ANALYTIC COMPACTIFICATIONS OF $C^2$ II: ONE IRREDUCIBLE CURVE AT INFINITY

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Abstract. We classify primitive normal compactifications of $C^2$ (i.e. normal analytic surfaces containing $C^2$ for which the curve at infinity is irreducible), compute the moduli space of these surfaces and their groups of automorphisms. As a result we show that in ‘most’ of these surfaces $C^2$ is ‘rigidly embedded’. We also give a description of ‘embedded isomorphism classes’ of planar curves with one place at infinity. We compute the canonical divisor and find numerical characterizations of primitive compactifications with at most rational and elliptic singularities, and those which are Gorenstein. This in particular recovers the classification in [BDP81] of Gorenstein primitive compactifications of $C^2$ with vanishing geometric genus. As an application of our results we classify all $G^2_a$-varieties with Picard number 1 - which completes the first step towards answering a question of Hassett and Tschinkel [HT99].

1. Introduction

Normal analytic compactifications (henceforth to be denoted by only ‘compactifications’) of $C^2$ have been studied by a number of authors from different perspectives, see e.g. [Mor72], [Bre73], [Bre80], [BDPS1], [MZ88], [Fur97], [Oht01], [Koj01], [KT09], [FJ11]. In [Mon13] a correspondence was established between following two sets of objects:

(*) normal algebraic compactifications of $C^2$ with one irreducible curve at infinity $\leftrightarrow$ algebraic curves in $C^2$ with one irreducible branch at infinity

More generally, a result of part I of this study [Mon11, Corollary 4.11] gives a description (in terms of the discrete valuation associated to the curve at infinity) of compactifications of $C^2$ with one irreducible curve at infinity, i.e. those for which the complement of $C^2$ is irreducible; following [Oht01] we call these primitive compactifications - see Proposition 3.3. The point of departure of this article is the observation that the correspondence (*) yields an explicit description of the defining equations of all algebraic primitive compactifications of $C^2$ (Proposition 3.7) and the singularities of the curve at infinity (Corollary 3.8) - in particular, all primitive algebraic compactifications of $C^2$ are weighted complete intersections and the curve at infinity has at most one singular point, which is at worst a toric (or monomial) singularity.

Our main tool is the (degree-wise) Puiseux series associated to generic curvettes that transversally intersect the curve at infinity (see Section 2.2) and the key forms (Section 2.1) of the associated discrete valuation on $C[x, y]$. The sequences of orders of pole of the key forms along the curve at infinity, which we denote as key sequences, generalize the notion of $\delta$-sequences used extensively (see e.g. [Sat77, Section 2.1], [Suz99, Section 3]) in the study of planar curves with one place at infinity. A main (technical) result of this article is the that the key sequence associated to a given primitive compactification of $C^2$ can be brought to a normal form with respect to appropriate systems of coordinates (Theorems 5.2, 5.3). Analyzing the properties of normal forms and automorphisms of $C^2$ which preserve normal forms, we derive a number of results in Section 5:

- We show that the embeddings of $C^2$ in ‘most’ primitive compactifications are ‘rigid’; more precisely, if $\bar{X}$ is a primitive compactification of $C^2$ which is not isomorphic to a weighted projective surface of the form $\mathbb{P}^2(1, 1, q)$ (for some positive integer $q$), then $\bar{X}$ has only one subset isomorphic to $C^2$ (Proposition 5.4).
• We compute the groups of automorphisms of primitive compactifications of $\mathbb{C}^2$ (Theorem 5.5). In particular, it turns out that ‘most’ primitive compactifications, including all the non-algebraic ones, admit only finitely many automorphisms (Corollary 5.7).

• We explicitly describe the moduli space of primitive compactifications, i.e. the space of (isomorphism classes of) compact analytic surfaces of Picard rank 1 which contain a copy of $\mathbb{C}^2$. The moduli space of primitive compactifications with a fixed key sequence (in the normal form) turns out to be of the form $(\mathbb{C}^*)^k \times \mathbb{C}^l$ for some $k, l \geq 0$ (Theorem 5.9). Using the correspondence we then give a description (originally due in another form to [Oka98]) of the moduli space of embedded isomorphism classes of planar curves with one place at infinity with a fixed $\delta$-sequence (in the normal form) (Corollary 5.11); in particular, the latter space is a quotient of $(\mathbb{C}^*)^k \times \mathbb{C}^l$ under an (explicitly described) action of $(\mathbb{C}^*)^2$.

In Section 4 we apply the results of Section 5 to study $G_a^2$ action on surfaces. Recall that $G_a^n$ is $\mathbb{C}^n$ with the additive group structure and a $G_a^n$-variety $Y$ is a variety with a fixed (left) $G_a^n$-action such that the stabilizer of a generic point is trivial and the orbit of a generic point is dense; in other words, $G_a^n$-varieties are equivariant compactifications of $G_a^n$.

**Question 1.1** ([HT99, Section 5.2, Question 3]). Classify $G_a^2$-structures on projective surfaces with log-terminal singularities and Picard number 1.

Since $G_a^2$-varieties are also compactifications of $\mathbb{C}^2$, Question 1.1 is about primitive compactifications of $\mathbb{C}^2$ which are also $G_a^2$-varieties. Using our description of automorphism groups of primitive compactifications, we classify all normal $G_a^2$-varieties with Picard number 1. In a forthcoming work we use this result to answer Question 1.1.

In Section 4 we compute the canonical divisor of primitive compactifications in terms of the associated key sequence. In particular, we characterize the primitive compactifications with rational and elliptic singularities and those which are Gorenstein. As a simple application we recover the classification in [BDPS] of primitive Gorenstein compactifications of $\mathbb{C}^2$ with vanishing geometric genus. In a sequel of this article, extending the work of [Fur97] and [Oht01], we give a complete classification of normal primitive compactifications of $\mathbb{C}^2$ which are hypersurfaces in $\mathbb{P}^3$.

**Notation 1.2.** Throughout the rest of the article we write $X := \mathbb{C}^2$ with polynomial coordinates $(x, y)$ and let $X(0) \cong \mathbb{P}^2$ be the compactification of $X$ induced by the embedding $(x, y) \mapsto [1 : x : y]$, so that the semidegree on $\mathbb{C}[x, y]$ corresponding to the line at infinity is precisely on $X(0)$ is deg, where deg is the usual degree in $(x, y)$-coordinates. Moreover, given elements $\omega_1, \ldots, \omega_k \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq 0}(\omega_1, \ldots, \omega_k)$ to denote the semigroup generated by $\omega_1, \ldots, \omega_k$.

2. Preliminaries

2.1. Divisorial discrete valuations, semidegrees, key forms, and associated compactifications.

**Definition 2.1** (Divisorial discrete valuations). A discrete valuation on $\mathbb{C}(x, y)$ is a map $\nu : \mathbb{C}(x, y) \setminus \{0\} \to \mathbb{Z}$ such that for all $f, g \in \mathbb{C}(x, y) \setminus \{0\}$,

1. $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$,
2. $\nu(fg) = \nu(f) + \nu(g)$.

A discrete valuation $\nu$ on $\mathbb{C}(x, y)$ is called **divisorial** iff there exists a normal algebraic surface $Y_\nu$ equipped with a birational map $\sigma : Y_\nu \to X(0)$ and a curve $C_\nu$ on $Y_\nu$ such that for all non-zero $f \in \mathbb{C}[x, y]$, $\nu(f)$ is the order of vanishing of $\sigma^*(f)$ along $C_\nu$. The center of $\nu$ on $X(0)$ is $\sigma(C_\nu)$. $\nu$ is said to be **centered at infinity** (with respect to $(x, y)$-coordinates) iff the center of $\nu$ on $X(0)$ is contained in $X(0) \setminus X$; equivalently, $\nu$ is centered at infinity iff there is a non-zero polynomial $f \in \mathbb{C}[x, y]$ such that $\nu(f) < 0$.

**Definition 2.2** (Divisorial semidegrees). A **divisorial semidegree** on $\mathbb{C}(x, y)$ is a map $\delta : \mathbb{C}(x, y) \setminus \{0\} \to \mathbb{Z}$ such that $-\delta$ is a divisorial discrete valuation centered at infinity.
Definition 2.3 (cf. definition of key polynomials in [FJ04 Definition 2.1]). Let $\delta$ be a divisorial semidegree on $\mathbb{C}[x,y]$ such that $\delta(x) > 0$. A sequence of elements $g_0, g_1, \ldots, g_{n+1} \in \mathbb{C}[x,x^{-1},y]$ is called the sequence of key forms for $\delta$ if the following properties are satisfied:

(P0) $g_0 = x$, $g_1 = y$.
(P1) Let $\omega_j := \delta(g_j)$, $0 \leq j \leq n+1$. Then

$$\omega_{j+1} < \alpha_j \omega_j = \sum_{i=0}^{j-1} \beta_{j,i} \omega_i \quad \text{for} \quad 1 \leq j \leq n,$$

where

(a) $\alpha_j = \min\{\alpha \in \mathbb{Z}_{>0} : \alpha \omega_j \in \mathbb{Z}\omega_0 + \cdots + \mathbb{Z}\omega_{j-1}\}$ for $1 \leq j \leq n$,
(b) $\beta_{j,i}$'s are integers such that $0 \leq \beta_{j,i} < \alpha_i$ for $1 \leq i < j \leq n$ (in particular, $\beta_{j,0}$'s are allowed to be negative).

(P2) For $1 \leq j \leq n$, there exists $\theta_j \in \mathbb{C}^*$ such that

$$g_{j+1} = g_j^{\alpha_j} - \theta_j g_0^{\beta_{j,0}} \cdots g_{j-1}^{\beta_{j-1,j-1}}.$$

(P3) Let $y_1, \ldots, y_{n+1}$ be indeterminates and $\omega$ be the weighted degree on $B := \mathbb{C}[x,x^{-1},y_1,\ldots,y_{n+1}]$ corresponding to weights $\omega_0$ for $x$ and $\omega_j$ for $y_j$, $0 \leq j \leq n+1$ (i.e. the value of $\omega$ on a polynomial is the maximum ‘weight’ of its monomials). Then for every polynomial $g \in \mathbb{C}[x,x^{-1},y]$,

$$(1) \quad \delta(g) = \min(\omega(G) : G(x,y_1,\ldots,y_{n+1}) \in B, \ G(x,g_1,\ldots,g_{n+1}) = g).$$

Theorem 2.4 (cf. [FJ04 Theorems 2.8 and 2.29]). There is a unique and finite sequence of key forms for any divisorial semidegree $\delta$. On the other hand, if $\bar{g} := (g_0, g_1, \ldots, g_{n+1})$ is a sequence of elements in $\mathbb{C}[x,x^{-1},y]$ which satisfies properties (P0)–(P2), there is a unique divisorial semidegree $\delta$ on $\mathbb{C}[x,y]$ which satisfies (P3), i.e. $\bar{g}$ is the sequence of key forms for $\delta$.

Example 2.5. Let $(p,q)$ are integers such that $p > 0$ and $\delta$ be the weighted degree on $\mathbb{C}(x,y)$ corresponding to weights $p$ for $x$ and $q$ for $y$. Then the key forms of $\delta$ are $x, y$.

Definition 2.6. Given a divisorial semidegree $\delta$ on $\mathbb{C}[X]$, we say that $\delta$ determines an algebraic (resp. analytic) compactification of $\mathbb{C}^2$ iff there exists a (necessarily unique) normal algebraic (resp. analytic) compactification $\bar{X}$ of $X := \mathbb{C}^2$ such that $\mathbb{C}_\infty := \bar{X} \setminus \bar{X}$ is an irreducible curve and $\delta$ is proportional to the order of pole along $\mathbb{C}_\infty$.

The following is the main result of [Mon13]:

Theorem 2.7. Let $\delta$ be a divisorial semidegree on $\mathbb{C}[x,y]$ such that $\delta(x) > 0$ and $g_0, \ldots, g_{n+1}$ be the key forms of $\delta$. Then

(1) $\delta$ determines a normal analytic compactification of $\mathbb{C}^2$ iff $\delta(g_{n+1}) > 0$.
(2) $\delta$ determines a normal algebraic compactification of $\mathbb{C}^2$ iff $\delta(g_{n+1}) > 0$ and $g_{n+1}$ is a polynomial.

Now assume $\delta$ determines a normal algebraic compactification $\bar{X}^\delta$ of $\mathbb{C}^2$. Then in fact $g_j$ is a polynomial for all $j$, $0 \leq j \leq n+1$, and $\bar{X}^\delta$ is isomorphic to the closure in the weighted projective space $W^\mathbb{P} := \mathbb{P}(1,\delta(g_0),\ldots,\delta(g_{n+1})$ of the image of the map $X \to W^\mathbb{P}$ given by

$$(x,y) \mapsto [1 : g_0(x,y) : \cdots : g_{n+1}(x,y)].$$

2.2. Degree-wise Puiseux series.

Definition 2.8 (Degree-wise Puiseux series). The field of degree-wise Puiseux series in $x$ is

$$\mathbb{C}((x)) := \bigcup_{p=1}^{\infty} \mathbb{C}((x^{-1/p})) = \left\{ \sum_{j \leq k} a_j x^{j/p} : k,p \in \mathbb{Z}, p \geq 1 \right\},$$

where for each integer $p \geq 1$, $\mathbb{C}((x^{-1/p}))$ denotes the field of Laurent series in $x^{-1/p}$. Let $\phi$ be a degree-wise Puiseux series in $x$. The polydromy order (terminology taken from [CA00]) of $\phi$ is the smallest positive integer $p$ such that $\phi \in \mathbb{C}((x^{-1/p}))$. For any $r \in \mathbb{Q}$, let us denote by $[\phi]_r$ (resp.
Proposition 2.10. Let \( \phi \) be a divisorial semidegree on \( \mathbb{C}(x,y) \) such that \( \delta(x) > 0 \). Then there exists a degree-wise Puiseux polynomial (i.e. a degree-wise Puiseux series with finitely many terms) \( \phi_\delta \in \mathbb{C}(\langle x \rangle) \) and a rational number \( r_\delta \) such that for every polynomial \( f \in \mathbb{C}[x,y] \),

\[
\delta(f) = \delta(x) \deg_x (f(x,y)|_{y=\phi_\delta(x)+\xi x^{r_\delta}}),
\]

where \( \xi \) is an indeterminate.

Definition 2.11. If \( \phi_\delta \) and \( r_\delta \) are as in Proposition 2.10, we say that \( \hat{\phi}_\delta(x,\xi) := \phi_\delta(x) + \xi x^{r_\delta} \) is the generic degree-wise Puiseux series associated to \( \delta \). Let the Puiseux pairs of \( \hat{\phi}_\delta \) be \((q_1,p_1), \ldots, (q_{l+1})\). Express \( r_\delta \) as \( q_{l+1}/(p_1 \cdots p_{l+1}) \) where \( p_{l+1} \geq 1 \) and \( \gcd(q_{l+1},p_{l+1}) = 1 \). Then the formal Puiseux pairs of \( \hat{\phi}_\delta \) are \((q_1,p_1), \ldots, (q_{l+1},p_{l+1})\). Note that

1. \( \delta(x) = p_1 \cdots p_{l+1} \),
2. it is possible that \( p_{l+1} = 1 \) (as opposed to other \( p_k \)'s, which are always \( \geq 2 \)).

Example 2.12. Let \( (p,q) \) are integers such that \( p > 0 \) and \( \delta \) be the weighted degree on \( \mathbb{C}(x,y) \) corresponding to weights \( p \) for \( x \) and \( q \) for \( y \). Then \( \hat{\phi}_\delta = \xi x^{r_\delta}/p \) (i.e. \( \hat{\phi}_\delta = 0 \)).

The following result, which is an immediate corollary of [Mon11, Proposition 4.2, Assertion 2], gives a connection between degree-wise Puiseux series of a semidegree with the geometry of the associated compactification.

Proposition 2.13. Let \( \hat{X} \) be a primitive compactification of \( X \), \( C_\infty := \hat{X} \setminus X \) be the curve at infinity, \( \delta \) be the associated semidegree on \( \mathbb{C}[x,y] \) and let \( \hat{\phi}_\delta(x,\xi) := \phi_\delta(x) + \xi x^{r_\delta} \) be the generic degree-wise Puiseux series associated to \( \delta \). Then there is a unique point \( P_\infty \in C_\infty \) such that for all \( P \in C_\infty \setminus P_\infty \) and all \( f \in \mathbb{C}[x,y] \setminus \{0\} \), \( P \) is on the curve (on \( \hat{X} \)) defined by \( f \) iff there is a degree-wise Puiseux root \( \phi(x) \) of \( f \) of the form

\[
\phi(x) = \phi_\delta(x) + c_{P_\infty} x^{r_\delta} + \text{l.o.t.}
\]

for some \( c_{P_\infty} \in \mathbb{C} \) (where l.o.t. denotes lower order terms in \( x \)). Moreover \( P \) is completely determined by \( c_{P_\infty} \); for each \( c \in \mathbb{C} \), we denote by \( P_c \) the unique point on \( C_\infty \) such that \( c_{P_\infty} = c \).

Now we list some properties of key forms and degree-wise Puiseux series of semidegrees that we use in this article. Assertion 6 of the proposition below follows from a straightforward computation (cf. [Mon11, Identity (4.7)]) and assertion 5b is a corollary of [Mon11, Corollary 4.11]. The others follow from [Mon13 Propositions 3.28 and 4.2] and Theorem 2.7.
Proposition 2.14. Consider the set up of Proposition 2.13. Let \(g_0 = x, g_1 = y, g_2, \ldots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]\) be the key forms of \(\delta\). For each \(j, 0 \leq j \leq n+1\), pick the smallest \(m_j \geq 0\) such that \(x^{m_j}g_j \in \mathbb{C}[x, y]\). Then

1. \(x^{m_j}g_j\) is an irreducible polynomial for each \(j\).
2. For each \(j \leq n\), no degree-wise Puiseux root of \(x^{m_j}g_j\) satisfies (4), i.e. \(V(x^{m_j}g_j) \cap C_{\infty} = P_{\infty}\).
3. \(x^{m_j}g_{n+1}\) has a degree-wise Puiseux factorization of the form
\[
x^{m_j}g_{n+1} = x^{m_j} \prod (y - \phi_i(x)), \quad \text{where } \phi\text{ satisfies (5)}
\]

\[
\phi(x) = \phi_\delta(x) + \text{terms of degree less than } r_i.
\]

In particular, if \(C_{n+1}\) is the curve on \(\bar{X}\) determined by \(g_{n+1}\), then

1. There is precisely one point \(P_0 \in (C_{n+1} \cap C_{\infty}) \setminus P_{\infty}\).
2. \(P_{\infty} \in C_{n+1}\) iff \(g_{n+1}\) is not a polynomial iff \(\bar{X}\) is not algebraic.
3. \(1 = \deg_g(g_j) \leq \deg_g(g_2) \leq \cdots \leq \deg_g(g_{n+1}) = p\), where \(p\) is the polydromy order of \(\phi_\delta\).
4. \(\delta(g_{j+1}) < p_1\delta(g_j), 0 \leq j \leq n\).
5. \(\delta(g_{n+1}) > 0\).
6. Let \(\phi\) be as in (3). Then \(\phi\) and \(\phi_\delta\) have the same Puiseux pairs. More precisely, if \((q_1, p_1), \ldots, (q_{l+1}, p_{l+1})\) are the formal Puiseux pairs of \(\phi_\delta\), then \((q_1, p_1), \ldots, (q_l, p_l)\) are the Puiseux pairs of \(\phi\). Moreover,
\[
\delta(g_{n+1}) = \sum_{k=1}^{l}(p_k - 1)\delta(g_k) + q_{l+1}
\]
(6)
\[
= q_{l+1} + p_1 \cdots p_l \sum_{k=1}^{l}(p_k - 1)p_{k+1} \cdots p_{l+1} \cdot \frac{q_k}{p_1 \cdots p_k}.
\]
(7)

Remark 2.15. The zero in the subscript of \(P_0\) of assertion (5a) of Proposition 2.14 is motivated by the observation that \(c_{P_0} = 0\), where \(c_{P_0}\) is defined by (4).

