Dropping Bodies

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Drop three bodies. Where can they go? When someone says they’ve dropped a cup, we imagine it falling down to the ground, attracted to the Earth by gravity. But please, in this thought experiment, take away the Earth. There’s no direction “down.” Our three bodies are alone in the universe, attracted only to each other.

Take each body to be a point mass. Dropping the bodies means letting them go from rest, subject only to the rules of Newtonian mechanics and the assumption that the only forces acting on a body are the gravitational inverse-square \(\frac{1}{r^2}\) pulls of the other two. Each body will then sweep out a plane curve, subject to the attractive pull of the other two moving bodies. Taken together, these three parameterized plane curves form a solution to the Newtonian three-body problem.

Henceforth, we will use the term “dropped solution” or “brake orbit” for any solution to the \(N\)-body problem in which all the bodies are instantaneously at rest at some instant. The adjective “brake” is meant as in “to stop a car.” One also finds the synonym “free-fall solution” for dropped solution in the literature.

Our initial question asks, “what do dropped solutions look like?” Below we have depicted a half-dozen answers. They vary widely depending on the starting triangle. We urge the reader to view some of the animations that can be found at [5] and [4]. In the sampled pictures and animations, the three masses are equal. (The answers seem to be prettier that way.) The solution depicted in Figure 1 was found by Lagrange. The other figures depict periodic solutions, for which the three bodies shuttle back and forth between two “brake triangles”—configurations of the three bodies at which they are instantaneously at rest, or “braked.” Either brake triangle can be supposed to be the initial configuration from which the bodies are dropped. Figures 2, 3, and 4 have been selected from a database of thirty collision-free equal-mass dropped solutions found by Xiaoming Li and Shijun Liao [9] and available to view at [10]. They are all collision-free. Figures 5 and 6 display two of the infinitely many periodic isosceles brake solutions found by Nai-Chia Chen [3], all of which suffer binary collisions and whose existence she established in her PhD thesis. Chen’s discoveries remind me of Paul Klee’s drawings. Her animations are as if Klee’s drawings had produced children with the mobiles of Alexander Calder.

The observer will see various symmetries within the orbits. For example, Figure 3 has a reflectional symmetry. If you watch the animation, you can see that reflectional symmetry become spatiotemporal. When we drop the bodies at time \(t = 0\) and if \(T\) is the period, then the other brake triangle arises at time \(T/2\). At time \(T/4\), all three bodies lie on the symmetry line of the figure. Why?

History

Lagrange showed that a dropped equilateral triangle remains equilateral until it has collapsed to a triple collision. This is not a surprise when the masses are equal, but his result holds for unequal masses. Five years before Lagrange, Euler uncovered a similar fact by placing three masses judiciously on a line: he could get their shape to remain the same on dropping them. It is easy to get the placement right for three equal masses: place one at the midpoint of the other two. The center of mass of the triple is then this midpoint, and the extremal two masses collapse symmetrically onto this midpoint. For general unequal masses, Euler needed to solve a quintic to get the mass placements right.

In 1893, a person named Meissel asserted, with little evidence, that if masses in the ratio of \(3 : 4 : 5\) are placed at the vertices of a Pythagorean \(3 : 4 : 5\) triangle, with mass 5 placed at the vertex opposite the side of length 5 and so on, and the bodies are dropped, then the resulting solution is periodic. Verifying Meissel’s assertion became a benchmark problem for numerical computation in celestial mechanics and was christened the “Pythagorean three-body problem” or “Burrau’s three-body problem.” Carl Burrau’s name [2] became attached to the problem after he took up Meissel’s challenge and published the inconclusive results of hand-cranked numerical integration for the problem in 1913. Let’s all raise our glasses to the long-suffering degree candidate Sigurd Kristensen, who, 109 years ago, did an essential part—“Ein wesentlicher Teil der Rechenarbeit”—of Burrau’s computational work (one guesses that he did it all) but whose name has faded from view, no coauthorship, only an honorable mention within the article. In 1967, the celestial mechanician Victor Szebehely and his team proved Meissel wrong using the new-fangled digital computers. They showed that the two smallest of the three masses eventually form a binary pair that escapes to infinity. Before the big escape, many interchanges and several near-collision incidents occur, incidents requiring Szebehely and his team to implement Levi-Civita’s regularization of binary collisions in order...
to get an accurate integration and proceed to follow the orbit to its end.¹

