LOCAL INTEGRAL MANIFOLDS FOR NONAUTONOMOUS AND ILL-POSED EQUATIONS WITH SECTORIALLY DICHOTOMOUS OPERATOR

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Abstract. We show the existence and $C^{k,\gamma}$ smoothness of local integral manifolds at an equilibrium point for nonautonomous and ill-posed equations with sectorially dichotomous operator, provided that the nonlinearities are $C^{k,\gamma}$ smooth with respect to the state variable. $C^{k,\gamma}$ local unstable integral manifold follows from $C^{k,\gamma}$ local stable integral manifold by reversing time variable directly. As an application, an elliptic PDE in infinite cylindrical domain is discussed.

1. Introduction. In this paper, we consider local integral manifolds for the nonautonomous differential equation

$$\frac{dz(t)}{dt} = Sz(t) + H(t, z(t)), \quad t \in \mathbb{R}, z(t) \in Z,$$

where $Z$ is a Banach space, $S : D(S) \subset Z \to Z$ is a linear, densely defined and hyperbolic bisectorial operator, and $H(t, \cdot) : \mathcal{O} \to Z$ is a smooth map with $H(t, 0) = 0$ and $DzH(t, 0) = 0$. Here, $\mathcal{O} \subset Z_0 \triangleq Z$, $Z_\alpha \triangleq D(S^\alpha)$ endows with the graph norm which satisfies the relation of continuous embedding $D(S) \hookrightarrow Z_\alpha \hookrightarrow Z$ for $\alpha \in (0, 1)$, and $S^\alpha$ is the $\alpha$-fractional power of $S$. In contrast to Henry’s work [9] which focuses on sectorial operator, the linear operator $S$ in (1) has infinite spectrum on both sides of the imaginary axis, the Hille-Yosida theorem follows that it is not the infinitesimal generator of a certain analytic (semi)group on $Z$ and so that the equation (1) does not generate a semi(flow) on $Z$ for a given initial value. So we say that (1) is ill-posed on $Z$.

Some elliptic PDEs in infinite cylindrical domain can be treated as an ill-posed dynamical system (1) in the unbounded spatial variable. Over the past several decades, invariant manifold theory has been a powerful tool in analyzing the asymptotic behavior of bounded solutions of elliptic equations in the vicinity of an equilibrium, involving autonomous and nonautonomous cases. When $S$ is not hyperbolic, (finite and infinite dimensional) center manifold theory has successfully
applied to this kinds of elliptic problem since Kirchgässner [10] pioneered the "spatial dynamics" method-viewing the elliptic problem in infinite cylindrical domain as an ill-posed dynamical system in the unbounded spatial variable-in order to find nontrivial bounded small solutions for elliptic PDEs in infinite cylindrical domain (see Mielke [15–19] and de la Llave [13]). It has been shown that the ill-posed Cauchy problem may not be solvable for arbitrary initial condition, but some solutions can be defined with the initial values in a center manifold if $S$ has the center subspace. When $S$ is hyperbolic, ElBialy in [6] showed the existence of local Lipschitzian stable and unstable manifolds for the following autonomous ill-posed system

\[
\begin{align*}
\begin{cases}
\frac{dx(t)}{dt} = Lx(t) + f(z(t)), \\
\frac{dy(t)}{dt} = Ry(t) + g(z(t)),
\end{cases}
\end{align*}
\]  

(2)

based on the existence of dichotomous mild solutions under the dichotomous initial conditions introduced by Latushkin and Layton [14]. Here $X$, $Y$ and $Z$ are all Banach spaces, $(L, R)$ is the infinitesimal generator of a hyperbolic and strongly continuous bisemigroup $(\{e^{Lt}\}_{t \geq 0}, \{e^{-Rt}\}_{t \geq 0})$, and the nonlinearity $(f, g)$ are locally Lipschitz continuous in $z$. Especially, ElBialy introduced the so-called dichotomous flow which could recover the symmetry between the past and the future in evolution time such that just reversing time is need rather than providing two separate proofs for stable and unstable manifolds.

Using the sufficient condition for a densely defined and hyperbolic bisectorial operator being sectorially dichotomous given by Deng and Xiao [4, Theorem 2.6], and under the dichotomous initial conditions

\[
\{ (\tilde{x}; t_1, t_2) | t_1 < t_2, \tilde{x} = (x_1, y_2) \in Z, x(t_1) = x_1 \in \mathcal{X}, y(t_2) = y_2 \in \mathcal{Y} \},
\]  

(3)

we can transform the ill-posed equation (1) into the coupled system:

\[
\begin{align*}
\begin{cases}
\frac{dx(t)}{dt} = S_+ x(t) + F(t, z(t)), \\
\frac{dy(t)}{dt} = S_- y(t) + G(t, z(t)),
\end{cases}
\end{align*}
\]  

(4)

where $S_+ := S|_{\mathcal{X}}$ and $-S_- := -S|_{\mathcal{Y}}$ are densely defined and sectorial operators on Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively, $F(t, z(t)) = P_+ H(t, z(t))$, $G(t, z(t)) = P_- H(t, z(t))$ and $P_+$ $(P_-)$ is the projection of $\mathcal{Z}$ onto $\mathcal{X}$ $(\mathcal{Y})$ along $\mathcal{Y}$ $(\mathcal{X})$. We are interested in the asymptotic behavior of the dichotomous solution in a small neighborhood of the equilibrium $z = 0$, and present local stable and unstable integral manifold theorem for the ill-posed system (4). The construction of local stable integral manifold is carried out by the Liapunov-Perron method, and the $C^{k, \gamma}$ smoothness proof is built on the Lemma 2.1 in Chow and Lu [3] and the Henry’s lemma [9, Lemma 6.1.6]. Furthermore, the results about local unstable integral manifold can follow from those about local stable integral manifold by reversing time. Note that the dichotomous solution we study always means dichotomous classical solution. Hence, in contrast to ElBialy’s results [6], the solutions we obtained on the manifolds have higher regularity under the same Lipschitz conditions attached to the nonlinear terms. Besides, Deng and Xiao [4] discussed the existence, uniqueness, continuous dependence on the dichotomous initial value, regularity and
\(Z_\alpha\)-estimate of dichotomous solutions for (4), provided that \(F(t,z)\) and \(G(t,z)\) are locally H"{o}lder continuous in \(t\) and locally Lipschitz continuous in \(z\).

It is worth pointing out that, the elliptic problem in infinite cylindrical domain formulated by the abstract form (1) could be transformed to a first order system consisting of a pair of semilinear coupled parabolic equations, and it is well known that the investigation of parabolic problem base on the theory of analytic semigroup has a great importance, we refer to the monographs [9, 12]. So the assumption of the investigation of parabolic problem base on the theory of analytic semigroup (1) in \(Z_\alpha\) which between \(Z\) and \(D(S)\).

As an application of our results, we study the following elliptic equation in infinite cylindrical domain \(\mathbb{R} \times \Omega\)

\[
\begin{align*}
  u_{xx} + \Delta_y u + f(x,y,u,u_x,\nabla_y u) &= 0, \quad (x,y,u) \in \mathbb{R} \times \Omega \times \mathbb{R}^m, \\
  u(x,y) &= 0, \quad x \in \mathbb{R}, \quad y \in \partial \Omega,
\end{align*}
\]

where \(\Omega\) is an open and bounded subset of \(\mathbb{R}^n\) with smooth boundary, \(\nabla_y\) is the gradient in the \(y\)-variable and \(\Delta_y\) is the Laplace operator in the \(y\)-variable. We shows that the existence and asymptotic behavior of solutions for system (5) under some boundary value conditions.

This paper is organized as follows. In Section 2, we first recall some notations and definitions. In Section 3, some hypothesis and lemmas are given. In Section 4, we devote to the \(C^{k,\gamma}\) local stable and unstable integral manifolds. In Section 5, an elliptic equation in infinite cylindrical domain is illustrated to the results.

2. Notations and definitions. Let \(S\) be a linear operator with the domain \(D(S)\) and range \(R(S)\). We denote the resolvent set and the spectrum of \(S\) in the complex plane by \(\rho(S)\) and \(\sigma(S)\), respectively, denote by \(R(\lambda,S) \triangleq (\lambda I - S)^{-1}\) its resolvent operator for \(\lambda \in \rho(S)\), and denote by \(\Re \lambda\) the real part of \(\lambda \in \sigma(S)\). For any constant \(\tilde{\sigma} \in (0, \pi)\), we define the sector regions \(\Sigma_{\tilde{\sigma}} \triangleq \{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \tilde{\sigma} \}\) and \(-\Sigma_{\tilde{\sigma}} \triangleq \{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg (-\lambda)| \leq \tilde{\sigma} \} \).

Let \(X\) and \(Y\) be Banach spaces and let \(U\) be an open subset of \(X\), we denote \(\mathcal{L}(X,Y)\) by the space of all bounded linear operators from \(X\) to \(Y\). Firstly we use the following notations to represent some function spaces:

(i) For any integer \(k \geq 0\), let

\[
C^k(U,Y) = \{ f | f : U \rightarrow Y \text{ is } k\text{-times continuously differentiable,} \}
\]

\[
\sup_{x \in U} \| D^j_x f(x) \|_{\mathcal{L}(X^j,Y)} < \infty \text{ for all } 0 \leq j \leq k, \]

where \(D^j_x\) denotes the \(j\)th differentiation operator with respect to the variable \(x\). \(C^k(U,Y)\) is a Banach space endowed with the norm

\[
\| f \|_k = \max_{0 \leq j \leq k} \sup_{x \in U} \| D^j_x f(x) \|_{\mathcal{L}(X^j,Y)},
\]

where \(X^j\) denotes the space \(\underbrace{X \times X \times \cdots \times X}_j\).

(ii) Let \(k \geq 0\) be an integer, \(\gamma \in (0,1]\), let

\[
C^{k,\gamma}(U,Y) = \{ f | f \in C^k(U,Y), \text{ and } H_\gamma(D^k_x f) < \infty \}.
\]

\(C^{k,\gamma}(U,Y)\) is a Banach space equipped with the norm

\[
\| f \|_{k,\gamma} = \max \{ \| f \|_k, H_\gamma(D^k_x f) \},
\]
where $H_\gamma(D^{k}_xf) = \sup_{x, \tilde{x} \in U, \tilde{x} \neq x} \|D^{k}_xf(x) - D^{k}_f(\tilde{x})\|_{C^{0}(\mathbb{R})} / \|x - \tilde{x}\|$.

(iii) Let $J = (-\infty, t_0]$ or $[t_0, \infty)$ for $t_0 \in \mathbb{R}$. For a constant $\beta \in \mathbb{R}$, let

$$C_\beta(J, Y) = \{ y \in C^0(J, Y) : \|y\|_{C_\beta} = \sup_{t \in J} e^{-\beta t} \|y(t)\|_Y < \infty \},$$

which is a Banach space equipped with the norm $\| \cdot \|_{C_\beta}$.

Note that we also define the space $C^{k, \gamma}(U, Y)$ for $\gamma = 0$, and specify $C^{k,0}(U, Y) := C^k(U, Y)$ and $\|f\|_k := \|f\|_k$. Next, let me recall the definitions of (bi)sectorial and sectorially dichotomous operator, see [23, 24].

**Definition 2.1.** Let $S : \mathcal{D}(S) \subset Z \to Z$ be a closed linear operator.

1. If there exists $\tilde{\sigma} \in (0, \pi/2)$ and $M > 0$ such that $\sigma(S) \subset \mathbb{C} \setminus \Sigma_{\pi/2 + \tilde{\sigma}}$, and

$$\|R(\lambda, S)\|_{\mathcal{L}(Z)} \leq \frac{M}{|\lambda|} \text{ for } \lambda \in \Sigma_{\pi/2 + \tilde{\sigma}},$$

then $S$ is called **sectorial**.

2. If there exists $\tilde{\sigma} \in (0, \pi/2)$ and $M > 0$ such that $\sigma(S) \subset \Sigma_{\tilde{\sigma}} \cup \{-\Sigma_{\tilde{\sigma}}\} \cup \{0\}$, and

$$\|R(\lambda, S)\|_{\mathcal{L}(Z)} \leq \frac{M}{|\lambda|} \text{ for } \lambda \in \mathbb{C} \setminus \left( \Sigma_{\tilde{\sigma}} \cup \{-\Sigma_{\tilde{\sigma}}\} \cup \{0\} \right),$$

then $S$ is called **bisectional**. Furthermore, $S$ is called **hyperbolic bisectional** if $0 \notin \rho(S)$.

3. For two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ satisfying $Z = \mathcal{X} \oplus \mathcal{Y}$, set $S_+ := S|_\mathcal{X}$ and $S_- := S|_\mathcal{Y}$, if the following four conditions hold:

   (i) $\mathcal{X} \subset \rho(S)$;

   (ii) $\mathcal{X}$ and $\mathcal{Y}$ are $S$-invariant, i.e., $S(\mathcal{D}(S) \cap \mathcal{X}) \subset \mathcal{X}$ and $S(\mathcal{D}(S) \cap \mathcal{Y}) \subset \mathcal{Y}$;

   (iii) $\sigma(S_+) \subset \{ \lambda \in \mathbb{C} : \Re \lambda < 0 \}$, $\sigma(S_-) \subset \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \}$;

   (iv) $S_+$ and $-S_-$ are densely defined and sectorial operators on $\mathcal{X}$ and $\mathcal{Y}$ respectively,

then $S$ is called **sectorially dichotomous** with respect to the decomposition $Z = \mathcal{X} \oplus \mathcal{Y}$.

For the sectorially dichotomous $S$ and a constant $\alpha \in (0, 1)$, we can define $\alpha$-fractional power of $-S_+$, $S_-$ and $S$ and obtain the relationship among them, refer to [4, section 3]. We denote $(-S_+)^\alpha$ and $(S_-)^\alpha$ by the $\alpha$-fractional powers of $-S_+$ and $S_-$, respectively, and denote

$$\mathcal{X}_\alpha := D((-S_+)^\alpha), \quad \mathcal{Y}_\alpha := D((S_-)^\alpha), \quad \mathcal{Z}_\alpha := D(S^\alpha)$$

by Banach spaces with norm $\|x\|_{\mathcal{X}_\alpha} = \|(-S_+)^\alpha x\|_\mathcal{X}$, $\|y\|_{\mathcal{Y}_\alpha} = \|(S_-)^\alpha y\|_\mathcal{Y}$ and $\|z\|_{\mathcal{Z}_\alpha} = \|S^\alpha z\|_Z$ respectively. Moreover, $\mathcal{Z}_\alpha = \mathcal{X}_\alpha \oplus \mathcal{Y}_\alpha$ and $\mathcal{D}(S) \to \mathcal{Z}_\alpha \to \mathcal{Z}$. Note that $\mathcal{Z}_0 = \mathcal{Z}$, $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{Y}_0 = \mathcal{Y}$. For all $r > 0$, we denote by $B^\alpha_r(x_0)$, $B^\alpha_r(y_0)$, $B^\alpha_r(z_0)$ the balls of radius $r$ around the origin in $\mathcal{X}_\alpha$, $\mathcal{Y}_\alpha$, $\mathcal{Z}_\alpha$ respectively, and satisfy that $B^\alpha_r \triangleq B^\alpha_r(x_0) \oplus B^\alpha_r(y_0)$. We define $\| \cdot \|_Z$ by $\|z\|_Z = \|x + y\| = \max\{\|x\|_\mathcal{X}, \|y\|_\mathcal{Y}\}$ for $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$.

Then we give the definitions of some dichotomous solutions and invariant manifolds for the ill-posed system (4).

**Definition 2.2.** A function $z(t) : [t_1, t_2] \to \mathcal{Z}_\alpha(\alpha \in [0, 1))$ is called **dichotomous solution** of the ill-posed system (4) for $-\infty < t_1 < t_2 < \infty$, if it satisfies the following three conditions:

1. $z(t) \in C^0([t_1, t_2], \mathcal{Z}_\alpha) \cap C^1((t_1, t_2), \mathcal{Z}_\alpha)$.
(ii) $x(t) \in \mathcal{D}(S_+), y(t) \in \mathcal{D}(S_-)$ and $z(t) \in \mathcal{O}$ for $t \in (t_1, t_2)$;
(iii) $x(t)$ and $y(t)$ satisfy the first and second equation in (4), respectively, for all $t \in (t_1, t_2)$.
Moreover, $x(t_1) = x_1, y(t_2) = y_2$.

