Dielectric Fundamental Strings in Matrix String Theory

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Abstract

Matrix string theory is equivalent to type IIA superstring theory in the light-cone gauge, together with extra degrees of freedom representing D-brane states. It is therefore the appropriate framework in which to study systems of multiple fundamental strings expanding into higher-dimensional D-branes. Starting from Matrix theory in a weakly curved background, we construct the linear couplings of closed string fields to type IIA Matrix strings. As a check, we show that at weak coupling the resulting action reproduces light-cone gauge string theory in a weakly curved background. Further dualities give a type IIB Matrix string theory and a type IIA theory of Matrix strings with winding. We comment on the dielectric effect in each of these theories, giving some explicit solutions describing fundamental strings expanding into various Dp-branes.

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1 Introduction

The idea that a collection of branes can undergo an “expansion” into a single higher-dimensional D-brane under the influence of a background Ramond-Ramond (R-R) potential was first explored by Emparan [1]. Although his main concern was with the description of $N$ fundamental strings expanding into a $D_p$-brane, he also realised that $D(p-2)$-branes could undergo such an expansion. Emparan’s analysis of this effect was entirely at the level of the abelian theory relevant to the description of the single $D_p$-brane. Switching on a background R-R ($p+2$)-form field strength, he found solutions of the combined Born-Infeld-Chern-Simons theory with topology $M^2 \times S^{p-1}$, where $M^n$ denotes an $n$-dimensional Minkowski space, and with $N$ units of dissolved electric flux, corresponding to the fundamental strings. Similar solutions, with topology $M^{p-1} \times S^2$, and with $N$ units of dissolved magnetic flux, corresponding to $D(p-2)$-branes, are easily found.

It was some years later that a description from the point of view of the lower-dimensional $D(p-2)$-branes was provided [2]. This involved an analysis of certain couplings in the non-abelian Born-Infeld-Chern-Simons action proposed by Taylor and van Raamsdonk [3] and Myers [2]. From this perspective, the expansion is due to the fact that the transverse coordinates of the $D(p-2)$-branes are matrix-valued. Myers’ original analysis was of a collection of $N$ $D0$-branes, in the presence of a background R-R four-form field strength: they spontaneously expand into a $D2$-brane of topology $\mathbb{R} \times S^2_{NC}$, where $S^2_{NC}$ denotes the non-commutative two-sphere. The $D2$-brane is uncharged with respect to the four-form, but has a non-zero dipole moment. Hence the name of a “dielectric” $D2$-brane. Configurations for arbitrary $p$, and more general configurations involving fuzzy cosets, were described in [4].

These two descriptions of the dielectric effect — abelian and non-abelian — are of course dual to one another. In the limit of large $N$, the non-commutative nature of the fuzzy two-sphere is lost, it becomes a smooth manifold, and the two descriptions do indeed agree.

In principle, all such dielectric branes should have a corresponding supergravity solution describing the back-reaction of the brane on spacetime. Technical difficulties, however, have limited the analysis of such solutions to that of $N$ $D4$-branes [1, 2] or $N$ F-strings [3] expanding into a $D6$-brane with topology $M^5 \times S^2$ and $M^2 \times S^5$ respectively. In both cases, there is a stable and an unstable radius of the dielectric sphere, the form of the effective potential matching the abelian worldvolume analysis precisely [1, 2].

There is thus much evidence, from both the worldvolume and supergravity perspectives, that the dielectric effect is not limited to D-branes, but that dielectric F-strings also exist. One can then pose the following question: can the expansion of F-strings into a $D_p$-brane be described from the point of view of the strings themselves? It is the purpose of this paper to answer precisely this question.

Since, from the strings’ perspective, the dielectric effect should be due to matrix-valued coordinates, we are led to a consideration of Matrix string theory [3, 4, 5]. Starting from the Matrix theory action [6, 7], one compactifies on a circle, reinterprets the resulting theory as a $(1+1)$-dimensional super Yang-Mills theory on the dual circle [8], and then performs the so-called 9-11 flip, which implements a further S- and T-duality [11]. It is easy to see that the result is a $(1+1)$-dimensional super Yang-Mills theory describing $N$ fundamental strings in the type IIA theory. In the weak string coupling limit, one recovers $N$ copies of the Green-Schwarz action describing light-cone gauge string theory.

To discuss possible dielectric solutions of Matrix string theory, we need to know how the R-R potentials couple to the worldvolume of the (Matrix) string. Since the Matrix theory action

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1 A recent attempt to answer this question from the perspective of Matrix string theory has already been made [8], although the results seem somewhat opaque to the authors and there is little overlap with this work.
in a weakly curved background is known \cite{15, 16}, one can in principle run through the above
chain of dualities to derive the Matrix string theory action in a corresponding weakly curved
background. Indeed, Schiappa has already considered this computation, and has written down an
obvious dielectric-like solution \cite{17}. However, it is not at all clear that this solution corresponds to
a dielectric \textit{string}, for reasons explained later in this paper. Moreover, we find couplings to various
components of the R-R fields over and above those found by Schiappa \cite{17} and, for this reason, we
consider the problem of deriving the Matrix string theory action in a weakly curved background
from scratch. We should further note that one of us has also already derived couplings of the type
IIA F-string to various background R-R potentials \cite{18}.

One might think that F-strings could expand into a Dp-brane with topology \(M^2 \times S^{p-1}\) for
any value of \(p\), but this is not the case. As mentioned above, one generically finds a stable and
an unstable spherical solution of the Dp-brane theory with dissolved electric flux. Expansion into
a D2-brane, however, is atypical: only an \textit{unstable} solution exists \cite{1}. Indeed, this observation
is mirrored in the corresponding (albeit smeared) supergravity solution \cite{1}. One might suspect,
therefore, that stable cylindrical D2-brane solutions of the type IIA Matrix string theory do not
exist, and we aim to address this issue here.

A further T-duality takes us to a type IIB Matrix string theory which, as the S-dual of the
D-string theory, describes \(N\) F-strings in the static gauge. Just as D-strings can expand into
D3-branes, so can F-strings and the dielectric solution presented in \cite{2} is equally applicable here.

In the following section, we consider Matrix theory in a weakly curved background \cite{15, 16}. It is easier to consider the chain of dualities leading to Matrix string theory in ten- rather than
eleven-dimensional language, so we choose to work with the D0-brane theory in a weakly curved
background. This has been derived from the Matrix theory results in \cite{19} — we show explicitly
that it reproduces the lowest order expansion of the non-abelian D0-brane theory of Taylor and
van Raamsdonk \cite{3} and Myers \cite{2}. T-duality, taking us to the D-string theory, is considered in
section 3, and the 9-11 flip in section 4. We show that the resulting linear action reproduces the
light-cone gauge string theory action in a weakly curved background, thus lending some weight
to our results. Indeed, we need the extra couplings, relative to the results of \cite{17}, for this to be
the case. A further T-duality taking us to the type IIB theory is considered in section 5 and,
although we cannot perform the S-duality rigorously, we argue that the result is equivalent to the
S-dual of the D-string theory. In section 6, we apply once more a T-duality transformation, this
time in a direction transverse to the IIB string, giving a theory of type IIA F-strings with winding
number rather than light-cone momentum. Some of the resulting couplings of both the type IIA
and the type IIB strings have already been considered by one of us \cite{18}. Dielectric solutions are
considered in section 7, where we give some explicit solutions describing the expansion of F-strings
into Dp-branes for different \(p\). We conclude in section 8.

2 Matrix theory

The bosonic sector of the Matrix theory action in a flat background is \cite{13}

\[
S_{\text{flat}} = \frac{1}{R} \int dt \Tr \left[ \frac{1}{2} D_t X^i D_t X^i + \frac{R^2}{16\pi^2 l_P^6} [X^i, X^j][X^i, X^j] \right],
\]

(2.1)

where \(l_P\) denotes the eleven-dimensional Planck length, \(R\) is the radius of the eleventh dimension
and \(D_t X^i = \partial_t X^i + i[A_t, X^i]\). We will choose the gauge \(A_t = 0\) throughout this paper at the
expense of losing explicit gauge invariance. With the string coupling set through the relation
\(R = g_s \sqrt{\alpha'}\), this action is the non-relativistic limit of the non-abelian Born-Infeld action describing
$N$ D0-branes, with mass quantized in units of $1/R$: the dimensional reduction to one dimension of ten-dimensional Yang-Mills theory with $g_{YM}^2 = g_s^2 / (4\pi^2\alpha'^{3/2})$. The connections are more subtle, however (see, e.g., [20]). After all, when $N$ is large, Matrix theory captures eleven-dimensional physics in the infinite momentum frame [13]. For finite $N$, it is equivalent to a DLCQ or null compactification of M-theory [21, 22, 23], whereas the type IIA D0-brane theory comes about via a spacelike compactification. The two actions are then related by an infinite boost in the eleventh dimension. At any rate, if we take $T_0 = 1/R$ and $R = g_s\sqrt{\alpha'}$, so that $l_P = g_s^{1/3}\sqrt{\alpha'}$, then we recover the Yang-Mills description of D0-branes.

2.1 Linear couplings in Matrix theory

Matrix theory in an arbitrary background is understood only rather poorly (although see [24, 25, 26] for early work on this subject) and, in the above sense, this is related to the question as to what is the form of the non-abelian Born-Infeld theory in curved space. Kabat and Taylor have derived the linear Matrix theory couplings to bosonic background fields [15] and Taylor and van Raamsdonk have extended these calculations to include fermionic backgrounds [16]. They have further derived the linear couplings of the D0-brane [13] and D$p$-brane [3] theories from Matrix theory. The results certainly seem to agree with the linear order expansion of the combined non-abelian Born-Infeld-Chern-Simons action proposed by Taylor and van Raamsdonk [4] and Myers [2], including the form of the overall symmetrized trace due to Tseytlin [27]. More precise expressions for the Matrix theory couplings, to all orders in both derivatives of the background fields and the fermionic coordinates, can be had by dimensional reduction of the eleven-dimensional supermembrane vertex operators constructed in [28, 29].

In other words, one can write the Matrix theory action as

$$S = S_{\text{flat}} + S_{\text{linear}},$$  \hspace{1cm} (2.2)

with $S_{\text{flat}}$ given by (2.1), and where the linear action has the form [15]

$$S_{\text{linear}} = \frac{1}{R} \int dt \, \text{STr} \left\{ \frac{1}{2} h_{AB} T^{AB} + A_{ABC} J^{ABC} + A_{ABCD} M^{ABCD} \right\}, \hspace{1cm} (2.3)$$

where $A, B = 0, \ldots, 10$ and STr denotes the symmetrized trace[2]. The eleven-dimensional metric, $h_{AB}$, 3-form potential, $A_{ABC}$, and its 6-form dual, $A_{ABCDEF}$, couple to the energy-momentum tensor, membrane current and 5-brane current respectively. The form of these currents has been worked out explicitly, and they match eleven-dimensional supergravity predictions [15]. There are also higher-order multipole couplings to derivatives of the background fields and the fermionic coordinates, but we will not be concerned with them here. One can relate this linear action to that relevant to the description of D0-branes, via the infinite boost mentioned above. The D0-brane currents, denoted $I$, are related in this manner to the eleven-dimensional currents $\mathcal{T}$, $\mathcal{J}$ and $\mathcal{M}$, as explained in [19].

