Intrinsic anyonic spin through deformed geometry

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Abstract

The properties of the deformed bosonic oscillator, and the quantum groups $U_q(SL(2))$ and $GL_q(2)$ in the limit as their deformation parameter $q$ goes to a root of unity are investigated and interpreted physically. These properties are seen to be related to fractional supersymmetry and intrinsic anyonic spin. A simple deformation of the Klein-Gordon equation is introduced, based on $GL_q(2)$. When $q$ is a root of unity this equation is a root of the undeformed Klein-Gordon equation.
1 Introduction

In a series of recent papers [1, 2, 3, 4, 5], a novel interpretation of supersymmetry and its generalization to fractional supersymmetry has been developed. The observation upon which these results are based is that when the deformation parameter $q$ is a root of unity, there are circumstances in which twice as many variables as usual are needed to fully describe the braided line [6, 7, 8], the extra variables being essential if one is to retain in this limit all of the structure associated with the braided line at generic $q$ (for the purposes of the present paper generic $q$ is taken to mean $|q| = 1$, with $q$ not equal to a root of unity). It was remarked in various places in these papers that this basic feature is likely to be widespread in the theory of Hopf algebras and braided Hopf algebras, and the present paper is devoted to working out some new and physically interesting examples.

We begin in section 3 by taking the $q \to \epsilon$ ($\epsilon$ a nontrivial primitive $n$th root of unity) limit of the $q$-deformed bosonic oscillator [3, 10, 11] or $q$-oscillator, and establish that in this limit it decomposes into two independent oscillators, one of them an ordinary (undeformed) boson and the other an anyon (we use the word anyon in the sense of [8]). The corresponding decomposition of the Fock space when $q \to \epsilon$ is also studied. For certain values of $q$ it turns out that in this limit both parts of the decomposed Fock space (it now has the form of a direct product) are physical (i.e. all states have positive definite norm). Since there exist $q$-oscillator realizations of all of the deformed enveloping algebras $\mathcal{U}_q(g)$ [12], it is reasonable to expect these to exhibit analogous decompositions when $q \to \epsilon$.

In section 4 we illustrate this for the simplest case, $\mathcal{U}_q(sl(2))$. Utilizing the $q$-Schwinger realization [10] we establish, for this realization, and the corresponding highest weight representations the decomposition of $\mathcal{U}_q(sl(2))$ into the direct product of undeformed $\mathcal{U}(sl(2))$, and $\mathcal{U}_\epsilon(sl(2))$ the naive version of $\mathcal{U}_q(sl(2))$ at $q = \epsilon$ obtained by simply setting $q = \epsilon$. To go further we draw on the detailed studies of $\mathcal{U}_q(sl(2))$ when $q$ is a root of unity, carried out by Kac [13], Lusztig [14, 15], Arnaudon [16], and collaborators. Using these we place restrictions on $\mathcal{U}_q(sl(2))$ as a Hopf algebra at $q = \epsilon$, finally obtaining a restricted form of the $\mathcal{U}_q(sl(2))$ Hopf algebra which has irreducible representations identical (up to a constant factor) to those associated with the Schwinger realization. We then extend this Hopf algebra, introducing new elements which are endowed with that part of the Hopf structure of $\mathcal{U}_q(sl(2))$ which is usually lost when we set $q = \epsilon$, and obtain thereby an FSUSY-like generalization of the $su(2)$ symmetry algebra. Although the extended algebra does not have direct product form, all of its highest weight representations do, and using our earlier work with the Schwinger realization, we are able to choose the basis in which this is most clearly manifested. This enables us to interpret such highest weight representations as being the direct product of an intrinsic spin degree of freedom and an orbital angular momentum degree of freedom.
It is reasonable to expect analogous results for \( SL_q(2) \), the Hopf algebra dual to \( U_q(sl(2)) \) \[19\]. In fact the extension to all of \( GL_q(2) \) is not difficult, and this is the topic of section 5. When \( q \) is a primitive \( \tilde{n} \)th root of unity, we find that the unrestricted quantum group \( GL_q(2) \), the matrices of which we denote by \( w \), has sub-Hopf algebra \( GL_{q^2}(2) \) (here \( n = \tilde{n} \) for odd \( \tilde{n} \) and \( n = \frac{\tilde{n}}{2} \) for even \( \tilde{n} \)), the matrices of which we denote by \( W \). In the case of odd \( \tilde{n} \) this sub-Hopf algebra is identical to the undeformed quantum group \( GL(2) \), and moreover, it is central. Setting \( \det_q w = 1 \) leads directly to \( \det_q W = 1 \), so that all of these results hold equally for \( SL_q(2) \). By imposing a suitable \( * \)-structure on \( GL_q(2) \), we find that for \( q^{n^2} = 1 \) we can identify the elements of \( W \) with linear combinations of the momenta of undeformed Minkowski spacetime. The quantum determinant of \( W \) is just the undeformed Laplacian, and the quantum determinant of \( w \) has the form of a \( q \)-deformed Laplacian.

This deformed Laplacian turns out to be an \( n \)th root of the undeformed Laplacian, an observation which enables us to construct (anyonic) Dirac-like square roots of the Klein-Gordon equation.

A restricted version of \( GL_q(2) \) can also be built. In this case \( W \) only contains two elements. These are grouplike and, for \( SL_q(2) \), mutually inverse. As in the case of \( U_q(sl(2)) \) we can extend the restricted version of \( GL_q(2) \) by elements which carry the Hopf structure lost from the generic case. We thereby obtain a new quantum group which involves a FSUSY-like coproduct.

## 2 Notation and conventions

In this section we establish our notation and a few results of which we will later have need. We begin by defining,

\[
[X] = \frac{q^X - q^{-X}}{q - q^{-1}} = q^{1-X}[X]_{q^2},
\]

A consequence of the relationship between \([X]\) and \([X]_{q^2}\), is that the \( q \) used in the present paper corresponds to the square root of the \( q \) used in \[1, 2, 3, 4, 5\], so that for example super/fermionic properties are associated here with \( q = \pm i \) rather than with \( q = -1 \). For nonnegative integer \( r \) we also define

\[
[r]! = \begin{cases} 
[r][r-1][r-2]...[2][1], & \text{for integer } r > 0, \\
1, & \text{for } r = 0 
\end{cases},
\]

\[
\left[ \begin{array}{c} m \\ r \end{array} \right] = \frac{[m]!}{[m-r]![r]!}.
\]

These factorials are related to the \([r]_{q^2}!\) factorials given in \[1, 2\] via

\[
[m]_{q^2}! = q^{\frac{m(m-1)}{2}}[m]! ,
\]

\[
[m]_{q^{-2}}! = q^{\frac{m(m-1)}{2}}[m]! .
\]
Let us define \( \epsilon \) to be a primitive \( \tilde{n} \)th root of unity so that \( \epsilon^{\tilde{n}} = 1 \), where \( \tilde{n} > 2 \) is a positive integer. The cases of odd and even \( \tilde{n} \) have to be treated in slightly different ways and because of this it is useful to introduce a variable \( n \), defined by

\[
n = \begin{cases} 
\tilde{n} & \text{for odd } \tilde{n} \\
\frac{\tilde{n}}{2} & \text{for even } \tilde{n}
\end{cases}
\]

so that for odd \( \tilde{n} \), \( \epsilon^n = 1 \) and for even \( \tilde{n} \), \( \epsilon^n = -1 \). For even \( n \) the roots \( q = \exp(\pm \frac{\pi i n}{n}) \) are of particular interest since for \( 0 < p < n \) the quantity \( \lfloor p \rfloor \) is positive definite,

\[
\lfloor p \rfloor = \frac{\sin \frac{\pi p}{n}}{\sin \frac{\pi}{n}},
\]

and this has important consequences for the construction of unitary representations. For similar reasons we also note that with \( q = \exp(\frac{2\pi i n}{n}) \) the quantity \( \lfloor m \rfloor! \) takes negative values for the first time when \( m > \frac{\tilde{n}}{2} \). Also, for \( r \geq 0 \) and \( n \geq p \geq 0 \) we have \( \lfloor rn \rfloor = 0 \), and the following useful identities,

\[
\frac{[rn + p]}{[p]} = q^{rn} \left( \frac{1 - q^{(2nr+2p)}}{1 - q^{2p}} \right) = q^{rn},
\]

and

\[
\lim_{\epsilon \to q} \frac{[rn]}{[r]} = \lim_{\epsilon \to q^{1-r}} \frac{1 - q^{2rn}}{1 - q^{2n}} = \lim_{\epsilon \to q^{1-r}} (1 + q^{2n} + q^{4n} + \ldots + q^{2n(r-1)}) = q^{n(1-r)r}.
\]

