**Lemma 3.1.** The eigenvalues of the matrix \(C\) (3.3) are

\[
0 \text{ and } c_1 + c_2 + c_3 \pm \sqrt{\frac{1}{2} \left((c_1 - c_2)^2 + (c_2 - c_3)^2 + (c_3 - c_1)^2\right)}.
\]

If \(c_1, c_2, c_3 \geq 0\), then the matrix \(C\) is non-negative definite. Furthermore, if \(c_1, c_2, c_3 > 0\), then the zero eigenvalue of \(C\) is simple.

**Proof of Lemma 3.1.** Since

\[
\det(\lambda I - C) = \lambda^3 - 2(c_1 + c_2 + c_3)\lambda^2 + 3(c_1 c_2 + c_2 c_3 + c_3 c_1),
\]

hence, one can easily obtained the eigenvalues (3.4). The second and the third statements of Lemma 3.1 are obtained by noting that,

\[
\frac{1}{2} \left((c_1 - c_2)^2 + (c_2 - c_3)^2 + (c_3 - c_1)^2\right) = (c_1 + c_2 + c_3)^2 - 3(c_1 c_2 + c_2 c_3 + c_3 c_1).
\]

\[\square\]

**Lemma 3.2.** If \(c_1, c_2, c_3 > 0\), then the kernel of the matrix \(C\) (3.3) is spanned by a single vector \((1, 1, 1)\).

**Proof of Lemma 3.2.** We can easily check that \((1, 1, 1)\) is an element of the kernel of \(C\). By Lemma 3.1, the kernel of \(C\) has dimension one, and hence it is spanned by \((1, 1, 1)\). \[\square\]

From Lemma 3.2 it follows that if \(|b^{(j)}_\infty| \neq 0\) for all \(j = 1, 2, 3\), then \(a^{(1)}_\infty = a^{(2)}_\infty = a^{(3)}_\infty\) as the solution of the system (3.2). In other words, in the equilibrium, if triple junction exists, then there is no lattice misorientation between neighboring grains that have grain boundaries meeting at that triple junction. As a consequence, the equilibrium system (3.2) becomes

\[
0 = \sum_{j=1}^{3} \frac{b^{(j)}_\infty}{|b^{(j)}_\infty|},
\]

(3.5)

\[
a_\infty + b^{(j)}_\infty = x^{(j)}, \quad j = 1, 2, 3.
\]
The equation (3.5) is related to the Fermat-Torricelli problem. More precisely, if we have that, for each \( i = 1, 2, 3, \)

\[
(3.6) \quad \left| \sum_{j=1, i \neq j}^{3} \frac{x^{(j)} - x^{(i)}}{|x^{(j)} - x^{(i)}|} \right| > 1,
\]

then \( a_\infty \) is the unique minimizer of the function,

\[
(3.7) \quad f(a) = \sum_{j=1}^{3} |a - x^{(j)}|, \quad a \in \mathbb{R}^2,
\]

and \( a_\infty \neq x^{(j)} \) for \( j = 1, 2, 3 \) (See [7, Theorem 18.28]). Note, that the assumption (3.6) satisfies if and only if all three angles of the triangle, formed by vertices located at the nodes \( x^{(1)}, x^{(2)}, x^{(3)}, \) are less than 120°.

Finally, we state one more property that we will need to use in Section 4:

**Lemma 3.3.** For \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3, \)

\[
(3.8) \quad |C\alpha| \leq 3(|c_1| + |c_2| + |c_3|)|\alpha|
\]

**Proof of Lemma 3.3** Since

\[
\frac{1}{2}[(c_1 - c_2)^2 + (c_2 - c_3)^2 + (c_3 - c_1)^2] \leq 2(c_1^2 + c_2^2 + c_3^2),
\]

the inequality (3.8) is obtained from Lemma 3.1.

\qed

4. **Local existence**

Here, we discuss local existence which validates the consistency of the proposed model. Let \( x^{(j)} \in \mathbb{R}^2, \alpha_0 \in \mathbb{R}^3, \) and \( a_0 \in \mathbb{R}^2 \) be given initial data and we consider the local existence of the problem of (3.1), namely

\[
\begin{aligned}
\frac{d\alpha}{dt} &= -B(t)\alpha, \quad t > 0, \quad \alpha(t) = (\alpha^{(1)}(t), \alpha^{(2)}(t), \alpha^{(3)}(t))^T, \\
B(t) &= \left( \begin{array}{ccc}
|b^{(1)}(t)| + |b^{(2)}(t)| & -|b^{(2)}(t)| & -|b^{(1)}(t)| \\
-|b^{(2)}(t)| & |b^{(2)}(t)| + |b^{(3)}(t)| & -|b^{(3)}(t)| \\
-|b^{(1)}(t)| & -|b^{(3)}(t)| & |b^{(3)}(t)| + |b^{(1)}(t)|
\end{array} \right), \\
\frac{da}{dt} &= \sum_{j=1}^{3} \left( 1 + \frac{1}{2}(\alpha^{(j-1)}(t) - \alpha^{(j)}(t))^2 \right) \frac{b^{(j)}}{|b^{(j)}|}, \quad t > 0, \\
a(t) + b^{(j)}(t) &= x^{(j)}, \quad t > 0, \quad j = 1, 2, 3, \\
\alpha(0) &= \alpha_0, \quad a(0) = a_0.
\end{aligned}
\]

Assume for each \( i = 1, 2, 3, \)

\[
(4.2) \quad \left| \sum_{j=1, i \neq j}^{3} \frac{x^{(j)} - x^{(i)}}{|x^{(j)} - x^{(i)}|} \right| > 1.
\]
We denote by $a_\infty \neq x^{(j)}$ for each $j = 1, 2, 3$, a solution to the system,

\begin{equation}
0 = \sum_{j=1}^{3} \frac{b^{(j)}_\infty}{|b^{(j)}_\infty|},
\end{equation}

\begin{equation}
a_\infty + b^{(j)}_\infty = x^{(j)}, \quad j = 1, 2, 3.
\end{equation}

The point $a_\infty$ is a triple junction point (see Section 5).

**Theorem 4.1** (Local existence). Let $x^{(1)}, x^{(2)}, x^{(3)} \in \mathbb{R}^2$, $a_0 \in \mathbb{R}^2$, and $a_0 \in \mathbb{R}^3$ be given initial data. Assume condition (4.2) for $i = 1, 2, 3$, and let $a_\infty$ be a solution of (4.3). Further, assume that for all $j = 1, 2, 3$,

\begin{equation}
|a_0 - a_\infty| < \frac{1}{2} |b^{(j)}_\infty|.
\end{equation}

Then, there exists a local in time solution $(\alpha, \beta)$ of (4.1).