Wait! What is Levi-Civita doing here? Didn’t he work on connections and Riemannian geometry and write a book with his advisor Gregorio Ricci-Curbastro on tensor calculus? A few years after Levi-Civita uncovered the famous connection now bearing his name, he established a surprisingly simple change of variables that “regularizes” the binary collisions of the $N$-body problem. Gravitational forces blow up at collisions, making it seem impossible to continue solutions through collision. However, Levi-Civita’s change of both time and space variables renders the defining ordinary differential equations (ODEs) analytic near isolated binary collisions. Chen’s solutions have binary collisions, and to make sense of them, or the isosceles

¹Nowadays, high-order Taylor methods can re-create Szebehely’s discovery without requiring regularization. See [5] and the final section of this paper.
The three-body problem in general, requires a variant of Levi-Civita’s regularization.

Soon after he established that Meissel was wrong, Szebehely decided to tweak Meissel’s initial conditions and found, remarkably, near Meissel’s initial conditions a periodic brake orbit. At the half-period, this solution suffers a binary collision in which the noncolliding mass has braked to a stop, and as a consequence of a closer look at Levi-Civita, the solution reverses its path, returning to the starting point. This solution provided inspiration for many more. At my own university, the astronomer Greg Laughlin (sadly for me, now moved to Yale) collaborated with a dance professor to choreograph Szebehely’s near Pythagorean periodic brake orbit. Three humans played the falling bodies. See [7] for more.

The Newtonian $N$-body problem is a special case of a Hamiltonian dynamical system whose energy, the Hamiltonian, has the form of a kinetic plus potential. See equation (1) below, allowing $V$ to be general. Brake orbits makes sense in this general context. Herbert Seifert, the topologist, wrote a beautiful paper [15] on brake orbits in this more general context. His paper provided some of the fuel for Alan Weinstein to make a now rather famous conjecture known as the Weinstein conjecture—a kind of contact version of the Arnol’d conjectures that drove the development of the field of symplectic topology. The Weinstein conjecture was proved for the case of 3-dimensional contact geometries by Clifford Taubes [18] about a decade ago.

Weinstein’s student Otto Raul Ruiz [13, 14] coined the name “brake orbit” in his thesis extending Seifert’s results. Ruiz insisted that brake orbits be periodic. We prefer to use the word in our less restrictive sense, requiring only one brake instant, while Ruiz’s definition requires two. Periodic brake orbits must of necessity shuttle back and forth between two brake configurations.

Nothing is sacred here about three bodies. Drop $N$ bodies. The analogues of the solutions found by Euler and Lagrange are nowadays called central configurations. In other words, if on being dropped, the $N$ bodies maintain their shape while collapsing to total collision, then that initial configuration is called a central configuration. If we apply an isometry or a scaling to a central configuration, we get another one. Modulo such isometries and scaling, is it true that the number of central configurations is finite? This problem made it onto Stephen Smale’s list of mathematical problems for the twenty-first century [16], published in this journal. In 2006, the problem was finally answered “yes” for the case $N = 4$ [6]. We have an “almost yes” for $N = 5$ (see [1]) and a “we’re basically clueless beyond numerical experiments that indicate yes” for $N > 5$.

The Problem

The ODE defining the three-body problem, some of whose solutions are depicted in our figures, can be written

$$\ddot{q} = -\nabla V(q).$$

Here $q = (q_1, q_2, q_3)$ records the positions of the three bodies, so that $q_a \in \mathbb{R}^2 \cong \mathbb{C}$, $a = 1, 2, 3$. The curves $q_a(t)$ are parameterized by Newtonian time $t$. The double dot over $q$ denotes acceleration, the second derivative with respect to time; $\nabla V$ denotes the gradient of the potential function (2) with respect to an inner product on the space $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ of $q$’s called the mass inner product:

$$\langle \dot{q}, \dot{q} \rangle = \sum m_a |\dot{q}_a|^2.$$

A dropped solution, or brake orbit, is one for which at the initial time $t = 0$, we have $\dot{q}_a = 0$, $a = 1, 2, 3$.