The linear D0-brane action is then

$$S_{\text{linear}} = \frac{1}{R} \int dt \, \text{STr} \left\{ \frac{1}{2} h_{\mu\nu} I_\mu^{\mu\nu} + \phi I_\phi + b_{\mu\nu} I_4^{\mu\nu} + b_{\mu_1\ldots\mu_5} I_5^{\mu_1\ldots\mu_5} + C^{(1)} I_0^{\mu} + C^{(3)} I_2^{\mu\rho} + \frac{1}{60} C^{(5)} I_4^{\mu_1\ldots\mu_5} + \frac{1}{336} C^{(7)} I_6^{\mu_1\ldots\mu_7} \right\} \hspace{1cm} (2.4)$$

We include both the overall factor of $1/R$ and the overall gauge trace in the action, rather than in the currents, for convenience.
where \( \mu, \nu = 0, \ldots, 9 \). The currents \( I_h, I_s, I_5 \) and \( I_p \) couple respectively to the metric, \( h_{\mu \nu} \), the Neveu-Schwarz-Neveu-Schwarz (NS-NS) 2-form potential \( b^{(2)} \), its 6-form Hodge dual \( b^{(6)} \), and the R-R \((p + 1)\)-form potentials \( C^{(p+1)} \). The potentials \( C^{(5)} \) and \( C^{(7)} \), the Hodge duals of \( C^{(3)} \) and \( C^{(1)} \), have been rescaled relative to \( [13, 3] \). Note that there should also be couplings to \( [19, 9] \), but these are not determined by the analysis of \( [13, 3] \). The currents appearing in (2.4) are given in terms of the dimensional reduction of the Born-Infeld field strength,

\[
F_{0i} = -F^{0i} = \partial_i X^i \equiv \dot{X}^i, \quad F_{ij} = F^{ij} = \frac{R}{2\pi l_P^2} i[X^i, X^j], \tag{2.5}
\]

where \( i, j = 1, \ldots, 9 \). Usually \( R \) is taken to be \( R = g_s \sqrt{\alpha'} \), and so one has \( \lambda \equiv 2\pi\alpha' = 2\pi l_P^2 / R \). After the 9-11 flip of section 4, however, the role of the ninth and the eleventh coordinate are interchanged, and this relation is no longer true. For this reason we define a general quantity \( \beta \equiv 2\pi l_P^2 / R \), the D0-brane theory and its duals being recovered upon taking \( \beta = \lambda \).

Substituting for (2.5), the NS-NS currents are [19]:

\[
I_\phi = 1 - \frac{1}{2} F^{0i} F^{0i} + \frac{1}{4} F^{ij} F^{ij} - \frac{1}{8} \left( F^{\mu \nu} F_{\rho \sigma} F^{\rho \sigma} - \frac{1}{4} (F^{\mu \nu} F_{\mu \nu})^2 \right),
\]

\[
I_h^{00} = 1 + \frac{1}{2} F^{0i} F^{0i} + \frac{1}{4} F^{ij} F^{ij} = 1 + \frac{1}{2} \dot{X}^2 - \frac{1}{4\beta^2} [X, X]^2,
\]

\[
I_h^{0i} = -F^{0i} \left( 1 + \frac{1}{2} F^{0j} F^{0j} + \frac{1}{4} F^{jk} F^{jk} \right) = F^{0i} F^{00} = X_i \left( 1 + \frac{1}{2} \dot{X}^2 - \frac{1}{4\beta^2} [X, X]^2 \right) - \frac{1}{\beta^2} [X^i, X^j] [X^j, X^k] \dot{X}^k,
\]

\[
I_h^{ij} = F^{0i} F^{0j} + F^{ik} F^{kj} = X^i \dot{X}^j - \frac{1}{\beta^2} [X^i, X^j] [X^k, X^j],
\]

\[
I_s^{0i} = \frac{1}{2} F^{ij} F^{0j} = -\frac{i}{2\beta} [X^i, X^j] \dot{X}^j,
\]

\[
I_s^{ij} = \frac{1}{2} F^{ij} \left( 1 + \frac{1}{4} F^{kl} F^{kl} - \frac{1}{2} F^{0j} F^{0j} \right) + \frac{1}{2} F^{0i} F^{0k} F^{kj} - \frac{1}{2} F^{0j} F^{0k} F^{ki} + \frac{1}{2} F^{ik} F^{kl} F^{lj} = \frac{i}{2\beta} [X^i, X^j] \left( 1 - \frac{1}{2\beta^2} [X, X]^2 \right) - \frac{i}{\beta^2} [X^i, X^j] [X^j, X^k] \dot{X}^k - \dot{X}^i [X^j, X^k] \dot{X}^k,
\]

where we have defined

\[
\dot{X}^2 \equiv \dot{X}^i \dot{X}_i, \quad [X, X]^2 \equiv [X^i, X^j] [X^i, X^j], \quad [X, X]^4 \equiv [X^i, X^j] [X^j, X^k] [X^k, X^l] [X^l, X^i].
\]

The R-R currents are [19]

\[
I_0^0 = 1, \tag{2.12}
\]
where T 3 and Myers 2. The first such caveat is that we must have linear order in the background fields, of the multiple D0-brane theory of Taylor and van Raamsdonk 3 and Myers 4. The pull-back is defined in terms of gauge covariant derivatives, such that to match 2. In other words, D0-branes cannot expand into NS5-branes. We will therefore drop action, as given in 3, 2, we have not been able to match these extra terms in expressions (2.7) and (2.9) match precisely the first-order expansion of the non-abelian Born-Infeld actually contain further terms 19, the form of which seems less certain. Indeed, although the D0-brane action is 27, 3, 2, we will now show that the above currents are recovered in the expansion to linear order in the background fields of the multiple D0-brane theory 3. We will, for this reason, ignore them.

2.2 Matrix theory vs. multiple D0-branes

We will now show that the above currents are recovered in the expansion to linear order in the background fields of the multiple D0-brane theory 4. If we set the NS-NS fields to zero, the multiple D0-brane action is 27, 8, 2.

\[
\begin{align*}
I_0^i & = -F_0^{0i} = \dot{X}^i, \\
I_2^{0ij} & = -\frac{1}{6} F^{ij} = -\frac{i}{6\beta}[X^i, X^j], \\
I_2^{ijk} & = \frac{1}{6} \left( F_0^{0i} F^{jk} + F_0^{0j} F^{ki} + F_0^{0k} F^{ij} \right) \\
& = -\frac{i}{6\beta} \left( \dot{X}^i [X^j, X^k] + X^j [X^k, X^i] + \dot{X}^k [X^i, X^j] \right), \\
I_4^{0ijkl} & = \frac{1}{2} \left( F^{ij} F^{kl} + F^{ik} F^{lj} + F^{il} F^{jk} \right) \\
& = -\frac{1}{2\beta^2} \left( [X^i, X^j][X^k, X^l] + [X^i, X^k][X^j, X^l] + [X^i, X^l][X^j, X^k] \right), \\
I_4^{ijklm} & = -\frac{15}{2} F^{0[i} F^{jk} F^{lm]} = -\frac{15}{2\beta^2} \dot{X}^{[i} [X^j, X^k][X^l, X^m]}, \\
I_6^{0ijklm} & = -F^{ij} F^{kl} F^{mn} = \frac{i}{\beta^3} [X^i, X^j][X^k, X^l][X^m, X^n], \\
I_6^{ijklmnp} & = 7F^{0[i} F^{jk} F^{lm} F^{np]} = \frac{7i}{\beta^3} \dot{X}^{[i} [X^j, X^k][X^l, X^m][X^n, X^p].
\end{align*}
\]

Up to a couple of caveats, we show below that the above currents agree with the expansion, to linear order in the background fields, of the multiple D0-brane theory of Taylor and van Raamsdonk 3 and Myers 4. The first such caveat is that we must have \( I_6^{[i1 \ldots p6} = 0 \) if the two actions are to match 2. In other words, D0-branes cannot expand into NS5-branes. We will therefore drop this term from now on. The second is that the currents \( I_6^{ij} \) and \( I_4^{ij} \) derived from Matrix theory actually contain further terms 13, the form of which seems less certain. Indeed, although the expressions (2.21) and (2.23) match precisely the first-order expansion of the non-abelian Born-Infeld action, as given in 3, 4, we have not been able to match these extra terms in \( I_6^{ij} \) and \( I_4^{ij} \). We will, for this reason, ignore them.

\[
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(2.24)
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\[\text{See also } 13.\]
with obvious generalizations to forms of higher degree. The interior multiplication is
\[(i_X \Sigma)_{\mu_1...\mu_p} = X^i \Sigma_{i\mu_1...\mu_p}, \quad (2.25)\]
giving rise to the relevant commutators in the action. The Chern-Simons action \((2.22)\) becomes
\[
S_{\text{CS}} = T_0 \int \text{STr} \left\{ \mathcal{P} \left[ C^{(1)} + \frac{i}{\lambda} (i_X i_X) C^{(3)} - \frac{1}{2\lambda^2} (i_X i_X)^2 C^{(5)} - \frac{i}{6\lambda^3} (i_X i_X)^3 C^{(7)} \right. \right.
\]
\[
+ \left. \frac{1}{24\lambda^4} (i_X i_X)^4 C^{(9)} \right] \left\}. \quad (2.26)\]

It is easy to see that the D0-brane R-R currents, \((2.12)-(2.19)\), of the previous subsection, can be written such that
\[
C^{(1)}_{\mu} I^\mu_{0} dt = P[C^{(1)}], \quad (2.27)
\]
\[
C^{(3)}_{\mu\nu\rho} I^{\mu\nu\rho}_{2} dt = \frac{i}{\lambda} P[i_X i_X C^{(3)}], \quad (2.28)
\]
\[
\frac{1}{60} C^{(5)}_{\mu_1...\mu_5} I^{\mu_1...\mu_5}_{4} dt = -\frac{1}{2\lambda^2} P[(i_X i_X)^2 C^{(5)}], \quad (2.29)
\]
\[
\frac{1}{336} C^{(7)}_{\mu_1...\mu_7} I^{\mu_1...\mu_7}_{6} dt = -\frac{i}{6\lambda^3} P[(i_X i_X)^3 C^{(7)}], \quad (2.30)
\]
which reproduce the Chern-Simons action \((2.26)\). Since we have set the NS-NS 2-form to zero, these linear couplings are actually exact. We should also note that one could derive the correct Matrix theory coupling to \(C^{(9)}\) in \((2.4)\) by comparing with the Chern-Simons action \((2.26)\).