To show Theorem 4.1 we construct a contraction mapping on a complete metric space. Let $C_1, C_2 > 0$ and $T > 0$ be positive constants that we will define later, and denote,

\[ X_T := \{ (\alpha, \beta) \in C([0, T]; \mathbb{R}^3 \times \mathbb{R}^2), \| \alpha \|_{C([0, T])} \leq C_1, \| \beta - a_\infty \|_{C([0, T])} \leq C_2 \}. \]

Next, define for $(\alpha, \beta) \in X_T$ and $t > 0$

\[ \Phi(\alpha, \beta)(t) := \alpha_0 - \int_0^t \mathbb{B}(\tau)\alpha(\tau) \, d\tau, \]

\[ \Psi(\alpha, \beta)(t) := \alpha_0 + \sum_{j=1}^{3} \int_0^t \left( 1 + \frac{1}{2} (\alpha^{(j-1)}(\tau) - \alpha^{(j)}(\tau))^2 \right) \frac{b^{(j)}(\tau)}{|b^{(j)}(\tau)|} \, d\tau, \]

where $b^{(j)}(\tau) = x^{(j)} - \alpha(\tau)$. Our goal now is to show that $(\Phi, \Psi)$ is a contraction mapping on $X_T$ for the appropriate choice of positive constants $C_1, C_2$, and $T > 0$.

**Lemma 4.2.** If the conditions below are satisfied,

\begin{equation}
2|\alpha_0| \leq C_1
\end{equation}

and

\begin{equation}
3(|b^{(1)}_\infty| + |b^{(2)}_\infty| + |b^{(3)}_\infty| + 3C_2)T \leq \frac{1}{2},
\end{equation}

then $|\Phi(\alpha, \beta)| \leq C_1$ for all $(\alpha, \beta) \in X_T$.

**Proof of Lemma 4.2** By the triangle inequality, for $0 \leq t \leq T$

\[ |\Phi(\alpha, \beta)(t)| \leq |\alpha_0| + \left| \int_0^t \mathbb{B}(\tau)\alpha(\tau) \, d\tau \right| \leq |\alpha_0| + \sup_{0 \leq \tau \leq T} |\mathbb{B}(\tau)\alpha(\tau)|T. \]

From Lemma 3.3

\[ |\mathbb{B}(\tau)\alpha(\tau)| \leq 3C_1(|b^{(1)}(\tau)| + |b^{(2)}(\tau)| + |b^{(3)}(\tau)|). \]

On the other hand, for $j = 1, 2, 3$

\begin{equation}
|b^{(j)}(t)| = |x^{(j)} - a_\infty + a_\infty - \alpha(t)| \leq |b^{(j)}_\infty| + C_2
\end{equation}

hence

\[ |\mathbb{B}(\tau)\alpha(\tau)| \leq 3C_1(|b^{(1)}_\infty| + |b^{(2)}_\infty| + |b^{(3)}_\infty| + 3C_2). \]
Therefore, from (4.5) and (4.6)

$$|\Phi(\alpha, a)(t)| \leq |\alpha_0| + 3C_1(|b_{(1)}^0| + |b_{(2)}^0| + |b_{(3)}^0| + 3C_2)T \leq C_1.$$ 

\[\square\]

**Lemma 4.3.** Assume for \( j = 1, 2, 3 \) we have that,

\[(4.8)\] 

\[C_2 < |b_{(j)}^0|,\]

Then, \( 0 < |b_{(j)}^0| - C_2 \leq |b^{(j)}(t)| \leq |b_{(j)}^0| + C_2, \) for all \( j = 1, 2, 3, \) \((\alpha, a) \in X_T, \) and \( 0 \leq t \leq T. \) Further if

\[(4.9)\] 

\[2|a_0 - a_{\infty}| \leq C_2,

and

\[(4.10)\] 

\[3(1 + 2C_1^2)T \leq \frac{1}{2}C_2,

then \( |\Psi(\alpha, a)(t) - a_{\infty}| \leq C_2, \) for all \((\alpha, a) \in X_T \) and \( 0 \leq t \leq T. \)

**Proof of Lemma 4.3** For \((\alpha, a) \in X_T, \) and \( 0 \leq t \leq T \)

\[|b_{(j)}^0| = |x^{(j)} - a(t) + a(t) - a_{\infty}| \leq |b^{(j)}(t)| + |a(t) - a_{\infty}| \leq |b^{(j)}(t)| + C_2,

thus we obtain \( 0 < |b_{(j)}^0| - C_2 \leq |b^{(j)}(t)|. \) And \( |b^{(j)}(t)| \leq |b_{(j)}^0| + C_2 \) follows from (4.7). To show estimate \( |\Psi(\alpha, a)(t) - a_{\infty}| \leq C_2, \) we use the assumption (4.9) and (4.10), to obtain that for any \((\alpha, a) \in X_T, \)

\[
|\Psi(\alpha, a)(t) - a_{\infty}| \leq |a_0 - a_{\infty}| + \sum_{j=1}^{3} \left| \int_0^t \left( 1 + \frac{1}{2} \left( a^{(j-1)}(\tau) - a^{(j)}(\tau) \right)^2 \right) \frac{b^{(j)}(\tau)}{|b^{(j)}(\tau)|} d\tau \right|
\]

\[
\leq \frac{1}{2} C_2 + \sum_{j=1}^{3} \left( 1 + \frac{1}{2} \sup_{0 \leq \tau \leq T} \left( a^{(j-1)}(\tau) - a^{(j)}(\tau) \right)^2 \right) T
\]

\[
\leq \frac{1}{2} C_2 + 3 \left( 1 + 2C_1^2 \right) T \leq C_2,
\]

for all \( 0 \leq t \leq T. \)

\[\square\]

**Lemma 4.4.** For \((\alpha_1, a_1), (\alpha_2, a_2) \in X_T, \) we have that

\[(4.11)\]

\[\|\Phi(\alpha_1, a_1) - \Phi(\alpha_2, a_2)\|_{C([0, T])} \leq 9C_1T\|a_1 - a_2\|_{C([0, T])} + 3(|b_{(1)}^0| + |b_{(2)}^0| + |b_{(3)}^0| + 3C_2)T\|\alpha_1 - \alpha_2\|_{C([0, T])}.

**Proof of Lemma 4.4** For \( 0 \leq t \leq T, \) let us define

\[
B_k(t) := \begin{pmatrix}
|b_{(1)}^0(t)| + |b_{(2)}^0(t)| & -|b_{(2)}^0(t)| & |b_{(2)}^0(t)| - |b_{(3)}^0(t)| & -|b_{(3)}^0(t)| \\
-|b_{(2)}^0(t)| & |b_{(2)}^0(t)| + |b_{(3)}^0(t)| & -|b_{(3)}^0(t)| & -|b_{(3)}^0(t)| \\
-|b_{(1)}^0(t)| & -|b_{(3)}^0(t)| & |b_{(3)}^0(t)| + |b_{(1)}^0(t)| & -|b_{(1)}^0(t)|
\end{pmatrix}, \text{ with } k = 1, 2.
\]
Then, we obtain that
\[
|\Phi(\alpha_1, a_1)(t) - \Phi(\alpha_2, a_2)(t)|
\leq \left| \int_0^t (B_2(\tau)\alpha_2(\tau) - B_1(\tau)\alpha_1(\tau)) \, d\tau \right|
\leq \sup_{0 \leq \tau \leq T} |(B_2(\tau) - B_1(\tau))\alpha_2(\tau)| T + \sup_{0 \leq \tau \leq T} |B_1(\tau)((\alpha_1 - \alpha_2)(\tau))| T.
\]

