LINEARIZATION OF COHOMOLOGY-FREE VECTOR FIELDS

LIVIO FLAMINIO AND MIGUEL PATERNAIN

ABSTRACT. We study the cohomological equation for a smooth vector field on a compact manifold. We show that if the vector field is cohomology free, then it can be embedded continuously in a linear flow on an Abelian group.

1. INTRODUCTION

We consider a smooth flow $\{\phi^t\}_{t \in \mathbb{R}}$ generated by a vector field $X$ on a smooth connected compact manifold $M$.

A major problem arising in many different contexts of the theory of dynamical systems is to solve the cohomological equation for the flow $\{\phi^t\}_{t \in \mathbb{R}}$, the equation given by

\[ L_X h = f. \]

Here $L_X$ denotes the Lie derivative in the direction of the vector field $X$, $f$ is a given function and $h$ is the solution we seek.

To make sense of this problem it is of course necessary to impose some regularity conditions on the data $f$ as well as on the solution $h$. In low regularity we shall interpret the equation (1) in a weak sense.

We endow the space $C^\infty(M)$ with the $C^\infty$-topology and define the $C^\infty$-cohomology of the flow $\{\phi^t\}_{t \in \mathbb{R}}$ as the quotient vector space $C^\infty(M) / L_X(C^\infty(M))$; the reduced $C^\infty$-cohomology is instead the topological vector space $C^\infty(M) / L_X(C^\infty(M))$, where the closure is taken in the $C^\infty$-topology.

2000 Mathematics Subject Classification. Primary: 37-XX, 37C15, 37C40.

Key words and phrases. Cohomological Equations, Greenfield-Wallach and Katok conjectures.

Miguel Paternain was partially supported by an ANII-grant. Both authors were partially supported by the PREMER franco-uruguayan grant.
Restricting Katok’s definition, ([Kat01], [Kat03]), to the $C^\infty$ setting, we say that the flow $(\phi^t)_{t \in \mathbb{R}}$ is $C^\infty$-stable if its cohomology and reduced cohomology coincide, that is if the image of the Lie derivative operator $L_X$ is a closed subspace of $C^\infty(M)$. By Hahn-Banach’s Theorem this is equivalent to saying that, for every function $f$ belonging to the kernel of all $X$-invariant Schwartz distributions, the equation (1) admits a solution $h \in C^\infty$.

We remark that since a continuous flow on a compact manifold admits always an invariant measure, the reduced cohomology of a flow is at least one-dimensional.

Stability in the $C^\infty$ setting has been established for a variety of flows; we mention just a few: [FF03], [Mie07], [FF07], in the homogeneous setting, and [GK80a], [GK80b], [dlLMM85] for systems of dynamical origin. Furthermore stability for action of higher rank Abelian group has also been established in many exemplary cases (cf. for example [DK04], [DK05], [Dam07], [DK07], [Mie07]).

In all the cases studied so far, the flow cohomology is infinite dimensional, with one notable exception. In fact, linear flows on $d$-dimensional tori provide the classical and only known examples with one-dimensional reduced flow cohomology. It is well known that such a flow is $C^\infty$-stable if and only if the direction numbers $\alpha \in \mathbb{R}^d$ of the vector field $X$ is a Diophantine vector, that is, if there exist positive constants $C$ and $\tau$ such that

\[
|n \cdot \alpha| \geq C\|n\|^{-\tau}, \quad \forall \, n \in \mathbb{Z}^d \setminus \{0\}.
\]

When $\alpha$ is not Diophantine then there exists $f \in C^\infty(\mathbb{T}^d)$, with $\int f \, d\mu = 0$, for which equation (1) admits no solution in $L^2(\mathbb{T}^d, \mu)$ ([Kat01], [Kat03]) of even measurable ([Her04]); thus fast approximation by periodic flows provides one of mechanisms through which $C^\infty$-stability fails to hold.

A $C^\infty$-stable flow is called rigid or cohomology free if its flow cohomology is one dimensional; A. Katok ([Hur85], [Kat01], [Kat03]) suggested the following:

**Conjecture 1.** Let $\{\phi^t\}_{t \in \mathbb{R}}$ be a cohomology-free flow generated by a vector field $X$ on a compact connected manifold $M$. Then, up to diffeomorphism, the manifold $M$ is a torus and $X$ is a Diophantine vector field.
Chen and Chi [CC00] have essentially proved that Katok’s conjecture is equivalent to a conjecture formulated by Greenfield and Wallach in [GW73] and which states that a globally hypoelliptic vector field is a linear diophantine flow on the torus (see [For08] for a review of the relation between the two conjectures).