Remark 2.16. [Mon11] Corollary 4.11] shows that the converse of assertion (5a) of Proposition 2.14 is also true, i.e. \(\delta(g_{n+1}) > 0\) iff \(\delta\) is the semidegree corresponding to a primitive compactification of \(\mathbb{C}^2\).

We will use the following lemma:

Lemma 2.17 (cf. [Abh77] Fundamental Theorem, Section 8.5). Let \(R := \mathbb{C}[x, x^{-1}, y]\). Let \(\delta\) be a semidegree on \(\mathbb{C}(x, y)\) such that \(\delta(x) > 0\), and \(g_0, \ldots, g_{n+1} \in R\) be the key forms of \(\delta\). Then

1. The following is a \(\mathbb{C}\)-vector space basis of \(R\):
\[
B := \{g_0^{\beta_0}g_1^{\beta_1} \cdots g_{n+1}^{\beta_{n+1}} : 0 \leq \beta_j < \alpha_j, 1 \leq j \leq n, \text{ and } \beta_{n+1} \geq 0\},
\]
where \(\alpha_j\)'s are as in Property (P1) of key forms.
2. If each \(g_j\) is a polynomial, then \(B \cap \mathbb{C}[x, y]\) is a \(\mathbb{C}\)-vector space basis of \(\mathbb{C}[x, y]\).
3. Let \(g \in R\). Write \(g\) as \(g = \sum a_\beta g_\beta\), where each \(a_\beta \in \mathbb{C}\) and \(g_\beta \in B\). Then
\[
\delta(g) = \max\{\delta(g_\beta) : a_\beta \neq 0\}.
\]

Proof. Pick \(g := \sum_\beta a_\beta \prod_{j=0}^{n+1} g_j^{\beta_j} \in R\) such that \(\prod_{j=0}^{n+1} g_j^{\beta_j} \in B\) for all \(\beta\) such that \(a_\beta \neq 0\). Assume there exists \(\beta\) such that \(a_\beta \neq 0\). We will show that \(g \neq 0\). Let
\[
d := \max\{\delta(\prod_{j} g_j^{\beta_j}) : a_\beta \neq 0\},
\]
\[
\bar{g} := \sum_\beta a_\beta \prod_{j} g_j^{\beta_j}.
\]
It suffices to show that $\delta(\hat{g}) = d$. Let $\hat{\phi}(x, \xi) := \phi_\beta(x) + \xi^r s$ be the generic degree-wise Puiseux series associated to $\delta$. Proposition 2.14 implies that

$$g_j|_{y=\hat{\phi}(x, \xi)} = \begin{cases} c_j x^{\omega_j/\omega_0} + \text{l.o.t.} & \text{for } 0 \leq j \leq n, \\ c_{n+1} x^{\omega_{n+1}/\omega_0} + \text{l.o.t.} & \text{for } j = n+1, \end{cases}$$

where $c_j \in \mathbb{C}^*$ and $\omega_j := \delta(g_j)$ for each $j$. Consequently,

$$\tilde{g}|_{y=\hat{\phi}(x, \xi)} = (\xi x^{d/\omega_0} + \text{l.o.t.}) \cdot c(\xi),$$

where $c(\xi) := \sum a_\beta \prod_j c_j^{\beta_j} \xi^{\beta_{n+1}}$.

If $c(\xi) \neq 0$, then $\delta(\tilde{g}) = d$ and we are done. So assume $c(\xi) = 0$. Then there must exist two distinct $\beta, \beta'$ such that $\beta_{n+1} = \beta_0' + 1$ and $\sum_{j=0}^n \beta_j \omega_j = \sum_{j=0}^n \beta_j' \omega_j = d - \beta_{n+1} \omega_{n+1}$. Let $\beta'$ be the maximal integer $\leq n$ such that $\beta' \neq \beta'$. Then it follows that $(\beta' - \beta') \omega'_{j-1}$ is in the group generated by $\omega_0, \ldots, \omega'_{j-1}$. Since $|\beta' - \beta'_{j-1}| \leq \alpha_{j-1}$, this contradicts the minimality in the definition of $\alpha_j$. Consequently $c(\xi) \neq 0$, and the proof of the lemma is complete.

**Corollary 2.18.** Consider the set up of Lemma 2.17. Assume that $\delta(g_j) > 0$ for all $j$, $0 \leq j \leq n+1$, and that there exists $f \in \mathbb{C}[x, y]$ such that $\delta(f) = 1$. Then

1. there exists $j \leq n+1$ such that $\delta(g_j) = 1$,
2. either $j = n+1$, or $\alpha_j = \gcd(\omega_0, \ldots, \omega_{j-1})$ (where $\alpha_j$ is as in Property 21 of key forms) and for all $k$, $j < k \leq n+1$, $g_k$ is of the form $g_k = g_{j+1} - h_k(g_j)$,

where $h_k$ is a polynomial in one variable with $\deg(h_k) \leq \alpha_j$. Moreover, $\delta(g_i) > 1$ for all $i < j$.

3. (a) either $f = ag_j + b$, for some $a \in \mathbb{C}^*$, $b \in \mathbb{C}$, in which case $g_j$ is a polynomial, or
   (b) $j < n+1$, $\delta(g_{n+1}) = 1$, and $f = ag_j + bg_{n+1} + c$ for some $b \in \mathbb{C}^*$, $a, c \in \mathbb{C}$. In this case both $g_{n+1}$ and $g_j$ are polynomials.

In any case, the curve $f$ has only one place at infinity.

**Proof.** Since $f \in \mathbb{C}[x, y]$, it follows that $f$ is in the $\mathbb{C}$-linear span of the subset $B_0$ of $B$ (from Lemma 2.17) consisting of all $g_0^{\gamma_0} g_1^{\beta_1} \cdots g_{n+1}^{\beta_{n+1}} \in B$ for which $\beta_0 \geq 0$. Then assertion 3 of Lemma 2.17 implies that

$$\delta(f) \neq 0$$

implying that

$$f|_{y=\hat{\phi}(x, \xi)} = (ac_j + bc_{n+1} \xi^{1/\omega_0}) x^{1/\omega_0} + \text{l.o.t.},$$

for some $c_j, c_{n+1} \in \mathbb{C}^*$. Since the coefficient of $x^{1/\omega_0}$ in the right hand side of (11) is not constant, it follows that there is a degree-wise Puiseux root $\phi$ of $f$ such that $\deg_s(\phi - \phi s) \leq r_s$. Assertion 3 of Proposition 2.14 then implies that $\phi$ is in fact the only degree-wise Puiseux root of $f$ (and, in particular, $f = 0$ has only one place at infinity). Since $f$ is a polynomial, it follows from [Mon13 Proposition 4.2] that $g_j$ is a polynomial for each $j$, $0 \leq j \leq n+1$, which proves that 3a holds. It remains to show that $f = 0$ has only one place at infinity in the case of 3b. But since $g_j$ is a polynomial in the case of 3b, this again follows from [Mon13 Proposition 4.2].
Moreover, it has the following properties:

A sequence \( \bar{\omega} := (\omega_0, \ldots, \omega_{n+1}) \), \( n \in \mathbb{Z}_{\geq 0} \), of integers is called a key sequence if it has the following properties:

1. \( \omega_0 \geq 1 \).
2. Let \( d_k := \gcd(\lvert \omega_0 \rvert, \ldots, \lvert \omega_k \rvert) \), \( 0 \leq k \leq n+1 \) and \( \alpha_k := d_{k-1}/d_k \), \( 1 \leq k \leq n+1 \). Then \( d_{n+1} = 1 \), and
3. \( \omega_{k+1} < \alpha_k \omega_k \), \( 1 \leq k \leq n \).

Moreover, \( \bar{\omega} \) is called primitive if

4. \( \omega_{n+1} > 0 \) (or equivalently, \( \omega_k > 0 \) for all \( k \), \( 0 \leq k \leq n+1 \)),
and it is called algebraic if in addition
5. \( \alpha_k \omega_k \in \mathbb{Z}_{\geq 0}(\omega_0, \ldots, \omega_{k-1}) \), \( 1 \leq k \leq n \).

Finally, \( \bar{\omega} \) is called essential if \( \alpha_k \geq 2 \) for \( 1 \leq k \leq n \). Note that

(a) Given an arbitrary key sequence \( (\omega_0, \ldots, \omega_{n+1}) \), it has an associated essential subsequence \( (\omega_{i_0}, \omega_{i_1}, \ldots, \omega_{i_k}, \omega_{n+1}) \) where \( \{i_j\} \) is the collection of all \( k \), \( 1 \leq k \leq n \), such that \( \alpha_k \geq 2 \).
(b) If \( \bar{\omega} \) is an algebraic key sequence, then its essential subsequence is also algebraic.

Remark 3.2. Let \( \bar{\omega} := (\omega_0, \ldots, \omega_{n+1}) \) be a key sequence. It is straightforward to see that property 3 implies the following: for each \( k, 1 \leq k \leq n \), \( \alpha_k \omega_k \) can be uniquely expressed in the form \( \alpha_k \omega_k = \beta_{k,0} \omega_0 + \beta_{k,1} \omega_1 + \cdots + \beta_{k,k-1} \omega_{k-1} \), where \( \beta_{k,j} \)'s are integers such that \( 0 \leq \beta_{k,j} < \alpha_j \) for all \( j \geq 1 \). If \( \bar{\omega} \) is in additional algebraic, then \( \beta_{k,0} \)'s of the preceding sentence are non-negative.

The motivation for the terminology in Definition 3.1 comes from the following observation:

Proposition 3.3.

1. Given a divisorial semidegree \( \delta \) on \( \mathbb{C}[x, y] \), the sequence \( \bar{\omega} \) of the \( \delta \)-values of its key forms is a key sequence. If \( \delta \) corresponds to a primitive (resp. primitive algebraic) compactification of \( \mathbb{C}^2 \), then \( \bar{\omega} \) is primitive (resp. primitive algebraic).
2. On the other hand, let \( \bar{\omega} := (\omega_0, \ldots, \omega_{n+1}) \) be a key sequence and \( \bar{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n \). Define \( g_0 := x \), \( g_1 := y \), and

\[
g_{k+1} = g_k^{\alpha_k} - \theta_k g_0 \delta_{k,0} \cdots \delta_{k,k-1}, \quad 1 \leq k \leq n,
\]

where \( \alpha_k \)'s and \( \beta_{k,j} \)'s are as in Remark 3.2. Then there is a unique divisorial semidegree \( \delta \) on \( \mathbb{C}[x, y] \) such that \( (g_0, \ldots, g_{n+1}) \) is the sequence of key forms for \( \delta \) and \( \delta(g_k) = \omega_k \), \( 0 \leq k \leq n+1 \). If \( \bar{\omega} \) is primitive (resp. primitive algebraic), then \( \delta \) corresponds to a primitive (resp. primitive algebraic) compactification of \( \mathbb{C}^2 \).

Proof. This is an immediate corollary of Theorems 2.4 and 2.7. \( \square \)

Definition 3.4. Let \( (x, y) \) be a fixed system of coordinates on \( X := \mathbb{C}^2 \). Let \( \bar{\omega} := (\omega_0, \ldots, \omega_{n+1}) \) be a primitive key sequence and \( \bar{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n \). We write \( \tilde{X}_{\bar{\omega}, \bar{\theta}} \) for the corresponding primitive compactification of \( X \) (as in assertion 2 of Proposition 3.3) and \( \delta_{\bar{\omega}, \bar{\theta}} \) for the associated semidegree on \( \mathbb{C}[x, y] \).

Remark 3.5. Note that \( \delta \)-sequences of plane curves with one place at infinity (defined e.g. in [Suz99, Section 3]) are special cases of key sequences: indeed, \( \bar{\omega} := (\omega_0, \ldots, \omega_{n+1}) \) is a \( \delta \)-sequence iff it is a primitive algebraic essential key sequence, and either \( n = 0 \) (in which case \( \bar{\omega} = (1) \)) or \( \alpha_{n+1} > 1 \) (where \( \alpha_{n+1} \) is defined as in Definition 3.1). In geometric terms the relation between \( \delta \)-sequences and key sequences can be explained in the following way: let \( C = V(f) \subseteq \mathbb{C}^2 \) be a curve
with one place at infinity, where \( f \in \mathbb{C}[x,y] \) is an irreducible polynomial. Let \( C_\xi := V(f - \xi) \subseteq \mathbb{C}^2 \) for each \( \xi \in \mathbb{C} \), and \( L_C := \{ C_\xi : \xi \in \mathbb{C} \} \) be the corresponding pencil of curves. It is a classical result of Moh [Moh74] that \( L_C \) is equisingular at infinity, i.e., given an embedding \( \mathbb{C}^2 \hookrightarrow \mathbb{P}^2 \), there is a unique point \( P \in \mathbb{P}^2 \setminus \mathbb{C}^2 \) such that \( P \) is the unique place at infinity on \( C_\xi \) for every \( \xi \in \mathbb{C} \) and the germs of \( C_\xi \) at \( P \) are equisingular for all \( \xi \in \mathbb{C} \). Let \( \delta_C \) be the semidegree on \( \mathbb{C}[x,y] \) that assigns to each \( h \in \mathbb{C}[x,y] \) the order of pole at \( P \) of \( h|_{C_\xi} \) for generic \( \xi \). Let \( \omega := (\omega_0, \ldots, \omega_{n+1}) \) be the essential subsequence of the key sequence of \( \delta_C \) in \((x, y)\)-coordinates. It is straightforward to see that \( \omega_{n+1} = 0 \). The \( \delta \)-sequence of \( C \) in \((x, y)\)-coordinates is \((\omega_0, \ldots, \omega_n)\).

3.2. General structure of primitive compactifications.

**Proposition 3.6.** Fix a system of polynomial coordinates \((x, y)\) on \( X := \mathbb{C}^2 \). Let \( \omega := (\omega_0, \ldots, \omega_{n+1}) \) be a primitive key sequence, \( \theta := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n \). Let \( g_0, \ldots, g_{n+1} \) be as in Proposition 3.3, and \( X := \hat{X}_{\omega, \theta} \) be the corresponding primitive compactification of \( X \). Let \( C_\infty \) be the curve at infinity in \( \hat{X}_{\omega, \theta} \), and \( P_\infty, P_0 \in C_\infty \) be as in Proposition 2.13.

1. \( X \setminus \{ P_0, P_\infty \} \) is non-singular.
2. If \( X \) is not a weighted projective space, then \( P_\infty \) is a singular point of \( X \).
3. Let \( \omega := \text{gcd}(\omega_0, \ldots, \omega_n) \). Then \( P_0 \) is a cyclic quotient singularity of type \( \frac{1}{k}(1, \omega_{n+1}) \).
4. Let \( \mathbb{W}_{\mathbb{P}} \) be the weighted projective space \( \mathbb{P}^{n+2}(1, \omega_0, \omega_1, \ldots, \omega_{n+1}) \) with \( \text{weighted homogeneous coordinates} \ [w : y_0 : \cdots : y_{n+1}] \). Then the map \( X \setminus V(x) \hookrightarrow \mathbb{W}_{\mathbb{P}} \) given by

\[
(x, y) \mapsto [1 : g_0(x, y) : g_1(x, y) : \cdots : g_{n+1}(x, y)]
\]

induces a closed embedding of \( X \setminus V(x) \) into \( \mathbb{W}_{\mathbb{P}} \setminus V(y_0) \).
5. \( C_\infty \) is non-singular off \( P_\infty \). In particular \( C_\infty \setminus P_\infty \cong \mathbb{C} \).

**Proof.** Assertion 4 is a consequence of assertion 2 of Lemma 2.17. The first two assertions follow from [Mon11, Proposition A.1]. Now we prove assertion 3.

Let \( \hat{W} := \mathbb{C}^{n+2} \) with coordinates \((\hat{w}, \hat{y}_1, \ldots, \hat{y}_{n+1})\). Then \( \mathbb{W}_{\mathbb{P}} \setminus V(y_0) \) is the quotient of \( \hat{W} \) by the action of the cyclic group of \( \omega_0 \) elements given by

\[
\zeta \cdot (\hat{w}, \hat{y}_1, \ldots, \hat{y}_{n+1}) := (\zeta \hat{w}, \zeta^{\omega_1} \hat{y}_1, \ldots, \zeta^{\omega_{n+1}} \hat{y}_{n+1})
\]

where \( \zeta \) is a primitive \( \omega_0 \)-th root of unity. Let \( \hat{\pi} : \hat{W} \twoheadrightarrow \mathbb{W}_{\mathbb{P}} \setminus V(y_0) \) be the quotient map

\[
\hat{\pi} : (\hat{w}, \hat{y}_1, \ldots, \hat{y}_{n+1}) \mapsto [\hat{w} : 1 : \hat{y}_1 : \cdots : \hat{y}_{n+1}]
\]

and \( \hat{X} := \hat{\pi}^{-1}(X) \). Then \( \hat{X} \subseteq V(\hat{G}_1, \ldots, \hat{G}_n) \), where for each \( k, 1 \leq k \leq n \), \( \hat{G}_k \) is of the form

\[
\hat{G}_k := \hat{w}^{\alpha_k \omega_{k-1}} \hat{y}_{k-1} - \left( \hat{y}_{k}^{\theta_k} \prod_{j=1}^{k-1} \hat{y}_{j}^{\beta_{k-j}} \right)
\]

**Claim 3.6.1.** \( \hat{X} = V(\hat{G}_1, \ldots, \hat{G}_n) \).

**Proof.** Let \( \hat{I} \) be the ideal generated by \( \hat{G}_1, \ldots, \hat{G}_n \) in \( \hat{A} := \mathbb{C}[\hat{w}, \hat{y}_1, \ldots, \hat{y}_{n+1}] \). Then \( (\hat{A}/\hat{I})_{\hat{w}} := \mathbb{C}[\hat{w}, \hat{w}^{-1}, \hat{y}_1] \). Since \( \text{dim} \hat{X} = 2 \), it follows that the ideal of \( \hat{X} \setminus V(\hat{w}) \) is generated by \( \hat{I} \) in \( \hat{A}[\hat{w}^{-1}] \). Pick an arbitrary \( \hat{\gamma} \) in the ideal of \( \hat{X} \). It then follows that \( \hat{w}^N \hat{\gamma} \in \hat{I} \) for \( N \gg 1 \). But then the last two assertions of the following lemma imply that \( \hat{\gamma} \in \hat{I} \), as required for the proof of the claim. \( \square \)

**Lemma 3.6.2.** Let \( \hat{\omega} := (0, \omega_1, \ldots, \omega_{n+1}) \) and \( \prec \) be the ordering on \( \mathbb{Z}_{\geq 0}^{n+2} \) by setting \( \beta \prec \beta' \) if

1. \( (\beta - \beta') \cdot \hat{\omega} < 0 \) (where \( \cdot \) denotes the usual dot product), or
2. \( (\beta - \beta') \cdot \hat{\omega} = 0 \) and the right-most non-zero entry of \( \beta - \beta' \) is negative.

Then

1. \( \prec \) induces a monomial ordering on \( \hat{A} \) (which we also denote by \( \prec \)),
2. the leading term of \( \hat{G}_k \) with respect to \( \prec \) is \( \hat{y}_k^{\theta_k} \), \( 1 \leq k \leq n \),
3. \( \hat{G}_1, \ldots, \hat{G}_n \) is a Gröbner basis of \( \hat{I} \) with respect to \( \prec \).
Proof. The first two assertions are straightforward to check (one needs to use assertion 5 of Proposition [2.14]). The last assertion follows from the second one via a straightforward application of Buchberger’s algorithm (as described in [CL07, Section 2.7]). □

Now we resume the proof of assertion 3 of the proposition. A computation shows that the matrix of partial derivatives $\frac{\partial G_i}{\partial y_j}, 1 \leq i, j \leq n$ has non-zero determinant on $C_\infty := \hat{X} \setminus \{\hat{w} = 0\}$ and therefore $\hat{X}$ is non-singular at every point on $C_\infty$. Moreover, the only ramification points of $\hat{\pi} | \hat{X}$ on $C_\infty$ are the points with zero $\hat{y}_{n+1}$-coordinate. Since $\hat{\pi}$ maps every such point to $P_0 \in C_\infty$, this gives another proof that $\hat{X}$ is non-singular at every point on $C_\infty \setminus \{P_0, P_\infty\}$.

