Hilbert space representations of cross product algebras II

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Abstract

In this paper, we study and classify Hilbert space representations of cross product $\ast$-algebras of the quantized enveloping algebra $U_q(e_2)$ with the coordinate algebras $\mathcal{O}(E_q(2))$ of the quantum motion group and $\mathcal{O}(\mathbb{C}_q)$ of the complex plane, and of the quantized enveloping algebra $U_q(su_{1,1})$ with the coordinate algebras $\mathcal{O}(SU_q(1,1))$ of the quantum group $SU_q(1,1)$ and $\mathcal{O}(U_q)$ of the quantum disc. Invariant positive functionals and the corresponding Heisenberg representations are explicitly described.

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1 Introduction

This is our second paper on $\ast$-representations of cross product $\ast$-algebras. While the first paper [13] deals mainly with coordinate algebras of compact quantum spaces, the present one is concerned with coordinate algebras of non-compact quantum spaces. We treat various cross product $\ast$-algebras related to the quantum motion group $E_q(2)$ and to the quantum group $SU_q(1,1)$. More precisely, we study cross product algebras of the Hopf $\ast$-algebra $U_q(e_2)$ with the coordinate algebras $\mathcal{O}(\mathbb{C}_q)$ of the quantum complex plane $\mathbb{C}_q$ and $\mathcal{O}(E_q(2))$ of the quantum group $E_q(2)$, and of the Hopf $\ast$-algebra $U_q(su_{1,1})$ with the coordinate algebras $\mathcal{O}(U_q)$ of the quantum disc $U_q$ and $\mathcal{O}(SU_q(1,1))$ of the quantum group $SU_q(1,1)$. 
The purpose of this paper is to study well-behaved Hilbert space representations of the cross product $\ast$-algebras, invariant positive functionals and Heisenberg representations. Our main aim is to describe representations and invariant positive functionals by explicit formulas in terms of generators and functions of them.

The quantum spaces considered in this paper are non-compact. Hence $\ast$-representations of their coordinate algebras involve unbounded operators. Thus, for $\ast$-representations of the corresponding cross product algebras, elements of quantized enveloping algebras and of coordinate algebras act by unbounded operators. During the classification and derivation of representations, we occasionally add regularity conditions concerning the unbounded operators in order to exclude pathological behaviour. In other words, we classify only “well-behaved” $\ast$-representations fulfilling these additional assumptions. In this paper, we use mainly the following regularity assumptions: For an algebraic relation $AB = BA$ with two hermitian elements $A$ and $B$ of a $\ast$-algebra, we assume that the corresponding Hilbert space operators are essentially self-adjoint and that their closures strongly commute (that is, their spectral projections mutually commute). If we have an algebraic relation $AB = pAB$ with $p$ real and $A$ hermitian, then we assume that $A$ is represented in the Hilbert space by an essentially self-adjoint operator and $\phi(\bar{A})B \subset B\phi(p\bar{A})$ for all $\phi \in L^\infty(\mathbb{R})$. If this is fulfilled, we say that the operator relation $AB = pBA$ holds in strong sense. As usual, $\bar{A}$ denotes the closure of an operator $A$. Finally, given a relation $N^*N = NN^*$ in a $\ast$-algebra, we require that $\bar{N}$ is normal when considered as a Hilbert space operator.

Our paper is organized as follows. In Section 2, we collect some preliminaries which are needed later. These are basic definitions and facts on cross product algebras, the definition of Heisenberg representations for non-unital $\ast$-algebras and two operator-theoretic lemmas. Representations of cross product algebras of $U_q(e_2)$ and $U_q(su_{1,1})$ are investigated in Sections 3 and 4, respectively. After developing some useful algebraic properties of cross product algebras, all $\ast$-representations satisfying our regularity assumptions are classified in terms of the actions of generators. At the end of both sections, invariant positive functionals for the corresponding quantum spaces and the Heisenberg representations are explicitly described. Since these quantum spaces are non-compact, invariant positive functionals are not finite on coordinate algebras. Hence we extend the actions of $U_q(e_2)$ and $U_q(su_{1,1})$ to larger function algebras $\mathcal{F}(\mathcal{X})$, where $\mathcal{X} = E_q(2)$, $\mathbb{C}_q$, $SU_q(1, 1)$, $U_q$, such that the invariant positive functionals are finite on appropriate subalgebras $\mathcal{F}_0(\mathcal{X})$. As a byproduct, we obtain explicit formulas for the actions of generators of $U_q(e_2)$ and $U_q(su_{1,1})$ on general elements of coordinate and function algebras.
For background information on these quantum spaces, we refer to [11, 15, 18] for SU\(_q(1, 1)\), [6, 8, 17, 18] for E\(_q(2)\) and [4, 14] for the quantum disc. Invariant functionals on the corresponding quantum spaces appear in [6, 8, 14, 15]. A cross product algebra related to a differential calculus on \(\mathcal{O}(\mathbb{C}_q)\), its representations and an invariant functional for the quantum complex plane \(\mathbb{C}_q\) were studied in [1].

In the reminder of this section, we set up some notation and terminology. By a ∗-representation of a ∗-algebra \(\mathcal{X}\), we mean a homomorphism \(\pi\) of \(\mathcal{X}\) into the algebra of endomorphisms of a dense linear subspace \(\mathcal{D}\) of a Hilbert space \(\mathcal{H}\) such that \(\langle \pi(x)\eta, \zeta \rangle = \langle \eta, \pi(x^*)\zeta \rangle\) for all \(\eta, \zeta \in \mathcal{D}\) and \(x \in \mathcal{X}\). Here \(\langle \cdot, \cdot \rangle\) denotes the scalar product of \(\mathcal{H}\). In order to simplify and shorten the notation, we shall drop the symbol \(\pi\) and use the same letter, say \(x\), for an element \(x\) of the abstract ∗-algebra and the corresponding Hilbert space operator \(\pi(x)\) under the representation \(\pi\). This should cause no confusion. By another abuse of notation, we occasionally denote an operator and its closure by the same symbol. Throughout we shall describe ∗-representations by formulas for the actions of algebra generators. In all cases, an invariant dense domain \(\mathcal{D}\) is easily constructed (for instance, by taking the linear span of base vectors).

For a self-adjoint operator \(T\), the notation \(\sigma(T) \subseteq (a, b)\) means that the spectrum \(\sigma(T)\) of \(T\) is contained in \([a, b]\) and that \(a\) is not an eigenvalue. The notations \(\sigma(T) \subseteq [a, b]\) and \(T > 0\) have a similar meaning. Let \(I\) be an index set, \(\mathcal{K}\) be a Hilbert space and \(\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i\), where each \(\mathcal{H}_i\) is the same Hilbert space \(\mathcal{K}\). With \(\eta\) in \(\mathcal{K}\) and \(j\) an index, \(\eta_j\) denotes the vector in \(\mathcal{H}\) which has \(\eta\) in the \(j\)th position and zero elsewhere. We set \(\eta_j = 0\) if \(j \notin I\).

Unless stated otherwise, \(q\) stands for a real number belonging to the open interval \((0, 1)\). We abbreviate \(\lambda := q - q^{-1}\) and

\[
\lambda_n := (1 - q^{2n})^{1/2}, \quad \alpha_k(A) := (1 + q^{2k}A^2)^{1/2}, \quad \beta_k(A) := (1 + q^{-2k}A^{-2})^{1/2},
\]

where \(n \in \mathbb{N}_0, k \in \mathbb{Z}\) and \(A\) is a self-adjoint operator.

2 Preliminaries

2.1 Basics of cross product algebras

This subsection reviews basic definitions and facts on cross product algebras. For further details, see [13]. We shall denote the comultiplication, the counit and the antipode of a Hopf algebra by \(\Delta, \varepsilon\) and \(S\), respectively, and use the Sweedler notations \(\varphi(x) = x_{(1)} \otimes x_{(2)}\) for a coaction \(\varphi\) and \(\Delta(x) = x_{(1)} \otimes x_{(2)}\).
Let $\mathcal{U}$ be a Hopf $\ast$-algebra. A $\ast$-algebra $\mathcal{X}$ is called a right $\mathcal{U}$-module $\ast$-algebra if $\mathcal{X}$ is a right $\mathcal{U}$-module with action $\triangleleft$ satisfying

$$(xy)\triangleleft f = (x\triangleleft f_1)(y\triangleleft f_2), \quad (x\triangleleft f)^* = x^*\triangleleft S(f)^*, \quad x, y \in X, \ f \in \mathcal{U}.$$ 

For an algebra $\mathcal{X}$ with unit 1, we additionally require $1\triangleleft f = \varepsilon(f)1$ for $f \in \mathcal{U}$.

The right cross product algebra $\mathcal{U} \ltimes \mathcal{X}$ is the linear space $\mathcal{U} \otimes \mathcal{X}$ equipped with product and involution defined by

$$(g \otimes x)(f \otimes y) = gf_1 \otimes (x\triangleleft f_2)y, \quad (f \otimes x)^* = f^*_1 \otimes (x^*\triangleleft f^*_2), \quad (1)$$

where $x, y \in \mathcal{X}$ and $g, f \in \mathcal{U}$. Let $\mathcal{U}_0$ be a $\ast$-subalgebra of $\mathcal{U}$ which is a right coideal of the Hopf algebra $\mathcal{U}$ (that is, $\Delta(\mathcal{U}_0) \subseteq \mathcal{U}_0 \otimes \mathcal{U}$). From (1), it follows that the linear subspace $\mathcal{U}_0 \otimes \mathcal{X}$ of $\mathcal{U} \otimes \mathcal{X}$ is a $\ast$-subalgebra of $\mathcal{U} \ltimes \mathcal{X}$. We shall denote this subalgebra by $\mathcal{U}_0 \ltimes \mathcal{X}$.

If $\mathcal{X}$ and $\mathcal{U}_0$ have unit elements, we can consider $\mathcal{X}$ and $\mathcal{U}_0$ as subalgebras of $\mathcal{U}_0 \ltimes \mathcal{X}$ by identifying $f \otimes 1$ and $1 \otimes x$ with $f$ and $x$, respectively. In this way, $\mathcal{U}_0 \ltimes \mathcal{X}$ can be viewed as the $\ast$-algebra generated by the two subalgebras $\mathcal{U}_0$ and $\mathcal{X}$ with respect to the cross commutation relations

$$xf = f_1(x\triangleleft f_2), \quad x \in \mathcal{X}, \ f \in \mathcal{U}_0. \quad (2)$$

Let $\mathcal{A}$ be a Hopf $\ast$-algebra, $\langle \cdot, \cdot \rangle$ a dual pairing of Hopf $\ast$-algebras $\mathcal{U}$ and $\mathcal{A}$, and $\mathcal{X}$ a left $\mathcal{A}$-comodule $\ast$-algebra. Then $\mathcal{X}$ becomes a right $\mathcal{U}$-module $\ast$-algebra with right action $\triangleleft$ given by

$$x\triangleleft f = \langle f, x_1 \rangle x_2, \quad x \in \mathcal{X}, \ f \in \mathcal{U}. \quad (3)$$

In this case, Equation (2) reads

$$xf = f_1(x\triangleleft f_2), \quad x \in \mathcal{X}, \ f \in \mathcal{U}. \quad (4)$$

The above definitions have their left-handed counterparts. Suppose that $\mathcal{X}$ is a left $\mathcal{U}$-module $\ast$-algebra, that is, $\mathcal{X}$ is a left $\mathcal{U}$-module with action $\triangleright$ satisfying

$$f\triangleright(xy) = (f_1\triangleright x)(f_2\triangleright y), \quad (f\triangleright x)^* = S(f)^*\triangleright x^*, \quad x, y \in X, \ f \in \mathcal{U},$$

and $f\triangleright 1 = \varepsilon(f)1$ for $f \in \mathcal{U}$ if $\mathcal{X}$ has a unit 1. Then the vector space $\mathcal{X} \otimes \mathcal{U}$ is a $\ast$-algebra, called left cross product algebra and denoted by $\mathcal{X} \rtimes \mathcal{U}$, with product and involution defined by

$$(y \otimes f)(x \otimes g) = y(f_1\triangleright x) \otimes f_2 g, \quad (x \otimes f)^* = (f^*_1\triangleright x^*) \otimes f^*_2, \quad x, y \in \mathcal{X}, \ f, g \in \mathcal{U}. \quad (5)$$
When $\mathcal{X}$ has a unit, $\mathcal{X} \rtimes \mathcal{U}$ can be considered as the $*$-algebra generated by the subalgebras $\mathcal{X}$ and $\mathcal{U}$ with cross relations
\[
fx = (f^{(1)} \triangleright x)f^{(2)}, \quad x \in \mathcal{X}, \; f \in \mathcal{U}.
\] (5)

If $\mathcal{X}$ is a right comodule $*$-algebra of a Hopf $*$-algebra $\mathcal{A}$ and $\langle \cdot, \cdot \rangle$ is a dual pairing of Hopf $*$-algebras $\mathcal{A}$ and $\mathcal{U}$, then $\mathcal{X}$ is a left $\mathcal{U}$-module $*$-algebra with left action $\triangleright$ given by
\[
f \triangleright x = x^{(1)}\langle f, x^{(2)} \rangle, \quad x \in \mathcal{X}, \; f \in \mathcal{U},
\] (6) and Equation (5) can be written
\[
f x = x^{(1)}\langle f^{(1)}, x^{(2)} \rangle f^{(2)}, \quad x \in \mathcal{X}, \; f \in \mathcal{U}.
\] (7)

In many cases, it suffices to consider the right-handed version only. The following simple lemma shows how one can pass under certain conditions from a left to a right action.

**Lemma 2.1** Let $\mathcal{U}$ be a Hopf $*$-algebra and $\mathcal{X}$ a left $\mathcal{U}$-module $*$-algebra with left action $\triangleright$. Suppose $\phi : \mathcal{U} \rightarrow \mathcal{U}$ is an algebra anti-automorphism and a coalgebra homomorphism, that is, $\phi$ is a bijective linear map satisfying $\phi(fg) = \phi(g)\phi(f)$ and $\Delta \circ \phi(f) = (\phi \otimes \phi) \circ \Delta(f)$ for all $f, g \in \mathcal{U}$. Assume that $\ast \circ S \circ \phi = \phi \circ \ast \circ S$. Then the formula
\[
x \triangleright f := \phi(f) \triangleright x, \quad f \in \mathcal{U}, \; x \in \mathcal{X},
\] (8) defines a right $\mathcal{U}$-action on $\mathcal{X}$ which turns $\mathcal{X}$ into a right $\mathcal{U}$-module $*$-algebra.

The idea of the next lemma is taken from the paper [3].

**Lemma 2.2** Let $\mathcal{U} \rtimes \mathcal{X}$ be a right cross product algebra. Let $\mathcal{V}$ be a right coideal of $\mathcal{U}$ and let $\mathcal{X}_0$ be a set of generators of the algebra $\mathcal{X}$. Suppose that there exists a linear mapping $\rho : \mathcal{V} \rightarrow \mathcal{X}$ such that
\[
x \rho(v) = \rho(v^{(1)})(x^sv^{(2)})
\] (9) for $x \in \mathcal{X}_0$ and $v \in \mathcal{V}$. Then, for each $v \in \mathcal{V}$, the element $\xi(v) := \rho(v^{(1)})S(v^{(2)})$ commutes with the algebra $\mathcal{X}$ in $\mathcal{U} \rtimes \mathcal{X}$.

**Proof.** First observe that if (9) holds for $x$ and $y$ in $\mathcal{X}$, then it holds also for the product $xy$. Thus, we can assume that $\mathcal{X}_0 = \mathcal{X}$. Let $x \in \mathcal{X}$ and $v \in \mathcal{V}$. Using Equations (2) and (9), we compute
\[
x \xi(v) = x \rho(v^{(1)})S(v^{(2)}) = \rho(v^{(1)})(x^sv^{(2)})S(v^{(3)}) \\
= \rho(v^{(1)})S(v^{(4)})(x^sv^{(2)})S(v^{(3)}) = \rho(v^{(1)})S(v^{(2)})(x^s1) = \xi(v)x.
\]

5
2.2 Heisenberg representations

Let $\mathcal{U}$ be a Hopf $\ast$-algebra and let $\mathcal{X}_1$ and $\mathcal{X}_0$ be right $\mathcal{U}$-module $\ast$-algebras such that $\mathcal{X}_0$ is a left $\mathcal{X}_1$-module satisfying

\[(a.x)^\ast y = x^\ast (a^\ast y), \quad (a.y)^\ast f = (a^\ast f(1)).(y^\ast f(2))\]  \hspace{1cm} (10)

for $x, y \in \mathcal{X}_0$, $a \in \mathcal{X}_1$ and $f \in \mathcal{U}$. Here, $a.x$ stands for the left action of $a \in \mathcal{X}_1$ on $x \in \mathcal{X}_0$. Note that the first condition of (10) appeared in [12]. Suppose that $h$ is a positive linear functional on $\mathcal{X}_0$ (i.e., $h(x^\ast x) \geq 0$ for all $x \in \mathcal{X}_0$) which is $\mathcal{U}$-invariant (i.e., $h(x^\ast f) = \varepsilon(f)h(x)$ for $x \in \mathcal{X}_0$ and $f \in \mathcal{U}$). We associate with such a functional $h$ a unique $\ast$-representation $\pi_h$ of $\mathcal{U} \ltimes \mathcal{X}_1$ called the Heisenberg representation associated with $h$. Its construction is similar to the one for unital $\ast$-algebras (see [13]). Let us review the basic ideas. Set $N = \{x \in \mathcal{X}_0; h(x^\ast x) = 0\}$ and $\tilde{\mathcal{X}}_0 = \mathcal{X}_0/N$. We write $x \mapsto \tilde{x}$ to denote the canonical mapping $\mathcal{X}_0 \rightarrow \tilde{\mathcal{X}}_0$. Then the linear space $\tilde{\mathcal{X}}_0$ is an inner product space with inner product

\[\langle \tilde{x}, \tilde{y} \rangle = h(y^\ast x), \quad x, y \in \mathcal{X}_0.\]  \hspace{1cm} (11)

The action of the cross product algebra $\mathcal{U} \ltimes \mathcal{X}_1$ on $\tilde{\mathcal{X}}_0$ is given by

\[\pi_h(f \otimes a)\tilde{x} := ((a.x)^\ast S^{-1}(f^\ast)), \quad x \in \mathcal{X}_0, \quad a \in \mathcal{X}_1, \quad f \in \mathcal{U}.\]

That $\pi_h$ is a well defined $\ast$-representation of the $\ast$-algebra $\mathcal{U} \ltimes \mathcal{X}_1$ has been proved in [13, Proposition 5.3] for unital $\ast$-algebras $\mathcal{X}_0 = \mathcal{X}_1$. With some necessary modifications, the proof remains valid in the present situation as well. If $\mathcal{X}_0 \subset \mathcal{X}_1$, then $\mathcal{X}_0$ is a $\ast$-subalgebra of $\mathcal{U} \ltimes \mathcal{X}_1$ and the restriction of $\pi_h$ to $\mathcal{X}_0$ is just the GNS-representation associated with $h$. In the sequel, we write simply $x$ instead of $\tilde{x}$.

