AN ESTIMATE OF THE MAXIMAL OPERATORS ASSOCIATED WITH GENERALIZED LACUNARY SETS

GRIGOR A. KARAGULYAN AND MICHAEL T. LACEY

Abstract. Let $\Omega$ be any set of directions (unit vectors) on the plane. Denote by $R_\Omega$ the set of all rectangles which have a side parallel to some direction from $\Omega$. In this paper we study maximal operators on the plane $\mathbb{R}^2$ defined by

$$M_\Omega f(x) = \sup_{x \in R \in R_\Omega} \frac{1}{|R|} \int_R |f(y)| dy.$$ 

We are interested in extensions of lacunary sets of directions, to collections we call $N$–lacunary, for integers $N$. We proceed by induction. Say that $\Omega = \{v_k \mid k \in \mathbb{N}\}$ is 1–lacunary iff for each integer $k$, $v_k$ and $v_{k+1}$ are neighboring points, and there is a direction $v_\infty$ so that

$$\frac{1}{2}|v_k - v_{k+1}| < |v_{k+1} - v_\infty| < |v_k - v_{k+1}|.$$ 

Every $N+1$–lacunary set can be obtained from some $N$–lacunary $\Omega_N$ adding some points to $\Omega_N$. Between each two neighbor points $a, b \in \Omega_N$ we can add a 1–lacunary sequence (finite or infinite). We show that for all $N$ lacunary sets $\Omega$,

$$\|M_\Omega f(x)\|_2 \lesssim N \|f\|_2.$$ 

Observe that every set $\Omega$ of $N$ points is $(C \log N)$–lacunary. We then obtain a Theorem of N. Katz \cite{18}. Both the current inequality, and Katz' result are consequence of a general result of Alfonseca, Soria, and Vargas \cite{3}. We offer the current proof as a succinct, self–contained approach to this inequality.

1. Introduction

Let $\Omega$ be any set of directions (unit vectors) on the plane. Denote by $R_\Omega$ the set of all rectangles which have a side parallel to some direction from $\Omega$. In this paper we study maximal operators on the plane $\mathbb{R}^2$ defined by

$$M_\Omega f(x) = \sup_{x \in R \in R_\Omega} \frac{1}{|R|} \int_R |f(y)| dy.$$ 

A. Nagel, E.M. Stein and S. Wainger \cite{19} using Fourier transform method proved the boundedness of $M_\Omega f(x)$ in spaces $L^p$, $1 < p < \infty$ for any lacunary set of directions $\Omega = \{\theta_k\}$, $(\arg \theta_{k+1} < \lambda \arg \theta_k, \lambda < 1)$.

We are interested in extensions of lacunary sets of directions, to collections we call $N$–lacunary, for integers $N$. We proceed by induction. Say that $\Omega = \{v_k \mid k \in \mathbb{N}\}$ is 1–lacunary
iff for each integer $k$, $v_k$ and $v_{k+1}$ are neighboring points, and there is a direction $v_\infty$ so that
\[
\frac{1}{2}|v_k - v_{k+1}| < |v_{k+1} - v_\infty| < |v_k - v_{k+1}|.
\]
Every $N+1$–lacunary set can be obtained from some $N$–lacunary $\Omega_N$ adding some points to $\Omega_N$. Between each two neighbor points $a, b \in \Omega_N$ we can add a 1–lacunary sequence (finite or infinite). So if $\Omega$ is some $N$–lacunary set we can fix a sequence of sets $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_{N-1} \subset \Omega$ such that each $\Omega_k$ is $k$–lacunary.

It is commonly known that maximal functions in $N$–lacunary directions are bounded for all integers $N$. For instance, the case of 2–lacunary is due to P. Sjögren and P. Sjölin [20]. We are interested in growth of the norm of $M_\Omega$ for $N$–lacunary, as $N$ tends to infinity.

