SPANNING CLASS IN THE CATEGORY OF BRANES

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Abstract. Given a generic anticanonical hypersurface \( Y \) of a toric variety determined by a reflexive polytope, we define a line bundle \( \mathcal{L} \) on \( Y \) that generates a spanning class in the bounded derivative category \( D^b(Y) \). From this fact, we deduce properties of some spaces of strings related with the brane \( \mathcal{L} \). We prove a vanishing theorem for the vertex operators associated to strings stretching from branes of the form \( \mathcal{L}^{\otimes i} \) to nonzero objects in \( D^b(Y) \). We also define a gauge field on \( \mathcal{L} \) which minimizes the corresponding Yang-Mills functional.

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1. Introduction

It is well-known that a compact toric manifold is not Calabi-Yau. However, Batyrev [3] showed the existence of anticanonical hypersurfaces, in the toric variety \( X \) determined by a reflexive polytope, that are Calabi-Yau. In this note, we will prove some particular properties of \( D \)-branes, strings, vertex operators and gauge fields on these hypersurfaces.

We will denote by \( Y \) such a hypersurface. The \( D \)-branes of type \( B \) on \( Y \) are the objects of \( D^b(Y) \), the bounded derived category of coherent sheaves on \( Y \) (see monograph [2]). In this context, given the branes \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), the strings with ghost number \( k \) stretching from the brane \( \mathcal{F}_1 \) to the brane \( \mathcal{F}_2 \) are the elements of the ext group \( \text{Ext}^k(\mathcal{F}_1, \mathcal{F}_2) \) [1, Section 5.2].

Let \( \Delta \) be a reflexive polytope in \( (\mathbb{R}^n)^* \) that defines the toric variety \( X \). The multiple \( (n-1)(-K_X) \) of the anticanonical divisor of \( X \) determines a line bundle on \( X \), and we put \( \mathcal{L} \) for the restriction of that bundle to \( Y \). In Propositions 2 and 3 we will prove that the family \( \{ \mathcal{L}^{\otimes i} \mid i \in \mathbb{Z} \} \) is a spanning class of \( D^b(Y) \) [5] [15]. Thus, given \( \mathcal{F} \) a nonzero brane on \( Y \), there exist two integers \( a, r \) such that the space of strings

\[
\text{Ext}^r(\mathcal{L}^{\otimes a}, \mathcal{F}) \neq 0.
\]

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Condition (1.1) implies relations between certain spaces of strings. If \( \mathcal{H} \) is a brane such that can bind with the above brane \( \mathcal{L} \otimes a \) to form a new brane \( \mathcal{G} \), then we deduce relations between the spaces of strings \( \text{Ext}^j(\mathcal{F}, \mathcal{G}) \) and \( \text{Ext}^j(\mathcal{F}, \mathcal{H}) \), which are gathered in Theorem 4.

In the case that a brane \( \mathcal{G} \) can decay to the above brane \( \mathcal{F} \) and other one \( \mathcal{J} \), then we prove relations between the spaces \( \text{Ext}^j(\mathcal{J}, \mathcal{L} \otimes a) \) and \( \text{Ext}^j(\mathcal{G}, \mathcal{L} \otimes a) \), which we have collected in Theorem 5.

We set \( \text{Hom}(\ldots, \ldots) \) for the sheaf functor \( \text{Hom} \) (see [16, page 87]) of the category \( \text{Coh}(Y) \) of coherent sheaves on \( Y \).

\[
\text{Hom}(\ldots, \ldots) : \text{Coh}(Y)^{\text{op}} \times \text{Coh}(Y) \to \mathcal{Sh},
\]

where \( \mathcal{Sh} \) is the category of sheaves of \( \mathbb{C} \)-vector spaces on \( Y \). Thus, one has the derived functor

\[
R\text{Hom}(\ldots, \ldots) : D^b(Y)^{\text{op}} \times D^b(Y) \to D(\mathcal{Sh}),
\]

where \( D(\mathcal{Sh}) \) is the derived category of \( \mathcal{Sh} \). By definition \( \text{Ext}^k(\mathcal{F}, \mathcal{G}) = H^k R\text{Hom}(\mathcal{F}, \mathcal{G}) \).

The space of vertex operators for strings, with ghost number \( k \), between the branes \( \mathcal{F} \) and \( \mathcal{G} \) is \([19]\)

\[
\bigoplus_q H^q(Y, \text{Ext}^k(\mathcal{F}, \mathcal{G})).
\]

We determine a rectangle \( R \) in \( \mathbb{R}^2 \) where “are located” the non-trivial spaces of vertex operators for strings from \( \mathcal{L} \otimes b \) and \( \mathcal{F} \), \( b \) being an arbitrary integer. That is, if the space of vertex operators \( H^q(Y, \text{Ext}^p(\mathcal{L} \otimes b, \mathcal{F})) \) is nonzero, then \((p, q)\) belongs to \( R \) (Theorem 7). From this theorem, it turns out the existence of an integer \( i_0 \), such that if \( i \leq i_0 \) the space of vertex operators \( H^q(Y, \text{Ext}^p(\mathcal{L} \otimes i, \mathcal{F})) \) corresponding to a some point of \( R \) is in fact nontrivial (Corollary 8).

Although \( Y \) is not smooth, we define in Section 5 connections on \( \mathcal{L} \) by using Kähler differentials. As \( (n-1)\Delta \) is very ample polytope, its lattice points determine an embedding in a projective space \( Y \hookrightarrow \mathbb{P}^N \), such that \( \mathcal{L} = j^*(\mathcal{O}_{\mathbb{P}^N}(1)) \). By pulling back the Chern connection on \( \mathcal{O}_{\mathbb{P}^N}(1) \), we define a particular connection on \( \mathcal{L} \). In Proposition 10 we prove that this gauge field minimizes the Yang-Mills action.

In the case that \( Y \) is a hypersurface invariant under the action of the torus \( T \) of the variety \( X \), then \( \mathcal{L} \) is a \( T \)-equivariant brane on \( X \) \([20]\). In Section 6 we determine the value of the localization of the \( T \)-equivariant Chern class of \( \mathcal{L} \). This result is stated in Theorem 12.
2. Calabi-Yau hypersurfaces

Reflexive polytopes. Let $\Delta$ be a lattice polytope of dimension $n$ in $(\mathbb{R}^n)^*$. Given a facet $F$ of $\Delta$, there is a unique vector $v_F \in \mathbb{Z}^n$ conormal to $F$ and inward to $\Delta$. So, $F$ is on the hyperplane in $(\mathbb{R}^n)^*$ of equation $\langle m, v_F \rangle = -c_F$, with $c_F \in \mathbb{Z}$ and $\Delta = \bigcap_F \{ m \in (\mathbb{R}^n)^* \mid \langle m, v_F \rangle \geq -c_F \}$.

We denote by $X$ the toric variety determined by $\Delta$ and by $D_F$ the divisor on $X$ associated to the facet $F$. The divisor $D_\Delta := \sum_F c_F D_F$ is Cartier, ample and base pointfree [7, page 269]. Moreover, the torus invariant divisor $K_X := -\sum_F D_F$ is a canonical divisor of $X$.

