CHIRAL DYNAMICS IN WEAK, INTERMEDIATE, AND STRONG COUPLING QED IN TWO DIMENSIONS

Y. HOSOTANI
School of Physics and Astronomy, University of Minnesota
Minneapolis, MN 55455, USA

$N$ flavor QED in two dimensions is reduced to a quantum mechanics problem with $N$ degrees of freedom for which the potential is determined by the ground state of the problem itself. The chiral condensate is determined at all values of temperature, fermion masses, and the $\theta$ parameter. In the single flavor case, the anomalous mass ($m$) dependence of the chiral condensate at $\theta = \pi$ at low temperature is found. The critical value is given by $m_c \sim 0.437 \cdot e/\sqrt{\pi}$.

1 QED$_2$ and Quantum Mechanics

Two-dimensional QED with massive fermions is not exactly solvable. The relevant parameter measuring the strength of the interaction is $e/m$ where $e$ and $m$ are the gauge coupling and fermion mass, respectively. The massless limit corresponds to the strong coupling, whereas the large fermion mass limit corresponds to the weak coupling. In both limits the model is exactly solvable. For the intermediate coupling $e/m = O(1)$, however, the model is highly interacting and some approximation methods need to be employed.

The model has many similarities to the four-dimensional QCD in such respects as chiral dynamics and confinement. In this article we would like to focus on the chiral condensate at all values of the coupling, temperature, and $\theta$ parameter.

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^{N} \overline{\psi}_a \left\{ \gamma^\mu (i \partial_\mu - e A_\mu) - m_a \right\} \psi_a . \quad (1)$$

We examine the model defined on a circle $S^1$ with a circumference $L$. Upon imposing periodic and anti-periodic boundary conditions on the bosonic and fermionic fields, respectively, the model is mathematically equivalent to a theory defined on a line ($R^1$) at finite temperature ($T$) by analytic continuation.

\[\text{---To appear in the Proceedings of 1996 International Workshop on Perspectives of Strong Coupling Gauge Theories, Nagoya, November 13 - 16, 1996---}\]
to imaginary time $\tau (= it)$ and the interchange of $\tau$ and $x$. Various physical quantities at $T \neq 0$ on $R^1$ are obtained from the corresponding ones at $T = 0$ on $S^1$ by substituting $T^{-1}$ for $L$.

In a series of papers it has been shown that the field theory system \([1]\) is effectively reduced to a quantum mechanical system of $N$ degrees of freedom.\(^{1-5}\)

The argument proceeds as follows.

Fermions are bosonized on a circle. Each two-component fermion ($\psi_a$) is expressed in terms of zero modes ($q_a^\pm, p_a^\pm$) and oscillatory modes ($\phi_a(x), \Pi_a(x)$). The boundary conditions enforce that $p_a^\pm$’s take integer eigenvalues.

The relevant parts of the Hamiltonian are expressed in terms of ($q_a=\frac{q_a^++q_a^-}{2}, p_a=\frac{1}{2}(p_a^++p_a^-)$), ($\phi_a, \Pi_a$), and ($\Theta_W, p_W$) where $\Theta_W$ is the Wilson line phase, the only physical degree of freedom associated with gauge fields on a circle. In the massless fermion theory the zero modes and oscillatory modes decouple. There appear one massive oscillatory mode, $\chi_1$ with $\mu_1^2 = Ne^2/\pi$, and $N - 1$ massless oscillatory modes, $\chi_\alpha$ with $\mu_\alpha^2 = 0$ ($\alpha = 2 \sim N$).

Fermion masses introduce nontrivial interactions among these variables. All of the oscillatory modes become massive. Fermion masses change the vacuum structure. The chiral condensate is induced and the boson mass spectrum is modified. Each affects the other, and must be determined self-consistently.

