Entropies and the derivatives of some Heun functions

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Abstract
This short note contains a list of new results concerning the Rényi entropy, the Tsallis entropy, and the Heun functions associated with positive linear operators.

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1 Introduction
This short note contains a list of new results supplementing the articles [10], [11], [12].

The Rényi entropy and the Tsallis entropy associated with positive linear operators have been investigated in [11] and [12]. Section 2 is concerned with two new examples in this direction.

The entropies are naturally related to some Heun functions, as explained in the mentioned articles. Using results from [4] and [13], the derivatives of these Heun functions are studied in Sections 3 and 4.

Detailed proofs will be presented in a forthcoming paper.

2 Integral operators
Let $B_{m-1}(x_0, x_1, \ldots, x_m; \cdot)$ be the $B$-spline function of degree $m-1$ associated to the equidistant points $x_0 < x_1 < \cdots < x_m$. Consider a given function $\sigma \in \mathcal{C}(\mathbb{R})$ such that $\sigma(x) > 0$, $x \in \mathbb{R}$. For $x, t \in \mathbb{R}$, let

$$W_n(x, t) := B_{n-1}\left(x - \sigma(x), x - \sigma(x) + \frac{2\sigma(x)}{n}, \ldots, x + \sigma(x); t\right), \quad n \geq 1.$$ 

The operator $L_n : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ will be defined by

$$L_n f(x) := \int_{\mathbb{R}} W_n(x, t) f(t) dt, \quad n \geq 1, f \in \mathcal{C}(\mathbb{R}), x \in \mathbb{R}. \quad (1)$$
Several properties of operators of this form are investigated, e.g., in [1], [2], [9].

The Rényi entropy and the Tsallis entropy associated with $L_n$ are, respectively,
\[ -\log \int_{\mathbb{R}} W_n^2(x, t) dt \quad \text{and} \quad 1 - \int_{\mathbb{R}} W_n^2(x, t) dt, \]
see, e.g., [11] and the references therein. For the operator described by (1), we have
\[ \int_{\mathbb{R}} W_n^2(x, t) dt = c_n \sigma(x), \quad x \in \mathbb{R}, \]
where $c_n > 0$ is a constant depending only on $n$; in particular $c_1 = \frac{1}{2}, c_2 = \frac{2}{3}, c_3 = \frac{33}{40}$.

Let $e_i(x) = x^i$, $i = 0, 1, 2$. The variance associated with $L_n$ is defined by
\[ V_n(x) = L_n e_2(x) - (L_n e_1(x))^2, \quad x \in \mathbb{R}; \]
in our case,
\[ V_n(x) = \sigma^2(x), \quad x \in \mathbb{R}. \]
Remark that the variance, the Rényi entropy and the Tsallis entropy are synchronous functions of $x$.

Let now $b_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j}, j = 0, 1, \ldots, n, x \in [0, 1]$. Then $B_n : C[0, 1] \to C[0, 1], B_n f(x) := \sum_{j=0}^n f \left( \frac{j}{n} \right) b_{n,j}(x)$, are the classical Bernstein operators.

The Kantorovich modifications of $B_n$ are defined by
\[ Q_{n[k]}^f := \frac{n^k(n-k)!}{n!} D^k B_n f^{(-k)}, \quad f \in C[0, 1], n \geq k \geq 0, \]
where $D$ is the differentiation operator and $f^{(-k)}$ is an antiderivative of order $k$ of $f$; see [3] and the references therein.

It can be proved that
\[ Q_{n[k]}^f = \sum_{j=0}^{n-k} b_{n-k,j} \int_0^1 f(t) B_{k-1} \left( \frac{j}{n}, \frac{j+1}{n}, \ldots, \frac{j+k}{n} ; t \right) dt. \]

Therefore, in order to compute the Rényi entropy and the Tsallis entropy associated with $Q_{n[k]}^f$, we need
\[ S_{n[k]}^f(x) := \int_0^1 \left( \sum_{j=0}^{n-k} b_{n-k,j}(x) B_{k-1} \left( \frac{j}{n}, \frac{j+1}{n}, \ldots, \frac{j+k}{n} ; t \right) \right)^2 dt. \]

