Integral functionals for spectrally positive
Lévy processes

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Abstract

We find necessary and sufficient conditions for almost sure finiteness of integral functionals of spectrally positive Lévy processes. Via Lamperti type transforms, these results can be applied to obtain new integral tests on extinction and explosion behaviors for a class of continuous-state nonlinear branching processes.

Key words and phrases. Lévy process, continuous-state branching process, stochastic integral equation, explosion, extinction, Lamperti transform, integral test.

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1. Introduction

For a diffusion process or a Lévy process $Z = (Z_t)_{t \geq 0}$ taking values in an open interval $I = (l, r)$, let $f$ be a non-negative, measurable and locally bounded function on $I$, and introduce

$$A_t(f) := \int_0^t f(Z_s)ds.$$  

Denote by $\zeta := \inf\{t > 0 : Z_t > r \text{ or } Z_t < l\}$ the exiting time of $Z$. The ultimate value of this additive functional, i.e., $A_\zeta(f)$, is often called a perpetual integral functional. In this short note we are interested in finding necessary and sufficient conditions for the a.s. finiteness of $A_\zeta(f)$ for certain choices of $\zeta$.

The perpetual integral functionals appear naturally in various applications. One of the common issues in studying Markov processes is the question that whether a random time-change of the process is again conservative. The time-changed process is usually realized as the generalized inverse of a positive additive functional. As a consequence, the time-changed process is conservative if and only if $A_\zeta(f) = \infty$ a.s. In insurance mathematics, the functional $A_\zeta(f)$ may be interpreted, in a suitable model, as the present value of a continuous stream of perpetuities, see Dufresne [8]. In the stochastic models of ecology, the functional $A_\zeta(f)$ can be regarded as the total population of a species.
The finiteness of (1.1) has been studied by various authors. The well known Engelbert-Schmidt zero-one law (see [17, Proposition 3.6.27]) states that for a Brownian motion \((B_t)_{t \geq 0}\) and any nonnegative Borel function \(f\) the following three statements are equivalent:

(i) \[ \mathbb{P} \left( \int_0^t f(B_s)ds < \infty \text{ for all } t \in [0, \infty) \right) > 0, \]

(ii) \[ \mathbb{P} \left( \int_0^t f(B_s)ds < \infty \text{ for all } t \in [0, \infty) \right) = 1, \]

(iii) the function \(f\) is locally integrable on \(\mathbb{R}\).

This important property has a plenty of applications. For example, it constitutes an important step in the Engelbert-Schmidt construction of weak solutions of one-dimensional SDEs. When \(Y\) is a Brownian motion with positive drift, it is known that \(A_\infty(f)\) is finite a.s. if and only if \(f\) is integrable at \(\infty\) (see Engelbert and Senf [9] and Salminen and Yor [18]). When \(Z\) is a diffusion process on an interval \(I = (r, l)\). The finiteness of the perpetual integral functional of \(Z\), i.e. \(A_\zeta(f)\), is considered in Khoshnevisan et al. [18] and where the proofs are based on Khasminskiis theorem.

When \(Z\) is a Lévy process, the fact that \(Z\) may have jumps makes things more involved. Suppose that \(f\) is a positive locally integrable function and \(Z\) is a Lévy process such that \(\mathbb{E}[Z_1] \in (0, \infty)\) and its local time exists. Döring and Kyprianou [10] find the following condition

\[ \int_0^\infty f(Z_s)ds < \infty \text{ a.s. } \iff \int_0^\infty f(x)dx < \infty. \]

Their proof is based on Jeulin’s lemma. Recently, Kolb and Savov [19] improve the above result by removing the assumption of the existence of local time. They establish that

\[ \int_0^\infty f(Z_s)ds < \infty \text{ a.s. } \iff \int_0^\infty f(x)U(dx) < \infty, \]

where \(U\) is the potential measure of \(Z\).

When the function \(f\) is an exponential function, i.e. \(f(x) = e^{-\theta x}\) for some \(\theta > 0\), the exponential integral functional has drawn the attention of many authors in recent years; see, e.g. [12, 20, 22, 30, 31]. The exponential integral functional plays an important role in many domains, e.g., mathematical finance, branching process in random environments, tail probability, especially in the case of Lévy process with one-sided jumps (see for example [32, 33]). We refer to Bertoin and Yor [6] for a survey of this area.

However, in the existing literature, when \(Z\) is a Lévy process, only functionals with deterministic upper limit have been considered so far. To the best of our knowledge, there has been no systematic discussion on the a.s. finiteness of functional with a stopping time as the upper limit for integral. In this work, we are going to fill this gap. For a spectrally positive Lévy process \(Z\), i.e. a one-dimensional Lévy process \(Z\) with no negative jumps, we obtain an explicit integral test on the finiteness of \(A_\zeta(f)\), where \(\zeta\) denotes the exiting time of \((0, \infty)\), i.e. \(\zeta = \inf\{t > 0 : Z_t = 0\}\) with the convention \(\inf\emptyset = \infty\).
The assumption of non-negative jump allows us to use the tools and techniques available for one-sided Lévy processes, such as scale functions. On the other hand, the spectrally positive Lévy process is of particular interest since in this case, the processes obtained by time-changing process \( Z \) forms a class of important processes with rich mathematical structures; see for example the continuous-state nonlinear branching processes. We will discuss the time-changed processes in details in the next section.

