A NONLOCAL BIHARMONIC OPERATOR AND ITS CONNECTION WITH THE CLASSICAL BI-LAPLACIAN

PETRONELA RADU, DANIEL TOUNDYKOV, AND JEREMY TRAGESER

Abstract. We introduce here a nonlocal operator as a natural generalization to the biharmonic operator that appears in plate theory. This operator is built in the nonlocal calculus framework defined in [3] and is connected with the recent theory of peridynamics. For the steady state equation coupled with different boundary conditions we show existence and uniqueness of solutions, as well as regularity of solutions. The boundary conditions considered are nonlocal counterparts of the classical clamped and hinged boundary conditions. For each system we show convergence of the nonlocal solutions to their local equivalents using compactness arguments developed in [2].

1. Introduction

The biharmonic operator arises in many sub-fields of continuum mechanics including elasticity (plate theory in particular). Classical results on elliptic PDEs guarantee that solutions to the corresponding homogeneous boundary value problem acquire two orders of weak differentiability with respect to the regularity of the interior forcing term. In two dimensions and with integrable forcing that yields at least $H^2 \supset W^{1,\infty}$ Sobolev regularity. In particular, such solutions (in 2D case) are necessarily continuous which makes it non-trivial to account for irregularities, e.g. cracks, in the structure. A nonlocal version of the classical Laplacian has been investigated [1] with applications in nonlocal diffusion, image processing, and biology. This nonlocal operator appears in the steady-state equations for theory of peridynamics [7] which follows the evolution of damage in solids in a unified framework, so that defects and fracture appear as a result of the deformation and it is not considered separately.

We take the nonlocal formulation of the Laplacian further and introduce a nonlocal version of the biharmonic operator. In the classical theory, plate equations which involve the biharmonic operator, take advantage of the disparity between the width of the plate and its planar dimensions. By replacing differential operators with integral operators we allow discontinuous (in fact, with lack of any Sobolev regularity) solutions, thus providing a direction for the study of dynamic fracture in plate theory. Prime examples of plate structures are suspension bridges, where the dynamic formation of cracks and their evolution is of great interest. However, the apparition of discontinuities corresponding to damage in the structure preclude the inclusion of any smoothness assumptions on the solutions. As we will show

---

2010 Mathematics Subject Classification. Primary: 45P05 Secondary: 35L35, 74K20.

Key words and phrases. keyword.

The research of the first author was partially supported by the National Science Foundation under Grant DMS-0908435.

The research of second author was partially supported by the National Science Foundation under Grant DMS-1211232.
here, solutions of the nonlocal biharmonic operator require minimal integrability assumptions.

The paper is concerned with the steady-state formulation and we show that in the limit of the vanishing non-locality, nonlocal solutions of $B[u] = f$ converge to classical solutions of $\Delta^2(u) = f$, when appropriate BC are considered.

Also, the results obtained here are transferrable to the vectorial case, as shown in [3].

If a nonlocal form of the biharmonic operator appears natural, the issue of boundary conditions is quite delicate. Two types of BC are prevalent in plate systems: hinged ($u = 0, \Delta u = 0$ on the boundary of the domain) and clamped ($u = 0, \frac{\partial u}{\partial \nu} = 0$). Since nonlocal operators are associated with collar-type domains (sets of non-zero Lebesgue measure that surround the domain) we need to find suitable ways to accommodate the nonlocal operator to its boundary. Moreover, to show that these extensions follow the classical theory we will prove that the nonlocal solution converges to the classical solution. To our knowledge, this is the first work that deals with integral approximations of higher order elliptic operators with first and second order boundary conditions.

1.1. Motivation and connection to previous work. The paper [3] extended this work to vector and tensor fields which allowed for an abstract formulation of the balance laws of momentum and energy as well as for the theory of solid mechanics in peridynamics that is analog the vector calculus formulation of the balance laws of elasticity. [3] conjectured their results were general enough to allow other applications such as to the laws of fluid mechanics and electromagnetics.

We will show that when certain conditions are imposed on $\alpha$, the local and nonlocal biharmonic coincide. We will additionally show well-posedness results for nonlocal problems involving the nonlocal biharmonic. Additionally, we will show that when the horizon goes to zero (i.e., when the volume of $\Gamma$ goes to 0) we get strong $L^2$ convergence of solutions to the nonlocal problem to the solution of the classical problem.

2. BACKGROUND

This section will contain definitions of several integral operators and some associated spaces. Throughout, $\Omega$ will be a connected bounded subset of $\mathbb{R}^2$. Because we will make connection with classical solutions to biharmonic problems we will take boundary to be of class $C^4$. The open subdomain $\Omega'$ will be compactly contained in $\Omega$. For convenience define $\Gamma := \Omega \setminus \Omega'$.

2.1. Operators. As in [3], we introduce the nonlocal peridynamic operators.

**Definition 2.1** (Nonlocal divergence). Given the function $\nu : \Omega \times \Omega \to \mathbb{R}^k$ and the antisymmetric vector valued function $\alpha : \Omega \times \Omega \to \mathbb{R}^k$, the nonlocal divergence operator $D_\alpha : \nu \mapsto D_\alpha[\nu]$ where $D_\alpha[\nu] : \Omega \to \mathbb{R}$ is defined by

$$D_\alpha[\nu](x) := -\int_{\Omega} (\nu(x,y) + \nu(y,x)) \cdot \alpha(y,x) \, dy$$

for $x \in \Omega$.

**Definition 2.2** (Nonlocal gradient). Given the function $u(x) : \Omega \to \mathbb{R}$, the formal adjoint of $D_\alpha$ is the nonlocal two-point gradient operator $G_\alpha : u \mapsto G_\alpha$ where $G_\alpha$:
\( \Omega \times \Omega \rightarrow \Omega \) is given by
\[
G_\alpha[u](x, y) = (u(y) - u(x))\alpha(x, y)
\]
for \((x, y) \in \Omega \times \Omega \).

**Definition 2.3** (Nonlocal normal). Given the function \( \nu : \Omega \times \Omega \rightarrow \mathbb{R}^k \) and the antisymmetric vector valued function \( \alpha : \Omega \times \Omega \rightarrow \mathbb{R}^k \), the nonlocal normal operator \( N_\alpha : \nu \rightarrow N_\alpha[\nu] \) where \( N_\alpha[\nu] : \Omega \rightarrow \mathbb{R} \) is defined by
\[
N_\alpha[\nu](x) := \int_\Omega (\nu(x, y) + \nu(y, x)) \cdot \alpha(y, x) dy
\]
for \( x \in \Gamma \).

