Einstein-Podolsky-Rosen uncertainty limits for bipartite multimode states

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Certification and quantification of correlations for multipartite states of quantum systems appear to be a central task in quantum information theory. We give here a unitary quantum-mechanical perspective of both entanglement and Einstein-Podolsky-Rosen (EPR) steering of continuous-variable multimode states. This originates in the Heisenberg uncertainty relations for the canonical quadrature operators of the modes. Correlations of two-party \((N \text{ vs } 1)\)-mode states are examined by using the variances of a pair of suitable EPR-like observables. It turns out that the uncertainty sum of these nonlocal variables is bounded from below by local uncertainties and is strengthened differently for separable states and for each one-way unsteerable ones. The analysis of the minimal properly normalized sums of these variances yields necessary conditions of separability and EPR unsteerability of \((N \text{ vs } 1)\)-mode states in both possible ways of steering. When the states and the performed measurements are Gaussian, then these conditions are precisely the previously-known criteria of separability and one-way unsteerability.

I. INTRODUCTION

The use of uncertainty arguments to understand quantum properties of composite systems was initiated in the seminal work of Einstein, Podolsky and Rosen [1] and deepened by Schrödinger, shortly thereafter [2]. Specifically, in Refs. [1, 2] the following paradox of quantum mechanics is described: two distant parties of a given system share an entangled state and one party, by measuring its subsystem, can remotely change (steer) the state of the other party’s subsystem, in contrast to the requirements of the local realism. Many years after, Bell proved that no local realistic theory can give a complete description of the predictions of quantum mechanics (Bell’s theorem) [3]. A further contribution in defining quantum correlations between the parties of a composite system was made by Werner [4]. What was subsequently known as a separable state, namely, a convex combination of product states, was termed by Werner as being classically correlated. Werner’s original terminology for all the states not having such an expansion has been Einstein-Podolsky-Rosen (EPR) correlated states. Nowadays they are called entangled. Soon after Werner’s definition, separability conditions were found for finite-dimensional systems in terms of 2-entropy inequalities [5, 6] or preservation of the non-negativity of the density matrix under partial transposion [7, 8]. Note that in 1989, a practical procedure to demonstrate the EPR paradox by using products of inferred variances was first proposed by Reid [9] for continuous-variable bipartite states.

A significant progress in understanding quantum correlations manifested through entanglement, EPR steering, and Bell non-locality, in the framework of a unified quantum-information description was given by Wiseman et al. as three rather different tasks for confirming inseparability [10–12]. First, for certification of two-party entanglement, use is made of the general aspect of a separable state [4] and quantum state tomography is performed with all-trusted measurement devices. Second, quantum steering corresponds to the task of verifiable entanglement distribution by one untrusted party. Third, in the case of Bell nonlocality, both parties do not trust their measuring devices. In the EPR steering scenario, the two parties, Alice and Bob, play a different role and thus EPR steering is asymmetric when interchanging the subsystems, unlike entanglement and Bell nonlocality [11]. Being perceived as another type of quantum correlations, EPR steering has recently attracted considerable interest especially related to its verification and quantification from a quantum information perspective [13–22]. Note that for continuous-variable settings, much attention was given from the very beginning to the conditions and quantifiers of steering for Gaussian states (GSs) [9, 11, 23–28].

In the present paper we take a different approach that treats on an equal footing entanglement and steering. This generalizes and enlarges the EPR-line of reasoning initiated by Reid for a two-mode state [9]. Essentially, Reid used the variances of two nonlocal observables linearly built with the one-mode canonical quadrature operators \(\hat{q}_j, \hat{p}_j, (j = 1, 2)\):

\[
\hat{Q}(\lambda) := \hat{q}_1 - \lambda \hat{q}_2, \quad \hat{P}(\mu) := \hat{p}_1 + \mu \hat{p}_2, \quad (1.1)
\]

where \(\lambda\) and \(\mu\) are adjustable positive parameters. The coordinates and momenta in Eq. (1.1) are defined in terms of the amplitude operators of the modes:

\[
\hat{q}_j := \frac{1}{\sqrt{2}}(\hat{a}_j + \hat{a}_j^\dagger), \quad \hat{p}_j := \frac{1}{\sqrt{2}i}(\hat{a}_j - \hat{a}_j^\dagger). \quad (1.2)
\]

Unless \(\lambda \mu = 1\), the operators (1.1) are not proper EPR observables since they do not commute. Consequently, we get the weak (Heisenberg) form of the uncertainty relation (UR)

\[
\Delta Q(\lambda) \Delta P(\mu) \geq \frac{1}{2} |1 - \lambda \mu|, \quad (1.3)
\]
which has to be fulfilled by any quantum state. In Eq. (1.3) and in the sequel as well, $\Delta A$ denotes the standard deviation of the observable $\hat{A}$ in the state $\hat{\rho}$, which is the square root of the variance

$$\left(\Delta A\right)^2 := \langle \left(\hat{A} - \langle\hat{A}\rangle\right)^2 \rangle = \langle \hat{A}^2 \rangle - \langle\hat{A}\rangle^2. \quad (1.4)$$

In Refs. [9, 23], a possible experimental observation of the inequality

$$\Delta Q(\lambda) \Delta P(\mu) < \frac{1}{2} \quad (1.5)$$

was interpreted as a signature of detecting an EPR paradox. Moreover, the EPR paradox as invoked by Reid and the concept of EPR steering as analyzed in Refs. [10–12] were proven to be equivalent. The way this equivalence arises from two different perspectives became a first reason for our interest in understanding the modification of the URs for unsteerable states. A second motivation for the present work stems from a recent result on EPR-like URs valid for separable two-mode states [29]. Essentially, in Ref. [29] it was proven that the minimum normalized product and sum of the uncertainties of appropriate EPR-like observables are separability indicators for two-mode GSs, due to the Peres-Simon separability theorem. According to this, such a state is separable if and only if its density matrix has a positive-semidefinite partial transpose (PPT) [8, 30]. Consequently, the remainder of this paper develops an approach for a parallel study of the separability and unsteerability of two-party ($N$ vs 1)-multimode states. This study exploits the sum-form URs for a pair of appropriate EPR-like observables. In Sec. II we extend the Reid treatment of two-mode states to the larger class of two-party ($N$ vs 1)-mode states. We thus define two linear combinations of canonical quadrature operators of the same kind, each one depending on $N + 1$ positive parameters, and then write their URs in an arbitrary bipartite ($N$ vs 1)-mode state. In Sec. III we apply an important theorem of Hofmann and Takeuchi [31] to find out the enhancement of the URs written for a separable ($N$ vs 1)-mode state. Accordingly, a modification of the lower bound of URs for separable states is imposed by observing the local uncertainty relations (LURs). By extremization of the normalized uncertainty sum with respect to the parameters, we then establish a necessary condition of separability which appears to reduce to the Peres PPT one. Section IV is devoted to an analysis of the steering process which aims to enable us to put forward the efficient way of applying the Hofmann-Takeuchi theorem in the case of one-way unsteerability. In Sec. V we perform the extremization of the properly normalized uncertainty sums in order to get necessary conditions of unsteerability for both ways of potential steering. It appears that they coincide with those found by Wiseman et al. [10, 11] in the Gaussian framework. We revisit in Sec. VI the special case of GSs, for which it is well known that the necessary conditions of separability and unsteerability we have derived here are also sufficient ones, as proven in Refs. [30, 32] and, respectively, in Refs. [10, 11]. Section VII contains a recapitulation and a discussion of the methods and results of the paper.