The remainder of this paper is organized as follows. In Section 2 we introduce the necessary notations and present the main results on the Engelbert-Schmidt type zero-one law for spectrally positive Lévy process. We also apply the main results to find integral tests on the boundary behaviors of the time-changed processes. The last section is devoted to the proofs of the main results.

2. Notations and main results

In this section we first recall the Lévy-Itô decomposition of spectrally positive Lévy process and some necessary notations. Let \( b \) and \( c \geq 0 \) be constants and \( \pi(dz) \) be a \( \sigma \)-finite measure on \((0, \infty)\) satisfying
\[
\int_{(0, \infty)} (1 \wedge z^2) \pi(dz) < \infty.
\]
Given a Brownian motion \((B_t)_{t \geq 0}\) and an independent Poisson random measure \( N(ds, dz) \) on \((0, \infty) \times (0, \infty)\) with intensity \( ds \pi(dz) \), both defined on a filtered probability space \((\mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), a spectrally positive Lévy process \((Z_t)_{t \geq 0}\) started at \( x \) can be represented as
\[
Z_t = x - bt + \sqrt{2}cB_t + \int_0^t \int_{[0, 1]} zN(ds, dz) + \int_0^t \int_{(1, \infty)} zN(ds, dz),
\]
where \( \tilde{N}(ds, dz) := N(ds, dz) - ds \pi(dz) \) denotes the compensated Poisson random measure on \((0, \infty) \times (0, 1]\). Throughout the paper we assume that process \( Z \) is not a subordinator.

Write \( \mathbb{P}_x(\cdot) := \mathbb{P}(\cdot|Z_0 = x) \) and \( \mathbb{E}_x \) for the corresponding expectation. The Laplace exponent \( \psi : [0, \infty) \to (-\infty, \infty) \) of the spectrally negative Lévy process \(-Z\) is specified by
\[
\mathbb{E}_x[e^{-\lambda Z_t}] = e^{-\lambda x + \psi(\lambda)t}, \quad \lambda, t \geq 0,
\]
where \( \psi \) has the expression of
\[
\psi(\lambda) = b\lambda + c\lambda^2 + \int_{(0, \infty)} (e^{-\lambda u} - 1 + \lambda u 1_{(0,1]}(u)) \pi(du), \quad \lambda \geq 0.
\]
By the assumption that \( Z \) is not a subordinator, we see that there exists a \( \lambda \in (0, \infty) \) such that \( \psi(\lambda) > 0 \) in that case, it is known that the Laplace exponent \( \psi \) is strictly convex and tends to \( \infty \) as \( \lambda \to \infty \). Let
\[
\Phi(0) := \inf \{ \lambda > 0 : \psi(\lambda) > 0 \} < \infty.
\]

Recall that we restrict \( Z \) on the positive half line. Then the life time of \( Z \) is just \( \zeta \), the first hitting time of 0. By [23, Theorem 3.12] we have
\[
\mathbb{P}_x(\zeta < \infty) = e^{-\Phi(0)x} \quad \text{and} \quad \mathbb{P}_x(\zeta = \infty) = 1 - e^{-\Phi(0)x}.
\]
Here, we stress that $\mathbb{P}_x(\zeta < \infty)$ is always strictly positive in our setting. On the other hand, we have $\mathbb{P}_x(\zeta = \infty) > 0$ if and only if $\Phi(0) > 0$.

We are concerned with the almost sure finiteness of $A_\zeta(f)$ under conditional probabilities $\mathbb{P}_x(\cdot|\zeta < \infty)$ and $\mathbb{P}_x(\cdot|\zeta = \infty)$, respectively. To illustrate our main contributions, we present the following statements for the Engelbert-Schmidt type zero-one law for spectrally positive Lévy processes. The proofs are deferred to Section 3.

**Theorem 2.1.** Let $f$ be a strictly positive function on $(0, \infty)$ satisfying $\sup_{x \geq \varepsilon} f(x) < \infty$ for any $\varepsilon > 0$. Then for any $x > 0$, the following four statements are equivalent:

(i) $\mathbb{P}_x(A_\zeta(f) < \infty|\zeta < \infty) > 0$;  

(ii) $\mathbb{P}_x(A_\zeta(f) < \infty|\zeta < \infty) = 1$;  

(iii) $\mathbb{E}_x\left[\int_0^\zeta f(Z_t)e^{-\lambda Z_t}dt|\zeta < \infty\right] < \infty$ for some/all $\lambda > 0$;  

(iv) $\int_0^\infty \frac{f(1/\lambda)}{\lambda \psi(\lambda)}d\lambda < \infty$.

For any $y \geq 0$, set $\tau_y := \inf\{t \geq 0 : Z_t \leq y\}$ with the convention $\inf\emptyset = \infty$. Note that $\zeta = \tau_y$ by definition.

**Theorem 2.2.** Suppose that $\Phi(0) > 0$ and $f$ is a strictly positive and decreasing function on $(0, \infty)$. Then the following three statements are equivalent:

(i) $\mathbb{P}_x(A_\infty(f) < \infty|\zeta = \infty) > 0$ for all $x > 0$;  

(ii) $\mathbb{P}_x(A_\infty(f) < \infty|\zeta = \infty) = 1$ for all $x > 0$;  

(iii) $\mathbb{E}_x[A_{\tau_y}(f)] < \infty$ for all $x > 0$ and all $y \in (0, x)$.  

Moreover, if we further assume that $\psi'(0) < \infty$, then the above three statements are all equivalent to

(iv) $\int_0^\infty f(y)dy < \infty$.

In the above theorem, the assumption $\psi'(0) < \infty$ could be removed at the price of an additional requirement on $f$, and the integral test becomes a little more complicated.

