On the geometry of global function fields,
the Riemann–Roch theorem,
and
finiteness properties of $S$-arithmetic groups

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1 Introduction

Harder’s reduction theory ([Har68], [Har69]) provides filtrations of Euclidean buildings that allow one to deduce cohomological ([Har77a]) and homological ([Stu80], [BW08]) properties of $S$-arithmetic groups over global function fields. In this survey I will sketch the main points of Harder’s reduction theory, starting from Weil’s geometry of numbers and the Riemann–Roch theorem. I will describe a filtration, used for example in [Beh98], that is particularly useful for deriving finiteness properties of $S$-arithmetic groups. Finally, I will state the recently established rank theorem and some of its earlier partial verifications that do not restrict the cardinality of the underlying field of constants. As a motivation for further research I also state a much more general conjecture on isoperimetric properties of $S$-arithmetic groups over global fields (number fields or function fields).

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2 Projective varieties

In this section I give a quick introduction to the concept of projective varieties. For further reading the sources [Har77b, I] and [NX09, 2] are highly recommended.

Definition 2.1 Let $k$ be a perfect field, let $\overline{k}$ be its algebraic closure, and let $S \subset \overline{k}[X] = \overline{k}[x_0, x_1, ..., x_n]$ be a set of homogeneous polynomials. The set

$$Z(S) = \{ P \in \mathbb{P}_n(\overline{k}) \mid f(P) = 0 \text{ for all } f \in S \}$$

is called a projective algebraic set. The algebraic set $Z(S)$ is defined over $k$, if $S$ can be chosen to be contained in $k[X]$. The Zariski topology on $\mathbb{P}_n(\overline{k})$ is defined by taking the closed sets to be the projective algebraic sets.

A non-empty projective algebraic set in $\mathbb{P}_n(\overline{k})$ is called a projective variety, if it is irreducible in the Zariski topology of $\mathbb{P}_n(\overline{k})$, i.e., if it is not equal to the union of two proper closed subsets. It is called a projective $k$-variety if it is defined over $k$. The dimension of a non-empty projective
variety is defined to be its dimension as a topological space in the induced Zariski topology, i.e., the supremum of all integers \( n \) such that there exists a chain \( Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n \) of distinct non-empty irreducible closed subsets.

**Theorem 2.2 ([NX09, 2.3.8])**
A projective algebraic set \( V \) is a projective variety if and only if the ideal \( I(V) \) of \( \mathbb{F}[X] \) generated by the set \( \{ f \in \mathbb{F}[X] \mid f \text{ homogeneous and } f(P) = 0 \text{ for all } P \in V \} \) is a prime ideal of \( \mathbb{F}[X] \).

**Definition 2.3** A non-empty intersection of a projective variety in \( \mathbb{P}_n(\mathbb{F}) \) with an open subset of \( \mathbb{P}_n(\mathbb{F}) \) is called a quasi-projective variety.

Let \( V \subseteq \mathbb{P}_n(\mathbb{F}) \) be a quasi-projective variety. For each \( P \in V \) there exists a hyperplane \( H \subset \mathbb{P}_n(\mathbb{F}) \) with \( P \notin H \). Then \( P \in V \setminus H = V \cap A_n(\mathbb{F}) \) for \( A_n(\mathbb{F}) := \mathbb{P}_n(\mathbb{F}) \setminus H \). Let \( r \) be a defining relation of the hyperplane \( H \) in the variables \( x_0, \ldots, x_n \) (cf. 2.1). A \( \mathbb{F} \)-valued function \( f \) on \( V \) is called regular at \( P \), if there exists a neighbourhood \( N \) of \( P \) in \( V \cap A_n(\mathbb{F}) \) such that there exist polynomials \( a, b \in \mathbb{F}[X]/(r) \) with \( b(Q) \neq 0 \) for all \( Q \in N \) and \( f|_N = \frac{a}{b} \). The function \( f \) is regular on a non-empty open subset \( U \) of \( V \), if it is regular at every point of \( U \).

**Definition 2.4** Let \( V \) be a quasi-projective variety and let \( P \in V \). Then the local ring \( \mathcal{O}_P = \mathcal{O}_P(V) \) at \( P \) is defined as the ring of germs \( [f] \) of functions \( f : V \to \mathbb{F} \) which are regular on a neighbourhood of \( P \). In other words, an element of \( \mathcal{O}_P \) is an equivalence class of pairs \( (U, f) \) where \( U \) is an open subset of \( V \) containing \( P \), and \( f \) is a regular function on \( U \), and two such pairs \( (U, f) \) and \( (V, g) \) are equivalent if \( f|_{U \cap V} = g|_{U \cap V} \). For a non-empty open subset \( U \) of \( V \) define \( \mathcal{O}_U = \mathcal{O}_U(V) := \bigcap_{P \in U} \mathcal{O}_P(V) \).

The ring \( \mathcal{O}_P \) is indeed a local ring in the sense of commutative algebra: its unique maximal ideal \( \mathfrak{m}_P \) is the set of germs of regular functions which vanish at \( P \). The \( i \)th power \( \mathfrak{m}_P^i \) of \( \mathfrak{m}_P \) consists of the germs of regular functions whose vanishing order at \( P \) is at least \( i \). Taking these powers \( \mathfrak{m}_P \) as a neighbourhood basis of \( \mathcal{O}_P \), this defines a topology on \( \mathcal{O}_P \), the \( \mathfrak{m}_P \)-adic topology.

The inverse limit \( \hat{\mathcal{O}}_P := \lim_{\leftarrow} \mathcal{O}_P / \mathfrak{m}_P^i \) is the completion of \( \mathcal{O}_P \): cf. [Eis95, 7.1], [Har77b, p. 33], [ZS75, VIII §2]. It is a local ring whose maximal ideal is denoted by \( \hat{\mathfrak{m}}_P \).

**Example 2.5** The projective line \( \mathbb{P}_1(\mathbb{C}) \cong \mathbb{C} \cup \{ \infty \} \) is a non-singular projective curve (see 3.1 below). For \( P \in \mathbb{C} \) one has

\[
\mathcal{O}_P = \left\{ \frac{a}{b} \in \mathbb{C}(t) \mid b(P) \neq 0 \right\},
\]

\[
\mathfrak{m}_P = \left\{ \frac{a}{b} \in \mathcal{O}_P \mid a(P) = 0 \right\},
\]

\[
\hat{\mathcal{O}}_P = \left\{ \frac{a}{b} \in \mathbb{C}((t)) \mid b(P) \neq 0 \right\},
\]

\[
\hat{\mathfrak{m}}_P = \left\{ \frac{a}{b} \in \hat{\mathcal{O}}_P \mid a(P) = 0 \right\}.
\]

**Definition 2.6** Let \( V \) be a variety defined over \( k \). Then the field \( \mathbb{F}(V) \) of rational functions of \( V \) consists of the equivalence classes of pairs \( (U, f) \) where \( U \) is a non-empty open subset of \( V \) and \( f \) is a regular function on \( U \), and two such pairs \( (U, f) \) and \( (V, g) \) are equivalent if \( f|_{U \cap V} = g|_{U \cap V} \). The field of \( k \)-rational functions of \( V/k \) is

\[
K := k(V) := \{ h \in \mathbb{F}(V) \mid \sigma(h) = h \text{ for all } \sigma \in \text{Gal}(\mathbb{F}/k) \}.
\]

**Definition 2.7** For a point \( P \in \mathbb{P}_n(\mathbb{F}) \) the set \( \{ \sigma(P) \mid \sigma \in \text{Gal}(\mathbb{F}/k) \} \) is called a \( k \)-closed point. A \( k \)-closed point of cardinality 1 is called \( k \)-rational.
For each \( P \in \mathbb{P}_n(\overline{k}) \), each homogeneous polynomial \( f \in \overline{k}[X] \), and each \( \sigma \in \text{Gal}(\overline{k}/k) \) one has \( f(P) = 0 \) if and only if \( \sigma(f(P)) = 0 \) if and only if \( \sigma(f)(\sigma(P)) = 0 \). Hence, given a projective \( k \)-variety \( V \subseteq \mathbb{P}_n(\overline{k}) \), the assertion \( P \in V \) is equivalent to the assertion \( \{ \sigma(P) \mid \sigma \in \text{Gal}(\overline{k}/k) \} \subseteq V \).

It therefore makes sense to speak of \( k \)-closed points of \( V \). The set of \( k \)-closed points of \( V \) is denoted by \( V^k = V^\circ/k \). The degree \( \deg(P) \) of a \( k \)-closed point \( P \) equals the number of points it contains.

Example 2.8 For an involutive automorphism \( \sigma \) of \( \mathbb{C} \), define \( \mathbb{K} := \text{Fix}_\mathbb{C}(\sigma) \). The set of \( \mathbb{K} \)-closed points of \( \mathbb{P}_1(\mathbb{C}) \) equals the set of \( \sigma \)-orbits of \( \mathbb{P}_1(\mathbb{C}) \), i.e., the \( \sigma \)-fixed/\( \mathbb{K} \)-rational points of \( \mathbb{P}_1(\mathbb{C}) \) and the pairs of \( \sigma \)-conjugate points of \( \mathbb{P}_1(\mathbb{C}) \).

3 Curves over finite fields, considered as varieties

In this section I turn to one of the main objects of study of this survey, non-singular projective curves. For further reading the sources [Har77b, I], [NX09, 3], and [Ser88, II] or, if one prefers a very algebraic approach, [Che51, I, II] and [Ros02, 5, 6] are highly recommended.

Definition 3.1 Let \( k \) be a perfect field. A projective variety of dimension 1 defined over \( k \) is called a projective curve over \( k \) (cf. 2.1). A projective curve \( Y \) is called non-singular at \( P \in Y \), if \( \mathcal{O}_P \) (cf. 2.4) is a discrete valuation ring. It is called non-singular, if it is non-singular at each of its points.

For \( Y \) non-singular at \( P \), the valuation of \( \mathcal{O}_P \) is given by

\[
\nu_P : \mathcal{O}_P \to \mathbb{Z} \cup \{ \infty \} : x \mapsto \sup \{ i \in \mathbb{N} \cup \{ 0 \} \mid x \in m_P^i \},
\]

with the understanding that \( m_P^0 = \mathcal{O}_P \). This valuation extends to the valuation \( \hat{\nu}_P : \mathbb{K}(Y) \to \mathbb{Z} \cup \{ \infty \} : x \mapsto \sup \{ i \in \mathbb{N} \cup \{ 0 \} \mid x \in m_P^i \} \), as \( m_P \cap \mathcal{O}_P = m_P \); cf. [Eis95, 7.1], [Har77b, I.5.4A].

The valuation \( \hat{\nu}_P : \mathcal{O}_P \to \mathbb{Z} \cup \{ \infty \} \) also extends to a valuation on its field of fractions \( \mathbb{K}(Y) \) via \( \hat{\nu}_P(x) := \nu_P(a) - \nu_P(b) \) for \( a, b \in \mathcal{O}_P \) with \( \frac{a}{b} = x \).

The geometric intuition is that \( \hat{\nu}_P(x) \) indicates whether \( P \) is a zero or a pole of \( x \) and counts its multiplicity (cf. 2.5).

An algebraic function field (in one variable over \( k \)) is an extension field \( \mathbb{F} \) of \( k \) that admits an element \( x \) that is transcendental over \( k \) such that \( \mathbb{F}/k(x) \) is a field extension of finite degree.

Theorem 3.2 ([NX09, 3.2.9])

There is a one-to-one correspondence between \( k \)-isomorphism classes of non-singular projective curves over \( k \) and \( k \)-isomorphism classes of algebraic function fields of one variable with full constant field \( k \), via the map \( Y/k \mapsto k(Y) \) (cf. 2.6).

Here, the full constant field of an algebraic function field \( K \) over \( k \) is the algebraic closure of \( k \) in \( K \). In case the full constant field is finite, \( K \) is a global function field. For example, \( \mathbb{F}_q(t) \) is a global function field with full constant field \( \mathbb{F}_q \).

Definition 3.3 Two discrete valuations \( \nu_1 \) and \( \nu_2 \) of an algebraic function field \( K \) are equivalent if there exists a constant \( c > 0 \) such that \( \nu_1(x) = \nu_2(x) \) for all \( 0 \neq x \in K \). An equivalence class of discrete valuations is called a place. The degree of a valuation/place is the degree of the residue class field \( \mathcal{O}_P/m_P \) over the constant field \( k \). This is always a finite number; cf. [NX09, 1.5.13].

Theorem 3.4 ([NX09, 3.1.15])

Let \( Y/\mathbb{F}_q \) be a non-singular projective curve. Then there exists a natural one-to-one correspondence between \( \mathbb{F}_q \)-closed points of \( Y \) and places (cf. 3.3) of the field \( K = \mathbb{F}_q(Y) \) of \( \mathbb{F}_q \)-rational functions (cf. 2.6). Moreover, the degree of an \( \mathbb{F}_q \)-closed point is equal to the degree of the corresponding place.
3.5 Given a point $P$ in an $\mathbb{F}_q$-closed point of $Y$, the valuation $\nu_P : O_P \rightarrow \mathbb{Z} \cup \{\infty\}$ (cf. 3.1) restricts to a valuation $\nu_P : O_{P,K} := O_P \cap \mathbb{F}_q(Y) \rightarrow \mathbb{Z} \cup \{\infty\}$. By [NX09, 3.1.14] this valuation only depends on the $\mathbb{F}_q$-closed point of $Y$ containing $P$ and not on the particular choice of $P$ inside that $\mathbb{F}_q$-closed point. As before, $\nu_P$ extends to the field of $\mathbb{F}_q$-rational functions $K = \mathbb{F}_q(Y)$, the completion $\hat{O}_{P,K} := \hat{O}_P \cap K$, and to the field of fractions $K_P$ of the completion $\hat{O}_{P,K}$. The field $K_P$ is the local function field at $P$. For example, $\mathbb{F}_q((t))$ is the local function field of the global function field $\mathbb{F}(t)$ at the place corresponding to the irreducible polynomial $t$.

