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Nonlinear bending and vibration analyses of FG nanobeams considering thermal effects

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Abstract

Nonlinear bending and nonlinear free vibration analysis are presented for FG nanobeams based on physical neutral surface concept and high-order shear deformation beam theory with a von Kármán-type equations and including thermal effects. The material properties are temperature-dependent and vary in the thickness direction. Nonlinear bending approximate solutions and free vibration solutions for present model with fixed supported boundary conditions are given out by a two-step perturbation method. Some comparisons are presented to valid the reliability of the present study. In numerical analysis, the effects of the volume fraction, nonlocal parameter, strain gradient parameter, and temperature changes on nonlinear bending and vibration are investigated.

1. Introduction

Nanostructure refers to the tiny structure with the size of 0.1 nm–100 nm, that is, a new system is constructed or assembled according to certain rules based on the material units at the nanoscale. It includes one-dimensional, two-dimensional, and three-dimensional systems. Because of its excellent mechanical and thermal properties of nanostructures in mechanical engineering, aerospace engineering has been widely applied in such fields as [1], such as nanoscale energy harvester, nano resonator, the structure of the generator [1], the use of nanostructures under the extrinsic motivation produces vibration and bending, which can lead to structure failure and destruction. Therefore, the study of vibration and bending of nanostructures has important theoretical value and engineering background. Generally speaking, the properties of materials involved in the nanoscale will be different due to the different locations of the points, resulting in different properties of the materials, so as to achieve better mechanical properties. The nanoscale structures of such materials that uniformly change in the spatial domain are called functional gradient (FG) nanostructures. FG nanostructures contribute to improving the performance of many nanoelectromechanical systems [2], FG nanostructures whose material properties vary in at least one direction. For example, in FG nanobeams, the material properties change gradually along the direction of length or thickness, and the designers can adjust the contours of the material properties change to achieve better mechanical properties of the nanostructures. There are a variety of nanostructures, such as nanorods, nanobeams, nanotubes and nanoplates [3].

For the linear analysis of isotropic nanobeams and FG nanobeams, based on different higher order beam theory, Lu et al [4, 5] analyzed the buckling, bending and vibration of nanobeams. Li et al [6] discussed the vibrations of FG nanobeams based on Euler beam theory. Ebrahimi and Barati [7] explored the hygrothermal effects on vibration behavior of viscoelastic FG nanobeams. Wang et al [8] presented the complex modal analysis of free vibrations for axially moving nanobeams. Mohammad et al [9] presented the damped forced vibration analysis of nanobeams resting on viscoelastic foundation in thermal environment. It should be pointed out that, in above works [4–9], the nonlocal strain gradient theory is adopted. Lu [10] performed the dynamic analysis of axially prestressed nanobeam. Alshorbagy [11] used the finite element method to analyze the linear vibration of Euler nanobeams. Lei et al [12] studied the vibrations of nonlocal Kelvin–Voigt viscoelastic damped...
Timoshenko nanobeams. Kiani and Wang [13] studied on the interaction of a nanobeams using nonlocal different beam theories. It should be pointed out that, in above works [10–13], the nonlocal theory is adopted. For the nonlinear analysis of isotropic nanobeams and FG nanobeams, Ke and his co-workers [14] discussed the nonlinear vibration of the piezoelectric nanobeams based on the nonlocal theory. Ansari et al [15] presented the nonlinear forced vibration analysis of magneto-electro-thermo-elastic Timoshenko nanobeams based on the nonlocal elasticity theory. Shafiee et al [16] investigated the nonlinear vibration of porous and imperfect FG tapered nanobeams. Li and Hu [17] presented the nonlinear bending and vibration analyses of FG nanobeams. Şimşek [18] discussed the nonlinear free vibration of FG nanobeams using the novel Hamiltonian approach and nonlocal strain gradient theory. Gao and his partners [19] analyzed the nonlinear vibration of different types of FG nanobeams using nonlocal strain gradient theory. Liu and his partners [20] discussed the nonlinear vibration of geometrically imperfect FG sandwich nanobeams. Zarepour et al [21] discussed the geometrically nonlinear vibrations of Timoshenko piezoelectric nanobeams. Sarafratz et al [22] studied the nonlinear secondary resonance of nanobeams subjected to subharmonic and superharmonic excitations considering surface free energy effects. Sun et al [23] performed the nonlinear frequency analysis of buckled nanobeams in longitudinal magnetic fields. Yang et al [24] discussed the nonlinear bending, buckling and vibration of bi-directional FG nanobeams. Tang et al [25] investigated the effects of neutral surface deviation on nonlinear resonance of embedded FG nanobeams. Gholami and Ansari [26] discussed the nonlinear resonance responses of geometrically imperfect nanobeams including surface stress effects. Lv and Liu [27] presented the nonlinear bending response of FG nanobeams with material uncertainties. Rouhi et al [28] discussed the nonlinear free and forced vibrations of Timoshenko nanobeams. Lv et al [29] presented the nonlinear free vibration analysis of defective FG nanobeams embedded in elastic medium. Shafiee et al [30] discussed the nonlinear vibration of axially FG non-uniform nanobeams. She et al [31] studied the snap-buckling of FG non-uniform nanobeams.

