MARKOV BASES AND GENERALIZED LAWRENCE LIFTINGS

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Abstract. Minimal Markov bases of configurations of integer vectors correspond to minimal binomial generating sets of the associated lattice ideal. We give necessary and sufficient conditions for the elements of a minimal Markov basis to be (a) inside the universal Gröbner basis and (b) inside the Graver basis. We study properties of Markov bases of generalized Lawrence liftings for arbitrary matrices $A \in \mathcal{M}_{m \times n}(\mathbb{Z})$ and $B \in \mathcal{M}_{p \times n}(\mathbb{Z})$ and show that in cases of interest the complexity of any two Markov bases is the same.

Introduction

Let $A$ be an element of $\mathcal{M}_{m \times n}(\mathbb{Z})$, for some positive integers $m, n$. The object of interest is the lattice $L(A) := \text{Ker}_\mathbb{Z}(A)$. A Markov basis $\mathcal{M}$ of $A$ is a finite subset of $L(A)$ such that whenever $w, u \in \mathbb{N}^n$ and $w - u \in L(A)$ (i.e. $Aw = Au$), there exists a subset $\{v_i : i = 1, \ldots, s\}$ of $\mathcal{M}$ that connects $w$ to $u$. This means that for $1 \leq p \leq s$, $w + \sum_{i=1}^{p} v_i \in \mathbb{N}^n$ and $w + \sum_{i=1}^{s} v_i = u$. A Markov basis $\mathcal{M}$ of $A$ gives rise to a generating set of the lattice ideal

$$I_{\mathcal{L}(A)} := \langle x^u - x^v : Au = Av \rangle.$$

Each $u \in \mathbb{Z}^n$ can be uniquely written as $u = u^+ - u^-$ where $u^+, u^- \in \mathbb{N}^n$. In the seminal work of Diaconis and Sturmfels in [4], it was shown that $\mathcal{M}$ is a Markov basis of $A$ if and only if the set $\{x^{u^+} - x^{u^-} : u \in \mathcal{M}\}$ is a generating set of $I_{\mathcal{L}(A)}$. A Markov basis $\mathcal{M}$ of $A$ is minimal if no subset of $\mathcal{M}$ is a Markov basis of $A$. We say that $\mathcal{L}(A)$ is positive if $\mathcal{L}(A) \cap \mathbb{N}^n = \{0\}$ and non positive if $\mathcal{L}(A) \cap \mathbb{N}^n \neq \{0\}$. When $\mathcal{L}(A)$ is positive then the graded Nakayama Lemma applies and all minimal Markov bases have the same cardinality. When $\mathcal{L}(A)$ is non positive, it is possible to have minimal Markov bases of $A$ of different cardinalities, see [3]. It is important to note that the study of non positive lattices has important implications in the study of positive ones, see for example the proof of [10, Theorem 3], [7, Lemma 5] and [8, Theorem 3.5]. The universal Markov basis of $A$ will be denoted by $\mathcal{M}(A)$ and is defined as the union of all minimal Markov bases of $A$ of minimal cardinality, where we identify a vector $u$ with $-u$, see [3, 10]. The sublattice of $\mathcal{L}(A)$ generated by all elements of $\mathcal{L}(A) \cap \mathbb{N}^n$ is called the pure sublattice of $\mathcal{L}(A)$ and is important when considering minimal Markov bases of $A$, see [3]. The pure sublattice of $\mathcal{L}(A)$ is zero exactly when $\mathcal{L}(A)$ is positive.

If $u = v + w$ we write $u = v_p + w_p$ to denote that this sum gives a conformal decomposition of $u$ i.e. $u^+ = v^+ + w^+$ and $u^- = v^- + w^-$. The set consisting of

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all elements of $\mathcal{L}(A)$ which have no conformal decomposition is denoted by $\mathcal{G}(A)$ and is called the Graver basis of $A$. When $u \in \mathcal{G}(A)$ the binomial $x^{u^+} - x^{u^-}$ is called primitive. $\mathcal{G}(A)$ is always a finite set, see [6, 11]. In this paper we examine in detail when an element of a minimal Markov basis belongs to $\mathcal{G}(A)$. We show in Theorem [13] that $\mathcal{M}(A) \subset \mathcal{G}(A)$ holds in just two cases: when $\mathcal{L}(A)$ is positive and when $\mathcal{L}(A)$ is pure of rank 1. We point out that even though the inclusion for positive lattices is well known, we could not locate its proof in the literature, so we provide it here for completeness of the exposition.