The NS-NS fields are somewhat harder to deal with. To linear order in the background fields, the non-abelian Born-Infeld action is
\[
S_{\text{NBI}} = S_{\text{NBI}}(\phi = 0) - \phi S_{\text{flat}}, \quad (2.31)
\]
with \(S_{\text{flat}}\) as in \((2.21)\). This gives the dilaton current:
\[
I_\phi = \sqrt{(1 - \dot{X} Q^{-1} \dot{X}) \det Q}, \quad (2.32)
\]
where \(\dot{X} Q^{-1} \dot{X} = \dot{X}^i (Q^{-1})_{ij} \dot{X}^j\). As shown already in \([27, 2]\), expanding the current \((2.32)\) to the relevant order, gives the Matrix theory result \((2.6)\).

The action containing the couplings to the metric and NS-NS 2-form is more complicated:
\[
S_{\text{NBI}}(\phi = 0) = -T_0 \int dt \text{STr} \left\{ \sqrt{-P} \left[ E_{00} + E_{0i} (Q^{-1} - \delta)^i_k E^{kj} E_{j0} \right] \det(Q^i_{\ j}) \right\}, \quad (2.33)
\]
where
\[
Q^i_{\ j} = \delta^i_j + \frac{i}{\lambda} [X^i, X^k] E_{kj}. \quad (2.34)
\]
To linear order in the background fields, \(E_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + b_{\mu\nu}\), so that
\[
Q^i_{\ j} = Q^i_{\ k} \left( \delta^i_k + (Q^{-1})^i_k \frac{i}{\lambda} [X^l, X^m] (h_{mj} + b_{mj}) \right), \quad (2.35)
\]
which gives, again to linear order,
\[
(Q^{-1})^i_{\ j} = \left( \delta^i_k - (Q^{-1})^i_k \frac{i}{\lambda} [X^l, X^m] (h_{mk} + b_{mk}) \right) (Q^{-1})^i_{\ j}. \quad (2.36)
\]
\[
\begin{align*}
I^{00}_h &= \sqrt{\text{det} Q} (1 - \dot{X} Q^{-1} \dot{X})^{-1/2}, \\
I^{0i}_h &= \sqrt{\text{det} Q} (1 - \dot{X} Q^{-1} \dot{X})^{-1/2} (Q^{-1})^{ij} \dot{X}^j, \\
I^{ij}_h &= \frac{1}{2} \sqrt{\text{det} Q} (1 - \dot{X} Q^{-1} \dot{X})^{-1/2} \left[ \ddot{X}^i (Q^{-1})^{jk} \dddot{X}^k - \frac{i}{\lambda} \dddot{X}^l (Q^{-1})_{lm} [X^m, X^j] (Q^{-1})^{jk} \dddot{X}^k \\
&\quad - \frac{i}{\lambda} (1 - \dot{X} Q^{-1} \dot{X}) (Q^{-1})^{jk} [X^k, X^i] + (i \leftrightarrow j) \right], \\
I^{0i}_s &= \frac{1}{2} \sqrt{\text{det} Q} (1 - \dot{X} Q^{-1} \dot{X})^{-1/2} (Q^{-1})^{ij} \dot{X}^j, \\
I^{ij}_s &= \frac{1}{4} \sqrt{\text{det} Q} (1 - \dot{X} Q^{-1} \dot{X})^{-1/2} \left[ \ddot{X}^i (Q^{-1})^{jk} \dddot{X}^k - \frac{i}{\lambda} \dddot{X}^l (Q^{-1})_{lm} [X^m, X^j] (Q^{-1})^{jk} \dddot{X}^k \\
&\quad - \frac{i}{\lambda} (1 - \dot{X} Q^{-1} \dot{X}) (Q^{-1})^{jk} [X^k, X^i] - (i \leftrightarrow j) \right].
\end{align*}
\]

One can verify that the lowest order expansion of these currents reproduces the results \((2.7)-(2.11)\).

3 T-duality: multiple D-strings

To construct the Matrix string theory of Dijkgraaf, Verlinde and Verlinde \[11\], one compactifies Matrix theory on a circle in, say, the \(x^9\) direction. Taylor has shown that this is equivalent to a \((1 + 1)\)-dimensional super Yang-Mills theory on the dual circle \[14\]. Taking now \(i, j = 1, \ldots, 8\), and denoting the dual coordinate by \(\hat{x}\), the worldvolume fields transform as \[14\]

\[
F_{ij} = F^{ij} = \frac{i}{\beta} [X^i, X^j] \quad \longrightarrow \quad \frac{1}{2\pi R_9} \int d\hat{x} \frac{i}{\beta} [X^i, X^j],
\]

\[
F_{9i} = F^{0i} = \frac{i}{\beta} [X^9, X^i] \quad \longrightarrow \quad \frac{1}{2\pi R_9} \int d\hat{x} \frac{\lambda}{\beta} D_\hat{x} X^i,
\]

\[
F_{0i} = - F^{0i} = \dot{X}^i \quad \longrightarrow \quad \frac{1}{2\pi R_9} \int d\hat{x} \dot{X}^i,
\]

\[
F_{99} = - F^{09} = \dot{X}^9 \quad \longrightarrow \quad \frac{1}{2\pi R_9} \int d\hat{x} \lambda \dot{A}_\hat{x},
\]

where \(R_9 = \alpha' / R_9\) is the radius of the dual circle. Of course, this is just T-duality applied to Matrix theory. By construction, the multiple D0-brane action \((2.20)\) considered in the previous section is covariant under T-duality, so an application of T-duality to the D0-brane action \((2.4)\) derived from Matrix theory should reproduce the non-relativistic limit of the multiple D-string action.

Certainly, the action \((2.1)\) in a flat background becomes, with \(D X^i \equiv D_\hat{x} X^i\) and \(A \equiv A_\hat{x}\),

\[
S_{\text{flat}} = - \frac{1}{2\pi R R_9} \int dt d\hat{x} \text{Tr} \left[ \frac{1}{2} \dot{X}^2 - \frac{\lambda^2}{2\beta^2} D X^2 + \frac{\lambda^2}{2} \dot{A}^2 + \frac{1}{4\beta^2} [X^i, X^j]^2 \right],
\]

which, with \(\beta = \lambda\), is just the non-relativistic limit of the flat space D-string action; with T-duality acting on the string coupling as

\[
g_s \longrightarrow \hat{g}_s = g_s \frac{\sqrt{\alpha'}}{R_9},
\]
the overall factor is $T_1 = 1/(\hat{g}_s \lambda)$ as required. Turning to the linear action \((2.4)\), we must consider T-duality applied to both the worldvolume and background fields. As far as the currents are concerned, we simply take the $I$s written in terms of the Born-Infeld field strength \((2.3)\), and reinterpret the relevant components through the action of T-duality as given by \((3.1)\), \((3.4)\) above\(^{[8]}\). This is a simple re-writing, the results being collected in the appendix.

To linear order, the action of T-duality on the background fields is:

$$h_{a9} \leftrightarrow -b_{a9}, \quad h_{9a} \leftrightarrow b_{9a}, \quad h_{99} \leftrightarrow -h_{99}, \quad \phi \rightarrow \phi - \frac{1}{\lambda} h_{99}, \quad (3.7)$$

$$C_{a_1 \ldots a_{p-1}9}^{(p)} \leftrightarrow C_{a_1 \ldots a_{p-1}9}^{(p-1)}, \quad (3.8)$$

where $a, b = 0, \ldots, 8$, and all other fields are invariant. A simple application of these rules gives

$$S_{\text{linear}} = \frac{1}{2\pi R R_9} \int dt d\hat{x} \text{STr} \left\{ \frac{1}{2} h_{ab} I^{ab}_h - 2 h_{a9} I^{a9}_h - \frac{1}{2} h_{99} I^{99}_h + \left( \phi - \frac{1}{2} h_{99} \right) I_\phi - b_{a9} I^{a9}_h + b_{ab} I^{ab}_s + C^{(0)} I^{9}_0 + C^{(2)} I^{a9}_0 + 3 C^{(2)} I^{ab9}_2 + C^{(4)} I^{abc9}_2 + \frac{1}{12} C^{(4)} I^{a_1 \ldots a_9}_4 + \frac{1}{60} C^{(6)} I^{a_1 \ldots a_9}_4 + \frac{1}{48} C^{(6)} I^{a_1 \ldots a_9}_4 + \frac{1}{336} C^{(8)} I^{a_1 \ldots a_9}_4 \right\}. \quad (3.9)$$

There should also be an extra coupling to $C^{(8)}$, coming from the unknown Matrix theory coupling to $C^{(9)}$.

The resulting action should be equivalent to the linearized version of the D-string action. With the results in the appendix, it is easy to see that the R-R terms can indeed be written as

$$S_{\text{R-R}} = T_1 \int \text{STr} \left\{ P \left[ \lambda C^{(0)} \wedge F + C^{(2)} + i (i X i X) C^{(2)} \wedge F + \frac{i}{\lambda} (i X i X) C^{(4)} - \frac{1}{2\lambda} (i X i X)^2 C^{(4)} \wedge F \right. \right.$$  

$$- \frac{1}{2\lambda^2} (i X i X)^2 C^{(6)} - \frac{i}{6\lambda^3} (i X i X)^3 C^{(6)} \wedge F - \frac{i}{6\lambda^3} (i X i X)^3 C^{(8)} + \frac{1}{24\lambda^3} (i X i X)^4 C^{(8)} \wedge F \right) \right\}, \quad (3.10)$$

where $F_{09} = \hat{A}$. We should note that only half of the terms necessary to form the pullback of $C^{(8)}$ are present in the linear action \((3.9)\). The missing terms come from the $C^{(9)}$ coupling in the D0-brane action \((2.4)\), as mentioned above. At any rate, this is just the expansion of the Chern-Simons action

$$S_{\text{CS}} = T_1 \int \text{STr} \left\{ P \left[ e^{i(i X i X)/\lambda} \left( \sum_n C^{(n)} \right) \right] \wedge e^{\lambda F} \right\}, \quad (3.11)$$

as expected. It is computationally more involved to check the NS-NS couplings in \((3.9)\) against the non-abelian Born-Infeld theory of D-strings, and we will not do this here.

