An Alternative Definition for Improper Integral with Finite Limits
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Introduction

There are two basic types of improper integrals. Integrals with an infinite limit are defined as the limit of a series of proper integrals as one of the limits approaches infinity. Improper integrals with finite limits are needed when the integrand does not have a finite limiting value as the variable of integration approaches a particular “critical” value. In this case, the improper integral is defined as the limit of a series of proper integrals as one of the limits approaches the critical value. Improper integrals with more than one critical value, or with interior critical values, can be found as a sum of these two basic types.

The improper integral with an infinite upper limit defined by
\[
\int_a^\infty f(x) \, dx \equiv \lim_{b \to \infty} \left\{ \int_a^b f(x) \, dx \right\}
\]
exists when the limit exists, with a similar definition for an integral with an infinite lower limit.

The finite limit improper integral with a critical lower limit, defined by
\[
\int_\alpha^\beta g(u) \, du \equiv \lim_{\delta \to 0^+} \left\{ \int_{\alpha+\delta}^\beta g(u) \, du \right\}
\]
events when that limit exists, with a similar definition for an integral with a critical upper limit. For any of the above cases, when the limit does not exist, the integrals are said to not exist, or to diverge. We will refer to the above as the conventional definitions.

In “An Alternative Definition for Improper Integral with Infinite Limit” [Blischke 2008], the following definition for an integral with an infinite limit was introduced:
\[
\int_a^\infty f(x) \, dx \equiv \lim_{b \to \infty} \left\{ \int_a^b f(x) \, dx + \int_b^{b+c} f(x)z(x-b) \, dx \right\}
\]
where \( f(x) \) is the function to be integrated, and where \( z(x) \) is a termination function, defined therein. The inclusion of the additional term inside the limit allows convergence to be rigorously shown for a greater range of functions, \( f(x) \). The overstruck \( Z \) on integrals using the alternate definition is included there to distinguish them from integrals that exist using conventional definitions.
A useful form that is equivalent to (1) is

\[
\int_{a}^{\infty} f(x) \, dx = -F(a) - \lim_{b \to \infty} \left\{ \int_{0}^{b} F(x + b) z'(x) \, dx \right\} \tag{2}
\]

where \(F(x)\) is defined (for some arbitrary lower limit \(\phi\)) by

\[
F(x) \equiv \int_{\phi}^{x} f(x') \, dx'. \tag{3}
\]

When the integrals and the limit in (1) or (2) exist, the integral of \(f(x)\) is said to exist under the alternate definition, and to have the value of the limit. In [Bli08], it was shown that for all termination functions for which the limit exists, the integral will have the same value, so that the definition gives a unique value. Many other properties of integrals found using the alternative definition were shown there.

**Definition for Finite Limit**

It is desirable to introduce a corresponding definition for an improper integral where the critical limit is finite. Using the conventional definitions, it is possible to convert between finite limit and infinite limit integrals using \(u\)-substitution. Our goal is to be able to do the same with our new definitions. For example, with \(u = 1/x\), we would like to be able to say,

\[
\int_{a}^{\infty} f(x) \, dx = \int_{0}^{1/a} \frac{f(\frac{1}{u})}{u^2} \, du = \int_{0}^{1/a} g(u) \, du
\]

where

\[
g(u) \equiv \frac{f(\frac{1}{u})}{u^2}.
\]

We introduce as our definition of improper integral for a function with a critical lower limit \(\alpha\) and a noncritical upper limit \(\beta\),

\[
\int_{\alpha}^{\beta} g(u) \, du \equiv \lim_{\delta \to 0^+} \left\{ \int_{\alpha}^{\alpha+\delta} g(u) w \left( \frac{u - \alpha}{\delta} \right) \, du + \int_{\alpha+\delta}^{\beta} g(u) \, du \right\} \tag{4}
\]

where \(w(v)\) is defined below. We will again follow the convention of using the overstruck \(\overline{Z}\) in this paper for the definition being presented. An improper integral with critical finite upper limit and noncritical lower limit is defined similarly as

\[
\int_{\alpha}^{\beta} g(u) \, du \equiv \lim_{\delta \to 0^+} \left\{ \int_{\alpha}^{\beta-\delta} g(u) \, du + \int_{\beta-\delta}^{\beta} g(u) w \left( \frac{\beta - u}{\delta} \right) \, du \right\}. \tag{5}
\]
For simplicity in the derivations, and without loss of generality, for the remainder of this paper we will take the critical limit to be the lower limit, and to be 0, giving

\[ \int_{0}^{\beta} g(u) \, du \equiv \lim_{\delta \to 0^+} \left\{ \int_{0}^{\delta} g(u) w(u/\delta) \, du + \int_{\delta}^{\beta} g(u) \, du \right\}. \] (6)

The function \( w(v) \) will be referred to as the initialization function, analogous to the termination functions in the infinite limit case. It is required to be finite, to not depend on \( \delta \), and to satisfy

\[ w(v) = \begin{cases} 0 & v \leq \epsilon \\ 1 & v \geq 1 \end{cases} \] (7)

and so

\[ w'(v) = \begin{cases} 0 & v < \epsilon \\ 0 & v > 1 \end{cases} \] (8)

for some \( \epsilon \) with

\[ 0 < \epsilon < 1. \] (9)

A less restrictive condition on \( w(v) \) may be possible, for example allowing \( \epsilon = 0 \) with \( w(v) \to 0 \) suitably fast as \( v \to 0 \). However, requiring nonzero \( \epsilon \) will allow us to show equivalence between the finite limit definition and the infinite limit definition from [Bli08].