From now on, we assume that $\Delta$ is a reflexive polytope; that is, $c_F = 1$ for all facet $F$. Then the divisor $K_X$ is Cartier and the variety $X$ is Gorenstein. Thus, the canonical sheaf $\omega_X = \mathcal{O}_X(K_X)$ is a line bundle and $X$ is Fano (that is, the anticanonical divisor $-K_X$ is ample).

On the other hand, a generic element $Y$ of the linear system $| -K_X |$ is an orbifold. By the adjunction formula, the canonical sheaf of $Y$, $\omega_Y$, satisfies $\omega_Y \simeq \omega_X \otimes_{\mathcal{O}_X} (\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$, where $\mathcal{I}_Y$ denotes the ideal sheaf of $Y$. As $\omega_X = \mathcal{O}_X(-Y)$ and $\mathcal{I}_Y/\mathcal{I}_Y^2 = \mathcal{O}_X(-Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, ones deduces the triviality of $\omega_Y$. The hypersurface $Y$ is in fact a Calabi-Yau variety (for more details see [6, Sec. 4.1]).

Homological dimension. Henceforth, $Y$ will be a generic anticanonical hypersurface of $X$. Since $Y$ is a Gorenstein variety a Serre functor $S : D^b(Y) \to D^b(Y)$, is $S = ( ) \otimes \omega_Y[n-1]$, where $[n-1]$ denotes the shifting of the complex by $n-1$ to the left [4]. By the Serre duality, we have functorial isomorphisms

$$\text{Hom}_{D^b(Y)}(\mathcal{F}, \mathcal{G}) \simeq \text{Hom}_{D^b(Y)}(\mathcal{G}, S(\mathcal{F}))^\vee$$

for every objects $\mathcal{F}, \mathcal{G}$ of $D^b(Y)$.

In particular, if $\mathcal{A}, \mathcal{B}$ are objects of the category $\mathbf{Coh}(Y)$ of coherent sheaves on $Y$, then

$$\text{Ext}^k(\mathcal{A}, \mathcal{B}) = \text{Hom}_{D^b(Y)}(\mathcal{A}, \mathcal{B}[k]) \simeq \text{Ext}^{n-1-k}(\mathcal{B}, \mathcal{A} \otimes \omega_Y)^\vee.$$

As $\text{Ext}^{n-1-k}(\mathcal{B}, \mathcal{A} \otimes \omega_Y) = 0$, for $n-1 < k$, the homological dimension of the category $\mathbf{Coh}(Y)$ is $\leq n-1$. On the other hand, taking $\mathcal{A} = \mathcal{O}_Y$ and $\mathcal{B} = \omega_Y$, one has

$$\text{Ext}^{n-1}(\mathcal{O}_Y, \omega_Y) = \text{Hom}(\omega_Y, \omega_Y)^\vee = \text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y)^\vee.$$
As this space is different from zero, we deduce that the mentioned homological dimension is \(n-1\). Therefore, given \(B\) an object of \(\mathfrak{Coh}(Y)\), there exists \(p_0\), such that

\[
\text{Hom}_{D^b(Y)}(A, B[p]) = 0,
\]

for all \(p > p_0\) and for any object \(A\) of \(\mathfrak{Coh}(Y)\).

Let \(F^\bullet\) be an object in the bounded derived category \(D^b(Y)\), then

\[
q_0 := \max\{m \mid H^{-m}(F^\bullet) \neq 0\} < \infty.
\]

On the other hand, the Grothendieck spectral sequence

\[
E_2^{pq} := \text{Hom}_{D^b(Y)}(H^{-q}(F^\bullet), B[p])
\]

abuts to \(\text{Hom}_{D^b(Y)}(F^\bullet, B[p+q])\). As \(E_2^{pq} = 0\) whether \(p > p_0\) or \(q > q_0\), we deduce the following lemma.

**Lemma 1.** Given an object \(F^\bullet\) of \(D^b(Y)\) and a coherent sheaf \(B\) on \(Y\), there exists an integer \(j_0\) such that

\[
\text{Hom}_{D^b(Y)}(F^\bullet, B[j]) = 0
\]

for all \(j > j_0\).

An analog reasoning proves the existence of an integer \(j_1\) such that

\[
\text{Hom}_{D^b(Y)}(F^\bullet, B[j]) = 0
\]

for all \(j < j_1\).

**Spanning class.** The multiple \(-(n-1)K_X\) of the anticanonical divisor \(-K_X\) is a very ample divisor on \(X\) [7, page 71]. We put \(L\) for the restriction to \(Y\) of the very ample invertible sheaf on \(X\) defined by the Cartier divisor \(-(n-1)K_X\).

Let \(F^\bullet\) be a nonzero object of the bounded derived category \(D^b(Y)\). Then there exists an integer \(r\), such that the cohomology \(H^r(F^\bullet) \neq 0\) and \(H^j(F^\bullet) = 0\), for \(j < r\). Thus, we may assume that \(F^\bullet\) is a complex of the form

\[
\cdots \rightarrow 0 \rightarrow 0 \rightarrow F^r \xrightarrow{\partial^r} F^{r+1} \rightarrow \cdots.
\]

As \(H^r(F^\bullet) = \text{Ker}(\partial^r) \subset F^r\), we have a monomorphism from the complex

\[
\cdots \rightarrow 0 \rightarrow H^r(F^\bullet) \rightarrow 0 \rightarrow 0 \ldots
\]

(with only one nonzero element at the position 0) to the complex \(F^\bullet[r]\).

Since the natural functor \(\mathfrak{Coh}(Y) \rightarrow D(Y)\) is fully faithful [12, page 165], we have the following injective map, induced by the above monomorphism,

\[
\text{Hom}_{\mathfrak{Coh}(Y)}(L^\otimes i, H^r(F^\bullet)) = \text{Hom}_{D^b(Y)}(L^\otimes i, H^r(F^\bullet)) \rightarrow \text{Hom}_{D^b(Y)}(L^\otimes i, F^\bullet[r]).
\]

(2.1)

As \(L\) is very ample and \(H^r(F^\bullet)\) is a coherent sheaf on \(Y\), there exists an integer \(m_0\) such that, for all \(m \geq m_0\), the sheaf \(H^r(F^\bullet) \otimes_{O_Y} L^\otimes m\)
is generated by a finite number of global sections [13 page 121]. The space of global sections

\[(2.2) \quad \Gamma(Y, H^r(\mathcal{F}^\bullet) \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes m}) = \Gamma(Y, \mathcal{H}om(\mathcal{L}^{\otimes -m}, H^r(\mathcal{F}^\bullet))) = \mathcal{H}om_{\xi_{ab}(Y)}(\mathcal{L}^{\otimes -m}, H^r(\mathcal{F}^\bullet)).\]

If $\mathcal{H}om_{D^b(Y)}(\mathcal{L}^{\otimes i}, \mathcal{F}^\bullet[r])$ were zero for for an integer $i \leq -m_0$, then from (2.1) it would follow $\mathcal{H}om_{\xi_{ab}(Y)}(\mathcal{L}^{\otimes i}, H^r(\mathcal{F}^\bullet)) = 0$, and from (2.2) we would deduce that the $\Gamma(Y, H^r(\mathcal{F}^\bullet) \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes m}) = 0$, for $m := -i \geq m_0$. But this is in contradiction with the fact that the non trivial sheaf $H^r(\mathcal{F}^\bullet) \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes m}$ is generated by its global sections, if $m \geq m_0$. We have proved the following proposition.