### 3 Deformed bosons at \( q \) a root of unity

The \( q \)-deformed bosonic oscillator (\( q \)-oscillator) has been discussed by numerous authors \[9, 10, 11\]. It is defined by the relations

\[
a_- a_+ - q^{\pm 1} a_+ a_- = q^{\pm N} \quad [N, a_{\pm}] = \pm a_{\pm}.
\]

It follows immediately from this definition that

\[
[q^N a_-, a_+]_{q^2} = 1 \quad [q^N a_-, (a_+)^m]_{q^{2m}} = [m]_{q^2} (a_+)^{m-1},
\]

and hence

\[
[q^N a_-,[q^N a_-,[\ldots [q^N a_-,(a_+)^r]\ldots]_{q^1}]_{q^2}] = [r]_{q^2}! = q^{\frac{(r-1)}{2}[r]!}.
\]
We now set \( r = n \) and take the \( q \to \epsilon \) limit, obtaining

\[
\lim_{q \to \epsilon} \frac{1}{[n]!} [q^N a_-, [q^N a_-, \ldots [q^N a_-, (a_+)^n]_q^n \ldots]_q^n] = \lim_{q \to \epsilon} \frac{q^{n(n-1)}}{[n]!} [q^N (a_-)^n, (a_+)^n]
\]

\[
= q^{n(n-1)}/2,
\]

which we can write as

\[
\lim_{q \to \epsilon} \left[ q^{n} \frac{(a_-)^n}{\sqrt{[n]}!}, \frac{(a_+)^n}{\sqrt{[n]}!} q^{nN/2} \right] = 1.
\]

(11)

Note that since \( q^{2n} = 1 \) we can write \( q^{nN} = q^{-nN} \). In consequence it is possible to change the signs on the exponents of the \( q^{nN/2} \) terms in the above and in the following definitions, a freedom of which we will shortly make use. Motivated by (12) we now define

\[
b_+ = \lim_{q \to \epsilon} q^{-N} \frac{(a_-)^n}{\sqrt{[n]}!} q^{nN/2}, \quad b_- = \lim_{q \to \epsilon} q^{-N} \frac{(a_+)^n}{\sqrt{[n]}!}. \]

(13)

Then from (12)

\[
[b_-, b_+] = 1,
\]

which is just the defining relationship of an ordinary boson.

If, in the following, we want to work at an algebraic level, we must assume that \( (a_+)^n \to 0 \) and \( (a_-)^n \to 0 \) when \( q \to \epsilon \) in such a way that \( b_+ \) and \( b_- \) are well defined, or put another way we must restrict our attention to the subalgebra for which this is true. We make such a restriction, and note that for Fock space representations it follows automatically. Let us now calculate the commutation relations between these new bosonic oscillators and our original \( q \)-oscillators. Trivially we have \([a_-, b_+]_{q^{\frac{1}{2}}} = 0\) and \([a_+, b_+]_{q^{\frac{1}{2}}} = 0\). The other two commutation relations can be obtained as follows,

\[
[a_-, b_+]_{q^{\frac{1}{2}}} = \lim_{q \to \epsilon} \left[ a_-, \frac{(a_-)^n}{\sqrt{[n]}!} q^{nN/2} \right]_q^{1/2},
\]

\[
= \lim_{q \to \epsilon} q^{-N} \left[ q^{N} a_-, \frac{(a_+)^n}{\sqrt{[n]}!} q^{nN/2} \right]_q^{1/2},
\]

\[
= \lim_{q \to \epsilon} q^{-N} \sqrt{[n]} \frac{(a_+)^{n-1}}{\sqrt{[n-1]}!} q^{nN/2} \]

\[
= 0.
\]

(15)

Here we have made use of (9) in going from the second line to the third. Similarly we find \([a_+, b_-]_{q^{\frac{1}{2}}} = 0\), so the complete set of commutation relations between the two sets of oscillators is as follows,

\[
[a_-, b_-]_{q^{\frac{1}{2}}} = 0, \quad [a_-, b_+]_{q^{\frac{1}{2}}} = 0,
\]

\[
[a_+, b_-]_{q^{\frac{1}{2}}} = 0, \quad [a_+, b_+]_{q^{\frac{1}{2}}} = 0.
\]

(16)
Introducing a number operator for these new bosonic oscillators, defined in the usual way as \( N_b = b_+ b_- \), we also have

\[
[N, a_\pm] = \pm a_\pm , \quad [N_b, a_\pm] = 0 , \\
[N, b_\pm] = \pm n b_\pm , \quad [N_b, b_\pm] = \pm b_\pm .
\] (17)

We can use these results to choose a more natural set of generators for the algebra. Let us define

\[
A_- = a_- q^{\frac{nN_b}{2}} , \\
A_+ = q^{\frac{nN_b}{2}} a_+ ,
\] (18)

and

\[ N_A = N - nN_b . \] (19)

Then in terms of these new generators we have,

\[
[A_-, A_+]_{q^\pm} = q^{\mp N_A} , \quad [N_A, A_\pm] = \pm A_\pm , \quad [b_-, b_+] = 1 , \quad [N_b, b_\pm] = \pm b_\pm ,
\] (20)

which are the defining relations of an anyon and an ordinary undeformed boson, as well as,

\[
[A_\pm, b_{(\pm)}] = 0 , \quad [N_A, b_\pm] = 0 , \quad [N_b, A_\pm] = 0 , \quad [N_b, N_A] = 0 ,
\] (21)

which show that the two algebras commute. Thus in the \( q \to \epsilon \) limit the deformed bosonic oscillator algebra decomposes into the direct product of the undeformed bosonic oscillator algebra and an anyonic oscillator algebra. A clearer understanding of the way in which the splitting of a single \( q \)-oscillator at generic \( q \), into two independent oscillators when \( q \to \epsilon \) occurs, can be obtained by examining the corresponding result for the Fock space. At generic \( q \), the normalized state \( |m\rangle \) is defined by

\[
|m\rangle = \frac{(a_+)^m}{\sqrt{[m]_q!}} |0\rangle ,
\] (22)

where \( a|0\rangle=0 \). If we write \( m = rn + p \), for integers \( 0 \leq p < n, r \geq 0 \), then after a little algebra \( |rn+p\rangle \) can be written as

\[
|rn+p\rangle = (a_+)^p \left( \frac{(a_+)^n q^{\frac{nN_b}{2}}}{\sqrt{[n]_q!}} \right)^r \left( \frac{([n]_q!)^r}{[rn+p]_q!} \right)^{\frac{1}{2}} \prod_{\alpha=0}^{r-1} q^{-n_\alpha^2} |0\rangle .
\] (23)

Also, from (8) and (9), we have the identity

\[
\lim_{q \to \epsilon} \left( \frac{([n]_q!)^r}{[rn+p]_q!} \right)^{\frac{1}{2}} = \frac{q^{-nP}}{\sqrt{r! [p]_q!}} \prod_{\alpha=0}^{r-1} q^{-2n^2} .
\] (24)
Note that we have used the sign ambiguity of the square root to choose the sign on the exponent in the $q^{-\frac{n\pi}{2}}$ term. This is of no physical significance, it just makes the equations tidier. Using this and definitions (13) and (18) we find that in the limit as $q \to \epsilon$, (23) becomes

$$\lim_{q \to \epsilon} |rn + p\rangle = \lim_{q \to \epsilon} \left( a_+^p q^{-\frac{n\pi}{2}} \frac{1}{\sqrt{|p|_q!}} \frac{1}{\sqrt{r!}} \left( \frac{(a_+)^{n} q^{\frac{n\pi}{2}}}{\sqrt{|n|_q!}} \right)^r \right) |0\rangle$$

$$= \left( a_+^p q^{-\frac{n\pi}{2}} \right) \frac{(b_+^r)}{\sqrt{|p|_q!}} |0\rangle$$

Clearly this result means that we can write

$$\lim_{q \to \epsilon} |rn + p\rangle = |r\rangle_{\text{bosonic}} \otimes |p\rangle_{\text{anyonic}} .$$

This is the Fock space analogue of the algebraic decomposition which led to (20). Thus we see that for each generic $q$ state, there is a corresponding state in the $q \to \epsilon$ limit. The difference is that when $q \to \epsilon$ these states are no longer part of a single $q$-oscillator irreducible representation, but are instead in the product of a bosonic irreducible representation and an anyonic irreducible representation. Let us also note that from (3) and (23), we know that for $q = \exp(\pm \frac{i\pi}{n})$ the states $\lim_{q \to \epsilon} |rn + p\rangle$ have positive definite norm (see [2] for a discussion of this point). In this case we have $(b_+)^{\dagger} = b_-$ and $(A_+)^{\dagger} = A_-$, so that we can use the notation

$$b^{\dagger} = b_+ , \quad b = b_- , \quad A^{\dagger} = a_+ , \quad A = a_- ,$$

since $\{b^{\dagger}, b, A^{\dagger}, A\}$ all have the implied hermiticity properties. It is particularly interesting to look at this result when $q \to i$ ($\bar{n}=4$, $n=2$), since in this case the $q$-oscillator decomposes into the physically observed bosonic and fermionic oscillators. To see this we note that when $q = i$ we have

$$AA^{\dagger} - i^{\pm 1} A^{\dagger} A = i^{\mp N_A} ,$$

which acting on the Fock space, for which $p = 0, 1$ reduces to the familiar fermionic algebra,

$$AA^{\dagger} + A^{\dagger} A = 1 .$$

4 Deformed angular momentum at $q$ a root of unity

In this section we examine the structure of $U_q(sl(2))$, and its subalgebra $U_q(su(2))$, the enveloping algebra of the $q$-deformed angular momentum algebra, in the $q \to \epsilon$ limit, with a view to finding
structures analogous to those found in association with fractional supersymmetry in [1, 2, 3, 4, 5].

