MULTI-WAY EXPANDERS AND IMPRIMITIVE GROUP ACTIONS ON GRAPHS.

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Abstract. For \( n \geq 2 \), the conception of \( n \)-way expanders was defined. Bigger \( n \) gives a weaker notion in general, and 2-way expanders coincide with expanders in usual sense. Koji Fujiwara has asked whether these concepts are equivalent to that of ordinary expanders for all \( n \) for a sequence of Cayley graphs. In this paper, we answer his question in the affirmative. Furthermore, we obtain universal inequalities on multi-way isoperimetric constants on any vertex-transitive finite graph, and show that gaps between these constants implies the imprimitivity of the group action on the graph. In the appendix, we make an estimation of Banach spectral gaps into noncommutative \( L_p \) spaces for \( p \in (1, \infty) \).

1. Introduction

In this paper, let \( n \) represent a natural number at least 2. We assume that all graphs \( G = (V, E) \) are finite, undirected, regular, and without multiple edges or self-loops. We use \( d \) for the regularity of \( G \). For a Cayley graph \( G = \text{Cay}(\Gamma, S) \), we use the right-multiplication to connect edges in order to have the left-action by graph isomorphisms. We allow the case where \( G \) is disconnected (for Cayley graphs, this amounts to the case where \( S \) does not generate the whole \( \Gamma \)). For disjoint subsets \( A, B \) of the vertex set \( V \), \( \partial(A, B) \) denotes the edge boundary (:= \( \{ e = (u, v) \in E : u \in A, v \in B \} \)), and \( \partial A \) denotes \( \partial(A, V \setminus A) \). In addition, \( \delta(A, B) \) denotes the symmetric vertex boundary (:= \( \{ u \in A : \exists e = (u, v) \in \partial(A, B) \} \sqcup \{ v \in B : \exists e = (u, v) \in \partial(A, B) \} \)), and \( \delta A \) denotes \( \delta(A, V \setminus A) \). For \( l \in \mathbb{N} \), by \( S_l \), we denote the symmetric group of degree \( l \). Let \( J_l \) be the set \( \{1, 2, \ldots, l\} \).

For \( (|V| \geq)n \geq 2 \), the following three quantities are defined.

**Definition 1.1.** Let \( G = (V, E) \), \( d = \text{deg}(G) \) and \( 2 \leq n \leq |V| \).

1. The \( n \)-way isoperimetric constant is defined by \( h_n(G) := \min \max_{1 \leq i \leq n} |\partial A_i|/|A_i| \).

Here the minimum is taken over all partitions of \( V \) into \( n \)-non-empty disjoint subsets \( V = \bigsqcup_{i=1}^n A_i \).

2. The \( n \)-way vertex isoperimetric constant is defined by \( g_n(G) := \min \max_{1 \leq i \leq n} |\delta A_i|/|A_i| \).

Here \( (A_1, \ldots, A_n) \) runs over the same partitions as above.

3. The \( \lambda_n(G) \) is the \( n \)-th nonnegative eigenvalue (with multiplicities) of the non-normalized combinatorial Laplacian \( L(G) := dI_V - A(G) \), where \( A(G) \) denotes the adjacency matrix of \( G \). Namely, the eigenvalues of \( L(G) \) is \( \lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_{|V|} \).
Note that in the standard literature \( \lambda_2 \) here is written as \( \lambda_1 \). We also note that in our definition above, we use nonnormalized ones.

The \( h_2, g_2, \lambda_2 \) are fundamental in spectral graph theory. They are non-zero if and only if \( G \) is connected, and \( 2h_2/d \leq g_2 \leq 2h_2 \). More deep relationships are Cheeger inequalities, which state as follows:

- (Alon–V. Milman [AM85]): \( \lambda_2/2 \leq h_2 \leq \sqrt{2d\lambda_2} \);
- (Bobkov–Houdré–Tetali [BHT00]): \( \lambda_2 \geq (\sqrt{g_2 + 1} - 1)^2/4 \).

The first one implies that \( \lambda_2/2 \leq h_2^2 \leq \sqrt{2\lambda_2} \), where \( h_2^2 := h_2/d \) and \( \lambda_2 := \lambda_2/d \) are the normalized versions. However, it is impossible to exclude contributions of \( d \) from the right-hand side of the first inequality, see an example in Section 2. The second inequality shows that we can bound \( \lambda_2 \) from below by \( g_2 \) without any affection of \( d \).

We say that an infinite sequence \( \{G_m = (V_m, E_m)\}_{m \in \mathbb{N}} \) is a sequence of expanders if \( \sup_m \deg(G_m) < \infty \); \( \lim_{m \to \infty} |V_m| = \infty \); and \( \inf_m h_2(G_m) > 0 \) hold true. The most important condition is the third one. By Cheeger inequalities above, that condition is equivalent (under the first condition) to \( \inf_m g_2(G_m) > 0 \), as well as to \( \inf_m \lambda_2(G_m) > 0 \).

In terms of multi-way expansions, the following notion is defined. The notion of 2-way expanders is identical to that of expanders.

**Definition 1.2.** For fixed \( n \), a sequence of finite graphs \( \{G_m\}_{m \in \mathbb{N}} \) is called a sequence of \( n \)-way expanders if \( \sup_m \deg(G_m) < \infty \); \( \lim_{m \to \infty} |V_m| = \infty \); and \( \inf_m h_n(G_m) > 0 \) hold true.

We note that under the first condition, the third one is equivalent to \( \inf_m g_n(G_m) > 0 \), as well as to \( \inf_m \lambda_n(G_m) > 0 \). Indeed, this follows from \( 2h_n/d \leq g_n \leq 2h_n \) and the following higher-order Cheeger inequality by Lee–Gharan–Trevisan [LGT12]:

\[
\frac{1}{2} \lambda_n(G) \leq h_n(G) \leq O(n^3)\sqrt{d\lambda_n(G)}.
\]

(Note that \( \rho_G \) in their paper and \( h_n(G) \) satisfy that \( \rho_G \leq h_n(G) \leq n\rho_G \). See the proof of [LGT12, Theorem 3.8].)

Our first result is a higher-order Cheeger inequality on multi-way vertex isoperimetric constants.

**Theorem A.** For a finite graph \( G \) and \( 2 \leq n \leq |V| \), we have that

\[
O(n^6)\lambda_n(G) \geq \left( \frac{g_n(G)}{n} + 1 - 1 \right)^2.
\]

Note that \( h_n, g_n, \lambda_n \) are non-decreasing for \( n \) (for first two, observe that \( |\partial(A \cup B)| \leq |\partial A| + |\partial B| \) and \( |\delta(A \cup B)| \leq |\delta A| + |\delta B| \) for disjoint \( A, B \subseteq V \), and hence that being \( (n+1) \)-way expanders are weaker than being \( n \)-expanders in general. This is strictly weaker. Indeed, pick some sequence of expanders \( \{H_k\}_{k \in \mathbb{N}} \) and construct a new family of graphs \( \{G_m\}_{m \in \mathbb{N}} \) as follows: connect components of the disjoint union \( \bigsqcup_{i=1}^m H_{m+i} \) each other by small number of edges (it can be done in such a way that resulting graphs are regular) and set it as \( G_m \). Then \( \{G_m\}_{m \in \mathbb{N}} \) are \( (n+1) \)-way
expanders but not $n$-way expanders. Conversely, M. Tanaka [Tan11, Theorem 2] has showed that if $h_{n+1}(G)$ is sufficiently larger than $h_n(G)$, then $G$ is constructed in the way above.

However, resulting graphs from the construction above do not seem homogeneous. In this point of view, Koji Fujiwara has asked the following question.

**Question 1.3. (K. Fujiwara)**

For a sequence of finite connected Cayley graphs, does the property of being $n$-way expanders in fact imply that of being expanders for every $n$?

We may ask stronger question as follows:

**Question 1.4.** (1) Does there exist a universal constant $C = C(n)$, depending only on $n$, such that for any finite connected Cayley graph $G$, $h_{n+1}(G) \leq C h_n(G)$ holds true?

(2) The same question with replacing $h_n$’s with $g_n$’s.

His original idea is to translate “thin” part to “thick” part by the group action and to lead a contradiction if there were some counterexample to Question 1.3. This idea is the first step to deal with these questions.

