Differentiability, Convenient Spaces and Smooth Diffeologies

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ABSTRACT.

We review the basic definitions and properties concerning smooth structures, convenient spaces, diffeological spaces and tangent structures. The relation between the first two is described. A tangent structure is constructed for each pre-convenient space. This one is proved to be convenient if and only if the space and the tangent fibres coincide.

1. Introduction

Infinite dimensional spaces appear in the description of many physical systems as configuration spaces, symmetry or gauge groups. However, most of these function spaces are not Banach and unfortunately, classical differentiation uses Banach spaces. The difficulty arises because the classical condition of Frchet differentiability requires an appropriate structure on the space $L(E,F)$ of linear morphisms, which does not exist if the spaces are not Banach (a straightforward generalization is not possible). However, geometrical difficulties arise in the category of manifolds which are not Banach, including the study of submanifolds, orbit spaces, exponential map, etc.

For this reason, many attempts, successful in different ways, have been made to substitute the Banach space theory of differentiation and manifolds, at different levels of generalization. In the area of differentiation it includes topology, convergence structures, bornologies, etc. Other approaches, more related with spaces of differentiable functions, include smooth structures (and convenient spaces) and (tangent) diffeological spaces.

Convenient spaces [7, 16, 21] are in many ways one of the most successful and natural generalizations of Banach spaces. The basic idea is based on the following two facts: (a) a curve in a Banach space $c: \mathbb{R} \to E$ is smooth iff $l \circ c$ is smooth iff $\forall l \in E'$, (b) a map between two Banach spaces $g: E \to F$ is smooth iff $g \circ c$ is smooth for every smooth curve $c: \mathbb{R} \to E$. This indicates that only $E', F'$ are required, without the use of norms and topologies. Diffeological spaces [22, 4, 23] instead, were defined in a more geometric setting, with the purpose of obtaining geometric quantization in infinite dimensional cases. In these spaces, the construction of geometric objects is based on a set of plaques, generalizing the basic constructions in finite dimensional manifolds (with the use of charts).

Our purpose here is to study some general and basic relations between these two approaches. In section 2 we recall the basic properties of the differentiable functions $f: E \to F$ in locally convex spaces with respect to several locally convex topologies and convergence structures in the space $\mathcal{L}(E,F)$ of continuous linear maps between these spaces. Section 3 describes the basic definitions and properties of smooth structures. The definition of a diffeological space is recalled in section 4, and section 5 describes a relation between the these two categories. Section 6 defines a tangent structure on a diffeological space. In section 7 we review the definition and basic properties of convenient spaces. Section 8 describes a relation between convenient spaces and tangent structures. The next section describes a relation between several definitions of a tangent vector in these categories and the basic relations between manifolds and tangent diffeological spaces.

2. Differentiation in locally convex spaces.

The definition of the derivative of a function $f: U \subset \mathbb{R} \to \mathbb{R}$, $U$ open, at a point $t_0$: $\lim_{t \to t_0} \frac{f(t_0+t) - f(t_0)}{t-t_0}$ is generalized to a function $f: U \subset \mathbb{R} \to F$ if $F$ is a topological vector space, by changing the topology of the range in the limit and if the domain is a topological vector space $E$, then the directional derivative is defined along each vector $v \in E$: $df(x,v) = (f \circ c)'(0)$ where $c(t) = x + tv$. It is better to require $E, F$ to be Hausdorff in order to have uniqueness of limits and also they are required to be locally convex vector spaces (l.c.s.), otherwise there may be functions whose derivatives are zero everywhere but that are not constant. Inductively, if $k \in \mathbb{N}$ and $v_i \in E, i = 1, \cdots k$ then

$$d^k f(x,v_1,\cdots,v_k) = d(d^{k-1} f(x,v_1,\cdots,v_{k-1}))(v_k)$$
The existence of all directional derivatives is a weak condition since it does not even imply continuity of the function (consider the function \( f(0,0) = 0 \) and \( f(x,y) = \frac{x^2+y^2}{x^2+y^2} \) otherwise).

The map \( v \to df(x,v) \) is not linear in general if \( df(x,v) \) exists for all \( v \in E \), not even in the finite dimensional case (consider the function \( f(0,0) = 0 \) and \( f(x,y) = \frac{x^2+y^2}{x^2+y^2} \) otherwise).

Besides the existence of \( d^k f(x,v_1,\cdots,v_k) \) for all \( x \in U \) and \( v_i \in E \), we usually require a continuity condition: \( f \) is called \( C^k \)-Gateaux differentiable if in addition the map \( d^k f : U \times E^n \to F \) is continuous. In this case \( d^k f \) is multilinear and symmetric and the space of \( C^k \)-maps is closed under composition [20]and chain rule holds. Inverse function theorem holds in the category of tamed Frchet spaces [10].

In the case of Banach spaces a stronger condition of differentiability may be used: denote, in the general case, by \( L^k(E,F) \) (resp. \( L(E,F) \)) the vector space of \( n \)-linear mappings (resp. continuous) from \( E \) to \( F \). Then the map \( f \) is called Frchet differentiable at \( x_0 \in U \) if there exists a linear continuous map (equal to \( df(x_0,\cdot) \)) which is tangent to \( f(x_0 + v) - f(x_0) \) at \( v = 0 \) (this is the remainder condition) and it is called Frchet \( C^1 \) on \( U \) if the map \( x_0 \to df(x_0,\cdot) \in L(E,F) \) is continuous (when \( L(E,F) \) is given the canonical Banach space structure), denoted \( Df(x_0) \).

The second derivative is a map \( x_0 \to L(E,L(E,F)) \) and inductively, the map \( D^k f : U \to L(E,F) \) is required to be continuous.

An important fact is that the remainder condition is a consequence of continuity of \( Df \) (this simplifies the study of generalizations to the study of different conditions of continuity of \( Df \) only).

Composition of \( C^k \) maps is again \( C^k \), chain rule, implicit function theorem and inverse function theorem holds, similar to the finite dimensional case [17]. Also good existence and uniqueness theorem of ordinary differential equations holds [3].

Frchet differentiability at a point implies the existence of all directional derivatives and continuity of \( f \) but does not imply continuity of directional derivatives (even if \( f \) is Frchet differentiable in an open set (take \( f(0,0) = 0 \) and \( f(x,y) = \frac{x^2+y^2}{x^2+y^2} \) otherwise). A weak converse is well known in the finite dimensional case (if all partial derivatives of \( f \) exist and are continuous in an open set \( U \) then \( f \) is Frchet differentiable at each point of \( U \)). However, existence of all directional derivatives and continuity of them along lines through a point does not imply Frchet differentiability at the point.

In the finite dimensional case Gateaux \( C^k \) and Frchet \( C^k \) are equivalent. In the more general case of Banach spaces both are equivalent if the condition ‘\( df(x_0,\cdot) \) continuous’ is replaced by ‘\( df(x_0,\cdot) \) \( Lip^k \)'(this means for normed spaces that in the definition of Gateaux differentiability the condition of continuous is replaced by Lipschitz locally at each point).

The notion of Frchet differentiability may not be generalized, not even to Frchet spaces, because the dual of a Frchet space which is not Banach is never a Frchet space.

For functions between Banach spaces, the continuity of \( df : E \times E \to F \) does not imply continuity of \( Df : E \to L(E,F) \) (however \( Df \) is locally Lipschitz iff \( df \) is locally Lipschitz). For this reason an alternative continuity condition for any \( E,F \) l.c.s. is constructed in the following way: \( f \) is called weakly \( p \)-times differentiable on \( U \) if

\[
D^{k+1}f(x)v := \lim_{t \to 0} t^{-1}[D^k f(x+tv) - D^k f(x)]
\]

exists \( \forall(x,v) \in U \times E \) and \( k = 0, \cdots, p-1 \), with respect to the topology of simple convergence in \( L^k(E,F) \) and \( D^k f(x)v \) is linear with respect to \( v \in E, \forall x \in U \).

A stronger condition is defined by providing a l.c.s. topology on \( L(E,F) \): a collection \( \sigma \) of bounded subsets of \( E \) defines a family of seminorms on \( L(E,F) \) (the supremum, for each seminorm of \( F \), over subsets \( S \in \sigma \)). The space with the topology \( \sigma \) is denoted \( L^n_{\sigma}(E,F) \). The most important examples are given by finite, compact, precompact, and bounded sets, whose topology is denoted by \( \wedge_s, \wedge_k, \wedge_{pk}, \wedge_b \). Also we define the spaces \( \mathcal{H}_{\sigma}^n(E,F) := F \) and

\[
\mathcal{H}_{\sigma}^n(E,F) := L_{\sigma}(E,\mathcal{H}_{\sigma}^{n-1}(E,F))
\]

If \( E \) is metrizable and barreled, or if \( E \) is metrizable and \( \wedge_k \subset \sigma \) then \( L^n_{\sigma}(E,F) = \mathcal{H}_{\sigma}^n(E,F) \) We may also define a convergence structure \( \wedge \) on \( \mathcal{H}_{\sigma}^n(E,F) \), even if \( E,F \) are convergence vector spaces. Some examples are the continuous convergence \( \wedge_c \) of Bastiani [1](this is the coarsest convergence vector space structure (c.v.s.) for which the evaluation is continuous), the quasi-bounded convergence \( \wedge_{qb} \) [9], the bounded convergence \( \wedge_b \) [28](which coincides with the topology of bounded topology if \( E,F \) are l.c.s.), the Marinescu convergence
∧_\Delta [19]. And the space with this structure is denoted \( \mathcal{L}_c^n(E, F) \). We have \( \Lambda_c \leq \Lambda_{qb} \leq \Lambda_{\pi} \leq \Lambda_\Delta \). For these c.v.s. and also if \( E \) is Banach, we have \( \mathcal{L}^m(E, \mathcal{L}_c^n(E, F)) \simeq \mathcal{L}^{m+n}(E, F) \).