With (7) and (8), we have that

\[ \int_{0}^{1} w'(v) = \int_{\epsilon}^{1} w'(v) = 1. \] (10)

We can get \( w(v) \) for \( 0 < v < 1 \) from \( w'(v) \) as

\[ w(v) = \int_{0}^{v} w'(u') \, du'. \] (11)

We now define

\[ G(u) \equiv \int_{\phi}^{u} g(s) \, ds \] (12)

and using integration by parts, (6) becomes

\[ \int_{0}^{\beta} g(u) \, du = \lim_{\delta \to 0^+} \left\{ G(u) w(u/\delta) \bigg|_{0}^{\delta} - \frac{1}{\delta} \int_{0}^{\delta} G(u) w'(u/\delta) \, du + G(u) \bigg|_{\delta}^{\beta} \right\} \\
= G(\beta) - \lim_{\delta \to 0^+} \left\{ \frac{1}{\delta} \int_{0}^{\delta} G(u) w'(u/\delta) \, du \right\}. \] (13)
This form is analogous to (2), and can be an easier form to work with than (3).

We now follow a path analogous to that taken with termination functions in [Bli08]. Given two initialization functions \( w_1(v) \) and \( w_2(v) \), we can combine them to obtain a third, via their derivatives, as

\[
w'(v) \equiv \int_0^1 \frac{w_1'(v/v') w'_2(v')}{v'} \, dv'.
\]  

(14)

Using (8), (14) can also be written as

\[
w'(v) = \int_v^1 \frac{w_1'(v/v') w'_2(v')}{v'} \, dv'.
\]  

(15)

If \( w_1'(v) \) satisfies (8), we can see that \( w'(v) \) also satisfies (8). We can also show that \( w_1'(v) \) satisfies (10), and therefore also (7):

\[
\int_0^1 w'(v) = \int_0^1 \left[ \int_0^1 \frac{w_1'(v/v') w'_2(v')}{v'} \, dv' \right] \, dv = \int_0^1 \left[ \int_0^v w_1'(v/v') \, dv \right] \frac{w'_2(v')}{v'} \, dv' = \int_0^1 \left[ \int_0^{v'} w_1'(v/v') \, dv \right] \frac{w'_2(v')}{v'} \, dv' = \int_0^1 v' \frac{w'_2(v')}{v'} \, dv' = 1.
\]  

(16)

We will denote the combined initialization function using the same notation we used for termination functions,

\[
w(v) = w_1(v) \odot w_2(v).
\]  

(17)

We also have, substituting \( v'' = v/v' \), that

\[
w'(v) = \int_0^1 \frac{w_1'(v/v') w'_2(v')}{v'} \, dv' = \int_0^1 \frac{w_1'(v/v'') w'_2(v/v'')}{v' v''} \, dv' = \int_0^1 \frac{w_1'(v/v'') w'_2(v/v'')}{v''} \, dv'
\]  

(18)

so the relation (17) satisfies commutivity.

We will now examine what happens when we switch between an infinite limit integral and a finite limit improper integral. We will use the general change of variable defined by

\[
u = \psi(x) \quad x = \psi^{-1}(u).
\]  

(19)
We require for all \(x > \psi^{-1}(\beta)\) that \(\psi(x)\) be finite, that it be strictly monotonic with
\[
\psi'(x) < 0,
\] (20)
and that
\[
\lim_{x \to \infty} \{\psi(x)\} \equiv 0.
\] (21)
Combining these properties, we obtain that
\[
\psi(x) > 0,
\] (22)
that
\[
0 \leq \frac{\psi(x + b)}{\psi(b)} \leq 1 \quad \text{for} \quad b \geq \psi^{-1}(\beta), \; x \geq 0
\] (23)
and that
\[
\psi(x) = -\int_{x}^{\infty} \psi'(x')dx'.
\] (24)
Using (19) in (12) gives
\[
G(u) = \int_{\psi^{-1}(\phi)}^{\psi^{-1}(u)} g(\psi(t))\psi'(t) \, dt.
\] (25)
Defining
\[
f(t) \equiv -g(\psi(t))\psi'(t)
\] (26)
and
\[
F(x) \equiv \int_{\psi^{-1}(\phi)}^{x} f(t) \, dt
\] (27)
we get
\[
G(u) = -\int_{\psi^{-1}(\phi)}^{\psi^{-1}(u)} f(t) \, dt = -F(\psi^{-1}(u))
\] (28)
or
\[
F(x) = -G(\psi(x)).
\] (29)
We will now define an infinite limit integral corresponding to our finite limit integral as
\[
\Xi \int_{\psi^{-1}(\beta)}^{\infty} f(x) \, dx \equiv \int_{0}^{\beta} g(u) \, du.
\] (30)
In (30) we use an overstruck \(\Xi\), instead of an overstruck \(\mathcal{Z}\), to allow us to distinguish the resulting integral (39) from the infinite limit integral definition introduced in [Bl08]. We will use the overstruck \(\Xi\) throughout this paper for the infinite limit improper integral that is obtained from our finite limit improper integral defined by (6) via a change of variable within the definition. Since the definition is in terms of conventional integrals, we know we can
safely perform those changes of variable. Note that we haven’t yet shown any relation between this and the infinite limit integral definition from [Bli08].