**Proposition 2.** If $\mathcal{F}$ is a nonzero object of the bounded derived category $D^b(Y)$, then there exist integers $i_0 := i_0(\mathcal{F})$ and $r = r(\mathcal{F})$ such that $\mathcal{E}xt^r(\mathcal{L}^{\otimes i}, \mathcal{F}) \neq 0$, for any $i \leq i_0$.

**Proposition 3.** If $\mathcal{G}$ is a nonzero object of the bounded derived category $D^b(Y)$, then there exist integers $n_0, l$ such that $\mathcal{E}xt^l(\mathcal{G}, \mathcal{L}^{\otimes i}) \neq 0$, for any $i \leq n_0$.

**Proof.** By the Serre duality, for all $l, i$

\[
\mathcal{E}xt^l(\mathcal{G}, \mathcal{L}^{\otimes i}) = \mathcal{H}om_{D^b(Y)}(\mathcal{G}, \mathcal{L}^{\otimes i}[l]) \simeq \mathcal{H}om_{D^b(Y)}(\mathcal{L}^{\otimes i}[l], \mathcal{S}(\mathcal{G}))^\vee
\simeq \mathcal{H}om_{D^b(Y)}(\mathcal{L}^{\otimes i}, \mathcal{S}(\mathcal{G})[-l])^\vee \simeq \mathcal{E}xt^{-l}(\mathcal{L}^{\otimes i}, \mathcal{S}(\mathcal{G}))^\vee.
\]

As the Serre functor $\mathcal{S}$ is an equivalence, $\mathcal{S}(\mathcal{G}) \neq 0$ and applying to it Proposition 2 we can take $n_0 = i_0(\mathcal{S}(\mathcal{G}))$ and $l = r(\mathcal{S}(\mathcal{G}))$.

In summary, the family of powers $\{\mathcal{L}^{\otimes a} | a \in \mathbb{Z}\}$ is a spanning class in $D^b(Y)$. That is, the vanishing $\mathcal{E}xt^k(\mathcal{F}, \mathcal{L}^{\otimes a}) = 0$ for all $a$ and all $k$ implies $\mathcal{F} = 0$. Equivalently, if $\mathcal{E}xt^k(\mathcal{L}^{\otimes a}, \mathcal{F}) = 0$ for all $a$ and all $k$, then $\mathcal{F} = 0$.

3. **$B$-branes on the Hypersurface**

As it was mentioned in the Introduction, the branes of type $B$ on the Calabi-Yau orbifold $Y$ are the objects of bounded derived category $D^b(Y)$, and the space of strings with ghost number $k$ stretching between the branes $\mathcal{F}_1$ and $\mathcal{F}_2$ is the group $\mathcal{E}xt^k(\mathcal{F}_1, \mathcal{F}_2)$.

Given a nonzero brane $\mathcal{F}$, by Proposition 3 there exists $a \in \mathbb{Z}$ such that the following set is nonempty

\[S := \{r \in \mathbb{Z}, \mid \mathcal{E}xt^r(\mathcal{F}, \mathcal{L}^{\otimes a}) \neq 0\} \neq \emptyset.\]
By Lemma 1 together with the comment after that lemma, $S$ is finite, and we denote by $k_1$ and $k_2$ the minimum and the maximum of $S$, respectively.

Let us assume that the branes $L \otimes a$ and $H$ may bind to form the brane $G$ through a potentially tachyonic open string. In mathematical terms, one has a distinguished triangle
\[
L \otimes a \rightarrow G \rightarrow H +1
\]
in the category $D^b(Y)$. We have the corresponding long exact Ext sequence
\[
\rightarrow Ext^j(F, L \otimes a) \rightarrow Ext^j(F, G) \rightarrow Ext^j(F, H) \rightarrow Ext^{j+1}(F, L \otimes a) \rightarrow
\]
By definition of $k_1$ and $k_2$, one deduces the following theorem.

**Theorem 4.** With the notations introduced above,

1. Whether $j < k_1 - 1$ or $k_2 < j$, $Ext^j(F, G) \simeq Ext^j(F, H)$.
2. For $j \in \{k_1, k_2 - 1\}$, the spaces $Ext^j(F, G)$ and $Ext^j(F, H)$ are not isomorphic.
3. For $j = k_1 - 1$, each string with ghost number $j$ from $F$ to $G$ admits a unique extension to a string between $F$ and $H$.
4. For $j = k_2$, each string with ghost number $j$ stretching from $F$ to $H$ is the extension of a string from $F$ to $G$.

In other words, for the numbers $j$ mentioned in the first item of theorem, each string with ghost number $j$ from $F$ to $H$ can be uniquely lifted to a string from $F$ to $G$.

On the other hand, if there is a string between the branes $F$ and $G$, then $G$ can decay to $F$ and other brane $J$. We have the corresponding distinguished triangle
\[
F \rightarrow G \rightarrow J +1,
\]
and the respective long exact sequence of Ext groups
\[
\rightarrow Ext^{j-1}(F, L \otimes a) \rightarrow Ext^j(J, L \otimes a) \rightarrow Ext^j(G, L \otimes a) \rightarrow Ext^j(F, L \otimes a) \rightarrow
\]

**Theorem 5.** With the above notations,

1. Whether $j < k_1$ or $k_2 + 1 < j$, $Ext^j(J, L \otimes a) \simeq Ext^j(G, L \otimes a)$.
2. For $j \in \{k_1 + 1, k_2\}$, $Ext^j(J, L \otimes a)$ and $Ext^j(G, L \otimes a)$ are not isomorphic.
3. For $j = k_1$, each string with ghost number $j$ between $J$ to $L \otimes a$ admits a unique lift of a string from $G$ to $L \otimes a$. 

4. Vertex operators

Let $F^\bullet$ be a nonzero object of the bounded derived category $D^b(Y)$. Then there exist two integers $r, s$ such that

\begin{equation}
H^r(F^\bullet) \neq 0 \neq H^s(F^\bullet), \quad H^j(F^\bullet) = 0 \quad \text{for} \quad j \notin [r, s].
\end{equation}

In other words, $F^\bullet$ is an object of the category $D^{\leq s} \cap D^{\geq r}$.

Given $b \in \mathbb{Z}$, we put $A := L^\otimes b$. The spectral sequence

\[ E_2^{p,q} = \text{Hom}_{D^b(Y)}(A, H^q(F^\bullet)[p]), \]

converges to

\[ \text{Hom}_{D^b(Y)}(A, F^\bullet[p+q]) = \text{Ext}^{p+q}(A, F^\bullet). \]

Since $H^q(F^\bullet)[p] = 0$ for all $(p, q)$ such that $p + q \notin [r, s]$, it follows that $\text{Ext}^k(A, F^\bullet) = 0$, for $k \notin [r, s]$. Thus, the result stated in Proposition 2 can be improved slightly.

**Proposition 6.** Let $F$ be above nonzero brane on $Y$. Then there exists an integer $i_0$ such that, if $i \leq i_0$, for some ghost number $k$, with $r \leq k \leq s$, the corresponding space of strings $\text{Ext}^k(L^\otimes i, F)$ is nonzero.

