Statistics of quantum transport in chaotic cavities with broken time-reversal symmetry

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The statistical properties of quantum transport through a chaotic cavity are encoded in the traces $T_n = \text{Tr}[(t t^\dagger)^n]$, where $t$ is the transmission matrix. Within the Random Matrix Theory approach, these traces are random variables whose probability distribution depends on the symmetries of the system. For the case of broken time-reversal symmetry, we use generalizations of Selberg’s integral and the theory of symmetric polynomials to present explicit closed expressions for the average value and for the variance of $T_n$ for all $n$. In particular, this provides the charge cumulants $\langle Q_n \rangle$ of all orders. We also compute the moments $\langle g^n \rangle$ of the conductance $g = T_1$. All the results obtained are exact, i.e. they are valid for arbitrary numbers of open channels.

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I. INTRODUCTION

We consider the problem of electronic transport through a ballistic quantum dot attached to two ideal leads supporting $N_1$ and $N_2$ open channels. Even at zero temperature, the electric current as a function of time has a fundamentally random nature associated with the granularity of charge, and can naturally be characterized by its statistics. Its average and variance, for example, are related respectively to the conductance and the shot-noise. These quantities have been under experimental investigation for already quite some time, and when the classical dynamics in the dot is chaotic they can display characteristic features like weak localization, universal conductance fluctuations and constant Fano factors. More recently, much attention has been devoted, both theoretically and experimentally, to higher cumulants comprising the full counting statistics, which contain more refined information about the transport process.

This problem can be described, in the Landauer-Büttiker scattering approach, in terms of the unitary scattering matrix $S = \begin{pmatrix} r & t^\dagger \\ t & r^\dagger \end{pmatrix}$ or in terms of a hermitian matrix $t t^\dagger$. The transmission matrix $t$ has dimension $N_2 \times N_1$, where $N_1$ and $N_2$ are the number of open quantum channels on the entrance and exit lead, respectively. The total number of channels is $N = N_1 + N_2$. The dimensionless conductance is given by $g = \text{Tr}(t t^\dagger)$, and more generally one is interested in the quantities $T_n = \text{Tr}[(t t^\dagger)^n]$.

A possible way to model chaotic cavities is to assume $S$ to be drawn at random from an appropriate ensemble, determined only from the existing symmetries. The average value of $T_n$ over such an ensemble is denoted $\langle T_n \rangle$. This is the random matrix theory approach. It treats the eigenvalues of $t t^\dagger$ as correlated random variables, and successfully predicts all the universal results already mentioned. However, even though experiments may be done with relatively small channel numbers, so far most explicit expressions are valid only in the asymptotic limit $N_1, N_2 \gg 1$ or to the first few terms in a perturbative expansion in $1/N$.

A few particular results exist that are valid for general $N_1, N_2$, like the average and variance of $g^3$, the density of transmission eigenvalues for equal leads $N_1 = N_2$. More recently, results were found for: $\langle T_2 \rangle$ and $\langle T_3 \rangle$, some nonlinear statistics like variance of shot-noise, skewness and kurtosis of conductance, the density of eigenvalues for unequal leads, as well as an expression for $\langle T_n \rangle$. The purpose of this paper is to establish general exact results, of which several of the above-mentioned ones are particular cases, for systems without time-reversal symmetry. It is well known that within this symmetry class leading-order perturbative results sometimes turn out to be exact (e.g. for conductance and for the form factor of closed systems). However, this is by no means usual, and most transport statistics are in general not identical with their $N \rightarrow \infty$ asymptotic limit.

We shall start by computing $\langle T_n \rangle$. Our formula is different from the one in Ref. [13] and computationally superior. We also present the variance of $T_n$ and all moments of the conductance, $\langle T_2 \rangle$. These advancements are made possible by making a connection with the theory of symmetric polynomials and generalizations of Selberg’s integral.

We must remark that the quantities more readily accessible to measurement are the charge cumulants $\langle Q_n \rangle$, which quantify the fluctuations in the amount of charge transmitted over an interval of time. They are related to $\langle T_n \rangle$ according to the generating function

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \langle Q_n \rangle = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \langle T_m \rangle (e^x - 1)^m. \quad (1)$$

Our exact expression for $\langle T_n \rangle$ therefore provides cumulants $\langle Q_n \rangle$ of arbitrary order.

