Spin networks are at the core of quantum gravity [1]. We have neither the space nor the competence to give an exhaustive list of the physical and philosophical interpretations of this notion (for these, see for example [2] and [3]). New (and old) approaches towards creating a finite quantum theory of general relativity would use combinatorial expressions in Feynman integrals, spin networks, spin foams and others combinatorial objects. The intention is to get out the standard “continuous” geometry. Our aim is to plug the mathematical community at large into these procedures as participants. For this, because of the different cultural backrounds, we would like to change tack: to relate discrete (combinatorial) objects to the standard “continuous” geometry. From the mathematical point of view, relations of this type give rise to identifications between the geometry of varieties and combinatorial objects, as exemplified by the relation between Lie algebras and root systems, or toric varieties and polytopes.

The general mathematical mechanism of such “interpretations” could be called the analytic theory of non-Abelian theta functions, since they run completely parallel to the original classical theory of theta functions. Recall that this classical theory has three parts:

1. One constructs a family of holomorphic functions \( \{ \theta^{\Omega}_{\alpha_k} \} \) on a fixed space (to be concrete, on \((\mathbb{C}^*)^g\)). Each function depends on a symmetric \( g \times g \) complex matrix \( \Omega \) with positive imaginary part \( \text{Im} \, \Omega > 0 \), and \( \alpha_k \) is some combinatorial data, the so-called “characteristics of level \( k \)”. We will see that this data is equivalent to a choice of a U(1)-spin network of genus \( g \) (see the final Section 4).
Three faces of SU(2)-spin networks

(2) If Ω is the period matrix of a marked Riemann surface Σ of genus g, the functions \{θ^Ω_α\} form a basis of \(H^0(JΣ, Θ^k)\), the space of holomorphic sections of the \(k\)th power of the polarizing line bundle Θ on the Jacobian of Σ. In particular, for \(k = 1\) there is just one section (up to scaling), whose zero set is the theta divisor of Σ; by Riemann’s theorem, this is birational to the symmetric power \(Σ^{[g-1]}\).

(3) The final part describes how these geometric objects behave under deformations of Ω. Here we have transformation rules governing changes of marking and projective flat connections under continuous deformation of parameters. (We would like to emphasize the best reference for these classical projective flat connections \[4\].)

Realizing this program in the case of SU(2) is the subject of current work \[5\] in collaboration with C. Florentino, J. Mourão and J.P. Nunes. The general picture is a mosaic consisting of many stones, or a many piece jigsaw puzzle. This paper describes one stone of the mosaic, taken from the first part of the program; namely, we describe the combinatorial data of every non-Abelian theta function, the analog of a theta characteristic, as an SU(2)-spin network of genus \(g\), and associate a “continuous” geometry with it. For this, we must show all three different mathematical faces of SU(2)-spin networks, although possibly only the third is new.

1 First face

A spin network is a labeled trivalent graph \(Γ\), with edges labeled by finite dimensional irreducible representations of SU(2) and vertices labeled by the associated intertwiners. Traditionally, in this case, the set \(\hat{SU}(2)\) of irreducible representations is the nonnegative half-integers

\[
\hat{SU}(2) = \frac{1}{2} \mathbb{Z}^+.
\]

However, from the combinatorial point of view, we find it convenient to multiply these numbers by 2, and call the result colors. Let \(E(Γ)\) be the set of edges, \(V(Γ)\) the set of vertices and \(F(Γ) = \{v \in e\}\) the set of flags, where a flag is an edge with fixed end. Clearly \(F(Γ) \subset E(Γ) \times V(Γ)\), and two projections

\[
e: F(Γ) \to E(Γ) \quad \text{and} \quad v: F(Γ) \to V(Γ)
\]
are ramified covers of degree 2 and 3 having the same ramification locus, consisting of pairs \( v \in e \) where the edge \( e \) is a loop. (Recall that a trivalent graph can contain loops, that is, an edge whose two ends coincide.) If \( L(\Gamma) \) is the set of loops in \( \Gamma \), and \(| \cdot |\) the number of elements of a finite set then

\[
2 \cdot |E(\Gamma)| - |L(\Gamma)| = |F(\Gamma)| = 3 \cdot |V(\Gamma)| - |L(\Gamma)|. \tag{1.3}
\]

Hence

\[
|V(\Gamma)| = 2g - 2; \quad |E(\Gamma)| = 3g - 3, \tag{1.4}
\]

where \( g > 1 \) is a certain integer called the genus of \( \Gamma \).

Thus a spin network defines a map

\[
j : E(\Gamma) \to \widehat{SU(2)}. \tag{1.5}\]

Recall that for a triple of representations \( j_1, j_2, j_3 \), an intertwiner is a trivial component of the tensor product \( j_1 \otimes j_2 \otimes j_3 \). Such a component exists iff the Clebsch–Gordan conditions

\[
j_1 + j_2 + j_3 \in \mathbb{Z} \quad \text{and} \quad |j_1 - j_2| \leq j_3 \leq j_1 + j_2 \tag{1.6}\]

hold for every ordering of edges around a vertex.

