A collection of cycling problems in linear programming

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Abstract

This paper provides a set of cycling problems in linear programming. These problems should be useful for researchers to develop and test new simplex algorithms. As a matter of the fact, this set of problems is used to test a recently proposed double pivot simplex algorithm for linear programming.

Keywords: linear programming, degenerate solution, cycling, Dantzig’s rule, steepest edge rule.

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1 Introduction

Since Dantzig formulated linear programming (LP) problem [5] and invented the simplex method, LP has been extensively studied for more than seven decades.

There are two important classes of methods that have been developed for solving LP. The first one is the simplex method which searches optimizer along edges of the polyhedra from one vertex to the next vertex. Many different pivot rules, such as Dantzig’s most negative rule [6], the best improvement rule [14], Bland’s least index pivoting rule [3], the steepest edge simplex rule [7], Zadeh’s rule [27], among others [20], have been developed. The second class is the interior-point method which searches optimizer from the interior of the polyhedra along the central path using either line search technique [22] or arc-search technique [24]. When solving LP problems, these two methods face different challenges. For the simplex method, the cycling, due to degeneracy of several basic feasible solutions for which the update cycles in a loop and stays away from the optimal solution, can be a problem. Although the degenerate/cycling case would seem (on mathematical grounds) to be rare, this is not so in practice [8]. As a matter of fact, many benchmark problems in Netlib repository have the degeneracy/cycling issue. Therefore, several anti-cycling methods were developed. The degeneracy/cycling, however, is not a severe problem for the interior-point method because the latter does not search among vertices and its iterates in a carefully designed algorithm do not approach the boundary before they approach to the optimal solution. A challenge for interior-point method is that many LP problems do not have an interior point, but this was resolved by using infeasible interior-point algorithms.

An obvious merit of the interior-point method is that many algorithms in this class are polynomial while none of the existing algorithms in simplex method has the property. However, the computational efficiency of the simplex method has been demonstrated by many years of experience. Moreover, if one finds a polynomial simplex algorithm, it will be strongly polynomial [18], a very attractive property that interior-point algorithms do not have. To develop and test potentially better simplex algorithms, such as recently proposed double pivot algorithms [21, 24], it is desired to test them on large size benchmark problems in Netlib repository. Since many problems in Netlib repository have the cycling issue, one may want to start working on small size cycling LP problems before testing more challenging large size problems in Netlib repository. This motivated us to collect small size cycling LP problems.

Cycling is a phenomenon that the iterates move in a cycle, i.e., when a basic feasible solution is degenerate, after a few iterations using the simplex algorithm, it may return to a previously constructed tableau [29] without improvement in objective function and they may stay away from the optimal solution. Cycling happens when several conditions are met. First, the problem has degenerate basic feasible solutions; the initial basic feasible solution is also a fact to determine if the cycling will happen; moreover, cycling

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1We realize that there exist several systematic methods to create simple cycling LP problems [12, 28], but significant efforts are required and we try to avoid repeating the efforts.

2Having some degenerate basic feasible solutions is necessary for a linear programming problem to cycle in iterations, but it is not sufficient for cycling to occur.
for one pivot rule does not mean that it will happen for other pivot rules; finally, when entering or leaving variables have a tie, different ways to break the tie also affect the cycling occurrence. Problems listed in this paper have a cycling behavior for either Dantzig’s pivot rule [6] or steepest edge rule [11]. We will also indicate which method is used to break the ties for cycling to happen.

The remainder of the paper is organized as follows. Section 2 provides the cycling problems for Dantzig’s pivot rule. Section 3 is the collection of the cycling problems for steepest edge rule. Section 4 discusses the test result for a recently developed double pivot simplex algorithm against the cycling problems presented in this paper. Conclusions are summarized in the last section.

2 Cycling problems for Dantzig’s pivot rule

We divide problems in this group by either using the least index rule or the largest pivot rule to break ties in leaving variables.

2.1 Cycling problems for the least index rule

Problem 1 The first cycling example was given in [9, 10, 13] by Hoffman as follows:

\[
\begin{align*}
\text{min} & \quad -2.2361x_4 + 2x_5 + 4x_7 + 3.6180x_8 + 3.236x_9 + 3.6180x_{10} + 0.764x_{11} \\
\text{s.t.} & \quad x_1 = 1 \\
& \quad x_2 + 0.3090x_4 - 0.6180x_5 - 0.8090x_6 - 0.3820x_7 \\
& \quad + 0.8090x_8 + 0.3820x_9 + 0.3090x_{10} + 0.6180x_{11} = 0 \\
& \quad x_3 + 1.4635x_4 + 0.3090x_5 + 1.4635x_6 - 0.8090x_7 \\
& \quad - 0.9045x_8 - 0.8090x_9 + 0.4635x_{10} + 0.309x_{11} = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 11.
\end{align*}
\]

The optimal solution is \(x_1 = 1\) and \(x_j = 0\) (\(j = 2 \ldots 11\)) with \(\text{obj} = 0\).

