Hyperbolic groups have flat-rank at most 1

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Abstract The flat-rank of a totally disconnected, locally compact group $G$ is an integer, which is an invariant of $G$ as a topological group. We generalize the concept of hyperbolic groups to the topological context and show that a totally disconnected, locally compact, hyperbolic group has flat-rank at most 1. It follows that the simple totally disconnected locally compact groups constructed by Paulin and Haglund have flat-rank at most 1.

Key words totally disconnected group, hyperbolic group, flat-rank, automorphism group, scale function

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1 Introduction

The concept of a hyperbolic group can be generalized to the realm of compactly generated, topological groups by a straightforward adaption of the definition in the discrete case (see Definition 4 on page 3). Such a generalization is an instance of ‘geometric group theory for topological groups’, which is a line of investigation proposed in [KM08].

This geometric approach is a natural one in the case of totally disconnected, locally compact groups, the subject of this paper, and has been

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pursued previously in [Mö02], [Mö], [Bau07b] and [BSW08]. We take a different line to these papers, however, by studying hyperbolicity and relating it to a structural invariant for totally disconnected, locally compact groups, namely, the flat-rank.

The flat-rank of a totally disconnected, locally compact group (see Definition 10) is a non-negative integer that is analogous to the $k$-rank of a semisimple algebraic group over a local field $k$. Indeed the flat-rank and $k$-rank coincide when $G$ is such a group, by Corollary 19 in [BRW07].

Just as the $k$-rank of a simple algebraic group determines many properties of the group, the flat-rank can be expected to convey important information about general simple totally disconnected groups. An indication of this is seen with the computation of the flat-rank of automorphism groups of buildings made in [BRW07], where it is shown, in conjunction with results from [CH09], that if the group action is sufficiently transitive then the flat-rank of the group equals the rank of its building. The following theorem, our main result, further demonstrates the relationship between the flat-rank and geometric properties of the group.

**Theorem 1.** The flat-rank of a totally disconnected, locally compact, hyperbolic group is at most 1.

The major part of this paper is devoted to the proof of Theorem 1. It is clear that the converse to this theorem does not hold, because discrete groups have flat-rank 0 and need not be hyperbolic. However, it may hold in the presence of further hypotheses that exclude discrete groups or non-discrete counterexamples based on them.

The properties of algebraic groups of $k$-rank 1 differ notably from the properties of groups of higher $k$-rank. In the expectation that the same will be true of the flat-rank, a secondary aim of this paper is to seek further geometric criteria for a totally disconnected, locally compact group to have flat-rank at most 1. We establish two such criteria.

One is based on the action of the group on the space of compact open subgroups of the group. The criterion and its proof are in the spirit of the papers [BRW07] and [BSW08]; the proof is contained in Section 9.

**Theorem 2.** Let $A$ be a group of automorphisms of the totally disconnected locally compact group $G$. Suppose that $A$ has a hyperbolic orbit in the space of compact open subgroups of $G$. Then the flat-rank of $A$ is at most 1.

The last criterion follows from results in [BW06], where the space of directions of a totally disconnected, locally compact group is defined; see page 12 for its proof.

**Theorem 3.** Let $G$ be a totally disconnected, locally compact group whose space of directions is discrete. Then the flat-rank of $G$ is at most 1. If the space of directions is not empty, then the flat-rank is exactly 1.

We do not know of a hyperbolic group whose space of directions is not discrete. In view of Theorem 3, it would be a strengthening of Theorem 1 to show that all hyperbolic groups have discrete spaces of directions.
2 Basic concepts

Definition 4 (hyperbolic group [topological version]). A topological group is called hyperbolic if and only if it is compactly generated and its Cayley graph with respect to some (hence any) compact generating set is Gromov-hyperbolic.

The definition makes sense, because all Cayley graphs with respect to compact generating sets are quasi-isometric by part (i) of Lemma 4.6 in [Möl]. The same definition is used in recent work by Yves Cornulier and Romain Tessera [CT09], where they characterize certain classes of non-discrete Gromov-hyperbolic groups.

In this paper we consider compactly generated, totally disconnected, locally compact topological groups only. These groups admit a locally finite, connected graph with a vertex-transitive action by the group such that vertex-stabilizers are compact and open. Such a graph with an action by the group is an instance of a so-called rough Cayley graph, a concept introduced in [KM08]. We now define this concept.

Definition 5. Let $G$ be a topological group. A connected graph $X$ is said to be a rough Cayley graph of $G$, if $G$ acts transitively on the vertex set of $X$ and the stabilizers of vertices are compact open subgroups of $G$.

The proof of our main result relies on the existence of a rough Cayley graph for the groups under consideration. The relevant result is Theorem 2.2+ in [KM08], or Corollary 1 in [Möl03], which we restate for ease of reference. In the formulation, $V_X$ denotes the vertex set of $X$.

Theorem 6 (Existence of a locally finite, rough Cayley graph). Let $G$ be a totally disconnected, compactly generated, locally compact group. Then there is a locally finite, connected graph $X$ such that:

(i) $G$ acts as a group of automorphisms on $X$ and is transitive on $V_X$;
(ii) for every vertex $v$ in $X$ the subgroup $G_v$ is compact and open in $G$;
(iii) if $\text{Aut}(X)$ is equipped with the permutation topology, then the homomorphism $\pi: G \to \text{Aut}(X)$ given by the action of $G$ on $X$ is continuous, the kernel of this homomorphism is compact and the image of $\pi$ is closed in $\text{Aut}(X)$.

Conversely, if $G$ acts as a group of automorphisms on a locally finite, connected graph $X$ such that $G$ is transitive on the vertex set of $X$ and the stabilizers of the vertices in $X$ are compact and open, then $G$ is compactly generated.

For a totally disconnected, locally compact group hyperbolicity can be formulated in terms of any of its rough Cayley graphs as follows.

Proposition 7 (hyperbolicity in terms of the rough Cayley graph). A totally disconnected, locally compact group is hyperbolic if and only if some (hence any) of its rough Cayley graphs is hyperbolic.
Proof. The claim is implied by the quasi-isometry of rough Cayley graphs; see Theorem 4.5 in [Möll] or Theorem 2.7 in [KM08]. □

The flat-rank of a group $\mathcal{A}$ of automorphisms of a totally disconnected locally compact group $G$ was introduced in [Wil04b], although it was not given that name there. Some auxiliary definitions and results will be required for its definition and in later sections.