It follows from the preceding paragraph that $(\hat{w}, \hat{y}_{n+1})$ defines an (analytic) system of coordinates on $\hat{X}$ near every point of $C_\infty$. Pick a point $\hat{P}_0 := (0, \hat{\theta}_1, \ldots, \hat{\theta}_n, 0) \in \hat{\pi}^{-1}(P_0)$ and choose a sufficiently small (analytic) neighborhood $U$ of $\hat{P}_0$ in $\hat{X}$ such that

1. $(\hat{w}, \hat{y}_{n+1})$ defines an (analytic) system of coordinates on $U$,
2. for every point $\hat{P}'_0 \in \pi^{-1}(P_0) \setminus \{\hat{P}_0\}$, there is a neighborhood $U'$ of $\hat{P}'_0$ such that $U'$ is disjoint from $U$ and $U' = \zeta^k \cdot U$ (where the action is the one induced from (12)).

Then $\hat{\pi}(U)$ is an open neighborhood of $P_0$ and $\hat{\pi}(U)$ is the quotient of $U$ under the subgroup of $\omega_0$-th roots of unity which preserves $U$. Now pick $k$ such that $\zeta^k \cdot U = U$. Then it follows that $\zeta^{(k)} = 1$ for all $j, 1 \leq j \leq n$, or equivalently, $\zeta^k$ is a $\hat{\omega}$-th root of unity, where $\hat{\omega} := \gcd(\omega_0, \ldots, \omega_n)$. Consequently $\hat{\pi}(U)$ is isomorphic to the quotient of $U$ (with coordinates $(w, y_{n+1})$) under the action of the group of $\hat{\omega}$-th roots of unity given by

$$\hat{\zeta} \cdot (w, y_{n+1}) = (\zeta w, \zeta^{\hat{\omega} n+1} y_{n+1})$$

This finishes the proof of assertion 3.

The arguments in the paragraph following (13) imply that $C_\infty \setminus \{P_0, P_\infty\}$ is non-singular. On the other hand, near $P_0, C_\infty$ is the image of $y_{n+1}$-axis under the action of (14). It then follows from standard theory of cyclic quotient singularities that $C_\infty$ is non-singular near $P_0$. This proves the last assertion, and therefore completes the proof of the proposition. □

3.3. Algebraic Case.

**Proposition 3.7.** Fix a system of polynomial coordinates $(x, y)$ on $X := \mathbb{C}^2$. Let $\hat{\omega} := (\omega_0, \ldots, \omega_{n+1})$ be a primitive algebraic key sequence, $\hat{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n$. Let $g_0, \ldots, g_{n+1}$ be as in Proposition 3.3 and $\hat{X}_{\hat{\omega}, \hat{\theta}}$ be the corresponding primitive algebraic compactification of $X$. Then the map

$$(x, y) \mapsto [1 : g_0(x, y) : \cdots : g_{n+1}(x, y)]$$

from $X$ into the weighted projective space $W^\mathbb{P} := \mathbb{P}^{n+2}(1, \omega_0, \ldots, \omega_{n+1})$ induces an isomorphism of $\hat{X}_{\hat{\omega}, \hat{\theta}}$ with the subvariety of $W^\mathbb{P}$ (with weighted homogeneous coordinates $[w : y_0 : y_1 : \cdots : y_{n+1}]$) defined by weighted homogeneous polynomials $G_k, 1 \leq k \leq n$, of the form

$$G_k := w^{\alpha_k} \omega_k - \omega_{k+1} y_{k+1} - \left( y_k^{\beta_k} - \theta_k \prod_{j=0}^{k-1} y_j^{\gamma_{k,j}} \right)$$

where $\alpha_k$'s and $\beta_{k,j}$'s are as in Remark 3.2.

**Proof.** The first assertion is an immediate corollary of the last statement of Theorem 2.7 and Claim 3.6.1. □

**Corollary 3.8** (The curve at infinity). Let $\hat{\omega} := (\omega_0, \ldots, \omega_{n+1})$ be a primitive algebraic key sequence, $\hat{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n$, and $X_{\hat{\omega}, \hat{\theta}}$ be the corresponding primitive algebraic compactification of $X$. Identify $X_{\hat{\omega}, \hat{\theta}}$ with the subvariety of $W^\mathbb{P}$ from Proposition 3.7. Let $C_\infty := X_{\hat{\omega}, \hat{\theta}} \setminus X$ be the curve at infinity and $P_\infty$ (resp. $P_0$) be the point on $C_\infty$ with coordinates $[0 : \cdots : 0 : 1]$ (resp. $[0 : 1 : \hat{\theta}_1 : \cdots : \hat{\theta}_n : 0]$), where $\hat{\theta}_k$ is an $\alpha_k$-th root of $\theta_k, 1 \leq k \leq n$. Then
(1) Let $S$ be the subsemigroup of $\mathbb{Z}^2$ generated by $\{(\omega_k,0) : 0 \leq k \leq n\} \cup \{(0,\omega_{n+1})\}$. Then $C_\infty \cong \text{Proj} \mathbb{C}[S]$, where $\mathbb{C}[S]$ is the semigroup algebra generated by $S$, and the grading in $\mathbb{C}[S]$ is induced by the sum of coordinates of elements in $S$.

(2) Let $\bar{S} := \mathbb{Z}_{\geq 0}(\alpha_{n+1}\omega_{n+1}) \cap \mathbb{Z}_{\geq 0}(\omega_0,\ldots,\omega_n)$. Then $\mathbb{C}[C_\infty \setminus P_0] \cong \mathbb{C}[\bar{S}]$. In particular, $C_\infty$ has at worst a (non-normal) toric singularity at $P_\infty$.

**Remark 3.9.** $P_0$ and $P_\infty$ of Corollary 3.8 are the same as $P_0$ and $P_\infty$ from Proposition 3.6

**Proof of Corollary 3.8.** Since $C_\infty = V(w) \cap \bar{X}_{\vec{\omega},\vec{\theta}}$, Proposition 3.7 implies that $C_\infty \cong \text{Proj} \mathbb{C}[w,y_0,\ldots,y_{n+1}] / \langle \bar{G}_1,\ldots,\bar{G}_n \rangle$, where $\bar{G}_k := y_0^{\alpha_k} - \theta_k \prod_{j=0}^{k-1} y_j^{\beta_{k,j}}$, $1 \leq k \leq n$. Lemma 6.5 then implies that there is an isomorphism of graded $\mathbb{C}$-algebras of the form

$$\mathbb{C}[y_0,\ldots,y_{n+1}] / \langle \bar{G}_1,\ldots,\bar{G}_n \rangle \cong \mathbb{C}[t^{\omega_0},\ldots,t^{\omega_n},y_{n+1}],$$

where $t$ is an indeterminate and the grading on the right hand side is given by assigning the degrees of $t$ and $y_{n+1}$ to be respectively 1 and $\omega_{n+1}$. Moreover, the isomorphism maps $y_k \mapsto t^{\omega_k}$ for $0 \leq k \leq n$, and $y_{n+1} \mapsto y_{n+1}$. This immediately implies assertion 1. Moreover, since $P_0 = C_\infty \cap V(y_{n+1})$, it follows that

$$\mathbb{C}[C_\infty \setminus P_0] \cong \mathbb{C} \left[ \sum_{k=0}^{n} \frac{\beta_k \omega_k}{y_{n+1}} : \beta_k \geq 0 \text{ for all } k, \sum_{k=0}^{n} \beta_k \omega_k = \beta_{n+1} \omega_{n+1} \right].$$

Assertion 2 now follows from the definition of $\alpha_{n+1}$. \hfill $\square$

4. **Canonical divisor**

The main result of this section is Theorem 4.1 which computes the canonical divisor of a primitive compactification $\bar{X}$ of $\mathbb{C}^2$ in terms of the associated key sequence. Using it we characterize when $X$ has simple types of singularities or when it is Gorenstein. The proof of Theorem 4.1 is in Subsection 4.1.

Fix a system of polynomial coordinates $(x,y)$ on $X := \mathbb{C}^2$. Let $\vec{\omega} := (\omega_0,\ldots,\omega_{n+1})$ be a primitive key sequence, $\vec{\theta} := (\theta_1,\ldots,\theta_n) \in (\mathbb{C}^*)^n$, $\bar{X} := \bar{X}_{\vec{\omega},\vec{\theta}}$ be the corresponding primitive compactification of $X$ and $C_\infty$ be the curve at infinity on $\bar{X}_{\vec{\omega},\vec{\theta}}$.

**Theorem 4.1.** The canonical divisor of $\bar{X}$ is

$$(16) \quad K_{\bar{X}} = - \left( \omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^{n} (\alpha_k - 1) \omega_k \right) [C_\infty],$$

where $[C_\infty]$ is the Weil divisor corresponding to $C_\infty$ and $\alpha_1,\ldots,\alpha_{n+1}$ are as in Definition 3.2.

**Remark 4.2.** Identity (16) in fact holds for all semidegrees on $\mathbb{C}[x,y]$ (i.e. not only for those corresponding to primitive compactifications of $X$) in the following sense: let $\bar{X}$ be an arbitrary normal analytic compactification of $X$ and $K_{\bar{X}}$ be the canonical divisor of $\bar{X}$ such that $\text{Supp}(K_{\bar{X}}) \subseteq \bar{X} \setminus X$. Let $C$ be an irreducible component of the curve at infinity and $\delta$ be the associated semidegree on $\mathbb{C}[x,y]$ (i.e. $\delta$ is the order of pole along $C$). Then the coefficient of $[C]$ in $K_{\bar{X}}$ is given by (16), where $(\omega_0,\ldots,\omega_{n+1})$ is the key sequence associated to $\delta$. Essentially the same proof works in the general case.

Recall that the geometric genus of an isolated singular point $P$ on a complex surface $Y$ is $p_g(P) := \dim_\mathbb{C}(R^1\pi_*\mathcal{O}_Y)_P$, where $\pi : \bar{Y} \to Y$ is a resolution of singularities. The singularity of $P$ is called rational (resp. elliptic) if $p_g(P) = 0$ (resp. $p_g(P) = 1$).

**Lemma 4.3** ([Fur97] Lemma 2.2). Assume $\bar{X}$ is algebraic. Then the sum of the geometric genera of singular points of $\bar{X}$ is equal to $\dim_\mathbb{C}(H^0(\bar{X},\mathcal{O}_{\bar{X}}(K_{\bar{X}})))$. 

Recall that $\bar{X}$ has at most two singular points, namely $P_0$ and $P_{\infty}$ from Theorem 3.6. Moreover, $P_0$ is a rational singularity, since all quotient singularities are rational. The following result characterizes when the singularity at $P_{\infty}$ is rational or elliptic.

**Corollary 4.4.** Set $k_{\bar{X}} := - (\omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^{n-1} (\alpha_k - 1)\omega_k)$.

1. The singularity at $P_{\infty}$ is rational iff $k_{\bar{X}} < 0$.
2. Assume $\bar{X}$ is algebraic. Then the singularity at $P_{\infty}$ is elliptic iff $0 \leq k_{\bar{X}} < \omega_{\min}$, where $\omega_{\min} := \min \{ \omega_0, \ldots, \omega_{n+1} \}$.
3. Assume $\bar{X}$ is algebraic. Then $p_g(P_{\infty}) = |\Sigma|$, where $\Sigma$ is the collection of all $(\beta_0, \ldots, \beta_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+2}$ such that $\beta_j < \alpha_j$, $1 \leq j \leq n$, and $\sum_{j=0}^{n+1} \omega_j \beta_j \leq k_{\bar{X}}$.

**Proof.** The first two assertions are immediate corollaries of Theorem 4.1 and Lemma 4.3. For the last assertion we also need Lemma 2.17. □

**Proposition 4.5.** Assume $\bar{X}$ is algebraic. Then the index of $\bar{X}$ (i.e. the smallest positive integer $m$ such that $mK_{\bar{X}}$ is Cartier) is

$$\text{ind}(\bar{X}) = \min \{ m \in \mathbb{Z}_{\geq 0} : mk_{\bar{X}} \in \mathbb{Z}_{\alpha_{n+1}} \cap \mathbb{Z}_{\omega_{n+1}} \},$$

where $k_{\bar{X}} := - (\omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^{n-1} (\alpha_k - 1)\omega_k)$. In particular, $\bar{X}$ is Gorenstein iff $k_{\bar{X}}$ is divisible by both $\omega_{n+1}$ and $\alpha_{n+1}$.

**Proof.** Since a normal surface is Gorenstein iff the canonical divisor is Cartier, the last assertion of the proposition follows from the first. For the first assertion it suffices to show that $mK_{\bar{X}}$ is a Cartier divisor iff $mk_{\bar{X}}$ is divisible by both $\omega_{n+1}$ and $\bar{\omega}$. At first we show the ($\Leftarrow$) direction. Since $\alpha_{n+1} = \gcd(\omega_0, \ldots, \omega_n)$, it follows that $|mk_{\bar{X}}| = km\omega_{n+1} = m\omega_0 - \sum_{i=1}^{n+1} m_i \omega_i$ for non-negative integers $k, m, m_0, \ldots, m_{n-1}$ consider the embedding of $\bar{X}$ given by Proposition 2.47. Then it follows that $1/\bar{h}_{n+1} + (\prod_{i=1}^{n+1} g_i^{m_i})/\bar{g}_0^{m_0}$ defines the Cartier divisor $|mk_{\bar{X}}| = C_{\infty}$ respectively near $P_{\infty}$ and $P_0$. Since $\bar{X} \setminus \{ P_0, P_{\infty} \}$ is non-singular, this proves the ($\Leftarrow$) direction. Now we prove the ($\Rightarrow$) direction. Let $h_{\infty} = h_{\infty,1}/h_{\infty,2}$ define $mk_{\bar{X}}$ near $P_{\infty}$, with $h_{\infty,1}, h_{\infty,2} \in \mathbb{C}[x, y]$. Then for each $\iota$, the closure in $\bar{X}$ of the curve on $X$ defined by $h_{\infty, \iota} = 0$ does not go through $P_{\infty}$. Proposition 2.13 then implies that $\delta(h_{\infty, \iota})$ is a multiple of $\omega_{n+1}$ for each $\iota$, where $\delta$ is the order of pole along $C_{\infty}$. It follows that $\omega_{n+1}$ divides $mk_{\bar{X}} = \delta(h_{\infty,2}) - \delta(h_{\infty,1})$. Now assume $\bar{h}_0 = h_{01}/h_{02}$ defines $mk_{\bar{X}}$ near $P_0$, with $h_{01}, h_{02} \in \mathbb{C}[x, y]$. Fix $i$, $1 \leq i \leq 2$. Let $h_{0i} = \sum a_{ij} \bar{g}_0^{\partial_i} \cdots \bar{g}_{n+1}^{\partial_i}$ be the expansion of $h_{0i}$ in terms of the basis $B$ of Lemma 2.17. Since $P_0$ is not on the closure on $\bar{X}$ of the curve $h_{0i} = 0$, it follows that among all $\beta$ such that $a_{ij} \neq 0$ and $\sum_{j=0}^{n+1} \omega_j \beta_j = \delta(h_{0i})$, there must be some $\beta$ such that $\beta_{n+1} = 0$. Since $\alpha_{n+1}$ divides $\omega_j$ for every $j$, $0 \leq j \leq n$, it follows that $\alpha_{n+1}$ divides $\delta(h_{01})$. Consequently, $\alpha_{n+1}$ divides $mk_{\bar{X}} = \delta(h_{02}) - \delta(h_{01})$, as required. □

**Corollary 4.6 (cf. [BDPSH Theorem 6]).** Let $\bar{X}$ be a Gorenstein primitive compactification of $X$ with vanishing geometric genus. Then one of the following is true:

1. $\bar{X} \cong \mathbb{P}^2$,
2. $\bar{X} \cong \mathbb{P}^2(1,1,2)$,
3. $\bar{X} \cong \mathbb{P}^2(1,2,3)$,
4. $\bar{X}$ is the hypersurface in $\mathbb{P}^2(1,2,3, r)$ (with weighted homogeneous coordinates $[w : x : y : z]$) for $5 \geq r \geq 1$ defined by the weighted homogeneous polynomial $F_r$ given by

$$F_r := \begin{cases} wz - (y^2 + 3x) & \text{if } r = 5, \\
w^2z - (y^2 + 3x + awxy) & \text{if } r = 4, \\
w^3z - (y^2 + 3x + awxy + bw^2x^2) & \text{if } r = 3, \\
w^4z - (y^2 + 3x + awxy + bw^2x^2 + cw^3y) & \text{if } r = 2, \\
w^5z - (y^2 + 3x + awxy + bw^2x^2 + cw^3y + dw^4y) & \text{if } r = 1, \\
h^6z - (y^2 + 3x + awxy + bw^2x^2 + cw^3y + dw^4y) & \text{if } r = 1, \
\end{cases}$$

where $a, b, c, d \in \mathbb{C}$.

**Proof.** W.l.o.g. assume $\bar{X} \not\cong \mathbb{P}^2$. Choose coordinates $(x, y)$ on $X$ such that the key sequence $\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})$ associated to $\bar{X}$ satisfies
(a) $\omega_0 < \omega_1$, and
(b) either $n = 0$ or $\alpha_1 > 1$.

These properties hold e.g. if $\bar{\omega}$ is in the normal form with respect to $(x, y)$-coordinates - see Section 3. Let $k_X := -(\omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^n (\alpha_k - 1)\omega_k)$. Corollary 4.3 implies that $|k_X| = \omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^n (\alpha_k - 1)\omega_k \geq 1$. If $n = 0$, then $\omega_{n+1} = \omega_1$, $\alpha_{n+1} = \omega_0$, and $|k_X| = \omega_0 + \omega_1 + 1$. In particular, $\omega_{n+1} < |k_X| \leq 2\omega_{n+1}$. Consequently, $\omega_{n+1}$ divides $k_X$ iff $\omega_1 = \omega_0 + 1$, i.e. $|k_X| = 2\omega_0 + 2$. But then $\alpha_{n+1} = \omega_0$ divides $k_X$ iff $\omega_0 = 2$ or $\omega_1 = 1$. Consequently we have two possibilities: $\omega_0 = 1$, $\omega_1 = 2$, which corresponds to case 2, or $\omega_0 = 2$, $\omega_1 = 3$, which corresponds to case 3 of the corollary. Now assume $n \geq 1$. Then

$$|k_X| = \omega_{n+1} + \omega_0 + 1 - \sum_{k=1}^n (\alpha_k - 1)\omega_k \leq \omega_{n+1} + \omega_0 + 1 - (\alpha_1 - 1)\omega_1$$

$$\leq \omega_{n+1} + \omega_0 + 1 - \omega_1 \leq \omega_{n+1}.$$

It follows that $\omega_{n+1}$ divides $k_X$ iff $\omega_{n+1} = |k_X|$ iff

(i) $\alpha_k = 1$ for all $k \geq 2$,

(ii) $\alpha_1 = 2$, and

(iii) $\omega_1 = \omega_0 + 1$.

But then, properties (i) and (ii) imply $\omega_0 = \alpha_1 = 2$, so that $\omega_1 = 3$ due to (iii). Case 3 of the corollary now follows from a straightforward examination of possibilities for equations (14).

