Hunting for quantum spikes

Andrzej Góźdź,1,* Włodzimierz Piechocki,2,†
Grzegorz Plewa,2,‡ and Tomasz Trześniewski3,§

1Institute of Physics, Maria Curie-Skłodowska University,
pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland
2Department of Fundamental Research,
National Centre for Nuclear Research,
Hoża 69, 00-681 Warszawa, Poland
3Institute of Theoretical Physics, Jagiellonian University,
Łojasiewicza 11, 30-348 Kraków, Poland
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Abstract

The quantum structures called quantum spikes, corresponding to known classical spikes are presented. The latter may occur in the evolution of the homogeneous sector of the Hamiltonian dynamics describing the Belinski-Khalatnikov-Lifshitz conjecture. The question if classical spikes may survive quantization is examined. The answer is in the affirmative. However, this is rather a subtle effect that needs further examination by making use of sophisticated analytical and numerical tools. The spikes seem to be of fundamental importance, both at classical and quantum levels, as they may serve as seeds of real structures in the universe.

* andrzej.gozd@umcs.lublin.pl
† wlodzimierz.piechocki@ncbj.gov.pl
‡ grzegorz.plewa@ncbj.gov.pl
§ t.trzesniewski@uj.edu.pl
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## I. INTRODUCTION

The Belinskii, Khalatnikov, and Lifshitz (BKL) conjecture says, among other things, that as a spacetime singularity is approached the dynamics can be locally approximated by neglecting spatial derivatives in the field equations in comparison to time derivatives. The dynamics underlying the BKL scenario is the evolution
of the homogeneous spacetimes (in particular the Bianchi IX model) towards the singularity [1–3].

The Hamiltonian formalism was proposed that may be used to address, after quantization, the BKL conjecture at quantum level [4]. Recently, the homogeneous sector of this approach has been used, near the surface of critical points, to derive a spike solution, and the possibility of the existence of a quantum spike was examined [5]. The preliminary results obtained in [5] suggest that quantum spikes do not exist. However, the issue of time has not been treated there satisfactory due to the fact that this paper deals mainly with the vacuum case. The present paper considers the case with the matter field described by a massless scalar field, which enables using this field as an evolution parameter both at classical and quantum levels. Apart from the time issue, only simplified analyses of quantum observables of the spike has been done in [5]. Our paper feels this gap as well.

The spikes, usually considered in inhomogeneous spacetimes, are steep structures (transient or permanent) which appear in the evolution towards singularities [6, 7]. However, a spike has been found recently in the homogeneous framework as well, in the context of the evolution of the Bianchi IX model applied to the phase space orbits of nearby worldlines within the Hamiltonian formalism (see sections I and II of the paper [5] for more details).

The spike-like structures, which we consider, are specific to nonlinear coupled system of ODEs in which one considers the mapping of a smooth curve of the initial data for the dynamics, into another curve via the propagation of the same amount of time of each point of the initial data. It may happen that such an initial curve evolves into an intriguing structure localized in some finite region of the space of solutions to the dynamics.

Such structures occur, for instance, in the context of the dynamics of the forced pendulum with damping. Known examples are the Ueda attractor and the Duffing oscillator. The damped driven (forced) pendulum models have many applications, for instance, in mathematical biology (see [8, 9] and references therein). The present paper is devoted to the examination of the possible structure formation, both at classical and quantum levels, in the context of gravitational physics.

The paper is organized as follows: In Sec. II we define the phase space variables satisfying the affine Lie algebra, Hamilton’s dynamics parameterized by massless scalar field, and derive the classical spikes. Section III deals with the quantum level. We specify the representation of the Lie algebra and introduce quantum evolution parameter. We present quantum propagation in terms of two eigenequations and determine solutions to these equations. Quantum dynamical constraint is imposed. In Sec. IV we determine the quantum spikes. Conclusions are presented in Sec. V.
II. CLASSICAL LEVEL

A. Phase space

The kinematical phase space of the homogeneous sector of the gravitational field in the Hamiltonian formalism [4] can be parameterized by the variables $C_I$ and $P_I$, where $I = 1, 2, 3$. We also introduce a massless matter field, described by the variables $\phi$ and $\pi$, where $\pi$ is the conjugate momentum. Poisson brackets for the total system read

$$\{ P^I, P^J \} = 0 = \{ C_I, C_J \}, \quad \{ P^I, C_J \} = 2\delta^I_J C_I, \quad \{ \phi, \pi \} = 1. \quad (1)$$

To connect with the notation that is more common for affine algebras, we perform the partial redefinition of variables $(C_I, P^J) =: (C_I, -2D^I)$, which leads to the affine Poisson brackets

$$\{ D^I, D^J \} = 0 = \{ C_I, C_J \}, \quad \{ C_J, D^I \} = \delta^I_J C_I. \quad (2)$$

An algebra with such brackets is called an affine Lie algebra.

The dynamics of the system is specified by the equations\(^1\):

$$\dot{D}_I = -C_I(C - 2C_I), \quad (3)$$
$$\dot{C}_I = 4C_I(D - 2D_I), \quad (4)$$
$$\dot{\pi} = 0, \quad (5)$$
$$\dot{\phi} = \kappa\pi, \quad (6)$$

where $D = D_1 + D_2 + D_3$ and $C = C_1 + C_2 + C_3$. There is no summation $\sum_I$ in the rhs of (3) and (4). The solutions to (3)–(6) have to satisfy the Hamiltonian constraint

$$H = \frac{1}{2} C^2 - \sum_I C_I^2 + 4 \left( \frac{1}{2} D^2 - \sum_I D_I^2 \right) + \frac{\kappa}{2} \pi^2 = 0, \quad (7)$$

where $\kappa = \pm 1$ defines two possible dynamics (two different signatures of the corresponding bilinear forms) with respect to the field $\phi$. Unlike the traditional momentum, which serves to translate the canonical coordinate $C_I$, the variable $D^I$ serves to dilate $C_I$.

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\(^1\) We have the extra factor 2 in Eqs. (3) and (4) that is missing in the corresponding equations of [5].
It results from Eqs. (5) and (6) that $\phi$ is a monotonic function of time. Thus, it can be used as an evolution parameter of the dynamics. Dividing both sides of (3) and (4) by $\dot{\phi} = \kappa \pi$, we obtain

$$\kappa \pi \frac{dD_I}{d\phi} = -C_I(C - 2C_I), \quad (8)$$

$$\kappa \pi \frac{dC_I}{d\phi} = 4C_I(D - 2D_I), \quad (9)$$

which defines the relative dynamics with respect to the variable $\phi$.

Since $C_I = 0$ is a fixed point of the system (3)–(4), the sign of each $C_I$ along any dynamical trajectory is fixed by the initial conditions. Consequently, the affine algebra divides into three sectors, with $C_I < 0$, $C_I > 0$ and $C_I = 0$. It is convenient to define the corresponding three sectors of the kinematical phase space for each direction $I$:

$$\Pi^I_- := \{(C_I, D_I) \mid C_I \in \mathbb{R}^-, D_I \in \mathbb{R}\}, \quad (10)$$

$$\Pi^I_+ := \{(C_I, D_I) \mid C_I \in \mathbb{R}^+, D_I \in \mathbb{R}\}, \quad (11)$$

$$\Pi^I_0 := \{(C_I, D_I) \mid C_I = 0, D_I \in \mathbb{R}\}. \quad (12)$$

These three sectors correspond to the three orbits of the dilation group generated by $D^I$ variable, acting on the space of values of the $C_I$ variable, which is the real line $\mathbb{R}$ [11]. The three orbits imply the three types of irreducible induced representations of the affine group [11].

**B. Classical spikes**

1. **Parametrization of dynamics by a scalar field**

   In order to derive the spike solutions, within the dynamics parameterized by the scalar field $\phi$, we follow the approach presented in Sec. II of Ref. [5].

   Let us assume that the initial conditions for $D_I$ and $C_I$ at $\phi = \phi_0$ have the form: $D_1 < D_2 < D_3 < 0$ and $1 \gg C_I > 0$. Then, it follows from (8)–(9) that $C_2$ and $C_3$ almost instantly vanish, while $D_2$ and $D_3$ turn out to be essentially constant. For later convenience we define $D_\pm := D_2 + D_3 \pm 2\sqrt{D_2 D_3}$. Therefore, the problem reduces to finding the evolution of $C_1$ and $D_1$, which is governed by the equations
(here we denote by prime the derivative with respect to $\phi$):

$$\kappa \pi C_1' = 4C_1(-D_1 + D_2 + D_3),$$  \hspace{1cm} (13)

$$\kappa \pi D_1' = C_1^2,$$  \hspace{1cm} (14)

$$-C_1^2 = 4(D_1 - D_+)(D_1 - D_-) - \kappa \pi^2,$$  \hspace{1cm} (15)

where the last one results from the constraint (7). Inserting the right-hand side of (15) into (14), we obtain an equation independent of $C_1$, whose solution can be written as

$$D_1(\phi) = D_2 + D_3 + \frac{1}{2} \sqrt{16D_2D_3 + \kappa \pi^2} \tanh \left( \frac{2}{\kappa \pi} \sqrt{16D_2D_3 + \kappa \pi^2} (\phi - \phi_0) \right) - \arctanh \left( \frac{16D_2D_3 - C_{10}^2 + \kappa \pi^2}{16D_2D_3 + \kappa \pi^2} \right).$$  \hspace{1cm} (16)

In the above expression the initial condition $D_1(\phi_0) = D_{10}$ has been replaced by

$$D_1(\phi_0) := D_{10} = D_2 + D_3 - \frac{1}{2} \sqrt{16D_2D_3 - C_{10}^2 + \kappa \pi^2},$$  \hspace{1cm} (17)

due to the relation (15) for $C_1(\phi_0) = C_{10}$. Furthermore, (16) and (15) give

$$C_1(\phi) = \text{sgn}(C_{10}) \sqrt{16D_2D_3 + \kappa \pi^2} \sech \left( \frac{2}{\kappa \pi} \sqrt{16D_2D_3 + \kappa \pi^2} (\phi - \phi_0) \right) - \arctanh \left( \frac{16D_2D_3 - C_{10}^2 + \kappa \pi^2}{16D_2D_3 + \kappa \pi^2} \right),$$  \hspace{1cm} (18)

where “sgn” denotes the sign function (its value for $C_{10} = 0$ is irrelevant since then $C_1(\phi) = 0$). One can verify that (18) together with (16) solve the equations (13)–(15).

