LOCAL WELL-POSEDNESS FOR A CLASS OF 1D BOUSSINESQ SYSTEMS

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Abstract. In this paper we study the local well-posedness for the Cauchy problem associated with a special class of one-dimensional Boussinesq systems that model the evolution of long water waves with small amplitude in the presence of surface tension.

1. Introduction and main result. It has been established that the evolution of 2D long water waves with small amplitude in the presence of surface tension is reduced to studying solutions \((\eta, \Phi)\) of the 1D-Boussinesq type system

\[
\begin{align*}
(I - \mu a \partial_x^2) \eta_t + \partial_x^2 \Phi - \mu b \partial_x^4 \Phi + \epsilon \partial_x (\eta \partial_x \Phi) &= 0, \\
(I - \mu c \partial_x^2) \Phi_t + \eta - \mu d \partial_x^2 \eta + \frac{\epsilon}{2} (\partial_x \Phi)^2 &= 0,
\end{align*}
\]

(1)

where \(\Phi = \Phi(x, t)\) represents the rescale nondimensional velocity potential on the bottom \(z = 0\), and the variable \(\eta = \eta(x, t)\) corresponds the rescaled free surface elevation, \(\epsilon\) is the amplitude parameter (nonlinearity coefficient), \(\mu\) is the long-wave parameter (dispersion coefficient), and the constants \(a, c \geq 0\) and \(b, d > 0\) are such that

\[a + c - (b + d) = \frac{1}{3} - \sigma,\]

where \(\sigma^{-1}\) is known as the Bond number (associated with the surface tension). This Boussinesq model is the 1D version of some system obtained in [22] and [23].

As happens in water wave models, there is a Hamiltonian type structure which is clever to characterize the space for the study of the Cauchy problem. In our

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particular Boussinesq system (1), the Hamiltonian functional $\mathcal{H} = \mathcal{H}(t)$ is defined as

$$\mathcal{H} \left( \frac{\eta}{\Phi} \right) = \frac{1}{2} \int \left( \eta^2 + \mu d(\partial_x \eta)^2 + (\partial_x \Phi)^2 + \mu b(\partial^2_x \Phi)^2 + c \eta (\partial_x \Phi)^2 \right) dx,$$

and the Hamiltonian type structure is given by

$$\left( \frac{\eta_t}{\Phi_t} \right) = J \mathcal{H}' \left( \frac{\eta}{\Phi} \right), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (I - \mu \partial_x^2)^{-1}.$$ 

Note that for $a = c = 0$ the operator $J$ becomes

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

We see directly that the functional $\mathcal{H}$ is well defined when $\eta(\cdot, t), \Phi_x(\cdot, t) \in H^1(\mathbb{R})$, for $t$ in some interval. These conditions already characterize the natural space for the study of solutions of the system (1). Certainly, J. Quintero and A. Montes in [24] showed for the model (1) the existence of solitary wave solutions which propagate with speed of wave $c > 0$, i.e., solutions of the form

$$\eta(x, t) = u(x - ct), \quad \Phi(x, t) = v(x - ct),$$

in the energy space $H^1 \times V^2$, where $H^1 = H^1(\mathbb{R})$ is the usual Sobolev space of order 1 and the space $V^2$ is defined with respect to the norm given by

$$\|v\|_{V^2}^2 = \int (|v'|^2 + (v'')^2) \, dx = \|v\|_{H^1}^2.$$ 

They also showed the locally well-posedness for the Cauchy problem associated to the system (1) in the Sobolev type space $H^s \times V^{s+1}$, where $H^s = H^s(\mathbb{R})$ is the usual Sobolev space of order $s$ defined as the completion of the Schwartz class $S(\mathbb{R})$ with respect to the norm

$$\|w\|_{H^s} = \| (1 + |\xi|)^s \hat{w}(\xi) \|_{L^2},$$

and $V^{s+1}$ denotes the completion of the Schwartz class with respect to the norm

$$\|w\|_{V^{s+1}} = \| (1 + |\xi|)^s |\xi| \hat{w}(\xi) \|_{L^2},$$

where $\hat{w}$ is the Fourier transform of $w$ in the space variable $x$ and $\xi$ is the variable in the frequency space related to the variable $x$. We will use the notation $\hat{w}^{(l)}$ for the Fourier transform of $w$ in the time variable $t$.

For $a, b, c, d > 0$, using a bilinear estimate obtained by J. Bona and N. Tzvetkov in [6], Quintero and Montes showed the local well-posedness for the Cauchy problem associated to the Boussinesq system (1) for initial data in the space $H^s \times V^{s+1}$ with $s \geq 0$. For $a = c = 0$ and $b, d > 0$, using the estimates for Kato’s commutator, they showed the local well-posedness for initial data in $H^s \times V^{s+1}$, with $s > 3/2$.

In the present work, when $a = c = 0$ and $b, d > 0$ we will prove that the Cauchy problem associated to system (1) is locally well-posed for initial data in the Sobolev space $H^s(\mathbb{R}) \times V^{s+1}(\mathbb{R})$ with $s \geq 0$. Hence we improve the result found in [24]. We refer to the concept of local well-posedness in the sense of Hadamard, that is, the solution uniquely exists in a certain time interval (unique existence), the solution has the same regularity as the initial data in a certain interval (persistence), and the solution varies continuously depending upon the initial data (continuous dependence).
To obtain our result we use Bourgain spaces and apply the same technique used by F. Linares in [21] to show the well-posedness for the Benjamin equation in Sobolev spaces with index $s \geq 0$ and by D. Berikanov, T. Ogawa and G. Ponce in [2] for the coupled Schrödinger-KdV system and coupled Benjamin-Ono-KdV system in Sobolev type spaces with index $s \geq 0$ (see also Farah [13] for the well-posedness of the Boussinesq equation, Farah-Esfahani [12] for the sixth-order Boussinesq equation, and Compaan-Tzirakis [11] and Li-Chen-Zhang [15] for the related boundary value problems). The scheme to prove the local well-posedness is the application of the contraction mapping principle to the corresponding integral problem in a suitable Banach function space $C\left(0,T\right) \cap Z^s$, where the appropriated space-time weight norm for $Z^s$ is determined by the knowledge of certain estimates for the solutions of the linear part. This method, introduced by J. Bourgain in [18]-[19], not only benefits of the above mentioned space time estimates, but also exploits structural properties of the nonlinearity.

We notice that taking $u = \partial_x \Phi$ the system (1) is related with the system considered by J. Bona, M. Chen and J. Saut (see [3]-[4]):

$$\begin{cases}
(I - \tilde{a} \partial_x^2) \eta_t + \partial_x u - \tilde{b} \partial_x^3 u + \partial_x (u\eta) = 0, \\
(I - \tilde{c} \partial_x^2) u_t + \partial_x \eta - \tilde{d} \partial_x^3 \eta + u \partial_x u = 0,
\end{cases}$$

with

$$\tilde{a} - \tilde{b} = \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right), \quad \tilde{c} - \tilde{d} = \frac{1}{2} \left( 1 - \theta^2 \right) \geq 0, \quad (2)$$

where $\theta$ is a fixed constant in the interval $[0,1]$. They studied in [3] the well-posedness with the parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ satisfying one of the following assumptions.

C1. $\tilde{a}, \tilde{c} \geq 0$, $\tilde{b}, \tilde{d} \geq 0$,

C2. $\tilde{a}, \tilde{c} \geq 0$, $\tilde{b} = \tilde{d} < 0$,

C3. $\tilde{a} = \tilde{c} < 0$, $\tilde{b} = \tilde{d} < 0$.

If $\tilde{a} = \tilde{c} = 0$, in view of the constraints (2), (C1) and (C2), the only admissible case is when $\tilde{b} = \tilde{d} < 0$, which means $\theta^2 = \frac{2}{3}$ and $\tilde{b} = \tilde{d} = -\frac{1}{9}$. By making a simple additional scaling, one may assume $\tilde{b} = \tilde{d} = -1$ and thus the system under consideration takes the form

$$\begin{cases}
\eta_t + \partial_x u + \partial_x^3 u + \partial_x (u\eta) = 0, \\
u_t + \partial_x \eta + \partial_x^3 \eta + u \partial_x u = 0.
\end{cases}$$

For this system, J. Bona, M. Chen and J. Saut in [4] showed the local well-posedness for the associated Cauchy problem in the space $H^s \times H^s$ with $s > 3/4$, by introducing $u = \frac{1}{4} \left( u + \eta \right)$ and $w = \frac{1}{4} \left( u - \eta \right)$ to obtain the equivalent system

$$\begin{cases}
v_t + \partial_x v + \partial_x^3 v + 3v \partial_x v + \partial_x (vw) - w \partial_x w = 0, \\
w_t - \partial_x w - \partial_x^3 w - v \partial_x v + \partial_x (vw) + 3w \partial_x w = 0.
\end{cases}$$

This is a system of two linear KdV equations coupled through nonlinear terms. One can apply the smoothing property theory developed by C. Kenig, G. Ponce and L. Vega in [16]-[17] for the scalar KdV to obtain the local well-posedness for $s > \frac{3}{4}$. Using a weighted Sobolev space of Bourgain type, J. Bona, Z. Grujić and H. Kalisch in [5] proved the local well-posedness of the previous system in the space $G_{\sigma,s} \times G_{\sigma,s}$. 

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for \( \sigma \geq 0 \) and \( s \geq 0 \), where \( G_{\sigma,s} \) is the analytic Gevrey space defined with respect to the norm
\[
\|w\|_{G_{\sigma,s}} = \| (1 + |\xi|)^s e^{\epsilon (1+|\xi|)} \hat{w}(\xi) \|_{L^2_\xi}.
\]

We point out that B. Alvarez-Samaniego and X. Carvajal in [1], using Bourgain spaces \( X^{s,\beta}_\alpha \) with associated norm
\[
\|w\|_{X^{s,\beta}_\alpha} = \| (1 + |\tau - \alpha \xi^3|)^\beta (1 + |\xi|)^s \hat{w}(\xi,\tau) \|_{L^2_{\xi,\tau}},
\]
showed local well-posedness for the system
\[
\begin{align*}
v_t + a_1 \partial_x^3 v + c_{11} v \partial_x v + c_{12} w \partial_x w + d_{11} \partial_x (vw) &= 0, \\
w_t + a_2 \partial_x^3 w + c_{21} v \partial_x v + c_{22} w \partial_x w + d_{22} \partial_x (vw) &= 0.
\end{align*}
\]
When \( |a_1| = |a_2| \neq 0 \), they proved that (3) is locally well-posed in \( H^s \times H^s \) for \( s > -\frac{3}{4} \), as in the single KdV equation case. The question whether (3) is well-posed when \( |a_1| \neq |a_2| \) is left open in [1].

In the work [27], X. Yang and B. Zhang present a complete classification for a class of coupled KdV-KdV systems
\[
\begin{align*}
v_t + a_{i} \partial_x^3 v + c_{11i} v \partial_x v + c_{12i} w \partial_x w + d_{11i} \partial_x (vw) &= 0, \\
w_t + a_{j} \partial_x^3 w + c_{21j} v \partial_x v + c_{22j} w \partial_x w + d_{22j} \partial_x (vw) &= 0,
\end{align*}
\]
in terms of the well-posedness in \( H^s \times H^s \) based on its coefficients \( a_i, c_{ij} \) and \( d_{ij} \) for \( i, j = 1, 2 \). The key ingredients in the proofs are the bilinear estimates in Bourgain spaces in both divergence and non-divergence forms (the system (4) is called in divergence form if \( d_{1,1} = d_{1,2} \) and \( d_{2,1} = d_{2,2} \)). In contrast to the lone critical index \(-\frac{3}{4}\) for the single KdV equation, the critical index for the class of KdV-KdV systems are \(-\frac{13}{12}, -\frac{3}{4}, 0 \) and \( \frac{3}{4} \). As a result, the systems (4) are classified into four classes, each of which corresponds to a unique index \( s^* \in \{-\frac{13}{12}, -\frac{3}{4}, 0, \frac{3}{4}\} \) such that any system in this class is locally well-posed if \( s > s^* \) (see Theorem 1.5 - Theorem 1.9 in [27]).

We notice that due the method used in this paper and the considerations above one could expect to obtain a similar result as the one for the KdV equation, i.e., local well-posedness for \( s > -3/4 \). We point out that Kenig et al. simplified the original Bourgain method and improved the bilinear estimates using only elementary techniques, such as Cauchy-Schwartz inequality and simple calculus inequality. This argument also uses some arithmetic facts involving the symbol of the linearized model; then splitting the domain of integration, Kenig et al. made some cancellations in the symbol in order to use his calculus inequalities (see Lemma 3.1) and a clever change of variables to establish their crucial estimates. Here, we will use this kind of argument, but unfortunately in our Boussinesq model we do not have good cancellations on the symbol for \( s < 0 \), in order to derive the relevant bilinear estimates. However, using suitable Bourgain spaces (see (7)-(8)), in the case \( a = c = 0 \) we establish the local well-posedness for the Cauchy problem associated to the system (1) with initial condition in \( H^s \times V^{s+1} \) for \( s \geq 0 \).