**Remark 2.1.** Note that the definition of the nonlocal divergence matches (up to a negative sign) the definition of the nonlocal normal. We will use the second operator for nonlocal boundary terms and show its connection to the classical normal derivative and the notation will be suggestive in that case.

**Definition 2.4.** Let \( u : \Omega \rightarrow \mathbb{R} \) and \( \mu = \alpha^2 \) where \( \alpha : \Omega \times \Omega \rightarrow \mathbb{R}^k \) is an antisymmetric vector valued function. The nonlocal Laplace operator is defined by:
\[
\mathcal{L}_\alpha[u](x) := D_\alpha[G_\alpha[u]] = 2 \int_\Omega (u(y) - u(x))\mu(x, y) dy
\]
where \( x \in \Omega \).

It was shown in [3, Prop. 5.4], that if \( \alpha^2 \) is replaced by distributional application of \( \frac{1}{2} \Delta_x \delta(y - x) \) then \( \mathcal{L}_\alpha \) can be identified, in the sense of distributions, with the Laplace operator \( -\Delta_x \).

Also, we have the following nonlocal “integration by parts” result:

**Proposition 2.1** (Nonlocal integration by parts, [3]). Let \( \Omega \subset \mathbb{R}^d \) be open, \( u : \Omega \rightarrow \mathbb{R} \), \( \alpha : \Omega \times \Omega \rightarrow \mathbb{R}^k \) be antisymmetric, and \( \nu : \Omega \times \Omega \rightarrow \mathbb{R}^k \). Then
\[
\int_\Omega u(x)\mathcal{D}_\alpha[\nu] dx = -\int_\Omega \int_\Omega \nu \cdot G_\alpha[u] dy dx.
\]

Following the classical local setting we define the nonlocal biharmonic operator:

**Definition 2.5.** Let \( \alpha : \Omega \times \Omega \rightarrow \mathbb{R}^k \) be an anti-symmetric vector valued function, \( \mu = \alpha^2 \), and \( u : \Omega \rightarrow \mathbb{R} \). The nonlocal biharmonic is defined by
\[
\mathcal{B}_\alpha[u](x) = \mathcal{L}_\alpha[\mathcal{L}_\alpha[u]], \quad \text{for } x \in \Omega.
\]  
(2.1)

We will need the following assumption on the family of kernels used in our nonlocal formulations.

**Assumption 2.1.** For \( \delta > 0 \) let \( \rho_\delta \) be a radial compactly-supported mollifier, i.e.,
\[
\rho_\delta : C^\infty(\mathbb{R}^d; \mathbb{R}^+), \quad \int_{\mathbb{R}^d} \rho_\delta(|x|) dx = 1, \quad \text{supp}(\rho_\delta) \subset (-\delta, \delta). \quad (2.2)
\]

Define
\[
\alpha_\delta(x, y) = \alpha_\delta(x - y) := \frac{\sqrt{\rho_\delta(|x - y|)}}{|x - y|^2}(x - y).
\]  
(2.3)
2.2. Spaces. Following [4] we will utilize the functional space
\[ \mathcal{H}_\alpha^1(\Omega) := \{u \in L^2(\Omega) : \|\mathcal{G}_\alpha[u]\|_{L^2(\Omega \times \Omega)} < \infty\}. \] (2.4)

Define bilinear forms
\[ \langle (u, w) \rangle_1 = \int_{\Omega} \int_{\Omega} \mathcal{G}_\alpha[u]\mathcal{G}_\alpha[w] d\sigma d\sigma' \] (2.5)
and
\[ (u, w)_{\mathcal{H}_\alpha^1} = (u, w)_{L^2(\Omega)} + \langle (u, w) \rangle_1. \] (2.6)

Note that if \(|\alpha|^2\) is integrable then \(\mathcal{H}_\alpha^1(\Omega)\) is equivalent to \(L^2(\Omega)\). However, under Assumption 2.1 this may not be the case when \(\Omega \subset \mathbb{R}^2\).

**Theorem 2.1** (c.f. [4, Thm 2.2]). Assume \(\alpha\) satisfies Assumption 2.1. Then, \(\mathcal{H}_\alpha^1(\Omega)\) is a Hilbert space with inner product (2.6).

**Definition 2.6.** For \(\Omega' \subset \subset \Omega\), define \(\mathcal{H}_\alpha^1(\Omega')\) to be the closed subspace of functions vanishing on \(\Omega \setminus \Omega'\)
\[ \mathcal{H}_\alpha^1(\Omega') = \{u \in \mathcal{H}_\alpha^1(\Omega) : u = 0 \ a.e. \ in \ \Omega \setminus \Omega'\}. \]

In the same spirit define “second-order” nonlocal spaces.

**Definition 2.7.** Let
\[ \mathcal{H}_\alpha^2(\Omega) := \{u \in \mathcal{H}_\alpha^1(\Omega) : \|\mathcal{L}_\alpha[u]\|_{L^2} < \infty\} \] (2.7)
and using
\[ \langle (u, w) \rangle_2 := \int_{\Omega} \int_{\Omega} \mathcal{L}[u]\mathcal{L}[w] d\sigma d\sigma' \] (2.8)
define an inner product on \(\mathcal{H}^2:\)
\[ (u, w)_{\mathcal{H}^2} = (u, w)_{\mathcal{H}_\alpha^1} + \langle (u, w) \rangle_2. \]

**Proposition 2.2.** The space \(\mathcal{H}^2(\Omega)\) is a Hilbert space with the indicated above inner product.

**Proof.** Details to be included later. \(\square\)

Finally, we define Hilbert spaces associated with the boundary conditions that we will consider.

**Definition 2.8** (Nonlocal versions of “hinged” and “clamped” spaces). Let
\[ \mathcal{H}_{\alpha,H}^2(\Omega') = \{u \in \mathcal{H}_{\alpha,0}^1(\Omega') \cap \mathcal{H}_\alpha^2(\Omega) : \mathcal{L}_\alpha[u] = 0 \ a.e. \ on \ \Omega \setminus \Omega'\} \] (2.9)
\[ \mathcal{H}_{\alpha,C}^2(\Omega') = \{u \in \mathcal{H}_{\alpha,0}^1(\Omega') \cap \mathcal{H}_\alpha^2(\Omega) : \int_{\Omega} \mathcal{L}_\alpha[u] v = - \int_{\Omega \times \Omega} \mathcal{G}_\alpha[u] \cdot \mathcal{G}_\alpha[v] \ for \ all \ v \in \mathcal{H}_\alpha^1(\Omega)\}. \] (2.10)

**Remark 2.2.** From the above definition note that if the nonlocal normal operator \(\mathcal{N}_\alpha[\mathcal{G}_\alpha[u]]\) is equal to zero then the identity in the nonlocal “clamped” space follows.
3. Connection between local and nonlocal operators

A key tool in our analysis will be the following version of a nonlocal Poincaré inequality.