Choosing the simple parametrization $C_{10} = \tilde{x}$, we can now draw $D_1$ and $C_1$ as functions of $\tilde{x}$ for different values of the evolution parameter $\phi$. Fig. 1 presents the corresponding plots for the choice $D_2 = -2$, $D_3 = -1$, $\kappa = 1$, $\pi = 1$ and $\phi_0 = 0$. One can see that $D_1(\phi)$ and $C_1(\phi)$ behave in the same way as $P_1(t)$ and $C_1(t)$ presented in Ref. [5], which is expected as the evolution parameter $\phi$ is a monotonic function of the evolution parameter $t$ owing to Eqs. (5)–(6).

For a different perspective, in Fig. 2 we show parametric plots of the function $(C_1(\tilde{x}), D_1(\tilde{x}))$, with $\tilde{x} \in [-5, 5]$ and $\phi = 0$ or $\phi = 0.2$. At the top of the right curve one can notice a gap, which reflects the asymptotic behavior of $C_1(\tilde{x})$ and $D_1(\tilde{x})$. For smaller $\phi$ the gap opens up, until only a piece of the curve is left.
2. **Parametrization of dynamics by the arc length**

The arc length of the curve $\vec{r}(\tilde{x}) \equiv (C_1(\tilde{x}), D_1(\tilde{x}))$ is given by the integral

$$s(\tilde{x}) = \int_{\tilde{x}_0}^{\tilde{x}} dy \sqrt{\left(\frac{dC_1(y)}{dy}\right)^2 + \left(\frac{dD_1(y)}{dy}\right)^2},$$

(19)

where $\tilde{x}_0$ is a certain chosen minimal value of $\tilde{x}$. Calculating (19) we obtain

$$s(x) = -\frac{1}{2} \sqrt{16D_2D_3 + \kappa \pi^2} \left( i E(i \zeta, 4) - i F(i \zeta, 4) + \sqrt{2 \cosh(2 \zeta)} - 1 \tanh \zeta \right),$$

$$\zeta \equiv \frac{2}{\kappa \pi} \sqrt{16D_2D_3 + \kappa \pi^2} (\phi - \phi_0) - \arctanh \frac{\tilde{x}}{\sqrt{16D_2D_3 + \kappa \pi^2}},$$

(20)

where $F$ denotes the elliptic integral of the first kind and $E$ of the second kind. This allows us to express the curve $\vec{r}(\tilde{x})$ as a function of $s$, which needs to be calculated.
numerically. Introducing the (normalized) Frenet vectors
\[
\hat{e}_1(s) := \frac{1}{|\vec{e}_1(s)|} \vec{r}'(s), \quad \hat{e}_2(s) := \frac{1}{|\vec{e}_2(s)|} (\vec{r}'''(s) - \vec{r}''(s) \cdot \vec{e}_1(s) \vec{e}_1(s)),
\]
(21) one can define the generalized curvature of \(\vec{r}(s)\) as follows (see, e.g., [10])
\[
\chi(s) := \frac{1}{|\vec{r}''(s)|} \hat{e}_1'(s) \cdot \hat{e}_2(s).
\]
(22)

In Fig. 3 we depict the generalized curvature of the curve \((C_1(s), D_1(s))\) as a function of the normalized arclength \(\bar{s}\) corresponding to \(\tilde{x} \in [-5, 5]\) (i.e. \(s\) divided by the maximal value \(s(\tilde{x} = 5)\), for a given \(\phi\)) for different values of the evolution parameter \(\phi\). The values of \(\kappa, \pi, \phi_0\) and \(D_2, D_3\) are kept the same as in the previous subsection. Moreover, dots on the horizontal axis denote the value of \(\bar{s}(\tilde{x} = 0)\) for a given \(\phi\), which naturally coincides with the middle of the spike. The double peak corresponds to the two inflection points of the curve visible on the right plot in Fig. 1.

Figure 3. The generalized curvature \(\chi(\bar{s})\) of \((C_1(\bar{s}), D_1(\bar{s}))\) for evolution parameters \(\phi = 0\) (red), \(\phi = 0.05\) (orange), \(\phi = 0.1\) (green), \(\phi = 0.2\) (blue), and \(\phi = 0.4\) (black)

Fig. 3 shows that the spike is created at some moment in the evolution of the gravitational system and seems to be permanent. The shape of the spike depends on time and changes from a plateau to singular structure.
III. QUANTUM LEVEL

A. Representation of the affine group

The quantum version of the Lie algebra (2) is defined by the algebraic quantization principle: $C_I \to \hat{C}_I$ and $D^I \to \hat{D}^I$, such that

$$[\hat{C}_I, \hat{C}_J] = 0 = [\hat{D}^I, \hat{D}^J], \quad [\hat{C}_J, \hat{D}^I] = i \delta^I_J \hat{C}_I. \quad (23)$$

where $I, J = 1, 2, 3$. The commutation relations (23) are the same as for the generators of the affine group [12].

The affine group $\text{Aff}(\mathbb{R}_+)_I$ generated by the pair $\hat{C}_I$ and $\hat{D}^I$ has two inequivalent unitary representations $U_-(p, q)_I$ and $U_+(p, q)_I$. They are constructed in two carrier spaces of square integrable functions $L^2(\mathbb{R}_-, d\nu(x^I))$ and $L^2(\mathbb{R}_+, d\nu(x^I))$, $d\nu(x^I) = dx^I/|x^I|$, which correspond to the negative and positive spectrum of the position operator $\hat{C}_I$, respectively. Because of physical interpretation we need the full spectrum of the position operator. This requirement enforces using the reducible representation of the affine group in the carrier space $\mathcal{K}_I := L^2(\mathbb{R}_-, d\nu(x^I)) \oplus L^2(\mathbb{R}_+, d\nu(x^I))$. The general form of the vector $f \in \mathcal{K}_I$ can be written as a direct sum of the functions $f_{\mp} \in L^2(\mathbb{R}_\mp, d\nu(x^I))$:

$$f = f_- \oplus f_. \quad (24)$$

The scalar product of such two vectors is the sum of the appropriate partial scalar products:

$$\langle f_1 \oplus f_2 | g_1 \oplus g_2 \rangle := \langle f_1 | g_1 \rangle_- + \langle f_2 | g_2 \rangle_+$$

$$= \int_{-\infty}^{0} d\nu(x^I) f_1(x^I)^* g_1(x^I) + \int_{0}^{\infty} d\nu(x^I) f_2(x^I)^* g_2(x^I). \quad (25)$$

The action of the affine group $\text{Aff}(\mathbb{R}_+)_I$ in this carrier space $\mathcal{K}_I$ can be written as

$$U(p, q)_I f = U_-(p, q)_I f_- \oplus U_+(p, q)_I f_+, \quad (26)$$

where $p \in \mathbb{R}, q \in \mathbb{R}_+$ and

$$U_\mp(p, q)_I f_{\mp}(x^I) = e^{ipx^I} f_{\mp}(qx^I). \quad (27)$$

This structure allows for extension of this affine action to the whole straight line. For this purpose it is enough to extend the appropriate functions from half-line to the whole line.

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2 Throughout the paper we choose $\hbar = 1$ and use Planck’s units except where otherwise stated.
full straight line: $f_-(x^I) = 0$ for $x^I \geq 0$ and $f_+(x^I) = 0$ for $x^I \leq 0$. Then, denoting by $|x^I \otimes x^I\rangle$ the “position” vector in the space $K_I$, every function belonging to $K_I$ can be represented as:

$$f(x^I) := \langle x^I \otimes x^I | f_- \otimes f_+ \rangle = \langle x^I | f_- \rangle + \langle x^I | f_+ \rangle = f_-(x^I) + f_+(x^I).$$

(28)

It is obvious that the space $K_I \subset L^2(\mathbb{R}, d\nu(x^I))$ and that the scalar product (25) can be rewritten as

$$\langle f_1 \oplus f_2 | g_1 \oplus g_2 \rangle_I = \langle f_1 | g_1 \rangle_I = \int_{-\infty}^{\infty} d\nu(x^I) f(x^I)^* g(x^I).$$

(29)

The action of the affine group $\text{Aff}(\mathbb{R}_+)_I$ in this new carrier space, which we denote again by $K_I$, can be written as

$$U(p, q)_I f(x^I) = e^{ipx^I} f(qx^I).$$

(30)

The explicit representation of the generators of this group are given by the following operators

$$\hat{D}_I f(x^I) := -ix^I \frac{\partial}{\partial x^I} f(x^I), \quad \hat{\mathcal{C}}_I f(x^I) := x^I f(x^I),$$

(31)

where $I = 1, 2, 3$.

The corresponding unitary operators representing elements of the affine group are:

$$\hat{U}(p, q)_I = e^{ip\hat{C}_I} e^{i\ln(q)\hat{D}_I}$$

(32)

where $-\infty < p < +\infty, 0 < q < +\infty$.