## 4 Matrix string theory: multiple IIA F-strings

### 4.1 The 9-11 flip

Having constructed the $(1 + 1)$-dimensional theory of the D-string, we are now in a position to perform the so-called 9-11 flip, a rotation

$$x^9 \rightarrow x^{11}, \quad x^{11} \rightarrow -x^9, \quad (4.1)$$

which will give us the type IIA Matrix string theory action in a linear background. Whereas Schiappa\(^{[17]}\) considers the action of the 9-11 flip on the currents, and leaves the background fields
invariant, we take the view here that it is the currents which are invariant under the 9-11 flip. After all, in the flat space case, the 9-11 flip does not change the worldvolume fields [11]. Moreover, Schiappa has argued that the currents are in fact invariant under the S- and T-dualities [17]. We simply take $R_9 = g_s \sqrt{\alpha'}$, so that $\tilde{R}_9 = \sqrt{\alpha'}/g_s$ and $l_P = g_s^{4/3} \tilde{R}_9$. We then define the dimensionless worldsheet coordinates

$$\sigma = \frac{\hat{x}}{R_9}, \quad \tau = \frac{R}{\alpha'} t,$$

and perform the rescalings $X^i \to \sqrt{\alpha'} X^i$ and $g_s \to g_s/(2\pi)$. The flat space action (3.3) becomes the Matrix string action of [11]:

$$S_{\text{flat}} = \frac{1}{2\pi} \int d\tau d\sigma \sum_{n=1}^{N} \left[ \frac{1}{2} \dot{X}^2 - \frac{1}{2} DX^2 + \frac{g_s^2}{2} \dot{A}^2 + \frac{1}{4g_s^2} [X, X]^2 \right],$$

where $-\infty < \sigma < 2\pi$. Weakly coupled string theory at $g_s = 0$ is recovered in the IR limit, so is described by strongly coupled (1+1)-dimensional Yang-Mills. The conformal field theory which describes this IR limit is a sigma model on an orbifold target space [11]. The matrix-valued coordinates must commute in this limit, so can be simultaneously diagonalized, the eigenvalues $x_1, \ldots, x_N$ corresponding to the positions of the $N$ strings. Then the action (4.3) reduces to a sum of Green-Schwarz actions for light-cone gauge string theory:

$$S_{\text{flat}} = \frac{1}{2\pi} \int d\tau d\sigma \sum_{n=1}^{N} \left[ \frac{1}{2} \dot{x}_n^2 - \frac{1}{2} \partial x_n^2 \right].$$

The couplings to linear background fields considered herein should thus tell us something about light-cone gauge string theory in weakly curved backgrounds.

The action of the 9-11 flip on the background fields is easy to derive. Consider, for example, the type IIB metric fluctuation $h_{ab}$. Its T-dual on the type IIA side is $-b_{a9}$, or in eleven-dimensional language $-A_{a911}$. Under the 9-11 flip $-A_{a911} \to A_{a119} = -b_{a9}$. In other words, $h_{a9} \to -b_{a9}$ under the 9-11 flip, which is just (linearized) S-duality followed by (linearized) T-duality in the $x^9$ direction. Arguing in a similar manner, one finds that the linear background fields transform in the following way under the 9-11 flip:

$$h_{ab} \to h_{ab} - \eta_{ab} (\phi - \frac{1}{2} h_{99}), \quad h_{a9} \to -b_{a9}, \quad h_{99} \to - (\phi + \frac{1}{2} h_{99}),$$

$$\phi \to -\phi + \frac{1}{2} h_{99}, \quad \phi \to -\phi + \frac{1}{2} h_{99}, \quad b_{ab} \to -C^{(3)}_{ab9}, \quad b_{a9} \to -C^{(1)}_{a},$$

$$C^{(0)} \to -C^{(1)}, \quad C^{(2)} \to b_{ab}, \quad b_{a9} \to -h_{a9},$$

$$C^{(4)}_{a1...a4} \to C^{(5)}_{a1...a49}, \quad C^{(6)}_{abc9} \to C^{(3)}_{abc}, \quad C^{(6)}_{a1...a6} \to N^{(7)}_{a1...a69},$$

$$C^{(6)}_{a1...a59} \to \tilde{b}_{a1...a59}, \quad C^{(8)}_{a1...a8} \to -C^{(9)}_{a1...a89}, \quad C^{(8)}_{a1...a79} \to -C^{(7)}_{a1...a7},$$

Here, $N^{(7)}_{a1...a69}$ is the field that couples minimally to a type IIA Kaluza-Klein monopole whose Taub-NUT direction is along $x^9$. It is easy to verify that the above transformations are precisely what one would find by performing a linearized S-duality (in the string frame)

$$h \to h - \eta \phi, \quad \phi \to -\phi, \quad b_{2} \to -C^{(2)},$$

$$C^{(0)} \to -C^{(0)}, \quad C^{(2)} \to b_{2}, \quad C^{(6)} \to \tilde{b}_{6}, \quad C^{(8)} \to -C^{(8)},$$

The couplings to linear background fields considered herein should thus tell us something about light-cone gauge string theory in weakly curved backgrounds.
followed by a linearized T-duality in the $x^9$ direction, as in (3.7) and (3.8), plus the linearized T-duality rules for the field $\tilde{b}_6$ [31]:

$$\tilde{b}_{a_1...a_6} \rightarrow N^{(7)}_{a_1...a_6}, \quad \tilde{b}_{a_1...a_59} \rightarrow \tilde{b}_{a_1...a_59}.$$  \hfill (4.8)

Performing the transformations (4.3) on the linear action (3.9), and substituting for (4.2), one finds

$$S_{\text{linear}}^\text{IIA} = \frac{1}{2\pi} \int d\tau d\sigma \frac{\alpha'}{R^2} \text{STr} \left\{ \left( \frac{1}{2} (h_{ab} - \eta_{ab}) \left( \phi - \frac{1}{2} h_{99} \right) \right) I^{ab}_h + 2b_{a9} I^{a9}_h + \frac{1}{2} \left( \phi + \frac{1}{2} h_{99} \right) I^{99}_h \right\}$$

$$- \frac{1}{2} \left( \phi - \frac{3}{2} h_{99} \right) I_\phi + C^{(1)}_a I^{a9}_h - C^{(3)}_{a9} I^{ab}_h - C^{(1)}_9 I^9_0 - h_{a9} I^a_0 + 3b_{a9} I^{ab9} + C^{(3)}_{abc} I^{abc}_2$$

$$+ \frac{1}{12} C^{(5)}_{a1a49} I^{a1a49}_4 + \frac{1}{60} b_{a_1...a_59} I^{a_1...a_59}_4 + \frac{1}{48} N^{(7)}_{a_1...a_69} r^{a_1...a_69}_4 - \frac{1}{336} C^{(7)}_{a_1...a_7} I^{a_1...a_7}_6 \right\}.$$  \hfill (4.9)

Writing the currents in the appendix in terms of the dimensionless quantities $\tau$ and $\sigma$, and after rescaling $X^i \rightarrow \sqrt{\alpha'} X^i$ and $g_s \rightarrow g_s/(2\pi)$, this is the action describing Matrix string theory in a weakly curved background. There is no need to write out the new couplings in full, suffice it to say that each term of the form $X, \lambda DX/\beta, [X, X]/\beta$ and $\lambda A$ appears multiplied by a factor of $R/\sqrt{\alpha'}$, each $[X, X]$ term appears multiplied by a factor of $1/g_s$, and each $A$ term appears with a factor of $g_s$. We should note that the couplings derived by Schiappa [17] are as above, but without those couplings to fields with a component in the $x^9$ direction. As we will see below, these latter are necessary to match with light-cone gauge string theory.

### 4.2 Light-cone gauge string theory in a general background

To examine the action (4.9), let us consider the weakly coupled string theory, $g_s = 0$. Just as in the flat case, the Born-Infeld field strength drops out entirely, and one is forced onto the space of commuting matrices as above. Moreover, the couplings to all Ramond-Ramond fields vanish, as one might expect, and we find

$$S_{\text{linear}}^\text{IIA} = \frac{1}{2\pi} \int d\tau d\sigma \sum_{n=1}^N \left\{ \frac{\alpha'}{R^2} \left( \frac{1}{2} h_{00} - h_{99} + \frac{1}{2} h_{99} \right) + \frac{\sqrt{\alpha'}}{R} \left( (h_{0i} - h_{9i}) \dot{x}^i_n + (b_{0i} - b_{9i}) \partial x^i_n \right) \right.$$  

$$+ \frac{1}{4} (h_{00} - h_{99}) (\dot{x}^2_n + \partial x^2_n) + \frac{1}{2} h_{ij} (\dot{x}^i_n \dot{x}^j_n - \partial x^i_n \partial x^j_n) + b_{ij} \dot{x}^i_n \partial x^j_n + b_{09} \dot{x}_n \cdot \partial x_n$$

$$+ \frac{R}{\sqrt{\alpha'}} \left( \frac{1}{2} h_{0i} \dot{x}^i_n (\dot{x}^2_n + \partial x^2_n) + b_{0i} \left( \frac{1}{2} \partial x^i_n (\dot{x}^2_n + \partial x^2_n) + \dot{x}^i_n \partial x_n \cdot \dot{x}_n \right) \right)$$

$$+ \frac{R^2}{\alpha'} \left( \frac{1}{2} \phi - \frac{3}{8} h_{99} \right) \left( (\dot{x}_n \cdot \partial x_n)^2 + \frac{1}{4} (\dot{x}^2_n + \partial x^2_n)^2 \right) \right\}.$$  \hfill (4.10)

If we now define the (somewhat non-standard) light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}} (X^0 + X^9),$$  \hfill (4.11)

then we have

$$S_{\text{linear}}^\text{IIA} = \frac{1}{2\pi} \int d\tau d\sigma \sum_{n=1}^N \left\{ \frac{\alpha'}{R^2} h_{++} + \frac{\sqrt{\alpha'}}{R} \sqrt{2} (h_{+i} \dot{x}^i_n + b_{+i} \partial x^i_n) + \frac{1}{2} h_{+-} (\dot{x}^2_n + \partial x^2_n) \right.$$  

$$+ \frac{R}{\sqrt{\alpha'}} \left( \frac{1}{2} \phi - \frac{3}{8} h_{99} \right) \left( (\dot{x}^2_n + \partial x^2_n)^2 \right) \right\}.$$
Since the worldsheet energy-momentum tensor vanishes, this must be supplemented with the constraint \(G \cdot (4.12)\). However, the constraint \(4.15\) can be solved for where we have defined \(G\). To linear order, then, we can replace \(\dot{x}^{\mu} \cdot \dot{x}^{\nu} B_{\mu\nu}(x)\) in the string action in a weakly curved background.