II. EPR-LIKE UNCERTAINTY RELATIONS FOR BIPARTITE ($N$ vs 1)-MODE STATES

Qualification and quantification of quantum correlations for multimode states continue to be a central task of continuous-variable information theory [13, 33, 34]. For simplicity and feasibility reasons, we consider here a two-party state $\hat{\rho}$ with ($N$ vs 1) modes shared by Alice and Bob as follows. Alice performs measurements on the $N$-mode part of the state with the canonical quadrature operators $\hat{q}_j, \hat{p}_j, (j = 1, \ldots, N)$, while the ($N + 1$)-th mode with the pair of canonical quadrature operators $\hat{q}_{N+1}, \hat{p}_{N+1}$ is controlled by Bob. In order to generalize Eq.(1.1), we introduce the following EPR-like observables depending on the positive parameters $\alpha_j > 0$, $\beta_j > 0$, $(j = 1, \ldots, N + 1)$:

$$\hat{Q}(\alpha) := \sum_{j=1}^{N} \alpha_j \hat{q}_j - \alpha_{N+1} \hat{q}_{N+1},$$

$$\hat{P}_\pm(\beta) := \sum_{j=1}^{N} \beta_j \hat{p}_j \pm \beta_{N+1} \hat{p}_{N+1}, \quad \text{where} \quad \alpha := \{\alpha_1, \ldots, \alpha_{N+1}\}, \quad \beta := \{\beta_1, \ldots, \beta_{N+1}\}. \quad (2.1)$$

Their commutation relations,

$$[\hat{Q}(\alpha), \hat{P}_\pm(\beta)] = i \left( \sum_{j=1}^{N} \alpha_j \beta_j \mp \alpha_{N+1} \beta_{N+1} \right) \hat{I}, \quad (2.2)$$

lead to the weak (Heisenberg) form of the URs in product form:

$$\Delta Q(\alpha) \Delta P_\pm(\beta) \geq \frac{1}{2} \left| \sum_{j=1}^{N} \alpha_j \beta_j \mp \alpha_{N+1} \beta_{N+1} \right|. \quad (2.3)$$

This implies the sum-form inequalities

$$|\Delta Q(\alpha)|^2 + |\Delta P_\pm(\beta)|^2 \geq \sum_{j=1}^{N} \alpha_j \beta_j \mp \alpha_{N+1} \beta_{N+1}. \quad (2.4)$$
The variances of the EPR-like observables (2.1),
\[
|\Delta Q(\alpha)|^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \sigma(q_j, q_k)
- 4\alpha_{N+1} \sum_{j=1}^{N} \alpha_j \sigma(q_j, q_{N+1}),
\]
\[
|\Delta P_+(\beta)|^2 = \sum_{j=1}^{N+1} \sum_{k=1}^{N+1} \beta_j \beta_k \sigma(p_j, p_k),
\]
\[
|\Delta P_-(\beta)|^2 = \sum_{j=1}^{N+1} \sum_{k=1}^{N+1} \beta_j \beta_k \sigma(p_j, p_k)
- 4\beta_{N+1} \sum_{j=1}^{N} \beta_j \sigma(p_j, p_{N+1})
\] (2.5)
are quadratic forms in the positive variables \(\alpha\) and \(\beta\), respectively. Their coefficients are covariances of the canonical quadrature operators of the \((N \text{ vs } 1)\)-mode state \(\hat{\rho}\):
\[
\sigma(q_j, q_k) := \langle \hat{q}_j \hat{q}_k \rangle - \langle \hat{q}_j \rangle \langle \hat{q}_k \rangle,
\]
\[
\sigma(p_j, p_k) := \langle \hat{p}_j \hat{p}_k \rangle - \langle \hat{p}_j \rangle \langle \hat{p}_k \rangle,
\]
\((j, k = 1, \ldots, N + 1)\). (2.6)
These are the entries of the covariance matrix (CM) of the two-party state \(\hat{\rho}\), which is denoted \(\mathcal{V} \in M_{2(N+1)}(\mathbb{R})\), is symmetric, and has the block structure:
\[
\mathcal{V} = \begin{pmatrix}
\mathcal{V}_A & \mathcal{C} \\
\mathcal{C}^T & \mathcal{V}_B
\end{pmatrix}.
\] (2.7)
The partition (2.7) allows one to visualize the contribution of each subsystem. Thus, \(\mathcal{V}_A\) is the \(2N \times 2N\) CM of the reduced \(N\)-mode state held by Alice, \(\mathcal{V}_B\) is the \(2 \times 2\) CM of the reduced one-mode state handled by Bob, while \(\mathcal{C}\) is the \(2N \times 2\) matrix describing the cross-correlations between the parties.

Notice that the covariances \(\sigma(q_j, p_k)\) of any coordinate-momentum pair do not appear in the quadratic forms (2.5). Apparently, the EPR-like inequalities (2.4) hold only for \((N \text{ vs } 1)\)-mode state having a standard-form CM with vanishing covariances \(\sigma(q_j, p_k)\). One could ask whether this might be a restriction on the class of states specified at the beginning of this section. This is not the case, because it was shown in Refs. [35–37] that any multimode CM can be transformed to the standard form described above with local symplectic matrices such as one-mode rotations and squeezings. Since two multimode states which are similar via a tensor product of one-mode unitaries possess the same amount of entanglement or steerability, they are equivalent as nonlocal resources. Therefore, our discussion applies to all the multimode states whose CMs are connected by one-mode symplectic transformations. The particular form acquired by the CM (2.7) discussed above is very convenient in many aspects. For instance, due to the identity \(\sigma(q_j, p_k) = 0\), it is more productive for subsequent evaluations to write the CM \(\mathcal{V}\) as the direct sum
\[
\mathcal{V} = \mathcal{V}^{(q)} \oplus \mathcal{V}^{(p)}.
\] (2.8)
The two terms of the above decomposition are the \((N + 1) \times (N + 1)\) CMs in the position and momentum spaces, written with the reordered canonical operators \(\{\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{N+1}\}\) and \(\{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_{N+1}\}\), respectively. A direct consequence of Eq. (2.8) is the factorization rule
\[
\det(\mathcal{V}) = \det \left(\mathcal{V}^{(q)}\right) \det \left(\mathcal{V}^{(p)}\right).
\] (2.9)
Let us recall the restrictions imposed by the canonical commutation relations to the CM \(\mathcal{V}\) of an arbitrary \(n\)-mode state \([30, 38]\). Essentially, any \textit{bona fide} CM \(\mathcal{V}\) fulfills the Robertson-Schrödinger matrix uncertainty relation, namely, the positive semidefiniteness condition
\[
\mathcal{V} + \frac{i}{2} J \succeq 0,
\] (2.10)
where \(J\) is the standard matrix of the symplectic form on \(\mathbb{R}^{2n}\):
\[
J := \bigoplus_{k=1}^{n} J_k, \quad J_k := \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad (k = 1, \ldots, n).
\] (2.11)
We stress that the conditions (2.3), (2.4), and (2.10) stem from the URs for the canonical observables and hence they have to be satisfied by any physical multimode state.