Definition 3.6 Let $Y/\mathbb{F}_q$ be a non-singular projective curve. The Weil divisor group $\text{Div}(Y) = \text{Div}(Y/\mathbb{F}_q)$ is the free abelian group over the set $Y^\circ$ of $\mathbb{F}_q$-closed points of $Y$ (cf. 2.7). An element $D = \sum_{P \in Y^\circ} n_P P \in \text{Div}(Y)$ is called a Weil divisor of $Y$. It is effective, in symbols $D \geq 0$, if $n_P \geq 0$ for all $P \in Y^\circ$. Two divisors $D_1$ and $D_2$ of $Y$ are equivalent if $D_1 - D_2$ is effective. The degree $\text{deg}(D)$ of a Weil divisor $D = \sum_{P \in Y^\circ} n_P P$ is given by $\text{deg}(D) := \sum_{P \in Y^\circ} n_P \text{deg}(P)$ (cf. 2.7). Also, define $\nu_P(D) := n_P$.

For $0 \neq x \in K = \mathbb{F}_q(Y)$, define the divisor $\text{div}(x)$ of $x$ by $\text{div}(x) := \sum_{P \in Y^\circ} \nu_P(x)P$. As, by [NX09, 3.3.2], [Ros02, 5.1], any $0 \neq x \in K$ admits only finitely many zeros (i.e., points $P \in Y^\circ$ with $\nu_P(x) > 0$) and poles (i.e., points $P \in Y^\circ$ with $\nu_P(x) < 0$), the divisor $\text{div}(x)$ indeed is a Weil divisor. A Weil divisor obtained in this way is principal, and has degree 0; cf. [NX09, 3.4.3], [Ros02, 5.1]. Two Weil divisors that differ by a principal divisor are called equivalent.

Definition 3.7 For a divisor $D$ of $Y/\mathbb{F}_q$, the Riemann–Roch space $L(D)$ is defined as

$$L(D) := \{0 \neq x \in K = \mathbb{F}_q(Y) \mid \text{div}(x) + D \geq 0\} \cup \{0\}.$$ 

The Riemann–Roch space $L(D)$ of a divisor $D$ is an $\mathbb{F}_q$-vector space of finite dimension (cf. [NX09, 3.4.1(iv)]).

Definition 3.8 Define the ring of repartitions as

$$\mathbb{A}_K := \{(x_P)_{P \in Y^\circ} \in \prod_{P \in Y^\circ} K \mid x_P \in O_P \text{ for almost all } P \in Y^\circ\}.$$ 

By [Che51, p. 25], [NX09, 3.3.2], the field $K$ embeds diagonally in $\mathbb{A}_K$. For any divisor $D \in \text{Div}(Y)$ define $\mathbb{A}_K(D) := \{x \in \mathbb{A}_K \mid \nu_P(x_P) + \nu_P(D) \geq 0 \text{ for all } P \in Y^\circ\}$. Note that $\mathbb{A}_K(D)$ is an $\mathbb{F}_q$-subvector space of $\mathbb{A}_K$.

Definition 3.9 A Weil differential of a non-singular projective curve $Y/\mathbb{F}_q$, resp. of its global function field $K$, is an $\mathbb{F}_q$-linear map $\omega : \mathbb{A}_K \rightarrow \mathbb{F}_q$ such that $\omega|_{\mathbb{A}_K(D)+K} = 0$ for some divisor $D \in \text{Div}(Y)$. Denote by $\Omega_K$ the set of all Weil differentials of $K$ and by $\Omega_K(D)$ the set of all Weil differentials of $K$ that vanish on $\mathbb{A}_K(D) + K$.

Note the difference between the ring of repartitions $\mathbb{A}_K$ and the ring of adèles $\hat{\mathbb{A}}_K$ defined in 4.3 below (completions). The concept of a Weil differential can equally well be introduced using the ring of adèles $\hat{\mathbb{A}}_K$, cf. [Ros02].

Observation 3.10 Let $D \in \text{Div}(Y)$. Then $\Omega_K(D) \cong \mathbb{A}_K/\mathbb{A}_K(D) + K$ as $\mathbb{F}_q$-vector spaces.

If $0 \neq \omega \in \Omega_K$, then by [NX09, 3.6.11], [Ros02, 6.8] the set $\{D \in \text{Div}(Y) \mid \omega|_{\mathbb{A}_K(D)+K} = 0\}$ has a unique maximal element with respect to $\geq$. This maximal element is called a canonical divisor of $Y$ and denoted by $(\omega)$. By [NX09, 3.6.10], [Ros02, 6.10]

$$\dim_K(\Omega_K) = 1$$

(1)

and by [NX09, 3.6.13], [Ros02, 6.9] the Weil differential $x \omega : \mathbb{A}_K \rightarrow \mathbb{F}_q : a \mapsto \omega(xa)$ satisfies

$$x \omega = \text{div}(x) + (\omega)$$

(2)

for any $0 \neq x \in K$, $0 \neq \omega \in \Omega_K$. Hence any two canonical divisors of $Y$ are equivalent (cf. 3.6).
Observation 3.11 Let \( D \) be a divisor of \( Y/\mathbb{F}_q \) and let \( W = (\omega) \) be a canonical divisor of \( Y \). Then
\[
L(W - D) \to \Omega_K(D) : x \mapsto x\omega
\]
is an isomorphism of \( \mathbb{F}_q \)-vector spaces.

**Proof.** For \( 0 \neq x \in L(W - D) \) we have \( (x\omega) \overset{(2)}{=} \text{div}(x) + (\omega) \geq -(W - D) + W = D \), so that indeed \( x\omega \in \Omega_K(D) \). Injectivity and \( \mathbb{F}_q \)-linearity of the map \( x \mapsto x\omega \) are clear. In order to prove surjectivity, let \( 0 \neq \omega_1 \in \Omega_K(D) \). By (1) there exists \( 0 \neq x \in K \) such that \( \omega_1 = x\omega \). Since we have \( \text{div}(x) + W \overset{(2)}{=} (x\omega) = (\omega_1) \geq D \), it follows that \( x \in L(W - D) \).

**Definition 3.12** Let \( Y/\mathbb{F}_q \) be a non-singular projective curve and let \( W \) be a canonical divisor of \( Y \). Define the **genus** \( g \) of the curve \( Y/\mathbb{F}_q \) by
\[
g := \dim_{\mathbb{F}_q}(L(W)) \overset{3.11}{=} \dim_{\mathbb{F}_q}(\Omega_K(0)) \overset{3.10}{=} \dim_{\mathbb{F}_q}(\mathcal{O}_K/K(0) + K).
\]

**Theorem 3.13** (Riemann–Roch Theorem, [Che51, p. 30], [NX09, 3.6.14], [Ros02, 5.4])
Let \( Y/\mathbb{F}_q \) be a non-singular projective curve of genus \( g \) and let \( W \) be a canonical divisor. Then for any divisor \( D \in \text{Div}(Y) \) one has
\[
\dim_{\mathbb{F}_q}(L(D)) - \dim_{\mathbb{F}_q}(L(W - D)) = \deg(D) + 1 - g.
\]

The Riemann–Roch theorem combined with the study of the zeta function of the global function field \( \mathbb{F}_q(Y) \) allows one to establish the following useful result.

**Proposition 3.14** ([NX09, 4.1.10])

Let \( Y/\mathbb{F}_q \) be a non-singular projective curve. Then there exists a Weil divisor of \( Y \) of degree 1.

Note that this Weil divisor of degree 1 need not be effective. In fact, for \( p \geq 5 \) prime the non-singular projective curve over \( \mathbb{F}_p \) given by \( x^{p-1} + y^{p-1} = 3z^{p-1} \) does not admit any \( \mathbb{F}_p \)-rational point and, thus, there cannot exist an effective Weil divisor of degree 1 of this curve.

**Definition 3.15** A vector bundle of rank \( r \) over a curve \( Y/k \) is a variety \( E/k \) together with a morphism \( \pi : E \to Y \) such that there exists an open covering \( \{U_i\} \) of \( Y \) and isomorphisms \( \phi_i : \pi^{-1}(U_i) \to U_i \times A_r(k) \) (where \( A_r(k) \) denotes the affine space over \( k \) of dimension \( r \)) such that for each pair \( U_i, U_j \) the composition \( \phi_j \circ \phi_i^{-1} |_{U_i \cap U_j} \) equals \( (\text{id}, \phi_{i,j}) \) for a linear map \( \phi_{i,j} \). A section of a vector bundle \( \pi : E \to Y \) over an open set \( U \subset Y \) is a map \( s : U \to E \) such that \( \pi \circ s = \text{id} \).

## 4 Geometry of numbers

In this section I describe Weil’s geometry of numbers. For further reading the sources [Wei82, 2], [Wei95, VI] are highly recommended.

### 4.1 Let \( Y/\mathbb{F}_q \) be a non-singular projective curve, let \( P \) be an \( \mathbb{F}_q \)-closed point of \( Y \), and let \( \nu_P \) be the valuation of \( K = \mathbb{F}_q(Y) \) discussed in 3.5. Then
\[
| \cdot |_P : K \to \mathbb{R} : x \mapsto (q^{\deg(P)})^{-\nu_P(x)}
\]
defines an absolute value on \( K \). The completion of \( K \) with respect to this absolute value equals \( K_P \) (cf. 3.5), which is locally compact as the zero neighbourhood \( \mathcal{O}_{P,K} \) is an inverse limit of finite rings (cf. 2.4), whence compact. On each field \( K_P \) define the canonical Haar measure to be the one with respect to which \( \mathcal{O}_{P,K} \) has volume 1.

\(^1\)The notation \( \overset{(2)}{=} \) means that (2) is a justification for the equality.
Theorem 4.2 ([AW45])
Let \(Y/\mathbb{P}_q\) be a non-singular projective curve. Then each \(0 \neq x \in K\) satisfies \(|x|_P = 1\) for almost all \(P \in Y^o\). Moreover, for each \(0 \neq x \in K\),
\[
\prod_{P \in Y^o} |x|_P = 1.
\]

Definition 4.3 In analogy to 3.8 define the \textbf{adèlle ring}
\[
\hat{K}_K := \{ (x_P)_{P \in Y^o} \in \prod_{P \in Y^o} K_P \mid x_P \in \hat{O}_P \text{ for almost all } P \in Y^o \}.
\]
It is a locally compact ring and contains \(K\) embedded diagonally as a discrete subring (cf. 4.2). Define
\[
| \cdot | : \hat{K}_K \to \mathbb{R} : x \mapsto \prod_{P \in Y^o} |x|_P.
\]
The product measure of the canonical Haar measures on each individual \(K_P\) (4.1) yields a Haar measure on the locally compact group \((\hat{K}_K, +)\). Let \(\omega_{\hat{K}_K}\) denote the corresponding volume form. For any divisor \(D \in \text{Div}(Y)\), define \(\hat{K}_K(D) := \{ x \in \hat{K}_K \mid \nu_P(x) + \nu_P(D) \geq 0 \text{ for all } P \in Y^o \}\). Note that \(\hat{O}_K := \hat{K}_K(0)\) is the maximal compact subring of \(\hat{K}_K\). It has volume 1 with respect to the chosen Haar measure.

4.4 For a finite subset \(S \subset Y^o\) define the \textbf{ring of S-adèles}
\[
\hat{K}_S := \prod_{P \in S} K_P \times \prod_{P \in Y^o \setminus S} \hat{O}_{P,K}
\]
(cf. 3.5). One has \(\hat{K}_K = \lim_{\to} \hat{K}_S\). For \(K\) embedded diagonally in \(\hat{K}_K\) as a discrete subring (cf. 4.3) define the \textbf{ring of S-integers}
\[
\hat{O}_S := K \cap \hat{K}_S = \bigcap_{P \in Y^o \setminus S} (K \cap \hat{O}_{P,K}) \quad \text{as a dense subring and, hence, also in } \hat{A}_S.
\]

4.5 Let \(G\) be a unimodular locally compact group, let \(\Gamma\) be a discrete subgroup of \(G\), let \(H\) be a compact open subgroup of \(G\), let \(\mu\) be a Haar measure on \(G\), and assume the double coset space \(X = H \backslash G/\Gamma\), considered as a system of representatives of the \(\Gamma\)-orbits on \(H \backslash G\), is countable. Note that \(\mu\) is also a Haar measure on \(H\), since \(H\) is open, and hence of positive volume. Then, cf. [Ser03, p. 84],
\[
\int_{G/\Gamma} \frac{d\mu}{|g\Gamma|} = \sum_{x \in X} \left( \int_{G_x/\Gamma_x} \frac{d\mu}{|g\Gamma_x|} \right) = \sum_{x \in X} \int_{G_x} \frac{d\mu}{|\Gamma_x|} = \int_{H} \int_{\Gamma} \frac{d\mu}{|\Gamma|}.
\]
Here, for choices of \(x_0 \in X\) and \((g_{x_0}^x)_{x \in X} \in G\) with \(g_{x_0}^x(x_0) = x\), the map \(\bigcup_{x \in X} G_x/\Gamma_x \to G/\Gamma : g\Gamma_x \mapsto g_{x_0}^x g\Gamma\) is an isomorphism of orbit spaces, where \(G_x\) and \(\Gamma_x\) denote the stabilizers of \(x\) in the respective group.