**Theorem 3.1** ([3] Thm. 1.2). Assume \( d = \dim \Omega \geq 2 \). Let \( (\delta_n) \) be a sequence of positive numbers decreasing to 0. Let \( (\rho_{\delta_n}) \) be a sequence of functions satisfying (2.2). There exists \( n_0 \geq 1 \) sufficiently large such that

\[
\|f - f_\Omega\|_{L^p(\Omega)} \leq C_{p,d,\Omega} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\delta_n}(|x - y|) \, dx \, dy
\]

for every \( f \in L^p(\Omega) \) and \( n \geq n_0 \). Here \( f_\Omega \) is the average value of \( f \) in \( \Omega \).

**Remark 3.1.** It should be noted that in [3], the result of Theorem 3.1 is extended to \( d = 1 \); however, in that case, it is necessary to place an additional constraint on \( \rho_{\delta_n} \).

In the nonlocal setting, embedding and compactness methods of Sobolev theory do not apply since the integrability of the difference quotient (even with a singular kernel) typically yields no gain in regularity of solutions. The crucial compactness result in this context will be provided by the following theorem by Brezis, Bourgain, and Mironescu:

**Theorem 3.2** ([2] Thm 4]). Suppose \( (f_n) \) is a sequence in \( L^p(\Omega), 1 \leq p < \infty \), of functions satisfying the uniform estimate

\[
\int_\Omega \int_\Omega \frac{|f_n(x) - f_n(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dx \, dy \leq C_0
\]

where \( (\rho_n) \) is a sequence of non-increasing mollifiers satisfying (2.2). If

\[
\int_\Omega f_n(x) \, dx = 0 \quad \text{for all} \quad n,
\]

then the sequence \( (f_n) \) is relatively compact in \( L^p(\Omega) \), so up to a subsequence we may assume \( f_n \to f \) in \( L^p(\Omega) \). Furthermore, if \( 1 < p \leq \infty \), then \( f \in W^{1,p}(\Omega) \).

The latter result has a useful corollary:

**Corollary 3.2.1.** In Theorem 3.2 replace the assumption (2.2) by the assertion that \( (f_n) \) are bounded in \( L^p(\Omega) \). Moreover, if \( 1 < p < \infty \) and esssupp \( f_n \subseteq \Omega \), then \( f \in W^{1,p}_0(\Omega) \).

**Proof.** Let \( a_n := \frac{1}{|\Omega|} \int_\Omega f_n(x) \, dx \). Since \( \Omega \) is bounded and \( (f_n) \) is bounded in \( L^p(\Omega) \), then the scalar sequence \( (a_n) \) is bounded. Define

\[
g_n(x) := f_n(x) - a_n.
\]

Then each \( g_n \) obeys (3.1) and has zero average. By Theorem 3.2 we know \( \{g_n\} \) is pre-compact in \( L^p(\Omega) \). Because \( (a_n) \) is a bounded scalar sequence, then \( \{f_n\} \) is also pre-compact in \( L^p(\Omega) \).

For the remainder, consider another other family of mollifiers \( (\tilde{\rho}_{\varepsilon_n}) \) with \( \varepsilon_n \searrow 0 \). Then \( f_n * \tilde{\rho}_{\varepsilon_n} \in C_c^\infty(\Omega) \) Define

\[
\tilde{f}_n(x) := \|f\|_{W^{1,p}(\Omega)} \frac{(f_n * \tilde{\rho}_{\varepsilon_n})(x)}{\|f_n * \tilde{\rho}_{\varepsilon_n}\|_{W^{1,p}(\Omega)}}
\]

For sufficiently small \( \varepsilon_n \) each \( \tilde{f}_n(x) \) is in \( C_c^\infty(\Omega) \). We argue that \( \tilde{f}_n(x) \) converges to \( f \) in \( L^p(\Omega) \) (details to be added later). Then this sequence is bounded in \( W^{1,p}_0(\Omega) \) and
converges weakly. Since $f$ must coincide with the weak limit, hence $f \in W^{1,p}_0(\Omega)$. □

3.1. Scaled operators. In this section we will demonstrate that with appropriate scaling, we have pointwise convergence of the nonlocal Laplace and Biharmonic operators to their local analogues.

Let

$$\Omega^\delta := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \quad \text{and} \quad \Gamma^\delta := \Omega \setminus \Omega^\delta$$

Throughout this subsection suppose Assumption 2.1 holds for kernel $\alpha_\delta$ and set

$$\mu_\delta(x, y) := |\alpha_\delta(x, y)|^2 = \frac{\rho_\delta(|x - y|)}{|x - y|^2}\frac{|x - y|^2}{s^2}.$$ 

Definition 3.1 (Scaling). Let

$$\pi_\delta(r) := \int_r^\delta s \mu_\delta(s) \, ds . \quad (3.3)$$

Let $\omega_{d-1}$ be the surface measure of unit sphere in $\mathbb{R}^d$ and define

$$C(\delta) := \frac{1}{2} \int_{B_\delta(0)} \pi(|z|) \, dz = \frac{1}{2} \omega_{d-1} \int_0^\delta \pi(r) r^{d-1} \, dr , \quad (3.4)$$

which is finite for $d \geq 2$. We will be using the scaling term

$$\sigma(\delta) := -\frac{1}{C(\delta)} . \quad (3.5)$$

This definition of $\sigma(\delta)$ placed here as a preview, whereas the expression for it comes out of the derivations presented below. Accordingly, we redefine the nonlocal Laplacian and biharmonic operators with the scaling term:

$$L_{\alpha_\delta} u(x) := \sigma(\delta) \int_{\Omega} |u(y) - u(x)| \mu_\delta(x, y) \, dy \quad (3.6)$$

$$B_{\alpha_\delta} u(x) := L_{\alpha_\delta} [L_{\alpha_\delta} u(x)] . \quad (3.7)$$

3.2. Pointwise convergence. We will show that the nonlocal operators approach uniformly their classical versions when acting on smooth functions as the peridynamic horizon $\delta$ goes to zero. The proofs of the following results were inspired by the strategy used in the upcoming paper [6].

