RECURRENCE TIMES AND RATES OF MIXING
FOR INVERTIBLE DYNAMICAL SYSTEMS

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Abstract. We consider invertible discrete-time dynamical systems having a hyperbolic
product structure in some region of the phase space with infinitely many branches and
variable recurrence time. We show that the decay of correlations of the SRB measure
associated to that hyperbolic structure is related to the tail of the recurrence times.
We also give sufficient conditions for the validity of the Central Limit Theorem. This
generalizes previous results by Benedicks and Young.

CONTENTS

1. Introduction .................................................. 1
2. Induced schemes ............................................. 7
3. Back to the original dynamics ............................. 12
4. A solenoid with intermittency ............................ 16
Appendix A. Mixing rates for tower maps ............... 19
References ....................................................... 29

1. Introduction

One of the most powerful ways of describing the dynamical features of chaotic dynamical
systems is through invariant probability measures. A map $f$ is said to be mixing with
respect to an invariant probability measure $\mu$ if

$$|\mu(f^{-n}(A) \cap B) - \mu(A)\mu(B)| \to 0, \quad \text{when } n \to \infty,$$

for any measurable sets $A, B$. Standard counterexamples show that in general there is
no specific rate at which this convergence to 0 occurs. However, defining the correlation
function of observables $\varphi, \psi \colon M \to \mathbb{R}$,

$$C_n(\varphi, \psi; \mu) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right|,$$

Date: October 10, 2018.

2000 Mathematics Subject Classification. 37A25, 37D25.

Key words and phrases. Recurrence times, Decay of correlations, Central Limit Theorem.

Work carried out at the Federal University of Bahia, University of Porto and IMPA. JFA was par-
tially supported by FCT through CMUP and POCI/MAT/61237/2004. VP was partially supported by
PADCT/CNPq and POCI/MAT/61237/2004.
it is sometimes possible to obtain specific rates of decay, which depend only on the map \( f \) (up to a multiplicative constant which is allowed to depend on \( \varphi, \psi \)), provided the observables \( \varphi, \psi \) have sufficient regularity. Notice that choosing these observables to be characteristic functions this gives exactly the definition of mixing. Still in this direction, the Central Limit Theorem states that the probability of a given deviation of the average values of an observable along an orbit from the spatial average is essentially given by a Normal Distribution.

Since the work of Sinai, Ruelle and Bowen [9, 8, 4] it is known that uniformly hyperbolic diffeomorphisms (Axiom A, Anosov) possess SRB (or physical) measures with exponential decay of correlations and satisfying the Central Limit Theorem. By physical measure we mean an invariant probability measure such that for a large set (positive volume) of initial states the asymptotic time average (with respect to a continuous observable) coincides with the spatial average of that observable (with respect to the measure). A key ingredient in the proofs of Sinai, Ruelle and Bowen are Markov partitions, which permit to codify the dynamics and from its codification to deduce the main statistical features of the dynamical system.

In the context of non-uniformly hyperbolic diffeomorphisms, Benedicks and Young introduced in [3] some kind of structures with Markov flavor in certain regions of the phase space with infinitely many branches and variable return times. This structures enabled them to obtain exponential decay of correlations and deduce the Central Limit Theorem for Hénon maps. Further developments by Young in [11] lead to a joint treatment of some non-uniformly hyperbolic diffeomorphisms, including Hénon maps, billiards with convex scatterers and Axiom A attractors. This kind of approach has also been successfully implemented by Young in [12] for studying the rates of mixing of non-invertible systems with some non-uniformly expanding behavior.

The frameworks developed by Young in [11] and [12] are certainly among the most powerful tools for studying the statistical properties of non-uniformly hyperbolic dynamical systems. In both approaches, there is an explicit relation between the tail of the recurrence times to the hyperbolic structure and the decay of correlations, at least for some specific rates. However, the results in both papers do not depict reasonably the whole scenario. On the one hand, the model in [11] can only be applied to systems whose decay of correlations is exponential. On the other hand, the model in [12], in spite of being suitable for other decay rates, is specific to non-invertible systems. Let us mention that such a simple diffeomorphism as the solenoid with intermittency that we present in Section 1.4 does not fit the model in [11]; see Remark 1.2.

The present work essentially aims at being a step farther in the construction of a theory on the statistical features of non-uniformly hyperbolic diffeomorphisms. We believe that hyperbolic structures with sub-exponential tail of recurrence times can play an important role in obtaining the rates of mixing for the diffeomorphisms introduced by Viana in [10]. Such hyperbolic structures can possibly be useful also in the study of some classes of billiards and Poincaré return maps for flows, for which the tails of recurrence frequently decay at sub-exponential rates.
1.1. **Hyperbolic structures.** Let \( f : M \to M \) be defined on a finite dimensional Riemannian manifold \( M \), and let \( \text{Leb} \) denote a normalized volume form on the Borel sets of \( M \) that we call it *Lebesgue measure*. Given a submanifold \( \gamma \subset M \) we use \( \text{Leb}_\gamma \) to denote the measure on \( \gamma \) induced by the restriction of the Riemannian structure to \( \gamma \).

An embedded disk \( \gamma \subset M \) is called an *unstable manifold* if \( \text{dist}(f^{-n}(x), f^{-n}(y)) \to 0 \) exponentially fast as \( n \to \infty \) for every \( x, y \in \gamma \). Similarly, \( \gamma \) is called a *stable manifold* if \( \text{dist}(f^n(x), f^n(y)) \to 0 \) exponentially fast as \( n \to \infty \) for every \( x, y \in \gamma \).

**Definition 1.** Let \( \text{Emb}^1(D^n, M) \) be the space of \( C^1 \) embeddings from \( D^n \) into \( M \). We say that \( \Gamma^u = \{ \gamma^u \} \) is a *continuous family of \( C^1 \) unstable manifolds* if there is a compact set \( K^u \), a unit disk \( D^n \) of some \( \mathbb{R}^n \), and a map \( \Phi^u : K^u \times D^n \to M \) such that

1. \( \gamma^u = \Phi^u(\{x\} \times D^n) \) is an unstable manifold;
2. \( \Phi^u \) maps \( K^u \times D^n \) homeomorphically onto its image;
3. \( x \mapsto \Phi^u(\{x\} \times D^n) \) defines a continuous map from \( K^u \) into \( \text{Emb}^1(D^n, M) \).

Continuous families of \( C^1 \) stable manifolds are defined similarly.

**Definition 2.** We say that \( \Lambda \subset M \) has a *hyperbolic product structure* if there exist a continuous family of unstable manifolds \( \Gamma^u = \{ \gamma^u \} \) and a continuous family of stable manifolds \( \Gamma^s = \{ \gamma^s \} \) such that

1. \( \Lambda = (\cup \gamma^u) \cap (\cup \gamma^s) \);
2. \( \dim \gamma^u + \dim \gamma^s = \dim M \);
3. each \( \gamma^s \) meets each \( \gamma^u \) in exactly one point;
4. stable and unstable manifolds are transversal with angles bounded away from 0.

Let \( \Lambda \subset M \) have a hyperbolic product structure, whose defining families are \( \Gamma^s \) and \( \Gamma^u \). A subset \( \Lambda_0 \subset \Lambda \) is called an *s-subset* if \( \Lambda_0 \) also has a hyperbolic product structure and its defining families \( \Gamma^s_0 \) and \( \Gamma^u_0 \) can be chosen with \( \Gamma^s_0 \subset \Gamma^s \) and \( \Gamma^u_0 = \Gamma^u \); *u-subsets* are defined analogously. Given \( x \in \Lambda \), let \( \gamma^s(x) \) denote the element of \( \Gamma^s \) containing \( x \), for \( * = s, u \). For each \( n \geq 1 \) let \( (f^n)^u \) denote the restriction of the map \( f^n \) to \( \gamma^u \)-disks, and let \( \text{det} D(f^n)^u \) be the Jacobian of \( D(f^n)^u \). We require that the hyperbolic product structure \( \Lambda \) satisfies several properties:

(P1) **Markov:** there are pairwise disjoint s-subsets \( \Lambda_1, \Lambda_2, \ldots \subset \Lambda \) such that

(1) \( \text{Leb}_\gamma (\Lambda \setminus \cup \Lambda_i) \cap \gamma) = 0 \) on each \( \gamma \in \Gamma^u_0 \);
(2) for each \( i \in \mathbb{N} \) there is \( R_i \in \mathbb{N} \) such that \( f^{R_i}_i(\Lambda_i) \) is u-subset, and for all \( x \in \Lambda_i \)

\[
 f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).
\]

In the statements of the remaining properties about the hyperbolic structure we assume that \( C > 0 \) and \( 0 < \beta < 1 \) are constants which only depend on \( f \) and \( \Lambda \).

(P2) **Contraction on stable leaves:** \( \text{dist}(f^n(y), f^n(x)) \leq C\beta^n, \forall y \in \gamma^s(x) \forall n \geq 1 \).

In spite of the uniform contraction in the stable direction, this condition is not too restrictive in systems having regions where the contraction fails to be uniform, since we are allowed to remove points in the unstable leaves, provided a subset with positive measure in those leaves remains at the end. This has been carried out in [3] for Hénon maps.
Next we introduce a return time function \( R : \Lambda \to \mathbb{N} \) and a return map \( f^R : \Lambda \to \Lambda \), defined for each \( i \in \mathbb{N} \) as
\[
R|_{\Lambda_i} = R_i \quad \text{and} \quad f^R|_{\Lambda_i} = f^{R_i}|_{\Lambda_i}.
\]
We consider the separation time \( s(x, y) \) for \( x, y \in \Lambda \) as
\[
s(x, y) = \min \{ n \geq 0 : (f^R)^n(x) \) and \( (f^R)^n(y) \) lie in distinct \( \Lambda_i \}.
\]
The last two properties involve information on the action of \( f^R \) on unstable leaves.

**Remark 1.1.** The Markov property we present here is weaker than the one in [11], since includes two extra assumptions: i) there are at most finitely many \( i \)'s with \( R_i = n \) for each \( n \in \mathbb{N} \); ii) \( R_i \geq R_0 \) for some \( R_0 > 1 \) depending on the constants \( C \) and \( \alpha \). These assumptions play a role in showing the existence of a spectral gap for a transfer operator associated to the dynamics. Here we use a more probabilistic argument, based on [12], which enables us to drop those extra assumptions. In particular, we are able to reobtain the conclusions of [11] under our weaker Markov condition.

**Remark 1.2.** We do not assume any uniform backward contraction along unstable leaves similar to (P4)(a) in [11]. This would be too restrictive for our purposes, since the application we make of our main results does not have this property. Properties (P3)(b) and (P4) are new if comparing our setup to the one in [11]. However, they can be easily obtained from (P4) and (P5) in [11]; see [11] Lemma 1.

### 1.2. Diameter control

Consider a sequence of stopping times defined for the points in \( \Lambda \) in the following way:
\[
S_0 = 0, \quad S_1 = R \quad \text{and} \quad S_{i+1} = S_i + R \circ f^{S_i}, \quad \text{for} \ i \geq 1.
\]
We also define a nested sequence \((\mathcal{P}_k)_{k \geq 0}\) of partitions of \( \Lambda \). Let \( \mathcal{P}_0 \) be the partition of \( \Lambda \) into the subsets \( \Lambda_i \). Given \( k \geq 1 \), we say that \( x \) and \( y \) belong to an element of \( \mathcal{P}_k \), if both \( f^R(x) \) and \( f^R(y) \) have the same stopping times \( S_0 < S_1 < \cdots < S_j \) up to time \( k - 1 \), and \( f^{S_i}(f^R(x)) \) and \( f^{S_i}(f^R(y)) \) belong to the same element of \( \mathcal{P}_0 \) for each \( 0 \leq i \leq j \). By construction we have that \( S_{j+1}(f^R(x)) = S_{j+1}(f^R(y)) \geq k \) and \( f^{S_{j+1}}(f^R(Q)) \) is a \( u \)-subset.
As it will become clear in the proof of Lemma 2.2, it will be necessary to have a control on the diameter of certain iterates of the elements in the partitions constructed above. Take any \( k \geq 1 \) and \( P \in \mathcal{P}_0 \). We consider separately the cases where \( k \) is bigger than \( R(P) - 1 \) or not. If \( k > R(P) - 1 \), then we define
\[
\delta_k(P) = \sup_{0 \leq \ell \leq R(P) - 1} \{ \text{diam} \left( f^\ell(Q \cap \gamma) \right) : \gamma \in \Gamma^u, Q \in \mathcal{P}_{k-R(P)+1+\ell}, Q \subset P \}.
\]
On the other hand, if \( k \leq R(P) - 1 \), then we define the quantities
\[
\delta_k^0(P) = \sup_{0 \leq \ell < R(P) - k} \{ \text{diam} \left( f^\ell(P \cap \gamma) \right) : \gamma \in \Gamma^u \},
\]
\[
\delta_k^+(P) = \sup_{R(P) - k \leq \ell \leq R(P) - 1} \{ \text{diam} \left( f^\ell(Q \cap \gamma) \right) : \gamma \in \Gamma^u, Q \in \mathcal{P}_{k-R(P)+1+\ell}, Q \subset P \},
\]
and
\[
\delta_k(P) = \sup \{ \delta_k^0(P), \delta_k^+(P) \}.
\]
Finally we define
\[
\delta_k = \sup_{P \in \mathcal{P}_0} \delta_k(P).
\]
Though the definition of \( \delta_k \) might seem somewhat technical, this is not so hard to calculate in practice, at least for some examples. One we have in mind is the one that we present at Section 1.4, for which we show that \( \delta_k \) decays polynomially fast with \( k \); see Section 4.2.

