LONGTIME BEHAVIOR OF A SECOND ORDER FINITE ELEMENT SCHEME SIMULATING THE KINEMATIC EFFECTS IN LIQUID CRYSTAL DYNAMICS

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Abstract. We consider an unconditional fully discrete finite element scheme for a nematic liquid crystal flow with different kinematic transport properties. We prove that the scheme converges towards a unique critical point of the elastic energy subject to the finite element subspace, when the number of time steps go to infinity while the time step and mesh size are fixed. A Lojasiewicz type inequality, which is the key for getting the time asymptotic convergence of the whole sequence furnished by the numerical scheme, is also derived.

1. Introduction. Discovered by Friedrich Reinitzer around 1888, the liquid crystalline have found nowadays a large impact in many areas of science and engineering by providing effective solutions to different problems. Liquid crystal technologies have a wide range of applications in which we could list, liquid crystal display (flat panel display), optical imaging, medical applications among others. Based on the characteristic of the molecular order, the liquid crystalline states can be roughly classified in two types: smectic and nematic. In this work, we are interested on nematic liquid crystal where the centers of mass of molecules are unordered as in isotropic fluids.

For the best of our knowledge, initiated by Oseen (1925), the continuum theory of liquid crystals has been developed mainly by Eriksen (1961) and Leslie (1968). As far as liquid crystal is concerned, a huge amount of works can be founded in the literature dedicated to diverse features of the system modeling the dynamic of the nematic liquid crystals flows so called Ericksen-Leslie model. The Ericksen-Leslie model describing dynamics of nematic liquid crystals consists of a coupling between the Navier-Stokes system and the Ginzburg-Landau equations. This system has been investigated widely through various aspects [5, 20, 2]. A comprehensive

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knowledge of the flow behavior and mechanism, a fundamental understanding of the process of such a fluid system are still relevant although a number of experimental and theoretical analysis and numerical studies have been addressed in the literature.

Authors in [2, 5] have given review about liquid crystal models. In [2], Ball gives a review of some mathematical contributions, ideas and questions concerning liquid crystals. On the same aspect, authors in [5] have presented an outstanding review of mathematical analysis of nematic and smectic-A liquids crystal models. They state some results about existence, regularity, time-periodicity and stability of solutions at infinite time for both model as well as the convergence of trajectories of strong solutions to equilibrium solutions. Elsewhere, Huang and collaborators [16] consider a numerical approximations for a model of smectic-A liquid. By an explicit-implicit treatments for nonlinear terms, they prove that the unconditional numerical scheme obeys the energy dissipation law at the discrete level. They also give a well supplied literature related to liquid crystal schemes. For nematic model, see [15] where the authors proved conditional stability of a fully discrete mixed scheme and show the conditional convergence towards measure-valued solutions of the Ericksen-Leslie model. See also [10, 4, 19] where different type of schemes of liquid crystal have been considered and their analysis provided. Further, the time asymptotic behavior towards equilibrium points for a number of continuous and discrete models has been considered in the literature. In [6] the Cahn-Hilliard-Navier-Stokes vesicle model is considered. In the same area, one has the works of [1] where the authors studied the convergence to equilibrium for a second-order time semi-discretization. In [23] is studied the asymptotic behavior at infinite time of the Backward Euler scheme applied to gradient flows (such as Ginzburg-Landau-Maxwell equations, Cahn-Hilliard equations etc). Grasseli et al. in [13] make use of two-dimensional Ginzburg-Landau-Maxwell model of superconductivity and show that for any initial datum in certain phase space, the corresponding global solution converges to an equilibrium as time goes to infinity. For the asymptotic behavior of solutions of liquid crystal models we have for example the works of [9, 14]. Fan et al. [9] show the convergence to equilibrium of a non homogeneous incompressible flows of nematic liquid crystals with trigonometric condition. Further, when the initial density is away from vacuum and bounded, they got exponential decay to an equilibrium state. Recently, In [9] the asymptotic behavior of the continuum model of liquid crystal has been considered while in [14] a conditional long-stable fully discrete finite element scheme for nematic liquid crystals flow is studied and the convergence of the whole sequence furnished by the numerical scheme towards a critical point is also shown. Mainly the convergence is obtained by using suitable Lojasiewicz-Simon type inequality either in infinite [24] or finite [21, 22] dimensional spaces related to the problem in hand. In [14] the authors have also given a suitable Lojasiewicz type inequality in finite dimension. In this paper, we follow the approach of one of the authors and his collaborator [14] and we apply it to a modified midpoint time discretization scheme obtained in [19]. In [19], a finite element method is used to simulate the hydrodynamical systems governing the motions of nematic liquid crystals in a bounded domain. Although we drew our inspiration from [14], the problem considered here is more complicated to handle due to the presence of another nonlinear term and a term expressing the effective stretching effect on the director vector. We obtain the convergence of the whole sequence of the unconditionally stable scheme to the discrete variational problem for critical point of the elastic energy. In addition, we have a second order time discretization
2. The model and its numerical scheme.

2.1. The model. In order to understand the interaction between the director field \( d \) and the flow in a nematic liquid crystal flow, let us consider the following simplified system, used in [19], derived from the original Ericksen-Leslie dynamics [7, 8, 17, 18] by means of a penalization via Ginzburg-Landau potential:

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \nu \nabla \cdot D(u) + \nabla p - \lambda \nabla \cdot ((\nabla u)^t \nabla d) \\
- \lambda \nabla \cdot (\beta(\Delta d - f(d)) d^t + (\beta + 1) d(\Delta d - f(d))^t) = 0, \\
\partial_t d + (u \cdot \nabla) d + \beta(\nabla u) d + (\beta + 1)(\nabla u)^t d - \gamma(\Delta d - f(d)) = 0,
\end{aligned}
\]

(2.1)

where the strain rate \( D(u) \) is defined by \( D(u) = (1/2)(\nabla u + (\nabla u)^t) \). We assume that a nematic liquid crystal is confined in an open bounded domain \( \Omega \subset \mathbb{R}^N \) (\( N = 2 \) or 3) with boundary \( \partial \Omega \). The macroscopic unknowns \( u : \Omega \times [0, +\infty) \rightarrow \mathbb{R}^N \) (velocity), and \( p : \Omega \times [0, +\infty) \rightarrow \mathbb{R} \) (the pressure) are coupled to the microscopic order parameter \( d : \Omega \times [0, +\infty) \rightarrow \mathbb{R}^N \) which is the preferential orientation vector of molecules. Here \( f(d) = \frac{1}{2}(|d|^2 - 1) \) is the penalization function. The coefficient \( \gamma, \lambda, \nu \) are positive constants standing for time relaxation, elasticity and viscosity, respectively and \( \beta \in [-1, 0] \).

Note that \( f(d) = \nabla_d F(d) \) where \( F(d) = \frac{1}{4\gamma}(|d|^2 - 1)^2 \) is the Ginzburg-Landau potential.

The problem (2.1) is completed with the initial and (Dirichlet) boundary conditions

\[
\begin{aligned}
u(x, 0) &= u_0(x), & d(x, 0) &= d_0(x) & \text{in } \Omega. \\
u(x, t) &= 0, & d(x, t) &= g(x) & \text{on } \partial \Omega \times (0, T).
\end{aligned}
\]

(2.2)

(2.3)

Note that Dirichlet boundary condition for the orientation vector is independent of the time, and the following compatibility condition

\[
g = d_0|_{\partial \Omega}
\]

must be imposed.

