Non-Bayesian Rate-Adaptive Wireless Communication Using ARQ-Feedback
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Abstract

To combat the detrimental effects of the variability in wireless channels, we consider cross-layer rate adaptation based on limited feedback. Based on limited feedback in the form of link-layer Automatic Repeat-reQuest (ARQ), we maximize the physical-layer transmission rate subject to an upper bound on the expected packet error rate without assuming any particular prior distribution on the channel state. We first analyze the fundamental limitations of such systems and derive an upper bound on the achievable rate for signalling schemes based on uncoded QAM and random Gaussian ensembles. We show that, for channel estimation based on binary ARQ feedback, it may be preferable to use a separate training sequence at high error rates, rather than to exploit low-error-rate data packets themselves. We also develop an adaptive recursive estimator, which is provably asymptotically optimal and asymptotically efficient.

Index Terms— adaptive modulation, rate adaptation, automatic repeat request, cross-layer strategies.

I. INTRODUCTION

The quality of the wireless communication channel depends on many different parameters and can vary significantly over time. Potential sources of channel variability include multipath propagation, mobility, and time-varying co-channel interference. These sources cause variabilities spanning a wide range of time scales, e.g., from bits to thousands of packets. For instance, relative movement of the transmitter-receiver pair may cause variations at relatively long time scales, since a very large number of packets can be transmitted during the time it takes for the stations to move far enough to cause significant change in the

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channel. On the other hand, co-channel interference can change significantly from one packet transmission to another. Finally, the multipath nature of the propagation medium can cause fast and/or slow fading in the channel, depending on the relative movement of the scatterers.

We propose physical-layer rate adaptation as a means of countering the detrimental effects of channel variability (as in, e.g., [1]–[15]). When the transmitter has perfect channel state information (CSI), rate adaptation is relatively straightforward. In practice, however, maintaining accurate CSI at the transmitter is highly nontrivial. One approach to obtaining CSI is via the inclusion of pilot symbols (as in, e.g., [1]–[4]). In such systems, the receiver’s channel estimates (or some quantized representations thereof) are fed back to the transmitter for use in rate adaptation. Because the inclusion of pilots and non-trivial feedback consumes bandwidth on both the forward and reverse links, we are motivated to consider other approaches that reduce this resource consumption. In this paper, we focus on rate adaptation schemes that infer the channel state by monitoring only packet acknowledgments (ACK/NACKs), i.e., the feedback information used for automatic repeat request (ARQ) (as in, e.g., [5]–[15]), and that adjust the physical-layer transmission rate accordingly. Since ARQ feedback is a standard provision of the link layer, its use by the physical layer comes essentially “for free.”

ARQ systems rely on the presence of error detection for each packet received. When an error is detected in a received packet, a request is sent to the transmitter to retransmit the packet. Hence, a given packet may be repeated multiple times until it is decoded correctly. While this transmission process is sometimes considered to be “adaptive,” in that the effective coding rate applied to the bits in a given packet varies with the number of retransmissions, we do not consider such standard-ARQ schemes as adaptive per se because the transmission rate per packet transmission may not change. In our work, we vary the transmission rate used at each packet interval according to the transmitter’s ARQ-based estimate of the CSI.

In particular, in this paper, we first derive conditions on the “quality” of CSI needed for a model-independent ARQ-based rate adaptation system to maximize data rate while keeping the packet error probability below a specified threshold. Based on these conditions, we derive fundamental bounds on the rate achievable under a given error probability constraint. Finally, we design an ARQ-feedback-based recursive CSI estimator with provable asymptotic optimality. Our findings are illustrated through both uncoded QAM and random Gaussian signaling.

Past works on physical-layer rate adaptation took, for the most part, one of two different approaches. In the first approach (e.g., [2]–[4]), a specific channel model is assumed and an adaptive coding scheme is optimized around this channel model. In the second approach (e.g., [5]–[8]), adaptation algorithms
are designed heuristically, based on practical experiences gained for a specific application in a specific
operating environment. From our viewpoint, neither of these approaches is fully satisfying, because the
nature of channel variability is highly dependent on the environment, the platform, and the application.
Furthermore, the characteristics of channel variability can fluctuate even under a fixed combination of
environment, platform, and application. Consequently, it is difficult to capture channel variability using a
single statistical model. The methods that we propose here strive for both statistical-model independence
and optimality. In this sense, they differ from both of the previously described approaches to physical-layer
rate adaptation.

The remainder of the paper is organized as follows. In Section II we detail the system model and
provide a mathematical statement of the problem. In Section III we derive conditions for successful
rate adaptation with imperfect CSI, and in Section IV we evaluate bounds on the achievable rates with
ARQ feedback. In Section V we develop an recursive channel estimator based on such feedback, and in
Section VI we conclude.

II. SYSTEM MODEL

A. System Components

Figure 1 depicts our model of the physical-layer adaptive communication system. At each discrete
packet index $t$, the transmitter transmits a packet $X_t = [X_{t,1}, \ldots, X_{t,n}]$ containing a fixed number, $n$, of
symbols $\{X_{t,k}\}_{k=1}^n$, which are encoded at a rate of $R_t$ bits/symbol, chosen by the rate controller from
the set of possible rates $\mathcal{R}$. We assume that the transmit power is constant and normalize all power levels
such that the energy per symbol is $E \left[ |X_{t,k}|^2 \right] = 1$. For this packet, the corresponding channel outputs are

$$ Y_{t,k} = H_{t} X_{t,k} + W_{t,k}, \quad k = 1, \ldots, n, $$

(1)
for complex-valued channel gain $H_t$ and additive white circularly symmetric complex Gaussian noise $W_{t,k}$ with two-sided power spectral density $N_o$. Some common models for $H_t$ include Rayleigh-, Rician- and Nakagami-fading (see e.g., [16]). However, we will not assume any specific statistical model for $H_t$ and we will make only weak assumptions on the distribution of $H_t$ in the sequel.

The quantity $\gamma_t = |H_t|^2/N_o$ can be interpreted as the $t^{th}$ packet’s channel SNR. Since each symbol has unit energy, $\gamma_t$ is also the received SNR for packet $t$. Thus, we will simply refer to $\gamma_t$ as the SNR.

We assume that, for all $t$, $\gamma_t$ takes on values from some prior distribution $p(\cdot) \in \mathcal{P}$, where $\mathcal{P}$ is a set of distributions with finite mean and variance. For the time being, we assume the distribution $p(\cdot)$ is known to both the receiver and the transmitter, though later we relax this assumption and consider a non-Bayesian framework. Due to lack of power adaptation, $\gamma_t$ is an exogenous quantity over which the system has no control.

We assume that the receiver has access to perfect CSI and uses a maximum likelihood decoder to decode the received packet. Let $\hat{X}_t$ denote the decoded estimate of packet $X_t$ based on received packet $Y_t = [Y_{t,1}, \ldots, Y_{t,n}]$, and the corresponding probability of decoding error be $\varepsilon(\gamma_t, R_t) = \Pr(\hat{X}_t \neq X_t \mid \gamma_t, R_t)$. Note that $\varepsilon(\cdot, \cdot)$ depends on the packet size $n$ and the coding/modulation schemes, which are assumed to be known at the decoder. For now, we assume only that the coding/modulation schemes are such that $\varepsilon(\gamma_t, R_t)$ is a convex, continuous, and increasing function of $R_t$ and a convex, continuous, and decreasing function of $\gamma_t$. Later, we detail the behavior of our proposed schemes for the specific cases of uncoded QAM and random Gaussian signaling.

Based on the received packet $Y_t$ and the decoded packet $\hat{X}_t$, the decoder generates a feedback packet $F_t$ which is communicated to the transmitter through a reverse channel. Assuming that the receiver is capable of perfect error detection, we take $F_t$ to be a binary ACK/NACK (i.e., $F_t = 0$ for ACK and $F_t = 1$ for NACK), so that

$$\Pr(F_t = f \mid \gamma_t, R_t) = \begin{cases} \varepsilon(\gamma_t, R_t), & f = 1 \\ 1 - \varepsilon(\gamma_t, R_t), & f = 0 \end{cases}.$$  

(2)

We assume that the reverse channel is error-free but introduces a delay of single packet interval. Thus, the “information” available to the transmitter when choosing rate $R_t$ is $I_t = [F_1, F_2, \ldots, F_{t-1}, R_1, R_2, \ldots, R_{t-1}]$. We find it convenient to explicitly include the previous rates $\{R_{\tau}\}_{\tau < t}$ in the information vector $I_t$ because the ACK/NACK feedback $F_{\tau}$ characterizes channel quality relative to the transmission rate $R_{\tau}$. Note that

\footnote{It is straightforward to generalize all of our results to a general delay of $d \geq 1$ packet intervals. While the generalization does not alter the fundamental nature of our results, it requires a more complex notation, which we avoid for clarity.}
the controller chooses the transmission rate at time $t$ solely based on the information vector $\mathbf{I}_t$, which is available at the receiver as well. We assume that the receiver is also aware of the controller’s rate allocation strategy, so that it can compute the current and previous values of $R_t$.

Finally, we assume in the sequel that the SNR is constant over each block of $T \gg 1$ packets, and that it changes independently from block to block, i.e., that the channel is “block fading.” In the sequel, we focus (without loss of generality) on the first block, for which $t \in \{1, \ldots, T\}$, and omit the $t$-dependence on the SNR, writing $\gamma_t$ as “$\gamma$.” In addition, we use $p(\gamma | \mathbf{I}_t)$ to denote the posterior SNR distribution, which can be associated with the prior distribution $p(\gamma)$ through the conditional mass function $P(F_t | \gamma, R_t)$ given in (2). Furthermore, we denote the set of possible posterior probability distributions using $\mathcal{P}(\mathbf{I}_t)$.

