Exact Density matrix of oscillator-bath system: Novel derivation

Fardin Kheirandish

\textsuperscript{1}Department of Physics, Faculty of Science, University of Kurdistan, P.O.Box 66177-15175, Sanandaj, Iran

Abstract

Starting from a total Lagrangian describing an oscillator-bath system, a novel derivation of exact quantum propagator is presented. Having the quantum propagator, the exact density matrix, reduced density matrix of the main oscillator and thermal equilibrium fixed point are obtained. The problem is generalised to the cases where the main oscillator is under the influence of a classical external force. By introducing generalised auxiliary classical fields, the generalised quantum propagator or generating functional of position correlation functions is obtained.

PACS numbers: 03.65.Yz; 05.40.Jc; 03.65.Ca
I. INTRODUCTION

The quantum propagator is the most important function in quantum theories [1, 2]. Knowing the quantum propagator, we can obtain all measurable quantities related to the physical system exactly, that is we have a complete physical description of the underline system in any time. Unfortunately, except for some simple physical systems, obtaining the exact form of quantum propagator for is usually a difficult task and we have to invoke perturbative methods. Among different approaches to find quantum propagator we can refer to three main approaches. In the first method, quantum propagator is written as a bilinear function in eigenvectors of the Schrödinger equation. The main task of this method is to find the eigenvectors of the Hamiltonian which are usually difficult to find and even having these eigenvectors, extracting a closed form quantum propagator from their bilinear expansion may be cumbersome. The second approach is based on Feynman path integral technique [3, 4]. One of the most efficient features of this method is its perturbative technique known as Feynman diagrams which extends the applicability of method to the era of non-quadratic Lagrangians. The path integral technique has been applied to oscillator-bath system in [5–11]. The third approach which we will describe here in detail is based on the position and momentum operators in Heisenberg picture. In this scheme, using elementary quantum mechanical relations two independent partial differential equations are found that quantum propagator satisfy in. The solution of these partial differential equations is easily found and unknown functions are determined from basic properties of quantum propagators. The first message of the present paper is that this method compared to other methods to derive quantum propagator of oscillator-bath systems with linear interaction or generally quadratic Lagrangians, is easier to apply and in particular, comparing with path integral technique, there is no need to introduce more advanced mathematical notions like infinite integrations, operator determinant and Weyl ordering. The second message is that since we will find a closed form for the total quantum propagator, we will find a closed form density matrix describing the combined oscillator-bath system. Also, by tracing out the bath degrees of freedom we find a reduced density matrix describing the main oscillator in any time. Then we generalise the oscillator-bath model by including external classical sources in Hamiltonian, and find the modified quantum propagator under the influence of classical forces. The modified quantum propagator can be interpreted also as a generating functional from which time-ordered correlation functions among different position operators can be determined [12]. Here we have not introduced basic functions describing a heat bath such as susceptibility function or spectral density matrix as is usual in describing an oscillator-bath system. All these data are encoded in a basic time-dependent matrix denoted by $F$.

II. LAGRANGIAN

In this section, we set the stage for what will be investigated in the following sections. We start with a total Lagrangian describing an interacting oscillator-bath system. Then from the corresponding Hamiltonian and Heisenberg equations of motion, we find explicit expressions for position and momentum operators as the main ingredients of an approach that will be applied in the next section. The Lagrangian describing a main oscillator interacting linearly with a bath of oscillators is given by [13]

$$L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega_0^2 x^2 + \sum_{i=1}^{N} \frac{1}{2} \left( \dot{X}_i^2 - \omega_i^2 X_i^2 \right) + \sum_{i=1}^{N} g_i X_i x,$$

Eq. (1) can be rewritten in a more compact form as

$$L = \frac{1}{2} \sum_{\mu=0}^{N} (\dot{\mathbf{Y}}_\mu^2 - \omega_\mu^2 \mathbf{Y}_\mu^2) + \frac{1}{2} \sum_{\mu,\nu=0}^{N} Y_\mu \Omega_{\mu\nu} Y_\nu,$$
where the matrix $\Omega^2_{\mu\nu}$ is given by

$$\Omega^2_{\mu\nu} = \begin{pmatrix}
0 & g_1 & g_2 & \cdots & g_N \\
g_1 & 0 & 0 & \cdots & 0 \\
g_2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
g_N & 0 & 0 & \cdots & 0 \\
\end{pmatrix},$$

and

$$Y_0 = x, \quad Y_k = X_k, \quad k = 1, \cdots, N.$$  (4)

