Existence and Weak* Stability for the Navier-Stokes System with Initial Values in Critical Besov Spaces

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Abstract

In 2016, Seregin and Šverák, conceived a notion of global in time solution (as well as proving existence of them) to the three dimensional Navier-Stokes equation with $L_3$ solenoidal initial data called 'global $L_3$ solutions'. A key feature of global $L_3$ solutions is continuity with respect to weak convergence of a sequence of solenoidal $L_3$ initial data. The first aim of this paper is to show that a similar notion of 'global $\dot{B}_{4,\infty}^{-\frac{1}{4}}$ solutions' exists for solenoidal initial data in the wider critical space $\dot{B}_{4,\infty}^{-\frac{1}{4}}$ and satisfies certain continuity properties with respect to weak* convergence of a sequence of solenoidal $\dot{B}_{4,\infty}^{-\frac{1}{4}}$ initial data. This is the widest such critical space if one requires the solution to the Navier-Stokes equations minus the caloric extension of the initial data to be in the global energy class.

For the case of initial values in the wider class of $\dot{B}_{p,\infty}^{-1+\frac{2}{p}}$ initial data ($p > 4$), we prove that for any $0 < T < \infty$ there exists a solution to the Navier-Stokes system on $\mathbb{R}^3 \times [0, T]$ with this initial data. We discuss how properties of these solutions imply a new regularity criteria for 3D weak Leray-Hopf solutions in terms of the norm $\|v(\cdot, t)\|_{\dot{B}_{p,\infty}^{-1+\frac{2}{p}}}$ (as well as certain additional assumptions).

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The main new observation of this paper, that enables these results, regards the decomposition of homogeneous Besov spaces $\dot{B}_{p,\infty}^{-1+\frac{2}{p}}$. This does not appear to obviously follow from the known real interpolation theory.

1 Introduction

In this paper, we consider the Cauchy problem for the Navier-Stokes system in the space-time domain $Q_S = \mathbb{R}^3 \times [0, S]$ ($0 < S \leq \infty$) for the vector-valued function $v = (v_1, v_2, v_3) = (v_i)$ and scalar function $q$, satisfying the equations

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q, \quad \text{div } v = 0$$

(1.1)

in $Q_S$, the boundary conditions

$$v(x, t) \to 0$$

(1.2)

as $|x| \to \infty$ for all $t \in [0, S]$, and the initial conditions

$$v(\cdot, 0) = u_0(\cdot)$$

(1.3)

In the recent paper [31], a notion of global in time solution to the Navier-Stokes equation was developed with $L_3$ solenoidal initial data called ‘global $L_3$ solutions’. A key feature of global $L_3$ solutions is as follows. Namely, if $u^{(n)}$ are global $L_3$ solutions corresponding the the initial datum $u_0^{(n)}$, and $u_0^{(n)}$ converge weakly in $L_3(\mathbb{R}^3)$ to $u_0$, then a suitable subsequence of $u^{(n)}$ converges to global $L_3$ solution $u$ corresponding to the initial condition $u_0$. To explain the notation of global weak $L_3$-solutions in [31] further, we introduce the notation

$$S(t)u_0(x) = \int_{\mathbb{R}^3} \Gamma(x - y, t)u_0(y)dy,$$

where $\Gamma$ is the three dimensional heat kernel. Throughout this paper we will often write $V(x, t) := S(t)u_0(x)$. In [31], any global weak $L_3$-solution of the Navier-Stokes equation, with initial data $u_0 \in L_3(\mathbb{R}^3)$, has the following structure. Namely,

$$v(x, t) = V(x, t) + u(x, t),$$

(1.4)

where $u$ is globally in the energy class in $\mathbb{R}^3 \times ]0, T[$ for any finite $T > 0$ and satisfies the global energy inequality
\begin{equation}
\|u(\cdot, t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, t')|^2 \, dx \, dt' \leq 2 \int_0^t \int_{\mathbb{R}^3} (V \otimes u + V \otimes V) : \nabla u \, dx \, dt'.
\end{equation}

In [31], the crucial estimate
\begin{equation}
\|u(\cdot, t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, t')|^2 \, dx \, dt' \leq t^{1/2} C(\|u_0\|_{L^3(\mathbb{R}^3)}).
\end{equation}

is proven for \( u \), with \( u \) and \( V \) as in (1.4)-(1.5). This estimate plays a central role in [31] in the proof of continuity of global weak \( L^3 \) solutions, with respect to the weak convergence of the initial data.

A natural question concerns whether an analogous notion of of global in time solutions (which we will refer to as 'global \( X \) solutions' or sometimes \( N(X) \)) with initial data \( u_0 \) in other critical spaces \( X \), that satisfy the properties

- **1) Global existence** for any \( u_0 \in X \) there exists a global in time solution in \( N(X) \);
- **2) Weak* stability**

\( u_0^{(k)} \rightharpoonup u_0 \text{ in } X \Rightarrow u^{(k)}(\cdot, u_0^{(k)}) \in N(X) \) converges up to subsequence (in sense of distributions) to \( u(\cdot, u_0) \in N(X) \);

As mentioned in [3], such properties are useful if one wants to show that critical norms of the Navier-Stokes equations tend to infinity at a potential blow up time. Let us now describe this in more detail. Suppose one wanted to prove that if \( v \) is a weak Leray-Hopf solution on \( \mathbb{R}^3 \times ]0, \infty[ \), with sufficiently regular initial data as well as a finite blow up time \( T \), then necessarily

\begin{equation}
\lim_{t \uparrow T} \|v(\cdot, t)\|_X = \infty.
\end{equation}

1 We say \( X \) is a critical space, if \( u_0 \in X \Rightarrow \lambda u_0(\lambda \cdot) \in X \) and \( \|u_0\|_X = \|u_0\|_X \)
Where $X$ is a critical space. One such strategy, given in [28] and subsequently used in [29], [5] and [1], for showing this is to assume for contradiction that there exists $t_n \uparrow T$ with

$$M := \sup_n \|v(\cdot, t_n)\|_X < \infty. \tag{1.8}$$

The next step is to perform the rescaling

$$u^{(n)}(y, s) = \lambda_n v(x, t), \quad p^{(n)}(y, s) = \lambda_n^2 q(x, t), \quad u_0^{(n)}(y) = \lambda_n v(\lambda_n y, t_n), \tag{1.9}$$

$$x = \lambda_n y, \quad t = T + \lambda_n^2 s, \quad \lambda_n = \sqrt{\frac{T - t_n}{2}}. \tag{1.10}$$

This gives that $(u^{(n)}, p^{(n)})$ are solutions to the Navier-Stokes equations on $\mathbb{R}^3 \times [2, 0]$ with

$$\sup_n \|u_0^{(n)}\|_X = \|v(\cdot, t_n)\|_X = M. \tag{1.11}$$

The final part of the strategy is to obtain a non-trivial ancient solutions to the Navier-Stokes equations and to then attempt to apply a Liouville type theorem to it, based on backward uniqueness and unique continuation for parabolic operators developed in [12]. The motivation for considering ‘global $X$ solutions’, with properties 1) Global existence- 2) Weak* stability, is that if such solutions exist then they are particularly useful in obtaining a non-trivial ancient solution in the above strategy.

Unfortunately, the method for proving the existence of global $L_3$ solutions, with properties 1) Global Existence- 2) Weak* stability, breaks down for $L^{3,\infty}$ initial data. In [3]-[4], this difficulty was overcome. In particular in [3]-[4], a notion of ‘global weak $L^{3,\infty}$ solutions’, satisfying 1) Global existence - 2) Weak* stability and having the structure (1.4)-(1.5), was conceived. In [3]-[4] the key is establishing that for any global $L^{3,\infty}$ solution $v$, with solenoidal initial data $u_0 \in L^{3,\infty}(\mathbb{R}^3)$, we have

$$\|v(\cdot, t) - S(t)u_0\|^2_{L_2} + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla v(x, t') - S(t)u_0|^2 dx dt' \leq t^{\frac{3}{2}} C(\|u_0\|_{L^{3,\infty}(\mathbb{R}^3)}). \tag{1.12}$$

A key part in showing this uses that for any $N > 0$ any divergence free function $u_0$ in $L^{3,\infty}(\mathbb{R}^3)$ can be decomposed into two divergence free pieces $\tilde{u}_0^N$ and $\tilde{u}_0^\gamma$ satisfying

$$\|\tilde{u}_0^N\|^2_{L_2} \leq CN^{-1} \|u_0\|_{L^{3,\infty}}^3. \tag{1.13}$$
and
\[ \|u_0^N\|_{L^p}^p \leq C N^{p-3} \|u_0\|_{L^{3,\infty}}^3 \] (1.14)
for any \(3 < p\). This, together with appropriate decompositions of the Navier-Stokes equations (inspired by [8]), imply (1.12).

Suppose one attempts constructs a global \(X\) solution to the Navier-Stokes equation, with \(X\) being a critical space, having the structure (1.4)-(1.5). To ensure the finiteness of the right hand side of the energy inequality (1.5), one should have that for any \(u_0 \in X\) that
\[ S(t)u_0 \in L^4_{4,loc}(0, \infty; L^4_{R^3}). \]

Hence, it is natural to consider \(X\) such that for any \(0 < T < \infty\) and \(0 < \epsilon < T\):
\[ \int_\epsilon^T \|S(t)u_0\|_{L^4} dt \leq c(T, \epsilon) \|u_0\|_X. \] (1.15)

This implies that
\[ \|S(1)u_0\|_{L^4} \leq c \|u_0\|_X. \] (1.16)

For \(\lambda > 0\) and \(u_{0\lambda}(x) := \lambda u_0(\lambda x)\), it is clear that \(S(1)u_{0\lambda}(x) = \lambda S(\lambda^2)u_0(\lambda x)\). Hence, (1.16) implies that for any \(\lambda > 0\)
\[ \lambda^{\frac{1}{4}} \|S(\lambda^2)u_0\|_{L^4} = \|S(1)u_{0\lambda}\|_{L^4} \leq C \|u_{0\lambda}\|_X = C \|u_0\|_X. \] (1.17)

Hence,
\[ \sup_{s > 0} s^{\frac{1}{4}} \|S(s)u_0\|_{L^4} \leq C \|u_0\|_X. \] (1.18)

In particular \(X \hookrightarrow \dot{B}_{4,\infty}^{-\frac{1}{4}}(R^3)\). Thus, if one seeks a global \(X\) solution, satisfying the requirements 1) Global existence- 2) Weak* Stability and having the structure (1.4)-(1.5), \(X = \dot{B}_{4,\infty}^{-\frac{1}{4}}(R^3)\) is the widest critical space for such a possibility.

The aim of this paper is to show that, for any divergence free initial data in \(\dot{B}_{4,\infty}^{-\frac{1}{4}}\), there exists ‘global \(\dot{B}_{4,\infty}^{-\frac{1}{4}}\) solution’ to the Navier-Stokes equations (1.1)-(1.3), which satisfies the requirements 1) Global existence- 2) Weak* Stability. To the best of the author’s knowledge, there is currently no notion of solutions to the Navier-Stokes equations with arbitrary \(\dot{B}_{4,\infty}^{-\frac{1}{4}}\) solenoidal initial data. Note that under certain smallness conditions on the \(\dot{B}_{4,\infty}^{-\frac{1}{4}}\) data,
solutions were constructed by means of the Banach contraction principle, in [26] and [9]. Let us mention that the recent preprint [7] implies that if \( u_0 \in \dot{B}_{4,\infty}^{\frac{1}{4}} \) is discretely self similar\(^2\), then there exists a discretely self similar solution to the Navier-Stokes equation. These solutions will also belong to our class of global \( \dot{B}_{4,\infty}^{\frac{1}{4}} \) solutions.

Before giving the definition of 'global \( \dot{B}_{4,\infty}^{\frac{1}{4}}(\mathbb{R}^3) \) solutions', we provide some relevant definitions and notation:

\( J \) and \( J^{\frac{1}{2}} \) are the completion of the space \( C^{\infty}_{0,0}(\mathbb{R}^3) := \{ v \in C^{\infty}_0(\mathbb{R}^3) : \text{div} \, v = 0 \} \) with respect to \( L_2 \)-norm and the Dirichlet integral

\[
\left( \int_{\mathbb{R}^3} |\nabla v|^2 \, dx \right)^{\frac{1}{2}},
\]
correspondingly. Additionally, we define the space-time domains \( Q_T := \mathbb{R}^3 \times ]0, T[ \) and \( Q_\infty := \mathbb{R}^3 \times ]0, \infty[ \).

For arbitrary vectors \( a = (a_i), b = (b_i) \) in \( \mathbb{R}^n \) and for arbitrary matrices \( F = (F_{ij}), G = (G_{ij}) \) in \( M^n \) we put

\[
a \cdot b = a_i b_i, \quad |a| = \sqrt{a \cdot a},
\]
\[
a \otimes b = (a_i b_j) \in M^n,
\]
\[
FG = (F_{ik} G_{kj}) \in M^n, \quad F^T = (F_{ji}) \in M^n,
\]
\[
F : G = F_{ij} G_{ij}, \quad |F| = \sqrt{F : F}.
\]

**Definition 1.1.** We say that \( v \) is a weak \( \dot{B}_{4,\infty}^{\frac{1}{4}} \)-solution to Navier-Stokes IVP in \( Q_T \) (with \( 0 < T < \infty \)) if

\[
v = V + u, \quad (1.19)
\]
with \( u \in L_\infty(0, T; J) \cap L_2(0, T; J^{\frac{1}{2}}) \) and there exists an \( \alpha > 0 \) such that

\[
\sup_{0 < t < T} \frac{\|u(\cdot, t)\|_{L_2}^2}{t^\alpha} < \infty. \quad (1.20)
\]

\(^2\) the discretely self similar solutions in [7] are shown to exist for any discretely self similar data in \( \dot{B}_{p,\infty}^{-\frac{1}{2}}(\mathbb{R}^3) \) with \( 3 < p < 6 \).
Additionally, it is required that there exists a \( q \in L^2_{2, \text{loc}}(Q_T) \) such that \( u \) and \( q \) satisfy the perturbed Navier-Stokes system in the sense of distributions:

\[
\partial_t u + v \cdot \nabla v - \Delta u = -\nabla q, \quad \text{div} u = 0 \tag{1.21}
\]

in \( Q_T \). Furthermore, it is required that for any \( w \in L^2 \):

\[
t \to \int_{\mathbb{R}^3} w(x) \cdot u(x, t) dx
\]

is a continuous function on \([0, T]\). Moreover, \( u \) satisfies the energy inequality:

\[
\|u(\cdot, t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, t')|^2 dx dt' \leq
\]

\[
\leq 2 \int_0^t \int_{\mathbb{R}^3} (V \otimes u + V \otimes V) : \nabla u dx dt' \tag{1.23}
\]

for all \( t \in [0, T] \).

Finally, it is required that \( v \) and \( q \) satisfy the local energy inequality. Namely, for almost every \( t \in ]0, T[ \) the following inequality holds for all nonnegative functions \( \varphi \in C^\infty_0(Q_T) \):

\[
\int_{\mathbb{R}^3} \varphi(x,t)|v(x,t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi |\nabla v|^2 dx dt' \leq
\]

\[
\leq \int_0^t \int_{\mathbb{R}^3} |v|^2 (\partial_t \varphi + \Delta \varphi) + v \cdot \nabla \varphi(|v|^2 + 2q) dx dt'. \tag{1.24}
\]

We define \( v \) to be a global \( \dot{B}_{2, \infty}^{-\frac{3}{4}}(\mathbb{R}^3) \) solution if it is a weak \( \dot{B}_{4, \infty}^{-\frac{1}{4}} \)-solution in \( Q_T \) for any \( 0 < T < \infty \).

**Remark 1.2.** The role of the requirement (1.20) is to ensure that the right hand side of the energy inequality (1.1) is finite. See Lemma 4.1 for more details. This therefore ensures that the function \( u \) satisfies the initial condition in the strong \( L^2 \)-sense, i.e., \( u(\cdot, t) \to 0 \) in \( L^2 \).

Next we state the main results of this paper.
Theorem 1.3. Let \( u^{(k)}_0 \overset{*}{\rightharpoonup} u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty} \) and let \( v^{(k)} \) be a sequence of a global weak \( \dot{B}^{-\frac{1}{4}}_{4,\infty} \)-solutions to the Cauchy problem for the Navier-Stokes system with initial data \( u^{(k)}_0 \). Then there exists a subsequence still denoted \( v^{(k)} \) that converges to a global weak \( \dot{B}^{-\frac{1}{4}}_{4,\infty} \)-solution \( v \) to the Cauchy problem for the Navier-Stokes system with initial data \( u_0 \), in the sense of distributions.