Since \((\alpha_k, a_k) \in X_T, a_k(t) + b_k^{(j)}(t) = x^j\) for \(k = 1, 2,\) and \(j = 1, 2, 3,\) we have from Lemma 3.3 and Lemma 4.3 that,
\[
|B_2(\tau) - B_1(\tau)| \leq 3 \left( |b_1^{(1)}(t)| - |b_2^{(1)}(t)| + |b_1^{(2)}(t)| - |b_2^{(2)}(t)| + |b_1^{(3)}(t)| - |b_2^{(3)}(t)| \right) |\alpha_2(\tau)|
\leq 3C_1 \left( |b_1^{(1)}(t) - b_2^{(1)}(t)| + |b_1^{(2)}(t) - b_2^{(2)}(t)| + |b_1^{(3)}(t) - b_2^{(3)}(t)| \right)
\leq 9C_1 ||a_1 - a_2||_{C([0,T])},
\]
and
\[
|B_1(\tau)((\alpha_1 - \alpha_2)(\tau))| \leq 3(|b_1^{(1)}(t)| + |b_2^{(2)}(t)| + |b_1^{(3)}(t)| + 3C_2) ||\alpha_1 - \alpha_2||_{C([0,T])}.
\]

Thus, we obtain the inequality (4.11). \(\square\)

**Lemma 4.5.** Assume condition (4.8) holds true. Then for \((\alpha_1, a_1), (\alpha_2, a_2) \in X_T,\) we have that
\[
||\Psi(\alpha_1, a_1)(t)-\Psi(\alpha_2, a_2)(t)||_{C([0,T])}
\leq 12C_1 T ||\alpha_1 - \alpha_2||_{C([0,T])}
+ 2(1 + 2C_1^2) \left( \frac{1}{|b_1^{(1)}(t)| - C_2} + \frac{1}{|b_1^{(2)}(t)| - C_2} + \frac{1}{|b_1^{(3)}(t)| - C_2} \right) T ||a_1 - a_2||_{C([0,T])}.
\]

**Proof of Lemma 4.5** For \(k = 1, 2,\) denote \(\sigma_k^{(j)}(t) := 1 + \frac{1}{2}(\alpha_k^{(j-1)}(t) - \alpha_k^{(j)}(t))^2.\) For \(0 \leq t \leq T,\) we can obtain the following estimate
\[
|\Psi(\alpha_1, a_1)(t)-\Psi(\alpha_2, a_2)(t)| = \left| \sum_{j=1}^{3} \int_0^T \left( \sigma_1^{(j)}(\tau) \frac{b_1^{(j)}(\tau)}{|b_1^{(j)}(\tau)|} - \sigma_2^{(j)}(\tau) \frac{b_2^{(j)}(\tau)}{|b_2^{(j)}(\tau)|} \right) \, d\tau \right|
\leq \sum_{j=1}^{3} \int_0^T \left| \sigma_1^{(j)}(\tau) \frac{b_1^{(j)}(\tau)}{|b_1^{(j)}(\tau)|} - \sigma_2^{(j)}(\tau) \frac{b_2^{(j)}(\tau)}{|b_2^{(j)}(\tau)|} \right| \, d\tau
\leq \sum_{j=1}^{3} \int_0^T \left| \sigma_1^{(j)}(\tau) - \sigma_2^{(j)}(\tau) \right| \, d\tau
+ \sum_{j=1}^{3} \int_0^T \sigma_2^{(j)}(\tau) \left| \frac{b_1^{(j)}(\tau)}{|b_1^{(j)}(\tau)|} - \frac{b_2^{(j)}(\tau)}{|b_2^{(j)}(\tau)|} \right| \, d\tau.
\]
Hence, we derive that
\[
\sigma_1^{(j)}(\tau) - \sigma_2^{(j)}(\tau) \leq \frac{1}{2} (\alpha^{(j-1)}_1(\tau) - \alpha^{(j)}_1(\tau))^2 - (\alpha^{(j-1)}_2(\tau) - \alpha^{(j)}_2(\tau))^2
\]
\[
\leq \frac{1}{2} \left| \alpha^{(j-1)}_1(\tau) - \alpha^{(j)}_1(\tau) + \alpha^{(j-1)}_2(\tau) - \alpha^{(j)}_2(\tau) \right| \left| \alpha^{(j-1)}_1(\tau) - \alpha^{(j)}_1(\tau) - \alpha^{(j-1)}_2(\tau) + \alpha^{(j)}_2(\tau) \right|
\]
\[
\leq 2C_1(\alpha^{(j-1)}_1(\tau) - \alpha^{(j-1)}_2(\tau)) + |\alpha^{(j)}_1(\tau) - \alpha^{(j)}_2(\tau)|)
\]

Hence, we derive that
\[
\sum_{j=1}^{3} \int_{0}^{T} \left| \sigma_1^{(j)}(\tau) - \sigma_2^{(j)}(\tau) \right| \, d\tau \leq 4C_1 \int_{0}^{T} \sum_{j=1}^{3} \left| \alpha^{(j)}_1(\tau) - \alpha^{(j)}_2(\tau) \right| \, d\tau
\]
\[
\leq 12C_1 T \| \alpha_1 - \alpha_2 \|_{C([0,T])}
\]

Next, due to condition (4.8), we can apply Lemma 4.3. Therefore, we have that \( |b^{(j)}_k(\tau)| \neq 0 \) for \( j = 1, 2, 3, k = 1, 2, \) and \( 0 \leq \tau \leq T \). By direct calculations, we have that
\[
\left| \frac{b^{(j)}_1(\tau)}{|b^{(j)}_1(\tau)|} - \frac{b^{(j)}_2(\tau)}{|b^{(j)}_2(\tau)|} \right| = \frac{1}{|b^{(j)}_1(\tau)|} \left| b^{(j)}_1(\tau) - \frac{|b^{(j)}_1(\tau)|}{|b^{(j)}_2(\tau)|} b^{(j)}_2(\tau) \right|
\]
\[
\leq \frac{1}{|b^{(j)}_1(\tau)|} \left( |b^{(j)}_1(\tau) - b^{(j)}_2(\tau)| + \left| 1 - \frac{|b^{(j)}_1(\tau)|}{|b^{(j)}_2(\tau)|} \right| \right)
\]
\[
\leq \frac{2}{|b^{(j)}_1(\tau)|} \left| b^{(j)}_1(\tau) - b^{(j)}_2(\tau) \right|
\]

