Near Optimal Stochastic Algorithms for Finite-Sum Unbalanced Convex-Concave Minimax Optimization

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Abstract

This paper considers stochastic first-order algorithms for convex-concave minimax problems of the form \( \min_x \max_y f(x, y) \), where \( f \) can be presented by the average of \( n \) individual components which are \( L \)-average smooth. For \( \mu_x \)-strongly-convex-\( \mu_y \)-strongly-concave setting, we propose a new stochastic algorithm which could find an \( \varepsilon \)-saddle point of the problem in \( \tilde{O}(\sqrt{n}(\sqrt{n} + \kappa_x)(\sqrt{n} + \kappa_y) \log(1/\varepsilon)) \) stochastic first-order complexity, where \( \kappa_x \triangleq L/\mu_x \) and \( \kappa_y \triangleq L/\mu_y \). This upper bound is near optimal with respect to \( \varepsilon, n, \kappa_x \) and \( \kappa_y \) simultaneously. In addition, the algorithm is easily implemented and works well in practice. Our methods can be extended to solve more general unbalanced convex-concave minimax problems and the corresponding upper complexity bounds are also near optimal.

1 Introduction

This paper studies the following finite-sum minimax problem:

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x, y),
\]

where \( \mathcal{X} \) and \( \mathcal{Y} \) are convex and closed. Our goal is to find the approximate saddle point of problem (1) to guarantee the duality gap no larger than \( \varepsilon \).

The formulation (1) includes a lot of machine learning applications such as AUC maximization, robust optimization, adversarial learning and reinforcement learning. We study the fundamental setting that each component \( f_i(x, y) \) is convex in \( x \) and concave in \( y \); \( \{f_i(x, y)\}_{i=1}^{n} \) are \( L \)-average smooth; and \( f(x, y) \) is \( \mu_x \)-strongly-convex in \( x \) and \( \mu_y \)-strongly-concave in \( y \). In particular, we focus on the unbalanced case where the condition numbers \( \kappa_x \triangleq L/\mu_x \) and \( \kappa_y \triangleq L/\mu_y \) could be quite different. Without loss of generality, we suppose \( \kappa_x \geq \kappa_y \).

Most existing algorithms for general strongly-convex-strongly-concave (SCSC) minimax optimization do not consider the difference between two condition numbers \( \kappa_x \) and \( \kappa_y \), which leads to their upper bound complexities depend on \( \max\{\kappa_x, \kappa_y\} \). Lin et al. [22] first proposed proximal point methods for the unbalanced problem with \( \tilde{O}(\sqrt{n}(\kappa_x + \kappa_y) \log(1/\varepsilon)) \) gradient calls, nearly matching the lower bound [15, 42] of the deterministic algorithm. Wang and Li [35], Xie et al. [38] improved Lin et al. [20]'s results under refined smoothness assumption and bilinear setting. Unfortunately, these methods [22, 35] are based on full gradient oracle and ignore the finite-sum structure in the objective function.

In practice, the number of components \( n \) could be very large. It is natural to use stochastic first-order oracle (SFO) algorithms to reduce the cost of gradient based methods. The SFO algorithms and their optimality are well-studied for minimization problems [2, 6, 8, 10, 16, 20, 36, 43], but the related theory for minimax optimization is still imperfect. In the balanced case of \( \kappa_x = \kappa_y = \kappa \), Palaniappan and Bach [20] first introduced SVRG/SAGA [6, 10] to solve the general formulation (1) in big data regime and obtained

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1We use notation \( \tilde{O}(\cdot) \) to hide logarithmic factors of \( n, \kappa_x \) and \( \kappa_y \) in complexities.
We can observe that the complexity of SVRG/SAGA [29] is near optimal in the case of (c). Tominin et al. [33]’s algorithm has the upper bound of the form $O\left(\frac{n}{\kappa_x\kappa_y} \log(1/\varepsilon)\right)$, but it requires the stronger assumption that each component $f_i$ is $L$-smooth. We should point out that Case (b) is also an important setting, which is useful to establish the efficient algorithms for ill-conditioned strongly-convex-nonconcave (or nonconvex-strongly-concave) minimax optimization [21, 22, 25].

In this paper, we propose a Catalyst-type algorithm that we call accelerated loopless stochastic variance reduced extragradient (AL-SVRE), whose iteration inexactly solves more well-conditioned minimax problems. We use loopless stochastic variance reduced extragradient (L-SVRE) [1] as the sub-problem solver and revise Alacaoglu and Malitsky [1]’s analysis to show L-SVRE solves balanced SCSC minimax problem in optimal SFO complexity. Combining an appropriate choice of parameters, AL-SVRE could find an approximate saddle point of our main problem (1) with SFO complexity nearly matching Han et al. [13]’s lower bound.

In fact, SCSC minimax problems with finite-sum structure can be classified by three types of relationships among $\kappa_x$, $\kappa_y$ and $n$ (recall we have assumed $\kappa_x \geq \kappa_y$):

(a) $f(x, y)$ is extremely ill-conditioned w.r.t two variables: $\kappa_x = \Omega(\sqrt{n})$ and $\kappa_y = \Omega(\sqrt{n})$;

(b) $f(x, y)$ is only extremely ill-conditioned w.r.t $y$: $\kappa_x = O(\sqrt{n})$ and $\kappa_y = \Omega(\sqrt{n})$;

(c) the number of components is extremely large: $\kappa_x = O(\sqrt{n})$ and $\kappa_y = O(\sqrt{n})$.

Han et al. [13] provided a general SFO lower bound for all three cases as follows

$$
\begin{align*}
\Omega\left(\left(n + \sqrt{n\kappa_x\kappa_y}\right) \log(1/\varepsilon)\right), & \quad \text{for } \kappa_x = \Omega(\sqrt{n}) \text{ and } \kappa_y = \Omega(\sqrt{n}); \\
\Omega\left(\left(n + n^{3/4}\sqrt{\kappa_y}\right) \log(1/\varepsilon)\right), & \quad \text{for } \kappa_x = O(\sqrt{n}) \text{ and } \kappa_y = \Omega(\sqrt{n}); \\
\Omega(n), & \quad \text{for } \kappa_x = O(\sqrt{n}) \text{ and } \kappa_y = O(\sqrt{n}).
\end{align*}
$$

(2) We can observe that the complexity of SVRG/SAGA [29] is near optimal in the case of (c). Tominin et al. [33]’s algorithm has the upper bound of the form $O\left(\left(n + \sqrt{n\kappa_x\kappa_y}\right) \log(1/\varepsilon)\right)$, but it requires the stronger assumption that each component $f_i$ is $L$-smooth. We should point out that Case (b) is also an important setting, which is useful to establish the efficient algorithms for ill-conditioned strongly-convex-nonconcave (or nonconvex-strongly-concave) minimax optimization [21, 22, 25].

In this paper, we propose a Catalyst-type algorithm that we call accelerated loopless stochastic variance reduced extragradient (AL-SVRE), whose iteration inexactly solves more well-conditioned minimax problems. We use loopless stochastic variance reduced extragradient (L-SVRE) [1] as the sub-problem solver and revise Alacaoglu and Malitsky [1]’s analysis to show L-SVRE solves balanced SCSC minimax problem in optimal SFO complexity. Combining an appropriate choice of parameters, AL-SVRE could find an approximate saddle point of our main problem (1) with SFO complexity nearly matching Han et al. [13]’s lower bound [2]. Additionally, AL-SVRE only applies one times Catalyst acceleration on AL-SVRE, which leads to the algorithm has two-loops of iterations in total and it is easily implemented. The empirical studies on AUC maximization [14, 30, 41] and wireless communication [3, 10, 40] problems show that AL-SVRE performs

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Table 1: Comparison of SFO complexities in the $(\mu_x, \mu_y)$-convex-concave setting.

| Algorithm          | SFO Complexity                                      | Loop | Reference |
|--------------------|-----------------------------------------------------|------|-----------|
| Extragradient      | $O(n\kappa_{\text{max}} \log(1/\varepsilon))$      | 1    | Gidel et al. [11], Korpelevich [17] |
| SVRG/SAGA/SVRE     | $O\left((n + \kappa_{\text{max}}^2) \log(1/\varepsilon)\right)$ | 1    | Chavdarova et al. [5], Palaniappan and Bach [29] |
| A-SVRG/A-SAGA      | $\tilde{O}\left((n + \sqrt{n}\kappa_{\text{max}}) \log(1/\varepsilon)\right)$ | 2    | Palaniappan and Bach [29] |
| L-SVRE             | $O\left((n + \sqrt{n}\kappa_{\text{max}}) \log(1/\varepsilon)\right)$ | 1    | Alacaoglu and Malitsky [11] + Theorem 2 |
| MINIMAX-APPA       | $\tilde{O}\left(n\sqrt{n\kappa_{\text{y}}} \log^3(1/\varepsilon)\right)$ | 3    | Lin et al. [22] |
| PBR                | $\tilde{O}\left(n\sqrt{n\kappa_{\text{y}}} \log(1/\varepsilon)\right)$ | 3    | Wang and Li [35] |
| AL-SVRE            | $\tilde{O}\left(\sqrt{n}(\sqrt{n} + \kappa_{\text{y}})(\sqrt{n} + \kappa_{\text{x}}) \log(1/\varepsilon)\right)$ | 2    | Corollary 2 |
| Lower Bound        | the expression of (2)                                | –    | Han et al. [13] |

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2Tominin et al. [33] assume each $f_i$ is $L_i$-smooth and $f$ is $L$-smooth, where $L = \frac{1}{n} \sum_{i=1}^{n} L_i$, while this paper only requires the weaker assumption that $\{f\}_{i=1}^{n}$ is $L$-average smooth as Definition 1. Please see the detailed discussion for these two different settings in Section 3.2.
To the best of our knowledge, the proposed AL-SVRE is the first algorithm which attains near optimal SFO complexities for all the above settings.

We present comparisons between our results and existing results in Table 1 for strongly-convex-strongly-concave and convex-concave settings, where the result of A-SVRG/A-SAGA [40] requires the assumption of $\varepsilon < \mu_y$.

better than baselines. In contrast, previous Catalyst-type methods [22, 33, 35] for such an unbalanced problem need twice Catalyst acceleration (three-loops of iterations), making these algorithms almost impractical.

We can also apply AL-SVRE to solve more general finite-sum convex-concave minimax problems. If we only suppose the objective function is convex in $x$ and concave in $y$, the algorithm could find an $\varepsilon$-saddle point of (1) in

$$
\tilde{O}\left( n + \frac{\sqrt{nL\kappa_y}}{\varepsilon} + n^{3/4} \sqrt{\kappa_y} + n^{3/4} \sqrt{\frac{L}{\varepsilon}} \right) \log\left( \frac{1}{\varepsilon} \right)
$$

SFO complexity, where $D_x$ is the diameter of $\mathcal{X}$. The upper bound (4) nearly matches the lower bounds with respect to $\varepsilon$, $L$, $\mu_y$, and $n$ simultaneously, and we do not require any additional assumption on $L$, $\mu_y$, $n$, and $\varepsilon$, while the state-of-the-art method [40] implicitly requires that $\varepsilon < \mu_y$.

More general, if we only suppose the objective function is convex in $x$ and concave in $y$, the algorithm could find an $\varepsilon$-saddle point in

$$
\tilde{O}\left( n + \frac{\sqrt{nLD_zD_y}}{\varepsilon} + n^{3/4}(D_z + D_y) \sqrt{\frac{L}{\varepsilon}} \right) \log\left( \frac{1}{\varepsilon} \right)
$$

SFO complexity, where $D_x$ and $D_y$ are diameters of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Note that the upper bound (4) nearly matches the lower bounds [13] in such a setting. Compared with the the best known stochastic SFO algorithm L-SVRE [11], our result additionally trades off the difference between $D_x$ and $D_y$.

We present comparisons between our results and existing results in Table 1 for strongly-convex-strongly-concave setting, in Table 2 for strongly-convex-concave setting, and in Table 3 for convex-concave settings. To the best of our knowledge, the proposed AL-SVRE is the first algorithm which attains near optimal SFO complexities for all the above settings.

| Algorithm          | SFO Complexity                                    | Loop | Reference        |
|--------------------|---------------------------------------------------|------|------------------|
| DIAG               | $\tilde{O}\left( n\kappa_y \sqrt{\frac{L}{\varepsilon}} \log^2 \left( \frac{1}{\varepsilon} \right) \right)$ | 3    | Thekumparampil et al. [32] |
| MINIMAX-APPA       | $\tilde{O}\left( n\sqrt{\frac{2L\kappa_y}{\varepsilon}} \log^3 \left( \frac{1}{\varepsilon} \right) \right)$ | 3    | Lin et al. [22]  |
| A-SVRG/A-SAGA      | $\tilde{O}\left( n + \sqrt{nL\kappa_y} \log \left( \frac{1}{\varepsilon} \right) \right)$ | 2    | Yang et al. [40] |
| AL-SVRE            | $\tilde{O}\left( n + \sqrt{nL\kappa_y} + n^{3/4} \sqrt{\kappa_y} + n^{3/4} \sqrt{\frac{L}{\varepsilon}} \log \left( \frac{1}{\varepsilon} \right) \right)$ | 2    | Corollary 3      |
| Lower Bound        | $\Omega\left( n + \sqrt{nL\kappa_y} + n^{3/4} \sqrt{\kappa_y} + n^{3/4} \sqrt{\frac{L}{\varepsilon}} \right)$ | –    | Han et al. [13]  |

Table 2: Comparison of SFO complexities in the $(0, \mu_y)$-convex-concave setting, where the result of A-SVRG/A-SAGA [40] requires the assumption of $\varepsilon < \mu_y$. 

2 Notation and Preliminaries

In this section, we present the notation and some definitions used in this paper.

Definition 1. For any differentiable function $\psi : Z \rightarrow \mathbb{R}$, we say $\psi$ is $L$-smooth for some $L > 0$ if for any $z, z' \in Z$, it holds that $\|\nabla \psi(z) - \nabla \psi(z')\|_2 \leq L\|z - z'\|_2$.

Definition 2. Suppose there are $n$ differentiable functions $\{\psi_i : Z \rightarrow \mathbb{R}\}_{i=1}^n$. We say $\{\psi_i\}_{i=1}^n$ is $L$-average smooth for some $L > 0$ if for any $z, z' \in Z$, it holds that

$$
\frac{1}{n} \sum_{i=1}^n \|\nabla \psi_i(z) - \nabla \psi_i(z')\|_2^2 \leq L^2 \|z - z'\|_2^2.
$$
Table 3: Comparison of SFO complexities in convex-concave setting.

| Algorithm      | SFO Complexity                     | Loop | Reference      |
|----------------|------------------------------------|------|----------------|
| Extragradient  | \(O\left(\max\{D_x^2, D_y^2\} \frac{n}{\mu^2}\right)\) | 1    | Gidel et al. [11], Korpelevich [17] |
| MINIMAX-APPA   | \(\tilde{O}\left(\frac{nLD_x D_y}{\epsilon} \log^4 \left(\frac{1}{\epsilon}\right)\right)\) | 3    | Lin et al. [22] |
| L-SVRE         | \(O\left(\max\{D_x^2, D_y^2\} (n + \frac{\sqrt{n}}{\epsilon})\right)\) | 1    | Alacaoglu and Malitsky [1] |
| AL-SVRE        | \(\tilde{O}\left((n + \frac{\sqrt{n}LD_x D_y}{\epsilon} + n^{3/4} (D_x + D_y) \sqrt{\frac{1}{\epsilon}}) \log^2 \left(\frac{1}{\epsilon}\right)\right)\) | 2    | Corollary 1 |
| Lower Bound    | \(\Omega\left(n + \frac{\sqrt{n}LD_x D_y}{\epsilon} + n^{3/4} (D_x + D_y) \sqrt{\frac{1}{\epsilon}}\right)\) | –    | Han et al. [13] |

**Definition 3.** For a differentiable function \(\psi: \mathcal{Z} \to \mathbb{R}\), we say \(\psi\) is convex if for any \(z, z' \in \mathcal{Z}\), it holds that \(\psi(z') \geq \psi(z) + \langle \nabla \psi(z), z' - z \rangle\). We say \(\psi\) is \(\mu\)-strongly-convex for some \(\mu > 0\) if \(\psi(\cdot) - \frac{\mu}{2}\|\cdot\|^2\) is convex. We also say \(\psi\) is concave (\(\mu\)-strongly-concave) if \(-\psi\) is convex (\(\mu\)-strongly-convex).

**Definition 4.** For any function \(f: \mathcal{X} \times \mathcal{Y} \to \mathbb{R}\) and \(\mu_x, \mu_y \geq 0\), we say \(f\) is \((\mu_x, \mu_y)\)-convex-concave if for any \(x \in \mathcal{X}\) and \(y \in \mathcal{Y}\), it holds that \(f(x, \cdot)\) is \(\mu_y\)-strongly-concave and \(f(\cdot, y)\) is \(\mu_x\)-strongly-convex.

