RIESZ SEQUENCES AND ARITHMETIC PROGRESSIONS

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Abstract. Given a set $S$ of positive measure on the circle and a set of integers $\Lambda$, one may consider the family of exponentials $E(\Lambda) := \{e^{i\lambda t}\}_{\lambda \in \Lambda}$ and ask whether it is a Riesz sequence in the space $L^2(S)$.

We focus on this question in connection with some arithmetic properties of the set of frequencies.

Improving a result of Bownik and Speegle (3, Thm. 4.16), we construct a set $S$ such that $E(\Lambda)$ is never a Riesz sequence if $\Lambda$ contains arbitrary long arithmetic progressions of length $N$ and step $\ell = O(N^{1-\epsilon})$. On the other hand, we prove that every set $S$ admits a Riesz sequence $E(\Lambda)$ such that $\Lambda$ does contain arbitrary long arithmetic progressions of length $N$ and step $\ell = O(N)$.

1. Introduction

We use below the following notation:

$\Lambda$ - a set of integers.

$S$ - a set of positive measure on the circle $\mathbb{T}$.

$|S|$ - the Lebesgue measure of $S$.

For $A, B \subset \mathbb{R}$, $x \in \mathbb{R}$ we let

$$A + B := \{\alpha + \beta | \alpha \in A, \beta \in B\}, \quad x \cdot A := \{x \cdot \alpha | \alpha \in A\}.$$

A sequence of elements in a Hilbert space $\{\varphi_i\}_{i \in I} \subset \mathcal{H}$ is called a Riesz sequence (RS) if there are positive constants $A, B$ s.t. the inequalities

$$A \cdot \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i \varphi_i \right\|^2 \leq B \cdot \sum_{i \in I} |c_i|^2,$$

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hold for every finite sequence of scalars \( \{c_i\}_{i \in I} \).

Given \( \Lambda \subset \mathbb{Z} \) we denote
\[
E(\Lambda) := \left\{ e^{i\lambda t} \right\}_{\lambda \in \Lambda}.
\]

The following result is classical (see [9], p.203, Lemma 6.5):

If \( \Lambda = \{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{Z} \) is lacunary in the sense of Hadamard, i.e. satisfies
\[
\frac{\lambda_{n+1}}{\lambda_n} \geq q > 1, \ n \in \mathbb{N}
\]

then \( E(\Lambda) \) forms a RS in \( L^2(S) \), for every \( S \subset \mathbb{T} \) with \( |S| > 0 \).

The following generalization is due to I.M. Miheev ([7], Thm. 7):

If \( E(\Lambda) \) is an \( S_p \)-system for some \( p > 2 \), i.e. satisfies
\[
\left\| \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \right\|_{L^p(\mathbb{T})} \leq C \cdot \left\| \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \right\|_{L^2(\mathbb{T})},
\]

with some \( C > 0 \), for every finite sequence of scalars \( \{a_\lambda\}_{\lambda \in \Lambda} \), then it forms a RS in \( L^2(S) \), for every \( S \subset \mathbb{T} \) with \( |S| > 0 \).

J. Bourgain and L. Tzafriri proved the following result as a consequence of their “restricted invertibility theorem” ([2], Thm. 2.2):

Given \( S \subset \mathbb{T} \), there is a RS \( E(\Lambda) \) s.t. \( \Lambda \) is a set of integers with positive asymptotic density
\[
\text{dens} \Lambda := \lim_{N \to \infty} \frac{\# \{\Lambda \cap [-N,N]\}}{2N} > C \cdot |S|.
\]

(Here and below \( C \) denotes positive absolute constants, which might be different from one another).

W. Lawton ([5], Cor. 2.1), assuming the Feichtinger conjecture for exponentials, proved the following proposition:

(\( L \)) For every \( S \) there is a RS \( E(\Lambda) \) s.t. the set of frequencies \( \Lambda \subset \mathbb{Z} \) is syndetic, that is \( \Lambda + \{0, \ldots, n-1\} = \mathbb{Z} \) for some \( n \in \mathbb{N} \).
Recall that the Feichtinger conjecture says that every bounded frame in a Hilbert space can be decomposed in a finite family of RS. This claim turned out to be equivalent to the Kadison-Singer conjecture (see [4]). The last conjecture has been proved recently by A. Marcus, D. Spielman and N. Srivastava (see [6]), so proposition (L) holds unconditionally.