Problem 2 The second cycling example was given in [2] by Beale as follows:

\[
\begin{align*}
\text{min} & \quad -3/4x_1 + 150x_2 - 1/50x_3 + 6x_4 \\
\text{s.t.} & \quad 1/4x_1 - 60x_2 - 1/25x_3 + 9x_4 + x_5 = 0 \\
& \quad 1/2x_1 - 90x_2 - 1/50x_3 + 3x_4 + x_6 = 0 \\
& \quad x_3 + x_7 = 1 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 7.
\end{align*}
\]

Starting with initial base \([x_5, x_6, x_7]\) shows the cycling for Dantzig’s pivot rule. The optimal solution is \(x_1 = 1/25, x_3 = 1, x_5 = 3/100\) with \(\text{obj} = -1/20\).

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3 The original paper [13] was not published. A detailed description of this problem was given in [9]. The form presented here was given in [10].
Problem 3  The following linear program problem was presented in [4, 10]4:

\[
\begin{align*}
\min & \quad 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
\text{s.t.} & \quad 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 = 0 \\
& \quad 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_6 = 0 \\
& \quad x_1 + x_7 = 1 \\
& \quad x_j \geq 0, \ j = 1, \ldots, 7.
\end{align*}
\]

In [10], the optimal solution was given as: \(x_1 = 1, x_3 = 1, x_5 = 2\); and \(\min = 1\). As a matter of fact, an unbounded solution of this problem is given as: \(x_1 = x_3 = x_5 = 0, x_7 = 1, x_2 = \frac{9}{5}x_4, \) and \(x_6 = \left(\frac{150}{3} - 1\right)x_4\); while \(x_4 \to -\infty\), the objective goes to \(-\infty\). Another unbounded solution is given as: \(x_1 = x_5 = x_7 = 0, x_2 = c, x_3 = 0.5c, x_4 = 0.75c, \) and \(x_6 = c\) with \(c \geq 0\); as \(c \to \infty\), the objective goes to \(-\infty\).

Problem 4  The following linear program problem was presented in [26]:

\[
\begin{align*}
\max & \quad x_3 - x_4 + x_5 - x_6 \\
\text{s.t.} & \quad x_1 + x_3 - 2x_4 - 3x_5 + 4x_6 = 0 \\
& \quad x_2 + 4x_3 - 3x_4 - 2x_5 + x_6 = 0 \\
& \quad x_3 + x_4 + x_5 + x_6 + x_7 = 1 \\
& \quad x_j \geq 0, \ j = 1, \ldots, 7.
\end{align*}
\]

The optimal solution was given in [10] as: \(x_1 = 3, x_2 = 2, x_5 = 1\); and \(\max = 1\). For this problem, there is another optimal solution which is given as \(x_1 = \frac{5}{3}, x_3 = \frac{1}{3}\), and \(x_5 = \frac{2}{3}\) with \(\text{obj} = 1\).

Problem 5  The following linear program problem was presented in [26]:

\[
\begin{align*}
\min & \quad -x_3 + x_4 - x_5 + x_6 \\
\text{s.t.} & \quad x_1 + 2x_3 - 3x_4 - 5x_5 + 6x_6 = 0 \\
& \quad x_2 + 6x_3 - 5x_4 - 3x_5 + 2x_6 = 0 \\
& \quad 3x_3 + x_4 + 2x_5 + 4x_6 + x_7 = 1 \\
& \quad x_j \geq 0, \ j = 1, \ldots, 7.
\end{align*}
\]

The optimal solution is given as: \(x_1 = 2.5, x_2 = 1.5, x_5 = 0.5\); and \(\text{obj} = -0.5\).

Problem 6  The following linear program problem was presented in [1]:

\[
\begin{align*}
\min & \quad -2x_4 - 3x_5 + x_6 + 12x_7 \\
\text{s.t.} & \quad x_1 - 2x_4 - 9x_5 + x_6 + 9x_7 = 0 \\
& \quad x_2 + 1/3x_4 + x_5 - 1/3x_6 - 2x_7 = 0 \\
& \quad x_3 + 2x_4 + 3x_5 - x_6 - 12x_7 = 2 \\
& \quad x_j \geq 0, \ j = 1, \ldots, 7.
\end{align*}
\]

Starting with initial base \([x_1, x_2, x_3]\) shows the cycling for Dantzig’s pivot rule. The optimal solution is given as: \(x_1 = 2, x_4 = 2, x_6 = 2\); and \(\text{obj} = -2\).

\[4\]The following three problems were given in [10] without providing initial points.
Problem 7 The following linear program problem was presented in [15]:

\[
\begin{align*}
\text{min} & \quad -0.4x_5 - 0.4x_6 + 1.8x_7 \\
\text{s.t.} & \quad x_1 + 0.6x_5 - 6.4x_6 + 4.8x_7 = 0 \\
& \quad x_2 + 0.2x_5 - 1.8x_6 + 0.6x_7 = 0 \\
& \quad x_3 + 0.4x_5 - 1.6x_6 + 0.2x_7 = 0 \\
& \quad x_4 + x_6 = 1 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 7.
\end{align*}
\]

Starting with initial base \([x_1, x_2, x_3, x_4]\) shows the cycling for Dantzig’s pivot rule. The optimal solution is given as: \(x_1 = 4, x_2 = 1, x_5 = 4, x_6 = 1\) with \(\text{obj} = -2\).