A function \( j \) (1.5) defines a spin network \( \Gamma_j \) iff these conditions hold for every triple \( j_{v,1}, j_{v,2}, j_{v,3} \) of representations around every vertex \( v \in V(\Gamma) \) and for every ordering of triple of edges. In this case any intertwiner

\[
i_v \in j_{v,1} \otimes j_{v,2} \otimes j_{v,3} \tag{1.7}\]

is defined uniquely. Thus we can omit any labeling of the vertices and denote a SU(2))-spin network by the symbol \( \Gamma_j \).

A spin network is of level \( k \) if \( j_i \leq k \) for every edge \( e_i \). There is a finite number \( N^k(\Gamma) \) of spin networks of level \( k \) with graph \( \Gamma \), and a finite number \( N^k_g \) of all spin networks of level \( k \) and genus \( g \).

Penrose prescribed a number \( V(\Gamma_j) \) for every spin network \( \Gamma_j \), its value. We omit the precise definition here for reasons of space: the “right” definition involves starting with loop representations, as in the beautiful paper [6].

A one-trivalent graph \( \Gamma \) (or a history) is a trivalent graph possibly having open ends, that is, half-edges with only one vertex and a boundary end-point. There is a simple operation, so-called doubling, that takes a one-trivalent
graph $\Gamma$ to a closed trivalent graph. It consists simply of gluing two mirror copies of $\Gamma$ along edges with one-vertices:

$$\Gamma \# \overline{\Gamma}.$$  \hfill (1.8)

For example, the simplest graph of genus 2 without loops in the shape of $\Theta$ is the double of the tristar, the one-trivalent graph having only one vertex.

This allows us to prescribe the value $V(\Gamma_j)$ of any colored trivalent graph $\Gamma_j$, even if the function $j$ doesn’t satisfy the Clebsch–Gordan conditions: removing all vertices not satisfying (1.6), we get a disjoint union of one-trivalent connected components, that we can double to get a disjoint union of spin networks. The sum of their values defines the value of any colored trivalent graph $\Gamma_j$.

The rest of the section comments briefly on the notion of spin network for mathematicians.

### 1.1 Why such an object?

A spin network realizes a simple model of quantum geometry that is at the same time discrete and purely combinatorial, and does not refer to any background notion of space, time or geometry. The system consists of a number of “units”, each having a total angular momentum (a representation $j \in \hat{SU}(2)$ if the system has symmetry group $SU(2)$). They interact in ways that conserve the symmetry (or the total angular momentum). Thus any interval $e$ (edge) is a propagator of this unit; an event is an end (vertex) of $e$ at which it meets two other edges whose labels satisfy (1.6). An intertwiner is necessarily nontrivial. A spin network is obtained by continuing in this way until we get a closed graph.

Hence such a model is described by an arbitrary trivalent graph with edges labeled by integers (colors = twice the total angular momentum). The vertices describe the interactions. An orientation is just an orientation of the procedure from past to future. A change of orientation is just a change of a “time direction”. A network with open ends (that is one-trivalent graph) is a history. The connected sum with its mirror image, that is, its double is a spin network (for details and other physical ideas behind this notion see [7]).
1.2 Why trivalent?

Multivalent graphs can be reduced to the trivalent case as follows. Take each $n$-valent vertex and replace it by a $n$-leaved tree with trivalent vertices. This tree has $n - 3$ new “internal edges”. A basis of intertwining operators for the original vertex is then given by all labelings of these internal edges by spins satisfying the Clebsch–Gordan condition. There are many different trees with $n$ leaves, and thus many such bases. To change from one basis to another requires repeated use of some standard matrices from recoupling theory based on the Racah sum rules and the Biedenharn–Elliott identity in terms of the 6j symbols (see [8] and [9]).

There are a number of applications of spin networks to “continuous” theories.

1.3 Lattice gauge theory

Spin networks are a generalization of knots and links if we consider them as graphs embedded in space. They can be used in place of the regular cubic lattice in lattice models of gauge theories. Moreover, the central kinematic concept in quantum gravity is that the space of diffeomorphism-invariant states is spanned by a basis in one-to-one correspondence with (orbits of) embeddings of spin networks (see [7]).

1.4 More algebra

The physical origin of spin networks dictates the labeling by representations of Lie groups. But in this definition, the main property that we need to switch on is the following:

any product of two labels can be decomposed into a sum of labels.

More precisely, the product of two labels defines a finite set of labels, and the Clebsch–Gordan condition is just a choice of an element of this set. This algebraic structure is usually called a category with tensor product.

