WELL-POSEDNESS AND GLOBAL IN TIME BEHAVIOR FOR MILD SOLUTIONS TO THE NAVIER-STOKES EQUATION ON THE HYPERBOLIC SPACE WITH INITIAL DATA IN $L^p$

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Abstract. We study mild solutions to the Navier-Stokes equation on the $n$-dimensional hyperbolic space $H^n$, $n \geq 2$. We use dispersive and smoothing estimates proved by Pierfelice on a class of complete Riemannian manifolds to extend the Fujita-Kato theory of mild solutions from $\mathbb{R}^n$ to $H^n$. This includes well-posedness results for $L^p$ initial data in the range $1 < p < \infty$, global in time results for small initial data, and $L^p$ norm decay results for both $u$ and $\nabla u$. As part of this, we extend to the hyperbolic space $H^n$ known facts in Euclidean space concerning the strong continuity and contractivity of the semigroup generated by the Laplacian. Also, we establish necessary boundedness and commutation properties for a certain projection operator in the setting of $H^n$ using spectral theory. This work, together with Pierfelice’s, contributes to providing a full theory for mild solutions on $H^n$. While the statements of the results are the same as in the Euclidean case, the methods of the proofs are at times different.

Contents

1. Introduction 2
   1.1. The Navier-Stokes equation on $H^n$ 3
   1.2. Overview of previous results 4
   1.3. Main results 7
   1.4. Comparing and contrasting the cases for $\mathbb{R}^n$ and $H^n$ 9
   1.5. Organization of the article 10

2. Notation and estimates used 11

3. Picard iteration on $H^n$ and proof of Theorem 1.3 17
   3.1. Local existence for $q > n$ 17
   3.2. Uniqueness 19
   3.3. Continuous dependence on initial data 20
   3.4. Global well-posedness 21
   3.5. The case $q=n$ 21
   3.6. Well-posedness for $\nabla u$ 22

4. Proof of Theorem 1.4 23

5. Proof of Theorem 1.5 26

6. Proof of Theorem 1.6 28
   6.1. Local well-posedness 28
   6.2. Continuous dependence on initial Data 29
   6.3. Well-posedness for $\nabla u$ 30

2010 Mathematics Subject Classification. 35Q30, 76D05, 76D03;
Key words and phrases. Mild solutions, Navier-Stokes, hyperbolic space, well-posedness, Fujita-Kato theory.
1. Introduction

In this work, we are concerned with the Navier-Stokes equation on the $n$-dimensional hyperbolic space $\mathbb{H}^n$ of constant sectional curvature $-1$, where $n \geq 2$. Specifically, we study the theory of mild solutions to Navier-Stokes on $\mathbb{H}^n$. To begin, let us recall the definition of mild solutions to Navier-Stokes in the context of Euclidean space $\mathbb{R}^n$.

On $\mathbb{R}^n$, the Navier-Stokes equation is given by

$$
\frac{\partial}{\partial t} u - \Delta_{\mathbb{R}^n} u + u \cdot \nabla u + \nabla p = 0, \quad \text{div } u = 0, \quad u(0, t) = a,
$$

(N-S$_{\mathbb{R}^n}$)

where $u = (u_1, u_2, \ldots, u_n)$ is the velocity of the fluid, $p$ a scalar field denoting the pressure, the condition $\text{div } u = 0$ means the fluid is incompressible, and the term $u \cdot \nabla u$ is given in coordinates by

$$(u \cdot \nabla u)^j = u^i \partial_i u^j,$$

where we sum over the repeated index $i$.

The mild solution approach was introduced by Fujita and Kato in [9] and Sobolevski˘ı in [32] and involves using the Leray projector $P_{\mathbb{R}^n}$ onto divergence free vector fields to rewrite equation (N-S$_{\mathbb{R}^n}$) as

$$
\frac{\partial}{\partial t} u - \Delta_{\mathbb{R}^n} u + P_{\mathbb{R}^n}(u \cdot \nabla u) = 0, \quad \text{div } u = 0, \quad u(0, t) = a,
$$

(N-S$_{\mathbb{R}^n}'$)

which can be considered as a non-linear perturbation of the heat equation on $\mathbb{R}^n$ (see, for example, [9] and [15]). Thus by using the heat semigroup $e^{t\Delta_{\mathbb{R}^n}}$ and Duhamel’s formula, the equation (N-S$_{\mathbb{R}^n}'$) can be converted into an integral equation

$$
u(t) = e^{t\Delta_{\mathbb{R}^n}} a - \int_0^t e^{(t-s)\Delta_{\mathbb{R}^n}} P_{\mathbb{R}^n}(u(s) \cdot \nabla u(s)) \, ds, \quad \text{div } a = 0,
$$

(N-S$_{\mathbb{R}^n, \text{int}}$)

which is then solved by means of fixed point methods in suitably chosen Banach spaces (or equivalently, by Picard iteration). Solutions to this integral equation are referred to as mild solutions.

We remark here that mild solutions were originally called “strong solutions” by Fujita and Kato in [9] and as we will see from Kato’s results in [15], this is because local existence, uniqueness, and smoothness results hold for mild solutions in all dimensions $n \geq 2$ with initial data coming from a wide range of $L^p$ spaces. Additionally, in the case of suitably small initial data, mild solutions are known to exist globally and decay estimates of certain $L^p$ norms have been established. We review more of the literature concerning mild solutions below.

The situation for mild solutions stands in contrast to the situation for Leray-Hopf weak solutions of the form

$$u \in L^\infty([0, T], L^2(\mathbb{R}^n)) \cap L^2([0, T], \dot{H}^1(\mathbb{R}^n)),
$$

(1.1)
which are weak solutions of Navier-Stokes satisfying the global energy inequality

\[ \|u(t)\|_{L^2(\mathbb{R}^n)}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2(\mathbb{R}^n)}^2 \, ds \leq \|a\|_{L^2(\mathbb{R}^n)}^2, \quad 0 \leq t \leq T. \] (1.2)

In the work of Leray [21] and Hopf [12], global existence of weak solutions was shown for dimensions \( n = 2, 3 \) and in the case of \( \mathbb{R}^2 \), smoothness and uniqueness of these solutions is well-known. However, unlike the situation for mild solutions and small initial data, questions concerning global existence, uniqueness, smoothness, and time decay for \( L^2 \) norms remain open for Leray-Hopf weak solutions in the case of \( \mathbb{R}^3 \).

1.1. The Navier-Stokes equation on \( \mathbb{H}^n \). Given that our goal is to extend the mild solution theory presented by Kato in [15] to the hyperbolic space \( \mathbb{H}^n \), we must first discuss the proper formulation of (N-S \( \mathbb{R}^n \)) on a complete Riemannian manifold \((M,g)\). Specifically, we must make a decision concerning how to generalize the Laplacian \( \Delta_{\mathbb{R}^n} \) to a Riemannian manifold, since in this context there is no canonical choice.

For instance, throughout the course of this work, we will consider the following operators: the Laplace-Beltrami operator on functions defined by

\[ \Delta_g f = \text{div}(\text{grad} f) = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j}), \]

where \( |g| \) denotes the determinant of the metric \( g \); the Bochner Laplacian on the space of rank \((k,0)\) tensor fields, \( \mathcal{T}^k_0(M) \), defined by

\[ \Delta_{B,k} u = -\nabla^* \nabla u \quad \text{for} \quad k = 0, 1, 2, \ldots, \]

where \( \nabla \) is the induced Levi-Civita connection on \( \mathcal{T}^k_0(M) \); and the Hodge Laplacian acting on the space of differential \( k \)-forms \( \Omega^k(M) \) defined by

\[ \Delta_{H,k} u = (d^* d + dd^*) u \quad \text{for} \quad k = 0, 1, 2, \ldots, n. \]

We remark here that with the sign conventions adopted above, \( \Delta_g \) and \( \Delta_{B,k} \) are negative operators and \( \Delta_{H,k} \) is positive.

For functions, one can show all of the above Laplacians coincide (up to a plus or minus sign).

However, though \( \Omega^k(M) \subset \mathcal{T}^k_0(M) \), the Bochner and Hodge Laplacians do not in general agree for \( k \)-forms with \( k \geq 1 \) and are instead related by the Bochner-Weitzenböck identity (see [14], for example). For our purposes, the 1-form version is sufficient, which is given by

\[ \Delta_{H,1} u = -\Delta_{B,1} u + \text{Ric} \, u, \] (1.3)

where \( \text{Ric} : T^*M \to T^*M \) is the Ricci operator on 1-forms.

Following Ebin-Marsden [6], we use yet another operator

\[ L = -2\text{Def}^* \text{Def} = -\Delta_{H,1} - dd^* + 2\text{Ric}, \] (1.4)

where \( \text{Def} \) is the deformation tensor defined by \( \text{Def} u = \frac{1}{2}((\nabla u) + (\nabla u)^T) \) and \( \text{Def}^* \) its adjoint.

On \( \mathbb{H}^n \) with \( d^* u = 0 \),

\[ Lu = \Delta_{B,1} u - (n - 1) u, \]

thus the Navier-Stokes equation becomes

\[ \partial_t u - \Delta_{B,1} u + (n - 1) u + \nabla_u u + d p = 0 \]

\[ d^* u = 0, \quad u(0,t) = a, \quad (\text{N-S}_{\mathbb{H}^n}) \]
where $u$ is now a differential 1-form on $M$, since $L$ as defined sends 1-forms to 1-forms. As well, $\#$ represents the musical isomorphism $\langle \cdot \rangle^\#: T^*M \to TM$. See [3] for more on the formulation of the Navier-Stokes equation on Riemannian manifolds.

As in [27] and in the spirit of equation (N-S$^\prime_{\mathbb{R}^n}$), we can write equation (N-S$^\prime_{\mathbb{H}^n}$) in pressure-free form. To do this, we take $d^*\,$ of (N-S$^\prime_{\mathbb{H}^n}$) and use the fact that $d^*u = 0$ for a smooth solution $(u, p)$ of (N-S$^\prime_{\mathbb{H}^n}$), which gives

$$-\Delta_g p + d^* \nabla_u u = 0. \quad (1.5)$$

Thus

$$dp = -d(-\Delta_g)^{-1}d^* \nabla_u u. \quad (1.6)$$

Therefore, we define an operator on $T^*\mathbb{H}^n$ by

$$P = I - d(-\Delta_g)^{-1}d^*, \quad (1.7)$$

which allows us to rewrite (N-S$^\prime_{\mathbb{H}^n}$) as follows:

$$\partial_t u - Lu + P \left( \text{div}(u^\# \otimes u^\#) \right)^\flat = 0 \quad (\text{N-S}^\prime_{\mathbb{H}^n})$$

where we have also used the fact that $\nabla_u u = (\text{div}(u^\# \otimes u^\#))^\flat$ when $d^*u = 0$. Here, $\flat$ represents the musical isomorphism $\langle \cdot \rangle^\flat: TM \to T^*M$.

Also, as in the case of $\mathbb{R}^n$, we can then convert (N-S$^\prime_{\mathbb{H}^n}$) into the following integral equation

$$u = u_0(t) + Gu(t), \quad (1.8)$$

where

$$u_0(t) = e^{tL}a, \quad Gu(t) = -\int_0^t e^{(t-s)L}P \left( \text{div}(u^\# \otimes u^\#) \right)^\flat(s) ds, \quad d^*a = 0. \quad (\text{N-S}^\prime_{\mathbb{H}^n, \text{int}})$$

Note that in defining (N-S$^\prime_{\mathbb{H}^n, \text{int}}$), we have used Strichartz’ result in [34] that $\Delta_{B,1}$ is self-adjoint on a complete Riemannian manifold $(M, g)$, so that the semigroup $e^{tL}$ is well defined. In subsequent sections, we will focus our attention on this integral formulation of the Navier-Stokes equation on $\mathbb{H}^n$.

1.2. Overview of previous results. First we will overview some classical results concerning mild solutions to Navier-Stokes on $\mathbb{R}^n$. Then we will examine some known results for mild solutions to Navier-Stokes in the setting of a complete Riemannian manifold $(M, g)$.

As mentioned, the theory of mild solutions to the Navier-Stokes equation goes back to both 1959 and the results of Sobolevski˘ı [32] and 1961 and the work of Fujita and Kato [9]. These works showed local existence and uniqueness results for the Navier-Stokes equation on bounded domains in $\mathbb{R}^2$ for initial data in $L^2$, and in $\mathbb{R}^3$ for sufficiently regular initial data.

Also in $\mathbb{R}^3$ and for initial data in $L^p$, $p > 3$, existence of local solutions in certain $L^sL^q$ spaces was shown for the full space $\mathbb{R}^3$ by Fabes, Jones, and Riviè re [7] in 1972, by Lewis [22] for $\mathbb{R}^3_+$ in 1973, and by Fabes, Lewis, and Riviè re [8] for bounded domains in 1977. For more on the case $\mathbb{R}^3$ and $L^p$ with $p > 3$, see also von Wahl [37] (1980), Miyakawa [26] (1981), and Giga [11] (1986).

In 1980, Weissler [38] constructed local solutions for initial data in $L^3(\mathbb{R}^3)$. Then in 1984, Kato [15] expanded on this result by presenting a rather complete theory for mild solutions to Navier-Stokes with initial data in $L^p$ spaces, $1 < p \leq \infty$ on the full space $\mathbb{R}^n$, $n \geq 2$. In
this work, Kato also showed global well-posedness and $L^p$-norm time decay properties in the case of small initial data.

Related to Kato’s 1984 paper is a 1983 preprint by Giga that was cited by Kato in \[15\] and eventually published in 1986 as \[11\], where Giga considered mild solutions to semilinear parabolic equations of the form

$$u_t + Au = Fu,$$

where $A$ is an elliptic operator and $Fu$ represents the nonlinearity of the equation. In this work, Giga proved well-posedness results in $L^p$ spaces, where $q$ and $p$ are chosen so that the $L^qL^p$ norm is either dimensionless or scaling invariant. Kato then used this to prove a decay result for the $L^p(R^n)$ norm of a mild solution to Navier-Stokes as part of the aforementioned theory presented in \[15\].

In 1985, Giga and Miyakawa extended the results of Fujita and Kato \[9\] on bounded domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ from $L^2$ to a full $L^p$ theory for $1 < p < \infty$, while also getting rid of the regularity requirement for the initial condition mentioned above for $\mathbb{R}^3$. The case $L^3(D)$, where $D \subset \mathbb{R}^3$ is an exterior domain, was established by Iwashita \[13\] in 1989.

For further information on mild solutions on $\mathbb{R}^n$, see also \[2, 7, 10, 17, 18, 19, 23, 36\].

Having surveyed some of the results concerning mild solutions to Navier-Stokes on Euclidean space, we now collect together the well-posedness and time decay results of Kato for $\mathbb{R}^n$ from \[15\], which we will subsequently extend to $\mathbb{H}^n$.

**Theorem 1.1.** \[11\] \[15\] Let $a \in L^n(\mathbb{R}^n)$ with $\text{div} a = 0$ and $n \geq 2$.

(K1) Then there is $T > 0$ and a unique solution $u$ of (N-S$_{\mathbb{R}^n}$) such that

\begin{align*}
&t^{\left(\frac{1}{2} - \frac{n}{2q}\right)} u \in BC([0, T), L^q(\mathbb{R}^n)) \quad \text{for} \quad n \leq q \leq \infty, \\
&t^{\left(\frac{3}{4} - \frac{n}{2q}\right)} \nabla u \in BC([0, T), L^q(\mathbb{R}^n)) \quad \text{for} \quad n \leq q \leq \infty,
\end{align*}

with values zero at $t = 0$ except for $q = n$ in (1.9), in which case $u(0) = a$. As well, $u$ has the additional property

$$u \in L^r((0, T_1), L^q(\mathbb{R}^n)) \quad \text{with} \quad \frac{1}{r} = \frac{1}{n} - \frac{n}{2q}, \quad n < q < \frac{n^2}{n-2},$$

for some $0 < T_1 \leq T$.

(K2) There is $\lambda > 0$ such that if $\|a\|_{L^n(\mathbb{R}^n)} \leq \lambda$, then the solution from (K1) is global and we may take $T = T_1 = \infty$.

(K2') In the situation (K2) where the solution $u$ is global, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|u(t)\|_{L^n(\mathbb{R}^n)} \, dt = 0.$$

(K3) If $a \in L^p(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$, where $1 < p < n$, then the solution given by (K1) has the following additional properties.

\begin{align*}
u \in BC([0, T_2), L^p(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)), \\
t^{\frac{1}{4}} \nabla u \in BC([0, T_2), L^p(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)),
\end{align*}

for some $0 < T_2 \leq T$. 
(K4) There is some \( 0 < \lambda_1 \leq \lambda \) such that if \( \|a\|_{L^q(\mathbb{R}^n)} \leq \lambda_1 \), then the solution in (K3) is global and we may set \( T = T_1 = T_2 = \infty \). Moreover, for any finite \( q \geq p \),

\[
t\left(\frac{2}{2^p} - \frac{n}{p^*}\right) u \in BC([1, \infty), L^q(\mathbb{R}^n)),
\]

(1.15)

\[
t\left(\frac{2}{2^p} - \frac{n}{p^*} + \frac{1}{2}\right) \nabla u \in BC([1, \infty), L^q(\mathbb{R}^n)),
\]

(1.16)

provided that the exponent of \( t \) is smaller than 1 (separately for each of \( u \) and \( \nabla u \)), otherwise the exponent should be replaced by an arbitrary number smaller than 1, but close to 1.