In general, well-posedness for the the system in consideration is not known in the energy space \( H^1 \times V^2 \), where existence of solitary wave solutions is showed. Moreover, in a work in preparation (see [26]) the authors show that the solitary wave solutions of (1) are stable in \( H^1 \times V^2 \) using the notion of stability introduced by Cazenave and Lions in [10]. To justify the stability result, the Cauchy problem
might be solved for data in $H^s \times \mathcal{V}^{s+1}$, $s \geq s_0$ with $s_0 \leq 1$, requiring the date being close to the solitary wave in the $H^s \times \mathcal{V}^{s+1}$-norm and thus establishing that the solutions will remain close to translations of the solitary wave in the same norm throughout their existence time. Here, we prove a result in this direction.

In order to state our result in a precise manner we introduce some definitions and notations. For the sake of simplicity and following the assumptions in the work [4] of Bona, Chen and Saut, we consider $a = c = 0$ with $b = d$. Moreover, the rescaling

$$
\eta(x,t) = \frac{1}{\epsilon} \eta^* \left( \frac{x}{\sqrt{\mu b}}, \frac{t}{\sqrt{\mu b}} \right), \quad \Phi(x,t) = \frac{\sqrt{\mu b}}{\epsilon} \Phi^* \left( \frac{x}{\sqrt{\mu b}}, \frac{t}{\sqrt{\mu b}} \right),
$$

gives an equivalent system for which $\epsilon = \mu b = \mu d = 1$, namely

$$\begin{align*}
\eta_t + \partial_x^2 \Phi - \partial_x^4 \Phi + \partial_x (\eta \partial_x \Phi) &= 0, \\
\Phi_t + \eta - \partial_x^2 \eta + \frac{1}{2} (\partial_x \Phi)^2 &= 0.
\end{align*}
$$

In this paper we study the Cauchy problem associated to (5) with the initial condition

$$
\eta(x,0) = \eta_0(x), \quad \Phi(x,0) = \Phi_0(x).
$$

We will search for the solutions of the Cauchy problem (5)-(6) in Bourgain spaces of the type $X^{s,\beta} \times Y^{s+1,\beta}$, $s, \beta \in \mathbb{R}$, where $X^{s,\beta}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$
\|w\|_{X^{s,\beta}} = \| (|\tau| - |\phi(\xi))^{\beta} |\xi|^s \tilde{w}(\xi, \tau) \|_{L^2_{\xi,\tau}},
$$

and $Y^{s+1,\beta}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$
\|w\|_{Y^{s+1,\beta}} = \| (|\tau| - |\phi(\xi))^{\beta} |\xi| \xi^s \tilde{w}(\xi, \tau) \|_{L^2_{\xi,\tau}},
$$

where $\langle a \rangle = 1 + |a|$ and $\phi(\xi) = |\xi|^3 + |\xi|$ is the symbol associated to the spacial linear part of the Boussinesq system (5). In addition, $\tilde{w}$ denotes the time-space Fourier transform of $w$ and $(\xi, \tau)$ is the variable in the frequency space with $\xi$ as before and $\tau$ corresponding to the time variable $t$, that is

$$
\tilde{w}(\xi, \tau) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-ix\xi - it\tau} w(x,t) dx dt.
$$

If $C(\mathbb{R} : H^s \times \mathcal{V}^{s+1})$ denotes the space of continuous functions from $\mathbb{R}$ in $H^s \times \mathcal{V}^{s+1}$, then we will see in Lemma 2.3 that $X^{s,\beta} \times Y^{s+1,\beta}$ is continuously embedded in $C(\mathbb{R} : H^s \times \mathcal{V}^{s+1})$. Moreover, we will establish in Lemma 3.2 two types of bilinear estimates.

For $T > 0$ we denote by $X_T^{s,\beta}$ the space of the restrictions to the interval $[0, T]$ of the elements $\eta \in X^{s,\beta}$ with norm defined by

$$
\|\eta\|_{X_T^{s,\beta}} = \inf_{w \in X^{s,\beta}} \{ \|w\|_{X^{s,\beta}} : w(t) = \eta(t) \text{ on } [0, T] \}
$$

and by $Y_T^{s+1,\beta}$ the space of the restrictions to the interval $[0, T]$ of the elements $\Phi \in Y^{s+1,\beta}$ with norm defined by

$$
\|\Phi\|_{Y_T^{s+1,\beta}} = \inf_{w \in Y^{s+1,\beta}} \{ \|w\|_{Y^{s+1,\beta}} : w(t) = \Phi(t) \text{ on } [0, T] \}.$$
Our concept of solution for the Cauchy problem comes from Duhamel’s formula. Formally, \((\eta, \Phi)\) in \(X_T^{s,\beta} \times Y_T^{s+1,\beta}\) is a solution of the Cauchy problem (5)-(6) in \([0,T]\) if only if for all \(t \in [0,T]\)
\[
(\eta(t), \Phi(t)) = S(t)(\eta_0, \Phi_0) - \int_0^t S(t-t') \left( \partial_x (\eta \partial_x \Phi), \frac{1}{2} (\partial_x \Phi)^2 \right) (t') dt',
\]
where \(S(t)(\eta_0, \Phi_0)\) is the solution of the linear problem associated to (5), that is
\[
S(t)(\eta_0, \Phi_0) = (S_1(t)(\eta_0, \Phi_0), S_2(t)(\eta_0, \Phi_0))
\]
with
\[
S_1(t)(\eta_0, \Phi_0) = \int_{\mathbb{R}} e^{ix \xi} \left[ \cos(\phi(\xi)t) \hat{\eta}_0(\xi) + |\xi| \sin(\phi(\xi)t) \hat{\Phi}_0(\xi) \right] d\xi,
\]
\[
S_2(t)(\eta_0, \Phi_0) = \int_{\mathbb{R}} e^{ix \xi} \left[ -\frac{\sin(\phi(\xi)t) \hat{\eta}_0(\xi)}{|\xi|} + \cos(\phi(\xi)t) \hat{\Phi}_0(\xi) \right] d\xi,
\]
and the function \(\phi\) defined as before
\[
\phi(\xi) = |\xi|^3 + |\xi|.
\]

In fact, to work in the context of the Bourgain spaces \(X_T^{s,\beta} \times Y_T^{s+1,\beta}\), we slightly modify the two terms in the right-hand of (9) by means of a cut off function. Let \(\psi \in C_0^\infty(\mathbb{R})\) with support in \((-2,2)\), such that \(0 \leq \psi \leq 1\), and \(\psi \equiv 1\) in \([-1,1]\) and for \(0 < T < 1\) define \(\psi_T(t) = \psi(t/T)\). Then we consider another version of (9), that is
\[
(\eta(t), \Phi(t)) = \psi(t) S(t)(\eta_0, \Phi_0)
- \psi_T(t) \int_0^t S(t-t') \left( \partial_x (\eta \partial_x \Phi), \frac{1}{2} (\partial_x \Phi)^2 \right) (t') dt' .
\]
We will show as usual the existence of a solution of the integral problem (10) using the Banach fixed point theorem and appropriate linear and nonlinear estimates.

To finish our introduction we state the main result in this work, the theorem of existence and uniqueness of local solutions for the Cauchy problem associated with the Boussinesq system (5), which will be proved in Section 4.

**Theorem 1.1.** Let \(s \geq 0\) and \(1/2 < \beta \leq 3/4\). Then for all \((\eta_0, \Phi_0) \in H^s \times V^{s+1}\), there exists a time \(T = T(\|\eta_0, \Phi_0\|_{H^s \times V^{s+1}}) > 0\) and a unique solution \((\eta, \Phi)\) of the Cauchy problem (5)-(6) such that

\[
\eta \in C ([0,T] : H^s) \cap X_T^{s,\beta} \quad \text{and} \quad \Phi \in C ([0,T] : V^{s+1}) \cap Y_T^{s+1,\beta}.
\]

Moreover, for all \(0 < T' < T\) there exists a neighborhood \(\mathcal{V}\) of \((\eta_0, \Phi_0)\) in \(H^s \times V^{s+1}\) such that the map data-solution is Lipschitz from \(\mathcal{V}\) in the class

\[
C ([0,T'] : H^s \times V^{s+1}) \cap (X_T^{s,\beta} \times Y_T^{s+1,\beta}).
\]

In addition, if \((\eta_0, \Phi_0) \in H^s' \times V^{s+1}\) with \(s' > s\), then the above results hold with \(s'\) instead of \(s\) in the same interval \([0,T]\) with

\[
T = T(\|\eta_0, \Phi_0\|_{H^s \times V^{s+1}}).
\]

The paper is organized as follows. Section 2 will be dedicated to establish all the linear estimates needed in the proof of the Theorem 1.1. In section 3 we will estimate the bilinear forms \(\partial_x (\eta \partial_x \Phi)\) and \((\partial_x \Phi)(\partial_x \Phi_1)\) associated to the nonlinear part of the Boussinesq system (5). In section 4 we prove the Theorem 1.1, via a standard fixed point argument.
2. Linear estimates. In this section we establish some estimates for the linear part of the truncated integral problem (10) in the space $X^{s, \beta} \times Y^{s+1, \beta}$. These estimates will be an important ingredient in the proof of local well-posedness result.

**Lemma 2.1.** Let $\beta \geq 0$. Then exists $C_1 > 0$ depending only on $\psi, s, \beta$ such that

$$\|\psi(t) S_1(t)(\eta_0, \Phi_0)\|_{X^{s, \beta}} \leq C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y^{s+1}}.$$ 

$$\|\psi(t) S_2(t)(\eta_0, \Phi_0)\|_{Y^{s+1, \beta}} \leq C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y^{s+1}}.$$ 

**Proof.** First we note that

$$\left[ \psi(t) \int e^{ix\xi} e^{\pm i\phi(\xi)t} \hat{\eta}_0(\xi) d\xi \right] \sim (\xi, \tau) = \hat{\psi}(t)(\tau \mp \phi(\xi)) \hat{\eta}_0(\xi).$$

Then, using that

$$||| \tau | - \phi(\xi) | \leq \min \{ | \tau - \phi(\xi) |, | \tau + \phi(\xi) | \} (11)$$

and also that $\hat{\psi}(t)$ is rapidly decreasing, we see that

$$\left\| \psi(t) \int e^{ix\xi} e^{\pm i\phi(\xi)t} \hat{\eta}_0(\xi) d\xi \right\|_{X^{s, \beta}}^2$$

$$= \left\| \left( ||| \tau | - \phi(\xi) | \right)^{\beta}(\xi, \tau) \psi(t) \int e^{ix\xi} e^{\pm i\phi(\xi)t} \hat{\eta}_0(\xi) d\xi \right\|_{L^2_{\xi, \tau}}^2$$

$$= \int \int \left( ||| \tau | - \phi(\xi) | \right)^{2\beta}(\xi, \tau) 2^{|\tau \mp \phi(\xi)|^2} | \hat{\psi}(t)(\tau \mp \phi(\xi)) |^2 | \hat{\eta}_0(\xi) |^2 d\xi d\tau$$

$$\leq C \int \int |\tau \mp \phi(\xi)|^{2\beta} \phi(\xi)^{2s} | \hat{\psi}(t)(\tau \mp \phi(\xi)) |^2 | \hat{\eta}_0(\xi) |^2 d\xi d\tau$$

$$= C \| \eta_0 \|_{H^s}^2.$$ 

In a similar fashion we see that

$$\left\| \psi(t) \int e^{ix\xi} e^{\pm i\phi(\xi)t} \hat{\Phi}_0(\xi) d\xi \right\|_{X^{s, \beta}}^2$$

$$= \int \int \left( ||| \tau | - \phi(\xi) | \right)^{2\beta}(\xi, \tau) 2^{|\tau \mp \phi(\xi)|^2} | \hat{\psi}(t)(\tau \mp \phi(\xi)) |^2 | \hat{\Phi}_0(\xi) |^2 d\xi d\tau$$

$$\leq C \| \Phi_0 \|_{Y^{s+1}}^2.$$ 

Then, from the previous estimates we obtain that

$$\|\psi(t) S_1(t)(\eta_0, \Phi_0)\|_{X^{s, \beta}} \leq C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y^{s+1}}.$$ 

