Full Length Article

Shifted Chebyshev wavelet-quasilinearization technique for MHD squeezing flow between two infinite plates and Jeffery–Hamel flows

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Article info

Article history:
Received 13 November 2014
Received in revised form 9 April 2015
Accepted 4 May 2015
Available online 9 July 2015

MSC:
35Q79
42C15
39B9

Keywords:
MHD flow
Chebyshev wavelets method
Quasilinearization technique
Jeffery–Hamel flow
Numerical solution

Abstract

In this article, shifted Chebyshev wavelets method is merged with quasilinearization technique to tackle with the nonlinearity of physical problems. The accuracy of the proposed method is verified by the help of two nonlinear physical models, one MHD squeezing flow between two infinite plates and other Jeffery–Hamel flow that is obtained using proper similarity transforms. Numerical solution is also sought using Runge–Kutta order 4 method. Results obtained for different iterations and different values of degree of polynomial are described in tables and graphs which verify the accuracy and stability of the proposed method.

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1. Introduction

After the pioneering works of Stefan [1], squeezing flows have been of much interest to the researchers due to their many practical and industrial applications. Many mechanical equipment work under the principle of moving pistons where two plates exhibit squeezing moment normal to their own surfaces. Electric motors, engines and hydraulic lifters also have this squeezing flow in some of their parts. Its biological applications are also of equal importance. Flow inside syringes and nasogastric tube is also a kind of squeezing flows. One can find more than enough literature on these flows in
Ref. [23] and references therein. Electrically conducting flows are also very important as a slight change in magnetic field may cause flow to disperse or often to go smoothly for some time. It was therefore essential to discuss the flow under the influence of magnetic field to see how it affects the flow behavior. Refs. [4–6] studied the effects of magnetic field on squeezing flow for different geometries and pointed out some important aspects of these flows.

Flows through nonparallel walls gain importance in early 19th century after the pioneering works of Jeffery [7] and Hamel [8]. Since then, there are many studies available that discussed the different practical and industrial applications of these flows and reported that flow characteristics vary by changing the angle between two channels [9–11]. Flows through rivers and channels, different biological flows such as blood and others, Legendre wavelets and Chebyshev wavelets. Islam et al. used Legendre wavelet method to obtain solutions for linear and nonlinear boundary layer flow problem [17] and Hariharan applied Haar wavelet method for solving Sine-Gordon and Klein–Gordon equations [18]. Rawashdeh implemented Legendre wavelet method to obtain solution of fractional integro-differential equations [16]. Ali et al. used Chebyshev wavelets to obtain solutions for linear and nonlinear boundary value problems [19]. Iqbal et al. obtained solutions for fractional delay differential equations using Chebyshev wavelets [20]. Since the models for which we are approximating solutions are of nonlinear in nature, so for better results we are also using quasilinearization technique.

The quasilinearization technique was first introduced by Bellman and Kalaba [21] as a generalization of the Newton–Raphson method [22] to tackle the single or systems of nonlinear ordinary or partial differential equations.

The proposed method formed by merging Chebyshev wavelets method with quasilinearization technique is fully compatible for solving such nonlinear physical models. To the best of our knowledge, this is the first article on Chebyshev wavelet methods in fluid mechanics. Two non-linear problems are taken into account. A well-known numerical method Runge–Kutta order 4 method is used to solve the same problems. Comparison is made among the solutions to verify the accuracy of the proposed solutions.

2. **Mathematical formulation**

2.1. **MHD squeezing flow between two infinite plates**

The equations of motion for the flow are given by [23],

\[ \nabla \cdot V = 0, \]  
(2.1)

\[ \rho \left[ \frac{\partial V}{\partial t} + (V \cdot \nabla)V \right] = \nabla \cdot T - f_3, \]  
(2.2)

where \( V \) is velocity vector, \( \rho \) density constant and \( T \) is the Cauchy Stress tensor given by,

\[ T = -\rho l + A_1, \]

where \( A_1 = (\nabla V) + (\nabla V)^T \).

