Uniqueness of a 3-D coefficient inverse scattering problem without the phase information

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Abstract

We use a new method to prove the uniqueness theorem for a coefficient inverse scattering problem without the phase information for the 3-D Helmholtz equation. We consider the case when only the modulus of the scattered wave field is measured and the phase is not measured. The spatially distributed refractive index is the subject of interest in this problem. Applications of this problem are in imaging of nanostructures and biological cells.

Keywords: phaseless data, inverse scattering problem, uniqueness theorem

1. Introduction

We consider an inverse problem of the determination of an unknown coefficient of the 3-D Helmholtz equation from measurements of only the modulus of the scattering part of the solution of this equation outside the scatterer. Since the phase of the complex valued wave field is not measured and since we search for an unknown coefficient of the 3-D Helmholtz equation, we call our problem coefficient phaseless inverse scattering problem (CPISP). The goal of this paper is to prove uniqueness theorem for this problem. The method of our proof is new. It was not used in proofs of previous uniqueness results for CPISPs \cite{13, 15--17, 23, 24}. Recall that it is assumed in the majority of publications about inverse scattering problems that both the phase and the modulus of the complex valued wave field are measured, see, e.g. \cite{4, 9, 10, 25, 28--30}. Note that publications \cite{28--30} are concerned with the scattering data measured at a single frequency.

Let $u_0$ be the incident wave field generated by a point source. Let $u_{sc}$ be the wave field, which occurs due to scattering by the scatterer. The total wave field is $u = u_0 + u_{sc}$. The goal of this paper is to prove uniqueness theorem for a CPISP in the case when the modulus $|u_{sc}|$ of the scattered wave field is measured on a certain surface. In the previous publication \cite{23}...
uniqueness was proven for the case when the modulus $|u|$ of the total wave field is measured. Compared with [23], the main difficulty here is caused by the interference of two wave fields: the total wave field and the incident wave field. In [24] uniqueness was also proven for the case when $|u_{sc}|^2$ is measured. However, there is one inconvenient condition of the uniqueness theorem of [24], see remark in section 2. Using the above mentioned new idea, we lift this condition here.

CPISPs have applications in imaging of nanostructures whose sizes are about 100 nm, which is 0.1 micron. Hence, the wavelength of the probing radiation must be also about 0.1 micron (0.1 $\mu$m). In this case the frequencies are millions of gigahertz [36]. It is well known that it is possible to measure only the intensity of the scattered wave at such huge frequencies, whereas the phase cannot be measured [8, 34, 40]. The intensity is the square of the modulus. To image a nanostructure, one needs to compute its unknown spatially distributed dielectric constant using measurements of only the intensity of the scattered wave field. Also, CPISPs have applications in optical imaging of biological cells, since their sizes are between 1−10 $\mu$m [35].

The light generated by lasers is used in this imaging. The diameter of the laser beam is a few millimeters (mm). Recall that 1 mm = $10^3$ $\mu$m. Hence, inside of the laser beam the intensity of this beam significantly exceeds the intensity of the portion of light scattered by nanostructures. In addition, it is well known that if a light detector is placed inside of this beam, then it is burned. Therefore, these detectors are placed outside of this beam. This means that so placed detectors measure only the intensity of the portion of light, which is scattered by those nanostructures, i.e. they measure $|u_{sc}|^2$. We point out that the precise mathematical model of the laser beam is outside of the scope of this publication. So, the above was given only to explain why it is important to consider the case when the function $|u_{sc}|^2$ rather than the function $|u|^2$ is assumed to be given outside of a scatterer.

For the first time, the uniqueness result for a CPISP was proven in [13]. This was done for the 1-D case. As to the 3-D case, first uniqueness theorems were proven in [15, 16] for the case when the Schrödinger equation

$$
\Delta v + k^2 v - q(x)v = -\delta(x-y), x \in \mathbb{R}^3 
$$

(1.1)

is the underlying one and the potential $q(x)$ is unknown. We note that equation (1.1) is easier to work with than with the Helmholtz equation. This is because, unlike the Helmholtz equation, the potential $q(x)$ is not multiplied by $k^2$ in (1.1). The multiplication of the unknown coefficient by $k^2$ prompts the use of the apparatus of the Riemannian geometry in the case of the Helmholtz equation, see [23, 24], which is unlike (1.1). In addition to uniqueness theorems, reconstruction procedures for 3-D CPISPs were developed by the authors both for the Schrödinger equation [20, 21] and for the Helmholtz equation [18, 19]. A modified reconstruction procedure of [18] was numerically implemented in [22].