For generic $q$ the algebraic part of $\mathcal{U}_q(sl(2))$ is

$$[J_+, J_-] = [2J_z], \quad q^{J_z} J_\pm q^{-J_z} = q^{\pm 1} J_\pm,$$

and it has coproduct, counit and antipode,

$$\Delta J_\pm = J_\pm \otimes q^{J_z} + q^{-J_z} \otimes J_\pm,$$
$$\Delta q^{\pm J_z} = q^{\pm J_z} \otimes q^{\pm J_z},$$
$$\varepsilon(J_\pm) = 0, \quad \varepsilon(q^{J_z}) = 1,$$
$$S(J_\pm) = -q^{\pm 1} J_\pm, \quad S(q^{J_z}) = q^{-J_z}. \quad (31)$$

$\mathcal{U}_q(sl(2))$ can be restricted to $\mathcal{U}_q(su(2))$ via the introduction of a $*$-structure. At algebraic level a $*$-structure is an anti-involution satisfying

$$(\beta a) = \beta^* (a), \quad a^* = a, \quad (ab)^* = (b)^* (a). \quad (32)$$

Here $\beta$ is a complex number with conjugate $\beta^*$, and $a$ is an arbitrary element of the algebra. In the case of $\mathcal{U}_q(sl(2))$ we impose the following $*$-structure, to make it into $\mathcal{U}_q(su(2))$,

$$* J_\pm = J_\mp, \quad *q^{\pm J_z} = q^{\mp J_z}, \quad (33)$$

which can be extended to the whole algebra using (32). In matrix realizations of $\mathcal{U}_q(su(2))$ we can identify $*$ as hermitian conjugation. This $*$-structure is compatible with the Hopf structure in the following sense

$$\tau \circ \Delta \circ * = * \otimes * \circ \Delta,$$
$$\varepsilon \circ * = * \circ \varepsilon,$$
$$* \circ S \circ * = S,$$

where $\tau$ is the transposition map $\tau(X \otimes Y) = Y \otimes X$. To avoid confusion we stress that the form of $\mathcal{U}_q(su(2))$ given here, and encountered in [7] for example, is distinct from the more frequently encountered form of $\mathcal{U}_q(su(2))$, which has $q$ real, a different $*$-structure, and compatibility relations which are different from those given in (34).

$\mathcal{U}_q(su(2))$ can be realized in terms of $q$-oscillators by means of the $q$-deformed analogue of the Schwinger realization [10]. This realization involves two copies $a_{1\pm}$ and $a_{2\pm}$ of the algebra (8) which
are mutually commutative, i.e. \([a_{1\pm},a_{2(\pm)}] = 0\), and is explicitly defined by

\[
J_+ = a_{1+} a_{2-} \quad , \quad J_- = a_{1-} a_{2+} \quad , \quad q^{J_z} = q^{\frac{N_1 - N_2}{2}} .
\] (35)

It is associated with those finite dimensional irreducible representations \([jm]\) of \(U_q(sl(2))\), which can be defined on the product of the Fock spaces of \(a_{1\pm}\) and \(a_{2\pm}\) as follows [10]

\[
|jm\rangle = \left( a_{1+} \right)^{j+m} \left( a_{2+} \right)^{j-m} \left( \frac{[j+m]! [j-m]!}{[j]!} \right)^{1/2} |0,0\rangle .
\] (36)

The action of the Schwinger realization (35) on these representations can be straightforwardly worked out to be

\[
J_- |jm\rangle = \frac{1}{n!} \left( a_{1+} \right)^n q^{\frac{nN_1}{2}} \frac{N_1}{2} |jm\rangle , \quad J_+ |jm\rangle = \frac{1}{n!} \left( a_{2+} \right)^n q^{\frac{nN_2}{2}} \frac{N_2}{2} |jm\rangle ,
\] (37)

When \(q \to \epsilon\) we can use (13) to define the corresponding bosonic oscillators

\[
b_{1+} = \lim_{q \to \epsilon} \left( a_{1+} \right)^n \frac{n N_1}{2} \frac{1}{\sqrt{[n]!}} , \quad b_{1-} = \lim_{q \to \epsilon} q^n \left( a_{1-} \right)^n \frac{n N_1}{2} \frac{1}{\sqrt{[n]!}} ,
\]

\[
b_{2+} = \lim_{q \to \epsilon} \left( a_{2+} \right)^n \frac{n N_2}{2} \frac{1}{\sqrt{[n]!}} , \quad b_{2-} = \lim_{q \to \epsilon} q^n \left( a_{2-} \right)^n \frac{n N_2}{2} \frac{1}{\sqrt{[n]!}} ,
\] (38)

and their number operators \(N_{kb} = b_k^* b_k\) for \(k = 1,2\). Note that we have made a change of sign \(q^{-\frac{nN_2}{2}} \to q^{-\frac{nN_2}{2}}\) in the second row, taking advantage of the freedom remarked upon previously. Using these we can construct a Schwinger realization of the undeformed \(sl(2)\) algebra.

\[
L_+ = b_{1+} b_{2-} = \lim_{q \to \epsilon} \left( \frac{1}{[n]!} \right) \left( a_{1+} \right)^n \left( a_{2-} \right)^n q^{n(N_1 - N_2 + n)} \frac{1}{2} (J_+)^n q^{nJ_+} q^{\frac{nJ_z}{2}} ,
\]

\[
L_- = b_{2+} b_{1-} = \lim_{q \to \epsilon} \left( \frac{1}{[n]!} \right) \left( a_{2+} \right)^n \left( a_{1-} \right)^n q^{n(N_1 - N_2 + n)} \frac{1}{2} (J_-)^n q^{nJ_-} q^{\frac{nJ_z}{2}} ,
\]

\[
L_z = \frac{N_{b1} - N_{b2}}{2} = \frac{1}{2} [L_+, L_-] ,
\] (39)

so that

\[
[L_+, L_-] = 2L_z \quad , \quad [L_z, L_\pm] = \pm L_\pm ,
\] (40)

and also

\[
[L_z, J_\pm] = 0 \quad , \quad [q^{J_z}, L_\pm] q^n .
\] (41)
As was the case with the $q$-oscillators, there is a more convenient basis for the algebra with generators $\{L_\pm, L_z, J_\pm, J_z\}$. We define

$$q^{S_z} = q^{J_z - nL_z}, \quad S_+ = q^{nL_z} J_+, \quad S_- = J_- q^{nL_z},$$

so that

$$[S_+, S_-] = [2S_z], \quad [S_z, S_\pm] = \pm S_\pm,$$

and

$$[L_z, S_\pm] = 0, \quad [q^{S_z}, L_\pm] = 0.$$ (43)

In this basis the enveloping algebras $U(sl(2))$ and $U_q(sl(2))$ spanned by $\{L_\pm, L_z\}$ and $\{S_\pm, q^{S_z}\}$ respectively are mutually commutative. In other words, we can write

$$\lim_{q \to \epsilon} U_q(sl(2)) = U_\epsilon(sl(2)) \otimes U(sl(2)),$$ (45)

where by $U_\epsilon(sl(2))$, we mean the enveloping algebra obtained by simply setting $q = \epsilon$, rather than by taking the limit as above.

It is important to note that we have only established the above decomposition for a particular realization.