Problem 8 The following linear program problem was presented in [10, 19]5:

\[
\begin{align*}
\text{min} & \quad -2x_3 - 2x_4 + 8x_5 + 2x_6 \\
\text{s.t.} & \quad x_1 - 7x_3 - 3x_4 + 7x_5 + 2x_6 = 0 \\
& \quad x_2 + 2x_3 + x_4 - 3x_5 - x_6 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]

The optimal solution is given as: for \(c \geq 0, x_1 = x_4 = x_6 = c,\) and the rest variables are zeros (while a special case is given in [10] that all variables are zeros) with \(\text{obj} = 0\).

Problem 9 The following linear program problem was presented in [17]:

\[
\begin{align*}
\text{min} & \quad -3x_1 + 80x_2 - 2x_3 + 24x_4 \\
\text{s.t.} & \quad x_1 - 32x_2 - 4x_3 + 36x_4 + x_5 = 0 \\
& \quad x_1 - 24x_2 - x_3 + 6x_4 + x_6 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]

The optimal solution is given as: \(x_1 = 1.8c, x_2 = x_6 = 0, x_3 = 3c, x_4 = 0.2c, x_5 = 3c,\) as \(c \to \infty,\) the \(\text{obj} \to -\infty,\) unbounded solution.

Problem 10 The following linear program problem was presented in [16]:

\[
\begin{align*}
\text{min} & \quad -3x_2 + x_3 - 6x_4 - 4x_6 \\
\text{s.t.} & \quad x_1 + x_2 + 1/3x_5 + 1/3x_6 = 2 \\
& \quad 9x_2 + x_3 - 9x_4 - 2x_5 - 1/3x_6 + x_7 = 0 \\
& \quad x_2 + 1/3x_3 - 2x_4 - 1/3x_5 - 1/3x_6 + x_8 = 2 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 8.
\end{align*}
\]

The optimal solution is given as: \(x_1 = x_2 = x_3 = x_5 = 0, x_4 = c/9, x_6 = 6, x_7 = c,\) \(x_8 = 2c/9,\) as \(c \to \infty,\) the \(\text{obj} \to -\infty,\) unbounded solution.

5The following three problems were given in [10] without providing initial points.
Problem 11 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -3x_1 - 59/20x_2 + 50x_3 + 2/5x_4 \\
\text{s.t.} & \quad 1/40x_1 + 1/400x_2 + 3x_3 + 2x_4 + x_5 = 0 \\
& \quad 1/20x_1 + 9/200x_2 - 1/2x_3 + 2/25x_4 + x_6 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]

Starting with initial base \([x_5, x_6]\) and breaking the tie by using the first pivot, the problem is cycling for Dantzig’s rule. The optimal solution is given as: all variables are zero with \(\text{obj} = 0\).

Problem 12 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -14x_1 + 25x_2 - 7/20x_3 + 20x_4 \\
\text{s.t.} & \quad x_1 - 2x_2 - 1/10x_3 + 5x_4 + x_5 = 0 \\
& \quad 7/10x_1 - 3/10x_2 - 1/100x_3 + 19/50x_4 + x_6 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]

Starting with initial base \([x_5, x_6]\) and breaking the tie by using the first pivot, the problem is cycling for Dantzig’s rule. The optimal solution is given as: \(x_2 = x_4 = x_6 = 0\), \(x_1 = c/70\), \(x_3 = c\), \(x_5 = 6c/70\), as \(c \to \infty\), the \(\text{obj} \to -\infty\), unbounded solution.

Problem 13 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -1/2x_1 - 2/5x_2 + 5x_3 + 1/5x_4 \\
\text{s.t.} & \quad 1/40x_1 - 1/100x_2 + 3x_3 + 2x_4 + x_5 = 0 \\
& \quad 1/20x_1 + 1/50x_2 + 1/50x_3 + 2/25x_4 + x_6 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]

Starting with initial base \([x_5, x_6]\) and breaking the tie by using the first pivot, the problem is cycling for Dantzig’s rule. The optimal solution is given as: all variables are zeros with \(\text{obj} = 0\).

Problem 14 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -14x_1 + 25x_2 - 7/20x_3 + 20x_4 \\
\text{s.t.} & \quad x_1 - 2x_2 - 1/10x_3 + 5x_4 + x_5 = 0 \\
& \quad 7/10x_1 - 3/10x_2 - 1/100x_3 + 19/50x_4 + x_6 = 0 \\
& \quad x_1 + x_2 + x_3 + x_4 + x_7 = 5 \\
& \quad x_1 + 2x_2 + 3x_3 + x_4 + x_8 = 10 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 8.
\end{align*}
\]

Starting with initial base \([x_5, x_6, x_7, x_8]\) and breaking the tie by using the first pivot, the problem is cycling for Dantzig’s rule. The optimal solution is given as: \(x_1 = 10/211\), \(x_2 = 0\), \(x_3 = 700/211\), \(x_4 = 0\), \(x_5 = 60/211\), \(x_6 = 0\), \(x_7 = 345/211\), and \(x_8 = 0\) with \(\text{obj} = -385/211\).
Problem 15 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -1/100x_1 + 1/100x_2 - 9/1000x_3 + 3/200x_4 - 1/500x_5 + 3/20x_6 \\
\text{s.t.} & \quad 1/20x_1 - 100x_2 - 2/5x_3 - 100x_4 - x_5 + 65x_6 + x_7 = 0 \\
& \quad 9/10x_1 - x_2 + 3/5x_3 - 3/2x_4 - 1/100x_5 + 1/100x_6 + x_8 = 0 \\
& \quad x_j \geq 0, \; j = 1, \ldots, 8.
\end{align*}
\]

Starting with initial base \([x_7, x_8]\) and breaking the tie by using the first pivot, the problem is cycling for Dantzig’s rule. The optimal solution is given as: \(x_1 = x_2 = x_4 = x_6 = x_8 = 0, x_3 = c/60, x_5 = 149c/150, x_7 = c, \) as \(c \to \infty, \) \(\text{obj} \to -\infty,\) unbounded.