From Definition 2.2, the dichotomous solution $z(t)$ of system (4) satisfies the following dichotomous system of integral equations:

\[
\begin{align*}
  x(t) &= T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s)F(s, z(s))\,ds, \\
  y(t) &= T_-(t-t_2)y_2 - \int_{t}^{t_2} T_-(t-s)G(s, z(s))\,ds.
\end{align*}
\]

If the limits of the integrals in (8) exist, we obtain a dichotomous solution on $I = \mathbb{R}$. In order to distinguishing some asymptotic behaviors of dichotomous solution clearly, two infinitely long dichotomous solutions deserve to be introduced similar to ElBily [6].

**Definition 2.3.** (1) We say that $z \in C^0([t_1, \infty), \mathcal{Z}_\alpha)$ is an infinitely long forward dichotomous solution if for all $t_1 \leq \tilde{t}_1 < t_2 < \infty$, the restriction $z|_{[\tilde{t}_1, t_2]} \in C^0([\tilde{t}_1, t_2], \mathcal{Z}_\alpha)$ is a dichotomous solution, where $\alpha \in [0, 1)$. If $z(t_1) = \zeta = (x_1, y(t_1); x_1)$, we write the infinitely long forward dichotomous solution $z$ as $z(\cdot; t_1, \zeta)$.

(2) We say that $z \in C^0((-\infty, t_2], \mathcal{Z}_\alpha)$ is an infinitely long backward dichotomous solution if for all $-\infty < t_1 < \tilde{t}_2 \leq t_2$, the restriction $z|_{[t_1, \tilde{t}_2]} \in C^0([t_1, \tilde{t}_2], \mathcal{Z}_\alpha)$ is a dichotomous solution, where $\alpha \in [0, 1)$. If $z(t_2) = \zeta = (x(t_2; y_2), y_2)$, we write the infinitely long backward dichotomous solution $z$ as $z(\cdot; t_2, \zeta)$.

Given a neighborhood $\hat{U} \subset \mathcal{Z}_\alpha$ with size $r$ of 0, we define local stable set $W^s_{loc}(0, r) \subset \mathbb{R} \times \hat{U}$ and the local unstable set $W^u_{loc}(0, r) \subset \mathbb{R} \times \hat{U}$ of the equilibrium 0 as

\[
W^s_{loc}(0, r) = \{(t_1, \zeta) \in \mathbb{R} \times \hat{U} : \exists \ a \ z(\cdot; t_1, \zeta) \in C^0([t_1, \infty), \hat{U}) \text{ and } \lim_{t \to \infty} z(t; t_1, \zeta) = 0\},
\]

\[
W^u_{loc}(0, r) = \{(t_2, \zeta) \in \mathbb{R} \times \hat{U} : \exists \ a \ z(\cdot; t_2, \zeta) \in C^0((-\infty, t_2], \hat{U}) \text{ and } \lim_{t \to -\infty} z(t; t_2, \zeta) = 0\}.
\]

It is trivial to check that $W^s_{loc}(0, r)$ and $W^u_{loc}(0, r)$ are both invariant to $\mathbb{R} \times \hat{U}$, and the purpose of this paper is to show that they are indeed integral manifolds, before that, we give the definitions of local stable and unstable integral manifolds.

**Definition 2.4.** Let $\alpha \in [0, 1)$, and let $\mathcal{U}_1 \subset \mathcal{X}_\alpha$ and $\mathcal{U}_2 \subset \mathcal{Y}_\alpha$ be neighbourhoods with size $r$ of the equilibrium 0. For $k \in \mathbb{N} \cup \{0\}$ and $\gamma \in [0, 1]$, (1) Let $h^s : \mathbb{R} \times \mathcal{U}_1 \to \mathcal{U}_2$ be $C^{k, \gamma}$ with $h^s(t, 0) = 0$, in addition, $D_x h^s(t, 0) = 0$ when $k \geq 1$, such that $W^s_{loc}(0, r) = \{(t, x, y) : y = h^s(t, x), t \in \mathbb{R}, x \in \mathcal{U}_1\}$. Then $W^s_{loc}(0, r)$ is called a $C^{k, \gamma}$ local stable integral manifold if $W^s_{loc}(0, r)$ is invariant under the infinitely long forward dichotomous solution of (4), i.e., $(t_1, x_1, h^s(t_1, x_1)) \in W^s_{loc}(0, r)$, then $z(t; t_1, \zeta) \in W^s_{loc}(0, r)$ and $\zeta = x_1 + h^s(t_1, x_1)$.
(2) Let $h^u : \mathbb{R} \times \mathbb{U}_2 \to \mathbb{U}_1$ be $C^{k,\gamma}$ with $h^u(t, 0) = 0$, in addition, $D_y h^u(t, 0) = 0$ when $k \geq 1$, such that $W^u_{\text{loc}}(0, r) = \{(t, x, y) : x = h^u(t, y), t \in \mathbb{R}, y \in \mathbb{U}_2\}$. Then $W^u_{\text{loc}}(0, r)$ is called a $C^{k,\gamma}$ local unstable integral manifold if $W^u_{\text{loc}}(0, r)$ is invariant under the infinitely long backward dichotomous solution of (4), i.e., $(t_2, h^u(t_2, y_2), y_2) \in W^u_{\text{loc}}(0, r)$, then $z(t; t_2, \zeta) \in W^u_{\text{loc}}(0, r)$ and $\zeta = h^u(t_2, y_2) + y_2$.

3. Lemmas and hypotheses. First we introduce two lemmas by Chow and Lu [3, Lemma 2.1] and Henry [9, page 151] respectively, which can be used to study $C^{k,\gamma}$ smoothness of invariant manifolds.

**Lemma 3.1** (Lemma 2.1, [3]). Let $X$, $Y$ be Banach spaces and $U$ be an open subset of $X$. Assume that $f : U \to X$ is locally Lipschitz continuous. Then $f$ is continuously differentiable if and only if for every $x_0 \in U$

$$
\|f(x + \Delta) - f(x) - f(x_0 + \Delta) + f(x_0)\|_Y = o(\|\Delta\|_X)
$$
as $(x, \Delta) \to (x_0, 0)$.

**Lemma 3.2** (Lemma 6.1.6, [9]). Let $X$, $Y$ be Banach spaces and $U$ be an open subset of $X$. Then a closed bounded ball in $C^{k,\gamma}(U, Y)(0 < \gamma \leq 1, k \in \mathbb{N} \cup \{0\})$ is also a closed bounded subset in $C^0(U, Y)$.

More precisely, Lemma 3.2 says that, a sequence $\{u_n\} \subset C^{k,\gamma}(U, Y)$ and a map $u : U \to Y$ such that $\|u_n - u\|_{C^0(U, Y)} \to 0$ as $n \to \infty$, then $u \in C^{k,\gamma}(U, Y)$.

Then two lemmas for hyperbolic bisectorial operator being sectorially dichotomous are given, see [4].

**Lemma 3.3** (Lemma 2.3, [4]). Let $S : \mathcal{D}(S) \subset Z \to Z$ be a linear operator. If there exist two constants $h > 0$ and $M_0 > 0$ such that a vertical strip around $i\mathbb{R}$ denoted by

$$
\mathbb{I}_h \triangleq \{\lambda \in \mathbb{C} : |\Re \lambda| \leq h\} \subset \rho(S)
$$

and

$$
||\lambda R(\lambda, S)||_{L(Z)} \leq M_0 \quad \text{for} \quad \lambda \in \mathbb{I}_h,
$$

then $S$ is hyperbolic bisectorial.

Consider two linear subspaces $Z_+, Z_- \subset Z$ of $Z$ for hyperbolic bisectorial operator $S$ as follows:

$Z_+ = \{z \in Z : R(\lambda, S)z \text{ has a bounded analytic extension to } \Re \lambda \geq -h\}$,

$Z_- = \{z \in Z : R(\lambda, S)z \text{ has a bounded analytic extension to } \Re \lambda \leq h\}$,

(11) and two operators $A_+$ and $A_-$ of the forms

$$
A_+ = \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} \frac{1}{\lambda} R(\lambda, S) d\lambda, \quad A_- = \frac{-1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda} R(\lambda, S) d\lambda.
$$

(12)

**Lemma 3.4** (Theorem 2.6, [4]). Let $S : \mathcal{D}(S) \subset Z \to Z$ be a linear, densely defined operator and satisfy (9) and (10). $Z_{\pm}$ are defined by (11) and $A_{\pm}$ are defined by (12). Then

(i) $P_{\pm} := A_{\pm} S$ are closed and complementary operators on $\mathcal{D}(P_{\pm})$, where

$$
P_+ z = \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} \frac{1}{\lambda} R(\lambda, S) S z d\lambda, \quad z \in \mathcal{D}(S),
$$

$$
P_- z = -\frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda} R(\lambda, S) S z d\lambda, \quad z \in \mathcal{D}(S).
$$

(13)
(ii) if $P_+$ (or $P_-$) is bounded on $\mathcal{D}(S)$, then $S$ is sectorially dichotomous. In this case, $P_+$ (or $P_-$) is the unique bounded extension of $A_+S (\text{or } A_-S)$ from $\mathcal{D}(S)$ to $\mathcal{Z}$, and $P_+$ (or $P_-$) is the bounded projection of $\mathcal{Z}$ onto $\mathcal{Z}_+ (\mathcal{Z}_-, \text{ resp.})$ along $\mathcal{Z}_- (\mathcal{Z}_+, \text{ resp.})$.

Based on the Lemma 3.4, there exist a pair of bounded, closed and complementary projections $P_+$ and $P_-$ on $\mathcal{Z}$ and two closed subspaces $\mathcal{X}, \mathcal{Y}$ of $\mathcal{Z}$ such that $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$ holds, where $P_+ : \mathcal{Z} \to \mathcal{X}$ and $P_- : \mathcal{Z} \to \mathcal{Y}$. Actually, $\mathcal{X}$ and $\mathcal{Y}$ have the forms of $\mathcal{Z}_+$ and $\mathcal{Z}_-$ respectively. Moreover, $S$ decomposes with respect to $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$. Namely, $S$ admits the block matrix representation $S = \begin{pmatrix} s_+ & 0 \\ 0 & s_- \end{pmatrix}$, $\mathcal{D}(S) = \mathcal{D}(S_+) \oplus \mathcal{D}(S_-)$, and $\mathcal{X}$ and $\mathcal{Y}$ are both $S$-invariant subspaces, where $S_+$ and $S_-$ are defined respectively by

$$S_+ z = S z, \forall z \in \mathcal{D}(S_+) := \{ z \in \mathcal{D}(S) \cap \mathcal{X} : S z \in \mathcal{X} \},$$

and

$$S_- z = S z, \forall z \in \mathcal{D}(S_-) := \{ z \in \mathcal{D}(S) \cap \mathcal{Y} : S z \in \mathcal{Y} \}.$$ 

In particular,

$$\sigma(S_-) \subset \{ \lambda \in \mathbb{C} : \Re \lambda > h \}, \quad \sigma(S_+) \subset \{ \lambda \in \mathbb{C} : \Re \lambda < -h \},$$

and $\sigma(S) = \sigma(S_+) \cup \sigma(S_-)$.

Furthermore, some results of $S_+$ and $-S_-$ and the semigroups generated by them as follows.

**Lemma 3.5** (Sections 2.5 and 2.6, [20]). $S_+$ and $-S_-$ generate uniformly exponentially stable, strongly continuous, analytic semigroups $\{T_+(t)\}_{t \geq 0}$ and $\{T_-(t)\}_{t \geq 0}$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively, where $T_+(t) := e^{S_+ t}$ and $T_-(t) := e^{-S_- t}$. Furthermore, there are real numbers $\beta_+ < 0 < \beta_-$ and some positive constants $M_0^+, M_0^-, M_\alpha^+, M_\alpha^-$ such that

$$\|T_+(t)\|_{\mathcal{L}(\mathcal{X})} \leq M_0^+ e^{\beta_+ t}, \quad \|T_-(t)\|_{\mathcal{L}(\mathcal{Y})} \leq M_0^- e^{-\beta_- t} \quad (14)$$

for $t \geq 0$, and

$$\|T_+(t)\|_{\mathcal{L}(\mathcal{X}, \mathcal{X}_u)} \leq M_\alpha^+ t^{-\alpha} e^{\beta_+ t}, \quad \|T_-(t)\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Y}_u)} \leq M_\alpha^- t^{-\alpha} e^{-\beta_- t} \quad (15)$$

for $t > 0$.

In particular, by the spectral mapping theorem [7, Corollary 4.3.12], $T_+(t)$ and $T_-(t)$ are contraction semigroups, i.e., $\|T_+(t)\|_{\mathcal{L}(\mathcal{X})} \leq 1$ and $\|T_-(t)\|_{\mathcal{L}(\mathcal{Y})} \leq 1$ for $t \geq 0$. Throughout this paper, we set

$$\widetilde{M}_k^+ := \sup_{t > 0} \|t^k S_+^k T_+(t)\|_{\mathcal{L}(\mathcal{X})}, \quad \widetilde{M}_k^- := \sup_{t > 0} \|t^k S_-^k T_-(t)\|_{\mathcal{L}(\mathcal{Y})}, \quad (16)$$

and

$$\widetilde{M}_\alpha^+ := \sup_{t > 0} \|t^\alpha (-S_+)^\alpha T_+(t)\|_{\mathcal{L}(\mathcal{X})}, \quad \widetilde{M}_\alpha^- := \sup_{t > 0} \|t^\alpha S_-^\alpha T_-(t)\|_{\mathcal{L}(\mathcal{Y})}, \quad (17)$$

From [12, Proposition 2.1.1(iii)], [22, Theorem 1.12], we have that $\widetilde{M}_k^+, \widetilde{M}_\alpha^+ < +\infty$ for every $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in [0, 1]$.

To study the existence and $C^k, \gamma$ smoothness of local integral manifolds, we give further hypothesis for the nonlinear map $H$ of the system (1) provided that Lemma 3.4 holds.
Hypothesis 3.6. Let \( F(t, z(t)) =: P_+ H(t, z(t)), G(t, z(t)) =: P_- H(t, z(t)), \) \( F \) and \( G \) are uniformly Hölder continuous in \( t, F(t, 0) = G(t, 0) = 0. \) Let \( k \in \mathbb{N} \cup \{0\}, \) \( \gamma \in [0, 1], \) and \( L_{0,1}(r), \hat{L}_{0,1}(r), L_{k,\gamma}(r), \hat{L}_{k,\gamma}(r) \) are all positive constants depending on \( r. \)

(i) For \( k = 0 \) and \( \gamma = 1, \) \( F(t, \cdot) \in C^{0,1}(B^{2\alpha}_r, \mathcal{X}) \) and \( G(t, \cdot) \in C^{0,1}(B^{2\alpha}_r, \mathcal{Y}) \) satisfy \( \|F\|_{0,1} \leq L_{0,1}(r) \) and \( \|G\|_{0,1} \leq \hat{L}_{0,1}(r), \) respectively, and \( L_{0,1}(r)\hat{L}_{0,1}(r) \to 0 \) as \( r \to 0. \)

(ii) For \( k \geq 1 \) and \( \gamma \in [0, 1], \) \( F(t, \cdot) \in C^{k,\gamma}(B^{2\alpha}_r, \mathcal{X}) \) and \( G(t, \cdot) \in C^{k,\gamma}(B^{2\alpha}_r, \mathcal{Y}) \) satisfy \( \|F\|_{k,\gamma} \leq L_{k,\gamma}(r) \) and \( \|G\|_{k,\gamma} \leq \hat{L}_{k,\gamma}(r), \) respectively, and \( L_{k,\gamma}(r) \hat{L}_{k,\gamma}(r) \to 0 \) as \( r \to 0. \) Besides, \( D_z F(t, 0) = D_z G(t, 0) = 0. \)

4. Local stable and unstable integral manifolds. In this section, we study local stable and unstable integral manifolds of equation (1) with dichotomous initial condition (3) provided that Lemma 3.4 holds, i.e., \( S \) in (1) is a sectorially dichotomous operator, which is equivalent to considering system (4). The local stable and unstable integral manifolds theorem for system (4) are as follows.

