The Sheffer stroke operation reducts of basic algebras

1 Introduction

Take into consideration the problem of expressing equational concepts as simply as possible with the least number of operations or the least number of axioms and so forth. For an example about the least number of axioms, Tarski solved the related problem for Abelian groups with the single axiom \( x = y = z = x = y = z \) in terms of the division operation in 1938 [1]. As a typical example, we tackle the use of Sheffer stroke operation in algebraic structures. The Sheffer stroke term operation was firstly given by H. M. Sheffer in 1913 [2]. He proved that all Boolean functions could be transplanted to a single binary operation for term operations. In recent years, the problem for Boolean algebra was solved with a single axiom in terms of the Sheffer Stroke operation [3].

The reduction attempts interest mathematicians who want to use less operations or axioms or formulas for the structures under their considerations. The first implication reduct of Boolean algebras connectives was studied by J. C. Abbott [4] under the name implication algebra. Once the logic of quantum mechanics was axiomatized by means of orthomodular lattices, Abbott obtained implication reducts of orthomodular lattices, called orthoimplication algebras in [5]. Later on, this work was generalized for implication reducts of ortholattices by Chajda and Halaš [6], and by Chajda [7] to orthomodular lattices but without the compatibility condition in Chajda, Halaš and Länger [8].

Basic algebras were introduced in Chajda and Emanovský [9], see also Chajda [10] and Chajda et al. [11] and [12, 13] for further information. Basic algebras are an important concept used in different non-classical logics since they contain orthomodular lattices \( \mathcal{L} = (L; \vee, \wedge, \perp, 0, 1) \), where \( x \oplus y = (x \wedge y) \vee \neg x = x \perp \), and constitute as well as provide an axiomatization of the logic of quantum mechanics along with MV-algebras [14], which get an axiomatization of many-valued Łukasiewicz logics; see Chajda [15] and Chajda et al. [16].

Given that the connective Sheffer stroke operation plays a central role in all mentioned logics above, in general, we would like to characterize this operation in basic algebras.
2 Preliminaries

The following fundamental notions are taken from [17] and [18].

**Definition 2.1.** A bounded lattice is an algebraic structure \( L = (L; \lor, \land, 0, 1) \) such that \( L = (L; \lor, \land) \) is a lattice having the following properties:

(i) for all \( x \in L \), \( x \lor 1 = 1 \) and \( x \land 1 = x \),
(ii) for all \( x \in L \), \( x \lor 1 = x \) and \( x \land 0 = 0 \).

The elements 0 and 1 are called the least element and the greatest element of the lattice, respectively.

**Definition 2.2.** Let \( L = (L; \lor, \land) \) be a lattice. A mapping \( x \mapsto x^\perp \) is called an antitone involution if it satisfies the following:

(i) \( x \leq y \) implies \( y^\perp \leq x^\perp \) (antitone),
(ii) \( x^\perp \perp = x \) (involution).

**Definition 2.3.** Let \( L \) be a bounded lattice with an antitone involution. If it satisfies \( x \land x^\perp = 1 \) and \( x \lor x^\perp = 0 \), then \( x^\perp \) is the complement of \( x \) and \( L = (L; \lor, \land, \perp, 0, 1) \) is an ortholattice.

**Lemma 2.4.** Let \( L = (L; \lor, \land, \perp) \) be a lattice with antitone involution. Then it satisfies the following De Morgan laws:

\[ x \lor y^\perp = (x \land y)^\perp \quad \text{and} \quad x \land y^\perp = (x \lor y)^\perp. \]

**Definition 2.5 ([19]).** Let \( A = (A; \cdot, \backslash) \) be a groupoid. The operation \( \cdot \) is said to be a Sheffer stroke operation if it satisfies the following conditions:

(S1) \( x \cdot y = y \cdot x \),
(S2) \( (x \cdot y) \cdot (x \cdot y) = x \cdot y \),
(S3) \( x \cdot (x \cdot y) \cdot (y \cdot z) = (x \cdot (x \cdot y) \cdot (y \cdot z)) \cdot (x \cdot (x \cdot y) \cdot (y \cdot z)) \cdot x \).