Remark 1.3. The argument in Section 4.2 can easily be adapted to show that \( \delta_k \) decays exponentially fast with \( k \), once we know that the diameter of the elements \( \Lambda_i \) decay exponentially fast with \( R_i \). This includes all the examples studied in [11], since property (P4)(a) in [11] gives the exponential decay for the diameters of the elements in the initial partition with respect to the return time.

Remark 1.4. In the light of [11, Definition 2.6] one may say that \( \delta_k \) decays exponentially fast with \( k \) whenever the return time \( R_i \) is a hyperbolic time for the points in \( \Lambda_i \) with respect to the derivative restricted to the tangent direction of the leaves in \( \Gamma^u \); see [11, Lemma 2.7] and recall Remark 1.3.

1.3. Main results. The first result we present here asserts the existence of SRB measures for systems having some hyperbolic structure, provided the return time is integrable with respect to the conditional of the Lebesgue measure on some local unstable leaf.

Definition 3. We say that an \( f \)-invariant probability measure \( \mu \) is a Sinai-Ruelle-Bowen (SRB) measure if \( f \) has no zero Lyapunov exponents \( \mu \) almost everywhere, and the conditional measures on local unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these manifolds.

The proof of the next result is quite standard and may be found in [11].

Theorem A. Assume that \( f \) has a hyperbolic structure \( \Lambda \) such that \( \text{Leb}_\gamma(\Lambda \cap \gamma) > 0 \) for some \( \gamma \in \Gamma^u \). If \( R \) is integrable with respect to \( \text{Leb}_\gamma \), then \( f \) has some SRB measure \( \mu \).
Theorem D. Let \( M \) be the solid torus map \( g \) such that \( \varphi(x) - \varphi(y) \leq C \operatorname{dist}(x, y)^\eta, \forall x, y \in M \).

Theorem B. Assume that \( f \) has a hyperbolic structure \( \Lambda \) for which \((P_1)-(P_4)\) hold, with \( \gcd\{R_i\} = 1 \) and \( \operatorname{Leb}_\gamma(\Lambda \cap \gamma) > 0 \) for some \( \gamma \in \Gamma^u \). Given \( \varphi, \psi \in H_\eta \),

1. if \( \operatorname{Leb}_\gamma\{R > n\} \lesssim n^{-\alpha} \) for some \( \alpha > 1 \), then \( C_n(\varphi, \psi; \mu) \lesssim \max\{n^{-\alpha+1}, \delta^n\} \);
2. if \( \operatorname{Leb}_\gamma\{R > n\} \lesssim e^{-n\zeta} \) for some \( c > 0 \) and \( 0 < \zeta \leq 1 \), then there exists \( c' > 0 \) such that \( C_n(\varphi, \psi; \mu) \lesssim \max\{e^{-cn^\zeta}, \delta^n\} \).

As shown in [11, Section 4.1], condition \( \gcd\{R_i\} = 1 \) can be replaced by the assumption that \( f^n \) is ergodic with respect to \( \mu \) for every \( n \geq 1 \). If we omit both assumptions, then the same conclusion holds for some power of \( f \). The next result gives the Central Limit Theorem for Hölder continuous observables which are not a coboundary with respect to the SRB measure \( \mu \).

Theorem C. Under the assumptions of Theorem B, if \( \operatorname{Leb}_\gamma\{R > n\} \lesssim n^{-\alpha} \) for some \( \alpha > 2 \), then given \( \varphi \in H_\eta \) for which there is no \( \psi \in L^2(\mu) \) with \( \varphi = \psi \circ f - \psi \) there exists \( \sigma > 0 \) such that for every interval \( J \subset \mathbb{R} \),

\[
\mu \left\{ x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left( \varphi(f^j(x)) - \int \varphi d\mu \right) \in J \right\} \xrightarrow{n \to \infty} \frac{1}{\sigma \sqrt{2\pi}} \int_J e^{-t^2/2\sigma^2} dt.
\]

1.4. Application. We give a diffeomorphism where we may apply our main results and deduce that it has an SRB measure with polynomial decay of correlations. This is obtained by perturbing the classical solenoid map in the unstable direction of one fixed point and transforming it into an indifferent fixed point. Let \( f : S^1 \to S^1 \) be a map of degree \( d \geq 2 \) with the following properties:

(i) \( f \) is \( C^2 \) on \( S^1 \setminus \{0\} \);
(ii) \( f \) is \( C^1 \) on \( S^1 \) and \( f' > 1 \) on \( S^1 \setminus \{0\} \);
(iii) \( f(0) = 0, f'(0) = 1 \), and there is \( \gamma > 0 \) such that

\[-xf''(x) \approx |x|^\gamma \quad \text{for all} \ x \neq 0.
\]

Consider the solid torus \( M = S^1 \times D^2 \), where \( D^2 \) is the unit disk in \( \mathbb{R}^2 \), and define the map \( g : M \to M \) by

\[
g(x, y, z) = \left( f(x), \frac{1}{10} y + \frac{1}{2} \cos x, \frac{1}{10} z + \frac{1}{2} \sin x \right).
\]

Let \( H_\eta \) be the space of Hölder continuous functions on \( M \) with exponent \( \eta > 0 \).

Theorem D. Let \( g : M \to M \) be as above and take \( \varphi, \psi \in H_\eta \).
(1) The map $g$ admits an SRB measure $\mu$ if and only if $\gamma < 1$.

(2) Assume that $\gamma < 1$. Then
   (a) for $\eta \geq 1 - \gamma$ we have $D_n(\varphi, \psi; \mu) \lesssim n^{1-1/\gamma}$;
   (b) for $\eta < 1 - \gamma$ we have $D_n(\varphi, \psi; \mu) \lesssim n^{-\eta/\gamma}$.

(3) If $\gamma < 1/2$, then the Central Limit Theorem holds for $\varphi \in H_\eta$, provided there is no $\psi \in L^2(\mu)$ with $\varphi = \psi \circ f - \psi$.

It is well known that for $\gamma \geq 1$ one has $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ converging in the weak* topology to the Dirac measure at 0 for Lebesgue almost every $x \in S^1$; see for example [6] and [7]. Using the fact that we have uniform contraction in the vertical direction, it is not hard to see that $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{g^j(x,y)}$ converges in the weak* topology to the Dirac measure at 0 for Lebesgue almost every $(x,y) \in S^1 \times D^2$. This observation justifies the “only if” part of the theorem above.

2. INDUCED SCHEMES

The objects that we introduce in this section have essentially been all presented in [3] and put into an abstract setting in [11].

2.1. The natural measure. Fix an arbitrary $\hat{\gamma} \in \Gamma_u$. Given $\gamma \in \Gamma_u$ and $x \in \gamma \cap \Lambda$ let $\hat{x}$ be the point in $\gamma^s(x) \cap \hat{\gamma}$. Defining for $x \in \gamma \cap \Lambda$

$$\hat{u}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\hat{x}))}$$

we have that $\hat{u}$ satisfies the bounded distortion property (P3)(b). For each $\gamma \in \Gamma_u$ let $m_\gamma$ be the measure in $\gamma$ such that

$$\frac{dm_\gamma}{d\text{Leb}_\gamma} = \hat{u} 1_{\gamma \cap \Lambda},$$

where $1_{\gamma \cap \Lambda}$ is the characteristic function of the set $\gamma \cap \Lambda$. These measures have been defined in such a way that if $\gamma, \gamma' \in \Gamma_u$ and $\Theta$ is obtained by sliding along stable leaves from $\gamma \cap \Lambda$ to $\gamma' \cap \Lambda$, then

$$\Theta_* m_\gamma = m_{\gamma'}.$$  (3)

To verify this let us show that the densities of these two measures with respect to $\text{Leb}_\gamma$ coincide. Take $x \in \gamma \cap \Lambda$ and $x' \in \gamma' \cap \Lambda$ such that $\Theta(x) = x'$. By (P3)(a) one has

$$\frac{d\Theta_* \text{Leb}_\gamma(x')} {d\text{Leb}_{\gamma'}(x')} = \frac{\hat{u}(x')} {\hat{u}(x)},$$

which implies that

$$\frac{d\Theta_* m_\gamma(x')} {d\text{Leb}_{\gamma'}(x')} = \hat{u}(x) \frac{d\Theta_* \text{Leb}_\gamma(x')}{d\text{Leb}_{\gamma'}(x')} = \hat{u}(x') = \frac{dm_{\gamma'}(x')} {d\text{Leb}_{\gamma'}(x')}.$$
Proof. (1) For Leb, almost every \( x \in \gamma \cap \Lambda \) we have

\[
J f^R(x) = |\text{det} D(f^R)^u(x)| \cdot \frac{\hat{u}(f^R(x))}{\hat{u}(x)}.
\]

Denoting \( \varphi(x) = \log |\text{det} Df^u(x)| \) we may write

\[
\log J f^R(x) = \sum_{i=0}^{R-1} \varphi(f^i(x)) + \sum_{i=0}^{\infty} \left( \varphi(f^i(f^R(x))) - \varphi(f^i(\hat{x})) \right) \]

\[
- \sum_{i=0}^{\infty} \left( \varphi(f^i(x)) - \varphi(f^i(\hat{x})) \right) \]

\[
= \sum_{i=0}^{R-1} \varphi(f^i(\hat{x})) + \sum_{i=0}^{\infty} \left( \varphi(f^i(f^R(\hat{x}))) - \varphi(f^i(\hat{x})) \right).
\]

Thus we have shown that \( J f^R(x) \) can be expressed just in terms of \( \hat{x} \) and \( \hat{x}^R \), which is enough for proving the first part of the lemma.

(2) It follows from (4) that

\[
\log \frac{J f^R(x)}{J f^R(y)} = \log \text{det} D(f^R)^u(x) + \log \hat{u}(f^R(x)) + \log \hat{u}(y) - \log \hat{u}(x)
\]

Observing that \( s(x, y) > s(f^R(x), f^R(y)) \) the conclusion follows from (P_3)(b) and (P_4). \( \square \)

2.2. A tower extension. We introduce a tower extension of the dynamical system \( f \) restricted to \( \cup_{n \geq 0} f^n(\Lambda) \); note that this space is preserved by \( f \). We define a tower

\[
\Delta = \{(x, \ell) : x \in \Lambda \text{ and } 0 \leq \ell < R(x)\},
\]

and a tower map \( F : \Delta \to \Delta \) as

\[
F(x, \ell) = \begin{cases} 
(x, \ell + 1), & \text{if } \ell + 1 < R(x); \\
(f^R(x), 0), & \text{if } \ell + 1 = R(x).
\end{cases}
\]

The \( \ell \text{th level of the tower} \) is by definition the set

\[
\Delta_\ell = \{(x, \ell) \in \Delta\}.
\]

The 0\textsuperscript{th} level of the tower \( \Delta_0 \) is naturally identified with \( \Lambda \) and we shall make no distinction between them. Under this identification it easily follows from the definitions that \( F^R = f^R \) for each \( x \in \Delta_0 \). Note that the \( \ell \text{th} \) level of the tower is a copy of the set \( \{R > \ell\} \subset \Delta_0 \).

Also, we easily obtain a partition \( \mathcal{P} \) of \( \Delta_0 \) into subsets \( \Delta_{0,i} \), with \( \Delta_{0,i} = \Lambda_i \) for \( i \geq 1 \). This partition gives rise to partitions \( \Delta_{\ell,i} \) on each tower level \( \ell \), considering

\[
\Delta_{\ell,i} = \{(x, \ell) \in \Delta_\ell : x \in \Delta_{0,i}\}.
\]
Collecting all these sets we obtain a partition \( Q = \{ \Delta_{\ell,i} \}_{\ell,i} \) of \( \Delta \). We introduce a sequence of partitions \( (Q_n)_{n \geq 0} \) of \( \Delta \) in the following way:

\[
Q_0 = Q, \quad \text{and} \quad Q_n = \cap_{i=0}^n F^{-i} Q \quad \text{for } n \geq 0. \tag{5}
\]

We shall denote by \( Q_n(x) \) the element in \( Q_n \) containing the point \( x \in \Delta \).

