Geometry
of
Slant Submanifolds

by

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Dedicated to
Professor Tadashi Nagano

on the occasion of his sixtieth birthday
The present volume is the written version of the series of lectures the author delivered at the Catholic University of Leuven, Belgium during the period of June-July, 1990. The main purpose of these talks is to present some of author’s recent work and also his joint works with Professor T. Nagano and Professor Y. Tazawa of Japan, Professor P. F. Leung of Singapore and Professor J. M. Morvan of France on geometry of slant submanifolds and its related subjects in a systematical way.

The main references of the results presented in this volume are the following articles:

[C2] Differential geometry of real submanifolds in a Kähler manifold, *Monatsh. für Math.*, 91 (1981), 257-274,

[C6] Slant immersions, *Bull. Austral. Math. Soc.*, 41 (1990), 135-147,

[CLN] Totally geodesic submanifolds of symmetric spaces, III, preprint, 1980,

[CM3] Cohomologie des sous-variétés α-obliques, *C. R. Acad. Sc. Paris*, 314 (1992), 931–934.

[CT1] Slant surfaces of codimension two, *Ann. Fac. Sc. Toulouse Math.*, 11 (1990), 29–43.

and

[CT2] Slant submanifolds in complex Euclidean spaces, *Tokyo J. Math.*, 14 (1991), 101–120.

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CHAPTER I

INTRODUCTION

§1. INTRODUCTION.

The theory of submanifolds of an almost Hermitian manifold, in particular of a Kaehlerian manifold, is one of the most interesting topics in differential geometry. According to the behaviour of the tangent bundle of a submanifold, with respect to the action of the almost complex structure $J$ of the ambient manifold, there are two well-known classes of submanifolds, namely, the complex submanifolds and the totally real submanifolds. In this volume we will present the geometry of another important class of submanifolds, called slant submanifolds.

The theory of submanifolds of an almost Hermitian manifold or of a Kaehlerian manifold began as a separate area of study in the last century with the investigation of algebraic curves and algebraic surfaces in classical algebraic geometry. Included among the principal investigators are Riemann, Picard, Enriques, Castelnuovo, Severi, and Segre.

It was E. Kähler ([Ka1]), J. A. Schouten, and D. van Dantzig ([SD1] and [SD2]) who first tried to study complex manifolds from the viewpoint of Riemannian geometry in the early 1930’s. In their studies a Hermitian space with the so-called symmetric unitary connection was introduced. A Hermitian space with such a connection is now known as a Kaehlerian manifold.

It was A. Weil [W1] who in 1947 pointed out there exists in a complex manifold a tensor field $J$ of type $(1,1)$ whose square is equal to the negative of the identity transformation of the tangent bundle, that is, $J^2 = -I$. In the same year, C. Ehresmann introduced the notion of an almost complex manifold as an even-dimensional differentiable manifold which admits such a tensor field $J$ of type $(1,1)$.

An almost complex manifold (respectively, complex manifold) is called an almost Hermitian manifold (respectively, a Hermitian manifold) if it admits a Riemannian structure which is compatible with the almost complex structure $J$. The theory of almost Hermitian manifolds, Hermitian mani-
The study of complex submanifolds of a Kaehlerian manifold from the differential geometrical points of view (that is, with emphasis on the Riemannian metric) was initiated by E. Calabi and others in the early of 1950's (cf. [Ca1], [Ca2]). Since then it has become an active and fruitful field in modern differential geometry. Many important results on Kaehlerian submanifolds have been obtained by many differential geometers in the last three decades. Two nice survey articles concerning this subject were given by K. Ogiue in [O1] and [O2].

In terms of the behaviour of the tangent bundle $TN$ of the submanifold $N$, complex submanifolds $N$ of an almost Hermitian manifold $(M,g,J)$ are characterized by the condition:

$$J(T_pN) \subseteq T_pN$$

for any point $p \in N$. In other words, $N$ is a complex submanifold of $(M,g,J)$ if and only if for any nonzero vector $X$ tangent to $N$ at any point $p \in N$, the angle between $JX$ and the tangent plane $T_pN$ is equal to zero, identically.

Besides complex submanifolds, there is another important class of submanifolds, called \textit{totally real submanifolds}.

A totally real submanifold $N$ of an almost Hermitian manifold $M$ or, in particular, of a Kaehlerian manifold, is a submanifold such that the (almost) complex structure $J$ of the ambient manifold $M$ carries each tangent vector of $N$ into the corresponding normal space of $N$ in $M$, that is,

$$J(T_pN) \subseteq T_p^\perp N$$

for any point $p \in N$. In other words, $N$ is a totally real submanifold of $(M,g,J)$ if and only if for any nonzero vector $X$ tangent to $N$ at any point $p \in N$, the angle between $JX$ and the tangent plane $T_pN$ is equal to $\frac{\pi}{2}$, identically.

The study of totally real submanifolds from the differential geometric points of view was intiated in the early 1970's (see [CO1] and [YK1]). Since then many differential geometers have contributed many interesting results in this subject.

In this volume I shall present the third important class of submanifolds of an almost Hermitian manifold $(M,g,J)$ (in particular, of a Kaehlerian mani-
A slant submanifold is defined in [C6] as a submanifold of \((M, g, J)\) such that, for any nonzero vector \(X \in T_pN\), the angle \(\theta(X)\) between \(JX\) and the tangent space \(T_pN\) is a constant (which is independent of the choice of the point \(p \in N\) and the choice of the tangent vector \(X\) in the tangent plane \(T_pN\)). It is obvious that complex submanifolds and totally real submanifolds are special classes of slant submanifolds. A slant submanifold is called \textit{proper} if it is neither a complex submanifold nor a totally real submanifold.

In the first section of Chapter 2, we present some basic definitions and basic formulas for later use. In particular, for any submanifold \(N\) of an almost Hermitian manifold \((M, g, J)\), the almost complex structure \(J\) of the ambient manifold induces a canonical endomorphism of the tangent bundle, denoted by \(P\), and a canonical normal-bundle-valued 1-form on the tangent bundle, denoted by \(F\). It will be shown in later sections that the endomorphism \(P\) plays a fundamental role in our study.

In Section 2 of Chapter 2, we give many examples of slant surfaces in the complex number space \(\mathbb{C}^2\) and give examples of Kaehlerian slant submanifolds (that is, proper slant submanifolds such that the canonical endomorphism \(P\) is parallel) in \(\mathbb{C}^m\).

Section 3 of Chapter 2 is devoted to the fundamental study of the endomorphism \(P\) and the normal-bundle-valued 1-form \(F\). In particular, we prove that every slant surface in any almost Hermitian manifold is a Kaehlerian slant submanifold, that is, \(\nabla P = 0\) (Theorem 3.4). Moreover, we prove that a proper slant submanifold \(N\) of a Kaehlerian manifold \(M\) is Kaehlerian slant if and only if the Weingarten map of \(N\) in \(M\) satisfies

\[(1.3) \quad A_{FX}Y = A_{FY}X\]

for any vectors \(X, Y\) tangent to the submanifold. In particular, by combining this result with Theorem 3.4, we obtain the important fact that formula (1.3) holds for any slant surface in any Kaehler manifold. By using this result we show that the Gauss curvature \(G\) and the normal curvature \(G^D\) of every slant surface in \(\mathbb{C}^2\) satisfies

\[(1.4) \quad G = G^D,\]

identically. In this section we also prove that every proper slant submanifold of a Kaehlerian manifold with \(\nabla F = 0\) is an austere submanifold (Theorem 3.8). We also obtain in this section two reduction theorems for submanifolds.
satisfying $\nabla F = 0$ (Theorem 3.9 and Theorem 3.10). Finally we show that for a surface in a real 4-dimensional Kaehlerian manifold, the parallelism of $F$ implies the parallelism of $P$ (Theorem 3.11).

In Section 4 of Chapter II, we establish some relations between minimal slant surfaces and totally real surfaces in a Kaehlerian manifold. For instance, we prove that if a proper slant surface in a real 4-dimensional Kaehlerian manifold is also totally real with respect to some compatible complex structure on the ambient manifold at the same time, then the surface must be a minimal surface (Theorem 4.2). Some applications of this result will also be given in this section (Theorem 4.3 and Theorem 4.4).

In Chapter III, we present some results of the author and Y. Tazawa [CT1]; in this work slant surfaces in $C^2$ were studied from the viewpoint of the Gauss map.

The first section of Chapter III reviews some basic geometry of the real Grassmannian $G(2,4)$ which consists of all oriented 2-planes in $E^4$ which will be used in later sections.

The second section is devoted to the detailed study of the set of compatible complex structures on $E^4$. Several lemmas are obtained in this section, in particular, Lemma 2.1 and Lemma 2.3 of this section, play some important roles in the later sections.

In Section 3 we study the following two geometric problems:

**Problem 3.1.** Let $N$ be a surface in $C^2 = (E^4, J_0)$. When is $N$ slant in $C^2$?

**Problem 3.2.** Let $N$ be a surface in $E^4$. If there exists a compatible complex structure $J$ on $E^4$ such that $N$ is slant in $(E^4, J)$. How many other compatible complex structures $\tilde{J}$ on $E^4$ are there such that $N$ is slant with respect to these complex structures?

Complete solutions to these two problems are obtained in this section.

Related to the problems studied in Section 3 is the notion of doubly slant surfaces. We prove that every doubly slant surface in $E^4$ has vanishing Gauss curvature and vanishing normal curvature (Theorem 4.1).

In the last section of Chapter III, we study slant surfaces in almost Hermitian manifolds. In fact we prove that if a surface in a real 4-dimensional
almost Hermitian manifold has no complex tangent points (that is, the surface is purely real), then, with respect to some suitable compatible almost complex structure on the ambient manifold, the surface is slant. This result shows that there exist ample examples of slant surfaces in almost Hermitian manifolds.

Chapter IV is devoted to the classification problems of slant surfaces in $\mathbb{C}^2$. In the first section of this chapter, we classify all slant surfaces in $\mathbb{C}^2$ with parallel mean curvature vector. In the second section we classify spherical slant surfaces in $\mathbb{C}^2$. In the third section, we classify slant surfaces in $\mathbb{C}^2$ whose Gauss map has rank $< 2$ at every point. The last section gives the classification of slant surfaces in $\mathbb{C}^2$ which are contained in a hyperplane of $\mathbb{C}^2$.

In the last Chapter, we study the topology and cohomology of slant submanifolds.

In the first section, we prove that every compact slant submanifold in $\mathbb{C}^m$ is totally real (Theorem 1.5) and every compact slant submanifold in an exact sympletic manifold is also totally real (Theorem 1.7).

In Section 2, we define a canonical 1-form $\Theta$ associated with a proper slant submanifold $N$ in an almost Hermitian manifold (by formula (2.7) of Chapter V). We prove in this section that for a slant surface in $\mathbb{C}^2$, this 1-form is closed, that is, $d\Theta = 0$ and if we put

$$(1.5) \quad \Psi = (2\sqrt{2}\pi)^{-1}(\csc \alpha)\Theta,$$

where $\alpha$ denotes the slant angle of $N$ in $\mathbb{C}^2$, then the 1-form $\Psi$ defines a canonical integral class on $N$ (Theorem 2.5):

$$(1.6) \quad [\Psi] \in H^1(N; \mathbb{Z}).$$

In this section we also prove that if $N$ is a complete, oriented, proper slant surface of $\mathbb{C}^2$ such that the mean curvature of $N$ is bounded below by some positive constant, then, topologically, the surface is either a circular cylinder or a 2-plane (Theorem 2.6).

In Section 3 of this chapter, we prove that in fact, for any $n$-dimensional proper slant submanifold of $\mathbb{C}^n$, the 1-form $\Theta$ is always closed. Thus for any $n$-dimensional proper slant submanifold $N$ in $\mathbb{C}^n$, we have a canonical cohomology class $[\Theta] \in H^1(N; \mathbb{R})$ (Theorem 3.1). Finally we prove in this section that every proper slant submanifold of a Kaehlerian manifold
has a canonical sympletic structure given by the 2-form induced from the canonical endomorphism $P$ (Theorem 3.4). The last result implies that if a compact $2k$-dimensional differentiable manifold $N$ satisfies $H^{2i}(N; \mathbb{R}) = 0$ for some $i \in \{1, \ldots, k\}$, then $N$ cannot be immersed in any Kaehlerian manifold as a proper slant submanifold (Theorem 3.5').

In Section 4, we recall some stability theorems of [CLN] obtained in 1980 and present some related results concerning the index form of compact minimal totally real submanifolds, a special class of slant submanifolds, of a Kaehlerian manifold.

In the last section, we present a general method introduced by the author, Leung and Nagano [CLN] for determining the stability of totally geodesic submanifolds in compact symmetric spaces. Since every irreducible totally geodesic submanifolds of a Hermitian symmetric space is a slant submanifold [CN1], the method can be used to determine the stability of such submanifolds.

Finally, I would like to mention that minimal surfaces of a complex projective space with constant Kaehlerian angle were recently studied from a different point of view by Bolton, Jensen, Rigoli, Woodward, Maeda, Ohnita and Udagawa (see, [BJRW], [MU1] and [Oh1]).
CHAPTER II

GENERAL THEORY

§1. PRELIMINARIES.

Let $N$ be an $n$-dimensional Riemannian manifold isometrically immersed in an almost Hermitian manifold $M$ with (almost) complex structure $J$ and almost Hermitian metric $g$. We denote by $<,>$ the inner product for $N$ as well as for $M$.

For any vector $X$ tangent to $N$ we put

\[(1.1) \quad JX = PX + FX,\]

where $PX$ and $FX$ are the tangential and the normal components of $JX$, respectively. Thus, $P$ is an endomorphism of the tangent bundle $TN$ and $F$ a normal-bundle-valued 1-form on $TN$.

For any nonzero vector $X$ tangent to $N$ at a point $x \in N$, the angle $\theta(x)$ between $JX$ and the tangent space $T_xN$ is called the Wirtinger angle of $X$. In the following we call an immersion $f : N \rightarrow M$ a slant immersion if the Wirtinger angle $\theta(X)$ is constant (which is independent of the choice of $x \in N$ and of $X \in T_xN$). Complex and totally real immersions are slant immersions with $\theta = 0$ and $\theta = \pi/2$, respectively. Moreover, it is easy to see that slant submanifolds of an almost Hermitian manifolds are characterized by the condition: $P^2 = \lambda I$, for some real number $\lambda \in [-1,0]$, where $I$ denotes the identity transformation of the tangent bundle $TN$ of the submanifold $N$.

The Wirtinger angle of a slant immersion is called the slant angle of the slant immersion. A slant submanifold is said to be proper if it is neither complex nor totally real.

A proper slant submanifold is said to be Kaehlerian slant if the canonical endomorphism $P$ defined above is parallel, that is, $\nabla P = 0$. A Kaehlerian slant submanifold of a Kaehlerian manifold with respect to the induced metric and with the almost complex structure given by $\tilde{J} = (\sec \theta)P$, $\theta = \pi/2$.
For a submanifold $N$ of an almost Hermitian manifold $M$, we denote by $\nabla$ and $\tilde{\nabla}$ the Levi-Civita connections of $N$ and $M$, respectively. Then the Gauss and Weingarten formulas of $N$ in $M$ are given respectively by
\begin{align}
(1.2) \quad \tilde{\nabla}_X Y &= \nabla_X + h(X, Y), \\
(1.3) \quad \tilde{\nabla}_X \xi &= -A_\xi X + D_X \xi,
\end{align}
for any vector fields $X, Y$ tangent to $N$ and any vector field $\xi$ normal to $N$, where $h$ denotes the second fundamental form, $D$ the normal connection, and $A$ the Weingarten map of the submanifold $N$ in $M$. The second fundamental form $h$ and the Weingarten map $A$ are related by
\begin{equation}
(1.4) \quad <A_\xi X, Y> = <h(X, Y), \xi>.
\end{equation}
For any vector field $\xi$ normal to the submanifold $N$, we put
\begin{equation}
(1.5) \quad J\xi = t\xi + f\xi,
\end{equation}
where $t\xi$ and $f\xi$ are the tangential and the normal components of $J\xi$, respectively. Then $f$ is an endomorphism of the normal bundle and $t$ is a tangent-bundle-valued 1-form on the normal bundle $T^\perp N$.

For a submanifold $N$ in a Riemannian manifold $M$, the mean curvature vector $H$ is defined by
\begin{equation}
(1.6) \quad H = \frac{1}{n} tr h = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)
\end{equation}
where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle $TN$ of $N$. A submanifold $N$ in $M$ is said to be totally geodesic if the second fundamental form $h$ of $N$ in $M$ vanishes identically.

Let $N$ be an $n$-dimensional submanifold in an $m$-dimensional Riemannian manifold $M$. We choose a local field of orthonormal frames
\[ e_1, \ldots, e_n, e_{n+1}, \ldots, e_m \]
such that, restricted to $N$, the vectors $e_1, \ldots, e_n$ are tangent to $N$ and hence $e_{n+1}, \ldots, e_m$ are normal to $N$. We shall make use of the following convention on the ranges of indices unless mentioned otherwise:
\[ 1 \leq A, B, C, \ldots \leq m; \quad 1 \leq i, j, k, \ldots \leq n; \]
With respect to the frame field of $M$ chosen above, let
\[ \omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^m \]
be the field of dual frames. Then the structure equations of $M$ are given by
\[
(1.7) \quad d\omega^A = -\sum_B \omega^A_B \wedge \omega^B, \quad \omega_B^A + \omega_A^B = 0,
\]
and
\[
(1.8) \quad d\omega^A_B = -\sum_C \omega^A_C \wedge \omega^C_B + \Phi^A_B, \quad \Phi^A_B = \frac{1}{2} \sum_{C,D} K^A_{BCD} \omega^C \wedge \omega^D,
\]
\[ K^A_{BCD} + K^A_{BDC} = 0. \]
We restrict these forms to $N$. Then
\[ \omega^r = 0. \]
Since
\[ 0 = d\omega^r = -\sum_i \omega^r_i \wedge \omega^i, \]
by Cartan’s lemma we have
\[
(1.9) \quad \omega^r_i = \sum_j h^r_{ij} \omega^j, \quad h^r_{ij} = h^r_{ji}.
\]
Formula (1.9) is equivalent to
\[
(1.9)' \quad \omega^r_i(X) = \langle A_{e_i}, e_i, X \rangle
\]
for any vector $X$ tangent to $N$. From these formulas we obtain
\[
(1.10) \quad d\omega^i = -\sum_j \omega^i_j \wedge \omega^j, \quad \omega^i_j + \omega^j_i = 0,
\]
\[
(1.11) \quad d\omega^j_k = -\sum_l \omega^j_k \wedge \omega^k_l + \Omega^j_l, \quad \Omega^j_l = \frac{1}{2} \sum_{k,\ell} R^j_{ik\ell} \omega^k \wedge \omega^\ell,
\]
\( R^i_{jkl} = K^i_{jkl} + \sum_r (h^r_{ik} h^r_{j\ell} - h^r_{ij} h^r_{\ell k}) \),

(1.13) \[ d\omega^r_i = -\sum_{j,k} h^r_{jk} \omega^j_i \wedge \omega^k_i + \frac{1}{2} \sum_{j,s} K^r_{ij} \omega^j_i \wedge \omega^s_i, \]

(1.14) \[ d\omega^r_s = -\sum_{t} \omega^r_t \wedge \omega^s_t + \Omega^r_s = \frac{1}{2} \sum_{k,\ell} R^r_{sk\ell} \omega^k_i \wedge \omega^\ell_i, \]

and

(1.15) \[ R^r_{sk\ell} = K^r_{sk\ell} + \sum_i (h^r_{ik} h^r_{\ell i} - h^r_{ij} h^r_{\ell k}). \]

For any vector field \( X \) tangent to the submanifold \( N \), these forms are also given by

(1.16) \[ \tilde{\nabla}_X e_i = \sum_{j=1}^n \omega^j_i (X) e_j + \sum_{r=n+1}^m \omega^r_i (X) e_r, \]

(1.17) \[ \tilde{\nabla}_X e_r = \sum_{j=1}^n \omega^j_i (X) e_j + \sum_{s=n+1}^m \omega^s_r (X) e_s. \]

These 1-forms \( \omega^j_i \), \( \omega^r_i \) and \( \omega^s_r \) are called the connection forms of \( N \) in \( M \). Denote by \( R \) and \( \tilde{R} \) the Riemann curvature tensors of \( N \) and \( M \), respectively, and by \( R^D \) the curvature tensor of the normal connection \( D \). Then the equation of Gauss and the equation of Ricci are given respectively by

(1.18) \[ \tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) + \langle h(X, Z), h(Y, W) \rangle - \langle h(X, W), h(Y, Z) \rangle, \]

(1.19) \[ R^D(X, Y; \xi, \eta) = \tilde{R}(X, Y; \xi, \eta) + \langle [A_{\xi}, A_{\eta}] (X), Y \rangle \]

for vectors \( X, Y, Z, W \) tangent to \( N \) and \( \xi, \eta \) normal to \( N \).
For the second fundamental form $h$, we define the covariant derivative $\bar{\nabla}h$ of $h$ with respect to the connection in $TN \oplus T^\perp N$ by

\begin{equation}
(1.20) \quad (\bar{\nabla}_X h)(Y, Z) = D_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\end{equation}

The equation of Codazzi is given by

\begin{equation}
(1.21) \quad (\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z),
\end{equation}

where $(\tilde{R}(X,Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X,Y)Z$.

A submanifold $N$ of a Riemannian manifold $M$ is called a parallel submanifold if the second fundamental form $h$ is parallel, that is, $\bar{\nabla}h = 0$, identically.
§2. SOME EXAMPLES.

In the following, $E^{2m}$ denotes the Euclidean $2m$-space with the standard metric. An almost complex structure $J$ on $E^{2m}$ is said to be compatible if $(E^{2m}, J)$ is complex analytically isometric to the complex number space $\mathbb{C}^m$ with the standard flat Kaehlerian metric. We denote by $J_0$ and $J_{-1}$ (when $m$ is even) the compatible almost complex structures on $E^{2m}$ defined respectively by

\begin{align}
J_0(a_1, \ldots, a_m, b_1, \ldots, b_m) &= (-b_1, \ldots, -b_m, a_1, \ldots, a_m) \\
J_{-1}(a_1, \ldots, a_m, b_1, \ldots, b_m) &= (-a_2, a_1, \ldots, -a_m, a_{m-1}, b_2, -b_1, \ldots, b_m, -b_{m-1}).
\end{align}

Example 2.1. For any $\alpha > 0$, $x(u, v) = (u \cos \alpha, v, u \sin \alpha, v, 0)$ defines a slant plane with slant angle $\alpha$ in $\mathbb{C}^2$.

Example 2.2. Let $N$ be a complex surface in $\mathbb{C}^2 = (E^4, J_0)$. Then for any constant $\alpha$, $0 < \alpha \leq \pi/2$, $N$ is slant surface in $(E^4, J_\alpha)$ with slant angle $\alpha$, where $J_\alpha$ is the compatible almost complex structure on $E^4$ defined by

\[ J_\alpha(a, b, c, d) = (\cos \alpha)(-c, -d, a, b) + (\sin \alpha)(-b, a, d, -c). \]

This example shows that there exist infinitely many proper slant minimal surfaces in $\mathbb{C}^2 = (E^4, J_0)$.

The following example provides us some non-minimal proper slant surfaces in $\mathbb{C}^2 = (E^4, J_0)$.

Example 2.3. [GVV] For any positive constant $k$,

\[ x(u, v) = (e^{ku} \cos u \cos v, e^{ku} \sin u \cos v, e^{ku} \cos u \sin v, e^{ku} \sin u \sin v). \]
defines a complete, non-minimal, pseudo-umbilical proper slant surface with slant angle $\theta = \cos^{-1}(k/\sqrt{1+k^2})$ and with non-constant mean curvature given by $|H| = e^{-ku/\sqrt{1+k^2}}$.

**Example 2.4.** For any positive number $k$,

$$x(u,v) = (u,k\cos v,v,k\sin v)$$

defines a complete, flat, non-minimal and non-pseudo-umbilical, proper slant surface with slant angle $\cos^{-1}(1/\sqrt{1+k^2})$ and constant mean curvature $k/2(1+k^2)$ and with non-parallel mean curvature vector.

**Example 2.5.** Let $k$ be any positive number and $(g(s),h(s))$ a unit speed plane curve. Then

$$x(u,s) = (-ks \sin u, g(s), ks \cos u, h(s))$$

defines a non-minimal, flat, proper slant surface with slant angle $k/\sqrt{1+k^2}$.

**Example 2.6 [CT1].** For any nonzero real numbers $p$ and $q$, we consider the following immersion form $\mathbb{R} \times (0, \infty)$ into $\mathbb{C}^2$ defined by

$$x(u,v) = (pv \sin u, pv \cos u, v \sin qu, v \cos qu).$$

Then the immersion $x$ gives us a complete flat slant surface in $\mathbb{C}^2$.

**Example 2.7.** For any $k > 0$,

$$x(u,v,w,z) = (u,v,k \sin w, k \sin z, kw, kz, k \cos w, k \cos z)$$

defines a Kaehlerian slant submanifold in $\mathbb{C}^4$ with slant angle $\cos^{-1} k$.

**Example 2.8.** Let $N$ be a complex submanifold of the complex number space $\mathbb{C}^{2m} = (E^{4m}, J_0)$. For any constant $\alpha$ we define $J_\alpha$ by

$$(2.3) \quad J_\alpha = (\cos \alpha)J_0 + (\sin \alpha)J^{-1}_0.$$

Then $J_\alpha$ is a compatible complex structure on $E^{4m}$ and $N$ is a Kaehlerian slant submanifold with slant angle $\alpha$ in $(E^{4m}, J_\alpha)$. 
§3. SOME PROPERTIES OF $P$ AND $F$.

Let $f : N \to M$ be an isometric immersion of an $n$-dimensional Riemannian manifold into an almost Hermitian manifold. Let $P$ and $F$ be the endomorphism and the normal-bundle-valued 1-form on the tangent bundle defined by (1.1). Since $M$ is almost Hermitian, we have

$$< PX, Y > = - < X, PY >$$

for any vectors $X, Y$ tangent to $N$. Hence, if we put $Q = P^2$, then $Q$ is a self-adjoint endomorphism of $TN$. Therefore, each tangent space $T_x N$ of $N$ at $x \in N$ admits an orthogonal direct decomposition of eigenspaces of $Q$:

$$T_x N = D^1_x \oplus \cdots \oplus D^k(x).$$

Since $P$ is skew-symmetric and $J^2 = -I$, each eigenvalue $\lambda_i$ of $Q$ lies in $[-1, 0]$ and, moreover, if $\lambda_i \neq 0$, then the corresponding eigenspace $D^i_x$ is of even dimension and it is invariant under the endomorphism $P$, that is $P(D^i_x) = D^i_x$. Furthermore, for each $\lambda_i \neq -1$, $\dim F(D^i_x) = \dim D^i_x$ and the normal subspaces $F(D^i_x)$, $i = 1, \ldots, k(x)$, are mutually perpendicular. From these we have

$$\dim M \geq 2\dim N - \dim H_x$$

where $H_x$ denotes the eigenspace of $Q$ with eigenvalue $-1$.

In this section we mention some results given in [C2] and [C6] concerning the endomorphism $P$ and the normal-bundle-valued 1-form $F$ associated with the immersion $f : N \to M$.

The following Lemma 3.1 follows from the definition of $\nabla Q$ which is defined by

$$\nabla_X QY = \nabla_X (QY) - Q(\nabla_X Y)$$

for $X$ and $Y$ tangent to $N$.

**Lemma 3.1.** Let $N$ be a submanifold of an almost Hermitian manifold $M$. Then the self-adjoint endomorphism $Q (= P^2)$ is parallel, that is, $\nabla Q = 0$, if and only if

1. each eigenvalue $\lambda_i$ of $Q$ is constant on $N$;
(2) each distribution $D^i$ (associated with the eigenvalue $\lambda_i$) is completely integrable and

(3) $N$ is locally the Riemannian product $N_1 \times \ldots \times N_k$ of the leaves of the distributions.

**Proof.** Since $Q$ is a self-adjoint endomorphism of the tangent bundle $TN$, there exist $n$ continuous functions $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ such that $\lambda_i, i = 1, \ldots, n$, are the eigenvalues of $Q$ at each point $p \in N$. Let $e_1, \ldots, e_n$ be a local orthonormal frame given by eigenvectors of $Q$. If $Q$ is parallel, then (3.1) implies

$$\nabla_X (\lambda_i e_i) = Q(\nabla_X e_i), \ i = 1, \ldots, n$$

for any vector $X$ tangent to $N$. Thus we have

$$(X \lambda_i)e_i + \lambda_i (\nabla_X e_i) = Q(\nabla_X e_i).$$

Since both $\nabla_X e_i$ and $Q(\nabla_X e_i)$ are perpendicular to $e_i$, we conclude that each eigenvalue of $Q$ is constant on $N$. This proves statement (1).

For statements (2) and (3) we let $\lambda_1, \ldots, \lambda_k$ denote the distinct eigenvalues of $Q$. For each $i; i = 1, \ldots, k$, let $D^i$ denote the distribution given by the eigenspaces of $Q$ with eigenvalue $\lambda_i$. For any two vector fields $X, Y$ in the distribution $D^i$, (3.1) and statement (1) imply

$$Q(\nabla_X Y) = \lambda_i(\nabla_X Y),$$

from which we conclude that $\nabla_X Y \in D^i$ for any $X, Y$ in $D^i$. Therefore, each distribution $D^i$ is completely integrable and each maximal integrable submanifold of $D^i$ is totally geodesic in $N$. Consequently, $N$ is locally the Riemannian product $N_1 \times \ldots \times N_k$ of the leaves of these distributions.

The converse of this is easy to verify.

