MULTILATERATION AND SIGNAL MATCHING WITH UNKNOWN EMISSION TIMES

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Abstract. Assume that a source emits a signal in 3-dimensional space at an unknown time, which is received by at least 5 sensors. In almost all cases the emission time and source position can be worked out uniquely from the knowledge of the times when the sensors receive the signal. The task to do so is the multilateration problem. But when there are several emission events originating from several sources, the received signals must first be matched in order to find the emission times and source positions. In this paper, we propose to use algebraic relations between reception times to achieve this matching. A special case occurs when the signals are actually echoes from a single emission event. In this case, solving the signal matching problem allows one to reconstruct the positions of the reflecting walls. We show that, no matter where the walls are situated, our matching algorithm works correctly for almost all positions of the sensors.

In the first section of this paper we consider the multilateration problem, which is equivalent to the GPS-problem, and give a simple algebraic solution that applies in all dimensions.

INTRODUCTION

Positioning is a ubiquitous problem in engineering. For example, one may want to determine the location of an object such as a vehicle, locate an event such as an earthquake, calibrate an array of devices such as microphones, or draw the map of an environment such as a building. In many scenarios, the objects to be located can emit a signal. In such case, one can use an array of receivers with known geometry to determine the objects location with respect to the position of the receiving array. Alternatively, the objects may be equipped with a receiver so to be located with an array of sources with known geometry. While both problems are dual to each other, their difficulty and conditioning can vary significantly depending on the specific setup scenario and constraints imposed. In some cases, the underlying mathematical problem may actually be ill-posed.

This paper is concerned with multilateration, which is the task of determining the position of one or more sources emitting a wave signal (e.g., electromagnetic, acoustic, or seismic waves). More specifically, we are trying to determine the position of sources sending out a signal from measurements of the times when this signal is received by various sensors situated at known positions. We assume that the clocks on the receivers are synchronized together, but not with the clock of the sources. In other words, the time of signal emission is unknown to the sensors, and thus the differences of arrivals (TDOAs) are the only meaningful available information. Therefore one also speaks of pseudo-range multilateration.

The literature on the well-posedness of multilateration problems is sparse. As far as we know, even the well-known GPS positioning problem, which we analyze in Section 1, has not been thoroughly studied. This corresponds to the problem of determining the position of one emission event (at an unknown time) received by several (synchronized) sensors at known locations. It turns out that, even in this simple scenario, the TDOAs may not uniquely determine the position of the source.

More generally, we consider the case of several sources emitting undistinguishable signals at unknown times (e.g., earthquakes [14] or gunshots). Clearly the order in which the signals...
arrive at the sensors can be wildly different from the order in which the signals were emitted. Therefore before feeding the reception times into a multilateration algorithm, a matching must be performed, identifying those received signals that come from the same emission event with each other. Ideally, this matching process should also discard spurious signals that are registered by just one sensor, i.e., not match such signals to any others. Note that a mistake in the matching process, or accidentally including a spurious signal in a match, will result in determining an emission event (time and position of emission) which never took place.

The problem of matching the sound events produced at a known time was previously studied in [7, 9, 8, 4]. In the case where the times of emission are unknown, the problem is known as “TDOA disambiguation.” Two sources of TDOA ambiguity are considered in the literature. The first one is the “multipath ambiguity” which is caused by the reverberation of the signal on objects in the environment and leads to spurious events. The triangle inequality, a zero cyclic sum condition, and characteristics of the cross-correlation and autocorrelation of the microphone signals are used to help disambiguate such cases in [11, 15]. The second one is the “multisource ambiguity” resulting from multiple sources emitting the same signal. A few of the false matches due to this can be ruled out using the triangle inequality. But, as far as we know, a more rigorous criterion for the case where the times of emission are unknown has not previously been proposed.

In this paper we show that if there are (at least) 5 sensors in 3-dimensional space, then the reception times (more precisely, TDOAs) of a signal coming from a single emission event satisfy a certain algebraic relation. We propose to use this relation to perform the signal matching. So if a selection of signal reception times, one for each sensor, satisfies this relation, then these reception times are accepted as coming from the same event (see Algorithm 2.3). Moreover, spurious signals registered by just one sensor will almost certainly not satisfy the relation and will therefore not be included in a match.

Of course the "almost" in the previous sentence is an important issue. In fact, there is no way to rule out the possibility that a spurious signal is registered at such an unlucky time that our matching algorithm, or any other algorithm based on the available information, falsely includes this signal in a match. This is true not only for spurious signals but also for signals coming from a real emission event that happened at an unlucky time. In particular, there is no way to position the sensors such that this possibility can be ruled out.

The situation becomes different, however, if the emission events are in fact just echoes from a single event. More precisely, consider an arrangement of flat surfaces ("walls") that reflect a sound signal emitted from an omnidirectional loudspeaker. Assuming the signal is of high frequency, we use the ray acoustics approximation. This means that the signals are virtually emitted from the point given by reflecting the loudspeaker position at the walls, and all (virtual) emissions happen simultaneously. Now we bring in 5 microphones at known positions. These record the echoes of the sound emission and feed them into the matching algorithm. After that, the wall positions can be determined by multilateration. Notice that in contrast to our previous paper [4], we do not assume that the loudspeaker and the microphones have synchronized clocks and communicate the times of signal emission time. So the common emission time is still unknown, and pseudo-range multilateration is required. In Theorem 3.1 in this paper, we show that in this situation almost all microphone positions are good, in the sense that no false matches can happen. As explained above, this is in contrast to the situation where the (virtual) emission events are not assumed to be simultaneous. The proof of the theorem uses methods from computational commutative algebra. This is something that it has in common with the proof of the main results from [4]. However, when we designed the proof of Theorem 3.1 we were surprised to find that the difficulties that arose were quite different from those in [4].

Even though the emphasis of this paper may lie on the matching problem and on multiple wall detection, we also study the pseudo-range multilateration problem itself. In fact, we present a simple algebraic solution algorithm. For simplicity, we formulate this in three dimensions, but it really works for all dimensions > 1. We give a self-contained and rigid proof for the validity. It is well-known that with just 4 sensors, the pseudo-range multilateration problem usually has 2 solutions. We show by examples that this may also happen if there are 5 sensors, even if no 4 of them are coplanar. This is not a shortcoming of our algorithm: in our example, the available
Theorem 1.1

Theorem 1.1

The paper is organized in three sections. The first section introduces the notation and discusses the pseudo-range multilateration problem, giving the solution algorithm in Theorem 1.1. In the second section we then turn our attention to the case of multiple emission events. We present and prove the relation that holds between reception times coming from the same event, and derive the matching algorithm (Algorithm 2.3) from this. The final section deals with the situation of matching echoes from a single sound event. The main result (Theorem 3.1) from that section says that, loosely speaking, almost all microphone positions are good.