Corollary 1.4. There exists at least one global weak \( \dot{B}^{-\frac{1}{4}}_{4,\infty} \)-solution to the Cauchy problem \((1.1)-(1.3)\).

Let us state our main observation, regarding decomposition of homogeneous Besov spaces, that enables us to show the global existence of global \( \dot{B}^{-\frac{1}{4}}_{4,\infty} \) solutions and that the property 2) Weak* stability holds true for them. Note that from this point onwards, for \( p_0 > 3 \), we will denote

\[
s_{p_0} := -1 + \frac{3}{p_0} < 0.
\]

Moreover, for \( 2 < \alpha \leq 3 \) and \( p_1 > \alpha \), we define

\[
s_{p_1,\alpha} := -\frac{3}{\alpha} + \frac{3}{p_1} < 0.
\]

Proposition 1.5. Suppose that \( 3 < p < \infty \),

\[
g \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)
\]

and \( \text{div} \, g = 0 \) in sense of tempered distributions. Then the above assumptions imply that there exists \( p < p_2 < \infty \), \( 0 < \delta_2 < -s_{p_2} \), \( \gamma_1 := \gamma_1(p) > 0 \) and \( \gamma_2 := \gamma_2(p) > 0 \) such that for any \( N > 0 \) there exists divergence free tempered distributions \( \bar{g}^N \in \dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \) and \( \tilde{g}^N \in L_2(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3) \) with the following properties. Namely,

\[
g = \bar{g}^N + \tilde{g}^N,
\]

\[
\| \bar{g}^N \|_{\dot{B}^{s_{p_2}+\delta_2}_{p_2,p_2}} \leq N^{\gamma_1} C(p, \| g \|_{\dot{B}^{s_p}_{p,\infty}}),
\]

\[
\| \tilde{g}^N \|_{L_2} \leq N^{\gamma_2} C(p, \| g \|_{\dot{B}^{s_p}_{p,\infty}}).
\]

Furthermore,

\[
\| \bar{g}^N \|_{\dot{B}^{s_p}_{p,\infty}} \leq C(p, \| g \|_{\dot{B}^{s_p}_{p,\infty}}),
\]

\[
\| \tilde{g}^N \|_{\dot{B}^{s_p}_{p,\infty}} \leq C(p, \| g \|_{\dot{B}^{s_p}_{p,\infty}}).
\]
This refines previous decompositions for homogeneous Besov spaces, proven by the author in [6]. This is the main new observation of this paper. Unlike the decompositions in [6], it is not clear if related decompositions are obtainable by the known real interpolation theory of homogeneous Besov spaces.

Once Proposition 1.5 is established, one can argue in a similar manner to the case of global weak $L^{3,\infty}$ solutions to obtain improved decay properties of $u(x,t) := v(x,t) - V(x,t)$ near the initial time, which we will state as a Lemma. Related properties have also been exploited by the author in establishing weak strong uniqueness results for the Navier-Stokes equation, see [6].

Lemma 1.6. Let $u$, $v$ and $u_0$ be as in Definition 1.1. Let $\gamma_1$, $\gamma_2$, $\delta_2$ and $p_2$ be as in Proposition 1.5. Then there exists a $\beta(\gamma_1, \gamma_2, \delta_2) > 0$ such that the following estimate is valid for any $0 < t < T$:

$$
\|u(\cdot, t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx dt' \leq C(T, \|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}, \delta_2)t^\beta. \quad (1.31)
$$

Once this Lemma is established, Theorem 1.3 and Corollary 1.4 follow by similar arguments presented for global $L^{3,\infty}$ solutions in [3]-[4].

We will also prove some conditional uniqueness and regularity statements for global weak $\dot{B}^{-\frac{1}{4}}_{4,\infty}$-solutions.

Proposition 1.7. Let $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$ be divergence free in the sense of tempered distributions. There exists an $\varepsilon > 0$ such that if

$$
\sup_{0 < t < T} t^\frac{1}{2} \|S(t)u_0\|_{L^4} < \varepsilon \quad (1.32)
$$

then all global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions, with initial data $u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}$, coincide on $Q_T$.

Unfortunately, the assumptions of Proposition 1.7 do not hold for arbitrary divergence free initial data in $\dot{B}^{-\frac{1}{4}}_{4,\infty}$. In particular, for large minus one solenoidal initial data. For this case, the conjectures in [17] and [18] suggest that there may be non uniqueness for global $\dot{B}^{-\frac{1}{4}}_{4,\infty}$ solutions, even for a short time.
This contrasts drastically with the global $L_3$ solutions to the Navier-Stokes equations, which are constructed in [31] for any $L_3(\mathbb{R}^3)$ solenoidal initial data. In [31], short time uniqueness and regularity results are shown to hold for arbitrary $L_3(\mathbb{R}^3)$ solenoidal initial data. A key feature in showing this is that for any $L_3(\mathbb{R}^3)$ solenoidal initial data, there exists a local in time mild solution to the Navier-Stokes equations. This was proven in [19], which crucially makes use of the fact that

$$\lim_{T \to 0} \sup_{0 < t < T} t^{\frac{1}{3}} \| S(t) u_0 \|_{L_3(\mathbb{R}^3)} = 0,$$

for any $u_0 \in L_3(\mathbb{R}^3)$. Such a property follows from the density of test functions in $L_3(\mathbb{R}^3)$.

Although local in time mild solutions to the Navier-Stokes equations have been constructed for wide classes of initial data (see for example, [9], [13], [16], [22], [23], [26], and [35]) it is not known if they can be constructed for arbitrary $\dot{B}^{-\frac{7}{4}}_{4,\infty}$ solenoidal initial data. The main obstacle is that, unlike $L_3(\mathbb{R}^3)$, test functions are not dense in $\dot{B}^{-\frac{7}{4}}_{4,\infty}$.

1.1 Solutions to the Navier-Stokes Equations with Initial Values in $\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and Applications to Regularity Criteria

For solenoidal initial data in $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ with $4 < p < \infty$, we do not know if there is a solution to the Navier-Stokes equations that satisfies 1) Global existence-2) Weak* Stability. Proposition 1.5, together with ideas from [8] and a Lemma from [1], is sufficient to give a notion of solution of the Navier-Stokes equations with solenoidal $\dot{B}^{s_p}_{p,\infty}$ initial data. To the best of our knowledge, there is no previous notion of solution to the Navier-Stokes equations with arbitrary solenoidal $\dot{B}^{s_p}_{p,\infty}$ initial data. Note that under certain smallness conditions on the $\dot{B}^{s_p}_{p,\infty}$ data, solutions were constructed by means of the Banach contraction principle, in [26] and [9]. We also mention an application of this solution in providing a new regularity criteria for weak Leray-Hopf solutions of the Navier-Stokes equations.

Let $3 < p < \infty$ and suppose $u_0 \in \dot{B}^{s_p}_{p,\infty}$ is a divergence free tempered distribution. Let $u_0 = \tilde{u}_0^N + \bar{u}_0^N$ denote the splitting from Proposition 1.5. In particular, $p < p_2 < \infty$, $0 < \delta_2 < -s_{p_2}$, $\gamma_1 := \gamma_1(p) > 0$ and $\gamma_2 := \gamma_2(p) > 0$
are such that for any $N > 0$ there exists weakly divergence free functions $\tilde{u}^N_0 \in \dot{B}^{s_p}_{p,2}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and $\tilde{u}^N_0 \in L_2(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ with

$$ u_0 = \tilde{u}^N_0 + \tilde{u}^N_0, \quad (1.34) $$

$$ \| \tilde{u}^N_0 \|_{\dot{B}^{s_p}_{p,2} + \frac{\delta_s}{2}} \leq N^{\gamma_1} C(p, \| u_0 \|_{\dot{B}^{s_p}_{p,\infty}}), \quad (1.35) $$

$$ \| \tilde{u}^N_0 \|_{L_2} \leq N^{-\gamma_2} C(p, \| u_0 \|_{\dot{B}^{s_p}_{p,\infty}}). \quad (1.36) $$

Furthermore,

$$ \| \tilde{u}^N_0 \|_{\dot{B}^{s_p}_{p,\infty}} \leq C(p, \| u_0 \|_{\dot{B}^{s_p}_{p,\infty}}), \quad (1.37) $$

$$ \| \tilde{u}^N_0 \|_{\dot{B}^{s_p}_{p,\infty}} \leq C(p, \| u_0 \|_{\dot{B}^{s_p}_{p,\infty}}). \quad (1.38) $$

**Theorem 1.8.** Let $u_0 \in \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ be divergence free, in the sense of tempered distributions, and let $4 < p < \infty$. For any finite $T > 0$ there exists an $N = N(\| u_0 \|_{\dot{B}^{s_p}_{p,\infty}}, T) > 0$ such that the following conclusions hold. In particular, there exists a solution to Navier-Stokes IVP in $Q_T$ such that

$$ v = W^N + u^N. \quad (1.39) $$

Here, $W^N$ is a mild solution to the Navier-Stokes equations, with initial data $\tilde{u}^N_0 \in \dot{B}^{s_p}_{p,2}(\mathbb{R}^3)$, such that

$$ \sup_{0 < t < T} t^{\frac{dp}{2} - \frac{dp}{2}} \| W^N(\cdot, t) \|_{L_p} \leq N^{\gamma_1} C(p, \| u_0 \|_{\dot{B}^{s_p}_{p,\infty}}). \quad (1.40) $$

Furthermore, $u^N \in L_\infty(0, T; J) \cap L_2(0, T; J^{\frac{1}{2}})$. Additionally, there exists a $q^N \in L_{\frac{3}{2}, \text{loc}}(Q_T)$ such that $u^N$ and $q^N$ satisfy the perturbed Navier-Stokes system in the sense of distributions:

$$ \partial_t u^N + v \cdot \nabla u^N - W^N \cdot \nabla W^N - \Delta u^N = -\nabla q^N, \quad \text{div} u^N = 0 \quad (1.41) $$

in $Q_T$. Furthermore, for any $w \in L_2$:

$$ t \to \int_{\mathbb{R}^3} w(x) \cdot u^N(x, t) dx \quad (1.42) $$

is a continuous function on $[0, T]$ and

$$ \lim_{t \to 0^+} \| u^N(\cdot, t) - \tilde{u}^N_0 \|_{L_2} = 0. \quad (1.43) $$

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Moreover, \( u \) satisfies the energy inequality:

\[
\| u^N(\cdot, t) \|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u^N(x, t')|^2 \, dx \, dt' \leq 2 \int_0^t \int_{\mathbb{R}^3} (W^N \otimes u^N) : \nabla u^N \, dx \, dt' + \| u_0^N \|_{L^2}^2
\]

(1.44)

for all \( t \in [0, T] \).

Finally, \( v \) and \( q \) satisfy the local energy inequality. Namely, for almost every \( t \in [0, T] \) the following inequality holds for all non-negative functions \( \varphi \in C_0^\infty(Q_T) \):

\[
\int_{\mathbb{R}^3} \varphi(x, t)|v(x, t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi|\nabla v|^2 \, dx \, dt' \leq \int_0^t \int_{\mathbb{R}^3} |v|^2(\partial_t \varphi + \Delta \varphi) + v \cdot \nabla \varphi(|v|^2 + 2q) \, dx \, dt'.
\]

(1.45)

**Remark 1.9.** It can be shown that (1.40), (1.8) and the estimates for \( \tilde{u}_0^N \) and \( \tilde{u}_0^N \) from Proposition 1.5 implies the following estimate for \( u^N \) in the above Theorem:

\[
\sup_{0 < t < T} \| u^N(\cdot, t) \|_{L^2}^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla u^N|^2 \, dx \, dt' \leq C(T, p, \delta_2, \| u_0 \|_{B^{s_p}_p, \infty}, N).
\]

(1.46)

Recall also that the Besov mild solution \( W^N \) from the above Theorem satisfies

\[
\sup_{0 < t < T} t^{\frac{\alpha p}{2} - \frac{\delta q}{2}} \| W^N(\cdot, t) \|_{L^p} \leq N^{\gamma_1} C(p, \| u_0 \|_{B^{s_p}_p, \infty}).
\]

(1.47)

### 1.1.1 Applications to Regularity Criteria for Weak Leray-Hopf solutions to the Navier-Stokes Equations

Let us now define the notion of ‘weak Leray-Hopf solutions’ to the Navier-Stokes system.
Definition 1.10. Consider $0 < S \leq \infty$. Let
\begin{equation}
    u_0 \in J(\mathbb{R}^3). \tag{1.48}
\end{equation}
We say that $v$ is a 'weak Leray-Hopf solution' to the Navier-Stokes Cauchy problem in $Q_S := \mathbb{R}^3 \times ]0, S[$ if it satisfies the following properties:
\begin{equation}
    v \in \mathcal{L}(S) := L_\infty(0, S; J(\mathbb{R}^3)) \cap L_2(0, S; J_{1/2}(\mathbb{R}^3)). \tag{1.49}
\end{equation}
Additionally, for any $w \in L_2(\mathbb{R}^3)$:
\begin{equation}
    t \to \int_{\mathbb{R}^3} w(x) \cdot v(x, t) dx \tag{1.50}
\end{equation}
is a continuous function on $[0, S]$ (the semi-open interval should be taken if $S = \infty$). The Navier-Stokes equations are satisfied by $v$ in a weak sense:
\begin{equation}
    \int_0^S \int_{\mathbb{R}^3} (v \cdot \partial_t w + v \otimes v : \nabla w - \nabla v : \nabla w) dxdt = 0 \tag{1.51}
\end{equation}
for any divergent free test function
\begin{equation}
    w \in C_0^\infty(Q_S) := \{ \varphi \in C_0^\infty(Q_S) : \div \varphi = 0 \}.
\end{equation}
The initial condition is satisfied strongly in the $L_2(\mathbb{R}^3)$ sense:
\begin{equation}
    \lim_{t \to 0^+} \| v(\cdot, t) - u_0 \|_{L_2(\mathbb{R}^3)} = 0. \tag{1.52}
\end{equation}
Finally, $v$ satisfies the energy inequality:
\begin{equation}
    \| v(\cdot, t) \|^2_{L_2(\mathbb{R}^3)} + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla v(x, t')|^2 dx dt' \leq \| u_0 \|^2_{L_2(\mathbb{R}^3)} \tag{1.53}
\end{equation}
for all $t \in [0, S]$ (the semi-open interval should be taken if $S = \infty$).

The corresponding global in time existence result, proven in [24], is as follows.
**Theorem 1.11.** Let $u_0 \in J(\mathbb{R}^3)$. Then, there exists at least one weak Leray-Hopf solution on $Q_\infty$.

There are two big open problems concerning weak Leray-Hopf solutions regarding uniqueness and regularity. Many regularity criteria exist for weak Leray-Hopf solutions. It was shown by Leray in [24] that if $v$ is a weak Leray-Hopf solution with sufficiently regular initial data and finite blow up time $T$, then there exists a $C(p) > 0$ such that for all $3 < p \leq \infty$:

$$\|v(\cdot, t)\|_{L^p} \geq \frac{c(p)}{(T-t)^{\frac{1}{2}(1-\frac{2}{p})}}. \quad (1.54)$$

The case $p = 3$ stood long open. It was shown in [12] that if $v$ is a weak Leray-Hopf solution with sufficiently regular initial data and finite blow up time $T$, then necessarily

$$\limsup_{t \uparrow T} \|v(\cdot, t)\|_{L^3(\mathbb{R}^3)}. \quad (1.55)$$

The proof in [12] is by a contradiction argument, involving a rescaling procedure producing a non trivial ancient solution and a Liouville theorem based on backward uniqueness for parabolic equations. For an alternative approach to this type of regularity criteria, we refer to [20], [14] and [15].

The criteria (1.55) was made more precise in [29], where it was shown that if $T$ is a finite blow up time then necessarily

$$\lim_{t \uparrow T} \|v(\cdot, t)\|_{L^3(\mathbb{R}^3)}. \quad (1.56)$$

The proof in [29] uses ideas in [12], as well as the fact that the local energy solutions of [25] on a fixed time interval, with $L^3(\mathbb{R}^3)$ initial data, are continuous with respect to weak convergence in $L^3(\mathbb{R}^3)$ of the initial data.