Through literature search, we can find that, most of the existing literature only focuses on the vibration and bending of simply supported nanobeams, and there is a lack of research on the nonlinear bending and vibration of FG nanobeams with fixed supports in thermal environments at both ends. Therefore, this paper is to solve this problem. In the present work, nonlinear bending and vibration of FG nanobeams with two clamped ends based on physical neutral surface including thermal effects are investigated via two-step perturbation method [32].

2. Mathematical model

Figure 1 shows an FG nanobeam with length L and thickness h, which is made from an anisotropic material of ceramics and metals.

The effective material properties, like Young’s modulus $E_f$, Poisson’s ratio $\nu_f$ and thermal expansion coefficient $\alpha_f$ are assumed to be the function of the uniform temperature distribution $T$ and the thickness $z$, as [33, 34]

$$E_f = E_0(E_{11}T^{-1} + 1 + E_{12}T + E_{13}T^2 + E_{14}T^3)$$

$$\rho_f = \rho_0(\rho_{11}T^{-1} + 1 + \rho_{12}T + \rho_{13}T^2 + \rho_{14}T^3)$$

$$\nu_f = \nu_0(\nu_{11}T^{-1} + 1 + \nu_{12}T + \nu_{13}T^2 + \nu_{14}T^3)$$

(1)

$$E_f = \left(\frac{1}{2} + \frac{z}{h}\right)^N [E_m(T) - E_c(T)] + E_m(T)$$

$$\rho_f = \left(\frac{1}{2} + \frac{z}{h}\right)^N [\rho_m(T) - \rho_c(T)] + \rho_m(T)$$

$$\nu_f = \left(\frac{1}{2} + \frac{z}{h}\right)^N [\nu_m(T) - \nu_c(T)] + \nu_m(T)$$

(2)

where $m, c$ represent metal and ceramic, and the volume fraction index is represented by $N (0 \leq N \leq +\infty)$. 

![Figure 1. An FG nanobeam.](image-url)
Due to being transversely non-uniform in the material properties, there exists tension-bending coupling effects in FG beams even in uniform temperature rise in the thickness direction, by introduction of physical neutral surface concept [33–40], there will be no stretching-bending coupling. As pointed out by Barretta et al [41], in the functional gradient section, the geometric center of the section does not coincide with the elastic center. This problem should be considered and used when determining the inertia term and stiffness parameters.