In Definition 4 we allow both \(\mu_x\) and \(\mu_y\) to be zero. The notation \((0, 0)\)-convex-concave means the function is general convex-concave, and \((0, \mu_y)\)-convex-concave means it is \(\mu_y\)-strongly-concave in \(y\) but possibly non-strongly-convex in \(x\). Similarly, we use \((\mu_x, 0)\)-convex-concave to present the function is \(\mu_x\)-strongly-convex in \(x\) but possibly non-strongly-convex in \(y\).

We are interested in finding an approximate saddle point which is defined as follows.

**Definition 5.** For the minimax optimization problem \((\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}\) is said to be an \(\varepsilon\)-saddle point if
\[
\max_{y \in \mathcal{Y}} f(\hat{x}, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}) \leq \varepsilon.
\]

We introduce the (stochastic) gradient operator for the ease of presentation.

**Definition 6.** For problem 4, we define the gradient operator and stochastic gradient operator as
\[
g(z) = \begin{bmatrix} \nabla_x f(z) \\ -\nabla_y f(z) \end{bmatrix} \quad \text{and} \quad g_i(z) = \begin{bmatrix} \nabla_x f_i(z) \\ -\nabla_y f_i(z) \end{bmatrix}
\]
where \(z = (x, y) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}\).

We also conduct the projection operator to address the constraints in the problem.

**Definition 7.** We define the projection operator \(P_C(z) = \arg \min_{u \in C} \|u - z\|^2\).

This paper focuses on solving finite-sum minimax problems by SFO algorithms and we give the formal definition as follows.

**Definition 8.** Consider a stochastic optimization algorithm \(A\) to solve Problem 4. Suppose an initial point \((x^{(0)}, y^{(0)})\) is given, and let \((x^{(t)}, y^{(t)})\) denote the point obtained by \(A\) at time-step \(t\). The algorithm is said to be an SFO algorithm if for any \(t > 0\), we have
\[
x^{(t)} = P_{\mathcal{X}}(\hat{x}^{(t)}) \quad \text{and} \quad y^{(t)} = P_{\mathcal{Y}}(\hat{y}^{(t)}),
\]
where
\[
\hat{x}^{(t)} \in \text{span} \left\{x^{(0)}, x^{(1)}, \ldots, x^{(t-1)}, \nabla_x f_{i_t}(x^{(0)}, y^{(0)}), \ldots, \nabla_x f_{i_t}(x^{(t-1)}, y^{(t-1)})\right\},
\]
\[
\hat{y}^{(t)} \in \text{span} \left\{y^{(0)}, y^{(1)}, \ldots, y^{(t-1)}, \nabla_y f_{i_t}(x^{(0)}, y^{(0)}), \ldots, \nabla_y f_{i_t}(x^{(t-1)}, y^{(t-1)})\right\},
\]
and \(i_t\) is drawn from \(\{1, \ldots, n\}\).
We start our discussion from the optimality of balanced SCSC problems such that the objective function is $O(2)$. 

We present the details of L-SVRE in Algorithm 1 and show its convergence behavior in Theorem 2. 

Algorithm 1 L-SVRE $(\{f_i(x,y)\}_{i=1}^n, (x_0,y_0), \tau,p,T)$

1: Initialize: $\alpha = 1 - p$, $w_0 = z_0$.
2: for $k = 0, 1, \ldots, T - 1$ do
3: $z_k = \alpha z_k + (1 - \alpha) w_k$
4: $z_{k+1/2} = P_Z(\bar{z}_k - \tau g(w_k))$
5: Draw an index $i \in [n]$ uniformly at random.
6: $z_{k+1} = P_Z(\bar{z}_k - \tau [g(w_k) + g_i(z_{k+1/2}) - g_i(w_k)])$
7: $w_{k+1} = \begin{cases} z_{k+1}, & \text{with probability } p \\ w_k, & \text{with probability } 1 - p \end{cases}$
8: end for
9: Output: $(x_T, y_T)$.

3 Accelerating Unbalanced Convex-Concave Optimization

In this section, we first revisit loopless stochastic variance reduced extragradient (L-SVRE) \cite{1}, and show it can be used to solve balanced strongly-convex-strongly-concave (SCSC) minimax with optimal SFO complexity.

We present the details of L-SVRE in Algorithm 1 and show its convergence behavior in Theorem 2.

Theorem 1. For any SFO algorithm $A$ and $L, \mu, n, \varepsilon$ such that $L/\mu > 2$ and $\varepsilon < 0.003$, there exist a dimension $d = O(n + \sqrt{\mu} \log(1/\varepsilon))$ and functions $\{f_i(x,y)\}_{i=1}^n: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ which satisfy that $\{f_i\}_{i=1}^n$ is $L$-average smooth, $f$ is $(\mu, \mu)$-convex-concave. In order to find an approximate saddle point $(\hat{x}, \hat{y})$ such that

$$E \left[ \|\hat{x} - x^*\|^2_2 + \|\hat{y} - y^*\|^2_2 \right] \leq \varepsilon,$$

algorithm $A$ needs at least $\Omega(n + \sqrt{\mu} \log(1/\varepsilon))$ SFO calls.

Recently, Alacaoglu and Malitsky \cite{1} proposed the L-SVRE algorithm, which combines the idea of loopless SVRG \cite{18} and extragradient \cite{17}. The original motivation of L-SVRE is to solve variational inequalities. We revise its analysis to adapt the standard finite-sum minimax problem \cite{1} under the average smooth assumptions. This result is slightly different from previous work which is based on duality gap convergence \cite{13} or under assumption that each component $f_i$ is $L$-smooth \cite{37}.

Theorem 2. Assume that $\mu_x = \mu_y = \mu$ and $\{f_i\}_{i=1}^n$ is $L$-average smooth. Then Algorithm 1 with probability parameter $p = 1/(2n)$ and stepsize $\tau = 1/(4\sqrt{\mu}L)$ satisfies

$$E \left[ \|z_k - z^*\|^2_2 \right] \leq 4 \left( \frac{1}{4(n + 2\sqrt{\mu}L/\mu)} \right)^k \left( \frac{1}{4(n + 2\sqrt{\mu}L/\mu)} \right)^k.$$

The result of inequality (5) means that L-SVRE could find $(\hat{x}, \hat{y})$ satisfying $E[\|\hat{x} - x^*\|^2_2 + \|\hat{y} - y^*\|^2_2] \leq \varepsilon$ with $O((n + \sqrt{\mu} \log(1/\varepsilon))$ iterations. Since we select $p = 1/2n$ in the theorem, each iteration requires $O(1)$ SFO calls in expectation. Hence, we have proved that the upper bound complexity of L-SVRE matches the lower bound shown in Theorem 1.
We introduce

Algorithm 2 AL-SVRE \( \{f_i(x, y)\}_{i=1}^n, (x_0, y_0), \beta, q, K, p, \{\tau_k\}_{k=1}^K, \{T_k\}_{k=1}^K \) 

1. Initialize: \( \gamma = \frac{1 - \sqrt{\eta}}{1 + \sqrt{\eta}} \) and \( u_0 = x_0 \).

2. for \( k = 1, \cdots, K \) do

3. \( (\tilde{x}_k, \tilde{y}_k) = \text{L-SVRE} \left( \{f_i(x, y) + \frac{\beta}{2} \|x - u_{k-1}\|^2\}_{i=1}^n, x_{k-1}, y_{k-1}, \tau_k, p, T_k \right) \)

4. \( x_k = \mathcal{P}_X (\tilde{x}_k - \tau_k \nabla_x F_k (\tilde{x}_k, \tilde{y}_k)) \)

5. \( y_k = \mathcal{P}_Y (\tilde{y}_k + \tau_k \nabla_y F_k (\tilde{x}_k, \tilde{y}_k)) \)

6. \( u_k = x_k + \gamma (x_k - x_{k-1}) \)

7. end for

8. Output: \( (x_K, y_K) \).

3.2 Acceleration for Unbalanced SCSC Minimax

Note that the convergence result of L-SVRE is not perfect when the objective function is unbalanced. For example, we consider problem (1) in the case of \( \kappa_x = \kappa > \sqrt{n} > \kappa_y = O(1) \). Then Theorem 2 cannot leverage the well-conditioned assumption on \( y \) and leads to SFO upper bound \( O(\sqrt{n} \kappa \log(1/\varepsilon)) \) achieved by deterministic algorithms [22].

Our key intuition to establish a better SFO algorithm for unbalanced SCSC minimax problems is taking the advantage of the optimality of L-SVRE in the balanced case. We present the new method AL-SVRE in Algorithm 2 whose iteration applies L-SVRE to solve the following sub-problem

\[
\min_{x \in X} \max_{y \in Y} F_k(x, y) = f(x, y) + \frac{\beta}{2} \|x - u_{k-1}\|^2.
\]

Since Theorem 2 only provides the convergence rate for L-SVRE by distance, we introduce additional projection gradient iterations (Lines 4-5 of Algorithm 2) on the output of the sub-problem solver. These steps help us control the accuracy with respect to primal and dual functions, which is helpful to establish the convergence results with respect to the duality gap in our main result Theorem 3.

Lemma 1. Suppose the function \( f(x, y): X \times Y \to \mathbb{R} \) is \( (\mu_x, \mu_y) \)-convex-concave and \( L \)-smooth. Denote the saddle point of \( \min_{x \in X} \max_{y \in Y} f(x, y) \) by \( (x^*, y^*) \) and let the condition numbers of \( f \) be \( \kappa_x = L/\mu_x \) and \( \kappa_y = L/\mu_y \). Assume that the point \( (\tilde{x}, \tilde{y}) \) satisfies

\[
\|\tilde{x} - x^*\|_2^2 + \|\tilde{y} - y^*\|_2^2 \leq \varepsilon.
\]

We introduce

\[
\tilde{x} = \mathcal{P}_X (\tilde{x} - \eta \nabla_x f(\tilde{x}, \tilde{y})) \quad \text{and} \quad \tilde{y} = \mathcal{P}_Y (\tilde{y} + \eta \nabla_y f(\tilde{x}, \tilde{y})) 
\]

then it holds that

\[
\max_{y \in Y} f(\tilde{x}, y) - f(x^*, y^*) \leq \left( \sqrt{2}(1 + \eta L) + 2(1 + \eta L)^2 + 2 \right) \kappa_y \varepsilon + \frac{\varepsilon}{2\eta}
\]

and

\[
f(x^*, y^*) - \min_{x \in X} f(x, \tilde{y}) \leq \left( \sqrt{2}(1 + \eta L) + 2(1 + \eta L)^2 + 2 \right) \kappa_x \varepsilon + \frac{\varepsilon}{2\eta}.
\]

The following lemma shows that we can upper bound the distance from given point to the saddle point by its primal-dual gap.

Lemma 2. Suppose the function \( f(x, y): X \times Y \to \mathbb{R} \) is \( (\mu_x, \mu_y) \)-convex-concave and \( L \)-smooth. Denote the saddle point of \( f \) by \( (x^*, y^*) \). Then for any \( \tilde{x} \in X, \tilde{y} \in Y \), we have

\[
\mu_x \|\tilde{x} - x^*\|_2^2 + \mu_y \|\tilde{y} - y^*\|_2^2 \leq 2 \left( \max_{y \in Y} f(\tilde{x}, y) - \min_{x \in X} f(x, \tilde{y}) \right).
\]
Then introduces four auxiliary quantities and provide their upper bounds, which are useful to the analyze the convergence of AL-SVRE.

**Lemma 3.** We use the notation of Algorithm 2 and denote $(x_k^*, y_k^*)$ is the saddle point of $F_k$ and

$$
\Delta_f = \max_{y \in Y} f(x, y) - \min_{x \in X} f(x, y).
$$

Then, for each $k \geq 1$, we have

\begin{align}
\mathbb{E}\left[\|x_k - x_k^*\|_2^2 + \|y_k - y_k^*\|_2^2\right] &\leq \varepsilon_k, \quad (7) \\
\max_{y \in Y} F_k(x_k, y) - F_k(x_k^*, y_k^*) &\leq \frac{2\Delta_f}{9}(1 - \rho)^k, \quad (8) \\
\|x_k - x^*\|_2^2 &\leq \frac{2\delta_k}{\mu_x}, \quad (9) \\
\mathbb{E}\left[\|x_k^* - x_{k+1}^*\|_2^2 + \|y_k - y_{k+1}\|_2^2\right] &\leq \frac{72\beta\delta_{k-1}}{\mu_x \min\{\mu_x, \mu_y\}}, \quad (10)
\end{align}

where

$$
\varepsilon_k \triangleq \frac{2\mu_y \Delta_f (1 - \rho)^k}{3(L + \beta)(7L + \beta) + 2\sqrt{n}\mu_y}
$$

and

$$
\delta_k \triangleq \frac{8\Delta_f (1 - \rho)^{k+1}}{(\sqrt{q} - \rho)^2}.
$$

Based on above lemmas, we obtain the main result of this paper as follows.

**Theorem 3.** Running Algorithm 2 with $q = \mu_x / (\mu_x + \beta)$, $\beta \geq 0$, $p = 1/2n$, $\tau_k = 1/(4\sqrt{n}(L + \beta))$ and

$$
T_k = \left[4\left(n + \frac{2\sqrt{n}(L + \beta)}{\min\{\mu_x + \beta, \mu_y\}}\right) \log \left(12\left(\frac{2}{1 - \rho} + \frac{1728\beta(L + \beta)(7(L + \beta) + 2\sqrt{n}\mu_y)}{\mu_x \mu_y \min\{\mu_x, \mu_y\}(1 - \rho)^2(\sqrt{q} - \rho)^2}\right)\right)\right],
$$

where $\rho < \sqrt{q}$, then we have

$$
\mathbb{E}\left[\max_{y \in Y} f(x_k, y) - \min_{x \in X} f(x, y_k)\right] \leq \frac{916\Delta_f (\kappa_x L + \sqrt{n}(L + \beta) + \kappa_y^2)}{\mu_x (\sqrt{q} - \rho)^2} (1 - \rho)^k,
$$

where $\Delta_f = \max_{y \in Y} f(x_0, y) - \min_{x \in X} f(x, y_0)$.

Theorem 3 shows the $\varepsilon$-saddle point can be obtained by calling $K = \tilde{O}\left((\mu_x + \beta)/\mu_x \log(1/\varepsilon)\right)$ times AL-SVRE to solve the sub-problem. By minimizing the product of dominant terms (ignore logarithmic terms) $\sqrt{(\mu_x + \beta)/\mu_x}$ and $(n + 2\sqrt{n}(L + \beta)/\min\{\mu_x + \beta, \mu_y\})$ in $K$ and $T_k$ respectively, we decide the choice of $\beta$ as follows (since we have suppose $\mu_x \leq \mu_y$)

$$
\beta = \begin{cases} 
\mu_y - \mu_x, & \text{if } \kappa_y \geq \sqrt{n}, \\
L/\sqrt{n} - \mu_x, & \text{if } \kappa_x > \sqrt{n} > \kappa_y, \\
0, & \text{otherwise},
\end{cases}
$$

Combining the result of Theorem 3 and equation (11), we immediately obtain the following corollary, which implies AL-SVRE has near optimal SFO upper bound for unbalanced SCSC minimax optimization.

**Corollary 1.** Suppose Problem (7) satisfies $\mu_x < \mu_y$. If running AL-SVRE with the setting of Theorem 3 and letting $\beta$ as (11), $\rho = 0.5\sqrt{q}$ and

$$
K = \left[\frac{2}{\sqrt{q}} \log \left(10992\sqrt{n}\Delta_f \kappa_y \kappa_x^3\right)\right],
$$
then we have
\[E \left[ \max_{y \in Y} f(x_K, y) - \min_{x \in X} f(x, y_K) \right] \leq \varepsilon \]
with the number of SFO calls at most
\[
O \left( \sqrt{n(\sqrt{n} + \kappa_x)(\sqrt{n} + \kappa_y)} \log(n\kappa_x) \log \left( \frac{n\Delta_f \kappa_y \kappa_x}{\varepsilon} \right) \right).
\]

The design of AL-SVRE aims to sufficiently take the advantage of AL-SVRE’s optimality for the balanced SCSC minimax problem. Hence, the choice of \( \beta \) makes the sub-problem \( \phi_1 \) more balanced than the main problem \( \phi \). In contrast, previous catalyst-type methods \cite{22,33,35} select a larger \( \beta \) to guarantee the sub-problem be well-conditioned in one of the variables. However, such strategy leads to that the sub-problem is still so unbalanced and we need two-loops of iterations to solve it. As a result, the implementation of the whole procedure is more complicated and almost cannot be used in practical.