Condition (10) is satisfied if there is a right $\mathcal{U}$-module $\ast$-algebra $\mathcal{X}$ such that $\mathcal{X}_1$ and $\mathcal{X}_0$ are right $\mathcal{U}$-module $\ast$-subalgebras and $\mathcal{X}_0$ is an ideal of $\mathcal{X}$. In all our examples below, $\mathcal{X}$ will be a $\ast$-algebra of functions on a non-compact quantum space which contains the coordinate algebra, here denoted by $\mathcal{X}_1$, as a subalgebra and $\mathcal{X}_0$ can be considered as a $\ast$-algebra of functions with compact support. In this situation, the left action $a.x$ of $a \in \mathcal{X}_1$ on $x \in \mathcal{X}_0$ becomes the product $ax$ in the algebra $\mathcal{X}$, and the restriction of the Heisenberg representation of the cross product $\ast$-algebra $\mathcal{U} \ltimes \mathcal{X}$ to the $\ast$-subalgebra $\mathcal{U} \ltimes \mathcal{X}_1$ is just the Heisenberg representation of $\mathcal{U} \ltimes \mathcal{X}_1$ (both associated with the same functional $h$ on $\mathcal{X}_0$).
2.3 Two auxiliary lemmas

Lemma 2.3 Let \( p \in (0, 1) \) and \( m \in \mathbb{N}_0 \). Suppose that \( w \) is a unitary operator and \( A_0, \ldots, A_m \) are self-adjoint operators on a Hilbert space \( \mathcal{H} \). Assume that

\[
A_0 > 0, \quad wA_0 \subseteq pA_0w, \quad (12)
\]

and, if \( m > 0 \),

\[
wA_k \subseteq A_kw, \quad k = 1, \ldots, m, \quad A_0, \ldots, A_m \text{ strongly commute.} \quad (13)
\]

Then, up to unitary equivalence, there exist a Hilbert space \( \mathcal{H}_0 \), strongly commuting self-adjoint operators \( B_0, \ldots, B_m \) on \( \mathcal{H}_0 \), and a dense subspace \( \mathcal{D}_0 \subseteq \mathcal{H}_0 \) such that \( \mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n \), where each \( \mathcal{H}_n \) is \( \mathcal{H}_0, \mathcal{D} := \text{Lin}\{\eta_n : n \in \mathbb{Z}, \eta \in \mathcal{D}_0\} \) is invariant under \( w, A_0, \ldots, A_m \), the operator \( B_0 \) satisfies \( \sigma(B_0) \subseteq (p, 1] \), and the actions of \( w, A_0, \ldots, A_m \) on \( \mathcal{H} \) are determined by

\[
w\eta_n = \eta_{n-1}, \quad A_0\eta_n = p^n B_0 \eta_n, \quad A_k\eta_n = B_k \eta_n, \quad k = 1, \ldots, m, \quad \eta \in \mathcal{D}_0.
\]

In particular,

\[
wA_0 = pA_0w, \quad wA_k = A_kw, \quad k = 1, \ldots, m, \quad A_jA_i = A_iA_j \text{ on } \mathcal{D}.
\]

If \( A_k \) satisfies \( \sigma(A_k) \subseteq (a, b] \) or \( \sigma(A_k) \subseteq [a, b) \), where \( a, b \in \mathbb{R} \cup \{\pm \infty\}, a < b \), and \( k \in \{1, \ldots, m\} \), then the same holds for \( B_k \).

Proof. Let \( e(\mu) \) denote the spectral projections of \( A_0 \). By (12), the self-adjoint operators \( wA_0w^* \) and \( pA_0 \) coincide, hence \( we(\mu)w^* = e(p^{-1})e(p^0) \). Define \( \mathcal{H}_n = (e(p^n) - e(p^{n+1}))\mathcal{H} \). Then \( w\mathcal{H}_n = (e(p^n) - e(p^{n+1}))w\mathcal{H} = \mathcal{H}_{n-1} \). After applying an obvious unitary transformation, we may assume that \( \mathcal{H}_n = \mathcal{H}_0 \) and \( w\eta_n = \eta_{n-1} \) for \( \eta \in \mathcal{H}_0 \). Denote by \( B_0 \) the restriction of \( A_0 \) to \( \mathcal{H}_0 \). Then, by the definition of \( \mathcal{H}_0 \), \( \sigma(B_0) \subseteq (p, 1] \) and \( A_0\eta_n = p^n a^w A_0w^* \eta_n = p^n a^w A_0\eta_0 = p^n B_0\eta_n \).

Let \( m > 0 \) and \( k \in \{1, \ldots, m\} \). The operator \( A_k \) commutes strongly with \( A_0 \) and thus it commutes with the spectral projections of \( A_0 \). Therefore \( A_k \) leaves each Hilbert space \( \mathcal{H}_n (= \mathcal{H}_0) \) invariant. Denote by \( A_{kn} \) the restriction of \( A_k \) to \( \mathcal{H}_n \). From (13), we conclude that \( A_{kn}\eta_{n-1} = A_{k,n-1}\eta_{n-1} \) for all \( \eta \) in the domain of \( A_{kn} \). Since both \( A_{kn} \) and \( A_{k,n-1} \) are self-adjoint operators on \( \mathcal{H}_0 \), we have \( A_{kn} = A_{k,n-1} \), so all \( A_{kn} \) are equal, say \( B_k := A_{k0} \).

Clearly, if \( A_k \) satisfies a spectral condition as stated in the lemma, then \( B_k \) does so. By (13), it is also clear that the self-adjoint operators \( B_0, \ldots, B_m \) strongly
commute. Let \( e(\lambda_0, \ldots, \lambda_m) \) denote the joint spectral projections. Take \( D_0 := \bigcup_{l \in \mathbb{N}} (e(l, \ldots, l) - e(-l, \ldots, -l)) \mathcal{H} \). Then \( D_0 \) is an invariant core for each \( B_j \) and \( D := \text{Lin}\{\eta_n; n \in \mathbb{Z}, \eta \in D_0\} \) is an invariant core for each \( A_j \). This completes the proof.

**Lemma 2.4** Let \( \epsilon \in \{\pm 1\} \). Assume that \( z \) is a closed operator on a Hilbert space \( \mathcal{H} \). Then we have \( D(zz^*) = D(z^*z) \), this domain is dense in \( \mathcal{H} \) and the relation

\[
z^*z - q^2zz^* = \epsilon(1 - q^2)
\]

holds if and only if \( z \) is unitarily equivalent to an orthogonal direct sum of operators of the following form.

\( \epsilon = 1 \): (I) \( z\eta_n = (1 - q^{2(n+1)})^{1/2}\eta_{n+1} \) on \( \mathcal{H} = \bigoplus_{n=0}^\infty \mathcal{H}_n \), where each \( \mathcal{H}_n \) is the same Hilbert space \( \mathcal{H}_0 \).

(II) \( A \) \( z\eta_n = (1 + q^{2(n+1)}A^2)^{1/2}\eta_{n+1} \) on \( \mathcal{H} = \bigoplus_{n=-\infty}^\infty \mathcal{H}_n \), \( \mathcal{H}_n = \mathcal{H}_0 \), where \( A \) denotes a self-adjoint operator on a Hilbert space \( \mathcal{H}_0 \) such that \( \sigma(A) \subseteq [q, 1] \) and either \( q \) or \( 1 \) is not an eigenvalue of \( A \).

(III) \( u \) \( z = u \), where \( u \) is a unitary operator on \( \mathcal{H} \).

\( \epsilon = -1 \): (IV) \( z\eta_n = (q^{-2n} - 1)^{1/2}\eta_{n-1} \) on \( \mathcal{H} = \bigoplus_{n=0}^\infty \mathcal{H}_n \), where each \( \mathcal{H}_n \) is \( \mathcal{H}_0 \).

The proof of Lemma 2.4 can be found in [9]. A version of this lemma appears also in [2], where irreducible \( * \)-representations of (14) are discussed.

### 3 Cross product algebras related to the quantum motion group \( E_q(2) \)

#### 3.1 Definitions

The coordinate Hopf \( * \)-algebra of the quantum motion group \( E_q(2) \) is the complex unital \( * \)-algebra \( \mathcal{O}(E_q(2)) \) with generators \( v, v^*, n, n^* \) and defining relations

\[
v^*v = vv^* = 1, \quad nn^* = n^*n, \quad nv = qvn.
\]

The Hopf algebra structure is given by

\[
\Delta(v) = v \otimes v, \quad \Delta(n) = v \otimes n + n \otimes v^*,
\]

\[
\epsilon(v) = 1, \quad \epsilon(n) = 0, \quad S(v) = v^*, \quad S(n) = -qn.
\]
Let $O(\mathbb{C}_q)$ denote the complex unital $\ast$-algebra with a single generator $z$ satisfying
\begin{equation}
z^*z = q^2zz^*. \tag{18}\end{equation}

We call $O(\mathbb{C}_q)$ the coordinate $\ast$-algebra of the quantum complex plane. The $\ast$-algebra $O(\mathbb{C}_q)$ is a left and a right $O(E_q(2))$-comodule $\ast$-algebra with left coaction $\varphi_L$ and right coaction $\varphi_R$ determined by
\begin{align*}
\varphi_L(z) &= v^2 \otimes z + vn \otimes 1, \\
\varphi_R(z) &= 1 \otimes vn^* + z \otimes v^2. \tag{19}\end{align*}

Using (15), we see that the map $z \mapsto vn$ extends to a $\ast$-isomorphism of $O(\mathbb{C}_q)$ onto the $\ast$-subalgebra of $O(E_q(2))$ generated by $vn$. By (16), (17) and (19), this $\ast$-isomorphism intertwines the left coaction of $O(E_q(2))$ and the comultiplication. Thus we can consider $O(\mathbb{C}_q)$ as a left $O(E_q(2))$-comodule $\ast$-subalgebra of $O(E_q(2))$ by identifying $z$ with $vn$. Analogously, we can consider $O(\mathbb{C}_q)$ as a right $O(E_q(2))$-comodule $\ast$-subalgebra of $O(E_q(2))$ by identifying $z$ with $vn^*$.

The quantized universal enveloping algebra $U_q(e_2)$ of the quantum motion group is generated by $E, F, K$ and $K^{-1}$ with relations
\begin{align*}
KK^{-1} &= K^{-1}K = 1, \\
KF &= qFK, \\
KE &= q^{-1}EK, \\
EF &= FE.
\end{align*}

It is a Hopf $\ast$-algebra with involution and Hopf algebra structure given by
\begin{align*}
K^\ast &= K, \\
E^\ast &= F, \\
\Delta(K) &= K \otimes K, \\
\Delta(E) &= E \otimes K + K^{-1} \otimes E, \\
\Delta(F) &= F \otimes K + K^{-1} \otimes F, \\
\varepsilon(K) &= 1, \\
\varepsilon(E) &= \varepsilon(F) = 0, \\
S(K) &= K^{-1}, \\
S(E) &= -q^{-1}E, \\
S(F) &= -qF.
\end{align*}

There is a dual pairing of Hopf $\ast$-algebras $\langle \cdot, \cdot \rangle : U_q(e_2) \times O(E_q(2)) \to \mathbb{C}$ which is zero on all pairs of generators except
\begin{align*}
\langle E, n^* \rangle &= -q^{-1}, \\
\langle F, n \rangle &= 1, \\
\langle K, v \rangle &= q^{1/2}, \\
\langle K, v^* \rangle &= q^{-1/2}.
\end{align*}

With the coaction induced by the comultiplication, $O(E_q(2))$ becomes a left and right $O(E_q(2))$-comodule $\ast$-algebra. By (3) and (6), $O(E_q(2))$ is a right and a left $U_q(e_2)$-module $\ast$-algebra. Simple computations show that the right action $\lhd$ and the left action $\rhd$ are given by
\begin{align*}
n^* \lhd E &= -q^{-1}v, \\
n \lhd E &= vE = v^* \lhd E = 0, \\
n^* \rhd F &= v^*, \\
n \rhd F &= vF = v^* \rhd F = 0, \\
n^* \rhd K &= q^{1/2}n, \\
n \lhd K &= q^{-1/2}n^*, \\
v \lhd K &= q^{1/2}v, \\
v^* \lhd K &= q^{-1/2}v^*;\end{align*}
Similarly, by (3), (6) and (19), \( \mathcal{O}(\mathbb{C}_q) \) is a left and a right \( \mathcal{U}_q(\mathfrak{e}_2) \)-module \(*\)-algebra with left and right action determined by

\[
\begin{align*}
E \triangleright n^* &= -q^{-1}v^*, & E \triangleright n &= E \triangleright v = E \triangleright v^* &= 0, \\
F \triangleright n &= v, & F \triangleright n^* &= F \triangleright v = F \triangleright v^* &= 0, \\
K \triangleright n &= q^{-1/2}n, & K \triangleright n^* &= q^{1/2}n^*, & K \triangleright v &= q^{1/2}v, & K \triangleright v^* &= q^{-1/2}v^*.
\end{align*}
\]

Similarly, by (3), (6) and (19), \( \mathcal{O}(\mathbb{C}_q) \) is a left and a right \( \mathcal{U}_q(\mathfrak{e}_2) \)-module \(*\)-algebra with left and right action determined by

\[
\begin{align*}
z \triangleright K &= qz, & z^* \triangleright K &= q^{-1}z^*, & z \triangleright E &= 0, & z^* \triangleright E &= -q^{-3/2}, & z \triangleright F &= q^{-1/2}, & z^* \triangleright F &= 0; \\
K \triangleright z &= qz, & K \triangleright z^* &= q^{-1}z^*, & E \triangleright z &= -q^{-3/2}, & E \triangleright z^* &= 0, & F \triangleright z &= 0, & F \triangleright z^* &= q^{-1/2}.
\end{align*}
\]

From (2), (5) and above formulas, we derive the following crosscommutation relations in the corresponding cross product algebras.

\[
\begin{align*}
\mathcal{U}_q(\mathfrak{e}_2) \ltimes \mathcal{O}(\mathbb{E}_q(2)) : & \quad vF = q^{1/2}Fv, \quad vE = q^{1/2}Ev, \quad vK = q^{1/2}Kv, \\
v^*F &= q^{-1/2}Fv^*, \quad v^*E = q^{-1/2}Ev^*, \quad v^*K = q^{-1/2}Kv^*, \\
nF &= q^{1/2}Fn + K^{-1}v^*, \quad nE = q^{1/2}En, \quad nK = q^{1/2}Kn, \\
n^*F &= q^{-1/2}Fn^*, \quad n^*E = q^{-1/2}En^* - q^{-1}K^{-1}v, \quad n^*K = q^{-1/2}Kn^*.
\end{align*}
\]

\[
\begin{align*}
\mathcal{O}(\mathbb{E}_q(2)) \ltimes \mathcal{U}_q(\mathfrak{e}_2) : & \quad Fv = q^{-1/2}vF, \quad Ev = q^{-1/2}vE, \quad Kv = q^{1/2}vK, \\
Fv^* &= q^{1/2}v^*F, \quad Ev^* = q^{1/2}v^*E, \quad Kv^* = q^{-1/2}v^*K, \\
Fn &= q^{1/2}Fn + vK, \quad En = q^{1/2}En, \quad Kn = q^{-1/2}Kn, \\
Fn^* &= q^{-1/2}Fn^*, \quad En^* = q^{-1/2}En^* - q^{-1}v^*K, \quad Kn^* = q^{1/2}Kn^*.
\end{align*}
\]

\[
\begin{align*}
\mathcal{U}_q(\mathfrak{e}_2) \ltimes \mathcal{O}(\mathbb{C}_q) : & \quad zK = qKz, \quad zE = qEq, \quad zF = qFz + q^{-1/2}K^{-1}, \\
z^*K &= q^{-1}Kz^*, \quad z^*E = q^{-1}Eq^* - q^{-3/2}K^{-1}, \quad z^*F = q^{-1}Fz^*.
\end{align*}
\]

\[
\begin{align*}
\mathcal{O}(\mathbb{C}_q) \ltimes \mathcal{U}_q(\mathfrak{e}_2) : & \quad Kz = qzK, \quad Ez = q^{-1}zE - q^{3/2}K, \quad Fz = q^{-1}zF, \\
Kz^* &= q^{-1}z^*K, \quad Ez^* = qz^*E, \quad Fz^* = qz^*F + q^{-1/2}K.
\end{align*}
\]

The next lemma shows that we can restrict ourselves to the right versions of these cross product algebras. It is proved by direct computations.

**Lemma 3.1** There is a \(*\)-isomorphism

\[
\theta : \mathcal{U}_q(\mathfrak{e}_2) \ltimes \mathcal{O}(\mathbb{E}_q(2)) \longrightarrow \mathcal{O}(\mathbb{E}_q(2)) \ltimes \mathcal{U}_q(\mathfrak{e}_2)
\]
determined by \( \theta(v) = v, \theta(n) = n^*, \theta(K) = K^{-1} \) and \( \theta(E) = F \).

There is a \(*\)-isomorphism

\[
\psi : U_q(e_2) \times O(C_q) \rightarrow O(C_q) \times U_q(e_2)
\]
determined by \( \psi(z) = z, \psi(K) = K^{-1} \) and \( \psi(E) = F \).

The inverse isomorphisms \( \theta^{-1} \) and \( \psi^{-1} \) are given by the same formulas.

Let \( U_0 \) denote the subalgebra of \( U_q(e_2) \) generated by the unit element and the linear span \( \mathcal{T}_0 \) of the elements

\[
X := q^{1/2} FK, \quad Y := -q^{3/2} EK.
\]

Clearly, \( YX = q^2 XY \) and \( X^* = -q^{-2} Y \). In particular, \( U_0 \) is a \(*\)-algebra. Since \( \mathbb{C} \cdot 1 + \mathcal{T}_0 \) is a right coideal of \( U_q(e_2), \mathcal{T}_0 \) is the quantum tangent space of a left-covariant first order differential calculus on \( O(E_q(2)) \) [5, Proposition 14.5]. It can be shown that this calculus induces a differential \(*\)-calculus on the \(*\)-subalgebra \( O(C_q) \) such that \( X \) and \( Y \) can be considered as partial derivatives. We shall not carry out the details because we are interested in the \(*\)-algebra \( U_0 \times O(C_q) \) only.

The \(*\)-algebra \( U_0 \times O(C_q) \) is the \(*\)-subalgebra of \( U_q(e_2) \times O(C_q) \) generated by \( U_0 \) and \( O(C_q) \) or, equivalently, the \(*\)-algebra with generators \( X, X^*, z, z^* \) and defining relations

\[
U_0 \times O(C_q) : \quad z^* z = q^2 zz^*, \quad X^* X = q^2 XX^*, \quad (20)
\]
\[
zX = q^2 Xz + 1, \quad zX^* = q^2 X^* z, \quad (21)
\]
\[
z^* X = q^{-2} Xz^*, \quad z^* X^* = q^{-2} X^* z^* - q^{-2}. \quad (22)
\]

### 3.2 Representations of the \(*\)-algebra \( U_0 \times O(C_q) \)

We set \( \gamma = (1 - q^2)^{-1} \) and define \( N = zX - \gamma \). In the \(*\)-algebra \( U_0 \times O(C_q) \), the element \( N \) satisfies the following relations

\[
zN = q^2 N z, \quad z^* N = N z^*, \quad N^* N = NN^*, \quad z^* z N N^* = N^* N z^* z. \quad (23)
\]

These four equations follow immediately from (20)–(22). As a sample, we verify the relation \( N^* N = N N^* \). Indeed, from

\[
X^* z^* z X = q^2 X^* z z^* X = X^* z X z^* = q^{-2} z X^* X z^* = z X X^* z^*,
\]

we conclude

\[
N^* N = X^* z^* z X - \gamma z X - \gamma X^* z^* + \gamma^2 = z X X^* z^* - \gamma z X - \gamma X^* z^* + \gamma^2 = NN^*.
\]
Now suppose we are given a $*$-representation of the $*$-algebra $\mathcal{U}_0 \rtimes \mathcal{O}(C_q)$ on a Hilbert space $\mathcal{H}$. As explained in the introduction, we assume that $N$ is a normal operator and that the self-adjoint operators $z^*z$ and $N^*N$ strongly commute.

We claim that $\ker z = \ker z^* = \{0\}$. To see this, observe that $z^*z = q^2zz^*$ yields $\ker z = \ker z^*$. Let $\eta \in \ker z$. Then, by (21),

$$\| \eta \|^2 = \langle (zX - q^2Xz)\eta, \eta \rangle = \langle X\eta, \eta \rangle = \langle X\eta, z^*\eta \rangle = 0$$

so that $\eta = 0$. Thus, $\ker z = \{0\}$.