4.6 One has
\[
\hat{K}_K(D) \hat{K}_K/K = \hat{K}_K/\hat{K}_K(D) + K \cong \hat{K}_K/\hat{K}_K(D) + K \cong \Omega_K(D) \cong L(W - D), \quad (3)
\]
\[
K \cap \hat{K}_K(D) = K \cap \{ x \in \hat{K}_K \mid \nu_P(x) + \nu_P(D) \geq 0 \text{ for all } P \in Y^o \} = L(D). \quad (4)
\]
The observation that the intersection of the lattice \(K\) with the compactum \(\hat{K}_K(D)\) equals \(L(D)\) is a classical theme in the geometry of numbers, cf. [Arm67, p. 387].
Proposition 4.7 ([Wei82, 2.1.3], [Wei95, p. 100])

Using the notation of 4.3, one has
\[ \int_{\hat{A}_K/K} \omega_{\hat{A}_K} = q^{g-1}. \]

Proof. For the compact open subgroup \( \hat{A}_K(0) \) of \( \hat{A}_K \), Serre’s formula (cf. 4.5) yields
\[
\int_{\hat{A}_K/K} \omega_{\hat{A}_K} = \int_{\hat{A}_K(0)/\hat{A}_K(0)/K} \frac{1}{|K \cap \hat{A}_K(0)|} \int_{\hat{A}_K(0)} \omega_{\hat{A}_K}.
\]

Therefore, as \( \int_{\hat{A}_K(0)} \omega_{\hat{A}_K} = 1 \) (cf. 4.3), the Riemann–Roch Theorem (cf. 3.13) or, in fact, its underlying definitions and observations 3.12 and 4.6 imply \( \int_{\hat{A}_K/K} \omega_{\hat{A}_K} = q^{g-1}. \)

Proposition 4.8 ([Har69, p. 39], [Wei95, p. 98])

One has
\[ \int_{\hat{A}_K(D)} \omega_{\hat{A}_K} = q^{\deg(D)}. \]

Proof. The strategy is to first use 4.7 and then apply 4.5 to the compact open subgroup \( \hat{A}_K(D) \) of \( \hat{A}_K \) in analogy to (5) in the proof of 4.7. This way one computes
\[
q^{g-1} = \int_{\hat{A}_K/K} \omega_{\hat{A}_K} = \int_{\hat{A}_K(D)/\hat{A}_K(D)/K} \frac{|\hat{A}_K(D) \cap \hat{A}_K/K|}{|K \cap \hat{A}_K(D)|} \int_{\hat{A}_K(D)} \omega_{\hat{A}_K}.
\]

Hence, by the Riemann–Roch Theorem (cf. 3.13), one has \( \int_{\hat{A}_K(D)} \omega_{\hat{A}_K} = q^{\deg(D)}. \)

5 Curves over finite fields, considered as schemes

In this section I return to the study of non-singular projective curves, in the more general context of schemes. For further reading the sources [Har77b, II, III, IV], [Liu02, 2, 6, 7] are highly recommended.

The basic idea of Harder’s reduction theory is to apply the geometry of numbers to the groups of \( \hat{A}_K \)-rational points of reductive \( K \)-isotropic algebraic \( K \)-groups, see 7.2 below. The concept of schemes, which generalizes the concept of varieties, allows one to do so in a very efficient way, as in this language one can consider reductive groups over projective curves, thus making the Riemann–Roch theorem applicable.

Definition 5.1 Let \( X \) be a topological space, let \( \mathfrak{Top}(X) \) be the category of open subsets of \( X \) with the inclusion maps as morphisms, and let \( \mathcal{C} \) be one of the categories \( \mathbf{Ab} \) of abelian groups, \( \mathcal{C} \mathbf{Ring} \) of commutative rings with 1, or \( \mathfrak{Mod}(A) \) of modules over the (commutative) ring \( A \). A contravariant functor \( F \) from \( \mathfrak{Top}(X) \) to \( \mathcal{C} \) which satisfies \( F(\emptyset) = 0 \) is called a presheaf on \( X \) with values in \( \mathcal{C} \). For presheaves \( F \), \( G \) on \( X \) with values in \( \mathcal{C} \), a natural transformation from \( F \) to \( G \) is called a morphism of presheaves.

A presheaf \( F \) is called a sheaf, if it satisfies the following conditions: (a) if \( U \) is an open subset of \( X \), if \( \{V_i\} \) is an open covering of \( U \), and if \( s \in \mathcal{F}(U) \) satisfies \( s|_{V_i} = 0 \) for all \( i \), then \( s = 0 \); and (b) if \( U \) is an open subset of \( X \), if \( \{V_i\} \) is an open covering of \( U \), and if the \( s_i \in \mathcal{F}(V_i) \) satisfy \( s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j} \) for all \( i, j \), then there exists \( s \in \mathcal{F}(U) \) such that \( s|_{V_i} = s_i \) for all \( i \).
For a presheaf $F$ on $X$ and an element $x \in X$, the stalk $F_x$ is defined as the direct limit $\lim_{U \subseteq X} F(U)$ over the open neighbourhoods $U$ of $x$ where $U \subseteq V$ if $U \supseteq V$, i.e., $F_x$ consists of the germ of elements of $F(U)$ at $x$.

**Example 5.2** Let $V$ be a quasi-projective variety. The functor $\mathcal{O}_V : U \mapsto \mathcal{O}_U(V)$ (cf. 2.4) is a sheaf on $V$. The stalk $\mathcal{O}_{V,P}$ at a point $P$ equals the local ring $\mathcal{O}_P$ at that point.

**Definition 5.3** Let $X$ be a topological space and let $F$ be a presheaf on $X$. The **sheaf associated to the presheaf** $F$, also called the **sheafification of the presheaf** $F$, is a pair $(\mathcal{F}^\dagger, \theta)$ consisting of a sheaf $\mathcal{F}^\dagger$ and a morphism $\theta : F \to \mathcal{F}^\dagger$ of presheaves that satisfies the following universal property: for every morphism $\alpha : F \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there exists a unique morphism $\tilde{\alpha} : \mathcal{F}^\dagger \to \mathcal{G}$ such that $\alpha = \tilde{\alpha} \circ \theta$.

The sheafification of a presheaf is unique for abstract reasons and exists by [Liu02, 2.2.15]; see also [Har08, 3.3], [Har77b, II.1.2], [Liu02, Ex. 2.2.3].

**Definition 5.4** A subfunctor $F'$ of a sheaf $F$ that is itself a sheaf is called a **subsheaf** of $F$. If $F$ takes values in $\mathcal{A}$, then the **quotient sheaf** $F/F'$ is the sheaf associated to the presheaf $U \mapsto F(U)/F'(U)$ (cf. 5.3). If $F$ is a sheaf on $X$ then, given an open cover $\{U_i\}$ of $X$, an element of $F/F'(X)$ can be described by elements $f_i \in F(U_i)$ such that $f_i \equiv f_j \pmod{F'}$ for all $i, j$.

**Definition 5.5** Let $f : X \to Y$ be a continuous map between topological spaces, let $F$ be a sheaf on $X$, and let $\mathcal{G}$ be a sheaf on $Y$. Then $Y \supseteq U \mapsto F(f^{-1}(U))$ defines a sheaf $f_*F$ on $Y$, the **direct image sheaf**. For each $x \in X$ there is a natural map $\epsilon_x : (f_*F)_{f(x)} \to F_x$. The **inverse image sheaf** $f^{-1}\mathcal{G}$ on $X$ is the sheaf associated to the presheaf $X \supseteq U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V)$ where the limit is taken over all open subsets $V$ of $Y$ that contain $f(U)$; cf. 5.3. For each $x \in X$ one has $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$ ([Liu02, p. 37]).

**Definition 5.6** A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space $X$ and a sheaf of rings $\mathcal{O}_X$, i.e., a sheaf on $X$ with values in $\mathcal{CRing}$. It is a **locally ringed space**, if for each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring.

**Example 5.7** Let $A$ be a commutative ring, define the set Spec$(A)$ to be the set of prime ideals of $A$, and define a topology on Spec$(A)$ by taking the closed sets to be sets of the type $\{p \in \text{Spec}(A) \mid a \subseteq p\}$, for arbitrary ideals $a$ of $A$; cf. [Har77b, II.2.1], [Liu02, 2.1.1]. For an open set $U \subseteq \text{Spec}(A)$ define $\mathcal{O}(U)$ to be the set of functions $s : U \to \bigcup_{p \in U} A_p$, where $A_p$ denotes the localization of $A$ at $p$, such that $s(p) \in A_p$ and for each $p \in U$ there exist a neighbourhood $V \subseteq U$ of $p$ and elements $a, f \in A$ with $f \not\in q$ and $s(q) = f/q$ in $A_q$ for each $q \in V$.

The resulting locally ringed space Spec$A = (\text{Spec}(A), \mathcal{O})$ is called the **spectrum** of $A$; cf. [Har77b, p. 70], [Liu02, 2.3.2].

**Definition 5.8** A **morphism of locally ringed spaces** $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$ the induced map $\mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp} (f_*\mathcal{O}_X)_{f(x)} \xrightarrow{\epsilon_x} \mathcal{O}_{X,x}$ is a local homomorphism (cf. 5.5). An **isomorphism** is an invertible morphism. A morphism $(f, f^\sharp) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ of ringed spaces is a **closed immersion** if $f$ is a topological closed immersion and if each $f^\sharp_x$ is surjective.

**Definition 5.9** Let $(X, \mathcal{O}_X)$ be a ringed space. An **$\mathcal{O}_X$-module** is a sheaf $\mathcal{F}$ on $X$ with values in $\mathcal{A}$ such that for each open subset $U \subseteq X$ the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and, for each inclusion of open subsets $V \subseteq U$, the group homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

If $U$ is an open subset of $X$, and if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}|_U$ is an $\mathcal{O}_{X|U}$-module. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules, the group of morphisms from $\mathcal{F}$ to $\mathcal{G}$ is denoted by $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. The presheaf $U \mapsto \text{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf, denoted by $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$; it is also an $\mathcal{O}_X$-module.
Definition 5.10 For any ringed space \((X, \mathcal{O}_X)\), define the Picard group \(\text{Pic}(X)\) of \(X\) to be the group of isomorphism classes of invertible \(\mathcal{O}_X\)-modules under the operation \(\otimes_{\mathcal{O}_X}\), cf. [Har77b, II.6.12]. The inverse of an invertible \(\mathcal{O}_X\)-module \(\mathcal{L}\) is \(\mathcal{L}' := \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)\), cf. [Har77b, Ex. II.5.1], [Liu02, 5.1.12].

Definition 5.11 An affine scheme is a locally ringed space \((X, \mathcal{O}_X)\) which is isomorphic to the spectrum of some ring. A locally ringed space \((X, \mathcal{O}_X)\) is a scheme, if every point of \(X\) has an open neighbourhood \(U\) such that \((U, \mathcal{O}_{X|U})\) is an affine scheme. Its dimension is the dimension of \(X\) as a topological space (cf. 2.1) and it is called irreducible, if \(X\) is irreducible.

Definition 5.12 A morphism of schemes is a morphism of locally ringed spaces. Likewise, an isomorphism of schemes is an isomorphism of locally ringed spaces. A closed subscheme \((Z, \mathcal{O}_Z)\) of a scheme \((X, \mathcal{O}_X)\) consists of a closed subset \(Z\) of \(X\) and a closed immersion \((j, j^\#)(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)\) where \(j\) is the canonical injection. (Cf. 5.8.)

Definition 5.13 Let \(S\) be a scheme. An \(S\)-scheme or a scheme over \(S\) is a scheme \(X\) endowed with a morphism of schemes \(\pi : X \to S\). If \(S = \text{Spec}A\) for a ring \(A\), then an \(S\)-scheme is also called an \(A\)-scheme.

Example 5.14 Let \((X, \mathcal{O}_X)\) be a scheme, for \(x \in X\) let \(\mathcal{O}_{X,x}\) be the stalk at \(x\) (cf. 5.1) and let \(m_x\) be its maximal ideal. The residue field of \(x\) on \(X\) is \(k(x) := \mathcal{O}_{X,x}/m_x\). For any field \(K\), by [Har77b, II.2.7], there exists a one-to-one correspondence between morphisms from the spectrum of \(K\) to \((X, \mathcal{O}_X)\) and pairs of a point \(x \in X\) and an inclusion \(k(x) \to K\). In other words, any spectrum of a field that contains the residue field of some point \(x \in X\) as a subfield can be considered as an \(X\)-scheme.

Definition 5.15 Let \(S\) be a scheme and let \(X, Y\) be \(S\)-schemes with respect to the morphisms \(\pi_X : X \to S\) and \(\pi_Y : Y \to S\). The fibre product \(X \times_S Y\) of \(X\) and \(Y\) over \(S\) is defined to be an \(S\)-scheme together with morphisms \(p_X : X \times_S Y \to X\) and \(p_Y : X \times_S Y \to Y\) satisfying \(\pi_X \circ p_X = \pi_Y \circ p_Y\), such that given any \(S\)-scheme \(Z\) and morphisms \(f : Z \to X\) and \(g : Z \to Y\) satisfying \(\pi_X \circ f = \pi_Y \circ g\), then there exists a unique morphism \(\theta : Z \to X \times_S Y\) satisfying \(f = p_X \circ \theta\) and \(g = p_Y \circ \theta\).