For broken time-reversal systems, random matrix theory predicts that the non-zero eigenvalues $\{T_1, \ldots, T_{N_1}\}$ of $t t^\dagger$ behave statistically like random numbers distributed between 0 and 1 according to the joint prob-
ability distribution\(^8\)

\[ \mathcal{P}(T) = \mathcal{N}(\Delta(T))^2 \prod_j T_j^a, \] \hspace{1cm} (2)

where \(\Delta(T) = \prod_{i<j} (T_i - T_j)\) is the Vandermonde determinant, \(\alpha = \sum_{i=1}^n \lambda_i\) (we assume \(N_1 \leq N_2\)) measures the asymmetry between the leads and \(\mathcal{N}\) is a normalization constant. Averages are therefore obtained as

\[ \langle f(T) \rangle = \int_{\mathcal{C}} \mathcal{P}(T)f(T)dT, \] \hspace{1cm} (3)

where \(\mathcal{C}\) is the hypercube \([0,1]^\otimes N_1\). By integrating out all but one of the variables one obtains the eigenvalue density \(\rho(T)\). An exact expression for \(\rho(T)\) can be written down using Jacobi polynomials\(^{13,15}\)

\[ \rho(T) = T^\alpha \sum_{j=0}^{N-1} (\alpha + 2j + 1)[P_j^{(\alpha,0)}(1-2T)]^2, \] \hspace{1cm} (4)

and hence \(\langle T_n \rangle = N_1 \int_0^1 T^n \rho(T)dT\). Our result \(^{13}\) may be seen as an explicit solution to this integral.

**II. GENERALIZATIONS OF THE SELBERG INTEGRAL**

We start by fixing some notation (a basic reference is \(^{16}\)). A non-increasing sequence of positive integers \(\lambda = (\lambda_1, \lambda_2, \ldots)\) is said to be a partition of \(n\) if \(\sum \lambda_i = n\). This is indicated by \(\lambda \vdash n\). The number of parts in \(\lambda\) is called its length and denoted by \(\ell(\lambda)\). The set of all partitions of \(n\) is \(P(n)\). With every partition \(\lambda\) we can associate a monomial \(x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots\). The symmetric polynomial in \(k\) variables \(m_{\lambda}^{(k)}(x)\) is the sum of all distinct monomials obtainable from \(x^\lambda\) by permutation of the \(x_i\)’s. For example, take \(n = k = 2\). Then \(m_{(2)}(x) = x_1^2 + x_2^2\) and \(m_{(1,1)}(x) = x_1 x_2\). The set of functions \(\{m_{\lambda}^{(k)}(x), \lambda \in P(n)\}\) forms a basis for the vector space of homogeneous symmetric polynomials of degree \(n\) in \(k\) variables.

Another basis for this space is composed by Schur functions \(s_{\lambda}^{(k)}(x)\). If \(\lambda \vdash n \leq k\),

\[ s_{\lambda}^{(k)}(x) = \frac{1}{\Delta(x)} \det \left( x_i^{k+\lambda_j-j} \right)_{1 \leq i,j \leq k}. \] \hspace{1cm} (5)

Let us use again \(n = k = 2\) as an example. We then have \(s_{(2)}(x) = x_1 x_2\) and \(s_{(1,1)}(x) = x_1^2 + x_2^2 + x_1 x_2\). These two basis are of course related by a linear transformation, \(s_{\lambda}^{(k)} = \sum_{\mu \vdash n} K_{\lambda,\mu} m_{\mu}^{(k)}\). The matrices \(K\) are called Kostka matrices.

Let us from now on take \(k = N_1\), i.e. consider only polynomials in \(N_1\) variables, and no longer write any superscript. Let

\[ [x]_\lambda = \prod_{i=1}^{\ell(\lambda)} \frac{(x+\lambda_i-i)!}{(x-i)!} \] \hspace{1cm} (6)

be a generalization of the usual rising factorial

\[ (x)_a = x(x+1) \cdots (x+a-1). \] \hspace{1cm} (7)

Our favorite examples are now \([x]_{(2)} = x(x+1)\) and \([x]_{(1,1)} = x(x-1)\). Let us also define the function

\[ H_\lambda = \prod_{i<j} (\lambda_i - \lambda_j + j - i). \] \hspace{1cm} (8)

This is equal to the product of all hook lengths of the partition \(\lambda\)\(^6\) but that is not relevant here. Kaneko\(^8\) and Kadell\(^12\) have proved a generalization of Selberg’s integral which implies that

\[ \langle s_\lambda \rangle = \int_{\mathcal{C}} s_\lambda(T) \mathcal{P}(T)dT = \frac{[N_1]! [N_2]!}{[N]! H_\lambda}. \] \hspace{1cm} (9)

Other generalizations are known. For example, Huang\(^{10}\) (see also \(^{20}\)) has shown that

\[ \int_{\mathcal{C}} s_\lambda(T)s_\mu(T) \mathcal{P}(T)dT = \langle s_\lambda \rangle \langle s_\mu \rangle \{N + 1\}_{\lambda,\mu}, \] \hspace{1cm} (10)

where

\[ \{N\}_{\lambda,\mu} = \prod_{i,j} \frac{(N-i+j+\lambda_i)(N-i-j+\mu_j)}{(N-i-j+\lambda_i+\mu_j)(N-i-j)}. \] \hspace{1cm} (11)