We define a projection map

\[
\pi : \Delta \times (x, \ell) \mapsto \bigcup_{n \geq 0} F^n(\Delta_0).
\tag{6}
\]

Observe that \( f \circ \pi = \pi \circ F \).

**Lemma 2.2.** There is \( C_2 > 0 \) such that for all \( k \geq 0 \) and \( Q \in Q_{2k} \)

\[
diam(\pi F^k(Q)) \leq C_2 \max\{b^k, \delta_k\}.
\]

**Proof.** Take \( k \geq 0 \) and \( Q \in Q_{2k} \). Given \( x, y \in Q \), there is \( z \in \gamma^n(x) \cap \gamma^s(y) \). Supposing that \( Q \subset \Delta_\ell \), then \( y_0 = \pi F^{-\ell}(y) \) and \( z_0 = \pi F^{-\ell}(z) \) are both in \( \Delta_0 \) and they lie on the same stable leaf. Hence

\[
dist(\pi F^k(y), \pi F^k(z)) = dist(\pi F^{k+\ell}(y_0), \pi F^{k+\ell}(z_0)) = dist(f^{k+\ell}(\pi y_0), f^{k+\ell}(\pi z_0)).
\]

Using \( (P_2) \) we get

\[
dist(\pi F^k(y), \pi F^k(z)) \leq C \beta^{k+\ell}. \tag{7}
\]

On the other hand, we have \( F^k(Q) \in Q_k \), which implies that \( F^k(x) \) and \( F^k(z) \) are both in an unstable leaf of some element of \( Q_k \). In particular, there are \( P \in P_0 \) and \( \ell < R(P) \) such that that element of \( Q_k \) is in the \( \ell \)-th level of the tower over \( P \). Moreover, the situations considered for defining \( \delta_k(P) \) correspond precisely to the possible cases for the elements of \( Q_k \) over \( P \). Taking into account the definition of \( \pi \), this gives

\[
dist(\pi F^k(x), \pi F^k(z)) \leq \delta_k(P),
\]

which together with \( (7) \) gives the desired conclusion. \( \square \)

Let \( m \) be the measure on \( \Lambda \) whose conditional measures on \( \gamma \cap \Lambda \) with \( \gamma \in \Gamma^u \) are the measures \( m_\gamma \) introduced in the previous section. This measure \( m \) allows us to introduce a measure on \( \Delta \) that we still denote \( m \), by letting \( m|_{\Delta_\ell} \) be the measure induced by the natural identification of \( \Delta_\ell \) with a subset of \( \Lambda \). We let \( JF \) denote the Jacobian of \( F \) with respect to this measure \( m \).

**Lemma 2.3.** There is \( C_F > 0 \) such that for all \( k \geq 1 \) and all \( x, y \in \Delta \) belonging to a same element of \( Q_{k-1} \)

\[
\left| \frac{JF^k(x)}{JF^k(y)} - 1 \right| \leq C_F \beta^{s(F^k(x), F^k(y))}.
\]

**Proof.** By Lemma 2.1 one knows that for all \( i \geq 1 \) and all \( x, y \in \Delta_{0,i} \)

\[
\left| \frac{JF^R(x)}{JF^R(y)} - 1 \right| \leq C_1 \beta^{s(F^R(x), F^R(y))}. \tag{8}
\]
It follows that there is a constant $C_F > 0$ such that for all $n \geq 1$ and all $x, y$ belonging to a same element of $\cs_{j=0}^{n-1}(F^R)^{-j}\mathcal{P}$

$$\left| \frac{J(F^R)^n(x)}{J(F^R)^n(y)} - 1 \right| \leq C_F \beta s((F^R)^n(x),(F^R)^n(y)),$$  \hfill (9)

In fact, if $x$ and $y$ belong to a same element of $\cs_{j=0}^{n-1}(F^R)^{-j}\mathcal{P}$, then $(F^R)^j(x)$ and $(F^R)^j(y)$ belong to a same element of $\mathcal{P}$ for every $0 \leq j < n$. Moreover,

$$s((F^R)^j(x),(F^R)^j(y)) = s((F^R)^n(x),(F^R)^n(y)) + (n - j).$$  \hfill (10)

Then

$$\log \frac{J(F^R)^n(x)}{J(F^R)^n(y)} = \sum_{j=0}^{n-1} \log \frac{J(\gamma F^R)^n((F^R)^j(x))}{J(\gamma F^R)^n((F^R)^j(y))} \leq \sum_{j=0}^{n-1} C_\gamma \beta s((F^R)^n(x),(F^R)^n(y)) + (n - j) - 1, \text{ by (9) and (10)} \leq C_F \beta s((F^R)^n(x),(F^R)^n(y)),$$  \hfill (11)

where $C_F > 0$ depends only on $C_\gamma$ and $\beta$. This implies that (11) holds.

From (9) we easily deduce that for all $k \geq 1$ and all $x, y \in \Delta$ belonging to a same element of $\mathcal{Q}_{k-1}$

$$\left| \frac{J(\gamma F)^k(x)}{J(\gamma F)^k(y)} - 1 \right| \leq C_F \beta s((\gamma F)^k(x),(\gamma F)^k(y)).$$  \hfill (12)

To see this, we consider $J(\gamma F)^k(x) = J(F^R)^n(x')$ and $J(\gamma F)^k(y) = J(F^R)^n(y')$, where $n$ is the number of visits of $x$ and $y$ to $\Delta_0$ prior to time $k$, and $x', y'$ are the elements in the bottom level $\Delta_0$ corresponding $x, y$, respectively. In this way, we have $x', y'$ belonging to a same element of $\cs_{j=0}^{n-1}(F^R)^{-j}\mathcal{P}$ and $s(x, y) = s(x', y')$. Using (9) we obtain (12). \hfill $\square$

**2.3. Quotient dynamics.** Let $\bar{\Lambda} = \Lambda / \sim$, where $x \sim y$ if and only if $y \in \gamma^s(x)$. This quotient space gives rise to a quotient tower $\bar{\Delta}$ with levels $\bar{\Delta}_i = \Delta_i / \sim$. A partition of $\bar{\Delta}$ into $\bar{\Delta}_{0,i}$, that we denote by $\bar{\mathcal{P}}$, and a sequence $\bar{\mathcal{Q}}_n$ of partitions of $\bar{\Delta}$ as in (5) are defined in a natural way.

As $f^R$ takes $\gamma^s$-leaves to $\gamma^s$-leaves and $R$ has been defined in such a way that it does not depend on the point we take in a same stable leaf, we may assume that we have defined the return time $\bar{R}: \bar{\Delta}_0 \to \mathbb{N}$, the tower map $\bar{F}: \bar{\Delta} \to \bar{\Delta}$ and the separation time $\bar{s}: \bar{\Delta}_0 \times \bar{\Delta}_0 \to \mathbb{N}$ naturally induced by the corresponding ones in $\Delta_0$ and $\Delta$. It will be convenient to have this separation time defined in the whole $\bar{\Delta}$. This may be done by taking $\bar{s}(x, y) = \bar{s}(x', y')$ if $x$ and $y$ belong in a same $\Delta_{0,i}$, where $x', y'$ are the corresponding elements of $\Delta_{0,i}$, and $\bar{s}(x, y) = 0$ otherwise.

Since (3) holds, we may introduce a measure $\bar{m}$ on $\bar{\Delta}$ whose representative on each $\gamma \in \Gamma^s$ is $m$. We let $J\bar{F}$ denote the Jacobian of $\bar{F}$ with respect to this measure $\bar{m}$. The
Theorem C. We postpone the proof of Theorem 2.6 to Appendix A.

Moreover, Theorem 2.5.

Assume that some of its properties. A proof of it is given in [11, Lemma 2] and [12, Theorem 1].

The following result gives the existence of an equilibrium measure for the tower map and

We introduce the spaces of Hölder functions in $\bar{\Delta}$

$$F_\beta = \{ \varphi : \bar{\Delta} \to \mathbb{R} | \exists C \varphi > 0 \text{ such that } |\varphi(x) - \varphi(y)| \leq C \varphi \beta^{\bar{\Delta}(x,y)} \text{ for all } x, y \in \bar{\Delta} \}$$

$$F_\beta^+ = \{ \varphi \in F_\beta | \exists C \varphi > 0 \text{ such that on each } \bar{\Delta}_{\ell,i}, \text{ either } \varphi \equiv 0, \text{ or } \varphi > 0$$

and

$${\bar{\nu}} > 0$$

and

$${\bar{\nu}}(x,y) - 1 \leq C \varphi \beta^{\bar{\Delta}(x,y)} \text{ for all } x, y \in \bar{\Delta}_{\ell,i} \}.$$}

The following result gives the existence of an equilibrium measure for the tower map and some of its properties. A proof of it is given in [11, Lemma 2] and [12, Theorem 1].

Theorem 2.5. Assume that $\bar{\nu}$ is integrable with respect to $\bar{\mu}$. Then

1. $\bar{\nu}$ has a unique absolutely continuous invariant probability $\nu$ equivalent to $\bar{\mu}$;
2. $d\nu/d\bar{\mu}$ belongs to $F_\beta^+$ and is bounded from below by some $c > 0$;
3. $(\bar{F}, \bar{\nu})$ is exact and, hence ergodic and mixing.

The decay of correlations for the measure $\bar{\nu}$ has been proved in [11]. This occurs at the same speed that the positive iterates under $\bar{F}_\ast$ of measures with densities in $F_\beta^+$ converge to the equilibrium $\bar{\nu}$. This speed is related to the decay of $\bar{\mu}\{\bar{R} > n\}$, at least for some specific rates.

Theorem 2.6. For $\varphi \in F_\beta^+$ let $\bar{\lambda}$ be the measure whose density with respect to $\bar{\mu}$ is $\varphi$.

1. If $\bar{\mu}\{\bar{R} > n\} \leq C n^{-\zeta}$, for some $C > 0$ and $\zeta > 1$, then there is $C' > 0$ such that

$$|\bar{F}_\ast^n \bar{\lambda} - \bar{\nu}| \leq C' n^{-\zeta + 1}.$$

2. If $\bar{\mu}\{\bar{R} > n\} \leq C e^{-cn^\eta}$, for some $C, c > 0$ and $0 < \eta \leq 1$, then there are $C', c' > 0$ such that

$$|\bar{F}_\ast^n \bar{\lambda} - \bar{\nu}| \leq C' e^{-c' n^\eta}.$$

Moreover, $c'$ does not depend on $\varphi$ and $C'$ depends only on $C_\varphi$.

A version of this theorem has been proved in [11, Theorem 2] but without establishing the dependence on the constants. This plays a crucial role in our proofs of Theorem B and Theorem C. We postpone the proof of Theorem 2.6 to Appendix A.
3. Back to the original dynamics

Let \( \pi \) be the map from \( \Delta \) to \( M \) defined in (3). Let also \( \bar{\pi} \) be the projection from \( \Delta \) to the quotient space \( \bar{\Delta} \). As observed in [11, Sections 2 & 4] we have \( \bar{\nu} = \bar{\pi}_* \nu \) and \( \mu = \pi_* \nu \).

Given \( \varphi, \psi \in H_\eta \) we define \( \tilde{\psi} = \psi \circ \pi \) and \( \tilde{\varphi} = \varphi \circ \pi \).

3.1. Decay of correlations. For proving Theorem 3.3 we start by noting that for \( \varphi, \psi \in H_\eta \) we have

\[
\int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu = \int (\tilde{\varphi} \circ F^n) \tilde{\psi} d\nu - \int \tilde{\varphi} d\nu \int \tilde{\psi} d\nu,
\]

which shows that it suffices to obtain the desired conclusions for \( C_n(\tilde{\varphi}; \tilde{\psi}; \nu) \). This will be done in several steps, firstly reducing it to a problem in \( \bar{\Delta} \) and then applying Theorem 2.6.

**Step 1.** Fix some positive integer \( k \leq n/4 \). Consider a discretization \( \tilde{\varphi}_k \) of \( \tilde{\varphi} \) defined on \( \Delta \) (or \( \bar{\Delta} \)) as

\[
\tilde{\varphi}_k|_A = \inf \{ \tilde{\varphi} \circ F^k(x) ; x \in A \}, \quad \text{for } A \in \mathcal{Q}_{2k}.
\]

We have

\[
|C_n(\tilde{\varphi}, \tilde{\psi}; \nu) - C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}; \nu)| \leq C_3 \delta_k^n
\]

for some \( C_3 \) depending only on \( C_\varphi \) and \( \|\psi\|_\infty \).