By [Har77b, II.3.3], [Liu02, 3.1.2] the fibre product of two \(S\)-schemes exists and is unique up to unique isomorphism.
Example 5.16 Let $A$ be a ring, let $B = \bigoplus_{d \geq 0} B_d$ be a graded $A$-algebra, and let $B_+ := \bigoplus_{d > 0} B_d$.
Define the set $\text{Proj}(B)$ to be the set of homogeneous prime ideals of $B$ which do not contain $B_+$, and define a topology on $\text{Proj}(B)$ by taking the closed sets to be sets of the form $V(a) = \{ p \in \text{Proj}(B) \mid a \subseteq p \}$, for arbitrary homogeneous ideals $a$ of $A$; cf. [Har77b, II.2.4], [Liu02, p. 50]. For each $p \in \text{Proj}(B)$ let $B_p$ be the ring of elements of degree zero in the localized ring $T^{-1}B$, where $T$ is the multiplicative system of all homogeneous elements of $B$ which are not in $p$. For an open set $U \subseteq \text{Proj}(B)$ define $\mathcal{O}(U)$ to be the set of functions $s : U \to \bigsqcup_{p \in U} B_p$ such that $s(p) \in B_p$ and for each $p \in U$ there exist a neighbourhood $V \subseteq U$ of $p$ and homogeneous elements $a, f \in B$ of the same degree with $f \not\equiv q$ for each $q \in V$ and $s(q) = a/f$ in $B_q$. The resulting ringed space $\text{Proj}B = (\text{Proj}(B), \mathcal{O})$ is a scheme, in fact an $A$-scheme, cf. [Har77b, II.2.5], [Liu02, 2.3.38].

Example 5.17 Let $A$ be a ring and let $B = A[x_0, x_1, ..., x_n]$ with grading by degree. Then the scheme $\mathbb{P}_n(A) := \text{Proj}B$ is the projective $n$-space over $A$.

5.18 The schemes together with their morphisms form a category $\mathfrak{Sch}$. It contains the category $\mathfrak{AffSch}$ of affine schemes as a subcategory. The functor $\text{Spec}$ from $\mathcal{CRing}$ to $\mathfrak{AffSch}$ that sends $A$ to Spec $A$ and a ring homomorphism $A \to B$ to the corresponding morphism of ringed spaces from the spectrum of $B$ to the spectrum of $A$ (cf. [Har77b, II.2.3]) is a contravariant equivalence (also called a duality) between the category of commutative rings and the category of affine schemes. The global section functor, i.e., the functor from $\mathfrak{AffSch}$ to $\mathcal{CRing}$ that sends $(\text{Spec}(A), \mathcal{O})$ to $\mathcal{O}(\text{Spec}(A))$, is an inverse functor to $\text{Spec}$; cf. [Har77b, II.2.2, II.2.3].

5.19 The affine scheme $\text{Spec} \mathbb{Z}$ is a final object in both categories of schemes and affine schemes; cf. [Har77b, Ex. II.2.5].

Definition 5.20 Let $k$ be a perfect field. A projective scheme over $k$ is a scheme over $k$ that is isomorphic to a closed subscheme of the projective space $\mathbb{P}_n(k)$, for some $n$ (cf. 5.12, 5.17). A projective curve over $k$ is an irreducible projective scheme of dimension 1. A projective curve $Y$ is called non-singular at $P \in Y$, if $\mathcal{O}_P$ (cf. 5.1) is a discrete valuation ring (cf. [Liu02, 4.2.9]). A projective curve is called non-singular, if it is non-singular at each of its points.

5.21 Comparing 3.1 with 5.20 it is clear that a non-singular projective curve, defined as a variety, yields a non-singular projective curve if considered as a scheme. In the context of 5.2 one can convert locally free $\mathcal{O}_Y$-modules (5.9) into vector bundles over $Y$ (3.15) and vice versa, cf. [Har08, p. 61], [Har77b, Ex. II.5.18], [Ser55]: a vector bundle $\pi : E \to Y$ yields a sheaf $\mathcal{E}_\pi$ by sending an open subset $U$ of $Y$ to the (fibre-wise) $\mathcal{O}_Y(U)$-module of sections $U \to E$ over $U$.

5.22 Let $A$ be a ring. The quotient of $A[x_0, x_1, ..., x_n]$ by a homogeneous ideal is a homogeneous $A$-algebra. By [Liu02, 2.3.41], for every homogeneous $A$-algebra $B$, the scheme $\text{Proj}B$ is a projective scheme and, by [Liu02, 5.1.30], conversely every projective scheme over $A$ is isomorphic to $\text{Proj}B$, for some homogeneous $A$-algebra $B$.

Definition 5.23 Let $X$ be a scheme. For each open subset $U$, let $\mathcal{S}(U)$ denote the set of elements $s \in \mathcal{O}_X(U)$ such that for each $x \in U$ the germ $s_x$ is not a zero divisor in the stalk $\mathcal{O}_x$. Then the sheaf $\mathcal{K}$ of total quotient rings of the sheaf $\mathcal{O}_X$ is the sheaf associated to the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$ (cf. 5.3). Let $\mathcal{K}^*$ and $\mathcal{O}^*$ denote the sheaves of groups of invertible elements in the sheaves of rings $\mathcal{K}$, resp. $\mathcal{O}_X$. An element of the group $(\mathcal{K}^*/\mathcal{O}^*)(X)$ is called a Cartier divisor. A Cartier divisor is principal if it is in the image of the natural map $\mathcal{K}^*(X) \to (\mathcal{K}^*/\mathcal{O}^*)(X)$. Two Cartier divisors are linearly equivalent if their difference is principal.

Proposition 5.24 ([Har77b, II.6.11], [Liu02, 7.2.16])

Let $Y$ be a non-singular projective curve over $\mathbb{F}_q$. Then there exists a natural isomorphism between the group $\text{Div}(Y)$ of Weil divisors (cf. 3.6) and the group $\mathcal{K}^*/\mathcal{O}^*(Y)$ of Cartier divisors and, furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.
In fact, as $\mathcal{O}_Y$ is integral ([Liu02, 2.4.17]), the sheaf $\mathcal{K}$ is the constant sheaf corresponding to the function field $K$ of $Y$. As in 5.4, a Cartier divisor is generated by a family $\{U_i, f_i\}$ where $\{U_i\}$ is an open cover of $Y$ and $f_i \in \mathcal{K}^*(U_i)$. For each $P \in Y$ and for all $i, j$ such that $P \in U_i, U_j$ one has $\nu_P(f_i) = \nu_P(f_j)$, as $\frac{1}{f_i}$ is invertible on $U_i \cap U_j$. One therefore obtains the well-defined Weil divisor $\sum_{P \in Y} \nu_P(f_i)P$.

**Definition 5.26** Let $D$ be a Cartier divisor on a non-singular projective curve $Y/\mathbb{F}_q$ represented by a family $\{U_i, f_i\}$ where $\{U_i\}$ is an open cover of $Y$ and $f_i \in \mathcal{K}^*(U_i)$. The sheaf $\mathcal{L}(D)$ associated to $D$ is the subsheaf of $\mathcal{K}$ given by taking $\mathcal{L}(D)$ to be the sub-$\mathcal{O}_Y$-module of $\mathcal{K}$ generated by $f_i^{-1}$ on $U_i$. Since $\frac{1}{f_i}$ is invertible on $U_i \cap U_j$, the elements $f_i^{-1}$ and $f_j^{-1}$ generate the same $\mathcal{O}_Y$-module, and hence $\mathcal{L}(D)$ is well defined.

**Proposition 5.27** ([Har77b, II.6.13, II.6.15], [Liu02, 7.1.19]) Let $Y$ be a non-singular projective curve over $\mathbb{F}_q$. Then for any Cartier divisor $D$, the sheaf $\mathcal{L}(D)$ is an invertible $\mathcal{O}_Y$-module. Moreover, the map $D \mapsto \mathcal{L}(D)$ induces a surjection from the group of Cartier divisors onto the Picard group $\text{Pic}(Y)$. The kernel of this surjection is the group of principal Cartier divisors.

**Definition 5.28** Let $Y/\mathbb{F}_q$ be a non-singular projective curve. Define the degree $c(\mathcal{L})$ of an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ to be the degree $\deg(D)$ of the corresponding Weil divisor; cf. 5.24, 5.25, 5.27; [Liu02, 7.3.1, 7.3.2].

**Example 5.29** Let $Y = \mathbb{P}^1 = \{(x : y)\}$ and let $U_1 = \{(x : y) \mid y \neq 0\}$ and $U_2 = \{(x : y) \mid x \neq 0\}$ be an open cover of $Y$ with local coordinates $s = \frac{x}{y}$ and $t = \frac{y}{x}$. For $d \in \mathbb{N}$ define the line bundle $L(d) = \{(x : y), (a, b)\} \in \mathbb{P}^1 \times A_2 \mid x^d y = a^d b\}$ with canonical projection $\pi : L(d) \rightarrow \mathbb{P}^1$. Then

$$U_1 \times A_1 \xrightarrow{f_1} \pi^{-1}(U_1) = \{(x : y), (a, b)\} \in U_1 \times A_2 \mid a = s^d b\}$$

$$(s, b) \mapsto ((s : 1, (s^d b, b))$$

$$U_2 \times A_1 \xrightarrow{f_2} \pi^{-1}(U_2) = \{(x : y), (a, b)\} \in U_1 \times A_2 \mid b = t^d a\}$$

$$(t, a) \mapsto ((1 : t), (a, t^d a)).$$

One has $f_1|_{U_1 \cap U_2} = s^d f_2|_{U_1 \cap U_2}$, so that the line bundle $L(d)$ corresponds to the Weil divisor $\sum_{P \in Y} \nu_P(f_i)P$ of degree $d$ (cf. 5.25).

**Theorem 5.30** (Riemann–Roch Theorem, [Har77b, IV.1.3], [Liu02, 7.3.26]) Let $Y/\mathbb{F}_q$ be a non-singular projective curve of genus $g$. Then for any invertible $\mathcal{O}_Y$-module $\mathcal{L}$

$$\dim_{\mathbb{F}_q}(H^0(Y, \mathcal{L})) - \dim_{\mathbb{F}_q}(H^1(Y, \mathcal{L})) = c(\mathcal{L}) + 1 - g.$$

Here $H^i(Y, \mathcal{L})$ denotes sheaf cohomology groups as defined in [Har77b, III], [Liu02, 5.2].

**5.31** For a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of rank $n$ define $c(\mathcal{E}) := \sum_{i=1}^n c(\mathcal{E}_i/\mathcal{E}_{i-1})$ where $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E}$ is a filtration such that each $\mathcal{E}_i/\mathcal{E}_{i-1}$ is invertible; cf. [Gro57, 2.1], [Har68, p. 122]. Then 5.30 implies $\dim_{\mathbb{F}_q}(H^0(Y, \mathcal{E})) - \dim_{\mathbb{F}_q}(H^1(Y, \mathcal{E})) = c(\mathcal{E}) + (1 - g)n$. (Cf. [Wei95, p. 99].)

6 Reduction theory for rationally trivial group schemes

In this section I describe the heart of Harder’s reduction theory based on [Har68]. From this section on I will assume the reader has a basic intuition for linear algebraic groups and feels comfortable with some of the standard terminology such as that of Borel subgroups and parabolic subgroups. For introductory reading the sources [Bor94] and [Hum73], for further reading the sources [Jan03], [Spa98], [Vos98], [Wal79] are highly recommended. I also recommend [Lau97, Appendix E].
Definition 6.1 An object \(X\) of a category \(\mathcal{C}\) is called a **group object**, if for each object \(Y\) of \(\mathcal{C}\) the set \(\text{Hom}_\mathcal{C}(Y, X)\) of morphisms from \(Y\) to \(X\) is a group and the correspondence \(Y \mapsto \text{Hom}_\mathcal{C}(Y, X)\) is a (contravariant) functor from the category \(\mathcal{C}\) to the category \(\mathcal{Gr}\) of groups; cf. [Vos98, p. 3].

Definition 6.2 A group object in the categories \(\mathbf{Sch}\) of schemes, resp. \(\mathbf{AffSch}\) of affines schemes (cf. 5.18) is called a **group scheme**, resp. an **affine group scheme**.

6.3 As the categories \(\mathbf{Sch}\) and \(\mathbf{AffSch}\) both have \(\text{Spec} \mathbb{Z}\) as a final object (cf. 5.19) and have finite products, being an (affine) group scheme is an intrinsic property; cf. [Vos98, p. 3]. In particular, any affine group scheme is a group scheme.

6.4 When restricting (and co-restricting accordingly) the functor \(\text{Spec}\) from \(\mathbf{CRing}\) to \(\mathbf{AffSch}\) (cf. 5.18) to the subcategory of commutative Hopf algebras, it yields a contravariant equivalence (duality) to the category of affine group schemes; cf. [Vos98, 1.3], [Wat79, p. 9].

Example 6.5 Let \(G_n\) be the covariant functor on the category \(\mathbf{CRing}\) defined by \(G_n(A) = (A, +)\). Since by means of duality (cf. 5.18)

\[
\text{Hom}_{\mathbf{CRing}}(\mathbb{Z}[T], A) = A \iff \text{Hom}_{\mathbf{AffSch}}(\text{Spec} A, \text{Spec} \mathbb{Z}[T]) = A,
\]

the functor \(G_n\), considered as the contravariant functor on the category \(\mathbf{AffSch}\) that sends \((\text{Spec}(A), \mathcal{O})\) to \((\mathcal{O}(\text{Spec}(A)), +)\), is represented by \(\text{Spec} \mathbb{Z}[T]\). Therefore, \(\text{Spec} \mathbb{Z}[T]\) is an affine group scheme. By 6.3, it is also a group scheme, which as a functor sends \((X, \mathcal{O}_X)\) to \((\mathcal{O}_X(X), +)\).