Let $z = w|z|$ be the polar decomposition of the closed operator $z$. As $\ker z = \ker z^* = \{0\}$, $w$ is unitary. The operator relation $z^*z = q^2zz^*$ is equivalent to $|z|^2 = q^2w|z|^2w^*$ and so to $|z| = qw|z|w^*$. Since $\ker |z| = \{0\}$, it follows from Lemma 2.3 that there is a Hilbert space $\mathcal{H}_0$ and a self-adjoint operator $z_0$ on $\mathcal{H}_0$ satisfying $\sigma(z_0) \subseteq (q, 1]$ such that

$$\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n = \mathcal{H}_0, \quad w\eta_n = \eta_{n-1}, \quad |z|\eta_n = q^{-n}z_0\eta_n.$$ 

As $z^*z$ and $N^*N$ strongly commute, $|z|$ and $|N|$ do so. Consequently, $|N|$ commutes with the spectral projections of $|z|$. Since $\sigma(z_0) \subseteq (q, 1]$, it follows that $|N|$ leaves each Hilbert space $\mathcal{H}_n$ invariant, so there are self-adjoint operators $N_n$ on $\mathcal{H}_0$ such that

$$|N|\eta_n = N_n\eta_n, \quad \eta_n \in \mathcal{H}_n.$$

An application of (23) yields $zN^*N = q^2N^*Nz$ which entails the operator identity $w|z||N|^2 = q^2|N|^2w|z|$. Using above commutation relations, we get $|z|w|N|^2 = q^2|z||N|^2w$. Since $\ker |z| = \{0\}$, it follows that $w|N|^2 = q^2|N|^2w$. Hence $w|N| = q|N|w$ because $w$ is unitary. This in turn gives $N_n\eta_{n-1} = w|N|w^*\eta_{n-1} = qN_{n-1}\eta_{n-1}$ and so $N_n = q^nN_0$. Accordingly, $|N|\eta_n = q^nN_0\eta_n$.

From (23), $zN^*N = q^2N^*Nz$ and $N^*Nz^* = q^2z^*N^*N$. We assume that these identities hold in strong sense. Since $\ker N = \ker N^*N$, we see that $\ker N$ is reducing. Hence the representation decomposes into a direct sum of two representations corresponding to the cases $N = 0$ and $\ker N = \{0\}$. We treat the two cases separately.

**Case I.** $N = 0$.

This means that $X = (1 - q^2)^{-\frac{1}{2}}z^{-1}$. Using the fact that $z^*z = q^2zz^*$, one immediately verifies that all relations (20)–(22) are fulfilled.

**Case II.** $\ker N = \{0\}$.

Using this additional condition, we continue the above operator-theoretic manipulations. Let $N = w|N|$ be the polar decomposition of $N$. Since $\ker N = \{0\}$
and $N$ is normal, $u$ is unitary and $u|N| = |N|u$. Applying (23) and the assumption that $|z|$ and $|N|$ strongly commute, we obtain

$$|z|^2u|N| = z^*zN = q^2Nz^*z = q^2u|N||z|^2 = q^2u|z|^2|N|.$$  

As $\ker|N| = \{0\}$, above equation implies $|z|^2u = q^2u|z|^2$. The operator $u$ is unitary, thus $|z|u = qu|z|$. Since $|z|w = qw|z|$, $w^*u$ commutes with the self-adjoint operator $|z|$ and hence with the spectral projections of $|z|$. Therefore, $w^*u$ leaves each space $H_n$ invariant. Hence there are unitary operators $u_n$ on $H_0$, $n \in \mathbb{Z}$, such that $w^*u_n_n = u_n\eta_n$ for $\eta_n \in H_n$. Accordingly, $u\eta_n = u_n\eta_{n-1}$. Using the relations $|N|w^* = qw^*|N|$, $u|N| = |N|u$, $|z|u = qu|z|$ and the fact that $|z|$ and $|N|$ strongly commute, we derive

$$|z||N|w^*u = q|z|w^*u|N| = qz^*N = qNz^* = qu|N||z|w^* = |z||N|uw^*.$$  

Thus, $w^*u = uw^*$ since $\ker|z| = \ker|N| = \{0\}$. This gives $u_n\eta_n = uw^*\eta_{n-1} = w^*u\eta_{n-1} = u_{n-1}\eta_n$, so $u_n = u_{n-1}$. Hence $u_n = u_0$ for all $n \in \mathbb{Z}$. Employing the relations $w^*u|z| = |z|w^*u$ and $u|N| = |N|u$, we derive $u_0z_0 = z_0u_0$ and $u_0N_0u_0^* = q^{-1}N_0$. Since $|N|$ and $|z|$ strongly commute, $N_0z_0 = z_0N_0$. As $|N|$ is a positive self-adjoint operator with trivial kernel, so is $N_0$. Inserting $z^{-1} = |z|^{-1}w^*$ and $N_0 = u_0|N|$ into $X = z^{-1}(N + \gamma)$, we can express the operator $X$ (and its adjoint $X^*$) in terms of $u_0$, $z_0$ and $N_0$.

Summarizing, we have in Case II

$$X\eta_n = q^{2n}z_0^{n-1}u_0N_0\eta_n + (1 - q^2)^{-1}q^{n+1}z_0^{n-1}\eta_{n+1}, \quad (24)$$
$$X^*\eta_n = q^{2n}z_0^{n-1}N_0u_0^*\eta_n + (1 - q^2)^{-1}q^{n+1}z_0^{n-1}\eta_{n-1}, \quad (25)$$
$$z\eta_n = q^{-1}z_0\eta_{n-1}, \quad z^*\eta_n = q^{-1}z_0^{-1}\eta_{n+1} \quad (26)$$

on the Hilbert space $H = \bigoplus_{n = -\infty}^{\infty}H_n$, $H_n = H_0$, where $z_0, N_0$ are self-adjoint operators and $u_0$ is a unitary operator on $H_0$ such that $\sigma(z_0) \subseteq (q, 1], N_0 > 0$, and

$$z_0N_0 = N_0z_0, \quad u_0z_0u_0^* = z_0, \quad u_0N_0u_0^* = q^{-1}N_0 \quad (27)$$

Conversely, if the latter is satisfied, then formulas (24)–(26) define a $*$-representation of $\mathcal{U}_0 \ltimes \mathcal{O}(\mathbb{C}_q)$.

The $*$-representations of the relations (27) are described by Lemma 2.3. Inserting the expressions for $z_0, N_0$ and $u_0$ from Lemma 2.3 into Equations (24)–(26) and renaming suitable, we obtain the following list of $*$-representation of
The parameters $A$ and $B$ denote self-adjoint operators on the Hilbert space $\mathcal{K}$ such that $\sigma(A) \subseteq (q, 1]$, $\sigma(B) \subseteq (q, 1]$, and, in the case $(II)_{A,B}$, $AB = BA$. Representations labeled by different sets of parameters (within unitary equivalence) or belonging to different series are not unitarily equivalent. A representation is irreducible if and only if $\mathcal{K}$ is isomorphic to the one-dimensional Hilbert space $\mathbb{C}$.

In this case, we can regard $A$ and $B$ as real numbers of the interval $(q, 1]$.

### 3.3 Representations of the $\ast$-algebra $\mathcal{U}_q(e_2) \ltimes \mathcal{O}(\mathbb{C}_q)$

Suppose we have a $\ast$-representation of the $\ast$-algebra $\mathcal{U}_q(e_2) \ltimes \mathcal{O}(\mathbb{C}_q)$ on a Hilbert space $\mathcal{H}$. Then the considerations of the preceding subsection apply to the restriction of the representation to the subalgebra $\mathcal{U}_0 \ltimes \mathcal{O}(\mathbb{C}_q)$. We freely use the facts and notations set up therein.

The new ingredient in the larger algebra $\mathcal{U}_q(e_2) \ltimes \mathcal{O}(\mathbb{C}_q)$ is the invertible generator $K$ satisfying the relations

$$
(1) A:\quad z\eta_n = q^{-n}A\eta_{n-1}, \quad z^*\eta_n = q^{-(n+1)}A\eta_{n+1}, \\
X\eta_n = (1 - q^2)^{-1}q^{(n+1)}A^{-1}\eta_{n+1}, \\
X^*\eta_n = (1 - q^2)^{-1}q^nA^{-1}\eta_{n-1} \quad \text{on} \quad \mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n = \mathcal{K}.
$$

$$(II)_{A,B}:\quad z\eta_{nk} = q^{-n}A\eta_{n-1,k}, \quad z^*\eta_{nk} = q^{-(n+1)}A\eta_{n+1,k}, \\
X\eta_{nk} = q^{2n-k}A^{-1}B\eta_{n,k-1} + (1 - q^2)^{-1}q^{n+1}A^{-1}\eta_{n+1,k}, \\
X^*\eta_{nk} = q^{2n-k-1}A^{-1}B\eta_{n,k+1} + (1 - q^2)^{-1}q^nA^{-1}\eta_{n-1,k} \\
\quad \text{on} \quad \mathcal{H} = \bigoplus_{n,k=-\infty}^{\infty} \mathcal{H}_{nk}, \quad \mathcal{H}_{nk} = \mathcal{K}.
$$

Again, we assume that these relations hold in strong operator-theoretic sense.

**Case I.** Let $z$, $X$ and $X^*$ be given as described in $(I)_A$. By (28), $K$ commutes with $z^*z$ and hence with the spectral projections of $z^*z$. Since $z^*z\eta_n = q^{-2n}A^2\eta_n$ and $\sigma(A) \subseteq (q, 1]$, we conclude that $K$ leaves each space $\mathcal{H}_n$ invariant, that is, there are operators $K_n$ on $\mathcal{H}_n$, $n \in \mathbb{Z}$, such that $K\eta_n = K_n\eta_n$. Inserting the expressions of $z$ and $K$ into the relations $zz^*K = Kz^*z$ and $zK = qKz$ gives
AK_n = K_n A and \( K_n = qK_{n-1} \). Setting \( H := K_0 \), we can write \( K_n = q^n H \), where \( H \) is an invertible self-adjoint operator on \( \mathcal{K} \) commuting with \( A \). Finally, \( N \equiv zX - \gamma = 0 \) and so \( F = q^{-1} X K^{-1} = q^{-1/2} \gamma z^{-1} K^{-1} \). This determines the actions of the operators \( z, z^*, K, F \) and \( E = F^* \) completely.

Case II. Suppose that the representation of the operators \( z, X \) and \( X^* \) takes the form described in \((II)_{A,B} \). As in the preceding paragraph, it follows from (28) that \( K \) is given on \( \mathcal{H}_n = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_{nk} \) by \( K \zeta_n = q^n K_0 \zeta_n \), \( \zeta_n \in \mathcal{H}_n \), where \( K_0 \) is an invertible self-adjoint operator acting on \( \mathcal{H}_0 \). Relations (28) and the definition of \( N \) yield \( NK = KN \) and \( N^* K = K N^* \). Thus we can assume that the self-adjoint operators \( K \) and \( |N| \) strongly commute. Observe that \( \eta \) acts on \( \mathcal{H}_0 \) by \( \eta \eta_{nk} = q^{-k} B \eta_{0k} \) and that \( \sigma(B) \subseteq (q, 1] \). Since \( K \) commutes with the spectral projections of \( |N| \), it leaves the Hilbert spaces \( \mathcal{H}_{nk} \) invariant. Hence there exist invertible self-adjoint operators \( K_{0k} \) on \( \mathcal{H}_{0k} \) such that \( K_0 \eta_{0k} = K_{0k} \eta_{0k} \) and \( BK_{0k} = K_{0k} B \).

From the last subsection, \( N \eta_{nk} = q^{-k} B \eta_{-1, k-1} \). Applying \( NK = KN \) to vectors \( \eta_{nk} \in \mathcal{H}_{nk} \) gives \( q^{2n-k} K_{0k} B \eta_{-1, k-1} = q^{2n-k} K_0B \eta_{-1, k-1} \), hence \( K_{0k} = q^{-k} K_0_{0, k-1} \). Denoting \( K_{00} \) by \( B \), we get \( K \eta_{nk} = q^{-k} H \eta_{nk} \), where \( H \) is an invertible self-adjoint operator on \( \mathcal{H}_{00} \) commuting with \( B \). Moreover, it follows from \( z^* K = K z^* \) that \( H \) commutes with \( A \). The actions of \( F \) and \( E \) on \( \mathcal{H} \) are obtained by computing \( F = q^{-1/2} z^{-1} (N + \gamma) K^{-1} \) and \( F^* = E \).

Summarizing, we have obtained the following series of \( * \)-representations of \( \mathcal{U}_q(e_2) \times \mathcal{O}(\mathbb{C}_q) \).

\[
(II)_{A,B,H} : \quad z \eta_{nk} = q^{-n} A \eta_{n-1, k}, \quad z^* \eta_{nk} = q^{-(n+1)} A \eta_{n+1, k}, \quad F \eta_{nk} = q^{1/2} (1 - q^2)^{-1} A^{-1} H^{-1} \eta_{n+1, k}, \quad E \eta_{nk} = q^{1/2} (1 - q^2)^{-1} A^{-1} H^{-1} \eta_{n-1, k}, \quad K \eta_{nk} = q^n H \eta_{nk} \quad \text{on} \quad \mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n = \mathcal{K}.
\]

Here \( A, B \) and \( H \) are commuting self-adjoint operators on the Hilbert space \( \mathcal{K} \) such that \( \sigma(A) \subseteq (q, 1] \), \( \sigma(B) \subseteq (q, 1] \), and \( H \) is invertible. Representations
labeled by different sets of parameters (within unitary equivalence) or belonging to different series are not unitarily equivalent. A representation is irreducible if and only if $K = \mathbb{C}$. Then $A$, $B$ and $H$ are real numbers such that $A, B \in (q, 1]$ and $H \neq 0$.

### 3.4 Representations of the $*$-algebra $\mathcal{U}_q(e_2) \ltimes \mathcal{O}(E_q(2))$

Recall from Subsection 3.1 that $\mathcal{O}(C_q)$ becomes a $*$-subalgebra of $\mathcal{O}(E_q(2))$ by identifying $z$ with $vn$. The relations from Subsection 3.1 show that the cross product algebra $\mathcal{U}_q(e_2) \ltimes \mathcal{O}(E_q(2))$ can be described as the $*$-algebra generated by $z, E, F, K \in \mathcal{U}_q(e_2) \ltimes \mathcal{O}(C_q)$ and an additional generator $v$ satisfying

$$vv^*v = v^*v = 1, \; zv = qvz, \; vK = q^{1/2}Kv, \; vE = q^{1/2}Ev, \; vF = q^{1/2}Fv. \; (29)$$

The element $n \in \mathcal{O}(E_q(2))$ is recovered by setting $n = v^*z$.

For a $*$-representations of $\mathcal{U}_q(e_2) \ltimes \mathcal{O}(E_q(2))$, we apply the results of the preceding subsection to its restriction to the $*$-subalgebra $\mathcal{U}_q(e_2) \ltimes \mathcal{O}(C_q)$. It only remains to determine the action of the additional operator $v$.

**Case I.** Assume that the operators $z, E, F, K$ are given by the formulas of the series $(I)_{A,H}$. Let $z = w|z|$ be the polar decomposition of the closed operator $z$. Then $w$ and $|z|$ act on $\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_0$, by $w\eta_n = \eta_{n-1}$ and $|z|\eta_n = q^{-n}A\eta_n$, where $\eta_n \in \mathcal{H}_n$. The relations $vv^* = v^*v = 1$ and $zv = qvz$ yield $z^*v = qvz^*$ and so $z^*zv = q^2vz^*z$. This implies $|z|v = qv|z|$ since $v$ is unitary. Note that $|z|w = qw|z|$. It follows that the unitary operator $w^*v$ commutes with $|z|$. As a consequence, $w^*v$ leaves each space $\mathcal{H}_n$ invariant. Hence there are unitary operators $v_n$ on $\mathcal{H}_n$ such that $w^*v\eta_n = v_n\eta_n$. Accordingly, $v_n = v_0\eta_n$. Evaluating $|z|v = qv|z|$ and $zv = qvz$ on vectors $\eta_n \in \mathcal{H}_n$ gives $Av_n = v_nA$ and $Av_n = v_{n-1}A$, respectively. Combining the first equation with the second shows that $v_n = v_{n-1}$ since $A$ is invertible. Hence we can write $v\eta_n = v_0\eta_{n-1}$, where $v_0$ is a unitary operator on $\mathcal{H}_0$ commuting with $A$. Finally, by applying $vK = q^{1/2}Kv$ to vectors $\eta_n$, one gets $v_0H = q^{-1/2}Hv_0$ on $\mathcal{H}_0$. It can easily be checked that, if the preceding conditions on the operator $v$ are satisfied, then the identities (29) hold.

In conclusion, we have to determine operators $v_0, A, H$ on $\mathcal{H}_0$ satisfying

$$Av_0 = v_0A, \; \; Hv_0 = q^{1/2}v_0H, \; \; AH = HA, \; (30)$$

where $v_0$ is unitary, $H$ is an invertible self-adjoint operator, and $A$ is a self-adjoint operator such that $\sigma(A) \subseteq (q, 1]$. Note that the subspaces of $\mathcal{H}_0$ where $H > 0$ and
$H < 0$ are reducing. Considering the cases $H > 0$ and $H < 0$ separately, we can apply Lemma 2.3 to establish the $*$-representations of the relations (30).

Case II. Suppose we are given operators $z$, $E$, $F$, $K$ as described by the formulas of the series $(I)_{A,B,H}$. As in Case I, one shows by using $zv = qvz$ and $|z|v = qv|z|$ that $v$ maps $\mathcal{H}_n = \bigoplus_{j=\infty}^{\infty} \mathcal{H}_{nj}$ into $\mathcal{H}_{n-1} = \bigoplus_{j=\infty}^{\infty} \mathcal{H}_{n-1,j}$. On the other hand, observe that $v$ commutes with $N = q^{1/2}zFK - (1 - q^2)^{-1}$ and its adjoint. This yields $v[N] = |N|v$ since $v$ is unitary. The action of $|N|$ on $\mathcal{H}_{nk}$ is given by $|N|\eta_{nk} = q^{n-k}B\eta_{nk}$, where $B$ is a self-adjoint operator on $\mathcal{H}_{00}$ such that $\sigma(B) \subseteq (q, 1]$. Since $v$ commutes with the spectral projections of $|N|$, we conclude that $v$ leaves each Hilbert space $\bigoplus_{j=\infty}^{\infty} \mathcal{H}_{n+1,k+j}$ invariant.

Combining these facts, it follows that $v$ maps $\mathcal{H}_{nk}$ into $\mathcal{H}_{n-1,k-1}$. Write $v\eta_{nk} = v_{nk}\eta_{n-1,k-1}$, where $v_{nk}$ denotes a unitary operator acting on $\mathcal{H}_{nk} = \mathcal{H}_{00}$. Applying $|z|v = qv|z|$ and $|N|v = v|N|$ to vectors $\eta_{nk} \in \mathcal{H}_{nk}$ gives $Av_{nk} = v_{nk}A$ and $Bv_{nk} = v_{nk}B$, respectively. Similarly, the relations $zv = qvz$ and $Nv = vN$ lead to $Av_{nk} = v_{n-1,k}A$ and $Bv_{nk} = v_{n-1,k-1}B$, respectively. This yields $v_{nk}A = v_{n-1,k}A$ and $v_{nk}B = v_{n-1,k-1}B$. Since $A$ and $B$ are invertible, we have $v_{nk} = v_{n-1,k}$ and $v_{nk} = v_{n-1,k-1}$. Hence all $v_{nk}$ are equal, say to $v_{00}$. Inserting the expression obtained for $v$ into $vK\eta_{nk} = q^{1/2}Kv\eta_{nk}$, we get $q^{n-k}v_{00}H\eta_{n-1,k-1} = q^{1/2}q^{n-k}Hv_{00}\eta_{n-1,k-1}$ so that $v_{00}H = q^{1/2}Hv_{00}$.

