Topological Meta-Materials: An Algorithmic Design

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Topological condensed matter systems from class A and class AII of the classification table have received classical electromagnetic and mechanical analogs and protected wave-guiding with such systems has been demonstrated experimentally. Here we introduce a map which generates classical analogs for all entries of the classification table, using only passive elements. Physical mechanical models are provided for all strong topological phases in dimension 2, as well as for three classes in dimension 3. This includes topological super-conducting phases, which have never been attempted with classical systems.

I. INTRODUCTION

Topological methods [1, 2] have revolutionized the way we approach the design of our materials and the functionalization of our devices. These days, condensed matter, photonic and mechanical/ acoustic crystals with demonstrated intrinsic topological properties are common. One of their common characteristics is the spontaneous emergence of wave-guiding modes whenever a boundary is cut into a material and, for strong topological materials, these modes are robust against disorder. The strong topological condensed matter systems have been classified at the end of the previous decade [3-5] and it emerged that there are only three fundamental symmetries that can stabilize these phases, namely, the time-reversal (TR), particle-hole (PH) and chiral (CH) symmetries. Their combinations led to the 10 distinct topological classes reproduced in Table I.

About the same time, the question of whether such exotic physical properties can be reproduced with classical systems was vehiculated within the physics community. For class A in dimension $d = 2$, the affirmative answer was provided in [7, 8] for photonic and mechanical systems, respectively, and experimental confirmations soon followed [9, 10]. Rapid progress happened afterwards. Electromagnetic [11, 12] as well as mechanical [13] systems exhibiting the physics of all class in $d = 2$ have been predicted and realized in laboratories. Classical mechanisms inspired from AII class has been demonstrated in [14, 15] and a linear mechanical chain from BDI class with locked-in particle-hole symmetry and Majorana edge excitations was observed in [16]. An electromagnetic emulation of a topological insulator from AII class in $d = 3$ was proposed in [17] and mechanical systems that emulate the quantum spin-Hall effect with passive materials have also been proposed [18] and implemented [19-21].

Among all, the Süsstrunk-Huber technique [13] and its generalization [22] stand out because they involve only passive components and the TR-like symmetry that stabilizes the edge modes is global and exact. In contradiction, in all the other approaches based on passive components, such as [12] or [18], this symmetry only holds at one point or between two points of the Brillouin torus, hence TRS protection occurs only in a small frequency interval where a certain effective model applies.

The mapping from the class A to class AII devised in [13, 22] carries an inherent additional $U(1)$ symmetry, hence the output is always a spin-Chern system with topological phases classified by $Z$ instead of $Z_2$. This is unfortunate for there is unique physics associated to latter that cannot be probed with spin-Chern insulators. For example, the generic systems from AII-class in $d = 2$ can support metallic phases in the presence of disorder, but such phases vanish when an additional $U(1)$-symmetry is present. For this reason, the critical quantum regimes at the topological transitions occurring in class AII are fundamentally different from the ones supported by spin-Chern insulators. Also, the protection mechanisms of the edge modes against Anderson localization are fundamentally different for these two classes of systems.

The most acute shortcoming of that map is its inability to produce topological classical systems from AII class in $d = 3$, where the analog of spin-Chern insulator does not exist. For the rest of the topological classes, the situation is even worse because no such map has ever been attempted. For example, classical analogs of topological systems from any of the BdG classes in dimension higher than one do not exist. Let us point out again that the distinct entries in the classification table implement the ten universal disordered classes, each of them displaying unique characteristics [23, 24], many never observed experimentally. Furthermore, each topological class displays unique bulk and boundary physical responses, which may enable important physical applications [25]. As such, despite the impetuous recent progress with classical systems, there are many missed opportunities, which we hope can be addressed in the near future with the tools provided by our present work.

Indeed, we demonstrate here how the map [13, 22] can be reformulated and generalized to ultimately cover the whole Table I. The final outcome is an algorithmic procedure to translate any strong topological condensed matter system from the classification table into an absolutely equivalent topological classical meta-material with passive components. To exemplify, we generate explicit mechanical analogs of all topological classes in $d = 2$, as well as three additional classes in $d = 3$, that have never been attempted before, classically.
TABLE I. Classification table of strong topological insulator and superconductors [3,4], listing all strong topological condensed matter phases according to their fundamental symmetries and space dimension. The phases highlighted in red will be explicitly mapped into classical systems.