Lemma 3.1. Let $u \in C^2(\Omega)$ and $x \in \Omega$. Further suppose $\alpha_\delta$ satisfies Assumption 2.7. For $\delta > 0$ sufficiently small,

$$L_{\alpha_\delta} [u](x) = -\sigma(\delta) \int_0^1 \int_{B_\delta(0)} s \left( \Delta u(x + sz) - \Delta u(x) \right) \pi(|z|) \, dz \, ds - \Delta u(x) \sigma(\delta) C(\delta) .$$
Proof. By definition and recalling the support of $\mu_{\delta_n}$ is contained in $B_{\delta_n}(x)$, we have

$$L_{\alpha_\delta}u(x) = \sigma(\delta) \int_{B_{\delta}(x)} [u(y) - u(x)] \mu_\delta(x, y) dy$$

$$= \sigma(\delta) \int_{B_{\delta}(x)} \int_0^1 \frac{d}{ds} [u(x + s(y - x))] \mu_\delta(x, y) ds dy$$

$$= \sigma(\delta) \int_{B_{\delta}(x)} \int_0^1 \nabla u(x + s(y - x)) \cdot (y - x) \mu_\delta(x, y) ds dy.$$  

Change the order of integration and then apply the change of variables $z = y - x$ to obtain

$$L_{\alpha_\delta}u(x) = \sigma(\delta) \int_0^1 \int_{B_{\delta}(0)} \nabla u(x + sz) \cdot [z \mu_\delta(|z|)] dz ds.$$  

With $\pi$ given by (5.3) we know

$$\nabla z \pi(|z|) = \pi'(|z|) \frac{z}{|z|} = \mu_\delta(|z|) z$$

and consequently,

$$L_{\alpha_\delta}u(x) = \sigma(\delta) \int_0^1 \int_{B_{\delta}(0)} \nabla u(x + sz) \cdot \nabla z \pi(|z|) dz ds.$$  

Since $\pi(\delta) = 0$, then integration by parts gives

$$L_{\alpha_\delta}u(x) = - \sigma(\delta) \int_0^1 \int_{B_{\delta}(0)} \text{div}_z [\nabla u(x + sz)] \pi(|z|) dz ds$$

$$= - \sigma(\delta) \int_0^1 \int_{B_{\delta}(0)} \Delta u(x + sz) s \pi(|z|) dz ds.$$  

Let $\omega_{d-1}$ be the surface measure of the unit sphere in $\mathbb{R}^d$. From the identity

$$\int_{B_{\delta}(0)} \pi(|z|) dz = \int_0^\delta \pi(r) \omega_{d-1} r^{d-1} dr$$

we may rewrite $L_{\alpha_\delta}u(x)$ in the desired form:

$$L_{\alpha_\delta}u(x) = - \sigma(\delta) \int_0^1 \int_{B_{\delta}(0)} (\Delta u(x + sz) - \Delta u(x)) s \pi(|z|) dz ds$$

$$- \sigma(\delta) \Delta u(x) \frac{\omega_{d-1}}{2} \int_0^\delta \pi(r) r^{d-1} dr.$$  

\[ \square \]

**Theorem 3.3.** Let $u \in C^4(\Omega) \cap H^4(\Omega), x \in \Omega \in \mathbb{R}^d$, and $\sigma(\delta) = -\frac{1}{\psi(\delta)}$. Suppose Assumption (8.1) holds. Then the nonlocal Laplacian operator $L_{\alpha_\delta}$ converges to the classical Laplacian $\Delta$ as $\delta \to 0$. Moreover,

$$\sup_{\delta \in (0, 1)} \sup_{x \in \Omega} |L[u]| \leq C(d, u)$$

Proof. By Lemma (5.3) we have

$$L_{\alpha_\delta}[u](x) = - \sigma(\delta) \int_0^1 \int_{B_{\delta}(0)} s[\Delta u(x + sz) - \Delta u(x)] \pi(|z|) dz ds + \Delta u(x).$$
Thus,
\[
L_{\alpha,s}[u](x) - \Delta u(x) = -\sigma(\delta) \int_0^1 \int_{B_\delta(z)} s [\Delta u(x + sz) - \Delta u(x)] \pi(|z|) dzds
\]
We will let \( P_1(s) \) be a polynomial of degree \( i \) in \( s \). Rewrite \( s = \frac{d}{ds} (\frac{s^2}{2}) \) and integrate by parts in \( s \) using \( \frac{d}{ds}(\Delta u(x + sz)) = \nabla \Delta u(x + sz) \cdot z \) along with
\[
\int_{B_\delta(z)} z \pi(|z|) dz = 0
\]
in order to obtain
\[
L_{\alpha,s}[u](x) - \Delta u(x) = -\sigma(\delta) \int_{B_\delta(0)} \int_0^1 P_1(s)[\Delta \nabla u(x + sz)] \cdot z \pi(|z|) dzds
\]
\[
= -\sigma(\delta) \int_{B_\delta(0)} \int_0^1 P_2(s)[\Delta \nabla u(x + sz) - \Delta \nabla u(x)] \cdot z \pi(|z|) dzds
\]
Since \( u \in C^4(\Omega) \cap H^4(\Omega) \) and \( \Omega \) is bounded, we know that \( |\nabla \Delta u(x + sz) - \nabla \Delta u(x)| \leq M|sz| \) for some constant \( M \). Thus we obtain
\[
|L_{\alpha,s}[u](x) - \Delta u(x)| \leq \sigma(\delta) M \int_{B_\delta(0)} \int_0^1 P_3(s)|z|^2 \pi(|z|) dzds
\]
\[
\lesssim \sigma(\delta) \omega_{d-1} \int_0^\delta \rho^{d+1} \pi(\rho) d\rho.
\]
Finally, use the fact that \( \rho^{d+1} \leq \rho^{d-1} \delta^2 \) for \( \rho \in (0, \delta) \) to obtain
\[
|L_{\alpha,s}[u](x) - \Delta u(x)| \lesssim \delta^2 \sigma(\delta) \omega_{d} = \delta^2.
\]
Taking \( \delta \searrow 0 \) verifies the result.