By $\mathcal{U}(A)$ we denote the universal Gröbner basis of $A$, i.e. the set which consists of all vectors $u \in \mathcal{L}(A)$ such that $x^{u^+} - x^{u^-}$ is part of a reduced Gröbner basis of $I_{\mathcal{L}(A)}$ for some term order on $\mathbb{N}^n$. The inclusion $\mathcal{U}(A) \subset \mathcal{G}(A)$ always hold, see [11, Lemma 4.6]. In this paper we examine the relation between $\mathcal{M}(A)$ and $\mathcal{U}(A)$. In general $\mathcal{M}(A)$ is not a subset of $\mathcal{U}(A)$ even when $\mathcal{L}(A)$ is positive as Example [13] shows. In Theorem [17] we give a necessary and sufficient condition for $\mathcal{M}(A)$ to be contained in $\mathcal{U}(A)$ when $\mathcal{L}(A)$ is positive.

In Section [2] for $r \geq 2$, $B \in \mathcal{M}_{p \times n}(\mathbb{Z})$, we study the generalized Lawrence lifting $\Lambda(A, B, r)$:

$$
\Lambda(A, B, r) = \begin{pmatrix}
A & 0 & 0 \\
0 & A & 0 \\
& \cdots & \\
0 & 0 & A \\
B & B & \cdots & B
\end{pmatrix}^{r\text{-times}}.
$$

When $B = I_n$ one gets the usual $r$-th Lawrence lifting $A^{(r)}$, see [10]. Such liftings were used to prove for example the finiteness of the Graver basis of $A$ and are connected to hierarchical models in Algebraic Statistics, see [10, 8]. We denote the columns of $A$ by $a_1, \ldots, a_n$ and the columns of $B$ by $b_1, \ldots, b_n$. The $(rm + p) \times rn$ matrix $\Lambda(A, B, r)$ has columns the vectors

$$
\{a_i \otimes e_j \oplus b_i : 1 \leq i \leq n, 1 \leq j \leq r\},
$$

where $e_1, \ldots, e_n$ represents the canonical basis of $\mathbb{Z}^n$. We note that $\mathcal{L}(\Lambda(A, B, r))$ is a sublattice of $\mathbb{Z}^{rn}$. Let $C \in \mathcal{L}(\Lambda(A, B, r))$. We can assign to $C$ an $r \times n$ matrix $\mathcal{C}$ such that $\mathcal{C}_{i,j} = C_{(i-1)n+j}$. Each row of $\mathcal{C}$ corresponds to an element of $\mathcal{L}(A)$ and the sum of the rows of $\mathcal{C}$ corresponds to an element in $\mathcal{L}(B)$. The number of nonzero rows of $\mathcal{C}$ is the type of $C$. The complexity of any subset of $\Lambda(A, B, r)$ is the largest type of any vector in that set.

The Graver complexity of $(A, B)$, denoted by $g(A, B)$ is the supremum of the complexities of $\mathcal{G}(\Lambda(A, B, r))$ as $r$ varies. By [8, Theorem 3.5], $g(A, B)$ is equal to the maximum 1-norm of an element in the Graver basis of the matrix $B \cdot \mathcal{G}(A)$, where $\mathcal{G}(A)$ represents the matrix whose columns are the vectors of the Graver basis of $A$ and is always finite. Moreover when $B = I_n$ or $A \in \mathcal{M}_{m \times n}(\mathbb{N})$, $m(A, B) \leq g(A, B)$, see [10] and [8] respectively. In both cases, it is not hard to see that $\mathcal{L}(\Lambda(A, B, r))$ is positive for $r \geq 2$. In general $\mathcal{L}(\Lambda(A, B, r))$ is positive for some $r \geq 2$ if and only if $\mathcal{L}(\Lambda(A, B, r))$ is positive for all $r \geq 2$ and to decide whether this holds it
We let $L = M/u$. By Theorem 1.1 we are in the case where $\text{rank}(L) = 1$. For the converse, assume first that $G$ is the hull of all elements of $G$. The elements of the connected components of $G$ are decisive in determining whether $u$ is in the universal Markov basis of $A$. The criterion of Theorem 1.2 is used in the proof of the next theorem.