In fact, it can be proved that
\[ S_{n+2}^{[2]}(x) = \frac{n+2}{3(n+1)4^n} \sum_{i=0}^n (3n-2i+2)4^i \binom{2i}{i} \binom{2n-2i}{n-i} \left( x - \frac{1}{2} \right)^{2i} \]
\[ = \frac{n+2}{3\pi} \int_0^\pi \left( 1 - 4x(1-x) \sin^2 \frac{\phi}{2} \right) \left( 1 + 2 \cos^2 \frac{\phi}{2} \right) d\phi. \quad (2) \]
The variance associated with $Q^{[2]}_{n+2}$ is

$$V^{[2]}_{n+2}(x) = \frac{n}{(n+2)^2}x(1-x) + \frac{1}{6(n+2)^2}, \quad x \in [0,1].$$

Again it follows that the variance, the Rényi entropy and the Tsallis entropy associated with $Q^{[2]}_{n+2}$ are synchronous functions.

### 3 Heun functions and their derivatives

With classical notation for Heun functions $Hl(a, q; \alpha, \beta; \gamma, \delta; x)$ and hypergeometric functions $\, _2F_1(a, b; c; x)$ we have (see [12])

$$Hl\left(\frac{1}{2}, q; 2q, 1; 1, 1; x\right) = (1-x)^{-2q} \, _2F_1\left(q, q; 1; \frac{x}{x-1}\right).$$

The integral representation can be compared with (2). Combined with

$$\, _2F_1(a, b; c; x) = (1-x)^{-a} \, _2F_1\left(a, c-b; c; \frac{x}{x-1}\right),$$

(3) leads to

$$Hl\left(\frac{1}{2}, q; 2q, 1; 1, 1; x\right) = (1-2x)^{-q} \, _2F_1\left(q, 1-q; 1; \frac{x^2}{2x-1}\right). \quad (4)$$

With notation from [10, 12], let

$$F_n(x) := \sum_{k=0}^{n} \left( \binom{n}{k} x^k (1-x)^{n-k} \right)^2,$$

$$G_n(x) := \sum_{k=0}^{\infty} \left( \binom{n+k-1}{k} x^k (1+x)^{-n-k} \right)^2,$$

$$U_n(x) := \sum_{k=0}^{n} \left( \binom{n}{k} x^k (1+x)^{-n} \right)^2,$$

$$J_n(x) := \sum_{k=0}^{\infty} \left( \binom{n+k}{k} x^k (1-x)^{n+1} \right)^2.$$

It was proved in [12] that

$$F_n(x) = Hl\left(\frac{1}{2}, -n; -2n, 1, 1; x\right),$$

(5)
\[ G_n(x) = Hl \left( \frac{1}{2}, n; 2n, 1; 1, -x \right), \]  

(6)

and, moreover, the polynomial Heun function \( F_n(x) \) and the rational Heun function \( G_n(-x) \) are related by

\[ G_n(-x) = (1 - 2x)^{1-2n} F_{n-1}(x). \]  

(7)

Similarly, \( U_n(x) \) and \( J_n(x) \) can be expressed as

\[ U_n(x) = F_n \left( \frac{x}{x + 1} \right), \]  

(8)

\[ J_n(x) = \left( \frac{1 - x}{1 + x} \right)^{2n+1} F_n \left( \frac{1}{1 - x} \right). \]  

(9)

Recall that the Legendre polynomials \( P_n(x) \) are related to \( 2F_1 \) by

\[ 2F_1(-n, n+1; 1; x) = P_n(1-2x). \]  

(10)

From (5), (4) and (10) we get

\[ F_n(x) = (1 - 2x)^n P_n \left( \frac{2x^2 - 2x + 1}{1 - 2x} \right). \]  

(11)

This formula was proved (with a different method) by Thorsten Neuschel [7] and Geno Nikolov [8]; it was used in order to prove a conjecture involving the polynomials \( F_n(x) \).