Notice that in some results above the system $E(\Lambda)$ serves as RS for all sets $S$; however the set of frequencies $\Lambda$ is quite sparse there. In others $\Lambda$ is rather dense but it works for $S$ given in advance.

It was asked in [8] whether one can somehow combine the density and "universality" properties. It turned out this is indeed possible. An exponential system has been constructed in that paper which forms a RS in $L^2(S)$ for any open set $S$ of a given measure, and the set of frequencies has optimal density, proportional to $|S|$. This is not true for nowhere dense sets (see [8]).

2. Results

In this paper we consider sets of frequencies $\Lambda$ which contain arbitrary long arithmetic progressions. Below we denote the length of a progression by $N$, by $\ell$ we denote its step. Given $\Lambda$ which contains arbitrary long arithmetic progressions there exists a set $S \subset \mathbb{T}$ of positive measure so that $E(\Lambda)$ is not a RS in $L^2(S)$ (see [7]).

In the case $\ell$ grows slowly with respect to $N$, one can define $S$ independent of $\Lambda$.

A quantitative version of such a result was proved in [3]:

There exists a set $S$ such that $E(\Lambda)$ is not a RS in $L^2(S)$ whenever $\Lambda$ contains arithmetic progression of arbitrary large length $N$, and $\ell = o\left(N^{1/2}\log^{-3}N\right)$.

The proof is based on some estimates of the discrepancy of sequences of the form $\{\alpha k\}_{k \in \mathbb{N}}$ on the circle.

Using a different approach we prove a stronger result:

Theorem 1. There exists a set $S \subset \mathbb{T}$ such that $E(\Lambda)$ is not a RS in $L^2(S)$ whenever $\Lambda$ contains arbitrary long arithmetic progressions with $\ell = O\left(N^\alpha\right)$ for some $\alpha < 1$. 
Here one can construct $S$ not depending on $\alpha$ and with arbitrary small measure of the complement.

The next theorem shows that the result is sharp.

**Theorem 2.** Given a set $S \subset \mathbb{T}$ of positive measure there is a set of frequencies $\Lambda \subset \mathbb{Z}$ such that

(i) $\Lambda$ contains arbitrary long arithmetic progressions with $\ell = O(N)$.

(ii) The system of exponentials $E(\Lambda)$ forms a RS in $L^2(S)$.

Increasing slightly the bound for $\ell$, one can get a version of Theorem 2 which admits a progression of any length:

**Theorem 3.** Given $S$ one can find $\Lambda$ with the property (ii) above and s.t.

(i') For every $\alpha > 1$ and for every $N$ it contains an arithmetic progression of length $N$ and step $\ell < C(\alpha) \cdot N^\alpha$.

3. Proof of Theorem 1

**Proof.** Fix $\varepsilon > 0$. Take a decreasing sequence of positive numbers $\{\delta(\ell)\}_{\ell \in \mathbb{N}}$ s.t.

\[(a) \quad \sum_{\ell \in \mathbb{N}} \delta(\ell) < \frac{\varepsilon}{2} \]

\[(b) \quad \delta(\ell) \cdot \ell^{1/\alpha} \to \infty \quad \ell \to \infty \quad \forall \alpha \in (0,1) \]

For every $\ell \in \mathbb{N}$ set $I_\ell = (-\delta(\ell), \delta(\ell))$ and let $\tilde{I}_\ell$ be the $2\pi$-periodic extension of $I_\ell$, i.e.

$$\tilde{I}_\ell = \bigcup_{k \in \mathbb{Z}} (I_\ell + 2\pi k).$$

We define

$$I_\ell = \left( \frac{1}{\ell} \cdot \tilde{I}_\ell \right) \cap [-\pi, \pi] \quad \text{and} \quad S = \mathbb{T} \setminus \bigcup_{\ell \in \mathbb{N}} I_\ell = \left( \bigcup_{\ell \in \mathbb{N}} I_\ell \right)^c.$$
whence we get that

$$|S| \geq 1 - \sum_{\ell=1}^{\infty} |I_{\ell}| = 1 - \sum_{\ell=1}^{\infty} 2\delta(\ell) > 1 - \varepsilon.$$  