Taking into account three variables $x^I$ ($I = 1, 2, 3$), the carrier space $K$ for the representation of the algebra (23) can be defined to be

$$K := K_1 \otimes K_2 \otimes K_3,$$

(33)

where

$$K_I = L^2(\mathbb{R}_-, d\nu(x^I)) \oplus L^2(\mathbb{R}_+, d\nu(x^I)) \subset L^2(\mathbb{R}, d\nu(x^I))$$

(34)

and the scalar product is constructed according to prescription for tensor product of Hilbert spaces:

$$\langle f | g \rangle = \int_{-\infty}^{\infty} d\nu(x^1) \int_{-\infty}^{\infty} d\nu(x^2) \int_{-\infty}^{\infty} d\nu(x^3) f(x^1, x^2, x^3)^* g(x^1, x^2, x^3).$$

(35)

The “total” affine group used in this paper is the direct product of the three affine groups $\text{Aff}_0 = \text{Aff}(\mathbb{R}_+)_1 \otimes \text{Aff}(\mathbb{R}_+)_2 \otimes \text{Aff}(\mathbb{R}_+)_3$. This realization of the affine group allows for physical interpretation of quantized $C_I$ and $D^I$ variables.
B. Quantum dynamics

The quantum dynamics of our system may be derived, to some extent, from the quantum version of the Hamiltonian constraint defined by Eq. (7). In a standard approach, one maps the dynamical constraint into an operator defined in kinematical Hilbert space. Its kernel may be used to construct physical Hilbert space. However, such an approach leads to the problem of time at the quantum level.

The reason for having the scalar field in the Hamiltonian (7), is the hope that it may resolve the problem of time both at classical and quantum levels. Such an approach works in the classical case as it leads to the relative dynamics, defined by Eqs. (8)–(9), parameterized by the scalar field \( \phi \). However, an extension of this strategy to the quantum level faces serious difficulty. Namely, quantization of the scalar field algebra \( \{ \phi, \pi \} = 1 \) as follows:

\[
\hat{\pi} f(\phi) := -i \frac{\partial}{\partial \phi} f(\phi), \quad \hat{\phi} f(\phi) := \phi f(\phi), \quad f \in L^2(\mathbb{R},d\phi),
\]

so that \([\hat{\phi}, \hat{\pi}] = i\mathbb{I}\), leads to the inconsistency. It is so because the operator \( \hat{\pi} \) is defined in a Hilbert space, whereas the operator \( \hat{\phi} \) is usually used as the classical variable \( \phi \), due to \( \hat{\phi} f(\phi) := \phi f(\phi) \), to play the role of time at the quantum level. In what follows we propose to resolve this commonly ignored problem.

Let us treat the field \( \phi \) as a classical clock, that is a parameter enumerating changes of our Hamiltonian system. We propose to follow an idea of the modified Schrödinger type unitary evolution operator \( U(\phi, \phi_0) \) defined by the following conditions:

- The operator \( U(\phi, \phi_0) \) evolves the quantum state of our gravitational system from the “time” \( \phi_0 \) to the “time” \( \phi \) as follows

\[
U(\phi, \phi_0)\Psi(\phi_0, x_1, x_2, x_3) = \Psi(\phi, x_1, x_2, x_3).
\]

- The evolution operator fulfils the conditions:

\[
\begin{align*}
U(\phi, \phi) &= \mathbb{I} \quad \text{(no shift in “time”)}, \\
U(\phi_2, \phi_0) &= U(\phi_2, \phi_1)U(\phi_1, \phi_0) \quad \text{(no “holes” in the evolution)}, \\
U(\phi_2, \phi_1)^\dagger &= U(\phi_2, \phi_1)^{-1} = U(\phi_1, \phi_2) \quad \text{(unitarity)}.
\end{align*}
\]

Let us now consider a formal shift operation with respect to the field \( \phi \). For this purpose we define a kind of adjoint action of the field \( \phi \) and its canonically conjugated
momentum $\pi$ on the classical phase space. For an arbitrary function $g(\phi, \pi)$ on this phase space the adjoint action is defined to be

$$\{ g(\phi, \pi), \cdot \} f(\phi, \pi) := \{ g(\phi, \pi), f(\phi, \pi) \} \quad \text{and} \quad \{ \cdot, g(\phi, \pi) \} f(\phi, \pi) := \{ f(\phi, \pi), g(\phi, \pi) \},$$

(41)

where the Poisson bracket is given by

$$\{ g(\phi, \pi), f(\phi, \pi) \} := \frac{\partial g(\phi, \pi)}{\partial \phi} \frac{\partial f(\phi, \pi)}{\partial \pi} - \frac{\partial g(\phi, \pi)}{\partial \pi} \frac{\partial f(\phi, \pi)}{\partial \phi}.$$  

(42)

One can directly check that

$$e^{\tau \{ \cdot, \pi \}} f(\phi, \pi) = e^{-\tau \{ \pi, \cdot \}} f(\phi, \pi) = f(\phi + \tau, \pi),$$  

(43)

where

$$e^{\tau \{ \pi, \cdot \}} = \sum_{n=0}^{\infty} \frac{\tau^n \{ \pi, \cdot \}^{(n)}}{n!}.$$  

(44)

The powers of the adjoint action are understood as

$$\{ \pi, \cdot \}^{(n)} f(\phi, \pi) = \{ \pi, \{ \pi, \ldots, \{ \pi, f(\phi, \pi) \} \ldots \},$$  

(45)

where $\{ \pi, \cdot \}^{(0)} f(\phi, \pi) = f(\phi, \pi)$.

A similar shift operation with respect to the field $\phi$, as an evolution parameter, may be constructed in a Hilbert space

$$\Psi(\phi + \tau, x) = e^{\tau \frac{\partial}{\partial \phi}} \Psi(\phi, x),$$  

(46)

where $x := (x_1, x_2, x_3)$.

The comparison of the operations (46) and (43) suggests that the shift generator $\frac{\partial}{\partial \phi} =: \pi$, defined in the quantum state space, may play similar role to the classical momentum $\pi$ acting (by the adjoint action) in the phase space. Working in the quantum state space we postulate the replacement of the classical momentum $\pi$ with the operation $\pi$.

Now, let us come back to the evolution operator defined by Eq. (37). Since the evolution parameter $\phi$ does not couple to the gravitational field in (7), we can factorize the evolution operator $U(\phi + \tau, \phi)$ into a product of the unitary operator $V_\mathcal{K}(\phi + \tau, \phi)$ acting on the spatial dependance of state vectors in the Hilbert space $\mathcal{K}$, and the operation $V_\pi(\phi + \tau, \phi)$ acting on the parametric dependence of the state.
vector of $\mathcal{K}$. In what follows, we assume the dependence of the evolution operator on the difference between the final and initial value of the field $\Psi$, i.e. we assume the translational invariance of the evolution operator with respect to the parameter $\phi$. This property allows to think about the clock as working with a constant speed. In this case all the evolution operators depend only on the single parameter $\tau$. Thus, the full evolution operator can be written as

$$U(\tau) = V_K(\tau)V_\pi(\tau). \quad (47)$$

Due to the unitarity, the operation $V_\pi(\tau)$ is expected to have the form:

$$V_\pi(\tau) = e^{-i\tau E(\hat{\pi})}, \quad (48)$$

where $E(\hat{\pi})$ is some real function of $\hat{\pi}$.

Making use of (46) and the factorization (47), we rewrite (37) as follows

$$e^{\tau \hat{\pi}} \Psi(\phi, x) = V_K(\tau)e^{-i\tau E(\hat{\pi})}\Psi(\phi, x) \quad (49)$$

Taking derivative of (49) with respect to $\tau$, at $\tau = 0$, leads to the local evolution equation:

$$\hat{\pi} \Psi(\phi, x) = \left[\left(\frac{\partial V_K(\tau)}{\partial \tau}\right)_{\tau=0} - iE(\hat{\pi})\right] \Psi(\phi, x). \quad (50)$$

Introducing

$$\hat{W} := i\left(\frac{\partial V_K(\tau)}{\partial \tau}\right)_{\tau=0}, \quad (51)$$

we can rewrite (50) in the form

$$i\frac{\partial \Psi(\phi, x)}{\partial \phi} = \left[\hat{W} + E \left(\frac{\partial}{\partial \phi}\right)\right] \Psi(\phi, x). \quad (52)$$

Assuming

$$\Psi(\phi, x) = \omega(\phi)\psi(x), \quad (53)$$

enables rewriting (52) in the separable form

$$\frac{1}{\omega(\phi)} \left[i \frac{\partial}{\partial \phi} - E \left(\frac{\partial}{\partial \phi}\right)\right] \omega(\phi) = \frac{1}{\psi(x)} \hat{W} \psi(x), \quad (54)$$

which leads to the two eigenequations:

$$\left[i \frac{\partial}{\partial \phi} - E \left(\frac{\partial}{\partial \phi}\right)\right] \omega_\lambda(\phi) = \lambda \omega_\lambda(\phi), \quad (55)$$
and
\[ \hat{W}\psi_\lambda(x) = \lambda \psi_\lambda(x). \] (56)

We assume, according to (47), that the quantum evolution operator corresponding to the classical constraint consists of the quantized position dependent part of (7) and the shifted parametric part of (7). This way we avoid quantization of the algebra \( \{\phi, \pi\} = 1 \), and consequently quantization of the classical time variable \( \phi \). Both classical and quantum evolutions are now parameterized by a single variable \( \phi \) that we call the time.