**4 Extensions to Non-SCSC Minimax**

We can also apply AL-SVRE to solve convex-concave minimax without SCSC assumption.

**4.1 Convex-Strongly-Concave Case**

Consider a simple class of functions \( \{g_i(z)\}_{i=1}^n \) such that \( g_i(z) = \frac{\sqrt{n}L}{2} (e_i^T z)^2 \), where \( \{e_i\}_{i=1}^n \) is the standard basis of \( \mathbb{R}^n \). It is easy to check that \( g_i \) is \( \sqrt{n}L \)-smooth and

\[
\frac{1}{n} \sum_{i=1}^n \| \nabla g_i(z_1) - \nabla g_i(z_2) \|_2^2 = \frac{1}{n} \sum_{i=1}^n \left( nL^2 \right) \| e_i e_i^T (z_1 - z_2) \|_2^2 = L^2 \sum_{i=1}^n (z_{1,i} - z_{2,i})^2 = L^2 \| z_1 - z_2 \|_2^2,
\]

which implies that \( \{g_i\}_{i=1}^n \) is \( L \)-average smooth.

Using notations of Tominin et al. \cite{33} on \( \{g_i\}_{i=1}^n \), we have \( L_i = \sqrt{n}L \) and \( L_G = \frac{1}{n} \sum_{i=1}^n L_i = \sqrt{n}L \). Hence, their smoothness parameter \( L_G \) is larger than the average-smoothness parameter \( L \) we used in this paper.

**4 Extensions to Non-SCSC Minimax**

We can also apply AL-SVRE to solve convex-concave minimax without SCSC assumption.

**4.1 Convex-Strongly-Concave Case**

If the objective function \( f(x, y) \) is \((0, \mu_y)\)-convex-concave and \( X \) is bounded with diameter \( D_x \), we construct the auxiliary SCSC function to approximate it as follows:

\[f_{\varepsilon, x_0}(x, y) \triangleq f(x, y) + \frac{\varepsilon}{4D_x^2} \| x - x_0 \|_2^2.\]  

Then the difference of duality gaps between \( f_{\varepsilon, x_0} \) and \( f \) should be small. The following lemma presents this fact formally.

**Lemma 4.** Suppose \( f(x, y) \) is \((0, \mu_y)\)-convex-concave, \( X \) is bounded with diameter \( D_x \) and \( x_0 \in X \). Consider the function \( f_{\varepsilon, x_0}(x, y) \) defined as \( f_{\varepsilon, x_0} \). Then for any \((\hat{x}, \hat{y}) \in X \times Y\), we have

\[
\max_{y \in Y} f(\hat{x}, y) - \min_{x \in X} f(x, y) \leq \varepsilon + \frac{\varepsilon}{2} + \max_{y \in Y} f_{\varepsilon, x_0}(\hat{x}, y) - \min_{x \in X} f_{\varepsilon, x_0}(x, \hat{y}).
\]

**Lemma 4** means that any \( \varepsilon/2\)-saddle-point of \( f_{\varepsilon, x_0} \) is also an \( \varepsilon \)-saddle-point of \( f \). Hence, we can directly run AL-SVRE on \( f_{\varepsilon, x_0} \) and connect Lemma 4 and Theorem 3 to establish the convergence result for \((0, \mu_y)\)-convex-concave minimax optimization as follows.
Corollary 2. Under settings of Lemma 4 for $\varepsilon \leq 4LD_x^2$, the total complexity of SFO calls for finding an $\varepsilon/2$-saddle-point of $f_{x,x_0}$, which is also an $\varepsilon$-saddle-point of $f$, is

$$
\tilde{O}\left( n\left(D_x \sqrt{\frac{nL_K_d}{\varepsilon}} + n^{3/4}\sqrt{K_y} + n^{3/4}D_x \sqrt{\frac{L}{\varepsilon}} \right) \log \left( \frac{1}{\varepsilon} \right) \right)
$$

Recently, Yang et al. [40] also studied Catalyst acceleration for strongly-convex-concave minimax optimization, but they only considered the special case of $\varepsilon < \mu_y$ which leads to that their result does not match the lower bound.

4.2 Convex-Concave Case

In general convex-concave setting, we suppose $X$ and $Y$ are bounded with diameters $D_x$ and $D_y$ respectively. Then unbalance of the problem comes from the difference between diameters $D_x$ and $D_y$. We introduce the auxiliary SCSC function as follows:

$$f_{\varepsilon,x_0,y_0}(x,y) \triangleq f(x,y) + \frac{\varepsilon}{8D_x^2} \|x - x_0\|^2 - \frac{\varepsilon}{8D_y^2} \|y - y_0\|^2.\quad(14)$$

Similarly, Lemma 5 shows that any $\varepsilon/2$-saddle-point of $f_{x,x_0,y_0}$ is an $\varepsilon$-saddle-point of $f$.

Lemma 5. Suppose that $f(x,y)$ is convex-concave, $X$ and $Y$ are bounded with diameters $D_x$ and $D_y$ respectively. Consider the function $f_{\varepsilon, x_0, y_0}(x,y)$ defined as (14). Then for any $(x, y) \in X \times Y$, we have

$$\max_{x \in X} f(x, \hat{y}) - \min_{x \in X} f(x, \hat{y}) \leq \frac{\varepsilon}{2} + \max_{y \in Y} f_{\varepsilon, x_0, y_0}(\hat{x}, y) - \min_{x \in X} f_{\varepsilon, x_0, y_0}(x, \hat{y}).$$

Combining Lemma 5 and Theorem 3, we obtain the convergence result of Corollary 3.

Corollary 3. Under settings of Lemma 5 for any $\varepsilon \leq 4L \min\{D_x^2, D_y^2\}$, the total complexity of SFO calls for finding an $\varepsilon/2$-saddle-point of $f_{x,x_0,y_0}$, which is also an $\varepsilon$-saddle-point of $f$, is

$$\tilde{O}\left( n + \sqrt{nLD_xD_y} \sqrt{\frac{L}{\varepsilon}} \right) \log \left( \frac{1}{\varepsilon} \right).$$

The result of Corollary 3 nearly matches the lower bound w.r.t. $\varepsilon$, $L$, $n$, $D_x$ and $D_y$ simultaneously [13]. Note that the best known upper bound in convex-concave setting [11] is optimal to $\varepsilon$, $L$ and $n$, but it regards the diameters as constants and does not consider the potential unbalance arose from $D_x$ and $D_y$.

5 Experiments

We conduct the experiments on AUC maximization [14, 36, 41] and wireless communication [3, 10, 40] problems. We evaluate the performance of AL-SVRE and compare it with baseline algorithms ExtraGradient (EG) [11, 17] and L-SVRE [1]. We summarize the datasets in Table 4. The empirical results in Figure 1 show that our proposed AL-SVRE performs better than the baselines.
We consider the wireless communication problem \cite{3, 40} as follows

\[
\min_{x \in \mathbb{R}^{d+2}} \max_{y \in \mathbb{R}^n} f(x, y) = \frac{1}{n} \sum_{i=1}^{n} f_i(x, y; a_i, b_i, \lambda),
\]

where \(x = [\theta; u; v] \in \mathbb{R}^{d+2}, \lambda > 0\) is the regularization parameter and each component is defined as

\[
f_i(x, y; a_i, b_i, \lambda) = \frac{\lambda}{2} \|x\|^2 - \hat{p}(1 - \hat{p}) y^2 + \hat{p}((\theta^T a_i - v)^2 + 2(1 + y) \theta^T a_i) I_{b_i = -1} + (1 - \hat{p})(\theta^T a_i - u)^2 - 2(1 + y) \theta^T a_i I_{b_i = 1}.
\]

We set \(\lambda = 10^{-10}\) and evaluate all algorithms on datasets “a9a” and “w8a” \cite{4}. We tune stepsize of EG and L-SVRE (or as sub-problem solver) from \(\{0.01, 0.05, 0.1, 0.2, 0.5\}\). For L-SVRE, we let \(p = 1/2n\) by the setting of Theorem \cite{4}. For AL-SVRE, we let \(\beta = 0.01, q = \lambda x / (\lambda + \beta); T_k = [0.3n]\) for “w8a” and \(T_k = [0.5n]\) for “a9a”. We present the results of epochs against \(\log \|\nabla f(x, y)\|_2\) in Figure \ref{fig:1} (a) and (b).

### 5.2 The Wireless Communication Problems

We consider the wireless communication problem \cite{3, 40} as follows

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) = \frac{1}{n} \sum_{i=1}^{n} \log \left(1 + \frac{b_i x_i}{a_i + y_i}\right),
\]

where constrained sets are \(\mathcal{X} = \{x : \|x\|_2 \leq R, x_i \geq 0\}\) and \(\mathcal{Y} = \{y : 1^T y = n, y_i \geq 0\}\). We can verify the objective function of problem \cite{16} is convex-concave but possibly be neither strongly-convex nor strongly-concave.

We generate two datasets with (i) \(n = 500, R = 1, b = 1, a \in \mathbb{R}^{500}\) uniformly from \([0, 10]^{500}\) and (ii) \(n = 1000, R = 1, b = 1, a \in \mathbb{R}^{1000}\) uniformly from \([0, 50]^{1000}\). We tune the stepsize for all the algorithms from \([0.01, 0.1, 1.0]\) and set \(p = 2/n\) like AUC maximization. For AL-SVRE, we use the auxiliary function as follows

\[
\hat{f}(x, y) = f(x, y) + \frac{\varepsilon}{s R^2} \|x\|^2 - \frac{\varepsilon}{8n^2} \|y - 1\|^2_2
\]

where \(\varepsilon = 10^{-6}/n\). We also set \(T_k = 500, q = R^2/n^2\) and \(\beta = \varepsilon/(4R^2) - \varepsilon/(4n^2)\). Since the problem \cite{16} is constrained, we use evaluation the performance by epochs against the logarithm of the magnitude of gradient mapping, which is defined as

\[
\|x - P_x(x - \hat{\tau} \nabla_x f(x, y))\|_2 + \|y - P_y(y + \hat{\tau} \nabla_y f(x, y))\|_2
\]

where \(\hat{\tau}\) is set to be 0.1. We present the empirical results in Figure \ref{fig:1} (c) and (d).
6 Conclusions

In this paper we have studied unbalanced convex-concave minimax problems with finite-sum structure. We have shown the optimality of L-SVRE for balanced SCSC minimax and proposed a near optimal algorithm AL-SVRE for unbalanced problems. AL-SVRE only contains two loops of iterations, making its implementation more simple and practical than the existing methods for unbalanced SCSC minimax. We have also extended our algorithm to more general convex-concave minimax problems and showed the near optimality.

It would be interesting to apply our ideas to solve specific convex-concave minimax with refined smoothness assumptions and specific bilinear settings [28, 35, 38]. It is also possible to design accelerated variance reduced algorithms to solve unbalanced minimax problems in online setting.

References

[1] Ahmet Alacaoglu and Yura Malitsky. Stochastic variance reduction for variational inequality methods. arXiv preprint arXiv:2102.08352, 2021.
[2] Zeyuan Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. Journal of Machine Learning Research, 18(1):8194–8244, 2017.
[3] Stephen Boyd, Stephen P Boyd, and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004.
[4] Chih-Chung Chang and Chih-Jen Lin. LIBSVM: A library for support vector machines. ACM Transactions on Intelligent Systems and Technology, 2:27:1–27:27, 2011. Software available at http://www.csie.ntu.edu.tw/~cjlin/libsvm
[5] Tatjana Chavdarova, Gauthier Gidel, François Fleuret, and Simon Lacoste-Julien. Reducing noise in GAN training with variance reduced extragradient. In NeurIPS, 2019.
[6] Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. In NIPS, 2014.
[7] John C. Duchi and Hongseok Namkoong. Variance-based regularization with convex objectives. Journal of Machine Learning Research, 20(68):1–55, 2019.
[8] Cong Fang, Chris Junchi Li, Zhouchen Lin, and Tong Zhang. SPIDER: Near-optimal non-convex optimization via stochastic path-integrated differential estimator. In NeurIPS, 2018.
[9] Roy Frostig, Rong Ge, Sham Kakade, and Aaron Sidford. Un-regularizing: approximate proximal point and faster stochastic algorithms for empirical risk minimization. In ICML, 2015.
[10] Andrey Garnaev and Wade Trappe. An eavesdropping game with SINR as an objective function. In International Conference on Security and Privacy in Communication Systems, pages 142–162. Springer, 2009.
[11] Gauthier Gidel, Hugo Berard, Gaëtan Vignoud, Pascal Vincent, and Simon Lacoste-Julien. A variational inequality perspective on generative adversarial networks. In ICLR, 2019.
[12] Zhishuai Guo, Mingrui Liu, Zhuoning Yuan, Li Shen, Wei Liu, and Tianbao Yang. Communication-efficient distributed stochastic AUC maximization with deep neural networks. In ICML, 2020.
[13] Yuze Han, Guangzeng Xie, and Zhihua Zhang. Lower complexity bounds of finite-sum optimization problems: The results and construction. arXiv preprint arXiv:2103.08280, 2021.
[14] James A Hanley and Barbara J McNeil. The meaning and use of the area under a receiver operating characteristic (ROC) curve. Radiology, 143(1):29–36, 1982.
[15] Adam Ibrahim, Wass Azizian, Gauthier Gidel, and Ioannis Mitliagkas. Linear lower bounds and conditioning of differentiable games. In ICML, 2020.
[16] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In NIPS, 2013.

[17] GM Korpelevich. Extragradient method for finding saddle points and other problems. Matekon, 13(4):35–49, 1977.

[18] Dmitry Kovalev, Samuel Horváth, and Peter Richtárik. Don’t jump through hoops and remove those loops: Srg and katyusha are better without the outer loop. In ALT, 2020.

[19] Guanghui Lan and Yi Zhou. An optimal randomized incremental gradient method. Mathematical programming, pages 1–49, 2017.

[20] Hongzhou Lin, Julien Mairal, and Zaid Harchaoui. Catalyst acceleration for first-order convex optimization: from theory to practice. Journal of Machine Learning Research, 18(212):1–54, 2018.

[21] Tianyi Lin, Chi Jin, and Michael I. Jordan. On gradient descent ascent for nonconvex-concave minimax problems. In ICML, 2020.

[22] Tianyi Lin, Chi Jin, and Michael I. Jordan. Near-optimal algorithms for minimax optimization. In COLT, 2020.

[23] Mingrui Liu, Zhuoning Yuan, Yiming Ying, and Tianbao Yang. Stochastic AUC maximization with deep neural networks. In ICLR, 2020.

[24] Luo Luo, Cheng Chen, Yujun Li, Guangzeng Xie, and Zhihua Zhang. A stochastic proximal point algorithm for saddle-point problems. arXiv preprint arXiv:1909.06946, 2019.

[25] Luo Luo, Haishan Ye, Zhichao Huang, and Tong Zhang. Stochastic recursive gradient descent ascent for stochastic nonconvex-strongly-concave minimax problems. In NeurIPS, 2020.

[26] Aryan Mokhtari, Asuman Ozdaglar, and Sarath Pattathil. A unified analysis of extra-gradient and optimistic gradient methods for saddle point problems: Proximal point approach. In AISTATS, pages 1497–1507. PMLR, 2020.

[27] Yurii Nesterov. Lectures on convex optimization, volume 137. Springer, 2018.

[28] Yuyuan Ouyang and Yangyang Xu. Lower complexity bounds of first-order methods for convex-concave bilinear saddle-point problems. arXiv preprint arXiv:1808.02901, 2018.

[29] Balamurugan Palaniappan and Francis Bach. Stochastic variance reduction methods for saddle-point problems. In NIPS, 2016.

[30] Zebang Shen, Aryan Mokhtari, Tengfei Zhou, Peilin Zhao, and Hui Qian. Towards more efficient stochastic decentralized learning: Faster convergence and sparse communication. In ICML, 2018.

[31] Aman Sinha, Hongseok Namkoong, and John Duchi. Certifying some distributional robustness with principled adversarial training. In ICLR, 2018.

[32] Kiran K. Thekumparampil, Prateek Jain, Praneeth Netrapalli, and Sewoong Oh. Efficient algorithms for smooth minimax optimization. In NeurIPS, 2019.

[33] Vladislav Tominin, Yaroslav Tominin, Ekaterina Borodich, Dmitry Kovalev, Alexander Gasnikov, and Pavel Dvurechensky. On accelerated methods for saddle-point problems with composite structure. arXiv preprint arXiv:2103.09344, 2021.

[34] Hoi-To Wai, Zhuoran Yang, Zhaoran Wang, and Mingyi Hong. Multi-agent reinforcement learning via double averaging primal-dual optimization. In NeurIPS, 2018.