Now let us prove the uniform estimate
\[
\sup_{\delta \in (0,1)} \sup_{x \in \Omega} |L_{\alpha,s}[u](x)| \leq C(d, u).
\]
Since \( u \in C^4(\Omega) \cap H^4(\Omega) \) and \( \Omega \) is bounded, we know \( \|\nabla u\|_{L^\infty(\Omega)} < \infty \). Along with \ref{3.8} it gives
\[
|L_{\alpha,s}[u]| \leq \|\nabla u\|_{L^\infty(\Omega)} + \delta^2.
\]

\begin{theorem}
Let \( u \in C^7(\Omega) \cap H^7(\Omega) \) (Rem: this condition is sufficient, but can likely be relaxed.), \( x \in \Omega \in \mathbb{R}^d \), and \( \sigma(\delta) = -\frac{1}{C(\delta)} \). Also assume that Assumption 1 and 2 holds. The nonlocal biharmonic operator \( B_{\alpha,s} \) converges to the classical biharmonic \( \Delta^2 \) as \( \delta \to 0 \). Moreover,
\[
\sup_{\delta > 0} \|B[u]\| \leq C\|\Delta^2 u\|_{L^\infty(\Omega)}
\]
\end{theorem}

\begin{proof}
Recall the definition of the nonlocal biharmonic with horizon \( \delta \):
\[
B_{\alpha,s}[u](x) = \sigma(\delta) \int_{B_{\delta}(x)} [L_\delta[u](x) - L_\delta[u](y)] \mu_\delta(x, y) dy.
\]
\end{proof}
Appealing to Lemma 3.1 and canceling $\sigma(\delta)$ with $C(\delta)$, yields

\[ B_{\alpha_s}[u](x) \]

\[ = \sigma(\delta) \int_{B_s(x)} \left[ -\Delta u(y) + \Delta u(x) \right. \]

\[ - \left. \sigma(\delta) \int_0^1 \int_{B_s(0)} s[\Delta u(y + sz) - \Delta u(y)]\pi(|z|)dzds \right. \]

\[ + \left. \sigma(\delta) \int_0^1 \int_{B_s(0)} s[\Delta u(x + sz) - \Delta u(x)]\pi(|z|)dzds \right] \mu_\delta(x, y)dy. \] \hspace{1cm} (3.9)

The first term in the above equation can be simplified using the definition of $\mathcal{L}_{\alpha_s}$:

\[- \sigma(\delta) \int_{B_s(x)} [\Delta u(y) - \Delta u(x)] \mu_\delta(x, y)dy = -\mathcal{L}_{\alpha_s} [\Delta u](x).\]

From Lemma 3.1 again, we obtain

\[- \mathcal{L}_{\alpha_s} [\Delta u](x) = \left\{ \sigma(\delta) \int_0^1 \int_{B_s(0)} s[\Delta^2 u(x + sz) - \Delta^2 u(x)]\pi(|z|)dzds \right\} + \Delta^2 u(x). \]

Substituting this back into (3.9) results in

\[ B_{\alpha_s}[u](x) - \Delta^2 u(x) \]

\[ = \sigma(\delta) \int_{B_s(x)} \left[ - \sigma(\delta) \int_0^1 \int_{B_s(0)} s[\Delta u(y + sz) - \Delta u(y)]\pi(|z|)dzds \right. \]

\[ + \left. \sigma(\delta) \int_0^1 \int_{B_s(0)} s[\Delta u(x + sz) - \Delta u(x)]\pi(|z|)dzds \right] \mu_\delta(x, y)dy \] \hspace{1cm} (3.10)

Demonstrating that the boxed term is of order $\delta$ is a simplified version of the argument necessary for the first two integrals (which incorporate the non-integrable kernel $\mu_\delta$). The rest of the proof will focus on the first two summands in (3.10).

For $s, z$

\[ F_{s,z}(x) := \Delta u(x + sz) - \Delta u(x) \]

then rewrite the first two terms in (3.10):

\[ \sigma(\delta) \int_0^1 s \int_{B_s(0)} -\mathcal{L}_\delta[F_{s,z}](x) \pi(|z|)dzds \]

Apply Lemma 3.1 to obtain

\[ \sigma(\delta) \int_0^1 \int_{B_s(0)} -\mathcal{L}_\delta[F_{s,z}](x) \pi(|z|)dzds \]

\[ = \sigma^2(\delta) \int_0^1 s \int_{B_s(0)} \left[ \int_0^1 \int_{B_s(0)} \tilde{s} (\Delta F_{s,z}(x + \tilde{s}z) - \Delta F_{s,z}(x)) \pi(|\tilde{s}z|)d\tilde{s}dz \right] \pi(|z|)dzds \]

\[ + \sigma(\delta) \int_0^1 \int_{B_s(0)} \Delta F_{s,z}(x) \pi(|z|)dzds \] \hspace{1cm} (3.11)
Recall \( u \in C^7(\Omega) \cap H^7(\Omega) \) and \( \Omega \) is bounded. So there exists a constant \( M \) such that
\[
\sup_{0 \leq i \leq 6} \| D^i (u(x + s\varepsilon) - u(x)) \|_{\mathbb{R}^d} \leq M|s\varepsilon|.
\]
Thus (3.11) can be estimated as
\[
\sigma(\delta) \int_0^1 s \int_{B_{\delta}(0)} -L_s[F_s,\varepsilon](x) \pi(|\varepsilon|)d\varepsilon ds
\leq \sigma(\delta) M \int_0^1 s \int_{B_{\delta}(0)} s\varepsilon \pi(|\varepsilon|)d\varepsilon ds
\]
\[
+ \sigma(\delta) \delta \int_0^1 \int_{B_{\delta}(0)} \pi(|\varepsilon|)d\varepsilon dz + \sigma(\delta) \delta \int_{B_{\delta}(0)} \pi(|\varepsilon|)d\varepsilon
\]
Recalling that \( C(\delta) = \frac{1}{2} \int_{B_{\delta}(0)} \pi(|\varepsilon|)d\varepsilon \) and \( \sigma(\delta) = \frac{1}{C(\delta)} \) results in:
\[
\sigma(\delta) \int_0^1 s \int_{B_{\delta}(0)} -L_s[F_s,\varepsilon](x) \pi(|\varepsilon|)d\varepsilon ds \lesssim \delta .
\]