Since there are no products of \( C_I \) and \( D_I \) in (7), and due to (23), the mapping of \( H \) defined by (7) into a Hamiltonian operator \( \hat{H} \) is straightforward. We get
\[ \hat{H} = E \left( \frac{\partial}{\partial \phi} \right) + \hat{W} \] (57)
\[ = 2 \left( -\frac{\kappa}{4} \frac{\partial^2}{\partial \phi^2} + \sum_i x_i^2 \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i,j} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i x_i \frac{\partial}{\partial x_i} \right) + \sum_{i,j} x_i x_j - \frac{1}{2} \sum_i x_i^2, \]
which implies that
\[ \hat{W} = 2 \left( \sum_i x_i^2 \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i,j} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i x_i \frac{\partial}{\partial x_i} \right) + \sum_{i,j} x_i x_j - \frac{1}{2} \sum_i x_i^2 \] (58)
\[ E \left( \frac{\partial}{\partial \phi} \right) = -\frac{\kappa}{2} \frac{\partial^2}{\partial \phi^2}. \] (59)

1. **Solving the eigenequation (55) analytically**

Making use of (59), we get (55) in the form
\[ \left( i \frac{d}{d\phi} + \frac{\kappa}{2} \frac{d^2}{d\phi^2} \right) \omega_\lambda(\phi) = \lambda \omega_\lambda(\phi). \] (60)

The solution to (60), for \( \kappa \lambda \neq 1/2 \), is found to be
\[ \omega_\lambda(\phi) = e^{-i\kappa \phi} \left\{ A_\lambda \exp(\kappa \sqrt{2\kappa \lambda - 1} \phi) + B_\lambda \exp(-\kappa \sqrt{2\kappa \lambda - 1} \phi) \right\}, \] (61)
whereas for \( \kappa \lambda = 1/2 \) one has
\[ \omega_\lambda(\phi) = (A_\lambda \phi + B_\lambda) e^{-i\kappa \phi}, \] (62)
where \( A_\lambda \) and \( B_\lambda \) are arbitrary constants. In what follows we denote the solutions (61)–(62) as \( \omega_\lambda(A_\lambda, B_\lambda; \phi) \).
2. Solving the eigenequation (56) by variational method

The eigenequation (56) can be solved numerically in terms of given finite basis of functions \( \{ \psi_n \}_{n=0}^N \), by taking the solution \( \psi_\lambda \) in the form

\[
\psi_\lambda \simeq \psi_\lambda^N = \sum_{n=0}^N c_n \psi_n ,
\]

where \( c_n \) are unknown coefficients to be determined. The functions \( \psi_n \) should be consistent with the boundary conditions. It means, they should vanish sufficiently fast at zero and infinity to satisfy the condition

\[
\| \psi_\lambda \|_2 = \int_{-\infty}^{\infty} \frac{dx_1}{|x_1|} \int_{-\infty}^{\infty} \frac{dx_2}{|x_2|} \int_{-\infty}^{\infty} \frac{dx_3}{|x_3|} |\psi_\lambda|^2 < \infty .
\]

The coefficients \( c_n \) can be found by considering the following functional:

\[
R[\psi_\lambda] := \frac{\| \hat{W} \psi_\lambda - \lambda \psi_\lambda \|_2^2}{\| \psi_\lambda \|_2^2}.
\]

It is clear that (65) vanishes identically if \( \psi_\lambda^N \) is an exact solution to the equation (56). If this is not the case but \( R[\psi_\lambda] \ll 1 \), then we have an approximate solution. The smaller \( R[\psi_\lambda] \), the better the approximation. The latter fact suggests a method of finding thenumerical solution. Namely, one can minimize (65) with respect to all unknown coefficients, including the eigenvalue \( \lambda \). This fixes all the parameters in Eq. (63) and determines the error \( R[\psi_\lambda] \).

To start the procedure one should fix the basis \( \{ \psi_n \} \). It is reasonable to incorporate the fact that the operator \( \hat{W} \) is invariant under \( S_3 \) group of permutations of the variables \( \{ x_1, x_2, x_3 \} \). Therefore, looking for the basis it is reasonable to consider functions sharing this symmetry, i.e. requiring they are symmetric with respect to the replacements \( x_i \leftrightarrow x_j \). A convenient choice is provided by the following ansatz

\[
(\psi_S)_\alpha^N = |x_1 x_2 x_3|^{\alpha} \sum_{n_1+n_2+n_3 \leq N} \frac{c(n_1n_2n_3)}{(n_1+n_2+n_3)!} (\ln x_1^{n_1}) (\ln x_2^{n_2}) (\ln x_3^{n_3}) \cdot \exp \left( -\frac{1}{2}(\gamma + i\tilde{\gamma})(|x_1| + |x_2| + |x_3|) \right) ,
\]

where \( \alpha \geq \frac{1}{2} \), \( \gamma > 0 \), \( \tilde{\gamma} \in \mathbb{R} \), while \( \sum_{n_1+n_2+n_3 \leq N} \) stands for the sum over \( n_1, n_2, n_3 \in [0, N] \) such that \( n_1 + n_2 + n_3 \leq N \), i.e. the series (66) is terminated at the \( N \)-th order \( (N = n_1 + n_2 + n_3) \). The bracket \( (n_1, n_2, n_3) \) denotes ordering operation, e.g.
\[ c_{(023)} = c_{023}; \quad c_{(203)} = c_{023}, \text{ etc.} \]

The operation guaranties that the function \( \psi^N_\lambda \) consists of symmetric terms with respect to the replacement \( x_i \leftrightarrow x_j \). For instance, there are two second-order \((N = 2)\) terms in \((66)\):

\[
\frac{1}{2}c_{011} (\ln x_1^2 \ln x_2^2 + \ln x_1^2 \ln x_3^2 + \ln x_2^2 \ln x_3^2)
\]

and

\[
\frac{1}{2}c_{002} ((\ln x_1^2)^2 + (\ln x_2^2)^2 + (\ln x_3^2)^2).
\]

The additional weights \(1/(n_1 + n_2 + n_3)!\) are introduced for technical simplicity. They guarantee that the coefficients \(c_{n_1n_2n_3}\), fixed by the minimization procedure, are of a similar order. The latter improve the minimization. Note that the number of \(c_{n_1n_2n_3}\) grows fast with order \(N\).

Taking \(\alpha = \frac{1}{2}\) and \(N = 10\), one finds a solution \(\psi_{\lambda_1} = (\psi_S)^{10}_{1/2}\) specified by numerical parameters \((\lambda, \gamma, \tilde{\gamma}, R[\psi_{\lambda_1}]) = (\lambda_1, \gamma_1, \tilde{\gamma}_1, R_1)\), where

\[
\lambda_1 \simeq -0.0821, \quad \gamma_1 \simeq 1.229, \quad \tilde{\gamma}_1 \simeq -4.05 \times 10^{-4} \quad R_1 \simeq 0.0172,
\]

while the coefficients \(c_{n_1n_2n_3}\) are given explicitly in Appendix A. The choice \(\alpha = \frac{1}{2}\) leads to the the smallest global error \(R_1\). The point-like precision defined as

\[
E[\psi_\lambda] := \sup_{x \in \mathbb{R}^3} (|\hat{W}\psi_\lambda - \lambda \psi_\lambda|)
\]

gives \(E[\psi_{\lambda_1}] \simeq 0.0028\). In \((68)\) \(\psi_\lambda\) stands for a normalized function.

For the second numerical solution we take the ansatz:

\[
(\psi_{\lambda})^N_\alpha = (\text{sign}(x_1) + \text{sign}(x_2) + \text{sign}(x_3)) (\psi_S)^N_\alpha.
\]

As the solution is antisymmetric, it is orthogonal to the previous one, i.e. \(\langle (\psi_{\lambda})^N_\alpha \rangle (\psi_S)^N_\alpha \rangle = 0\). Taking as before \(\alpha = \frac{1}{2}\) and \(N = 10\), we get the function \(\psi_{\lambda_2} = (\psi_A)^{10}_{1/2}\). Applying our method of fixing the coefficients in the ansatz leads to

\[
\lambda_2 \simeq -0.0957, \quad \gamma_2 \simeq 1.369, \quad \tilde{\gamma}_2 \simeq 4.46 \times 10^{-3}, \quad R_2 \simeq 0.0218.
\]

The coefficients \(c_{n_1n_2n_3}\) are listed in Appendix A. As in the case of \(\psi_{\lambda_1}\) the point-like precision \((68)\) gives \(E[\psi_{\lambda_2}] \simeq 0.0058\).

3. **Solving the eigenequation (56) by spectral method**

We start with the ansatz

\[
\psi = |x_1x_2x_3|^\alpha f(x_1, x_2, x_3) \exp \left( -\frac{\gamma}{2}(|x_1| + |x_2| + |x_3|) \right),
\]

where \(\gamma > 0, \alpha \geq 1/2\). The eigenequation \((56)\) can be rewritten as

\[
\hat{W}\psi - \lambda \psi = |x_1x_2x_3|^\alpha \left( \hat{F}_\alpha \gamma (f(x_1, x_2, x_3)) - \lambda f(x_1, x_2, x_3) \right) \exp \left( -\frac{\gamma}{2}(|x_1| + |x_2| + |x_3|) \right),
\]

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where

\[
\hat{F}_{\alpha\gamma} = 2 \sum_{I} x_I^2 \frac{\partial^2}{\partial x_I^2} - 4 \sum_{I<J} x_I x_J \frac{\partial^2}{\partial x_I \partial x_J} \\
+ 2 \sum_{I} (1 - 2\alpha + \gamma(|x_1| + |x_2| + |x_3| - 2|x_I|)) x_I \frac{\partial}{\partial x_I} \\
+ \sum_{I} \left(\frac{\gamma^2 - 1}{2} x_I^2 + \gamma(2\alpha - 1)|x_I|\right) + \sum_{I<J} (x_I x_J + \gamma^2 |x_I||x_J|) - 6\alpha^2.
\]

(73)

With the ansatz (71), the problem of solving the eigenequation (56) reduces to the problem of solving the corresponding eigenequation for \( \hat{F} \) operator, i.e.

\[
\hat{F}_{\alpha\gamma} f(x_1, x_2, x_3) = \lambda f(x_1, x_2, x_3).
\]

(74)

We solve Eq. (74) by using the spectral methods [13]. This form is much more convenient from numerical point of view because of the lack of two terms \(|x_1 x_2 x_3|^\alpha \) and \(\exp(-\gamma/2(|x_1| + |x_2| + |x_3|))\), causing additional numerical errors\(^3\).