There are $N + 1$ relevant zero modes, $\Theta_W$ and $q_a$. The $\theta$ vacuum structure, which follows from the gauge invariance, eliminates one degree. With an appropriate choice of the basis, we have $N$ zero mode degrees which we denote by $p_W$ and $\phi_a$ ($a = 1 \sim N - 1$). The vacuum wave function must solve the Schrödinger equation

$$H f(p_W, \varphi) = \epsilon f(p_W, \varphi)$$
$$H = -\frac{\partial^2}{\partial p_W^2} - \frac{4\pi^2(N-1)}{N^2} \Delta + V_N(p_W, \varphi)$$

(2)

where

$$\Delta_N = \sum_{a=1}^{N-1} \frac{\partial^2}{\partial \varphi_a^2} - \frac{2}{N-1} \sum_{a<b} \frac{\partial^2}{\partial \varphi_a \partial \varphi_b}$$
$$V_N = \left(\frac{\pi \mu L p_W}{N}\right)^2 - \frac{4\pi}{N} \sum_{a=1}^{N} m_a L B_a \cos \left(\varphi_a - \frac{2\pi p_W}{N}\right)$$

(3)

In the potential $V_N$, $B_a$ depends on the boson mass spectrum $\mu_a$ as well as
$e$, $L$, and $m$. The detailed form has been given in refs. 1 and 3. The spectrum $\mu_\alpha$ is to be determined from the vacuum wave function solving (3). We have a routine

$$V(p_W, \varphi) \to f(p_W, \varphi) \to \mu^2_\alpha \to V(p_W, \varphi) .$$

(4)

This is a rather non-trivial condition to be satisfied. The Hamiltonian is determined by its ground state wave function.

2 Chiral condensates — $T$, $m$, and $\theta$-dependence

In the strong ($e/m \gg 1$) and weak ($e/m \ll 1$) coupling limits Eq. (2) and the condition (4) can be solved analytically. For general values of the parameters, however, they must be solved numerically. We have numerically solved the Schrödinger equation on workstations.

Let us focus on the $N = 1$ (single flavor) model. In this case there is no $\varphi$ degree, the Schrödinger equation (2) becoming

$$\left\{ -\frac{d^2}{dp^2_W} + V(p_W) \right\} f(p_W) = \epsilon f(p_W)$$

$$V(p_W) = (\pi \mu L p_W)^2 - \kappa \cos(2\pi p_W)$$

$$\kappa = 4\pi m B(\mu L)$$

$$B(z) = \frac{z}{4\pi} \exp \left( \gamma + \frac{\pi}{z} - 2 \int_1^\infty \frac{du}{(e^{uz} - 1)\sqrt{u^2 - 1}} \right)$$

(5)

where $\mu^2 = e^2 / \pi$. Note that $B(0) = 1$. $\mu_1$ is the boson mass which is determined by the vacuum wave function $f(p_W)$:

$$\mu^2_1 = \mu^2 + \frac{8\pi m B(\mu_1 L)}{L} \langle \cos(2\pi p_W - \theta) \rangle_f .$$

(6)

In the weak coupling limit $e/m \ll 1$ at $T = 0$ ($L = \infty$), $\langle \cos(2\pi p_W - \theta) \rangle_f = 1$ and the boson mass $\mu_1 = 2e^2m$. On the other hand, in the strong coupling limit $e/m \gg 1$ or in the massless fermion theory, $\langle \cos(2\pi p_W - \theta) \rangle_f = e^{-\pi/\mu L} \cos \theta$.

Eqs. (3) and (4) must be solved simultaneously.

The chiral condensate is given by

$$\langle \bar{\psi} \psi \rangle_0 = -\frac{2B(\mu_1 L)}{L} \langle \cos(2\pi p_W - \theta) \rangle_f + \frac{e^{2\gamma}}{\pi} m .$$

(7)

The condensate is normalized such that it vanishes in the weak coupling limit at $T = 0$. (In the earlier references$^1$-$^5$ the chiral condensate has been defined
without the last term in (7). This reflects ambiguity in defining composite operators.) In the massless theory \((m = 0), \mu_1 = \mu\) and
\[
\langle \bar{\psi} \psi \rangle_\theta = \frac{2}{L} B(\mu L) e^{-\pi/\mu L} \cos \theta
\]  
(8)

Figure 1: \(T\) dependence of the chiral condensate in the \(N = 1\) model. Lines and points are for \(\theta = 0\) and \(\pi\), respectively. The mass \(m\) in the figure is measured in the unit of \(\mu\).

In fig. 1 the condensate \(\langle \bar{\psi} \psi \rangle_\theta/\mu\) is plotted as a function of the temperature \(T/\mu = 1/\mu L\). There is a crossover transition around \(T/\mu \sim 1\). At high temperature the condensate becomes \(\theta\) independent, irrespective of \(m/\mu\). At low temperature, however, there appears a significant difference between \(\theta = 0\) and \(\theta = \pi\).

