A semi-numerical computation for the added mass coefficients of an oscillating hemi-sphere at very low and very high frequencies

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Abstract

A floating hemisphere under forced harmonic oscillation at very high and very low frequencies is considered. The problem is reduced to an elliptic one, that is, the Laplace operator in the exterior domain with standard Dirichlet and Neumann boundary conditions, so the flow problem is simplified to standard ones, with well known analytic solutions in some cases. The general procedure is based in the use of spherical harmonics and its derivation is based on a physics insight. The results can be used to test the accuracy achieved by numerical codes as, for example, by finite elements or boundary elements.

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1 Classification (AMS/MSC)

76Bxx Incompressible inviscid fluids

76B07 Free-surface potential flows

33C55 Spherical harmonics

33F05 Numerical approximation

2 Introduction

A semi-numerical computation of the added mass for a floating hemisphere is shown, under an harmonic forced oscillation on the free surface of an irrotational, incompressible fluid, and linearized boundary conditions. Two standard problems are considered: the heave and the surge modes, that is, vertical and horizontal oscillations, respectively. For simplicity, our attention is restricted to a fluid of infinite depth. The added mass coefficients at very-high and very-low frequencies found in this work are in agreement with those given in literature by another strategies.

As it is noted by Hulme [4], the hydrodynamic formulation of a floating hemisphere is analogous to the two-dimensional circular cylinder. The added mass coefficients are computed as $A'_{kk} = A_{kk}/(\rho V)$, and the damping ones, as $D'_{kk} = D_{kk}/(\rho V \omega)$, where $V = (2/3)\pi R^3$ is the hemisphere volume, $\rho$ is fluid density, and $\omega$ is the circular frequency of the oscillation. The asymptotic values of these coefficients, for very slow and very high frequencies, can be obtained by analytical calculus, for instance, by a variable separation or image methods. For the surge/sway mode at very slow frequency, the boundary condition $\phi_z = 0$, where $\phi$ is the velocity potential, is equivalent to a symmetry operation respect the plane $z = 0$ and, then, corresponds to the solution of a sphere oscillating in an infinity medium. The added mass for the last case is half of the displaced volume, e.g. see [8]: then, the surge/sway added mass coefficient is $A'_{11} = 1/2$, respect to the true displaced mass $(2/3)\pi R^3 \rho$, where the half factor is due to the analytic prolongation. On the other hand, the asymptotic values of the added mass in heave mode are not too easy to obtain, and they may be obtained by a semi-numerical computation with spherical harmonics.

The reason for doing this work is twofold. First, the method of solution adopted here is based by a physics insight rather a mathematical one. Second, the computation is near-exact, so the results can be used to test the accuracy achieved by numerical codes, as finite-element and boundary-element ones, adapted to wave-drag and seakeeping flow problems, e.g. see [5, 6, 7, 8, 9, 10, 11].
3 An oscillating hemisphere

An oscillating hemisphere in a forced motion is considered. The flat face of the hemisphere is on the free surface of an irrotational and incompressible fluid, without a mean flow, where the fluid depth is assumed as infinity. The upward direction is $z$ and the hydrostatic equilibrium plane is $z = 0$. Due to the symmetry, a spherical coordinate system is chosen as

$$
\begin{align*}
    z &= r \cos \theta \\
    x &= r \sin \theta \cos \phi \\
    y &= r \sin \theta \sin \phi
\end{align*}
$$

At very high frequencies, the free surface boundary condition shrinks to the simple form $\phi = 0$. But, by anti-symmetry respects to the plane $z = 0$, the flow problem is equivalent to solve the modified one

$$
\begin{align*}
    \Delta \phi &= 0 \quad \text{in } \Omega' \\
    \phi_n &= \cos \theta \quad \text{at } \Gamma'_e \\
    \phi &= 0 \quad \text{at } \Gamma'_0
\end{align*}
$$

where $\phi$ is the velocity potential, $\Delta$ is the Laplace operator, $\Omega'$, $\Gamma'_e$ and $\Gamma'_0$ are the extended flow domain, extended hemisphere surface and extended free surface, respectively, through the reflection plane $z = 0$. As the free surface boundary condition for very high frequencies is $\phi = 0$, then, its right hand side term has been extended in an anti-symmetric way. That is, the extended problem at very high frequencies is the same that a sphere in infinite medium. On the other hand, the free surface boundary condition for very slow frequencies is $\phi_n = 0$, and for the extended problem, its right hand side must be extended in a symmetrical way and, then,

$$
\begin{align*}
    \Delta \phi &= 0 \quad \text{in } \Omega' \\
    \phi_n &= |\cos \theta| \quad \text{at } \Gamma'_e \\
    \phi &= 0 \quad \text{at } \Gamma'_0
\end{align*}
$$

where, due the module on $|\cos \theta|$, the slow frequency radiation problem does not have, in general, a closed solution and, then, it must be found with another resources like spherical harmonics, as it is considered in this work.