Progress towards this conjecture has been limited. A major advance has been made by F. and J. Rodriguez-Hertz [RHRH06] who showed that a cohomology-free flow on a manifold \( M \) is up to semi-conjugate to a Diophantine linear flow on a torus of dimension equal to the first Betti number of \( M \) (the Albanese variety of \( M \)); furthermore the semi-conjugacy is smooth. More substantial progress has made for low dimensional manifolds. In [LdS98] the analogous problem for diffeomorphisms is solved for tori of dimension four or less. Recently independent work of G. Forni [For08], A. Kocsard [Koc09] and S. Matsumoto [Mat09], have proved the Katok-Greenfield-Wallach conjecture when \( \text{dim} \ M \leq 3 \), using Taubes’ proof of Weinstein’s Conjecture [Tau07].

More recently Avila and Kocsard have recently announced in [AK10] that the reduced cohomology of every minimal circle diffeomorphism — hence of very minimal flow on the two-torus — is one-dimensional; from this it follows easily that, up to a diffeomorphism, the only \( C^\infty \)-stable minimal flows on the two-torus are the diophantine linear flows.

**Theorem 1.1.** Let \( \{ \phi^t \}_{t \in \mathbb{R}} \) be a cohomology-free flow generated by a vector field \( X \) on a compact connected manifold \( M \). Then there are a (possibly non-separated) topological Abelian group \( A \), a continuous homomorphism \( t \in \mathbb{R} \mapsto at \in A \), and a continuous injection \( l : M \to A \), such that \( l(\phi^t(y)) = l(y) + at \) for every \( y \in M \) and \( t \in \mathbb{R} \). Furthermore there is a continuous projection \( \pi \) of \( A \) onto the Albanese torus of \( M \) such that \( \pi \circ l \) is F. and J. Rodriguez-Hertz’ semi-conjugacy.

**Acknowledgments.** The authors would like to thank A. Verjowsky and the Instituto de Matemáticas de Cuernavaca for their hospitality. The second author is grateful to Armando Treibich for his invitation to the Université d’Artois and to Federico Rodriguez Hertz for explaining his work on this problem.
From now our standing hypothesis is that $(\phi^t)_{t \in \mathbb{R}}$ is a cohomology-free flow generated by a vector field $X$ on a compact connected manifold $M$.

Let $I$ denote the unit interval $[0, 1]$. With the term “curves”, we shall mean piecewise $C^1$-immersed parametrized curves $I \to M$, modulo the equivalence relation given by piecewise $C^1$-reparametrization. The initial point and end point of a curve $\gamma$ are the points $\alpha(\gamma) = \gamma(0)$ and $\omega(\gamma) = \gamma(1)$.

Two curves $\gamma_1$ and $\gamma_2$ can be concatenated if $\omega(\gamma_1) = \alpha(\gamma_2)$; with $\gamma = \gamma_1 \gamma_2$ we shall denote the usual concatenation $\gamma(t) = \gamma_1(2t)$ for $t \leq 1/2$ and $\gamma(t) = \gamma_2(2t - 1)$ for $t \geq 1/2$, $t \in I$.

For the sequel we restrict our consideration to the set $\Gamma$ of curves which are finite concatenations of unoriented flow segments and geodesic segments (for some fixed Riemannian metric on $M$) transverse to the flow. We endow $\Gamma$ with the topology of uniform convergence (say for the uniform speed parametrization of curves).

Curves in $\Gamma$ have the following regularity property:

**Lemma 2.1.** Two curves in $\Gamma$ meet a finite collections of intervals (which may reduce to points). Furthermore there is a compact set $K \subset \Gamma$ such that every two points of $M$ can be joined by an element of $K$.

**Proof.** The first statement is an immediate consequence of the choice of $\Gamma$: if two geodesic segments intersect, they do so on an interval; the same is true for flow segments. For the set $K$ we can choose the set paths which are concatenation of $N$ geodesic segments of length less than some positive $\epsilon_0$, where $N = \lfloor 2\epsilon_0^{-1} \text{diam}(M) \rfloor + 1$. \hfill $\Box$

For $x \in M$, we denote by $\Gamma_x$ the curves $\gamma \in \Gamma$ with $\alpha(\gamma) = x$, that is the curves starting at $x$. Finally we let $\Delta$ be the set of curves $\gamma \in \Gamma$ with $\alpha(\gamma) = \omega(\gamma)$, the set of closed loops in $\Gamma$.

