Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
A novel extended model with versatile shaped failure rate: Statistical inference with Covid-19 applications

Anum Shafiq a,b,1, Tabassum Naz Sindhu b,1, Naif Alotaibi c,1

a School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
b Department of Statistics, Quaid-i-Azam University, 45320, Islamabad 44000, Pakistan
c Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh, 11432, Saudi Arabia

A R T I C L E I N F O

Keywords:
MKNPF model
Least Square Estimates
Lehmann Type I
Mean square error
Weighted Least Square Estimates
Estimation techniques

A B S T R A C T

Statistical models perform an essential role in data analysis, and statisticians are constantly looking for novel or pretty recent statistical models to fit data sets across a broad variety of fields. In this study, we used modified Kies generalized transformation and the new power function to suggest an unique statistical model. We present and discuss a linear illustration of the density function. Theoretically, quantile function, characteristic function, stochastic ordering, mean, and moments are just a few of the structure properties we discuss. By defining an ideal statistical distribution for assessing the COVID-19 mortality rate, an attempt is performed to determine the model of COVID-19 spread in different nations like the United Kingdom and Italy. In some countries, the novel distribution have been shown to be more appropriate than existing competing models when fitted to COVID-19.

Introduction

Over the last three decades, scholars’ enthusiasm for developing new generalized models has grown as they seek to uncover the hidden characteristics of baseline models. Newly diversified models provide up new potential for solving real-world problems and fitting complicated and asymmetric random occurrences. As a consequence, a variety of models have been designed and investigated in the literature. The Lehmann Type I (L-I) and Type II models [1] are two of the simplest and most useful lifespan models elicited in the statistical history. The L-I model is widely addressed in support of the power function (PF) model in the literature. The L-I technique was used by Gupta et al. [2] to create a generalized form of the exponential distribution. Cordeiro et al. [3], on the other hand, suggested a dual transformation of the L-I technique and formed the Lehmann Type II (L-II) G class of models. Because of their simplistic closed CDFs, the L-I and L-II techniques have been widely employed to provide more adaptable and improved versions of classical models. A simplified version of the L-I distribution is the PF model. The PF model’s simplicity and utility have motivated numerous scholars to explore in depth its potential advancements and implementations in a variety of fields. Tavangar [4] employed dual generalized order statistics to characterize the PF distribution, while Ahsanullah et al. [5] used lower record values to explain the PF distribution. The assessment of the PF parameters using different evaluation methods was examined by Akhter [6], and the estimation of the PF parameters using the trimmed L moments was discussed by Shahzad et al. [7]. The beta-PF [8], weighted-PF [9], Weibull-PF [10,11], odd generalized exponential PF [12], Transmuted Weibull-PF [13], and other famous extensions of the PF distribution are just a few examples. Arshad et al. [14,15] recently formed the exponentiated-PF distribution, a finite bathtub shaped failure rate PF model using L-II class. Iqbal et al. [16] investigated a two-parameter model known as the new power function (NPF) model. The CDF and PDF of NPF model with scale (µ) and shape (λ) are

\[ G(t|\mu, \lambda) = 1 - \left( \frac{1-t}{1+\mu} \right)^\lambda, \]

\[ g(t|\mu, \lambda) = \lambda (\mu + 1) (1-t)^{\lambda-1} \left(1 + \mu t\right)^{\lambda-1}, -1 < \mu < \infty, \lambda > 0, t \in (0, 1). \]

The primary goal of this investigation is to create a flexible extremely versatile failure rate model known as the modified Kies new power function (MKNPF) model. It may simulate hazard rates that are growing, decreasing, bathtub, and inverted-U and U-shaped. It offers some appealing features, such as a simple, closed-form PDF, CDF, and Quantile Function, all of which are straightforward to use. This model is simple to use from an application perspective. The properties of the model make it a good fit for actuarial data, biomedical life monitoring,
and reliability implementations. The new model is designed to model the COVID-19 data. We have used COVID-19 real data from the United Kingdom and Italy to assess the model’s efficacy. For the United Kingdom and Italy, we used the COVID-19 daily death rate. For modeling COVID-19 data, many researchers, such as Sindhu et al. [17–19] used a new class of models.

The modified (reduced) Kies (MK) model was presented by Kumar and Dharmaja [20] as a specific case of the Kies model. In [21] authors presented the exponentiated reduced Kies model and investigated some of its features. Dey et al. [22] used a progressive censoring strategy to develop recurrence relations for single and product moments of the MK model. Al-Babtain et al. [23] develop a novel family of generators termed as the modified Kies generalized (MK-G) on basis of the MK model. The MK-G model can be employed efficiently for analysis when given a baseline model. If \( G(t|\theta) \) is the reference CDF for a parameter vector \( \theta \), then CDF of MK-G model is defined as

\[
F(t|\Psi) = 1 - \exp \left\{ - \left( \frac{G(t|\theta)}{1 - G(t|\theta)} \right)^{\xi} \right\}, \xi > 0, \tag{3}
\]

\( \Psi = (\xi, \Theta) \), The PDF of (3) is

\[
f(t|\Psi) = \xi \left( \frac{G(t|\theta)}{1 - G(t|\theta)} \right)^{\xi-1} \exp \left\{ - \left( \frac{G(t|\theta)}{1 - G(t|\theta)} \right)^{\xi} \right\}, \xi > 0. \tag{4}
\]

Parameter estimation is fundamental when studying any probability distribution. To optimize the parameters of any model, Maximum likelihood Estimation (MLE) is often used, due to its appealing properties. They are asymptotically consistent, unbiased, and normally distributed asymptotically (check [24]). Various estimating strategies for distributions have evolved over time, such as Least-squares estimation (LSE), maximum probability, Weighted least square estimation (WLSE), and L-moments approaches and minimal distance estimation (check [25–29]). The parameters of generalized power Weibull (GPW) model were determined using the maximum likelihood and maximum product spacing procedures in [30].