4.1 Proof of Theorem 4.1 Consider the set up of Proposition 3.6. Let $\hat{\pi} : \hat{X} \rightarrow \hat{X} \setminus V(x)$ be as in the proof of Proposition 3.6. For every $P \in C_\infty \setminus \{P_0, P_\infty\}$ and every $\hat{Q} \in \hat{\pi}^{-1}(P)$, $\hat{\pi}$ restricts to an isomorphism near $Q$. It follows that $(\hat{w}, \hat{y}_{n+1})$ defines a system of coordinates near every $P \in C_\infty \setminus \{P_0, P_\infty\}$. Now note that $x = y_0/w^{\omega_0} = \hat{w}^{\omega_0}$ and the last key form for $\delta$ is $g_{n+1}(x, y) = \hat{y}_{n+1}/\hat{w}^{\omega_{n+1}+1}$. It follows that

$$d\hat{w} = -(\hat{w}^{\omega_{n+1}+1}/\omega_0)dx$$

$$d\hat{y}_{n+1} = \hat{w}^{\omega_{n+1}+1}dg_{n+1} + g_{n+1}d(\hat{w}^{\omega_{n+1}}), \text{ and}$$

$$d\hat{w}d\hat{y}_{n+1} = -\hat{w}^{\omega_0+\omega_{n+1}+1} \frac{\partial g_{n+1}}{\partial y} dx dy, \text{ so that}$$

$$K_X = -(\omega_0 + \omega_{n+1} + 1 - \delta (\partial g_{n+1}/\partial y)) [C_\infty],$$

where $\delta$ is as usual the order of pole along $C_\infty$. We now compute $\delta (\partial g_{n+1}/\partial y)$ using a result of [KP00].

Definition 4.7. Let $q \in \mathbb{Q}$, $f \in \mathbb{C}[u, v]$, and $\phi(u)$ be a Puiseux series in $u$. We say that

(1) $\phi(u)$ is a root mod $q$ of $f = 0$ iff there exists a Puiseux series $\tilde{\phi}(u)$ such that $f(u, \tilde{\phi}(u)) \equiv 0 \mod q$ and $\phi(u) - \tilde{\phi}(u) = cu^q + \text{h.o.t.}$ for some $c \in \mathbb{C}$;

(2) $\phi(u)$ is a root exactly mod $q$ of $f = 0$ iff there exists a Puiseux series $\tilde{\phi}(u)$ such that $f(u, \tilde{\phi}(u)) \equiv 0 \mod q$ and $\phi(u) - \tilde{\phi}(u) = cu^q + \text{h.o.t.}$ for some $c \in \mathbb{C}, c \neq 0$.

A mod $q$ root $\phi(u)$ of $f = 0$ has multiplicity $m$ iff there are exactly $m$ distinct Puiseux series $\phi_1(u), \ldots, \phi_m(u)$ such that $f(u, \phi_k(u)) \equiv 0$ and $\phi(u) - \phi_k(u) = cu^q + \text{h.o.t.}$ for some $c_k \in \mathbb{C}$, $1 \leq k \leq m$. Similarly, $\phi(u)$ is an exactly mod $q$ root of multiplicity $m$ of $f = 0$ iff there are exactly $m$ distinct Puiseux series $\phi_1(u), \ldots, \phi_m(u)$ which satisfy the conditions of the preceding sentence with $c_k \neq 0, 1 \leq k \leq m$.

Remark 4.8. Note that the notion of mod $q$ roots and exactly mod $q$ roots have (obvious) analogues in the case of degree-wise Puiseux series: namely in Definition 4.7 replace every occurrence of ‘Puiseux series’ with ‘degree-wise Puiseux series’, and ‘h.o.t.’ with ‘l.o.t.’.

Theorem 4.9 ([KP00 Theorem 1.1]). Let $q \in \mathbb{Q}_{>0}$ and $\phi(u)$ be a Puiseux series which is a mod $q$ root of $f = 0$ of multiplicity $m \geq 1$. Then $\phi(u)$ is a mod $q$ root of $\partial f/\partial v = 0$ of multiplicity $m - 1$. 


Remark 4.10. It was assumed throughout [KP00] that $f$ is mini-regular in $v$, i.e. if $d := \text{ord}(f)$ then there is a monomial term in $f$ of the form $cv^d$ with $c \in \mathbb{C}^*$. However, the proof of Theorem 1.1 of [KP00] does not use this assumption.

Corollary 4.11. Let $g \in \mathbb{C}[x, x^{-1}, y]$ and $\phi(x)$ be a degree-wise Puiseux series in $x$ which is a mod $q$ root of $g = 0$ of multiplicity $m \geq 1$. Then $\phi(x)$ is a mod $q$ root of $\partial g/\partial y = 0$ of multiplicity $m - 1$.

Proof. Consider the (birational) change of coordinates $(u, v) = (1/x, y/x^d)$, where $d \gg 1$. Let $\tilde{g} := u^d \deg_{y}(g) g(1/u, v/u^d) \in \mathbb{C}[u, v]$. Now note that $y = \phi(x)$ is a mod $q$ root of $g = 0$ iff $v = u^d \phi(1/u)$ is a mod $d - q$ root of $\tilde{g}$. Theorem 4.9 implies that $v = u^d \phi(1/u)$ is a mod $d - q$ root of $\partial \tilde{g}/\partial v$ of multiplicity $m - 1$. Since $\partial \tilde{g}/\partial v = u^d \deg_{y}(g-1) \partial g/\partial y$, it follows that $y = \phi(x)$ is a mod $q$ root of $\partial g/\partial y = 0$ of multiplicity $m - 1$, as required.

Corollary 4.12. Let $f \in \mathbb{C}[x, x^{-1}, y]$ and $\phi(x)$ be a degree-wise Puiseux series in $x$. Let the multiplicity of $\phi(x)$ as a mod $q$ and exactly $q$ root of $f = 0$ be respectively $m'$ and $m$ (so that $m' \geq m$). Assume $m' > m \geq 1$. Then the multiplicity of $\phi(x)$ as an exactly mod $q$ root of $\partial f/\partial y = 0$ is also $m$.

Proof. Let $n := m' - m \geq 1$. Then for all sufficiently small $\epsilon > 0$, $\phi(x)$ is a mod $(q - \epsilon)$ root of $f = 0$ of multiplicity $n$, so that Corollary 4.11 implies that it is a mod $(q - \epsilon)$ root of $\partial f/\partial y = 0$ of multiplicity $n - 1$. On the other hand Corollary 4.11 also implies that $\phi(x)$ is a mod $q$ root of $\partial f/\partial y = 0$ of multiplicity $m' - 1$. It follows that $\phi(x)$ is an exactly mod $q$ root of $\partial f/\partial y = 0$ of multiplicity $m' - 1 - (n - 1) = m$.

Corollary 4.13. Let $f \in \mathbb{C}[x, x^{-1}, y]$ be monic in $y$ and have an analytically irreducible branch at infinity for which $|x| \to \infty$; in other words, assume that

$$f = \prod_{\phi_i \text{ is a conjugate of } \phi} (y - \phi_i(x)),$$

where $\phi(x)$ is a degree-wise Puiseux series in $x$. Let the Puiseux pairs of $\phi$ be $(\tilde{q}_1, \tilde{p}_1), \ldots, (\tilde{q}_k, \tilde{p}_k)$, $k \geq 1$. Set $\tilde{p}_{k+1} := 1$. Then

$$\min\{\deg_x(\phi(x) - \psi(x)) : \psi(x) \text{ is a degree-wise Puiseux root of } \partial f/\partial y = 0\} = \frac{\tilde{q}_k}{\tilde{p}_1 \cdots \tilde{p}_k}.$$

Proof. Let $\tilde{p} := \tilde{p}_1 \cdots \tilde{p}_k$. Then $f = 0$ has precisely $\tilde{p}$ degree-wise Puiseux roots in $x$, and for each $j$, $1 \leq j \leq k$, the multiplicity of $\phi(x)$ as a mod $\tilde{q}_j/(\tilde{p}_1 \cdots \tilde{p}_j)$ root (resp. as an exactly mod $\tilde{q}_j/(\tilde{p}_1 \cdots \tilde{p}_j)$ root) of $f = 0$ is $\tilde{p}_j\tilde{p}_{j+1} \cdots \tilde{p}_{k+1}$ (resp. $\tilde{p}_j\tilde{p}_{j+1} \cdots \tilde{p}_{k+1}$). Corollary 4.12 implies that for each $j$, $1 \leq j \leq k$, the multiplicity of $\phi(x)$ as an exactly mod $\tilde{q}_j/(\tilde{p}_1 \cdots \tilde{p}_j)$ root of $\partial f/\partial y = 0$ is $\tilde{p}_j\tilde{p}_{j+1} \cdots \tilde{p}_{k+1}$. The corollary follows since $\sum_{j=1}^k (\tilde{p}_j - 1)\tilde{p}_{j+1} \cdots \tilde{p}_{k+1} = \tilde{p} - 1$, and since $\partial f/\partial y$ has precisely $\tilde{p} - 1$ degree-wise Puiseux roots in $x$. 

Now we go back to the proof of Theorem 4.11. Let $\hat{\delta}(x, \xi) := \phi(x) + \xi x^r$ be the generic degree-wise Puiseux series of $\delta$ and $(q_1, p_1), \ldots, (q_{k+1}, p_{k+1})$ be the formal Puiseux pairs of $\hat{\delta}$. At first consider the case that $l = 0$, i.e. $\delta \circ \hat{\delta} \in \mathbb{C}(x(x))$. Then $g_{n+1} = y - \delta(x)$, so that $\delta(\partial g_{n+1}/\partial y) = 0$. On the other hand, $\alpha_k = 1$ for all $k \geq 1$, so that (16) follows from (18). Now assume $l \geq 1$. Recall from (3) that

$$\delta(\partial g_{n+1}/\partial y) = p \deg_x(\partial g_{n+1}/\partial y)_{y = \hat{\delta}(x, \xi)},$$

where $p := p_1 \cdots p_{l+1} = \delta(x)$. Let $\phi(x)$ be the degree-wise Puiseux root of $g_{n+1}$ from assertion 3 of Proposition 2.4. Proposition 2.4 implies that $g_{n+1}$ satisfies the assumption of Corollary 4.13.
Since \( \phi_\delta \) is a mod \( r_\delta \) root of \( g_{n+1} \) and since \( r_\delta < q_l/\langle p_1 \cdots p_l \rangle \), it follows that

\[
\deg_x \left( (\partial g_{n+1}/\partial y) \bigg|_{y=\phi_\delta(x,\xi)} \right) = \deg_x \left( (\partial g_{n+1}/\partial y) \bigg|_{y=\phi(x)} \right).
\]

It then follows from Corollary 11.13 that

\[
\delta (\partial g_{n+1}/\partial y) = p_1 \cdots p_l \sum_{k=1}^{l} (p_k - 1)p_{k+1} \cdots p_{l+1} \frac{q_k}{p_1 \cdots p_k} = \omega_{n+1} - q_{l+1} = \sum_{k=1}^{n} (\alpha_k - 1) \omega_k,
\]

where the last two equalities follows from (7) and (6). The theorem then follows from identity 118. \( \square \)

5. Groups of automorphisms and moduli spaces

**Definition 5.1** (Normal form for primitive key sequences). Let \( \vec{\omega} := (\omega_0, \ldots, \omega_{n+1}) \) be a key sequence. We prove these theorems in Sections 7.2 and 7.3. Let \( \delta \) be as in Definition 3.1. We say that \( \vec{\omega} \) is in the normal form if it satisfies one of the following (mutually exclusive) conditions:

(N0) \( n = 0 \) and

(a) \( \omega_0 = \omega_1 = 1 \), or

(b) \( \omega_0 = 1, \omega_1 = 0 \), or

(c) \( 0 < \omega_0 < \omega_1 \),

or

(N1) \( n \geq 1 \)

(a) \( 0 < \omega_0 < \omega_1 \),

(b) \( 0 < \omega_0 < \omega_1 \),

(c) \( \alpha_1 \geq 2 \), and

(d) there does not exist \( j, 1 < j \leq n \), such that either of the following holds:

(i) \( \omega_j = \sum_{i=1}^{j-1} (\alpha_i - 1) \omega_i + k \omega_0 \) for some non-negative integer \( k \), or

(ii) \( \omega_j = \sum_{i=2}^{j-1} (\alpha_i - 1) \omega_i + \alpha_1 \omega_1 - \omega_0 \).

The utility of the notion of normal forms comes from Theorems 5.2 and 5.3 below which show in particular that the semidegree associated to a primitive compactification of \( C^2 \) has a unique key sequence which is in the normal form. We prove these theorems in Sections 11.2 and 11.3.

**Theorem 5.2.** Let \( X := C^2 \) and \( \delta \) be a semidegree on \( C[X] \). Then there exist polynomial coordinates \( (x, y) \) on \( X \) such that \( \delta(x) > 0 \) and the key sequence \( \vec{\omega} := (\omega_0, \ldots, \omega_{n+1}) \) of \( \delta \) with respect to \( (x, y) \)-coordinates satisfies one of the following (mutually exclusive) properties:

(1) either \( \omega_1 < 0 < \omega_0 \), or

(2) \( \vec{\omega} \) is in the normal form.

In particular, if \( \delta \) is the semidegree corresponding to a primitive compactification of \( X \), then \( \vec{\omega} \) is in the normal form.

**Theorem 5.3.** Let \( X := C^2 \) and \( \delta \) be a semidegree on \( C[X] \). Assume there exist polynomial coordinates \( (x, y) \) on \( X \) such that the key sequence \( \vec{\omega} := (\omega_0, \ldots, \omega_{n+1}) \) of \( \delta \) with respect to it is in the normal form. Let \( (x', y') \) be another system of coordinates on \( X \) such that \( \delta(x') > 0 \), and \( \vec{\omega}' \) be the key sequence of \( \delta \) with respect to \( (x', y') \)-coordinates. Then

(1) \( \omega_0' \geq \omega_0 \).

(2) Assume \( \vec{\omega}' \) is in the normal form. Then

(a) \( \vec{\omega}' = \vec{\omega} \).

(b) Let \( \tilde{\phi}_\delta(x,\xi) = \phi(x) + \xi x^r = \sum_{\beta \leq \beta_0} a_\beta x^\beta + \xi x^r \) be the generic degree-wise Puiseux series of \( \delta \) in \( (x, y) \)-coordinates. Then the generic degree-wise Puiseux series of \( \delta \) in \( (x', y') \)-coordinates is \( \tilde{\phi}_\delta(x',\xi) = e \sum_{\beta \leq \beta_0} a_\beta a^{-\tilde{\omega}_0} x'^{\beta} + \xi x'^r \) for some \( e, a \in C^* \), where \( \tilde{\omega}_0 \) is the polydromy order of \( \phi(x) \).

(c) Let \( F: C[x, y] \to C[x, y] = C[x', y'] \) be the change of coordinates from \( (x, y) \) to \( (x', y') \) (i.e. \( (x', y') := (F_1(x, y), F_2(x, y)) \)).

(i) If \( \vec{\omega} \) satisfies (N0) with \( \omega_0 > 0 \), then \( F \) is an affine automorphism.
(ii) Otherwise \( F \) is of the form below:
\[
F : (x, y) \mapsto (a^\phi x + b, cy + f(x)), \quad \text{where},
\]
\[
f(x) \in \mathbb{C}[x], \quad \deg(f) \leq r, \quad \text{and}
\]
\[
b = \begin{cases} 
0 & \text{if } \phi \neq 0 \text{ and } \beta_0 - 1 > r, \\
\text{an arbitrary element in } \mathbb{C} & \text{otherwise}.
\end{cases}
\]

In particular, \( f \equiv 0 \) if \( r < 0 \).

(3) Conversely let \( F : \mathbb{C}[x, y] \to \mathbb{C}[x, y] \) be as in assertion \((\mathbb{R})\), i.e. either 
(a) \( \omega \) satisfies \((\mathbb{M})\) with \( \omega_0 > 0 \) and \( F \) is an affine automorphism, or 
(b) \( F \) is as in assertion \((\mathbb{R})\) for some \( a, c \in \mathbb{C}^* \).

Let \((x', y') := (F_1(x, y), F_2(x, y))\). Then \( \delta(x') > 0 \) and \((x', y')\) satisfies assertions \((\mathbb{R})\) and \((\mathbb{R})\).

Our first application of normal forms is the following result about ‘rigidity’ of \( \mathbb{C}^2 \) in a primitive compactification:

**Proposition 5.4.** Let \( \bar{X} \) be a primitive compactification of \( X := \mathbb{C}^2 \). Let \( U \) be an (open) subset of \( \bar{X} \) isomorphic to \( \mathbb{C}^2 \).

(1) If \( \bar{X} \not\cong \mathbb{P}^2(1, 1, q) \) for any integer \( q \), then \( U = X \), i.e. there exists only one open subset of \( \bar{X} \) isomorphic to \( \mathbb{C}^2 \).

(2) Assume \( \bar{X} \cong \mathbb{P}^2(1, 1, q) \), \( q \geq 1 \), with weighted homogeneous coordinates \([z : x : y]\).

(a) If \( q = 1 \), then \( U = \bar{X} \setminus v(h) \) where \( h \) is a homogeneous polynomial of degree one in \( x, y, z \).

(b) If \( q > 1 \), then \( U = \bar{X} \setminus v(h) \) where \( h \) is a homogeneous polynomial of degree one in \( x, y, z \).

**Proof.** Let \( X_0 := \bar{X} \setminus \{P_\infty\} \), where \( P_\infty \) is as in Theorem 3.6. Note that \( X_0 \) is a quasi-projective variety. We start with an (obvious!) observation:

\[(21) \quad \text{every irreducible curve on } X_0 \text{ is linearly equivalent (as a Weil divisor) to an integer multiple of } C_\infty \cap X_0, \text{ where } C_\infty := \bar{X} \setminus X. \]

Let \( \delta \) be the semidegree on \( \mathbb{C}[X] \) corresponding to \( C_\infty := \bar{X} \setminus X \). Assume there exists an open subset \( U \subseteq \bar{X} \) such that \( U \cong X \), but \( U \neq X \). We will show that assertion \((\mathbb{a})\) of the proposition holds.

Indeed, under our assumption \( C := \bar{X} \setminus U \) is the closure of an irreducible curve on \( X \) defined by some \( h \in \mathbb{C}[x, y] \). Since \( C \) is the curve at infinity with respect to \( U \), observation \((21)\) applied to \( U \) implies in particular that \( C_\infty \cap X_0 \) is linearly equivalent to an integer multiple of \( C \cap X_0 \). This implies that \( \delta(h) = 1 \) (since \( \text{div}(h) = C - \delta(h)C_\infty \)). Corollary \((2.4)\) implies that

(i) \( h \) is a linear combination of some key forms and constants, and
(ii) the curve \( h = 0 \) has only one place at infinity.

**Claim 5.4.1.** \( C \cap X \cong \mathbb{C} \).

**Proof.** If \( \bar{X} \cong \mathbb{P}^2 \), then \( C \cong \mathbb{P}^1 \), and the claim is true. So assume \( \bar{X} \not\cong \mathbb{P}^2 \). Then \( \bar{X} \) is singular, and the singular points of \( \bar{X} \) are contained in \( C_\infty \cap C \). Since \( |C_\infty \cap C| = 1 \) (due to \((\mathbb{a})\)), it follows that \( \bar{X} \) has only one singular point, which we may assume (by changing the coordinates on \( X \) if necessary) to be \( P_\infty \). Then \( P_\infty = C_\infty \cap C \), and consequently, assertion \((\mathbb{a})\) of Theorem 3.6 implies that \( C \cap X \cong \mathbb{C} \), as required. \(\square\)

Choose coordinates \((x, y)\) on \( X \) such that the key sequence of \( \delta \) is in the normal form. Claim \((5.4.1)\) and the Abhyankar-Moh-Suzuki theorem imply that

(iii) either \( h \) is a polynomial of degree 1, or
(iv) one of the integers among \( \{\deg_x(h), \deg_y(h)\} \) divides the other.

**Claim 5.4.2.** \((\mathbb{M})\) holds with \( \omega_0 = 1 \). In particular, \( \bar{X} \cong \mathbb{P}^2(1, 1, \omega_1) \).
Proof. If (N0a) occurs, then either \( \omega_0 = \delta(x) = 1 \) or \( \omega_1 = \delta(y) = 1 \). An inspection of normal forms shows that only possibility is (N0) with \( \omega_0 = 1 \). This implies that \( \bar{X} \cong \mathbb{P}^2(1,1,\omega_1) \). On the other hand, if (N0b) does not hold, it can be shown using assertion (i) and defining properties of key forms that \( \deg_x(h)/\deg_y(h) = \omega_1/\omega_0 \). Assertion (X) and the properties of normal forms then imply again that \( \omega_0 = 1 \) and \( \bar{X} \cong \mathbb{P}^2(1,1,\omega_1) \). \( \square \)

Now, if \( \omega_1 = 1 \), then assertion (2a) holds, so assume \( \omega_1 > 1 \). Then assertion (2b) follows from observation (i) and the observation that (the closure of) \( h = 0 \) passes through the (unique) singular point on \( X \). \( \square \)

**Theorem 5.5.** Fix a system of coordinates \((x,y)\) on \( X := \mathbb{C}^2 \). Let \( \tilde{\omega} := (\omega_0, \ldots, \omega_{n+1}) \) be a primitive key sequence in normal form, \( \tilde{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n \), and \( \tilde{X} := X_{\tilde{\omega}, \tilde{\theta}} \) be the corresponding primitive compactification of \( X \). Let \( G \) be the group of automorphisms of \( X \).