Actually, by Lemma 2.2 one knows that \( |\tilde{\varphi} \circ F^k - \tilde{\varphi}_k| \leq C_\varphi (C_2 \delta_k)^n \). To be precise, one should consider the case \( \beta^k > \delta_k \), but this would only be relevant in the second part of Theorem 3.3. However, it does not play any special role for the conclusion.

Observing that \( C_n(\tilde{\varphi}, \tilde{\psi}; \nu) = C_{n-k}(\tilde{\varphi} \circ F^k, \tilde{\psi}; \nu) \), the left hand side of inequality (13) is

\[
\leq \left| \int (\tilde{\varphi} \circ F^k - \tilde{\varphi}_k) \circ F^{n-k} \cdot \tilde{\psi} d\nu \right| + \left| \int (\tilde{\varphi} \circ F^k - \tilde{\varphi}_k) d\nu \cdot \int \tilde{\psi} d\nu \right|
\leq 2C_\varphi (C_2 \delta_k)^n \|\psi\|_\infty
\]

We just have to take \( C_3 = 2C_\varphi C_2^2 \|\psi\|_\infty \).

**Step 2.** Consider \( \tilde{\psi}_k \) defined similarly to \( \tilde{\varphi}_k \) above. Let \( \tilde{\psi}_k \nu \) denote the signed measure whose density with respect to \( \nu \) is \( \tilde{\psi}_k \), and let \( \tilde{\psi}_k \) denote the density of \( F^k(\tilde{\psi}_k \nu) \) with respect to \( \nu \). Then

\[
|C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}; \nu) - C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}_k; \nu)| \leq C_4 \delta_k^n,
\]

for some \( C_4 \) depending only on \( C_\psi \) and \( \|\varphi\|_\infty \).

In fact, the left hand side of (14) is

\[
\leq \left| \int (\tilde{\varphi}_k \circ F^{n-k})(\tilde{\psi} - \tilde{\psi}_k) d\nu \right| + \left| \int \tilde{\varphi}_k d\nu \int (\tilde{\psi} - \tilde{\psi}_k) d\nu \right|
\leq 2\|\varphi\|_\infty \cdot \left| \int (\tilde{\psi} - \tilde{\psi}_k) d\nu \right|
\]

Letting \( |\cdot| \) denote the total variation of a signed measure, and noting that

\[
F^k((\tilde{\psi} \circ F^k) \nu) = \tilde{\psi}_k \nu,
\]

we have

\[
|\tilde{\psi}_k \nu| = |\tilde{\psi}_k| \leq \|\tilde{\psi}_k\|_\infty \leq \|\tilde{\psi}\|_\infty \leq \|\psi\|_\infty,
\]

which shows that it suffices to obtain the desired conclusions for \( C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}_k; \nu) \).
we have
\[
\left| \int (\tilde{\psi} - \tilde{\psi}_k) d\nu \right| = |\tilde{\psi} \nu - \tilde{\psi}_k \nu|
\]
\[
= |F_k^k((\tilde{\psi} \circ F^k) \nu) - F_k^k(\tilde{\psi}_k \nu)|
\]
\[
\leq |(\tilde{\psi} \circ F^k - \tilde{\psi}_k) \nu|
\]
\[
= \int |\tilde{\psi} \circ F^k - \tilde{\psi}_k| d\nu.
\]
By Lemma 2.2 one has \(|\tilde{\psi} \circ F^k - \tilde{\psi}_k| \leq C_\psi (C_2 \delta_k)^n\). Take \(C_4 = 2C_\psi C_2^n \|\varphi\|_\infty\).

**Step 3.** Now we show that
\[
C_{n-k}(\tilde{\varphi}_k, \tilde{\psi}_k; \nu) = C_n(\tilde{\varphi}_k, \tilde{\psi}_k; \bar{\nu})
\]
(15)
Indeed,
\[
\int (\tilde{\varphi}_k \circ F^{n-k}) \tilde{\psi}_k d\nu = \int \tilde{\varphi}_k d(F^{n-k}_*(\tilde{\psi}_k \nu)) = \int \tilde{\varphi}_k d(F^n_*(\tilde{\psi}_k \nu)),
\]
and since \(\tilde{\varphi}_k\) is constant on \(\gamma^s\) leaves and \(F\) and \(\bar{F}\) are semi-conjugated by \(\tilde{\pi}\), we have
\[
\int \tilde{\varphi}_k d(F^n_*(\tilde{\psi}_k \nu)) = \int \tilde{\varphi}_k d(\bar{F}_* F^n_*(\tilde{\psi}_k \nu)) = \int \tilde{\varphi}_k d(\bar{F}_* \tilde{F}_k^* \tilde{\psi}_k \bar{\nu}) = \int (\tilde{\varphi}_k \circ F^n) \tilde{\psi}_k d\bar{\nu}.
\]
Thus we have proved that
\[
\int (\tilde{\varphi}_k \circ F^{n-k}) \tilde{\psi}_k d\nu = \int (\tilde{\varphi}_k \circ F^n) \tilde{\psi}_k d\bar{\nu}.
\]
On the other hand,
\[
\int \tilde{\varphi}_k d\nu \cdot \int \tilde{\psi}_k d\nu = \int \tilde{\varphi}_k d\bar{\nu} \cdot \int d(F_k^k(\tilde{\psi}_k \nu)) = \int \tilde{\varphi}_k d\bar{\nu} \cdot \int \tilde{\psi}_k d\bar{\nu}.
\]
These last to formulas give precisely (15).

**Step 4.** With no loss of generality we assume that \(\tilde{\psi}_k\) is not the null function. Taking
\[
b_k = \left( \int (\tilde{\psi}_k + 2 \|\tilde{\psi}_k\|_\infty) d\bar{\nu} \right)^{-1}
\]
and \(\tilde{\psi}_k = b_k (\tilde{\psi}_k + 2 \|\tilde{\psi}_k\|_\infty)\),
we then have
\[
\int \tilde{\psi}_k \bar{\rho} dm = 1, \quad \text{where} \quad \bar{\rho} = \frac{d\bar{\nu}}{dm}.
\]
Moreover,
\[
\frac{1}{3 \|\psi\|_\infty} \leq b_k \leq \frac{1}{\|\psi\|_\infty} \quad \text{and} \quad 1 \leq \|\tilde{\psi}_k\|_\infty \leq 3.
\]
Observe that \( \hat{\psi}_k \) is constant on elements of \( Q_{2k} \), since \( \tilde{\psi}_k \) has this property. Let \( \hat{\lambda}_k \) be the probability measure on \( \Delta \) whose density with respect to \( \bar{m} \) is \( \hat{\psi}_k \bar{\rho} \). Then,

\[
\left| \int (\tilde{\varphi}_k \circ \tilde{F}^n) \hat{\psi}_kd\bar{\nu} - \int \tilde{\varphi}_kd\bar{\nu} \int \tilde{\psi}_kd\bar{\nu} \right| = \frac{1}{b_k} \left| \int (\tilde{\varphi}_k \circ \tilde{F}^n) \hat{\psi}_kd\bar{\nu} - \int \tilde{\varphi}_kd\bar{\nu} \int \hat{\psi}_kd\bar{\nu} \right| \\
\leq \frac{1}{b_k} \int |\tilde{\varphi}_k| \left| d(\hat{F}^n \hat{\lambda}_k) - \bar{\rho} \right| d\bar{m}.
\]

(16)

Letting \( \hat{\lambda}_k = \hat{F}^{2k}_* \hat{\lambda}_k \), we have

\[
d \frac{d}{d\bar{m}} F^{2k}_* \hat{\lambda}_k = \frac{d}{d\bar{m}} \tilde{F}^{n-2k}_* \hat{\lambda}_k,
\]

which together with (16) gives

\[
C_n(\tilde{\varphi}_k, \tilde{\psi}_k; \nu) \leq \frac{1}{b_k} \|\tilde{\varphi}_k\|_\infty \left| \hat{F}^{n-2k}_* \hat{\lambda}_k - \nu \right| \leq 3 \|\nu\|_\infty \|\varphi\|_\infty \left| \hat{F}^{n-2k}_* \hat{\lambda}_k - \nu \right|.
\]

Let \( \phi_k \) represent the density of the measure \( \hat{\lambda}_k \) with respect to \( \bar{m} \). The next lemma shows that \( \phi_k \in \mathcal{F}_\beta^+ \), with the constant \( C_{\phi_k} \) not depending on \( \phi_k \). This is enough for using Theorem [2.6] and conclude the proof of Theorem [3]. Recall that we have taken \( k \leq n/4 \).

**Lemma 3.1.** There is \( C > 0 \), not depending on \( \phi_k \), such that

\[
|\phi_k(\bar{x}) - \phi_k(\bar{y})| \leq C\beta^{q(\bar{x}, \bar{y})}, \quad \text{for all } \bar{x}, \bar{y} \in \Delta.
\]

**Proof.** Since \( \hat{F}^{2k}_* \nu = \nu \) and \( \bar{\rho} = d\nu/d\bar{m} \), we may write

\[
\bar{\rho}(\bar{x}) = \sum_{Q \in \mathcal{Q}_{2k}} \frac{\bar{\rho}((\hat{F}^{2k}_*|Q)^{-1}(\bar{x}))}{\hat{F}^{2k}_*((\hat{F}^{2k}_*|Q)^{-1}(\bar{x}))}.
\]

(17)

Recall that we have by definition

\[
\phi_k = \frac{d\hat{\lambda}_k}{d\bar{m}} = \frac{d}{d\bar{m}} \hat{F}^{2k}_* \hat{\lambda}_k \quad \text{and} \quad \frac{d\hat{\lambda}_k}{d\bar{m}} = \hat{\psi}_k \bar{\rho}.
\]

Since \( \hat{\psi}_k \) is constant on elements of \( Q_{2k} \), we have

\[
\phi_k(\bar{x}) = \sum_{Q \in \mathcal{Q}_{2k}} c_Q \frac{\bar{\rho}((\hat{F}^{2k}_*|Q)^{-1}(\bar{x}))}{\hat{F}^{2k}_*((\hat{F}^{2k}_*|Q)^{-1}(\bar{x}))},
\]

where \( c_Q \) is constant on each \( Q \in \mathcal{Q}_{2k} \). Hence,

\[
\phi_k(\bar{x}) - \phi_k(\bar{y}) = \sum_{Q \in \mathcal{Q}_{2k}} c_Q \left( \frac{\bar{\rho}((\hat{F}^{2k}_*|Q)^{-1}(\bar{x}))}{\hat{F}^{2k}_*((\hat{F}^{2k}_*|Q)^{-1}(\bar{x}))} - \frac{\bar{\rho}((\hat{F}^{2k}_*|Q)^{-1}(\bar{y}))}{\hat{F}^{2k}_*((\hat{F}^{2k}_*|Q)^{-1}(\bar{y}))} \right).
\]

(18)

Fixing \( Q \in \mathcal{Q}_{2k} \), let \( \bar{x}', \bar{y}' \in Q \) be such \( \hat{F}^{2k}_*(\bar{x}') = \bar{x} \) and \( \hat{F}^{2k}_*(\bar{x}') = \bar{x} \). We have

\[
\frac{\bar{\rho}(\bar{x}')}{\hat{F}^{2k}_*(\bar{x}') - 1} - \frac{\bar{\rho}(\bar{y}')}{\hat{F}^{2k}_*(\bar{y}') - 1} = \left( \frac{\bar{\rho}(\bar{y}')}{\hat{F}^{2k}_*(\bar{y}') - 1} \right) \left( \frac{\bar{\rho}(\bar{x}')}{\hat{F}^{2k}_*(\bar{x}') - 1} \right). \]

(19)
It follows from Theorem 2.5 that there is $C_\beta > 0$ such that
\[
\log \left| \frac{\rho(\bar{x}')}{\rho(\bar{y}')} \right| \leq C_\beta \beta(\bar{x}', \bar{y}') .
\]

On the other hand, by Lemma 2.4 there is $C_\hat{F} > 0$ such that
\[
\log \left| \frac{\hat{F}^{2k}(\bar{y}')}{\hat{F}^{2k}(\bar{x}')} \right| \leq C_\hat{F} \beta(\hat{F}^{2k}(\bar{x}'), \hat{F}^{2k}(\bar{y}')).
\]

Since $s(\bar{x}', \bar{y}') \geq s(\hat{F}^{2k}(\bar{x}'), \hat{F}^{2k}(\bar{y}')) = s(\bar{x}, \bar{y})$, we have
\[
\log \left| \frac{\hat{\rho}(\bar{x}')}{\hat{\rho}(\bar{y}')} \right| \frac{\hat{F}^{2k}(\bar{y}')}{\hat{F}^{2k}(\bar{x}')} \leq (C_\beta + C_\hat{F}) \beta(\bar{x}, \bar{y}).
\] (20)