While \( f_3 \) is a source term arising due to applied magnetic field, i.e., the so called magnetic or Lorentz force. This force is known to be a function of the imposed magnetic field \( B \), the induced electric field \( E \) and the fluid velocity vector \( V \), that is

\[ f_3 = \sigma(E + V \times V) \times B. \]

Detailed derivation of the considered model is discussed in Ref. [23]. Using compatibility equation

\[ \rho \left[ \frac{1}{r} \frac{\partial (rE \psi)}{\partial r} - \frac{1}{r^2} \frac{\partial \psi}{\partial z} \right] = \mu \frac{\partial^2 \psi}{\partial z^2} - \sigma \frac{\partial \psi}{\partial z}, \]  
(2.3)

where \[ E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} = 0, \]

from Eq. (2.3), after simplification we get

\[ -\rho \left[ \frac{1}{r} \frac{\partial (rE \psi)}{\partial r} - \frac{1}{r^2} \frac{\partial \psi}{\partial z} \right] = \mu \frac{\partial^2 \psi}{\partial z^2} - \sigma \frac{\partial \psi}{\partial z}, \]  
(2.4)

with associated auxiliary conditions

\[ z = H, \ \text{then} \ u = 0, \ \w = -V, \]  
(2.5a)

\[ z = 0, \ \text{then} \ w = 0, \ \frac{\partial u}{\partial z} = 0. \]  
(2.5b)

We can now define stream function as

\[ \psi(r, z) = r^2 f(z). \]  
(2.6)

Replacing value from Eq. (2.6) into Eq. (2.4) reduces Eq. (2.4) into a nonlinear ordinary differential equation

\[ f''(z) + \frac{2 \rho}{\mu} f(z) f'(z) - \frac{\sigma B_0^2}{\mu} f'(z) = 0. \]  
(2.7)

Subject to the boundary conditions

\[ f(0) = 0, \ \ f'(0) = 0, \]  
\[ f(H) = \frac{V}{2} f''(H) = 0. \]  
(2.8)

Nonlinear differential equation in Eq. (2.7) along with boundary conditions in Eq. (2.8) can be made dimensionless by using the following non-dimensional parameters

\[ F = \frac{f}{\sqrt{V/2}}, \ \ x = \frac{z}{H}, \ \ Re = \frac{\rho H}{\mu/V}, \ \ M = \sqrt{\sigma HB_0^2 \mu}, \]  
(2.9)

\[ F''(x) + Re F(x) F''(x) - M^2 F'(x) = 0, \]  
(2.10)

with boundary conditions converted into the form
From Eqs. (2.12) and (2.13) after eliminating pressure terms and using Eqs. (2.15) and (2.16), we get a nonlinear ordinary differential equation for normalized velocity profile \( F(x) \)

\[
F''(x) + 2aReF(x)F'(x) + 4a^2F'(x) = 0.
\]

Accordingly the boundary conditions (2.14) are

\[
F(0) = 1, \quad F'(0) = 0, \quad F(1) = 0,
\]

where \( Re \) is Reynolds number given by

\[
Re = \frac{U_{max}}{v} = \begin{cases} \frac{a + b}{b} & \text{Divergent Channel: } a > 0, \ U_{max} > 0 \\ \frac{a - b}{b} & \text{Convergent Channel: } a < 0, \ U_{max} < 0 \end{cases}.
\]

\( U_{max} \) here is center line velocity.

3. Shifted Chebyshev wavelets

In the present work, we use the shifted Chebyshev polynomials on \([a, b]\), so the shifted Chebyshev nodes are

\[
x_k = \frac{b - a}{2} \cos \left( \frac{(2k + 1)\pi}{2M} \right) + \frac{a + b}{2}, \quad k = 0, 1, 2, \ldots, M - 1,
\]

where \( a \) and \( b \) are real numbers with \( a < b \). The shifted Chebyshev polynomials \( T_m(x) \) of order \( m \) are defined on the interval \([a, b]\) and are given by the following recurrence formulae,

\[
T_0(x) = 1, \quad T_1(x) = \frac{2x - (b + a)}{b - a}, \quad T_{m+1}(x) = 2\frac{2x - (b + a)}{b - a}T_m(x) - T_{m-1}(x), \quad m = 1, 2, 3, \ldots
\]