**Corollary 2.3.** Suppose that $\Phi(0) > 0$ and function $f$ can be represented as the Laplace transform of a nonnegative function $g$, i.e. $f(x) = \int_0^\infty e^{-xz}g(z)dz$ for $x > 0$. Then the statements (i), (ii) and (iii) in Theorem 2.2 are all equivalent to

$$\int_{0+} g(\Lambda) \psi(\lambda) d\lambda > -\infty.$$
The integral functionals are very useful in the study of explosion and extinction behaviors of the following SDE, whose solution can be represented as a time changed spectrally positive \( \text{Lévy} \) process. Consider a filtered probability space \((\Omega, \mathcal{F}, \mathcal{G}_t, \mathbb{P})\) satisfying the usual hypotheses. Let \( \{W_t\}_{t\geq 0} \) be an \((\mathcal{F}_t)\)-Brownian motion. Let \( M(ds, dz, du) \) be an independent \((\mathcal{F}_t)\)-Poisson random measure on \((0, \infty) \times (0, \infty) \times (0, \infty) \) with intensity \( ds \pi (dz)du \) and let \( M(ds, dz, du) \) be the corresponding compensated Poisson random measure. We consider the nonnegative solution of the stochastic integral equation

\[
X_t = x + \sqrt{2c} \int_0^t \sqrt{1/f(X_s)} dW_s + \int_0^t \int_0^{1/f(X_s)} z M(ds, dz, du)
\]

(2.5)

\[
- b \int_0^t 1/f(X_s) ds + \int_0^t \int_{(1, \infty)} 1/f(X_s) z M(ds, dz, du).
\]

By a solution \( X = (X_t)_{t \geq 0} \) to equation (2.5) we mean a càdlàg \((0, \infty)\)-valued \((\mathcal{F}_t)\)-adapted process satisfying (2.5) up to times

\[
\zeta_n := \inf \{ t \geq 0 : X_t \geq n \text{ or } X_t \leq 1/n \}
\]

for all \( n > 0 \) and both \( 0 \) and \( \infty \) are absorbing boundaries for \( X \). Applying Ikeda and Watanabe [15] Theorem 9.1 we see that, if \( 1/f \) is locally Lipschitz, then SDE (2.5) has a pathwise unique solution. Note that if \((Z_t)_{t \geq 0}\) is a spectrally one-sided \( \alpha \)-stable \( \text{Lévy} \) process with \( \alpha \in (1, 2) \), then SDE (2.5) can be transformed into the following form:

\[
X_t = x + \int_0^t \sigma(X_{s-}) dZ_t,
\]

where \( \sigma(X_{s-}) = 1/f(X_{s-})^{1/\alpha} \). In Döring and Kyprianou [11], they study the boundary behavior of the solution to (2.6), and in their setting, \((Z_t)_{t \geq 0}\) is a \( \alpha \)-stable \( \text{Lévy} \) process with \( \alpha \in (0, 2) \).

The following result on random time change can be proved using techniques similar to those in Caballero et al. [7], where the function \( f(x) = x^{-1} \) is considered.

**Proposition 2.4.** Given any locally bounded and strictly positive function \( f \) on \((0, \infty)\), for any \( t \geq 0 \) let

\[
\eta_f(t) := \inf \{ s \geq 0 : A_{s+\zeta}(f) > t \}
\]

with the convention \( \inf \emptyset = \infty \). Then \((X_t)_{t \geq 0} := (Z_{\eta_f(t) \wedge \zeta})_{t \geq 0} \) with convention \( Z_\infty \equiv \infty \) for \( \tau^-_0 = \infty \) is a weak solution to equation (2.5).

For the case that \( f(x) = 1/x \), the solution to (2.5) is called **continuous-state branching process**, which has attracted the attention of many researchers in the past decades. This process is a model for the evolution of populations. A well known result is that such a process satisfies the branching property, i.e.

\[
\mathbb{E}_{x_1+x_2}[e^{-\lambda X_t}] = \mathbb{E}_{x_1}[e^{-\lambda X^{(1)}_t}]\mathbb{E}_{x_2}[e^{-\lambda X^{(2)}_t}]
\]

for any \( \lambda, t, x_1, x_2 > 0 \), where \( X^{(1)}_t \) and \( X^{(2)}_t \) denote two independent copies of \( X \) with initial values \( x_1 \) and \( x_2 \), respectively. The connection of those processes with spectrally positive \( \text{Lévy} \) processes through random time changes is pointed out by Lampertii [24]. The continuous-state branching processes also appear in the study of \( \text{Lévy} \) trees; see, e.g., Duquesne [9]. A remarkable theory of flows of such processes with applications to flows of \( \text{Bessel} \) bridges and coalescents with multiple collisions has been developed by Bertoin.
and Le Gall [2, 3, 4, 5]. The interested readers are referred to Kyprianou [23], Li [28] and Pardoux [29] for reviews of the literature in this subject.

For general locally bounded positive function $f$, the solution to (2.5) called nonlinear branching process is thus a natural generalization of the continuous-state branching process in which the underlying population evolves non-linearly governed by the function $1/f$. Thus, a general nonlinear branching process no longer satisfies the additive branching property. On the other hand, it could be used to describe the interaction and competition between particles. Such a process allows richer boundary behaviors. We refer to Li [26] and Li et al. [27] for the criteria of boundary behaviours of this process. The speed of coming down from infinity for such process is studied in Foucart et al. [13] and the speed of explosion is discussed by Li and Zhou [25].