If additionally it satisfies the identity

(S5) \( y \cdot (x \cdot (x \cdot y)) = y \cdot y \), it is called an ortho Sheffer stroke operation.

**Lemma 2.6 ([19]).** Let \( A = (A; \cdot, \backslash) \) be a groupoid. The binary relation \( \leq \) defined on \( A \) as below

\[ x \leq y \quad \text{if and only if} \quad x \cdot y = x \cdot x \]

is an order on \( A \).

**Lemma 2.7 ([19]).** Let \( \cdot \) be a Sheffer stroke operation on \( A \) and \( \leq \) the induced order of \( A = (A; \cdot) \). Then

(i) \( x \leq y \) if and only if \( y \cdot y \leq x \cdot x \),
(ii) \( x \cdot (y \cdot (x \cdot y)) = x \cdot x \) is the identity of \( A \),
(iii) \( x \leq y \) implies \( y \cdot z \leq x \cdot z \) for all \( z \in A \),
(iv) \( a \leq x \) and \( a \leq y \) imply \( x \cdot y \leq a \cdot a \).

In order to obtain a construction of Sheffer stroke reduction of basic algebras, we firstly give the definition of a basic algebra:

**Definition 2.8 ([20]).** A basic algebra is an algebra \( A = (A; \oplus, \neg, 0) \) of type \((2, 1, 0)\) which satisfies the following axioms:

(BA1) \( x \oplus 0 = x \),
(BA2) \( \neg \neg x = x \).
By using (BA3), we can define the following concept. When we substitute for basic algebras. Now, we can define the following concept. role of logic connective implication. There are also some interesting results related to implication operation used

As mentioned in [21], we can consider \( A = (A; \oplus, \neg, 0) \) alternatively in signature \( \{ \rightarrow, 0 \} \), where \( x \rightarrow y = \neg x \oplus y \). When regarding a logic axiomatized by a basic algebra, especially if \( A \) is an MV-algebra, then it is a many valued fuzzy logic, or if \( A \) is an orthomodular lattice, it is a logic of quantum mechanics. Then the operation \( \rightarrow \) plays the role of logic connective implication. There are also some interesting results related to implication operation used (see [22–27]). Starting from this point of view, we construct an alternative signature \( \{ \} \) which consists of only the Sheffer stroke operation. Hence, it is of some importance to investigate the Sheffer stroke reduct in a general setting for basic algebras. Now, we can define the following concept.

**Definition 3.1.** An algebra \((A, \cdot)\) of type \((2)\) is called a Sheffer stroke basic algebra if the following identities hold:

\begin{align*}
(\text{SH1}) \quad & x(x|x) = x, \\
(\text{SH2}) \quad & x(\cdot(y|y))(y|y) = (y|x)(x|x), \\
(\text{SH3}) \quad & ((x(y|y))(y|y))(z|z)(z|z)(x(z|z)) = x|x).
\end{align*}

First of all, we give some simple properties of Sheffer Stroke basic algebras. Then, we demonstrate that every such algebra has an algebraic constant \( 1 \) as in the case for implication basic algebras [20].

**Lemma 3.2.** Let \((A, \cdot)\) be a Sheffer Stroke basic algebra. Then there exists an algebraic constant element \( 1 \in A \) and \((A, \cdot, 1)\) meets the following identities:

\begin{enumerate}[(i)]
\item \( x(1|x) = 1 \),
\item \( x(1)1 = 1 \),
\item \( 1|x = x \),
\item \( ((x(y|y))(y|y))(y|y) = x|y|y) \),
\item \( (y|x(y|y))(x(y|y)) = 1 \).
\end{enumerate}

**Proof.** (i) : In (SH3), we substitute \([z := y]\) and \([y := x]\) simultaneously and we carry out (SH1). Then we obtain

\[ x|x = ((x|x)(x|x))(y|y)((y|y))(y|y)) = (x|y|y)(x|y|y)(x|y|y). \]

When we substitute \([x := (x|y|y)]\) in the above equation, we derive

\[ (x|y|y)((x|y|y))(y|y)) = ((x|y|y))(y|y)(((x|y|y)|(y|y))((x|y|y)|(y|y))((x|y|y)|(y|y))). \]