Using (30) in (13) we get

$$\int_{\psi^{-1}(\beta)}^{\infty} f(x) \, dx = G(\beta) - \lim_{\delta \to 0^+} \left\{ \frac{1}{\delta} \int_0^\delta G(u) w'(u/\delta) \, du \right\}$$

$$= G(\beta) - \lim_{\delta \to 0^+} \left\{ \frac{1}{\delta} \int_{\psi^{-1}(\delta)}^{\infty} G(\psi(x)) w' \left( \frac{\psi(x)}{\delta} \right) \psi'(x) \, dx \right\}$$

$$= -F \left( \psi^{-1}(\beta) \right) + \lim_{\delta \to 0^+} \left\{ \frac{1}{\delta} \int_{\psi^{-1}(\delta)}^{\infty} F(x) w' \left( \frac{\psi(x)}{\delta} \right) \psi'(x) \, dx \right\}$$

$$= -F \left( \psi^{-1}(\beta) \right) + \lim_{\delta \to 0^+} \left\{ \frac{1}{\psi(b)} \int_b^{\infty} F(x) w' \left( \frac{\psi(x)}{\psi(b)} \right) \psi'(x) \, dx \right\}$$

(31)

where $b \equiv \psi^{-1}(\delta)$.

We now define $\zeta(x, b)$ such that

$$\zeta(0, b) \equiv 1$$

(32)

and

$$\zeta'(x - b, b) \equiv \begin{cases} w' \left( \frac{\psi(x)}{\psi(b)} \right) & x \geq b \\ 0 & x < b \end{cases}$$

(33)

or

$$\zeta'(x, b) = \begin{cases} w' \left( \frac{\psi(x+b)}{\psi(b)} \right) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(34)

From (33), $\zeta'(x, b) = 0$ when $\psi(x + b) < \epsilon \psi(b)$. We will define $e(\epsilon, b)$ by

$$\frac{\psi(e(\epsilon, b) + b)}{\psi(b)} \equiv \epsilon$$

(35)

so we get

$$e(\epsilon, b) = \psi^{-1}(\epsilon \psi(b)) - b$$

(36)

and then

$$\zeta'(x, b) = 0 \, \text{ for } x > e(\epsilon, b).$$

(37)

Since $w'(v)$ and $\frac{w'(x+b)}{\psi(b)}$ are finite, we have that $\zeta'(x, b)$ is finite, and from (32), (34), and (37) it is seen that

$$\int_{-\infty}^{\infty} \zeta'(x, b) \, dx = \int_0^{e(\epsilon, b)} \zeta'(x, b) \, dx = -1.$$  (38)

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The integral in (38) is proper and absolutely convergent.

In a reciprocal fashion, if we have \( \zeta(x, b) \) that can be shown to satisfy (32) through (34) and (37) for some \( w'(v) \), with \( \zeta'(x, b) \) finite, then (7) through (10) can be satisfied.

Substituting, (31) becomes

\[
\int_0^\infty f(x) \, dx = -F\left(\psi^{-1}(\beta)\right) + \lim_{b \to \infty} \left\{ \int_b^\infty F(x) \zeta'(x - b, b) \, dx \right\}.
\]  

Because of the similarity of (39) and (2), we will refer to \( \zeta(x, b) \) also as a termination function, even though it is not constant WRT \( b \). When it is not obvious from context which termination function we are referring to, we will call the \( \zeta(x, b) \) a termination function of the second type, and will call termination functions as described in [Bli08] a termination function of the first type.

Using the relations between \( w(v) \) and \( \zeta(x, b) \), we can freely switch between the finite limit integral (13) and its corresponding infinite limit integral (39). We’ll denote the relation between an initialization function \( w(v) \) used in the finite limit integral and the termination function \( \zeta(x, b) \) used in the corresponding infinite limit integral, and given by (33) or (34), as

\[
\zeta(x) \Leftrightarrow w(v).
\]  

We next show that combining initialization functions and combining termination functions of the second type are equivalent. Beginning with (15) and using

\[
v = \frac{\psi(x + b)}{\psi(b)}
\]

and

\[
v' = \frac{\psi'(x' + b)}{\psi(b)}
\]

we get

\[dv' = \frac{\psi'(x' + b)}{\psi(b)} \, dx'.\]  

and

\[
w' \left( \frac{\psi(x + b)}{\psi(b)} \right) = -\int_0^x w'_1 \left( \frac{\psi(x + b)}{\psi(x' + b)} \right) w'_2 \left( \frac{\psi(x' + b)}{\psi(b)} \right) \frac{\psi'(x' + b)}{\psi(b)} \, dx'.
\]

\[
= -\int_0^x \left[ w'_1 \left( \frac{\psi(x + b)}{\psi(x' + b)} \right) \frac{\psi'(x' + b)}{\psi'(x') \psi(b)} \right] \left[ w'_2 \left( \frac{\psi(x' + b)}{\psi(b)} \right) \frac{\psi'(x')}{\psi(b)} \right] \, dx'.
\]

\[
= -\int_0^x \left[ \psi'(x + b)/\psi(b) \right] \psi'(x') \psi(b) \, dx'.
\]

\[
= -\int_{x+b}^b \left[ \psi'(x + b)/\psi(b) \right] \psi'(x') \psi(b) \, dx'.
\]  