Unfortunately, it is not known if the notion of local energy solutions in [25] carries over to the half space. Consequently, the proof of [29] doesn’t apply to the case of weak Leray-Hopf solutions on $\mathbb{R}^3_+ \times [0, \infty[$. This was overcome in [5]. In particular it was shown that the $L^{3,q}(\mathbb{R}^3_+)$ of a weak Leray-Hopf solution $v$ on $\mathbb{R}^3_+ \times [0, \infty[ \times [0, \infty[ \$ must tend to infinity with $3 \leq q < \infty$.

A further refinement has been recently obtained in [11], who showed that if $T$ is a finite blow up time and $3 < p < \infty$, then necessarily

$$\lim_{t \uparrow T} \|v(\cdot, t)\|_{B^s_{p,q}(\mathbb{R}^3)}. \quad (1.57)$$
Note that for $3 < p < \infty$

$$L^3(\mathbb{R}^3) \hookrightarrow L^{3,p}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_p}_{p,p}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3).$$

Theorem 1.8 and the above Remark (specifically (1.46)-(1.47)), together with ideas from [1] and [5], allow us to obtain a new of a regularity criteria for weak Leray-Hopf solutions to the Navier-Stokes equations.

**Theorem 1.12.** Let $v$ be a global in time weak Leray-Hopf solution to the Navier-Stokes equations. Assume $0 < T < \infty$ is such that for any $0 < \epsilon < T$

$$v \in L_{\infty}(\epsilon, T - \epsilon, L_{\infty}(\mathbb{R}^3)). \quad (1.57)$$

Additionally assume that there exists an increasing sequence $t_k$ tending to $T$ such that

$$\sup_k \|v(\cdot, t_k)\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)} = M < \infty. \quad (1.58)$$

Furthermore, assume that for any $\varphi \in C^\infty_0(\mathbb{R}^3)$

$$\lim_{\lambda \to 0} \frac{1}{\lambda^2} \int_{\mathbb{R}^3} v(y, T) \cdot \varphi(y/\lambda) dy = 0. \quad (1.59)$$

The assumptions (1.57)-(1.59) then imply that for any $0 < \epsilon < T$

$$v \in L_{\infty}(\epsilon, T - \epsilon, L_{\infty}(\mathbb{R}^3)). \quad (1.60)$$

Once we have Theorem 1.8 and Remark 1.9 (specifically (1.46)-(1.47)) the proof the above refined regularity criteria can be completed by verbatim arguments of [1].

It should be noted that removing the assumption (1.59) in the above Theorem is a challenging open problem. A positive resolution would provide regularity of $v$ at time $T$ if

$$\sup_{x \in \mathbb{R}^3, 0 < t < T} (|x| + (T - t)^{\frac{1}{2}}) |v(x, t)| < \infty.$$  

For axisymmetric solutions of the Navier-Stokes equations, Type I blow up has been ruled out. See [10]-[11], [21] and [30].

---

3Specifically, Theorem 1.1 of [1].

4By regularity of $v$ at time $T$, we mean $v \in L_\infty(T - \delta, T; L_\infty(\mathbb{R}^3))$ for some $0 < \delta < T/2$.
2 Preliminaries

2.1 Relevant Function Spaces

We first introduce the frequency cut off operators of the Littlewood-Paley theory. The definitions we use are contained in [2]. For a tempered distribution $f$, let $\mathcal{F}(f)$ denote its Fourier transform. Let $C$ be the annulus

$$\{ \xi \in \mathbb{R}^3 : 3/4 \leq |\xi| \leq 8/3 \}.$$ 

Let $\chi \in C_0^\infty(B(4/3))$ and $\varphi \in C_0^\infty(C)$ be such that

$$\forall \xi \in \mathbb{R}^3, \ 0 \leq \chi(\xi), \varphi(\xi) \leq 1,$$

$$\forall \xi \in \mathbb{R}^3, \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1$$

and

$$\forall \xi \in \mathbb{R}^3 \setminus \{0\}, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1.$$ 

For $a$ being a tempered distribution, let us define for $j \in \mathbb{Z}$:

$$\dot{\Delta}^j a := \mathcal{F}^{-1}(\varphi(2^{-j} \xi) \mathcal{F}(a)) \quad \text{and} \quad \dot{S}^j a := \mathcal{F}^{-1}(\chi(2^{-j} \xi) \mathcal{F}(a)).$$

Now we are in a position to define the homogeneous Besov spaces on $\mathbb{R}^3$. Let $s \in \mathbb{R}$ and $(p, q) \in [1, \infty] \times [1, \infty]$. Then $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is the subspace of tempered distributions such that

$$\lim_{j \to -\infty} \| \dot{S}^j u \|_{L^\infty(\mathbb{R}^3)} = 0,$$

$$\| u \|_{\dot{B}^s_{p,q}(\mathbb{R}^3)} := \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \dot{\Delta}^j u \|_{L^p(\mathbb{R}^3)}^q \right)^{\frac{1}{q}}.$$ 

Remark 2.1. This definition provides a Banach space if $s < \frac{3}{p}$, see [2].

Remark 2.2. It is known that if $1 \leq q_1 \leq q_2 \leq \infty$, $1 \leq p_1 \leq p_2 \leq \infty$ and $s \in \mathbb{R}$, then

$$\dot{B}^s_{p_1,q_1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s-3(\frac{1}{p_1} - \frac{1}{p_2})}_{p_2,q_2}(\mathbb{R}^3).$$
Remark 2.3. It is known that for \( s = -2s_1 < 0 \) and \( p, q \in [1, \infty] \), the norm can be characterised by the heat flow. Namely there exists a \( C > 1 \) such that for all \( u \in \dot{B}_{p,q}^{-2s_1}(\mathbb{R}^3) \):

\[
C^{-1}\|u\|_{\dot{B}_{p,q}^{-2s_1}(\mathbb{R}^3)} \leq \|t^{s_1}S(t)u\|_{L_p(\mathbb{R}^3)} \leq C\|u\|_{\dot{B}_{p,q}^{-2s_1}(\mathbb{R}^3)}.
\]

Here, \( S(t)u(x) := \Gamma(\cdot, t) * u \)

where \( \Gamma(x, t) \) is the kernel for the heat flow in \( \mathbb{R}^3 \).

We will also need the following Proposition, whose statement and proof can be found in [2] (Proposition 2.22 there) for example. In the Proposition below we use the notation

\[
S_h' := \{ \text{tempered distributions } u \text{ such that } \lim_{j \to -\infty} \|S_j u\|_{L_\infty(\mathbb{R}^3)} = 0 \}. \tag{2.7}
\]

**Proposition 2.4.** A constant \( C \) exists with the following properties. If \( s_1 \) and \( s_2 \) are real numbers such that \( s_1 < s_2 \) and \( \theta \in [0, 1] \), then we have, for any \( p \in [1, \infty] \) and any \( u \in S_h' \),

\[
\|u\|_{\dot{B}^\theta s_1 + (1-\theta)s_2(\mathbb{R}^3)} \leq \frac{C}{s_2 - s_1} \left( \frac{1}{\theta} + \frac{1}{1-\theta} \right) \|u\|^\theta_{\dot{B}^{s_1}_p(\mathbb{R}^3)} \|u\|^{1-\theta}_{\dot{B}^{s_2}_p(\mathbb{R}^3)}. \tag{2.8}
\]

**Proposition 2.5.** Let \( u_0 \in \dot{B}_4^{-\frac{3}{4}}(\mathbb{R}^3) \) be divergence free, in the sense of tempered distributions. Then there exists a weakly divergence free sequence \( u_0^{(k)} \in L_3(\mathbb{R}^3) \) such that

\[ u_0^{(k)} \rightharpoonup u_0 \]

in \( \dot{B}_4^{-\frac{1}{4}} \).

**Proof.** Next, it is well known that for any \( u_0 \in \dot{B}_4^{-\frac{3}{4}}(\mathbb{R}^3) \) we have that \( u_{0, <|k|} := \sum_{j=-k}^k \Delta_j u_0 \) converges to \( u_0 \) in the sense of tempered distributions. Furthermore, we have that \( \Delta_j \Delta_j'u_0 = 0 \) if \( |j - j'| > 1 \). Thus,

\[
\|u_{0, <|k|}\|_{\dot{B}_4^{-\frac{1}{4}}} \leq C\|u_0\|_{\dot{B}_4^{-\frac{3}{4}}} \tag{2.9}
\]

and

\[
\Delta_N u_{0, <|k|} = 0 \text{ if } |N| > k + 1. \tag{2.10}
\]
It then follows from \[2\]\(^5\) that there exists a Schwartz function \(g^{(k)}\), whose Fourier transform is supported away from the origin, such that
\[
\|g^{(k)} - u_{0,<|k|}\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} < \frac{1}{k}.
\] (2.11)

Define \(u^{(k)}_0\) to be the Leray Projector \(P\) applied to \(g^{(k)}\), which is continuous on \(\dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)\). Obviously since \(g^{(k)}\) is Schwartz, we have that \(u^{(k)}_0\) is a weakly divergence free function in \(L^3(\mathbb{R}^3)\). Since \(u_0\) is divergence free in the sense of tempered distributions, we have that
\[
P u_0^{(k)} - u_0, <|k| | = 0.
\] (2.12)

So for any Schwartz function \(\varphi\) we have
\[
| < u^{(k)}_0 - u_0, \varphi > | \leq | < u^{(k)}_0 - u_{0,<|k|}, \varphi > | + | < u_0 - u_{0,<|k|}, \varphi > | \leq \frac{C}{k} \|\varphi\|_{\dot{B}^{\frac{1}{4}}_{3,1}} + | < u_0 - u_{0,<|k|}, \varphi > |.
\]

Thus, \(u^{(k)}_0\) satisfies the desired properties of the Proposition.

\[\square\]

**Proposition 2.6.** Let \(u_0 \in \dot{B}^{-\frac{1}{4}}_{4,\infty}(\mathbb{R}^3)\). Then we have
\[
\sup_{0 < t < \infty} t^{\frac{1}{2}} \|S(t)u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}} \leq C\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}. \tag{2.13}
\]

Moreover for \(4 \leq r < \infty, m, k \in \mathbb{N}\):
\[
\|\partial_t^m \nabla^k S(t)u_0\|_{L^r} \leq \frac{C(m, k, r)\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}}{t^{m + \frac{k}{2} + \frac{1}{2}(1 - \frac{3}{r})}}. \tag{2.14}
\]

\(^5\) Specifically, Remark 2.2 pg 69 of \([2]\).
3 Decomposition of Homogeneous Besov Spaces

Before proving Proposition 3.2, we take note of a useful Lemma presented in [2] (specifically, Lemma 2.23 and Remark 2.24 in [2]).

Lemma 3.1. Let $C'$ be an annulus and let $(u^{(j)})_{j \in \mathbb{Z}}$ be a sequence of functions such that
\[
\text{Supp } \mathcal{F}(u^{(j)}) \subset 2^j C'
\]
and
\[
\left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|u^{(j)}\|_{L^p} \right)^{\frac{1}{r}} < \infty.
\]
Moreover, assume in addition that
\[
s < \frac{3}{p}.
\]

Then the following holds true. The series
\[
\sum_{j \in \mathbb{Z}} u^{(j)}
\]
converges (in the sense of tempered distributions) to some $u \in \dot{B}^{s}_{p,r}(\mathbb{R}^3)$, which satisfies the following estimate:
\[
\|u\|_{\dot{B}^{s}_{p,r}} \leq C_s \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|u^{(j)}\|_{L^p} \right)^{\frac{1}{r}}.
\]

Let us state a useful Lemma, proven in [6], regarding decomposition of homogeneous Besov spaces.

Proposition 3.2. For $i = 1, 2, 3$ let $p_i \in ]1, \infty[\, s_i \in \mathbb{R}$ and $\theta \in ]0, 1[$ be such that $s_1 < s_0 < s_2$ and $p_2 < p_0 < p_1$. In addition, assume the following relations hold:
\[
s_1(1 - \theta) + \theta s_2 = s_0,
\]
\[
\frac{1 - \theta}{p_1} + \frac{\theta}{p_2} = \frac{1}{p_0},
\]
\[
s_i < \frac{3}{p_i}
\]
Suppose that $u_0 \in \dot{B}^{s_0}_{p_0,p_0}(\mathbb{R}^3)$. Then for all $\epsilon > 0$ there exists $u^{1,\epsilon} \in \dot{B}^{s_1}_{p_1,p_1}(\mathbb{R}^3)$, $u^{2,\epsilon} \in \dot{B}^{s_2}_{p_2,p_2}(\mathbb{R}^3)$ such that

$$u = u^{1,\epsilon} + u^{2,\epsilon},$$

$$\|u^{1,\epsilon}\|_{\dot{B}^{s_1}_{p_1,p_1}} \leq \epsilon^{p_1-p_0}\|u_0\|_{\dot{B}^{s_0}_{p_0,p_0}},$$

$$\|u^{2,\epsilon}\|_{\dot{B}^{s_2}_{p_2,p_2}} \leq C(p_0,p_1,p_2,\|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{p_2-p_0}\|u_0\|_{\dot{B}^{s_0}_{p_0,p_0}}. \quad (3.10)$$

In order to prove Proposition 1.5, we must first state and prove two Lemmas. Here is the first.

**Lemma 3.3.** Let $3 < p < \infty$ and suppose $u_0 \in \dot{B}^{s_0}_{p,\infty}(\mathbb{R}^3)$. Then there exists $p < p_0 < \infty$ and $0 < \delta < -s_{p_0}$ such that for any $N > 0$ there exists functions $\bar{u}^{1,N} \in \dot{B}^{s_{p_0}+\delta}_{p_0,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_{p_0}}_{p,\infty}(\mathbb{R}^3)$ and $\bar{u}^{2,N} \in \dot{B}^{s_0}_{2,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_0}_{p,\infty}(\mathbb{R}^3)$ with

$$u_0 = \bar{u}^{1,N} + \bar{u}^{2,N}, \quad (3.11)$$

$$\|\bar{u}^{1,N}\|_{\dot{B}^{s_{p_0}+\delta}_{p_0,\infty}} \leq N^{p_0-p}\|u_0\|_{\dot{B}^{s_{p_0}}_{p_0,\infty}}, \quad (3.12)$$

$$\|\bar{u}^{2,N}\|_{\dot{B}^{s_0}_{2,\infty}} \leq C(p,p_0,\|\mathcal{F}^{-1}\varphi\|_{L_1})N^{2-p}\|u_0\|_{\dot{B}^{s_0}_{p,\infty}}, \quad (3.13)$$

$$\|\bar{u}^{1,N}\|_{\dot{B}^{s_{p_0}}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{\dot{B}^{s_{p_0}}_{p_0,\infty}}, \quad (3.14)$$

and

$$\|\bar{u}^{2,N}\|_{\dot{B}^{s_{p_0}}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{\dot{B}^{s_{p_0}}_{p_0,\infty}}. \quad (3.15)$$

**Proof.** If $p < p_0 < \infty$, there exists a $\theta \in ]0,1[$ such that

$$\frac{1-\theta}{p_0} + \frac{\theta}{2} = \frac{1}{p}. \quad (3.16)$$

If we define

$$\delta := \frac{\theta}{2(1-\theta)} > 0 \quad (3.17)$$

we see that

$$(s_{p_0} + \delta)(1-\theta) = s_p. \quad (3.18)$$

Denote,

$$f^{(j)} := \Delta_j u,$$

$$f^{(j)M} := f^{(j)} \chi_{|f^{(j)}| \leq M}$$
and
\[ f_{+}^{(j)M} := f^{(j)}(1 - \chi_{|f^{(j)}| \leq M}). \]

It is easily verified that the following holds:
\[ \| f_{-}^{(j)M} \|_{L^{p_{0}}} \leq M^{p_{0}-p} \| f^{(j)} \|_{L^{p}}, \]
\[ \| f_{-}^{(j)M} \|_{L^{p}} \leq \| f^{(j)} \|_{L^{p}}, \]
\[ \| f_{+}^{(j)M} \|_{L^{2}}^{2} \leq M^{2-p} \| f^{(j)} \|_{L^{p}}^{p}. \]

and
\[ \| f_{+}^{(j)M} \|_{L^{p}} \leq \| f^{(j)} \|_{L^{p}}. \]