Thanks to Proposition 2.4, we can apply Theorems 2.1 and 2.2 to obtain the following new integral tests on the boundary behaviors for the continuous-state nonlinear branching process $X$ that solves SDE (2.5). Define the extinction time and the explosion time of $X$ by

$$T_0^- := \inf\{t \geq 0 : X_t = 0\} \quad \text{and} \quad T_\infty^+ := \inf\{t \geq 0 : X_t = \infty\},$$

respectively, with the convention $\inf \emptyset = \infty$. We say that process $X$ becomes extinguishing if $T_0^- = \infty$ and $X_t \to 0$ as $t \to \infty$. By Proposition 2.4 we see that $\eta_f(A_\zeta(f)) = \zeta$ and $X_{A_\zeta(f)} = Z_\zeta$. Then we obtain the following corollary.

**Corollary 2.5.** Given any locally bounded and strictly positive function $f$ on $(0, \infty)$. We have

$$T_0^- = A_\zeta(f) \quad \text{if} \quad \zeta < \infty \quad \text{and} \quad T_\infty^+ = A_\zeta(f) \quad \text{if} \quad \zeta = \infty.$$

Thus, the extinction and explosion of the solution $X$ within finite time correspond to the almost sure finiteness of $A_\zeta(f)$ conditioned on events $\{\zeta < \infty\}$ and $\{\zeta = \infty\}$, respectively. Similarly, extinguishing occurs for $X$ if $A_\zeta(f) = \infty$ given $\{\zeta < \infty\}$. Then, using Theorems 2.1 and 2.2 we can find necessary and sufficient conditions for the process $X$ to die out, to become extinguishing or to explode within finite time.

**Corollary 2.6.** Given $\theta > 0$, let $f(x) = x^{-\theta}$ for $x > 0$.

(i) The process $(X_t)_{t \geq 0}$ started at $x > 0$ goes extinct in finite time with a positive probability if and only if

$$\int_0^\infty \frac{s^{\theta-1}}{\psi(s)} ds < \infty;$$

and the process does not go extinct but become extinguishing with a positive probability if and only if

$$\int_0^\infty \frac{s^{\theta-1}}{\psi(s)} ds = \infty.$$

(ii) The process $(X_t)_{t \geq 0}$ started at $x > 0$ explodes in finite time with a positive probability if and only if $\Phi(0) > 0$ and

$$\int_{0+} \frac{s^{\theta-1}}{\psi(s)} ds > -\infty.$$
The above corollary generalizes the results in Grey [14] and Kawazu and in Watanabe [15], where the classical continuous-state branching process corresponding to $\theta = 1$ is considered.

Proofs of the main results are given in Section 3.

3. Proofs of the main results

As the fluctuation theory of the Lévy process $Z$ plays an important role in our proofs, we recall the definition of the scale function and some of its basic properties. There exists a strictly increasing and positive continuous function $W$ on $[0, \infty)$, called scale function, such that

$$\int_0^\infty e^{-\lambda y} W(y) dy = \frac{1}{\psi(\lambda)} , \quad \lambda > \Phi(0).$$

(3.7)

Define $W(x) = 0$ for $x < 0$ and write $W = W^{(0)}$ for short.

Some classical results for the scale functions for spectrally positive Lévy processes are summarized in the following lemmas. We refer to Bertoin [1, Chapter VII] and Kyprianou [23, Chapter 8] for the general theory of scale function.

Lemma 3.1. For $x > 0$, the potential measure can be represented as

$$\mu_x(dy) := \mathbb{E}_x \left[ \int_0^\zeta 1_{\{Z_t \in dy\}} dt \right] = [e^{-\Phi(x)\zeta}W(y) - W(y-x)]dy.$$

Moreover, given $\varepsilon > 0$ there exist $c_1, c_2 > 0$ such that for any $x \in (0, \varepsilon)$

$$c_1 \frac{1}{x\psi(1/x)} \leq W(x) \leq c_2 \frac{1}{x\psi(1/x)}.$$

(3.9)

Proof. We refer to [23, Theorem 8.1] for the existence of the scale function and [21, Theorem 2.7 (ii)] for (3.8). In the following, we are going to prove (3.9) for completeness. Let

$$\psi^\natural(\lambda) := \psi(\lambda + \Phi(0)), \quad \lambda > 0 \quad \text{and} \quad W^\natural(x) := e^{-\Phi(0)x}W(x).$$

Note that, by the definition we have

$$\lim_{x \to 0} \frac{W^\natural(x)}{W(x)} = 1.$$ 

(3.10)

From [1, p.193] we have that $\psi^\natural$ is again a Laplace exponent and that $W^\natural$ is the corresponding scale function. By the proof of [1, Propositions VII.10], we see that there exist $c_1', c_2' > 0$ such that for any $x > 0$

$$c_1' \frac{1}{x\psi^\natural(1/x)} \leq W^\natural(x) \leq c_2' \frac{1}{x\psi^\natural(1/x)}.$$

(3.11)

Using the following form of Taylor’s formula

$$g(y+z) - g(y) - zg'(y) = z^2 \int_0^1 g''(y+zv)(1-v)dv \quad z > 0,$$

The Laplace exponent $\psi$ can be expressed as

$$\psi(\lambda) = b\lambda + c\lambda^2 + \lambda^2 \int_{(0,1]} u^2 \pi(du) \int_0^1 e^{-\lambda u}(1-v)dv + \int_{(1,\infty)} (e^{-\lambda u} - 1)\pi(du), \quad \lambda \geq 0.
Then
\[
\psi^2(\lambda) \leq b(\lambda + \Phi(0)) + c(\lambda + \Phi(0))^2 + (\lambda + \Phi(0))^2 \int_{[0,1]} u^2 \pi(du) \int_0^1 e^{-\lambda u}(1 - v)dv \\
+ \int_{(0,\infty)} (e^{-\lambda u} - 1)\pi(du).
\]