Again, using Lemma 4.3 and due to uniqueness of the point \( a_\infty \) (see Section 3), we have that \( 0 < |b^{(j)}_\infty| - C_2 \leq |b^{(j)}_1(\tau)| \) for \( j = 1, 2, 3, \) and \( 0 \leq \tau \leq T \). Thus, we derive that
\[
\left| \frac{b^{(j)}_1(\tau)}{|b^{(j)}_1(\tau)|} - \frac{b^{(j)}_2(\tau)}{|b^{(j)}_2(\tau)|} \right| \leq \frac{2}{|b^{(j)}_\infty| - C_2} \| a_1 - a_2 \|_{C([0,T])}
\]
and,
\[
\sum_{j=1}^{3} \int_{0}^{T} \sigma_2^{(j)}(\tau) \left| \frac{b^{(j)}_1(\tau)}{|b^{(j)}_1(\tau)|} - \frac{b^{(j)}_2(\tau)}{|b^{(j)}_2(\tau)|} \right| \, d\tau
\]
\[
\leq \sum_{j=1}^{3} \int_{0}^{T} \frac{2}{|b^{(j)}_\infty| - C_2} \| a_1 - a_2 \|_{C([0,T])} \, d\tau
\]
\[
\leq 2(1 + 2C^2_1) \left( \frac{1}{|b^{(1)}_\infty| - C_2} + \frac{1}{|b^{(2)}_\infty| - C_2} + \frac{1}{|b^{(3)}_\infty| - C_2} \right) T \| a_1 - a_2 \|_{C([0,T])} \, d\tau.
\]
Hence, we obtain the desired estimate,
\[
|\Psi(\alpha_1, a_1)(t) - \Psi(\alpha_2, a_2)(t)| \\
\leq 12C_1T\|\alpha_1 - \alpha_2\|_{C([0,T])} \\
+ 2(1 + 2C_1^2) \left( \frac{1}{|b_0^{(1)}| - C_2} + \frac{1}{|b_0^{(2)}| - C_2} + \frac{1}{|b_0^{(3)}| - C_2} \right) T\|\alpha_1 - \alpha_2\|_{C([0,T])}.
\]

\[\square\]

**Proof of Theorem 4.1.** Let us define constants \(C_1\) and \(C_2\) as, \(C_1 := 2|\alpha_0|\) and \(C_2 := 2|\alpha_0 - a_\infty|\). Note, that due to assumption (4.4), we obtain that \(C_2 < |b_\infty^{(j)}|\) for all \(j = 1, 2, 3\), and hence, we have that,
\[
|b_\infty^{(1)}| + |b_\infty^{(2)}| + |b_\infty^{(3)}| + 3C_2 \leq 2(|b_\infty^{(1)}| + |b_\infty^{(2)}| + |b_\infty^{(3)}|).
\]

Next, let us take time \(T > 0\) to be
(4.14)
\[
T := \min \left\{ \frac{1}{12 \sum_{j=1}^3 |b_\infty^{(j)}|}, \frac{|a_0 - a_\infty|}{3(1 + 8|\alpha_0|^2)}, \frac{1}{96|\alpha_0|^2}, \frac{1}{12(1 + 8|\alpha_0|^2) \sum_{j=1}^3 \frac{1}{|b_\infty^{(j)}| - |a_0 - a_\infty|}} \right\}.
\]
Recall, that the space \(X_T\) (see Section 4) is a complete metric space endowed with a distance
\[
d_{X_T}((\alpha_1, a_1), (\alpha_2, a_2)) = \|\alpha_1 - \alpha_2\|_{C([0,T])} + \|a_1 - a_2\|_{C([0,T])}.
\]
In addition, definition of constants \(C_1\) and \(C_2\) above implies condition (4.5), (4.8), and (4.9) in Lemmas 4.2-4.3. Moreover, since we selected \(T\), as
\[
T \leq \frac{1}{12 \sum_{j=1}^3 |b_\infty^{(j)}|} \quad \text{and} \quad T \leq \frac{|a_0 - a_\infty|}{3(1 + 8|\alpha_0|^2)} = \frac{C_2}{6(1 + 2C_1^2)},
\]
we also have that,
\[
3(|b_\infty^{(1)}| + |b_\infty^{(2)}| + |b_\infty^{(3)}| + 3C_2)T \leq 6(|b_\infty^{(1)}| + |b_\infty^{(2)}| + |b_\infty^{(3)}|)T \leq \frac{1}{2},
\]
and
\[
3(1 + 2C_1^2)T \leq \frac{1}{2}C_2.
\]
Thus, the other conditions (4.6) and (4.10) in Lemmas 4.2-4.3 are also satisfied. Therefore, we can employ Lemmas 4.2 and 4.3 to show that the mapping
\[
X_T \ni (\alpha, a) \mapsto (\Phi(\alpha, a), \Psi(\alpha, a)) \in X_T
\]
is well-defined. Next, combining estimates (4.11) and (4.12) in Lemmas 4.4-4.5 together, we obtain that,
(4.15)
\[
d_X((\Phi(\alpha_1, a_1), \Psi(\alpha_1, a_1)), (\Phi(\alpha_2, a_2), \Psi(\alpha_2, a_2))) \\
\leq \left( 6(|b_\infty^{(1)}| + |b_\infty^{(2)}| + |b_\infty^{(3)}|) + 12C_1 \right) T\|\alpha_1 - \alpha_2\|_{C([0,T])} \\
+ \left( 9C_1 + 6(1 + 2C_1^2) \left( \frac{1}{|b_\infty^{(1)}| - C_2} + \frac{1}{|b_\infty^{(2)}| - C_2} + \frac{1}{|b_\infty^{(3)}| - C_2} \right) \right) T\|\alpha_1 - \alpha_2\|_{C([0,T])}.
\]
for \((\alpha_1, a_1), (\alpha_2, a_2) \in X_T\). Next, since we selected time \(T\) as in (4.14) and constants \(C_1 = 2|\alpha_0|, C_2 = 2|a_0 - a_\infty|\), we have that,

\[
(4.16) \quad T \leq \frac{1}{12} \sum_{j=1}^{3} |b_\infty^{(j)}|, \quad T \leq \frac{1}{96|\alpha_0|} \leq \frac{1}{48C_1},
\]

and,

\[
(4.17) \quad T \leq \left(12(1 + 8|\alpha_0|^2) \sum_{j=1}^{3} \frac{1}{|b_\infty^{(j)}| - 2|a_0 - a_\infty|} \right)^{-1} = \left(12(1 + 2C_1^2) \sum_{j=1}^{3} \frac{1}{|b_\infty^{(j)}| - C_2} \right)^{-1}.
\]

Using the above estimates on time \(T\), (4.16)-(4.17) in (4.15) we obtain that,

\[
d\chi((\Phi(\alpha_1, a_1), \Psi(\alpha_1, a_1)), (\Phi(\alpha_2, a_2), \Psi(\alpha_2, a_2)))
\leq \frac{3}{4} \|\alpha_1 - \alpha_2\|_{C([0,T])} + \frac{11}{16} \|a_1 - a_2\|_{C([0,T])}
\leq \frac{3}{4} d\chi((\alpha_1, a_1), (\alpha_2, a_2)).
\]

Therefore, by the contraction mapping principle, there is a fixed point \((\alpha, a) \in X_T\), such that

\[
\alpha = \Phi(\alpha, a), \quad a = \Psi(\alpha, a),
\]

which is a solution of the system of differential equations (4.1). \(\square\)

**Remark 4.6.** From the proof of Theorem 4.1 we obtain the following estimates:

\[
\|\alpha\|_{C([0,T])} \leq 2|\alpha_0|, \quad \|a - a_\infty\|_{C([0,T])} \leq 2|a - a_\infty|,
\]

\[
T_{\text{max}} \geq \min \left\{ \frac{1}{12} \sum_{j=1}^{3} |b_\infty^{(j)}|, \frac{|a_0 - a_\infty|}{3(1 + 8|\alpha_0|^2)}, \frac{1}{96|\alpha_0|} \right\}
\frac{1}{12(1 + 8|\alpha_0|^2) \sum_{j=1}^{3} \frac{1}{|b_\infty^{(j)}| - 2|a_0 - a_\infty|}}.
\]

where \(T_{\text{max}}\) is a maximal existence time of the solution \((\alpha, a)\). Note, that once some a priori estimates for \(\|\alpha\|_{C([0,T])}\) and \(\|a - a_\infty\|_{C([0,T])}\) are deduced, a global solution of (4.1) can be obtained.