Thus, we may write
\[ 2^{p_{0}(s_{p_{0}}+\delta)j} \| f_{-}^{(j)M} \|_{L^{p_{0}}} \leq M^{p_{0}-p} 2^{(p_{0}(s_{p_{0}}+\delta) - p_{p})j} 2^{ps_{p}j} \| f^{(j)} \|_{L^{p}}. \] (3.19)
\[ \| f_{+}^{(j)M} \|_{L^{2}}^{2} \leq M^{2-p} 2^{-p_{p}j} 2^{ps_{p}j} \| f^{(j)} \|_{L^{p}}^{p}. \] (3.20)

With (3.19) in mind, we define
\[ M(j, N, p, p_{0}, \delta) := N^{(p_{p} - p_{0}(s_{p_{0}}+\delta))j}. \]

For the sake of brevity we will write \( M(j, N) \). Using the relations of the Besov indices given by (3.16)-(3.17), we can infer that
\[ M(j, N)^{2-p} 2^{-ps_{p}j} = N^{p_{2}-p_{0}}. \]

The crucial point being that this is independent of \( j \). Thus, we infer
\[ 2^{p_{0}(s_{p_{0}}+\delta)j} \| f_{-}^{(j)M(j,N)} \|_{L^{p_{0}}} \leq N^{p_{0}-p} 2^{ps_{p}j} \| f^{(j)} \|_{L^{p}}. \] (3.21)
\[ 2^{s_{p}j} \| f_{-}^{(j)M(j,N)} \|_{L^{p}} \leq 2^{s_{p}j} \| f^{(j)} \|_{L^{p}}. \] (3.22)
\[ \| f_{+}^{(j)M(j,N)} \|_{L^{2}}^{2} \leq N^{2-p} 2^{ps_{p}j} \| f^{(j)} \|_{L^{p}}^{p}. \] (3.23)

and
\[ 2^{s_{p}j} \| f_{+}^{(j)M(j,N)} \|_{L^{p}} \leq 2^{s_{p}j} \| f^{(j)} \|_{L^{p}}. \] (3.24)

Next, it is well known that for any \( u \in \dot{B}_{p,\infty}^{s_{p}}(\mathbb{R}^{3}) \) we have that \( \sum_{j=-m}^{m} \Delta_{j}u \) converges to \( u \) in the sense of tempered distributions. Furthermore, we have
that $\hat{\Delta}_j \hat{\Delta}_{j'} u = 0$ if $|j - j'| > 1$. Combining these two facts allows us to observe that

$$\hat{\Delta}_j u = \sum_{|m - j| \leq 1} \hat{\Delta}_m f^{(j)} = \sum_{|m - j| \leq 1} \hat{\Delta}_m f^{(j)}_{-} M(j, N) + \sum_{|m - j| \leq 1} \hat{\Delta}_m f^{(j)}_{+} M(j, N). \quad (3.25)$$

Define

$$u_{j, N}^{1, N} := \sum_{|m - j| \leq 1} \hat{\Delta}_m f^{(j)}_{-} M(j, N), \quad (3.26)$$

$$u_{j, N}^{2, N} := \sum_{|m - j| \leq 1} \hat{\Delta}_m f^{(j)}_{+} M(j, N) \quad (3.27)$$

It is clear, that

$$\text{Supp } F(u_{j, N}^{1, N}), \text{Supp } F(u_{j, N}^{2, N}) \subset 2^j C'.$$ \quad (3.28)

Here, $C'$ is the annulus defined by $C' := \{ \xi \in \mathbb{R}^3 : 3/8 \leq |\xi| \leq 16/3 \}$. Using $(3.21)-(3.24)$ we can obtain the following estimates:

$$2^{p_0(s_0 + \delta)j} \| u_{j, N}^{1, N} \|_{L_{p_0}} \leq \lambda_1(p_0, \| F^{-1} \varphi \|_{L_1}) 2^{p_0(s_0 + \delta)j} \| f^{(j)}_{-} M(j, N) \|_{L_{p_0}} \leq \lambda_1(p_0, \| F^{-1} \varphi \|_{L_1}) N^{p_0 - p_2 s_0 j} \| f^{(j)} \|_{L_p}^p, \quad (3.29)$$

$$2^{s_0 j} \| u_{j, N}^{1, N} \|_{L_p} \leq \lambda_2(\| F^{-1} \varphi \|_{L_1}) 2^{s_0 j} \| f^{(j)}_{-} M(j, N) \|_{L_p} \leq \lambda_2(\| F^{-1} \varphi \|_{L_1}) 2^{s_0 j} \| f^{(j)} \|_{L_p}, \quad (3.30)$$

$$\| u_{j, N}^{2, N} \|_{L_2}^2 \leq \lambda_3(\| F^{-1} \varphi \|_{L_1}) \| f^{(j)}_{+} M(j, N) \|_{L_2} \leq \lambda_3(\| F^{-1} \varphi \|_{L_1}) N^{2 - 2 p_2 s_0 j} \| f^{(j)} \|_{L_p}^p \quad (3.31)$$

and

$$2^{s_0 j} \| u_{j, N}^{2, N} \|_{L_p} \leq \lambda_4(\| F^{-1} \varphi \|_{L_1}) 2^{s_0 j} \| f^{(j)}_{+} M(j, N) \|_{L_p} \leq \lambda_4(\| F^{-1} \varphi \|_{L_1}) 2^{s_0 j} \| f^{(j)} \|_{L_p}. \quad (3.32)$$

It is then the case that $(3.28)-(3)$ allow us to apply the results of Lemma 3.1. This allows us to achieve the desired decomposition with the choice

$$u_{1, N} = \sum_{j \in \mathbb{Z}} u_{j, N}^{1, N},$$

$$u_{2, N} = \sum_{j \in \mathbb{Z}} u_{j, N}^{2, N}.$$
Lemma 3.4. Fix $2 < \alpha < 3$.

- For $2 < \alpha < 3$, take $p$ such that $3 < p < \frac{\alpha}{3-\alpha}$.

For $p$ and $\alpha$ satisfying these conditions, suppose that

$$u_0 \in \dot{B}^{s_{p,\alpha}}_{p,p}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$$

(3.33)

Then the above assumptions imply that there exists $p < p_1 < \infty$ and $0 < \delta_1 < -s_{p_1}$ such that for any $\epsilon > 0$ there exists functions $U^{1,\epsilon} \in \dot{B}^{s_{p,p_1}+\delta_1}_{p_1,p_1}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and $U^{2,\epsilon} \in L^2(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ with

$$u_0 = U^{1,\epsilon} + U^{2,\epsilon},$$

(3.34)

$$\|U^{1,\epsilon}\|_{\dot{B}^{s_{p,p_1}+\delta_1}_{p_1,p_1}} \leq \epsilon^{p_1-p} \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}},$$

(3.35)

$$\|U^{2,\epsilon}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|F^{-1} \varphi\|_{L^1}) \|u_0\|_{\dot{B}^{s_p}_{p,\infty}},$$

(3.36)

$$\|U^{2,\epsilon}\|_{L^2} \leq C(p, p_1, \|F^{-1} \varphi\|_{L^1}) \epsilon^{2-p} \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}},$$

(3.37)

and

$$\|U^{2,\epsilon}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|F^{-1} \varphi\|_{L^1}) \|u_0\|_{\dot{B}^{s_{p,\alpha}}_{p,p}}.$$  

(3.38)

Proof. Under this condition, we can find $p < p_1 < \infty$ such that

$$\theta := \frac{\frac{1}{2} - \frac{1}{p_1}}{\frac{1}{2} - \frac{1}{p}} > \frac{6}{\alpha} - 2.$$  

(3.39)

Clearly, $0 < \theta < 1$ and moreover

$$\frac{1 - \theta}{p_1} + \frac{\theta}{2} = \frac{1}{p}.$$  

(3.40)

Define

$$\delta_1 := \frac{\frac{1}{2} - \frac{\delta}{\alpha}}{1 - \theta}.$$  

(3.41)

From (3.39), we see that $\delta_1 > 0$. One can also see we have the following relation:

$$(1 - \theta)(s_{p_1} + \delta_1) = s_{p,\alpha}.$$  

(3.42)
The above relations allow us to apply Proposition 3.2 to obtain the following decomposition: (we note that $\dot{B}^{0}_{2,2}(\mathbb{R}^3)$ coincides with $L_2(\mathbb{R}^3)$ with equivalent norms)

\[ u_0 = U^{1,\epsilon} + U^{2,\epsilon}, \quad (3.43) \]

\[ \|U^{1,\epsilon}\|_{\dot{B}^{s_{p_1}+\delta_1}_{p_1,1}} \leq \epsilon^{p_1-p}\|u_0\|_{\dot{B}^{s_{p_0},p}_p}, \quad (3.44) \]

\[ \|U^{2,\epsilon}\|_{L_2}^2 \leq C(p, p_1, \|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{2-p}\|u_0\|_{\dot{B}^{s_{p},p}_p}. \quad (3.45) \]

For $j \in \mathbb{Z}$ and $m \in \mathbb{Z}$, it can be seen that

\[ \|\hat{\Delta}_m (\hat{\Delta}_j u_0)\chi_{|\hat{\Delta}_j u_0| \leq N(j,\epsilon)}\|_{L_p} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|\hat{\Delta}_j u_0\|_{L_p}. \quad (3.46) \]

and

\[ \|\hat{\Delta}_m (\hat{\Delta}_j u_0)\chi_{|\hat{\Delta}_j u_0| \geq N(j,\epsilon)}\|_{L_p} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|\hat{\Delta}_j u_0\|_{L_p}. \quad (3.47) \]

From [6], we see that the definitions of $U^{1,\epsilon}$ and $U^{2,\epsilon}$ used in Proposition 3.2 are of the following form:

\[ U^{1,\epsilon} := \sum_j \sum_{|m-j| \leq 1} \hat{\Delta}_m \left( (\hat{\Delta}_j u_0)\chi_{|\hat{\Delta}_j u_0| \leq N(j,\epsilon)} \right) \]

and

\[ U^{2,\epsilon} := \sum_j \sum_{|m-j| \leq 1} \hat{\Delta}_m \left( (\hat{\Delta}_j u_0)\chi_{|\hat{\Delta}_j u_0| \geq N(j,\epsilon)} \right) \]

Using this, along with (3.46)-(3.47), we can infer that

\[ \|U^{1,\epsilon}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{\dot{B}^{s_p}_{p,\infty}} \]

and

\[ \|U^{2,\epsilon}\|_{\dot{B}^{s_p}_{p,\infty}} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{\dot{B}^{s_p}_{p,\infty}}. \]

\[ \square \]

**Proof of Proposition 1.5**

*Proof.* Applying Lemma 3.3 we see that there exists $p < p_0 < \infty$ and $0 < \delta < -s_{p_0}$ such that for any $N > 0$ there exists functions $u^{1,N} \in \dot{B}^{s_0}_{p_0,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ and $u^{2,N} \in \dot{B}^{0}_{2,\infty}(\mathbb{R}^3) \cap \dot{B}^{s_p}_{p,\infty}(\mathbb{R}^3)$ with
In particular, there exists $p < p_1$. Noting (3.53), along with (3.52) and (3.56), we may now apply Lemma 3.4.

and there exists $\beta$ with $3 < p < \infty$, it is clear that there exists an $\alpha := \alpha(p)$ such that $2 < \alpha < 3$ and

$$3 < p < \frac{\alpha}{3 - \alpha}.$$ (3.53)

With this $p$ and $\alpha$, we may apply Proposition 2.4 with $s_1 = -\frac{3}{2} + \frac{3}{p}$, $s_2 = -1 + \frac{3}{p}$ and $\theta = 6\left(\frac{1}{\alpha} - \frac{1}{3}\right)$. In particular this gives for any $f \in S'_h$:

$$\|f\|_{\dot{B}^{s,\alpha}_{p,\infty}} \leq C\|f\|_{L_1}\|f\|_{L_1}^\alpha \|f\|_{L_1}^{-\frac{\alpha}{2}}.$$ (3.54)

From Remark 2.2, we see that $\dot{B}^{s,\alpha}_{p,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s,\alpha}_{p,0}(\mathbb{R}^3)$ and $\dot{B}^0_{2,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s,\alpha}_{p,\infty}(\mathbb{R}^3)$. Thus, we have the inclusion

$$\dot{B}^{s,\alpha}_{p,0}(\mathbb{R}^3) \cap \dot{B}^0_{2,\infty}(\mathbb{R}^3) \subset \dot{B}^{s,\alpha}_{p,0}(\mathbb{R}^3) \cap \dot{B}^{s,\alpha}_{p,\infty}(\mathbb{R}^3).$$ (3.55)

Now (3.54)-(3.55), together with (3.51)-(3.52), imply that $u^2, N \in \dot{B}^{s,\alpha}_{p,0}(\mathbb{R}^3)$ and there exists $\beta(p) > 0$ such that

$$\|u^{2,N}\|_{\dot{B}^{s,\alpha}_{p,0}} \leq N^{-\beta(p)}C(p, p_0, \|F^{-1}\varphi\|_{L_1}, \|g\|_{\dot{B}^{s,\alpha}_{p,\infty}}).$$ (3.56)

Noting (3.53), along with (3.52) and (3.56), we may now apply Lemma 3.4. In particular, there exists $p < p_1 < \infty$ and $0 < \delta_1 < -s_{p_1}$ such that for any $\lambda > 0$ there exists functions $U^{1,\lambda} \in \dot{B}^{s_{p_1} + \delta_1}_{p_1,1}(\mathbb{R}^3) \cap \dot{B}^{s_{p_1}}_{p,\infty}(\mathbb{R}^3)$ and $U^{2,\lambda} \in L_2(\mathbb{R}^3) \cap \dot{B}^{s_{p_1}}_{p,\infty}(\mathbb{R}^3)$ with

$$u^{2,N} = U^{1,\lambda} + U^{2,\lambda},$$ (3.57)

$$\|U^{1,\lambda}\|_{\dot{B}^{s_{p_1} + \delta_1}_{p_1,1}} \leq \lambda^{p_1 - p}\|u^{2,N}\|_{\dot{B}^{s_{p_1}}_{p,\infty}} \leq$$
\[
\begin{align*}
\|U^1,1\|_{\dot{B}^{s_p}} & \leq C (\|F^{-1}\varphi\|_{L^1}) \|u^{2,N}\|_{\dot{B}^{s_p}}^{p,1} \\
& \leq C (\|F^{-1}\varphi\|_{L^1}) |g|_{\dot{B}^{s_p}}^{p,1},
\end{align*}
\]
(3.59)
\[
\begin{align*}
\|U^2,λ\|^2_{L^2} & \leq C(p, p_1, \|F^{-1}\varphi\|_{L^1}) \lambda^2 - p \|u^{2,N}\|_{\dot{B}^{s_p}}^{p,1} \\
& \leq C(p, p_0, p_1, \|F^{-1}\varphi\|_{L^1}, |g|_{\dot{B}^{s_p}}^{p,1})
\end{align*}
\]
(3.60)
and
\[
\begin{align*}
\|U^2,λ\|_{\dot{B}^{s_p}} & \leq C (\|F^{-1}\varphi\|_{L^1}) \|u^{2,N}\|_{\dot{B}^{s_p}}^{p,1} \\
& \leq C (\|F^{-1}\varphi\|_{L^1}) |g|_{\dot{B}^{s_p}}^{p,1}.
\end{align*}
\]
(3.61)
Taking \(λ = N^K\) gives that \(u_0 = u^{1,1}_N + U^1,1 + U^2,λ\) with \(u^{1,1}_N ∈ \dot{B}_{p,∞}^{sp} (\mathbb{R}^3) \cap \dot{B}_{p,∞}^{sp} (\mathbb{R}^3), U^{1,N}_n ∈ \dot{B}_{p_1,p_1}^{sp,δ} (\mathbb{R}^3) \cap \dot{B}_{p_1,p_1}^{sp,δ} (\mathbb{R}^3) \) and \(U^{2,N}_n ∈ L_2 (\mathbb{R}^3) \cap \dot{B}_{p,∞}^{sp} (\mathbb{R}^3)\). Furthermore,
\[
\begin{align*}
\|u^{1,1}_N\|_{\dot{B}_{p_0,∞}^{p_0}}^{p_0} & \leq N^{p_0 - p} |g|_{\dot{B}^{s_p}}^{p_1} \\
\|u^{1,1}_N\|_{\dot{B}_{p_0,∞}^{s_p}} & \leq C (\|F^{-1}\varphi\|_{L^1}) |g|_{\dot{B}^{s_p}}^{p_1} \\
\|U^{1,N}_n\|_{\dot{B}_{p_1,p_1}^{s_p+δ}}^{p_1} & \leq N^{(p_1-p) - p} C(p, p_0, \|F^{-1}\varphi\|_{L^1}, |g|_{\dot{B}^{s_p}}^{p_1}),
\end{align*}
\]
(3.62)
\[
\begin{align*}
\|U^{1,N}_n\|_{\dot{B}_{p_1,p_1}^{s_p}} & \leq C (\|F^{-1}\varphi\|_{L^1}) |g|_{\dot{B}^{s_p}}^{p_1} \\
\|U^{2,N}_n\|_{\dot{B}_{p_1,p_1}^{s_p}}^{p_1} & \leq C (\|F^{-1}\varphi\|_{L^1}) |g|_{\dot{B}^{s_p}}^{p_1}
\end{align*}
\]
(3.63)
and
\[
\begin{align*}
\|U^{2,N}_n\|_{\dot{B}_{p_1,p_1}^{s_p}}^{p_1} & \leq C (\|F^{-1}\varphi\|_{L^1}) |g|_{\dot{B}^{s_p}}^{p_1}.
\end{align*}
\]
(3.64)
Let \(p_2 = 2 \max(p_0, p_1)\) and \(δ_2 = \min(δ_1, δ_2)\). From Remark 2.2 and (3.62)-(3.63), we have that \(u^{1,N}_n ∈ \dot{B}_{p_2,∞}^{sp_2} (\mathbb{R}^3) \cap \dot{B}_{p_2,∞}^{sp_2} (\mathbb{R}^3)\) with estimates
\[
\begin{align*}
\|u^{1,N}_n\|_{\dot{B}_{p_2,∞}^{sp_2+δ}}^{p_2} & \leq C(p_2, N^{p_2 - p_0} |g|_{\dot{B}^{s_p}}^{p_0})
\end{align*}
\]
(3.65)
and
\[
\begin{align*}
\|u^{1,N}_n\|_{\dot{B}_{p_2,∞}^{sp_2}} & \leq C (\|F^{-1}\varphi\|_{L^1, p_2}) |g|_{\dot{B}^{s_p}}^{p_1}.
\end{align*}
\]
(3.66)
we may apply Proposition 2.4 with \( s_1 = s_{p_2}, s_2 = s_{p_2} + \delta \) and \( \theta = 1 - \frac{\delta}{\delta_p} \in ]0, 1[. \)