There are algebras whose representation theory has this property; these more general objects are Hopf algebras. The algebraic structure of their representation theory can be described in terms of monoidal categories. There is a still more general class of spin network associated with these objects. It is also traditional to use representations of quantum groups (deformations
of Lie algebras) as labels. This is technically quite reasonable: they satisfy a modified set of recoupling identities – quantum $6j$ symbols that depend on the parameter $q$ that can be specified. New invariants of 3-manifolds can be constructed using these \cite{10}. However the labeling by representations of quantum groups differs from ordinary spin networks in several ways. They do not correspond to groups, and thus do not correspond to gauge invariants of classical connections. (However the limit $q \to 1$ corresponds to the classical limit $k \to \infty$ of Chern–Simons theory, sending the Kauffman bracket $K^k(\Gamma)$ to the Penrose value $V(\Gamma)$. This reflects a deep mathematical relationship between the representation theory of quantum groups at roots of unity and the representation theory of the corresponding loop group of level $k$.)

Other geometric objects that could be used as labels are exceptional bundles \cite{11}. The many mathematical possibilities for labeling spin networks stimulates approaches to realize the main expectation of experts in physics:

there is a quantum theory “X”, \textit{defined purely algebraically}, not involving any background geometry, whose classical limit is 3 + 1 general relativity coupled to certain matter fields.

The theory “X” realizes directly the holographic conjecture and the Bekenstein bound. We recall that 'tHooft and Susskind’s holographic conjecture states that such a theory is defined in terms of state spaces and observables on surfaces (see for example \cite{12}). It is mathematically quite reasonable to develop a correspondence sending spin networks to the geometry (topology) of surfaces; but first we describe the second (well known) face of spin networks.

\section{Second face}

\subsection{Harmonic analysis}

The main motivation behind spin networks is to quantize general relativity. For this, we need a Hilbert space of states with a collection of operators as observables. It is reasonable to expect that the states are functions on the configuration space of a system (or more generally, sections of a bundle). Following this thread, we start by sending our spin networks to functions on some space corresponding to a spin network, and prove later (in Section 3) that the space is actually independent of the spin network. As before, we
construct this space using only the first component – the trivalent graph. Here we need to use harmonic analysis on groups.

Consider the product

$$\text{SU}(2)^{E(\Gamma)} = \prod_{e \in E(\Gamma)} \text{SU}(2)_e$$

(2.1)

with \(\text{SU}(2)\) components enumerated by edges of \(\Gamma\), and the product

$$\text{SU}(2)^{V(\Gamma)} = \prod_{v \in V(\Gamma)} \text{SU}(2)_v$$

(2.2)

with components enumerated by the vertices. Let \(dx\) be the Haar measure on \(\text{SU}(2)\) normalized by the condition \(\int_{\text{SU}(2)} dx = 1\) and \(\vec{dx}\) the product measure on \(\text{SU}(2)^{E(\Gamma)}\) normalized by \(\int \vec{dx} = 1\). Then by the Peter–Weyl formula, any function \(f \in L^2(\text{SU}(2)^{E(\Gamma)}, \vec{dx})\) has the decomposition

$$f(x) = \sum_{\vec{\rho} \in \hat{\text{SU}(2)}^{E(\Gamma)}} \text{Tr}[B_{\vec{\rho},f} \vec{\rho}(x)],$$

(2.3)

where \(\hat{\text{SU}(2)}^{E(\Gamma)}\) is the space of irreducible representations of \(\text{SU}(2)^{E(\Gamma)}\), and \(B_{\vec{\rho},f}\) are endomorphisms of the space \(V_{\vec{\rho}}\) of the representation \(\vec{\rho}\), given by

$$B_{\vec{\rho},f} = \frac{1}{\dim V_{\vec{\rho}}} \int_{\text{SU}(2)^{E(\Gamma)}} f(x) \vec{\rho}^{-1}(x) \vec{dx}.$$

(2.4)

Recall that every irreducible representation of \(\text{SU}(2)^{E(\Gamma)}\) is given by tensor product of irreducible representations of \(\text{SU}(2)\):

$$\vec{\rho} = \rho_1 \otimes \cdots \otimes \rho_{3g-3}.$$  

(2.5)

This is of course an analog of the standard Fourier decomposition. Here a representation \(\vec{\rho}\) is a label of a frequency and an endomorphism \(B_{\vec{\rho},f}\) is the Fourier coefficient, that is, a number. The last formula is nothing other than the integral formula for a Fourier coefficient.