In general the irreducible representations of an algebra are characterized by the eigenvalues of its central elements. When $q^{\tilde{n}} = 1$, it follows directly from (30) that $U_q(sl(2))$ has an expanded centre, now including $(J_\pm)^{\tilde{n}}$ and $q^{nJ_z}$ as well as the usual Casimir operator. The structure of the representations is consequently richer for such values of $q$. We now summarize the finite dimensional irreducible representations of the $U_q(sl(2))$ algebra for $q^{\tilde{n}} = 1$. For the derivation of this classification see [10]. There are two basic cases. Type A irreducible representations are labelled by two parameters, a half integral spin $j$ and a discrete parameter $\omega = \pm 1, \pm i$. They have dimension $2j + 1$ where $0 < 2j + 1 \leq n$. If we use a basis $\{|-j\rangle, |-j + 1\rangle, ... |j\rangle\}$, then the action of the algebra on these irreducible representations is given by

$$J_- |m\rangle = \omega (|j + m\rangle |j + m + 1\rangle)^{\frac{1}{2}} |m - 1\rangle, \quad J_+ |m\rangle = \omega (|j - m\rangle |j + m + 1\rangle)^{\frac{1}{2}} |m + 1\rangle,$$

$$J_- |-j\rangle = 0, \quad J_+ |j\rangle = 0,$$

$$q^{J_z} |m\rangle = \omega q^m |m\rangle, \quad q^{-J_z} |m\rangle = \omega^* q^{-m} |m\rangle.$$ (46)

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Type B representations have dimension \( n \), and are characterized by three complex parameters \( j, x, y \). If we use a basis \( \{| -j \rangle, | -j + 1 \rangle, \ldots | -j + n - 1 \rangle \} \), then the action of the algebra on these irreducible representations is given by

\[
\begin{align*}
J_- |m\rangle &= (\lfloor j + m \rfloor \lfloor j - m + 1 \rfloor + xy) \frac{1}{2} |m - 1\rangle, \\
J_+ |m\rangle &= (\lfloor j - m \rfloor \lfloor j + m + 1 \rfloor + xy) \frac{1}{2} |m + 1\rangle, \\
J_- | -j\rangle &= y | -j + n - 1\rangle, \\
J_+ | -j + n - 1\rangle &= x | -j\rangle, \\
q^{J_z} |m\rangle &= q^m |m\rangle.
\end{align*}
\]

(47)

When \( x, y \neq 0 \) the irreducible representations (47) are said to be cyclic, and when \( x = 0, y \neq 0 \) or \( y = 0, x \neq 0 \) they are said to be semicyclic. Two independent restrictions which we can impose upon \( \mathcal{U}_q(sl(2)) \), both of them compatible with the coproduct are

\[
J_+^n = 0 \quad , \quad J_-^n = 0 \quad .
\]

(48)

When these restrictions are imposed the algebra plainly loses the cyclic and semicyclic irreducible representations from (47). When the *-structure (33) is also imposed the only remaining irreducible representations are those type A irreducible representations with \( \omega = \pm 1 \). For these irreducible representations we have

\[
J_\pm = J_\mp \quad , \quad (q^{\pm J_z})^* = q^{\mp J_z} \quad ,
\]

(49)

which reflects the *-structure (33). We will refer to this restricted *-Hopf algebra as \( \mathcal{U}_q(sl(2), r) \). Note also that on any irreducible representation \( q^{J_z} = \omega q^m \) where \( m \) takes on half integer values in the range \(-j\) to \(+j\), \( 2j + 1 \leq n \). Consequently \( q^{J_z} \) satisfies the characteristic polynomial

\[
\prod_{m=0}^{n-1} (q^{4mJ_z} - q^{2m}) = 0 \quad ,
\]

(50)

which is equivalent to both

\[
\prod_{m=0}^{n-1} [2J_z - m]_q = 0 \quad ,
\]

(51)

and \( q^{4mJ_z} = 1 \). Apart from the \( \omega \) factor, the irreducible representations (46) are identical to those associated with the Schwinger realization (47). For \( \mathcal{U}_q(sl(2)) \) we cannot use the *-structure to restrict the algebra, and instead introduce \( q^{4mJ_z} = 1 \) directly. This condition also restricts us to the type A irreducible representations, only now \( \omega \) can take on any of the values \( \omega = \pm 1, \pm i \). For more on restricted forms of \( \mathcal{U}_q(sl(2)) \) and other \( \mathcal{U}_q(g) \) enveloping algebras see [7, 17] and the references.
We can use them to determine the algebraic properties of $J$. From \[18\], we have the identities

$$J_{i}^{(n)} = \lim_{q \to \epsilon} \frac{J_{i}^{n}}{[n]!}, \quad J_{z},$$

(52)

is well defined. We now extend $U_{q}(sl(2), r)$ by the elements $J_{\pm}^{(n)}$, $J_{z}$ to make $U_{q}(sl(2), f)$ ($f$ for fractionally supersymmetric). $J_{\pm}^{(n)}$, $J_{z}$ are to be endowed with the Hopf algebraic structure associated with the $q \to \epsilon$ limit of $J_{\pm}^{n}$, $J_{z}$, so that they add to $U_{q}(sl(2), r)$ some more of the structure associated with the generic $q$ case. We do not make the explicit identification $J_{\pm}^{(n)} = \lim_{q \to \epsilon} \frac{J_{\pm}^{n}}{[n]!}$, since the latter quantity is not well defined algebraically (although it is for all representations). From \[13\], we have the identities

$$[J_{+}^{m}, J_{-}^{s}] = \sum_{p=1}^{\text{min}(m, s)} \left[ \begin{array}{c} m \\ p \end{array} \right] \left[ \begin{array}{c} s \\ p \end{array} \right] \frac{2^{p-s-m}}{[2J_{z} + k]} \frac{1}{[2J_{z}]_{k=p-m-s+1}^{\infty}} J_{-}^{s-p} J_{+}^{m-p}.$$

(53)

We can use them to determine the algebraic properties of $J_{\pm}^{(n)}$, which we obtain by taking the $q \to \epsilon$ limits of the algebraic properties of $J_{\pm}^{n}$:

$$[J_{+}^{(n)}, J_{-}] = q^{-n} \frac{[2J_{z} + 1]^{n-1}}{[n-1]!},$$

$$[J_{+}, J_{-}^{(n)}] = q^{-n} \frac{J_{z}^{n-1} [2J_{z} + 1]}{[n-1]!}.$$

(54)

In addition there are the trivial results $[J_{\pm}^{(n)}, J_{z}] = 0$. We also have

$$[J_{+}^{(n)}, J_{-}^{(n)}] = \sum_{p=1}^{n-1} \frac{1}{([n-p]!)^{2}[p]!} \frac{2^{p}}{[2J_{z} + k]} J_{+}^{n-p} \lim_{q \to \epsilon} \frac{1}{[n]!} \prod_{k=1-n}^{0} \prod_{k=1-n}^{0} [2J_{z} + k]^{n-p}.$$

(55)

As a consequence of (51), the last term in this equation is algebraically well defined. Its explicit value, which is worked out in the appendix, is

$$\lim_{q \to \epsilon} \frac{1}{[n]!} \prod_{k=1-n}^{0} [2J_{z} + k] = q^{2nJ_{z} - n^{2}} \left( \frac{2J_{z} - \frac{3n}{2} - \frac{1}{2}}{n} \right) + q^{2nJ_{z} - n^{2}} \sum_{k=1}^{n-1} \frac{(-q^{1-4J_{z} + n})^{k}}{[n-k]![k]!},$$

(56)

and we will make use of this below. The commutation relations of $J_{z}$ are easily found to be

$$[J_{z}, J_{\pm}] = \pm J_{\pm} \quad \text{and} \quad [J_{z}, J_{\pm}^{(n)}] = \pm n J_{\pm}^{(n)}.$$
The Hopf structure given by (31), (48) and $q^{4nJ_z} = 1$ leads in a similar fashion to the following Hopf structure for $J^{(n)}_\pm$, $J_z$,

$$
\Delta J^{(n)}_\pm = J^{(n)}_\pm \otimes q^{nJ_z} + q^{-nJ_z} \otimes J^{(n)}_\pm + \sum_{k=1}^{n-1} q^{(k-n)J_z} J^{(n)}_\pm \otimes q^{kJ_z} J^{n-k}_\pm \left/ \left[ k! [n-k]! \right] \right.,
$$

$$
\Delta J_z = 1 \otimes J_z + J_z \otimes 1 ,
$$

$$
S(J^{(n)}_\pm) = -q^{-n^2} J^{(n)}_\pm , \quad S(J_z) = -J_z ,
$$

$$
\varepsilon(J^{(n)}_\pm) = 0 , \quad \varepsilon(J_z) = 0 ,
$$

and for $U_q(sl(2), f)$ the *-structure

$$
*(J^{(n)}_\pm) = J^{(n)}_\mp , \quad *(J_z) = J_z .
$$

Equations (54)-(58) show that as expected the additional elements $J^{(n)}_\pm$, $J_z$ can be added to the Hopf algebra without contradiction, so that $U_q(sl(2), f)$ is well defined as a Hopf algebra. When (59) is also imposed, the corresponding result for $U_q(su(2), f)$ is obtained. $U_q(sl(2), r)$ is a sub-Hopf algebra of $U_q(sl(2), f)$, for which we already know all of the irreducible representations (46). Let us denote a state in an highest weight representation of $U_q(sl(2), f)$ by $|m, ?\rangle$, in which the $m$ is associated with the $U_q(sl(2), r)$ subalgebra, and the $?$ denotes any, as yet unknown, extra labeling, which is needed to fully describe the highest weight representations of the extended algebra. Using (54) and (60) we find

$$
[J^{(n)}_+, J_-]|m, ?\rangle = q^n \frac{[2J_z + 1]J^{n-1}_+] \left/ [n-1]_q ! \right.|m, ?\rangle
$$

$$
= 0 ,
$$

and similarly

$$
[J_+, J^{(n)}_-]|m, ?\rangle = 0 ,
$$

as well as the trivial results,

$$
[J_\pm, J^{(n)}_\pm]|m, ?\rangle = 0 .
$$

Note that these results hold regardless of the full form of $|m, ?\rangle$, since they depend only on the $m$ part. Using (54) and (60)-(62) we can also evaluate the following without knowing the full form of $|m, ?\rangle$.