Furthermore, the MKNPF parameters are evaluated using three traditional estimation techniques MLE, LSE and WLSE. We have provided a thorough explanation for study of three approaches for guessing underdetermined parameters of MKNPF model, also a study of their performance for different sample sizes in this article. We run thorough simulations to investigate behaviors of different estimators focused on bias and mean squared error because it is tough to match characteristics of different estimation strategies theoretically. The new model is an excellent competitor with certain well-known and trendy models like two-bias and mean squared error because it is tough to match characteristics for different sample sizes in this article. We run thorough simulations to investigate behaviors of different estimators focused on bias and mean squared error because it is tough to match characteristics of different estimation strategies theoretically. The new model is an excellent competitor with certain well-known and trendy models like two-bias and mean squared error because it is tough to match characteristics for different sample sizes in this article. We run thorough simulations to investigate behaviors of different estimators focused on bias and mean squared error because it is tough to match characteristics of different estimation strategies theoretically.

RHRF of \( F(t|\Psi) \) is

\[
\xi \frac{\lambda (\mu + 1) \left( (1 + \mu)^{\xi} - (1 - t)^{\xi} \right)}{\lambda \xi - 1} \left( \frac{1}{1 - G(t|\theta)} \right)^{\xi - 1} \exp \left\{ - \left( \frac{G(t|\theta)}{1 - G(t|\theta)} \right)^{\xi} \right\}, \tag{6}
\]

where \( \Psi = (\lambda, \mu, \xi) \), where \( t \in (0, 1) \), \( \mu > -1 \), is a scale and \( \lambda, \xi > 0 \) are the two shape parameters, respectively. In the literature, terms like "failure rate function", is widely mentioned. This term is employed to indicate an element’s failure rate over a given time period \( t \) and mathematically formulated as \( h(t|\Psi) = f(t|\Psi)/(1 - F(t|\Psi)) \). The failure rate function is

\[
h(t|\Psi) = \frac{\xi \lambda (\mu + 1) \left( (1 + \mu)^{\xi} - (1 - t)^{\xi} \right)}{\lambda \xi - 1} \left( \frac{1}{1 - G(t|\theta)} \right)^{\xi - 1}, \tag{7}
\]

an ideal mechanism in reliability study. The chance that a component will survive at time \( t \) can be described as the reliability function \( S(t|\Psi) \). Analytically, it is characterized as \( S(t|\Psi) = 1 - F(t|\Psi) \), here, \( S(t|\Psi) \) functions of MKNPF(\( \Psi \)) model is

\[
S(t|\Psi) = \exp \left\{ - \left( \frac{(1 + \mu)^{\xi} - (1 - t)^{\xi}}{(1 - t)^{\xi}} \right)^{\xi} + 1 \right\}. \tag{8}
\]

One of valuable reliability indicators is the CHRF. The CHRF is a measure of risk: higher the \( H(t|\Psi) \), elevate the risk of collapse by \( r \)-time.

\[
H(t|\Psi) = \int_{0}^{t} h(y|\Psi) dy = - \log S(t|\Psi). \tag{9}
\]

\[
H(t|\Psi) = \left( \frac{(1 + \mu)^{\xi} - (1 - t)^{\xi}}{(1 - t)^{\xi}} \right)^{\xi}. \tag{10}
\]

Mills ratio is defined by \( M(t|\Psi) = S(t|\Psi)/f(t|\Psi) \). Mills ratio of \( T \) is given by

\[
M(t|\Psi) = \frac{(1 + \mu)^{-1} \left( (1 - t)^{\xi} - (1 - t)^{\xi+1} \right) \xi}{\lambda}. \tag{11}
\]

The odd function is defined by \( O(t|\Psi) = F(t|\Psi)/S(t|\Psi) \). The odd function of \( T \) is

\[
O(t|\Psi) = \exp \left\{ - \left( \frac{(1 + \mu)^{\xi} - (1 - t)^{\xi}}{(1 - t)^{\xi}} \right)^{\xi} \right\} - 1. \tag{12}
\]

The reverse hazard rate function (RHRF) is defined by \( f(t|\Psi)/F(t|\Psi) \). The RHRF of \( T \) is given by

\[
RHRF(t|\Psi) = \frac{\xi \lambda (\mu + 1) \left( (1 + \mu)^{\xi} - (1 - t)^{\xi} \right)}{\lambda \xi - 1} \left( \frac{1}{1 - G(t|\theta)} \right)^{\xi - 1} \exp \left\{ - \left( \frac{(1 + \mu)^{\xi} - (1 - t)^{\xi}}{(1 - t)^{\xi}} \right)^{\xi} \right\} - 1. \tag{13}
\]

Useful expansion of PDF of MKNPF

\[
(1 - t)^{-(\xi + 1)} = \sum_{k=0}^{m} \left( \frac{\xi + k}{k} \right)^{\xi}, \text{for } |t| < 1. \tag{14}
\]

and

\[
e^{-t} = \sum_{j=0}^{m} (-1)^j \frac{t^j}{j!}. \tag{15}
\]

Then, using (14) and (15) to expand \( 1 - G(t|\theta)^{-1}(\xi + 1) \) and exp \( \left\{ - \left( \frac{G(t|\theta)}{1 - G(t|\theta)} \right)^{\xi} \right\} \) respectively, it (4) follows as,

\[
f(t|\Psi) = \sum_{l, m=0}^{+\infty} \frac{(-1)^l}{l!} \left( \frac{(1 + \mu)^{\xi} - (1 - t)^{\xi}}{(1 - t)^{\xi}} \right)^{\xi} \left( \frac{(1 + \mu)^{\xi} - (1 - t)^{\xi}}{(1 - t)^{\xi}} \right)^{\xi} \exp \left\{ - \left( \frac{(1 + \mu)^{\xi} - (1 - t)^{\xi}}{(1 - t)^{\xi}} \right)^{\xi} \right\}. \tag{16}
\]