**Theorem 1.1.** [3 Theorem 4.18] If $\text{rank}(L_{\text{pure}}) > 1$ or $\text{rank}(L_{\text{pure}}) = 1$ and $L \neq L_{\text{pure}}$ then $M$ is infinite.

Next we consider the fibers $F_u$ of $I_L$ for any $u \in L$. We let $F_u := \{ t \in \mathbb{N}^n : u^+ - t \in L \}$. We note that if $L$ is positive then $F_u$ is a finite set. We construct a graph $G_u$ with vertices the elements of $F_u$. Two vertices $w_1, w_2$ are joined by an edge if there is an index $i$ such that $i$-th component of $w_1$ and $w_2$ are nonzero. Thus $w_1, w_2$ are joined by an edge if and only if $(w_1 - w_2)^+ > u^+$ is componentwise smaller than $w_1$, meaning that at least one component of their difference is strictly positive. The following necessary condition for $u \in L$ to be in $M$ was observed in [2 Theorem 2.7] and [3 Theorem 1.3.2] when $L$ is positive.

**Theorem 1.2.** If $L$ is positive then $u$ is in the universal Markov basis of $A$ if and only if $u^+$ and $u^-$ belong to different connected components of $G_u$.

$F_u$ is called a Markov fiber when there exists an element $v$ in the universal Markov basis of $A$ such that $v^+ \in F_u$. The Markov polyhedra of $F_u$ are the convex hulls of the elements of the connected components of $G_u$. When $L$ is positive, the Markov polyhedra of $F_u$ are actually polytopes. For $u \in L$ we let $P[u]$ be the convex hull of all elements of $G_u$. When $u \in M$ the vertices of $P[u]$ are vertices of the Markov polyhedra of $F_u$ and we will see that the vertices of the Markov polyhedra are decisive in determining whether $u$ belongs to the universal Gr"obner basis of $A$. The criterion of Theorem 1.2 is used in the proof of the next theorem.

**Theorem 1.3.** The universal Markov basis of $A$ is a subset of the Graver basis of $A$ if and only if $L$ is positive or $L = L_{\text{pure}}$ and $\text{rank}(L) = 1$.

**Proof.** Assume first that $M \subseteq \mathcal{G}$. Since the Graver basis of $A$ is finite we get the desired conclusion from Theorem 1.1. For the converse, assume first that $L$ is non positive. By Theorem 1.1 we are in the case where $\text{rank}(L_{\text{pure}}) = 1$ and $L = L_{\text{pure}}$. We let $0 \neq w \in \mathbb{N}^n$ be such that $L = \langle w \rangle$. It is immediate that $w \in G$ and thus $M = G$. Next we examine the case where $L$ is positive. We will show that if $u \in L$, $u \notin G$ then $u \notin M$. Since $u \notin G$ there exist nonzero vectors $v, w \in L$ such that

$$\text{rank}(L) = 1.$$
\[ u = v + c \cdot w. \] Thus \( u^+ = v^+ + w^+ \) and \( u^- = v^- + w^- \). It follows that \( u^+ , u^- \) and \( u^+ - v = w^+ + v^- \) are all in \( F_u \). Next we show that \( v^- \) is nonzero. Indeed suppose not. Since \( v^- = v - v^+ = v \in L \cap \mathbb{N}^n = \{0\} \), it follows that \( v = 0 \), a contradiction. Similarly \( w^+ , v^- , w^- \) are nonzero. Thus in the fiber \( F_u \), the elements \( u^+ , w^+ + v^- \) are connected by an edge because \((u^+ - (w^+ + v^-))^+ = v^+\), which is smaller than \( u^+ \). Similarly \( w^+ + v^- \) are connected by an edge because \((w^+ + v^-) - u^-)^+ = w^+\), which is smaller than \( w^+ + v^- \). It follows that \( u^+ , u^- \) belong to the same connected component of \( G_u \) and thus \( u \notin M \). □