By using Lemma 3.1 we have the following characterization of submanifolds with $\nabla P = 0$. 

---

*Geometry of Slant Submanifolds*
Lemma 3.2. Let $N$ be a submanifold of an almost Hermitian manifold $M$. Then $\nabla P = 0$ if and only if $N$ is locally the Riemannian product $N_1 \times \ldots \times N_k$, where each $N_i$ is either a complex submanifold, a totally real submanifold, or a Kaehlerian slant submanifold of $M$.

Proof. Under the hypothesis, if $P$ is parallel, then $Q = P^2$ is parallel. Thus, by applying Lemma 3.1, we see that $N$ is locally the Riemannian product $N_1 \times \ldots \times N_k$ of leaves of distributions defined by eigenvectors of $Q$ and moreover each eigenvalue $\lambda_i$ is constant on $N$. If an eigenvalue $\lambda_i$ is zero, the corresponding leaf $N_i$ is totally real. If $\lambda_i$ is $-1$, then $N_i$ is a complex submanifold. If $\lambda_i \neq 0, -1$, then because $D^i$ is invariant under the endomorphism $P$ and $\langle PX, Y \rangle = -\lambda_i \langle X, Y \rangle$ for any $X, Y$ in $D^i$, we have $|PX| = \sqrt{-\lambda_i} |X|$. Thus the Wirtinger angle $\theta(X)$ satisfying $\cos \theta(X) = \sqrt{-\lambda_i}$, which is a constant $\neq 0, -1$. Therefore, $N_i$ is a proper slant submanifold.

Assume $\lambda_i \neq 0$. We put $P_i = P|_{TN_i}$. Then $P_i$ is nothing but the endomorphism of $TN_i$ induced from the almost complex structure $J$. Let $\nabla^i$ denote the Riemannian connection of $N_i$. Since $N_i$ is totally geodesic in $N$, we have

$$(\nabla^i_X P_i)Y = (\nabla^i_X P)Y = 0$$

for any $X, Y$ tangent to $N_i$. This shows that if $N_i$ is a complex submanifold, $N_i$ is a Kaehlerian manifold. And if $N_i$ is proper slant, then $N_i$ is a Kaehlerian slant submanifold of $M$ by definition.

The converse can be verified directly.

From Lemma 3.2 we may obtain the following

Proposition 3.3. Let $N$ be an irreducible submanifold of an almost Hermitian manifold $M$. If $N$ is neither complex nor totally real, then $N$ is a Kaehlerian slant submanifold if and only if the endomorphism $P$ is parallel, that is, $\nabla P = 0$.

Theorem 3.4. Let $N$ be a surface in an almost Hermitian manifold $M$. Then the following three statements are equivalent:

(1) $N$ is neither totally real nor complex in $M$ and $\nabla P = 0$, that is, $P$ is parallel;

(2) $N$ is a Kaehlerian slant surface;
(3) $N$ is a proper slant surface.

**Proof.** Since every proper slant submanifold is of even dimension, Lemma 3.2 implies that if the endomorphism $P$ is parallel, then $N$ is a Kaehlerian surface, or a totally real surface, or a Kaehlerian slant surface. Thus, if $N$ is neither totally real nor complex, then statements (1) and (2) are equivalent by definition. It is obvious that (2) implies (3). Now, we prove that (3) implies (2). Let $N$ be a proper slant surface in $M$ with slant angle $\theta$. If we choose an orthonormal frame $e_1, e_2$ tangent to $N$ such that

$$Pe_1 = (\cos \theta)e_2, \quad Pe_2 = - (\cos \theta)e_1.$$ 

then we have

$$(\nabla_X P)e_1 = \cos \theta (\omega^1_2(X) + \omega^2_1(X))e_1.$$ 

Since $\omega^2_1 = \omega^1_2$, we obtain $\nabla P = 0$.

For submanifolds of a Kaehlerian manifold we have the following general lemma.

**Lemma 3.5** Let $N$ be a submanifold of a Kaehlerian manifold $M$. Then

(i) For any vectors $X, Y$ tangent to $N$, we have

$$(3.2) \quad (\nabla_X P)Y = th(X, Y) + A_{FY}X.$$ 

Hence $\nabla P = 0$ if and only if $A_{FX}Y = A_{FY}X$ for any $X, Y$ tangent of $N$.

(ii) For any vectors $X, Y$ tangent to $N$, we have

$$(3.3) \quad (\nabla_X F)Y = f h(X, Y) - h(X, PY)$$

Hence $\nabla F = 0$ if and only if $A_{f\xi}X = - A_{\xi}(PX)$, for any normal vector $\xi$ and tangent vector $X$. 

**Proof.** Since $M$ is Kaehlerian, $J$ is parallel. Thus, by applying the formulas of Gauss and Weingarten and using formulas (1.1) and (1.5), we may obtain

\[(3.2)' \quad (\nabla_X P)Y = \nabla_X (PY) - P(\nabla_X Y) = th(X,Y) + A_{FY}X,\]

and

\[(3.3)' \quad (\nabla_X F)Y = D_X(FY) - F(\nabla_X Y) = fh(X,Y) - h(X,PY).\]

Thus, $P$ is parallel if and only if

\[< th(X,Y) + A_{FY}X, Z > = 0\]

which is equivalent to

\[< A_{FY}X, Z > = -< th(X,Y), Z >= < h(X,Y), FZ > =< A_{FZ}X, Y > =< A_{FZ}Y, X > .\]

This proves statement (i).

Statement (ii) follows easily from (3.3)'.

**Remark 3.1.** If $N$ is either a totally real or complex submanifold of a Kaehlerian manifold, then $\nabla P = \nabla F = 0$, automatically.

Combining Theorem 3.4 and Lemma 3.5 we obtain the following characterization of slant surfaces in terms of Weingarten map.

**Corollary 3.6.** Let $N$ be a surface in a Kaehlerian manifold $M$. Then $N$ is slant if and only if $A_{FY}X = A_{FX}Y$ for any $X, Y$ tangent to $N$.

Let $N$ be a slant surface in the complex number space $C^2$ with slant angle $\theta$. For a unit tangent vector field $e_1$ of $N$, we put

\[(3.4) \quad e_2 = (\sec \theta)Pe_1, \quad e_3 = (\csc \theta)Fe_1, \quad e_4 = (\csc \theta)Fe_2.\]

Then $e_1 = -(\sec \theta)Pe_2$, and $e_1, e_2, e_3, e_4$ form an orthonormal frame such that $e_1, e_2$ are tangent to $N$ and $e_3, e_4$ are normal to $N$. As before we put

\[(3.5) \quad h^r_{ij} = < h(e_i, e_j), e_r >, \quad i, j = 1, 2; \quad r = 3, 4.\]
Let $G$ and $G^D$ denote the Gauss curvature and the normal curvature of $N$ in $\mathbf{C}^2$, respectively. Then we have

\begin{equation}
G = h_{11}^3 h_{22}^3 - (h_{12}^3)^2 + h_{11}^4 h_{22}^4 - (h_{12}^4)^2 \tag{3.6}
\end{equation}

and

\begin{equation}
G^D = h_{11}^3 h_{12}^4 + h_{12}^3 h_{22}^4 - h_{12}^3 h_{11}^4 - h_{22}^3 h_{12}^4. \tag{3.7}
\end{equation}

From Corollary 3.6 we obtain the following

**Theorem 3.7.** If $N$ is a slant surface in $\mathbf{C}^2$, then $G = G^D$, identically.

**Proof.** Let $N$ be a slant surface in $\mathbf{C}^2$. Then Corollary 3.6 implies $A_{FY} X = A_{FX} Y$ for any vectors $X, Y$ tangent to $N$. Let $e_1, e_2, e_3, e_4$ be an orthonormal frame satisfying (3.4). Then we have

$$h_{12}^3 = h_{11}^4, \quad h_{22}^3 = h_{12}^4.$$

Therefore, by (3.6) and (3.7), we obtain $G = G^D$.

In the remaining part of this section we mention some properties of the normal-bundle valued 1-form $F$. In order to do so, we recall the following definition.

**Definition 3.1.** Let $N$ be a submanifold of a Riemannian manifold $M$. Then $N$ is called a minimal submanifold if $tr h = 0$, identically. And it is called austere (cf. [HL1]) if for each normal vector $\xi$ the set of eigenvalues of $A_\xi$ is invariant under multiplication by $-1$; this is equivalent to the condition that all the invariants of odd order of the Weingarten map at each normal vector of $N$ vanish identically.

Of course every austere submanifold is a minimal submanifold.

**Theorem 3.8.** Let $N$ be a proper slant submanifold of a Kaehlerian manifold $M$. If $\nabla F = 0$, then $N$ is autere.

**Proof.** Let $N$ be a proper slant submanifold of a Kaehlerian manifold $M$. If $\nabla F = 0$, then we have from formula (3.3)

$$fh(X, Y) = h(X, PY).$$
Let $X$ be any unit eigenvector of $Q = P^2$ with eigenvalue $\lambda \neq 0$. Then $X_* = PX/\sqrt{-\lambda}$ is a unit vector perpendicular to $X$. Thus, we have

$$h(X, X) = h(PX, PX)/\lambda = -h(X_*, X_*)$$

which implies that $N$ is autere.

If the ambient space $M$ is a complex-space-form, then we have the following reduction theorem.

**Theorem 3.9.** Let $N$ be an $n$-dimensional proper slant submanifold of a complex $m$-dimensional complex-space-form $M^m(c)$ with constant holomorphic sectional curvature $c$. If $\nabla F = 0$, then $N$ is contained in a complex $n$-dimensional complex totally geodesic submanifold of $M^m(c)$ as an autere submanifold.

**Proof.** Let $N$ be an $n$-dimensional proper slant submanifold of $\mathbb{C}^m$. Assume that $\nabla F = 0$. Then the normal bundle $T^\perp N$ has the following orthogonal direct decomposition:

$$T^\perp N = F(TN) \oplus \nu, \nu_p \perp F(T_pN)$$

for any point $p \in N$. For any vector field $\xi$ in $\nu$ and any vector fields $X, Y$ in $TN$, we have

$$<AJ_{\xi}X, Y > = <h(X, Y), J_\xi > = <V_XY, J_\xi > = -<PY, A_\xi X > + <FY, D_X\xi >,$$

from which we find

$$<D_X(FY), \xi > = -<A_\xi(PY) + A_\xi Y, X >.$$

On the other hand, for any $\xi$ normal to $N$, if we denote by $t\xi$ and $f\xi$ by using (1.5), then Lemma 3.5 gives

$$A_{f\xi}Y + A_\xi(PY) = 0.$$

Since $f = J$ on the normal subbundle $\nu$, formulas (3.8) and (3.9) imply $<D_X(FY), \xi > = 0$ for any $\xi$ in $\nu$. From this we conclude that the normal subbundle $F(N)$ is a parallel normal subbundle.
Next, we claim that the first normal subbundle \( \text{Im} h \) is contained in \( F(TN) \). This can be proved as follows.

Since \( \nabla F = 0 \), statement (ii) of Lemma 3.5 implies
\[
< h(X, Y), J\xi > = -< h(X, PY), \xi >
\]
for any normal vector \( \xi \) in \( \nu \). Thus, for any eigenvector \( Y \) of the self-adjoint endomorphism \( Q \) with eigenvalue \( \lambda \) and any normal vector \( \xi \) in \( \nu \), we have
\[
< h(X, Y), \xi > = -\lambda < h(X, Y), \xi >.
\]

Since \( N \) is a proper slant submanifold, \(-1 < \lambda < 0\). Thus, we obtain \( \text{Im} h \subset F(TN) \). Consequently, by applying the reduction theorem, we obtain the result.

In particular if the ambient space \( M \) is the complex number space \( \mathbb{C}^m \), then we have the following

**Theorem 3.10.** Let \( N \) be an \( n \)-dimensional proper slant submanifold of \( \mathbb{C}^m \). If \( \nabla F = 0 \), then \( N \) is contained in a complex linear subspace \( \mathbb{C}^n \) of \( \mathbb{C}^m \) as an austere submanifold.

For surfaces in a real 4-dimensional Kaehlerian manifold we have the following

**Theorem 3.11.** Let \( N \) be a surface in a real 4-dimensional Kaehlerian manifold \( M \). Then \( \nabla F = 0 \) if and only if either \( N \) is a complex surface, or a totally real surface, or a minimal proper slant surface of \( M \).

**Proof.** (\( \Rightarrow \)) Let \( N \) be a surface in \( M \) with \( \nabla F = 0 \). Assume that \( N \) is neither complex nor totally real. Then both \( P : T_xN \to T_xN \) and \( F : T_xN \to T^\perp_xN \) are surjective. Denote by \( \theta \) the Wirtinger angle. Define \( e_1, e_2, e_3, e_4 \) by (3.4). Then, by using \( J^2 = -I \), we have
\[
(3.10) \quad te_3 = -\sin \theta e_1, \quad te_4 = -\sin \theta e_2, \quad fe_3 = -\cos \theta e_4, \quad fe_4 = \cos \theta e_3.
\]

Since \( F \) is parallel, Lemma 3.5 implies \( A_{f\xi}X = -A_{\xi}(PX) \). Therefore, we find
\[
A_{f_{e_1}}e_2 = (\csc \theta)^{-1} \sec \theta A_{e_3}(Pe_1) = (\tan \theta) A_{e_3}(Pe_1) = -\tan \theta A_{f_{e_3}}e_1
\]
Therefore, by applying Corollary 3.6, \( N \) is slant. Furthermore, by applying Theorem 3.8, we know that \( N \) is minimal.

\((\Leftarrow)\) It is clear that if \( N \) is a complex or totally real surface in \( M \), then \( F \) is parallel. Therefore, we may assume that \( N \) is a minimal proper slant surface in \( M \). We choose \( e_1, e_2, e_3, e_4 \) according to (3.4). Then by Corollary 3.6, we have

\[
\begin{align*}
h_{11}^3 &= -h_{22}^3 = -h_{12}^4, & h_{12}^3 &= h_{11}^4 = -h_{22}^4.
\end{align*}
\]

From (3.11) and direct computation we may prove that \( A_{f\xi}X = -A_\xi(PX) \) for any tangent vector \( X \) and normal vector \( \xi \) of \( N \). Therefore, by applying Lemma 3.5, we conclude \( F \) is parallel.
§4. MINIMAL SLANT SURFACES AND TOTALLY REAL SURFACES.

In this section we want to establish some relations between minimal slant surfaces and totally real surfaces in a Kaehlerian manifold, in particular, in $\mathbb{C}^2$.

Let $N$ be a proper slant surface with slant angle $\theta$ in a real 4-dimensional Kaehlerian manifold $M$. Let $e_1$ be a local vector field tangent to $N$. We choose a canonical orthonormal local frame $e_1, e_2, e_3, e_4$ defined by

\begin{equation}
(4.1) \quad e_2 = (\sec \theta)Pe_1, \quad e_3 = (\csc \theta)Fe_1, \quad e_4 = (\csc \theta)Fe_2.
\end{equation}

We call such an orthonormal frame $e_1, e_2, e_3, e_4$ an adapted slant frame.

For an adapted slant frame we have

\begin{equation}
(4.2) \quad te_3 = -\sin \theta e_1, \quad te_4 = -\sin \theta e_2, \quad fe_3 = -\cos \theta e_4, \quad fe_4 = \cos \theta e_3.
\end{equation}

As before we put

\[
D e_r = \sum_s \omega^s_r \otimes e_s, \quad \nabla e_i = \sum_j \omega^j_i \otimes e_j,
\]

\[
h = \sum_r h^r e_r, \quad i,j = 1,2; \quad r,s = 3,4.
\]

We have the following

**Lemma 4.1.** Let $N$ be a proper slant surface in a real 4-dimensional Kaehlerian manifold $M$. Then, with respect to an adapted slant frame, we have

\begin{equation}
(4.3) \quad \omega_3^2 - \omega_1^2 = -\cot \theta \{(trh^3)\omega^1 + (trh^4)\omega^2\},
\end{equation}

where $\omega^1, \omega^2$ is the dual frame of $e_1, e_2$.

**Proof.** Since $J$ is parallel, we have

\[
D_X(FY) - F(\nabla_X Y) = fh(X,Y) - h(X,PY).
\]

Thus, we find

\[
D_{e_1} e_3 = D_{e_1} (\csc \theta Fe_1) = (\csc \theta)De_{e_1}(Fe_1) =
\]
\[
= (\csc \theta) \{ F(\nabla e_1 e_1) + fh(e_1, e_1) - h(e_1, Pe_1) \} = \\
= (\csc \theta) \{ \omega_1^3(e_1) Fe_2 + h_1^3 f e_3 + \\
nh_1^1 f e_4 - \cos \theta(h_1^3 e_3 + h_1^4 e_4) \} = \\
= \omega_1^2(e_1)e_4 - (\cos \theta)(trh^3)e_4.
\]

This implies

\[
\omega_3^4(e_1) - \omega_2^2(e_1) = -(\cot \theta)(trh^3).
\]

Similarly, we may obtain

\[
\omega_3^4(e_2) - \omega_2^2(e_2) = -(\cot \theta)(trh^4).
\]

These prove the lemma.

**Theorem 4.2.** Let \( N \) be a proper slant surface in a real 4-dimensional Kaehlerian manifold \((M, J, g)\). If there exists a compatible complex structure \( J_1 \) such that \( N \) is totally real with respect to the Kaehlerian manifold \((M, J_1, g)\), then \( N \) is minimal in \( M \).

**Proof.** Since \( N \) is assumed to be totally real in \((M, J_1, g)\), there exists a function \( \varphi \) such that

\[
(4.4) \quad e_3 = (\cos \varphi)J_1 e_1 + (\sin \varphi)J_1 e_2
\]

and

\[
(4.5) \quad e_4 = -(\sin \varphi)J_1 e_1 + (\cos \varphi)J_1 e_2.
\]

Since \( J_1 \) is parallel, these imply

\[
\omega_3^4(X) = < \nabla_X e_3, e_4 > = \\
< -(\sin \varphi)(X \varphi)J_1 e_1 + (\cos \varphi)(X \varphi)J_1 e_2 + (\cos \varphi)\omega_1^2(X)(X \varphi)J_1 e_1 + \\
(\sin \varphi)\omega_1^2(X)J_1 e_1, -(\sin \varphi)J_1 e_1 + (\cos \varphi)J_1 e_2 >.
\]
Therefore, we have
\[ \omega^4_3(X) = \sin^2 \varphi(X \varphi) + \cos^2 \varphi(X \varphi) + \cos^2 \varphi \omega^2_1(X) - \sin^2 \varphi \omega^1_2(X). \]
This implies
\[ (4.6) \quad \omega^4_3 - \omega^2_1 = d\varphi. \]
Combining (4.4) and Lemma 4.1 we obtain
\[ \cot \theta \{(trh^3)\omega^1 + (trh^4)\omega^2\} = -d\varphi. \]
Also from (4.4) and (4.5) we find
\[ h^3_{11} = - < \nabla_{e_1} e_3, e_1 > = \]
\[ = - < (\cos \varphi) \nabla_{e_1} (J_1 e_1) + (\sin \varphi) \nabla_{e_2} (J_1 e_2), e_1 > = \]
\[ < \cos \varphi h(e_1, e_1) + (\sin \varphi) h(e_1, e_2), J_1 e_1 > = \]
\[ = < (\cos \varphi) h(e_1, e_1) + (\sin \varphi) h(e_1, e_2), \cos \varphi e_3 - \sin \varphi e_4 > = \]
\[ = \cos^2 \varphi h^3_{11} - \sin^2 \varphi h^3_{22}. \]
This implies
\[ \sin^2 \varphi (h^3_{11} + h^3_{22}) = 0. \]
Similarly, we have
\[ \sin^2 \varphi (h^4_{11} + h^4_{22}) = 0. \]
Let \( U = \{ x \in N : H(x) \neq 0 \} \). Then \( U \) is an open subset of \( N \). If \( U \neq \emptyset \), then \( \varphi \equiv 0 \) (mod \( \pi \)) on \( U \). Thus,
\[ \cos \theta \{(trh^3)\omega^1 + (trh^4)\omega^2\} = -d\varphi = 0 \]
on \( U \), which implies \( \cos \theta = 0 \). This implies that \( N \) is totally real in \( (M, J, g) \) which is a contradiction. Thus \( U = \emptyset \), that is, \( N \) is minimal.

**Theorem 4.3.** Let \( N \) be a proper slant surface in \( \mathbb{C}^2 \). Then \( N \) is minimal if and only if there exists a compatible almost complex structure \( J_1 \) on \( E^4 \) such that \( N \) is totally real in \( (E^4, J_1) \).
This Theorem follows from Example 2.2, Theorem 4.3 and the following [C6, Theorem 5.2]

**Theorem 4.4.** Let $N$ be a proper slant surface in $\mathbb{C}^2$. Then $N$ is minimal if and only if there exists a compatible almost complex structure $J_2$ on $E^4$ such that $N$ is a complex surface in $(E^4, J_2)$.

This theorem was proved by studying the relation between Gauss map and slant immersions. In the next chapter we will treat these problems by using the notion of Gauss map.
CHAPTER III

GAUSS MAP AND SLANT IMMERSIONS

In this chapter we present some results of the author and Y. Tazawa [CT1]; in this work slant surfaces in $\mathbb{C}^2$ were studied from the point of view of the Gauss map.

§1. GEOMETRY OF $G(2, 4)$.

In this section we review the geometry of the real Grassmannian $G(2, 4)$ which consists of all oriented 2-planes in $E^4$.

Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ be the canonical orthonormal basis of $E^4$. Then $\Psi_0 =: \varepsilon_1 \wedge \cdots \wedge \varepsilon_4$ gives the canonical orientation of $E^4$. Let $\wedge^2 (E^4)^\ast$ denote the 6-dimensional real vector space with inner product, also denoted by $<,>$, defined by

$$
< X_1 \wedge X_2, Y_1 \wedge Y_2 >= \det (X_i, Y_j)
$$

and extended bilinearly. The two vector spaces $\wedge^2 (E^4)^\ast$ and $(\wedge^2 E^4)^\ast$ are identified in a natural way by

$$
(1.1) \quad \Phi(X_1 \wedge X_2) =: \Phi(X_1, X_2)
$$

for any $\Phi \in \wedge^2 (E^4)^\ast$. The Grassmannian $G(2, 4)$ was identified with the set $D_1(2, 4)$ which consists of all unit decomposable 2-vectors in $\wedge^2 E^4$ via $\phi : G(2, 4) \to D_1(2, 4)$ given by $\phi(V) = X_1 \wedge X_2$, for any positive orthonormal basis $\{X_1, X_2\}$ of $V \in G(2, 4)$.

The Hodge star operator $*: \wedge^2 E^4 \to \wedge^2 E^4$ is defined by

$$
(1.3) \quad <*\xi, \eta> = \Psi_0 = \xi \wedge \eta,
$$

for any $\xi, \eta \in \wedge^2 E^4$. So, if we regard an oriented 2-plane $V \in G(2, 4)$ as an element in $D_1(2, 4)$ via $\phi$, then we have $*V = V^\perp$, where $V^\perp$ denotes the oriented orthogonal complement of the oriented 2-plane $V$ in $E^4$.

Since $*^2 = 1$ and $*$ is a self-adjoint endomorphism of $\wedge^2 E^4$, we have the following orthogonal decomposition:

$$
(1.4) \quad \wedge^2 E^4 = \wedge^2_+ E^4 \oplus \wedge^2 E^4
$$
of eigenspaces of $*$ with eigenvalues 1 and $-1$, respectively.

Denote by $\pi_+$ and $\pi_-$ the natural projections: $\pi_{\pm} : \wedge^2 E^4 \to \wedge^2_{\pm} E^4$, respectively.

We put

$$\eta_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), \quad \eta_4 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4),$$

$$\eta_2 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4), \quad \eta_5 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_2 \wedge e_4),$$

$$\eta_3 = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3), \quad \eta_6 = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 - e_2 \wedge e_3).$$

Then $\{\eta_1, \eta_2, \eta_3\}$ and $\{\eta_4, \eta_5, \eta_6\}$ form canonical orthonormal bases of $\wedge^2_+ E^4$ and $\wedge^2_- E^4$, respectively. We shall orient the spaces $\wedge^2_+ E^4$ and $\wedge^2_- E^4$ such that these two bases are positive, that is, they give positive orientations for the oriented spaces $\wedge^2_+ E^4$ and $\wedge^2_- E^4$.

For any $\xi \in D_1(2,4)$ we have

$$\pi_+(\xi) = \frac{1}{2}(\xi + *\xi), \quad \pi_-(\xi) = \frac{1}{2}(\xi - *\xi).$$

Denote by $S^2_+$ and $S^2_-$ the 2-spheres in $\wedge^2_+ E^4$ and $\wedge^2_- E^4$ centered at the origin with radius $1/\sqrt{2}$, respectively. Then we have

$$\pi_+ : D_1(2,4) \to S^2_+, \quad \pi_- : D_1(2,4) \to S^2_-$$

and

$$D_1(2,4) = S^2_+ \times S^2_-.$$
§2. COMPLEX STRUCTURES ON $E^4$.

Let $C^2 = (E^4, J_0)$ be the complex 2-plane with the canonical complex structure $J_0$ and the canonical metric. It is well-known that $J_0$ is an orientation preserving isomorphism. We denote by $\mathcal{J}$ the set of all almost complex structures (or simply called complex structures) on $E^4$ which are compatible with the inner product $\langle , \rangle$, that is,

$$\mathcal{J} = \{ J : E^4 \to E^4 : J \text{ is linear}, J^2 = -I, \text{ and } \langle JX, JY \rangle = \langle X, Y \rangle, \text{ for any } X,Y \in E^4 \}.$$ 

An orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $E^4$ is called a $J$-basis if we have $Je_1 = e_2, Je_3 = e_4$. Two $J$-bases of the same $J$ have the same orientation.

Using the canonical orientation on $E^4$ we divide $\mathcal{J}$ into two disjoint subsets of $\mathcal{J}$:

$$\mathcal{J}^+ = \{ J \in \mathcal{J} : J\text{-bases are positive} \},$$

$$\mathcal{J}^- = \{ J \in \mathcal{J} : J\text{-bases are negative} \}.$$

For any $J \in \mathcal{J}$, there exists a unique 2-vector $\zeta_J \in \wedge^2 E^4$ defined as follows:

(2.1) \[ < \zeta_J, X \wedge Y > = -\Omega_J(X,Y) =: - < X, JY >, \]

for any $X,Y \in E^4$. In other words, $\zeta_J$ is nothing but the metrical dual of $-\Omega_J$, where $\Omega_J$ is the Kaehler form associated with $J$.

**Lemma 2.1.** The mapping $\zeta : \mathcal{J} \to \wedge^2 E^4$ defined by $\zeta(J) = \zeta_J$, gives rise to two bijections:

(2.2) \[ \zeta^+ : \mathcal{J}^+ \to S^2_+ (\sqrt{2}), \quad \zeta^- : \mathcal{J}^- \to S^2_- (\sqrt{2}), \]

where $S^2_+ (\sqrt{2})$ and $S^2_- (\sqrt{2})$ are the 2-spheres centered at the origin with radius $\sqrt{2}$ in $\wedge^2_+ E^4$ and $\wedge^2_- E^4$, respectively.

**Proof.** If $J \in \mathcal{J}^+$ and $e_1, e_2, e_3, e_4$ is a $J$-basis, then $e_1, e_2, e_3, e_4$ is positive. From (2.1) we have

$$\zeta_J = e_1 \wedge e_2 + e_3 \wedge e_4.$$ 

Thus, we have $\zeta_J \in S^2_+ (\sqrt{2})$. Similarly, if $J \in \mathcal{J}^-$, then we have $\zeta_J \in S^2_- (\sqrt{2})$. 

If $J$ and $J'$ are two distinct compatible complex structures on $E^4$, then their corresponding Kaehler forms $\Omega_J$ and $\Omega_{J'}$ are distinct. Thus, $\zeta_J \neq \zeta_{J'}$. This proves the injectivity of $\zeta$.

Conversely, for any element $\xi \in S^2_+(\sqrt{2})$, we have $\frac{1}{2}\xi \in S^2_+$. Since $\pi_+: D_1(2,4) \to S^2_+$ is isomorphic, there exists an element $V \in D_1(2,4)$ such that $\pi_+(V) = \frac{1}{2}\xi$. Let $e_1, e_2, e_3, e_4$ be a positive orthonormal basis of $E^4$ such that $e_1 \wedge e_2 = V$. Define $J \in \mathcal{J}^+$ such that $Je_1 = e_2, Je_3 = e_4$. Then we have $\zeta_J = \xi$.

This completes the proof of the Lemma.

By applying Lemma 2.1 we may make the following identifications via $\zeta, \zeta^+$ and $\zeta^-$, respectively:

$$\mathcal{J}^+ \cong S^2_+((\sqrt{2})),$$  

$$\mathcal{J}^- \cong S^2_-((\sqrt{2})),$$  

$$\mathcal{J} \cong S^2_+((\sqrt{2})) \cup S^2_-((\sqrt{2}))$$

For any $V \in G(2,4)$ and for any $J \in \mathcal{J}$, we choose a positive orthonormal basis $\{e_1, e_2\}$ of $V$ and put

$$\alpha_J(V) = \cos^{-1}(\langle Je_1, e_2 \rangle).$$

Then $\alpha_J(V) \in [0, \pi]$ and $\theta(X) = \min \{\alpha_J(V), \pi - \alpha_J(V)\}$. A 2-plane $V \in G(2,4)$ is said to be $\alpha$-slant if $\alpha_J(V) = \alpha$, identically.