The orthogonality conditions is

\[
\int_a^b \frac{1}{\sqrt{1 - (\frac{2x - (b + a)}{b - a})^2}} T_m(x)T_n(x)dx = \begin{cases} 0, & m \neq n; \\ \frac{\pi}{b - a}, & m = n. \end{cases}
\]

Shifted Chebyshev wavelets defined on the interval \([a, b]\) as

\[
\psi_{n,m}(x) = \begin{cases} \frac{4}{(b - a)\pi}T_m(2x - \hat{n}), & \frac{a + (b - a)\hat{n}}{2} \leq x \leq \frac{a + (b - a)\hat{n} + 1}{2} \\ 0, & \text{otherwise} \end{cases}
\]

\[
\hat{n} = 2^k, \quad k = 1, 2, 3, \ldots, \text{is the level of resolution}, \quad n = 1, 2, 3, \ldots, 2^k - 1 \text{ is the order of the Chebyshev polynomials}, \quad M > 0. \text{ The solution obtained by Chebyshev wavelets is of the form}
\]

\[
y(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{2^k - 1} c_{k,m}\psi_{n,m}(x),
\]

where \( \psi_{n,m}(x) \) is given by equation (3.1). We approximate \( y(x) \) by the truncated series.
\[ y(x) = \sum_{n=0}^{N-1} c_n \psi_{n,m}(x). \]  

(3.2)

Then a total number of conditions \(2k^{-1}M\) should exist for determination of \(2k^{-1}M\) coefficients \(c_{01}, c_{02}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, c_{k-10}, c_{k-11}, \ldots, c_{kM-1}.\)

Some conditions are furnished by the initial or boundary conditions, while for remaining conditions we replace \(y_{k,M}\) in our differential equation to recover the unknown coefficients \(c_{n,m}\).

\[ y(a) = \alpha \quad \text{and} \quad y(b) = \beta, \quad a \leq x \leq b. \]  

(4.2)

where \(f\) may be a function of \(x\) or \(y(x)\). Let \(y_0(x)\), be an initial approximation of the function \(y(x)\). The Taylor’s series expansion of \(f\) about \(y_0(x)\) is

\[ f(y(x), x) = f(y_0(x), x) + (y(x) - y_0(x)) f_y (y_0(x), x) + O((y(x) - y_0(x))^2). \]  

(4.3)

Ignoring second and higher order terms and replacing in Eq. (4.1), we get

\[ y'(x) = f(y_0(x), x) + (y(x) - y_0(x)) f_y (y_0(x), x). \]  

(4.4)

solving Eq. (4.4) and calling answer \(y_1(x)\). Using \(y_1(x)\) and again expanding Eq. (4.1) about \(y_1(x)\), we have

\[ y'(x) = f(y_1(x), x) + (y(x) - y_1(x)) f_y (y_1(x), x). \]  

(4.5)

after simplification we get \(y_2(x)\), second approximation to \(y(x)\). Continuing this process we obtain the desired accuracy if the problem converges. Generally we can write the recurrence relation in the following form

\[ y_{n+1}(x) = f(y_n(x), x) + (y(x) - y_n(x)) f_y (y_n(x), x) \]  

(4.6)

in which \(y_n(x)\) is known and after solving we get \(y_{n+1}(x)\). The boundary condition in (4.2) is also converted into the form \(y_{n+1}(a) = \alpha \quad \text{and} \quad y_{n+1}(b) = \beta\). Same procedure can be applied on other higher order nonlinear problems also.

5. Convergence analysis

Since we are using both quasilinearization technique and Chebyshev wavelets method so we discuss the convergence for both of these.

5.1. Convergence of quasilinearization technique

The convergence of quasilinearization technique is derived and discussed in Ref. [25] which shows that the convergence of quasilinearization technique is second order if there is convergence at all.

5.2. Convergence of Chebyshev wavelets method

The convergence of Chebyshev wavelets method is derived in Ref. [20] which shows that the series solution by Chebyshev wavelets method converges to \(y(x)\) for differential equation of any order.