As $F^\bullet$ satisfies (4.1), there exists an complex $G^\bullet$ quasi-isomorphic to $F^\bullet$, such that $G^j = 0$ for $j \notin [r, s]$. Thus $F^\bullet$ and $G^\bullet$ are isomorphic objects in the derived category $D^b(Y)$. In particular

\[ \mathcal{E}xt^p(A, F^\bullet) = \mathcal{E}xt^p(A, G^\bullet). \]

Since $A$ is a locally free $\mathcal{O}_Y$-module,

\[ A_* : \quad \rightarrow 0 \rightarrow 0 \rightarrow A \rightarrow \mathcal{A} \]

is a locally free resolution of $A$. By definition of the Ext sheaves

\[ \mathcal{E}xt^p(A, G^\bullet) = H^p(\mathcal{H}om^\bullet(A_*, G^\bullet)), \]

where

\[ \mathcal{H}om^k(A_*, G^\bullet) = \prod_i \mathcal{H}om(A_i, G^{i+k}) = \mathcal{H}om(A, G^k). \]

Hence,

\begin{equation}
\mathcal{E}xt^p(A, F^\bullet) = 0, \quad \text{for} \quad p \notin [r, s]
\end{equation}
By Grothendieck’s vanishing theorem $H^q(Y, \mathcal{E}xt^p(A, F^*)) = 0$, for $q \notin [0, n - 1]$. We have proved the following theorem.

**Theorem 7.** If the space of vertex operators $H^q(Y, \mathcal{E}xt^p(L \otimes b, F))$ for strings between $L \otimes b$ and $F \in D^{\leq s} \cap D_{\geq r}$ is nonzero, then $(p, q)$ is a point of the rectangle $[r, s] \times [0, n - 1]$.

Since $L$ generates a spanning class in $D^b(Y)$, there are integer $a$ and $c$, such that $\mathcal{E}xt^c(L \otimes a, F^*) \neq 0$. From the local-to-global spectral sequence

$$H^q(Y, \mathcal{E}xt^p(L \otimes a, F^*)) \Rightarrow \mathcal{E}xt^{p+q}(L \otimes a, F^*),$$

we conclude that the following space of vertex operators

$$\bigoplus_{p+q=c} H^q(Y, \mathcal{E}xt^p(L \otimes a, F^*)) \neq 0.$$

Thus, Proposition 2 gives the following corollary.

**Corollary 8.** Given a nonzero object $F$ of $D^{\leq s} \cap D_{\geq r}$, there exists an integer $i_0$ satisfying the following property: For each $i \leq i_0$ there exists a point $(p, q) \in [r, s] \times [0, n - 1]$ such that the vertex space $H^q(Y, \mathcal{E}xt^p(L \otimes i, F))$ is nontrivial.

5. Yang-Mills fields on $Y$

**Embedding of $Y$ in $\mathbb{P}^N$.** We denote by $N$ the number of lattice points of $(n - 1)\Delta$. Each $m = (a_1, \ldots, a_n) \in ((n - 1)\Delta) \cap \mathbb{Z}^n$ determines a character of the torus $T$ of the variety $X$,

$$\chi_m : T = (\mathbb{C}^*)^n \to \mathbb{C}^n, \quad \chi_m(z_1, \ldots, z_n) = \prod_{i=1}^n z_i^{a_i}.$$

Since the divisor $D = (n - 1) \sum F.D_F$ is very ample [7, page 269], we have the embedding $\varphi : X \to \mathbb{P}^N$ determined by the map

$$x \in T \mapsto (\chi_{m_1}(x), \ldots, \chi_{m_N}(x)),$$

where $m_1, \ldots, m_N$ are the points of the set $((n - 1)\Delta) \cap \mathbb{Z}^n$. Moreover, the inverse image $\varphi^*(\mathcal{O}_{\mathbb{P}^N}(1))$ is the line bundle $\mathcal{O}_X(D)$ determined by the divisor $D$. We put $j$ for the composition $Y \hookrightarrow X \xrightarrow{\varphi} \mathbb{P}^N$. Then $\mathcal{L}$ is the inverse image

$$\mathcal{L} = \mathcal{O}_Y \otimes j^{-1}(\mathcal{O}_{\mathbb{P}^N}) j^{-1}\mathcal{O}_{\mathbb{P}^N}(1).$$

Since $Y$ may be a singular variety, one defines the first Chern class $c_1(\mathcal{L})$ as an element of the ring Chow $A^*(Y)$ of $Y$; more precisely, $c_1(\mathcal{L})$ is the class in $A^*(Y)$ of the divisor of any nonzero rational section of
Thus, if $B$ is a subvariety of $Y$, then $c_1(L) \cap [B] = c_1(L|_B)$. As $L = \mathcal{O}_{\mathbb{P}^N}(1)|_Y$, 

$$c_1(L) \cap [B] = [H] \cap [B],$$

$H$ being a hyperplane divisor in $\mathbb{P}^N$.

$C^\infty$ sheaves. From the mathematical point of view, the gauge fields are connections on complex vector bundles over real manifolds. The existence of partitions of the unity in the $C^\infty$ category allows to patch local connections [17]. So, to study gauge fields on $L$ we do not need neither the holomorphic structure of $L$ nor the complex structure of $Y$.

We denote by $A_{\mathbb{P}^N}$ the sheaf of differentiable functions on $\mathbb{P}^N$. We put $A_Y$ for the sheaf on $Y$ which defines the differentiable structure of $Y$. In view of the above observation, we will consider connections on the $C^\infty$ line bundle

$$L = A_Y \otimes j^{-1}(A_{\mathbb{P}^N}) \otimes j^{-1}\mathcal{O}_{\mathbb{P}^N}(1).$$

A gauge field on a line bundle over a smooth manifold is given locally by differential 1-forms. As $Y$ is not necessary smooth, to define gauge fields on $Y$, we will consider Kähler differentials.

To simplify notations, in this Section we write $\mathbb{P} := \mathbb{P}^N$, and $\mathcal{R}$ for the pullback $j^{-1}A_{\mathbb{P}^N}$ of $A_{\mathbb{P}^N}$ by the inclusion $j : Y \hookrightarrow \mathbb{P}$. We also put $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^N}(1)$.

Given $U$ an open subset of $Y$, $\Omega_{A_Y(U)/\mathbb{C}}$ denotes the module of Kähler differentials of $A_Y(U)$ over $\mathbb{C}$ [9, Chapter 16]. Let $A_Y^1$ be the sheaf of $A_Y$-modules defined by

$$A_Y^1(U) = \Omega_{A_Y(U)/\mathbb{C}}.$$

Similarly, we define the sheaves of Kähler differentials

$$A_{\mathcal{R}}^1(U) := \Omega_{(j^{-1}A_{\mathbb{P}^N})(U)/\mathbb{C}}, \quad A_{\mathcal{R}}^1(W) := \Omega_{A_{\mathbb{P}^N}(W)/\mathbb{C}}.$$

In general, given a sheaf of rings $\mathcal{C}$ on a space $Z$ and the $\mathcal{C}$-modules $\mathcal{H}$ and $\mathcal{G}$, we denote by $\theta$ the natural morphism

$$\theta : \mathcal{H} \otimes_{\mathcal{R},\mathcal{C}} \mathcal{G} \to \mathcal{H} \otimes_{\mathcal{C}} \mathcal{G},$$

between the tensor product presheaf and the tensor product. Since the stalks of $\mathcal{H} \otimes_{\mathcal{R},\mathcal{C}} \mathcal{G}$ and $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{G}$ at any point of $Z$ are canonically isomorphic, there exists an open covering of $Z$, such that the restriction of $\theta$ to each element of this covering is an isomorphism.