1. If (N0a) holds, then \( \bar{X} \cong \mathbb{P}^2 \) and \( G \cong \text{PGL}(3, \mathbb{C}) \).
2. If (N0c) holds, then \( \bar{X} \cong \mathbb{P}^2(1, \omega_0, \omega_1) \). Fix weighted homogeneous coordinates \([z : x : y]\) on \( X \).
   (a) If \( \omega_0 = 1 \), then \( G = \{ F : [z : x : y] \mapsto [az + bx : cz + dx : ey + f(x, z)] : a, b, c, d, \in \mathbb{C}, (ad - bc) \neq 0, e \in \mathbb{C}^* \} \), \( f \) is an immediate corollary of Proposition 2.13.
   (b) If \( \omega_0 > 1 \), then \( G = \{ F : [z : x : y] \mapsto [az + bx + \omega_0^2 : cy + f(x, z)] : a, c \in \mathbb{C}^* \} \), \( f \) is a primitive polynomial in \( x, z \) of degree \( \omega_0 \).
3. Assume (N1) holds. Let \( \tilde{\omega}_k := \omega_k/\omega_{n+1}, 0 \leq k \leq n \), and \( \tilde{\omega}_k := \alpha_1 \omega_1 + \sum_{j=2}^{k-1} (\alpha_j - 1) \omega_j - \tilde{\omega}_k, 2 \leq k \leq n \), where \( \alpha_1, \ldots, \alpha_{n+1} \) are as in Definition 2.11. Set \( \tilde{\omega} := \text{gcd}(\omega_2, \ldots, \omega_n) \) (note that \( \tilde{\omega} \) is defined only if \( n \geq 2 \)) and \( \tilde{K}_X := - (\omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^{n} (\alpha_k - 1) \omega_k) \).
   Then \( G \) consists of all \( F : \bar{X} \to \bar{X} \) such that \( F|_X : (x, y) \mapsto (a\omega_0 x + b, a\omega_1 y + f(x)) \), where
   \[
   a = \begin{cases} 
   \text{an arbitrary element of } \mathbb{C}^* & \text{if } n = 1, \\
   \text{an } \omega^\text{th root of unity} & \text{if } n \geq 2.
   \end{cases}
   \]
   \[
   b = \begin{cases} 
   0 & \text{if } \omega_1 + \tilde{K}_X \geq 0, \\
   \text{an arbitrary element of } \mathbb{C} & \text{otherwise}.
   \end{cases}
   \]
   \[
   f(x) \in \mathbb{C}[x], \quad \text{deg}(f) \leq (\omega_{n+1} - \sum_{k=1}^{n} (\alpha_k - 1) \omega_k)/\omega_0.
   \]

Proof. Let \( \delta \) be the semidegree on \( \mathbb{C}[x, y] \) corresponding to \( C_\infty := \bar{X} \setminus X \) and \( \tilde{\phi}_\delta(x, \xi) = \phi(x) + \xi x^r = \sum_{\beta \leq \beta_0} \alpha_\beta x^\beta + \xi x^r \) be the generic degree-wise Puiseux series of \( \delta \). The following lemma is an immediate corollary of Proposition 2.13.

**Lemma 5.6.** Let \( G \) be an automorphism of \( X \) and \( (x', y') := G(x, y) \). Then \( G \) extends to an automorphism of \( \bar{X} \) iff the generic degree-wise Puiseux series \( \tilde{\phi}_\delta(x', \xi) \) of \( \delta \) with respect to \( (x', y') \)-coordinates is conjugate to \( \tilde{\phi}_\delta(x', \xi) \). \( \square \)

Assertions (i) and (ii) are straightforward applications of Lemma 5.10, Theorem 5.3, and Proposition 5.4. We now prove assertion (iii). So assume (N1) holds. Let \( G' \) be the group consisting of all \( F \) described in assertion (i). We have to show that \( G = G' \). Pick \( F \in G \). Proposition 5.2 implies that \( F \) restricts to an automorphism of \( X \). Set \( (x', y') := F(x, y) \). Lemma 5.6 implies that the key sequence of \( \delta \) in \( (x', y') \)-coordinates is also \( \tilde{\omega} \). Theorem 5.5 then imply that \( F : (x, y) \mapsto (a\omega_0 x + b, cy + f(x)) \) where \( \tilde{\omega}_0, b, f(x) \) are as in assertion (i) of Theorem 5.5 and \( a, c \in \mathbb{C}^* \) satisfy:

\[
\sum_{\beta \leq \beta_0} a_{\beta} a^{-\tilde{\omega}_0} x^\beta \text{ is conjugate to } \phi(x) = \sum_{\beta \leq \beta_0} a_{\beta} x^\beta, \text{ or equivalently,}
\]

\[
a_{\beta} a^{-\tilde{\omega}_0} c^{-\mu} = \zeta^m \text{ for all } m \text{ such that } a_{\mu}/\omega_0 \neq 0,
\]

where \( \zeta \) is an \( \omega_0 \)-th root of unity. Note that \( \phi \neq 0 \) (due to property (N1a) of normal forms). In particular, since \( \beta_0 = \omega_0/\omega_0 \), it follows that \( c = (\zeta a)^{\omega_1} = a^{-\tilde{\omega}_1} \), where \( a' := \zeta a \). Since \( a_{\omega_0} = a^\omega_0 \), this implies that \( G \subseteq G' \) in the case that \( \phi \) has only one monomial term, or equivalently, if \( n = 1 \).
On the other hand if \( \delta \) has more than one monomial term, then it follows that \( a^{\tilde{\omega}_1 - m} = 1 \) for all \( m \) such that \( a_{m/\tilde{\omega}_0} \neq 0 \), i.e. \( a \) is an \( \tilde{\omega} \)-th root of unity \((\tilde{\omega} \) as in assertion \( 3 \)), so that \( G \supseteq G' \) in this case as well.

It remains to show \( G' \subseteq G \). So pick \( F \in G' \) and set \((x', y') := F \mid_{X(x,y)} \). Then assertion \( 3 \) of Theorem 5.3 implies that the generic degree-wise Puiseux series of \( \delta \) in \((x', y')\)-coordinates is

\[
\tilde{\phi}_\delta(x', \xi) = a^{\tilde{\omega}_1} \sum_{\beta \leq \delta_0} a_\beta a^{\tilde{\omega}_0 \beta} x^\beta + \xi x^r = \sum_{m \leq \omega_1} a_{m/\omega_0} a^{\tilde{\omega}_1 - m} x^\beta + \xi x^r
\]

Lemma 5.0 then implies \( G' \subseteq G \), as required to prove Theorem 5.6.

**Corollary 5.7.** Adopt the notations of Theorem 5.5. If \( k_X \geq -1 \) and \( n \geq 2 \), then \( X \) admits only finitely many automorphisms. In particular, every non-algebraic primitive compactification of \( \mathbb{C}^2 \) admits only finitely many automorphisms.

**Proof.** The first statement follows from assertion \( 3 \) of Theorem 5.5. For the last statement, we show that if \( X \) is non-algebraic, then \( k_X > -1 \) and \( n \geq 2 \). Indeed, if \( k_X \leq -1 \), then Corollary 4.4 implies that \( X \) has only rational singularities, so that a result of Artin [Art66] implies that \( X \) is algebraic. On the other hand, if \( n \leq 1 \), then it is straightforward to see that \( \omega \) is an algebraic key sequence, so that \( X \) is algebraic (Proposition 5.3).

**Definition 5.8.** Let \( \omega \) be a primitive key sequence in normal form. We denote by \( \mathcal{Y}_\omega \) the moduli space of primitive normal compactifications of \( \mathbb{C}^2 \) with key sequence \( \omega \). More precisely, \( \mathcal{Y}_\omega \) is the set of (isomorphism classes of) compact normal analytic surfaces \( Y \) of Picard rank 1 such that

1. \( Y \) has a subset \( X \) isomorphic to \( \mathbb{C}^2 \), and
2. if \( \delta \) is the semidegree on \( \mathbb{C}[X] \) corresponding to \( C_\infty := Y \setminus X \), then there exists a system of coordinates \((x, y)\) on \( X \) such that \( \delta(x) > 0 \) and \( \omega \) is the key sequence of \( \delta \) in \((x, y)\) coordinates.

In case that \( \omega \) is also an essential key sequence, we denote by \( \mathcal{Y}_\omega \) the union of all \( \mathcal{Y}_\omega \) such that

1. \( \omega \) is a primitive key sequence in normal form, and
2. the essential subsequence of \( \omega' \) is \( \omega \).

Finally \( \mathcal{Y}_\omega^{\text{alg}} \) is the subset of \( \mathcal{Y}_\omega \) consisting of all \( Y \in \mathcal{Y}_\omega \) such that \( Y \) is algebraic.

**Theorem 5.9.** Let \( \omega := (\omega_0, \ldots, \omega_{n+1}) \) be a primitive key sequence in normal form. Define \( \alpha_i \)'s and \( \beta_{i,j} \)'s as in Remark 5.2. Moreover, set \( \alpha_0 := 1 \). Let \( \omega := (\omega_0, \ldots, \omega_{k+1}) \) be the essential subsequence of \( \omega \). Recall that \( i_0 = 0 \) and \( i_{k+1} = n + 1 \). The normal form of \( \omega \) further implies that \( i_1 = 1 \). Let \( \mu_1, \ldots, \mu_n \) be defined as in Corollary 7.4 i.e. if \( i_k \leq i < i_{k+1} \), then \( \mu_i := \alpha_{i_0} \cdots \alpha_{i_k} \). Define \( \lambda_1, \ldots, \lambda_n \) by (22),

\[
(\lambda_1, \lambda_2, \ldots, \lambda_n) := (\lambda_1^{\mu_1}, \lambda_2^{\alpha_1}, \ldots, \lambda_n^{\alpha_n}).
\]

In particular, \( \mathcal{Y}_\omega \subseteq (\mathbb{C}^*)^{\max\{n-2,0\}} \).

3. Assume \( \omega \) is also essential, i.e. \( \omega = \omega \). Let \( \Omega \) resp. \( \Omega^{\text{alg}} \) be the collection of all integers \( \omega \) such that \( \omega \) is an element of some key sequence (resp. algebraic key sequence) \( \omega \) in normal form with essential subsequence \( \omega \). More precisely, for each \( k, 1 \leq k \leq n \), define

\[
\Omega_k := (Z(\omega_0, \ldots, \omega_k) \cap (\omega_{k+1}, k \omega_k)) \setminus \Omega^X_k,
\]

\[
\Omega^{\text{alg}}_k := (Z(\omega_0, \ldots, \omega_k) \cap (\omega_{k+1}, k \omega_k)) \setminus \Omega^X_k.
\]
Corollary 5.11. Let $\mathcal{C} \subseteq C^2$ be the space of all planar curves with one place at infinity. For $C,D \in \mathcal{C}$, write $C \sim D$ if $D = \phi(C)$ for some automorphism $\phi$ of $C^2$. The ‘embedded isomorphism classes’ of planar curves with one place at infinity is the quotient $\mathcal{C}_\sim$ of $\mathcal{C}$ by the equivalence relation $\sim$.

\begin{enumerate}
\item \begin{align}
\Omega_k^\delta \subseteq \{ \omega \in C^2 \mid (x,y) \in \overline{X}_{\omega,\theta} \}
\end{align}
\item \begin{align}
\Omega^\delta \subseteq \{ \omega \in C^2 \mid (x,y) \in \overline{X}_{\omega,\theta} \}
\end{align}
\item \begin{align}
\Omega^\delta \subseteq \{ \omega \in C^2 \mid (x,y) \in \overline{X}_{\omega,\theta} \}
\end{align}
\end{enumerate}

**Proof.** Assertion 1 follows from Theorem 5.9. Assertion 3 is an immediate corollary of assertions 1 and 2 and the definition of normal forms. We now prove assertion 2.

Assume $\omega$ satisfies (M1), and consider the surjective map $(C^*)^n \to \mathcal{X};$ which maps $\theta \mapsto \overline{X}_{\omega,\theta}$. Pick $\overline{X}_{\omega,\theta} \in (C^*)^n$ such that $\overline{X}_{\omega,\theta} \approx \overline{X}_{\omega',\theta'}$. Proposition 5.2 implies that the isomorphism $\Phi$ between $\overline{X}_{\omega,\theta}$ and $\overline{X}_{\omega',\theta'}$ fixes the (unique) $C^2$ (which we denote by $X$) inside them. The curves at infinity on $\overline{X}_{\omega,\theta}$ and $\overline{X}_{\omega',\theta'}$ induce the same semidegree (which we denote by $\delta$) on $C[X]$. By assumption there exist coordinates $(x,y)$ (resp. $(x',y')$) on $X$ such that with respect to it the key sequence of $\omega$ is $\phi$ and the key forms of $\delta$ are constructed inductively as in Proposition 3.3 by means of $x,y,\theta$ (resp. $x',y',\theta'$). Let $\phi_{x,y}(x,0) = \phi(x)$ then $\phi = \sum_{x,y} a_{x,y} x y + \xi x y$ be the generic degree-wise Puiseux series of $\omega$ with respect to $(x,y)$-coordinates. Corollary 5.3 then implies that the generic degree-wise Puiseux series of $\delta$ in $(x',y')$-coordinates is $\phi'_{x,y}(x',0) = c \sum_{x,y} a_{x,y} x y + \xi x y$ for some $c,a \in C^*$, where $\omega_0$ is the polydormy order of $\phi(x)$. Corollary 5.3 then implies that $(\theta_1',\ldots,\theta_n') = (\phi_{x,y}(x',0),\ldots,\phi_{x,y}(x',0))$. This proves assertion 2 and finishes the proof of the theorem. \hfill \Box

As an application of Theorem 5.6 we describe the moduli space of embedded isomorphism classes of planar curves with one place at infinity. Recall the notion of $\delta$-sequences from Remark 5.9. In the definition and corollary below, we write delta sequence instead of $\delta$-sequence, in order to reduce confusion arising from both the terminology ‘$\delta$-sequence’ and our convention of writing $\delta$ to denote semidegrees.

**Definition 5.10.** Let $\mathcal{C}$ be a delta sequence in normal form. We denote by $\mathcal{C}_\sim$ the space of all planar curves $C \subseteq C^2$ with one place at infinity such that the delta sequence of poles at its place at infinity is $\omega$. For $C,D \in \mathcal{C}_\sim$, we write $C \sim D$ if $D = \phi(C)$ for some automorphism $\phi$ of $C^2$. The ‘embedded isomorphism classes’ of planar curves with one place at infinity is the quotient $\mathcal{C}_\sim$ of $\mathcal{C}$ by the equivalence relation $\sim$.

**Corollary 5.11.** Let $\mathcal{C} := (\omega_0,\ldots,\omega_n)$ be a delta sequence in normal form. Set $\omega_{n+1} := 0$ and let $\Omega^\delta$ be the collection of all integers $\omega'$ such that $\omega'$ is an element of some algebraic key sequence $\omega'$ in normal form with essential subsequence $(\omega_0,\ldots,\omega_{n+1})$. More precisely, define $\alpha_i$’s and $\beta_i$’s as in Remark 5.3 and define $\alpha_0 := 1$. For each $k, 1 \leq k \leq n$, define

$$\Omega_k^\delta := (\mathbb{Z}_{\geq 0}(\omega_0,\ldots,\omega_k) \cap (\omega_{n+1},\alpha_k \omega_k)) \setminus \Omega_k,$$
where \(\mathbb{Z}_{\geq 0}(\omega_0, \ldots, \omega_k)\) denotes the semigroup generated by \(\omega_0, \ldots, \omega_k\), and

\[
\Omega_k^\times := \left\{ \sum_{j=1}^{k} (\alpha_j - 1) \omega_j + m \omega_0 : m \geq 0 \right\} \bigcup \left\{ \sum_{j=2}^{k} (\alpha_j - 1) \omega_j + \alpha_1 \omega_1 - \omega_0 \right\}.
\]

Let \(\Omega_k^{\text{alg}} := \bigcup_{k=1}^{n} \Omega_k^\times\). Let \(\Omega_k^{\text{alg}} = \{\omega'_1, \ldots, \omega'_m\}\), where \(m := |\Omega_k^{\text{alg}}|\), and set \(\omega_{m+1} := 0\).

Fix \(i, 1 \leq i \leq m + 1\). Pick (the unique) \(k\) such that \(\omega'_i \in \Omega_k^\times\). Then there exist unique integers \(\beta_{i,0}, \ldots, \beta_{i,k}\) such that \(0 \leq \beta_{i,j} < \alpha_j\) for \(1 \leq j \leq k\) and \(\omega'_i = \sum_{j=0}^{k} \beta_{i,j} \omega_j\). Define \(\mu' := \alpha_1 \cdots \alpha_k - \sum_{j=1}^{k} \alpha_0 \cdots \alpha_j \alpha_{j+1} \beta_{i,j}\). Then \(\bar{C}_i \cong ((\mathbb{C}^*)^n \times \mathbb{C}^{m+1})/(\mathbb{C}^*)^2\), where the action of \((\mathbb{C}^*)^2\) is given by

\[
(\lambda_1, \lambda_2) \cdot (\theta, \theta') := ((\lambda_1^{\mu_1} \lambda_2^{-\beta_{1,0}} \theta_1, \ldots, \lambda_1^{\mu_{m+1}} \lambda_2^{-\beta_{m+1,0}} \theta_{m+1})),
\]

where \((\theta, \theta') := (\theta_1, \ldots, \theta_n), (\theta_1', \ldots, \theta_{m+1}') \in (\mathbb{C}^*)^n \times \mathbb{C}^{m+1}.

Proof. Let \(C \subseteq \mathbb{C}^2\) be a curve with one place at infinity. Define \(L_C\) and \(\delta_C\) as in Remark 3.5. It is straightforward to see that \(L_C\) and \(\delta_C\) are independent of our choice of coordinates. Since \(\delta_C\) is non-negative on all non-zero polynomials, Theorems 5.2 and 5.3 imply that there is a unique key sequence \(\omega' := (\omega'_0, \ldots, \omega'_{n+1})\) in normal form, \(\omega'_{n+1} = 0\), and there is \(\bar{\theta} \in (\mathbb{C}^*)^{n'}\) such that \(\delta_C = \delta_{\omega'}\). Therefore, \(C\) corresponds to a unique delta sequence \(\bar{\omega} := (\omega_0, \ldots, \omega_n)\) in normal form, and the essential subsequence of \(\bar{\omega}\) is \((\omega_0, \ldots, \omega_n, 0)\). Let \(g_0', \ldots, g_{n+1}'\) be the key forms of \(\delta_C\) (in \((x', y')\)-coordinates). It follows from Proposition 2.14 that each \(g_j'\) is a polynomial. Moreover, since \(\delta_C(f) = 0 \leq \delta_C(h)\) for all \(h \in \mathbb{C}[x', y'] \setminus \{0\}\), it is follows from definition of key forms and Lemma 2.17 that \(C = V(g_{n+1}' - b')\) for some \(b' \in \mathbb{C}\). To \(C\) we associate the point \((\bar{\theta}', b') \in (\mathbb{C}^*)^{n'} \times \mathbb{C}\). The corollary will now follow from Theorem 5.9 provided we can show that the action of \((\mathbb{C}^*)^2\) on \(\bar{\theta}'\) defined in 24 extends to the required action on \((\bar{\theta}', b')\).