Now, having a topology or convergence structure \( \Lambda \), a map \( f:U \subset E \rightarrow F \) is called differentiable of class \( C^p_\Lambda \) if \( f \) is weakly \( p \)-times differentiable and \( D^k f(x) \in \mathcal{H}^\Lambda_k(E, F) \) for all \( x \in U \) and \( 0 \leq k \leq p \), and \( D^k f:U \rightarrow \mathcal{H}^\Lambda_k(E, F) \) is continuous. If \( \Lambda \geq \Lambda_c \) we may take \( \mathcal{L}^\Lambda_k \) instead.

We have the inclusions [12]:

\[
C^p_\Delta \subset C^p_\pi \subset C^p_{qb} \subset \{C^0_c, C^0_\pi\} \subset C^0_b \subset C^0_{pk} \subset C^0_k \subset C^0_s
\]

If \( E \) is normable then \( C^p_\Delta = C^p_\pi = C^p_{qb} = C^p_b \) and \( C^p_{pk} = C^p_k \). Also \( C^p_{pk} = C^p_k = C^p_s \) if \( E \) is Fréchet; however, \( C^1_\pi \neq C^1_\Delta \) even if \( E, F \) are Banach (all of them are equivalent however if \( E \) is finite dimensional). Therefore we have two notions of differentiability in Fréchet spaces: Gateaux-Levi \( (C^0_\pi) \) and Fréchet \( (C^0_s) \), which coincide if \( E \) is finite dimensional or if \( E \) is Fréchet-Schwartz and \( F \) is normable [27].

Differentiability in l.c.s. preserves some properties: if \( f \in C^p_\pi \) then \( D^k f(x) \) is symmetric for all \( 0 \leq k \leq p \) and Taylor approximation holds. Also \( f \in C^p_\pi \) implies that \( D^k f(x)v \) exists uniformly with respect to \( v \in S \in \sigma \) and it is linear with respect to \( f \). In addition, the remainder satisfies: if \( f \in C^p_\pi \) then

\[
\lim_{t \to 0} t^{-1}[D^k f(x + tv) - D^k f(x) - D^{k+1} f(x)tv] = 0
\]

uniformly with respect to \( v \in S \in \sigma, \forall k \leq p - 1 \).

There are many limitations however: the evaluation map

\[
\mathcal{L}^n_\sigma(E, F) \times E^n \rightarrow F
\]

is not continuous for any l.c.s. topology on \( \mathcal{L}^n_\sigma(E, F) \) unless \( E \) is normable [13](it is continuous however for convergence structures stronger or equal to \( \Lambda_c \)). Composition

\[
\mathcal{L}_\sigma(F, G) \times \mathcal{L}_\sigma(E, F) \rightarrow \mathcal{L}_\sigma(E, G)
\]

is not continuous for any l.c.s. topology on \( \mathcal{L}(E, F) \), except for special \( E, F \), such as Banach spaces [14] (it is continuous for convergence structures \( \Lambda_{qb}, \Lambda_\Delta \)). Differentiable functions with respect to convergence structures stronger or equal to \( \Lambda_c \) are closed under composition, but it does not hold in the category of l.c.s. unless they are Banach. Some restricted cases hold: if \( E \) is metrizable, \( f:U \rightarrow F \) is \( C^1_\pi \) and \( g:V \subset F \rightarrow G \) is \( C^1_k \) then \( g \circ f \in C^1_k \). In addition the continuity of \( D^k f:U \rightarrow \mathcal{L}_\Lambda(E, F) \) does not imply the continuity of \( D^k f:U \times E^n \rightarrow F \) in general. It holds however if \( E \) is metrizable and barrelled and \( \Lambda = \Lambda_b \) or if \( E \) is metrizable and \( \Lambda = \Lambda_k \) or for general l.c.s. and \( \Lambda \) finer than \( \Lambda_c \). Also the continuity of \( D^p f:U \rightarrow \mathcal{L}^p_\pi(E, F) \) does not imply the continuity of \( D^k f \) if \( k < p \) (it does if \( E \) is finite dimensional).

A map \( f:U \rightarrow F \) is called of class \( C^{\infty}_{\Lambda} \) if it is of class \( C^p_\Lambda \) for all \( p \in \mathbb{N} \). The inclusions, described above, between spaces \( C^p_{\Lambda} \) for different \( \Lambda \), also holds for \( p = \infty \). If \( E \) is Banach, \( F \) l.c.s. or \( E \) Fréchet and \( F \) normable then all spaces \( C^{\infty}_{\Lambda} \) coincide, except for \( C^{\infty}_s \) and therefore we speak of the space \( C^{\infty} \) instead, which is closed under composition. In addition the composition of two \( C^{\infty} \) functions is a \( C^s_{\Lambda} \) function in the category of Fréchet spaces. The spaces are closed under composition in other special cases: if \( f \in C^{\infty}_{\Lambda}(E, F) \) and \( g \in C^{\infty}_s(F, G) \) then \( g \circ f \in C^{\infty}_{\Lambda}(E, G) \) if \( \Lambda = \Lambda_c \).

3. Smooth structures.

In this section we recall some basic definitions and results concerning smooth structures: a theory that provides a generalization to the construction of \( Lip^k \) and \( C^{\infty} \) curves in Banach spaces.

In [2] Boman proves that a function \( \mathbb{R}^n \rightarrow \mathbb{R} \) is smooth if and only if it is smooth along smooth curves. A result that is generalized [7] for any classical smooth manifold or any Banach space \( X \): the family of smooth curves \( c: \mathbb{R} \rightarrow X \) and the family of smooth functions \( f: X \rightarrow \mathbb{R} \) determine each other (a map belongs to one of the families if the composites with the members of the other family are functions in \( C^{\infty}(\mathbb{R}, \mathbb{R}) \)). Another generalization of Boman’s result is the following: a map between two Banach spaces is a smooth
function if and only if its composite with smooth curves of the source and the smooth real valued functions of the range are functions in $C^\infty(R, R)$.

These two results, together with the need of a differentiation theory in general spaces that include spaces of functions such as diffeomorphisms (which in general are not Banach) is the reason for a generalization: instead of $C^\infty(R, R)$, a general set $\mathcal{M}$ of functions from a set $S$ to another set $R$ is considered, a Banach space is substituted by a general set $X$, together with a family $\mathcal{C}_X$ of curves $c: S \to X$ and a family $\mathcal{F}_X$ of functions $f: X \to R$ such that one of the families determine the other one: given $X$, $\mathcal{M}$ we observe that

1. A set $\mathcal{C}$ of curves in $X$ determines a set $\mathcal{F}$ of functions on $X$: $\{f: X \to R / f \circ c \in \mathcal{M}, \forall c \in \mathcal{C}\}$, denoted $\Phi_c$.
2. A set $\mathcal{F}$ of functions on $X$ determines a set $\mathcal{C}$ of curves in $X$: $\{c: I \to X / f \circ c \in \mathcal{M}, \forall f \in \mathcal{F}\}$, denoted $\Gamma_{\mathcal{F}}$.

An $\mathcal{M}$-structure on $X$ [7], [15] is a pair $(\mathcal{C}, \mathcal{F})$ of curves in $X$ and functions $\mathcal{F}$ on $X$ such that they determine each other: $\Phi_c = \mathcal{F}$ and $\Gamma_{\mathcal{F}} = \mathcal{C}$. Also $(\mathcal{X}, \mathcal{C}, \mathcal{F})$ is called an $\mathcal{M}$-space. A map $f: X \to Y$ between two spaces $X, Y$ with $\mathcal{M}$-structures $(\mathcal{C}_X, \mathcal{F}_X)$ and $(\mathcal{C}_Y, \mathcal{F}_Y)$ respectively, is called an $\mathcal{M}$-map if $h_s(C_X) \subset C_Y$ (this is equivalent to $h^*(\mathcal{F}_Y) \subset \mathcal{F}_X$ and also is equivalent to $g \circ h \circ c \in \mathcal{M}, \forall c \in C_X, \forall g \in \mathcal{F}_Y$). The set of $\mathcal{M}$-maps $f: X \to Y$ is denoted $\mathcal{M}(X, Y)$. The elements of $\mathcal{C}$ are called structure curves and the elements of $\mathcal{F}$ are called structure functions.

We may have several $\mathcal{M}$-structures on the same set $X$. In this case a structure $(\mathcal{C}_1, \mathcal{F}_1)$ on $X$ is called smaller than $(\mathcal{C}_2, \mathcal{F}_2)$ if $\mathcal{C}_1 \subset \mathcal{C}_2$.

For any set $C_0$ of curves on $X$ there exists a smallest $\mathcal{M}$-structure $(\mathcal{C}, \mathcal{F})$ on $X$ such that $C_0 \subset C$ (given by $\mathcal{F} = \Phi_{C_0}$ and $\mathcal{C} = \Gamma_{\mathcal{F}}$). With this $\mathcal{M}$-structure any map $h: X \to Y$ is an $\mathcal{M}$-map if and only if $h_s(C_0) \subset C_Y$.