II. COUPLED MECHANICAL RESONATORS

We consider a collection of coupled passive resonators placed on a lattice $\mathcal{L}$, such as the one described in Fig. 1, with the following essential features: 1) existence of $N$ degrees of freedom $q_x^\ell$ localized around the vertices $x$ of $\mathcal{L}$ and 2) the ability to couple $q_x^\ell$'s, one pair at a time (see [26, 27] for a laboratory realization). In this setting, the configuration space consists of column matrices $Q = \{q_x^\ell\}_{x,\ell}^{N}$. and, in the regime of small oscillations relative to a stable equilibrium point, the dynamics is determined by a quadratic Lagrangian $L = \frac{1}{2} Q^T T Q - \frac{1}{2} Q^T W Q$, leading to the equations of motion $\dot{Q} = -DQ$, with the dynamical matrix given by $D = \hat{T}^{-1} W \hat{T}^{-1}$. Since all the topological examples can be constructed with identical resonators, we will assume that $\hat{T}$ is diagonal, hence the dynamical matrix can be read-off directly from our diagrammatic models, as explained below.

Due to our assumptions, the equations of motion have a particular structure. Indeed, $Q$ can be viewed as a function $Q : \mathcal{L} \rightarrow \mathbb{C}^N$, $Q(x) = (q_1^x, \ldots, q_N^x)$, and this space of functions can be organized as a complex Hilbert space, usually denoted by $\ell^2(\mathcal{L}, \mathbb{C}^N) \approx \mathbb{C}^N \otimes \ell^2(\mathcal{L}, \mathbb{C})$, via the scalar product $\langle Q, Q' \rangle = \sum_{x,\ell} Q_x^\ell \cdot Q_x'^\ell$. With the ansatz $Q(t) = \text{Re}[e^{i\omega t} Q]$, all the resonant modes of the system can be found by solving the eigen-system $\omega^2 Q = DQ$ on $\ell^2(\mathcal{L}, \mathbb{C}^N)$. On this particular Hilbert space, the most general form of a dynamical matrix takes the form:

$$D = \sum_{x, x' \in \mathcal{L}} \tilde{d}_{x,x'} \otimes |x\rangle \langle x'|$$

(1)

where $|x\rangle$ denotes the standard orthonormal basis of $\ell^2(\mathcal{L}, \mathbb{C})$, $\psi_x(\ell) = \delta_{x, \ell}$. Throughout, we will use $\ell^2(\mathcal{L})$ instead of $\ell^2(\mathcal{L}, \mathbb{C})$ and we will also denote by $\xi_1, \ldots, \xi_N$ the standard basis of $\mathbb{C}^N$. Since we avoid active materials, the entries of $\tilde{d}$'s are all real and they can be easily read-off from diagrams like Fig. 1B. For example, $d_{x,x'}(1,3) = d_{x',x}(3,1)$ is equal to the numerical value associated to the red connection in Fig. 1B. In the concrete examples, these values will be placed explicitly on the connecting lines or specified in the captions. Reciprocally, once given the $\tilde{d}$ matrices, one can quickly sketch a physical representation of the mechanical system driven by $\hat{Q}$, by reversing the process we just described. Because of these reasons, while somewhat mathematical, we find the notation in (1) extremely useful. Let us also mention that if the system is translational invariant, then:

$$D = \sum_q \sum_{x \in \mathbb{Z}^d} \tilde{d}_q \otimes |x + q\rangle \langle x| = \sum_q \tilde{d}_q \otimes S_q,$$

(2)

with $S_q|x\rangle = |x + q\rangle$, from where the momentum representation $D(k) = \sum_q e^{i\mathbf{k} \cdot \mathbf{q}} \tilde{d}_q$, $\mathbf{k} \in [-\pi, \pi]^d$, follows.

III. THE ALGORITHM

We start from a generic hermitian Hamiltonian $H$ on the complex Hilbert space $\mathbb{C}^N \otimes \ell^2(\mathcal{L})$, which necessarily takes the form (1) but the hopping matrices are not real in general:

$$H = \sum_{x, x' \in \mathcal{L}} \hat{h}_{x,x'} \otimes |x\rangle \langle x'|, \quad \hat{h}_{x,x'} = \hat{h}^*_{x',x} \in M_{N \times N}(\mathbb{C}).$$

(3)

Any model Hamiltonian from the classification table of topological condensed matter systems, periodic or disordered, can be written as in (3). The symmetries induce certain additional constraints on $\hat{h}$-matrices.