**Theorem 1.** For all integers $N$, and all $N$–lacunary sets $\Omega$ we have
\[
\|M_\Omega f(x)\|_2 \lesssim N\|f\|_2.
\]

It is easy to check that each set of directions of cardinality $N$ is $(C \log N)$–lacunary, for an absolute constant $C$. Therefore, as a corollary, we see that for finite collections $\Omega$, we have
\[
(1.2) \quad \|M_\Omega f\|_2 \lesssim (\log \# \Omega)\|f\|_2.
\]
This inequality is due to N. Katz [18]. This estimate is sharp as the power of $(\log \# \Omega)$, and so in the Theorem, our estimate is sharp as to the power of $N$.

Both Katz’ result and our Theorem is a consequence of a more general result of Alfonseca, Soria, and Vargas [3], a result we recall in more detail below. The current proof is succinct, and self–contained, and so may prove to be of some independent interest.

We close this section with a more detailed, but far from complete, description of the history of this question, and the relationship of our result to the literature. In 1977, A. Cordoba [7] considered the maximal function formed over all rectangles that are 1 by $N$, obtaining a slow increase in the norm on $L^2$. Thus, the set $\Omega$ is uniformly distributed, but one only considers rectangles of one aspect ratio. The method of proof employed a geometric method to prove a covering lemma. The method, as described in A. Cordoba and R. Fefferman [9], was broadly influential. The point of view adopted in this paper was formalized in an article from 1979 by S. Wainger [24]. The estimate (1.2) in the instance of uniformly distributed directions was proved by J. Stromberg [22], in 1978.

On the other hand, there were natural reasons to expect that the instance of lacunary directions would behave differently, and was investigated by J. Stromberg [21]. The full range of $L^p$, $1 < p < \infty$, inequalities in this instance was established by Fourier analysis, and square function methods by A. Nagel, S. Wainger, and E.M. Stein [19], a method that also proved to be influential. These results are related to interesting results on multipliers, as shown by A. Cordoba and R. Fefferman [10]. For extensions of this, see A. Carbery [6].
An interesting question was if Stromberg’s result [22] in the uniformly distributed case extended to the case of \( N \) distinct directions. A partial result was treated by Barrionuevo [4,5]. And the definitive result was obtained by N. Katz [18]. His method of proof is a clever duality argument, relying on an John–Nirenberg type to obtain the required estimate.

At this point, we note that there is a distinction between the case of rectangles of all aspect ratios, as we do, and the case of a fixed aspect ratio. It is the later case that is considered by e.g. A. Cordoba [7], and in Katz’ paper [17].

An interesting question concerns the maximal function computed in a set of directions specified by a Cantor set of directions. For the ordinary middle third Cantor set, there is a partial result on \( L^2 \) by A. Vargas [23]. Yet, this full maximal function is unbounded on \( L^2 \), as proved by N. Katz [16]. It would be interesting to obtain meaningful information about this maximal operator on \( L^p \), for \( p > 2 \). K. Hare [13] uses Katz’ argument, with more general Cantor sets.

Recently, A. Alfonesca, F. Soria and A. Vargas [2,3], also see Alfonseca [1], have proved an interesting orthogonality principle for these maximal functions. Let \( \Omega = \{v_k \mid k \in \mathbb{N}\} \) be a set of directions, and between two neighboring directions \( v_k, v_{k+1} \), let \( \Omega_k \) be an arbitrary set of directions. Then, (2) it is the case that

\[
\|M\|_{2 \rightarrow 2} \leq C \|M_\Omega\|_{2 \rightarrow 2} + \sup_k \|M_{\Omega_k}\|_{2 \rightarrow 2}.
\]

What is essential is that the second term occurs with constant 1. This proves our Theorem. Let \( \eta(N) \) be the maximum of \( \|M_{\Omega_n}\|_{2 \rightarrow 2} \), with the maximum taken over all \( N \)-lacunary sets of directions. The inequality above clearly implies that \( \eta(N) \leq C \eta(1) + \eta(N - 1) \). Iterating the inequality \( N - 1 \) times proves the Theorem.

General necessary and sufficient conditions on \( \Omega \) for the boundedness of \( M_{\Omega} \) have been sought by J. Duoandikoetxea, and A. Vargas [11], with extensions by K. Hare, and J. Rönning [14,15].

A paper by M. Christ [8] includes examples of sets of directions \( \Omega \), and partial results on the norm boundedness of \( M_{\Omega} \) which are not incorporated into the theories associated with this subject. K. Hare and F. Ricci [12] have established an interesting variant of the lacunary directional maximal function.