For any set $\mathcal{F}_0$ of functions on $X$ there is a largest $\mathcal{M}$-structure $(\mathcal{C}, \mathcal{F})$ such that $\mathcal{F}_0 \subset \mathcal{F}$ (given by $\mathcal{C} = \Gamma_{\mathcal{F}_0}$ and $\mathcal{F} = \Phi_c$). With this $\mathcal{M}$-structure a map $h: Y \to X$ is an $\mathcal{M}$-map if and only if $h^*(\mathcal{F}_0) \subset \mathcal{F}_Y$.

In particular if $X$ is a vector space and $\mathcal{F}_0$ is a set of linear functions, we say that $(X, \mathcal{C}, \mathcal{F})$ is linearly generated and the $\mathcal{M}$-space may be denoted by $(X, F_0)$ instead.

Some fundamental examples of sets $\mathcal{M}$ related with differentiability are:

(a) $C^k(R, R)$, denoted $\mathcal{M}^k$, for any $k \in \mathbb{N} \cup \{\infty\}$.
(b) $\text{Lip}^k(R, R)$ (the set of differentiable functions $f: R \to R$ of order $k$ whose derivatives are locally Lipschitz), denoted $\mathcal{M}^k_L$, for any $k \in \mathbb{N} \cup \{\infty\}$.
(c) $l^\infty(N, R)$, the set of bounded real sequences, denoted $l^\infty$.

For any set $\mathcal{M}$, the class of $\mathcal{M}$-spaces as objects with the $\mathcal{M}$-maps as morphisms forms a category denoted $\mathcal{M}$. This category has initial and final structures: if $(X_i)_{i \in I}$ is a family of $\mathcal{M}$-spaces and $g_i: X_i \to X_i$ is given, the initial structure on $X$ is generated by the set of functions $\{f \circ g_i : i \in I, f \in \mathcal{F}_i\}$. The set of structure curves is $\{c: S \to X / g_i \circ c \in C_i, \forall i \in I\}$.

Instead, if $g_i: X_i \to X$ is given, the final structure on $X$ is generated by the set of curves $\{g_i \circ c : i \in I, c \in C_i\}$, and the set of structure functions is $\{f: X \to R / f \circ g_i \in \mathcal{F}_i, \forall i \in I\}$.

Initial structures provides an $\mathcal{M}$-structure on products and subsets and final structures on quotients.

Cartesian closeness provides a way to define $\mathcal{M}$-structures on spaces of functions: let $\mathcal{M}$ containing all constant maps and let

$$\mathcal{C}_M = \{c: S \to \mathcal{M} / \bar{c}: S \times S \to R \text{ is an } \mathcal{M}\text{-map}\}$$

where $\bar{c}(s, t) = c(s)(t)$. If $\mathcal{M}$ satisfies that $\Gamma \Phi \mathcal{C}_M \subset \mathcal{C}_M$, then for any $\mathcal{M}$-spaces $Y, Z$, the set $\mathcal{M}(Y, Z)$ is an $\mathcal{M}$-space [8] with structure curves

$$\mathcal{C}_{\mathcal{M}(Y, Z)} = \{h: S \to \mathcal{M}(Y, Z) / \bar{h}: S \times Y \to Z \text{ is an } \mathcal{M} - \text{map}\}$$

This condition is satisfied by $\mathcal{M}^\infty$ and $l^\infty$ and both are linearly generated [18]. An application of this fact is that evaluation and composition maps are $\mathcal{M}^\infty$-maps in this category.

In addition, if $E, F$ are normed spaces, the set of continuous linear maps $E^*, F^*$ generates $\mathcal{M}$-structures on $E, F$ for any $\mathcal{M}$ and in particular if we take $\mathcal{M} = \text{Lip}^k(R, R)$ and $U$ an open subset of $E$, then a map $g: U \to F$ is $\text{Lip}^k$ (in the classical sense) if and only if $g \in \mathcal{M}(U, F)$: the composition with the structure curves on $U$ and structure functions on $F$ are members of $\mathcal{M}$ [7]. This result shows that the theory of $\mathcal{M}$-spaces provides a generalization of differentiation theory in Banach spaces (in the case of $\text{Lip}^k$ and smooth maps).
Cartesian closeness provides a simple application to the theory of groups of diffeomorphisms: if $X$ is any $\mathcal{M}^\infty$-space, then $C^\infty(X, X)$ is an $\mathcal{M}^\infty$-space by cartesian closeness and the two maps

$$i, j: Diff(X) \to C^\infty(X, X)$$

given by $i(f) = f, j(f) = f^{-1}$ provides an $\mathcal{M}^\infty$-structure on $Diff(X)$ (the initial one), which makes it a smooth group (composition and inversion are $\mathcal{M}^\infty$-maps).

Even though we may define $\mathcal{M}^k$-spaces with $k < \infty$, they do not provide a generalization of $C^k$-differentiation theory, since there are maps $g: E \to F$ between Banach spaces which are not $C^k$ but $g \in \mathcal{M}^k(E, F)$, even for finite dimensional spaces.

4. Diffeological spaces.

The category of diffeological spaces [22], [4], [5], contains smooth manifolds, finite and infinite dimensional, and also finite dimensional $C^k$ manifolds as subcategories. They were defined by Souriau in order to extend quantization through coadjoint orbits to groups of diffeomorphisms. In this section we recall its definition.

The notation $PM$ will mean a class of $C^k$ functions $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ for some $0 \leq k \leq \infty$, and any $n, m \in \mathbb{N}$, containing constants and closed under composition, such as:

(a) The set of $C^k$ maps $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$, for any $U$ open and any $n, m \in \mathbb{N}$. This set is denoted $PM^k$, with $k \in \mathbb{N} \cup \{\infty\}$.

(b) The set of $C^k$ maps $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ whose derivatives are locally Lipschitz, for any $U$ open and any $n, m \in \mathbb{N}$. Denoted $PM^k_L$ with $k \in \mathbb{N} \cup \{\infty\}$.

Given a set $X$, an $n$-plaque on $X$ is a function $p: U \to X$ where $U$ is an open subset of $\mathbb{R}^n$. A $PM$-diffeology on $X$ is a set $P(X)$ of $n$-plaques, for each $n \in \mathbb{N}$ such that the images of the plaques covers $X$ and

i. If a set $(p_i)$ of $n$-plaques admits a common extension, then the smallest such extension is also an $n$-plaque in $P(X)$.

ii. For each $\phi \in PM$, $\phi: U' \to U$, where $U', U$ are open in $\mathbb{R}^m, \mathbb{R}^n$ respectively and for every plaque $p: U \to X$, the map $p \circ \phi$ is also in $P(X)$.

The pair $(X, P(X))$ is called a $PM$-diffeological space. The set of $n$-plaques is denoted $P_n(X)$.

The standard $PM$-diffeology on $\mathbb{R}$ is defined as the set of maps $f: U \subset \mathbb{R}^n \to \mathbb{R}$ in $PM$ and it is denoted by $P(\mathbb{R})$.

Given two diffeological spaces $(X, P(X))$ and $(Y, P(Y))$, a map $f: X \to Y$ is called a $PM$ morphism if $p \in P(X)$ implies $f \circ p \in P(Y)$. The set of such maps is denoted $D(P(X), P(Y))$ or $D(X, Y)$ if possible. In particular if $Y = \mathbb{R}$ with the standard diffeology, then we use the notation $D(X)$ instead. We shall say that two diffeologies $P^1(X), P^2(X)$ on $X$ are equivalent if $D(P^1(X)) = D(P^2(X))$, and it is denoted by $(X, P_1(X)) \sim (X, P_2(X))$.

The class of $PM$ diffeological spaces as objects with the differentiable functions as morphisms forms a category, denoted $PM$.

Remarks 4.1.

[1] Any set $P_0$ of plaques on $X$, whose images covers $X$, generates a diffeology $\tilde{P}_0$, formed by plaques $p: U \to X$ such that $\forall r \in U$ there exists an open neighborhood $U_r$, $\phi \in PM$, and $p_0 \in P_0$ such that $p|_{U_r} = p_0 \circ \phi$.

[2] Any set $F_0$ of functions on $X$ generates a set $P$ of plaques $p$ on $X$ such that $f \circ p \in PM$ for all $f \in F$. This set of plaques is denoted $\Gamma F_0$. Also, any set $P_0$ of plaques defines a set $\Phi F$ of functions on $X$ such that $f \circ p \in PM$ for all $p \in P_0$. Given $P_0$, then $(X, \tilde{P}_0)$ is a diffeology whose set of smooth functions is $\Phi P_0 = \Phi P_0$. Notice that $\tilde{P}_0 = \Gamma \Phi P_0$. Also, notice that given $F_0$, we have that $(X, \Gamma F_0)$ is a diffeology whose set of differentiable functions is $\Phi F_0$.

[3] Given a collection $(X_j, P_j)_{j \in J}$ of diffeological spaces and maps $g_j: X_j \to X$, the final diffeology on $X$ is generated by $(g_j \circ p)_{p \in P_j, j \in J}$. Instead, if $g_j: X \to X_j$, the initial diffeology on $X$ is the set of plaques $p: U \subset \mathbb{R}^n \to X$ such that $g_j \circ p \in P_j$ for all $j \in J$. In particular this allows the definition of diffeologies on products, quotients and subsets.
Cartesian closeness follows in general: if \((Y, P(Y)), (Z, P(Z))\) are diffeological spaces, the \textit{functional diffeology} on \(D(Y, Z)\) is formed by all plaques \(p: U \to D(Y, Z)\) such that \(\bar{p}: U \times Y \to Z\) is differentiable, where \(\bar{p}(r, y) = p(r)(y)\). It follows that, given \((X, P(X))\), a map \(f: X \to D(Y, Z)\) is differentiable if and only if \(\bar{f}\) is differentiable, and therefore \(D(X, D(Y, Z))\) is diffeomorphic to \(D(X \times Y, Z)\).