Problem 16 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -3/100x_1 - 1/100x_2 - 1/100x_3 + x_4 + 3/10x_5 + 1/10x_6 + 2/125x_7 + 1/2x_8 \\
\text{s.t.} & \quad 1/10x_1 - 100x_2 - 13x_3 - 3/20x_4 - 6x_5 + 23/100x_6 + 1/100x_7 + 10x_8 + x_9 = 0 \\
& \quad 1/2x_1 + 3/5x_2 + 2/25x_3 - 8x_4 - 5x_5 - 13/10x_6 - 2/5x_7 + 1/10x_8 + x_{10} = 0 \\
& \quad x_j \geq 0, \; j = 1, \ldots, 10.
\end{align*}
\]

Starting with initial base \([x_9, x_{10}]\) and breaking the tie by using the first pivot, the problem is cycling for Dantzig’s rule. The optimal solution is given as: \(x_2 = x_4 = x_5 = x_6 = x_8 = x_{10} = 0, x_1 = 0.786601106330670c, x_3 = 0.083743085433313c, x_7 = x_9 = c, \) as \(c \to \infty, \) \(\text{obj} \to -\infty,\) unbounded.

Problem 17 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -14x_1 + 25x_2 - 7/20x_3 + 20x_4 \\
\text{s.t.} & \quad x_1 - 2x_2 - 1/10x_3 + 5x_4 + x_5 = 0 \\
& \quad 7/10x_1 - 3/10x_2 - 1/100x_3 + 19/50x_4 + x_6 = 0 \\
& \quad x_j \geq 0, \; j = 1, \ldots, 6.
\end{align*}
\]

Starting with initial base \([x_5, x_6]\) and breaking the tie by using the first pivot, the problem is cycling for steepest edge rule. The optimal solution is given as: \(x_2 = x_4 = x_6 = 0, x_1 = c, x_3 = 70c, x_5 = 6c, \) as \(c \to \infty, \) \(\text{obj} \to -\infty,\) unbounded.

Problem 18 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -1/4x_4 - 23/100x_5 - 4849/20000x_6 + 21/50x_7 \\
& \quad -123/2000x_8 - 1809/10^6x_9 + 4511/1250x_{10} \\
\text{s.t.} & \quad x_1 + 2x_4 + 6/5x_5 + 13/10x_6 + 1/100x_7 + 7/10x_8 + 1/1000x_9 + 3/50x_{10} = 0 \\
& \quad x_2 + 7/5x_4 + 13/10x_5 + 34/25x_6 + 1/20x_7 + 6/5x_8 + 13/10000x_9 + 23/10x_{10} = 0 \\
& \quad x_3 - 4x_4 - 3/2x_5 - 17/10x_6 - 28/5x_7 - 2x_8 - 1/100x_9 + 15x_{10} = 0 \\
& \quad x_j \geq 0, \; j = 1, \ldots, 10.
\end{align*}
\]

Starting with initial base \([x_1, x_2, x_3]\) and breaking the tie by using the first pivot, the problem is cycling for Dantzig’s rule. The optimal solution is given as: all variables are zeros with \(\text{obj} = 0.\)
2.2 Cycling problems for the largest pivot

Problem 19 The following linear program problem was presented in [15]:

\[
\begin{align*}
\text{min} & \quad -x_3 + 7x_4 + x_5 + 2x_6 \\
\text{s.t.} & \quad x_1 + 0.5x_3 - 5.5x_4 - 2.5x_5 + 9x_6 = 0 \\
& \quad x_2 + 0.5x_3 - 1.5x_4 - 0.5x_5 + x_6 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]

Starting with initial base \([x_1, x_2]\) and breaking the tie by using the largest pivot, (as is normal for numerical stability), the problem is cycling for Dantzig’s pivot rule. The optimal solution is given as: for \(c \geq 0, x_1 = 2c, x_3 = x_5 = c, \) and the rest variables are zeros (while [15] gives a special solution that all variables are zero) with obj = 0.

Problem 20 The following linear program problem was also presented in [12] which introduces two additional constraints to make it a bounded problem:

\[
\begin{align*}
\text{min} & \quad -2.3x_1 - 2.15x_2 + 13.55x_3 + 0.4x_4 \\
\text{s.t.} & \quad 0.4x_1 + 0.2x_2 - 1.4x_3 - 0.2x_4 + x_5 = 0 \\
& \quad -7.8x_1 - 1.4x_2 + 7.8x_3 + 0.4x_4 + x_6 = 0 \\
& \quad x_1 + x_7 = 1 \\
& \quad x_2 + x_8 = 1 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 8.
\end{align*}
\]

Starting with initial base \([x_5, x_6, x_7, x_8]\) and breaking the tie by using the largest pivot, (as is normal for numerical stability), the problem is cycling for Dantzig’s pivot rule. The optimal solution is given as: \(x_1 = 1, x_2 = 1, x_4 = 3, x_6 = 8, \) and \(x_3 = x_5 = x_7 = x_8 = 0\) with obj = -3.25.

