Monomial invariants in codimension two

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Abstract
We define the monomial invariants of a projective variety $Z$; they are invariants coming from the generic initial ideal of $Z$.

Using this notion, we generalize a result of Cook [C]:
If $Z$ is an integral variety of codimension two, satisfying the additional hypothesis $s_Z = s_1$, then its monomial invariants are connected.

Introduction

The generic initial ideal of a projective variety $Z$, $\text{gin}(I_Z)$, preserves some informations about $Z$, in particular it has the same Hilbert function; on the other hand, $\text{gin}(I_Z)$ is a combinatorial object, which can be “finitely” described in terms of its monomial generators; thus any limitation on the “shape” of $\text{gin}(I_Z)$, i.e. any relation among its generators, translates into a limitation on the possible Hilbert functions of projective varieties.

Generic initial ideals are particularly well suited to study codimension two varieties; in this situation $\text{gin}(I_Z)$ seems to have just the right amount of information. Since we shall mostly deal with sections of $Z$, we use the reverse lexicographic order on monomials; also, $\text{gin}(I_Z)$ being a saturated ideal, its monomial generators do not contain the last variable $x_n$—here $Z$ is a (nondegenerate) subvariety of $\mathbb{P}^n$.

The first instance of such a variety is a set $\Gamma$ of points in $\mathbb{P}^2$, for which $\text{gin}(I_\Gamma) \subseteq \mathbb{K}[x_0, x_1, x_2]$ is minimally generated by monomials of type $x_0^{\lambda_0} x_1^{\lambda_1}$; a classical result of Gruson and Peskine [GP] implies that, if the points of $\Gamma$ are in general position, then $\lambda_{i+1} + 1 \geq \lambda_i \geq \lambda_{i+1} + 2$, for all $i = 0, 1, ..., s - 2$; we describe this situation by saying that the invariants $\lambda_i$ are connected, i.e. in passing from a generator $x_0^{\lambda_0} x_1^{\lambda_1}$ to the next $x_0^{\lambda_0+1} x_1^{\lambda_1+1}$, the invariant $\lambda_i$ can...

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“jump” downward to \( \lambda_i + 1 \) by one or two steps, but no more (the fact that it jumps no less than that is a consequence of another property of \( \text{gin} \), its Borel–fixedness).

Next case is a space curve \( C \subseteq \mathbb{P}^3 \). Now \( \text{gin}(I_C) \) can be thought of as a stack of slices in the \( x_2 \) direction, the slice at level \( p \geq 0 \) being the set of all monomials \( x_0^\alpha x_1^\beta x_3^\gamma \) belonging to \( \text{gin}(I_C) \). Since the slice at level \( p \gg 0 \) represents \( \text{gin}(I_{\Gamma}) \), where \( \Gamma \) is the general plane section of \( C \), then the invariants of such a slice are connected; the question is what happens when \( p \) is small. The main result of a paper of Cook [C] is that the invariants of \( \text{gin}(I_C) \) are connected at all levels \( p \geq 0 \). Unfortunately, in the proof of this result there are some gaps. In order to fix those gaps, Decker and Schreyer made the hypothesis that \( s_C = s_{\Gamma} \), where \( s_C \) (or \( s_{\Gamma} \)) is the minimal degrees of a polynomial vanishing on \( C \) (or \( \Gamma \)); this hypothesis seems to be unavoidable, but the proof given in [DS] is still incomplete. Finally Amasaki [A] gave a complete proof of the connectedness, under the same hypothesis, using mostly algebraic techniques.

The present paper translates Amasaki’s ideas in a more geometric language, generalizing the result about the connectedness of invariants to higher dimensional varieties, in the following sense.

Let \( Z \subseteq \mathbb{P}^n \) be an integral (i.e. reduced and irreducible) nondegenerate variety of codimension two and suppose that \( s_Z = s_{\Gamma} \), then the slice of \( \text{gin}(I_Z) \) at level \( p_j \) with respect to the variable \( x_j, j = 2, ..., n-1 \), has connected invariants, for all \( p_j \geq 0 \).