**3. Currents**

Let $\Omega^1(M)$ denote the Fréchet space of $C^\infty$ differential one-forms with the $C^\infty$-topology and let $C^0(M)$ and $C^1(M)$ be the Fréchet spaces of de Rham currents on $M$ of degree zero and one, that is the dual
space of $\Omega^0(M) = \mathcal{C}^\infty(M)$ and the dual space of $\Omega^1(M)$, respectively. The spaces $\mathcal{C}^0(M)$ and $\mathcal{C}^1(M)$ are endowed with the vague topology.

To each parametrized curve $\gamma$ it corresponds an integration one-current $\tilde{\gamma}$ given by

$$\tilde{\gamma}(\eta) = \int_\gamma \eta, \quad \forall \eta \in \Omega^1(M).$$

Clearly the current $\tilde{\gamma}$ does not depend upon the choice of a parametrization of $\gamma$ so that the map $\gamma \in \Gamma \mapsto \tilde{\gamma} \in \mathcal{C}^1(M)$ is well defined and we set

$$\tilde{\Gamma} = \{ \tilde{\gamma} \in \mathcal{C}^1(M) \mid \gamma \in \Gamma \};$$

we also denote by $\tilde{\Gamma}_x$ and $\tilde{\Delta}$ the sets of currents images of curves in $\Gamma_x$ and loops in $\Delta$.

It is obvious from the definition that for any $\gamma_1$ and $\gamma_2$ in $\tilde{\Gamma}$ for which the concatenation is defined we have

$$\tilde{\gamma}_1 \gamma_2 = \tilde{\gamma}_1 + \tilde{\gamma}_2.$$

The following elementary propositions are stated here for further reference. We set $\mathcal{M} = \{ \delta_x \in \mathcal{C}^0(M) \mid x \in M \}$.

**Proposition 3.1.** The map $x \in M \mapsto \delta_x \in \mathcal{M}$ is continuous (in fact differentiable) and injective and hence a homeomorphism $M \approx \mathcal{M}$.

**Proposition 3.2.** For any $x \in M$ we have

$$(3) \quad \tilde{\Delta} = \{ T \in \tilde{\Gamma} \mid \partial T = 0 \} = \tilde{\Gamma}_x \cap \Delta.$$

Hence $\tilde{\Delta}$ is a closed set both in $\tilde{\Gamma}$ and in $\tilde{\Gamma}_x$. Furthermore $\tilde{\Gamma}$ and $\tilde{\Gamma}_x$ are both invariant by translation by $\tilde{\Delta}$.

**Proof.** Let $T = \tilde{\gamma}$ with $\gamma \in \Gamma$. Since $\partial T = \delta_{\omega(\gamma)} - \delta_{\alpha(\gamma)}$ we have $\partial T = 0$ iff $\alpha(\gamma) = \omega(\gamma)$, i.e. iff $\gamma \in \Delta$. This shows that $\Delta = \{ T \in \tilde{\Gamma} \mid \partial T = 0 \}$.

Suppose that $T = \tilde{\gamma}$ with $\gamma \in \Delta$. Letting $\delta \in \Gamma$ be a curve such that $\alpha(\delta) = x$ and $\omega(\delta) = \alpha(\gamma)$, we have $\gamma_1 = \delta \gamma \delta^{-1} \in \Delta \cap \Gamma_x$ and $\tilde{\gamma}_1 = \tilde{\gamma} = T$. Thus $T \in \tilde{\Gamma}_x \cap \Delta$ and $\tilde{\Delta} = \tilde{\Gamma}_x \cap \Delta$.

If $\gamma \in \Gamma_x$ and $\delta \in \Delta$ let $\gamma_1 \in \Gamma_x$ be a curve with $\omega(\gamma_1) = \alpha(\delta)$; then the curve $\gamma_1 \delta \gamma_1^{-1} \gamma$ belongs to $\Gamma_x$ and the current associated to it is equal to $\tilde{\gamma} + \tilde{\delta}$. This shows that $\tilde{\Gamma}_x$ (and hence $\tilde{\Gamma}$) is invariant by translation by $\tilde{\Delta}$. \qed
Remark 3.3. Observe that the above proposition implies that $\tilde{\Delta}$ is an additive subgroup of space of the currents $C^1(M)$. The Abelian group $A$ appearing in Theorem 1.1 will be defined as the quotient group $C^1(M)/\tilde{\Delta}$.

Proposition 3.4. For $x \in M$, let $\pi_x : C^1(M) \to C^0(M)$ be the continuous affine map defined by

$$\pi_x(T) = \partial T + \delta_x.$$ 

For $\gamma \in \Gamma_x$, we have $\pi_x(\gamma) = \delta_{\omega(\gamma)}$ and hence $\pi_x$ maps $\tilde{\Gamma}_x$ onto $\mathcal{M}$.