Indeed, assume \((\bar{\theta}', b')\) corresponds to a curve \(C''\) in some coordinates \((x'', y'')\) such that \(C'' = F(C)\), where \(F : (x', y') \mapsto (x'', y'')\) is the change of coordinates. Then \(\delta_{C''} = F^* \delta_C\), and therefore Theorem 5.9 implies that \(\bar{\theta}'' := (\lambda_1, \lambda_2) \cdot \bar{\theta}'\) for some \((\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2\), where the action is as in 24. Furthermore, since \(\omega'_{n+1} = 0\), the same arguments as in the proof of Theorem 5.9 show that \(F(x', y') = (\lambda_2 x + b, \lambda_1 y + d)\) for some \(b, d \in \mathbb{C}\), and moreover, if \(\omega' \neq (1, 0)\), then \(b = d = 0\). Now if \(\omega' = (1, 0)\), then \(n' = 0\) and \(g_{n+1}' = y'\), so that the embedded isomorphism type of \(C\) is that of the line \(y' = 0\). In particular, in this case \(\bar{C}_i\) is a singleton as prescribed by the corollary. So assume \(\omega' \neq (1, 0)\). Then \(C'' = F(C) = V(g_{n+1}' - b')\), where \(g_{n+1}'\) is the last key form of \(\delta_C\) in \((x', y')\)-coordinates. Since \(g_{n+1}'\) is monic in \(y'\) and \(g_{n+1}'\) is monic in \(y''\), it follows that \(g_{n+1}' = \lambda_1^2 g_{n+1}'(\lambda_1^{-1} x'', \lambda_1^{-1} y'')\) and consequently, \(b' = \lambda_1^2 b''\), where \(p := \deg y'(g_{n+1}')\). Now note that \(b'\) corresponds to the \(\theta_{n+1}'\)-coordinate of \(\bar{C}_i\). Since \(\omega_{n+1} = 0\), it follows that \(\beta_{n+1,j} = 0\) for all \(1 \leq j \leq n\), which implies in turn that \(\mu'_{n+1} = p\). Consequently the action of \((\mathbb{C}^*)^2\) on \(b'\) is precisely the action on \(\theta_{n+1}'\)-coordinate as stated in the corollary. This completes the proof of the corollary.

6. \(G_a^2\)-actions

The main result of this section is Theorem 6.1 below, which describes \(G_a^2\)-actions on primitive compactifications of \(\mathbb{C}^2\) which are not isomorphic to \(\mathbb{P}^2\). As a corollary we classify normal analytic \(G_a^2\)-varieties with Picard number 1.

Theorem 6.1. Let \(X := \mathbb{C}^2\) with coordinates \((x, y)\), \(\bar{X} := \bar{X}_{\omega, \bar{\theta}}\) where \(\bar{\omega} := (\omega_0, \ldots, \omega_{n+1})\) is a primitive key sequence in the normal form and \(\bar{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n\).

1. The following are equivalent:
   (a) \(X\) can be equipped with a non-trivial \(G_a^2\)-action,
   (b) \(g_{\bar{\omega}} := \omega_{n+1} - \sum_{k=1}^{n} (\alpha_k - 1) \omega_k \geq 0\).
(c) either
(i) \( \bar{X} \cong \mathbb{P}^2(1, p, q) \) for positive integers \( p \leq q \), or
(ii) \( \omega \) satisfies (X1) and \( q_2 > 0 \).

(2) Now assume \( \bar{X} \) has a non-trivial \( \mathbb{G}_a^2 \)-action and \( \bar{X} \not\cong \mathbb{P}^2 \). Then there exists a \( \mathbb{G}_a^2 \)-invariant open subset \( X' \cong \mathbb{C}^2 \) of \( \bar{X} \) and a system of coordinates \( (x, y) \) on \( X' \) such that the key sequence of \( (\text{the semidegree on } \mathbb{C}[X']) \) associated to the curve \( \bar{X} \setminus X' \) on \( \bar{X} \) in \( (x', y') \)-coordinates is \( \omega \). Set \( k_{\bar{X}} := (-\omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^{\infty} (\alpha_k - 1)\omega_k) \).

(a) If \( \omega_1 + k_{\bar{X}} \geq 0 \), then the action of \( \mathbb{G}_a^2 \) on \( X' \) is of the form

\[
(a, b) \cdot (x', y') = \left( x', y' + \sum_{i=0}^{\lfloor q_2/\omega_2 \rfloor} L_i(a, b)x^i \right),
\]

where \( L_i \)'s are linear functions on \( \mathbb{G}_a^2 \).

(b) If \( \omega_1 + k_{\bar{X}} < 0 \), then it is also possible that the action of \( \mathbb{G}_a^2 \) on \( X' \) is of the form

\[
(a, b) \cdot (x', y') = (x' + L(a, b), y' + L'(a, b)),
\]

where \( L, L' \) are linear functions on \( \mathbb{G}_a^2 \).

(c) Conversely,
(i) (24) defines a valid \( \mathbb{G}_a^2 \)-action for each choice of linear \( L_i \)'s.
(ii) If \( \omega_1 + k_{\bar{X}} < 0 \), then (25) also defines a valid \( \mathbb{G}_a^2 \)-action for each choice of linear \( L, L' \).

Proof. The equivalence of assertions (11b) and (11c) follows directly from Theorem 5.5. We now show (11a) \( \Rightarrow \) (11b). Indeed, assume (11a) holds but \( q_2 < 0 \). In particular, then (X1) holds and assertion (3) of Theorem 5.5 implies that the action of \( \mathbb{G}_a^2 \) on \( X \) is given by \( (a, b) \cdot (x, y) = (c_1(a, b)x, c_2(a, b)y) \). But then \( c_1 \) and \( c_2 \) would have to be algebraic maps from \( \mathbb{G}_a^2 \) to \( \mathbb{C}^* \). Since no such map exists, it follows that (11b) \( \Rightarrow \) (11a). The implication (11b) \( \Rightarrow \) (11a) follows from the observation that if \( q_2 \geq 0 \), then the action of \( \mathbb{G}_a^2 \) on \( X \) defined by either (24) or (25) (with \( X' := X \) and \( (x', y') := (x, y) \)) extends to \( \bar{X} \).

Now we prove assertion (2). So assume \( \bar{X} \not\cong \mathbb{P}^2 \) and there is a non-trivial \( \mathbb{G}_a^2 \)-action \( \sigma : \mathbb{G}_a^2 \times \bar{X} \rightarrow \bar{X} \).

Claim 6.1.1. There is an open subset \( X' \subseteq \bar{X} \) such that \( X' \cong \mathbb{C}^2 \) and \( X' \) is invariant under \( \sigma \).

Proof. If \( X' \not\cong \mathbb{P}^2(1, 1, q) \) for any \( q \geq 1 \), then Claim 6.1.1 holds with \( X' := X \) (due to Proposition 5.4). Now assume \( \bar{X} \cong \mathbb{P}^2(1, 1, q) \), \( q \geq 2 \). Theorem 5.5 implies that there is a weighted homogeneous system of coordinates \( [z : x : y] \) on \( \bar{X} \) such that \( \sigma \) is of the form:

\[
(a, b) \cdot [z : x : y] = \left[ a_0(a, b)x + \beta_0(a, b)z : a_1(a, b)x + \beta_1(a, b)z : c(a, b)y + \sum_{i=0}^{q} c_i(a, b)x^i y^{q-i} \right],
\]

where \( a_0, a_1, \beta_0, \beta_1, c \) and \( c_i \)'s are polynomials in \( (a, b) \). Let \( D \) be the (Cartier) divisor at infinity on \( \bar{X} \) defined by \( y^q \).

Find open affine \( U, U' \subseteq \bar{X} \setminus \text{Supp}(D) \cong \mathbb{C}^2 \) and \( V \subseteq \mathbb{G}_a^2 \) such that the action maps \( V \times U \) into \( U' \). Then the action is given on \( V \times U \) as \( (a, b) \cdot (x, y) = \left( \frac{f_1(a, b, x, y)}{g_1(a, b, x, y)}, \frac{f_2(a, b, x, y)}{g_2(a, b, x, y)} \right) \) for polynomials \( f_1, f_2, g_1, g_2 \) in \( (a, b, x, y) \). Theorem 5.5 implies that for all \( (a, b) \), \( f_1 / f_2 \) is of the form \( (ax + \beta) / (\gamma x + \delta) \) for some \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \). It follows that each \( f_2 \) is of the form \( \alpha x + \beta_1 \) for polynomials \( \alpha_i, \beta_i \) in \( (a, b) \). Similarly it follows that \( g_1 = c(a, b)y + \sum_{i=0}^{q} c_i(a, b)x^i \) and \( g_2 = f_2^q \).

Claim 6.1.2. \( \sigma^*(D) \) is the divisor of \( (a_0(a, b)x + \beta_0(a, b)z)^q \).

Proof. Let \( \gamma_i := \gcd(\alpha_i, \beta_i), i = 0, 1 \), and let \( \gamma := \gcd(c, c_0, \ldots, c_q) \). Dividing each \( g_i \) by appropriate powers of \( \gamma := \gcd(\gamma_0, \gamma_1, \gamma g) \), we may assume that \( \gamma^q \) does not divide \( g \) for each irreducible
factor $\gamma'$ of $\gamma$.

We claim that $\gamma$ is constant. Indeed, otherwise let $H := V(\gamma) \subseteq G^2$. Then for each $(a, b) \in H$, for almost all $[z : x : y] \in \bar{X}$ (namely all $[x : y : z]$ such that $c_i/\gamma'^{r}y + \sum c_i/\gamma^r x^q y^{q-r} \neq 0$, where $\gamma'$ is an irreducible factor of $\gamma$ such that $(a, b) \in V(\gamma')$ and $r$ is the highest power such that $\gamma'$ divides $g$), $(a, b) \cdot [z : x : y] = P$, where $P$ is the origin in $U_y$. But this is impossible, since each $(a, b)$ induces an automorphism. This contradiction shows that $\gamma$ is constant.

It follows that $V := V(\alpha_0(a, b)x + \beta_0(a, b)z, \alpha_1(a, b)x + \beta_1(a, b)z, c(a, b)y + \sum c_i(a, b)x^i y^{q-r})$ has codimension bigger than 1 in $G^2$. Last part of the claim follows now from identity (29a). □

Proof of Claim 6.1.3 contd. Now, since Pic($\bar{X}$) is parametrized by degree and automorphisms preserve degree of a divisor, it follows that $G^2_a$ acts trivially on Pic($\bar{X}$). In particular, $D$ is a fixed point under that action. [MPK94, Proposition 1.5] then implies that the sheaf $\mathcal{L}_D$ of sections of $D$ is linearizable, i.e. there is an automorphism $\Phi : \sigma^* \mathcal{L}_D \to \pi^* \mathcal{L}_D$ which satisfies the following ‘compatibility condition’: if $P \in \bar{X}$, $Q \in G^2_a$ and $\Phi_{P,Q}$ is the isomorphism between $(\mathcal{L}_D)_Q \to (\mathcal{L}_D)_P$ induced by $\Phi$, then

$$\Phi_{Q+Q',P} = \Phi_{Q,Q'} \circ \Phi_{Q',P} \quad (27)$$

for all $Q, Q' \in G^2_a$. Let $U_z, U_x, U_y$ be respectively be the complements of $z = 0, x = 0, y = 0$ in $\bar{X}$. Let $(u, v) := (x/z, y/z)$ be coordinates on $U_z := \bar{X} \setminus V(z) \cong \mathbb{C}^2$. Claim 6.1.2 implies that for all $(a, b) \in G^2_a$ and $(u, v) \in U_z$,

$$(\sigma^* \mathcal{L}_D)(a,b,u,v) = \frac{1}{(\alpha_0(a, b)u + \beta_0(a, b))^q} (\mathcal{O}_{G^2_a \times \bar{X}})(a,b,u,v) \quad (28)$$

so that $\Phi_{(a,b),(u,v)}$ is multiplication by $\zeta/(\alpha(a, b)u + \beta(a,b))^q$ for some $\zeta \in \mathbb{C}^*$. Then for all $(a, b), (a', b') \in G^2_a$ and $(u, v) \in U_z$ such that $(a + a', b + b') \in U_z$ and $(a', b') \cdot (u, v) \in U_z$, identities (29) and (27) imply that

$$\frac{\zeta}{(\alpha_0(a + a', b + b')u + \beta_0(a + a', b + b')^q)} = \frac{\zeta}{(\alpha_0(a', b')u + \beta_0(a', b')^q)} \quad (30)$$

for some $\zeta' \in \mathbb{C}^*$. Since $(a + a', b + b') \cdot (u, v) = (a, b) \cdot ((a', b') \cdot (u, v))$, identity (29) then implies that

$$\alpha_1(a + a', b + b') = \zeta'(\alpha_1(a, b)\alpha_1(a', b') + \beta_1(a, b)\beta_0(a', b'))$$

(28a)

$$\beta_0(a + a', b + b') = \zeta'(\alpha_0(a, b)\alpha_1(a', b') + \beta_0(a, b)\beta_0(a', b'))$$

(28b)

for some $\zeta' \in \mathbb{C}^*$. Since $(a + a', b + b') \cdot (u, v) = (a, b) \cdot ((a', b') \cdot (u, v))$, identity (29) then implies that

$$\alpha_1(a + a', b + b') = \zeta'(\alpha_1(a, b)\alpha_1(a', b') + \beta_1(a, b)\alpha_0(a', b'))$$

(29a)

$$\beta_1(a + a', b + b') = \zeta'(\alpha_1(a, b)\beta_1(a', b') + \beta_1(a, b)\beta_0(a', b'))$$

(29b)

Claim 6.1.3. $\alpha_1, \beta_0$ and $\alpha_0 \beta_1$ are constant functions on $G^2_a$.

Proof. Assume at first (to the contrary of the claim) that $\alpha_1$ a non-constant polynomial. Pick $(a'_0, b'_0) \neq (0, 0) \in G^2_a$ such that $\alpha_1(a'_0, b'_0) = 1/\zeta'$. Then (29a) implies that

$$\zeta' \alpha_0(a'_0, b'_0) \beta_1(a, b) = \alpha_1(a + a'_0, b + b'_0) - \alpha_1(a, b) \quad (30)$$

for all $(a, b) \in G^2_a$. Since $(a'_0, b'_0) \neq (0, 0)$, it follows that the right hand side of (30) is not identically zero. It then follows that $\deg(\beta_1)$ is the same as the degree of the right hand side of (30); in particular, $\deg(\beta_1) < \deg(\alpha_1)$. It follows via similar arguments that $\deg(\alpha_0) < \deg(\alpha_1)$. But then the degree in $(a, b, a', b')$ of the left hand side of (29a) is $\deg(\alpha_1)$, whereas the degree of the right hand side of (29a) is $2 \deg(\alpha_1)$, which is a contradiction. Therefore $\alpha_1$ is constant on $G^2_a$. It follows via similar arguments with identity (28b) that $\beta_0$ is also a constant function. The last part of the claim follows now from identity (29a). □
Proof of Claim 6.1.1 contd. If \( \alpha_0 \equiv 0 \) (resp. \( \beta_1 \equiv 0 \)), then Claim 6.1.1 holds with \( X' := U \) (resp. \( X' := U \)), so assume neither \( \alpha_0 \) nor \( \beta_1 \) is identically zero on \( G^2_a \). But then Claim 6.1.1 implies that both \( \alpha_0 \) and \( \beta_1 \) are constants. Let \( (t_0, t_1) \) be an eigenvector of the matrix \( \begin{pmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{pmatrix} \). Then Claim 6.1.1 holds with \( X' := \tilde{X} \setminus V(t_0x + t_1z) \). \( \square \)

Proof of Theorem 6.1 contd. Let \( X' \) be as in Claim 6.1.1. Then Claim 6.1.1 implies that with respect to some coordinate system \( (x, y) \) of \( X' \), the restriction of the action of \( G^2_a \) on \( X' \) is of the form

\[
(a, b) \cdot (x, y) = \left( \alpha(a, b)x + \beta(a, b), c(a, b)y + \sum_{i=0}^{s} c_i(a, b)x^i \right),
\]

where \( s := \lfloor g_d/\omega_0 \rfloor \). The compatibility of the action then implies that

\[
\begin{align*}
(32a) \quad & \alpha(a + a', b + b') = \alpha(a, b)\alpha(a', b') \\
(32b) \quad & c(a + a', b + b') = c(a, b)c(a', b'), \\
(32c) \quad & \beta(a + a', b + b') = \alpha(a, b)\beta(a', b') + \beta(a, b), \\
(32d) \quad & f(a + a', b + b', x) = c(a, b)f(a', b', x) + f(a, b, \alpha(a', b')x + \beta(a', b')), \text{ where} \\
(32e) \quad & f(a, b, x) := \sum_{i=0}^{s} c_i(a, b)x^i.
\end{align*}
\]

Since \( c, \alpha \) non-zero polynomials in \( (a, b) \), (32a) and (32b) imply that \( \alpha(a, b) = c(a, b) = 1 \) for all \( (a, b) \in G^2_a \). (32c) then imply that \( \beta \) is a linear function in \( (a, b) \), i.e. \( \beta(a, b) = \beta_1 a + \beta_2 b \) for some \( \beta_1, \beta_2 \in \mathbb{C} \). In particular, (32d) implies that

\[
\sum_{i=0}^{s} (c_i(a + a', b + b') - c_i(a', b'))x^i = \sum_{i=0}^{s} c_i(a, b)(x + \beta_1 a' + \beta_2 b')^i.
\]

If \( \deg_x(f) \leq 0 \), then the action is of the form of assertion 2a. On the other hand, if \( \beta \equiv 0 \), then (33) implies that \( c_i \)'s are linear and therefore the action is of the form of assertion 2b. Consequently the following claim implies assertion 2a.

Claim 6.1.4. Assume \( \beta \not\equiv 0 \). Then \( \deg_x(f) \leq 0 \).