Similarly we have that

$$\left\| \psi(t) \int \frac{e^{ix\xi} e^{\pm i\phi(\xi)t} \hat{\eta}_0(\xi)}{||| \xi | \right\|_{Y^{s+1, \beta}}^2 d\xi$$

$$= \int \int \left( ||| \tau | - \phi(\xi) | \right)^{2\beta}(\xi, \tau) 2^{|\tau \mp \phi(\xi)|^2} | \hat{\psi}(t)(\tau \mp \phi(\xi)) |^2 | \hat{\eta}_0(\xi) |^2 d\xi d\tau$$

$$\leq C \| \eta_0 \|_{H^s}^2.$$
Lemma 2.2. Let \( -1/2 < \beta' \leq 0 \leq \beta \leq \beta' + 1 \) and \( 0 < T \leq 1 \). Then we conclude that
\[
\| \psi(t) S_2(t)(\eta_0, \Phi_0) \|_{Y^{s+1, \beta}} \leq C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y^{s+1}}.
\]

Proof. For (i) see Lemma 2.1 in [14]. For (ii) we note that
\[
\left( \psi_T(t) \int_0^t \right) e^{ix \xi} e^{\pm i(t-t')\phi(\xi)\eta(\xi, t')d\xi} dt' (\xi, t)
\]
\[
= \psi_T(t) \int_0^t e^{\pm i(t-t')\phi(\xi)\eta(\xi, t')d\xi} dt'
\]
\[
= e^{\pm i\phi(\xi)t} \psi_T(t) \int_0^t e^{\mp i\phi(\xi)t'} \eta(\xi, t')dt'
\]
\[
= e^{\pm i\phi(\xi)t} \tilde{w}(\xi, t),
\]
where \( w(x, t) = \psi_T(t) \int_0^t e^{\mp i\phi(\xi)t'} \eta(x, t')dt' \). Then we obtain that
\[
\left[ \psi_T(t) \int_0^t \right] e^{ix \xi} e^{\pm i(t-t')\phi(\xi)\eta(\xi, t')d\xi} dt' (\xi, \tau) = \tilde{w}(\xi, \tau \mp \phi(\xi)).
\]
Using the fact that for all \( \xi \in \mathbb{R}, \phi(\xi) \geq 0 \) and
\[
\max\{||\tau + \phi(\xi)| - \phi(\xi)|, ||\tau - \phi(\xi)| - \phi(\xi)||\} \leq ||\tau||
\]
we have that
\[
\| \psi_T(t) \int_0^t e^{ix \xi} e^{\pm i(t-t')\phi(\xi)\eta(\xi, t')d\xi} dt' \|_{X^{s, \beta}}^2
\]
\[
= \int \int R \int \int \int \int \int (||\tau| - \phi(\xi)|^2 \xi |\tilde{w}(\xi, \tau \mp \phi(\xi))|^2 d\xi d\tau
\]
\[
= \int \int \int \int \int (||\tau \pm \phi(\xi)| - \phi(\xi)|^2 \xi |\tilde{w}(\xi, \tau)|^2 d\xi d\tau
\]
\[
\leq \int \int \int \int \int (\tau)^{2\beta} \xi^{2s} |\tilde{w}(\xi, \tau)|^2 d\xi d\tau
\]
\[
= \int \xi^{2s} ||\tilde{w}||_{H^s}^2 d\xi.
\]
Thus, we obtain

\[
\int_R (\xi)^{2s} \| \hat{\omega} \|^2_{H^s} \, d\xi = \int_R (\xi)^{2s} \| \psi_T (t) \int_0^t e^{i \phi (\xi) (t-t')} \hat{\eta} (\xi, t') \, d\xi' \|^2_{H^s} \, d\xi
\]

\[
\leq T^{2[1-(\beta'-\beta)]} \int_R (\xi)^{2s} \| e^{i \phi (\xi) t} \hat{\eta} (\xi, t) \|^2_{H^s} \, d\xi
\]

\[
= T^{2[1-(\beta'-\beta)]} \int_R \int_R (t \pm \phi (\xi))^{2\beta'} (\xi)^{2s} |\hat{\eta} (\xi, \tau)|^2 \, d\xi \, d\tau
\]

\[
\leq T^{2[1-(\beta'-\beta)]} \int_R \int_R (|\tau| - \phi (\xi))^{2\beta'} (\xi)^{2s} |\hat{\eta} (\xi, \tau)|^2 \, d\xi \, d\tau
\]

\[
= T^{2[1-(\beta'-\beta)]} \| \eta \|^2_{X^{s,\beta'}}.
\]

Thus, we obtain

\[
\| \psi_T (t) \int_0^t e^{i z \xi e^{\pm i (t-t')} \phi (\xi) } \hat{\eta} (\xi, t') \, d\xi' \|^2_{X^{s,\beta}} \leq T^{2[1-(\beta'-\beta)]} \| \eta \|^2_{X^{s,\beta'}}.
\]

In a similar way we see that

\[
\| \psi_T (t) \int_0^t e^{i z \xi e^{\pm i (t-t')} \phi (\xi) } \hat{\Phi} (\xi, t') \, d\xi' \|^2_{X^{s,\beta}} \leq C T^{2[1-(\beta'-\beta)]} \| \Phi \|^2_{Y^{s+1,\beta'}}.
\]

So, we conclude that

\[
\| \psi_T (t) \int_0^t S_1 (t-t') (\eta, \Phi) (t') \, d\xi' \|^2_{X^{s,\beta}} \leq C_2 T^{1-(\beta'-\beta)} \left( \| \eta \|_{X^{s,\beta'}} + \| \Phi \|_{Y^{s+1,\beta'}} \right).
\]

Similarly we obtain the inequality in (iii). \( \square \)

In the following lemma we show the embedding of the space \( X^{s,\beta} \times Y^{s+1,\beta} \) in \( C(\mathbb{R} : H^s \times V^{s+1}) \), for \( \beta > 1/2 \).

**Lemma 2.3.** Let \( \beta > 1/2 \). Then there exists \( C > 0 \) depending only on \( \beta \) such that

\[
\| (\eta, \Phi) \|_{C(\mathbb{R} : H^s \times V^{s+1})} \leq C \| (\eta, \Phi) \|_{X^{s,\beta} \times Y^{s+1,\beta}}.
\]

**Proof.** First we will prove that \( X^{s,\beta} \subseteq L^\infty (\mathbb{R} : H^s) \). Let \( \eta_1, \eta_2 \) such that

\[
\eta = \eta_1 + \eta_2, \quad \eta_1 (\xi, \tau) = \tilde{\eta} (\xi, \tau) \chi_A (\tau), \quad \eta_2 (\xi, \tau) = \tilde{\eta} (\xi, \tau) \chi_B (\tau),
\]
where \( A = \{ \tau : \tau < 0 \} \) and \( B = \{ \tau : \tau \geq 0 \} \). Then for all \( t \in \mathbb{R} \) we see that

\[
\| \eta_1(t) \|_{H^s} = \left\| \left( e^{it\phi(\xi)} (\eta_1)^\wedge \right)^\vee (x, t) \right\|_{H^s}
= \left\| \int_\mathbb{R} e^{it\tau} \left( \left( e^{it\phi(\xi)} (\eta_1)^\wedge \right)^\vee \right)^\wedge(t) (x, \tau) \, d\tau \right\|_{H^s}
\leq \int_\mathbb{R} \left\| \left( \left( e^{it\phi(\xi)} (\eta_1)^\wedge \right)^\vee \right)^\wedge(t) (x, \tau) \right\|_{H^s} \, d\tau
= \int_\mathbb{R} \left( \int_\mathbb{R} \left( e^{-it(\tau-\phi(\xi))} \eta_1(\xi, t) \right) \, d\xi \right)^2 \, d\tau
= \int_\mathbb{R} \left( \int_\mathbb{R} e^{-it\tau} (\eta_1(\xi, \tau-\phi(\xi))) \, d\xi \right)^2 \, d\tau.
\]

Since \( |\tau + \phi(\xi)| = |\tau| - |\phi(\xi)| \) for \( \tau \leq 0 \), using Cauchy-Schwarz inequality and the fact that \( \beta > 1/2 \), we have that

\[
\| \eta_1(t) \|_{H^s} \leq \left( \int_\mathbb{R} \left( \int_\mathbb{R} \left( \int_\mathbb{R} \langle \tau + \phi(\xi) \rangle^{2\beta} \left( \eta_1(\xi, \tau-\phi(\xi)) \right)^2 \, d\xi \right)^2 \, d\tau d\xi \right)^{1/2}
\leq C \left( \int_\mathbb{R} \left( \int_\mathbb{R} \left( \int_\mathbb{R} \langle \tau + \phi(\xi) \rangle^{2\beta} \left( \eta_1(\xi, \tau) \right)^2 \, d\xi \right)^2 \, d\tau d\xi \right)^{1/2}
\leq C \left( \int_\mathbb{R} \left( \int_\mathbb{R} \langle |\tau| - \phi(\xi) \rangle^{2\beta} \left( \eta(\xi, \tau) \right)^2 \, d\tau d\xi \right)^{1/2}
= C \| \eta \|_{X^{*, \beta}}.
\]

In a similar fashion we have that

\[
\| \eta_2(t) \|_{H^s} = \left\| \left( e^{-it\phi(\xi)} (\eta_2)^\wedge \right)^\vee (x, t) \right\|_{H^s}
\leq \int_\mathbb{R} \left( \int_\mathbb{R} \left( \int_\mathbb{R} \langle \tau + \phi(\xi) \rangle^{2\beta} \left( \eta_2(\xi, \tau + \phi(\xi)) \right)^2 \, d\xi \right)^2 \, d\tau d\xi \right)^{1/2}
\leq \left( \int_\mathbb{R} \langle \tau \rangle^{-2\beta} \, d\tau \right)^{1/2} \left( \int_\mathbb{R} \left( \int_\mathbb{R} \langle \tau + \phi(\xi) \rangle^{2\beta} \left( \eta_2(\xi, \tau + \phi(\xi)) \right)^2 \, d\tau d\xi \right)^{1/2}
\leq C \left( \int_\mathbb{R} \left( \int_\mathbb{R} \langle |\tau| - \phi(\xi) \rangle^{2\beta} \left( \eta(\xi, \tau) \right)^2 \, d\tau d\xi \right)^{1/2}
= C \| \eta \|_{X^{*, \beta}}.
\]

Thus, we obtain

\[
\| \eta \|_{L^\infty(\mathbb{R} : H^s)} \leq C \| \eta \|_{X^{*, \beta}}.
\]
Next, we show that $\eta \in C(\mathbb{R} : H^s)$. In fact,

\[
\|\eta_1(t) - \eta_1(t')\|_{H^s} = \left\| \int_{\mathbb{R}} \left( e^{it\tau} - e^{it'\tau} \right) \left( e^{-it\phi(x)} (\eta_1)^{\Lambda(t)} \right) (x, \tau) d\tau \right\|_{H^s} \\
= \left( \int_{\mathbb{R}} (\tau)^{2s} \left( \left\| (\eta_1)^{\Lambda(t)} \right\|_{H^s} \right)^2 d\tau \right)^{1/2} \\
\leq \left( \int_{\mathbb{R}} (\tau)^{2s} \left( \left\| (\eta_1)^{\Lambda(t)} \right\|_{H^s} \right)^2 d\tau \right)^{1/2} \\
\leq C \left( \int_{\mathbb{R}} \int_{\mathbb{R}} (\tau)^{2s} \left\| (\eta_1)^{\Lambda(t)} \right\|_{H^s}^2 d\tau d\xi \right)^{1/2}.
\]

Letting $t \to t'$ and using the Dominated Convergence Theorem we obtain that

\[
\|\eta_1(t) - \eta_1(t')\|_{H^s} \to 0.
\]

In a similar fashion we have that

\[
\|\eta_2(t) - \eta_2(t')\|_{H^s} = \left\| \int_{\mathbb{R}} \left( e^{it\tau} - e^{it'\tau} \right) \left( e^{-it\phi(x)} (\eta_2)^{\Lambda(t)} \right) (x, \tau) d\tau \right\|_{H^s} \\
= \left( \int_{\mathbb{R}} (\tau)^{2s} \left( \left\| (\eta_2)^{\Lambda(t)} \right\|_{H^s} \right)^2 d\tau \right)^{1/2} \\
\leq \left( \int_{\mathbb{R}} (\tau)^{2s} \left( \left\| (\eta_2)^{\Lambda(t)} \right\|_{H^s} \right)^2 d\tau \right)^{1/2} \\
\leq C \left( \int_{\mathbb{R}} \int_{\mathbb{R}} (\tau)^{2s} \left\| (\eta_2)^{\Lambda(t)} \right\|_{H^s}^2 d\tau d\xi \right)^{1/2}.
\]

Then we conclude that $\|\eta_2(t) - \eta_2(t')\|_{H^s} \to 0$ and $\|\eta\|_{C(\mathbb{R}; H^s)} \leq C\|\eta\|_{X_{\tau,B}}$.