Problem 21 The following linear program problem was presented in [12]:

\[
\begin{align*}
\text{min} & \quad -2.3x_1 - 2.15x_2 + 13.55x_3 + 0.4x_4 \\
\text{s.t.} & \quad 0.4x_1 + 0.2x_2 - 1.4x_3 - 0.2x_4 + x_5 = 0 \\
& \quad -7.8x_1 - 1.4x_2 + 7.8x_3 + 0.4x_4 + x_6 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]

Starting with initial base \([x_5, x_6]\) and breaking the tie by using the largest pivot, (as is normal for numerical stability), the problem is cycling for Dantzig’s pivot rule. The optimal solution is given as: \(x_1 = x_3 = x_5 = 0, x_2 = c, x_4 = c, x_6 = c, \) as \(c \to \infty, \) \(\text{obj} \to -\infty, \) unbounded.

Problem 22 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -1/100x_1 - 1/1000x_2 - 1/2000x_3 + 63/100x_4 \\
\text{s.t.} & \quad 20x_1 + 7/100x_2 - 7/100x_3 + 100x_4 + x_5 = 0 \\
& \quad -100x_1 - 3/10x_2 - 1/100x_3 + 1/4x_4 + x_6 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 6.
\end{align*}
\]
Starting with initial base \([x_5, x_6]\) and breaking the tie by using the largest pivot, (as is normal for numerical stability), the problem is cycling for Dantzig’s pivot rule. The optimal solution is given as: \(x_1 = x_4 = x_5 = 0, x_2 = c, x_3 = c, x_6 = 0.31c,\) as \(c \to \infty,\) \(\text{obj} \to -\infty,\) unbounded.

**Problem 23** The following linear program problem was presented in [28]:

\[
\begin{align*}
\min & \quad -1/250x_1 + 1/200x_2 - 1/1000x_3 + 3/20x_4 + 1/2000x_5 + 1/10x_6 \\
\text{s.t.} & \quad x_1 - 100x_2 - 1/25x_3 - 1/250x_4 - 1/25x_5 + 3/2x_6 + x_7 = 0 \\
& \quad 1/2x_1 - 1/10x_2 + 1/1000x_3 - 1/2x_4 - 2/25x_5 + 1/2x_6 + x_8 = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 8.
\end{align*}
\]

Starting with initial base \([x_7, x_8]\) and breaking the tie by using the largest pivot, (as is normal for numerical stability), the problem is cycling for Dantzig’s pivot rule. The optimal solution is given as: \(x_2 = x_4 = x_6 = x_7 = x_8 = 0, x_1 = 0.054c, x_3 = c, x_5 = 0.35c,\) as \(c \to \infty,\) \(\text{obj} \to -\infty,\) unbounded.

**Problem 24** The following linear program problem was presented in [28]:

\[
\begin{align*}
\min & \quad -8x_1 + 25x_2 - 39/5x_3 + 3182/5x_4 - 37/25x_5 \\
& \quad + 4713/1000x_6 - 2447/2500x_7 + 247367/5000x_8 \\
\text{s.t.} & \quad 11/25x_1 - 50x_2 - 3x_3 + 12x_4 - 8x_5 + 1/2x_6 - x_7 + 50x_8 + x_9 = 0 \\
& \quad 2/5x_1 + x_2 + 9/100x_3 - 5/2x_4 - 1/100x_5 - 1/50x_6 - 1/2500x_7 + 1/100x_8 + x_{10} = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 10.
\end{align*}
\]

Starting with initial base \([x_9, x_{10}]\) and breaking the tie by using the largest pivot, (as is normal for numerical stability), the problem is cycling for Dantzig’s pivot rule. The optimal solution is given as: \(x_1 = x_2 = x_4 = x_5 = x_6 = x_7 = x_8 = x_{10} = 0, x_3 = 0.004385964912281c, x_7 = 0.986842105263158c, x_9 = c,\) as \(c \to \infty,\) \(\text{obj} \to -\infty,\) unbounded.

**Problem 25** The following linear program problem was presented in [28]:

\[
\begin{align*}
\min & \quad -x_1 + 340x_2 + 71/5x_3 + 107/2x_4 - 7/40x_5 + 1469/500x_6 - 99/100x_7 + 15627/100x_8 \\
& \quad + 2/5x_1 - 145x_2 - 39/5x_3 + 6/5x_4 - 11/50x_5 + 11/5x_6 - 11/50x_7 - 1/2x_8 + x_9 = 0 \\
& \quad 9/25x_1 + 28x_2 + 10x_3 - 120x_4 - 1/2x_5 - 2x_6 - x_7 + 100x_8 + x_{10} = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 10.
\end{align*}
\]