Based on physical neutral surface concept, the displacement fields have the form [33, 34]:

\[
\begin{align*}
    u(x, z) &= u_0 + \left( z - \frac{\int_{-h/2}^{h/2} z E(z, T) dz}{\int_{-h/2}^{h/2} E(z, T) dz} \right) \psi - \frac{4}{3h^2} \left( z^3 - \frac{\int_{-h/2}^{h/2} z^3 E(z, T) dz}{\int_{-h/2}^{h/2} E(z, T) dz} \right) \left( \frac{\partial w}{\partial x} + \psi \right) \\
    w(x, z) &= w(x)
\end{align*}
\]

Considering nonlinear strain-displacement relationships and nonlocal strain gradient theory, the stress resultant and couples can be obtained as

\[
[1 - (ea)^2 \nabla^2] \begin{bmatrix} N_{xx} \\ M_{y(11)} \\ M_{y(12)} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} e^{(0)}_x \\ e^{(1)}_x \\ e^{(2)}_x \end{bmatrix} - \begin{bmatrix} N_T \\ M_T \end{bmatrix} S_T
\]

\[
[1 - (ea)^2 \nabla^2] Q = [1 - l^2 \nabla^2]Q_{44}^{-1} \left( z \frac{\partial z}{\partial z} \right)
\]

In which, \(ea\) is the nonlocal parameter, \(f\) is the strain gradient parameter, \(N_T, M_T\) and \(S_T\) are the forces, moment and higher order moments caused by elevated temperature, and these appeared symbols are defined as

\[
\begin{align*}
Q_{12} &= Q_{21} = 0, \quad Q_{13} = Q_{31} = 0, \quad Q_{23} = Q_{32}, \\
(Q_{11}, Q_{12}, Q_{13}, Q_{22}, Q_{23}, Q_{33}) &= \int_{-h/2}^{h/2} E(z, T) \left[ 1, f, g, f^2, fg, g^2 \right] dz,
\end{align*}
\]

\[
(Q_{44}) = \int_{-h/2}^{h/2} E(z, T) \left[ 12 + 2v(z, T) \right] g \frac{\partial z}{\partial z} dz,
\]

\[
(N_T, M_T, S_T) = \int_{-h/2}^{h/2} E(z, T) \alpha \Delta T (1, f, g) dz
\]

and

\[
\begin{align*}
    f(z, T) &= \left( z - \frac{\int_{-h/2}^{h/2} z E(z, T) dz}{\int_{-h/2}^{h/2} E(z, T) dz} \right) \\
    g(z, T) &= \left( z - \frac{\int_{-h/2}^{h/2} z E(z, T) dz}{\int_{-h/2}^{h/2} E(z, T) dz} \right) - \frac{4}{3h^2} \left( z^3 - \frac{\int_{-h/2}^{h/2} z^3 E(z, T) dz}{\int_{-h/2}^{h/2} E(z, T) dz} \right)
\end{align*}
\]

Using the energy method, the motion equations can be obtained as

\[
\begin{align*}
\frac{\partial N_{xx}}{\partial x} &= 0, \quad [1 - l^2 \nabla^2] \left( \frac{\partial^2 M_x}{\partial x^2} - N_x \frac{\partial^2 w}{\partial x^2} - \frac{\partial Q}{\partial x} - q \right) \\
+ [1 - (ea)^2 \nabla^2] \left( l_0 \frac{\partial^4 w}{\partial t^2} - l_1 \frac{\partial^4 w}{\partial x^2 \partial t^2} - l_2 \frac{\partial^3 \varphi}{\partial x \partial t^2} \right) &= 0,
\end{align*}
\]

\[
[1 - l^2 \nabla^2] \left( \frac{\partial M_{y(11)}}{\partial x} - Q \right) - [1 - (ea)^2 \nabla^2] \left( l_0 \frac{\partial^4 w}{\partial x \partial t^2} - l_1 \frac{\partial^3 \varphi}{\partial x \partial t^2} \right) = 0
\]

