ON THE $q$-EXTENSION OF HIGHER-ORDER EULER POLYNOMIALS

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ABSTRACT. The purpose of this paper is to present a systemic study of some families of the generalized $q$-Euler numbers and polynomials of higher-order. In particular, by using multivariate $p$-adic invariant integral on $\mathbb{Z}_p$, we construct the generalized $q$-Euler numbers and polynomials of higher-order.

1. Introduction

Let $p$ be a fixed odd prime and let $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. For $d$ a fixed positive odd integer with $(p, d) = 1$, let

$$X = X_d = \lim_{\rightarrow} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, (see [3-19]).

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When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1 - q|_p < 1$. In this paper we use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$  

Let $\chi$ be the Dirichlet’s character with conductor $d (= \text{odd}) \in \mathbb{N}$. Then the generalized Euler polynomials, $E_{n, \chi}(x)$, are defined as

$$F_{\chi}(x, t) = \frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) e^{lt}}{e^{dt} + 1} e^{xt} = \sum_{l=0}^{\infty} E_{n, \chi}(x) \frac{t^n}{n!}, \quad (\text{see [3, 6]}).$$

We note that, by substituting $x = 0$ in (1), $E_{n, \chi}(0) = E_n, \chi$ is the familiar $n$-th Euler number defined by

$$F_{\chi}(0, t) = \frac{2 \sum_{l=0}^{d-1} (-1)^l \chi(l) e^{lt}}{e^{dt} + 1} = \sum_{l=0}^{\infty} E_{n, \chi} \frac{t^n}{n!}.$$  

For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\sum_{0 \leq j < p^N} (-1)^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu(j + p^N \mathbb{Z}_p)$$

representing analogue of Riemann’s sums for $f$, cf.[1-10].

The fermionic $p$-adic invariant integral of $f$ on $\mathbb{Z}_p$ will be defined as the limit ($N \to \infty$) of these sums, which it exists. The fermionic $p$-adic invariant integral of a function $f \in UD(\mathbb{Z}_p)$ is defined in [1, 3, 5, 7, 10] as follows:

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \sum_{0 \leq j < p^N} f(j) \mu(j + p^N \mathbb{Z}_p) = \lim_{N \to \infty} \sum_{0 \leq j < p^N} f(j) (-1)^j.$$  

Thus, we have

$$I(f_1) + I(f) = 2f(0), \quad \text{where } f_1(x) = f(x + 1).$$

By using integral iterative method, we also easily see that

$$I(f_n) + (-1)^n I(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad \text{where } f_n(x) = f(x + n) \text{ for } n \in \mathbb{N}.$$
From (3), we note that

\[
\int_X \chi(x)e^{xt}d\mu(x) = \frac{2\sum_{l=0}^{d-1}(-1)^l e^{lt}\chi(l)}{e^{dt} + 1} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}.
\]

By (4), we see that

\[
\int_X \chi(x)x^nd\mu(x) = E_{n,\chi}, \quad \text{and} \quad \int_X \chi(y)(x+y)^nd\mu(y) = E_{n,\chi}(x), \quad \text{(see [6])}.
\]

The \(n\)-th generalized Euler polynomials of order \(k\), \(E_{n,\chi}^{(k)}(x)\), are defined as

\[
\left(\frac{2\sum_{l=0}^{d-1}(-1)^l e^{lt}\chi(l)}{e^{dt} + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}^{(k)}(x) \frac{t^n}{n!} \quad \text{(see [6, 7])}.
\]

In the special case \(x = 0\), \(E_{n,\chi}^{(k)}(0) = E_{n,\chi}^{(k)}\) are called the \(n\)-th generalized Euler numbers of order \(k\). Now, we consider the multivariate \(p\)-adic invariant integral on \(Z_p\) as follows:

\[
\int_X \cdots \int_X \chi(x_1) \cdots \chi(x_k)e^{(x_1+\cdots+x_k+x)t}d\mu(x_1) \cdots d\mu(x_k)
\]

\[
= \left(\frac{2\sum_{l=0}^{d-1}(-1)^l e^{lt}\chi(l)}{e^{dt} + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}^{(k)}(x) \frac{t^n}{n!}.
\]