Then we ultimately arrive at
\[
|\mathbf{B}_{\alpha}[u](x) - \Delta^2 u(x)| \leq C\delta
\]
Finally, since \( u \in C^7(\Omega) \cap H^7(\Omega) \) and \( \Omega \) is bounded, we know \( \|\Delta^2 u\|_{L^\infty(\Omega)} < \infty \). Combing this with (3.13) results in
\[
|\mathbf{B}_{\alpha}[u]| \leq \|\Delta^2 u\|_{L^\infty(\Omega)} + \delta .
\]

\[
\Box
\]

4. Wellposedness of the nonlocal static problem

In this section we will look at various nonlocal boundary value problems. The first problem we will look at is the nonlocal elliptic biharmonic equation
\[
\mathbf{B}_{\alpha}[u] = f \quad \text{in} \quad \Omega'
\]
with nonlocal equivalent of hinged or clamped boundary conditions (Definition 2.8): \( u \in \mathbf{H} = \mathcal{H}^{2}_{\alpha,H} \quad \text{or} \quad \mathcal{H}^{2}_{\alpha,C} \quad (4.2) \)

Note that both the spaces are topologized by the norm in \( \mathcal{H}^{2}_{\alpha} \).

**Proposition 4.1.** Suppose \( f \in L^2(\Omega') \), and \( \alpha \) satisfies Assumption 2.4 There exists a unique (weak) solution \( u \in \mathbf{H} \) of the nonlocal PDE (1.1).

**Proof.** We prove this with the Lax-Milgram Theorem. For \( v \in \mathbf{H} \) the associated weak formulation is
\[
\int_{\Omega} \mathbf{B}_{\alpha}[u] v \, dx = \int_{\Omega} f v \, dx.
\]

Using the fact that \( v = 0 \) on \( \Omega \setminus \Omega' \) (regardless of the definition of \( \mathbf{H} \)) and through repeated application of Proposition 2.1 we obtain
\[
a[u, v] = \int_{\Omega} f v .
\]
where

\[ a[u, v] = \int_{\Omega} L_\alpha[u] L_\alpha[v] \, dx. \]

This form is continuous since

\[ |a[u, v]| \leq \|L_\alpha[u]\|_{L^2(\Omega)} \|L_\alpha[v]\|_{L^2(\Omega)} \leq \|u\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)}. \]

It remains to show ellipticity. By Theorem 3.1 we have

\[ \|u\|_{L^2(\Omega)}^2 \leq C \|G_\alpha[u]\|_{L^2(\Omega \times \Omega)}. \]

Nonlocal integration by parts gives

\[ \|G_\alpha[u]\|_{L^2(\Omega \times \Omega)}^2 = \int_{\Omega} |u L_\alpha[u]|^2 \, dx \leq \|L_\alpha[u]\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}. \]

Combining these two we conclude that

\[ \|u\|_{L^2(\Omega)} \leq C_1 \|L_\alpha[u]\|_{L^2(\Omega)} \]

and, consequently

\[ \|G_\alpha[u]\|_{L^2(\Omega \times \Omega)} \leq C \|L_\alpha[u]\|_{L^2(\Omega)}. \]

We conclude there exists a positive constant \( c_2 \) such that

\[ a(u, u) = \|L_\alpha[u]\|_{L^2(\Omega)}^2 \geq c_2 \|u\|_{H^2(\Omega)}^2. \]

By Lax-Milgram theorem there exists a unique element \( u \in H \) such that

\[ B[u] = f. \]

□

5. Convergence to classical solution: Hinged (iterated approach)

In this section we will discuss the significance of the horizon. Our goal will be to show that with appropriate scaling when the horizon approaches 0, the solution to the nonlocal equation approximates the solution of the classical equation. This method has been applied to the Navier system in [4]. Below we have the classical Laplace equation with Dirichlet boundary:

\[ \begin{align*}
\Delta v &= f, \quad x \in \Omega \\
v &= 0, \quad x \in \partial \Omega 
\end{align*} \quad (5.1) \]

Next we define the nonlocal analogue of the Laplace equation with Dirichlet boundary:

\[ \begin{align*}
L_{\alpha_{s_n}}[v_{s_n}] &= f, \quad x \in \Omega_{s_n} \\
v &= 0, \quad x \in \Omega \setminus \Omega_{s_n} 
\end{align*} \quad (5.2) \]

Keep in mind we are using the scaled version of the Laplacian \([3.4]\). Our first step will be to show that solution \( v_{s_n} \in H_{\alpha_{s_n}}^1(\Omega_{s_n}) \) of \((5.2)\) converges strongly in \( L^2(\Omega) \) to the solution of \((5.1)\) when the horizon goes to 0.

The following theorem is an altered version of [3] Thm 5.4.

**Theorem 5.1.** For all \( f \in L^2(\Omega) \), the sequence of solutions \( \{v_{s_n}\} \subset H_{\alpha_{s_n}}^1(\Omega_{s_n}) \) to \((5.2)\) converges strongly in \( L^2(\Omega) \) to \( v \in W^1_0(\Omega) \), where \( v \) solves the classical Laplace equation \((5.1)\).

**Proof.** To invoke Theorem 3.2 we need to show upper bounds on \( \|G_{\alpha_{s_n}}[v_{s_n}]\|_{L^2(\Omega \times \Omega)} \) and on \( \|v_{s_n}\|_{L^2(\Omega)} \), independently of \( \delta \). Apply Theorem 3.1 and nonlocal integration by parts:

\[ \|v_{s_n}\|_{L^2(\Omega)}^2 \leq C \|G_{\alpha_{s_n}}[v_{s_n}]\|_{L^2(\Omega \times \Omega)}^2 \leq C \|f\|_{L^2(\Omega)} \|v_{s_n}\|_{L^2(\Omega)}, \]
where $C$ is independent of $\delta$. Since $f \in L^2(\Omega)$ then $v_{\delta_n}$ is bounded in $L^2(\Omega)$. We conclude by Corollary 3.2.1 that $\{v_{\delta_n}\}$ is relatively compact in $L^2(\Omega)$ and if $v$ is a limit point of $\{v_{\delta_n}\}$, then $v \in W^{1,2}_0(\Omega)$.