Therefore, we get

\[ x|x = ((x|y|y))(y|y)(((x|y|y)|(y|y))((x|y|y)|(y|y))((x|y|y)|(y|y))). \]

By using (SH2), we deduce

\[ x|x = ((x|y|y))(y|y)(((x|y|y)|(y|y))((x|y|y)|(y|y))((x|y|y)|(y|y))\ldots) \]

3 The Sheffer Stroke Reduction of Basic Algebras

As mentioned in [21], we can consider \( A = (A; \oplus, \neg, 0) \) alternatively in signature \( \{ \rightarrow, 0 \} \), where \( x \rightarrow y = \neg x \oplus y \).
Thus \((A; |)\) satisfies the identity \(x|x|z = y|y|y\) for all \(x, y \in A\). It means that \((A; |)\) contains an algebraic constant which will be denoted by 1 and hence this system satisfies the identity \(x|x|1 = 1\).

\((iii)\) : Applying (i) in \((SH1)\), we obtain

\[
x = (x|x)(x|x) = 1|x|x.
\]

\((v)\) : By using (i) in \((SH3)\), we get

\[
(((x|y)(y|y))((y|y))((z|z))((z|z))) = 1.
\]

Firstly, the substitution \([x := y]\) and \([y := x]\) gives

\[
(((x|x)(x|x))((z|z))((y|y)) = 1.
\]

Substituting \([x := (x|(y|y))]\) and \([z := (x|(y|y))]\) in the latter equation yields

\[
(((y|(x|(y|y)))((x|(y|y)))((y|y)))(x|(y|y)))((x|(y|y)))(x|(y|y))) = 1.
\]

By using \((SH3)\) and the identity \((iii)\), we conclude

\[
y|(x|(y|y))((y|y)) = 1.
\]

\((iv)\) : In \((iii)\), the substitution \([x := (x|(y|y))]\) implies

\[
1|(x|(y|y))((x|(y|y))) = x|(y|y).
\]

And using \((v)\) for the value 1 in the above equation, we get

\[
(y|(x|(y|y)))(x|(y|y)))((x|(y|y)))(x|(y|y))) = x|(y|y).
\]

By using \((SH2)\), we obtain

\[
((x|(y|y))|(y|y)) = x|(y|y).
\]

\((ii)\) : In \((v)\) , if we choose \(x = y\) then we get

\[
x|(x|x)(x|(x|x)) = 1.
\]

From the identity \((i)\), we obtain

\[
x|(1|1) = 1.
\]

**Theorem 3.3.** The axioms of Sheffer Stroke basic algebra are independent.

**Proof.** To prove this claim, we construct a model for each axiom in which that axiom is false while the others are true. Let \(A = \{0, 1\}, |^A\) be our model defined in the following tables:

\((1)\) Independence of \((SH1)\):

We define the operation \(|^A\) in the following table:

**Table 1. Operation Table for Independence of \((SH1)\)**

| \(|^A\) | 0 | 1 |
|-------|---|---|
| 0     | 0 | 0 |
| 1     | 0 | 0 |
Then $|A|$ satisfies $(SH2)$ and $(SH3)$, but not $(SH1)$ since $(1|1|1)|1|1 = 0 \neq 1$.

(2) **Independence of $(SH2)$:**

Consider the operation $|A|$, defined as in the following table:

**Table 2. Operation Table for Independence of $(SH2)$**

| $|A|$ | 0 | 1 |
|------|---|---|
| 0    | 0 | 1 |
| 1    | 1 | 0 |

Then $|A|$ satisfies $(SH1)$ and $(SH3)$, but not $(SH2)$ because if we choose $x = 1$ and $y = 0$, then $(1|0|0)|0|0 = 1 \neq 0 = (0|1|1)|1|1$.

(3) **Independence of $(SH3)$:**

Define the operation $|A|$ as in Table 3:

**Table 3. Operation Table for Independence of $(SH3)$**

| $|A|$ | 0 | 1 |
|------|---|---|
| 0    | 0 | 1 |
| 1    | 1 | 1 |

The model $|A|$ satisfies $(SH1)$ and $(SH2)$, but not $(SH3)$. When we choose $x = 0$ and $y = z = 1$, we get $((0|1|1)|(1|1))|(1|1))((0|1|1)|(1|1)) = 1 \neq 0 = (0|0|0)$.