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1 Background

Let \( K \) be the base field and assume that it be algebraically closed.

Let \( W \) be a \( K \)-vector space of dimension \( n+1 \) and denote by \( K(x_0, ..., x_n) \) the \((n+1)\)-dimensional \( K \)-vector space generated by \( x_0, ..., x_n \).

Our environment is the projective space \( \mathbb{P}^n = \mathbb{P}(W^*) \), whose ring of polynomials is the symmetric algebra \( S_W = \otimes_{d \geq 0} S^d W. \) Of course, starting with \( \mathbb{K}(x_0, ..., x_n) \), we get \( \mathbb{P}(\mathbb{K}) \) and \( \mathbb{K}[x_0, ..., x_n] \) respectively. We use italic \( I, J, ... \) for ideals in \( SW \) and gothic i, j, ... for ideals in \( K[x_0, ..., x_n] \).

Another standard piece of notation is the multiindex one: \( x^p = x_0^{p_0} ... x_n^{p_n} \), with \( |p| = \deg x^p = p_0 + ... + p_n \).

Definition 1.1 A coordinate system (or simply coordinates) on \( W \) is an isomorphism of \( K \)-vector spaces \( W \to K(x_0, ..., x_n) \). We denote by \( \text{R}(W) \) the set of all coordinate systems on \( W \).
Clearly, a coordinate system on $W$ gives coordinates in $SW$, i.e. an isomorphism of $K$-algebras $SW \cong K[x_0, ..., x_n]$. Since $W \cong K^{n+1}$, there is a bijection $R(W) \leftrightarrow GL(n + 1, K)$, thus $R(W)$ has a natural structure of algebraic variety, with its attendant Zariski topology.

**Definition 1.2** (i) We consider an ordering of the set of monomials of $K[x_0, ..., x_n]$, the reverse lexicographic order, denoted by $>_rlex$, and defined as follows: $x^p >_rlex x^q$ if either $\deg x^p < \deg x^q$, or $\deg x^p = \deg x^q$ and $p_n = q_n, \ldots, p_k = q_k, p_{k+1} < q_{k+1}$, for some $k$.

(ii) If $f \in K[x_0, ..., x_n]$, the initial monomial of $f, \text{in}(f)$, is the greatest monomial, with respect to $>_rlex$, that appears in $f$.

(iii) Let $\mathfrak{i}$ be a homogeneous ideal in $K[x_0, ..., x_n]$, we define the initial ideal of $\mathfrak{i}$, denoted by $\text{in}(\mathfrak{i})$, as the monomial ideal generated by all $\text{in}(f)$, $f \in \mathfrak{i}$.

(iv) The $j$-th elementary move, $j = 1, ..., n$ is defined on monomials by $e_j(x^p) = x_0^{p_0} \cdots x_{j-1}^{p_{j-1}} x_j x_{j+1}^{p_{j+1}} \cdots x_n^{p_n}$, if $p_j > 0$, otherwise $e_j(x^p) = 0$.

(v) A monomial ideal $\mathfrak{b}$ is Borel-fixed if, for all monomials $x^p \in \mathfrak{b}$ and all elementary moves $e_j$, $e_j(x^p) \in \mathfrak{b}$.

If $I$ is a homogeneous ideal in $SW$ and $g \in R(W)$, then $gI$ is an ideal of $K[x_0, ..., x_n]$, so it makes sense to consider $\in_g(I) := \text{in}(gI)$. Although $\in_g(I)$ depends, by its very definition, on the choice of $g$, it turns out that, for general coordinates $g \in R(W)$, $\in_g(I)$ stays constant, i.e. does not depend on $g$; this is the content of the following theorem, due to Galligo.

**Theorem 1.3** Let $I \subseteq SW$ be a homogeneous ideal. There exists a Zariski open subset $U_I \subseteq R(W)$ such that $\forall g \in U_I \in_g(I)$ is constant; it is called the generic initial ideal of $I$, $\text{gin}(I)$.

Furthermore, $\text{gin}(I)$ is Borel-fixed.