Definition 8. Let $G$ be a totally disconnected, locally compact group.

(i) The **scale** of the automorphism, $\alpha$, of $G$ is the positive integer

$$s_G(\alpha) := \min \{|\alpha(O)\cap O| : O \subseteq G \text{ compact and open}\}.$$  \hfill (1)

(ii) The compact, open subgroup $O$ is **minimizing** for $\alpha$ if the minimum index in (1) is attained at $O$.

(iii) The group, $\mathcal{A}$ of automorphisms of $G$ is **flat** if there is a compact open subgroup $O \subseteq G$ that is minimizing for every $\alpha \in \mathcal{A}$.

Theorem 9 ([Wil04b], Corollary 6.15). Let $\mathcal{A}$ be a flat group of automorphisms of $G$ and $O$ be minimizing for $\mathcal{A}$. Then $A_1 := \{\alpha \in \mathcal{A} : \alpha(O) = O\}$ is a normal subgroup of $\mathcal{A}$, and $\mathcal{A}/A_1$ is a free abelian group.

The group $A_1$ is independent of the minimizing subgroup used to define it.

Definition 10. Let $G$ be a totally disconnected, locally compact group.

(i) The **rank** of the flat group, $\mathcal{A}$ of automorphisms of $G$ is the rank of the free abelian group $\mathcal{A}/A_1$.

(ii) The **flat-rank** of a group $\mathcal{A}$ of automorphisms of $G$ is the supremum of the ranks of all the flat subgroups of $\mathcal{A}$.

(iii) The **flat-rank** of $G$ is the flat-rank of the group of inner automorphisms.

3 Constructing hyperbolic topological groups

The following proposition provides a method to construct totally disconnected, locally compact, hyperbolic groups. For example one might take for $X$ the Cayley graph of any discrete hyperbolic group and let $G$ be its full automorphism group with the permutation topology (equivalently, the compact-open topology). Further applications of this result will be provided in Section 8.

Proposition 11. Let $G$ be a totally disconnected, locally compact group acting cocompactly and with compact, open point stabilizers on a locally finite, connected Gromov-hyperbolic complex, $X$ say. Then the 1-skeleton of $X$ is quasi-isometric to a locally finite, connected, rough Cayley graph for $G$ which is also Gromov-hyperbolic, and $G$ is a hyperbolic group.
Proof. Since the group $G$ acts cocompactly on $X$, there is a finite subcomplex, $F$ say, of $X$ whose $G$-translates cover $X$. The group $G$ is then generated by its subset $\{ x \in G : x.F \cap F \neq \emptyset \}$; this subset is compact, hence $G$ is compactly generated.

Denote the 1-skeleton of $X$ by $\Gamma$. The graph $\Gamma$ is a locally finite, connected $G$-set with compact, open point stabilizers. We use a standard argument, see [M"ol, Theorem 4.9] for instance, to complete the proof. Since the $G$-translates of the 1-skeleton of the finite complex, $F$, of the previous paragraph cover $\Gamma$, there are finitely many $G$-orbits in $\Gamma$ and there is a constant $k$ (for example, the diameter of $F$) such that every vertex is within distance $k$ of each orbit. Fix a vertex $x$ of $\Gamma$ and define a graph structure on the orbit $G.x$ by drawing an edge between vertices $g.x$ and $h.x$ if they are within distance $2k + 1$. Then the graph $G.x$ is connected, locally finite and quasi-isometric to $\Gamma$. Hence $G.x$ is Gromov-hyperbolic. Furthermore, $G$ acts transitively with compact vertex stabilizers on this graph. Therefore the graph $G.x$ is a rough Cayley graph for $G$ and $G$ is a hyperbolic group.

While the action of a group on its Cayley graph is always faithful, the action of a compactly generated topological group on its rough Cayley graph need not be. The following result explains what happens when passing to the quotient of the group by the kernel of this action. The proof is straightforward, and is left to the reader.

**Proposition 12.** Let $G$ be a compactly generated topological group that contains a compact, open subgroup and let $\Gamma$ be any of its locally finite, connected, vertex-transitive rough Cayley graphs with compact, open vertex-stabilizers. Let $\hat{G}$ be the quotient of $G$ by the compact kernel of its action on $\Gamma$. Then $\Gamma$ with its induced action by $\hat{G}$ is again a rough Cayley graph for $\hat{G}$ with the same properties. In particular, if the group $G$ is hyperbolic, then so is the group $\hat{G}$.

We also have the following transfer result for flat subgroups under continuous, open, surjective homomorphism with compact kernel.

**Proposition 13.** Let $\pi : G \to \hat{G}$ be a continuous, open, surjective homomorphism with compact kernel between totally disconnected, locally compact groups. Further, let $H$ be a flat subgroup of $G$ and $\hat{H}$ a flat subgroup of $\hat{G}$. Then $\pi(H)$ is a flat subgroup of $\hat{G}$ of the same rank as $H$ while $\pi^{-1}(\hat{H})$ is a flat subgroup of $G$ of the same rank as $\hat{H}$. The groups $G$ and $\hat{G}$ have the same flat rank.

**Proof.** Consider $h \in G$ and suppose that $O$ is minimizing for $h$, that is, is minimizing for the inner automorphism $\alpha_h : x \mapsto hxh^{-1}$. The index $|hOh^{-1} : hOh^{-1} \cap O|$ is unchanged if $O$ is replaced by $O\ker(\pi)$ and so it may be assumed that $\ker(\pi) \subseteq O \cap hOh^{-1}$. The subgroup $\pi(O)$ of $\hat{G}$, which is also compact and open, then satisfies

$$|hOh^{-1} : hOh^{-1} \cap O| = |\pi(h)\pi(O)\pi(h)^{-1} : \pi(h)\pi(O)\pi(h)^{-1} \cap \pi(O)|,$$

(2)
from which it follows that \( s_G(\pi(h)) \leq s_G(h) \). On the other hand, if \( \hat{O} \leq \hat{G} \) is minimizing for \( \hat{h} \in \hat{G} \), then \( \pi^{-1}(\hat{O}) \) is compact and open in \( G \) and

\[
|h\pi^{-1}(\hat{O})h^{-1} : \pi^{-1}(\hat{O})h^{-1} \cap \pi^{-1}(\hat{O})| = |\hat{h}\hat{O}\hat{h}^{-1} : \hat{h}\hat{O}\hat{h}^{-1} \cap \hat{O}|, \tag{3}
\]

for any \( h \in G \) with \( \pi(h) = \hat{h} \) and it follows that \( s_G(h) \leq s_G(\pi(h)) \). Therefore the scales are equal and \( O \) is minimizing for \( h \) if and only if \( \pi(O) \) is minimizing for \( \pi(h) \).