In this paper, we provide the satisfactory answers to all of these questions. the answer to Question 1.3 is affirmative. Item (1) of Question 1.4, however, has the negative answer. Surprisingly, nevertheless, we answer item (2) in the affirmative. These answers follow from the following universal inequalities for finite connected vertex-transitive graphs (observe that $g_{n+1}(G) \leq 2n + 1$ always holds).

**Theorem B** (Main Theorem). Let $G$ be a finite connected vertex-transitive graph and $2 \leq n \leq |V| - 1$. Then we have that

$$h_n(G) \geq \frac{h_{n+1}(G)}{10n + h_{n+1}(G)} \quad \text{and} \quad g_n(G) \geq \frac{2g_{n+1}(G)}{20n + g_{n+1}(G)}.$$  

In particular, $g_{n+1}(G) \leq (11n + 1) g_n(G)$.

**Corollary 1.5.** Let $\{G_m\}_{m \geq N}$ be a sequence of finite connected vertex-transitive graphs such that $\lim_{m \to \infty} |V_m| = \infty$ (we do not assume that $\sup_m \deg(G_m) < \infty$). Then for any $n \geq 2$, $\inf_m h_{n+1}(G_m) > 0$ implies $\inf_m h_n(G_m) > 0$; and $\inf_m g_{n+1}(G_m) > 0$ implies $\inf_m g_n(G_m) > 0$.

In particular, if $\{G_m\}_{m \in \mathbb{N}}$ are $n$-way expanders for some $n \geq 2$, then they are in fact expanders.

We remark that since $h_{n+1}(G) \leq d := \deg(G)$, Theorem B implies that $h_{n+1}(G) \leq (10n + d) h_n(G)$. However it is impossible to avoid the contribution of the degree form the right-hand side of this inequality. Also we note that Theorem B implies that if $h_n(G) < 1 - \epsilon$ for some $\epsilon > 0$, then $h_{n+1}(G) < \frac{10m}{r} h_n(G)$. We however have no hope to obtain any nontrivial estimate of $h_{n+1}(G)$ as soon as $h_n(G) \geq 1$. We show that this vaulue 1 is the optimal critical value. We also warn that if we consider normalized $h_n$ (by dividing $h_n$ by $d$) or normalized $\lambda_n$, then the corresponding assertion to Corollary 1.5 is no longer true. We discuss in Section 3 for the assertions here with counterexamples. Our main results may suggest that $g_n$ behaves better than
$h_n$, and that nonnormalized one behaves better than normalized one. To the best knowledge of the author, similar results to above for (nonnormalized) $\lambda_n$’s seem to be open. More precisely, Li \cite{Li80} has showed a universal inequality for homogeneous manifold, but a naive application of his result to a vertex-transitive graph fails to be true, see Section 3. The problem here may be because the vertex-transitivity of a graph can be regarded as a weaker assumption than the homogeneity of a manifold in the corresponding setting. Our inequalities in Theorem 3.4 may have some similarity to inequalities in \cite{FS13} and in \cite{Fun13}: their inequalities are universal, independent of dimensions of manifolds with nonnegative Ricci curvature; and ours are universal, independent of degrees of vertex-transitive graphs.

We furthermore show that if the group action $\Gamma \acts G$ possesses certain “homogeneity”, then the answer to (1) of Question 1.4 is affirmative. This condition is stated in terms of primitive group actions (for the definition of a system of imprimitivity of size $n$, see Definition 5.1). More precisely, we show that gaps between $n$-way isoperimetry and $(n+1)$-way one implies the existence of a system of imprimitivity of size $n$ “sufficiently near” from a fixed realizer of $n$-way isoperimetry.

**Theorem C.** Let $G$ be a finite vertex-transitive graph (possibly disconnected) and $2 \leq n \leq |V| - 1$. If $h_{n+1}(G) > 2(n+1)h_n(G)$, then there exists decompositions $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_n$ and $V = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ into $n$ non-empty sets which satisfy the following properties:

(i) The $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_n$ is a system of imprimitivity (of size $n$) for $\text{Aut}(G) \acts V$.

(ii) The $V = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ achieves $h_n(G)$.

(iii) For any $1 \leq i \leq n$, $|V_1 \triangle A_i| \leq \frac{h_n(G)}{h_{n+1}(G)} |V|$. In fact, we may obtain $(V_i)_i$ with (i) and (iii) for any given $(A_i)_i$ with (ii).

In particular, for $\Gamma$ a group which acts on $G$ vertex-transitively, if there exists no system of imprimitivity of size $n$ for $\Gamma \acts V$, then $h_{n+1}(G) \leq 2(n+1)h_n(G)$ holds.

The same results hold true if we replace $h_n(G)$ and $h_{n+1}(G)$, respectively, with $g_n(G)$ and $g_{n+1}(G)$.

This theorem may relate to the famous problem of M. Kac, “Can one hear the shape of a drum?”, which asks whether we can detect shapes from spectral data. A baby case of Theorem C is where $h_n = 0$ and $h_{n+1} > 0$. Then $G$ has exactly $n$ connected components, and we can take the associated decomposition for both $(V_i)_i$ and $(A_i)_i$. We will see in Corollary 6.1 that for a connected vertex and edge transitive graph, item (1) of Question 1.4 has the positive answer. To prove Theorem 3.4 first we verify Theorem C (and a weak form of it). Then we deal with the general case, that covers the case where $h_{n+1} > 2(n+1)h_n$ (or $g_{n+1} > 2(n+1)g_n$).

**Organization of this paper.** In Section 2 we prove Theorem A. Section 3 is for counterexamples to (1) of Question 1.4. In Section 4 we prove Theorem 3.4, which is a weak form of Theorem C. Section 5 is devoted to the proof of Theorem C. In Section 6, Theorem B shall finally be established. In Appendix, we obtain an
application of the results in a previous paper [Mim14] of the author to bound Banach spectral gaps of finite graphs into noncommutative \( L_p \) spaces, from the classical ones.

2. Proof of Theorem A

Proof. Theorem A essentially follows from the works in [BHT00] and [LGT12]. For a non-zero \( f \in \ell_2(V, \mathbb{R}) \), the Rayleigh quotient of \( f \) is given by

\[
\text{Ray}_G(f) := \frac{\sum_{(u,v) \in E} |f(u) - f(v)|^2}{\sum_{v \in V} |f(v)|^2}.
\]

Note that we consider nonnormalized one. Then the following is easily derived from arguments in [BHT00] (compare with Lemma 2.2 in [LGT12]).

Lemma 2.1. For any \( 0 \neq f \in \ell_2(V, \mathbb{R}) \), there exists a subset \( \emptyset \neq S \subseteq \text{supp}(f) \) such that

\[
4\text{Ray}_G(f) \geq \left( \sqrt{\frac{|\delta S|}{|S|}} + 1 - 1 \right)^2.
\]

This lemma together with Theorem 1.5 in [LGT12] ends our proof (see also the proof of Theorem 3.8 in [LGT12]). \( \square \)

Theorem A together with the higher-order Cheeger inequality by [LGT12] implies that for a fixed \( n \), and for \( \{G_m\}_{m \in \mathbb{N}} \),

\[
\inf_m g_n(G_m) > 0 \implies \inf_m \lambda_n(G_m) > 0 \implies \inf_m h_n(G_m) > 0.
\]

These three conditions are all equivalent if \( \sup_m \text{deg}(G_m) < \infty \). However in general case, no two of these three are equivalent. We explain constructions of counterexamples for \( n = 2 \) in more detail. Consider the \( m \)-cycle with \( k \)-multiple edges and call it \( C_{m,k} \). Then we have that \( g_2(C_{m,k}) \sim \frac{1}{m}, \lambda_2(C_{m,k}) \sim \frac{k}{m}, \text{ and } h_2(C_{m,k}) \sim \frac{k}{m} \). Therefore, \( \{G_m\} := \{C_{m,m}\}_m \) serves as a counterexample to “\( \inf_m \lambda_2(G_m) > 0 \implies \inf_m g_2(G_m) > 0 \)” and \( \{G_m\}_m := \{C_{m,m}\}_m \) serves as a counterexample to “\( \inf_m h_2(G_m) > 0 \implies \inf_m \lambda_2(G_m) > 0 \).” These examples do not consist of Cayley graphs, and we may take the following modification: let \( \Gamma_{m,k} := \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}, S_{m,k} := \{1\} \times \mathbb{Z}/k\mathbb{Z}, \text{ and } C'_{m,k} := \text{Cay}(\Gamma_{m,k}, S_{m,k}) \). Then \( \{C'_{m,m}\}_m \) and \( \{C'_{m,m}\}_m \), respectively, serve as counterexamples of the above two assertions. The second counterexample in particular implies that in the Alon–Milman inequality, we cannot remove contribution of \( d \) from the right-hand side, see Section 1.