(K4’) In the situation (K4) where the solution \( u \) is global, we have

\[
\lim_{t \to \infty} \|u(t)\|_{L^p(\mathbb{R}^n)} = 0.
\]

(1.17)

Remark 1.1. The property (1.11) in (K1) for the free solution \( u_0 \) is owed to Giga, who proved the result in [11], Kato then extended the result to the full solution \( u \) and used this to prove the norm decay result (1.12) in (K2’).

Remark 1.2. The result (K2’), combined with the energy inequality, which holds for any smooth solution \((u, p)\) of (N-S) by the structure of the equation, can be used to show

\[
\lim_{n \to \infty} \|u(t)\|_{L^2(\mathbb{R}^n)} = 0,
\]

(1.18)

which answered an open question of Leray’s in [20] and [21] concerning decay of the \( L^2 \) norm for dimension 2, at least for the case of mild solutions and small initial data.

Remark 1.3. Leray’s question concerning decay of the \( L^2 \) norm for solutions in \( \mathbb{R}^n \) for \( n \geq 3 \) is answered, at least for mild solutions and small initial data, by Kato’s result (K4’). In particular, we have

\[
\lim_{t \to \infty} \|u(t)\|_{L^2(\mathbb{R}^3)} = 0.
\]

(1.19)

Moving now to the setting of a complete Riemannian manifold, we first mention the work of Mitrea and Taylor in [25], where they considered the inhomogenous form of (N-S) in the setting of Lipschitz subdomains of compact Riemannian manifolds and proved local existence and uniqueness results for mild solutions (see also [35]).

Then in [27], Pierfelice considered the Navier-Stokes equation on a class of non-compact and complete Riemannian manifolds with positive injectivity radius and satisfying certain curvature bounds (see [27] for details). In this setting, Pierfelice proved dispersive and smoothing estimates for the semigroup \( e^{tL} \) that parallel known dispersive and smoothing estimates for the heat semigroup on \( \mathbb{R}^n \) (see [15], for example). These estimates then allowed Pierfelice to extend Kato’s well-posedness result (K1), specifically (1.9), and the global in time result (K2) for small initial data to the setting of complete Riemannian manifolds where the Ricci operator is a negative constant scalar multiple of the metric. We state the result as follows, where \( c_n(t) = C_n \max \left( t^{-n/2}, 1 \right) \) for some constant \( C_n \) depending only on the dimension.

**Theorem 1.2.** [27] Let \( a \in L^n(M) \) with \( d^*a = 0 \) and \( M \) a complete Riemannian manifold where the Ricci operator is a negative constant scalar multiple of the metric \( g \). Then there exists \( T > 0 \) and a unique solution \( u \) of Navier-Stokes on \( M \) such that

\[
c_n(t)^{-\left(\frac{1}{n} - \frac{1}{4}\right)} e^{\beta t} u \in BC([0, T], L^q(M)), \quad n \leq q < \infty,
\]

(1.20)
where $\beta > 0$ depends on $n$ and $q$ and such that we have continuous dependence on the initial data. Moreover, there is $\lambda > 0$ such that if $\|a\|_{L^q(\mathbb{H}^n)} \leq \lambda$, then the solutions are global in time. For $n = 2$, the solutions are global in time without any restriction on the size of the initial data.

Examining (1.9) and (1.20), we see that for $0 \leq t \leq 1$, Pierfelice’s result for $\mathbb{H}^n$ coincides with Kato’s (1.9) in Theorem 1.1 as well as the global in time result (K2). However, for the large time regime $t > 1$, Pierfelice’s result offers an improvement on Kato’s, giving exponential decay in time as opposed to decay according to an inverse power of $t$. We also note that Pierfelice’s result does not include $q = \infty$, whereas Kato’s does.

In fact, Pierfelice does more and extends Theorem 1.2 to arbitrary non-compact and complete Riemannian manifolds with positive injectivity radius and satisfying certain curvature properties (see [27] for more information). In this case, it is required that the initial data also satisfy

$$a \in L^n(M) \cap L^2(M).$$

(1.21)

As well, for global existence in this scenario, both the $L^n$ and the $L^2$ norms are required to be small, rather than just the $L^n$ norm, though in dimension 2, global existence and uniqueness is still possible for large data (see [27] for further details).

In what follows, we will show that Pierfelice’s dispersive and smoothing estimates proved for the semigroup $e^{tL}$ on a specific class of Riemannian manifolds allow extension of Kato’s Theorem 1.1 to the setting of $\mathbb{H}^n$, including all results about the total covariant derivative $\nabla u$ and all global in time decay results. However, as is the case for Pierfelice in [27], we do not get the case $q = \infty$ in (1.9) and (1.10).

1.3. Main results. The following theorems collect together the statements to be proven and are essentially restatements of the results of Theorem 1.1 in the context of $\mathbb{H}^n$.

**Theorem 1.3.** Let $a \in L^n(\mathbb{H}^n)$, $n \geq 2$, with $d^*a = 0$. Then there exists $T > 0$ and a unique solution $u$ of (N-S$_{\mathbb{H}^n}$) such that

$$t\left(\frac{1}{2} - \frac{n}{2q}\right) u \in BC([0,T), L^q(\mathbb{H}^n)), \quad n \leq q < \infty,$$

(1.22)

$$t^{1 - \frac{n}{2q}} \nabla u \in BC([0,T), L^q(\mathbb{H}^n)), \quad n \leq q < \infty,$$

(1.23)

both with values zero at $t = 0$ except for $q = n$ in (1.22), in which case $u(0) = a$, and such that we have continuous dependence on the initial data. Moreover, there is $\lambda > 0$ such that if $\|a\|_{L^n(\mathbb{H}^n)} \leq \lambda$, then the solution is global in time.

**Remark 1.4.** This is Pierfelice’s Theorem 1.2 given above for the case of $\mathbb{H}^n$, though now expanded to include information about $\nabla u$. As mentioned above, unlike the case of Kato’s Theorem 1.1, we do not have $q = \infty$ in Theorem 1.3 though we hope to study this case further in future works.

**Theorem 1.4.** For the solution $u$ and $T > 0$ from Theorem 1.3 there exists some $0 < T_1 \leq T$ such that

$$u \in L^r((0,T_1), L^2(\mathbb{H}^n)) \quad \text{with} \quad \frac{1}{r} = \frac{1}{2} - \frac{n}{2q}, \quad n < q < \frac{n^2}{n - 2}.$$  

(1.24)

As well, there is $\lambda_1 > 0$ such that if $\|a\|_{L^n(\mathbb{H}^n)} \leq \lambda_1$, then $T_1$ can be extended to $+\infty$. 
Remark 1.5. As we will see in the course of the proof of Theorem 1.4, the requirements on $r$ and $q$ listed in (1.24) are a consequence of the Marcinkiewicz interpolation theorem. We also mention here that, as remarked by Stein in appendix B of [33], the Marcinkiewicz interpolation theorem is valid if the underlying measure space $\mathbb{R}^n$ of $L^p(\mathbb{R}^n)$ is replaced by a more general measure space, such as $\mathbb{H}^n$.

**Theorem 1.5.** For the case in Theorem 1.3 when the solution $u$ is global, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|u(t)\|_{L^n(\mathbb{H}^n)} \, dt = 0. \quad (1.25)$$

Remark 1.6. Theorem 1.5 implies, with the same argument as outlined by Kato in [15] and discussed above, that

$$\lim_{t \to \infty} \|u(t)\|_{L^2(\mathbb{H}^2)} = 0. \quad (1.26)$$

**Theorem 1.6.** Let $a \in L^p(\mathbb{H}^n) \cap L^q(\mathbb{H}^n)$ with $d^*a = 0$ and where $1 < p < n$, $n \geq 2$. Then the solution given by Theorem 1.3 has the following additional properties.

$$u \in BC([0, T), L^p(\mathbb{H}^n) \cap L^q(\mathbb{H}^n)), \quad t^{1/2}u' \in BC([0, T), L^p(\mathbb{H}^n) \cap L^q(\mathbb{H}^n)), \quad (1.27)$$

and such that we have continuous dependence on the initial data. Moreover, if $\|a\|_{L^n(\mathbb{H}^n)} < \lambda$, then the solution is global in time.

Remark 1.7. Though $\|a\|_{L^n(\mathbb{H}^n)}$ is required to be small for global existence and uniqueness in Theorem 1.6, there is no restriction on the size of $\|a\|_{L^p(\mathbb{H}^n)}$. This is the same as the situation for $\mathbb{H}^n$.

**Theorem 1.7.** For the case in Theorem 1.6 when the solution $u$ and its derivative $\nabla u$ are global, the following decay estimates on $u$ hold for any $p \leq q < \infty$:

$$t\left(\frac{n}{2p'} - \frac{n}{2q'}\right) u \in BC([1, \infty), L^q(\mathbb{H}^n)) \quad \text{if} \quad \frac{n}{2p'} - \frac{n}{2q'} < 1, \quad (1.29)$$

$$t\left(\frac{n}{2p} - \frac{n}{2q}\right) u \in BC([1, \infty), L^q(\mathbb{H}^n)) \quad \text{if} \quad \frac{n}{2p} - \frac{n}{2q} \geq 1, \quad (1.30)$$

where $p'$ is chosen such that $p < p' < n$, $p' \leq q$, and $\frac{n}{2p} - \frac{n}{2q} < 1$, but such that $\frac{n}{2p'} - \frac{n}{2q'}$ is arbitrarily close to 1.

Furthermore, we have the following decay estimates on $\nabla u$ for any $p \leq q < \infty$:

$$t\left(\frac{n}{2p'} - \frac{n}{2q'} + 1\right) \nabla u \in BC([1, \infty), L^q(\mathbb{H}^n)) \quad \text{if} \quad \frac{n}{2p'} - \frac{n}{2q'} + \frac{1}{2} < 1, \quad (1.31)$$

$$t\left(\frac{n}{2p} - \frac{n}{2q} + 1\right) \nabla u \in BC([1, \infty), L^q(\mathbb{H}^n)) \quad \text{if} \quad \frac{n}{2p} - \frac{n}{2q} + \frac{1}{2} \geq 1, \quad (1.32)$$

where $p'$ is chosen such that $p < p' < n$, $p' \leq q$, and $\frac{n}{2p'} - \frac{n}{2q'} + \frac{1}{2} < 1$, but such that $\frac{n}{2p'} - \frac{n}{2q'} + \frac{1}{2}$ is arbitrarily close to 1.

**Theorem 1.8.** For the case in Theorem 1.6 when the $L^p$ solution $u$ and its derivative $\nabla u$ are global,

$$\lim_{t \to \infty} \|u(t)\|_{L^p(\mathbb{H}^n)} = 0. \quad (1.33)$$
Remark 1.8. Theorem 1.8 implies, as in the case of \( \mathbb{R}^3 \), that
\[
\lim_{t \to \infty} \| u(t) \|_{L^2(\mathbb{H}^3)} = 0.
\] (1.34)

As well, in the course of proving Theorem 1.8, we will explicitly show
\[
\| u(t) - e^{tL}a \|_{L^p(\mathbb{H}^n)} = O(t^{-\omega/2}),
\] (1.35)
where \( \omega > 0 \) is any real number satisfying
\[
\omega < \min \left\{ 1, n - \frac{n}{p}, \frac{n}{p} - 1 \right\}.
\] (1.36)

Thus, as shown by Kato in [15] for the case of \( \mathbb{R}^n \), on the hyperbolic space \( \mathbb{H}^n \), we have that the decay rate of \( \| u(t) \|_{L^p(\mathbb{H}^n)} \) is at least as fast as the slower of \( \| e^{tL}a \|_{L^p(\mathbb{H}^n)} \) and \( t^{-\omega/2} \).

Remark 1.9. It is enough to prove the results of Theorems 1.3 and 1.6 for \( L^\infty([0,T]) \) in time. Indeed, the main ingredient in extending from \( L^\infty([0,T]) \) to \( BC([0,T]) \) in time in the Euclidean setting \( \mathbb{R}^n \) is the strong continuity of the heat semigroup \( e^{t\Delta_{\mathbb{R}^n}} \) on \( L^p(\mathbb{R}^n) \) for \( 1 \leq p < \infty \) (see, for example, [9], [11], [15], and Chapter 5 of [36]). Thus if it is known that the corresponding semigroup \( e^{tL} \) in the setting of \( \mathbb{H}^n \) is strongly continuous, the extension from \( L^\infty([0,T]) \) to \( BC([0,T]) \) follows just as it does in the Euclidean case. Therefore, in the appendix, we prove the semigroup \( e^{tL} \) is not only strongly continuous, but contractive on \( L^p(\mathbb{H}^n) \) for \( 1 \leq p < \infty \).

1.4. Comparing and contrasting the cases for \( \mathbb{R}^n \) and \( \mathbb{H}^n \). As we have remarked, in [27], Pierfelice proved dispersive and smoothing estimates for the semigroup \( e^{tL} \) that are analogous to well-known estimates for the heat semigroup on \( \mathbb{R}^n \), and then expanded Kato’s results (1.9) and (K2) from Theorem 1.1 to the setting of a class of complete Riemannian manifolds containing \( \mathbb{H}^n \).

Thus it is our hope that this current work, where we extend the rest of Kato’s Theorem 1.1 to the setting of \( \mathbb{H}^n \), can be taken together with Pierfelice’s [27] to provide as complete a picture as possible of the mild solution theory in \( L^p \) for the Navier-Stokes equation \( \text{N-S}_{\mathbb{H}^n} \).

However, though the obtained statements of the results on \( \mathbb{H}^n \) are the same as the ones on \( \mathbb{R}^n \), there are some notable differences in our methods that are worth mentioning.

One such difference involves how to handle the term \( Gu \). The estimates used by Kato in [15] for this term differ from our estimates to be presented in Section 2 in that, Kato’s estimates involve norms of both \( u \) and \( \nabla u \). Thus in running his Picard iteration argument for the solution \( u \), Kato necessarily requires information about norm bounds and decay rates for both \( u \) and \( \nabla u \).

In our approach, we found this unnecessary, at least for proving Theorems 1.3 - 1.6 for \( u \) itself. More precisely, by exploiting the fact that
\[
\nabla_{\#} u = (\text{div}(u_{\#} \otimes u_{\#}))^\flat
\]
whenever \( d^*u = 0 \) and using a general smoothing estimate of Pierfelice stated in [27] for the divergence of tensors in \( TM \otimes TM \), we are able to derive estimates on the term \( Gu \) involving only norms of \( u \), and thus are able to prove all of the results for \( u \) in Theorems 1.3 - 1.6 without knowing any information about \( \nabla u \).

Moreover, because Kato’s estimates for \( Gu \) intertwine both \( u \) and \( \nabla u \), Kato has to run concurrent iteration arguments for both \( u \) and \( \nabla u \) in establishing (1.9) and (1.23) from Theorem 1.1. Our approach is simplified in that, once (1.22) from Theorem 1.3 is known.
for finite $q > n$, both the result (1.22) for $q = n$ and the derivative result (1.23) for all finite $q \geq n$ can be established by a straightforward application of Grönwall’s inequality.

Similarly, for proving (K3) in Theorem 1.1, Kato again returns to the sequence of Picard iterates and runs convergence arguments for both $u$ and $\nabla u$. In the same way as discussed above, we are able to avoid this by again using (1.22) from Theorem 1.3 for the case $q > n$ and Grönwall’s inequality.

Another benefit of this simplified approach based on Grönwall’s inequality is that, whereas the times $T$ in (K1) and $T_1$ in (K3) from Theorem 1.1 may be different, with $T_1 < T$ possible, we are able to prove our Theorems 1.4 and 1.6 with the same time of existence $T$. Additionally, while the smallness requirements $\lambda$ in (K2) and $\lambda_1$ in (K4) of Theorem 1.1 may satisfy $\lambda_1 < \lambda$, we are able to prove our Theorems 1.4 and 1.6 with the same $\lambda$.

However, for proving Theorem 1.4, we have so far found no way around either a fixed point argument or Picard iteration, for the proof requires different techniques altogether, such as the Marcinkiewicz interpolation theorem and the Hardy-Littlewood-Sobolev lemma.

Also, as far as we can tell, to prove Theorems 1.7 and 1.8, it is necessary to use estimates for the term $Gu$ in the spirit of Kato that involve norms and decay rates of both $u$ and $\nabla u$, for at certain moments, the powers of $t$ appearing on $\nabla u$ in Theorems 1.3 and 1.6 are required for the convergence of various time integrals appearing throughout the work.