Given \((X, P(X)), (Y, P(Y))\) as before with corresponding sets of smooth functions \(D(X), D(Y)\), and a map \(f: X \to Y\), we might consider three natural definitions of a differentiable function:

1. \(f\) is differentiable if \(p \in P(X)\) \(\Rightarrow f \circ p \in P(Y)\).
2. \(f\) is differentiable if \(g \in D(Y)\) \(\Rightarrow g \circ f \in D(X)\).
3. \(f\) is differentiable if \(p \in P(X), g \in D(Y)\) \(\Rightarrow g \circ f \circ p \in PM\).

It is easy to check that \(i \Rightarrow ii\) and \(ii \Rightarrow iii\), in addition, \(ii \Rightarrow i\) if and only if \(\Gamma \Phi P(Y) = P(Y)\). This last condition may be added to the definition of a diffeology \(P\) on \(X\). In this case the set \(D(X)\) does not change, \(P(X)\) becomes larger, and also the set of differentiable functions \(D(X, Y)\) for any \((Y, P(Y))\), but cartesian closeness would not follow in general.

Notice that definition (i) corresponds to the category \(\underline{PM}\), definitions (ii) and (iii) will define another category \(PM_1\) with the same set \(D(X)\) but with different set of morphisms \(D_1(X, Y)\). The map \(\gamma: PM \to PM_1\) given by the identity on morphisms and \(\gamma(X, P(X)) = (X, \Gamma \Phi P(X))\) is a functor such that

\[
\gamma(X, P(X)) = \gamma(X, \Gamma \Phi P(X))
\]

The functor \(\beta: PM_1 \to PM\) given by the identity on objetts and morphisms satisfies \(\gamma \circ \beta = id\). Final diffeologies are the same under both categories, and also the product diffeology. Cartesian closeness also follows in \(PM_1\) if \(PM\) satisfies \(\Gamma \Phi (PM) = PM\).

If \((X, P^1(X)) \simeq (X, P^2(X))\) and \((Y, P(Y))\) satisfies \(\Gamma \Phi P(Y) = P(Y)\) (\(P(Y)\) is maximal) then

\[
D(P^1(X), Y) = D(P^2(X), Y)
\]

If \(P(Y)\) is not maximal then the equality does not hold. For example, let \(X = Y = \mathbb{R}^2\) and \(P^1(X) = P^2 M, P^2(X) = P(Y) = P^2 M \circ P^1 M\), then \(id \in D(P^1(X), Y) \setminus D(P^2(X), Y)\).

5. Smooth structures and diffeologies.

\textbf{Lemma 5.1.} ([7]).

Let \(E, F\) be Banach spaces and \(k \in \mathbb{N} \cup \{\infty\}\). A map \(g: E \to F\) is \(k\)-Lipschitz (in the classical sense) if and only if for every \(k\)-Lipschitz curve \(\gamma: \mathbb{R} \to E\) the map \(g \circ \gamma\) is a \(k\)-Lipschitz curve on \(F\).

\textbf{Proposition 5.2.}

Let \(PM\) be \(PM_k^\perp\) for some \(k \in \mathbb{N} \cup \{\infty\}\).

a. Every \(M\)-structure \((\mathcal{C}, \mathcal{F})\) on \(X\) defines a \(PM\)-diffeology \(P(X)\) on \(X\) with \(P_1(X) = \mathcal{C}, D(X) = \mathcal{F}\). This defines a functor \(\Psi: M \to PM\) which is 1-1 on objects and it is an embedding.

b. Every \(PM\)-diffeology \((X, P(X))\) defines an \(M\)-structure \((\mathcal{C}, \mathcal{F})\) on \(X\) generated by \(\mathcal{C}_0 = P_1(X)\) such that \(\mathcal{F} = D(X)\). This defines a faithful functor \(\Upsilon: PM \to M\) which is onto on objects and satisfies \(\Upsilon \circ \Psi = id\) and \(\Psi \circ \Upsilon(X, P(X)) \simeq (X, P(X))\).

c. \(M \simeq PM_1\) (and also for any \(PM\) satisfying \(*\), given below).

\textit{Proof:} for any \(PM\) let us use the notation

\[
P^n_m M = \{p: U \subset \mathbb{R}^m \to \mathbb{R}^n / p \in PM\}
\]

Observe that lemma 5.1. implies that \(h \in P^n_m M\) if and only if

\[
h \circ \gamma \in M, \forall \gamma \in P^n_m M
\]

\((*)\)

(a): Given \((X, \mathcal{C}, \mathcal{F})\) define

\[
P(X) = \{p: U \subset \mathbb{R}^n \to X / f \circ p \in P^n_m M, \forall f \in \mathcal{F}, n \in \mathbb{N}\}
\]
Then \( P(X) \) is a diffeology satisfying \( P_1(X) = C \) and \( \mathcal{F} \subset D(X) \). Conversely, if \( f \in D(X) \) then \( f \circ p \in P^1_m \mathcal{M}, \forall p \in P(X) \) and therefore \( f \circ p \in P^1_1 \mathcal{M} = \mathcal{M}, \forall p \in P_1(X) = C \). This proves that \( f \in \mathcal{F} \) and that \( \mathcal{F} = D(X) \).

This defines a map \( \Psi: \mathcal{M} \to PM \) which is 1-1: if

\[
\Psi(X, C_1, F_1) = \Psi(X, C_2, F_2) = (X, P(X))
\]

then \( C_1 = P_1(X) = C_2 \) and therefore \( F_1 = F_2 \). Given \( f \in \mathcal{M}(X, Y) \) define \( \Psi(f) = f \). This is a map which is 1-1 and onto since \( D(X, Y) = \mathcal{M}(X, Y) \): if \( f \in \mathcal{M}(X, Y) \) and \( p \in P(X) \), then \( h \circ f \in F_X, \forall h \in F_Y \). It follows that \( h \circ f \circ p \in P^1_m \mathcal{M}, \forall h \in F_Y \) this means that \( f \circ p \in P(Y) \) and therefore that \( f \in D(X, Y) \). Conversely, if \( f \in D(X, Y) \), then \( f \circ p \in P(Y), \forall p \in P(X) \); thus

\[
f \circ c \in P_1(Y) = C_Y, \forall c \in P_1(X) = C_X
\]

and this proves that \( f \in \mathcal{M}(X, Y) \).

(b) Let \( (X, P(X)) \) be a \( PM \)-diffeology and let \( \Upsilon(X, P(X)) \) be the smooth \( \mathcal{M} \)-structure generated by \( C_0 = P_1(X) \). Then

\[
\mathcal{F} = \{(f: X \to R/f \circ p \in \mathcal{M}, \forall p \in P_1(X))\}
\]

It follows that \( D(X) \subset \mathcal{F} \). Now let \( f \in \mathcal{F} \). Let \( p \in P_m(X) \), then \( p \circ \gamma \in P_1(X), \forall \gamma \in P^m \mathcal{M} \), then \( (f \circ p) \circ \gamma \in \mathcal{M}, \forall \gamma \in P^m \mathcal{M} \), and taking \( h = f \circ p \) in * we obtain \( f \circ p \in P^m \mathcal{M} \) and then \( f \in D(X) \). This proves that \( D(X) = \mathcal{F} \). If \( (Y, P(Y)) \) is another diffeology, we define \( \Upsilon(f) = f \) if \( f \in D(X, Y) \). We have that \( D(X, Y) \subset \mathcal{M}(X, Y) \) and therefore \( \Upsilon \) is also a faithful functor: if \( f \in D(X, Y) \) and \( h \in \mathcal{F}_Y \) then \( f \circ p \in P_2(Y), \forall p \in P_1(X) \), then \( (h \circ f) \circ p \in \mathcal{M}, \forall p \in P_1(X) \), consequently \( (h \circ f) \in \mathcal{F}_X \).

Given \( (X, \mathcal{C}, \mathcal{F}) \), let \( \Psi(X, \mathcal{C}, \mathcal{F}) = (X, P(X)) \) and

\[
\Upsilon(X, P(X)) = (X, \mathcal{C}, \mathcal{F})
\]

Then \( D(X) = \mathcal{F} = \mathcal{F} \) and therefore \( \mathcal{C} = \mathcal{C} \). This proves that \( \Upsilon \) is onto and \( \Upsilon \circ \Psi = Id \). Conversely, given \( (X, P(X)) \), let \( \Upsilon(X, P(X)) = (X, \mathcal{C}, \mathcal{F}) \) and \( \Psi(X, \mathcal{C}, \mathcal{F}) = (X, P(X)) \), then \( D_P(X) = \mathcal{F} = D_P(X) \) and therefore \( \Psi \circ \Upsilon(X, P(X)) \sim (X, P(X)) \).

(c) The proof is also straightforward.

Remarks 5.3.