We will show that any limit point solves the classical Poisson problem equation (5.1) and is unique. Let $\{v_{\delta_n}\}$ be a convergent subsequence. Consider a test function $w \in C^\infty_c(\Omega)$. By Proposition 2.1 we obtain

$$\int_{\Omega} L_{\alpha_{\delta_n}}[v_{\delta_n}]w = \int_{\Omega} L_{\alpha_{\delta_n}}[w]v_{\delta_n} = \int_{\Omega} w f.$$

Using the fact that $v_{\delta_n} \to v$ strongly in $L^2(\Omega)$, and the result of Theorem 3.3 we obtain

$$\int_{\Omega} v \Delta w \, dx = \int_{\Omega} w f \, dx$$

as $\delta_n \to 0$, verifying that $v$ is a weak solution of the classical Poisson problem. □

Now let us consider a slightly different family of equations

$$\begin{cases}
L_{\alpha_{\delta_n}}[v_{\delta_n}] = f_n, & x \in \Omega \\
v_{\delta_n} = 0, & x \in \Omega \setminus \Omega_{\delta_n}
\end{cases}
$$

(5.3)

where $v_{\delta_n} \in \mathcal{H}_{\alpha_{\delta_n},0}(\Omega_{\delta_n})$.

**Corollary 5.1.1.** Let $(f_n) \subset L^2(\Omega)$ with $f_n \to f$ in $L^2(\Omega)$. The associated sequence of solutions $(v_{\delta_n}) \subset \mathcal{H}_{\alpha_{\delta_n},0}(\Omega_{\delta_n})$ converges strongly in $L^2(\Omega)$ to $v \in W^{1,2}_0(\Omega)$ which solves (5.1).

**Proof.** This result is a straightforward extension. Details to be added. □

Repeated applications of Corollary 5.1.1 shows that solutions of the non-local hinged problem

$$\begin{cases}
L_{\alpha_{\delta_n}}[u_{\delta_n}] = f, & x \in \Omega_{\delta_n} \\
u_{\delta_n} = L_{\alpha_{\delta_n}}[u_{\delta_n}] = 0, & x \in \Omega \setminus \Omega_{\delta_n}
\end{cases}
$$

(5.4)

converge in $L^2(\Omega)$ and in the sense of distributions $\mathcal{D}'(\Omega)$ to the solution $u$ of

$$\begin{cases}
\Delta^2 u = f, & x \in \Omega \\
u = \Delta u = 0, & x \in \partial \Omega
\end{cases}
$$

(5.5)

when the horizon $\delta \to 0$. Moreover from the elliptic theory for (5.5) we a posteriori conclude $u \in W^{4,2}(\Omega)$.

6. **Convergence to classical solution for Hinged problem (direct approach)**

This section reproduces the result of the preceding one via a very similar technique, however, it is instructive to go through this particular version of the argument, because it sets the stage for the treatment of the clamped problem in the next section.

Consider again (5.1) which is the nonlocal analogue of hinged plate problem (5.5).

**Theorem 6.1.** For all $f \in L^2(\Omega)$, the sequence of solutions $(v_{\delta_n}) \subset \mathcal{H}_{\alpha_{\delta_n},0}(\Omega_{\delta_n})$ to (5.1) converges strongly in $L^2(\Omega)$ to solution $u \in W^{1,2}_0(\Omega) \cap W^{4,2}(\Omega)$ of (5.5).
Proof. We will invoke Theorem 3.2, so we need to demonstrate an upper bound on \( \|G_{\delta_n} [u_{\delta_n}]\|_{L^2(\Omega)} \), independent of \( \delta \). First, nonlocal integration by parts (Proposition 2.1) gives

\[
c \|u_{\delta_n}\|_{L^2(\Omega)}^2 \leq \|G_{\delta_n} [u_{\delta_n}]\|_{L^2(\Omega \times \Omega)}^2 = - \int_{\Omega} L_{\delta_n} [u_{\delta_n}] u_{\delta_n} \, dx \\
\leq \|L_{\delta_n} [u_{\delta_n}]\|_{L^2(\Omega)} \|u_{\delta_n}\|_{L^2(\Omega)}
\]

(6.1)

for some \( c > 0 \). Apply Theorem 3.1 and (6.1):

\[
\|u_{\delta_n}\|_{L^2(\Omega)}^2 \leq C \|G_{\delta_n} [u_{\delta_n}]\|_{L^2(\Omega)}^2 \\
\leq C \|L_{\delta_n} [u_{\delta_n}]\|_{L^2(\Omega)}^2 \\
= C \|\langle B_{\delta_n} [u_{\delta_n}], u_{\delta_n} \rangle_{L^2(\Omega)}\| \\
= C \|\langle f, u_{\delta_n} \rangle_{L^2(\Omega)}\| \\
\leq C \|f\|_{L^2(\Omega)} \|u_{\delta_n}\|_{L^2(\Omega)}
\]

where \( C \) is independent of \( \delta \). Hence \( \{u_{\delta_n}\} \) is bounded in \( L^2(\Omega) \) and consequently so is \( \{G_{\delta_n} [u_{\delta_n}]\} \). By Theorem 5.2 \( \{u_{\delta_n}\} \) is relatively compact in \( L^2(\Omega) \) and if \( u \) is a limit point of \( \{u_{\delta_n}\} \), then \( u \in W_0^{1,2}(\Omega) \).

We will show that any limit point of \( \{u_{\delta_n}\} \) solves (5.6) and consequently is unique since the latter elliptic problem has a unique solution. Pick a test function \( w \in C^\infty_c(\Omega) \). By Proposition 2.1 we have

\[
\langle B_{\delta_n} [w], u_{\delta_n} \rangle = \langle w, B_{\delta_n} [u_{\delta_n}] \rangle = \langle w, f \rangle
\]

Use the fact that \( u_{\delta_n} \to u \) strongly in \( L^2 \), Theorem 3.4 \( u_{\delta_n} = L_{\delta_n} [u_{\delta_n}] = 0 \) outside \( \Omega_{\delta_n} \) and \( \Omega_{\delta_n} \subset \text{supp}(w) \) as \( \delta \to 0 \), we obtain for all \( w \in C^\infty_c(\Omega) \),

\[
\int_{\Omega} \Delta^2 w u \, dx = \int w f \, dx
\]

as \( \delta_n \to 0 \). This verifies that \( u \in W_0^{1,2}(\Omega) \) is a distributional solution to

\[
\Delta^2 u = f
\]

Unlike in the iterated version, the above approach does not yield outright \( \Delta u = 0 \) on the boundary. It can, however, be inferred as follows. We have \( z_{\delta_n} := L_{\delta_n} [u_{\delta_n}] \) is bounded in \( L^2(\Omega) \) and via Theorem 3.2 it has a weak subsequential limit \( \eta \). This limit can be readily identified with \( \Delta u \). At the same time, the result for the nonlocal Poisson problem satisfied by \( z_{\delta_n} \) converges strongly in \( L^2(\Omega) \) to a function \( W_0^{1,2}(\Omega) \). Thus \( \Delta u = 0 \) on \( \partial \Omega \). (Further details to be added).