Therefore every spin network \(\Gamma_j\) of genus \(g\) defines a representation of \(\text{SU}(2)^{E(\Gamma)}\) by the tensor product of all labels

$$\vec{j} = \bigotimes_{e \in E(\Gamma)} j_e$$

(2.6)
(the label of a frequency) and to get a function on $\text{SU}(2)^{E(\Gamma)}$ we must define an endomorphism $B(\Gamma_j)$ using a labeling of a spin network. However, any endomorphism of the space $V_j$ is a vector in the tensor product

$$
\left( \bigotimes_{e \in E(\Gamma)} j_e \right) \otimes \left( \bigotimes_{e \in E(\Gamma)} j_e \right)^* = \left( \bigotimes_{e \in E(\Gamma)} j_e \right) \otimes \left( \bigotimes_{e \in E(\Gamma)} j_e \right),
$$

(2.7)
since we are dealing with $\text{SU}(2)$-representations. But components of the final product can be labeled by elements of the set $F(\Gamma)$, and by (1.2–3) we can decompose it as

$$
\left( \bigotimes_{e \in E(\Gamma)} j_e \right) \otimes \left( \bigotimes_{e \in E(\Gamma)} j_e \right)^* = \bigotimes_{v \in V(\Gamma)} (j_{v,1} \otimes j_{v,2} \otimes j_{v,3}).
$$

(2.8)

For every triple representations around a vertex $v \in V(\Gamma)$ we have a vector

$$
i_v \in j_{v,1} \otimes j_{v,2} \otimes j_{v,3}
$$

(2.9)
as in (1.7) and their tensor product gives us the vector

$$
B(\Gamma_j) = \bigotimes_{v \in V(\Gamma)} i_v \in \text{End } V_j.
$$

(2.10)

As we saw, this endomorphism is an analog of a number – a Fourier coefficient. But in some sense this number is an integer. Indeed, to construct the endomorphism, we use integer blocks of representations and its matrix must be an integer with respect to the multiplicative components of the representation.

### 2.2 Fourier term with integer coefficient

Hence a spin network defines a Fourier term with integer coefficient. Moreover, we may identify a spin network with this Fourier term with integer coefficient, and vice versa. Of course, every spin network $\Gamma_j$ as a Fourier term with integer coefficient defines a state, that is, a function

$$
f_{\Gamma_j}(x) = \text{Tr}[B(\Gamma)j(x)] \in L^2(\text{SU}(2)^{E(\Gamma)}, d\bar{\xi}).
$$

(2.11)

To switch on the action of $\text{SU}(2)^{V(\Gamma)}$ (2.2) on the space $\text{SU}(2)^{E(\Gamma)}$ (2.1), consider an orientation of $\Gamma$, that is, orientations of the edges such that every
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A vertex has edges both “in” and “out”. Such an orientation always exists. Now for every oriented edge \( e \) with ends \( v_{\text{in}} \) and \( v_{\text{out}} \), we set

\[
g(t_e) = g_{v_{\text{in}}} \circ t_e \circ g_{v_{\text{out}}}^{-1},
\]

(2.12)

where

\[
g = (g_{v_1}, \ldots, g_{v_{2g-2}}) \in \text{SU}(2)^{2g-2},
\]

(2.13)

\[
t = (t_1, \ldots, t_{3g-3}) \in \text{SU}(2)^{3g-3}.
\]

(2.14)

As a standard result of harmonic analysis on groups, we get the following

**Proposition 2.1** If the endomorphism \( B(\Gamma_j) \) (2.10) is intertwining, then the function \( f_{\Gamma_j} \) (2.11) is invariant under the action (2.12):

\[
f_{\Gamma_j} \in L^2(\text{SU}(2)^{E(\Gamma)})^{\text{SU}(2)^V(\Gamma)}.
\]

(2.15)

That is, \( f_{\Gamma_j} \) is a function on the homogeneous space

\[
Q_\Gamma = \text{SU}(2)^{E(\Gamma)}/\text{SU}(2)^{V(\Gamma)}.
\]

(2.16)

Now \( Q_\Gamma \) does not depend on the choice of the orientation and on the labeling \( j \) of the graph, but the function \( f_{\Gamma_j} \) is equivalent to the full spin network: the labeling is recognized by its Fourier decomposition.

Thus the second face of every spin network \( \Gamma_j \) is an integer. How to add two such “numbers”? How to multiply them? These operations are related to the “interaction” of the combinatorial string “events” (see below).

The practical experiences behind such number theoretic intuition of spin networks led to the technique of Feynman diagrams in perturbative quantum field theories and quantum computers.

Many years ago Jacobi, with his theory of theta functions of one variable, and Riemann related number theory (a theory with discrete objects) with complex analysis of one variable (a theory with nondiscrete objects, and admitting many limits). What we want to do in the full program is similar: to send the theory of spin networks as a combinatorial theory to “continuous” theories (the Chern–Simons and WZW theories). The first step in this is the third face of spin networks.
3 Third face

3.1 From spin networks to surfaces

There are two ways of transforming a spin network $\Gamma_j$ embedded in some space $Y$ into a surface. The first is well known from the point of view of deformation quantization (see for example [13]); let us call it the ribbon method. We describe it briefly because [13] does not treat it in our way. Our spin network is a generalization of a knot or link, and it can be framed in the same vein. Any framed trivalent graph can be lifted to a ribbon by the same trick as a knot. Now our ribbon is an oriented Riemann surface $S$ with finite set of holes. We can apply the technique that is well known in the theory of framed graphs to get some topological invariants of the pair $S \subset Y$. Following this standard method of knot theory, we can deform the representation theory of the algebra $\mathfrak{su}(2)$ to the representation theory of the quantum group $\mathfrak{su}_q(2)$ and so on. Using our analogy between spin networks and integers, we can compare the theory of spin networks with quantum groups with the theory over function fields in algebraic geometry. Everybody knows that this theory is easier and simpler.