If \( s \in \mathbb{R}^+, \) and \( |t| < 1 \), then it holds

\[
(1 - t)^{\xi - 1} = \sum_{l=0}^{+\infty} (-1)^l \left( \frac{\xi - 1}{l!} \right)^{\xi}. \tag{17}
\]
Applying (17) to expand \[1 - \left(\frac{1-q}{1+\mu}\right)\left(1+\lambda t\right)^{\lambda m}m^{\lambda+1}\], we get PDF of MKNPF in (16) is with 
\[
\kappa_{l,m,p} = \frac{(-1)^{l+p} \xi^l}{\lambda^{l+1}} \left(\lambda^m + m \right) \left(\lambda^l + 1\right),
\]
\[
f(t|\Psi) = \sum_{l,m,p=0}^{\infty} \kappa_{l,m,p} \frac{\lambda^{l+1} \left(\lambda^{m+1} + 1\right)}{(1+\mu)^{l+p+1+1}}.
\]
As seen in (18) MKNPF density can be represented as a linear conjunction of NPF densities. As a result, the features of MKNPF model can be extrapolated from those of NPF model. In the hereafter, the result in (18) will be used to calculate numerous mathematical features of the MKNPF distribution.

**Shape**

Figs. 1–3 show possible shapes for the MKNPF density, CDF and FR functions based on different parameter values. The potential shapes of the PDF corresponding to the parameter \(\lambda\), that regulates the distribution’s scale, as well as the two shape parameters \(\mu\) and \(\xi\), which govern the distribution’s shapes, include growing, bathtub, symmetric, asymmetric, inverted U, decreasing, and J forms. Fig. 1(a–h) demonstrate such shapes. Fig. 2(a–h) also shows CDF shapes for the MKNPF model. The failure rate function (FRF) forms, which include rising, U, and bathtub shapes, are shown in Fig. 3(a–h). These adaptable FRF shapes are appropriate for both monotonic and non-monotonic hazard rate behaviors, which are most common in real-time scenarios. Non-stationary lifespan phenomena frequently exhibit these types of forms.

**Simulation**

Hyndman and Fan \[42\] first proposed the notion of a quantile function (QF). The \(q\)th QF of MKNPF is obtained by inverting the CDF (5). The MKNPF model can be easily simulated from (19). The generated variate having PDF (6) is
\[
\tau(q|\Psi) = \left\{1 + \left[-\log(1-q)\right]^\frac{1}{\xi}\right\}^\frac{1}{\lambda} - 1
\]
\[
\mu + \left\{1 + \left[-\log(1-q)\right]^\frac{1}{\xi}\right\}^\frac{1}{\lambda}, 0 < q < 1.
\]
As a consequence, the median as well as lower and upper quantiles, are computed as follows:
\[
\tilde{T} = \left\{1 + \left[\log(2)\right]^\frac{1}{\xi}\right\}^\frac{1}{\lambda} - 1
\]
\[
\mu + \left\{1 + \left[\log(2)\right]^\frac{1}{\xi}\right\}^\frac{1}{\lambda},
\]
\[
\tau(0.25|\Psi) = \left\{1 + \left[\log(0.75)\right]^\frac{1}{\xi}\right\}^\frac{1}{\lambda} - 1
\]
\[
\mu + \left\{1 + \left[\log(0.75)\right]^\frac{1}{\xi}\right\}^\frac{1}{\lambda},
\]
\[
\tau(0.75|\Psi) = \left\{1 + \left[\log(0.25)\right]^\frac{1}{\xi}\right\}^\frac{1}{\lambda} - 1
\]
\[
\mu + \left\{1 + \left[\log(0.25)\right]^\frac{1}{\xi}\right\}^\frac{1}{\lambda}.
\]
The stronger the change in the median curve, the lower the inputs of the parameters \( \mu \), \( \lambda \), and \( \zeta \). Also, when the \( \mu \) approaches 6, the median function provides decreasing values. On the other hand there is a noticeable shift in the skewness trend along \( \mu \) at lesser characteristics of \( \zeta \), but as \( \zeta \) rises, it comes up to nearly 0.1. As \( \zeta \) increases, the extent of peakedness of the model increases and may also be platykurtic.

**Moments**

Moments are utilized in statistics to explain the different features of a model. The central tendency, skewness, dispersion, and kurtosis of a model can all be examined using moments.

**Theorem 1.** If \( \text{T MKNPF}(\Psi) \), then the moment \( \mu_r \) of \( \text{T} \) is

\[
\mu_r = \lambda (p+1) (\mu+1) \sum_{l,m,p=0}^{+\infty} \sum_{i=0}^{\infty} k_{l,m,p} B \{ r + i + 1, \lambda (p+1) \} .
\]

where \( B(p,\sigma) = \int_0^{\infty} z^{p-1} (1 - e^{-\sigma z}) \, dz \), and \( B(p,\sigma) = \frac{\Gamma(p+\sigma)}{\Gamma(p+1)} \).

**Proof.** (18) can be used to write \( \mu_r \) as

\[
\mu_r = \int_0^\infty t^r dF_t(t; \Psi); r = 1, 2, \ldots
\]

\[
\mu_r = \int_0^1 \sum_{l,m,p=0}^{+\infty} \sum_{i=0}^{\infty} k_{l,m,p} \frac{\lambda (p+1) (\mu+1) (1 - i)^{i(p+1)-1}}{(1 + \mu i)^{i(p+1)+1}} \, dt.
\]

After some algebra, we get

\[
\mu_r = \lambda (p+1) \sum_{l,m,p=0}^{+\infty} \sum_{i=0}^{\infty} (-1)^i \mu^i \left( \lambda (p+1) \right)_i k_{l,m,p} \int_0^1 t^r (1 - i)^{i(p+1)-1} \, dt,
\]
Fig. 3. Fluctuations of CDF of MKNPF with $\mu$, $\lambda$, and $\zeta$.

Fig. 4. Variations of QF and quantile density function of MKNPF with $\zeta$ and $q$ at different extents of $\lambda$ with $\mu = 0.7$.