By (6) and (7), we obtain the Witt’s formula for the \(n\)-th generalized Euler polynomials of order \(k\) as follows:

\[
\int_X \cdots \int_X \left(\prod_{i=1}^{k} \chi(x_i)\right)(x_1 + \cdots + x_k + x)^nd\mu(x_1) \cdots d\mu(x_k) = E_{n,\chi}^{(k)}(x).
\]

In the viewpoint of the \(q\)-extension of (8), we will consider the \(q\)-extension of generalized Euler numbers and polynomials of order \(k\). The purpose of this paper is to present a systemic study of some families of the generalized \(q\)-Euler numbers and polynomials of higher-order. In particular, by using multivariate \(p\)-adic invariant integral on \(Z_p\), we construct the generalized \(q\)-Euler numbers and polynomials of higher-order.
2. On the $q$-extension of higher-order Euler numbers and polynomials

In this section we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, let $\chi$ be the Dirichlet's character with conductor $d$. For $h \in \mathbb{Z}, k \in \mathbb{N}$, let us consider the generalized $q$-Euler numbers and polynomials of order $k$ in the viewpoint of the $q$-extension of (8). First, we consider the $q$-extension of (1) as follows:

\begin{equation}
\sum_{n=0}^{\infty} E_{n,\chi,q}(x) \frac{t^n}{n!} = \int_X e^{[x+y]q^t} \chi(y) d\mu(y) = 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m e^{[m]q^t}, \quad (1, \text{ cf. [1, 4]}).
\end{equation}

By (9), we have

\begin{equation}
\int_X [x+y]^n \chi(y) d\mu(y) = 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m [m]_q^n
\end{equation}

\begin{equation}
= 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a \frac{1}{(1-q)^n} \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) (-1)^l q^{l(a+x)} + \frac{1}{1+q^d}t.
\end{equation}

From the multivariate $p$-adic invariant integral on $\mathbb{Z}_p$, we can also derive the $q$-extension of the generalized Euler polynomials of order $k$ as follows:

\begin{equation}
E_{n,\chi,q}^{(k)}(x) = \int_X \cdots \int_X \left(\prod_{i=1}^{k} \chi(x_i)\right) [x_1 + \cdots + x_k + x]^n q d\mu(x_1) \cdots d\mu(x_k)
= \sum_{a_1,\cdots,a_k=0}^{d-1} \left(\prod_{i=1}^{k} \chi(a_i)\right) (-1)^{\sum_{j=1}^{k} a_j} \frac{2^k}{(1-q)^n} \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) (-1)^l q^{l(x+\sum_{j=1}^{k} a_j)}
\end{equation}

\begin{equation}
= 2^k \sum_{a_1,\cdots,a_k=0}^{d-1} \left(\prod_{i=1}^{k} \chi(a_i)\right) (-1)^{\sum_{j=1}^{k} a_j} \sum_{m=0}^{\infty} \left(\begin{array}{c} m+k-1 \\ m \end{array}\right) (-1)^m [x + \sum_{j=1}^{k} a_j + dm]_q^n
\end{equation}

Let $F_{q,\chi}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(k)}(x) t^n/n!$. Then we have

\begin{equation}
F_{q,\chi}^{(k)}(t, x) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(k)}(x) t^n/n!
= \sum_{a_1,\cdots,a_k=0}^{d-1} \left(\prod_{i=1}^{k} \chi(a_i)\right) (-1)^{\sum_{j=1}^{k} a_j} \sum_{m=0}^{\infty} \left(\begin{array}{c} m+k-1 \\ m \end{array}\right) (-1)^m e^{[x+\sum_{j=1}^{k} a_j + dm]}.\end{equation}
From (12), we obtain the following theorem.