If $N$ is an oriented surface in $\mathbf{C}^2$, then $N$ has a unique complex structure determined by its orientation and its induced metric. With respect to the angle $\alpha_J$, $(J \in \mathcal{J})$, we have

$N$ is holomorphic $\iff \alpha_J(TN) = 0,$

$N$ is antiholomorphic $\iff \alpha_J(TN) = \pi,$

$N$ is totally real $\iff \alpha_J(TN) = \frac{\pi}{2}.$

The following lemma obtained in [CT1] establishes the fundamental relations between slant angle and the projections $\pi_+$ and $\pi_-.$

**Lemma 2.2.**

(i) If $J \in \mathcal{J}^+$, then $\alpha_J(V)$ is the angle between $\pi_+(V)$ and $\zeta_J$ and

(ii) If $J \in \mathcal{J}^-$, then $\alpha_J(V)$ is the angle between $\pi_-(V)$ and $\zeta_J$. 

Proof. (i) If $J \in \mathcal{J}^+$, then we have
\[
\cos(\alpha_J(V)) = -\Omega_J(V) = <\zeta_J, V>
\]
\[
= <\zeta_J, \pi_+(V) + \pi_-(V) > = <\zeta_J, \pi_+(V) >.
\]
Since $||\zeta_J|| = \sqrt{2}$ and $||\pi_+(V)|| = 1/\sqrt{2}$, this implies
\[
\alpha_J(V) = \angle(\pi_+(V), \zeta_J).
\]
(ii) can be proved in a similar way.

For any $a \in [0, \pi]$ and for any $J \in \mathcal{J}$, we define

\[
G_{J,a} = \{V \in G(2,4) : \alpha_J(V) = a\},
\]
which is equivalently to say that $G_{J,a}$ is the set consisting of all oriented $a$-slant oriented 2-planes in $E^4$ with respect to the complex structure $J$.

For any $a \in [0, \pi]$ and any $V \in G(2,4)$, we define

\[
\mathcal{J}_{V,a} = \{J \in \mathcal{J} : \alpha_J(V) = a\}.
\]

We put
\[
\mathcal{J}_{V,a}^+ = \mathcal{J}_{V,a} \cap \mathcal{J}^+, \quad \mathcal{J}_{V,a}^- = \mathcal{J}_{V,a} \cap \mathcal{J}^-.
\]

From Lemma 2.2 we may obtain [CT1]

Lemma 2.3.

(1) If $J \in \mathcal{J}^+$, then $G_{J,a} = S_{J,a}^+ \times S_2^-$, where $S_{J,a}^+$ is the circle on $S_2^+$ consisting of 2-vectors which makes constant angle $a$ with $\zeta_J$.

(2) If $J \in \mathcal{J}^-$, then $G_{J,a} = S_2^+ \times S_{J,a}^-$, where $S_{J,a}^-$ is the circle on $S_2^-$ consisting of 2-vectors which makes constant angle $a$ with $\zeta_J$.

(3) Via the identification given by Lemma 2.1, $\mathcal{J}_{V,a}^+$ is the circle on $S_2^+(\sqrt{2})$ consisting of 2-vectors in $S_2^+(\sqrt{2})$ which makes constant angle $a$ with $\pi_+(V)$, and $\mathcal{J}_{V,a}^-$ is the circle on $S_2^-(\sqrt{2})$ consisting of 2-vectors in $S_2^-(\sqrt{2})$ which makes constant angle $a$ with $\pi_-(V)$.

This Lemma can be regarded as a generalization of Proposition 2 of [CM2].
For later use we give the following

**Notation.** Let $V$ be an oriented 2-plane in $G(2,4)$. $J_V^+$ and $J_V^-$ are defined by

\[ J_V^+ = (\zeta^+)^{-1}(\pi_+(V)) \in J^+, \quad J_V^- = (\zeta^-)^{-1}(\pi_-(V)) \in J^- \text{.} \]

**Remark. 2.1.** It is easy to see that $J_V^+$ (respectively, $J_V^-$) defined by (2.6) is the complex structure in $J^+$ (respectively, in $J^-$) such that $V$ is a holomorphic plane with respect to the complex structure $J$. 

§3. SLANT SURFACES AND GAUSS MAP.

In this section we study the following problems:

**Problem 3.1.** Let $N$ be a surface in $\mathbb{C}^2 = (E^4, J_0)$. When is $N$ slant in $\mathbb{C}^2$?

**Problem 3.2.** Let $N$ be a surface in $E^4$. If there exists a compatible complex structure $J$ on $E^4$ such that $N$ is slant in $(E^4, J)$, how many other compatible complex structures $\tilde{J}$ on $E^4$ are there such that $N$ is slant with respect to these complex structures?

Let $f : N \to E^4$ be an immersion from an oriented surface $N$ into $E^4$. Denote by $\nu : N \to \mathbb{G}(2, 4)$ be the Gauss map associated with the immersion $f$ defined by $\nu(p) = T_p N, \ p \in N$ (or, equivalently, by $\nu(p) = (e_1 \wedge e_2)(p)$).

We put

$$(3.1) \quad \nu_+ = \pi_+ \circ \nu, \ \nu_- = \pi_- \circ \nu.$$

Then we have

$$(3.2) \quad \nu_\pm : N \to \mathbb{G}(2, 4) \to S^2_\pm.$$

We give the following [CT1]

**Proposition 3.1.** Let $f : N \to E^4$ be an immersion of an oriented surface $N$ into $E^4$. Then

1. $f$ is slant with respect to a complex structure $J \in \mathcal{J}^+$ (respectively, $J \in \mathcal{J}^-$) if and only if $\nu_+(N)$ (respectively, $\nu_-(N)$) is contained in a circle on $S^2_+$ (respectively, on $S^2_-$).

2. $f$ is $\alpha$-slant with respect to a complex structure $J \in \mathcal{J}^+$ (respectively, $J \in \mathcal{J}^-$) if and only if $\nu_+(N)$ (respectively, $\nu_-(N)$) is contained in a circle $S^2_{J,\alpha}$ on $S^2_+$ (respectively, $S^2_{J,\alpha}$ on $S^2_-$), where $S^2_{J,\alpha}$ (respectively, $S^2_{J,\alpha}$) is the circle on $S^2_+$ (respectively, $S^2_-$) consisting of all 2-vectors which makes constant angle $\alpha$ with $\zeta_J$.

**Proof.** If $f : N \to E^4$ is $\alpha$-slant with respect to a compatible complex structure $J \in \mathcal{J}^+$, then, by (2.5) and Lemma 2.3, we have $\alpha_J(T_p N) \in$
$S^+_J \times S^2$ for any point $p \in N$. Thus, $\nu_+(N)$ is contained in the circle $S^+_J$ on the 2-sphere $S^2_\alpha$ consisting of 2-vectors in $S^2_\alpha$ which makes constant angle $\alpha$ with $\zeta_J$.

Conversely, if $f : N \to E^4$ is an immersion such that $\nu_+(N)$ is contained in a circle $S^1$ on the 2-sphere $S^2_\alpha$. Let $\eta$ be a vector of length $\sqrt{2}$ in $\wedge^2 E^4$ perpendicular to the 2-plane in $\wedge^2 E^4$ containing $S^1$. Then $\eta \in S^2_\alpha(\sqrt{2})$. By Lemma 2.1, there is a unique complex structure $J \in \mathcal{J}^+$ such that $\zeta_J = \eta$.

It is clear that $S^1$ is a circle $S^+_J$ for some constant angle $\alpha$. Therefore, by Lemma 2.3, the immersion $f$ is $\alpha$-slant with respect to this compatible complex structure $J \in \mathcal{J}^+$.

Similar argument applies to the other cases.

Now we may give the main result of this section [CT1].

**Theorem 3.2.**

1. Let $f : N \to E^4$ be a minimal immersion. If there exists a compatible complex structure $J \in \mathcal{J}^+$ (respectively, $J \in \mathcal{J}^-$) such that the immersion $x$ is slant with respect to $J$, then

   1-a) for any $\alpha \in [0, \pi]$, there is a compatible complex structure $J_\alpha \in \mathcal{J}^+$ (respectively, $J_\alpha \in \mathcal{J}^-$) such that $f$ is $\alpha$-slant with respect to the complex structure $J_\alpha$.

   1-b) the immersion $f$ is slant with respect to any complex structure $J \in \mathcal{J}^+$ (respectively, $J \in \mathcal{J}^-$).

2. If $f : N \to E^4$ is a non-minimal immersion, then there exist at most two complex structures $\pm J^+ \in \mathcal{J}^+$ and at most two complex structures $\pm J^- \in \mathcal{J}^-$ such that the immersion $f$ is slant with respect to them.

**Proof.** (a) Assume that $f : N \to E^4$ is a minimal immersion. Then both $\nu_+$ and $\nu_-$ are anti-holomorphic (cf. Lawson’s book [L1]). So, both $\nu_+$ and $\nu_-$ are open maps if they are not constant maps. If the immersion $f$ is slant with respect to a complex structure $J \in \mathcal{J}^+$ (respectively, $J \in \mathcal{J}^-$), then $\nu_+$ (respectively, $\nu_-$) cannot be an open map by Proposition 3.1. Thus $\nu_+(N)$ (respectively, $\nu_-(N)$) is a singleton, since a singleton is contained in every circle on the 2-sphere $S^2_\alpha$ (respectively, $S^2_\alpha$), the immersion $f$ is slant with respect to every complex structure $J \in \mathcal{J}^+$ (respectively, $J \in \mathcal{J}^-$). So, for any constant $\alpha \in [0, \pi]$, there is a complex structure $J_\alpha$ which makes
the immersion $f$ to be $\alpha$-slant.

(b) Assume that $f: N \to E^4$ is a non-minimal immersion. If the immersion $f$ is $\alpha$-slant with respect to a complex structure $J \in J^+$ (respectively, $J \in J^-$), then $\nu_+(N)$ (respectively, $\nu_-(N)$) is contained in the circle $S^+_{J,\alpha}$ (respectively, $S^-_{J,\alpha}$). If $f$ is $\alpha$-holomorphic with respect to some compatible complex structure on $E^4$ which implies that $N$ is minimal in $E^4$. Therefore, the complex structures $\pm J^+$ and $\pm J^-$ are the only possible complex structures on $E^4$ which may make the immersion $f$ slant.

From Theorem 3.2 we obtain the following corollaries of [CT1] immediately.

**Corollary 3.3.**

(a) If $f: N \to C^2 = (E^4, J_0)$ is holomorphic, then the immersion $f$ is slant with respect to every complex structure $J \in J^+$.

(b) If $f: N \to C^2 = (E^4, J_0)$ is anti-holomorphic, then the immersion $f$ is slant with respect to every complex structure $J \in J^-$.

**Corollary 3.4.** Let $f: N \to E^4$ be a minimal immersion. Then the immersion $f$ is slant with respect to some complex structure $J \in J^+$ if and only if $f$ is holomorphic (respectively, anti-holomorphic) with respect to some complex structure structure $J \in J^+$ (respectively, $J \in J^-$).

**Corollary 3.5.** If $f: N \to E^3$ is a non-totally geodesic minimal immersion, then $f: N \to E^3 \subset E^4$ is not slant with respect to every compatible complex structure on $E^4$.

**Proof.** If $f: N \to E^3 \subset E^4$ is slant, then, by the minimality of $f$, Theorem 3.2 implies that the immersion $f$ must be a proper slant immersion with respect to some compatible complex structure $J$ on $E^4$. Therefore, by Theorem 3.7 of Chapter II, we have $G = G^D = 0$, identically. Because the only flat minimal surfaces in a Euclidean space are totally geodesic ones, this is impossible.

**Definition 3.1.** An immersion $f: N \to E^4$ is called doubly slant if it is slant with respect to some complex structure $J^+ \in J^+$ and at the same time it is slant with respect to another complex structure $J^- \in J^-$. 
From Theorem 3.2 we have the following [CT1]

**Corollary 3.6.** Every non-minimal immersion $f : N \to E^4$ which is slant with respect to more than two complex structures are doubly slant.

**Remark 3.1.** From Theorem 3.2 we also know that for any immersion $f : N \to E^4$, exactly one of the following four cases occurs:

(a) $f$ is not slant with respect to every compatible complex structure on $E^4$.
(b) $f$ is slant with respect to infinitely many compatible complex structures on $E^4$.
(c) $f$ is slant with respect to exactly two compatible complex structures on $E^4$.
(d) $f$ is slant with respect to exactly four compatible complex structures on $E^4$.

Corollary 3.5 shows that if $N$ is a non-totally geodesic minimal surface in $E^3$, then, by regarding $E^3$ as a linear subspace of $E^4$, $N$ is not slant with respect to every compatible complex structures on $E^4$. This provides us many examples for case (a) of Remark 3.1.

Here we remark that every totally real immersion of a 2-sphere $S^2$ into $C^2$ gives us an example of surface in $E^4$ which is slant with respect to exactly two compatible complex structures on $E^4$. This is due the fact that the Gauss curvature of any Riemannian metric on $S^2$ is non-flat.

**Example 3.1.** Let $f : E^3 \to E^4$ be the map from $E^3$ into $E^4$ defined by

$$ f(x_0, x_1, x_2) = (x_1, x_2, 2x_0x_1, 2x_0x_2). $$

Then $f$ induces an immersion $\hat{f} : S^2 \to E^4$ from the unit 2-sphere $S^2$ into $E^4$, called the Whitney immersion which has a unique self-intersection point $\hat{f}(-1,0,0) = \hat{f}(1,0,0)$. It is know that this immersion $\hat{f} : S^2 \to E^4$ is a totally real immersion with respect to two suitable compatible complex structures on $E^4$. Moreover, since the surface is non-flat, $\hat{f}$ is a slant immersion with respect to only two compatible complex structures (cf. Theorem 4.1.)

In Section 1 of Chapter V we will prove that there exist no compact proper slant submanifolds in any complex number space $C^n$. 
Example 3.2. Let $N$ be the surface in $E^4$ defined by

$$x(u, v) = (u, v, k \cos v, k \sin v) \quad (3.4)$$

Then $N$ is the Riemannian product of a line and a circular helix in a hyperplane $E^3$ of $E^4$. Let $J_1, J_2$ be the compatible complex structures on $E^4$ defined respectively by

$$J_1(a, b, c, d) = (-b, a, -d, c), \quad J_2(a, b, c, d) = (b, -a, -d, c).$$

Then $J_1 \in J^+$ and $J_2 \in J^-$. Moreover, by direct computation, we can prove that the surface $N$ is slant with respect to the following four complex structures: $J_1, -J_1, J_2, -J_2$, with slant angles given by

$$\cos^{-1}\left(\frac{1}{\sqrt{1 + k^2}}\right), \cos^{-1}\left(\frac{-1}{\sqrt{1 + k^2}}\right), \cos^{-1}\left(\frac{-1}{\sqrt{1 + k^2}}\right), \cos^{-1}\left(\frac{1}{\sqrt{1 + k^2}}\right),$$

respectively.

Remark 3.2. In view of Corollary 3.6, it is interesting to point out that the only doubly slant minimal immersion from a surface into a complex 2-plane is the totally geodesic one.
III-4. DOUBLY SLANT SURFACES IN $\mathbb{C}^2$

§4. DOUBLY SLANT SURFACES IN $\mathbb{C}^2$.

As we defined in Section 3 an immersion $f : N \to E^4$ is called doubly slant if it is slant with respect to a complex structure in $J^+$ and at the same time it is slant with respect to another complex structure in $J^-$. Equivalently, the immersion $f$ is doubly slant if and only if there exists an oriented 2-plane $V \in G(2,4)$ such that $f$ is slant with respect to both $J^+_V$ and $J^-_V$, where $J^+_V$ and $J^-_V$ are defined by (2.6).

In this section we give the following [CT1]

**Theorem 4.1.** If $f : N \to E^4$ is a doubly slant immersion, then

\begin{equation}
G = G^D = 0
\end{equation}

identically.

**Proof.** If $f$ is doubly slant, then, by Proposition 3.1, we know that both $\nu_+(N)$ and $\nu_-(N)$ lie in some circles on $S^2_+$ and $S^2_-$, respectively. Thus, both $(\nu_+)_*$ and $(\nu_-)_*$ are singular maps at every point $p \in N$. Therefore, the result follows from the following

**Lemma 4.2.** For any immersion $f : N \to E^4$ of an oriented surface $N$ into $E^4$ we have

\begin{equation}
\det (\nu_+)_* = \frac{1}{2}(G + G^D), \quad \det (\nu_-)_* = \frac{1}{2}(G - G^D).
\end{equation}

**Proof.** Let $x : N \to E^4$ be an immersion of an oriented surface $N$ into $E^4$. Denote by $\{e_1, e_2\}$ be a positive orthonormal basis of $N$. Then at each point $p \in N$ the Gauss map $\nu$ is given by

\begin{equation}
\nu(p) = (e_1 \wedge e_2)(p).
\end{equation}

Thus, for any vector $X$ tangent to $N$, we have

\[\nu_* X = (\nabla_X e_1) \wedge e_2 + e_1 \wedge (\nabla_X e_2)\]
\[
\omega_1^3(X)e_3 \wedge e_2 + \omega_1^4(X)e_4 \wedge e_2 + e_1 \wedge \omega_2^3(X)e_3 + e_1 \wedge \omega_2^4(X)e_4 \\
= -\omega_1^3(X)e_2 \wedge e_3 - \omega_1^4(X)e_2 \wedge e_4 + \omega_2^3(X)e_1 \wedge e_3 + \omega_2^4(X)e_1 \wedge e_4 \\
= \frac{1}{2}\{(\omega_1^4 + \omega_2^3)(X)(e_1 \wedge e_3 - e_2 \wedge e_4) + (-\omega_1^3 + \omega_2^4)(X)(e_1 \wedge e_4 - e_2 \wedge e_3) \\
+(-\omega_1^4 + \omega_2^3)(X)(e_1 \wedge e_3 + e_2 \wedge e_4) + (\omega_1^3 + \omega_2^4)(X)(e_1 \wedge e_4 - e_2 \wedge e_3)\} \\
= \frac{1}{\sqrt{2}}\{(\omega_1^4 + \omega_2^3)(X)\eta_2 + (-\omega_1^3 + \omega_2^4)(X)\eta_3 + \\
+(-\omega_1^4 + \omega_2^3)(X)\eta_5 + (\omega_1^3 + \omega_2^4)(X)\eta_6\}. \\
\]

Therefore we have

(4.4) \quad (\nu_+)_* = \frac{1}{\sqrt{2}}\{(\omega_1^4 + \omega_2^3)\eta_2 + (-\omega_1^3 + \omega_2^4)\eta_3\},

and

(4.5) \quad (\nu_-)_* = \frac{1}{\sqrt{2}}\{(-\omega_1^4 + \omega_2^3)\eta_5 + (\omega_1^3 + \omega_2^4)\eta_6\}.

This proves Lemma 4.2.

**Remark 4.1.** Examples 2.1-2.6 given in Section 2 of Chapter II are examples of doubly slant surfaces.

**Remark 4.2.** Lemma 4.2 can be found in [HO1]. Our proof of Lemma 4.2 is different from theirs.
§5. SLANT SURFACES IN ALMOST HERMITIAN MANIFOLDS.

Let \( f : N \rightarrow (M, J) \) be an immersion of a differentiable manifold \( N \) into an almost complex manifold \( (M, J) \). Then a point \( p \in N \) is called a complex tangent point if the tangent plane of \( N \) at \( p \) is invariant under the action of the almost complex structure \( J \).

The purpose of this section is to prove the following [CT1]

**Theorem 5.1.** Let \( f : N \rightarrow (M, g, J) \) be an imbedding of an oriented surface \( N \) into a real 4-dimensional almost Hermitian manifold \( (M, g, J) \). If the immersion \( f \) has no complex tangent points, then, for any prescribed angle \( \alpha \in [0, \pi] \), there exists an almost complex structure \( \tilde{J} \) on \( M \) satisfying the following conditions:

(a) \((M, g, \tilde{J})\) is an almost Hermitian manifold and

(b) the immersion \( f \) is \( \alpha \)-slant with respect to \( \tilde{J} \).

**Proof.** \((M, g, J)\) has the natural orientation determined by the almost complex structure \( J \) and, at each point \( p \in M \), the tangent space \( T_pM \) together with the metric \( g_p \) is a Euclidean 4-space. So we may apply the argument given in Sections 1 and 2 of this chapter.

According to (1.4) the vector bundle \( \wedge^2(M) \) of 2-vectors on the ambient manifold \( M \) is the direct sum of two vector subbundles:

\[
\wedge^2(M) = \wedge^2_+(M) \oplus \wedge^2_-(M). \tag{5.1}
\]

We define two sphere-bundles over \( M \) by

\[
S^2_+(M) = \{ \xi \in \wedge^2_+(M) : ||\xi|| = \frac{1}{\sqrt{2}} \},
\]

\[
S^2_-(M) = \{ \xi \in \wedge^2_+(M) : ||\xi|| = \sqrt{2} \}.
\]

By using Lemma 2.1 we can identify a cross-section

\[
\gamma : M \rightarrow S^2_+(M) \tag{5.2}
\]

with an almost complex structure \( J_\gamma \) on \( M \) such that \((M, g, J_\gamma)\) is an almost Hermitian manifold.
In the following we denote by \( \rho \) the cross-section corresponding to the almost complex structure \( J \) and we want to construct another cross-section \( \tilde{\sigma} \) to obtain the desired almost complex structure \( \tilde{J} \) on the ambient manifold \( M \).

We consider the pull-backs of these bundles via the imbedding \( f : N \to M \), that is,

\[
\wedge^2_+(N) = f^*(\wedge^2_+(M)), \quad S^2_+(N) = f^*(S^2_+(M)),
\]

\[
\bar{S}^2_+(N) = f^*(\bar{S}^2_+(M)).
\]

The tangent bundle \( TN \) of \( N \) determines a cross-section \( \tau : N \to S^2_+(N) \) defined by

\[
\tau(p) = \pi_+(T_pN)
\]

for any point \( p \in N \), where \( \pi_+ \) denotes the natural projection from \( \wedge^2(T_pM) \) onto \( \wedge^2_+(T_pM) \). Notice that \( 2\tau \) is a cross-section of \( \bar{S}^2_+(N) \):

\[
2\tau : N \to \bar{S}^2_+(N).
\]

We denote \( f^*\rho \) also by \( \rho \) for simplicity. We have the following cross-section:

\[
\rho = f^*\rho : N \to \bar{S}^2_+(N).
\]

Since the imbedding \( f \) is assumed to have no complex tangent points,

\[
\rho(p) \neq \pm 2\tau(p)
\]

for any point \( p \in N \). Therefore, \( \rho(p) \) and \( 2\tau(p) \) determine a 2-plane in \( \wedge^2_+(T_pM) \) which intersects the circle \((J^+_{\tau,a})_p \) at two points. Where \((J^+_{\tau,a})_p \) is a circle on \((\bar{S}^2_+(N))_p \) defined in Lemma 2.3 with \( V = T_pN \).

Let \( \sigma(p) \) be one of these two points which lies on the half-great-circle on \((\bar{S}^2_+(N))_p \) starting from \( 2\tau(p) \) and passing through \( \rho(p) \). Since \( \rho \) and \( \tau \) are differentiable, so is \( \sigma \). Thus we obtain a third cross-section:

\[
\sigma : N \to \bar{S}^2_+(N)
\]
and we want to extend \( \sigma \) to a cross-section \( \tilde{\sigma} \) of \( \bar{S}^2_+(M) \).

For each point \( p \in N \), we choose an open neighborhood \( U_p \) of \( p \in M \) such that \( \sigma|_{U_p \cap N} \) can be extended to a cross-section of \( \bar{S}^2_+(M) \) on \( U_p \):

\[
(5.9) \quad \sigma_p : U_p \to \bar{S}^2_+(M)|_{U_p}.
\]

Here we identify the manifold \( N \) with its image \( f(N) \) of \( N \) under the imbedding \( f \). We put

\[
(5.10) \quad \mathcal{U} = \cup_{p \in N} U_p
\]

and pick a locally finitely countable refinement \( \{U_i\} \) of the open covering \( \{U_p : p \in N\} \) of \( U \).

For each \( i \) we pick a point \( p \in N \) such that \( U_i \) is contained in \( U_p \) and we put

\[
(5.11) \quad \sigma_i = \sigma_p|_{U_i}.
\]

Let \( \phi_i \) be a differentiable partition of unity on \( \mathcal{U} \) subordinate to the covering \( \{U_i\} \). We define a cross-section \( \tilde{\sigma} \) of \( \wedge^2_+(M)|_{\mathcal{U}} \) by

\[
(5.12) \quad \tilde{\sigma} : \mathcal{U} \to \wedge^2_+(M)|_{\mathcal{U}}; \quad \tilde{\sigma} = \sum_i \phi_i \sigma_i.
\]

From the constructions of \( \sigma_i \) and \( \tilde{\sigma} \) we have

\[
(5.13) \quad \tilde{\sigma}|_N = \sigma.
\]

Since the angle \( \angle(\tilde{\sigma}(p), \rho(p)) \) between \( \tilde{\sigma}(p) \) and \( \rho(p) \) is less that \( \pi \) for any point \( p \in N \), we have

\[
\tilde{\sigma}(p) \neq 0, \quad \angle(\tilde{\sigma}(p), \rho(p)) < \pi
\]

for any point \( p \in N \). By continuity of \( \tilde{\sigma} \) we can pick an open neighborhood \( W \) of \( N \) contained in \( \mathcal{U} \) such that

\[
\tilde{\sigma}(q) \neq 0, \quad \angle(\tilde{\sigma}(q), \rho(q)) < \pi
\]

for any point \( q \in W \). We define a cross-section \( \hat{\sigma} \) of \( \bar{S}^2_+(M)|_W \) by

\[
\hat{\sigma} : W \to \bar{S}^2_+(M)|_W; \quad \hat{\sigma} = \tilde{\sigma}/\sqrt{2||\tilde{\sigma}||}.
\]
Then we have \( \angle (\hat{\sigma}(q), \rho(q)) < \pi \) for any point \( q \in M \), too.

Finally, we consider the open covering \( \{W, M - N\} \) of \( M \) and local cross-section \( \hat{\sigma} \) and \( \rho|_{M-N} \) and repeat the same argument using an partition of unity subordinate to \( \{W, M - N\} \) to obtain a cross-section \( \tilde{\sigma} : M \to \mathbb{S}^2(M) \) satisfying \( \tilde{\sigma}|_N = \sigma \). Now it is clear that the almost complex structure \( \tilde{J} \) corresponding to \( \tilde{\sigma} \) is the desired almost complex structure. This completes the proof of the theorem.
CHAPTER IV
CLASSIFICATIONS OF SLANT SURFACES

§1. SLANT SURFACES WITH PARALLEL MEAN CURVATURE VECTOR.

The main purpose of this section is to present the following classification theorem of [C6].

**Theorem 1.1.** Let $N$ be a surface in $\mathbb{C}^2$. Then $N$ is a slant surface with parallel mean curvature vector, that is, $DH = 0$, if and only if $N$ is one of the following surfaces:

(a) an open portion of the product surface of two plane circles, or
(b) an open portion of a circular cylinder which is contained in a hyperplane of $\mathbb{C}^2$, or
(c) a minimal slant surface in $\mathbb{C}^2$.

Moreover, if either case (a) or case (b) occurs, then $N$ is a totally real surface in $\mathbb{C}^2$.

**Proof.** Let $N$ be a slant surface in $\mathbb{C}^2$ with parallel mean curvature vector. Then the length of the mean curvature vector $H$ is constant. If the length is zero, $N$ is a minimal slant surface. So Case (c) occurs. Now assume that $N$ is not minimal in $\mathbb{C}^2$. Then one may choose a unit tangent vector $e_1$ such that $e_3 = (\csc \theta) F e_1$ is in the direction of $H$, where $\theta$ denotes the slant angle of $N$ in $\mathbb{C}^2$. Such an $e_1$ can be chosen if we choose $e_1$ to be in the direction of $-tH$. Since the mean curvature vector is parallel, $\omega^2_1 = 0$. Thus the normal curvature $G^D = 0$, identically. Hence, by applying Theorem 3.7 of Chapter II, we have $G = 0$, that is, $N$ is flat. Let $V = \{ p \in N : A_{e_3} \neq 0 \text{ at } p \}$. Then $V$ is an open subset of $N$.

**Case (i):** $V = \emptyset$. In this case we have $Im h \subset Span\{H\}$. Thus, by applying the condition $DH = 0$, we conclude that the surface $N$ lies in a hyperplane $E^3$ of $\mathbb{C}^2$. Since $N$ is a flat surface in $E^3$ with nonzero constant mean curvature, $N$ is an open portion of a circular cylinder (cf. Proposition 3.2.
of [C1, p.118]).

Case (ii): $V \neq \emptyset$. In this case, let $W$ be a connected component of $V$. Then $e_4$ is a parallel minimal non-geodesic section on $W$ (cf. [C1]). Since $W$ is flat, Proposition 5.4 of [C1, p.128] implies that $W$ is an open piece of the product surface of two plane circles. Since $\text{det}(A_{e_4})$ is a nonzero constant on $W$, by continuity we have $V = N$. Thus the whole surface is an open portion of the product surface.

Finally, if $N$ is an open portion of the product surface of two plane circles or an open portion of a circular cylinder contained in a hyperplane of $\mathbb{C}^2$, then $N$ is totally real in $E^4$ with respect to some compatible complex structure, say $J'$, on $E^4$. Therefore, by applying Theorem 4.3 of Chapter II, we know that $N$ must be totally real in $\mathbb{C}^2$, since $N$ is non-minimal. This completes the proof of the theorem.

By applying Theorem 1.1 we may obtain the following classification of parallel slant surfaces, that is, slant surfaces with parallel second fundamental form [C4].

**Theorem 1.2.** Let $N$ be a surface in $\mathbb{C}^2$. Then $N$ is a slant surface in $\mathbb{C}^2$ with parallel second fundamental form, that is, $\nabla h = 0$, if and only if $N$ is one of the following surfaces:

(a) an open portion of the product surface of two plane circles;

(b) an open portion of a circular cylinder which is contained in a hyperplane of $\mathbb{C}^2$;

(c) an open portion of a plane in $\mathbb{C}^2$.