B. Ideal Rate Selection

We define the ideal controller as the one that, at time $t$, based on the available information $\mathbf{I}_t$, jointly optimizes the transmission rates $R_t, \ldots, R_T$ to maximize the sum-rate $\sum_{\tau=t}^{T} R_{\tau}$ subject to a constraint on expected error probability. In doing so, we allow any packet to be declared a probe packet, which is exempt from the expected-error-probability constraint but contributes nothing to sum rate. Probe packets are used exclusively to learn about the SNR $\gamma$, in the hope of more efficient allocation of future data packets. In particular, the ideal controller chooses rates according to the following constrained optimization problem:

$$\max_{(D_t, \ldots, D_T) \in \{0,1\}^{T-t+1}, \ (R_t, \ldots, R_T) \in \mathbb{R}^{T-t+1}} \sum_{\tau=t}^{T} D_{\tau} R_{\tau}$$

subject to

$$D_{\tau} E_p[\varepsilon(\gamma, R_{\tau}) | \mathbf{I}_t] \leq e^{-\alpha} \quad \text{for all } \tau = t, \ldots, T. \quad (4)$$

Here, $D_{\tau} \in \{0,1\}$ indicates whether the $\tau$th packet is a data packet ($D_{\tau} = 1$) or a probe packet ($D_{\tau} = 0$), and $\alpha > 0$ is an application-dependent quality-of-service (QoS) parameter. Note that the expectation $E_p[\cdot]$ in (4) is taken over the conditional distribution $p(\gamma | \mathbf{I}_t)$. The Lagrangian for the time-$t$ constrained optimization problem (3)-(4) is

$$\sum_{\tau=t}^{T} L_{\tau}(D_{\tau}, R_{\tau}, \lambda_{\tau} | \mathbf{I}_t)$$

for

$$L_{\tau}(D_{\tau}, R_{\tau}, \lambda_{\tau} | \mathbf{I}_t) = D_{\tau} R_{\tau} - \lambda_{\tau} (D_{\tau} E_p[\varepsilon(\gamma, R_{\tau}) | \mathbf{I}_t] - e^{-\alpha}), \quad (5)$$

where $\lambda_1, \ldots, \lambda_T$ are the Lagrange multipliers. The dual problem can then be stated as

$$\max_{(D_t, \ldots, D_T) \in \{0,1\}^{T-t+1}, \ (R_t, \ldots, R_T) \in \mathbb{R}^{T-t+1}} \sum_{\tau=t}^{T} \min_{\lambda_{\tau} \geq 0} L_{\tau}(D_{\tau}, R_{\tau}, \lambda_{\tau} | \mathbf{I}_t). \quad (6)$$
With ACK/NACK feedback, recall that $I_t = [F_1, F_2, \ldots, F_{t-1}, R_1, R_2, \ldots, R_{t-1}]$. Thus, the choice of $R_t$ affects not only the contribution to the sum-rate but also the “quality” of the conditional SNR distribution $p(\gamma \mid I_t)$ at times $\tau \geq t + 1$. As these future SNR estimates get worse, the controller is forced to choose more conservative (i.e., lower) rates in order to satisfy the expected error-rate constraint. (We justify this statement in the sequel.) Thus, the selection of $R_t$ has both short-term and long-term consequences, which may be in conflict. Consequently, the solution to the ideal rate adaptation problem under ACK/NACK feedback is a partially observable Markov decision process (POMDP) [17]. For practical horizons $T$, it is computationally impractical to implement this POMDP, as now described.

Firstly, notice that the state of the channel is continuous. Even if the channel state was discretized (at the expense of some loss in performance), the required memory to implement the optimal scheme would grow exponentially with the horizon $T$. Indeed, this POMDP lies in the space of PSPACE-complete problems, i.e., it requires both complexity and memory that grow exponentially with the horizon $T$ [18].

Next, consider the (genie-aided) case of perfect CSI, i.e., $I_t = \gamma$ for all $t$. When the channel is known, there is no need for probe packets, and thus the optimal solution chooses $D_\tau = 1 \forall \tau$. Furthermore, since the rate choice does not affect the quality of the SNR estimate, the ideal rate assignment problem decouples, so that the best choice for $R_t$ becomes

$$R_t^{\text{perf-CSI}}(\gamma) \triangleq \arg \max_{R_t \in \mathbb{R}} R_t \text{ s.t. } \varepsilon(\gamma, R_t) \leq e^{-\alpha}. \tag{7}$$

Indeed, with perfect CSI, constraint (4) is active for all $t = T_p + 1, \ldots, T$, since $\varepsilon(\gamma, R_t)$ is a convex increasing function of $R_t$ and the objective function is linear in $R_t$. Notice that, in this case, ideal rate selection is greedy and $R_t^{\text{perf-CSI}}(\gamma)$ is invariant to time $t$.

C. Practical Rate Selection

Based on the observations above, we consider a practical (non-ideal) approach, motivated by techniques from the field of adaptive control [19], which deviates from the ideal approach in two principal ways:

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2 This invariance holds as long as $\varepsilon(\cdot, \cdot)$ is $t$-invariant, i.e., the coding/modulation scheme does not change with time.
1) the probe packet locations are set at the first $T_p$ packets in each $T$-block, and
2) the controller is split into two components: a channel estimator, which produces an SNR estimate $\hat{\gamma}(I_{T_p+1})$ based on the probe-packet feedback $I_{T_p+1}$, and a rate allocator, which assigns the data packet rate based on $\hat{\gamma}(I_{T_p+1})$. (See Fig. 2)

As before, the rate allocator chooses the data-packet rates $(R_{T_p+1}, \ldots, R_T)$ in order to maximize sum-rate under an expected-error-probability constraint. In particular, at each time $t \in \{T_p + 1, \ldots, T\}$, the rate $R_t$ is chosen via:

$$\max_{(R_t, \ldots, R_T) \in \mathbb{R}^{T-t+1}} \sum_{\tau=t}^{T} R_{\tau}$$

subject to $E_p[\varepsilon(\gamma, R_{\tau}) \mid \hat{\gamma}(I_{T_p+1})] \leq e^{-\alpha}$, for all $\tau = t, \ldots, T$, (9)

where the expectation in (9) is taken over the posterior distribution $p(\gamma \mid \hat{\gamma}(I_{T_p+1}))$. While related, the constraints (4) and (9) have an important difference: the information contained by $I_t$ in (4) is summarized by the possibly incomplete statistic $\hat{\gamma}(I_{T_p+1})$ in (9). Consequently, satisfaction of (9) does not necessarily guarantee satisfaction of (4) or vice versa.

Due to the fact that $R_{t\mid t \geq T_p+1}$ does not affect the quality of future SNR estimates, the rate assignment problem (8)-(9) decouples, and the value of $R_t$ satisfying (8)-(9) reduces to

$$R^*_t \triangleq \arg \max_{R_t \in \mathcal{R}} R_t \quad \text{s.t.} \quad E_p[\varepsilon(\gamma, R_t) \mid \hat{\gamma}(I_{T_p+1})] \leq e^{-\alpha}.$$ (10)

Moreover, (10) implies that $R^*_t$ is invariant to time $t$. Note that the decoupling that occurs here is reminiscent of the decoupling that occurred with ideal rate selection (3)-(4) under perfect channel state information, i.e., (7).

In the next section, we shall see that the choice of estimator plays a key role in the overall performance of the practical rate adaptation scheme. Recall that the estimator determines $p(\gamma \mid \hat{\gamma}(I_{T_p+1}))$, which determines the expected error probability constraint. Under certain scenarios, we shall see that a solution to (10) does not exist, i.e., that no rates within $\mathcal{R}$ satisfy the expected error probability constraint.

III. RATE ADAPTATION WITH IMPERFECT CSI

Before studying the practical rate allocator (10), we first consider a particular “naive” data-rate allocator, in order to draw intuition on how estimation errors affect system performance. Given SNR estimate $\hat{\gamma}$, generated from a particular unbiased estimator, the naive allocator assigns the data rate

$$R^\text{naive}_t(\hat{\gamma}) \triangleq \arg \max_{R_t \in \mathcal{R}} R_t \quad \text{s.t.} \quad \varepsilon(\hat{\gamma}, R_t) \leq e^{-\alpha}.$$ (11)
for all \( t = T_p + 1, \ldots, T \). Due to the lack of expectation in the error-probability constraint of (11), the naive rates may violate the desired expected-error-probability constraint in (10). This follows from the fact that, when the posterior distribution \( p(\gamma_t | \hat{\gamma}_t) \) is non-atomic (i.e., \( \sigma^2_{\gamma | \hat{\gamma}} > 0 \)), Jensen’s inequality implies that

\[
E_p [\epsilon(\gamma, R_t) | \hat{\gamma}] > \epsilon(\hat{\gamma}, R_t) \quad \forall R_t.
\] (12)

Therefore, to ensure the expected-error-probability constraint in (10), the practical allocator must “back-off” the rate relative to \( R_{t \text{naive}}(\hat{\gamma}) \). To do so, it chooses \( R^*_t(\hat{\gamma}) \leq R_{t \text{naive}}(\hat{\gamma}) \), where equality occurs if and only if the estimation error \( N \triangleq \gamma - \hat{\gamma} \) is zero-valued (with probability one).

When the estimator is perfect (i.e., \( \hat{\gamma} = \gamma \)), we note that the naive rate coincides with the ideal rate under perfect CSI (i.e., \( R_{t \text{naive}}(\gamma) = R_{t \text{perf-CSI}}(\gamma | \gamma = \hat{\gamma}) \)). In this case, \( R_{t \text{naive}} \) acts as an upper bound on the ideal \( R_t \) under ACK/NACK feedback, as specified by (3)-(4). Accordingly, we make the following two definitions.