The light-cone gauge is defined by taking \(x^{+}(\tau,\sigma) = \tau\). Since we are only interested in linear backgrounds, we set \(G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\), where \(\eta_{+-} = -1\) and \(\eta_{ij} = \delta_{ij}\), and \(B_{\mu\nu} = b_{\mu\nu}\). As usual in a light-cone treatment of gravity, we can further use spacetime diffeomorphisms to set \(G_{--} = 0 = G_{-i}\). The string action is then

\[
S = \frac{1}{2\pi \alpha'} \int d\tau d\sigma \left\{ -2\dot{x}^{2} - \dot{x}^{2} + \dot{x}^{2} + h_{++} + h_{++} \dot{x}^{i} + h_{+} \dot{x}^{i} + \frac{1}{2} h_{ij} (\dot{x}^{i} \dot{x}^{j} - \dot{x}^{i} \dot{x}^{j}) \right.
\]
\[
+ b_{+} \dot{x}^{i} \dot{x}^{i} + b_{+} \dot{x}^{i} \dot{x}^{j} + b_{-} \left( \frac{1}{2} (\dot{x}^{2} + \dot{x}^{2}) \dot{x}^{i} - \dot{x}^{i} \dot{x}^{i} \right) \right\}.
\] (4.16)

At first sight, the linear couplings to the background fields do not seem to match those in the action \(4.12\). However, the constraint \(4.13\) can be solved for \(x^{-}\) giving, again to linear order in \(h_{\mu\nu}\),

\[
\dot{x}^{-} = \frac{1}{2} (\dot{x}^{2} + \dot{x}^{2}) + \frac{1}{2} h_{++} + \dot{x}^{i} h_{++} + h_{+} \dot{x}^{i} \dot{x}^{i} \dot{x}^{j} h_{ij} + \frac{1}{2} (\dot{x}^{2} + \dot{x}^{2}) h_{++},
\] (4.17)
\[
\dot{x}^{i} \dot{x}^{j} = \dot{x}^{i} \dot{x}^{j} h_{++} + \dot{x}^{i} \dot{x}^{j} h_{ij} + \dot{x} \cdot \dot{x} h_{i},
\] (4.18)

To linear order, then, we can replace \(\dot{x}^{-}\) and \(\dot{x}^{i}\) in the string action with \((\dot{x}^{2} + \dot{x}^{2})/2\) and \(\dot{x} \cdot \dot{x}\) respectively. In that case, we find exact agreement with the Matrix string theory result \(4.12\). In other words, we have succeeded in reproducing the correct form of the light-cone gauge type IIA string action in a weakly curved background.

### 4.3 Ramond-Ramond couplings

Turning to the R-R couplings in the action \(4.9\), let us consider the 3-form couplings, which could potentially give rise to D2-brane solutions. If we set the Born-Infeld field to zero, we find

\[
S_{C(3)} = \frac{i}{2\pi g_{s}} \int d\tau d\sigma \text{STr} \left\{ \frac{\sqrt{\alpha'}}{R} C^{(3)}_{+} + X^{i} C^{(3)}_{i} \right\},
\] (4.19)

where we have defined

\[
C^{(3)}_{+} = \frac{1}{\sqrt{2}} C^{(3)}_{+ij} [X^{j}, X^{i}],
\] (4.20)
\[
C^{(3)}_{i} = \frac{1}{2} C^{(3)}_{ijk} [X^{k}, X^{j}] + C^{(3)}_{+-j} [X^{j}, X^{i}].
\] (4.21)
The couplings given in [17] differ considerably from the ones given here, since in [17] only the $C_{0ij}^{(3)}$ and $C_{ijk}^{(3)}$ terms were considered. Yet from our analysis it is clear that the other terms not only contribute, but are necessary in order to be able to write the $C^{(3)}$ couplings in the light-cone gauge.

A nice check of the 3-form couplings (4.19) is against the eleven-dimensional supermembrane theory. The connection between the light-cone treatment of the eleven-dimensional supermembrane in flat space and Matrix theory is well-known [33]. One can further argue that Matrix string theory is found by compactifying the light-cone supermembrane theory on a circle, the off-diagonal elements of the Matrix string theory fields being related to the infinite tower of Kaluza-Klein modes along the compact direction [34]. Given further that the light-cone supermembrane theory in an arbitrary supergravity background is known [35, 36, 37], one could in principle derive Matrix string theory in an arbitrary background using these techniques. Indeed, the Lagrangian density for the light-cone supermembrane in a curved (but not entirely general) background has been derived [36] and it is easy to see that the methods of [34] would generate some of the couplings in (4.19) in our Matrix string theory action. More specifically, the couplings to $C_{ij}^{(3)}$ and $C_{ijk}^{(3)}$ are reproduced although, since the gauge $C_{+ij}^{(3)} = 0$ was chosen in [36], the coupling to these components of $C^{(3)}$ cannot be checked. Of course, neither can the couplings to the remaining R-R fields, since it is not known how or, indeed, if their eleven-dimensional counterparts couple to the supermembrane.

It can be seen from (1.9) that as regards the R-R 5-form potential, only the terms of the form $C_{ijklmn}^{(5)}$ contribute. It is easy to check that the terms involving $C_{a_1...a_9}^{(5)}$ would couple to the current $I_5$, which was previously set to zero in order to match the action (2.4) with the action of the multiple D0-brane theory of [1, 2]. Setting the Born-Infeld vector to zero, the remaining 5-form R-R field couplings can be written as:

$$S_{C^{(5)}} = \frac{i}{4\pi g_s} \frac{R}{\sqrt{\alpha'}} \int d\tau d\sigma \text{STr} \left\{ \frac{\sqrt{\alpha'}}{R} C_{+i}^{(5)} + \dot{X}^i C_{-i}^{(5)} - \dot{X}^i C_{-i}^{(5)} \right\},$$

(4.22)

where we have defined

$$C_{\mu\nu}^{(5)} = [X^k, X^j] DX^i C_{ijkl\mu\nu}^{(5)}.$$  

(4.23)

Similarly, the $C^{(7)}$ couplings only have contributions involving terms of the form $C_{a_1...a_7}^{(7)}$, since terms of the form $C_{a_1...a_9}^{(7)}$ couple to currents for which the explicit expression is not known, corresponding to $N^{(7)}$ couplings in the D0-brane action (2.4). The $C_{a_1...a_7}^{(7)}$ terms can be written as

$$S_{C^{(7)}} = \frac{i}{96\pi g_s^2} \frac{R^2}{\alpha'} \int d\tau d\sigma \text{STr} \left\{ \frac{\sqrt{\alpha'}}{R} C_{+i}^{(7)} + \frac{\sqrt{\alpha'}}{R} C_{-i}^{(7)} + \dot{X}^i C_{i}^{(7)} \right\},$$

(4.24)

with

$$C_{\mu}^{(7)} = [X^n, X^m][X^l, X^k][X^j, X^i] C_{ijklmn\mu}^{(7)}.$$  

(4.25)

Let us for the sake of completeness also consider the couplings to $\tilde{b}^{(6)}$ and $N^{(7)}$. As for $C^{(5)}$, only the terms with a 9-component appear in the action. The other components couple to currents for which the explicit expression is not known. The $\tilde{b}^{(6)}$ couplings can be written as:

$$S_{\tilde{b}^{(6)}} = \frac{1}{16\pi g_s^2} \frac{R}{\sqrt{\alpha'}} \int d\tau d\sigma \text{STr} \left\{ \frac{\sqrt{\alpha'}}{R} \tilde{b}_{+i}^{(6)} + \dot{X}^i \tilde{b}_{+i}^{(6)} - \dot{X}^i \tilde{b}_{-i}^{(6)} \right\},$$

(4.26)

where

$$\tilde{b}_{\mu\nu}^{(6)} = [X^l, X^k][X^j, X^i] \tilde{b}_{ijkl\mu\nu}^{(6)}.$$  

(4.27)
and the $N^{(7)}$ couplings are given by

$$S_{N^{(7)}} = -\frac{1}{16\pi g_s} R^2 \int d\tau d\sigma \text{STr} \left\{ \sqrt{\alpha'} \frac{N^{(7)}_{+i} - \dot{X}^i N^{(7)}_{+i} + \dot{X}^i N^{(7)}_{-i}}{R} \right\}, \quad (4.28)$$

where

$$N^{(7)}_{\mu\nu} = [X^m, X^l][X^k, X^j]DX^{iN^{(7)}_{ijklm\mu\nu}}. \quad (4.29)$$

Note that the couplings of the different fields occur at a different order of the expansion parameter $R/\sqrt{\alpha'}$.

## 5 T-duality once more: multiple IIB F-strings

To describe fundamental strings in the type IIB theory, we perform another T-duality in the $x^9$ direction, as in $(3.7)$ and $(3.8)$. As before, we assume that the worldvolume fields (the currents) do not change, so the flat action $(4.3)$ is unchanged. The linear action $(4.9)$ becomes

$$S_{IIB\text{ linear}} = \frac{1}{2\pi} \int d\tau d\sigma \frac{\alpha'}{R^2} \text{STr} \left\{ \frac{1}{2} (h_{ab} - \phi \eta_{ab}) I_h^{ab} + C^{(2)} I_{\phi} + \frac{1}{2} (\phi - h_{99}) I_9^{99} - \frac{1}{2} (\phi + h_{99}) I_{\phi} 
- C^{(2)} I_s - 2 h_{a9} I_{a9}^{a9} + b_{a9} I_9^{a9} - C^{(0)} I_0^{99} + C^{(4)} I_2^{abc} + 3 b_{ab} I_{a9}^{b9} + \frac{1}{12} C^{(4)} I_{a1...a4}^{a1...a49} 
+ \frac{1}{60} b_{a1...a5} I_{a1...as}^{a1...as} + \frac{1}{48} b_{a1...a4} I_{a1...as}^{a1...as9} - \frac{1}{336} C^{(8)} I_{a1...a7}^{a1...a79} \right\}, \quad (5.1)$$

which should describe strongly coupled Matrix strings in the IIB theory, with the currents as in the appendix, up to the coordinate transformations $(4.2)$, and the rescalings $X^i \rightarrow \sqrt{\alpha'} X^i$ and $g_s \rightarrow g_s/(2\pi)$.

Note that precisely the same action is obtained if one applies the S-duality rules $(4.6)$ and $(4.7)$ to the D1-brane action $(3.9)$. Although this might be expected from the S-duality connection between D- and F-strings in the type IIB theory, it is not clear a priori how such an S-duality should be done directly, since we are dealing with non-abelian fields; as in Yang-Mills theory, we cannot rigorously perform a worldvolume duality transformation to show the S-duality equivalence between the two.

It is worth pointing out however that the S-duality transformation of the NS-NS 2-form:

$$b^{(2)} \rightarrow -C^{(2)}, \quad C^{(2)} \rightarrow b^{(2)}, \quad (5.2)$$

implies that the dynamics will now be governed by open D-strings, given that the invariant 2-form field strength that will couple in the worldvolume is constructed with $C^{(2)}$ instead of $b^{(2)}$. Therefore the Born-Infeld field $A$ should transform into a new worldvolume vector field $A'$ whose abelian component forms a gauge invariant field strength with the R-R 2-form. The right S-duality transformation rule should be then

$$A \rightarrow -A', \quad A' \rightarrow A, \quad (5.3)$$

in order to match $(5.2)$. This cannot however be seen explicitly at the level of the linearized actions that we have considered.
The action above can be rewritten in a more convenient form, filling in the expressions for the currents as given in the appendix (upon the rescaling). In particular, for the Chern-Simons action we have:

\[ S_{\text{CS}}^{\text{IIB}} = \frac{1}{2\pi} \int d\tau d\sigma \text{STr} \left\{ P \left[ \frac{\alpha'}{R^2} b^{(2)} + \frac{g_s \sqrt{\alpha'}}{R} C^{(0)} \wedge F + \frac{i \sqrt{\alpha'}}{g_s R} (ixi_x) C^{(4)} + \frac{i (ixi_x) b^{(2)} \wedge F}{g_s} \right] \right\}. \]

Again we note that only half of the terms necessary to form the pullback of \( C^{(8)} \) are present in the linear action. Some of the above couplings have been given before in [8].