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1. Multilateration and the GPS-problem

In this section we look at the pseudo-range multilateration problem, which is equivalent to the GPS-problem (see below), and present two simple algebraic (“direct”, as opposed to iterative) solutions, see Theorem 1.1. This topic has received considerable interest in the literature (see Bancroft [1], Krause [10], Chaffee and Abel [6], Li et al. [12], Lundberg [13], and Beck and Pan [2]). But our results are general and appear to be new.

Our primary interest is in the following situation: A source at an unknown position \( \mathbf{x} \in \mathbb{R}^3 \) emits a signal, in our applications usually by sound, at an unknown time \( t \). (In fact, everything we are about to say can easily be adapted to \( \mathbb{R}^n \) with \( n \geq 2 \), but not to \( \mathbb{R}^1 \), see Remark 1.2.) There are \( m \) sensors at known positions \( \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^3 \). They receive the signal at times \( t_1, \ldots, t_m \). We choose the unit of time such that the signal propagation speed becomes 1. So we have

\[
\| \mathbf{a}_i - \mathbf{x} \| = |t_i - t| \quad (i = 1, \ldots, m) \tag{1.1}
\]

The task now is to work out the position \( \mathbf{x} \) and the emission time \( t \). The very same equations arise if there are \( m \) sources at known positions \( \mathbf{a}_i \), emitting signals at known times, which are then received by a device at an unknown position \( \mathbf{x} \). In this case the \( t_i \) are the differences between the reception times according to the clock on the device and the emission times according to the (near-perfect) clocks on the sources, and \( t \) is the (unknown) bias between the clock on the receiver and the clocks on the sources. This is the GPS-problem.

We will work with the slightly weaker equations

\[
\| \mathbf{a}_i - \mathbf{x} \| = |t_i| \quad (i = 1, \ldots, m). \tag{1.2}
\]

Writing \( L := \begin{pmatrix} -1 & \mathbf{a}_i^T \\ 1 & 1 \end{pmatrix} \), \( \mathbf{a}_i := (t_i, \mathbf{a}_i) \) and \( \mathbf{\bar{x}} := (\mathbf{\bar{x}}, t) \), we have

\[
\| \mathbf{a}_i - \mathbf{x} \|^2 - (t_i - t)^2 = (\mathbf{\bar{a}}_i - \mathbf{\bar{x}})^T \cdot L \cdot (\mathbf{\bar{a}}_i - \mathbf{\bar{x}}) = \mathbf{\bar{a}}_i^T L \mathbf{\bar{a}}_i - 2 \mathbf{\bar{a}}_i^T L \mathbf{\bar{x}} + \mathbf{\bar{x}}^T L \mathbf{\bar{x}} = \| \mathbf{a}_i \|^2 - t_i^2 + 2t_i t - 2 \mathbf{a}_i^T \mathbf{x} + \| \mathbf{x} \|^2 - t^2,
\]

so (1.2) is equivalent to

\[
-2t_i t + 2 \mathbf{a}_i^T \mathbf{x} - \| \mathbf{x} \|^2 + t^2 = \| \mathbf{a}_i \|^2 - t_i^2 \quad (i = 1, \ldots, m). \tag{1.3}
\]

We form the matrix

\[
A := \begin{pmatrix} -2t_1 & 2 \mathbf{a}_1^T \\ & \vdots & \vdots & \vdots \\ & & \vdots & \vdots \\ -2t_m & 2 \mathbf{a}_m^T \\ & & -1 & -1 \end{pmatrix} \in \mathbb{R}^{m \times 5}, \tag{1.4}
\]
which contains only known quantities. With this, (1.3) can be expressed as a system of linear equations for the unknown quantities:

$$A \cdot \begin{pmatrix} t \\ x \\ \|x\|^2 - t^2 \end{pmatrix} = \begin{pmatrix} \|a_1\|^2 - t_1^2 \\ \vdots \\ \|a_m\|^2 - t_m^2 \end{pmatrix}. \quad (1.5)$$

Then this has the same solutions as (1.2). Now we make the assumption that $A$ has rank 5, so (1.5) has a unique solution. (The existence of a solution follows from the fact that the point $x$ and time $t$ of emission exist.) In the case $m = 5$ we can then simply invert $A$. If $m > 5$, we could delete all but 5 linearly independent equations from (1.5), which would give an algebraically equivalent system with invertible matrix. But in the real world there are inaccurate measurements, so it should be wiser to apply the Moore-Penrose inverse $(A^T A)^{-1} A^T \in \mathbb{R}^{5 \times m}$. Specifically, if $B \in \mathbb{R}^{4 \times m}$ is obtained by deleting the last row from $(A^T A)^{-1} A^T$, then

$$\begin{pmatrix} t \\ x \end{pmatrix} = B : \begin{pmatrix} \|a_1\|^2 - t_1^2 \\ \vdots \\ \|a_m\|^2 - t_m^2 \end{pmatrix}. \quad (1.6)$$

So we have obtained a unique solution for the emission time and place. This is our first solution, which is available only if $A$ has rank 5. In Proposition 1.4 we will say something about how likely this is.

But first we will consider the case that $A$ has rank $< 5$, and derive our second solution. What we do assume is that the $a_i$ are not coplanar. This makes sense, since if the $a_i$ all lay in the same plane, then even with a known emission time $t$ the location $x$ of the source could not be distinguished from the point obtained by reflecting $x$ at this plane. Our assumption amounts to saying that the matrix

$$\tilde{A} := \begin{pmatrix} 2a_1^T & -1 \\ \vdots & \vdots \\ 2a_m^T & -1 \end{pmatrix} \in \mathbb{R}^{m \times 4} \quad (1.7)$$

has rank 4 (see Assumption A in [2]), so in particular we need $m \geq 4$. The Moore-Penrose inverse is $\tilde{B} := (\tilde{A}^T \tilde{A})^{-1} \tilde{A}^T \in \mathbb{R}^{4 \times m}$, so $\tilde{B}A = I_4$. Now (1.3) can be restated as

$$\tilde{A} \cdot \begin{pmatrix} x \\ \|x\|^2 - t^2 \end{pmatrix} = 2t \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} + \begin{pmatrix} \|a_1\|^2 - t_1^2 \\ \vdots \\ \|a_m\|^2 - t_m^2 \end{pmatrix}, \quad (1.8)$$

and multiplying by $\tilde{B}$ yields

$$\begin{pmatrix} x \\ \|x\|^2 - t^2 \end{pmatrix} = t \begin{pmatrix} u \\ \alpha \end{pmatrix} + \begin{pmatrix} v \\ \beta \end{pmatrix}, \text{ where } \begin{pmatrix} u \\ \alpha \end{pmatrix} := 2\tilde{B} \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \text{ and } \begin{pmatrix} v \\ \beta \end{pmatrix} := \tilde{B} \begin{pmatrix} \|a_1\|^2 - t_1^2 \\ \vdots \\ \|a_m\|^2 - t_m^2 \end{pmatrix}. \quad (1.9)$$

Extracting components, we obtain the equivalent equations

$$x = tu + v \quad \text{and} \quad (\|u\|^2 - 1)t^2 + (2u^T v - \alpha)t + \|v\|^2 - \beta = 0. \quad (1.10)$$

Observe that $u, v, \alpha, \text{ and } \beta$ are all derived from known quantities, so (1.10) can be resolved. In the following theorem, part (a) summarizes our result in the rank-5 case, (b) tells us that (1.10) is actually equivalent to (1.2), and (c) says that the quadratic equation in (1.10) never degenerates.

