INEQUALITIES OF LEVIN-STEČKIN, CLAUSING AND CHEBYSHEV REVISITED

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Abstract. We prove the Levin-Stečkin inequality using Chebyshev’s inequality and symmetrization. Symmetry and slightly modified Chebyshev’s inequality are also the key to an elementary proof of Clausing’s inequality.

1. Introduction

It seems that the Levin-Stečkin inequality appeared first in an appendix to the Russian edition of the famous Hardy, Littlewood and Pólya’s Bible on inequalities [3]. The translator (Levin) enumerates the appendices written by Stečkin, by Levin and by both of them. The inequality we consider here comes from Appendix I written by Stečkin. But the English version of the appendix [4] did not probably make this distinction clear enough, so all inequalities cited in the literature are called Levin-Stečkin.

Theorem 1.1 (Levin-Stečkin’s inequality). Let the function \( p : (0, 1) \rightarrow \mathbb{R} \) satisfies the conditions

\begin{enumerate}
\item \( p \) is non-decreasing in \( (0, \frac{1}{2}) \),
\item \( p \) is symmetric, i.e. \( p(x) = p(1 - x) \),
\end{enumerate}

then for every convex function \( \varphi \) the inequality

\[
\int_0^1 p(x) \varphi(x) dx \leq \int_0^1 p(x) dx \int_0^1 \varphi(x) dx.
\]

The original proof is elementary, but quite complicated. Recently Mercer [5] published a proof that uses the notion of extremal points of the set of concave positive functions satisfying \( \int_0^1 f(x) dx \leq 1 \). His method, not very elementary, has an advantage: leads to a simple proof of the Clausing inequality.

Theorem 1.2 (Clausing’s inequality [2]). Let \( p \) be nonnegative functions on \( (0, 1) \) satisfying the following conditions:

\begin{itemize}
\item \( p \) are symmetric (i.e. \( p(x) = p(1 - x) \)),
\item \( p \) is non-decreasing on \( [0, 1/2] \),
\end{itemize}

Then for every concave, positive function \( \varphi \) the inequality

\[
\int_0^1 p(x) \varphi(x) dx \leq \int_0^1 \varphi(x) dx \int_0^1 4 \min\{x, 1 - x\} p(x) dx
\]
Both inequalities make the reader think of the inequality of Chebyshev, linking the integral of a product of functions with the product of integrals.

**Theorem 1.3** (Chebyshev’s inequality). If the functions \( f, g : [a, b] \to \mathbb{R} \) are monotone in the same direction, then

\[
\frac{1}{b-a} \int_a^b f(x)g(x) dx \geq \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx
\]

The inequality is reversed if the monotonicities are opposite.

Our aim is to give elementary proofs of Levin-Stečkin’s and Clausing’s inequalities. The proofs we offer here are sponsored by the word *symmetrization*.

2. **The Levin-Stečkin Inequality**

We prove this inequality in two steps: firstly we show that Theorem 1.1 is valid for symmetric functions:

**Lemma 2.1.** Under the assumptions of Theorem 1.1 if \( \varphi \) is symmetric and convex, then the inequality (1.1) holds.

**Proof.** Suppose \( \varphi \) is convex. Its symmetry implies that it is non-increasing in the interval \((0, \frac{1}{2})\), and using Chebyshev’s inequality we get

\[
\int_0^{1/2} p(x) dx \int_0^{1/2} \varphi(x) dx = \left( \int_0^{1/2} p(x) dx + \int_0^{1/2} p(x) dx \right) \left( \int_0^{1/2} \varphi(x) dx + \int_0^{1/2} \varphi(x) dx \right) = 4 \int_0^{1/2} p(x) dx \int_0^{1/2} \varphi(x) dx \geq 2 \int_0^{1/2} p(x) \varphi(x) dx \int_0^{1/2} \varphi(x) dx.
\]

Now consider arbitrary \( \varphi \).

**Proof of the Levin-Stečkin inequality.** Once more we shall explore the symmetry. Note that for convex \( \varphi \) the function \( \frac{\varphi(x) + \varphi(1-x)}{2} \) is convex and symmetric, so we can use Lemma 2.1

\[
\int_0^1 p(x) \varphi(x) dx = \int_0^1 p(x) \frac{\varphi(x) + \varphi(1-x)}{2} dx
\]
\[
\leq \int_0^1 p(x) dx \int_0^1 \frac{\varphi(x) + \varphi(1-x)}{2} dx
\]
\[
= \int_0^1 p(x) dx \int_0^1 \varphi(x) dx.
\]
3. Chebyshev’s Inequality

To prove the Clausing inequality we need a bit stronger version of Chebyshev’s inequality, where the monotonicity of one function get replaced by a weaker condition. Note that this result is somewhat similar to the result of Brunn [1].