The quantumness requirement (2.10) implies, via Williamson’s theorem \([39]\), the positive definiteness of the CM \(\mathcal{V}\),
\[
\mathcal{V} > 0,
\] (2.12)
as well as the inequality
\[
\det(\mathcal{V}) \geq \frac{1}{2^{2n}}, \quad (n = 1, 2, 3, \ldots).
\] (2.13)
It is worth mentioning that the inequality (2.13) can be recovered by selecting from the whole class of \(n\)-mode states possessing the same CM \(\mathcal{V}\) a Gaussian one, denoted \(\hat{\rho}_G\), which is determined up to a translation in the phase space. Indeed, its purity depends only on the determinant of the CM \(\mathcal{V}\), as follows \([40, 41]\):
\[
\text{Tr} \left(\left[\hat{\rho}_G\right]^2\right) = \frac{1}{2^n \sqrt{\det(\mathcal{V})}} \leq 1.
\] (2.14)
This universal property of the purity, when applied to any \(n\)-mode Gaussian state, Eq. (2.14), coincides therefore with the general inequality (2.13).
III. EPR-LIKE SEPARABILITY BOUNDS FOR BIPARTITE (N vs 1)-MODE STATES

Use of uncertainty relations between nonlocal operators of the type (1.1) appeared to be a fruitful idea in quest of separability conditions for continuous-variable two-party states and an enforcement of the role of uncertainty principle in understanding quantum correlations. Besides the original approach of Duan et al. [35], some other necessary conditions of separability which employ pairs of more general nonlocal observables depending on one or more parameters have been written [30, 42]. Essentially, all these treatments rely on the important finding that for separable states the uncertainty relations should modify to become stronger. This was first shown in Refs. [30, 35] by directly exploiting the expansion of two-party states and an enforcement of the role of uncertainty principle in understanding quantum correlations.

In continuous-variable systems, we have the privilege to employ in this approach the one-mode canonical observables \( \hat{q}_j \) and \( \hat{p}_j \), whose non-commutativity generates the Heisenberg uncertainty relations that are at the heart of quantum mechanics since its early days. The system of (N vs 1) modes we are dealing with is a perfect test bed for an elegant application of the Hofmann-Takeuchi inequality (3.4).

We now focus on a separable (N vs 1)-mode state \( \hat{\rho}_\text{sep} \), with a partitioned CM \( \mathcal{V} \), Eq. (2.7), and choose as nonlocal observables \( \hat{\mathcal{M}}_k \) the pair of generalized Reid operators (2.1):

\[
\hat{\mathcal{M}}_1 = \hat{Q}(\alpha), \quad \hat{\mathcal{M}}_2 = \hat{P}_\pm(\beta),
\]

whose one-party components are respectively:

\[
\hat{A}_1 = \sum_{j=1}^{N} \alpha_j \hat{q}_j, \quad \hat{B}_1 = -\alpha_{N+1} \hat{q}_{N+1};
\]

\[
\hat{A}_2 = \sum_{j=1}^{N} \beta_j \hat{p}_j, \quad \hat{B}_2 = \pm \beta_{N+1} \hat{p}_{N+1},
\]

\[
(\alpha_j > 0, \beta_j > 0 : j = 1, \ldots, N + 1).
\]

When inserting into the inequality (3.4) the sum URs (3.2) associated to the commutators

\[
\begin{bmatrix} \hat{A}_1, \hat{A}_2 \end{bmatrix} = i C_A \hat{I}_A, \quad C_A := \sum_{j=1}^{N} \alpha_j \beta_j,
\]

\[
\begin{bmatrix} \hat{B}_1, \hat{B}_2 \end{bmatrix} = \mp i C_B \hat{I}_B, \quad C_B := \alpha_{N+1} \beta_{N+1},
\]

we get a pair of sum-form necessary conditions of separability:

\[
[\Delta \hat{Q}(\alpha)]^2 + [\Delta \hat{P}_\pm(\beta)]^2 \geq \sum_{l=1}^{N+1} \alpha_l \beta_l.
\]

The upper Eq.(3.8) has a larger separability bound than the upper physicality bound in the corresponding sum UR (2.4). Interestingly, the lower separability condition (3.8) coincides with the lower sum-form UR (2.4). Therefore, in order to be separable, an (N vs 1)-mode state \( \hat{\rho} \) has to obey both conditions (3.8) for any values of the involved parameters \( (\alpha, \beta) \).

The separability conditions (3.8) are always met when the absolute minima of the functions

\[
\Sigma_{\pm}(\mathcal{V}; \alpha, \beta) := \frac{[\Delta \hat{Q}(\alpha)]^2 + [\Delta \hat{P}_\pm(\beta)]^2}{\sum_{l=1}^{N+1} \alpha_l \beta_l}
\]

with respect to all their variables \( \alpha_j > 0, \beta_j > 0, (j = 1, \ldots, N + 1) \) are greater than or at least equal to 1. Let
us denote the corresponding minimum points \( \{ \alpha_{\pm}, \beta_{\pm} \} \) in order to write concisely the EPR-like necessary conditions of separability (3.8):

\[
\Sigma_\pm (V; \alpha_{\pm}, \beta_{\pm}) \geq 1. \quad (3.10)
\]

Except for the simplest case \( N = 1 \) [29], an analytic evaluation of such a minimum is hampered by algebraic difficulties. Nevertheless, in view of the formulas (2.5), we apply Euler’s theorem on homogeneous functions to the fractions (3.9) and find the identities:

\[
\sum_{j=1}^{N+1} a_j \frac{\partial}{\partial a_j} |\Sigma_\pm (V; \alpha, \beta)| = - \sum_{j=1}^{N+1} b_j \frac{\partial}{\partial b_j} |\Sigma_\pm (V; \alpha, \beta)| = \frac{[\Delta Q(\alpha)]^2 - [\Delta P_\pm(\beta)]^2}{\sum_{j=1}^{N+1} a_\pm b_j}. \quad (3.11)
\]

Accordingly, any \((N \text{ vs } 1)\)-mode separable state \( \hat{\rho}_{\text{sep}} \) fulfills the conditions

\[
[\Delta Q(\alpha_{\pm})]^2 = [\Delta P_\pm(\beta_{\pm})]^2 \geq \frac{1}{2} \sum_{j=1}^{N+1} (a_{\pm}) (b_j)_{\pm}. \quad (3.12)
\]

The difference between the necessary conditions of separability (3.12) for an \((N \text{ vs } 1)\)-mode state \( \hat{\rho} \) arises from the opposite signs of the \( \hat{\rho}_{N+1} \) terms in the nonlocal observables \( \hat{P}_\pm(\beta) \) introduced in Eq. (2.1). As shown by Simon [30], the change of sign \( \hat{P}_{N+1} \rightarrow -\hat{P}_{N+1} \) in the Wigner function \( W(q_1, \ldots, q_{N+1}, p_1, \ldots, p_{N+1}) \) of the \((N \text{ vs } 1)\)-mode state \( \hat{\rho} \) amounts to its transformation into the Wigner function of the partially transposed operator \( \hat{\rho}^{\text{PT}}(B) \) with respect to Bob’s party:

\[
W^{\text{PT}}(B)(q_1, \ldots, q_N, q_{N+1}, p_1, \ldots, p_N, p_{N+1}) = W(q_1, \ldots, q_N, q_{N+1}, p_1, \ldots, p_N, -p_{N+1}).
\]

By virtue of the decomposition (2.8), the CM of the partially transposed operator \( \hat{\rho}^{\text{PT}}(B) \) is positive definite: \( \Sigma^{\text{PT}}(B) > 0 \). Moreover, in view of Ref. [48], it fulfills the Robertson-Schrödinger matrix UR:

\[
\Sigma^{\text{PT}}(B) + \frac{i}{2} J \geq 0. \quad (3.13)
\]

As a matter of fact, Eq. (3.13) is a consequence of the Peres theorem [8], which asserts that any bipartite separable state \( \hat{\rho} \) is a PPT state: this simply means that its partial transpose is positive, \( \hat{\rho}^{\text{PT}}(B) \geq 0 \). Therefore, Eq. (3.12) expresses the quantumness requirements for the couple of \((N \text{ vs } 1)\)-mode states \( \hat{\rho} \) and \( \hat{\rho}^{\text{PT}}(B) \).