Recalling (17) and the fact that $|c_Q| \leq \|\hat{\psi}_k\|_\infty \leq 3$, it follows from (13), (19) and (20) that there is some constant $C > 0$ not depending on $\phi_k$ such that
\[
|\phi_k(\bar{x}) - \phi_k(\bar{y})| \leq C \beta(\bar{x}, \bar{y}).
\]

Actually, we may take $C = 3\|\hat{\rho}\|_\infty (C_\beta + C_\hat{F})$. \(\square\)

3.2. Central Limit Theorem. Let $\varphi \in H_0$, and consider its lift to $\Delta$ defined as $\hat{\varphi} = \varphi \circ \pi$. Similarly to what we have done at the beginning of Section 3.1, we easily see that for proving Theorem C it is enough to obtain the Central Limit Theorem for $\hat{\varphi}$ with respect to $\nu$ on $\Delta$. As in the study of the correlations decay, the proof uses results from the quotient dynamics $\hat{F}: \hat{\Delta} \to \hat{\Delta}$. Let $\hat{B}$ be the Borel $\sigma$-algebra on $\Delta$. Define
\[
\hat{B}_0 = \{ \hat{\pi}^{-1} \hat{A}: \hat{A} \in \hat{B} \} \quad \text{and} \quad \hat{\varphi}_0 = E_{\nu}(\hat{\varphi} \mid \hat{B}_0).
\]

Putting together the information from [11, Section 5.1.B] and Claim 1 in [11, Section 5.2] we easily see that it is enough to show that
\[
\sum_{j \geq 0} \int |P^j(\hat{\varphi}_0 \hat{\rho})| d\hat{m} < \infty,
\]
where $P$ is the transfer operator associated to $(\hat{F}, \hat{\nu})$. The proof of the Sublemma in [11, Section 5.2] gives that $\hat{\varphi}_0 \in F_\beta$. Thus, if we consider $\hat{\lambda}$ the measure whose density with respect to $\hat{m}$ is $\hat{\varphi}_0 \hat{\rho}$, then $P^j(\hat{\varphi}_0 \hat{\rho})$ is by definition the density of $F^j \hat{\lambda}$ with respect to $\hat{m}$. Hence, we just have to show that
\[
\sum_{j \geq 0} \int \left| \frac{d}{d\hat{m}} F^j \hat{\lambda} \right| d\hat{m} < \infty.
\]

First we “renormalize” $\hat{\lambda}$. Let
\[
b = \left( \int (\hat{\varphi}_0 + \|\hat{\varphi}_0\|_\infty) d\hat{\nu} \right)^{-1} \quad \text{and} \quad \hat{\varphi}_0 = b(\hat{\varphi}_0 + 2\|\hat{\varphi}_0\|_\infty),
\]
and consider $\hat{\lambda}$ the probability measure whose density with respect to $\hat{m}$ is $\hat{\varphi}_0 \hat{\rho}$. We have
\[
\int \hat{\varphi}_0 d\hat{\nu} = \int \hat{\varphi}_0 \hat{\rho} d\hat{m} = 1. \quad (21)
\]
Recalling that
\[ \int \bar{\varphi}_0 d\bar{\nu} = \int \varphi_0 \rho d\bar{m} = 0, \]
we may write
\[ \int \left| \frac{d}{d\bar{m}} \bar{F}_j \bar{\lambda} \right| d\bar{m} \]

Using (21) and the fact that
\[ \frac{d}{d\bar{m}} \bar{F}_j \bar{\nu} = \frac{d}{d\bar{m}} \bar{\nu} = \bar{\rho}, \]
we obtain
\[ \int \left| \frac{d}{d\bar{m}} \bar{F}_j \bar{\lambda} \right| d\bar{m} = \frac{1}{b} \int \left| \frac{d}{d\bar{m}} \bar{F}_j \bar{\lambda} - 2\bar{\rho} \| \bar{\varphi}_0 \|_\infty - \bar{\rho} \int \bar{\varphi}_0 d\bar{\nu} + 2\bar{\rho} \| \bar{\varphi}_0 \|_\infty \right| d\bar{m} \]
\[ = \frac{1}{b} \int \left| \frac{d}{d\bar{m}} \bar{F}_j \bar{\lambda} - \bar{\nu} \right| d\bar{m} \]

Under the hypotheses of Theorem [C] this last quantity is clearly summable, by Theorem [2.6]

4. A SOLENOID WITH INTERMITTENCY

Here we construct a hyperbolic structure for the map \( g \) defined in Section 1.4 which satisfies the assumptions of our main theorems. Concerning (P1)-(P4), we just have to show that (P1) and (P4) hold, since (P2) and (P3) are trivially satisfied due to the uniform contraction of \( g \) in the vertical direction and the skew-product form of \( g \). We also need to give suitable estimates for the decay of return times and the diameters in (2). The conclusions of Theorem [D] are then a consequence of our main results.

The map \( g \) possesses an attractor in \( M \) which is precisely
\[ \Sigma = \bigcap_{n \geq 0} g^n(M). \]
\( \Sigma \) is locally a product of an interval by a Cantor set. Topologically this set coincides with the solenoid attractor for the classical case where \( f \) is taken uniformly expanding in \( S^1 \).

For defining the hyperbolic structure we are going to construct a (mod 0) countable partition \( \mathcal{P}_0 \) of an interval \( I_1 \subset S^1 \) and associate to each element of \( \mathcal{P}_0 \) a suitable return time \( R^* \) with respect to the map \( f \). Then we take
\[ \Lambda = \Sigma \cap (I_1 \times D^2). \]
For each \( (x, y) \in \Lambda \) we define \( \gamma^s(x, y) = \{(x, y) : y \in D^2\} \) and \( \gamma^u(x, y) \) as the connected component of \( \Lambda \) that contains \( (x, y) \). The \( s \)-subsets are precisely the sets \( \Sigma \cap (P \times D^2) \) with \( P \in \mathcal{P}_0 \) and the return times are taken accordingly.
4.1. **Partition and return times.** Here we recall some objects and results from [12, Section 6] related to the map $f$. Let $I_1, \ldots, I_d$ be the partition of $S^1$ made by the fundamental domains of $f$ arranged in a natural order, and assume for definiteness that 0 is the common endpoint of $I_1$ and $I_d$. Letting $x_0$ be the other endpoint of $I_1$ we define a sequence $(x_n)_n$ in $I_1$ with the property that $f(x_{n+1}) = x_n$ for $n \geq 0$. Likewise, we consider $x'_0$ the endpoint of $I_d$ distinct from 0 and define a sequence $(x'_n)_n$ in $I_d$ so that $f(x'_{n+1}) = x'_n$ for $n \geq 0$.

Let $J_n = [x_{n+1}, x_n]$ and $J'_n = [x'_n, x'_{n+1}]$ for $n \geq 0$. Consider the (mod 0) partition of $S^1$

\[ \mathcal{A} = \{I_2, \ldots, I_{d-1}\} \cup \{J_n, J'_n; n \geq 0\}. \]

Let $R = 1$ on $I_2 \cup \cdots I_{d-1} \cup J_0 \cup J'_0$ and let $R|J_n = R|J'_n = n+1$ for $n \geq 1$. We have $f^R(I_j) = S^1$ for $2 \leq j \leq d-1$ and the $f^R$ images of all other elements of $\mathcal{A}$ are either $I_1 \cup \cdots \cup I_{d-1}$ or $I_2 \cup \cdots \cup I_d$. The following results were proved in [12, Sections 6.2 & 6.3]:

1. **Tail decay:** $\text{Leb}\{R > n\} \approx n^{-1/\gamma}$;
2. **Expansion:** there is $0 < \beta < 1$ such that $(f^R)'(x) \geq \beta^{-1}$ for every $x \in S^1 \setminus \{0\}$;
3. **Bounded distortion:** there is $C > 0$ such that for every $1 \leq i \leq n$ and $x, y \in J_n$

\[
\log \left( \frac{(f)'(x)}{(f)'(y)} \right) \leq C \frac{|f^i(x) - f^i(y)|}{|J_n - i|}.
\]

This function $R$ does not qualify as a return time for a hyperbolic structure of $f$ satisfying the Markov property. That role will be played by the function $R^*$ we introduce below. We use the time function $R$ to define a sequence of stopping times $(S_i)_i$ as in (1). We also define the sequence of return times

\[ r_1 = S_1 = R, \quad r_{i+1} = S_{i+1} - S_i, \quad \text{for } i \geq 1. \]

Using this sequence of stopping times we define the first return time $R^*$ to $I_1$ as follows. We simply take $R^*(x) = S_i(x)$, where $i \geq 1$ is the minimum such that $f^{S_i}(x) \in I_1$. As shown in [12, Section 6.2] we have

\[ \text{Leb}\{R^* > n\} \lesssim n^{-1/\gamma}. \] (22)

Let $\mathcal{P}_0$ be the Markov partition of $I_1$ associated to $R^*$. Naturally associating the return times to the $s$-subsets described above, then (22) gives the tail estimate that we need. Property (P4) is an easy consequence of the bounded distortion and expansion above, since the estimates on the derivative of $g$ in the unstable direction are given by $f$.

4.2. **Diameter estimate.** Now we are going to show that $\delta_k \lesssim 1/k^{1/\gamma}$, where $\delta_k$ is the quantity defined in (2). Taking into account the uniform contraction on the stable direction, we just have to obtain the desired control on the unstable one. We start by proving the following auxiliary result.

**Lemma 4.1.** Let $X$ be an interval in $S^1$ whose points have the same stopping times $S_1, \ldots, S_N$, for some $N \geq 1$, with $S_N \geq k$, and such that $f^{S_N}(X) = I_1$. Then $|X| \lesssim 1/k^{1/\gamma}$.

**Proof.** Let $r_1, \ldots, r_N$ be the return times of points in $X$. Since we are assuming that $S_N = r_1 + \cdots + r_N \geq k$, there must be some $1 \leq m \leq N$ such that $r_m \geq k/N$. Let

\[ Y = f^{r_1 + \cdots + r_{m-1}}(X). \]
Considering the interval $I \in \mathcal{A}$ such that $Y \subset I$ we have $R|I| = r_m$. Bounded distortion yields

$$|Y| \lesssim \frac{|f^{r_m}(Y)|}{|f^{r_m}(I)|} \cdot |I|.$$  

On the other hand, the tail decay and expansion estimates give

$$|I| \lesssim \left( \frac{1}{r_m} \right)^{1/\gamma}, \quad |X| \leq \beta^{m-1}|Y| \quad \text{and} \quad |f^{r_m}(Y)| \leq \beta^{N-m}|I_1|.$$  

Taking into account the choice of $m$ we obtain

$$|X| \lesssim \beta^N \left( \frac{N}{k} \right)^{1/\gamma} \lesssim \frac{1}{k^{1/\gamma}},$$

and so we are done.  

Take $P \in \mathcal{P}_0$ and $k \geq 1$. We consider the three possible cases of sets whose diameters have to be controlled. The first two correspond to $k \leq R^*(P) - 1$, and the last one corresponds to $k > R^*(P) - 1$; recall the definition of $\delta_k$ in Section 1.2.

Case 1. Assume first that $k \leq R^*(P) - 1$ and $0 \leq \ell < R^*(P) - k$. There is $m \geq 1$ such that $R^*(P) = r_1 + \cdots + r_m$. Considering $r_0 = 0$, let $0 \leq p < m$ be such that

$$r_0 + \cdots + r_p \leq \ell < r_0 + \cdots + r_{p+1}.$$  

Letting $\ell' = \ell - (r_0 + \cdots + r_p)$, we have

$$\ell' < r_{p+1} \quad \text{and} \quad \ell' < r_{p+1} + \cdots + r_m - k.$$  

Now, if $i + 1 = m$, then

$$|f^\ell(P)| = |f^{\ell'}(f^{r_0 + \cdots + r_p}(P))| \lesssim \left( \frac{1}{r_m - \ell'} \right)^{1/\gamma} \leq \frac{1}{k^{1/\gamma}}.$$  

Otherwise, for $p + 1 < m$ we use Lemma 4.1 with $X = f^{r_0 + \cdots + r_p + \ell'}(P)$, $N = m - p$, and

$$S_j = r_{p+1} + \cdots + r_{p+j} - \ell', \quad \text{for} \ 1 \leq j \leq N,$$

thus obtaining

$$|f^\ell(P)| = |X| \lesssim \frac{1}{k^{1/\gamma}}.$$  

Observe that $r_{p+1} - \ell'$ is still a return time, which then implies that $S_1$ is well defined.
Case 2. Assume now that \( k \leq R^*(P) - 1 \) and \( R^*(P) - k \leq \ell \leq R^*(P) - 1 \). We simply write \( R^* \) for \( R^*(P) \). Take \( Q \in \mathcal{P}_{k-R^*+1} \) with \( Q \subset P \). By construction, there is \( j \geq 0 \) such that points in \( f^{R^*}(Q) \) have the same stopping times \( S^*_1, \ldots, S^*_j \) up to time \( k - R^* + \ell \). Moreover, \( f^{S^*_{j+1}}(f^{R^*}(Q)) = I_1 \) and

\[
S^*_{j+1} \geq k - R^* + 1 + \ell. \tag{23}
\]

There are integers \( m, n \geq 1 \) and return times \( r_1, \ldots, r_m+n \) such that

\[
R^* = r_1 + \cdots + r_m \quad \text{and} \quad S^*_{j+1} = r_m+1 + \cdots + r_m+n. \tag{24}
\]

It follows from (23) and (24) that

\[
\ell < r_1 + \cdots + r_{m+n} - k.
\]

Considering \( 0 \leq p < m \) such that

\[
r_0 + \cdots + r_p \leq \ell < r_0 + \cdots + r_{p+1},
\]

where \( r_0 = 0 \) as before, and taking \( \ell' = \ell - (r_0 + \cdots + r_p) \), we have

\[
\ell' < r_{p+1} \quad \text{and} \quad \ell' < r_{p+1} + \cdots + r_{m+n} - k.
\]

The proof now follows as in the previous case.