Starting with initial base \([x_9, x_{10}]\) and breaking the tie by using the largest pivot, (as is normal for numerical stability), the optimal problem is cycling for Dantzig’s pivot rule. The solution is given as: \(x_2 = x_4 = x_6 = x_8 = x_9 = 0, x_1 = c, x_3 = 0.003483870967742c, x_5 = 0.789677419354839c, x_7 = x_{10} = c,\) as \(c \to \infty,\) \(\text{obj} \to -\infty,\) unbounded.
Problem 26 The following linear program problem was presented in [28]:

\[
\begin{align*}
\text{min} & \quad -1/10x_1 - 1/20x_2 - 2/25x_3 + 1/10x_4 - 1/25x_5 - 2/25x_6 + 33/200x_7 \\
\text{s.t.} & \quad 2/5x_1 - 13/10x_2 + 5x_3 + 8/5x_4 + 1/5x_5 + 3/5x_6 + 3/2x_7 + x_8 = 0 \\
& \quad 12/5x_1 + 2/5x_2 + 2x_3 - 13/20x_4 + 1/25x_5 + 7/10x_6 - 1/2x_7 + x_9 = 0 \\
& \quad -25x_1 - 8/5x_2 - 24x_3 - 1/5x_4 - 2x_5 - 3x_6 + 5x_7 + x_{10} = 0 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 10.
\end{align*}
\]

Starting with initial base \([x_8, x_9, x_{10}]\) and breaking the tie by using the largest pivot, (as is normal for numerical stability), the problem is cycling for Dantzig’s pivot rule. The optimal solution is given as: \(x_1 = x_3 = x_6 = x_7 = x_8 = x_9 = 0, x_2 = 0.259775040171398c, x_4 = 0.176754151044456c, x_5 = 0.274504552758436c, x_{10} = c,\) as \(c \to \infty, \text{obj} \to -\infty,\) unbounded.

3 Cycling problems for steepest edge rule

The following problems shows cycling behavior for steepest edge rule.

Problem 27 This linear program problem was presented in [12] which introduces two additional constraints in Problem 20 to make it a bounded problem:

\[
\begin{align*}
\text{min} & \quad -1.0x_1 - 1.75x_2 + 12.25x_3 + 0.5x_4 \\
\text{s.t.} & \quad 0.4x_1 + 0.2x_2 - 1.4x_3 - 0.2x_4 + x_5 = 0 \\
& \quad -7.8x_1 - 1.4x_2 + 7.8x_3 + 0.4x_4 + x_6 = 0 \\
& \quad -20x_2 + 156x_3 + 8.0x_4 + x_7 = 1 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 7.
\end{align*}
\]

Starting with initial base \([x_5, x_6, x_7]\) and breaking the tie by using the larger pivot, (as is normal for numerical stability), the problem is cycling for steepest edge rule. The optimal solution is given as: \(x_1 = x_3 = x_5 = 0.070512820513462c, x_2 = x_4 = x_6 = x_7 = c,\) as \(c \to \infty, \text{obj} \to -\infty,\) unbounded.

Problem 28 The following linear program problem was presented in [23] which is verified by the author:

\[
\begin{align*}
\text{min} & \quad -10x_1 + 57x_2 + 9x_3 + 24x_4 \\
\text{s.t.} & \quad 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 = 0 \\
& \quad 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_6 = 0 \\
& \quad x_1 + x_2 + x_3 + x_4 + x_7 = 1 \\
& \quad x_j \geq 0, \quad j = 1, \ldots, 7.
\end{align*}
\]

Starting with initial base \([x_5, x_6, x_7]\) and breaking the tie by using the first pivot, the problem is cycling for steepest edge rule. The optimal solution is given as: \(x_1 = 0.5, x_3 = 0.5, x_5 = 1,\) and \(x_2 = x_4 = x_6 = x_7 = 0\) with \(\text{obj} = -0.5.\)
Problem 29 This linear program problem was also presented in [28]:

\[
\begin{align*}
\min & \quad -7/100x_1 - 3/50x_2 + 11/50x_3 + 133/12500x_4 \\
\text{s.t.} & \quad 33/25x_1 + 1/2x_2 - 53/25x_3 - 9/20x_4 + x_5 = 0 \\
 & \quad -44x_1 - 24/5x_2 + 6x_3 + 33/100x_4 + x_6 = 0 \\
 & \quad 1319/1000x_1 - 40x_2 + 2123/10x_3 + 1173/100x_4 + x_7 = 0 \\
 & \quad x_j \geq 0, \quad j = 1, \ldots, 7.
\end{align*}
\]

Starting with initial base \([x_5, x_6, x_7]\) and breaking the tie by using the larger pivot, (as is normal for numerical stability), the problem is cycling for steepest edge rule. The optimal solution is given as: \(x_1 = x_3 = 0, x_2 = 0.264476614699332c, x_4 = 0.816629547141797c, x_5 = 0.235244988864143c, x_6 = x_7 = c, \) as \(c \to \infty, \) \(\text{obj} \to -\infty,\) unbounded.

Problem 30 This linear program problem was also presented in [28]:

\[
\begin{align*}
\min & \quad 36x_1 - 3/5x_2 + 20x_3 + 1/4x_4 - 1/20x_5 - 1/20x_6 \\
\text{s.t.} & \quad 2x_1 + 1/5x_2 - 5x_3 - 9/10x_4 + x_5 + 23/1000x_6 + x_7 = 0 \\
 & \quad -41x_1 - 6/5x_2 + 12x_3 + 1/5x_4 - 14/5x_5 - 1/500x_6 + x_8 = 0 \\
 & \quad 165000x_1 + 2600x_2 + 9600x_3 + 125x_4 - 100x_5 - 300x_6 + x_9 = 0 \\
 & \quad x_j \geq 0, \quad j = 1, \ldots, 9.
\end{align*}
\]

Starting with initial base \([x_7, x_8, x_9]\) and breaking the tie by using the larger pivot, (as is normal for numerical stability), the problem is cycling for steepest edge rule. The optimal solution is given as: \(x_1 = x_3 = x_5 = x_7 = x_9 = 0, x_2 = 0.112949260042283c, x_4 = 0.050655391120507c, x_6 = c, x_8 = 0.127408033826638c, \) as \(c \to \infty, \) \(\text{obj} \to -\infty,\) unbounded.