Applying the monotonicity of \(\psi^2\), one can check that
\[
\lim_{\lambda \to \infty} \frac{\psi^2(\lambda)}{\psi(\lambda)} = 1.
\]

(3.12)

Combining (3.10), (3.11) and (3.12) we prove (3.9). \(\square\)

Scale functions rarely have explicit expressions. In the critical stable case, for which \(\Psi(\lambda) = \lambda^\alpha\) with \(\alpha \in (1, 2]\), the scale function can be found for instance in [21, Example 4.17], and \(W(x) = x^{\alpha-1}/\Gamma(\alpha)\), for \(x > 0\).

\textbf{Lemma 3.2.} For any \(x > 0\) and \(\lambda > 0\), we have
\[
\mathbb{E}_x \left[ \int_0^\zeta e^{-\lambda Z_t}dt \mid \zeta < \infty \right] = \frac{1 - e^{-\lambda x}}{\psi(\lambda + \Phi(0))} < \infty,
\]

Proof. By the Markov property, we have
\[
\mathbb{E}_x \left[ \int_0^\zeta e^{-\lambda Z_t}dt \mid \zeta < \infty \right] = \frac{1}{\mathbb{P}_x(\zeta < \infty)} \int_0^\infty \mathbb{E}_x[e^{-\lambda Z_t}1_{\{t < \zeta < \infty\}}]dt \\
= \frac{1}{\mathbb{P}_x(\zeta < \infty)} \int_0^\infty \mathbb{E}_x[e^{-\lambda Z_t}1_{\{t < \zeta\}} \mathbb{E}_x[1_{\{\zeta < \infty\}} \mid \mathcal{F}_t]]]dt \\
= \frac{1}{\mathbb{P}_x(\zeta < \infty)} \mathbb{E}_x \left[ \int_0^\zeta e^{-(\lambda + \Phi(0))Z_t}dt \right] \\
= \int_0^\infty e^{-(\lambda + \Phi(0))y} \left[ W(y) - e^{\Phi(0)y}W(y - x) \right] dy,
\]

(3.14)

where for the last equality we used (2.4) and (3.8). Finally, thanks to (3.7) we see that (3.13) holds. \(\square\)

\textbf{Lemma 3.3.} Let \(f\) be a locally bounded function on \((0, \infty)\). Then for any \(x > 0\) and \(\lambda > 0\), we have
\[
\mathbb{E}_x \left[ \int_0^{\tau_y} f(Z_t)dt \right] < \infty \iff \int_0^\infty f(z + y)\left[e^{-\Phi(0)(x-y)}W(z) - W(z - x + y)\right]dz < \infty.
\]

Proof. By the spatial homogeneousness of Lévy process and (3.8) we have for \(x > y > 0\),
\[
\mathbb{E}_x \left[ \int_0^{\tau_y} f(Z_t)dt \right] = \mathbb{E}_{x-y} \left[ \int_0^\zeta f(Z_t + y)dt \right] \\
= \int_0^\infty f(z + y)\mu_{x-y}(dz) \\
= \int_0^\infty f(z + y)\left[e^{-\Phi(0)(x-y)}W(z) - W(z - x + y)\right]dz.
\]

(3.15)
By the continuity of $W$, we have that function
\[ f(z + y)[e^{-\Phi(0) (x - y)} W(z) - W(z - x + y)] \]
is bounded for $z$ near 0. Then we only need to show the finiteness of the integral near $\infty$. □

We are now ready to prove the main results.

Proof of Theorem 2.1. (i)⇒(iii) For $x > 0$ let $f_x(y) := f(y)1_{\{y \leq x\}}$. Given $x > 0$, choose $\varepsilon > 0$ small enough so that $\varepsilon < x$.

From the non-negative jump property we see, for $t \leq \tau^-_\varepsilon$
\[ f_x(Z_t) \leq \sup_{y \geq \varepsilon} f(y) < \infty. \]

Therefore, by (3.13) we have
\[
\mathbb{E}_x \left[ \int_0^\tau^-_\varepsilon f_x(Z_t) e^{-\lambda Z_t} dt \right] < \infty
\]
\[
\mathbb{E}_x \left[ \int_0^\tau^-_\varepsilon f_x(Z_t) e^{-\lambda Z_t} dt \right] \leq \sup_{y \geq \varepsilon} f(y) \mathbb{E}_x \left[ \int_0^\tau f_x(Z_t) e^{-\lambda Z_t} dt \right] < \infty
\]
(3.16)

If (i) holds, then there exists a $N > 0$ such that
\[
\mathbb{P}_x(A_\varepsilon(f_x) \leq N | \zeta < \infty) > 0.
\]

Define a stopping time
\[ T := \inf\{t \geq 0 : A_t(f_x) > N\} \]
with the convention $\inf \emptyset = \infty$. Then there exists a constant $\alpha \in (0,1]$ such that
(3.17)
\[ \mathbb{P}_x(T \geq \zeta | \zeta < \infty) = \alpha. \]