[35] Yuanhao Wang and Jian Li. Improved algorithms for convex-concave minimax optimization. arXiv preprint arXiv:2006.06359, 2020.
[36] Blake Woodworth and Nathan Srebro. Tight complexity bounds for optimizing composite objectives. In *NIPS*, 2016.

[37] Guangzeng Xie, Luo Luo, Yijiang Lian, and Zhihua Zhang. Lower complexity bounds for finite-sum convex-concave minimax optimization problems. In *ICML*, 2020.

[38] Guangzeng Xie, Yuze Han, and Zhihua Zhang. DIPPA: An improved method for bilinear saddle point problems. *arXiv preprint arXiv:2103.08270*, 2021.

[39] Yan Yan, Yi Xu, Qihang Lin, Lijun Zhang, and Tianbao Yang. Stochastic primal-dual algorithms with faster convergence than $O(1/\sqrt{t})$ for problems without bilinear structure. *arXiv preprint arXiv:1904.10112*, 2019.

[40] Junchi Yang, Siqi Zhang, Negar Kiyavash, and Niao He. A catalyst framework for minimax optimization. In *NeurIPS*, 2020.

[41] Yiming Ying, Longyin Wen, and Siwei Lyu. Stochastic online AUC maximization. In *NIPS*, 2016.

[42] Junyu Zhang, Mingyi Hong, and Shuzhong Zhang. On lower iteration complexity bounds for the saddle point problems. *arXiv preprint arXiv:1912.07481*, 2019.

[43] Dongruo Zhou and Quanquan Gu. Lower bounds for smooth nonconvex finite-sum optimization. In *ICML*, 2019.
Supplemental Materials

The supplemental materials are organized as follows. Section A presents several lemmas for further theoretical analysis. Section B provides the proof of Theorem 2, which is the convergence result of L-SVRE for balanced SCSC setting. Section C gives the detailed analysis for AL-SVRE, including the proof of our main results Theorem 3, Corollary 1 and other lemmas. Section D provides the proof of Theorem 1, which implies L-SVRE is an optimal SFO algorithm for balanced finite-sum SCSC minimax. Section E provides the detailed proofs for the results in Section D.

A Technical Lemmas

We first present some useful tools for the analysis of constrained optimization.

Lemma 6 (Nesterov [27] Theorem 2.2.9 and 2.2.12). Let \( z^* = \arg \min_{z \in C} Q(z) \), where \( Q \) is smooth and strongly-convex; \( C \) is convex and compact. Then, for any \( z \in C \) and \( \eta > 0 \), we have

\[
(\nabla Q(z^*), z - z^*) \geq 0 \quad \text{and} \quad z^* = P_C(z^* - \eta \nabla Q(z^*)).
\]

Lemma 7 (Nesterov [27] Corollary 2.2.3). Given a convex and compact set \( C \subseteq \mathbb{R}^d \); and any \( u, v \in \mathbb{R}^d \), we have \( \|P_C(u) - P_C(v)\|_2 \leq \|u - v\|_2 \).

Lemma 8. Given a convex and compact set \( C \subseteq \mathbb{R}^d \), for any \( u \in \mathbb{R}^d \) and \( v \in C \), we have

\[
\langle P_C(u) - u, P_C(u) - v \rangle \leq 0.
\]

Proof. Let \( Q(z) = \frac{1}{2} \|z - u\|_2^2 \), then \( P_C(u) = \arg \min_{z \in C} Q(z) \). Using Lemma 6, we have

\[
\langle P_C(u) - u, v - P_C(u) \rangle = \langle \nabla Q(P_C(u)), v - P_C(u) \rangle \geq 0.
\]

Then we provides some properties for convex-concave functions.

Lemma 9 (Lin et al. [22] Lemma B.2]). Assume that \( f(x, y) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) is L-smooth and \((\mu_x, \mu_y)\)-convex-concave. We define

\[
y^*_f(\cdot) \triangleq \arg \max_{y \in \mathcal{Y}} f(\cdot, y), \quad \Phi_f(\cdot) \triangleq \max_{y \in \mathcal{Y}} f(\cdot, y),
\]

\[
x^*_f(\cdot) \triangleq \arg \min_{x \in \mathcal{X}} f(x, \cdot), \quad \Psi_f(\cdot) \triangleq \min_{x \in \mathcal{X}} f(x, \cdot).
\]

Then, there holds that

(a) the function \( y^*_f(\cdot) \) is \( \mu_y \)-Lipschitz,

(b) the function \( \Phi_f(\cdot) \) is \( 2\mu_y L \)-smooth and \( \mu_x \)-strongly convex with \( \nabla \Phi_f(\cdot) = \nabla_x f(\cdot, y^*_f(\cdot)) \),

(c) the function \( x^*_f(\cdot) \) is \( \mu_x \)-Lipschitz,

(d) the function \( \Psi_f(\cdot) \) is \( 2\mu_y L \)-smooth and \( \mu_y \)-strongly concave with \( \nabla \Psi_f(\cdot) = \nabla_x f(x^*_f(\cdot), \cdot) \).

Lemma 10. Assume that \( f(x, y) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) is differentiable and \((\mu_x, \mu_y)\)-convex-concave, then for any \( z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathcal{X} \times \mathcal{Y} \), we have

\[
(g(z_1) - g(z_2), z_1 - z_2) \geq \mu_x \|x_1 - x_2\|_2^2 + \mu_y \|y_1 - y_2\|_2^2.
\]

where \( g(x, y) = (\nabla_x f(x, y), -\nabla_y f(x, y)) \).

14
Proof. According to $f(\cdot, y)$ is $\mu_y$-strongly-convex, we have
\[ f(x_2, y_1) - f(x_1, y_1) \geq \langle \nabla_x f(x_1, y_1), x_2 - x_1 \rangle + \frac{\mu_y}{2} \|x_2 - x_1\|_2, \]
\[ f(x_1, y_2) - f(x_2, y_2) \geq \langle \nabla_x f(x_2, y_2), x_1 - x_2 \rangle + \frac{\mu_y}{2} \|x_2 - x_1\|_2. \]
Similarly, by strongly-convexity of $-f(x, \cdot)$, we have
\[ -f(x_1, y_2) + f(x_1, y_1) \geq -\langle \nabla_y f(x_1, y_1), y_2 - y_1 \rangle + \frac{\mu_y}{2} \|y_2 - y_1\|_2, \]
\[ -f(x_2, y_1) + f(x_2, y_2) \geq -\langle \nabla_y f(x_2, y_2), y_1 - y_2 \rangle + \frac{\mu_y}{2} \|y_2 - y_1\|_2. \]
The desired result just follows from adding above four inequalities together. \qed

The following two lemmas show that the relationship between the smoothness of each component $f_i$, the average-smoothness of $\{f_i\}_{i=1}^n$ and the smoothness of $\frac{1}{n} \sum_{i=1}^n f_i$.

Lemma 11. Assume that $f_i$ is $L_i$-smooth, then $\{f_i\}_{i=1}^n$ is $\sqrt{\frac{1}{n} \sum_{i=1}^n L_i^2}$-average-smooth.

Proof. By the definition of smoothness, we have
\[ \frac{1}{n} \sum_{i=1}^n \|f_i(z_1) - f_i(z_2)\|_2^2 \leq \frac{1}{n} \sum_{i=1}^n L_i^2 \|z_1 - z_2\|_2^2, \]
which is our desired result. \qed

Lemma 12. Assume that $\{f_i\}_{i=1}^n$ is $L$-average-smooth, then $f(z) = \frac{1}{n} f_i(z)$ is $L$-smooth.

Proof. Note that
\[
\|\nabla f(z_1) - \nabla f(z_2)\|_2 = \left\| \frac{1}{n} \sum_{i=1}^n (\nabla f_i(z_1) - \nabla f_i(z_2)) \right\|_2 \\
\leq \frac{1}{n} \sum_{i=1}^n \|\nabla f_i(z_1) - \nabla f_i(z_2)\|_2 \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(z_1) - \nabla f_i(z_2)\|_2^2} \\
\leq \sqrt{L^2 \|z_1 - z_2\|_2^2} = L \|z_1 - z_2\|_2,
\]
where the first inequality is according to triangle inequality, the second inequality follows from AM-QM inequality, and the last inequality is due to average-smoothness of $\{f_i\}_{i=1}^n$. \qed

B Proof of Theorem 2

Following proof is adapted from the proof of Theorem 4.9 in [1].

Proof. By Lemma 8 we know that
\[
\langle z_{k+1/2} - z_k + \tau g(w_k), z_{k+1} - z_{k+1/2} \rangle \geq 0, \\
\langle z_{k+1} - z_k + \tau [g(w_k) + g_i(z_{k+1/2}) - g_i(w_k)], z^* - z_{k+1} \rangle \geq 0.
\]
Summing above two inequalities, we have
\[
\langle z_{k+1} - z_k, z^* - z_{k+1} \rangle + \langle z_{k+1/2} - z_k, z_{k+1} - z_{k+1/2} \rangle \\
+ \tau \langle g(w_k) + g_i(z_{k+1/2}) - g_i(w_k), z^* - z_{k+1/2} \rangle \\
+ \tau \langle g_i(z_{k+1/2}) - g_i(w_k), z_{k+1/2} - z_{k+1} \rangle \geq 0. \tag{18}
\]
Since $2(a, b) = ||a + b||^2_2 - ||a||^2_2 - ||b||^2_2$, the first term in inequality (18) can be written as
\begin{align}
2(z_{k+1} - z_k, z^* - z_{k+1}) \\
= 2(z_{k+1} - \alpha z_k - (1 - \alpha)w_k, z^* - z_{k+1}) \\
= 2\alpha(z_{k+1} - z_k, z^* - z_{k+1}) + 2(1 - \alpha)(z_{k+1} - w_k, z^* - z_{k+1}) \\
= \alpha(\|z_k - z^*\|^2_2 - \|z_{k+1} - z^*\|^2_2 - \|z_{k+1} - z_k\|^2_2) \\
+ (1 - \alpha)(\|w_k - z^*\|^2_2 - \|z_{k+1} - z^*\|^2_2 - \|z_{k+1} - w_k\|^2_2) \\
= \alpha \|z_k - z^*\|^2_2 + (1 - \alpha)\|w_k - z^*\|^2_2 - \|z_{k+1} - z^*\|^2_2 - \|z_{k+1} - w_k\|^2_2
\end{align}
(19)

Similarly, the second term in inequality (18) can be written as
\begin{align}
2(z_{k+1/2} - z_k, z_{k+1} - z_{k+1/2}) \\
= \alpha(\|z_k - z_{k+1/2}\|^2_2 + (1 - \alpha)\|w_k - z_{k+1/2}\|^2_2 - \|z_{k+1/2} - z_{k+1}\|^2_2) \\
- \alpha(\|z_{k+1/2} - z_k\|^2_2 - (1 - \alpha)\|z_{k+1} - w_k\|^2_2)
\end{align}
(20)

Using the fact $E_k[g_i(z)] = g(z)$, the expectation of third term in inequality (18) can be bounded as
\begin{align}
2E_k \left[(g(w_k) + g_i(z_{k+1/2}) - g_i(w_k), z^* - z_{k+1/2})\right] \\
= 2(g(z_{k+1/2}), z^* - z_{k+1/2}) \\
\leq 2(g(z^*), z^* - z_{k+1/2}) - 2\mu \|z_{k+1/2} - z^*\|^2_2 \\
\leq -\mu E_k \|z_{k+1} - z^*\|^2_2 + 2\mu E_k \|z_{k+1/2} - z_{k+1}\|^2_2
\end{align}
(21)

where the first inequality follows from Lemma 10 and the last inequality is according to Lemma 6 and $\|a + b\|^2_2 \leq 2\|a\|^2_2 + 2\|b\|^2_2$.

Moreover, by Young’s inequality $2(a, b) \leq \beta \|a\|^2_2 + \frac{1}{2}\|b\|^2_2$ and $L$-average-smoothness of $\{f_i\}_{i=1}^n$, there holds
\begin{align}
E_k \left[2\tau(g_i(z_{k+1/2}) - g_i(w_k), z_{k+1/2} - z_{k+1})\right] \\
\leq 2\tau^2 E_k \|g_i(z_{k+1/2}) - g_i(w_k)\|^2_2 + \frac{1}{2} E_k \|z_{k+1/2} - z_{k+1}\|^2_2 \\
\leq 2\tau^2 L^2 \|z_{k+1/2} - w_k\|^2_2 + \frac{1}{2} E_k \|z_{k+1/2} - z_{k+1}\|^2_2
\end{align}
(22)

Plugging results of (19), (20), (21) and (22) into inequality (18), we obtain that
\begin{align}
\alpha \|z_k - z^*\|^2_2 + (1 - \alpha)\|w_k - z^*\|^2_2 - E_k \|z_{k+1} - z^*\|^2_2 - E_k \|z_{k+1/2} - z_{k+1}\|^2_2 \\
- \alpha \|z_{k+1/2} - z_k\|^2_2 - (1 - \alpha)\|z_{k+1/2} - w_k\|^2_2 \\
- \tau\mu E_k \|z_{k+1} - z^*\|^2_2 + 2\tau\mu E_k \|z_{k+1/2} - z_{k+1}\|^2_2 \\
+ 2\tau^2 L^2 \|z_{k+1/2} - w_k\|^2_2 + \frac{1}{2} E_k \|z_{k+1/2} - z_{k+1}\|^2_2 \geq 0
\end{align}
that is
\begin{align}
(1 + \tau\mu)E_k \|z_{k+1} - z^*\|^2_2 \\
\leq \alpha \|z_k - z^*\|^2_2 + (1 - \alpha)\|w_k - z^*\|^2_2 - \left(\frac{1}{2} - 2\tau\mu\right) E_k \|z_{k+1/2} - z_{k+1}\|^2_2 \\
- (1 - \alpha - 2\tau^2 L^2) \|z_{k+1/2} - w_k\|^2_2.
\end{align}
Consequently, with setting $\alpha = 1 - p$, $p = \frac{1}{2n}$ and $\tau = \frac{1}{\sqrt{\bar{m}}} L$, we know that $\frac{1}{2} - 2\tau \mu \geq 0$, $1 - \alpha - 2\tau^2 L^2 \geq 0$ and

$$(1 + \tau \mu)\mathbb{E}_k \left[ \|z_{k+1} - z^*\|^2 \right] \leq (1 - p) \|z_k - z^*\|^2 + p \|w_k - z^*\|^2. \quad (23)$$

On the other hand, by definition of $w_{k+1}$, we have

$$\mathbb{E}_k \left[ \|w_{k+1} - z^*\|^2 \right] = (1 - p) \|w_k - z^*\|^2 + p \mathbb{E}_k \left[ \|z_{k+1} - z^*\|^2 \right]. \quad (24)$$

Following from results of (23) and (24), there holds

$$(1 + \tau \mu)\mathbb{E}_k \left[ \|z_{k+1} - z^*\|^2 \right] + c\mathbb{E}_k \left[ \|w_{k+1} - z^*\|^2 \right] \leq (1 - p) \|z_k - z^*\|^2 + p \mathbb{E}_k \left[ \|z_{k+1} - z^*\|^2 \right],$$

that is

$$(1 + \tau \mu - cp)\mathbb{E}_k \left[ \|z_{k+1} - z^*\|^2 \right] + c\mathbb{E}_k \left[ \|w_{k+1} - z^*\|^2 \right] \leq (1 - p) \|z_k - z^*\|^2 + p \mathbb{E}_k \left[ \|z_{k+1} - z^*\|^2 \right]. \quad (25)$$

Letting $c = \frac{2\tau^2 \mu + 2p}{\tau^2 \mu + 2p}$ and noticing that $\tau \mu < 1, p < 1$, we have

$$1 - p \frac{1 - \frac{\tau \mu - p(c - 1)}{1 + \tau \mu - cp}}{1 - \frac{\tau \mu - p\tau^2 \mu}{1 + \tau \mu - p\tau^2 \mu + 2p}} = 1 - \frac{\tau^2 \mu^2 + p\tau \mu}{\tau \mu(1 + \tau \mu) + 2p(1 - p)} \leq 1 - \frac{\tau \mu}{2(\tau \mu + p)} \leq 1 - \frac{\tau \mu}{2(\tau \mu + p)}.$$

and

$$\frac{p + c(1 - p)}{c} = 1 - p \left( 1 - \frac{\tau \mu + 2p}{\tau \mu + 2p} \right) = 1 - \frac{\tau \mu}{2(\tau \mu + p)}.$$ 

Together with inequality (25), there holds

$$(1 + \tau \mu - cp)\mathbb{E}_k \left[ \|z_{k+1} - z^*\|^2 \right] + c\mathbb{E}_k \left[ \|w_{k+1} - z^*\|^2 \right] \leq \left( 1 - \frac{\tau \mu}{2(\tau \mu + p)} \right) \left( (1 + \tau \mu - cp) \|z_k - z^*\|^2 + c \|w_k - z^*\|^2 \right),$$

which implies that

$$(\tau \mu + p) \mathbb{E} \left[ \|z_k - z^*\|^2 \right] \leq (\tau \mu + 2p(1 - p)) \mathbb{E} \left[ \|z_k - z^*\|^2 \right] + 2(\tau \mu + p) \|w_k - z^*\|^2 \leq \left( 1 - \frac{\tau \mu}{2(\tau \mu + p)} \right)^k \left( (\tau \mu + 2p(1 - p)) \mathbb{E} \left[ \|z_0 - z^*\|^2 \right] + 2(\tau \mu + p) \|w_0 - z^*\|^2 \right) \leq 4(\tau \mu + p) \|z_0 - z^*\|^2 \left( 1 - \frac{\tau \mu}{2(\tau \mu + p)} \right)^k,$$

where we have recalled that $2(1 - p) \geq 1$ due to $p = \frac{1}{2n} \leq \frac{1}{2}$ and $\tau \mu < 1$. 

\[\square\]
C Convergence Analysis of AL-SVRE

In this section, we aim to give the convergence rate of AL-SVRE. We first give the proof of Lemma 1 and establish the connection between the distance to saddle point and primal-dual gap. Then, we show upper bounds of some auxiliary quantities, which is useful to the analysis for AL-SVRE. Finally, we provide the formal proofs of Theorem 3 and Corollary 4.