**Remark 1.4.** Let \( E \) be the union of all minimal Markov bases of \( A \), not necessarily of minimal cardinality, where we identify a vector \( u \) with \( -u \). Note that \( M \subset E \) and that \( M = E \) exactly when \( L \) is positive, see [3]. If \( L = \langle w \rangle \) with \( 0 \neq w \in \mathbb{N}^n \) then one can easily see that \( \{kw, lw\} \) is a minimal Markov basis of \( A \) for any two relatively prime integers \( k, l \geq 2 \) and thus \( E \) is infinite. Therefore if \( L \) is positive then \( E \) is not a subset of the Graver basis of \( A \).

Next we examine the relation between the universal Markov basis of \( A \) and the universal Gröbner basis of \( A \). First we note the following: if \( L \) is non positive and rank \( L_{\text{pure}} > 1 \), then \( M \) is infinite by Theorem 1.1 and thus \( M \notin U \). On the other hand if \( L = L_{\text{pure}} \) and rank \( L = 1 \) then \( M = U = \{w\} \) where \( w \) is the generator of \( L \). We also recall the following characterization given in [11].

**Theorem 1.5.** [11, Theorem 7.8] If \( L \) is positive and \( u \in L \) then \( u \) is in the universal Gröbner basis of \( A \) if and only if the greatest common divisor of the coordinates of \( u \) is one and the line segment \([u^+, u^-]\) is an edge of the polytope \( P[u] \).

For \( u \in \mathbb{R}^n \), we let \( \text{supp}(u) := \{i : u_i \neq 0\} \). For \( X \subset \mathbb{R}^n \), we let
\[ \text{supp}(X) := \bigcup_{u \in X} \text{supp}(u). \]

We note that if \( u \in M \) and \( L \) is positive then it is not hard to prove that the supports of different connected components of \( G_u \) are disjoint. Hence the Markov polytopes of \( F_u \) are disjoint.

**Lemma 1.6.** Let \( L \) be positive. An element \( u \) of the universal Markov basis of \( A \) belongs to the universal Gröbner basis of \( A \) if and only if \( u^+ \) and \( u^- \) are vertices of two different Markov polytopes.

**Proof.** Suppose that \( u \in U \). Since \( u \in M \), it follows by Theorem 1.2 that \( u^+ \) and \( u^- \) are elements of two different Markov polytopes of \( F_u \). Since \([u^+, u^-]\) is an edge of \( P[u] \) it follows that \( u^+ , u^- \) are vertices of \( P[u] \) and thus of their Markov polytopes as well.

For the converse assume \( u^+ , u^- \) are vertices of the disjoint Markov polytopes \( P_1 , P_2 \) respectively. Since \( u^+ , u^- \) are vertices of \( P_1 , P_2 \) we can find vectors \( c_1 , c_2 \) such that \( \text{supp}(c_i) \subset \text{supp}(P_i) \) for \( i = 1, 2 \) with the property that \( c_1 \cdot u^+ = 0 \), \( c_1 \cdot v > 0 \) for all \( v \in P_1 \setminus \{u^+\} \) and \( c_2 \cdot u^- = 0 \), \( c_2 \cdot v > 0 \) for all \( v \in P_2 \setminus \{u^-\} \). We
define $c$ as follows

$$c_i = \begin{cases} (c_1)_i, & \text{if } i \in \text{supp}(c_1), \\ (c_2)_i, & \text{if } i \in \text{supp}(c_2), \\ 1, & \text{otherwise.} \end{cases}$$

From the definition of $c$ it follows that $c \cdot v = 0$ for all $v \in [u^+, u^-]$ and $c \cdot v > 0$ for all $v \in \text{conv}(P_1 \cup P_2) \setminus \{[u^+, u^-]\}$. On the other hand $c \cdot v > 0$ for all $v \notin \text{conv}(P_1 \cup P_2)$, since $P[u] \subset \mathbb{R}_+^n$. Therefore $[u^+, u^-]$ is an edge of $P[u]$ and by Theorem 1.5 we have $u \in \mathcal{U}$, as desired.