For convenience of writing, hereafter we define \( D_\beta(u) = \beta(\nabla u) + (\beta + 1)(\nabla u)^t \).

2.2. Notation. We use similar notations as in [14].

- We denote \( Q = (0, +\infty) \times \Omega \) and \( \Sigma = (0, +\infty) \times \partial \Omega \).
- In general the notation will be abridged. We set \( L^p = L^p(\Omega), p \geq 1, H^1_0 = H^1_0(\Omega) \), etc. Boldface letters will be used for vectorial spaces, for instance \( L^2 = L^2(\Omega)^N \).

In Sect. 2, the model is presented as well as the finite elements spaces, the numerical scheme and some preliminary results. A Lojasiewicz type inequality is shown in Sect. 3 while the convergence in infinite time and some convergence rates estimates of the numerical scheme are considered in Sect. 4.
2.3. Global in time weak solutions. We consider a weak solution in the following sense:

Definition 2.1. A vector \((\mathbf{u}, \mathbf{d})\) is a weak solution of (2.1)-(2.3) in \((0, +\infty)\) if

\[
\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{\Sigma} = 0, \quad \mathbf{d}|_{\Sigma} = \mathbf{g},
\]

\[
(\mathbf{u}, \mathbf{d}) \in L^\infty(0, +\infty; L^2 \times H^1), \quad \int_\Omega F(\mathbf{d})dx \in L^\infty(0, +\infty),
\]

\[
(\nabla \mathbf{u}, \partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla)\mathbf{d} + D_\beta(\mathbf{u})\mathbf{d}) \in L^2(0, +\infty; L^2 \times L^2),
\]

verifies

\[
(\partial_t \mathbf{u}, \mathbf{v}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v})
\]

\[
+ \frac{\lambda}{\gamma} (\partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla)\mathbf{d} + D_\beta(\mathbf{u})\mathbf{d}, (\mathbf{v} \cdot \nabla)\mathbf{d} + D_\beta(\mathbf{v})\mathbf{d}) = 0, \quad \forall \mathbf{v} \in \mathcal{V},
\]

\[
\partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla)\mathbf{d} + D_\beta(\mathbf{u})\mathbf{d} - \gamma(\Delta \mathbf{d} - f(\mathbf{d})) = 0, \quad \text{a.e. in } Q.
\]

\[
\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{d}(0) = \mathbf{d}_0 \quad \text{in } \Omega.
\]

Moreover, for almost every \(t \in [0, +\infty)\) there holds

\[
E(\mathbf{u}(t), \mathbf{d}(t)) + \int_0^t \left(\nu|\nabla \mathbf{u}(s)|^2 + \frac{\lambda}{\gamma} |\partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla)\mathbf{d} + D_\beta(\mathbf{u})\mathbf{d}|_2^2\right)ds = E(\mathbf{u}_0, \mathbf{d}_0),
\]

where

\[
E(\mathbf{u}(t), \mathbf{d}(t)) = E_h(\mathbf{u}(t)) + \lambda E_e(\mathbf{d}(t)) := \frac{1}{2} |\mathbf{u}(t)|_2^2 + \lambda \int_\Omega \left(\frac{1}{2} |\nabla \mathbf{d}(t)|_2^2 + F(\mathbf{d}(t))\right)dx
\]

is the total energy which is a sum of the kinetic energy \(E_h\) and the elastic energy \(E_e\).

The existence of global in time weak and regular solutions of (2.1) with periodic boundary conditions, have been obtained in [12] for both 2D and 3D cases, but with the assumption of big viscosity for the 3D case.

2.4. Finite element spaces. Let \(T_h\) be a quasi-uniform triangulation of a bounded polygonal or polyhedral domain \(\Omega \subset \mathbb{R}^N\) into triangles or tetrahedrons of maximal diameter \(h > 0\), i.e. \(\Omega = \bigcup_{K \in T_h} K\). Let \((\mathbf{X}_h, M_h, \mathbf{D}_h) \subset (H^1_0(\Omega), L^2_0(\Omega), H^1(\Omega))\) be finite element spaces defined as follow

\[
M_h = \{ \mathbf{p}_h \in L^2_0(\Omega) : \mathbf{p}_h|_K \in P_0(K) \}\n\]

\[
\mathbf{X}_h = \{ \mathbf{v}_h \in C(\bar{\Omega}, \mathbb{R}^N) \cap H^1_0 : \mathbf{v}_h|_K \in P_2(K, \mathbb{R}^N) \}.
\]

• The \(L^p\) norm is denoted by \(|\cdot|_p, 1 \leq p \leq \infty\), the \(H^m\) norm by \(||\cdot||_m\) (in particular \(|\cdot|_2 = ||\cdot||_0\) and the product norm in \(H^m \times H^m\) by \(||\cdot||_{m \times m}\). The inner product of \(L^2(\Omega)\) is denoted by \((\cdot, \cdot)\).

• We set \(\mathcal{V}\) the space formed by all fields \(\mathbf{u} \in C^\infty(\Omega)^N\) satisfying \(\nabla \cdot \mathbf{u} = 0\). We denote \(\mathcal{V}\) the closure of \(\mathcal{V}\) in \(H^1\). \(\mathcal{V}\) is a Hilbert space for the norm \(||\cdot||_1\). Furthermore,

\[
\mathcal{V} = \left\{ \mathbf{u} \in H^1; \nabla \cdot \mathbf{u} = 0, \mathbf{u} = 0 \text{ on } \partial \Omega \right\}
\]

\[
L^2_0(\Omega) = \left\{ p : p \in L^2(\Omega), \int_\Omega p(x)dx = 0 \right\}
\]
\[ D_h = \{ \bar{d}_h \in C(\bar{\Omega}, \mathbb{R}^N) : \bar{d}_h|_K \in P_1(K, \mathbb{R}^N) \}, \quad D_{0h} = D_h \cap H^1_0(\Omega). \]

Let
\[ V_h = \{ \bar{v}_h \in X_h : (\text{div}\bar{v}_h, q_h) = 0 \quad \forall \, q_h \in M_h \}. \]

Note that the pair \((X_h, M_h)\) verifies the compatibility Inf-Sup condition [11]: There exists \(\beta > 0\) (independent of \(h\)) such that
\[ \|q_h\|_{L^2(\Omega)} \leq \beta \sup_{\bar{v}_h \in X_h} \frac{(q_h, \nabla \cdot \bar{v}_h)_\Omega}{\|\bar{v}_h\|_1}, \quad \forall \, q_h \in M_h. \]

The triangulation of \(\Omega\) and the discrete spaces verify the following inverse inequalities
\begin{align}
\|\bar{v}_h\|_{H^1} &\leq C h^{-1}\|\bar{v}_h\|_2, \quad \forall \, \bar{v}_h \in X_h, \\
\|\bar{d}_h\|_{W^{1,\infty}} &\leq C h^{-1}\|\bar{d}_h\|_1, \quad \forall \, \bar{d}_h \in D_h, \\
\|\bar{d}_h\|_{W^{1,\infty}} &\leq C h^{-1}\|\bar{d}_h\|_1, \quad \forall \, \bar{d}_h \in D_h.
\end{align}

Here and in the sequel, \(C\) denotes a generic constant which can change even in a single line. Let \(N_h = \{\bar{a}_j\}_{j \in J}\) denote the set of all nodes of \(T_h\), we choose the nodal interpolation operator \(\mathcal{I}_{D_h} : C(\bar{\Omega}, \mathbb{R}^N) \to D_h\), such that \(\mathcal{I}_{D_h} \psi := \sum_{z \in N_h} \psi(z) \varphi_z\), where \(\{\varphi_z : z \in N_h\} \subset D_h\) denotes the nodal basis for \(D_h\) and \(\psi \in C(\bar{\Omega}, \mathbb{R}^N)\).