Fig. 5. Fluctuation of $\tilde{T}$, skewness and kurtosis of MKNPF with $\mu$ and $\zeta$ at different extents of $\lambda$. 
Table 1

| n | Bias (μ) | Bias (λ) | Bias (ζ) | MSE (μ) | MSE (λ) | MSE (ζ) |
|---|---|---|---|---|---|---|
| 30 | 0.4407 | 0.9886 | -0.1381 | 0.3440 | 1.5420 | 0.3524 |
| 65 | 0.2801 | 1.03936 | -0.1451 | 0.1895 | 1.2529 | 0.2076 |
| 100 | 0.1540 | 0.94022 | -0.1187 | 0.1327 | 0.9886 | 0.0852 |
| 135 | 0.1506 | 0.82407 | -0.0867 | 0.0852 | 0.7575 | 0.0316 |
| 170 | 0.1695 | 0.73511 | -0.0613 | 0.0734 | 0.5918 | 0.0263 |
| 205 | 0.1838 | 0.64434 | -0.0571 | 0.0719 | 0.4400 | 0.0262 |
| 310 | 0.2123 | 0.66378 | -0.0695 | 0.0596 | 0.3827 | 0.0188 |
| 400 | 0.2793 | 0.56925 | -0.1170 | 0.0414 | 0.3425 | 0.0312 |

Table 2

| n | Bias (μ) | Bias (λ) | Bias (ζ) | MSE (μ) | MSE (λ) | MSE (ζ) |
|---|---|---|---|---|---|---|
| 30 | 0.10238 | 0.3900 | 0.0868 | 8.5295 | 1.0509 | 0.3548 |
| 65 | 0.0874 | 0.2665 | 0.0886 | 8.6184 | 0.7614 | 0.2571 |
| 100 | 0.0727 | 0.2946 | 0.0457 | 2.9078 | 0.5108 | 0.1845 |
| 135 | 0.0630 | 0.1963 | 0.0625 | 2.4030 | 0.4818 | 0.1524 |
| 170 | 0.0532 | 0.2289 | 0.0478 | 1.7104 | 0.4140 | 0.1450 |
| 205 | 0.0432 | 0.1767 | 0.0475 | 1.6661 | 0.3590 | 0.1204 |
| 310 | 0.0410 | 0.1184 | 0.0486 | 1.5938 | 0.2895 | 0.0891 |
| 400 | 0.028588 | 0.1628 | 0.0148 | 1.5104 | 0.2689 | 0.0746 |

Remark 1. The Moment Generating Function (MGF) is commonly employed in model characterization. The MGF of MKNPF model using the Maclaurin series expansion of an exponential function is mentioned as

\[ M_r (d | \Psi) = E (e^{dt}) = \sum_{r=0}^{\infty} \frac{d^r}{r!} \mu_r. \]

Substituting by (29) into (30), we get

\[ M_r (d | \Psi) = \lambda (p + 1) (\mu + 1) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\lambda=0}^{\infty} k_j \kappa_{j, \mu, \lambda} B \{ r + i + 1, \lambda (p + 1) \}. \]

Proposition 1. Suppose \( T \) be a random variable following MKNPF model, then central moments is

\[ \mu_r (t | \Psi) = \lambda (p + 1) (\mu + 1) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\lambda=0}^{\infty} k_j \kappa_j \kappa_{j, \mu, \lambda} B \{ s - j + i + 1, \lambda (p + 1) \}. \]

Substituting (18) into (33) and after simple arithmetic, we get

\[ \mu_r (t | \Psi) = \lambda (p + 1) (\mu + 1) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\lambda=0}^{\infty} k_j \kappa_j \kappa_{j, \mu, \lambda} B \{ s - j + i + 1, \lambda (p + 1) \}. \]

Remark 2. The moment formula (34) facilitates in the formulation of statistical measures that are helpful. In (34), for example, the variance of \( T \) follows with \( s = 2 \). Furthermore, by putting \( s = 2, 3, 4 \) in (34), the skewness \((\theta_1 = \mu^3 / \mu^2)\) and kurtosis \((\theta_2 = \mu^4 / \mu^2)\) of \( T \) can well be calculated using (34).

Characteristic function

\[ \tilde{\Delta}(t | \Psi) = \int_0^1 e^{it} dF(t | \Psi). \]

After using exponential series, we have

\[ \tilde{\Delta}(t | \Psi) = \frac{1}{t!} \int_0^1 t^r dF(t | \Psi). \]

Hence, we obtain

\[ \tilde{\Delta}(t | \Psi) = \lambda (p + 1) (\mu + 1) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\lambda=0}^{\infty} \frac{(it)^r}{r!} \kappa_j \kappa_{j, \mu, \lambda} B \{ r + i + 1, \lambda (p + 1) \}. \]

Factorial generating function

\[ F_r (\delta | t | \Psi) = \int_0^1 e^{(\delta + t) \delta} dF(t | \Psi), \]

so, we can put together the integral as follows:

\[ F_r (\delta | t | \Psi) = \lambda (p + 1) (\mu + 1) \sum_{r=0}^{\infty} \frac{(\log(1 + \delta))^r}{r!}. \]
\[ \sum_{l,m,p=0}^{+\infty} B(r+i+1,\lambda(p+1)) \times \sum_{i=0}^{+\infty} \sum_{l,m,p=0}^{+\infty} \kappa_{l,m,p} B(r+i+1,\lambda(p+1)) \] (39)