Example 6.6 Let \(G_m\) be the covariant functor on the category \(\mathbf{CRing}\) defined by \(G_m(A) = (A^*, \cdot)\). Since

\[
\text{Hom}_{\mathbf{CRing}}(\mathbb{Z}[T, T^{-1}], A) = A^* \iff \text{Hom}_{\mathbf{AffSch}}(\text{Spec} A, \text{Spec} \mathbb{Z}[T, T^{-1}]) = A^*,
\]

the functor \(G_m\), considered as a contravariant functor on the category \(\mathbf{AffSch}\), is represented by \(\text{Spec} \mathbb{Z}[T, T^{-1}]\). Therefore, \(\text{Spec} \mathbb{Z}[T, T^{-1}]\) is an affine group scheme and, by 6.3, also a group scheme.

Example 6.7 Let \(\text{GL}_n\) be the covariant functor on the category \(\mathbf{CRing}\) defined by

\[
A \mapsto \text{GL}_n(A) = \{M \in A^{n \times n} \mid \det(M) \neq 0\}.
\]

It is represented by the affine scheme \(\text{GL}_n = \text{Spec} \mathbb{Z}[T_{11}, ..., T_{nn}, \det(T_{ij})^{-1}]\) and, hence, is an affine group scheme. By 6.3, \(\text{GL}_n\) is also a group scheme, and induces the contravariant functor on \(\mathbf{Sch}\) that sends \((X, \mathcal{O}_X)\) to \(\text{GL}_n(\mathcal{O}_X(X))\).

An alternative, basis-free way of defining \(\text{GL}_n\) is to define a contravariant functor on \(\mathbf{Sch}\) that sends

\[
(X, \mathcal{O}_X) \to \text{Aut}_{\mathcal{O}_X(X)} \left( \bigoplus_{i=1}^{n} \mathcal{O}_X(X) \right).
\]

Example 6.8 Let \(\text{SL}_n\) be the covariant functor on the category \(\mathbf{CRing}\) defined by

\[
A \mapsto \ker \left( \text{GL}_n(A) \to \text{G}_m(A) \right) = \{M \in A^{n \times n} \mid \det(M) = 1\}.
\]

By Yoneda’s lemma ([Wat79, p. 6]) the natural transformation \(\text{GL}_n \to \text{G}_m\) can be described by a homomorphism \(\mathbb{Z}[T, T^{-1}] \to \mathbb{Z}[T_{11}, ..., T_{nn}, \det(T_{ij})^{-1}]\), the one that sends \(T\) to \(\det(T_{ij})\). Therefore \(\text{SL}_n\) is represented by

\[
\mathbb{Z}[T_{11}, ..., T_{nn}, \det(T_{ij})^{-1}] \otimes_{\mathbb{Z}[T, T^{-1}]} \mathbb{Z} = \mathbb{Z}[T_{11}, ..., T_{nn}]/(\det(T_{ij}) - 1).
\]

In other words, \(\text{Spec} \mathbb{Z}[T_{11}, ..., T_{nn}]/(\det(T_{ij}) - 1)\) is an (affine) group scheme.
**Definition 6.9** An $S$-scheme (cf. 5.13) that is an (affine) group scheme is called an **(affine) group $S$-scheme**. As the categories of (affine) $S$-schemes have finite products and have the scheme $S$ as a final object, as in 6.3 being an (affine) group $S$-scheme is an intrinsic property. In particular, any affine group $S$-scheme is a group $S$-scheme.

**Example 6.10** Generalizing 6.8, let $S$ be a scheme, let $G$ and $H$ be group $S$-schemes, and let $f : G \to H$ be a homomorphism of group $S$-schemes, i.e., a morphism of $S$-schemes from $G$ to $H$ such that for each $S$-scheme $Y$ the induced map $\text{Hom}_S(Y, G) \to \text{Hom}_S(Y, H)$ is a group homomorphism. Define a functor on the category of $S$-schemes by sending

$$Y \to \ker(\text{Hom}_S(Y, G) \to \text{Hom}_S(Y, H)).$$

It is represented by the fibred product $S$-scheme $G \times_H S$, cf. 5.15.

**Example 6.11** Let $(X, \mathcal{O}_X)$ be an (irreducible) scheme and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank (5.9). Following [DG70a, I 4.5] define a functor $\text{GL}(\mathcal{E})$ on the category of $X$-schemes that sends

$$(Y, \mathcal{O}_Y) \to \text{Aut}_{\mathcal{O}_Y}(Y)\left(\left(\mathcal{O}_Y \otimes_{\mathcal{O}_X} f^{-1}\mathcal{E}\right)(Y)\right),$$

where $f$ denotes the morphism of schemes from the $X$-scheme $(Y, \mathcal{O}_Y)$ to the scheme $(X, \mathcal{O}_X)$ (5.13) and $f^{-1}\mathcal{O}_X$ and $f^{-1}\mathcal{E}$ denote the respective inverse image sheaves (5.5). For $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{O}_X$ this is just the affine group $X$-scheme $\text{GL}_n/X$ from 6.7. The present more general example is obtained from that example by a twisting process; cf. [Har68, p. 134], [Har77b, p. 117]. In analogy to 6.8 define the (affine) group $X$-scheme $\text{SL}(\mathcal{E})$ as $\ker(\text{GL}(\mathcal{E}) \to \text{G}_m/X)$ (cf. 6.10).

**Definition 6.12** Let $Y/\mathbb{F}_q$ be a non-singular projective curve (5.21), let $K = \mathbb{F}_q(Y)$ be its field of $\mathbb{F}_q$-rational functions (2.6) considered as the residue field of $Y$ at its generic point (5.14 and [Har77b, II.3.6]), let $G/Y$ be an affine group $Y$-scheme (6.9), and consider $\text{Spec} K$ as a $Y$-scheme via the identity map on $K$ (5.14). The scheme $G/Y$ is called a **rationality trivial group $(Y)$-scheme** if $G/K := G \times_Y \text{Spec} K$ (5.15) is a Chevalley scheme ([Gro60, I.8.3.1]). It is called **reductive**, if $G/K = G \times_Y \text{Spec} K$ is reductive ([DG70b, XIX 1.6]).

**Proposition 6.13** ([DG70b, XX 1.1.3, 1.15, 1.16, 1.17; XXII 5.6.5, 5.9.5; XXVI 1.12, 2.1]) Let $Y/\mathbb{F}_q$ be a non-singular projective curve, let $G/Y$ be a reductive group $Y$-scheme, and let $P/Y$ be a parabolic subgroup of $G/Y$ (cf. [DG70b, XXI 1.1]). Then the unipotent radical $R_u(P)$ (cf. [DG70b, XIX 1.2]) admits a filtration $R_u(P) = U_0 \supset U_1 \supset \cdots \supset U_k = \{e\}$ such that each $U_i/U_{i+1}$ is a vector bundle over $Y$. More precisely, if $P_\alpha$ denotes the vector bundle over $Y$ corresponding to the root space $g^\alpha$ (cf. [DG70b, XIX 1.10]), then

$$U_i = \prod_{\alpha \in \Delta^+, l(\alpha) > i} P_\alpha$$

and $U_i/U_{i+1} \cong \prod_{\alpha \in \Delta^+, l(\alpha) = i+1} P_\alpha$.

**Example 6.14** Let $A_1 \cong \mathcal{O}_Y \cong A_2$, let $E = A_1 \oplus A_2$ and let $\text{SL}(\mathcal{E}) = \text{SL}_2/Y$ be the group $Y$-scheme defined in 6.8, 6.11. Each of $\text{Hom}_{\mathcal{O}_Y}(A_1, A_2)$ and $\text{Hom}_{\mathcal{O}_Y}(A_2, A_1)$ equals the unipotent radical of a Borel subgroup of $\text{SL}(A_1 \oplus A_2)$ and is an invertible $\mathcal{O}_Y$-module, because $\text{Hom}_{\mathcal{O}_Y}(A_i, A_j) = A_1^\vee \otimes_{\mathcal{O}_Y} A_j$. By 5.21, this corresponds to a vector bundle (of dimension 1), confirming 6.13 for $\text{SL}_2$.

More generally, let $E$ be a locally free $\mathcal{O}_Y$-module of rank 2 with an invertible $\mathcal{O}_Y$-submodule $\mathcal{L}$ such that $E/\mathcal{L}$ is also invertible and let $\text{SL}(\mathcal{E})$ be the group $Y$-scheme defined in 6.11. There is a one-to-one correspondence between the Borel subgroups $B$ of $\text{SL}(\mathcal{E})$ and the invertible $\mathcal{O}_Y$-submodules $\mathcal{L}$ such that $E/\mathcal{L}$ is also invertible, given by $\mathcal{L} \mapsto \text{Stab}_{\text{SL}(\mathcal{E})}(\mathcal{L})$. The unipotent radical $R_u(B)$ is isomorphic to $\text{Hom}_{\mathcal{O}_Y}(E/\mathcal{L}, \mathcal{L})$. For any invertible $\mathcal{O}_Y$-module $\mathcal{H}$, the equality $\text{Hom}_{\mathcal{O}_Y}(E/\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{H} \otimes_{\mathcal{O}_Y} \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{H}) = (E/\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{H})^\vee \otimes_{\mathcal{O}_Y} (\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{H}) = \text{Hom}_{\mathcal{O}_Y}(E/\mathcal{L}, \mathcal{L})$ implies $\text{SL}(E \otimes_{\mathcal{O}_Y} \mathcal{H}) \cong \text{SL}(\mathcal{E})$. (Cf. [DG70a, I.4.5],[DG70b, XX, 5.1].)
Definition 6.15 Let $G/Y$ be a rationally trivial group $Y$-scheme, let $B/Y$ be a Borel subgroup of $G/Y$ and let $\{\alpha_1, ..., \alpha_r\}$ be the simple roots of $B$. Using the notation introduced in 5.21, 5.28 and 6.13, define

$$n_i(B) := c_\left(\mathcal{L}_{B_{\alpha_i}}\right).$$

Note that, since $G/Y$ is rationally trivial, each $B_{\alpha_i}$ is a vector bundle of dimension 1, so that $\mathcal{L}_{B_{\alpha_i}}$ is an invertible $\mathcal{O}_Y$-module.

Theorem 6.16 ([Har68, 2.2.6])
Let $G/Y$ be a rationally trivial group $Y$-scheme of genus $g$ (cf. 3.12). Then there exists a Borel subgroup $B/Y$ such that $n_i(B) \geq -2g$ for all $i \in \{1, ..., r\}$.

Proof. In case $G/Y$ is of type $A_1$, i.e., if there exists a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of rank 2 such that $G \cong \text{SL}(\mathcal{E})$, one can proceed as follows. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-submodule such that $\mathcal{E}/\mathcal{L}$ is also invertible and let $B = \text{Stab}\text{SL}(\mathcal{E})(\mathcal{L})$ (cf. 6.14).

Then $n_1(B) = c(R_0(B)) = c(\text{Hom}_{\mathcal{O}_Y}(\mathcal{E}/\mathcal{L}, \mathcal{L})) = c((\mathcal{E}/\mathcal{L})^\vee \otimes_{\mathcal{O}_Y} \mathcal{L}) = c(\mathcal{L}) - c(\mathcal{E}/\mathcal{L}) = 2c(\mathcal{L}) - c(\mathcal{E})$. By 3.14 and 5.28 there exists an invertible $\mathcal{O}_Y$-module $\mathcal{H}$ with $c(\mathcal{H}) = 1$. Since $\text{SL}(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H}^\vee) \cong \text{SL}(\mathcal{E}) \cong \text{SL}(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H})$ by 6.14, the formula $c(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H}^\vee) = c(\mathcal{E}) - 2c(\mathcal{H}) = 0$ allows us to assume without loss of generality that $2g - 2 < c(\mathcal{E}) \leq 2g$.

By the Riemann–Roch theorem (cf. 5.31) there exists $0 \neq t \in H^0(Y, \mathcal{E}) \cong \mathcal{E}(Y)$. Since $\mathcal{E}$ is locally free of rank 2 there exists an open cover $\{U_i\}$ of $Y$ such that $\mathcal{E}_{|U_i} \cong \mathcal{O}_{Y|U_i} \oplus \mathcal{O}_{Y|U_i}$, for each $U_i$. Therefore $t$ is contained in an invertible $\mathcal{O}_Y$-submodule $\mathcal{L}_t$ such that $\mathcal{E}/\mathcal{L}_t$ is also invertible. As $t \in H^0(Y, \mathcal{L}_t)$, one has $c(\mathcal{L}_t) \geq 0$ by [Har77b, Lemma IV.1.2]. Hence

$$n_1(B_t) = c(\text{Hom}_{\mathcal{O}_Y}(\mathcal{E}/\mathcal{L}_t, \mathcal{L}_t)) = 2c(\mathcal{L}_t) - c(\mathcal{E}) \geq -2g,$$

where $B_t = \text{Stab}\text{SL}(\mathcal{E})(\mathcal{L}_t)$ (cf. 6.14).

The general case can be reduced to the case $A_1$ via a local to global argument. □

Definition 6.17 Let $c_1 < -2g$. A Borel subgroup $B/Y$ of $G/Y$ is reduced, if $n_i(B) \geq c_1$ for all $i$.