**Theorem 1.1.** In the above situation and with the notation introduced, we have:

(a) If the matrix $A$ from (1.4) has rank 5, then (1.2) has a unique solution $(\begin{pmatrix} t \\ x \end{pmatrix})$ given by (1.6).

(b) Assume that $\text{rank}(A) \leq 4$ and that the $a_i$ are not coplanar, so in particular $m \geq 4$. Then the equations (1.2) are satisfied by the same $(\begin{pmatrix} t \\ x \end{pmatrix})$ as (1.10).

(c) Moreover, the coefficients of $t^2$ and $t$ in the quadratic equation in (1.10) are not both zero, so (1.10) has one or, more likely, two solutions.
Proof. Part (a) has already been shown, so we turn our attention to (b). By renumbering we may assume that \( a_1, \ldots, a_4 \) are not coplanar. Then \( \tilde{A}_4 \), the upper 4 \( \times \) 4-part of \( \tilde{A} \), is invertible, and \( A_4 \), the upper 4 \( \times \) 5-part of \( A \), has rank 4. So we have a matrix \( C \in \mathbb{R}^{m \times 4} \), with upper 4 \( \times \) 4-part the identity matrix, such that \( A = C \cdot A_4 \) and \( \tilde{A} = C \cdot \tilde{A}_4 \). Extracting the first column from the first equation gives
\[
\begin{pmatrix}
    t_1 \\
    \vdots \\
    t_m \\
    t_4
\end{pmatrix} = C \cdot \begin{pmatrix}
    t_1 \\
    \vdots \\
    t_4
\end{pmatrix}.
\]
(1.11)

Since (1.5) has a solution \( \left( \frac{t}{\|x\|^2} \right) \), we obtain
\[
\begin{pmatrix}
    \|a_1\|^2 - t_1^2 \\
    \vdots \\
    \|a_m\|^2 - t_m^2 \\
    \|a_4\|^2 - t_4^2
\end{pmatrix} = A \begin{pmatrix}
    t \\
    \vdots \\
    t \\
    \|x\|^2 - t^2
\end{pmatrix} = CA_4 \begin{pmatrix}
    t \\
    \vdots \\
    t \\
    \|x\|^2 - t^2
\end{pmatrix} = C \cdot \begin{pmatrix}
    \|a_1\|^2 - t_1^2 \\
    \vdots \\
    \|a_4\|^2 - t_4^2
\end{pmatrix}.
\]
(1.12)

We have
\[
\tilde{B} = (\tilde{A}^T A)^{-1} \tilde{A}^T = (\tilde{A}_4^T C^T C \tilde{A}_4)^{-1} \tilde{A}_4^T C^T = \tilde{A}_4^{-1} (C^T C)^{-1} C^T,
\]
so
\[
\tilde{A} \tilde{B} C = I_4 \quad \text{and} \quad \tilde{A} \tilde{B} C = C \tilde{A}_4 \tilde{B} C = C.
\]
(1.13)

To prove (b), let \( \left( \frac{t}{\|x\|^2} \right) \) satisfy (1.10). Then it satisfies (1.9), so
\[
\begin{pmatrix}
    \|a_1\|^2 - t_1^2 \\
    \vdots \\
    \|a_m\|^2 - t_m^2 \\
    \|a_4\|^2 - t_4^2
\end{pmatrix} = 2t \tilde{B} \begin{pmatrix}
    t_1 \\
    \vdots \\
    t_m \\
    t_4
\end{pmatrix} + \tilde{B} \begin{pmatrix}
    \|a_1\|^2 - t_1^2 \\
    \vdots \\
    \|a_m\|^2 - t_m^2 \\
    \|a_4\|^2 - t_4^2
\end{pmatrix} = 2t \tilde{B} C \begin{pmatrix}
    t_1 \\
    \vdots \\
    t_m \\
    t_4
\end{pmatrix} + \tilde{B} C \begin{pmatrix}
    \|a_1\|^2 - t_1^2 \\
    \vdots \\
    \|a_m\|^2 - t_m^2 \\
    \|a_4\|^2 - t_4^2
\end{pmatrix},
\]
where we used (1.11) and (1.12). With (1.13) this implies
\[
\tilde{A} \begin{pmatrix}
    \|x\|^2 - t^2 \\
    \|x\|^2 - t^2
\end{pmatrix} = 2t C \begin{pmatrix}
    t_1 \\
    \vdots \\
    t_m \\
    t_4
\end{pmatrix} + C \begin{pmatrix}
    \|a_1\|^2 - t_1^2 \\
    \vdots \\
    \|a_m\|^2 - t_m^2 \\
    \|a_4\|^2 - t_4^2
\end{pmatrix} = 2t \begin{pmatrix}
    t_1 \\
    \vdots \\
    t_m \\
    t_4
\end{pmatrix} + \begin{pmatrix}
    \|a_1\|^2 - t_1^2 \\
    \vdots \\
    \|a_m\|^2 - t_m^2 \\
    \|a_4\|^2 - t_4^2
\end{pmatrix}.
\]

Thus (1.8) holds, which is equivalent to (1.2). So every solution of (1.10) satisfies (1.2), and the converse is true since (1.10) was derived from (1.2).

To prove part (c), assume that the coefficients of \( t^2 \) and \( t \) in the quadratic equation in (1.10) are both zero. Since (1.2) has a solution, this means that also the constant coefficient vanishes, so there is a solution of (1.10), and therefore of (1.2), for every \( t \). In particular, we have \( x_0 \in \mathbb{R}^4 \) such that \( \left( \frac{0}{0} \right) \) satisfies (1.2). Substituting each \( a_i \) by \( a_i - x_0 \) preserves the noncoplanarity of \( a_1, \ldots, a_4 \) and yields new systems (1.2) and (1.10) of equations. For each solution \( \left( \frac{t}{\|x\|^2} \right) \) of the old system (1.2), the new one now has the solution \( \left( \frac{t}{\|x\|^2 - x_0} \right) \), so the new quadratic equation (1.10) still has infinitely many solutions. Moreover, \( \left( \frac{0}{0} \right) \) is a solution of (1.2), so \( \|a_i\| = \|t\| \). Hence (1.9) implies \( v = 0 \) and \( \beta = 0 \). The vanishing of the coefficient of \( t \) in the quadratic equation therefore means \( \alpha = 0 \), therefore means \( \alpha = 0 \), so
\[
2 \begin{pmatrix}
    2 a_i^T u \\
    \vdots \\
    2 a_i^T u
\end{pmatrix} = \begin{pmatrix}
    2 a_i^T -1 \\
    \vdots \\
    2 a_i^T -1
\end{pmatrix} \cdot \begin{pmatrix}
    u \\
    \vdots \\
    u
\end{pmatrix} = \tilde{A}_4 \begin{pmatrix}
    u \\
    \vdots \\
    u
\end{pmatrix} = \begin{pmatrix}
    \varepsilon_i \|a_i\| \\
    \vdots \\
    \varepsilon_i \|a_i\|
\end{pmatrix}.
\]
(1.19)