In particular this gives for any \( f \in S'_h \):

\[
\| f \|_{B^{s_{p_2} + \delta_2}_{p_1, 1}} \leq c(p_2, \delta, \delta_2) \| f \|_{B^{s_{p_2} + \delta_2}_{p_1, \infty}} \| f \|_{\dot{B}^{s_{p_2} + \delta}_{p_1, \infty}}. \quad (3.70)
\]

From Remark 2.2, we see that \( \dot{B}^{s_{p_2} + \delta_2}_{p_1, 1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{p_2} + \delta_2}_{p_2, 2}(\mathbb{R}^3) \). This, and (3.68)-(3.70) imply that

\[
\| u^{1,N} \|_{\dot{B}^{s_{p_2} + \delta_2}_{p_2, p_2}} \leq N \frac{\delta_2(p_0 - p)}{\delta p_0} C(p_2, p_0, \delta, \delta_2, \| \mathcal{F}^{-1} \varphi \|_{L_1}, \| g \|_{\dot{B}^{s_p}_{p, \infty}}). \quad (3.71)
\]

Using identical reasoning, it can also be inferred that

\[
\| U^{1,Nk} \|_{\dot{B}^{s_{p_2} + \delta_2}_{p_2, p_2}} \leq N \frac{\delta_2(p_0 - p)}{\delta p_0} C(p_0, p_0, p_2, p_1, \delta, \delta_1, \| \mathcal{F}^{-1} \varphi \|_{L_1}, \| g \|_{\dot{B}^{s_p}_{p, \infty}}). \quad (3.72)
\]

The choice

\[
\kappa = \frac{1}{p_1 - p} \left( p \beta(p) + \frac{\delta_1 p_1 (p_0 - p)}{\delta p_0} \right)
\]

implies

\[
\| u^{1,N} + U^{1,Nk} \|_{\dot{B}^{s_{p_2} + \delta_2}_{p_2, p_2}} \leq N \frac{\delta_2(p_0 - p)}{\delta p_0} C(p_0, p_0, p_2, p_1, \delta, \delta_1, \delta_2, \| \mathcal{F}^{-1} \varphi \|_{L_1}, \| g \|_{\dot{B}^{s_p}_{p, \infty}}). \quad (3.73)
\]

It is also the case that

\[
\| u^{1,N} + U^{1,Nk} \|_{\dot{B}^{s_p}_{p, \infty}} \leq C(\| \mathcal{F}^{-1} \varphi \|_{L_1}) \| g \|_{\dot{B}^{s_p}_{p, \infty}}. \quad (3.74)
\]

To establish the decomposition of Theorem we define \( \tilde{g}^N \) to be the Leray projector applied to \( u^{1,N} + U^{1,Nk} \) and \( \tilde{g}^N \) to be the Leray projector applied to \( U^{2,Nk} \). Note that the Leray projector is a continuous linear operator on the homogeneous Besov spaces under consideration.

\[\Box\]
4 Weak* Stability of Global Weak $\dot{B}_{4,\infty}^{-\frac{1}{4}}(\mathbb{R}^3)$-Solutions

4.1 Apriori Estimates

Let $L_{s,l}(Q_T)$, $W_{s,l}^{1,0}(Q_T)$, $W_{s,l}^{2,1}(Q_T)$ be anisotropic (or parabolic) Lebesgue and Sobolev spaces with norms

$$
\|u\|_{L_{s,l}(Q_T)} = \left(\int_0^T \|u(\cdot, t)\|_{L_s}^l \, dt\right)^{\frac{1}{l}}, \quad \|u\|_{W^{1,0}_{s,l}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},
$$

$$
\|u\|_{W^{2,1}_{s,l}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)}.
$$

**Lemma 4.1.** Assume that $u \in L_\infty(0, T; J) \cap L_2(0, T; \dot{J}^{\frac{1}{2}})$ and that there exists an $\alpha > 0$ such that

$$
ess \sup_{0 < t < T} \frac{\|u(\cdot, t)\|_{L_2}^2}{t^\alpha} < \infty. \quad (4.1)
$$

Let $u_0 \in \dot{B}_{4,\infty}^{-\frac{1}{4}}$ be divergence free and let $V(x, t) := S(t)u_0$. Then

$$
V \cdot \nabla V \in L_{2,\frac{4}{5}}(Q_T), \quad (4.2)
$$

$$
V \cdot \nabla u + u \cdot \nabla V \in L_{\frac{4}{3},\frac{6}{5}}(Q_T), \quad (4.3)
$$

and

$$
V \otimes u : \nabla u \in L_1(Q_T). \quad (4.4)
$$

**Proof.** By the Hölder inequality and Proposition 2.6

$$
\|V \cdot \nabla V\|_{L_2} \leq \|V\|_{L_4} \|\nabla V\|_{L_4} \leq \frac{c}{t^{\frac{4}{5}}} \|u_0\|_{\dot{B}_{4,\infty}^{-\frac{1}{4}}}^2.
$$

From here, (4.2) is easily established. Again, by the Hölder inequality and Proposition 2.6

$$
\|u \cdot \nabla V\|_{L_{\frac{4}{3}}} \leq \|\nabla V\|_{L_6} \|u\|_{L_2} \leq \frac{c}{t^{\frac{4}{5}}} \|u_0\|_{\dot{B}_{4,\infty}^{-\frac{1}{4}}}.
$$

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From this it is immediate that \( u \cdot \nabla V \in L_{2, \frac{6}{5}}^2(Q_T) \). Using Hölder’s inequality once more, one can verify that

\[
\int_0^T \| V \cdot \nabla u \|_{L_2}^{\frac{6}{5}} \, dt \leq \left( \int \| \nabla u \|_{L_2}^2 \, dt \right)^{\frac{3}{5}} \left( \int \| V \|_{L_6}^3 \, dt \right)^{\frac{2}{5}}.
\]

The desired conclusion is reached by noting that Proposition 2.6 gives:

\[
\| V \|_{L_6}^3 \leq \frac{c}{t^{\frac{1}{4}}} \| u_0 \|_{B^{\frac{1}{4}}_4, \infty}^3.
\]

The last estimate shows why there are difficulties to prove energy estimate for \( u \).

By Proposition 2.6 (4.1) and the Hölder inequality:

\[
\int_0^T \int_{\mathbb{R}^3} |V \otimes u : \nabla u| \, dx \, dt \leq \left( \int \int |\nabla u|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int \int |V \otimes u|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \| u_0 \|_{B^{\frac{1}{4}}_4, \infty} \left( \int \int |\nabla u|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int \int \frac{1}{\tau^{1-\alpha}} \text{ess sup}_{0<s<T} \left( \frac{\| u(\cdot, s) \|_{L_2}^2}{s^\alpha} \right) \, dx \, dt \right)^{\frac{1}{2}}.
\]

The next statement is a direct consequence of Lemma 4.1 and coercive estimates of solutions to the Stokes problem.

**Lemma 4.2.** Let \( v \) be a global weak \( B^{\frac{1}{4}}_4, \infty \)-solution with functions \( u \) and \( q \) as in Definition 1.1. Then

\[
(u, q) = \sum_{i=1}^2 (u^i, p_i)
\]

such that for any finite \( T \):

\[
(u^i, \nabla p_i) \in W_{s_i, l_i}^{2, 1}(Q_T) \times L_{s_i, l_i}(Q_T)
\]

and

\[
(s_1, l_1) = (9/8, 3/2), (s_2, l_2) = (2, 5/4), (s_2, l_2) = (3/2, 6/5).
\]
In addition \((u^i, p_i)\) satisfy the following:

\[
\partial_t u^1 - \Delta u^1 + \nabla p_1 = -u \cdot \nabla u, \tag{4.8}
\]

\[
\partial_t u^2 - \Delta u^2 + \nabla p_2 = -V \cdot \nabla V \tag{4.9}
\]

\[
\partial_t u^3 - \Delta u^3 + \nabla p_3 = -V \cdot \nabla u - u \cdot \nabla V \tag{4.10}
\]

in \(Q_{\infty}\), and

\[
\text{div} \ u^i = 0 \tag{4.11}
\]

in \(Q_{\infty}\) for \(i = 1, 2, 3\),

\[
u^i(\cdot, 0) = 0 \tag{4.12}
\]

for all \(x \in \mathbb{R}^3\) and \(i = 1, 2, 3\).

Before the next Lemma let us introduce some notation. Let \(u, v\) and \(u_0\) be as in Definition 1.1. Let \(u_0^N = \bar{u}_0^N + \tilde{u}_0^N\) denote the splitting from Proposition 1.5. In particular, \(4 < p_2 < \infty\), \(0 < \delta_2 < -s_{p_2}\), \(\gamma_1 > 0\) and \(\gamma_2 > 0\) are such that for any \(N > 0\) there exists weakly divergence free functions \(\bar{u}_0^N \in \dot{B}^{s_{p_2} + \delta_2}_{p_2, p_2}(\mathbb{R}^3) \cap \dot{B}^{-\frac{1}{4}}_{4, \infty}(\mathbb{R}^3)\) and \(\tilde{u}_0^N \in L^2(\mathbb{R}^3) \cap \dot{B}^{-\frac{1}{4}}_{4, \infty}(\mathbb{R}^3)\) with

\[
u_0 = \bar{u}_0^N + \tilde{u}_0^N \tag{4.13}
\]

\[
\|\bar{u}_0^N\|_{\dot{B}^{s_{p_2} + \delta_2}_{p_2, p_2}} \leq N^{\gamma_1} C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4, \infty}}), \tag{4.14}
\]

\[
\|\tilde{u}_0^N\|_{L^2} \leq N^{-\gamma_2} C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4, \infty}}). \tag{4.15}
\]

Furthermore,

\[
\|\bar{u}_0^N\|_{\dot{B}^{-\frac{1}{4}}_{4, \infty}} \leq C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4, \infty}}), \tag{4.16}
\]

\[
\|\tilde{u}_0^N\|_{\dot{B}^{-\frac{1}{4}}_{4, \infty}} \leq C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4, \infty}}). \tag{4.17}
\]

Let us define the following:

\[
\bar{V}^N(\cdot, t) := S(t)\bar{u}_0^N(\cdot, t) \tag{4.18}
\]

\[
\tilde{V}^N(\cdot, t) := S(t)\tilde{u}_0^N(\cdot, t) \tag{4.19}
\]

and

\[
w^N(x, t) := u(x, t) + \bar{V}^N(x, t). \tag{4.20}
\]
Lemma 4.3. In the above notation, we have the following global energy inequality

$$\|w^N(.; t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla w^N(x, t')|^2 dx dt' \leq$$

$$\leq \|\tilde{u}_0^N\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} (\tilde{V}^N \otimes w^N + \tilde{V}^N \otimes \tilde{V}^N) : \nabla w^N dx dt'$$

that is valid for positive $N$ and $t$.

Proof. Let us mention that with Lemma 4.1 in hand, the proof of Lemma 4.3 follows from very similar reasoning as presented in [3]-[4]. We provide all the details here for the convenience of the reader.

The first stage is showing that $w^N$ satisfies the local energy inequality. Let us briefly sketch how this can be done. Let $\varphi \in C_0^\infty(Q_\infty)$ be a positive function. Observe that the assumptions in Definition 1.1 imply that the following function

$$t \to \int_{\Omega} w^N(x, t) \cdot \tilde{V}^N(x, t) \varphi(x, t) dx$$

is continuous for all $t \geq 0$. It is not so difficult to show that this term has the following expression:

$$\int_{\mathbb{R}^3} w^N(x, t) \cdot \tilde{V}^N(x, t) \varphi(x, t) dx = \int_0^t \int_{\mathbb{R}^3} (w^N \cdot \tilde{V}^N)(\Delta \varphi + \partial_t \varphi) dx dt' -$$

$$-2 \int_0^t \int_{\mathbb{R}^3} \nabla w^N : \nabla \tilde{V}^N \varphi dx dt' + \int_0^t \int_{\mathbb{R}^3} \tilde{V}^N \cdot \nabla \varphi q dx dt' +$$

$$+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} (|v|^2 - |w^N|^2)v \cdot \nabla \varphi dx dt' -$$

$$- \int_0^t \int_{\mathbb{R}^3} (\tilde{V}^N \otimes w^N + \tilde{V}^N \otimes \tilde{V}^N) : \nabla w^N \varphi dx dt' -$$

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\begin{equation}
- \int_0^t \int_{\mathbb{R}^3} (\hat{V}^N \otimes \hat{V}^N + \hat{V}^N \otimes w^N) : (w^N \otimes \nabla \varphi) dx dt'. \tag{4.23}
\end{equation}

It is also readily shown that

\begin{equation}
\int_{\mathbb{R}^3} |\hat{V}^N(x,t)|^2 \varphi(x,t) dx = \int_0^t \int_{\mathbb{R}^3} |\hat{V}^N(x,t')|^2 (\Delta \varphi(x,t') + \partial_t \varphi(x,t')) dx dt' - \\
- 2 \int_0^t \int_{\mathbb{R}^3} |\nabla \hat{V}^N|^2 \varphi dx dt'. \tag{4.24}
\end{equation}

Using (1.1), together with (4.1)-(4.1), we obtain that for all \( t \in [0, \infty) \) and for all non negative functions \( \varphi \in C^\infty_0 (Q_{\infty}) \):

\begin{equation}
\int_{\mathbb{R}^3} \varphi(x,t) |w^N(x,t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi |\nabla w^N|^2 dx dt' \leq \\
\leq \int_0^t \int_{\mathbb{R}^3} |w^N|^2 (\partial_t \varphi + \Delta \varphi) + 2 q w^N \cdot \nabla \varphi + |w^N| v \cdot \nabla \varphi dx dt' + \\
+ 2 \int_0^t \int_{\mathbb{R}^3} (\hat{V}^N \otimes \hat{V}^N + \hat{V}^N \otimes w^N) : (\nabla w^N \varphi + w^N \otimes \nabla \varphi) dx dt'. \tag{4.25}
\end{equation}

In the next part of the proof, let \( \varphi(x,t) = \varphi_1(t) \varphi_R(x) \). Here, \( \varphi_1 \in C^\infty_0 (0, \infty) \) and \( \varphi_R \in C^\infty_0 (B(2R)) \) are positive functions. Moreover, \( \varphi_R = 1 \) on \( B(R) \), \( 0 \leq \varphi_R \leq 1 \),

\[ |\nabla \varphi_R| \leq c/R, \quad |\nabla^2 \varphi_R| \leq c/R^2. \]

Since \( \tilde{u}^N_0 \in [C^\infty_0(\mathbb{R}^3)]^{L^2(\mathbb{R}^3)} \), it is obvious that for \( \hat{V}^N(\cdot, t) := S(t) \tilde{u}^N_0(\cdot, t) \) we the energy equality:

\begin{equation}
\| \hat{V}^N(\cdot, t) \|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} |\nabla \hat{V}^N|^2 dx dt' = \| \tilde{u}^N_0 \|^2_{L^2}. \tag{4.26}
\end{equation}

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By semigroup estimates, we have for \(2 \leq p \leq \infty, 4 \leq q \leq \infty\):

\[
\| \tilde{V}^N (\cdot, t) \|_{L^p} \leq \frac{C(p)}{t^{\frac{1}{2} (\frac{1}{2} - \frac{1}{p})}} \| \tilde{u}_0^N \|_{L^2},
\]

\[
\| \tilde{V}^N (\cdot, t) \|_{L^q} \leq \frac{C(q)}{t^{\frac{1}{2} (\frac{1}{4} - \frac{1}{q}) + \frac{1}{8}}} \| \tilde{u}_0^N \|_{B^\frac{1}{4}_4, \infty}.
\]

Hence, we have \(w^N \in C_w([0, T]; J) \cap L_2(0, T; J^\frac{1}{2})\). Here, \(T\) is finite and \(C_w([0, T]; J)\) denotes continuity with respect to the weak topology. Using Hölder’s inequality and Sobolev embeddings, this implies that\(w^N \in L^p(Q_T)\) for \(2 \leq p \leq 10/3\).