The corresponding Hamiltonian is

$$H = \frac{1}{2} \sum_{\mu=0}^{N} (\dot{P}_\mu^2 + \omega^2_\mu Y^2_\mu) - \frac{1}{2} \sum_{\mu,\nu=0}^{N} Y_\mu \Omega^2_{\mu\nu} Y_\nu,$$

where $P_\mu = \dot{Y}_\mu$ is the canonical conjugate momentum corresponding to the canonical position $Y_\mu$. The system is quantized by imposing the equal-time commutation relations

$$[\hat{Y}_\mu, \hat{P}_\nu] = i\hbar \delta_{\mu\nu},$$
$$[\hat{Y}_\mu, \hat{Y}_\nu] = [\hat{P}_\mu, \hat{P}_\nu] = 0,$$

and from Heisenberg equations of motion one finds

$$\ddot{\hat{Y}}_\mu + \omega^2_\mu \hat{Y}_\mu = \sum_{\nu} \Omega^2_{\mu\nu} \hat{Y}_\nu.$$  (7)

Note that $(\hat{Y}_0, \hat{P}_0)$ refer to the position and momentum of the main oscillator and $(\hat{Y}_k, \hat{P}_k)$, $(k = 1, \cdots, N)$ refer to position and momentum operators of bath oscillators. Taking the Laplace transform from both sides of Eq. (7) we find

$$\sum_{\nu} \Lambda_{\mu\nu}(s) \hat{Y}_\nu(s) = s\hat{Y}_\mu(0) + \hat{P}_\mu(0),$$

where the $N+1$-dimensional matrix $\Lambda$ is defined by

$$\Lambda_{\mu\nu}(s) = [(s^2 + \omega^2_\mu)\delta_{\mu\nu} - \Omega^2_{\mu\nu}].$$

Therefore, applying the inverse matrix, we find

$$\hat{Y}_\mu(s) = \sum_{\nu}[s\Lambda^{-1}_{\mu\nu}(s)\hat{Y}_\nu(0) + \Lambda^{-1}_{\mu\nu}(s)\hat{P}_\nu(0)],$$

and a formal solution is obtained by inverse Laplace transform as

$$\hat{Y}_\mu(t) = \hat{F}_{\mu\nu}(t)\hat{Y}_\nu(0) + F_{\mu\nu}(t)\hat{P}_\nu(0),$$

where we defined

$$F_{\mu\nu}(t) = \mathcal{L}^{-1}[\Lambda^{-1}(s)]_{\mu\nu}.$$  (12)

The matrix $\Lambda$ is explicitly given by

$$\Lambda(s) = \begin{pmatrix}
 s^2 + \omega^2_0 & -g_1 & -g_2 & \cdots & -g_N \\
-g_1 & s^2 + \omega^2_1 & 0 & \cdots & 0 \\
-g_2 & 0 & s^2 + \omega^2_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-g_N & 0 & 0 & \cdots & s^2 + \omega^2_N \\
\end{pmatrix},$$

(13)
which can be rewritten as
\[ \Lambda(s) = s^2 I + B, \]  
wherein
\[
B = \begin{pmatrix}
\omega_0^2 & -g_1 & -g_2 & \cdots & -g_N \\
-g_1 & \omega_1^2 & 0 & \cdots & 0 \\
-g_2 & 0 & \omega_2^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-g_N & 0 & 0 & \cdots & \omega_N^2
\end{pmatrix}.
\]  
The inverse matrix can be formally written as
\[
\Lambda^{-1}(s) = \frac{1}{s^2 B} = \frac{1}{s^2} \left( I - \frac{1}{s^2} B \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{2n+2}} B^n, \quad (B^0 = I).
\]  
Therefore, from Eq. (12) we have
\[
F_{\mu\nu}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} (B^n)_{\mu\nu},
\]
\[
\dot{F}_{\mu\nu}(t) = \left( \frac{dF}{dt} \right)_{\mu\nu} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} (B^n)_{\mu\nu}.
\]  
The equations Eqs. (17) can be formally written as
\[
F(t) = \frac{1}{\sqrt{B}} \sin(\sqrt{B} t),
\]
\[
\dot{F}(t) = \cos(\sqrt{B} t).
\]  