Gathering the facts of Case II together shows that the representation is determined by operators $v_{00}$, $A$, $B$, $H$ on $\mathcal{H}_{00}$ satisfying

$$Av_{00} = v_{00}A, \quad v_{00}H = q^{1/2}Hv_{00}, \quad AH = HA,$$
$$Bv_{00} = v_{00}B, \quad BH = BH, \quad BA = AB,$$

where $v_{00}$ is unitary and $A$, $B$, $H$ are self-adjoint operators such that $\sigma(A) \subseteq (q, 1]$, $\sigma(B) \subseteq (q, 1]$, and $H$ is invertible. Similarly to Case I, we employ Lemma 2.3 to describe the representations of the above relations.

After a slight change of notation, the preceding discussion leads to the following $*$-representations of the cross product algebra $\mathcal{U}_q(e_2) \rtimes \mathcal{O}(E_q(2))$.

$$(I)_{A,H,\epsilon}: \quad \eta_{nk} = \eta_{m-1,k-1}, \quad m\eta_{nk} = q^{-m}A\eta_{m,k+1},$$
$$F\eta_{nk} = (1 - q^2)^{-1}q^{(k+1)/2}\epsilon A^{-1}H^{-1}\eta_{m+1,k},$$
$$E\eta_{nk} = (1 - q^2)^{-1}q^{(k+1)/2}\epsilon A^{-1}H^{-1}\eta_{m-1,k},$$
$$K\eta_{nk} = q^{m-k/2}\epsilon H\eta_{nk} \quad \text{on} \quad \mathcal{H} = \bigoplus_{m,k=-\infty}^{\infty} \mathcal{H}_{mk}, \quad \mathcal{H}_{mk} = \mathcal{K}.$$
\[ (II)_{A,B,H,\epsilon} : \quad v \eta_{mkl} = \eta_{m-1,k-1,l+1}, \quad n \eta_{mkl} = q^{-m}A \eta_{m,k+1,l-1}, \]
\[ F \eta_{mkl} = q^{m+(l-1)/2} \epsilon A^{-1} BH^{-1} \eta_{m,k-1,l} + (1 - q^2)^{-1} q^{k+(l+1)/2} \epsilon A^{-1} H^{-1} \eta_{m+1,kl}, \]
\[ E \eta_{mkl} = q^{m+(l-1)/2} \epsilon A^{-1} BH^{-1} \eta_{m,k+1,l} + (1 - q^2)^{-1} q^{k+(l+1)/2} \epsilon A^{-1} H^{-1} \eta_{m-1,kl}, \]
\[ K \eta_{mkl} = q^{m-k-1/2} \epsilon H \eta_{mkl} \quad \text{on} \quad \mathcal{H} = \bigoplus_{m,k,l=-\infty}^{\infty} \mathcal{H}_{mkl}, \quad \mathcal{H}_{mkl} = \mathcal{K}. \]

The parameters \( A, B, H \) denote commuting self-adjoint operators on a Hilbert space \( \mathcal{K} \) such that \( \sigma(A) \subseteq (q, 1], \sigma(B) \subseteq (q, 1], \sigma(H) \subseteq (q^{1/2}, 1], \) and \( \epsilon \) takes the values 1 and \(-1\). Representations labeled by different sets of parameters (within unitary equivalence) or belonging to different series are not unitarily equivalent. A representation of this list is irreducible if and only if \( \mathcal{K} = \mathbb{C} \). In this case, \( A, B, H \) are real numbers such that \( A \in (q, 1], B \in (q, 1], \) and \( H \in (q^{1/2}, 1], \) respectively.

### 3.5 Heisenberg representations of the cross product algebras \( \mathcal{U}_q(e_2) \rtimes \mathcal{O}(E_q(2)) \) and \( \mathcal{U}_q(e_2) \rtimes \mathcal{O}(\mathbb{C}_q) \)

Let \( \mathbb{C}[u,v] \) be the algebra of complex Laurent polynomials \( p(u,v) = \sum_{n,k} \alpha_{nk} u^n v^k \) in two commuting variables \( u, v \) and let \( \mathcal{F}(\mathbb{R}^+) \) be the algebra of all complex Borel functions \( f = f(r) \) on \( \mathbb{R}^+ = (0, +\infty) \) such that \( f \) is locally bounded, that is, the restriction of \( f \) to any compact subset contained in \( \mathbb{R}^+ \) is bounded.

We denote by \( \mathcal{F}(E_q(2)) \) the \( * \)-algebra generated by the two algebras \( \mathbb{C}[u,v] \) and \( \mathcal{F}(\mathbb{R}^+) \) with cross commutation relations and involution

\[ u^j v^k f(r) = f(q^{-k}r) u^j v^k, \quad (u^j v^k f(r))^* = \bar{f}(r) v^{-k} u^{-j}, \quad j, k \in \mathbb{Z}, \quad f \in \mathcal{F}(\mathbb{R}^+). \quad (31) \]

We introduce a right action \( \ast \) of \( \mathcal{U}_q(e_2) \) on \( \mathcal{F}(E_q(2)) \) by

\[ u^j v^k f(r) \ast E = q^{(j-k-3)/2} \lambda^{-1} u^{j+1} v^{k+1} (f(r) - q^{-j} f(q r)) r^{-1}, \quad (32) \]
\[ u^j v^k f(r) \ast F = q^{(j-k+1)/2} \lambda^{-1} u^{j-1} v^{k-1} (q^j f(r) - f(q^{-1} r)) r^{-1}, \quad (33) \]
\[ u^j v^k f(r) \ast K = q^{(j+k)/2} u^j v^k f(r), \quad (34) \]

where \( j, k \in \mathbb{Z} \) and \( f \in \mathcal{F}(\mathbb{R}^+) \). Straightforward computations show that these formulas define indeed a right action of \( \mathcal{U}_q(e_2) \) such that \( \mathcal{F}(E_q(2)) \) becomes a right \( \mathcal{U}_q(e_2) \)-module \( * \)-algebra.

From (31), it follows that there is a \( * \)-isomorphism \( \phi \) from \( \mathcal{O}(E_q(2)) \) onto the \( * \)-subalgebra of \( \mathcal{F}(E_q(2)) \) generated by \( v \) and \( u r \) such that \( \phi(v) = v \) and
\( \phi(n) = ur. \) From (32)–(34), we conclude that \( \phi \) intertwines the \( \mathcal{U}_q(\mathfrak{e}_2) \)-action, that is, \( x \cdot f = \phi(x) \cdot f \) for all \( f \in \mathcal{U}_q(\mathfrak{e}_2) \) and \( x \in \mathcal{O}(E_q(2)) \). Identifying \( x \in \mathcal{O}(E_q(2)) \) with \( \phi(x) \in \mathcal{F}(E_q(2)) \), we can consider \( \mathcal{O}(E_q(2)) \) as a right \( \mathcal{U}_q(\mathfrak{e}_2) \)-module \(*\)-subalgebra of \( \mathcal{F}(E_q(2)) \), and \( \mathcal{U}_q(\mathfrak{e}_2) \rtimes \mathcal{O}(E_q(2)) \) becomes a \(*\)-subalgebra of the corresponding cross product algebra \( \mathcal{U}_q(\mathfrak{e}_2) \rtimes \mathcal{F}(E_q(2)) \).

Let us briefly indicate how Equations (32)–(34) have been derived. Using the definition of the right actions of the generators \( E, F, K \) on \( n, n^* \) and the fact that \( \mathcal{O}(E_q(2)) \) is a \( \mathcal{U}_q(\mathfrak{e}_2) \)-module algebra, one obtains (32)–(34) by straightforward calculations if \( j = k = 0 \) and \( f(r) \) is a polynomial in \( r^2 := n^*n \). Now we postulate that the formulas hold for arbitrary functions \( f(r) \). Setting \( u = nr^{-1} \)

and \( u^{-1} = r^{-1}n^* \), one can compute the actions of \( E, F, K \) on arbitrary elements \( u^j v^k f(r) \) obtaining Equations (32)–(34). On the other hand, since \( n^j = u^j r^j \), \( n^j v^k = u^{-j} r^j \), we rediscover the actions of the generators \( E, F, K \) on the vector space basis \( \{ n^j v^k, n^* v^k : j \in \mathbb{N}_0, l \in \mathbb{N}, k \in \mathbb{Z} \} \) of \( \mathcal{O}(E_q(2)) \) from Equations (32)–(34).

We turn now to the description of a \( \mathcal{U}_q(\mathfrak{e}_2) \)-invariant positive linear functional \( h_{\mu_0} \). Let \( \mu_0 \) be a finite positive Borel measure on the interval \((q, 1]\). We extend \( \mu_0 \) to a positive Borel measure on \( \mathbb{R}^+ \), denoted by \( \mu \), such that \( \mu(q^k M) = q^{k_0} \mu_0(M) \) for \( k \in \mathbb{Z}, M \subseteq (q, 1] \). Let \( \mathcal{F}_0(E_q(2)) \) denote the \(*\)-subalgebra of \( \mathcal{F}(E_q(2)) \) generated by all elements \( p(u, v) f(r) \), where \( p \in \mathbb{C}[u, v] \) and \( f \in \mathcal{F}(\mathbb{R}^+) \) has compact support. Define

\[
 h_{\mu_0}(p(u, v) f(r)) = \int p(u, v) dudv \int_0^{+\infty} f(r) r d\mu(r). \tag{35}
\]

Using the above formulas (32)–(34) for the actions of \( E, F \) and \( K \), one easily verifies that \( h_{\mu_0}(x^* Z) = \varepsilon(Z) h_{\mu_0}(x) \) for \( x = u^j v^k f(r) \) and \( Z = E, F, K \) \( K^{-1}. \)

Since \( \mathcal{F}_0(E_q(2)) \) is a \( \mathcal{U}_q(\mathfrak{e}_2) \)-module algebra, this implies \( h_{\mu_0}(x^* Z) = \varepsilon(Z) h_{\mu_0}(x) \) for all \( x \in \mathcal{F}_0(E_q(2)) \) and \( Z \in \mathcal{U}_q(\mathfrak{e}_2) \). Carrying out the integration over \( \mathbb{R}^2 \), we obtain for \( x = \sum j, k \alpha_{jk} u^j v^k f_{jk}(r) \) and \( y = \sum j, k \beta_{jk} u^j v^k g_{jk}(r) \) from \( \mathcal{F}_0(E_q(2)) \)

\[
 h_{\mu_0}(y^* x) = \sum j, k \alpha_{jk} \beta_{jk} \int_0^{+\infty} f_{jk}(r) g_{jk}(r) r d\mu(r) \tag{36}
\]

This shows that \( h_{\mu_0}(x^* x) \geq 0 \) for all \( x \in \mathcal{F}_0(E_q(2)) \). Thus \( h_{\mu_0} \) is a \( \mathcal{U}_q(\mathfrak{e}_2) \)-invariant positive linear functional on the right \( \mathcal{U}_q(\mathfrak{e}_2) \)-module \(*\)-algebra \( \mathcal{F}_0(E_q(2)) \).

Finally, we describe the Heisenberg representation of the cross product algebra \( \mathcal{U}_q(\mathfrak{e}_2) \rtimes \mathcal{F}(E_q(2)) \) associated with the invariant positive linear functional \( h \equiv h_{\mu_0} \). From (36), it follows that the underlying Hilbert space is the tensor product of the
Hilbert spaces $L^2(\mathbb{T}^2)$ and $L^2(\mathbb{R}^+, rd\mu)$. The actions of the generators $n, v \in O(E_q(2))$, $f(r) \in F(E_q(2))$ and $Z \in U_q(e_2)$ on $L^2(\mathbb{T}^2) \otimes L^2(\mathbb{R}^+, rd\mu)$ are given by

$$\pi_h(n)(u^j v^k \zeta(r)) = q^k u^j v^k (r \zeta(r)), \quad \pi_h(v)(u^j v^k \zeta(r)) = u^j v^k+1 (r \zeta(r)),$$

$$\pi_h(f(r))(u^j v^k \zeta(r)) = u^j v^k f(q^k r) \zeta(r), \quad \pi_h(Z)(u^j v^k \zeta(r)) = (u^j v^k \zeta(r)) \circ S^{-1}(Z),$$

where $j, k \in \mathbb{Z}$ and $\zeta(r) \in F_0(E_q(2))$.

Let $\mathcal{H} = \bigoplus_{m,k,l} \mathcal{H}_{mkl}$, where each Hilbert space $\mathcal{H}_{mkl}$ is $L^2((q, 1], rd\mu)$.

Define a linear operator $W: \mathcal{H} \to L^2(\mathbb{T}^2) \otimes L^2(\mathbb{R}^+, rd\mu)$ by

$$W_{\zeta_{mkl}} := q^m u^k v^l \zeta(q^m r), \quad \zeta \in L^2((q, 1], rd\mu), \ m, k, l \in \mathbb{Z},$$

From

$$\int_{q^{-m+1}}^{q^{-m}} |q^m \zeta(q^m r)|^2 r \, d\mu(r) = \int_{q^{-m+1}}^{q^{-m}} |\zeta(q^m r)|^2 q^{-m} r \, d\mu(q^m r) = \int_1^q \zeta(1)^2 r \, d\mu(r),$$

it follows that $W$ is isometric. Since $\text{Lin}\{W_{\zeta_{mkl}}; \zeta \in L^2((q, 1], rd\mu), \ m, k, l \in \mathbb{Z}\}$ is dense in $L^2(\mathbb{T}^2) \otimes L^2(\mathbb{R}^+, rd\mu)$, we conclude that $W$ is a unitary operator. Hence the Heisenberg representation on $L^2(\mathbb{T}^2) \otimes L^2(\mathbb{R}^+, rd\mu)$ is unitarily equivalent to a $*$-representation on $\mathcal{H}$. Straightforward calculations show that the action of $U_q(e_2) \rtimes O(E_q(2))$ is determined by the following formulas:

$$v_{\zeta_{mkl}} = \zeta_{m,k+1,l}, \quad n_{\zeta_{mkl}} = q^{-m+1} Q \zeta_{m,k+1,l}, \quad K_{\zeta_{mkl}} = q^{-(k+1)/2} \zeta_{mkl},$$

$$F_{\zeta_{mkl}} = -q^{m+(k-1)/2} \lambda^{-1} Q^{-1} \zeta_{m,k-1,l-1} + q^{-m-(k+1)/2} \lambda^{-1} Q^{-1} \zeta_{m-1,k-1,l-1},$$

$$E_{\zeta_{mkl}} = -q^{m+(k-1)/2} \lambda^{-1} Q^{-1} \zeta_{m,k+1,l+1} + q^{-m-(k+1)/2} \lambda^{-1} Q^{-1} \zeta_{m+1,k+1,l+1},$$

where $Q$ denotes the multiplication operator on the Hilbert space $L^2((q, 1], rd\mu)$, that is, $Q \zeta(r) := r \zeta(r)$. We carry out the computations only for the generator $E$, the other formulas are proved analogously. Recall from Subsection 2.2 that

$$\pi_h(E)(p(u, v) f(r)) = (p(u, v) f(r)) \circ S^{-1}(E).$$

This gives

$$E_{\zeta_{mkl}} = W^{-1}(-q^{m+1}(u^k v^l \zeta(q^m r)) \circ E)$$

$$= W^{-1}(-q^{m+(k-1)/2} \lambda^{-1} u^k v^l (r^{-1} \zeta(q^m r) - q^{-k} r^{-1} \zeta(q^{m+1} r)))$$

$$= -q^{m+(k-1)/2} \lambda^{-1} Q^{-1} \zeta_{m,k+1,l+1} + q^{-m-(k+1)/2} \lambda^{-1} Q^{-1} \zeta_{m+1,k+1,l+1}$$

as asserted. Now let $\beta \in \mathbb{Z}$ such that $-q^{-\beta-1} \lambda^{-1} \in (q, 1]$. We define a unitary transformation $U$ on $\mathcal{H}$ by renaming the indices in the following way:

$$U_{\eta_{mkl}} := (-1)^m \zeta_{m-k,l, \beta-m+k, \beta-m+k+l}, \quad U^{-1}_{\zeta_{mkl}} = (-1)^m \eta_{m-l, -\beta+m-k-l, -k+l}.$$
Computing $U^{-1}fU\eta_{mkl}$ for the generators $f = v, n, K, E, F$ shows the representation of $U_q(e_2) \times O(E_q(2))$ on $\mathcal{H}$ is unitarily equivalent to the $*$-representation $\left(JI\right)_{q^{-1}-q^{-\beta-1}l,m}^{l,m}$ from the preceding subsection.

Our next aim is the construction of a $U_q(e_2)$-invariant positive linear functional for the quantum complex plane $\mathbb{C}_q$. We proceed in a similar manner as in the case of $E_q(2)$. Let $\mathbb{C}[w]$ denote the algebra of complex Laurent polynomials in $w$ and let $\mathcal{F}(\mathbb{C}_q)$ be the $*$-algebra generated by the two algebras $\mathcal{F}(\mathbb{R}^+)$ and $\mathbb{C}[w]$ with cross commutation relation $w^k f(r) = f(q^{-k}r)w^k$ and involution $(w^k f(r))^* = \bar{f}(r)w^{-k}$, where $k \in \mathbb{Z}$ and $f \in \mathcal{F}(\mathbb{R}^+)$. We turn $\mathcal{F}(\mathbb{C}_q)$ into a right $U_q(e_2)$-module $*$-algebra with right $U_q(e_2)$-action $*$ by setting

\begin{align*}
  w^k f(r) & \mapsto E = q^{-3/2}\lambda^{-1}w^{k+1}(f(r) - q^{-k}f(qr))r^{-1}, \quad (37) \\
  w^k f(r) & \mapsto F = q^{1/2}w^{k-1}(f(r) - q^{-k}f(q^{-1}r))r^{-1}, \quad (38) \\
  w^k f(r) & \mapsto K = q^k w^k f(r), \quad (39)
\end{align*}

where $k \in \mathbb{Z}$ and $f \in \mathcal{F}(\mathbb{R}^+)$. There is a $*$-isomorphism $\phi$ from $O(\mathbb{C}_q)$ onto the $*$-subalgebra of $\mathcal{F}(\mathbb{C}_q)$ generated by $wr$ such that $\phi(z) = wr$ which intertwines the $U_q(e_2)$-action. Again, let $\mathcal{F}_0(\mathbb{C}_q)$ denote the $*$-subalgebra of $\mathcal{F}(\mathbb{C}_q)$ which is generated by $w$ and all functions $f \in \mathcal{F}(\mathbb{R}^+)$ with compact support. Recall that $O(\mathbb{C}_q)$ is a right $U_q(e_2)$-module $*$-subalgebra of $O(E_q(2))$ by identifying $z$ with $vn$. Comparing (31) with the defining relations of $\mathcal{F}(\mathbb{C}_q)$ and (32)–(34) with (37)–(39) shows that we can consider $\mathcal{F}(\mathbb{C}_q)$ as a right $U_q(e_2)$-module $*$-subalgebra of $\mathcal{F}(E_q(2))$ by identifying $w$ with $uv$. From this, we deduce that the linear functional $\hat{h}_{\mu_0}$ defined by

\begin{align*}
  \hat{h}_{\mu_0}(p(w)f(r)) = \int_T p(w)dw \int_0^{\infty} f(r)rd\mu(r)
\end{align*}

is a $U_q(e_2)$-invariant positive linear functional on $\mathcal{F}_0(\mathbb{C}_q)$, where $\mu_0$ and $\mu$ are given as above. Moreover, the Heisenberg representation of $U_q(e_2) \times \mathcal{F}(\mathbb{C}_q)$ associated with $\hat{h}_{\mu_0}$ is unitarily equivalent to the restriction of the Heisenberg representation of $U_q(e_2) \times \mathcal{F}(E_q(2))$ associated with $h_{\mu_0}$ to $U_q(e_2) \times \mathcal{F}(\mathbb{C}_q)$. In particular, the Heisenberg representation of $U_q(e_2) \times \mathcal{F}(\mathbb{C}_q)$ is unitarily equivalent to the representation on the subspace $\mathcal{H} := \oplus_{m,k=-\infty}^{\infty} \mathcal{H}_{mkk}$ of $\mathcal{H}$, where the actions of $E$, $F$ and $K$ on $\zeta_{mkk} \in \mathcal{H}_{mkk}$ are given by above formulas and the action of $z$ is determined by $z\zeta_{mkk} = q^{-m+k}\zeta_{m,k+1,k+1}$. Setting $\eta_{hk} = (-1)^m q^{\beta+k,\beta-m+k,\beta-m+k}$ and computing the actions of the generators $z, E, F$ and $K$ on $\eta_{hk}$, we obtain the formulas of the $*$-representation $\left(JI\right)_{q^{-1}-q^{-\beta-1}l,m}^{l,m}$ from Subsection 3.3, where $\beta$ is defined as before.
We summarize the main results of this subsection in the next proposition.