3.2 Pumping up trick

The “compact” method is much more interesting for us: we pump up the edges of $\Gamma$ to tubes and the vertices to small 2-spheres. We get a Riemann surface $\Sigma_\Gamma$ of genus $g$ marked by a tube $\{\tilde{e}\}$ for every $e \in E(\Gamma)$ and a trinion $\{\tilde{v}\}$ for every $v \in V(\Gamma)$, where each trinion is a 2-sphere with three holes. The isotopy classes of meridian circles of tubes define $3g - 3$ disjoint, non-contractible, pairwise nonisotropic classes. Let us consider any representations of these classes as simple loops (or circles) $\{C_i\}$ on $\Sigma$. The complement is the union

$$\Sigma_\Gamma \setminus \{C_1, \ldots, C_{3g-3}\} = \bigcup_{i=1}^{2g-2} \tilde{v}_i \quad (3.1)$$

of $2g - 2$ trinions corresponding to vertices of our graph $\Gamma$.

Moreover, we can construct a map

$$m: \Sigma_\Gamma \to \Gamma \quad (3.2)$$
such that for every point \( p \) in the open edge \( m^{-1}(p) = S^1 \), and for every vertex \( v \) the preimage \( m^{-1}(v) = \cdot \infty \) (that is, a bouquet of 2 circles). Thus the result of this pumping up can be viewed as a \textit{dynamics of circles} in \( Y \).

This is the geometric idea of string dynamics, rather than the particle dynamics described in 1.2. We call it \textit{combinatorial string dynamics}. In this natural picture an event or an interaction of two surfaces (or string families) is its intersection. A result of an interaction of two surfaces \( \Sigma', \Sigma'' \) at a point \( p \in \Sigma', \Sigma'' \) is just the connected sum

\[ \Sigma = \Sigma' \#_p \Sigma'' \tag{3.3} \]

and to get a trinion decomposition of \( S \) we have to solve a local problem. Indeed, it is enough to consider the case when a point \( p \) is contained in two tubes \( p \in \tilde{e}' \subset \Sigma', \ p \in \tilde{e}'' \subset \Sigma'' \). Thus we only need to decompose a connected sum \( \tilde{e}' \#_p \tilde{e}'' \). The connected sum operation gives us a new tube \( \tilde{p} \) (the “neck” of the connected sum) with its meredian and the trinion decomposition of \( \tilde{e}' \#_p \tilde{e}'' \). This procedure is well known and parallel to the construction of a trivalent graph \( \Gamma \) from two trivalent graphs \( \Gamma', \Gamma'' \) by joining the midpoints of two edges \( m' \in e' \in \Gamma' \) and \( m'' \in e'' \in \Gamma'' \) by a new edge \( p \). We get two new trivalent vertices \( m' \) and \( m'' \) with edges \( e'_{1,2}, p \) around \( m' \) and \( e''_{1,2}, p \) around \( m'' \). Pairs of edges \( e'_1, e'_2 \) and \( e''_1, e''_2 \) are “halves” of the edges \( e' \) and \( e'' \). They inherit colors from these edges: \( j(e'_i') = j(e') \) and \( j(e''_i') = j(e'') \).

The final problem is to prescribe the color of the last tube \( \tilde{p} \). We can do this by prescribing any number \( j(p) \) not contradicting the Clebsch–Gordan conditions around \( m' \) and \( m'' \).

There are interpretations in this vein of the area operator (see \cite{2}, Fig. 2), the intersection of spin networks with a boundary surface in a 3-dimensional cylinder and so on, although we don’t have space to go into this here. (It is interesting to try to extract these combinatorial objects from nonperturbative string theories, for example from \cite{14}).

This approach to constructing a good theory leads to an amazing geometry of embedded graphs: a pumped up surface can itself be knotted, and we have to develop a theory of 2-dimensional knots. Secondly there are others versions writing down the amplitude of an event, that is, some intersection index of surfaces.

In the modern (super) string theory \cite{13}, two surfaces interact via a metric on \( Y \). In some dimensions, such theories do not have anomalies, but of course we have lost background-independence at the same time. The relation
between this string theory and the expected background-independent string theory is reflected in the quantum theory “X” at the end of 1.4. Namely the perturbative theory around the classical limit of “X”’ must be described by the modern perturbative string theory.