Theorem 6.18 ([Har68, 2.2.13, 2.2.14])
Let $G/Y$ be a rationally trivial group $Y$-scheme. Then there exist constants $c_2 > \gamma > c_1$ such that the following hold: Let $B/Y$ be a reduced Borel subgroup of $G/Y$ and let $\alpha_{\omega_0}$ be a simple root of $B$ such that $n_{\omega_0}(B) \geq c_2$. Then each reduced Borel subgroup $B'$ of $G/Y$ satisfies $n_{\omega_0}(B') \geq \gamma$ and is contained in $P_{\omega_0}(B)$, where $P_{\omega_0}(B)$ denotes the maximal parabolic of type $\alpha_{\omega_0}$ containing $B$.

Proposition 6.19 ([Har68, 2.2.3, 2.2.11])
To each reduced Borel subgroup $B/Y$ of $G/Y$, let $I^B := \{i \in I \mid n_i(B) \geq c_2\}$, and let $P^B := \bigcap_{i \in I^B} P_i(B)$. Then

$$P := \bigcap_{B \subseteq G/Y \text{ reduced}} P^B$$

either equals $G/Y$ or is a parabolic subgroup of $G/Y$.

Proof. Let $B$ be a reduced Borel subgroup of $G$. By 6.18 each reduced Borel subgroup $B'$ of $G$ is contained in $P^B$ and conversely, again by 6.18, $B \subseteq P^B$. Hence $P$ contains a Borel group and therefore either equals $G/Y$ or is a parabolic subgroup of $G/Y$. □

Theorem 6.18 is the heart of Harder’s reduction theory. The only proof of 6.18 known to me in fact uses 6.19, as can be guessed from the numbering used in [Har68]. However, as the length of the proof of 6.18 by far surpasses anything I can reasonably include in this survey and as I will need to refer to 6.19 later, I took the liberty of deducing 6.19 from 6.18 for the sake of this exposition.
Proposition 6.20 ([Har68, p. 120], [Har69, p. 39])

Let $G/Y$ be a reductive group $Y$-scheme, let $P/Y$ be a parabolic $K$-subgroup of $G/Y$, let $d := \dim(R_u(P))$, and let $\Delta^+_P$ be the set of positive roots of $P$. Then the character

$$\chi_P : P \xrightarrow{\text{Ad}} \text{GL}(\text{Lie}(R_u(P))) \xrightarrow{\det} \text{GL} \left( \bigwedge^d \text{Lie}(R_u(P)) \right) \cong G_m,$$

considered as a character of a maximal split torus contained in $P$, is given by $\chi_P = \sum_{\alpha \in \Delta^+_P} \dim(P_\alpha) \alpha$.

Definition 6.21

Let $B$ be a parabolic $K$-subgroup of $G/Y$, let $\{\alpha_1, \ldots, \alpha_r\}$ be the simple roots of $B$, and let $(P_1)_i$ be the maximal parabolic $K$-subgroups of $G$ of type $\alpha_i$. Using the notation of 5.21, 5.31 and 6.13, define

$$p_i(B) := p(P_1)_i := \sum_{\alpha \in \Delta^+_P} c(\mathcal{L}_{P_\alpha}).$$

Let $B$ be a minimal parabolic $K$-subgroup of $G$, let $R(B)$ be the radical of $B$, let $R_u(B)$ be the unipotent radical of $B$, let $T = R(B)/R_u(B)$, let $S \subseteq T$ be the maximal $K$-split subtorus of $T$, let $\pi = \{\alpha_1, \ldots, \alpha_r\} \subset X(S)$ be the system of simple roots, and let $X(B) = \text{Hom}_K(B, G_m)$ be the module of $K$-rational characters of $B$ so that $X(B) \otimes \mathbb{Q} = X(S) \otimes \mathbb{Q}$, let $P_i \supseteq B$ be the maximal $K$-parabolic of type $\alpha_i$, let $\chi_{P_i} : P_i \rightarrow G_m$ be the sum of roots of $P_i$ (cf. 6.20), and let $\chi_i := \chi_{P_i}|_B$. The $\chi_i$ form a basis of $X(B) \otimes \mathbb{Q}$ and, if $(\cdot, \cdot)$ is a positive definite bilinear form on $X(B) \otimes \mathbb{Q}$ which is invariant under the action of the Weyl group, then

$$(\chi_i, \chi_j) = 0, \quad \text{if } i \neq j, \quad \text{and} \quad (\chi_i, \chi_i) > 0 \quad \text{for all } i \in I.$$  \hspace{1cm} (6)

If $G$ is rationally trivial, for $m_{i,j} \in \mathbb{Z}$ such that $\chi_i = \sum_{j=1}^r m_{i,j} \alpha_j$, then

$$p_i(B) = \sum_{j=1}^r m_{i,j} n_j(B).$$

Furthermore, for $c_{i,j} \in \mathbb{Q}$ such that $\alpha_i = \sum_{j=1}^r c_{i,j} \chi_j$, then

$$n_i(B) = \sum_{j=1}^r c_{i,j} p_j(B).$$

7 Reduction theory for reductive groups over the adèles

In this section I describe the interplay between Harder’s reduction theory and Weil’s geometry of numbers based on [Har69] in order to arrive at a geometric version of Harder’s reduction theory.

7.1 Let $G/Y$ be a reductive group $Y$-scheme, let $P/Y$ be a parabolic $K$-subgroup of $G/Y$, and let $\omega$ be a non-trivial volume form on $R_u(P)$ defined over $K$. This volume form yields a Haar measure of $R_u(P(\hat{K}))$ which is independent of the choice of $\omega$, i.e., for each $0 \neq \lambda \in K$ the volume forms $\omega$ and $\lambda \omega$ yield identical Haar measures; cf. [Har69, p. 37], [Wei82, 2.3.1].

This measure differs from the classical Tamagawa measure by the factor $q^{(1-g)d}$ where $d = \dim(R_u(P))$, cf. [Har69, p. 38], [Tam66], [Vos98, § 14.4], [Wei82, 2.3], [Wei95, p. 113].

Theorem 7.2 ([Har69, 1.3.1])

Let $G/Y$ be a reductive group $Y$-scheme, let $P/Y$ be a maximal parabolic $K$-subgroup of $G/Y$, let $\omega$ be a non-trivial volume form of $R_u(P)$ defined over $K$, and let $\mathfrak{r} := G(\hat{O}_K)$, cf. 4.3. Then

$$\int_{R_u(P(\hat{K})) \setminus \mathfrak{r}} \omega^{\mathfrak{r}_K} = q^{p(P)}.$$
Proof. For each $\alpha \in \Delta_p^+$ let $0 \subset P_\alpha^{(1)} \subset \cdots \subset P_\alpha^{(\dim(P_\alpha))} = P_\alpha$ be a filtration such that each $P_\alpha^{(i)}/P_\alpha^{(i-1)}$ is a vector bundle of dimension 1; cf. [Gro57, 2.1], [Har68, p. 122]. One then computes

$$\int_{R_u(P(\hat{K})) \cap \mathcal{O}} \omega_{\hat{K}} \overset{6.13}{=} \prod_{\alpha \in \Delta^+_p} \left( \int_{P_u(\hat{K}) \cap \mathcal{O}_K^\dim(P_\alpha)} \omega_{\hat{K}} \right)^{6.13} \prod_{i=1}^{\dim(P_\alpha)} \left( \int_{(P_\alpha^{(i)}/P_\alpha^{(i-1)}) \cap \mathcal{O}_K} \omega_{\hat{K}} \right)^{5.31} \prod_{\alpha \in \Delta^+_p} q^{(\mathcal{O}_P)} \overset{4.8}{=} q^{P(P)}.$$ 

\[ \Box \]

**Theorem 7.3** ([Har69, 1.3.2])

Let $G/Y$ be a reductive group $Y$-scheme, let $P/Y$ be a maximal parabolic $K$-subgroup, and let $\mathfrak{a} := G(\hat{O}_K)$. Then, for each $x \in P(\hat{K})$, one has

$$\int_{R_u(P(\hat{K})) \cap \mathcal{O}} \omega_{\hat{K}} = |\chi_P(x)| \int_{R_u(P(\hat{K})) \cap \mathcal{O}} \omega_{\hat{K}}.$$ 

**Proof.** The absolute value of the determinant of the derivative of conjugation by $x$

$$|\chi_P| : P(\hat{K}) \overset{\text{Ad}}{\rightarrow} \text{GL}(\text{Lie}(R_u(P(\hat{K})))) \overset{\text{det}}{\rightarrow} \text{GL} \left( \bigwedge^d \text{Lie}(R_u(P(\hat{K}))) \right) \overset{|\cdot|}{\rightarrow} \mathbb{R}$$

(cf. 6.20) measures the ratio of the volumes of $R_u(P(\hat{K})) \cap \mathfrak{a}$ and of $R_u(P(\hat{K})) \cap \mathfrak{x} \mathfrak{a}$. \[ \Box \]

**Definition 7.4** Using the notation of 6.22, let

$$\pi_1(B, x \mathfrak{a}) := \pi(P, x \mathfrak{a}) := \int_{R_u(P(\hat{K})) \cap \mathcal{O}} \omega_{\hat{K}} \quad \text{and}$$

$$\nu_1(B, x \mathfrak{a}) := \prod_{j=1}^{r} \pi_j(B, x \mathfrak{a}).$$

**Corollary 7.5** ([Har69, p. 40])

For each $x \in B(\hat{K})$, one has $\nu_1(B, \mathfrak{a}) = |\alpha_1(x)| \nu_1(B, x \mathfrak{a})$.

**Observation 7.6** For each $x \in G(K)$, one has $\pi_1(B, \mathfrak{a}) = \pi_1(xB, x \mathfrak{a})$ and $\nu_1(B, \mathfrak{a}) = \nu_1(xB, x \mathfrak{a})$.

**Proof.** Conjugation by $x \in G(K)$ maps the $K$-volume form $\omega$ on $R_u(P)$ onto a $K$-volume form $x \omega$ on $R_u(xP)$, whence

$$\pi_1(B, \mathfrak{a}) = \int_{R_u(P(\hat{K})) \cap \mathcal{O}} \omega_{\hat{K}} = \int_{R_u(xP(\hat{K})) \cap \mathcal{O}} x \omega_{\hat{K}} \overset{7.1}{=} \pi_1(xB, x \mathfrak{a}).$$

By 7.4 the second identity follows from the first. \[ \Box \]

**Theorem 7.7** ([Har69, 2.1.1])

Let $G/Y$ be a rationally trivial group $Y$-scheme and let $\mathfrak{a} := G(\hat{O}_K)$. Then there exists a constant $C_1 > 0$ such that for each $x \in G(\hat{K})$ there exists a Borel subgroup $B/Y$ of $G/Y$ such that

$$\nu_1(B, x \mathfrak{a}) \geq C_1.$$
Proof. By [Har69, 1.1.2], for each \( x \in G(\hat{A}_K) \), there exists a rationally trivial group scheme \( G^{(x)}/Y \) such that \( x \mathcal{R} = G^{(x)}(\hat{O}_K) \). Let \( B^{(x)}/Y \) be a Borel subgroup of \( G^{(x)}/Y \). Then, by 6.22, 7.2, and 7.4,

\[
\nu_i(B^{(x)}, G^{(x)}(\hat{O}_K)) = q^\alpha_i(B^{(x)}).
\]

Therefore, if \( c_1 < -2g \), then by 6.16 the conclusion of the theorem holds for \( C_1 := q^{c_1} \).

\[\square\]

Corollary 7.8 ([Har69, 2.1.2])

Let \( G/Y \) be a rationally trivial group \( Y \)-scheme, let \( \mathcal{R} := G(\hat{O}_K) \), and let \( C_1 = q^{c_1} \) be a constant for which the conclusion of 7.7 holds. Then there exist constants \( C_2 > \Gamma > C_1 \) such that the following hold: let \( x \in G(\hat{A}_K) \), let \( B/Y \) be a Borel subgroup of \( G/Y \) such that \( \nu_i(B, x \mathcal{R}) \geq C_1 \) for all \( i \in I \), and let \( \alpha_i \) be a simple root of \( B \) such that \( \nu_i(\alpha, x \mathcal{R}) \geq C_2 \). Then each Borel subgroup \( B'/Y \) of \( G/Y \) with \( \nu_i(B', x \mathcal{R}) \geq C_1 \) for all \( i \in I \) satisfies \( \nu_i(B', x \mathcal{R}) \geq \Gamma \) and is contained in \( P_{\alpha}(B) \).

Proof. By 6.18 the conclusion holds for \( C_2 := q^{c_2} \) and \( \Gamma := q^\gamma \).

\[\square\]

Corollary 7.9 ([Har69, 2.3.2])

Let \( G/Y \) be a reductive group \( Y \)-scheme and let \( \mathcal{R} := G(\hat{O}_K) \). Then there exists a constant \( C_1 > 0 \) such that for each \( x \in G(\hat{A}_K) \), there exists a minimal \( K \)-parabolic subgroup \( B/Y \) of \( G/Y \) with

\[
\nu_i(B, x \mathcal{R}) \geq C_1 \quad \text{for all } i \in I.
\]

Definition 7.10

Once and for all fix \( C_1 \in (0, 1) \) such that the conclusion of 7.9 holds. A pair consisting of a minimal parabolic subgroup \( B \) of \( G/Y \) and an element \( x \in G(\hat{A}_K) \) is called **reduced**, if \( \nu_i(B, x \mathcal{R}) \geq C_1 \) for all \( i \).

Corollary 7.11 ([Har69, 2.3.3])

Let \( G/Y \) be a reductive group \( Y \)-scheme and let \( \mathcal{R} := G(\hat{O}_K) \). Then there exist constants \( C_2 > \Gamma > C_1 \) such that the following hold: let \( x \in G(\hat{A}_K) \), let \( B/Y \) be a minimal parabolic \( K \)-subgroup of \( G/Y \) such that \( (B, x) \) is reduced, and let \( \alpha_i \) be a simple root of \( B \) such that \( \nu_i(\alpha, x \mathcal{R}) \geq C_2 \). Then each minimal parabolic \( K \)-subgroup \( B'/Y \) of \( G/Y \) with \( (B', x) \) reduced satisfies \( \nu_i(B', x \mathcal{R}) \geq \Gamma \) and \( B' \subset P_{\alpha}(B) \).