4 The very-high and very-low frequencies limits

The free-surface boundary condition in the limits of very-low and very frequencies

$$
\phi_n = \frac{\omega^2}{g} \phi
$$

is reduced to the homogeneous Neumann and Dirichlet boundary conditions, respectively, where $g$ is the gravity acceleration. Also, the radiation boundary condition at infinity imposes that
the velocity potential \( \phi \) tends to zero, so the corresponding Partial Differential Equation (PDE) system is

\[
\begin{align*}
\Delta \phi & = 0 \quad \text{in } \Omega ; \\
\phi, n & = i\omega h \quad \text{at } \Gamma_B ; \\
\phi, n & = 0 \quad \text{at } \Gamma_F \text{ for low frequencies}; \\
\phi & = 0 \quad \text{at } \Gamma_F \text{ for high frequencies}; \\
|\phi| & \to 0 \quad \text{for } |x| \to \infty ;
\end{align*}
\]

where the load \( h \) is the normal displacement of the mode under consideration. It can be seen that, under these conditions, the flow problem is transformed in a standard elliptic one, whose solution is real valued (in reality it is an imaginary one but it can be transformed by means of a simple re-definition). It is assumed that the load \( h \) is real (that is, the body motion is in phase). The added mass for the mode motion is found from

\[
a_{jj} = -\frac{1}{\omega^2} \int_{\Gamma} d\Gamma i\omega \phi, n = -\int_{\Gamma} d\Gamma \psi, n ;
\]

where now \( \psi = -(i/\omega)\phi \), so in the limits \( \omega \to 0 \) and \( \omega \to \infty \), the function \( \psi \) is real.

By symmetry, the Eqns (5 c-d) can be reproduced extending the flow problem to \( z > 0 \), by means of a mirror body image and extending the load \( h \) in an appropriate way. For instance, the homogeneous Neumann boundary condition is obtained extending the load in a symmetrical way with respect to \( z = 0 \), that is,

\[
h(x, y, z) = +h(z, y, -z) ;
\]

while the Dirichlet boundary condition can be obtained extending in a skew-symmetrical way

\[
h(x, y, z) = +h(z, y, -z) ;
\]

For example, the boundary boundary condition for the hemisphere in the heave-mode is

\[
\psi, n = \cos \varphi ;
\]
where \((r, \varphi, \theta)\) is the spherical coordinate system with origin in the center of the hemisphere such as

\[
\begin{align*}
    z &= r \cos \varphi \\
    x &= r \sin \varphi \cos \theta \\
    y &= r \sin \varphi \sin \theta
\end{align*}
\]

(10)

here the Hildebrand’s convention \[1\] is used. Then, the limits at very-low and very-high frequencies for the hemisphere can be computed from the sphere ones, but with the loads

\[
\begin{align*}
    \psi_n &= |\cos \varphi| \quad \text{for low frequencies} \\
    \psi_n &= \cos \varphi \quad \text{for high frequencies}
\end{align*}
\]

(11)

see figure 2. Similarly, for the surge mode (oscillation along the \(x\)-axis) the equivalent load is

\[
h(\varphi \theta) = \cos \theta
\]

(12)

and the extensions for very-low and very-high frequencies are

\[
\begin{align*}
    \psi_n &= \cos \theta \quad \text{for low frequencies} \\
    \psi_n &= \text{sign} (\cos \varphi) \cos \theta \quad \text{for high frequencies}
\end{align*}
\]

(13)
Figure 4: Extension of the load $h = \text{sign} (\cos \varphi) \cos \theta$ for surge-mode at very-high frequencies.

5 Solution of the flow problems

The previous flow problems can be solved, for instance, by an analytical way or by series. The heave solution at very-high frequencies and the surge one at very-low frequencies are the same of a sphere in an infinity medium and uniform velocity, so the additional mass is a half of the displaced mass, that is,

$$
\begin{align*}
\{ \alpha_3 (\omega \to \infty) &= \frac{\pi}{3} \rho R^3 ; \\
\alpha_1 (\omega \to 0) &= \frac{\pi}{3} \rho R^3 .
\end{align*}
$$

(14)

In the other two cases it is necessary to make an expansion of the sources by means of spherical harmonics.