For each \(\phi, \psi \in C(\bar{\Omega}, \mathbb{R}^N)\), we define
\[ ((\phi, \psi))_h := \int_\Omega \mathcal{I}_{D_h}((\phi, \psi)) \, dx = \sum_{z \in N_h} \rho_z \langle \phi(z), \psi(z) \rangle, \quad ||\phi||_h^2 := (\phi, \phi)_h, \]
where \(\rho_z = \int_\Omega \varphi_z \, dx\) for \(z \in N_h\). Note that for all \(\phi_h, \psi \in D_h\),
\[ ||\phi_h||_h \leq ||\phi_h||_h \leq (N+2)^{1/2}||\phi_h||_2. \]

Let \(\Delta_h : D_h \to D_h\) be the discrete Laplace operator defined as:
\[ (\Delta_h \bar{d}_h, \bar{d}_h) = - (\nabla \bar{d}_h, \nabla \bar{d}_h), \quad \forall \, \bar{d}_h \in D_h. \]

Note that (see [3]) there exists a constant \(C > 0\) such that for all \(\bar{d}_h \in D_h\), there holds
\[ ||\Delta_h \bar{d}_h||_2 \leq Ch^{-2}||\bar{d}_h||_2. \]

### 2.5. Numerical scheme.

The numerical scheme will be based on the following mixed weak formulation of problem (2.1)
\begin{align}
(\partial_t \bar{u}, \bar{u}) + (u \cdot \nabla)\bar{u}, \bar{u}) + \nu(\nabla \bar{u}, \nabla \bar{u}) &\quad (\nabla \cdot \bar{u}, \bar{p}) \\
+ \frac{\lambda}{\gamma} (\partial_t \bar{d} + (u \cdot \nabla)\bar{d} + D_\beta(u)\bar{d}, (\nabla \cdot \bar{d}) &\quad (\nabla \cdot \bar{u}, \bar{p}) = 0 \quad \forall \, \bar{p} \in L^2_0(\Omega), \\
\partial_t \bar{d}, \bar{\bar{d}} &\quad (u \cdot \nabla)\bar{d}, \bar{d} + (D_\beta(u)\bar{d}, \bar{d}) \\
+ \gamma (\nabla \bar{d}, \nabla \bar{d}) &\quad (f(\bar{d}), \bar{d}) = 0 \quad \forall \, \bar{d} \in H^1_0(\Omega), \\
\bar{d}|_{\partial \Omega} &\quad g(x). \tag{2.11}
\end{align}

The numerical scheme is a modified midpoint scheme due to the unit length relaxation of the penalization function. This approximate unit length relaxation term \(f_h\) is so designed to get an accurate discrete energy law for the fully discrete system. The numerical scheme is coupled, nonlinear and based on approximating the weak
formulation (2.11) by a semi-implicit Euler scheme in time and finite element in space.

Initialization: Let \((u^n_h, d^n_h) \in (X_h, D_h)\) be a suitable approximation of \((u_0, d_0)\) satisfying
\[
(\nabla \cdot u^n_h, \tilde{p}_h) = 0 \quad \forall \tilde{p}_h \in M_h.
\]

Step \(n+1\): Given \((u^n_h, d^n_h) \in X_h \times D_h\) and find \((u^{n+1}_h, d^{n+1}_h, p^{n+1}_h) \in X_h \times D_h \times M_h\), solving the system:
\[
\begin{align*}
(d_t u^{n+1}_h, \bar{u}_h) + ((u^{n+\frac{1}{2}} \cdot \nabla) u^n_h, \bar{u}_h) + \frac{1}{2} (\nabla \cdot u^{n+\frac{1}{2}}) u^n_h, \bar{u}_h) & + \frac{\lambda}{\gamma} \left( d_t d^{n+1}_h + (u^{n+\frac{1}{2}} \cdot \nabla) d^n_h + D_\beta(u^{n+\frac{1}{2}}) d^n_h + (\bar{u}_h \cdot \nabla) d^{n+\frac{1}{2}} + D_\beta(\bar{u}_h) d^{n+\frac{1}{2}} \right) \\
+ \nu (\nabla u^{n+\frac{1}{2}}, \nabla \bar{u}_h) - (p^{n+\frac{1}{2}}, \nabla \cdot \bar{u}_h) & = 0 \quad \forall \bar{u}_h \in X_h, \quad (2.12) \\
(d_t d^{n+1}_h + (u^{n+\frac{1}{2}} \cdot \nabla) d^n_h + D_\beta(u^{n+\frac{1}{2}}) d^n_h + d^{n+\frac{1}{2}_h}, \bar{d}_h) & + \gamma (\nabla d^{n+\frac{1}{2}}, \nabla \bar{d}_h) + \gamma (f_h(d^n_h, d^{n+1}_h), \bar{d}_h) = 0 \quad \forall \bar{d}_h \in D_{0h}, \quad (2.13) \\
d^{n+1}_{h, \partial\Omega} = g_h, \quad (2.14)
\end{align*}
\]

where \(g_h \in D_h\) is an approximation of \(g\).
\[
\begin{align*}
d_t u^{n+1}_h &= \frac{1}{k} (u^{n+1}_h - u^n_h), \quad d_t d^{n+1}_h = \frac{1}{k} (d^{n+1}_h - d^n_h), \\
u^n_{n+\frac{1}{2}} &= \frac{1}{2} (u^n_{n+1} + u^n_h), \quad d^n_{n+\frac{1}{2}} = \frac{1}{2} (d^n_{n+1} + d^n_h), \quad p^n_{n+\frac{1}{2}} = \frac{1}{2} (p^n_{n+1} + p^n_h)
\end{align*}
\]

and
\[
f_h(d^n_h, d^{n+1}_h) = \frac{1}{2} \left( \frac{|(d^{n+1}_h)^2 - 1|}{2} + \frac{|(d^n_h)^2 - 1|}{2} \right)
\]
is an approximation to the penalization function. The existence of a solution to this coupled and nonlinear scheme is a consequence (cf. [11]) of a point-fixed argument jointly with the following result about the unconditional stability of the discrete energy. A solution to the scheme may not be unique.