2. Notations

By \( A \lesssim B \) we mean that there is an absolute constant \( K \) so that \( A \leq KB \). By \( \hat{f}(\xi) \), we mean the Fourier transform of \( f \), thus

\[
\hat{f}(\xi) = \int f(x)e^{ix \cdot \xi} \, dx
\]
We use a well–known reduction to parallelograms. It is clear that we can associate directions in $\Omega$ to points in e.g. $(0, 1/4)$. Denote
\begin{equation}
P_\alpha f(x) = \sup_{\delta_1, \delta_2} \frac{1}{4\delta_1 \delta_2} \int_{x_1 - \delta_1}^{x_1 + \delta_1} \int_{x_2 - x_1 \alpha - \delta_2}^{x_2 + x_1 \alpha + \delta_2} |f(t_1, t_2)| \, dt_2 \, dt_1.
\end{equation}
This is a maximal function over parallelograms, with one side parallel to the $x$ axis, and the other side forming an angle of slope $\alpha$ with the $x$ axis. Then in order to prove the theorem it is sufficient to prove
\[ \| \sup_{\alpha \in \Omega} P_\alpha f \|_2 \leq CN \| f \|_2 \]
where $\Omega$ is any $N$–lacunary set from $(0, 1)$.

Our method of proof is Fourier analytic, and we shall find it convenient to use the the Fejer kernel
\[ K_r(x) = \int_{-r}^{r} \left( 1 - \frac{|t|}{r} \right) e^{-itx} \, dt = \frac{4 \sin \frac{N\pi}{2}}{N\pi} \]
For any $r, R$ with $0 \leq r < R/2$ we define the following functions
\[ \psi_r(x) = 2K_{2r}(x) - K_r(x), \quad \psi_{r,R}(x) = \psi_R(x) - \psi_r(x) \]
Sometimes we will write $\psi_{0,r}$ instead of $\psi_r(x)$. We have
\begin{equation}
\hat{\psi}_{r,R}(\xi) = \begin{cases} 
1 & \text{if } |\xi| \in [2r, R] \\
0 & \text{if } 0 \leq |\xi| \leq r \text{ or } |\xi| > 2R \\
\text{linear on each } \pm [r, 2r], \pm [R, 2R] 
\end{cases}
\end{equation}
From a property of Fejer kernel we have
\[ |\psi_{r,R}(x)| \leq C \left( \max \left\{ \frac{1}{R \pi^2}, R \right\} + \max \left\{ \frac{1}{r \pi^2}, r \right\} \right) \]
Thus for some sequence of intervals $\omega_k = \omega_{k,r,R}$ with centers at 0.
\begin{equation}
|\psi_{r,R}(x)| \leq C \sum_k \gamma_k \frac{\mathbb{I}_{\omega_k}(x)}{|\omega_k|} = \zeta_{r,R}(x)
\end{equation}
\[ \gamma_k > 0, \quad \sum_k \gamma_k < 1, \quad \omega_k \supset (1/R, 1/R). \]

Choose a Schwartz function $\phi$ with
\begin{equation}
\phi \geq 0, \quad \text{supp } \hat{\phi} \subset [-1, 1].
\end{equation}
We can fix an even function $\lambda$ with
\begin{equation}
\max \{|\phi(x)|, |x\phi(x)|\} \leq \lambda(x), \quad \int_{\mathbb{R}} \lambda(x) \, dx \leq C,
\end{equation}
Then define a Fourier analog of the average over parallelograms by
\[
\Gamma_{r,R,h}^\alpha f(x) = (\psi_{r,R}(x_2 - x_1 \alpha) \phi_h(x_1)) \ast f(x), \quad x = (x_1, x_2) \in \mathbb{R}^2.
\]
where
\[
\phi_h(x) = \frac{1}{h} \phi \left( \frac{x}{h} \right).
\]
From (2.6) and (2.1) it follows that
\[
P_{\alpha} f(x) \leq C \sup_{R,h} \Gamma_{R,h}^\alpha f(x).
\]
and therefore to prove our Theorem we need to verify the inequality
\[
\| \sup_{R,h,\alpha \in \Omega} \Gamma_{R,h}^\alpha f(x) \|_2 \leq C N \| f \|_2.
\]