It is convenient, for numerical treatment, to assume

\[
f(x_1, x_2, x_3) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \sum_{n_3=1}^{N} c_{n_1, n_2, n_3} f_{n_1 n_2 n_3}(x_1, x_2, x_3),
\]

(75)

where \(N > 1\) is the cut-off, while \(f_{n_1 n_2 n_3}\) stand for a fixed basis of functions. The standard procedure involves cosine function, however, it will be convenient to adopt a different choice. The solution to the eigenequation (74) is specified by fixing the unknown coefficients \(c_{n_1, n_2, n_3}\). They can be determined demanding the eigenequation to be satisfied at a lattice composed of fixed points. To illustrate this, let us restrict for simplicity to the one-dimensional case, rewriting the ansatz (75) as

\[
f(x) = \sum_{n=1}^{N} c_n f_n(x).
\]

(76)

The eigenequation reads

\[
\hat{F}_{\alpha\gamma} f(x) = \lambda f(x).
\]

(77)

\(^3\) More precisely, within the spectral method, these terms result in combination of small and large numbers (components of the matrix representing an approximate form of the eigenequation at a lattice).
Let \( \{x_n\}_{n=1}^N \) stands for the lattice. For instance, one could consider Chebyshev’s nodes. These are defined as roots of the Chebyshev polynomial of the first kind of the degree \( n \) [13]. On the finite interval \([-1, 1]\), they read

\[
x_n = \cos\left(\frac{2n - 1}{2N} \pi\right), \quad n = 1, \ldots, N.
\]

This can be extended on \([a, b]\) defining

\[
x_n = \frac{a + b}{2} + \frac{b - a}{2} \cos\left(\frac{2n - 1}{2N} \pi\right), \quad n = 1, \ldots, N.
\]

Here, it is important that the number of points should match the number of coefficients \( c_n \). At the lattice Eq. (77) can be rewritten as

\[
F^{\alpha \gamma}_{nm} c^m = \lambda f_{nm} c^m,
\]

where \( \vec{c} = (c^n) = \{c_1, \ldots, c_N\} \) is a vector built out of unknown coefficients, while \((f_{nm})\) and \((F^{\alpha \gamma}_{nm})\) stand for \(N \times N\) matrices defined as

\[
f_{nm} := f_m(x_n), \quad F^{\alpha \gamma}_{nm} := \hat{F}_{\alpha \gamma} f_m(x)|_{x=x_n}.
\]

Solving Eq. (80) one guarantees the combination (76) satisfies the eigenequation (77) at \( x = x_n, \ n = 1, \ldots, N \). Eq. (80) is a generalized eigenequation: having specified matrices \((f_{nm})\) and \((F^{\alpha \gamma}_{nm})\) one obtains unknown coefficients \( c_n \) and the eigenvalue \( \lambda \) solving algebraic eigenequation (80). The coefficients specify approximate solution of the differential equation. The denser the grid \( \{x_n\} \), the better the precision.

Now, we consider 3-dimensional case. The first element of the construction is the functional basis \( f_{n_1 n_2 n_3}(x_1, x_2, x_3) \). It turns out, a convenient choice is the basis

\[
f_{n_1 n_2 n_3}(x_1, x_2, x_3) = \sin\left(1 + \frac{\ln |x_1|}{n_1} + \frac{\ln |x_2|}{n_2} + \frac{\ln |x_3|}{n_3}\right).
\]

We can now search for solutions to the eigenequation (74). Finding the numerical solution \( f_{n_1 n_2 n_3} \) of Eq. (74), one finds the solution to the eigenequation (56), given by Eq. (71). In order to do so, consider a tree-dimensional grid \( \{x_n\} = \{x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\} \). As we are interested in covering both positive and negative \( x_i \in \mathbb{R} \) in (82), it is reasonable to allow negative \( n_i \) (we exclude \( n_i = 0 \) because of the form of the right hand side of Eq. (82)). The sum (76) becomes \( \sum_{n=-N}^{-1} c_n f_n(x) + \sum_{n=1}^{N} c_n f_n(x) \). Due to the presence of logarithmic function in Eq. (82), this choice should respect the fact that terms in Eq. (82) become highly oscillating in the limit \( x_i \to 0 \). Hence, it
is reasonable to make the grid denser close to zero. However, this is not the case for the original Chebyshev’s nodes (79). This can be achieved adopting the following, modified Chebyshev’s nodes:

\[ x_n = b_\pm \left(1 + \cos \left(\frac{2n - 1}{4N} \pi\right)\right), \quad n = N + 1, \ldots, 2N, \quad (83) \]

where \( b_\pm \) stands for two real parameters, positive \( b_+ \) and negative \( b_- \). They provide respectively positive and negative nodes. Clearly, this holds for all three dimensions. Because of terms \(|x_1x_2x_3|^\alpha \exp(-\gamma/2(|x_1| + |x_2| + |x_3|))\), the function (71) vanishes close to zero, \(|x_f| \ll 1\), and close to infinity, \(|x_f| \gg 1\). Therefore, one gets a good approximation restricting to a relatively small finite number of nodes. This justifies the choice (83). Having specified the grid and functional basis, we are ready to find the solutions. Choosing

\[ b_- = -3, \quad b_+ = 3.5, \quad N = 5, \quad (84) \]

and adopting the basis (82) with

\[ \alpha = \frac{1}{2}, \quad \gamma = 1, \quad (85) \]

we get approximate solutions with discrete spectrum of positive and negative eigenvalues. Restricting to negative values and starting with the highest \( \lambda \), one finds:

\[ \lambda \simeq -2.193, -2.193, -6.470, -6.470, -14.34, -17.71, -27.34, -27.34, \]
\[ -32.05, -36.26, -39.65, -47.50, -47.50, -62.62, -62.62, -63.83, \]
\[ -74.97, -102.0, -110.7, -125.1, -125.1, -125.7, -155.9, -249.6, \]
\[ -249.6, -325.3, -325.3, -358.0, -358.0 \quad (86) \]

For instance, choosing the third eigenvalue one finds the solution \( \psi_s \), and the corresponding numerical error given by Eq. (68):

\[ \psi_s : \quad \lambda_s = -6.470, \quad E[\psi_s] \simeq 2.86 \times 10^{-6}. \quad (87) \]

Here \( \psi_s \) stands for a normalized function found by renormalization of the numerical solution \( \psi_s \to \psi_s/\|\psi_s\|^{1/2} \). We have obtained a fairly good precision despite considering a small grid. More precisely, taking \( N = 5 \) means we adopted ten points per each dimension (the whole three dimensional lattice is composed of 1000 points).

\[^4\text{It turns out that taking } |b_-| \neq |b_+| \text{ significantly improves numerical precision.}\]
The rationale for this are the following. First, the functions (82) provides a good basis in the sense that a combination involving small number of terms results in good approximation to the solution of the eigenequation (in the sense the numerical error turns out to be small). This is because of the presence of logarithmic function; something that has already been observed discussing variational method. Second, the eigenfunction vanishes fast for \( |x_I| \gg 1 \), and so we can restrict our analysis to covering a small, finite region \( x_I \in [b_-, b_+] \).

In addition to the symmetric function (82), one can consider the antisymmetric one, adopting the basis

\[
f_{n_1n_2n_3}(x_1, x_2, x_3) = (\text{sign}(x_1) + \text{sign}(x_2) + \text{sign}(x_3)) \cdot 
\sin \left( 1 + \frac{\ln |x_1|}{n_1} + \frac{\ln |x_2|}{n_2} + \frac{\ln |x_3|}{n_3} \right). \tag{88}
\]

Adopting the choices (84)–(85) leads to the following spectrum of negative eigenvalues:

\[
\lambda \simeq -2.193, -2.193, -6.470, -6.470, -14.34, -17.71, -32.05, -36.26, \\
-39.65, -47.50, -47.50, -63.83, -74.97, -96.98, -96.98, -102.0, \\
-110.7, -125.7, -155.9, -249.6, -249.6, -325.3, -325.3, -358.0, -358.0. \tag{89}
\]

Choosing, for instance, the first eigenvalue, one finds the antisymmetric solution \( \psi_a \) and the corresponding error (68):

\[
\psi_a : \quad \lambda_a = -2.193, \quad E[\psi_a] \simeq 2.68 \times 10^{-6}. \tag{90}
\]

The functions \( \psi_s \) and \( \psi_s \) are orthogonal and they both were constructed as normalized.

C. Imposition of the dynamical constraint

Eq. (52) is the Schrödinger-like equation corresponding to the classical dynamics defined by Eqs. (8)–(9). However, the latter is constrained by the condition \( H = 0 \), with \( H \) given by (7). The Dirac quantization scheme applied in this paper consists in mapping the classical constraint to the quantum constraint \( \hat{H} = 0 \), which according to Eq. (57) reads:

\[
\hat{H}\Psi(\phi, x) := \left[ E \left( \frac{\partial}{\partial \phi} \right) + \hat{W} \right] \Psi(\phi, x) = 0. \tag{91}
\]
Therefore, not all solutions to (52) are physical but only the ones satisfying (91). It turns out, however, that the solution to (91), in the form (53) with \( \omega_\lambda \) defined by (61)–(62), can only be the trivial one \( \Psi(x) = 0 \). To address this difficulty, we propose to impose, instead of (91), the weak form of the Dirac condition:

\[
\langle \Psi | \hat{H} | \Psi \rangle =: \langle \hat{H} \rangle_\Psi = 0 , \tag{92}
\]

which has to be satisfied by a given linear combination of eigenfunctions

\[
\Psi(\phi, x) = \sum_{\lambda} \omega_\lambda(A_\lambda, B_\lambda; \phi) \psi_\lambda(x) , \tag{93}
\]

where \( \omega_\lambda \) are defined by (61)–(62) (up to arbitrary constants \( A_\lambda \) and \( B_\lambda \)), and \( \psi_\lambda \) is determined numerically via (63) and (71). The symbol \( \sum_{\lambda} \) denotes summation or integration depending on the solutions to the eigenequations (55)–(56).

