Precursory Metal-Insulator transition in a small cluster of the ‘$t$-$J$’ model: Exact analytic results

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Abstract

We study the effect of hole hopping in a doped antiferromagnet described by the ‘$t$-$J$’ model, using exact analytic solutions for small clusters. In spite of the small size, they reveal interesting details about the magnetic order, which are not apparent in Mean Field treatments or in numerical calculations. The 4-site cluster with one hole yields the most interesting physics, displaying different behaviors for the ground state: i) an antiferromagnetic phase for \( t \ll J \), where the hole seems to be localized, not affecting the order of the Heisenberg spins; ii) another regime for \( t \sim J \) that presents mixed ferro and antiferro correlations and coexistence of metallic and insulating behaviors, with the presence of charge and spin density waves, in what may be the analog of the spiral phase obtained in Mean Field solutions; and finally iii) for \( t \gg J \), we obtain strong ferromagnetic correlations (maximum spin) and no density waves, with quantum fluctuations precluding the saturation of the magnetic moment. This behavior shows traces of a metal-insulator transition as the hole kinetic energy competes with the antiferromagnetic interactions.

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The physics of the normal state of High-$T_c$ compounds is yet to be understood after nearly 12 years of great experimental and theoretical developments. The antiferromagnetic long range order at $T = 0$ is rapidly destroyed when the CuO planes are slightly doped with holes ($x \sim 0.02$ for $La_{2-x}Sr_xCuO_4$) [1]. These holes form $O^-$ ions which coupled to the $Cu^{2+}$ spin $1/2$ local moment can be regarded as singlets centered at the coppers. Looking at this singlet as a hole on a $Cu^{2+}$ square lattice, Zhang and Rice [2] derived an effective Hamiltonian, the so called ‘$t - J$ ’ model, which describes the hopping of holes on a spin lattice, with the spins coupled via an antiferromagnetic Heisenberg interaction. Mean field theory (MFT) was extensively used by numerous authors to treat the hole motion on the interacting spin background [3–5]. They found an evolution from an antiferro to a ferromagnetic phase, when the parameter $t/J$ is increased at low or intermediate hole concentrations. The most interesting feature of those calculations is the spiral phase that they predict for $t \sim J$, a trait that seems to have been observed in $La_{2-x}Sr_xCuO_4$ by neutron scattering experiments [6].

The question then arises on what would be the effect of quantum fluctuations (which are neglected in MFT) on this magnetic order, and what is the nature of this spiral state. Since this phase is induced by hole mobility, it should be accompanied by an insulator-to-metal transition.

In this paper we calculate exact eigenfunctions for the ‘$t - J$ ’ model Hamiltonian

$$H_{t-J} = -t \sum_{\sigma<i,j>} P_0 \left( c^\dagger_{i\sigma} c_{j\sigma} + \text{h.c.} \right) P_0 + J \sum_{<i,j>} P_0 \left( \vec{S}_i \cdot \vec{S}_j - \frac{n_i n_j}{4} \right) P_0,$$

where $c^\dagger_{i\sigma}$ is the fermionic operator that creates an electron (hole) with spin $\sigma$ on site $i$, $n_i = \sum_\sigma c^\dagger_{i\sigma} c_{i\sigma}$ is the number operator, $P_0$ projects out double occupied states in the lattice, and the brackets $\langle \ldots \rangle$ under the summation signs mean nearest neighbors. The parameter $t$ measures the mobility of carriers which hop from site to site in an antiferromagnetic Heisenberg background, with exchange $J > 0$. We were able to obtain exact analytic solutions for the eigenfunctions of (1) in a 4-site cluster, which despite its small size, shows traces of a metal-insulator transition for the ground state ($T = 0$), when the parameter $\lambda = t/J$ is varied. We also propose a way to analyze the subjacent magnetic order for $\lambda \sim 1$,
searching for a precursory spiral phase in this small cluster. We solve the problems with one and two holes on a square cluster. The two-hole case is simpler, since the remaining spins accommodate to form a Resonating-Valence-Bonding state for small $\lambda$, and holes develop Charge Density Waves (CDW) states. The other phase ($\lambda \gg 1$) is a spin triplet and the two holes are delocalized. The physics of the one-hole problem is much richer, since there is a spin that cannot be compensated in spite of the antiferromagnetic correlations. As a result, we obtained an intermediate phase, with mixed insulating and metallic behavior, displaying a variety of magnetic textures, one of which we identify with the spiral phase predicted by MFT. We are aware that a small cluster cannot display true phase transitions. However, broken spin and charge symmetry states can be constructed for the ground level that, at least qualitatively, can be associated with macroscopic phases.

