HIDDEN SUB-HYPERGROUP PROBLEM

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Abstract. The Hidden Subgroup Problem is used in many quantum algorithms such as Simon’s algorithm and Shor’s factoring and discrete log algorithms. A polynomial time solution is known in case of abelian groups, and normal subgroups of arbitrary finite groups. The general case is still open. An efficient solution of the problem for symmetric group $S_n$ would give rise to an efficient quantum algorithm for Graph Isomorphism Problem. We formulate a hidden sub-hypergroup problem for finite hypergroups and solve it for finite commutative hypergroups. The given algorithm is efficient if the corresponding QFT could be calculated efficiently.

1. Background

Peter Shor in his seminal paper presented efficient quantum algorithms for computing integer factorizations and discrete logarithms. These algorithms are based on an efficient solution to the hidden subgroup problem (HSP) for certain abelian groups. HSP was already appeared in Simon’s algorithm implicitly in form of distinguishing the trivial subgroup from a subgroup of order $2$ of $\mathbb{Z}_{2^n}$.

The efficient algorithm for the abelian HSP uses the Fourier transform. Other methods have been applied by Mosca and Ekert [12]. The fastest currently known (quantum) algorithm for computing the Fourier transform over abelian groups was given by Hales and Hallgren [7]. Kitaev [10] has shown us how to efficiently compute the Fourier transform over any abelian group (see also [9]).

For general groups, Ettinger, Hoyer and Knill [5] have shown that the HSP has polynomial query complexity, giving an algorithm that makes an exponential number of measurements. Several specific non-abelian HSP have been studied by Ettinger and Hoyer [4], Rotteler and Beth [15], and Puschel,

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Rotteler, and Beth [14]. Ivanyos, Mangniez, and Santha [9] have shown how to reduce certain non-abelian HSP’s to an abelian HSP. The non-abelian HSP for normal subgroups is solved by Hallgren, Russell, and Ta-Shma [8].

As for the Graph Isomorphism Problem (GIP), which is a special case of HSP for the symmetric group $S_n$, Grigni, Schulman, Vazirani and Vazirani [6] have independently shown that measuring representations is not enough for solving GIP. However, they show that the problem can be solved when the intersection of the normalizers of all subgroups of $G$ is large. Similar negative results are obtained by Ettinger and Hoyer [4]. At the positive side, Beals [3] showed how to efficiently compute the Fourier transform over the symmetric group $S_n$ (see also [11]).

**Definition 1.1. (Hidden Subgroup Problem (HSP)).** Given an efficiently computable function $f : G \rightarrow S$, from a finite group $G$ to a finite set $S$, that is constant on (left) cosets of some subgroup $H$ and takes distinct values on distinct cosets, determine the subgroup $H$.

An efficient quantum algorithms for abelian groups is as follows.

**Algorithm 1.2. (abelian HSP).**

1. Prepare the state

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g|f(g)$$

and measure the second register, the resulting state is

$$\frac{1}{\sqrt{|H|}} \sum_{h \in H} |ch|f(ch)$$

where $c$ is an element of $G$ selected uniformly at random.

2. Compute the Fourier transform of the "coset" state above, resulting in

$$\frac{1}{\sqrt{|H|(|G|)}} \sum_{\rho \in \hat{G}} \sum_{h \in H} \rho(ch)|\rho(f(ch))$$

where $\hat{G}$ denotes the Pontryagin dual of $G$, namely the set of homomorphisms $\rho : G \rightarrow \mathbb{C}$.

3. Measure the first register and observe a homomorphism $\rho$. 
Note that the resulting distribution over $\rho$ is independent of the coset $cH$ arising after the first stage, as the support of the first register in (1). Thus, repetitions of this experiment result in the same distribution over $\hat{G}$. Also by the principle of delayed measurement, measuring the second register in the first step can in fact be delayed until the end of the experiment.