4 Preliminary test for a double pivot algorithm

Although Bland’s rule can prevent cycling from happening, it is not as efficient as the Dantzig’s pivot rule [8]. A double pivot simplex algorithm was recently proposed [24] aiming at dealing with degeneracy/cycling problem and improving the efficiency of Dantzig’s pivot rule. This algorithm updates two pivots at a time which is different from the traditional simplex algorithms. An important feature of the algorithm is that one of the pivot takes the longest step among all possible entering variables. This feature makes it possible to avoid cycling problems. The promising result of the double pivot simplex algorithm for large size LP problems has been discussed in [24]. We consider the primal linear programming problem in the standard form:

\[
\begin{align*}
\min & \quad c^T x, \\
\text{subject to} & \quad A x = b, \quad x \geq 0,
\end{align*}
\]

where \(A = [a_1, \ldots, a_n] \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, \) \(c \in \mathbb{R}^n\) are given, and \(x \in \mathbb{R}^n\) is the vector to be optimized. To save space, we will write the column vector \(x = [x_1^T, x_2^T]^T\) as
\( x = (x_1, x_2) \). We denote by \( a_i \), the \( i \)th column of \( A \), for \( i \in I = \{1, \ldots, n\} \), where the subscript \( i \) is the index of column of \( A \). We denote by \( B \subset I \) the index set with cardinality \( |B| = m \) and \( N = I \setminus B \) the complementary set of \( B \) with cardinality \( |N| = n - m \) such that matrix \( A \) and vector \( x \) can be partitioned as \( A = [A_B, A_N] \) and \( x = (x_B, x_N) \), moreover the columns of \( A_B \) are linearly independent and \( A_B x_B = b \), hence \( x_N = 0 \). We call this \( x = (x_B, 0) \geq 0 \) as the basic feasible solution. Similarly, we have the partition \( c = (c_B, c_N) \).

Using the \( B - N \) partition, we can rewrite the problem (1) as

\[
\begin{align*}
\min & \quad c_B^T x_B + c_N^T x_N, \\
\text{subject to} & \quad A_B x_B + A_N x_N = b, \quad x_B \geq 0, \quad x_N \geq 0.
\end{align*}
\] (2)

Since \( A_B \) is non-singular, we can rewrite (2) as

\[
\begin{align*}
\min & \quad c_B^T A_B^{-1} b + (c_N - A_N^T A_B^{-1} c_B)^T x_N, \\
\text{subject to} & \quad x_B = A_B^{-1} b - A_B^{-1} A_N x_N, \quad x_B \geq 0, \quad x_N \geq 0.
\end{align*}
\] (3)

Let \( c_N^T = c_B^T - c_B^T A_B^{-1} A_N \) be the reduced cost, notice that the index of \( c_N \) is the same as the index of \( A_N \). Denote the basic feasible solution

\[ \bar{b} = A_B^{-1} b = (\bar{b}_1, \ldots, \bar{b}_m) = x_B. \] (4)

The first entering column \( p \) of \( A_N \) is determined by \( \bar{c}_p = \min \{ \bar{c}_N < 0 \} \). Since \( a_t \in A_N \) can be expressed as \( a_t = A_B y_t \), therefore, we can write

\[ y_t = A_B^{-1} a_t = (y_{t1}, \ldots, y_{tm}). \] (5)

The second entering column \( q \) is determined by considering all \( a_t \) corresponding to \( c_t \in c_N < 0 \) such that

\[ \bar{b}_j / y_{tj} = \max \{ \min_{t \in \mathcal{C}_N < 0} \bar{b}_j / y_{tj}, \text{ subject to } y_{tj} > 0 \}. \] (6)

The leaving columns in \( A_B \) are obtained by solving a two dimensional linear programming problem.

\[
\begin{align*}
\min & \quad c_p x_p + c_q x_q, \\
\text{subject to} & \quad [a_p \ a_q](x_p, x_q) = \bar{b}, \quad x_p \geq 0, \quad x_q \geq 0.
\end{align*}
\] (7)

The indexes corresponding to vertex that achieves the optimum are the indexes of the leaving columns.