6 T-duality along a transverse direction: multiple IIA strings with winding number

Let us now also describe type IIA fundamental strings with winding number. The IIA strings that are described by Matrix string theory carry momentum \( p^+ \). This is the charge which is related by the 9-11 flip to the number of D-particles. T-duality in the \( x^9 \) direction gives IIB strings with winding number, which we have just checked are S-dual to multiple D-strings. Strongly coupled IIA F-strings with winding number can then be obtained from IIB strings by performing a T-duality transformation in a direction transverse to the IIB strings. These strings with winding number have interesting dielectric properties that we will discuss in the next section.

Calling \( z \) the T-duality direction and \( a = (0, i) \), where now \( i = 1, \ldots, 7 \), the linear action that is obtained from [5.1] is given by:

\[ S_{\text{linear}}^{\text{IIA}} = \frac{1}{2\pi} \int d\tau d\sigma \frac{\alpha'}{R^2} \text{STr} \left\{ \frac{1}{2} \left( h_{ab} + \eta_{ab} (\phi - \frac{1}{2} h_{zz}) \right) I_{I}^{ab} - b_{az} r_{az} - \frac{1}{2} \left( \frac{1}{2} h_{zz} + \phi \right) I_{I}^{zz} \right. \]

\[ + C_{a_9 z}^{(3)} I_{I}^{a_9 z} - \frac{1}{2} \left( h_{99} - \phi + \frac{1}{2} h_{zz} \right) I_{I}^{99} - C_{9_9}^{(1)} I_{I}^{99} - \frac{1}{2} \left( \phi - \frac{1}{2} h_{zz} + h_{99} \right) I_{I}^{zz} - C_{a b z}^{(3)} I_{I}^{a b z} - 2 C_{a}^{(1)} I_{I}^{a z} \]

\[ - 2 h_{a_9 g} r_{a_9 g} - 2 b_{a_9} I_{I}^{a_9 g} + b_{a_0} r_{a_0} + h_{a_9 z} I_{I}^{a_9 z} - C_{a_9 z}^{(1)} I_{I}^{a_9 z} + C_{a b c z}^{(5)} I_{I}^{a b c z} - 3 C_{a b g} I_{I}^{a b g} + 3 b_{a_9} I_{I}^{a_9} - 6 h_{a 9 z} I_{I}^{a 9 z} \]

\[ + \frac{1}{60} N_{a_1 \ldots a_9 z} r_{a_1 \ldots a_9 z} I_{I}^{a_1 \ldots a_9 z} + \frac{1}{12} \delta_{a_1 \ldots a_4 z} I_{I}^{a_1 \ldots a_4 z} + \frac{1}{12} C_{a_1 \ldots a_4 z} I_{I}^{a_1 \ldots a_4 z} + \frac{1}{3} C_{a b c e z} I_{I}^{a b c e z} - \frac{1}{336} C_{a_1 \ldots a_7 z} I_{I}^{a_1 \ldots a_7 z} \].

The direction \( z \) in which the T-duality is performed appears as an isometry direction in the transverse space of the strings. We denote the corresponding Killing vector as

\[ k^\mu = \delta^\mu_z, \quad k_\mu = \eta_{z \mu} + h_{z \mu}. \]

In a manner similar to the Kaluza-Klein monopole, the non-abelian strings do not see this special direction, the embedding scalar \( X^z \) is not a degree of freedom of the strings, but is transformed under T-duality into a world volume scalar \( \omega \) [11]. This worldvolume scalar forms an invariant field strength with \( b_{a z} \) (see [18] for the details), and can therefore be associated to fundamental strings wrapped around the isometry direction \( z \), which themselves end on the Matrix strings.
The action \( (6.1) \) can be written in a covariant way as a gauged sigma model, where gauge covariant derivatives \( \mathcal{D}_\alpha X^\mu \) are used to gauge away the embedding scalar corresponding to the isometry direction \( \beta \):

\[
\mathcal{D}_\alpha X^\mu = D_\alpha X^\mu - k_\rho D_\alpha X^{\rho k^\mu},
\]

with \( \alpha = \sigma, \tau \). These gauge covariant derivatives reduce to the standard covariant derivatives \( D_\alpha X^\mu \) for \( \mu \neq z \) and are zero for \( \mu = z \). The pull-backs that appear in the action of the F-strings with winding are constructed from these gauge covariant derivatives. For example, \( F \rightarrow \alpha \), \( D \rightarrow \alpha \), \( \rho \rightarrow \alpha \), and so on.

\[
P \left[ b^{(2)} \right] = b_{\mu \nu} {\mathcal{D}}^{\mu} X^\nu \, dt \, dx
\]

\[
= \left( b_{09} + \frac{R}{\sqrt{\alpha'}} b_{0i} DX^i + \frac{R}{\sqrt{\alpha'}} b_{29} X^9 + \frac{R^2}{\alpha'} b_{ij} X^i DX^j \right) dt \, dx.
\]

Filling in the expressions for the currents as given in the appendix (upon the rescaling) we can write the Chern-Simons action as:

\[
S_{\text{linear}} = \frac{1}{2\pi} \int d\tau d\sigma \, \text{STr} \left\{ \mathcal{P} \left[ -\frac{g_s \sqrt{\alpha'}}{R} i_k C^{(1)} \right. \right.
\]

\[
+ \frac{i}{g_s} \frac{\sqrt{\alpha'}}{R} (i_k x^i) C^{(3)} + \frac{i}{g_s} (i_k x^i) C^{(3)} \right) \wedge D\omega + \frac{R}{g_s \sqrt{\alpha'}} (i_k x^i) \right) C^{(3)} \wedge F - \frac{i}{g_s R} (i_k x^i) i_k C^{(5)}
\]

\[
+ \frac{i}{g_s} \frac{\sqrt{\alpha'}}{R} (i_k x^i) C^{(5)} \wedge F + \frac{1}{2g_s^3} (i_k x^i) i_k \right) C^{(6)} + \frac{R}{2g_s^2 \sqrt{\alpha'}} (i_k x^i) i_k \right) C^{(6)} \wedge D\omega
\]

\[
+ \frac{R}{2g_s^2 \sqrt{\alpha'}} (i_k x^i) C^{(7)} + \frac{i}{g_s^3} (i_k x^i) C^{(7)} + \frac{R}{6g_s^3 \sqrt{\alpha'}} (i_k x^i) C^{(9)} \right) \left\}
\]

where we have introduced the following types of interior multiplication:

\[
\begin{align*}
(i_k \Sigma)_{\mu_1 \ldots \mu_p} &= k^\rho \Sigma_{\mu_1 \ldots \mu_p} = \Sigma_{z \mu_1 \ldots \mu_p}, \\
(i_k x^i \Sigma)_{\mu_1 \ldots \mu_p} &= [X^i, \omega] \Sigma_{\mu_1 \ldots \mu_p}.
\end{align*}
\]

Again only half of the terms to form the pullback of \( C^{(7)} \) and \( C^{(9)} \) are present in the linear action \( (6.1) \). Some of the couplings in \( (6.5) \) have been derived earlier in \( (6.8) \), applying T- and S-duality relations to the actions of D-branes. We will see in the next section that the presence of the isometry direction will enable us to find dielectric solutions of F-strings expanding into Dp-branes with \( p \) even.

7 Dielectric solutions

It is well-known in the case of D-branes that some of the linear couplings in \( (2.3) \) and \( (1.9) \) give rise to stable, dielectric configurations. In particular it was shown \( (2.3) \) that a set of coinciding Dp-branes in the presence of a \( (p + 4) \)-form R-R field strength will expand into a non-commutative or fuzzy two-sphere, \( S_{NC}^2 \). This configuration is stable and can be identified with a fuzzy D(\( p + 2 \))-brane of topology \( \mathbb{M}^{p + 1} \times S_{NC}^2 \). This dielectric D(\( p + 2 \))-brane has no net (monopole) charge with respect to the \( (p + 4) \)-form R-R field strength, but it does have a dipole moment.

In this section we will comment on various solutions describing F-strings expanding into D-branes.
7.1 Dielectric D-strings

It is instructive to review the case of $N$ D-strings expanding into a dielectric D3-brane in our notation. The action for $N$ coinciding D-strings is given by the action (3.5) for a flat background plus the action (3.9) for the linear couplings, which in the static case ($X^i = DX^i = \dot{A} = 0$) gives rise to the following potential:

$$V_{D1} = \text{STr} \left\{ -\frac{1}{4\lambda^2} [X, X]^2 + \frac{i}{3\lambda} X^k X^j X^i F^{(5)}_{09ijk} \right\}, \quad (7.1)$$

where, in order to obtain the couplings to the five-form field strength, we performed a partial integration and a non-abelian Taylor expansion \[39, 2\]

$$C^{(4)}_{\mu\nu\rho\lambda}(X) = C^{(4)}_{\mu\nu\rho\lambda}(0) + \partial_k C^{(4)}_{\mu\nu\rho\lambda}(0) X^k + ... \quad (7.2)$$

The equation of motion

$$[[X^i, X^j], X^j] + \frac{i\lambda}{2} [X^k, X^j] F^{(5)}_{09ijk} = 0, \quad (7.3)$$

is clearly satisfied for the following choice of $F^{(5)}$ and $X^i$:

$$F^{(5)}_{09ijk} = f\varepsilon_{ijk}; \quad X^i = -\frac{\lambda}{4} f\sigma^i \quad \text{where} \quad [\sigma^i, \sigma^j] = 2i \varepsilon^{ijk}\sigma^k, \quad (7.4)$$

where $i, j, k = 1, 2, 3$. In other words, three of the transverse coordinates of the D-string form a $N \times N$ matrix representation of $SU(2)$. The D-strings have expanded into a fuzzy two-sphere with radius

$$R^2 \equiv \frac{1}{N} \text{Tr}(X^i X^i) = \frac{\lambda^2}{16} f^2(N^2 - 1), \quad (7.5)$$

representing a (fuzzy) D3-brane of topology $M^2 \times S^2_{NC}$, with no net D3-brane charge, but with a dipole moment with respect to $F^{(5)}$. The potential (7.1) for the solution (7.4) gives

$$V_{D1} = -\frac{\lambda^2}{384} f^4 N(N^2 - 1), \quad (7.6)$$

which is clearly negative. The dielectric D3-brane thus has a lower energy than the original configuration of $N$ coinciding D-strings, and the latter will spontaneously decay into the former.