Our CPISP is over-determined: the data depend on five variables, whereas the unknown coefficient depends on three variables. On the other hand, the authors are unaware about uniqueness results for 3-D coefficient inverse scattering problems which would not use over-determined data even in the case when both the phase and the modulus of the scattered waves are measured. As to the uniqueness theorems for non over-determined 3-D coefficient inverse problems without the phase information, we refer to [17] for the case of the Helmholtz equation with single measurement data. The price to pay for this is the assumption that the right hand side of the Helmholtz equation is a non-vanishing function $r(x)$, whereas the right hand side in [23, 24] and the current paper is the $\delta-$function. In addition, similar results for the Schrödinger equation are in theorems 3 and 4 of [15] and in theorem 2 of [16]. In all these
latter results for single measurement data the method of [7] is applied on the last step of the proof. This method is based on Carleman estimates, also see, e.g. the recent survey of this method in [14].

CPISPs were also considered by Novikov in [31, 32]. Statements of CPISPs in [31, 32] differ from ours in some respects. In these publications, uniqueness theorems are proven and reconstruction procedures are developed.

Recall that a CPISP is about the reconstruction of an unknown coefficient from phaseless measurements. Along with reconstructions of unknown coefficients in CPISPs, phaseless inverse problems of the reconstruction of unknown surfaces of scatterers are also attractive. In this regard, we refer to [1–3, 11, 12, 26] for numerical solutions of some inverse scattering problems without the phase information in the case when the surface of a scatterer was reconstructed. In addition, in [44] the phaseless inverse problem of the reconstruction of a source was considered.

In section 2 we formulate our CPISP as well as the uniqueness theorem of our paper. In section 3 we prove this theorem.

2. Statement of the problem

Below \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Consider a non-magnetic and non-conductive medium, which occupies the entire space \( \mathbb{R}^3 \). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain. Let \( S \in C^2 \) be a surface, which is the boundary of another bounded convex domain \( G \subset \mathbb{R}^3, S = \partial G \). We assume that \( \Omega \subseteq G \). Hence, \( S \cap \Omega = \emptyset \), although the surface \( S \) might be the boundary of \( \Omega \). Let \( n^2(x) \) be the spatially varying dielectric constant of the medium, where \( n(x) \) is the refractive index. We assume below that the function \( n(x) \) satisfies the following conditions:

\[
\begin{align*}
n(x) &\in C^{15}(\mathbb{R}^3), \\
n(x) &\geq 1 \text{ in } \mathbb{R}^3, \\
n(x) &= 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega.
\end{align*}
\]  

Condition (2.2) means that the refractive index of the medium is not less than the refractive index of the vacuum, which equals 1. Condition (2.3) means that the vacuum is outside of the domain \( \Omega \). To explain the smoothness condition (2.1), we note that lemma 1 formulated below follows from results of [18, 23]. On the other hand, those results of [18] use the fundamental solution of the hyperbolic equation

\[
n^2(x)\nabla^2 v = \Delta v.
\]  

The construction of this solution works only if \( n(x) \in C^{15}(\mathbb{R}^3) \) [18, 38]. In addition, the constructions of [18, 38] require the regularity of geodesic lines, see Condition below. We also note that the minimal smoothness requirements for unknown coefficients are rarely a significant concern in uniqueness theorems for multidimensional coefficient inverse problems, see, e.g. [29, 30], theorem 4.1 in section 4 of [37] and [14].