**Definition 1:** The *rate penalty* associated with estimator \( \hat{\gamma} \) is the smallest \( \delta \) (in bits/symbol) that satisfies

\[
E_p [\epsilon(\gamma, R_{t \text{naive}}(\hat{\gamma}) - \delta) | \hat{\gamma}] \leq e^{-\alpha}.
\] (13)

**Definition 2:** The *power penalty* associated with estimator \( \hat{\gamma} \) is the smallest scale factor \( \mu \) that satisfies

\[
E_p [\epsilon(\mu \gamma, R_{t \text{naive}}(\hat{\gamma})) | \hat{\gamma}] \leq e^{-\alpha}.
\] (14)

Next, we analyze two different scenarios for the described rate adaptation system. In the first scenario, the \( n \) symbols in the packet are assumed to be uncoded QAM symbols, while in the second scenario, the \( n \) symbols are a Gaussian random coded ensemble. Within the second scenario, we focus on the high-SNR and low-SNR cases separately. For both scenarios, we use the analysis presented next, in Sec. III-A.

**A. Gaussian Approximation of the Estimation Error**

Under the posterior distribution \( p(\gamma | \hat{\gamma}) \), let the estimation error \( N = \gamma - \hat{\gamma} \) have the distribution \( q(N | \hat{\gamma}) = p(N + \hat{\gamma} | \hat{\gamma}) \). Let \( g_{N|\hat{\gamma}}(r) \) and \( \Lambda_{N|\hat{\gamma}}(r) \) denote the moment generating function and the semi-invariant log moment generating function of \( N \) given \( \hat{\gamma} \), respectively. We assume that there exists some

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3 For unbiased \( \hat{\gamma} \), (12) immediately follows from Jensen’s inequality. For biased \( \hat{\gamma} \), (12) still holds but requires some effort to derive. We skip these details since our focus is on unbiased \( \hat{\gamma} \).
\( r_{\text{max}} > 0 \) such that \( \Lambda_{N|\gamma}(r) < \infty \) for all \( |r| < r_{\text{max}} \). It is well known [20] that \( \Lambda_{N|\gamma}(0) = 0, \Lambda'_{N|\gamma}(0) = E_q[N|\gamma], \) and \( \Lambda''_{N|\gamma}(0) = \sigma^2_{N|\gamma} \). Then, for any \( |r| < r_{\text{max}} \),

\[
E_q[\exp(rN) | \gamma] = g_{N|\gamma}(r) = \exp \left( \Lambda_{N|\gamma}(r) \right)
\]

(15)

for some \( r' \) between 0 and \( r \) (having the same sign as \( r \)), where (16) follows from Taylor’s theorem. From Jensen’s inequality, we know that

\[
E_q[\exp(rN) | \gamma] \geq \exp \left( E_q[N|\gamma] r \right).
\]

(17)

From Eq. (16), one can deduce that the ratio of the two sides of (17) is \( \exp \left( \frac{1}{2} \Lambda''_{N|\gamma}(r') r^2 \right) \), which is strictly greater than 1, since the log moment generating function is convex [20]. Furthermore, applying Taylor’s theorem to the third-order expansion, we get

\[
g_{N|\gamma}(r) = \exp \left( E_q[N|\gamma] r + \frac{1}{2} \sigma^2_{N|\gamma} r^2 + \frac{1}{6} \Lambda'''_{N|\gamma}(r'') r^3 \right)
\]

(18)

for some \( r'' \) between 0 and \( r \).

In many cases, the first two terms of the expansion (18) not only closely capture the variability of the estimation error \( N \), but also lead to insightful expressions to illustrate the impact of the first- and second-order statistics of “channel variability.” This will be referred to as the Gaussian approximation, since, when \( N|\gamma \) is Gaussian, the cumulants of higher order than the variance vanish. Note also that, whenever the distribution of \( N|\gamma \) is symmetric around its mean, its skewness is zero and thus \( \Lambda'''_{N|\gamma}(0) = 0 \). The Gaussian approximation works well in this case too, since the effects of the higher-order terms in the Taylor series expansion often diminish quickly (e.g., for many distributions, the cumulants decay as the order of the cumulants increase).

Recall that we focus on unbiased estimators, for which \( E_q[N|\gamma] = 0 \). In this case, the Gaussian approximation yields the simple second-order approximation

\[
\Lambda_{N|\gamma}(r) \approx \frac{1}{2} \sigma^2_{N|\gamma} r^2.
\]

(19)

The approximation (19) is asymptotically accurate for the estimator proposed in Section [V] which is asymptotically unbiased and asymptotically normal, as will be seen.

**B. Rate Adaptation with Uncoded QAM**

Here, we study the scenario in which the \( n \) symbols \( \{X_{t,k}\}_{k=1}^n \) of packet \( t \) are uncoded and selected from a QAM constellation of size \( M_t \). Since the constellation size is constant over the packet, the rate
equals $R_t = \log_2 M_t$ bits/symbol. The following is a tight approximation [1, p. 289] on the symbol error rate associated with minimum-distance decision making [21, p. 280]:

$$\varepsilon_k(\gamma, R_t) \approx 0.2 \exp \left( - \frac{3}{2} \frac{\gamma}{2R_t - 1} \right).$$  
(20)

The associated packet error rate is

$$\varepsilon(\gamma, R_t) = 1 - (1 - \varepsilon_k(\gamma, R_t))^n,$$  
(21)

since $\varepsilon_k(\gamma, R_t)$ remains constant for all $k$, as $\gamma$ and $R_t$ remain constant over the packet.

Since we can write

$$(1 - \varepsilon_k(\gamma, R_t))^n \leq 1 - n\varepsilon_k(\gamma, R_t) + \frac{1}{2} n(n-1)\varepsilon_k^2(\gamma, R_t),$$  
(22)

it follows that $\varepsilon(\gamma, R_t) > \frac{n}{2} \varepsilon_k(\gamma, R_t)$ for all $(\gamma, R_t)$ such that $\varepsilon_k(\gamma, R_t) < \frac{1}{n-1}$. Similarly, (21) implies that $\varepsilon(\gamma, R_t) < 1 - (1 - \frac{1}{n-1})^n$ for the same $(\gamma, R_t)$. This latter bound is an increasing function of $n$, and, for $n \gg 1$, it approximately equals $1 - e^{-1}$, which is much higher than typical error rates. We assume that $n$ is large enough and the possible outcomes of $(\gamma, R_t)$ are such that $\varepsilon(\gamma, R_t) > \frac{n}{2} \varepsilon_k(\gamma, R_t)$ for all $t$ with probability close to 1. We further elaborate on this assumption after deriving the condition next.

To meet the expected-error-probability constraint (10), it is necessary that

$$\frac{n}{2} \mathbb{E}_p [\varepsilon_k(\gamma, R_t) \mid \dot{\gamma}] \approx \frac{n}{2} \mathbb{E}_p \left[ 0.2 \exp \left( - \frac{3}{2} \frac{\gamma}{2R_t - 1} \right) \mid \dot{\gamma} \right] = \frac{n}{2} \mathbb{E}_q \left[ 0.2 \exp \left( - \frac{3}{2} \frac{\dot{\gamma}}{2R_t - 1} + \frac{\sigma_N^2}{\frac{\dot{\gamma}}{2R_t - 1}} \right) \mid \dot{\gamma} \right] \leq e^{-\alpha}.$$  
(23)

Using the unbiased Gaussian approximation [19], condition (23) can be rewritten as follows, after taking the natural log of both sides:

$$- \frac{3}{2} \frac{\dot{\gamma}}{2R_t - 1} + \frac{\sigma_N^2}{2} \left( \frac{3}{2} \frac{\dot{\gamma}}{2R_t - 1} \right)^2 \leq -\alpha - \ln 0.1n.$$  
(24)

Condition (24) implies that, for the existence of a feasible rate $R_t$, it is necessary that

$$\frac{\dot{\gamma}^2}{\sigma_N^2} \geq 2(\alpha + \ln 0.1n).$$  
(25)

Condition (25) implies that $\dot{\gamma}^2/\sigma_N^2$, the effective SNR of estimator $\dot{\gamma}$, must be at least $2(\alpha + \ln 0.1n)$ to guarantee an expected error rate of $e^{-\alpha}$. Using similar steps\footnote{The bound holds within approximately 1 dB from the true value for a wide range of SNRs [1, p. 289].} a sufficient condition $\dot{\gamma}^2/\sigma_N^2 \geq 2(\alpha + \ln 0.1n)$.

\footnotetext{From (21) and the fact that $(1 - \varepsilon_k(n))^n > 1 - n\varepsilon_k$, we have $\varepsilon(\gamma, R_t) \leq n\varepsilon_k(\gamma, R_t)$ for all $(t, k)$ with probability 1. Consequently, for satisfaction of (10), it is sufficient that $n\mathbb{E}_p [\varepsilon_k(\gamma, R_t) \mid \dot{\gamma}] \leq e^{-\alpha}$. Replicating (23)-(25), we obtain the sufficiency condition.}
\(2(\alpha + \ln 0.2n)\) can also be derived, illustrating the tightness of (25). We will investigate the difficulty of achieving this condition in the next section.