The map defined. Let $\mathcal{K}$ be the complex conjugation on the space $\mathbb{C}^N \otimes \ell^2(\mathcal{L})$. Since the standard basis of $\ell^2(\mathcal{L})$ is real, $\mathcal{K}$ acts on the Hamiltonians as $\hat{h}_{x,x'} \rightarrow \hat{h}^*_{x',x}$, where $*$ stands for the ordinary complex conjugation. Now, let
\[ B[C^N \otimes \ell^2(L)] \] denote the algebra of continuous linear operators over \( C^N \otimes \ell^2(L) \), of which \( H \) is part of. We are going to define an algebra morphism:

\[ \rho : B[C^N \otimes \ell^2(L)] \to B[C^{2N} \otimes \ell^2(L)], \]

such that \( X^{-1} \rho(A)X = \rho(A) \) for arbitrary and possibly non-hermitean \( A = \sum_{x,x' \in L} \hat{a}_{x,x'} \otimes |x\rangle\langle x'| \). Explicitly:

\[ \rho(A) = \sum_{x,x' \in L} \left( \begin{array}{c|c} \text{Re}[\hat{a}_{x,x'}] & \text{Im}[\hat{a}_{x,x'}] \\ \hline -\text{Im}[\hat{a}_{x,x'}] & \text{Re}[\hat{a}_{x,x'}] \end{array} \right) \otimes |x\rangle\langle x'|. \]

It is straightforward to verify that \( \rho \) respects the algebraic operations, e.g. \( \rho(AB) = \rho(A)\rho(B) \), sends the identity operator into the identity operator, \( \rho(1) = 1 \), respects the operation of taking the adjoint, \( \rho(A^\dagger) = \rho(A) \), and is into, \( \rho(A) = \rho(B) \) if and only if \( A = B \). Among many things, the properties we just listed ensure that the spectral properties are preserved:\[ \text{Spec}(\rho(H)) = \text{Spec}(H). \]

Let us point out that, when applied on the Hofstadter model with with 1/3 flux quanta per cell, the map returns the mechanical system analyzed in [13].

**Transfer of symmetries.** Consider the transformation:

\[ U : C^{2N} \otimes \ell^2(L) \to C^{2N} \otimes \ell^2(L), \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1, \]

which satisfies \( U^{-1} = U^T = -U \) and hence \( U^2 = -1 \). This implies that \( \text{Spec}(U) = \{ \pm 1 \} \) and the spectral projectors on the eigen-subspaces are given by:

\[ \Pi_{\pm} = \frac{1}{2}(1 \mp iU) = \frac{1}{2}(1 \pm i) \otimes 1. \]

It is immediate to verify that \( [\rho(H), U] = 0 \), hence \( \rho(H) \) has automatically a built-in \( U(1) \) symmetry. Furthermore, \( \rho(H) = \Pi_+ \rho(H) \Pi_+ + \Pi_- \rho(H) \Pi_- \), and \( \Pi_+ \rho(H) \Pi_+ \) and \( \Pi_- \rho(H) \Pi_- \) are unitarily equivalent to \( H \) and \( H^\dagger \), respectively. Note that \( \mathbb{Z}_2 X^{-1} = \Pi_\pm \) and if:

\[ J : C^{2N} \otimes \ell^2(L) \to C^{2N} \otimes \ell^2(L), \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1, \]

then \( J\rho(H)J^{-1} = \rho(H^\dagger) \) and \( J\Pi_\pm J^{-1} = \Pi_\mp \).

Since \( \rho(H) \) is real, it also has a built-in fermionic TR-symmetry \( \Sigma \rho(H) \Sigma^{-1} = \rho(H) \), where \( \Sigma = U \mathcal{K} = \mathcal{K} U \), \( \Sigma^2 = -1 \). As such, Kramer pairing is automatically present for the classical system and the spectrum is necessarily doubly degenerate. It is important to note that this time-reversal symmetry and the \( U(1) \) symmetry derived from \( U \) are always simultaneously present. As such, the map \( \rho \), just by itself, cannot generate systems from \( \text{AI}I \) class classified by a \( \mathbb{Z}_2 \) invariant. It then becomes clear that \[ [13] \] produced a spin-Chern insulator rather than a true quantum spin-Hall insulator. The conclusion is that \( \Sigma \) is not the anti-unitary operation that will ultimately stabilize the true topological phases from \( \text{AI}I \) class. Indeed, as we shall demonstrate next, we can correctly transfer all the symmetries, hence generate all the topological strong phases, if we restrict to the \( \Pi_\pm \) sectors.