To be more explicit, without using any information concerning $\nabla u$, it is possible to prove a version of Theorem 1.7 that is almost identical, but where the right hand side of each inequality defining $n$, $p$, and $q$ in (1.29) - (1.32) has a $1/2$ in place of the $1$. Then splitting the term $Gu = G_0u + G_1u$, where

$$G_0u(t) = -\int_0^1 e^{-(t-s)L_P} \left( \nabla (u \otimes u) \right)(s) \, ds,$$

$$G_1u(t) = -\int_1^t e^{-(t-s)L_P} \left( \nabla (u \otimes u) \right)(s) \, ds,$$

we were able to use this modified version of Theorem 1.7 to show that $\|G_0u(t)\|_{L^p(H^n)} \to 0$ as $t \to \infty$.

But the difference of $1/2$ mentioned above in the right hand sides of the defining inequalities of this modified version of Theorem 1.7 proved insufficient to show the convergence $\|G_1u(t)\|_{L^p(H^n)} \to 0$ as $t \to \infty$, and thus necessitated a method similar to that used by Kato in [15]. We explore this issue further in the author’s thesis to see if the convergence of $G_1u$ to $0$ in $L^p$ can still be established in the modified setting discussed above.

We also mention here some differences in the semigroup theory between the cases for $\mathbb{R}^n$ and $H^n$. On Euclidean space $\mathbb{R}^n$, it is well-known that the Laplacian $\Delta_{\mathbb{R}^n}$ generates a strongly continuous semigroup $e^{t\Delta_{\mathbb{R}^n}}$ (see, for example, Chapter 9 of [16]). However, for the semigroup $e^{tL}$ studied in this current work, the strong continuity is not automatic, and, as far as we are aware, has not been shown elsewhere for the case $H^n$. Therefore, we prove this fact in the appendix, as mentioned above.

Additionally, in the more general setting of $H^n$, necessary facts such as the $L^p$ boundedness of the projection $P$ defined by (1.7) and the commutation of $P$ and the semigroup $e^{tL}$ are not obvious and without the availability of Fourier transform methods, we must instead resort to spectral theory to prove these results.

1.5. Organization of the article. The rest of the article is structured in the following way. In section 2 we establish notation, present modified versions of Pierfelice’s dispersive
and smooth estimates for the semigroup $e^{tL}$, state and prove functional theoretic properties concerning the operators $P$ and $L$, and prove necessary estimates on the term $Gu$ in $(N-S_{H_n',\text{int}})$. 

With these estimates, we then prove Theorem 1.3 in Section 3 using the method of Picard iteration. Specifically, we use Picard iteration to show (1.22) for the case $q > n$, and once this is established, we prove (1.22) for the case $q = n$ and the derivative result (1.23) using Grönwall’s inequality, as discussed above.

To prove Theorem 1.4, we first follow the ideas of Giga in [11], which involves the Marcinkiewicz interpolation theorem to prove (1.24) for the free solution $u_0$ in $(N-S_{H_n',\text{int}})$. Then, we return to the sequence of Picard iterates and use induction and the Hardy-Littlewood-Sobolev lemma to extend the result to the solution $u$.

Following this, in Section 5 we use Theorem 1.4 to prove Theorem 1.5. Then, as previously discussed, we again apply the result (1.22) for $q > n$ together with Grönwall’s inequality in Section 6 to prove Theorem 1.6.

Finally, in Section 8 we combine the results of Theorems 1.3, 1.6, and 1.7 to prove Theorem 1.8 and to do this, we must derive estimates on the term $Gu$ different than those appearing in Section 2 and that are analogous to the estimates used by Kato in [15].

Acknowledgments. The author would like to sincerely thank his thesis advisor Magdalena Czubak for suggesting the problem, for her patience, and for the many helpful conversations throughout the course of the research.

2. Notation and estimates used

In this section, we first establish notation and then state the aforementioned dispersive and smoothing estimates of Pierfelice from [27] for the semigroup $e^{tL}$. Following this, we must discuss and prove some functional analytic properties of the operator $P$ defined by (1.7), specifically that it is a bounded operator from $L^p$ to $L^p$ and that it commutes with the semigroup $e^{tL}$.

Both of these facts are discussed in [27], though in the more general setting of that work, Pierfelice must rely on the general Riesz transform boundedness results derived by Lohoué in [24]. For our more specific setting of $H^n$, which is a rank-one symmetric space, things are simpler and we can instead rely on the boundedness results for Riesz transforms shown by Strichartz in [34].

With these dispersive and smoothing estimates stated and the required functional analytic properties of $P$ established, we then prove estimates on the term $Gu$ in $(N-S_{H_n',\text{int}})$ and its derivatives, which will be essential to our proofs of Theorems 1.3-1.8.

To begin, we define some constants that appear in the subsequent dispersive and smoothing estimates. Here and in the rest of the paper, a constant depending on the fixed parameters $a_1, a_2, \ldots, a_k$ is denoted by $C(a_1, a_2, \ldots, a_k)$. For fixed $n, p, q > 0$, define

\begin{align}
\gamma(n, p, q) &= \frac{\delta_n}{2} \left[ \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{8}{q} \left( 1 - \frac{1}{p} \right) \right], \\
\beta_1(n, p, q) &= \gamma(n, p, q) + c_0, \\
\beta_2(n, p) &= \frac{4 \delta_n}{p} \left( 1 - \frac{1}{p} \right) + c_0, \\
\beta_3(n, p, q) &= \frac{1}{2} [\gamma(n, q, q) + \gamma(n, p, q)] + c_0, \tag{2.1}
\end{align}
where $c_0$ is a positive constant bounding the Ric operator and the constant $\delta_0 > 0$ depends only on the dimension $n$ (for more information on $c_0$ and $\delta_0$, see [27] and [91]).

As well, going forward, we will use the notation $L^p(\mathbb{H}^n)$ (or sometimes just $L^p$) to denote $L^p$ spaces for functions, forms, and tensors alike and if the distinction is important or necessary at any point, it will be noted.

With this notation established, we can now present the following dispersive and smoothing estimates shown by Pierfelice in [27] for $u_0(t)$ and the specific case $M = \mathbb{H}^n$.

**Theorem 2.1.** [27] For all times $t > 0$ and $a \in L^p(\mathbb{H}^n)$,

$$
\|u_0(t)\|_{L^q(\mathbb{H}^n)} \leq c_n(t) \left(\frac{1}{p} - \frac{1}{q}\right) e^{-t\beta_1(n,p,q)} \|a\|_{L^p(\mathbb{H}^n)}, \quad 1 \leq p \leq q \leq \infty; \quad (2.2)
$$

$$
\|\nabla u_0(t)\|_{L^p(\mathbb{H}^n)} \leq C \max(t^{-1/2},1) e^{-t\beta_2(n,p,q)} \|a\|_{L^p(\mathbb{H}^n)}, \quad 1 < p < \infty; \quad (2.3)
$$

$$
\|\nabla u_0(t)\|_{L^q(\mathbb{H}^n)} \leq c_n(t) \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{n}\right) e^{-t\beta_3(n,p,q)} \|a\|_{L^p(\mathbb{H}^n)}, \quad 1 < p \leq q < \infty; \quad (2.4)
$$

where $c_n(t) = C(n) \max(t^{-1/2},1)$. Moreover, for all tensors $T_0 \in L^p(T^*\mathbb{H}^n \otimes T^*\mathbb{H}^n)$, we have the following general smoothing estimate

$$
\|e^{tL} \nabla^* T_0\|_{L^q(\mathbb{H}^n)} \leq c_n(t) \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{n}\right) e^{-t\beta_1(n,p,q)} \|T_0\|_{L^p(\mathbb{H}^n)}, \quad 1 < p \leq q < \infty. \quad (2.5)
$$

To prove Theorems 1.3-1.8, it suffices to use a simplified form of Theorem 2.1. Specifically, by definition of Pierfelice’s constant $c_n(t)$, estimates using Theorem 2.1, especially global in time estimates, will require considering separate cases where $0 < t < 1$ and $t \geq 1$. To avoid the technical difficulties presented by this and given that the inverse powers of $t$ found in Theorem 2.1 are sufficient to prove our desired results, we instead use the following, which is in the spirit of the estimates used by Kato in [15].

**Theorem 2.2.** For all times $t > 0$ and $a \in L^p(\mathbb{H}^n)$,

$$
\|u_0(t)\|_{L^q(\mathbb{H}^n)} \leq C(n,p,q) t^{-n/p} \left(\frac{1}{p} - \frac{1}{q}\right) \|a\|_{L^p(\mathbb{H}^n)}, \quad 1 \leq p \leq q \leq \infty; \quad (2.6)
$$

$$
\|\nabla u_0(t)\|_{L^p(\mathbb{H}^n)} \leq C(n,p) t^{-1/2} \|a\|_{L^p(\mathbb{H}^n)}, \quad 1 < p < \infty; \quad (2.7)
$$

$$
\|\nabla u_0(t)\|_{L^q(\mathbb{H}^n)} \leq C(n,p,q) t^{-n/p} \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{n}\right) \|a\|_{L^p(\mathbb{H}^n)}, \quad 1 < p \leq q < \infty. \quad (2.8)
$$

Moreover, for all tensors $T_0 \in L^p(T^*\mathbb{H}^n \otimes T^*\mathbb{H}^n)$, we have the following general smoothing estimate

$$
\|e^{tL} \nabla^* T_0\|_{L^q(\mathbb{H}^n)} \leq C(n,p,q) t^{-n/p} \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{n}\right) \|T_0\|_{L^p(\mathbb{H}^n)}, \quad 1 < p \leq q < \infty. \quad (2.9)
$$

**Proof.** To prove this modification of Theorem 2.1 it suffices to show that for $\sigma, \beta > 0$, there exists a uniform constant $C(\sigma, \beta)$ such that

$$
\frac{t^\sigma}{e^{\beta t}} \leq C(\sigma, \beta) \quad \text{for all } t > 0. \quad (2.10)
$$
Letting \( f(t) = t^{\sigma}e^{\beta t} \), elementary calculus shows that \( f \) achieves a global maximum on the interval \((0, \infty)\) at \( t = \sigma/\beta \) with value
\[
C(\sigma, \beta) := f(\sigma/\beta) = \left( \frac{\sigma}{\beta} \right)^{\sigma} e^{-\sigma}.
\] (2.11)

Thus to conclude, we apply this result to the various exponents of \( t \) and the \( \beta_i \)'s from (2.2) - (2.5), using the fact that \( e^{-\beta_i t} \leq 1 \) when handling the case \( 0 \leq t \leq 1 \).

Finally, we must prove estimates on the term \( Gu \) appearing in \((N-S\,H'_{\text{int}})\), which will be crucial to our subsequent application of Picard iteration. To do this, we must prove some useful facts about the operator \( P \) defined in (1.7), the first of which is that it is a bounded operator on \( L^p(T^*\mathbb{H}^n) \). To that end, we recall the following theorem of Strichartz from [34], which concerns the boundedness of Riesz transforms for functions on rank one symmetric spaces.

**Theorem 2.3.** [34] Let \( M \), a complete Riemannian manifold of dimension \( n \), be a rank-one symmetric space. Then for any \( 1 < p < \infty \), \( \nabla(-\Delta_g)^{-1/2} \) is a bounded operator from \( L^p(M) \), the space of \( L^p \) functions on \( M \), to \( L^p(T^0_1 M) \), the space of \( L^p \) tensor fields of rank \((1,0)\).

Using Theorem 2.3, we state and prove the following boundedness result for \( P \) on \( \mathbb{H}^n \) as a corollary.

**Corollary 2.4.** The operator \( P = I - d(-\Delta_g)^{-1/2}d^* \) is a bounded operator on \( L^p(\Omega^1(\mathbb{H}^n)) \), \( 1 < p < \infty \).

**Proof.** Fix \( 1 < p < \infty \). Since \( \mathbb{H}^n \) is a symmetric space of rank one (see for example [5]), Theorem 2.3 applies for this manifold. Moreover, since \( P \) is defined as the identity minus the differential operator \( d(-\Delta_g)^{-1/2}d^* \), it suffices to prove the \( L^p \) boundedness of this latter term.

First we observe that the adjoint of \( d(-\Delta_g)^{-1/2} \) is
\[
(d(-\Delta_g)^{-1/2})^* = ((-\Delta_g)^{-1/2})^*d^* = (-\Delta_g)^{-1/2}d^* = ((-\Delta_g)^{-1/2})d^*,
\]
where we have used that \((-\Delta_g)\) is positive and self-adjoint by [34], which in turn implies its square root \((-\Delta_g)^{-1/2}\) is self-adjoint by the Spectral Theorem (see [4], [28], or [30], for example).

As we have previously shown, for a function \( f \), the total covariant derivative \( \nabla f \) is given by
\[
\nabla f = df,
\]
so that
\[
\|\nabla f\|_{L^p(\mathbb{H}^n)} = \int_{\mathbb{H}^n} g(\nabla f, \nabla f)^{p/2} dV = \int_{\mathbb{H}^n} g(df, df)^{p/2} dV = \|df\|_{L^p(\mathbb{H}^n)}.
\]
(2.12)

Hence Theorem 2.3 also shows \( d(-\Delta_g)^{-1/2} \) is a bounded operator from \( L^p \) functions to \( L^p \) differential 1-forms on \( \mathbb{H}^n \). Explicitly, we have the following
\[
\|d(-\Delta_g)^{-1/2}\|_{op} \leq C(p),
\]
(2.13)
where $\| \cdot \|_{op}$ denotes the operator norm.

As well, letting $p'$ denote the Hölder conjugate of $p$, we also have by Theorem 2.3 that $d(-\Delta_g)^{-1/2}$ is a bounded operator from $L^p$ functions to $L^{p'}$ differential 1-forms. And since $\|d(-\Delta_g)^{-1/2}\|_{op} = \|(d(-\Delta_g)^{-1/2})^*\|_{op}$, the dual operator

$$(-\Delta_g)^{-1/2} d^* = (d(-\Delta_g)^{-1/2})^*$$ (2.14)

is thus a bounded operator from $L^p$ differential 1-forms (the dual space of $L^{p'}$ differential 1-forms) to $L^p$ functions (the dual space of $L^{p'}$ functions), which therefore gives

$$\|(d(-\Delta_g)^{-1/2} d^*)\|_{op} \leq C(p').$$ (2.15)

Finally, since the Spectral Theorem and functional calculus allow us to write

$$(\Delta_g)^{-1} = (\Delta_g)^{-1/2}(\Delta_g)^{-1/2},$$

by (2.14) and (2.15), we have that for $u \in L^p(\Omega^1(\mathbb{H}^n))$,

$$\|d(-\Delta_g)^{-1} d^* u\|_{L^p(T^*\mathbb{H}^n)} \leq C(p)C(p')\|u\|_{L^p(\Omega^1(\mathbb{H}^n))}.$$ (2.16)

We conclude that $d(-\Delta_g)^{-1} d^*$ is a bounded operator on $L^p(\Omega^1(\mathbb{H}^n))$, and thus so is $\mathbb{P}$.

Having shown $\mathbb{P}$ is bounded on $L^p(\Omega^1(\mathbb{H}^n))$ for $1 < p < \infty$, we can now show $\mathbb{P}$ commutes with the semigroup $e^{tL}$.

**Lemma 2.5.** On $\mathbb{H}^n$, $e^{tL} \mathbb{P} = \mathbb{P} e^{tL}$ for any $t \in \mathbb{R}$.

**Proof.** For a fixed $t \in \mathbb{R}$, the function $e^t : \mathbb{R} \to \mathbb{R}$ is Borel measurable on $\mathbb{R}$ and since we have shown in Corollary 2.4 that $\mathbb{P}$ is bounded on the Hilbert space $L^2(T^*\mathbb{H}^n)$, it suffices by the Spectral Theorem to show

$$L \mathbb{P} = \mathbb{P} L,$$ (2.17)

where the domains $D(\mathbb{P}), D(L), D(L \mathbb{P})$, and $D(\mathbb{P} L)$ are all taken to be $C_c^\infty(T^*\mathbb{H}^n)$ (see, for instance, Theorem 4.11 in Chapter X of [4]).

Since we are not assuming any divergence free condition here, we must use the full definition of $L$ given in (1.4), which for $\mathbb{H}^n$ can be written as

$$L = -\Delta_{H,1} - dd^* - 2(n-1).$$ (2.18)

Therefore, letting $u \in C_c^\infty(T^*\mathbb{H}^n)$, and using that $d\Delta_{H,k} = \Delta_{H,k+1}d$ and $d^*\Delta_{H,k} = \Delta_{H,k-1}d^*$, we have

$$L \mathbb{P} u = L(u - d(-\Delta_g)^{-1}d^* u)$$

$$= Lu + \Delta_{H,1}d(-\Delta_g)^{-1}d^* u + dd^* d(-\Delta_g)^{-1}d^* u + 2(n-1)d(-\Delta_g)^{-1}d^* u$$

$$= Lu + \Delta_{H,0}d(-\Delta_g)^{-1}d^* u + d(-\Delta_g)(-\Delta_g)^{-1}d^* u + d(-\Delta_g)^{-1}d^* (2(n-1)u)$$

$$= Lu + d(-\Delta_g)(-\Delta_g)^{-1}d^* u + d(-\Delta_g)(-\Delta_g)^{-1}d^* u + d(-\Delta_g)^{-1}d^* (2(n-1)u)$$

$$= Lu + d(-\Delta_g)^{-1}\Delta_{H,0}d^* u + d(-\Delta_g)^{-1}d^* dd^* u + d(-\Delta_g)^{-1}d^* (2(n-1)u)$$

$$= Lu + d(-\Delta_g)^{-1}d^* \Delta_{H,1} u + d(-\Delta_g)^{-1}d^* dd^* u + d(-\Delta_g)^{-1}d^* (2(n-1)u)$$

$$= Lu - d(-\Delta_g)^{-1}d^* (\Delta_{H,1} u - dd^* u - 2(n-1)u)$$

$$= Lu - d(-\Delta_g)^{-1}d^* Lu$$

$$= \mathbb{P} Lu.$$
With Lemma \ref{lemma:2.5} in hand, we next show the following useful and necessary result for the divergence of a solution $u$ defined by the integral equation (1.8).