6 Spherical harmonics

The solution of the exterior potential problem

$$
\begin{align*}
\{ \Delta \psi &= 0 \quad \text{for } r > 1 ; \\
\psi &= f(\varphi, \theta) \quad \text{at } r = 1 ;
\end{align*}
$$

(15)

where $\psi = \psi(\varphi, \theta)$, it can be solved expanding the function $f(\varphi, \theta)$ in terms of the harmonics

$$
f(\varphi, \theta) = \sum_{n=0}^{\infty} a_{n0} P_n(\cos \varphi) + \sum_{n=0}^{\infty} \sum_{m=1}^{n} [a_{nm} \cos(m\theta) + b_{nm} \sin(m\theta)] P_n^m(\cos \varphi) ;
$$

(16)

where

$$
a_{n0} = \frac{2n + 1}{4\pi} \int_{r=1} \psi d\Gamma f(\varphi, \theta) P_n(\cos \varphi) ;
$$

(17)

$$
a_{nm} = \frac{2n + 1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{r=1} \psi d\Gamma f(\varphi, \theta) P_n^m(\cos \varphi) \cos m\theta ;
$$

(18)

$$
b_{nm} = \frac{2n + 1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{r=1} \psi d\Gamma f(\varphi, \theta) P_n^m(\cos \varphi) \sin m\theta ;
$$

(19)
where \( d\Gamma = \sin \varphi \, d\theta \, d\varphi \) is the solid angle differential in spherical coordinates. Once this expansion is computed, the exterior potential can be written as

\[
\psi (r, \varphi, \theta) = \sum_{n=0}^{\infty} Y_n(\varphi, \theta) \, r^{-(n+1)}
\]

where

\[
Y_n(\varphi, \theta) = a_{n0} P_n(\cos \varphi) + \sum_{m=1}^{n} [a_{nm} \cos(m\theta) + b_{nm} \sin(m\theta)] P_m^{n}(\cos \varphi)
\]

now, the Neumann problem can be solved taking derivatives with respect to \( r \) and evaluating at \( r = 1 \), obtaining the expression

\[
h(\varphi, \theta) = -\sum_{n=0}^{\infty} (n+1) \, Y_n(\varphi, \theta)
\]

from which analogous relations are obtained

\[
a_{n0} = \frac{2n+1}{4\pi(n+1)} \int_{r=1} \, d\Gamma \, h(\varphi, \theta) \, P_n(\cos \varphi)
\]

\[
a_{nm} = \frac{2n+1}{2\pi(n+1)(n+m)!} \int_{r=1} \, d\Gamma \, h(\varphi, \theta) \, P_n^m(\cos \varphi) \, \cos(m\theta)
\]

\[
b_{nm} = \frac{2n+1}{2\pi(n+1)(n+m)!} \int_{r=1} \, d\Gamma \, h(\varphi, \theta) \, P_n^m(\cos \varphi) \, \sin(m\theta)
\]

Once obtained the coefficients of the expansion, the additional mass is obtained from

\[
A_{jj} = \int_{r=1} \, d\Gamma \, \psi \, \psi_r
\]

\[
= \int_{\varphi=0}^{\pi} \, d\varphi \left\{ a_{n0}^2 [P_n(\cos \varphi)]^2 + \sum_{m=1}^{n} (a_{nm}^2 + b_{nm}^2) [P_m^m(\cos \varphi)]^2 \right\}
\]

where the orthogonality property of the spherical harmonics was taken into account, and

\[
A_{jj} = \int_{r=1} \, d\Gamma \, \psi \, \psi_r
\]

\[
= \int_{\varphi=0}^{\pi} \, d\varphi \left\{ a_{n0}^2 [P_n(\cos \varphi)]^2 + \sum_{m=1}^{n} (a_{nm}^2 + b_{nm}^2) [P_m^m(\cos \varphi)]^2 \right\}
\]

and using the properties of the Legendre polynomials

\[
A_{jj} = \sum_{n=1}^{\infty} \frac{2}{2n+1} \left[ \frac{a_{n0}^2 + \sum_{m=1}^{n} (a_{nm}^2 + b_{nm}^2)(n-m)!}{(n+m)!} \right]
\]