Moreover, if either case (a) or case (b) occurs, then $N$ is totally real in $\mathbb{C}^2$.

**Proof.** If $N$ is a surface in $\mathbb{C}^2$ with parallel second fundamental form, then $N$ has parallel mean curvature vector. Thus, by Theorem 1.1, it suffices to prove that slant planes are the only minimal slant surfaces with parallel second fundamental form. But this follows from the facts that every surface in $\mathbb{C}^2$ with parallel second fundamental form has constant Gauss curvature and every minimal surface in $\mathbb{C}^2$ with constant Gauss curvature is totally geodesic.

By using Theorem 1.1, we obtain
Corollary 1.3. Let $N$ be a slant surface in $\mathbb{C}^2$ with constant mean curvature. Then $N$ is spherical if and only if $N$ is an open portion of the product surface of two plane circles.

Proof. If $N$ is a spherical surface with constant mean curvature, then the mean curvature vector of $N$ in $\mathbb{C}^2$ is parallel. Thus, by applying Theorem 1.1, $N$ is one of the surfaces mentioned in Theorem 1.1. Among them, surfaces of type (a) are the only spherical surfaces in $\mathbb{C}^2$.

The converse is obvious.

Similarly we may prove the following

Corollary 1.4. Let $N$ be a slant surface in $\mathbb{C}^2$ with constant mean curvature. Then $N$ lies in a hyperplane of $\mathbb{C}^2$ if and only if $N$ is either an open portion of a 2-plane or an open portion of a circular cylinder.

Remark 1.1. In views of Theorem 1.1, Corollary 1.3 and Corollary 1.4, it seems to be interesting to propose the following open problem:

Problem 1.1. Classify all slant surfaces in $\mathbb{C}^2$ with nonzero constant mean curvature.
§2. SPHERICAL SLANT SURFACES.

In the remaining part of this chapter we want present some classification theorems obtained in [CT2].

In this section we want to classify slant surfaces of $\mathbb{C}^2$ which lie in a hypersphere of $\mathbb{C}^2$. In order to do so, we need to review the geometry of the unit hypersphere $S^3 = S^3(1)$ in $\mathbb{C}^2$ centered at the origin.

It is known that $S^3$ is the Lie group consisting of all unit quaternions 

$$\{ u = a + ib + jc + kd : ||u|| = 1 \}$$

which can also be regarded as a subgroup of the orthogonal group $O(4)$ in a natural way. Let 1 denote the identity element of the Lie group $S^3$ given by

$$(2.1) \quad 1 = (1, 0, 0, 0) \in S^3 \subset E^4.$$ 

We put

$$(2.2) \quad X_1 = (0, 1, 0, 0), \quad X_2 = (0, 0, 1, 0), \quad X_3 = (0, 0, 0, 1) \in T_1 S^3.$$ 

We denote by $\tilde{X}_i, i = 1, 2, 3,$ the left-invariant vector fields obtained from the extensions of $X_i, i = 1, 2, 3,$ on $S^3$, respectively. Let $\phi : S^3 \to S^3$ be the orientation-reversing isometry defined by

$$(2.3) \quad \phi(a, b, c, d) = (a, b, d, c).$$ 

We recall that the left-translation $L_p$ and the right-translation $R_p$ on $S^3$ are isometries which are analogous to the parallel translations on $E^3$ and they are given by

$$(2.4) \quad \iota (L_p q) = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

and
\[ t(R_p q) = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \]

for \( p = (a, b, c, d), q = (x, y, z, w) \in S^3 \subset E^4 \), where \( tA \) denotes the transpose of \( A \).

Let \( \eta \) denote the unit outer normal of \( S^3 \) in \( E^4 \) and \( J_1 \) and \( J_{-1} \) the complex structures on \( E^4 \) defined respectively by

\[ J_1(a, b, c, d) = (-b, a, -d, c), \]
\[ J_{-1}(a, b, c, d) = (-b, a, d, -c). \]

**Lemma 2.1.** For any \( q \in S^3 \), we have

\[ (J_1 \eta)(q) = R_q X_1, \]
\[ (J_{-1} \eta)(q) = L_q X_1 = \tilde{X}_1(q). \]

Hence \( J_1 \eta \) and \( J_{-1} \eta \) are right-invariant and left-invariant vector fields on \( S^3 \), respectively.

**Proof.** Let \( q = (a, b, c, d) \in S^3 \). Then

\[ \eta(q) = (a, b, c, d) \in T_q S^3, \]
\[ (J_1 \eta)(q) = (-b, a, -d, c), \quad (J_{-1} \eta)(q) = (-b, a, d, -c). \]

Let \( \gamma \subset S^3 \) be a curve on \( S^3 \) parametrized by arclength \( s \) given by

\[ \gamma(s) = (\cos s, \sin s, 0, 0). \]

Then we have
\[ R_q X_1 = \frac{d}{ds}(R_q (\gamma(s)))_{s=0} = \]
\[
\frac{d}{ds}((\cos s + i \sin s)(a + ib + jc + kd))_{|s=0} = (-b, a, -d, c) = (J_1 \eta)(q).
\]
Similarly we may prove \(L_q X_1 = (J^- \eta)(q)\).

As in Section 2 of Chapter III, we denote by \(\mathcal{J}\) the set of all complex structures on \(E^4\) compatible with the inner product \(\langle , \rangle\). By using the natural orientation of \(E^4\) we divide \(\mathcal{J}\) into two disjoint subsets:

\[
\mathcal{J}^+ = \{J \in \mathcal{J} | J-\text{bases are positive}\},
\]

\[
\mathcal{J}^- = \{J \in \mathcal{J} | J-\text{bases are negative}\}.
\]
Thus we have \(\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^-\) (cf. Section 2 of Chapter III).

**Lemma 2.2.** Let \(W \in G(3,4)\) and \(V \in G(2,4)\) such that \(V \subset W\). Then \(V\) is \(\alpha\)-slant with respect to a complex structure \(J \in \mathcal{J}^+\) (respectively, \(J \in \mathcal{J}^-\)) if and only if

\[
<\xi_V, J\eta_W> = -\cos \alpha \quad \text{(respectively,} \quad <\xi_V, J\eta_W> = \cos \alpha),
\]
where \(\xi_V\) and \(\eta_W\) are positive unit normal vectors of \(V\) in \(W\) and of \(W\) in \(E^4\), respectively.

**Proof.** We choose an orthonormal \(J\)-basis \(\{e_1, \ldots, e_4\}\) of \(E^4\) such that

\[
e_1, e_2 \in W \cap JW, \quad e_4 = Je_3 = \eta_W.
\]
We also choose a positive orthonormal basis \(\{X_1, X_2\}\) of \(V\). Let \(\zeta_J\) be the 2-vector defined as before as the metrical dual of \(-\Omega_J \in \wedge^2 E^4^*\), that is,

\[
<\zeta_J, X \wedge Y> = -\Omega_J(X,Y)
\]
for any \(X, Y \in E^4\). Then, by formula (2.3) of Section 2 of Chapter III, we see that the slant angle \(\alpha_J(V)\) of \(V\) with respect to \(J\) satisfies

\[
\cos \alpha_J(V) = <\zeta_J, X_1 \wedge X_2> = <e_1 \wedge e_2 + e_3 \wedge e_4, X_1 \wedge X_2> = <e_1 \wedge e_2, X_1 \wedge X_2>
\]
\[
= <\pm e_3, \xi_V> = \mp <J\eta_W, \xi_V>
\]
Geometry of Slant Submanifolds

for \( J \in J^\pm \). This proves the lemma.

Let \( f : N \to S^3 \subset E^4 \) be a spherical immersion of an oriented surface \( N \) into \( S^3 \) and \( \xi \) the positive unit normal of \( f(N) \) in \( S^3 \).

It is easy to see that every spherical surface in \( \mathbb{C}^2 \) is non-minimal. Hence, every spherical surface in \( \mathbb{C}^2 \) cannot be a complex surface of \( \mathbb{C}^2 \).

**Lemma 2.3.** Let \( f : N \to S^3 \subset E^4 \) be an immersion of an oriented surface \( N \) into the hypersphere \( S^3 \) of \( E^4 \). Then the following three statements hold.

(i) The immersion \( f \) is \( \alpha \)-slant with respect to the complex structure \( J_1 \) if and only if

\[
< \xi(p), J_1 \eta(f(p)) > = -\cos \alpha \quad \text{for any } p \in N.
\]

(ii) The immersion \( f \) is \( \alpha \)-slant with respect to the complex structure \( J_1^- \) if and only if

\[
< \xi(p), \tilde{X}_1(f(p)) >= \cos \alpha \quad \text{for any } p \in M.
\]

(iii) The immersion \( f \) is \( \alpha \)-slant with respect to the complex structure \( J_1 \) if and only if the composition \( \phi \circ f \) is \( \alpha \)-slant with respect to the complex structure \( J_1^- \).

**Proof.** Statement (i) follows from Lemma 2.2. Statement (ii) follows from Lemma 2.1 and Lemma 2.2. Finally, the last statement follows from statements (i) and (ii) and from the fact that \( \phi \) is an isometric involution which reverses the orientation of \( E^4 \).

We define two maps \( g_+ \) and \( g_- \) from \( N \) into the unit sphere \( S^2 \) in \( T_1S^3 \) by

\[
g_+(p) = (L_{\phi(f(p))})^{-1}(\phi_* \xi(p)), \quad g_-(p) = (L_{f(p)_*})^{-1}(\xi(p))
\]

for \( p \in N \). In fact, \( g_+ \) and \( g_- \) are the analogues of the classical Gauss map of a surface in \( E^3 \) in which the parallel translations in \( E^3 \) are replaced by the left-translations \( L_q \) on \( S^3 \).

We also define a circle \( S^1_\alpha \) for \( \alpha \in [0, \pi] \) on the unit sphere \( S^2 \) in \( T_1S^3 \) by

\[
S^1_\alpha = \{ X \in T_1S^3 \mid \| X \| = 1, < X, X_1 > = -\cos \alpha \}.
\]
Now we prove the following result of [CT2] which characterizes spherical slant surfaces in $\mathbb{C}^2$.

**Proposition 2.4.** Let $f : N \to S^3 \subset E^4$ be an immersion of an oriented surface $N$. Then we have

(i) $f$ is $\alpha$-slant with respect to the complex structure $J_1$ if and only if

$$g_+(N) \subset S^1_\alpha \subset T_1S^3.$$  

(ii) $f$ is $\alpha$-slant with respect to the complex structure $J_1^-$ if and only if

$$g_-(N) \subset S^1_{-\alpha} \subset T_1S^3.$$  

**Proof.** Proposition 2.4 follows from Lemma 2.1, Lemma 2.3 and the definitions of $g_+$ and $g_-$.  

Proposition 2.4 can be regarded as the spherical version of Proposition 3.1 in Section 3 of Chapter III.

Concerning the images of $N$ under the spherical Gauss maps $g_+$ and $g_-$, we give the following two simple examples (cf. [CT2]).

**Example 2.1.** If $N = S^1 \times S^1$ is the flat torus in $E^4$ defined by

$$f(u, v) = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v),$$

then the images of $N$ under the spherical Gauss maps $g_+$ and $g_-$ are the great circles perpendicular to $X_1 = (0, 1, 0, 0)$.

**Example 2.2.** If $N = S^2$ is the totally geodesic 2-sphere of $S^3$ parametrized by

$$f(u, v) = (\cos u \cos v, \sin u \cos v, \sin v, 0),$$

then

$$g_+(u, v) = (0, -\sin v, -\cos u \cos v, \sin u \cos v),$$

$$g_-(u, v) = (0, \sin v, \sin u \cos v, -\cos u \cos v).$$

Hence, both $g_+$ and $g_-$ are isometries.
In order to describe slant surfaces in $S^3$ geometrically, we give the following

**Definition 2.1.** Let $c(s)$ be a curve in $S^3$ parametrized by arclength $s$ and let

$$c'(s) = \sum_{i=1}^{3} f_i(s) \tilde{X}_i(c(s)).$$

We call the curve $c(s)$ a *helix in $S^3 with axis vector field $\tilde{X}_1$* if

$$f_1(s) = b, \ f_2(s) = a \cos(ks + s_0), \ f_3(s) = a \sin(ks + s_0)$$

for some constants $a, b, k$ and $s_0$ satisfying

$$a^2 + b^2 = 1.$$

We call the curve $c(s)$ a *generalized helix in $S^3 with axis vector field $\tilde{X}_1$* if

$$\langle c'(s), \tilde{X}_1(c(s)) \rangle = \text{constant}.$$

Helices in $S^3$ defined above are the analogues of Eucliden helices in the Euclidean 3-space $E^3$.

**Definition 2.2.** We call an immersion $f : D \to S^3$ of a domain $D$ around the origin $(0, 0)$ of $R^2$ into $S^3$ a *helical cylinder in $S^3* if

$$f(s, t) = \gamma(t) \cdot c(s),$$

for some helix $c(s)$ in $S^3$ with axis $\tilde{X}_1$ satisfying $k = -2/b$ and $ab < 0$, and for some curve $\gamma(t)$ in $S^3$ which is either a geodesic or a curve of constant torsion 1 parametrized by arclength such that

(i) $c(0) = \gamma(0)$, and

(ii) the osculating planes of $c(s)$ and of $\gamma(t)$ coincide at $t = s = 0$.

We note that the binormal of $c(s)$ is normal to $f(D)$ in $S^3$. Here we orient the curve $c$ in such a way that the binormal of $c(s)$ is the positive unit normal of $f(D)$.

The main purpose of this section is to prove the following classification theorem for spherical slant surfaces [CT2].
Theorem 2.5. Let \( f : N \to S^3 \subset C^2 = (E^4, J_1) \) be a spherical immersion of an oriented surface \( N \) into the complex 2-plane \( C^2 = (E^4, J_1) \). Then \( f \) is a proper slant immersion if and only if \( f(N) \) is locally of the form \( \{ \phi(\gamma(t) \cdot c(s)) \} \) where \( \phi \) is the isometry on \( S^3 \) defined by (2.3) and \( \{ \gamma(t) \cdot c(s) \} \) is a helical cylinder in \( S^3 \) (cf. Definition 2.2).

In order to prove this theorem so we need several lemmas.

First, we note that curves in \( S^3 \) can be described in terms of the orthonormal left-invariant vector fields \( \{ \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \} \). Let \( I \) be an open interval containing 0 and \( c : I \to S^3 \) a curve parametrized by arclength \( s \). Let \( t(s), n(s), b(s), \kappa(s), \) and \( \tau(s) \) be the unit tangent vector, the unit principal normal vector, the unit binormal vector, the curvature, and the torsion of \( c \) in \( S^3 \), respectively. We put

\[
(2.26) \quad t(s) = \sum_{i=1}^{3} f_i(s) \tilde{X}_i(c(s)).
\]

Then

\[
(2.27) \quad (f_1(s))^2 + (f_2(s))^2 + (f_3(s))^2 = 1.
\]

Conversely, we have the following

Lemma 2.6. Let \( f_i(s), i = 1, 2, 3 \), be differentiable functions on \( I \) satisfying (2.27). Then, for any point \( p_0 \in S^3 \), there exists a curve \( c(s) \) in \( S^3 \) defined on an open subinterval \( I' \) of \( I \) containing 0 and satisfying (2.26) and the initial condition \( c(0) = p_0 \).

Proof. Considering the curve \( L_{p_0}^{-1} \circ c \) instead of \( c \), if necessary, we can assume without loss of generality that \( p_0 = 1 \). The solution of the following system of the first order linear differential equations:

\[
(2.28) \quad \begin{pmatrix} x' \\ y' \\ z' \\ w' \\ \end{pmatrix} = \begin{pmatrix} x & -y & -z & -w \\ y & x & -w & z \\ z & w & x & -y \\ w & -z & y & x \end{pmatrix} \begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}
\]

with the initial condition \( (x(0), y(0), z(0), w(0)) = (1, 0, 0, 0) \) satisfies the condition:

\[
xx' + yy' + zz' + ww' = 0
\]
and the curve $c(s) = (x(s), y(s), z(s), w(s))$ is in fact the desired one.

Lemma 2.6 guarantees the existence of helices in $S^3$.

**Lemma 2.7.** The following two statements are equivalent:

(i) The curve $c(s)$ is a helix in $S^3$ with axis vector $\tilde{X}_1$ satisfying

\begin{equation}
    f_1(s) = b,
\end{equation}

\begin{equation}
    f_2(s) = a \cos\left(-\frac{2}{b}s + s_0\right),
\end{equation}

\begin{equation}
    f_3(s) = a \sin\left(-\frac{2}{b}s + s_0\right),
\end{equation}

\begin{equation}
    a^2 + b^2 = 1, \quad ab < 0.
\end{equation}

(ii) The curve $c(s)$ is a curve in $S^3$ satisfying

\begin{equation}
    \tau(s) \equiv -1,
\end{equation}

\begin{equation}
    < b(s), \tilde{X}_1(c(s)) > \equiv a.
\end{equation}

\begin{equation}
    a \neq \pm 1, 0.
\end{equation}

**Proof.** (ii) $\Rightarrow$ (i): Suppose $c$ is a curve in $S^3$ parametrized by arclength and the unit tangent vector $t$ of $c$ is given by (2.26). Let

$$g_1 = f_2f_3' - f_3f_2', \quad g_2 = f_3f_1' - f_1f_3',$$

$$g_3 = f_1f_2' - f_2f_1'.$$

By Frenet-Serret formulas and (2.33) we have

\begin{equation}
    \left(\frac{g_i}{\kappa}\right)' = \frac{2f_i'}{\kappa}, \quad i = 1, 2, 3.
\end{equation}
By using (2.34) and the identity $b = t \times n$, we may obtain

\[(2.36) \quad a = \frac{g_1}{\kappa}.\]

Hence, we find $2f_1'/\kappa = a' = 0$. Let $b$ denote the constant $f_1$. Then, from (2.36) and $b = t \times n$, we may find

\[(2.37) \quad b = a\tilde{X}_1 - \left(\frac{bf_3'}{\kappa}\right)\tilde{X}_2 + \left(\frac{bf_2'}{\kappa}\right)\tilde{X}_3,\]

\[(2.38) \quad \kappa^2 = (f_2')^2 + (f_3')^2,\]

\[(2.39) \quad n = \left(\frac{f_1'}{\kappa}\right)\tilde{X}_2 + \left(\frac{f_3'}{\kappa}\right)\tilde{X}_3.\]

Since $\|b\| = 1, \|n\| = 1$, (2.37) and (2.39) imply $a^2 + b^2 = 1$. Thus, from $\|t\| = 1$ and (2.26), we get $f_2^2 + f_3^2 = a^2$. So we may put

\[(2.40) \quad f_2 = a\cos \theta, \quad f_3 = a\sin \theta, \quad \theta = \theta(s).\]

Thus, by applying the definition of $g_1, g_2, g_3$, we have

\[(2.41) \quad g_1 = a\kappa, \quad g_2 = -bf_3', \quad g_3 = bf_2'.\]

By using (2.35), (2.38), (2.40), and $\tau \neq 0$, we get

\[(2.42) \quad \kappa = |a\theta'| \neq 0.\]

From (2.26), (2.40), (2.41) and (2.42) we find $\sin \theta(b\theta' + 2) = 0$. Since $\sin \theta(s)$ has only isolated zeros by (2.42), $b\theta' + 2 = 0$. Thus, $b \neq 0$. So $\theta = -\frac{2}{b}s + s_0, s_0 = \text{const}$. Hence, by (2.37) and $\kappa > 0$, we get $ab < 0$.

$(i) \Rightarrow (ii)$ follows from straight-forward computation.

**Lemma 2.8.** A helical cylinder $f(N) = \{\gamma(t) \cdot c(s)\}$ in $S^3$ is a proper slant surface with respect to the complex structure $J_0^-$ with the slant angle equal to $\cos^{-1} a$, where $a$ is the constant given by (2.22) of Definition 2.1.
Proof. Let $\xi$ be the positive unit normal of $f(N)$ in $S^3$ and $b$ the binormal vector of $c$ in $S^3$. Then we have

$$\xi(t) \cdot c(s) = L_\gamma(t) \cdot b(s).$$

Lemma 2.8 now follows from Lemma 2.3 and Lemma 2.7.

Lemma 2.9. For any point $p_0 \in S^3$ and any oriented 2-plane $P_0 \subset T_{p_0}S^3 \subset E^4$ which is proper slant with respect to the complex structure $J_1^-$, there exist helical cylinders in $S^3$ passing through $p_0$ and whose tangent planes at $p_0$ are $P_0$.

Proof. Let $\xi$ be the positive unit normal of $P_0$ in $T_{p_0}S^3$ and $\alpha$ the slant angle of $P_0$ with respect to $J_1^-$. Put

$$a = \cos \alpha \ (\neq 0, \pm 1), \ b = \pm (1 - a^2)^{1/2},$$

where $\pm$ was chosen so that $ab < 0$. Pick $s_0 \in [0, 2\pi)$ such that

$$\cos s_0 = -\frac{1}{b} < \xi, \tilde{X}_2(p_0) >, \ \sin s_0 = -\frac{1}{b} < \xi, \tilde{X}_3(p_0) >.$$

We define $f_i$ by (2.29)-(2.52). Then they satisfy (2.27) and we can choose a curve $c(s)$ satisfying the conditions mentioned in Lemma 2.6.

Let $\gamma(t)$ be either a geodesic in $S^3$ satisfying

$$\gamma(0) = p_0, \ \gamma'(0) \in P_0, \ \gamma'(0) \neq c'(0),$$

or a curve in $S^3$ satisfying this condition and also the condition $\tau \equiv 1$ (see Theorem 3 of [Sp1, p.35] for the existence of such curves). Then we can verify that $\{\gamma(t) \cdot c(s)\}$ is a desired surface.

Proof of Theorem 2.5. First, we note that the isometry $\phi$ of $S^3$ has the following properties:

$$\phi(p \cdot q) = \phi(q) \cdot \phi(p), \ \text{for} \ \forall p, q \in S^3,$$

$$X \in \mathcal{X}(S^3) \text{ is left–(respectively, right–) invariant}$$

$$\iff \phi_* X \text{ is right–(respectively, left–) invariant},$$
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(2.47) \[ \tau_{\phi \circ c} = -\tau_c \] for a curve \( c \) in \( S^3 \).

(2.48) \( b \) is the binormal of a curve \( c \) in \( S^3 \) \( \iff \) \( -\phi_* b \) is the binormal of \( \phi \circ c \) in \( S^3 \),

where \( \tau_c \) denotes the torsion of the curve \( c \) in \( S^3 \).

Let \( \alpha \) be the slant angle of \( f(N) \) with respect to \( J_1 \). Since \( f(N) \) is spherical, the normal curvature \( G_D \) of the slant immersion \( f \) vanishes. Thus, by Theorem 3.2 in Section 3 of Chapter II, \( N \) is a flat surface in \( S^3 \). Therefore, \( f(N) \) is locally a flat translation surface \( f(N) = \{ c(s) \cdot \gamma(t) \} \) (cf. [Sp1, pp.149-157]), where \( c \) and \( \gamma \) are curves in \( S^3 \) parametrized by arclength satisfying one of the following conditions:

(i) \[ \tau_c \equiv +1 \quad \text{and} \quad \tau_\gamma \equiv -1, \]

(ii) \[ \tau_c \equiv +1 \quad \text{and} \quad \gamma \text{ is a geodesic}, \]

(ii') \[ c \text{ is a geodesic and} \quad \tau_\gamma \equiv -1, \]

(iii) \( c \) and \( \gamma \) are distinct geodesics.

**Cases (i) and (ii):** Let \( b \) be the binormal of \( c \). With a suitable choice of the orientation, \( b \) is the positive unit normal of \( f(N) \) in \( S^3 \). By Lemma 2.3, Lemma 2.7, (2.47), and (2.48), \( \phi \circ c \) is a helix in \( S^3 \) with \( a \) and \( b \) in (2.29)-(2.32) determined by

(2.49) \[ a = \cos \alpha, \quad b = \pm \sin \alpha, \quad ab < 0, \]

where either \( \tau_{\phi \gamma} \equiv +1 \) or \( \phi \circ \gamma \) is a geodesic. So, by (2.45), \( (\phi \circ f)(N) \) is a helical cylinder in \( S^3 \).

The converse is given by Lemma 2.8. Moreover, Lemma 2.9 guarantees the existence of such surfaces.

Next, we want to show that both cases (ii') and (iii) cannot occur. Without loss of generality we can assume

(2.50) \[ c(0) = \gamma(0) = 1 \in S^3, \]
because Lemmas 3.1 and 3.3 imply that the slantness of a surface in $S^3$ with respect to $J_1$ is right-invariant, that is, if $f$ is $\alpha$-slant with respect to $J_1$, so is $R_q \circ f$ for any $q \in S^3$, and hence we can replace $f, c$ and $\gamma$ by $R_{c(0)} \cdot \gamma(0) \circ f, R_{c(0)} \circ c$ and $L_{c(0)} \circ R_{\gamma(0)}^{-1} \circ c \circ \gamma$, respectively, if necessary.

Case (ii'): Let $b$ be the binormal of $\gamma$. We can choose the orientation so that

$$(2.51) \quad \xi(c(s) \cdot \gamma(t)) = L_{c(s)} \cdot b(t), \quad \text{for any } s, t.$$ 

So, by Lemmas 2.1 and 2.3 and also (2.50), we have

$$(2.52) \quad < L_{c(s)} \cdot b(0), R_{c(s)} \cdot X_1 > = -\cos \alpha \quad \text{for any } s.$$ 

Put

$$(2.53) \quad c'(0) = (0, a_1, a_2, a_3), \quad b(0) = (0, b_1, b_2, b_3) \in T_c S^3 \subset E^4.$$ 

Then

$$(2.54) \quad c(s) = (\cos s, a_1 \sin s, a_2 \sin s, a_3 \sin s).$$

Putting $s = 0$ in (2.52), we find

$$(2.55) \quad b_1 = -\cos \alpha \neq 0, \pm 1,$$

since $f(N)$ is properly slant. On the other hand, by (2.4), (3.5), (2.45), and (2.46), we have

$$(2.56) \quad < L_{c(s)} \cdot b(0), R_{c(s)} \cdot X_1 > = b_1 \cos 2s + (-a_3 b_2 + a_2 b_3) \sin 2s.$$ 

So, from (2.52) and (2.56), we get $b_1 = 0$ which contradicts to (2.55). Consequently, case (ii') cannot occur.

Case (iii'): Let $\{c(s) \cdot \gamma(t)\}$ be defined by using two distinct geodesics $c$ and $\gamma$ and assume

$$(2.57) \quad f : I_1 \times I_2 \rightarrow S^3 ; \quad (s, t) \mapsto c(s) \cdot \gamma(t)$$

is properly slant. Since the geodesics $c$ and $\gamma$ can be extended for all $s, t \in \mathbb{R}$, we can extend the immersion $x$ to a global mapping:

$$(2.58) \quad y : \mathbb{R}^2 \rightarrow S^3 ; \quad (s, t) \mapsto c(s) \cdot \gamma(t).$$
Now, we claim that $y$ is in fact an immersion and it is properly slant. To see this, we recall (2.50) and put
\begin{equation}
(2.59) \quad c'(0) = (0, a_1, a_2, a_3), \quad \gamma'(0) = (0, b_1, b_2, b_3) \in T_1S^3.
\end{equation}

Let $\tilde{X}, \tilde{Y}$ be the vector fields along $y(\mathbb{R}^2)$ defined by
\begin{equation}
(2.60) \quad \tilde{X}(s,t) = R_{\gamma(t)} c'(s), \quad \tilde{Y}(s,t) = L_{c(s)} \gamma'(t).
\end{equation}

Then it follows from (2.4) and (2.5) that
\begin{equation}
(2.61) \quad < \tilde{X}(s,t), \tilde{Y}(s,t) > \text{ is a polynomial of } \sin s, \cos s, \sin t \text{ and } \cos t.
\end{equation}

On the other hand, since the $s$-curves and the $t$-curves on $f(I_1 \times I_2)$ intersect at a constant angle (cf. [Sp1, p.157]), we have
\begin{equation}
(2.62) \quad < \tilde{X}(s,t), \tilde{Y}(s,t) > = const. \neq \pm 1, \text{ for any } (s,t) \in I_1 \times I_2.
\end{equation}

From (2.61), we see that (2.62) holds for all $(s,t) \in \mathbb{R}^2$ and hence $y$ is an immersion. Since
\[ \xi(c(s) \cdot \gamma(t)) = ||\tilde{X}(s,t) \times \tilde{Y}(s,t)||^{-1}(\tilde{X}(s,t) \times \tilde{Y}(s,t)), \]
where $\times$ denotes the usually vector product in $T_{c(s) \cdot \gamma(t)}S^3$ determined by the metric and the orientation, so, by (2.4), (2.5) and (2.6), we know that $< \xi(c(s) \cdot \gamma(t)), J_0 \eta(c(s) \cdot \gamma(t)) >$ is a polynomial of $\sin s, \cos s, \sin t$ and $\cos t$. By Lemma 2.3 we conclude that this polynomial is a constant on $I_1 \times I_2$ and hence $y$ is a proper slant immersion defined globally on $\mathbb{R}^2$.

Now, by the double periodicity, $y$ induces a proper slant immersion:
\[ \tilde{y} : T^2 = (\mathbb{R}/2\pi \mathbb{Z}) \times (\mathbb{R}/2\pi \mathbb{Z}) \to \mathbb{C}^2 = (E^4, J_0) \]

of a torus into $\mathbb{C}^2$, which contradicts to Theorem 1.5 of Chapter V. Consequently, case (iii) cannot occur.