**III. RESULTS**

**A. Average of \(T_n\)**

Making use of \(^{10}\) one can in principle obtain the average value of any symmetric polynomial of the transmission eigenvalues. For example, it is known\(^{16}\) that

\[ T_n = m_{(n)}(T) = \sum_{p=0}^{n-1} (-1)^p s_{(n-p,1^p)}(T), \] \hspace{1cm} (12)

where the notation \(1^p\) means that the number 1 appears \(p\) times. Substituting into \(^{10}\) gives the remarkably simple result for the counting statistics

\[ \langle T_n \rangle = \sum_{p=0}^{n-1} G_{n,p}, \] \hspace{1cm} (13)

where

\[ G_{n,p} = \frac{(-1)^p}{n!} \binom{n-1}{p} \frac{(N_1 - p)_{n_2 - p} n_1}{(N - p)_{n_1}}. \] \hspace{1cm} (14)

The sum \(^{13}\) allows in principle the calculation of any linear statistic. It has the merit of having only \(n\) terms, in contrast with the result in \(^{12}\) where the number of terms in the sum grows with \(N_1, N_2\). In Fig.1 (top) we plot \(\langle T_n \rangle / N\) for a few values of \(n\), as a function of \(N_1\) for
FIG. 1: Top: some of the counting statistics for $N_2 = 20$. The value at the maximum $N_1 \approx N_2$ is close to $(2^n/n)^4$. Bottom: Comparison between exact result (13) and asymptotic limit (15) for $N_2 = 1$.

$N_2 = 20$. All curves have a maximum around $N_1 \approx N_2$, where they are approximately equal to $(2^n/n)^4$.

Let us denote by $\langle T_n \rangle_\infty$ the asymptotic form of $\langle T_n \rangle$ when both channel numbers are large, $N_1, N_2 \gg 1$. These functions are given by

$$\langle T_n \rangle_\infty = N \xi \sum_{m=0}^{n-1} \frac{(-1)^m (2m)!}{m+1} \binom{n-1}{m} \xi^m,$$

where we have defined the finite variable $\xi = N_1 N_2 / N^2$.

Interestingly, a derivation of this expression directly from (13) does not seem to be trivial. Notice that $\langle T_n \rangle_\infty = (2^n/n)^4$ for $N_1 = N_2$. In Fig.1 (bottom) we compare the exact result (13) with the asymptotics (15) for $N_2 = 1$, in which case $\langle T_n \rangle = N_1/(N_1 + n)$. We can see that these quantities may differ significantly if both channel numbers are close to unity, specially for higher values of $n$.

We can obtain from (13) the set of charge cumulants $\langle \langle Q_n \rangle \rangle$, which are given by

$$\langle \langle Q_n \rangle \rangle = \sum_{m=1}^{n} C_{n,m} \langle T_m \rangle,$$

with the coefficients

$$C_{n,m} = \sum_{j=0}^{n-1} (-1)^j \binom{m-1}{j} (j + 1)^{n-1}.$$

Figure 2 shows some plots of $\langle \langle Q_n \rangle \rangle$. The odd ones are always zero for equal leads, $N_1 = N_2$, and contain a factor $(N_2 - N_1)^2$ in general. Except for the second one (shot-noise), all $\langle \langle Q_n \rangle \rangle$ oscillate and even change sign as functions of $N_1$ for fixed $N_2$. The asymptotic form of these quantities for large channel numbers, which we denote $\langle \langle Q_n \rangle \rangle_\infty$, can be derived by using $\langle T_n \rangle_\infty$ instead of $\langle T_n \rangle$ in (16). That yields

$$\langle \langle Q_n \rangle \rangle_\infty = -N \xi \sum_{m=0}^{n-1} (-1)^{n+m} \binom{2m}{m} S_{n-1,m} \xi^m,$$

where $S_{n,m}$ are the Stirling numbers of the second kind.

In the last panel of Fig.2 we compare the exact and asymptotic results for $N_1 = N_2 = N$. In contrast to $\langle T_n \rangle$, for which the asymptotic value is always above the exact one, here we see that the difference $\langle \langle Q_n \rangle \rangle - \langle \langle Q_n \rangle \rangle_\infty$ oscillates as a function of $N$, with a number of zeros that increases with $n$. 
Using (10) we see that \( \text{var}(T) \) has proved (see also \[23\]) that \( \text{var}(T) \) is Gaussian, while for small \( N \), it increases mildly with \( n \), and is monotonic in \( N \). They all saturate at a finite value as \( N \to \infty \). For \( N_1 = N_2 \) the many-channels limit can be obtained explicitly by other means\(^{21}\) and is given by

\[
\text{var}(T) = \sum_{p,q=0}^{n-1} \frac{n^2 G_{n,p} G_{n,q}}{(N + 2n - p - q - 1)(N - p - q - 1)}.
\]

(20)

This generalizes to \( n > 1 \) the well known result for conductance fluctuations, \( \text{var}(T_1) = N_1^2 N_2^2 / N^2 (N^2 - 1) \). Of course, we could just as easily have computed the value of \( \langle T_n T_m \rangle \) for any \( n, m \).