6. Solution procedure

6.1. For MHD squeezing flow between two infinite plates

Differential equation of MHD squeezing flow (2.10) after applying quasilinearization technique becomes

\[ \frac{d^4}{dx^4} F_{n+1}(x) + \Re F_n(x) \frac{d^3}{dx^3} F_{n+1}(x) + \Re F_{n-1}(x) \frac{d^3}{dx^3} F_n(x) \]  

\[ - M^2 \frac{d^2}{dx^2} F_{n+1}(x) = \Re F_n(x) \frac{d^2}{dx^2} F_n(x). \]  

(6.1)

with boundary conditions converted into the form

\[ F_{n+1}(0) = 0, \quad \frac{d^2}{dx^2} F_{n+1}(0) = 0, \quad F_{n+1}(1) = 1, \quad \frac{d}{dx} F_{n+1}(1) = 0. \]

For applying Chebyshev wavelets technique now substitute,

\[ F_{n+1}(x) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{n,m} \psi_{n,m}(x) \]

Eq. (6.1) becomes (see Tables 1 and 2, Figs. 1 and 2),

\[ \frac{d^4}{dx^4} \left( \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{n,m} \psi_{n,m}(x) \right) + \Re F_n(x) \frac{d^3}{dx^3} \left( \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{n,m} \psi_{n,m}(x) \right) \]

\[ + \Re \left( \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{n,m} \psi_{n,m}(x) \right) \frac{d^3}{dx^3} F_n(x) \]

\[ - M^2 \frac{d^2}{dx^2} \left( \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{n,m} \psi_{n,m}(x) \right) = \Re F_n(x) \frac{d^2}{dx^2} F_n(x). \]  

(6.2)

6.2. For Jeffery–Hamel flow

Eq. (2.17) representing Jeffery–Hamel flow after applying quasilinearization becomes,

\[ \frac{d^3}{dx^3} F_{n+1}(x) + 2a \Re F_{n+1}(x) \frac{d}{dx} F_n(x) + 2a \Re F_n(x) \frac{d}{dx} F_{n+1}(x) \]

\[ + 4a^2 \frac{d^2}{dx^2} F_{n+1}(x) = 2a \Re F_n(x) \frac{d}{dx} F_n(x). \]  

(6.3)

Accordingly the boundary conditions are converted into
\[ F_{n+1}(0) = 1, \quad \frac{d}{dx} F_{n+1}(0) = 0, \quad F_{n+1}(1) = 0. \]

For applying Chebyshev wavelets technique now substitute,
\[ F_{n+1}(x) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x). \]

Eq. (6.3) becomes (see Tables 3 and 4, Figs. 3 and 4)

\[
\frac{d^3}{dx^3} \left( \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \right) + 2\alpha Re \left( \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \right) \frac{d}{dx} F_n(x) + 2\alpha Re F_n(x) \frac{d}{dx} \left( \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \right) + 4\alpha \frac{d}{dx} \left( \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \right) = 2\alpha Re F_n(x) \frac{d}{dx} F_n(x). \]

(6.4)
Comparison of solutions for Jeffery–Hamel flow (6.4) when $\alpha = 3$ and $Re = 10$ (diverging channel), for different polynomial values of proposed method with RK-4.

| $x$ | $y_{m=5}$ | $y_{m=10}$ | $y_{m=15}$ | Error at $y_{m=5}$ | Error at $y_{m=10}$ | Error at $y_{m=15}$ |
|-----|------------|-------------|-------------|--------------------|--------------------|--------------------|
| 0.0 | 1.00000000 | 1.00000000 | 1.00000000 | 0.00000000        | 0.00000000        | 0.00000000        |
| 0.5 | 0.98927742 | 0.98927742 | 0.98927742 | 2.3017E-03         | 1.0674E-05         | 4.2882E-07         |
| 1.0 | 0.95722187 | 0.95722187 | 0.95722187 | 8.4531E-03         | 4.4222E-03         | 1.2284E-07         |

Conclusion

This article investigates the MHD flow between two parallel plates and Jeffery–Hamel flow. Shifted Chebyshev wavelets-quasilinearization technique is applied to solve the equations of flow. In order to check the accuracy of the solution obtained by shifted Chebyshev wavelets-quasilinearization technique at different values of polynomials figures and tables are drawn which shows that the accuracy of method is increased when we increase the order of the polynomial. Numerical solution is also sought out for the sake of comparison. It is clear from Figures and Tables that this method can be applied successfully to different problems of physical nature.

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