$$
[[J^{(n)}_+, J^{(n)}_-], J^{(n)}_+]|m, ?\rangle = \sum_{p=1}^{n-1} \frac{1}{([n-p]!)^2 [p]!} J^{n-p}_+ \left( \prod_{k=p+1}^{2p} [2J_z + k] \right) J^{n-p}_+ J^{(n)}_+ |m, ?\rangle
$$

$$
+ \left[ q^{2nJ_z + n^2} \left( \frac{2J_z - 3n^2 - 1}{n} \right), J^{(n)}_+ \right] |m, ?\rangle
$$

$$
+ \left[ q^{2nJ_z} q^{-\frac{3(n+1)}{2}}, \sum_{k=1}^{n-1} \frac{(-q(1-4J_z+n))^k}{[n-k]! [k]!}, J^{(n)}_+ \right] |m, ?\rangle .
$$
From (30)-(32) the first and third terms term are zero. The second term then yields,

\[
[[J_+^{(n)}, J_-^{(n)}], J_+^{(n)}] |m, \gamma\rangle = 2J_+^{(n)} q^{2n J_z + n^2} |m, \gamma\rangle .
\]  

(64)

Similarly we find that

\[
[[J_+^{(n)}, J_-^{(n)}], J_-^{(n)}] |m, \gamma\rangle = -2J_-^{(n)} q^{2n J_z + n^2} |m, \gamma\rangle .
\]  

(65)

These results have their most natural form in the \(L, S\) basis introduced earlier. We can change to this using

\[
L_+ = J_+^{(n)} q^{n J_z} q^{\frac{n^2}{2}}, \quad L_- = q^{n J_z} q^{\frac{n^2}{2}} J_-^{(n)}, \quad L_z = \frac{[L_+, L_-]}{2},
\]  

(66)

and

\[
S_+ = q^{n L_z} J_+, \quad S_- = J_- q^{n L_z}, \quad q^{S_z} = q^{J_z - n L_z},
\]  

(67)

which follow from (39) and (42). Note that from (33) and (59), the \(*\)-structure of the elements in \(U_q(sl(2), f)\), the \(*\)-structure of the elements in this modified basis is

\[
*(L_+) = q^{-2n S_z - 2n^2 L_z - n^2} L_-, \quad *(S_+) = S_- q^{-2n L_z},
\]

\[
*(L_-) = L_+ q^{-2n S_z - 2n^2 L_z - n^2}, \quad *(S_-) = q^{-2n L_z} S_+, \quad *(L_z) = q^{-4n (S_z + n L_z)} L_z,
\]

\[
*(q^{S_z}) = q^{-S_z} q^{-n (L_z - q^{-4n (S_z + n L_z)} L_z)}.
\]  

(68)

Although the algebra described by (54)-(57) does not have the direct product form seen in (45), it follows none the less from (60)-(65) that for all highest weight representations,

\[
[S_+, S_-] = [2 S_z], \quad [S_z, S_\pm] = \pm S_\pm, \quad [L_+, L_-] = 2 L_z, \quad [L_z, L_\pm] = \pm L_\pm,
\]

\[
[S_\pm, L_\pm] = 0, \quad [S_z, L_\pm] = 0, \quad [L_z, S_\pm] = 0.
\]  

(69)

In consequence, the highest weight representations of \(U_q(sl(2), f)\) are a direct product of the form

\[
U_q(sl(2), r) \otimes U(sl(2))
\]  

(70)

in which \(U(sl(2))\) denotes the enveloping algebra of undeformed \(sl(2)\). It remains to establish the equivalent results for \(U_q(su(2), f)\) with the reality properties implied by (68). There are three basic cases:

i) \(\hat{n}\) odd. In this case the \(*\)-structure (68) implies the following hermiticity properties for the algebraic elements on Fock space representations.

\[
S_\pm^\dagger = S_\mp, \quad (q^{S_z})^\dagger = q^{-S_z}, \quad L_\pm^\dagger = L_\mp, \quad L_z^\dagger = L_z.
\]  

(71)
so that the analogue of (70) is
\[ \mathcal{U}_q(su(2), r) \otimes \mathcal{U}(su(2)) \quad . \] (72)

If we take \( q = \exp\left( \frac{2a\pi i}{n} \right) \) with \( a \) an integer, then since \( [m] \) goes negative for \( m > \frac{n}{2a} \), i.e. those with \( j < \frac{n-2a}{4a} \) are unitary. The simplest non-trivial example has \( a = 1, \tilde{n} = n = 5 \), for which this unitarity condition gives \( j_s < \frac{3}{4} \), i.e. \( j_s = 0, \frac{1}{2} \). The \( j_s = 0 \) irreducible representation is trivial, and the action of the algebraic elements on the \( j_s = \frac{1}{2} \) irreducible representation is given by
\[
J_+ \left| \frac{1}{2}, \frac{1}{2} \right> = \left| \frac{1}{2}, \frac{1}{2} \right>, \quad J_- \left| \frac{1}{2}, \frac{1}{2} \right> = 0, \quad J_\mp \left| \frac{1}{2}, \frac{1}{2} \right> = 0, \quad (73)
\]

which is identical to the undeformed \( s = \frac{1}{2} \) irreducible representation. Thus (72) can be interpreted as a single wave function consisting of the direct product of an intrinsic spin degree of freedom with an orbital angular momentum part. For higher odd \( \tilde{n} \) we can extend this interpretation so that in general (72) can be viewed as

Intrinsic anyonic spin \( \otimes \) Orbital angular momentum. (74)

Note that to make (74) strictly accurate we have to exclude non integer values of \( j_l \). This can be done in the usual way by imposing \( \psi(\theta + 2\pi) = \psi(\theta) \) on the wave functions.

ii) \( \tilde{n} \) even, \( n \) even. In this case the hermiticity properties implied by (58) are
\[
S^\dagger = (-1)^{2L_z} S, \quad (qS_z)^\dagger = q^S_z, \quad L^\dagger = (-1)^{2S_z} L, \quad L^\dagger = L, \quad (75)
\]
so that in this case the analogue of (70) is
\[
\mathcal{U}_q(su(2)) \otimes \mathcal{U}(su(2)) \quad \text{for } 2j_s \text{ even, } 2j_l \text{ even},
\]
\[
\mathcal{U}_q(su(2)) \otimes \mathcal{U}(su(1, 1)) \quad \text{for } 2j_s \text{ odd, } 2j_l \text{ even},
\]
\[
\mathcal{U}_q(su(1, 1)) \otimes \mathcal{U}(su(2)) \quad \text{for } 2j_s \text{ even, } 2j_l \text{ odd},
\]
\[
\mathcal{U}_q(su(1, 1)) \otimes \mathcal{U}(su(1, 1)) \quad \text{for } 2j_s \text{ odd, } 2j_l \text{ odd}. \quad (76)
\]

It follows from (17) that for \( 2j_l \) odd
\[
S_+S_- |m_s\rangle = -[j_s - m_s][j_s + m_s + 1]|m_s\rangle, \quad (77)
\]
so that there are always negative norm states in these irreducible representations, and consequently they are not physical (i.e. not unitary). On the other hand when \( 2j_l \) is even, and with \( q = \exp(\pm \frac{\pi i}{n}) \),
all of the irreducible representations are unitary (for other values of $q$ there are restrictions on the value of $j_s$ as in case (i)). Thus the unitary highest weight representations of $U_q(su(2), f)$ all have integer $j_l$. In this case the interpretation of the $L$ part as orbital angular momentum follows without the need for any further restrictions. Note that there are also highest weight representations in which the $L$ part is a $U(su(1, 1))$ algebra with integer $j_l$. These correspond to orbital angular momentum in a space of $1 + 2$ dimensions.