The derivative of a Heun function \( Hl(a, q; \alpha, \beta; \gamma, \delta; x) \) satisfying

\[ q = a\alpha\beta \]  

(12)

was studied in [4]. From the corresponding results we infer, for \( \gamma \not= 0, -1, -2, \ldots, \)

\[ \frac{d}{dx} Hl \left( \frac{1}{2}, \frac{1}{2} a\beta; a, \beta; \gamma, \gamma; x \right) \]  

(13)

\[ = \alpha\beta \gamma (1 - 2x) Hl \left( \frac{1}{2}, \frac{1}{2}; (\alpha + 2) (\beta + 2); \alpha + 2, \beta + 2, \gamma + 1, \gamma + 1; x \right), \]

(14)

\[ = \alpha\beta \gamma (1 - 2x)^{2\gamma - \alpha - \beta - 1} Hl \left( \frac{1}{2}, \frac{1}{2}; (2\gamma - \alpha)(2\gamma - \beta); 2\gamma - \alpha, 2\gamma - \beta, \gamma + 1, \gamma + 1; x \right). \]

The equality between the right-hand sides of (13) and (14) follows also from line 3 in Table 2 of [6]. Let us remark that the Heun functions in these right-hand sides satisfy also the condition (12), so that it is possible to express their derivatives in terms of other Heun functions.
From (5) and (13) we get

\[ \frac{d}{dx} F_n(x) = 2n(2x - 1) Hl \left( \frac{1}{2}, 3 - 3n; 2 - 2n, 3; 2, 2; x \right), \]  \tag{15}

and finally, for \( i = 0, 1, \ldots, n, \)

\[ Hl \left( \frac{1}{2}, (i - n)(2i + 1); 2(i - n), 2i + 1; i + 1; x \right) \]  \tag{16}

\[ = \frac{(2i)!!}{(2i - 1)!!} \frac{1}{4^n} \binom{n}{i}^{-1} \sum_{j=0}^{n-i} \frac{4^j}{i} \binom{2i + 2j}{i} \binom{2n - 2i - 2j}{n - i - j} \left( x - \frac{1}{2} \right)^{2j}. \]

To conclude this section, let us remark that the Heun function from (3) satisfies the condition (33) from [5]; the consequence of this fact will be investigated elsewhere.

4 A confluent Heun function

Let \( u(x) = HC(p, \gamma, \delta, \alpha, \sigma; x) \) be the solution of the confluent Heun equation

\[ u''(x) + \left( 4p + \frac{\gamma}{x} + \frac{\delta}{x - 1} \right) u'(x) + \frac{4p\alpha x - \sigma}{x(x - 1)} u(x) = 0, \]  \tag{17}

with \( u(0) = 1 \) (See [13]). From [13 (21)] we get

\[ \frac{d}{dx} HC(p, \gamma, 0, \alpha, 4p\alpha; x) = -\frac{\sigma}{\gamma} HC(p, \gamma + 1, 0, \alpha + 1, 4p(\alpha + 1); x), \]  \tag{18}

\[ \frac{d}{dx} HC(p, \gamma, 0, \alpha, 4p\alpha; x) = \frac{\sigma}{\gamma}(x - 1)HC(p, \gamma + 1, 2, \alpha + 2, 4p(\alpha + 1) - \gamma - 1; x). \]  \tag{19}

With notation from [10], [12], let

\[ K_n(x) := \sum_{k=0}^{\infty} \left( e^{-nx} \frac{(nx)^k}{k!} \right)^2. \]

Then (see [10], [12]),

\[ xK_n''(x) + (4nx + 1)K_n'(x) + 2nK_n(x) = 0. \]  \tag{20}

We get immediately

\[ K_n(x) = HC \left( n, 1, 0, \frac{1}{2}, 2n; x \right). \]  \tag{21}

From (19) and (21) it follows that

\[ HC \left( n, 2, 2, \frac{5}{2}, 6n - 2; x \right) = \frac{1}{2n(x - 1)} K_n'(x). \]  \tag{22}
Similarly, from (18) and (21),

\[
HC\left(n, 2, 0, \frac{3}{2}6n; x \right) = -\frac{1}{2n}K_n'(x).
\] (23)

Now applying repeatedly (18) we can generalize (23) to

\[
HC\left(n, j + 1, 0, \frac{2j + 1}{2}, 2n(2j + 1); x \right) = \frac{K_n^{(j)}(x)}{K_n^{(j)}(0)}, \quad j \geq 0,
\] (24)

where

\[
K_n^{(j)}(0) = (-2n)^j \sum_{i=0}^{[j/2]} \binom{j}{2i} \binom{2i}{i} x^{-i}.
\] (25)

From (20) we get

\[
\left(K_n^{(j)}\right)^{''} + \left(4n + \frac{j + 1}{x}\right) \left(K_n^{(j)}\right)^{'} + \frac{2n(2j + 1)}{x}K_n^{(j)} = 0,
\] (26)

and this provides an alternative proof of (24).

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