\section{III. QUANTUM PROPAGATOR}

In this section a novel scheme to derive the quantum propagator of the combined oscillator-bath system is introduced in detail. Let \( |y_0\rangle \) be an eigenket of \( \hat{Y}_0 \) and \( |y_k\rangle \) an eigenket of \( \hat{Y}_k \), then in Heisenberg picture, we can write
\[
\hat{Y}_\mu(t) |y, t\rangle = y_\mu |y, t\rangle,
\]  
where for notational simplicity the tensor product is abbreviated as
\[
|y, t\rangle = |y_0, t\rangle \otimes |y_1, t\rangle \otimes \cdots \otimes |y_N, t\rangle = |y_0, \cdots, y_N, t\rangle.
\]  
Multiplying Eq. (19) from the left by \( \langle y'| \) and using Eq. (11), we find
\[
\sum_{\nu} \left( \hat{F}_{\mu\nu}(t) y'_\nu - i\hbar F_{\mu\nu}(t) \frac{\partial}{\partial y'_\nu} \right) \mathcal{X}(y'|y, t) = y_\mu \mathcal{X}(y'|y, t),
\]  
where we have defined the function \( \mathcal{X} \) as
\[
\langle y'|y, t\rangle = \mathcal{X}(y'|y, t).
\]
and made use of the identities
\[<y'|\hat{H}_\mu(0) = y'_\mu|y'|,\]
\[<y'|\hat{P}_\mu(0) = -i\hbar \frac{\partial}{\partial y_\mu}(y').\]  
(23)

Eq. (21) can be rewritten as
\[\sum_\nu F_{\mu\nu}(t) \frac{\partial}{\partial y_\nu} \ln J(y'|y,t) = \frac{i}{\hbar} \left( y_\mu - \sum_\nu F_{\mu\nu}(t) y_\nu \right).\]  
(24)

The right hand side of Eq. (24) is linear in \(y'_\mu\), so the following quadratic form can be assumed for \(\ln J\)
\[\ln J(y'|y,t) = A(y,t) + \sum_\mu A_\mu(y,t) y'_\mu + \frac{1}{2} \sum_{\mu,\nu} y'_\mu C_{\mu\nu}(y,t) y'_\nu,\]  
(25)

where \(C_{\mu\nu} = C_{\nu\mu}\). By inserting Eq. (25) into Eq. (24), we easily find
\[A_\mu(y,t) = \frac{i}{\hbar} \sum_\nu F_{\mu\nu}^{-1}(t) y_\nu,\]
\[C_{\mu\nu}(t) = -\frac{i}{\hbar} \sum_\sigma F_{\mu\sigma}^{-1}(t) \hat{F}_{\sigma\nu}(t),\]  
(26)

therefore, in dyadic notation, we can write
\[J(y'|y,t) = e^{A(y,t)} e^{\frac{i}{\hbar} y \cdot \hat{F}^{-1}(t) y} e^{-\frac{i}{\hbar} y' \cdot \hat{F}^{-1}(t) \hat{F}(t) y'}.\]  
(27)

The form of \(A(y,t)\) can be determined from the properties of propagators. Since the Hamiltonian Eq. (5) is time-independent, we can write
\[J(y'|y,t) = <y'|y,t) = <y'|e^{\frac{i}{\hbar} \hat{H}}|y).\]  
(28)

Eq. (28), is invariant under successive transformations (i) complex conjugation (ii) \(y \leftrightarrow y'\) (iii) \(t \rightarrow -t\), therefore,
\[J(y'|y,t) = J^*(y|y',-t),\]  
(29)

leading to
\[e^{A(y,t)} = e^{\varphi(t)} e^{-\frac{i}{\hbar} y \cdot \hat{F}^{-1}(t) \hat{F}(t) y},\]
\[\varphi^*(-t) = \varphi(t).\]  
(30)

Note that in Sec.VI, the Hamiltonian will be time-dependent and to find \(A(y,t)\) we can not use these transformations and we will follow another approach. Up to now the form of the propagator is as follows
\[J(y'|y,t) = e^{\varphi(t)} e^{-\frac{i}{\hbar} y \cdot \hat{F}^{-1}(t) \hat{F}(t) y} e^{-\frac{i}{\hbar} y' \cdot \hat{F}^{-1}(t) \hat{F}(t) y'},\]
\[= e^{\varphi(t)} e^{-\frac{i}{\hbar} (y \cdot \hat{F}^{-1}\hat{F} \cdot y + y' \cdot \hat{F}^{-1}\hat{F} \cdot y' - 2y \cdot \hat{F}^{-1}\hat{F} \cdot y')}\]  
(31)

From Eqs. (18) we find the following asymptotic behaviours of Matrices \(\hat{F}, \hat{F}^{-1},\) and \(\hat{F}\)
\[
\lim_{t \to 0} \hat{F}(t) \approx t \mathbb{I},
\]
\[
\lim_{t \to 0} \hat{F}^{-1}(t) \approx \frac{1}{t} \mathbb{I},
\]
\[
\lim_{t \to 0} \hat{F}(t) \approx \mathbb{I},
\]  
(32)

By inserting these asymptotic behaviours into Eq. (31) we find
\[\lim_{t \to 0} J(y'|y,t) = \delta(y' - y) = \lim_{t \to 0} e^{\varphi(t)} e^{-\frac{i}{\hbar} (y' - y)^2},\]  
(33)
The function \( K \) to find \( \lambda \) where \( \theta \) is a real function that will be determined from a limiting case where the coupling constants are turned off \((g_1 = \cdots = g_N = 0)\) and also the fact that the propagator should satisfy the Schrödinger equation.