The function \( n(x) \) generates the conformal Riemannian metric,

\[
d\tau = n(x) \|dx\|, \quad |dx| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}.
\]  

We now formulate the assumption of the regularity of geodesic lines:
2.1. Assumption of the regularity of geodesic lines

Geodesic lines generated by the metric (2.5) are regular. In other words, each pair of points \( x, y \in \mathbb{R}^3 \) can be connected by a single geodesic line \( \Gamma(x, y) \).

A sufficient condition of the regularity of geodesic lines is [39]

\[
\sum_{i,j=1}^{3} \frac{\partial^2 \ln n(x)}{\partial x_i \partial x_j} \xi_i \xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^3, \forall x \in \mathbb{R}^3.
\]

For an arbitrary pair of points \( x, y \in \mathbb{R}^3 \) consider the travel time \( \tau(x, y) \) between them due to the Riemannian metric (2.5). Then [37]

\[
\tau(x, y) = \int_{\Gamma(x, y)} n(\xi) d\sigma,
\]

where \( d\sigma \) is the Euclidean arc length.

Let \( y \in \mathbb{R}^3 \) be the position of the point source, \( r = |x - y| \) and \( k > 0 \) be the wavenumber. We consider the Helmholtz equation with the radiation condition at the infinity

\[
\Delta u + k^2 n^2(x) u = -\delta(x - y), \quad x \in \mathbb{R}^3,
\]

\[
\partial_r u - i k u = o\left(1/r\right), \quad r \to \infty.
\]

Let \( u_0 \) be the incident spherical wave and \( u_{sc} \) be the scattered wave,

\[
u_{0}(x, y, k) = A_0(x, y) e^{ik|x-y|}, \quad A_0(x, y) = \frac{1}{4\pi |x-y|},
\]

\[
u_{sc}(x, y, k) = u(x, y, k) - u_0(x, y, k).
\]

We model the propagation of the electric wave field in \( \mathbb{R}^3 \) by the solution of the problem (2.7) and (2.8). This model was justified numerically in [6] in the case of the time domain. Numerical results of section 7.2.2 of [6] demonstrate that this model can replace the modeling via the full Maxwell’s system, provided that only a single component of the electric field is incident upon the medium. Then this component dominates two other components while propagating through the medium. Furthermore, the propagation of this component is well governed by the single PDE (2.4), which is the time domain analog of equation (2.7), see figure 19(b) in [6] and the discussion in the paragraph just above section 8 of [6]. This conclusion was verified via accurate imaging using electromagnetic experimental data in, e.g. section 5 of [5] and [41, 42].

Let \( (a, b) \subset \{ k : k > 0 \} \) be an arbitrary interval. Our interest in this paper is in the following CPISP.

2.2. Coefficient phaseless inverse scattering problem

Suppose that the function \( n(x) \) satisfies conditions (2.1)–(2.3). Assume that the following function \( f(x, y, k) \) is given

\[
f(x, y, k) = |u_{sc}(x, y, k)|^2, \quad \forall x, y \in S, x \neq y, \forall k \in (a, b).
\]

Determine the function \( n(x) \) for \( x \in \Omega \).
For an arbitrary number $\theta > 0$ denote $\mathbb{C}_\theta = \{ z \in \mathbb{C} : \text{Im} \, z > -\theta \}$.

**Lemma 1.** Choose an arbitrary bounded domain $G_1 \subset \mathbb{R}^3$ such that $G \subset G_1$ and $S \cap \partial G_1 = \emptyset$. Then there exists a number $\theta = \theta (G_1) > 0$ such that for each $y \in \mathbb{R}^3$ and for all $x \in G_1$, $x \neq y$ the function $u_\infty (x,y,k)$ is analytic as the function of $k \in \mathbb{C}_\theta$ and the function $f (x,y,k)$ is analytic as the function of $k \in \mathbb{R}$. Next, for any pair $x, y \in \mathbb{R}^3$, $x \neq y$ the asymptotic behavior of the function $u_\infty (x,y,k)$ as $k \to \infty$ is

$$u_\infty (x,y,k) = A(x,y)e^{ik\tau (x,y)} - A_0 (x,y)e^{ik|x-y|} + \hat{u} (x,y,k),$$