The two-mode states \((N = 1)\) are important in their own right. They are present in many areas, generating a productive research in quantifying bipartite quantum correlations especially in the Gaussian scenario [40, 41, 49–51]. In the two-mode case, the EPR-like observables (2.1) depend on two pairs of positive parameters,

\[
\alpha := \{\alpha_1, \alpha_2\}, \quad \beta := \{\beta_1, \beta_2\},
\]

as follows:

\[
\hat{Q}(\alpha) := \alpha_1 \hat{q}_1 - \alpha_2 \hat{q}_2, \quad \hat{P}_\pm(\beta) := \beta_1 \hat{p}_1 \pm \beta_2 \hat{p}_2. \quad (3.14)
\]

Clearly, Reid’s pair of nonlocal observables (1.1) built with two independent parameters, \( \lambda \) and \( \mu \), as well as the single-parameter ones employed by Duan et al. in Ref.[35] are particular forms of EPR-like observables (3.14).

Let us consider an arbitrary separable two-mode state \( \hat{\rho}_{\text{sep}} \), whose \( 4 \times 4 \) CM \( V \), Eq. (2.7), is always characterized by four standard-form parameters [35]:

\[
b_1 := \sigma(q_1, q_1) = \sigma(p_1, p_1), \quad b_2 := \sigma(q_2, q_2) = \sigma(p_2, p_2),
\]

\[
c := \sigma(q_1, q_2), \quad d := \sigma(p_1, p_2). \quad (3.15)
\]

With no loss of generality, one can choose:

\[
b_1 \geq b_2 \geq \frac{1}{2}, \quad c \geq |d|.
\]

Then the \( 2 \times 2 \) submatrices in the partition (2.7) of the CM \( V \) are diagonal:

\[
\begin{align*}
\mathcal{V}_A &= b_1 I_2, \quad \mathcal{V}_B = b_2 I_2, \quad C = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \quad (3.16)
\end{align*}
\]

where \( I_2 \) is the \( 2 \times 2 \) identity matrix. We specialize Eq. (2.5) to get the variances of the EPR-like observables (3.14):

\[
\begin{align*}
[\Delta Q(\alpha)]^2 &= b_1 \sigma_1^2 + b_2 \sigma_2^2 - 2c \alpha_1 \alpha_2, \\
[\Delta P_\pm(\beta)]^2 &= b_1 \beta_1^2 + b_2 \beta_2^2 \pm 2d \beta_1 \beta_2. \quad (3.17)
\end{align*}
\]

The next step is to evaluate the absolute minima of the functions (3.9) written for two-mode states,

\[
\Sigma_\pm (V; \alpha, \beta) := \frac{[\Delta Q(\alpha)]^2 + [\Delta P_\pm(\beta)]^2}{\alpha_1 \beta_1 + \alpha_2 \beta_2}, \quad (3.18)
\]

with respect to their four positive variables \( \{\alpha, \beta\} \). These minima are found to be equal to the doubles of the smallest symplectic eigenvalues \( \kappa_{\pm}^{\text{PT}} \) and \( \kappa_{-} \) [52] of the CMs \( \Sigma^{\text{PT}} \) and \( V \), respectively:

\[
\begin{align*}
\Sigma_+ (V; \alpha_+, \beta_+) &= 2\kappa_+^{\text{PT}}, \quad \Sigma_- (V; \alpha_-, \beta_-) = 2\kappa_- \quad (3.19)
\end{align*}
\]

In order to derive the formulas (3.19) one needs the symplectic eigenvalues \( \kappa_{\pm}^{\text{PT}} \) and \( \kappa_{\pm}^{\text{PT}}' \) expressed in terms of the standard-form parameters (3.15) [53]:

\[
(\kappa_\pm)^2 = \frac{1}{2} \left( b_1^2 + b_2^2 + 2cd \pm \sqrt{\Delta} \right), \quad (3.20)
\]

where \( \Delta \) is the discriminant of a quadratic trinomial:

\[
\Delta = (b_1^2 + b_2^2 + 2cd)^2 - 4 \det(V)
\]

\[
= (b_1^2 - b_2^2)^2 + 4(b_1 c + b_2 d)(b_2 c + b_1 d) \geq 0. \quad (3.21)
\]
Similarly,

$$(\kappa_{\pm}^{\text{PT}})^2 = \frac{1}{2} \left( b_1^2 + b_2^2 - 2cd \pm \sqrt{\Delta_{\text{PT}}} \right), \quad (3.22)$$

with the discriminant

$$\Delta_{\text{PT}} = (b_1^2 + b_2^2 - 2cd)^2 - 4 \det(V)$$

$$= (b_1^2 - b_2^2)^2 + 4(b_1c - b_2d)(b_2c - b_1d) \geq 0. \quad (3.23)$$

Although recalled in Ref. [29] just in the special case of separable two-mode Gaussian states, Eqs. (3.20)-(3.23) hold for any separable two-mode state, either Gaussian or non-Gaussian. It is worth mentioning that they imply the following sign rule:

$$\text{sgn} \left( \kappa_{\pm}^{\text{PT}} - \kappa_- \right) = \text{sgn}(d). \quad (3.24)$$

According to Eq. (3.19), the EPR-like necessary conditions of separability (3.10) acquire in the two-mode case their simplest form:

$$\kappa_{\pm}^{\text{PT}} \geq \frac{1}{2}, \quad \kappa_- \geq \frac{1}{2}. \quad (3.25)$$

The first above inequality manifestly illustrates Peres’ criterion of inseparability [8], while the second one is just the quantumness requirement for the given separable two-mode state $\hat{\rho}_{\text{sep}}$. Note that the first minimal normalized UR sum in Eq. (3.19) was derived in Ref. [29] by using a special set of independent variables, in the framework of a comprehensive analysis.

IV. UNCERTAINTY-RELATION CRITERION FOR EPR STEERING

Our aim in this section is to establish the sum-form URs valid for unsteerable two-party states. To this end, we briefly recall the meaning of the steering scenario, as emerging from its interpretation in Refs.[11, 12, 14–20]. Accordingly, two separate observers, Alice and Bob, share a bipartite state $\hat{\rho}_{AB}$ of a composite quantum system, such that its Hilbert space is the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$. Alice performs a local unknown measurement $x$ whose positive operators $\hat{M}_{a|x}$ add up to identity:

$$\sum_a \hat{M}_{a|x} = \hat{I}_A.$$ 

Here $a$ denotes a possible outcome of the measurement chosen by Alice on her side, having the probability $p(a|x)$. The conditional unnormalized state on Bob’s side, $\hat{\sigma}^B(a|x) = \text{Tr}_A \left[ \hat{\rho}_{AB} \left( \hat{M}_{a|x} \otimes \hat{I}_B \right) \right], \quad (4.1)$

has the trace $\text{Tr}_B \left[ \hat{\sigma}^B(a|x) \right] = p(a|x)$. Bob’s collection of states $\{\hat{\sigma}^B(a|x)\}_{a,x}$ built with all the possible pairs $\{a, x\}$ is called an assemblage [14]. Let us write the sum rule

$$\sum_a \hat{\sigma}^B(a|x) = \text{Tr}_A (\hat{\rho}_{AB}) =: \hat{\rho}_B. \quad (4.2)$$

giving the reduced state at Bob’s hand, prior to Alice’s measurement. Equation (4.2) shows that Bob’s reduced state $\hat{\rho}_B$ is not modified whatever the nonselective measurement $x$ made by Alice.