If we substitute the expression for force and force couple into the above equation, we get

\[
\begin{align*}
[1 - l^2 \nabla^2] \left( \frac{d^2 w}{dx^2} + \frac{d^2 \varphi}{dx^2} \right) + [1 - (ea)^2 \nabla^2] \left( l_0 \frac{\partial^4 w}{\partial x^2 \partial t^2} - l_1 \frac{\partial^3 \varphi}{\partial x^2 \partial t} \right) &= 0
\end{align*}
\]
For static bending problems, the governing equations have the form

\[ [1 - \rho^2 \nabla^2] \left( Q_{23} \frac{d^4 w}{dx^4} + Q_{33} \frac{d^2 \phi}{dx^2} - Q_{44} \left( \frac{\partial w}{\partial x} + \frac{d \phi}{dx} \right) \right) - [1 - (\rho^2 \eta)^2 \nabla^2] \left( E \frac{\partial^4 w}{\partial \xi^4} + \frac{\partial^2 \phi}{\partial \xi^2} - E \frac{\partial^2 \phi}{\partial \xi^2} \right) = 0 \]  

(9)

Defining:

\[
\xi = \frac{\pi x}{L}, \quad W = \frac{w}{L}, \quad \Phi = \frac{\phi}{\pi}, \quad (\gamma_0, \gamma_2, \gamma_3) = \frac{1}{D} (Q_{23}, Q_{33}, Q_{44}), \quad \lambda_T = \Delta T, \\
\gamma_T = E_0 \frac{L^2}{D \rho_0}, \quad (\gamma_0, \gamma_0) = \frac{1}{D \rho_0} (Q_{11}, Q_{44}), \quad \gamma_{10} = E_0 \frac{L^2 \rho_0}{D}, \quad \alpha = \frac{\pi a}{L}, \quad \beta = \frac{\pi l}{L}, \\
(\gamma_{11}, \gamma_{12}, \gamma_{13}) = \frac{E_0}{D \rho_0} (I_1, I_2, I_3), \quad \lambda_q = \frac{qL^3}{D \pi^4}, \quad \tau = \frac{\pi t}{L \sqrt{E_0}}, \quad \omega_L = \frac{\Omega L}{\pi \sqrt{E_0}}
\]

(10)

where \( E_0 \) is Young's modulus and \( \rho_0 \) is mass density of SUS304 at 300 K,

\[ D = \int_{-h/2}^{h/2} z^2 Edz, \quad A_{C} = \int_{-h/2}^{h/2} Eodz. \]

Substituting equations (10) into (8) and (9), we have

\[
\gamma_1 \frac{\partial^4 W}{\partial \xi^4} + \gamma_2 \frac{\partial^2 \Phi}{\partial \xi^2} - \left[ \int_0^\pi \left( \frac{\partial W}{\partial \xi} \right)^2 d\xi - \gamma_T \lambda_T \right] \frac{\partial^2 W}{\partial x^2} = \gamma_0 \left( \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi} \right) - \lambda_q \]

\[-\beta^2 \frac{\partial}{\partial \xi^2} \left( \gamma_1 \frac{\partial^4 W}{\partial \xi^4} + \gamma_2 \frac{\partial^2 \Phi}{\partial \xi^2} - \left[ \int_0^\pi \left( \frac{\partial W}{\partial \xi} \right)^2 d\xi - \gamma_T \lambda_T \right] \frac{\partial^2 W}{\partial x^2} = \gamma_0 \left( \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi} \right) - \lambda_q \right) = 0\]

\[
\gamma_2 \frac{\partial^4 W}{\partial \xi^4} + \gamma_3 \frac{\partial^2 \Phi}{\partial \xi^2} - \left[ \int_0^\pi \left( \frac{\partial W}{\partial \xi} \right)^2 d\xi - \gamma_T \lambda_T \right] \frac{\partial^2 W}{\partial x^2} = \gamma_0 \left( \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi} \right) - \lambda_q \right) = 0\]

(11)