**Proposition 2.6.** If $u$ is defined by $[\text{N-S}_{\mathbb{H}^n}]$, then $d^* u = 0$.

**Proof.** It follows by the definition of the operator $P$ that for a 1-form $w$,

$$d^* (Pw) = d^* w - d^* (d(-\Delta_g)^{-1} d^* w)$$

$$= d^* w - (d(-\Delta_g))^{-1} d^* w$$

$$= d^* w - d^* w$$

$$= 0.$$  \hfill (2.20)

As well, for our initial condition $a \in L^p(\mathbb{H}^n)$,

$$P a = a - d(-\Delta_g)^{-1} d^* a = a,$$  \hfill (2.21)

since $d^* a = 0$. Thus by Lemma \ref{lemma:2.5}

$$d^* u(t) = d^* e^{tL} a - d^* \int_0^t e^{(t-s)L} P \left( \text{div}(u \otimes u) \right)(s) \, ds$$

$$= d^* e^{tL} (Pa) - \int_0^t d^* e^{(t-s)L} P \left( \text{div}(u \otimes u) \right)(s) \, ds$$

$$= d^* P e^{tL} a - \int_0^t d^* P e^{(t-s)L} \left( \text{div}(u \otimes u) \right)(s) \, ds$$

$$= 0,$$

where we have used (2.20) in the last step. \hfill \square

With these facts about the operator $P$ established, we can now state and prove the following important estimate on the term $G u(t)$, which will be used often in the sequel.

**Lemma 2.7.** Let $u$ be defined by equation (1.8). Then $G u$ and $\nabla G u$ satisfy the following estimates for $0 < \gamma \leq \alpha + \zeta < n$.

1. $$\|G u(t)\|_{L^{n/\gamma}} \leq C(n, \alpha, \gamma, \zeta) \int_0^t (t-s)^{-\frac{\alpha + \zeta - n+1}{2}} \|u(s)\|_{L^{n/\alpha}} \|u(s)\|_{L^{n/\zeta}} \, ds,$$  \hfill (2.23)

2. $$\|G u(t) - G v(t)\|_{L^{n/\gamma}} \leq C(n, \alpha, \gamma, \zeta) \int_0^t (t-s)^{-\frac{\alpha + \zeta - n+1}{2}} (\|u\|_{L^{n/\alpha}} + \|v\|_{L^{n/\alpha}}) \|u - v\|_{L^{n/\zeta}} \, ds,$$  \hfill (2.24)

3. $$\|\nabla G u(t)\|_{L^{n/\gamma}} \leq C(n, \alpha, \gamma, \zeta) \int_0^t (t-s)^{-\frac{\alpha + \zeta - n+1}{2}} \|u(s)\|_{L^{n/\alpha}} \|\nabla u(s)\|_{L^{n/\zeta}} \, ds,$$  \hfill (2.25)

4. $$\|\nabla G u(t) - \nabla G v(t)\|_{L^{n/\gamma}} \leq C(n, \alpha, \gamma, \zeta) \int_0^t (t-s)^{-\frac{\alpha + \zeta - n+1}{2}} (\|u\|_{L^{n/\alpha}} \|\nabla u - \nabla v\|_{L^{n/\zeta}}$$

$$+ \|\nabla v\|_{L^{n/\zeta}} \|u - v\|_{L^{n/\alpha}}) \, ds.$$  \hfill (2.26)
Proof. We first observe that by Proposition 2.6 if $u$ is defined by (1.8), then $d^*u = 0$, so that we may move freely back and forth between
\[ \nabla u^\# \quad \text{and} \quad (\text{div}(u^\# \otimes u^\#))^b \]
as needed, since these expressions are equal whenever $d^*u = 0$. Next we observe that for $T \in \mathbb{T}^n \otimes T^*\mathbb{T}^n$,
\[ \nabla^* T = -\text{div}(T^\#), \quad \text{(2.27)} \]
so that (2.9) implies
\[ \| e^{tL} \text{div}(T^\#) \|_{L^q(\mathbb{H}^n)} \leq C(n, p, q) t^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} + \frac{1}{n} \right)} \| T^\# \|_{L^p(\mathbb{H}^n)} \quad 1 < p \leq q < \infty, \quad \text{(2.28)} \]
for all tensors $T^\# \in TM \otimes TM$, where now $\#$ represents the induced musical isomorphism $(\cdot)\# : \mathbb{T}^n \otimes T^*\mathbb{T}^n \to T^*\mathbb{T}^n \otimes \mathbb{T}^n$ (see, for example, 2.4).

By commuting $\mathbb{P}$ and $e^{(t-s)L}$ according to Lemma 2.5, using the $L^q$ boundedness of $\mathbb{P}$ from Corollary 2.4 and applying (2.28) along with Hölder’s inequality, we have that
\[ \| Gu(t) \|_{L^q(\mathbb{H}^n)} \leq C(n, p, q) \int_0^t (t-s)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} + \frac{1}{n} \right)} \| u^\# \otimes u^\#(s) \|_{L^p(\mathbb{H}^n)} ds \]

\[ \leq C(n, \mu, \lambda, q) \int_0^t (t-s)^{-\frac{n}{2} \left( \frac{1}{p} + \frac{1}{q} - \frac{1}{n} + \frac{1}{\lambda} \right)} \| u(s) \|_{L^q(\mathbb{H}^n)} \| u(s) \|_{L^{\lambda}(\mathbb{H}^n)} ds, \quad \text{(2.29)} \]
where $\lambda, \mu > 0$ and $\frac{1}{p} + \frac{1}{\lambda} = \frac{1}{q}$ and where we have also used that $(\cdot)\#$ and $(\cdot)^b$ are isomorphisms.

Taking this inequality and setting
\[ q = \frac{n}{\gamma}, \quad \mu = \frac{n}{\alpha}, \quad \lambda = \frac{n}{\zeta}, \quad \text{(2.30)} \]
where $\gamma \leq \alpha + \zeta < n$, proves (2.23). As well, using the fact that
\[ u^\# \otimes u^\# - v^\# \otimes v^\# = u^\# \otimes (u - v)^\# - (v - u)^\# \otimes v^\#, \]
a slight modification of the argument leading to (2.23) gives the difference estimate (2.24).

The derivative estimate (2.25) is simpler, for in this case we only need the $L^1$ boundedness of $\mathbb{P}$ from Corollary 2.4 and the following pointwise bound for $C^1$ 1-forms $v_1, v_2$:
\[ g(\nabla_{v_1} v_1, \nabla_{v_2} v_2) \leq |\nabla v_1|^2 |v_2|^2, \quad \text{(2.31)} \]
which can be shown much like it would in the Euclidean case by using geodesic normal coordinates (here, $|\cdot| = \sqrt{g(\cdot, \cdot)}$).

Indeed, using Corollary 2.4 and the pointwise estimate (2.31), passing the covariant derivative through the time integral defining $Gu$, applying estimate (2.8) from Theorem 2.2 and using Hölder’s inequality, we have
\[ \| \nabla Gu(t) \|_{L^q(\mathbb{H}^n)} \leq \int_0^t \| \nabla e^{(t-s)L}(\mathbb{P} \nabla u^\# u(s)) \|_{L^q(\mathbb{H}^n)} ds \]

\[ \leq C(n, p, q) \int_0^t (t-s)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} + \frac{1}{n} \right)} \| \nabla u^\# \|_{L^p(\mathbb{H}^n)} ds \]

\[ \leq C(n, \mu, \lambda, q) \int_0^t (t-s)^{-\frac{n}{2} \left( \frac{1}{p} + \frac{1}{q} - \frac{1}{n} + \frac{1}{\lambda} \right)} \| u(s) \|_{L^q(\mathbb{H}^n)} \| \nabla u(s) \|_{L^{\lambda}(\mathbb{H}^n)} ds, \quad \text{(2.32)} \]
where as before, \( \lambda, \mu > 0 \) and \( \frac{1}{\mu} + \frac{1}{\lambda} = \frac{1}{p} \). We finish the proof of (2.25) by once again setting
\[
q = \frac{n}{\gamma}, \quad \mu = \frac{n}{\alpha}, \quad \lambda = \frac{n}{\zeta}.
\]

(2.33)

It remains to show (2.26) and for this, we can proceed much as in the proof of (2.25), again using the pointwise estimate (2.31) and the fact that
\[
\nabla u^\# u - \nabla v^\# v = \nabla u^\# (u - v) - \nabla (v - u)^\# v.
\]

□

3. Picard Iteration on \( H^n \) and Proof of Theorem 1.3

3.1. Local existence for \( q > n \). We will solve (1.8) and thus prove Theorem 1.3 by Picard iteration. Starting with \( u_0(t) = e^{tL}a \), where \( a \in L^n(H^n) \) is the initial condition, we construct the following sequence
\[
u_{k+1} = u_0 + G u_k, \quad k = 0, 1, 2, 3, \ldots
\]

(3.1)

Let \( 0 < \delta < 1 \) be fixed. We will first show by induction that the sequence defined by (3.1) exists and satisfies
\[
t^{\left(\frac{1}{2} - \frac{\delta}{2}\right)} u_k \in L^n([0, T], L^{n/\delta}(H^n)),
\]

(3.2)

with norm
\[
M_k := \sup_{0 \leq t < T} t^{\left(\frac{1}{2} - \frac{\delta}{2}\right)} \| u_k(t) \|_{L^{n/\delta}(H^n)}, \quad k = 0, 1, 2, 3, \ldots
\]

(3.3)

where \( T > 0 \) is to be chosen. For \( k = 0 \), we set \( p = n \) and \( q = n/\delta \) in (2.6) to get
\[
\| u_0(t) \|_{L^{n/\delta}(H^n)} \leq C(n, \delta) t^{-\left(\frac{1}{2} - \frac{\delta}{2}\right)} \| a \|_{L^n(H^n)}.
\]

(3.4)

Thus
\[
M_0 \leq C(n, \delta) \| a \|_{L^n(H^n)}
\]

(3.5)

and (3.2) is satisfied for \( k = 0 \).

Assuming now that (3.2) is true for \( k > 0 \), we next show it holds for \( k + 1 \). Since the first term in (3.1) has just been estimated, we must estimate the term \( Gu_k(t) \) in the \( L^{n/\delta}(H^n) \) norm. To do this, we set \( \alpha = \gamma = \zeta = \delta \) in (2.23), which gives
\[
\| Gu_k(t) \|_{L^{n/\delta}(H^n)} \leq C(n, \delta) \int_0^t (t - s)^{-\left(1 + \delta\right)/2} \left( \| u_k(s) \|_{L^{n/\delta}(H^n)} \right)^2 ds
\]
\[
= C(n, \delta) M_k^2 \int_0^t (t - s)^{-\left(1 + \delta\right)/2} s^{-\left(\frac{1}{2} - \frac{\delta}{2}\right)} ds
\]
\[
\leq C(n, \delta) M_k^2 \int_0^t (t - s)^{-\left(1 + \delta\right)/2} s^{-1 - \delta} ds.
\]

(3.6)

To compute the terminal integral in this inequality, we use the Beta function (see, for instance, [1]), which is defined for \( x, y \in \mathbb{C} \) with \( \text{Re}(x), \text{Re}(y) > 0 \) by
\[
B(x, y) = \int_0^1 \tau^{x-1}(1 - \tau)^{y-1} d\tau.
\]

(3.7)
After performing the substitution \( \tau = s/t \), the terminal integral in (3.6) can be rewritten as
\[
\int_0^t (t - s)^{(1+\delta)/2}s^{-1+\delta} ds = t^{-\left(\frac{1}{2} + \frac{\delta}{2}\right)} \int_0^1 \tau^{\delta-1}(1-\tau)^{-\left(\frac{1}{2} + \frac{\delta}{2}\right)} d\tau
\]
\[
= t^{-\left(\frac{1}{2} + \frac{\delta}{2}\right)} B\left(\delta, \frac{1-\delta}{2}\right). \tag{3.8}
\]
Hence
\[
\|G u_k(t)\|_{L^{n/\delta}(\mathbb{R}^n)} \leq C(n, \delta) M_k^2 t^{-\left(\frac{1}{2} + \frac{\delta}{2}\right)}, \tag{3.9}
\]
where we have absorbed the convergent Beta function \( B\left(\delta, \frac{1-\delta}{2}\right) \) into the constant \( C(n, \delta) \). Thus we have shown
\[
t^{\left(\frac{1}{2} + \frac{\delta}{2}\right)} \|u_{k+1}(t)\|_{L^{n/\delta}(\mathbb{R}^n)} \leq M_0 + C(n, \delta) M_k^2,
\]
which proves (3.2) for \( k + 1 \) and leads to the recurrence inequality
\[
M_{k+1} \leq M_0 + C(n, \delta) M_k^2. \tag{3.11}
\]
We claim this defines a bounded sequence \( \{M_k\} \) where
\[
M_k < M := \frac{1}{2C(n, \delta)}, \quad k = 0, 1, 2, 3, \ldots, \tag{3.12}
\]
provided
\[
M_0 < \frac{1}{4C(n, \delta)}, \tag{3.13}
\]
which is possible according to (3.3) by choosing \( T > 0 \) sufficiently small. Indeed, for the base case, we have
\[
M_0 < \frac{1}{4C(n, \delta)} < \frac{1}{2C(n, \delta)},
\]
so that if (3.12) is true for \( k > 0 \), then by the recurrence inequality (3.11),
\[
M_{k+1} \leq M_0 + C(n, \delta) M_k^2
\]
\[
< \frac{1}{4C(n, \delta)} + C(n, \delta) \left(\frac{1}{2C(n, \delta)}\right)^2
\]
\[
= \frac{1}{2C(n, \delta)}, \tag{3.14}
\]
which verifies (3.12) for \( k + 1 \) and thus the entire sequence of norms \( \{M_k\} \).

With \( T > 0 \) chosen so that (3.13) holds, we next show (3.12) implies the sequence \( u_k \) defined by (3.1) converges uniformly. Define a new sequence
\[
w_k(t) = u_k(t) - u_{k-1}(t), \quad k = 0, 1, 2, \ldots, \tag{3.15}
\]
where \( u_{-1}(t) = 0 \). We will prove by induction that for each \( k = 0, 1, 2, \ldots, \)
\[
\|w_k(t)\|_{L^{n/\delta}(\mathbb{R}^n)} \leq \frac{M(2C(n, \delta)M)^k}{t^{(1-\delta)/2}} \tag{3.16}
\]
for some constant \( C(n, \delta) \). For \( k = 0 \), we have by (3.12) that \( M_0 \leq M \). Hence for \( 0 \leq t \leq T, \)
\[
t^{(1-\delta)/2}\|w_0(t)\|_{L^{n/\delta}(\mathbb{R}^n)} = t^{(1-\delta)/2}\|u_0(t)\|_{L^{n/\delta}(\mathbb{R}^n)} \leq M, \tag{3.17}
\]
so that
\[
\|w_0(t)\|_{L^{n/\delta}(\mathbb{H}^n)} \leq t^{-1-\delta/2}M \\
\leq \frac{M(2C(n, \delta)M)^0}{t^{(1-\delta/2)}}.
\] (3.18)

Assuming that (3.16) holds for \(k > 0\), we estimate the \((k+1)\)-th term. Setting \(\alpha = \gamma = \zeta = \delta\) in (2.24) and using (3.12), we can proceed as in (3.6) to get
\[
\|w_{k+1}(t)\|_{L^{n/\delta}(\mathbb{H}^n)} = \|u_{k+1}(t) - u_k(t)\|_{L^{n/\delta}(\mathbb{H}^n)} \\
= \|Gu_k(t) - Gu_{k-1}(t)\|_{L^{n/\delta}(\mathbb{H}^n)} \\
\leq C(n, \delta) \int_0^t (t-s)^{-1+\delta/2}(\|u_k(s)\|_{L^{n/\delta}} + \|u_{k-1}(s)\|_{L^{n/\delta}})\|w_k(s)\|_{L^{n/\delta}} ds \\
\leq 2C(n, \delta)M \int_0^t (t-s)^{-1+\delta/2}s^{-1+\delta/2} \|w_k(s)\|_{L^{n/\delta}(\mathbb{H}^n)} ds \\
\leq (2C(n, \delta)M)(M(2C(n, \delta)M)^k) \int_0^t (t-s)^{-1+\delta/2}s^{-1+\delta/2} ds \\
= (2C(n, \delta)M)(M(2C(n, \delta)M)^k) \int_0^t (t-s)^{-1+\delta/2} ds \\
= \frac{M(2C(n, \delta)M)^{k+1}}{t^{(1-\delta/2)}},
\] (3.19)

where in the last step, we have computed the integral using the same Beta function computation as employed in (3.8) and have absorbed the resulting convergent Beta function into the general constant \(C(n, \delta)\).