5. **A priori estimates**

We first derive the energy dissipation principle for the system (4.1). The system does not depend on parametrization \(s\), hence the energy of the system (4.1) is given by

\[
E(t) = \sum_{j=1}^{3} \left(1 + \frac{1}{2}(\alpha^{(j-1)}(t) - \alpha^{(j)}(t))^2 \right) |b_\infty^{(j)}(t)|.
\]

**Proposition 5.1** (Energy dissipation). Let \((\alpha, a)\) be a solution of (4.1) for \(0 \leq t \leq T\). Then, for all \(0 < t \leq T\), we have the local dissipation equality,

\[
E(t) + \int_{0}^{t} \left| \frac{d\alpha}{dt}(\tau) \right|^2 d\tau + \int_{0}^{t} \left| \frac{da}{dt}(\tau) \right|^2 d\tau = E(0).
\]
Proof of Proposition 5.1. Let us first compute the rate of the dissipation of the energy of the system (4.1) at time $t$,

$$
\frac{d}{dt} E(t) = \sum_{j=1}^{3} (\alpha^{(j-1)} - \alpha^{(j)}) \left( \frac{d\alpha^{(j-1)}}{dt} - \frac{d\alpha^{(j)}}{dt} \right) |b^{(j)}|
$$

(5.3)

$$
+ \sum_{j=1}^{3} \left( 1 + \frac{1}{2} (\alpha^{(j-1)} - \alpha^{(j)})^2 \right) \frac{b^{(j)}}{|b^{(j)}|} \cdot \frac{db^{(j)}}{dt}.
$$

Since $(\alpha, a)$ is a solution of the system (4.1), the right hand side of (5.3) can be calculated as,

$$
\sum_{j=1}^{3} (\alpha^{(j-1)} - \alpha^{(j)}) \left( \frac{d\alpha^{(j-1)}}{dt} - \frac{d\alpha^{(j)}}{dt} \right) |b^{(j)}|
$$

$$
= \sum_{j=1}^{3} \left( |b^{(j+1)}| + |b^{(j)}| \alpha^{(j)} - |b^{(j+1)}| \alpha^{(j+1)} - |b^{(j)}| \alpha^{(j-1)} \right) \frac{d\alpha^{(j)}}{dt}
$$

$$
= - \sum_{j=1}^{3} \left| \frac{d\alpha^{(j)}}{dt} \right|^2 = - \left| \frac{da}{dt} \right|^2,
$$

and

$$
\sum_{j=1}^{3} \left( 1 + \frac{1}{2} (\alpha^{(j-1)} - \alpha^{(j)})^2 \right) \frac{b^{(j)}}{|b^{(j)}|} \cdot \frac{db^{(j)}}{dt} = - \left| \frac{da}{dt} \right|^2.
$$

Thus, we obtain the energy dissipation for the system,

$$
\frac{d}{dt} E(t) = - \left| \frac{da}{dt} \right|^2 - \left| \frac{da}{dt} \right|^2.
$$

(5.4)

Next, integrating (5.4) with respect to $t$, we have the local dissipation equality (5.2). □

Proposition 5.2 (Maximum principle). Let $(\alpha, a)$ be a solution of the system (4.1) for $0 \leq t \leq T$. Then, for all $0 < t \leq T$, we have,

$$
|\alpha(t)|^2 \leq |\alpha_0|^2.
$$

(5.5)

Proof of Proposition 5.2. Due to Lemma 3.1, the matrix $B(t)$ is non-negative definite, hence we have that,

$$
|\alpha(t)|^2 \leq |\alpha_0|^2.
$$

(5.6)

Next, taking an inner product with $\alpha$ on both sides of the first equation of (4.1), integrating with respect to $t$, and using the estimate (5.6), we obtain the result (5.5). □

Now, let us define,

$$
A := \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
$$

Then, we also have that,

$$
\begin{pmatrix}
\alpha^{(3)} - \alpha^{(1)} \\
\alpha^{(1)} - \alpha^{(2)} \\
\alpha^{(2)} - \alpha^{(3)}
\end{pmatrix} = (A - I)\alpha.
$$
**Proposition 5.3** (Misorientation estimates). Let \((\alpha, a)\) be a solution of \((4.1)\) for \(0 \leq t \leq T\). Then, for all \(0 < t \leq T\), we have the following estimate for the misorientation,

\[
|(A - \mathbb{I})\alpha(t)|^2 \leq |(A - \mathbb{I})\alpha_0|^2.
\]

**Proof of Proposition 5.3.** We multiply the first equation of \((4.1)\) by \(A - \mathbb{I}\),

\[
\frac{d}{dt}(A - \mathbb{I})\alpha = -(A - \mathbb{I})B\alpha,
\]

and, take an inner product with \((A - \mathbb{I})\alpha\), to obtain,

\[
\frac{1}{2} \frac{d}{dt} |(A - \mathbb{I})\alpha|^2 = -(A - \mathbb{I})B\alpha \cdot (A - \mathbb{I})\alpha = -(A - \mathbb{I})(A - \mathbb{I})B\alpha \cdot \alpha = -3(B\alpha \cdot \alpha).
\]

Note that, the last equality is obtained by direct calculation,

\[
\begin{align*}
&\quad \frac{d}{dt} (A - \mathbb{I})B \\
&= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} |b^{(1)}| + |b^{(2)}| - |b^{(2)}| & -|b^{(1)}| \\ -|b^{(2)}| & |b^{(2)}| + |b^{(3)}| - |b^{(3)}| \\ -|b^{(1)}| & -|b^{(3)}| + |b^{(1)}| \end{pmatrix} \\
&= 3 \begin{pmatrix} |b^{(1)}| + |b^{(2)}| - |b^{(2)}| & -|b^{(1)}| \\ -|b^{(2)}| & |b^{(2)}| + |b^{(3)}| - |b^{(3)}| \\ -|b^{(1)}| & -|b^{(3)}| + |b^{(1)}| \end{pmatrix} = 3B.
\end{align*}
\]

Next, integrating \((5.8)\) with respect to \(t\), we obtain

\[
|(A - \mathbb{I})\alpha(t)|^2 + 6 \int_0^t (B\alpha \cdot \alpha) \, dt = |(A - \mathbb{I})\alpha_0|^2
\]

Similar to the Proposition 5.2 we use that the matrix \(B(t)\) is non-negative definite, hence we obtain final result \((5.7)\).

\[\square\]

### 6. Uniqueness and continuous dependence

In this section, we show uniqueness and continuous dependence on the initial data of the solution of the system \((4.1)\).