We first consider the case of two holes on a square cluster. This problem has dimension $6 \times 2^2 = 24$. The ground state energy shows a crossover at $\lambda = \frac{1}{2(\sqrt{2}-1)} \approx 1.21$. The small $\lambda$ phase has an antiferromagnetic character, with vanishing total spin and total $S_z$ component (note that \( [I] \) commutes with $\sum_i S_i^z$ and $\left( \sum_i \vec{S}_i \right)^2$ for periodic boundary conditions). It is double degenerate and can be written as a dimerization of singlets on the cluster, as shown in Fig. 1, where the double links stand for spin singlets and the lone circles for holes. This resembles closely the Resonating-Valence-Bond state (RVB), as prescribed by Anderson for the Heisenberg model \[8\]. Broken symmetry states in the form of Charge Density Waves (CDW) can be obtained by linear combinations of the above (for example, $\frac{1}{\sqrt{2}} (|a_1 > + |a_2 >)$ localizes the holes on one of the diagonals). The above features are characteristic of the insulating state. In contrast, the ground state for $\lambda \gtrsim 1.21$ shows a qualitatively different behavior: it is a spin triplet ($S_z = 1, 0, -1$, see the states \( |b_i > \) below), with admixtures from all the 24 kets of our base, suggesting some type of disorder induced by hole hopping. They are given by:

\[
|b_1 > = \frac{1}{4} \left( \left| 0 \ 0 \right> + \left| 0 \ 0 \right> + \left| \uparrow \ \downarrow \right> + \left| \downarrow \ \uparrow \right> + \left| \downarrow \ 0 \right> + \left| 0 \ \downarrow \right> + \left| \uparrow \ 0 \right> + \left| 0 \ \uparrow \right> \right) \]
critical value. For low
The ground state for this latter problem has energy
resemble that of the Heisenberg model for a 3-site chain with open b oundary conditions.

\(|0\rangle = \frac{1}{4} \begin{pmatrix} 0 \downarrow \rangle + \frac{1}{2\sqrt{2}} \left( \begin{pmatrix} 0 \downarrow \rangle + \begin{pmatrix} 0 \uparrow \rangle + \begin{pmatrix} 0 \uparrow \rangle + \begin{pmatrix} \uparrow \uparrow \rangle + \begin{pmatrix} 0 \rangle + \begin{pmatrix} \uparrow \rangle + \begin{pmatrix} 0 \rangle \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \right)

\(|b_2\rangle = \frac{1}{2\sqrt{2}} \left( \begin{pmatrix} 0 \downarrow \rangle + \begin{pmatrix} 0 \rangle + \begin{pmatrix} 0 \rangle + \begin{pmatrix} \uparrow \rangle + \begin{pmatrix} 0 \rangle + \begin{pmatrix} \uparrow \rangle + \begin{pmatrix} 0 \rangle \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \right)

\(|b_3\rangle = \frac{1}{2\sqrt{2}} \left( \begin{pmatrix} 0 \downarrow \rangle + \begin{pmatrix} 0 \rangle + \begin{pmatrix} 0 \rangle + \begin{pmatrix} \downarrow \rangle + \begin{pmatrix} 0 \rangle + \begin{pmatrix} \downarrow \rangle + \begin{pmatrix} 0 \rangle \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \right)

with an obvious notation for the basis functions that depicts the geometry of the square, the symbol 0 standing for holes. Note that basis kets entering in one of the states |\(b_i\rangle\) do not appear in any other |\(b_j\rangle\), meaning that linear combinations of these degenerate states will not lead to CDW’s (the probability of measuring a hole in one of the sites is the same for all sites). Therefore the |\(b_i\rangle\) states can be seen as precursors of the metallic regime. It comes as a surprise that we can see traces of a metal-insulator transition on such a small cluster.