**Stochastic ordering**

In this subsection, we compare the MKNPF \( F_1(t|\Psi_1) \) and the MKNPF \( F_2(t|\Psi_2) \) with respect to stochastic ordering information. Assume that \( T_1 \) and \( T_2 \) be two random variables with reliability functions, cdfs, and pdfs \( S_1(t|\Psi_1) \) and \( S_2(t|\Psi_2) \); \( F_1(t|\Psi_1) \) and \( F_2(t|\Psi_2) \); and \( f_1(t|\Psi_1) \) and \( f_2(t|\Psi_2) \), respectively, where \( \Psi_1 = (\mu_1; \xi_1; \lambda_1) \) and \( \Psi_2 = (\mu_2; \xi_2; \lambda_2) \) respectively. A random variable \( T_1 \leq T_2 \) in the following ordering (see [43]), if: (i) Stochastic order \( T_1 \triangleleft_{st} T_2 \) if \( S_1(t|\Psi_1) \triangleleft S_2(t|\Psi_2) \) \( \forall t \); (ii) Hazard rate order \( T_1 \leq_{hr} T_2 \) if \( h_1(t|\Psi_1) \geq h_2(t|\Psi_2) \) \( \forall t \); (iii) likelihood ratio order \( X \leq_{lr} T \) if \( \frac{f_1(t|\Psi_1)}{f_2(t|\Psi_2)} \) decreases in \( t \). Among the various partial orderings discussed above, the following chain of implications follows.

\[ T_1 \leq_{st} T_2 \Rightarrow T_1 \leq_{hr} T_2 \Rightarrow T_1 \leq_{lr} T_2. \] (40)

As stated in the following theorem, MKNPF\((t|\Psi)\) models are ranked according to the strongest "likelihood ratio" ordering.

**Theorem 2.** Let \( T_1 \sim MKNPF_1(\mu_1; \xi_1; \lambda_1) \), and \( T_2 \sim MKNPF_2(\mu_2; \xi_2; \lambda_2) \), if \( \mu_1 = \mu_2 \), \( \lambda_1 = \lambda_2 \) and \( \xi_1 \leq \xi_2 \), then \( T_1 \leq_{lr} T_2 \) \( \forall t \); \( T_1 \leq_{hr} T_2 \) \( \forall \lambda \); \( T_1 \leq_{st} T_2 \) in all three cases exist.

**Proof.** It is sufficient to show \( \frac{f_1(t|\Psi_1)}{f_2(t|\Psi_2)} \) is a decreasing function of \( t \); the likelihood ratio is

\[ \frac{f_{MKNPF_1}(t)}{f_{MKNPF_2}(t)} = \frac{\xi_1}{\xi_2} (1-t)^{\xi_1-1} (1+t)^{-\xi_1-1} (1-t)^{\xi_2-1}. \]
Thus if $\mu = \mu_1 = \lambda_2$, and $\xi_1 \leq \xi_2$, then

$$
\frac{\partial}{\partial t} f_{\text{MKNPF}}(t) = 0.2584536, 36.1414106 1.163282, 55.314676
$$

Hence it shows that $T_1 \leq T_2$, and according to (40) these both are

$$
T_1 \leq T_2, T_1 \leq T_2, T_2 \leq T_2 \text{ also hold.}
$$

The certain estimation techniques with simulation

In this part, we focus on the three techniques for estimating MKNPF model parameters: MLE ordinary LSE, and WLSE techniques. Simulation studies are used to investigate the effectiveness of certain techniques. From now, $t_1, t_2, \ldots, t_n$ indicate $n$ observed characteristics from $T$ and their ascending ordering values $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$.

### Table 4

| Models | MLs | Standard errors |
|--------|-----|-----------------|
| MKNPF ($\mu, \lambda, \xi$) | 102.9867375, 0.4283564, 1.7290155 | 85.1797881, 0.1702234, 0.4221061 |
| MKITL ($\mu, \beta$) | 0.4436225, 223.1059521 | 0.04081851, 26.78938003 |
| MKIEx ($\xi, \beta$) | 0.8623125, 15.0042180 | 0.07684104, 1.28409033 |
| NPF ($\theta, \gamma$) | -0.2584536, 36.1414106 | 1.163282, 55.314676 |

### Table 5

| Models | MLs | Standard errors |
|--------|-----|-----------------|
| MKNPF ($\mu, \lambda, \xi$) | 27.4046947, 0.3418181, 2.9171457 | 6.39169, 0.03375, 0.28293 |
| MKITL ($\mu, \beta$) | 0.9046125, 49.7649243 | 0.0809519, 41.671355 |
| MKIEx ($\xi, \beta$) | 0.9011534, 24.3806218 | 0.0698631, 1.7450298 |
| NPF ($\theta, \gamma$) | 1.561634, 3.542441 | 0.1146064, 0.1586595 |

### MLE approach

The maximum likelihood strategy is the most extensively used methodology for estimating parameters. Let $T_1, T_2, \ldots, T_n$ be a random sample and the corresponding observed values, $t_1, t_2, \ldots, t_n$ from MKNPF model with parameter vector $\Psi = (\mu, \lambda, \xi)$. Then the joint probability function $L(t | \Psi) = \prod_{i=1}^{n} f(t_i | \Psi)$ of $T_1, T_2, \ldots, T_n$ as a log-likelihood function is

$$
l(t | \Psi) = \ln \prod_{i=1}^{n} f(t_i | \Psi),
$$

where

$$
= n \log (\xi) + n \log (\lambda) + n \log (\mu + 1)
$$

$$
+ \sum_{i=1}^{n} \log \left( \left[ (1 + \mu t_i) \right]^\xi - (1 - t_i) \right) \right]^{\xi - 1}.
$$
\[ \frac{\partial L_S}{\partial \mu} = \frac{n}{(\mu + 1) + (\xi - 1) \sum_{i=1}^{n} (1 + \mu_i) + \sum_{i=1}^{n} (1 + \mu_i)^{-1}} \]

\[ \frac{\partial L_S}{\partial \lambda} = \frac{-n}{(\mu + 1) + \lambda \sum_{i=1}^{n} (1 - t_i) + \sum_{i=1}^{n} (1 + \mu_i)} \]

\[ \frac{\partial L_S}{\partial \xi} = \frac{-n}{(\mu + 1) + (\xi - 1) \sum_{i=1}^{n} (1 + \mu_i) + \sum_{i=1}^{n} (1 + \mu_i)^{-1}} \]

\[ \frac{\partial L_S}{\partial \lambda} + \frac{\partial L_S}{\partial \xi} = 0 \]