It should be noted that according to the definition Eq. (22), the Feynman propagator has the following relation to \( \mathcal{K} \) Eq. (32).

By inserting Eq. (38) and its complex conjugation into Eq. (39) and doing the integral we will find

\[
e^\theta = \frac{i^{N+1}}{\sqrt{|\det F(t)|}} e^{i\theta},
\]

where \( \theta \) is a real function that will be determined from a limiting case where the coupling constants are turned off \((g_1 = \cdots = g_N = 0)\) and also the fact that the propagator should satisfy the Schrödinger equation.

The function \( \mathcal{K} \) now has the form

\[
\mathcal{K}(\mathbf{y}', \mathbf{y}, t) = e^{\lambda(t)} \left( \frac{i}{2\pi\hbar t} \right)^{\frac{N+1}{2}} e^{-\frac{i}{\hbar} \left[ \mathbf{y} \cdot \mathbf{F}^{-1} \mathbf{F} \mathbf{y} + \mathbf{y}' \cdot \mathbf{F}^{-1} \mathbf{F} \mathbf{y}' - 2\mathbf{y} \cdot \mathbf{F}^{-1} \mathbf{F} \mathbf{y}' \right]}. \tag{38}
\]

To find \( \lambda(t) \) we make use of the following identity

\[
\delta(\mathbf{y}' - \mathbf{y}) = \int d\mathbf{y}'' \mathcal{K}(\mathbf{y}'|\mathbf{y}'', t) \mathcal{K}^*(\mathbf{y}|\mathbf{y}'', t), \tag{39}
\]

which can be easily checked using the definition of \( \mathcal{K} \), Eq. (32). By inserting Eq. (38) and its complex conjugation into Eq. (39) and doing the integral we will find

\[
e^{\lambda(t)} = \frac{i^{N+1}}{\sqrt{|\det F(t)|}} e^{i\theta}, \tag{40}
\]

After spatial differentiations we set \( y = 0 \) and by comparing both sides of Eq. (33) we find that \( \theta \) is a constant. To find the constant \( \theta \), we turn off the coupling constants, \((g_1 = g_2 = \cdots = g_N = 0)\), and from consistency condition we should recover the quantum propagator of \( N \) noninteracting oscillators. When the coupling constants are turned off, we have

\[
\mathbf{F}^{-1}(t) = \text{diag} \left( \frac{\omega_0}{\sin(\omega_0 t)}, \frac{\omega_1}{\sin(\omega_1 t)}, \cdots, \frac{\omega_N}{\sin(\omega_N t)} \right), \tag{41}
\]

\[
\mathbf{F}(t) = \text{diag} \left( \cos(\omega_0 t), \cos(\omega_1 t), \cdots, \cos(\omega_N t) \right), \tag{42}
\]

\[
\left( \mathbf{F}^{-1} \right)_{\mu \nu} = \sum_{\mu', \nu'} F_{\mu \mu', \nu \nu'} \left[ \begin{array}{c} F^-_{\mu \nu} - \frac{\omega_{\mu'}}{\sin(\omega_{\mu'} t)} y_{\mu'} y_{\nu'} 
\end{array} \right]. \tag{43}
\]
Inserting Eqs. (14) into Eq. (12) we find
\[ K(y, t; y', 0) = e^{-i\theta} \prod_{\mu=0}^{N} \frac{\omega_{\mu}}{2\pi i} \sin(\omega_{\mu} t) e^{\frac{i\omega_{\mu}}{2\pi} \int_{0}^{t} \left[ (y_{\mu}^{2} + y_{\mu}^{2}') \cos(\omega_{\mu} t) - 2y_{\mu} y_{\mu}' \right] dt}, \]
which is the propagator of \( N \) noninteracting oscillators if we set \( \theta = 0 \). Finally, we find the quantum propagator of oscillator-bath system as
\[ K(y, t; y', 0) = \frac{1}{\sqrt{\det P(t)}} \left( \frac{1}{2\pi i} \right)^{\frac{N+1}{2}} e^{\frac{i}{2\pi} \int_{0}^{t} \left[ \sum_{\mu=0}^{N} \omega_{\mu} y_{\mu}^{2} + y_{\mu} y_{\mu}' \right] dt}. \]