Proof. Assume w.l.o.g. that \( \beta_2 \not\equiv 0 \). Then setting \( b' := -\frac{\beta_1 a'}{\beta_2} \) in (33) yields

\[
\sum_{i=0}^{s} \left( c_i(a + a', b - \beta_1 a') - c_i(a', -\beta_1 a') - c_i(a, b) \right)x^i = 0, \text{ so that}
\]

\[
(34) \quad c_i(a + a', b - \beta_1 a') - c_i(a', -\beta_1 a') - c_i(a, b) = 0, \quad 0 \leq i \leq s.
\]

Fix \( i, 0 \leq i \leq s' \). Let \( \tilde{c}_i := c_i \circ \sigma^{-1} \), where \( \sigma : \mathbb{C}^2 \to \mathbb{C}^2 \) maps \( (a, b) \mapsto (a, \frac{\beta_1}{\beta_2}a + b) \). Then (34) implies that

\[
(35) \quad \tilde{c}_i(a + a', b + \beta_1 a') - \tilde{c}_i(a', 0) - \tilde{c}_i(a, b + \beta_1 a) = 0.
\]

Differentiating (35) with respect to \( a' \) implies that

\[
\frac{\partial \tilde{c}_i}{\partial a}(a + a', b + \beta_1 a') - \frac{\partial \tilde{c}_i}{\partial a}(a', 0) = 0,
\]

which in turn implies that \( \frac{\partial \tilde{c}_i}{\partial a} \) is a constant. It follows that \( \tilde{c}_i(a, b) = g_i(b) + \mu_i a \) for some \( g_i \in \mathbb{C}[b] \) and \( \mu_i \in \mathbb{C} \). Plugging this into (35) further implies that

\[
g_i(0) = 0, \quad 0 \leq i \leq s.
\]
Plugging $c_1(a, b) = \tilde{c}_1 \circ \sigma(a, b) = g_i(\frac{\beta_1}{\beta_2}a + b) + \mu_i a$ into (33) gives $\sum_{i=0}^{s}(g_i(\frac{\beta_1}{\beta_2}a + b) + \mu_i a)x^i = \sum_{i=0}^{s}(g_i(\frac{\beta_1}{\beta_2}a + b) + \mu_i a)(x + \beta_1 a' + \beta_2 b')^i$, so that

$$\sum_{i=0}^{s}(\tilde{g}_i(e + e') - \tilde{g}_i(e') + \mu_i a)x^i = \sum_{i=0}^{s}(\tilde{g}_i(e) + \mu_i a)(x + e')^i,$$

where $e := \beta_1 a + \beta_2 b$, $e' := \beta_1 a' + \beta_2 b'$, and $\tilde{g}_i(b) := g_i(b/\beta_2)$, $0 \leq i \leq s$. Setting $a := 0$ in (37) and differentiating with respect to $e$ yields

$$\sum_{i=0}^{s} \tilde{g}_i'(e')x^i = \sum_{i=0}^{s} \tilde{g}_i'(e)(x + e')^i.$$ 

Substituting $e = 0$ into (38) gives

$$\sum_{i=0}^{s} \tilde{g}_i'(0)x^i = \sum_{i=0}^{s} \tilde{g}_i'(0)(x + e')^i.$$ 

In particular, $\tilde{g}_0'$ is a constant. On the other hand, substituting $x = 0$ into (38) gives $\tilde{g}_0'(e + e') = \sum_{i=0}^{s} \tilde{g}_i'(0)(e')^i$, so that $\tilde{g}_i'(e) = 0$ for $1 \leq i \leq s$. Combining this with (39) implies that

$$g_i(e) = \begin{cases} \mu_i e & \text{for } i = 0, \text{ where } \mu_i \in \mathbb{C}, \\ 0 & \text{for } 1 \leq i \leq s. \end{cases}$$

It then follows from (37) that $\sum_{i=0}^{s} \mu_i x^i = \sum_{i=0}^{s} \mu_i (x + e')^i$, which implies that $\mu_i = 0$, $1 \leq i \leq s$. It follows that $f = \mu_1 a + \mu_2 b$ for some $\mu_1, \mu_2 \in \mathbb{C}$. This completes the proof of the claim. \hfill \Box

**Corollary 6.2.** Normal analytic $\mathbb{G}_2$-varieties with Picard number 1 are precisely the surfaces $\tilde{X}_{\omega, \theta}$ for some primitive key sequence $\omega := (\omega_0, \ldots, \omega_{n+1})$ in the normal form such that $k_{\omega} + \omega_1 < 0$, where $k_{\omega} := -(\omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^{n}(\alpha_k - 1)\omega_k)$. \hfill \Box

### 7. Appendix

#### 7.1. Some properties of key sequences.

Let $\delta$ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Let $\tilde{\delta}_\phi(x, \xi) = \phi(x) + \xi x^r = \sum_{i=1}^{N} a_i x^{\beta_i} + \xi x^r$ be the generic degree-wise Puiseux series and $\tilde{\omega}$ be the key sequence of $\delta$ (in $(x, y)$-coordinates).

**Lemma 7.1.** Let $k_{\tilde{\omega}} := -(\omega_0 + \omega_{n+1} + 1 - \sum_{k=1}^{n}(\alpha_k - 1)\omega_k)$. Then

1. $k_{\tilde{\omega}} \leq 1$ iff $r \geq -1$.
2. $\beta_0 - 1 > r$ iff $\omega_1 + k_{\tilde{\omega}} \geq 0$.

**Proof.** Follows from a straightforward computation using (4). \hfill \Box

**Notation 7.2.** Let $g_0 = x, g_1 = y, g_2, \ldots, g_{n+1}$ be the key forms of $\delta$ in $(x, y)$-coordinates. For $G(y_0, \ldots, y_{n+1}) \in \mathbb{C}[y_0, y_0^{-1}, y_1, \ldots, y_{n+1}]$ and $\psi \in \mathbb{C}[x^{1/p}, x^{-1/p}, \xi]$, we write $G' = \psi$ if $\psi$ is obtained from $G(g_0, \ldots, g_{n+1})|_{y=\tilde{g}_\phi(x, \xi)}$ by discarding all terms with degree in $x$ smaller than the highest $x$-degree of all terms in which the exponent of $\xi$ is non-zero.

Let the **formal Puiseux pairs** (see Definition 2.11) of $\tilde{\delta}_\phi$ be $(q_1, p_1), \ldots, (q_{l+1}, p_{l+1})$, $l \geq 0$. Note that

(a) the essential subsequence of $\tilde{\omega}$ is of the form $\tilde{\omega}_e := (\omega_{i_0}, \ldots, \omega_{i_l})$, and
(b) $a_{i_k} = p_k$, $1 \leq i \leq l + 1$, where $\alpha_k$’s are defined as in Definition 5.1.

Let $\tilde{\theta} := (\theta_1, \ldots, \theta_n) \in (\mathbb{C}^*)^n$ be such that $\delta = \delta_{\tilde{\phi}, \tilde{\omega}, \tilde{\theta}}$ in the notation of Definition 3.4. The following lemma describes how $\tilde{\theta}$ affects the coefficients $a_1, \ldots, a_N$ of $\tilde{g}_\phi$.

**Lemma 7.3.** Let $\eta$ be the weighted degree on $\mathbb{C}[a_1, \ldots, a_N, a_1^{-1}, \ldots, a_l^{-1}]$ which assigns weight $\beta_i$ to $a_i$. Set $p_0 := 1$, $q_0 := 1$, and $\tilde{\omega}_j := \omega_j/\omega_0$, $0 \leq j \leq n + 1$. Fix $k$, $0 \leq k \leq l$.

1. For each $i$, $1 \leq i \leq k$, $y_i := \sum_{j \leq \tilde{\omega}_e} b_{i, \beta} x^{\beta} + L_i(\xi)x^r$, where each $b_{i, \beta}$ satisfies
   (a) $b_{i, \beta} \in \mathbb{C}[a_1, \ldots, a_N, a_1^{-1}, a_1^{-1}, \ldots, a_l^{-1}]$,
   (b) $b_{i, \beta}$ is homogeneous of degree $p_0 \cdots p_k$,
(c) $b_{i,\beta}$ is weighted homogeneous with respect to $\eta$ with weighted degree $\beta$.
(d) Fix $j$, $k+1 \leq j \leq l$, and define $\bar{\omega}_{i_1,j}$ to be the largest rational number $\beta$ such that $b_{i,\beta} \neq 0$ and $\beta \not\in \frac{1}{p_0 \cdots p_{j-1}} \mathbb{Z}$ (in particular, if $k+1 \leq l$, then $\bar{\omega}_{i_k+1,i_{k+1}} = \bar{\omega}_{i_{k+1}}$). Then

$$
\bar{\omega}_{i,j} = \sum_{s=1}^{k-1} p_k \cdots p_{s+1}(p_s - 1) \frac{q_s}{p_1 \cdots p_s} + (p_k - 1) \frac{q_k}{p_0 \cdots p_k} + \frac{q_j}{p_0 \cdots p_j},
$$

$$
b_{i,\bar{\omega}_{i,j}} = p_k \left( \prod_{s=1}^{k-1} p_{s+1} \cdots p_k \right) a_{ik}^{-1} \left( \prod_{s=1}^{k-1} a_{is}^{-1} \prod_{s=1}^{k-1} a_{ps+1} \cdots p_k \right).
$$

(2) Fix $i$, $\max\{1, i_k\} \leq i < i_{k+1}$. Let $\beta_{i,0}, \ldots, \beta_{i,i-1}$ be as in Remark. Set $\beta_{i,i} := 0$. Then

(a) $\theta_i \in \mathbb{C}[a_1, \ldots, a_N, a_{i-1}^{-1}, a_{i}^{-1}, \ldots, a_{i_k}^{-1}]$.

(b) $\theta_i$ is homogeneous in $a_1, \ldots, a_N$ of degree $\mu_i := p_0 \cdots p_k - \sum_{j=1}^{k} p_0 \cdots p_j - 1 - \beta_{i,i}$. Set $\beta_{i,i} := 0$.

(c) $\theta_i$ is weighted homogeneous with respect to $\eta$ with $\eta(\theta_i) = \beta_{i,0}$.

Proof. We prove the lemma by induction on $k$. It follows by definition of essential subsequence that

(i) $\beta_i \in \mathbb{Z}$ for all $i$, $1 \leq i < i_1$,

(ii) $g_i = y - \sum_{j=1}^{i-1} a_j x^\beta_j$ for all $i$, $0 < i \leq i_1$.

This implies that

(i') $y_i = \sum_{j=1}^{N} a_j x^\beta_j + \xi x^r$ for all $i$, $0 < i \leq i_1$,

(ii') $\theta_i = a_i$ for all $i$, $1 \leq i < i_1$.

In particular, it follows that Lemma holds for $k = 0$. Now assume it holds for $k$, $0 \leq k < l$.

Then

$$
y_{k+1}^{l_1,1} = \tilde{y}_{k+1}^{l_1,1} x^{l_1,1} + \text{d.t.}
$$

Recall that $p_{k+1} \bar{\omega}_{i_{k+1}} = \beta_{i_{k+1},0} + \beta_{i_{k+1},i_{k+1}} \bar{\omega}_{i_{k+1}} + \cdots + \beta_{i_{k+1},i_{k+1}} \bar{\omega}_{i_{k+1}}$. By induction hypothesis, for each $j$, $1 \leq j \leq k$,

$$
y_{i,j} = b_{i,j} x^{\bar{\omega}_{i,j}} + \text{d.t.}
$$

It follows that

$$
\theta_{i_{k+1}} = \frac{y_{k+1}^{l_1,1} \bar{\omega}_{i_{k+1}}}{b_{i_{k+1},i_{k+1}} x^{\bar{\omega}_{i_{k+1}}} + \text{d.t.}}
$$

in particular $i = i_{k+1}$ satisfies (2). Moreover,

$$
\deg(\theta_{i_{k+1}}) = p_{k+1} \deg(\bar{\omega}_{i_{k+1}}) = \sum_{j=1}^{k} \beta_{i_{k+1},i_{j}} \deg(b_{i,j} x^{\bar{\omega}_{i,j}}) = p_{k+1} \cdots p_0 - \sum_{j=1}^{k} \beta_{i_{k+1},i_{j}} p_0 \cdots p_{j-1} \cdots p_k = \mu_{i_{k+1}},
$$

$$
\eta(\theta_{i_{k+1}}) = p_{k+1} \eta(\bar{\omega}_{i_{k+1}}) = \sum_{j=1}^{k} \beta_{i_{k+1},i_{j}} \eta(b_{i,j} x^{\bar{\omega}_{i,j}}) = p_{k+1} \bar{\omega}_{i_{k+1}} = \beta_{i_{k+1},0},
$$

so that assertion (2) holds for $i = i_{k+1}$.

Now fix $i$, $i_{k+1} \leq i < i_{k+2}$, and assume both assertions (1) and (2) hold for $i$. We will show that assertion (1) holds for $i+1$. Indeed,

$$
g_{i+1} = \tilde{g}_{i+1}(x, \xi) = (g_i^{\alpha_i} - \theta_i x^{\beta_i,0} g_i^{\beta_i,1} \cdots g_i^{\beta_i,1} + \text{d.t.})_{y = \bar{\omega}_{i}(x, \xi)}
$$

\begin{equation}
= \left( \sum_{\beta} b_{i,\beta} x^{\bar{\beta}} + L_1(\xi) x^{r_1} + \text{d.t.} \right) - \theta_i x^{\beta_i,0} \prod_{j=1}^{k-1} \left( \sum_{\beta \leq \omega_{i,j}} b_{i,j} x^{\beta} + L_{i,j}(\xi) + \text{d.t.} \right) \prod_{j=1}^{k} \beta_{i,j} x^{\bar{\beta}_{i,j}}.
\end{equation}
The induction hypothesis now immediately implies assertions \((1a)\) and \((1c)\). Now note that the coefficient of each term in the expansion of the second product in \((40)\) is of the form \(\theta_i \prod_{j=1}^{k+1} \prod_{s=1}^{\beta_{i,j}} b_{i,j,s} \cdot \), and therefore has degree

\[
\deg(\theta_i) + \sum_{j=1}^{k+1} \sum_{s=1}^{\beta_{i,j}} \deg(b_{i,j,s}) = p_k \cdots p_{k+1} - \sum_{j=1}^{k+1} p_0 \cdots p_{j-1} \beta_{i,j} + \sum_{j=1}^{k+1} \beta_{i,j} p_k \cdots p_{j-1} = p_k \cdots p_{k+1}.
\]

Since

\[
\alpha_i = \begin{cases} p_k+1 & \text{if } i = i_{k+1}, \\ 1 & \text{if } i_{k+1} < i < i_{k+2}, \end{cases}
\]

the coefficient of each term from the expansion of the first product in \((40)\) is also \(\alpha_i\) degree-wise Puiseux series of \(\tilde{\text{formal Puiseux pairs}}\). Finally, let \(i_{k+1}, i_{k+2}\) be as in Remark \(3.2\)

Now we show that assertion \((2)\) holds for \(i+1\). Indeed, in that case \(\alpha_{i+1} = 1\), so that \(\tilde{\omega}_{i+1} = \beta_{i+1,0} + \beta_{i+1,1} \tilde{\omega}_i + \cdots + \beta_{i+1,i_{k+2}} \tilde{\omega}_{i_{k+1}}\). Consequently,

\[
\theta_{i+1} = \frac{b_{i+1,0} \tilde{\omega}_{i+1}}{b_{i+1,1} \tilde{\omega}_i \cdots b_{i+1,i_{k+2}} \tilde{\omega}_{i_{k+1}}},
\]

so that assertion \((2)\) for \(i+1\) now follows from the inductive hypothesis. \(\square\)

**Corollary 7.4.** Let \(\delta, \delta'\) be semidegrees on \(\mathbb{C}[x,y]\) such that \(\delta(x) > 0\) and \(\delta'(x) > 0\). Let \(\tilde{\phi}_\delta(x, \xi) = \sum_{\beta \leq \delta_0} a_{\beta} x^\beta + \xi x^r\) (resp. \(\tilde{\phi}'_{\delta}(x, \xi) = \sum_{\beta \leq \delta_0} a'_{\beta} x^\beta + \xi x^r\)) be the generic degree-wise Puiseux series, and \(\tilde{\omega} := (\omega_0, \ldots, \omega_{n+1})\) (resp. \(\tilde{\omega}' := (\omega_0', \ldots, \omega_{n+1}')\)) be the key sequence of \(\delta\) (resp. \(\delta'\)) in \((x,y)\)-coordinates. Finally, let \(\tilde{\delta} \in (\mathbb{C}^*)^n\) (resp. \(\tilde{\delta}' \in (\mathbb{C}^*)^n\)) be such that \(\delta = \tilde{\omega}_\delta\) (resp. \(\delta' = \tilde{\omega}'_{\delta'}\)) under the correspondence of Definition \(3.4\). Assume there exists \(c, a \in \mathbb{C}^*\) such that \(a_{\beta} = ca^{-\omega_0 \beta} a_{\beta}\) for all \(\beta \in \mathbb{Q}\). Then

\(1\) \(n = n'\) and \(\tilde{\omega} = \tilde{\omega}'\).

\(2\) Let the essential subsequence of \(\tilde{\omega}\) be \(\tilde{\omega}_e := (\omega_i, \ldots, \omega_{i_{k+1}})\). Recall that \(i_0 = 0\) and \(i_{l+1} = n + 1\). Let \(\alpha_1, \ldots, \alpha_{n+1}\) be as in Definition \(7.1\). Define \(\alpha_0 := 1\). Fix \(k, j, 0 \leq k \leq l, \max \{1, i_k\} \leq i < i_{k+1}\). Define \(\mu_i := \alpha_{i_0} \cdots \alpha_{i_k} - \sum_{j=1}^{k} \alpha_{i_0} \cdots \alpha_{i_{j-1}} \beta_{i,j}\). Then \(\mu_i \geq 1\) and \(\theta_i = c^{-\omega_0 \beta_{i,0}} a_{\beta_{i,0}}\), where \(\beta_{i,0}\) is as in Remark \(3.3\). \(\square\)

### 7.2. Proof of Theorem \(5.2\)

It is more or less straightforward to see (see e.g. \[Mon11\] Lemma 3.1) using Jung’s theorem on polynomial automorphisms of \(\mathbb{C}^2\) that one of the following is true:

(A) either there exist polynomial coordinates \((x, y)\) on \(X\) such that \(\delta(x) > 0 > \delta(y)\), or

(B) there exist polynomial coordinates \((x, y)\) on \(X\) such that \(\delta(x) > 0\) and the key sequence \(\tilde{\omega}\) of \(\delta\) with respect to \((x, y)\)-coordinates satisfies either \(\text{\eqref{N0}\ or properties \(\eqref{V1a}\ or \(\text{\eqref{V1c}\).\)}}\)

So w.l.o.g. we may (and will) assume that \((x, y)\) and \(\tilde{\omega}\) are as in \(\text{\eqref{B}\).\)

We will show that after a change of coordinates if necessary, the key sequence of \(\delta\) satisfies \(\text{\eqref{N1}\). \\text{\end{proof}\)}

\[
\tilde{\phi}_\delta(x, \xi) = \sum_{\beta \leq \delta_0} a_{\beta} x^\beta + \xi x^r.
\]

Let the formal Puiseux pairs (see Definition \(2.11\)) of \(\tilde{\phi}_\delta\) be \((q_1, p_{1}), \ldots, (q_{l+1}, p_{l+1})\). Then note that

\(a\) \(\alpha_{i_{k}} = p_k, 1 \leq i \leq l + 1\), where \(\alpha_{j}\)’s are defined as in Definition 3.1.
Define
\[ S_k := \left\{ \frac{q}{p_1 \cdots p_k} : q > q_k + 1/p_{k+1} \right\}, \quad 1 \leq k \leq l, \]
\[ S := \bigcup_{k=1}^{l} S_k. \]

For each \( \beta \in S_k, 1 \leq k \leq l \), define \( \hat{\omega}_\beta := \sum_{j=1}^{k} (p_j - 1) \omega_{i_j} + p_1 \cdots p_{l+1} \beta \). Claim \(7.5\) below follows from Lemma \(7.3\) via straightforward but long computations - here we omit the proof.

**Claim 7.5.** Fix \( \beta \in S \). For each \( \lambda \in \mathbb{C} \), let \( \delta_\lambda \) be the semidegree on \( \mathbb{C}[x, y] \) corresponding to the generic degree-wise Puiseux series \( \tilde{\phi}_\lambda(x, \xi) := \tilde{\phi}_\beta(x, \xi) + \lambda x^\beta \); in particular, \( \delta_0 = \delta \). Then

1. For each \( \lambda \in \mathbb{C} \), the formal Puiseux pairs of \( \tilde{\phi}_\lambda \) are identical to those of \( \tilde{\phi}_\beta \). Equivalently, for each \( \lambda \in \mathbb{C} \), the essential subsequence of the key sequence of \( \delta_\lambda \) is identical to that of \( \delta \).
2. There is a unique \( \lambda^* \in \mathbb{C} \) such that key sequence \( \tilde{\omega}_\lambda^* := (\omega_{\lambda, 0}, \ldots, \omega_{\lambda, n_{\lambda^*} + 1}) \) of \( \delta_\lambda \) satisfies the following: there does not exist \( j, 0 \leq j \leq n_{\lambda^*} + 1 \), such that \( \omega_{\lambda, j} = \omega_\beta \).
3. Assume \( \beta \in S_k, 1 \leq k \leq l \). Then \( \lambda^* = \frac{\hat{\omega}_\beta}{p} - a_\beta \), where
   a) \( \Psi \) is a homogeneous polynomial of degree \( p_k^* := p_1 \cdots p_k \) for \( \beta > \beta \) such that each monomial \( a_{\beta^*} \), which appears in \( \Psi \) with non-zero coefficient satisfies:
   \[ \beta_1 + \cdots + \beta_{p_k^*} = \omega_\beta/p, \]
   where \( p := p_1 \cdots p_{l+1} \).
   b) \( \mu \) is a monomial of degree \( p_1 \cdots p_k - 1 \) in \( a_{\beta^*}, \ldots, a_{\beta_{p_k^*}^*} \), where \( \beta_{j}^* := q_j/(p_1 \cdots p_j) \), \( 1 \leq j \leq k \), are the first \( k \) Puiseux exponents of \( \tilde{\phi}_\beta \). Moreover, if the exponent of \( \beta_{j}^* \) in \( \mu \) is \( e_j \), \( 1 \leq j \leq k \), then \( \sum e_j \beta^*_j = \omega_\beta/p - \beta \).
4. There is a unique positive integer \( j^* \geq 2 \) such that
   a) for each \( \lambda \neq \lambda^* \in \mathbb{C}, \omega_{\lambda, j^*} = \omega_\beta \),
   b) for each \( \lambda \in \mathbb{C} \), the initial \( j^* \)-element subsequence \( (\omega_{\lambda, 0}, \ldots, \omega_{\lambda, j^* - 1}) \) of \( \tilde{\omega}_\lambda \) is identical.