**Proof:** [G] 1.27. ■

Let $I$ be a homogeneous ideal, then $I_{<d}$ (resp. $I_{\leq d}$) denotes the ideal generated by the elements of $I$ of degree $< d$ (resp. $\leq d$); furthermore, we say that $\text{gin}(I)$ has a gap in degree $\delta$ if $\text{gin}(I)$ has no minimal generator of degree $\delta$.

The following result is due to Green.

**Proposition 1.4** If $\text{gin}(I)$ has a gap in degree $\delta$, then $\text{gin}(I_{<\delta}) = (\text{gin}(I))_{<\delta}$.

**Proof:** [DS] 2.12. ■

**Corollary 1.5** If $\text{gin}(I)$ has a gap in degree $\delta$, then $\text{gin}(I_{\leq \delta}) = (\text{gin}(I))_{\leq \delta}$.

**Proof:** Since $\text{gin}(I)$ has a gap in degree $\delta$, then $(\text{gin}(I))_{\leq \delta} = (\text{gin}(I))_{<\delta}$, because there is no generator in degree $\delta$, but also $I_{\leq \delta} = I_{<\delta}$, because a minimal Gröbner basis of $I$ (cf. [CLO] ch. 2) contains no polynomial of degree $\delta$ either. It follows that $\text{gin}(I_{\leq \delta}) = \text{gin}(I_{<\delta}) = (\text{gin}(I))_{<\delta} = (\text{gin}(I))_{\leq \delta}$. ■

For ease of reference, we collect here a few facts that will be needed later on.
Proposition 1.6 ([G] 2.30) The following are equivalent:
(i) $I$ is saturated;
(ii) $\text{gin}(I)$ is saturated;
(iii) no generator of $\text{gin}(I)$ contains the last variable $x_n$.

Proposition 1.7 ([G] 2.14) For a general linear form $h$, the equality
$$\text{gin}((I : h^p)|_h) = (\text{gin}(I) : x_n^p)|_{x_n}$$
holds for all $p \geq 0$.

Theorem 1.8 ([S] theorem I.6.8) Let $f : X \to B$ be a surjective morphism of projective varieties. If $B$ is irreducible and all the fibers $f^{-1}(b)$ are irreducible and of the same dimension, then $X$ is irreducible.

2 Connectedness

If $W' \subseteq W$ is a linear subspace, the natural projection $W \to W/\langle h \rangle$ induces a map on the corresponding symmetric algebras $f : SW \to f|_{W'} \in S(\frac{W}{\langle h \rangle})$; especially, for any linear form $h \in W$, $f|_h \in S(\frac{W}{\langle h \rangle})$.

Note that $\frac{\mathbb{K}[x_0, ..., x_n]}{(x_n)} \cong \mathbb{K}(x_0, ..., x_{n-1})$ canonically, hence, for any $f \in \mathbb{K}[x_0, ..., x_n]$, we think of $f|_{x_n}$ as a polynomial in $\mathbb{K}[x_0, ..., x_{n-1}]$, and similarly, if $i \subseteq \mathbb{K}[x_0, ..., x_n]$ is a (homogeneous) ideal, then $i|_{x_n}$ is a (homogeneous) ideal of $\mathbb{K}[x_0, ..., x_{n-1}]$.

Let $S = V(W') \subseteq \mathbb{P}(W^*)$ be the subspace determined by $W'$, and let $Z \subseteq \mathbb{P}(W^*)$ be a projective variety, with ideal $I = I_Z$, then, set-theoretically, $V(I|_{W'}) = Z \cap S$; especially, if $H = V(h)$ is a hyperplane, then $V(I|_h) = Z \cap H$ is the hyperplane section of $Z$.

Note that, for arbitrary $h$, $I|_h$ is a proper subset of $I_{Z \cap H}$; however, for general $h$, their saturations coincide, i.e. for a general linear form $h \in W$, $(I|_h)^{\text{sat}} = I_{Z \cap H}$ (recall that $I_{Z \cap H}$ is prime, hence saturated).