To construct a bridge between basic algebras and Sheffer stroke basic algebras, we need the following theorem:

**Theorem 3.4.** Let $A = (A; \oplus, \neg, 0)$ be a basic algebra. We define $x|y = \neg x \oplus \neg y$. Then $A; |$ is a Sheffer Stroke basic algebra.

**Proof.** By using $(BA1) - (BA4)$, Lemma 2.9 and Lemma 3.2, we can verify the identities $(SH1) - (SH3)$ as follows:

$(SH1)$

$$(x|(x|x))(x|x) = \neg(\neg x \oplus \neg(\neg x \oplus \neg x)) \oplus \neg(\neg x \oplus \neg x)$$

$$= \neg(\neg x \oplus x) \oplus x = 0 \oplus x = 1.$$  

$(SH2)$

$$(x|(y|y))(y|y) = \neg(\neg x \oplus \neg(\neg y \oplus \neg y)) \oplus \neg(\neg y \oplus \neg y)$$

$$= \neg((\neg y \oplus \neg y) \oplus x$$

$$= \neg(\neg y \oplus x) \oplus x,$$

$$= \neg(\neg y \oplus \neg(\neg x)) \oplus \neg(\neg x)$$

$$= \neg(\neg y \oplus \neg(\neg x \oplus \neg x)) \oplus \neg(\neg x \oplus \neg x)$$

$$= (y|(x|x))(x|x).$$

$(SH3)$

$$(((x|(y|y))(y|y))(z|z))((x|(z|z))(z|z)$$

$$= (x|(x|z))(z|z).$$
Suppose that

\[ \neg(\neg(x \oplus \neg(y)) \oplus \neg(z)) \oplus \neg(\neg(x \oplus \neg(z)) \oplus \neg(x \oplus \neg(z))) \]

is defined on \( A \).

To reveal the structure of Sheffer stroke basic algebras, we introduce a partial order relation on \( A \).

Lemma 3.5. Let \( (A; |) \) be a Sheffer stroke basic algebra. A binary relation \( \leq \) is defined on \( A \) as follows:

\[ x \leq y \quad \text{if and only if} \quad x|(y|y) = 1. \]

Then the binary relation \( \leq \) is a partial order on \( A \) such that \( x \leq 1 \) for each \( x \in A \). Moreover, we have

\[ z \leq (x|(z|z)) \quad \text{and} \quad x \leq y \quad \text{implies} \quad y|(z|z) \leq x|(z|z) \]

for all \( x, y, z \in A \).

Proof.

\begin{itemize}
  \item Reflexivity follows from Lemma 3.2 (i).
  \item Assume that \( x \leq y \) and \( y \leq x \). Then \( x|(y|y) = 1 \) and \( y|(x|x) = 1 \). From the hypothesis, \((SH2)\) and Lemma 3.2 (iii), we obtain
    \[ x = 1|(x|x) = (y|(x|x))(x|x) = (x|(y|y))(y|y) = 1|(y|y) = y. \]
  \item Suppose that \( x \leq y \) and \( y \leq z \). Then we have \( x|(y|y) = 1 \) and \( y|(z|z) = 1 \). Using this, \((SH3)\) and Lemma 3.2 (iii), we get
    \begin{align*}
      1 &= (((x|(y|y))(y|y))(z|z))(x|x)(y|(z|z)) \\
      &= ((y|(z|z))(x|x))(y|(z|z)) \\
      &= 1|(x|(z|z))(y|(z|z)).
    \end{align*}
\end{itemize}

Thus \( x \leq z \), showing that \( \leq \) is a partial order on \( A \). From Lemma 3.2 (ii), we get \( x \leq 1 \) for each \( x \in A \).

Moreover, assume that \( x \leq y \) and \( z \in A \). Then

\[ 1 = (((x|(y|y))(y|y))(z|z))(x|x)(y|(z|z)) \\
    = ((y|(z|z))(x|x))(y|(z|z)) \\
    = 1|(x|(z|z))(y|(z|z)).\]

which means \( y|(z|z) \leq x|(z|z) \). Putting here \( y := 1 \), we obtain \( z = 1|(z|z) \leq x|(z|z) \). \( \square \)

The partial order \( \leq \) introduced in the above lemma will be called induced partial order of the Sheffer stroke basic algebra \( (A; |) \).