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Using (33) and (34)
\[
w' \left( \frac{\psi(x + b)}{\psi(b)} \right) = -\frac{\psi(b)}{\psi'(x + b)} \int_b^{x + b} \zeta_1'(x + b - x', b) \zeta_2'(x' - b, b) \, dx' \\
= -\frac{\psi(b)}{\psi'(x + b)} \int_x^x \zeta_1'(x - x', b) \zeta_2'(x', b) \, dx' \tag{45}
\]
and so (34) gives
\[
\zeta'(x, b) = -\int_0^x \zeta_1'(x - x', b) \zeta_2'(x', b) \, dx' \tag{46}
\]
Thus we have that the combination of two termination functions of the second type is also a termination function of the second type. Note that each correspondence between termination function and initialization function uses the same change of variable, \( u = \psi(x) \).
We write this more compactly as
\[
\zeta'(x, b) = \zeta_1'(x, b) \otimes \zeta_2'(x, b) \tag{47}
\]
and denote the combined termination function using the notation
\[
\zeta(x, b) = \zeta_1(x, b) \odot \zeta_2(x, b). \tag{48}
\]
Thus, given
\[
\zeta_1(x, b) \Leftrightarrow w_1(v) \quad \text{and} \quad \zeta_2(x, b) \Leftrightarrow w_2(v) \tag{49}
\]
we find that
\[
\zeta_1(x, b) \odot \zeta_2(x, b) \Leftrightarrow w_1(v) \odot w_2(v). \tag{50}
\]
That is, given two initialization functions, the combination of their corresponding termination functions is equal to the termination function corresponding to their combination.

**Equivalence with Infinite Limit Definition**

We are now ready to show equivalence between our finite limit and infinite limit definitions. Equation (39) is almost the same form as (2) (and Eq. (9) of [Bli08]). The only difference is that \( \zeta(x, b) \) is a function of \( b \), unlike the termination functions described in [Bli08]. For a particular choice of \( \psi(x) \), however, the dependence of \( \zeta(x, b) \) on \( b \) vanishes. In that case, the termination function of the second type is also a termination function of the
first type. The infinite limit integral corresponding to our finite limit improper integral, (39), is then identical to the infinite limit integral presented in [Bli08].

Choosing, with $\alpha > 0$,

$$\psi(x) = e^{-\alpha x}$$  \hspace{1cm} (51)

we get

$$\psi'(x) = -\alpha e^{-\alpha x}.$$  \hspace{1cm} (52)

Substituting these into (34) we can write

$$\zeta'(x,b) = -\alpha w' \left( e^{-\alpha x} \right) e^{-\alpha x} = z'(x).$$  \hspace{1cm} (53)

Since this is not a function of $b$, it satisfies the requirements for a termination function of the first type. Using this transformation, we can see that for every finite limit improper integral there is a corresponding infinite-limit improper integral using the definition from [Bli08]. Also, for any termination function $z(x)$, the corresponding initialization function can be explicitly found as

$$w' \left( e^{-\alpha x} \right) = -z'(x) \frac{e^{\alpha x}}{\alpha}$$  \hspace{1cm} (54)

$$w' \left( u \right) = -z' \left( \frac{-\ln(u)}{\alpha} \right) \frac{1}{\alpha u}.$$  \hspace{1cm} (55)

We thus have, when the change of variable (19) is given by (51), that

$$\int_{-\ln(\beta)}^{\infty} f(x) \, dx = \int_{-\ln(\beta)}^{\infty} f(x) \, dx$$  \hspace{1cm} (56)

and the definition for the infinite limit case is seen to be equivalent to the definition for the finite limit case,

$$\int_{0}^{\beta} g(u) \, du = \int_{-\ln(\beta)}^{\infty} f(x) \, dx$$  \hspace{1cm} (57)

We find that $c$ and $\epsilon$ are related as

$$\epsilon = e^{-\alpha c}.$$  \hspace{1cm} (58)

The equivalence between the finite limit and infinite limit definitions given by (57) and the correspondence between initialization and termination functions given by (40) and (53) means that the properties found for the infinite limit case in [Bli08] all have corresponding properties in the finite limit case. We will list those properties here.
If $w_1(\nu)$ is an initialization function for $g(u)$, then for any other initialization function $w_2(\nu)$, $w(\nu)$ given by (14) or (15) is also an initialization function for $g(u)$, and gives the same value for the integral. The integral defined using (6) produces a unique value for all initialization functions for which the limit exists. When the integral exists using the conventional definition, our definition gives the same answer,

$$
\int_0^\beta g(u) \, du = \int_0^\beta g(u) \, du.
$$

(59)

Our definition satisfies linearity,

$$
a \int_0^\beta g(u) \, du + b \int_0^\beta h(u) \, du = \int_0^\beta [ag(u) + bh(u)] \, du
$$

(60)

when the integrals on the LHS both exist.

Differentiation under the integral sign can be performed when both of the integrals exist, and we have

$$
\frac{d}{dy} \left[ \int_0^\beta g(u, y) \, du \right] = \left[ \int_0^\beta \frac{\partial}{\partial y} g(u, y) \, du \right].
$$

(61)

When one or the other integrals exists with the conventional definition, we also get

$$
\frac{d}{dy} \left[ \int_0^\beta g(u, y) \, du \right] = \left[ \int_0^\beta \frac{\partial}{\partial y} g(u, y) \, du \right]
$$

(62)

and

$$
\frac{d}{dy} \left[ \int_0^\beta g(u, y) \, du \right] = \left[ \int_0^\beta \frac{\partial}{\partial y} g(u, y) \, du \right].
$$

(63)

Interchange of the order of iterated integrations is also allowed. Here we assume that

$$
\int_0^\beta g(u, y) \, dx
$$

exists over the domain $\gamma \leq y \leq \delta$ for some initialization function $w(v, y)$, and that

$$
h(u, t) \equiv \int_0^t g(u, y) s(y) \, dy
$$

(65)

exists, using the Riemann definition, over the domain $\gamma \leq t \leq \delta$, for $0 < u \leq \beta$. The function $s(y)$ is arbitrary. We further assume that

$$
\int_0^\beta h(u, y) \, dx
$$

(66)

exists over the domain $\gamma \leq y \leq \delta$ for some initialization function $\tilde{w}(v, y)$. 