C.1 The proof of Lemma 1

Proof. The definition of $\hat{x}$ means

$$\hat{x} = \arg\min_{x \in \mathcal{X}} \left( \langle \nabla_x f(\hat{x}, \hat{y}), x - \hat{x} \rangle + \frac{1}{2\eta} \|x - \hat{x}\|_2^2 \right).$$

Therefore, for any $x \in \mathcal{X}$, we have

$$\langle \nabla_x f(\hat{x}, \hat{y}), x - \hat{x} \rangle + \frac{1}{2\eta} \|x - \hat{x}\|_2^2 \geq \langle \nabla_x f(\hat{x}, \hat{y}), \hat{x} - \hat{x} \rangle + \frac{1}{2\eta} \|\hat{x} - \hat{x}\|_2^2,$n

that is

$$\langle \nabla_x f(\hat{x}, \hat{y}), x - \hat{x} \rangle \geq \frac{1}{2\eta} \left( \|\hat{x} - \hat{x}\|_2^2 - \|x - \hat{x}\|_2^2 \right) \geq -\frac{1}{2\eta} \|x - \hat{x}\|_2^2.$$ (26)

Let $\Phi_f(x) = \max_{y \in Y} f(x, y)$ and $y_f^*(x) = \arg \max_{y \in Y} f(x, y)$, then it holds that

$$\begin{align*}
\Phi_f(\hat{x}) - \Phi_f(x^*) &\geq \Phi_f(\hat{x}) - \Phi_f(\hat{x}) - (\Phi_f(x^*) - \Phi_f(\hat{x})) \\
&\leq \langle \nabla \Phi_f(\hat{x}), \hat{x} - x^* \rangle + \kappa_y L \|\hat{x} - x^*\|_2^2 - \langle \nabla \Phi_f(\hat{x}), x^* - \hat{x} \rangle \\
&= \langle \nabla_x f(\hat{x}, y_f^*(\hat{x})), \hat{x} - x^* \rangle + \kappa_y L \|\hat{x} - x^*\|_2^2 \\
&= \langle \nabla_x f(\hat{x}, y_f^*(\hat{x})) - \nabla_x f(\hat{x}, \hat{y}), \hat{x} - x^* \rangle + \langle \nabla_x f(\hat{x}, \hat{y}), \hat{x} - x^* \rangle + \kappa_y L \|\hat{x} - x^*\|_2^2 \\
&\leq L \|\hat{y} - y_f^*(\hat{x})\|_2 \|\hat{x} - x^*\|_2 + \frac{1}{2\eta} \|\hat{x} - x^*\|_2^2 + \kappa_y L \|\hat{x} - x^*\|_2^2.
\end{align*}$$ (27)

where the first inequality is according to $\Phi_f$ is $2\kappa_y L$-smooth and convex, the last inequality is due to inequality (26) with setting $x = x^*$.

According to the $\|a + b\|_2^2 \leq 2 \|a\|_2^2 + 2 \|b\|_2^2$ and the Lipschitz continuity of $y_f^*(\cdot)$ (cf. Lemma 9), we observe that

$$\begin{align*}
\|\hat{y} - y_f^*(\hat{x})\|_2^2 &\leq 2 \|\hat{y} - y^*\|_2^2 + 2 \|y^* - y_f^*(\hat{x})\|_2^2 \\
&\leq 2 \|\hat{y} - y^*\|_2^2 + 2\kappa_y^2 \|\hat{x} - x^*\|_2 \\
&\leq 2\kappa_y^2 \varepsilon.
\end{align*}$$ (28)

Next, following from optimality of $x^*$ such that $x^* = \arg\min_{x \in \mathcal{X}} \Phi_f(x)$ and Lemma 6, we have

$$x^* = \mathcal{P}_\mathcal{X}(x^* - \eta \nabla_x \Phi_f(x^*)) = \mathcal{P}_\mathcal{X}(x^* - \eta \nabla_x f(x^*, y^*)).$$

Hence, by smoothness of the function $f$ and Lemma 7, we have

$$\begin{align*}
\|\hat{x} - x^*\|_2 &\leq \|\mathcal{P}_\mathcal{X}(x^* - \eta \nabla_x f(\hat{x}, \hat{y})) - \mathcal{P}_\mathcal{X}(x^* - \eta \nabla_x f(x^*, y^*))\|_2 \\
&\leq \|\hat{x} - x^* - \eta (\nabla_x f(\hat{x}, \hat{y}) - \nabla_x f(x^*, y^*))\|_2 \\
&\leq \|\hat{x} - x^*\|_2 + \eta L \sqrt{\|\hat{x} - x^*\|_2^2 + \|\hat{y} - y^*\|_2^2} \\
&\leq (1 + \eta L) \sqrt{\varepsilon}.
\end{align*}$$ (29)
Consequently, plugging inequalities (28) and (29) into (27) yields that
\[
\max_{y \in \mathcal{Y}} f(\tilde{x}, y) - f(x^*, y^*) \leq \sqrt{2(1 + \eta L)}\kappa_y L\varepsilon + \frac{\varepsilon}{2\eta} + 2\kappa_y L \left(\|\tilde{x} - x^*\|_2^2 + \|\tilde{x} - x^*\|_2^2\right)
\leq \left(\sqrt{2(1 + \eta L) + 2(1 + \eta L)^2 + 2}\right)\kappa_y L\varepsilon + \frac{\varepsilon}{2\eta}
\]

Similarly, by definition of \(\tilde{y}\), we also have
\[
\|\tilde{y} - y^*\|_2 = \|\tilde{y} - y^* + (\nabla_y f(\tilde{x}, \tilde{y}) - \nabla_y f(x^*, y^*))\|_2.
\]
and
\[
g(x^*, y^*) - \min_{x \in \mathcal{X}} g(x, \tilde{y}) \leq \left(\sqrt{2(1 + \eta L) + 2(1 + \eta L)^2 + 2}\right)\kappa_x L\varepsilon + \frac{\varepsilon}{2\eta}.
\]

Furthermore, by inequality (29) and (30), we have
\[
\|\tilde{x} - x^*\|_2^2 + \|\tilde{y} - y^*\|_2^2
= \|\mathcal{P}_x (\tilde{x} - \eta\nabla_x f(\tilde{x}, \tilde{y})) - \mathcal{P}_x (x^* - \eta\nabla_x f(x^*, y^*))\|_2^2
+ \|\mathcal{P}_y (\tilde{y} - \eta\nabla_y f(\tilde{x}, \tilde{y})) - \mathcal{P}_y (y^* + \eta\nabla_y f(x^*, y^*))\|_2^2
\leq 2\|\tilde{x} - x^*\|_2^2 + 2\|\tilde{y} - y^*\|_2^2 + 2\eta^2 \left(\|\nabla_x f(\tilde{x}, \tilde{y}) - \nabla_x f(x^*, y^*)\|_2^2 + \|\nabla_y f(\tilde{x}, \tilde{y}) - \nabla_y f(x^*, y^*)\|_2^2\right)
\leq 2\varepsilon + 2\eta^2 L^2 \left(\|\tilde{x} - x^*\|_2^2 + \|\tilde{y} - y^*\|_2^2\right) \leq 2(1 + \eta^2 L^2)\varepsilon,
\]
where the second inequality is according to the smoothness of \(f\).

\[\square\]

C.2 The Proof of Lemma 2

Proof. Following the convexity of function \(f(\cdot, y^*)\), we know that
\[
\frac{\mu_y}{2} \|\tilde{x} - x^*\|_2^2 \leq f(\tilde{x}, y^*) - f(x^*, y^*).
\]
Similarly, the concavity of function \(f(x^*, \cdot)\) leads to
\[
\frac{\mu_y}{2} \|\tilde{y} - y^*\|_2^2 \leq f(x^*, \tilde{y}) - f(x^*, y^*).
\]
Together with these pieces, it holds that
\[
\mu_x \|\tilde{x} - x^*\|_2^2 + \mu_y \|\tilde{y} - y^*\|_2^2 \leq 2(f(\tilde{x}, y^*) - f(x^*, \tilde{y})) \leq 2 \left(\max_{y \in \mathcal{Y}} f(\tilde{x}, y) - \min_{x \in \mathcal{X}} f(x, \tilde{y})\right).
\]

\[\square\]

C.3 The Proof of Lemma 3

Proof. It is easy to verify that that \(F_k\) is \((L + \beta)\)-smooth and \((\mu_x + \beta, \mu_y)\)-convex-concave. We will use induction to prove inequalities (7) to (10) hold for each \(k \geq 1\).

We first assume that inequalities (7) to (10) hold for any \(k = 1, 2, \ldots, K - 1\), then we prove the statements for \(k = K\).

Part (a): Inequalities (7) and (8) hold for \(k = K\).

By definition of \(T_K\), we know that
\[
e^{\theta T_k} \geq 12 \left(\frac{2}{1 - \rho} + \frac{1728\beta(L + \beta)}{\mu_x \mu_y \min\{\mu_x, \mu_y\}} \frac{(7(L + \beta) + 2\sqrt{n}\mu_y)}{(1 - \rho)^2(\sqrt{q} - \rho)^2}\right) \triangleq R,
\]

19
where \( \frac{1}{\theta} = 4 \left( n + \frac{2\sqrt{n}(L+\beta)}{\min(\mu_x, \mu_y)} \right) \).

Let \( \{u_{K,t}, v_{K,t}\}_{t \geq 0} \) be the sequence of using L-SVRE to solve minimax problem

\[
\min \max_{u \in X} F_K(u, v)
\]

with initial point \( (u_{K,0}, v_{K,0}) = (x_{K-1}, y_{K-1}) \) and stepsize \( \tau_K = \frac{1}{4\sqrt{n}(L+\beta)} \). Based on Theorem 2 we have

\[
\mathbb{E} \left[ \|u_{K,T_K} - x_K^*\|_2^2 + \|v_{K,T_K} - y_K^*\|_2^2 \right] 
\leq 4 \left( 1 - \theta \right)^{T_K} \mathbb{E} \left[ \|x_{K-1} - x_K^*\|_2^2 + \|y_{K-1} - y_K^*\|_2^2 \right] 
\leq 4e^{-\theta T_K} \mathbb{E} \left[ \|x_{K-1} - x_K^*\|_2^2 + \|y_{K-1} - y_K^*\|_2^2 \right] 
\leq \frac{4}{R} \mathbb{E} \left[ \|x_{K-1} - x_K^*\|_2^2 + \|y_{K-1} - y_K^*\|_2^2 \right].
\]  

(32)

Note that

\[
\mathbb{E} \left[ \|x_{K-1} - x_K^*\|_2^2 + \|y_{K-1} - y_K^*\|_2^2 \right] 
\leq 2 \mathbb{E} \left[ \|x_{K-1} - x_{K-1}\|_2^2 + \|y_{K-1} - y_{K-1}\|_2^2 \right] 
\leq 2 \mathbb{E} \left[ \|x_{K-1} - x_{K-1}\|_2^2 + \|y_{K-1} - y_{K-1}\|_2^2 \right] + 2 \mathbb{E} \left[ \|x_{K-1} - x_{K-1}\|_2^2 + \|y_{K-1} - y_{K-1}\|_2^2 \right]
\]  

(33)

where we have used induction hypothesis (7) and (10).

Plugging (33) into (32), we have

\[
\mathbb{E} \left[ \|u_{K,T_K} - x_K^*\|_2^2 + \|v_{K,T_K} - y_K^*\|_2^2 \right] 
\leq \frac{4}{R} \left( 2\varepsilon_{K-1} + \frac{144\beta \delta_{K-3}}{\mu_x \min(\mu_x, \mu_y)} \right) \mathbb{E} \left[ \|u_{K,T_K} - x_K^*\|_2^2 + \|v_{K,T_K} - y_K^*\|_2^2 \right] 
\leq \frac{4}{R} \left( 2\varepsilon_{K-1} + \frac{144\beta}{\mu_x \min(\mu_x, \mu_y)} \right) \frac{8\Delta_f (1 - \rho)^{K-2}}{(\sqrt{\rho} - \rho)^2} 
\leq \frac{4}{R} \left( \frac{2}{1 - \rho} + \frac{1728\beta (L + \beta)(7(L + \beta) + 2\sqrt{\mu_y})}{\mu_x \mu_y \min(\mu_x, \mu_y) (1 - \rho)^2 (\sqrt{\rho} - \rho)^2} \right) \varepsilon_K \leq \frac{1}{3} \varepsilon_K.
\]  

(34)

Consequently, combing inequality (34) with equation (31) in the proof of Lemma 1 we obtain

\[
\mathbb{E} \left[ \|x_K - x_K^*\|_2^2 + \|y_K - y_K^*\|_2^2 \right] \leq \frac{2(1 + \tau_K^2 (L + \beta)^2)}{3} \varepsilon_K \leq \frac{2(1 + \frac{1}{16})}{3} \varepsilon_K \leq \varepsilon_K,
\]

and

\[
\max_{y \in Y} F_K(x_K, y) - F_K(x_K^*, y_K) 
\leq \left( \sqrt{2} (1 + \tau_K (L + \beta)) + 2(1 + \tau_K (L + \beta))^2 + 2 \right) \frac{(L + \beta)^2 \varepsilon_K}{\mu_y} + \frac{\varepsilon_K}{6 \tau_K} 
\leq \left( \frac{5}{4} \sqrt{2} + \frac{50}{16} + 2 \right) \frac{(L + \beta)^2 \varepsilon_K}{\mu_y} + 2 \sqrt{n} (L + \beta) \frac{\varepsilon_K}{3} 
\leq \left( \frac{7}{16} + 2 \sqrt{n} \right) (L + \beta) \frac{\varepsilon_K}{3} 
\leq \frac{2\Delta_f}{9} (1 - \rho)^K,
\]

where we have used \( \tau_K = \frac{1}{4\sqrt{n}(L+\beta)} \) and

\[
\left\{ \begin{array}{l}
x_K = P_X (u_{K,T_K} - \tau_K \nabla_x F_K(u_{K,T_K}, v_{K,T_K})) \\
y_K = P_Y (v_{K,T_K} + \tau_K \nabla_y F_K(u_{K,T_K}, v_{K,T_K}))
\end{array} \right.
\]

Therefore, we have proved inequalities (7) and (8) hold for \( 1 \leq k \leq K \).
Part (b): Inequality (9) holds for $k = K$.

Let \( \Phi(x) \triangleq \max_{y \in Y} f(x, y) \), and \( \Phi_k(x) \triangleq \max_{y \in Y} F_k(x, y) = \Phi(x) + \frac{\beta}{2} \|x - u_{k-1}\|^2_2 \). It is easy to check that \( x_k^* = \arg \min_{x \in X} \Phi_k(x) \) and \( \Phi_k \) is \( \mu_k \)-strongly convex by Lemma 9.