3. For the bending problems

For static bending problems, the governing equations have the form

\[
\gamma_1 \frac{\partial^4 W}{\partial \xi^4} + \gamma_2 \frac{\partial^2 \Phi}{\partial \xi^2} - \left[ \int_0^\pi \left( \frac{\partial W}{\partial \xi} \right)^2 d\xi - \gamma_T \lambda_T \right] \frac{\partial^2 W}{\partial x^2} = \gamma_0 \left( \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi} \right) - \lambda_q \]

\[-\beta^2 \frac{\partial}{\partial \xi^2} \left( \gamma_1 \frac{\partial^4 W}{\partial \xi^4} + \gamma_2 \frac{\partial^2 \Phi}{\partial \xi^2} - \left[ \int_0^\pi \left( \frac{\partial W}{\partial \xi} \right)^2 d\xi - \gamma_T \lambda_T \right] \frac{\partial^2 W}{\partial x^2} = \gamma_0 \left( \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi} \right) - \lambda_q \right) = 0\]

\[
\gamma_2 \frac{\partial^4 W}{\partial \xi^4} + \gamma_3 \frac{\partial^2 \Phi}{\partial \xi^2} - \left[ \int_0^\pi \left( \frac{\partial W}{\partial \xi} \right)^2 d\xi - \gamma_T \lambda_T \right] \frac{\partial^2 W}{\partial x^2} = \gamma_0 \left( \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi} \right) - \lambda_q \right) = 0\]

(12)

Using the two step perturbation method, the following asymptotic solutions up to third order can be obtained as

\[ W(\xi, \phi) = \varepsilon A_{10}^{(1)} (1 - \cos 2m \xi) + O(\xi^4) \]  

(13)

\[ \Phi(\xi, \phi) = -\frac{2m(4m^2\gamma_2 + \gamma_4)}{4m^2\gamma_3 + \gamma_4} \varepsilon \sin 2m \xi + O(\xi^4) \]  

(14)

\[ \lambda_q = \frac{qL^3}{D \pi^4} = \lambda_q^{(1)} (\varepsilon A_{10}^{(1)} + \lambda_q^{(2)} (\varepsilon^2 A_{10}^{(2)})^3) + O(\xi^4) \]  

(15)

Taking [\xi = \pi/(2m)] in equation (13), one can obtain

\[ W_m = W|_{\xi = \pi/(2m)} = 2A_{10}^{(1)} = \frac{W_m}{L} \]  

(16)
Putting equations (16) into (15), then, the load-central deflection relationship can be obtained as

$$\frac{qL^3}{D\pi^4} = A_w^{(1)} \left( \frac{w_m}{2L} \right) + A_w^{(3)} \left( \frac{w_m}{2L} \right)^3 + ...$$

(17)

in which

$$A_w^{(1)} = 2m^3m^2(1 + m^2\beta^2) \left[ 4m^2\gamma_1 - \gamma_2 - \frac{4m^2\gamma_2 + \gamma_4}{4m^2\gamma_3 + \gamma_4} - \gamma_4 \frac{\gamma_2 - \gamma_3}{4m^2\gamma_3 + \gamma_4} \right] - \lambda T \gamma T$$

$$A_w^{(3)} = 2m_0 \pi^2m^4$$

(18)

To prove the authenticity of the present study, Figure 2 compare the load-deflection curves of the isotropic beam ($L = 100$ in, $b = h = 1$ in) in with two clamped ends. The results of Ranjan [42] and Reddy [43] by using finite element method are also displayed, from which a good agreement can be seen. In the following study, the material are adopted as in table 1.

Figure 2 shows effect of the temperature variation on the nonlinear bending behavior of FG nanobeams with two clamped ends subjected to a uniform pressure. During the calculation, $I = \int_{h/2}^{-h/2} z^2 dz$. $E_0$ is the value of $E_m$ at 300 K. It can be seen that the deflections are increased with increase in temperature.