Now, we turn to the problem of four sites with one hole, which is closer to the half filled band condition and displays more interesting magnetic properties. Before obtaining the exact ground state, we propose a variational solution for the case of low hole mobility (\(\lambda \approx 0\)). This way, we expect to shed light on approximations for larger cluster problems, since analytic solutions can only be obtained in very special cases. As shown below, it turns out that our variational state corresponds to the exact solution when \(\lambda\) is smaller than a critical value. For low \(t\) (or high magnetic coupling \(J\)), we expect the magnetic order to resemble that of the Heisenberg model for a 3-site chain with open boundary conditions.

The ground state for this latter problem has energy \(-J\) and is

\(|g_o\rangle = \frac{1}{\sqrt{6}} \left\{ |\uparrow\uparrow\rangle - 2 |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right\} \). \hspace{1cm} (2)

Next, we define the state |\(\phi_j\rangle\) as having a hole on the \(j\)-th site with the magnetic order given by (2) for the remaining sites, and propose the variational wave function
\[ |\Psi\rangle = \sum_{j=1}^{4} \frac{1}{2} e^{i\theta_j} |\phi_j\rangle , \]
suggesting that the hole motion just introduces a phase \( \theta_j \) as it resonates through the cluster.

We find the mean energy \( E = \langle \Psi | H_{t-J} | \Psi \rangle \) to be
\[ E = -J + \frac{t}{4} \left[ \cos (\theta_1 - \theta_2) + \cos (\theta_2 - \theta_3) + \cos (\theta_3 - \theta_4) + \cos (\theta_1 - \theta_4) \right] , \]
which is minimal for \( \theta_1 = 0 \), \( \theta_2 = \pi \), \( \theta_3 = 2\pi \), \( \theta_4 = 3\pi \), with the value \(-J - t\). For what range of \( \lambda \) is this variational solution valid? We will show below that for \( 0 < \lambda < \frac{1}{2} \), this is indeed the exact ground state for the cluster. This state is degenerate with the one obtained by reversing all the spins, in the negative sector of the magnetization.

To get the exact solution, we note that the Hilbert space has dimension \( 4 \times 2^3 = 32 \) (the hole can occupy each of the 4 sites and each spin can be up or down), but the \( H_{t-J} \) matrix divides up in 2 blocks of \( S_z = \pm \frac{3}{2} \) (each of dimension \( 4 \times 4 \)) and 2 of \( S_z = \pm \frac{1}{2} \) (each of dimension \( 12 \times 12 \)). We only diagonalize the positive magnetization blocks, the other eigenfunctions being obtained by flipping the spins of the basis (from now on, we always work on the positive magnetization sector). In addition, the \( 12 \times 12 \) matrix, when written in the total spin basis, breaks up in 2 blocks of \( 8 \times 8 \) (spin 1/2) and \( 4 \times 4 \) (spin 3/2). The energy levels can be seen in Fig. 2 and the eigenstates are given in the Appendix. The first feature we note in Fig.2 is the great number of level crossings at \( J = 2t \). This is the Supersymmetric point, where an exact solution via the Bethe Ansatz has been obtained in one dimension \([4]\), with exact separation of the charge and spin degrees of freedom \([10]\).