Now, let $\Phi = \Phi_1 + \Phi_2$ where $\Phi_1(\xi, \tau) = \Phi(\xi, \tau)\chi_A(\tau)$, $\Phi_2(\xi, \tau) = \Phi(\xi, \tau)\chi_B(\tau)$. Then for all $t \in \mathbb{R}$ we have that

\[
\|\Phi_1(t)\|_{X_{\tau,B}} = \left\| \left( e^{it\phi(x)} (\Phi_1)^{\Lambda(t)} \right) (x, t) \right\|_{X_{\tau,B}} \\
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\xi|^2 (\xi)^{2s} |\Phi_1(\xi, \tau - \phi(\xi)|^2 d\xi \right)^{1/2} d\tau \\
\leq \left( \int_{\mathbb{R}} (\tau)^{-2\beta} d\tau \right)^{1/2} \left( \int_{\mathbb{R}} \left\| (\Phi_1)^{\Lambda(t)} \right\|_{H^s}^2 d\tau d\xi \right)^{1/2} \\
\leq C \left( \int_{\mathbb{R}} \int_{\mathbb{R}} (\tau)^{2\beta} (\xi)^{2s} |\Phi_1(\xi, \tau - \phi(\xi)|^2 d\tau d\xi \right)^{1/2} \\
= C\|\Phi\|_{Y_{\tau,B}}^{1, \beta}.
\]
Lemma 3.2. and considerably improved by Kenig, Ponce and Vega (see [18]-[20]). We prove these estimates using a method originally due to Bourgain (see [7]-[9]).

3. Bilinear estimates

First we state some inequalities that will be useful later.

Lemma 3.1. If \( p, q > 0 \) and \( r = \min\{p, q, p + q - 1\} \) with \( p + q > 1 \), we have that

\[
\int_{\mathbb{R}} \frac{dx}{(x - \lambda)^r (x - \mu)^r} \leq \frac{C}{(\lambda - \mu)^r}
\]

and

\[
\int_{\mathbb{R}} \frac{dx}{(x - \lambda)^{2\beta} |\sqrt{\lambda - x}|} \leq \frac{C}{(\lambda)^{1/2}}.
\]

Proof. See Lemma 4.2 in [14] and Lemma 2.3 in [19].

The following nonlinear estimates constitute an important result for this work. We prove these estimates using a method originally due to Bourgain (see [7]-[9]) and considerably improved by Kenig, Ponce and Vega (see [18]-[20]).

Lemma 3.2. Let \( s \geq 0, \alpha \geq 1/4 \) and \( \beta > 1/2 \). Then there exists a constant \( C_3 > 0 \) depending only on \( s, \alpha, \beta \) such that

(i) \( \| \partial_x (\eta \partial_x \Phi) \|_{X^{s, -\alpha}} \leq C_3 \| \eta \|_{X^{s, \alpha}} \| \Phi \|_{Y^{s+1, \beta}} \)

(ii) \( \| (\partial_x \Phi)(\partial_x \Phi_1) \|_{Y^{s+1, -\alpha}} \leq C_3 \| \Phi \|_{Y^{s+1, \beta}} \| \Phi_1 \|_{Y^{s+1, \beta}} \).

Proof. First, notice that

\[
\| \partial_x (\eta \partial_x \Phi) \|_{X^{s, -\alpha}} = \left\| (|\tau| - \phi(\xi))^{-\alpha} \phi(\xi)^{s} \tilde{\Phi}(\xi, \tau) \right\|_{L^2_{\xi, \tau}}.
\]

Thus, by letting

\[
f(\xi, \tau) = (|\tau| - \phi(\xi))^\beta \phi(\xi)^s \tilde{\Phi}(\xi, \tau), \quad g(\xi, \tau) = (|\tau| - \phi(\xi))^\beta \phi(\xi)^s \tilde{\Phi}(\xi, \tau),
\]

and also that

\[
\| \Phi_2(t) \|_{Y^{s+1}} = \left\| e^{-it\phi(\xi)} (\Phi_2)^v(x, t) \right\|_{Y^{s+1}}
\]

\[
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\xi|^2 |\xi|^{2\alpha} \tilde{\Phi}_2(\xi, \tau + \phi(\xi))|^2 d\xi \right)^{1/2} d\tau
\]

\[
\leq \left( \int_{\mathbb{R}} |(\tau - 2\beta) d\tau \right)^{1/2} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} (\tau - 2\beta) |\xi|^2 |\xi|^{2\alpha} \tilde{\Phi}_2(\xi, \tau + \phi(\xi))|^2 d\tau d\xi \right)^{1/2}
\]

\[
\leq C \left( \int_{\mathbb{R}} \int_{\mathbb{R}} (|\tau| - \phi(\xi))^{2\beta} |\xi|^2 |\xi|^{2\alpha} \tilde{\Phi}(\xi, \tau)|^2 d\tau d\xi \right)^{1/2}
\]

\[
= C \| \tilde{\Phi} \|_{Y^{s+1, \beta}}.
\]

Therefore

\[
\| \tilde{\Phi} \|_{L^\infty(\mathbb{R} : Y^{s+1})} \leq C \| \Phi \|_{Y^{s+1, \beta}},
\]

and using Dominated Convergence Theorem we have that \( \tilde{\Phi} \in C(\mathbb{R} : Y^{s+1}) \). Then we conclude that

\[
\| (\eta, \Phi) \|_{C(\mathbb{R} : H^s \times Y^{s+1})} \leq C \| (\eta, \Phi) \|_{X^{s, \beta} \times Y^{s+1, \beta}}.
\]

\[\square\]
where

$$J(f, g, h) = \int_{\mathbb{R}^4} \frac{\xi(\xi)^{s} f(\xi - \xi_1, \tau - \tau_1) g(\xi_1, \tau_1) h(\xi, \tau) d\xi d\tau d\xi_1 d\tau_1}{(|\tau| - \phi(\xi))^{1/2} (|\tau_1| - \phi(\xi_1))^{1/2}}.$$ 

(14)

For to perform the desired estimate we need to analyze all possible cases for the sign of \(\tau, \tau_1, \tau - \tau_1\). To do this we split \(\mathbb{R}^4\) into the regions

\[
\begin{align*}
\Gamma_1 &= \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : \tau_1 < 0, \tau - \tau_1 < 0\}, \\
\Gamma_2 &= \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : \tau_1 \geq 0, \tau - \tau_1 < 0, \tau \geq 0\}, \\
\Gamma_3 &= \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : \tau_1 \geq 0, \tau - \tau_1 < 0, \tau < 0\}, \\
\Gamma_4 &= \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : \tau_1 < 0, \tau - \tau_1 \geq 0, \tau \geq 0\}, \\
\Gamma_5 &= \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : \tau_1 < 0, \tau - \tau_1 \geq 0, \tau < 0\}, \\
\Gamma_6 &= \{(\xi, \tau, \xi_1, \tau_1) \in \mathbb{R}^4 : \tau_1 \geq 0, \tau - \tau_1 \geq 0\}.
\end{align*}
\]

We note that \(\tau_1 < 0\) and \(\tau - \tau_1 < 0\) implies \(\tau < 0\), and \(\tau_1 \geq 0\) and \(\tau - \tau_1 \geq 0\) implies \(\tau \geq 0\). Then the cases \(\tau_1 < 0, \tau - \tau_1 < 0, \tau \geq 0\) and \(\tau_1 \geq 0, \tau - \tau_1 \geq 0, \tau < 0\) cannot occur. Now, since

$$1 + |\xi| \leq (1 + |\xi_1|) (1 + |\xi - \xi_1|),$$

then for \(s \geq 0\) we see that

$$\frac{\langle \xi \rangle^{2s}}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s}} \leq 1.$$

So, we will prove the inequality (14) with \(Z(f, g, h)\) instead of \(J(f, g, h)\) where

$$Z(f, g, h) = \int_{\mathbb{R}^4} \frac{\xi f(\xi_2, \tau_2) g(\xi_1, \tau_1) h(\xi, \tau) d\xi d\tau d\xi_1 d\tau_1}{(\sigma)^{\alpha} (\sigma_1)^{\beta} (\sigma_2)^{\beta}},$$

with \(\xi_2 = \xi - \xi_1, \tau_2 = \tau - \tau_1\) and \(\sigma, \sigma_1, \sigma_2\) belonging to one of the following cases

\[
\begin{align*}
(C_1) &\quad \sigma = \tau + |\xi|^3 + |\xi|, \quad \sigma_1 = \tau_1 + |\xi_1|^3 + |\xi_1|, \quad \sigma_2 = \tau_2 + |\xi_2|^3 + |\xi_2|, \\
(C_2) &\quad \sigma = \tau - |\xi|^3 - |\xi|, \quad \sigma_1 = \tau_1 - |\xi_1|^3 - |\xi_1|, \quad \sigma_2 = \tau_2 + |\xi_2|^3 + |\xi_2|, \\
(C_3) &\quad \sigma = \tau + |\xi|^3 + |\xi|, \quad \sigma_1 = \tau_1 - |\xi_1|^3 - |\xi_1|, \quad \sigma_2 = \tau_2 + |\xi_2|^3 + |\xi_2|, \\
(C_4) &\quad \sigma = \tau - |\xi|^3 - |\xi|, \quad \sigma_1 = \tau_1 + |\xi_1|^3 + |\xi_1|, \quad \sigma_2 = \tau_2 - |\xi_2|^3 - |\xi_2|, \\
(C_5) &\quad \sigma = \tau + |\xi|^3 + |\xi|, \quad \sigma_1 = \tau_1 + |\xi_1|^3 + |\xi_1|, \quad \sigma_2 = \tau_2 - |\xi_2|^3 - |\xi_2|, \\
(C_6) &\quad \sigma = \tau - |\xi|^3 - |\xi|, \quad \sigma_1 = \tau_1 - |\xi_1|^3 - |\xi_1|, \quad \sigma_2 = \tau_2 - |\xi_2|^3 - |\xi_2|.
\end{align*}
\]

We first consider \(\sigma, \sigma_1, \sigma_2\) as in the case \((C_1)\). Using inequality \((12)\) in Lemma 3.1 we have that

$$\int_{\mathbb{R}} \frac{d\tau_1}{(\sigma_1)^{2\beta} (\sigma_2)^{2\beta}} \leq \frac{C}{(\tau + |\xi_1|^3 + |\xi_1| + |\xi_2|^3 + |\xi_2|)^{2\beta}}.$$
Thus, from the Cauchy-Schwarz inequality we have that

\[ |Z(f, g, h)|^2 \leq \|h\|^2_{L^2_{\xi, \tau}} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(\xi_2, \tau_2)g(\xi_1, \tau_1)|^2 \, d\xi_1 \, d\tau_1 \right) \left( \int_{\mathbb{R}^2} \frac{|\xi|^2 \, d\xi_1 \, d\tau_1}{(\sigma + |\xi_1|^3 + |\xi_1| + |\xi_2|^3 + |\xi_2|)^{2\alpha}} \right) \, d\xi \, d\tau \]

\[ \leq C \|h\|^2_{L^2_{\xi, \tau}} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(\xi_2, \tau_2)g(\xi_1, \tau_1)|^2 \, d\xi_1 \, d\tau_1 \right) \times \left( \frac{|\xi|^2}{(\sigma + |\xi_1|^3 + |\xi_1| + |\xi_2|^3 + |\xi_2|)^{2\alpha}} \right) \, d\xi \, d\tau \]

\[ \leq C \|f\|^2_{L^2_{\xi, \tau}} \|g\|^2_{L^2_{\xi, \tau}} \|h\|^2_{L^2_{\xi, \tau}} \times \left\| \frac{|\xi|^2}{(\sigma + |\xi_1|^3 + |\xi_1| + |\xi_2|^3 + |\xi_2|)^{2\alpha}} \int_{\mathbb{R}^2} \frac{d\xi_1}{(\sigma + |\xi_1|^3 + |\xi_1| + |\xi - \xi_1|^3 + |\xi - \xi_1|)^{2\beta}} \right\|_{L^2_{\xi, \tau}}. \]

Then, for \( \beta > 1/2 \) and \( \alpha \geq 1/4 \) we will prove that there exists \( C > 0 \) such that

\[ G(\xi, \tau) = \frac{|\xi|^2}{(\sigma + |\xi_1|^3 + |\xi_1| + |\xi - \xi_1|^3 + |\xi - \xi_1|)^{2\alpha}} \leq C. \]

For \( \xi > 0 \) we split \( \mathbb{R} \) into three parts

\[ A_1 = (-\infty, 0], \ A_2 = [0, \xi], \ A_3 = [\xi, +\infty). \]