**Proposition 3.2** The Heisenberg representation of $\mathcal{U}_q(e_2) \ltimes O(q(SL_q(2)))$ associated with $h_{\mu_0}$ is unitarily equivalent to the $*$-representation $(II)_Q,-q^{-\beta-1}\lambda^{-1}I,I$ from Subsection 3.4 and the Heisenberg representation of $\mathcal{U}_q(e_2) \ltimes O(C_q)$ associated with $\hat{h}_{\mu_0}$ is unitarily equivalent to the $*$-representation $(II)_Q,-q^{-\beta-1}\lambda^{-1}I,I$ from Subsection 3.3, where $Q$ denotes the multiplication operator on the Hilbert space $L^2((q,1],rd\mu_0)$.

4 Cross product algebras related to the quantum group $SU_q(1, 1)$

4.1 Definitions and “decoupling” of cross product algebras

For a moment, let $q$ be a complex number such that $q \neq 0, \pm 1$. First we repeat the definitions of the left and right cross product algebras of the Hopf algebra $\mathcal{U}_q(sl_2)$ with the coordinate Hopf algebra $O(SL_q(2))$ from [SW]. Recall that the algebra $O(SU_q(1,1))$ has generators $a, b, c, d$ with defining relations

\[ ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \quad ad - qbc = da - q^{-1}bc = 1, \]

(40)

The Hopf algebra $\mathcal{U}_q(sl_2)$ is generated by $E, F, K, K^{-1}$ with relations

\[ KK^{-1} = K^{-1}K = 1, \quad KE = qEK, \quadKF = q^{-1}FK, \quad EF - FE = \lambda^{-1}(K^2 - K^{-2}), \]

(41)

and Hopf algebra structure

\[ \Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F, \]
\[ \varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0, \quad S(K) = K^{-1}, \quad S(E) = -qE, \quad S(F) = -q^{-1}F. \]

There is a dual pairing of Hopf algebras $\langle \cdot, \cdot \rangle : \mathcal{U}_q(sl_2) \times O(SL_q(2)) \rightarrow \mathbb{C}$ given on generators by

\[ \langle K^\pm, d \rangle = \langle K^\mp, a \rangle = q^{\pm 1/2}, \quad \langle E, c \rangle = \langle F, b \rangle = 1 \]

and zero otherwise. Using Equations (4) and (7), one derives the following cross
commutation relations in the corresponding left and right cross product algebras.

\[ \mathcal{U}_q(\mathfrak{sl}_2) \ltimes \mathcal{O}(\text{SL}_q(2)) : \]
\[
\begin{align*}
    aK &= q^{-1/2} Ka, & aE &= q^{-1/2} Ea, & aF &= q^{-1/2} Fa + K^{-1} c, \\
    bK &= q^{-1/2} Kb, & bE &= q^{-1/2} Eb, & bF &= q^{-1/2} Fb + K^{-1} d, \\
    cK &= q^{1/2} Kc, & cE &= q^{1/2} Ec + K^{-1} a, & cF &= q^{1/2} Fc, \\
    dK &= q^{1/2} Kd, & dE &= q^{1/2} Ed + K^{-1} b, & dF &= q^{1/2} Fd.
\end{align*}
\]

\[ \mathcal{O}(\text{SL}_q(2)) \rtimes \mathcal{U}_q(\mathfrak{sl}_2) : \]
\[
\begin{align*}
    Ka &= q^{-1/2} aK, & Ea &= q^{1/2} aE + bK, & Fa &= q^{1/2} aF, \\
    Kb &= q^{1/2} bK, & Eb &= q^{-1/2} bE, & Fb &= q^{-1/2} bF + aK, \\
    Kc &= q^{-1/2} cK, & Ec &= q^{1/2} cE + dK, & Fc &= q^{1/2} cF, \\
    Kd &= q^{1/2} dK, & Ed &= q^{-1/2} Ed, & Fd &= q^{-1/2} dF + cK.
\end{align*}
\]

Next we want to embed \( \mathcal{O}(\text{SL}_q(2)) \) into a larger algebra where \( b \) and \( c \) are invertible. For this reason, we consider the localization of \( \mathcal{O}(\text{SL}_q(2)) \) at the set \( \mathcal{S} := \{ b^n c^m ; n, m \in \mathbb{N}_0 \} \). Clearly, \( \mathcal{S} \) is a left and right Ore set of \( \mathcal{O}(\text{SL}_q(2)) \), and the algebra \( \mathcal{O}(\text{SL}_q(2)) \) has no zero divisors. Therefore the localization \( \hat{\mathcal{O}}(\text{SL}_q(2)) \) of \( \mathcal{O}(\text{SL}_q(2)) \) at \( \mathcal{S} \) exists and contains \( \mathcal{O}(\text{SL}_q(2)) \) as a subalgebra. Note that all elements \( b^{-n} c^m, n \in \mathbb{Z} \), belong to the center of \( \hat{\mathcal{O}}(\text{SL}_q(2)) \). From Theorem 3.4.1 in [10] or from Theorem 1.2 in [7], it follows that the algebra \( \hat{\mathcal{O}}(\text{SL}_q(2)) \) is a right (resp. left) \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module algebra which contains \( \mathcal{O}(\text{SL}_q(2)) \) as a \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module subalgebra. The actions of generators of \( \mathcal{U}_q(\mathfrak{sl}_2) \) on inverses \( b^{-1}, c^{-1} \) are given by

\[
\begin{align*}
    b^{-1} &\cdot E = 0, & b^{-1} &\cdot F = -q^{-1} db^{-2}, & b^{-1} &\cdot K = q^{1/2} b^{-1}, \\
    c^{-1} &\cdot E = -qac^{-2}, & c^{-1} &\cdot F = 0, & c^{-1} &\cdot K = q^{-1/2} c^{-1}, \\
    E &\cdot b^{-1} = 0, & F &\cdot b^{-1} = -qab^{-2}, & K &\cdot b^{-1} = q^{1/2} b^{-1}, \\
    E &\cdot c^{-1} = -q^{-1} dc^{-2}, & F &\cdot c^{-1} = 0, & K &\cdot c^{-1} = q^{1/2} c^{-1}.
\end{align*}
\]

Taking these formulas as definitions, one may also verify directly that \( \hat{\mathcal{O}}(\text{SL}_q(2)) \) is a \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module algebra without using the results from [10], [7].

Since \( \hat{\mathcal{O}}(\text{SL}_q(2)) \) is a left and right \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module algebra, the cross product algebras \( \mathcal{U}_q(\mathfrak{sl}_2) \ltimes \hat{\mathcal{O}}(\text{SL}_q(2)) \) and \( \hat{\mathcal{O}}(\text{SL}_q(2)) \rtimes \mathcal{U}_q(\mathfrak{sl}_2) \) are well defined. We can regard \( \mathcal{U}_q(\mathfrak{sl}_2) \ltimes \hat{\mathcal{O}}(\text{SL}_q(2)) \) and \( \hat{\mathcal{O}}(\text{SL}_q(2)) \rtimes \mathcal{U}_q(\mathfrak{sl}_2) \) as algebras generated by \( a, b, c \), etc.
\[ b^{-1}, c, c^{-1}, d \text{ and } E, F, K, K^{-1} \text{ with the defining relations of } \mathcal{U}_q(sl_2) \times \mathcal{O}(SL_q(2)) \text{ and } \mathcal{O}(SL_q(2)) \times \mathcal{U}_q(sl_2), \text{ respectively, and the additional relations} \]
\[ bb^{-1} = b^{-1}b = cc^{-1} = c^{-1}c = 1. \quad (42) \]

The left cross product algebra is isomorphic to its right-handed counterpart. An isomorphism \( \theta \) is given by \( \theta(a) = a, \theta(b) = -qc, \theta(c) = -q^{-1}b, \theta(d) = d, \theta(E) = F, \theta(F) = E \) and \( \theta(K) = K^{-1} \).

Next we show that, by passing to different generators, the defining relations of the cross product algebras simplify remarkably. The following lemma is proved by direct calculations. We restrict ourselves to the right-handed version.

**Lemma 4.1** Set
\[ Q := -q^{1/2} \lambda K^{-1} E - K^{-2} c^{-1} a, \quad R := q^{1/2} \lambda FK^{-1} - qdb^{-1}K^{-2}. \quad (43) \]

Then
\[ xQ = Qx, \quad xR = RX, \quad x \in \mathcal{O}(SU_q(1,1)), \]
\[ KQ = qQK, \quad KR = q^{-1}RK, \quad QR - q^2 RQ = 1 - q^2. \quad (45) \]

By (43), we can write
\[ E = -q^{-1/2} \lambda^{-1}(KQ + K^{-1}c^{-1}a), \quad F = q^{1/2} \lambda^{-1}(RK + qdb^{-1}K^{-1}). \quad (46) \]

It is straightforward to check that \( a, b, b^{-1}, c, c^{-1}, d \text{ and } Q, R, K, K^{-1} \) are generators of the cross product algebra \( \mathcal{U}_q(sl_2) \times \hat{\mathcal{O}}(SL_q(2)) \) satisfying the defining relations (40), (42), (44), (45) and
\[ aK = q^{-1/2}Ka, \quad bK = q^{-1/2}Kb, \quad cK = q^{1/2}Kc, \quad dK = q^{1/2}Kd. \quad (47) \]

Let \( \mathcal{U} \) denote the subalgebra of \( \mathcal{U}_q(sl_2) \times \hat{\mathcal{O}}(SL_q(2)) \) generated by \( Q, R, K, K^{-1} \). Then \( \mathcal{U}_q(sl_2) \times \hat{\mathcal{O}}(SL_q(2)) \) can be considered as the algebra generated by the subalgebras \( \hat{\mathcal{O}}(SL_q(2)) \) and \( \mathcal{U} \) with “almost decoupled” cross relations (44) and (47).

The main advantage of the new generators \( Q \) and \( R \) is that they commute with the elements of the algebra \( \hat{\mathcal{O}}(SL_q(2)) \) by (44). This fact and the form of these generators can also be derived from Lemma 2.2 applied to the right coideals \( \mathcal{V} = \text{Lin}\{EK, \varepsilon\}, \mathcal{V}' = \text{Lin}\{FK, \varepsilon\} \) and the set of generators \( \mathcal{X}_0 = \{a, b, c, d\} \).

Indeed, for \( v = EK \), condition (9) means that
\[ a\rho(EK) = q^{-1} \rho(EK)a, \quad b\rho(EK) = q^{-1} \rho(EK)b, \]
\[ c\rho(EK) = q\rho(EK)c + q^{-1/2}a, \quad d\rho(EK) = q\rho(EK)d + q^{-1/2}b. \]
Therefore, setting \( \rho(EK) = -q^{-3/2} \lambda^{-1} c \) and \( \rho(\varepsilon) = 1 \), (9) is satisfied for \( \mathcal{V} = \text{Lin}\{EK, \varepsilon\} \), and we get

\[
\xi(EK) = \rho(EK)S(K^2) + \rho(\varepsilon)S(EK) = -q^{-3/2} \lambda^{-1} c K^{-2} - q K^{-1} E
= q^{1/2} \lambda^{-1} Q.
\]

Similarly, with \( \rho(FK) = q^{1/2} \lambda^{-1} d \), we obtain \( \xi(FK) = -q^{-1/2} \lambda^{-1} R \).

It might be worth to mention that there is no algebra homomorphism \( \varphi \) from \( \mathcal{U}_q(\mathfrak{sl}_2) \) into \( \hat{\mathcal{O}}(\mathfrak{sl}_q(2)) \) such that \( \varphi(x) = x \) for all \( x \in \hat{\mathcal{O}}(\mathfrak{sl}_q(2)) \). Indeed, one can easily show that there is no element \( \varphi(K^2) \in \hat{\mathcal{O}}(\mathfrak{sl}_q(2)) \) such that \( b \varphi(K^2) = q^{-1} \varphi(K^2) b \) and \( c \varphi(K^2) = q \varphi(K^2) c \). Hence the results of [3] do not apply to the cross product algebra \( \mathcal{U}_q(\mathfrak{sl}_2) \times \hat{\mathcal{O}}(\mathfrak{sl}_q(2)) \).

Let us remark that the generators \( Q \) and \( R \) behave nicely under the involutions of the three real forms of \( \mathcal{U}_q(\mathfrak{sl}_2) \) and \( \mathcal{O}(\mathfrak{sl}_q(2)) \). For \( q \) real, we have \( Q^* = -R \) and \( R^* = R \) in the \(*\)-algebras \( \mathcal{U}_q(\mathfrak{su}_2) \times \mathcal{O}(\mathfrak{su}_q(2)) \) and \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \times \mathcal{O}(\mathfrak{su}_q(1, 1)) \), respectively. The third relation of (45) reads then \( QQ^* - q^2 Q^*Q = \pm(1 - q^2) \) with the minus sign in the first case and plus in the second. The representations of this relation are described in Lemma 2.2. For \( |q| = 1 \), we have \( (q^{1/2}Q)^* = q^{1/2}Q \) and \( (q^{-1/2}R)^* = q^{-1/2}R \) in the \(*\)-algebra \( \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R})) \times \hat{\mathcal{O}}(\mathfrak{sl}_q(2, \mathbb{R})) \). Here the involutions of the Hopf \(*\)-algebras \( \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R})) \) and \( \mathcal{O}(\mathfrak{sl}_q(2, \mathbb{R})) \) are defined by \( E^* = -qE, \ F^* = -q^{-1}F, \ K^* = K \) and \( a^* = a, \ b^* = b, \ c^* = c, \ d^* = d \) so that \( \langle \cdot, \cdot \rangle \) is a dual pairing of Hopf \(*\)-algebras and \( \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R})) \times \mathcal{O}(\mathfrak{sl}_q(2, \mathbb{R})) \) is indeed a \(*\)-algebra. In all three cases, the algebra \( \mathcal{U} \) generated by \( Q, R, K, K^{-1} \) is a \(*\)-algebra.

From now, we suppose again that \( q \in (0, 1) \). We are interested in the real forms \( \mathcal{O}(\mathfrak{su}_q(1, 1)) \) and \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \). On generators, the involution is given by

\[
a^* = d, \quad b^* = qc, \quad E^* = -F, \quad K^* = K.
\]

The pairing \( \langle \cdot, \cdot \rangle \) defined above is a dual pairing of Hopf \(*\)-algebras, so the cross product algebras \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \times \hat{\mathcal{O}}(\mathfrak{su}_q(1, 1)) \) and \( \hat{\mathcal{O}}(\mathfrak{su}_q(1, 1)) \times \mathcal{U}_q(\mathfrak{su}_{1,1}) \) are \(*\)-algebras with involutions induced from the \(*\)-algebras \( \hat{\mathcal{O}}(\mathfrak{su}_q(1, 1)) \) and \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \). The mapping \( \theta \) realizing the isomorphism of the left and right cross product algebras is a \(*\)-isomorphism. Hence the \(*\)-algebras \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \times \hat{\mathcal{O}}(\mathfrak{su}_q(1, 1)) \) and \( \hat{\mathcal{O}}(\mathfrak{su}_q(1, 1)) \times \mathcal{U}_q(\mathfrak{su}_{1,1}) \) are \(*\)-isomorphic.

Now we turn to cross product algebras related to the quantum disc. The quantum disc algebra \( \mathcal{O}(\mathcal{U}_q) \) is defined as the \(*\)-algebra generated by \( z \) and \( z^* \) with relation

\[
z^* z - q^2 zz^* = 1 - q^2.
\]
A left action \( \triangleright \) which turns \( \mathcal{O}(U_q) \) into a \( \mathcal{U}_q(su_{1,1}) \)-module \( \ast \)-algebra appears in [4] and [14]. On generators, it takes the form

\[
\begin{align*}
K^{\mp 1} z &= q^{\mp 1} z, & E z &= q^{1/2}, & F z &= -q^{-1/2} z^2, \\
K^{\mp 1} z^* &= q^{\mp 1} z^*, & E z^* &= -q^{1/2} z^{*2}, & F z^* &= q^{-1/2}.
\end{align*}
\]

A right action \( \bowtie \) can be obtained from \( \triangleright \) by applying Lemma 2.1 with the algebra anti-automorphism and coalgebra homomorphism \( \phi : \mathcal{U}_q(su_{1,1}) \to \mathcal{U}_q(su_{1,1}) \) given by \( \phi(K) = K, \phi(E) = qF, \phi(F) = q^{-1}E \). From (8), we derive

\[
\begin{align*}
z \bowtie K^{\mp 1} &= q^{\mp 1} z, & z \bowtie E &= -q^{1/2} z^2, & z \bowtie F &= q^{-1/2}, \\
z^* \bowtie K^{\mp 1} &= q^{\mp 1} z^*, & z^* \bowtie E &= q^{1/2}, & z^* \bowtie F &= -q^{-1/2} z^{*2}.
\end{align*}
\]

These formulas lead to the following cross commutation relations in the corresponding cross product algebras.

\[
\mathcal{U}_q(su_{1,1}) \bowtie \mathcal{O}(U_q) :
\begin{align*}
zK &= q^{-1} K z, & zE &= q^{-1} Ez - q^{1/2} K^{-1} z^2, & zF &= q^{-1} F z + q^{-1/2} K^{-1}, \\
z^* K &= q K z^*, & z^* E &= q Ez^* + q^{1/2} K^{-1}, & z^* F &= q F z^* - q^{-1/2} K^{-1} z^{*2}.
\end{align*}
\]

\[
\mathcal{O}(U_q) \bowtie \mathcal{U}_q(su_{1,1}) :
\begin{align*}
Kz &= q^{-1} z K, & Ez &= qzE + q^{1/2} K, & Fz &= qzF - q^{-1/2} z^2 K, \\
Kz^* &= qz^* K, & Ez^* &= q^{-1} z^* E - q^{1/2} z^{*2} K, & Fz^* &= q^{-1} z^* F + q^{-1/2} K.
\end{align*}
\]

The \( \ast \)-algebras \( \mathcal{U}_q(su_{1,1}) \bowtie \mathcal{O}(U_q) \) and \( \mathcal{O}(U_q) \bowtie \mathcal{U}_q(su_{1,1}) \) are \( \ast \)-isomorphic with a \( \ast \)-isomorphism \( \psi \) determined by \( \psi(z) = z, \psi(K) = K^{-1} \) and \( \psi(E) = -F \).