7.2 Dielectric F-strings in type IIB

It is clear that for the type IIB F-strings, a similar effect will occur. If we compare the linear couplings (5.4) of the type IIB F-strings with those of the D-strings (3.10), we notice that, although the former now has a monopole charge with respect to $b^{(2)}$, the dielectric coupling to $C^{(4)}$ is identical to that for the D-strings. Hence the action (5.4) for $N$ coinciding type IIB F-strings gives rise to the same type of potential:

$$V_{F1}^{\text{IB}} = \text{STr} \left\{ -\frac{1}{4g_s^2} [X, X]^2 + \frac{i}{3g_s} X^k X^j X^i F^{(5)}_{09ijk} \right\}, \quad (7.7)$$

where now we performed a non-abelian Taylor expansion

$$C^{(4)}_{\mu\nu\rho\lambda}(X) = C^{(4)}_{\mu\nu\rho\lambda}(0) + \frac{R}{\sqrt{\alpha'}} \partial_k C^{(4)}_{\mu\nu\rho\lambda}(0) X^k + ... \quad (7.8)$$
Clearly, the potential (7.4) has a solution of the same type as (7.3):

$$F^{(5)}_{09ijk} = f \varepsilon_{ijk},$$

$$X^i = -\frac{g_s}{4} f \sigma^i \quad \text{where} \quad [\sigma^i, \sigma^j] = 2i \varepsilon^{ijk} \sigma^k. \quad (7.9)$$

The interpretation now is that the $N$ $F$-strings have expanded to form a fuzzy D3-brane with topology $\mathbb{R}^2 \times S^2_{\text{NC}}$, the S-dual of the D1-D3 configuration described above. Again this is a stable configuration with radius and energy as in (7.5) and (7.6) respectively.

Similarly, the $(i X_i X)_{2 \tilde{b}}$ couplings in (5.4) indicate that $F$-strings can expand into an NS5-brane with quadrupole moments, this being the S-dual of a D1-D5 configuration. A solution of $N$ D-strings expanding into a D5-brane was given in [4], using the gamma matrices of $SO(5)$. This solution however turned out to be unstable, and we expect to find a similar unstable solution here.

### 7.3 Matrix string theory: dielectric F-strings with momentum in type IIA

The case of type IIA Matrix string theory is more subtle. A solution, claimed to describe $N$ F-strings expanding into a spherical D2-brane, has been given in [7]. There, the $C^{(3)}$-couplings were considered in the static gauge $t, \sigma = x^9$, rather then in the light cone gauge as in section 4. The potential is then

$$V_{\text{FIA}}^{\text{IIA}} = \text{STr} \left\{ -\frac{1}{4g_s^2} [X, X]^2 + \frac{i}{2g_s} X^k X^j X^i F^{(4)}_{0ijk} \right\}, \quad (7.10)$$

which clearly has a minimum of the form (7.3) with $F^{(4)}_{0ijk} = f \varepsilon_{ijk}$. However, the question arises as to whether this solution really corresponds to a string expanding into a D2-brane, given that the worldvolume of the D2-brane $((t, x^i)$ in Cartesian coordinates) does not contain the worldsheet direction, $x^9$, of the string. How a collection of strings could give rise to such a solution, where no trace of the spatial component of the worldvolume of the strings is found, is unclear. Moreover, the dipole coupling $(i X_i X)_{C^{(3)}}$ in the Chern-Simons action, that would describe such a situation, cannot appear as a pull-back to a two-dimensional worldvolume. Nor is it possible to construct from the action (4.19) a R-R four-form field strength coupling of the form $F_{09ij}$ or $F_{+-ij}$, which would include the worldsheet coordinate of the string in the dielectric brane.

Let us take static gauge $X^9 = \sigma$ in the linear action (4.9) so that we can better identify the dielectric couplings that appear in the Chern-Simons part of this action. In that case, one immediately realizes that the couplings cannot be written as a pull-back to a two-dimensional worldvolume. Moreover, the different terms can be organized as pull-backs to a one-dimensional worldvolume which has $X^9$ as an isometric direction. It turns out that the resulting action describes multiple pp-waves carrying momentum along the $X^9$ direction, as in [10]. Pp-waves carrying momentum in a compactified direction are, of course, T-dual to fundamental strings wound around this direction [11, 12]. There is a sense, then, in which waves can be considered as fundamental strings carrying momentum (see [13] for a discussion of this point). We have found that the action that describes type IIA Matrix strings coupled to background fields reduces to that describing multiple waves when one goes to the static gauge, which is in agreement with the fact that Matrix strings carry momentum along the compactified direction.

The interpretation in terms of waves sheds some light on the dielectric solution for IIA Matrix strings constructed in [7]. Recall that the description of multiple waves is in terms of a gauged sigma model in which the direction of propagation of the waves is isometric [14]. Indeed, in this action one finds a coupling $(i X_i X)_{C^{(3)}}$ to the worldvolume [15], but now the pull-backs include
gauge covariant derivatives (see the discussion about gauged sigma models in section 6) which
gauge away $X^9$ as a world volume scalar. The resulting dielectric D2-brane is therefore constrained
to move in the space transverse to the isometric direction.

In summary, we believe that the solution

$$F_{0ijk}^{(4)} = f \varepsilon_{ijk},$$

$$X^i = -\frac{g_s}{4} f \sigma^i \; \text{where} \; \{\sigma^i, \sigma^j\} = 2i \varepsilon^{ijk} \sigma^k,$$

(7.11)
presented in [17] corresponds to a set of $N$ gravitational waves expanding into a spherical D2-brane,
which again is a stable configuration with radius and energy as in (7.5) and (7.6).

We have argued in the introduction that we do not expect to find stable solutions describing
F-strings expanding into a cylindrical D2-brane. Since we cannot find a coupling of the F-strings to
either $F_{00ij}$ or $F_{+-ij}$, which would give rise to such a dielectric brane, this expectation is confirmed.
Such a configuration simply does not exist in the type IIA Matrix string theory. Although we find a
possible cylindrical D2-brane solution below, it is unstable. Whether it corresponds to the unstable
abelian D2-brane solution is unclear, however.

### 7.4 Dielectric F-strings with winding in type IIA

Coinciding F-strings with winding can expand into a spherical D4-brane in the presence of an R-R
6-form field strength. We can see this by analysing the $C^{(5)}$ couplings in the action (6.1), which can be rewritten as the following coupling to the R-R six-form field strength:

$$S_{F1}^{IIA} \sim \frac{i}{3g_s} \int d\tau d\sigma \; \text{Str} \left\{ X^k X^j X^i F_{09zijk}^{(6)} \right\} .$$

(7.12)

It is then clear that a stable solution of the potential

$$V_{F1}^{IIA} = \text{Str} \left\{ \frac{1}{4g_s^2} [X, X]^2 + \frac{i}{3g_s} X^k X^j X^i F_{09zijk}^{(6)} \right\} ,$$

(7.13)
is given by

$$F_{09zijk}^{(6)} = f \varepsilon_{ijk},$$

$$X^i = -\frac{g_s}{4} f \sigma^i \; \text{where} \; \{\sigma^i, \sigma^j\} = 2i \varepsilon^{ijk} \sigma^k,$$

(7.14)
and corresponds to $N$ F-strings expanding into a D4-brane, which is itself wrapped around
the isometry direction $z$. We believe this solution corresponds to the (smeared) supergravity solution
of an F-string expanding into a D4-brane, the T-dual of the fully localized F1-D6 solution given in [6, 7].

Coinciding F-strings with winding also have couplings to the R-R 3-form field potential, which
seems to suggest an expansion into a cylindrical D2-brane. Yet these $C^{(3)}$-couplings have an unusual
form, due to the presence of the world volume scalar field $\omega$. In the static case the potential is
given by

$$V_{F1}^{IIA} = \text{Str} \left\{ -\frac{1}{4g_s^2} [X, X]^2 - \frac{1}{2g_s^2} [X, \omega]^2 - \frac{i}{6g_s} \left( [X^i, X^j] \omega + [X^j, \omega] X^i + [\omega, X^i, X^j] \right) F_{09ij}^{(4)} \right\} .$$

(7.15)

$^4$A similar type of solution, for a magnetic 4-form field strength, has been constructed in [14].
It is easy to check that the following Ansatz

\[ F_{09ij}^{(4)} = f \varepsilon_{ij}, \]
\[ [X^i, X^j] = 0, \quad [X^i, \omega] = ig_s f \varepsilon^{ij} X^j \]  

(7.16)

where \( i, j = 1, 2 \), is a solution of the equations of motion

\[ [[X^i, X^j], X^j] + [[X^i, \omega], \omega] - ig_s [X^j, \omega] F_{09ij}^{(4)} = 0, \]
\[ [[X^i, \omega], X^i] - \frac{i}{2} g_s [X^i, X^j] F_{09ij}^{(4)} = 0. \]  

(7.17)

The solution (7.16) is the algebra of the symmetry group of a cylinder. A similar solution was recently given in [45] using D0-brane matrix degrees of freedom. This brane has no net monopole charge with respect to the \( F^{(4)} \) field strength but carries a dipole moment in the XZ and YZ planes.

From (7.16), the worldvolume of the D2-brane is \((t, x^9, x^i)\) in Cartesian coordinates, which suggests a cylindrical worldvolume. However, the proper interpretation is not entirely clear since the solution explicitly involves the worldvolume scalar field \( \omega \). It is possible that this solution is dual to the unstable solution of the abelian D2-brane theory, since it turns out to correspond to a maximum of the potential and is therefore unstable. The only stable radius for the cylinder is

\[ R^2 = \frac{1}{N} \text{Tr}(X^i X^i) = 0, \]  

(7.18)

as might be expected from the analysis of [1, 7]. There is another remark we would like to make: an explicit representation of the cylindrical algebra (7.16) is

\[ \omega_{ab} = g_s f \delta_{ab}, \quad X^1 = R(a + a^\dagger), \quad X^2 = iR(a^\dagger - a) \]  

(7.19)

\[ a_{ab} = e^{-i\Omega} \delta_{a-1,b}, \quad a_{ab}^\dagger = e^{i\Omega} \delta_{a+1,b}, \]  

(7.20)

where \( a, b = 1, \ldots, N \) and for some arbitrary phase \( \Omega \), is only possible for the infinite-dimensional case. There are no finite-dimensional representations of (7.16), which makes the solution rather unphysical, requiring an infinite number of F-strings for the effect to take place.

### 8 Discussion

The dielectric effect of Myers [2] is not limited to D-branes, but should be observable in a theory of fundamental strings too. To describe such dielectric F-strings, we are led to a consideration of Matrix string theory, which captures type IIA superstring theory in the light-cone gauge, together with extra degrees of freedom representing D-brane states.

We have shown explicitly that the Matrix theory action in a weakly curved background reproduces the lowest order expansion of the combined non-abelian Born-Infeld-Chern-Simons of Taylor and van Raamsdonk [3] and Myers [2]. Using T-duality and the 9-11 flip, we have derived the Matrix string theory action in a weakly curved background and shown that, in the weak string coupling limit, the result reproduces light-cone gauge string theory. We have not been able to find explicit solutions describing dielectric Matrix strings expanding into a D2-brane in this theory, and have argued that this is to be expected. We have further argued that the solution found in [17]

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\(^5\)This can be done by adding a small number of D0-branes to the F1-D2-brane system and taking then the D0-brane density to zero.
(in which the static, as opposed to the light-cone, gauge was considered) in some sense describes
dielectric particles or pp-waves, carrying momentum.