**Algorithm 1.3.** (non-abelian HSP, normal case) 1. Prepare the state $\sum_{g \in G} |g\rangle|f(g)\rangle$ and measure the second register $|f(g)\rangle$. The resulting state is $\sum_{h \in H} |ch\rangle|f(ch)\rangle$ where $c$ is an element of $G$ selected uniformly at random. As above, this state is supported on a left coset $cH$ of $H$.

2. Let $\hat{G}$ denote the set of irreducible representations of $G$ and, for each $\rho \in \hat{G}$, fix a basis for the space on which $\rho$ acts. Let $d_\rho$ denote the dimension of $\rho$. Compute the Fourier transform of the coset state, resulting in

$$\sum_{\rho \in \hat{G}} \sum_{1 \leq i,j \leq d_\rho} \frac{\sqrt{d_\rho}}{\sqrt{|H||G|}} \sum_{h \in H} \rho(ch)|\rho, i,j\rangle|f(ch)\rangle$$

3. Measure the first register and observe a representation $\rho$.

As before, one wishes the resulting distribution to be independent of the actual coset $cH$ and depend only on the subgroup $H$. This is guaranteed by measuring only the name of the representation $\rho$ and leaving the matrix indices unobserved. The fact that $O(\log(|G|))$ samples of this distribution are enough to determine $H$ with high probability is proved in [8].

2. Hypergroup Representations

A finite hypergroup is a set $K = \{c_0, c_1, \ldots, c_n\}$ together with a *-algebra structure on the complex vector space $\mathbb{C}K$ spanned by $K$ which satisfies the following axioms. The product of elements is given by the structure equations

$$c_i \ast c_j = \sum_k n_{i,j}^k c_k,$$

with the convention that summations always range over $\{0,1,\ldots,n\}$. The axioms are

1. $n_{i,j}^k \in \mathbb{R}$ and $n_{i,j}^k \geq 0$,

2. $\sum_k n_{i,j}^k = 1$,

3. $c_0 \ast c_i = c_i \ast c_0 = c_i$, 


(4) $K^* = K$, $n^0_{i,j} \neq 0$ if and only if $c_i^* = c_j$,
for each $0 \leq i, j, k \leq n$.

If $c_i^* = c_i$, for each $i$, then the hypergroup is called hermitian. If $c_i * c_j = c_j * c_i$, for each $i, j$, then the hypergroup is called commutative. Hermitian hypergroups are automatically commutative.

In harmonic analysis terminology, we have a convolution structure on the measure algebra $M(K)$. This means that we can convolve finitely additive measures on $K$ and, for $x, y \in K$, the convolution $\delta_x * \delta_y$ is a probability measure. Indeed $\delta_{c_i} * \delta_{c_j} \{c_k\} = n^k_{i,j}$. We follow the convention of harmonic analysis texts and denote the involution by $x \mapsto \bar{x}$ (instead of $x^*$), and the identity element by $e$ (instead of $c_0$). For a function $f : K \to \mathbb{C}$, and sets $A, B \subseteq K$ we put

$$f(x * y) = \sum_{z \in K} f(z)(\delta_x * \delta_y)\{z\}, \quad (x, y \in K),$$

and

$$A * B = \cup\{\text{supp}(\delta_x * \delta_y) : x \in A, y \in B\}.$$ 

A finite hypergroup $K$ always has a left Haar measure (positive, left translation invariant, finitely additive measure) $\omega = \omega_K$ given by

$$\omega\{x\} = \left(\{\delta_x * \delta_e\}\{e\}\right)^{-1} \quad (x \in K).$$

A function $\rho : K \to \mathbb{C}$ is called a character if $\rho(e) = 1, \rho(x * y) = \rho(x)\rho(y)$, and $\rho(\bar{x}) = \overline{\rho(x)}$. In contrast with the group case, characters are not necessarily constant on conjugacy classes. Let $K$ be a finite commutative hypergroup, then $\hat{K}$ denotes the set of characters on $K$. In this case, for $\mu \in M(K)$ and $f \in \ell^2(K)$, we put

$$\hat{\mu}(\rho) = \sum_{x \in K} \rho(x)\mu\{x\}, \quad \hat{f}(\rho) = \sum_{x \in K} f(x)\rho(x)\omega\{x\} \quad (\rho \in \hat{K}).$$