2.6. Unconditional long-time stability. From the local energy law obtained in [19] one has the following:

Lemma 2.2 (Local energy law). Let \((u^n_h, d^n_h, p^n_h)\) be any solution of the algorithm (2.12)-(2.15). Then one has for all \(n \geq 0\),
\[
k \nu |\nabla u^{n+\frac{1}{2}}|_2^2 + k \lambda |d_t d^{n+1}_h + (u^{n+\frac{1}{2}} \cdot \nabla) d^n_h + D_\beta(u^{n+\frac{1}{2}}) d^n_h + (\bar{u}_h \cdot \nabla) d^{n+\frac{1}{2}} + D_\beta(\bar{u}_h) d^{n+\frac{1}{2}}|_2^2 \\
+ E(u^{n+1}_h, d^{n+1}_h) = E(u^n_h, d^n_h). \quad (2.16)
\]

Adding (2.16) for \(n = 0, \cdots, r - 1\) (for any \(r \in \mathbb{N}^+\)), one has the following discrete global energy law:
\[
k \sum_{n=0}^{r-1} \left( \nu |\nabla u^{n+\frac{1}{2}}|_2^2 + \frac{\lambda}{\gamma} |d_t d^{n+1}_h + (u^{n+\frac{1}{2}} \cdot \nabla) d^n_h + D_\beta(u^{n+\frac{1}{2}}) d^n_h + (\bar{u}_h \cdot \nabla) d^{n+\frac{1}{2}} + D_\beta(\bar{u}_h) d^{n+\frac{1}{2}}|_2^2 \right) \\
+ E(u^r_h, d^r_h) = E(u^0_h, d^0_h). \quad (2.17)
\]
In the reminder of the paper, we assume that there exists a constant $C_0$ independent of $h$, $k$ and $\epsilon$, such that

$$E(u_h^0, d_h^0) \leq C_0.$$  

We deduce from (2.17) that for all $n \geq 0$,

$$E(u_h^n, d_h^n) \leq C_0.$$  

3. A Lojasiewicz type inequality. In this section, we derive a useful result in order to get the convergence of the whole sequence furnished by the numerical scheme to a steady state system. To this end, denote

$$S = \{ d_h^\infty \in D_h \mid d_h^\infty |_{\partial \Omega} = g_h, \langle \nabla d_h^\infty, \nabla \bar{d}_h \rangle + (f(d_h^\infty), \bar{d}_h) = 0 \ \forall \ d_h \in D_{0h} \}$$

the set of critical points for the elastic energy $E_\epsilon(d_h)$ subject to $d_h \in D_h$ such that $d_h_{|\partial \Omega} = g_h$. In order to prove the convergence to equilibrium, one needs the following suitable Lojasiewicz type inequality proved in [14].

Lemma 3.1 (Lojasiewicz type inequality). Let $d_h^\infty \in S$. Then there exists $\theta \in (0, 1/2]$, $\sigma > 0$, $\rho > 0$ and $K(h) > 0$ such that, for any $d_h \in D_h$ with $d_h_{|\partial \Omega} = g_h$ such that $|d_h - d_h^\infty|^2 < \rho/K(h)$, it holds

$$|E_\epsilon(d_h) - E_\epsilon(d_h^\infty)|^{1-\theta} \leq \sigma K(h)|P_{D_{0h}}(-\Delta_h d_h + f(d_h))|_2$$

with $P_{D_{0h}}$ the $L^2$-projector onto $D_{0h}$.  

The constant $\theta \in (0, 1/2)$ is called the Lojasiewicz exponent of $d_h^\infty$. For the scheme concerned in the paper, Lemma 3.1 itself is not sufficient to ensure the convergence of the whole sequence $d_h^n$ furnished by the scheme (2.12)-(2.15) to equilibrium. Lemma 3.1 has been derived for a scheme corresponding to a model where an auxiliary unknown $w$ has been introduced in the following way:

$$w = -\Delta d + f(d).$$

The implicit treatment of the penalization function in [14], ensures that

$$w_h(d_h) := P_{D_{0h}}(-\Delta_h d_h + f(d_h))$$

with $P_{D_{0h}}$ the $L^2$-projector onto $D_{0h}$ and $\Delta_h : D_h \to D_h$ is the discrete Laplace operator with Dirichlet boundary condition defined as:

$$\Delta_h : (d_h)_{|\partial \Omega} = g_h \text{ such that } (\Delta_h d_h, \bar{d}_h) = -\langle \nabla d_h, \nabla \bar{d}_h \rangle \ \forall \ \bar{d}_h \in D_{0h}.$$  

For our concern, since there is a semi-implicit treatment of the penalization function in (2.14), we have that

$$w_h(d_h^{n+1}) = P_{D_{0h}}(-\Delta_h d_h^{n+1} + f(d_h^{n+1})) \neq P_{D_{0h}}(-\Delta_h d_h^{n+1} + f(d_h^n, d_h^{n+1}))$$

$$= d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}.$$  

Then the previous Lojasiewicz type inequality could not be applied directly to show the convergence of the whole sequence to equilibrium. Instead one needs the following

Lemma 3.2. There are two constants $C > 0$ and $I(h,k) > 0$ such that

$$|P_{D_{0h}}(-\Delta_h d_h^{n+1} + f(d_h^{n+1}))|_2 \leq I(h,k)C[|P_{D_{0h}}(-\Delta_h d_h^{n+\frac{1}{2}} + f(d_h^n, d_h^{n+1}))|_2$$

$$+ |\nabla u_h^{n+\frac{1}{2}} |_2].$$
Throughout the rest of the paper, we will use $C$ to denote a universal constant that may vary in different places but independent of $k, h$ and $n$.

Proof. Since $\lambda |\nabla d_h^n|^2/2 \leq C_0$, let us show that there exists $C > 0$, depending only on $C_0$ and the Dirichlet data $g$, such that

$$||d_h^n||_1 \leq C. \tag{3.1}$$

Indeed, considering a lifting $G_h$ of $g_h$ stable with respect to the $H^1$-norm, that is $G_h \in D_h$ such that $G_h = g_h$ on $\partial\Omega$ and $\|G_h\|_{H^1} \leq C$, we have $(d_h^n - G_h)_{|\partial\Omega} = 0$, hence using Poincaré inequality, we get

$$|d_h^n - G_h|_2 \leq C|\nabla d_h^{n+1} - \nabla G_h|_2 \leq C(|\nabla d_h^n|_2 + |\nabla G_h|_2) \leq C \left(\frac{2C_0}{\lambda}\right)^{1/2} + C,$$

where $C$ is independent of $n$. Therefore

$$|d_h^n|_2 \leq |d_h^n - G_h|_2 + |G_h|_2 \leq C, \tag{3.2}$$

and (3.1) follows from (3.2) and from $\lambda |\nabla d_h^n|^2/2 \leq C_0$. By the same proof as the one of (3.1), we obtain

$$||d_h^{n+1}||_1 \leq C. \tag{3.3}$$

We relate $P_{Doh}(\Delta_h d_h^n + f(d_h^n))$ and $P_{Doh}(\Delta_h d_h^{n+1} + f_h(d_h^n, d_h^{n+1}))$ as follows:

$$P_{Doh}(\Delta_h d_h^n + f(d_h^n)) = P_{Doh}(\Delta_h d_h^{n+1} + f_h(d_h^n, d_h^{n+1})) + P_{Doh}(\Delta_h d_h^{n+1} + f(d_h^{n+1}) + \Delta_h d_h^{n+1} - f_h(d_h^n, d_h^{n+1})).$$