Then we have for $0 < \varepsilon < x$,
\[
\mathbb{E}_x \left[ \int_0^\tau^-_\varepsilon f_x(Z_t) e^{-\lambda Z_t} dt \right] < \infty
\]
\[
\mathbb{E}_x \left[ \int_0^\tau^-_\varepsilon f_x(Z_t) e^{-\lambda Z_t} dt ; T \geq \tau^-_\varepsilon | \zeta < \infty \right]
\]
\[+ \mathbb{E}_x \left[ \int_0^T f_x(Z_t) e^{-\lambda Z_t} dt ; T < \tau^-_\varepsilon | \zeta < \infty \right]
\]
\[+ \mathbb{E}_x \left[ \int_\tau^-_\varepsilon^T f_x(Z_t) e^{-\lambda Z_t} dt ; T < \tau^-_\varepsilon | \zeta < \infty \right]
\]
\[
\leq N \mathbb{P}_x(T \geq \tau^-_\varepsilon | \zeta < \infty) + N \mathbb{P}_x(T < \tau^-_\varepsilon | \zeta < \infty)
\]
\[
\mathbb{E}_x \left[ \int_\tau^-_\varepsilon^T f_x(Z_t) e^{-\lambda Z_t} dt ; T < \tau^-_\varepsilon | \zeta < \infty \right]
\]
(3.18)

For each $\varepsilon < y \leq x$,
\[
\mathbb{E}_x \left[ \int_0^\tau^-_\varepsilon f_x(Z_t) e^{-\lambda Z_t} dt | \zeta < \infty \right] = \mathbb{E}_x \left[ \int_0^\tau^-_\varepsilon f_x(Z_t) e^{-\lambda Z_t} dt | \zeta < \infty \right]
\]
where in the last inequality we used (3.19) and the fact that
\[ (3.20) \]
Combining (3.16), (3.18) and (3.20) we have
\[ 10 \]
By using Markov property, the second term of the above right hand side of the above equality can be written by
\[ \mathbb{E}_x \left[ \int_{\tau_y^-}^{\tau^-} f_x(Z_t)e^{-\lambda Z_t}dt \mid \zeta < \infty \right] = \mathbb{E}_x \left[ \int_{\tau_y^-}^{\tau^-} f_x(Z_t)e^{-\lambda Z_t}dt1_{\{\zeta < \infty\}} \right] \]
where \( \theta \) is the shift operator. Then for each \( \varepsilon < y < x \) we have
\[ (3.19) \]
Since \( \varepsilon < Z_T < x \) on the event \( \{T < \tau^-_\varepsilon\} \) under \( \mathbb{P}_x \), the strong Markov property and (3.19) together yields
\[ (3.20) \]
where in the last inequality we used (3.19) and the fact that
\[ \mathbb{P}_x(T < \tau^-_\varepsilon, \zeta < \infty) = \mathbb{E}_x \left[ \mathbb{E}_x[1_{\{T<\tau^-_\varepsilon, \zeta<\infty\}}] \mid \mathcal{F}_T \right] \]
Combining (3.16), (3.18) and (3.20) we have
\[ \mathbb{E}_x \left[ \int_{0}^{\tau^-_\varepsilon} f_x(Z_t)e^{-\lambda Z_t}dt \mid \zeta < \infty \right] \leq N + \mathbb{P}_x(T < \tau^-_\varepsilon \mid \zeta < \infty) \mathbb{E}_x \left[ \int_{0}^{\tau^-_\varepsilon} f_x(Z_t)e^{-\lambda Z_t}dt \mid \zeta < \infty \right]. \]
It follows that
\[ P_x(T \geq \tau^-_\varepsilon | \zeta < \infty) \mathbb{E}_x \left[ \int_0^{\tau^-_\varepsilon} f_x(Z_t)e^{-\lambda Z_t} dt \right] \leq N. \]

Letting \( \varepsilon \to 0 \) and using (3.17) yields
\[ \mathbb{E}_x \left[ \int_0^\zeta f_x(Z_t)e^{-\lambda Z_t} dt | \zeta < \infty \right] \leq N/\alpha < \infty. \]

In addition,
\[ \mathbb{E}_x \left[ \int_0^\zeta (f - f_x)(Z_t)e^{-\lambda Z_t} dt | \zeta < \infty \right] \leq \sup_{y \geq x} f(y) \mathbb{E}_x \left[ \int_0^\zeta e^{-\lambda Z_t} dt | \zeta < \infty \right] < \infty, \]
where the last inequality holds by Lemma 3.2. Combing the above two inequalities gives (iii).

(iii) \( \Rightarrow \) (ii) From (iii) we have
\[ P_x \left[ \int_0^\zeta f(Z_t)e^{-\lambda Z_t} dt < \infty | \zeta < \infty \right] = 1. \]

Since \( \inf_{t \leq \zeta} e^{-\lambda Z_t} > 0 \) on event \( \{ \zeta < \infty \} \), we have a.s.
\[ \int_0^\zeta f(Z_t) dt < (\inf_{t \leq \zeta} e^{-\lambda Z_t})^{-1} \int_0^\zeta f(Z_t)e^{-\lambda Z_t} < \infty \]
on event \( \{ \zeta < \infty \} \). Then we obtain (ii).

(ii) \( \Rightarrow \) (i) Obvious.