**Theorem 1.** For \( k \in \mathbb{N}, n \geq 0 \), we have

\[
E^{(k)}_{n, \chi, q} = \frac{2^k}{(1 - q)^n} \sum_{a_1, \ldots, a_k = 0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) \left( -1 \right)^{\sum_{j=1}^{k} a_j} \sum_{l=0}^{n} \frac{(n)_l (-1)^l q^{l(x + \sum_{j=1}^{k} a_j)}}{(1 + q^d)^k} \\
= 2^k \sum_{a_1, \ldots, a_k = 0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) \left( -1 \right)^{\sum_{j=1}^{k} a_j} \sum_{m=0}^{\infty} \left( m + k - 1 \right) \sum_{j=1}^{k} a_j \cdot m \left( -1 \right)^m [x + \sum_{j=1}^{k} a_j + md]_q^n.
\]

For \( h \in \mathbb{Z}, k \in \mathbb{N} \), let us consider the extension of \( E^{(k)}_{n, \chi, q}(x) \) as follows:

\[
E^{(h,k)}_{n, \chi, q}(x) = \int_X \cdots \int_X q^{\sum_{j=1}^{k} (h-j) x_j} \left( \prod_{j=1}^{k} \chi(x_j) \right) \left[ x + \sum_{j=1}^{k} x_j \right]^n d\mu(x_1) \cdots d\mu(x_k)
\]

\[
= \sum_{a_1, \ldots, a_k = 0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) \left( -1 \right)^{\sum_{j=1}^{k} a_j} q^{\sum_{j=1}^{k} a_j (h-j)} \int_X \cdots \int_X \left[ x + \sum_{j=1}^{k} (dx_j + a_j) \right]^n d\mu(x_1) \cdots d\mu(x_k)
\]

\[
= \sum_{a_1, \ldots, a_k = 0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) \left( -1 \right)^{\sum_{j=1}^{k} a_j} q^{\sum_{j=1}^{k} a_j (h-j)} \frac{2^k}{(1 - a)^n} \sum_{l=0}^{n} \frac{(n)_l (-1)^l q^{l(x + \sum_{j=1}^{k} a_j)}}{(1 - q)^n \prod_{i=1}^{d} (1 - a q^{h-k+i}) : q^d)^k},
\]

where \( (a : q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1}) \), (see [1, 4]).

It is well known that the Gaussian binomial coefficient is defined as

\[
\binom{n}{k}_q = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdots [2]_q \cdot [1]_q}, \quad \text{(see [1, 4]).}
\]
By (13) and (14), we easily see that

\[ E^{(h,k)}_{n,\chi,q}(x) = \frac{2^k}{(1-q)^n} \sum_{a_1,\ldots,a_k=0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) (-1)^{\sum_{i=1}^{k} a_i q^{\sum_{j=1}^{k} (h-j)a_j}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \]

\[ q^{l(x+\sum_{j=1}^{k} a_j)} \sum_{m=0}^{\infty} \binom{m+k-1}{m} q^d (-1)^m q^{d(h-k)m} q^{dim} \]

\[ = 2^k [d]^n \sum_{m=0}^{\infty} \binom{m+k-1}{m} q^d (-1)^m q^{d(h-k)m} \sum_{a_1,\ldots,a_k=0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) (-1)^{\sum_{j=1}^{k} a_j} q^{\sum_{j=1}^{k} (h-j)a_j} [m + \frac{x+\sum_{j=1}^{k} a_j}{d}] q^d. \]

Let \( F^{(h,k)}_{\chi,q}(t,x) = \sum_{n=0}^{\infty} E^{(h,k)}_{n,\chi,q}(x) \frac{t^n}{n!}. \) From (15), we note that

\[ F^{(h,k)}_{\chi,q}(t,x) = 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} q^d (-1)^m q^{d(h-k)m} \sum_{a_1,\ldots,a_k=0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) (-1)^{\sum_{j=1}^{k} a_j} q^{\sum_{j=1}^{k} (h-j)a_j} e^{md+x+\sum_{j=1}^{k} a_j q}. \]

By (16), we obtain the following theorem.