Let $b$ be a Borel-fixed monomial ideal in $\mathbb{K}[x_0, ..., x_n]$, then, for all $\mathbf{p} = (p_2, p_3, ..., p_n) \in \mathbb{N}^{n-1}$ and $\mathbf{x}^\mathbf{p} = x_2^{p_2}x_3^{p_3}...x_n^{p_n}$, the ideal $(b : \mathbf{x}^\mathbf{p}) \cap \mathbb{K}[x_0, x_1]$ is Borel-fixed too, hence it has a minimal system of generators of type $x_0^{s_0}, x_0^{s-1}x_1, ..., x_0x_1^{s_1}, x_1^{s_1}$, with $s = s(\mathbf{p})$ and $\lambda_i = \lambda_i(\mathbf{p})$, for $i = 0, 1, ..., s - 1$.

Definition 2.1 The monomial invariants of $b$ are the integers $\lambda_i(\mathbf{p})$, for all $i$ and $\mathbf{p}$.

Remark 2.2 (i) By noetherianity, there are finitely many monomial invariants.
(ii) Borel-fixedness implies that $\lambda_i(\mathbf{p}) \geq \lambda_{i+1}(\mathbf{p}) + 1$, for all $\mathbf{p}$.
(iii) Note that $s(\mathbf{0}) = \text{deg}(a)$ is the smallest degree of a polynomial of $b$; furthermore, if $p_j \leq q_j$ for all $j = 2, ..., n$, then $s(\mathbf{p}) \geq s(\mathbf{q})$. 


Let $Z \subseteq \mathbb{P}^n$ be a projective variety and let $I_Z$ be its ideal, then no generator of $\text{gin}(I_Z)$ contains the last variable $x_n$, so its monomial invariants depend only on $\hat{\mathbf{p}} := (p_2, p_3, \ldots, p_{n-1}) \in \mathbb{N}^{n-2}$, i.e. they are independent of $p_n$.

**Definition 2.3** The monomial invariants of $Z$ are the monomial invariants of $\text{gin}(I_Z)$, i.e. the integers $\lambda_i(\hat{\mathbf{p}})$, for all $\hat{\mathbf{p}} \in \mathbb{N}^{n-2}$, and $i = 0, 1, \ldots, s(\hat{\mathbf{p}}) - 1$.

**Remark 2.4**
(i) For $p_{n-1}$ big enough, $\lambda_i(\hat{\mathbf{p}})$ are the monomial invariants of the generic hyperplane section of $Z$. Especially, if $Z$ has codimension two and all $p_j$ are big enough, $j = 2, \ldots, n - 1$, then $\lambda_i(\hat{\mathbf{p}})$ are independent of $\hat{\mathbf{p}}$ and are the monomial invariants of the section $\Gamma = Z \cap \Pi$ of $Z$ with a general 2-plane $\Pi$.
(ii) Note that $s(0) = s_Z = \text{smallest degree of a polynomial vanishing on } Z$ and $s(\hat{\mathbf{p}}) = s_{\Gamma} = \text{smallest degree of a polynomial vanishing on } \Gamma$, when $\hat{p}_j \gg 0$ for all $j$.

**Theorem 2.5** Let $Z$ be an integral nondegenerate projective variety of codimension two, and let $\Gamma$ be its generic section with a linear subspace of dimension two. If $s_Z = s_{\Gamma}$, then the monomial invariants of $Z$ are connected i.e. they satisfy the following inequality:

$$\lambda_{i+1}(\hat{\mathbf{p}}) + 2 \geq \lambda_i(\hat{\mathbf{p}}) \geq \lambda_{i+1}(\hat{\mathbf{p}}) + 1.$$ 

for all $\hat{\mathbf{p}}$ and $i = 0, 1, \ldots, s(\hat{\mathbf{p}}) - 2$.

**Proof:** As already remarked, the second inequality is a consequence of Borel-fixedness, so one needs to prove only the first inequality. To avoid too cumbersome notations, we restrict ourselves to the case of a surface $\Sigma$ in $\mathbb{P}^4$, the general case being completely similar.

