On Fundamental Domains and Volumes of Hyperbolic Coxeter–Weyl Groups

PHILIPP FLEIG\textsuperscript{1,2}, MICHAEL KOEHN\textsuperscript{1}, and HERMANN NICOLAI\textsuperscript{1}

\textsuperscript{1}Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, 14476 Potsdam, Germany. e-mail: nicolai@aei.mpg.de

\textsuperscript{2}Université de Nice-Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France

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Abstract. We present a simple method for determining the shape of fundamental domains of generalized modular groups related to Weyl groups of hyperbolic Kac–Moody algebras. These domains are given as subsets of certain generalized upper half planes, on which the Weyl groups act via generalized modular transformations. Our construction only requires the Cartan matrix of the underlying finite-dimensional Lie algebra and the associated Coxeter labels as input information. We present a simple formula for determining the volume of these fundamental domains. This allows us to re-produce in a simple manner the known values for these volumes previously obtained by other methods.

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1. Introduction

Constructions of fundamental domains of generalized modular groups usually rely on geometric considerations. By considering the different possible symmetry transformations acting on some generalized upper-half plane, the precise shape of the fundamental domain is narrowed down step-by-step until one arrives at its final shape. Especially for higher rank groups (such as $SL_n(\mathbb{Z})$) this poses a considerable computational and combinatorial problem since one has to consider a large number of possible successive symmetry transformations (already the determination of the fundamental domain of the standard modular group $PSL_2(\mathbb{Z})$ along these lines takes more than two pages of computations, see e.g. [1]). Although one can show that the precise shape of the fundamental domain can be determined within a finite number of steps, in the actual computation of a domain it is not always clear how many steps are actually necessary.

In this paper we show that, at least for modular groups arising as (even) Weyl groups of certain hyperbolic Kac–Moody algebras, such cumbersome constructions
can be altogether avoided. More specifically, we present an easy method for obtaining the complete geometric information about the associated fundamental domains. All we require as information for determining the explicit shape and volume is the Cartan matrix of the corresponding Kac–Moody algebra and its Coxeter labels. As we will demonstrate this construction works for all hyperbolic Kac–Moody algebras of over-extended type, which are generally obtained by extending a given finite dimensional simple Lie algebra via its affine extension by adding two nodes to the Dynkin diagram in a specified way. Likewise, it applies to the twisted algebras obtained by inverting the arrows in the Dynkin diagram, because their Weyl groups are the same (but note that these twisted algebras, while being indefinite Kac–Moody algebras, in general are not of over-extended type). In particular, our construction also applies to those hyperbolic Kac–Moody algebras whose even Weyl groups can be identified with generalized modular groups defined over rings of integers in division algebras. The first example of such an identification was given in where it was shown that the rank-3 hyperbolic Kac–Moody algebra $A_1^{++}$ (also denoted $AE_3$ or $F$ in the literature) has the usual modular group $PSL_2(\mathbb{Z})$ as its even Weyl group, the full Weyl group being $W(A_1^{++}) = PGL_2(\mathbb{Z}).$ In more complicated examples were given, involving, for instance, the quaternionic integers (Hurwitz numbers), and admitting a M"{o}bius-like realization. The most interesting (and most complicated) example is the even Weyl group $W^+(E_{10})$ which can be identified with the arithmetic group $PSL_2(\mathbb{O})$ (where $\mathbb{O}$ are octonionic integers, also called octavians). For this example, we will explicitly display the coordinates of the vertices of the fundamental domain of the Weyl group.

Knowledge of the shape of the fundamental domain allows one to compute its volume. In the non-linear realization of the hyperbolic Weyl group on some generalized upper half plane (a hyperbolic space of constant negative curvature) the fundamental domains are realized as higher dimensional simplices. We present a very simple general formula for the volume of the domain in terms of integrals involving a quadratic form which contains all the information about the Lie algebra $g^{++}$ (see (32) below). We note that our considerations would also apply to cases where analogs of the so-called congruence subgroups of $PSL_2(\mathbb{Z})$ can be defined: the volume is then simply a multiple of the original volume, with the factor equal to the index of the congruence subgroup in the given generalized modular group. Such congruence subgroups presumably do exist for the generalized arithmetic groups studied in [7], but we are not aware of any concrete results along these lines.

As an historic aside, we mention that the first computation of hyperbolic volumes in terms of the dihedral angles of the simplex under consideration is due to one of the inventors of hyperbolic geometry, Lobachevsky. His results were

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1An indefinite Kac–Moody algebra is called hyperbolic if the removal of any one node from its Dynkin diagram leaves an algebra which is either affine or finite [11].