### IV. DENSITY MATRIX

In this section we will find the density matrix for the oscillator-bath system using the explicit form of the quantum propagator Eq. (46) of the combined system. If we denote the evolution operator by \( \hat{U}(t) \) then the density matrix at time \( t \) can be obtained from the initial density matrix at \( t = 0 \) as
\[ \hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^{\dagger}(t), \]
in position representation we have
\[ \rho(y, y'; t) = \langle y | \rho(0) | y' \rangle = \int dy_{1} dy_{2} \langle y | \hat{U}(t) | y_{1} \rangle \langle y_{1} | \rho(0) | y_{2} \rangle \langle y_{2} | \hat{U}^{\dagger}(t) | y' \rangle, \]
\[ = \int dy_{1} dy_{2} K(y, t; y_{1}, 0) \rho(y_{1}, y_{2}; 0) K^{*}(y', t; y_{2}, 0). \]

We can assume an arbitrary initial state for oscillator-bath system, but for simplicity we assume that the initial state is a product state as
\[ \rho(y_{1}, y_{2}; 0) = \rho_{s}(y_{10}, y_{20}; 0) \otimes \rho_{B}(\vec{y}_{1}, \vec{y}_{2}; 0), \]
where \( y_{1} = (y_{10}, \vec{y}_{1}) \) and \( y_{2} = (y_{20}, \vec{y}_{2}) \). To find the reduced density matrix of the main oscillator, we should take trace over the degrees of freedom of the bath oscillators. After some straightforward calculations we find
\[ \rho_{s}(y_{0}, y_{0}'; t) = \frac{1}{2\pi \det F \det D} \int dy_{01} dy_{02} e^{i\frac{\omega_{01}^{2} - \omega_{02}^{2}}{2\pi} t} e^{-\frac{i}{2\pi} \int_{0}^{t} \sum_{k=1}^{N} \left( \sum_{l=1}^{N} \left[ q_{k}^{l} A_{kl} q_{l} + 2 \sum_{k=1}^{N} \left[ y_{01} B_{k} - y_{02} C_{k} \right] q_{k}^{l} \right] \right) \hat{\rho}_{B}(\vec{p}, \vec{q} + \vec{q}_{0}), \]
where the Fourier transform \( \hat{\rho}_{B} \) and the vectors \( \vec{q} = (q_{1}, \cdots, q_{N}), \vec{p} = (p_{1}, \cdots, p_{N}) \) are defined by \( (k, l = 1 \cdots N) \)
\[ \hat{\rho}_{B}(\vec{p}, \vec{q} + \vec{q}_{0}) = \int d\vec{y}_{1} e^{i\vec{q}_{0} \cdot \vec{y}_{1}} \rho_{B}(\vec{y}_{1}, \vec{y}_{1} + \vec{q}_{0}), \]
\[ q_{k} = (y_{0}^{0} - y_{0}^{0})(\delta_{k0} - bF_{k0}) - (y^{0} - y^{0})(\delta_{k0} - aF_{k0}), \]
\[ p_{k} = (y_{0}^{0} - y_{0}^{0})B_{k} + (y^{0} - y^{0})C_{k} - \sum_{l=1}^{N} A_{kl} q_{l}. \]
The time dependent functions \( (a, b) \), vectors \( (C_{k}, B_{k}) \) and matrices \( (A_{kl}, D_{kl}) \) are defined by
\[ a(t) = (F^{-1} \dot{F})_{00}, \]
\[ b(t) = (F^{-1})_{00}, \]
\[ C_{k}(t) = (F^{-1})_{k0} = (F^{-1})_{0k}, \]
\[ B_{k}(t) = (F^{-1})_{0k} = (F^{-1})_{k0}, \]
\[ A_{kl} = (A)_{kl} = (F^{-1})_{kl}, \]
\[ D_{kl} = (D)_{kl} = (F^{-1})_{kl}. \]
which can be rewritten more compactly in matrix form as

\[ F^{-1} \dot{F} = \begin{pmatrix} a & B^T \\ B & A \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} b & C^T \\ C & D \end{pmatrix}, \tag{53} \]

\[ B^T = [B_1, B_2, \cdots, B_N], \quad C^T = [C_1, C_2, \cdots, C_N]. \tag{54} \]

V. THERMAL EQUILIBRIUM: FIXED POINT

In the equilibrium state, the density matrix of oscillator-bath system can be obtained from the quantum propagator using the correspondence

\[ \rho(y_0, \vec{y}; y'_0, \vec{y}', \beta) = \frac{1}{Z(\beta)} K(y_0, \vec{y}, -i\hbar \beta; y'_0, \vec{y}', 0), \tag{55} \]

where \( \beta = 1/\kappa_B T \) is the inverse of temperature and \( \kappa_B \) is Boltzmann constant. The function \( Z(\beta) \) is the total partition function

\[ Z(\beta) = \int dy_0 d\vec{y} K(y_0, \vec{y}, -i\hbar \beta; y'_0, \vec{y}', 0), \]

\[ = \frac{1}{2^{N+1}} \sqrt{\det(\dot{F} - I)} \bigg|_{t=-i\hbar \beta}, \tag{56} \]

and \( I \) is a \( N \)-dimensional unit matrix.