Taking the Fourier transform both sides of (2.6) we get
\[
\hat{\Gamma}_{r,R,h}^\alpha f(\xi) = \hat{\phi}(h(\xi_1 + \xi_2 \alpha)) \hat{\psi}_{r,R}(\xi_2) \hat{f}(\xi).
\]

3. Proof of Theorem

Lemma 1. Let \( \alpha, \beta \in (0,1) \) be any numbers and \( 0 < r < R, h > 0 \). The operator \( \Gamma_{r,R,h}^\alpha f(x) \)
defined in (2.7) satisfies pointwise estimate
\[
|\Gamma_{r,R,h}^\alpha f(x)| \leq C (hR|\alpha - \beta| + 1) P_{\beta} f(x), \quad x \in \mathbb{R}^2.
\]

Proof. From (2.3) we have
\[
\psi_{r,R}(x_2 - x_1 \alpha) \leq C \sum_{k} \frac{\gamma_k}{|\omega_k|} \mathbb{1}_{\omega_k}(x_2 - x_1 \alpha)
\]
where we have \( |\omega_k| > 2/R \). Denote \( \lambda(x_1) = 2Rx_1|\alpha - \beta| + 2 \) and assume
\[
x_2 - x_1 \alpha \in \omega_k
\]
for some \( k \). Then taking account of (2.3) we get
\[
\frac{|x_2 - x_1 \beta|}{\lambda(x_1)} = \frac{|x_2 - x_1 \alpha + x_1(\alpha - \beta)|}{\lambda(x_1)} \leq \frac{|x_2 - x_1 \alpha|}{2} + \frac{1}{2R} \leq \frac{|\omega_k|}{2},
\]
which means
\[
\frac{x_2 - x_1 \beta}{\lambda(x_1)} \in \omega_k.
\]
Hence we conclude that (3.2) implies (3.4). Therefore
\[
\mathbb{1}_{\omega_k}(x_2 - x_1 \alpha) \leq \mathbb{1}_{\omega_k} \left( \frac{x_2 - x_1 \beta}{\lambda(x_1)} \right).
\]
Finally we get
\[ \psi_{r,R}(x_2 - x_1\alpha) \leq C \sum_k \frac{\gamma_k}{|\omega_k|} \|\omega_k\| \left( \frac{x_2 - x_1\beta}{\lambda(x_1)} \right) \leq \zeta_{r,R} \left( \frac{x_2 - x_1\beta}{\lambda(x_1)} \right) \]

Thus taking account of (2.5) we obtain
\[ \frac{1}{h} \phi \left( \frac{x_1}{h} \right) \zeta_{r,R} \left( \frac{x_2 - x_1\beta}{\lambda(x_1)} \right) \leq C (hR|\alpha - \beta| + 1) \frac{1}{h} \xi \left( \frac{x_1}{h} \right) \frac{1}{\lambda(x_1)} \zeta_{r,R} \left( \frac{x_2 - x_1\beta}{\lambda(x_1)} \right) \]

from which we easily get (3.1).

For any interval \( J = (a, b) \) we denote by \( S(J) \) the sector \( \{ ax_2 \leq x_1 \leq bx_2 \} \). For any sector \( S \) define by \( 2S \) the sector which has same bisectrix with \( S \) and twice bigger angle. Denote by \( T_S f \) the multiplier operator defined \( \hat{T}_S f = \mathbb{I}_S \hat{f} \).

**Lemma 2.** Let \( J_1 \supset J_2 \supset \cdots \supset J_n \) be some sequence of intervals with
\[
J_k = [\alpha_k, \beta_k] \subset (0, 1), \quad \text{dist}((J_k)^c, J_{k+1}) \leq |J_{k+1}|, \quad 1 \leq k \leq n
\]

Then for any \( \theta \in \bigcap J_k \) and any function \( f \in L^2(\mathbb{R}^2) \) we have
\[
P_\theta f \lesssim P_0 f + P_0(T_{2S(J_n)} f)
\]
\[
+ \sum_{k=1}^{n-1} P_{\alpha_k}(T_{2S(J_k)} f) + P_{\beta_k}(T_{2S(J_k)} f)
\]

where \( P_0 \) is a \( P_\alpha \) with \( \alpha = 0 \).