\[
2 \tilde{A}_4 \tilde{B} \begin{pmatrix}
    t_1 \\
    \vdots \\
    t_m \\
    t_4
\end{pmatrix} = 2 \tilde{A}_4 \tilde{B} C \begin{pmatrix}
    t_1 \\
    \vdots \\
    t_m \\
    t_4
\end{pmatrix} = 2 \begin{pmatrix}
    \varepsilon_1 \|a_1\| \\
    \vdots \\
    \varepsilon_4 \|a_4\|
\end{pmatrix}
\]
(1.11)

for some \( \varepsilon_i \in \{ \pm 1 \} \). The vanishing of the \( t^2 \)-coefficient in (1.10) means that \( \|u\| = 1 \), so
\[
\|a_i^T u\| = \|a_i\| \cdot \|u\| \quad (i = 1, \ldots, 4).
\]
Thus the Cauchy–Schwarz inequality is actually an equality, implying that \( \mathbf{a}_i \) and \( \mathbf{u} \) are linearly dependent. This shows that \( \mathbf{a}_1, \ldots, \mathbf{a}_4 \) are collinear, contradicting the hypothesis that they are not coplanar. \( \square \)

**Remark 1.2.** Everything in this section carries over directly to \( n \)-dimensional space, with \( n > 1 \). Just replace every instance of 3, 4, and 5 by \( n, n+1, \) and \( n+2 \), and replace “coplanar” by “contained in a common \( (n-1) \)-dimensional affine subspace.”

What happens for \( n = 1 \)? Everything works, except for the very last sentence in the proof of Theorem 1.1: In dimension 1, collinearity does not contradict being noncopunctual (i.e., not being the same point). But this makes everything break down. In fact, the mathematics bears out what has always been clear about the one-dimensional case: If the source lies on the same side of every sensor, then there is no way to find out its position from the time differences of signal arrivals; and indeed in this case the coefficients of the quadratic equation in (1.10) are all 0, and the matrix \( \mathbf{A} \in \mathbb{R}^{m \times 3} \) has rank 2.

Even though the one-dimensional case is not interesting in itself, it shows, as do (5) and (6) in Example 1.3 below, that the effort of proving Theorem 1.1(c) was not irrelevant: this is not a truism.

Up to now, we have worked with the equations (1.2), and seen that they may have two solutions. But according to (1.1) (which expresses that signals arrive after having been sent) we have \( t_i \geq t \). If one of the solutions does not satisfy this, it is spurious and can be discarded. But if both do, the given data do not uniquely determine \( \mathbf{x} \) and \( t \). The first two of the following examples show that this can actually happen. The third example has a spurious solution, and the others exemplify some special cases.

**Example 1.3.** In the following, we chose the coordinates in a way to keep all numbers rational, so the examples are quite Pythagorean-triple-prone.

(1) Of the five sensor positions

\[
\mathbf{a}_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_4 = \begin{pmatrix} 2/21 \\ -24/7 \\ 0 \end{pmatrix}, \quad \mathbf{a}_5 = \begin{pmatrix} 1/21 \\ 76/7 \end{pmatrix},
\]

no four are coplanar. If we assume that the source is at the origin \( \mathbf{x} = 0 = (0,0,0)^T \) and emits a signal at time \( t = 0 \), then the \( i \)-th sensor will receive this at time \( t_i = \|\mathbf{a}_i\| \). We have \( (t_1, t_2, t_3, t_4, t_5) = (5, 3, 1, 50/21, 76/21) \). We chose the \( \mathbf{a}_i \) in such a way that the affine relation \( 2a_1 + 2a_2 + 2a_3 - 3a_4 - 3a_5 = 0 \) is also satisfied by their lengths \( \|\mathbf{a}_i\| \). So the matrix \( \mathbf{A} \in \mathbb{R}^{5 \times 5} \) is not invertible. The computations according to (1.9) yield

\[
\mathbf{u} = \frac{1}{55} \begin{pmatrix} 21 \\ 34 \\ 199 \end{pmatrix}, \quad \mathbf{v} = 0, \quad \alpha = -\frac{152}{55}, \quad \text{and} \quad \beta = 0,
\]

so (1.10) becomes

\[
\mathbf{x} = t\mathbf{u} \quad \text{and} \quad \frac{38173}{3025} t^2 + \frac{152}{55} t = 0,
\]

which is solved by

\[
t = 0, \quad \mathbf{x} = 0, \quad \text{and} \quad t' = \frac{-8360}{38173}, \quad \mathbf{x}' = \frac{-152}{38173} \begin{pmatrix} 21 \\ 34 \end{pmatrix} \frac{199}{199}.
\]

Now it can easily be verified directly that (1.1) holds with \( t \) and \( \mathbf{x} \) replaced by \( t' \) and \( \mathbf{x}' \).

In other words, had the signal been emitted from the position \( \mathbf{x}' \) at time \( t' \) rather than from \( \mathbf{x} = 0 \) at \( t = 0 \), it would have arrived at the exact same times \( t_i \) at the sensors. So even if the system (1.1) is overdetermined (5 equations for 4 unknowns) and the sensor positions are not chosen in an obviously clumsy way, it may still be impossible to uniquely determine the source position.
(2) A simpler example of the same type can be constructed in 2 dimensions. Take the sensor positions
\[ a_1 = \begin{pmatrix} 9 \\ 12 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 9 \\ -12 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 10 \\ -24 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 10 \\ 24 \end{pmatrix}. \]
The source is again located at \( x = 0 \) and emits its signal at time \( t = 0 \). It is then received at times \((t_1, t_2, t_3, t_4) = (15, 15, 26, 26)\). The same computation as above (but with smaller numbers) shows that emission time and place
\[ t' = \frac{7}{5} \quad \text{and} \quad x' = \frac{77}{5} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
would have led to the exact same arrival times \( t_i \).

(3) An example with a spurious solution (again in dimension 2) is given by
\[ a_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \quad a_3 = \begin{pmatrix} -3 \\ -4 \end{pmatrix}, \quad a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad t = 0. \]
Reception times are \((t_1, t_2, t_3) = (4, 5, 5)\). Here another solution of \((1.2)\) is \( t' = 28/3 \) and \( x' = -\frac{4}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). This is spurious, since \( t_i - t' < 0 \). In fact, if the signal had been sent from \( x' \) at time \( t' \), it would have arrived at times \((\frac{4}{3}, \frac{1}{3}, \frac{4}{3})\). The absolute differences between the arrival times are the same, but the sequence is reversed.