**Lemma 6.1.** For \(x^{(1)}, x^{(2)}, x^{(3)} \in \mathbb{R}^2\), \(a_{01}, a_{02} \in \mathbb{R}^2\), and \(\alpha_{01}, \alpha_{02} \in \mathbb{R}^3\), assume that \((\alpha_1(t), a_1(t))\) and \((\alpha_2(t), a_2(t))\) are classical solutions of \((4.1)\) on time interval \(0 \leq t \leq T\), associated with the given initial data \((\alpha_{01}, a_{01})\) and \((\alpha_{02}, a_{02})\), respectively. Next, assume that there exists a constant \(C_3 > 0\) such that \(|b^{(j)}_k(t)| \geq C_3\) for \(0 \leq t \leq T\), \(j = 1, 2, 3\) and \(k = 1, 2, 3\). Here, \(b^{(j)}_k(t) := x^{(j)} - a_k(t)\), \(j = 1, 2, 3\) and \(k = 1, 2, 3\). Then,

\[
\frac{d}{dt} \left(|\alpha_1 - \alpha_2|^2 + |a_1 - a_2|^2\right) \leq C_4(2|\alpha_1 - \alpha_2|^2 + 2|a_1 - a_2|^2)
\]

holds, where \(C_4 > 0\) is a positive constant that depends only on \(\alpha_{01}, \alpha_{02}\) and \(C_3\).
Proof of Lemma 6.1 Denote for $k = 1, 2$,

$$
\mathbb{B}_k(t) := \begin{pmatrix}
|b_k^{(1)}(t)| + |b_k^{(2)}(t)| & -|b_k^{(2)}(t)| & -|b_k^{(3)}(t)| \\
-|b_k^{(2)}(t)| & |b_k^{(2)}(t)| + |b_k^{(3)}(t)| & -|b_k^{(3)}(t)| \\
-|b_k^{(2)}(t)| & -|b_k^{(3)}(t)| & |b_k^{(2)}(t)| + |b_k^{(3)}(t)|
\end{pmatrix}.
$$

Using the equation (4.1), we have that,

$$
\frac{d}{dt}(\alpha_1 - \alpha_2) = -\mathbb{B}_1(t)\alpha_1(t) + \mathbb{B}_2(t)\alpha_2(t)
$$

$$
= -(\mathbb{B}_1(t) - \mathbb{B}_2(t))\alpha_1(t) - \mathbb{B}_2(t)(\alpha_1(t) - \alpha_2(t)),
$$

and, hence, taking an inner product with $\alpha_1 - \alpha_2$, we obtain,

$$
\frac{1}{2} \frac{d}{dt}|\alpha_1 - \alpha_2|^2 = -(\mathbb{B}_1(t) - \mathbb{B}_2(t))\alpha_1(t) \cdot (\alpha_1 - \alpha_2) - \mathbb{B}_2(t)(\alpha_1 - \alpha_2) \cdot (\alpha_1 - \alpha_2)
$$

$$
\leq 9|a_1 - a_2||\alpha_1||\alpha_1 - \alpha_2|.
$$

The estimate for the right hand side of (6.2) is obtained using Lemmas 3.1 and 3.3

$$
(-\mathbb{B}_1(t) - \mathbb{B}_2(t))\alpha_1(t) \cdot (\alpha_1 - \alpha_2) - \mathbb{B}_2(t)(\alpha_1 - \alpha_2) \cdot (\alpha_1 - \alpha_2)
$$

$$
\leq |(\mathbb{B}_1(t) - \mathbb{B}_2(t))\alpha_1||\alpha_1 - \alpha_2|
$$

$$
\leq 3(|b_1^{(1)} - b_2^{(1)}| + |b_1^{(2)} - b_2^{(2)}| + |b_1^{(3)} - b_2^{(3)}|)|\alpha_1||\alpha_1 - \alpha_2|
$$

$$
\leq 9|a_1 - a_2||\alpha_1||\alpha_1 - \alpha_2|.
$$

Next, using the maximum principle (5.5) and the Young’s inequality for the estimate in the right-hand side of (6.2), we deduce,

$$
\frac{d}{dt}|\alpha_1 - \alpha_2|^2 \leq 9|\alpha_0|||a_1 - a_2|^2 + |\alpha_1 - \alpha_2|^2|.
$$

Similarly, from the equation (4.1), we have that,

$$
\frac{d}{dt}(\alpha_1 - \alpha_2) = \sum_{j=1}^{3} \left( 1 + \frac{1}{2}(\alpha_1^{(j-1)} - \alpha_1^{(j)})^2 \right) \frac{b_1^{(j)}}{|b_1^{(j)}|} - \left( 1 + \frac{1}{2}(\alpha_2^{(j-1)} - \alpha_2^{(j)})^2 \right) \frac{b_2^{(j)}}{|b_2^{(j)}|}
$$

$$
= \sum_{j=1}^{3} \left( \frac{1}{2}(\alpha_1^{(j-1)} - \alpha_1^{(j)})^2 - \frac{1}{2}(\alpha_2^{(j-1)} - \alpha_2^{(j)})^2 \right) \frac{b_1^{(j)}}{|b_1^{(j)}|}
$$

$$
+ \sum_{j=1}^{3} \left( 1 + \frac{1}{2}(\alpha_2^{(j-1)} - \alpha_2^{(j)})^2 \right) \left( \frac{b_1^{(j)}}{|b_1^{(j)}|} - \frac{b_2^{(j)}}{|b_2^{(j)}|} \right).$$
Similarly, for the second term in the right-hand side of (6.4), applying (4.13), and using that we apply the maximum principle (5.5), and the Young’s inequality, to obtain,

\[
(6.4)
\]

\[
\frac{1}{2} \frac{d}{dt} |a_1 - a_2|^2
= \sum_{j=1}^{3} \left( \frac{1}{2} (a_1^{(j-1)} - a_1^{(j)})^2 - \frac{1}{2} (a_2^{(j-1)} - a_2^{(j)})^2 \right) \left( \frac{b_1^{(j)}}{|b_1^{(j)}|} - \frac{b_2^{(j)}}{|b_2^{(j)}|} \right) \cdot (a_1 - a_2)
\]

\[
\leq \sum_{j=1}^{3} \left( 1 + \frac{1}{2} (a_2^{(j-1)} - a_2^{(j)})^2 \right) \left( \frac{b_1^{(j)}}{|b_1^{(j)}|} - \frac{b_2^{(j)}}{|b_2^{(j)}|} \right) |a_1 - a_2|,
\]

Hence, we have that,

\[
Hence, we obtain,
\]

\[
(6.4)
\]

\[
\]

Next, let us estimate the two terms in the right-hand side of the (6.4). To estimate the first term, we apply the maximum principle (5.5), and the Young’s inequality, to obtain,

\[
\left| a_1^{(j-1)} - a_1^{(j)} + a_2^{(j-1)} - a_2^{(j)} \right| \left| (a_1^{(j-1)} - a_2^{(j-1)}) - (a_1^{(j)} - a_2^{(j)}) \right| |a_1 - a_2|
\]

\[
\leq 2(|a_{01}| + |a_{02}|) \left( |a_1^{(j-1)} - a_2^{(j-1)}| + |a_1^{(j)} - a_2^{(j)}| \right) |a_1 - a_2|
\]

\[
\leq (|a_{01}| + |a_{02}|) \left( 2|a_1^{(j-1)} - a_2^{(j-1)}|^2 + 2|a_1^{(j)} - a_2^{(j)}|^2 + |a_1 - a_2|^2 \right).
\]