**Proof.** Regard \( \theta \in \bigcap J_k \) as fixed. For any \( R, h \) we have
\[
\hat{\Gamma}_{R,h}^\theta f(\xi) = \hat{\psi}_R(\xi_2) \hat{\phi}(h(\xi_1 + \xi_2\theta)) \hat{f}(x)
\]

Denote
\[
r_0 = 0, \quad r_k = \frac{2}{h|J_k|} \quad 1 \leq k \leq n.
\]

From (2.2) it follows that
\[
\hat{\psi}_R(\xi_2) = \sum_{k=1}^m \hat{\psi}_{2r_{k-1},r_k}(\xi_2) + \hat{\psi}_{2r_m,R}(\xi_2)
\]

where \( m = \max\{k : r_k < 2R\} \). Denote
\[
\Gamma_k f(x) = \Gamma_{2r_{k-1},r_k}^\theta f(x) \quad 0 \leq k < m,
\]
\[
\Gamma_m f(x) = \Gamma_{2r_m,R}^\theta f(x).
\]
Then by (2.8) we have
\[ \hat{\Gamma}_k f(\xi) = \hat{\psi}_{2r_k - 1, r_k}(\xi_2) \hat{\phi}(h(\xi_1 + \xi_2 \theta)) \hat{f}(x) \quad 1 \leq k < m \]
\[ \hat{\Gamma}_m f(x) = \hat{\psi}_{2r_m, r}(\xi_2) \hat{\phi}(h(\xi_1 + \xi_2 \theta)) \hat{f}(x) \]
and therefore using (3.9) we obtain
\[ (3.10) \quad \Gamma_{R,h}^\theta f = \sum_{k=0}^{m} \Gamma_k f \]

Let us show
\[ \text{supp } \hat{\psi}_{2r_k, r_k+1}(\xi_2) \hat{\phi}(h(\xi_1 + \xi_2 \theta)) \subset 2S(J_k), \quad 1 \leq k < m, \]
\[ \text{supp } \hat{\psi}_{2r_m, r}(\xi_2) \hat{\phi}(h(\xi_1 + \xi_2 \theta)) \subset 2S(J_m) \]
From which it follows that
\[ \Gamma_k f = \Gamma_k (T_{2S(J_k)} f), \quad 1 \leq k \leq m \]
Indeed, from (2.4) and (2.2) it follows that
\[ \text{supp } \hat{\psi}_{2r_k, r_k+1}(\xi_2) \hat{\phi}(h(\xi_1 + \xi_2 \theta)) \]
\[ = \{(\xi_1, \xi_2) : r_k \leq \xi_2 \leq 2r_{k+1}, |\xi_1 + \xi_2 \theta| < \frac{1}{h} \} \]
The last set is a parallelogram with vertexes \((r_k \theta \pm \frac{1}{h}, r_k)\) and \((2r_{k+1} \theta \pm \frac{1}{h}, 2r_{k+1})\). These vertexes are from \(2S(J_k)\) because
\[ \frac{r_k \theta \pm \frac{1}{h}}{r_k} = \theta \pm \frac{|J_k|}{2} \]
which means \((r_k \theta \pm \frac{1}{h}, r_k) \in 2S(J_k)\). The same conclusion is true for next the pair of vertexes. This implies (3.11).