As a preliminary step, recall that, in general, $s_{\Sigma} = s(0, 0) \geq s(p, q) \geq s_{\Gamma}$ for all $p, q$, hence the hypothesis $s_{\Sigma} = s_{\Gamma}$ implies that all $s(p, q)$ are equal; denote this common value by $s$.

Let $I := I_\Sigma$ be the ideal of $\Sigma$ and assume that the monomial invariants are not connected, i.e. there are integers $j, \hat{p}, \hat{q}$, with $0 \leq j < s(\hat{p}, \hat{q}) - 1$, such that

$$\lambda_j(\hat{p}, \hat{q}) > \lambda_{j+1}(\hat{p}, \hat{q}) + 2.$$

Set $\delta := j + \lambda_{j+1}(\hat{p}, \hat{q}) + 2$ and define, for general linear forms $h, l, m$,

$$J = J(h, l, m) := ((|I_h| : l^\hat{\mathbf{p}} | \lambda_\delta) : m^{\hat{\mathbf{p}}})|_m.$$

**Step 1** $\text{gin}(J)$ has a gap in degree $\delta$.

A repeated application of proposition 1.7, shows that

$$\text{gin}(J) = (\{(\text{gin}(I)|_{x_4} : x_3^\hat{q})|_{x_3} : x_2^{\hat{\mathbf{p}}})|_{x_2}.$$ 

It follows that $\text{gin}(J)$ is a Borel–fixed monomial ideal in $\mathbb{K}[x_0, x_1]$ and has invariants $\lambda_i = \lambda_i(\hat{\mathbf{p}}, \hat{\mathbf{q}})$, $i = 0, 1, \ldots, s-1$; thus $\text{gin}(J) = (x_0^{\hat{p}}, x_0^{s-1} x_1^{\hat{p}-1}, \ldots, x_0^{j+1} x_1^{\hat{p}+1}, x_0^j x_1^\lambda, \ldots, x_1^\lambda).$
where the degrees of the generators are in increasing order; especially, since 
\(\deg(x_0^{j+1} x_1^{\lambda_j+1}) = j+1+\lambda_j+1 = \delta - 1\) and \(\deg(x_0^{j} x_1^{\lambda_j}) = j+\lambda_j > j+\lambda_j+1+2 = \delta,\)
then \(\text{gin}(J)\) has a gap in degree \(\delta.\)

**Step 2** There exists a homogeneous polynomial \(F = F_{h,l,m}\) of degree \(j+1\) such that
\[
J_{\leq \delta} \subseteq (F).
\]
Since \(\text{gin}(J)\) has a gap in degree \(\delta,\) then, by corollary 1.5
\[
\text{gin}(J_{\leq \delta}) = \text{gin}(J)_{\leq \delta} = (x_0^{\lambda_0}, x_0^{s-1} x_1^{\lambda_1-1}, \ldots, x_0^{j+1} x_1^{\lambda_j+1}) \subseteq (x_0^{j+1}).
\]
This relation shows that the Hilbert function \(\text{gin}(J_{\leq \delta}),\) \(P_{\text{gin}(J_{\leq \delta})}(t) = j+1,\) for all \(t \gg 0;\) on the other hand, \(J_{\leq \delta}\) and \(\text{gin}(J_{\leq \delta})\) have the same Hilbert function, hence \(P_{J_{\leq \delta}}(t) = j+1\) definitively, so the variety \(V(J_{\leq \delta})\) is a group of \(j+1\) points in \(\mathbb{P}^1;\) it follows that there is a homogeneous polynomial \(F = F_{h,l,m}\) of degree \(j+1\) such that \(J_{\leq \delta} \subseteq (F).\) Note that, as \(j+1\) is the maximal degree of such a polynomial, \(F\) is also the greatest common divisor (gcd) of the generators of \(J_{\leq \delta}.\)

**Step 3** \(F_{h,l,m}\) form an algebraic family.

It means that there exists a polynomial \(F(x, \xi, \eta, \theta),\) separately homogeneous in all groups of variables, of degree \(j+1\) with respect to \(x,\) such that, for general \(a, b, c \in \mathbb{K}^5, F(x, a, b, c)\) restricted to \(h = a \cdot x, l = b \cdot x, m = c \cdot x,\) is \(F_{h,l,m};\)
here \(a \cdot x := \sum_{i=0}^4 a_i x_i,\) and similar meaning have \(b \cdot x\) and \(c \cdot x.\) This can be seen using the following argument.