In Fig.3 we plot \( \text{var}(T_n) \) as a function of \( N_1 \) for the case \( N_2 = 20 \), for a few values of \( n \). The function increases mildly with \( n \), and is monotonic in \( N_1 \). They all saturate at a finite value as \( N_1 \to \infty \). For \( N_1 = N_2 \) the many-channels limit can be obtained explicitly by other means\(^{21}\) and is given by

\[
\text{var}(T_n)_{\infty} = \frac{(2n-1)\Gamma(n+1/2)\Gamma(n-1/2)}{8\pi n (\Gamma(n))^2}.
\]

(21)

This quantity starts at the value 1/16 (variance of the conductance), and saturates at 1/4\(\pi\) for large \( n \).

**C. Moments of the conductance**

It is known that for large channel numbers the probability distribution of the conductance \( g = T_1 \) approaches a Gaussian, while for small \( N_1, N_2 \) there are deviations. The last quantity we shall compute is the set of all moments \( \langle g^n \rangle \) characterizing this distribution. Forrester\(^{22}\) has proved (see also \[23\]) that

\[
\langle \prod_j (1 - xT_j)^{-a} \rangle = \sum_{\lambda} \frac{[a]_\lambda}{[N]_\lambda H_\lambda^a} s_\lambda(x),
\]

(22)

where we have an infinite sum over partitions. Let us perform the change of variables \( x = y/a \) and take the limit \( a \to \infty \). It is well known\(^{16}\) that, if \( \lambda \vDash n \), then

\[
[a]_\lambda \approx a^n, \quad s_\lambda(y/a) \approx \left( \frac{y}{a} \right)^n \frac{[N_1]_\lambda}{H_\lambda}, \quad a \to \infty.
\]

(23)

Using \( \lim_{a \to \infty} (1 - yT/a)^{-a} = e^{yT} \) on the left hand side and the simplifications above on the right hand side, equation (22) provides a generating function for the quantities we are looking for. The result is

\[
\langle g^n \rangle = n! \sum_{\lambda \vDash n} \frac{[N_1]_\lambda [N_2]_\lambda}{[N]_\lambda H_\lambda^2}.
\]

(24)

From these moments one may compute the corresponding cumulants \( \langle (g^n) \rangle \). For a Gaussian distribution,
\[ \langle g^n \rangle = 0 \text{ for } n > 2. \]  From the symmetry of the distribution \[^2\text{2}\], all odd cumulants with \( n > 1 \) vanish identically if \( N_1 = N_2 \) (in general, they contain the factor \((N_1 - N_2)^2\)). Figure 4 (top) presents some of the even cumulants \( \langle g^{2n} \rangle \) for \( N_1 = N_2 = N \). We see that \( \langle g^2 \rangle \) saturates at the expected value 1/6. Although the others seem to have somewhat singular behaviour for \( N < n \), only integer values of \( N \) are actually physical. They all decrease rapidly after \( N > n \), and already for \( N = 6 \) the distribution function of the conductance is Gaussian to a good approximation. In Fig.4 (middle) we see the first few cumulants as functions of \( N_1 \) for \( N_2 = 20 \), while Fig.4 (bottom) shows how fast the value of \( \langle g^n \rangle \) decays with \( n \) for \( N_2 = 2N_1 = 20 \).

IV. CONCLUSIONS

We have extended the random matrix theory of quantum transport in ballistic chaotic cavities connected to two ideal leads, by obtaining exact results for several statistical quantities. All our formulas as expressed in simple closed form. Even though we consider systems with broken time-reversal (TR) symmetry, we have seen that exact results are in general different from their large-N asymptotic limit.

Unfortunately, the calculations presented here are not immediately generalizable to TR symmetric systems, for which we expect the difference between exact and asymptotic results to be more important. This is because the role played here by Schur functions is played in that case by zonal polynomials, and much less is known about the analogues of Kostka matrices. This certainly deserves further investigation (actually, relation \[^2\text{2}\] has been proven\[^2\text{4}\] for other symmetry classes, so the moments of the conductance can in fact be obtained in those cases). Also, it would be interesting to obtain systematic exact results for more general nonlinear statistics.

After this work was completed, another paper appeared\[^2\text{4}\] in which the probability distribution of conductance is studied for broken TR. In particular, they obtain an efficient recurrence relation for the cumulants of \( g \). It would be important to clarify what are the relations between those results and the present work.

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