Henceforth, assume that \( \Theta H \Theta^{-1} = eH, e = \pm 1 \), with \( \Theta \) anti-unitary and \( \Theta^2 = \gamma, \gamma = \pm 1 \), and recall that, always, such \( \Theta \) can be written as \( \mathcal{K} \mathcal{W} = \mathcal{W} \mathcal{K} \) with \( \mathcal{W} \) unitary. The map \( \rho \) applies only on linear operators, hence we do not know, as of yet, how to transfer anti-unitary symmetries. Nevertheless, based on the remarkable identity:

\[ \epsilon \Pi_+ \rho(H) \Pi_\pm = (\mathcal{K} \rho(W)) \Pi_\pm \rho(H) \Pi_\pm (\mathcal{K} \rho(W))^{-1}, \]

we find that the proper mapping of \( \Theta \) is:

\[ \Theta \to \tilde{\Theta} = \mathcal{K} \rho(W), \mathcal{K}^2 = \gamma. \]

At this point, we reached the important conclusion: If the original Hamiltonian enjoys any of the symmetries in the classification table, then the mapped Hamiltonian does too when projected on either of the \( \Pi_\pm \) sectors.

**IV. BULK-BOUNDARY CORRESPONDENCE**

We divide the lattice as \( L = L_+ \cup L_- \) and we restrict \( H \) on the Hilbert space \( C^N \otimes \ell^2(L_+) \) via:

\[ \tilde{H} = \sum_{x,x' \in L_+} \hat{h}_{x,x'} \otimes |x\rangle\langle x'|. \]

By doing so, we cleanly separate all the connections between the resonators in sub-lattice \( L_+ \) and the ones in sub-lattice \( L_- \), but our arguments apply for more general boundary conditions, as we shall see later. This operation defines a linear map:

\[ \text{Cut} : B[C^N \otimes \ell^2(L)] \to B[C^N \otimes \ell^2(L_+)]. \]

which in general does not respect the multiplication. A similar map can be defined on the Hilbert spaces with doubled internal degrees of freedom. Now, note that \( \rho \) acts ultra-locally, in the sense that the internal degrees of freedom at a point \( x \) are not mixed with the ones at another point \( x' \). Due to this ultra-locality, \( \rho \circ \text{Cut} =
Cut ◦ ρ and, since the original lattice was arbitrary, \( \text{Spec}(\hat{H}) = \text{Spec}(\rho(\hat{H})) \). Furthermore, since \( U \), hence also \( \Pi_x \), are ultra-local, we can be much sharper:

\[
\text{Spec}(\hat{H}) = \text{Spec}\left( \Pi_x \rho(\hat{H}) \Pi_x \right).
\]

(14)

The important conclusion is that, if \( \hat{H} \) displays gapless boundary spectrum, so does \( \rho(\hat{H}) \) as well as the individual components \( \Pi_x \rho(\hat{H}) \Pi_x \), where the latter are viewed as acting on the appropriate truncated Hilbert spaces.

The above statement assures us that the bulk-boundary correspondence transfers if the cuts are clean. In realistic conditions, however, we need to add a boundary term to \( H = \sum_{x,x' \in \mathcal{L}} \hat{b}_{x,x'} \otimes |x \rangle \langle x' | \), with \( \hat{b}_{x,x'} = 0 \) for \( x \) and \( x' \) faraway from the boundary, to take into account relaxations of or damages to the lattice during the cutting process. Let us recall that if \( H \) is a topological condensed matter system from the classification table, then the bulk-boundary correspondence principle holds if and only if \( \hat{H} \) enjoys the same symmetries as \( H \). Given \( \rho \), we can state at once that

\[
\text{Spec}(\hat{H} + \hat{H}) = \text{Spec}\left( \Pi_x \rho(\hat{H}) + \rho(\hat{H}) \Pi_x \right).
\]

(15)

Hence, as long as the boundary potential of the classical system is of the form \( \Pi_x \rho(\hat{H}) \Pi_x \otimes \Pi_x \rho(\hat{H}) \Pi_x \), i.e. it respects the \( U(1) \) symmetry, the bulk-boundary principle transfers. Note that no assumption was made above about the translation invariance of the system, hence the statements apply also in the presence of strong disorder as long as it respects proper symmetries.