Then we see that

\[ \int_{A_1} \frac{d\xi_1}{(\tau + |\xi_1|^3 + |\xi_1| + |\xi - \xi_1|^3 + |\xi - \xi_1|)^{2\beta}} \leq \int_{\mathbb{R}} \frac{d\xi_1}{(2\xi_1^3 + 2\xi_1 - \tau - \xi^3 + 3\xi^2\xi_1 - 3\xi\xi_1^2 - \xi)^{2\beta}} = I_1. \]

Using the change of variable

\[ \mu = 2\xi_1^3 + 2\xi_1 - \tau - \xi^3 + 3\xi^2\xi_1 - 3\xi\xi_1^2 - \xi, \ d\mu = (6\xi_1^2 - 6\xi\xi_1 + 3\xi^2 + 2) \, d\xi_1 \]

and that

\[ 6\xi_1^2 - 6\xi\xi_1 + 3\xi^2 + 2 \geq \xi^2 \]

we have that

\[ \frac{|\xi|^2}{(\tau + |\xi_1|^3 + |\xi_1|)^{2\alpha}} I_1 \leq \frac{1}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \int_{\mathbb{R}} \frac{d\mu}{(\mu)^{2\beta}} \leq C. \]

Remark 1. Following the calculations in previous argument, for \( \xi < 0 \) and \( B_1 = (-\infty, \xi) \) we see that

\[ \frac{|\xi|^2}{(\tau + |\xi_1|^3 + |\xi_1|)^{2\alpha}} \int_{B_1} \frac{d\xi_1}{(\tau + |\xi_1|^3 + |\xi_1| + |\xi - \xi_1|^3 + |\xi - \xi_1|)^{2\beta}} \leq \frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \int_{\mathbb{R}} \frac{d\xi_1}{(2\xi_1^3 + 2\xi_1 - \tau - \xi^3 + 3\xi^2\xi_1 - 3\xi\xi_1^2 - \xi)^{2\beta}} \]

\[ = \frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} I_1 \leq C. \]
Next, note that
\[
\int_{A_2} \frac{d\xi_1}{(\tau + |\xi_1|^3 + |\xi_1| + |\xi - \xi_1|^3 + |\xi - \xi_1|)^{2\beta}} \leq \int_{\mathbb{R}} \frac{d\mu}{(\tau + \xi^3 - 3\xi^2\xi_1 + 3\xi\xi_1^2 + \xi^2\xi_1) + |\xi - \xi_1|^{2\beta}} = I_2. \tag{15}
\]

We make the change of variable
\[
\mu = \tau + \xi^3 - 3\xi^2\xi_1 + 3\xi\xi_1^2 + \xi, \quad d\mu = 3\xi(2\xi_1 - \xi)d\xi_1, \quad \xi_1 = \frac{1}{2} \left[ \xi \pm \sqrt{4\mu - \xi^3 - 4\tau - 4\xi} \right].
\]

Then, using that \(2\alpha \geq 1/2\), inequality (13) in Lemma 3.1 and
\[
|3\xi(2\xi_1 - \xi)| = \sqrt{3\xi^2} \sqrt{4\mu - \xi^3 - 4\tau - 4\xi}, \quad d\xi_1 = \frac{d\mu}{\sqrt{3\xi^2} \sqrt{4\mu - \xi^3 - 4\tau - 4\xi}}
\]
and also that
\[
\frac{3\xi^3}{4} \leq (\tau + \xi^3 + |\xi|) \left( \tau + \xi + \frac{\xi^3}{4} \right)
\]
we obtain that
\[
\frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} I_2 \leq \frac{\xi^{3/2}}{(\tau + \xi^3 + \xi)^{2\alpha}} \int_{\mathbb{R}} \frac{d\mu}{\sqrt{4\mu - \xi^3 - 4\tau - 4\xi} (\mu)^{2\beta}} \leq \frac{\xi^{3/2}}{(\tau + \xi^3 + \xi)^{2\alpha}} \int_{\mathbb{R}} \frac{d\mu}{\sqrt{\mu - \left( \tau + \xi + \frac{\xi^3}{4} \right)} (\mu)^{2\beta}} \leq \frac{C\xi^{3/2}}{(\tau + \xi^3 + \xi)^{2\alpha} \left( \tau + \xi + \frac{\xi^3}{4} \right)^{1/2}} \leq C.
\]

We see that
\[
\int_{A_3} \frac{d\xi_1}{(\tau + |\xi_1|^3 + |\xi_1| + |\xi - \xi_1|^3 + |\xi - \xi_1|)^{2\beta}} \leq \int_{\mathbb{R}} \frac{d\xi_1}{(\tau + 2\xi_1^3 + 2\xi_1 - \xi^3 + 3\xi^2\xi_1 - 3\xi\xi_1^2 - \xi) + |\xi - \xi_1|^{2\beta}} = I_3.
\]

Using the change of variable
\[
\mu = \tau + 2\xi_1^3 + 2\xi_1 - \xi^3 + 3\xi^2\xi_1 - 3\xi\xi_1^2 - \xi, \quad d\mu = (6\xi_1^2 - 6\xi\xi_1 + 3\xi^2 + 2)d\xi_1
\]
we have that
\[
\frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} I_3 \leq \frac{1}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \int_{\mathbb{R}} \frac{d\mu}{(\mu)^{2\beta}} \leq C.
\]

**Remark 2.** Following the previous calculations, for \(\xi < 0\) and \(B_3 = [0, +\infty)\) we see that
\[
\frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \int_{B_3} \frac{d\xi_1}{(\tau + |\xi_1|^3 + |\xi_1| + |\xi - \xi_1|^3 + |\xi - \xi_1|)^{2\beta}} \leq \frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} I_3 \leq C.
\]
Now, from Remark 1 and Remark 2, for $\xi < 0$ we only consider $B_2 = [\xi, 0]$. Then we have that
\[
\int_{B_2} \frac{d\xi_1}{(\tau + |\xi|^3 + |\xi_1|^3 + |\xi - \xi_1|^3)^{2\beta}} \leq \int_{\mathbb{R}} \frac{d\xi_1}{(\tau + \xi^3 - 3\xi^2\xi_1 + 3\xi\xi_1^2 + \xi_1^3)^{2\beta}}.
\]
Using the change of variable $\mu = \tau - \xi^3 + 3\xi^2\xi_1 - 3\xi\xi_1^2 - \xi$ and $d\mu = 3\xi(\xi - 2\xi_1)d\xi_1$ we obtain that
\[
\xi_1 = \frac{1}{2} \left[ \frac{\xi}{3} \right] \left[ \frac{\xi}{3} - 4\xi - 4\mu \right] \right] \int_{B_3} \frac{d\xi}{(\tau + |\xi|^3 + |\xi_1|^3 + |\xi - \xi_1|^3)^{2\beta}} \leq \int_{\mathbb{R}} \frac{d\mu}{(4\mu + 4\xi + \xi^3 - 4\tau)^{2\beta}} \leq \int_{\mathbb{R}} \frac{d\mu}{\mu - \left( \tau - \xi - \xi^3 \right)^{2\beta}} \leq \frac{C|\xi|^{3/2}}{(\tau - \xi^3 - \xi^2)^{2\alpha}} \left( \tau - \xi - \xi^3 \right)^{1/2} \leq C.
\]
Therefore, for $\beta > 1/2$, $a \geq 1/4$ and $\sigma, \sigma_1, \sigma_3$ as in the case (C1) we have that there exists $C > 0$ such that
\[
|Z(f, g, h)| \leq C\|f\|_{L^2_{\xi}} \|g\|_{L^2_{\xi}} \|h\|_{L^2_{\xi}}.
\]
Now, we consider $\sigma, \sigma_1, \sigma_2$ as in the case (C3). From inequality (12) in Lemma 3.1 we see that
\[
|Z(f, g, h)|^2 \leq C\|f\|_{L^2_{\xi}}^2 \|g\|_{L^2_{\xi}}^2 \|h\|_{L^2_{\xi}}^2 \|H\|_{L^\infty_{\tau}},
\]
where
\[
H(\xi, \tau) = \frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi_1|^3)^{2\alpha}} \int_{B_3} \frac{d\xi_1}{(\tau - |\xi|^3 - |\xi_1|^3 - |\xi - \xi_1|^3)^{2\beta}}.
\]
Following the previous argument, for $\xi > 0$ we split $\mathbb{R}$ again into the intervals
\[
A_1 = (-\infty, 0], \quad A_2 = [0, \xi], \quad A_3 = [\xi, +\infty).
\]
First we see that
\[
\int_{A_1} \frac{d\xi_1}{(|\xi_1|^3 + |\xi_1| - \tau - |\xi - \xi_1|^3 - |\xi_1|^3)^{2\beta}} \leq \int_{\mathbb{R}} \frac{d\xi_1}{(\tau + \xi^3 - 3\xi^2\xi_1 + 3\xi\xi_1^2 + \xi_1^3)^{2\beta}} = I_2,
\]
where $I_2$ is defined in (15). Therefore
\[
\frac{|\xi|^2}{\langle \tau + |\xi|^3 + |\xi| \rangle^{2\alpha}} I_2 \leq C.
\]
Now, note that
\[
\int_{A_2} \frac{d\xi_1}{\langle |\xi_1|^3 + |\xi_1| - \tau - |\xi - \xi_1|^3 - |\xi - \xi_1| \rangle^{2\beta}} \leq \int_{R} \frac{d\xi_1}{\langle 2\xi_1^3 + 2\xi_1 - \tau - \xi^3 + 3\xi_1^2 \xi_1 - 3\xi \xi_1^2 - \xi \rangle^{2\beta}} = I_1,
\]
so that
\[
\frac{|\xi|^2}{\langle \tau + |\xi|^3 + |\xi| \rangle^{2\alpha}} I_1 \leq C.
\]
Next, we see that
\[
\int_{A_3} \frac{d\xi_1}{\langle |\xi_1|^3 + |\xi_1| - \tau - |\xi - \xi_1|^3 - |\xi - \xi_1| \rangle^{2\beta}} \leq \int_{R} \frac{d\xi_1}{\langle \xi^3 - \tau - 3\xi^2 \xi_1 + 3\xi \xi_1^2 + \xi \rangle^{2\beta}} = I_4.
\]
Using the change of variable $\mu = \xi^3 - \tau - 3\xi^2 \xi_1 + 3\xi \xi_1^2 + \xi$ and $d\mu = 3\xi(2\xi_1 - \xi)d\xi_1$ we have that
\[
\xi_1 = \frac{1}{2} \left[ \xi \pm \sqrt{\frac{4\tau - \xi^3 - 4\xi + 4\mu}{3\xi}} \right]
\]
and also that
\[
|3\xi(2\xi_1 - \xi)| = \sqrt{3\xi} \sqrt{4\mu + 4\tau - \xi^3 - 4\xi}, \quad d\xi_1 = \frac{d\mu}{\sqrt{3\xi} \sqrt{4\mu + 4\tau - \xi^3 - 4\xi}}.
\]
Then, using inequality (13) in Lemma 3.1 and that
\[
\left| \frac{5\xi^3}{4} \right| \leq \langle \tau + \xi^3 + \xi \rangle \langle \tau - \xi^3 - \xi \rangle = \langle \tau + \xi^3 + \xi \rangle \langle \tau - \xi^3 - \xi \rangle
\]
we obtain that
\[
\frac{|\xi|^2}{\langle \tau + |\xi|^3 + |\xi| \rangle^{2\alpha}} I_4 \leq \frac{\xi^{3/2}}{\langle \tau + \xi^3 + \xi \rangle^{2\alpha}} \int_{R} \frac{\mu^{3/2}}{\sqrt{4\mu + 4\tau - \xi^3 - 4\xi} (\mu)^{2\beta}} \frac{d\mu}{\sqrt{\mu - \left( -\tau + \xi + \frac{\xi^3}{\tau} \right) (\mu)^{2\beta}}}
\]
\[
\leq \frac{C\xi^{3/2}}{\langle \tau + \xi^3 + \xi \rangle^{2\alpha} \left( \tau - \xi^3 - \xi \right)^{1/2}} \leq C.
\]
Now, for $\xi < 0$ we consider $B_1 = (-\infty, \xi)$, $B_2 = [\xi, 0]$ and $B_3 = [0, +\infty]$. Then we see that
\[
\int_{B_1} \frac{d\xi_1}{\langle |\xi_1|^3 + |\xi_1| - \tau - |\xi - \xi_1|^3 - |\xi - \xi_1| \rangle^{2\beta}} \leq \int_{R} \frac{d\xi_1}{\langle \tau + \xi^3 - 3\xi^2 \xi_1 + 3\xi \xi_1^2 + \xi \rangle^{2\beta}} = I_2.
\]
Using the change of variable $\mu = \tau + \xi^3 - 3\xi^2\xi_1 + 3\xi\xi_1^2 + \xi$ and $d\mu = 3\xi(2\xi_1 - \xi)d\xi_1$
we obtain that

$$\xi_1 = \frac{1}{2} \left[ \xi \pm \sqrt{\frac{4\mu - 4\xi - 4\xi^3}{3\xi}} \right]$$

and also

$$|3\xi(2\xi_1 - \xi)| = \sqrt{3|\xi|\sqrt{4\tau + 4\xi + \xi^3} - 4\mu}, \quad d\xi_1 = \frac{d\mu}{\sqrt{3|\xi|\sqrt{4\tau + 4\xi + \xi^3} - 4\mu}}.$$ 