Definition 3.1. We shall say that an integrable function \( f : [a, b] \to \mathbb{R} \) belongs to the class \( M^+ \) if there is a \( c \in [a, b] \) such that

1. if \( f(x) < \frac{1}{b-a} \int_a^b f(x)dx \), then \( x < c \) and
2. if \( f(x) > \frac{1}{b-a} \int_a^b f(x)dx \), then \( x > c \).

We say that \( f \) belongs to \( M^- \) if the inequalities in (1) and (2) are reversed.

Obviously every non-decreasing function belongs to the class \( M^+ \) and a non-increasing one is a member of \( M^- \).

Theorem 3.1. If \( f, g : [a, b] \to \mathbb{R} \) are integrable, \( g \) is non-decreasing and \( f \in M^+ \) or \( g \) is non-increasing and \( f \in M^- \), then

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \geq \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx.
\]

Exchanging \( M^+ \) and \( M^- \) toggles the inequality.

Proof. Let \( f \in M^+ \) and \( g \) be non-decreasing (the proof in other cases is similar). Denote \( f^* = \frac{1}{b-a} \int_a^b f(x)dx \). Then

\[
\int_a^b [f(x) - f^*] g(x)dx = \int_a^c [f(x) - f^*] g(x)dx + \int_c^b [f(x) - f^*] g(x)dx \geq g(c) \int_a^c [f(x) - f^*] dx + g(c) \int_c^b [f(x) - f^*] dx = 0. \quad \square
\]

4. Clausing’s Inequality

In this section we present an elementary proof of a generalization of the Clausing inequality.

Theorem 4.1. Let \( p, q \) be nonnegative functions on \((0, 1)\) satisfying the following conditions:

- \( p \) and \( q \) are symmetric (i.e. \( p(x) = p(1-x) \)),
- \( p \) is increasing on \([0, 1/2]\),
- \( q \) is convex on \([0, 1/2]\),
- \( q(0) = 0 \) and \( \int_0^1 q(x)dx = 1 \).

Then for every concave function \( \varphi \) with \( \varphi(0) + \varphi(1) \geq 0 \) the inequality

\[
\int_0^1 p(x)\varphi(x)dx \leq \int_0^1 \varphi(x)dx \int_0^1 p(x)q(x)dx
\]

holds.

Proof. Assume first that \( \varphi \) is symmetric and denote \( \int_0^1 \varphi(x)dx = K \). The inequality (4.1) can be rewritten as

\[
0 \leq \int_0^{1/2} [Kq(x) - \varphi(x)]p(x)dx.
\]
The Hermite-Hadamard inequality yields $K \geq 0$ and the symmetry of $\varphi$ implies $\varphi(0) \geq 0$, thus the function $Kq - \varphi$ is convex, $Kq(0) - \varphi(0) \leq 0$ and $\int_0^{1/2} [Kq(x) - \varphi(x)] dx = 0$, therefore it belongs to the class $M^+$, and by Theorem 4.1 $\int_0^{1/2} [Kq(x) - \varphi(x)] p(x) dx \geq 0$ which proves (4.2).

Now let $\varphi$ be arbitrary. We have

$$\int_0^1 p(x)\varphi(x) dx = \int_0^1 p(x)\frac{\varphi(x) + \varphi(1 - x)}{2} dx$$

$$\leq \int_0^1 \frac{\varphi(x) + \varphi(1 - x)}{2} dx \int_0^1 p(x)q(x) dx \quad \text{(by (4.1))}$$

$$= \int_0^1 \varphi(x) dx \int_0^1 p(x)q(x) dx$$

which completes the proof. □

The function $q_0(x) = 4 \min\{x, 1 - x\}$ is a borderline between admissible $q$’s and sample functions $\varphi$. Setting $\varphi = q_0$ in (4.1) we obtain

$$\int_0^1 p(x)q_0(x) dx \leq \int_0^1 p(x)q(x) dx$$

which means that $q_0$ provides the best bound in (4.1).

References

[1] Brunn H., Nachtrag zu dem Aufsatz über Mittelwertsätze für bestimmte Integrale, Münchener Berichte 1903, 205-212.

[2] Clausing A., Disconjugacy and Integral Inequalities, Trans. Amer. Math. Soc. 260 (1980), 293–307.

[3] Харди Г.Г., Литлвуд Дж.Е., Пойа Г., Неравенства, Moscow, 1948

[4] Levin V.I and Stečkin S.B., Inequalities, Amer. Math. Soc. Transl. (2) 14 (1960), 1–29

[5] Mercer P.R., A note on inequalities due to Clausing and Levin-Stečkin, J. Math. Ineq. (to appear)

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