3. Counterexamples to (1) in Question 1.4

First we give a counterexample for \( n = 2 \). Let \( H = \text{Cay}(\Lambda, T) \) have very big \( h_2 \) (for instance, take a complete graph). This implies that \( |T| \) is also very big. Let \( \Gamma := \Lambda \times \mathbb{Z}/2\mathbb{Z} \), and set a generating set \( S := (T \times \{0\}) \cup \{(e_{\Lambda}, 1)\} \). Then the Cayley graph \( G := \text{Cay}(\Gamma, S) \) is a counterexample (note that this graph is the graph product of \( H \) and \( \text{Cay}(\mathbb{Z}/2\mathbb{Z}, \{1\}) \)). Indeed, by decomposing as \( \Gamma = (\Lambda \times \{0\}) \cup (\Lambda \times \{1\}) \), we have that \( h_2(G) \leq 1 \). However, Lemma 1 in [Tan11] implies that \( h_3(G) \geq h_2(H) \),
and this shows that we can have \( h_3(G) \) as large as we wish with appropriate choices of \((\Lambda, T)\).

To show that \(1\) is the critical value for \(h_n\) to bound \(h_{n+1}\) (see Section 11), we modify this construction if \(n \geq 3\). Take a dihedral group \(D_{2n} := \langle a, b \mid a^2 = b^2 = (ab)^n = e_{D_{2n}} \rangle\), and from \((\Lambda, T)\) construct \((\Gamma, S)\) as follows: \(\Gamma := \Lambda \times D_{2n}\), and \(S := (T \times \langle e_{D_{2n}}, a \rangle) \cup \{(e, a)\}\). Then for \(G = \text{Cay}(\Gamma, S)\), a similar argument to above tells that \(h_n(G) \leq 1\) but that \(h_{n+1}(G)\) can be arbitrarily big. To see this, more precisely, decompose \(V(G) = \Gamma\) as \(\Gamma = \bigsqcup_{i=0}^{n-1} (\Lambda \times \{(ab)^i, (ab)^i a\})\). Then \(h_n(G) \leq 1\), and \([\text{Tan11, Lemma}1]\) shows that \(h_{n+1}(G) \geq h_2(H)\).

In the view of \([\text{Tan11, Lemma}6]\), these are also counterexamples to the corresponding question to \(\lambda_n\)’s in Question 1.4. In particular, we cannot naively apply Li’s results in \([\text{Li80}]\) on \((\text{normalized})\) \(\lambda_n\)’s to the case of connected vertex-transitive graphs (because otherwise \([\text{Li80, Theorem}11]\) would imply that \(\lambda_{n+1}(G) < 5\lambda_n(G)\)). We in addition note that if we consider the normalized \(h_n\) (or the normalized \(\lambda_n\)) or weighted cases, then the corresponding assertions in Corollary 1.5 fail to be true. More precisely, in weighted cases, we put a weight on \((e_A, b)\) relative to the other elements in \(S\) in the example in the paragraph above, then this construction serves as counterexamples to the assertions both on weighted \(h\) and weighted \(\lambda\). These counterexamples for weighted cases may be constructed even in such a way of that the degrees are uniformly bounded.

4. Universal inequality for graphs without transitive action on \(n\) point set

Here we prove the following theorem, which is a weak form of Theorem C. For each \(n \geq 2\), we define condition \((*_{n})\) for a finite group \(\Gamma\) as follows:

\[ (*_{n}) : \text{no action } \Gamma \text{ on an } n\text{-point set is transitive.} \]

This condition is characterized by the non-existence of sugroups of index \(n\).

**Theorem 4.1.** Let \(G\) be a finite vertex-transitive graph (possibly disconnected) and \(n \geq 2\). Take any group \(\Gamma\) that acts on \(G\) vertex-transitively. If \(\Gamma\) satisfies condition \((*_{n})\), then \(h_{n+1}(G) \leq 2(n+1)h_n(G)\) and \(g_{n+1}(G) \leq 2(n+1)g_n(G)\) hold true.

One example to which the theorem above applies is a Cayley graph of \(S_N\) for \(N \geq 5\). Because \(S_N\) has only three normal subgroups: \(\{e\}\), the alternating group \(A_N\) of degree \(N\), and \(S_N\) itself, Theorem 1.1 applies to any Cayley graph of \(S_N\) for all \(3 \leq n \leq N - 1\).

In the proof of this theorem, we use the following lemmata. First one is obvious, and we will use it in this paper without mentioning. Last one is a key lemma.

**Lemma 4.2.** Let \(G = (V, E)\) be a finite graph and \(A, B \subseteq V\).

1. Let \(A \cap B = \emptyset\). Then for any \(\gamma \in \text{Aut}(G)\), \(|\partial(\gamma \cdot A, \gamma \cdot B)| = |\partial(A, B)|\). In particular, \(|\partial(\gamma \cdot A)| = |\partial A|\).
2. For \(A \subseteq A' \subseteq V\) and \(B \subseteq B' \subseteq V\) with \(A' \cap B' = \emptyset\), \(|\partial(A, B)| \leq |\partial(A', B')|\).
3. We have that \(|\partial(A \cap B)| \leq |\partial A| + |\partial B|\).
All of the corresponding statements remain true if we replace all $\partial$ with $\delta$ in the setting above.

**Lemma 4.3.** Let $\varepsilon > 0$. Let a finite group $\Lambda$ act on a finite set $W$ transitively. Assume that a non-empty subset $C \subseteq W$ satisfies that for any $\lambda \in \Lambda$, $|C \Delta \lambda \cdot C| \leq \varepsilon |C|$. Then $|W \setminus C| \leq \frac{\varepsilon}{2} |W|$. In particular, if $|C| \leq 1/2 |W|$, then $\varepsilon \leq 1$.

**Proof.** (Lemma 4.3) On the (finite dimensional) Banach space $\ell_{1,0}(\Lambda) := \{\xi \in \ell_{1}(W) : \sum_{w \in W} \xi(w) = 0\}$, a linear isometric $\Lambda$-representation $\pi$ is induced by the permutations $\Lambda \curvearrowright W$, namely, we set as $\pi(\gamma)\xi(x) := \xi(\gamma^{-1} \cdot x)$. Note that there does not exist nonzero $\pi(\Lambda)$-invariant vector in $\ell_{1,0}(W)$ because $\Lambda \curvearrowright W$ is transitive. Set $\xi := |W \setminus C|\chi_{C} - |C|\chi_{W \setminus C}(= |W|\chi_{C} - |C|1) \in \ell_{1,0}(W)$, where $\chi_{A}$ denotes the characteristic function of $A$ and $1$ means the constant $1$ function. Then $\|\xi\| = 2|C||W \setminus C|$, where $\|\cdot\|$ is the $\ell_{1}$-norm. By the assumption of the lemma, for any $\lambda \in \Lambda$, $\|\xi - \pi(\lambda)\xi\| \leq \varepsilon \|C\| |W|$.

Set $\eta := |\Lambda|^{-1} \sum_{\lambda \in \Lambda} \pi(\lambda)\xi$. Because $\eta$ is $\pi(\Lambda)$-invariant, $\eta$ must be $0$. We also have that

$$\|\xi - \eta\| = \frac{1}{|\Lambda|} \left\| \sum_{\lambda \in \Lambda} (\xi - \pi(\lambda)\xi) \right\| \leq \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \|\xi - \pi(\lambda)\xi\|.$$ 

Therefore we conclude that $2|C||W \setminus C| \leq \varepsilon |C||W|$. \( \square \)

**Proof of Theorem 4.1.** We will only show the assertion for $h_{n}$ (the proof for $g_{n}$ goes along exactly the same way). Suppose, to the contrary, that $h_{n+1}(G) > 2(n + 1)h_{n}(G)$. Note that this in particular implies that $6h_{n}(G) < h_{n+1}(G)$. Let $(A_{1}, \ldots, A_{n})$ be a (non-empty) $n$-partition of $V$ which achieves $h_{n}(G)$. Without loss of generality, we may assume that $|A_{1}|$ is the largest among $|A_{1}|, \ldots, |A_{n}|$.