Letting \( H \) be a flat subgroup of \( G \) and \( \hat{H} \) be a flat subgroup of \( \hat{G} \), it follows that \( \pi(H) \) and \( \pi^{-1}(\hat{H}) \) are flat subgroups of \( G \) and \( \hat{G} \) respectively as claimed. Moreover, \( h \in H_1 \) if and only if \( \pi(h_1) \in \pi(H)_1 \) and so the map \( \pi : H \to \pi(H) \) induces an isomorphism \( H/H_1 \to \pi(H)/\pi(H)_1 \). Hence the ranks of \( H \) and \( \pi(H) \) are equal as claimed. That the ranks of \( \hat{H} \) and \( \pi^{-1}(\hat{H}) \) also agree may be seen similarly. Therefore \( G \) and \( \hat{G} \) have the same flat-rank.

\[\square\]

4 Method of proof

The proof of the main result uses the classification of group actions by isometries on Gromov-hyperbolic spaces in terms of fixed point properties versus existence of free subgroups. This is combined with topological properties of elliptic, parabolic and hyperbolic isometries and an analysis of the dynamics of actions of flat subgroups on the boundary of the hyperbolic space. For the geometric ideas we follow the approach in [Woe93].

We begin by extending the necessary concepts of hyperbolic geometry to encompass topological groups.

Definition 14 (boundary of a hyperbolic group). Let \( G \) be a hyperbolic topological group. The hyperbolic boundary of \( G \) is the Gromov-boundary of the Cayley graph of \( G \) with respect to some compact generating set of \( G \).

The usual properties of the hyperbolic boundary carry over from the discrete case. That it is independent of the rough Cayley graph chosen will be important in subsequent arguments.

Proposition 15. The hyperbolic boundary of a hyperbolic topological group, \( G \), is independent of the choice of compact generating set used for its definition. It is a metric space which admits an action of \( G \) by bi-Lipschitz maps. If \( G \) admits a compact, open subgroup, then metrics can be chosen such that its hyperbolic boundary is equivariantly isometric to the Gromov-boundary of any of its rough Cayley graphs; in particular, in that case the hyperbolic boundary of \( G \) is compact.

Proof. Using standard results about Gromov-hyperbolic spaces, the above statements follow from Theorem 4.5 and Lemma 4.6 in [Möl]. \[\square\]
The classification of isometries of hyperbolic spaces also plays a central role in what follows. Since the action of a topological group on a rough Cayley graph need not be faithful, we extend the usual definitions as follows.

**Definition 16 (elliptic, parabolic and hyperbolic elements).** Let $G$ be a group, $X$ be a Gromov-hyperbolic space and $\alpha : G \to \text{Aut}(X)$ be an action of $G$ on $X$ by isometries. An element $g$ in $G$ is called

1. $\alpha$-elliptic if there is a point of $X$ whose $\alpha(g)$-orbit is bounded; in that case every other point of $X$ has the same property;
2. $\alpha$-parabolic if it is not $\alpha$-elliptic and $\alpha(g)$ fixes a unique boundary point;
3. $\alpha$-hyperbolic if it is not $\alpha$-elliptic and $\alpha(g)$ fixes precisely two boundary points, which, for arbitrary $x \in X$ are then of the form $\lim_{n \to \infty} \alpha^n(g).x$, called attracting for $g$, and $\lim_{n \to \infty} \alpha^{-n}(g).x$, called repelling for $g$.

An element of a hyperbolic topological group is called elliptic, parabolic or hyperbolic respectively, if it is $\alpha$-elliptic, $\alpha$-parabolic or $\alpha$-hyperbolic respectively for $\alpha$ equal to the natural action of the group on its Cayley graph with respect to a compact generating set.

Since Cayley graphs with respect to compact generating sets are quasi-isometric, the notions elliptic, parabolic or hyperbolic do not depend on the particular Cayley graph chosen and reference to the homomorphism $\alpha$ is usually omitted in the following.

The elliptic elements of a locally compact, hyperbolic group can be characterized by an intrinsic topological property which generalizes the corresponding characterization in the discrete case.

**Proposition 17.** An element of a locally compact, hyperbolic, topological group is elliptic if and only if it is topologically periodic.

**Proof.** By definition, an element, $g$ say, of the given group $G$ is elliptic if and only if its orbits in the Cayley graph, $\Gamma$ say, of $G$ with respect to a compact set of generators is bounded in the graph metric. The property of being bounded is independent of the orbit chosen. Hence $g$ is elliptic if and only if its orbit $\langle g \rangle. e = \langle g \rangle$ is bounded. Abels’ result 2.3 in [Abe74] (Heine-Borel-Eigenschaft) implies that a subset of $\Gamma$ is bounded in the graph-metric if and only if it is a relatively compact subset of $G$. The latter condition is satisfied by the set $\langle g \rangle$ if and only $g$ is topologically periodic. \qed

### 5 Properties of elliptic, parabolic and hyperbolic elements

#### 5.1 The scale of an elliptic element is 1

An elliptic element in a totally disconnected, locally compact group is topologically periodic, by Proposition 17. Hence the set of conjugates of an open subgroup by powers of such an element is finite and the intersection of these conjugates is an open subgroup normalized by the element.

**Proposition 18.** The scale of every elliptic element in a totally disconnected, locally compact, hyperbolic group is 1. \qed
5.2 Totally disconnected, hyperbolic groups contain no parabolics

Discrete hyperbolic groups do not contain parabolic elements. This is proved in each one of the following sources: [Gro87, Corollary 8.1.D] (together with the obvious observation that torsion elements are elliptic), [CDP90, Chapitre 9, Théorème 3.4] and [GdlH90, Chapitre 8, Théorème 29].

Theorem 20 below extends this result to totally disconnected, locally compact, hyperbolic groups, the proof of which is modelled on the argument from [CDP90]. The following property of non-hyperbolic elements is of central importance in the proof.

Lemma 19. Let $X$ be a geodesic $\delta$-hyperbolic space. Then there is a constant $k$, depending only on $\delta$, such that: given a non-hyperbolic isometry, $g$, of $X$ that fixes a boundary point $\omega$, and a geodesic ray, $x$, ending in $\omega$, all points on $x$ which are sufficiently far out are moved by a distance of at most $k$ by $g$.