We have derived a, strongly coupled, type IIB Matrix string theory, S-dual to the D-string
theory, and a, strongly coupled, type IIA theory of Matrix strings with winding. As regards
the former, the standard D1-D3 dielectric solution of the D-string theory equally well describes
a collection of F-strings expanding into a D3-brane and other, albeit unstable, solutions of the
D-string theory carry through in a similar way. The theory of Matrix strings with winding exhibits
an isometry direction, in a manner similar to the theory of the Kaluza-Klein monopole, and one
can find dielectric D4-brane solutions which are smeared over this direction. Finally, we have
mentioned a cylindrical solution, the interpretation of which is somewhat unclear since it involves
the worldvolume scalar $\omega$.

On a different note, one might expect to find solutions of the type IIA Matrix string theory
corresponding to the cylindrical supertubes of [46, 47]. As far as both the D2-brane theory and the
supergravity solutions are concerned, these involve a cylindrical D2-brane with dissolved F-string
and D0-brane charge. The configuration is supported from collapse by an angular momentum in
the circle direction. It should be clear that possible solutions of Matrix string theory corresponding
to such supertubes would then involve couplings to both $C^{(3)}$ and $C^{(1)}$. As far as the D0-branes
are concerned, however, the D2-brane worldvolume is a non-commutative cylinder [48, 49]. Then,
since the F-strings lie in the direction along the cylinder, the worldsheet direction would also have
to be non-commutative — and this cannot occur in Matrix string theory.

The proper determination of both the Matrix and the Matrix string theories in an arbitrary
background remains an open problem. Of course, they could be determined by duality transforma-
tions of multiple D-brane theories in an arbitrary background, but the correct form of the latter is
also only poorly understood; and so are the duality transformations for these non-abelian actions.
We are tempted to think that one might make progress via a comprehensive treatment of light-cone
gauge supermembrane theory in a general background.

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A Appendix: D-string currents

T-duality applied to the D0-brane currents is explained in the text. We give the final results here
for completeness. With $i, j = 1, \ldots, 8$, we have

\begin{align}
I_0^0 &= \mathbb{1}, \\
I_0^i &= \dot{X}^i, \\
I_0^\alpha &= \lambda \dot{\tilde{A}}, \\
I_{0ij}^b &= -\frac{i}{6\beta} [X^i, X^j], \\
I_{2b}^b &= -\frac{\lambda}{6\beta} DX^i, \\
I_{2ji}^b &= -\frac{i}{6\beta} \left( \dot{X}^i [X^j, X^k] + \dot{X}^j [X^k, X^i] + \dot{X}^k [X^i, X^j] \right),
\end{align}
In terms of pullbacks of background fields, we have (for $\beta = \lambda$ and $a, b, = 0, ..., 8$)

$$I_0^{ij} = -\frac{\lambda}{6\beta} \left( i \dot{A} [X^i, X^j] - \dot{X}^i DX^j + \dot{X}^j DX^i \right),$$

(A.7)

$$I_4^{ijkl} = -\frac{1}{2\beta^2} \left( [X^i, X^j][X^k, X^l] + [X^i, X^k][X^j, X^l] + [X^i, X^l][X^j, X^k] \right),$$

(A.8)

$$I_4^{0ijk} = \frac{i\lambda}{2\beta^2} \left( DX^i[X^j, X^k] + DX^j[X^k, X^i] + DX^k[X^i, X^j] \right),$$

(A.9)

$$I_4^{ijklm} = -\frac{15}{2\beta^2} \dot{X}^{[i}[X^j, X^k][X^l, X^m]},$$

(A.10)

$$I_4^{0ijkl} = -\frac{3\lambda}{2\beta^2} \left( \dot{A}[X^i, X^j][X^k, X^l] + 4i\dot{X}^{[i}[X^j, X^k]DX^{l]} \right),$$

(A.11)

$$I_6^{0ijklm} = \frac{i}{\beta^3} [X^i, X^j][X^k, X^l][X^m, X^n],$$

(A.12)

$$I_6^{0ijkl} = \frac{\lambda}{\beta^3} DX^{[i}[X^j, X^k][X^l, X^m],$$

(A.13)

$$I_6^{ijklmn} = \frac{7i}{\beta^3} \dot{X}^{[i}[X^j, X^k][X^l, X^m][X^n, X^p],$$

(A.14)

$$I_6^{ijklmn} = \frac{\lambda}{\beta^3} \left( i\dot{A}[X^i, X^j][X^k, X^l][X^m, X^n] - 6\dot{X}^{[i}[X^j, X^k][X^l, X^m]DX^{n]} \right).$$

(A.15)

In terms of pullbacks of background fields, we have (for $\beta = \lambda$ and $a, b, = 0, ..., 8$)

$$C^{(0)} I_0^g \, dt \, dx = \lambda P[C^{(0)}] \wedge F,$$

(A.16)

$$\left( C_{ab}^{(2)} I_2^a + 3C_{ac}^{(2)} I_2^{abg} \right) \, dt \, dx = P[C^{(2)}] + \frac{i}{\lambda} P[(i\chi_{iX}) C^{(2)}] \wedge F,$$

(A.17)

$$\left( C_{abc}^{(4)} I_2^{abc} + 12C_{a_1 \ldots a_4}^{(4)} I_4^{a_1 \ldots a_4g} \right) \, dt \, dx = \frac{i}{\lambda} P[(i\chi_{iX}) C^{(4)}] - \frac{1}{2\lambda} P[(i\chi_{iX})^2 C^{(4)}] \wedge F,$$

(A.18)

$$\left( \frac{1}{60} C_{a_1 \ldots a_9}^{(6)} I_4^{a_1 \ldots a_9} + \frac{1}{48} C_{a_1 \ldots a_9}^{(6)} I_4^{a_1 \ldots a_9} \right) \, dt \, dx = -\frac{i}{2\lambda^2} P[(i\chi_{iX})^2 C^{(6)}] - \frac{i}{6\lambda^2} P[(i\chi_{iX})^3 C^{(6)}] \wedge F,$$

(A.19)

giving the action (3.10) in the text. The dots stand for the unknown coupling to $C^{(9)}$ which, of course, could be determined by demanding that it can be rewritten in terms of the pullback. The dilaton current is

$$I_\phi = 1 - \frac{1}{2} \dot{X}^2 \left( 1 + \frac{1}{4} \dot{X}^2 + \frac{\lambda^2}{4\beta^2} DX^2 + \frac{\lambda^2}{4} \dot{A}^2 - \frac{1}{8\beta^2} [X, X]^2 \right) - \frac{1}{8\beta^4} [X, X]^4 - \frac{\lambda^2}{2\beta^2} \left( \dot{X} \cdot DX \right)^2$$

$$+ \frac{\lambda^2}{2\beta^2} DX^2 \left( 1 - \frac{1}{4} \dot{X}^2 - \frac{\lambda^2}{4\beta^2} DX^2 + \frac{\lambda^2}{4} \dot{A}^2 - \frac{1}{8\beta^2} [X, X]^2 \right) - \frac{\lambda^2}{2\beta^2} DX^i[X^i, X^j][X^j, X^k]DX^k$$

$$- \frac{\lambda^2}{2} \dot{A}^2 \left( 1 + \frac{1}{4} \dot{X}^2 - \frac{\lambda^2}{4\beta^2} DX^2 + \frac{\lambda^2}{4} \dot{A}^2 - \frac{1}{8\beta^2} [X, X]^2 \right) + \frac{1}{2\beta^2} \dot{X}^i[X^i, X^j][X^j, X^k] \dot{X}^k$$

$$- \frac{1}{4\beta^2} [X, X]^2 \left( 1 - \frac{1}{4} \dot{X}^2 + \frac{\lambda^2}{4\beta^2} DX^2 - \frac{\lambda^2}{4} \dot{A}^2 - \frac{1}{8\beta^2} [X, X]^2 \right) - \frac{i\lambda^2}{\beta^2} ADX^i[X^i, X^j] \dot{X}^j.$$

(A.21)
and the remaining NS-NS currents are

\[
I_{h}^{0i} = \frac{1}{2} \dot{X}^2 + \frac{\lambda^2}{2\beta^2} DX^2 + \frac{\lambda^2}{2} A^2 - \frac{1}{4\beta^2} [X, X]^2, \\
I_{h}^{0i} = \dot{X}^i \left( \frac{1}{2} \dot{X}^2 + \frac{\lambda^2}{2} A^2 + \frac{\lambda^2}{2\beta^2} DX^2 - \frac{1}{4\beta^2} [X, X]^2 \right) - \frac{i}{\beta^2} [X^i, X^j] \left( \dot{A} \dot{X}^j - i[X^j, X^k] \dot{X}^k \right), \\
I_{h}^{0i} = \lambda \dot{A} \left( \frac{1}{2} \dot{X}^2 + \frac{\lambda^2}{2} A^2 - \frac{\lambda^2}{2\beta^2} DX^2 - \frac{1}{4\beta^2} [X, X]^2 \right) + \frac{i\lambda}{\beta^2} DX^i [X^i, X^j] \dot{X}^j, \\
I_{h}^{0i} = \lambda \dot{A} \dot{X}^i - \frac{i\lambda}{\beta^2} [X^i, X^j] DX^j, \\
I_{h}^{0} = \lambda^2 \dot{A}^2 - \frac{\lambda^2}{\beta^2} DX^2, \\
I_{h}^{0} = -\frac{\lambda}{2\beta} \dot{X} \cdot DX, \\
I_{s}^{i} = \frac{i\lambda}{2\beta} [X^i, X^j] \left( \frac{1}{2} \dot{X}^2 - \frac{\lambda^2}{2} A^2 + \frac{\lambda^2}{2\beta^2} DX^2 - \frac{1}{4\beta^2} [X, X]^2 \right) - \frac{1}{\beta} \dot{X}^i [X^j, X^k] \dot{X}^k \\
+ \frac{\lambda^2}{\beta} \dot{A} \dot{X}^i [X^j] + \frac{i\lambda}{\beta^2} DX^i [X^j, X^k] DX^k - \frac{i}{2\beta} [X^i, X^k] [X^k, X^j] [X^j, X^l], \\
I_{s}^{0i} = \frac{\lambda}{2\beta} DX^i \left( \frac{1}{2} \dot{X}^2 + \frac{\lambda^2}{2} A^2 - \frac{\lambda^2}{2\beta^2} DX^2 - \frac{1}{4\beta^2} [X, X]^2 \right) \\
- \frac{i\lambda}{2\beta} \dot{A} [X^i, X^j] \dot{X}^j + \frac{\lambda}{2\beta} \dot{X}^i DX \cdot X - \frac{\lambda}{2\beta} [X^i, X^j] [X^j, X^k] DX^k.
\]

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