From inequality (13) in Lemma 3.1 and that

$$\int \frac{5\xi^3}{4} \leq \langle \tau - \xi^3 - \xi \rangle \langle \tau + \xi + \frac{\xi^3}{4} \rangle$$

we obtain that

$$\frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha} I_2 \leq \frac{|\xi|^3/2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha} \int_{\mathbb{R}} \sqrt{4\tau + 4\xi + \xi^3} - 4\mu (\mu)^{2\beta} \frac{d\mu}{(\tau + |\xi|^3 + |\xi|)^{2\alpha} \int_{\mathbb{R}} \sqrt{4\tau + 4\xi + \xi^3} - 4\mu (\mu)^{2\beta}} \leq \frac{C|\xi|^3/2}{(\tau - \xi^3 - \xi)^{2\alpha} \langle \tau + \xi + \frac{\xi^3}{4} \rangle^{1/2} \leq C}.$$ 

Next, note that

$$\int_{B_2} \frac{d\xi_1}{(\langle |\xi_1|^3 + |\xi_1| - \tau - |\xi - \xi_1|^3 - |\xi - \xi_1|) \leq \int_{\mathbb{R}} \frac{d\xi_1}{(\tau + 2\xi_1 + 2\xi_1 - \xi^3 + 3\xi^2\xi_1 - 3\xi\xi_1^2 - \xi^3)^{2\beta}} = I_3,$$

then

$$\frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} I_3 \leq C.$$ 

Now, note that

$$\int_{B_3} \frac{d\xi_1}{(\langle |\xi_1|^3 + |\xi_1| - \tau - |\xi - \xi_1|^3 - |\xi - \xi_1|) \leq \int_{\mathbb{R}} \frac{d\xi_1}{(\tau - \xi^3 - 3\xi^2\xi_1 + 3\xi\xi_1^2 + \xi) \leq I_4.}$$

Using the change of variable $\mu = \xi^3 - 3\xi^2\xi_1 + 3\xi\xi_1^2 + \xi$ and $d\mu = 3\xi(2\xi_1 - \xi)d\xi_1$
we have that

$$\xi_1 = \frac{1}{2} \left[ \xi \pm \sqrt{\frac{4\tau - 4\xi^3 - 4\xi + 4\mu}{3\xi}} \right]$$

and also

$$|3\xi(2\xi_1 - \xi)| = \sqrt{3|\xi|\sqrt{\xi^3 + \xi - 4\tau - 4\mu}}, \quad d\xi_1 = \frac{d\mu}{\sqrt{3|\xi|\sqrt{\xi^3 + \xi - 4\tau - 4\mu}}.$$ 

Thus, using inequality (13) in Lemma 3.1 and that

$$\frac{3\xi^3}{4} \leq \langle \tau - \xi^3 - \xi \rangle \langle \tau - \xi - \frac{\xi^3}{4} \rangle$$
we obtain that
\[
\frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} I_4 \leq \frac{|\xi|^{3/2}}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \int_{\mathbb{R}} \frac{d\mu}{\sqrt{\xi^3 + \xi - 4\tau - 4\mu(\mu)^{2\beta}}} \\
\leq \frac{|\xi|^{3/2}}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \int_{\mathbb{R}} \frac{d\mu}{\sqrt{(\xi - \tau + \frac{3\xi}{4}) - \mu(\mu)^{2\beta}}} \\
\leq \frac{C|\xi|^{3/2}}{(\tau - \xi^3 - \xi)^{2\alpha}(\tau - \xi - \frac{3\xi}{4})^{1/2}} \leq C.
\]

Therefore, for \(\sigma, \sigma_1, \sigma_2\) as in the case (C3) we have that there exists \(C > 0\) such that
\[
|Z(f, g, h)| \leq C\|f\|_{L^2_{\xi, \tau}} \|g\|_{L^2_{\xi, \tau}} \|h\|_{L^2_{\xi, \tau}}.
\]

Now, we consider \(\sigma, \sigma_1, \sigma_2\) as in the case (C4). Using the change of variable
\((\xi, \tau, \xi_1, \tau_1) \mapsto - (\xi, \tau, \xi_1, \tau_1)\) we have that
\[
|Z(f, g, h)|^2 \leq \|h\|_{L^2_{\xi, \tau}}^2 \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(\xi_2, \tau_2)g(\xi_1, \tau_1)|^2 d\xi_1 d\tau_1 \right) \\
\times \left( \int_{\mathbb{R}^2} \frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \left( \tau_1 - |\xi_1|^3 - |\xi_1| \right)^\beta (\tau_2 + |\xi_2|^3 + |\xi_2|)^\beta \right) d\xi d\tau \\
\leq C\|h\|_{L^2_{\xi, \tau}}^2 \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(\xi_2, \tau_2)g(\xi_1, \tau_1)|^2 d\xi_1 d\tau_1 \right) \\
\times \left( \frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \int_{\mathbb{R}} \left( |\xi_1|^3 + |\xi_1| - \tau - |\xi_2|^3 - |\xi_2| \right)^{2\beta} \right) d\xi_1 d\tau_1 \\
\leq C\|f\|_{L^2_{\xi, \tau}}^2 \|g\|_{L^2_{\xi, \tau}}^2 \|h\|_{L^2_{\xi, \tau}}^2 \|H\|_{L^2_{\xi, \tau}}^2,
\]
then, the proof is reduced to the case (C3). Similarly, using the change of variable
\((\xi, \tau, \xi_1, \tau_1) \mapsto - (\xi, \tau, \xi_1, \tau_1)\), we see that the proof with \(\sigma, \sigma_1, \sigma_2\) as in the cases (C5) and (C6) can be reduced, respectively, to the cases (C2) and (C1).

Finally, we consider \(\sigma, \sigma_1, \sigma_2\) as in the case (C2). Making the change of variable
\(\tau_2 = \tau - \tau_1, \xi_2 = \xi - \xi_1\) and then \((\xi, \tau, \xi_2, \tau_2) \mapsto - (\xi, \tau, \xi_2, \tau_2)\) we obtain that
\[
|Z(f, g, h)|^2 \\
\leq \|h\|_{L^2_{\xi, \tau}}^2 \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(\xi_2, \tau_2)g(\xi_1, \tau_1)|^2 d\xi_2 d\tau_2 \right) \\
\times \left( \int_{\mathbb{R}^2} \frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} (\tau - \tau_2 + |\xi - \xi_2|^3 + |\xi - \xi_2|)^\beta \right) d\xi d\tau_2 \\
\leq C\|h\|_{L^2_{\xi, \tau}}^2 \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(\xi_2, \tau_2)g(\xi_1, \tau_1)|^2 d\xi_2 d\tau_2 \right) \\
\times \left( \frac{|\xi|^2}{(\tau + |\xi|^3 + |\xi|)^{2\alpha}} \int_{\mathbb{R}} \left( |\xi_2|^3 + |\xi_2| - \tau - |\xi - \xi_2|^3 - |\xi - \xi_2| \right)^{2\beta} \right) d\xi_2 d\tau_2 \\
\leq C\|f\|_{L^2_{\xi, \tau}}^2 \|g\|_{L^2_{\xi, \tau}}^2 \|h\|_{L^2_{\xi, \tau}}^2 \|H\|_{L^2_{\xi, \tau}}^2,
\]
and the proof can be reduced to the case (C3).
Now, note that
\[
\|\langle \partial_x \Phi \rangle (\partial_x \Phi_1) \|_{Y^{\alpha+1,-\alpha}} = \sup_{\|h\|_{L^2_{\xi,\tau}}} \left| \int_{\mathbb{R}^4} \xi\langle \xi \rangle^{\alpha} \langle |\tau| - \varphi(\xi) \rangle^{-\alpha} \langle \xi - \xi_1 \rangle \bar{\Phi}(\xi - \xi_1, \tau - \tau_1) \xi_1 \bar{\Phi}_1(\xi_1, \tau_1) \times h(\xi, \tau) \, d\xi d\tau d\xi_1 d\tau_1 \right|.
\]
Then, by letting
\[
f(\xi, \tau) = \langle |\tau| - \varphi(\xi) \rangle^\beta \langle \xi \rangle^s \bar{\Phi}(\xi, \tau), \quad f_1(\xi, \tau) = \langle |\tau| - \varphi(\xi) \rangle^\beta \langle \xi \rangle^s \bar{\Phi}_1(\xi, \tau)
\]
we have that (ii) is equivalent to
\[
|K(f, f_1, h)| \leq C\|f\|_{L^2_{\xi,\tau}} \|f_1\|_{L^2_{\xi,\tau}} \|h\|_{L^2_{\xi,\tau}},
\]
where
\[
K(f, f_1, h) = \int_{\mathbb{R}^4} \xi\langle \xi \rangle^{s} f(\xi - \xi_1, \tau - \tau_1) f_1(\xi_1, \tau_1) h(\xi, \tau) \, d\xi d\tau d\xi_1 d\tau_1.
\]
and then the proof of (16) is analogous to the proof of (14).
\[ \square \]

**Corollary 1.** Let \( s \geq 0 \) and \( \alpha, \beta \in \mathbb{R} \) given in Lemma 3.2. Then for \( s' > s \) we have that
\[
(i) \|\partial_x (\eta \partial_x \Phi)\|_{X_{s',-\alpha}} \leq C_4 (\|\eta\|_{X_{s',\alpha}} \|\Phi\|_{Y^{\alpha+1,\beta}} + \|\eta\|_{X_{s,\beta}} \|\Phi\|_{Y^{\alpha+1,\beta}}),
\]
\[
(ii) \|\partial_x (\eta \partial_x \Phi)\|_{X_{s',-\alpha}} \leq C_4 (\|\Phi\|_{Y^{\alpha+1,\beta}} \|\Phi_1\|_{Y^{\beta+1,\alpha}} + \|\Phi\|_{Y^{\alpha+1,\beta}} \|\Phi_1\|_{Y^{s',\beta+1}}).
\]

**Proof.** First we note that
\[
\|\partial_x (\eta \partial_x \Phi)\|_{X_{s',-\alpha}} = \sup_{\|h\|_{L^2_{\xi,\tau}}} \left| \int_{\mathbb{R}^4} \xi\langle \xi \rangle^{s'} \langle |\tau| - \varphi(\xi) \rangle^{-\alpha} \eta(\xi - \xi_1, \tau - \tau_1) \xi_1 \bar{\Phi}(\xi_1, \tau_1) h(\xi, \tau) \, d\xi d\tau d\xi_1 d\tau_1 \right|.
\]
Since \( s' > s \), we have the inequality
\[
\langle \xi \rangle^{s'} \leq \langle \xi \rangle^{s} \langle \xi - \xi_1 \rangle^{s'-s} + \langle \xi \rangle^{s} \langle \xi - \xi_1 \rangle^{s'-s}.
\]
Thus,
\[
\int_{\mathbb{R}^4} \xi\langle \xi \rangle^{s} \langle |\tau| - \varphi(\xi) \rangle^{-\alpha} \eta(\xi - \xi_1, \tau - \tau_1) \xi_1 \bar{\Phi}(\xi_1, \tau_1) h(\xi, \tau) \, d\xi d\tau d\xi_1 d\tau_1
\]
\[
\leq \int_{\mathbb{R}^4} \xi\langle \xi \rangle^{s} \langle |\tau| - \varphi(\xi) \rangle^{\alpha} \langle |\tau| - \varphi(\xi_1) \rangle^{\beta} \langle |\tau_1| - \varphi(\xi - \xi_1) \rangle^{\beta} \langle |\tau_1| - \varphi(\xi - \xi_1) \rangle^{\beta} h(\xi, \tau) \, d\xi d\tau d\xi_1 d\tau_1
\]
\[
+ \int_{\mathbb{R}^4} \xi\langle \xi \rangle^{s} \langle |\tau| - \varphi(\xi) \rangle^{\alpha} \langle |\tau_1| - \varphi(\xi_1) \rangle^{\beta} \langle |\tau_1| - \varphi(\xi - \xi_1) \rangle^{\beta} \langle |\tau_1| - \varphi(\xi - \xi_1) \rangle^{\beta} h(\xi, \tau) \, d\xi d\tau d\xi_1 d\tau_1
\]
\[
= W_1(f_1, f_2, h) + W_2(g_1, g_2, h),
\]
where
\[
f_1(\xi, \tau) = \langle |\tau| - \varphi(\xi) \rangle^\beta \langle \xi \rangle^{s} \eta(\xi, \tau), \quad f_2(\xi, \tau) = \langle |\tau| - \varphi(\xi) \rangle^\beta \langle \xi \rangle^{s} \bar{\Phi}(\xi, \tau)
\]
and
\[
g_1(\xi, \tau) = \langle |\tau| - \varphi(\xi) \rangle^\beta \langle \xi \rangle^{s'} \eta(\xi, \tau), \quad g_2(\xi, \tau) = \langle |\tau| - \varphi(\xi) \rangle^\beta \langle \xi \rangle^{s'} \bar{\Phi}(\xi, \tau).
\]
Notice that the inequality (i) is equivalent to
\[
|W_1(f_1, f_2, h)| + |W_2(g_1, g_2, h)| \leq C(\|f_1\|_{L^2_{\xi,\tau}} \|f_2\|_{L^2_{\xi,\tau}} + \|g_1\|_{L^2_{\xi,\tau}} \|g_2\|_{L^2_{\xi,\tau}}) \|h\|_{L^2_{\xi,\tau}}.
\]
We will show that the correspondence \( (\partial_t \Phi)(\partial_x \Phi) \|_{Y^{s+1}, -\alpha} \)