Returning to our case. By the compactness of $Y$, $Y$ can be covered by a finite family $U = \{U\}$ of “small enough” open sets such that, for any $U \subseteq U$

(a) The restriction $L|_U$ is a trivial $C^\infty$ line bundle.
(b) The epimorphism \( \pi : R \to A_Y \) of sheaves of rings corresponding to the embedding \( j \), defines a surjective map
\[
\pi_U : b \in \mathcal{R}(U) \mapsto \pi_U(b) \in A_Y(U).
\]

(c) From the following part of exact conormal sequence \([9, \text{ page } 387]\),\( q : A_Y \otimes_R A^1_{\mathbb{P}} \to A^1_Y \to 0 \), one obtains a surjective \( A_Y(U) \)-linear map
\[
q_U : A_Y(U) \otimes_R A^1_{\mathbb{P}}(U) \to A^1_Y(U), \quad c \otimes db \mapsto cd(\pi_U(b)).
\]

Let \( D \) be any of the following sheaves on \( \mathbb{P} : \mathcal{A}_P, \mathcal{O}(1), \mathcal{A}_{\mathbb{P}}^1 \). Given an open subset \( U \subset Y \), let \( E \) denote the set consisting of of open \( W \subset \mathbb{P} \) such that \( W \cap Y \supset U \). Any section of \( j^{-1}D(U) \) is defined by a family
\[
\{ \tilde{c}_W \in \mathcal{D}(W) \} \}_{W \in C},
\]
where \( C \) is a cofinal subset of \( E \), satisfying \( \tilde{c}_W|_{W'} = \tilde{c}_{W'} \), for \( W' \subset W \). Conversely, such a family defines a section of \( j^{-1}D(U) \), which will be denoted \( j^{-1}(\tilde{c}) \).

**Connections on \( L \).** A connection on \( L \) is a morphism \( \nabla \) of the category \( \mathcal{Sh} \)
\[
\nabla : L \to A^1_Y \otimes_{A_Y} L,
\]
satisfying the following property: Given an open subset \( U \subset Y \), a function \( g \in A_Y(U) \) and \( s \in L(U) \) then
\[
\nabla_U(gs) = \theta_U(ds \otimes s) + g \nabla_U s.
\]
Here, \( \theta_U \) is the canonical map \( A^1_Y(U) \otimes_{A_Y(U)} L(U) \to (A^1_Y \otimes_{A_Y} L)(U) \).

If \( U \) is a small enough open subset of \( Y \), then condition \( (5.4) \) implies
\[
\nabla_U(gs) = dg \otimes s + g \nabla_U s.
\]

Given a trivialization of \( L \) on a small enough open \( U \subset Y \), a connection on \( L \) defines a section \( \alpha_U \in A^1_Y(U) \). A general connection on \( \mathcal{O}(1) \) gives rise to local 1-forms on \( \mathbb{P} \), but the restriction to \( Y \) of those forms are not necessarily sections of the sheaf of Kähler differentials \( A^1_Y \).

Nevertheless, some connections on \( \mathcal{O}(1) \) determine connections on \( L \). Let \( U \) be the above mentioned open covering of \( Y \). On each \( U \in \mathcal{U} \) there is a local frame \( \sigma \) of \( L \). Then \( \sigma \in j^{-1}\mathcal{O}(1)(U) \) is defined by a family indexed by a cofinal subset of \( \mathcal{E} \)
\[
\{ \tilde{\sigma}_W \in \mathcal{O}(1)(W) \} \}_{W \in C},
\]
satisfying the compatibility condition.

Let us assume that \( \tilde{\nabla} \) is a connection on \( \mathcal{O}(1) \), such that there exist sections \( \tilde{\alpha}_W \in A^1_{\mathbb{P}}(W) \), satisfying \( \tilde{\nabla} \tilde{\sigma}_W = \tilde{\alpha}_W \otimes \tilde{\sigma}_W \). The compatibility
condition of the $\tilde{\sigma}_W$ implies $\tilde{\alpha}_W|_{W'} = \tilde{\alpha}_{W'}$, for $W' \subset W$. That is, the $\tilde{\alpha}_W$ determine an element $j^{-1}(\tilde{\alpha}) \in A^1_{R}(U)$. Thus,

$$1 \otimes j^{-1}(\tilde{\alpha}) \in A_{Y}(U) \otimes_{R(U)} A^1_{R}(U).$$

We set

$$\alpha_U := q_U (1 \otimes j^{-1}(\tilde{\alpha})). \tag{5.6}$$

**Proposition 9.** The family $\{\alpha_U \in A^1_{Y}(U)\}_{U \in \mathcal{U}}$ defines a connection on $\mathcal{L}$.

**Proof.** Let $\sigma'$ be other frame of $L|_{U}$. Then there exists a function $b \in A_{Y}(U)$ different from zero everywhere, such that $\sigma = b\sigma'$. The frame $\sigma'$ determines the corresponding sections $\tilde{\sigma}'_W$ for each $W$ belonging to a cofinal subset of $\mathcal{E}$. Thus, for each $W$ belonging to a cofinal subset, there is a function $\tilde{b}_W \in A_{\mathcal{E}}(W)$ satisfying $\tilde{\sigma}_W = \tilde{b}_W \tilde{\sigma}'_W$ and $\tilde{b}_W$ is nonzero everywhere. So, one has $j^{-1}(\tilde{b}) \in \mathcal{R}(U)$ and

$$\pi_U(j^{-1}(\tilde{b})) = b, \tag{5.7}$$

since $\sigma = \pi_U(j^{-1}(\tilde{b}))\sigma'$.

The connection form $\tilde{\alpha}'_W$ of $\tilde{\nabla}$ in the frame $\tilde{\sigma}'_W$ is defined by the relation $\tilde{\nabla} \tilde{\sigma}'_W = \tilde{\alpha}'_W \otimes \tilde{\sigma}'_W$. We have the form $\alpha'_U$ defined from the $\tilde{\alpha}'_W$ as in (5.6)

$$\alpha'_U := q_U (1 \otimes j^{-1}(\tilde{\alpha}')). \tag{5.8}$$

Since $\tilde{\nabla}$ is a connection

$$\tilde{\alpha}_W = \tilde{\alpha}'_W + \tilde{b}_W^{-1} \tilde{d}\tilde{b}_W,$$

for $W$ belonging to a cofinal subset of $\mathcal{E}$. Therefore,

$$j^{-1}(\tilde{\alpha}) = j^{-1}(\tilde{\alpha}') + j^{-1}(\tilde{b})^{-1}j^{-1}(\tilde{d}\tilde{b}),$$

and

$$q_U(1 \otimes j^{-1}(\tilde{\alpha})) = q_U(1 \otimes j^{-1}(\tilde{\alpha}')) + q_U((1 \otimes (j^{-1}(\tilde{b}))^{-1}d(j^{-1}(\tilde{b})))�).$$

From (5.8), (5.6) and (5.7), we deduce

$$\alpha_U = \alpha'_U + b^{-1} db. \tag{5.9}$$

Thus, the Kähler differentials $\alpha_U$, defined for the open sets of the covering $\mathcal{U}$ of $Y$, satisfy the compatibility condition for defining a connection on $\mathcal{L}$. □