**Theorem 2.** For \( h \in \mathbb{Z}, \ k \in \mathbb{N}, \) we have

\[ E^{(h,k)}_{n,\chi,q}(x) = 2^k [d]^n \sum_{m=0}^{\infty} \binom{m+k-1}{m} q^d (-1)^m q^{d(h-k)m} \sum_{a_1,\ldots,a_k=0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) \]

\[ (-1)^{\sum_{j=1}^{k} a_j q^{\sum_{j=1}^{k} (h-j)a_j}} [m + \frac{x+a_1+a_2+\cdots+a_k}{d}] q^d \]

\[ = \sum_{a_1,\ldots,a_k=0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) (-1)^{\sum_{j=1}^{k} a_j q^{\sum_{j=1}^{k} (h-j)a_j}} \left( \frac{2^k}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l(x+\sum_{j=1}^{k} a_j)} \right). \]
For $h = k$, we have

\[(17)\]

\[E_{n,\chi,q}^{(k,k)}(x) = \frac{2^k}{(1-q)^n} \sum_{a_1, \ldots, a_k=0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) (-1)^{\sum_{j=1}^{k} a_j} q^{\sum_{j=1}^{k} (h-j)a_j} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} q^l (\sum_{j=1}^{k} a_j + x) \]

\[= 2^k [d]_q^n \sum_{m=0}^{\infty} \left( \begin{array}{c} m+k-1 \end{array} \right)_q m \sum_{a_1, \ldots, a_k=0}^{d-1} \left( \prod_{i=1}^{k} \chi(a_i) \right) (-1)^{\sum_{j=1}^{k} a_j} q^{\sum_{j=1}^{k} (k-j)a_j} \cdot \left[ m + \frac{x + a_1 + a_2 + \cdots + a_k}{d} \right]_q^n \]

It is not difficult to show that

\[(18)\]

\[\int_X \cdots \int_X \left( \prod_{j=1}^{k} \chi(x_j) \right) q^{\sum_{j=1}^{k} (m-j)x_j + mx} d\mu(x_1) \cdots d\mu(x_k) = \sum_{a_1, \ldots, a_k=0}^{d-1} \left( \prod_{j=1}^{k} \chi(a_j) \right) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^d \sum_{j=1}^{k} (m-j)x_j d\mu(x_1) \cdots d\mu(x_k) \]

\[= 2^k q^{mx} \sum_{a_1, \ldots, a_k=0}^{d-1} \left( \prod_{j=1}^{k} \chi(a_j) \right) q^{\sum_{j=1}^{k} (m-j)a_j} (-1)^{\sum_{j=1}^{k} a_j} \]

\[= \frac{2^k q^{mx} \sum_{a_1, \ldots, a_k=0}^{d-1} \left( \prod_{j=1}^{k} \chi(a_j) \right) q^{\sum_{j=1}^{k} (m-j)a_j} (-1)^{\sum_{j=1}^{k} a_j}}{(-q^d(m-k) : q^d)_k} \]

From (18), we can derive the following equation (19).