Using these facts, it is obvious that the following limits hold:

\[
\lim_{R \to \infty} \int_{\mathbb{R}^3} \varphi_R(x) \varphi_1(t) |w^N(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi_R \varphi_1 |\nabla w^N|^2 dx dt' =
\]

\[
= \int_{\mathbb{R}^3} \varphi_1(t) |w^N(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi_1 |\nabla w^N|^2 dx dt',
\]

\[
\lim_{R \to \infty} \int_0^t \int_{\mathbb{R}^3} (|w^N|^2 \partial_t \varphi_1 \varphi_R + 2(\tilde{V}^N \otimes w^N + \tilde{V}^N \otimes \tilde{V}^N) : \nabla w^N \varphi_1 \varphi_R) dx dt' =
\]

\[
= \int_0^t \int_{\mathbb{R}^3} (|w^N|^2 \partial_t \varphi_1 + 2(\tilde{V}^N \otimes w^N + \tilde{V}^N \otimes \tilde{V}^N) : \nabla w^N \varphi_1) dx dt',
\]

\[
\lim_{R \to \infty} \int_0^t \int_{\mathbb{R}^3} (|w^N|^2 \varphi_1 \Delta \varphi_R + \varphi_1 |w^N|^2 v \cdot \nabla \varphi_R +
\]

\[
+ 2 \varphi_1 (\tilde{V}^N \otimes w^N + \tilde{V}^N \otimes \tilde{V}^N) : (w^N \otimes \nabla \varphi_R)) dx dt' = 0.
\]

Let us focus on the term containing the pressure, namely

\[
\int_0^t \int_{\mathbb{R}^3} q w^N \cdot \nabla \varphi_R \varphi_1 dx dt'.
\]
It is known that the pressure $q$ can be represented as the composition of Riesz transforms $\mathcal{R}$. In particular,

$$q = q_1 + q_2,$$  \hspace{1cm} (4.30)  

$$q_1 = \mathcal{R}_i \mathcal{R}_j (u_i u_j)$$  \hspace{1cm} (4.31)  

and

$$q_2 = \mathcal{R}_i \mathcal{R}_j (u_i V_j + V_i u_j).$$  \hspace{1cm} (4.32)  

Since $u \in C_w([0, T]; J) \cap L^2(0, T; J^\frac{1}{2})$, it follows from the Hölder inequality and Sobolev embeddings that $u \in L^p(Q_T)$ for $2 \leq p \leq 10/3$. Using this, Proposition 2.6 and continuity of the Riesz transforms on Lebesgue spaces, we infer that

$$q_1 \in L^{\frac{2}{3}}(Q_T)$$  \hspace{1cm} and  \hspace{1cm} $$q_2 \in L^2(\mathbb{R}^3 \times [\epsilon, T])$$  \hspace{1cm} for any $0 < \epsilon < T$ and $T > 0.$  \hspace{1cm} (4.33)  

From this and (4.29), we infer that

$$\lim_{R \to \infty} \int_0^t \int_{T(R)} q w^N \cdot \nabla \varphi_R \varphi_1 \ dx \ dt' = 0.$$  

Thus, putting everything together, we get for arbitrary positive function $\phi_1 \in C_0^\infty(0, \infty)$:

$$\int_{\mathbb{R}^3} \varphi_1(t) |w^N(x, t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi_1(t') |\nabla w^N|^2 \, dx \, dt' \leq$$

$$\leq \int_0^t \int_{\mathbb{R}^3} |w^N|^2 \partial_t \varphi_1 + 2 (\mathcal{V}^N \otimes w^N + \mathcal{V}^N \otimes \overline{\mathcal{V}}^N) : \nabla w^N \varphi_1 \, dx \, dt'$$  \hspace{1cm} (4.34)  

From Remark 1.2 we see that

$$\lim_{t \to 0} \|w^N(\cdot, t) - \tilde{u}_0^N(\cdot)\|_{L^2} = 0.$$  \hspace{1cm} (4.35)  

For $\overline{\mathcal{V}}^N$, we have

$$\|\overline{\mathcal{V}}^N(\cdot, t)\|_{L^4} \leq \frac{1}{t^\frac{1}{5}} \|\tilde{u}_0^N\|_{B_{4, \infty}^{\frac{1}{2}}} \leq \frac{1}{t^\frac{1}{5}} C(\|u_0\|_{B_{4, \infty}^{\frac{1}{2}}}).$$  \hspace{1cm} (4.36)  

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Using that $\dot{B}_{p_2}^{\alpha_2+\delta_2}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+\delta_2}_{\infty,\infty}(\mathbb{R}^3)$, together with the heat flow characterisation of homogeneous Besov spaces with negative upper index, we infer that

$$\|\nabla^N(\cdot, t)\|_{L^\infty} \leq \frac{1}{t^{\frac{1}{2} - \frac{\delta_2}{2}}} \|\tilde{u}_0^N\|_{\dot{B}^{\alpha_2+\delta_2}_{p_2}(\mathbb{R}^3)} \leq \frac{N^{\gamma_1}}{t^{\frac{1}{2} - \frac{\delta_2}{2}}} C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}) \quad (4.37)$$

Thus, we have the following estimates:

$$\int_0^t \int_{\mathbb{R}^3} |\nabla^N \otimes w^N : \nabla w^N| \, dx \, dt' \leq \int_0^t \int_{\mathbb{R}^3} \left| \nabla w^N \right|^2 \, dx \, dt' \frac{N^{\gamma_1}}{t^{\frac{1}{2} - \frac{\delta_2}{2}}} C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}) \left( \frac{t^{\frac{1}{2}}}{\tau^{1-\delta_2}} \right) \left( \int_0^t \int_{\mathbb{R}^3} \left| \nabla w^N \right|^2 \, dx \, dt' \right)^{\frac{1}{2}}, \quad (4.38)$$

$$\int_0^t \int_{\mathbb{R}^3} |\nabla^N \otimes \nabla^N : \nabla w^N| \, dx \, dt' \leq C t^{\frac{1}{2}} C(\|u_0\|_{\dot{B}^{-\frac{1}{4}}_{4,\infty}}) \left( \frac{t^{\frac{1}{2}}}{\tau^{1-\delta_2}} \right) \left( \int_0^t \int_{\mathbb{R}^3} \left| \nabla w^N \right|^2 \, dx \, dt' \right)^{\frac{1}{2}}. \quad (4.39)$$

Let

$$\varphi_\varepsilon(s) := \begin{cases} 0 & \text{if } 0 \leq s \leq \varepsilon/2, \\ 2(s - (\varepsilon/2))/\varepsilon & \text{if } \varepsilon/2 \leq s \leq \varepsilon, \\ 1 & \text{if } \varepsilon \leq s. \end{cases}$$

Using (4.1)-(4.39) and by taking suitable approximations of $\varphi_\varepsilon$, it can be shown that $\varphi_1 = \varphi_\varepsilon$ is admissible in (4.1). From this we obtain that the following inequality is valid for any $\varepsilon > 0$ and $t > 0$:

$$\int_{\mathbb{R}^3} |w^N(x, t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi_\varepsilon(t') |\nabla w^N|^2 \, dx \, dt' \leq$$

$$\leq \int_0^t \int_{\mathbb{R}^3} |w^N|^2 \partial_t \varphi_\varepsilon + 2(\nabla^N \otimes w^N + \nabla^N \otimes \nabla^N : \nabla w^N \varphi_\varepsilon) \, dx \, dt'. \quad (4.40)$$

Using (4.35), we can obtain (4.3) by letting $\varepsilon$ tend to zero in (4.1).
Proof of Lemma 1.6

Proof. First observe that \( u = w^N - \tilde{V}^N \). Thus, using (4.26) we see that

\[
\|u(\cdot, t)\|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt' \leq 2\|\tilde{u}^N_0\|^2_{L^2} + 2\|w^N(\cdot, t)\|^2_{L^2} + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 \, dx \, dt'.
\]

By (4.15):

\[
\|\tilde{u}^N_0\|^2_{L^2} \leq N^{-2\gamma_2} C(\|u_0\|_{B_{4,\infty}^{-\frac{1}{4}}}).
\] (4.41)

From now on, denote

\[ y_N(t) := \|w^N(\cdot, t)\|^2_{L^2}. \]

Using (4.3), estimates (4.1)-(4.39), (4.41) and the Young’s inequality obtain that

\[
y_N(t) + \int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 \, dx \, dt' \leq N^{2\gamma_1} C(\|u_0\|_{B_{4,\infty}^{-\frac{1}{4}}}) \left( \int_0^t \frac{y_N(\tau)}{\tau^{1-\delta_2}} \, d\tau \right) + (N^{-2\gamma_2} + t^{\frac{3}{4}}) C(\|u_0\|_{B_{4,\infty}^{-\frac{1}{4}}}).
\] (4.42)

By Gronwall’s Lemma we obtain

\[
y_N(t) \leq (N^{-2\gamma_2} + t^{\frac{3}{4}}) C(\|u_0\|_{B_{4,\infty}^{-\frac{1}{4}}}) \times \exp \left( N^{2\gamma_1} C(\|u_0\|_{B_{4,\infty}^{-\frac{1}{4}}}, \delta_2) t^{\delta_2} \right).
\] (4.43)

Hence,

\[
\|u(\cdot, t)\|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt' \leq (N^{-2\gamma_2} + t^{\frac{3}{4}}) C(\|u_0\|_{B_{4,\infty}^{-\frac{1}{4}}}, \delta_2) \times \exp \left( N^{2\gamma_1} C(\|u_0\|_{B_{4,\infty}^{-\frac{1}{4}}}, \delta_2) t^{\delta_2} \right) + 1.
\] (4.44)

The conclusion is then easily reached by taking \( N = t^{-\kappa} \) with \( 0 < \kappa < \delta_2/2\gamma_1 \).
4.2 Proof of Weak* Stability and Existence of Global Weak $\dot{B}_{4,\infty}^{-\frac{1}{4}}(\mathbb{R}^3)$-Solutions

Once Lemma 1.6 and Lemma 4.1 are established, the proof of Theorem 1.3 is along similar lines to arguments in [3]-[4]. We present the full details for completeness.

**Proof of Theorem 1.3**

*Proof.* We have

$$u^{(k)}_0 \rightharpoonup u_0$$

in $\dot{B}_{4,\infty}^{-\frac{1}{4}}$ and may assume that

$$M := \sup_k \|u^{(k)}_0\|_{\dot{B}_{4,\infty}^{-\frac{1}{4}}} < \infty.$$  

Firstly, define

$$V^{(k)}(\cdot, t) := S(t)u^{(k)}_0(\cdot, t), \quad V(\cdot, t) := S(t)u_0(\cdot, t).$$

We see that $V^{(k)}$ converges to $V$ on $Q_\infty$ in the sense of distributions. By Proposition 2.6, we see that

$$\|V^{(k)}(\cdot, t)\|_{L^4} \leq \frac{CM}{t^{\frac{1}{8}}}, \quad (4.45)$$

$$\|\partial^{m}_t \nabla^l V^{(k)}(\cdot, t)\|_{L^r} \leq \frac{CM}{t^{m+\frac{l}{2}+\frac{1}{r}(1-\frac{1}{r})}}. \quad (4.46)$$

Here $r \in [4, \infty]$. For $T < \infty$ and $l \in ]1, \infty[$, we have the compact embedding

$$W^{2,1}_l(B(n) \times ]0, T[) \hookrightarrow C([0, T]; L_l(B(n))).$$

From this and (4.46) one immediately infers that for every $n \in \mathbb{N}$ and $l \in ]1, \infty[$:

$$\partial^{m}_t \nabla^l V^{(k)} \to \partial^{m}_t \nabla^l V \text{ in } C([1/n, n]; L_l(B(n))). \quad (4.47)$$

From Lemma 1.6 we have that for any $0 < T < \infty$:

$$\sup_{0 < t < T} \|u^{(k)}(\cdot, t)\|^2_{L^2} + \int_0^T \int_{\mathbb{R}^3} |\nabla u^{(k)}|^2 dx dt' \leq f_0(M, T, \beta, \delta_2). \quad (4.48)$$
By Hölder’s inequality and the Sobolev inequality, this implies that
\[ \| u^{(k)} \|_{L^p(Q_T)} \leq f_1(M, T, \beta, \delta_2) \quad \text{for any} \quad 2 \leq p \leq 10/3. \quad (4.49) \]
By means of a Cantor diagonalisation argument, we can find a subsequence such that for any finite \( T > 0 \):
\[ u^{(k)} \rightharpoonup u \quad \text{in} \quad L^2(Q_T), \quad (4.50) \]
\[ \nabla u^{(k)} \rightharpoonup \nabla u \quad \text{in} \quad L^2(Q_T). \quad (4.51) \]
Using (4.50), together with (1.31), we also get that for \( 0 < t < 1 \):
\[ \| u \|_{L^2(Q_t)} \leq c(M, \delta_2) t^{\frac{\beta}{2}}. \quad (4.52) \]
From (4.48) it is easily inferred that
\[ \| u^{(k)} \cdot \nabla u^{(k)} \|_{L^{\frac{9}{8}, \frac{3}{2}}(Q_T)} \leq f_2(M, T, \beta, \delta_2). \quad (4.53) \]
By the same reasoning as in Lemma 4.1, we obtain:
\[ \| V^{(k)} \cdot \nabla V^{(k)} \|_{L^{\frac{5}{4}, \frac{5}{4}}(Q_T)} \leq f_3(M, T), \quad (4.54) \]
\[ \| V^{(k)} \cdot \nabla u^{(k)} + u^{(k)} \cdot \nabla V^{(k)} \|_{L^{\frac{3}{4}, \frac{3}{4}}(Q_T)} \leq f_4(M, T, \beta, \delta_2). \quad (4.55) \]
Split \( u^{(k)} = \sum_{i=1}^{3} u^{i(k)} \) according to Definition 1.1, namely (4.5). By coercive estimates for the Stokes system, along with (4.53) obtain:
\[ \| u^{1(k)} \|_{W^{1, 1}_{\frac{9}{8}, \frac{3}{2}}(Q_t)} + \| \nabla P^{1(k)} \|_{L^{\frac{9}{8}, \frac{3}{2}}(Q_T)} \leq C f_2(M, T, \beta, \delta_2), \quad (4.56) \]
\[ \| u^{2(k)} \|_{W^{1, 1}_{\frac{5}{4}, \frac{5}{4}}(Q_T)} + \| \nabla P^{2(k)} \|_{L^{\frac{5}{4}, \frac{5}{4}}(Q_T)} \leq C f_3(M, T), \quad (4.57) \]
\[ \| u^{3(k)} \|_{W^{1, 1}_{\frac{3}{2}, \frac{3}{2}}(Q_T)} + \| \nabla P^{3(k)} \|_{L^{\frac{3}{2}, \frac{3}{2}}(Q_T)} \leq C f_4(M, T, \beta, \delta_2). \quad (4.58) \]
By the previously mentioned embeddings, we infer from (4.56)-(4.58) that for any \( n \in \mathbb{N} \) we have the following convergence for a certain subsequence:
\[ u^{(k)} \rightarrow u \quad \text{in} \quad C([0, n]; L^2_8(B(n))). \quad (4.59) \]
Hence, using (4.49), we infer that for any \( s \in [1, 10/3[ \)
\[ u^{(k)} \rightarrow u \quad \text{in} \quad L^s(B(n) \times [0, n[). \quad (4.60) \]
It is also not so difficult to show that for any $f \in L^2$ and for any $n \in \mathbb{N}$:

\[
\int_{\mathbb{R}^3} u^{(k)}(x, t) \cdot f(x) dx \to \int_{\mathbb{R}^3} u(x, t) \cdot f(x) dx \ \text{in} \ C([0, n]). \tag{4.61}
\]