Fix $\gamma \in \Gamma$. For each $1 \leq k \leq n$, decompose $V$ into $\gamma^{-1} \cdot A_{k} \cap A_{1}, A_{1} - \gamma^{-1} \cdot A_{k},$ and $A_{2}, \ldots, A_{n}$. Because

$$|\partial(\gamma^{-1} \cdot A_{k} \cap A_{1}, A_{1} - \gamma^{-1} \cdot A_{k})| \leq |\partial A_{k}| \leq h_{n}(G)|A_{k}| \leq h_{n}(G)|A_{1}|,$$

we have that

$$|\partial(\gamma^{-1} \cdot A_{k} \cap A_{1})| \leq h_{n}(G)|A_{1}| + |\partial(A_{1})| \leq 2h_{n}(G)|A_{1}|,$$

and that $|\partial(A_{1} - \gamma^{-1} \cdot A_{k})| \leq 2h_{n}(G)|A_{1}|$. From the condition of $h_{n+1}(G)$, we conclude the following: for fixed $\gamma \in \Gamma$, for each $1 \leq k \leq n$, either of the following $(i)_{1}$ and $(ii)_{1}$ holds true:

$$(i)_{1} : |\gamma \cdot A_{1} \cap A_{k}| \geq \left( 1 - \frac{2h_{n}(G)}{h_{n+1}(G)} \right) |A_{1}|;$$

$$(ii)_{1} : |\gamma \cdot A_{1} \cap A_{k}| \leq \frac{2h_{n}(G)}{h_{n+1}(G)} |A_{1}|.$$ 

(Note that if either of two sets in the decomposition is empty, then the assertion above trivially holds.) Because $4h_{n}(G) < h_{n+1}(G)$, these two options are exclusive.

We claim that for each $\gamma \in \Gamma$, there exists a unique $k \in J_{n}(:= [1, n] \cap \mathbb{Z}$, recall in Section 1) which satisfies $(i)_{1}$. Indeed, if there exists at least 2 such $k$’s, then

$$|A_{1}| \geq \bigcup_{k=1}^{n} (\gamma \cdot A_{1} \cap A_{k}) \geq 2 \left( 1 - \frac{2h_{n}(G)}{h_{n+1}(G)} \right) |A_{1}|,$$
Suppose, to the contrary, that there exists $k > l$ for any $i$ and it is absurd. Also if there is no such $k$ such that $\Im(I_1) = J_i$ (note that $I_1(e) = 1$). An important observation is that for any $2 \leq j \leq l$, we have that

$$|A_j| \geq \left(1 - \frac{2h_n(G)}{h_{n+1}(G)}\right)|A_1| \left(\geq \frac{n}{n+1}|A_1|\right) \cdots (\circ)$$

because $I_1^{-1}(j) \neq \emptyset$. In the next paragraph, we proceed to an argument which is needed if $l \geq 2$. If $l = 1$, then we do not do anything there.

Fix $2 \leq j \leq l$. For fixed $\gamma \in \Gamma$, in a similar argument to one above, we have that for any $1 \leq k \leq n$,

$$|\partial(\gamma^{-1} \cdot A_k \cap A_j)| \leq h_n(G)(|A_1| + |A_j|), \quad |\partial(A_j - \gamma^{-1} \cdot A_k)| \leq h_n(G)(|A_1| + |A_j|).$$

Hence we similarly conclude that (for each $\gamma \in \Gamma$ and) for any $1 \leq k \leq n$, either of the following $(i)_j$ and $(ii)_j$ holds true:

$(i)_j : |\gamma \cdot A_j \cap A_k| \geq |A_j| - \frac{h_n(G)}{h_{n+1}(G)}(|A_1| + |A_j|) \left(\geq |A_j| - \frac{2h_n(G)}{h_{n+1}(G)}|A_1|\right);$

$(ii)_j : |\gamma \cdot A_j \cap A_k| \leq \frac{h_n(G)}{h_{n+1}(G)}(|A_1| + |A_j|) \left(\leq \frac{2h_n(G)}{h_{n+1}(G)}|A_1|\right).$

Note that from $(\circ)$ these two options are exclusive. In a similar argument to the one above, we can show that (for fixed $2 \leq j \leq l$ and) for each $\gamma \in \Gamma$, there exists a unique $k$ which satisfies $(i)_j$. Thus for each $2 \leq j \leq l$, we get a map $I_j : \Gamma \to J_n$ by sending $\gamma \in \Gamma$ to $k$ for which $(i)_j$ is satisfied. We shall show the following lemma:

**Lemma 4.4.** Let $1 \leq j \leq l$.

1. The $\Im I_j$ satisfies that $\Im I_j \subseteq J_l$.
2. For each $\gamma \in \Gamma$, we define $\sigma_\gamma : J_l \to J_l$ by $\sigma_\gamma(j) := I_j(\gamma)$. Then for any $\gamma \in \Gamma$, $\sigma_\gamma \in \Aut(J_l) \cong \mathcal{S}_l$.
3. For any $\gamma, \gamma' \in \Gamma$, $\sigma_\gamma \sigma_{\gamma'} = \sigma_{\gamma' \gamma}$.
4. If $I_j(\gamma) = k$, then we have that $|A_k \triangle \cdot A_j| \leq \frac{h_n(G)}{h_{n+1}(G)}(2|A_1| + |A_j| + |A_k|) \left(\leq \frac{4h_n(G)}{h_{n+1}(G)}|A_1|\right)$.

**Proof.** (Lemma 4.4)

1. Suppose, to the contrary, that there exists $k > l$ such that $k \in \Im I_j$. Because $I_j^{-1}(k) \neq \emptyset$, there exists $\gamma \in \Gamma$ such that $|\gamma \cdot A_j - A_k| \leq \frac{2h_n(G)}{h_{n+1}(G)}|A_1|$. Because $j \in \Im I_1$, there again exists $\gamma' \in \Gamma$ such that

$$|\gamma' \cdot A_1 - \gamma \cdot A_j| = |\gamma \cdot A_1 - A_j| \leq \frac{2h_n(G)}{h_{n+1}(G)}|A_1|.$$ 

By combining these two inequalities, we obtain that $|\gamma' \cdot A_1 - A_k| \leq \frac{4h_n(G)}{h_{n+1}(G)}|A_1|$. Recall that by the assumption of the proof, in particular $6h_n(G) < h_{n+1}(G)$.
This implies that $k$ cannot satisfy option (ii) for $\gamma'$. Therefore $I_1(\gamma') = k$ and this is a contradiction.

(2) In a similar argument to one in the proof of (1), we have that for any $\gamma \in \Gamma$, $\sigma_l \sigma_{\gamma^{-1}} = \sigma_{\gamma^{-1}} \sigma_l = \text{id}_{\{1, \ldots, l\}}$. Hence $\sigma_l \in \text{Aut}(J_l)$.

(3) This can be also showed in a similar argument to one in the proof of (1).

(4) First, because $I_j(\gamma) = k$, we have that $|\gamma \cdot A_j - A_k| \leq \frac{h_n(G)}{h_{n+1}(G)}(|A_1| + |A_j|)$.

Secondly, from item (2) in this lemma we have that for any $\gamma \cdot A_j \cap A_k$.

$|A_k - \gamma \cdot A_j| = |\gamma^{-1} \cdot A_k - A_j| \leq \frac{h_n(G)}{h_{n+1}(G)}(|A_1| + |A_j|)$.

By combining these two inequalities, we get the conclusion.

\[
\begin{align*}
|A_k - \gamma \cdot A_j| &= |\gamma^{-1} \cdot A_k - A_j| \\
&\leq \frac{h_n(G)}{h_{n+1}(G)}(|A_1| + |A_j|).
\end{align*}
\]

From this lemma, we have obtained a group homomorphism

$$\Phi_n : \Gamma \to \mathfrak{S}_l; \quad \gamma \mapsto \sigma_\gamma.$$ 

Now we shall employ condition $(\ast)_n$ on $\Gamma$. Because $\text{Im}\Phi_n$ is a transitive subgroup of $\mathfrak{S}_l$, under this assumption we conclude that $l < n$. We set $A := \bigcup_{j=1}^l A_j$, and $B := V \setminus A$; and rename $A_{l+1}, \ldots, A_n$ respectively $B_1, \ldots, B_{n-l}$. Note that from the argument above, $B$ is non-empty. We also note that from (5), $|A| = |A_1| + \sum_{j=2}^l |A_j| \geq \frac{ln}{n+1}|A_1|$.