Note that 7.9 and 7.11 follow from 7.7 and 7.8 by a standard field extension argument using [Har69, 2.3.5].

7.12

Let \( G/Y \) be a reductive group \( Y \)-scheme, let \( B/Y \) be a minimal parabolic \( K \)-subgroup of \( G \), and let \( \mathcal{R} := G(\hat{O}_K) \). Since \( G(\hat{A}_K)/B(\hat{A}_K) \) is compact and \( \mathcal{R} \) is open ([Har69, p. 36]), the double coset space \( \mathcal{R}\backslash G(\hat{A}_K)/B(\hat{A}_K) \) is finite. As \( B(\hat{A}_K) \) is self-normalizing in \( G(\hat{A}_K) \), one can consider \( G(\hat{A}_K)/B(\hat{A}_K) \) as the space of conjugates of \( B(\hat{A}_K) \) in \( G(\hat{A}_K) \), and \( \mathcal{R}\backslash G(\hat{A}_K)/B(\hat{A}_K) \) as the space of \( \mathcal{R} \)-orbits via conjugation on these.

Theorem 7.13 ([Har69, p. 40])

Let \( G/Y \) be a reductive group \( Y \)-scheme, let \( B/Y \) be a minimal parabolic \( K \)-subgroup of \( G \), let \( \mathcal{R} := G(\hat{O}_K) \), let \( B^{(1)}, ..., B^{(t)} \) be a system of representatives of the \( \mathcal{R} \)-orbit space \( \mathcal{R}\backslash G(\hat{A}_K)/B(\hat{A}_K) \) (cf. 7.12), let \( \xi \in G(K) \) with \( B^{(s)} = \xi^{-1}B^{(1)}\xi \), and for \( c \in \mathbb{R} \) define \( B^{(s)}(c) = \{ x \in B^{(s)}(\hat{A}_K) \mid |\alpha_i(x)| \leq c \text{ for all } i \in I \} \). Then there exists \( r \in \mathbb{R} \) such that

\[
\bigcup_{s=1}^t B^{(1)}(r)\xi_s\mathcal{R}
\]

is a fundamental domain for \( G(K)\backslash G(\hat{A}_K) \).
8 FILTRATIONS OF EUCLIDEAN BUILDINGS

Proof. For $x \in G(\hat{\mathbb{A}}_K)$, by 7.9, there exists a minimal parabolic $K$-subgroup $B$ of $G$ such that $\nu_1(B, x, \mathfrak{r}) \geq C_1$ for all $i \in I$. Let $B^{(s)}$ be the representative of the $\mathfrak{r}$-orbit of $x^{-1}Bx$ in $G(\hat{\mathbb{A}}_K)/B(\hat{\mathbb{A}}_K)$, let $u \in \mathfrak{r}$ such that $u^{-1}x^{-1}Bu = B^{(s)}$, let $a \in G(K)$ with $aBa^{-1} = B^{(s)}$, and define $y := axu \in N_{G(\hat{\mathbb{A}}_K)}(B^{(s)}(\hat{\mathbb{A}}_K)) = B^{(s)}(\hat{\mathbb{A}}_K)$. Since $a \in G(K)$ one has

$$\nu_i(B, x, \mathfrak{r}) = \nu_i(aB, ax \mathfrak{r}) = \nu_i(B^{(s)}, y \mathfrak{r}).$$

By 7.5, for each $i \in I$ one has $|\alpha_i(y)| = \nu_i(B^{(s)}, \mathfrak{r}) (\nu_i(B^{(s)}, y \mathfrak{r}))^{-1} \leq \nu_i(B^{(s)}, \mathfrak{r})C_1^{-1}$. Hence, for $r := \max\{\nu_i(B^{(s)}, \mathfrak{r})C_1^{-1} | 1 \leq s, i \in I\}$, to each $x \in G(\hat{\mathbb{A}}_K)$ there exists $a \in G(K)$ and $u \in \mathfrak{r}$ such that $ax = yu^{-1} \in B^{(s)}(r)\mathfrak{r} = \xi_s^{-1}B^{(s)}(r)\xi_s\mathfrak{r}$. The claim follows because $a, \xi_s \in G(K)$. \qed

8 Filtrations of Euclidean buildings

In this section I translate the geometric version of Harder’s reduction theory into the setting of Euclidean buildings based on [Har77a]. From this section on the survey is intended for the reader familiar with the concept of Euclidean buildings, as simplicial complexes and as CAT(0) spaces. For both introductory and further reading the sources [AB08], [BH99], [Bro89], [Wei03], [Wei09] are highly recommended.

8.1 Let $Y/S_q$ be a non-singular projective curve, let $S \subset Y^0$ be finite, and let $G/Y$ be a reductive group $Y$-scheme. Clearly, each of 7.7, 7.8, 7.9, 7.11 holds for $x \in G(\hat{\mathbb{A}}_S) \subset G(\hat{\mathbb{A}}_K)$ (cf. 4.4). Since

$$G(K \cap \hat{\mathbb{A}}_S)/G(\hat{\mathbb{A}}_K) \cong G(O_S)\backslash G(\prod_{P \in S} K_P)/G(\prod_{P \in S} \hat{\mathbb{O}}_{P,K}) = G(O_S)\backslash \prod_{P \in S} G(K_P)/G(\hat{\mathbb{O}}_{P,K}),$$

the functions $\pi_i(B, x, \mathfrak{r})$ and $\nu_i(B, x, \mathfrak{r})$ (cf. 7.4) allow one to define $G(O_S)$-invariant (cf. 7.6) filtrations on the Euclidean building $X$ of $\prod_{P \in S} G(K_P)$. The group $G(O_S)$ is called an $S$-arithmetic group. The set of special vertices of $X$ is $X_v := \prod_{P \in S} G(K_P)/G(\hat{\mathbb{O}}_{P,K})$. The diagonal embedding of $K$ in $\prod_{P \in S} K_P$ (cf. 4.4) yields a diagonal embedding of the spherical building of $G(K)$ into the spherical building at infinity of $X$ with respect to the complete system of apartments. Note that this embedding is in general not simplicial.

Definition 8.2 Let $X$ be a CAT(0) space, e.g., a Euclidean building, and let $\gamma : [0, \infty) \to X$ be a unit speed geodesic ray, i.e., $d(\gamma(t), \gamma(0)) = t$ for all $t \geq 0$. The function

$$b_\gamma : X \to \mathbb{R} : x \mapsto \lim_{t \to \infty} (t - d(x, \gamma(t)))$$

is the Busemann function with respect to $\gamma$. Note that $t = d(\gamma(0), \gamma(t)) \leq d(\gamma(0), x) + d(x, \gamma(t))$ and $t - d(x, \gamma(t))$ non-decreasing in $t$, so that the limit always exists. A linear reparametrization of a Busemann function is called a generalized Busemann function. For a (generalized) Busemann function $b_\gamma$, a sub-level set of $-b_\gamma$ is a horoball centred at $\gamma(\infty)$. The boundary of a horoball is a horosphere, centred at $\gamma(\infty)$.

Some sources define the Busemann function with respect to a geodesic ray as $b_\gamma : X \to \mathbb{R} : x \mapsto \lim_{t \to \infty} (d(x, \gamma(t)) - t)$. This does not affect the concept of a horoball, i.e., in that case a horoball is defined as a sub-level set of $b_\gamma$.

Theorem 8.3
Let $P$ be a maximal $K$-parabolic and let $\mathfrak{r} := G(O_K)$. Then for each $g \in \prod_{P \in S} P(K_P)$ one has

$$\log_q(\pi(P, g \mathfrak{r})) = \log_q(\pi(P, \mathfrak{r})) + \sum_{P \in S} \deg(P) \nu_P(\chi_P(g)).$$

In particular, there exists a generalized Busemann function $p(P, \cdot) : X \to \mathbb{R}$ whose restriction to the set $X_v$ of special vertices of $X$ equals $\log_q(\pi(P, \cdot))$. 


The maximal $K$-fundamental weight corresponding to $P$ is defined as $\chi_P(g) = \sum_{\alpha \in R_+} \frac{1}{\alpha} \log \frac{\gamma_0}{\gamma - \alpha}$, where $\gamma_0$ is a constant for which the conclusion of 8.6 holds, then replacing $c$ by $c - d$ for all $i \in I$ and $c \in \mathbb{R}$ and

$$n_i(B, x') \geq c \quad \text{implies} \quad n_i(B, x) \geq c - d \quad \text{for all} \quad i \in I \quad \text{and} \quad c \in \mathbb{R}.$$
Proof. In case one only considers special vertices \( x \in X \), such constants \( c_2 := \log_q(C_2) \) and \( \gamma := \log_q(\Gamma) \) exist by 7.11. Choose again a constant \( d \) for which the conclusion of 8.6 holds, define \( c'_1 := c_1 - d \) and \( C'_1 := q^{c'_1} \), use this constant \( C'_1 \) in 7.10 to define reduced pairs, let \( C'_2 \) and \( \Gamma' \) be constants for which the conclusion of 7.11 holds for this definition of a reduced pair, and let \( c'_2 := \log_q(C'_2) \). For \( c_2 := c'_2 + d \) by 8.6 there exists a special vertex \( x' \in X \) such that

\[
\begin{align*}
\forall i & \in I, \quad \forall i \in I, \\
n_i(B, x) & \geq c_1 \quad \text{implies} \quad n_i(B, x') \geq c'_1 \\
n_i(B', x) & \geq c_1 \quad \text{implies} \quad n_i(B', x') \geq c'_1 \\
n_j(B, x) & \geq c_2 \quad \text{implies} \quad n_j(B, x') \geq c'_2.
\end{align*}
\]

We conclude from 7.11 that \( B' \) is contained in \( P_j \) and that \( n_j(B', x') \geq \log_q(\Gamma') \), whence \( n_j(B', x) \geq \log_q(\Gamma') - d =: \gamma \).

\[ \square \]

Notation 8.9 Once and for all fix constants \( c_1, c_2, \gamma \in \mathbb{R} \) such that \( c_1 \) is negative, \( c_2 > \gamma > c_1 \) and such that the conclusions of 8.7 and 8.8 hold.

Definition 8.10 A pair \((B, x)\) consisting of a minimal \( K \)-parabolic subgroup \( B \) of \( G \) and an element \( x \in X \) such that \( n_i(B, x) \geq c_1 \) for all \( i \in I \) is called reduced. For a minimal parabolic \( K \)-subgroup \( B \) of \( G \) and a maximal \( K \)-parabolic \( P \supseteq B \), following [Har74, p. 254], an element \( x \in X \) is called close to the boundary of \( X \) with respect to \( P \), if \((B, x)\) is a reduced pair and \( n_j(B, x) \geq c_2 \). An element \( x \in X \) is called close to the boundary of \( X \), if there exists a maximal \( K \)-parabolic \( P \) such that \( x \) is close to the boundary of \( X \) with respect to \( P \). For \( x \in X \) close to the boundary of \( X \), define

\[
P_x := \bigcap \{ P \subseteq G \mid x \text{ is close to the boundary of } X \text{ with respect to } P \}.
\]

By 8.8 the group \( P_x \) is a \( K \)-parabolic subgroup of \( G \) (cf. 6.19). Following [Har68, p. 138], it is called the isolated parabolic subgroup of \( G \) corresponding to \( x \). For each \( K \)-parabolic \( Q \supset P_x \), the element \( x \in X \) is called close to the boundary of \( X \) with respect to \( Q \).

Proposition 8.11 ([Beh04, p. 35])

Let \( x \in X \) be close to the boundary of \( X \) and let \( P_x \) be the corresponding isolated parabolic subgroup of \( G \). Let \( \gamma \) be a geodesic ray in \( X \) with \( \gamma(0) = x \) and whose end point lies in the simplex of the building at infinity corresponding to \( P_x \). Then each \( y \in \gamma([0, \infty)) \) is close to the boundary of \( X \). Moreover, one has \( P_y = P_x \) for each minimal \( K \)-parabolic \( B \) the pair \((B, y)\) is reduced if and only if \((B, x)\) is reduced.

Proof (Bux, Gramlich, Witzel). First notice that by (6) and (7) for each reduced pair \((B, x)\) and all \( i \in I \) one has \( n_i(B, y) \geq n_i(B, x) \). In particular, each reduced pair \((B, x)\) gives rise to a reduced pair \((B, y)\) and, moreover, \( y \) lies close to the boundary of \( X \) with respect to \( P_x \). This implies \( P_y \subseteq P_x \).