This verifies (3.16) for all \(k = 0, 1, 2 \ldots\) and since \(2C(n, \delta)M < 1\) by assumption, we have that \(\sum_{j=0}^{\infty} w_k\) converges uniformly and absolutely to 1-form \(u\) such that \(t^{\alpha/2}u \in L^\infty([0, T], L^{n/\delta}(\mathbb{H}^n))\) and such that
\[
u = \lim_{k \to \infty} \sum_{j=0}^{k} w_j \\
= \lim_{k \to \infty} u_k.
\] (3.20)

3.2. Uniqueness. Having concluded the local existence part of Theorem 1.2, we now deal with uniqueness. Let \(a, a' \in L^n(\mathbb{H}^n)\) and suppose \(u\) and \(v\) are the solutions to (1.8) for these initial data, with respective times of existence \(T\) and \(T'\). Then
\[
u(t) = u_0(t) + Gu(t), \\
v(t) = v_0(t) + Gv(t),
\] (3.21)

where \(u_0(t) = e^{tL}a\) and \(v_0(t) = e^{tL}a'\). Letting
\[
w(t) = u(t) - v(t),
\]
using dispersive estimate \((2.6)\) and difference estimate \((2.24)\) with \(\alpha = \gamma = \zeta = \delta\), and performing similar calculations as above, we have
\[
\|w(t)\|_{L^{n/\delta}([0,T])} \leq \|e^{tL}(a - a')\|_{L^{n/\delta}([0,T])} + C(n,\delta)M \int_0^t (t-s)^{-\frac{1}{2}} \|w(s)\|_{L^{n/\delta}([0,T])}ds
\]
\[
= C(n,\delta)t^{-\frac{1}{2}} \|a - a'\|_{L^n([0,T])} + t^{-\frac{1}{2}} \int_0^t g(t,s)s^{-\frac{1}{2}} \|w(s)\|_{L^{n/\delta}([0,T])}ds,
\]
(3.22)
where \(g(t,s) = C(n,\delta)M(t-s)^{-(\delta+1)/2}s^{-(1-\delta)/2}\). Multiplying through by \(t^{(1-\delta)/2}\) in the above gives
\[
t^{\left(\frac{1}{2}-\frac{\delta}{2}\right)} \|w(t)\|_{L^{n/\delta}([0,T])} \leq C(n,\delta)\|a - a'\|_{L^n([0,T])}\exp\left(\int_0^t g(t,s)ds\right),
\]
(3.23)
and after applying Grönwall’s inequality, we get
\[
t^{\left(\frac{1}{2}-\frac{\delta}{2}\right)} \|w(t)\|_{L^{n/\delta}([0,T])} \leq C(n,\delta)\|a - a'\|_{L^n([0,T])}\exp\left(\int_0^t g(t,s)ds\right),
\]
(3.24)
so that defining \(\tilde{T} = \min\{T, T'\}\),
\[
\sup_{0 \leq t < \tilde{T}} t^{\left(\frac{1}{2}-\frac{\delta}{2}\right)} \|w(t)\|_{L^{n/\delta}([0,T])} \leq C(n,\delta)\|a - a'\|_{L^n([0,T])}\exp\left(\int_0^t g(t,s)ds\right)
\]
\[
= C(n,\delta)\|a - a'\|_{L^n([0,T])}\exp\left(C(n,\delta)Mt^{(1-\delta)/2}\int_0^t (t-s)^{-(\delta+1)/2}s^{-(1-\delta)/2}ds\right)
\]
\[
= C(n,\delta)\|a - a'\|_{L^n([0,T])}\exp\left(C(n,\delta)Mt^{(1-\delta)/2}t^{-(1-\delta)/2}B\left(\delta, \frac{1-\delta}{2}\right)\right)
\]
\[
= C(n,\delta)\|a - a'\|_{L^n([0,T])}\exp\left(C(n,\delta)MB\left(\delta, \frac{1-\delta}{2}\right)\right)
\]
\[
= C(n,\delta, M)\|a - a'\|_{L^n([0,T])},
\]
(3.25)
where we have used the same Beta function computation as in \((3.8)\). Thus in the case where \(a = a'\), \((3.25)\) shows \(w = 0\) in \(L^\infty([0,\tilde{T}), L^{n/\delta}([0,T]))\), so that \(u = v\) and we can take \(T = T' = \tilde{T}\). This proves uniqueness.

3.3. Continuous dependence on initial data. Let \(\varepsilon > 0\) and suppose \(a, a' \in L^n([0,T])\). If \(u\) is the solution constructed in the preceding sections for \(a\) with time of existence \(T\) and if \(v\) is the corresponding solution for \(a'\) with time of existence \(T'\), then letting \(\tilde{T} = \min\{T, T'\}\), our Grönwall estimate \((3.25)\) shows
\[
\sup_{0 \leq t < \tilde{T}} t^{\left(\frac{1}{2}-\frac{\delta}{2}\right)} \|u(t) - v(t)\|_{L^{n/\delta}([0,T])} \leq C(n,\delta, M)\|a - a'\|_{L^n([0,T])}.
\]
(3.26)
Therefore, if \(a\) and \(a'\) are such
\[
\|a - a'\|_{L^n([0,T])} < \frac{\varepsilon}{C(n,\delta, M)},
\]
(3.27)
then
\[
\sup_{0 \leq t < \tilde{T}} t^{\left(\frac{1}{2}-\frac{\delta}{2}\right)} \|u(t) - v(t)\|_{L^{n/\delta}} < \varepsilon,
\]
(3.28)
which establishes continuous dependence on the initial data.

3.4. **Global well-posedness.** For global existence and uniqueness, we next observe by virtue of (3.5), (3.12), and (3.13) that if

\[ \|a\|_{L^n(H^n)} < \frac{1}{4(C(n, \delta))^2}, \]  

then \( M_0 < 1/4C(n, \delta) \) for any choice of \( T > 0 \). In this case, the uniform convergence of \( u_k \) to \( u \) can be shown on the interval \([0, \infty)\).

3.5. **The case \( q = n \).**

3.5.1. **Local Existence.** So far, we have shown existence and uniqueness of a solution \( u \) of (1.8) satisfying

\[ \tau t^{\left(\frac{1}{2} - \frac{n}{2q}\right)} u \in L^\infty([0, T), L^q(H^n)), \quad n < q < \infty, \]  

since, given any \( q > n, \delta = n/q \) satisfies 0 < \( \delta < 1 \) as assumed above. As well, we have shown we can extend \( T \) to infinity if \( \|a\|_{L^n} \) is sufficiently small. Therefore, it remains to show that the limit \( u \) satisfies \( u \in L^\infty([0, T), L^n(H^n)) \). To that end, by (2.6) with \( q = p = n \),

\[ \|u_0(t)\|_{L^n(H^n)} \leq C(n)\|a\|_{L^n(H^n)}. \]  

Next, if \( T \) is chosen so that (3.13) holds, then (3.2) and (3.12) imply

\[ \|u(t)\|_{L^n(H^n)} \leq M T^{-\left(\frac{1}{2} - \frac{n}{2q}\right)}, \quad \text{for } 0 \leq t < T. \]  

Hence, using (2.23) with \( \alpha = \delta \) for a fixed 0 < \( \delta < 1 \) and \( \gamma = \zeta = 1 \), it follows that

\[ \|Gu(t)\|_{L^n(H^n)} \leq C(n, \delta) \int_0^t (t - s)^{-\frac{(\delta + 1)}{2}} \|u(s)\|_{L^n(H^n)} \|s\|_{L^n(H^n)} ds \leq C(n, \delta)M \int_0^t (t - s)^{-\frac{(\delta + 1)}{2}} s^{-\frac{(1 - \delta)}{2}} \|u(s)\|_{L^n(H^n)} ds. \]  

By combining this estimate with (3.31) and using that \( u(t) = u_0(t) + Gu(t) \), we have shown that

\[ \|u(t)\|_{L^n(H^n)} \leq C(n)\|a\|_{L^n(H^n)} + C(n, \delta)M \int_0^t (t - s)^{-\frac{(\delta + 1)}{2}} s^{-\frac{(1 - \delta)}{2}} \|u(s)\|_{L^n(H^n)} ds, \]  

so that by Grönwall’s inequality,

\[ \|u(t)\|_{L^n(H^n)} \leq C(n)\|a\|_{L^n(H^n)} \exp\left(C(n, \delta)M \int_0^t (t - s)^{-\frac{(\delta + 1)}{2}} s^{-\frac{(1 - \delta)}{2}} ds\right) \leq C(n)\|a\|_{L^n(H^n)} \exp\left(C(n, \delta)MB\left(\frac{\delta + 1}{2}, \frac{1 - \delta}{2}\right)\right) \]  

where we have made the substitution \( \tau = s/t \). This shows

\[ \sup_{0 \leq t < T} \|u(t)\|_{L^n(H^n)} \leq C(n)\|a\|_{L^n(H^n)} \exp\left(C(n, \delta)MB\left(\frac{\delta + 1}{2}, \frac{1 - \delta}{2}\right)\right), \]  

so that \( u \in L^\infty([0, T), L^n(H^n)) \). Moreover if \( \|a\|_{L^n(H^n)} \) satisfies the smallness condition (3.29), then

\[ \|u(t)\|_{L^n(H^n)} < Mt^{-\left(\frac{1}{2} - \frac{n}{2}\right)}, \quad \text{for } 0 \leq t < \infty \]  

and the same arguments show \( u \in L^\infty([0, \infty), L^n(H^n)) \).
3.5.2. Uniqueness and continuous dependence on initial data. For the case \( q = n \), the proofs for uniqueness and continuous dependence on initial data are analogous to the proofs for the case \( q > n \).

3.6. Well-posedness for \( \nabla u \). It remains to prove (1.23) holds for \( n \leq q \) and for this, we first take \( p = n \) in smoothing estimate (2.8) to get

\[
\|\nabla u_0(t)\|_{L^q(\mathbb{H}^n)} \leq C(n, q) t^{-\frac{q}{2}} \left( \frac{q}{2} \right) \|a\|_{L^n(\mathbb{H}^n)} \\
\leq C(n, q) t^{-1+\frac{n}{2q}} \|a\|_{L^n(\mathbb{H}^n)}. \tag{3.36}
\]

Next, choose \( 0 < \delta < 1 \) such that

\[
n < q < \frac{n}{1-\delta}. \tag{3.37}
\]

Note that if \( q = n \), then any \( 0 < \delta < 1 \) will work, whereas if \( q > n \), we can choose \( 0 < \varepsilon < \frac{n}{q} \) and define

\[
\delta = 1 + \varepsilon - \frac{n}{q}.
\]

Then \( 0 < \delta < 1 \) and

\[
q = \frac{n}{1-\delta + \varepsilon} < \frac{n}{1-\delta}.
\]

Therefore, if \( T > 0 \) is chosen so that (3.13) holds, then (3.2) and (3.12) imply for our chosen \( \delta \) that

\[
\|u(t)\|_{L^{n/\delta}(\mathbb{H}^n)} \leq M t^{-(1-\delta)/2}, \text{ for } 0 \leq t < T. \tag{3.38}
\]

Combining this fact and by taking \( \gamma = \zeta = n/q \) and \( \alpha = \delta \) in (2.25), it follows that

\[
\|\nabla G u(t)\|_{L^q(\mathbb{H}^n)} \leq C(n, q, \delta) \int_0^t (t-s)^{-(1+\delta)/2} \|u(s)\|_{L^{n/\delta}(\mathbb{H}^n)} \|\nabla u(s)\|_{L^q(\mathbb{H}^n)} ds \leq C(n, q, \delta) M \int_0^t (t-s)^{-(\delta+1)/2} s^{-(1-\delta)/2} \|\nabla u(s)\|_{L^q(\mathbb{H}^n)} ds. \tag{3.39}
\]

Applying this estimate together with (3.36) and using that \( \nabla u(t) = \nabla u_0(t) + \nabla G u(t) \), we have

\[
t^{-\frac{n}{2q}} \|\nabla u(t)\|_{L^q(\mathbb{H}^n)} \leq C(n, q, \delta) \|a\|_{L^q(\mathbb{H}^n)} + C(n, q, \delta) M \int_0^t g(t, s)s^{1-\frac{n}{2q}} \|\nabla u(s)\|_{L^q(\mathbb{H}^n)} ds, \tag{3.40}
\]

where \( g(t, s) = (t-s)^{-(\delta+1)/2} s^{-(1-\delta)/2} t^{1-\frac{n}{2q}} s^{1-\frac{n}{2q}} \). By Grönwall’s inequality,

\[
t^{-\frac{n}{2q}} \|\nabla u(t)\|_{L^q(\mathbb{H}^n)} \leq C(n, q, \delta) \|a\|_{L^q(\mathbb{H}^n)} \exp \left( C(n, q, \delta) M \int_0^t g(t, s) ds \right) \]

\[
= C(n, q, \delta) \|a\|_{L^q(\mathbb{H}^n)} \exp \left( C(n, q, \delta) M t^{-\frac{n}{2q}} \int_0^t (t-s)^{-(\delta+1)/2} s^{-(1-\delta)/2} s^{-1+\frac{n}{2q}} ds \right) \]

\[
= C(n, q) \|a\|_{L^q(\mathbb{H}^n)} \exp \left( C(n, q, \delta) M t^{-\frac{n}{2q}} \int_0^t (t-s)^{-(\delta+1)/2} s^{\left(\frac{\delta}{2} + \frac{n}{2q} - \frac{1}{2}\right)} ds \right) \]

\[
= C(n, q) \|a\|_{L^q(\mathbb{H}^n)} \exp \left( C(n, q, \delta) M t^{-\frac{n}{2q}} t^{-1+\frac{n}{2q}} B \left( \frac{\delta}{2} + \frac{n}{2q} - \frac{1}{2}, 1 - \frac{\delta}{2} \right) \right) \]

\[
= C(n, q) \|a\|_{L^q(\mathbb{H}^n)} \exp \left( C(n, q, \delta) MB \left( \frac{\delta}{2} + \frac{n}{2q} - \frac{1}{2}, 1 - \frac{\delta}{2} \right) \right). \tag{3.41}
\]
As well, the assumption (3.37) implies
\[ \frac{\delta}{2} + \frac{n}{2q} - \frac{1}{2} > 0, \]
so that \( B\left(\frac{\delta}{2} + \frac{n}{2q} - \frac{1}{2}, \frac{1-\delta}{2}\right) \) converges.

We conclude that
\[
\sup_{0 \leq t < T} t^{-\frac{n}{2q}} \| \nabla u(t) \|_{L^q(\mathbb{H}^n)} \leq C(n, q) \| a \|_{L^n(\mathbb{H}^n)} \exp\left( C(n, q, \delta) MB\left(\frac{\delta}{2} + \frac{n}{2q} - \frac{1}{2}, \frac{1-\delta}{2}\right) \right),
\]
so that \( t^{-\frac{1}{2q}} \nabla u \in L^\infty([0, T), L^q(\mathbb{H}^n)) \). Moreover, as we have previously argued, if \( \| a \|_{L^n(\mathbb{H}^n)} \) satisfies the smallness condition (3.29), then we can take \( T = \infty \).