7 Hemisphere in heave at very-low frequencies

In this case the load \( h(\varphi, \theta) = |\cos \varphi| \), so

\[
a_n = \frac{2n+1}{4\pi(n+1)} \int_{-1}^{1} \, d\mu \, |\mu| \, P_n(\mu)
\]
as the $P_k$ are even (odd) for $k$ even (odd), only remains the even terms and then
\[ a_n = \frac{2n+1}{2\pi(n+1)} \int_0^1 \mu \, d\mu \, P_n(\mu) \quad \text{for } n \text{ even.} \tag{32} \]

For computing the integral, the $P_n$ terms are generating in a recursive way from $P_0 = 1$, $P_1 = \mu$ and $P_2$, ..., $P_n$ are obtained solving
\[ (n+1) \, P_{n+1} - (2n+1) \, \mu \, P_n(\mu) + n \, P_{n-1}(\mu) = 0 . \tag{33} \]

The coefficients of the polynomials $\mu P_n(\mu)$ are obtained from the $P_n$ ones, and the integral is made in a direct way. The final result is
\[ a_{33} = 1.7403 \, \rho \, R^3 ; \tag{34} \]
corresponding to $a'_{33} = 0.83093$, that is, the non-dimensional coefficient respects to the hemisphere mass $2/3\pi\rho R^3$.

### 8 Hemisphere in heave at very-high frequencies

In this case, the factor $\cos \theta$ made that the only non-null coefficients are the $a_{n1}$ ones. For obtaining these, an integral from $\mu = 1$ to $-1$ must be done, with a function that contains the $P_n^1$ ones. These terms having a factor $\sqrt{1-\mu^2}$, so it is convenient to compute the integral in a numerically way. Our final numerical result is
\[ a_{11}(\omega \to \infty) = 0.29806... \quad (0.14231...) ; \tag{35} \]
where the value between parentheses is referred, always, to the non-dimensional value respect to displaced mass.

In brief, our estimates for the asymptotic values at very-low and very-high frequencies (subindex 1 for the surge mode and 3 for the heave one), of the non-dimensional coefficients respects to the hemisphere mass $2/3\pi\rho R^3$ (denoted with primes) are
\[
\begin{align*}
A'_{11} & \to 0.5 \quad \text{for } Ka \to 0 ; \\
A'_{11} & \to 0.14231 \quad \text{for } Ka \to \infty ; \\
A'_{33} & \to 0.83093 \quad \text{for } Ka \to 0 ; \\
A'_{33} & \to 0.5 \quad \text{for } Ka \to \infty . 
\end{align*} \tag{36}
\]

On the other hand, some literature values found for the surge/sway mode, e.g. see Sierevogel \[9\], Prins \[3\] (where only the intervals $[0.25, 1.50]$ and $[0.6, 1.5]$ are considered, respectively, so the extrapolations are rather doubtful), and for the heave one, e.g. see Korsmeyer \[2\] and Liapis \[13\], are
\[
\begin{align*}
A'_{11} & \to 0.50 \quad \text{for } Ka \to 0 ; \\
A'_{11} & \to 0.25 \quad \text{for } Ka \to \infty ; \\
A'_{33} & \to 0.80 \quad \text{for } Ka \to 0 ; \\
A'_{33} & \to 0.45 \quad \text{for } Ka \to \infty . 
\end{align*} \tag{37}
\]
and the Hulme’s ones are

\[
\begin{cases}
    A_{11}' = 0.5 & \text{for } Ka \to 0; \\
    A_{11}' = 0.273\ 239 & \text{for } Ka \to \infty; \\
    A_{33}' = 0.830\ 951 & \text{for } Ka \to 0; \\
    A_{33}' = 0.5 & \text{for } Ka \to \infty;
\end{cases}
\]

(38)

Korsmeyer used a panel method with Fourier transform and complex impedance extended to very slow frequencies, while the Hulme’s numerical results are obtained by spherical harmonics. The Sierevogel, Prins and Liapis results are obtained with a panel method and Kelvin source. In general, the concordance between our estimates and the literature ones is good, except in our surge mode coefficient $A_{11}'$, which is about a half.

9 Conclusions

We have considered a semi-numerical computation by spherical harmonics for added mass coefficients of an oscillating hemi-sphere at very low and very high frequencies. The method is based by a physics insight and the computation is near-exact, so the results can be used to test the accuracy achieved by numerical codes, as finite-element and boundary-element ones.

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