Since \( \hat{W} \) is a Hermitian operator, we have

\[
\langle \psi_\lambda' | \hat{W} | \psi_\lambda \rangle = \lambda \delta(\lambda', \lambda) , \quad \lambda \in \mathbb{R} ,
\]

so Eq. (92) takes the form

\[
\langle \hat{H} \rangle_\Psi = \sum_{\lambda} \omega_\lambda^*(A_\lambda, B_\lambda; \phi) \left( \lambda - \frac{\kappa}{2} \frac{d^2}{d\phi^2} \right) \omega_\lambda(A_\lambda, B_\lambda; \phi) = 0 . \tag{94}
\]

For the case \( \kappa \lambda \neq 1/2 \), Eq. (94) leads to

\[
\sum_{\lambda} \omega_\lambda^*(A_\lambda, B_\lambda; \phi) \left( \omega_\lambda(A_\lambda, B_\lambda; \phi) + i \sqrt{2\kappa\lambda - 1} \omega_\lambda(A_\lambda, -B_\lambda; \phi) \right) = 0 , \tag{95}
\]

whereas for the case \( \kappa \lambda = 1/2 \) we get

\[
\omega_\lambda^*(A_\lambda, B_\lambda; \phi) \omega_\lambda(\kappa A_\lambda, \kappa B_\lambda + i A_\lambda; \phi) = 0 . \tag{96}
\]

In what follows we consider \( \kappa = 1 \) and \( \lambda < 1/2 \), in which case \( 2\kappa\lambda - 1 < 0 \) so that Eq. (61) presents an oscillatory solution.

For \( \lambda < 0 \), one has

\[
1 + \sqrt{|2\lambda - 1|} > 0 \quad \text{and} \quad 1 - \sqrt{|2\lambda - 1|} < 0 . \tag{97}
\]

Eq. (95) leads to the condition

\[
\sum_{\lambda} \left[ (1 - \sqrt{|2\lambda - 1|}) |A_\lambda|^2 + (1 + \sqrt{|2\lambda - 1|}) |B_\lambda|^2 \right] = 0 , \tag{98}
\]
where \( A_\lambda B_\lambda = 0 \). Assuming the orthonormality condition \( \langle \psi_\lambda | \psi_\lambda' \rangle = \delta(\lambda', \lambda) \), we get
\[
\sum_\lambda \left| \omega_\lambda(A_\lambda, B_\lambda; \phi) \right|^2 = 1.
\] (99)

Eqs. (97)–(99) lead to the condition
\[
\sum_{\lambda \in \mathcal{O}_1} |A_\lambda|^2 + \sum_{\lambda \in \mathcal{O}_2} |B_\lambda|^2 = 1, \quad \text{where} \quad \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset.
\] (100)

Let us consider a special solution including only two eigenvalues \( \lambda \neq \lambda' \). In such a case Eqs. (98)–(100) give
\[
\left( 1 - \sqrt{2\lambda - 1} \right) |A_\lambda|^2 + \left( 1 + \sqrt{2\lambda' - 1} \right) |B_{\lambda'}|^2 = 0,
\] (101)
and
\[
|A_\lambda|^2 + |B_{\lambda'}|^2 = 1.
\] (102)

The solution to (101)–(102) reads
\[
|A_\lambda|^2 = \frac{1 + \sqrt{2\lambda' - 1}}{\sqrt{2\lambda - 1} + \sqrt{2\lambda' - 1}}, \quad |B_{\lambda'}|^2 = \frac{\sqrt{2\lambda - 1} - 1}{\sqrt{2\lambda - 1} + \sqrt{2\lambda' - 1}}.
\] (103)

Therefore, one of the possible solutions to the constraint (92) has the form
\[
\Psi(\phi, x) = \omega_\lambda(A_\lambda, B_\lambda; \phi) \psi_\lambda(x) + \omega_{\lambda'}(A_{\lambda'}, B_{\lambda'}; \phi) \psi_{\lambda'}(x),
\] (104)
which is defined by the specification of any pair of \( \lambda \neq \lambda' \).

**IV. QUANTUM SPIKES**

The expectation values of our basic observables read
\[
\langle \hat{C}_I \rangle(\phi) = \int_{\mathbb{R}^3_+} d\nu(x_1, x_2, x_3) \Psi^*(\phi, x_1, x_2, x_3) \hat{C}_I \Psi(\phi, x_1, x_2, x_3)
\]
\[
= \sum_{\lambda_1, \lambda_2} \omega_{\lambda_1}^*(A_{\lambda_1}, B_{\lambda_1}; \phi) \omega_{\lambda_2}(A_{\lambda_2}, B_{\lambda_2}; \phi) \langle \psi_{\lambda_1} | \hat{C}_I | \psi_{\lambda_2} \rangle,
\] (105)
and similarly
\[
\langle \hat{D}_I \rangle(\phi) = \sum_{\lambda_1, \lambda_2} \omega_{\lambda_1}^*(A_{\lambda_1}, B_{\lambda_1}; \phi) \omega_{\lambda_2}(A_{\lambda_2}, B_{\lambda_2}; \phi) \langle \psi_{\lambda_1} | \hat{D}_I | \psi_{\lambda_2} \rangle,
\] (106)
where $\Psi$ is defined by (92)--(93), and the observables are specified by Eqs. (31). The coefficients $A_\lambda$ and $B_\lambda$, which are present in Eq. (105)--(106), can be fixed by imposing the initial conditions:

$$
\langle \hat{C}_1 \rangle (\phi_0) = \bar{x}_{C_1},
$$

$$
\langle \hat{D}_1 \rangle (\phi_0) = \bar{x}_{D_1},
$$

where $\bar{x}_{C_1}, \bar{x}_{D_1} \in \mathbb{R}$. The value $\phi_0$ is arbitrary. However, keeping in mind classical results, we take $\phi_0 = 0$.

A. Using the results of the variational method

Using the numerical results (66)--(70), we get

$$
(C_1)_{ij} = \begin{pmatrix}
0 & c_0 + i\delta c \\
c_0 - i\delta c & 0
\end{pmatrix}
$$

where $(C_1)_{ij} := \langle \psi_\lambda | \hat{C}_1 | \psi_\lambda \rangle$, and where

$$
c_0 \simeq -0.0174, \quad \delta c \simeq -2.21 \times 10^{-4}.
$$

Denote for convenience $\bar{x} := \bar{x}_{C_1}$. Recalling the condition $A_\lambda B_\lambda = 0$ we assume $A_{\lambda_1} = B_{\lambda_2} = 0$. Hence, there are two independent complex parameters left, $A_{\lambda_1}$ and $B_{\lambda_2}$. Let

$$
A_{\lambda_1} = |A_{\lambda_1}| e^{i\phi_1}, \quad B_{\lambda_2} = |B_{\lambda_2}| e^{i\phi_2}.
$$

The absolute values $|A_{\lambda_1}|, |B_{\lambda_2}|$ are fixed by Eqs. (103):

$$
|A_{\lambda_1}| = \sqrt{\frac{1 + \sqrt{|2\lambda_1 - 1|}}{\sqrt{2\lambda_1 - 1} + \sqrt{2\lambda_2 - 1}}}, \quad |B_{\lambda_2}| = \sqrt{\frac{\sqrt{|2\lambda_1 - 1|} - 1}{\sqrt{2\lambda_1 - 1} + \sqrt{2\lambda_2 - 1}}},
$$

where $\lambda_1 = \lambda \simeq -0.0821$, $\lambda_2 = \lambda' \simeq -0.0957$. The numerical values come from numerical solutions (67)-(70). The function $\Psi(\phi, x)$ is given by Eq. (104). As discussed, we are interested in oscillatory solutions, letting $\kappa = 1$ in Eq. (61). Plugging $A_{\lambda_2} = B_{\lambda_1} = 0$ and $\kappa = 1$ into Eq. (61) gives

$$
\omega_{\lambda_1} = A_{\lambda_1} e^{i\sqrt{|2\lambda_1 - 1|} \phi}, \quad \omega_{\lambda_2} = B_{\lambda_2} e^{-i\sqrt{|2\lambda_2 - 1|} \phi}.
$$

These are complex amplitudes. The final form of the wave function reads

$$
\Psi(\phi, x) = \omega_{\lambda_1} \psi_{\lambda_1}(x) + \omega_{\lambda_2} \psi_{\lambda_2}(x).
$$
Inserting (114) into Eq. (105) one finds

$$\langle \hat{C}_1 \rangle (\phi) = \beta \cos(\Delta \varphi + \chi \phi) + \delta \beta \sin(\Delta \varphi + \chi \phi),$$

(115)

where

$$\Delta \varphi = \varphi_1 - \varphi_2,$$

(116)

$$\beta := 2c_0|A_{\lambda_1}| |A_{\lambda_2}|,$$

(117)

$$\delta \beta = 2\delta c |A_{\lambda_1}| |A_{\lambda_2}|,$$

(118)

$$\chi := \sqrt{|2\lambda_1 - 1|} + \sqrt{|2\lambda_2 - 1|}.$$ 

(119)

Here $\beta$ and $\gamma$ are number coefficients. Recalling numerical solutions (66)–(70) gives

$$\beta \simeq -0.0065, \quad \delta \beta \simeq -8.3 \times 10^{-6}, \quad \chi \simeq 2.17.$$ 

(119)

The parameter $\Delta \varphi$ in Eq. (115) can be eliminated imposing the boundary condition (107). Combining with Eq. (115) the condition reads

$$\beta \cos(\Delta \varphi + \chi \phi) + \delta \beta \sin(\Delta \varphi + \chi \phi) = \tilde{x}.$$ 

(120)

Solving Eq. (120) one expresses $\Delta \varphi$ as a function of $\tilde{x}$. There are two different solutions $\Delta \varphi^{(\pm)} = \Delta \varphi^{(\pm)}(\tilde{x})$:

$$\Delta \varphi^{(\pm)} = \text{atan2} \left( \frac{\beta \tilde{x} \mp |\delta \beta| \sqrt{\beta^2 + \delta \beta^2 - \tilde{x}^2}}{\beta^2 + \delta \beta^2}, \frac{\delta \beta \tilde{x} \pm |\beta| \text{sign}(\delta \beta) \sqrt{\beta^2 + \delta \beta^2 - \tilde{x}^2}}{\beta^2 + \delta \beta^2} \right) + 2n\pi, \quad n \in \mathbb{Z},$$ 

(121)

where $\text{atan2}(.,.)$ stands for two-argument arctangent function. Substituting $\Delta \varphi^{(\pm)}$ given by Eq. (121) into Eq. (115) gives

$$\langle \hat{C}_1 \rangle^{(\pm)} = \tilde{x} \cos(\chi \phi) \mp \text{sign}(\delta \beta) \sin(\chi \phi) \sqrt{\beta^2 + \delta \beta^2 - \tilde{x}^2}.$$ 