4. Proof of Theorem 1.4

We now show the solution \( u \) found in the previous section satisfies Theorem 1.4. Following closely the methods of Giga in [11], the first step is to show by induction that for some \( T_1 > 0 \) to be chosen later, the sequence (3.1) satisfies
\[
(4.1)
\]
For the case \( k = 0 \), fix \( r \) and \( n < q \) such that
\[
(4.2)
\]
and define a map \( U \) from \( L^q(\mathbb{H}^n) \) to functions on \( (0, T_1) \) by
\[
U f = \| e^{tL} f \|_{L^q(\mathbb{H}^n)}.
\]
Let \( \tilde{n} = n - \varepsilon \), where \( \varepsilon > 0 \) is to be chosen below and such that \( n - \varepsilon > 0 \). As well, define \( \tilde{r} \) by
\[
(4.3)
\]
We claim that \( U \) is of weak type \((\tilde{n}, \tilde{r})\). To see this, we must show
\[
(4.4)
\]
Let \( \tau \in \{ \tau : |U f(\tau)| > t \} \). Then by (2.6) with \( \tilde{n} = p \), we have
\[
\tau < \left( \frac{C(n, \varepsilon, q) \| f \|_{L^\tilde{n}(\mathbb{H}^n)}}{t} \right)^{\tilde{r}},
\]
which gives
\[
\tau < \left( \frac{C(n, \varepsilon, q) \| f \|_{L^\tilde{n}(\mathbb{H}^n)}}{t} \right)^{\tilde{r}}.
\]
From this it follows that

\[ \{ \tau : |Uf(\tau)| > t \} \subset \left[ 0, \left( \frac{C(n, \varepsilon, q) \|f\|_{L^q(\mathbb{H}^n)}}{t} \right)^{\frac{n}{q}} \right] \]

and thus (4.4) is verified. We also get that \( U \) is of weak type \((q, \infty)\) by taking \( p = q \) in (2.6), which gives

\[ |Uf(t)| = \|e^{tL}f\|_{L^q(\mathbb{H}^n)} \leq C(n, q) \|f\|_{L^q(\mathbb{H}^n)}. \]

Next we note that if \( q < \frac{n^2}{n - 2} \), then \( 0 < n - \frac{q(n - 2)}{n} \). Thus if we choose \( \varepsilon \) so that

\[ \varepsilon < n - \frac{q(n - 2)}{n}, \]

then \( n - \varepsilon > \frac{q(n - 2)}{n} \geq 0 \), and it follows that \( \tilde{n} < \tilde{r} \). Having shown that \( U \) is of weak type \((\tilde{n}, \tilde{r})\) and \((q, \infty)\), noting that \( U \) is subadditive by Minkowski’s integral inequality, and using that \( \tilde{n} < n < q \) and \( \tilde{n} < \tilde{r} \), we can then apply the Marcinkiewicz interpolation theorem as follows (see, for example, \([33]\)): For \( 0 < \theta < 1 \), \( U \) is of strong-type \((n_1, r_1)\), where

\[ \frac{1}{n_1} = 1 - \theta + \theta \frac{q}{n}, \quad \frac{1}{r_1} = 1 - \theta \frac{n}{\tilde{r}}. \quad (4.5) \]

Solving for \((1 - \theta)\) in the first equation gives

\[ 1 - \theta = \left( \frac{1}{q} - \frac{1}{n_1} \right) \left( \frac{1}{q} - \frac{1}{\tilde{n}} \right)^{-1}, \quad (4.6) \]

and combining this with the fact that

\[ \frac{1}{\tilde{r}} = \left( \frac{1}{\tilde{n}} - \frac{1}{q} \right) \frac{n}{2}, \]

the second equation in (4.5) can be rewritten as

\[ \frac{1}{r_1} = \left( \frac{1}{n_1} - \frac{1}{q} \right) \frac{n}{2}. \quad (4.7) \]

By definition, the pair \((n, r)\) satisfies (4.7). Moreover, since \( \tilde{n} < n < q \) is assumed and since

\[ \frac{1}{\tilde{r}} = \left( \frac{1}{\tilde{n}} - \frac{1}{q} \right) \frac{n}{2} > \left( \frac{1}{n} - \frac{1}{q} \right) \frac{n}{2} = \frac{1}{r}, \quad (4.8) \]

we have \( \tilde{r} < r < \infty \). From this we conclude \( U \) is of strong type \((n, r)\).

With this fact established, we can then estimate the \( L^r(0, T_1), L^q(\mathbb{H}^n)) \) norm of \( u_0 \) in the following way:

\[ \|u_0(t)\|_{L^r(0, T_1), L^q(\mathbb{H}^n))} = \left[ \int_0^{T_1} \|e^{tL}a\|_{L^q(\mathbb{H}^n)}^r \, dt \right]^{1/r} \]

\[ = \|Ua\|_{L^r(0, T_1))} \leq C(n, q) \|a\|_{L^q(\mathbb{H}^n)}, \quad (4.9) \]

which gives \( u_0 \in L^r(0, T_1), L^q(\mathbb{H}^n)) \).
Assuming now that \(4.1\) is true for \(k\), we prove it for for \(k + 1\). To do this, we will use the following simple corollary to the one-dimensional form of the Hardy-Littlewood-Sobolev lemma (see [33], for example).

**Lemma 4.1.** Let \(0 < T_1\), \(0 < \eta < 1\), and \(1 < \lambda < \mu < \infty\) such that
\[
\frac{1}{\mu} = \frac{1}{\lambda} - \eta
\]
and define \(I_\eta\) by
\[
I_\eta(f)(x) = C(\eta) \int_0^{T_1} \frac{f(s)\chi_{[0,T_1]}(s)}{|t-s|^{1-\eta}} ds.
\]
Then
\[
\|I_\eta(f)\|_{L^\mu((0,T_1))} \leq C(\mu, \lambda)\|f\|_{L^\lambda((0,T_1))}.
\]

**Proof.** Letting \(\chi_{[0,T_1]}\) denote the indicator function for the interval \([0, T_1]\), we have by the Hardy-Littlewood-Sobolev lemma that
\[
\|I_\eta(f)\|_{L^\mu((0,T_1))} = \left\| C(\eta) \int_{\mathbb{R}} \frac{f(s)\chi_{[0,T_1]}(s)}{|t-s|^{1-\eta}} ds \right\|_{L^\mu(\mathbb{R})}
\leq C(\mu, \lambda)\|f\|_{L^\lambda((0,T_1))}. \tag{4.13}
\]

To apply this lemma, we take \(\alpha = \zeta = \gamma = n/q\) in \(2.23\) to get
\[
\|G_{u_k}(t)\|_{L^\mu(\mathbb{H}^n)} \leq C(n, q) \int_0^t \left( t - s \right)^{-\left( \frac{1}{2} + \frac{n}{2q} \right)} \left( \|u_k(s)\|_{L^\mu(\mathbb{H}^n)} \right)^2 ds
\leq C(n, q) C(1/2 - n/2q) \int_0^{T_1} \left( \|u_k(s)\|_{L^\mu(\mathbb{H}^n)} \right)^2 ds
\leq C(n, q) I_{(1/2 - n/2q)}(\|u_k(\cdot)\|_{L^\mu(\mathbb{H}^n)}^2(t)). \tag{4.14}
\]
Taking the preceding inequality and applying the \(L^r((0, T_1))\) norm as well as \(4.12\), we get
\[
\|G_{u_k}\|_{L^r((0, T_1), L^\mu(\mathbb{H}^n))} \leq C(n, q) \left\| I_{(1/2 - n/2q)}(\|u_k(\cdot)\|_{L^\mu(\mathbb{H}^n)}^2) \right\|_{L^r((0, T_1))}
\leq C(n, q) \|\|u_k(\cdot)\|_{L^\mu(\mathbb{H}^n)}^2\|_{L^r((0, T_1))}, \tag{4.15}
\]
where
\[
\frac{1}{\lambda} = \frac{1}{r} + \left( \frac{1}{2} - \frac{n}{2q} \right) = \frac{2}{r}. \tag{4.16}
\]
by \(4.2\). Thus \(4.15\) becomes
\[
\|G_{u_k}\|_{L^r((0, T_1), L^\mu(\mathbb{H}^n))} \leq C(n, q) \|\|u_k(\cdot)\|_{L^\mu(\mathbb{H}^n)}^2\|_{L^r/2((0, T_1))}
= C(n, q) \|u_k\|_{L^r((0, T_1), L^\mu(\mathbb{H}^n))}^2. \tag{4.17}
\]
Using \(3.1\), we so far have shown
\[
\|u_{k+1}\|_{L^r((0, T_1), L^\mu(\mathbb{H}^n))} \leq \|u_0\|_{L^r((0, T_1), L^\mu(\mathbb{H}^n))} + C(n, q) \|u_k\|_{L^r((0, T_1), L^\mu(\mathbb{H}^n))}^2, \tag{4.18}
\]
which is structurally the same recurrence inequality as (3.11). This has been shown to
define a bounded sequence as long as
\[ \|u_0\|_{L^r((0,T_1),L^q(\mathbb{H}^n))} < \frac{1}{4C(n,q)}, \] (4.19)
and since \(\|u_0\|_{L^r((0,T_1),L^q(\mathbb{H}^n))}\) is bounded as shown in (4.19), we can choose \(T_1 \leq T\) sufficiently
small so that (4.19) holds. As well, since our sequence \(u_k \in L^r((0,T_1),L^q(\mathbb{H}^n))\) and since
\(u_k\) converges to \(u\) in \(L^\infty((0,T),L^q(\mathbb{H}^n))\) by (1.2), we conclude \(u \in L^r((0,T_1),L^q(\mathbb{H}^n))\).

As for the global in time result, by (4.9) it follows that if
\[ \|a\|_{L^n(\mathbb{H}^n)} < \frac{1}{4C(n,q)}, \] (4.20)
where this constant \(C(n,q)\) is the product of the constants appearing in (4.15) and (4.9),
respectively, then
\[ \|u_0\|_{L^r((0,T_1),L^n(\mathbb{H}^n))} < \frac{1}{4C(n,q)}, \] (4.21)
for any choice of \(T_1 > 0\) and in this case, we conclude \(u \in L^r((0,\infty),L^n(\mathbb{H}^n))\).

5. Proof of Theorem 1.5

Writing the solution \(u\) from Theorem 1.3 as
\[ u(t) = u_0(t) + Gu(t) \]
in the usual way, we can estimate the term \(u_0\) using dispersive estimate (2.2) with \(p = q = n\),
which gives
\[ \|u_0(t)\|_{L^n(\mathbb{H}^n)} \leq C(n)e^{-\beta_1(n)t}\|a\|_{L^n(\mathbb{H}^n)}. \] (5.1)
Therefore,
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \|u_0(t)\|_{L^n(\mathbb{H}^n)} dt \leq \lim_{T \to \infty} \frac{C(n)\|a\|_{L^n(\mathbb{H}^n)}}{\beta_1(n)T}(1 - e^{-\beta_1(n)T}) = 0. \] (5.2)
It remains to show (1.25) for \(Gu\). To do this, we fix \(0 < \kappa < 1/n\), so that by Theorem 1.4
\[ u \in L^{1/\kappa}((0,\infty),L^{n/(1-2\kappa)}(\mathbb{H}^n)). \] (5.3)
Thus writing \(\|u(t)\|_{L^{n/(1-2\kappa)}(\mathbb{H}^n)} = f(t) \in L^{1/\kappa}((0,\infty))\) and taking \(\gamma = 1\) and \(\alpha = \zeta = 1 - 2\kappa\) in (2.23), we get
\[ \|Gu(t)\|_{L^n(\mathbb{H}^n)} \leq C(n,\kappa) \int_0^t (t-s)^{-(1-2\kappa)}\|u(s)\|_{L^{\frac{n}{1-2\kappa}}(\mathbb{H}^n)} f(s) ds. \] (5.4)
As well, by the results of Section 3 we have that
\[ \sup_{0 \leq t < \infty} t^{\left(\frac{1}{2} - \frac{1-2\kappa}{2}\right)} \|u(t)\|_{L^{\frac{n}{1-2\kappa}}(\mathbb{H}^n)} \leq M, \] (5.5)
so that (5.4) becomes
\[ \|Gu(t)\|_{L^n(\mathbb{H}^n)} \leq C(n,\kappa)M \int_0^t (t-s)^{-(1-2\kappa)}s^{\left(-\frac{1}{2} - \frac{1-2\kappa}{2}\right)} f(s) ds \]
\[ = C(n,\kappa) \int_0^t (t-s)^{-(1-2\kappa)}s^{-\kappa} f(s) ds. \] (5.6)
Next we use the following formula for computing the derivative of an integral involving the variable of differentiation both in the bounds of the integral and in the integrand:

\[
\frac{d}{dt} \int_0^{g(t)} h(t, s) \, ds = h(t, g(t))g'(t) + \int_0^{g(t)} \frac{\partial}{\partial t} h(t, s) \, ds.
\]  

(5.7)

Noting that the integrand in the final integral appearing in (5.6) can be rewritten as

\[
(t - s)^{-\alpha} s^{-\kappa} f(s) = \frac{\partial}{\partial t} \left( \frac{(t - s)^{2\kappa}}{2\kappa} s^{-\kappa} f(s) \right),
\]

(5.8)

we can then integrate (5.6) from 0 to \(T\) and apply the differentiation result (5.7) to get

\[
\int_0^T \|Gu(t)\|_{L^{n}(\mathbb{H}^n)} \, dt \leq C(n, \kappa) M \int_0^T \int_0^t \frac{\partial}{\partial t} \left( \frac{(t - s)^{2\kappa}}{2\kappa} s^{-\kappa} f(s) \right) \, ds \, dt
\]

\[
= C(n, \kappa) \int_0^T \frac{d}{dt} \int_0^t (t - s)^{2\kappa} s^{-\kappa} f(s) \, ds \, dt
\]

\[
= C(n, \kappa) \int_0^T (T - s)^{2\kappa} s^{-\kappa} f(s) \, ds
\]

\[
= C(n, \kappa) (I_\tau + II_\tau),
\]

(5.9)

where \(0 < \tau < T\) and

\[
I_\tau = \int_0^\tau (T - s)^{2\kappa} s^{-\kappa} f(s) \, ds,
\]

\[
II_\tau = \int_\tau^T (T - s)^{2\kappa} s^{-\kappa} f(s) \, ds.
\]

(5.10)

We estimate these integrals separately, starting with \(I_\tau\). Applying Hölder’s inequality and using that \(f(t) \in L^{1/\kappa}((0, \infty))\), we have

\[
I_\tau = \int_0^\tau (T - s)^{2\kappa} s^{-\kappa} f(s) \, ds
\]

\[
\leq T^{2\kappa} \int_0^\tau s^{-\kappa} f(s) \, ds
\]

\[
\leq T^{2\kappa} \left[ \int_0^\tau |s^{-\kappa}|^{\frac{1}{1-\kappa}} \, ds \right]^{1-\kappa} \left[ \int_0^\tau |f(s)|^{\frac{\kappa}{\kappa}} \, ds \right]^\kappa
\]

(5.11)

\[
\leq T^{2\kappa} \left[ \int_0^\tau s^{-\kappa} \, ds \right]^{1-\kappa} \left[ \int_0^\infty |f(s)|^{\frac{\kappa}{\kappa}} \, ds \right]^\kappa
\]

\[
= C(\kappa) T^{2\kappa-2\kappa} \|f\|_{L^{1/\kappa}((0, \infty))}.
\]
Next we estimate $II_\tau$. Applying Hölder’s inequality again and integrating,

$$II_\tau = \int_\tau^T (T-s)^{2\kappa} s^{-\kappa} f(s) \, ds.$$

$$\leq T^{2\kappa} \|f\|_{L^{1/\kappa}((\tau,T))} \left[ \int_\tau^T s^{-\frac{n}{1-\kappa}} \, ds \right]^{1-\kappa}$$

$$= T^{2\kappa} \|f\|_{L^{1/\kappa}((\tau,T))} \left[ T^{\frac{1}{1-\kappa}} - \tau^{\frac{1}{1-\kappa}} \right] \left[ 1^{1-\kappa} \right]$$

$$\leq T^{2\kappa} \|f\|_{L^{1/\kappa}((\tau,T))} \left[ T^{\frac{1}{1-\kappa}} - \tau^{\frac{1}{1-\kappa}} \right] \left[ 1^{1-\kappa} \right]$$

$$= T \|f\|_{L^{1/\kappa}((\tau,T))}.$$

Thus dividing (5.9) by $T$ and taking the limit supremum, we have by the above estimates on $I_\tau$ and $II_\tau$ that

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T ||Gu(t)||_{L^n(\mathbb{H}^n)} \, dt \leq C(n,\kappa) \limsup_{T \to \infty} \left[ T^{2\kappa-1} T^{1-2\kappa} \|f\|_{L^{1/\kappa}((0,\infty))} + \|f\|_{L^{1/\kappa}((\tau,T))} \right]$$

$$= C(n,\kappa) \|f\|_{L^{1/\kappa}((\tau,\infty))},$$

(5.13)

where we have used that $2\kappa - 1 < 0$ since $\kappa$ was chosen such that $0 < \kappa < 1/2$. Moreover, the lefthand side of (5.14) is independent of $\tau$, thus

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T ||Gu(t)||_{L^n(\mathbb{H}^n)} \, dt = \limsup_{\tau \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T ||Gu(t)||_{L^n(\mathbb{H}^n)} \, dt$$

$$\leq \lim_{\tau \to \infty} C(n,\kappa) \|f\|_{L^{1/\kappa}((\tau,\infty))}$$

$$= 0,$$

since $f(t) \in L^{1/\kappa}((0,\infty))$.