When all the conditional states (4.1) have the special structure

$$\hat{\sigma}_\text{us}^B(a|x) = \sum_{\lambda} p(a|x, \lambda) q(\lambda) \hat{\rho}_B(\lambda), \quad (4.3)$$

then the assemblage $\{\hat{\sigma}_\text{us}^B(a|x)\}_{a,x}$ is termed unsteerable. In Eq.(4.3), $\{\hat{\rho}_B(\lambda)\}_\lambda$ is a preexisting ensemble of local hidden states (LHs), each one depending on the local hidden variable (LHV) $\lambda$ and having the associated weight $q(\lambda)$. The weights are normalized, i.e., they sum to one: $\sum_\lambda q(\lambda) = 1$. Further, $p(a|x, \lambda)$ is the probability of the outcome $a$ in Alice’s measurement $x$, provided that Bob’s party is described by the LHS $(\hat{\rho}_B(\lambda))$. The trace of the operator (4.3) acting on the Hilbert space $\mathcal{H}_B$ is just the probability of the outcome $a$ in Alice’s measurement $x$:

$$p(a|x) = \sum_{\lambda} p(a|x, \lambda) q(\lambda). \quad (4.4)$$

On the other hand, the initial Bob’s reduced state has the convex decomposition

$$\hat{\rho}_B = \sum_{\lambda} q(\lambda) \hat{\rho}_B(\lambda). \quad (4.5)$$

When the local measurement $x$ performed by Alice has the outcome $a$, then it changes Bob’s reduced state (4.5) into an updated convex combination of LHs:

$$\hat{\rho}_B(a|x) = \frac{1}{p(a|x)} \hat{\sigma}_\text{us}^B(a|x) = \sum_{\lambda} \frac{p(a|x, \lambda) q(\lambda)}{p(a|x)} \hat{\rho}_B(\lambda). \quad (4.6)$$

The key quantity in the analysis of unsteerability is the joint probability of a pair of outcomes $\{a, b\}$ in two successive measurements, $x$, performed by Alice, and $y$, made by Bob:

$$P(a|x, b|y, \hat{\rho}_{AB}) = \text{Tr}_B \left[ \hat{\sigma}^B(a|x) \hat{M}_{b|y} \right]. \quad (4.7)$$

In view of Eq. (4.3), for any unsteerable assemblage $\{\hat{\sigma}_\text{us}^B(a|x)\}_{a,x}$, Eq. (4.7) acquires a specific form [10, 11]:

$$P(a|x, b|y, \hat{\rho}_{AB}) = \sum_{\lambda} q(\lambda)p(a|x, \lambda)P[|y, \hat{\rho}_B(\lambda)|]. \quad (4.8)$$

Thus, the joint probability (4.7) has the asymmetric structure (4.8) whenever the measurement performed by Alice cannot steer (pilot) Bob’s party. Essentially, Eq. (4.8) can be interpreted in terms of a LHS model for Bob and a LHV model for Alice, which is correlated with Bob’s reduced state (4.5) via the conditional probabilities $p(a|x, \lambda)$. The lack of such a model even for a single
pair \( \{a, x\} \) of the assemblage means steering from Alice to Bob \([10–12]\).

It can be directly seen that any measurement performed by Alice on a separable state, Eq. (3.1), leads to an unsteerable assemblage (4.3). It follows that entanglement is a necessary condition for EPR steerability. For instance, an unsteerable assemblage can be obtained from the separable state \( \hat{\rho}_{AB} = \sum_{\lambda} g(\lambda) |\lambda\rangle_A \otimes \hat{\rho}_B(\lambda) \).

Here, Alice performs projective measurements of mutually commuting observables whose common eigenvectors are generically denoted \( |\lambda\rangle \). Moreover, it was proven that unsteerable assemblages can be obtained by joint local measurements \([16, 17, 20]\). Joint measurability is an extension of commutativity to general measurements described by positive operator valued measures (POVMs).

Note that observation of one-way EPR steering requires the presence of both entanglement and incompatible measurements. The connection between steering and incompatibility was recently extended to continuous-variable systems \([20, 27]\). To conclude this discussion, an unsteerable assemblage can be obtained from Alice via local measurements of commuting or jointly measurable observables.

The main question we have now to answer is as follows: how might URs be written analogously to the Hofmann-Takeuchi inequality (3.4) such as to be specifically valid for two-party unsteerable states? The above-sketched tableau of one-way EPR steering offers an idea for answering this question. First, let us remark that the structure (4.3) of an unsteerable assemblage from Alice to Bob imposes the existence of a local uncertainty limit \( C_B \). Second, this is not the case for Alice’s side, where no uncertainty relation is available since her measurements are unknown. In other words, our lack of knowledge about the nature of Alice’s measurements amounts to ignoring the uncertainty limit \( C_A \) of her party \([47, 54, 55]\). We therefore suggest that the right necessary condition for unsteerability of Bob’s reduced state through Alice’s local measurement is the sum-form UR

\[
\sum_k (\Delta M_k)^2 \geq C_B, \tag{4.9}
\]

which is weaker than the similar one for separability, Eq. (3.4). Any violation of the inequality (4.9) is thus a pertinent signature of steering from Alice to Bob.

Accordingly, with the EPR-like observables (2.1), one gets the sum-form UR which is compulsory for any \((N \text{ vs } 1)\)-mode state which is unsteerable \((us)\) from Alice to Bob:

\[
[\Delta Q(\alpha)]^2 + [\Delta P(\beta)]^2 \geq \alpha_{N+1} \beta_{N+1}. \tag{4.10}
\]

Similarly, for the same state, the necessary condition of unsteerability from Bob to Alice reads:

\[
[\Delta Q(\alpha)]^2 + [\Delta P(\beta)]^2 \geq \sum_j \alpha_j \beta_j. \tag{4.11}
\]

Conditions (4.10) and (4.11) display the asymmetry of steering. They have to be fulfilled for all the positive parameters \( \{\alpha, \beta\} \). If the conditions (4.10) or (4.11) are not met for some values of these parameters, then the \((N \text{ vs } 1)\)-mode state is steerable from Alice to Bob or from Bob to Alice, respectively.

In the next section we intend to find the extremal normalized URs derived from Eqs. (4.10) and (4.11). Analytically, these distinct extremization problems for the two ways of steering depend on \(2N + 2\) parameters and appear to be less complicated than those of the functions (3.9) for separability.

V. EPR-LIKE UNCERTAINTY LIMITS FOR STEERING

We carry out here the extremization of the two properly normalized uncertainty sums required for the unsteerability of a \((N \text{ vs } 1)\)-mode state in both possible ways of steering.