Hence $\hat{f} = (f\omega)^*$. If $H \subseteq K$ is a subhypergroup (i.e. $H = H$ and $H * H \subseteq H$), then $\hat{\omega}_H = \chi_{H^\perp}$ [2, 2.1.8], where the right hand side is the indicator (characteristic) function of

$$H^\perp = \{\rho \in \hat{K} : \rho(x) = 1 \ (x \in H)\}.$$ 

If $K/H$ is the coset hypergroup (which is the same as the double coset hypergroup $K//H$ in finite case [2, 1.5.7]) with hypergroup epimorphism (quotient map) $q : K \to H/K$ [2, 1.5.22], then $(K/H)^* \simeq H^\perp$.
(with isomorphism map \( \chi \mapsto \chi \circ q \)) \cite[2.2.26, 2.4.8]{2}. Moreover, for each \( \mu \in M(K) \), \( q(\mu \ast \omega_H) = q(\mu) \) \cite[1.5.12]{2}. We say that \( K \) is strong if \( \hat{K} \) is a hypergroup with respect to some convolution satisfying

\[
(\rho \ast \sigma)^\ast = \rho^\ast \sigma^\ast \quad (\rho, \sigma \in \hat{K}),
\]

where

\[
\hat{k}(x) = \sum_{\rho \in \hat{K}} k(\rho) \rho(x) \pi\{\rho\} \quad (x \in K, k \in \ell^2(\hat{K}, \pi))
\]

is the inverse Fourier transform. In this case, for \( \rho, \sigma \in \hat{K} \), we have \( \rho \in \sigma \ast H^\perp \) if and only if \( \text{Res}_H \rho = \text{Res}_H \sigma \), where \( \text{Res}_H : \hat{K} \to \hat{H} \) is the restriction map \cite[2.4.15]{2}. Also \( H \) is strong and \( \hat{K}/H^\perp \simeq \hat{H} \) \cite[2.4.16]{2}. Moreover \( (\hat{K})^\ast \simeq K \) \cite[2.4.18]{2}.

Let us quote the following theorem from \cite[2.2.13]{2} which is the cornerstone of the Fourier analysis on commutative hypergroups.

**Theorem 2.1. (Levitan)** If \( K \) is a finite commutative hypergroup with Haar measure \( \omega \), there is a positive measure \( \pi \) on \( \hat{K} \) (called the Plancherel measure) such that

\[
\sum_{x \in K} |f(x)|^2 \omega\{x\} = \sum_{\rho \in \hat{K}} |\hat{f}(\rho)|^2 \pi\{\rho\} \quad (f \in \ell^2(K, \omega)).
\]

Moreover \( \text{supp}(\pi) = \hat{K} \) and \( \pi\{\rho\} = \pi\{\bar{\rho}\} \). In particular the Fourier transform \( \mathcal{F} \) is a unitary map from \( \ell^2(K, \omega) \) onto \( \ell^2(\hat{K}, \pi) \).

In quantum computation notation,

\[
\mathcal{F} : |x\rangle \mapsto \frac{1}{\tau(x)} \sum_{\rho \in \hat{K}} \rho(x) \pi\{\rho\} |\rho\rangle,
\]

where

\[
\tau(x) = \left( \sum_{\rho \in \hat{K}} |\rho(x)|^2 \pi^2\{\rho\} \right)^{\frac{1}{2}} \quad (x \in K).
\]