iii) $\tilde{n}$ even, $n$ odd. In this case the hermiticity properties implied by (68) are

$$
S^\dagger_\pm = (-1)^{2L_z} S^\pm_\mp, \quad (q^{S_z})^\dagger = q^{S_z},
$$

$$
L^\dagger_\pm = (-1)^{2S_z+2L_z+1} L^\pm_\mp, \quad L^\dagger_\mp = L_\mp
$$

so that in this case the analogue of (70) is

$$
U_q(su(2)) \otimes U(su(2)) \quad \text{for } 2j_s \text{ odd, } 2j_l \text{ even},
$$

$$
U_q(su(2)) \otimes U(su(1, 1)) \quad \text{for } 2j_s \text{ even, } 2j_l \text{ even},
$$

$$
U_q(su(1, 1)) \otimes U(su(2)) \quad \text{for } 2j_s \text{ even, } 2j_l \text{ odd},
$$

$$
U_q(su(1, 1)) \otimes U(su(1, 1)) \quad \text{for } 2j_s \text{ odd, } 2j_l \text{ odd},
$$

As for case (ii), only the first two classes of highest weight representation in the list given above have members which are unitary.

In [14, 15] a more rigorous and elegant, though less direct, approach to some of the material covered here is introduced in a purely mathematical context. In particular, this method leads to a generalization of the results in this section to all $U_q(g)$ enveloping algebras. Let us conclude this section by noting that the direct product structure of the highest weight representations of $U_q(sl(2))$ at $q = \epsilon$ was anticipated by the result (15) for the deformed Schwinger realization. There are deformed bosonic realizations of all $U_q(g)$ enveloping algebras [12], for which at $q = \epsilon$, similar decompositions are to be expected. Perhaps such realizations will serve as useful tools for studying fractionally supersymmetric extensions of $U_q(g)$.

5 \textbf{GL}_q(2) \textbf{ at } q \textbf{ a root of unity}

The quantum group $SL_q(2)$ is dual [19] to $U_q(sl(2))$, and so it is reasonable to expect that there are analogous extensions and decompositions for this Hopf algebra when $q$ is a root unity. To begin with we will work with the complete $GL_q(2)$ Hopf algebra. This quantum group is generated by the
elements \( \{a, b, c, d\} \) of the matrices
\[
w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
which have the nontrivial commutation relations given below
\[
[a, b]_q = 0 \quad , \quad [a, c]_q = 0 \quad , \quad [a, d] = \lambda bc \quad , \\
[b, c] = 0 \quad , \quad [b, d]_q = 0 \quad , \quad [c, d]_q = 0
\]
with \( \lambda = q - q^{-1} \). The coproduct is
\[
\Delta(a) = a \otimes a + b \otimes c \quad , \quad \Delta(b) = a \otimes b + b \otimes d \\
\Delta(c) = c \otimes a + d \otimes c \quad , \quad \Delta(d) = c \otimes b + d \otimes d
\]
and the counit and antipode are given by
\[
\varepsilon(a) = \varepsilon(d) = 1 \quad , \quad \varepsilon(b) = \varepsilon(c) = 0 \\
S(a) = (\det_q w)^{-1}d \quad , \quad S(b) = -q^{-1}(\det_q w)^{-1}b \\
S(c) = -q(\det_q w)^{-1}c \quad , \quad S(d) = (\det_q w)^{-1}a
\]
where
\[
\det_q w := ad - qbc
\]
is a central and grouplike element known as the ‘quantum determinant’ of \( w \). Here the word grouplike is used to indicate that \( \Delta g = g \otimes g \) for \( g = \det_q w \). \( GL_q(2) \) can be restricted to \( SL_q(2) \) by setting \( \det_q w = 1 \). Two Hopf algebras are said to be dually paired if
\[
\langle xy, \alpha \rangle = \langle x \otimes y, \Delta \alpha \rangle \\
\langle x, \alpha \beta \rangle = \langle \Delta x, \alpha \otimes \beta \rangle \\
\langle 1, \alpha \rangle = \varepsilon(\alpha) \\
\langle x, 1 \rangle = \varepsilon(x) \\
\langle S(x), \alpha \rangle = \langle x, S(\alpha) \rangle
\]
This duality can be extended to the \( * \)-structure in more than one way \([7, 20]\). For our purposes the most convenient form is
\[
\langle x, \alpha \rangle = \langle *x, *\alpha \rangle^*.
\]
This pairing is often degenerate. For \( SL_q(2) \) and \( \mathcal{U}_q(sl(2)) \) there is a pairing of type \((85)\) given by
\[
\langle q^{\pm J_z}, a \rangle = \langle q^{\pm J_z}, d \rangle = q^{\pm \frac{1}{2}} \\
\langle q^{\pm J_z}, b \rangle = \langle q^{\pm J_z}, c \rangle = 1 \\
\langle J_+, b \rangle = \langle J_-, c \rangle = 1 \\
\langle J_+, a \rangle = \langle J_-, d \rangle = 0
\]
and extended to products using (85). For details see [19, 7, 17]. At generic $q$, we have

\begin{align*}
\langle J_n^+, b^n \rangle &= [n]_q^{-2} = q^{-(n-1)}[n]! , \\
\langle J_n^-, c^n \rangle &= [n]_q^{2} = q^{n(n-1)}[n]! ,
\end{align*}

which are easily found using induction. Note that $b^n, c^n$ from $SL_q(2)$ are not paired to any elements in $U_q(sl(2))$ other than $J_n^+$ and $J_n^-$ respectively, so that when $q = \epsilon$, they have a null pairing with all of $U_q(sl(2))$. There are two interesting ways of retaining the non-null pairing from the generic case. Our work in section 4, with $U_q(su(2, f))$ provides one. In this case the null pairing is a straightforward consequence of $J_n^\pm = 0$, so that by rearranging (88) and taking the $q \to \epsilon$ limit we obtain the following pairings for $J_n^{(n)}$,

\begin{align*}
\langle J_n^{(n)}^+, b^n \rangle &= q^{n(n-1)} , \\
\langle J_n^{(n)}^-, c^n \rangle &= q^{-n(n-1)} ,
\end{align*}

and also find that they have null pairings with the rest of $SL_q(2)$. An immediate consequence of this is that the form of $SL_q(2)$ dual to $U_q(su(2, f))$ has $b^n, c^n \neq 0$. A second way of retaining the non-null generic $q$ pairings will be discussed later in this section. From (87) it follows immediately that

\begin{align*}
\langle ad - qbc, 1 \rangle &= \langle a \otimes d - qb \otimes c, 1 \otimes 1 \rangle \\
&= \langle a, 1 \rangle \langle d, 1 \rangle - q \langle b, 1 \rangle \langle c, 1 \rangle \\
&= 1 ,
\end{align*}

which is compatible with $\det_q w = 1$. If we impose the $*$-structure (33) on $U_q(sl(2))$ to make it into $U_q(su(2))$ then from (88) we obtain the following hermitian $*$-structure on $\omega$

\[
* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} .
\]

It is easy to check that this is compatible with (32) and (34). We will refer to $SL_q(2)$ with this $*$-structure as $*SL_q(2)$ and to $GL_q(2)$ with the same $*$-structure as $*GL_q(2)$. Note that although $*SL_q(2)$ is dual to our $U_q(su(2))$, it is not the quantum group usually referred to as $SU_q(2)$, which involves real $q$ and a different $*$-structure. Let us now define

\[
A = a^n , \quad B = b^n , \quad C = c^n , \quad D = d^n .
\]