Proof. Choose any point $p$ of $X$ and denote the midpoint of a chosen geodesic segment connecting $p$ to $g.p$ by $m$. By Lemme 9.3.1 in [CDP90]

\[ d(g.m, m) \leq 6\delta. \]

Applying Lemme 9.3.6 in [CDP90] to the entities $x$ and $m$ chosen, we obtain that there is a number $t_0 \geq 0$ such that for each $t \geq t_0$

\[ d(g.x(t), x(t)) \leq 72\delta + d(g.m, m) \leq 72\delta + 6\delta = 78\delta, \]

and so we may take $k$ to be $78\delta$. \qed

The following theorem is the main result of this subsection.

Theorem 20. Suppose that a group $G$ acts cocompactly and by automorphisms on a connected, locally finite, metric, Gromov-hyperbolic complex. Then $G$ does not contain parabolic elements. In particular, a hyperbolic topological group with a compact, open subgroup does not contain parabolic elements.

Proof. We argue by contradiction, and assume that some element, $g$ say, of $G$ acts by a parabolic isometry. All positive powers of $g$ are again parabolic and have the same unique fixed point on the boundary, $\omega$ say.

The diameter of cells in our space is bounded from above, say by the positive number $D$, because the group $G$ is assumed to act cocompactly. Denote by $n$ a natural number larger than the maximal number of vertices of the space that are contained in any closed ball whose radius is $k + 2D$, where $k$ is the number introduced in the statement of Lemma 19. Such a number $n$ exists, because $G$ acts cocompactly.

Choose a geodesic ray, $x$ say, that ends in $\omega$. Lemma 19 applied to the elements $g, g^2, \ldots, g^n$ and this $x$ implies that there is a number $T$ such that for $t \geq T$ and $i = 1, \ldots, n$ we have $d(g^i.x(t), x(t)) \leq k$. By the definition
of $D$, there is a vertex, $v$, say, at distance at most $D$ from $x(t)$. Then the above bound on the displacement of the point $x(t)$, implies that $v$ is moved a distance at most $k + 2D$ by each of the elements $g, g^2, \ldots g^n$.

By our choice of $n$, there are exponents $i < j$ such that $g^i.v = g^j.v$. But then $g^{j-i}$ fixes $v$, and hence is elliptic. Hence the element $g$ is also elliptic, in contradiction to the assumption that $g$ is parabolic. This contradiction shows that there is no parabolic element, finishing the proof.

6 Proof of the Main Result

We will use the classification of actions on hyperbolic spaces established in [Woe93], as already mentioned. The bound on the rank of a flat subgroup is proved on a case-by-case basis according to this classification.

The following two lemmas prepare Proposition 23, which provides this bound if the flat group contains a non-abelian free group consisting of hyperbolic elements.

**Lemma 21.** Let $G$ be a totally disconnected, locally compact group acting cocompactly and with compact, open point stabilizers on a locally finite, connected $\delta$-hyperbolic complex. If $h$ is an element of $G$ of non-trivial scale (which is thus necessarily hyperbolic), then the orbit of $\omega_h$, the repelling boundary point of $h$, under every open subgroup is infinite. In particular, no open subgroup of $G$ fixes $\omega_h$.

**Proof.** It follows from Proposition 18 and Theorem 20 that an element $h$ in $G$ of non-trivial scale must indeed be hyperbolic as stated. Proposition 11 implies that $h$ also acts as a hyperbolic automorphism of the given complex.

We now begin the proof proper; we will prove the contraposition. Assume then that $V$ is an open subgroup of $G$ such that the orbit of $\omega_h$ under $V$ is finite. Then a closed subgroup of finite index in $V$ fixes $\omega_h$ and we can assume that $V$ fixes $\omega_h$. Intersecting the group $V$ with the stabilizer of a vertex, $v$, say, we may assume that $V$ fixes a given vertex $v$ also.

Applying Theorem 7.7 in [Mö102] with $V$ equal to the group of the same name, and $x$ equal to $h$, we see that the scale of $h$ is given by the limit

$$\lim_{n \to \infty} |V.(h^{-n}.v)|^{1/n}.$$  

We will use our assumptions to show that there is a bound on the diameter of the orbits $V.(h^{-n}.v)$ that is uniform in $n \in \mathbb{N}$. Because $G$ acts cocompactly, this implies that there is a uniform bound on the number of the vertices in these orbits. The displayed formula above will then show that the scale of $h$ is 1, and establish our claim.

The map $f$ that sends an integer $n$ to the vertex $h^{-n}.v$ is a quasi-geodesic ray that converges to $\omega_h$. By part (i) of Théorème 5.25 in [GdlH90], there is a geodesic ray, $r$, say, that starts at $v$ and is at Hausdorff-distance at most $H$ from $f$; the ray $r$ therefore ends in $\omega_h$ also. Then part (i) of Corollaire 7.3 in [GdlH90] implies that $d(r(t), g.r(t)) \leq 8\delta$ for all $g \in V$ and all $t \geq 0$. 

□
Let \( n \) be an integer. On the geodesic ray \( r \) choose a point, \( r(t_n) \) say, such that the distance between \( h^{-n}.v \) and \( r(t_n) \) is at most \( H \). For every \( g \) in \( V \) the distance between \( g.(h^{-n}.v) \) and \( g.r(t_n) \) is then at most \( H \) also. We conclude that for all \( g \) in \( V \) the distance between \( h^{-n}.v \) and \( g.(h^{-n}.v) \) is at most \( 2H + 8\delta \); hence the diameter of the orbits \( V.(h^{-n}.v) \) is indeed uniformly bounded, and we are done.

The claim of the following Lemma is false for flat groups of rank 0.

**Lemma 22.** Let \( G \) be a totally disconnected, locally compact, hyperbolic group and \( H \) a flat subgroup of \( G \) of rank at least 1. Then the group \( H_1 \) does not contain hyperbolic elements.

**Proof.** We will derive a contradiction to Lemma 21 from the assumption that there is a hyperbolic element in \( H_1 \) and an (automatically hyperbolic) element in \( H \setminus H_1 \), thus establishing the claim.