At low \(T\) the condensate approaches \(-e^\gamma \mu \cos \theta/2\pi\) in the massless theory. It, however, vanishes in the large mass limit as it approaches a free theory.

At high \(T\) the condensate vanishes in the massless theory, whereas it approaches a non-vanishing value in the massive theory. Thermal excitations yield finite condensates in the massive theory, as there is no chiral symmetry.

The \(\theta\) dependence of the condensate originates from the potential. At low \(T/\mu = 1/\mu L \ll 1\) it is given by
\[
V = \pi^2 \mu^2 L^2 p_W^2 - m_\mu L^2 \cos(\theta - 2\pi p_W).
\]  
(9)

At \(\theta = 0\) the potential is always minimized at \(p_W = 0\) and everything smoothly changes. As the fermion mass becomes larger, the condensate \(\langle \bar{\psi} \psi \rangle\) increases.
On the other hand at $\theta = \pi$ there appear two degenerate minima for large $m$. The problem becomes delicate for a moderate value of $m/\mu \sim 1$. $\mu_1$ is determined by (6) which involves the vacuum wave function $f(p_W)$ determined by the potential (4) itself. There appear two or three consistent solutions.

For small $m/\mu$, the condensate decreases as $T/\mu$ increases. The behavior is opposite for large $m/\mu$. The condensate increases as $T/\mu$. For intermediate $m/\mu$, the condensate initially increases as $T/\mu$, but eventually starts to decrease. We recognize profound structure in the behavior of the condensate for the intermediate coupling $e/m = O(1)$ at $\theta = \pi$.

### 3 Anomalous behavior at $\theta = \pi$

To get more insight into the anomalous behavior of the condensate near $\theta = \pi$, we have depicted the $m$ dependence of the condensate at $T/\mu = .03$ in fig. 2.

At $\theta = 0$ the absolute value of the chiral condensate at $T = 0$ decreases as $m/\mu$ increases, and vanishes at $m/\mu = \infty$. This result has been previously obtained by Tomachi and Fujita by the Bogoliubov transformation. Our result agrees with theirs numerically.

However, at $\theta = \pi$, there appears a discontinuity in the $m/\mu$ dependence. To magnify this point, more detailed study is presented in fig. 3, in which the condensate is plotted for $0.4 < m/\mu < .48$ at $T/\mu = .03$. The discontinuity occurs at $m/\mu = .437$.

Why and how can such a discontinuity arise in a theory with a Hamiltonian
having the smooth dependence on $m$? To understand it, we have to go back to the equation (3).

Suppose that $\mu L = \mu / T$, $m/\mu$, and $\theta$ are given. The potential $V(p_W)$ in (3) is then fixed by the coefficient $\kappa$ of the $\cos(\theta - 2\pi p_W)$ term. With a given $\kappa_{\text{in}}$, the vacuum wave function $f(p_W)$ is determined, solving the Schrödinger equation. Now the boson mass $\mu_1$ (or $\mu_1 L$ numerically) is obtained by solving (6) from $f(p_W)$. The output $\kappa_{\text{out}} = 4\pi mLB(\mu_1 L)$ is determined with this new $\mu_1$. This process defines a mapping $\kappa_{\text{out}} = g(\kappa_{\text{in}})$:

$$
\kappa_{\text{in}} \rightarrow V(p_W) \rightarrow f(p_W) \rightarrow \mu_1 L \rightarrow \kappa_{\text{out}}. \tag{10}
$$

We are looking for a solution $\kappa_{\text{out}} = \kappa_{\text{in}}$, namely a fixed point of $g(\kappa)$. Although the function $g(\kappa)$ depends on $m/\mu$ very smoothly, there takes place bifurcation in the fixed point structure as $m/\mu$ varies.

This problem was investigated in ref. 5. It has been found that for $T/\mu < 0.12$ and $\theta = \pi$, there appears two attracting and one repelling fixed points for $m_c/\mu < m/\mu < m_c'/\mu$. Among these three fixed points, the biggest $\kappa$ corresponds to the lowest energy density and is chosen. Hence the lower critical mass $m_c/\mu$ appears as the location of the discontinuity in the $m/\mu$ dependence of the various quantities. In fig. 4 we have displayed the $m$ dependence of the $\kappa$ parameter at $T/\mu = 0.03$.