We reached the main conclusion of the section: Let \( H \) be a Hamiltonian from the classification table of topological condensed matter systems. Then \( \Pi_x \rho(\hat{H}) \Pi_x \) displays identical bulk-boundary principle as the original Hamiltonian, provided the boundary conditions do not mix the \( \Pi_x \) sectors and preserve the original symmetries, as transferred through \( \rho \).

V. FILTERING THE TOPOLOGICAL SECTORS

We describe here one concrete way to excite modes only in one of the \( \Pi_x \) sectors. The time varying generalized forces can be specified by functions \( F_i : \mathcal{L} \to \mathbb{C}^{2N} \), \( F_i \in \mathcal{L}(\mathcal{L}, \mathbb{C}^{2N}) \), such that the evaluation at some \( x \in \mathcal{L} \) gives the instantaneous generalized forces on the degrees of freedom at \( x \), \( F_i(x) = (F_{i1}(t), \ldots, F_{i2N}(t))^T \). When such driving forces are present, the equations of motion in time and frequency domains become:

\[
\dot{\mathbf{q}}_t = -\mathbf{D} \mathbf{q}_t + \mathbf{F}_t, \quad [D - \omega^2]Q(\omega) = F(\omega),
\]

(16)

respectively. The solution belongs to one and only one of the \( \Pi_x \) sectors if and only if \( \Pi_x F(\omega) = 0 \), respectively. We reached the main conclusion of the section: Starting from a generic force field \( F_t \), one can excite and drive modes from one and only one of the two \( \Pi_x \) sectors by simply reworking the force field to:

\[
F_i \to \int_0^\infty \text{Re}\left[ \Pi_x F(\omega)e^{i\omega t} \right] d\omega,
\]

(17)

respectively.

As an example, let us consider the case where we start with only one force \( F_0 \cos(\omega t) \), acting on the degree of freedom \( \alpha \in \{1, \ldots, N\} \) located at vertex \( x_0 \). It is understood that \( \omega \) belongs to the spectrum of the dynamical matrix. In the standard basis, this force field is written as \( F_i = F_0 \cos(\omega t) \xi_\alpha \otimes |x_0 \rangle \). To excite only the modes in the \( \Pi_x \) sector, we need to modify this force field to \( F_i \to F_0 \text{Re}\left[ e^{i\omega t} \Pi_x \xi_\alpha \otimes |x_0 \rangle \right] \), which translates into:

\[
F_i \to F_0 \left[ \cos(\omega t) \xi_\alpha - \sin(\omega t) \xi_{N+\alpha} \right] \otimes |x_0 \rangle.
\]

(18)

This tells us that all we have to do is to apply the same generalized force also on the degree of freedom \( N + \alpha \) of the same site, and de-phase the force by a quarter of the period.

VI. EXAMPLES

The map is applied here to the generating models of the topological classes in 2 and 3 dimensions highlighted in Table I. One goal is to verify explicitly all our theoretical predictions, particularly the identity (10), for both bulk and half-space Hamiltonians. Another goal is to provide concrete physical representations of the mapped
Hamiltonians. We will use Pauli’s matrices $\sigma_i$ together with $\sigma_0 = \mathbb{1}_2$, as well as the $4 \times 4$ Gamma matrices:

$$\Gamma_{1, 2, 3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},$$

and $\Gamma_E = \Gamma_0 \Gamma_2 \Gamma_4$. Since all the Hamiltonians are translation invariant, we adopt the notation from \([2]\), involving the shift operators $S_j|x\rangle = |x + e_j\rangle$, with $e_j$’s being the generators of $\mathbb{Z}^d$.

**Dimension 2. D-A Classes.** The strong topological phases from these classes in $d = 2$ are classified by topological invariants that take integer values. Chern insulators and topological fermionic excitations in spinless $p_x + ip_y$ BdG superconductors are condensed matter examples from classes A and D, respectively. At the single particle level, the only difference between class A and D is the presence of a particle-hole symmetry $\Theta_{PH}H\Theta_{PH}^{-1} = -H$ for the latter, with $\Theta_{PH}$ anti-unitary and $\Theta_{PH}^2 = e^{i\pi}$. All topological phases from these two classes can be generated from the Hamiltonian:

$$H = \frac{1}{2} \sum_{j=1,2} \sigma_j \otimes (S_j - S_j^\dagger) + \sigma_3 \otimes (m + \frac{1}{2} \sum_{j=1,2} (S_j + S_j^\dagger)).$$