Theorem 4.1. Assume that Hypothesis 3.6 is satisfied for the ill-posed system (4), and \( L_{k,\gamma}(r), \hat{L}_{k,\gamma}(r) \) are sufficiently small. Then

1. \( W_{\text{loc}}^s(0, r) \) is a unique \( C^{k,\gamma} \) local stable integral manifold of the ill-posed system (4). Moreover, the infinitely long forward dichotomous solutions on \( W_{\text{loc}}^s(0, r) \) take the form

\[
\begin{align*}
x(t) &= T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s)F(s,x(s),h^s(s,x(s))) \, ds, \\
y(t) &= h^s(t,x(t)) = - \int_{t}^{\infty} T_-(t-s)G(s,x(s),h^s(s,x(s))) \, ds.
\end{align*}
\]

for \( -\infty < t_1 \leq t < \infty. \)

2. \( W_{\text{loc}}^u(0, r) \) is a unique \( C^{k,\gamma} \) local unstable integral manifold of the ill-posed system (4). Moreover, the infinitely long backward dichotomous solutions on \( W_{\text{loc}}^u(0, r) \) take the form

\[
\begin{align*}
x(t) &= h^u(t,y(t)) = \int_{-\infty}^{t} T_+(t-s)F(s,h^u(s,y(s)),y(s)) \, ds, \\
y(t) &= T_-(t-t_2)y_2 - \int_{t}^{t_2} T_-(t-s)G(s,h^u(s,y(s)),y(s)) \, ds.
\end{align*}
\]

for \( -\infty < t \leq t_2 < \infty. \)

Set

\[
\theta_{k,\gamma}^+ := M_{\alpha}^+ L_{k,\gamma}(r) \int_{0}^{\infty} \mu^{-\alpha} e^{\zeta \mu} \, d\mu, \quad \theta_{k,\gamma}^- := M_{\alpha}^- \hat{L}_{k,\gamma}(r) \int_{0}^{\infty} \mu^{-\alpha} e^{\zeta \mu} \, d\mu,
\]

for \( \alpha \in (0, 1), \) \( \zeta < 0, \) \( k \in \{0\} \cup \mathbb{N} \) and \( \gamma \in [0, 1]. \) Note that \( \theta_{k,\gamma}^+, \theta_{k,\gamma}^- < +\infty \) since \( \int_{0}^{\infty} \mu^{-\alpha} e^{\zeta \mu} \, d\mu \) converges. Moreover, take \( \beta \in (-\eta, 0), \) where \( \eta = \min\{-\beta_+, \beta_-.\}. \) To prove the Theorem 4.1, we first give some lemmas in the following.

Lemma 4.2. Assume that the nonlinear term \( F(t, z(t)) \) and \( G(t, z(t)) \) of system (4) on arbitrary finite time interval \([t_1, t_2]\) satisfy the Hypothesis 3.6(i), and \( \max\{\theta_{0,1}^+, \theta_{0,1}^-\} < 1/2. \) Then there is a unique local dichotomous solution \( z(t) \in B^{2\alpha}_r \)
for any $\tilde{z} \in \overline{D(S)}^\alpha \cap B_{r_\alpha}^\infty$ and $\|\tilde{z}\|_\alpha \leq \frac{r}{2 \min\{M_0; M_0}\},$ where $\overline{D(S)}^\alpha$ is the closure of $D(S)$ in $Z_\alpha$.

Proof. Let

$$E \triangleq \left\{ u \in C^0([t_1, t_2]; B_r^Z) : u_1(t_1) = (-S_+)^\alpha x_1, u_2(t_2) = (S_-)^\alpha y_2 \right\},$$

where $u(t) = u_1(t) + u_2(t)$, $u_1(t) \in B_r^X$ and $u_2(t) \in B_r^Y$. Obviously, $E$ is a non-empty closed subset of $C^0([t_1, t_2]; B_r^Z)$ in the uniform $C^0$ norm, so it is a complete metric space with the induced metric $d_E(u, v) = \max_{t \in [t_1, t_2]} \|u(t) - v(t)\|_Z$.

We define a mapping $\Psi : E \to E$ as follows:

$$(\Psi u)(t) = T_+(t - t_1)S^\alpha x_1 + \int_{t_1}^t S^\alpha T_+(t - s)F(s, S^{-\alpha}u(s))ds$$

$$+ T_-(t - t_2)S^\alpha y_2 - \int_{t_1}^{t_2} S^\alpha T_-(t - s)G(s, S^{-\alpha}u(s))ds,$$

that is,

$$(\Psi u_1)(t) = T_+(t - t_1)(-S_+)^\alpha x_1 + \int_{t_1}^t (-S_+)^\alpha T_+(t - s)F(s, S^{-\alpha}u(s))ds,$$  \hspace{1cm} (21)

$$(\Psi u_2)(t) = T_-(t - t_2)(S_-)^\alpha y_2 - \int_{t_1}^{t_2} (S_-)^\alpha T_-(t - s)G(s, S^{-\alpha}u(s))ds.$$  \hspace{1cm} (22)

Now we check the well-definedness of mapping $\Psi$ firstly, so we consider the formulas (21) and (22).

Obviously,

$$\|\Psi u_1\|_X \leq M_0^+ e^{\beta_1(t-t_1)} \|x_1\|_\alpha^X + M_0^+ L_{0,1}(r) \int_{t_1}^t (t - s)^{-\alpha} e^{\beta_1(t-s)} \|u(s)\|_Z ds < r$$

provided that $\|x_1\|_\alpha^X \leq r/(2M_0^+)$ and $\beta_0^+ < 1/2.$ Similarly, $\|\Psi u_2\|_Y < r$ provided that $\|y_2\|_\alpha^Y \leq r/(2M_0^+)$ and $\beta_0^+ < 1/2.$ Thus, $\|\Psi u\|_Z < r,$ and then $(\Psi u)(t) \in B_r^Z$.

From [1, Corollary 3.7.21], and since $T_+(t)$ and $T_-(-t)$ are contraction semigroups, we know that

$$T_+(t - t_1)(-S_+)^\alpha x_1 \in C^0([t_1, t_2]; B_r^X) \cap C^\infty((t_1, t_2]; B_r^X),$$

$$T_-(t - t_2)(S_-)^\alpha y_2 \in C^0([t_1, t_2]; B_r^Y) \cap C^\infty((t_1, t_2]; B_r^Y).$$

On the other hand, we claim that there exists constants $\beta_1, \beta_2 \in (0, 1 - \alpha)$ such that

$$F_\alpha(t) := \int_{t_1}^t (-S_+)^\alpha T_+(t - s)F(s, S^{-\alpha}u(s))ds \in C^{0, \beta_1}([t_1, t_2]; B_r^X),$$  \hspace{1cm} (23)

and

$$G_\alpha(t) := -\int_{t_1}^{t_2} (S_-)^\alpha T_-(t - s)G(s, S^{-\alpha}u(s))ds \in C^{0, \beta_2}([t_1, t_2]; B_r^Y),$$  \hspace{1cm} (24)

which proves that

$$(\Psi u)(t) \in C^0([t_1, t_2]; B_r^Z) \cap C^{0, \beta_3}((t_1, t_2]; B_r^Z),$$  \hspace{1cm} (25)

where $\beta_3 = \min\{\beta_1, \beta_2\}.
Indeed, the continuity of \( u(t) \) and the Hypothesis 3.6 (i) imply that \( F(t, S^{-\alpha} u(t)) \) and \( G(t, S^{-\alpha} u(t)) \) are bounded on \([t_1, t_2]\), there exist two positive constants \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) such that

\[
\|F(t, S^{-\alpha} u(t))\|_X \leq \mathcal{N}_1, \quad \|G(t, S^{-\alpha} u(t))\|_Y \leq \mathcal{N}_2.
\] (26)

for \( t \in [t_1, t_2] \). Besides, assume that there exists \( \beta_1, \beta_2 \) and \( h_0 \) such that \( \beta_1, \beta_2 \in (0, 1 - \alpha) \), \( h_0 \in (0, 1) \) and \( t, t + h_0 \in [t_1, t_2] \). We discuss that whether (23) and (24) hold in two cases of \( \alpha \).

If \( \alpha \in (0, 1) \), by [20, Theorem 2.6.13(d)], there exists positive constants \( C^+_{\beta_1} \), \( C^-_{\beta_2} \), \( \overline{M}^+_{\alpha+\beta_1} \) and \( \overline{M}^-_{\alpha+\beta_2} \) such that

\[
\|(T_+(h_0) - I)(-S_+)^{\alpha} T_+(t-s)\|_{\mathcal{L}(X)} \leq C^+_{\beta_1} h_0^{\beta_1} \|(-S_+)^{\alpha+\beta_1} T_+(t-s)\|_{\mathcal{L}(X)} \leq C^+_{\beta_1} h_0^{\beta_1} \overline{M}^+_{\alpha+\beta_1}(t-s)^{-\alpha - \beta_1},
\] (27)

and

\[
\|(I - T_-(-h_0))(S_-)^{\alpha} T_-(t + h_0 - s)\|_{\mathcal{L}(Y)} \leq C^-_{\beta_2} h_0^{\beta_2} \|(S_-)^{\alpha+\beta_2} T_-(t + h_0 - s)\|_{\mathcal{L}(Y)} \leq C^-_{\beta_2} h_0^{\beta_2} \overline{M}^-_{\alpha+\beta_2}(s - t - h_0)^{-\alpha - \beta_2}.
\] (28)

Then

\[
\|\mathcal{F}_\alpha(t + h_0) - \mathcal{F}_\alpha(t)\|_X 
\leq \left\| \int_{t_1}^{t + h_0} (T_+(h_0) - I)(-S_+)^{\alpha} T_+(t-s)F(s, S^{-\alpha} u(s))ds \right\|_X 
\leq \left\| \int_{t_1}^{t + h_0}(t-s)^{-\alpha - \beta_1} ds + \int_{t}^{t + h_0}(t + h_0 - s)^{-\alpha - \beta_1} ds \right\|_X 
\leq C^+_{\beta_1} \overline{M}^+_{\alpha+\beta_1} \mathcal{N}_1 h_0^{\beta_1} \int_{t_1}^{t + h_0} (t-s)^{-\alpha - \beta_1} ds + \overline{M}^+_{\alpha} \mathcal{N}_1 h_0^{1 - \alpha}
\leq \left( C^+_{\beta_1} \overline{M}^+_{\alpha+\beta_1} (t_2 - t_1)^{1 - \alpha - \beta_1} + \overline{M}^+_{\alpha} h_0^{1 - \alpha} \right) \mathcal{N}_1 h_0^{\beta_1},
\] (29)

and

\[
\|\mathcal{G}_\alpha(t + h_0) - \mathcal{G}_\alpha(t)\|_Y 
\leq \left\| \int_{t + h_0}^{t + h_0} (T_-(h_0) - I)(S_-)^{\alpha} T_-(t + h_0 - s)G(s, S^{-\alpha} u(s))ds \right\|_Y 
\leq \left\| \int_{t}^{t + h_0}(t-s)^{-\alpha - \beta_2} ds + \int_{t}^{t + h_0}(s - t - h_0)^{-\alpha - \beta_2} ds \right\|_Y 
\leq C^-_{\beta_2} \overline{M}^-_{\alpha+\beta_2} \mathcal{N}_2 h_0^{\beta_2} \int_{t}^{t + h_0} (s-t)^{-\alpha} ds + \overline{M}^-_{\alpha} \mathcal{N}_2 h_0^{1 - \alpha}
\leq \left( C^-_{\beta_2} \overline{M}^-_{\alpha+\beta_2} (t_2 - t_1)^{1 - \alpha - \beta_2} + \overline{M}^-_{\alpha} h_0^{1 - \alpha} \right) \mathcal{N}_2 h_0^{\beta_2}.
\] (30)
If $\alpha = 0$, by the Proposition 2.1.4(ii) in [12] and (16), we have

$$
\| \mathcal{F}_0(t + h_0) - \mathcal{F}_0(t) \|_\mathcal{X} \leq \int_{t_1}^t ds \int_{t-s}^{t+h_0-s} S_+ T_+(\tau) F(s, u(s)) d\tau \|_\mathcal{X} \\
+ \int_0^{t+h_0} T_+(t + h_0 - s) F(s, u(s)) ds \|_\mathcal{X}
$$

$$
\leq \hat{M}_1^+ N_1 \int_{t_1}^t ds \int_{t-s}^{t+h_0-s} \frac{1}{\tau} d\tau + \hat{M}_1^+ N_1 h_0
$$

$$
\leq \hat{M}_1^+ N_1 \int_{t_1}^t ds \int_{t-s}^{t+h_0-s} \frac{1}{\tau^{1-\beta_1}} d\tau + \hat{M}_0^+ N_1 h_0
$$

$$
\leq \hat{M}_1^+ \left( \frac{t_2 - t_1}{\beta_1(1 - \beta_1)} + \hat{M}_0^+ \right) N_1 h_0^{\beta_1}, \tag{31}
$$

and

$$
\| \mathcal{G}_0(t + h_0) - \mathcal{G}_0(t) \|_\mathcal{Y} \leq \int_{t_1}^t ds \int_{s-t-h_0}^{s-t} -S_- T_-(s, s, u(s)) d\tau \|_\mathcal{Y} \\
+ \int_0^{t+h_0} T_-(t - s) G(s, u(s)) ds \|_\mathcal{Y}
$$

$$
\leq \hat{M}_1^+ N_2 \int_{t_1}^t ds \int_{s-t-h_0}^{s-t} \frac{1}{\tau} d\tau + \hat{M}_0^+ N_2 h_0
$$

$$
\leq \hat{M}_1^+ N_2 \int_{t_1}^t ds \int_{s-t-h_0}^{s-t} \frac{1}{\tau^{1-\beta_2}} d\tau + \hat{M}_0^+ N_2 h_0
$$

$$
\leq \hat{M}_1^+ N_2 \int_{t_1}^t ds \int_{s-t-h_0}^{s-t} \frac{1}{\tau^{1-\beta_2}} d\tau + \hat{M}_0^+ N_2 h_0
$$

$$
\leq \left( \frac{t_2 - t_1}{\beta_2(1 - \beta_2)} + \hat{M}_0^+ \right) N_2 h_0^{\beta_2}. \tag{32}
$$

Therefore, (23) holds by the estimations (29) and (31), and (24) follows from the estimations (30) and (32).

Hence, $\Psi$ maps $\mathcal{E}$ into itself. In the following, we show that $\Psi$ is a contraction map on $\mathcal{E}$, which prove that $\Psi$ has a unique fixed point on $\mathcal{E}$.

Indeed, for $u, v \in \mathcal{E}$,

$$
\|(\Psi u)(t) - (\Psi v)(t)\|_Z
$$

$$
= \max \left\{ \int_{t_1}^t (-S_+)^\alpha T_+(t-s)[F(s, S^{-\alpha} u(s)) - F(s, S^{-\alpha} v(s))] ds \|_\mathcal{X}, \right.

$$

$$
\int_0^{t+h_0} T_+(t + h_0 - s) F(s, u(s)) ds \|_\mathcal{X} \right\}
$$

$$
\leq \max \left\{ \frac{\hat{M}_1^+ L_{0,1}(r)}{1 - \alpha} (t - t_1)^{1-\alpha}, \frac{\hat{M}_0^- L_{0,1}(r)}{1 - \alpha} (t_2 - t)^{1-\alpha} \right\} d\mathcal{E}(u, v)
$$
By induction, we have
\[ d_{\mathcal{E}}(\Psi^n u, \Psi^n v) \leq R(n)d_{\mathcal{E}}(u,v), \]
where \( R(n) = \frac{1}{n} \left( \frac{\max\{M^+_{\alpha,1}(r), M^-_{L_0,1}(r)\}}{1-\alpha} \right)^n \). \( R(n) < 1 \) for \( n \) being large enough, one can apply the extension of the contraction mapping theorem to \( \Psi \) on \( \mathcal{E} \) to obtain that there exists a unique fixed point \( u \in \mathcal{E} \) of the mapping \( \Psi \), that is, \( \Psi u = u \).