(4) It can happen that the signal arrives at the same time \( t_1 = \cdots = t_m \) at all sensors, so the source has the same distance from all of them. Intuition tells us that there can only be one such point, and again the mathematics bears this out. In fact, the equation \( BA = I_{n+1} \) before \((1.8)\) and the definition of \( \tilde{A} \) imply that all rows from \( \tilde{B} \), except for the last one, have coefficient sum 0. Therefore \( u = 0 \), so indeed there is only one solution for \( x \). Of the solutions for \( t \), one is spurious.

(5) Assume that \( x \) is collinear with two of the \( a_i \), say \( a_1 \) and \( a_2 \), but does not lie between them. Choosing the coordinate system suitably, we may assume \( x = 0 \) and \( t = 0 \). Then our assumption means \( a_1 = \lambda a_2 \) with \( 1 \neq \lambda > 0 \). Now \( BA = I_{n+1} \) implies that the last row of \( \tilde{B} \) is \( \begin{pmatrix} 1 \\ \lambda a_3 \\ \vdots \\ \lambda a_m \end{pmatrix} \). We have \( t_1 = \|a_1\| = \lambda \|a_2\| = \lambda t_2 \), so \( \alpha = 0 \) by \((1.9)\).
Since \( v = 0 \) and \( \beta = 0 \), the quadratic equation becomes \( t^2 = 0 \).

(6) Here is an example where the coefficient of \( t^2 \) becomes 0:
\[ a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad t = 0. \]
The computation shows \( u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \alpha = -2 \), and \( \beta = 0 \). So here the quadratic equation degenerates to \( 2t = 0 \). This can be interpreted as follows: As the positions \( a_i \) approach the values given above, the alternative solution \( \begin{pmatrix} t' \\ x' \end{pmatrix} \) of \((1.10)\), apart from the solution \( \begin{pmatrix} 1991 \\ 0 \end{pmatrix} \), tends to infinity. For example, taking \( a_3 = (3, 3, 99) \) and leaving \( a_1 \) and \( a_2 \) unchanged leads to \( t' \approx -1991 \) and \( x' \approx \begin{pmatrix} 0 \\ 992 \end{pmatrix} \).

Having seen from Example 1.3(1) that even with 5 sensors such that no 4 of them are coplanar, it may happen that the matrix \( A \), defined in \((1.4)\) is not invertible, we wonder how often this happens. The following result says that under mild hypotheses, the answer is “very rarely”; i.e., almost certainly the formula \((1.6)\) can be applied for finding \( x \) and \( t \). Notice that \( A \) is formed with times \( t_i \) given by \((1.1)\), so, because of the last column of \( A \), its rank only depends on the positions \( a_i \) and \( x \).

**Proposition 1.4.** Assume we have \( m = 5 \) sensors such that
\[
\det \left( \begin{array}{c|c|c}
\varepsilon_1 \cdot \|a_1\| & a_1^T & 1 \\
\vdots & \vdots & \vdots \\
\varepsilon_5 \cdot \|a_5\| & a_5^T & 1 \\
\end{array} \right) \neq 0 \quad \text{for all} \quad \varepsilon_1, \ldots, \varepsilon_5 = \pm 1. \tag{1.14}
\]
Then the set of all \( x \in \mathbb{R}^3 \) such that the matrix \( A \) has rank \( < 5 \) is contained in a 2-dimensional subvariety of \( \mathbb{R}^3 \).
Proposition 1.4

The hypothesis (1.14) in Proposition 1.4 looks a bit messy and lacks geometric content, but is readily verifiable. It would be desirable to have some more geometric conditions under which the assertion of Proposition 1.4 holds. The following conjecture would be the best possible result, since its converse is clearly true.

**Conjecture 1.5.** The assertion of Proposition 1.4 holds under the milder hypothesis that the $a_i$ are not coplanar and pairwise distinct.

We managed to prove the conjecture in the 2-dimensional case by considering the polynomial $f$ used in the above proof as a polynomial in the coordinates of $x$ as main variables, and forming the ideal generated by the coefficients. A computation in the computer algebra system MAGMA [3] then shows that the equations that express that the $a_i$ do not satisfy the hypothesis of the conjecture all lie in the radical of this ideal. However, in the 3-dimensional case our computations ran into an impasse. The 1-dimensional case of the conjecture is false.

**Remark 1.6.** The first, and most cited, algebraic solution of the GPS-problem appears to have been given by Bancroft [1] in 1985. Let us point out some differences between his approach and ours.

- Bancroft reaches a quadratic equation even if there are more than 4 sensors, and does not offer an explicit formula such as (1.6).
- For reaching the equations (1.10) in the case that $A$ has rank $< 5$, we assume that $\tilde{A}$, defined in (1.7), has rank 4 or, equivalently, that the $a_i$ are not coplanar. On the other hand, Bancroft assumes that the matrix
  \[
  \begin{pmatrix}
  a_1^T & t_1 \\
  \vdots & \vdots \\
  a_m^T & t_m
  \end{pmatrix} \in \mathbb{R}^{m \times 4}
  \]
  has rank 4. But whether this is the case does not only depend on the positions $a_i$ and $x$, but also on $t$, which in Bancroft’s situation is the bias between the clocks. For example, if there are $m = 4$ noncoplanar sensors, there is always a value for $t$ such that the above matrix has determinant 0.
- There is no proof that Bancroft’s quadratic equation does not degenerate.

2. Relations and matching

We consider the same situation as in the previous section: A source at position $x \in \mathbb{R}^3$ emits a signal at time $t$, which is received by $m$ sensors at positions $a_1, \ldots, a_m \in \mathbb{R}^3$ and at times $t_1, \ldots, t_m$. The unit of time is chosen such that the signal propagates with speed 1, so $\|a_i - x\| = t_i - t$. The $a_i$ and $t_i$ are considered as known, and Section 1 was about how to find $x$ and $t$ from them. Now if there are 5 sensors or more, then, according to the following result, there are algebraic relations between the known quantities.

**Theorem 2.1.** In the above situation, write $d_{i,j} := \|a_i - a_j\|$ and $t_{i,j} := t_i - t_j$. Then the matrix

\[
D = (t_{i,j}^2 - d_{i,j}^2)_{i,j=1,\ldots,m} = \begin{pmatrix}
0 & t_{1,2}^2 - d_{1,2}^2 & \ldots & t_{1,m}^2 - d_{1,m}^2 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots \\
t_{m,1}^2 - d_{m,1}^2 & \ldots & t_{m,2}^2 - d_{m,2}^2 & 0
\end{pmatrix} \in \mathbb{R}^{m \times m}
\]
Theorem 2.1 in the 3-dimensional case for the sake of simplicity. But it actually holds in any dimension $n$, saying that $\text{rank}(D) \leq n + 1$.

For the proof we will use the Cayley-Menger matrix. The following result about its rank is well known (see Cayley [5]) in the case of Euclidean spaces, but we need a more general version given by the following proposition. We will not need the exact value of the rank, but include it for the sake of completeness.