For each $1 \leq j \leq l$, we have that for each $\gamma \in \Gamma$, $|\gamma \cdot A_j \cap B| \leq \frac{2h_n(G)}{h_{n+1}(G)}|A_1|$ (consider $\Phi_n(\gamma)(j)$), and that

$$|\gamma \cdot A \cap B| \leq \frac{2l h_n(G)}{h_{n+1}(G)}|A_1| \leq \frac{2l (2n + 3)}{(l+1)(2n+1)} \frac{h_n(G)}{h_{n+1}(G)}|A| < \frac{1}{3}|A|.$$ 

Hence we obtain that for any $\gamma \in \Gamma$, $|\gamma \cdot A \Delta A| < \frac{2}{3}|A|$. Therefore from Lemma 4.3 we conclude that $|B| = |V \setminus A| < \frac{1}{3}|V|$. In particular, $|B| < |A|$.

In what follows, we shall show that for any $\gamma \in \Gamma$, $|B \Delta \gamma \cdot B| < |B|$ holds true.

To see this, fix $\gamma \in \Gamma$. For any $1 \leq k \leq n-l$, we have that

$$|\partial(\gamma \cdot B_k \cap A, A - \gamma \cdot B_k)| \leq h_n(G)|B_k|$$

and that

$$|\partial(\gamma \cdot B \cap A)| \leq \sum_{k=1}^{n-l} h_n(G)|B_k| + \sum_{m=1}^{n-l} |\partial B_m| \leq 2h_n(G)|B|.$$ 

Hence for any $1 \leq j \leq l$,

$$|\partial(A_j - \gamma \cdot B)| \leq 2h_n(G)|B| + h_n(G)|A_j| < h_n(G)|A| + h_n(G)|A_j|$$

$$\leq (l+1)h_n(G)|A_1| \leq nh_n(G)|A_1|$$

(recall that we have verified that $|A| > 2|B|$). We also observe that according to $\gamma$ and $j, j' := \Phi_n(\gamma^{-1})(j)$ satisfies that $|\gamma \cdot A_{j'} \cap A_j| \geq |A_j| - \frac{2h_n(G)}{h_{n+1}(G)}|A_1|$. This implies
that
\[ |A_j - \gamma \cdot B| \geq |A_j| - \frac{2h_n(G)}{h_{n+1}(G)}|A_1| \geq \left( \frac{n}{n+1} - \frac{2h_n(G)}{h_{n+1}(G)} \right)|A_1|. \]

Therefore we have the following inequalities:
\[ \frac{|\partial(A_j - \gamma \cdot B)|}{|A_j - \gamma \cdot B|} < \frac{nh_n(G)}{n+1 - \frac{2h_n(G)}{h_{n+1}(G)}} < (2n+2)h_n(G) < h_{n+1}(G). \]

Finally, for \( \gamma \in \Gamma \), we decompose \( V \) into \((n+1)\) disjoint subsets \( \gamma \cdot B \cap A, B_1, \ldots, B_{n-l} \), and \( A_j - \gamma \cdot B \) \((1 \leq j \leq l)\). Note that the argument above shows that \( A_j - \gamma \cdot B \neq \emptyset \) for all \( j \). If \( \gamma \cdot B \cap A = \emptyset \), then \( |B \Delta \gamma \cdot B| = 0 \) and we are done. Hence we may assume that all of the \((n+1)\) subsets are non-empty. Then from the condition of \( h_{n+1} \), at least one subset \( C \) of these must satisfy that \( \frac{|\partial C|}{|C|} \geq h_{n+1}(G) \). However by construction, neither of \( B_1, \ldots, B_{n-l} \) satisfies this condition. From the inequalities above, all of the \( A_j - \gamma \cdot B \)’s, \( 1 \leq j \leq l \) also fail to do so. Therefore \( C = \gamma \cdot B \cap A \) must satisfy that condition. This amounts to saying that \( |\gamma \cdot B \cap A| \leq \frac{2h_n(G)}{h_{n+1}(G)}|B| \), and hence we have that
\[ |\gamma \cdot B \Delta B| \leq \frac{4h_n(G)}{h_{n+1}(G)}|B| < |B|. \]

This completes the proof of the assertion stated in the very first part of this paragraph. This contradicts Lemma 4.3 because \( 0 \neq |B| < |A| \), and ends our proof. \( \square \)

5. Proof of Theorem \( \mathcal{C} \)

We recall the definition of a system of imprimitivity (of size \( n \)).

**Definition 5.1.** Let \( \Gamma \curvearrowright V \) be a finite group action on a finite set that is transitive.

Let \( n \geq 2 \). A non-empty decomposition \((V_1, \ldots, V_n)\) of \( V \) \((V = V_1 \sqcup \cdots \sqcup V_n)\) is called a system of imprimitivity (of size \( n \)) if for any \( \gamma \in \Gamma \) there exists \( \sigma_\gamma \in \mathfrak{S}_n \) such that \( \gamma \cdot V_i = V_{\sigma_i(\gamma)} \) for all \( 1 \leq i \leq n \). Each \( V_i \) is called a block.

Intuitively, if a system of imprimitivity exists, then the group action does not “break” the partitions given by blocks. It is well-known that \( \Gamma \curvearrowright V \) admits a system of imprimitivity of size \( n \) if and only if there exists a subgroup of \( \Gamma \) of index \( n \) between \( \Gamma \) and a point stabilizer. For instance, see [DM96].

**Proof of Theorem \( \mathcal{C} \).** Let \( h_{n+1}(G) > 2(n+1)h_n(G) \) and take a decomposition \( V = A_1 \sqcup \cdots \sqcup A_n \) which achieves \( h_n(G) \) with \( |A_1| \geq |A_2| \geq \ldots \geq |A_n| \). Consider the action \( \Gamma := \text{Aut}(G) \curvearrowright V \). Take the group homomorphism \( \Phi_n : \Gamma \to \mathfrak{S}_n \) obtained by the proof of Theorem 4.1. Then by Theorem 4.1, the resulting action \( \Gamma \curvearrowright J_n \) by \( \Phi_n \) is transitive. For each \((i, j)\) \( \in J_n \times J_n \), we define \( \Gamma_{i,j} := \{ \gamma \in \Gamma : \Phi_n(\gamma)(j) = i \} \) (the condition on \( \Gamma_{i,j} \) may be understood as “\( i \overset{\Phi_n(\gamma)}{\leftarrow} j \)”.” Note that \( |\Gamma_{i,j}| = |\Gamma|/n \).

Consider the Banach space \( \ell_1(V) \), and denote by \( \rho \) the isometric linear representation of \( \Gamma \) on \( \ell_1(V) \) by permutations: \( \rho(\gamma)\eta(v) := \eta(\gamma^{-1} \cdot v) \). For each \((i, j)\) \( \in J_n \times J_n \), define \( M_{i,j} \) as the averaging operator on \( \rho(\Gamma_{i,j}) \), namely, \( M_{i,j} \eta := (\sum_{\gamma \in \Gamma_{i,j}} \rho(\gamma)\eta)/|\Gamma_{i,j}| \). Note that for any \( i, j, k \in J_n \), \( M_{i,j}M_{j,k} = M_{i,k} \) holds.
Set $\xi_1 = \chi_{A_1}, \ldots, \xi_n = \chi_{A_n}$, and for each $i, j \in J_n$ define
\[
\zeta_i := \frac{1}{n} (M_{i,1} \xi_1 + M_{i,2} \xi_2 + \cdots + M_{i,n} \xi_n).
\]

We claim the following:

1. The $\sum_{i=1}^n \zeta_i = 1$ and $\zeta_i(v) \in [0, 1]$ for any $v \in V$ and $i \in J_n$.
2. For any $\gamma \in \Gamma_{i,j}$, $\rho(\gamma) \zeta_i = \zeta_i$.
3. For any $i, \|\zeta_i - \xi_i\| \leq \frac{n^2 - 1}{(n^2 + 1) h_{n+1}(G)} |A_i| \leq \frac{n^2 - 1}{n(n^2 + 1) h_{n+1}(G)} |V|$. Here $\|\cdot\|$ means the $\ell_1$-norm.

Indeed, item (1) follows from $\sum_{i=1}^n \zeta_i = 1$ and the construction. Item (2) is by $M_{i,j} M_{j,k} = M_{i,k}$ and $|\Gamma_{i,j}| = |\Gamma|/n$. Item (3) can be confirmed by item (4) of Lemma 4.4, the triangle inequality, and $\langle \phi \rangle$ in the proof of Theorem 4.1.