When \( K \) is a group, \( \tau(x) = |\hat{K}|^{\frac{1}{2}} \), for each \( x \in K \). It is essential for quantum computation purposes to associate a unitary matrix to each quantum gate. however, if we write the matrix of \( \mathcal{F} \) naively using the above formula we don’t get a unitary matrix. The reason is that, in contrast with the group case, the discrete measures on \( \ell^2 \) spaces are not counting measure. More specifically, when \( K \) is a group, \( \ell^2(K) = \bigoplus_{x \in K} \mathbb{C} \), where as here \( \ell^2(K, \omega) = \bigoplus_{x \in K} \omega\{x\}^{\frac{1}{2}} \mathbb{C} \) and \( \ell^2(K) = \bigoplus_{\rho \in \hat{K}} \pi\{\rho\}^{\frac{1}{2}} \mathbb{C} \). The exponent
½ is needed to get the same inner product on both sides. If we use change of bases $|x\rangle = \omega\{x\} \frac{1}{\sqrt{2}} |x\rangle$ and $|\rho\rangle = \pi\{|\rho\rangle\}$, the Fourier transform can be written as

$$\tilde{\mathcal{F}}: |x\rangle \mapsto \omega\{x\} \sum_{\rho \in \hat{K}} \rho(x)\pi\{|\rho\rangle\} ,$$

and the corresponding matrix turns out to be unitary.

There are not many finite hypergroups whose character table is known [Wil]. Here we give two classical examples (of order two and three and compute the corresponding Fourier matrix.

**Example 2.2** (Ross). The general form of an hypergroup of order 2 is known. It is denoted by $K = \mathbb{Z}_2(\theta)$ and consists of two elements 0 and 1 with multiplication table

|   | $\delta_0$ | $\delta_1$ |
|---|------------|------------|
| $\delta_0$ | $\delta_0$ | $\delta_1$ |
| $\delta_1$ | $\theta\delta_0 + (1 - \theta)\delta_1$ | $\delta_1$ |

and Haar measure and character table

|   | $\chi_0$ | $\chi_1$ |
|---|---------|---------|
| $\omega$ | 0 | 1 |
| $\chi_0$ | 1 | $\frac{1}{\sqrt{2}}$ |
| $\chi_1$ | 1 | $\theta$ |

When $\theta = 1$ we get $K = \mathbb{Z}_2$. The dual hypergroup is again $\mathbb{Z}_2(\theta)$ with the plancherel measure

|   | $\chi_0$ | $\chi_1$ |
|---|---------|---------|
| $\pi$ | $\theta$ | $\frac{1}{1 + \theta}$ |
| $\frac{1}{1 + \theta}$ | $\chi_1$ |

The unitary matrix of the corresponding Fourier transform is given by

$$\tilde{\mathcal{F}}_2 = \frac{1}{\sqrt{1 + \theta^2}} \begin{pmatrix} \theta & 1 \\ 1 & -\theta \end{pmatrix}$$

**Example 2.3** (Wildberger). The general form of hypergroups of order 3 is also known. We know that it is always commutative, but in this case, the Hermitian and non Hermitian case should be treated
separately. Let \( K = \{0, 1, 2\} \) be a Hermitian hypergroup of order three and put \( \omega_i = \omega \{i\} \), for \( i = 0, 1, 2 \).

Then the multiplication table of \( K \) is

| \( \times \) | \( \delta_0 \) | \( \delta_1 \) | \( \delta_2 \) |
|----------------|----------------|----------------|
| \( \delta_0 \) | \( \delta_0 \) | \( \delta_1 \) | \( \delta_2 \) |
| \( \delta_1 \) | \( \frac{1}{\omega_1} \delta_0 + \alpha_1 \delta_1 + \beta_1 \delta_2 \) | \( \gamma_1 \delta_1 + \gamma_2 \delta_2 \) |
| \( \delta_2 \) | \( \gamma_1 \delta_1 + \gamma_2 \delta_2 \) | \( \frac{1}{\omega_2} \delta_0 + \beta_2 \delta_1 + \alpha_2 \delta_2 \) |

where

\[
\beta_1 = \frac{\gamma_1 \omega_2}{\omega_1}, \quad \beta_2 = \frac{\gamma_2 \omega_1}{\omega_2}, \quad \alpha_1 = 1 - \frac{1 + \gamma_1 \omega_2}{\omega_1}, \quad \alpha_2 = 1 - \frac{1 + \gamma_2 \omega_1}{\omega_2}, \quad \gamma_2 = 1 - \gamma_1,
\]

and \( \gamma_1, \omega_1 \) and \( \omega_2 \) are arbitrary parameters subject to conditions \( 0 \leq \gamma_1 \leq 1, \omega_1 \geq 1, \omega_2 \geq 1 \), and

\[
1 + \gamma_1 \omega_2 \leq \omega_1
\]

\[
1 + (1 - \gamma_1) \omega_1 \leq \omega_2.
\]