A. Steering from Alice to Bob

Guided by the UR (4.10) we define the normalized sum

\[
\Sigma^{(A \rightarrow B)}(\alpha, \beta) := \frac{[\Delta Q(\alpha)]^2 + [\Delta P(\beta)]^2}{\alpha_{N+1} \beta_{N+1}} \geq 1. \tag{5.1}
\]

On account of the expressions (2.5) of the variances involved in the above function, we first write its extremization conditions with respect to the variables \( \alpha_j, \beta_j \):

\[
\frac{\partial \Sigma^{(A \rightarrow B)}}{\partial \alpha_j} = 0, \quad \frac{\partial \Sigma^{(A \rightarrow B)}}{\partial \beta_j} = 0, \quad (j = 1, \ldots, N).
\]

They amount to a couple of independent systems of \(N\) linear equations:

\[
\sum_{k=1}^{N} \alpha_k \sigma(q_j, q_k) - \alpha_{N+1} \sigma(q_j, q_{N+1}) = 0, \tag{5.2}
\]

and

\[
\sum_{k=1}^{N} \beta_k \sigma(p_j, p_k) + \beta_{N+1} \sigma(p_j, p_{N+1}) = 0. \tag{5.3}
\]

The system (5.2) with the variables \( \alpha_j \), \((j = 1, \ldots, N)\) has the coefficient matrix \( Y_A^{(q)} \), while the system (5.3) with the variables \( \beta_j \), \((j = 1, \ldots, N)\) has the coefficient matrix \( Y_A^{(p)} \).

We start to minimize the function (5.1) by using the
conditions (5.2) and (5.3):

\[
\Sigma_{\text{us}}^{(A \rightarrow B)} = \frac{1}{\beta_{N+1}} \left[ \alpha_{N+1} \sigma(q_{N+1}, q_{N+1}) - \sum_{j=1}^{N} \alpha_j \sigma(q_j, q_{N+1}) \right] + \frac{1}{\alpha_{N+1}} \left[ \beta_{N+1} \sigma(p_{N+1}, p_{N+1}) + \sum_{j=1}^{N} \beta_j \sigma(p_j, p_{N+1}) \right].
\]

(5.4)

When inserting into Eq.(5.4) the solutions \(\alpha_j\) and \(\beta_j\) of the linear systems (5.2) and (5.3), written with Cramer’s rule, we get two fractions whose numerators are proportional to the Laplace expansions of \(\det(\mathcal{V}(\rho))\) and \(\det(\mathcal{V}(\rho))\) along the \((N + 1)\)-th columns of their matrices. We find thus the function

\[
\Sigma_{\text{us}}^{(A \rightarrow B)}(\epsilon) = \epsilon \frac{\det(\mathcal{V}(\rho))}{\det(\mathcal{V}(\rho))} + \frac{1}{\epsilon} \frac{\det(\mathcal{V}(\rho))}{\det(\mathcal{V}(\rho))},
\]

(5.5)

of a single positive variable, \(\epsilon := \alpha_{N+1}/\beta_{N+1}\). Its unique minimum point,

\[
\epsilon_m = \left[ \frac{\det(\mathcal{V}(\rho)) \det(\mathcal{V}(\rho))}{\det(\mathcal{V}(\rho)) \det(\mathcal{V}(\rho))} \right]^{\frac{1}{2}},
\]

gives, via Eq.(2.9), the minimal uncertainty sum

\[
\min_{(\alpha, \beta)} \Sigma_{\text{us}}^{(A \rightarrow B)} = 2 \sqrt{\frac{\det(\mathcal{V})}{\det(\mathcal{V}(\rho))}}.
\]

(5.6)

In view of the URs (4.10) satisfied by unsteerable states from Alice to Bob we get the following necessary condition of unsteerability:

\[
\frac{\det(\mathcal{V})}{\det(\mathcal{V}(\rho))} \geq \frac{1}{4}.
\]

(5.7)

Therefore, in order to be unsteerable from Alice, who operates on an \(N\)-mode state, to Bob, who holds a single-mode state, an \((N \text{ vs } 1)\)-mode state has to obey the condition (5.7). Its violation is a signature of steerability of any \((N \text{ vs } 1)\)-mode state and an expression of the EPR-paradox as stated by Reid in Ref.[9].

B. Steering from Bob to Alice

For the same setting of \((N \text{ vs } 1)\)-mode we exploit the alternative way of unsteerability (4.11) by using the condition imposed to the normalized sum-form UR

\[
\Sigma_{\text{us}}^{(B \rightarrow A)}(\alpha, \beta) := \frac{[\Delta Q(\alpha')]^2 + [\Delta P(\beta')]^2}{\sum_{j=1}^{N} \alpha_j \beta_j} \geq 1.
\]

(5.8)

It is convenient to start from its extremization conditions with respect to the variables \(\alpha_{N+1}\) and \(\beta_{N+1}\):

\[
\frac{\partial \Sigma_{\text{us}}^{(B \rightarrow A)}}{\partial \alpha_{N+1}} = 0, \quad \frac{\partial \Sigma_{\text{us}}^{(B \rightarrow A)}}{\partial \beta_{N+1}} = 0,
\]

which read, respectively:

\[
\alpha_{N+1} \sigma(q_{N+1}, q_{N+1}) = \sum_{k=1}^{N} \alpha_k \sigma(q_k, q_{N+1})
\]

and

\[
\beta_{N+1} \sigma(p_{N+1}, p_{N+1}) = -\sum_{k=1}^{N} \beta_k \sigma(p_k, p_{N+1}).
\]

(5.9)

(5.10)

Taking account of Eqs. (5.9) and (5.10), the variances (2.5) acquire reduced forms that are characteristic for an \(N\)-mode state:

\[
[\Delta Q(\alpha')]^2 := \sum_{j=1}^{N} \alpha_j \alpha_j \sigma(q_j, q_k),
\]

\[
[\Delta P(\beta')]^2 := \sum_{j=1}^{N} \beta_j \beta_j \sigma(p_j, p_k),
\]

with \(\alpha' := \{\alpha_1, \ldots, \alpha_N\}, \beta' := \{\beta_1, \ldots, \beta_N\}\).

(5.11)

Here we have denoted:

\[
\sigma(q_j, q_k) := \sigma(q_j, q_k) - \sigma(q_j, q_{N+1}) \sigma(q_{N+1}, q_k) \sigma(q_{N+1}, q_{N+1})^{-1} \sigma(q_{N+1}, q_k),
\]

\[
\sigma(p_j, p_k) := \sigma(p_j, p_k) - \sigma(p_j, p_{N+1}) \sigma(p_{N+1}, p_{N+1})^{-1} \sigma(p_{N+1}, p_k),
\]

\((j, k = 1, \ldots, N)\).

(5.12)

Therefore, the unsteerability condition (5.8) takes a simpler form:

\[
\min_{(\alpha, \beta)} \Sigma_{\text{us}}^{(B \rightarrow A)} = \min_{(\alpha', \beta')} \frac{[\Delta Q(\alpha')]^2 + [\Delta P(\beta')]^2}{\sum_{j=1}^{N} \alpha_j \beta_j} \geq 1.
\]

(5.13)

To see its significance, let us introduce two nonlocal operators built with the same positive parameters \(\alpha'\) and \(\beta'\) as in Eq. (5.11):

\[
\hat{Q}(\alpha') := \sum_{j=1}^{N} \alpha_j \hat{q}_j, \quad \hat{P}(\beta') := \sum_{j=1}^{N} \beta_j \hat{p}_j.
\]

(5.14)

Equations (5.11) appear to specify the variances of the observables (5.14) for the class of \(N\)-mode states which share the CM with the only non-vanishing entries (5.12).