(122)

Searching for quantum spikes, we now examine $\langle \hat{C}_1 \rangle$ as function of $\tilde{x}$ for a fixed $\phi$. For $\phi = 0$ the dependence is trivial. For $\phi \neq 0$ the domain is restricted to the interval $\tilde{x} \in [-\sqrt{\beta^2 + \delta \beta^2}, \sqrt{\beta^2 + \delta \beta^2}]$ and the function $\langle \hat{C}_1(\tilde{x}) \rangle$ has a non-trivial derivative:

$$\frac{d\langle \hat{C}_1 \rangle^{(\pm)}}{d\tilde{x}} = \cos(\chi \phi) \pm \frac{\text{sign}(\delta \beta) \sin(\chi \phi) \tilde{x}}{\sqrt{\beta^2 + \delta \beta^2 - \tilde{x}^2}}.$$ 

(123)

At $\tilde{x} = 0$ one gets

$$\left. \frac{d\langle \hat{C}_1 \rangle^{(\pm)}}{d\tilde{x}} \right|_{\tilde{x}=0} = \cos(\chi \phi).$$ 

(124)
The latter vanishes if
\[ \chi \phi = \frac{\pi}{2} + k\pi. \] (125)

The second derivative of Eq. (122) reads
\[
\frac{d^2 \langle \hat{C}_1 (\pm) \rangle}{d\tilde{x}^2} = \pm \frac{(\beta^2 + \delta \beta^2)}{(\beta^2 + \delta \beta^2 - \tilde{x}^2)^{3/2}} \text{sign}(\delta \beta) \sin(\chi \phi)
\] (126)

Eqs. (124)–(126) show that, depending on the values \( \chi \phi \) and \( \delta \beta \), \( \langle \hat{C}_1 (\pm) \rangle \) reach a local maximum or minimum at \( \tilde{x} = 0 \). In particular, taking \( k = 0 \) in (125) one finds this would be the case for \( \phi \approx 0.72 \). This is a quantum analogue of classical spikes. In general, quantum spikes only occur at the specific single moments in time \( \phi \), determined by Eq. (125). They repeat again with the period \( \Delta \phi = \pi/\chi \).

The dependence \( \langle \hat{C}_1 (\pm) \rangle \) is presented in Fig. 4. In both cases spikes corresponding to \( \phi \approx 0.72 \) are represented by solid lines.

Figure 4. \( \langle \hat{C}_1 (\pm) \rangle \) (left) and \( \langle \hat{C}_1 (\pm) \rangle \) (right) for evolution parameters \( \phi = 0.2 \) (red), \( \phi = 0.4 \) (green), \( \phi = 0.72 \) (blue), \( \phi = 1 \) (purple). The same results are for \( \langle \hat{C}_{2,3} (\pm) \rangle \).

Having specified \( \langle \hat{C}_1 \rangle \), we can repeat the analysis for \( \langle \hat{D}_1 \rangle \). Defining \( (D_1)_{ij} := \langle \psi_{\lambda_i} | \hat{D}_1 | \psi_{\lambda_j} \rangle \), one finds
\[
(D_1)_{ij} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}
\] (127)

where
\[
d_1 \approx -5.71 \times 10^{-6}, \quad d_2 \approx 7.35 \times 10^{-5}.
\] (128)
This leads to
\[
\langle \hat{D}_1 \rangle = \frac{d_1 - d_2 + d_1 \sqrt{|1 - 2\lambda_2|} + d_2 \sqrt{|1 - 2\lambda_1|}}{\sqrt{|1 - 2\lambda_1|} + \sqrt{|1 - 2\lambda_2|}}
\approx -2.83 \times 10^{-6}.
\]  
(129)

Here \( \langle \hat{D}_1 \rangle = \text{const} \) because there are no off-diagonal components in the matrix (127). More precisely, the off-diagonal components are non-zero, but are small of order \( 10^{-20} \). This result is almost unaffected by the change of the order \( N \), however, off-diagonal components are becoming smaller with growing \( N \). In conclusion, there is no evolution for \( \langle \hat{D}_1 \rangle \).

It is worth underlying that, at least in the case of restricting to the superposition of two eigenstates (104), the presence of the spike-like structure, related either to \( \hat{C}_1 \) or \( \hat{D}_1 \), is unaffected by the choice of a pair of numerical solutions composed of one symmetric and one antisymmetric functions. Adopting different ones will modify the values of the coefficients \( \beta, \delta\beta, \chi \). Still, one could find the value of \( \phi \) corresponding to the quantum spike. For \( \hat{C}_1 \) operator the latter is given by Eq. (125); this is a simple function of two eigenvalues \( \lambda_1, \lambda_2 \). In fact, the crucial requirement making spikes to be present is the presence of non-zero components of the matrixes \( (C_1)_{ij} \) or \( (D_1)_{ij} \).

B. Using the results of the spectral method

We can now repeat the procedure starting with numerical solutions based on spectral method. Taking \( \psi_1 = \psi_s \) and \( \lambda_1 = \lambda_s \) as given by Eq. (87), and \( \psi_2 = \psi_a \) corresponding to \( \lambda_2 = \lambda_a \) as given by Eq. (90), gives
\[
(C_1)_{ij} = \begin{pmatrix} 0 & c_0 \\ c_0 & 0 \end{pmatrix},
\]  
(130)
where \( c_0 = -1.28 \times 10^{-5} \) and \( (D_1)_{ij} = 0 \). The matrices \( (C_1)_{ij} \) and \( (D_1)_{ij} \) are defined, respectively, by Eq. (109) and Eq. (127). There is no contribution to \( (D_1)_{ij} \) because the numerical solutions are real. On the other hand, now we get much better precision.

Following the steps described above, one can find expectation value \( \langle \hat{C}_1 \rangle \) as function of \( \tilde{x} \). Since there is no imaginary term in Eq (130), \( \delta c = 0 \). The expectation

\[5\] For instance, for \( N = 8 \) one finds them to be equal \( \approx 7.7 \times 10^{-20} \), while for \( N = 10 \) one gets \( \approx 2.2 \times 10^{-20} \).
value $\langle \hat{C}_1 \rangle$ simplifies becoming
\begin{equation}
\langle \hat{C}_1 \rangle (\phi) = \beta \cos(\Delta \varphi + \chi \phi),
\end{equation}
where $\Delta \varphi$, $\beta$ and $\chi$ are given by Eqs. (116)–(118). Supplementing by numerical values one finds
\begin{equation}
\beta \simeq 1.28 \times 10^{-5}, \quad \chi \simeq 6.05.
\end{equation}
As in the case of Eq. (115), the parameter $\Delta \varphi$ can be eliminated imposing the boundary condition (107), which now reads
\begin{equation}
\beta \cos(\Delta \varphi + \chi \phi) = \tilde{x}.
\end{equation}
Solving Eq. (133) with respect to $\Delta \varphi$ gives
\begin{equation}
\Delta \varphi = \pm \arccos \left( \frac{\tilde{x}}{\beta} \right) + 2n\pi.
\end{equation}
Substituting $\Delta \varphi$ given by Eq. (134) into Eq. (131) gives
\begin{equation}
\langle \hat{C}_1 \rangle (\phi) = \tilde{x} \cos(\chi \phi) \mp \beta \sqrt{1 - \frac{\tilde{x}^2}{\beta^2}} \sin(\chi \phi),
\end{equation}
where $\langle \hat{C}_1 \rangle = \langle \hat{C}_1 \rangle (\phi)$. As in case of Eq. (126), there are two solutions. Both they are depicted in Fig. 5. In both cases quantum spikes correspond to $\phi \simeq 0.26$; they are represented by solid lines. The resulting spikes are periodic in time $\phi$ with a period
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{\(\langle \hat{C}_1 \rangle (\phi)\) (left) and \(\langle \hat{C}_2 \rangle (\phi)\) (right) for evolution parameters $\phi = 0.2$ (red), $\phi = 0.26$ (green), $\phi = 0.4$ (blue), $\phi = 1$ (purple). The same results are for $\langle \hat{C}_2,3 \rangle (\phi)$.}
\end{figure}
\begin{equation}
\Delta \phi = \pi / \chi \simeq 0.52.
\end{equation}
As in the case of the variational method, the crucial requirement
guarantying spikes to be present is the presence of non-zero components of the matrix $C_{iji}$. Starting with different pair of symmetric and antisymmetric functions $\psi_s$ and $\psi_a$, corresponding to different eigenvalues $\lambda_1$ and $\lambda_2$, one also expects to get spikes (unless $C_{iji} = 0$).

V. CONCLUSIONS

We define quantum spikes to be distinctive structures, which occur in quantum propagation of given system. They may correspond to steep structures, called classical spikes, that appear in classical evolution of corresponding system. The local structures like the extremum (maximum, minimum) or inflection point of the classical level may turn into similar structures at the quantum level. Quantization does not need to suppress the classical spikes, as it was conjectured in the article [5], but may turns them into some quantum structures.

In our case, the quantum spike turns out to be the extremum in the propagation of the expectation value of considered quantum observable. The inflection point like classical spike of Fig. 1 becomes extremum like quantum spike of Fig. 4 or 5. Another difference is that our classical spike is a monotonic function of time, whereas the corresponding quantum spike is periodic. To be specific, we have restricted our analyses to just one pair of observables $(C_1, D_1)$ (the results for $D_1$ have insufficient accuracy), but one may try to make a generalization to more observables.

For simplicity, we have identified quantum spikes by making use of only two classes, i.e. obtained by two different numerical methods, of solutions to the quantum dynamics. Many other classes of solutions are possible so that other types of quantum spikes may be found. The quantum wave packet we have used consists of only two particular solutions of a given class but it could be extended to many solutions and might lead to a new type of quantum spikes.

We have used the massless scalar field to play the role of a clock both at the classical and quantum levels. This way we have avoided quantization of the scalar field. It decouples from the gravitational field in Eq. (7) so that such a treatment seems to be justified. In fact, this is our hypothesis we use to avoid the inconsistency that is commonly ignored in literature. It is related to the problem of time that occurs in the process of quantization of gravitational systems.