Denoting

$$I = -\Delta_h d_h^n + f(d_h^n) + \Delta_h d_h^{n+1} - f_h(d_h^n, d_h^{n+1}),$$

one has, using $d_h^{n+1} = \frac{1}{2}(d_h^n + d_h^{n+1})$,

$$I = -\Delta_h d_h^n + \frac{1}{2}\Delta_h d_h^n + \frac{1}{2}\Delta_h d_h^n + f(d_h^n) + f(d_h^{n+1}) - f_h(d_h^n, d_h^{n+1}),$$

$$= -\frac{1}{2}\Delta_h (d_h^n + d_h^{n+1}) + f(d_h^n) - f_h(d_h^n, d_h^{n+1}) := I_1 + I_2.$$}

In the sequel, we bound $I_1$ and $I_2$ by $d_h^{n+1} - d_h^n$. In light of (2.10), $I_1$ is bounded as follows

$$I_1 \leq C|\Delta_h (d_h^{n+1} - d_h^n)|_2 \leq Ch^{-2}d_h^{n+1} - d_h^n|_2. \tag{3.4}$$

For $I_2$, one has

$$f_h(d_h^n, d_h^{n+1}) = \frac{1}{\epsilon^2}((d_h^n)^2 - 1 + (d_h^n)^2 - (d_h^{n+1})^2)(d_h^{n+1} + d_h^n - d_h^{n+1}),$$

$$= f(d_h^n) + \frac{1}{2\epsilon^2}((d_h^n)^2 - 1)(d_h^n - d_h^{n+1}),$$

$$+ \frac{1}{4\epsilon^2}((d_h^n)^2 - (d_h^{n+1})^2)(d_h^{n+1} + d_h^n),$$

$$= f(d_h^n) + \frac{(d_h^n)^2 - 1 + (d_h^{n+1})^2}{4\epsilon^2}(d_h^n - d_h^{n+1}),$$

hence

$$|I_2| \leq C((d_h^n)^2 - 1 + (d_h^{n+1} + d_h^n)^2)|d_h^{n+1} - d_h^n|_2,$$

$$\leq C(\|d_h^{n+1}\|_\infty^2 + \|d_h^n\|_\infty^2 + 1)|d_h^{n+1} - d_h^n|_2.$$
Then using the inverse inequality (2.9) and the bounds (3.1) and (3.3), one ends with

$$|I_2|_2 \leq C(h)|d_h^{n+1} - d_h^n|_2. \quad (3.5)$$

Combining (3.4) and (3.5), we get

$$|P_{D_{ob}} (\Delta_h d_h^{n+1} + f(d_h^{n+1}))|_2 \leq |P_{D_{ob}} (\Delta_h d_h^{n+1} + f_h(d_h^n, d_h^{n+1}))|_2 + C(h)|d_h^{n+1} - d_h^n|_2. \quad (3.6)$$

Now it remains to bound \(d_h^{n+1} - d_h^n\) by \(\Delta_h d_h^{n+1} + f_h(d_h^n, d_h^{n+1})\) and \(|\nabla u_h^{n+1/2}|_2\).

From (2.14), one has

$$d_h^{n+1} = P_{D_{ob}}((u_h^{n+1/2} \cdot \nabla) d_h^{n+1/2} + D\beta(u_h^{n+1/2} d_h^{n+1/2}))$$

$$+ \gamma P_{D_{ob}} (\Delta_h d_h^{n+1/2} + f_h(d_h^n, d_h^{n+1})), \quad \text{then}$$

$$|d_h^{n+1}|_2 \leq \gamma |P_{D_{ob}} (\Delta_h d_h^{n+1/2} + f_h(d_h^n, d_h^{n+1}))|_2$$

$$+ |(u_h^{n+1/2} \cdot \nabla) d_h^{n+1/2}|_2 + |D\beta(u_h^{n+1/2} d_h^{n+1/2})|_2. \quad (3.7)$$

Now, let us bound the two last terms of the RHS of (3.7). One has from (3.1) and (3.3) that

$$||d_h^{n+1/2}||_1 \leq C, \quad (3.8)$$

then

$$|(u_h^{n+1/2} \cdot \nabla) d_h^{n+1/2}|_2 \leq |u_h^{n+1/2}||\nabla d_h^{n+1/2}|_3 \leq C|\nabla u_h^{n+1/2}|_2|d_h^{n+1/2}|_{W^{1,2}(\Omega)},$$

$$\leq \sqrt{h} C|\nabla u_h^{n+1/2}|_2, \quad (3.9)$$

where we have used the inverse inequality (2.8). On the other hand

$$|D\beta(u_h^{n+1/2} d_h^{n+1/2})|_2 = |\beta(\nabla u_h^{n+1/2} d_h^{n+1/2} + (\beta + 1)(\nabla u_h^{n+1/2}) d_h^{n+1/2})|_2,$$

$$\leq |\beta(\nabla u_h^{n+1/2}) d_h^{n+1/2}|_2 + |(\beta + 1)(\nabla u_h^{n+1/2}) d_h^{n+1/2}|_2,$$

$$\leq C|\nabla u_h^{n+1/2}|_2|d_h^{n+1/2}|_\infty \leq \sqrt{h} C|\nabla u_h^{n+1/2}|_2, \quad (3.10)$$

where we have used the inverse inequality (2.9). Using (3.9) and (3.10) in (3.7), one gets

$$|d_h^{n+1} - d_h^n|_2 \leq kC(h) (|P_{D_{ob}} (\Delta_h d_h^{n+1/2} + f_h(d_h^n, d_h^{n+1}))|_2 |\nabla u_h^{n+1/2}|_2),$$

$$\leq kC(h) (|d_h^{n+1} + (u_h^{n+1/2} \cdot \nabla) d_h^{n+1/2} + D\beta(u_h^{n+1/2} d_h^{n+1/2})|_2 |\nabla u_h^{n+1/2}|_2).$$

and taking in account (3.6), the proof of Lemma 3.2 is finished.

4. **Convergence at infinite time.** In this section, we show the convergence at infinite time of the whole sequence \(d_h^n\) furnished by scheme (2.12)-(2.15) to the discrete variational problem for critical points of the elastic energy. To this end, we firstly show that this result is true for subsequence of \(d_h^n\) (**Step 1**) and finally for the whole sequence (**Step 2**) by using Lemma 3.2 which is a key point in obtaining such a result.
Theorem 4.1. In the conditions of Lemma 2.2, there exists a unique \( d_h^\infty \in \mathcal{S} \) such that the following limits hold as \( n \) tends to \( +\infty \):

\[
E(u_h^n, d_h^n) \searrow E^\infty_h = E_\infty(d_h^\infty),
\]

\[
u u_h^n \rightarrow 0 \text{ in } L^2(\Omega) \quad \text{and} \quad d_h^n \rightarrow d_h^\infty \text{ in } H^1(\Omega).
\]

Proof. Step 1: \( d_h^\infty \in \mathcal{S} \). It follows from Lemma 2.2 that the real sequence \( E(u_h^n, d_h^n) \) is decreasing, and since \( E(u_h^n, d_h^n) \) is bounded from below by zero, \( E(u_h^n, d_h^n) \) tends to some \( E^\infty_h \in \mathbb{R}_+ \) as \( n \) tends to \( +\infty \). On the other hand we also get from the same Lemma that the following convergences hold as \( n \) tends to \( +\infty \):

\[
\nabla u_h^{n+\frac{1}{2}} \rightarrow 0 \quad \text{in } L^2(\Omega),
\]

\[
d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}} \rightarrow 0 \quad \text{in } L^2(\Omega).
\]

Moreover, it holds the bounds

\[
|\nabla d_h^{n+1}|_2 \leq C, \quad |u_h^{n+1}|_2 \leq C, \quad |\nabla u_h^{n+\frac{1}{2}}|_2 \leq C.
\]

Let show that \( u_h^{n+1} \rightarrow 0 \) in \( H^1(\Omega) \cap L^p(\Omega) \) \( \forall p \), as \( n \rightarrow \infty \).