From (25) and the fact that the composition of Hölder continuous functions is a Hölder continuous function, we know that \( F(S^{-\alpha}u(t)) \) and \( G(S^{-\alpha}u(t)) \) are uniformly Hölder continuous in \( t \) on \( (t_1, t_2) \). Let us consider the following linear non-homogeneous system for \( t \in (t_1, t_2) \):
\[
\begin{align*}
\frac{dx(t)}{dt} &= S_+x(t) + f(t), \\
\frac{dy(t)}{dt} &= S_-y(t) + g(t),
\end{align*}
\]
where \( f(t) = F(t, S^{-\alpha}u(t)) \) and \( g(t) = G(t, S^{-\alpha}u(t)) \) are continuous in \( t \). By [20, Corollary 4.3.3], the linear non-homogeneous system (33) has a unique solution pair \((x, y)\), where \( x \in C^0([t_1, t_2]; \mathcal{X}) \cap C^1((t_1, t_2); \mathcal{X}) \) and \( y \in C^0([t_1, t_2]; \mathcal{Y}) \cap C^1((t_1, t_2); \mathcal{Y}) \) is given respectively by
\[
\begin{align*}
x(t) &= T_+(t - t_1)x_1 + \int_{t_1}^{t} T_+(t - s)F(s, S^{-\alpha}u(s))\,ds, \quad (34) \\
y(t) &= T_-(t - t_2)y_2 - \int_{t}^{t_2} T_-(t - s)G(s, S^{-\alpha}u(s))\,ds. \quad (35)
\end{align*}
\]
It remains to show that \( z(t) = x(t) + y(t) = S^{-\alpha}u(t), \ t \in (t_1, t_2), \) (36) which proves that the function \( z(t) \) is a local dichotomous solution of the system (4), and \( z \in C^0([t_1, t_2]; B_r^{\mathcal{X}_\alpha}) \cap C^1((t_1, t_2); B_r^{\mathcal{X}_\alpha}) \). Especially, \( z \in C^0([t_1, t_2]; B_r^{\mathcal{Y}_\alpha}) \cap C^1((t_1, t_2); B_r^{\mathcal{Y}_\alpha}) \) is obvious when \( \alpha = 0 \).

Indeed, for \( t \in (t_1, t_2) \), each term of (34) and (35) is in \( \mathcal{D}(S) \) and is also in \( \mathcal{D}(S^\alpha) \), then operating on both sides of (34) and (35) with \( S^\alpha \) and adding them we obtain
\[
\begin{align*}
S^\alpha(x(t) + y(t)) &= T_+(t - t_1)S^\alpha x_1 + \int_{t_1}^{t} S^\alpha T_+(t - s)F(s, S^\alpha u(s))\,ds \\
&\quad + T_-(t - t_2)S^\alpha y_2 - \int_{t}^{t_2} S^\alpha T_-(t - s)G(s, S^\alpha u(s))\,ds \\
&= u(t), \quad (37)
\end{align*}
\]
This proves the formula (36).

Concerning the continuity of \( x \) with values in \( B_r^{\mathcal{X}_\alpha} \) for \( \alpha \in (0, 1) \) up to \( t = t_1 \), from [12, Lemma 7.1.1], the function \( t \mapsto x(t) - T_+(t - t_1)x_1 \) belongs to \( C^0([t_1, t_2]; B_r^{\mathcal{X}_\alpha}) \), while \( t \mapsto T_+(t - t_1)x_1 \) belongs to \( C^0([t_1, t_2]; B_r^{\mathcal{X}_\alpha}) \) if and only if \( x_1 \in \overline{\mathcal{D}(S^\alpha)} \cap B_r^{\mathcal{X}_\alpha} \). Therefore, \( x \in C^0([t_1, t_2]; B_r^{\mathcal{X}_\alpha}) \) if and only if \( x_1 \in \overline{\mathcal{D}(S^\alpha)} \cap B_r^{\mathcal{X}_\alpha} \) and \( \|x_1\|_\alpha \leq \frac{r}{2M_0^+} \), where \( \overline{\mathcal{D}(S_+)} \) is the closure of \( \mathcal{D}(S_+) \) in \( \mathcal{X}_\alpha \). The similar argument
follows the continuity of $y$ with values in $B^{2\alpha}$ for $\alpha \in (0, 1)$ up to $t = t_2$. Thus, $z \in C^0([t_1, t_2], B^{2\alpha})$ if and only if $\bar{z} \in \mathcal{D}(S)^\alpha \cap B^{2\alpha}$ and $\|\bar{z}\|_{\alpha} \leq \frac{r}{2\min(M^+, M^-)}$. 

At last, the uniqueness of $z(t)$ follows from the uniqueness for solution of linear inhomogeneous system (33) and fixed point of the mapping $\Psi$. 

**Lemma 4.3.** (i) $z(t) \in B^{2\alpha}$, $t \geq t_1$ is an infinitely long forward dichotomous solution of system (4) if and only if $z(t)$ satisfies the integral equation

$$z(t) = T_+(t - t_1)x_1 + \int_{t_1}^{t} T_+(t - s)F(s, z(s))ds - \int_{t}^{\infty} T_-(t - s)G(s, z(s))ds \quad (38)$$

and $x_1 \in \mathcal{D}(S_+)^\alpha \cap B^{2\alpha}, \|x_1\|_{\alpha} \leq r/(2M^+)$. 

(ii) $z(t) \in B^{2\alpha}$, $t \leq t_2$ is an infinitely long backward dichotomous solution of system (4) if and only if $z(t)$ satisfies the integral equation

$$z(t) = T_-(t - t_2)y_2 - \int_{t_2}^{t} T_-(t - s)G(s, z(s))ds + \int_{-\infty}^{t} T_+(t - s)F(s, z(s))ds \quad (39)$$

and $y_2 \in \mathcal{D}(S_-)^\alpha \cap B^{2\alpha}, \|y_2\|_{\alpha} \leq r/(2M^-)$. 

**Proof.** (i) Since $z(t), t \geq t_1$ is an infinitely long forward dichotomous solution of (4), by the Definition 2.3, for each $t_2 \in (t_1, \infty)$, we have

$$y(t) = T_-(t - t_2)y_2 - \int_{t}^{t_2} T_-(t - s)G(s, z(s))ds.$$ 

Let $t_2 \to \infty$, from the estimation (14), we obtain

$$y(t) = -\int_{t}^{\infty} T_-(t - s)G(s, z(s))ds.$$ 

Since

$$x(t) = T_+(t - t_1)x_1 + \int_{t_1}^{t} T_+(t - s)F(s, z(s))ds,$$

then from $z(t) = x(t) + y(t)$, we obtain (38). By Lemma 4.2, it is obvious that $x_1 \in \mathcal{D}(S_+)^\alpha \cap B^{2\alpha}$ and $\|x_1\|_{\alpha} \leq r/(2M^+)$. 

For converse part, Let $z(t) = x(t) + y(t)$, where

$$x(t) = T_+(t - t_1)x_1 + \int_{t_1}^{t} T_+(t - s)F(s, z(s))ds,$$

and

$$y(t) = -\int_{t}^{\infty} T_-(t - s)G(s, z(s))ds.$$ 

Since the right side of (38) is continuously differentiable, differentiate (38) with respect to $t$, we will have

$$\begin{aligned}
\frac{dx(t)}{dt} &= S_+x(t) + F(t, z(t)), \\
\frac{dy(t)}{dt} &= S_-y(t) + G(t, z(t)).
\end{aligned}$$
Moreover, for any $t_2 \in (t_1, \infty)$,

$$y(t) = -T_-(t - t_2) \int_{t_2}^{\infty} T_-(t_2 - s) G(s, z(s)) ds - \int_{t}^{t_2} T_-(t - s) G(s, z(s)) ds$$

$$= T_-(t - t_2) y(t_2) - \int_{t}^{t_2} T_-(t - s) G(s, z(s)) ds.$$  

From above, $x(t_1) = x_1, y(t_2) = y_2$. In particular, for each $[t_1, t_2] \subset [t_1, \infty)$, $z(t)$ satisfies the Definition 2.2. The statement follows.

(ii) The case for infinitely long backward dichotomous solution can be evidenced by the same token.

Lemma 4.4. Assume that $\max\{\theta^+_0, \theta^-_1\} < 1/2$ holds. Then, for each $x_1 \in B_r^{X_0} \cap \overline{D(S^\alpha)}$ and $\|x_1\|_\alpha \leq r/(2M_0^+)$, system (4) exists a unique infinitely long forward dichotomous solution $z(t)$ in $C_\beta([t_1, \infty), B_r^{Z_\alpha})$ such that $\|z\|_{C_\beta} < \infty$.

Proof. Fix $x_1 \in B_r^{X_0}$ and $\|x_1\|_\alpha \leq r/(2M_0^+)$, and define the operator $T_{x_1}$ on the Banach space $C_\beta([t_1, \infty), B_r^{Z_\alpha})$ as

$$(T_{x_1}z)(t) = T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s) F(s, z(s)) ds - \int_{t}^{\infty} T_-(t-s) G(s, z(s)) ds \quad (40)$$

for $t \geq t_1$. Obviously, $(T_{x_1}z)(t) \in B_r^{Z_\alpha}$ is a continuous function of $t$ for all $z \in C_\beta([t_1, \infty), B_r^{Z_\alpha})$. We shall show that $T_{x_1}$ is a contraction in $C_\beta([t_1, \infty), B_r^{Z_\alpha})$, then the contraction mapping theorem yields the integral equation (38) has a unique solution $z(t)$ in $C_\beta([t_1, \infty), B_r^{Z_\alpha})$ such that $\|z\|_{C_\beta} < \infty$.

Since $F(t, 0) = G(t, 0) = 0$, we have

$$\|(T_{x_1}z)(t)\|_\alpha^Z$$

$$\leq \max \left\{ M_0^+ e^{\beta_+ (t-t_1)} \|x_1\|_\alpha^X + \int_{t_1}^{t} M_0^+ L_{0,1}(r)(t-s)^{-\alpha} e^{\beta_+(t-s)} \|z(s)\|_\alpha^Z ds, \right.$$  

$$\left. \int_{t}^{\infty} M_0^- \hat{L}_{0,1}(r)(s-t)^{-\alpha} e^{-\beta_-(s-t)} \|z(s)\|_\alpha^Z ds \right\}.$$  

Then,

$$\|T_{x_1}z\|_{C_\beta}$$

$$\leq \sup_{t \geq t_1} \max \left\{ M_0^+ e^{\beta_+ - \beta t} e^{-\beta t_1} \|x_1\|_\alpha^X + M_0^+ L_{0,1}(r) \int_{t_1}^{t} (t-s)^{-\alpha} e^{(\beta_+ - \beta)(t-s)} \|z\|_{C_\beta} ds, \right.$$  

$$\left. M_0^- \hat{L}_{0,1}(r) \int_{t}^{\infty} (s-t)^{-\alpha} e^{(\beta - \beta_-)(s-t)} \|z\|_{C_\beta} ds \right\}$$

$$\leq M_0^+ e^{-\beta t_1} \|x_1\|_\alpha^X + \frac{1}{2} \|z\|_{C_\beta} < \infty.$$  

Hence, $T_{x_1}$ maps $C_\beta([t_1, \infty), B_r^{Z_\alpha})$ into itself.
Furthermore, for any $z, \hat{z} \in C_{\beta}([t_1, \infty), B_{r^\alpha}'(\mathbb{R}))$, we have
\[
\|T_{x_1}z - T_{x_1}\hat{z}\|_{C_{\beta}} \leq \sup_{t \geq 0} \left\{ M_{\alpha}^+ L_{0,1}(r) \int_{t}^{t_1} (t-s)^{-\alpha} e^{(\beta+\beta)(t-s)} ds, \right. \\
\left. M_{\alpha}^- L_{0,1}(r) \int_{t}^{\infty} (s-t)^{-\alpha} e^{(\beta-\beta)(s-t)} ds \right\} \|z - \hat{z}\|_{C_{\beta}} \\
< \frac{1}{2} \|z - \hat{z}\|_{C_{\beta}}.
\]
Thus, $T_{x_1} : C_{\beta}([t_1, \infty), B_{r^\alpha}'(\mathbb{R})) \to C_{\beta}([t_1, \infty), B_{r^\alpha}'(\mathbb{R}))$ is a contraction.

Hence, from the lemma 4.3, if $x_1 \in \overline{D(S_+)}^\alpha$, the fixed point $z(t)$ of $T_{x_1}$ is the unique infinitely long forward dichotomous solution to system (4) in $C_{\beta}([t_1, \infty), B_{r^\alpha}'(\mathbb{R}))$ such that $P_+ z(t_1) = x_1$. The proof is completed. \hfill \Box

Obviously, $\|z\|_{C_{\beta}} < \infty$ implies $\|z(t)\|_{Z_{\alpha}} \to 0$ as $t \to \infty$. In the following we will show that all these infinitely long forward dichotomous solutions involved in $B_{r^\alpha}'$ lie on the graph of a map $h^\alpha : \mathbb{R} \times B_{r^\alpha}' \to B_{r^\alpha}'$. Prior to this, a generalized Gronwall’s inequality with singular kernel will be given, which generalizes the case $\alpha = 1/2$ in [8, Lemma 6, p.33] to $\alpha \in (0, 1)$.

**Lemma 4.5.** Let $\phi \in L^\infty([t_1, t_2], \mathbb{R}_+)$. Assume that there exist $a > 0$ and $\alpha \in (0, 1)$ such that $\phi$ satisfies
\[
\phi(t) \leq \phi(t_1) + a \int_{t_1}^{t} (t-s)^{-\alpha} \phi(s) ds, \quad t \in [t_1, t_2],
\]
Then there exists $K_0 > 1$ such that $\phi(t) \leq K_0 \phi(t_1)$ on $[t_1, t_2]$.

**Proof.** Let $\phi^*(t) := \text{ess sup}_{s \in [t_1, t]} \phi(s)$. For $s \in [t_1, t]$, $\varepsilon$ is small such that $1 - \alpha - a \varepsilon^{-1 - \alpha} > 0$. We have
\[
\phi(s) \leq \phi(t_1) + a \int_{t_1}^{s-\varepsilon} (s-\tau)^{-\alpha} \phi(\tau) d\tau + a \int_{s-\varepsilon}^{s} (s-\tau)^{-\alpha} \phi(\tau) d\tau \\
\leq \phi(t_1) + a \frac{a}{\varepsilon^\alpha} \int_{t_1}^{s} \phi(\tau) d\tau + a \frac{a^{1-\alpha}}{1-\alpha} \phi^*(s) \\
\leq \phi(t_1) + a \frac{a}{\varepsilon^\alpha} \int_{t_1}^{t} \phi^*(\tau) d\tau + a \frac{a^{1-\alpha}}{1-\alpha} \phi^*(t).
\]
Thus, $\left(1 - a \frac{a^{1-\alpha}}{1-\alpha}\right) \phi^*(t) \leq \phi(t_1) + a \frac{a}{\varepsilon^\alpha} \int_{t_1}^{t} \phi^*(\tau) d\tau$. By the classical Gronwall’s inequality, we get $\phi(t) \leq K_0 \phi(t_1)$, where $K_0 = \frac{1-a}{1-\alpha - a \varepsilon^{-1-\alpha}} \exp \left( \frac{a^{1-\alpha}}{\varepsilon^\alpha (1-\alpha - a \varepsilon^{-1-\alpha}) (t_2 - t_1)} \right)$.

The following lemma gives the existence of local stable integral manifold of system (4).