The effects of temperature (volume fraction index $N$, and nonlocal parameter $\alpha$) on the nonlinear bending of FG nanobeams are analyzed in figures 3–5. As seen, the deflection increases as the temperature (volume fraction index, nonlocal parameter $\alpha$) increases, which is due to the reduction of stiffness as the temperature (volume fraction index, nonlocal parameter $\alpha$) rises. Apparently, the strain gradient parameter $l$ have the opposite effect compared to the nonlocal parameter when we see figure 6.
4. For nonlinear vibrations

In the present case, the displacement and the transverse load $q$ can be expanded as [32]

$$W(\xi, t, \varepsilon) = \sum_{k=1} \varepsilon^k w_k(\xi, t), \quad \Phi(\xi, t, \varepsilon) = \sum_{k=1} \varepsilon^k \varphi_k(\xi, t), \quad \lambda_q(\xi, t, \varepsilon) = \sum_{k=1} \varepsilon^k \lambda_k(\xi, t)$$

(19)

With the use of perturbation expansion, the corresponding first-order, the second-order and third-order equation can be obtained, solve those equations step by step, the asymptotic solutions can be obtained as

$$W(\xi, t, \varepsilon) = \varepsilon A_{10}^{(1)} (1 - \cos 2m\xi) + O(\xi^4)$$

(20)

$$\Phi(\xi, t, \varepsilon) = \varepsilon B_{10}^{(1)} \sin 2m\xi + \varepsilon^3 B_{10}^{(3)} \sin 2m\xi + O(\xi^4)$$

(21)

$$\lambda_q(\xi, t, \varepsilon) = \varepsilon A_{10}^{(1)} g_{30} (1 - \cos 2m\xi) + \varepsilon A_{10}^{(1)} g_{31} \cos 2m\xi + (\varepsilon^3 A_{10}^{(1)})^3 \cos 2m\xi + O(\xi^4)$$

(22)
Applying the Galerkin procedure to equation (22), one has

\[
\left(2\pi m^2\gamma_{11} + \frac{3\pi}{2} \gamma_{10} + \frac{2\pi m^2\gamma_{12}(4m^2\gamma_2 + \gamma_4)^2}{(4m^2\gamma_3 + \gamma_4)^2} - \frac{4\pi m^2\gamma_{12}(4m^2\gamma_2 + \gamma_4)}{4m^2\gamma_3 + \gamma_4}\right) \frac{d^2\varepsilon A_{10}^{(1)}}{dt^2} \\
+ \frac{(1 + m^2\beta^2)}{(1 + m^2\alpha^2)} 8\pi m^4 \left(\gamma_1 - \gamma_2 \frac{4m^2\gamma_2 + \gamma_4}{4m^2\gamma_3 + \gamma_4} - \gamma_4 \frac{\gamma_2 - \gamma_3}{4m^2\gamma_3 + \gamma_4} \right) - 2\pi m^2\lambda_0 \gamma_1 = 0
\]

\[ \omega_{NL} = \omega_L \sqrt{\frac{1 + \frac{3}{4} \frac{(1 + m^2\beta^2)}{(1 + m^2\alpha^2)} 8\pi m^4 \left(\gamma_1 - \gamma_2 \frac{4m^2\gamma_2 + \gamma_4}{4m^2\gamma_3 + \gamma_4} - \gamma_4 \frac{\gamma_2 - \gamma_3}{4m^2\gamma_3 + \gamma_4} \right) - 2\pi m^2\lambda_0 \gamma_1}{2L}} \]

The solution of equation (23) can be written as, that is to say, the dimensionless nonlinear frequency can be obtained as

\[
\omega_{NL} = \omega_L \sqrt{\frac{1 + \frac{3}{4} \frac{(1 + m^2\beta^2)}{(1 + m^2\alpha^2)} 8\pi m^4 \left(\gamma_1 - \gamma_2 \frac{4m^2\gamma_2 + \gamma_4}{4m^2\gamma_3 + \gamma_4} - \gamma_4 \frac{\gamma_2 - \gamma_3}{4m^2\gamma_3 + \gamma_4} \right) - 2\pi m^2\lambda_0 \gamma_1}{2L}}
\]
In which, the dimensional linear frequency can be expressed as,