Fix $\alpha < 1$ and let $\Lambda \subset \mathbb{Z}$ be such that one can find arbitrary large $N \in \mathbb{N}$ for which

$$\{M + \ell, \ldots, M + N \cdot \ell \} \subset \Lambda,$$

with some $M = M(N) \in \mathbb{Z}$, $\ell = \ell(N) \in \mathbb{N}$ and

$$\ell < C(\alpha) \cdot N^\alpha.$$  

Recall that by (1) we have $t \in I_{\ell}$ if and only if $t \cdot \ell \in \tilde{I}_\ell \cap [-\pi \ell, \pi \ell]$. Since $S$ lies inside the complement of $I_{\ell}$ we get

$$\left| \sum_{k=1}^{N} c(k) e^{i(M + k\ell)t} \right|^2 \leq \int_{I_{\ell}} \left| \sum_{k=1}^{N} c(k) e^{i(M + k\ell)t} \right|^2 \frac{dt}{2\pi} =$$

$$= \int_{[-\pi \ell, \pi \ell] \setminus \tilde{I}_\ell} \left| \sum_{k=1}^{N} c(k) e^{ik\tau} \right|^2 \frac{d\tau}{2\pi \ell} = \int_{I_{\ell}} \left| \sum_{k=1}^{N} c(k) e^{ik\tau} \right|^2 \frac{d\tau}{2\pi}.$$  

Therefore, in order to complete the proof, it is enough to show that $\left\| \sum_{k=1}^{N} c(k) e^{ik\tau} \right\|_{L^2(I_{\ell}^c)}$ can be made arbitrary small while keeping $\sum_{k=1}^{N} |c(k)|^2$ bounded away from zero. This observation allows us to reformulate the problem as a norm concentration problem of trigonometric polynomials of degree $N$ on the interval $I_{\ell}$.

Let $P_N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{ikt}$, so $\|P_N\|_{L^2(T)} = 1$. Moreover, for every $t \in T$ we have $|P_N(t)| \leq \frac{C}{\sqrt{N \sin \frac{\pi}{2}}}$, hence

$$\int_{I_{\ell}^c} |P_N(t)|^2 \frac{dt}{2\pi} \leq \frac{C}{N} \int_{\delta(\ell)}^{\pi} \frac{dt}{\sin^2 \frac{\pi t}{2}} < \frac{C}{N} \int_{\delta(\ell)}^{\pi} \frac{dt}{t^2} < \frac{C}{\delta(\ell) N} < \frac{C(\alpha)}{\delta(\ell) \ell^\alpha},$$
where last inequality holds for every $N$ for which (2) holds. Using condition (b) we see that indeed last term can be made arbitrary small, and so $E(\Lambda)$ is not a RS in $L^2(S)$.

\[\square\]

4. Proof of theorem 2

For $n \in \mathbb{N}$ we define
\[B_n := \{n, 2n, \ldots, n^2\}.\]

**Lemma 4.** Let $\mathcal{P}$ be the set of all prime numbers. Then the blocks $\{B_p\}_{p \in \mathcal{P}}$ are pairwise disjoint.

**Proof.** Let $p < q$ be prime numbers. Notice that a number $a \in B_p \cap B_q$ if and only if there exist $1 \leq m \leq p$ and $1 \leq k \leq q$ s.t.
\[a = m \cdot p = k \cdot q,
\]
which is possible only if $q$ divides $m$. But since $m < q$ this cannot happen and so such $a$ does not exist. \[\square\]

**Lemma 5.** Let $\{a(n)\}_{n \in \mathbb{N}}$ a sequence of non-negative numbers s.t. \[\sum_{n=1}^{\infty} a(n) \leq 1.\] Then for every $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ s.t.
\[\sum_{\ell=1}^{n} a(\ell \cdot n) < \frac{\varepsilon}{n}.
\]

**Proof.** By Lemma 4 we may write
\[\sum_{n=1}^{\infty} a(n) \geq \sum_{p \in \mathcal{P}} \sum_{\ell=1}^{p} a(\ell \cdot p).
\]
Assuming the contrary for some $\varepsilon$, i.e. for all but finitely many $p \in \mathcal{P}$ we have $\sum_{\ell=1}^{p} a(\ell \cdot p) \geq \frac{\varepsilon}{p}$, we get a contradiction to the well-known fact that $\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty$. \[\square\]
Corollary 6. For every $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ s.t.