Let \( O \) be a minimizing subgroup for \( H \). The group \( O \) is the stabilizer in \( G \) of the point \( O \) in any rough Cayley graph of \( G \) constructed from \( O \). The group \( H_1 \) normalizes the subgroup \( O \). Hence the group \( O \) fixes the whole orbit \( H_1.O \) pointwise. All boundary points fixed by hyperbolic elements in \( H_1 \) are limit points of this orbit, hence the set of these points, \( L_{hyp}(H_1) \) say, is fixed by the group \( O \) also.

Choose a hyperbolic element \( k \) in \( H_1 \). Denote the repelling boundary point of \( k \) by \( \omega_k \in L_{hyp}(H_1) \). Further choose an element \( h \) in \( H \setminus H_1 \); which is possible because the rank of \( H \) is at least 1. Replacing, if necessary, \( h \) by its inverse, we may assume that \( h \) has non-trivial scale. Proposition 18 and Theorem 20 imply that the element \( h \) is hyperbolic. Denote the repelling boundary point of \( h \) by \( \omega_h \).

Since the subgroup \( O \) fixes \( \omega_k \) and \( h \) has non-trivial scale, \( \omega_k \) is different from \( \omega_h \), for otherwise we would have a contradiction to Lemma 21. Because \( H_1 \) is normal in \( H \), the group \( H \) leaves the set \( L_{hyp}(H_1) \) invariant. In particular, the sequence \( (h^{-n}(\omega_k))_{n \in \mathbb{N}} \) which converges to \( \omega_h \) since \( \omega_k \neq \omega_h \), is contained in \( L_{hyp}(H_1) \). By continuity of the action of \( G \) on the hyperbolic compactification, \( \omega_h \) is contained in \( L_{hyp}(H_1) \) and hence is fixed by \( O \). This is the anticipated contradiction to Lemma 21.

We are now ready to treat the first case in the classification.

**Proposition 23.** Let \( G \) be a totally disconnected, locally compact, hyperbolic group and \( H \) a flat subgroup of \( G \) that contains a non-abelian free group consisting of hyperbolic elements. Then the rank of \( H \) is 0.

**Proof.** Let \( F \) be a non-abelian free subgroup of \( H \) consisting of hyperbolic elements. Assume by way of contradiction that the rank of \( H \) is at least 1. Using Lemma 22, we then conclude that the subgroup \( H_1 \) contains no hyperbolic elements. Then the restriction of the canonical map \( H \to H/H_1 \) to the subgroup \( F \) of \( H \) has trivial kernel. It follows that the abelian group \( H/H_1 \) contains a non-abelian free group, which is absurd. Therefore the rank of \( H \) is 0 as claimed.
The second case of the classification is easy.

**Proposition 24.** Let $G$ be a totally disconnected, locally compact, hyperbolic group and $H$ a flat subgroup of $G$ that stabilizes a non-empty, compact subset of some rough Cayley graph of $G$. Then the rank of $H$ is 0.

*Proof.* The condition satisfied by $H$ implies that all elements of $H$ are elliptic. By Proposition 18, $H$ is contained in its subgroup $H_1$ and the flat-rank of $H$ is 0 as claimed. □

The next lemma proves that a quasi-geodesic ray converging to a boundary point, $\omega$ say, is uniformly close to any geodesic ray converging to $\omega$. This lemma is used in the last cases in the classification, Proposition 26 below.

**Lemma 25.** Given real numbers $\delta \geq 0$, $\lambda \geq 1$ and $c \geq 0$ there is a constant $R$ (depending on $\delta$, $\lambda$ and $c$ only) such that for any proper $\delta$-hyperbolic space, $X$: given a geodesic ray, $r$, and a $(\lambda, c)$-quasi-geodesic ray, $f$, in $X$ that converge to the same boundary point $\omega$, the image of $f$ intersects the ball of radius $R$ centred on any point sufficiently far out on $r$.

*Proof.* By part (i) of Théorème 5.25 in [GdlH90] there is a geodesic ray $g$ at Hausdorff distance at most $H$ from $f$, where $H$ depends on $\delta$, $\lambda$ and $c$ only. The geodesic ray $g$ also converges to the boundary point $\omega$. Hence, according to Proposition 7.2 in [GdlH90] appropriate subrays $r'$ and $g'$ of the respective rays $r$ and $g$ have Hausdorff distance at most $16\delta$. Then for each point on $r'$, the ball with radius $16\delta$ centred on that point intersects $g'$, and for each point on $g'$ the ball with radius $H$ centred on that point intersects $f$. Hence the claim holds with $R = H + 16\delta$. □

Finally, we cover the last two cases of the classification.

**Proposition 26.** Let $G$ be a totally disconnected, locally compact, hyperbolic group and $H$ a flat subgroup of $G$ that fixes a boundary point or a pair of boundary points (not necessarily pointwise). Then the rank of $H$ is at most 1.

*Proof.* We first reduce to the case where the flat subgroup $H$ fixes a boundary point. The other case is where $H$ fixes a pair of distinct boundary points without fixing the points. Then the subgroup of $H$ that fixes both points is also flat and has index 2 in $H$. Hence this subgroup has the same rank as $H$ and it suffices to prove the claim for it.

Next we show that the images of any two hyperbolic elements, $g$ and $h$ say, that both fix a boundary point satisfy a nontrivial relation in $H/H_1$, thus showing that $H$ can not contain two elements mapping to linearly independent elements in the quotient and thereby finishing the proof.

Inverting one of $g$, $h$ if necessary, we may assume that $g$ and $h$ have the same attracting boundary point, $\omega$ say. Choose a vertex, $v$, in a rough Cayley graph, $\Gamma$, for $G$. The map $f_h: \mathbb{N} \to \Gamma$ defined by $f_h(n) := h^n.v$ is
a quasi-geodesic ray, with quasi-isometry-constants \((\lambda, c)\) say. Then all the maps \(g^i.f_h\) with \(i \geq 0\) are \((\lambda, c)\)-quasi-geodesic rays and converge to \(\omega\).

Choose a geodesic ray, \(r\) say, ending in \(\omega\). Denote by \(n\) a natural number larger than the maximal number of vertices of \(\Gamma\) that are contained in any closed ball whose radius is the constant \(R\) provided by Lemma 25. Such a number \(n\) exists, because \(G\) acts cocompactly on its rough Cayley graph \(\Gamma\).