Finally, define $V_1, \ldots, V_n$ by setting for every $i \in J_n$
\[
V_i := \{ v \in V : \zeta_i(v) > 1/2 \}.
\]

We shall show that $(V_1, \ldots, V_n)$ and $(A_1, \ldots, A_n)$ satisfy all of the conclusions (i)–(iii) in Theorem C. First, we discuss (i) and (ii). Item (ii) is by definition. To see (i), observe that $V_i \neq \emptyset$ by items (1) and (3) above, and that for any $\gamma \in \Gamma_{i,j}$, $\gamma \cdot V_j = V_i$ by item (2). Also, $V_i$'s are pairwise disjoint because otherwise $\sum_{i=1}^n \zeta_i \neq 1$. By the transitivity of the action, we see that $\bigcup_{i=1}^n V_i = V$. Hence $(V_1, \ldots, V_n)$ is a decomposition of $V$, and moreover is a system of imprimitivity of size $n$.

Finally, we deal with the proof of item (iii). Since $\xi_i$ is $\rho(\Gamma_{i,j})$-invariant (by item (2)), items (1) and (3) shows that for every $i \in J_n$ and $v \in V$,
\[
\zeta_i(v) \in \left[ 0, \frac{n^2 - 1}{(n^2 + 1) h_{n+1}(G)} \right] \cup \left[ 1 - \frac{n^2 - 1}{(n^2 + 1) h_{n+1}(G)}, 1 \right] \quad (\subset \mathbb{R})
\]
holds (note that $\chi_{A_i}$ takes values only in $\{0, 1\}$). Therefore for every $i \in J_n$,
\[
|V_i \triangle A_i| \leq \frac{n^2 - 1}{1 - \frac{n^2 - 1}{(n^2 + 1) h_{n+1}(G)}} |V| \leq \frac{4 h_n(G)}{h_{n+1}(G)} |V|,
\]
as desired.

\section{6. proof of Theorem B}

\textit{Proof of Theorem B} First we prove the inequality for $h_n$'s. If $h_{n+1}(G) \leq 2(n + 1)h_n(G)$, then we are done. Otherwise, by Theorem B we may take $V = V_1 \sqcup \cdots \sqcup V_n$ and $V = A_1 \sqcup \cdots \sqcup A_n$ in the statement. For $\Gamma := \text{Aut}(G)$, take $\Gamma_{i,j}$ for $(i, j) \in J_n \times J_n$ in the previous section.

Now we use the assumption of that $G$ is connected. This implies that for any $i$, there exist $v_i \in V_i$ and an edge which connects $v_i$ to a vertex $w_i$ lying in other $V_j$. Then by translating by $\Gamma_{i,j}$-action, we observe that any $v \in V_i$, there exists at least one edge $(v, w)$ with $w \notin V_i$.

Here we claim that we can take $w = w(v)$ in such a way that a different $v \in V_i$ gives a different $w$. To prove this, take a pair $v_i \in V_i$ and $w_i \in V \setminus V_i$ as above and
fix them. Take any $\emptyset \neq K \subseteq V_i$ and define $\Lambda_K := \{g \in \Gamma : g \cdot v_i \in K\} \subseteq \Gamma_{i,i}$ (we may replace $\Gamma$ with $\Gamma_{i,i}$ above). Then from the construction, we have that
\[
|K| = \sum_{v \in K} \frac{|\{g \in \Gamma_{i,i} : g \cdot v_i = v\}|}{|\text{Stab}_v \cap \Gamma_{i,i}|}.
\]
Here for $y \in V$, $\text{Stab}_y \leq \Gamma$ denotes the stabilizer of $y$ for $\Gamma \acts V$. Because $v \in V_i$ and the $\Gamma$-action is transitive, we have that $\text{Stab}_x \leq \Gamma_{i,i}$ for any $x \in V_i$ and that $|\text{Stab}_y| = |\text{Stab}_{v_i}|$ for any $y \in V$. We obtain that
\[
|K| = \sum_{v \in K} \frac{|\{g \in \Gamma_{i,i} : g \cdot v_i = v\}|}{|\text{Stab}_v|} = \sum_{v \in K} \frac{|\{g \in \Gamma_{i,i} : g \cdot v_i = v\}|}{|\text{Stab}_{v_i}|} = \frac{|\Lambda_K|}{|\text{Stab}_{v_i}|}.
\]
Let $V(K) \subseteq V \setminus V_i$ be the set $\{g \cdot w_i : g \in \Lambda_K\}$. In a similar way to one above, we have that
\[
|V(K)| = \sum_{w \in V(K)} \frac{|\{g \in \Lambda_K : g \cdot w_i = w\}|}{|\text{Stab}_w \cap \Lambda_K|}.
\]
Therefore we conclude that for any $\emptyset \neq K \subseteq V$,
\[
|V(K)| \geq \sum_{w \in V(K)} \frac{|\{g \in \Lambda_K : g \cdot w_i = w\}|}{|\text{Stab}_w|} = \sum_{w \in V(K)} \frac{|\{g \in \Lambda_K : g \cdot w_i = w\}|}{|\text{Stab}_{v_i}|} = \frac{|\Lambda_K|}{|\text{Stab}_{v_i}|} = |K|.
\]
The marriage theorem therefore verifies our claim (note that $V(K)$ coincides with the set $\bigcup_{v \in K}\{g \cdot w_i : g \in \Gamma_{i,i}, g \cdot v_i = v\}$).

Fix $i \in J_n$. Set $A_i^{(1)} := A_i \cap V_i$ and $A_i^{(2)} := A_i \setminus V_i$. Note that by item $(iii)$ in Theorem $\text{[C]}$ $|A_i^{(2)}| \leq \frac{4h_n(G)}{h_{n+1}(G)}|V|$. Then the claim above implies that
\[
|\partial(A_i^{(1)}, V \setminus A_i)| \geq |\partial(A_i^{(1)}, V \setminus (V_i \cup A_i^{(2)}))| \geq |A_i| - \frac{8h_n(G)}{h_{n+1}(G)}|V|.
\]
We hence have that
\[
\frac{|\partial A_i|}{|A_i|} \geq 1 - \frac{8h_n(G)}{h_{n+1}(G)} \frac{|V|}{|A_i|}.
\]
Take the minimum over all $i \in J_n$. Then by definition the minimum of the left-hand side equals $h_n(G)$. By $(\diamond)$ in the proof of Theorem $\text{[B]}$ we conclude that
\[
h_n(G) \geq 1 - \frac{n^2 + 1}{n} \frac{8h_n(G)}{h_{n+1}(G)} \geq 1 - 10n \cdot \frac{h_n(G)}{h_{n+1}(G)}
\]
because $n \geq 2$. These inequalities lead us to the desired inequality.

For the inequalities on $g_n$’s, in a similar manner to the one above, we can show that for every $i \in J_n$,
\[
\frac{|\delta A_i|}{|A_i|} \geq 2 - \frac{16g_n(G)}{g_{n+1}(G)} \frac{|V|}{|A_i|}.
\]
This ends our proof of Theorem $\text{[B]}$. \qed
Corollary 6.1. Let $G$ be a finite connected graph. If $G$ is vertex and edge transitive, then for any $2 \leq n \leq |V| - 1$, we have that $h_{n+1}(G) \leq (10n + 1)h_n(G)$.

Proof. Suppose that $h_{n+1}(G) > 2(n + 1)h_n$. Then by Theorem [C] there exists a system $(V_1, \ldots, V_n)$ of imprimitivity of size $n$ for $\text{Aut}(G) \lhd G$. If there exists an edge inside $V_i$ for some $i$, then it contradicts the assumption. Indeed, since $G$ is connected and the group action is vertex-transitive, then there must exist $v, v' \in V_i$ and $w \in V \setminus V_i$ such that $(v, v')$ and $(v, w)$ are in $E$. By the edge-transitivity, this contradicts the imprimitivity of the system.