Furthermore, if $T_1, T_2 \in \tilde{\Gamma}_x$, we have $\pi_x(T_1) = \pi_x(T_2)$ if and only if $T_1 - T_2 \in \tilde{\Delta}$.

Proof. The first affirmation is obvious. Suppose that $\pi_x(T_1) = \pi_x(T_2)$ where $T_i = \gamma_i$, $\gamma_i \in \Gamma_x$. Then $\gamma^{-1}_1 \gamma_2 \in \Gamma$, thus $(T_1 - T_2) \in \tilde{\Gamma}$ and $\partial(T_1 - T_2) = 0$. By Proposition 3.2 we have $(T_1 - T_2) \in \tilde{\Delta}$.

Corollary 3.5. The map $\pi_x$ induces a homeomorphism

$$p_x : \tilde{\Gamma}_x/\tilde{\Delta} \to \mathcal{M}.$$ 

Hence

$$\tilde{\Gamma}_x/\tilde{\Delta} \approx \mathcal{M} \approx M.$$ 

Proof. The induced map $p_x$ is continuous for the quotient topology and injective by Proposition 3.4. Let $K$ be as in Lemma 2.1. The set of currents images of elements in $K \cap \Gamma_x$ is compact in the weak topology and surjects onto $\tilde{\Gamma}_x/\tilde{\Delta}$; hence $\tilde{\Gamma}_x/\tilde{\Delta}$ is compact. Since $\mathcal{M}$ is a Hausdorff space, the map $p_x$ is a homeomorphism.

4. Twisting the Embedding

The hypothesis of Theorem 1.1 imply that for every $\eta \in \Omega^1(M)$ there exist a $C^\infty$ function $h_\eta : M \to \mathbb{R}$ and a constant $c_\eta \in \mathbb{R}$ such that

$$L_X h_\eta = \eta(X) - c_\eta.$$ 

The function $h_\eta$ is only defined up to a constant, by the unique ergodicity of the flow $(\phi^t)$. We shall need the following observations.

Remark 4.1. If $\eta_1, \eta_2 \in \Omega^1(M)$ and $\eta_1(X) - \eta_2(X) = C$ is a constant function then $h_{\eta_1} - h_{\eta_2}$ is a constant and $c_{\eta_1} - c_{\eta_2} = C$. 
Proposition 4.2. Let $C^\infty_0(M)$ be the quotient space $C^\infty(M)$ modulo constants (which we can also identify to the space $C^\infty$ functions of $\mu$-average zero). Then the maps

$$\eta \in \Omega^1(M) \mapsto h_\eta \in C^\infty_0(M)$$

and

$$\eta \in \Omega^1(M) \mapsto dh_\eta \in \Omega^1(M)$$

are continuous if $C^\infty_0(M)$ and $\Omega^1(M)$ are endowed with the $C^\infty$ Fréchet topology.

Proof. The Lie derivative $L_X : C^\infty_0(M) \to C^\infty_0(M)$, is a continuous linear operator which, by hypothesis and by the unique ergodicity of $\mu$, is also bijective. Since $C^\infty_0(M)$ is a Fréchet space, the open mapping theorem implies that this map is an isomorphism. The map $\Omega^1(M) \to C^\infty(M), \eta \mapsto \eta(X)$, being clearly continuous, our claim follows. \qed

We shall use the family of functions $h_\eta$ to “twist” our homeomorphism $M \approx \tilde{\Gamma}_x/\tilde{\Delta}$.

Definition 4.3. We define the map $L : C^1(M) \to C^1(M)$

$$(5) \quad L(T)(\eta) = T(\eta) - \partial T(h_\eta) = T(\eta - dh_\eta)$$

(which is well defined since $h_\eta$ is defined up to a constant).

The following lemma is obvious.

Lemma 4.4. The map $L$ is a continuous linear operator (in fact a projection). The restriction of $L$ to $\tilde{\Delta}$ is the identity map of $\tilde{\Delta}$.

Let $L_x$ be the restriction of $L$ to $\tilde{\Gamma}_x$.

5. Injectivity of $L_x$

The aim of this section is to show the following proposition.

Proposition 5.1. The map $L_x : \tilde{\Gamma}_x \to C^1(M)$ is a continuous injection.

Definition 5.2. We say that a finite family $\{\gamma_1, \ldots, \gamma_m\} \subset \Gamma$ is broadly equivalent to a finite family $\{\beta_1, \ldots, \beta_n\} \subset \Gamma$, if

$$\sum_{i=1}^m \tilde{\gamma}_i = \sum_{i=1}^n \tilde{\beta}_i.$$
Clearly broad equivalence is an equivalence relation.