(2.12)

where the function $\hat{u} (x,y,k) = O (1/k)$ as $k \to \infty$ and the function $A(x,y) > 0$. Furthermore, the function $\hat{u} (x,y,k)$ is such that

$$\frac{\partial}{\partial k} \hat{u} (x,y,k) = O \left( \frac{1}{k} \right), \quad k \to \infty.$$  

(2.13)

**Remark 1.**

1. Since by lemma 1 the function $f (x,y,k)$ is analytic with respect to $k \in \mathbb{R}$ for any fixed pair of points $x, y \in S$, $x \neq y$, then the uniqueness of the analytic continuation implies that the values of this function for $k \in (a,b)$ uniquely define its values for all $k \in \mathbb{R}$. This fact is used in our proof of theorem 1.

2. Since $S \cap \Omega = \emptyset$, then it follows from (2.3) and (2.11) that measurements are performed outside of the domain $\Omega$ where possible heterogeneities are. Thus, (2.11) removes an inconvenient assumption of [24], which requires to perform measurements at a surface, which is located inside of the domain with heterogeneities.

We assume below that conditions (2.1)–(2.3) as well as the assumption of the regularity of geodesic lines hold true and devote the rest of this paper to the proof of theorem 1.

### 3. Proof

Fix a pair of points $x, y \in S$, $x \neq y$. Then the function $f (x,y,k)$ is determined uniquely for all $k \in \mathbb{R}$, see item 1 of remark 1. Consider the asymptotic expansion of the function $f (x,y,k)$ at $k \to \infty$. Denote

$$\alpha = \alpha (x,y) = \tau (x,y) - |x-y|.$$  

(3.1)

By lemma 1 the function $f (x,y,k)$ is known for all $k > 0$. It follows from (2.11)–(3.1) that

$$f (x,y,k) = A^2 (x,y) + A_0^2 (x,y) - 2A(x,y)A_0 (x,y) \cos (k\alpha) + \hat{f} (x,y,k),$$

(3.2)

where the functions $\hat{f} (x,y,k) = O (1/k)$ and $\partial_k \hat{f} (x,y,k) = O (1/k)$ as $k \to \infty$. We prove below that the number $\alpha$ can be uniquely recovered from the function $f (x,y,k)$.
First, we find $A(x, y)$. To do this, we prove first that

$$\lim_{k' \to \infty} \sup_{k \in (k', \infty)} \left[ -2A(x, y)A_0(x, y) \cos (k \alpha) + \hat{f} (x, y, k) \right] = 2A(x, y)A_0(x, y). \tag{3.3}$$

Indeed, since $A(x, y) > 0$ and $A_0(x, y) > 0$, we have

$$\sup_{k \in (k', \infty)} \left[ 2A(x, y)A_0(x, y) \cos (k \alpha) + \hat{f} (x, y, k) \right]$$

$$\geq \sup_{k \in (k', \infty)} \left[ 2A(x, y)A_0(x, y) \cos (k \alpha) \right] - \sup_{k \in (k', \infty)} |\hat{f} (x, y, k)|$$

$$= 2A(x, y)A_0(x, y) - \sup_{k \in (k', \infty)} |\hat{f} (x, y, k)|.$$

On the other hand,

$$\sup_{k \in (k', \infty)} \left[ 2A(x, y)A_0(x, y) \cos (k \alpha) + \hat{f} (x, y, k) \right] \leq 2A(x, y)A_0(x, y) + \sup_{k \in (k', \infty)} |\hat{f} (x, y, k)|.$$

Hence, we have obtained that

$$2A(x, y)A_0(x, y) - \sup_{k \in (k', \infty)} |\hat{f} (x, y, k)|$$

$$\leq \sup_{k \in (k', \infty)} \left[ 2A(x, y)A_0(x, y) \cos (k \alpha) + \hat{f} (x, y, k) \right] \leq 2A(x, y)A_0(x, y) + \sup_{k \in (k', \infty)} |\hat{f} (x, y, k)|. \tag{3.4}$$