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From Eq. (59) in [Bli08], we have that

$$
\int_\gamma^\delta s(y) \left[ \int_{\gamma}^\infty f(x, y) \, dx \right] \, dy = \int_{\gamma}^\infty \left[ \int_\gamma^\delta s(y) f(x, y) \, dy \right] \, dx. \tag{67}
$$

Using (26), (65), and (56) and (57), the RHS of (67) becomes

$$
\int_{\gamma}^\infty \left[ \int_\gamma^\delta s(y) f(x, y) \, dy \right] \, dx = \int_{\gamma}^\infty \left[ \int_\gamma^\delta -s(y) g(\psi(x), y) \psi'(x) \, dy \right] \, dx \\
= \int_{\gamma}^\infty \left[ -h(\psi(x), y) \psi'(x) \right]_\gamma^\delta \, dx \\
= \int_0^\delta \left[ \int_\gamma^\delta s(y) g(u, y) \, dy \right] \, du. \tag{68}
$$

Using (56) and (57) in the LHS of (67) we then get

$$
\int_\gamma^\delta s(y) \left[ \int_0^\delta g(u, y) \, du \right] \, dy = \int_0^\delta \left[ \int_\gamma^\delta s(y) g(u, y) \, dy \right] \, du. \tag{69}
$$

When one side exists using the conventional definition, we also get either

$$
\int_\gamma^\delta s(y) \left[ \int_0^\delta g(u, y) \, du \right] \, dy = \int_0^\delta \left[ \int_\gamma^\delta s(y) g(u, y) \, dy \right] \, du \tag{70}
$$

or

$$
\int_\gamma^\delta s(y) \left[ \int_0^\delta g(u, y) \, du \right] \, dy = \int_0^\delta \left[ \int_\gamma^\delta s(y) g(u, y) \, dy \right] \, du. \tag{71}
$$

A change of variable of integration of the form $u' = cu$ for nonzero constant $c$ can be performed. An arbitrary change of the variable of integration, however, is not necessarily valid.

**Examples**

The following two examples show the evaluation of integrals which do not exist using the conventional definition.

**Example 1:**

$$
g(u) = \frac{\sin(1/u)}{u^2}
$$

$$
G(u) = \int_0^u \frac{\sin(1/s)}{s^2} \, ds = \cos(1/u)
$$
Using (13) we have
\[ \int_0^{1/a} \frac{\sin(1/u)}{u^2} \, du = G(1/a) - \lim_{\delta \to 0^+} \left\{ \frac{1}{\delta} \int_0^\delta G(u)w(u/\delta) \, du \right\}. \]

We can use
\[ w(v) = \begin{cases} 
(2v - 1) & 1/2 \leq v \leq 1 \\
0 & v < 1/2
\end{cases} \]
so
\[ w'(v) = \begin{cases} 
2 & 1/2 \leq v \leq 1 \\
0 & \text{else}
\end{cases} \]
giving us
\[ \int_0^{1/a} \frac{\sin(1/u)}{u^2} \, du = \cos(a) - \lim_{\delta \to 0^+} \left\{ \frac{1}{\delta} \int_0^\delta 2 \cos(1/u) \, du \right\}. \]

Evaluating the integral on the RHS gives
\[ \int_{\delta/2}^\delta \cos(1/u) \, du = \frac{2}{\delta} \left( u \cos(1/u) + Si(1/u) \right) \bigg|_{\delta/2}^\delta \]
where \( Si \) is the sine integral. Expanding this to the necessary order of argument in the large argument \((u \to 0^+)\) limit,
\[ Si(1/u) \approx \frac{\pi}{2} - u \cos(1/u) - u^2 \sin(1/u). \]
The cosine terms cancel, and the \( \pi/2 \) term doesn’t contribute when the limits are taken. Thus, the integral is of order \( \delta^2 \), and we get for the limit
\[ \lim_{\delta \to 0^+} \{O(\delta)\} = 0 \]
so we obtain that
\[ \int_0^{1/a} \frac{\sin(1/u)}{u^2} \, du = \cos(a). \]

Example 2:
\[ g(u) = \frac{\cos(1/u)}{u^3} \]
\[ G(u) = \int_0^u \frac{\cos(1/s)}{s^3} \, ds = - \cos(1/u) - \frac{\sin(1/u)}{u} \]