The implementation of the dynamical constraint at the quantum level has been done in a weak sense. Such a way of imposing constraints is practised in other branches of quantum physics and quantum chemistry, especially in variational methods (see, e.g. [14–16] and references therein).

The spikes seem to be of fundamental importance since they may be treated as seeds of real structures in our universe. It has been shown that at the classical
level they are seeds of inhomogeneities arising in the dynamics of the family of homogeneous spacetimes [4]. They may finally evolve into large-scale structures (i.e. (super)clusters of galaxies, in which stars are later born). At the quantum level, spikes indicate the oscillations of quantum field that may have something to do with the creation and annihilation of excitations of matter fields. Spikes may also initiate the creation of primordial black holes. The examination of the corresponding issues in the case of inhomogeneous spacetimes would be highly interesting as they favour naturally inhomogeneities, as has already been found (see, e.g. [6, 7] and references therein).

Our paper is about possible existence of quantum spikes. The theoretical framework has been established and more effort is needed to prove that quantum spikes are generic features of quantum gravitational systems. In fact, our preliminary results can be extended by further examination of the eigenequation problem (56). New classes of solutions to this problem may contribute to obtaining new types of quantum spikes. An extension of this project needs definitely making use of sophisticated analytical and numerical tools so that it is far from being completed. New activity in this context is expected.

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Appendix A: Numerical solutions

All coefficients have been found using Mathematica computer software.
The coefficients \( c_{n_1 n_2 n_3} \) of \( \psi_S \) corresponding to (66) read:

\[
c_{000} = 0.0000971398, c_{001} = -0.0000668377, c_{002} = 0.0000576278, \\
c_{011} = 0.000145685, c_{003} = -0.0000408203, c_{012} = -0.000175153, \\
c_{111} = -0.000093447, c_{004} = 5.14783 \times 10^{-6}, c_{013} = 0.000120379, \\
c_{022} = 3.38894 \times 10^{-6}, c_{112} = 0.000204403, c_{005} = 6.14349 \times 10^{-6}, \\
c_{014} = -0.0000357561, c_{023} = -0.0000709548, c_{113} = -0.000189175, \\
c_{122} = 0.0000118927, c_{006} = 0.0000130402, c_{015} = -7.68871 \times 10^{-6}, \\
c_{024} = -0.000037775, c_{033} = -0.0000601953, c_{114} = 0.0000173017, \\
c_{123} = -0.00011458, c_{222} = 0.0000716101, c_{007} = 9.86615 \times 10^{-6}, \\
c_{016} = 0.0000427615, c_{025} = -0.0000538152, c_{034} = 0.0000404286, \\
c_{115} = 0.000035388, c_{124} = 0.000202186, c_{133} = 0.0000730395, \\
c_{223} = 0.00011985, c_{008} = 3.4044 \times 10^{-6}, c_{017} = 0.0000249608, \\
c_{026} = -0.0000690364, c_{035} = -0.0000391754, c_{044} = -0.0000453004, \\
c_{116} = -0.000104671, c_{125} = -0.0000730649, c_{134} = -0.0000535717, \\
c_{224} = 0.0000549831, c_{233} = -0.0000756014, c_{009} = 5.96683 \times 10^{-7}, \\
c_{018} = 5.7956 \times 10^{-6}, c_{027} = -0.0000289886, c_{036} = -0.0000465147, \\
c_{045} = -0.0000499186, c_{117} = -0.0000613235, c_{126} = -0.000175011, \\
c_{135} = -0.000150828, c_{144} = -0.0000280041, c_{225} = -0.000230914, \\
c_{234} = 0.0000408805, c_{333} = 0.0000680184, c_{0010} = 4.36818 \times 10^{-8}, \\
c_{019} = 5.0905 \times 10^{-7}, c_{028} = -4.19215 \times 10^{-6}, c_{037} = -0.000128081, \\
c_{046} = -0.0000245689, c_{055} = -0.0000213292, c_{118} = -9.56085 \times 10^{-6}, \\
c_{127} = -0.0000502355, c_{136} = -0.000111775, c_{145} = -0.0000995701, \\
c_{226} = -0.000193417, c_{235} = -0.000350751, c_{244} = -0.000167525. \quad (A1)
\]

Similarly, the coefficients \( c_{n_1 n_2 n_3} \) of \( \psi_A \) corresponding to (70) read:
\[c_{000} = 0.0000154524, c_{001} = -7.25196 \times 10^{-7}, c_{002} = -6.31044 \times 10^{-6},\]
\[c_{011} = -1.84189 \times 10^{-6}, c_{003} = 7.81442 \times 10^{-6}, c_{012} = 4.7519 \times 10^{-6},\]
\[c_{111} = -6.32555 \times 10^{-6}, c_{004} = -6.01093 \times 10^{-6}, c_{013} = -0.000016969,\]
\[c_{022} = -6.35142 \times 10^{-6}, c_{112} = 8.55031 \times 10^{-6}, c_{005} = -7.06458 \times 10^{-7},\]
\[c_{014} = 0.000023153, c_{023} = -2.43363 \times 10^{-6}, c_{113} = -0.0000170932,\]
\[c_{122} = -0.0000151945, c_{006} = -5.85877 \times 10^{-6}, c_{015} = -6.02818 \times 10^{-6},\]
\[c_{024} = 2.59351 \times 10^{-6}, c_{033} = 7.14048 \times 10^{-6}, c_{114} = 0.0000165033,\]
\[c_{123} = 0.0000425176, c_{222} = 5.83009 \times 10^{-6}, c_{007} = -3.85805 \times 10^{-6},\]
\[c_{016} = -3.95188 \times 10^{-7}, c_{025} = -0.0000125786, c_{034} = -0.0000220554,\]
\[c_{115} = -0.0000163389, c_{124} = -0.0000196522, c_{133} = -0.0000324696,\]
\[c_{223} = -0.0000273951, c_{008} = -8.02081 \times 10^{-7}, c_{017} = 0.0000107185,\]
\[c_{026} = 1.27489 \times 10^{-6}, c_{035} = 9.31717 \times 10^{-6}, c_{044} = 5.99047 \times 10^{-6},\]
\[c_{116} = -5.39372 \times 10^{-7}, c_{125} = -1.92239 \times 10^{-6}, c_{134} = 0.0000483597,\]
\[c_{224} = 0.0000255206, c_{233} = 0.0000448492, c_{009} = -3.69269 \times 10^{-8},\]
\[c_{018} = 4.74694 \times 10^{-6}, c_{027} = 5.24472 \times 10^{-6}, c_{036} = 7.44385 \times 10^{-6},\]
\[c_{045} = 6.93398 \times 10^{-6}, c_{117} = 6.12731 \times 10^{-6}, c_{126} = 0.0000300494,\]
\[c_{135} = -0.0000468598, c_{144} = -0.0000161307, c_{225} = -0.0000284781,\]
\[c_{234} = -0.000107147, c_{333} = -0.0000231205, c_{0010} = 3.87598 \times 10^{-9},\]
\[c_{019} = 6.21427 \times 10^{-7}, c_{028} = 1.58561 \times 10^{-6}, c_{037} = 1.98172 \times 10^{-6},\]
\[c_{046} = 0.0000147037, c_{055} = -3.09361 \times 10^{-6}, c_{118} = 2.09284 \times 10^{-6},\]
\[c_{127} = 0.0000119165, c_{136} = 0.0000362559, c_{145} = 8.76622 \times 10^{-6},\]
\[c_{226} = 0.0000623312, c_{235} = 0.000147871, c_{244} = 0.000069584.\]
[1] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, “Oscillatory approach to a singular point in the relativistic cosmology”, Adv. Phys. 19, 525 (1970).
[2] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, “A general solution of the Einstein equations with a time singularity”, Adv. Phys. 31, 639 (1982).
[3] V. Belinski and M. Henneaux, The Cosmological Singularity (Cambridge University Press, Cambridge, 2017).
[4] A. Ashtekar, A. Henderson, and D. Sloan, Hamiltonian formulation of the Belinskii-Khalatnikov-Lifshitz conjecture, Phys. Rev. D 83, 084024 (2011).
[5] E. Czuchry, D. Garfinkle, J. R. Klauder and W. Piechocki, Do spikes persist in a quantum treatment of spacetime singularities?, Phys. Rev. D 95, 024014 (2017).
[6] W. C. Lim, L. Andersson, D. Garfinkle, and F. Pretorius, Spikes in the Mixmaster regime of G(2) cosmologies, Phys. Rev. D 79, 123526 (2009).
[7] W. C. Lim and A. A. Coley, General relativistic density perturbations, Class. Quant. Grav. 31, 015020 (2014).
[8] T. Hillen, A classification of spikes and plateaus, SIAM Review, 49, 35 (2007).
[9] B. Gutkin, G. B. Ermentrout, and M. Rudolph, Spike generating dynamics and the conditions for spike-time precision in cortical neurons, Journal of Computational Neuroscience 15, 91 (2003).
[10] W. Kühnel, Differential Geometry: Curves - Surfaces - Manifolds (American Mathematical Society, 2002), sec. edition.
[11] A. Wawrzyńczyk, Współczesna teoria funkcji specjalnych, (PWN, Warszawa, 1978).
[12] E. W. Aslaksen, J. R. Klauder, Unitary Representation of the Affine Group, J. Math. Phys. 9, 206 (1968).
[13] P. Grandclement and J. Novak, Spectral methods for numerical relativity, Living Rev. Rel. 12, 1 (2009).
[14] D. P. Chong and Y. Rasiel, Constrained-Variation Method in Molecular Quantum Mechanics. Comparison of Different Approaches, Journal of Chemical Physics 44 1819 (1966).
[15] A. V. Smilga, Lectures on Quantum Chromodynamics (World Scientific, 2001).
[16] H. J. Mang, B. Samadi, and P. Ring On the Solution of Constrained Hartree-Fock-Bogolyubov Equations, Z. Physik A 279, 325 (1976).