Indeed, this matrix is precisely the Schur complement of the \(2 \times 2\) CM \(\mathcal{V}_B\) of the one-mode state held by Bob according to the partition (2.7):

\[
\mathcal{V}/\mathcal{V}_B = \mathcal{V}_A - \mathcal{C}(\mathcal{V}_B)^{-1} \mathcal{C}^T.
\]

(5.15)

Recall [48] that the Schur complement (5.15) satisfies the physicality condition

\[
\mathcal{V}/\mathcal{V}_B + \frac{i}{2} J_A \geq 0,
\]

(5.16)
which qualifies it as a *bona fide* CM of an $N$-mode state [30, 38].

By the same token, the inequality (5.13) holds for the above-mentioned class of $N$-mode states. In other words, Eqs. (5.13) and (5.16) are two *equivalent* necessary conditions of unsteerability from Bob to Alice. They are simultaneously obtained by taking the minimum of the normalized sum-form UR (5.8).

Furthermore, we employ the Aitken factorization formula [56] for the partitioned CM (2.7):

$$V = TDT^T.$$  

(5.17)

The matrix product (5.17) consists of a central factor, namely, the positive definite block-diagonal matrix,

$$D = (V/V_B) \oplus V_B,$$  

(5.18)

sandwiched between a pair of unimodular triangular matrices,

$$T = \begin{pmatrix} I_A & CV_B^{-1} \\ 0 & I_B \end{pmatrix}, \quad T^T = \begin{pmatrix} I_A & 0 \\ V_B^{-1}C^T & I_B \end{pmatrix}.$$  

(5.19)

The Aitken formula (5.17) is therefore a LDU decomposition, exhibiting the $T$ congruence of the CM (2.7) to the direct sum $D$, Eq. (5.18), via the upper triangular matrix $T$, Eq. (5.19). The above discussion is valid for any bipartite ($N$ vs $M$) CM. A straightforward consequence of Eqs. (5.17)-(5.19) is the Schur determinantal formula:

$$\det(V/V_B) = \frac{\det(V)}{\det(V_B)}. \quad (5.20)$$

Equations (5.17)-(5.19) clearly imply the equivalence between the matrix inequality (5.16) and the following one:

$$V + \frac{i}{2} J_A \oplus 0_B \geq 0.$$  

(5.21)

To sum up, we have proven that two necessary conditions of unsteerability from Bob to Alice, Eqs. (5.13) and (5.21), are fully equivalent, because each of them is equivalent to Eq. (5.16). This is the main result of the current subsection.

Owing to Williamson’s theorem [39], the physicality condition (5.16) implies the inequality

$$\det(V/V_B) \geq \frac{1}{2^{2N}}.$$  

(5.22)

By combining Eq. (5.22) with the Schur determinantal formula (5.20), one gets a numerical necessary condition of unsteerability from Bob to Alice:

$$\frac{\det(V)}{\det(V_B)} \geq \frac{1}{2^{2N}}.$$  

(5.23)

Except for the two-mode states ($N = 1$), this is weaker than the matrix condition (5.21).

The inequality (5.22) is analog to the necessary condition of unsteerability from Alice to Bob (5.7), written with Schur’s formula (5.20) as

$$\det(V/V_A) \geq \frac{1}{4}.$$  

(5.24)

Note, however, that the numerical inequality (5.24) is the only one which is equivalent to the matrix necessary conditions of unsteerability from Alice to Bob similar to the equivalent matrix inequalities (5.16) and (5.21):

$$V/V_A + \frac{i}{2} J_B \geq 0 \iff V + 0_A \oplus \frac{i}{2} J_B \geq 0.$$  

(5.25)

The above extremizations of the normalized sums of EPR uncertainties do not allow us to find explicitly the corresponding minima. Nevertheless, violation of at least one of the similar conditions (5.7) and (5.23) is sufficient for the one-way steerability of any ($N$ vs $1$)-mode state. Such violations are therefore signatures of steerability from Alice to Bob and, respectively, from Bob to Alice.

VI. SEPARABILITY AND UNSTEERABILITY CRITERIA FOR ($N$ vs $1$)-MODE GAUSSIAN STATES REVISITED

The class of Gaussian states (GSs) is a very special one in the whole set of continuous-variable multimode states. Any $n$-mode GS $\hat{\rho}_G$ is completely determined by a displacement vector in the phase space, $u \in \mathbb{R}^{2n}$, and the CM $V \in M_{2n}(\mathbb{R})$, which is subjected to the quantumness requirement (2.10). This characteristic property of the GSs is the main reason for the central role they play in quantum optics, as well as in quantum information theory.

A. Separability criteria for Gaussian states

Let us start with the case of the two-mode GSs. As recalled in Sec. III, the first EPR-type inequality in Eq. (3.10),

$$\kappa^{PT}_- \geq \frac{1}{2},$$  

(6.1)

is manifestly equivalent to the Peres-Simon PPT necessary condition of separability [8, 30]. Moreover, in Ref. [30], Simon proved that the physicality requirement (6.1) for the existence of the two-mode GS $\hat{\rho}_G^{PT}$ is also a criterion (sufficient condition) for the separability of a given two-mode GS $\hat{\rho}_G$. Accordingly, the symplectic eigenvalue $\kappa^{PT}$ is a PPT-type separability indicator at hand for any two-mode GS. Furthermore, its equivalence with a less convenient EPR-type separability indicator was explicitly proven [57].

The next accomplishment is the treatment of the separability problem for ($N$ vs $1$)-mode GSs by Werner and
Wolf in Ref. [32]. Their main result is that the PPT property of such a GS implies its separability. Although this important Werner-Wolf theorem generalizes to an arbitrary $N$ the remarkable Simon’s one for $N = 1$ [30], it does not provide any analytic separability indicator for $N > 1$. A second complementary result obtained in Ref. [32] is that the PPT property and separability are not equivalent concepts for any $(N \text{ vs } M)$-mode GS with $M > 1$. Indeed, the authors get an example of a $(2 \text{ vs } 2)$-mode GS which is PPT, but entangled. Such a bipartite state is termed \textit{bound entangled} [58]. A comprehensive analysis of both above-mentioned results is recently developed in Ref. [48].

Coming back to the present work, we stress that, by virtue of the Werner-Wolf theorem, the EPR-like necessary conditions of separability (3.12) are also \textit{sufficient} ones. They could therefore be employed to get computable separability indicators for $(N \text{ vs } 1)$-mode GSs.