**Proposition 2.2.** Let $v_0, \ldots, v_m \in V$ be vectors in a Euclidean space or, more generally, in a vector space over a field of characteristic $\neq 2$ equipped with a quadratic form $q$. Set $\delta_{i,j} := q(v_i - v_j)$, which in the special case of a Euclidean space is the squared distance between $v_i$ and $v_j$. Then the Cayley-Menger matrix

$$
C := \begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & \delta_{0,1} & \delta_{0,2} & \cdots & \delta_{0,m} \\
1 & \delta_{1,0} & 0 & \delta_{1,2} & \cdots & \delta_{1,m} \\
1 \delta_{2,0} & \delta_{2,1} & 0 & \cdots & \delta_{2,m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \delta_{m,0} & \delta_{m,1} & \cdots & 0 \\
\end{pmatrix} \in K^{(m+2) \times (m+2)}
$$

has rank $\leq \dim(V) + 2$. More precisely, if $r$ is the rank of $q$ restricted to the subspace $U \subseteq V$ generated by $v_1 - v_0, \ldots, v_m - v_0$, then rank$(C) = r + 2$. Notice that $r \leq \dim(U)$, with equality in the special case of a Euclidean space. Also notice that $\dim(U)$ is equal to the dimension of the affine subspace generated by $v_0, \ldots, v_m$.

**Proof.** We only need to show rank$(C) = r + 2$ in the more general case of a quadratic space over a field $K$. Replacing each $v_i$ by $v_i - v_0$ does not change $C$ or $U$, so by doing this we may assume $v_0 = 0$. Then $U$ is generated (as a vector space) by $v_1, \ldots, v_m$, and we may replace $V$ by $U$. Now $V$ is generated by the $v_i$ and in particular finite-dimensional. By choosing a basis of $V$ we may then replace $V$ by $K^n$. Then $q$ is given by $q(v) = v^T A v$ for $v \in V = K^n$, with $A \in K^{n \times n}$ a symmetric matrix of rank $r$. $A$ also defines the bilinear form $\langle \cdot, \cdot \rangle$ belonging to $q$. After these reductions, the main part of the proof rests on the following matrix computation.

With

$$
E := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & -2A \\
\end{pmatrix} \in K^{(n+2) \times (n+2)}, \quad F := \begin{pmatrix}
I_2 & \delta_{0,1} \cdots & \delta_{0,m} \\
\delta_{1,0} \cdots & 1 \\
\vdots & \ddots & \vdots & \vdots \\
\delta_{m,0} \cdots & 0 \\
v_1 & \cdots & v_m \\
\end{pmatrix} \in K^{(n+2) \times (m+2)}
$$

we have

$$
F^T E F = I_2 \begin{pmatrix}
0 & \delta_{0,1} \cdots & \delta_{0,m} \\
\delta_{1,0} \cdots & 1 \\
\vdots & \ddots & \vdots & \vdots \\
\delta_{m,0} \cdots & 0 \\
v_1 & \cdots & v_m \\
\end{pmatrix} = \begin{pmatrix}
0 & 1 & \cdots & 1 \\
1 & 0 & \cdots & \delta_{0,1} \cdots & \delta_{0,m} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
1 & \delta_{1,0} \cdots & \delta_{1,m} & \cdots & B \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \delta_{m,0} \cdots & \delta_{m,1} & \cdots & \delta_{m,m} \\
\end{pmatrix},
$$

where the $(i,j)$-th entry of the matrix $B$ is

$$
\delta_{0,i} + \delta_{0,j} - 2v_i^T A v_j = \langle v_i, v_j \rangle + \langle v_j, v_j \rangle - 2\langle v_i, v_j \rangle = \langle v_i - v_j, v_i - v_j \rangle = \delta_{i,j}.
$$

So $F^T E F = C$. Since the $v_i$ span $V = K^n$, $F$ has rank $n + 2$, so the linear map $K^{m+2} \rightarrow K^{n+2}$ given by $F$ is surjective. Likewise, the map given by $F^T$ is injective, and the image of the map
given by $E$ has dimension equal to $\text{rank}(A) + 2 = r + 2$. It follows that the map given by $C$ has an image of dimension $r + 2$, which is our claim. \hfill \Box

Proof of Theorem 2.1. We apply Proposition 2.2 with $V = \mathbb{R}^4$ being Minkowski space, so for $v = (t^u) \in V$ with $t \in \mathbb{R}$, $u \in \mathbb{R}^3$ the quadratic form is $q(v) = t^2 - \|u\|^2$, with $\|\|$ the usual Euclidean norm. For $i = 1, \ldots, m$, set $v_i := \left(\frac{t_i}{s_i}\right)$, and also set $v_0 := \left(\frac{t}{s}\right)$ (the time and place of emission). Then the equation $\|a_i - x\| = t_i - t$ implies $\delta_{0,i} = q(v_0 - v_i) = 0$ for all $i$. So the Cayley-Menger matrix becomes

$$C = \begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & D \\
\vdots & \vdots & & \ddots & \vdots \\
1 & 0 & & & 0
\end{pmatrix},$$

with $D$ as defined in the theorem. Since $\text{rank}(C) \leq 6$ by Proposition 2.2, we get $\text{rank}(D) \leq 4$. \hfill \Box

By Theorem 2.1 there is a relation between the reception times of a signal coming from a single emission event. If there are multiple emission events (from different source locations and/or at different times), this relation can be used to match those reception times that come from the same event. Matching reception times can then be used to determine the time and place of emission, making use of the methods from Section 1. Algorithm 2.3 makes this idea precise.

Algorithm 2.3 Detect source positions and emission times of multiple emission events

Input: For $i = 1, \ldots, 5$, a set $T_i$ containing the points in time when the $i$th sensor received a signal. The units of time and distance should be chosen such that signals travel with speed 1. The positions $a_1, \ldots, a_5 \in \mathbb{R}^3$ of the sensors need to be known.

1: Let $d_{i,j} := \|a_i - a_j\|$ be the distances between the sensors. Set $E := \emptyset$. The detected emission events will be collected as pairs $(t, x)$ in the set $E$.

2: for $(t_1, t_2, t_3, t_4, t_5) \in T_1 \times T_2 \times T_3 \times T_4 \times T_5$ do

3: Set up the matrix $D = ((t_i - t_j)^2 - d_{i,j}^2)_{i,j=1,\ldots,5}$.

4: if $\text{det}(D) = 0$ then

5: Use Theorem 1.1, with the current $t_1, \ldots, t_5$ and $a_1, \ldots, a_5$ as input, to compute the emission time $t$ and the source position $x$. Include $(t, x)$ in the set $E$. In the unlikely event that Theorem 1.1 yields two solutions and neither can be discarded (as spurious or from other context), include both.

6: end if

7: end for

8: Output the set $E$ of detected emission events.