**Lemma 4.6.** Assume in addition to the hypotheses of Lemma 4.4 hold and
\[
\max \left\{ \frac{1}{l_0^{1-\alpha}}, K \right\} M_{0,1}^+ \theta_{0,1}^- \leq \kappa, \quad \theta_{0,1}^-[1 + (1+\kappa)K_1] < 1
\]
for some positive constants $K, K_1, \kappa$. Then there exists a unique $C^{0,1}$ local stable integral manifold for system (4).
Proof. Let \( \kappa \in (0, 1] \) be an arbitrary and fixed constant. Define the space \( \Gamma(r) \):

\[
\Gamma(r) = \{ h^\ast : \mathbb{R} \times B_r^{X_\alpha} \to B_r^{X_\alpha} | \text{ } h^\ast(t, \cdot) \in C^{0,1}(B_r^{X_\alpha}, B_r^{2\alpha}), h^\ast(t, 0) = 0, \| h^\ast \|_{0,1} \leq \kappa \}. \quad (41)
\]

Observe that \( \| h^\ast \|_{0,1} \leq \kappa \) implies that \( \| h^\ast(t, x) \|_{2\alpha} \leq \kappa \| x \|_{\alpha} \) and \( \| h^\ast(t, x) - h^\ast(t, \hat{x}) \|_{2\alpha} \leq \kappa \| x - \hat{x} \|_{\alpha} \) for \( x, \hat{x} \in B_r^{X_\alpha} \). \( \Gamma(r) \) is complete metric space with the induced metric

\[
d_{\Gamma}(h^\ast, \hat{h}^\ast) = \sup_{x \in B_r^{X_\alpha}, t \in \mathbb{R}} \{ \| h^\ast(t, x) - \hat{h}^\ast(t, x) \|_{2\alpha} \}.
\]

Define the Lyapunov-Perron operator \( \mathcal{L} \) on the Lipschitz function \( h^\ast \) in \( \Gamma(r) \) as follows:

\[
\mathcal{L}(h^\ast)(t_1, x_1) = - \int_{t_1}^{\infty} \mathcal{T}_{-s}(t_1 - s) \mathcal{G}(s, x(s), h^\ast(s, x(s))) ds,
\]

where \( x(t) = x(t; t_1, x_1, h^\ast) \) is the unique solution of the following system

\[
\begin{cases}
\frac{dx(t)}{dt} = S_+ x(t) + F(t, x(t), h^\ast(t, x(t))), & t \geq t_1, \\
x(t_1) = x_1 \in \overline{D(S_+)^{\alpha}} \cap B_r^{X_\alpha}.
\end{cases}
\]

Since \( S_+ \) is the infinitesimal generator of strongly continuous and analytic semigroup \( \{ T_+(t) \}_{t \geq 0} \), from the Hypothesis 3.6 (i), \( x(t) \) is well defined for all \( t \geq t_1 \), and \( x(t) \) has the form

\[
x(t) = T_+(t - t_1) x_1 + \int_{t_1}^{t} T_+(t - s) F(s, x(s), h^\ast(s, x(s))) ds.
\]

On the one hand, since \( \| T_+(t - t_1) x_1 \|_{2\alpha} = \| T_+(t - t_1) (-S_+)^{\alpha} x_1 \|_{X} \leq \| T_+(t - t_1) \|_{\mathcal{L}(X)} \| x_1 \|_{2\alpha} \), by (44) and (14), we have

\[
\| x(t) \|_{2\alpha} \leq \frac{M^+_{\alpha}}{1 - \theta_{0,1}} e^{\beta(t-t_1)} \| x_1 \|_{2\alpha},
\]

and note that \( \| x(t) \|_{2\alpha} \leq r \) can be verified by \( \| x_1 \|_{2\alpha} \leq r/(2M^+_{\alpha}) \) and \( \theta_{0,1} < 1/2 \).

Then the integral on the right side of (42) converges and belongs to \( B_r^{2\alpha} \). Indeed, it follows from

\[
\| \mathcal{L}(h^\ast)(t_1, x_1) \|_{2\alpha} \leq \int_{t_1}^{\infty} M^\alpha_{\alpha} \mathcal{L}_{0,1}(r)(s-t_1)^{-\alpha} e^{-\beta_{\ast}(s-t_1)} \| x(s) \|_{2\alpha} ds \leq \theta_{0,1} r < r/2.
\]

On the other hand, set \( x(t) = x(t; t_1, x_1, h^\ast) \), \( \hat{x}(t) = x(t; t_1, \hat{x}_1, h^\ast) \). From (14) and (15), we have

\[
\| x(t) - \hat{x}(t) \|_{2\alpha} \leq M^+_{\alpha} e^{\beta_{\ast}(t-t_1)} \| x_1 - \hat{x}_1 \|_{2\alpha} + M^+_{\alpha} \mathcal{L}_{0,1}(r) \int_{t_1}^{t} (t-s)^{-\alpha} e^{\beta_{\ast}(t-s)} \| x(s) - \hat{x}(s) \|_{2\alpha} ds.
\]

Since \( e^{\beta_{\ast}(t-s)} < 1 \), by the Lemma 4.5, it yields that

\[
\| x(t) - \hat{x}(t) \|_{2\alpha} \leq K M^+_{\alpha} e^{\beta_{\ast}(t-t_1)} \| x_1 - \hat{x}_1 \|_{2\alpha},
\]

where \( K > 1 \) depends on \( M^+_{\alpha}, \mathcal{L}_{0,1}(r) \) and \( \alpha \).
In addition, we use the notation $x(t, h^*)$ to signify the dependence of $x(t)$ on $h^*$. For $h^*, \tilde{h}^* \in \Gamma(r)$,
\[
\|x(t, h^*) - x(t, \tilde{h}^*)\|_\alpha^X \\
\leq \int_{t_1}^t \|T_+(t-s)\|_{L(X, X_\alpha)} \left[ \|F(s, x(s, h^*), h^*(s, x(s, h^*))) - F(s, x(s, \tilde{h}^*), \tilde{h}^*(s, x(s, \tilde{h}^*))))\|_X \\
+ \|F(s, x(s, h^*), \tilde{h}^*(s, x(s, h^*))) - F(s, x(s, \tilde{h}^*), \tilde{h}^*(s, x(s, \tilde{h}^*))))\|_X \\
+ \|F(s, x(s, \tilde{h}^*), \tilde{h}^*(s, x(s, \tilde{h}^*))) - F(s, x(s, h^*), \tilde{h}^*(s, x(s, \tilde{h}^*))))\|_X \right] ds \\
\leq \int_{t_1}^t M_{\alpha}^{-}\Gamma L_{0,1}(r)(t-s)^{-\alpha}e^{-\beta(-(s-t_1))} \left[ d_{\Gamma}(h^*, \tilde{h}^*) + (1 + \kappa)\|x(s, h^*) - x(s, \tilde{h}^*)\|_\alpha^X \right] ds,
\]
then by Lemma 4.5, there exists a constant $K_1 > 1$ that depends on $M_{\alpha}^{-}\Gamma L_{0,1}(r)(1 + \kappa)$ and $\alpha$, such that
\[
\|x(t, h^*) - x(t, \tilde{h}^*)\|_\alpha^X \leq K_1\theta_{0,1}^{-}\Gamma d_{\Gamma}(h^*, \tilde{h}^*). \quad (48)
\]
If $h^*$ is a fixed point of $\mathcal{L}$ in $\Gamma(r)$, then the graph of $h^*$ is the local stable manifold. In the follows, we prove $\mathcal{L}$ is a contraction map in $\Gamma(r)$. First we verify that $\mathcal{L} (\Gamma(r)) \subset \Gamma(r)$.

Choose $r > 0$ so small that $\max\{\frac{1}{1-\theta_{0,1}}, K\} M_{\alpha}^{-}\theta_{0,1} \leq \kappa$. By (45) and (46), we have $\|\mathcal{L}(h^*)(t_1, x_1)\|_\alpha^X \leq \kappa\|x_1\|_\alpha^X$. Furthermore, we have
\[
\|\mathcal{L}(h^*)(t_1, x_1) - \mathcal{L}(\tilde{h}^*)(t_1, \tilde{x}_1)\|_\alpha^X \\
\leq \int_{t_1}^{\infty} M_{\alpha}^{-}\Gamma \tilde{L}_{0,1}(r)(s-t_1)^{-\alpha}e^{-\beta-(s-t_1)} \|x(s) - \tilde{x}(s)\|_\alpha^X ds \leq \kappa\|x_1 - \tilde{x}_1\|_\alpha^X.
\]
Besides, since $h^*(t, 0) = 0$ and $G(t, 0) = 0$, from (42), $\mathcal{L}(h^*)(t_1, x_1) \in \mathcal{Y}_\alpha$ and $\mathcal{L}(\tilde{h}^*)(t_1, x_1) = 0$ are obvious. Thus, $\mathcal{L} (\Gamma(r)) \subset \Gamma(r)$.

Furthermore, for $h^*, \tilde{h}^* \in \Gamma(r)$, by (48), we obtain
\[
\|\mathcal{L}(h^*)(t_1, x_1) - \mathcal{L}(\tilde{h}^*)(t_1, x_1)\|_\alpha^Y \\
\leq \int_{t_1}^t \||T_-(t_1 - s)||_{L(\mathcal{Y}_\alpha, \mathcal{Y}_\alpha)} [\|G(s, x(s, h^*), h^*(s, x(s, h^*))) \\
- G(s, x(s, \tilde{h}^*), \tilde{h}^*(s, x(s, \tilde{h}^*))))\|_Y \\
+ \|G(s, x(s, h^*), \tilde{h}^*(s, x(s, \tilde{h}^*))) - G(s, x(s, \tilde{h}^*), \tilde{h}^*(s, x(s, \tilde{h}^*))))\|_Y \\
+ \|G(s, x(s, \tilde{h}^*), \tilde{h}^*(s, x(s, \tilde{h}^*))) - G(s, x(s, h^*), \tilde{h}^*(s, x(s, \tilde{h}^*))))\|_Y \right] ds \\
\leq \int_{t_1}^t M_{\alpha}^{-}\Gamma \tilde{L}_{0,1}(r)(s-t_1)^{-\alpha}e^{-\beta-(s-t_1)} \left[ d_{\Gamma}(h^*, \tilde{h}^*) + (1 + \kappa)\|x(s, h^*) - x(s, \tilde{h}^*)\|_\alpha^X \right] ds \\
\leq \theta_{0,1}^{-} \left[ 1 + (1 + \kappa)K_1\theta_{0,1}^{-}\Gamma d_{\Gamma}(h^*, \tilde{h}^*) \right] d_{\Gamma}(h^*, \tilde{h}^*) < d_{\Gamma}(h^*, \tilde{h}^*) .
\]
Hence, $\mathcal{L}$ is a contraction map in $\Gamma(r)$, then Banach fixed point theorem follows that there exists a unique fixed point $h^*$ of $\mathcal{L}(h^*) = h^*$ in $\Gamma(r)$. From the Lemma
Lemma 4.7. and the Taylor expansion, we have $z^{(4)}$ with

$$z^{(4)} = T_{-}(t_1 - s)G(s, x(s), h^s(s, x(s)))ds. \quad (50)$$

This means that $W^s_{loc}(0, r) = \{(x(t), y(t)) : y(t) = h^s(t, x(t)), t \in \mathbb{R}, x(t) \in B^X_r\}$. To prove that $W^s_{loc}(0, r) = \{(x(t), y(t)) : y(t) = h^s(t, x(t)), t \in \mathbb{R}, x(t) \in B^X_r\}$ is $C^{0,1}$ local stable integral manifold, it remains to prove invariance of $W^s_{loc}(0, r)$. Let $x(t), t \geq t_1$, be a solution of (43), and $(t_1, x_1, h^s(t_1, x_1)) \in W^s_{loc}(0, r)$, then denote $y(t) := h^s(t, x(t))$ for $t \geq t_1$. This defines a curve $(t, x(t), y(t)) \in W^s_{loc}(0, r)$, $t \geq t_1$, through the point $(t_1, x_1, h^s(t_1, x_1)) \in W^s_{loc}(0, r)$. It suffices to prove that $y(t)$ satisfies

$$\frac{dy(t)}{dt} = S_+y(t) + G(t, x(t), h^s(t, x(t))), \ t \in [t_1, t_2], \quad (51)$$

for all $t_2 \in (t_1, \infty)$. The equation (51) indeed has a unique solution $y(t)$ which remains bounded as $t_2 \to \infty$, namely

$$y(t) = -\int_{t}^{\infty} T_{-}(t - s)G(s, x(s), h^s(s, x(s)))ds.$$ 

Thus, $z(t) = x(t) + h^s(t, x(t))$ is the unique infinitely long forward dichotomous solution of the system (4) with $z(t_1) = x_1 + h^s(t_1, x_1)$.

The proof is complete. □

In the following, we shall focus on the smoothness of $h^s$ obtained in Lemma 4.6.

**Lemma 4.7.** Assume in addition to the hypotheses of Lemma 4.6 that Hypothesis 3.6 (ii) holds for $k = 1$ and $\gamma = 0$, and that $(1 + \kappa)K_2\theta_{1,0}^+\theta_{1,0}^+ < 1$ for some positive constants $K_2$. Then there exists a unique $C^1$ local stable integral manifold for system (4).

**Proof.** By Lemma 4.6, we obtain a unique $C^{0,1}$ local stable integral manifold characterized by the graph of $h^s \in \Gamma(r)$, where $h^s$ and $\Gamma(r)$ are as in to (50) and (41) respectively. We shall proceed to prove that $h^s$ is $C^1$ provided $F$ and $G$ are $C^1$ in $z$ and $(1 + \kappa)K_2\theta_{1,0}^+\theta_{1,0}^+ < 1$ for some positive constant $K_2$.

Fix $t$, and defined

$$\lambda(h^s, x_0)$$

$$= \limsup_{(x_1, \Delta) \to (x_0, 0)} \frac{\|h^s(t, x_1 + \Delta) - h^s(t, x_1) - h^s(t, x_0 + \Delta) + h^s(t, x_0)\|_\alpha}{\|\Delta\|_\alpha^{\gamma}}. \quad (52)$$

By Lemma 3.1, $h^s$ is $C^1$ in $x$ if and only if $\lambda(h^s, x_0) = 0$ for every $x_0 \in \overline{D(S^+) \cap B^X_r}$. Here we use the notation $x(t, x_1, h^s)$ to represent the solution of (43). By (44) and the Taylor expansion, we have

$$x(t_1 + \Delta, h^s) - x(t_1, h^s) - x(t, x_0 + \Delta, h^s) + x(t, x_0, h^s)$$

$$= \int_{t_1}^{t} T_{+}(t - s)\{D_xF(s, x(s, x_0, h^s), h^s(s, x(s, x_0, h^s)))$$

$$[x(s, x_1 + \Delta, h^s) - x(s, x_1, h^s) - x(s, x_0 + \Delta, h^s) + x(s, x_0, h^s)]$$

$$+ D_hF(s, x(s, x_0, h^s), h^s(s, x(s, x_0, h^s)))h(s, x(s, x_1 + \Delta, h^s))$$

$$- h(s, x(s, x_1, h^s)) - h(s, x(s, x_0 + \Delta, h^s)) + h(s, x(s, x_0, h^s)]$$

$$+ \mathcal{R}_2(x))ds, \quad (53)$$

4.4, (44) and (42), all infinitely long forward dichotomous solutions of the system (4) with $z(t) \in B^R_\infty$ and $\|z\|_{C^0} < \infty$ are contained in the graph of $h^s$ defined by

$h^s(t_1, x_1) = -\int_{t_1}^{\infty} T_{-}(t_1 - s)G(s, x(s), h^s(s, x(s)))ds.$
where $\mathcal{R}_2(x)$ represents the sum of higher order Taylor expansions of $F$ in (53) at the point $(x(s, x_0, h^s), h^s(x(s, x_0, h^s)))$.

By Lemma 4.5, it follows that
\[
\|x(t, x_1 + \Delta, h^s) - x(x_1, h^s) - x(t, x_0 + \Delta, h^s) + x(t, x_0, h^s)\|_\alpha^X \\
\leq K_2[\theta^+_1, 0] \Delta_{\alpha, X}^\| \sup_{t \geq t_1} [\lambda(h^s, x(t, x_0, h^s))] + o(\|\Delta\|_\alpha^X) \tag{54}
\]
as $(x_1, \Delta) \to (x_0, 0)$, where $K_2 > 1$ which depends on $M^+_1 L_{1,0}(r)$ and $\alpha$.

Then, similarly as (53), by (50) and the Taylor expansion, we have
\[
h^s(t, x_1 + \Delta) - h^s(t, x_1) - h^s(t, x_0 + \Delta) + h^s(t, x_0)
\]
\[
= - \int_{t_1}^\infty T_{-}(t_1 - s) \left\{ D_x G(s, x(s, x_0, h^s), h^s(x(s, x_0, h^s))) \\
+ [h(s, x(s, x_1 + \Delta, h^s)) - h(s, x(s, x_1, h^s)) + x(s, x_0 + \Delta, h^s) - x(s, x_0, h^s))] \\
+ [x(s, x_1, h^s) + x(s, x_0 + \Delta, h^s) - x(s, x_0, h^s)) - h^s(s, x(s, x_1, h^s))] \\
- [h(s, x(s, x_0 + \Delta, h^s)) - h(s, x(s, x_0, h^s))] \right\} ds,
\]
where $\hat{\mathcal{R}}_2(x)$ represents the sum of higher order Taylor expansions of $G$ in (55) at the point $(x(s, x_0, h^s), h^s(x(s, x_0, h^s)))$.