\[
\omega_L = \sqrt{\frac{(1 + m^2)^3}{(1 + m^2a^2)^3}} \left( \frac{\pi m^4}{2}\frac{\gamma_1 - \gamma_2}{4m^2\gamma_3 + \gamma_4} - \frac{\gamma_2 - \gamma_3}{4m^2\gamma_3 + \gamma_4} \right) \frac{2\pi m^2\gamma_{11}}{L^2h} + \frac{3\pi}{2}\frac{m^2}{h} + \frac{2\pi m^2\gamma_{13}(4m^2\gamma_2 + \gamma_4)}{(4m^2\gamma_3 + \gamma_4)^2} - \frac{4\pi mn^2\gamma_{12}(4m^2\gamma_2 + \gamma_4)}{4m^2\gamma_3 + \gamma_4}
\]

To guarantee the correctness of the present research, some examples are presented for nonlinear vibration analysis.

As the first example, the relations of nonlinear-to-linear fundamental frequency ratio \(\omega_{NL}/\omega_L\) and dimensionless vibration amplitudes \(w_{\text{max}}/h\) for Si\(_3\)N\(_4\)/SUS304 beams (\(ea = l = 0\)) with two clamped ends are presented in table 2. In this example, the dimensionless linear fundamental frequency is defined by \(\omega_L^* = \Omega_L/(L^2/h)\). The Ritz method results of Zhang [33] are also displayed for comparisons, it is clear that our results agree well with the results of Zhang [33].

As the second example, the fundamental linear dimensionless frequency is compared with the results of Ebrahimi and Salari [44] based on the differential transforms method in figures 7-8, in this example, the strain gradient parameter is set to zero, it is clear that our results agree well with the results of Ebrahimi and Salari [44].

Table 3 presents the natural frequencies of Si\(_3\)N\(_4\)/SUS304 beams in thermal environments. It can be seen that, with the increase of the temperature, the modal frequencies are decreased. Also, the natural frequencies decreases as the volume fraction index increase. The relations of nonlinear-to-linear fundamental frequency ratio \(\omega_{NL}/\omega_L\) and dimensionless vibration amplitudes \(w_{\text{max}}/h\) for Si\(_3\)N\(_4\)/SUS304 beams with two clamped ends are presented in table 4. It can be concluded that the amplitude have great effect on nonlinear frequencies.
The effects of volume fraction index $N$ (nonlocal parameter $ea$) on the nonlinear vibration of FG nanobeams are analyzed in tables 5 and 6. The figure shows that the frequency decrease when the nonlocal parameter $ea$ rises, which is due to the reduction of stiffness as nonlocal parameter $ea$ rises. Apparently, the strain gradient parameter have the opposite effect compared to the nonlocal parameter when we see table 6.

| Table 3. Natural frequencies $\omega^2_k = \Omega_k(L^2/h)(\rho_b/E_b)^{1/2}$ for Si3N4/SUS304 beams with two clamped ends ($L = 20h, ea = 0$). |
|---|
| $N$ | Modes |
|---|---|---|---|---|---|---|---|---|---|
| 300 | 1 | 6.5756 | 5.3300 | 4.5265 | 3.9677 | 3.5641 | 3.2418 | 3.0880 |
| | 2 | 26.1764 | 21.2113 | 18.0109 | 15.7867 | 14.1799 | 12.8893 | 12.2884 |
| | 3 | 58.4359 | 47.3281 | 40.1773 | 35.2127 | 31.6259 | 28.7755 | 27.4147 |
| 400 | 1 | 5.0832 | 3.8710 | 3.0795 | 2.5277 | 2.1345 | 1.8143 | 1.6370 |
| | 2 | 24.6078 | 19.7483 | 16.6179 | 14.4482 | 12.8891 | 11.6465 | 11.0421 |
| | 3 | 56.5571 | 45.6330 | 38.6042 | 33.7135 | 30.2158 | 27.4222 | 26.0793 |