\[
= \sup_{\|h\|_{L^2_{\xi, r}} = 1} \int \xi (\xi)^{\gamma} (|\tau| - \phi(\xi))^{-\alpha} (\xi - \xi_1) \tilde{\Phi}(\xi - \xi_1, \tau - \tau_1) \tilde{\Phi}_1(\xi, \tau) \]

\[
\times h(\xi, \tau) d\xi d\tau d\xi_1 d\tau_1.
\]

Then we have that

\[
\int_{\mathbb{R}^4} |\xi| (\xi)^{\gamma} (|\tau| - \phi(\xi))^{-\alpha} |\xi - \xi_1| \|\tilde{\Phi}(\xi - \xi_1, \tau - \tau_1)\|_{L^2_{\xi, r}}\|\tilde{\Phi}_1(\xi, \tau)\|_{L^2_{\xi, r}} d\xi d\tau d\xi_1 d\tau_1
\]

\[
\leq \int_{\mathbb{R}^4} |\xi| \|g_2(\xi - \xi_1, \tau - \tau_1) h_1(\xi, \tau)\|_{L^2_{\xi, r}} d\xi d\tau d\xi_1 d\tau_1
\]

\[
+ \int_{\mathbb{R}^4} |\xi| f_2(\xi - \xi_1, \tau - \tau_1) h_2(\xi, \tau)\|_{L^2_{\xi, r}} d\xi d\tau d\xi_1 d\tau_1
\]

\[
= W_3(g_2, h_1, h) + W_4(f_2, h_2, h),
\]

where

\[
h_1(\xi, \tau) = (|\tau| - \phi(\xi))\beta (\xi, \tau) \|	ilde{\Phi}(\xi, \tau)\|_{L^2_{\xi, r}}\|\tilde{\Phi}_1(\xi, \tau)\|_{L^2_{\xi, r}}
\]

We note that the inequality (ii) is equivalent to

\[
|W_3(g_2, h_1, h)| + |W_4(f_2, h_2, h)| \leq C(\|g_2\|_{L^2_{\xi, r}}\|h_1\|_{L^2_{\xi, r}} + \|f_2\|_{L^2_{\xi, r}}\|h_2\|_{L^2_{\xi, r}})\|h\|_{L^2_{\xi, r}}.
\]

Therefore, using the argument in the proof of Lemma 3.2 we obtain the desired inequality.

\[\square\]

4. Proof of Theorem 1.1. For \((\eta_0, \Phi_0) \in H^s \times V^{s+1}\) and \(0 < T < 1\) we consider the operator \(\Gamma = (\Gamma_1, \Gamma_2)\) where

\[
\Gamma_1(\eta, \Phi)(t) = \psi(t) S_1(t) (\eta_0, \Phi_0) - \psi_T(t) \int_0^t S_1(t - t') \left( \partial_t (\eta \partial_x \Phi), \frac{1}{2} (\partial_x \Phi)^2 \right)(t') dt'
\]

and

\[
\Gamma_2(\eta, \Phi)(t) = \psi(t) S_2(t) (\eta_0, \Phi_0) - \psi_T(t) \int_0^t S_2(t - t') \left( \partial_t (\eta \partial_x \Phi), \frac{1}{2} (\partial_x \Phi)^2 \right)(t') dt'.
\]

Let \(Z_M\) the closed ball of radius \(M\) centered at the origin in \(X^{s, \beta} \times Y^{s+1, \beta}\),

\[
Z_M = \{ (\eta, \Phi) \in X^{s, \beta} \times Y^{s+1, \beta} : \| (\eta, \Phi) \|_{X^{s, \beta} \times Y^{s+1, \beta}} \leq M \}.
\]

We will show that the correspondence \((\eta, \Phi) \mapsto \Gamma(\eta, \Phi)\) maps \(Z_M\) into itself and defines a contraction if \(M\) and \(T\) are well chosen. In fact, let \(\alpha \in [1/4, 1/2]\) such that \(\beta \leq 1 - \alpha\) and \(\delta = 1 - (\beta + \alpha)\). Then using Lemma 2.1, Lemma 2.2 with \(\beta' = -\alpha\) and Lemma 3.2 we have that

\[
\| \Gamma_1(\eta, \Phi) \|_{X^{s, \beta}}
\]

\[
\leq C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y^{s+1}} + C_2 T^\delta \left( \| \partial_x (\eta \partial_x \Phi) \|_{X^{s, -\alpha}} + \| (\partial_x \Phi)^2 \|_{Y^{s+1, -\alpha}} \right)
\]

\[
\leq C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y^{s+1}} + C_2 C_3 T^\delta \left( \| \eta \|_{X^{s, \beta}} \Phi \|_{Y^{s+1, \beta}} + \| \Phi \|_{Y^{s+1, \beta}}^2 \right)
\]

\[
\leq C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y^{s+1}} + C_2 C_3 T^\delta \| (\eta, \Phi) \|_{X^{s, \beta} \times Y^{s+1, \beta}}^2.
\]
and also that
\[
\| \Gamma_2(\eta, \Phi) \|_{Y^{s+1, \beta}} \\
\leq C_1 \| (\eta_0, \Phi_0) \|_{H^{s, \beta} Y^{s+1, \beta}} + C_2 T^\delta \left( \| \partial_x (\eta \partial_x \Phi) \|_{X^{s, -\alpha}} + \| (\partial_x \Phi)^2 \|_{Y^{s+1, -\alpha}} \right) \\
\leq C_1 \| (\eta_0, \Phi_0) \|_{H^{s, \beta} Y^{s+1, \beta}} + C_2 C_3 T^{\delta} \| (\eta, \Phi) \|_{X^{s, \beta} Y^{s+1, \beta} X^{s, \beta} Y^{s+1, \beta}},
\]
so that
\[
\| \Gamma(\eta, \Phi) \|_{X^{s, \beta} Y^{s+1, \beta}} \leq C_1 \| (\eta_0, \Phi_0) \|_{H^{s, \beta} Y^{s+1, \beta}} + C_2 C_3 T^{\delta} \| (\eta, \Phi) \|_{X^{s, \beta} Y^{s+1, \beta}}. \tag{17}
\]
Choosing \( M = 2C_1 \| (\eta_0, \Phi_0) \|_{H^{s, \beta} Y^{s+1, \beta}} \) and \( 0 < T < 1 \) such that
\[
K_1 = 4C_1 C_2 C_3 \| (\eta_0, \Phi_0) \|_{H^{s, \beta} Y^{s+1, \beta}} T^{\delta} < 1,
\]
we obtain that
\[
\| \Gamma(\eta, \Phi) \|_{X^{s, \beta} Y^{s+1, \beta}} \leq C_1 \| (\eta_0, \Phi_0) \|_{H^{s, \beta} Y^{s+1, \beta}} (1 + 4C_1 C_2 C_3 \| (\eta_0, \Phi_0) \|_{H^{s, \beta} Y^{s+1, \beta}} T^{\delta}) \\
\leq 2C_1 \| (\eta_0, \Phi_0) \|_{H^{s, \beta} Y^{s+1, \beta}} = M
\]
and that \( \Gamma \) maps \( Z_M \) to itself. Now, let us prove that \( \Gamma \) is a contraction. In fact, if \( (\eta, \Phi), (\eta_1, \Phi_1) \in Z_M \), using Lemma 2.2 and Lemma 3.2 we have that
\[
\| \Gamma_1(\eta, \Phi) - \Gamma_1(\eta_1, \Phi_1) \|_{X^{s, \beta}} \\
\leq C_2 T^\delta \left( \| \partial_x (\eta \partial_x \Phi - \eta_1 \partial_x \Phi_1) \|_{X^{s, -\alpha}} + \| (\partial_x \Phi)^2 - (\partial_x \Phi_1)^2 \|_{Y^{s+1, -\alpha}} \right) \\
\leq C_2 T^\delta \left( \| \partial_x (\eta \partial_x \Phi - \Phi_1) \|_{X^{s, -\alpha}} + \| (\eta - \eta_1) \|_{X^{s, \beta}} \| \partial_x \Phi_1 \|_{Y^{s+1, -\alpha}} \\
+ \| \partial_x (\Phi - \Phi_1) \partial_x (\Phi + \Phi_1) \|_{Y^{s+1, -\alpha}} \right) \\
\leq C_2 C_3 T^{\delta} \left( \| \eta \|_{X^{s, \beta}} \| \Phi - \Phi_1 \|_{Y^{s+1, \beta}} + \| \eta - \eta_1 \|_{X^{s, \beta}} \| \Phi_1 \|_{Y^{s+1, \beta}} \\
+ \| \Phi - \Phi_1 \|_{Y^{s+1, \beta}} \| \Phi + \Phi_1 \|_{Y^{s+1, \beta}} \right) \\
\leq C_2 C_3 T^{\delta} \| (\eta, \Phi) \|_{X^{s, \beta} Y^{s+1, \beta}} \left( \| (\eta, \Phi) \|_{X^{s, \beta} Y^{s+1, \beta}} \\
+ \| (\eta_1, \Phi_1) \|_{X^{s, \beta} Y^{s+1, \beta}} \right).
\]
In a similar fashion we see that
\[
\| \Gamma_2(\eta, \Phi) - \Gamma_2(\eta_1, \Phi_1) \|_{Y^{s+1, \beta}} \\
\leq C_2 T^\delta \left( \| \partial_x (\eta \partial_x \Phi - \eta_1 \partial_x \Phi_1) \|_{X^{s, -\alpha}} + \| (\partial_x \Phi)^2 - (\partial_x \Phi_1)^2 \|_{Y^{s+1, -\alpha}} \right) \\
\leq C_2 T^\delta \left( \| \partial_x (\eta \partial_x \Phi - \Phi_1) \|_{X^{s, -\alpha}} + \| (\eta - \eta_1) \partial_x \Phi_1 \|_{X^{s, -\alpha}} \\
+ \| \partial_x (\Phi - \Phi_1) \partial_x (\Phi + \Phi_1) \|_{Y^{s+1, -\alpha}} \right) \\
\leq C_2 C_3 T^{\delta} \| (\eta, \Phi) \|_{X^{s, \beta} Y^{s+1, \beta}} \left( \| (\eta_1, \Phi_1) \|_{X^{s, \beta} Y^{s+1, \beta}} \right).
Then, we conclude
\[ ||\Gamma(\eta, \Phi) - \Gamma(\eta_1, \Phi_1)||_{X^{s,\beta} \times Y^{s+1,\beta}} \]
\[ \leq C_2 C_3 T^\delta ||(\eta, \Phi) - (\eta_1, \Phi_1)||_{X^{s,\beta} \times Y^{s+1,\beta}} \]
\[ + ||(\eta_1, \Phi_1)||_{X^{s,\beta} \times Y^{s+1,\beta}}. \] (19)

So, choosing \( 0 < T < 1 \) enough small so that (18) holds we obtain that
\[ ||\Gamma(\eta, \Phi) - \Gamma(\eta_1, \Phi_1)||_{X^{s,\beta} \times Y^{s+1,\beta}} \leq K_1 ||(\eta, \Phi) - (\eta_1, \Phi_1)||_{X^{s,\beta} \times Y^{s+1,\beta}} \]
and then \( \Gamma \) is a contraction in \( Z_M \). Thus, the contraction mapping principle guarantees the existence of a unique fixed point \((\eta, \Phi)\) of \( \Gamma \) in \( Z_M \), which is solution of the truncated integral problem (10). Now, if \((\eta_1, \Phi_1)\) is a restriction of \((\eta, \Phi)\) on \([0, T]\), then using Lemma 2.3 we have that
\[ \eta_1 \in C([0, T] : H^s) \cap X^{s,\beta}_T, \quad \Phi_1 \in C([0, T] : Y^{s+1}) \cap Y^{s+1,\beta}_T \]
and \((\eta_1, \Phi_1)\) is a solution of the integral problem (9) on \([0, T]\).