Using (13) we have
\[ \int_0^{1/a} \frac{\cos(1/u)}{u^3} \, du = G(1/a) - \lim_{\delta \to 0^+} \left\{ \frac{1}{\delta} \int_0^\delta G(u)w'(u/\delta) \, du \right\}. \]
We will use
\[ w(v) = \begin{cases} 
3(2v - 1)^2 - 2(2v - 1)^3 & \frac{1}{2} \leq v \leq 1 \\
0 & v < \frac{1}{2}
\end{cases} \]
so
\[ w'(v) = \begin{cases} 
12(2v - 1)(2 - 2v) & \frac{1}{2} \leq v \leq 1 \\
0 & \text{else}
\end{cases} \]
giving
\[
\int_0^{1/a} \frac{\cos(1/u)}{u^3} du = -\cos(a) - a \sin(a) + \\
\lim_{\delta \to 0^+} \left\{ \frac{12}{\delta} \int_{\delta/2}^{\delta} \left( \cos(1/u) + \frac{\sin(1/u)}{u} \right) (2u/\delta - 1)(2 - 2u/\delta) du \right\}.
\]
Evaluating the integral on the RHS gives
\[
\int_{\delta/2}^{\delta} \left( \cos(1/u) + \frac{\sin(1/u)}{u} \right) (2u/\delta - 1)(2 - 2u/\delta) du \\
= -\frac{2}{\delta^2} \left[ \frac{u}{6} (6\delta^2 - 9u\delta + 4u^2 + 4) \cos(1/u) + \\
\frac{u}{6} (4u - 9\delta) \sin(1/u) + \frac{3\delta}{2} \text{Ci}(1/u) + \frac{2}{3} \text{Si}(1/u) \right]_{\delta/2}^{\delta}
\]
where \( Si \) and \( Ci \) are the sine integral and cosine integral. Expanding these to the necessary order in the argument in the large argument \((u \to 0^+)\) limit, we obtain
\[
\text{Ci}(1/u) \approx u \sin(1/u) - u^2 \cos(1/u) - 2u^3 \sin(1/u)
\]
\[
\text{Si}(1/u) \approx \frac{\pi}{2} - u \cos(1/u) - u^2 \sin(1/u) + 2u^3 \cos(1/u).
\]
With a little algebra there is much cancellation of terms, and the integral can be shown to be at least of order \( \delta^2 \). We thus get for the limit
\[
\lim_{\delta \to 0^+} \{ O(\delta) \} = 0
\]
so we get
\[
\int_0^{1/a} \frac{\cos(1/u)}{u^3} du = -\cos(a) - a \sin(a).
\]
General Transform between Infinite and Finite Limit Integrals

We now return to the general transformation between finite and infinite limit integrals given by (19). The particular change of variable of integration given by (51) can always be used to convert between the two types of improper integral, showing that the two definitions are equivalent. We would like to be able to show that the infinite limit improper integral given in [Bli08] and the finite limit improper integral are more generally equivalent, that we can always use (19) through (21) to switch between finite limit and infinite limit integrals. For the general case, however, we will only be able to show the weaker condition that, if both the finite limit and infinite limit integrals exist, they have equal values.

Our goal is to show when
\[
\int_{\Psi^{-1}(x)}^{\infty} \psi^{-1}(x) f(x) \, dx = \int_{\Psi^{-1}(x)}^{\infty} \psi^{-1}(x) f(x) \, dx
\]
holds for any \( \psi(x) \) satisfying (19) through (21).

We are able to combine termination functions of the two types, \( z(x) \) and \( \zeta(x,b) \), using
\[
\zeta'(x,b) \otimes z'(x) = - \int_0^{c(e,b)} z'(x-x') \zeta'(x',b) \, dx' = - \int_0^c z'(x') \zeta'(x-x',b) \, dx'.
\]
We will denote the combined termination function using the \( \otimes \) notation as
\[
\zeta(x) \otimes z(x) = z(x) \otimes \zeta(x).
\]
Note that this combined termination function is in general not a termination function of either the first or the second type.

We will first show that the integral using a termination function of the second kind, \( \zeta(x,b) \), gives the same value as integration using a combined termination function with \( \zeta(x,b) \) as one if its components and a termination function of the first kind \( z(x) \) as its other component.

Recalling from [Bli08] that for a termination function of the first kind
\[
\int_{-\infty}^{\infty} z'(x) \, dx = \int_0^c z'(x) \, dx = -1,
\]
the limit term from the RHS of (39) can be written as
\[
\lim_{b \to \infty} \left\{ \int_0^b F(x+b) \zeta'(x,b) \, dx \right\} = - \left[ \lim_{b \to \infty} \left\{ \int_0^b F(x+b) \zeta'(x,b) \, dx \right\} \right] \int_0^c z'(x') \, dx'.
\]
The limiting operation is brought inside of the integration over \( x' \), giving
\[
= - \int_0^c z'(x') \left[ \lim_{b \to \infty} \left\{ \int_0^\infty F(x + b) \zeta'(x, b) \, dx \right\} \right] \, dx'. \tag{77}
\]

The limit as \( b \) approaches \( \infty \) is the same as the limit as \( x' + b \) approaches \( \infty \) for any finite value of \( x' \), so (77) can be written as
\[
= - \int_0^c z'(x') \left[ \lim_{b \to \infty} \left\{ \int_0^\infty F(x + x' + b) \zeta'(x, b) \, dx \right\} \right] \, dx'. \tag{78}
\]

The limit WRT \( b \) converges uniformly in \( x' \), so the order of the limit WRT \( b \) and the integration WRT \( x' \) can be safely interchanged (See for example [Spi06] for this and subsequent interchanges of limiting operations), giving
\[
= - \lim_{b \to \infty} \left\{ \int_0^c z'(x') \left[ \int_0^\infty F(x + x' + b) \zeta'(x, b) \, dx \right] \right\} \, dx'. \tag{79}
\]

Since \( x' \geq 0 \), and since \( \zeta'(x, b) = 0 \) for \( x \leq 0 \), the lower limit of the integration over \( x \) can be changed to \(-x'\). Thus
\[

= - \lim_{b \to \infty} \left\{ \int_0^c z'(x') \left[ \int_{-x'}^\infty F(x + x' + b) \zeta'(x, b) \, dx \right] \right\} \, dx' \\
= - \lim_{b \to \infty} \left\{ \int_0^c z'(x') \left[ \int_0^\infty F(x + b) \zeta'(x' - x, b) \, dx \right] \right\} \, dx' \\
= \lim_{b \to \infty} \left\{ \int_0^\infty F(x + b) \left[ - \int_0^c \zeta'(x' - x, b) \zeta'(x') \, dx' \right] \right\} \, dx \tag{80}
\]
or
\[
\lim_{b \to \infty} \left\{ \int_0^c F(x + b) \zeta'(x, b) \, dx \right\} = \lim_{b \to \infty} \left\{ \int_0^\infty F(x + b) \zeta'(x, b) \otimes z'(x) \, dx \right\}. \tag{81}
\]

We thus obtain the same value for the limit using a combined termination function which \( \zeta(x, b) \) is a component of, as we get using the termination function \( \zeta(x, b) \) itself.