The phase diagram of \((22)\) consists of a topological phase for $m \in (-2, 2)$ and of a trivial phase in rest. Upon adding trivial bands, any Hamiltonian from DIII or AII classes can be adiabatically connected to the phases of \((22)\). TR-symmetry is implemented by $\Theta_{TR} = (i\Gamma_2 \otimes \mathbb{1})\mathcal{X}$, $\Theta_{TR}^2 = -1$, and PH-symmetry by $\Theta_{PH} = (i\Gamma_1 \Gamma_2 \Gamma_4 \otimes \mathbb{1})\mathcal{X}$, $\Theta_{PH}^2 = -1$. The combined TR and PH symmetries result in an additional CH-symmetry, given by $W_{CH} = \Gamma_0 \otimes \mathbb{1}$.

The mapped Hamiltonian is given by:

$$\rho(H) = -\frac{1}{2} \sum_{j=1,2} \rho(\sigma_j) \otimes (S_j - S_j^\dagger) + \rho(\sigma_3) \otimes (m + \frac{1}{2} \sum_{j=1,2} (S_j + S_j^\dagger)), $$

which can be used to attain the dynamical matrix $D = \rho(H) + \rho(\sigma) \otimes |m\rangle \langle m|$. The transferred PH-symmetry operator is $\Theta_{PH} = \rho(\sigma_1 \otimes \mathbb{1})\mathcal{X}$ and we verified numerically that \((10)\) holds and $\Theta_{PH}^2 = -1$.
which can be used to attain the dynamical matrix $D = \rho(H) + (2 + |m|)\mathbb{I}$). The transferred symmetry operators are $\hat{\Theta}_{\text{TR}} = \rho(i\Gamma_2 \otimes \mathbb{I})f\mathcal{K}$ and $\hat{\Theta}_{\text{PH}} = \rho(i\Gamma_1 \Gamma_3 \otimes \mathbb{I})f\mathcal{K}$ respectively, and we have verified numerically that they satisfy the identity \cite{10}. Furthermore, the transferred CH-symmetry operator is $\hat{W}_{\text{CH}} = \rho(\Gamma_0 \otimes \mathbb{I})$.

C Class. In $d = 2$, the topological phases from this class are classified by $2\mathbb{Z}$ and the generating model is:

$$H = \frac{1}{2}(\Gamma_3 \otimes (S_1 - S_1^\dagger)) + \Gamma_3 \otimes (S_2 - S_2^\dagger) + \Gamma_1 \otimes (m + \frac{1}{2}\sum_{j=1,2} (S_j + S_j^\dagger)). \quad (24)$$

The phase diagram of \cite{24} consists of the topological phases with Chern number $-2$ for $m \in (-2,0)$ and $+2$ for $m \in (0,2)$, as well as of a trivial phase if $m \notin (-2,2)$. By staking \cite{24}, one can generate all the topological phases from class C. The PH-symmetry operators are $\hat{\Theta}_{\text{PH}} = (i\Gamma_2 \otimes \mathbb{I})\mathcal{K}$ with $\hat{\Theta}_{\text{PH}}^2 = -I$.

The mapped Hamiltonian is:

$$\rho(H) = -\frac{1}{2}\rho(i\Gamma_1) \otimes (S_1 - S_1^\dagger) - \frac{1}{2}\rho(i\Gamma_3) \otimes (S_2 - S_2^\dagger)$$

$$+ \rho(i\Gamma_4) \otimes (m + \frac{1}{2}\sum_{j=1,2} (S_j + S_j^\dagger)), \quad (25)$$

which can be used to attain the dynamical matrix $D = \rho(H) + (2 + |m|)\mathbb{I}$. The transferred PH-symmetry operation is given by $\hat{\Theta}_{\text{PH}} = \rho(i\Gamma_2 \Gamma_4 \otimes \mathbb{I})\mathcal{K}$ and we verified numerically that it satisfies the identity \cite{10}.

Physical Models. The symmetry operators as well as the mappings in \cite{21, 23, 25} are all computed explicitly in the supplemental material. They can be used in conjunction with the procedure explained in section II to translate \cite{21, 23, 25} in physical models, as shown in Figs. 2, 3 and 4 respectively. Since the models are periodic, we only need to specify $\hat{a}_{x,x}$ and $\hat{a}_{x,x+e_j}$, $j = 1,2$.

Edge Modes. The rendering of the spectra of $\hat{\rho}(H)$ as function of quasi-momentum parallel to the boundary is shown in Fig. 5, for all the above models. The overlapping spectra corresponding to the $\Pi_k$ sectors have been artificially displaced in order to be visualized. The numerical calculations confirm that $\Pi_k, \hat{\rho}(H)\Pi_k$ displays topological edge modes with the expected symmetries and degeneracies.