The reduced density matrix of the oscillator is obtained by integrating out the bath degrees of freedom as

\[ \rho_{\text{red}}(y_0, y'_0; \beta) = \int d\vec{y} K(y_0, \vec{y}, -i\hbar \beta; y'_0, \vec{y}, 0), \]

\[ = \sqrt{\frac{\det(\dot{F} - I)}{i\pi \hbar \det F \det(A - D)}} e^{\frac{i}{\hbar} \left( y_0^2 + y'_0^2 (a - \frac{\eta}{2}) - 2y_0y'_0 (b + \frac{\eta}{2}) \right)}, \tag{57} \]

where

\[ \eta = \sum_{k,l=1}^N (B_k - C_k)(A - D)^{-1} (B_l - C_l) \bigg|_{t=-i\hbar \beta}. \tag{58} \]

From Eq. (53) we have

\[ F^{-1} (\dot{F} - I) = \begin{pmatrix} a - b & B^T - C^T \\ B & C \end{pmatrix}, \tag{59} \]

by making use of the identity \[14]\n
\[ \det[F^{-1}(F - I)] = \det(A - D) \det[a - b - (B^T - C^T)(A - D)^{-1}(B - C)]_{\eta}, \]

\[ = \det(A - D) (a - b - \eta), \tag{60} \]

Eq. (57) can be rewritten as

\[ \rho_{\text{red}}(y_0, y'_0; \beta) = \sqrt{\frac{a - b - \eta}{i\hbar \pi}} e^{\frac{i}{\hbar} \left( y_0^2 + y'_0^2 (a - \frac{\eta}{2}) - 2y_0y'_0 (b + \frac{\eta}{2}) \right)}, \tag{61} \]

From Eq. (61) we find the thermal mean square of position and momentum as

\[ \langle y_0^2 \rangle = \frac{i\hbar}{2(a - b - \eta)} \bigg|_{t=-i\hbar \beta}, \]

\[ \langle p_0^2 \rangle = \frac{-i\hbar}{2} \left( \frac{a + b}{2} \right) \bigg|_{t=-i\hbar \beta}, \tag{62} \]
therefore,
\[
\rho_{\text{red}}(y_0, y'_0; \beta) = \frac{1}{\sqrt{2\pi(y'_0)^2}} e^{-\frac{1}{2\pi}(y_0 - y'_0)^2 - \frac{1}{\hbar^2}(y_0 + y'_0)^2},
\]
(63)
for another derivation, see [13].

VI. MAIN OSCILLATOR INTERACTS WITH AN EXTERNAL FIELD

Now assume that the main oscillator is under the influence of an external classical field \(f(t)\). In this case the total Lagrangian is written as
\[
L = \frac{1}{2} \sum_{\mu=0}^{N} (\dot{Y}_\mu^2 - \omega_\mu^2 Y_\mu^2) + \frac{1}{2} \sum_{\mu,\nu=0}^{N} Y_\mu \Omega_{\mu\nu}^2 Y_\nu - f(t)Y_0,
\]
(64)
and the corresponding Hamiltonian is
\[
H = \frac{1}{2} \sum_{\mu=0}^{N} (P_\mu^2 + \omega_\mu^2 Y_\mu^2) - \frac{1}{2} \sum_{\mu,\nu=0}^{N} Y_\mu \Omega_{\mu\nu}^2 Y_\nu + f(t)Y_0.
\]
(65)
Note that the Hamiltonian is now time-dependent and we cannot use Eqs. (28,29) in this case. We can find another partial differential equation satisfied by \(K(y'\mid y, t)\) as follows. From Heisenberg equations of motion we find
\[
\ddot{Y}_\mu + \omega_\mu^2 Y_\mu - \sum_\nu \Omega_{\mu\nu}^2 Y_\nu = -f(t) \delta_{\mu0}.
\]
(66)
The Green tensor corresponding to Eq. (66) is defined by
\[
\sum_\nu \left( [\partial^2_t + \omega_\mu^2] \delta_{\mu\nu} - \Omega_{\mu\nu}^2 \right) G_{\nu\alpha}(t - t') = \delta_{\mu\alpha} \delta(t - t').
\]
(67)
By making use of Laplace transform and definitions Eqs. (9,12), we find the retarded Green tensor as
\[
G_{\mu\nu}(t - t') = F_{\mu\nu}(t - t'),
\]
(68)
and the position and momentum operators are respectively given by
\[
\hat{Y}_\mu(t) = \sum_\nu \left[ \hat{F}_{\mu\nu}(t)\hat{Y}_\nu(0) + F_{\mu\nu}(t)\hat{P}_\nu(0) \right] - R_\mu(t),
\]
\[
\hat{P}_\mu(t) = \hat{\dot{Y}}_\mu = \sum_\nu \left[ \hat{F}_{\mu\nu}(t)\hat{Y}_\nu(0) + \hat{F}_{\mu\nu}(t)\hat{P}_\nu(0) \right] - \dot{R}_\mu(t),
\]
(69)
where we defined
\[
R_\mu(t) = \int_{0}^{t} dt' F_{\mu0}(t - t') f(t').
\]
(70)
We can rewrite the identity
\[
\hat{P}_\mu(t) = \hat{U}^\dagger(t)\hat{P}_\mu(0)\hat{U}(t),
\]
(71)
as
\[
\hat{P}_\mu(t)\hat{U}^\dagger(t) = \hat{U}^\dagger(t)\hat{P}_\mu(0),
\]
(72)
then
\[
\langle y'\mid \hat{P}_\mu(t)\hat{U}^\dagger(t)\mid y \rangle = \langle y'\mid \hat{U}^\dagger(t)\hat{P}_\mu(0)\mid y \rangle.
\]
(73)
By inserting the momentum operator from the second line of Eqs. (69) into Eq. (73), we easily find