From (81) it follows directly that

\begin{align*}
[A, b]_{q^n} &= [A, c]_{q^n} = [D, b]_{q^n} = [D, c]_{q^n} = 0 , \\
[B, a]_{q^n} &= [B, d]_{q^n} = [C, a]_{q^n} = [C, d]_{q^n} = 0 , \\
[A, d] &= [D, a] = [B, c] = [C, b] = 0 .
\end{align*}
in regard to which we recall that $q^n = \pm 1$. Also

\begin{align*}
[A, B]_{q^n} &= 0 , 
[A, C]_{q^n} &= 0 , 
[A, D] &= 0 , 
[B, C] &= 0 , 
[B, D]_{q^n} &= 0 , 
[C, D]_{q^n} &= 0 , 
\end{align*} 
(94)

where likewise it should be noted that $q^{n^2} = \pm 1$. Let us now go on to consider the rest of the Hopf structure. From (82), the coproduct of $A$ is given by

$$\Delta(A) = [\Delta(a^n) = (a \otimes a + b \otimes c)^n = a^n \otimes a^n + b^n \otimes c^n + \sum_{m=1}^{n-1} \left[ \begin{array}{c} n \\ m \end{array} \right] q^{m^2-nm}(a \otimes a)^{n-m}(b \otimes c)^m$$ 
(95)

$$= a^n \otimes a^n + b^n \otimes c^n = A \otimes A + B \otimes C .$$

The coproducts of $B, C$ and $D$ can be similarly derived, and we find that

$$\Delta(A) = A \otimes A + B \otimes C \quad , \quad \Delta(B) = A \otimes B + B \otimes D \quad ,$$

$$\Delta(C) = C \otimes A + D \otimes C \quad , \quad \Delta(D) = C \otimes B + D \otimes D \quad ,$$

which has the same form as (82). The counit and antipode of $\{A, B, C, D\}$ are easily deduced to be

$$\varepsilon(A) = \varepsilon(D) = 1 \quad , \quad \varepsilon(B) = \varepsilon(C) = 0 \quad ,$$
(97)

and

$$S(A) = (\det_q w)^{-n}D \quad , \quad S(B) = -q^{n^2}(\det_q w)^{-n}B \quad ,$$

$$S(C) = -q^{-n^2}(\det_q w)^{-n}C \quad , \quad S(D) = (\det_q w)^{-n}A .$$
(98)

Writing

$$AD = a^n d^n$$

$$= a^{n-1}(\det_q w + qbc)\alpha^{n-1}$$

$$= \prod_{r=1}^{n} (\det_q w + (q^{-1}bc)q^{2r})^n ,$$
(99)

and using the identity \[2\]

$$\prod_{r=1}^{n} (\alpha + p^r \beta) = \alpha^n + p^{\frac{n(n+1)}{2}} \beta^n ,$$
(100)

in which $p$ is a root of unity, we have

$$AD = (\det_q w)^n + q^{n(n-1)}q^{-n}b^n c^n$$

$$= (\det_q w)^n + q^{n^2}BC .$$
(101)
Now, if we define

\[ W := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \]

so that

\[ \det_{q^{n^2}} W = AD - q^{n^2} BC, \]

we have

\[ (\det_q w)^n = \det_{q^{n^2}} W. \]

Using this, we can rewrite (98) as

\[ S(A) = (\det_{q^{n^2}} W)^{-1} D, \quad S(B) = -q^{n^2}(\det_{q^{n^2}} W)^{-1} B, \]

\[ S(C) = -q^{-n^2}(\det_{q^{n^2}} W)^{-1} C, \quad S(D) = (\det_{q^{n^2}} W)^{-1} A. \]

By comparing (94), (96), (97), (103) and (105) with (81)-(84), we see that the elements \{A, B, C, D\} generate a \( GL_{q^{n^2}}(2) \) sub-Hopf algebra of \( GL_q(2) \). In the case of odd \( n \), \( q^{n^2} = 1 \), so that this sub-Hopf algebra is just undeformed \( GL(2) \), and moreover since \( q^n = 1 \), it is central. We can specialize to \( SL_q(2) \) by fixing \( \det_q w = 1 \). From (104) this implies that \( \det_{q^{n^2}} W = 1 \), so that the subalgebra generated by \{A, B, C, D\} is itself restricted to \( SL_{q^{n^2}}(2) \). Similarly, when we impose the \(*\)-structure (94) we find from (92) that a \(*\)-structure of the same form is induced on \( W \), i.e.

\[ * \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \]

so that \(*GL_q(2)\) and \(*SL_q(2)\) have, respectively, sub-Hopf algebras \(*GL_{q^{n^2}}(2)\) and \(*SL_{q^{n^2}}(2)\). The following observations are intended to assist in the development of a physical interpretation of these results.

i) The \(*\)-structure preserves the determinant, i.e.

\[ *\det_q w = \det_q w, \]

\[ *\det_{q^{n^2}} W = \det_{q^{n^2}} W. \]

ii) The theory of covariant transformations of \( GL_q(2) \), has been discussed by several authors, e.g. [21, 22, 23, 24]. As one would expect, these preserve the determinant \( \det_q w \), and thus from (104) \( \det_{q^{n^2}} W \) as well.

iii) If we write

\[ w = \begin{pmatrix} p_0 - p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 + p_3 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \]

20
and
\[
W = \begin{pmatrix}
P_0 - P_3 & P_1 - iP_2 \\
P_1 - iP_2 & P_0 + P_3
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\] (109)
then the reality of \(\{p_\mu\}\) and \(\{P_\mu\}\) follow from the \(\ast\)-structure (91) and (106). Also, from (81).

\[
[p_0, p_3] = \frac{1}{4} [a + d, d - a] = \frac{1}{2} [a, d] ,
\]
\[
= \frac{1}{2} \lambda bc = \frac{1}{2} \lambda (p_2^2 + p_1^2) ,
\] (110)
so that
\[
det_q w = p_0^2 - p_3^2 + [p_0, p_3] - q(p_2^2 + p_1^2)
\]
\[
= p_0^2 - \frac{q + q^{-1}}{2} (p_2^2 + p_1^2) - p_3^2 ,
\] (111)
and similarly
\[
det_{q^2} W = P_0^2 - q^2 (P_2^2 + P_1^2) - P_3^2 .
\] (112)

iv) When \(q = 1\), we have \(w = W\), and \(\det W\) is the Laplacian on undeformed Minkowski space, which appears in the Klein-Gordon equation, \textit{i.e.} \((\det w^2 - M^2) |\psi\rangle = 0\).

Based on these observations we interpret \(\ast GL_q(2)\) as the algebra of quantized momenta (\textit{i.e.} derivatives up to a factor of \(i\)) on \(q\)-deformed Minkowski-space, with \(\det w\) as the \(q\)-deformed Laplacian. A central feature of this deformed momentum-space is that for \(q^2 = 1\) it contains undeformed Minkowski momentum-space, \textit{i.e.} \(\ast GL(2)\) with coordinate functions \(\{P_i\}\) as a sub-Hopf algebra. To make this interpretation more explicit we introduce the notation
\[
p^2 = \det q w , \quad P^2 = \det q^2 W .
\] (113)

Our work in section 4 with \(U_q(su(2, f))\), the dual to \(SL_q(2)\) suggests that we view the additional structure due to \(\{p_\mu\}\) as in some way connected to anyonic degrees of freedom. Considering the case of \(q = +i\) we find from (104), (111) and (112) that
\[
(p_0^2 - p_3^2)^2 = P_0^2 - P_1^2 - P_2^2 - P_3^2 , \quad \text{i.e.} \quad (p^2)^2 = P^2 .
\] (114)
Thus the deformed Laplacian is the square root of the undeformed Laplacian. This means that we can use it to construct an equation which is Dirac-like in the sense that it is a square root of the Klein-Gordon equation.
\[
(p^2 \pm m^2) |\psi\rangle = 0 ,
\] (115)
where $m^2$ is assumed real. Using (114), and defining $M = m^2$, we find that for $q = i$

$$(p^2 + m^2)(p^2 \pm m^2)|\psi\rangle = ((p^2)^2 - m^4)|\psi\rangle$$

$$= (P^2 - M^2)|0\rangle = 0$$

and thereby recover the Klein-Gordon equation from the undeformed case. More generally, if $q = \exp^{2\pi i \tilde{n}}$, with $n$ prime relative to $r$ we have

$$p^2 = \det_q w = p_0^2 - \cos \frac{2\pi r}{\tilde{n}} (p_1^2 + p_2^2) - p_3^2$$

which by (104) is the $n$th root of the Laplacian on undeformed Minkowski space. Using this we can construct $n$th roots of the Klein-Gordon equation. For even $\tilde{n}$ these roots are given by

$$(p^2 - q^{2s}m^2)|\psi\rangle = 0$$

for any integer $s$, so that there are $n$ distinct roots. Each of these equations implies the Klein-Gordon equation because

$$(p^2)^n|0\rangle = q^{2sn}m^{2n}|0\rangle = m^{2n}|0\rangle$$

or in the form analogous to (116),

$$\prod_{r=s}^{n+s-1} (p^2 - q^{2r}m^2)|\psi\rangle = \prod_{r=1}^{n} (p^2 - q^{2r}m^2)|\psi\rangle$$

$$= ((p^2)^n - m^{2n})|\psi\rangle$$

$$= (P^2 - M^2)|\psi\rangle$$

where we have made use of (100), and defined $M = m^n$. For odd $\tilde{n}$ there is an analogous argument. This time the $n$th roots of the Klein-Gordon equation are given by

$$(p^2 - q^s m^2)|\psi\rangle = 0$$

and we verify that they imply the Klein-Gordon equation by using

$$(p^2)^n|0\rangle = q^{sn}m^{2n}|0\rangle = m^{2n}|0\rangle = M^2|0\rangle$$

Equations (118) and (121) have a clear interpretation as anyonic Dirac-like equations.