[1] The functor \( \Psi \) is not onto on objects; for example, take \( X = R^2 \) with \( P^1_1(R^2) = P^2_1 \mathcal{M} \) and \( P^1_1(R^2) = P^1_1(R^2) \circ P^1_1 \mathcal{M} \). Then the image of every plaque is a curve. If \( \Psi(R^2, \mathcal{C}, \mathcal{F}) = (R^2, P^2(R^2)) \) then \( \mathcal{C} = P^1_1(R^2) \) and therefore \( P^2_1(R^2) = P^2_1 \mathcal{M} \), which is different from \( P^1_1(R^2) \). However, given \( (X, P(X)) \) we have

\[
\Phi(X, P_1(X), D(X)) \sim (X, P(X))
\]

Notice that if the space \( PM \) satisfies * then a smooth structure is identical with an equivalence class of diffeological spaces. Otherwise there are two types of diffeological spaces: those generated, or not by curves \( P_1(X) \). In the first place they are equivalent to the smooth structure \( (X, P_1(X), D(X)) \).

[2] The functor \( \Upsilon \) is not 1-1 on objects: both \( P^i(R^2), i = 1, 2 \) given above have the same image \( (R^2, \mathcal{C}, \mathcal{F}) \).

[3] If we consider \( PM^k \) with \( k \neq \infty \), then (a) remains valid in the proposition. In part (b), \( \Upsilon \) is still a faithful functor, \( C \) is generated by \( P_1(X) \) and \( D(X) \subset \mathcal{F} \) (only), \( \Upsilon \) is onto on objects, but \( \Upsilon \circ \Psi \neq Id \). Also \( \mathcal{M} \) is embedded in \( PM_1 \) but the last one is larger.

[4] \( \mathcal{M}^k_1 \)-structures provide a generalization of \( \mathcal{M}^k_1 \)-Banach manifolds, but do not generalize \( \mathcal{M}^k \)-manifolds, not even in the finite dimensional case. Diffeological spaces generalizes \( \mathcal{M}^k_1 \)-Banach manifolds and finite dimensional \( \mathcal{M}^k \)-manifolds.

[5] Some disadvantages of those diffeologies outside \( PM_1 \) are: \( P(X) \) determines \( D(X) \) but \( D(X) \) does not determine \( P(X) \), but instead, it determines a class of diffeologies. Also, consider the following example: \( PM = PM^k_1 \) and \( X = Y = R^2, P^1(X) = P^1(Y) = PM^k \) and \( P^2(X) = P^2(Y) = PM^k_1 \). Then \( D(X) = D(Y) = P^2_1 \mathcal{M}^k \) (because a function \( f: R^2 \to R \) is Lip^k if it is Lip^k along smooth curves iff it is Lip^k along Lip^k curves. Therefore \( D(P^2(X), P^2(Y)) = P^2_1 \mathcal{M}^k \) but \( D(P^1(X), P^1(Y)) = P^2_1 \mathcal{M}^\infty \) (by Boman’s result [2]). This is unexpected because we probably hope to obtain spaces of Lip^k functions.
There are important examples (such as \(PM^k\)) for which * does not hold and \(PM_1\) is larger than \(M\).

6. Tangent structures.

Both, smooth structures and diffeologies provides a relation between the spaces of curves (plaques) and spaces of functions. In the first case, convenient spaces are the most general vector spaces, with a smooth structure which is linearly generated for which the structure curves have derivatives (the definitions are recalled in the next section). In the second case, tangent diffeologies (with a tangent structure) provides n-dimensional directions at every point, allowing the definition of tangent spaces, derivatives of functions, vector fields and differential forms, in which the tangent bundle itself is a tangent diffeology. The category of tangent diffeologies \([24], [25], [26]\) provides a generalization of smooth and \(C^k\) manifolds. The definition is recalled in this section. For simplicity of notation we may assume that all plaques \(p\) through a point \(F \in X\) are defined in a neighborhood of 0 and \(p(0) = F \) and \(p \circ \phi\) means that \(\phi: U' \subset \mathbb{R}^m \to U \subset \mathbb{R}^n\) and \(p: U \to X\).

(a) Given a \(PM\)-diffeology \((X, P(X))\), a tangent structure on \(X\) is a collection \(\sim\) of equivalence relations \(\sim_p^F\) on \(P^F_p(X)\), for each \(F \in X\) and \(n \leq k\) satisfying the two consistency conditions:

(i) \(p_1 \circ \phi \sim_p^F p_2 \circ \phi\) whenever \(p_1 \sim_p^F p_2\) and \(\phi \in P^m_p\).

(ii) \(p_1 \sim_{F_1} p_1 \mid V\) if \(p_1: U \to X\) and \(V \subset U\) is an open neighborhood of 0.

(b) The tangent structure is called linear if the set \(P^F_p(X)/\sim_p^F\) carries a vector space structure for each \(F \in X\) with the following consistency conditions:

(i) If \(p_1 \in [p_1] + [p_2]\) and \(\phi \in PM\) then \(p_1 \circ \phi \in [p_1 \circ \phi] + [p_2 \circ \phi]\).

(ii) If \(p \in c[p_1]\) then \(p \circ \phi \in c[p_1 \circ \phi]\) for all \(\phi \in PM\).

(c) The linear structure is called continuous if given two \((n + m)\)-plaques \(p_1(r, s)\) with \(i = 1, 2\) such that \(p_1(r, 0) = p_2(r, 0)\) for each \(r\), then there exists a plaque \(p_{12}(r, s)\) such that

\[
[p_{12}(r, s)]_s = [p_1(r, s)]_s + [p_2(r, s)]_s \quad \text{for all } r.
\]

The standard \(PM\) tangent structure on \((R, P(R))\) is defined by \(p_1 \sim_{p_2} p_2\) if \(D^i p_1(0) = D^i p_2(0)\) for all \(i \leq n\).

We shall say also that \((X, P(X), \sim)\) is a \(PM\) (linear) tangent diffeological space (TDS). The class of the plaque \(p(t)\) at a point \(F \in X\) will be denoted by \([p]\) or by \([p]_t\). The set \(T^n_F X := P^F_p(X)/\sim_p^F\) is called the \(n\)-th tangent space at \(F\), and the disjoint union \(T^n X := \bigsqcup_{F \in X} T^n_F X\) is called the \(n\)-th tangent bundle over \(X\).

Let \((X, P(X), \sim)\), \((Y, P(Y), \sim)\) be two tangent diffeologies; a \(PM\) morphism \(f: X \to Y\) is called \(PM\)-differentiable at \(F\) if for all \(n \leq k\), \(f \circ p_1 \sim_{f^*(p)} f \circ p_2\) whenever \(p_1 \sim p_2\). The set of differentiable functions is denoted \(C^k(X, Y)\) or \(C(X)\) if \(Y = R\), with the standard tangent diffeology (it is denoted by \(C^k(P(X), P(Y))\)) if necessary). If the tangent diffeologies are linear, \(f\) is called a \(C^k\)-map (or a smooth map if \(k = \infty\)) if, in addition,

\[
D^i_F f : T^n_F X \to T^n_{f(F)} Y : [p] \mapsto [f \circ p]
\]

is linear for each \(F \in X\). The set of \(C^k\)-maps is denoted by \(C^k(X, Y)\).

Remark 6.1.

[1] Notice that \(C^k(X, Y) \subset D(X, Y)\).

[2] The set of smooth diffeological spaces as objects with the differentiable maps as morphisms is a category, since, if \(f: A \to B\) and \(g: B \to C\) are differentiable then \(g \circ f\) is a \(PM\) morphism and preserves the relations. In addition

\[
D^i_F (g \circ f)[p] = (D^i_{f(F)} g)(D^i_F f[p])
\]

(and this is linear), therefore \(g \circ f\) is \(C^k\) (a \(C^k\)-map).

[3] Let \((X, P(X))\) be a diffeological space. If \(p\) is an \(n\)-plaque at \(F\) and \(f \in D(X)\), we shall denote by \(D^n_p f\) the sequence \(D^i (f \circ p)|_{a_i}, i = 1, \cdots, n\). Let \(\mathcal{F}\) a subspace of \(D(X)\). The canonical TDS defined by \(\mathcal{F}\) is defined by:

\[
p_1 \sim_{\mathcal{F}} p_2 \iff D^n_{p_1} f = D^n_{p_2} f, \forall f \in \mathcal{F}
\]

and \([p_1] + [p_2]\) is defined by: \(p_{12} \in [p_1] + [p_2]\) if and only if \(D^{p_1}_{p_2} f = D^{p_1} f + D^{p_2} f\), for all \(f \in \mathcal{F}\) and if \(c \in R\), define \(c[p_1(t)] = [p_1(ct)]\). The set of these classes is denoted \(\sim_{\mathcal{F}}\). In particular if \(\mathcal{F} = D(X)\)
then $D(X) = C^k(X)$. (called the canonical TDS). (This is an equivalence relation that satisfies the two consistency conditions. Then $(X, P(X), \sim_P)$ is an TDS with a partial vector space structure). Since $c[p]$ always exists, any difeology has a canonical TDS for which each tangent space is a cone, and satisfies $D_{p, c}(f) = cD_p f$ for every $c, p, f$. However, not every canonical TDS is linear, for example, the intersection of two transverse smooth curves in $\mathbb{R}^n$ is not linear: at the point of intersection the tangent space is the sum of two lines, everywhere else it is a line.

[4] Notice also that if $P(X)$ satisfies the condition: $\forall p_1, p_2$ there exists $p_{12}$ such that $D_{p_1, 2} f = D_{p_1} f + D_{p_2} f, \forall f \in F$ then $\mathcal{T}_F X$ is a vector space and the structure is linear. In particular this occurs if $P(X)$ generates all derivations $\phi_P: D(X) \to \mathbb{R}$, that is, there exists $p \in P(X)$ such that

\[
\phi_P(f) = D(f \circ p)|_0, \forall f \in D(X)
\]

7. Convenient spaces.

In this section we recall the definition of a convenient space and some of the most important properties of these spaces. They are the most general vector spaces with a smooth structure in which $\text{Lip}^k$-curves have unique derivatives, forming subcategory of $\mathcal{M}_R^k$. (They are the most convenient spaces for differentiation in many ways).