The proof of the next theorem follows immediately by the remarks preceding Theorem 1.5 and Lemma 1.6.

**Theorem 1.7.** Let $A$ be an arbitrary integer matrix. The universal Markov basis of $A$ is a subset of the universal Gröbner basis of $A$ if and only if one of the following two conditions holds

1. $\mathcal{L}$ is positive and every element of a Markov fiber is a vertex of a Markov polytope,
2. $\mathcal{L} = \mathcal{L}_{\text{pure}}$ and rank $\mathcal{L} = 1$.

We finish this section with an example which shows specific elements of $\mathcal{M}$ not in $\mathcal{U}$.

**Example 1.8.** Let

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 4 & 0 & 0 & 4 & 0 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 6 & 0 & 6 & 0 \\ 2 & 2 & 2 & 2 & 6 & 0 & 0 & 6 \end{pmatrix}.$$  

Then $\mathcal{L}$ is a positive lattice. One can prove that $\{u, v, w\}$ is a a minimal Markov basis of $A$ where $u = (1, 1, -1, -1, 0, 0, 0, 0)$, $v = (0, 0, 0, 0, 1, 1, -1, -1)$ and $w = (2, 2, 1, 1, -1, -1, -1, -1)$. The corresponding Markov fibers are $\mathcal{F}_u = \{u^+, u^-\}$, $\mathcal{F}_v = \{v^+, v^-\}$ and $\mathcal{F}_w = \{3u^+, w^+, u^+ + 2u^-, 3u^-, 2v^+, w^-, 2v^-\}$. The Markov polytopes of $\mathcal{F}_u$ and $\mathcal{F}_v$ are zero dimensional, they consist of points. The Markov polytopes of $\mathcal{F}_w$ are one dimensional: they are the line segments $\text{conv}$(3$u^+$, 3$u^-$) and $\text{conv}(2v^+, 2v^-)$. We note that $w^+ = 2u^+ + u^-$ is in $\text{conv}(3u^+, 3u^-)$, $w^- = v^+ + v^-$ is in $\text{conv}(2v^+, 2v^-)$ but they are not vertices. Thus $w$ is not in the universal Gröbner basis of $A$. Note that $A$ has 12 different minimal Markov bases. Of those bases, exactly 4 are subsets of $\mathcal{U}$. Moreover $|\mathcal{M}| = 14$ and $|\mathcal{M} \cap \mathcal{U}| = 6$. Moreover, computing with 4ti2 [H], we get that $\mathcal{G} = \mathcal{M}$ and $|\mathcal{U}| = 6$.

2. **Generalized Lawrence liftings**

Let $\mathcal{L} \subset \mathbb{Z}^n$ be a lattice. We say that $0 \neq u$ is $\mathcal{L}$-primitive if $\mathbb{Q}u \cap \mathcal{L} = \mathbb{Z}u$. Suppose that $\mathcal{L}$ is non positive. In [K] it was shown that there exists an $\mathcal{L}$-primitive element $u \in \mathcal{L} \cap \mathbb{N}^n$ such that $\text{supp}(u) = \text{supp} \mathcal{L}_{\text{pure,}}$ [K] Proposition 2.7, Proposition 2.10. If $\mathcal{L} = \mathcal{L}(A)$ then this element can be extended to a minimal basis of $\mathcal{L}_{\text{pure}}$ and then to a minimal Markov basis of $A$ of minimal cardinality by [K] Theorem 2.12, Theorem 4.1, Theorem 4.11]. This is the point of the next lemma.
Lemma 2.1. If \( \mathcal{L} \) is non positive, there exists an \( \mathcal{L} \)-primitive element \( \mathbf{v} \in \mathbb{N}^n \) such that \( \mathbf{v} \) is in the universal Markov basis of \( A \) and \( \text{supp}(\mathbf{v}) = \text{supp} \mathcal{L}_{\text{pure}} \).