(iii) \( \Leftrightarrow \) (iv) By (3.8), we have
\[
\begin{align*}
\mathbb{E}_x \left[ \int_0^\zeta f(Z_t)e^{-\lambda Z_t} dt | \zeta < \infty \right] &= \frac{\int_0^\zeta f(Z_t)e^{-\lambda Z_t} dt}{\mathbb{P}_x(\zeta < \infty)} \\
&= e^{\Phi(0)x} \int_0^\infty \mathbb{E}_x[f(Z_t)e^{-\lambda Z_t}1_{\{t < \zeta < \infty\}}] dt \\
&= e^{\Phi(0)x} \int_0^\infty \mathbb{E}_x[f(Z_t)e^{-\lambda Z_t}1_{\{t < \zeta\}}] \mathbb{E}_x[1_{\{\zeta < \infty\}} | \mathcal{F}_t] dt \\
&= e^{\Phi(0)x} \int_0^\infty \mathbb{E}_x[f(Z_t)e^{-\lambda Z_t}1_{\{t < \zeta\}}] dt \\
&= e^{\Phi(0)x} \int_0^\infty f(y)e^{-(\lambda + \Phi(0))y} \mu_x(dy) \\
&= \int_0^\infty f(y)e^{-(\lambda + \Phi(0))y}[W(y) - e^{\Phi(0)x}W(y - x)] dy. \\
\end{align*}

(3.21)

It follows from (3.7) that
\[
\int_0^\infty e^{-(\lambda + \Phi(0))y}[W(y) - e^{\Phi(0)x}W(y - x)] dy = \frac{1 - e^{-\lambda x}}{\psi(\lambda + \Phi(0))} < \infty.
\]
Since for any \( \varepsilon > 0 \), \( f \) is bounded on \( (\varepsilon, \infty) \) and \( W(x) = 0 \) for \( x < 0 \), the integral in (3.21) is finite if and only if
\[
\int_{0+} f(y)W(y)dy < \infty.
\]
From (3.9), the above inequality is equivalent to
\[
\int_{0+} \frac{f(y)}{y\psi(1/y)}dy < \infty.
\]
By a change of variable \( y = 1/\lambda \), we see the above inequality holds if and only if
\[
\int_{\infty-} \frac{f(1/\lambda)}{\lambda\psi(\lambda)}d\lambda < \infty.
\]
\[\square\]

**Proof of Theorem 2.2** (i)⇒(ii) We only need to prove that if \( \mathbb{P}_x(A_\infty(f) < \infty, \zeta = \infty) > 0 \), then \( \mathbb{P}_x(A_\infty(f) = \infty, \zeta = \infty) = 0 \). If \( \mathbb{P}_x(A_\infty(f) < \infty, \zeta = \infty) > 0 \), then there exists a \( d > 0 \) such that
\[
\alpha := \mathbb{P}_x(A_\infty(f) \geq d) < 1,
\]
where we make the convention that \( f(x) = 0 \) for \( x < 0 \). Let \( \sigma_0 := 0 \) and for \( n = 0, 1, 2, \ldots \) define
\[
\sigma_{n+1} := \inf \{ t \geq \sigma_n : A_t(f) > A_{\sigma_n}(f) + d, Z_t \wedge \zeta > x \}, \quad n \geq 0.
\]
Since \( Z_{\sigma_k} \geq x \) if \( \sigma_k < \infty \), by the stochastic monotonicity of Lévy process and the monotonicity of \( f \), we have for \( \sigma_k < \infty \),
\[
\mathbb{P}_{Z_{\sigma_k}}(\sigma_{k+1} < \infty) \leq \mathbb{P}_{Z_{\sigma_k}}(A_\infty(f) \geq d) \leq \mathbb{P}_x(A_\infty(f) \geq d) = \alpha < 1.
\]
Notice that \( Z_t \to \infty \) if \( \zeta = \infty \). Using the strong Markov property and by induction, for any \( n \geq 1 \) we have
\[
\mathbb{P}_x(A_\infty(f) = \infty, \zeta = \infty) \leq \mathbb{P}_x \left( \bigcap_{k=1}^{n} \{ \sigma_k < \infty \} \right) = \mathbb{E}_x \left[ \prod_{k=1}^{n-1} \mathbb{1}_{\{ \sigma_k < \infty \}} \mathbb{P}_{Z_{\sigma_{k-1}}}(\sigma_n < \infty) \right] \leq \alpha \mathbb{E}_x \left[ \prod_{k=1}^{n-1} \mathbb{1}_{\{ \sigma_k < \infty \}} \right] = \alpha \mathbb{P}_x \left( \bigcap_{k=1}^{n-1} \{ \sigma_k < \infty \} \right) \leq \cdots \leq \alpha^n.
\]
Letting \( n \to \infty \), we have \( \mathbb{P}_x(A_\infty(f) = \infty, \zeta = \infty) = 0 \).