3.3 Representations space

We stress again that up to now we have only used the trivalent graph \( \Gamma \). We forget for a moment the function \( j: E(\Gamma) \to SU(2) \) (1.5). Now consider the space \( R_g \) of gauge classes of flat SU(2)-connections on \( \Sigma \), that is, the space

\[
R_g = \text{Hom}(\pi_1(\Sigma_\Gamma), SU(2))/PU(2)
\]  

of conjugacy classes of SU(2) representations of the fundamental group of our Riemann surface. Our trinion decomposition (3.1) defines a map

\[
\pi_\Gamma: R_g \to \mathbb{R}^{3g-3},
\]  

where the last space is just Euclidean space with the special coordinate system \((c_1, \ldots, c_{3g-3})\). For a class of representations \( \rho \in R_g \)

\[
c_i(\rho) = \frac{1}{\pi} \cdot \cos^{-1}\left(\frac{1}{2} \text{Tr} \rho([C_i])\right) \subset [0,1].
\]

It is well known that the functions \( c_i \) on \( R_g \) are continuous on all \( R_g \) and smooth over \((0,1)\). Moreover, the image of \( R_g \) under \( \pi_\Gamma \) is a convex polyhedron

\[
\Delta_\Gamma \subset [0,1]^{3g-3}.
\]

Now we use the second component \( j \) of our spin network \( \Gamma_j \). For this, suppose that our spin network is of level \( k \). Then for every label, we can consider the number \( \frac{j_i}{k} \) as the \( i \)th coordinate of a point in \( \mathbb{R}^{3g-3} \). So the function \( \frac{j}{k} \) as coordinates defines a point \( p_j \in \mathbb{R}^{3g-3} \). It was proved in \([16]\) that

\[
p_j \in \Delta_\Gamma.
\]

So we get the subcycle

\[
\pi_\Gamma^{-1}(p_j) \subset R_g
\]
This is a serious geometric object deserving careful study.

First of all, the space \( \mathbb{R}^{3g} \) is equipped with the canonical symplectic form \( \Omega_G \). We call it the Goldman form \(^{[17]}\). Thus the pair \((\mathbb{R}^{3g}, \Omega_G)\) is a phase space of a classical mechanical system. This system is completely integrable, and the complete set of first integrals is given by the coordinates \(c_i\) in \(\mathbb{R}^{3g-3}\). So the map \(\pi_\Gamma (3.5)\) is a real polarization of this mechanical system, and the general fiber of this map is a \((3g-3)\)-dimensional Lagrangian torus.

Moreover, our phase space has canonical prequantization data \((\Theta, a_{CS})\), where \(\Theta\) is Hermitian line bundle with unitary connection \(a_{CS}\) (the so-called Chern–Simons connection), whose curvature form satisfies

\[
F_{a_{CS}} = 2\pi i \Omega_G. \tag{3.10}
\]

Recall that a Lagrangian cycle \(S \subset \mathbb{R}^{3g}\) is a Bohr–Sommerfeld cycle of level \(k\) iff the restriction \((\Theta^k, ka_{CS})|_S\) admits a covariant constant section (see \(^{[18]}\)). Let

\[
\text{SNW}_k(\Gamma) = \{\Gamma_j\} \tag{3.11}
\]

be the set of all spin networks of level \(k\) over a graph \(\Gamma\). Then we have a set of Lagrangian fibers of the fibration (3.5):

\[
\text{BS}_k(\Gamma) = \{\pi_\Gamma^{-1}(p_j)\}. \tag{3.12}
\]

The following result was proved in \(^{[16]}\):

**Proposition 3.1** The collection \(\text{BS}_k(\Gamma) = \{\pi_\Gamma^{-1}(p_j)\}\) is the set of all Bohr–Sommerfeld fibers of level \(k\).

Now we can fix the third face of a spin network \(\Gamma_j\) of genus \(g\) and level \(k\), its geometric equivalent: this is a Bohr–Sommerfeld fiber of the real polarization of \(\mathbb{R}^{3g}\) given by the trinion decomposition of Riemann surface \(\Sigma_\Gamma\). Thus the number \(|\text{BS}_k(\Gamma)|\) of Bohr–Sommerfeld fibers of level \(k\) is equal to the number \(N^k(\Gamma)\) of spin networks of level \(k\) over a graph \(\Gamma\). But this number is equal to the Verlinde number (see \(^{[16]}\) and \(^{[18]}\)):

\[
N^k(\Gamma) = |\text{BS}_k(\Gamma)| = \frac{(k + 2)^{g-1}}{2^{g-1}} \sum_{n=1}^{k+1} \frac{1}{(\sin(n\pi/k+2))^{2g-2}}.
\]
3.4 More “bulk” geometry

In the same vein as $\Sigma$, our trivalent graph $\Gamma$ defines a handlebody $\tilde{\Sigma}_\Gamma$, that is, a 3-manifold with boundary $\Sigma$. We have the epimomorphism

$$r: \pi_1(\Sigma) \to \pi_1(\tilde{\Sigma}). \quad (3.13)$$

The fundamental group $\pi_1(\tilde{\Sigma}_\Gamma)$ is the free group with generators, say $\{b_i\}$ for $i = 1, \ldots, g$. We can consider them as elements of $\pi_1(\Sigma)$. And the kernel of $r$ is the free group with generators, say $\{a_i\}$ for $i = 1, \ldots, g$. Then

$$\pi_1(\Sigma) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \left| \prod_{i=1}^{g} [a_i, b_i] = \text{id} \right. \right\rangle. \quad (3.14)$$

is the standard presentation of the fundamental group.