Because the double pivot algorithm achieves at least the same cost reduction as the longest step rule, if the longest step rule results in a positive step size, the double pivot rule will also achieve a positive step size in the same iteration. This strategy greatly increases the chance of bringing a non-degenerate variable (whose step length is greater than zero) into the next basic feasible solution. We use Beale’s problem [2] (Problem 2) to support this claim.
**Beale’s problem** (Problem 2) For this problem, \( c^T = [-3/4, 150, -1/50, 6, 0, 0, 0] \)

\[
A = \begin{bmatrix}
  1/4 & -60 & -1/25 & 9 & 1 & 0 & 0 \\
  1/2 & -90 & -1/50 & 3 & 0 & 1 & 0 \\
\end{bmatrix}
\quad b = \begin{bmatrix}
  0 \\
  0 \\
\end{bmatrix}
\]

the initial base is \( B^0 = \{5, 6, 7\} \) and \( N^0 = \{1, 2, 3, 4\} \), \( c^T_{N^0} = [−3/4, 150, -1/50, 6] \). There are only two elements in \( c^0_N < 0 \), the entering columns are \( \{1, 3\} \). Therefore, we need to solve

\[
\begin{align*}
\text{min} & \quad -3/4x_1 - 1/50x_3, \\
\text{subject to} & \quad [a_1 \ a_3](x_1, x_3) = b, \quad x_1 \geq 0, \quad x_3 \geq 0.
\end{align*}
\]

Using graphic method, one can find the optimal solution of (8) as \((x_1, x_3) = (1/25, 1)\) with the last two rows composed of the vertex. Therefore, the leaving variables are \(x_6\) and \(x_7\). This gives \( B^1 = \{1, 3, 5\} \), \( N^1 = \{2, 4, 6, 7\} \), and \( c^T_{N^1} = [15, 10.5, 1.5, 0.05] > 0 \). The optimal solution of this problem is found in one iteration. While applying Dantzig’s rule, after 6 iterations, one will get \( B^6 = B^0 \), cycling occurs.

For large \( m \), an efficient algorithm was developed in [21] to solve (7). More test shows that the double pivot algorithm finds the optimal solutions for all cycling problems provided in this paper. The test result is summarized in the table below. In this table, only basic feasible solutions and optimal basic solutions are provided. All non-basic variables are zeros.

| Problem | Initial point | Iteration | Optimal solution | Objective function |
|---------|---------------|-----------|------------------|-------------------|
| 1       | \([x_1, x_2, x_3] = [1, 0, 0]\) | 2         | \([x_1, x_4, x_5] = [1, 0, 0]\) | 0                 |
| 2       | \([x_1, x_6, x_7] = [0, 0, 1]\) | 1         | \([x_1, x_3, x_4] = [0.04, 1, 0.03]\) | -0.05             |
| 3       | \([x_1, x_6, x_7] = [0, 0, 1]\) | 1         | \([x_1, x_2, x_3] = [3, 4, 1]\) | -∞                |
| 4       | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | \([x_1, x_2, x_3] = [2, 5, 0, 0]\) | -0.9              |
| 5       | \([x_1, x_2, x_3] = [0, 0, 1]\) | 6         | \([x_1, x_2, x_3] = [2, 4, 1]\) | -∞                |
| 6       | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | \([x_1, x_2, x_3] = [4, 1, 4, 1]\) | -2                |
| 7       | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | \([x_1, x_2, x_3] = [4, 1, 4, 1]\) | -2                |
| 8       | \([x_1, x_2, x_3] = [0, 0]\) | 3         | \([x_1, x_2, x_3] = [0, 0]\) | 0                 |
| 9       | \([x_1, x_2, x_3] = [0, 0]\) | 1         | unbounded        | -∞                |
| 10      | \([x_1, x_2, x_3] = [1, 0, 0]\) | 2         | unbounded        | -∞                |
| 11      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 4         | \([x_1, x_2, x_3] = [0, 0]\) | 0                 |
| 12      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | unbounded        | -∞                |
| 13      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 4         | \([x_1, x_2, x_3] = [0, 0]\) | 0                 |
| 14      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | \([x_1, x_2, x_3] = [0, 0]\) | 1.8246            |
| 15      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | unbounded        | -∞                |
| 16      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | unbounded        | -∞                |
| 17      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | unbounded        | -∞                |
| 18      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 7         | \([x_1, x_2, x_3] = [0, 0]\) | 0                 |
| 19      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 4         | \([x_1, x_2, x_3] = [0, 0]\) | 0                 |
| 20      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 2         | \([x_1, x_2, x_3] = [1, 3, 8]\) | -4.2600           |
| 21      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 2         | unbounded        | -∞                |
| 22      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | unbounded        | -∞                |
| 23      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 2         | unbounded        | -∞                |
| 24      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 1         | unbounded        | -∞                |
| 25      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 6         | unbounded        | -∞                |
| 26      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 3         | unbounded        | -∞                |
| 27      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 5         | \([x_1, x_2, x_3] = [0, 0, 0, 1]\) | -0.5             |
| 28      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 3         | unbounded        | -∞                |
| 29      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 3         | unbounded        | -∞                |
| 30      | \([x_1, x_2, x_3] = [0, 0, 1]\) | 3         | unbounded        | -∞                |

Table 1: Numerical test for double pivot method
5 Conclusions

In this paper, we collected a set of cycling problems in linear programming. This set of problems is used to test a double pivot simplex algorithm. The test result indeed shows that the double pivot algorithm prevents the cycling from happening for all the problems presented in this paper. Our ultimate goal in the next project is to show that this double pivot algorithm, together with some other simple strategies, will be able to solve a majority set of very challenging Netlib benchmark problems, if not all them. These problems can be downloaded from Matlab file exchange cite https://www.mathworks.com/matlabcentral/fileexchange/91985-a-collection-of-cycling-problems-in-linear-programming.

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