This completes the proof of the theorem.
§3. SLANT SURFACES WITH $rk(\nu) < 2$.

For an immersion $f : N \to \mathbb{C}^m$, the Gauss map $\nu$ of the immersion $f$ is given by

$$\nu : N \to G(n, 2m) \equiv D_1(n, 2m) \subset S^{K-1} \subset \wedge^n E^{2m},$$

where $n = \dim N$, $K = (2m)^n$, $D_1(n, m)$ is the set of all unit decomposable $n$-vectors in $\wedge^n E^{2m}$, identified with the real Grassmannian $G(n, 2m)$ in a natural way, and $S^{K-1}$ is the unit hypersphere of $\wedge^n (E^{2m})$ centered at the origin, and $\{e_1, \ldots, e_{2m}\}$ is a local adapted orthonormal tangent frame along $f(N)$.

The main purpose of this section is to prove the following classification theorem [CT2].

**Theorem 3.1.** If $f : N \to \mathbb{C}^2 = (E^4, J_1)$ is a slant immersion such that the rank of its Gauss map is less than 2, then the image $f(N)$ of $f$ is a union of some flat ruled surfaces in $E^4$. Therefore, locally, $f(N)$ is a cylinder, a cone or a tangential developable surface in $\mathbb{C}^2$. Furthermore,

(i) A cylinder in $\mathbb{C}^2$ is a slant surface if and only if it is of the form $\{c(s) + te\}$, where $e$ is a fixed unit vector and $c(s)$ is a (Euclidean) generalized helix with axis $J_1 e$ contained in a hyperplane of $E^4$ and with $e$ as its hyperplane normal,

(ii) A cone in $\mathbb{C}^2$ is a slant surface if and only if, up to translations, it is of the form $\{tc(s)\}$, where $(\phi \circ c)(s)$ is a generalized helix in $S^3$ with axis $\tilde{X}_1$ (cf. Definition 2.1), and

(iii) A tangential developable surface $\{c(s) + (t-s)c'(s)\}$ in $\mathbb{C}^2$ is a slant surface if and only if, up to rigid motions, $(\phi \circ c')(s)$ is a generalized helix in $S^3$ with axis $\tilde{X}_1$.

As before, let $*$ denote the Hodge star operator $*: \wedge^2 E^4 \to \wedge^2 E^4$ induced from the natural orientation and the canonical inner product of $E^4$. Denote by $\wedge_+^2 E^4$ and $\wedge_-^2 E^4$ the eigenspaces of $*$ with eigenvalues $+1$ and
−1, respectively. And denote by \( S_2^2 \) and \( S_2^2 \) the 2-spheres in \( \wedge^2 E^4 \) and \( \wedge^2 E^4 \) centered at the origin with radius \( 1/\sqrt{2} \), respectively. Then we have \( D_1(2, 4) = S_2^2 \times S_2^2 \). Let

\[
\pi_+ : D_1(2, 4) \to S_2^2, \quad \pi_- : D_1(2, 4) \to S_2^2
\]

denote the natural projections. We define as before the two maps \( \nu_+ \) and \( \nu_- \) given respectively by

\[
\nu_+ = \pi_+ \circ \nu \quad \text{and} \quad \nu_- = \pi_- \circ \nu.
\]

Suppose that the slant immersion \( f : N \to C^2 = (E^4, J_1) \) satisfying the condition \( \text{rank}(\nu) < 2 \). Then we also have \( \text{rank}(\nu_\pm) < 2 \). Hence, by Lemma 4.2 of Chapter III, \( f(N) \) is a flat surface in \( E^4 \).

Furthermore, we have \([\text{CT}2]\)

**Lemma 3.2.** If \( f \) is a slant immersion with \( \text{rank}(\nu) < 2 \), then \( f(N) \) is a union of flat ruled surfaces in \( E^4 \).

**Proof.** Since the normal curvature \( R^D = 0 \), identically, we can choose local orthonormal frame \( \{e_1, e_2\} \) such that with respect to it the second fundamental form \( \{h_{ij}\} \) is simultaneously diagonalized, that is, we have

\[
(h^3_{ij}) = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} d & 0 \\ 0 & e \end{pmatrix}.
\]

Put

\[
N_1 = \{ p \in N \mid H(p) \neq 0 \}, \quad N_0 = \text{Interior of } (N - N_1).
\]

Then

\[
N = N_0 \cup \partial N_1 \cup N_1,
\]

where \( H \) is the mean curvature vector of \( N \) in \( C^2 \).

Since \( f(N) \) is flat and \( H = 0 \) on \( N_0 \), \( f(N_0) \) is a union of portions of 2-planes in \( E^4 \) with the same slant angle.

On \( N_1 \), we put \( e_3 = H/\|H\| \). Since \( \text{rank}(\nu) < 2 \), we have \( bc = 0 \) and \( d = e = 0 \). We may choose \( \{e_1, e_2\} \) such that \( b \neq 0, c = d = e = 0 \) on \( N_1 \). From these we may prove that the integral curves of \( f_\ast e_2 \) are open.
portions of straight lines and therefore \( f(N_1) \) is a union of flat ruled surfaces. Consequently, \( f(N) \) is a union of flat ruled surfaces possibly guled along \( \partial N_1 \). This proves the lemma.

For the local classification of flat ruled surfaces in \( E^4 \), see [Sp1].

Now we give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** The first part of the theorem is giving in Lemma 3.2. Now, we prove the remaining part of the theorem.

**Case (i):** If \( f(N) \) is a slant cylinder, then we may assume that \( f(N) \) is of the form:
\[
\text{(3.6)} \\
f(N) = \{c(s) + te\},
\]
where \( e \) is a fixed unit vector in \( E^4 \) and \( c(s) \) is a curve parametrized by arclength which lies in the orthogonal complement (up to sign), say \( W \in G(3, 4) \), of \( e \). Since \( \{c'(s), e\} \) is a positive orthonormal basis of \( T_{c(s)+te}N \), we obtain \( \cos \alpha = \langle c'(s), -J_1 e \rangle \) by (1.4). Hence, \( c(s) \) is a generalized helix lies in the hyperplane \( W(\equiv E^3) \) whose tangents make a constant angle \( \alpha \) with \( -J_1 e \in W \).

**Case (ii):** If \( f(N) \) is a slant cone in \( E^4 \), then, without loss of generality, we may assume that the vertex of the cone is the origin of \( E^4 \). So we can write
\[
\text{(3.7)} \\
f(N) = \{tc(s)\},
\]
where \( c(s) \) is a curve in \( S^3 \) parametrized by arclength. Since \( \{c'(s), \eta(c(s))\} \) is a positive orthonormal basis of \( T_{tc(s)}N \), \( \cos \alpha = \langle c'(s), -J_1 \eta(c(s)) \rangle \) for all \( s \). Thus, by Lemmas 2.1 and 2.3, we conclude that \( (\phi \circ c)(s) \) is a generalized helix in \( S^3 \) with axis \( X_1 \) (cf. Definition 2.1).

**Case (iii):** If \( f(N) \) is a slant tangential developable surface in \( E^4 \), the surface has the form:
\[
\text{(3.8)} \\
f(N) = \{c(s) + (t - s)c'(s)\},
\]
where \( c(s) \) is a curve parametrized by arclength. We put
\[
\text{(3.9)} \\
v_1(s) = c'(s), \quad \kappa_1(s) = \|v_1'(s)\|, \quad v_2(s) = \left( \frac{1}{\kappa_1(s)} \right)v_1'(s).
\]
Note that \( \kappa_1 \neq 0 \), since \( c(s) \) generates a tangential developable surface. \( \{v_2(s), v_1(s)\} \) forms a positive orthonormal basis of \( T_{c(s)+(t-s)c'(s)}N \), and so we have
\[
\cos \alpha = \langle \frac{v_1'(s)}{\|v_1'(s)\|}, -J_1v_1(s) \rangle
\]
for all \( s \). If we consider \( v_1(s) \) as a curve in \( S^3 \), then (3.9) means that
\[
\cos \alpha = \langle t(s), -J_1\eta(v_1(s)) \rangle
\]
where \( t(s) \) is the unit tangent of \( v_1(s) \). So, as in Case (ii), \( (\phi \circ v_1)(s) \) is a generalized helix in \( S^3 \) with axis \( \tilde{X}_1 \).

It is easy to verify that in each of the cases (i)-(iii), the converse is also true.

This completes the proof of the theorem.
§4. SLANT SURFACES WITH CODIMENSION ONE

In this section we want to classify slant surfaces which are contained in a hyperplane $W$ of $E^4$.

**Lemma 4.1.** Let $f : N \to \mathbb{C}^2 = (E^4, J_1)$ be a slant immersion of an oriented surface $N$ into $\mathbb{C}^2$. If $N$ is contained in some hyperplane $W \in G(3,4)$, then

1. $\text{rank}(\nu) < 2$
2. The immersion $f$ is doubly slant with the same slant angle.

**Proof.** We choose a positive orthonormal $J_1$-basis $\{e_1, e_2, e_3, e_4\}$ such that $e_1, e_2 \in W \cap J_1W$, $e_4 = J_1 e_3 = \eta_W$, where $\eta_W$ is the positive unit normal vector of the hyperplane $W$ in $E^4$. We put

$$G_W = G(2,4) \cap \wedge^2 W \subset \wedge^2 E^4.$$  

Then $G_W$ is the unit 2-sphere in the 3-dimensional Euclidean space $\wedge^2 W$. For $\alpha \in [0, \pi]$ we put

$$G_{W,\alpha} = G_{J_1, \alpha} \cap G_W,$$

where $G_{J_1, \alpha}$ is the set of all 2-planes in $E^4$ with slant angle $\alpha$ with respect to $J_1$. We recall that a 2-plane $V$ was identified with a unit decomposable 2-vector $e_1 \wedge e_2$ in $\wedge^2 E^4$ with $\{e_1, e_2\}$ as a positively oriented orthonormal basis of $V$. From the proof of Lemma 2.2, we see that $G_{W,\alpha}$ is the circle on $G_W = S^2 \subset \wedge^2 W$ defined by

$$G_{W,\alpha} = \{V \in G_W \mid <V, e_1 \wedge e_2> = \cos \alpha\}.$$  

For each $J \in \mathcal{J}$, we denote by $\zeta_J$ the 2-vector which is the metrical dual of $-\Omega_J$ as defined in Section 2. Let $\zeta : \mathcal{J} \to \wedge^2 E^4$ be the mapping defined by $\zeta(J) = \zeta_J$. Then $\zeta$ gives rise to two bijections (cf. Lemma 2.1 of this chapter):

$$\zeta^+ : \mathcal{J}^+ \to S^2_+ \quad \text{and} \quad \zeta^- : \mathcal{J}^- \to S^2_-.$$
For each oriented 2-plane $V \in G(2, 4)$ we define two complex structures $J^+_V \in J^+$ and $J^-_V \in J^-$ by

$$J^+_V = (\zeta^+)^{-1}(\pi_+(V)) \quad \text{and} \quad J^-_V = (\zeta^-)^{-1}(\pi_-(V)).$$

Let $\hat{J} = J_{e_1 \wedge e_2}$. Then we have

$$\pi_+(GW, \alpha) = S^+_J, \alpha \subset S^+_2, \quad \pi_-(GW, \alpha) = S^-_J, \alpha \subset S^-_2,$$

where $S^+_J, \alpha$ are the circles (possibly singletons) on $S^+_2$, respectively, consisting of all 2-vectors which make constant angle $\alpha$ with $\zeta_J$. If $f$ is $\alpha$-slant with respect to $\hat{J}$ and $f(N) \subset W$, then $\nu(M) \subset GW$. Therefore, $\text{rank}(\nu) < 2$ and, by (4.3), $f$ is $\alpha$-slant with respect to $\hat{J}$.

This proves the lemma.

We note here that if we identify $\wedge^2 W$ with the Euclidean 3-space $E^3 \equiv W$ (where $W$ is spanned by $\{e_1, e_2, e_3\}$) via the isometry $X \wedge Y \rightarrow X \times Y$, then $\nu : M \rightarrow GW \subset \wedge^2 W$ is nothing but the classical Gauss map $g : M \rightarrow S^2 \subset E^3$.

Since $e_1 \times e_2 = e_3 = -J_1 \eta W$, $f$ is $\alpha$-slant if and only if

$$g(M) \subset S^1_\alpha = \{Z \in S^2 \mid <Z, -J_1 \eta W> = \cos \alpha\} \subset S^2 \subset W.$$

Now we give the following classification theorem [CT2].

**Theorem 4.2.** Let $f : N \rightarrow C^2 = (E^4, J_1)$ be a proper slant immersion of an oriented surface $M$ into $C^2$. If $f(N)$ is contained in a hyperplane $W$ of $E^4$, then $f$ is a doubly slant immersion and $f(N)$ is a union of some flat ruled surfaces in $W$. Therefore, locally, $f(N)$ is a cylinder, a cone or a tangential developable surface in $W$. Furthermore,

(i) A cylinder in $W$ is a proper slant surface with respect to the complex structure $J_1$ on $E^4$ if and only if it is a portion of a 2-plane.

(ii) A cone in $W$ is a proper slant surface with respect to the complex structure $J_1$ on $E^4$ if and only if it is a circular cone.

(iii) A tangential developable surface in $W$ is a proper slant surface with respect to the complex structure $J_1$ on $E^4$ if and only if it is a tangential developable surface obtained from a generalized helix in $W$. 
Proof. Assume \( f : N \rightarrow \mathbb{C}^2 = (E^4, J_1) \) is a proper slant immersion of an oriented surface \( N \) such that \( f(N) \) is contained in some hyperplane \( W \in G(3, 4) \). The first part of Theorem 4.2 is given by Lemma 4.1. For the remaining part it suffices to check the three cases of Theorem 3.1.

Suppose \( f \) is properly slant with slant angle \( \alpha \). Denote by \( \xi \) the local unit normal of \( f(N) \) in \( W \). We put

\[
(4.5) \quad e_1 = t\xi/\|t\xi\|, \quad e_2 = (\sec \alpha)Pe_1, \quad e_3 = (\csc \alpha)Fe_1, \quad e_4 = (\csc \alpha)Fe_2,
\]

where \( PX \) and \( FX \) denote the tangential and the normal components of \( J_1X \), respectively, and \( t\xi \) is the tangential component of \( J_1\xi \). Then \( \{e_1, \cdots, e_4\} \) is an adapted orthonormal frame along \( f(N) \) and it satisfies

\[
(4.6) \quad e_3 = \text{unit normal of } f(N) \text{ in } W, \quad e_4 \in W^\perp,
\]

\[
(4.7) \quad te_3 = -(\sin \alpha)e_1, \quad te_4 = -(\sin \alpha)e_2,
\]

\[
fe_3 = -(\cos \alpha)e_4, \quad fe_4 = (\cos \alpha)e_3,
\]

where \( fe_3 \) is the normal component of \( J_1e_3 \). Since \( e_4 \) is a constant vector in \( E^4 \), Corollary 3.6 of Chapter II implies that the second fundamental form \( (h_{ij}^r) \) is of the following form:

\[
(4.8) \quad (h_{ij}^3) = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \quad (h_{ij}^4) = 0,
\]

which shows that the orthonormal frame \( \{e_1, \cdots, e_4\} \) coincides with that chosen in the proof of Lemma 3.2 (up to orientations). Since \( J_1e_4 \) is also a constant vector in \( E^4 \), from (4.7), we have

\[
(4.9) \quad -\sin \alpha \nabla_X e_2 - \cos \alpha A_{e_3}X = 0, \quad \text{for } X \in TM.
\]

Hence we get

\[
(4.10) \quad \omega_2^1(e_1) = -b \cot \alpha, \quad e_2b = b^2 \cot \alpha.
\]

Case (i): In this case, the curve \( c(s) \) of (3.6) lies in a 2-plane \( W' = \{e\}^\perp \cap W \subset W \) which is perpendicular to \( e \). So, \( f(N) \) is totally real with respect
to the complex structures $\pm J_{W}^{\pm}$, defined above. If $\nu_{+}(N)$ is not a singleton, then $J_{1}$ is one of the complex structures $\pm J_{W}^{\pm}$, according to Proposition 3.1 and Theorem 3.2 of Chapter III. Hence we get $\alpha = \pi/2$, which contradicts to the assumption. So, $\nu_{+}(N)$ is a singleton and hence $f(N)$ is minimal (cf. Theorem 3.2 of Chapter III). Thus, by (4.8), $f(N)$ is an open portion of an $\alpha$-slant 2-plane.

In Cases (ii) and (iii), we may assume $N = N_{1}$

**Case (ii):** In this case the curve $c(s)$ in (3.7) lies in the unit 2-sphere $S^{2} = S^{3} \cap W$. Choose $\{e_{1}, \cdots, e_{4}\}$ according to (4.5) and let $t, n, b, \kappa, \tau$ be the unit tangent vector, unit principal normal vector, the unit binormal vector, the curvature, and the torsion of $c(s)$ in $W = E^{3}$, respectively. We want to show that $\tau \equiv 0$.

Since

$$e_{1}(s, t) = \frac{1}{t} \frac{\partial}{\partial s}, e_{2}(s, t) = \frac{\partial}{\partial t},$$

$$e_{3}(s, t) = e_{1}(s, t) \times e_{2}(s, t),$$

where $\times$ denotes the vector product in $W$, we have

$$b = -\left(\frac{\kappa}{t}\right) < b, c >.$$

From $\|c\| = 1$, we get

$$\kappa < n, c >= -1.$$

Differentiating (4.13) with respect to $s$, we get

$$\kappa^{2} \tau < b, c >= \kappa'.$$

From (4.12) we obtain

$$-t \tau \kappa b = \kappa'.$$

Differentiating (4.15) with respect to $t$ and using (4.10) and (4.15), we obtain

$$\kappa'(\tau \kappa \tan \alpha - \kappa') = 0.$$
By (4.12), (4.14) and $< t, c > = 0$, we find

\[(6.17) \quad \kappa^2 \tau c = -\kappa \tau n + \kappa' b.\]

Since $||c|| = 1$, we also get

\[(4.18) \quad \tau^2 \kappa^4 = \tau^2 \kappa^2 + (\kappa')^2.\]

If $\kappa'(s_0) = 0$ at a point $s = s_0$, then, by (4.15), we have $\tau(s_0) = 0$, since $b(s, t) \neq 0$ by assumption and also $\kappa(s) \neq 0$ because $c(s)$ is spherical.

If $\kappa'(s_0) \neq 0$, we choose a neighborhood $U$ of $s_0$ on which $\kappa'$ never vanishes. By (4.16), (4.18) and $\kappa \neq 0$, we get

\[(4.19) \quad (\tau(s))^2 ((\kappa(s))^2 - 1 - \tan^2 \alpha) = 0 \quad \text{for } \forall s \in U.\]

If $\tau(s_0) \neq 0$ in addition, we choose another neighborhood $U'$ of $s_0$ contained in $U$ on which $\tau$ never vanishes. Then, by (4.19), we get

\[(\kappa(s))^2 - 1 - \tan^2 \alpha = 0 \quad \text{for all } s \text{ in } U'.\]

By continuity we get $\kappa(s) = \text{constant on } U'$ which contradicts to $\kappa'(s) \neq 0$ on $U'$. So, again we have $\tau(s_0) = 0$. Therefore, $\tau \equiv 0$, which means that $c(s)$ is a circle on $S^2$ and thus $f(N)$ is a circular cone. According to the remark after Lemma 6.1, the axis of the cone is given by $-J_1 e_4$.

**Case (iii):** We assume the surface is given by (3.8) and $\{e_1, \ldots, e_4\}$, $t$, $n$, $b$, $\kappa$ and $\tau$ are given as in Case (ii). We have

\[(4.20) \quad e_1(s, t) = n(s) = \frac{1}{(t - s)\kappa} \frac{\partial}{\partial s}, \quad e_2(s, t) = t(s) = \frac{\partial}{\partial t}, \quad e_3(s, t) = e_1 \times e_2 = -b(s).\]

Hence

\[(4.21) \quad \nabla_{e_1} e_1 = -\frac{1}{(t - s)} e_2 - \frac{\tau}{(t - s)\kappa} e_3.\]

So, by (4.8), we find

\[(4.22) \quad b = -\frac{\tau}{(t - s)\kappa}.\]
By (4.20) and (4.22), we obtain
\[ e_2 b = \frac{\tau}{\kappa (t - s)^2}. \]

This formula together with (4.10) and (4.21) imply that \( \tau = \tan\alpha \) is a constant. This means that the curve \( c(s) \) is a generalized helix in \( W \). The axis of the helix is given by \(-J_1 e_4\).

In each of the cases (i), (ii) and (iii), the converse is easy to verify. For example, if \( f(N) \) is a circular cone with the axis vector \( e \) in a 3-plane \( W \) perpendicular to a unit vector \( \eta \) in \( E^4 \), then, by picking the complex structure \( J \) so that \( J = J_\eta e_4 \), \( f(N) \) is properly slant with respect to \( J \).

This completes the proof of the theorem.

**Remark 4.1.** In the classifications of slant surfaces given in Sections 2, 3 and 4 of this chapter, we avoid the messy argument of glueing.
CHAPTER V

TOPOLOGY AND STABILITY OF SLANT SUBMANIFOLDS

§1. NON-COMPACTNESS OF PROPER SLANT SUBMANIFOLDS.

Let $E^{2m} = (\mathbb{R}^{2m}, <, >)$ and $C^m = (E^{2m}, J_0)$ be the Euclidean $2m$-space and the complex Euclidean $m$-space, respectively, with the canonical inner product $<, >$ and the canonical (almost) complex structure $J_0$ given by

\begin{equation}
J_0(x_1, \ldots, x_m, y_1, \ldots, y_m) = (-y_1, \ldots, -y_m, x_1, \ldots, x_m).
\end{equation}

Denote by $\Omega_0$ the Kaehler form of $C^m$, that is,

\begin{equation}
\Omega_0(X, Y) = <X, J_0 Y>, \quad X, Y \in E^{2m}, \quad \Omega_0 \in \wedge^2(E^{2m})^*.
\end{equation}

For an immersion $f : N \to C^m$, the Gauss map $\nu$ of the immersion $f$ is given by

\begin{equation}
\nu : N \to G(n, 2m) \equiv D_1(n, 2m) \subset S^{K-1} \subset \wedge^n(E^{2m}),
\end{equation}

\[ \nu(p) = e_1(p) \wedge \ldots \wedge e_n(p), \quad p \in N, \]

where $n = \dim N$, $K = \binom{2m}{n}$, $D_1(n, 2m)$ is the set of all unit decomposable $n$-vectors in $\wedge^n E^{2m}$, identified with the real Grassmannian $G(n, 2m)$ in a natural way, and $S^{K-1}$ is the unit hypersphere of $\wedge^n(E^{2m})$ centered at the origin, and $\{e_1, \ldots, e_{2m}\}$ is a local adapted orthonormal tangent frame along $f(N)$.

Before we give the main result of this section, we give the following lemmas.

**Lemma 1.1.** For $X_1, \ldots, X_{2k} \in E^{2m}$ ($k < m$), we have

\begin{equation}
(2k)! \Omega_0^k(X_1 \wedge \ldots \wedge X_{2k}) = \sum_{\sigma \in S_{2k}} \text{sign}(\sigma) \Omega_0(X_{\sigma(1)}, X_{\sigma(2)}) \cdot \ldots \cdot \Omega_0(X_{\sigma(2k-1)}, X_{\sigma(2k)}).
\end{equation}
where $S_{2k}$ is the permutation group of order $2k$, sign denotes the signature of permutations and $\Omega_0^k \in \wedge^{2k}(E^{2m})^* \equiv (\wedge^{2k}E^{2m})^*$.

**Proof.** Let $e_1, \ldots, e_m$ be an orthonormal frame of $E^{2m}$ with its dual coframe given by $\omega^1, \ldots, \omega^{2m}$. Let
\[ \Omega_0 = \sum_{A,B=1}^{2m} \varphi_{AB} \omega^A \wedge \omega^B. \]
Then by direct computation we have
\[ \Omega_0^k(X_1, \ldots, X_{2k}) = \frac{1}{(2k)!} \sum_{\sigma} \text{sign}(\sigma) \left( \sum \varphi_{A_1 A_2} \omega^{A_1} (X_{\sigma(1)}) \omega^{A_2} (X_{\sigma(2)}) \ldots \right. \]
\[ \left. \ldots \left( \sum \varphi_{A_{2k-1} A_{2k}} \omega^{A_{2k-1}} (X_{\sigma(2k-1)}) \omega^{A_{2k}} (X_{\sigma(2k)}) \right) \right). \]
From these we obtain (1.4).

**Lemma 1.2.** Let $V \in G(n,2m)$ and $\pi_V : E^{2m} \to V$ be the orthogonal projection. If $V$ is $\alpha$-slant, that is, $V$ is slant with slant angle $\alpha \neq \pi/2$, in $C^m = (E^{2m}, J_0)$, then the linear endomorphism $J_V$ of $V$ defined by
\[ J_V = (\sec \alpha)(\pi_V \circ J_{|V}) \]
is a complex structure compatible with the inner product $<,>_{|V}$. In particular, $n$ is even.

**Proof.** Let
\[ P = \pi_V \circ (J_{|V}) : V \to V, \]
\[ P^\perp = J_{|V} - P : V \to V^\perp \]
and
\[ Q = P^2 : V \to V. \]
Then
\[ J_{|V} = P + P^\perp. \]
By simple computation and using (1.7), we have

\[(1.10) \quad \langle QX, Y \rangle = \langle X, QY \rangle \]

and

\[(1.11) \quad \langle PX, Y \rangle = -\langle X, PY \rangle \]

for any \(X, Y \in V\). Since \(V\) is assumed to be \(\alpha\)-slant,

\[\angle (JX, V) = \angle (JX, PX) = \alpha\]

for any nonzero vector \(X \in V\). Hence we have

\[(1.12) \quad ||PX|| = \cos \alpha ||X||\]

for any nonzero vector \(X \in V\). This proves the lemma.

Let \(\hat{\zeta}_0\) be the metrical dual of \((-\Omega_0)^k\) with respect to the inner product \(<, >\) naturally defined on \(\wedge^{2k}E^{2m}\), that is,

\[(1.13) \quad \langle \hat{\zeta}_0, \eta \rangle = (-1)^k \Omega_0^k(\eta) \quad \text{for any} \ \eta \in \wedge^{2k}E^{2m},\]

then we have the following

**Lemma 1.3.** Let \(V \in G(2k,2m)\). If \(V\) is \(\alpha\)-slant in \(C^m\) with \(\alpha \neq \pi/2\), then

\[(1.14) \quad \langle \hat{\zeta}_0, V \rangle = \mu_k \cos^k \alpha,\]

where \(\mu_k\) is a nonzero constant depending only on \(k\).

**Proof.** Let \(J_V\) be the complex structure on \(V\) defined by Lemma 1.2. For a unit vector \(X \in V\), we put \(Y = J_VX \in V\). Then we have

\[(1.15) \quad \Omega_0(X, J_VX) = \langle -J_VY, J_0Y \rangle = -\cos \alpha.\]
If \(X, Y \in V\) and \(Z\) is perpendicular to \(J_V X\), then

\[
\Omega_0(X, Z) = \cos \alpha \langle X, J_V Z \rangle = 0.
\]

Therefore, if we choose an orthonormal \(J_V\)-basis \(\{e_1, \ldots, e_{2k}\}\) on \(V\), that is,

\[
e_{2i} = J_V e_{2i-1}, \quad i = 1, \ldots, k,
\]

and

\[
V = e_1 \wedge \ldots \wedge e_{2k},
\]

via the natural identification of \(G(2k, 2m)\) with \(D_1(2k, 2m)\), then we have

\[
\Omega_0(e_a, e_b) = -\delta_{a^* b} \cos \alpha \quad \text{for } a < b,
\]

where

\[
(2i)^* = 2i - 1, \quad (2i - 1)^* = 2i \quad \text{for } i = 1, \ldots, k.
\]

By (1.18), Lemma 1.1, and (1.19), we find

\[
(2k)! \Omega_0^k(V) = (2k)! \Omega_0(e_1 \wedge \ldots \wedge e_{2k})
\]

\[
= \sum_{\sigma \in S_{2k}} \text{sign}(\sigma) \Omega_0(e_{\sigma(1)}, e_{\sigma(2)}) \cdots \Omega_0(e_{\sigma(2k-1)}, e_{\sigma(2k)})
\]

\[
= \sum_{a_1, \ldots, a_{2k} = 1}^{2k} \delta_{a_1 \cdots a_{2k}} \Omega_0(e_{a_1}, e_{a_2}) \cdots \Omega_0(e_{a_{2k-1}}, e_{a_{2k}})
\]

\[
= 2^k \sum_{a_1 < a_1^*} \cdots \sum_{a_k < a_k^*} \delta_{a_1 a_1^* \cdots a_k a_k^*} \Omega_0(e_{a_1}, e_{a_1^*}) \cdots \Omega_0(e_{a_k}, e_{a_k^*})
\]

\[
= 2^k (-\cos \alpha)^k \sum_{a_1 < a_1^*} \cdots \sum_{a_k < a_k^*} \delta_{a_1 a_1^* \cdots a_k a_k^*}.
\]
\[ 2^k (-\cos \alpha)^k k!. \]

Hence, by (1.13) and (1.21), we obtain (1.14) with \( \mu_k = 2^k k!/(2k)! \).

**Lemma 1.4.** Let \( f : N \to E^m \) be an isometric immersion of an \( n \)-dimensional compact oriented manifold \( N \) into \( E^m \). Then the Gauss map \( \nu : N \to \wedge^n(E^m) \) is mass-symmetric in \( S^{K-1} \), \( K = \binom{m}{n} \), that is, the center of gravity of \( \nu \) coincides with the center of the hypersphere \( S^{K-1} \) in \( \wedge^n(E^m) \).