Remark 2.4. In fact, Algorithm 2.3 does more than just matching reception times of signals. It also discards spurious signals registered by sensors. By this we mean erroneous registrations of signals, or registrations of signals that originate very near to a sensor and are irrelevant since they cannot be perceived by other sensors. Indeed, it is almost impossible for such a spurious signal to satisfy the relation $\text{det}(D) = 0$ together with other "legitimate" reception times. \hfill \triangleleft

In a later paper we will study the behavior (and modifications) of the algorithm in situations where the input data is inexact because of measurement errors.

3. WALL DETECTION BY ECHOES

Theorem 2.1 guarantees that the relation $\text{det}(D) = 0$ always holds if the signals received at times $t_1, \ldots, t_5$ come from the same emission event. So all events for which a signal is received by every sensor will be detected. But it is possible that the determinant becomes 0 even if the signals do not come from the same event. For instance, if $t_1, \ldots, t_4$ do come from the same event,
computes the mirror point, and it is easy to find the wall position from this. With Algorithm 2.3 there are points \( s_i \). These points are all equal. So “excellent” implies “very good” and “good”. Now the set \( M \) is excellent which is easily seen to be a polynomial in the coordinates of its arguments. Let us say that \( M \) is \textit{good} if \( M \) is good, meaning that from a single sound emitted by the loudspeaker, \( M \) will be an actual mirror point, meaning that \( M \) depends on the coordinates of the mirror points, but not on the time of the sound emission, since only differences of reception times go into the matrix \( D \). Even more hopeful, we say that \textit{almost all} positions are good if the bad positions are contained in a lower-dimensional subvariety of the configuration space \( (\mathbb{R}^3)^D \) of all possible microphone positions. Intuitively “almost all” can be thought of as “with probability one.”

**Theorem 3.1.** Consider a given room, by which we understand an arrangement of walls, which may include ceilings, floors, and sloping walls. Assume there is a loudspeaker at a given position in the room. Now five microphones are positioned in the room. Then almost all loudspeaker positions are good, meaning that from a single sound emitted by the loudspeaker, Algorithm 2.3 detects all walls from which an echo is received by every microphone, but it detects no ghost walls.

**Proof.** We are given a finite set \( S \subset \mathbb{R}^3 \) of mirror points, obtained by reflecting the given position of the loudspeaker at the various walls. From now on we can forget about the walls and the loudspeaker, since the signals will be received by the microphones as if they were all simultaneously emitted from the mirror points. The microphone positions can be represented as the columns of a matrix \( M = (a_1, \ldots, a_5) \subset \mathbb{R}^{3 \times 5} \). We can thus speak of good or bad matrices \( M \).

We will also say that \( M \) is \textit{very good} if the following is true: For any five points \( s_1, \ldots, s_5 \in S \), the relation

\[
\det \left( (\|s_i - a_i\| - \|s_j - a_j\|)^2 - \|a_i - a_j\|^2 \right)_{i,j=1,\ldots,5} = 0
\]  

only holds if \( s_1 = s_2 = s_3 = s_4 = s_5 \). A very good position is a good one, since for a quintuple \( (t_1, \ldots, t_5) \in T_1 \times \cdots \times T_5 \) considered in Algorithm 2.3 there are points \( s_i \in S \) such that \( t_i - t = \|s_i - a_i\| \) (with \( t \) the time of simultaneous emission). So the determinant considered in the algorithm is just the one in (3.1). If this is zero, the hypothesis of “very good” implies that the \( s_i \) are all the same point \( s \in S \). So indeed if the algorithm includes a point into the set \( E \), this will be an actual mirror point, meaning that \( M \) is good.

The main difficulty of the proof arises from the fact that the determinant in (3.1) is not a polynomial in the coordinates of the \( a_i \) and the \( s_i \), because the norms involve square roots. To circumvent this problem, we form the product

\[
f(a_1, \ldots, a_5, s_1, \ldots, s_5) := \prod_{\varepsilon_1, \ldots, \varepsilon_5 = \pm 1} \det \left( \left( \varepsilon_i \|s_i - a_i\| - \varepsilon_j \|s_j - a_j\| \right)^2 - \|a_i - a_j\|^2 \right)_{i,j=1,\ldots,5},
\]

which is easily seen to be a polynomial in the coordinates of its arguments. Let us say that \( M \) is \textit{excellent} if for any \( s_1, \ldots, s_5 \in S \) the relation \( f(a_1, \ldots, a_5, s_1, \ldots, s_5) = 0 \) implies that the \( s_i \) are all equal. So “excellent” implies “very good” and “good”. Now the set

\[
U_{s_1, \ldots, s_5} := \{ M = (a_1, \ldots, a_5) \in \mathbb{R}^{3 \times 5} | f(a_1, \ldots, a_5, s_1, \ldots, s_5) \neq 0 \}
\]
is Zariski open in \( \mathbb{R}^{3 \times 5} \), and
\[
U := \bigcap_{s_1, \ldots, s_5 \in S \text{ such that not all } s_i \text{ are equal}} U_{s_1, \ldots, s_5}
\]
is the set of excellent matrices. So if we can show that \( U_{s_1, \ldots, s_5} \neq \emptyset \) for all \( s_1, \ldots, s_5 \in S \) that are not all equal, by the Zariski openness the theorem follows. Since we have no control over the given set \( S \), we need to show the nonemptiness for any five points \( s_i \in \mathbb{R}^3 \) that are not all equal, and this also suffices. Equivalently, we need to prove the following.

**Claim.** If \( s_1, \ldots, s_5 \in \mathbb{R}^3 \) are points such that
\[
f(a_1, \ldots, a_5, s_1, \ldots, s_5) = 0 \quad \text{for all} \quad M = (a_1, \ldots, a_5) \in \mathbb{R}^{3 \times 5},
\]
then \( s_1 = s_2 = \cdots = s_5 \).

Having reduced the proof to the claim, we can forget about the situation of the theorem. In principle, the claim could be proved by explicitly forming the polynomial \( f \), regarding the variables for the \( a_i \) as main variables, extracting the coefficients (which are polynomials in the \( s_i \)-variables), and showing that the ideal generated by the coefficients defines the variety given by \( s_1 = s_2 = \cdots = s_5 \). Unfortunately, \( f \) has 15 variables and is homogeneous of degree 160. The first step towards making the computation feasible is choosing suitable Cartesian coordinates as follows. The vector \( s_1 \) can be taken as the origin of the coordinate system. This turns the matrix \( S := (s_1, \ldots, s_5) \in \mathbb{R}^{5 \times 5} \) into \( S = (0, s_2, \ldots, s_5) \). We can now apply QR-decomposition, i.e., write
\[
S = Q \cdot \begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 & b_9 \end{pmatrix} =: QB \tag{3.2}
\]
with \( Q \in \text{SO}(3) \). So using the columns of \( Q \) as a new basis of \( \mathbb{R}^3 \), \( S \) becomes the above upper triangular matrix \( B: (s_1, \ldots, s_5) = S = B \).