Case 3. The case \( k > R(P) - 1 \geq \ell \) is treated as Case 2.

Appendix A. Mixing rates for tower maps

The goal of this section is to prove Theorem 2.6. We follow the scheme of [12] with a delicate control on the constants. The only exception is Subsection A.2.2 where we use results from [5]. The setting will be the same of Subsection 2.3. For the sake of notational simplicity we shall drop all bars.

Let \( \lambda \) and \( \lambda' \) be probability measures in \( \Delta \) whose densities with respect to \( m \) belong to \( \mathcal{F}^+_\beta \). Let

\[
\varphi = \frac{d\lambda}{dm} \quad \text{and} \quad \varphi' = \frac{d\lambda'}{dm},
\]

and consider \( C_\varphi, C_{\varphi'} \) as in the definition of \( \mathcal{F}^+_\beta \).

A.1. Main estimates. Consider the product map \( F \times F : \Delta \times \Delta \to \Delta \times \Delta \), and \( P = \lambda \times \lambda' \) the product measure on \( \Delta \times \Delta \). Let \( \pi, \pi' : \Delta \times \Delta \to \Delta \) be the projections on the first and second coordinates respectively. Note that \( F^n \circ \pi = \pi \circ (F \times F)^n \). Consider the partition \( Q := \{ \Delta_{l,i} \} \) of \( \Delta \), and the partition \( Q \times Q \) of \( \Delta \times \Delta \). Note that each element of \( Q \times Q \) is sent bijectively by \( F \times F \) onto a union of elements of \( Q \times Q \). For each \( n \geq 1 \), let

\[
(Q \times Q)_n := \bigcup_{i=0}^{n-1} (F \times F)^{-i}(Q \times Q),
\]

and let \( (Q \times Q)_n(x, x') \) be the element of \( (Q \times Q)_n \) that contains \( (x, x') \in \Delta \times \Delta \).
Since $(F, \nu)$ is mixing and the density of $\nu$ with respect to $m$ belongs to $L^\infty(m)$, we may find $n_0 \in \mathbb{N}$ and $\gamma_0 > 0$ such that $m(F^{-n}(\Delta_0) \cap \Delta_0) \geq \gamma_0$ for all $n \geq n_0$. Then we introduce a sequence of stopping times $0 \equiv \tau_0 < \tau_1 < \tau_2 < \ldots$ in $\Delta \times \Delta$ given by

\[
\begin{align*}
\tau_1(x, x') &= n_0 + \hat{R}(F^{n_0}(x)), \\
\tau_2(x, x') &= \tau_1 + n_0 + \hat{R}(F^{n_1}(x')), \\
\tau_3(x, x') &= \tau_2 + n_0 + \hat{R}(F^{n_2}(x)), \\
\tau_4(x, x') &= \tau_3 + n_0 + \hat{R}(F^{n_3}(x')), \\
&\vdots
\end{align*}
\]

with the falls to the ground level $\Delta_0$ alternating between $x$ and $x'$. This implies that $\tau_{i+1} - \tau_i \geq n_0$ for all $i \geq 1$. We define the simultaneous return time $T : \Delta \times \Delta \to \mathbb{N}$ as

\[
T(x, x') = \min \{\tau_i : (F^n(x), F^n(x')) \in \Delta_0 \times \Delta_0, \text{ with } i \geq 2\}.
\]

Note that we have $T \geq 2n_0$. Since $(F, \nu)$ is mixing, then $(F \times F, \nu \times \nu)$ is ergodic, and so $T$ is well-defined $m \times m$ almost everywhere. Observe that if $T(x, x') = n$, then

\[
T|_{(Q \times Q)_n(x, x')} = n \quad \text{and} \quad (F \times F)^n((Q \times Q)_n(x, x')) = \Delta_0 \times \Delta_0.
\]

Now we define a sequence $\xi_1 < \xi_2 < \xi_3 < \ldots$ of partitions of $\Delta \times \Delta$. First we take $\xi_1(x, x') = (F^{-\tau_1(x)+1}Q)(x) \times \Delta$. The partition $\xi_1$ is formed by sets of the form $\Gamma = A \times \Delta$ where $\tau_1$ is constant on $\Gamma$ and $F^{\tau_1}$ sends $A$ bijectively to $\Delta_0$. For $i > 1$, if $i$ is even (resp. odd), we define $\xi_i$ as the refinement of $\xi_{i-1}$ obtained by partitioning $\Gamma \in \xi_{i-1}$ in the $x'$ direction (resp. $x$ direction) into sets $\tilde{\Gamma}$ such that $\tau_i$ is constant on each $\tilde{\Gamma}$ and $F^{\tau_i}$ sends $\pi'(<\tilde{\Gamma})$ (resp. $\pi(<\tilde{\Gamma})$) bijectively to $\Delta_0$. It will be useful to consider $\xi_0 = \{\Delta \times \Delta\}$. Let us mention two useful properties about the measurability of the functions with respect to the partitions defined above:

- $\tau_1, \tau_2, \ldots, \tau_i$ are $\xi_i$-measurable for each $i \geq 1$;
- $\{T = \tau_i\}$ and $\{T > \tau_i\}$ are $\xi_{i+1}$-measurable for each $i \geq 1$.

This follows from the construction of the objects. Now we present the main estimates we need on $\{\tau_i\}$ and $T$, whose proofs we postpone to Section A.3.

\begin{enumerate}
\item[(E1)] There is $\varepsilon_0 = \varepsilon_0(C', C'') > 0$ such that $P\{T = \tau_i \mid \Gamma\} \geq \varepsilon_0$ for $i \geq 2$ and $\Gamma \in \xi_i$ with $T \mid \Gamma > \tau_i-1$. The dependence of $\varepsilon_0$ on $C'$ and $C''$ can be removed if we consider $i \geq \min(C', C'')$.

\item[(E2)] There is $K_0 = K_0(C', C'') > 0$ such that $P\{\tau_{i+1} - \tau_i > n_0 + n \mid \Gamma\} \leq K_0 m\{\hat{R} > n\}$ for $i \geq 0$, $\Gamma \in \xi_i$ and $n \geq 0$. The dependence of $K_0$ on $C'$ and $C''$ can be removed if we consider $i \geq \min(C', C'')$.
\end{enumerate}

Let $0 \equiv T_0 < T_1 < T_2 < \cdots$ be stopping times in $\Delta \times \Delta$ given by

\[
T_1 = T, \quad \text{and} \quad T_n = T_{n-1} + T \circ (F \times F)^{T_{n-1}}, \quad \text{for } n \geq 2.
\]

\begin{equation}
\tag{25}
\end{equation}

\begin{enumerate}
\item[(E3)] There are $K_1 = K_1(C', C'') > 0$ and $\varepsilon_1 > 0$ (not depending on $\varphi$ or $\varphi'$) such that $|F^n\lambda - F^n\lambda'| \leq 2P\{T > n\} + K_1 \sum_{i=1}^{\infty}(1 - \varepsilon_1)^i P\{T_i \leq n < T_{i+1}\}$ for $n \geq 1$.
\end{enumerate}
(E₄) There is $K₂ = K₂(Cφ, Cφ') > 0$ such that $P\{T_{i+1} - T_i > n\} \leq K₂(m \times m)\{T > n\}$ for $i \geq 0$.

A.2. Convergence to the equilibrium. We shall use (E₁)-(E₄) to prove Theorem 2.6. Let $ν$ be the measure given by Theorem 2.5. Observe that $ν$ is a fixed point for $F^*$, whose density with respect to $m$ belongs to $F^+_β$. Theorem 2.6 follows just by taking $λ' = ν$, once we obtain the upper bound for $|F^*_nλ - F^*_nλ'|$.

We start by observing that for each $i \geq 1$ we have

$$P\{T_i \leq n < T_{i+1}\} \leq \sum_{j=0}^{i} P\left\{T_{j+1} - T_j > \frac{n}{i+1}\right\}.$$  \hfill (26)

Actually, since we have $T_{i+1} > n$, there must be some $0 \leq j \leq i$ with $T_{j+1} - T_j > n/(i+1)$. For otherwise

$$T_{i+1} = \sum_{j=0}^{i} (T_{j+1} - T_j) \leq \sum_{j=0}^{i} \frac{n}{j+1} = n,$$

which is an absurd. Hence

$$\{T_i \leq n < T_{i+1}\} \subset \bigcup_{j=0}^{i} \left\{T_{j+1} - T_j > \frac{n}{i+1}\right\},$$

which gives (26). It follows respectively from (E₃), (26) and (E₄) that

$$|F^*_nλ - F^*_nλ'| \leq 2P\{T > n\} + K₁ \sum_{i=1}^{∞} (1 - ε₁)^i P\{T_i \leq n < T_{i+1}\},$$

$$\leq 2P\{T > n\} + K₁ \sum_{i=1}^{∞} (1 - ε₁)^i \sum_{j=0}^{i} P\left\{T_{j+1} - T_j > \frac{n}{j+1}\right\},$$

$$\leq 2P\{T > n\} + K₁ K₂ \sum_{i=1}^{∞} (1 - ε₁)^i (i+1)(m \times m) \left\{T > \frac{n}{i+1}\right\}.$$

Observe that both in the polynomial and stretched exponential cases, as long we obtain the desired decay for $P\{T > n\}$, then taking $P = m \times m$ it immediately follows that

$$\sum_{i=1}^{∞} (1 - ε₁)^i (i+1)(m \times m) \left\{T > \frac{n}{i+1}\right\}$$

decays at the same speed of $P\{T > n\}$. Consequently, we are left to estimate $P\{T > n\}$. At this point we distinguish the polynomial and stretched exponential cases.

A.2.1. Polynomial decay. Assume there are $C > 0$ and $α > 1$ such that $m\{R > n\} \leq Cn^{-α}$ for all $n \geq 1$. Then, there is $\hat{C} > 0$ (depending only on $C$ and $α$) such that

$$m\{\hat{R} > n\} = \sum_{l>n} m\{R > l\} \leq \hat{C}n^{-α+1}.$$  \hfill (27)
Recall that $T \geq 2n_0$ by construction. We write

$$P\{T > n\} = \sum_{1 \leq i < \left\lfloor \frac{n}{2n_0} \right\rfloor} P\{T > n: \tau_i \leq n < \tau_{i+1}\} + P\{T > n: \tau_{\left\lfloor \frac{n}{2n_0} \right\rfloor} \leq n\}. \quad (28)$$

Since $\{T > \tau_{i-1}\}$ is $\xi_i$-measurable, conditioning on the elements of the partition $\xi_i$ and using (E1) it yields for $i \geq 2$

$$P\{T > \tau_i | T > \tau_{i-1}\} = 1 - P\{T = \tau_i | T > \tau_{i-1}\} \geq 1 - \varepsilon_0. \quad (29)$$

From (29) we obtain for $n \geq 4n_0$

$$P\left\{T > \tau_i: \tau_{i+1} < n \leq \tau_i \right\} \leq \left(1 - \varepsilon_0\right)^{\left\lfloor \frac{n}{2n_0} \right\rfloor - 1}. \quad (30)$$

Since the dependence of $\varepsilon_0$ on $P$ can be removed if we consider $i \geq i_0$ for some $i_0 = i_0(P)$, we are left to compute the decay of

$$\sum_{1 \leq i < \left\lfloor \frac{n}{2n_0} \right\rfloor} P\{T > n: \tau_i \leq n < \tau_{i+1}\}. \quad (31)$$