**Proof.** Let \( e_1, \ldots, e_n \) be an oriented orthonormal local frame of \( TN \) with its dual coframe given by \( \omega^1, \ldots, \omega^n \). Then we have

\[
(1.22) \quad dx = e_1 \omega^1 + \ldots + e_n \omega^n.
\]

By direct computation we have

\[
dx \wedge \ldots \wedge dx = n! (e_1 \wedge \ldots \wedge e_n) \omega^1 \wedge \ldots \wedge \omega^n = n! \nu(\ast 1),
\]

where \( dx \) on the left-hand-side is repeated \( n \)-times. Therefore, by applying the divergence theorem, we have

\[
n! \int_N \nu \ast 1 = \int_N dx \wedge \ldots \wedge dx = \int_N dx \wedge dx \wedge \ldots \wedge dx = 0.
\]

This proves that the center of gravity of \( \nu \) is the origin of \( \wedge^n(E^m) \), that is, the Gauss map is mass-symmetric in \( S^{K-1} \).

Now we give the following [CT2]

**Theorem 1.5.** Let \( f : N \to C^m \) be a slant immersion of an \( n \)-dimensional differentiable manifold \( N \) into the complex Euclidean \( m \)-space \( C^m \). If \( N \) is compact, then \( f \) is totally real.

**Proof.** Without loss of generality we may assume that \( N \) is oriented because otherwise we may simply replace \( N \) by its two-fold covering. Assume \( f \) is \( \alpha \)-slant with \( \alpha \neq \pi/2 \). Then, by Lemma 1.2, \( n \) is even. Put \( n = 2k \). Since \( N \) is compact, Lemma 1.4 implies that the Gauss map \( \nu \) is mass-symmetric in \( S^{K-1} \), \( K = \binom{2m}{2k} \). Therefore

\[
(1.23) \quad \int_{p \in N} < \nu(p), \zeta > \ast 1 = 0
\]
Geometry of Slant Submanifolds

for any fixed 2k-vector $\zeta \in \wedge^{2k}(E^{2m})$, where *1 is the volume element of $N$ with respect to the metric induced from the immersion $f$. Let $\zeta = \zeta_0$, where $\zeta_0$ is defined by (1.13). Then Lemma 1.3 and (1.23) imply

\begin{equation}
\mu_k \text{vol}(N) \cos^k \alpha = 0.
\end{equation}

But this contradicts the assumption $\cos \alpha \neq 0$. Hence $\alpha = \pi/2$ and $f$ is a totally real immersion.

We recall that a submanifold $N$ of an almost complex manifold $(M, J)$ is said to be purely real [C2] if every eigenvalue of $Q = P^2$ lies in $(-1, 0]$. In fact, by using a method similar to the proof of Theorem 1.5, we may prove the following

**Theorem 1.6.** Let $f : N \rightarrow \mathbb{C}^m$ be a purely real immersion from an $n$-dimensional differentiable manifold $N$ into $\mathbb{C}^m$. If $N$ is compact, then $f$ is totally real.

**Definition 1.1.** An almost Hermitian manifold $(M, g, J)$ is called an exact sympletic manifold if the sympletic form (or, equivalently, the Kaehler form) $\tilde{\Omega} = \tilde{\Omega}_J$ of $(M, g, J)$ is an exact 2-form.

For compact submanifolds in an exact sympletic manifold, we have the following

**Theorem 1.7.** Every compact slant submanifold $N$ in an exact sympletic manifold $(M, g, J)$ is totally real.

**Proof.** Let $(M, g, J)$ be an exact sympletic manifold with sympletic form $\tilde{\Omega}$. Then there exists a 1-form $\tilde{\varphi}$ on $M$ such that $\tilde{\Omega} = d\tilde{\varphi}$. Let $f : N \rightarrow M$ be the immersion and we put

\begin{equation}
\Omega = f^*\tilde{\Omega}, \quad \varphi = f^*\tilde{\varphi}.
\end{equation}

Then we have

\begin{equation}
\Omega = f^*\tilde{\Omega} = f^*d\tilde{\varphi} = d(f^*\varphi) = d\varphi.
\end{equation}

If $N$ is either a proper slant submanifold or a complex submanifold of $M$, then, by using Lemmas 1.1 and 1.2, we may prove that there is a nonzero constant $C$ (which depends only on the dimension of $N$ and the slant angle)
such that $\Omega^k = C(\ast 1)$ where $\text{dim } N = 2k$ and $\ast 1$ is the volume element of the slant submanifold $N$. From (1.26) we know that $\Omega^k$ is exact. Therefore, by the Stokes theorem, we have

$$\text{vol}(N) = \int_N (\ast 1) = C^{-1} \int_N \Omega^k = 0.$$ 

This is a contradiction.
§2. TOPOLOGY OF SLANT SURFACES.

Let $N$ be an $n$-dimensional proper slant submanifold with slant angle $\alpha$ in a Kaehlerian manifold $M$ of complex dimension $m$. Then $n$ is even, say $n = 2k$. Let $e_1$ be a unit tangent vector of $N$. We put

$$e_2 = (\sec \alpha) Pe_1, \quad e_1^* := e_{n+1} = (\csc \alpha) Fe_1,$$

$$e_2^* := e_{n+2} = (\csc \alpha) Fe_2.$$

If $n > 1$, then, by induction, for each $\ell = 1, \ldots, k - 1$, we may choose a unit tangent vector $e_{2\ell+1}$ of $N$ such that $e_{2\ell+1}$ is perpendicular to $e_1, e_2, \ldots, e_{2\ell-1}, e_{2\ell}$. We put

$$e_{2\ell+2} = (\sec \alpha) Pe_{2\ell+1}, \quad e_{(2\ell+1)^*} := e_{n+2\ell+1} = (\csc \alpha) Fe_{2\ell+1},$$

$$e_{(2\ell+2)^*} := e_{n+2\ell+2} = (\csc \alpha) Fe_{2\ell+2}.$$

If $N$ is totally real in $M$, that is, if the slant angle $\alpha = \frac{\pi}{2}$, then we can just choose $e_1, \ldots, e_n$ to be any local orthonormal frame of $TN$ and put

$$e_{n+1} := e_1^* = Je_1, \ldots, e_{2n} := e_n^* = Je_n.$$

If $m > n$, then at each point $p \in N$ there exist a subspace $\nu_p$ of the normal space $T_p^\perp N$ such that $\nu_p$ is invariant under the action of the complex structure $J$ of $M$ and

$$T_p^\perp N = F(T_pN) \oplus \nu_p, \quad \nu_p \perp F(T_pN).$$

We choose a local orthonormal frame $e_{4k+1}, \ldots, e_{2m}$ of $\nu$ such that

$$e_{2n+2} =: e_{(4k+1)^*} = Je_{4k+1}, \ldots, e_{2m} =: e_{(2m-1)^*} = Je_{2m-1},$$

that is, $e_{2n+1}, \ldots, e_{2m}$ is a $J$-frame of $(\nu, J)$.

We call such an orthonormal frame

$$e_1, e_2, \ldots, e_{2k-1}, e_{2k}, e_1^*, e_2^*, \ldots, e_{(2k-1)^*}, e_{(2k)^*},$$
Lemma 2.1. Let $N$ be an $n$-dimensional proper slant submanifold of a Kaehlerian manifold $M$. If $N$ is a Kaehlerian slant submanifold, then, with respect to an adapted slant frame (2.4), we have

\[ \omega_i^* = \omega_j^*, \quad \text{or equivalently,} \quad h^i_{ik} = h^i_{jk} \]

for any $i, j, k = 1, \ldots, n$, where $\omega_B^A$ are the connection forms associated with the adapted slant frame.

Proof. Since $N$ is a Kaehlerian slant submanifold, $\nabla P = 0$ by definition. Thus, by applying by Lemma 3.5 of Chapter II, we have $A_F X Y = A_F Y X$ for any $X, Y$ tangent to $N$. Therfore, we have (2.5) according to the definition of adapted slant frame.

Remark 2.1. If $N$ is a totally real submanifold of a Kaehlerian manifold, Lemma 2.1 was given in [CO1].

Corollary 2.2. If $N$ is a proper slant surface of a Kaehlerian manifold $M$, then, with respect to an adapted slant frame of $N$ in $M$, we have

\[ \omega_i^* = \omega_j^*, \quad \text{or equivalently,} \quad h^i_{ik} = h^i_{jk} \]

for any $i, j, k = 1, 2$.

Proof. This Corollary follows immediately from Lemma 2.1 and Theorem 3.4 of Chapter II.

For an $n$-dimensional proper slant submanifold $N$ of an almost Hermitian manifold $M$, we define a canonical 1-form $\Theta$ on $N$ by

\[ \Theta = \sum_{i=1}^{n} \omega_i^* \]

Lemma 2.3. Let $N$ be an $n$-dimensional proper slant submanifold with slant angle $\alpha$ in a Kaehlerian manifold $M$. Then we have

\[ \Theta = \sum_{i} (\text{tr} h^i) \omega^i \]
and

\( (2.9) \quad \Theta(X) = -n(\csc \alpha) < tH, X > \)

for any vector \( X \) tangent to \( N \), where \( H \) denotes the mean curvature vector of \( N \) in \( M \).

**Proof.** Equation (2.8) follows from equation (1.9) of Chapter II, (2.7) and Corollary 2.2.

Since

\[
 n < tH, e_j > = -n < H, Fe_j > = -n(\sin \alpha) < H, e_j^* > = -(\sin \alpha) \text{tr} h^j,
\]

(2.8) implies (2.9).

Now we give the following [CM3]

**Lemma 2.4.** Let \( N \) be an \( n \)-dimensional proper slant submanifold of \( \mathbb{C}^n \). If \( N \) is Kahlerian slant, then the canonical 1-form \( \Theta \) is closed, that is, \( d\Theta = 0 \). Hence, \( \Theta \) defines a canonical cohomology class on \( N \):

\( (2.10) \quad [\Theta] \in H^1(N; \mathbb{R}) \).

**Proof.** Under the hypothesis, formula (1.8) of Chapter II gives

\( (2.11) \quad d\Theta = - \sum_{i,j=1}^{n} \omega_i^j \wedge \omega_j^* - \sum_{i,j=1}^{n} \omega_i^* \wedge \omega_j^j. \)

Since \( \omega_i^j = \omega_j^i \) by Lemma 2.1 and \( \omega_i^j = -\omega_j^i, \omega_i^* = -\omega_j^*, \omega_i^* = -\omega_j^j \), we have

\( (2.12) \quad \sum_{i,j} \omega_i^j \wedge \omega_j^* = \sum_{i,j} \omega_i^* \wedge \omega_j^j = 0. \)

From (2.11) and (2.12) we obtain the lemma.

**Remark 2.2.** In fact, this lemma holds without the condition that \( N \) is Kahlerian slant (see Theorem 3.1.). However, the proof for the general case is much more complicated.
Now we give the main result of this section [CM3].

**Theorem 2.5.** Let $N$ be a proper slant surface in $\mathbb{C}^2$ with slant angle $\alpha$. Put $\Psi = (2\sqrt{2}\pi)^{-1}(\text{csc} \, \alpha)\Theta$. Then $\Psi$ defines a canonical integral class on $N$:

\begin{equation}
\psi = [\Psi] \in H^1(N; \mathbb{Z}).
\end{equation}

**Proof.** Let $N$ be a proper slant surface with slant angle $\alpha$ in $\mathbb{C}^2$. Denote by $\nu$ the Gauss map $\nu : N \to G(2, 4) \cong S^2_+ \times S^2_-$ and by $\nu_+$ and $\nu_-$ the projections: $\nu_\pm : N \to G(2, 4) \to S^2_\pm$ (cf. Section 3 of Chapter III).

From formula (4.3) of Chapter III, we have

\begin{equation}
(\nu_-)_* = \frac{1}{\sqrt{2}}\{(-\omega_1^4 + \omega_2^3)\eta_6 + (\omega_1^3 + \omega_2^4)\eta_6\}.
\end{equation}

By Lemma 2.1 we obtain $\omega_1^4 = \omega_2^3$. Thus, (2.14) implies

\begin{equation}
(\nu_-)_* = \frac{1}{\sqrt{2}}\Theta \eta_6,
\end{equation}

where $\eta_6 = \frac{1}{2}(e_1 \wedge e_4 - e_2 \wedge e_3)$. Now because $\nu_-$ is given by

\[(\nu_-)(p) = \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4)(p)\]

for any point $p \in N$, the slantness of $N$ in $\mathbb{C}^2$ implies that the image $\nu_-(N)$ lies in the small circle $S^1_\alpha$ of $S^2_-$. Moreover, it is easy to see that $\sqrt{2}\eta_6$ is a unit vector tangent to $S^1_\alpha$. Let $\omega = dS^1_\alpha$ be the arclength element of $S^1_\alpha$. Then, for any vector $X$ tangent to $N$, we have

\begin{equation}
((\nu_-)^*\omega)(X) = \omega((\nu_-)_*(X)) = \omega(\frac{1}{\sqrt{2}}\Theta(X)\eta_6) = \frac{1}{2}\Theta(X).
\end{equation}

Hence we have

\begin{equation}
(\nu_-)^*\omega = \frac{1}{2}\Theta.
\end{equation}

Therefore, for any closed loop $\gamma$ on $N$, we have

\[
\int_\gamma \frac{1}{2}\Theta = \int_\gamma (\nu_-)^*\omega = \int_{\nu_-(S^1_\alpha)} \omega = (\text{index of } \nu_-) \text{vol}(S^1_\alpha)
\]
This implies that for any closed loop \( \gamma \) in \( N \), \( \int_{\gamma} \Psi \in \mathbb{Z} \). Thus by Lemma 2.4 we obtain (2.13). This completes the proof of the theorem.

As an application we obtain the following [CM3]

**Theorem 2.6.** Let \( N \) be a complete, oriented, proper slant surface in \( \mathbb{C}^2 \). If the mean curvature of \( N \) is bounded below by some positive constant \( c > 0 \), then, topologically, \( N \) is either a circular cylinder or a 2-plane.

**Proof.** Consider the map \( \nu_- : N \to S^2 \). Assume the slant angle of \( N \) in \( \mathbb{C}^2 \) is \( \alpha \). Then \( \nu_-(N) \subset S^1_{\alpha} \). Since

\[
(\nu_-)(X) = \frac{1}{\sqrt{2}} (\omega_1^3 + \omega_2^4)(X) \eta_6 = \frac{1}{\sqrt{2}} \left( \sum_i h_{i1}^3 \omega_1 + \sum_i h_{i1}^4 \omega_2 \right)
\]

for any vector \( X \) tangent to \( N \) by Lemma 2.1, the assumption on the mean curvature implies that the map \( \nu_- \) is an onto map. Furthermore, since the rank of \( \nu_- \) is equal to one, a result of Ehresmann implies that \( \nu_- \) is in fact a fibration. Because \( N \) is not compact by Theorem 1.5, topologically, \( N \) is either the product of a line and a circle or a 2-plane.
§3. COHOMOLOGY OF SLANT SUBMANIFOLDS.

One of the purposes of this section is to improve Lemma 2.4 to obtain the following Theorem 3.1 of [CM3]. The other purpose is to prove that every proper slant submanifold in any Kaehlerian manifold is a sympletic manifold (Theorem 3.4) with the sympletic structure induced from the canonical endomorphism $P$.

**Theorem 3.1.** Let $N$ be an $n$-dimensional proper slant submanifold of $\mathbb{C}^n$. Then the canonical 1-form $\Theta$ defined by (2.7) is closed, that is, $d\Theta = 0$. Hence, $\Theta$ defines a canonical cohomology class on $N$:

$$[\Theta] \in H^1(N; \mathbb{R}).$$

If $N$ is an $n$-dimensional proper slant submanifold of $\mathbb{C}^n$ with slant angle $\alpha$, then as we already known the dimension $n$ is even. Let $n = 2k$ and $e_1, \ldots, e_n, e_1^*, \ldots, e_n^*$ an adapted slant (orthonormal) frame of $N$ in $\mathbb{C}^n$. Then we have

$$e_2 = (\sec \alpha)Pe_1, \ldots, e_{2k} = (\sec \alpha)Pe_{2k-1},$$

$$e_1^* = (\csc \alpha)Fe_1, e_2^* = (\csc \alpha)Fe_2, \ldots, e_{(2k)}^* = (\csc \alpha)Fe_{2k}.$$  

By direct computation we also have

$$te_i^* = -(\sin \alpha)e_i, \quad i = 1, \ldots, 2k,$$

$$Pe_{2j} = -(\cos \alpha)e_{2j-1},$$

$$fe_{(2j-1)}^* = -(\cos \alpha)e_{(2j)}^*, \quad fe_{(2j)}^* = (\cos \alpha)e_{(2j-1)}^*, \quad j = 1, \ldots, k.$$  

In order to prove Theorem 3.1, we need the following lemmas which can be regarded as generalizations of Lemma 2.1.

**Lemma 3.2.** Let $N$ be an $n$-dimensional $(n = 2k)$ proper slant submanifold of $\mathbb{C}^n$. Then, with respect to an adapted slant frame, we have
Geometry of Slant Submanifolds

(3.5) \( \omega_{2j-1}^{(2i-1)*} - \omega_{2i-1}^{(2j-1)*} = \cot \alpha (\omega_{2i-1}^{2j} - \omega_{2j-1}^{2i}) \),

(3.6) \( \omega_{2j}^{(2i-1)*} - \omega_{2i-1}^{(2j)*} = \cot \alpha (\omega_{2i}^{2j} - \omega_{2j-1}^{2i}) \),

(3.7) \( \omega_{2i}^{(2j)*} - \omega_{2j}^{(2i)*} = \cot \alpha (\omega_{2i}^{2j-1} - \omega_{2j}^{2i-1}) \),

(3.8) \( \omega_{2j-1}^{(2i-1)*} - \omega_{2i-1}^{(2j-1)*} = \cot \alpha (\omega_{2i-1}^{2j} - \omega_{2j-1}^{2i}) \),

(3.9) \( \omega_{2j}^{(2j)*} - \omega_{2j-1}^{(2j)*} = \cot \alpha (\omega_{2j}^{2j-1} - \omega_{2j}^{2i}) \),

(3.10) \( \omega_{2i-1}^{(2j-1)*} - \omega_{2i-1}^{(2j-1)*} = \cot \alpha (\omega_{2i-1}^{2j} - \omega_{2i}^{2j}) \),

(3.11) \( \omega_{2j}^{(2j-1)*} - \omega_{2j-1}^{(2j)*} = \cot \alpha (\omega_{2i}^{2j-1} - \omega_{2i}^{2j}) \),

(3.12) \( \omega_{2i}^{(2j-1)*} - \omega_{2i}^{(2j-1)*} = \cot \alpha (\omega_{2i}^{2j} - \omega_{2i}^{2i}) \),

(3.13) \( \omega_{2j-1}^{(2i)*} - \omega_{2i-1}^{(2j-1)*} = \cot \alpha (\omega_{2i}^{2j} - \omega_{2i}^{2i}) \),

for any \( i, j = 1, \ldots, k \).

**Proof.** From the definition of adapted slant frames we have

(3.14) \(< Je_{2i-1}, e_{2j-1}> = 0, \quad i, j = 1, \ldots, k.\)

By taking the derivative of (3.13) with respect to a tangent vector \( X \) of \( N \) and applying (3.2), we have

\[
0 = < J\nabla_X e_{2i-1}, e_{2j-1} > + < J e_{2i-1}, \nabla_X e_{2j-1} > \\
= -< \nabla_X e_{2i-1}, Pe_{2j-1} > - < h(e_{2i-1}, X), Fe_{2j-1} >
\]
\[ + \langle P e_{2i-1}, \nabla_X e_{2j-1} \rangle = \langle F e_{2i-1}, h(e_{2j-1}, X) \rangle \]
\[= - (\cos \alpha) \langle \nabla_X e_{2i-1}, e_{2j} \rangle - \langle \nabla_X e_{2j-1}, e_{2i} \rangle \]
\[+ (\sin \alpha) \langle e_{(2i-1)*}, h(X, e_{2j-1}) \rangle - \langle e_{(2j-1)*}, h(X, e_{2i-1}) \rangle. \]

This implies
\[ (3.15) \]
\[(\cot \alpha) (\omega_{2i-1}^{2j} - \omega_{2j-1}^{2i}) (X) \]
\[= \langle A e_{(2i-1)*}, e_{2j-1} - A e_{(2j-1)*}, e_{2i-1}, X \rangle. \]

Therefore, by applying formula (1.9)' of Chapter II, we get formula (3.5).

Similarly, by taking the derivatives of the following equations:
\[ < J e_{2i-1}, e_{2j} >= (\cos \alpha) \delta_{ij}, \quad < J e_{2i}, e_{2j} >= 0, \]
\[ < J e_{(2i-1)*}, e_{(2j-1)*} >= 0, \quad < J e_{(2j)*}, e_{(2j)*} >= 0, \]
\[ < J e_{2i-1}, e_{(2j-1)*} >= (\sin \alpha) \delta_{ij}, \quad < J e_{2i-1}, e_{(2j)*} >= 0, \]
\[ < J e_{2i}, e_{(2j)*} >= (\sin \alpha) \delta_{ij}, \quad < J e_{(2i)*}, e_{(2j)*} >= 0, \]

we obtain formulas (3.6)-(3.13), respectively.

This proves the lemma.

**Lemma 3.3.** Let \( N \) be an \( n \)-dimensional \( (n = 2k) \) proper slant submanifold of \( \mathbb{C}^n \). Then, with respect to an adapted slant frame, we have

\[ (3.16) \]
\[ \omega_{2i}^{(2j)*} + \omega_{2j-1}^{(2j-1)*} = \omega_{2i}^{(2i)*} + \omega_{2j-1}^{(2i-1)*}, \]
\[ (3.17) \]
\[ \omega_{(2i)*}^{(2j)} - \omega_{(2i-1)*}^{(2j-1)} = \omega_{2i}^{2j} - \omega_{2i-1}^{2j-1}, \]
\[ (3.18) \]
\[ \omega_{2j}^{2i-1} - \omega_{(2j)*}^{(2i-1)} = \omega_{2i}^{2j-1} - \omega_{(2i)*}^{(2j-1)}. \]
for any $i, j = 1, \ldots, k$.

**Proof.** Formula (3.16) follows from (3.5) and (3.7). Formula (3.17) follows from (3.10) and (3.12). And formula (3.18) follows from (3.7) and (3.9).

Now we give the proof of Theorem 3.1.

**Proof.** From the definition we have

$$
\Theta = \sum_{\ell=1}^{2k} \omega_\ell^*.
$$

Thus from the structure equations we have

$$
d\Theta = \sum_{i,j=1}^{k} \omega_{2i}^{2j} \wedge \omega_{2j}^{(2i)^*} + \sum_{i,j=1}^{k} \omega_{2i}^{2j-1} \wedge \omega_{2j-1}^{(2i)^*}
$$

$$
+ \sum_{i,j=1}^{k} \omega_{2i-1}^{2j} \wedge \omega_{2j}^{(2i-1)^*} + \sum_{i,j=1}^{k} \omega_{2i-1}^{2j-1} \wedge \omega_{2j-1}^{(2i-1)^*}
$$

$$
+ \sum_{i,j=1}^{k} \omega_{2i}^{(2j-1)^*} \wedge \omega_{2j}^{2i} + \sum_{i,j=1}^{k} \omega_{2i-1}^{(2j-1)^*} \wedge \omega_{2j}^{2i-1}.
$$

By using formula (3.17) we have

$$
\sum_{i,j=1}^{k} \omega_{2i}^{(2j)^*} \wedge \omega_{2j}^{(2i)^*} + \sum_{i,j=1}^{k} \omega_{2i}^{2j-1} \wedge \omega_{2j-1}^{(2i)^*}
$$

$$
+ \sum_{i,j=1}^{k} \omega_{2i-1}^{2j} \wedge \omega_{2j}^{(2i-1)^*} + \sum_{i,j=1}^{k} \omega_{2i-1}^{2j-1} \wedge \omega_{2j-1}^{(2i-1)^*}
$$

$$
= \sum_{i,j=1}^{k} \omega_{2i}^{(2j)^*} \wedge \omega_{2j}^{(2i)^*} + \sum_{i,j=1}^{k} \omega_{2i-1}^{2j} \wedge \omega_{2j-1}^{(2i-1)^*}.$$
Moreover, by (3.18), we have

\[
\sum_{i,j=1}^{k} \omega^{(2j-1)*}_{i,j} \wedge (\omega^{2i-1}_{2j-1} - \omega^{2j}_{2j} + \omega^{(2i)*}_{(2j)}) + \sum_{i,j=1}^{k} \omega^{2j}_{2i} \wedge \omega^{(2i)*}_{2j} + \sum_{i,j=1}^{k} \omega^{2j}_{2i} \wedge \omega^{(2i)*}_{2j} = 0
\]

Since \(\omega^{(2j-1)*}_{2i-1} + \omega^{(2j)*}_{2i}\) is symmetric in \(i\) and \(j\) by Lemma 3.3 and \(\omega^{(2j)*}_{2i}\) and \(\omega^{(2i)*}_{(2j)}\) are skew-symmetric in \(i\) and \(j\), we obtain

\[
(3.21) \quad \sum_{i,j=1}^{k} \omega^{(2j)*}_{2i} \wedge \omega^{(2i)*}_{(2j)} + \sum_{i,j=1}^{k} \omega^{2j-1}_{2i-1} \wedge \omega^{(2i-1)*}_{2j-1} + \sum_{i,j=1}^{k} \omega^{2j}_{2i} \wedge \omega^{(2i)*}_{2j} = 0.
\]

Moreover, by (3.18), we have

\[
\sum_{i,j=1}^{k} \omega^{2j-1}_{2i} \wedge \omega^{(2i)*}_{2j-1} + \sum_{i,j=1}^{k} \omega^{(2j-1)*}_{2i} \wedge \omega^{(2i)}_{(2j-1)*} + \sum_{i,j=1}^{k} \omega^{2j}_{2i} \wedge \omega^{(2i)*}_{2j} + \sum_{i,j=1}^{k} \omega^{2j}_{2i} \wedge \omega^{(2i-1)*}_{2j-1} = 0.
\]
\[ k \sum_{i,j=1}^{k} \omega_{2i}^{2j-1} \wedge \omega_{2j-1}^{(2i)^*} - \sum_{i,j=1}^{k} \omega_{2i}^{2j-1} \wedge \omega_{2j}^{(2i)^*} \]

\[ = \sum_{i,j=1}^{k} (\omega_{2i}^{2j-1} - \omega_{2j}^{(2i)^*}) \wedge (\omega_{2i}^{2j-1} - \omega_{2j}^{(2i)^*}). \]

Since \( \omega_{2i}^{2j-1} - \omega_{2j}^{(2i)^*} \) is symmetric in \( i \) and \( j \) by Lemma 3.3 and \( \omega_{2i-1}^{(2j)^*} - \omega_{2j-1}^{(2i)^*} \) is skew-symmetric in \( i \) and \( j \) by formula (3.6) of Lemma 3.2, we obtain

\[ (3.22) \sum_{i,j=1}^{k} \omega_{2i}^{2j-1} \wedge \omega_{2j-1}^{(2i)^*} + \sum_{i,j=1}^{k} \omega_{2i}^{(2j-1)^*} \wedge \omega_{2j-1}^{(2i)^*} \]

\[ + \sum_{i,j=1}^{k} \omega_{2i-1}^{2j} \wedge \omega_{2j}^{(2i-1)^*} + \sum_{i,j=1}^{k} \omega_{2i-1}^{(2j)^*} \wedge \omega_{2j}^{(2i-1)^*} = 0 \]

From (3.20), (3.21) and (3.22) we obtain the theorem.

**Remark 3.1.** If \( N \) is not a slant submanifold of \( \mathbb{C}^n \), then the 1-form \( \Theta \) is not closed in general. For example, if \( N \) is the standard unit 2-sphere \( S^2 \) in \( E^3 \subset E^4 \), then \( N \) is not slant with respect to any compatible complex structure on \( E^4 \).

Now, assume that \( S^2 \) is parametrized by

\[ (3.23) \quad x(\theta, \varphi) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi, 0). \]

Put

\[ (3.24) \quad e_1 = (-\sin \theta, \cos \theta, 0, 0), \]

\[ e_2 = (\cos \varphi \cos \theta, \cos \varphi \sin \theta, -\sin \varphi, 0). \]
Then, with respect to the complex structure $J_0$, we have
\begin{equation}
(3.25) \quad Fe_1 = \left(-\frac{1}{4}\sin 2\theta \sin 2\varphi, -\frac{1}{2}\sin^2 \theta \sin 2\varphi, -\sin \theta \cos^2 \varphi, \cos \theta\right),
\end{equation}
and
\begin{equation}
(3.26) \quad Fe_2 = \left(\cos^2 \theta \sin \varphi, \sin \theta \cos \theta \sin \varphi, \cos \theta \cos \varphi, \sin \theta \cos \varphi\right).
\end{equation}

Let $Fe_1 = ||Fe_1||e_1^\ast$, $Fe_2 = ||Fe_2||e_2^\ast$. Then, with respect to the local orthonormal frame $e_1, e_2, e_1^\ast, e_2^\ast$ on the open subset of $S^2$ on which $\theta \not\equiv \frac{\pi}{2}$ and $\varphi \not\equiv 0 (mod \pi)$, we have
\begin{equation}
\Theta = \omega_1^1 + \omega_2^2 = (1 - \sin^2 \theta \cos^2 \varphi)^{-1}\left(\frac{1}{2}\sin \theta \sin 2\varphi d\theta - \cos \theta d\varphi\right).
\end{equation}

It is easy to see that $d\Theta \neq 0$.