Part (a) means we have (9) holds for \( 1 \leq k \leq K \), which implies that

\[
E[\Phi_k(x_k)] - \Phi_k^* \leq \frac{2\Delta_f}{9}(1 - \rho)^k, \quad \text{for} \quad 1 \leq k \leq K.
\]

Consequently, according to the analysis of Catalyst for convex minimization in (20) it holds that

\[
E\Phi(x_k) - \Phi^* \leq \frac{8\Delta_f}{(\sqrt{q} - \rho)^2}(1 - \rho)^{k+1} = \delta_k, \quad \text{for} \quad 0 \leq k \leq K.
\]  

(35)

Combining above inequality with the strong convexity of \( \Phi \), we have

\[
\frac{\mu_x}{2} E\|x_k - x^*\|^2_2 \leq E\Phi(x_k) - \Phi^* \leq \delta_k, \quad \text{for} \quad 0 \leq k \leq K.
\]

Part (c): Inequality (10) holds for \( k = K \).

For \( K \geq 2 \), the definition of \( u_k \) and \( \gamma < 1 \) means

\[
\|u_{K-1} - u_K\|^2_2 = \|x_{K-1} + \gamma(x_{K-1} - x_{K-2}) - x_K - \gamma(x_K - x_{K-1})\|^2_2
\]

\[
\leq (1 + \gamma) \|x_K - x_{K-1}\|^2_2 + \gamma \|x_{K-1} - x_{K-2}\|^2_2
\]

\[
\leq 3 \max\{\|x_K - x_{K-1}\|^2_2, \|x_{K-1} - x_{K-2}\|^2_2\}.
\]

Thus, we have

\[
\|u_{K-1} - u_K\|^2_2 \leq 9 \max\{\|x_K - x_{K-1}\|^2_2, \|x_{K-1} - x_{K-2}\|^2_2\}
\]

\[
\leq 9 \max\{2\|x_K - x^*_K\|^2_2 + 2\|x_{K-1} - x^*_K\|^2_2, 2\|x_{K-1} - x^*_K\|^2_2 + 2\|x_{K-2} - x^*_K\|^2_2\}
\]

\[
\leq 36 \max\{\|x_K - x^*_K\|^2_2, \|x_{K-1} - x^*_K\|^2_2, \|x_{K-2} - x^*_K\|^2_2\}
\]

\[
\leq \frac{72\delta_{K-1}}{\mu_x}.
\]

where the last inequality is according to the fact that (9) holds for \( 0 \leq k \leq K \).

Next, by strong convexity of \( F_k(x, y^*_K) \) and \( x_k^* = \arg \min_{x \in X} F_k(x, y^*_K) \), we have

\[
\frac{\mu_x + \beta}{2} \|x^*_K - x_{K+1}^*\|^2_2 \leq F_k(x^*_K + y^*_K) - F_k(x^*_K, y^*_K).
\]

Similarly, by strong concavity of \( F_k(x^*_K, y) \) and \( y_k^* = \arg \max_{y \in Y} F_k(x^*_K, y) \), we have

\[
\frac{\mu_y}{2} \|y^*_K - y_{K+1}^*\|^2_2 \leq F_k(x^*_K, y^*_K) - F_k(x^*_K, y^*_K + y^*_K).
\]

Therefore, we can conclude that

\[
\frac{\mu_x + \beta}{2} \|x^*_K - x_{K+1}^*\|^2_2 + \frac{\mu_y}{2} \|y^*_K - y_{K+1}^*\|^2_2
\]

\[
\leq F_k(x^*_K + y^*_K) - F_k(x^*_K, y^*_K)
\]

\[
= f(x^*_K, y^*_K) + \frac{\beta}{2} \|x^*_K + u_{K-1} - u_K\|^2_2 - \left(f(x^*_K, y^*_{K+1}) + \frac{\beta}{2} \|x^*_K - u_{K-1}\|^2_2\right)
\]

\[
= f(x^*_K, y^*_K) + \frac{\beta}{2} \left(\|x^*_K + u_{K-1} - u_K\|^2_2 + \|u_K - u_{K-1}\|^2_2 + 2\|x^*_K + u_{K-1} - u_K, u_K - u_{K-1}\|^2\right)
\]

\[
- \left(f(x^*_K, y^*_{K+1}) + \frac{\beta}{2} \left(\|x^*_K - u_{K-1}\|^2_2 + \|u_K - u_{K-1}\|^2_2 + 2\|x^*_K - u_K, u_K - u_{K-1}\|^2\right)\right)
\]

\[
= F_{K+1}(x^*_K + y^*_K) - F_{K+1}(x^*_K, y^*_K) + \beta(x^*_K + u_{K-1} - x^*_K, u_K - u_{K-1})
\]

\[
\leq \beta(x^*_K + u_{K-1} - x^*_K, u_K - u_{K-1}) \leq \frac{\beta}{2} \|x^*_K + u_{K-1} - x^*_K\|^2_2 + \frac{\beta}{2} \|u_K - u_{K-1}\|^2_2,
\]

\[
\text{\textsuperscript{3}The original proof of Proposition 5 in Pages 45-46 of [20] should be slightly modified by employing } \Phi(x_0) - \Phi^* = \max_y f(x_0, y) - f(x^*, y^*) \leq \Delta_f. \text{ In fact, this result means inequality (9) also holds for } k = 0.
\]
where we have used that $F_{K+1}(x_{K+1}, y_{K+1}) \leq F_{K+1}(x_{K+1}', y_{K+1}) \leq F_{K+1}(x_{K}', y_{K+1}')$. Hence, it holds that

$$E \left[ \|x_k' - x_{K+1}'\|^2_2 + \|y_k' - y_{K+1}'\|^2_2 \right] \leq \frac{\beta}{\min\{\mu_x, \mu_y\}} E \|u_{K-1}\|^2_2 \leq \frac{72\beta \delta_{K-2}}{\mu_x \min\{\mu_x, \mu_y\}}.$$

We also need to show the induction base to finish the proof of inequalities (7) to (10).

**Part (d): Induction base.**

Finally, we present the induction base that inequalities (7) to (10) hold for $k = 1$.

Since $F_1$ is $(\mu_x + \beta, \mu_y)$-convex-concave, we have

$$\frac{\mu_x + \beta}{2} \|x_0 - x_1^*\|^2_2 + \frac{\mu_y}{2} \|y_0 - y_1^*\|^2_2 \leq F_1(x_0, y_1^*) - F_1(x_1^*, y_1^*) + F_1(x_1^*, y_1^*) - F_1(x_1^*, y_0)$$

$$= f(x_0, y_1^*) - f(x_1^*, y_0) - \frac{\beta}{2} \|x_1^* - x_0\|^2_2$$

$$\leq \max_{y \in Y} f(x_0, y) - \min_{y \in Y} f(x, y_0) = \Delta_f,$$

which implies

$$E \left[ \|u_1, r_1 - x_1^*\|^2_2 + \|v_1, r_1 - y_1^*\|^2_2 \right] \leq 4e^{-\theta T} \frac{\mu_x \min\{\mu_x, \mu_y\}}{ \min\{\mu_x, \mu_y\}} \leq \frac{1}{3} \varepsilon_1. \quad (37)$$

Similar to the proof in Part (a), we can use Lemma 12 and inequality (37) to show that (7) and (8) hold for $k = 1$.

Moreover, according to Proposition 5 in [20] and the strong convexity of $F_1$, we know that (9) holds for $k = 1$.

Note that $z_1 - z_0 = (1 + \gamma)(y_1 - y_0)$, thus we have

$$E \|u_0 - u_1\|^2_2 \leq 4E \|x_1 - x_0\|^2_2 \leq \frac{32\delta_0}{\mu_x} \leq \frac{72\delta_{K-1}}{\mu_x}.$$

Therefore, by inequality (36) in Part (c), we have

$$E \left[ \|x_1^* - x_2^*\|^2_2 + \|y_1^* - y_2^*\|^2_2 \right] \leq \frac{\beta}{\min\{\mu_x, \mu_y\}} E \|u_1 - u_0\|^2_2 \leq \frac{72\beta \delta_{K-1}}{\mu_x \min\{\mu_x, \mu_y\}}.$$

As a conclusion, we have proved inequalities (7) to (10) hold for any $k \geq 1$.

**C.4 The proof of Theorem 3**

**Proof.** Consider that the fact

$$y_k^* = \arg \max_{y \in Y} f(x_k^*, y) + \frac{\beta}{2} \|x_k^* - u_{k-1}\|^2_2 = \arg \max_{y \in Y} f(x_k^*, y) = y_j^*(x_k^*),$$

then we have

$$\|y_k^* - y^*\|_2 = \|y_j^*(x_k^*) - y_j^*(x^*)\|_2 \leq \kappa_y \|x_k^* - x^*\|_2 \quad (38)$$

by using Lemma 9.
Therefore, we conclude that

\[
\mathbb{E} \left[ \left\| u_{k,T_k} - x^* \right\|^2 + \left\| v_{k,T_k} - y^* \right\|^2 \right] \\
\leq 2 \mathbb{E} \left[ \left\| u_{k,T_k} - x_k^* \right\|^2 + \left\| v_{k,T_k} - y_k^* \right\|^2 \right] + 2 \mathbb{E} \left[ \left\| x_k^* - x^* \right\|^2 + \left\| y_k^* - y^* \right\|^2 \right] \\
\leq \frac{2}{3} \varepsilon_k + 2 (\kappa_y^2 + 1) \mathbb{E} \left\| x_k^* - x^* \right\|^2 \\
\leq \varepsilon_k + 4 (\kappa_y^2 + 1) \mathbb{E} \left[ \left\| x_k^* - x_k \right\|^2 + \left\| x_k - x^* \right\|^2 \right] \\
\leq \varepsilon_k + 4 (\kappa_y^2 + 1) \left( \varepsilon_k + \frac{2 \delta_k}{\mu_x} \right) \leq \varepsilon_k,
\]

where the second inequality is due to (34) and (38) and the last one is according to (7) and (19).

Note that

\[
y_k = \mathcal{P}_Y (v_{k,T_k} + \tau_k \nabla_y F_k(u_{k,T_k}, v_{k,T_k})) = \mathcal{P}_Y (v_{k,T_k} + \tau_k \nabla_y f(u_{k,T_k}, v_{k,T_k})).
\]

Then, Lemma [1] and [12] implies

\[
\mathbb{E} \left[ f(x^*, y^*) - \min_{x \in X} g(x, y_k) \right] \\
\leq \left( \sqrt{2} + \frac{L}{4 \sqrt{n} (L + \beta)} \right) + 2 \left( 1 + \frac{L}{4 \sqrt{n} (L + \beta)} \right)^2 \kappa_x L \varepsilon + 2 \sqrt{n} (L + \beta) \varepsilon \\
\leq 7 \kappa_x L \varepsilon + 2 \sqrt{n} (L + \beta) \varepsilon \leq (7 \kappa_x L + 2 \sqrt{n} (L + \beta)) \left( 4 \kappa_y^2 + 5 \right) \varepsilon_k + \frac{8 (\kappa_y^2 + 1)}{\mu_x} \delta_k \\
\leq (7 \kappa_x L + 2 \sqrt{n} (L + \beta)) \left( \frac{2 \Delta_f (4 \kappa_y^2 + 5) \mu_y}{3 (L + \beta)(7 (L + \beta) + 2 \sqrt{n} \mu_y)} + \frac{64 \Delta_f (\kappa_y^2 + 1)(1 - \rho)}{\mu_x (\sqrt{q} - \rho)^2} \right)^k \\
\leq \left( \frac{2 (4 \kappa_y^2 + 5) (\mu_x + \mu_y)}{3 \mu_x} + \frac{128 \kappa_y^2 (7 \kappa_x L + 2 \sqrt{n} (L + \beta))}{\mu_x (\sqrt{q} - \rho)^2} \right) \Delta_f (1 - \rho)^k
\]

Together with inequality (35), we obtain that

\[
\mathbb{E} \left[ \max_{y \in Y} f(x_k, y) - \min_{x \in X} f(x, y_k) \right] \\
\leq \Delta_f (1 - \rho)^k \left( \frac{2 (4 \kappa_y^2 + 5) (\mu_x + \mu_y)}{3 \mu_x} + \frac{128 \kappa_y^2 (7 \kappa_x L + 2 \sqrt{n} (L + \beta))}{\mu_x (\sqrt{q} - \rho)^2} \right) + \frac{8}{(\sqrt{q} - \rho)^2} \\
\leq \Delta_f (1 - \rho)^k \left( 12 \kappa_y^2 \kappa_x + \frac{128 \kappa_y^2 (7 \kappa_x L + 2 \sqrt{n} (L + \beta))}{\mu_x (\sqrt{q} - \rho)^2} \right) + \frac{8}{(\sqrt{q} - \rho)^2} \\
\leq \frac{916 \Delta_f (\kappa_x L + \sqrt{n} (L + \beta)) \kappa_y^2}{\mu_x (\sqrt{q} - \rho)^2} (1 - \rho)^k.
\]

\[\square\]

C.5 The Proof of Corollary [1]

Proof. First, note that \( \beta = \min \{ \mu_y - \mu_x, \max \{ L/\sqrt{n} - \mu_x, 0 \} \} \leq \mu_y - \mu_x \leq L \) and

\[
\frac{1}{q} = \frac{\mu_x + \beta}{\mu_x} \leq \frac{\mu_y}{\mu_x}.
\]
Together with Theorem 3, we have

\[
\mathbb{E} \left[ \max_{y \in \mathcal{Y}} f(x, y) - \min_{x \in \mathcal{X}} f(x, y_K) \right] \leq e^{-\rho K} \frac{916\Delta_f (\kappa_x L + \sqrt{n}(L + \beta)) \kappa_y^2}{\mu_x (\sqrt{q} - \rho)^2}
\]

\[
\leq e^{-\rho K} \left( \frac{2748 \sqrt{n} \Delta_f \kappa_y^2 \kappa_x^2}{(\sqrt{q} - \rho)^2} \right) \leq e^{-\rho K} \left( 10992 \sqrt{n} \Delta_f \kappa_y \kappa_x^3 \right) \leq \epsilon,
\]

where we use that \( \rho = 0.5\sqrt{q} \).

Recall that

\[
T_k = 4 \left( n + \frac{2\sqrt{n}(L + \beta)}{\min\{\mu_x + \beta, \mu_y\}} \right) \log \left( 12 \left( \frac{2}{1 - \rho} + \frac{1728\beta(L + \beta)(7(L + \beta) + 2\sqrt{n}\mu_y)}{\mu_x \mu_y \min\{\mu_x, \mu_y\}(1 - \rho)^2(\sqrt{q} - \rho)^2} \right) \right).
\]

\[
\leq 4 \left( n + \frac{4\sqrt{n}L}{\min\{\mu_x + \beta, \mu_y\}} \right) \log \left( 12 \left( 4 + \frac{884736\sqrt{n}L^3}{\mu_x^2 \mu_y q} \right) \right).
\]

\[
\leq 4 \left( n + \frac{4\sqrt{n}L}{\min\{\mu_x + \beta, \mu_y\}} \right) \log \left( 10616880 \sqrt{n} \kappa_x^3 \right),
\]

where we have noticed that \( \rho \leq 0.5 \).

Therefore, the total SFO complexity of AL-SVRE is

\[
\sum_{k=1}^{K} (T_k + n) = \mathcal{O} \left( \left( \frac{2n}{\sqrt{q}} + \frac{2\sqrt{n}}{\sqrt{q}} \left( n + \frac{4\sqrt{n}L}{\min\{\mu_x + \beta, \mu_y\}} \right) \log(\sqrt{n} \kappa_x^3) \right) \log(\sqrt{n} \Delta_f \kappa_y \kappa_x^3/\varepsilon) \right).
\]

Observe that

1. if \( \kappa_y \geq \sqrt{n} \), we have \( \mu_y \leq L/\sqrt{n} \), \( \beta = \mu_y - \mu_x \), which means

\[
\frac{n}{\sqrt{q}} = n \sqrt{\frac{\mu_y}{\mu_x}} \leq n^{3/4} \sqrt{\kappa_x}
\]

and

\[
\frac{1}{\sqrt{q}} \frac{\sqrt{n}L}{\min\{\mu_x + \beta, \mu_y\}} = \frac{\sqrt{n}L}{\mu_y} \sqrt{\frac{\mu_y}{\mu_x}} = \sqrt{n} \sqrt{\kappa_x} \kappa_y;
\]

2. if \( \kappa_x > \sqrt{n} > \kappa_y \), we have \( \beta = L/\sqrt{n} - \mu_x \), \( \mu_x + \beta = L/\sqrt{n} < \mu_y \), which means

\[
\frac{n}{\sqrt{q}} = n \sqrt{\frac{L}{\sqrt{n} \mu_x}} = n^{3/4} \sqrt{\kappa_x}
\]

and

\[
\frac{1}{\sqrt{q}} \frac{\sqrt{n}L}{\min\{\mu_x + \beta, \mu_y\}} = \frac{\sqrt{n}L}{L/\sqrt{n}} \sqrt{\frac{L}{\sqrt{n} \mu_x}} = n^{3/4} \sqrt{\kappa_x};
\]

3. if \( \sqrt{n} \geq \kappa_x \), we have \( \beta = 0 \), which means

\[
\frac{n}{\sqrt{q}} = n \quad \text{and} \quad \frac{1}{\sqrt{q}} \frac{\sqrt{n}L}{\min\{\mu_x + \beta, \mu_y\}} = \sqrt{n} \kappa_x \leq n.
\]

Hence, we can conclude that the total SFO complexity of AL-SVRE is

\[
\sum_{k=1}^{K} (T_k + n) = \mathcal{O} \left( \sqrt{n} \sqrt{(\sqrt{n} + \kappa_x)(\sqrt{n} + \kappa_y)} \log(\sqrt{n} \kappa_x^3) \log(\sqrt{n} \Delta_f \kappa_y \kappa_x^3/\varepsilon) \right),
\]

\( \square \)
D The Proof of Theorem 1

The construction for the lower bound in Theorem 1 follows from the idea of “zero-chain” property \cite{13, 42}. Since we focus on SFO algorithm without proximal operator, our analysis is simpler than Han et al. \cite{13} and Xie et al. \cite{37}’s.