Let \(I = (\Phi_1, \ldots, \Phi_r), \Phi_i \in \mathbb{K}[x_0, \ldots, x_4],\) set \(H = \xi \cdot x, L = \eta \cdot x, M = \theta \cdot x,\) and define \(I|_H = (\Phi_1|_H, \ldots, \Phi_r|_H),\) where \(\Phi_i|_H = \Phi_i(x_0, \ldots, x_3, -\frac{x_0}{\xi}, \frac{x_1}{\xi}, \ldots, \frac{x_3}{\xi}) \in \mathbb{K}(\xi)[x_0, \ldots, x_4]\) (of course, the substitution \(x_4 = -\frac{x_0}{\xi} x_0 \ldots - \frac{x_3}{\xi} x_3\) comes from \(\xi \cdot x = 0);\) with a similar substitution define the image of \(L, M\) in \(\mathbb{K}(\xi, \eta)[x_0, \ldots, x_3],\) image that we still denote by \(L.\) Since the generators of the ideal quotient \((I_H : L^3)\) are algorithmically obtainable from the \(\Phi_i’s\) via rational operations in \(\mathbb{K}(\xi, \eta)[x_0, \ldots, x_3]\) (cf. [CLO] theorem 4.4.1 and remark afterwards), we can therefore compute \((I_H : L^3) = (\Psi_1, \ldots, \Psi_t),\) where \(\Psi_i \in \mathbb{K}(\xi, \eta)[x_0, \ldots, x_3].\) Iterating the argument, we have \(J := ((I_H : L^3)|_L : M^5)|_M = (\Omega_1, \ldots, \Omega_u),\) with \(\Omega_i \in \mathbb{K}(\xi, \eta, \theta)[x_0, x_1] \text{ and } J_{\leq \delta} = (\Omega_1, \ldots, \Omega_u), v \leq u,\) taking only the generators of degree \(\leq \delta.\) Now we compute gcd\((\Omega_1, \ldots, \Omega_u)\) in \(\mathbb{K}(\xi, \eta, \theta)[x_0, x_1]\) using the euclidean algorithm; clear the denominators of gcd\((\Omega_1, \ldots, \Omega_u)\) and the result is \(F(x, \xi, \eta, \theta).\) Summing up, \(F(x, \xi, \eta, \theta)\) is obtained from the \(\Phi_i’s\) via an algorithm \(\mathcal{A}\) that uses only rational operations; specializing \(\xi \sim a, \eta \sim b, \theta \sim c,\) where \(a, b, c\) are such that none of the denominators that appear in the algorithm \(\mathcal{A}\) vanishes (hence \(a, b, c\) are general in \(\mathbb{K}^5),\) we get as a result \(F(x, a, b, c),\) which, when restricted to \(h, l, m,\) is the gcd of the generators of \(J_{\leq \delta},\) hence
Let $G \in I$ be a polynomial of minimal degree $s$, then $G\mid_{(h,l,m)} \in J_{\leq \delta}$, because $s \leq \delta$, hence $F_{h,l,m}$ divides $G\mid_{(h,l,m)}$, because $J_{\leq \delta} \subseteq (F_{h,l,m})$, and furthermore it is a proper factor, for $\deg F_{h,l,m} = j + 1 < s = \deg G\mid_{(h,l,m)}$.

Note that we use here the hypothesis $s_{\Sigma} = s_{\Gamma}$; indeed, in general, $s(\bar{p}, \bar{q}) < \delta$, but, without the hypothesis $s_{\Sigma} = s_{\Gamma}$, it could happen that $s(0,0) > \delta$, so $G\mid_{(h,l,m)} \notin J_{\leq \delta}$.