**Remark.** We have constructed a connection on $\mathcal{L}$ by restricting a connection on $\mathcal{O}(1)$ whose connection forms, relative to local frames on the members of $\mathcal{U}$, are Kähler differentials. Conversely, the connections on $\mathcal{L}$ can be extended to connections on $\mathcal{O}(1)$. 
Let \( \{V_i\}, i = 0, \ldots, N \) be the standard affine covering of \( \mathbb{P} \) and let \( g_{ij} \) denote the transition functions of \( \mathcal{O}(1) \) corresponding to the usual trivializations on the \( V_i \). A connection on \( \mathcal{L} \) is defined by a collection \( \alpha_0, \ldots, \alpha_N \), with \( \alpha_k \in \mathcal{A}^1_\mathbb{C}(Y \cap V_k) \), satisfying on \( Y \cap V_j \cap V_i \)

\[
\alpha_i = \alpha_j + g_{ji}^{-1}dg_{ji},
\]

since the transition functions of \( \mathcal{L} \) are the restrictions of the corresponding transition functions of \( \mathcal{O}(1) \). By (5.10), together with the fact that \( g_{ji}^{-1}dg_{ji} \) is already a 1-form on \( V_j \cap V_k \), there exists for each \( j \) an open subset of \( W_j \subset V_j \) and \( \tilde{\alpha_j} \in \mathcal{A}^1_\mathbb{C}(W_j) \), such that

\[
\tilde{\alpha_i} = \tilde{\alpha_j} + g_{ji}^{-1}dg_{ji},
\]

on \( W_j \cap W_i \). In this way, the forms \( \tilde{\alpha}_j \) define a connection on \( \mathcal{O}(1)|_Y \), where \( Y \) is an open subset of \( \mathbb{P} \) containing \( Y \). This connection can be extended to a connection on \( \mathcal{O}(1) \) \[17\] page 67.

**Chern connection on** \( \mathcal{O}_{\mathbb{P}^N}(1) \). On \( \mathcal{O}(1) \) is defined the Chern connection \( \nabla^0 \), compatible with the Hermitian and holomorphic structures of this line bundle. The homogeneous coordinates \( z_0, \ldots, z_N \) of \( \mathbb{P} \) define sections of \( \mathcal{O}(1) \). We denote by \( \mathbf{v} = (v_1, \ldots, v_N) \) the inhomogeneous coordinates on an affine subset \( V_k \) belonging to the standard covering of \( \mathbb{P} \). The Hermitian metric on \( \mathcal{O}(1)|_{V_k} \) is given by the function

\[
h = (1 + |\mathbf{v}|^2)^{-1}.
\]

Thus, the form of the connection \( \nabla^0 \) on \( V_k \) is

\[
\tilde{\alpha}_k = h^{-1}dh = -(1 + |\mathbf{v}|^2)^{-1}(\mathbf{v} \cdot d\mathbf{v}).
\]

The curvature of \( \nabla^0 \) is \( \mathbb{C} \cdot \omega_{FS} \), where \( \omega_{FS} \) is the Fubini-Study form. Thus,

\[
c_1(\mathcal{O}_{\mathbb{P}^N}(1)) = [w_{FS}] \in H^2(\mathbb{P}, \mathbb{Z}).
\]

In particular, \( \tilde{\alpha}_k \) is a section of \( \mathcal{A}^1_\mathbb{C}(V_k) \). Let \( U \) be the covering of \( Y \) consisting of the open set \( U_k = Y \cap V_k \), with \( k = 0, 1, \ldots, N \). The form \( \tilde{\alpha}_k \) determines a Kähler differential \( \alpha_k \in \mathcal{A}^1_\mathbb{C}(U) \), and by Proposition \[9\] the family \( \{\alpha_k\} \) defines a connection on \( \mathcal{L} \).

We write \( \nabla^0 \) for the connection on \( \mathcal{L} \) determined by \( \nabla^0 \). If \( U \) is an open subset of \( Y \) contained in the affine subset \( V_k \subset \mathbb{P} \), then taking into account (5.6), (5.3) and (5.11), the connection form of \( \nabla^0 \) can be written on \( U \subset Y \) in terms of \( \pi_U(v_i) \), the functions on \( Y \) induced by the inhomogeneous coordinates \( v_i \), as

\[
- \pi_U(h)((\pi_U\mathbf{v}) \cdot d(\pi_U\mathbf{v})).
\]

Analogously, for the curvature \( d\alpha + \alpha \wedge \alpha \) of \( \nabla^0 \), one has

\[
\pi_U(h^{-1})d\pi_U\mathbf{v} \wedge d\pi_U\mathbf{v} - (\pi_U\mathbf{v} \cdot d\pi_U\mathbf{v}) \wedge (\pi_U\mathbf{v} \cdot d\pi_U\mathbf{v})
\]

\[
\pi_U(h^{-2})
\]
where the dot product is also involved in \(d\pi_U \mathbf{v} \wedge d\pi_U \tilde{\mathbf{v}}\). That is, on the smooth locus of \(Y\) the curvature \(F_{\nabla^0}\) of \(\nabla^0\) is

\[
F_{\nabla^0} = \frac{2\pi}{i} \hat{\omega}_{FS},
\]

where \(\hat{\omega}_{FS}\) is the restriction of the Fubini-Study form to that locus.

**Yang-Mills functional.** We denote by \(\hat{g}\) the metric on \(Y \setminus \text{Sing}(Y)\) induced by the Fubini-Study one of \(\mathbb{P}\). With \(\ast\) we denote the corresponding Hodge star operator. On the affine space of connections of \(L\), one defines the Yang-Mills functional

\[
\nabla \mapsto \YM(\nabla) = -\int_Y F_{\nabla} \wedge \ast F_{\nabla},
\]

where \(F_{\nabla}\) is the curvature form of \(\nabla\).

From the variational principle, one deduces that the fields \(\nabla\) that are stationary points for the functional \(\YM\) satisfy the equation

\[
\nabla^* F_{\nabla} = 0,
\]

\(\nabla^*\) being the formal adjoint operator of \(\nabla\). As \(L\) is a line bundle, Bianchi identity and (5.17) reduce to

\[
dF_{\nabla} = 0, \quad d^* F_{\nabla} = 0,
\]

where \(d^*\) is the codifferential operator. That is, the Yang-Mills fields are those whose curvature is a 2-form closed and coclosed in \(Y \setminus \text{Sing}(Y)\).

By (5.15),

\[
\YM(\nabla^0) = 4\pi^2 \int_Y \hat{\omega}_{FS} \wedge \ast \hat{\omega}_{FS},
\]

On the other hand,

\[
\hat{\omega}_{FS} \wedge \ast \hat{\omega}_{FS} = \hat{g}(\hat{\omega}_{FS}, \hat{\omega}_{FS}) \text{vol} = \text{vol},
\]

where \(\text{vol}\) is the volume form determined by \(g\) on the smooth locus of \(Y\). As \(\hat{\omega}_{FS}\) and \(\text{vol}\) are closed forms, \(\hat{\omega}_{FS} \wedge d(\ast \hat{\omega}_{FS}) = 0\). Since \(\hat{\omega}_{FS}\) is non degenerate, it follows \(d^* \hat{\omega}_{FS} = 0\). Thus, \(\nabla^0\) is a Yang-Mills field.