Given that (25) is satisfied, one can solve (24) to find the upper bound

\[
R^*_t \leq \bar{R}^*_t(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}),
\]

where

\[
\bar{R}^*_t(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}) \triangleq \log_2 \left( 1 + \frac{3}{2} \frac{\sigma^2_{N|\hat{\gamma}}}{\hat{\gamma}^2} \left( 1 - 2(\alpha + \ln 0.1n) \frac{\sigma^2_{N|\hat{\gamma}}}{\hat{\gamma}^2} \right)^{-1} \right). \tag{26}
\]

Fig. 3(a) plots the upper bound (26) as a function of the estimator’s effective SNR \(\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}\) for \(\hat{\gamma} \in \{13, 20, 25\}\) dB, a desired packet error rate of \(e^{-\alpha} = 10^{-3}\), and a packet size of \(n = 500\) symbols. The naive rate allocation

\[
R^\text{naive}_t(\hat{\gamma}) = \log_2 \left( 1 + \frac{3}{2} \frac{1}{\alpha + \ln 0.1n} \right) \tag{27}
\]

(derived from (23) with \(N = 0\)) is also shown on the same plot. The required effective SNR \(\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}\), as imposed by (25), is 21.6 here. Fig. 3(a) shows that \(R^\text{naive}_t(\hat{\gamma}) < 2\) bits/symbol for \(\hat{\gamma} \leq 13\) dB. Since 2 bits/symbol is the minimum possible rate for uncoded QAM, we conclude that it is impossible to meet the target packet-error rate of \(10^{-3}\) when \(\hat{\gamma} \leq 13\) dB, even with perfect CSI.

By definition, the rate penalty is the smallest \(\delta\) that satisfies

\[
\delta = R^\text{naive}_t(\hat{\gamma}) - R^*_t(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}).
\]

Thus, an upper bound on \(\delta\) is given by

\[
\bar{\delta}(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}) \triangleq R^\text{naive}_t(\hat{\gamma}) - \bar{R}^*_t(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}). \tag{28}
\]
From Fig. 3(a) we can see that \( \overline{\delta}(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}) \) depends on the effective SNR \( \hat{\gamma}^2 / \sigma^2_{N|\hat{\gamma}} \); it is significant when the effective SNR is near the minimum value established by (25), but shrinks as \( \hat{\gamma}^2 / \sigma^2_{N|\hat{\gamma}} \) gets large. In addition, \( \overline{\delta}(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}) \) grows in proportion to \( \hat{\gamma} \).

By definition, the power penalty is the smallest \( \mu \) that satisfies \( R^*_t(\hat{\gamma}) = R^\text{naive}_t(\hat{\gamma}/\mu) \). Thus, a lower bound \( \underline{\mu}(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}) \) on the power penalty can be found by solving \( R^*_t(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}) = R^\text{naive}_t(\hat{\gamma}/\mu) \) for \( \mu \). The power penalty lower bound \( \underline{\mu}(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}) \) is plotted in Fig. 3(b) as a function of effective SNR \( \hat{\gamma}^2 / \sigma^2_{N|\hat{\gamma}} \) for the same expected packet-error rate, \( 10^{-3} \), and packet size, \( n = 500 \), as in Fig. 3(a). The power penalty is seen to be as high as 3 dB when the effective SNR is near the minimum value established by (25), but shrinks as \( \hat{\gamma}^2 / \sigma^2_{N|\hat{\gamma}} \) gets large.

C. Rate Adaptation with Random Gaussian Ensembles

Next, we study the random coding [22], [23] scenario in which the codewords are selected from a Gaussian ensemble. Let \( R_{\max} \) be the maximum rate in \( \mathcal{R} \). Then the Gaussian ensemble consists of \( 2^{nR_{\max}} \) possible packets, where each symbol, \( X_{t,k} \), of packet \( t \) is chosen independently from a \( \mathcal{N}(0,1) \) distribution.\footnote{We use real-valued symbols, instead of complex-valued symbols, for simplicity. Consequently, the data rates will be represented in units of bits per real-symbol. For fair comparison with uncoded QAM, one should simply double these data rates.} (We use unit variance here because earlier we assumed \( E[|X_{t,k}|^2] = 1 \).) At time \( t \), say that transmission rate \( R_t \in \mathcal{R} \) is chosen. Then one packet from a size-\( 2^{nR_t} \) subset of the initially generated set of \( 2^{nR_{\max}} \) packets is chosen arbitrarily for transmission.

The receiver is assumed to know the subsets of possible packets corresponding to each admissible rate \( R_t \in \mathcal{R} \). Based on its observation of the \( t^{th} \) packet, the receiver finds the most likely packet within the subset of \( 2^{nR_t} \) possible packets. Note that, unlike the uncoded QAM scenario, where each symbol is decoded separately, here the entire packet is decoded as a unit. An upper bound for the associated decoding error probability is (e.g., [22])

\[
\varepsilon(\gamma, R_t) \leq \exp \left( n \rho \left[ R_t \ln 2 - \frac{1}{2} \ln \left( 1 + \frac{\gamma}{1 + \rho} \right) \right] \right),
\]

where \( \rho \in [0,1] \) is the union bound parameter. One can minimize (29) over \( \rho \in [0,1] \) to find the tightest bound, if so desired. To satisfy the expected-error-probability constraint (10), it suffices that there exists a \( \rho \in [0,1] \) for which

\[
\mathbb{E}_p \left[ \exp \left( n \rho \left[ R_t \ln 2 - \frac{1}{2} \ln \left( 1 + \frac{\gamma}{1 + \rho} \right) \right] \right) \right] \mid \hat{\gamma} \leq e^{-\alpha}.
\]
1) **Low-SNR Regime:** When $\Pr(\gamma \ll 1 \mid \hat{\gamma}) \approx 1$, we can write

$$
\ln \left(1 + \frac{\gamma}{1 + \rho}\right) \approx \frac{\gamma}{1 + \rho} = \frac{\hat{\gamma} + N}{1 + \rho}.
$$

For an unbiased estimator, $E_q[\hat{\gamma} + N | \frac{\gamma}{1 + \rho}] = \frac{\gamma}{1 + \rho}$ and $\text{var}_q(\hat{\gamma} + N | \frac{\gamma}{1 + \rho}) = \sigma_{N \hat{\gamma}}^2$. Thus, using the Gaussian approximation (19), the constraint (30) is satisfied if there exists a $\rho \in [0, 1]$ for which

$$
\alpha \leq -n\rho \left(R_t \ln 2 - \frac{\hat{\gamma}}{2(1 + \rho)} + \frac{1}{8} \frac{n\rho^2}{(1 + \rho)^2} \sigma_{N \hat{\gamma}}^2\right),
$$

or, equivalently, for which

$$
R_t \leq \frac{1}{\ln 2} \left(\frac{\alpha}{n\rho} + \frac{\hat{\gamma}}{2(1 + \rho)} - \frac{1}{8} \frac{n\rho^2}{(1 + \rho)^2} \sigma_{N \hat{\gamma}}^2\right).
$$

Thus, if there exists some $\rho \in [0, 1]$ for which the right side of (33) is positive, then any $R_t$ below it is feasible. For this to be possible, we need

$$
2\alpha(1 + \rho)^2 - \hat{\gamma}n\rho(1 + \rho) + \frac{1}{4}(n\rho)^2 \sigma_{N \hat{\gamma}}^2 \leq 0
$$

for some $\rho \in [0, 1]$, which leads to the following necessary condition for the estimator:

$$
\frac{\hat{\gamma}^2}{\sigma_{N \hat{\gamma}}^2} \geq 2\alpha.
$$

One can then find an upper bound on $R_t \in \mathcal{R}$ satisfying (30) as follows:

$$
\tilde{R}_t^*(\hat{\gamma}, \sigma_{N \hat{\gamma}}^2) = \max_{\rho \in [0, 1]} \frac{1}{\ln 2} \left(\frac{\alpha}{n\rho} + \frac{\hat{\gamma}}{2(1 + \rho)} - \frac{1}{8} \frac{n\rho^2}{(1 + \rho)^2} \sigma_{N \hat{\gamma}}^2\right).
$$

Likewise, one can deduce from (29) and (31) that the naive rate is

$$
R_t^\text{naive}(\hat{\gamma}) = \max_{\rho \in [0, 1]} \frac{1}{\ln 2} \left(\frac{\alpha}{n\rho} + \frac{\hat{\gamma}}{2(1 + \rho)}\right).
$$

The rate upper bound $\tilde{R}_t^*$ is plotted in Fig. 4(a) as a function of the estimator’s effective SNR $\frac{\hat{\gamma}^2}{\sigma_{N \hat{\gamma}}^2}$ for $\hat{\gamma} \in \{-3, -8, -12\}$ dB, a desired packet error rate of $e^{-\alpha} = 10^{-3}$, and a packet size of $n = 500$ symbols. The rate $R_t^\text{naive}$ from (36) is also shown on the same plot. Every point on the rate curves was computed using the optimal value of $\rho \in [0, 1]$, found numerically. We note that, with these parameters, (34) implies that $\frac{\hat{\gamma}^2}{\sigma_{N \hat{\gamma}}^2}$ must be at least 13.8. Figure 4(a) also shows that the rate penalty $\tilde{\delta}(\hat{\gamma}, \sigma_{N \hat{\gamma}}^2) = R_t^\text{naive}(\hat{\gamma}) - \tilde{R}_t^*(\hat{\gamma}, \sigma_{N \hat{\gamma}}^2)$ is significant when $\frac{\hat{\gamma}^2}{\sigma_{N \hat{\gamma}}^2}$ is near the lower bound established by (34), but that the rate penalty shrinks as $\frac{\hat{\gamma}^2}{\sigma_{N \hat{\gamma}}^2}$ increases.