**Step 4** The set

$$X := \{(P, h, l, m) \in \mathbb{P}^4 \times (\mathbb{P}^4)^3 | P \in V(G, h, l, m)\}$$

is an irreducible variety.

Consider the canonical projections of $X$ on $V(G)$ and $(\mathbb{P}^4)^3$,

$$\begin{align*}
X & \xrightarrow{\pi_1} V(G) \\
& \xrightarrow{\pi_2} (\mathbb{P}^4)^3
\end{align*}$$

The fiber of $\pi_1$ over a point $P \in V(G)$ consist of all $(h, l, m) \in (\mathbb{P}^4)^3$ such that $P \in V(h, l, m)$, i.e. $\pi_1^{-1}(P) = \Pi_P \times \Pi_P \times \Pi_P$, where $\Pi_P := \{h \in \mathbb{P}^4 | P \in h\}$ is a hyperplane of $\mathbb{P}^4$, hence all fibers of $\pi_1$ are irreducible and of the same dimension; $V(G)$ is irreducible too, because $G$ is a polynomial of minimal degree vanishing on an irreducible variety $\Sigma$; thus $X$ is irreducible by theorem 1.8. Note that $\dim X = \dim(\mathbb{P}^4)^3 = 12$, because the generic fiber of $\pi_2$ is zero dimensional.

To conclude the proof of the theorem, let $Y := \{(P, h, l, m) \in \mathbb{P}^4 \times (\mathbb{P}^4)^3 | P \in V(h, l, m), F(P, h, l, m) = 0\}$. Considering again the canonical projections

$$\begin{align*}
X \cap Y & \xrightarrow{\pi_1} \mathbb{P}^4 \\
& \xrightarrow{\pi_2} (\mathbb{P}^4)^3
\end{align*}$$

we notice that the general fiber $\pi_2^{-1}(h, l, m)$ is the zero set of $F(x, a, b, c) = F_{h,l,m}$, i.e. $j + 1$ points on the line $V(h, l, m)$ that belong to $V(G)$ too; thus $\dim X \cap Y = \dim(\mathbb{P}^4)^3 = 12$, and furthermore, since those points are only $j + 1$ out of the $s$ points of $V(G, h, l, m)$, $X \cap Y$ is a proper subvariety of $X$ of the same dimension; this of course contradicts the irreducibility of $X$.

This contradiction proves the theorem. $\blacksquare$
Remark 2.6 To get a contradiction in the previous proof, one only needs that $\delta \geq s(0,0)$. This observation makes possible to push the statement of theorem 2.5 a little further, as follows. (For sake of simplicity, we only consider the case of a curve $C \subseteq \mathbb{P}^3$.)

(i) Even dropping the hypothesis $s_C = s_\Gamma$, the invariants at level zero, i.e. the invariants $\lambda_0(p)$, are connected, because, at this level $\delta \geq s(0)$; the invariants at level one, $\lambda_1(p)$, are connected too, because $s(1) \geq s(0) - 1$ by Borel-fixedness, so $\delta \geq s(1) + 1 \geq s(0)$; also, the invariants are connected at level $p \gg 0$, as already noticed in the introduction.

(ii) The monomial invariants of $C$ are likewise the monomial invariants of $I_C|_h$: this follows from the fact that $\text{gin}(I_C)$ has no generator containing $x_3$ (proposition 1.6). Since $I_\Gamma = (I_C|_h)^{sat}$, for all $k \geq 0$, $I_C|_h \subseteq (I_C|_h : m^k) \subseteq I_\Gamma$, where $m = (x_0, x_1, x_2)$ is the irrelevant maximal ideal of $\mathbb{K}[x_0, x_1, x_2]$, and the hypothesis $s_C = s_\Gamma$ implies that a similar condition is satisfied also by $(I_C|_h : m^k)$, and for such an ideal the proof of theorem 2.5 carries through, so we can conclude that the monomial invariants of $(I_C|_h : m^k)$ are connected as well.

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