Using (4.52) with (4.61), we establish that

\[
\lim_{t \to 0} \|u(\cdot, t)\|_{L^2} = 0. \tag{4.62}
\]

Using (4.52), along with the fact that $u \in L_{2,\infty}(Q_T)$ for any $0 < T < \infty$, we see that for any $0 < T < \infty$

\[
\sup_{0 < t < T} \frac{\|u(\cdot, t)\|_{L^2}^2}{t^3} < \infty. \tag{4.63}
\]

All that remains to show is establishing the local energy inequality (1.1) for the limit and establishing the energy inequality (1.1) for $u$. Verifying the local energy inequality is not so difficult and hence omitted. Let us focus on showing (1.1) for $u$. By identical reasoning to Lemma 4.3, we have that for an arbitrary positive function $\phi_1(t) \in C_0^\infty(0, \infty)$:

\[
\int_{\mathbb{R}^3} \phi_1(t)|u(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \phi_1(t)|\nabla u|^2 dx dt' \leq \\
\leq \int_0^t \int_{\mathbb{R}^3} |u|^2 \partial_t \phi_1 + 2(V \otimes u + V \otimes V) : \nabla u \phi_1 dx dt'. \tag{4.64}
\]

From (4.63), Lemma 4.1 and semigroup estimates, we have that

\[
(V \otimes u + V \otimes V) : \nabla u \in L_1(Q_T)
\]

for any positive finite $T$. Using these facts and (4.62), the conclusion is reached in a similar way to the final steps of the proof of Lemma 4.3.$\square$

Let us comment on Corollary 1.4. Recall that by Proposition 2.5, there exists a solenoidal sequence $u_0^{(k)} \in L_3(\mathbb{R}^3)$ such that

\[
u^{(k)}_0 \rightharpoonup u_0
\]
in $\dot{B}_{4,\infty}^{-\frac{1}{4}}$. It was shown in [31] that for any $k$ there exists a global $L_3$-weak solution $v^{(k)}$, which satisfies
\[
\sup_{0 < t < \infty} \frac{\|v^{(k)} - S(t)u_0^{(k)}\|_{L^2_{x,t}}}{t^{\frac{1}{2}}} < \infty.
\]
It is also the case that $v^{(k)}$ is a global $\dot{B}_{4,\infty}^{-\frac{1}{4}}$ weak solution. Now, Corollary 1.4 follows from Theorem 1.3.

5 Uniqueness Results

5.1 Construction of Strong Solutions

Before constructing mild solutions we will briefly explain the relevant kernels and their pointwise estimates.

Let us consider the following Stokes problem:
\[
\partial_t v - \Delta v + \nabla q = -\text{div} F, \quad \text{div} v = 0
\]
in $Q_T$, $v(\cdot, 0) = 0$.

Furthermore, assume that $F_{ij} \in C^\infty_0(Q_T)$. Then a formal solution to the above initial boundary value problem has the form:
\[
v(x, t) = \int_0^t \int_{\mathbb{R}^3} K(x - y, t - s) : F(y, s) dy ds.
\]

The kernel $K$ is derived with the help of the heat kernel $\Gamma$ as follows:
\[
\Delta_y \Phi(y, t) = \Gamma(y, t),
\]
\[
K_{mjs}(y, t) := \delta_{mj} \frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_s}(y, t) - \frac{\partial^3 \Phi}{\partial y_m \partial y_j \partial y_s}(y, t).
\]

Moreover, the following pointwise estimate is known:
\[
|K(x, t)| \leq \frac{C}{(|x|^2 + t)^2}. \tag{5.1}
\]
Define
\[ G(f \otimes g)(x, t) := \int_0^t \int_{\mathbb{R}^3} K(x - y, t - s) : f \otimes g(y, s) dy ds. \] (5.2)

It what follows, we will use the notation \( \pi_{u \otimes u} := R_i R_j(u_i u_j) \), where \( R \) is a Riesz transform and the summation convention is adopted.

**Proposition 5.1.** Suppose that \( u_0 \in \dot{B}^{-\frac{1}{4}}_{4, \infty} \) is weakly divergence free. There exists a universal constant \( \varepsilon_3 \) such that if
\[ \text{ess sup}_{0 < t < T} t^{\frac{1}{5}} \| S(t) u_0 \|_{L^4} < \varepsilon_3, \] (5.3)
then there exists a weak \( \dot{B}^{-\frac{1}{4}}_{4, \infty} \) solution \( \tilde{v} := V + \tilde{u} \) to the Navier-Stokes IVP on \( Q_T \) that satisfies the following properties. The first property is that \( \tilde{v} \) satisfies the estimates
\[ \sup_{0 < t < T} t^{\frac{1}{5}} \| \tilde{v} \|_{L^4} < 2 \sup_{0 < t < T} t^{\frac{1}{5}} \| V \|_{L^4} := 2 M^{(0)} \] (5.4)
The other property is that for \( \lambda \in ]0, T[ \) and \( k = 0, 1 \ldots, l = 0, 1 \ldots \):
\[ \sup_{(x, t) \in Q_{\lambda, T}} | \partial_t \nabla^k \tilde{v} | + | \partial_t \nabla^k \pi_{\tilde{v} \otimes \tilde{v}} | \leq c(\lambda, \| u_0 \|_{\dot{B}^{-\frac{1}{4}}_{4, \infty}, k, l}). \] (5.5)

**Proof.** Let us introduce the space
\[ X_4(T) := \{ f \in \mathcal{S}'(\mathbb{R}^3 \times ]0, T[) : \text{ess sup}_{0 < t < T} t^{\frac{1}{5}} \| f(\cdot, t) \|_{L^4(\mathbb{R}^3)} < \infty \}. \]
\[ \| f \|_{X_4(T)} := \text{ess sup}_{0 < t < T} t^{\frac{1}{5}} \| f(\cdot, t) \|_{L^4(\mathbb{R}^3)}. \] (5.6)

We use the method of successive iterations. Let us define the following, for \( k = 1, 2, \ldots, \)
\[ v^{(1)} = V, \quad v^{(k+1)} = V + u^{(k+1)}, \]
where \( u^{(k+1)} := G(u^{(k)} \otimes v^{(k)}) \) solves the following problem
\[ \partial_t u^{(k+1)} - \Delta u^{(k+1)} + \nabla q^{(k+1)} = -\text{div} u^{(k)} \otimes v^{(k)}, \quad \text{div} u^{k+1} = 0 \]
in \( Q_T, \)
\[ u^{(k+1)}(\cdot, 0) = 0 \]
in $\mathbb{R}^3$. Using (5.1), it is easy to check that for solutions to the above linear problem the following estimate is true

$$\|u^{(k+1)}\|_{X_4(T)} \leq c\|v^{(k)}\|_{X_4(T)}^2,$$

and thus we have

$$\|v^{(k+1)}\|_{X_4(T)} \leq \|V\|_{X_4(T)} + c\|v^{(k)}\|_{X_4(T)}^2$$

for all $k = 1, 2, \ldots$. Using arguments in [19], one can show that if

$$\|V\|_{X_4(T)} < \varepsilon < \frac{1}{4c},$$

then we have

$$\|v^{(k)}\|_{X_4(T)} < 2\|V\|_{X_4(T)} \quad (5.7)$$

for all $k = 1, 2, \ldots$.

Furthermore, Kato’s arguments also give that there is a $\tilde{v} = V + \tilde{u}$ such that

$$\|v^{(k)} - \tilde{v}\|_{X_4(T)}, \|u^{(k)} - \tilde{u}\|_{X_4(T)} \to 0, \quad (5.8)$$

We also can exploit our equation, together with the pressure equation, to derive the following estimate for the energy and pressure:

$$\|u^{(k)} - u^{(m)}\|_{2, \infty, Q_T}^2 + \|\nabla u^{(k)} - u^{(m)}\|_{2, Q_T}^2 + \|\pi_{v^{(k)}}\otimes v^{(k)} - \pi_{v^{(m)}}\otimes v^{(m)}\|_{2, Q_T}^2 \leq$$

$$\leq c \int_0^T \int_{\mathbb{R}^3} |v^{(k)} \otimes v^{(k)} - v^{(m)} \otimes v^{(m)}|^2 dx \, dt. \quad (5.9)$$

Using (5.8), we immediately see the following

$$u^{(k)} \to \tilde{u} \text{ in } W^{1,0}_2(Q_T) \cap C([0, T]; L_2(\mathbb{R}^3)), \quad (5.10)$$

$$\pi_{v^{(k)}} \otimes v^{(k)} \to \pi_{v} \otimes \tilde{v} \text{ in } L_2(Q_T), \quad (5.11)$$

$$\partial_t \tilde{u} - \Delta \tilde{u} + \nabla \pi_{v} \otimes \tilde{v} = -\text{div} \tilde{v} \otimes \tilde{v}, \text{ div } \tilde{u} = 0 \quad (5.12)$$

in $Q_T$ and

$$\tilde{u}(\cdot, 0) = 0. \quad (5.13)$$
Furthermore, it is not difficult to show that for $0 < t < T$

$$
\|\tilde{u}(\cdot, t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \tilde{u}(x, t')|^2 \, dx \, dt' \leq \int_0^t \int_{\mathbb{R}^3} |\tilde{v} \otimes \tilde{v}|^2 \, dx \, dt' \leq 4t^2 \|\nabla \tilde{u}\|_{X_4(T)}^4.
$$

(5.14)

Thus

$$
\sup_{0 < t < T} \frac{\|\tilde{u}(\cdot, t)\|_{L_2}^2}{t^2} < \infty.
$$

(5.15)

Clearly, the pair $\tilde{v}$ and $\pi_{\tilde{v} \otimes \tilde{v}}$ satisfies the Navier-Stokes equations, in a distributional sense. Showing that $(\tilde{v}, \pi_{\tilde{v} \otimes \tilde{v}})$ satisfies the local energy inequality and (5.5) follows from arguments in [6].

Specifically Theorem 3.1 of [6].

Before proceeding further, we state a known Lemma found in [27] and [32], for example.

**Lemma 5.2.** Let $p \in [3, \infty]$ and

$$
\frac{3}{p} + \frac{2}{r} = 1.
$$

(5.17)

Suppose that $w \in L_{p,r}(Q_T)$, $v \in L_{2,\infty}(Q_T)$ and $\nabla v \in L_2(Q_T)$. Then for $t \in [0, T]$:

$$
\int_0^t \int_{\mathbb{R}^3} |\nabla v||v||w| \, dx \, dt' \leq C \int_0^t \int_{\mathbb{R}^3} \|w\|_{L_p}^r \|v\|_{L_2}^2 \, dt' + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 \, dx \, dt'.
$$

(5.18)
5.2 Proof of Proposition 1.7

Proof. The proof of Proposition 1.7 is based on ideas developed by the author in [6].

Suppose,

\[ \sup_{0 < t < T} t^{\frac{1}{4}} \| S(t) u_0 \|_{L^4} := \| V \|_{X_T} < \varepsilon. \]

Furthermore, suppose \( \varepsilon < \varepsilon_3 \), where \( \varepsilon_3 \) is as in the statement of Proposition 5.1. Then by Proposition 5.1 there exists a weak \( \dot{B}^{-\frac{1}{4}} \) solution \( (\tilde{v} := \tilde{u} + V) \) on \( Q_T \) that satisfies

\[ \| \tilde{v} \|_{X_T} < 2 \| V \|_{X_T}. \]

(5.19)

Here we recall that

\[ \| f \|_{X_T} := \text{ess sup}_{0 < t < T} t^{\frac{1}{4}} \| f(\cdot, t) \|_{L^4}, \]

(5.20)

Let us now consider any global \( \dot{B}^{-\frac{1}{4}} \) solution \( u := v + V \), defined on \( Q_\infty \), with the same initial data \( u_0 \in \dot{B}^{-\frac{1}{4}}(\mathbb{R}^3) \). We define

\[ w = v - \tilde{v} = u - \tilde{u} \in W^{1,0}_2(Q_T) \cap C_w([0, T]; J(\mathbb{R}^3)). \]

(5.21)

Moreover, \( w \) satisfies the following equations

\[ \partial_t w + w \cdot \nabla w + \tilde{v} \cdot \nabla w + w \cdot \nabla \tilde{v} - \Delta w = -\nabla q, \quad \text{div} \ w = 0 \]

in \( Q_T \), with the initial condition satisfied in the strong \( L^2 \) sense:

\[ \lim_{t \to 0^+} \| w(\cdot, 0) \|_{L^2} = 0 \]

(5.23)

in \( \mathbb{R}^3 \).

Using minor adjustments to arguments used in Proposition 14.3 of [25], one can deduce that for \( t \in [\delta, T] \):

\[
\int_{\mathbb{R}^3} u(y, t) \cdot \tilde{u}(y, t) dy = \int_{\mathbb{R}^3} u(y, \delta) \cdot \tilde{u}(y, \delta) dy - 2 \int_{\delta}^{t} \int_{\mathbb{R}^3} \nabla u : \nabla \tilde{u} dy d\tau + \\
+ \int_{\delta}^{t} \int_{\mathbb{R}^3} V \otimes V : \nabla (u + \tilde{u}) dy d\tau - \int_{\delta}^{t} \int_{\mathbb{R}^3} \tilde{v} \otimes w : \nabla w dy d\tau + \\
+ \int_{\mathbb{R}^3} V \otimes V : \nabla (u + \tilde{u}) dy \\
- \int_{\mathbb{R}^3} \tilde{v} \otimes w : \nabla w dy,
\]

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Using Lemma 5.2 and (5.19), we see that

\[
+ \int_\delta^t \int_{\mathbb{R}^3} V \otimes \tilde{u} : \nabla \tilde{u} + V \otimes u : \nabla u \, dyd\tau
\]

(5.24)

Similarly, \( u \) and \( \tilde{u} \), implies that there exists

\[ \beta(\gamma_1, \gamma_2, \delta_2) > 0 \]

\[ \beta(\gamma_1, \gamma_2, \delta_2) > 0 \]

a such that for \( 0 < t < T \):

\[
\|u(\cdot, t)\|_{L^2}^2 \leq t^\beta c(T, \|u_0\|_{\tilde{B}^{\frac{4}{3}, \infty}_{4, \infty}}, \delta_2),
\]

(5.28)
\[ \|\tilde{u}(\cdot, t)\|_{L^2}^2 \leq t^3 c(T, \|u_0\|_{B^\frac{3}{4}_{4,\infty}}, \delta_2) \] (5.29)

and

\[ \|w(\cdot, t)\|_{L^2}^2 \leq t^3 c(T, \|u_0\|_{B^\frac{3}{4}_{4,\infty}}, \delta_2). \] (5.30)

Hence,

\[ \sup_{0 < t < T} \frac{\|w(\cdot, t)\|_{L^2}^2}{t^3} < \infty. \] (5.31)

Then (5.28)-(5.30) allows us to take \( \delta \to 0 \) in (5.2) infer that for any \( t \in [0, T] \) we have

\[
\int_{\mathbb{R}^3} u(y, t) \cdot \tilde{u}(y, t) dy = -2 \int_{0}^{t} \int_{\mathbb{R}^3} \nabla u : \nabla \tilde{u} dyd\tau + \\
+ \int_{0}^{t} \int_{\mathbb{R}^3} V \otimes V : \nabla (u + \tilde{u}) dyd\tau - \int_{0}^{t} \int_{\mathbb{R}^3} \tilde{v} \otimes w : \nabla w dyd\tau + \\
+ \int_{0}^{t} \int_{\mathbb{R}^3} V \otimes \tilde{u} : \nabla \tilde{u} + V \otimes u : \nabla u dyd\tau \] (5.32)