**Definition 5.3.** We say that a curve $\gamma \in \Gamma$ contains a retraced arc $r$ if $\gamma = arb r^{-1}c$, with $a, b, c$ and $r \in \Gamma$. Similarly, we say that two given curves $\gamma_1, \gamma_2, \in \Gamma$ contain a retraced arc $r$ if $\gamma_1 = arb$ and $\gamma_2 = cr^{-1}d$ with $a, b, c, d$ and $r \in \Gamma$.

From a set of curves in $\Gamma$ we can, in an iterative way, obtain a new set of curves, broadly equivalent to the given set and without retraced arcs.

In fact if $\{\gamma = arb r^{-1}c\}$ we say that the set of curves $\{\gamma_1 = ac, \gamma_2 = b\}$ is the set obtained from $\{\gamma\}$ by simple excision of the retraced arc $r$. (Observe that $\gamma_2$ is a closed curve).

If a set of two curves is given, $\{\gamma_1, \gamma_2\}$, such that $\gamma_1 = arb$ and $\gamma_2 = cr^{-1}d$ we have two cases: if $\gamma_2$ is closed, the simple excision of the retraced arc $r$ will yield a single curve set $\{\gamma_3 = adcb\}$; if $\gamma_2$ is open the simple excision of $r$ will result in the set of two curves $\{\gamma_3 = ad, \gamma_4 = cb\}$.

It is clear in the above procedure that after a simple excision the new set of curves is broadly equivalent to the original set.

The simple excision of a retraced arc from one or two arcs in a family of broken arcs $\gamma_0^1, \gamma_0^2, \ldots, \gamma_0^n$ yields a new family of broken arcs; successive simple excisions will lead to a family of broken arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ which we say obtained by maximal excision from $\gamma_1^0, \gamma_2^0, \ldots, \gamma_n^0$ if it does not contain any further retraced arcs.

The proof of the following two elementary lemmata is obtained by induction on the number of excisions and by a direct application of the definition of excision.

**Lemma 5.4.** If the sequence $G_j = \{\gamma_1^j, \gamma_2^j, \ldots, \gamma_n^j\}$ of finite families of curves is obtained by successive excisions from $G_0 = \{\gamma_1^0, \gamma_2^0, \ldots, \gamma_n^0\}$ then

1. we have $\partial G_j = \partial G_0$, for all $j \geq 0$, where $\partial G_j = \sum_{i=1}^{n_j} \partial \gamma_i^j$.
2. $G_j = \{\gamma_1^j, \gamma_2^j, \ldots, \gamma_n^j\}$ is broadly equivalent to $G_0$, for all $j \geq 0$.

**Lemma 5.5.** Let $\gamma \in \Gamma$. Then there exists a set of curves obtained by maximal excision $G = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ from $\gamma$ such that $\gamma_2, \ldots, \gamma_n$ are closed and $\gamma_1$ is closed if and only if $\gamma$ is.
The choice of $\Gamma$ as a set of curves obtained as concatenations of geodesic segments and flow segments yields a simple proof the following lemma which holds true in greater generality (cf. [BS97]).

**Lemma 5.6.** Let $G = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a finite subset of $\Gamma$. Set $Y = \bigcup_{i=1}^n \gamma_i(1)$. Recall that a point $y \in Y$ is regular if it satisfies the following two conditions:

1. every $t \in \bigcup_{i=1}^n \gamma_i^{-1}\{y\}$ is a regular point and
2. there exists an open neighborhood $W$ of $y$ such that $Y \cap W$ is an embedded arc.

The set of regular points $y \in Y$ is an open and dense subset of $Y$.

**Proof of Proposition 5.1.** Suppose $L_x(\tilde{\gamma}_1) = L_x(\tilde{\gamma}_2)$ with $\gamma_1, \gamma_2 \in \Gamma_x$.

First, suppose that the points $\omega(\gamma_1)$ and $\omega(\gamma_2)$ are distinct, i.e. $\partial \tilde{\gamma}_1 \neq \partial \tilde{\gamma}_2$.