We next show that the improper integral using a termination function of the first kind \( z(x) \) gives the same value as integration using a combined termination function having \( z(x) \) as one if its components and a termination function of the second kind \( \zeta(x, b) \) as its other component. Using (10), the limit term from the RHS of (2) becomes
\[
\lim_{b \to \infty} \left\{ \int_0^c F(x + b) z'(x) \, dx \right\} = \lim_{b \to \infty} \left\{ \int_0^\infty F(x + b) \zeta'(x, b) \otimes z'(x) \, dx \right\}. \tag{82}
\]
The limiting operation is invariant WRT \( v' \), and can be pulled inside of the integration over \( v' \), giving

\[
\int_{\epsilon}^{1} w'(v') \left[ \lim_{b \to \infty} \left\{ \int_{0}^{c} F(x + b + \bar{x}'(v', b)) z'(x) \, dx \right\} \right] \, dv'. 
\]

(83)

For \( \bar{x}'(v', b) \) such that

\[
\bar{x}'(v', b) \text{ is finite for } \epsilon \leq v' \leq 1,
\]

(84)

and such that, for some constant \( d \),

\[
\bar{x}'(v', b) \geq d \text{ for } \epsilon \leq v' \leq 1,
\]

(85)

then

\[
\lim_{b \to \infty} \{ \bar{x}'(v', b) + b \} = \infty,
\]

(86)

and the limit in (83) as \( b \) approaches \( \infty \) is the same as the limit as \( \bar{x}' + b \) approaches \( \infty \). In that case, (83) can be written as

\[
\int_{\epsilon}^{1} w'(v') \left[ \lim_{b \to \infty} \left\{ \int_{0}^{c} F(x + b + \bar{x}'(v', b)) z'(x) \, dx \right\} \right] \, dv'.
\]

(87)

with the limit \( b \to \infty \) converging uniformly in \( v' \).

We’ll define \( \bar{x}'(v', b) \) by

\[
\psi(\bar{x} + b) \equiv v' \psi(b),
\]

(88)

or

\[
\bar{x}'(v', b) = \psi^{-1}(v' \psi(b)) - b.
\]

(89)

Using (23), we find that \( \bar{x}' \geq 0 \) for \( \epsilon \leq v' \leq 1 \). Using (20) and (21), for \( v' \geq \epsilon \), the largest \( \bar{x}' \) can be is when

\[
\psi(\bar{x}' + b) = \epsilon \psi(b),
\]

(90)

so using (35) we find that, for \( \epsilon \leq v' \leq 1 \),

\[
0 \leq \bar{x}'(v', b) \leq e(\epsilon, b)
\]

(91)

and (84) and (85) are satisfied for \( \epsilon > 0 \). Thus the step from (83) to (87) is valid for a change of variable satisfying (19) through (21).

Since the limit WRT \( b \) in (87) converges uniformly in \( v' \), the order of the limit WRT \( b \) and the integration WRT \( v' \) can be safely interchanged, giving

\[
\lim_{b \to \infty} \left\{ \int_{\epsilon}^{1} w'(v') \left[ \int_{0}^{c} F(x + b + \bar{x}') z'(x) \, dx \right] \, dv' \right\}.
\]

(92)
Since \( z'(x) = 0 \) for \( x < 0 \) and for \( x > c \), the lower limit of the integration over \( x \) can be changed to \(-\bar{x}'\), and using (91), the upper limit can be changed to \( c + e(\epsilon, b) - \bar{x}'\). Thus

\[
\lim_{b \to \infty} \left\{ \int_{c+e(\epsilon, b)}^{\infty} \int_{\epsilon}^{1} w' (v') \left[ \int_{-\bar{x}'}^{c+e(\epsilon, b)} F(x + b + \bar{x}')z'(x) \, dx \right] \, dv' \right\} = \lim_{b \to \infty} \left\{ \int_{c+e(\epsilon, b)}^{\infty} \int_{\epsilon}^{1} w' (v') \left[ \int_{0}^{c+e(\epsilon, b)} F(x + b)z'(x - \bar{x}') \, dx \right] \, dv' \right\}
\]

or

\[
\lim_{b \to \infty} \left\{ \int_{c+e(\epsilon, b)}^{\infty} \int_{\epsilon}^{1} w' (v') \left[ \int_{0}^{1} w' (v') z'(x - \bar{x}') \, dv' \right] \, dx \right\} = \lim_{b \to \infty} \left\{ \int_{c+e(\epsilon, b)}^{\infty} F(x + b) \left[ \int_{\epsilon}^{1} w' (v') z'(x - \bar{x}') \, dv' \right] \, dx \right\}
\]

Now we perform a change of variable of integration with

\[
v' = \frac{\psi(x' + b)}{\psi(b)},
\]

so

\[
dv' = \frac{\psi'(x' + b)}{\psi(b)} \, dx'
\]

and \( \bar{x}' \) is simply

\[
\bar{x}'(v', b) = x'.
\]