We see that the ground state energy undergoes two level crossovers, with the total spin being 1/2, 1/2 and 3/2, as the parameter \( \lambda \) increases. This cascade effect on the total spin has already been reported in numerical calculations \([11,12]\). The first ground state function holds for \( 0 < \lambda < \frac{1}{2} \), has energy \( E = -J - t \), and it is exactly equal to our variational solution shown above. The hole motion does not affect the magnetic order, just changes the relative phase between antiferromagnetic components. Although we expected CDW’s, this expectation is not realized, since there is not enough degeneracy in the ground manifold.
The absence of a broken symmetry state might be ascribed to the small size of the system. In this case, size effects are lifting the degeneracy with the first excited level, degeneracy that is restored at the Supersymmetric point.

The second ground state is double degenerate ($|c_1\rangle$ and $|c_2\rangle$ in the Appendix) and holds for $\frac{1}{2} < \lambda < \frac{4+\sqrt{13}}{2} \approx 3.8$, with energy $E = -\frac{J}{4}(1 + \sqrt{1 + 12\lambda^2})$. What makes those states non trivial is the fact that they depend on $\lambda$ (see the functions $a, b, c, \ldots$ in the Appendix), making them harder to analyze. Both of them obey a curious rule of sign when the hole hops, the same pattern repeating with a negative sign, in the same way as in the variational solution. It is easy to see that we can build linear combinations of $|c_1\rangle$ and $|c_2\rangle$ (which are linearly independent but not orthogonal), with real coefficients, that concentrate the hole on the diagonals, i.e. CDW’s can be obtained with those states. Fig 3 shows the probability $p_i$ of finding the hole at the $i$-th site as $\lambda$ varies, for the two kets $|c_1\rangle$ and $|c_2\rangle$. In the high mobility limit ($\lambda \to \infty$), all $p_i$’s go asymptotically to $1/4$, as expected.

As noted in the introduction, MFT’s yield a spiral magnetic order for $\lambda \sim 1$. We would like to test if our solution resembles this spiral in any way. We note that the states $|c_1\rangle$ and $|c_2\rangle$ have no uniform spin distribution. Broken spin symmetry states in the form of Spin Density Waves (SDW) can be build with the above states in the form

$$|SDW\rangle = \cos\left(\frac{\theta}{2}\right)|c_1+\rangle + \exp(i\phi)\sin\left(\frac{\theta}{2}\right)|c_2-\rangle,$$

where $\pm$ refer to the manifolds $S_z = \pm1/2$, respectively, and $(\theta, \phi)$ are arbitrary parameters which can be related with the geometry and pitch of the density wave. As noted before, states $|c_1\pm\rangle$ and $|c_2\pm\rangle$ are degenerate and linear independent and were chosen for their simple pattern. Calculation of the site magnetization $\langle SDW | \vec{S}_j | SDW \rangle$ shows a spiral spin distribution within the cluster, with a net ferromagnetic component that precesses with $\phi$, and whose amplitude is modulated as $\lambda$ and $\theta$ vary. The detailed calculation yields:

$$\langle \vec{S}_1 \rangle = \langle \vec{S}_3 \rangle = [h(\lambda) \cos \theta - z(\lambda)]\hat{z} - g(\lambda) \sin \theta \hat{n}_\perp$$

$$\langle \vec{S}_2 \rangle = \langle \vec{S}_4 \rangle = [j(\lambda) \cos \theta + z(\lambda)]\hat{z} + k(\lambda) \sin \theta \hat{n}_\perp$$

(3)
where $\hat{n}_\perp = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$ is a unit vector on the $xy$-plane and $(h, z, g, j, k)$ are functions given in the Appendix. Fig. 4 depicts this distribution for various choices of $(\theta, \phi)$. Some of the configurations shown closely resemble the double-spiral state found in Ref. [5] through a mean-field calculation. Canting of the spins in doped one-dimensional antiferromagnets has also been reported previously by one of the authors using a variational approach [13].