For each $i \geq 1$ we have $P\{T > n: \tau_i \leq n < \tau_{i+1}\} \leq P\{T > \tau_i: n < \tau_{i+1}\}$. As in (26) we may show that

$$\{T > \tau_i: n < \tau_{i+1}\} \subset \bigcup_{j=0}^{i} \left\{T > \tau_i: \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\},$$

which then gives

$$P\{T > n: \tau_i \leq n < \tau_{i+1}\} \leq \sum_{j=0}^{i} P\left\{T > \tau_i: \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\}. \quad (32)$$

Our next goal is to estimate the terms in the sum (32). Consider first the terms with $i, j \geq 2$. We write

$$P\left\{T > \tau_i: \tau_{j+1} - \tau_j > \frac{n}{i+1}\right\} = A \cdot B \cdot C, \quad (33)$$
with
\[
A = P\{T > \tau_1\} \cdot \prod_{k=2}^{j-1} P\{T > \tau_k \mid T > \tau_{k-1}\}
\]
\[
B = P\Big\{T > \tau_j: \tau_{j+1} - \tau_j > \frac{n}{i+1} \mid T > \tau_{j-1}\Big\};
\]
\[
C = \prod_{k=j+1}^{i} P\{T > \tau_k \mid T > \tau_{k-1}: \tau_{j+1} - \tau_j > \frac{n}{i+1}\}.
\]

Observe that \(A = P\{T > \tau_1\}\) when \(j = 2\), and \(C\) is void when \(j = i\). Arguing as in (29), from estimate \((E_1)\) one gets
\[
A \leq (1 - \varepsilon_0)^{i-2}.
\]
Conditioning on \(\xi_k\) and using \((E_1)\), we have that each term in \(C\) is also bounded from above by \(1 - \varepsilon_0\), which then gives
\[
C \leq (1 - \varepsilon_0)^{i-j}.
\]
Since \(\{T > \tau_{i-1}\}\) is \(\xi_i\)-measurable, conditioning on elements of \(\xi_i\) and using \((E_2)\) we get
\[
B \leq P\Big\{\tau_{j+1} - \tau_j > \frac{n}{i+1} \mid T > \tau_{j-1}\Big\} \leq K_0 \hat{C} \left\{\hat{R} > \frac{n}{i+1} - n_0\right\}.
\]
Using (27) and the fact that \(i < \left\lfloor \frac{n}{2n_0} \right\rfloor\) we obtain
\[
B \leq K_0 \hat{C} \left(\frac{n}{i+1} - n_0\right)^{-\alpha+1} \leq K_0 \hat{C} 2^{1-\alpha} \left(\frac{n}{i+1}\right)^{-\alpha+1}.
\]
From (33), (34), (35) and (36) we deduce for \(i, j \geq 2\)
\[
P\Big\{T > \tau_i: \tau_{j+1} - \tau_j > \frac{n}{i+1}\Big\} \leq K_0 \hat{C} 2^{1-\alpha} \left(\frac{n}{i+1}\right)^{-\alpha+1} (1 - \varepsilon_0)^{i-2}.
\]
Let us consider now the small terms in the sum (32). For \(i \geq 2\) and \(j = 0, 1\) we write
\[
P\Big\{T > \tau_i: \tau_{j+1} - \tau_j > \frac{n}{i+1}\Big\}
\]
\[
\leq P\Big\{T > \tau_1: \tau_{j+1} - \tau_j > \frac{n}{i+1}\Big\} \cdot \prod_{k=2}^{i} P\Big\{T > \tau_k \mid T > \tau_{k-1}: \tau_{j+1} - \tau_j > \frac{n}{i+1}\Big\}
\]
\[
\leq P\big\{\tau_{j+1} - \tau_j > \frac{n}{i+1}\big\} \cdot \prod_{k=2}^{i} P\Big\{T > \tau_k \mid T > \tau_{k-1}: \tau_{j+1} - \tau_j > \frac{n}{i+1}\Big\}
\]
We treat this case arguing as before, thus obtaining
\[
P\Big\{T > \tau_i: \tau_{j+1} - \tau_j > \frac{n}{i+1}\Big\} \leq K_0 \hat{C} 2^{1-\alpha} \left(\frac{n}{i+1}\right)^{-\alpha+1} (1 - \varepsilon_0)^{i-1}.
\]
Finally, for \( i = 1 \) and \( j = 0, 1 \), we have
\[
P\left\{ T > \tau_1: \tau_{j+1} - \tau_j > \frac{n}{i+1} \right\} \leq P\left\{ \tau_{j+1} - \tau_j > \frac{n}{i+1} \right\} \leq K_0 \hat{C} 2^{1-n} \left( \frac{n}{i+1} \right)^{-\alpha+1}.
\]

Using (32), (37), (38) and (39) we get \( P\left\{ T > n: \tau_i \leq n < \tau_{i+1} \right\} \leq C_0 (1-\varepsilon_0)^i(i+1)\alpha n^{-\alpha+1} \), where \( C_0 \) is a constant depending only on \( K_0, \hat{C}, \alpha \) and \( \varepsilon_0 \). This yields the desired bound for (31) in the polynomial case.

### A.2.2. Stretched exponential decay

Assume that there are \( C, c > 0 \) and \( 0 < \eta \leq 1 \) such that \( \text{Leb}\{ R > n \} \leq C e^{-c n^\eta} \) for all \( n \geq 1 \). Then there is \( \hat{C} > 0 \) such that
\[
m\{ \hat{R} > n \} = \sum_{l>n} m\{ R > l \} \leq \hat{C} e^{-c n^\eta}.
\]

The conclusion in this case is a consequence of \((E_1)-(E_2)\) and the next lemma, which can easily be obtained from [5, Lemma 4.2] by taking \( L = 1, \tau = T, \mu = P \) and \( t_j = \tau_j \).

**Lemma A.1.** Assume that there are \( \varepsilon_0 > 0 \) and \( K_0 > 0 \) such that for all \( i \geq 2 \) and \( \Gamma \in \xi_i \) with \( T \mid \Gamma > \tau_{i-1} \) we have
\[
(1) \ P\{ T = \tau_i \mid \Gamma \} \geq \varepsilon_0;
\]
\[
(2) \ P\{ \tau_{i+1} - \tau_i > n \mid \Gamma \} \leq K_0 e^{-c n^\eta}.
\]

Then there exist \( C', c' > 0 \) such that \( P\{ T > n \} \leq C' e^{-c' n^\eta} \).

For the sake of completeness one must verify that the constants \( C' \) and \( c' \) obey the final requirement of Theorem 2.6. Actually, it is proved in [5, Lemma 4.2] that there are a measurable function \( k \) and a measurable set \( B_n \) such that for \( q(n) = \lfloor \alpha n^\eta \rfloor \), with small \( \alpha > 0 \), we have \( \{ T > n \} \subset \{ k > q(n) \} \cup B_n \) (recall estimate (27) in [5]) with \( P\{ k > q(n) \} \leq (1-\varepsilon_0)^q(n) \), and for some positive integer \( K \) only depending on \( K_0 \) and \( \eta \),
\[
P(B_n) \leq 2^{q(n)} \sum_{p \geq n/2} C e^{-c p^\eta}, \quad \text{for} \ n \geq K.
\]

Hence, taking \( \alpha > 0 \) sufficiently small we obtain the desired conclusion.

### A.3. Main estimates

Here we obtain estimates \((E_1)-(E_4)\). We start with some preliminary results on distortion control that will enable us to prove \((E_1)\) and \((E_2)\).

**Lemma A.2.** There is \( C_0 = C_0(C_\varphi) > 0 \) such that for every \( k \geq 1 \) and \( A \in \bigvee_{i=0}^{k-1} F^{-i} \Omega \) with \( F^k(A) = \Delta_0 \), we have for \( \mu = F^k_*(\lambda | A) \) and all \( x, y \in \Delta_0 \)
\[
\left| \frac{d\mu}{dm}(x) \right| / \left( \frac{d\mu}{dm}(y) \right) \leq C_0.
\]

Moreover, the dependence of \( C_0 \) on \( C_\varphi \) can be removed if we assume that the number of visits \( j \leq k \) of \( A \) to \( \Delta_0 \) is bigger than some \( j_0 = j_0(C_\varphi) \).
Proof. Let \( x_0, y_0 \in A \) be such that \( F^k(x_0) = x \) and \( F^k(y_0) = y \). Using (12) and the fact that \( \varphi \in F_\beta \) we have

\[
\left| \frac{d\mu}{dm}(x) \right| \left| \frac{d\mu}{dm}(y) \right| = \left| \frac{\varphi(x)}{JF^k(x)} \right| \left| \frac{\varphi(y)}{JF^k(y)} \right| \leq \left| \frac{\varphi(x)}{\varphi(y)} \right| \left| \frac{JF^k(y)}{JF^k(x)} \right| \leq (1 + C_\varphi \beta^j)(1 + C_F),
\]

where \( j \) is the number of visits of \( A \) to \( \Delta_0 \) prior to \( k \). \( \square \)

The next result is proved in [12, Sublemma 1].

Lemma A.3. There is \( M_0 > 0 \) such that \( \frac{dF_n m}{dm} \leq M_0 \) for all \( n \geq 1 \).

A.3.1. Proof of \((E_1)\). Assume without loss of generality that \( i \) is even, and take \( \Gamma \in \xi_i \) as in the statement of \((E_1)\). We have \( \Gamma = A \times B \) with \( A, B \subset \Delta \), where \( A \) is sent bijectively by \( F^{\tau_i} \) to \( \Delta_0 \) and \( F^{\tau_i}(B) \) is contained in some \( \Delta_{l,j} \). At time \( \tau_i \) we have \( F^{\tau_i}(B) = \Delta_0 \) and \( F^{\tau_i}(A) \) is spread over several parts of \( U \{ \Delta_l : l \leq \tau_i - \tau_{i-1} \} \). The set \( \{ T = \tau_i \} \cap \Gamma \) has the form \( A' \times B \) where \( A' \) is the set of points in \( A \) which are sent to \( \Delta_0 \) by \( F^{\tau_i} \) and return to \( \Delta_0 \) by \( F^{\tau_i-n} \). Letting \( \mu = F^{\tau_i-n}(\lambda|A) \) we may write

\[
P\{ T = \tau_i \mid \Gamma \} = \frac{\lambda(A')}{\lambda(A)} = \frac{\mu(F^{-(\tau_i-n)}(\Delta_0) \cap \Delta_0)}{\mu(\Delta_0)}.
\]

Note that Lemma A.2 applies to \( \mu \), thus giving

\[
P\{ T = \tau_i \mid \Gamma \} \geq C_0^{-2} \frac{m(F^{-(\tau_i-n)}(\Delta_0) \cap \Delta_0)}{m(\Delta_0)}.
\]

Recall that \( n_0 \) has been chosen in such a way that there is \( \gamma_0 \) such that \( m(F^{-n}(\Delta_0) \cap \Delta_0) \geq \gamma_0 > 0 \), for all \( n \geq n_0 \). By construction we have \( \tau_i - \tau_{i-1} \geq n_0 \). This is enough for concluding that there is some \( \epsilon_0 = \epsilon_0(C_\varphi) > 0 \) for which \( P\{ T = \tau_i \mid \Gamma \} \geq \epsilon_0 \). The other case \((i \text{ odd})\) gives the dependence of \( \epsilon_0 \) also on \( C_\varphi \). These dependencies can be removed if we take \( i \) large enough, according to Lemma A.2.

A.3.2. Proof of \((E_2)\). For \( i = 0 \) we have

\[
P\{ \tau_1 > n_0 + n \} = (F^0_\pi \lambda)\{ \hat{R} > n \} \leq \left\| \frac{d\lambda}{dm} \right\| \infty M_0 m\{ \hat{R} > n \},
\]

and for \( i = 1 \)

\[
P\{ \tau_2 - \tau_1 > n_0 + n \} = (F^{\tau_1} + n_0 \lambda)\{ \hat{R} > n \} \leq \left\| \frac{d\lambda}{dm} \right\| \infty M_0 m\{ \hat{R} > n \},
\]

which obviously give upper bounds depending on \( C_\varphi \) and \( C_\varphi' \).

Let us consider now the case \( i \geq 2 \). Assume for definiteness that \( i \) is even. Considering the probability measure

\[
\mu = \frac{1}{P(\Gamma)} F^{\tau_i-1} \pi_*(P|\Gamma)
\]
we have

\[ P\{\tau_{i+1} - \tau_i > n_0 + n \mid \Gamma \} = \left( F^{(\tau_i - \tau_{i-1})+n_0}_* \mu \right) \{ \hat{\mathcal{R}} > n \} \leq \left\| \frac{d}{dm} \left( F^{(\tau_i - \tau_{i-1})+n_0}_* \mu \right) \right\|_\infty \ m\{\hat{\mathcal{R}} > n \} \leq M_0 \left\| \frac{d\mu}{dm} \right\|_\infty \ m\{\hat{\mathcal{R}} > n \}, \quad \text{by Lemma A.3} \]

Using Lemma A.2 one has that \( \left\| \frac{d\mu}{dm} \right\|_\infty \) is bounded from above by some constant only depending on \( C_0 \). Moreover, according to Lemma A.2, this dependency can be removed if we take \( i \) large enough.