Let $\gamma$ be the curve $\gamma = \gamma_1^{-1}\gamma_2$; $\gamma$ is an open curve in $\Gamma$ with $\partial \gamma = \partial \tilde{\gamma}_2 - \partial \tilde{\gamma}_1 = \delta \omega(\gamma_2) - \delta \omega(\gamma_1) \neq 0$. The hypothesis $L_x(\tilde{\gamma}_1) = L_x(\tilde{\gamma}_2)$ yields $L(\gamma) = 0$, that is

$$\tilde{\gamma}(\eta) = h_\eta(\omega(\gamma)) - h_\eta(\alpha(\gamma)), \quad \forall \eta \in \Omega^1(M).$$

By considering a maximal excision of the set $\{\gamma\}$, we obtain a set of curves $G = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset \Gamma$ satisfying

1. the set $G$ is broadly equivalent to $\{\gamma\}$;
2. $\partial \gamma_1 = \partial \gamma, \partial \gamma_2 = \cdots = \partial \gamma_n = 0$;
3. every $\gamma_i \in G$ admits no further excisions, that is no sub-arc of the collection $G$ is retraced.

The first condition means

$$\tilde{\gamma} = \sum_i \tilde{\gamma}_i.$$

We have two cases.

In the first case there exists $t_0 \in I$ such that $\gamma_i(t_0)$ is regular and the velocity $\dot{\gamma}_i(t_0)$ is not collinear to $X(\gamma_i(t_0))$. By the previous Lemma, there exists an open neighborhood $W$ of $\gamma_i(t_0)$, such that $W \cap \cup \gamma_i(I) = W \cap \gamma_i(I)$ is an embedded arc. Since $\gamma_i(I)$ and $X$ are transverse, there exists a one-form $\theta$, supported in $W$, such that $\theta(X)$ vanishes identically and such that $\int_{\gamma_i} \theta \neq 0$. Since there are no retraced arcs in the
set $G$ and by the choice of $W$ and $\theta$ we have $\sum_i \int_{\gamma_i} \theta \neq 0$. By the Remark 4.1 we have that $h_\theta$ is identically constant. But this contradicts (6) since $\int_{\gamma} \theta = \sum_i \int_{\gamma_i} \theta \neq 0$ and $h_\theta(\omega(\gamma)) - h_\theta(\alpha(\gamma)) = 0$. This case is impossible.

We are left with the case where, at all regular points $\gamma(t_0)$, the velocity $\dot{\gamma}(t_0)$ is collinear to $X(\gamma(t_0))$. Then, since the flow of $X$ does not admit closed orbits the collection of curves $G$ is reduced to a singleton $\{\gamma_1\}$. Since $\gamma_1$ does not contain retraced arcs it is in fact a segment of orbit, and we obtain that the original curve $\gamma$ is broadly equivalent to a segment of orbit $\gamma_1 : t \in [0,1] \mapsto \phi^{st}(x)$, from $\alpha(\gamma_1) = \alpha(\gamma)$ to $\omega(\gamma_1) = \omega(\gamma)$. Let $\eta_0$ be a one form such that $\eta_0(X) = 1$ so that $\int_{\gamma} \eta_0 = \int_{\gamma_1} \eta_0 = s$. Again, by the Remark 4.1, we observe that $h_{\eta_0}$ is a constant function. Hence $h_{\eta_0}(\omega(\gamma)) - h_{\eta_0}(\alpha(\gamma)) = 0$, a contradiction if $s \neq 0$, i.e. if $\alpha(\gamma) \neq \omega(\gamma)$. Thus the second case is also impossible.

We showed that we must have $\alpha(\gamma) = \omega(\gamma)$. Then the curve $\gamma = \gamma_1 \gamma_2^{-1}$ is a closed loop (based at $x$); by hypothesis $L_x(\tilde{\gamma}_1) = L(\tilde{\gamma}_2)$, hence $L(\tilde{\gamma}) = 0$. Finally by Lemma 4.4 we obtain $0 = L(\tilde{\gamma}) = \tilde{\gamma}$. Thus $\tilde{\gamma}_1 = \tilde{\gamma}_2$ and we conclude that $L_x$ on $\tilde{\Gamma}_x$ is injective. \hfill \Box

6. Proof of Theorem 1.1

Let $x \in M$ be a point fixed once for all. Consider the Abelian group $A = C^1(M)/\tilde{\Delta}$. The map $L : C^1(M) \rightarrow C^1(M)$ of Definition 4.3 defines a quotient map of $A$ into itself since, by Lemma 4.4, $L|\tilde{\Delta}$ is the identity mapping of $\tilde{\Delta}$. The restriction of $L$ to $\tilde{\Gamma}_x$, which we have denoted by $L_x$, induces a mapping of $\tilde{\Gamma}_x/\tilde{\Delta}$ into $A$, which, by Proposition 5.1, is in fact a continuous bijection of $\tilde{\Gamma}_x/\tilde{\Delta}$ onto $L_x(\tilde{\Gamma}_x)/\tilde{\Delta}$. Using the identification $M \approx \tilde{\Gamma}_x/\tilde{\Delta}$ of Corollary 3.5 we then conclude that $L_x$ induces a continuous injection $l : M \rightarrow A$, mapping $M$ onto $L_x(\tilde{\Gamma}_x)/\tilde{\Delta}$.