The presence of CDW and SDW in the above states suggests that they are precursors of the insulating regime. What is a remarkable peculiarity of the region $\frac{1}{2} < \lambda < \frac{4+\sqrt{13}}{2}$, is that we can also construct a ground state with uniform charge and spin distributions, i.e. with the same probability $p_i$ of finding the hole in any site and correspondingly, with the same site magnetization. We call them Uniform Hole Probability (UHP) states. Interestingly enough, the only two combinations that yield this result are complex and are only feasible for $\lambda > 1/2$:

$$ |uhp \sigma \rangle = \sqrt{1-u^2} |c_1 \sigma \rangle \pm iu |c_2 \sigma \rangle, \text{ with } u(\lambda) = \frac{1}{6\lambda} \sqrt{12\lambda^2 - 1 - \sqrt{1 + 12\lambda^2}}, \quad (4) $$

where the $\langle ... \rangle$ symbol stands to indicate normalized states and $\sigma = \pm$ refers to the manifolds $S_z = \pm1/2$, respectively. For these states, we get

$$ p_1 = p_2 = p_3 = p_4 = \frac{1}{4},$$

$$ \langle S_z (1) \rangle = \langle S_z (2) \rangle = \langle S_z (3) \rangle = \langle S_z (4) \rangle = \sigma \frac{1}{8},$$

We can generalize the above state to

$$ |uhp \hat{n} \rangle = \cos\left(\frac{\theta}{2}\right) |uhp+\rangle + \exp{(i\phi)} \sin\left(\frac{\theta}{2}\right) |uhp-\rangle, \quad (5) $$

as a state with uniform charge distribution and with the uniform magnetization pointing along the direction $\hat{n}$ ($\hat{n}$ being a unitary vector with latitude $\theta$ and azimuthal angle $\phi$).

Clearly, the $|uhp\rangle$ kets are precursors of the metallic state. Thus, in the region $\frac{1}{2} < \lambda < \frac{4+\sqrt{13}}{2}$, beginning at the Supersymmetric point, the system displays a coexistence of metallic and insulating behaviors. Two main questions remain: $i)$ why is the coexistence region so wide; and $ii)$ how can we extrapolate these results to the infinite-size limit? The degeneracy
of the ground state can be ascribed to the vicinity of the Supersymmetric point \((J = 2t)\) and to the square symmetry (in a sense, it may be a particular feature of the 4-site problem). The large region spanned by this precursory phase is certainly a size effect, and it will extrapolate to a single point for the infinite-size limit, or to a line when the doping is introduced as a variable.

The third ground state, for \(\lambda \gtrsim 3.8\), presents the same metallic character as in the two-hole problem solved at the beginning of this paper: it presents ferromagnetic spin correlations but holes are delocalized, all the base kets having the same admixture, thus avoiding the saturation of the magnetic moment (this is clearly due to quantum fluctuations, contrary to what is observed in MFT’s). This state is double degenerate and its energy is \(\frac{J}{2} - 2t\). CDW’s cannot be obtained, as in the two-hole case.

In conclusion, we described traces of a metal insulator transition in the ground state as we increase \(\lambda = \frac{J}{t}\). Although this treatment can only be done in small clusters, we were able to do a clear analysis of the wave function, showing properties that are not apparent in a number of numerical calculations performed for the same problem [12]. We analyzed the evolution of the magnetic properties by explicitly writing the wave functions, in contrast to the usual self-consistent calculation of magnetic averages. Broken charge and spin symmetry states are proposed as precursors of macroscopic phases. In particular, we found Spin Density Wave states that can be identified with the spiral phase obtained by Mean-Field procedures.