Furthermore, by (55), (47) and (54), it follows that
\[
\|h^s(t, x_1 + \Delta) - h^s(t, x_1) - h^s(t, x_0 + \Delta) + h^s(t, x_0)\|_\alpha^X
\]
\[
\leq \int_{t_1}^\infty M_{\alpha, \tilde{L}_{1,0}}(r)(s - t_1)^{-a} e^{-\beta_-(s-t_1)} (1 + \kappa) \\\n\times \left\{ \|x(s, x_1 + \Delta, h^s) - x(s, x_1, h^s) - x(s, x_0 + \Delta, h^s) + x(s, x_0, h^s))\|_\alpha^X \\
+ 2\kappa \|x(s, x_0 + \Delta, h^s) - x(s, x_0, h^s))\|_\alpha^X \right\} ds + o(\|\Delta\|_\alpha^X) \tag{56}
\]
\[
\leq (1 + \kappa) K_2 \hat{\theta}^+_1, 0 \tilde{\theta}^{-1, 0}_{1,0} \Delta_{\alpha, X}^\| \sup_{t \geq t_1} [\lambda(h^s, x(t, x_0, h^s))] + o(\|\Delta\|_\alpha^X)
\]
as $(x_1, \Delta) \to (x_0, 0)$, and it yields
\[
\lambda(h^s, x_0) \leq (1 + \kappa) K_2 \hat{\theta}^+_1, 0 \tilde{\theta}^{-1, 0}_{1,0} \sup_{t \geq t_1} \lambda(h^s, x(t, x_0, h^s)) < \infty. \tag{57}
\]
Because $x(t, x(s, x_0, h^s), h^s) = x(t, x, x_0, h^s)$ for $t + s \geq t_1$, we have
\[
\sup_{t \geq t_1} \lambda(h^s, x(t, x_0, h^s)) \leq (1 + \kappa) K_2 \hat{\theta}^+_1, 0 \tilde{\theta}^{-1, 0}_{1,0} \sup_{t \geq t_1} \lambda(h^s, x(t, x_0, h^s)).
\]
Since $(1 + \kappa) K_2 \hat{\theta}^+_1, 0 \tilde{\theta}^{-1, 0}_{1,0} < 1$, then $\sup_{t \geq t_1} \lambda(h^s, x(t, x_0, h^s)) = 0$. By (57), $\lambda(h^s, x_0) = 0$. Thus, $h^s$ is $C^1$ in $x$. Moreover, by (50), $D_x h^s(t, 0) = 0$.

The proof is complete. $\square$

**Remark 1.** With the same arguments as Lemma 4.7 and Lemma 3.1, the conclusion of Lemma 4.7 can be improved to the $C^k$ ($k \geq 2$) case provided in addition to the existence of $C^{k-1,1}$ local stable integral manifold that $F$ and $G$ are $C^k$ in $z$ along with sufficiently small $L_{k,0}(r)$ and $\tilde{L}_{k,0}(r)$. While we omit the details and shift attention to the $C^{k,\gamma}$ ($k \geq 1, \gamma \in (0, 1]$) smoothness later.
Lemma 4.8. For the ill-posed system (4), assume in addition to the hypotheses of Lemma 4.6 that \( \beta_+ < (1 + \gamma)\beta \) and Hypothesis 3.6 (ii) holds for \( \gamma \in (0, 1] \) with sufficiently small \( L_{k, \gamma}(r) \) and \( L_{k, \gamma}(r) \). Then there exists a unique \( C^{k, \gamma} \) local stable integral manifold.

Proof. Set \( \Gamma_k(r) = \{ h^s(t, \cdot) \in \Gamma(r) \cap C^{k, \gamma}(X_\alpha, Y_\alpha) : D_x h^s(t, 0) = 0, \| h^s \|_{k, \gamma} \leq \kappa \} \) for \( k \geq 1 \), where \( \Gamma(r) \) refers to (41). By Lemma 4.6, we obtain a unique \( C^{0,1} \) local stable integral manifold characterized by the graph of \( h^s \in \Gamma \), where \( h^s \) refers to (50). In the following, we shall continue to prove \( h^s \) is \( C^{k, \gamma} \) in \( x \) for \( \gamma \in (0, 1] \). From (42) and Lemma 3.2, it suffices to show \( \mathfrak{L}(\Gamma_k(r)) \subset \Gamma_k(r) \) in the \( C^0 \) norm.

Step I: Prior to this, we need to prove that \( x(t) \) defined by (44) is \( C^{k, \gamma} \) in \( x_1 \) and satisfies the estimates on the derivatives up to order \( k \) and Hölder derivatives of \( D^k_{x_1} x(t) \).

Now we define the space

\[
L_\beta(r) = \left\{ \phi : [t_1, \infty) \times (\overline{D(S_+)^\alpha} \cap B_{\alpha}^\gamma) \to B_1^\gamma : \phi(t_1, \xi) = \xi, \\
\phi(\cdot, \xi) \in C^1([t_1, \infty)), \phi(t, \cdot) \in C^{k, \gamma}(\overline{D(S_+)^\alpha} \cap B_{\alpha}^\gamma), \right. \\
\left. \| \phi(t, \cdot) \|_{C^0(x_\alpha, x_\alpha)} \leq \frac{M_0^+}{1 - \theta_{0,1}} e^{\beta(t-t_1)}, \right. \\
\left. \| D^i_\xi \phi(t, \cdot) \|_{C^0(x_\alpha, x_\alpha)} \leq \frac{M_0^+}{1 - \theta_{0,1}} e^{\beta(t-t_1)}, \quad i = 1, \ldots, k, \right. \\
H_\gamma(D^k_\xi \phi(t, \cdot)) \leq \frac{M_0^+}{1 - \theta_{0,1}} e^{\beta(t-t_1)} \right\},
\]

where \( M_0^+ \) refers to (14). From the Proposition A2 in [11, p182], we obtain that \( L_\beta(r) \) is a complete metric space endowed with the induced metric

\[
d_L(\phi, \tilde{\phi}) \equiv \| \phi - \tilde{\phi} \|_{C_\beta} = \sup_{t \in [t_1, \infty)} \sup_{\xi \in \overline{D(S_+)^\alpha} \cap B_{\alpha}^\gamma} e^{-\beta t} \| \phi(t, \xi) - \tilde{\phi}(t, \xi) \|_{X_\alpha}.
\]

For any \( x \in L_\beta(r) \), define

\[
[\mathfrak{S}]x(t, x_1) = T_+(t - t_1)x_1 + \int_{t_1}^t T_+(t - s)F(s, x(s, x_1), h^s(s, x(s, x_1)))ds. \tag{59}
\]

To prove that \( x(t) \) defined by (44) is \( C^{k, \gamma} \) in \( x_1 \) provided \( h^s \in \Gamma_k(r) \), it suffices to prove \( \mathfrak{S} \) is a contraction map in \( L_\beta(r) \). We first prove that \( \mathfrak{S}(L_\beta(r)) \subset L_\beta(r) \).

Obviously, \( [\mathfrak{S}]x(t_1, x_1) = x_1 \). For any \( x \in L_\beta(r) \), \( [\mathfrak{S}]x \) being \( C^1 \) in \( t \) and \( C^{k, \gamma} \) in \( x_1 \), follow from the fact that, for any \( l \in \mathbb{N} \) and \( \gamma \in (0, 1] \), the composition of \( C^{k, \gamma} \) functions is a \( C^{\gamma} \) function. Moreover, by (45), we know that \( \|[\mathfrak{S}]x\|_{C^0(x_\alpha, x_\alpha)} \leq \frac{M_0^+}{1 - \theta_{0,1}} e^{\beta(t-t_1)} \) and \( \|[\mathfrak{S}]x(t, x_1)\|_{X_\alpha} \leq r \).

Differentiating (59) in \( x_1 \), it yields

\[
D_{x_1}[\mathfrak{S}]x(t, x_1) = T_+(t - t_1) + \int_{t_1}^t T_+(t - s)\left[ D_x F(s, x_1)D_{x_1} x(s, x_1) + D_h^s F(s, x_1)D_x h^s(s, x(s, x_1))D_{x_1} x(s, x_1) \right]ds, \tag{60}
\]

where we use the notation \( F(s, x_1) \) for \( F(s, x(s, x_1), h^s(s, x(s, x_1))) \). Note that the integral in (60) converges. Indeed, since \( x \in L_\beta(r) \) and \( h^s \in \Gamma_k(r) \), choose \( L_{1, \gamma}(r) \)}
so small that \( \theta_{1,0}^{+} < \frac{\theta_{0,1}^{+}}{1+\kappa} \), we have

\[
\| D_{x_1} [\mathcal{I}x] \|_{C^0(x_0, x_0)} \\
\leq M_0^{+} e^{\beta_1 (t-t_1)} + \int_{t_1}^{t} M_0^{+} (t-s)^{-\alpha} e^{\beta_1 (t-t_1)} L_{1,0} (r)(1+\kappa) K M_0^{+} e^{\beta (s-t_1)} ds \\
\leq [1 + M_0^{+} L_{1,0} (r) (1+\kappa) \int_{t_1}^{t} (t-s)^{-\alpha} e^{(\beta_1-\beta)(t-s)} ds] M_0^{+} e^{\beta (t-t_1)} \\
\leq \frac{M_0^{+}}{1-\theta_{0,1}^{+}} e^{\beta (t-t_1)}.
\]

Furthermore, for \( 2 \leq i \leq k \),

\[
D_{x_1}^{i} [\mathcal{I}x] (t, x_1) = \int_{t_1}^{t} T_{+} (t-s) [D_{x} F(s, x_1) D_{x_1}^{i} x(s, x_1) \\
+ D_{h} F(s, x_1) D_{x} h^{*} (s, x(s, x_1)) D_{x_1}^{i} x(s, x_1) + \mathcal{R}_{i}(s, x_1)] ds,
\]

where \( \mathcal{R}_{i}(s, x_1) \) is a sum of monomials whose factors are derivatives of \( F \) and of \( h^{*} \) up to order \( i \) and of \( x \) up to order \( i-1 \). Note that (61) is well-defined because \( [\mathcal{I}x] \) is \( C^{k, \gamma} \) in \( x_1 \) and the integral in (61) converges. Moreover, by choosing \( L_{i,\gamma} (r) \) sufficiently small, we can obtain

\[
\| D_{x_1}^{i} [\mathcal{I}x] \|_{C^0(x_0, x_0)} \leq \frac{M_0^{+}}{1-\theta_{0,1}^{+}} e^{\beta (t-t_1)}.
\]

The only thing that remains to ensure that \( \mathcal{I} (L_{\beta} (r)) \subset L_{\beta} (r) \) is the estimate on \( H_{\gamma} (D_{x_1}^{k} [\mathcal{I}x]) \). For all \( x_1, \hat{x}_1 \in \overline{D(S_{+}^{\alpha})} \cap B_{r}^{\alpha, \gamma}, x \in L^{\beta} (r) \) and \( h^{*} \in \Gamma_{k}(r) \), we have

\[
D_{x_1}^{k} [\mathcal{I}x] (t, x_1) - D_{x_1}^{k} [\mathcal{I}\hat{x}] (t, \hat{x}_1) \\
\leq \int_{t_1}^{t} T_{+} (t-s) [D_{x} F(s, x_1) D_{x_1}^{k} x(s, x_1) - D_{x} F(s, \hat{x}_1) D_{x_1}^{k} x(s, \hat{x}_1) \\
+ D_{h} F(s, x_1) D_{x} h^{*} (s, x(s, x_1)) D_{x_1}^{k} x(s, x_1) \\
- D_{h} F(s, \hat{x}_1) D_{x} h^{*} (s, x(s, \hat{x}_1)) D_{x_1}^{k} x(s, \hat{x}_1) \\
+ \mathcal{R}_{i}(s, x_1) - \mathcal{R}_{i}(s, \hat{x}_1)] ds
\]

Since each difference term in the right side of (62) contain the factors \( D_{x} x(s, x_1) \) and \( D_{x_1} x(s, \hat{x}_1) \), we use the triangle inequality to estimate (62) in the \( C^{0} \) norm and assume \( \beta_+ < (1+\gamma) \beta \) and \( L_{k,\gamma} (r) \) being sufficiently small, then we can obtain

\[
H_{\gamma} (D_{x_1}^{k} [\mathcal{I}x]) \leq \frac{M_0^{+}}{1-\theta_{0,1}^{+}} e^{eta (t-t_1)}.
\]

This finishes the verification of \( \mathcal{I} (L_{\beta} (r)) \subset L_{\beta} (r) \).

In addition, for any \( x, \hat{x} \in L_{\beta} (r) \), we have

\[
\| [\mathcal{I}x] (t, x_1) - [\mathcal{I}\hat{x}] (t, x_1) \|_{\alpha}^X \\
\leq \| \int_{t_1}^{t} T_{+} (t-s) [F(x(s, x_1), h^{*} (s, x(s, x_1))) - F(\hat{x}(s, x_1), h^{*} (s, \hat{x}(s, x_1))))] \|_{\alpha}^X \\
\leq \int_{t_1}^{t} M_0^{+} (t-s)^{-\alpha} e^{\beta_1 (t-s)} L_{1,0} (r) \max \{1, \kappa\} \| x(s, x_1) - \hat{x}(s, x_1) \|_{\alpha}^X ds \\
\leq \theta_{1,0}^{+} e^{\beta_1 ds} L (x, \hat{x}).
\]
Since \( \theta_{0,1}^+ < \frac{\theta_{0,1}^+}{1 + \kappa} < \theta_{0,1}^+ \) and \( \theta_{0,1}^+ < \frac{1}{2} \), it follows that \( d_L([\bar{\Sigma} x], [\bar{\Sigma} x]) < \frac{1}{2} d_L(x, \tilde{x}) \).
Thus, the contraction mapping theorem yields that \( \bar{\Sigma} \) has a fixed point \( x \) on \( L(x) \).

**Step II:** Now we prove that \( \mathcal{L}(\Gamma_k(r)) \subset \Gamma_k(r) \).
Since \( x \in \Gamma_k(r) \) and \( h^s \in \Gamma_k \), \( \mathcal{L}(h^s) \) is \( C_k \) in \( x_1 \). This is again a result that the composition of \( \mathcal{L}(h^s) \) functions is \( C_k \) functions. Differentiable \( \mathcal{L}(h^s) \) with respect to \( x_1 \), we have

\[
D_{x_1} \mathcal{L}(h^s)(t_1, x_1) = - \int_{t_1}^{\infty} T_-(t_1 - s) \left[ D_x G(s, x_1) D_{x_1} x(s, x_1) + D_h G(s, x_1) D_{x} h^s(s, x_1) D_{x_1} x(s, x_1) \right] ds,
\]

where we use the notation \( G(s, x_1), h^s(x, x_1) \). Choosing \( \hat{L}_{1,0}(r) \) such that \( \theta_{0,1}^- \leq \frac{\kappa (1 - \theta_{0,1}^+)}{1 + \kappa} M_{0}^+ \), it yields

\[
\|D_{x_1} \mathcal{L}(h^s)\| \leq \int_{t_1}^{\infty} M_{1}^- \left( s - t_1 \right)^{-\alpha} e^{-\beta_{-} \left( s - t_1 \right)} \hat{L}_{1,0}(r) (1 + \kappa) \frac{M_{0}^+}{1 - \theta_{0,1}^+} e^{\beta(s - t_1)} ds \leq \kappa.
\]
Besides, \( D_{x_1} \mathcal{L}(h^s)(t_1, 0) = 0 \).
Furthermore, for \( 2 \leq i \leq k \),

\[
D_{x_1}^i \mathcal{L}(h^s)(t_1, x_1) = - \int_{t_1}^{\infty} T_-(t_1 - s) \left[ D_x G(s, x_1) D_{x_1}^i x(s, x_1) + D_h G(s, x_1) D_{x} h^s(s, x_1) D_{x_1}^i x(s, x_1) + \hat{\mathcal{R}}_{i}(x_1) \right] ds,
\]

\( \hat{\mathcal{R}}_{i}(x_1) \) is a sum of monomials whose factors are derivatives of \( G \) and \( x \) up to order \( i - 1 \) and \( h^s \) up to \( i \). Note that all the terms in \( \hat{\mathcal{R}}_{i}(x_1) \) contain at least one factor which is a derivative of \( G \). Hence, assuming \( \hat{L}_{i,\gamma}(r) \) is sufficiently small, we can obtain \( \|D_{x_1}^i \mathcal{L}(h^s)\| \leq \kappa \).