On one hand, we have \( u_h^{n+1} = u_h^{n+\frac{1}{2}} - \frac{1}{2}(u_h^{n+1} - u_h^n) \), therefore

\[
|u_h^{n+1}|_2 \leq C(|u_h^{n+\frac{1}{2}}|_2 + |u_h^{n+1} - u_h^n|_2).
\]

On the other hand, (2.12) can be rewritten as

\[
\begin{aligned}
(d_t u_h^{n+1}, \bar{u}_h) \\
= - \frac{1}{2} \left( (\nabla \cdot u_h^{n+\frac{1}{2}}) u_h^{n+\frac{1}{2}}, \bar{u}_h \right) - \nu \left( \nabla u_h^{n+\frac{1}{2}}, \nabla \bar{u}_h \right) \\
- \frac{\lambda}{\gamma} \left( d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}, (\bar{u}_h \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(\bar{u}_h) d_h^{n+\frac{1}{2}} \right) \\
:= \sum_{i=1}^{4} I_i.
\end{aligned}
\]

Recall that in the initialization we have

\[
(\nabla \cdot u_h^0, \bar{p}_h) = 0 \quad \forall \bar{p}_h \in M_h.
\]

Thus by induction, we deduce from equation (2.13) that for all \( n \geq 0 \), we have

\[
(\nabla \cdot u_h^n, \bar{p}_h) = 0 \quad \forall \bar{p}_h \in M_h.
\]

Since \( \bar{u}_h \) will be chosen as \( u_h^{n+1} - u_h^n \), therefore the term with the pressure is omitted. Now let us bound \( I_i, \ i = 1, \ldots, 4 \). Concerning \( I_1 \), we have

\[
I_1 = \left( (u_h^{n+\frac{1}{2}} \cdot \nabla) u_h^{n+\frac{1}{2}}, \bar{u}_h \right) \leq |u_h^{n+\frac{1}{2}}|_2 |\nabla u_h^{n+\frac{1}{2}}|_2 |\bar{u}_h|_6 \\
\leq C |u_h^{n+\frac{1}{2}}|_2^{\frac{1}{2}} |\nabla u_h^{n+\frac{1}{2}}|_2^{\frac{3}{2}} |\bar{u}_h|_2 \leq C |\nabla u_h^{n+\frac{1}{2}}|_2 |\bar{u}_h|_2.
\]

By virtue of (2.7) and (4.5) we get

\[
I_1 \leq C(h) |\nabla u_h^{n+\frac{1}{2}}|_2 |\bar{u}_h|_2 \leq C(h) |\bar{u}_h|_2.
\]

Likewise, \( I_2 \) can be bounded as \( I_1 \). The term \( I_3 \) can be bounded as follows

\[
I_3 = \left( \nabla u_h^{n+\frac{1}{2}}, \nabla \bar{u}_h \right) \leq |\nabla u_h^{n+\frac{1}{2}}|_2 |\nabla \bar{u}_h|_2 \leq C(h) |\nabla u_h^{n+\frac{1}{2}}|_2 |\bar{u}_h|_2,
\]
where we have used (2.7). The remaining term $I_4$ is treated as follows
\[
I_4 = C(d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}} , (\bar{u}_h \cdot \nabla) d_h^{n+\frac{1}{2}} \\
+ D_\beta(\bar{u}_h) d_h^{n+\frac{1}{2}})
\]
\[
\leq C|d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2 (|\bar{u}_h|_2 |\nabla d_h^{n+\frac{1}{2}}|_2)
\]
\[
\leq C|d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2 (|\bar{u}_h|_6 |\nabla d_h^{n+\frac{1}{2}}|_3
\]
\[
+ |D_\beta(\bar{u}_h) d_h^{n+\frac{1}{2}}|_2).
\]
Using (2.8), (3.8), (3.10) and (2.7), we end up with
\[
I_4 \leq C(h)|d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2
\]
\[
\leq C(h)|d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2 |\bar{u}_h|_2.
\]
Gathering all the obtained bounds for $I_i$, $i = 1, \ldots, 4$, one gets
\[
(d_t u_h^{n+1}, u_h) \leq C(h)|\bar{u}_h|_2 (|\nabla u_h^{n+\frac{1}{2}}|_2
\]
\[
+ |d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2).
\]
Therefore taking $\bar{u}_h = u_h^{n+1} - u_h^n$, we get
\[
|u_h^{n+1} - u_h^n|_2 \leq kC(h)(|\nabla u_h^{n+\frac{1}{2}}|_2
\]
\[
+ |d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2). 
\]
(4.7)
Thus, one has $|u_h^{n+1} - u_h^n|_2 \to 0$ as $n \to \infty$ and finally from Poincaré inequality and (4.6), one gets
\[
u_h^{n+1} \to 0 \quad \text{in } L^2(\Omega).
\]
(4.8)
Therefore
\[
u_h^{n+1} \to 0 \quad \text{in } H^1(\Omega) \cap L^p(\Omega) \quad \forall \, p,
\]
(4.9)
since in finite dimension, weak convergence and strong convergence are equivalent, and all the norms are equivalent (in particular $L^p$-convergence and $H^1$-convergence are equivalents).

On the other end, from (3.3), there exists a subsequence denoted by $(d_h^{m+1})$ and a function $d_h^\infty$ in $D_h$ such that the following convergences hold as $m \to +\infty$:
\[
d_h^{m+1} \to d_h^\infty \quad \text{in } H^1(\Omega) \cap L^p(\Omega) \quad \forall \, p.
\]
(4.10)
From (4.1)-(4.4) and (4.10), it follows that
\[
d_h^{m} \to d_h^\infty \quad \text{in } H^1(\Omega) \cap L^p(\Omega) \quad \forall \, p.
\]
(4.11)
Recall that $u_h^{m+1} \to 0$, $u_h^{m+\frac{1}{2}} \to 0$ and $u_h^m \to 0$ as $m \to \infty$ then passing to the limit as $m \to \infty$ in (2.14), it follows that $d_h^\infty \in D_h$ verifies the discrete variational problem for critical points of the elastic energy:
\[
d_h^\infty |_{\partial \Omega} = g_h, \
(\nabla d_h^\infty, \nabla \bar{d}_h) + (f(d_h^\infty), \bar{d}_h) = 0 \quad \forall \bar{d}_h \in D_{bh},
\]
(4.12)
that is $d_h^\infty \in S$. Step 2: The whole sequence $d_h^{n+1}$ converges to the same limit.
We distinguish two possibilities: **Possibility 1:** Assume that there exists \( n_0 \) such that \( E(u_h^{n_0}, d_h^{n_0}) = E_c(d_h^{\infty}) \). In this case, for all \( n \geq n_0, \) \( E(u_h^{n+1}, d_h^{n+1}) = E_c(d_h^{\infty}) \). Thus, from the local energy law (2.16) one gets

\[
\nabla u_h^{n+\frac{1}{2}} \equiv 0, \quad d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}} \equiv 0. \tag{4.13}
\]

By Poincaré inequality we have from (4.13) that

\[
u_h^{n+\frac{1}{2}} \equiv 0. \tag{4.14}
\]

and then \( d_t d_h^{n+1} \equiv 0 \) that is \( d_h^{n+1} = d_h^n \), hence \( d_h^{n+\frac{1}{2}} = d_h^{n+1} = d_h^n = d_h^\infty \).