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A. Appendix: Eigenfunctions for the 4-site cluster with 1 hole

The basis for $S_z = \frac{1}{2}$ is (note the order):

$$
\begin{align*}
&\left\{ \begin{array}{c}
\uparrow \downarrow \\
\uparrow 0 \\
\uparrow \uparrow \\
0 \uparrow \\
0 \downarrow \\
\downarrow \uparrow \\
\downarrow 0 \\
\uparrow \downarrow \\
\uparrow \uparrow
\end{array} \right\},
&\left\{ \begin{array}{c}
\downarrow \uparrow \\
\uparrow 0 \\
\downarrow \uparrow \\
0 \uparrow \\
0 \downarrow \\
\downarrow \uparrow \\
\downarrow 0 \\
\uparrow \downarrow \\
\uparrow \uparrow
\end{array} \right\},
&\left\{ \begin{array}{c}
\uparrow \uparrow \\
\downarrow \uparrow \\
\downarrow \uparrow \\
\downarrow \uparrow \\
\downarrow 0
\end{array} \right\},
&\left\{ \begin{array}{c}
\uparrow \downarrow \\
\uparrow 0 \\
\downarrow \uparrow \\
\uparrow \downarrow \\
\uparrow \uparrow
\end{array} \right\},
&\left\{ \begin{array}{c}
\uparrow \downarrow \\
\uparrow 0 \\
\downarrow \uparrow \\
\uparrow \downarrow \\
\uparrow \uparrow
\end{array} \right\},
&\left\{ \begin{array}{c}
\uparrow \downarrow \\
\uparrow 0 \\
\downarrow \uparrow \\
\uparrow \downarrow \\
\uparrow \uparrow
\end{array} \right\},
&\left\{ \begin{array}{c}
\uparrow \downarrow \\
\uparrow 0 \\
\downarrow \uparrow \\
\uparrow \downarrow \\
\uparrow \uparrow
\end{array} \right\},
&\left\{ \begin{array}{c}
\uparrow \downarrow \\
\uparrow 0 \\
\downarrow \uparrow \\
\uparrow \downarrow \\
\uparrow \uparrow
\end{array} \right\},
&\left\{ \begin{array}{c}
\uparrow \downarrow \\
\uparrow 0 \\
\downarrow \uparrow \\
\uparrow \downarrow \\
\uparrow \uparrow
\end{array} \right\}
\end{align*}
$$

and the eigenstates can be classified by their total spin $S$ and total component $S_z$:

| $S_z = \frac{1}{2}$ | $S = \frac{3}{2}$ |
|---------------------|------------------|
| Eigenenergies       | Eigenfunctions (not normalized) |
| $\frac{J}{2}$       | $(1, 1, 1, 0, 0, 0, 0, 0, 0, -1, -1, -1)$ |
| $\frac{J}{2}$       | $(0, 0, 0, 1, 1, -1, -1, -1, 0, 0, 0)$ |
| $\frac{J}{2} - 2t$  | $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ |
| $\frac{J}{2} + 2t$  | $(1, 1, 1, -1, -1, -1, -1, -1, 1, 1, 1)$ |

| $S_z = \frac{1}{2}$ | $S = \frac{1}{2}$ |
|---------------------|------------------|
| Eigenenergies       | Eigenfunctions (not normalized) |
| $-J - t$            | $(-1, 2, -1, 1, -2, 1, 1, -2, 1, -1, 2, -1)$ |
| $-J + t$            | $(-1, 2, -1, 1, -2, 1, 1, -1, 2, -1, -1, 2, -1)$ |
| $-t$                | $(-1, 0, 1, 1, 0, -1, 1, 0, -1, -1, 0, 1)$ |
| $t$                 | $(-1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 1)$ |
| $-\frac{J}{2}(1 + \sqrt{1 + 2\lambda^2})$ | $|c_1\rangle = (-1, 1, 0, a, b, c, -a, -b, -c, 1, -1, 0)$ |
| $-\frac{J}{2}(1 + \sqrt{1 + 2\lambda^2})$ | $|c_2\rangle = (-1, 0, 1, b, -b, a, -b, -c, 1, 0, -1)$ |
| $\frac{J}{2}(1 + \sqrt{1 + 12\lambda^2})$ | $(-1, 1, 0, -a, -b, -f, d, e, f, 1, -1, 0)$ |
| $\frac{J}{2}(1 + \sqrt{1 + 12\lambda^2})$ | $(-1, 0, 1, e, -2e, e, -e, 2e, -e, 1, 0, -1)$ |
\[ a(\lambda) = \frac{-2 + \sqrt{1 + 12\lambda^2}}{6\lambda} \; ; \; b(\lambda) = \frac{1 + \sqrt{1 + 12\lambda^2}}{6\lambda} \; ; \; c(\lambda) = \frac{1 - 2\sqrt{1 + 12\lambda^2}}{6\lambda} \]