Note that condition (34) is not exactly analogous to condition (25). Condition (34) is necessary for a non-empty solution set to exist for inequality (33), whereas (25) is necessary for the existence of a feasible rate that satisfies the expected-error bound. In order to derive an analogous necessary condition, one can use a sphere-packing (SP) bound for the Gaussian channel (see, e.g., [24]). With the SP lower bound, our findings would be qualitatively similar, but the derivation would be extremely tedious. For this reason, we assume that the upper bound is a good approximation for the actual error rate.
For the same target packet error rate ($10^{-3}$) and packet size ($n = 500$), Fig. 4(b) plots $\bar{R}_t^*$ versus $\hat{\gamma}$ for estimator effective SNR $\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}} \in \{60, 100\}$. In the same figure, $R_{t}^{\text{naive}}$ and the “naive” Shannon limit (i.e., ergodic capacity) $\frac{1}{2} \log_2 (1 + \hat{\gamma})$ bits/real-symbol are shown. By comparing the naive Shannon limit with $R_{t}^{\text{naive}}$, one can observe that, in the low-SNR regime, the power penalty of Gaussian signaling scheme can be significant, especially at small values of $\hat{\gamma}$. From the same plot, one can observe that the additional power penalty due to imperfect SNR estimation, $\mu(\hat{\gamma})$, is quite small: less than 0.5 dB when $\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}} = 100$ and less than 1 dB when $\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}} = 60$.

2) **High-SNR Regime:** When $\Pr(\gamma \gg 1 | \hat{\gamma}) \approx 1$, we can write

\[
\ln \left(1 + \frac{\gamma}{1 + \rho}\right) \approx \ln \left(\frac{\hat{\gamma} + N}{1 + \rho}\right) = \ln \left(\frac{\hat{\gamma}}{1 + \rho}\right) + \ln \left(1 + \frac{N}{\hat{\gamma}}\right). \tag{37}
\]

Thus, for an unbiased estimator, $E_q[\ln \frac{\gamma}{1 + \rho}|\hat{\gamma}] \approx \ln \frac{\hat{\gamma}}{1 + \rho}$ and $\text{var}_q[\ln \frac{\gamma}{1 + \rho}|\hat{\gamma}] \approx \frac{\sigma^2_{\hat{\gamma}}}{\hat{\gamma}}$. Similar to the low SNR scenario, we can use the Gaussian approximation (19) to claim that (30) is satisfied if there exists a $\rho \in [0, 1]$ for which

\[
\alpha \geq -n\rho \left(R_t \ln 2 - \frac{1}{2} \ln \left(\frac{\hat{\gamma}}{1 + \rho}\right) + \frac{1}{8} \frac{n\rho}{\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}} \right), \tag{38}
\]

or, equivalently,

\[
R_t \leq \frac{1}{\ln 2} \left(\frac{-\alpha}{n\rho} + \frac{1}{2} \ln \left(\frac{\hat{\gamma}}{1 + \rho}\right) - \frac{1}{8} \frac{n\rho}{\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}} \right). \tag{39}
\]
Hence, if there exists some \(\rho \in [0, 1]\) for which the right side of (39) is positive, then any \(R_t\) below it is feasible. In the high-SNR regime, we have \(\hat{\gamma} \gg 1\) with high probability, and thus there almost always exists some \(\rho \in [0, 1]\) for which a feasible \(R_t > 0\) exists. One can deduce from this observation that, a principal difference between the high-SNR and low-SNR regimes is that, in the high-SNR regime, the expected error probability constraint is satisfied much more easily, with nearly any SNR estimator. One can then find an upper bound on \(R_t \in \mathbb{R}\) satisfying (30) as follows:

\[
\bar{R}_t^* (\hat{\gamma}, \sigma^2_{N|\hat{\gamma}}) = \max_{\rho \in [0, 1]} \frac{1}{2} \ln \left( \frac{1}{n \rho + \frac{1}{2} \ln \left( \frac{\hat{\gamma} + \rho}{1 + \rho} \right) - \frac{1}{8} \hat{\gamma}^2 / \sigma^2_{N|\hat{\gamma}}} \right)
\]

Likewise, one can deduce from (29) and (37) that the naive rate is

\[
R_t^{naive} (\hat{\gamma}) = \max_{\rho \in [0, 1]} \frac{1}{2} \ln \left( \frac{1}{n \rho + \frac{1}{2} \ln \left( \frac{\hat{\gamma} + \rho}{1 + \rho} \right)} \right)
\]

Fig. 5. For Gaussian signaling at high SNR, rates \(\bar{R}_t^*\) and \(R_t^{naive}\) versus (a) estimator’s effective SNR \(\hat{\gamma}^2 / \sigma^2_{N|\hat{\gamma}}\) and (b) estimated SNR \(\hat{\gamma}\).

The rate upper bound \(\bar{R}_t^* (\hat{\gamma}, \sigma^2_{N|\hat{\gamma}})\) is plotted in Fig. 5(a) as a function of the estimator’s effective SNR \(\hat{\gamma}^2 / \sigma^2_{N|\hat{\gamma}}\) for \(\hat{\gamma} \in \{13, 20, 25\}\) dB, a desired packet error rate of \(e^{-\alpha}=10^{-3}\), and a packet size of \(n=500\) symbols. The rate \(R_t^{naive} (\hat{\gamma})\) from (41) is also shown on the same plot. Every point on the rate curves was computed using the optimal value of \(\rho \in [0, 1]\), found numerically. We emphasize that the rates plotted in Fig. 5(a) are expressed in bits per real-symbol, and thus should be doubled for fair comparison with the QAM rates presented in Fig. 5(a). For Gaussian signaling, if we compare the high-SNR results in Figs. 5(a)-5(b) to the low-SNR results in Figs. 4(a)-4(b), we can see that the normalized rate penalty...
\(\bar{\delta}/\bar{R}^*\) is much smaller in the high-SNR regime. For instance, at \(\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}} = 20\), \(\bar{\delta}\) is no more than 0.5 bits/symbol and \(\bar{\delta}/\bar{R}^*\) is less than 25% for all three values of \(\hat{\gamma}\). This decrease in rate penalty is expected, since, in the high-SNR regime, the rate scales roughly with the log of the SNR.

For the same target packet error rate \((10^{-3})\) and packet size \((n = 500)\), Fig. 5(b) plots \(\bar{R}^*_t(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}})\) versus \(\hat{\gamma}\) for estimator effective SNR \(\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}\in \{60, 100\}\). In the same figure, \(R^*_{t,naive}(\hat{\gamma})\) and the naive Shannon limit \(\frac{1}{2}\log_2(1+\hat{\gamma})\) are shown. There we observe that, in the high-SNR regime, the power penalty for Gaussian signaling is constant with \(\hat{\gamma}\), and no more than 1.5 dB. The additional power penalty due to imperfect SNR estimation, \(\mu(\hat{\gamma}, \sigma^2_{N|\hat{\gamma}})\), is approximately 1 dB when \(\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}} = 60\) and approximately 2.5 dB when \(\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}} = 20\).

IV. FUNDAMENTAL LIMITATIONS OF ACK/NACK-BASED RATE ADAPTATION

In the previous section, we studied the performance of the rate adaptation system for a generic unbiased estimator. We analyzed the feasible rates with particular coding/modulation schemes as a function of the “quality” of the estimation provided by the estimator, for which the relevant metric was the estimator’s effective SNR \(\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}\). Note that we assumed no knowledge of the prior SNR distribution \(p(\gamma)\).

In this section, we view the SNR of the current block, \(\gamma\), as an unknown parameter and pose the estimation of \(\gamma\) as a non-Bayesian parameter estimation problem. We first investigate the fundamental limitations of SNR estimators that are based on packet-level ACK/NACK feedback, e.g., \(\hat{\gamma} = \hat{\gamma}(I_{T_p+1})\). Using that analysis, we show that it is difficult to make good SNR estimates while simultaneously keeping packet-error-rate low. This latter property motivates SNR-estimation via probe packets that come without error-rate constraints (in contrast to data packets, which are error-rate constrained) as assumed in Sec. II.

Finally, we discuss optimization of the probing period \(T_p\), and we derive an upper bound on the optimal sum rate \(R^*_{sum}\).

A. Fundamental Limitations of ACK/NACK-Based SNR Estimation

Consider the SNR estimator \(\hat{\gamma}(I_{T_p+1})\), based on the \(T_p\) ACK/NACKs in

\[I_{T_p+1} = [F_1, F_2, \ldots, F_{T_p}, R_1, R_2, \ldots, R_{T_p}],\]

where \(R_t\) denotes the rate and \(F_t\) denotes the ACK/NACK feedback for packet \(t\). In the sequel, we abbreviate \(\hat{\gamma}(I_{T_p+1})\) by \(\hat{\gamma}\). Recall that \(R_t\) and \(F_t\) are connected through the packet error probability \(\varepsilon(\gamma, R_t)\), as specified in (2).

\(^8\)We assume that \(\gamma\) is a random variable, taking on an independent value for each block, but that the distribution of \(\gamma\) is unknown to the transmitter.
Theorem 1: For true SNR $\gamma$ and any unbiased estimator $\hat{\gamma}$ based on $T_p$ ACK/NACKs, the estimation error variance, $\sigma^2_{N|\hat{\gamma}} \triangleq \text{var}(\gamma - \hat{\gamma})$, is lower bounded by:

$$
\sigma^2_{N|\hat{\gamma}} \geq \left( \sum_{t=1}^{T_p} \frac{(\varepsilon'(\gamma, R_t))^2}{\varepsilon(\gamma, R_t) [1 - \varepsilon(\gamma, R_t)]} \right)^{-1},
$$

where $\varepsilon(\gamma, R_t)$ is continuously differentiable in $\gamma$ and $\varepsilon'(\gamma, R_t) \triangleq \frac{\partial}{\partial \gamma} \varepsilon(\gamma, R_t)$.