We thus have

\[
\lim_{b \to \infty} \left\{ \int_{0}^{c+e(\epsilon, b)} F(x + b) \left[ \int_{\epsilon}^{1} w' \left( \frac{\psi(x' + b)}{\psi(b)} \right) \frac{\psi'(x' + b)}{\psi(b)} z'(x - x') \, dx' \right] \, dx \right\} = \lim_{b \to \infty} \left\{ \int_{0}^{c+e(\epsilon, b)} F(x + b) \left[ -\int_{0}^{\epsilon} \zeta'(x', b)z'(x - x') \, dx' \right] \, dx \right\}
\]

or

\[
\lim_{b \to \infty} \left\{ \int_{0}^{c} F(x + b)z'(x) \, dx \right\} = \lim_{b \to \infty} \left\{ \int_{0}^{\infty} F(x + b)\zeta'(x, b) \odot z'(x) \, dx \right\}.
\]

We thus obtain the same value for the limit using a combined termination function which \( z(x) \) is a component of, as we get using the termination function \( z(x) \) alone.

Note that the interchange of the order of integration relies on \( \epsilon > 0 \) in (90). If \( \epsilon = 0 \) were allowed, interchange of the order of integration would still be possible if the integrals have sufficient convergence. This would require
an integrand-dependent requirement on \( \zeta(x, b) \) for \( x \to \infty \), and therefore a condition on \( w(v) \) as \( v \to 0 \).

Comparing (81) and (100) we see that

\[
\lim_{b \to \infty} \left\{ \int_0^\infty F(x + b) \zeta'(x, b) \, dx \right\} = \lim_{b \to \infty} \left\{ \int_0^c F(x + b) \zeta'(x) \, dx \right\}
\]

whenever both limits exist. Using (101) in (39) and (2), we get the important result that (72) holds when both integrals exist.

It can be shown that an analogous relation holds when starting with an improper integral with an infinite limit and using (19) to transform to an improper integral with a finite limit. For this case, one obtains a finite limit integral, using a “second type” of initialization function, which can be shown to give the same value as the finite limit improper integral using the “first type” of initialization function (one satisfying (7) through (10)), as long as both integrals exist.

Thus, when transforming between an infinite limit improper integral and a finite limit improper integral, using a change of variable satisfying (19) through (21), and then using (26), we have that

\[
\int_\infty^{\tilde{x}} f(x) \, dx = \int_0^{\beta} g(u) \, du
\]

holds whenever the integrals on the LHS and RHS exist under the definition presented here and in [Bli08] respectively.

A change of variable which one would expect to be useful is

\[
u = \psi(x) = 1/x^r \quad (r > 0).
\]

For this change of variable, (7) through (10) are satisfied. We have

\[
\tilde{x}'(v', b) = (v'b^{-r})^{-1/r} - b = (v'^{-1/r}-1)b,
\]

so for \( \epsilon \leq v' \leq 1 \)

\[
b \leq \tilde{x}' + b \leq \frac{b}{\epsilon^{1/r}},
\]

and (84) and (85) are also satisfied. As long as the integrals both exist, then

\[
\int_a^\infty f(x) \, dx = \int_0^{1/a^r} \frac{f(\frac{1}{\sqrt{u}})}{u^{1/r}} \, du = \int_0^{1/a^r} g(u) \, du.
\]

For examples of (102), when using (103) with \( r = 1 \), we can compare our Example 1 above with Example 1 from [Bli08]. We see that

\[
\int_a^\infty \cos(x) \, dx = \int_0^{1/a} \cos(1/u) \, du
\]

\[
\int_a^\infty \frac{\cos(x)}{x^2} \, dx = \int_0^{1/a} \frac{\cos(1/u)}{u^2} \, du
\]
as we’d expect from using $u$-substitution with $u = 1/x$. Similarly, comparing Example 2 with Example 2 from [Bli08] we get that

$$
\int_a^\infty x \cos(x) \, dx = \int_0^{1/a} \cos(1/u) \frac{u^3}{u^3} \, du,
$$

again, as we expect. For both examples, then, (102) is seen to hold.

Conclusion

A definition for an improper integral with finite bounds has been presented. The definition presented here is a more general alternative to the conventional definition. The range of functions which are integrable under this definition is expanded as compared with the conventional definition.

This definition gives the same result as the conventional definition when that applies, and preserves uniqueness and linearity. The new definition allows interchange of the order of differentiation and integration whenever the two integrals exist under the new definition. Also allowed is interchange of the order of integration of iterated integration, again when the integrations exist under this definition. The ability to rigorously interchange order of integrations, or order of integration and differentiation, in cases where integrals under the conventional definition do not converge, provides an added tool for manipulation of complicated integrals. An arbitrary change of the variable of integration is not necessarily valid, although scaling the variable of integration by a constant is.

The finite bounds definition presented here has been shown to be equivalent to the infinite bounds definition presented in [Bli08]. For a particular change of variable transforming between finite limit and infinite limit integrals, the existence of either the finite or the infinite limit integral implies existence of the other, with the same value. For a more general class of change of variable, the definitions presented here and in [Bli08] give the same answers whenever both integrals exist.
References

[Bli08] Michael Blischke. An alternative definition for improper integral with infinite limit. arXiv:0805.3559 [math.CA], 2008.

[Spi06] Jack Spielberg. Interchange of limits and uniform convergence. http://math.asu.edu/~jss/courses/fall06/mat472/limit_interchange.pdf, 2006.