Form the matrices
\[
X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,5} \\ \vdots & \vdots & \vdots \\ x_{3,1} & \cdots & x_{3,5} \end{pmatrix}
\]
and
\[
Y = (Y_{i,j}) = \begin{pmatrix} 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & 0 & y_5 & y_6 & y_7 \\ 0 & 0 & 0 & y_8 & y_9 \end{pmatrix}
\]
with \( x_{i,j} \) and \( y_i \) indeterminates. With additional indeterminantes \( z_1, \ldots, z_5 \), form the ideal
\[
J := \left( z_j^2 - \sum_{i=1}^{3} (Y_{i,j} - x_{i,j})^2 \right)_{j = 1, \ldots, 5},
\]
with the idea that the \( z_j \) stand for \( ||s_i - a_i|| \). Modulo \( J \), the product
\[
\prod_{\varepsilon_i = \pm 1} \det \left( (\varepsilon_i z_i - \varepsilon_j z_j)^2 - \sum_{k=1}^{3} (x_{k,i} - x_{k,j})^2 \right)_{i,j=1,\ldots,5} \tag{3.3}
\]
reduces to a polynomial \( F(x_{1,1}, \ldots, x_{3,5}, y_1, \ldots, y_9) \) which does not involve the \( z_j \). If we specialize the variables in the matrix \( X \) to the entries in a matrix \( M = (a_1, \ldots, a_5) = (a_{i,j}) \in \mathbb{R}^{3 \times 5} \), and the variables in \( Y \) to the matrix \( B \) in (3.2), we obtain
\[
F(a_1, \ldots, a_5, b_1, \ldots, b_9) = f(a_1, \ldots, a_5, s_1, \ldots, s_5).
\]
So to prove the above claim and thus the theorem, we need to form \( F(x_{1,1}, \ldots, x_{3,5}, y_1, \ldots, y_9) \), regard the \( x_{i,j} \) as main variables and consider the ideal \( L \subseteq \mathbb{R}[y_1, \ldots, y_9] \) generated by the coefficients. Then we need to show that \( L \) has \( y_1 = \cdots = y_9 = 0 \) as the only real solution, which says that all \( s_i \) are zero and hence equal.

Alas, even after reducing the number of variables by our choice of coordinates, computing the polynomial \( F \) is still utterly impossible. What we did instead was setting almost all of the variables \( x_{i,j} \) to zero and compute the product (3.3) with these specializations, always reducing modulo \( J \). This turns out to be possible in many cases, and extracting coefficients gave us some generators of \( L \). Doing this for many choices of specialized variables provides ever more
generators of $L$. Each time, we also reduced modulo the generators of $L$ already known, which is permissible and accelerates the computation. When we found a sum of squares of variables in the ideal, we substituted this by the variables themselves since only real solutions need to be considered.

With this technique, using random specializations of variables, we eventually arrived at an ideal whose only solution is the origin $y_i = 0$. We recorded exactly which sequence of specializations led to this result and from this produced a deterministic, reproducible procedure for verifying the claim. All computations were done in MAGMA [3].

Remark. Theorem 3.1 gives the theoretical justification for a procedure that detects walls by solving the pseudo-range multilateration problem for each wall. The following alternative method comes to mind. Since the sound traveling directly from the loudspeaker to the microphones always arrives first, before any echoes, the very first signals registered by the microphones must be the ones coming directly from the loudspeaker. Therefore these can be used, without any matching process, to determine the emission time $t$ by multilateration. Since the echoes virtually come from the mirror points and are emitted at the same time $t$ which is now known, the methods from Dokmanić et al. [7] or Boutin and Kemper [4] can then be used for the wall detection.

While this alternative would presumably work in many cases, it has some drawbacks. For one thing, obstacles may in some cases prevent one or more microphones from hearing the direct signal from the loudspeaker. Secondly, and perhaps more importantly, a spurious signal may be registered by one or more of the microphones before the true signal from the loudspeaker arrives. In this case the alternative method would mistake this spurious signal as the direct signal. Therefore the multilateration would produce a drastically wrong time of emission, and all subsequent wall detections, based on this erroneous time, would become false.

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References

[1] Stephen Bancroft, An algebraic solution of the GPS equations, IEEE Transactions on Aerospace and Electronic Systems 7 (1985), 56–59.

[2] Amir Beck, Dror Pan, On the solution of the GPS localization and circle fitting problems, SIAM J. Optim. 22 (2021), 108–134.

[3] Wieb Bosma, John J. Cannon, Catherine Playoust, The Magma Algebra System I: The User Language, J. Symb. Comput. 24 (1997), 235–265.

[4] Mireille Boutin, Gregor Kemper, A Drone Can Hear the Shape of a Room, SIAM J. Appl. Algebra Geometry 4 (2020), 123–140.

[5] Arthur Cayley, On a theorem in the geometry of position, Cambridge Mathematical Journal II (1841), 267–271.

[6] James Chaffee, Jonathan S. Abel, On the Exact Solutions of Pseudorange Equations, IEEE Transactions on Aerospace and Electronic Systems 30 (1994), 1021–1030.

[7] Ivan Dokmanić, Reza Parhizkar, Andreas Walther, Yue M. Lu, Martin Vetterli, Acoustic echoes reveal room shape, Proceedings of the National Academy of Sciences 110 (2013).

[8] Youssef El Baba, Andreas Walther, Emamiel AP Habeta, Time of arrival disambiguation using the linear Radon transform, in: 2017 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pp. 106–110, IEEE, 2017.

[9] Ingmar Jager, Richard Heusdens, Nikolay D Gaubitch, Room geometry estimation from acoustic echoes using graph-based echo labeling, in: 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pp. 1–5, IEEE, 2016.

[10] Lloyd O. Krause, A Direct Solution to GPS-Type Navigation Equations, IEEE Transactions on Aerospace and Electronic Systems 23 (1987), 225–232.

[11] Martin Kreiligg, Bin Yang, Fast and reliable TDOA assignment in multi-source reverberant environments, in: 2013 IEEE International Conference on Acoustics, Speech and Signal Processing, pp. 355–359, IEEE, 2013.

[12] Wei Li, Deng Li, Shuhui Yang, Zhiwei Xu, Wei Zhao, Design and analysis of a new gps algorithm, in: 2010 IEEE 30th International Conference on Distributed Computing Systems, pp. 40–51, IEEE, 2010.

[13] John B Lundberg, Alternative algorithms for the GPS static positioning solution, Applied mathematics and computation 119(1) (2001), 21–34.

[14] AZ Sha’ameri, WA Wan Aris, S Sadiah, TA Musa, Reliability of Seismic Signal Analysis for Earthquake Epicenter Location Estimation Using 1 Hz GPS Kinematic Solution, Measurement 182 (2021), 109669.

[15] Cecilia Maria Zannini, Albenzio Cirillo, Raffaele Parisi, Aurelio Uncini, Improved TDOA disambiguation techniques for sound source localization in reverberant environments, in: Proceedings of 2010 IEEE International Symposium on Circuits and Systems, pp. 2666–2669, IEEE, 2010.