The Newtonian potential is given by

$$V = -G \sum_{a,b=1 \atop a \neq b}^3 \frac{m_a m_b}{r_{ab}},$$

where $r_{ab} = |q_a - q_b|$ is the distance between the bodies and $G > 0$ is the gravitational constant, needed if the units on both sides of (1) are to match up. The positive constants $m_a$, $m_b$ are the masses. We set them equal to each other for the equal-mass three-body problem.

The dynamics defined by (1) leave the center-of-mass subspace $\sum m_a q_a = 0$ invariant. This means that if we start with initial conditions $q_a(0)$, $\dot{q}_a(0)$, $a = 1, \ldots, N$, for which $\sum m_a q_a(0) = 0$ and $\sum m_a \dot{q}_a(0) = 0$, then for all time, the resulting solution $q_a(t)$ lies in the subspace.
\[ \sum m_i q_a(t) = 0. \] There is a standard undergraduate physics trick that reduces any solution to such a center-of-mass zero solution. We invoke this reduction to center of mass several times below, perhaps once or twice without saying so.

The mass inner product was built so that the usual kinetic energy is written as \( K(\dot{q}) = \frac{1}{2} \ddot{q} \). The total energy \( K(\dot{q}) + V(q) \) is conserved, meaning that it is constant along solutions to (1). In addition to this energy, the linear momentum and angular momentum are conserved. The linear momentum we have already seen. It is \( \sum m_i \dot{q}_a. \) The angular momentum is given by \( \sum m_i q_a \wedge \dot{q}_a, \) where \( \wedge \) denotes the two-dimensional version of the cross product:

\[
(x, y) \wedge (\dot{x}, \dot{y}) = x\dot{y} - y\dot{x} \quad \text{for} \quad (x, y), (\dot{x}, \dot{y}) \in \mathbb{R}^2. \]

The linear and angular momenta are linear in velocities, while the energy is of the form (kinetic) \( + \) (potential), where the kinetic energy is positive definite in velocities and the potential is an everywhere negative function. It follows that all our dropped solutions have zero linear momentum, zero angular momentum, and negative energy.

The physically literate reader may have protested at my potential and said, “The potential you wrote down is the potential coming from the fundamental solution of the Laplacian in \( \mathbb{R}^3 \), not \( \mathbb{R}^2 \).” It yields the gravitational force for bodies moving in space, not in the plane. Your \( q_a \) must lie in Euclidean space \( \mathbb{R}^3 \). Your insistence on \( q_a \in \mathbb{R}^2 \) is unsightly and wrong.” I will counter by reminding my literate reader that since dropped solutions have zero angular momentum, they necessarily remain in the plane containing their initial triangle with vertices \( q_a(0), a = 1, 2, 3, \) at time \( t = 0. \) Identify this plane with \( \mathbb{R}^2, \) and the dynamics are correct.

### Symmetry Puzzles

The seeds of this paper were two mysteries and a paradox that Alex Gofen brought to my attention. Gofen had been going through a database of 30 periodic collision-free brake orbits for the equal-mass three-body problem that are compiled in [9]. He was verifying and exploring those solutions using his own “Taylor Center” integration scheme [5]. A periodic brake orbit must shuttle back and forth between two distinct brake configurations, which is to say two distinct brake triangles for the three-body problem. He noticed that in 12 out of the 30 cases, the two brake triangles were related by an isometry. Thus mystery number one: why such a large number? And is that the “right” number? For example, if we could make a database of the “next” 300 or 3,000 equal-mass periodic collision-free brake orbits, would we continue to find that roughly one-third of them had congruent brake triangles?