There is also a “decoupling” for the cross commutation relations of the cross product algebra \( \mathcal{U}_q(su_{1,1}) \bowtie \mathcal{O}(U_q) \). The elements

\[
S := q^{1/2} \lambda F K^{-1} - q z^* K^{-2}, \quad S^* = -q^{-1/2} \lambda K^{-1} E - q K^{-2} z, \quad T := K^{-2}(1 - z^* z)
\]

of the algebra \( \mathcal{U}_q(su_{1,1}) \bowtie \mathcal{O}(U_q) \) satisfy the relations

\[
\begin{align*}
zS &= S z, & z^* S &= S z^*, & z S^* &= S^* z, & z^* T &= T z, & z^* T &= T z^* \quad (49) \\
ST &= q^{-2} T S, & S^* T &= q^2 T S^*, & S^* S - q^2 S S^* &= 1 - q^2. \quad (50)
\end{align*}
\]

Thus, the \( \ast \)-subalgebra generated by \( S, S^* \) and \( T = T^* \) commutes with the \( \ast \)-subalgebra \( \mathcal{O}(U_q) \) of \( \mathcal{U}_q(su_{1,1}) \bowtie \mathcal{O}(U_q) \). These two \( \ast \)-subalgebras generate a large
part but not the whole of the cross product algebra \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \times \mathcal{O}(U_q) \). An alternative set of generators of \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \times \mathcal{O}(U_q) \) is \( z, z^*, S, S^*, K, K^{-1} \). Then the defining relations are (48), the corresponding relations of (49) and (50), and

\[ KK^{-1} = K^{-1}K = 1, \quad Kz = qzK, \quad z^*K = qKz^*, \quad SK = qKS, \quad KS^* = qS^*K. \]

The form of the generators \( S, S^*, T \) and the fact that they commute with the algebra \( \mathcal{O}(U_q) \) can also be obtained from Lemma 2.2 applied to right coideals \( \mathcal{V} = \operatorname{Lin}\{FK, \varepsilon\}, \mathcal{V}' = \operatorname{Lin}\{EK, \varepsilon\}, \mathcal{V}'' = \operatorname{Lin}\{K^2\} \) and the set \( \mathcal{X}_0 = \{z, z^*\} \). Equation (9) is satisfied if we set \( \rho(FK) = q^{1/2}\lambda^{-1}z^*, \rho(EK) = -q^{-1/2}\lambda^{-1}z, \rho(K^2) = 1 - z^*z \) so that \( \xi(FK) = -q^{-1/2}\lambda^{-1}S, \xi(EK) = q^{1/2}\lambda^{-1}S^* \) and \( \xi(K^2) = T \).

Analogously to Subsection 3.1, one can also consider a cross product \(*\)-subalgebra \( \mathcal{U}_0 \times \mathcal{O}(SU_q(1, 1)) \) of \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \times \mathcal{O}(SU_q(1, 1)) \), where \( \mathcal{U}_0 \subset \mathcal{U}_q(\mathfrak{su}_{1,1}) \) is the unital \(*\)-algebra generated by the quantum tangent space of a left-covariant first order differential \(*\)-calculus on \( \mathcal{O}(SU_q(1, 1)) \). For Woronowicz’s 3D-calculus, this cross product algebra and its representations have been studied in [16].

### 4.2 Representations of the \(*\)-algebra \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \times \mathcal{O}(SU_q(1, 1)) \)

By applying Lemma 2.4 to the relation \( aa^* - q^2a^*a = 1 - q^2 \), using the identity \( c^*c = a^*a - 1 \), taking the polar decomposition of the closed operator \( c \), and arguing as in Subsection 3.2, one easily shows that any \(*\)-representation of \( \mathcal{O}(SU_q(1, 1)) \) is unitarily equivalent to a representation on the orthogonal sum \( \mathcal{G} \oplus \mathcal{H} \) of Hilbert spaces \( \mathcal{G} \) and \( \mathcal{H} \) determined by

\[
\begin{align*}
a &= v, \quad d = v^*, \quad b = c = 0 \quad \text{on } \mathcal{G}, \\
a\eta_n &= (1 + q^{2n}A^2)^{1/2}\eta_{n-1}, \quad d\eta_n = (1 + q^{2(n+1)}A^2)^{1/2}\eta_{n+1}, \\
b\eta_n &= q^{n+1}Aw^*\eta_n, \quad c\eta_n = q^nAw\eta_n \quad \text{on } \mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n = \mathcal{K}.
\end{align*}
\]

Here, \( w \) is a unitary and \( A \) is a self-adjoint operator on a Hilbert space \( \mathcal{K} \) satisfying \( wA = Aw \) and \( \sigma(A) \subseteq (q, 1) \), and \( v \) is a unitary operator on \( \mathcal{G} \) (see [16]).

On \(*\)-representations of \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \times \mathcal{O}(SU_q(1, 1)) \), we impose the following regularity conditions. We assume that the restriction to \( \mathcal{O}(SU_q(1, 1)) \) is of the form described above and that there exist dense linear subspaces \( \mathcal{E} \) of \( \mathcal{G} \) and \( \mathcal{D}_0 \) of \( \mathcal{H}_0 \) such that \( v\mathcal{E} = \mathcal{E}, \quad w\mathcal{D}_0 = \mathcal{D}_0, \quad A\mathcal{D}_0 = \mathcal{D}_0, \quad \) and \( \mathcal{E} \oplus \mathcal{D} \) is invariant under the actions of \( a, b, c, d, E, F, K \) and \( K^{-1} \), where \( \mathcal{D} = \operatorname{Lin}\{\eta_n; \eta \in \mathcal{D}_0, \ n \in \mathbb{Z}\} \).
First we show that $G = \{0\}$. Note that $G = \ker b = \ker c$. From $KEb = qbKE$, it follows that the operator $KE$ leaves $E$ invariant. Thus $\nu \eta = a \eta = (q^{-1/2}cKE - q^{1/2}KEc)\eta = 0$ for all $\eta \in E$. Since $\nu$ is unitary and $E$ is dense in $G$, we have $G = \{0\}$.

We assume that the commutation relations of $K$ with $E$, $F$ and the generators of $O(SU_q(1, 1))$ hold in strong sense. Then it follows that the subspaces of $H$ on which $K > 0$ and $K < 0$ are reducing. Therefore we may assume that $K = \epsilon |K|$ with $\epsilon \in \{1, -1\}$. From $c^*cK = Kc^*c$, it follows that $K$ leaves each Hilbert space $H_n$ invariant. Hence there exist positive self-adjoint operators $H_n$ on $H_0$ such that the action of $K$ on $H = \bigoplus_{n=-\infty}^{\infty}H_n$, $H_n = H_0$, is given by $H\eta_n = \epsilon H_n\eta_n$, and each $H_n$ commutes strongly with $A$. The relation $aK = q^{-1/2}Ka$ applied to vectors $\eta_{n+1} \in D \cap H_{n+1}$ implies $H_{n+1} = q^{-1/2}H_n$ since $H_n\alpha_n(A) = \alpha_n(A)H_n$ and $\ker \alpha_n(A) = \{0\}$. Hence $H_n = q^{-n/2}H_0$. Moreover, from $cK = q^{1/2}c$, we derive $wH_0 = q^{1/2}H_0w$.

As $G = \{0\}$, $b$ and $c$ are invertible. With $Q$ defined in (43), we assume that the relation $c^*cQ = Qc^*c$ holds in strong sense. Then it follows by arguments similar to those used in the preceding paragraph that $Q$ acts on $H$ by $Q\eta_n = Q_0\eta_n$, where $Q_0$ denotes an operator on $H_0$ satisfying $Q_0A = AQ_0$ and $wQ_0 = Q_0w$. In addition, $KQ = qQK$ gives $H_0Q_0 = qQ_0H_0$ and the last equation of (45) yields $Q_0Q_0^* - q^2Q_0^*Q_0 = (1 - q^2)$.

Summarizing, the *-representations of $U_q(su_{1,1}) \times O(SU_q(1, 1))$ are obtained by solving the following operator equations on a dense linear subspace $D_0$ of $H_0$:

\[
\begin{align*}
wa &= Aw, & AQ_0 &= Q_0A, & wQ_0 &= Q_0w, \\
AH_0 &= H_0A, & wH_0 &= q^{1/2}H_0w, & H_0Q_0 &= qQ_0H_0, \\
Q_0Q_0^* - q^2Q_0^*Q_0 &= (1 - q^2),
\end{align*}
\]

where $w$ is a unitary, $H_0$ is a positive self-adjoint operator, and $A$ is a self-adjoint operator subjected to the spectral condition $\sigma(A) \subset \{q, 1\}$. In addition, we require that (52) and the first two relations of (51) hold in strong sense, and that $AD_0 = D_0$ and $H_0D_0 = D_0$.

The representations of (53) are listed in Lemma 2.4. If $Q_0$ is given by the series $(I)$ or $(II)_B$, then $H_0$ is a direct sum $\bigoplus_{k \in J}H_{0k}$, $H_{0k} = H_{00}$, where $J = \mathbb{N}_0$ and $J = \mathbb{Z}$ for representations of type $(I)$ and $(II)_B$, respectively. A similar analysis as used to derive the identities (51) and (52) shows that the operators $w$, $A$ and $H$ act on $H_0 = \bigoplus_{k \in J}H_{0k}$, $H_{0k} = H_{00}$, by $w\zeta_k = w_0\zeta_k$, $A\zeta_k = A_0\zeta_k$ and $H_0\zeta_k = q^{-k}H_{00}\zeta_k$, where $w_0$ is unitary, $A_0$ is a self-adjoint operator satisfying
\( \sigma(A_0) \subseteq (q, 1) \) and \( H_{00} \) is a positive self-adjoint operator on \( \mathcal{H}_{00} \) such that
\[
w_0A_0 = A_0w_0, \quad A_0H_{00} = H_{00}A_0, \quad w_0H_{00} = q^{1/2}H_{00}w_0,
\]
and, for representations of type \( (II)_B \),
\[
w_0B = Bw_0, \quad A_0B = BA_0, \quad H_{00}B = BH_{00}.
\]
The representations of these relations are described in Lemma 2.3. We choose the self-adjoint operator \( B \) such that \( 1 \) is not an eigenvalue, that is, \( \sigma(B) \subseteq [q, 1) \).

Finally, suppose that \( Q_0 \) is given by the series \( (III)_u \). Then \( Q_0 = u \) is a unitary operator on \( \mathcal{H}_0 \). Now we apply Lemma 2.3 to the relations \( wA = Aw \), \( wH_0 = q^{1/2}H_0w \) and \( AH_0 = H_0A \). It states that there exist commuting self-adjoint operators \( A_0 \) and \( H_{00} \) on a Hilbert space \( \mathcal{H}_{00} \) satisfying \( \sigma(A_0) \subseteq (q, 1] \) and \( \sigma(H_{00}) \subseteq [q^{1/2}, 1] \) such that \( H_{00} = \oplus_{k=-\infty}^{\infty} H_{0k}, \mathcal{H}_{0k} = \mathcal{H}_{00} \), and the actions of \( w \) and \( H_0 \) on \( \mathcal{H}_0 \) are determined by \( w\zeta_k = \zeta_{k-1}, H_0\zeta_k = q^{k/2}H_{00}\zeta_k \) and \( A\zeta_k = A_0\zeta_k \). From \( w^2uH_0 = Hw^2u \), it follows that \( w^2u \) leaves each Hilbert space \( \mathcal{H}_{0k} \) invariant. Hence we can write \( w^2u\zeta_k = u_k\zeta_k \), where \( u_k \) denotes a unitary operator on \( \mathcal{H}_{0k} (= \mathcal{H}_{00}) \). Accordingly, \( u\zeta_k = u_k\zeta_{k+2} \). Since \( uu = uu \), we have \( u_k = u_{k-1} \), hence all \( u_k \) are equal.

Inserting the expressions derived for \( a, b, c, d, Q \) and \( R = Q^* \) into Equation (43), using the abbreviations \( \alpha_n(t) \) and \( \beta_n(t) \) introduced in the introduction, and renaming the operators, we obtain the following list of \(*\)-representations of \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \rtimes \mathcal{O}(\mathfrak{SU}_q(1, 1)) \).

\[
(I.1)_{A,H,E} : \quad a_{n,k,l} = \alpha_n(A)\eta_{n-1,k,l}, \quad b_{n,k,l} = q^{n+1}A\eta_{n,k,l+1}, \quad c_{n,k,l} = q^nA\eta_{n,k,l-1},
\]
\[
F_{n,k,l} = \lambda^{-1}q^{(-n-2k+l+1)/2}\lambda_{n,k+1}\epsilon H_1\eta_{n,k+1,l+1} + \lambda^{-1}q^{(n+2k-l+1)/2}\epsilon H_1^{-1}\beta_{n+1}(A)\eta_{n+1,k+1,l-1},
\]
\[
E_{n,k,l} = -\lambda^{-1}q^{(-n-2k+l+1)/2}\lambda_{n,k-1}\epsilon H_1\eta_{n,k-1,l} - \lambda^{-1}q^{(n+2k-l+1)/2}\epsilon H_1^{-1}\beta_{n}(A)\eta_{n-1,k+l+1},
\]
\[
K_{n,k,l} = q^{(-n-2k+l+1)/2}\epsilon H_1\eta_{n,k,l} \quad \text{on} \quad \mathcal{H} = \bigoplus_{n,l=-\infty}^{\infty} \bigoplus_{k=0}^{\infty} \mathcal{H}_{n,k,l}, \quad \mathcal{H}_{n,k,l} = \mathcal{K}.
\]

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\[(I.2)_{A,B,H,\epsilon} : \quad \alpha_{nkl} = \alpha_{n}(A)\eta_{n-1,k}, \quad d\eta_{nkl} = \alpha_{n+1}(A)\eta_{n+1,kl}, \quad b\eta_{nkl} = q^{n+1}A\eta_{nk,l+1}, \quad c\eta_{nkl} = q^n A\eta_{nk,l-1}, \quad F\eta_{nkl} = \lambda^{-1}q^{-n-2k+l-1}/2\alpha_{k+1}(B)\epsilon H\eta_{n,k+1,l} \]
\[+ \lambda^{-1}q^{n+2k-l+1}/2\epsilon H^{-1}\beta_{n+1}(A)\eta_{n+1,k,l-1}, \quad E\eta_{nkl} = -\lambda^{-1}q^{-n-2k+l+1}/2\alpha_{k}(B)\epsilon H\eta_{n,k-1,l} \]
\[+ \lambda^{-1}q^{n+2k-l-1}/2\epsilon H^{-1}\beta_{n}(A)\eta_{n-1,k+1,l}, \quad K\eta_{nkl} = q^{(-n-2k+l)/2}\epsilon H\eta_{nkl} \quad \text{on} \quad \mathcal{H} = \bigoplus_{n,k,l=-\infty}^{\infty} \mathcal{H}_{nk}, \quad \mathcal{H}_{nkl} = \mathcal{K}. \]

\[(I.3)_{A,H,v,\epsilon} : \quad \alpha_{nk} = \alpha_{n}(A)\eta_{n-1,k}, \quad d\eta_{nk} = \alpha_{n+1}(A)\eta_{n+1,k}, \quad b\eta_{nk} = q^{n+1}A\eta_{n,k+1}, \quad c\eta_{nk} = q^n A\eta_{n,k-1}, \quad F\eta_{nk} = \lambda^{-1}q^{-(n+k-1)/2}\epsilon Hv\eta_{n-1,k-2} + \lambda^{-1}q^{(n+k+1)/2}\epsilon H^{-1}\beta_{n+1}(A)\eta_{n+1,k-1}, \quad E\eta_{nk} = -\lambda^{-1}q^{-(n+k+1)/2}\epsilon Hv^{*}\eta_{n,k+2} - \lambda^{-1}q^{(n-k-1)/2}\epsilon H^{-1}\beta_{n}(A)\eta_{n-1,k+1}, \quad K\eta_{nk} = q^{-(n+k)/2}\epsilon H\eta_{nk} \quad \text{on} \quad \mathcal{H} = \bigoplus_{n,k=-\infty}^{\infty} \mathcal{H}_{nk}, \quad \mathcal{H}_{nk} = \mathcal{K}. \]

Here, \(A, B, H\) are commuting self-adjoint operators acting on a Hilbert space \(\mathcal{K}\) such that \(\sigma(A) \subseteq (q, 1], \sigma(B) \subseteq [q, 1)\), and \(\sigma(H) \subseteq (q^{1/2}, 1]\). In the last series, \(v\) is a unitary operator on \(\mathcal{K}\) satisfying \(Av = vA\) and \(Hv = vH\). The parameter \(\epsilon\) takes values in \([-1, 1]\). Representations labeled by different sets of parameters (within unitary equivalence) or belonging to different series are not unitarily equivalent. A representation of this series is irreducible if and only if \(\mathcal{K} = \mathbb{C}\). In this case, we can regard the parameters \(A, H, B\) and \(v\) as complex numbers such that \(A \in (q, 1], H \in (q^{1/2}, 1], B \in [q, 1)\) and \(|v| = 1\).

### 4.3 Representations of the \(*\)-algebra \(\mathcal{U}_q(\mathfrak{su}_{1,1}) \rtimes \mathcal{O}(\mathbb{U}_q)\)

Clearly, \(\mathcal{O}(\mathbb{U}_q)\) is a \(*\)-subalgebra of \(\mathcal{U}_q(\mathfrak{su}_{1,1}) \rtimes \mathcal{O}(\mathbb{U}_q)\). By Lemma 2.4, there are three series of \(*\)-representations of \(\mathcal{O}(\mathbb{U}_q)\). Our aim is to extend these representations to \(*\)-representations of the cross product algebra \(\mathcal{U}_q(\mathfrak{su}_{1,1}) \rtimes \mathcal{O}(\mathbb{U}_q)\).

To begin, let us determine the action of \(K\) on the Hilbert space \(\mathcal{H}\) from Lemma 2.4. Assuming that the commutation relations of \(K\) with \(E, z\) and \(z^*\) hold in strong sense, \(\mathcal{H}\) decomposes into two reducing subspaces on which \(K > 0\) and \(K < 0\). Studying both cases separately, we can write \(K = \epsilon |K|\), where \(\epsilon \in \{1, -1\}\).

Note that \(z^*\) and \(K\) satisfy the same relations as \(Q_0\) and \(H_0\) in Equations (52) and (53). If \(z\) is given by the formulas of the series (I) or (II)\(_A\), then the same
reasoning as in Subsection 4.2 shows that $K$ acts on $H$ by $K\eta_n = q^n\epsilon H\eta_n$, where $H$ denotes an invertible positive self-adjoint operator on $H_0$. In addition, we have $AH = HA$ in the case $II$.

In the third series $(III)_v$, $z = v$ is a unitary operator. Thus we can apply Lemma 2.3 to describe the representations of the relation $zK = q^{-1}zK$. It states that $H = \oplus_{n=\infty}^{\infty} H_n$, where each $H_n$ is $H_0$, $z\eta_n = \eta_{n+1}$ and $K\eta_n = q^n\epsilon H\eta_n$, where $H$ denotes a self-adjoint operator on $H_0$ such that $\sigma(H) \subseteq (q, 1]$.

In the first two series, it follows from $S\eta z = z^*zS$, $Sz = zS$ and $SK = qKS$ by standard arguments used repeatedly in this paper that $S$ acts on $H$ by $S\eta_n = S_0\eta_n$, where $S_0$ is an operator on $H_0$ satisfying

$$S_0H = qHS_0, \quad S_0^*S_0 - q^2S_0S_0^* = 1 - q^2$$

and, in the second series, $S_0A = AS_0$. Hence $S_0$, $S_0^*$ and $H^{-1}$ satisfy on $H_0$ the same relations as $z$, $z^*$ and $K$ on $H$ so that above results concerning $z$, $z^*$ and $K$ apply. The operator $A$ can be handled in the same way as the operator $A$ in Equations (52) and (53), where $Q_0$ plays the role of $S_0$. This determines the first two series of representations of $U_q(su_{1,1}) \ltimes O(U_q)$.