On the other hand, from (4.7), \( u_h^{n+1} = u_h^n \), hence \( u_h^{n+\frac{1}{2}} = u_h^{n+1} = u_h^n = 0 \) and the proof of the Theorem is trivial. If \( \nabla u_h^{n+\frac{1}{2}} \equiv 0 \) and \( -\Delta_h d_h^{n+\frac{1}{2}} + f_h(d_h^n, d_h^{n+1}) = 0 \) for some \( n \geq 0 \), one gets as above \( u_h^{n+1} \equiv 0 \) and \( d_h^{n+1} = d_h^n \equiv d_h^\infty \). Namely \( E(u_h^{n+1}, d_h^{n+1}) = E_c(d_h^{\infty}) \).

**Possibility 2:** Assume that \( E(u_h^{n+1}, d_h^{n+1}) > E_c(d_h^{\infty}) \) for all \( n \geq 0 \). Then

\[
|\nabla u_h^{n+\frac{1}{2}}|_2 + |d_t d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta(u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2 > 0,
\]

in particular

\[
|\nabla u_h^{n+\frac{1}{2}}|_2 + |P_{D_h} (\Delta_h d_h^{n+\frac{1}{2}} + f_h(d_h^n, d_h^{n+1}))|_2 > 0 \quad \text{for all} \quad n \geq 0.
\]

Indeed, if we assume by contradiction that \( -\Delta_h d_h^{n+\frac{1}{2}} + f_h(d_h^n, d_h^{n+1}) = 0 \) and \( \nabla u_h^{n+\frac{1}{2}} \equiv 0 \) for some \( n \geq 0 \), one gets as above \( u_h^{n+1} \equiv 0 \) and \( d_h^{n+1} = d_h^n \equiv d_h^\infty \). Namely \( E(u_h^{n+1}, d_h^{n+1}) = E_c(d_h^{\infty}) \).

On the other hand, from Lemma 3.1, since \( d_h^{\infty} \in S, E_c \) satisfies the Lojasiewicz inequality at \( d_h^{\infty} \), there exist \( \theta \in (0, 1/2], \sigma > 0, \rho > 0 \) and \( K(h) > 0 \) such that, if \( |d_h^{n+1} - d_h^{\infty}|_2 < \rho/K(h) \) then

\[
|E_c(d_h^{n+1}) - E_c(d_h^{\infty})|^{1-\theta} \leq \sigma K(h)|P_{D_h} (\Delta_h d_h^{n+1} + f_h(d_h^n, d_h^{n+1}))|_2.
\]

Then owing to Lemma 3.2, there exists \( C > 0 \) and \( I(h, k) > 1 \) such that

\[
|E_c(d_h^{n+1}) - E_c(d_h^{\infty})|^{1-\theta} \leq \sigma C K(h)|P_{D_h} (\Delta_h d_h^{n+1} + f_h(d_h^n, d_h^{n+1}))|_2 + |\nabla u_h^{n+\frac{1}{2}}|_2. \tag{4.15}
\]

In the following, we let \( L(h, k) = \sigma C K(h)|I(h, k) \).

Using the fact that \( 2(1-\theta) - 1 \geq 0 \) (because \( \theta \in (0, 1/2) \)) and Poincaré inequality, one has from (4.5) that

\[
|u_h^{n+1}|_2^{2(1-\theta)} = |u_h^{n+1}|_2^{2(1-\theta)} - 1 |u_h^{n+1}|_2 \leq C|u_h^{n+1}|_2 \leq C|\nabla u_h^{n+1}|_2. \tag{4.16}
\]

From the basic inequality

\[
(a + b)^{1-\theta} \leq a^{1-\theta} + b^{1-\theta}, \quad \forall a, b \geq 0, \tag{4.17}
\]

one gets

\[
(E(u_h^{n+1}, d_h^{n+1}) - E_c(d_h^{\infty}))^{1-\theta} = \left( \frac{1}{2} |u_h^{n+1}|_2^2 + |E_c(d_h^{n+1}) - E_c(d_h^{\infty})| \right)^{1-\theta} \leq \left( \frac{1}{2} |u_h^{n+1}|_2^2 + L(h, k)(|P_{D_h} (\Delta_h d_h^{n+1} + f_h(d_h^n, d_h^{n+1}))|_2 + |\nabla u_h^{n+\frac{1}{2}}|_2) \tag{4.15, 4.16, 4.17}
\]

It follows that
Hence
\[ C(h,k) \left( |E_h^{n+1} - E^\infty| \right) \leq C(h,k) \left( |d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta (u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2 + |\nabla u_h^{n+\frac{1}{2}}|_2 \right). \]

To prove the convergence of the whole sequence, we proceed as in the proof of Theorem 4 in [14] and Theorem 2.4 in [23].

Denoting \( E_n^{n,\infty} = E(u_h^{n}, d_h^{n}) - E_c(d_h^{\infty}) \), one has \( (E_n^{n,\infty}) \subseteq \mathbb{R}_+ \) and \( E_n^{n,\infty} \searrow 0 \) as \( n \to +\infty \). Moreover we get from (2.14), the local energy inequality (2.16) and (4.1) that
\[ \frac{C(h)}{k} |d_h^{n+1} - d_h^n|_2^2 + E_n^{n+1,\infty} \leq E_n^{n,\infty} \]
and
\[ k \left( \nu |\nabla u_h^{n+\frac{1}{2}}|_2^2 + \frac{\lambda}{\gamma} |d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta (u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2 \right) \leq E_n^{n,\infty}, \]

hence
\[ k \frac{C(h)}{k} [(|\nabla u_h^{n+\frac{1}{2}}|_2 + |d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta (u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2)^2 + E_n^{n+1,\infty}] \leq E_n^{n,\infty}. \]

Let \( n \) big enough such that \( |d_h^{n+1} - d_h^{\infty}|_2 < \beta/K(h) \), then inequality (4.18) holds.