Without loss of generality, we assume that the SFO algorithm starts iteration at \((x^{(0)}, y^{(0)}) = (0_d, 0_d)\). Otherwise, we can take the objective function
\[
\hat{f}(x, y) = \frac{1}{n} \sum_{i=1}^{n} f_i(x + x^{(0)}, y + y^{(0)})
\]
into consideration.

Consider following function \(H: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}\) defined as
\[
H(x, y; \alpha, d) = \frac{\alpha}{2} \|x\|^2 + x^\top (By - c) - \frac{\alpha}{2} \|y\|^2,
\] (39)
where
\[
B = \begin{bmatrix}
1 & -1 & 1 & \ldots & 1 \\
-1 & 1 & -1 & \ldots & -1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-1 & -1 & 1 & \ldots & -1 \\
\end{bmatrix} \in \mathbb{R}^{d \times d},
\]
c = \((\omega, 0, 0, \ldots, 0)\) and \(\omega = \sqrt{\alpha^2 + 1 - \alpha}\).

Furthermore, we define subspaces as follows
\[
\mathcal{F}_k = \begin{cases}
\text{span}\{e_1, e_2, \ldots, e_k\}, & k = 1, \ldots, d, \\
\{0_d\}, & k = 0,
\end{cases}
\]
where \(\{e_1, \ldots, e_d\}\) is the standard basis of \(\mathbb{R}^d\).

Here, we state some properties of the Function \(H\) in the above definition.

**Lemma 13.** For the function \(H\) defined in Equation (39), following properties hold.

1. \(H\) is \(\sqrt{8 + 2\alpha^2}\)-smooth.

2. The saddle point of function \(H\) is
\[
\begin{align*}
x^* &= (q, q^2, \ldots, q^d)^	op, \\
y^* &= \omega \left( q, q^2, \ldots, q^{d-1}, \frac{1}{\sqrt{1-q}} q^d \right)^	op,
\end{align*}
\]
where \(q = \frac{2 + \alpha^2 - \alpha \sqrt{\alpha^2 + 1}}{2} \).

3. For \(k < d\), if \((x, y) \in \mathcal{F}_k \times \mathcal{F}_k\), then \((\nabla_x H(x, y), \nabla_y H(x, y)) \in \mathcal{F}_{k+1} \times \mathcal{F}_{k+1}\).

4. For \(k \leq d/2\) and \((x, y) \in \mathcal{F}_k \times \mathcal{F}_k\), we have
\[
\frac{\|x - x^*\|^2_2 + \|y - y^*\|^2_2}{\|x^*\|^2_2 + \|y^*\|^2_2} \geq \frac{1}{2} q^{2k}.
\]

**Proof.** 1. Note that \(\alpha \omega = \frac{2 \alpha}{2 \omega + 4 + \alpha} < 1\). Hence \(\|B\|_2 \leq \|B\|_1 = 2\).

Then for any \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d\), we have
\[
\frac{\|\nabla_x H(x_1, y_1) - \nabla_x H(x_2, y_2)\|_2^2}{\|x^*\|^2_2} = \frac{\|\nabla_y H(x_1, y_1) - \nabla_y H(x_2, y_2)\|_2^2}{\|y^*\|^2_2}.
\]
Therefore we have

\[ \leq 2\alpha^2 \|x_1 - x_2\|_2^2 + 2 \|B\|_2^2 \|y_1 - y_2\|_2^2 + 2 \|B\|_2^2 \|x_1 - x_2\|_2^2 + 2\alpha^2 \|y_1 - y_2\|_2^2 \]

\[ \leq (8 + 2\alpha^2)\left(\|x_1 - x_2\|_2^2 + \|y_1 - y_2\|_2^2\right), \]

where the first inequality is according to \(\|a + b\|_2^2 \leq 2\|a\|_2^2 + 2\|b\|_2^2\).

2. Letting the gradient of \(H\) equal to zero, we get that

\[ \nabla_x H(x, y) = \alpha x + By - c = 0, \]
\[ \nabla_y H(x, y) = B^\top x - \alpha y = 0. \]

Hence, the saddle point of \(H\) satisfies

\[ y^* = \frac{1}{\alpha} B^\top x^*, \quad \text{(40)} \]
\[ (\alpha^2 I + BB^\top) x^* = \alpha c. \quad \text{(41)} \]

Equation (41) are equivalent to

\[
\begin{bmatrix}
1 + \alpha^2 & -1 \\
-1 & 2 + \alpha^2 & -1 \\
& \ddots & \ddots & \ddots \\
& & -1 & 2 + \alpha^2 & -1 \\
& & & -1 & 1 + \alpha \omega + \alpha^2
\end{bmatrix}
\begin{bmatrix}
\alpha \omega \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix},
\]

Note that \(q\) is a root of the equation \(s^2 - (2 + \alpha^2)s + 1 = 0\), then we have

\[ (1 + \alpha^2)q - q^2 = 1 - q = \frac{\alpha \sqrt{\alpha^2 + 4} - \alpha^2}{2} = \alpha \omega, \]
\[ -q^{d-1} + (1 + \alpha \omega + \alpha^2)q^d = q^{d-1} (-1 + (2 + \alpha^2)q - \alpha \omega q) \]
\[ = q^{d-1} (-1 + (2 + \alpha^2)q - (1 - q)q) = 0, \]

and

\[ -q^k + (2 + \alpha^2)q^{k+1} - q^{k+2} = 0, \quad \text{for } k = 1, 2, \ldots, d - 2, \]

which implies the solution to Equation (41) is \(x^* = (q, q^2, \ldots, q^d)^\top\). Additionally, we have

\[ y^* = \frac{1}{\alpha} B^\top x^* = \omega \left( q, q^2, \ldots, q^{d-1}, \frac{1}{\sqrt{1 - q}} q^d \right)^\top, \]

where we have used that \(\alpha \omega = 1 - q\).

3. For \((x, y) \in F_k\) for \(k < d\), note that

\[ c \in F_1 \subseteq F_{k+1}, \quad B y \in F_{k+1}, \quad B^\top x \in F_{k+1}. \]

Therefore we have \(\nabla_x H(x, y), \nabla_y H(x, y) \in F_{k+1}\).

4. For \((x, y) \in F_k\), we have \(x_{k+1} = \cdots = x_d = y_{k+1} = \cdots = y_d = 0\) and

\[ \|x - x^*\|_2^2 + \|y - y^*\|_2^2 \geq \sum_{i=k+1}^d q^{2i} + \omega^2 \sum_{i=k+1}^{d-1} q^{2i} + \frac{\omega^2}{1 - q} q^{2d} \]
\[ = (1 + \omega^2) \frac{q^{2(k+1)}(1 - q^{2(d-k)})}{1 - q^2} + \frac{\omega^2 q^{2d}}{1 - q^2}. \]

Consequently, there holds

\[ \frac{\|x - x^*\|_2^2 + \|y - y^*\|_2^2}{\|x^*\|_2^2 + \|y^*\|_2^2} \geq \frac{(1 + \omega^2) \frac{q^{2(k+1)}(1 - q^{2(d-k)})}{1 - q^2} + \omega^2 q^{2d}}{(1 + \omega^2) \frac{q^{2(k+1)}(1 - q^{2(d-k)})}{1 - q^2} + \omega^2 q^{2d}}. \]
where we have used that $1 + q^{2d} \geq 2q^d$ and $2(d - k) \geq d$ according to $k \leq d/2$.

With these pieces in hand, we now construct our adversary problem as follows:

$$
\min_{x} \max_{y} f(x, y; \alpha, \lambda, d) = \frac{1}{n} \sum_{i=1}^{n} f_i(x, y; \alpha, \lambda, d) = \frac{1}{n} \sum_{i=1}^{n} \lambda H(U_i x, U_i y),
$$

where $f : \mathbb{R}^{nd} \times \mathbb{R}^{nd} \to \mathbb{R}$, and matrices $U_1, \ldots, U_n \in \mathbb{R}^{d \times nd}$ consist a partition of the identity matrix of order $nd$ such that $I = [U_1^T, \ldots, U_n^T]$. The trick of constructing worse objective function for SFO algorithms by matrices $U_1^\top, \ldots, U_n^\top$ also can be found in the analysis for minimization problem \cite{19, 43}.

The smoothness, convexity and concavity of $f$ can be characterized as follows.

**Lemma 14.** The class of functions $\{f_i\}$ defined in equation (42) is $\sqrt{\frac{4 + 2\alpha^2}{n}}$-average-smooth; and the function $f$ is $(\frac{\alpha}{n}, \frac{\alpha}{n})$-convex-concave.

**Proof.** Using the fact $\sum_{i=1}^{n} \|U_i x\|^2 = \|x\|^2$, we have

$$
f(x, y) = \frac{\alpha}{n} \|x\|^2 - \frac{\lambda}{n} x^\top \left( \sum_{i=1}^{n} U_i^\top c \right) + \frac{\lambda}{n} x^\top \left( \sum_{i=1}^{n} U_i^\top B U_i \right) y - \frac{\lambda \alpha}{n} \|y\|^2.
$$

It is clear that $f$ is $(\frac{\alpha}{n}, \frac{\alpha}{n})$-convex-concave.

Moreover, for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^{nd} \times \mathbb{R}^{nd}$, it holds that

$$
\frac{1}{n} \sum_{i=1}^{n} \left( \|\nabla_x f_i(x_1, y_1) - \nabla_x f_i(x_2, y_2)\|^2 + \|\nabla_y f_i(x_1, y_1) - \nabla_y f_i(x_2, y_2)\|^2 \right)
= \frac{\lambda^2}{n} \sum_{i=1}^{n} \left( \|U_i^\top (\nabla_x H(U_i x_1, U_i y_1) - \nabla_x H(U_i x_2, U_i y_2))\|^2 
+ \|U_i^\top (\nabla_y H(U_i x_1, U_i y_1) - \nabla_y H(U_i x_2, U_i y_2))\|^2 \right)
= \frac{\lambda^2}{n} \sum_{i=1}^{n} \left( \|\nabla_x H(U_i x_1, U_i y_1) - \nabla_x H(U_i x_2, U_i y_2)\|^2 + \|\nabla_y H(U_i x_1, U_i y_1) - \nabla_y H(U_i x_2, U_i y_2)\|^2 \right)
\leq \frac{\lambda^2(8 + 2\alpha^2)}{n} \sum_{i=1}^{n} \left( \|x_1 - x_2\|^2 + \|y_1 - y_2\|^2 \right)
= \frac{\lambda^2(8 + 2\alpha^2)}{n} \left( \|x_1 - x_2\|^2 + \|y_1 - y_2\|^2 \right),
$$

where the first inequality follows from smoothness of $H$ (cf. Property 1 in Lemma 13).

Each component function $f_i$ has “zero-chain” property for stochastic first-order oracle. That is, the information provided by an SFO call at the current point $(x, y)$ can at most increase the dimension of the linear space which contains $(x, y)$ by 1. We present the formal statement in Lemma 15.

**Lemma 15.** Let $(x^{(t)}, y^{(t)})$ be the point obtained by an SFO algorithm $A$ at time-step $t$ and denote $k_i^{(t)} \triangleq |\{s \leq t : i_s = i\}|$. Then there holds

$$
U_i x^{(t)}, U_i y^{(t)} \in F_{k_i^{(t)}}, \quad \text{for } t \geq 0, i = 1, \ldots, n.
$$

(43)
Proof. For \( t = 0 \), Equation (43) holds apparently by \( \mathbf{x}(0) = \mathbf{y}(0) = \mathbf{0} \).

Now we suppose Equation (43) holds for \( t < T \). It is easily to check that \( k_i(T) = k_i(T-1) \) for \( i \neq i_T \) and \( k_{i_T}(T) = k_{i_T}(T-1) + 1 \).

Observe that for \( i \neq i_T \) we have

\[
\mathbf{U}_i \nabla_{\mathbf{x}} f_{i_T} (\mathbf{x}, \mathbf{y}) = \lambda \mathbf{U}_i \mathbf{U}_{i_T} \nabla_{\mathbf{x}} H (\mathbf{U}_{i_T} \mathbf{x}, \mathbf{U}_{i_T} \mathbf{y}) = \mathbf{0},
\]

which implies

\[
\mathbf{U}_i \mathbf{x}(T) \in \text{span} \{ \mathbf{U}_i \mathbf{x}(0), \ldots, \mathbf{U}_i \mathbf{x}(T-1), \mathbf{U}_i \nabla_{\mathbf{x}} f_{i_T} (\mathbf{x}(0), \mathbf{y}(0)), \ldots, \mathbf{U}_i \nabla_{\mathbf{x}} f_{i_T} (\mathbf{x}(T-1), \mathbf{y}(T-1)) \}
\]

\[
= \text{span} \{ \mathbf{U}_i \mathbf{x}(0), \ldots, \mathbf{U}_i \mathbf{x}(T-1) \} \subseteq \mathcal{F}_{k_{i_T}^{(T-1)}} = \mathcal{F}_{k_{i_T}^{(T)}}.
\]

Next, Following from Property 3 in Lemma 13, we know that for \( t \leq T - 1 \)

\[
\mathbf{U}_{i_T} \nabla_{\mathbf{x}} f_{i_T} (\mathbf{x}^{(t)}, \mathbf{y}^{(t)}) = \lambda \mathbf{U}_{i_T} \mathbf{U}_{i_T} \nabla_{\mathbf{x}} H (\mathbf{U}_{i_T} \mathbf{x}^{(t)}, \mathbf{U}_{i_T} \mathbf{y}^{(t)})
\]

\[
= \lambda \nabla_{\mathbf{x}} H (\mathbf{U}_{i_T} \mathbf{x}^{(t)}, \mathbf{U}_{i_T} \mathbf{y}^{(t)}) \in \mathcal{F}_{k_{i_T}^{(T-1)}} = \mathcal{F}_{k_{i_T}^{(T)}},
\]

where we have used the inductive hypothesis that

\[
\mathbf{U}_{i_T} \mathbf{x}^{(t)}, \mathbf{U}_{i_T} \mathbf{y}^{(t)} \in \mathcal{F}_{k_{i_T}^{(T)}} \subseteq \mathcal{F}_{k_{i_T}^{(T-1)}}.
\]

Therefore, we can conclude that

\[
\mathbf{U}_{i_T} \mathbf{x}^{(T)} \in \text{span} \{ \mathbf{U}_{i_T} \mathbf{x}(0), \ldots, \mathbf{U}_{i_T} \mathbf{x}(T-1), \mathbf{U}_{i_T} \nabla_{\mathbf{x}} f_{i_T} (\mathbf{x}(0), \mathbf{y}(0)), \ldots, \mathbf{U}_{i_T} \nabla_{\mathbf{x}} f_{i_T} (\mathbf{x}(T-1), \mathbf{y}(T-1)) \}
\]

\[
\subseteq \mathcal{F}_{k_{i_T}^{(T)}}.
\]

The reason for the of result \( \mathbf{U}_i \mathbf{x}(T) \in \mathcal{F}_{k_{i_T}^{(T)}} \) is same. \( \square \)

Now we can show the lower bound for finding an approximate saddle point of problem (42) by SFO algorithms when \( L/\mu = \Omega(\sqrt{n}) \).