### 6. Proof of Theorem 1.6

#### 6.1. Local well-posedness. If $a \in L^p(\mathbb{H}^n) \cap L^p(\mathbb{R}^n)$ for $1 < p < n$, then a unique solution $u$ exists for some $T > 0$ and such that $u$ satisfies (1.22) with $q = n$. Thus to prove Theorem 1.6, we must first show $u$ satisfies (1.27) and for this, we first set $q = p$ in (2.6), which gives

$$\|u_0(t)\|_{L^p(\mathbb{H}^n)} \leq C(n,p)\|a\|_{L^p(\mathbb{H}^n)}.$$  (6.1)

If $T$ is chosen so that (3.13) holds, then for any $0 < \delta < 1$ and any $0 < t < T$,

$$\|u(t)\|_{L^{n/\delta}(\mathbb{H}^n)} \leq Mt^{-(1-\delta)/2}.  \quad (6.2)$$

Thus using (2.23) with $\alpha = \gamma = \frac{n}{p}$ and $\zeta = \delta$, with $0 < \delta < 1$ small enough so that $n/p + \delta < n$, we have

$$\|Gu(t)\|_{L^p(\mathbb{H}^n)} \leq C(n,p,\delta) \int_0^t (t-s)^{-(\delta+1)/2} \|u(s)\|_{L^{n/\delta}(\mathbb{H}^n)} \|u(s)\|_{L^p(\mathbb{H}^n)} \, ds$$

$$\leq C(n,p,\delta) M \int_0^t (t-s)^{-(\delta+1)/2} s^{-(1-\delta)/2} \|u(s)\|_{L^p(\mathbb{H}^n)} \, ds.$$  (6.3)
Taking this estimate together with (6.1) and using that \(u(t) = u_0(t) + Gu(t)\), we have shown that
\[
\|u(t)\|_{L^p(\mathbb{H}^n)} \leq C(n, p)\|a\|_{L^n(\mathbb{H}^n)} + C(n, p, \delta)M \int_0^t (t-s)^{-\frac{(\delta+1)}{2}} s^{-\frac{(1-\delta)}{2}} \|u(s)\|_{L^n(\mathbb{H}^n)} ds, \tag{6.4}
\]
so that by Grönwall’s inequality,
\[
\|u(t)\|_{L^p(\mathbb{H}^n)} \leq C(n, p)\|a\|_{L^n(\mathbb{H}^n)} \exp\left( C(n, p, \delta)M \int_0^t (t-s)^{-\frac{(\delta+1)}{2}} s^{-\frac{(1-\delta)}{2}} ds \right) \tag{6.5}
\]
\[
\leq C(n, p)\|a\|_{L^n(\mathbb{H}^n)} \exp\left( C(n, p, \delta)MB\left(\frac{\delta + 1}{2}, \frac{1-\delta}{2}\right) \right),
\]
where we have made the substitution \(t = s/t\) to compute the Beta integral. This shows
\[
\sup_{0 \leq t < T} \|u(t)\|_{L^p(\mathbb{H}^n)} \leq C(n)\|a\|_{L^n(\mathbb{H}^n)} \exp\left( C(n, \delta)MB\left(\frac{\delta + 1}{2}, \frac{1-\delta}{2}\right) \right), \tag{6.6}
\]
so that \(u \in L^\infty([0, T), L^p(\mathbb{H}^n))\). As well, if \(\|a\|_{L^n(\mathbb{H}^n)}\) satisfies the smallness condition \(3.29\), then
\[
\|u(t)\|_{L^{n/\delta}(\mathbb{H}^n)} < Mt^{-\left(\frac{s}{2} - \frac{\delta}{2}\right)}, \quad \text{for } 0 \leq t < \infty
\]
and we conclude \(u \in L^\infty([0, \infty), L^p(\mathbb{H}^n))\).

6.2. Continuous dependence on initial Data. Let \(a \in L^p(\mathbb{H}^n) \cap L^n(\mathbb{H}^n)\) and let \(u\) be corresponding solution from Theorem 1.6 with time of existence \(T\). As well, let \(a' \in L^p(\mathbb{H}^n) \cap L^n(\mathbb{H}^n)\) and let \(v\) be the corresponding solution from Theorem 1.6 with time of existence \(T'\). Let \(\tilde{T} = \min\{T, T'\}\).

Let \(0 < \delta < 1\) be such that \(n/p + \delta < n\). By (3.2), both \(u\) and \(v\) satisfy
\[
\sup_{0 \leq t < \tilde{T}} t^{-\frac{(1-\delta)}{2}} \|u\|_{L^{n/\delta}(\mathbb{H}^n)} \leq \tilde{M}, \tag{6.7}
\]
\[
\sup_{0 \leq t < \tilde{T}} t^{-\frac{(1-\delta)}{2}} \|v\|_{L^{n/\delta}(\mathbb{H}^n)} \leq \tilde{M}, \tag{6.8}
\]
where, letting \(M_u\) denote the uniform bound on the sequence \(\{u_k\}\) from Theorem 1.3 and \(M_v\) that for \(\{v_k\}\), we define \(\tilde{M} = \max\{M_u, M_v\}\). Using these facts and applying (2.24) with \(\alpha = \delta\) and \(\zeta = \gamma = n/p\) as well as (2.6) with \(p = q = n\), we have
\[
\|u(t) - v(t)\|_{L^p} \leq \|e^{tL}(a - a')\|_{L^p} + C(n, p, \delta) \int_0^t (t-s)^{-\frac{(\delta+1)}{2}} (\|u\|_{L^n/\delta} + \|v\|_{L^n/\delta}) \|u - v\|_{L^p} ds
\]
\[
\leq C(n, p)\|a - a'\|_{L^p} + 2C(n, p, \delta)\tilde{M} \int_0^t (t-s)^{-\frac{(\delta+1)}{2}} s^{-\frac{(1-\delta)}{2}} \|u - v\|_{L^p} ds. \tag{6.9}
\]
Apply Grönwall’s inequality, the above computation implies
\[
\|u(t) - v(t)\|_{L^p(\mathbb{H}^n)} \leq C(n, p)\|a - a'\|_{L^p(\mathbb{H}^n)} \exp\left( 2C(n, p, \delta)\tilde{M} \int_0^t (t-s)^{-\frac{(\delta+1)}{2}} s^{-\frac{(1-\delta)}{2}} ds \right)
\]
\[
= C(n, p)\|a - a'\|_{L^p(\mathbb{H}^n)} \exp\left( 2C(n, p, \delta)\tilde{M}B\left(\frac{\delta + 1}{2}, \frac{1-\delta}{2}\right) \right)
\]
\[
= C(n, p, \delta, \tilde{M})\|a - a'\|_{L^p(\mathbb{H}^n)}. \tag{6.10}
\]
Therefore, given \( \varepsilon > 0 \), if
\[
\|a - a'\|_{L^p(\mathbb{H}^n)} < \frac{\varepsilon}{C(n, p, \delta, \hat{M})},
\]
then
\[
\|u(t) - v(t)\|_{L^p(\mathbb{H}^n)} < \varepsilon.
\]
This shows continuous dependence on the initial data in the space \( L^\infty([0, \bar{T}), L^p(\mathbb{H}^n)) \) and since continuous dependence on the initial data was already shown for \( L^\infty([0, \bar{T}), L^n(\mathbb{H}^n)) \) in Theorem 1.3, we conclude it holds in \( L^\infty([0, \bar{T}), L^p(\mathbb{H}^n) \cap L^n(\mathbb{H}^n)) \) as well.

6.3. Well-posedness for \( \nabla u \). The proof for showing (1.28) proceeds almost exactly as in the proof for (1.23) in Theorem 1.3 though here we choose \( q = p \) in (2.7) to estimate \( u_0 \) and to estimate \( Gu \), we apply (3.2) and (3.12) together with (2.25), where \( \gamma = \zeta = n/p \) and \( \alpha = \delta \), where \( 0 < \delta < 1 \) is chosen so that \( \alpha + \zeta = \delta + n/p < n \)

7. Proof of Theorem 1.7

Assuming the solution \( u \) and its derivative \( \nabla u \) from Theorem 1.6 are global, we first prove (1.29) and in order to do this, we must prove a modified version of estimate (2.23) appearing in Lemma 2.7. Using that \( d^* u = 0 \) to write
\[
Gu(t) = -\int_0^t e^{-(t-s)\mathbb{L}}(\nabla u#u)(s) \, ds
\]
and letting \( 1 < r \leq q < \infty \), we have by our dispersive estimate (2.6), the \( L^r \) boundedness of \( \mathbb{P} \) shown in Corollary 2.4, the pointwise estimate (2.31), and Hölder’s inequality that
\[
\|Gu(t)\|_{L^r(\mathbb{H}^n)} \leq C(n, r, q) \int_0^t (t-s)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \|\nabla u(s)\|_{L^p(\mathbb{H}^n)} \|u(s)\|_{L^q(\mathbb{H}^n)} \, ds
\]
\[
\leq C(n, \mu, \lambda, q) \int_0^t (t-s)^{-\frac{n}{2} \left( \frac{1}{p} + \frac{1}{q} - \frac{1}{\gamma} \right)} \|u(s)\|_{L^p(\mathbb{H}^n)} \|\nabla u(s)\|_{L^\lambda(\mathbb{H}^n)} \, ds,
\]
where \( \mu, \lambda > 0 \) and \( \frac{1}{p} + \frac{1}{\lambda} = \frac{1}{r} \). Then, setting
\[
q = \frac{n}{\gamma}, \quad \mu = \frac{n}{\alpha}, \quad \lambda = \frac{n}{\zeta},
\]
where \( \gamma \leq \alpha + \zeta < n \), we get
\[
\|Gu(t)\|_{L^n(\mathbb{H}^n)} \leq C(n, \alpha, \gamma, \zeta) \int_0^t (t-s)^{-\frac{n}{2p} \left( \frac{1}{2} - \frac{1}{\gamma} \right)} \|u(s)\|_{L^n(\mathbb{H}^n)} \|\nabla u(s)\|_{L^n(\mathbb{H}^n)} \, ds. \quad (7.3)
\]

With the estimate (7.3) established, we fix \( q \) such that \( p \leq q < \infty \) and first suppose
\[
\frac{n}{2p} - \frac{n}{2q} < 1.
\]
It follows by (2.6) that
\[
t^\frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \|u_0(t)\|_{L^q(\mathbb{H}^n)} \leq C(n, p, q) \|a\|_{L^p(\mathbb{H}^n)}, \quad (7.4)
\]
which proves the desired decay rate for \( u_0 \). For estimating \( Gu \), we chose \( 0 < \delta < 1 \) such
\[
\frac{n}{p} - \frac{n}{q} + \delta < 2
\]
(7.5)
and observe that by taking \( q = n/\delta \) in (1.23) from Theorem 1.3

\[
\|\nabla u(t)\|_{L^{n/\delta}(\mathbb{H}^n)} \leq C(n, \delta) t^{-1+\delta/2}.
\]

As well, by Theorem 1.6 and our assumption that \( u \) is global in time, \( \|u(t)\|_{L^p(\mathbb{H}^n)} \) is bounded for \( 0 \leq t < \infty \). Thus setting

\[
\alpha = \frac{n}{p}, \quad \gamma = \frac{n}{q}, \quad \zeta = \delta,
\]

in (7.3), we have

\[
\|G(t)\|_{L^q(\mathbb{H}^n)} \leq C(n, p, q, \delta) \int_0^t (t-s)^{-1/2} \left( \frac{n+p+\delta-\frac{n}{q}}{2p} \right) s^{-1+\delta/2} \, ds
\]

\[
\leq C(n, p, q, \delta) \int_0^t (t-s)^{-1/2} \left( \frac{n+p+\delta-\frac{n}{q}}{2p} \right) s^{-1+\delta/2} \, ds
\]

\[
\leq C(n, p, q, \delta) t^{-\left( \frac{n}{2p} - \frac{n}{2q} \right)} \int_0^1 (1-\tau)^{-\frac{1}{2} \left( \frac{n+p+\delta}{q} - \frac{n}{q} \right)} \tau^{\delta/2-1} \, d\tau
\]

\[
= C(n, p, q, \delta) B\left( \frac{\delta}{2}, 1 - \frac{1}{2} \left( \frac{n}{p} - \frac{n}{q} + \delta \right) \right) t^{-\left( \frac{n}{2p} - \frac{n}{2q} \right)},
\]

where we have used the substitution \( \tau = s/t \) to compute the Beta integral. As well, by our assumption (7.5),

\[
1 - \frac{1}{2} \left( \frac{n}{p} - \frac{n}{q} + \delta \right) > 0,
\]

so both arguments appearing in the Beta function in (7.6) are greater than zero, as required for convergence. Thus \( G \) has the desired decay rate and this fact combined with (7.4) shows (1.29).

Next we prove (1.30), where in this case, we assume

\[
\frac{n}{2p} - \frac{n}{2q} \geq 1.
\]

Choose \( 0 < \varepsilon < \min\left\{ 1, \frac{1}{2} \left( 1 + \frac{n}{q} \right) \right\} \) and define

\[
p' = \frac{qn}{2(1-\varepsilon)q + n}.
\]

Then

\[
\frac{n}{2p'} - \frac{n}{2q} = 1 - \varepsilon < 1,
\]

and this quantity can be taken arbitrarily close to 1 by choosing \( \varepsilon \) small enough. We next claim \( p < p' < n \). For the first inequality, note that by (7.7),

\[
0 < 1 - \varepsilon < 1 \leq \frac{n}{2p} - \frac{n}{2q}.
\]
Rearranging this, we get
\[
\frac{1}{p} > \frac{2(1 - \varepsilon)}{n} + \frac{1}{q} = \frac{2(1 - \varepsilon)q + n}{qn} = \frac{1}{p'}, \tag{7.8}
\]
which implies \( p' > p \). To show the second inequality \( p' < n \), we use that
\[
\varepsilon < \frac{1}{2} \left( 1 + \frac{n}{q} \right),
\]
which after rearrangement implies
\[
\frac{1}{n} < \frac{2(1 - \varepsilon)}{n} + \frac{1}{q} = \frac{2(1 - \varepsilon)q + n}{qn} = \frac{1}{p'}, \tag{7.9}
\]
so that \( p' < n \) follows.

Having shown that \( p < p' < n \), it follows by interpolation that, if the initial condition \( a \in L^p(\mathbb{H}^n) \cap L^n(\mathbb{H}^n) \), then \( a \in L^{p'}(\mathbb{H}^n) \). So as long as
\[
p' \leq q, \tag{7.10}
\]
then the steps used to derive (1.29) are valid for \( p' \) in place of \( p \), which verifies (1.30). Thus it remains to show (7.10), but since \( 0 < \varepsilon < 1 \) by definition, then \( 0 < 2(1 - \varepsilon)/n \), giving
\[
\frac{1}{q} < \frac{2(1 - \varepsilon)}{n} + \frac{1}{q} = \frac{2(1 - \varepsilon)q + n}{qn} = \frac{1}{p'}, \tag{7.11}
\]
so that \( p' < q \).

The decay estimates for \( \nabla u \) are proved in an analogous way.

8. Proof of Theorem 1.8

To prove Theorem 1.8, we must show (1.33). By taking \( p = q \) in Pierfelice’s original dispersive estimate (2.2), we have
\[
\| u_0(t) \|_{L^p(\mathbb{H}^n)} \leq e^{-t \beta_1(n,p,p)} \| a \|_{L^p(\mathbb{H}^n)}. \tag{8.1}
\]
Taking a limit as \( t \) goes to infinity and using that
\[
\beta_1(n,p,p) = \delta_n \left[ \frac{8}{p} \left( 1 - \frac{1}{p} \right) \right] + c_0 > 0,
\]
it then follows that
\[
\lim_{t \to \infty} \| u_0(t) \|_{L^p(\mathbb{H}^n)} = 0. \tag{8.2}
\]
To handle the limit of the \( L^p \) norm of \( Gu \), we first claim that
\[
\| Gu(t) \|_{L^p(\mathbb{H}^n)} = \mathcal{O}(t^{-\omega/2}) \quad \text{as} \quad t \to \infty, \tag{8.3}
\]
where \( \omega \) is any positive number satisfying
\[
\omega < \min\left\{ 1, n - \frac{n}{p}, \frac{n}{p} - 1 \right\}. \tag{8.4}
\]
To prove (8.3), we begin by splitting \( Gu(t) = G_0 u(t) + G_1 u(t) \), where
\[
G_0 u(t) = - \int_0^t e^{-(t-s)L_p} (\text{div}(u \otimes u))(s) \, ds
\]
\[
G_1 u(t) = - \int_1^t e^{-(t-s)L_p} (\text{div}(u \otimes u))(s) \, ds. \tag{8.5}
\]
Employing the same steps used to derive (8.22), we have a similar estimate for \( G_0 u \) given by
\[
\| G_0 u(t) \|_{L^{n/\gamma}(\mathbb{H}^n)} \leq C(n, \alpha, \gamma, \zeta) \int_0^1 (t-s)^{-\frac{\alpha + \gamma - \zeta}{2}} \| u(s) \|_{L^{n/\alpha}(\mathbb{H}^n)} \| \nabla u(s) \|_{L^{n/\zeta}(\mathbb{H}^n)} \, ds, \tag{8.6}
\]
where again, \( \gamma \leq \alpha + \zeta < n \). Setting
\[
\alpha = \omega + \frac{n}{p} - 1, \quad \gamma = \frac{n}{p}, \quad \zeta = 1, \tag{8.7}
\]
then \( \alpha > 0 \) since \( p < n \) is assumed and
\[
\alpha + \zeta = \omega + \frac{n}{p} < n, \tag{8.8}
\]
since \( \omega < n - \frac{n}{p} \) is assumed. As well,
\[
\alpha + \zeta = \omega + \frac{n}{p} > \frac{n}{p} = \gamma. \tag{8.9}
\]
Applying these choices for \( \alpha, \gamma \) and \( \zeta \) in (8.6) and using (1.23) from Theorem 1.3 with \( p = q \),
\[
\| G_0 u(t) \|_{L^p(\mathbb{H}^n)} \leq C(n, \omega, p) \int_0^1 (t-s)^{-\frac{\omega}{2}} \| u(s) \|_{L^{n/\alpha}(\mathbb{H}^n)} \| \nabla u(s) \|_{L^{n/\zeta}(\mathbb{H}^n)} \, ds
\leq C(n, \omega, p) \int_0^1 (t-s)^{-\frac{\omega}{2}} \| u(s) \|_{L^{n/\alpha}(\mathbb{H}^n)} s^{-1/2} \, ds. \tag{8.10}
\]
Therefore, we must estimate the term \( \| u(s) \|_{L^{n/\alpha}(\mathbb{H}^n)} \) and for this we claim
\[
\| u(s) \|_{L^{n/\alpha}(\mathbb{H}^n)} = \mathcal{O}(s^{-(1-\omega)/2}) \quad \text{as} \quad s \to 0. \tag{8.11}
\]
To show (8.11), we consider first the case where $0 < \alpha \leq 1$. Here, $n/\alpha \geq n$, so that by applying Theorem 1.3 with $q = n/\alpha$,

\[
\|u(s)\|_{L^{n/\alpha}(\mathbb{R}^n)} \leq C(n, \omega, p)s^{\frac{1}{2}(1-\alpha)} = C(n, \omega, p)s^{\frac{1}{2}(1-\omega - \frac{n}{p} + 1)} = C(n, \omega, p)s^{\frac{1}{2}(1-\omega)}s^{\frac{n}{2p} - \frac{1}{2}} \leq C(n, \omega, p)s^{\frac{n}{2p} - \frac{1}{2}} \text{ as } s \to 0,
\]

where in the last line we have used that $p < n$, which in turn implies $\frac{n}{2p} - \frac{1}{2} > 0$, so that

\[
s^{\frac{n}{2p} - \frac{1}{2}} < 1 \text{ for } 0 < s < 1.
\]