Proof: Given $\gamma$ and the rates $R_1, \ldots, R_{T_p}$, the feedback $F_1, \ldots, F_{T_p}$ satisfies

$$
\Pr(F_1 = f_1, \ldots, F_{T_p} = f_{T_p} | \gamma, R_1, \ldots, R_{T_p}) = \prod_{t=1}^{T_p} \Pr(F_t = f_t | \gamma, R_t). \tag{43}
$$

Then

$$
V(\gamma, R_t, f_t) \triangleq \frac{\partial}{\partial \gamma} \ln \Pr(F_t = f_t | \gamma, R_t) = \frac{\partial}{\partial \gamma} \ln \left( \frac{[\varepsilon(\gamma, R_t)]^{f_t} [1 - \varepsilon(\gamma, R_t)]^{1-f_t}}{1 - \varepsilon(\gamma, R_t)} \right) = \frac{\varepsilon'(\gamma, R_t)}{1 - \varepsilon(\gamma, R_t)} \left( \frac{f_t}{\varepsilon(\gamma, R_t)} - 1 \right). \tag{44}
$$

The Fisher information [25] associated with $F_t$ is:

$$
\Phi(\gamma, R_t) = \text{var}(V(\gamma, R_t, f_t) | \gamma, R_t) = \frac{(\varepsilon'(\gamma, R_t))^2}{\varepsilon(\gamma, R_t) [1 - \varepsilon(\gamma, R_t)]}, \tag{45}
$$

and the cumulative Fisher information is $\sum_{t=1}^{T_p} \Phi(\gamma, R_t)$. Theorem 1 follows since the Cramer-Rao lower bound (CRLB) for unbiased estimators is the reciprocal of the Fisher information [25].

B. Lower Bounds on the Required Probing Period $T_p$

In Sec. [III], we derived lower bounds (25) and (34) on the value of $\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}$ (i.e., the estimator’s effective SNR) required to facilitate the use of data transmission via uncoded QAM signaling and randomly coded Gaussian signaling, respectively. In this section, we translate those lower bounds (on required $\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}$) into lower bounds on required probe-duration $T_p$, recognizing that the quality of SNR estimates (and thus $\hat{\gamma}^2/\sigma^2_{N|\hat{\gamma}}$) increases with $T_p$. From these bounds, we shall see that the required value of $T_p$ depends heavily on the probe error rate, and in particular that the required value of $T_p$ grows very large as the probe error rate grows small. This motivates the optimization of probe error rate, which requires the decoupling of probe error rate from data error rate (since the latter is usually constrained by the application).
In this section, we assume that both the modulation/coding scheme and the rate is fixed over the probe interval, i.e., that $R_t = R_p$ for $t \in \{1, \ldots, T_p\}$. In this case, the CRLB (42) reduces to

$$\sigma^2_{N|\hat{\gamma}} \geq \frac{1}{T_p} \varepsilon(\gamma, R_p) \left[ 1 - \varepsilon(\gamma, R_p) \right],$$  

(46)

which is inversely proportional to $T_p$.

Recall that, to make uncoded QAM signaling feasible, condition (25) must be satisfied, and to make random Gaussian signaling feasible in the low-SNR regime, condition (34) must be satisfied. Though (25) and (34) are expressed in terms of the estimator’s effective SNR, we can rewrite them as

$$\sigma^2_{N|\hat{\gamma}} \leq \frac{1}{2} \hat{\gamma}^2 / (\alpha + \ln 0.1n)$$

and

$$\sigma^2_{N|\hat{\gamma}} \leq \frac{1}{2} \hat{\gamma}^2 / \alpha,$$

respectively, and apply the CRLB (46) to arrive (see Appendix A) at the following. For uncoded QAM, we need $T_p \geq T_p^{\text{min}}$, where

$$T_p^{\text{min}} = \frac{2(\alpha + \ln 0.1n) \varepsilon(\gamma, R_p)}{(1 - \varepsilon(\gamma, R_p))(1 - \varepsilon(\gamma, R_p))^{1/n} - 1} \ln^2 \left( 5 \varepsilon(\gamma, R_p) (n \hat{\gamma} / \gamma)^2 \right),$$

(47)

and for random Gaussian signaling in the low-SNR regime, we need $T_p \geq T_p^{\text{min}}$, where

$$T_p^{\text{min}} = \frac{8\alpha (1 - \varepsilon(\gamma, R_p)) (1 + \rho^* + \gamma)^2}{\varepsilon(\gamma, R_p)/(n \rho^* \hat{\gamma})^2}$$

(48)

and where $\rho^*$ is the union bound parameter corresponding to the tightest error bound (29), which itself depends on $\gamma, R_p, \text{ and } n$.

---

Fig. 6. Lower bound on required probing duration $T_p^{\text{min}}$ versus probe packet-error rate $\varepsilon(\gamma, R_p)$ for (a) uncoded QAM and (b) random Gaussian signaling.

Figures 6(a)-(b) plot $T_p^{\text{min}}$ as a function of the probe error rate $\varepsilon(\gamma, R_p)$ for uncoded QAM signaling and random Gaussian signaling, respectively. For the plots, we assume $\hat{\gamma} \approx \gamma$, which eliminates the
dependence of $T_p^{\text{min}}$ on $\hat{\gamma}$ and $\gamma$ in the QAM case; for the Gaussian case, we show $T_p$ for the values $\hat{\gamma} \in \{-3, -7, -10\}$ dB. As in our previous plots, we assumed $n = 500$ and $e^{-\alpha} = 10^{-3}$. The key observation to make from these plots is that the number of probe packets increases quickly as $\varepsilon(\gamma, R_p)$ shrinks. In fact, the plots suggest that $T_p$ is roughly proportional to $1/\varepsilon(\gamma, R_p)$. This inverse relationship is somewhat intuitive because, given a probe packet-error rate of $\varepsilon(\gamma, R_p)$, one must wait for $1/\varepsilon(\gamma, R_p)$ packets (on average) to see a single NACK. Recall, however, that Fig. 6 shows only a lower bound $T_p^{\text{min}}$ on the probe duration required for communication with positive rate; the optimal value of $T_p$ is expected to be even larger.

The main conclusion to draw from this section is that, to keep the probing period small, one must allow relatively high probe error rate $\varepsilon(\gamma, R_p)$. For systems which estimate SNR using only ACK/NACK feedback from data packets, this implies that if the data error rate $e^{-\alpha}$ is small, then the number of packets required to get a decent SNR estimate will be large. Such systems would only be suitable for channels that are very slowly fading.

C. An Upper Bound on the Optimal Sum-Rate

Recall that, in our practical rate adaptation system, the data packet rates $\{R_t\}_{t=1}^{R_T}$ are chosen based on the SNR estimated using ACK/NACKs from probe packets with rates $\{R_t\}_{t=1}^{T_p}$. To complete the system design, we must choose the rates $\{R_t\}_{t=1}^{T}$ as well as the probe duration $T_p$. In doing so, we aim to maximize the sum data rate $R_{\text{sum}} = \sum_{t=T_p+1}^{T} R_t$ while satisfying the expected error-probability constraint in (10). Intuitively, we know that increasing $T_p$ improves the SNR estimate which, in turn, allows a higher data rate (since less rate “back-off” is needed to satisfy the error constraint). On the other hand, for a fixed block length $T$, the number of data packets, $T - T_p$, shrinks as $T_p$ increases. Therefore, the choice of $T_p$ involves a tradeoff between these two objectives. In this section, we discuss the choice of $\{T_p, R_1, \ldots, R_T\}$ and derive an upper bound on the sum rate $R_{\text{sum}}$ that leverages the rate bounds from Sec. [III] and the CRLB from Sec. [IV-A].

In Sec. [II-C] we recognized that the data-rate assignment problem decouples in such a way that the optimal data rates $\{R^*_t\}_{t=1}^{T}$ become independent of time $t$. Thus, in the sequel, we focus on choosing a single data rate $R_d$, whose optimal value will be denoted by $R^*_d$. The system design problem then reduces to the following sum-rate maximization:

$$R^*_\text{sum} \triangleq \max_{T_p \leq T, (R_1, \ldots, R_T, R_d) \in R_{T_p+1}^T} (T - T_p)R_d \quad \text{s.t.} \quad \mathbb{E}_p \left[ \varepsilon(\gamma, R_d) \mid \hat{\gamma}(I_{T_p}) \right] \leq e^{-\alpha}. \quad (49)$$

As argued in Sec. [III] the optimal data rate $R^*_d$ increases monotonically with the quality of the SNR
estimate, i.e., with the inverse of the estimator variance $1/\sigma_{N|\hat{\gamma}}^2$. Thus, the optimal probe parameters $\{T_p, R_1, \ldots, R_T\}$ are those that minimize $\sigma_{N|\hat{\gamma}}^2$. From the CRLB in Theorem [1], we know that $\sigma_{N|\hat{\gamma}}^2 \geq \underline{\sigma}_{N|\hat{\gamma}}^2(\gamma)$, where

$$
\underline{\sigma}_{N|\hat{\gamma}}^2(\gamma) \triangleq \min_{(R_1, \ldots, R_T) \in \mathcal{R}} \left( \frac{\sum_{t=1}^{T_p} [\varepsilon'(\gamma, R_t)]^2}{\sum_{t=1}^{T_p} \varepsilon(\gamma, R_t)[1 - \varepsilon(\gamma, R_t)]} \right)^{-1}
$$

(50)

$$
= \left( \frac{\sum_{t=1}^{T_p} \max_{R_t \in \mathcal{R}} \left[ \frac{[\varepsilon'(\gamma, R_t)]^2}{\varepsilon(\gamma, R_t)[1 - \varepsilon(\gamma, R_t)]} \right] }{\varepsilon(\gamma, R)\bar{T}p} \right)^{-1}.
$$

(51)

Thus, if $\gamma$ was provided by a genie, and if the SNR estimator was efficient (i.e., CRLB achieving), then (51) suggests to set the probe rate at

$$
R_{p}^{\text{genie}}(\gamma) = \arg \max_{R_t \in \mathcal{R}} \frac{[\varepsilon'(\gamma, R_t)]^2}{\varepsilon(\gamma, R_t)[1 - \varepsilon(\gamma, R_t)]},
$$

(52)

which is invariant to both time $t$ and probe duration $T_p$. This yields

$$
\underline{\sigma}_{N|\hat{\gamma}}^2(\gamma) = \frac{1}{T_p} \varepsilon(\gamma, R_{p}^{\text{genie}}(\gamma))[1 - \varepsilon(\gamma, R_{p}^{\text{genie}}(\gamma))].
$$

(53)

Using the genie-aided probe rate $R_{p}^{\text{genie}}(\gamma)$, we can upper bound the optimal sum rate (49) by

$$
R_{\text{opt}}^{\ast} \triangleq \max_{T_p \leq T, R_d \in \mathcal{R}} (T - T_p)R_d \ \text{s.t.} \ \mathbb{E}_p \left[ \varepsilon(\gamma, R_d) \mid \hat{\gamma}(R_{p}^{\text{genie}}(\gamma), T_p) \right] \leq e^{-\alpha},
$$

(54)

where we explicitly denote the dependence of the estimate $\hat{\gamma}$ on both $T_p$ and $R_{p}^{\text{genie}}(\gamma)$.