\[ \frac{\partial L_S}{\partial \xi} = \frac{\partial L_S}{\partial \lambda} = 0 \]

\[ L_S(\Psi) = \sum_{i=1}^{n} F(t_{i(\Psi)}; \Psi) - \frac{i}{n+1} \]

\[ L_S(\Psi) = \sum_{i=1}^{n} \left[ \left\{ 1 - \exp \left( \frac{(1 + \mu_i)^{-2} - (1 - t_i)^{-2}}{(1 - t_i)^{-2}} \xi \right) \right\} - \frac{i}{n+1} \right]^2 \]
\[ \frac{\partial L_S(\Psi)}{\partial \mu} = 2 \sum_{i=1}^{n} \nu_i^{(1)}(\Psi) \left\{ \left(1 - \exp \left[ \left( \frac{1 + \mu t_i - (1 - t_i)^2}{(1 - t_i)^2} \right)^\zeta \right] \right) \right\} - \frac{i}{n+1} \]
The decision has been made using the given algorithm:

1. Generate a thousand samples of size $n$ from (6). QF accomplished all of the work and gleaned the data from a uniform distribution.
2. The Set: $(0.95, 1.15, 2.5)$ of true parameter values $\mu, \lambda$ and $\xi$ is employed. The simulated and theoretical model for true parameter values of $\mu, \lambda$ and $\xi$ is shown in Fig. 6.
3. Compute the values for 5000 samples, say $(\hat{\mu}_k, \hat{\lambda}_k, \hat{\xi}_k)$ for $k = 1, 2, \ldots, 5000$.
4. Appraise average bias values and MSEs. These targets are acquired with following formulas:

$$
\text{Bias}_n(\psi) = \frac{1}{5000} \sum_{i=1}^{5000} (\hat{\psi}_i - \psi),
$$

$$
\text{MSE}_n(\psi) = \frac{1}{5000} \sum_{i=1}^{5000} (\hat{\psi}_i - \psi)^2,
$$

where $\psi = (\lambda, \mu, \xi)$.

5. These processes have been replicated with the defined parameters for MLEs, LSEs, and WLSEs for $n = 30, 35, \ldots, 400$. The bias $\text{Bias}_n(\psi)$ and MSE$_n(\psi)$ have both been computed. We utilized optimum function of $R$ to assess the quality of estimates. Tables 1–3 and Fig. 7 illustrate the findings of the simulations. These biases and MSEs fluctuate with respect to $n$ in Fig. 7 (left panels and right panels).

Because as $n$ increases, the bias approaches zero, we may infer that estimators exhibit the attribute of asymptotic unbiasedness. Meanwhile, the trend in the MSE indicates consistency because the error approaches zero as $n$ rises.

Conclusions on the Simulation Results

The outcomes of the study are interpreted through graphs and tables as described in the results and discussion. The main findings of study can be stated as follows:

- The biases of $\hat{\mu}, \hat{\lambda}$ and $\hat{\xi}$ decrease as $n$ rises in all estimating methods.
- The biases of and are relatively positive for MLEs, however there exist negative biases for $\hat{\xi}$.
- The MLEs, LSEs and WLSEs are overestimated however, MLEs of $\hat{\xi}$ are underestimated (see left panel of Fig. 7).
- The LSE and WLSE have analogous results, while the MLE has a slightly different result.
- It is noticed that Bias of $\hat{\mu}_{\text{MLE}} < \text{Bias of } \hat{\mu}_{\text{LSE}} < \text{Bias of } \hat{\mu}_{\text{WLS}}$: Bias of $\hat{\lambda}_{\text{MLE}} > \text{Bias of } \hat{\lambda}_{\text{LSE}} > \text{Bias of } \hat{\lambda}_{\text{WLS}}$ and $\hat{\xi}_{\text{MLE}} < \text{Bias of } \hat{\xi}_{\text{LSE}} < \text{Bias of } \hat{\xi}_{\text{WLS}}$.
- Generally, the smallest MSEs of $\hat{\mu}$ and $\hat{\xi}$ are noticed under MLE, although the least MSE of $\hat{\lambda}$ is recorded under LSE (check right panel of Fig. 7).
• As shown in right panel of Fig. 7, the maximum likelihood technique of estimation outperforms alternative approaches in terms of MSE.
• The LSE is usually the next best estimator, followed by MLE in other situations. According to Fig. 7, when the \( n \) grows, all bias and MSE plots for all parameters eventually reach zero. That highlights the accuracy of various estimation techniques.

Real data practices
In this portion, the MKNPF model’s usefulness for two real data sets is presented. The MKITL (modified Kies inverted Topp-Leone) model [25], MKEx (modified Kies exponential) model [26], and NPF model are all considered viable alternatives to the MKNPF model. The analytical measures, including, -Log-likelihood (-LL), the AIC (Akaike Information Criterion), HQIC (Hannan-Quinn information) and the BIC (Bayesian information criterion) have all been used to compare these models. We also analyze the Kolmogorov–Smirnov (K–S) statistic and its P-value (PV).

As shown in Figs. 8 and 9 MKNPF is the most effective model for fitting datasets I and II.

Table 6
The values of the considered goodness-of-fit indicators for Data I.

| Distribution | -LL   | AIC   | CAIC  | BIC   | HQIC  | \( K - S \) | PV       |
|--------------|-------|-------|-------|-------|-------|------------|---------|
| MKNPF \( (\mu, \lambda, \zeta) \) | -195.9468 | -385.8936 | -385.5859 | -378.6734 | -382.9948 | 0.04680 | 0.9939 |
| MKITL \( (\mu, \beta) \) | -178.85 | -353.7001 | -353.5482 | -348.8866 | -351.7675 | 0.22956 | 0.0002 |
| MKEx \( (\zeta, \beta) \) | -186.7006 | -369.4012 | -369.2493 | -364.5877 | -367.4687 | 0.13983 | 0.0810 |
| NPF \( (\theta, \kappa) \) | -191.6261 | -379.2522 | -379.1003 | -374.4387 | -377.3197 | 0.11996 | 0.1887 |

As shown in Figs. 10 and 11, MKNPF is the best model for modeling datasets I and II, as noticed in Figs. 10 and 11.