The Plancherel measure and character table are given by

| \( \pi \) | \( 0 \) | \( 1 \) | \( 2 \) |
|----------|----------|----------|
| \( \chi_0 \) | \( \frac{\alpha_1 - \gamma_1}{\omega_1} \) | \( 1 \) | \( 1 \) | \( 1 \) |
| \( \chi_1 \) | \( \frac{\alpha_2 - \gamma_2}{\omega_1} \) | \( 1 \) | \( x \) | \( z \) |
| \( \chi_2 \) | \( \frac{\alpha_2 - \gamma_2}{\omega_2} \) | \( 1 \) | \( y \) | \( v \) |

where

\[
x = \frac{\alpha_1 - \gamma_1}{2} + \frac{D}{2 \omega_2}, \quad y = \frac{\alpha_1 - \gamma_1}{2} - \frac{D}{2 \omega_2}
\]

\[
z = \frac{\alpha_2 - \gamma_2}{2} - \frac{D}{2 \omega_2}, \quad v = \frac{\alpha_2 - \gamma_2}{2} + \frac{D}{2 \omega_2}
\]

\[
D = \sqrt{(1 + \gamma_1 \omega_2 - \gamma_2 \omega_1)^2 + 4 \gamma_2 \omega_1}
\]

and

\[
s_1 = x^2 + \frac{y^2}{\omega_2} + \frac{z^2}{\omega_1} - \left( y^2 z^2 + \frac{x^4}{\omega_2} + \frac{y^4}{\omega_1} \right)
\]

\[
s_2 = y^2 + \frac{z^2}{\omega_1} + \frac{1}{\omega_2} - \left( v^2 + \frac{y^2}{\omega_2} + \frac{1}{\omega_1} \right)
\]

\[
s_3 = z^2 + \frac{x^2}{\omega_2} + \frac{1}{\omega_1} - \left( x^2 + \frac{z^2}{\omega_1} + \frac{1}{\omega_1} \right)
\]
\[ t = x^2 v^2 + y^2 + z^2 - (x^2 + y^2 z^2 + v^2). \]

Let \( \pi_i = \pi \{ \chi_i \} = \frac{\omega_i}{4} \) and \( w_{ij} = \sqrt{\omega_i \omega_j} \), for \( i, j = 0, 1, 2 \), then the Fourier transform is given by the unitary matrix

\[
\mathcal{F}_3 = \begin{pmatrix}
 w_{00} & w_{10} & w_{20} \\
 w_{01} & x w_{11} & z w_{21} \\
 w_{02} & y w_{12} & v w_{22}
\end{pmatrix}
\]

One concrete example is the normalized Bose Mesner algebra of the square. In this case, \( \omega_1 = 1, \omega_2 = 2, \gamma_1 = \beta_1 = \alpha_1 = \alpha_2 = 0, \gamma_2 = 1 \), and \( \beta_2 = \frac{1}{2} \). A simple calculation gives \( D = 2, x = 1, y = z = -1, v = 0 \), and if we put \( \pi_1 = \frac{1}{4} \), we get \( \pi_2 = \frac{1}{4} \) and \( \pi_3 = \frac{1}{2} \). In this case, the Fourier transform matrix is

\[
\mathcal{F}_3 = \frac{1}{2} \begin{pmatrix}
 1 & 1 & \sqrt{2} \\
 1 & 1 & -\sqrt{2} \\
 \sqrt{2} & -\sqrt{2} & 0
\end{pmatrix}
\]