Furthermore, from (5.18), it follows

\[
\YM(\nabla^0) = 4\pi^2 \text{vol}(Y).
\]

Similarly, the functional \(\YM_P\) defined on the connections over \(O(1)\) takes at the Chern connection \(\hat{\nabla}^0\) the value,

\[
\YM_P(\hat{\nabla}^0) = 4\pi^2 \text{vol}(\mathbb{P}),
\]

and it is the minimum value taken by \(\YM_P\).
If \( \nabla \) were a gauge field on \( Y \) such that \( \text{YM}(\nabla) < \text{YM}(\nabla^0) \), then one could construct on an small open neighborhood \( V \) of \( Y \) an extension \( \tilde{\nabla} \) of \( \nabla \), such that

\[
\text{YM}_V(\nabla^0) - \text{YM}_V(\tilde{\nabla}) = \epsilon > 0.
\]

On the other hand, \( \tilde{\nabla} \) can be extended to \( P \), so that

\[
|\text{YM}_{P \backslash V}(\nabla^0) - \text{YM}_{P \backslash V}(\tilde{\nabla})| < \epsilon/2.
\]

Thus

\[
\text{YM}_P(\tilde{\nabla}) < \text{YM}_P(\nabla^0).
\]

But this inequality contradicts the fact that \( \text{YM}_P(\nabla^0) \) is the minimum value of \( \text{YM}_P \). Thus, we can summarize the above results in the following proposition.

**Proposition 10.** The Yang-Mills functional (5.16) reaches its minimum value at the connection \( \nabla^0 \).

### 6. Equivariant Chern class

**Universal \( T \)-bundle.** Let \( T \) denote the torus \( T = (\mathbb{C}^\times)^n \). The space \((\mathbb{P}^\infty)^n\) can be taken as the classifying space \( BT \) and the natural \( T \)-bundle

\[
ET = (\mathbb{C}^\infty \setminus \{0\})^n \to BT = (\mathbb{P}^\infty)^n,
\]

as the the universal \( T \)-bundle. As it is well-known, the rational cohomology \( H^*(BT, \mathbb{Q}) \) is the polynomial ring \( \mathbb{Q}[t_1, \ldots, t_n] \), where each \( t_i \) has degree 2.

Given \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \), one has the group homomorphism

\[
(6.1) \quad \chi_m : g = (z_1, \ldots, z_n) \in T \mapsto \prod_i (z_i)^{m_i} \in \mathbb{C}^\times.
\]

In Appendix, we will prove the following lemma.

**Lemma 11.** The character \( \chi_m \) determines:

1. A map \( \xi : BT \to BC^\times \), such that the morphism of algebras induced by \( \xi \) on the cohomologies

\[
\xi^* : \mathbb{Q}[t] = H^*(\mathbb{P}^\infty, \mathbb{Q}) \to H^*(BT, \mathbb{Q}) = \mathbb{Q}[t_1, \ldots, t_n]
\]

satisfies \( \xi^*(t) = \sum_{i=1}^n m_i t_i \).

2. A bundle map \( \psi : ET \to EC^\times \) over \( \xi \), satisfying \( \psi(e \cdot g) = \chi_m(g)\psi(e) \), for all \( e \in ET \) and all \( g \in G \).
Associated with the $T$-action on the toric variety $X$, there is the homotopy quotient space $ET \times_T X$, whose cohomology is the $T$-equivariant cohomology $H^*_T(X)$ of $X$. The universal $T$-bundle $ET$ does not admit a finite dimensional model. Nevertheless, as $C^\infty = \cup_r C^r$, there are finite dimensional obvious approximations $ET_r \to BT_r$ to $ET \to BT$ that are algebraic varieties. Furthermore, $H^i(ET_r \times_T X) = H^i_T(X)$ for $r > r(i)$. Similarly, one defines the $i$-th $T$-equivariant Chow group $A_{i,T}(X)$ as the usual Chow group $A_{i+r-n}(ET_r \times_T X)$.

The anticanonical divisor of $X$ is $T$-invariant, hence the invertible sheaf $\mathcal{L}'$ is $T$-equivariant. The $T$-equivariant first Chern class $c_1^T(L')$ is by definition the first Chern class of the line bundle $\mathcal{L}' := ET \times_T \mathcal{L}' \to ET \times_T X$.

That is, $c_1(\mathcal{L}'_T)$ is the class in $A_{1,T}(X)$ of divisor defined by any nonzero rational section of $\mathcal{L}'_T$. As $X$ is a rationally smooth variety, then $c_1^T(L')$ can be considered as a cohomology class in $H^2_T(X, \mathbb{Q})$.

**$T$-invariant hypersurface.** Let us assume that $Y$ is a $T$-invariant subvariety of $X$, then $\mathcal{L}$ is a $T$-equivariant line bundle over $Y$, and the inclusion $i : Y \hookrightarrow X$ induces natural map $\tilde{i} : ET \times_T Y \to ET \times_T X$. Over $ET \times_T Y$ we have the line bundles $\tilde{i}^*(\mathcal{L}'_T)$ and $\mathcal{L}_T$. As $\mathcal{L} = i^*\mathcal{L}'$, one has $\mathcal{L}_T = \tilde{i}^*(\mathcal{L}'_T)$.

By the above equality and the functoriality of the Chern class

$$c_1^T(\mathcal{L}) = c_1(\mathcal{L}_T) = \tilde{i}^*(c_1^T(\mathcal{L}')).$$

The set $X^T$ of fixed point of $X$ for the $T$-action is in bijective correspondence with the set of vertices of the polytope $\Delta$. Since the action of $T$ on $X^T$ is trivial and the rational cohomology of the classifying space $ET/T$ is the polynomial ring $\mathbb{Q}[t_1, \ldots, t_n]$, the inclusion $X^T \subset X$ gives rise to the localization map

$$\lambda : H^*_T(X, \mathbb{Q}) \to H^*(ET \times_T X^T, \mathbb{Q}) = \prod_v \mathbb{Q}[t_1, \ldots, t_n],$$

where $v$ ranges the set of vertices of $\Delta$.

The obvious inclusion $Y^T \subset X^T$ gives rise to the natural projection on the respective equivariant cohomologies

$$\prod_v \mathbb{Q}[t_1, \ldots, t_n] \overset{pr}{\longrightarrow} \prod_w \mathbb{Q}[t_1, \ldots, t_n],$$

$w$ ranging on the fixed points in $Y$.

Similarly, one has a localization map

$$\lambda : H^*_T(Y, \mathbb{Q}) \to H^*(ET \times_T Y^T, \mathbb{Q}) = \prod_w \mathbb{Q}[t_1, \ldots, t_n].$$
From (6.2), it follows
\[ \text{pr}(\lambda'(c_1^T(L'))) = \lambda(c_1^T(L)). \]

In what follows, we will determine \( c_1^T(L') \) in order to obtain the localization \( \lambda(c_1^T(L)) \) by means of (6.3). \( c_1(L') \) is the class \([D] \in H^2(X, \mathbb{Q})\) of the divisor \( \sum_F (n-1)D_F \). This divisor is the one defined for the character
\[ \chi : T \to \mathbb{C}^\times, \quad (z_1, \ldots, z_n) \mapsto \prod_i (z_i)^{n-1}. \]

Thus, if \( pt \) is any point of the set \( D \cap X^T \), the restriction of \( L' \) to \( \{pt\} \) is the vector space \( V \simeq \mathbb{C} \) endowed with the \( T \)-action
\[ g \cdot z = \chi(g)z, \]
with \( z \in \mathbb{C} \) and \( g \in T \). So, one can construct the \( T \)-equivariant line bundle \( V_T = ET \times_T V \to BT \).