For obtaining (E3) and (E4) we consider the dynamical system \( \hat{F} = (F \times F)^T : \Delta \times \Delta \to \Delta \times \Delta \). It follows from the definition of the sequence \( \{ T_n \} \) in (25) that

\[ \hat{F}^n = (F \times F)^{T_n}, \quad \text{for all } n \geq 1. \tag{40} \]

Let \( \hat{\xi}_1 \) denote the partition into rectangles \( \hat{\Gamma} \) of \( \Delta \times \Delta \) on which \( T \) is constant and \( \hat{F}^n \) maps \( \hat{\Gamma} \) bijectively to \( \Delta_0 \times \Delta_0 \). Next we define inductively partitions \( \hat{\xi}_2, \hat{\xi}_3, \ldots \) of \( \Delta \times \Delta \) by \( \hat{\xi}_n := \hat{F}^{-(n-1)} \hat{\xi}_1 \), for \( n \geq 2 \). Each \( \hat{\xi}_n \) is the partition into subsets \( \hat{\Gamma} \) of \( \Delta \times \Delta \) on which \( T_n \) is constant and \( \hat{F} \) maps \( \hat{\Gamma} \) bijectively to \( \Delta_0 \times \Delta_0 \). We consider the reference measure \( m \times m \) for the dynamical system \( \hat{F} \) and \( J \hat{F} \) the Jacobian of \( \hat{F} \) with respect to \( m \times m \). We define a separation time \( \hat{s} : (\Delta \times \Delta) \times (\Delta \times \Delta) \to \mathbb{N}_0 \) for \( \hat{F} \) in the following way: given \( w, z \in \Delta \times \Delta \), take

\[ \hat{s}(w, z) = \min \left\{ n \geq 0 : \hat{F}^nw \text{ and } \hat{F}^nz \text{ lie in distinct elements of } \hat{\xi}_1 \right\}. \]

Denoting

\[ \Phi = \frac{dP}{d(m \times m)}, \]

we have \( \Phi(x, x') = \varphi(x) \varphi'(x') \). With no loss of generality we assume from here on that \( \varphi(x) > 0 \) and \( \varphi(y) > 0 \). The next two results are proved in [12, Sublemma 3].

**Lemma A.4.** Let \( C_1 > 0 \) be as in Lemma A.4. Given \( w, z \in \Delta \times \Delta \) with \( \hat{s}(w, z) \geq n \geq 1 \)

\[ \log \frac{J \hat{F}^n(w)}{J \hat{F}^n(z)} \leq 2C_1 \beta^{\hat{s}(F^n(w), F^n(z))}. \]

**Lemma A.5.** Let \( C_\Phi = C_\varphi + C_{\varphi'} \). Given \( w, z \in \Delta \times \Delta \)

\[ \log \frac{\Phi(w)}{\Phi(z)} \leq C_\Phi \beta^{\hat{s}(w, z)}. \]

The next result gives a distortion control similar to that of Lemma A.2.
We claim that
\[ \left| \frac{dQ}{dm}(x) / \frac{dQ}{dm}(y) \right| \leq C_*. \]

**Proof.** Let \( x_0, y_0 \in \Gamma \) be such that \( \hat{F}^i(x_0) = x \) and \( \hat{F}^i(y_0) = y \). Recall that \( \hat{s}(x_0, y_0) \geq i \).
Using Lemma A.5 and Lemma A.4 we obtain
\[ \left| \frac{dQ}{dm}(x) / \frac{dQ}{dm}(y) \right| = \left| \frac{\Phi(x_0)}{J\hat{F}^i(x_0)} \cdot \frac{J\hat{F}^i(y_0)}{\Phi(y_0)} \right| \leq \frac{\Phi(x_0)}{\Phi(y_0)} \cdot \left| \frac{J\hat{F}^i(y_0)}{J\hat{F}^i(x_0)} \right| \leq \exp(C_\Phi + 2C_1). \]

We just have to take \( C_* = \exp(C_\Phi + C_{\varphi'} + 2C_1) \). \( \square \)

Now we are going to define a sequence of densities \( \hat{\Phi}_0 \geq \hat{\Phi}_1 \geq \hat{\Phi}_2 \geq \cdots \) in \( \Delta \times \Delta \) with the property that for all \( i \geq 0 \) and all \( \hat{\Gamma} \in \hat{\xi}_i \)
\[ \pi_* \hat{F}_*^i((\hat{\Phi}_{i-1} - \hat{\Phi}_i)((m \times m)|\hat{\Gamma})) = \pi_* \hat{F}_*^i((\hat{\Phi}_{i-1} - \hat{\Phi}_i)((m \times m)|\hat{\Gamma})) \]  
(41)

Let \( \varepsilon = \varepsilon(F) > 0 \) be a small number to be determined later (see Lemma A.7 below). Let \( i_1 = i_1(\Phi) \) be such that
\[ C_\Phi \beta^{i_1} < C_\hat{F}. \]  
(42)

For \( i < i_1 \), we take \( \hat{\Phi} \equiv \Phi \). For \( i \geq i_1 \), let
\[ \hat{\Phi}_i(z) = \left[ \frac{\hat{\Phi}_{i-1}(z)}{J\hat{F}^i(z)} - \varepsilon \min_{w \in \hat{\xi}_i} \frac{\hat{\Phi}_{i-1}(w)}{J\hat{F}^i(w)} \right] J\hat{F}^i(z), \]  
(43)

where \( \hat{\xi}_i(z) \) is the element of \( \hat{\Gamma} \) of \( \hat{\xi}_i \) that contains \( z \). One can easily see that the sequence \( \{\hat{\Phi}_i\} \) satisfies condition (41). The next result is proved in [12, Lemma 3]. As observed in [12, page 166], \( \varepsilon \) depends only on \( \beta \).

**Lemma A.7.** If \( \varepsilon > 0 \) is sufficiently small, then there is \( 0 < \varepsilon_1 < 1 \) (not depending on \( \Phi \)) such that \( \hat{\Phi}_i \leq (1 - \varepsilon_1)\hat{\Phi}_{i-1} \) for all \( i \geq i_1 \).

**A.3.3. Proof of (E3).** Let \( \varepsilon_1 > 0 \) be as in Lemma A.7. Let \( \Phi_0, \Phi_1, \Phi_2, \ldots \) be defined in the following way: given \( n \geq 0 \) and \( z \in \Delta \times \Delta \), let
\[ \Phi_n(z) = \hat{\Phi}_i(z) \quad \text{for} \quad T_i(z) \leq n < T_{i+1}(z). \]  
(44)

We claim that
\[ |F_n^\lambda - F_n^\lambda| \leq 2 \int \Phi_n d(m \times m) \quad \text{for all} \quad n \geq 1. \]  
(45)
Actually, taking \( \Phi = \Phi_n + \sum_{k=1}^{n} (\Phi_{k-1} - \Phi_k) \) we have

\[
|F^n_x \lambda - F^n_x \lambda'| = |\pi_x(F \times F)^n_x(\Phi(m \times m)) - \pi_x'(F \times F)^n_x(\Phi(m \times m))| \\
\leq |\pi_x(F \times F)^n_x(\Phi_n(m \times m)) - \pi_x'(F \times F)^n_x(\Phi_n(m \times m))| \\
+ \sum_{k=1}^{n} |(\pi - \pi')_x[(F \times F)^n_x((\Phi_{k-1} - \Phi_k)(m \times m))]|.
\]

For the first term in the last sum we have

\[
|\pi_x(F \times F)^n_x(\Phi(m \times m)) - \pi_x'(F \times F)^n_x(\Phi(m \times m))| \leq 2 \int \Phi_n d(m \times m).
\]

Let us see that all the other terms vanish. Define \( A_{k,i} = \{ z \in \Delta \times \Delta : k = T_i(z) \} \) and \( A_k = \cup A_{k,i} \). Each \( A_{k,i} \) is a union of elements of \( \Gamma \in \xi_i \) and \( A_{k,i} \neq A_{k,j} \) for \( i \neq j \). By \((41)\) we have \( \Phi_{k-1} - \Phi_k = \hat{\Phi}_{i-1} - \hat{\Phi}_i \) on \( \Gamma \in \xi_i | A_{k,i} \), and \( \Phi_k = \Phi_{k-1} \) on \( \Delta \times \Delta - A_k \). For \( k \geq 1 \)

\[
\pi_x(F \times F)^n_x((\Phi_{k-1} - \Phi_k)(m \times m)) \\
= \sum_{i} \sum_{\Gamma \subset A_{k,i}} F^{n-k}\pi_x(F \times F)^T_x((\hat{\Phi}_{i-1} - \hat{\Phi}_i)(m \times m)|\Gamma)) \\
= \sum_{i} \sum_{\Gamma \subset A_{k,i}} F^{n-k}\pi'_x(F \times F)^T_x((\hat{\Phi}_{i-1} - \hat{\Phi}_i)(m \times m)|\Gamma), \quad \text{by \((41)\)} \\
= \pi'_x(F \times F)^n_x((\Phi_{k-1} - \Phi_k)(m \times m)).
\]

This completes the proof of \((45)\). To finish \((E_5)\) we write

\[
\int \Phi_n d(m \times m) = \int_{\{T_i > n\}} \Phi_n d(m \times m) + \sum_{i=1}^{\infty} \int_{\{T_i \leq n < T_{i+1}\}} \Phi_n d(m \times m).
\]

Observe that

\[
\int_{\{T_i > n\}} \Phi_n d(m \times m) = \int_{\{T_i > n\}} \Phi d(m \times m) = P\{T_i > n\},
\]

while for \( i \geq i_i \),

\[
\int_{\{T_i \leq n < T_{i+1}\}} \Phi_n d(m \times m) = \int_{\{T_i \leq n < T_{i+1}\}} \hat{\Phi}_i d(m \times m) \\
\leq \int_{\{T_i \leq n < T_{i+1}\}} (1 - \varepsilon_1)^{i-i_i+1} \Phi d(m \times m) \\
= (1 - \varepsilon_1)^{i-i_i+1} P\{T_i \leq n < T_{i+1}\}.
\]
Hence

\[ |F^n \lambda - F^n' \lambda| \leq 2P\{T_{i_1} > n\} + 2 \sum_{i=i_1}^{\infty} (1 - \varepsilon_1)^{i-i_1+1} P\{T_i \leq n < T_{i+1}\} \]

\[ \leq 2P\{T_{i_1} > n\} + 2(1 - \varepsilon_1)^{-i_1+1} \sum_{i=i_1}^{\infty} (1 - \varepsilon_1)^i P\{T_i \leq n < T_{i+1}\}. \]  

(46)

We may write

\[ P\{T_{i_1} > n\} = P\{T > n\} + (1 - \varepsilon_1)^{-i_1+1} \sum_{i=1}^{i_1-1} (1 - \varepsilon_1)^i P\{T_i \leq n < T_{i+1}\} \]

\[ \leq P\{T > n\} + (1 - \varepsilon_1)^{-i_1+1} \sum_{i=1}^{i_1-1} (1 - \varepsilon_1)^i P\{T_i \leq n < T_{i+1}\}, \]

which together with (46) yields

\[ |F^n \lambda - F^n' \lambda| \leq 2P\{T > n\} + K_1 \sum_{i=1}^{\infty} (1 - \varepsilon_1)^i P\{T_i \leq n < T_{i+1}\}, \]

with \( K_1 \) depending only on \( \varepsilon_1 \) and \( i_1 \). From (42) and Lemma A.7 one easily obtains the desired dependence of \( K_1 \) on \( \varphi \) and \( \varphi' \).

A.3.4. Proof of (E4). This estimate is obviously true for \( i = 0 \). Take an arbitrary \( i \geq 1 \) and \( \Gamma \in \hat{\xi}_i \). Recall that \( \hat{F}^i \) maps \( \hat{\Gamma} \) bijectively to \( \Delta_0 \times \Delta_0 \). Letting \( Q = F^i_\lambda(P|\Gamma) \) and observing that from (25) and (40) we have \( T_{i+1} - T_i = T \circ \hat{F}^i \), we may write

\[ P\{T_{i+1} - T_i > n \mid \Gamma\} = \frac{Q\{T > n\}}{Q(\Delta_0 \times \Delta_0)}. \]

Using Lemma A.6

\[ P\{T_{i+1} - T_i > n \mid \Gamma\} \leq C^2 \frac{(m \times m)\{T > n\}}{(m \times m)(\Delta_0 \times \Delta_0)}. \]

From this last inequality one easily obtains (E4) with \( K_2 = C^2/(m \times m)(\Delta_0 \times \Delta_0) \).

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