In addition, for \( x_1, \tilde{x}_1 \in \overline{D(S_0)} \cap B_{r} X_0 \),

\[
D_{x_1}^k \mathcal{L}(h^s)(t_1, x_1) - D_{x_1}^k \mathcal{L}(h^s)(t_1, \tilde{x}_1) \leq \int_{t_1}^{t} T_+(t - s) \left[ D_x G(s, x_1) D_{x_1}^k x(s, x_1) - D_x G(s, \tilde{x}_1) D_{x_1}^k x(s, \tilde{x}_1) + D_h G(s, x_1) D_{x} h^s(s, x_1) D_{x_1}^k x(s, x_1) \right. \\
- D_h G(s, \tilde{x}_1) D_{x} h^s(s, \tilde{x}_1) D_{x_1}^k x(s, \tilde{x}_1) + \hat{\mathcal{R}}_{k}(x_1) - \hat{\mathcal{R}}_{k}(\tilde{x}_1)] ds
\]

Since each difference terms in the right side of (66) contain the factors \( D_{x_1} x(s, x_1) \) and \( D_{x_1} x(s, \tilde{x}_1) \), we use the triangle inequality to estimate (66) in the \( C^0 \) norm and assume \( \beta_{+} \leq (1 + \gamma) \beta \) and \( \hat{L}_{k,\gamma}(r) \) being sufficiently small, we obtain that \( H_{\gamma}(D_{x_1}^k \mathcal{L}(h^s)) \leq \kappa \). Thus, \( \mathcal{L}(\Gamma_k(r)) \subset \Gamma_k(r) \).

By Lemma 3.2, \( \Gamma_k(r) \) is a non-empty closed subset of \( \Gamma(r) \subset C^0(X_0, Y_0) \) in the \( C^0 \) norm, and since \( \mathcal{L} \) has a fixed point \( h^s \) in \( \Gamma(r) \), \( \mathcal{L}(\Gamma_k(r)) \subset \Gamma_k(r) \) implies that the fixed point \( h^s \) of \( \mathcal{L} \) also lies in \( \Gamma_k(r) \) and is therefore of class \( C^{k,\gamma} \). Thus, \( W_{\text{loc}}^{k,\gamma}(0) \) is the unique local stable integral manifold.

The proof is complete. \( \square \)

We are now in the position to prove Theorem 4.1.
Proof of Theorem 4.1. (1) From Lemma 4.2, Lemma 4.3, Lemma 4.4 and Lemma 4.6, for each $x_1 \in \mathcal{D}(S_+^\alpha) \cap B_r^{Y_\alpha}$ and $\|x_1\|_\alpha \leq r/(2M_0^\alpha)$, there is a unique point $\zeta = (x_1, h^s(t_1, x_1))$ such that system (4) has a unique infinitely long forward dichotomous solution $z(t; t_1, \zeta) \in C_\beta([t_1, \infty), Z)_{t_1}$ defined by

$$z(t) = T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s)F(s, x(s), h^s(x(s)))ds - \int_{t}^{\infty} T_-(t-s)G(s, x(s), h^s(x(s)))ds$$

(67)

for $t \geq t_1$, and $\lim_{t \to \infty} z(t; t_1, \zeta) = 0$. Then, Lemma 4.6, Lemma 4.7 and Lemma 4.8 imply that

$W^s_{loc}(0, r) = \{(t, x(t), y(t)) : y(t) = h^s(t, x(t)), t \in \mathbb{R}, x(t) \in B_r^{X_\alpha}\}$

contains $z(t; t_1, \zeta)$, and is the unique $C^{k, \gamma}$ local stable integral manifold of system (4), where $h^s$ is defined by (50). Moreover, $x(t)$ and $y(t)$ on $W^s_{loc}(0, r)$ have the form (18).

(2) In order to obtain a unique local unstable integral manifold for system (4), we begin by considering (8) written in the following form:

$$x(t) = e^{S_+(t-t_1)}x_1 + \int_{t_1}^{t} e^{S_+(t-s)}F(s, z(s))ds,$$

$$y(t) = e^{S_-(t-t_2)}y_2 - \int_{t}^{t_2} e^{S_-(t-s)}G(s, z(s))ds$$

(68)

for $-\infty < t_1 \leq t \leq t_2 < \infty$.

Let $\tau = t, \tau_1 = -t_2$ and $\tau_2 = -t_1$. Set $\bar{x}(\tau) := x(-\tau), \bar{y}(\tau) := y(-\tau)$ and $\bar{z}(\tau) := z(-\tau)$. Then $z(t) = (x(t), y(t)) : [t_1, t_2] \to Z_\alpha$ is the dichotomous solution of (68) if and only if $\bar{z}(\tau)(\bar{x}(\tau) + \bar{y}(\tau)) : [\tau_1, \tau_2] \to Z_\alpha$ is the dichotomous solution of the following system

$$\bar{x}(\tau) = e^{(-S_+)(\tau-\tau_2)}\bar{x}(\tau_2) - \int_{\tau}^{\tau_2} e^{(-S_+)(\tau-s)}(-F)(s, \bar{z}(s))ds,$$

$$\bar{y}(\tau) = e^{(-S_-)(\tau-\tau_1)}\bar{y}(\tau_1) + \int_{\tau_1}^{\tau} e^{(-S_-)(\tau-s)}(-G)(s, \bar{z}(s))ds,$$

(69)

for $-\infty < \tau_1 \leq \tau \leq \tau_2 < \infty$, where $(-F)(\tau, z) = -F(\tau, z)$ and $(-G)(\tau, z) = -G(\tau, z)$. By Theorem 4.1(1), system (69) exists a $C^{k, \gamma}$ unique local stable integral manifold $W^s_{loc}(0, r) = \{(\tau, \bar{y}(\tau), \bar{x}(\tau)) : \bar{x}(\tau) = h^s(\tau, \bar{y}(\tau)), \bar{y}(\tau) \in B_r^{Y_\alpha}\}$, where

$$h^s(\tau_1, \bar{y}(\tau_1)) = -\int_{\tau_1}^{\infty} e^{(-S_+)(\tau_1-s)}(-F)(s, h^s(s, \bar{y}(s)), \bar{y}(s))ds.$$

This implies that system (4) has a $C^{k, \gamma}$ unique local unstable integral manifold

$W^u_{loc}(0, r) = \{(t, x(t), y(t)) : x(t) = h^u(t, y(t)), t \in \mathbb{R}, y(t) \in B_r^{Y_\alpha}\}$, where

$$h^u(t_2, y_2) = \int_{-\infty}^{t_2} T_+(t_2-s)F(s, h^u(s, y(s)), y(s))ds.$$

Moreover, for each $y_2 \in \mathcal{D}(S_-^\alpha) \cap B_r^{Y_\alpha}$, there is a unique point $\zeta = (h^u(t_2, y_2), y_2)$ such that system (4) has a unique infinitely long backward dichotomous solution
compact inverse. Moreover, 
\[ z(t; t_2, \zeta) \in C_\beta((-\infty, t_2], B^Z_r) \] which defined by
\[ z(t) = T_-(t - t_2)y_2 - \int_{t_2}^{t} T_-(t - s)G(s, h^n(y(s)), y(s)) \, ds \]
\[ + \int_{-\infty}^{t} T_+(t - s)F(s, h^n(y(s)), y(s)) \, ds \]
for \( t \leq t_2 \), and \( \lim_{t \to -\infty} z(t; t_2, \zeta) = 0 \). In addition, \( x(t) \) and \( y(t) \) on \( W_{loc}^n(0, r) \) has the form \( \text{(19)} \).
The proof is complete. \( \square \)

5. Elliptic equations in infinite cylindrical domain. Consider the following elliptic equation with Dirichlet boundary condition on \( \partial \Omega \)
\[ u_{xx} + \Delta_y u + f(x, y, u, u_x, \nabla_y u) = 0, \quad (x, y, u) \in \mathbb{R} \times \Omega \times \mathbb{R}^m, \]
\[ u(x, y) = 0, \quad x \in \mathbb{R}, \quad y \in \partial \Omega \]
(70)
in infinite cylindrical domain \( \mathbb{R} \times \Omega \), where \( \Omega \) is an open and bounded subset of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( \nabla_y \) is the gradient in the \( y \)-variable and \( \Delta_y \) is the Laplace operator in the \( y \)-variable. The function \( (x, y, \varrho, \chi, \omega) \mapsto f(x, y, \varrho, \chi, \omega) \) is defined in \( \mathbb{R} \times \Omega \times \mathcal{V} \), and has value in \( \mathbb{R}^m \), where \( \mathcal{V} \subset \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{mn} \) is an open subset and it is also the higher order term of \((\varrho, \chi, \omega)\).
Assume that \( f \) is locally Hölder continuous with respect to \( x \), and locally Lipschitz continuous with respect to \((\varrho, \chi, \omega)\). Precisely, we assume that for arbitrary finite time interval \([t_1, t_2]\), there exist \( \gamma \in (0, 1], \ r > 0 \) and \( K_0 = K_0(r) > 0 \) such that
\[ \|f(\hat{x}, \hat{\varrho}, \hat{\chi}_1, \omega_1) - f(\bar{x}, \bar{\varrho}, \bar{\chi}_2, \omega_2)\|_{\mathbb{R}^m} \]
\[ \leq K_0(|\hat{x} - \bar{x}|_2^\gamma + |\hat{\varrho} - \bar{\varrho}|_{\mathbb{R}^m} + |\hat{\chi}_1 - \bar{\chi}_2|_{\mathbb{R}^m} + |\omega_1 - \omega_2|_{\mathbb{R}^{mn}}) \]
(71)
for \( \hat{x}, \bar{x} \in [x_1, x_2] \) and \((\varrho_1, \chi_1, \omega_1), (\varrho_2, \chi_2, \omega_2) \in B^Z_{r^{2m+mn}}\). Moreover, \( K_0 \to 0 \) as \( r \to 0 \).

Based on the idea from Kirchhoff [10] who considered (70) as an evolution equation by treating the unbounded spatial variable \( x \) as time variable. We first transfer the problem of elliptic equation (70) to the abstract semilinear problem.

Let \( A := -\Delta_y \). Then \( A \) is a closed operator on \( X := L^2(\Omega, \mathbb{R}^m) \) with dense domain \( X^1 := D(A) = H^2(\Omega, \mathbb{R}^m) \cap H^1_0(\Omega, \mathbb{R}^m) \), and it is positive, symmetric with compact inverse. Moreover, \( \sigma(A) = \{\lambda_n : n \in \mathbb{Z}_+\} \) is a discrete set such that \( \lambda_n \geq \lambda_{n-1} > 0 \) and \( \lambda_n \to \infty \) as \( n \to \infty \). Furthermore, the corresponding eigenfunctions \( \{e_n : n \in \mathbb{Z}_+\} \) of \( A \) can be chosen to form an orthonormal basis for \( X \), and in terms of this basis, the operator \( A \) can be represented by \( Au = \sum_{n=1}^{\infty} \lambda(u, e_n) e_n \), where \((\cdot, \cdot)\) is a inner product on \( X \). In particular, \( -A \) is sectorial on \( X \).

Note that \( X \) has an orthonormal basis of eigenfunctions of \( A \), by [21, Exercise 3.10, p87], for \( \alpha \in (0, 1] \), we have
\[ A^{-\alpha/2} u = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-At} u \, dt = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-\lambda t} dt \cdot u \]
\[ = \lambda^{-\alpha/2} \sum_{n=1}^{\infty} (u, e_n) e_n, \]
(72)
where \( \Gamma(x) = \int_{0}^{\infty} t^{-1} e^{-t} dt \). Thus, by (72), \( A^{\alpha/2} \) can be defined as \( A^{\alpha/2} := (A^{-\alpha/2})^{-1} \), that is,

\[
A^{\alpha/2} u = \sum_{n=1}^{\infty} \lambda^{\alpha/2}(u, e_n)e_n, \tag{73}
\]

and

\[ X^{\alpha/2} := \mathcal{D}(A^{\alpha/2}) = \{ u : \|A^{\alpha/2}u\|_X < +\infty \} \]

is a Hilbert space when endowed with the inner product

\[ (u_1, u_2)_{\alpha/2} := (A^{\alpha/2}u_1, A^{\alpha/2}u_2), \]

which gives rise to a corresponding norm \( \|u\|_{\mathcal{D}(A^{\alpha/2})} = \|A^{\alpha/2}u\|_X \).

Note that \( X^{\alpha_2/2} \subset X^{\alpha_1/2} \) if \( 0 < \alpha_1 < \alpha_2 < 1 \). In particular, \( \mathcal{D}(A^{1/2}) = H^1_0(\Omega) \), \( \sigma(A^{1/2}) = \{ \sqrt{\lambda_n} : n \in \mathbb{Z}_+ \} \) and \(-A^{1/2}\) is the infinitesimal generator of a strongly continuous and analytic semigroup \( \{ e^{-tA^{1/2}} \}_{t \geq 0} \) on \( X \).

The linear part of elliptic equation (70) has the following form

\[ -u_{xx} + Au = 0, \tag{74} \]

where \( u \) represents a function of \( x \in \mathbb{R} \) with values in \( X \). By utilizing the factorized method in [2, section 2], (88) can be rewritten as

\[ (-d/dx + A^{1/2})(d/dx + A^{1/2})u = 0. \tag{75} \]

Set \( Z := X \times X \) endowed with the norm \( \|z\|_Z = \max\{\|z_1\|_X, \|z_2\|_X \} \) for \( z_1 \in X, z_2 \in X \) and \( z = (z_1, z_2)^T \in Z \). Let \( v = u_x \) and

\[ p := -v + A^{1/2}u, \quad q := v + A^{1/2}u. \tag{76} \]

Then (86) can be transformed into the following abstract linear equation

\[ \frac{dz(x)}{dx} = Sz(x), \quad x \in \mathbb{R}, \quad z(x) \in Z, \tag{77} \]

where \( z = (p, q)^T \). Moreover, there exists a bounded and boundedly invertible transform

\[ \mathcal{T} : X^{1/2} \times X \rightarrow X \times X, \quad \mathcal{T} \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} p \\ q \end{array} \right), \quad \mathcal{T}^{-1} \left( \begin{array}{c} p \\ q \end{array} \right) = \left( \begin{array}{c} u \\ v \end{array} \right), \]

where

\[ \mathcal{T} = \begin{pmatrix} A^{1/2} & -I \\ A^{1/2} & I \end{pmatrix}, \quad \mathcal{T}^{-1} = \frac{1}{2} \begin{pmatrix} A^{-1/2} & A^{-1/2} \\ -I & I \end{pmatrix}. \]

Note that \( u \in X^{1/2} \) and \( v \in X \) signify \( u(x, \cdot) \in X^{1/2} \) and \( v(x, \cdot) \in X \). Besides,

\[ S = \begin{pmatrix} -A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \tag{78} \]

with \( \mathcal{D}(S) = X^{1/2} \times X^{1/2} \). Obviously, \( \overline{\mathcal{D}(S)} = Z \) and \( \sigma(S) = \sigma(-A^{1/2}) \cup \sigma(A^{1/2}) \). Thus, \( S \) is a densely defined and hyperbolic bisectorial operator on \( Z \). Furthermore, we have the following lemma.

**Lemma 5.1.** The operator \( S \) in (77) is sectorially dichotomous on \( Z \).
Proof. For \( z \in \mathbb{Z} \), there exist two bounded and complementary projections \( P_+: (p, q)^T \mapsto (p, 0)^T \) and \( P_-: (p, q)^T \mapsto (0, q)^T \) (we refer to [5, Appendix A.7]), such that

\[
\mathcal{Z} = \mathcal{Z}_+ \oplus \mathcal{Z}_-,
\]

where \( \mathcal{Z}_+ = \{ z \in \mathbb{Z} : q = 0 \} = X \times \{ 0 \} \) and \( \mathcal{Z}_- = \{ z \in \mathbb{Z} : p = 0 \} = \{ 0 \} \times X \).

Since \( \mathcal{D}(S) = [\mathcal{D}(S) \cap \mathcal{Z}_+] \oplus [\mathcal{D}(S) \cap \mathcal{Z}_-] \), \( S \) maps \( \mathcal{D}(S) \cap \mathcal{Z}_+ \) and \( \mathcal{D}(S) \cap \mathcal{Z}_- \) into \( \mathcal{Z}_+ \) and \( \mathcal{Z}_- \) respectively, then \( \mathcal{Z}_+ \) and \( \mathcal{Z}_- \) are \( S \)-invariant.