Theorem 3.6. Let \( (A; |) \) be a Sheffer stroke basic algebra and \( \leq \) its induced partial order. Then \( (A; \leq) \) is a join semi-lattice with the greatest element 1 where \( x \lor y = (x|(y|y))(y|y) \).

Proof. Using Lemma 3.5 and \((SH2)\), we obtain \( x \leq (x|(y|y))(y|y) \) and \( y \leq (y|(x|x))(x|x) = (x|(y|y))(y|y) \). Hence \( (x|(y|y))(y|y) \) is an upper bound for \( x \) and \( y \).

We assume \( x, y \leq z \). Then by using Lemma 3.5 twice we get

\[ (x|(y|y))(y|y) \leq (z|(y|y))(y|y) = (y|(z|z))(z|z) = 1|(z|z) = z. \]

Therefore, \( (x|(y|y))(y|y) \) is the least upper bound for \( x \) and \( y \), i.e., \( x \lor y = (x|(y|y))(y|y) \) is the supremum of \( x, y \). \( \square \)
Let \((A;|)\) be a Sheffer stroke basic algebra. The semilattice \((A;\lor)\) derived in the above theorem will be called the induced semilattice of \((A;|)\).

**Theorem 3.7.** Let \((A;|)\) be a Sheffer Stroke basic algebra and \((A;\lor)\) its induced semi-lattice. For all \(p \in A\), the closed interval \([p, 1]\) is a lattice \(([p, 1]; \lor, \land_p, p)\) with an antitone involution \(x \mapsto x^p\) where

\[
x^p = x|(p|p) \quad \text{and} \quad x \land_p y = ((x|(p|p)) \lor (y|(p|p)))|(p|p)
\]

for all \(x, y \in A\).

**Proof.** Let \(x\) be in \([p, 1]\). Assume that \(x \leq y\). Then \(x|(y|y) = 1\). Now substituting \([z := p]\) into \((SH3)\) we get

\[
((x|(y|y)|(y|y))|(p|p))((x|(p|p))|(x|(p|p)))) = 1.
\]

Since \(x|(y|y) = 1\), by Lemma 3.2 \((iii)\) we obtain

\[
((y|(p|p))|(x|(p|p))|(x|(p|p))) = 1.
\]

By the definition of \(\leq\), we get \(y|(p|p) \leq x|(p|p)\), hence \(y^p \leq x^p\). So \(x \mapsto x^p\) is a partial order reversing mapping. In \((SH3)\), substitute \([y := 1]\) and \([z := p]\). Then

\[
(((x|(1|1)|(1|1))|(p|p))((x|(p|p))|(x|(p|p)))) = 1.
\]

By Lemma 3.2 \((ii)\) and \((iii)\) we have

\[
p|(x|(p|p))|(x|(p|p))) = 1.
\]

Hence \(p \leq x|(p|p)\), i.e., \(p \leq x^p\). Then \(x \mapsto x^p\) is a mapping of \([p, 1]\) into itself. By Theorem 3.6, \(x^{pp} = (x|(p|p))|(p|p) = x \lor p = x\) and so it is an involution of \([p, 1]\). Then we can carry out De Morgan laws to show that

\[
(x^p \lor y^p)^p = ((x|(p|p)) \lor (y|(p|p)))^p = ((x|(p|p)) \lor (y|(p|p)))|(p|p) = x \land_p y.
\]

This is the infimum of \(x, y \in [p, 1]\). Consequently, \(([p, 1]; \lor, \land_p, p)\) is a lattice with an antitone involution. \(\square\)

**Corollary 3.8.** Let \((A;|)\) be a Sheffer Stroke basic algebra and \(\leq\) is the induced partial order on this system. Then \((A;\leq)\) is a join-semilattice which has the greatest element 1. For each \(p \in A\) the closed interval \([p, 1]\) is a basic algebra \(([p, 1]; \oplus_p, \neg_p, p)\) if \(x \oplus_p y = (x|(p|p))|(y|y)\) and \(\neg_p x = x|(p|p)\) are defined for all \(x, y \in A\).