$$
\sum_{\nu} \left( \hat{F}_{\mu\nu}(t) y_{\nu} - i\hbar \frac{\partial}{\partial y_{\nu}} - \dot{y}_{\mu} \right) \mathcal{K}(y', y, t) = y_{\mu} \mathcal{K}(y', y, t).
$$

(74)

By making use of Eqs. (21,33,39,74), and following the same process as we did in Sec.III, we will find

$$
K^{(f)}(y, t; y', 0) = \frac{e^{-i\zeta(t)}}{\sqrt{\det F(t)}} \left( \frac{1}{2\pi\hbar} \right)^{\frac{N+1}{2}} e^{\frac{i\pi}{4} \left[ y F^{-1} T y + y' F^{-1} T y' - 2y' F^{-1} y \right]} \\
\times e^{\frac{-i\pi}{4} \left[ y F^{-1} T R + y' F^{-1} T R \right]},
$$

(75)

where we have defined $R$ as

$$
\hat{R}_{\mu}(t) = \int_0^t dt' F_{\mu\nu}(t') f(t'),
$$

(76)

and the function $\zeta(t)$ can be determined from the Schrödinger equation

$$
i\hbar \frac{\partial K^{(f)}(y, t; 0, 0)}{\partial t} \bigg|_{y = 0} = \left[ \frac{1}{2} \sum_{\mu=0}^N \left( -\hbar^2 \frac{\partial^2}{\partial y_{\mu}^2} + \omega_{\mu}^2 y_{\mu}^2 \right) - \frac{1}{2} \sum_{\mu,\nu=0}^N y_{\mu} \Omega^2_{\mu\nu} y_{\nu} + f(t) y_{\mu} \right] K^{(f)}(y, t; 0, 0) \bigg|_{y = 0},
$$

(77)

as

$$
\zeta(t) = \frac{1}{2\hbar} \int_0^t ds \hat{R}(s) \cdot F^{-2}(s) \cdot \hat{R}(s),
$$

$$
= \frac{1}{\hbar} \int_0^t ds \int_0^s du f(s) \left[ \frac{\sin(\sqrt{B}t) \sin(\sqrt{B}(t - s))}{\sqrt{B} \sin(\sqrt{B}t)} \right]_{00}.
$$

(78)

Finally, the quantum propagator for oscillator-bath system under the influence of an external classical force on the main oscillator, is obtained as

$$
K^{(f)}(y, t; y', 0) = \frac{1}{\sqrt{\det F(t)}} \left( \frac{1}{2\pi\hbar} \right)^{\frac{N+1}{2}} e^{\frac{i\pi}{4} \left[ y F^{-1} T y + y' F^{-1} T y' - 2y' F^{-1} y \right]} \\
\times e^{\frac{-i\pi}{4} \left[ y F^{-1} T R + y' F^{-1} T R \right]} e^{\frac{-i\pi}{4} \int_0^t ds \hat{R}(s) \cdot F^{-2}(s) \hat{R}(s)}.
$$

(79)

**A. A generalization: generating function**

We can generalize the Lagrangian Eq. (64) as

$$
L = \frac{1}{2} \sum_{\mu=0}^N \left( \dot{Y}_{\mu}^2 - \omega_{\mu}^2 Y_{\mu}^2 \right) + \frac{1}{2} \sum_{\mu,\nu=0}^N Y_{\mu} \Omega_{\mu\nu}^2 Y_{\nu} - \sum_{\mu=0}^N f_{\mu}(t) Y_{\mu},
$$