\[ d(\lambda) = \frac{2 + \sqrt{1 + 12\lambda^2}}{6\lambda} \; ; \; e(\lambda) = \frac{-1 + \sqrt{1 + 12\lambda^2}}{6\lambda} \; ; \; f(\lambda) = \frac{-1 + 2\sqrt{1 + 12\lambda^2}}{6\lambda} \]

Base \( S_z = \frac{3}{2} \): (all eigenfunctions with spin \( S = \frac{3}{2} \))

\[
\left\{ \begin{array}{c}
 0 \uparrow \rangle, \; \uparrow 0 \rangle, \; \uparrow \uparrow \rangle, \; \uparrow \uparrow \rangle, \; \uparrow \uparrow \rangle, \; 0 \uparrow \rangle
\end{array} \right\}
\]

| Eigenenergies | Eigenfunctions |
|---------------|----------------|
| \( \frac{1}{2} J \) | \((0, -1, 0, 1)\) |
| \( \frac{1}{2} J \) | \((-1, 0, 1, 0)\) |
| \( \frac{1}{2} J - 2t \) | \((1, 1, 1, 1)\) |
| \( \frac{1}{2} J + 2t \) | \((-1, -1, -1, 1)\) |

Coefficients for the SDW state given in (3)

\[ z(\lambda) = \frac{1 + 32\lambda^2 - \frac{1 + 8\lambda^2}{\sqrt{1 + 12\lambda^2}}}{16 + 256\lambda^2} \; , \; \quad h(\lambda) = \frac{11 + 128\lambda^2 + \frac{5 + 56\lambda^2}{\sqrt{1 + 12\lambda^2}}}{48(1 + 16\lambda^2)} \; , \]

\[ j(\lambda) = \frac{1 + 64\lambda^2 - \frac{5 + 56\lambda^2}{\sqrt{1 + 12\lambda^2}}}{48(1 + 16\lambda^2)} \; , \; \quad g(\lambda) = \frac{2 + \frac{2 + 21\lambda^2}{\sqrt{1 + 12\lambda^2}}}{6\sqrt{2}\sqrt{1 + 24\lambda^2} + \sqrt{1 + 12\lambda^2}} \]

\[ k(\lambda) = \frac{1 + \frac{1 + 6\lambda^2}{\sqrt{1 + 12\lambda^2}}}{12\sqrt{2}\sqrt{1 + 24\lambda^2} + \sqrt{1 + 12\lambda^2}} \]
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FIGURES

FIG. 1. Bi-degenerate ground state for $t/J < 1.21$. The double links are singlets which characterize this state as a RVB insulator.

FIG. 2. Energy levels of the 4-site cluster with 1 hole. As we increase $\lambda$ the ground state energy undergoes two level crossings. Note the amount of degeneracy for $\lambda = 0.5$, the Supersymmetric point.

FIG. 3. Probability of measuring the hole on the $i$-th site; solid line is for $|c_1\rangle$ and dotted for $|c_2\rangle$. Those states span the ground manifold when $1/2 < \lambda < 3.8$.

FIG. 4. Spiral magnetic order of SDW states for $t \sim J$. In all cases, the angle $\phi$ gives the orientation on the $xy$-plane: a) $\theta = \pi/2$; b) $\theta = \pi$; c) $\theta = 3\pi/2$. The thin ends of the spins are meant to point into the page.