For an $n$-dimensional $(n = 2k)$ proper slant submanifold $N$ of a Kaehlerian manifold $(M, g, J)$, we put
\begin{equation}
(3.27) \quad \Lambda(X, Y) = \langle X, PY \rangle
\end{equation}
for any vectors $X, Y$ tangent to $N$, that is, $\Lambda = \Omega_{|TN}$. Then, from Lemmas 1.1 to 1.3, $\Lambda$ is a non-degenerate 2-form on $N$, that is, $\Lambda^k \neq 0$.

We recall that an even-dimensional manifold is called a sympletic manifold if it has a non-degenerate closed 2-form.

Now we give the second main result of this section.

**Theorem 3.4.** Let $N$ be an $n$-dimensional proper slant submanifold of a Kaehlerian manifold $M$. Then $\Lambda$ is closed, that is, $d\Lambda = 0$. Hence $\Lambda$ defines a canonical cohomology class of $N$:

\begin{equation}
(3.28) \quad [\Lambda] \in H^2(N; \mathbb{R}).
\end{equation}

In particular, $(N, \Lambda)$ is a sympletic manifold.

**Proof.** By definition of the exterior differentiation we have
\begin{equation}
\begin{aligned}
d\Lambda(X, Y, Z) &= \frac{1}{3}\{X\Lambda(Y, Z) + Y\Lambda(Z, X) + Z\Lambda(X, Y)\}
\end{aligned}
\end{equation}
Thus, by the definition of \( \Lambda \), we obtain
\[
d\Lambda(X,Y,Z) = \frac{1}{3}\left\{ <\nabla_X Y, PZ> + < Y, \nabla_X (PZ)> \\
+ <\nabla_Y Z, PX> + < Z, \nabla_Y (PX)> + < \nabla_Z X, PY> \\
+ < X, \nabla_Z (PY)> - < [X,Y], PZ> - < [Z,X], PY> \\
- < [Y,Z], PX> \right\}
\]
= \frac{1}{3}\left\{ <Y, \nabla_X (PZ)> + < Z, \nabla_Y (PX)> + < X, \nabla_Z (PY)> \\
+ < \nabla_X Z, PY> + < \nabla_Y X, PZ> + < \nabla_Z Y, PX> \right\}.
\]

Therefore, by the definition of \( \nabla P \), we obtain
\[
(3.29) \quad d\Lambda(X,Y,Z) = \frac{1}{3}\left\{ <X, (\nabla_Z P)Y> + < Y, (\nabla_X P)Z> \\
+ < Z, (\nabla_Y P)X> \right\}.
\]

Therefore, by applying formula (3.2) of Chapter II and formula (3.29), we get
\[
d\Lambda(X,Y,Z) = \frac{1}{3}\left\{ <X, th(Y,Z)> + < Y, AF_Y Z> \\
+ < Y, th(Z,X)> + < Y, AF_Z X> \\
+ < Z, th(X,Y)> + < Z, AF_X Y> \right\}.
\]

Consequently, by applying formulas (1.1), (1.4), (1.5) of Chapter II and formula (3.30), we obtain (3.28).

This proves the theorem.

As an immediate consequence of Theorem 3.4 we obtain the following

**Theorem 3.5.** If \( N \) is a compact \( 2k \)-dimensional proper slant submanifold of a Kaehlerian manifold \( M \), then
\[
(3.31) \quad H^{2i}(N; \mathbb{R}) \neq 0
\]
for any $i = 1, \ldots, k$.

In other words we have the following non-immersion theorem.

**Theorem 3.5’.** Let $N$ be a compact $2k$-dimensional differentiable manifold such that $H^{2i}(N; R) = 0$ for some $i \in \{1, \ldots, k\}$. Then $N$ cannot be immersed in any Kaehlerian manifold as a proper slant submanifold.
§4. STABILITY AND INDEX FORM.

The main purpose of this section is to present some results concerning the stability and index of totally real submanifolds, a special class of slant submanifolds, in a Kaehlerian manifold.

Let $f : N \to M$ be an immersion from a compact $n$-dimensional manifold $N$ into an $m$-dimensional Riemannian manifold $M$. Let $\{f_t\}$ be a 1-parameter family of immersions of $N \to M$ with the property that $f_0 = f$. Assume the map $F : N \times [0,1] \to M$ defined by $F(p,t) = f_t(p)$ is differentiable. Then $\{f_t\}$ is called a variation of $f$. A variation of $f$ induces a vector field in $M$ defined along the image of $N$ under $f$. We shall denote this field by $\zeta$ and it is constructed as follows:

Let $\partial/\partial t$ be the standard vector field in $N \times [0,1]$. We set

$$\zeta(p) = F_\ast(\frac{\partial}{\partial t}(p,0)).$$

Then $\zeta$ gives rise to cross-sections $\zeta^T$ and $\zeta^N$ in $TN$ and $T^\perp N$, respectively. If we have $\zeta^T = 0$, then $\{f_t\}$ is called a normal variation of $f$. For a given normal vector field $\xi$ on $N$, $\exp t\xi$ defines a normal variation $\{f_t\}$ induced from $\xi$. We denote by $V(t)$ the volume of $N$ under $f_t$ with respect to the induced metric and by $V'(\xi)$ and $V''(\xi)$, respectively, the values of the first and the second derivatives of $V(t)$ with respect to $t$, evaluated at $t = 0$.

The following formula is well-known:

$$V'(\xi) = -n \int_N \langle \xi, H \rangle \ast 1.$$

For a compact minimal submanifold $N$ of a Riemannian manifold $M$, the second variation formula is given by

$$V''(\xi) = \int_N \{||D\xi||^2 - \tilde{S}(\xi, \xi) - ||A\xi||^2 \} \ast 1,$$

where $\tilde{S}(\xi, \eta)$ is defined by

$$\tilde{S}(\xi, \eta) = \sum_{i=1}^n \tilde{R}(\xi, e_i, e_i, \eta),$$

$$\tilde{R}(\xi, e_i, e_i, \eta) = \sum_{j=1}^m \tilde{R}^{ij}(\xi, e_i, e_j, \eta).$$
\(e_1, \ldots, e_n\) a local orthonormal frame of \(TN\) and \(\tilde{R}\) the Riemann curvature tensor of the ambient manifold \(M\).

Applying the Stokes theorem to the integral of the first term of (4.2) (as Simons did in [Si1]), we have

\[(4.4)\]
\[I(\xi, \xi) =: V''(\xi) = \int_N \langle L\xi, \xi \rangle \ast 1,\]

in which \(L\) is a self-adjoint, strongly elliptic linear differential operator of the second order acting on the space of sections of the normal bundle given by

\[(4.5)\]
\[L = -\Delta^D - \hat{A} - \hat{S},\]

where \(\Delta^D\) is the Laplacian operator associated with the normal connection, \(\langle A\xi, \eta \rangle = \text{trace} \langle A\xi, A\eta \rangle\), and \(\langle S\xi, \eta \rangle = \hat{S}(\xi, \eta)\).

The differential operator \(L\) is called the Jacobi operator of \(N\) in \(M\). The differential operator \(L\) has discrete eigenvalues \(\lambda_1 < \lambda_2 < \ldots \neq \infty\). We put \(E_\lambda = \{\xi \in \Gamma(T^\perp N) : L(\xi) = \lambda \xi\}\). The number of \(\sum_{\lambda<0} \dim(E_\lambda)\) is called the index of \(N\) in \(M\). A vector field \(\xi\) in \(E_0\) is called a Jacobi field.

A minimal submanifold \(N\) of \(M\) is said to be stable if \(V''(\xi) \geq 0\) for any normal vector field \(\xi\) of \(N\) in \(M\). Otherwise, \(N\) is said to be unstable. It is clear that \(N\) in \(M\) is stable if and only if the index of \(N\) in \(M\) is equal to 0.

Concerning the stability of totally real submanifold we mention the following result of [CLN] obtained in 1980.

**Proposition 4.1.** Let \(N\) be a compact \(n\)-dimensional minimal totally real submanifold in a real \(2n\)-dimensional Kaehlerian manifold \(M\). Then \(N\) is stable if and only if

\[(4.6)\]
\[I(JX, JX) = \int_N \{||\nabla X||^2 + S(X, X) - \tilde{S}(X, X)\} \ast 1\]

is non-negative for every tangent vector field \(X\) on \(N\), where \(S\) and \(\tilde{S}\) denote the Ricci forms of \(N\) and \(M\), respectively.

**Proof.** Under the hypothesis, we have

\[(4.7)\]
\[
D_X JY = J\nabla_X Y, \quad A_J X Y = A_J Y X, \quad \tilde{S} \cdot J = J \cdot \tilde{S}
\]
(see, pp.145-146 of [CO1]). From this we may obtain

$\bar{S}(X,X) = \tilde{S}(X,X) - \sum_{i=1}^{n} \tilde{R}(X,e_{i},e_{i},X)$.

From the equation of Gauss and (4.7) we may also obtain

$\bar{S}(JX,JX) = \tilde{S}(X,X) - S(X,X) - ||A_{JX}||^{2}$

since $N$ is minimal. From these we obtain the result.

By using Proposition 4.1 we have the following results of [CLN] also obtained in 1980.

**Theorem 4.2.** Let $N$ be a compact, totally real submanifold of a Kaehlerian manifold $(M,g,J)$ with $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} M$. Then we have

1. If $M$ has positive Ricci tensor and $N$ is stable, then $H^{1}(N;\mathbb{R}) = 0$ and
2. If $M$ has nonpositive Ricci tensor, then $N$ is always stable.

**Proof.** Let $\varphi$ be the 1-form dual to a vector field $X$ tangent to $N$. Then we have

$$\int_{N} \left( ||\nabla X||^{2} + S(X,X) \right) \ast 1 = \int_{N} \left\{ \frac{1}{2} ||d\varphi||^{2} + ||\delta X||^{2} \right\} \ast 1,$$

where $\delta$ is the codifferential operator. Thus (4.6) becomes

$$I(JX,JX) = \int_{N} \left\{ \frac{1}{2} ||d\varphi||^{2} + ||\delta X||^{2} - \tilde{S}(X,X) \right\} \ast 1$$

which implies the theorem.

**Proposition 4.3.** Let $N$ be a compact $n$-dimensional minimal totally real submanifold in a real $2n$-dimensional Kaehlerian manifold $M$. Then $N$ is stable if $N$ satisfies condition (1) or (2) below and $N$ is unstable if $N$ satisfies condition (3):

1. $i^{*} \tilde{S} \leq S$ where $i$ is the inclusion: $N \rightarrow M$.
2. $i^{*} \tilde{S} \leq 2S$ and the identity map of $N$ is stable as a harmonic map.
3. $i^{*} \tilde{S} > 2S$ and $N$ admits a nonzero Killing vector field.
Proof. Stability follows from (1) immediately by Proposition 4.1 from the (2) by Proposition 4.1 and the fact that the second variation for the identity map is $\int_N \{||\nabla X||^2 - S(X, X)\} \ast 1$ (cf. [S1]). Sufficiency of (3) follows from Proposition 4.1 and the formula $\int_N \{||\nabla X||^2 - S(X, X)\} \ast 1 = 0$ for Killing vector field $X$.

For two normal vector fields $\xi, \eta$ to a minimal submanifold $N$ in $M$, their index form is defined by

$$I(\xi, \eta) = \int_N \langle L\xi, \eta \rangle \ast 1.$$  \hspace{1cm} (4.8)

It is easy to see that the index form $I$ is a symmetric bilinear form: $I : T^\perp N \times T^\perp N \to \mathbb{R}$. For a vector subbundle $V$ of the normal bundle $T^\perp N$, we denote by $I_V$ the restriction of the index form on $V$. Thus, $I_V$ is a symmetric bilinear form on $V : I_V : V \times V \to \mathbb{R}$.

By the index of $I_V$, denoted by $\text{index}(I_V)$, we mean the number of negative eigenvalues of the index form $I_V$.

The normal bundle of a totally real submanifold $N$ in a Kaehlerian manifold $(M, g, J)$ has the following orthogonal decomposition:

$$T^\perp N = J(TN) \oplus \nu, \; J(TN) \perp \nu.$$  

For totally real minimal submanifold of a Kaehlerian manifold of higher codimension we have the following result of the author and J. M. Morvan.

**Proposition 4.4** Let $N$ be a compact, $n$-dimensional, minimal, totally real submanifold of a Kaehlerian manifold of complex dimension $n + p (p > 0)$. If $M$ has non-positive holomorphic bisectional curvatures, then the index form satisfies

$$I(\xi, \xi) + I(J\xi, J\xi) \geq 0$$  \hspace{1cm} (4.9)

for any normal vector field $\xi$ of $N$ in $M$.

**Proof.** Let $N$ be a compact, $n$-dimensional, minimal, totally real submanifold of a Kaehlerian manifold $M$ of complex dimension $n + p$ with $p > 0$. Then, for any normal vector field $\xi$ in the normal subbundle $\nu$ and vector fields $X, Y$ tangent to $N$, we have

$$\langle D_X \xi, JY \rangle = -\langle \tilde{\nabla}_X J\xi, Y \rangle = \langle A_{J\xi} X, Y \rangle.$$  \hspace{1cm} (4.10)
This implies
\[(4.11) \quad ||D\xi||^2 \geq ||A\xi||^2, \quad ||DJ\xi||^2 \geq ||A\xi||^2,\]
for any normal vector field $\xi$ in $\nu$. By using (4.2), (4.4), (4.10) and (4.11) we find
\[I(\xi,\xi) + I(J\xi,J\xi) \geq -\int_{\mathcal{N}} \sum_{i=1}^{n} \{ \tilde{R}(\xi,e_i,e_i,\xi) + \tilde{R}(J\xi,e_i,e_i,J\xi) \} * 1.\]
Therefore, if $M$ has non-positive holomorphic bisectional curvature, then, for any normal vector field $\xi$ in $\nu$, we have
\[I(\xi,\xi) + I(J\xi,J\xi) \geq 0.\]
This proves the Proposition.

**Example 4.1.** Let $N$ be any non-totally geodesic, minimal hypersurface of an $(n+1)$-dimensional flat real torus $\mathbb{R}T^{n+1}$ which is imbedded in a complex $(n+1)$-dimensional flat complex torus $\mathbb{C}T^{n+1}$ as a totally geodesic, totally real submanifold. Denote by $\xi$ a unit normal vector field of $N$ in $\mathbb{R}T^{n+1}$. Then we have
\[A\xi \neq 0, \quad D\xi = DJ\xi = 0, \quad A\xi = 0.\]
Since $\mathbb{C}T^{n+1}$ is flat, (4.2) and (4.3) yield
\[(4.12) \quad I(\xi,\xi) < 0, \quad I(J\xi,J\xi) > 0.\]
For the index of the index form $I_{J(TN)}$ we have the following result of the author and Morvan.

**Theorem 4.5.** Let $N$ be a compact, $n$-dimensional, totally real, minimal submanifold of a Kaehlerian manifold $M$ of complex dimension $n+p$. If $M$ has non-positive holomorphic bisectional curvatures, then
\[(4.13) \quad \text{index}(I_{J(TN)}) = 0.\]

**Proof.** Under the hypothesis, let $e_1, \ldots, e_{2n+2p}$ be a local orthonormal frame along the submanifold $N$ such that
\[e_{n+1} = Je_1, \ldots, e_{2n} = Je_n, \quad e_{2n+p+1} = Je_{2n+1}, \ldots, e_{2n+2p} = Je_{2n+p}.\]
Then $e_{n+1}, \ldots, e_{2n}$ form a local orthonormal frame of $J(TN)$ and $e_{2n+1}, \ldots, e_{2n+2p}$ form a local orthonormal frame of $\nu$. By applying Lemma 3.5 of Chapter II and the equation of Gauss, we may obtain

\begin{equation}
\sum_{i=1}^{n} \tilde{R}(X, e_i, e_i, X) - S(X, X) = \sum_{i=1}^{n} <h(X, e_i), h(X, e_i)>
\end{equation}

\begin{equation}
= ||A_{JX}||^2 + \sum_{i=1}^{2n+2p} \sum_{r=2n+1} \langle h(X, e_i), e_r \rangle^2.
\end{equation}

Combining (4.2) with (4.14) we may find

\begin{equation}
I(JX, JX) = \int_{N} \{||D JX||^2 + S(X, X) + \sum_{i,r} \langle h(X, e_i), e_r \rangle^2
\end{equation}

\begin{equation}
- \sum_{i=1}^{2n} \tilde{R}(X, e_t, e_t, X) \} * 1.
\end{equation}

As we did in [CLN] (see [C3, p.51]), put

$W = \nabla_X X + (\text{div} X) X$

where $\text{div} X$ denotes the divergence of $X$. Let $\varphi$ be the 1-form associated with $X$. Then, by computing the divergence of $W$ and applying the divergence theorem, we obtain

\begin{equation}
0 = \int_{N} \langle \text{div} W \rangle * 1 = \int_{N} \{S(X, X) + ||\nabla X||^2 - \frac{1}{2} ||d \varphi||^2 - (\delta \varphi)^2 \} * 1.
\end{equation}

Combining (4.15) and (4.16) we find

\begin{equation}
I(JX, JX) = \int_{N} \{||D JX||^2 + \sum_{i,r} \langle h(X, e_i), e_r \rangle^2
\end{equation}

\begin{equation}
- \sum_{i=1}^{2n} \tilde{R}(X, e_t, e_t, X) - ||\nabla X||^2 + \frac{1}{2} ||d \varphi||^2 + (\delta \varphi)^2 \} * 1,
\end{equation}

for any vector field $X$ tangent to $N$. 
For a totally real submanifold $N$ of a Kaehlerian manifold, the Gauss and Weingarten formulas imply

\begin{equation}
D_X JY = J\nabla_X Y + h(X, Y)
\end{equation}

which yields $||DJY|| \geq ||\nabla Y||$. Thus, by applying (4.17), (4.18) and the definition of the index, we obtain the theorem.

**Remark 4.1.** An $n$-dimensional totally real submanifold of a real $2n$-dimensional Kaehlerian manifold is also called a *Lagrangian submanifold* by some mathematicians.
§5. STABILITY OF TOTALLY GEODESIC SUBMANIFOLDS.

In this section we would like to present a general method introduced by the author, Leung and Nagano [CLN] obtained in 1980 for determining the stability of totally geodesic submanifolds in compact symmetric spaces. Since every irreducible totally geodesic submanifold of a Hermitian symmetric space is a slant submanifold [CN1], the method can be used to determine the stability of such slant submanifolds.

Recall that the second variation formula of a compact minimal submanifold $N$ in a Riemannian manifold is given by formula (4.2) (or, equivalently, by formula (4.4)). If $N$ is totally geodesic, then $A = 0$. So the stability obtained trivially when $\bar{S}$ is non-positive. For this reason we are interested in the case $M$ is of compact type. Also we assume $M$ is irreducible partly to preclude tori as $M$.

We need to fix some notations. Since $N$ is totally geodesic, there is a finitely covering group $G_N$ of the connected isometry group $G_oN$ of $N$ such that $G_N$ is a subgroup of the connected isometry group $G_M$ of $M$ and leaves $N$ invariant, provided that $G_oN$ is semi-simple. Let $P$ denote the orthogonal complement of the Lie algebra $g_N$ in the Lie algebra $g_M$ with respect to the bi-invariant inner product on $g_M$ which is compatible with the metric of $M$.

Every member of $g_M$ is though of as a Killing vector field because of the action $G_M$ on $M$. Let $\hat{P}$ denote the space of the vector fields corresponding to the member of $P$ restricted to the submanifold $N$.

**Lemma 5.1.** To every member of $P$ there corresponds a unique (but not canonical) vector field $v \in \hat{P}$, $v$ is a normal vector field and hence $\hat{P}$ is a $G_N$-invariant subspace of the space, $\Gamma(T^\perp N)$, of the sections of the normal bundle to $N$. Moreover, $\hat{P}$ is homomorphic with $P$ as a $G_N$-module.

**Proof.** Let $o$ be an arbitrary point of $N$. Let $K_M$ and $K_N$ denote the isotropy subgroups of $G_M$ and $G_N$ at $o$, respectively. Then $g_M/\kappa_M$ and $g_N/\kappa_N$ and $P/(P \cap \kappa_M)$ are identified with $T_oM$, $T_oN$ and $T_o^\perp N$ by isomorphisms induced by the evaluation of vector fields in $g_M$ at $o$. In particular, the value $v(o)$ of $v$ is normal to $N$. This proves the lemma.

Now we are ready to explain the method of [CLN].

The group $G_N$ acts on sections in $\Gamma(T^\perp N)$ and hence on the differential
operators: $\Gamma(T^\perp N) \to \Gamma(T^\perp N)$. $G_N$ leaves $L$ fixed since $L$ is defined with $N$ and the metric of $M$ only. Therefore, each eigenspace of $L$ is left invariant by $G_N$. Let $V$ be one of its $G_N$-invariant irreducible subspaces. We have a representation $\rho : G_N \to GL(V)$. We denote by $c(V)$ or $c(\rho)$ the eigenvalue of the corresponding Casimir operator.

To define $c(V)$ we fix an orthonormal basis $(e_\lambda)$ for $g_N$ and consider the linear endomorphism $C$ or $C_V$ of $V$ defined by

$$C = -\sum \rho(e_\lambda)^2.$$  

It is known that $C$ is $c(V)I_V$ (see Chapter 8 of [Bo1]), where $I_V$ is the identity map on $V$. In our case, the Casimir operator $C_V = -\sum [e_\lambda, [e_\lambda, V]]$ for every member $v$ of $V$ (after extending to a neighborhood of $N$).

Now, we state the following theorem of [CLN] which says, modulo details, that $N$ is stable if and only if $c(V) \geq c(P)$ for every $G_N$-invariant irreducible space $V$.

**Theorem 5.2.** A compact totally geodesic submanifold $N (= G_N/K_N)$ of a compact symmetric space $M (= G_M/K_M)$ is stable as a minimal submanifold if and only if one has $c(V) \geq c(P')$ for the eigenvalue of the Casimir operator of every simple $G_N$-module $V$ which shares as a $K_N$-module some simple $K_N$-submodule of the $K_N$-module $T^\perp_0 N$ in common with some simple $G_N$-submodule $P'$ of $\hat{P}$.

**Proof.** Given a point $p$ of $N$, we choose a basis of $g_N$ given by

$$(e_\lambda) = (\ldots, e_i, \ldots, e_\alpha, \ldots)$$

and a finite system $(e_r)$ of vectors in $P \subset g_M$ such that (1) $(e_i(p))_{1 \leq i \leq n}$ is an orthonormal basis for the tangent space $T_p N$, (2) $\nabla e_i = 0$, $1 \leq i \leq n$, at $p$, (3) $e_\alpha(p) = 0$, $n < \alpha \leq \dim g_N$, and (4) $(e_r(p))$ is an orthonormal basis for the normal space $T_p^\perp N$, which we can do as is well-known.

An arbitrary normal vector field $\xi$ is written as $\xi = \sum e_r e_r$ on a neighborhood of $p$ by Lemma 5.1. Since $N$ is totally geodesic in $M$, we have $D_X \xi = \bar{\nabla}_X \xi$ for $X$ tangent to $N$ and $\xi \in \Gamma(T^\perp N)$, where $\bar{\nabla}$ is the Riemannian connection of $M$. As the curvature tensor of a symmetric space we also have $\bar{R}(X,Y)Z = -[[X,Y],Z]$ for $X,Y,Z \in m =: g_m/k_m$. Therefore, by evaluating $L\xi$ and $C\xi$ at $p$, we obtain

$$L\xi = -\sum D_{e_i} D_{e_i} \xi - \sum \bar{S}_{rs} \xi^r e_s.$$
\[ = - \sum \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \xi - \sum \tilde{S}_{rs} \xi^r e_s, \]

where \( \tilde{S}_{rs} \) are the components of \( \tilde{S} \) in (4.3) and

\[ C \xi = - \sum [e_\lambda, [e_\lambda, \xi]] \]

\[ = - \sum \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \xi + \sum \xi^r (\tilde{\nabla}_{e_i} e_\alpha)^r \tilde{\nabla}_{e_i} e_\alpha \]

\[ = - \sum \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \xi - \sum \xi^r (\tilde{\nabla}_{e_i} e_\alpha)^r \tilde{\nabla}_{e_i} e_\alpha, \]

where for the vanishing of the second term we use the fact that \( e_i \) is a Killing vector field. Thus we find

\[ (L - C) \xi = \sum \xi^r (\tilde{\nabla}_{e_i} e_\alpha)^r \tilde{\nabla}_{e_i} e_\alpha - \sum \tilde{S}_{rs} \xi^r e_s \]

\[ = \sum (A_\alpha)^2 \xi - \tilde{S}(\xi, \xi), \]

where \( A_\alpha \) is the Weingarten map given by the restriction of the operator: \( X \to -\tilde{\nabla}_X e_\alpha \) on \( T_p N \) to the normal space \( T_{pN} \). This proves the following two statements:

(a) The difference \( L - C \) is an operator of order one and

(b) The difference \( L - C \) is given by a self-adjoint endomorphism \( \tilde{S} \) of the normal bundle \( T^\perp N \).

Now, because both \( L \) and \( C \) are \( GN \)-invariant, statement (b) implies

(c) The endomorphism \( \tilde{S} \) is \( GN \)-invariant.

The theorem follows from statements (a) through (c) easily when the isotropy subgroup \( K_N \) is irreducible on the normal space \( T_{pN} \equiv g_{st} / (g_N \oplus k_{st}) \).

In fact, \( \tilde{S} \) is then a constant scalar multiple of the identity map of \( T^\perp N \); \( \tilde{S} = k \cdot I \), by statements (a) through (c) and Schur’s lemma. \( N \) is stable if and only if the eigenvalues of \( L \) are all non-negative. Since \( L = (c(P') + k) \cdot I \) on the normal Killing fields (see Lemma 5.1), this is equivalent to say that \( 0 \leq c(V) + k = c(V) - c(P') \) for every simple \( GN \)-module \( V \) in \( \Gamma(T_{pN}^\perp) \) (which is necessarily contained in an eigenspace of \( L \) by \( \tilde{S} = k \cdot I \)), and the Bott-Frobenius theorem completes the proof.

In the general case, we decompose the normal space into the direct sum of simple \( K_N \)-modules: \( P' \oplus P'' \oplus \ldots \). Accordingly we have \( T^\perp N = E' \oplus E'' \oplus \ldots \), where \( E', E'', \ldots \), etc. are obtained from \( P', P'', \ldots \), etc., in the usual way by applying the action of \( GN \) to the vectors in \( P', P'', \ldots \).
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etc. Since $G_N$ leaves invariant $E', E'', \ldots$, the normal connection leaves invariant the section spaces $\Gamma(E'), \Gamma(E''), \ldots$. Hence, $L$ and $C$ leave these spaces invariant. (For this, the irreducible subspaces $P', P'', \ldots$ must be taken within eigenspaces of the symmetric operator $\bar{S}$ at the point $o$). In particular, the projections of $\Gamma(T^\perp N)$ onto $E', E'', \ldots$, etc. commute with $C$ and $L$. Thus one can repeat the argument for irreducible case to each of $E', E'', \ldots$ to finish the proof of the theorem.

Theorem 5.2 provides us an algorithm for stability which goes like this: One can compute $c(V)$ by the Freudenthal formula (cf. Chapter 8, p.120 of [Bo1]) once one knows the action $\rho$ of $G_N$ on $V$. So the rest is to know all the simple $G_N$-modules $V$ in $\Gamma(T^\perp N)$. This is done by means of the Frobenius theorem as reformulated by Bott, which asserts in our case that a simple $G_N$-module $V$ appears in $\Gamma(T^\perp N)$ if and only if it is a $K_N$-module which is isomorphic with a $K_N$-module of $T^\perp_o N$.

Remark 5.1. A reformulation of the method of [CLN] was given by Y. Ohnita in [Oh2].

Proposition 5.3. A compact totally geodesic submanifold $N$ of a compact symmetric space $M$ is unstable as a minimal submanifold if the normal bundle admits a nonzero $G_N$-invariant section and if the centralizer of $G_N$ in $G_M$ is discrete.

Proof. Let $\xi$ be a nonzero $G_N$-invariant normal vector field on $N$. We have $D\xi = 0$. In view of (4.2) we will show that $\bar{S}(\xi, \xi)$ is positive. The sectional curvature of a tangential 2-plane at a point $p \in N$ equals $||[e, f]||^2$ if (i) $e$ is a member of $g_N$, (ii) $f$ is that of $g_M$, (iii) $e(p)$ and $f(p)$ form an orthonormal basis for the 2-plane, and (iv) $\nabla e = \nabla f = 0$ at $p$. Therefore, $\bar{S}(\xi, \xi)$ fails to be positive only if $[e, f] = 0$ for every such $e$ and $f$ satisfying $\xi(p) \wedge f(p) = 0$. Since the isotropy subgroup $K_N$ at $p$ leaves the normal vector $\xi(p)$ invariant, we have $[e', f] = 0$ for every member $e'$ of $k_M$ and hence $[g_N, f] = 0$ if $\bar{S}(\xi, \xi) = 0$ at $p$. Such an $f$ generates a subgroup in the centralizer of $g_N$ in $g_M$. This contradicts to the assumption.

Example 5.1. Let $N$ be the equator in the sphere $M = S^n$. That $N$ is unstable follows from the Proposition 5.3 if one consider a unit $G_N$-invariant normal vector field to it. The centralizer in this case is generated by the antipodal map: $x \rightarrow -x$. Its orbit space is the real projective space $M'$. 
The projection: $M \to M'$ carries $N$ onto a hypersurface $N'$. The reflection in $N'$ is a member of $G_N$ by our general agreement on $G_N$ (if $n > 1$) and precludes the existence of non-vanishing $G_N$-invariant normal vector field to $N'$. It is clear by Theorem 5.2 that $N'$ is stable.