According to Lemma 25, we may choose a point sufficiently far out on the ray \(r\) such that all the quasi–geodesic rays \(g.f_h, \ldots, g^n.f_h\) intersect the ball \(B\) of radius \(R\) around it. All points of intersection of \(g.f_h, \ldots, g^n.f_h\) with \(B\) are vertices and so, by our choice of \(n\), there are integers \(i\) and \(j\) with \(0 < i < j\) such that \(g^i(h^p.v) = g^j(h^q.v)\) for some integers \(p\) and \(q\).

The element \(h^{-q}g^{-j}h^p\) fixes \(v\), hence is elliptic and has scale 1. The relation \((p - q)hH_1 + (i - j)gH_1 = 0\) therefore holds in \(H/H_1\) and, since \(j - i \neq 0\), \(hH_1\) and \(gH_1\) are linearly dependent. \(\square\)

The main result of the paper is now obtained by combining these cases.

**Theorem 27.** The flat-rank of a totally disconnected, locally compact, hyperbolic group is at most 1.

**Proof.** Let \(G\) be a totally disconnected, locally compact, hyperbolic group and \(H\) a flat subgroup of \(G\). Choose a connected, locally finite, rough Cayley graph, say \(\Gamma\), for \(G\). The graph \(\Gamma\) is Gromov-hyperbolic and §4C of [Woe93] explains how the results of that paper apply to \(\Gamma\) and its hyperbolic compactification. In particular, [Woe93, Theorem 3] lists the possible types of actions for \(H\) on \(\Gamma\). Each of these possible types is covered by either Proposition 23 (type (a)), Proposition 24 (type (b)) or Proposition 26 (types (c) and (d)) and the rank of \(H\) is seen to be at most 1 in all cases. \(\square\)

**7 Flat subgroups, space of directions and hyperbolic boundary**

In this section we present the proof of Theorem 3, which shows that discreteness of the space of directions (defined in [BW06]) also imposes a bound of 1 on the flat-rank. The proof is followed by two conjectures that propose further links between flat subgroups of a hyperbolic, totally disconnected, locally compact group, its space of directions and hyperbolic boundary.

**Proof (of Theorem 3).** The space of directions of a totally disconnected, locally compact group of flat-rank \(k\) contains a \(k\)-cell by Proposition 23 in [BW06]. Hence a group with a discrete space of directions can have flat-rank at most 1. Furthermore, since a group has flat-rank 0 if and only if its space of directions is empty, we even have that a group with a non-empty, discrete space of directions has flat-rank equal to 1. \(\square\)

The argument in the above proof in fact shows that the flat-rank of a group is bounded by the dimension of the space of directions. This pseudo-metric space need not be finite dimensional manifold however, as the example (a group having flat-rank 1) in [BW06, 5.2.3] illustrates.
Theorem 3 applies to closed subgroup of the automorphism group of a locally finite tree, by Proposition 36(2) in [BW06]. In fact, the bound on the flat-rank of such groups also follows from Theorem 1 in case they are compactly generated, because they are then hyperbolic.

**Proposition 28.** Let $G$ be a compactly generated topological group acting minimally on a tree, $X$, such that the stabilizers of vertices are open subgroups of $G$. Then $G$ acts with finitely many orbits on the vertices.

*Proof.* The argument follows that of [Bass93, Proposition 7.9(b)] which establishes the claim for discrete $G$. Let $A$ denote the graph of groups arising from the action of $G$ on $X$. Let $G_B$ denote the fundamental group of a subgraph of groups $B$ of $A$. Note that $G_B$ is always an open subgroup of $G$. Clearly the groups $G_B$, where $G_B$ ranges over all finite subgraphs of groups form an open covering of $G$. Because $G$ has a compact generating set we see that finitely many of the groups $G_B$, with $B$ a finite subgraph of groups, cover the generating set. Indeed, one sees from this that there is a finite subgraph of groups, $A'$, such that the fundamental group of $A'$ contains the generating set and thus the fundamental group of $A'$ is equal to $G$. By [Bass93, Proposition 7.12] we can now conclude that because the action is minimal that $A' = A$. The graph of groups $A'$ is finite and thus $A$ is also finite and hence the group $G$ has only finitely many orbits on both the vertices and edges of $X$. \( \square \)

**Corollary 29.** Let $G$ be a compactly generated group acting on a tree, $X$, such that the stabilizers of vertices are compact open subgroups of $G$. Then $G$ is hyperbolic.

*Proof.* If $G$ consists only of elliptic elements, then, since $G$ is compactly generated, [Bass93, Proposition 7.2] implies that it is compact. Hence $G$ is in this case trivially hyperbolic.

Otherwise, $G$ contains a hyperbolic element and the union of all axes of all hyperbolic elements is a minimal $G$-invariant subtree of $X$. Replacing $X$ by this subtree and $G$ by its quotient by the (compact) stabilizer of this subtree, it may be assumed that the action is minimal. By Proposition 28, $G$ acts with only finitely many orbits on the edges of $X$. Hence $X$ is locally finite and is quasi-isometric to a rough Cayley graph of $G$, which must therefore be hyperbolic. \( \square \)

The following conjecture asks for a common extension of our Theorem 1 and Proposition 36(2) in [BW06].

**Conjecture 30.** Let $G$ be a hyperbolic, totally disconnected, locally compact group. Then the following holds.

1. The map which assigns each element of non-trivial scale to its attracting boundary point defines an injection of the set of directions of $G$ into the hyperbolic boundary.
2. Elements of $G$ with distinct directions have pseudo-distance 2.

The next conjecture asks whether the relationship between flat subgroups and the geometry of the rough Cayley graph that may be observed in automorphism groups of trees or in the setting of [BRW07] holds for hyperbolic totally disconnected, locally compact groups in general.

**Conjecture 3.** Suppose that $G$ is a hyperbolic, totally disconnected, locally compact group which does not fix a point of its hyperbolic boundary. Then every flat subgroup of flat-rank 1, $H$ say, of $G$ has a limit set that contains 2 elements, which are both fixed by $H$. The group $H_1$ is relatively compact and is equal to the set of elliptic elements of $H$.

The necessity of the hypothesis that the group should not fix a point on the hyperbolic boundary is shown by the following example.