Next, recall that we established, in Sec. III, upper bounds on the largest data rate that satisfies an expected error constraint of the type in (54). In particular, (26) gave an upper bound for uncoded QAM signaling, and (35) and (40) gave upper bounds for Gaussian signaling in the low-SNR and high-SNR regimes, respectively. These data-rate upper bounds, $R_d^{\ast}(\hat{\gamma}, \sigma_{N|\hat{\gamma}}^2)$, can be applied to (54) to bound the optimal sum rate as $R_{\text{sum}}^{\ast} \leq \bar{R}_{\text{sum}}^{\ast}$, where

$$
\bar{R}_{\text{sum}}^{\ast} \triangleq \max_{T_p \leq T} (T - T_p)\bar{R}_d(\hat{\gamma}, \sigma_{N|\hat{\gamma}}^2),
$$

(55)

and where $\hat{\gamma}$ and $\sigma_{N|\hat{\gamma}}^2$ are dependent on both $T_p$ and $R_p(\gamma)$. Since $\bar{R}_d^{\ast}(\hat{\gamma}, \sigma_{N|\hat{\gamma}}^2)$ increases monotonically in $1/\sigma_{N|\hat{\gamma}}^2$, we can upper bound $\bar{R}_{d}^{\ast}$ using the lower bound on $\sigma_{N|\hat{\gamma}}^2$ established in (53). This yields $\bar{R}_{\text{sum}}^{\ast} \leq R_{\text{sum}}^{\max}$ for

$$
R_{\text{sum}}^{\max} \triangleq \max_{T_p \leq T} (T - T_p)\bar{R}_d^{\ast}(\hat{\gamma}, \underline{\sigma}_{N|\hat{\gamma}}^2) = \left( \frac{T_p - 1}{T_p} \right) R_{d}^{\ast}(\gamma, \sigma_{N|\hat{\gamma}}^2).
$$

(56)

Figures 7(a) and 7(b) plot the normalized sum-rate bound $\frac{1}{T}R_{\text{sum}}^{\max}$ as a as a function of the estimated SNR $\hat{\gamma}$ for uncoded QAM and Gaussian ensembles, respectively, at $T = 5$ and $T = 50$. As before, we
Fig. 7. Normalized sum-rate bound $\frac{1}{T} R_{\text{sum}}^{\max}$ as a function of SNR $\hat{\gamma}$ for (a) QAM and (b) Gaussian signaling in the low-SNR regime.

use target error rate $10^{-3}$ and packet size $n = 500$. For the genie-aided probe rate $R_p^{\text{genie}}(\gamma)$ used to calculate $\mathcal{E}_N^{2|\hat{\gamma}}(\gamma)$, we assumed that $\gamma \approx \hat{\gamma}$. The figures also show $R_d^{\text{naive}}$ and the naive Shannon limit $\frac{1}{T} \log_2 (1 + \hat{\gamma})$, for comparison. Note that the difference between the naive rate $R_d^{\text{naive}}$ and the upper bound $\frac{1}{T} R_{\text{sum}}^{\max}$ increases significantly as $T$ decreases. This is due to the fact that, as $T$ decreases, it is too costly to allocate a long probing interval, implying that the quality of SNR estimates decreases, so that more rate back-off is required. Note also that the difference between the naive rate and the upper bound increases as the SNR increases. This implies that the lack of perfect CSI becomes more costly as the SNR increases.

V. AN ASYMPTOTICALLY OPTIMAL SNR ESTIMATOR

The quality of SNR estimates based on ACK/NACKs from a probe interval is strongly dependent on both the probe rates $\{R_t\}_{t=1}^{T_p}$ and the probe interval $T_p$. For the sum-rate upper bound derived in Sec. IV-C, the probe rate $R_p^{\text{genie}}(\gamma)$ in (52) was selected in a genie-aided manner, assuming knowledge of the true SNR $\gamma$. Clearly, $\gamma$ is not known in practice.

In this section, we develop a practical SNR estimator that, during the probing interval $t \in \{1, \ldots, T_p\}$, recursively updates the probe rate $R_t$ and $\hat{\gamma}_t$ (i.e., the time-$t$ estimate of $\gamma$) using the latest feedback pair.

We emphasize that $\hat{\gamma}_t$ is the time-$t$ estimate of the time-invariant SNR $\gamma$, and should not be confused with the time-varying SNR $\gamma_t$ that was briefly used in Sec. II before the time-invariance assumption was introduced.
\{F_{t-1}, R_{t-1}\}. We show that the probe rate adaptation is \textit{asymptotically optimal}, in that \(R_t\) converges to \(R_{p,\text{genie}}(\gamma)\) for any initial probe rate \(R_1\). Moreover, we show that our SNR estimator is \textit{asymptotically efficient} and \textit{asymptotically normal}, i.e., that the corresponding estimation error \(N_t \triangleq \hat{\gamma}_t - \gamma\) converges to a zero-mean Gaussian random variable whose variance is identical to the CRLB achieved with the genie-aided probe rate \(R_{p,\text{genie}}(\gamma)\). The normality of the error helps to justify the Gaussian approximation used to derive the rate bounds (47) and (48) for the uncoded QAM and Gaussian cases, respectively.

\textbf{The SNR Estimator:}

1) At time \(t = 1\), choose an arbitrary rate \(R_1 \in \mathcal{R}\) and an arbitrary estimate \(\hat{\gamma}_1\).

2) At each time \(t = 2, \ldots, T_p\), update the estimate as

\[
\hat{\gamma}_t = \hat{\gamma}_{t-1} + \frac{F_{t-1} - \varepsilon(\hat{\gamma}_{t-1}, R_{t-1})}{(t-1)\varepsilon'(\hat{\gamma}_{t-1}, R_{t-1})},
\]

and choose the rate \(R_t\) as:

\[
R_t = \arg\max_{R \in \mathcal{R}} \Phi(\hat{\gamma}_t, R),
\]

where \(\Phi(\cdot, \cdot)\) is the Fisher information as defined in (45).

We prove the following for our estimator.

\textbf{Theorem 2:} For both uncoded QAM and Gaussian ensembles, as \(T_p \to \infty\),

\[
\sqrt{T_p} (\hat{\gamma}_{T_p} - \gamma) \xrightarrow{d} N_{T_p} \sim N\left(0, \Phi^{-1}(\gamma, R_{p,\text{genie}}(\gamma))\right).
\]

\textbf{Proof:} See Appendix B.

Theorem 2 implies that our estimator (57) is asymptotically efficient and consistent. Moreover, without any prior information on \(\gamma\), rate allocation (58) guarantees the performance achieved with the genie-aided probe rate \(R_{p,\text{genie}}(\gamma)\). Next, we simulate the estimator. Instead of the \(t-1\) on the denominator, we use \((t-1)^\beta\) for various values of \(\beta \in (0,1]\).

In Fig. 8 a single realization of the estimator and the corresponding assigned rate are illustrated for different values of \(\gamma\), over a block of \(T_p = 500\) probe packets of size \(n = 500\) symbols. The value of \(\gamma\) and the asymptotic rate \(R_{p,\text{genie}}(\gamma)\) are also shown on the associated graphs. The initial points for the estimator are \(\hat{\gamma}_1 = 3\) dB, \(R_1 = 1\) bit/symbol, and the set of possible rates are \(\mathcal{R} = \{1, 2, \ldots, 10\}\) in bits/complex-symbol, i.e., the possible constellation sizes are integer powers of 2. For \(\beta = 0.5\), one can observe that the optimal rate is reached with approximately 20 probe packets for all values of SNR. Once that point is reached, the estimation error variance decays fairly slowly due to the low decay rate \(\beta = 0.5\). With a higher \(\beta\), it takes longer to approach the vicinity of \(\gamma\), from the initial value \(\hat{\gamma}_1\), but the estimation error variance is lower once in steady state. This observation is illustrated in Fig. 8(c), where
Fig. 8. Examples trajectories of the recursive SNR estimator when uncoded QAM is used.

\(\beta = 1\) and the probing block size is \(T_p = 2000\) packets. In the realization corresponding to \(\gamma = 20\) dB, the “steady state” is yet to be reached after 2000 packets. On the other hand, the amplitude of the fluctuations around the final point decay much faster, as one can observe in the realization corresponding to \(\gamma = 10\) dB. Different choices for \(\beta\) and the associated tradeoffs involved in stochastic approximation algorithms are studied in [26].