From now on, \(([p, 1]; \oplus_p, \neg_p, p)\) is said to be an interval basic algebra with the greatest element 1 and the least element \(p\). Therefore, Theorem 3.7 corresponds to the semilattice structure of a Sheffer stroke basic algebra. We prove that this explanation is complete, in other words, the other direction of Theorem 3.7 can be obtained.

**Theorem 3.9.** Let \((A;\lor)\) be a join-semilattice with the greatest element 1 such that for every \(p \in A\), the closed interval \([p, 1]\) is a lattice with antitone involution \(x \mapsto x^p\). If we define \(x|y = (x \lor y)^p\), then \((A;|)\) is a Sheffer Stroke basic algebra.

We say that \((A;|)\) is a Sheffer Stroke basic algebra which has the least element if there exists an element \(0 \in A\) such that \(0 \leq a\) for all \(a \in A\), where \(\leq\) is the induced partial order. So, any Sheffer Stroke basic algebra with the least element 0 satisfies the identity \(0|(x|x) = 1\).

The proof of the following theorem is straightforward.

**Theorem 3.10.** Let \((A;|)\) be a Sheffer Stroke basic algebra which has the least element 0. If we define \(\neg x = x|(0|0)\) and \(x \oplus y = (x|(0|0))|(y|y)\), then the system \((A;\oplus, \neg, 0)\) is a basic algebra and \(x|y = \neg x \oplus \neg y\).

In the remaining part of this work, we show that there is a bridge between Sheffer stroke basic algebras and Boolean algebras.
Lemma 3.11. Let \((A; \wedge)\) be a Sheffer stroke basic algebra and \((A; \vee)\) its induced complemented semilattice. For each \(p \in A\), the closed interval \([p, 1]\) is a lattice \([\{p, 1\}; \vee, \wedge, p, \_]\) with an antitone involution. Then the following identities hold
\[
x \vee x^p = 1 \quad \text{and} \quad x \wedge_p x^p = p
\]
for each \(x \in [p, 1]\).

Proof. The definition of Sheffer stroke operation gives
\[
(x | y | (x | x) = x.
\]
Substituting \(y := (p | p)\) into the latter equation yields
\[
(x | (p | p)) | (x | x) = x.
\]
From the definition of \(\leq\) we have
\[
((x | (p | p)) | (x | x)) | (x | x) = 1.
\]
From (SH2) we get
\[
(x | ((x | (p | p)) | (x | (p | p)))) | ((x | (p | p)) | (x | (p | p))) | (x | (p | p)) = 1.
\]

So we can conclude that \(x \vee (x | (p | p)) = 1\) and \(x \vee x^p = 1\). Now we have
\[
x \wedge_p x^p = x \wedge_p (x | (p | p))
= ((x | (p | p)) \vee ((x | (p | p)) | (p | p)) | (p | p)
\]
Since \(p\) is the least element, we have \(x = x \vee p = (x | (p | p)) | (p | p)\). We use this equality in the latter equation and by Lemma 3.2 (iii) we obtain
\[
(x^p \vee x) | (p | p) = 1 | (p | p) = p.
\]

Corollary 3.12. Let \((A; \wedge)\) be a Sheffer stroke basic algebra with the least element \(0\) and the greatest element \(1\). Then \((A; \vee, \wedge, 0, 1)\) is a lattice with an antitone involution.

Lemma 3.13. Let \((A; \wedge)\) be a Sheffer stroke basic algebra with the least element \(0\) and the greatest element \(1\), and \((A; \vee, \wedge, 0, 1)\) is a lattice with an antitone involution \(x \mapsto x^0\). Then it satisfies the following properties for all \(x \in A\):

(i) \(x | (0 | 0) = 0\) if and only if \(x = 1\),

(ii) \(x | (k | k) = k\) if and only if \(x \vee k = 1\),

(iii) \(0 | 0 = 1\),

(iv) \(x | 1 = x | x\),

(v) \(x^0 = x | x\),

(vi) \((k | k) | (x | x) = 1\) if and only if \(x \vee k = 1\).