**Theorem 4.** For the parameter \( L, \mu, n, \varepsilon \) such that \( L/\mu > \sqrt{10n} \) and \( \varepsilon < \frac{1}{2} \varepsilon^{-1} \approx 0.00337 \), we set

\[
\alpha = \sqrt{\frac{8n}{L^2/\mu^2 - 2n}}, \quad \lambda = \frac{n\mu}{\alpha}, \quad d = \left\lfloor \frac{1}{\alpha \ln \left( \frac{1}{2\varepsilon} \right)} \right\rfloor - 4.
\]

Then the functions \( f(x, y; \alpha, \lambda, d) \) and \( f_i(x, y; \alpha, \lambda, d) \) defined in Problem (43) satisfy \( f \) is \((\mu, \mu)\)-convex-concave and \( \{f_i\}_{i=1}^n \) is \( L\)-average-smooth. Moreover, when we employ any SFO algorithm \( A \) to solve the Problem (42), there holds

\[
\mathbb{E} \left[ \|x^{(t)} - x^*\|^2 + \|y^{(t)} - y^*\|^2 \right] > \varepsilon \quad \text{for } t \leq nd/2.
\]

**Proof.** By Lemma 14 and definition of \( \alpha, \lambda, \) it is clear that \( f \) is \((\mu, \mu)\)-convex-concave and \( \{f_i(x, y; \alpha, \lambda, d)\}_{i=1}^n \) is \( L\)-average-smooth.

For \( T = nd/2 \), let \( i = \arg \min_j \{k_j^{(T)}\} \). It is clear that \( k_i^{(T)} \leq d/2 \). Then by Property 4 in Lemma 13 for \( t \leq T \), we have

\[
\mathbb{E} \left[ \|x^{(t)} - x^*\|^2 + \|y^{(t)} - y^*\|^2 \right] \geq \mathbb{E} \left[ \|U_i(x^{(t)} - x^*)\|^2 + \|U_i(y^{(t)} - y^*)\|^2 \right]
\]

\[
\geq \frac{\varepsilon^{k_{i_T}^{(T)}}}{2} \left( \|U_i x^*\|^2 + \|U_i y^*\|^2 \right)
\]

28
where \( q = \frac{2 + \alpha^2 - \alpha \sqrt{\alpha^2 + 1}}{2}, \omega = \frac{\sqrt{\alpha^2 + 1} - \alpha}{2}, \) and \( U_i x^* = (q, q^2, \ldots, q^d)^T \) by Property 2 in Lemma 13.

Next, note that

\[
(2 + d/2) \ln(1/q) = (2 + d/2) \ln \left( 1 + \frac{\alpha(\alpha + \sqrt{\alpha^2 + 4})}{2} \right) \\
\leq (2 + d/2) \frac{\alpha(\alpha + \sqrt{\alpha^2 + 4})}{2} \\
\leq (4 + d) \alpha \\
\leq \ln(1/\varepsilon)
\]

which means that \( \frac{d/2 + 2}{\alpha} \geq \varepsilon \). The first inequality is according to \( \ln(1 + a) \leq a \), the second inequality follows from \( \sqrt{\alpha + \beta} < \sqrt{\alpha} + \sqrt{\beta} \), the third inequality is due to \( L/\mu \geq \sqrt{10n} \) and \( \alpha = \sqrt{\frac{8n}{L^2/\mu^2 - 2n}} \leq 1 \), and the last inequality is based on the definition of \( d \).

We remark that the condition \( \varepsilon < \frac{1}{2} \varepsilon^5 \) can ensure that \( d \geq 1 \).

Furthermore, Theorem 4 implies that for any SFO algorithm \( \mathcal{A} \) and \( L, \mu, n, \varepsilon \) such that \( L/\mu \geq \sqrt{10n} \) and \( \varepsilon < 0.003 \), there exist a dimension \( d = \mathcal{O}(\sqrt{nL/\mu \log(1/\varepsilon)}) \) and functions \( \{f_i(x, y)\}_{i=1}^{n} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) which satisfy \( \{f_i\}_{i=1}^{n} \) is \( L \)-average smooth, \( f \) is \((\mu, \mu)\)-convex-concave. In order to find an approximate saddle point \((\hat{x}, \hat{y})\) such that

\[
\mathbb{E} \left[ \|\hat{x} - x^*\|_2^2 + \|\hat{y} - y^*\|_2^2 \right] \leq \varepsilon,
\]

algorithm \( \mathcal{A} \) needs at least \( \Omega((\sqrt{nL}/\mu \log(1/\varepsilon)) \) steps.

For the case \( L/\mu = \mathcal{O}(\sqrt{n}) \), we consider following problem

\[
\min_{x} \max_{y} \hat{f}(x, y) = \frac{1}{n} \sum_{i=1}^{n} \hat{f}_i(x, y) \\
= \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\mu}{2} \|x\|_2^2 + \sqrt{nL} (x_i - 1)^2 - \frac{\mu}{2} \|y\|_2^2 - \sqrt{nL} (y_i - 1)^2 \right),
\]

where \( \hat{f} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \), \( \hat{L} = \sqrt{\frac{L^2}{2} - \mu^2} \).

And we provide the lower bound for finding approximate saddle point of Problem (44) as follows.

**Theorem 5.** For the parameter \( L, \mu, n, \varepsilon \) such that \( L/\mu > 2 \) and \( \varepsilon < \frac{1}{8} \), the functions \( f(x, y) \) and \( \hat{f}_i(x, y) \) defined in Problem (44) satisfy \( \hat{f} \) is \((\mu, \mu)\)-convex-concave and \( \{\hat{f}_i\}_{i=1}^{n} \) is \( L \)-average-smooth. Moreover, when we employ any SFO algorithm \( \mathcal{A} \) to solve the Problem (44), there holds

\[
\mathbb{E} \left[ \|x^{(t)} - x^*\|_2^2 + \|y^{(t)} - y^*\|_2^2 \right] > \varepsilon \quad \text{for } t \leq n/2.
\]

**Proof.** It is easily to check that

\[
\hat{f}(x, y) = \frac{\mu}{2} \|x\|_2^2 + \frac{\hat{L}}{2\sqrt{n}} \|x - 1\|_2^2 - \frac{\mu}{2} \|y\|_2^2 - \frac{\hat{L}}{2\sqrt{n}} \|y - 1\|_2^2.
\]
Hence \( \hat{f} \) is \((\mu, \mu)\)-convex-concave and the saddle point \((x^*, y^*)\) of \( \hat{f} \) satisfies

\[
x^* = y^* = \frac{\hat{L}}{L + \sqrt{n}\mu} - 1.
\]

Next, observe that

\[
\nabla_x \hat{f}(x, y) = \mu x + \sqrt{n}\hat{L}(e_i^T x - 1)e_i,
\]

\[
\nabla_y \hat{f}(x, y) = -\mu y - \sqrt{n}\hat{L}(e_i^T y - 1)e_i.
\]

Then we have

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \left\| \nabla_x \hat{f}_i(x_1, y_1) - \nabla_x \hat{f}_i(x_1, y_1) \right\|^2_2 + \left\| \nabla_y \hat{f}_i(x_1, y_1) - \nabla_y \hat{f}_i(x_1, y_1) \right\|^2_2 \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \left\| \mu(x_1 - x_2) + \sqrt{n}\hat{L}e_i^T (x_1 - x_2) \right\|^2_2 + \left\| \mu(y_1 - y_2) + \sqrt{n}\hat{L}e_i^T (y_1 - y_2) \right\|^2_2 \right)
\]

\[
\leq \frac{1}{n} \sum_{i=1}^{n} \left( 2\mu^2 \left\| x_1 - x_2 \right\|^2_2 + 2n\hat{L}^2 \left\| e_i e_i^T (x_1 - x_2) \right\|^2_2 + 2\mu^2 \left\| y_1 - y_2 \right\|^2_2 + 2n\hat{L}^2 \left\| e_i e_i^T (y_1 - y_2) \right\|^2_2 \right)
\]

\[
= 2(\hat{L}^2 + \mu^2) \left( \left\| x_1 - x_2 \right\|^2_2 + \left\| y_1 - y_2 \right\|^2_2 \right) = L^2 \left( \left\| x_1 - x_2 \right\|^2_2 + \left\| y_1 - y_2 \right\|^2_2 \right),
\]

where the first inequality is according to \( \left\| a + b \right\|^2_2 \leq 2 \left\| a \right\|^2_2 + 2 \left\| b \right\|^2_2 \) and the last equation follows from

\[
\sum_{i=1}^{n} \left\| e_ie_i^T u \right\|^2_2 = \sum_{i=1}^{n} u_i^2 = \left\| u \right\|^2_2.
\]

Hence, \( \{\hat{f}_i\}_{i=1}^{n} \) is \( L \)-average-smooth.

Moreover, by Equation (45) and definition of SFO algorithm, we know that

\[
x^{(t)}, y^{(t)} \in \text{span} \{e_1, \ldots, e_n\}.
\]

Consequently, for \( t \leq n/2 \), there holds

\[
\mathbb{E} \left[ \left\| x^{(t)} - x^* \right\|^2_2 + \left\| y^{(t)} - y^* \right\|^2_2 \right]
\]

\[
\geq (n-t) \cdot \frac{\hat{L}^2}{(L + \sqrt{n}\mu)^2} \geq \frac{n}{2} \cdot \frac{\hat{L}^2}{2L^2 + 2n\mu^2}
\]

\[
= \frac{n}{2} \cdot \frac{L^2/2 - \mu^2 + 2n\mu^2}{L^2/2 - \mu^2} = \frac{n}{4} \left( 1 - \frac{2n}{L^2/2 - \mu^2} \right)
\]

\[
\geq \frac{n}{4(n+1)} \geq \frac{1}{8} > \varepsilon,
\]

where we have recalled that \( \hat{L}^2 = L^2/2 - \mu^2, L/\mu > 2 \) and \( n \geq 1 \).

Combining Theorem 4 and 5, we obtain the result of Theorem 1.

### E The Detailed Proof for Extension Case

In this section, we provide the detailed proof for the results in Section 4, including the convex-strongly-concave case and the convex-concave cases.
E.1 The Proof of Lemma 4

**Proof.** Just note that

$$\max_{(x,y) \in X \times Y} |f(x,y) - f_{\epsilon, x_0}(x, y)| = \frac{\epsilon}{4D_x^2} \max_{x \in X} \|x - x_0\|_2^2 \leq \frac{\epsilon}{4}.$$  

Therefore, we can conclude that

$$\max_{y \in Y} f(\hat{x}, y) - \min_{x \in X} f(x, \hat{y}) = f(\hat{x}, \hat{y}) - f(\hat{x}, \hat{y}) \leq \left( f_{\epsilon, x_0}(\hat{x}, y^*_j(\hat{x})) + \frac{\epsilon}{4} \right) - \left( f_{\epsilon, x_0}(x^*_j(\hat{y}), \hat{y}) - \frac{\epsilon}{4} \right) \leq \frac{\epsilon}{2} + \max_{y \in Y} \epsilon \max_{x \in X} f_{\epsilon, x_0}(\hat{x}, y) - \min_{x \in X} f_{\epsilon, x_0}(x, \hat{y}),$$

where $y^*_j(\hat{x}) = \arg \max_{y \in Y} f(\hat{x}, y)$ and $x^*_j(\hat{y}) = \arg \min_{x \in X} f(x, \hat{y})$. □

E.2 The Proof of Corollary 2

**Proof.** By the definition of $f_{\epsilon, x_0}$, we know that $f_{\epsilon, x_0}$ is $(\frac{\epsilon}{4D_x^2}, \mu_y)$-convex-concave. The smoothness of $f_{\epsilon, x_0}$ can be verified by

$$\frac{1}{n} \sum_{i=1}^{n} \|\nabla f_{\epsilon, x_0, i}(x, y) - \nabla f_{\epsilon, x_0, i}(x', y')\|_2^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left\| \nabla f_i(x, y) - \nabla f_i(x', y') + \frac{\epsilon}{4D_x^2}(x - x') \right\|_2^2$$

$$\leq \frac{2}{n} \sum_{i=1}^{n} \|\nabla f_i(x, y) - \nabla f_i(x', y')\|_2^2 + \frac{\epsilon^2}{8D_x^2} \|x - x'\|_2^2$$

$$\leq 2L^2 \left( \|x - x'\|_2^2 + \|y - y'\|_2^2 \right) + \frac{\epsilon^2}{8D_x^2} \|x - x'\|_2^2$$

$$\leq 4L^2 \left( \|x - x'\|_2^2 + \|y - y'\|_2^2 \right),$$

where the first inequality is according to $(a + b)^2 \leq 2a^2 + 2b^2$ and the second inequality follows from the smoothness of $\{f_i\}_{i=1}^{n}$ and $\epsilon \leq 4LD_x^2$. Hence, $\{f_{\epsilon, x_0, i}\}_{i=1}^{n}$ is $2L$-average smooth.

Then by Corollary 1, the number of SFO calls of AL-SVRE for finding $\epsilon/2$-saddle point of $f_{\epsilon, x_0}$ corresponds to finding an $\epsilon$-saddle point of $f$, that is

$$\tilde{O} \left( \sqrt{n \left( \sqrt{n + \frac{2L}{4D_x^2}} \right)^{\frac{3}{2}}} \right) = \tilde{O} \left( \left( n + D_x \sqrt{\frac{n L \kappa_y}{\epsilon} + n^{3/4} \sqrt{\kappa_y} + n^{3/4} D_x \sqrt{\frac{L}{\epsilon}}} \right) \right).$$  

□

E.3 The Proof of Lemma 5

**Proof.** Note that

$$\max_{(x,y) \in X \times Y} |f(x,y) - f_{\epsilon, x_0, y_0}(x, y)| = \max_{(x,y) \in X \times Y} \left\| \frac{\epsilon}{8D_x^2} \|x - x_0\|_2^2 - \frac{\epsilon}{8D_y^2} \|y - y_0\|_2^2 \right\| \leq \frac{\epsilon}{4}.$$

For the same reason we got the result in the proof of Lemma 4 we also have

$$\max_{y \in Y} f(\hat{x}, y) - \min_{x \in X} f(x, \hat{y}) = f(\hat{x}, y^*_j(\hat{x})) - f(x^*_j(\hat{y}), \hat{y})$$
\[
\leq \left( f_{\varepsilon,x_0}(\hat{x},y^*_f(\hat{x})) + \frac{\varepsilon}{4} \right) - \left( f_{\varepsilon,x_0}(\hat{x}^*_f(\hat{y}),\hat{y}) - \frac{\varepsilon}{4} \right)
\]

\[
\leq \frac{\varepsilon}{2} + \max_{y \in Y} f_{\varepsilon,x_0}(\hat{x},y) - \min_{x \in X} f_{\varepsilon,x_0}(x,\hat{y}),
\]

where \( y^*_f(\hat{x}) = \arg\max_{y \in Y} f(\hat{x},y) \) and \( x^*_f(\hat{y}) = \arg\min_{x \in X} f(x,\hat{y}) \).

**E.4 The Proof of Corollary 3**

**Proof.** By the definition of \( f_{\varepsilon,x_0,y_0} \), we know that \( f_{\varepsilon,x_0,y_0} \) is \( \left( \frac{\varepsilon}{8D^2_x}, \frac{\varepsilon}{8D^2_y} \right) \)-convex-concave. The smoothness of \( f_{\varepsilon,x_0} \) can be verified by

\[
\frac{1}{n} \sum_{i=1}^{n} \| \nabla f_{\varepsilon,x_0,y_0,i}(x,y) - \nabla f_{\varepsilon,x_0,y_0,i}(x',y') \|^2
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left\| \nabla f_i(x,y) - \nabla f_i(x',y') + \frac{\varepsilon}{8D^2_x} (x-x') - \frac{\varepsilon}{8D^2_y} (y-y') \right\|^2
\]

\[
\leq \frac{3}{n} \sum_{i=1}^{n} \| \nabla f_i(x,y) - \nabla f_i(x',y') \|^2 + \frac{3\varepsilon^2}{64D^2_x} \| (x-x') \|^2 + \frac{3\varepsilon^2}{64D^2_y} \| (y-y') \|^2
\]

\[
\leq 4L^2 \left( \| x-x' \|^2 + \| y-y' \|^2 \right),
\]

where the first inequality is according to \( (a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2 \) and the second inequality follows from the smoothness of \( \{f_i\}_{i=1}^{n} \) and \( \varepsilon \leq 4L \min\{D^2_x, D^2_y\} \). Hence, \( \{f_{\varepsilon,x_0,i}\}_{i=1}^{n} \) is \( 2L \)-average smooth.

Then by Corollary [1], the number of SFO calls of AL-SVRE for finding \( \varepsilon/2 \)-saddle point of \( f_{\varepsilon,x_0} \) corresponds to finding an \( \varepsilon \)-saddle point of \( f \), is

\[
\tilde{O} \left( \sqrt{n} \left( \sqrt{n} + \frac{2L}{\varepsilon D^2_x} \right) \left( \sqrt{n} + \frac{2L}{\varepsilon D^2_y} \right) \right) = \tilde{O} \left( n + \frac{\sqrt{\varepsilon L D_x D_y}}{\varepsilon} + n^{3/4} (D_x + D_y) \sqrt{\frac{L}{\varepsilon}} \right).
\]