We also have $\sup_{k \in (k', \infty)} |\hat{f} (x, y, k)| = O (1/k')$. Hence, taking the limit in first and third lines of (3.4) as $k' \to \infty$, we obtain (3.3). Therefore,

$$f^*(x, y) := \lim_{k' \to \infty} \sup_{k \in (k', \infty)} f (x, y, k) = (A(x, y) + A_0(x, y))^2. \tag{3.5}$$

The number $A(x, y) > 0$ can be uniquely found from (3.5) as

$$A(x, y) = \sqrt{f^*(x, y)} - A_0(x, y).$$

Introduce the functions $g$ and $p$,

$$g(x, y, k) = \frac{A^2(x, y) + A_0^2(x, y) - f (x, y, k)}{2A(x, y)A_0(x, y)},$$

$$p(x, y, k) = \frac{1}{2A(x, y)A_0(x, y)} \hat{f}(x, y, k).$$

Then equation (3.2) can be rewritten in the form

$$g(x, y, k) = \cos(k \alpha) - p(x, y, k), \tag{3.6}$$

where the function $g(x, y, k)$ is known and $p(x, y, k) = O(1/k), \partial_k p(x, y, k) = O(1/k).$

First, if $\alpha \neq 0$, then it follows from (3.6) that the limit
\[
\lim_{k \to \infty} g(x, y, k) \tag{3.7}
\]
does not exist. On the other hand, if \( \alpha = 0 \), then \( \lim_{k \to \infty} g(x, y, k) = 0 \). Thus, we have established that the limit (3.7) exists if and only if (see (3.1))

\[
\tau(x, y) = |x - y|. \tag{3.8}
\]

Since the function \( g \) is known, then one can establish whether or not the limit (3.7) exists, thus establishing whether or not (3.8) holds.

Assume now that (3.8) does not hold. Hence, \( \alpha \neq 0 \). By (2.6) and (3.1)

\[
\alpha(x, y) > 0. \tag{3.9}
\]

We show now how to find the number \( \alpha(x, y) \). First we show that there exist a countable number of zeros \( k_n \) of the function \( g(x, y, k) \) and

\[
\lim_{n \to \infty} k_n = \infty. \tag{3.10}
\]

We have:

\[
g(x, y, k) = 0 \iff \cos(\alpha(x, y)) = p(x, y, k). \tag{3.11}
\]

Let

\[
k\alpha \in (\pi(n-1), \pi n) \tag{3.12}
\]

for a sufficiently large integer \( n > 1 \). Then \( p(x, y, k) = O(1/n) \) and also \( \partial_k p(x, y, k) = O(1/n) \) as \( n \to \infty \). Rewrite the equation \( \cos(\alpha(x, y)) = p(x, y, k) \) in the form

\[
\sin(\pi/2 + k\alpha) = -p(x, y, k). \tag{3.13}
\]

Since \( |p(x, y, k)| < 1 \) for sufficiently large \( n \) for \( k \) satisfying (3.12), then (3.13) is equivalent with

\[
k\alpha = \frac{\pi}{2} + (-1)^{(n+1)} \arcsin p(x, y, k) + n\pi. \tag{3.14}
\]

For \( k \) satisfying (3.12), consider the function \( F_n(k, x, y) \),

\[
F_n(k, x, y) = k\alpha + \frac{\pi}{2} + (-1)^{(n)} \arcsin p(x, y, k) - n\pi.
\]

Then

\[
\partial_k F_n(x, y, k) = \alpha + O\left(\frac{1}{n}\right), \quad n \to \infty.
\]

Hence, \( \partial_k F_n(x, y, k) > 0 \) for sufficiently large \( n \). This means that the function \( F_n(x, y, k) \) is monotonically increasing with respect to \( k \) on the interval (3.12). Next,

\[
F_n(x, y, (n-1)\pi/\alpha) = -\frac{\pi}{2} + O\left(\frac{1}{n}\right), \quad F_n(x, y, n\pi/\alpha) = \frac{\pi}{2} + O\left(\frac{1}{n}\right). \tag{3.15}
\]