Proof. (i) \((\Rightarrow)\) Assume that \(x | (0 | 0) = 0\). Then \(x | (0 | 0) | (0 | 0) = 1\). Since \(x \mapsto x^0\) is an involution, we have \(1 = (x | (0 | 0)) | (0 | 0) = x^{00} = x\).

(\(\Leftarrow\)) It follows from Lemma 3.2 (iii).

(ii) \((\Rightarrow)\) Let \(x | (k | k) = k\). Then we have \((x | (k | k)) | (k | k) = 1\); hence \(x \vee k = 1\).

(\(\Leftarrow\)) Assume that \(x \vee k = 1\). Then we have \((x | (k | k)) | (k | k) = 1\). Now, we substitute \(y := k\) in Lemma 3.2 (iv). Thereafter by using the hypothesis and Lemma 3.2 (iii) we obtain
\[
x | (k | k) = ((x | (k | k)) | (k | k)) | (k | k) = 1 | (k | k) = k.
\]
Assume that \( x \in A \). Then
\[ 0 \leq x \leq (x|x) \geq 0 \]
and
\[ 0 \leq 1 \geq 0 \].

It follows from Lemma 2.6.

By (ii), we have
\[ x^0 = x(0|0) = x|1 = x|x. \]

In [28], we substitute \([x := (k|k)]\) and \([y := x]\) in the equation \( x(y|y) = x(x|y) \) to
\[ 1 = (k|k)|(x|x) = (k|k)|(k|k)|x) = x \lor k. \]

It is verified similarly.

**Theorem 3.14.** Let \((A; |)\) be a Sheffer Stroke basic algebra with the least element \(0\) and the greatest element \(1\), and \((A; \lor, \land, 0^0, 0, 1)\) its induced complemented lattice with an antitone involution \( x \mapsto x^0 \). Then, there exists unique \( x^0 \) such that
\[ x \land x^0 = 1 \quad \text{and} \quad x \lor x^0 = 0 \]
for all \( x \in A \).

**Proof.** We show that there exists \( x \mapsto x^0 \) an antitone involution of \( A \) such that \( x^0 \lor x = 1 \) and \( x^0 \land x = 0 \) in Lemma 3.11. For the uniqueness, assume that
\[ x^0 = k \quad \text{and} \quad x^0 = l \]
in Lemma 3.13 (vi). Then by Lemma 3.13 (vi), we have
\[ k = x^0 = (x|x) \quad \text{and} \quad l = x^0 = (x|x). \]

Then from these equalities we get
\[ x \lor k = 1 \Rightarrow (x|x)(k|k) = 1 \]
\[ \Rightarrow l(k|k) = 1 \]
\[ \Rightarrow l \leq k. \]

Using the same technique, we can obtain \( k \leq l \). Therefore, \( k = l \). Hence, we have unique \( x^0 \) such that \( x^0 \lor x = 1 \) for each \( x \in A \). The identity \( x \land x^0 = 0 \) is verified similarly.

**Definition 3.15.** A Sheffer Stroke basic algebra \((A; |)\) is commutative if it satisfies
\[ (x|(p|p))((y|y) = (y|(p|p)))(x|x) \]
for all \( x, y \in [p, 1] \).

**Lemma 3.16.** Let \((A; |)\) be a commutative basic algebra. Then the interval basic algebra \([p, 1]; \oplus, \neg_p, p\) is commutative for each \( p \in A \).

From Theorem 2.8 in [20] we obtain the following corollary:

**Corollary 3.17.** Let \((A; |)\) be a commutative Sheffer Stroke basic algebra and \((A; \lor)\) its induced semilattice. Then
(i) the interval basic algebra \([p, 1]; \oplus, \neg_p, p\) is commutative basic algebra for each \( p \in A \),
(ii) the interval lattice \([p, 1]; \lor, \land_p\) is distributive for each \( p \in A \).

**Theorem 3.18.** Let \((A; |)\) be a Sheffer Stroke basic algebra with the least element \(0\) and the greatest element \(1\), and \((A; \lor, \land, 0, 0^0, 0, 1)\) its induced lattice with an antitone involution \( x \mapsto x^0 \). Then \((A; \lor, \land, 0^0, 0, 1)\) is Boolean algebra.

**Proof.** It follows from Theorem 3.10 and Corollary 2.10.

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