Fix $y \in M$ and let $\gamma$ be a curve starting at $x$ and ending at $y$. For $t \in \mathbb{R}$, let $\gamma_t$ be the arc of orbit $s \mapsto \phi^{ts}(y)$. Integrating the equation (4) along the orbit $\gamma_t$ of $y \in M$ we have

$$\int_{\gamma_t} \eta = c_\eta t + h_\eta(\phi^t(y)) - h_\eta(y)$$
Let \( c \in C^1(M) \) be given by \( c(\eta) = c_\eta \). Then the equations (8), in view of the Definition 4.3, can be rewritten as the following equation in \( C^1(M) \) for the currents \( \tilde{\gamma} \) and \( \tilde{\gamma}_t \) associated to the arcs \( \gamma \) and \( \gamma_t \):

\[
L(\tilde{\gamma} + \tilde{\gamma}_t) = L(\tilde{\gamma}) + ct.
\]

As the endpoints of the arcs \( \tilde{\gamma} \tilde{\gamma}_t \) and \( \tilde{\gamma} \) are respectively equal to \( \phi_t(y) \) and \( y \), passing to the quotient by \( \tilde{\Delta} \), we obtain that \( l(\phi_t(y)) = l(y) + ta \), where \( t \mapsto ta \) is the projection to \( A \) of the line subgroup \( t \mapsto ct \).

Finally notice that, by restricting the space of currents \( C^1(M) \) to the finite dimensional space \( H^1(M) \) of harmonic one-forms on \( M \), we obtain a projection \( q : C^1(M) \to H^1(M)^* \). Taking a further quotient by the lattice \( \mathcal{P} \) of periods of \( M \), we obtain a projection \( q' \) of \( C^1(M) \) onto the Albanese torus \( H^1(M)^*/\mathcal{P} \); the map \( q' \) factors through a map \( \pi : A \to H^1(M)^*/\mathcal{P} \), as \( q \) sends the loop group \( \tilde{\Delta} \) onto \( \mathcal{P} \). Thus we have

\[
C^1(M) \to C^1(M)/\tilde{\Delta} \to H^1(M)^*/\mathcal{P}.
\]

Since the maps \( L \) and \( q' \) are smooth, the composite map \( q' \circ L \) is also smooth. For any smooth local lift \( \phi : U \subset M \to \tilde{\Gamma}_x \) of the projection map \( \tilde{\Gamma}_x \to M \), we have that \( q' \circ L \circ \phi = \pi' \circ l \mid_U \); we conclude that \( \pi' = \pi \circ l \) is a smooth map of \( M \) into \( H^1(M)^*/\mathcal{P} \).

From the continuity of \( \pi' = \pi \circ l \) and the minimality of the flow it follows that the image of \( \pi' \) is a rational sub-torus of \( H^1(M)^*/\mathcal{P} \). However the map associating to a closed loop in \( M \) the period along this loop induces a surjection of \( \tilde{\Delta} \) onto \( \mathcal{P} \); it follows that \( \pi' \) is surjective in homology and thus (smooth and) surjective and indeed equal to F. and J. Rodriguez Hertz’ semi-conjugacy.

**Remark 6.1.** The proof above shows that in fact there is, to some degree, a differential structure on the space \( L_x(\tilde{\Gamma}_x)/\tilde{\Delta} \), inherited from the linear structure of \( C^1(M) \); in fact one could show that the map \( l \) is a morphism of differential (or diffeological) spaces in the sense of Chen and Iglesias ([Che77], [Igl]). As the major problem is to tie the topological structure with the differentiable one, we omit any discussion of this point.
REFERENCES

[AK10] Artur Avila and Alejandro Kocsard, Cohomological equations and invariant distributions for minimal circle diffeomorphisms, available on ArXiv, 2010.