**Remark 5.2.** In general, if $N$ is a stable minimal submanifold of a Riemannian manifold $M$ and $M$ is a covering Riemannian manifold of $M'$, then the projection $N'$ of $N$ in $M$ is stable too. The example above shows the converse is false.

**Definition 5.1.** For a compact connected symmetric space $M$, there is a unique symmetric space $M^*$ of which $M$ and every connected symmetric space which is locally isomorphic with $M$ are covering Riemannian manifold of $M^*$. We call $M^*$ the bottom space of $M$. If $M$ is a group manifold, $M^*$ is the adjoint group $ad(M)$.

By applying Theorem 5.2 and Proposition 5.3 above, we may obtain the following result of [CLN].

**Theorem 5.4.** A compact subgroup $N$ of a compact Lie group $M$ is stable with respect to a bi-invariant metric on $M$ if (a) $N$ has the same rank as $M$ and (b) $M = M^*$, that is, $M$ has no nontrivial center.

**Proof.** The compact group manifold $M$ has $G_M = M_L \times M_R$, where $M_L$ is the left translation group $M \times \{1\}$ and $M_R$ the right translation group; here $M_R$ acts “to the left” too, that is, $(1, a)$ carries $x$ into $xa^{-1}$. Similarly for $G_N$. $G_N$ is effective on every invariant neighborhood of $N$ in $M$ by (b). We first consider the case where $N$ is a maximal toral subgroup $T$ of $M$. Let $A_T$ denote the subgroup $\{(a, a^{-1}) : a \in T\}$ of $G_N$. We have an epimorphism $\epsilon : K_N \times A_T \to G_T$ by the multiplication whose kernel is the subgroup of elements of order 2. In order to use Theorem 5.2, we look at an arbitrary simple $G_T$-module $V$ in $\Gamma(E')$ where $E'$ is, as before, the vector bundle $G_N P'$ defined from the simple $K_T$-submodule $P'$ of the normal space. $P'$ is a root space corresponding to a root $\alpha$ of $g_M$. With $V$ we compare the space $P'$, a simple $K_N$-module in $\Gamma(E')$ which is defined from the members of the Lie algebra of $M_L$ taking values in $P'$ at a point of $N$. We want to show $c(V) \geq c(P')$. Since $\alpha \neq 0$ by (a), both $V$ and $P'$ have dimension 2 and these are isomorphic as $K_T$-modules. The relationship between $V$ and $P'$ can be made more explicitly. Namely, a basis for $P'$ is a
global frame of $E'$ and so the sections in $V$ are linear combinations of the basis vectors whose components are functions on $N$. These functions form a simple $G_T$-module $F$ of dimension 2 and $V$ is a $G_T$-submodule of $F \otimes P'$. By the Bott-Frobenius theorem, $K_T$ acts trivially on a 1-dimensional subspace of $F$. Every weight $\varphi$ of $F$ is a linear combination of roots of $g_M$ whose coefficients are even numbers. In fact all the weights of the representations of $G_T$ are linear combinations of those roots over the integers by (a) and (b) and, since $K_T \cap A_T \cong \text{ker} \epsilon$ is trivial on $F$, the coefficients must be even.

On the other hand, if one looks at the definition of Casimir operator, $C = -\sum \rho(e_\lambda)^2$, one sees that the eigenvalue $c(V)$ is a sort of average of the eigenvalues of $-\rho(x)^2$, $||x|| = 1$, or more precisely, $c(V) = -(\text{dim } V)^{-1} \int \text{trace}(\rho(x))^2$, where the integral is taking over the unit sphere of the Lie algebra with an appropriately normalized invariant measure. For this reason, showing $c(V) \geq c(P')$, or equivalently, $(\varphi + \alpha)^2 - \alpha^2 \geq 0$ amounts to showing the inner product $<2\alpha + \varphi, \alpha> = <\varphi + \alpha, \varphi + \alpha> - <\alpha, \alpha> \geq 0$ in which we may assume that $\varphi$ is dominant (with respect to the Weyl group of $g_M$). This proves the case $N = T$.

We turn to the general case $N \supset T$. Assume $N$ is unstable and will show this contradicts the stability of $T$. There is then a simple $G_N$-module $V$ such that the second variation (4.2) is negative for some member $\xi$ of $V$. If we restrict $\xi$ to $T$ we still have a normal vector field but the integrand in (4.2) for $\xi|_T$ will differ from the restriction of the integrand for $\xi$ by the terms corresponding to the tangential directions to $N$ which are normal to $T$. However, a remedy comes from the group action. First (4.2) with $A = 0$ is invariant under $G_N$ acting on $V$. Second, every tangent vector to $N$ is carried into a tangent vector to $T$ by some isometry in $G_N$. Third, $N$ and $T$ are totally geodesic in $G$, but more importantly the connection and the curvature restrict to the submanifolds comfortably. And finally, the isotropy subgroup $K_N$ acts irreducibly on the tangent space to each simple or circle normal subgroup of $G_N$. From all these it follows that (4.2) for $\xi$ is a positive constant multiple of (4.2) for $\xi|_T$, as one sees by integrating (4.2) for $g(\xi)|_{g(T)}$, $g \in G$, over the group $G$ and over the unit sphere of $V$. This is a contradiction which completes the proof of the theorem.

**Remark 5.3.** Neither the assumption (a) or (b) can be omitted from Theorem 5.4 as the examples of $M = SU(2)$ with $N = SO(2)$ and $M = G_2$ with $N = SO(2)$ show. Also the Theorem will be false if $M$ is not a group.
manifold, a counter-example being \( M = M^* = GI \) with \( N = S^2 \cdot S^2 \) (local product).

By applying Theorem 5.2, Proposition 5.3 and Theorem 5.4 we may obtain the following results of [CLN].

**Proposition 5.5.** (a) Among the compact connected simple Lie groups \( M^* \), the only ones that have unstable \( M_+ \) are \( SU(n)^* \), \( SO(2n)^* \) with \( n \) odd, \( E_6^* \) and \( G_2 \).

(b) The unstable \( M_+ \) are \( G^C(k, n-k) \), \( 0 < k < n-k \), for \( SU(n)^* \); \( SO(2n)/U(n)^* \) for \( SO(2n)^* \); \( EIII^* \) for \( E_6^* \); and \( M_+^* \) for \( G_2 \).

(c) Every \( M_- \) is stable for the group \( M^* \).

**Comments on the Proof.** (I) The stability of \( M_- \) is immediate from Theorem 5.4 since \( M_- \) has the same rank as \( M \) (cf. [CN1, II]). Otherwise the proof is based on scrutinizing all the individual cases and omitted except for a few cases to illustrate our methods. (II) Take \( M^* = SO(2n+1) \). Then \( M_+ = G_+ / K_+ = G^R(k, 2n+1-k) \), \( 0 < k < n-k \), the Grassmannians of the unoriented \( k \)-planes in \( E^{2n+1} \) by Table I in [CN1, II]. The action of \( G_+ = SO(2n+1) \) on \( P \) (in the notation of Lemma 5.1) is the adjoint representation corresponding to the highest weight \( \tilde{\omega}_2 \) in Bourbaki’s notation [Bo1]. By Freudenthal’s formula, one finds that \( \tilde{\omega}_1 \) is the only representation that has a smaller eigenvalue than \( \tilde{\omega}_2 \); \( c(\tilde{\omega}_1) < c(\tilde{\omega}_2) \). But \( \tilde{\omega}_1 \) does not meet the Bott-Frobenius condition simply because its dimension \( 2n+1 \) is too small. Therefore \( M_+ \) is stable by Theorem 5.2. (III) Take \( SU(n)^* \) for another example. We know \( M_+ = G_+ / K_+ = G^C(k, n-k) \), the complex Grassmann manifold. If \( k \neq n-k \), \( M_+ \) is 1-connected and hence \( K_+ \) is connected. On the other hand, \( M_- = K_+ = SU(k) \times U(n-k) \), which contains a circle group as the center. Therefore, \( M_+ \) admits a unit \( G_+ \)-invariant normal vector field. Moreover, the centralizer of \( G_+ \) in \( G_M \) is trivial. Hence Proposition 5.3 applies to conclude that \( M_+ \) is unstable. This argument fails in the case \( k = n-k \) and we can conclude the stability of \( M_+ \) by Theorem 5.2 as in (I). (IV) Unstability is established by means of Proposition 5.3 except for the case of \( G_2 \). In this case we have \( c(\tilde{\omega}_1) < c(\tilde{\omega}_2) = c \) (the adjoint representation). This \( \tilde{\omega}_1 \) gives a monomorphism of \( G_2 \) into \( SO(7) \) which restricts to a monomorphism of \( K_+ = SO(4) \) into \( SO(4) \times SO(3) \) in \( SO(7) \) and then projects to \( SO(3) \). This implies that \( \tilde{\omega}_1 \) appears in a space of normal vector fields.
**Proposition 5.6.** Let $M^*$ be a compact symmetric space $G/K$ with $G$ simple. Then, among the $M_+$ and $M_-$, the unstable minimal submanifolds are $G^R(k, n-k), k < n-k,$ in $AI(n)^*$; $G^H(k, n-k), k < n-k,$ in $AII(n)^*$; $SO(k)$ in $G^R(k, k)$ with $k$ odd; $M_+ = M_- = SO(2) \times AI(n)$ in $CI(n)^*$; $M_+ = M_- = SO(2n)$ in $DIII^* = SO(2n)/U(n)$ with $n$ even; $G^H(2, 2)$ in $EI^*$; $FII$ in $EIV^*$; $AII(4)$ in $EV^*$; and $M_+ = M_- = S^2 \cdot S^2$ in $GI$.

**Comments on the Proof.** (I) In some cases, one can use another method to get the results quickly. For instance, if $M^*$ is Kaehlerian, then it is well-known that every compact complex submanifold is stable. (II) Mostly, unstability is established by using Proposition 5.3. In the cases, $M_+ = M_- = SO(2) \times L$, this proposition does not literally apply but unstability is proven in the same spirit. Consider, say $SO(2) \times AI(n)$ in $CI(n)^*$. This space in $M^*$ is $U(n)/O(n)$. The normal space is isomorphic with the space of the symmetric bilinear forms on $E^n$ as an $O(n)$-module. Therefore, there is a $U(n)$-invariant unit normal vector field $\xi$ on $M_+$. We have $\nabla \xi = 0$. We have to show $S(\xi, \xi) > 0$ in view of (4.2). Since $M_+ = M_-$ has the same rank as $M$, there is a tangent vector $X$ in $T_x M_-$ such that the curvature of the 2-plane spanned by $X$ and $\xi(y)$ is positive. (III) The case of $M_+ = M_- = S^2 \cdot S^2$ in $GI$. Precisely, $M_+ = M_-$ is obtained from $S^2 \times S^2$ = (the unit sphere in $E^3$) × (the unit sphere in $E^3$) ⊂ $E^3 \times E^3$ by identifying $(x, y)$ with $(−x, −y)$. The group $G_-$ for $M_- = G_-/K_-$ is the adjoint group but we have to take its double covering group $SO(4)$ to let it act on a neighborhood of $M_-$. The identity representation of $SO(4)$ on $E^4$ restricts to the normal representation of $K_-$ = $SO(2) \times SO(2)$ as somewhat detailed examination of the root system reveals. Therefore, $M_-$ is unstable. Similarly for $M_+$ which is congruent with $M_-.$

**Remark 5.4.** From the known facts about geodesics, one would not expect a simple relationship between stability and homology. More specifically, we remark that $M_+$ is homologous to zero for a group manifold $M^*$. The proof may go like this. Consider the quadratic map $f : x \mapsto s_x(o)$ on a symmetric space $M = G/K$ for a fixed point $o$, where $s_x$ is the symmetry at $x$. Assume $M$ is compact and orientable. Then $f$ has a nonzero degree if and only if the cohomology ring $H^*(M)$ is a Hopf algebra (cf. M. Clancy’s thesis, University of Notre Dame, 1980). On the other hand, the inverse image $f^{-1}(o)$ is exactly $M_+$ and $\{o\}$. Since $H^*(M)$ is a Hopf algebra for a group $M^*$, it follows that every $M_+$ is homologous to zero.
Finally, we give the following \cite{CLN}

**Proposition 5.7.** The minimal totally real totally geodesic submanifold $G^{R}(p, q)$ is unstable in $G^{C}(p, q)$.

**Proof.** Let $N = G_{N}/K_{N}$ be a totally real and totally geodesic submanifold of a compact Kaehlerian symmetric space $M = G_{M}/K_{M}$. Then $N$ will be unstable if we find $c(V) < c(P')$ as in Theorem 5.2. For each simple $P'$ in $\mathcal{P}$, there is a simple $g_{N}$-module $V$ in $\Gamma(E')$ whose members are normal vector fields $\xi = JX$ for some Killing vector field $X$ in $g_{N}$. This is obvious from the definition of a totally real submanifold. In the case of $G^{R}(p, q)$ in $G^{C}(p, q)$, $\mathcal{P}$ is simple and $c(P) = c(2\tilde{\omega}_{1}) > c(\tilde{\omega}_{2}) = c(g_{N})$, where $\tilde{\omega}_{2}$ denotes the highest weight in Bourbaki's notation (see, \cite{Bo1}) and $\tilde{\omega}_{1}$ is the only representation that has a smaller eigenvalue than $\tilde{\omega}_{2}$. This proves the Proposition.

**Remark 5.5.** If $p = 1$, Proposition 5.7 was due to \cite{LS1}.

**Remark 5.6.** The method of \cite{CLN} was used by several mathematicians in their recent studies. For results in this direction see, for instances, \cite{MT1}, \cite{MT2}, \cite{Oh2} and \cite{Ta1}.

**Remark 5.7.** For the fundamental theory of $(M_{+}, M_{-})$ and some of its applications see \cite{C3}, \cite{C7}, \cite{CN1}, \cite{CN3} and \cite{N2}.

Y. Ohnita (1987) improved the above algorithm to include the formulas for the index, the nullity and the Killing nullity of a compact totally geodesic submanifold in a compact symmetric space.

Let $f : N \to M$ be a compact totally geodesic submanifold of a compact Riemannian symmetric space. Then $f : N \to M$ is expressed as follows: There are compact symmetric pairs $(G, K)$ and $(U, L)$ with $N = G/K$, $M = U/L$ so that $f : N \to M$ is given by $uK \mapsto \rho(u)L$, where $\rho : G \to U$ is an analytic homomorphism with $\rho(K) \subset L$ and the injective differential $\rho : g \to u$ which satisfies $\rho(m) \subset p$. Here $u = l + p$ and $g = \mathfrak{k} + \mathfrak{m}$ are the canonical decompositions of the Lie algebras $u$ and $g$, respectively.

Let $m \perp$ denote the orthogonal complement of $\rho(m)$ with $p$ relative to the $\text{ad}(U)$-invariant inner product $(\ , \ )$ on $u$ such that $(\ , \ )$ induces the metric of $M$. Let $l \perp$ be the orthogonal complement of $\rho(l)$ in $l$. Put $g \perp = l \perp + m \perp$. Then $g \perp$ is the orthogonal complement of $\rho(g)$ in $u$ relative to $(\ , \ )$, and $g \perp$ is $\text{ad}_{\rho}(G)$-invariant. Let $\theta$ be the involutive automorphism of the symmetric
pair \((U, L)\). Choose an orthogonal decomposition \(\mathfrak{g}^\perp = \mathfrak{g}_1^\perp \oplus \cdots \oplus \mathfrak{g}_t^\perp\) such that each \(\mathfrak{g}_i^\perp\) is an irreducible \(\text{ad}_\rho(G)\)-invariant subspace with \(\theta(\mathfrak{g}_i^\perp) = \mathfrak{g}_i^\perp\). Then, by Schur’s lemma, the Casimir operator \(C\) of the representation of \(G\) on each \(\mathfrak{g}_i^\perp\) is \(a_i I\) for some \(a_i \in \mathbb{C}\).

Put \(m_i^\perp = m \cap \mathfrak{g}_i^\perp\) and let \(D(G)\) denote the set of all equivalent classes of finite dimensional irreducible complex representations of \(G\). For each \(\lambda \in D(G)\), \((\rho_\lambda, V_\lambda)\) is a fixed representation of \(\lambda\).

For each \(\lambda \in D(G)\), we assign a map \(A_\lambda\) from \(V_\lambda \otimes \text{Hom}_K(V_\lambda, W)\) to \(C^\infty(G, W)_K\) be the rule \(A_\lambda(v \otimes L)(u) = L(\rho_\lambda(u^{-1})v)\). Here \(\text{Hom}_K(V_\lambda, W)\) denotes the space of all linear maps \(L\) of \(V_\lambda\) into \(W\) so that \(\sigma(k) \cdot L = L \cdot \rho_\lambda(k)\) for all \(k \in K\).

Y. Ohnita’s formulas for the index \(i(f)\), the nullity \(n(f)\), and the Killing nullity \(n_k(f)\) are given respectively by:
(a) \(i(f) = \sum_{i=1}^t \sum_{\lambda \in D(G), a_\lambda < a_i} m(\lambda)d_\lambda\),
(b) \(n(f) = \sum_{i=1}^t \sum_{\lambda \in D(G), a_\lambda = a_i} m(\lambda)d_\lambda\),
(c) \(n_k(f) = \sum_{i=1, m_i^\perp \neq \{0\}}^t \dim \mathfrak{g}_i^\perp\),

where \(m(\lambda) = \dim \text{Hom}_K(V_\lambda, (m_i^\perp)^C)\) and \(d_\lambda\) denotes the dimension of the representation \(\lambda\).

By applying his formulas, Ohnita determined the indices, the nullities and the Killing nullities for all totally geodesic submanifolds in compact rank one symmetric spaces.
REFERENCES

A. Bejancu
[Be1] Geometry of CR-submanifolds, D. Reidel Publishing Co., Dordrecht-Boston-Lancaster-Tokyo, 1986

D. E. Blair
[Bl1] Contact Manifolds in Riemannian Geometry, Lecture Notes in Math., 509, Springer-Verlag, Berlin-New York, 1976.

D. E. Blair and B. Y. Chen
[BC] On CR-submanifolds of Hermitian manifolds, Israel J. Math., 34 (1979), 353-363.

J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward
[BJRW] On conformal minimal immersion of $S^2$ into $CP^n$, Math. Ann., 279 (1988), 599-620.

N. Bourbaki
[Bo1] Groupes et Algebres de Lie, Chapters 7 & 8, Hermann, Paris, 1975.

E. Calabi
[Ca1] Isometric embeddings of complex manifolds, Ann. of Math., 58 (1953), 1-23.
[Ca2] Metric Riemann surfaces, Ann. of Math. Studies, 58 (1953), 1-23.

B. Y. Chen,
[C1] Geometry of Submanifolds, Mercel Dekker, New York, 1973.
[C2] Differential geometry of real submanifolds in a Kaehler manifold, Monatsh. für Math., 91 (1981), 257-274.
[C3] Geometry of Submanifolds and Its Applications, Science University of Tokyo, 1981.
[C4.1] CR-submanifolds of a Kaehler manifolds, I, J. Differential Geometry,
[C4.2] CR-submanifolds of a Kaehler manifolds, II, J. Differential Geometry, 16 (1981), 493-509.

[C5] Total Mean Curvature and Submanifolds of Finite Type, World Scientific, Singapore-New Jersey-London-Hong Kong, 1984.

[C6] Slant immersions, Bull. Austral. Math. Soc., 41 (1990), 135-147.

[C7] A New Approach to Compact Symmetric Spaces and Applications: A report on joint work with Professor T. Nagano, Katholieke Universiteit Leuven, 1987.

B. Y. Chen, P. F. Leung and T. Nagano

[CLN] Totally geodesic submanifolds of symmetric spaces, III, preprint, 1980.

B. Y. Chen and G. D. Ludden

[CL1] Surfaces with mean curvature vector parallel in the normal bundle, Nagoya Math. J., 47 (1972), 161-167.

B. Y. Chen, G. D. Ludden and S. Montiel [CLM] Real submanifolds of a Kaehler manifold, Algebras, Groups and Geometries, 1 (1984), 176-212.

B. Y. Chen, C. S. Houh and H. S. Lue

[CHL] Totally real submanifolds, J. Differential Geometry, 12 (1977), 473-480.

B. Y. Chen and J. M. Morvan

[CM1] Propriété riemanniennes des surfaces lagrangiennes, C.R. Acad. Sc. Paris, Ser. I, 301 (1985), 209-212.

[CM2] Géométrie des surfaces lagrangiennes de $\mathbb{C}^2$, J. Math. Pures et Appl., 66, (1987), 321-335.

[CM3] Cohomologie des sous-variétés $\alpha$-obliques, C. R. Acad. Sc. Paris, 314 (1992), 931–934.

B. Y. Chen and T. Nagano

[CN1] Totally geodesic submanifolds of symmetric spaces, I & II, Duke
Math. J., 44 (1977), 745-755 & 45 (1978), 405-425.

[CN2] Harmonic metrics, harmonic tensors and Gauss maps, J. Math. Soc. Japan, 36 (1984), 295-313.

[CN3] A Riemannian geometric invariant and its applications to a problem of Borel and Serre, Trans. Amer. Math. Soc., 308 (1988), 273-297.

B. Y. Chen and K. Ogiue,

[CO1] On totally real submanifolds, Trans. Amer. Math. Soc., 193 (1974), 257-266.

B. Y. Chen and P. Piccinni

[CP1] Submanifolds with finite type Gauss map, Bull. Austral. Math. Soc., 35 (1987), 321-335.

B. Y. Chen and Y. Tazawa

[CT1] Slant surfaces of codimension two, Ann. Fac. Sc. Toulouse Math., 11 (1990), 29–43.

[CT2] Slant submanifolds in complex Euclidean spaces, Tokyo J. Math. 14 (1991), 101–120.

B. Y. Chen and L. Vanhecke

[CV] Differential geometry of geodesic spheres, J. Reine Angew. Math., 325 (1981), 28-67.

S. S. Chern

[Ch1] Complex Manifolds without Potential Theory, 2nd Ed., Springer-Verlarg, Berlin-New York, 1979.

P. Dazord

[D1] Une interprétation géométrique de la classe de Maslov-Arnold, J. Math. Pures et Appl., 56 (1977), 231-150.

F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken

[DOVV] On totally real 3-dimensional submanifolds of the nearly Kaehler 6-sphere, Proc. Amer. Math. Soc., 99 (1987), 741-749.
J. Eells and L. Lemaire
[EL1] A report on harmonic maps, *Bull. London Math. Soc.*, **10** (1978), 1-68.

[EL2] Another report on harmonic maps, *Bull. London Math. Soc.*, **20** (1988), 385-524.

C. Ehresmann

[Eh1] Sur la théorie des variétés feuilletées, *Rend. di Mat.*, **10** (1951), 64-82.

J. A. Erbacher

[Er1] Reduction of the codimension of an isometric immersion, *J. Differential Geometry*, **5** (1971), 333-340.

L. Gheysens, P. Verheyen and L. Verstraelen

[GVV] Characterization and examples of Chen submanifolds, *Jour. of Geometry*, **20** (1983), 47-72.

S. Greenfield

[G1] Cauchy-Riemann equations in several variables, *Ann. della Scuola Norm. Sup. Pisa*, **22** (1968), 275-314.

S. Helgason

[He1] *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New-York-San Francisco-London, 1978.

R. Harvey and H. B. Lawson

[HL1] Calibrated foliation, *Amer. J. Math.*, **104** (1982), 607-633.

D. A. Hoffman and R. Osserman,

[HO1] The Gauss map of surfaces in $R^3$ and $R^4$, *Proc. London Math. Soc.*, (3) **50** (1985), 27-56.

C. S. Houh

[Ho1] Some totally real minimal surfaces in $CP^2$, *Proc. Amer. Math. Soc.*, **40** (1973), 240-244.
E. Kähler

[Ka1] Über eine bemerkenswerte Hermitische Metrik, *Abh. Math. Sem. Univ. Hamburg*, 9 (1933), 173-186.

S. Kobayashi

[Ko1] Recent results in complex differential geometry, *Jahresber. Deut-sch. Math.-Verein.*, 83 (1981), 147-158.

[Ko2] *Differential Geometry of Complex Vector Bundles*, Iwanami & Princeton University Press, 1987.

S. Kobayashi and K. Nomizu

[KN1] *Foundations of Differential Geometry*, Volume I, John Wiley and Sons, 1963.

[KN2] *Foundations of Differential Geometry*, Volume II, John Wiley and Sons, 1969.

H. B. Lawson

[L1] *Lectures on Minimal Submanifolds*, Publish or Perish, Berkeley, 1980.

H. B. Lawson and J. Simons

[LS1] On stable currents and their applications to global problems in real and complex geometry, *Ann. of Math.*, 98 (1973), 427-450.

S. Maeda and S. Udagawa

[MU1] Surfaces with constant Kähler angle all of whose geodesics are circles in a complex space form, preprint, 1990.

K. Mashimo and H. Tasaki

[MT1] Stability of maximal tori in compact Lie groups, *Algebras, Groups and Geometries*, to appear.

[MT2] Stability of closed Lie subgroups in compact Lie groups, *Kodai Math. J.*, to appear.

Y. Miyaoka [Mi1] Inequalities on Chern numbers, *Sūgaku*, 41 (1989), 193-207.
J. D. Moore [Mr1] Isometric immersions of Riemannian products, *J. Differential Geometry*, 5 (1971), 159-168.

J. M. Morvan

[Mo1] Classe de Maslov d’une immersion lagrangienne et minimalité, *C.R. Acad. Sc. Paris*, 292 (1981), 633-636.

T. Nagano

[N1] Stability of harmonic maps between symmetric spaces, *Lectures Notes in Math.*, Springer-Verlag, 949 (1982), 130-137.

[N2] The involutions of compact symmetric spaces, *Tokyo J. Math.*, 11 (1988), 57-79.

K. Ogiue

[O1] Differential geometry of Kaehler submanifolds, *Advan. in Math.*, 13 (1974), 73-114.

[O2] Recent topics in submanifold theory, *Sûgaku*, 39 (1987), 305-319.

Y. Ohnita

[Oh1] Minimal surfaces with constant curvature and Kaehler angle in complex space forms, *Tsukuba J. Math.*, 13 (1989), 191-207.

[Oh2] On stability of minimal submanifolds in compact symmetric spaces, *Compositio Math.*, 64 (1987), 157-189.

B. O’Neill

[On1] *Semi-Riemannian Geometry*, Academic Press, New York-San Francisco-London, 1983.

B. Opozda

[Op1] Generic submanifolds in almost Hermitian manifolds, *Ann. Polon. Math.*, XLIX (1988), 115-128.

R. Osserman

[Os1] *Survey of Minimal Surfaces*, Van Nostrand Reinhold, New York, 1969.

[Os2] Curvature in the eighties, *Amer. Math. Monthly*, 1990.
A. Ros and L. Verstraelen

[RV1] On a conjecture of K. Ogiue, *J. Differential Geometry*, 19 (1984), 561-566.

P. Ryan

[Ry1] Homogeneity and some curvature conditions for hypersurfaces, *Tohoku Math. J.*, 21 (1969), 363-388.

J. A. Schouten and D. van Dantzig

[SD1] Über unitäre Geometrie, *Math. Ann.*, 103 (1930), 319-346.
[SD2] Über unitäre Geometrie konstanter Krümmung, *Proc. Kon. Nederl. Akad. Amsterdam*, 34 (1931), 1293-1314.

J. Simons

[Si1] Minimal varieties in Riemannian manifolds, *Ann. of Math.*, 88 (1968), 62-105.

I. M. Singer and J. A. Thorpe

[ST1] The curvature of 4-dimensional Einstein spaces, *Global Analysis*, Princeton University Press, 1969, 73-114.

R. T. Smyth

[S1] The second variation formula for harmonic mappings, *Proc. Amer. Math. Soc.*, 47 (1975), 229-236.

B. Smyth

[Sm1] Differential geometry of complex hypersurfaces, *Ann. of Math.*, 85 (1967), 246-266.

M. Spivak

[Sp1] *A Comprehensive Introduction to Differential Geometry*, Vol. 4, Publish or Perish, Berkeley 1979.

M. Takeuchi

[Ta1] Stability of certain minimal submanifolds of compact Hermitian sym-
metric spaces, *Tohoku Math. J.*, **36** (1984), 293-314.

[Ta2] On the fundamental group of the group of isometries of a symmetric space, *J. Fac. Sci. Univ. Tokyo, Sect. I*, **10** (1964), 88-123.

I. Vaisman

[V1] *Symplectic Geometry and Secondary Characterisstic Classes*, Birkhäuser, Boston, 1987.

A. Weil

[W1] Sur la théorie des formes différentielles attaché analytique complexe, *Comm. Math. Helv.*, **20** (1947), 110-116.

J. L. Weiner

[We1] The Gauss map for surfaces in 4-space, *Math. Ann.*, **269** (1984), 541-560.

A. Weinstein

[Wn1] *Lectures on Sympletic Manifolds*, Regional Conference Series in Mathematics, No. **29**, American Mathematical Society, Providence, 1977.

R. O. Wells, Jr.

[Wl1] *Differential Analysis on Complex Manifolds*, Springer-Verlag, Berlin-New York, 1980.

W. Wirtinger,

[Wi1] Eine Determinantenidentität und ihre Anwendung auf analytische Gebilde in Euclidischer und Hermitischer Massbestimmung, *Monatsch. Math. Phys.*, **44** (1936), 343-365.

K. Yano

[Y1] *Integral Formulas in Riemannian Geometry*, Mercel Dekkker, New York, 1970.

K. Yano and M. Kon

[YK1] *Anti-invariant Submanifolds*, Mercel Dekker, New York, 1976.
[YK2] CR-submanifolds of Kaehlerian and Sasakian Manifolds, Birkhäuser, Boston, 1983.
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