Let \( A \in \mathcal{M}_{m \times n}(\mathbb{Z}) \), \( B \in \mathcal{M}_{p \times n}(\mathbb{Z}) \) and an integer \( r \geq 2 \). We let

\[ \mathcal{L}_r := \mathcal{L}(\Lambda(A, B, r)), \quad \mathcal{L}_{A, B} := \text{Ker}_\mathbb{Z}(A) \cap \text{Ker}_\mathbb{Z}(B). \]

We note that \( \mathcal{L}_r \subset \mathbb{Z}^{zn} \) while \( \mathcal{L}_{A, B} \subset \mathbb{Z}^n \).

Proposition 2.2. \( \mathcal{L}_{A, B} \) is positive if and only if \( \mathcal{L}_r \) is positive for any \( r \geq 2 \).

Proof. Let \( C \in \mathcal{L}_{A, B} \cap \mathbb{N}^n \). We think of the elements of \( \mathcal{L}_r \) as \( r \times n \) matrices, as explained in the introduction. We have that \([C \cdots C]^T \in \mathcal{L}_r \cap \mathbb{N}^m\). Conversely, if \([C_1 \cdots C_r]^T \in \mathcal{L}_r \cap \mathbb{N}^{rn}\) then \( C_1 + \cdots + C_r \in \mathcal{L}_{A, B} \cap \mathbb{N}^n \). \( \square \)

Suppose that \( \mathcal{L}_r \) is positive. Let \( W \in \mathcal{L}_r \) and let \( \mathcal{W} \) the corresponding \( r \times n \) matrix with \( w_i \) as its \( i \)-th row. We define \( \text{supp}(W) = \{ i : w_i \neq 0, \, 1 \leq i \leq r \} \). Thus the type of \( \mathcal{W} \) is the cardinality of \( \text{supp}(W) \). The \( \Lambda(A, B, r) \)-degree of \( W \) is the vector \( \Lambda(A, B, r)W^+ \). Thus the \( \Lambda(A, B, r) \)-degree of \( W \) is in the span \( \mathbb{N}(a_i \otimes e_j \oplus b_i : 1 \leq i \leq n, j \in \text{supp}(W)) \). It is well known that the \( \Lambda(A, B, r) \)-degrees of any minimal Markov basis of \( \Lambda(A, B, r) \) are invariants of \( \Lambda(A, B, r) \), see [8].

Theorem 2.3. When \( \mathcal{L}_r \) is positive the complexity of a minimal Markov basis of \( \Lambda(A, B, r) \) is an invariant of \( \Lambda(A, B, r) \).

Proof. Let \( M_1, M_2 \) be two minimal Markov bases of \( \mathcal{L}_r \). It is enough to show that the complexity of \( M_1 \) is less than or equal to the complexity of \( M_2 \). Let \( \mathcal{W} = [w_1 \cdots w_r]^T \in M_1 \) be such that the type of \( \mathcal{W} \) is equal to the complexity of \( M_1 \). We let \( \mathcal{V} = [v_1 \cdots v_r]^T \in M_2 \) be such that the \( \Lambda(A, B, r) \)-degree of \( \mathcal{V} \) is the same as the \( \Lambda(A, B, r) \)-degree of \( \mathcal{W} \). Thus the \( \Lambda(A, B, r) \)-degree of \( \mathcal{V} \) is in \( \mathbb{N}(a_i \otimes e_j \oplus b_i : 1 \leq i \leq n, j \in \text{supp}(\mathcal{W})) \). This implies that \( \mathbf{v}_i^+ = 0 \) for every \( i \notin \text{supp}(\mathcal{W}) \). Since every nonzero element in \( \text{Ker}_\mathbb{Z}(A) \) has a nonzero positive part (and a nonzero negative part) it follows that \( \mathbf{v}_i = 0 \) for every \( i \notin \text{supp}(\mathcal{W}) \). Thus \( \text{supp}(\mathcal{V}) \subset \text{supp}(\mathcal{W}) \). Reversing the argument we get that \( \text{supp}(\mathcal{W}) = \text{supp}(\mathcal{V}) \). Therefore the complexity of \( M_1 \) is less than or equal to the complexity of \( M_2 \). \( \square \)

As in [8] Theorem 3.5] one can prove the following statement for arbitrary integer matrices \( A \in \mathcal{M}_{m \times n}(\mathbb{Z}) \), \( B \in \mathcal{M}_{p \times n}(\mathbb{Z}) \). We denote by \( \mathcal{G}_r(A) \) the matrix whose columns are the vectors of the Graver basis of \( A \).