Recall that \( u \) and \( \tilde{u} \) satisfy the following energy inequalities for \( 0 < t < T \):

\[ \|u(\cdot, t)\|_{L^2}^2 \leq t^3 c(T, \|u_0\|_{B^\frac{3}{4}_{4,\infty}}, \delta_2) \] (5.33)

\[ \|\tilde{u}(\cdot, t)\|_{L^2}^2 \leq t^3 c(T, \|u_0\|_{B^\frac{3}{4}_{4,\infty}}, \delta_2). \] (5.34)

From this and (5.2), we deduce that for any \( t \in [0, T] \)

\[ \|w(\cdot, t)\|_{L^2}^2 \leq 2 \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla w|^2 dyd\tau \leq 2 \int_{0}^{t} \int_{\mathbb{R}^3} \tilde{v} \otimes w : \nabla w dyd\tau. \] (5.35)
Using (5.2) and (5.31) we see that for \( t \in [0, T] \):

\[
\|w(\cdot, t)\|_{L^2}^2 \leq C\|V\|_{X_4(T)}^8 \int_0^t \frac{\|w(\cdot, \tau)\|_{L^2}^2}{\tau} d\tau \leq \frac{C}{\beta} t^\beta \|V\|_{X_4(T)}^8 \sup_{0 < \tau < T} \left( \frac{\|w(\cdot, \tau)\|_{L^2}^2}{\tau^\beta} \right). \tag{5.36}
\]

Thus,

\[
\sup_{0 < \tau < T} \left( \frac{\|w(\cdot, \tau)\|_{L^2}^2}{\tau^\beta} \right) \leq \frac{C'}{\beta} \|V\|_{X_4(T)}^8 \sup_{0 < \tau < T} \left( \frac{\|w(\cdot, \tau)\|_{L^2}^2}{\tau^\beta} \right). \tag{5.37}
\]

If we assume further that that \( \varepsilon \leq \min \left( \left( \frac{\beta}{2C'} \right)^{\frac{1}{8}}, \varepsilon_3 \right) \),

then

\[
\|V\|_{X_4(T)} \leq \left( \frac{\beta}{2C'} \right)^{\frac{1}{8}}.
\]

It then immediately follows that \( w = 0 \) on \( Q_T \).

\[\square\]

### 6 Existence of solutions with \( \dot{B}^{s_p}_{p, \infty}(\mathbb{R}^3) \) initial values

In what follows we will need the following two Propositions. The proof of Proposition 6.1 can be found in [6] and the proof of Proposition 6.2 can be found in [1], for example.

**Proposition 6.1.** Consider \( p_2 \) and \( \delta_2 \) such that \( 4 < p_2 < \infty \), \( \delta_2 > 0 \) and \( s_{p_2} + \delta_2 < 0 \). Suppose that \( u_0 \in \dot{B}^{s_{p_2} + \delta_2}_{p_2, p_2}(\mathbb{R}^3) \) is a divergence free tempered distribution. There exists a constant \( c = c(p_2) \) such that if \( 0 < T < \infty \) is such that

\[
4cT^{\frac{\delta_2}{p_2}} \|u_0\|_{\dot{B}^{s_{p_2} + \delta_2}_{p_2, p_2}} < 1, \tag{6.1}
\]
then there exists a \( w \), which solves the Navier-Stokes equations (1.1)-(1.3) on \( Q_T \) in the sense of distributions and satisfies the following properties. The first property is that \( w \) satisfies the estimate
\[
\sup_{0 < t < T} t^{s_p - \frac{1}{2}} \| w(\cdot, t) \|_{L^p_{\infty}} < 2 \sup_{0 < t < T} t^{s_p - \frac{1}{2}} \| S(t) u_0 \|_{L^p_{2}} := 2 M(0) \quad (6.2)
\]
The other property is that for \( \lambda \in ]0, T[ \) and \( k=0,1,\ldots, l=0,1,\ldots \):
\[
\sup_{(x,t) \in Q_{\lambda,T}} |\partial_t^{l} \nabla^k w| + |\partial_t^{l} \nabla^k \pi w \otimes w| \leq c(p_2, \delta_2, \lambda, \| u_0 \|_{B^{s_p + \delta_2}_{2,2}}, l). \quad (6.3)
\]

**Proposition 6.2.** Let \( u_0 = U_0 + V_0 \), with \( U_0 \in B^{s_p + \delta_2}_{2,2}(\mathbb{R}^3) \) (\( 0 < \delta_2 < -s_p_2 \)) and \( V_0 \in L^2(\mathbb{R}^3) \) being divergence free tempered distributions. Moreover, assume \( 0 < T < \infty \) is such that
\[
4c T^{\frac{1}{2}} \| U_0 \|_{B^{s_p + \delta_2}_{2,2}} < 1. \quad (6.4)
\]

Then there exists a solution to Navier-Stokes IVP in \( Q_T \) with the following properties. In particular,
\[
v = W + u. \quad (6.5)
\]
Here, \( W \) is a mild solution to the Navier-Stokes equations from Proposition 6.1, with initial data \( U_0 \in B^{s_p + \delta_2}_{2,2}(\mathbb{R}^3) \), such that
\[
\sup_{0 < t < T} t^{s_p - \frac{1}{2}} \| W(\cdot, t) \|_{L^p_{2}} \leq 2 \sup_{0 < t < T} t^{s_p - \frac{1}{2}} \| S(t) u_0 \|_{L^p_{2}}. \quad (6.6)
\]
Furthermore, \( u \in L^\infty(0, T; J) \cap L^2(0, T; J^{\frac{1}{2}}) \). Additionally, there exists a \( q \in L^{\frac{3}{2}}_{1,0}(Q_T) \) such that \( u \) and \( q \) satisfy the perturbed Navier-Stokes system in the sense of distributions:
\[
\partial_t u + v \cdot \nabla v - W \cdot \nabla W - \Delta u = -\nabla q, \quad \text{div} u = 0 \quad (6.7)
\]
in \( Q_T \). Furthermore, for any \( w \in L^2 \):
\[
t \rightarrow \int_{\mathbb{R}^3} w(x) \cdot u(x, t) dx \quad (6.8)
\]
is a continuous function on \([0, T]\) and
\[
\lim_{t \rightarrow 0^+} \| u(\cdot, t) - V_0 \|_{L^2} = 0. \quad (6.9)
\]
Moreover, $u$ satisfies the energy inequality:

$$
\|u(\cdot,t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,t')|^2 dx dt' \leq 
$$

$$
\leq 2 \int_0^t \int_{\mathbb{R}^3} (W \otimes u) : \nabla u dx dt' + \|V_0\|_{L^2}^2
$$

(6.10)

for all $t \in [0,T]$.

Finally, $v$ and $q$ satisfy the local energy inequality. Namely, for almost every $t \in [0,T]$ the following inequality holds for all non negative functions $\varphi \in C_0^\infty(Q_T)$:

$$
\int_{\mathbb{R}^3} \varphi(x,t) |v(x,t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi |\nabla v|^2 dx dt' \leq 
$$

$$
\leq \int_0^t \int_{\mathbb{R}^3} |v|^2 (\partial_t \varphi + \Delta \varphi) + v \cdot \nabla \varphi (|v|^2 + 2q) dx dt'.
$$

(6.11)

## Proof of Theorem 1.8

**Proof.** Let $u_0 \in \dot{B}_p^{s_0,p}$ be divergence free. Let $u_0 = \bar{u}_0^N + \tilde{u}_0^N$ denote the splitting from Proposition 1.5. In particular, $p < p_2 < \infty$, $0 < \delta_2 < -s_{p_2}$, $\gamma_1 := \gamma_1(p) > 0$ and $\gamma_2 := \gamma_2(p) > 0$ are such that for any $N > 0$ there exists weakly divergence free functions $\bar{u}_0^N \in \dot{B}_2^{s_{p_2}+\delta_2}(\mathbb{R}^3) \cap \dot{B}_p^s(\mathbb{R}^3)$ and $\tilde{u}_0^N \in L_2(\mathbb{R}^3) \cap \dot{B}_p^s(\mathbb{R}^3)$ with

$$
u = \bar{u}_0^N + \tilde{u}_0^N,
$$

(6.12)

$$
\|\bar{u}_0^N\|_{\dot{B}_2^{s_{p_2}+\delta_2}} \leq N^{\gamma_1} C(p, \|u_0\|_{\dot{B}_p^{s_0,p}}),
$$

(6.13)

$$
\|\tilde{u}_0^N\|_{L_2} \leq N^{-\gamma_2} C(p, \|u_0\|_{\dot{B}_p^{s_0,p}}).
$$

(6.14)

Furthermore,

$$
\|\bar{u}_0^N\|_{\dot{B}_p^{s_0,p}} \leq C(p, \|u_0\|_{\dot{B}_p^{s_0,p}}),
$$

(6.15)

$$
\|\tilde{u}_0^N\|_{\dot{B}_p^{s_0,p}} \leq C(p, \|u_0\|_{\dot{B}_p^{s_0,p}}).
$$

(6.16)
From (6.13) we see that for every $0 < T < \infty$ we can choose $N(T, p, \delta_2, p_2, \gamma_1, \|u_0\|_{\dot{B}^{\sigma_2}_{p,\infty}}) > 0$ such that
\[
4c(p_2)T^{\frac{\delta_2}{2}}\|u_0^N\|_{\dot{B}^{\sigma_2}_{p,\infty} + \delta_2} < 1.
\]

The proof is now completed by means of Proposition 6.2. \hfill \Box

References

[1] Albritton, D. Blow-up criteria for the Navier-Stokes equations in non-endpoint critical Besov spaces. December 2016, arXiv:1612.04439.

[2] Bahouri, H.; Chemin, J.-Y.; Danchin, R. Fourier analysis and non-linear partial differential equations. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. Springer, Heidelberg, 2011. xvi+523 pp. ISBN: 978-3-642-16829-1

[3] Barker, T.; Seregin, G. On global solutions to the Navier-Stokes system with large $L^{3,\infty}$ initial data. arXiv:1603.03211, March 2016.

[4] Barker, T.; Seregin, G.; Šverák, V. On stability of weak Navier-Stokes solutions with large $L^{3,\infty}$ initial data. Submitted

[5] Barker, T.; Seregin, G. A necessary condition of potential blowup for the Navier-Stokes system in half-space. Math. Ann. (2016). doi:10.1007/s00208-016-1488-9

[6] Barker, T. Uniqueness Results for Weak Leray-Hopf Solutions of the Navier-Stokes System with Initial Values in Critical Spaces. J. Math. Fluid Mech. (2017). doi:10.1007/s00021-017-0315-8.

[7] Bradshaw, Z.; Tsai, T.-P. Discretely self-similar solutions to the Navier-Stokes equations with Besov space data. March 2017, arXiv:1703.03480

[8] Calderón, C. P. Existence of weak solutions for the Navier-Stokes equations with initial data in $L^p$. Trans. Amer. Math. Soc. 318 (1990), no. 1, 179-200.

[9] Cannone, M. A generalization of a theorem by Kato on Navier-Stokes equations. Rev. Mat. Iberoamericana 13 (1997), no. 3, 515-541.
[10] Chen, C.-C.; Strain, R. M.; Yau, H.-T.; Tsai, T.-P. Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations. Int. Math. Res. Not. IMRN 2008, no. 9, Art. ID rnm016, 31 pp.

[11] Chen, C.-C.; Strain, R. M.; Tsai, T.-P.; Yau, H.-T. Lower bounds on the blow-up rate of the axisymmetric Navier-Stokes equations. II. Comm. Partial Differential Equations 34 (2009), no. 1-3, 203-232.

[12] Escauriaza, L.; Seregin, G.; Šverák, V. $L_3,\infty$-solutions of Navier-Stokes equations and backward uniqueness. (Russian) Uspekhi Mat. Nauk 58 (2003), no. 2(350), 3–44; translation in Russian Math. Surveys 58 (2003), no. 2, 211-250.

[13] Fujita, H.; Kato, T. On the Navier-Stokes initial value problem. I. Arch. Rational Mech. Anal. 16 1964 269-315.

[14] Gallagher, I.; Koch, G.; Planchon, F. A profile decomposition approach to the $L_\infty^t L_3^x$ Navier-Stokes regularity criterion. Math. Ann. 355 (2013), no. 4, 1527-1559.

[15] Gallagher, I.; Koch, G.; Planchon, F. Blow-up of critical Besov norms at a potential Navier-Stokes singularity. Comm. Math. Phys. 343 (2016), no. 1, 39-82.

[16] Giga, Y.; Miyakawa, T. Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces. Comm. Partial Differential Equations 14 (1989), no. 5, 577-618.

[17] Jia, H.; Šverák, V. Local-in-space estimates near initial time for weak solutions of Navier-Stokes equations and forward self-similar solutions, Invent. Math. 196 (2014), no.1, 233–265.

[18] Jia, H.; Šverák V. Are the incompressible 3d Navier-Stokes equations locally ill-posed in the natural energy space? J. Funct. Anal. 268 (2015), no. 12, 3734-3766.

[19] Kato, T. Strong $L^p$-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. Math. Z. 187 (1984), no. 4, 471-480.

[20] Kenig, C. E.; Koch, G. An alternative approach to regularity for the Navier-Stokes equations in critical spaces. Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011), no. 2, 159-187.
[21] Koch, G.; Nadirashvili, N.; Seregin, G.; Šverák, V. Liouville theorems for the Navier-Stokes equations and applications. Acta Math. 203 (2009), no. 1, 83-105.

[22] Koch, H.; Tataru, D. Well-posedness for the Navier-Stokes equations. Adv. Math. 157 (2001), no. 1, 22-35.

[23] Kozono, H.; Yamazaki, M. Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. Comm. Partial Differential Equations 19 (1994), no. 5-6, 959-1014.

[24] J. Leray, *Sur le mouvement d’un liquide visqueux emplissant l’espace*, Acta Math. 63 (1934), 193-248.

[25] Lemarié-Rieusset, P.G. Recent developments in the Navier-Stokes problem. Chapman&Hall/CRC Research Notes in Mathematics, 431. Chapman & Hall/CRC, Boca Raton, FL, 2002. xiv+395 pp.

[26] Planchon, F. Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$. Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), no. 3, 319-336.

[27] Prodi, G. Un teorema di unicità per le equazioni di Navier-Stokes. (Italian) Ann. Mat. Pura Appl. (4) 48 1959 173-182.

[28] Seregin, G. A. Necessary conditions of potential blow up for Navier-Stokes equations. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 385 (2010), Kraevye Zadachi Matematichesko Fiziki i Smezhnye Voprosy Teorii Funktsii 41, 187–199, 236; translation in J. Math. Sci. (N.Y.) 178 (2011), no. 3, 345-352

[29] Seregin, G. A., A certain necessary condition of potential blow up for Navier-Stokes equations. Comm. Math. Phys. 312 (2012), no. 3, 833-845.

[30] Seregin, G.; Šverák, V. On type I singularities of the local axi-symmetric solutions of the Navier-Stokes equations. Comm. Partial Differential Equations 34 (2009), no. 1-3, 171-201.

[31] Seregin, G.; Šverák, V. On global weak solutions to the Cauchy problem for the Navier-Stokes equations with large $L^3$-initial data. Nonlinear Analysis (2016), [http://dx.doi.org/10.1016/j.na.2016.01.018](http://dx.doi.org/10.1016/j.na.2016.01.018)
[32] Serrin, J. The initial value problem for the Navier-Stokes equations. 1963 Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962) pp. 69-98 Univ. of Wisconsin Press, Madison, Wis.

[33] Solonnikov, V. A., *Estimates of solutions to the non-stationary Navier-Stokes system*, Zapiski Nauchn. Seminar. LOMI 28(1973), 153–231.

[34] Solonnikov, V. A. Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator. (Russian) Uspekhi Mat. Nauk 58 (2003), no. 2(350), 123–156; translation in Russian Math. Surveys 58 (2003), no. 2, 331-365

[35] Taylor, M. E. Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations. Comm. Partial Differential Equations 17 (1992), no. 9-10, 1407-1456.