In general $m_c$ depends on $T$. At $T = 0$ it can be determined analytically to be $m_c/\mu = 0.435$. At finite $T$ we have found numerically $m_c/\mu=0.437$ and $0.454$ at $T/\mu=0.03$ and $0.07$, respectively.
This gives us a puzzle. The Mermin-Wagner theorem ensures that there is no discontinuity in the $T$ dependence of any physical quantities. It implies that if there is a discontinuity in the $m$ dependence at $m_c$, $m_c$ must be independent of $T$. In our approximation we have found that $m_c$ is almost universal, but has small $T$ dependence. Indeed, we see a jump in the $T$ dependence of the chiral condensate for $m/\mu = 0.437$ in fig. 1, though it may be a smooth transition when the approximation is improved.

There are two possible scenarios. The first possibility is that $m_c$ is universal and $T$ independent in the full theory. The second possibility is that the discontinuity in $m$ disappears and is replaced by a rapid crossover in the full theory. It is a challenge to know which picture is real.

4 Generalization and summary

The generalization to the $N$ flavor case is straightforward. Extensive analysis for small fermion masses has been given in refs. 1-4. For instance, the chiral condensate at low $T$ for degenerate fermion masses ($m_a = m \ll \mu$) is given by

$$\frac{1}{\mu} \langle \bar{\psi} \psi \rangle_\theta = -\frac{1}{4\pi} \left( 2e^\gamma \cos \frac{\theta}{N} \right) \frac{2\pi}{\tilde{\theta}} \left( \frac{m}{\mu} \right)^{\frac{N-1}{N+1}} \text{ for } T \ll \frac{N}{N+1} \mu^{\frac{1}{N+1}}$$

where $\tilde{\theta}$ is defined in the interval $-\pi \leq \tilde{\theta} \leq +\pi$ by $\tilde{\theta} = \theta - 2\pi[(\theta + \pi)/2\pi]$. Recently Smilga has obtained an exact result for $N = 2$ at $\theta = 0$, $T = 0$, and
$m \ll \mu$, which agrees with our result within 5%. At $T = 0$ the cusp singularity appears at $\theta = \pi$. The $N = 3$ model with fermion masses $m_1 < m_2 \ll m_3$ mimics the four-dimensional QCD.

In this article we have presented an alternative method to explore QED in two dimensions. QED is reduced to a quantum mechanical system of finite degrees of freedom whose Hamiltonian needs to be determined by its ground state wave function itself. The associated Schrödinger equation has been solved numerically on workstations. With this algorithm one can determine various physical quantities at any temperature and $\theta$ and with arbitrary fermion masses. Our method supplements other numerical methods such as the lattice gauge theory and light-front quantization method.\(^8\sim 9\) More results by these methods are wanted for comparison.

Acknowledgments

This work was supported in part by the U.S. Department of Energy under contracts DE-AC02-83ER-40105.

References

1. J.E. Hetrick, Y. Hosotani, and S. Iso, Phys. Lett. B 350, 92 (1995); Phys. Rev. D 53, 7255 (1996).
2. Y. Hosotani, in the Proceedings of The 4-th Workshop on Thermal Field Theories and Their Applications, (World Scientific, 1995), page 355.
3. R. Rodriguez and Y. Hosotani, Phys. Lett. B 375, 273 (1996).
4. Y. Hosotani, R. Rodriguez, J.E. Hetrick, and S. Iso, in the Proceedings of Continuous Advances in QCD 1996, (World Scientific 1996), page 382.
5. R. Rodriguez and Y. Hosotani, Phys. Lett. B 389, 121 (1996).
6. T. Tomachi and T. Fujita, Ann. Phys. 223, 197 (1993).
7. A. Smilga, Phys. Rev. D 55, 443 (1997).
8. A. Irving and J. Sexton, Nucl. Phys. B (Proc. Suppl.) 47, 679 (1996); I. Horvath, Nucl. Phys. B (Proc. Suppl.) 47, 683 (1996); Phys. Rev. D 53, 3808 (1996); V. Azcoiti et al., Nucl. Phys. B (Proc. Suppl.) 47, 687 (1996).
9. G. McCartor, Z. Phys. C 64, 349 (1994); K. Harada, A. Okazaki, and M. Taniguchi, Phys. Rev. D 54, 7656 (1996); A.C. Kalloniatis and D.G. Robertson, Phys. Lett. B 381, 209 (1996); G. McCartor, D. G. Robertson, and S. S. Pinsky, [hep-th/9612053](http://arxiv.org/abs/hep-th/9612053)