There are hence no edges inside $V_i$ for each $i$. Then by item $(iii)$ of Theorem [C] in a similar argument to one in the proof of Theorem [B] we have that

$$h_n(G) \geq d - d \cdot \frac{n^2 + 1}{n} \frac{8h_n(G)}{h_{n+1}(G)},$$

where $d := \text{deg}(G)$. This implies that

$$h_n(G) \geq \frac{dh_{n+1}(G)}{10dn + h_{n+1}(G)}.$$

Because $h_{n+1}(G) \leq d$, we obtain the conclusion. \qed

In the comparison with results in [Li80], it might be reasonable to ask whether there exists a universal constant $C$, independent even of $n \geq 2$, such that $h_{n+1}(G) \leq C h_n(G)$ for any finite, connected, and vertex and edge transitive graph $G$. Also, it might be interesting to ask a similar problem for finite, connected, and distance regular graphs.

**Appendix: Banach spectral gaps into noncommutative $L_p$ spaces**

Let $G = (V, E)$ be a finite graph (not necessarily regular, and possibly with self-loops or multiple edges, and we consider the oriented edges for $E$ in this appendix), and $(X, p)$ be a pair of a Banach space and an exponent in $[1, \infty]$. In [Mim14], a one form of $(X, p)$-Banach spectral gap is defined by the following formula:

$$\lambda_2(G; X, p) := \frac{1}{2} \inf_{f: V \to X} \frac{\sum_{v \in V} \sum_{e=(v,w) \in E} \|f(w) - f(v)\|^p_X}{\sum_{v \in V} \|f(v) - m(f)\|^p_X}.$$

Here $m(f) := \sum_{v \in V} f(v)/|V|$ and $f$ runs over all nonconstant maps. Note that in [Mim14] this quantity is written as $\lambda_1(G; X, p)$, but we use the symbol $\lambda_2(G; X, p)$ because this is a generalization of $\lambda_2(G)$ in the sense of the current paper (also note that 1/2 in the right-hand side comes from the setting where $E$ is the set of oriented edges in this appendix). For fixed $(X, p)$, we say that a sequence of graphs $\{G_m = (V_m, E_m)\}_m$ is a sequence of $(X, p)$-anders if the following three conditions are all satisfied: $\sup_m \Delta(G_m) < \infty$; $\lim_{m \to \infty} |V_m| = \infty$; and $\inf_m \lambda_2(G_m; X, p) > 0$. Here $\Delta(G)$ denotes the maximal degree of $G$. In particular, a sequence of (2-way) expanders in classical sense is that of $(H, 2)$-anders, where $H$ is any Hilbert space.

A Banach space $X$ is said to be uniformly convex if for every $t \in (0, 2]$,

$$d_X(t) := \inf \{1 - \|x + y\|/2 : \|x\|, \|y\| \leq 1, \text{ and } \|x - y\| \leq t\} > 0,$$
and \( X \) is said to be \textit{uniformly smooth} if \( \lim_{t \to +0} r_X(t)/t = 0 \) holds, where for \( t \in (0, 2] \),
\[
r_X(t) := \sup\{||x + y||/2 + ||x - y||/2 - 1 : ||x|| \leq 1, \text{ and } ||y|| \leq t\}.
\]
A great reference on geometries of Banach spaces is [BL00]. For instance, see Appendix A for properties of these conceptions.

In [Mim14], we say that two Banach spaces \( X, Y \) are \textit{sphere equivalent}, written as \( X \sim_S Y \), if there exists a uniformly homeomorphism (namely, a bi-uniformly continuous map) from \( S(X) \) to \( S(Y) \), where \( S(Z) \) means the unit sphere of \( Z \).

Theorem A.1. \( (X, q) = (\mathcal{H}, 2) \) and \( (Y, p) = (L_p(M, \tau), p) \), where \( L_p(M, \tau) \) denotes the noncommutative \( L_p \) space associated with a semifinite von Neumann algebra \((M, \tau)\) (\( \tau \) is a normal, faithful, and semifinite trace). We remark that here we discuss noncommutative \( L_p \) spaces associated with semifinite von Neumann algebras only for simplicity, and that we may obtain the same results as below for general cases with a similar (but more complicated) proof. For a comprehensive treatise on noncommutative \( L_p \) spaces, we refer the reader to [PX03]. Basic examples of such \( Y \) are \( L_p([0, 1]) \), and \( C_p \), which denotes the ideal of all Schatten \( p \)-class operators acting on a (fixed) infinite dimensional separable Hilbert space.

**Theorem A.1.** Let \( p \in (1, \infty) \) and \((M, \tau)\) be as in the paragraph above.

(i) There exists a uniform homeomorphism
\[
\Psi_p : S(\ell_p(N, L_p(M, \tau))) \to S(\ell_2(N, L_2(M, \tau)))
\]
that is \text{Sym}(N)-equivariant. Here for a Banach space \( Z \), \( \ell_q(N, Z) \) denotes the \( \ell_q \)-sequence space of \( Z \) over \( N \), and \( \Psi_p \) is said to be \text{Sym}(N)-equivariant if the map is equivariant under all the permutations on \( N \), including ones of infinite support (a detailed definition is given in [Mim14] Definition 3.7). Moreover, \( \Psi_p \) is a \( (1/2 - \epsilon) \)-Hölder map for any \((1/2 >)\epsilon > 0\).

(ii) Let \( \epsilon > 0 \) and \( k \in \mathbb{Z}_{>0} \). Then there exists a constant \( C = C((M, \tau), p, \epsilon, k) > 0 \), that only depends on \((M, \tau), p, \epsilon, \) and \( k \), such that for any finite graph \( G \) with \( \Delta(G) \leq k \), we have that
\[
\lambda_2(G)^{p/2} \geq \lambda_2(G; L_p(M, \tau), p) \geq C\lambda_2(G)^{p(1+\epsilon)}.
\]
Here \( \lambda_2(G) := \lambda_2(G; \mathbb{R}, 2) \), and it coincides with \( \lambda_2(G) \) in Definition 1.1 if \( G \) is regular.

Remark A.2. (1) Recall that \((Y, p)\)-Banach spectral gap is computed in terms of the \( p \)-powers of certain norms. Hence if we change \( p \), then we need to take the \( p \)-th root of \( \lambda_2(G; Y, p) \) to compare with other \((X, q)\)-Banach spectral gaps. In this
point of view, the inequalities in item (ii) above may be rewritten as
\[
\lambda_2(G)^{1/2} \geq \lambda_2(G; L_p(M, \tau), p)^{1/p} \geq C' \{\lambda_2(G)^{1/2}\}^{2+\epsilon}
\]
for some \( C' = C'(\lambda(G), p, \epsilon, \Delta(G)) > 0 \).

It has been asked by several experts on coarse and metric geometry whether noncommutative \( L_p \) spaces, specially \( C_p \), are “much flexible” (in certain sense) compared with a Hilbert space when \( p < \infty \) is much larger than 2. However, the above inequalities imply that, concerning Banach spectral gaps, behaviors of noncommutative \( L_p \) spaces are controlled by that of Hilbert spaces, “uniformly” on \( p \in (1, \infty) \) (in some sense related to decay-orders for sequences of finite graphs with uniformly bounded degree).

(2) The space \( \ell_p(N, L_p(M, \tau)) \) coincides with the noncommutative \( L_p \) space associated with \( \hat{M}, \hat{\tau} := (M \otimes \ell_\infty(N), \tau \otimes \mathrm{Tr}) \), where \( \mathrm{Tr} \) is the canonical trace on \( \ell_\infty(N) \). In this point of view, we can have the map \( \Psi_p: S(\ell_p(N, L_p(M, \tau))) \to S(\ell_2(N, L_2(M, \tau))) \) in item (i) of Theorem A.1 as the one identical to the (noncommutative version of) Mazur map:
\[
S(L_p(M, \tau)) \to S(L_2(M, \tau)); \quad a = u|a| \mapsto u|a|^{p/2},
\]
where \( a = u|a| \) is a polar decomposition of \( a \in L_p(M, \tau) \).

**Proof.** The proof is based on geometry of noncommutative \( L_p \) spaces and complex interpolation theory. We refer the reader, respectively, to [PX03] and [BL76] for comprehensive treatments on these topics.

(i) We devide the proof into two cases.