[BS97] John C. Baez and Stephen Sawin, Functional integration on spaces of connections, J. Funct. Anal. 150 (1997), no. 1, 1–26, available on ArXiv. MR1473623

[CC00] Wenyi Chen and M. Y. Chi, Hypoelliptic vector fields and almost periodic motions on the torus $T^n$, Comm. Partial Differential Equations 25 (2000), no. 1-2, 337–354. MR1737551

[Che77] Kuo Tsai Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977), no. 5, 831–879. MR0454968

[Dam07] Danijela Damjanović, Central extensions of simple Lie groups and rigidity of some abelian partially hyperbolic algebraic actions, J. Mod. Dyn. 1 (2007), no. 4, 665–688. MR2342703

[DK04] Danijela Damjanović and Anatole Katok, Local rigidity of actions of higher rank abelian groups and KAM method, Electron. Res. Announc. Amer. Math. Soc. 10 (2004), 142–154 (electronic). MR2119035

[DK05] Danijela Damjanović and Anatole Katok, Periodic cycle functionals and cocycle rigidity for certain partially hyperbolic $\mathbb{R}^k$ actions, Discrete Contin. Dyn. Syst. 13 (2005), no. 4, 985–1005. MR2166714

[DK07] Danijela Damjanović and Anatole Katok, Local rigidity of restrictions of Weyl chamber flows, C. R. Math. Acad. Sci. Paris 344 (2007), no. 8, 503–508. MR2324486

[dlLMM85] R. de la Llave, J. M. Marco, and R. Moriyón, Canonical perturbation theory of Anosov systems, and regularity results for the Livsic cohomology equation, Topology 19 (1980), no. 3, 291–299. MR579578

[FF03] Livio Flaminio and Giovanni Forni, Invariant distributions and time averages for horocycle flows, Duke Math. J. 119 (2003), no. 3, 465–526. MR2003124

[FF07] Livio Flaminio and Giovanni Forni, On the cohomological equation for nilflows, J. Mod. Dyn. 1 (2007), no. 1, 37–60, available on ArXiv. MR2261071

[For08] Giovanni Forni, On the Greenfield-Wallach and Katok conjectures in dimension three, Geometric and probabilistic structures in dynamics, Contemp. Math., vol. 469, Amer. Math. Soc., Providence, RI, 2008, pp. 197–213. MR2478471

[GK80a] Victor Guillemin and David Kazhdan, On the cohomology of certain dynamical systems, Topology 19 (1980), no. 3, 291–299. MR579578

[GK80b] Victor Guillemin and David Kazhdan, Some inverse spectral results for negatively curved $n$-manifolds, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 153–180. MR573432

[GW73] Stephen J. Greenfield and Nolan R. Wallach, Globally hypoelliptic vector fields, Topology 12 (1973), 247–254. MR0320502

[Her04] Michael Robert Herman, $L^2$ regularity of measurable solutions of a finite-difference equation of the circle, Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1277–1281. MR2104585

[Hur85] S. Hurder, Problems on rigidity of group actions and cocycles, Ergodic Theory Dynam. Systems 5 (1985), no. 3, 473–484. MR805843
[Igl] Patrick Iglesias, Diffeology, manuscript.

[Kat01] Anatole Katok, Cocycles, cohomology and combinatorial constructions in ergodic theory, Smooth ergodic theory and its applications (Seattle, WA, 1999), Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, In collaboration with E. A. Robinson, Jr., pp. 107–173. MR1858535

[Kat03] Anatole Katok, Combinatorial constructions in ergodic theory and dynamics, University Lecture Series, vol. 30, American Mathematical Society, Providence, RI, 2003. MR2008435

[Koc09] Alejandro Kocsard, Cohomologically rigid vector fields: the Katok conjecture in dimension 3, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 4, 1165–1182. MR2542719

[LdS98] Richard U. Luz and Nathan M. dos Santos, Cohomology-free diffeomorphisms of low-dimension tori, Ergodic Theory Dynam. Systems 18 (1998), no. 4, 985–1006. MR1645342

[Mat09] Shigenori Matsumoto, The parameter rigid flows on 3-manifolds, Foliations, Geometry, and Topology: Paul Schweitzer Festschrift, Contemp. Math., vol. 498, Amer. Math. Soc., Providence, RI, 2009, pp. 135–139.

[Mie07] David Mieczkowski, The first cohomology of parabolic actions for some higher-rank abelian groups and representation theory, J. Mod. Dyn. 1 (2007), no. 1, 61–92. MR2261072

[RHR06] Federico Rodriguez Hertz and Jana Rodriguez Hertz, Cohomology free systems and the first Betti number, Discrete Contin. Dyn. Syst. 15 (2006), no. 1, 193–196. MR2191392

[Tau07] Clifford Henry Taubes, The Seiberg-Witten equations and the Weinstein conjecture, Geom. Topol. 11 (2007), 2117–2202, available on ArXiv. MR2350473

UFR de Mathématiques, Université de Lille 1 (USTL), F59655 Villeneuve d’Asq CEDEX, FRANCE

Centro de Matemática, Facultad de Ciencias, Iguá 4225, 11400 Montevideo, URUGUAY
E-mail address: livio.flaminio@math.univ-lille1.fr
E-mail address: miguel@cmat.edu.uy