Now to mystery two. One observes in the database that a symmetry relating the two “end” brake triangles induces a symmetry of the entire spatiotemporal structure of the orbit. Why? An instance of mystery number two was discussed above in relation to Figure 3.

Finally, to the paradox. Gofen observed that the solution depicted in Figure 4 shuttles back and forth between two brake triangles that are congruent by a rotation, indeed by a rotation of 180°. My own “shape space” perspective (see [12]) on the three-body problem told me that what he observed was impossible. Hence the paradox.

To resolve the paradox, let’s begin by understanding the shuttling back and forth. Newton’s equations (1) enjoy time-reversal symmetry: if \( q(t) \) solves Newton’s equations, then so does \( q(-t) \). Now, if \( q(t) \) is a brake orbit with brake instant \( t = 0 \), then \( q(-t) \) has precisely the same initial conditions—the same configuration and same zero velocity—at time \( t = 0 \) as does \( q(t) \). It follows by the unique dependence of solutions on initial conditions that we must have \( q(t) = q(-t) \). Suppose in addition that the brake orbit is periodic with period \( T \). Then we have the additional temporal symmetry \( q(t + T) = q(t) \). By substituting \( t = h - T/2 \) into this periodicity relation, we get \( q(h + T/2) = q(h - T/2) \). Use the time-reversal symmetry to get \( q(h + T/2) = q(T/2 - h) \) and then differentiate with respect to \( h \) at \( h = 0 \) to see that \( t = T/2 \) is another brake instant. Thus periodic brake orbits must shuttle back and forth between two brake configurations.

This second brake configuration must be different from the first. If not, we can cut our period in half and repeat the argument. By this process, eventually either we arrive at a fundamental minimal period with two distinct configurations or the periods go to zero, which means that our original brake orbit was in fact a fixed point, that is, a critical point of the potential. For Newton’s potential, this last possibility is excluded. The potential has no critical points: \( N \) stars cannot just sit in space, attracting each other but not moving.

Let me proceed now to shape-space thinking. The Newtonian \( N \)-body problem enjoys spatial symmetries in addition to its temporal symmetries. These are the isometries of space, or, in the case of the planar \( N \)-body problem, the isometries of the plane. What this means is that if \( R \) is any isometry of the plane and \( q(t) = (q_1(t), q_2(t), q_3(t)) \) solves the planar three-body problem, then so does \( Rq(t) \), where by \( R(q_1, q_2, q_3) \) we mean \( (Rq_1(t), Rq_2(t), Rq_3(t)) \). We can use these isometries to push the ODEs defining the three-body problem down to a space I call “shape space.” The points of shape space are oriented congruence classes of planar triangles. Two triangles represent the same “shape,” or oriented congruence class, if there is an orientation-preserving isometry \( R \), i.e., a rotation composed with a translation, taking one to the other.

Here is the salient point of the paradox: when the angular momentum is zero, these pushed-down dynamics are also of Newtonian type, so the argument of the preceding paragraph holds, and brake orbits all have zero angular momentum. A periodic brake orbit in configuration space remains a periodic brake orbit when projected to shape space, and so it must shuttle back and forth between two shapes, and these shapes must be distinct. But Gofen told me that he had found orbits for which the two brake triangles were related by rotation, and hence down in shape space it was shuttling back and forth between the same point!

The resolution of this paradox is that Gofen was viewing his triangles as unlabeled, while my triangles have to be labeled triangles in order for me to construct shape space with its dynamics. When a rotation \( R \) takes a labeled...
triangle to another, it must, by definition, preserve the (mass or vertex) labelings. But in the example depicted in Figure 4, the two brake triangles are congruent as unlabeled triangles, not as labeled ones. A 180° rotation takes one triangle to the other, but in so doing, it messes up their labelings.

When the masses are all equal, the N-body problem enjoys additional symmetries beyond the Galilean symmetries of time and space isometries. We may interchange any two masses: if \((q_1(t), q_2(t), q_3(t))\) solves the equal-mass three-body problem, then so does \((q_3(t), q_1(t), q_2(t))\), etc. The operation of interchanging masses defines a representation of the permutation group on the configuration space, so that we could write the above interchange of masses 1 and 2 as \(\sigma_{12}\).