Dimension 3. DIII-AIII-AII Class. We start with the DIII class, which is the most symmetric. The topological phases from this class in $d = 3$ are classified by $\mathbb{Z}$ and the generating model is:

$$H = \frac{1}{2}\sum_{j=1}^{3} \Gamma_j \otimes (S_j - S_j^\dagger) + \Gamma_4 \otimes (m + \frac{1}{2}\sum_{j=1}^{3} (S_j + S_j^\dagger)). \quad (26)$$

The phase diagram of \cite{26} consist of the topological phases with winding number $+1$ for $m \in (-3,-1) \cup (1,3)$, and with winding $-1$ for $m \in (-1,1)$, and a trivial phase in rest. By staking \cite{25} we can generate all the topological phases from DIII-class in $d = 3$.

The TR-symmetry is implemented by $\hat{\Theta}_{\text{TR}} = (i\Gamma_2 \otimes \mathbb{I})\mathcal{K}$ with $\hat{\Theta}_{\text{TR}}^2 = -1$, the PH-symmetry is implemented by $\hat{\Theta}_{\text{PH}} = (i\Gamma_1 \Gamma_3 \otimes \mathbb{I})\mathcal{K}$, while the CH-symmetry operator can be expressed as $\hat{W}_{\text{CH}} = \Gamma_0 \otimes \mathbb{I}$. All class can be generated from the DIII class by breaking PH-symmetry but keeping the TR-symmetry. The AII class contains the well known 3D topological insulator which exhibits and odd number of surface Dirac cones and the bulk magneto-electric effect. Breaking the TR-symmetry of the DIII class gives the AIII class, which is marked by the presence of an integer number of surface Dirac cones.

The mapped Hamiltonian takes the form:

$$\rho(H) = -\frac{1}{2}\sum_{j=1}^{3} \rho(i\Gamma_j) \otimes (S_j - S_j^\dagger)$$

$$+ \rho(i\Gamma_4) \otimes (m + \frac{1}{2}\sum_{j=1}^{3} (S_j + S_j^\dagger)). \quad (27)$$

which can be used to attain the dynamical matrix $D = \rho(H) + (3 + |m|)\mathbb{I}$. The transferred TR-symmetry and PH-symmetry operators are given by $\hat{\Theta}_{\text{TR}} = \rho(i\Gamma_2 \otimes \mathbb{I})\mathcal{K}$ and $\hat{\Theta}_{\text{PH}} = \rho(i\Gamma_1 \Gamma_3 \otimes \mathbb{I})\mathcal{K}$, respectively, while chiral symmetry is implemented by the operator $\hat{W}_{\text{CH}} = \rho(\Gamma_0 \otimes \mathbb{I})$. We have verified numerically that these operators satisfy the identity \cite{10}. The symmetry operations as well as the mappings in \cite{27} are all computed explicitly in the supplemental material. They can be used in conjunction with the procedure explained in section II to translate \cite{27} in a physical model, as shown in Figs. 6.

The spectrum of \cite{27} with a clean surface and appropriate symmetry breaking potentials is reported in Fig. 7. It confirms the presence of topological singularities, as well as the predicted symmetries and degeneracies.
VII. CONCLUSION

The algorithm introduced in section 11 enables one to make identical classical copies of any topological condensed matter system from the classification table. This provides an alternative venue to probe, confirm and apply the rich and exotic physical properties of the topological phases. Hence, our algorithm may resolve some outstanding open problems such as resolving the critical regime and measuring the critical exponents of the topological transitions in the presence of disorder, verifying first hand the robustness of the edge/surface states against disorder, confirming the quantized bulk physical responses, such as the magneto-electric effect in AII-class and stabilizing and manipulations of majorana states.