A dual vector space is a pair $(X, X')$ where $X$ is a vector space and $X'$ is a subspace of the algebraic dual. Each dual vector space determines several structures: (a) the weak topology $\tau_W$ is the initial one induced by $X'$ (it is the coarsest locally convex topology yielding $X'$ as topological dual), (b) the Mackey topology $\tau_M$ is the finest locally convex topology yielding $X'$ as topological dual [11], (c) $X'$ generates a $\text{Lip}^k$-structure, (d) the Mackey closure topology $\tau_{MC}$ is the final topology generated by the set of $\text{Lip}^k$-curves. In addition it determines a bornology, a $\text{L}^\infty$ vector space and a convergence structure. A good description of the functors that relate these structures is given in [7].

A preconvenient space is a dual vector space $(X, X')$ such that the $\text{Lip}^k$ structure that $X'$ determines (linearly generated) has exactly $X'$ as set of linear structure functions. An equivalent condition is that $X'$ is the topological dual under $\tau_{MC}$. The category of preconvenient spaces is denoted $\mathcal{Pre}$.

Remarks 7.1.

[1] Every preconvenient space have the three locally convex topologies defined above and each of these topologies have $X'$ as the space of continuous linear functionals. Every l.c.s. $X$ determines a dual pair with $X'$ the topological dual, however, not every l.c.s. forms a preconvenient space with the topological dual. For example, an uncountable direct sum of $\mathbb{R}$ with the box topology is not pre-convenient [11]. However, every metrizable space is pre-convenient and in this case the Mackey topology and the Mackey closure topology coincide. (in general $\tau_W \subset \tau_M \subset \tau_{MC}$). A dual vector space is preconvenient iff the Mackey topology is an inductive limit of seminormable spaces.

[2] If a d.v.s. $(X, X')$ is preconvenient, the space determines and is determined by each of the structures mentioned above: a topological vector bornology $(B \subset X$ is bounded iff $l(X)$ is bounded $\forall l \in X'$), a bornological locally convex topology $\tau_M$, a linearly generated $\text{L}^\infty$ structure (generated by $X'$), a linearly generated $\text{Lip}^k$ structure (generated by $X'$), a convergence structure (a sequence $(x_n)$ converges to 0 if there exists reals $t_n \to \infty$ such that $\{t_n a_n/n \in \mathbb{N}\}$ is bounded), an arc-determined topology (the Mackey closure topology). The space $X'$ is recovered by taking the set of linear functions respecting the structures. Preconvenient spaces form the category that is identified with subcategories of these categories, for example, $\mathcal{Pre}$ is isomorphic to the category of linearly generated $\text{Lip}^k$ vector spaces.

[3] Given $X, Y$ pre-convenient, a map $f: X \to Y$ is a $\text{Lip}^k$ map ( or a $\mathcal{Pre}$ morphism) iff it is continuous for the Mackey closure topology [15], iff $f'(Y') \subset X'$, iff it is continuous for the Mackey topology.

[4] Given a vector space $X$ and $X_j \in \mathcal{Pre}, \forall j \in I$ and linear maps $m_j: X \to X_j$ (respectively $m_j: X_j \to X$), the initial (resp. final) $\text{Lip}^k$-structure on $X$ defines an initial (resp. final) preconvenient structure on $X$.

Given $(X, X') \in \mathcal{Pre}$ and $c: \mathbb{R} \to X$. We say that $c'(t) \in X$ is the weak derivative of $c$ at $t$ if $(l \circ c)'(t)$ exists and equals $l(c'(t)), \forall l \in X'$: $c'(t)$ is the derivative of $c$ with respect to $\tau_W$. Also, $\int_0^t c$ is the weak integral of $c$ if $\int_0^t (l \circ c)(u)du$ exists and equals $l(\int_0^t c), \forall l \in X'$. 

A preconvenient space is called separated if every $\text{Lip}^k$-curve has at most one weak derivative. The subcategory of these spaces is denoted $\text{Spre}$.

A space $(X, X') \in \text{Spre}$ is called complete if every $\text{Lip}^k$-curve has a weak derivative. The subcategory of them is denoted $\text{Con}$.

Remarks 7.2.

[1] Given $(X, X') \in \text{Pre}$, the following are equivalent:

a. $X'$ separates points.

b. Every $\text{Lip}^k$-curve has at most one weak derivative.

c. Every $\text{Lip}^k$-curve has at most one weak integral.

d. $\tau_{MC}$ is Hausdorff.

e. $\tau_M$ is Hausdorff.

[2] Given $(X, X') \in \text{Spre}$, the following are equivalent:

a. $(X, X')$ is convenient.

b. For every $\text{Lip}^k$-curve $c$, the weak integral $\int_0^k c$ exists.

c. If $\{l(x_n)/n \in \mathbb{N}\}$ is bounded $\forall l \in X'$ and $(t_n) \in I^1$, then $\sum t_n x_n$ converges weakly.

d. Every Mackey Cauchy sequence converges (a sequence $(x_n)$ converges to $x_0$ if there exists $t_n \in \mathbb{R}^*, \forall n \in \mathbb{N}$ such that $\lim_{n \to \infty} t_n = \infty$ and $\{l(t_n(x_n - x_0))/n \in \mathbb{N}\}$ is bounded $\forall l \in X'$.

The sequence is called a Mackey Cauchy sequence if there exists $t_{n, m} \in \mathbb{R}^*, \forall n, m \in \mathbb{N}$ with $\lim_{n, m \to \infty} t_{n, m} = \infty$ and the set $\{(l(t_{n, m}(x_n - x_m))/n, m \in \mathbb{N}\}$ is bounded $\forall l \in X'$.

[3] The product $\pi X_j$ of convenient spaces $X_j$ is convenient (the $\text{Lip}^k$ curves on $X$ are those whose coordinates are $\text{Lip}^k$). The coproduct $\coprod X_j$ is also convenient (the vector space is the direct sum, the dual is the product of the duals, the structure curves $c: \mathbb{R} \to X$ are those which are locally representable as finite sums of $\text{Lip}^k$ curves $c_j: \mathbb{R} \to X_j$). Inductive limits may also be convenient: if $X_j$ is convenient, and it is a pre-subspace of $X_{j+1}$, closed with $\tau_{MC}$, then the inductive limit in $\text{Pre}$ is convenient (the $\text{Lip}^k$ curves are locally $\text{Lip}^k$ curves into some $X_j$). The tensor product of convenient spaces is also convenient.

[4] Subspaces are convenient in some cases: If $V$ is a vector space, $X$ is convenient and $f: V \to X$ is injective, linear and $f(V)$ is closed with the Mackey closure topology, then the initial pre-structure of $f(V)$ is convenient. Quotients may also be formed in some cases: if $f: X \to V$ where $X$ is convenient and $V$ is a vector space, then the final pre-structure on $V$ is convenient iff the kernel of $f$ is closed in the Mackey topology and the final locally convex topology of $V$ is Mackey complete.

[5] If $X$ is convenient and $Y$ is preconvenient then $L(X,Y)$ is a convenient space with the initial structure induced by the evaluations. More generally $L^n(X,Y)$ is convenient and

$L(X, L^{n-1}(X,Y)) = L^n(X,Y)$

[6] If $X$ is a convenient space and $k \in \mathbb{N} \cup \{\infty\}$, then $\text{Lip}^k(\mathbb{R}, X)$ is also a convenient space: the $\text{Pre}$-structure is the initial one induced by $\delta^0, \ldots, \delta^{k+1}$ where $\delta^k: \text{Lip}^k(\mathbb{R}, X) \to \ell^\infty(\mathbb{R}^{<\rho'}), X$), (and $\ell^\infty(\mathbb{R}^{<\rho'}), X$) has the initial structure defined by evaluations) is defined as follows: if $f: D \subset \mathbb{R} \to X$, let

$$D^{<k>} = \{(t_0, \ldots, t_k) \in D^{k+1}: t_j \neq t_i \text{ if } i \neq j\}$$

define $\delta^0 f = f$ and

$$\delta^k f(t_0, \ldots, t_k) = \frac{k}{t_0 - t_k} (\delta^{k-1} f(t_0, \ldots, t_{k-1}) - \delta^{k-1} f(t_1, \ldots, t_k))$$

Given $k \in \mathbb{N} \cup \{\infty\}, (Y, C, F)$ a $\text{Lip}^k$-space and $X$ a convenient space, then $\text{Lip}^k(Y, X)$ is also a convenient space, where $\text{Lip}^k(Y, X)$ denotes the vector space formed by the $\text{Lip}^k$-maps $Y \to X$ together with the initial $\text{Pre}$-structure induced by the linear maps $c^*: \text{Lip}^k(Y, X) \to \text{Lip}^k(\mathbb{R}, Y)$, for $c \in C$.