Theorem 2.4. The Graver complexity \( g(A, B) \) is the maximum 1-norm of any element in the Graver basis \( \mathcal{G}(B \cdot \mathcal{G}_r(A)) \). In particular, we have \( g(A, B) < \infty \).

Suppose that \( \mathcal{L}_r \) is non positive. Next we show that \( \Lambda(A, B, r) \) has a minimal Markov basis (of minimal cardinality) whose complexity is \( r \).

Lemma 2.5. Suppose that \( \mathcal{L}_r \) is non positive. There exists a minimal Markov basis of \( \Lambda(A, B, r) \) of minimal cardinality, that contains an element of type \( r \).

Proof. We first show that \( \mathcal{L}_r \cap \mathbb{N}^{rn} \) has an element of type \( r \). By Lemma 2.2, \( \mathcal{L}_{A, B} \) is non positive. We let \( \mathbf{w} \in \mathcal{L}_{A, B} \cap \mathbb{N}^n \) be such that \( \text{supp}(\mathbf{w}) = \text{supp}((\mathcal{L}_{A, B})_{\text{pure}}) \). It
follows that
\[
\begin{pmatrix}
w \\
\vdots \\
w
\end{pmatrix} \in \mathcal{L}_r \cap \mathbb{N}^n
\]
has type \( r \). Since \( \mathcal{L}_r = \langle \mathcal{L}_r \cap \mathbb{N}^n \rangle \), we are done by Lemma 2.1. □

**Remark 2.6.** Suppose that \( r \geq 2 \) and \( \mathcal{L}_r \) is non positive. Let \( \mathbf{v} \) be the \( \mathcal{L}_r \)-primitive element of Lemma 2.1. By adding positive multiples of \( \mathbf{v} \) to the other elements of the Markov basis of Lemma 2.1 the new set is still a minimal Markov basis of \( \Lambda(A, B, r) \), see [3], with the property that all of its elements are of type \( r \).

As the next example shows if \( \mathcal{L}(\Lambda(A, B, r)) \) is non positive, then all is possible when considering the complexities of individual minimal Markov bases of \( \Lambda(A, B, r) \).

**Example 2.7.** We let \( A_1 \in \mathcal{M}_2(\mathbb{Z}) \), \( A_2 \in \mathcal{M}_2(\mathbb{Z}) \), \( B_2 \in \mathcal{M}_2(\mathbb{Z}) \) and \( A, B \in \mathcal{M}_{2 \times 4}(\mathbb{Z}) \) be the following matrices:

\[
A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad A = (A_1|A_2), \quad B_2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad B = (I_2|B_2).
\]

We consider the matrix \( \Lambda(A, B, r) \). After column permutations it follows that

\[
\Lambda(A, B, r) = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & A \\ B & B & \cdots & B \end{pmatrix} \rightarrow \begin{pmatrix} A_1 & 0 & 0 & A_2 & 0 & 0 \\ 0 & A_1 & 0 & 0 & A_2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & A_1 & 0 & 0 & A_2 \\ I_2 & I_2 & \cdots & I_2 & B_2 & B_2 & \cdots & B_2 \end{pmatrix}
\]

We note that the lattice \( \mathcal{L}(\Lambda(A_1, I_2, r)|\Lambda(A_2, B_2, r)) \) is isomorphic to the direct sum of the lattices \( \mathcal{L}(\Lambda(A_1, I_2, r)) \) and \( \mathcal{L}(\Lambda(A_2, B_2, r)) \) and thus there is a one to one correspondence between the Markov bases of \( \Lambda(A, B, r) \) and unions of the Markov bases of \( \mathcal{L}(\Lambda(A_1, I_2, r)) \) and \( \mathcal{L}(\Lambda(A_2, B_2, r)) \).