Consider now the third series. From $SK = qKS$, it follows that $S$ maps $H_n$ into $H_{n-1}$ since the relation is assumed to hold in strong sense. Write $S\eta_n = S_n\eta_{n-1}$. The identity $Sz = zS$ implies $S_{n+1} = S_n$, therefore $S_n = S_0$ for all $n$. Applying $SK = qKS$ to vectors $\eta_n \in H_n$ shows that $S_0H = HS_0$. On $H_0$, we have again $S_0^*S_0 - q^2S_0S_0^* = 1 - q^2$. The representations of this relation are described in Lemma 2.4. The operator $H$ is treated just as the operator $A$ in the preceding paragraph. This completes the discussion of the third series.

Carrying out all details, we obtain the following nine series of $*$-representations of $U_q(su_{1,1}) \ltimes O(U_q)$. Let $K$ be a Hilbert space. Suppose that $H_i$, $A_i$, $i = 1, 2$, are self-adjoint operators acting on $K$ such that $\ker H_i = \{0\}$, $\sigma(H_2) \subseteq (q, 1]$ and $\sigma(A_i) \subseteq (q^2, 1]$, $i = 1, 2$, and suppose that $v$ is a unitary operator on $K$. Assume that $A_1A_2 = A_2A_1$, $H_iA_j = A_jH_i$, $i, j = 1, 2$, and $H_2v = vH_2$. Let $\epsilon \in \{\pm 1\}$. The representations will be labeled by $(I.1)_{H_1}$, $(II.1)_{A_1, H_1}$, etc. Define the Hilbert spaces $H = \oplus_{n,k=0}^{\infty} H_{nk}$ in the case $(I.1)_{H_1}$; $H = \oplus_{n=0}^{\infty} \oplus_{k=-\infty}^{\infty} H_{nk}$ in the cases $(I.2)_{A_2, H_1}$ and $(I.3)_{H_1, \epsilon}$; $H = \oplus_{n=-\infty}^{\infty} \oplus_{k=0}^{\infty} H_{nk}$ in the cases $(II.1)_{A_1, H_1}$, $(III.1)_{H_2, \epsilon}$; $H = \oplus_{n,k=-\infty}^{\infty} H_{nk}$ in the cases $(II.2)_{A_1, A_2, H_1}$, $(II.3)_{A_1, H_1, \epsilon}$ and $(III.2)_{A_2, H_2, \epsilon}$; and $H = \oplus_{n=-\infty}^{\infty} H_n$ in the case $(III.3)_{v, H_2, \epsilon}$, where each $H_{nk}$ and
each $\mathcal{H}_n$ is equal to $\mathcal{K}$. The operators $z$ and $z^*$ act as follows.

$$
(I.1)_{H_1}, (I.2)_{A_2,H_1}, (I.3)_{H_2}\varepsilon : 
\begin{align*}
z\eta_{nk} &= \lambda_{n+1}\eta_{n+1,k}, \\
z^*\eta_{nk} &= \lambda_n\eta_{n-1,k},
\end{align*}

$$

$$
(II.2)_{A_1,H_1}, (II.2)_{A_1,A_2,H_1}, (III.3)_{A_1,H_2}\varepsilon : 
\begin{align*}
z\eta_{nk} &= \alpha_{n+1}(A_1)\eta_{n+1,k}, \\
z^*\eta_{nk} &= \alpha_n(A_1)\eta_{n-1,k},
\end{align*}

$$

$$
(III.1)_{H_2}\varepsilon, (III.2)_{H_2,A_2}\varepsilon : 
\begin{align*}
z\eta_{nk} &= \eta_{n+1,k}, \\
z^*\eta_{nk} &= \eta_{n-1,k},
\end{align*}

$$

$$
(III.3)_{H_2,v}\varepsilon : 
\begin{align*}
z\eta_n &= \eta_{n+1}, \\
z^*\eta_n &= \eta_{n-1}.
\end{align*}

The operators $E$, $F$ and $K$ are given by

$$
(I.1)_{H_1} : 
\begin{align*}
K\eta_{nk} &= q^{n-k}H_1\eta_{nk}, \\
F\eta_{nk} &= q^{n-k-1/2}\lambda^{-1}\lambda_{k+1}H_1\eta_{nk,k+1} + q^{-(n-k)+1/2}\lambda^{-1}\lambda_nH_1^{-1}\eta_{n-1,k}, \\
E\eta_{nk} &= -q^{n-k+1/2}\lambda^{-1}\lambda_kH_1\eta_{nk,k-1} - q^{-(n-k)-1/2}\lambda_nH_1^{-1}\eta_{n+1,k}.
\end{align*}

$$

$$
(I.2)_{A_2,H_1} : 
\begin{align*}
K\eta_{nk} &= q^{n-k}H_1\eta_{nk}, \\
F\eta_{nk} &= \lambda^{-1}q^{n-k-1/2}H_1\alpha_{k+1}(A_2)\eta_{nk,k+1} + \lambda^{-1}q^{-(n-k)+1/2}\lambda_nH_1^{-1}\eta_{n-1,k}, \\
E\eta_{nk} &= -\lambda^{-1}q^{n-k+1/2}H_1\alpha_k(A_2)\eta_{nk,k-1} - \lambda^{-1}q^{-(n-k)-1/2}\lambda_nH_1^{-1}\eta_{n+1,k}.
\end{align*}

$$

$$
(I.3)_{H_2}\varepsilon : 
\begin{align*}
K\eta_{nk} &= q^{n-k}\epsilon H_2\eta_{nk}, \\
F\eta_{nk} &= \lambda^{-1}q^{n-k-1/2}\epsilon H_2\eta_{nk,k+1} + \lambda^{-1}q^{-(n-k)+1/2}\lambda_n\epsilon H_2^{-1}\eta_{n-1,k}, \\
E\eta_{nk} &= -\lambda^{-1}q^{n-k+1/2}\epsilon H_2\eta_{nk,k-1} - \lambda^{-1}q^{-(n-k)-1/2}\lambda_n\epsilon H_2^{-1}\eta_{n+1,k}.
\end{align*}

$$

$$
(II.1)_{A_1,H_1} : 
\begin{align*}
K\eta_{nk} &= q^{n-k}H_1\eta_{nk}, \\
F\eta_{nk} &= \lambda^{-1}q^{n-k-1/2}\lambda_{k+1}H_1\eta_{nk,k+1} + \lambda^{-1}q^{-(n-k)+1/2}H_1^{-1}\alpha_n(A_1)\eta_{n-1,k}, \\
E\eta_{nk} &= -\lambda^{-1}q^{n-k+1/2}\lambda_kH_1\eta_{nk,k-1} - \lambda^{-1}q^{-(n-k)-1/2}H_1^{-1}\alpha_{n+1}(A_1)\eta_{n+1,k}.
\end{align*}

$$

$$
(II.2)_{A_1,A_2,H_1} : 
\begin{align*}
K\eta_{nk} &= q^{n-k}H_1\eta_{nk}, \\
F\eta_{nk} &= \lambda^{-1}q^{n-k-1/2}H_1\alpha_{k+1}(A_2)\eta_{nk,k+1} + \lambda^{-1}q^{-(n-k)+1/2}H_1^{-1}\alpha_n(A_1)\eta_{n-1,k}, \\
E\eta_{nk} &= -\lambda^{-1}q^{n-k+1/2}H_1\alpha_k(A_2)\eta_{nk,k-1} - \lambda^{-1}q^{-(n-k)-1/2}H_1^{-1}\alpha_{n+1}(A_1)\eta_{n+1,k}.
\end{align*}

$$

$$
(II.3)_{A_1,H_2}\varepsilon : 
\begin{align*}
K\eta_{nk} &= q^{n-k}\epsilon H_2\eta_{nk}, \\
F\eta_{nk} &= \lambda^{-1}q^{n-k-1/2}\epsilon H_2\eta_{nk,k+1} + \lambda^{-1}q^{-(n-k)+1/2}\epsilon H_2^{-1}\alpha_n(A_1)\eta_{n-1,k}, \\
E\eta_{nk} &= -\lambda^{-1}q^{n-k+1/2}\epsilon H_2\eta_{nk,k-1} - \lambda^{-1}q^{-(n-k)-1/2}\epsilon H_2^{-1}\alpha_{n+1}(A_1)\eta_{n+1,k}.
\end{align*}

$$
4.4 Heisenberg representations of the cross product algebra $\mathcal{U}_q(\mathfrak{su}_{1,1}) \ltimes \mathcal{O}(\text{SU}_q(1, 1))$

We proceed in a similar manner as in Subsection 3.5. Let $\mathcal{F}(\text{SU}_q(1, 1))$ denote the *-algebra generated by the algebra $\mathbb{C}[u, v]$ of complex Laurent polynomials in commuting variables $u, v$ and the algebra $\mathcal{F}(\mathbb{R}^+)$ of locally bounded Borel functions on $\mathbb{R}^+ = (0, +\infty)$ with cross commutation relations and involution given by

$$u^n v^k f(r) = f(q^k r) u^n v^k, \quad (u^n v^k f(r))^* = \bar{f}(r) v^{-k} u^{-n},$$

where $n, k \in \mathbb{Z}$ and $f \in \mathcal{F}(\mathbb{R}^+)$. Define

$$u^n v^k f(r) \cdot E = q^{\frac{n+k+1}{2}} \lambda^{-1} u^{n-1} v^{k+1} \left( f(r) \sqrt{1+q^{-2k}r^2 - q^{-n} f(q^{-1} r) \sqrt{1+r^2}} \right) r^{-1},$$

$$u^n v^k f(r) \cdot F = q^{\frac{n+k-1}{2}} \lambda^{-1} u^{n+1} v^{k-1} \left( f(qr) \sqrt{1+q^2r^2 - q^n f(r) \sqrt{1+q^{-2k+2}r^2}} \right) r^{-1},$$

$$u^n v^k f(r) \cdot K = q^{\frac{n-k}{2}} u^n v^k f(r).$$

Representations labeled by different sets of parameters (within unitary equivalence) or belonging to different series are not unitarily equivalent. A representation of this list is irreducible if and only if $K = \mathbb{C}$. In this case, the parameters $A_i, H_i, i = 1, 2,$ become real numbers such that $H_1 \neq 0, H_2 \in (q, 1], A_i \in (q^2, 1], i = 1, 2,$ and $v$ becomes a complex number of modulus 1.
Straightforward computations show that these formulas define indeed a right action of the Hopf \(\ast\)-algebra \(U_q(su_{1,1})\) on \(\mathcal{F}(SU_q(1,1))\) such that the \(\ast\)-algebra \(\mathcal{F}(SU_q(1,1))\) becomes a right \(U_q(su_{1,1})\)-module \(\ast\)-algebra. We omit the details of this lengthy and tedious verification.

Further, one easily checks that there is an injective \(\ast\)-homomorphism \(\phi\) from \(\mathcal{O}(SU_q(1,1))\) into \(\mathcal{F}(SU_q(1,1))\) given by \(\phi(a) = v\sqrt{1+r^2}\) and \(\phi(c) = ur\) such that \(x\ast f = \phi(x)f\) for \(x \in \mathcal{O}(SU_q(1,1))\) and \(f \in U_q(su_{1,1})\). We shall identify \(x \in \mathcal{O}(SU_q(1,1))\) with \(\phi(x) \in \mathcal{F}(SU_q(1,1))\). Then \(\mathcal{O}(SU_q(1,1))\) is a right \(U_q(su_{1,1})\)-module \(\ast\)-subalgebra of \(\mathcal{F}(SU_q(1,1))\) and the cross product algebra \(U_q(su_{1,1}) \times \mathcal{O}(SU_q(1,1))\) is a \(\ast\)-subalgebra of \(U_q(su_{1,1}) \times \mathcal{F}(SU_q(1,1))\). Using the identities \(a = v\sqrt{1+r^2}\), \(d = a^\ast = \sqrt{1+r^2}u^{-1}\), \(b = qc^\ast = qu^{-1}r\), \(c = ur\), it follows from the definition of the action \(\ast\) on \(\mathcal{F}(SU_q(1,1))\) that

\[
\begin{align*}
   a^k b^l c^n f(r) \ast E &= q^{\frac{k+l+n+1}{2}} r^{-1} a^{k+1} b^l c^{n+1} (f(r) - q^{-2n} f(q^{-1}r)), \\
   d^k b^l c^n f(r) \ast E &= q^{\frac{n-k+l+1}{2}} r^{-1} d^{k+1} b^l c^{n+1} ((1+q^{-2k} r^2) f(r) - (1+r^2) q^{-2n} f(q^{-1}r)),
\end{align*}
\]

for \(k, l, n \in \mathbb{N}_0\) and \(f \in \mathcal{F}(\mathbb{R}^+)\). If we set \(f \equiv 1\) and \(r^2 = q^{-1}bc\), we recover the action of \(E\) on the vector space basis \(\{a^k b^l c^n ; k, l, n \in \mathbb{N}_0, j \in \mathbb{N}\}\) of \(\mathcal{O}(SU_q(1,1))\). The action of \(F\) on this basis is easily obtained from action of \(E\) by applying \(x \ast F = q^{-1} x \ast S(E) = q^{-1} (x^\ast \circ E)\).

The construction of a \(U_q(su_{1,1})\)-invariant linear functional \(h_{\mu_0}\) and of the corresponding Heisenberg representation is completely similar to Subsection 3.5. We fix a finite positive Borel measure \(\mu_0\) on \((q, 1]\) and extend it to a Borel measure \(\mu\) on \(\mathbb{R}^+\) such that \(\mu(q^k \mathcal{M}) = q^k \mu_0(\mathcal{M})\) for \(k \in \mathbb{Z}\) and \(\mathcal{M} \subseteq (q, 1]\). Let \(\mathcal{F}_0(SU_q(1,1))\) be the subalgebra of \(\mathcal{F}(SU_q(1,1))\) generated by the elements \(p(u, v) f(r)\), where \(p(u, v) \in \mathbb{C}[u, v]\) and \(f(r) \in \mathcal{F}(\mathbb{R}^+)\) has compact support. Then the formula

\[
   h_{\mu_0}(p(u, v) f(r)) = \int_{\mathbb{T}^2} p(u, v) dudv \int_0^\infty f(r) r \, d\mu(r),
\]

defines a \(U_q(su_{1,1})\)-invariant positive linear functional \(h_{\mu_0}\) on the right \(U_q(su_{1,1})\)-module \(\ast\)-algebra \(\mathcal{F}_0(SU_q(1,1))\).

The Heisenberg representation \(\pi_h\) of \(U_q(su_{1,1}) \times \mathcal{F}(SU_q(1,1))\) associated with \(h \equiv h_{\mu_0}\) acts on the Hilbert space \(L^2(\mathbb{T}^2) \otimes L^2(\mathbb{R}^+, r d\mu)\). The actions of the generators \(a, b, c, d \in \mathcal{O}(SU_q(1,1))\), \(f(r) \in \mathcal{F}(\mathbb{R}^+)\) and \(X \in U_q(su_{1,1})\) are given...
Let $\mathcal{H} = \bigoplus_{n,k,l=-\infty}^{\infty} \mathcal{H}_{nk,l}$, where each Hilbert space $\mathcal{H}_{nk,l}$ is $L^2((q, 1], r d\mu_0)$. Define a linear operator $W : \mathcal{H} \to L^2(\mathbb{T}^2) \otimes L^2(\mathbb{R}^+, r d\mu)$ by

$$W\zeta_{nk,l} := q^k u^{-l} v^{-n} \zeta(q^k r), \quad \zeta \in L^2((q, 1], r d\mu_0), \ n, k, l \in \mathbb{Z}.$$

The reasoning from Subsection 3.5 which shows that $W$ is unitary applies verbatim. Hence the Heisenberg representation on $L^2(\mathbb{T}^2) \otimes L^2(\mathbb{R}^+, r d\mu)$ is unitarily equivalent to a $*$-representation on $\mathcal{H}$ determined by the following formulas:

$$a\zeta_{nk,l} = \alpha_{n-k}(Q)\zeta_{n-1,k,l}, \quad b\zeta_{nk,l} = q^{n-k+1} Q\zeta_{nk,l+1},$$

$$d\zeta_{nk,l} = \alpha_{n-k+1}(Q)\zeta_{n+1,k,l}, \quad c\zeta_{nk,l} = q^{-n-k} Q\zeta_{nk,l-1},$$

$$F\zeta_{nk,l} = \lambda^{-1}(q^{(n+l+1)/2} \beta_{n-k+1}(Q)\zeta_{n+1,k,l+1} - q^{-n-l-3/2} \alpha_k(Q^{-1})\zeta_{n+1,k+1,l-1}),$$

$$E\zeta_{nk,l} = \lambda^{-1}(q^{-(n+l+1)/2} \alpha_{n-k-1}(Q^{-1})\zeta_{n-1,k-1,l+1} - q^{-n-1-3/2} \beta_{n-k-1}(Q)\zeta_{n-1,k,l+1},$$

$$K\zeta_{nk,l} = q^{-(n+l)/2}\zeta_{nk,l},$$

where $Q$ is the multiplication operator on $L^2((q, 1], r d\mu_0)$. As a sample, we verify the formula for the action of $E$ on $\mathcal{H}$ and compute

$$E\zeta_{nk,l} = W^{-1}(-q^{-k-1}(u^{-l} v^{-n} \zeta(q^k r)) \circ E)$$

$$= W^{-1}(q^{-(n+l+1)/2} \lambda^{-1} q^k u^{-l-1} v^{-n+1}(q^{-1} v^{-1} r^{-2} \zeta(q^k r) - \sqrt{q^{2n+1} r^{-2}} \zeta(q^k r)$$

$$= \lambda^{-1}(q^{-(n+l+1)/2} \alpha_{k-1}(Q^{-1})\zeta_{n-1,k-1,l+1} - q^{-n-1-3/2} \beta_{n-k-1}(Q)\zeta_{n-1,k,l+1}).$$

Applying the unitary transformation $U\eta_{nk,l} := (-1)^k \zeta_{n+k+2,k+2,l-k+2}$ and rewriting above formulas in terms of $\eta_{nk,l}$ proves the following proposition.

**Proposition 4.2** The Heisenberg representation of $\mathcal{U}_q(\mathfrak{su}_{1,1}) \ltimes \mathcal{O}(\mathfrak{su}_q(1, 1))$ associated with $h_{\mu_0}$ is unitarily equivalent to the representation $(II)_{Q,q \phi_{-1/2}}$ from Subsection 4.2, where $Q$ denotes the multiplication operator on the Hilbert space $L^2((q, 1], r d\mu_0)$. 

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4.5 Heisenberg representations of the cross product algebra $U_q(\text{su}_{1,1}) \ltimes \mathcal{O}(U_q)$

Analogously to Subsections 3.5 and 4.4, we denote by $\mathcal{F}(\mathbb{R}\{0\})$ the $*$-algebra of all locally bounded Borel functions on $\mathbb{R}\{0\}$. Let $\mathcal{F}(U_q)$ be the $*$-algebra generated by the two $*$-algebras $\mathcal{O}(U_q)$ and $\mathcal{F}(\mathbb{R}\{0\})$ with relations

$$z^*z = 1 - y, \quad z^n z^{*k} f(y) = f(q^{2(k-n)} y) z^n z^{*k}, \quad n, k \in \mathbb{N}_0, \quad f \in \mathcal{F}(\mathbb{R}\{0\}).$$

From the defining relations of $\mathcal{F}(U_q)$, it follows that each element $x \in \mathcal{F}(U_q)$ can be written as a finite sum

$$x = \sum_{n \geq 1} z^n f_n(y) + f_0(y) + \sum_{k \geq 1} f_{-k}(y) z^{*k}, \quad (54)$$

where the functions $f_j(y) \in \mathcal{F}(\mathbb{R}\{0\})$ are uniquely determined by $x$.