(80)

in this case the quantum propagator is given by Eq. (70) but now the definitions Eqs. (70,76) have to be replaced by the new definitions

$$
R_{\mu}(t) = \int_0^t dt' F_{\mu\nu}(t - t') f_{\nu}(t'),
$$

$$
\hat{R}_{\mu}(t) = \int_0^t dt' F_{\mu\nu}(t') f_{\nu}(t').
$$

(81)

The path integral representation of quantum propagator Eq. (72) is

$$
K^{(f)}(y, t; y', 0) = \int d[x] e^{\frac{i}{\hbar} \int_0^t dt' L},
$$

(82)
where $L$ is the Lagrangian Eq. (80). Having the closed form expression Eq. (79), we can find ordered correlation functions among position operators of the oscillator-bath system. In this case, the external source $f_\mu(t)$ is an auxiliary force that should be set zero at the end of functional derivatives [12], we have

$$
\langle y, t | \hat{T} \hat{Y}_\mu_1(t_1) \hat{Y}_\mu_2(t_2) \cdots \hat{Y}_\mu_N(t_N) | y', 0 \rangle = \left. \int D[x] y_\mu_1(t_1) y_\mu_2(t_2) \cdots y_\mu_N(t_N) e^{i \int_0^T dt' L} \frac{(i \hbar)^N}{K^{(0)}(y, t; y', 0) \delta f_\mu_1(t_1) \cdots \delta f_\mu_N(t_N)} K^{(f)}(y, t; y', 0) \right|_{f=0}, \quad (83)
$$

where $\hat{T}$ is a time ordering operator acting on bosonic operators as

$$
\hat{T}(\hat{A}(t) \hat{B}(t')) = \begin{cases} 
\hat{A}(t) \hat{B}(t'), & t > t'; \\
\hat{B}(t') \hat{A}(t), & t' > t.
\end{cases} \quad (84)
$$

VII. CONCLUSION

Using elementary quantum mechanical calculations and basic properties of quantum propagators, a novel derivation of exact quantum propagator for the oscillator-bath system was introduced. The method compared to other methods to derive quantum propagator of an oscillator-bath system with linear interaction or generally quadratic Lagrangians, was easier to apply and in particular, compared to path integral approach, there was no need to introduce more advanced mathematical notions like infinite integrations, operator determinant and Weyl ordering. From quantum propagator, a closed form density matrix describing the combined oscillator-bath system was obtained from which reduced density matrix could be derived. The problem was generalised to the case where the main oscillator was under the influence of an external classical source. By introducing auxiliary classical fields the modified quantum propagator or generating functional of position correlation functions was found. The main unknown matrix in this approach was the inverse of the matrix $F$ describing the physical properties of the bath relevant to the main oscillator. These physical properties are susceptibility function or spectral density function in other approaches. Therefore, from numerical or simulation point of view, the only challenge was to find the inverse of the matrix $F$. The efficiency of the method in determining the exact form of the quantum propagator for quadratic Lagrangians, inspires the idea of developing a perturbative approach to include non-quadratic Lagrangians too.

[1] W. H. Dickhoff and D. V. Neck, *Many-Body Theory Exposed! Propagator description of quantum mechanics in many-body systems* (World Scientific, Singapore, 2005).
[2] J. Linderberg and Y. Öhrn, *Propagators in quantum chemistry* (John Wiley, New Jersey, 2004).
[3] R. P. Feynman, A. R. Hibbs, D. F. Styer, *Quantum Mechanics and Path Integrals*, Emended Ed. (McGrawHill, New York, 2005).
[4] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009).
[5] R. P. Feynman and F. L. Vernon, Ann. Phys. (NY) 24, 118-173 (1963).
[6] A. O. Caldeira and A. J. Leggett, Physica A121, 587 (1983).
[7] V. Hakim and V. Ambegaokar, Phys. Rev. A32, 423 (1985).
[8] C. Morais Smith and A. O. Caldeira, Phys. Rev. A 36, 3509 (1987).
[9] H. Grabert, P. Schramm, and G.-L. Ingold, Phys. Rep. 168, 115 (1988).
[10] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D45, 2843 (1992).
[11] B. L. Hu and A Matacz, Phys. Rev. D49, 6612 (1994).
[12] W. Greiner and J. Reinhardt, *Field Quantization* (Springer-Verlag, Berlin, 1996).
[13] U. Weiss, *Dissipative Quantum Systems* (World Scientific, Singapore, 1993).
[14] F. Zhang, *Matrix Theory: Basic Results and Techniques* (Springer-Verlag, Berlin 2010).