Differentials of functions in preconvenient spaces are defined as follows: let $U \subset \mathbb{R}$ be open, $(X, X') \in \text{Pre}$. A curve $c: U \to X$ is called differentiable if the weak derivative of $c$ at $t \in U$ exists $\forall t \in U$. And it is called $k$-times differentiable if $c$ is differentiable and $c(t) = (k - 1)$-times differentiable. In particular if $X$ is a convenient space then $c$ is a $\text{Lip}^k$-curve iff $c$ is $j$-times differentiable and $c^j$ is a $\text{Lip}^{k-j}$ curve if $0 \leq j \leq k$. 
The differential of a function \( f: U \subset X \to Y \), where \((Y, Y')\) is also preconvenient is defined as follows: \( f \) is called differentiable at \( x \) in direction \( v \in X \) if \( c(t) = f(x + tv) \) is differentiable. This derivative is denoted \( df(x, v) \) and \( f \) is called differentiable if \( df(x, v) \) exists \( \forall x \in U, \forall v \in X \); (if \( df \) is a \( \text{Lip}^0 \)-map also, then \( f \) is called \( \text{Lip-differentiable} \)). The differential of \( f \) is the map \( df: U \times X \to Y \). And \( f \) is called \textit{k-times differentiable} if \( f \) is differentiable and \( df(\cdot, v) \) is \((k-1)\)-times differentiable \( \forall v \in X \); (if \( f \) is \( \text{Lip-differentiable} \) and \( df \) is \((k-1)\)-times \( \text{Lip-differentiable} \) \( \forall v \in X \) then \( f \) is called \textit{k-times \( \text{Lip-differentiable} \)}. In particular if \( X, Y \) are convenient spaces then these two definitions and the definition of a \( \text{Lip}^k \) map are equivalent. It is also equivalent to a stronger condition of differentiability in which the limit exists uniformly with respect to \( x \) and \( v \) and \( f'(x) \in L(X, Y) \) where \( f'(x)(v) = df(x, v) \) [21](with respect to \( \text{b-compact sets}: K \) is \( \text{b-compact} \) in \( X \) if there exists \( B \) absolutely convex such that \( K \) is compact in \( X_B = \bigcup_{n \in \mathbb{N}} nB \)).
then $c$ is a $\text{Lip}^k$-curve if $c$ is $j$-times differentiable and $c^j$ is a $\text{Lip}^{k-j}$ curve if $0 \leq j \leq k$.

The differential of a function $f: U \subset X \rightarrow Y$, where $(Y, Y')$ is also preconvenient is defined as follows: $f$ is called differentiable at $x$ in direction $v \in X$ if $c(t) = f(x + tv)$ is differentiable. This derivative is denoted $df(x, v)$ and $f$ is called differentiable if $df(x, v)$ exists $\forall x \in U$, (if $df$ is a $\text{Lip}^k$-map also, then $f$ is called $\text{Lip}$-differentiable) $\forall v \in X$. The differential of $f$ is the map $df: U \times X \rightarrow Y$. And $f$ is called $k$-times differentiable if $f$ is differentiable and $df(\cdot, v)$ is $(k-1)$-times differentiable $\forall v \in X$, (if $f$ is Lip-differentiable and $df$ is $(k-1)$-times Lip-differentiable $\forall v \in X$ then $f$ is called $k$-times Lip-differentiable). Intuitively, if $X, Y$ are convenient spaces then these two definitions and the definition of a $\text{Lip}^k$ map are equivalent. It is also equivalent to a stronger condition of differentiability in which the limit exists uniformly with respect to $x$ and $v$ and $f'(x) \in L(E, F)$ where $f'(x)(v) = df(x, v)$ [21] with respect to $b$-compact sets: $K$ is $b$-compact in $E$ if there exists $B$ an absolutely convex such that $K$ is compact in $E_B = \bigcup_{n \in \mathbb{N}} nB$.

8. Convenient spaces and tangent diffeologies.

Given a pre-convenient space $(X, X')$, a linear tangent diffeology is defined on $X$ as follows: let $(X, P(X))$ be the diffeology defined by $(\mathcal{C}, \mathcal{F})$ and let $\sim$ be the canonical tangent diffeology defined by $X'$. $(X, P(X), \sim)$ is linear: given $[c_1], [c_2] \in T^n_p X$, we define $[c_1] + [c_2] = [c]$ where $c(t) = c_1(t) + c_2(t), c_1 \in [c_1], i = 1, 2$, this is well defined since $D(f \circ c)_0 = D(f \circ c_1)_0 + D(f \circ c_2)_0, \forall f \in X'$.

Let $\alpha: X \rightarrow T_F X$ be defined by $\alpha(v_0) = [F + tv_0]$. Then $\alpha$ is linear by definition of the linear structure on $T_F X$.

Remark 8.1.

1. $\alpha$ is 1-1 (or $X \subset T_F X$) if and only if $(X, X')$ is separated. Proof: If $(X, X')$ is not Hausdorff then $X'$ does not separate points $\Leftrightarrow \exists x_1, x_2 \in X$ different such that $l(x_1) = l(x_2), \forall l \in X'$ then $l \circ c_1 = l \circ c_2, \forall l \in X'$, where $c_1(t) = F + tx_1, i = 1, 2$, therefore $Dl \circ c_1|_0 = Dl \circ c_2|_0$ and then $[c_1] = [c_2]$. This proves that $\alpha$ is not 1-1. Assume now that $\alpha$ is not 1-1, then: $\alpha(x_1) = \alpha(x_2)$ for some $x_1, x_2 \in X$. That is $F + tx_1 = [F + tx_2], then Dl \circ c_1 = Dl \circ c_2$ for all $l \in X'$, therefore $Dl(F + tx_1)|_0 = Dl(F + tx_2)|_0, \forall l \in X'$, then $l(x_1) = l(x_2)$forall $l \in X'$, then $X'$ does not separate points.

2. $\alpha$ is 1-1 and onto (or $X = T_F X$) if and only if $(X, X')$ is a convenient space. Proof: Assume $(X, X')$ is complete, then $\forall c \in \mathcal{C}$ the weak derivative $c'(t)$ exists, then $Dl(l \circ c)(t_0) = l(c'(t_0)) = Dl(F + tc'(t)|_{t_0}, \forall l \in X'$ then $[c] = [F + tc'(t_0)]$ and therefore $\alpha$ is onto. Assume now that $\alpha$ is onto. Let $c \in \mathcal{C}$ and let $[c] = \alpha(v_0) = [F + tv_0]$ where $v_0 \in X$. Then $Dl(l \circ c) = l(v_0), \forall l \in X'$. Then $c'(t)$ exists.

3. $(X, P(X), \sim)$ is continuous if $(X, X')$ is convenient. Proof: Let $P_i(r, s), i = 1, 2$ be two 2-plaques such that $p_1(r, 0) = p_2(r, 0)$. Identify $T_F X$ with $X$ and define

$$p(r, t) = p_1(r, 0) + t([p_1(r, s)]_s + [p_2(r, s)]_s)$$

then $p(r, 0) = p_1(r, 0)$ and $[p(r, t)]_t = [p_1(r, s)]_s + [p_2(r, s)]_s$. Using the definitions it is proved that $p(r, t)$ is a 2-plaque.

4. If $(X, X'), (Y, Y')$ are convenient then $D(X, Y) = \mathcal{M}(X, Y) = C^k_X(X, Y)$: we have that $D(X, Y) = \mathcal{M}(X, Y)$ because $PM = PM^k$ and $C^k_X(X, Y) \subset D(X, Y)$ from definition. If $f \in D(X, R)$, then $f \in Lip^k(X)$, therefore $Dl f(x + tv)|_0$ exists.

5. Conversely, a $PM^k$ tangent diffeology $(X, P(X), \sim)$ is a convenient space iff it is linear and $\forall F \in X$ the space $T_F X$ is bijective with $X$, defining a sum on $X$ which is independent of $F$, and with this operation $P(X)$ is generated by a subspace $X'$ of linear functions $l: X \rightarrow R$.

9. Manifolds.

Let $\mathcal{V}$ be a subcategory of the category of locally convex vector spaces such that for any two objects $V, W$ the set of morphisms $D^k(V, W)$ is a subspace of $k$-times weakly differentiable maps $f: U \subset V \rightarrow W$, $U$ open. In particular some examples are:

a. Objetcs are Banach spaces and $D^k(V, W)$ is formed by $C^k$ Frechet differentiable maps $f: U \subset V \rightarrow W$, $U$ open.
b. Objects are locally convex vector spaces and $D^k(V, W)$ is the set of Gateaux $C^k$ maps $f: U \subset V \to W$, $U$ open (k-times Gateaux differentiable and $f^k(x, v_1, \cdots, v_k)$ is continuous as a map $U \times V^k \to W$).

c. Objects are (pre)convenient spaces and $D^k(V, W) = Lip_k^k(V, W)$.

d. Objects are the spaces $\mathbb{R}^n$, for each $n \in \mathbb{N}$ and $D^k(\mathbb{R}^n, \mathbb{R}^m) = P^m_nM^k$, denoted $PY^k$.

A $\mathcal{V}k$-manifold modelled on a vector space $V$ is a set $M$ together with a set of bijective maps (called \underline{charts}) $\phi_i: M_i \subset M \to V_i \subset V$, $i \in I$ such that

1. $V_i$ is open for each $i \in I$ and $\bigcup_{i \in I} M_i = M$.
2. $\phi_i(M_i \cap M_j)$ is open and $\phi_i \circ \phi_j^{-1} \in D^k(V(V), V)$ for all $i, j \in I$.
3. The set of maps $\phi_i$ is maximal with respect to conditions 1, 2. A set of charts $\mathcal{A}$ satisfying 1, 2 is called an atlas.

A map $f: X_1 \to X_2$ between two $\mathcal{V}k$-manifolds modelled on $V_i, W$ respectively is called a $C^k$-map if $\phi_j \circ f \circ \phi_i^{-1} \in D^k(V, W)$ for every pair of charts $\phi_i, \phi_j$ in $X_1, X_2$ respectively. The set of such $C^k$-maps is denoted $C^k(X_1, X_2)$.