Fig. 9. Examples trajectories of the recursive SNR estimator when Gaussian signaling is used.

We illustrate our estimator response for Gaussian ensembles in Fig. 9. As the set of rates \(\mathcal{R}\), we picked 100 points, equally spaced between 0 and 5 bits/real-symbol. The initial SNR estimate, \(\hat{\gamma}_1 = 0\) dB, was much smaller than the initial one in the QAM simulations, but the initial rate, \(R_1 = 0.5\) bits/complex-symbol, was identical to the one in the QAM simulations. Here, we analyze SNR realizations \(\gamma = 3, 10\) and 20 dB. With Gaussian ensembles, the convergence speed is slightly lower than that with QAM. While the convergence is almost immediate for \(\gamma = 3\) dB, it takes 30-40 packets for 10 dB and 130-140 packets for \(\gamma = 20\) dB. This difference is mainly due to the difference in the distances between the initial and
final points. On the other hand, due to the large size of the set of possible rates (unlike QAM, where only a few discrete points are possible), there exists some $R_t \in \mathcal{R}$ that is very close to the genie-aided probe rate $R_{p,\text{genie}}(\gamma)$. Consequently, the estimation error variance decays much faster once $R_t$ comes near the vicinity of $R_{p,\text{genie}}(\gamma)$. We also illustrate the estimator with $\beta = 1$ in Fig. 9(c) and one can notice the slow convergence, similar to the QAM simulations.

VI. Conclusion

In this paper, we studied rate adaptation based on ACK/NACK feedback. In particular, we studied methods that maximize data rate subject to a constraint on expected packet-error probability, assuming that the transmitter has no knowledge of the SNR distribution. Because optimal rate allocation was identified as a POMDP, which is impractical to implement, we focused on a suboptimal framework where a channel estimate is calculated based on previous feedback and a rate is chosen based on this channel estimate. To aid the initial rate allocation, we allowed the use of $T_p$ probe packets at the start of each data block. First we considered a so-called “naive” rate allocator that maximizes rate subject to a constraint on instantaneous packet-error probability, calculated from a given unbiased estimate $\hat{\gamma}$ of the true SNR $\gamma$. Due to the inevitable error in SNR estimation, we argued that one must either back-off the naive rate, or correspondingly increase the SNR, to meet the stricter expected error probability constraint. Based on a Gaussian approximation of the estimation error $N = \gamma - \hat{\gamma}$, we derived conditions on the “effective estimator SNR” $\hat{\gamma}^2/\sigma^2_{N|\gamma}$ that are necessary for the existence of a feasible transmission rate, as well as an upper bound on the transmission rate when this necessary condition is satisfied. This latter analysis was carried out for both uncoded QAM signaling and random Gaussian signaling (the latter in both the low-SNR and high-SNR regimes). Next, we considered unbiased SNR estimation via ACK/NACK feedback. First, we lower bounded the error variance of those estimates (for general signaling schemes), and based on that bound, we lower bounded the necessary probing duration $T_p$ and upper bounded the sum data rate (for both uncoded QAM signaling and random Gaussian signaling). Finally, we proposed a practical unbiased ACK/NACK-based SNR estimator and showed that (as the probe duration increases) our estimator is asymptotically efficient and asymptotically normal.

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APPENDIX A

DERIVATION OF $T_p^{\text{min}}$ FOR UNCODED QAM AND GAUSSIAN SIGNALLING

In this section, we derive (47) and (48). For brevity, we write $\varepsilon \triangleq \varepsilon_p(\gamma, R_p)$ and $\varepsilon' \triangleq \varepsilon'_p(\gamma, R_p)$. Recall that, from (25) and (46), we have for uncoded QAM,

$$T_p^{\text{min}} = \frac{\varepsilon (1 - \varepsilon)}{(\varepsilon')^2} \frac{2(\alpha + \ln 0.1n)}{\hat{\gamma}^2}$$

where, from (21),

$$\varepsilon' = \frac{\partial}{\partial \gamma} \left[ 1 - \left[ 1 - 0.2 \exp \left( -\frac{1.5\gamma}{2R_p} - 1 \right) \right]^{\frac{1}{1-\varepsilon}}} \right]^{\frac{1}{n}}$$

$$= n(1 - \varepsilon)^{\frac{1}{n}-1} \frac{0.2}{1-(1-\varepsilon)^{1/n}} \left[ 1 - 0.2 \exp \left( -\frac{1.5\gamma}{2R_p} - 1 \right) \right] \frac{1.5}{2R_p - 1}$$

$$= (1 - \varepsilon) \left( (1 - \varepsilon)^{-1/n} - 1 \right) \frac{1.5\gamma}{2R_p - 1} \frac{n}{\ln(5(1-(1-\varepsilon)^{1/n}))}.$$

Thus

$$T_p^{\text{min}} = \frac{\varepsilon}{1 - \varepsilon} \left( \frac{\varepsilon'}{(\varepsilon')^2} \right) \frac{2(\alpha + \ln 0.1n)}{\hat{\gamma}^2}$$

$$= \frac{\varepsilon}{(1 - \varepsilon) \left( (1 - \varepsilon)^{-1/n} - 1 \right) \ln^2 \left( 5 \left( 1 - (1 - \varepsilon)^{1/n} \right) \right)} \frac{2(\alpha + \ln 0.1n)}{(n\hat{\gamma}/\gamma)^2}.$$

From (34) and (46), we have for Gaussian signaling in the low-SNR regime,

$$T_p^{\text{min}} = \frac{\varepsilon (1 - \varepsilon)}{(\varepsilon')^2} \frac{2\alpha}{\hat{\gamma}^2}$$

where, from (29),

$$\varepsilon' = \frac{\partial}{\partial \gamma} \exp \left( n\rho^* \left[ R_p \ln 2 - \frac{1}{2} \ln \left( 1 + \frac{\gamma}{1 + \rho^*} \right) \right] \right)$$

$$= \frac{-n\rho^*}{2} \frac{1}{1 + \frac{\gamma}{1 + \rho^*}} \frac{1}{1 + \rho^*} = \frac{-n\rho^*}{2(1 + \rho^* + \gamma)}.$$

Thus

$$T_p^{\text{min}} = \frac{(1 - \varepsilon) 2\alpha}{\varepsilon (\varepsilon')^2 \hat{\gamma}^2} = \frac{8\alpha (1 - \varepsilon) (1 + \rho^* + \gamma)^2}{\varepsilon (n\rho^*\hat{\gamma})^2}.$$
We will directly apply Theorem 2.1 [27, p. 223]. The necessary conditions for asymptotic normality and asymptotic efficiency to hold in our system are:

1) The expectation, $E[F_t]$, of observation $F_t$ must exist and must be bounded:

$$E[F_t] = \varepsilon(\gamma, R_t)$$
exists and is clearly bounded by 1 for all $t$.

2) The partial derivative $\left| \frac{\partial E[F_t]}{\partial \gamma} \right|$ must be jointly continuous (in $\gamma$ and $R_t$) and bounded.

For both QAM (20) and Gaussian (29) signals, $\left| \frac{\partial E[F_t]}{\partial \gamma} \right| = \left| \frac{\partial \varepsilon(\gamma, R_t)}{\partial \gamma} \right|$ is continuous and bounded for $\gamma \geq 0$ and $R_t \geq 0$.

3) The variance $\text{var}(F_t)$ of observation $F_t$ must be continuous in $\gamma$ and $R_t$.

For both QAM and Gaussian signaling, $\text{var}(F_t) = \varepsilon(\gamma, R_t)(1 - \varepsilon(\gamma, R_t))$ is continuous and bounded for $\gamma \geq 0$ and $R_t \geq 0$.

4) Fisher information $\Phi(\gamma, R_t)$ must be continuous, positive and for each $\gamma$, it must have a unique maximum in $R_t$.

For both QAM and Gaussian signaling, the Fisher information $\Phi(\gamma, R_t)$ as given in (45) is continuous and positive for $\gamma \geq 0$ and $R_t \geq 0$. Moreover, it has a unique maximum $R_t = R_{p\text{genie}}^\gamma(\gamma)$ for each $\gamma \geq 0$, since $\Phi(\gamma, R_t)$ is a strictly concave and continuous function of $R_t$.

5) For some $b > 2$, $E[|F_t|^b]$ must be bounded for all possible values of $\gamma$ and associated rate $R_{p\text{genie}}^\gamma(\gamma)$.

Since $F_t \in \{0, 1\}$, we know that $E[|F_t|^b]$ is bounded for all $b > 2$ and for all values of $(\gamma, R_{p\text{genie}}^\gamma(\gamma))$.

Furthermore, the asymptotic efficiency [27, p. 186,224] of the estimator is

$$\Phi(\gamma, R_t) \cdot \left. \left( \frac{\partial E[F_t]}{\var(F_t)} \right)^2 \right|_{R_t = R_{p\text{genie}}^\gamma} = \Phi(\gamma, R_{p\text{genie}}^\gamma(\gamma)) \cdot \left( \frac{\varepsilon'(\gamma, R_{p\text{genie}}^\gamma(\gamma))}{\varepsilon(\gamma, R_{p\text{genie}}^\gamma(\gamma))(1 - \varepsilon(\gamma, R_{p\text{genie}}^\gamma(\gamma)))} \right)^2$$

$$= 1.$$ 

The asymptotic optimality, i.e., $T_p \sigma^2_{X_t|\gamma_p} \rightarrow \left[ \Phi(\gamma, R_{p\text{genie}}^\gamma(\gamma)) \right]^{-1}$ as $T_p \rightarrow \infty$ follows as a consequence of Theorem 2.1 [27, p. 223].