In the non-Hermitian case, the multiplication table of \( K \) is

\[
\begin{array}{c|ccc}
* & \delta_0 & \delta_1 & \delta_2 \\
\hline
\delta_0 & \delta_0 & \delta_1 & \delta_2 \\
\delta_1 & \gamma \delta_1 + (1-\gamma) \delta_2 & \alpha \delta_0 + \gamma \delta_1 + \gamma \delta_2 \\
\delta_2 & \alpha \delta_0 + \gamma \delta_1 + \gamma \delta_2 & (1-\gamma) \delta_1 + \gamma \delta_2
\end{array}
\]

where \( \gamma = \frac{1-\alpha}{2} \), and \( \alpha \) is an arbitrary parameter with \( 0 < \alpha \leq 1 \). When \( \alpha = 1 \), we get \( K = \mathbb{Z}_3 \). The dual hypergroup is again \( K \) and the Plancherel measure and character table are given by

\[
\begin{array}{c|ccc}
\pi & \chi_0 & \chi_1 & \chi_2 \\
\hline
\chi_0 & \frac{s_1}{4} & 1 & 1 \\
\chi_1 & \frac{s_2}{4} & 1 & \bar{z} \\
\chi_2 & \frac{s_2}{4} & 1 & z
\end{array}
\]

where

\[
 z = -\frac{\alpha \pm i \sqrt{\alpha^2 + 2\alpha}}{2}.
\]

\( s_1 = 2 - \omega_1 (\alpha^2 + \alpha), \quad s_2 = \omega_1 - 1, \quad t = \omega_1 (2 - \alpha^2 - \alpha). \)
Put \( \pi_i = \pi\{\chi_i\} \) and \( w_{ij} = \sqrt{\omega_i \pi_j} \), for \( i, j = 0, 1, 2 \), then the Fourier transform is given by the unitary matrix

\[
\mathbf{\hat{F}}_3 = \begin{pmatrix}
w_{00} & w_{10} & w_{20} \\
w_{01} & zw_{11} & zw_{21} \\
w_{02} & zw_{12} & zw_{22}
\end{pmatrix}
\]

As a concrete example, let us put \( \omega_1 = \omega_2 = 2, \gamma = \frac{1}{4} \) and \( \alpha = \frac{1}{2} \) to get \( z = \frac{-1 + i\sqrt{5}}{4} \) and \( \pi_1 = \frac{1}{5} \), \( \pi_2 = \pi_3 = \frac{2}{5} \). In this case, the Fourier transform matrix is

\[
\mathbf{\hat{F}}_3 = \frac{1}{\sqrt{5}} \begin{pmatrix}1 & \sqrt{2} & \sqrt{2} \\\sqrt{2} & -\frac{1 + i\sqrt{5}}{4} & -\frac{1 - i\sqrt{5}}{4} \\\sqrt{2} & -\frac{1 - i\sqrt{5}}{4} & -\frac{1 + i\sqrt{5}}{4} \end{pmatrix}
\]

Lemma 2.4. Let \( K \) be commutative and \( H \) be a sub-hypergroup of \( K \) and \( \rho \in \hat{K} \), then the following are equivalent.

(i) \( \rho \in H^\perp \),

(ii) \( \sum_{m \in c*H} \omega\{m\} \rho(m) \neq 0 \), for each \( c \in K \),

(iii) \( \sum_{m \in c*H} \omega\{m\} \rho(m) \neq 0 \), for some \( c \in K \).

Proof. (i) \( \Rightarrow \) (ii) If \( \rho \in H^\perp \) and \( q : K \to K/H \) is the quotient map, then given \( c \in K \), \( q(\mu * \omega_H) = q(\mu) \) for \( \mu = \delta_c \omega \in M(K) \). But clearly

\[
q(\delta_c \omega) = \delta_{c*H} \omega = \sum_{m \in c*H} \delta_m \omega.
\]

Hence \( \rho(\delta_{c*H}) \omega = \rho \circ q(\delta_c \omega) \neq 0 \), where the last equality is because \( \rho \circ q \in (K/H)^\circ \) and a character is never zero.