4. The MKITL and NPF distributions demonstrate poor fit for the first dataset I and II respectively, as shown in Tables 6 and 7.

Concluding remarks
The three-parameter modified Kies new power function (MKNPF) model is proposed in this content as a novel 3-parameter model. The MKNPF model is more adaptable than other known models when it comes to studying lifespan data. The linear version of PDF, SF, hrf, chr, QF and moments of the MKNPF model are obtained. The estimation methodologies like MLE, LSE, and WLSE are employed to evaluate the parameters of MKNPF model and compared. A emulation study is used to assess the model’s performance under various estimating approaches. We represent two accomplishment based on the COVID 19 mortality rate, proving that the MKNPF distribution is the finest model for fitting this type of data among its counterparts. Based on two real-life examples, the model gives a good fit than the MKITL, MKIEx, and NPF distributions.

Nomenclature

| Symbols | Description |
|---------|-------------|
| \( f (t | \Psi) \) | PDF |
| \( S(t | \Psi) \) | SF |
| \( H(t | \Psi) \) | CHRF |
| \( Q^0(q; \Psi) \) | Quantile Density Function |
| \( \rho_{r,s} \) | Probability weighted moments |

| Abbreviations | Description |
|---------------|-------------|
| MLE | Maximum likelihood Estimation |
| MK | Modified Kies |
| PDF | Probability Density Function |
| MPSM | Maximum product spacing method |
| SF | Survival Function |

Table 7
The values of the considered goodness-of-fit indicators for Data II.

| Distribution | -LL   | AIC   | CAIC  | BIC   | HQIC  | \( K - S \) | PV       |
|--------------|-------|-------|-------|-------|-------|------------|---------|
| MKNPF \( (\mu, \lambda, \zeta) \) | -127.363 | -248.7258 | -248.5015 | -240.5972 | -245.4282 | 0.07692 | 0.5273 |
| MKITL \( (\mu, \beta) \) | -122.942 | -241.8847 | -241.7735 | -236.4656 | -239.6863 | 0.12186 | 0.0740 |
| MKEx \( (\zeta, \beta) \) | -124.913 | -245.826 | -245.7148 | -240.4069 | -243.6276 | 0.09839 | 0.2328 |
| NPF \( (\theta, \kappa) \) | -110.930 | -199.8599 | -199.7488 | -194.4409 | -197.6616 | 0.24203 | 0.0000 |

Symbols
\[ f (t | \Psi) \] PDF
\[ S(t | \Psi) \] SF
\[ H(t | \Psi) \] CHRF
\[ Q^0(q; \Psi) \] QF
\[ M (t | \Psi) \] MGF

Abbreviations
MLE | Maximum likelihood Estimation
MK | Modified Kies
PDF | Probability Density Function
MPSM | Maximum product spacing method
SF | Survival Function