Next we consider the case $\alpha > 1$. Since $n/\alpha < n$, we can apply the results of Theorem 1.6 with $p = n/\alpha$ to get

\[
\|u(s)\|_{L^{n/\alpha}(\mathbb{R}^n)} \leq C(n, \omega, p) < C(n, \omega, p)s^{\frac{1}{2}(1-\omega)}s^{\frac{n}{2p} - \frac{1}{2}} \text{ as } s \to 0,
\]

since for $0 < s < 1$ and $\omega < 1$, $1 < s^{\frac{n}{2p} - \frac{1}{2}}$.

Having shown (8.11), we return to estimating $G_0u$ and to do so, we will use the incomplete Beta and Hypergeometric functions, which are defined in [1] as follows.

**Definition 8.1.** The incomplete Beta function $B_x(a, b)$ is defined for $x \geq 0$ and $a, b > 0$ by

\[
B_x(a, b) = \int_0^x \eta^{a-1}(1-\eta)^{b-1} d\eta.
\]

**Definition 8.2.** The Hypergeometric function $\, _2F_1$ is defined for for $|z| < 1$ by

\[
\, _2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},
\]

where $(q)_n$ is the rising Pochhammer symbol

\[
(q)_n = \begin{cases} 1 & n = 0 \\ q(q + 1) \cdots (q + n - 1) & n > 0. \end{cases}
\]

Also shown in [1] is the following relationship between the incomplete Beta function and the Hypergeometric function, which we state as a lemma.

**Lemma 8.1.** [1] The incomplete Beta function $B_x(a, b)$ and the Hypergeometric function $\, _2F_1$ are related by the following identity

\[
B_x(a, b) = \frac{x^a}{a} \, _2F_1(a, 1-b; a+1; x).
\]
Thus starting from (8.10) and employing (8.11), we have

\[ \|G_0 u(t)\|_{L^p(H^n)} \leq C(n, \omega, p) \int_0^1 (t - s)^{-\frac{\omega}{2}} \|u(s)\|_{L^\infty(H^n)} s^{-1/2} \, ds \]

\[ \leq C(n, \omega, p) \int_0^1 (t - s)^{-\frac{\omega}{2} - \frac{1 - \omega}{2}} s^{-1/2} \, ds \]

\[ = C(n, \omega, p) \int_0^1 (1 - \gamma)^{-\frac{\omega}{2} - 1} \gamma^{-1/2} \, ds \]

\[ = C(n, \omega, p) \int_0^1 (1 - \gamma)^{-\frac{\omega}{2} - 1}(1 - \gamma)^{-1} \, ds \]

\[ = C(n, \omega, p) \frac{2t^{-\omega/2}}{\omega} \frac{1 + \frac{\omega}{2}}{t} \]

\[ = C(n, \omega, p) t^{-\omega/2} \frac{1 + \frac{\omega}{2}}{t} \]

\[ = C(n, \omega, p) t^{-\omega/2} \frac{1 + \frac{\omega}{2}}{t} \]

\[ = C(n, \omega, p) t^{-\omega/2} \frac{1 + \frac{\omega}{2}}{t} \]

\[ (8.18) \]

where we have used the substitution \( \tau = s/t \), the definition of the incomplete Beta function (8.1), and Lemma 8.1. Given that we will eventually take a limit as \( t \to \infty \), we can safely assume \( t > 1 \) so that the Hypergeometric function appearing in the last line of (8.18) converges. Moreover, by combining the series definition of \( 2F1 \) in (8.2) with the estimate (8.18), we have

\[ \|G_0 u(t)\|_{L^p(H^n)} \leq C(n, \omega, p) t^{-\frac{\omega}{2}} \frac{1 + \frac{\omega}{2}}{t} \]

\[ = C(n, \omega, p) t^{-\frac{\omega}{2}} \frac{1 + \frac{\omega}{2}}{t} \]

\[ (8.19) \]

As well, since

\[ \lim_{t \to \infty} \sum_{n=1}^{\infty} \frac{(\omega/2)_n (\omega/2)_n}{(1 + \omega/2)_n} \frac{t^{-n}}{n!} = 0, \]

\[ (8.20) \]

we conclude that

\[ \|G_0 u(t)\|_{L^p(H^n)} = O(t^{-\omega/2}) \text{ as } t \to \infty. \]

\[ (8.21) \]

It still remains to estimate \( G_1 u(t) \) and the same steps leading to (8.6) show that

\[ \|G_1 u(t)\|_{L^p(H^n)} \leq C(n, \alpha, \gamma, \zeta) \int_1^t (t - s)^{-\frac{\alpha + \gamma - \zeta}{2}} \|u(s)\|_{L^\infty(H^n)} \|\nabla u(s)\|_{L^\infty(H^n)} \, ds. \]

\[ (8.22) \]

Thus by setting

\[ \alpha = \omega, \quad \gamma = \zeta = \frac{n}{p}, \]
in (8.22), it follows that
\[ \|G_1 u(t)\|_{L^p(\mathbb{H}^n)} \leq C(n, p, \omega) \int_1^t (t-s)^{-\omega/2} \|u(s)\|_{L^q(\mathbb{H}^n)} \|\nabla u(s)\|_{L^p(\mathbb{H}^n)} ds \] (8.23)
where in the second step we used Theorem 1.6, specifically result (1.28). Next we will apply the results (1.29) and (1.30) from Theorem 1.7 to estimate the term \( \|u(s)\|_{L^n/\omega(\mathbb{H}^n)} \) and in doing so, there are two cases to consider.

For the first case, suppose
\[ \frac{n}{2p} - \frac{\omega}{2} < 1, \] (8.24)
which would allow us to estimate \( \|u(s)\|_{L^n/\omega(\mathbb{H}^n)} \) according to (1.29) by taking \( q = n/\omega \). By our assumptions on \( \omega \), we must also have \( \omega < n - \frac{n}{p} \). (8.25)
But by rearranging the inequality (8.24), we get the following lower bound for \( \omega \),
\[ \frac{n}{p} - 2 < \omega, \] (8.26)
thus for \( \omega \) to satisfy both (8.25) and (8.25), we require
\[ \frac{n}{p} - 2 < n - \frac{n}{p}. \] (8.27)
This latter inequality, after rearrangement, is equivalent to
\[ \frac{2n}{n+2} < p. \] (8.28)
But \( \frac{2n}{n+2} \leq 1 \) if and only if \( n \leq 2 \), so if we want all \( 1 < p \leq n \), the original assumption (8.24) forces \( n = 2 \). As we are aiming to prove Theorem 1.7 for general \( n \geq 2 \), we must then omit the possibility (8.24).

Having ruled out the first case, we now consider the second case by supposing
\[ \frac{n}{2p} - \frac{\omega}{2} \geq 1. \] (8.29)
Applying result (1.30) in this case gives
\[ \|u(s)\|_{L^n/\omega(\mathbb{H}^n)} = \mathcal{O}(s^{-\left(\frac{n}{2p'} - \frac{\omega}{2}\right)}), \] (8.30)
where \( p' \) is chosen such that \( p < p' < n \), \( p' \leq q \), and \( \frac{n}{2p'} - \frac{n}{2q} < 1 \), but such that \( \frac{n}{2p'} - \frac{n}{2q} \) is arbitrarily close to 1. Returning to the computation in (8.23), we then have
\[ \|G_1 u(t)\|_{L^p(\mathbb{H}^n)} \leq C(n, p, \omega) \int_1^t (t-s)^{-\omega/2} s^{-1/2} \|u(s)\|_{L^n/\omega(\mathbb{H}^n)} ds \]
\[ \leq C(n, p, \omega) \int_1^t (t-s)^{-\omega/2} s^{-1/2} s^{-\left(\frac{n}{2p'} - \frac{\omega}{2}\right)} ds \] (8.31)
\[ \leq C(n, p, \omega) \int_1^t (t-s)^{-\omega/2} s^{-\left(\frac{n}{2p'} - \frac{\omega}{2} + \frac{1}{2}\right)} ds. \] (8.32)
Since the term
\[ \frac{n}{2p'} - \frac{\omega}{2} \]
is less than 1 but arbitrarily close to 1 by construction, we can achieve
\[ \frac{n}{2p'} - \frac{\omega}{2} + \frac{1}{2} > 1. \]
(8.33)
So to finish verifying the claim (8.3), consider the following lemma used by Kato in [15].

**Lemma 8.2.** If \( 0 < a < 1 < b \) and \( t \geq 1 \), then
\[
\int_1^t (t - s)^{-a} s^{-b} ds \leq C(a, b) t^{-a}. \tag{8.34}
\]

We apply this lemma to the final integral in (8.31) with
\[
\begin{align*}
a &= \frac{\omega}{2} < 1 \\
b &= \frac{n}{2p'} - \frac{\omega}{2} + \frac{1}{2} > 1,
\end{align*}
\]
so that
\[
\|G_1 u(t)\|_{L^p(H^n)} \leq C(n, p, \omega) \int_1^t (t - s)^{-\omega/2} s^{-\left(\frac{\omega}{2p'} - \frac{\omega}{2} + \frac{1}{2}\right)} ds \leq C(n, p, p', \omega) t^{-\omega/2}. \tag{8.35}
\]
This shows
\[
\|G_1 u(t)\|_{L^p(H^n)} = O(t^{-\omega/2}) \quad \text{as} \quad t \to \infty, \tag{8.36}
\]
implying
\[
\|G u(t)\|_{L^p(H^n)} \leq \|G_0 u(t)\|_{L^p(H^n)} + \|G_1 u(t)\|_{L^p(H^n)} = O(t^{-\omega/2}) \quad \text{as} \quad t \to \infty, \tag{8.37}
\]
which verifies (8.3).

Thus to conclude the proof Theorem 1.7, it follows by (8.2) and (8.3) that
\[
\lim_{t \to \infty} \|u(t)\|_{L^p(H^n)} \leq \lim_{t \to \infty} \left(\|u_0(t)\|_{L^p(H^n)} + \|G u(t)\|_{L^p(H^n)}\right) = \lim_{t \to \infty} O(t^{-\omega/2}) = 0. \tag{8.38}
\]
More over, as discussed in the remark following the statement of Theorem 1.8 the result (8.37) above implies
\[
\|u(t) - e^{tL}a\|_{L^p(H^n)} = \|G u(t)\|_{L^p(H^n)} = O(t^{-\omega/2}) \quad \text{as} \quad t \to \infty, \tag{8.39}
\]
so that \( u \) decays at least as fast as the slower of \( t^{-\omega/2} \) and \( \|e^{tL}a\|_{L^p(H^n)} \).

**Appendix A. Strong continuity of the semigroup \( e^{tL} \)**

In this section, we prove \( e^{tL} \) is a contractive and strongly continuous semigroup on \( L^p(T^*H^n), 1 \leq p < \infty \). To do so, we first state two definitions, a theorem, and a corollary from [29].

**Definition.** [29] Let \( X \) be a Banach space, \( \varphi \in X \). An element \( \ell \in X^* \) that satisfies \( \|\ell\|_{X^*} = \|\varphi\|_X \) and \( \ell(\varphi) = \|\varphi\|_X^2 \) is called a normalized tangent functional to \( \varphi \).
Remark A.1. As mentioned in [29], every \( \varphi \in X \) has at least one normalized tangent functional by the Hahn-Banach Theorem.

Remark A.2. For a \( \sigma \)-finite measure space \( (X, \mathcal{B}, m) \), it is straightforward to check that for any \( u \in L^p(X), 1 \leq p < \infty \), there exists a normalized tangent functional given by

\[
\ell = c|u|^{p-2}u, \tag{A.1}
\]

where \( c = \|u\|^{1-p/p'}_{L^p(X)} \) and \( p' \) is the Hölder conjugate to \( p \).

Definition. [29] A densely defined operator \( A \) on a Banach space \( X \) is called accretive if for each \( \varphi \in D(A), \)

\[
\Re(\ell(A\varphi)) \geq 0 \tag{A.2}
\]

for some normalized tangent functional \( \ell \) to \( \varphi \).

With these definitions in hand, we can now state a Theorem from [29], as well as a useful corollary, which will be the main tools in showing contractivity and strong continuity of \( e^{tL} \) on \( L^p(\mathbb{H}^n) \), \( 1 \leq p < \infty \).

Theorem A.1. [29] A closed operator \( A \) on a Banach space \( X \) is the generator of a strongly continuous and contractive semigroup \( e^{-tA} \) if and only if \( A \) is accretive and

\[
\text{Ran}(\lambda_0 + A) = X \tag{A.3}
\]

for some \( \lambda_0 > 0 \).

Corollary A.2. [29] Let \( A \) be a closed operator on a Banach space \( X \) such that both \( A \) and its adjoint \( A^* \) are accretive operators. Then \( A \) generates a strongly continuous and contractive semigroup \( e^{-tA} \).

We now state the main result to be proven.

Theorem A.3. For \( \mathbb{H}^n, n \geq 2 \), the unique self-adjoint extension of \( L \), which will also be denoted by \( L \), generates a contractive and strongly continuous semigroup \( e^{tL} \) on \( L^p(T^*\mathbb{H}^n) \) for all \( 1 \leq p < \infty \).

Proof. Since the extension \( L \) is self-adjoint by Strichartz’ work in [33], it suffices by Corollary A.2 to show \( -L \) is accretive. To that end, let \( u \in L^p(T^*M), 1 \leq p < \infty \). Then

\[
\langle |u|^{p-2}u, Lu \rangle = \langle |u|^{p-2}u, \Delta_{B,1}u \rangle - (n-1)\langle |u|^{p-2}u, u \rangle, \tag{A.4}
\]

where \( \langle \cdot, \cdot \rangle \) represents the duality pairing. The term \( (n-1)\langle |u|^{p-2}u, u \rangle \geq 0 \), so if \( -\Delta_{B,1}u \) is accretive for \( 1 \leq p < \infty \), then \( \langle |u|^{p-2}u, \Delta_{B,1}u \rangle \leq 0 \) and it follows that

\[
\langle |u|^{p-2}u, Lu \rangle \leq 0. \tag{A.5}
\]

Therefore, if \( -\Delta_{B,1} \) is accretive on \( L^p(T^*M) \) for \( 1 \leq p < \infty \), then \( -L \) is also accretive on \( L^p(T^*M) \) for all \( 1 \leq p < \infty \) and by Corollary A.2, \( e^{tL} \) extends to a strongly continuous semigroup on \( L^p(T^*M) \) for all \( 1 \leq p < \infty \).

So to conclude the proof, we must show that \( -\Delta_{B,1}u \) is accretive for \( 1 \leq p < \infty \). In the author’s thesis, we prove that \( \Delta_{B,k} \) generates a contractive and strongly continuous semigroup on \( L^p(T^*_M) \) for \( 1 \leq p < \infty \) and any complete Riemannian manifold \( (M, g) \), where \( k = 0, 1, 2, 3, \ldots, n \). Therefore, by Theorem A.1, \( -\Delta_{B,1} \) is accretive on \( L^p(T^*\mathbb{H}^n) \) for \( 1 \leq p < \infty \).

□
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