By the fixed point argument used we have the uniqueness of the solution of the truncated integral problem (10) in the set \( Z_M \). We will use an argument as in [2] to obtain the uniqueness of the integral problem (9) in the space \( X^{s,\beta}_T \times Y^{s+1,\beta}_T \).

Let \( T > 0 \) and \((\eta, \Phi) \in X^{s,\beta}_T \times Y^{s+1,\beta}_T \) be the solution of the truncated integral problem (10) obtained above and \((\eta_1, \Phi_1) \in X^{s,\beta}_T \times Y^{s+1,\beta}_T \) a solution of the integral problem (9) with the same initial data \((\eta_0, \Phi_0) \in H^s \times Y^{s+1}_T \). Fix an extension \((\eta_2, \Phi_2) \in X^{s,\beta}_T \times Y^{s+1,\beta}_T \) of \((\eta_1, \Phi_1)\), then, for some \( T^* < T < 1 \) to be fixed later, we have that
\[ \eta_2(t) = \psi(t)S_1(t)(\eta_0, \Phi_0) - \psi_T(t) \int_0^t S_1(t - t') \left( \frac{1}{2} \partial_x \partial_x (\eta_2 \partial_x \Phi_2) \right) (t') dt', \]
and
\[ \Phi_2(t) = \psi(t)S_2(t)(\eta_0, \Phi_0) - \psi_T(t) \int_0^t S_2(t - t') \left( \frac{1}{2} \partial_x \partial_x (\eta_2 \partial_x \Phi_2) \right) (t') dt', \]
for all \( t \in [0, T^*] \).

Now, by definition of \( X^{s,\beta}_T \times Y^{s+1,\beta}_T \) we have that for any \( \epsilon > 0 \), there exists \((\omega, \vartheta) \in X^{s,\beta}_T \times Y^{s+1,\beta}_T \) such that for all \( t \in [0, T^*] \),
\[ \omega(t) = \eta(t) - \eta_2(t), \quad \vartheta(t) = \Phi(t) - \Phi_2(t) \]
and
\[ ||\omega||_{X^{s,\beta}_T} \leq ||\eta - \eta_2||_{X^{s,\beta}_T} + \epsilon, \quad ||\vartheta||_{Y^{s+1,\beta}_T} \leq ||\Phi - \Phi_2||_{Y^{s+1,\beta}_T} + \epsilon. \] (20)

We define
\[ \omega_1(t) = -\psi_T(t) \int_0^t S_1(t - t') \left( \frac{1}{2} \partial_x \partial_x (\eta_2 \partial_x \vartheta) + \partial_x (\omega \partial_x \Phi_2) \right) (t') dt', \]
and
\[ \vartheta_1(t) = -\psi_T(t) \int_0^t S_2(t - t') \left( \frac{1}{2} \partial_x \partial_x (\eta_2 \partial_x \vartheta) + \partial_x (\omega \partial_x \Phi_2) \right) (t') dt'. \]
Thus, from Lemma 2.2 and Lemma 3.2 we obtain that
\[ \| \eta - \eta_2 \|_{X_{\beta}^\ast Y_{\beta + 1}} \leq \| \omega \|_{X_{\beta}^\ast Y_{\beta + 1}} \]
\[ \leq C_2 C_3 T^{s\delta} \| (\omega, \theta) \|_{X_{\beta}^\ast Y_{\beta + 1}} \{ \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}} + \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}} \}
\]
\[ \leq 2 C_2 C_3 N T^{s\delta} \| \omega, \theta \|_{X_{\beta}^\ast Y_{\beta + 1}} \]
where we assume that
\[ \max \{ \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}}, \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}} \} \leq N. \]

In a similar fashion we have that
\[ \| \Phi - \Phi_2 \|_{Y_{\beta + 1}} \]
\[ \leq \| \theta_1 \|_{Y_{\beta + 1}} \]
\[ \leq C_2 C_3 T^{s\delta} \| (\omega, \theta) \|_{X_{\beta}^\ast Y_{\beta + 1}} \{ \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}} + \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}} \}
\]
\[ \leq 2 C_2 C_3 N T^{s\delta} \| (\omega, \theta) \|_{X_{\beta}^\ast Y_{\beta + 1}}. \]

If we choose \( T^* > 0 \) such that
\[ 4 C_2 C_3 N T^{s\delta} \leq 1/2, \]
we obtain, using (20), (21) and (22), that
\[ \| \eta - \eta_2 \|_{X_{\beta}^\ast Y_{\beta + 1}} + \| \Phi - \Phi_2 \|_{Y_{\beta + 1}} \leq 4 C_2 C_3 N T^{s\delta} \| (\omega, \theta) \|_{X_{\beta}^\ast Y_{\beta + 1}} \]
\[ \leq \frac{1}{2} \left( \| \eta - \eta_2 \|_{X_{\beta}^\ast Y_{\beta + 1}} + \| \Phi - \Phi_2 \|_{Y_{\beta + 1}} + 2 \epsilon \right). \]

So, we see that
\[ \| \eta - \eta_2 \|_{X_{\beta}^\ast Y_{\beta + 1}} + \| \Phi - \Phi_2 \|_{Y_{\beta + 1}} \leq 2 \epsilon. \]

Therefore \( \eta = \eta_2 \) and \( \Phi = \Phi_2 \) on \([0, T^*]\). Now, since the argument does not depend on the initial data, we can iterate this process a finite number of times to extend the uniqueness result in the whole existence interval \([0, T]\).

Combining an identical argument to the one used in the existence proof with Lemma 2.3, one can easily show that the map data-solution is locally Lipschitz.

Finally, we will prove that if \( s' > s \) the above results hold with \( s' \) instead of \( s \) in the same interval \([0, T]\). For this, we define the space
\[ W = \{(\eta, \Phi) \in X_{\beta}^\ast Y_{\beta + 1} : \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}} + \lambda \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}} < \infty \} \]
with
\[ \lambda = \frac{\| (\eta_0, \Phi_0) \|_{H^s \times Y_{\beta + 1}}}{\| (\eta_0, \Phi_0) \|_{H_{\beta'} \times Y_{\beta' + 1}}} \]
and the closed ball
\[ W_{M'} = \{(\eta, \Phi) \in W : \| (\eta, \Phi) \|_{s'} \leq M'\} \]
with \( M' = 2 C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y_{\beta + 1}} \). From (17) we see that
\[ \| \Gamma(\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}} \leq C_1 \| (\eta_0, \Phi_0) \|_{H^s \times Y_{\beta + 1}} + C_2 C_3 T^{s\delta} \| (\eta, \Phi) \|_{X_{\beta}^\ast Y_{\beta + 1}}. \]
Using Lemma 2.1, Lemma 2.2 and Corollary 1, we have that
\[ \|\Gamma_1(\eta, \Phi)\|_{X^{s', \beta}} \]
\[ \leq C_1 \left( \|\eta_0, \Phi_0\|_{H^{s'} \times Y^{s'+1}} + C_2 C_4 T^\delta \left( \|\eta, \Phi\|_{X^{s, \beta} \times Y^{s'+1, \beta}} + \|\eta, \Phi\|_{X^{s, \beta} \times Y^{s'+1, \beta}} \right) \right) \]
\[ \leq \frac{1}{\lambda} \left( C_1 \|\eta_0, \Phi_0\|_{H^{s} \times Y^{s+1}} + C_2 C_4 T^\delta \|\eta, \Phi\|_{X^{s, \beta} \times Y^{s'+1, \beta}} \right) \]
and also that
\[ \|\Gamma_2(\eta, \Phi)\|_{Y^{s'} + 1, \beta} \]
\[ \leq C_1 \|\eta_0, \Phi_0\|_{H^{s'} \times Y^{s'+1}} + C_2 C_4 T^\delta \left( \|\eta, \Phi\|_{X^{s, \beta} \times Y^{s'+1, \beta}} + \|\eta, \Phi\|_{X^{s, \beta} \times Y^{s'+1, \beta}} \right) \]
\[ \leq \frac{1}{\lambda} \left( C_1 \|\eta_0, \Phi_0\|_{H^{s} \times Y^{s+1}} + C_2 C_4 T^\delta \|\eta, \Phi\|_{X^{s, \beta} \times Y^{s'+1, \beta}} \right) \]
Therefore
\[ \|\Gamma(\eta, \Phi)\|_{s'} \leq C_1 \|\eta_0, \Phi_0\|_{H^{s} \times Y^{s+1}} + C_2 (C_3 + C_4) T^\delta \|\eta, \Phi\|_{X^{s, \beta} \times Y^{s'+1, \beta}} \]
Choosing 0 < T < 1 such that
\[ K_2 = 4C_1 C_2 (C_3 + C_4) \|\eta_0, \Phi_0\|_{H^{s} \times Y^{s+1}} T^\delta < 1, \]
we have that
\[ \|\Gamma(\eta, \Phi)\|_{s'} \leq C_1 \|\eta_0, \Phi_0\|_{H^{s} \times Y^{s+1}} (1 + 4C_1 C_2 (C_3 + C_4)) \|\eta_0, \Phi_0\|_{H^{s} \times Y^{s+1}} T^\delta \]
\[ \leq 2C_1 \|\eta_0, \Phi_0\|_{H^{s} \times Y^{s+1}} = M'. \]
So, we obtain that \( \Gamma \) maps \( W_{M'} \) to itself. Now, if \((\eta, \Phi), (\eta_1, \Phi_1) \in W_{M'}\), using Corollary 1, we have that
\[ \|\Gamma_1(\eta, \Phi) - \Gamma_1(\eta_1, \Phi_1)\|_{X^{s', \beta}} \]
\[ \leq C_2 T^\delta \|\partial_x (\eta (\partial_x \Phi)) - \partial_x (\eta_1 (\partial_x \Phi_1))\|_{X^{s'-\alpha, 0}} + \| \partial_x \Phi \|^2 - \| \partial_x \Phi_1 \|^2 \|_{Y^{s'+1, -\alpha}} \]
\[ \leq \frac{C_2 C_4 T^\delta}{\lambda} \left( \|\eta, \Phi\|_{s'} + \|\eta_1, \Phi_1\|_{s'} \right) \|\eta, \Phi\|_{s'} - \|\eta_1, \Phi_1\|_{s'}. \]
In a similar fashion we have that
\[ \|\Gamma_2(\eta, \Phi) - \Gamma_2(\eta_1, \Phi_1)\|_{Y^{s'+1, \beta}} \]
\[ \leq C_2 T^\delta \|\partial_x (\eta (\partial_x \Phi)) - \partial_x (\eta_1 (\partial_x \Phi_1))\|_{X^{s'-\alpha, 0}} + \| \partial_x \Phi \|^2 - \| \partial_x \Phi_1 \|^2 \|_{Y^{s'+1, -\alpha}} \]
\[ \leq \frac{C_2 C_4 T^\delta}{\lambda} \left( \|\eta, \Phi\|_{s'} + \|\eta_1, \Phi_1\|_{s'} \right) \|\eta, \Phi\|_{s'} - \|\eta_1, \Phi_1\|_{s'}. \]
Thus, from (19) and previous estimates we obtain that
\[ \|\Gamma(\eta, \Phi) - \Gamma(\eta_1, \Phi_1)\|_{s'} \]
\[ = \|\Gamma(\eta, \Phi) - \Gamma(\eta_1, \Phi_1)\|_{X^{s, \beta} \times Y^{s'+1, \beta}} + \lambda \|\Gamma(\eta, \Phi) - \Gamma(\eta_1, \Phi_1)\|_{X^{s, \beta} \times Y^{s'+1, \beta}} \]
\[ \leq C_2 (C_3 + C_4) T^\delta \left( \|\eta, \Phi\|_{s'} + \|\eta_1, \Phi_1\|_{s'} \right) \|\eta, \Phi\|_{s'} - \|\eta_1, \Phi_1\|_{s'} \]
\[ \leq K_2 \left( \|\eta, \Phi\|_{s'} - \|\eta_1, \Phi_1\|_{s'} \right). \]
Then we have that \( \Gamma \) is a contraction and therefore there exists a solution with \( T = T(\|\eta_0, \Phi_0\|_{H^{s} \times Y^{s+1}}) \).

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