**Corollary 7.6.** Let \( S' := \{ \beta_1, \ldots, \beta_k \} \subseteq S \). For each \( \lambda := (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k \), let \( \delta_\lambda \) be the semidegree on \( \mathbb{C}[x, y] \) corresponding to the generic degree-wise Puiseux series of the form
\[ \tilde{\phi}_\lambda(x, \xi) := \tilde{\phi}_\beta(x, \xi) + \sum_{j=1}^{k} \lambda_j x^{\beta_j} + \sum_{\beta \in S' \setminus S} c_\beta (\lambda_1, \ldots, \lambda_k) x^\beta, \]
where for each \( j, 1 \leq j \leq k \), \( c_\beta \) does not depend on \( \lambda_j \) for each \( \beta \in S \) such that \( \beta > \beta_j \). Then there exists unique \( \lambda^* := (\lambda^*_1, \ldots, \lambda^*_k) \in \mathbb{C}^k \) such that none of the elements in the key sequence of \( \delta_{\lambda^*} \) equals \( \omega_{\beta_j} \) for any \( j, 1 \leq j \leq k \).

**Proof.** Assume w.l.o.g. \( \beta_1 > \cdots > \beta_k \). The existence of \( \lambda^* \) follows from applying Claim \(7.5\) to \( (\delta, \beta_1) \), then to \( (\delta_{\lambda^*_1}, \beta_2) \), and so on. The uniqueness then follows from another application of Claim \(7.5\) to \( \delta_{\lambda^*_2} \) \( \square \)

**Proof of Theorem 7.2 contd.** Recall that by assumption \( \omega \) satisfies properties \( \{1.1, 1.1\} \). It follows that the key sequence of \( \delta \) with respect to another system \( (x', y') \) of coordinates continues to satisfy properties \( \{1.1, 1.1\} \) provided \( x' = x + \delta_2 \) and \( y' = y + f(x) \) for some \( \delta_2 \in \mathbb{C} \) and \( f \in \mathbb{C}[x] \) with \( \deg(f) < q_1/p_1 = \deg_{x} (\tilde{\phi}_\beta) \). Let \( S' := S \cap (\mathbb{Z}_{\geq 0} \cup \{q_1/p_1 - 1\}) \). Then, as \( f \) and \( \beta \) varies through all possible choices, the family of generic degree-wise Puiseux series \( \tilde{\phi}_\beta(x', \xi) \) satisfies the hypothesis of Corollary \(7.4\) for \( S' \). Corollary \(7.4\) then implies that there is a choice of \( f \) and \( \beta \) such that the no key form of \( \delta \) with respect to \( (x', y') \)-coordinates has \( \delta \)-value equal to \( \omega_\beta \) for any \( \beta \in S' \). But then Lemma \(7.7\) below (which follows from a straightforward computation) completes the proof of Theorem 7.2. \( \square \)

**Lemma 7.7.** Let \( S' := S \cap (\mathbb{Z}_{\geq 0} \cup \{q_1/p_1 - 1\}) \), where \( S \) is as in Claim \(7.3\) Then \( \omega := (\omega_0, \ldots, \omega_{n+1}) \) satisfies condition \( \{1.1\} \) iff \( \omega_j \neq \omega_\beta \) for any \( \beta \in S' \), and each \( j, 0 \leq j \leq n + 1 \). \( \square \)
7.3. Proof of Theorem 5.3.

Proof of assertion 3 of Theorem 5.3 Consider the set up of assertion 2 of Theorem 5.3. Let $F: \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y] = \mathbb{C}[x', y']$ be the change of coordinates from $(x, y)$ to $(x', y')$ (i.e. $(x', y') := (F_1(x, y), F_2(x, y))$). We start with a claim:

Claim 7.8.

1. If $\omega$ satisfies $(V0a)$, then $\omega'$ also satisfies $(V0a)$ and $F$ is an affine automorphism.
2. If $\omega$ does not satisfy $(V0a)$, then $F(x, y) = (ax + b, cy + f(x))$ for some $a, c \in \mathbb{C}^*$, $b, c \in \mathbb{C}$, and $f \in \mathbb{C}[x]$ with $\deg(f) \leq \deg_x(\phi_b)$.
3. If $\omega$ satisfies $(V0a)$, then $\omega' = \omega$.

Proof. It follows from Jung’s theorem [Jun42] on polynomial automorphisms of $\mathbb{C}[x, y]$ that $F$ has a factorization of the form

$$F = F_k \circ \cdots \circ F_1$$

where each $F_j$ is either an affine map of the form

(Type I) \hspace{1cm} (u, v) \mapsto (au + bv + c, a'u + b'v + c'), \hspace{1cm} a, b, c, a', b', c' \in \mathbb{C}

or a map of the form

(Type II) \hspace{1cm} (u, v) \mapsto (au + b, cv + f(u)), \hspace{1cm} a, b, c \in \mathbb{C}, \hspace{0.5cm} f(u) \in \mathbb{C}[u]

Let us take a factorization of $F$ as in (11) such that the length $k$ of the factorization is the minimum. It then follows that

1. (min-1) (Type I) and (Type II) maps alternate, i.e. $F_j$ is (Type I) if $F_{j+1}$ is (Type II) for all $j$.
2. (min-2) if $F_j$ is (Type II), then $\deg(f) > 1$, and
3. (min-3) if there exists $j$ such that $F_j$ is (Type I) with $b = 0$, then in fact $j = k = 1$ and $F = F_1$.

Now we show that $k = 1$. For all $j$, $1 \leq j \leq k$, let us denote by $(x_j, y_j)$ the coordinates of $F_j \circ \cdots \circ F_1$, by $\omega_j := (\omega_0^j, \ldots, \omega_{n_j}^j)$ the key sequence of $\delta$ in $(x_j, y_j)$ coordinates, and by $\tilde{\phi}_j(x_j, \xi) = \phi_j(x_j) + \xi x_j^{\gamma}$ the generic degree-wise Puiseux series of $\delta$ in $(x_j, y_j)$ coordinates. Note that by assumption

1. $\delta(x) = \omega_0 \leq \omega_1 = \delta(y)$, or equivalently,
2. $\deg_x(\tilde{\phi}) \geq 1$.

Claim 7.8.1. Assume $k > 1$. Then there exists $j$, $1 \leq j \leq k$, such that

1. $F_j$ is (Type II).
2. $s_j := \deg_x(\tilde{\phi}_j) = \omega_j^1/\omega_0^1$ is an integer $\geq 2$, and
3. $\tilde{\phi}_j \neq 0$, or equivalently, $n_j \geq 1$.

Proof. If $F_1$ is (Type I), then (min-3) implies that $b \neq 0$, so that $\delta(x_1) = \delta(y)$. If $b' \neq 0$, then $\delta(x_1) = \delta(y) = \delta(y_1)$ and therefore $\deg_{x_1}(\phi_1) = 1$. It then follows from (min-2) that $j = 1$ satisfies the claim. On the other hand, if $b' = 0$, then $\delta(y_1) = \delta(x) \leq \delta(x_1)$, so that $\deg_{x_1}(\tilde{\phi}_1) = \delta(y_1)/\delta(x_1) \leq 1$. Consequently (min-2) implies that $j = 2$ satisfies the claim.

Now assume $F_1$ is (Type II). If $\deg(f) > \deg_x(\tilde{\phi}_b)$, then $j = 1$ satisfies the claim. So assume $\deg(f) \leq \deg_x(\tilde{\phi}_b)$. Then assertion (min-2) and the defining properties of normal forms imply that $\delta(y_1) = \delta(y) > \delta(x) = \delta(x_1)$. Now $F_2$ is of (Type I), and it follows as in the preceding paragraph that

1. (a) if $b' \neq 0$, then $\deg_x(\tilde{\phi}_2) = 1$, or equivalently, $\omega_0^2 = \omega_1^2$, and
2. (b) if $b' = 0$, then $\deg_x(\tilde{\phi}_2) < 1$, or equivalently, $\omega_0^2 > \omega_1^2$.

Moreover, since $\delta(y_1) > \delta(x_1)$, it follows that in the situation of Case (11), we must have $\phi_2 \neq 0$, or equivalently, $n_2 \geq 1$, i.e. $\omega^2$ violates condition (V0). On the other hand, in Case (13) $\omega^2$ violates condition (V10). In particular, it follows that $\omega^2$ is not in the normal form, so that $k \geq 3$. Now (min-2) implies that $j = 3$ satisfies the claim.
Claim 7.8.2. Let \( j \) be as in Claim 7.8.1. Then \( k \geq j + 2 \). Moreover \( j + 2 \) also satisfies Claim 7.8.1. 

Proof. Let \( j \) be as in Claim 7.8.1. Then \( \omega_j \) is not in the normal form, so that \( k \geq j + 1 \). \( F_{j+1} \) is [Type I] with \( b \neq 0 \), so that \( \delta(x_{j+1}) = \delta(y_j) = s_j \delta(x_j) \). At first assume \( b' = 0 \). Then \( \delta(y_{j+1}) = \delta(x_j) \). It follows that \( \omega_j^{j+1} = s_j \omega_j^{j+1} \). Consequently, \( \omega_j^{j+1} \) violates \((N0a)\) and \((N1a)\), so that \( \omega_j^{j+1} \) is not in the normal form. In particular, \( k \geq j + 1 \). Then \((\text{min-2})\) ensures that \( j + 2 \) satisfies Claim 7.8.1. On the other hand, if \( b' \neq 0 \), then \( \delta(y_{j+1}) = \delta(y_j) \), which implies in turn that \( \omega_j^{j+1} = \omega_j^{j+1} \) and \( n_{j+1} \geq 1 \). It then follows by the same reasoning as the \( b' = 0 \) case that \( k > j + 1 \) and \( j + 2 \) satisfies Claim 7.8.1. \( \square \)

Since \( (x_k, y_k) = (x', y') \) and \( \phi' \) is in the normal form, Claims 7.8.1 and 7.8.2 imply that \( k = 1 \), i.e. \( F = F_1 \). Now we consider two cases:

Case 1: \( \omega \) satisfies \((N0a)\). In this case, if \( F \) is of \([\text{Type I}] \), then \( \deg(f) = 1 \) (for otherwise \( \omega_0 \omega' \omega(0) = \delta(y') \delta(x') \) would be an integer \( \geq 0 \) and \( n' \) would be \( \geq 1 \), so that \( (x', y') \) would not be in the normal form). This implies assertion \([\text{I}]\) of Claim 7.8.1.

Case 2: \( \omega \) does not satisfy \((N0a)\). In this case \( \delta(x) < \delta(y) \), and therefore, if \( F \) is of \([\text{Type I}] \), then \( b = 0 \) (for otherwise \( \omega_0 = \omega_1 \) and \( n' \geq 1 \), so that \( (x', y') \) would not be in the normal form). It follows that \( F \) is \([\text{Type II}] \), as required to prove assertion \([\text{II}]\) of Claim 7.8.1 follows from assertion \([\text{II}]\) in a straightforward manner. \( \square \)

Proof of assertion \([\text{II}]\) of Theorem 5.3 contd. In the case that \( \omega \) satisfies \((N0) \), assertion \([\text{II}]\) of Theorem 5.3 follows from Claim 7.8.1. So assume \( \omega \) satisfies \((N1) \). Let \( F : (x, y) \to (x', y') \) be as in Claim 7.8.1. We now compute \( \phi'_b(x', \xi) \).

\[
\phi'_b(x', \xi) = c \sum_{\beta \leq \beta_0} a_\beta a^{-\beta} x'^{\beta} \left( 1 - \beta b/x' + \beta (1 - b^2/2x'^2) + \cdots \right) \sum_{s = [r]} \lambda_s x'^s + \xi x'^r,
\]

where \( \lambda_s \in \mathbb{C} \) for all each \( s \), \( [r] \leq s \leq [\beta_0] \). Now, if \( [\beta_0] > r \), then there is nothing left to prove, so assume \( [\beta_0] > r \). Then

\[
\phi'_b(x', \xi) = c \sum_{\beta \leq \beta_0} a_\beta a^{-\beta} x'^{\beta} + \left( ca_{[\beta_0]} a^{-[\beta_0]} + \left[ \lambda_{[\beta_0]} \right] a^{[\beta_0]} \right) x'^{[\beta_0]} + 1 \text{d.t.}
\]

Now, since \( \omega \) satisfies \((N1d) \), Lemma 7.7 implies that \( \omega_j \neq \omega_{[\beta_0]} \) for any \( j \), \( 0 \leq j \leq n + 1 \). Assertion \([\text{III}]\) of Claim 7.8.1 implies that \( a_{[\beta_0]} = \Psi(a_{[\beta_0]}, \ldots, a_{[\beta_s]})/\mu(a_{[\beta_0]}, \ldots, a_{[\beta_s]}) \), where \( \beta_0, \ldots, \beta_s \) are the exponents \( > [\beta_0] \) that appear in \( \Psi_b \) with non-zero coefficients. Since \( \omega \) also satisfies \((N1d) \), it follows via assertion \([\text{III}]\) of Claim 7.8.1 that

\[
ca_{[\beta_0]} a^{-[\beta_0]} + \lambda_{[\beta_0]} = \frac{\Psi(a_{[\beta_0]}, \ldots, a_{[\beta_0]})}{\mu(a_{[\beta_0]}, \ldots, a_{[\beta_0]})} = \frac{c^{[\beta_0]} a^{-\omega_{[\beta_0]}/p} \Psi(a_{[\beta_0]}, \ldots, a_{[\beta_0]})}{c^{[\beta_0]} a^{-\omega_{[\beta_0]}/p + [\beta_0]} \mu(a_{[\beta_0]}, \ldots, a_{[\beta_0]})} \ (\text{due to homogeneity of } \Psi \text{ and } \mu)
\]

ca_{[\beta_0]} a^{-[\beta_0]},

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so that $\lambda_{|\beta_0|} = 0$. This implies that the coefficient in $\tilde{\partial}_\delta(x, \xi) \frac{\partial}{\partial x^\beta}$ is $ca^{-\beta}$ for all $\beta \geq |\beta_0|$. Note that $|\beta_0|$ is the maximum of all elements in $S'$ (where $S'$ is from Lemma 7.17). Applying the preceding arguments one by one (in linear order from the largest to the smallest) to the elements of $S'$, it follows that the coefficient in $\tilde{\partial}_\delta(x, \xi) \frac{\partial}{\partial x^\beta}$ is $ca^{-\beta}$ for all $\beta \in \mathbb{Q}$. This completes the proof of assertion 2b of Theorem 5.3. The rest of assertion 2 now follows via straightforward computations. □

Proof of assertion 3 of Theorem 5.3. Let $F$ and $(x', y')$ be as in assertion 3 of Theorem 5.3. Then it is immediate to see that $(x', y')$ satisfies assertion 2b of Theorem 5.3 which in turn implies via Corollary 7.4 that the key sequence of $\delta$ in $(x', y')$ coordinates is $\bar{\omega}$.

Proof of assertion 1 of Theorem 5.3. Assertion 1 of Theorem 5.3 follows from uniqueness of normal forms (assertion 2a) and the following observation: “If $\omega'$ is a key sequence of $\delta$ (with respect to some coordinate system on $X$), and if $\omega''$ is a key sequence of $\delta$ that satisfies either (N0) or properties (N1a)–(N1c), then $\omega_0'' \leq \omega_0'$” (see e.g. the proof of [Mon11 Lemma 3.1]). □

REFERENCES

[Abh77] S. S. Abhyankar. Lectures on expansion techniques in algebraic geometry, volume 57 of Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Tata Institute of Fundamental Research, Bombay, 1977. Notes by Balwant Singh. [Art66] Michael Artin. On isolated rational singularities of surfaces. Amer. J. Math., 88:129–136, 1966. [BDP81] Lawrence Brenton, Daniel Drucker, and Geert C. E. Prins. Graph theoretic techniques in algebraic geometry and commutative algebra. 9 Mathematics. Springer-Verlag, New York, second edition, 1997. An introduction to computational algebraic geometry and commutative algebra. [Bre73] Lawrence Brenton. A note on compactifications of $C^2$. Math. Ann., 206:303–310, 1973. [Bre80] Lawrence Brenton. On singular complex surfaces with negative canonical bundle, with applications to 2-dimensional rational singularities. Math. Ann., 248(2):117–124, 1980. [CA00] Eduardo Casas-Alvero. Singularities of plane curves, volume 276 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2000. [Clo97] David Cox, John Little, and Donal O’Shea. Ideals, varieties, and algorithms. Undergraduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1997. An introduction to computational algebraic geometry and commutative algebra. [FJ04] Charles Favre and Mattias Jonsson. The valuative tree, volume 1853 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2004. [FJ11] Charles Favre and Mattias Jonsson. Dynamical compactifications of $C^2$. Ann. of Math. (2), 173(1):211–248, 2011. [Fur97] Mikio Furushima. On minimal compactifications of $C^2$. Math. Nachr., 186:115–129, 1997. [HT99] Brendan Hassett and Yuri Tschinkel. Geometry of equivariant compactifications of $C^2$. Internat. Math. Res. Notices, (22):1211–1230, 1999. [Jun42] Heinrich W. E. Jung. Über ganze birationale Transformationen der Ebene. J. Reine Angew. Math., 184:161–174, 1942. [Koj01] Hideo Kojima. Minimal singular compactifications of the affine plane. Nihonkai Math. J., 12(2):165–195, 2001. [KPT] T.-C. Kuo and A. Parusiński. Newton polygon relative to an arc. In Real and complex singularities (São Carlos, 1998), volume 412 of Chapman & Hall/CRC Res. Notes Math., pages 76–93. Chapman & Hall/CRC, Boca Raton, FL, 2000. [KT09] Hideo Kojima and Takeshi Takahashi. Notes on minimal compactifications of the affine plane. Ann. Mat. Pura Appl. (4), 188(1):153–169, 2009. [MFK94] D. Mumford, J. Fogarty, and F. Kirwan. Geometric invariant theory, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, third edition, 1994. [Moh74] T. T. Moh. On analytic irreducibility at infinity of a pencil of curves. Proc. Amer. Math. Soc., 44:22–24, 1974. [Mon11] Pinaki Mondal. Analytic compactifications of $C^2$ part I - curvettes at infinity. http://arxiv.org/abs/1110.6905 2011. [Mon13] Pinaki Mondal. An effective criterion for algebraic contracibility of rational curves. http://arxiv.org/abs/1301.0126 2013. [Mor72] James A. Morrow. Compactifications of $C^2$. Bull. Amer. Math. Soc., 78:813–816, 1972. [MZ88] Masayoshi Miyanishi and De-Qi Zhang. Gorenstein log del Pezzo surfaces of rank one. J. Algebra, 118(1):63–84, 1988.
[Oht01] Tomoaki Ohta. Normal hypersurfaces as a compactification of $\mathbb{C}^2$. *Kyushu J. Math.*, 55(1):165–181, 2001.

[Oka98] Mutsuo Oka. Moduli space of smooth affine curves of a given genus with one place at infinity. In *Singularities (Oberwolfach, 1996)*, volume 162 of *Progr. Math.*, pages 409–434. Birkhäuser, Basel, 1998.

[Sat77] Avinash Sathaye. On planar curves. *Amer. J. Math.*, 99(5):1105–1135, 1977.

[Suz99] Masakazu Suzuki. Affine plane curves with one place at infinity. *Ann. Inst. Fourier (Grenoble)*, 49(2):375–404, 1999.