Starting with a trivalent graph $\Gamma$ we get a Riemann surface $\Sigma$ with the collection of elements $\{[C_i]\}$ of $\pi_1(\Sigma)$ corresponding to disjoint loops (3.1). Between these elements we can fix the first $g$ loops to be $[C_1] = a_1, \ldots, [C_g] = a_g$ by changing the numbering, so that we can add elements $b_1, \ldots, b_g \in \{C_i\}$ to make a standard basis (3.14) of the fundamental group $\pi_1(\Sigma)$.

A graph with such additional choice is a marked graph $\Gamma_m$. Returning to the space $R_g$ (3.4), we get the subcycle

$$uS_g = \{\rho \in R_g \mid \rho([C_i]) = 1 \text{ for } i = 1, \ldots, g\}, \quad (3.15)$$

that we call the unitary Schottky space of genus $g$.

This space can be presented as a homogeneous space

$$uS_g = \text{SU}(2)^g / \text{Ad}_{\text{diag}} \text{SU}(2) \quad (3.16)$$

where $\text{Ad}_{\text{diag}} \text{SU}(2)$ is the diagonal adjoint action on the direct product.

The following statements are proved in [16] (see also [18]).

**Proposition 3.2** The unitary Schottky space (3.15) is a fiber of the real polarization $\pi_{\Gamma_m}$ (3.5). More precisely

$$uS_g = \pi_{\Gamma_m}^{-1}(0, \ldots, 0). \quad (3.17)$$

In particular

$$uS_g \in \text{BS}_k(\Gamma) = \text{SNW}_k(\Gamma) \quad (3.18)$$

for every level $k$. (See (3.11) and (3.12)).
The interpretation of the space $R_g$ as the space of gauge classes of flat connections on $\Sigma_\Gamma$ gives the description of all fibers of $BS_k(\Gamma)$: let $\pi^{-1}_\Gamma(p_j)$ be such a fiber. For each loop $C_i$ for $i = 1, \ldots, 3g - 3$, write

$$Z^{j}_{e_i} = \text{stabilizer of } \pi^{-1}_\Gamma(p_j)|_{C_i} \quad (3.19)$$

and

$$Z^{j}_{v_i} = \text{stabilizer of } \pi^{-1}_\Gamma(p_j)|_{P_i} \quad (3.20)$$

where $P_i$ is a trinion of the decomposition (3.1).

We have two groups

$$Z^{j}_{E(\Gamma)} = \prod_{e \in E(\Gamma)} Z^{j}_e \quad (3.21)$$

$$Z^{j}_{V(\Gamma)} = \prod_{v \in V(\Gamma)} Z^{j}_v \quad (3.22)$$

and the action of the second group on the first given by the formula (2.12). Then the fiber is

$$\pi^{-1}_\Gamma(p_j) = Z^{j}_{E(\Gamma)}/Z^{j}_{V(\Gamma)}. \quad (3.23)$$

For general $j$

$$Z^{j}_e = U(1) \text{ for every } e \in E(\Gamma) \quad (3.24)$$

$$Z^{j}_v = Z_2 \text{ is the center of } SU(2) \quad (3.25)$$

Thus for general $p_j \in BS_k(\Gamma)$ the fiber

$$\pi^{-1}_\Gamma(p_j) = U(1)^{3g-3} \quad (3.26)$$

is a $(3g - 3)$-torus.

But for the special point,

$$\pi^{-1}_\Gamma(0, \ldots, 0) = SU(2)^{3g-3}_{\Gamma}/SU(2)^{2g-2}_{\Gamma} = Q_{\Gamma} \quad (3.27)$$

(see (3.23)), since

$$\pi^{-1}_\Gamma(0, \ldots, 0)|_{C_i} = 1 \quad \text{for } i = 1, \ldots, 3g - 3 \quad (3.28)$$
and

\[ \pi^{-1}_{\Gamma}(0, \ldots, 0)|_{P_i} = 1, \quad \text{for } i = 1, \ldots, 2g - 2. \]  

(3.29)

Thus we have the identification

\[ u_S g = Q_\Gamma. \]  

(3.30)

Of course it depends on the marking (3.15) of \( \Gamma \). Comparing the probabilistic measures on components of products (2.1), (2.2) and (3.16) gives the following result:

**Proposition 3.3**  
*Under the identification (3.30)*

\[ f_{\Gamma_j} \in L^2(SU(2)^g, d\vec{x})^{Ad_{diag}SU(2)} \]  

(3.31)

(see (2.15)).