Since the function \( F_n(x, y, k) \) is continuous with respect to \( k \) and has different signs on the edges of the interval (3.12), then the mon monotonicity of this function implies that it has unique zero inside of this interval. Denote this zero by \( \hat{k}_n \). The asymptotic formula for these zeros follows from (3.14):
\[ k_n \alpha = \frac{\pi}{2} + n\pi + O\left(\frac{1}{n}\right), \quad n \to \infty. \]

Hence,
\[ \alpha (k_{n+1} - k_n) = \pi + O\left(\frac{1}{n}\right), \quad n \to \infty. \quad (3.16) \]

Therefore the number \( \alpha \) is uniquely determined as
\[ \alpha = \lim_{n \to \infty} \frac{\pi}{k_{n+1} - k_n}. \quad (3.17) \]

Finally, since we know the function \( g(x,y,k) \) as the function of \( k \) and numbers \( k_n \) are zeros of this function, then we know the numbers \( k_n \) as well. Hence, (3.1) and (3.17) imply that the number \( \tau(x,y) \) is determined uniquely from the function \( f(x,y,k) \) in (2.11) for any fixed pair of points \( x,y \in S, x \neq y \). Therefore, the first assertion of theorem 1 is proved.

We now prove the second assertion of theorem 1. To do this, we apply theorem 3.4 of section 3 of the book [37]. This is the theorem about stability and uniqueness of the so-called inverse kinematic problem of seismic in the 3-D case. We use notations of that theorem for the convenience of the reader. It follows from the assumption of the regularity of geodesic lines and condition (2.3) that the Riemannian metric is simple in any convex domain \( P \subset \mathbb{R}^3, \Omega \subset P \). Recall that a metric is called simple in \( P \) if any two points \( x \) and \( y \) can be connected by a single geodesic line lying in \( P \) and the boundary \( \partial P \) is convex with respect to geodesic lines. As to the 2-D case, for the first time the stability and uniqueness theorem was proved in [27]. The most general 2-D result in this direction is the one of [33] about the rigidity of a simple 2-D metric.

Suppose that there exist two coefficients \( n_1(x) \) and \( n_2(x) \), which generate the same function \( f(x,y,k) \) in (2.11). Consider corresponding functions \( \tau_1(x,y) \) and \( \tau_2(x,y) \) defined via (2.6),
\[ \tau_1(x,y) = \int_{\Gamma_1(x,y)} n_1(\xi)d\sigma, \quad \tau_2(x,y) = \int_{\Gamma_2(x,y)} n_2(\xi)d\sigma, \forall x,y \in \mathbb{R}^3, \]
where \( \Gamma_1(x,y) \) and \( \Gamma_2(x,y) \) are geodesic lines generated by functions \( n_1(x) \) and \( n_2(x) \) respectively. Then the first assertion of theorem 1 implies that
\[ \tau_1(x,y) = \tau_2(x,y), \forall x,y \in S. \quad (3.18) \]
It follows from (2.2) that
\[ n_1(x), n_2(x) \geq 1. \quad (3.19) \]
Using (2.1) and (3.19), we obtain that there exists a number \( n_{00} > 1 \) such that
\[ \|n_1(x)\|_{C^1(\Gamma)}, \|n_2(x)\|_{C^1(\Gamma)} \leq n_{00}. \quad (3.20) \]

Let \( \Lambda(1,n_{00}) \) be the set of functions \( n(x) \) defined in \( \overline{G} \) and satisfying the following conditions:

1. the function \( n(x) \in C^{15}(\overline{G}) \), \( \|n(x)\|_{C^1(\overline{G})} \leq n_{00} \) and also \( n(x) \geq 1 \) in \( \overline{G} \).

2. The function \( n(x) \) satisfies condition of section 2 about the regularity of geodesic lines generated by metric (2.5).

By (3.18)–(3.20) functions \( n_1(x), n_2(x) \in \Lambda(1,n_{00}) \). Thus, (3.18) and the estimate (3.66) of theorem 3.4 of section 3 of the book [37] imply that \( n_1(x) \equiv n_2(x) \). \( \square \)
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