Conversely, let \((B, y)\) be a reduced pair. Then by 8.8 one has \( B \subseteq P_y \subseteq P_x \). Let \((P_i)_{i \in I}\) be the family of maximal parabolic \( K \)-subgroups of \( G \) containing \( B \) and let \( I' \subseteq I \) such that \( P_x = \bigcap_{i \in I'} P_i \). If there exists \( j \in I \) such that \( n_j(B, y) \geq c_2 \), but \( P_j \nsubseteq P_x \), then \( j \in I \setminus I' \). As for each \( i \in I \setminus I' \) one has \( n_i(B, x) = n_i(B, y) \), in particular \( n_j(B, x) = n_j(B, y) \geq c_2 \), and in view of 8.10 the pair \((B, x)\) cannot be reduced. As being a reduced pair is a closed condition (see 8.10), there therefore exists a minimal \( a \in (0, \infty) \) such that \((B, \gamma(a))\) is reduced. The first paragraph of this proof implies that \( P_{\gamma(a)} \nsubseteq P_x \). Thus 8.8 applied to the reduced pair \((B, \gamma(a))\) yields \( n_i(B, \gamma(a)) \geq c_1 \), and hence \( n_i(B, \gamma(a)) > c_1 \) by 8.9, for all \( i \in I' \). As the \( n_i(B, \gamma(a)) \) are continuous in \( b \) and as for each \( i \in I \setminus I' \) one has \( n_i(B, x) = n_i(B, \gamma(b)) = n_i(B, y) \geq c_1 \), this contradicts the minimality of \( a \). Therefore \((B, x)\) has to be reduced. Consequently, each \( j \in I \) with the property that \( n_j(B, y) \geq c_2 \) satisfies \( P_j \supseteq P_x \), whence \( j \in I' \). Thus we have \( P_y = P_x \). \( \square \)
\textbf{Definition 8.12} For }c \in \mathbb{R} \text{ define }
\begin{align*}
X^n(c) &= \{ x \in X \mid (B, x) \text{ reduced implies } n_i(B, x) \leq c \text{ for all } i \in I \}, \\
X^p(c) &= \{ x \in X \mid (B, x) \text{ reduced implies } p_i(B, x) \leq c \text{ for all } i \in I \}, \quad \text{and} \\
X^\mathcal{P}(c) &= \{ x \in X \mid (B, x) \text{ reduced implies } \mathcal{P}_i(B, x) \leq c \text{ for all } i \in I \}.
\end{align*}

These filtrations are }G(O_S)\text{-invariant (cf. 7.6) and }G(O_S)\text{-cocompact (cf. [Har69, 2.2.2]). There exists }c_3 \in \mathbb{R} \text{ such that } X^n(c_2) \subseteq X^\mathcal{P}(c_3).

\textbf{Proposition 8.13 ([BGW11, Section 5])} 
\begin{enumerate}
\item Let }c \geq c_3, \text{ let } x \in X \setminus X^\mathcal{P}(c) \text{ and let } \mathcal{P} \in X^\mathcal{P}(c) \text{ be an element at which the function } X^\mathcal{P}(c) \to \mathbb{R} : z \mapsto d(x, z) \text{ assumes a global minimum. Then } P_\mathcal{P} = P_x. \text{ Furthermore, there exists a unique unit speed geodesic ray } \gamma_x^\mathcal{P} : [0, \infty) \to X \text{ with } \gamma_x^\mathcal{P}(0) = x \text{ along which the function } X \setminus X^\mathcal{P}(c) \to \mathbb{R} : x \mapsto d(x, X^\mathcal{P}(c)) \text{ assumes its steepest ascent; its end point lies in the simplex at infinity corresponding to } P_x.
\end{enumerate}

The preceding proposition shows that one can measure the distance from the set }X^\mathcal{P}(c_3)\text{ in a neat way. It also shows that there is a substantial difference between }K\text{-rank 1 and higher }K\text{-rank. Indeed, by 8.13, in the case of }K\text{-rank 1 the boundary of }X^\mathcal{P}(c_3)\text{ consists of hypersurfaces of codimension one which must have pairwise empty intersections, as otherwise one would need non-minimal isolated }K\text{-parabolics, which do not exist. The following result makes this heuristic argument more concrete.}

\textbf{Theorem 8.14 ([BW08, 3.7])} 
\begin{enumerate}
\item If }rk_K(G) = 1, \text{ then there exists a collection } \mathcal{H} \text{ of pairwise disjoint horoballs of } X \text{ such that } X^\mathcal{P}(c_3) = X \setminus \mathcal{H} \text{ is } G(O_S)\text{-invariant and } G(O_S)\text{-cocompact.}
\end{enumerate}

\textbf{Proof.} Since the functions }\mathcal{P}\text{ are Busemann functions (cf. 8.3, 8.4), there clearly exists a collection } \mathcal{H} \text{ of horoballs of } X \text{ such that } X^\mathcal{P}(c_3) = X \setminus \mathcal{H}. \text{ By 8.12 the set } X^\mathcal{P}(c_3) \text{ is } G(O_S)\text{-invariant and } G(O_S)\text{-cocompact. It therefore remains to prove that the horoballs in } \mathcal{H} \text{ can be chosen to be either disjoint or equal. Let } H_1, H_2 \in \mathcal{H} \text{ have non-trivial intersection and let } B_1, B_2 \text{ be the minimal } K\text{-parabolic subgroups corresponding to their respective centres (cf. 8.2). Then there exists } x \in X \text{ such that } \mathcal{P}_{\alpha}(B_1, x), \mathcal{P}_{\alpha}(B_2, x) \geq c_3, \text{ where } \alpha \text{ denotes the unique simple root. Therefore, by 8.12, } n_{\alpha}(B_1, x), n_{\alpha}(B_2, x) \geq c_3, \text{ and by 8.8 one has } B_1 = B_2. \text{ Hence, } H_1 \subseteq H_2 \text{ or } H_2 \subseteq H_1. \text{ If } H_1 \neq H_2, \text{ one can remove one of the two from } \mathcal{H} \text{ without changing } X \setminus \mathcal{H}. \quad \square

\section{Applications and conjectures}

\textit{In this section I sketch the applicability of the geometric version of Harder’s reduction theory to the study of finiteness properties of }S\text{-arithmetic groups over global function fields and state a very general conjecture on isoperimetric properties of }S\text{-arithmetic groups over arbitrary global fields. For reduction theory over number fields I strongly recommend [PR94].}

\subsection{Finiteness properties of }S\text{-arithmetic groups}

\textbf{Definition 9.1} A group } \Gamma \text{ is of type } F_m, \text{ if it admits a free action on a contractible CW complex } X \text{ with finitely many orbits on the } m\text{-skeleton of } X.

A group action on a CW complex is called \textbf{cellular}, if the action preserves the cell structure, and \textbf{rigid}, if each group element that elementwise fixes the skeleton of a cell in fact elementwise fixes the whole cell.

\textbf{Theorem 9.2 ([Bro87, 1.1])} 
\begin{enumerate}
\item Let } m \in \mathbb{N}, \text{ let } X \text{ be an } (m-1)\text{-connected CW complex, and let } \Gamma \to \text{ Aut}(X) \text{ act cellularly, rigidly and cocompactly on } X \text{ such that the stabilizer of each } i\text{-cell is of type } F_{m-i}. \text{ Then } \Gamma \text{ is of type } F_m.
\end{enumerate}
Theorem 9.3 ([BW08, 7.7])
Let \( X = X_1 \times \cdots \times X_t \) be an affine building, decomposed into its irreducible factors, let \( \partial X \) be the spherical building at infinity of \( X \), let \( \partial X \) be the spherical building at infinity of \( \prod_{i=1, i\neq j} X_i \), and let \( \xi \in \partial X \setminus \bigcup_{1 \leq j \leq t} \partial X \). Then any horosphere centred at \( \xi \) \((8.2)\) is \((\dim(X) - 2)\)-connected.

Theorem 9.4 ([BW08, 8.1])
Let \( K \) be a global function field, let \( G \) be an absolutely almost simple \( K \)-group of \( K \)-rank 1, let \( \emptyset \neq S \subset Y^o \) be finite, let \( X \) be the Euclidean building of \( \prod_{P \in S} G(K_P) \), and let \( m = \dim(X) = \sum_{P \in S} \text{rk}_{K_P}(G) \). Then \( G(O_S) \) is of type \( F_{m-1} \), but not \( F_m \).

Proof. The group \( G(O_S) \) clearly acts rigidly and cellularly on the (contractible) building \( X \). By 8.14, the group \( G(O_S) \) acts cocompactly on \( \mathbb{X}^t(c_3) = X \setminus \mathcal{H} \) and, in view of 8.1, each of the horoballs \( H \in \mathcal{H} \) is centred at some \( \xi \) which satisfies the hypothesis of 9.3. Therefore, by 9.3, \( \mathbb{X}^t(c_3) \) is \((n-2)\)-connected. As cell stabilizers in \( G(O_S) \) are finite, whence of type \( F_\infty \), the claim follows from 9.2.

For \( K \)-rank greater than 1 this strategy cannot work, as 8.14 becomes false. It can be adapted, however, which leads to a couple of technical difficulties that have first been overcome in [BGW09], [Wit10] for the \( S \)-arithmetic groups \( G(\mathbb{F}_q[t]) \) and \( G(\mathbb{F}_q[t, t^{-1}]) \) where \( G \) is an absolutely almost simple \( \mathbb{F}_q \)-group of rank \( n \geq 1 \). In this situation one can in fact make use of the theory of Euclidean twin buildings, as in [Abr96].

Theorem 9.5 ([BGW09, A], [Wit10, Main Theorem])
Let \( G \) be an absolutely almost simple \( \mathbb{F}_q \)-group of rank \( n \geq 1 \). Then \( G(\mathbb{F}_q[t]) \) is of type \( F_{n-1} \), but not \( F_n \), and \( G(\mathbb{F}_q[t, t^{-1}]) \) is of type \( F_{2n-1} \), but not \( F_{2n} \).

Recently, a combination of the ideas developed in [BW08], [BGW09], [Wit10] with Harder’s reduction theory allowed the authors of [BGW11] to prove the following theorem, providing a positive answer to the question asked in [AB08, 13.20], [Beh98, p. 80], [Bro89, p. 197].

Theorem 9.6 ([BGW11, Rank Theorem])
Let \( K \) be a global function field, let \( G \) be an absolutely almost simple \( K \)-isotropic \( K \)-group, let \( \emptyset \neq S \subset Y^o \) be finite, let \( X \) be the Euclidean building of \( \prod_{P \in S} G(K_P) \), and let \( m = \dim(X) = \sum_{P \in S} \text{rk}_{K_P}(G) \). Then \( G(O_S) \) is of type \( F_{m-1} \), but not \( F_m \).

Remark 9.7 Using the notation introduced above, already [BW07, 1.1] established that \( G(O_S) \) is not of type \( F_m \).

Remark 9.8 \( S \)-arithmetic groups over number fields are known to be of type \( F_\infty \), cf. [BS76, §11].

9.2 Isoperimetric properties of \( S \)-arithmetic groups
In the number field case, results similar to 7.9 and 7.11 hold, cf. [BH62], [Har69, p. 53]. To the best of my knowledge, the question posed by Harder ([Har69, p. 54]) whether it is possible to prove the results from [BH62] using methods similar (or at least closer) to the approach used by Harder is still open. The key, of course, would be to prove 6.18 for number fields.

As this problem by itself probably will not attract sufficient attention, I will finish this survey by stating a very general conjecture on properties of \( S \)-arithmetic groups over arbitrary global fields, i.e., global function fields or number fields, which, if verified, provides another proof of 9.6.

Definition 9.9 A coarse \( n \)-manifold \( \Sigma \) in a metric space \( X \) is a function from the vertices of a triangulated \( n \)-manifold \( M \) into \( X \). The homeomorphism type of the manifold \( M \) is called the topological type of \( \Sigma \). The boundary \( \partial \Sigma \) is the restriction to \( \partial M \) of the function \( \Sigma \). The coarse manifold \( \Sigma \) has scale \( r \in \mathbb{R}_+ \), if \( d(\Sigma(x), \Sigma(y)) \leq r \) for all adjacent vertices \( x, y \) of \( M \). The volume \( \text{vol}(\Sigma) \) equals the number of vertices in \( M \).
Conjecture 9.10 ([BW07])
Let $K$ be a global field, i.e., global function field or a number field, let $G$ be an absolutely almost simple $K$-isotropic $K$-group, let $S$ be a non-empty finite set of places of $K$ containing all archimedean ones, and let $n < \sum_{P \in S} \text{rk}_{K_P}(G)$. To any $r_1 > 0$ there exists a linear polynomial $f$ and $r_2 > 0$ such that, if $\Sigma \subseteq \prod_{P \in S} G(K_P)$ is a coarse $n$-manifold of scale $r_1$ with $\partial \Sigma \subseteq G(O_S)$, then there exists a coarse $n$-manifold $\Sigma' \subseteq G(O_S)$ of scale $r_2$ and identical topological type as $\Sigma$ such that $\partial \Sigma' = \partial \Sigma$ and $\text{vol}(\Sigma') \leq f(\text{Vol}(\Sigma))$.

In case $n < |S|$, the existence of a polynomial $f$ of unspecified degree and $r_2 > 0$ as in the conjecture have been established by Bestvina, Eskin and Wortman. The precise statement of their result is as follows.

Theorem 9.11 ([Wor10, 2])
Let $K$ be a global field, i.e., global function field or a number field, let $G$ be an absolutely almost simple $K$-isotropic $K$-group, let $S$ be a non-empty finite set of places of $K$ containing all archimedean ones, and let $n < |S|$. To any $r_1 > 0$ there exists a polynomial $f$ and $r_2 > 0$ such that, if $\Sigma \subseteq \prod_{P \in S} G(K_P)$ is a coarse $n$-manifold of scale $r_1$ with $\partial \Sigma \subseteq G(O_S)$, then there exists a coarse $n$-manifold $\Sigma' \subseteq G(O_S)$ of scale $r_2$ and identical topological type as $\Sigma$ such that $\partial \Sigma' = \partial \Sigma$ and $\text{vol}(\Sigma') \leq f(\text{Vol}(\Sigma))$.

Note that this result provides an alternative proof that the $S$-arithmetic group $G(O_S)$ in 9.6 is of type $F_{|S|-1}$.

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