We define a right action $\cdot$ of the Hopf algebra $U_q(\text{su}_{1,1})$ on $\mathcal{F}(U_q)$ by

$$z^n f(y) \cdot E = q^{1/2} \lambda^{-1} z^{n+1} (q^{-n} f(y) - q^n f(q^2 y)), \quad (55)$$

$$f(y) z^{*k} \cdot E = q^{1/2} \lambda^{-1} [z q^k (f(y) - f(q^2 y)) z^{*k} + (q^k - q^{-k}) f(y) z^{*k-1}], \quad (56)$$

$$z^k f(y) \cdot F = q^{-1/2} \lambda^{-1} [z q^k (f(y) - f(q^2 y)) z^* + (q^k - q^{-k}) z^{k-1} f(y)], \quad (57)$$

$$f(y) z^{*n} \cdot F = q^{-1/2} \lambda^{-1} (q^n f(y) - q^{-n} f(q^2 y)) z^{*(n+1)}, \quad (58)$$

$$z^n f(y) \cdot K = q^{-n} z^n f(y), \quad f(y) z^{*k} \cdot K = q^{-k} f(y) z^{*k}. \quad (59)$$

for $n \in \mathbb{N}_0, k \in \mathbb{N}$ and $f \in \mathcal{F}(\mathbb{R}\{0\})$. Then $\mathcal{F}(U_q)$ is a right $U_q(\text{su}_{1,1})$-module $*$-algebra and $\mathcal{O}(U_q)$ is a right $U_q(\text{su}_{1,1})$-module $*$-subalgebra of $\mathcal{F}(U_q)$. As in the previous subsections, we omit the details of these verifications.

Our next aim is to define a $U_q(\text{su}_{1,1})$-invariant positive linear functional on an appropriate $U_q(\text{su}_{1,1})$-module $*$-subalgebra of $\mathcal{F}(U_q)$. Let $\mathcal{F}_0(U_q)$ denote the $*$-algebra generated by the elements $z^k z^{*n} f(y)$, where $k, n \in \mathbb{N}_0$ and $f(y) \in \mathcal{F}(\mathbb{R}\{0\})$ has compact support. With the $U_q(\text{su}_{1,1})$-action given by (55)–(59), $\mathcal{F}_0(U_q)$ becomes a $U_q(\text{su}_{1,1})$-module $*$-algebra. Let $h$ be a linear functional on $\mathcal{F}_0(U_q)$. From (59), it follows that $h$ is invariant under $K$ if and only if

$$h(z^n f(y)) = h(f(y) z^{*n}) = 0 \quad \text{for all } n \in \mathbb{N}. \quad (60)$$

If (60) holds, then (55)–(58) imply that $h$ is invariant under $E$ and $F$ if and only if $h(f(y) z^{*k} \cdot E) = h(z f(y) \cdot F) = 0$. By (56) and (57), the latter is equivalent to

$$h(f(q^{-2} y)(1 - q^{-2} y)) = q^{-2} h(f(y)(1 - y)). \quad (61)$$
That is, \( h \) is invariant if and only if (60) and (61) are satisfied. Now we turn to the positivity condition. Let \( x \) be as in (54). The relations \( z^*z - q^2zz^* = 1 - q^2 \) and \( z^*z = 1 - y \) imply that

\[
z^n z'^n = \prod_{l=0}^{n-1} (1 - q^{2l} y), \quad z^* z'^* = \prod_{l=1}^{n} (1 - q^{-2l} y), \quad n \in \mathbb{N}.
\]

Combining the latter with condition (60), we obtain

\[
h(x^*x) = \sum_{n \geq 1} h \left( |f_n(y)|^2 \prod_{l=0}^{n-1} (1 - q^{2l} y) \right) + h(|f_0(y)|^2) + \sum_{k \geq 1} h \left( |f_{-k}(y)|^2 \prod_{l=1}^{k} (1 - q^{-2l} y) \right).
\]

(62)

Let us suppose that \( h \) is given by a positive measure on \( \mathbb{R} \setminus \{0\} \). From (62), we conclude that \( h(x^*x) \geq 0 \) for all \( x \) provided that the support of the measure is contained in the set \( (-\infty, 0) \cup \{ q^{2n}; n \in \mathbb{N} \} \). We write the measure as a sum of measures with supports contained in \((-\infty, 0) \) and \( \{ q^{2n}; n \in \mathbb{N} \} \), respectively. That is, we write \( h = h_I + h_{\mu_0} \), where \( h_I \) and \( h_{\mu_0} \) are defined by

\[
h_I(z^n f(y)) = h_I(f(y) z^n) = \delta_{n0} \sum_{k=1}^{\infty} f(q^{2k}) q^{-2k},
\]

\[
h_{\mu_0}(z^n f(y)) = h_{\mu_0}(f(y) z^n) = \delta_{n0} \int_{-\infty}^{0} f(y) y^{-2} \, d\mu(y).
\]

Here \( \mu_0 \) is a finite positive Borel measure on \([-1, -q^2] \) and \( \mu \) denotes its extension to a Borel measure on \((-\infty, 0) \) such that \( \mu(q^{2k} \mathcal{M}) = q^{2k} \mu_0(\mathcal{M}) \) for \( k \in \mathbb{Z}, \mathcal{M} \subseteq [-1, -q^2] \). One easily checks that Equation (61) is satisfied for both functionals \( h_I \) and \( h_{\mu_0} \). Hence \( h_I \) and \( h_{\mu_0} \) are \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \)-invariant positive linear functionals on the right \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \)-module \( \ast \)-algebra \( \mathcal{F}_0(U_q) \).

Let \( h = h_I \) or \( h = h_{\mu_0} \). For the generators \( z \) and \( z^* \), the Heisenberg representation \( \pi_h \) acts on the domain \( \mathcal{F}_0(U_q) \) by

\[
\pi_h(z)x = \sum_{n \geq 0} z^{n+1} f_n(y) + \sum_{k \geq 1} f_{-k}(q^{-2} y) (1 - q^{-2} y) z^{*(k-1)},
\]

\[
\pi_h(z^*)x = \sum_{n \geq 1} z^{n-1} (1 - q^{2n-2} y) f_n(y) + \sum_{k \geq 0} f_{-k}(q^2 y) z^{*(k+1)},
\]

where \( x \in \mathcal{F}_0(U_q) \) is given by (54).
Let \( u \) be the partial isometry from the polar decomposition of the closure of the operator \( \pi_h(z) \). When \( h = h_{\mu_0} \), the operator \( u \) is a bilateral shift, so \( u \) is unitary. Using this fact, we present another approach to the Heisenberg representation \( \pi_h \). For notational convenience, we replace \( y \) by \( -y \).

Let \( \mathcal{F}^+(U_q) \) be the \(*\)-algebra generated by the algebra \( \mathbb{C}[u] \) of complex Laurent polynomials in \( u \) and the algebra \( \mathcal{F}((\mathbb{R}^+)) \) of locally bounded Borel functions on \( \mathbb{R}^+ \) with cross commutation relation \( u^n f(y) = f(q^{-2n}y)u^n \) and involution \( (u^n f(y))^* = \bar{f}(y)u^{-n} \), where \( n \in \mathbb{Z} \) and \( f \in \mathcal{F}((\mathbb{R}^+)) \). Then \( \mathcal{F}^+(U_q) \) is a right \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \)-module \(*\)-algebra with respect to the action \( \cdot \) given by the formulas

\[
\begin{align*}
u^n f(y) \cdot E &= q^{1/2} \lambda^{-1} u^{n+1} (q^{-n} f(y) \sqrt{1 + q^{2n} y - q^n f(q^2 y) \sqrt{1 + y}}), \\
u^n f(y) \cdot F &= q^{-1/2} \lambda^{-1} u^{n-1} (q^n f(q^{-2} y) \sqrt{1 + q^{-2} y - q^{-n} f(y) \sqrt{1 + q^{-2n} y}}), \\
u^n f(y) \cdot K &= q^{-n} u^n f(y)
\end{align*}
\]

for \( n \in \mathbb{Z} \) and \( f \in \mathcal{F}((\mathbb{R}^+)) \). There is an injective \(*\)-homomorphism \( \phi : \mathcal{O}(U_q) \to \mathcal{F}^+(U_q) \) such that \( \phi(z) = u \sqrt{1 + y} \) and \( x \cdot f = \phi(x) \cdot f \) for \( x \in \mathcal{O}(U_q) \) and \( f \in \mathcal{U}_q(\mathfrak{su}_{1,1}) \). Thus, we can consider \( \mathcal{O}(U_q) \) as a \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \)-module \(*\)-subalgebra of \( \mathcal{F}^+(U_q) \) by identifying \( x \in \mathcal{O}(U_q) \) with \( \phi(x) \).

Let \( \mu_0 \) be a finite positive Borel measure on \( (q^2, 1] \). We extend \( \mu_0 \) to a Borel measure \( \mu \) on \( \mathbb{R}^+ \) such that \( \mu(q^{2k} \mathcal{M}) = q^{2k} \mu_0(\mathcal{M}) \) for \( k \in \mathbb{Z} \), \( \mathcal{M} \subseteq (q^2, 1] \). Let \( \mathcal{F}^+_0(U_q) \) denote the \(*\)-subalgebra of \( \mathcal{F}^+(U_q) \) generated by the elements \( u^k f(y) \), where \( k \in \mathbb{Z} \) and \( f \in \mathcal{F}((\mathbb{R}^+)) \) has compact support. Define a linear functional \( \hat{h}_{\mu_0} \) on \( \mathcal{F}^+_0(U_q) \) by

\[
\hat{h}_{\mu_0}(p(u)f(y)) = \int \frac{p(u) \text{d}u}{\mathcal{T}} \int_0^{+\infty} f(y) y^{-2} \text{d}\mu(y).
\]

From the definitions of the actions of the generators \( E, F, K \), it follows immediately that \( \hat{h}_{\mu_0}(x \cdot f) = \varepsilon(f) \hat{h}_{\mu_0}(x) \) for \( x \in \mathcal{F}^+_0(U_q) \) and \( f = E, F, K \). If \( x = \sum_k u^k f_k(y) \), then

\[
\begin{align*}
\hat{h}_{\mu_0}(x^* x) &= \hat{h}_{\mu_0} \left( \sum_{k,l} \bar{f}_k(y) u^{-k} u^l f_l(y) \right) \\
&= \sum_k \hat{h}_{\mu_0}(u^{l-k} \bar{f}_k(y) q^{2(l-k)} y f_l(y)) = \sum_k \int_0^{+\infty} |f_k(y)|^2 y^{-2} \text{d}\mu(y) \geq 0.
\end{align*}
\]

Therefore, \( \hat{h}_{\mu_0} \) is a \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \)-invariant positive linear functional on \( \mathcal{F}^+_0(U_q) \). If \( \hat{\mu}_0(\mathcal{M}) = \mu_0(-\mathcal{M}) \) for all \( \mathcal{M} \subseteq (q^2, 1] \), then the Heisenberg representations associated with \( \hat{h}_{\mu_0} \) and \( \hat{\mu}_0 \) are unitarily equivalent.
Since \( h = h_I + h_{\mu_0} \) is given by a direct sum of measures with supports contained in \( (-\infty, 0) \) and \( \{q^{2n}; n \in \mathbb{N}\} \), and since the action of the cross product algebra \( U_q(\mathfrak{su}_{1,1}) \otimes \mathcal{F}(U_q) \) respects this decomposition, the Heisenberg representation associated with \( h \) decomposes into a direct sum of Heisenberg representations associated with \( h_I \) and \( h_{\mu_0} \).

Recall that the Heisenberg representations of \( U_q(\mathfrak{su}_{1,1}) \otimes \mathcal{F}(U_q) \) associated with \( h_I \) acts on \( \mathcal{F}_0(U_q) := \mathcal{F}_0(U_q) / \mathcal{N}_I \), where \( \mathcal{N}_I := \{x \in \mathcal{F}_0(U_q); h_I(x^*x) = 0\} \). For \( k \in \mathbb{N}_0 \), let \( \delta_k(r) \) denote the characteristic function of the point \( \{q^{2k}\} \), that is, \( \delta_k(r) = 1 \) if \( r = q^{2k} \) and zero otherwise. Note that \( z^m \delta_k(y) = \delta_k(q^{2n}y)z^m = 0 \) in \( \mathcal{F}_0(U_q) \) for \( n \geq k \). Set

\[
\zeta_{nk} := \left( \prod_{l=0}^{n-1} (1 - q^{2(k+l)}) \right)^{-1/2} q^k z^m \delta_k(y), \quad \zeta_{0k} := q^k \delta_k(y), \quad k \in \mathbb{N}, \ n \in \mathbb{N},
\]

\[
\zeta_{nk} := \left( \prod_{l=1}^{n} (1 - q^{2(k-l)}) \right)^{-1/2} q^k z^m \delta_k(y), \quad k > 1, \ n = -k + 1, \ldots, -1.
\]

Then \( \{\zeta_{nk}; k \in \mathbb{N}, n = -k + 1, -k + 2, \ldots\} \) is a set of orthonormal vectors which span \( \mathcal{F}_0(U_q) \). Computing the actions of \( z, z^*, E, F, K \) and \( f(r) \in \mathcal{F}(\mathbb{R} \setminus \{0\}) \) on \( \zeta_{nk} \) gives

\[
z\zeta_{nk} = \lambda_{n+k-1} \zeta_{n+1, k}, \quad z^* \zeta_{nk} = \lambda_{n+k-1} \zeta_{n+1, k}, \quad f(r) \zeta_{nk} = f(q^{2(n+k)}) \zeta_{nk},
\]

\[
E \zeta_{nk} = \lambda^{-1} q^{n+1/2} \lambda_{n-1} \zeta_{n+1, k-1} - \lambda^{-1} q^{-n-1/2} \lambda_{n+k} \zeta_{n+1, k},
\]

\[
F \zeta_{nk} = -\lambda^{-1} q^{-n+1/2} \lambda_{n-1} \zeta_{n-1, k-1} + \lambda^{-1} q^{-n+1/2} \lambda_{n+k} \zeta_{n-1, k}, \quad K \zeta_{nk} = q^n \zeta_{nk}.
\]

For instance, since \( \delta_k(q^{2n}y) = \delta_k(-y) \) and \( z \delta_k(y) z^* = (1 - q^{-2}y) \delta_{k+1}(y) = (1 - q^{2k}) \delta_{k+1}(y) \), we have

\[
z^m \delta_k(y) E = \delta_{k-m}(y) z^m E = q^{1/2} \lambda^{-1} [q^m (1 - q^{2(k-m)}) \delta_{k-m+1}(y) - q^{m}(1 - q^{2(k-m-1)}) \delta_{k-m}(y)] z^{m-1}
\]

\[
= q^{1/2} \lambda^{-1} [q^m (1 - q^{2(k-m)})] z^{m-1} \delta_k(y) - q^{m}(1 - q^{2(k-1)}) z^{m-1} \delta_k(y).
\]

From the latter expression, we derive the action of \( E \) on \( \zeta_{m,k} \), \( k, m \in \mathbb{N}, m < k \), by inserting the definition of the vectors \( \zeta_{nk} \). If we set \( \eta_{nk} := (-1)^k \zeta_{n-k,k+1}, \ n, k \in \mathbb{N}_0 \), then the actions of \( z, z^*, K, E \) and \( F \) on \( \eta_{nk} \) coincide with the formulas of the series (I.1) from Subsection 4.3.

Next we turn to the the Heisenberg representation of \( U_q(\mathfrak{su}_{1,1}) \otimes \mathcal{F}(U_q) \) associated with \( h_{\mu_0} \). As noted above, it is unitarily equivalent to the Heisenberg representation associated with \( \hat{h}_{\mu_0} \), where \( \hat{\mu_0} \) is a Borel measure on \( (q^2, 1] \) such
that \( \hat{\mu}_0(\mathcal{M}) = \mu_0(-\mathcal{M}) \) for \( \mathcal{M} \subseteq (q^2, 1] \). The latter representation acts on the Hilbert space \( \mathcal{L}^2(\mathbb{T}) \otimes \mathcal{L}^2(\mathbb{R}^+, y^{-2}d\mu) \) by

\[
z(u^n\zeta(y)) = u^{n+1}\sqrt{1 + q^{2n}y}\zeta(y), \quad z^*(u^n\zeta(y)) = u^{n-1}\sqrt{1 + q^{2(n-1)}y}\zeta(y),
\]

\[
f(y)(u^n\zeta(y)) = u^n f(q^{2n}y)\zeta(y), \quad Z(u^n\zeta(y)) = (u^n\zeta(y))oS^{-1}(Z),
\]

where \( f(y) \in \mathcal{F}(\mathbb{R}\setminus\{0\}) \) and \( Z \in U_q(\mathfrak{su}_{1,1}) \).

Let \( \mathcal{H} = \bigoplus_{n,k=-\infty}^{\infty} \mathcal{H}_{nk} \), where each \( \mathcal{H}_{nk} \) is \( \mathcal{L}^2((q^2, 1], y^{-2}d\mu_0) \), and consider the linear operator \( W : \mathcal{H} \rightarrow \mathcal{L}^2(\mathbb{T}) \otimes \mathcal{L}^2(\mathbb{R}^+, y^{-2}d\mu) \) defined by

\[
W\zeta_{nk} := q^k u^n\zeta(q^{-2k}y), \quad \zeta \in \mathcal{L}^2((q^2, 1], y^{-2}d\mu_0), \quad n, k, l \in \mathbb{Z}.
\]

Similarly to Subsection 4.4, one shows that \( W \) is a unitary operator. Hence the Heisenberg representation associated with \( \hat{\mu}_0 \) is unitarily equivalent to a \(*\)-representation on \( \mathcal{H} \). Let \( Q \) denote the multiplication operator on \( \mathcal{L}^2((q^2, 1], y^{-2}d\mu_0) \). Computing the actions of \( z, z^*, K, E \) and \( F \) on vectors \( \zeta_{nk} \) gives

\[
z\zeta_{nk} = \sqrt{1 + q^{2(n+k)}Q}\zeta_{n+1,k}, \quad z^*\zeta_{nk} = \sqrt{1 + q^{2(n-k-1)}Q}\zeta_{n-1,k}, \quad K\zeta_{nk} = q^n\zeta_{nk};
\]

\[
E\zeta_{nk} = \lambda^{-1}q^{n+1/2}\sqrt{1 + q^{2(k-1)}Q}\zeta_{n+1,k-1} - \lambda^{-1}q^{-n-1/2}\sqrt{1 + q^{2(n+k)}Q}\zeta_{n+1,k};
\]

\[
F\zeta_{nk} = -\lambda^{-1}q^{-n-1/2}\sqrt{1 + q^{2k}Q}\zeta_{n-1,k+1} + \lambda^{-1}q^{-n+1/2}\sqrt{1 + q^{2(n+k-1)}Q}\zeta_{n-1,k}.
\]

Renaming \( \eta_{nk} := (-1)^k\zeta_{-n-k,k+1} \) and computing the actions of \( z, z^*, K, E \) and \( F \) on \( \eta_{nk} \), we obtain the formulas of the series \((II.2)_{Q,Q,1}\).

We summarize the preceding results in the following proposition.

**Proposition 4.3** The Heisenberg representation of \( U_q(\mathfrak{su}_{1,1}) \times O(\mathbb{U}) \) associated with \( h \) is unitarily equivalent to the direct sum of the irreducible \(*\)-representation \((I.1)_I\) and the representation \((II.2)_{Q,Q,1}\) from Subsection 4.3, where \( Q \) denotes the multiplication operator on \( \mathcal{K} = \mathcal{L}^2((q^2, 1], y^{-2}d\mu_0) \).

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