\[(19)\]

\[\frac{2^k q^{mx} \sum_{a_1, \ldots, a_k=0}^{d-1} \left( \prod_{j=1}^{k} \chi(a_j) \right) q^{\sum_{j=1}^{k} (m-j)a_j} (-1)^{\sum_{j=1}^{k} a_j}}{(-q^d(m-k) : q^d)_k} = \int_X \cdots \int_X \left( |x + x_1 + \cdots + x_k|_q (q-1) + 1 \right)^m q^{-\sum_{j=1}^{k} jx_j} \left( \prod_{j=1}^{k} \chi(x_j) \right) d\mu(x_1) \cdots d\mu(x_k) \]

\[= \sum_{l=0}^{m} \binom{m}{l} q^{-1} \int_X \cdots \int_X \left( \prod_{j=1}^{k} \chi(x_j) \right) [x + x_1 + \cdots + x_k]^l q^{-\sum_{j=1}^{k} jx_j} d\mu(x_1) \cdots d\mu(x_k) \]

\[= \sum_{l=0}^{m} \binom{m}{l} (q-1)^l E_{l,\chi,q}^{(0,k)}(x). \]

By (19), we obtain the following theorem.
Theorem 3. For $d, k \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have
\[
2^k q^{mx} \sum_{a_1, \ldots, a_k = 0}^{d-1} \left( \prod_{j=1}^{k} \chi(a_j) \right) q^{\sum_{j=1}^{k} (m-j) a_j} (-1)^{\sum_{j=1}^{k} a_j} (-q^d)^{(m-k)} = \sum_{l=0}^{m} \binom{m}{l} (q-1)^l E^{(0,k)}_{l,x,q}(x).
\]

From the definition of $p$-adic invariant integral on $\mathbb{Z}_p$, we note that
\[(20) \quad q^d (h-1) \int_{X} \cdots \int_{X} [x + d + x_1 + \cdots + x_k]^{n} q^{\sum_{j=1}^{k} (k-j)x_j} \left( \prod_{j=1}^{k} \chi(x_j) \right) d\mu(x_1) \cdots d\mu(x_k)
\]
\[= - \int_{X} \cdots \int_{X} [x + x_1 + \cdots + x_k]^{n} q^{\sum_{j=1}^{k} (k-j)x_j} \left( \prod_{j=1}^{k} \chi(x_j) \right) d\mu(x_1) \cdots d\mu(x_k) + 2 \sum_{l=0}^{d-1} \chi(l)
\]
\[(-1)^l \int_{X} \cdots \int_{X} [x + \sum_{j=1}^{k-1} x_{j+1}]^{n} \left( \prod_{j=1}^{k-1} \chi(x_{j+1}) \right) q^{\sum_{j=1}^{k-1} (h-1-j)x_{j+1}} d\mu(x_2) \cdots d\mu(x_k).
\]

By (20), we obtain the following theorem.

Theorem 4. For $h \in \mathbb{Z}$, $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have
\[(21) \quad q^d (h-1) E^{(h,k)}_{n,x,q}(x + d) + E^{(h,k)}_{n,x,q}(x) = 2 \sum_{l=0}^{d-1} \chi(l)(-1)^l E^{(h-1,k-1)}_{n,q}(x).
\]

Moreover,
\[q^x E^{(h+1,k)}_{n,x,q}(x) = (q - 1) E^{(h,k)}_{n+1,x,q}(x) + E^{(h,k)}_{n,x,q}(x).
\]

Let
\[F^{(h,1)}_{x,q}(t, x) = \sum_{n=0}^{\infty} E^{(h,1)}_{n,x,q}(x) \frac{t^n}{n!}.
\]

Then we have
\[(22) \quad F^{(h,1)}_{x,q}(t, x) = 2 \sum_{n=0}^{\infty} \chi(n) q^{(h-1)n} (-1)^n e^{[n+x]_q t}.
\]
By (22), we see that

\[ E_{n,h,1}^{(h,1)}(x) = 2 \sum_{m=0}^{\infty} \chi(m)q^{(h-1)m}(-1)^{m}[m+x]_q^n \]

\[ = \frac{2}{(1-q)^n} \sum_{a_1=0}^{d-1} \chi(a_1)(-1)^{a_1} \sum_{l=0}^{d-1} \frac{\binom{n}{l}}{q^l(l+x+a_1)} \]

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