**Example 3.** Let $G$ be the semidirect product $\mathbb{Z} \ltimes \mathbb{F}_q((t))$, where $\mathbb{F}_q((t))$ is the ring of formal Laurent series over the finite field $\mathbb{F}_q$ and $\mathbb{Z}$ acts by multiplication by $t$. This group is isomorphic to the group of matrices of the form $\begin{pmatrix} t^n & f \\ 0 & 1 \end{pmatrix}$ where $n \in \mathbb{Z}$ and $f \in \mathbb{F}_q((t))$. Then $G$ acts faithfully and co-compactly on the Bruhat-Tits-tree of $SL_2(\mathbb{F}_q((t)))$, a homogeneous tree where every vertex has valency $q+1$, and so is a hyperbolic group by Proposition 11. (Although $G$ is not a subgroup of $SL_2(\mathbb{F}_q((t)))$, it does act on this group by conjugation and this action induces an action on the tree.)

Put $O$ equal to $\mathbb{F}_q((t))$, a compact open subgroup of $G$. Direct calculation shows that, if $g = (n, f) \in G$, then $|gOg^{-1} : gOg^{-1} \cap O|$ is equal to 1 if $n \geq 0$ and to $q^{-n}$ if $n < 0$ and that these are the minimum possible. Hence $O$ is minimizing for $G$ and $G$ is flat.

However $G$ does not satisfy the last hypothesis of the conjecture because it fixes a point on the hyperbolic boundary, in this case the set of ends of the tree. It does not satisfy the conclusions because there is no other end of the tree fixed by $G$. Furthermore, as the above calculation shows, $G_1 = \mathbb{F}_q((t))$ which is the set of elliptic elements in $G$, and this group is not compact.

**8 Examples of simple groups of flat rank at most 1**

Here we list examples of simple, totally disconnected, locally compact groups whose flat rank is at most 1. We expect none of the listed groups to have flat-rank 0. Indeed, in any given case it is usually easy to exhibit a hyperbolic element of non-trivial scale.

In [Tit70], Tits showed that many closed subgroups of automorphism groups of locally finite trees are simple and provided concrete constructions of examples in terms of $a$-coverings. As seen in the previous section, these groups have flat-rank at most 1.

Haglund and Paulin, in [HP98], adapted Tits’ methods to automorphism groups of negatively curved complexes, thus providing many more totally
disconnected, locally compact, non-discrete, non-linear, simple groups. In order to be able to formulate an analogue of Tits’ property (P), a central assumption in [Tit70], they introduced an axiomatic framework, namely, spaces with walls, which allowed them to generalize Tits’ result to groups acting on hyperbolic spaces with walls; see Théorème 6.1 in their paper.

The groups studied by Haglund and Paulin are non-discrete under fairly general conditions; compare their Lemme 3.6. We suspect that all non-discrete groups $G^+$, where $G$ satisfies the conditions of [HP98, Théorème 6.1], act cocompactly on the hyperbolic graph associated to the space with walls in the statement of that theorem. If so, such a group $G^+$ is a totally disconnected, locally compact, non-discrete, simple, hyperbolic group as a consequence of Proposition 11 and thus has flat-rank at most 1 by Theorem 1.

While there is some uncertainty as to whether the group $G^+$ associated to a general group satisfying the conditions of Théorème 6.1 in Haglund and Paulin’s paper acts cocompactly, all concrete examples given in that paper do act cocompactly. These examples are

1. the group of type-preserving automorphisms of a Bourdon building (Théorème 1.1);
2. a subgroup of finite index in the automorphism group of a Benakli-Haglund building (Théorème 1.2);
3. a subgroup of finite index in the automorphism group of the Cayley graph of a hyperbolic, non-rigid Weyl group with respect to its standard system of generators (Théorème 1.3);
4. subgroups of finite index in the automorphism groups of certain even polyhedral complexes (Théorème 1.4).

That these concrete examples do act cocompactly is no accident. It is difficult to construct complexes with uncountable automorphism groups; most constructions of such complexes start from a discrete group acting cocompactly on some complex. A notable exception is the horocyclic product of two locally finite trees with different valencies; while the automorphism group of this complex acts cocompactly, there is no discrete subgroup doing the same as shown in [EFW07].

9 Hyperbolic orbits for groups of automorphisms

The idea of the proof of Theorem 2 is straightforward. If $\mathcal{A}$ contains a flat subgroup whose rank is 2, then there is an orbit of $\mathcal{A}$ which contains a subset ‘looks like’ $\mathbb{Z}^2$; this is inconsistent with the assumption that $\mathcal{A}$ has a hyperbolic orbit. Since we are now dealing with spaces which are not geodesic, we must use the general definition of $\delta$-hyperbolic space in terms of the Gromov-product; recall that this states that a metric space $X$ is $\delta$-hyperbolic if and only if

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta$$
for all \( w, x, y, z \in X \) \([BH99, III.H.1.20]\).

For our argument, we first need to be able to choose the orbit at our convenience, as the extent to which a flat subgroup of flat-rank 2 "looks like" \( \mathbb{Z}^2 \) depends on the orbit. The following two results take care of that problem.

**Lemma 33.** Let a group \( A \) act by isometries on a metric space \( B \). Then any two orbits of \( A \) in \( B \) are \((1, \epsilon)\)-quasi-isometric, with \( \epsilon \) only depending on the pair of orbits.

**Proof.** Fix two points, \( M \) and \( N \) say, in \( B \). For every \( P \in A.M \) choose an element \( \alpha_P \in A \) such that \( P = \alpha_P.M \). Once this choice is made, define a map \( \tau_{M,N}: A.M \to A.N \) which maps \( P \) to \( \alpha_P.N \). Then, for \( A \) and \( B \) in \( A.M \) we have

\[
d(A, B) \leq d(\alpha_A.M, \alpha_A.N) + d(\alpha_A.N, \alpha_B.N) + d(\alpha_B.N, \alpha_B.M) \\
= d(\alpha_A.N, \alpha_B.N) + 2d(M, N),
\]

and

\[
d(\alpha_A.N, \alpha_B.N) \leq d(\alpha_A.N, \alpha_A.M) + d(\alpha_A.M, \alpha_B.M) + d(\alpha_B.M, \alpha_B.N) \\
= d(A, B) + 2d(M, N)
\]

hence \( |d(\tau_{M,N}(A), \tau_{M,N}(B)) - d(A, B)| \leq 2d(M, N) \), which shows that \( \tau_{M,N} \) is a \((1, 2d(M, N))\)-quasi-isometric embedding. Further, we have

\[
d(\tau_{M,N}(\alpha.M), \alpha.N) = d(\alpha_{\alpha.M}.N, \alpha.N) \\
\leq d(\alpha_{\alpha.M}.N, \alpha_{\alpha.M}.M) + d(\alpha_{\alpha.M}.M, \alpha.M) + d(\alpha.M, \alpha.N) \\
= 2d(M, N),
\]

showing that \( \tau_{M,N} \) is \( 2d(M, N) \)-dense. \( \square \)

**Corollary 34.** If one orbit of \( A \) in \( B(G) \) is hyperbolic, then all are. \( \square \)

**Lemma 35.** Let \( f: B_1 \to B_2 \) be a \((1, \epsilon)\)-quasi-isometric embedding. If \( B_2 \) is \( \delta \)-hyperbolic, then \( B_1 \) is \((\delta + 3\epsilon)\)-hyperbolic.