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VIII. SUPPLEMENTAL SECTION

A. Proof of (10)

Note that $U = \rho(iI)$, hence $U$ automatically commutes with $\rho(W)$, because:

$$U \rho(W) = \rho(U) \rho(W) = \rho(iI) W = \rho(W(iI)) = \rho(W) U,$$

where the fact that $\rho$ is an algebra morphism was used in an essential way. An immediate consequence is that $\rho(W)$ commutes with $\Pi_\pm$. Furthermore:

$$e^{\Pi_\pm \rho(H) \Pi_\pm} = \Pi_\pm \rho(\Theta H \Theta^{-1}) \Pi_\pm = \Pi_\pm \rho(W H^* W^\dagger) \Pi_\pm,$$

and, using again the fact that $\rho$ is a morphism, we continue:

$$e^{\Pi_\pm \rho(H) \Pi_\pm} = \rho(W) \Pi_\pm \rho(H^*) \Pi_\pm \rho(W)^\dagger = \left( \mathcal{K} \rho(W) \right) \Pi_\pm \rho(H^*) \Pi_\pm \left( \mathcal{K} \rho(W) \right)^{-1}.$$

At this point, we use the transformation $J$ and its properties to conclude:

$$e^{\Pi_\pm \rho(H) \Pi_\pm} = \left( \mathcal{K} \rho(W) \right) J \Pi_\pm \rho(H) \Pi_\pm \left( \mathcal{K} \rho(W) \right)^{-1}.$$

B. Proof of (11)

The second identity in (11) holds because $\mathcal{K}$ and $W$ commute, hence $W^* = W$ and, as such, $\rho(W)$ and $J$ also commute.

C. Explicit matrix mapping

In this section we provide explicit expressions for the mapped matrices. The mappings of the Pauli matrices are:

$$\rho(-iI_2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho(i_1) = \begin{pmatrix} i_1 & 0 \\ 0 & i_1 \end{pmatrix}, \quad \rho(i_2) = \begin{pmatrix} 0 & i_2 \\ -i_2 & 0 \end{pmatrix}. $$

For D-class in $d = 2$, the mapped PH-symmetry is:

$$\Theta_{PH} = \rho(\sigma_1 \otimes I) J \mathcal{K} = \begin{pmatrix} 0 & \sigma_1 \otimes 0 \\ \sigma_1 \otimes 0 & 0 \end{pmatrix} \otimes I \mathcal{K}.$$

The mappings of $\Gamma$ matrices are:

$$\rho(-iI_4) = \begin{pmatrix} 0 & -i_4 \\ i_4 & 0 \end{pmatrix}, \quad \rho(\Gamma_1) = \begin{pmatrix} i_1 & 0 \\ 0 & i_1 \end{pmatrix}, \quad \rho(\Gamma_0) = \begin{pmatrix} 0 & i_0 \\ i_0 & 0 \end{pmatrix},$$

and

$$\rho(\Gamma_1) = \begin{pmatrix} 0 & 0 & 0 & i_2 \\ 0 & 0 & 0 & -i_2 \\ 0 & 0 & -i_2 & 0 \\ 0 & i_2 & 0 & 0 \end{pmatrix}, \quad \rho(\Gamma_4) = \begin{pmatrix} 0 & 0 & 0 & -i_2 \\ 0 & 0 & -i_2 & 0 \\ 0 & i_2 & 0 & 0 \\ -i_2 & 0 & 0 & 0 \end{pmatrix}.$$

The explicit expression for the mapped TR-symmetry operator for the classes DIII-AII in $d = 2$ and for the classes DIII-AIII in $d = 3$ are given by:

$$\tilde{\Theta}_{TR} = \rho(\sigma_2 \otimes I) J \mathcal{K} = \begin{pmatrix} 0 & 0 & 0 & i_2 \\ 0 & 0 & 0 & -i_2 \\ 0 & 0 & -i_2 & 0 \\ i_2 & 0 & 0 & 0 \end{pmatrix} \otimes I \mathcal{K}.$$

The mapped PH-symmetry operator for the DIII class in $d = 2$ and 3 is given by:

$$\tilde{\Theta}_{PH} = \rho(i_1 \Gamma_3 \Gamma_4) J \mathcal{K} = \begin{pmatrix} 0 & 0 & 0 & -i_2 \\ 0 & 0 & -i_2 & 0 \\ 0 & i_2 & 0 & 0 \\ -i_2 & 0 & 0 & 0 \end{pmatrix} \otimes I \mathcal{K}.$$

The mapped PH-symmetry operator for the class C in two-dimensions is given by

$$\tilde{\Theta}_{PH} = \rho(\Gamma_2 \Gamma_4) J \mathcal{K} = \begin{pmatrix} 0 & 0 & 0 & i_2 \\ 0 & 0 & 0 & -i_2 \\ 0 & i_2 & 0 & 0 \\ -i_2 & 0 & 0 & 0 \end{pmatrix} \otimes I \mathcal{K}. \tag{28}$$