Similarly, for the second term in the right-hand side of (6.4), applying (4.13), and using that \(|b_k^{(j)}(t)| \geq C_3, b_k^{(j)}(t) = x^{(j)} - a_k(t)\) for \(j = 1, 2, 3\) and \(k = 1, 2\), we have that,

\[
\left( 1 + \frac{1}{2} (a_2^{(j-1)} - a_2^{(j)})^2 \right) \left( \frac{b_1^{(j)}}{|b_1^{(j)}|} - \frac{b_2^{(j)}}{|b_2^{(j)}|} \right) |a_1 - a_2|
\]

\[
= \frac{2}{|b_1^{(j)}|} \left( 1 + \frac{1}{2} (a_2^{(j-1)} - a_2^{(j)})^2 \right) |b_1^{(j)} - b_2^{(j)}| |a_1 - a_2|
\]

\[
\leq \frac{2}{C_3} \left( 1 + \frac{1}{2} (a_2^{(j-1)} - a_2^{(j)})^2 \right) |a_1 - a_2|^2.
\]

Hence, we have that,

\[
\frac{d}{dt} |a_1 - a_2|^2 \leq \sum_{j=1}^{3} (|a_{01}| + |a_{02}|) \left( 2|a_1^{(j-1)} - a_2^{(j-1)}|^2 + 2|a_1^{(j)} - a_2^{(j)}|^2 + |a_1 - a_2|^2 \right)
\]

\[
+ \sum_{j=1}^{3} \frac{4}{C_3} \left( 1 + \frac{1}{2} (a_2^{(j-1)} - a_2^{(j)})^2 \right) |a_1 - a_2|^2.
\]

(6.5)
Thus, we can simplify (6.5), to have,

\[
\frac{d}{dt} |a_1 - a_2|^2 \leq 4(|\alpha_{01}| + |\alpha_{02}|) |a_1 - a_2|^2 + 3(|\alpha_{01}| + |\alpha_{02}|) |a_1 - a_2|^2
\]

(6.6)

\[+ \frac{4}{C_3} |a_1 - a_2|^2 \sum_{j=1}^{3} \left( 1 + \frac{1}{2} (\alpha_{02}^{(j-1)} - \alpha_{02}^{(j)})^2 \right).\]

Finally, using the misorientations estimate (5.7) in the third term of the right-hand side (6.6), we deduce that,

\[
\frac{d}{dt} |a_1 - a_2|^2 \leq 4(|\alpha_{01}| + |\alpha_{02}|) |a_1 - a_2|^2 + 3(|\alpha_{01}| + |\alpha_{02}|) |a_1 - a_2|^2
\]

(6.7)

\[+ \frac{4}{C_3} |a_1 - a_2|^2 \sum_{j=1}^{3} \left( 1 + \frac{1}{2} (\alpha_{02}^{(j-1)} - \alpha_{02}^{(j)})^2 \right).\]

Therefore, by (6.3) and (6.7), we have,

\[
\frac{d}{dt}(|\alpha_1 - \alpha_2|^2 + |a_1 - a_2|^2) \leq C_5|\alpha_1 - \alpha_2|^2 + C_6|a_1 - a_2|^2,
\]

where,

\[C_5 := 13(|\alpha_{01}| + |\alpha_{02}|), \quad C_6 := 12(|\alpha_{01}| + |\alpha_{02}|) + \frac{4}{C_3} \sum_{j=1}^{3} \left( 1 + \frac{1}{2} (\alpha_{02}^{(j-1)} - \alpha_{02}^{(j)})^2 \right).\]

Thus, we obtain (6.1) by taking \(C_4 = C_5 + C_6\). \(\square\)

By the neighboring inequality, we can now show uniqueness of the classical solution to the system (4.1).

**Theorem 6.2** (Uniqueness). Consider \(x^{(1)}, x^{(2)}, x^{(3)} \in \mathbb{R}^2\), and initial data \(a_0 \in \mathbb{R}^2\) and \(\alpha_0 \in \mathbb{R}^3\). Assume also, that there exists a constant \(C_7 > 0\), such that \(|b_k^{(j)}(t)| \geq C_7\) for \(0 \leq t \leq T, j = 1, 2, 3\) and \(k = 1, 2\). Then, there exists a unique classical solution \((\alpha(t), a(t))\) for \(0 \leq t \leq T\) of the system (4.1).

Note that, \(C_4\) stays bounded when \((\alpha_{01}, a_{01}) \to (\alpha_{02}, a_{02})\). Thus, we obtain,

**Theorem 6.3** (Continuous dependence on the initial data). For \(x^{(1)}, x^{(2)}, x^{(3)} \in \mathbb{R}^2\), \(a_{01}, a_{02} \in \mathbb{R}^2\) and \(\alpha_{01}, \alpha_{02} \in \mathbb{R}^3\), let \((\alpha_1, a_1)\) and \((\alpha_2, a_2)\) be two classical solutions of the system (4.1) on \(0 \leq t \leq T\), associated with the given initial data \((\alpha_{01}, a_{01})\) and \((\alpha_{02}, a_{02})\), respectively. Assume, that there exists a constant \(C_8 > 0\), such that \(|b_k^{(j)}(t)| \geq C_8\) for \(0 \leq t \leq T, j = 1, 2, 3\) and \(k = 1, 2\). Then,

\[
|\alpha_1 - \alpha_2|^2 + |a_1 - a_2|^2 \leq e^{C_4t}(|\alpha_{01} - \alpha_{02}|^2 + |a_{01} - a_{02}|^2)
\]

(6.8)

holds, where \(C_4 > 0\) is a positive constant given in Lemma 6.1. In particular, continuous dependence on the initial data holds, namely,

\[\|\alpha_1 - \alpha_2\|_{C([0,T])} + \|a_1 - a_2\|_{C([0,T])} \to 0\]

as \((\alpha_{01}, a_{01}) \to (\alpha_{02}, a_{02})\).
7. Evolution of grain boundary network

In this section, we extend the results obtained above for a system with a single junction to a network of grains that have lattice orientations \( \{a^{(k)}\}_{k=1}^{N^{SG}} \), grain boundaries \( \{\Gamma^{(j)}\}_{j=1}^{N^{GB}} \) and the triple junctions \( \{a^{(l)}\}_{l=1}^{N^{TJ}} \). We identify the lattice neighboring \( a^{(k)} \) with the single grain \( k \). Hence, the grain boundary energy of the entire network is defined now as,

\[
E(t) = \sum_{j=1}^{N^{GB}} \int_{\Gamma^{(j)}} \sigma(n^{(j)}, \Delta^{(j)} \alpha) \, d\mathcal{H}^1,
\]

where \( \Delta^{(j)} \alpha \) is a difference between the lattice orientations of the two grains that share the same grain boundary \( \Gamma^{(j)} \). The difference \( \Delta^{(j)} \alpha \) is called a misorientation of the grain boundary \( \Gamma^{(j)} \).