Conclude that \( u \in W_0^{1,2}(\Omega) \) and \( \Delta u \in W_0^{1,2}(\Omega) \). Thus \( u \) is a solution to (5.5), and has regularity \( W_0^{4,2}(\Omega) \).

\[ \square \]

7. Convergence to Classical Solution: Clamped Problem

Now let’s look at a nonlocal version of the hinged biharmonic problem.

\[
\begin{align*}
& \mathcal{B}_{\alpha_n} [u_{\delta_n}] = f, \quad x \in \Omega_{\delta} \\
& u_{\delta_n} = \mathcal{N}_{\alpha_n} [u_{\delta_n}] = 0, \quad x \in \Omega \setminus \Omega_{\delta_n}
\end{align*}
\]

(7.1)
with the classical analog:
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\Delta^2 u = f, & x \in \Omega \\
u = \frac{du}{dv} = 0, & x \in \partial \Omega.
\end{array} \right.
\end{aligned}
\] (7.2)

**Theorem 7.1.** For all \( f \in L^2(\Omega) \), the sequence of solutions \( \{u_{\delta_n}\} \subset \mathcal{H}^{2,\alpha}_{\alpha_{\delta_n},C}(\Omega_{\delta_n}) \) to (7.1) converges strongly in \( L^2(\Omega) \) to \( u \in W^{2,2}_0(\Omega) \), where \( v \) solves (7.2).

**Proof.** In the same manner as for the hinged problem, we have relative compactness of \( \{u_{\delta_n}\} \) in \( L^2(\Omega) \) and also that any limit point is contained in \( W^{1,2}_0(\Omega) \). Every limit point is a distributional solution to \( \Delta^2 u = f \).

It remains to verify the zero normal boundary condition. For every \( n \) the boundary condition of the space \( \mathcal{H}^{2,\alpha}_{\alpha_{\delta_n},C}(\Omega_{\delta_n}) \) in (2.10) provides
\[
\int_{\Omega_{\delta_n}} \mathcal{L}_{\alpha_{\delta_n}}[u]v = -\int_{\Omega} \mathcal{G}_{\alpha_{\delta_n}}[u] \cdot \mathcal{G}_{\alpha_{\delta_n}}[v],
\] (7.3)
for every \( v \in \mathcal{H}^{1,0}_{\alpha_{\delta_n},0}(\Omega_{\delta_n}) \). Let’s pass to the limit in this identity.

First, we have \( \mathcal{L}_{\alpha_{\delta_n}}[u_{\delta_n}] \rightarrow \eta \) weakly in \( L^2(\Omega) \). Let \( \phi \in C^\infty_c(\Omega) \) and take the limit \( \int_{\Omega} \mathcal{L}_{\alpha_{\delta_n}}[u_{\delta_n}] \phi = \int_{\Omega} u_{\delta_n} \mathcal{L}_{\alpha_{\delta_n}}[\phi] \) to conclude identify \( \eta = \Delta u \). Since \( \eta \in L^2(\Omega) \) we get by elliptic regularity that \( u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \).

Next, fix \( v \in C^\infty(\Omega) \cap H^4(\Omega) \), then \( v \in \mathcal{H}^{1,0}_{\alpha_{\delta_n}}(\Omega_{\delta_n}) \). We have
\[
\int_{\Omega_{\delta_n}} \mathcal{L}_{\alpha_{\delta_n}}[u_{\delta_n}]v \rightarrow \int_{\Omega} \eta v = \int_{\Omega} \Delta uv
\]
Next notice
\[
-\int_{\Omega} \mathcal{G}_{\alpha_{\delta_n}}[u_{\delta_n}] \cdot \mathcal{G}_{\alpha_{\delta_n}}[v] = \int_{\Omega} u_{\delta_n} \mathcal{L}_{\alpha_{\delta_n}}[v]
\]
Since \( v \in C^\infty(\Omega) \cap H^4(\Omega) \) we can pass to the limit on the right invoking Theorem 3.3 to obtain
\[
-\int_{\Omega} u \Delta v
\]
Integrate by parts using \( u \in H^1(\Omega) \) to arrive at
\[
\int_{\Omega} \nabla u \cdot \nabla v
\]
Thus, passing to the limit in (7.3) yields
\[
\int_{\Omega} \Delta uv = \int_{\Omega} \nabla u \cdot \nabla v \quad \text{for all} \quad v \in C^\infty \cap H^4(\Omega)
\]
Because traces of \( v \in C^\infty \cap H^4(\Omega) \) functions are dense in \( L^2(\Omega) \), it follows that
\[
\frac{\partial u}{\partial v} = 0 \quad \text{in} \quad L^2(\partial \Omega)
\]
Consequently \( u \) is the solution to (7.2) and belongs to \( W^{2,4}(\Omega) \cap W^{2,2}_0(\Omega) \). \( \square \)
REFERENCES

[1] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, and J. J. Toledo-Melero. Nonlocal diffusion problems, volume 165 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

[2] J. Bourgain, H. Brezis, and P. Mironescu. Another look at sobolev spaces. In Optimal Control and Partial Differential Equations, pages 439–455, 2001.

[3] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. Mathematical Models and Methods in Applied Sciences, 23(03):493–540, 2013.

[4] T. Mengesha and Q. Du. The bond-based peridynamic system with Dirichlet-type volume constraint. Proc. Roy. Soc. Edinburgh Sect. A, 144(1):161–186, 2014.

[5] A. C. Ponce. An estimate in the spirit of Poincaré’s inequality. J. Eur. Math. Soc. (JEMS), 6(1):1–15, 2004.

[6] P. Radu and M. Foss. Sobolev regularity for solutions to nonlocal systems with weakly singular kernels. Preprint., 2014.

[7] S. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. Journal of the Mechanics and Physics of Solids, 48:175–209, 2000.

E-mail address: pradu@unl.edu

E-mail address: dtoundyk@unl.edu

E-mail address: s-trages1@math.unl.edu

University of Nebraska-Lincoln, Department of Mathematics, Lincoln, NE 68588