Using Lemma 1 we conclude
\[ |\Gamma_k f| \lesssim (hr_{k+1} \min\{|\theta - \alpha_k|, |\theta - \beta_k|\} + 1) \times \]
\[ (P_{\alpha_k} (T_{2S(J_k)} f) + P_{\beta_k} (T_{2S(J_k)} f)) \quad 1 \leq k < m \]
Notice also
\[ |\Gamma_0 f| \leq P_0 f \]
\[ (3.14) \quad |\Gamma_m f| \leq P_0 T_{2S(J_m)} f \]
By \( \theta \in J_{k+1} \subset J_k \) and (3.5) we have
\[ \min\{|\theta - \alpha_k|, |\theta - \beta_k|\} \leq 2|J_{k+1}| \]
The last with (3.8) implies
\[ hr_{k+1} \min\{|\theta - \alpha_k|, |\theta - \beta_k|\} \leq 4 \]
Hence by (3.12) we observe
\[ |\Gamma_k f| \lesssim P_{\alpha_k} (T_{2S(J_k)} f) + P_{\beta_k} (T_{2S(J_k)} f), \quad 1 \leq k < m. \]
Finally taking account also (3.13) and (3.14) we get Lemma 2.

**Proof of Theorem 1.** Let \( \Omega \subset (0, 1) \) be any \( N \)-lacunary set. We fix the sets \( \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_{N-1} \subset \Omega_N = \Omega \) from definition of \( N \)-lacunarity. Fix any angle \( \theta \in \Omega \) and \( R, h > 0 \). Suppose

\[ \theta \in \Omega_m \setminus \Omega_{m-1}, \text{ for some } m \leq N. \]

Denote by \( G_k \) the set of all intervals whose vertexes are neighbor points in \( \Omega_k \). We can choose a sequence of intervals \( J_k = [\alpha_k, \beta_k] \in G_k \) \( k = 1, 2, \cdots, m \) such that
\[ \theta \in \bigcap_{1 \leq k \leq m} J_k, \quad \theta = \alpha_m \text{ (or } \theta = \beta_m) \]

It is clear that sequence \( J_k \) satisfies conditions of Lemma 2. Hence,
\[ |M_{\theta} f|^2 \lesssim \left\{ |M_0 f| + \sum_{k=1}^m (M_{\alpha_k} (T_{2S(J_k)} f) + M_{\beta_k} (T_{2S(J_k)} f)) \right\}^2 \]
\[ \lesssim |M_0 f|^2 + m \sum_{k=1}^m \left( |M_\alpha (T_{2S(J_k)} f)|^2 + |M_\beta (T_{2S(J_k)} f)|^2 \right) \]

and therefore, summing over every interval \( J = (\alpha, \beta) \in G_k \),
\[ \sup_{\theta \in \Omega} |M_{\theta} f|^2 \lesssim |M_0 f|^2 + N \sum_{k=1}^N \sum_{J=(\alpha, \beta) \in G_k} |M_\alpha (T_{2S(J)} f)|^2 + |M_\beta (T_{2S(J)} f)|^2 \]

On the other hand using the \((2,2)\) bound of strong maximal operator we get for each \( 1 \leq k \leq N \),
\[ \int_{\mathbb{R}^2} \sum_{J=(\alpha, \beta) \in G_k} |M_\alpha (T_{2S(J)} f)|^2 + |M_\beta (T_{2S(J)} f)|^2 \, dx \leq \int_{\mathbb{R}^2} \sum_{J=(\alpha, \beta) \in G_k} \|2S(J) \hat{f}\|^2 \, d\xi \]
\[ \lesssim \int_{\mathbb{R}^2} |\hat{f}|^2 \, d\xi \]
\[ = \int_{\mathbb{R}^2} |f|^2 \, dx \]
Finally taking account of (3.16) we obtain
\[ \int_{\mathbb{R}^2} \sup_{\theta \in \Omega} |M_{\theta} f|^2 \, dx \lesssim N^2 \int_{\mathbb{R}^2} |f|^2 \, dx \]
\[ \square \]
AN ESTIMATE OF THE MAXIMAL OPERATORS ASSOCIATED WITH GENERALIZED LACUNARY SETS

REFERENCES

[1] Angeles Alfonseca, *Strong type inequalities and an almost-orthogonality principle for families of maximal operators along directions in \( \mathbb{R}^2 \)*, J. London Math. Soc. (2) **67** (2003), 208–218. MR 942 421

[2] Angeles Alfonseca, Fernando Soria, and Ana Vargas, *A remark on maximal operators along directions in \( \mathbb{R}^2 \)*, Math. Res. Lett. **10** (2003), 41–49. MR 960 122

[3] Angeles Alfonseca, *An Almost–Orthogonality Principle in \( L^2 \) for Directional Maximal Functions*, Contemp. Math. (2003).