The brake triangles of Figure 4 are related by a symmetry of the form \(F = R \sigma_{12}\), where \(\sigma\) is one of the transpositions. Although \(R\) acts as the identity on shape space, such an \(F\) does not. The shapes of \(q(0)\) and \(F(q(0))\) are different, allowing us to avoid the paradox.

Having extricated ourselves from our mathematical paradox, we move on to mystery two. If we have a brake orbit whose ends—the two brake triangles—are related by a symmetry \(F\) as above, we can use that symmetry to extract nontrivial information about the configuration \(q(T/4)\) at the quarter-period, with \(T/4\) being halfway between the two brake times of \(t = 0\) and \(t = T/2\).

Suppose, then, that \(F(q(0)) = q(T/2)\). Consider the new solution \(F(q(t))\). At time \(t = 0\), its initial conditions consist of the position \(q(T/2)\) and the velocity \(0\) (since \(q(0) = 0\)). Thus \(F(q(t))\) shares initial conditions with the curve \(q(t) + T/2\), which also solves Newton's equations. (Like any autonomous ODE, \(F\) enjoys time-translational symmetry: if \(q(t)\) solves Newton's equations, so does \(q(t + t_0)\) for any time \(t_0\).) It follows that
\[
F(q(t) + T/2) = F(q(t)).
\]

Now take \(t = -T/4\) and use the time-reversal symmetry to conclude that
\[
q(T/4) = F(q(T/4)).
\]

The midpoint must be a fixed point of our symmetry \(F\)!

Let us return to Figure 3. A reflection \(F\) relates its two brake triangles. Reflections are symmetries of the three-body problem. Take \(F\) to be this reflection and \(\ell\) its line of reflection. Then \(F\)’s fixed points are collinear “triangles” in which all three masses lie on \(\ell\). In this way, we have solved the mystery around that figure, that is, why all three masses form a syzygy at the mid time. And indeed, at the same time, a moment’s contemplation of (3) shows that we have also accounted for the overall reflectional symmetry of that orbit: it is a consequence of the symmetric relation between its endpoints.

In Figure 4, the two brake triangles are related by \(F = R \sigma_{12}\), where \(R\) is rotation by 180°, and \(\sigma\) interchanges two of the masses. The fixed-point set of such an \(F\) is the set of “Euler configurations”: degenerate collinear triangles with the noninterchanged mass forming the midpoint of the other two. This fact, together with, of course, (3), matches Gofen’s data.

A Hole in Shape Space and Harmonic Oscillators

Our mathematical hero Vladimir Arnold had a saying he was fond of sprinkling into his lectures that was a variation of the phrase “the exception that proves the rule.” We present our exception.

Observe that we can run our shape-space argument for any potential invariant under isometries. One such potential is the harmonic oscillator potential \(V = \sum k_{ab}q_ab\) with \(k_{ab} > 0\) spring constants. The act of replacing Newton’s potential (2) by this quadratic potential corresponds to replacing the gravitational force by Hooke’s spring forces. The corresponding ODEs are linear of the form \(\dot{q} = -\omega^2 q\), where \(\omega\) is a matrix depending on the masses and springs that is positive definite on the center-of-mass subspace \(\sum q_i = 0\) and leaves this subspace invariant. Choose an eigenbasis \(E_i\) for \(\omega\) restricted to this subspace such that \(AE_i = -\omega^2 E_i\), with \(\omega^2 > 0\) the eigenvalues. Then \(q(t) = \cos(\omega t)E_i\) is a brake solution shutting back and forth between \(E_i\) and \(-E_i\). But \(-E_i\) corresponds to rotating \(E_i\) by 180°. Pushed down to shape space, this brake solution connects the shape corresponding to \(E_i\) to itself, contradicting my alleged “shape-space thinking” theorem that such a solution is impossible. This is the exception that proves the rule.