**Proof.** From our assumption \( |d(x, y) - d(f(x), f(y))| \leq \epsilon \) for any pair, \( x \) and \( y \), of points in \( B_1 \), we infer that

\[
|\langle x \cdot y \rangle_o - \langle f(x) \cdot f(y) \rangle_{f(o)}| \leq 3\epsilon/2.
\]

We conclude that \( \langle x \cdot z \rangle_o - \min\{\langle x \cdot y \rangle_o, \langle y \cdot z \rangle_o\} \) differs from

\[
\langle f(x) \cdot f(z) \rangle_{f(o)} - \min\{\langle f(x) \cdot f(y) \rangle_{f(o)}, \langle f(y) \cdot f(z) \rangle_{f(o)}\}
\]

by at most \( 3\epsilon \). \( \square \)

Taking \( f \) in Lemma 35 to be the inclusion of a subset we also obtain the following corollary.
Corollary 36. Every subspace of a hyperbolic space is hyperbolic. □

Corollaries 34 and 36 will prove Theorem 2, once we show that the presence of a flat group of rank \( r \geq 2 \) implies the existence of a subspace of \( B(G) \) that is an integer lattice in a normed space of dimension \( r \). For completeness, the proof that such a lattice is not hyperbolic is outlined after the next result.

Lemma 37. Let \( \mathcal{H} \) be a flat group of automorphisms of the totally disconnected locally compact group \( G \) with rank \( r \). Let \( O \) be minimizing for \( \mathcal{H} \). Then \( (\mathcal{H}.O, d) \) is isometric to a lattice in in a normed real linear space of dimension \( r \).

Proof. Since \( O \) is minimizing for \( \mathcal{H} \), an automorphism \( \alpha \in \mathcal{H} \) satisfies \( \alpha.O = O \) if and only if \( \alpha \in \mathcal{H}_1 \) and the map \( \alpha \mapsto \alpha.O \) induces a bijection \( \mathcal{H}.O \to \mathcal{H}/\mathcal{H}_1 \).

Composition with the isomorphism \( \mathcal{H}/\mathcal{H}_1 \to \mathbb{Z}^r \) then produces a bijection \( \mathcal{H}.O \to \mathbb{Z}^r \). The metric on \( \mathcal{H}.O \) pushes forward to \( \mathbb{Z}^r \) via this bijection and the resulting metric on \( \mathbb{Z}^r \) is translation-invariant because \( \mathcal{H}/\mathcal{H}_1 \) and \( \mathbb{Z}^r \) are isomorphic as groups.

An explicit formula may be given for the distance \( d(O, \alpha.O) \) for \( \alpha \in \mathcal{H} \). There are: a finite set \( \Phi = \Phi(\mathcal{H}, G) \) of surjective homomorphisms \( \rho: \mathcal{H} \to \mathbb{Z} \) such that the intersection of the kernels of elements in \( \Phi \) equals \( \mathcal{H}_1 \); and a set \( \{t_\rho | \rho \in \Phi \} \) of integers greater than one such that

\[
  s_G(\alpha) = \prod_{\rho \in \Phi, \rho(\alpha) > 0} t_\rho^{\rho(\alpha)}, \quad (\alpha \in \mathcal{H}),
\]

see [Wil04b, Theorems 6.12 and 6.14]. Since \( O \) is minimizing for \( \mathcal{H} \), we further have that \( d(\alpha(O), O) = \log(s_G(\alpha) \cdot s_G(\alpha^{-1})) \), whence

\[
  d(\alpha(O), O) = \sum_{\rho \in \Phi} \log(t_\rho \rho(\alpha)), \quad (\alpha \in \mathcal{H}).
\]

Composition of each \( \rho \in \Phi \) with the isomorphism \( \mathcal{H}/\mathcal{H}_1 \to \mathbb{Z}^r \) yields homomorphisms \( \tilde{\rho}: \mathbb{Z}^r \to \mathbb{Z} \) and there are \( z_\rho \in \mathbb{Z}^r \) such that \( \tilde{\rho}(\mathbf{z}) = z_\rho, \mathbf{z} \) for each \( \rho \in \Phi \). Hence the bijection \( \mathcal{H}.O \to \mathbb{Z}^r \) becomes an isometry if \( \mathbb{Z}^r \) is equipped with the translation-invariant metric

\[
  d(\mathbf{w}, \mathbf{z}) := \sum_{\rho \in \Phi} \log(t_\rho) \| z_\rho \cdot (\mathbf{w} - \mathbf{z}) \|, \quad (\mathbf{w}, \mathbf{z} \in \mathbb{Z}^r).
\]

This metric extends to \( \mathbb{R}^r \) by the same formula. □
We conclude with a sketch of the argument that a lattice $X$ in a normed space of dimension $r \geq 2$ is not hyperbolic. Given 4 points $x, y, z$ and $w$ in $X$ let $\delta(x, y, z)_w$ denote the quantity $\min\{(y \cdot x)_w, (x \cdot z)_w\} - (y \cdot z)_w$. The number $\delta(x, y, z)_w$ is a lower bound for any $\delta$ such that $X$ could be $\delta$-hyperbolic. But $\delta(\lambda x, \lambda y, \lambda z)_{\lambda w} = |\lambda| \delta(x, y, z)_w$ for any $\lambda \in \mathbb{Z}$, showing that no such $\delta$ can exist, if we can find a quadruple $(x, y, z, w)$ such that $\delta(x, y, z)_w$ is positive. If $x$ and $y$ are vectors for which $\|x + y\| + \|x - y\| > \|x\| + \|y\|$ then $\delta(x, y, z)_0$ is positive. Such vectors exist for any normed linear space of dimension at least 2; vectors in the lattice can be found by rational approximation followed by scaling by integers.

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