[4] Jose Barrionuevo, *Estimates for some Kakeya-type maximal operators*, Trans. Amer. Math. Soc. **335** (1993), 667–682. MR 93f:42038

[5] *A note on the Kakeya maximal operator*, Math. Res. Lett. **3** (1996), 61–65. MR 98k:42023

[6] Anthony Carbery, *Differentiation in lacunary directions and an extension of the Marcinkiewicz multiplier theorem*, Ann. Inst. Fourier (Grenoble) **38** (1988), 157–168. MR 89h:42026 (English, with French summary)

[7] Antonio Cordoba, *The Kakeya maximal function and the spherical summation multipliers*, Amer. J. Math. **99** (1977), 1–22. MR 56 #6259

[8] Michael Christ, *Examples of singular maximal functions unbounded on \( L^p \)*, Publ. Mat. **35** (1991), 269–279, Conference on Mathematical Analysis (El Escorial, 1989). MR 92f:42024

[9] A. Córdoba and R. Fefferman, *On differentiation of integrals*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 2211–2213. MR 57 #16522

[10] A. Cordoba and R. Fefferman, *On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 423–425. MR 55 #6096

[11] Javier Duoandikoetxea and Ana Vargas, *Directional operators and radial functions on the plane*, Ark. Mat. **33** (1995), 281–291. MR 97c:42031

[12] Kathryn Hare and Fulvio Ricci, *Maximal functions with polynomial densities in lacunary directions*, Trans. Amer. Math. Soc. **355** (2003), 1135–1144 (electronic). 1 938 749

[13] Kathryn E. Hare, *Maximal operators and Cantor sets*, Canad. Math. Bull. **43** (2000), 330–342. MR 2003f:42027

[14] Kathryn E. Hare and Jan-Olav Ronning, *Applications of generalized Perron trees to maximal functions and density bases*, J. Fourier Anal. Appl. **4** (1998), 215–227. MR 2000a:42035

[15] *The size of Max\((p)\) sets and density bases*, J. Fourier Anal. Appl. **8** (2002), 259–268. MR 2003b:42032

[16] Nets Hawk Katz, *A counterexample for maximal operators over a Cantor set of directions*, Math. Res. Lett. **3** (1996), 527–536. MR 98b:42032

[17] *Remarks on maximal operators over arbitrary sets of directions*, Bull. London Math. Soc. **31** (1999), 700–710. MR 2001g:42041

[18] *Maximal operators over arbitrary sets of directions*, Duke Math. J. **97** (1999), 67–79. MR 2000a:42036

[19] A. Nagel, E. M. Stein, and S. Wainger, *Differentiation in lacunary directions*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 1060–1062. MR 57 #6349

[20] P. Sjögren and P. Sjölin, *Littlewood-Paley decompositions and Fourier multipliers with singularities on certain sets*, Ann. Inst. Fourier (Grenoble) **31** (1981), vii, 157–175. MR 82g:42014 (English, with French summary)

[21] Jan-Olav Strömberg, *Weak estimates on maximal functions with rectangles in certain directions*, Ark. Mat. **15** (1977), 229–240. MR 58 #6911

[22] *Maximal functions associated to rectangles with uniformly distributed directions*, Ann. Math. (2) **107** (1978), 399–402. MR 58 #1978
[23] Ana M. Vargas, *A remark on a maximal function over a Cantor set of directions*, Rend. Circ. Mat. Palermo (2) **44** (1995), 273–282. MR 96m:42033

[24] Stephen Wainger, *Applications of Fourier transforms to averages over lower-dimensional sets*, Harmonic Analysis in Euclidean Spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979, pp. 85–94. MR 82g:42018

INSTITUTE OF MATHEMATICS, ARMENIAN NATIONAL ACADEMY OF SCIENCES, MARSHAL BAGHRAMIAN AVE. 24B., YEREVAN, 375019, ARMENIA,

*E-mail address: karagul@instmath.sci.am*

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA GA 30332

*E-mail address: lacey@math.gatech.edu*