Although (97) prevents us from setting $A$ or $D$ equal to zero, there is nothing to prevent us from defining a restricted, form of $GL_q(2)$ with $B = C = 0$. We will refer to this as simply restricted $GL_q(2)$. Apart from dropping the $B$ and $C$ parts, the only change to the Hopf structure is that (94) becomes

$$\Delta(A) = A \otimes A$$

$$\Delta(D) = D \otimes D$$
so that \( A \) and \( D \) are grouplike. For the sub-case of restricted \( SL_q(2) \), we also have from (94) and (103),
\[
AD = DA = 1 ,
\]
so that we can write \( D = A^{-1} \). Using an approach similar to that adopted in section 4 we can add generators to restricted \( GL_q(2) \) (equally \( SL_q(2) \)) to obtain extended \( GL_q(2) \). Specifically, we introduce extra elements \( B \) and \( C \), which we endow with the Hopf algebraic structure of respectively, \( B_{[n]} \) and \( C_{[n]} \) in the \( q \to \epsilon \) limit of the generic case. Straightforwardly we obtain
\[
\begin{align*}
[A, b]_{q^n} &= [A, c]_{q^n} = [D, b]_{q^n} = [D, c]_{q^n} = 0 , \\
[B, a]_{q^n} &= [B, d]_{q^n} = [C, a]_{q^n} = [C, d]_{q^n} = 0 , \\
[A, d] &= [D, a] = [B, c] = [C, b] = 0 ,
\end{align*}
\]
and
\[
\begin{align*}
[A, B]_{q^n^2} &= 0 , & [A, C]_{q^n^2} &= 0 , & [A, D] &= 0 , \\
[B, C] &= 0 , & [B, D]_{q^n^2} &= 0 , & [C, D]_{q^n^2} &= 0 ,
\end{align*}
\]
as well as
\[
\begin{align*}
\varepsilon(A) &= \varepsilon(D) = 1 , & \varepsilon(B) = \varepsilon(C) &= 0 ,
\end{align*}
\]
and
\[
\begin{align*}
S(A) &= (\det_{q^n^2 W}^{-1})D , & S(B) &= -q^{-n^2} (\det_{q^n^2 W}^{-1})B , \\
S(C) &= q^{n^2} (\det_{q^n^2 W}^{-1})C , & S(D) &= (\det_{q^n^2 W}^{-1})A .
\end{align*}
\]
The *-structure of the elements of the extended *\( GL_q(2) \) is
\[
\ast \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} .
\]
These are all the same as the corresponding results for \( \{ A, B, C, D \} \) in the unrestricted algebra. However, the coproduct is distinct since for generic \( q \)
\[
\Delta \left( \frac{b^n}{[n]!} \right) = a^n \otimes \frac{b^n}{[n]!} + \frac{b^n}{[n]!} \otimes d^n + \sum_{r=1}^{n-1} q^{r^2 - nr} (a \otimes b)^{n-m} (b \otimes d)^m \frac{[r]! [n-r]!}{[m]! [n-m]!} .
\]
Taking the \( q \to \epsilon \) limit we obtain the coproduct structure of \( B \), and similarly \( C \). The results are
\[
\Delta(A) = A \otimes A , & \Delta(D) = D \otimes D , \\
\Delta(B) = A \otimes B + B \otimes D + \sum_{m=1}^{n-1} q^{m^2 - nm} (a \otimes b)^{n-m} (b \otimes d)^m \frac{[m]! [n-m]!}{[m]! [n-m]!} , \\
\Delta(C) = C \otimes A + D \otimes B + \sum_{m=1}^{n-1} q^{m^2 - nm} (c \otimes a)^{n-m} (d \otimes c)^m \frac{[m]! [n-m]!}{[m]! [n-m]!} .
\]
The cocommutativity of this coproduct can be verified by a lengthy but straightforward calculation.

The fact that the Hopf structure of extended $GL_q(2)$ is well defined is really just a consequence of its status as a natural limit of a well defined generic $q$ Hopf algebra, in which the retention of all of the generic $q$ structure required the introduction of new elements $B$ and $C$. From (89) we find that the pairings of $B$ and $C$ with $U_q(sl(2))$ are all null except for

$$\langle J^n_+, B \rangle = q^{n(n-1)}$$
$$\langle J^n_-, C \rangle = q^{-n(n-1)} .$$

Note that this implies that $J^n_+ \neq 0$, so that extended $SL_q(2)$ is not dual to $U_q(sl(2), f)$ or $U_q(sl(2))$, but rather to a form of $U_q(sl(2))$ which includes the cyclic irreducible representations from (47).

Thus if we want to maintain the generic $q$ pairing we can choose to restrict, then extend, either of $SL_q(2)$ and $U_q(sl(2))$, but not both. Note that there are intermediate forms with $b^n = 0$, $c^n \neq 0$ dual to $J^n_+ \neq 0, J^n_- = 0$, etc.

Coproducts similar to those for $B$ and $C$ appeared in relation to fractional supersymmetry in [2, 3, 4, 5], suggesting that these elements can be viewed as a generalized supersymmetric extension of restricted $GL_q(2)$.

It seems reasonable to expect that other $Fun_q(G)$ algebras will have analogous properties when their deformation parameters are roots of unity. It would also be interesting to see if any such properties are exhibited by braided Hopf algebras such as $BM_q(2)$ [6, 7].

Appendix A

Here we derive (56). First of all note that

$$\lim_{q \to \epsilon} \frac{1}{[n]!} \prod_{k=1-n}^{0} [2J_z + k] = \lim_{q \to \epsilon} \frac{1}{[n]!} \prod_{k=1}^{n} [2J_z - n + k]$$

$$= \lim_{q \to \epsilon} \frac{q^{2J_z n - n^2} q^{\frac{n(n-1)}{2}}}{(1 - q^{-2})^{n} [n]!} \prod_{k=1}^{n} (1 - q^{-4j_z + 2n - 2k}) .$$

From [2] we have the identity

$$\frac{1}{[m]!} \prod_{k=1}^{m} (1 - \alpha q^{-2k}) = \frac{1}{[m]!} \sum_{k=0}^{m} \frac{(-\alpha)^k q^{k(k+1)} [m]_{q^2}}{[m-k]_{q^2}! [k]_{q^2}} ! .$$

Setting $m = n$ and taking the limit as $q \to \epsilon$ we find that

$$\lim_{q \to \epsilon} \frac{1}{[n]!} \prod_{k=1}^{n} (1 - \alpha q^{-2k}) = \lim_{q \to \epsilon} \frac{1}{[n]!} (1 + (-1)^n q^{n(n+1)\alpha^n}) + \sum_{k=1}^{n-1} \frac{(-\alpha)^k q^{k(k+1)} q^{\frac{n(n+1)}{2}}}{[n-k]_{q^2}! [k]_{q^2}!} .$$
If we now set $\alpha = q^{-4J_z+2n}$, then the first term is well defined, and using (133) we find that
\[
\lim_{q \to 0} \frac{1}{[n]!} \prod_{k=1-n}^{0} [2J_z + k] = \lim_{q \to 0} \frac{q^{2nJ_z-n^2}}{\prod_{k=1}^{n-1} (1-q^{-2k})} \left( \frac{1 + (-1)^n q^{n(n+1)} q^{-4J_z n + 2n^2}}{1 - q^{-2n}} \right) \prod_{k=1}^{n-1} \frac{(-q^{1-4J_z+n})^k}{[n-k]![k]!} + \frac{q^{2nJ_z} q^{-a(n+1)/2}}{(1-q^{-2})^n} \sum_{k=1}^{n-1} \frac{(-q^{1-4J_z+n})^k}{[n-k]![k]!} \prod_{k=1}^{n-1} \frac{(-q^{1-4J_z+n})^k}{[n-k]![k]!}. \tag{136}
\]
Finally, using the identity $\prod_{k=1}^{n-1} (1-q^{-2k}) = n [2]$, and (3) this reduces to
\[
\lim_{q \to 0} \frac{1}{[n]!} \prod_{k=1-n}^{0} [2J_z + k] = q^{2nJ_z-n^2} \left( \frac{2J_z - \frac{3n}{2} - \frac{1}{2}}{n} \right) + \frac{q^{2nJ_z} q^{-a(n+1)/2}}{(1-q^{-2})^n} \sum_{k=1}^{n-1} \frac{(-q^{1-4J_z+n})^k}{[n-k]![k]!} \prod_{k=1}^{n-1} \frac{(-q^{1-4J_z+n})^k}{[n-k]![k]!} \tag{137}
\]
in agreement with (56).

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