(iii) \( \Rightarrow \) (i) If \( \rho \notin H^\perp \) then the multiplicative map \( \rho \circ q \) should be identically zero on \( K/H \) (otherwise it is a character and \( \rho \in H^\perp \)). Hence \( \sum_{m \in c*H} \rho(m) \omega = \rho(\delta_{c*H}) \omega = 0 \), for each \( c \in K \). \( \square \)

3. HSHP

In this section we give an algorithm for solving hidden sub-hypergroup problem (HSHP) for abelian (strong) hypergroups. This algorithm is efficient for those finite commutative hypergroups whose Fourier transform is efficiently calculated. It is desirable that, following Kitaev [10], one shows that the Fourier
transform could be efficiently calculated on each finite commutative hypergroup. This could be difficult, as there is yet no complete structure theory for finite commutative hypergroups (see chapter 8 of [2]).

**Definition 3.1.** (Hidden Sub-hypergroup Problem (HSHP)). Given an efficiently computable function \( f : K \to S \), from a finite hypergroup \( K \) to a finite set \( S \), that is constant on (left) cosets of some subhypergroup \( H \) and takes distinct values \( \lambda_c \) on distinct cosets \( c \ast H \), for \( c \in K \). Determine the subhypergroup \( H \).

**Algorithm 3.2.** (abelian HSHP).

1. Prepare the state \(|\chi_0\rangle\rangle 0\rangle\).
2. Apply \( \mathbb{F}^{-1} \) to the first register to get
   \[
   \sum_{x \in K} \omega\{x\}^{1/2} |x\rangle\langle 0|.
   \]
3. Apply the black box to get
   \[
   \sum_{x \in K} \omega\{x\}^{1/2} |x\rangle\langle f(x)|,
   \]
   and measure the second register, to get
   \[
   \frac{\sqrt{|K|}}{\sqrt{|c \ast H|}} \sum_{m \in c \ast H} \omega\{m\}^{1/2} |m\rangle\langle \lambda_c|,
   \]
where \( c \) is an element of \( K \) selected uniformly at random, and \( \lambda_c \) is the value of \( f \) on the coset \( c \ast H \).

4. Apply \( \mathbb{F} \) to the first register to get
   \[
   \frac{\sqrt{|K|}}{\sqrt{|c \ast H|}} \sum_{m \in c \ast H} \sum_{\rho \in K} \omega\{m\} \pi\{\rho\}^{1/2} \rho(m)|\rho\rangle\langle \lambda_c| = \frac{\sqrt{|K|}}{\sqrt{|c \ast H|}} \sum_{\rho \in K} \pi\{\rho\}^{1/2} \left( \sum_{m \in c \ast H} \omega\{m\} \rho(m) \right)|\rho\rangle\langle \lambda_c|
   \]
   5. Measure the first register and observe a character \( \rho \).

Note that the resulting distribution over \( \rho \) is independent of the coset \( c \ast H \) arising after the first step. Also note that by Lemma 2.2, the character observed in step 3 is in \( H^\perp \).

**Theorem 3.3.** If the Fourier transform could be efficiently calculated on a finite commutative hypergroup \( K \), then the above algorithm solves HSHP for \( K \) in polynomial time.
There are a variety of examples of (commutative hypergroups) whose dual object is known. One might hope to relate the HSP on a (non-abelian) group $G$ to the HSHP on a corresponding commutative hypergroup like $\hat{G}$ (see next example). The main difficulty is to go from a function $f$ which is constant on cosets of some subgroup $H \leq G$ to a function which is constant on cosets of a subhypergroup of $\hat{G}$. The canonical candidate $\hat{f}$ fails to be constant on costs of $H^\perp \leq \hat{G}$.

We list some of the examples of commutative hypergroups and their duals, hoping that one can get such a relation in future.