The matrix \( \Lambda(A_1, I_2, r) \) is the defining matrix of the toric ideal of the complete bipartite graph \( K_{2,r} \) and has a unique minimal Markov basis corresponding to cycles of length 4: all its elements have type 2, see [9] and [12]. We denote by \( C_i \) the columns of \( \Lambda(A_2, B_2, r) \), for \( i = 1, \ldots, 2r \). We note that \( C_1, C_3, \ldots, C_{2r-1} \) are linearly independent while \( C_{2l-1} = -C_{2l} \) for \( 1 \leq l \leq r \). It follows that the lattice \( \mathcal{L}(\Lambda(A_2, B_2, r)) \) has rank \( r \) and is pure. Thus it has infinitely many Markov bases, see [3]. We consider the following minimal Markov basis of \( \Lambda(A_2, B_2, r) \) consisting of elements of type 1:

\[
\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \}.
\]
For fixed $1 \leq a \leq r$ and $1 \leq b \leq 4$ we let $E_{a,b}$ be the matrix of $\mathcal{M}_{r \times 4}(\mathbb{Z})$ which has 1 on the $(a,b)$-th entry and 0 everywhere else. Moreover for $1 \leq i < j \leq r$ and $1 \leq s \leq r$, we let $P_{i,j} \in \mathcal{M}_{r \times 4}(\mathbb{Z})$ and $T_s \in \mathcal{M}_{r \times 4}(\mathbb{Z})$ be the matrices

$$P_{i,j} = E_{i,1} - E_{i,2} - E_{j,1} + E_{j,2}, \quad T_s = E_{s,3} + E_{s,4}.$$  

It follows that the set $\mathcal{M} = \{T_1, \ldots, T_r\} \cup \{P_{i,j} : 1 \leq i < j \leq r\}$ is a minimal Markov basis of $\Lambda(A, B, r)$ of cardinality $r + \binom{r}{2}$. The elements of $\mathcal{M}$ have type 1 and 2.

Note that the set

$$\{T_1, T_1 + T_2, \ldots, T_1 + \cdots + T_r\} \cup \{P_{i,j} : 1 \leq i < j \leq r\}$$

is a minimal Markov basis of $\Lambda(A, B, r)$ and the type of its elements range from 1 to $r$. Moreover if $T = \sum_{s=1}^{r} T_s$, then the set

$$\{T, T + T_2, \ldots, T + T_r\} \cup \{P_{i,j} : 1 \leq i < j \leq r\}$$

is a minimal Markov basis of $\Lambda(A, B, r)$ such that all its elements are of type $r$, see [3].

We remark that if $S$ is any integer linear combination of the elements $T_s, 1 \leq s \leq r$ and $1 \leq i < j \leq r$ then the element $S + P_{i,j}$ belongs to the infinite universal Markov basis of $\Lambda(A, B, r)$.

**Remark 2.8.** In the literature there are two definitions of the Markov complexity. The first was introduced in [10], as the smallest integer $m$ such that there exists a Markov basis of $\Lambda(A, B, r)$ of type less than or equal to $m$ for any $r \geq 2$. This Markov complexity is always finite, since there exists one minimal Markov basis inside the Graver basis and thus this Markov complexity is always smaller than the Graver complexity. Essential in the computation of this Markov complexity is Theorem 2.3 that guarantees all minimal Markov bases have the same complexity in the case that $\mathcal{L}_{A,B}$ is a positive lattice. The second definition was given in [8] where the Markov complexity of $(A, B)$ is the largest type of any element in the universal Markov basis of $\Lambda(A, B, r)$ as $r$ varies. In the second case, it follows from the previous discussions that the Markov complexity $m(A, B)$ is finite if and only if $\mathcal{L}_{A,B}$ is a positive lattice and in this case all minimal Markov bases of $\Lambda(A, B, r)$ have the same complexity for every $r \geq 2$. In the Example 2.7 the first Markov complexity is equal to two, while the second is infinite.

We point out that in the case that $\mathcal{L}_{A,B}$ is a positive lattice, which is the main case of interest in Algebraic Statistics, it follows from Theorem 2.3 that both definitions of the Markov complexity are equivalent.

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