At this point, we consider two cases:

**Case 1:** \( E_n^{n+1,\infty} > E_n^{n,\infty} / 2 \). We get
\[ \left[ E_n^{n,\infty} \right]^\theta - \left[ E_n^{n+1,\infty} \right]^\theta = \int_{E_n^{n,\infty}}^{E_n^{n+1,\infty}} \theta x^{\theta-1} dx \geq \int_{E_n^{n,\infty}}^{E_n^{n+1,\infty}} \theta \left[ E_n^{n,\infty} \right]^{\theta-1} dx \geq 2^{\theta-1} \theta \left[ E_n^{n+1,\infty} \right]^{\theta-1} \left[ E_n^{n,\infty} - E_n^{n+1,\infty} \right], \]
\[ \geq 2^{\theta-1} \theta \frac{k}{C(h,k)} |d_h^{n+1} + (u_h^{n+\frac{1}{2}} \cdot \nabla) d_h^{n+\frac{1}{2}} + D_\beta (u_h^{n+\frac{1}{2}}) d_h^{n+\frac{1}{2}}|_2 \]
\[ \geq 2^{\theta-1} \theta \frac{k}{C(h,k)} |d_h^{n+1} - d_h^{n}|_2. \]

**Case 2:** Assume \( E_n^{n+1,\infty} \leq E_n^{n,\infty} / 2 \). In this case, we bound directly as follows
\[ |d_h^{n+1} - d_h^n|_2 \leq \frac{\sqrt{k}}{C(h)} \left[ E_n^{n,\infty} - E_n^{n+1,\infty} \right]^{1/2} \leq C \left[ E_n^{n,\infty} \right]^{1/2} \]
\[ = \frac{\sqrt{k}}{C(h)} \left( 1 - \frac{1}{\sqrt{2}} \right)^{-1} \left( 1 - \frac{1}{\sqrt{2}} \right) \left[ E_n^{n,\infty} \right]^{1/2} \]
\[ = \frac{\sqrt{k}}{C(h)} \left( 1 - \frac{1}{\sqrt{2}} \right)^{-1} \left[ E_n^{n,\infty} \right]^{1/2} - \frac{1}{\sqrt{2}} \left[ E_n^{n,\infty} \right]^{1/2} \]
Then, in both cases, for all \( n \) such that \( |d_{n+1}^{\infty} - d_{h}^{\infty}|_2 < \rho/K(h) \), we have
\[
|d_{n+1}^{\infty} - d_{h}^{\infty}|_2 \leq C(h,k)([E_{n,\infty}]^\theta - [E_{n+1,\infty}]^\theta) \\
+ \frac{\sqrt{E}}{C(h)}([E_{n,\infty}]^{1/2} - [E_{n+1,\infty}]^{1/2}).
\]

(4.21)

Let \( \tilde{E} > 0 \) be small enough such that
\[
\frac{C(h,k)}{2^{q-1}2} \tilde{E}^{\theta} + \frac{\sqrt{E}}{C(h)} \tilde{E}^{1/2} \leq \frac{\rho}{3K(h)}.
\]

In particular, if \( E_{n,\infty} \leq \tilde{E} \) then \( |d_{n+1}^{\infty} - d_{h}^{\infty}|_2 < \rho/(3K(h)) \). Let \( n_m \) large enough such that \( |d_{n+1}^{n+m} - d_{h}^{\infty}|_2 < \rho/(3K(h)) \) and \( E_{n+m,\infty} \leq \tilde{E} \). Therefore
\[
|d_{n+1}^{n+m} - d_{h}^{\infty}|_2 < \rho/(3K(h)).
\]

Let \( \tilde{N} \geq n_m \) be the largest integer (including \( +\infty \)) such that \( |d_{h}^{\infty} - d_{h}^{n}|_2 < 2\rho/(3K(h)) \) for all \( n \) such that \( n_m \leq n \leq \tilde{N} \). From (4.19)
\[
\frac{C(h)}{k} |d_{h}^{\tilde{N}+2} - d_{h}^{\tilde{N}+1}|_2 \leq E_{\tilde{N},\infty} - E_{\tilde{N}+2,\infty} \leq E_{\tilde{N}+1,\infty} \leq \tilde{E}.
\]

In addition to (4), \( \tilde{E} \) is also chosen small enough such that
\[
\sqrt{\tilde{E}} \leq \frac{\rho}{3K(h)}.
\]

Therefore \( |d_{h}^{\tilde{N}+2} - d_{h}^{\tilde{N}+1}|_2 \leq \frac{\rho}{3K(h)} \).

Assume by contradiction that \( \tilde{N} \) is finite. In this case,
\[
|d_{h}^{\tilde{N}+2} - d_{h}^{\tilde{N}+1}|_2 \leq |d_{h}^{\tilde{N}+2} - d_{h}^{\tilde{N}+1}|_2 + |d_{h}^{\tilde{N}+1} - d_{h}^{\infty}|_2 < \frac{\rho}{3K(h)} + \frac{2\rho}{3K(h)} = \frac{\rho}{K(h)}.
\]

Applying (4.21) to every \( n_m \leq n \leq \tilde{N} \) and since \( E_{n,\infty} \) is decreasing, we obtain
\[
|d_{h}^{\tilde{N}+2} - d_{h}^{n+1}|_2 \leq \sum_{n=n_m+1}^{\tilde{N}+1} |d_{h}^{n+1} - d_{h}^{n}|_2 \leq \sum_{n=n_m+1}^{\tilde{N}+1} \frac{C(h,k)}{2^{q-1}2} \tilde{E}^{\theta} + \frac{\sqrt{E}}{C(h)} \tilde{E}^{1/2} \\
\leq \frac{\rho}{3K(h)}.
\]

(4.22)

Thus, \( |d_{h}^{\tilde{N}+2} - d_{h}^{\infty}|_2 \leq \frac{\rho}{3K(h)} \leq \frac{\rho}{3K(h)} + \rho/(3K(h)) < 2\rho/(3K(h)) \), and this contradicts the definition of \( \tilde{N} \). Therefore \( \tilde{N} = +\infty \) and from (4.22)
\[
\sum_{n \geq n_m+1} |d_{h}^{n+1} - d_{h}^{n}|_2 \leq \rho/(3K(h))
\]

hence the whole sequence \( (d_{n}^{h})_{n} \) converges towards \( d_{h}^{\infty} \).

**Remark 1.** With appropriate changes (according to Lemma 3.2) of the proof of Theorem 5 in [14], we are also able to estimate the following convergence rates to equilibrium.
Theorem 4.2. Under assumptions of Theorem 4.1, let \( \theta \in (0, 1/2) \) be the Łojasiewicz exponent of \( d_h^{\infty} \in S \) given in Lemma 3.1, where \( d_h^{\infty} \) is the limit of the sequence \( (d_h^n)_{n \geq 0} \subset D_h \).

1. If \( \theta = 1/2 \), the convergence is geometric, i.e. there exists \( \tilde{n} \in \mathbb{N} \), and \( \tau(k, h) \), \( \mu(k, h) > 0 \), such that for all \( n > \tilde{n} \),

\[
|d_h^n - d_h^{\infty}|_2 \leq \mu(k, h) \exp \left( -\tau(k, h)nk \right).
\]

2. If \( 0 < \theta < 1/2 \), the convergence is polynomial of order \( -\theta/(1-2\theta) \), i.e. there exists \( \tilde{n} \in \mathbb{N} \) and \( \mu(k, h) > 0 \) such that for all \( n > \tilde{n} \),

\[
|d_h^n - d_h^{\infty}|_2 \leq \frac{\mu(k, h)}{(nk)^{\theta/(1-2\theta)}}.
\]

\[\Box\]

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