Thus we have collected all spin network states as functions on the same space, that is, on the unitary Schottky space

\[ f_{\Gamma_j} \in L^2(sU_g, d\vec{x}). \]  

(3.32)

This is just what we need to come to the general theory [19]. Here we send \( g \) to infinity to get all differential invariant states. On the other hand, we have a bridge to well known theories [20] and [21]. The following partial case of the constructions under consideration can be added to [22].

### 4 Illustration: the Abelian case

For U(1)-spin networks

\[ \widehat{U(1)} = \mathbb{Z}^*, \]  

(4.1)

and the triangle inequality (1.6) becomes the equality

\[ j_{v,1} + j_{v,2} + j_{v,3} = 0. \]  

(4.2)

The harmonic analysis is just the classical Fourier decomposition. Now

\[ U(1)^{E(\Gamma)} = \prod_{e \in E(\Gamma)} U(1)_e \]  

(4.3)
and

$$U(1)^{V(\Gamma)} = \prod_{v \in V(\Gamma)} U(1)_v$$  \hspace{1cm} (4.4)$$

It is easy to see that under the action (2.12), the diagonal of $U(1)^{V(\Gamma)}$ acts trivially. Thus in this case

$$Q_\Gamma = U(1)^{E(\Gamma)}/U(1)^{V(\Gamma)} = U(1)^g.$$  \hspace{1cm} (4.5)$$

Now the representation space (3.4)

$$J_{\Sigma} = \text{Hom}(\pi_1(\Sigma), U(1))$$  \hspace{1cm} (4.6)$$
is the Jacobian of our surface. This group is the direct product of $2g$ copies of $U(1)$, but components can be labeled by the basis (3.14)

$$J_{\Sigma} = \prod_{i=1}^g U(1)_{a_i} \times \prod_{i=1}^g U(1)_{b_i}$$  \hspace{1cm} (4.7)$$

We can view the coordinates (3.6) of the map (3.5) just as elements of target group $U(1)$. Thus we have the map (3.5) in this case

$$\pi_\Gamma: J_{\Sigma} \rightarrow U(1)^{E(\Gamma)}.$$  \hspace{1cm} (4.8)$$

It is easy to see that the image is a $g$-torus

$$\Delta_\Gamma = T^g \subset U(1)^{E(\Gamma)}$$  \hspace{1cm} (4.9)$$
such that the projection

$$\text{pr}: U(1)^{E(\Gamma)} \rightarrow \prod_{i=1}^g U(1)_{c_i}$$  \hspace{1cm} (4.10)$$
defines the isomorphism of $T^g$ to $\prod_{i=1}^g U(1)_{c_i}$.

Recall that we have the identification $a_i = c_i$ (3.14). Thus the map $\pi_\Gamma$ (3.5) is just the projection of the direct product (4.7) to the component

$$\pi_\Gamma: \prod_{i=1}^g U(1)_{a_i} \times \prod_{i=1}^g U(1)_{b_i} \rightarrow \prod_{i=1}^g U(1)_{a_i} = T^g.$$  \hspace{1cm} (4.11)$$
But the intersections of 1-cycles on our Riemann surface defines the integral symplectic form $\Omega$ on $J_{\Sigma_{\Gamma}}$, that is, the polarisation of the Jacobian. It is easy to see that the fibration (4.11) is Lagrangian, that is, each fiber is a Lagrangian torus.

Now consider a $U(1)$-spin network of level $k$, that is, for every $e \in E(\Gamma)$, $0 \leq j_e \leq k - 1$. Then the function $\frac{j}{k}$ defines a point $p_j \in U(1)^{E(\Gamma)}$ and moreover $p_j \in \Delta_\Gamma = T^g_{\frac{k}{g}}$. This point is a point of order $k$ on the torus $T^g_{\frac{k}{g}}$.

It is easy to see that the function $j$ (1.5) can be reconstructed from this point and every point of order $k$ of $T^g_{\frac{k}{g}}$ is an image of some commutative spin network. To say nothing of the fact that all fibers over these points are Bohr–Sommerfeld fibers of the real polarization $\pi_{\Gamma}$ (see [18]).

Thus the number $|BS_k(\Gamma)|$ of Bohr–Sommerfeld fibers of level $k$ is equal to number $N^k_A(\Gamma)$ of Abelian spin networks of level $k$ over a graph $\Gamma$ and is equal to the number of points of order $k$ on $g$-torus $T^g_{\frac{k}{g}}$. But this number is equal to (see [18]):

$$N^k_A(\Gamma) = |BS_k(\Gamma)| = |(T^g_{\frac{k}{g}})_k| = k^g.$$ 

The full program (all the stones of the mosaic) of the theory of classical theta functions for Abelian case is developed in [23].

Acknowledgments

I would like to express my gratitude to my collaborators C. Florentino, J. Mourão, J.P. Nunes and to the Instituto Superior Tecnico of Lisbon for support and hospitality. Special thanks to Miles Reid for his permanent help.

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