**Example 3.4.** If $G$ is a finite group, then $\hat{G} := (G^G)^*$ is a commutative strong (and so Pontryagin [2, 2.4.18]) hypergroup [2, 8.1.43]. The dual hypergroups of the Dihedral group $D_n$ and the (generalized) Quaternion group $Q_n$ are calculated in [2, 8.1.46,47].

**Example 3.5.** If $G$ is a finite group and $H$ is a (not necessarily normal) subgroup of $G$ then the double coset space $G//H$ (which is basically the same as the homogeneous space $G/H$ in the finite case) is a hypergroup whose dual object is $A(\hat{G},H)$ [2, 2.2.46]. It is easy to put conditions on $H$ so that $G//H$ is commutative.

There are also a vast class of special hypergroups (see chapter 3 of [2] for details) which are mainly infinite hypergroups, but one might mimic the same constructions to get similar finite hypergroups in some cases.

**References**

[1] Robert Beals, Quantum computation of Fourier transforms over symmetric groups, in Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing, pages 48-53, El Paso, Texas, 4-6 May 1997.

[2] Walter R Bloom, Herbert Heyer, *Harmonic analysis of probability measures on hypergroups*, Walter de Gruyter, Berlin, New York, 1995.

[3] Persi Diaconis and Daniel Rockmore, Efficient computation of the Fourier transform on finite groups, J. Amer. Math. Soc. 3(2)(1990), 297-332.

[4] Mark Ettinger and Peter Hoyer, On quantum algorithms for noncommutative hidden subgroups, Advances in Applied Mathematics 25 (2000), 239-251.

[5] Mark Ettinger and Peter Hoyer and Emanuel Knill, Hidden subgroup states are almost orthogonal, Technical report, quant-ph/9901034, 1999.
[6] Michaelangelo Grigni, Leonard Schulman, Monica Vazirani and Umesh Vazirani, Quantum mechanical algorithms for the non-Abelian hidden subgroup problem, in Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing, Crete, Greece, 6-8 July 2001.

[7] Lisa Hales and Sean Hallgren, An Improved Quantum Fourier Transform Algorithm and Applications, in Proceedings of the 41st Annual Symposium on Foundations of Computer Science, pages 515-525, Redondo Beach, California, 12-14 November 2000.

[8] Sean Hallgren, Alexander Russell, and Amnon Ta-Shma, Normal subgroup reconstruction and quantum computation using group representations, in Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, pages 627-635, Portland, Oregon, 21-23 May 2000.

[9] Gabor Ivanyos, Frederic Magniez, and Miklos Santha, Efficient quantum algorithms for some instances of the non-Abelian hidden subgroup problem, in Proceedings of the Thirteenth Annual ACM Symposium on Parallel Algorithms and Architectures, pages 263-270, Heraklion, Crete Island, Greece, 4-6 July 2001.

[10] Alexi Yu. Kitaev, Quantum computations: algorithms and error correction, Russian Mathematical Surveys 52(6)(1997), 1191-1249.

[11] Johannes Kohler, Uwe Schoning, and Jacobo Toran, The graph isomorphism problem: its structural complexity, Birkhauser Boston Inc., Boston, MA, 1993.

[12] Michele Mosca and Artur Ekert, The hidden subgroup problem and eigenvalue estimation on a quantum computer, in C.P. Williams, editor, Proceedings of the 1st NASA International Conference on Quantum Computing and Quantum Communications, volume 1509 of Lecture Notes in Computer Science, Springer-Verlag, pages 174-188, 1999.

[13] Michael A. Nielsen and Isaac L. Chuang, Quantum computation and quantum information, Cambridge University Press, Cambridge, 2000.

[14] M. Puschel, M. Rotteler, and T. Beth, Fast quantum Fourier transforms for a class of non-Abelian groups, in Proc. 13th AAECC, volume 1719, LNCS, pages 148-159, 1999.

[15] M. Rotteler and T. Beth, Polynomial-time solution to the hidden subgroup problem for a class of non-abelian groups, in quant-ph/9812070

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