Next, using the same argument as in Section 2 for a system with a single triple junction, we obtain similar expression for the dissipation rate of the energy of the grain boundary network,

\[
\frac{d}{dt} E(t) = -\sum_{j=1}^{N^{GB}} \int_{\Gamma^{(j)}} \frac{d}{ds} T^{(j)} \, d\mathcal{H}^1 + \sum_{k=1}^{N^{SG}} \frac{\partial E}{\partial \alpha^{(k)}} \frac{d\alpha^{(k)}}{dt} - \sum_{l=1}^{N^{TJ}} \sum_{a^{(l)} \in \Gamma^{(j)}} T^{(j)} \cdot \frac{da^{(l)}}{dt},
\]

Here,

\[
T^{(j)} = \sigma^{(j)}_{\theta} \hat{n}^{(j)} + \sigma^{(j)} \hat{b}^{(j)},
\]

and \( a^{(l)} \) denotes the triple junction where three grain boundaries meet (we assume in our model that only triple junctions are stable). Note that, the line tension vector \( T^{(j)} \) points toward an inward direction of the grain boundary at the triple junction \( a^{(l)} \).

Next, similar to Section 2 we obtain the following system of differential equations to ensure that the entire system is dissipative:

\[
\dot{\alpha}^{(k)} = \mu \frac{d}{ds} T^{(j)} \cdot \hat{n}^{(j)}, \quad j = 1, \ldots, N^{GB},
\]

\[
\frac{d\alpha^{(k)}}{dt} = -\gamma \frac{\delta E}{\delta \alpha^{(k)}}, \quad k = 1, \ldots, N^{SG},
\]

\[
\frac{da^{(l)}}{dt} = \eta \sum_{a^{(l)} \in \Gamma^{(j)}} T^{(j)}, \quad l = 1, \ldots, N^{TJ},
\]

where \( \mu, \gamma, \eta > 0 \) are positive constants. For simplicity of the calculations below, we further assume that the energy density \( \sigma(n, \alpha) \) is an even function with respect to the misorientation \( \alpha \), that is, the misorientation effects are symmetric with respect to the difference between the lattice orientations. For the two grains \( k_1 \) and \( k_2 \) with orientations \( \alpha^{(k_1)} \) and \( \alpha^{(k_2)} \), respectively, we introduce notation that will be helpful for calculations below, \( \Gamma^{(j)} := \Gamma^{(j(k_1,k_2))} \) a grain boundary which is formed by grains \( k_1 \) and \( k_2 \) (See Figure 3). We also assume, that if grains \( k_1 \) and \( k_2 \) have no common interface/grain boundary, then we just set \( \Gamma^{(j(k_1,k_2))} = \emptyset \). Then,

\[
\frac{\delta E}{\delta \alpha^{(k)}} = \sum_{k' \neq k}^{N^{SG}} \int_{\Gamma^{(j(k,k'))}} \sigma_{\alpha}(n^{(j(k,k'))}, \alpha^{(k)} - \alpha^{(k')}) \, d\mathcal{H}^1.
\]
We let $\mu \to \infty$, $\gamma = \eta = 1$, and define,

\begin{equation}
\sigma(n, \alpha) = 1 + \frac{1}{2} \alpha^2.
\end{equation}

Then, the problem (7.4) is turned into,

\begin{equation}
\Gamma_i^{(j)} \text{ is a line segment between some } a^{(l,j_1)} \text{ and } a^{(l,j_2)}, \quad j = 1, \ldots, N^{GB},
\end{equation}

\begin{align*}
\frac{d\alpha^{(k)}}{dt} &= - \sum_{k' \neq k}^{N^{SG}} |\Gamma_i^{(j(k,k'))}|(\alpha^{(k)} - \alpha^{(k')}), \\
\frac{d a^{(l)}}{dt} &= \sum_{a^{(l)} \in \Gamma_i^{(j)}} T^{(j)}, \\
\end{align*}

\begin{equation}
\text{for } k = 1, \ldots, N^{SG}, \quad l = 1, \ldots, N^{TG}.
\end{equation}

The coefficient matrix for $\alpha^{(k)}$ of (7.7) is semi-positive definite. In fact, for a fixed $j = 1, \ldots, N^{GB}$, there are only two grains $k_{j_1}, k_{j_2} \in \{1, \ldots, N^{SG}\}$ such that $\Gamma^{(j)}$ is formed between grains $k_{j_1}$ and $k_{j_2}$. Using this fact, we find that,

\begin{align*}
\sum_{k=1}^{N^{SG}} \sum_{k' \neq k}^{N^{SG}} |\Gamma_i^{(j(k,k'))}|(\alpha^{(k)} - \alpha^{(k')})\alpha^{(k)} &= \sum_{j=1}^{N^{GB}} \sum_{k=1}^{N^{SG}} |\Gamma_i^{(j)}| \frac{\delta E}{\delta \alpha^{(k)}} \alpha^{(k)} \\
&= \sum_{j=1}^{N^{GB}} \sum_{k=1}^{N^{SG}} |\Gamma_i^{(j)}| \frac{\delta E}{\delta \alpha^{(k)}} \alpha^{(k)} \\
&= \sum_{j=1}^{N^{GB}} |\Gamma_i^{(j)}| \frac{\delta E}{\delta \alpha^{(k_{j_1})}} \alpha^{(k_{j_1})} + |\Gamma_i^{(j)}| \frac{\delta E}{\delta \alpha^{(k_{j_2})}} \alpha^{(k_{j_2})} \\
&= \sum_{j=1}^{N^{GB}} |\Gamma_i^{(j)}| \left(\alpha^{(k_{j_1})} - \alpha^{(k_{j_2})}\right)^2 \geq 0.
\end{align*}

Thus, we can proceed now using the same arguments as in Sections 4 and 6. To show the existence of solution of (7.7), we integrate (7.7) and rewrite in the form of integral equations. After that, we can make a contraction mapping argument as it was done in Section 4 for a single triple.
junction. The key ingredient in this approach is to show a priori lower bounds for the distance of two triple junctions, similar to Lemma 4.3. If an initial grain boundary network is sufficiently close to some equilibrium state, then any triple junction is close to its associated initial position (moreover, no critical events happen during short enough time interval). Thus, we can obtain a priori lower bounds for the distance between the two triple junctions.

To show the uniqueness and continuous dependence on the initial data of the solution, maximum principle for orientations plays an important role. Since the coefficient matrix for $\alpha^{(k)}$ of (7.7) is semi-positive definite, we can obtain the maximum principle like in Proposition 5.2 and hence we can proceed with the same argument as in Section 6. Therefore, we obtain,

**Theorem 7.1.** In a grain boundary network with lattice orientations, if triple junctions at the initial state are sufficiently close to triple junctions at the equilibrium state, then the problem (7.7) has a unique time local solution.

**Remark 7.2.** Note, that the proposed model of dynamic orientations (7.4) (and, hence, dynamic misorientations, (7.7), or Langevin type equation if critical events/grain boundaries disappearance events are taken into account) is reminiscent of the recently developed theory for the grain boundary character distribution (GBCD) [3, 4, 5, 2], which suggests that the evolution of the GBCD satisfies a Fokker-Planck Equation (GBCD is an empirical distribution of the relative length (in 2D) or area (in 3D) of interface with a given lattice misorientation and normal). More details will be presented in future studies.

Large time asymptotic analysis of the model proposed in the current work will be presented in the forthcoming paper.

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