Another paradox. What is happening? Is our theorem true or not? The resolution of this apparent paradox involves the projection of 0 to shape space, 0 representing total collision. The map from configuration space to shape space, on restriction to the center-of-mass subspace, fails to be a submersion exactly at 0. We should view the shape of total collision as a singularity in shape space. (Indeed, for \(N > 3\), it is a topological singularity, since the shape space for the planar N-body problem is the cone over complex projective space of complex dimension \(N - 2\).) The relation between the dynamics upstairs and downstairs breaks down at total collision. Our eigenvector-based solution above passes through 0 at time \(t = \pi/(2\omega)\), and our shape-space argument fails for solutions passing through 0.

We can derive an alternative resolution to this paradox by following the implications of (3). A rotation \(F = R\) is a symmetry, and so that equation holds for any brake solution whose ends triangles are related by a rotation \(R\). Then (4) asserts that \(q(T/4)\) is a fixed point of the rotation. But the only centered configuration invariant under a nontrivial rotation is the zero configuration \(0 = (0, 0, 0)\), the configuration representing total collision. Our periodic brake solution must pass through total collision halfway between its two ends! That’s fine for the harmonic oscillator. No problem. For the gravitational N-body problem, total collision acts like an essential singularity—a hole in shape space if you will—through which there is no consistent way to travel, and we have to stop the dynamics at total collision and call it quits. So there’s no such brake orbit for the planar N-body problem.

More information can be extracted from (3) by evaluating the equation at \(t = -T/2\) and \(t = 0\) to get \(q(0) = F(q(T/2))\) and \(q(T/2) = F(q(0))\), so that \(q(0) = F^2(q(0))\). If the triangle \(q(0)\) is in general position, or even if it is a degenerate collinear triangle but \(F\) is of
the form $R$ or $R\sigma$, then this fixed-point relation implies that $F^2 = Id$, where $Id$ is the identity mapping. (In other words, the rotation group acts freely on configuration space away from a triple collision, so that $R^2(q(0)) = q(0)$ implies that $R^2 = Id$.) When $F = R\sigma$, we have $F^2 = R^2$, so either way, when $F = R$ or $R\sigma$, we get $R^2 = Id$. Thus the only $R$’s that solve the identity $R^2 = Id$ are rotations by $180^\circ$! This $R$ is also known as central inversion: $Rq = -q$. The original solution that Gofen showed me, Figure 4 above, has its brake triangles related by central inversion, but again, related as unlabeled triangles.

I sure hope that neither Gofen nor Li nor Liao nor one of my readers wanders out into the land of equal-mass periodic brake orbits in summer and returns with a solution whose brake triangles are related by an $F$ of the form $R\sigma$ with $R$ a $45^\circ$ rotation! I will not know how to resolve the resulting paradox. The remainder of my summer vacation with family would be threatened with ruin! Wait till the following spring, please, to alert me to the next such paradox.

Mystery one, the mystery of 12 out of 30 of these “first” equal-mass collision-free periodic brake orbits having extra symmetries, remains a mystery.

End Note: Gofen’s Taylor Center
Alex Gofen, whose figures grace this paper, asked me to say a few things about the “Taylor Center” that he runs that generated three of the displayed figures.

The name Taylor Center stands for two things. First, it is a comprehensive resource dedicated to particular mathematical problems. Second, it provides software running under Windows, the advanced ODE solver, based on modern Taylor integration. This ODE solver offers several unique features, numerical and graphical. As a numerical tool, it employs the most accurate Intel extended-precision floating-point type, with 63-bit mantissa, integrating with the aid of Taylor expansions to order 30 or higher and providing several methods of accuracy control up to all available 63 binary digits.

As a graphical tool, it offers high-resolution graphics, plotting trajectories as a real-time animation, both in 2D and 3D stereo (viewable via red/blue glasses). Thanks to such graphics, this software may serve as a laboratory in various fields of applied mathematics. A few such lab topics have already been posted, for example, the three types of rigid body motion and selected samples in celestial mechanics. The library continues to grow.

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2You can find the Taylor Center at http://taylorcenter.org/.
3See http://taylorcenter.org/Gofen/TaylorMethod.htm.
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