(ii)⇒(iii) Fix a \( x > 0 \) and \( 0 < y < x \), by (ii) we have \( \mathbb{P}_y(A_\infty(f) < \infty|\zeta = \infty) = 1 \). Since \( f \) is non-increasing, we can make the convention that \( f(0) = \lim_{x \to 0} f(x) > 0 \). Hence, if \( \zeta < \infty \), then \( A_\infty(f) = \infty \). Therefore, there exists a constant \( d > 0 \) and a constant \( \alpha \in (0, 1) \) such that
\[
\mathbb{P}_y(A_\infty(f) < d) = \mathbb{P}_y(A_\infty(f) < d, \zeta = \infty) > \alpha.
\]
Then by the stochastically monotonicity of Lévy process and the monotonicity of $f$ we have
\[ P_z(A_\infty(f) < d) \geq P_y(A_\infty(f) < d) > \alpha > 0 \quad \text{for each } z \geq y. \]
It follows that
\[ P_z(A_\infty(f) \geq d) \leq 1 - \alpha < 1 \quad \text{for each } z \geq y. \]
Then we have
\[ 1 > 1 - \alpha \geq P_z(A_\infty(f) \geq d) \geq P_z(A_{y^-}(f) \geq d) \quad \text{for each } z \geq y. \]
Put $\bar{\sigma}_0 := 0$ and
\[ \bar{\sigma}_{n+1} := \inf \left\{ t > \bar{\sigma}_n : \int_{\bar{\sigma}_n}^t f(Z_s) ds = d, Z_s > y \text{ for } s \leq t \right\} \]
for $n \geq 0$. Then we see that
\[ \{ A_{y^-}(f) \geq nd \} = \{ \bar{\sigma}_n < \infty \}. \]
If for some $k \geq 2$, $\bar{\sigma}_{k-1} < \infty$, then we have $Z_{\bar{\sigma}_{k-1}} \geq y$ and
\[ P_{Z_{\bar{\sigma}_{k-1}}}(\bar{\sigma}_k < \infty) = P_{Z_{\bar{\sigma}_{k-1}}}(A_{y^-}(f) \geq d) \leq 1 - \alpha. \]
Using the strong Markov property and combing the above formulas, by induction we have for $n \geq 1$,
\[ P_x(A_{y^-}(f) \geq nd) = P_x \left( \bigcap_{k=1}^n \{ \bar{\sigma}_k < \infty \} \right) \]
\[ = \mathbb{E}_x \left[ P_x \left( \bigcap_{k=1}^n \{ \bar{\sigma}_k < \infty \} \bigg| \mathcal{F}_{\bar{\sigma}_{n-1}} \right) \right] \]
\[ = \mathbb{E}_x \left[ P_{Z_{\bar{\sigma}_{n-1}}}(\bar{\sigma}_n < \infty) \prod_{k=1}^{n-1} 1_{\{ \bar{\sigma}_k < \infty \}} \right] \]
\[ \leq (1 - \alpha)^n. \]
It follows that $\mathbb{E}_x[A_{y^-}(f)] < \infty$.

(iii) $\Rightarrow$ (i) Suppose that $\mathbb{E}_x[A_{y^-}(f)] < \infty$ for all $x > y > 0$. Then $P_x(A_{y^-}(f) < \infty) = 1$. Since $\{ \tau_y^- = \infty \} \subset \{ \zeta = \infty \}$ and
\[ P_x(\tau_y^- = \infty) = 1 - e^{-\Phi(0)(x-y)} > 0 \]
for $x > y \geq 0$, we have
\[ P_x(A_\infty(f) < \infty, \zeta = \infty) \geq P_x(A_\infty(f) < \infty, \tau_y^- = \infty) > 0. \]
It follows that $P_x(A_\infty(f) < \infty | \zeta = \infty) > 0$.

Finally, under the condition $\psi'(0) < \infty$, we are going to show $(iii) \Leftrightarrow (iv)$. Thanks to Lemma 3.3, we only need to show that
\[ \int_0^\infty f(z+y)[e^{-\Phi(0)(x-y)}W(z) - W(z-x+y)]dz < \infty \Leftrightarrow \int_0^\infty f(y)dy < \infty. \]
By [25] Lemma 2, we see that for any $x > y > 0$,
\[ \lim_{z \to \infty} e^{-\Phi(0)(x-y)}W(z) - W(z-x+y) = \frac{1 - e^{\Phi(0)(x-y)}}{\psi'(0)}. \]
Then we can immediately obtain the desired result. □

Proof of Corollary 2.3. By (5.15), we have
\[
\int_0^\infty f(z + y) [e^{-\Phi(0)(x - y)} W(z) - W(z - x + y)] dz
\]
\[
= \int_0^\infty \int_0^\infty e^{-(z+y)} g(\lambda) [e^{-\Phi(0)(x - y)} W(z) - W(z - x + y)] d\lambda dz
\]
\[
= \int_0^\infty e^{-\lambda y} g(\lambda) d\lambda \int_0^\infty e^{-\lambda z} [e^{-\Phi(0)(x - y)} W(z) - W(z - x + y)] dz
\]
\[
= \int_0^\infty e^{-\lambda y} (e^{-\Phi(0)(x - y)} - e^{-\lambda(x-y)}) \frac{g(\lambda)}{\psi(\lambda)} d\lambda.
\]
Notice that
\[
\lim_{\lambda \to 0} e^{-\lambda y} (e^{-\Phi(0)(x - y)} - e^{-\lambda(x-y)}) = e^{-\Phi(0)(x - y)} - 1
\]
and
\[
e^{-\lambda y} (e^{-\Phi(0)(x - y)} - e^{-\lambda(x-y)}) \sim e^{-\lambda y} e^{-\Phi(0)(x - y)} \quad \text{as} \quad \lambda \to \infty.
\]
Then we can finish the proof. □

Proof of Corollary 2.6. (i) By letting \( f(x) = x^{-\theta} \) in Theorem 2.1, we immediately obtain the desired result.

(ii) For \( f(x) = x^{-\theta} \), we have \( f(x) = \frac{1}{\Gamma(\theta)} \int_0^\infty e^{-\lambda x} \lambda^{\theta-1} d\lambda \). Then by letting \( g(\lambda) = \frac{1}{\Gamma(\theta)} \lambda^{\theta-1} \) in Corollary 2.3, we complete the proof. □

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