\section*{B. Unsteerability criteria for Gaussian states}

We recall the noteworthy results concerning the special case of Gaussian unsteerability established by Jones, Wiseman, and Doherty in Ref. [11]. Let us consider a bipartite $(N \text{ vs } M)$-mode GS $\hat{\rho}_G$ shared by Alice and Bob, whose CM $\mathcal{V}$ has an adequate partition of the type (2.7). Based on the assumption that Alice makes only Gaussian measurements, the necessary and sufficient condition for the Alice-to-Bob unsteerability of the GS $\hat{\rho}_G$ found in Refs. [10, 11] is

$$\mathcal{V} + 0_A \oplus \frac{i}{2} J_B \geq 0. \quad (6.2)$$

Obviously, a similar necessary and sufficient condition holds for the Bob-to-Alice unsteerability of the GS $\hat{\rho}_G$:

$$\mathcal{V} + \frac{i}{2} J_A \oplus 0_B \geq 0. \quad (6.3)$$

As shown in Refs. [24, 26, 48], the unsteerability inequality (6.2) is equivalent to the physicality requirement

$$\mathcal{V} / \mathcal{V}_A + \frac{i}{2} J_B \geq 0 \quad (6.4)$$

for the Schur complement of the CM $\mathcal{V}_A$ in a partition of the type (2.7),

$$\mathcal{V} / \mathcal{V}_A = \mathcal{V}_B - C^T (\mathcal{V}_A)^{-1} C. \quad (6.5)$$

Accordingly, the condition (6.2) of Alice-to-Bob Gaussian unsteerability holds no matter whether Alice’s party does or does not fulfill the requirement of quantumness

$$\mathcal{V}_A + \frac{i}{2} J_A \geq 0. \quad (6.6)$$

Note that this aspect of steering of Bob’s party by Alice’s measurement is also at the heart of our UR treatment according to Eqs. (4.9) and (4.10).

The inequality (6.4) implies a numerical necessary condition of Alice-to-Bob Gaussian unsteerability,

$$\frac{\det(\mathcal{V})}{\det(\mathcal{V}_A)} \geq \frac{1}{2^M}. \quad (6.7)$$

which is similar to the condition (5.23) for Bob-to-Alice unsteerability. Except for the case $M = 1$, the necessary condition (6.7) is weaker than the matrix inequality (6.4), so that it is not a sufficient condition for Alice-to-Bob Gaussian unsteerability.

In this work our interest is restricted to bipartite $(N \text{ vs } 1)$-mode states. By using suitable EPR-like URs, we have derived the \textit{necessary} matrix conditions (6.2) and (6.3) for their one-way unsteerability, \textit{irrespective} of their Gaussian or non-Gaussian nature. Concerning the well-known sufficiency of these conditions for GSs and Gaussian measurements [10, 11], we just take advantage of the above discussion to stress that the ratio $\det(\mathcal{V})/\det(\mathcal{V}_A)$ is a valuable indicator of Gaussian unsteerability from Alice to Bob, when compared to the number $1/4$, as pointed out by Eq. (5.7).

\section*{VII. SUMMARY AND DISCUSSION}

A main issue we are concerned with in the present work is to give a unified description of both entanglement and EPR steering for a class of bipartite multimode states of continuous-variable systems by using an efficient theoretical method. Looking at the impressive volume of recent work on steering matters one can notice that the question on the relationship between criteria of entanglement and EPR steering is scarcely addressed. Accordingly, the present work on inseparability and steering conditions for two-party multimode quantum states is built on the idea of the fundamental role the sum-form URs of nonlocal EPR-like observables play in defining and describing these two types of quantum correlations.

A widespread classification of quantum correlations termed as entanglement, EPR steering, and Bell nonlocality, from a quantum-information perspective was given by Wiseman et al. [10–12]. Adopting here this fruitful point of view, we extend the quantum-mechanical line of reasoning initiated by Reid for a two-mode state [9] by applying it to the whole class of two-party $(N \text{ vs } 1)$-mode states. A parallel analysis of their entanglement and steering makes use of the same pair of nonlocal EPR-like observables introduced in Eq. (2.1). Essentially, we take two steps. First, we employ the theorem of Hofmann and Takeuchi [31] to write the uncertainty relations with the sum of variances of the EPR-like observables (2.1) for separable states, Eq. (3.8), and, respectively, for unsteerable ones, Eqs. (4.10) and (4.11).

The second step consists in looking for the extremal normalized sums of uncertainties with respect to the involved parameters $\alpha, \beta$. We start with the necessary conditions of separability (3.10). Although not exploited analytically for $N > 1$, they exhibit the PPT property,
as expected from Peres’ general theorem [8]. Nevertheless, in the case of two-mode states (N = 1), we readily recover the minimum 2νPT for d < 0 [29], as specified in Eqs. (3.19) and (3.24).

Then we focus on the steering scenario which, unlike entanglement, is known to be asymmetric with respect to the interchange of the two parties. The necessary condition of unsteerability from Alice to Bob, Eq. (5.7), is the only numerical one to be equivalent to matrix inequalities having the form (5.25). By contrast, when N > 1, the equivalent matrix inequalities (5.16) and (5.21), requested by the Bob-to-Alice unsteerability, are stronger than the numerical one, Eq. (5.23). However, the similar inequalities (5.7) and (5.23) are valuable since their violation allows one to recognize the steerability from Alice to Bob and, respectively, from Bob to Alice.

We finally recall the privileged position of the two-party (N vs 1)-mode GSs, for whom the above necessary conditions regarding both kinds of quantum correlations are equally sufficient ones. We start with Simon’s seminal work [30], who essentially proved that the first inequality (3.25), κPT ≥ 1/2, is a criterium of separability for two-mode GSs. The result of Werner and Wolf [32] that the PPT property of any (N vs 1)-mode GS implies its separability is the maximal generalization of Simon’s theorem for N = 1 [30]. However, by contrast to the latter, it provides no analytic separability indicator for N > 1.

Then we apply the unsteerability criterion for two-party (N vs M)-mode GSs, under Gaussian measurements, found by Wiseman et al. in Refs. [10, 11]. Accordingly, the matrix inequalities (6.2) and (6.3), written for a bipartite (N vs 1)-mode GS ̂ρG shared by Alice and Bob, are necessary and sufficient conditions of its unsteerability from Alice to Bob and, respectively, from Bob to Alice. We emphasize that the necessity part is proven here with no reference to the Gaussian or non-Gaussian nature of the state, as well as to that of the one-party measurements involved.

Let us make a comment about three sets of (N vs 1)-mode states having the same CM V in the standard form (2.8): the set S of separable states, as well as the sets U(A→B) and U(B→A) of unsteerable states from Alice to Bob and from Bob to Alice, respectively. All the states belonging to S fulfill the EPR-like necessary conditions of separability (3.10), while those belonging to the sets U(A→B) and U(B→A) satisfy the necessary matrix conditions of unsteerability (5.25) and, respectively, (5.21). However, by virtue of the Werner-Wolf theorem [32], for the GSs in S, defined up to a phase-space translation, the conditions of separability (3.10) are, in addition, sufficient ones. Similarly, in view of a theorem proven by Wiseman et al. [11], for the GSs belonging to the sets U(A→B) and U(B→A) and under one-party Gaussian measurements, the matrix conditions of unsteerability (5.25) and (5.21) are also sufficient ones. These remarks are just some aspects of the special role played by the GSs among all (N vs 1)-mode states with the same CM. The best known example of extremal property of the GSs within the multimode states with a given CM is that they attain the maximum von Neumann entropy [59]. Other relevant examples are mentioned and discussed in Ref. [60].

We finally emphasize that we have developed in this work a unified treatment of inseparability and steering of two-party (N vs 1)-mode states, based on specific sum-type URs of suitable EPR-like observables. Extremization of the associated normalized sums of variances provides the right necessary conditions for separability and one-way unsteerability. In the Gaussian framework, these conditions are, at the same time, the well-known sufficient ones.

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