By decomposition of \( \mathcal{Z} \), \( S \) can be reduced to the block matrix representation

\[
S = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix},
\]

where

\[
S_+ := S|_{\mathcal{Z}_+} = \begin{pmatrix} -A^{1/2} & 0 \\ 0 & 0 \end{pmatrix}, \quad S_- := S|_{\mathcal{Z}_-} = \begin{pmatrix} 0 & 0 \\ 0 & A^{1/2} \end{pmatrix}.
\]

Obviously, \( \mathcal{D}(S_+) = \mathcal{D}(S) \cap \mathcal{Z}_+ \), \( \mathcal{D}(S_-) = \mathcal{D}(S) \cap \mathcal{Z}_- \). In particular,

\[
\mathcal{D}(S_+) = X^{1/2} \times \{ 0 \}, \quad \mathcal{D}(S_-) = \{ 0 \} \times X^{1/2}.
\]

Moreover, \( \sigma(S) = \sigma(S_+) \cup \sigma(S_-) \), \( \sigma(S_+) = \sigma(-A^{1/2}) \) and \( \sigma(S_-) = \sigma(A^{1/2}) \).

Furthermore, since \( S_+ \) and \( -S_- \) are sectorial operators on \( \mathcal{Z}_+ \) and \( \mathcal{Z}_- \), respectively,

\[
\mathcal{D}(S_+ \mathcal{Z}_+) = \mathcal{Z}_+, \quad \mathcal{D}(S_- \mathcal{Z}_-) = \mathcal{Z}_-.
\]

\( S_+ \) generates strongly continuous and analytic semigroup

\[
\left\{ \begin{pmatrix} e^{-A^{1/2}t} & 0 \\ 0 & 0 \end{pmatrix} \right\}_{t \geq 0}
\]
on \( \mathcal{Z}_+ \) and \( -S_- \) generates strongly continuous and analytic semigroup

\[
\left\{ \begin{pmatrix} 0 & 0 \\ 0 & e^{-A^{1/2}t} \end{pmatrix} \right\}_{t \geq 0}
\]
on \( \mathcal{Z}_- \), both are uniformly exponentially stable. Hence, \( S \) is sectorially dichotomous on \( \mathcal{Z} \). \( \square \)

Based on the Lemma \( 5.1 \), we can construct the fractional power \( S^\alpha \) and obtain its domain. The property of \( A \) follows that sectorial operator \( -S_+ \) is also positive, symmetric with compact inverse, and the corresponding eigenfunctions \( \{ \tilde{e}_n : n \in \mathbb{Z}_+ \} \) of \( -S_+ \) can be chosen to form an orthonormal basis for \( X^{1/2} \times \{ 0 \} \). In terms of this basis, \( \tilde{u} = \sum_{n=1}^\infty (\tilde{u}, \tilde{e}_n)\tilde{e}_n \) for \( \tilde{u} \in X^{1/2} \times \{ 0 \} \), and the operator \( -S_+ \) can be represented by \( -S_+ \tilde{u} = \sum_{n=1}^\infty \lambda(\tilde{u}, \tilde{e}_n)\tilde{e}_n \). Similarly as (72) and (73), we can define fractional power \( (-S_+)^\alpha \) as

\[
(-S_+)^\alpha \tilde{u} = \sum_{n=1}^\infty \lambda^{\alpha/2}(\tilde{u}, \tilde{e}_n)\tilde{e}_n.
\]

By (73), we can rewrite \( (-S_+)^\alpha \) as \( \begin{pmatrix} A^{\alpha/2} & 0 \\ 0 & 0 \end{pmatrix} \) \begin{pmatrix} 0 & 0 \\ 0 & A^{\alpha/2} \end{pmatrix} \) can be defined in the same way above. From above, we can define the fractional power

\[
S^\alpha := \begin{pmatrix} (-S_+)^\alpha & 0 \\ 0 & (-S_-)^\alpha \end{pmatrix} \quad (79)
\]
for \( \alpha \in (0, 1) \), along with \( \mathcal{D}(S^\alpha) = \mathcal{D}((-S_+)^\alpha) \oplus \mathcal{D}((-S_-)^\alpha) = X^{\alpha/2} \times X^{\alpha/2} \), \( \mathcal{D}((-S_+)^\alpha) = X^{\alpha/2} \times \{0\} \) and \( \mathcal{D}((-S_-)^\alpha) = \{0\} \times X^{\alpha/2} \). Moreover, \( \mathcal{D}(S_+) \subset \mathcal{D}((-S_+)^\alpha) \subset \mathcal{Z}_+ \) and \( \mathcal{D}(S_-) \subset \mathcal{D}((-S_-)^\alpha) \subset \mathcal{Z}_- \). Therefore, we can define

\[
Z_\alpha := \mathcal{D}(S^\alpha) \text{ with the norm } \|z\|_\alpha = \|S^\alpha z\|_\mathcal{Z}
\]

for \( \alpha \in (0, 1) \).

Let \( \hat{\mathcal{O}} \) be the open subset in \( Z_\alpha \) consisting of the functions \( u \) and \( v \) such that the range of \((u,v)\) is contained in \( \mathcal{V} \). From (71), we set the function

\[
\tilde{f} : \mathbb{R} \times \hat{\mathcal{O}} \to \mathcal{X}, \quad \tilde{f}(x,u,v)(y) = f(x,y,u(y),v(y),\nabla_y u(y)),
\]

so \( \tilde{f} \) is continuous. By (71), the boundedness of \( \Omega \) and Minkowski inequality, there exists \( \tilde{K}_0 > 0 \) depends on \( K_0 \) such that

\[
\|\tilde{f}(\widehat{x}, \widehat{u}, \widehat{v}) - \tilde{f}(\widehat{x}, u, v)\|_\mathcal{X} \\
\leq \tilde{K}_0(\|\widehat{x} - x\|_\mathcal{X} + \|\widehat{u} - u\|_{X^{\alpha/2}} + \|\widehat{v} - v\|_{X^{\alpha/2}} + \|\nabla_y \widehat{u} - \nabla_y u\|_{X^{\alpha/2}})
\]

for \((\widehat{x}, \widehat{u}, \widehat{v}), (x,u,v) \in [x_1,x_2] \times B_{\mathcal{Z}^\alpha}^\alpha \).

Combining with (77), the equation (70) on \( Z \) can be written as the following abstract semilinear equation

\[
\frac{dz}{dx} = Sz + H(x,z), \quad x \in \mathbb{R}, \quad z(x) \in \mathcal{Z},
\]

where the map \( H : \mathbb{R} \times \hat{\mathcal{O}} \to \mathcal{Z} \) has the form \( H(x,z) = \left( \tilde{f}(x,u,v), u \right) = \mathcal{T}^{-1}(p, z) = (p, q)^T \). By (80), there exists a \( \tilde{K}_0 > 0 \) which depends on \( \hat{K}_0 \) such that

\[
\|H(\widehat{x}, \widehat{z}) - H(x, z)\|_\mathcal{Z} \leq \tilde{K}_0(\|\widehat{x} - x\|_\mathcal{X} + \|\widehat{z} - z\|_\alpha)
\]

for \((\widehat{x}, \widehat{z}), (x,z) \in [t_1,t_2] \times B_{\mathcal{Z}^\alpha}^\alpha \), and \( H(x,0) = 0 \).

We now state the following results for (70) as follows.

**Theorem 5.2.** Assume that system (70) satisfies the condition (71) with a sufficiently small \( K_0 \). Then,

(i) for each \([x_1,x_2]\) and \((u(x_1,\cdot), u_x(x_1,\cdot)), (u(x_2,\cdot), u_x(x_2,\cdot)) \in H_0^1(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m)\) such that \( p(x_1), q(x_2) \in B_{\mathcal{X}^\alpha} \), the system (70) has a unique solution \( u(x, y) : [x_1,x_2] \times \Omega \to \mathbb{R}^m \), such that \( u, u_x \in C([x_1,x_2] \times \Omega, \mathbb{R}^m) \) and \( u_{xx} \in C((x_1,x_2) \times \Omega, \mathbb{R}^m) \), where \( p \) and \( q \) refer to (76).

(ii) There is a unique infinite dimensional \( C^{0,1} \) local stable integral manifold \( W^s \) in \( H_0^1(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m) \) such that any solution of (70) with initial condition \((u, u_x) \in W^s\) satisfies \( \|u(x, \cdot)\|_{H_0^1(\Omega, \mathbb{R}^m)} \to 0 \) and \( \|u_x(x, \cdot)\|_{L^2(\Omega, \mathbb{R}^m)} \to 0 \) as \( x \to \infty \).

(iii) There is a unique infinite dimensional \( C^{0,1} \) local unstable integral manifold \( W^u \) in \( H_0^1(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m) \) such that any solution of (70) with initial condition \((u, u_x) \in W^u\) satisfies \( \|u(x, \cdot)\|_{H_0^1(\Omega, \mathbb{R}^m)} \to 0 \) and \( \|u_x(x, \cdot)\|_{L^2(\Omega, \mathbb{R}^m)} \to 0 \) as \( x \to -\infty \).
Proof. Let \( \bar{p} = (p, 0)^T \) and \( \bar{q} = (0, q)^T \). Based on the discussion above, equation (5.3) can be transformed into the following semilinear ill-posed system:

\[
\begin{aligned}
\frac{d}{dx} \bar{p} &= S_+ \bar{p} + P_+ H(x, z), \\
\frac{d}{dx} \bar{q} &= S_- \bar{q} + P_- H(x, z),
\end{aligned}
\]  

(83)

where \( z = \bar{p} + \bar{q} \in Z_+^p \oplus Z_-^q = Z, \bar{p} \in Z_+^p \) and \( \bar{q} \in Z_-^q \). Under the hypotheses for \( f \) in (71), \( P_\pm H(x, z) \) in (83) satisfies the hypothesis 3.6(i).

(i) For each \([x_1, x_2]\), by (76), we take

\[
(u(x_1, \cdot), u_x(x_1, \cdot), u(x_2, \cdot), u_x(x_2, \cdot)) \in H_0^1(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m)
\]

in (70) such that \( p(x_1), q(x_2) \in B_{\infty}^\alpha/2 \), which is equivalent to the choice of the dichotomous initial condition

\[
(\bar{p}(x_1), \bar{q}(x_2)) := \{(p(x_1), 0)^T, (0, q(x_2))^T\} \in B_{\infty}^\alpha
\]  

(84)

on \([x_1, x_2]\) for (83). Note that the closure of \( D(S) \) in \( Z_\alpha \) is \( Z_\alpha \). Hence, Lemma 4.2 yields that, system (83) with the above dichotomous initial condition (84) has a unique dichotomous solution \( z(x) \) on \( Z_\alpha \), \( x \in [x_1, x_2] \) such that \( p, q \in C([t_1, t_2], X^\alpha/2) \) and \( p, q \in C^1([t_1, t_2], X^\alpha) \). It follows that elliptic equation (70) has a local solution \( u(x, y) : [x_1, x_2] \times \Omega \rightarrow \mathbb{R}^m \) such that \( u, u_x \in C([x_1, x_2] \times \Omega, \mathbb{R}^m) \) and \( u_{xx} \in C([x_1, x_2] \times \Omega, \mathbb{R}^m) \).

(ii) From Lemma 4.6, under the condition (84), it follows that system (83) has an infinite dimensional \( C^{0,1} \) local stable integral manifold \( W^s_{loc}(0) \) given by the graph of a \( C^{0,1} \) map

\[
h^s : \mathbb{R} \times B_{\infty}^\alpha \rightarrow B_{\infty}^\alpha, \quad \bar{q} = h^s(x, \bar{p}),
\]  

(85)

where \( \mathcal{X}_\alpha = X^\alpha/2 \times \{0\} \) and \( \mathcal{Y}_\alpha = \{0\} \times X^\alpha/2 \). Besides, we take \( \mathcal{K}_0 \) so small such that \( \|h^s\|_{0,1} < 1 \). In fact, the map \( h^s \) can be viewed as a local map from \( \mathbb{R} \times X^\alpha/2 \) to \( X^\alpha/2 \), and \( q = h^s(x, p) \).

Corresponding to \((u, u_x)\)-coordinates, \( \mathcal{X}_\alpha \) and \( \mathcal{Y}_\alpha \) can be written as

\[
\mathcal{X}_\alpha = \{(2A^{1/2}u, 0), u \in X^{1/2}\}, \quad \mathcal{Y}_\alpha = \{(0, 2A^{1/2}u), u \in X^{1/2}\}.
\]  

(86)

Note that the above statement (i) follows that \( A^{1/2}u \in X^{1/2} \subset X \) in (86). Thus, \( h^s \) can be represented by a local \( C^{0,1} \) map

\[
h^s : \mathbb{R} \times X^{1/2} \rightarrow X^{1/2}, \quad h^s(x, u) = (2A^{1/2})^{-1}h^s(x, 2A^{1/2}u).
\]  

(87)

Moreover, by (87) and direct calculation, we have

\[
\|h^s(x, u_1) - h^s(x, u_2)\|_{X^{1/2}} \leq \|h^s\|_{0,1} \|u_1 - u_2\|_{X^{1/2}}
\]

for \( u_1, u_2 \in X^{1/2} \cap B_r(0_{\mathbb{R}^m}, 0_{\mathbb{R}^m}, 0_{\mathbb{R}^m}) \), it implies that \( \|h^s\|_{0,1} \leq \|h^s\|_{0,1} < 1 \).

Let \( z_0 = (p_0, h^s(x_0, p_0)) \), \( p_0 = 2A^{1/2}u_0 \), be a point on the stable manifold of system (83) as given by (85). By \( \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A^{-1/2} & A^{-1/2} \\ -I & I \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \), and (87), \( z_0 \) in the \((u, u_x)\)-coordinates is given by

\[
u = u_0 + \tilde{h}^s(x_0, u_0), \quad u_x = A^{1/2}(-u_0 + \tilde{h}^s(x_0, u_0)).
\]  

(88)

Then the Lipschitz inverse function theorem follows that there exists a local \( C^{0,1} \) map \( \tilde{h}^s \triangleq (I + \hat{h}^s)^{-1} : \mathbb{R} \times X^{1/2} \rightarrow X^{1/2} \) such that \( u_0 = \tilde{h}^s(x, u) \).
Since the composition of \( C^{0,1} \) functions is a \( C^{0,1} \) function, there exists a local \( C^{0,1} \) map \( \tilde{h} \) such that

\[
\tilde{h}^s : \mathbb{R} \times X^{1/2} \to X, \quad u_x = \tilde{h}^s(x, u) = A^{1/2}(-\tilde{h}^s(x, u) + \tilde{h}^s(\tilde{h}^s(x, u))).
\]  

(89)

Hence, system (70) has an infinite dimensional \( C^{0,1} \) local stable integral manifold \( \mathcal{W}^s = \{(u, \tilde{h}^s(x, u)) \mid x \in \mathbb{R}, u \in H^1_0(\Omega, \mathbb{R}^m) \} \) in \( H^1_0(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m) \), where \( \tilde{h}^s \) refers to (89). Moreover, any solution of (70) with initial condition \((u, u_x) \in \mathcal{W}^s \) satisfies \( \|u(x, \cdot)\|_{H^1_0(\Omega, \mathbb{R}^m)} \to 0 \) and \( \|u_x(x, \cdot)\|_{L^2(\Omega, \mathbb{R}^m)} \to 0 \) as \( x \to \infty \).

(iii) The assertions about the \( C^{0,1} \) local unstable integral manifold follows from those about the \( C^{0,1} \) local stable integral manifolds by reversing the direction of “time” variable \( x \).

The proof is complete. \( \square \)

**Remark 2.** (1) In contrast to the Appendix A in ElBialy [6], under the same Lipschitz condition for the state variables of nonlinear term of elliptic equation in infinite cylinders, we obtain the continuous differentiability of \( u_x \) additionally.

(2) If \( f \) is \( C^{k,\gamma} \) with respect to \((u, u_x, \nabla_y u)\) for \( k \geq 1 \) and \( \gamma \in [0, 1] \), with the same arguments, there is an unique infinite dimensional \( C^{k,\gamma} \) local stable/unstable integral manifold \( \mathcal{W}^s(\mathcal{W}^u) \) in \( H^1_0(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m) \) such that any solution of (70) with initial condition \((u, u_x) \in \mathcal{W}^s(\mathcal{W}^u) \) satisfies \( \|u(x, \cdot)\|_{H^1_0(\Omega, \mathbb{R}^m)} \to 0 \) and \( \|u_x(x, \cdot)\|_{L^2(\Omega, \mathbb{R}^m)} \to 0 \) as \( x \to \infty \).

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