| Table 4. $(\omega_{NL}/\omega_k)$ for Si3N4/SUS304 beams with two clamped ends ($L = 20h, ea = 0$). |
|---|
| $T(\text{K})$ | $N$ | $\omega_k^{2}$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| 300 | 1 | 9.1414 | 1.0114 | 1.0449 | 1.0985 | 1.1695 | 1.2548 | 1.5141 | 1.8160 |
| | 0.2 | 12.2825 | 1.0114 | 1.0450 | 1.0986 | 1.1696 | 1.2549 | 1.5144 | 1.8164 |
| | 0.5 | 10.4500 | 1.0115 | 1.0453 | 1.0993 | 1.1708 | 1.2567 | 1.5177 | 1.8213 |
| 400 | 1 | 8.3663 | 1.0134 | 1.0525 | 1.1146 | 1.1962 | 1.2936 | 1.5859 | 1.9219 |
| | 0.2 | 11.4353 | 1.0129 | 1.0507 | 1.1109 | 1.1900 | 1.2846 | 1.5693 | 1.8975 |
| | 0.5 | 9.6233 | 1.0133 | 1.0520 | 1.1137 | 1.1947 | 1.2913 | 1.5817 | 1.9157 |

| Table 5. Natural frequencies $\omega^2_k = \Omega_k(L^2/h)(\rho_b/E_b)^{1/2}$ for Si3N4/SUS304 nanobeams with different nonlocal parameter $ea$ ($L = 20h, l = 0, T = 300 \text{K}$). |
|---|
| Modes | $ea$ | 0 | 1 nm | 2 nm | 3 nm |
|---|---|---|---|---|---|
| 1 | 5.3300 | 5.2178 | 5.0963 | 4.8730 |
| 2 | 21.2113 | 20.7957 | 20.0472 | 19.1269 |
| 3 | 47.3281 | 45.5742 | 43.2291 | 41.1186 |
**5. Concluding remarks**

The nonlinear bending and vibration analyses for FG nanobeams with two clamped ends have been presented based on neutral surface and higher order shear deformation beam theory are analyzed in this paper. The nonlinear strain-displacement relation is introduced and the nonlinear governing equations are solved by two-step perturbation method, many examples are solved numerically and compared with existed results. The effects of volume fraction index, environmental temperature, nonlocal parameter and strain gradient parameter are discussed in detail. Through the above analysis, we can draw the following conclusions:

For nonlinear bending, firstly, the method adopted in this paper is very close to the results obtained by finite element method adopted in the existing literature. Secondly, the deflection increases as the temperature (volume fraction index, nonlocal parameter ea) increases, which is due to the reduction of stiffness as the temperature (volume fraction index, Apparently, the strain gradient parameter l have the opposite effect compared to the nonlocal parameter.

For nonlinear vibration, the method adopted in this paper is very close to the results obtained by Ritz method and the Differential transforms method adopted in the previous literature. Second, the frequency decreases as the temperature (volume fraction index, nonlocal parameter ea) increases, which is due to the reduction of stiffness as the temperature (volume fraction index, Apparently, the strain gradient parameter l have the opposite effect compared to the nonlocal parameter.

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**Table 6.** Natural frequencies $\omega^2 = \Omega l^2 / (h)(\rho_i / E_i)^{1/2}$ for Si$_3$N$_4$/SUS304 nanobeams with different strain gradient parameter $l$ ($L = 20h$, $ea =$0, $T = 300$ K).

| Modes | 0 | 1 nm | 2 nm | 3 nm |
|-------|---|------|------|------|
| 1     | 5.3300 | 5.4536 | 5.6001 | 5.9254 |
| 2     | 21.2113 | 21.6105 | 22.5582 | 24.2647 |
| 3     | 47.3281 | 49.1853 | 51.0852 | 53.4402 |
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