On the other hand, we can endow \( V \) with a structure of \( \mathbb{C}^\times \)-equivariant bundle, by means of the natural action of \( \mathbb{C}^\times \) on \( \mathbb{C} \). So, one has \( V_{\mathbb{C}^\times} \to \mathbb{P}^\infty \), where \( V_{\mathbb{C}^\times} \) is the set of classes \([(x_k), z] \), with
\[ [(x_k), z] = [(\mu x_k), \mu^{-1}z], \]
where \( (x_k) \in \mathbb{C}^\infty \setminus \{0\}, z \in \mathbb{C} \) and \( \mu \in \mathbb{C}^\times \). Denoting with \( \{x_k\} \) the point \([x_0 : x_1 : \ldots] \in \mathbb{P}^\infty \), the fiber \( p^{-1}(\{x_k\}) \) over \( \{x_k\} \) is the set
\[ \{[\mu \cdot (x_k), 1] | \mu \in \mathbb{C}^\times \} \cup \{0\}. \]

Thus, \( V_{\mathbb{C}^\times} \) is the tautological bundle \( \mathcal{O}_{\mathbb{P}^\infty}(-1) \). The following diagram, where \( h \) is defined by \( h([e, z]) = [\psi(e), z] \), shows that \( V_T \) is the pullback of \( \mathcal{O}_{\mathbb{P}^\infty}(-1) \) by means of \( \xi \).

\[
\begin{array}{ccc}
V_T & \xrightarrow{h} & V_{\mathbb{C}^\times} \\
\downarrow & & \downarrow p \\
BT & \xrightarrow{\xi} & \mathbb{P}^\infty.
\end{array}
\]

From the functoriality of the Chern class together with Lemma \[\text{III}\] it follows
\[ c_1(V_T) = \xi^*(c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))) = -\xi^*(i) = -(n-1) \sum_i t_i \in \mathbb{Q}[t_1, \ldots, t_n]. \]

The preceding arguments are valid for each point of \( D \cap X^T \). Thus, the localization \( \lambda' \) maps \( c_1^T(L') \in H^2_T(X, \mathbb{Q}) \) to
\[ (\xi, \ldots, \xi) \in \prod_v \mathbb{Q}[t_1, \ldots, t_n], \]
From (6.3), it follows the following theorem.

**Theorem 12.** Assumed that $Y$ is a $T$-invariant subvariety of $X$, the localization $\lambda(c^T_1(L))$ of the $T$-equivariant Chern class of $L$ is the element

$$\left(\overline{c}, \ldots , \overline{c}\right) \in \prod_w \mathbb{Q}[t_1, \ldots , t_n],$$

where $w$ ranges on the set of fixed points of $Y$ and $s$ is the cardinal of this set.

**Appendix**

In this section we will sketch a proof of Lemma 11. The universal bundle $ET$ is homotopically equivalent to $EU(1)^n$, the one of the compact torus $U(1)^n$. According to the Milnor construction (see [18], [14, page 54]), given a topological group $G$, one can take as $EG$ the infinite join

$$EG = G \ast G \ast G \cdots .$$

Following Husemoller, we write $\langle g, t \rangle$ for the elements of $EG$, where $\langle g, t \rangle$ is given by the sequence $(t_0g_0, t_1g_1, \ldots )$ with $t_i \in [0, 1]$, $g_i \in G$ satisfying the properties detailed in [14]. The quotient of $EG$ by the right $G$-action

$$\langle g, t \rangle g' = \langle t_0g_0g', t_1g_1g', \ldots \rangle$$

is the classifying space $BG$.

Denoting with $EG_r$, the subspace of $EG$ consisting of the elements $\langle g, t \rangle$ such that $t_i = 0$ for $i > r$, and by $BG_r$ the corresponding quotient, the fibre bundle $EG_r \rightarrow BG_r$ is a finite dimensional approximation to the universal bundle.

When $G = U(1)$, then $EG_r$ is the sphere $S^{2r+1}$ and $BG_r = \mathbb{P}^r$. Moreover, $BG_1 = \mathbb{P}^1 \subset \mathbb{P}^\infty = BG$ is the generator of $H_2(\mathbb{P}^\infty, \mathbb{Q})$. If $G = U(1)^n$, then $BG_1 = (\mathbb{P}^1)^n \subset (\mathbb{P}^\infty)^n = BG$.

The group homomorphism (6.1) defines a map

$$\psi := E\chi_m : \langle g, t \rangle \in EU(1)^n \mapsto \langle \chi_m(g), t \rangle \in EU(1).$$

That is, for $(y_1, \ldots , y_n), (z_1, \ldots , z_n) \in U(1)^n$

$$\psi \langle t_0(y_1, \ldots , y_n), t_1(z_1, \ldots , z_n), \ldots \rangle = \langle t_0 \prod_i y_i^{m_i}, t_1 \prod_i z_i^{m_i}, \ldots \rangle.$$

The induced mapping $\xi := B\chi_m : BU(1)^n \rightarrow BU(1)$ maps

$$\xi \langle t_0(y_1, 1, \ldots , 1), t_1(z_1, 1, \ldots , 1), 0, 0, \ldots \rangle = \langle t_0y_1^{m_1}, t_1z_1^{m_1}, 0, 0, \ldots \rangle.$$
In other words, $\xi$ restricted to $\mathbb{P}^1 \times \text{pt} \times \cdots \times \text{pt} \subset (\mathbb{P}^\infty)^n$ is defined by

$$([y_1: z_1], \text{pt}, \ldots, \text{pt}) \in (\mathbb{P}^1)^n \mapsto [y_1^m_1: z_1^m_1] \in \mathbb{P}^1.$$  

In terms of the homology, $\xi$ maps the generator $a_1 \in H_2((\mathbb{P}^\infty)^n, \mathbb{Q})$ determined by first factor in the product $(\mathbb{P}^1)^n$ to $m_1 b$, where $b$ is the generator of $H_2(BU(1), \mathbb{Q})$.

Analogously,

$$\xi(t_0(1, \ldots, 1, y_n), t_1(1, \ldots, 1, z_n), 0, 0, \ldots) = \langle t_0 y_n^m_n, t_1 z_n^m_n, 0, 0, \ldots \rangle.$$  

Thus, $\xi$ maps the generator $a_n \in H_2((\mathbb{P}^\infty)^n, \mathbb{Q})$ determined by the $n$-th factor in the product $(\mathbb{P}^1)^n$ to $m_n b$. That is,

$$\xi: H_2((\mathbb{P}^\infty)^n, \mathbb{Q}) = \mathbb{Q}[t_1, \ldots, t_n] \to H_2(\mathbb{P}^\infty, \mathbb{Q}) = \mathbb{Q}[t],$$

maps $t_i$ to $m_i t$.

The map induced between the cohomologies $\xi^*: \mathbb{Q}[t] \to \mathbb{Q}[t_1, \ldots, t_n]$ is the dual of the preceding one, hence $\xi^*(t) = \sum_i m_i t_i$.  

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