Given a $\mathcal{V}k$-manifold $M$ modelled on $V$ with atlas $\mathcal{A}$, a \underline{tangent vector} at $x_0 \in M$ may be defined in one of the following ways:

[a] It is an equivalence class of pairs $(\phi, v)$ where $\phi \in \mathcal{A}$, $v \in V$ and $(\phi, v_1) \sim (\phi, v_2)$ iff $d(\phi^{-1} \circ \phi_1)|_{x_0}(v_1) = v_2$.

[b] It is an equivalence class of $C^k$-curves $c: I \to M$ with $c(0) = x_0$: choose any chart $\phi \in \mathcal{A}$ with $\phi(0) = x_0$, fixed, and if $c_1, c_2$ are two curves, define $c_1 \sim c_2$ iff

$$d(\phi^{-1} \circ c_1)|_{x_0} = d(\phi^{-1} \circ c_2)|_{x_0}$$

[c] It is an equivalence class of curves $c: I \to M$ with $c(x) = x_0$: $c_1 \sim c_2$ iff $d(f \circ c_1)|_{x_0} = d(f \circ c_2)|_{x_0}$ for all $f \in D^k(M, V)$.

[d] It is a derivation on the set of germs of functions in $D^k(M, \mathbb{R})$ at $x_0$.

Remark 9.1.

[1] A topology is defined on a manifold $M$: $U \subset M$ is open iff $\phi_i(U \cap M_i)$ is open $\forall i \in I$. Usually the topology is required to be Hausdorff and normal, and perhaps also paracompact in order to have partitions of unity.

[2] Definitions 1, 2, 3 of a tangent vector are equivalent and the set of tangent vectors at $x_0$, denoted $T_{x_0}M$, is bijective with $V$. The bijection is given by $\tau: V \to T_{x_0}M$, where $\tau(v) = [\phi, v]$ for a fixed $\phi$ in definition 1, $[t \mapsto \phi(\phi^{-1}(x_0) + tv)]$ in definitions 2 and 3. Under definition 4, the space $T_{x_0}M$ contains $V$: $\tau(v)f := d(f \circ \phi^{-1}(x_0) + tv)|_0$, and it is equal to $V$ in the category of Banach manifolds.

[3] If $M$ is a compact finite dimensional manifold, then the manifold structure may be recovered using only the algebra $A = C^k(M)$: the set $M$ and its topology comes from Gelfand-Naimark theorem (the set is the class of irreducible representations of $C^k(M)$ and its topology is the pointwise limit topology); having $M$, we define the diffeology $P(M)$ as the set of plaques $p: U \subset \mathbb{R}^n \to M$, where $U$ is open and $n \in \mathbb{N}$ such that $f \circ p \in PY^k$ for each $f \in A$. The canonical tangent structure defined by $A$ allows the definition of $C^k$-maps. With this, the atlas $\mathcal{A}$ is the set of plaques $p: U \to M_p$ such that $M_p \subset M$ is open, $p$ is a homeomorphism and $p^{-1}$ is smooth. The same construction works if $M$ is any manifold and we know not only $A$ but also the set $M$.

[4] A similar problem is: given a set $M$ and an algebra $A$ of functions on $M$, is $M$ a $\mathcal{V}k$ manifold with algebra $A$? A solution is: the topology on $M$ and the set $M$ is constructed as before, then the answer is positive iff the collection of sets $M_p$ covers $M$. We may notice that if $M$ is a manifold then the tangent structure is linear and continuous, but it is not a sufficient condition (consider the sphere $S^2$ with the diffeology $P(S^2)$ generated by $M^k$ plaques along parallels, then the algebra $A$ of $M^k$ functions generates $P(S^2)$. The tangent structure is linear and continuous, but it is not a manifold.) It may be noticed also that if the tangent structure is linear and continuous then the set $\Gamma(M)$ of smooth vector fields of $M$ is an $A$ module (even if $M$ is not a manifold) and it must be a projective module if $M$ is a compact smooth manifold.

[5] Every $\mathcal{V}k$ manifold $M$ modelled on a vector space $V$ defines a linear, continuous $TDS$ in which $P_1(M)$ is the set of $C^k$-maps $p: U \subset \mathbb{R}^k \to M$, with the canonical tangent structure defined by $C^k(M, \mathbb{R})$. In general

$$C^k(M) = C^k_f(M) \subset C^k(M)$$

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and they are equal if $V$ satisfies that

$$\Phi\{D^k(R^n, V), n \in \mathbb{N}\} = D^k(M)$$

In this case $P_{k}$ is a subcategory of $\mathcal{PM}$ (for any two manifolds $M_1, M_2$ we have $C^k(M_1, M_2) = C^k(M_1, M_2)$). This occurs if $V$ is a TDS, in particular if $V$ is a convenient space, such as a Fréchet space. From remark 2 it follows that at each point both tangent spaces are the same and isomorphic to $V$. Another TDS is defined by taking

$$P_2(M) = \{p: u \subset R^n \to M/f \circ p \in D^k(R^n, R), \forall f \in D^k(M)\}$$

We have $P_1(M) \subset P_2(M)$ and they are equal if

$$\Gamma(D^k(V, R) = \{D^k(R^n, V), \forall n \in \mathbb{N}\}$$

In particular this holds if $V$ is a convenient space.

Acknowledgments

I gratefully acknowledge the support of the Vicerrectoría de Investigación of the University of Costa Rica. I would like also to thank Joseph C. Várilly for helpful discussions.

References

[1] Differentiabilit dans les espaces localement convexes. Districtures, These Doct. Sci. Math. Fac. Sci. Univ. Paris 1962.
[2] J. Boman, Differentiability of a function and of its compositions with functions of one variable, Math. Scand. 20, (1967) 249–68.
[3] Choquet-Bruhat, De Witt-Morette, Dillard-Bleick, Analysis, Manifolds and Physics, Amsterdam, North Holland, 1982.
[4] P. Donato, Géométrie des orbites coadjointes des groupes de difféomorphismes, in: C. Albert, ed., Géométrie Symplectique et Mécanique, Lecture Notes in Mathematics 1416 (Springer, Berlin, 1988) 84–104.
[5] P. Donato, Les difféomorphismes du circle comme orbit symplectique dans les moments de Virasoro, Preprint CPT–92/P.2681, CNRS–Luminy, 1992.
[6] P. Donato and P. Iglesias, Cohomologie des formes dans les espaces difféologiques, Preprint CPT–87/P.1986, CNRS–Luminy, 1987.
[7] A. Frolicher and A. Kriegl, Linear Spaces and Differentiation Theory, (John Wiley, 1987).
[8] A. Frolicher, Durch Monoide erzeugte kartesisch abgeschlossene Kategorien, Seminarberichte Fachbereich Math., Fernuniversitat Hagen, (1979) 7–48.
[9] A. Frolicher and W. Bucher, Calculus in Vector Spaces Without Norm, Lect. Notes in Math. 30, Springer, Berlin, 1966. N° 4723.
[10] H. Jarchow, Locally convex spaces, Teubner, Stuttgart, 1981.
[11] H. Keller, Differential calculus in locally convex spaces, Lect. Notes Math. 417, 1974.
[12] H. Keller, Raume stetiger multilinearer Abbildungen als Limesräume, Math. Ann. 159, (1965) 259-270.
[13] H. Keller, Ueber Probleme, die bei einer Differentialrechnung in topologischen Vektorraumen auftreten, Festaband z. 70. Geburtstag v. Rolf Nevanlinna, Springer Berlin (1966) 49-57.
[14] A. Kriegl, Dierichtigen Raume fur Analysis im Unendlich-Dim., Monatshefte Math 94, (1982) 109–124.
[16] A. Kriegl, A convenient setting of differential calculus in locally convex spaces, Abh. Akad. Wiss. DDR, 2N, (1984) 135–40.

[17] S. Lang, Introduction to Differentiable Manifolds, Interscience Publ., Wiley, New York, 1962.

[18] Lawvere, L. W., Schanuel, S. H., Zame, W. R., On $C^\infty$-function spaces, Preprint (1981).

[19] G. Marinescu, Espaces vectoriels pseudotopologiques et theorie des distributions. VEB Deutsch. Berlag d. Wissensch., Berlin, 1963.

[20] J. Milnor, Remarks on infinite dimensional Lie groups in Relativity, groups and Topology, Les Houches (1983), 1007-1057; North Holland, Amsterdam-New York (1984).

[21] U. Seip, A convenient setting for differential calculus. J. of Pure and Appl. Algebra 14 (1979) 73-100.

[22] J. M. Souriau, Un algoritme générant de structures quantiques, Astérisque, hors série (1985) 341–399.

[23] J. M. Souriau, Groupes différentiels, Lecture notes in math., 838 (1980) 91-128.

[24] C. Torre, A Tangent bundle on diffeological spaces, math/9801046 (1998).

[25] C. Torre, Examples of smooth diffeological spaces, Preprint (1999).

[26] C. Torre and A. Banyaga, A symplectic structure on coadjoint orbits of diffeomorphism subgroups, Ciencia y Tecnologa 17 No 2, (1993) 1–14.

[27] Y. Wong, Differential Calculus and Differentiable Partitions of Unity in Locally Convex Spaces, Univ. of Toronto, 1974.

[28] S. Yamamuro, Differential Calculus in Topological Linear Spaces, Lect. Notes in Math. 374, Springer, Berlin, 1974.