Results in Physics 36 (2022) 105398
References

[1] Lehmann EL. The power of rank tests. Ann Math Stat 1953;23:23–43.
[2] Gupta RC, Gupta PL, Gupta RD. Modeling failure time data by Lehman alternatives. Comm Statist Theory Methods 1998;27(4):887–904.
[3] Cordeiro GM, de Castro M. A new family of generalized distributions. J Stat Comput Simul 2011;81(7):983–98.
[4] Tavangar M. Power function distribution characterized by dual generalized order statistics. 2011.
[5] Ahsanullah M, Shakil M, Golam Kibria BMG. A characterization of the power function distribution based on lower records. In ProLibStat Forum 2013:213–24.
[6] Akhter AS. Methods for estimating the parameters of the power function distribution. Pak J Stat Oper Res 2013:213–24.
[7] Naveed-Shahzad M, Asghar Z, Shehzad F, Shahzadi M. Parameter estimation of power function distribution with TL-moments. Revista Colombiana de Estadistica 2015;38(2):321–34.
[8] Cordeiro GM, dos Santos Brito R. The beta power distribution. Braz J Probab Stat 2012;26(1):88–112.
[9] Al Mutairi Alya O. Truncated weighted power function distribution: properties and applications. Pak J Statist Anal 2011;7(3):491–8.
[10] Tahiri MH, Alizadeh M, Mansoor M, Cordeiro GM, Zabbar M. The Weibull-power function distribution with applications. Hacet J Math Stat 2019;48(1):332–50.
[11] Tahiri MH, Cordeiro GM, Alizadeh M, Mansoor M, Zabbar M, Hamedani GG. The odd generalized exponential family of distributions with applications. J Stat Distrib Appl 2015;2(1):1–28.
[12] Hassan A, Elshabrawy M, Mohamed R. Odd generalized exponential power function distribution: properties & applications. J Stat Distrib Appl 2015;2(1):351–70.
[13] Haq MA, Elgarhy M, Hashmi S, Ozel G, ul Ain Q. Transmuted Weibull power function distribution: Its properties and applications. J Data Sci 2018;16(2):297–418.
[14] Arshad MG, Iqbal MZ. Ahmed M. Exponentiated power function distribution: Properties and applications. J Stat Theory Appl 2020;19(2):297–313.
[15] Arshad MZ, Iqbal MZ, Anees A, Ahmad Z, Balogun OS. A new bathtub shaped failure rate model: properties, and applications to engineering sector. Pak J Stat 2021;37(1).
[16] Iqbal MZ, Arshad MZ, Özel G, Balogun OS. A better approach to discuss medical science and engineering data with a modified Lehmann type-II model. F1000. Research 2021;19(823):823.
[17] Sindhup TN, Shafiq A, Al-Mdallal QM. On the analysis of number of deaths due to covid-19 outbreak data using a new class of distributions. Results Phys 2021;21:103747.
[18] Sindhup TN, Shafiq A, Al-Mdallal Q. Exponentiated transformation of Gumbel type-II distribution for modeling COVID-19 data. Alex Eng J 2020.
[19] Shafiq A, Lone SA, Sindhup TN, Khathib YEI, Al-Mdallal QM, Muhammad T. A new modified Kies Fréchet distribution: Applications of mortality rate of Covid-19. Results Phys 2021;28:104638.
[20] Kamar CS, Dharmaja SHS. The exponentiated reduced kies distribution: Properties and applications. Commun. Stat - Theory Methods 2017;46:8778–890.
[21] Kamarusyamsy SHS, Dharmaja SHS. The exponentiated reduced Kies distribution: Properties and applications. Stat Statist Math Comput 2017;6:315–337.
[22] Shankar DEY, Nassar M, Kumar D. Moments and estimation of reduced kies distribution based on progressive type-II right censored order statistics. Hacet J Math Stat 2019;48(1):332–50.
[23] Al-Babtain AA, Shakhatreh MK, Nassar M, Afify AZ. A new modified kies family: Properties, estimation under complete and type-II censored samples, and engineering applications. Mathematics 2020;8:1345.
[24] Lehmann E. Elements of large-sample theory. New York: Springer-Verlag; 1999.
[25] Gupta RD, Kundu D. Generalized exponential distribution: Different method of estimation. J Stat Comput Simul 2001;69(4):315–337.
[26] Machiel J, Louzada F, Ghitany ME. Comparison of estimation methods for the parameters of the weighted lindley distribution. Appl Math Comput 2013;220:463–71.
[27] Kundu D, Raqab MZ. Generalized Rayleigh distribution: different methods of estimations. J Stat Comput Simul 2005;75(1):187–200.
[28] Dey S, Machiel J, Nadarajah S. Kumaraswamy distribution: different methods of estimation. Appl Math Comput 2018;37(2):2094–111.
[29] Machiel J, Menacer AFB. L-moments and maximum likelihood estimation for the complementary beta distribution with applications on temperature extremes. J Data Sci 2019;17(2):391–406.
[30] E.M. Almetwally, Almongy HM. Maximum product spacing and Bayesian method for parameter estimation for generalized power Weibull distribution under censoring scheme. J Data Sci 2019;17:407–44.
[31] Almetwally EM, Alharbi R, Alnagar D, Hafez EH. A new inverted topp-leone distribution: Applications to the COVID-19 mortality rate in two different countries. J Math Stat 2021;19(3):423–50.
[32] Al-Babtain AA, Shakhatreh MK, Nassar M, Afify AZ. A new modified kies family: Properties, estimation under complete and type-II censored samples, and engineering applications. Mathematics 2020;8(1):1345.
[33] Sindhup TN, Saleem A, Aslam M. Bayesian estimation for topp leone distribution under truncated samples. J Basic Appl Stat Res 2013;2(3):307–46.
[34] Sindhup TN, Aslam M. Preference of prior for Bayesian analysis of the mixed burr type X distribution under type I censored samples. Pak J Stat Oper Res 2014;37–39.
[35] Sindhup TN, Aslam M, Hussain Z. A simulation study of parameters for the censored shifted Gompertz mixture distribution: A Bayesian approach. J Stat Manag Syst 2016;19(3):423–50.
[36] Sindhup TN, Fereoz N, Aslam M, Shafiq A. BayesIan inference of mixture of two Rayleigh distributions: a new look. J Math 2016;48(2):49–64.
[37] Sindhup TN, Hussain Z, Aslam M. Parameter and reliability estimation of inverted Maxwell distribution model. J Stat Manag Syst 2019;22(3):459–93.
[38] Sindhup TN, Khan HM, Hussain Z, Al-Zahrani B. Bayesian inference from the mixture of half-normal distributions under censoring. J Stat Manag Syst 2019;22(3):459–93.
[39] Sindhup TN, Hussain Z, Aslam M. Parameter and reliability estimation of inverted Maxwell distribution model. J Stat Manag Syst 2019;22(3):459–93.
[40] Sindhup TN, Alzahrani B. A new bathtub shaped failure rate model: properties, and applications to engineering sector. Pak J Stat 2021;37(1).
[41] Iqbal MZ, Arshad MZ, Özel G, Balogun OS. A better approach to discuss medical science and engineering data with a modified Lehmann type-II model. F1000. Research 2021;19(823):823.
[42] Sindhup TN, Alzahrani B. A new bathtub shaped failure rate model: properties, and applications to engineering sector. Pak J Stat 2021;37(1).
[43] Iqbal MZ, Arshad MZ, Özel G, Balogun OS. A better approach to discuss medical science and engineering data with a modified Lehmann type-II model. F1000. Research 2021;19(823):823.
[44] Sindhup TN, Shafiq A, Al-Mdallal QM. On the analysis of number of deaths due to covid-19 outbreak data using a new class of distributions. Results Phys 2021;21:103747.
[45] Sindhup TN, Shafiq A, Al-Mdallal Q. Exponentiated transformation of Gumbel type-II distribution for modeling COVID-19 data. Alex Eng J 2020.
[45] Abu El Azm WS, Almetwally EM, Naji AL-Aziz S, El-Bagoury AAAH, Alharbi R, Abo-Kasem OE. A new transmuted generalized lomax distribution: Properties and applications to COVID-19 data. Comput Intell Neurosci 2021;(2021).

[46] Hassan AS, Almetwally EM, Ibrahim GM. Kumaraswamy inverted topp-leone distribution with applications to COVID-19 data. CMC-Comput Mater Continua 2021;68(1):337–358.

[47] Abu El Azm WS, Almetwally EM, Naji AL-Aziz S, El-Bagoury AAAH, Alharbi R, Abo-Kasem OE. A new transmuted generalized lomax distribution: Properties and applications to COVID-19 data. Comput Intell Neurosci 2021;(2021).

[48] Lone SA, Sindhu TN, Jarad F. Additive trinomial fréchet distribution with practical application. Results Phys 2021;(105087).