**Case 1:** \( p \in (1, 2) \). Take any \( q \in (1, p) \) and \( r \in (2, \infty) \), and fix them. It is known that \( \ell_q(N, L_q(M, \tau)) \) and \( \ell_r(N, L_r(M, \tau)) \), respectively, are isometrically isomorphic to \( E_0 \) and \( E_1 \) for some complex interpolation pair \((E_0, E_1)\) (for instance, see Section 2 in [PX03] and (2) of Remark A.2), and \( \ell_p(N, L_p(M, \tau)) \) and \( \ell_2(N, L_2(M, \tau)) \), respectively, are isometrically isomorphic to intermediate points of \((E_0, E_1)\). Then by [Mim14] Theorem 3.8, there exists an \( \mathrm{Sym}(N) \)-equivariant uniform homeomorphism
\[
\Psi_p: S(\ell_p(N, L_p(M, \tau))) \to S(\ell_2(N, L_2(M, \tau))).
\]
To make an estimate of the modulous of continuity of \( \Psi_p \), we go back to the proofs of Theorem 9.2 (and Proposition 1.3) of [BL00]. The order of modulous of continuity of \( \Psi_p \) is bounded from above, up to positive scalar multiplication, by \( d_{W^{-1}}(t) \) (recall \( d_X(t) \) in the definition of the uniform convexity), where \( W = F_2(\sigma) \) in [BL00] Appendix I, that is a subspace of \( L_2(\mu_0, E_0) \oplus_2 L_2(\mu_1, E_1) \) for some measures \( \mu_0, \mu_1 \). Results in [PX03, Section 5] tell us that for any \( t \in (0, 2] \),
\[
d_{E_0}(t) \geq \frac{q-1}{8} t^2 \quad \text{and} \quad d_{E_1}(t) \geq \frac{1}{r_2} t^r.
\]
hols true (see also item (2) of Remark [A.2]). From this, we conclude that there exists \( K = K(r, q) > 0 \) such that for any \( t \in (0, 2] \),
\[
d_W(t) \geq K \cdot t^r.
\]
Therefore, by the proof of [BL00] Theorem 9.2, there exists \( K' = K'((M, \tau), q, r) > 0 \) such that for any \( t \in (0, 2] \) and for any \( x, y \in S(\ell_p(\mathbb{N}, L_p(M, \tau))) \),
\[
\|x - y\|_{\ell_p(\mathbb{N}, L_p(M, \tau))} \leq t \quad \implies \quad \|
\Psi_p(x) - \Psi_p(y)\|_{\ell_2(\mathbb{N}, L_2(M, \tau))} \leq K' t^{1/r}.
\]
Since \( r \in (2, \infty) \) is arbitrary, this verifies the assertion. Here we also mention that by item (2) of Remark [A.2] \( \Psi_p \), constructed in the way above, is independent of the choices of \((q, r)\).

**Case 2:** \( p \in (2, \infty) \). Set \( p' \in (1, 2) \) is the conjugate of \( p \) (\( 1/p + 1/p' = 1 \)). Then by Case 1, there exists an \( \text{Sym}(\mathbb{N}) \)-equivariant uniform homeomorphism
\[
\Psi_{p'} : S(\ell_{p'}(\mathbb{N}, L_{p'}(M, \tau))) \rightarrow S(\ell_2(\mathbb{N}, L_2(M, \tau)))
\]
that is \((1/2 - \epsilon)\)-Hölder for any \((1/2 >) \epsilon > 0\). In the view of item (2) of Remark [A.2] there is the duality mapping \( j_p : S(\ell_p(\mathbb{N}, L_p(M, \tau))) \rightarrow S(\ell_{p'}(\mathbb{N}, L_{p'}(M, \tau))) \), obtained by the uniform smoothness of \( \ell_p(\mathbb{N}, L_p(M, \tau)) \) (for the precise definition, see [BL00] Appendix A)]. Set \( i_p := * \circ j_p : S(\ell_p(\mathbb{N}, L_p(M, \tau))) \rightarrow S(\ell_{p'}(\mathbb{N}, L_{p'}(M, \tau))) \), where \(* : S(\ell_{p'}(\mathbb{N}, L_{p'}(M, \tau))) \rightarrow S(\ell_p(\mathbb{N}, L_p(M, \tau))) \) is the adjoint operation. Then by Proposition A.5 in [BL00], we obtain that for any \( t \in (0, 2] \) and for any \( x, y \in S(\ell_p(\mathbb{N}, L_p(M, \tau))) \),
\[
\|x - y\|_{\ell_p(\mathbb{N}, L_p(M, \tau))} \leq t \quad \implies \quad \|i_p(x) - i_p(y)\|_{\ell_{p'}(\mathbb{N}, L_{p'}(M, \tau))} \leq \frac{2r_{\ell_p(\mathbb{N}, L_p(M, \tau))}(2t)}{t}.
\]
Here recall \( r_X(t) \) in the definition of the uniform smoothness (the statement of [BL00] Proposition A.5] seems to contain a minor error, and above we correct it). By Corollary 5.4 in [PX03], we conclude that \( i_p \) is a Lipschitz map because \( p > 2 \). Finally, set
\[
\Psi_p := \Psi_{p'} \circ i_p : S(\ell_p(\mathbb{N}, L_p(M, \tau))) \rightarrow S(\ell_2(\mathbb{N}, L_2(M, \tau))).
\]
Then this map satisfies all of the conditions in the assertion (note that the uniform smoothness of \( \ell_{p'}(\mathbb{N}, L_{p'}(M, \tau)) \) implies that \( i_p^{-1} \) is also uniformly continuous).

(ii) Observe that \( \ell_2(\mathbb{N}, L_2(M, \tau)) \) is nothing but a Hilbert space. Hence by [Mim14, Lemma 2.1], we have that for any \( G \),
\[
\lambda_2(G; \ell_2(\mathbb{N}, L_2(M, \tau)), 2) = \lambda_2(G; \mathbb{R}, 2)(= \lambda_2(G)).
\]
From item (i), we can have the conclusion in a similar argument in [Mim14, Section 4], with the aid of the Gross trick.

\[ \square \]

Note that if we consider a commutative \( L_p \) space, then for \( p \in (2, \infty) \), we have that
\[
\lambda_2(G)^{1/2} \geq \lambda_2(G; L_p([0, 1]), p)^{1/p} \geq C' \lambda_2(G)^{1/2},
\]
for some $C' = C'(p, \Delta(G)) > 0$. In other words, $\lambda_2(G; L_p([0,1]), p)^{1/p}$ has the same other as $\lambda_2(G)^{1/2}$ if we fix $p$ and $\Delta(G)$. (This is a part of Matoušek’s extrapolation. For instance, see [Mim14, Theorem 1.4].) It, hence, might be interesting to ask whether the same order-estimate holds for any noncommutative $L_p$ space for $p \in (2, \infty)$.

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REFERENCES

[AM85] N. Alon and V. D. Milman, $\lambda_1$ isoperimetric inequalities for graphs and superconcetrators, J. Comb. Theory B 8 (1985), 73–88.

[BHT00] S. Bobkov, C. Houdré, and P. Tetali, $\lambda_\infty$, vertex isoperimetry and concentration, Combinatorica 20 (2000), no. 2, 153–172.

[BL76] J. Bergh and J. Löström, Interpolation spaces. An introduction., Grundlehren der Mathematischen Wissenschaften, vol. 223, Springer-Verlag, Berlin, 1976.

[BL00] Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis. Vol. 1., American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[DM96] J.D. Dixon and B. Mortimer, Permutation groups, Graduate Texts in Mathematics, vol. 163, Springer Verlag, 1996.

[FS13] K. Funano and T. Shioya, Concentration, Ricci curvature, and eigenvalues of Laplacian, Geom. Funct. Anal. 23 (2013), no. 3, 886–936.

[Fun13] K Funano, Eigenvalues of Laplacian and multi-way isoperimetric constants on weighted Riemannian manifolds, preprint, arXiv:1307.3919v1 (2013).

[LGT12] J.R. Lee, S. O. Gharan, and L. Trevisan, Multi-way spectral partitioning and higher-order Cheeger inequalities, in Proceedings of the 44th Annual Symposium on Theory of Computing (STOC) (2012), 1117–1130.

[Li80] P. Li, Eigenvalue estimates on homogeneous manifolds, Comm. Math. Helv. 55 (1980), no. 1, 347–363.

[Mim14] M. Mimura, Sphere equivalence, Banach expanders, and extrapolation, published online in Int. Math. Res. Notices (2014), 20pp.

[PX03] G. Pisier and Q. Xu, Non-commutative $L^p$-spaces, in Handbook of the geometry of Banach spaces, vol. 2, North-Holland, Amsterdam, 2003.

[Tan11] M. Tanaka, Multi-way expansion constants and partitions of a graph, preprint, arXiv:1112.3434v5 (2011).

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