$$\sum_{\lambda, \mu \in B_n \atop \mu < \lambda} a(\lambda - \mu) < \varepsilon.$$  

Proof. Every $\mu < \lambda$ from $B_n$ must take the form

$$\lambda = k \cdot n, \quad 1 \leq k' < k \leq n,$$

$$\mu = k' \cdot n,$$

hence $\lambda - \mu = \ell \cdot n$, $\ell \in \{1, 2, \ldots N - 1\}$. From Lemma 5 we get for infinitely many $n \in \mathbb{N}$

$$\sum_{\lambda, \mu \in B_n \atop \mu < \lambda} a(\lambda - \mu) = \sum_{\ell=1}^{n} (n - \ell) \cdot a(\ell \cdot n) \leq n \cdot \sum_{\ell=1}^{n} a(\ell \cdot n) < \varepsilon \quad \Box$$

Let $B \subset \mathbb{R}$, we say that a positive number $\gamma$ is a lower Riesz bound (in $L^2(S)$) for a sequence $E(B)$ if the inequality

$$\left\| \sum_{\lambda \in B} c(\lambda) e^{i\lambda t} \right\|_{L^2(S)}^2 \geq \gamma \cdot \sum_{\lambda \in B} |c(\lambda)|^2,$$

holds for every finite sequence of scalars $\{c(\lambda)\}_{\lambda \in B}$.

Lemma 7. Given $S \subset T$ of positive measure, there exist a constant $\gamma = \gamma(S) > 0$ for which the following holds: For infinitely many $n \in \mathbb{N}$ $\gamma$ is a lower Riesz bound (in $L^2(S)$) for $E(B_n)$.

Proof. Let $S \subset T$, with $|S| > 0$. Apply Corollary 5 to the sequence $\{a(n)\}_{n \in \mathbb{N}} := \left\{ \mathbb{1}_S(n) \right\}_{n \in \mathbb{N}}$ (where $\mathbb{1}_S$ is the indicator function of the set $S$), we get for every $\varepsilon > 0$ infinitely many $n \in \mathbb{N}$ for which (3) holds. We write

$$\int_S \left| \sum_{\lambda \in B_n} c(\lambda) e^{i\lambda t} \right|^2 \frac{dt}{2\pi} = \int_S \left( \sum_{\lambda \in B_n} |c(\lambda)|^2 + \sum_{\lambda, \mu \in B_n \atop \lambda \neq \mu} c(\lambda) \overline{c(\mu)} e^{i(\lambda - \mu)t} \right) \frac{dt}{2\pi} =$$
By Cauchy-Schwarz inequality, the last term does not exceed
\[
\left| \sum_{\lambda, \mu \in B_n, \lambda \neq \mu} c(\lambda) \overline{c(\mu)} \widehat{1_S}(\mu - \lambda) \right| \leq \left( \sum_{\lambda, \mu \in B_n, \lambda \neq \mu} |c(\lambda) c(\mu)|^2 \right)^{1/2} \cdot \left( \sum_{\lambda, \mu \in B_n, \lambda \neq \mu} |\widehat{1_S}(\mu - \lambda)|^2 \right)^{1/2} = \\
= \sum_{\lambda \in B_n} |c(\lambda)|^2 \cdot \left( \sum_{\lambda, \mu \in B_n, \lambda \neq \mu} |\widehat{1_S}(\mu - \lambda)|^2 \right)^{1/2}.
\]
By (3) we get
\[
\sum_{\lambda, \mu \in B_n, \lambda \neq \mu} |\widehat{1_S}(\mu - \lambda)|^2 = 2 \sum_{\lambda, \mu \in B_n, \mu < \lambda} |\widehat{1_S}(\mu - \lambda)|^2 < 2 \varepsilon,
\]
hence
\[
\int_S \left| \sum_{\lambda \in B_n} c(\lambda) e^{i\lambda t} \right|^2 dt \geq \frac{(|S| - 2\varepsilon) \sum_{\lambda \in B_n} |c(\lambda)|^2}{2} \geq \frac{|S|}{2} \sum_{\lambda \in B_n} |c(\lambda)|^2.
\]
Fixing some \( \varepsilon < \frac{|S|}{4} \), we get the last inequality holds for infinitely many \( n \in \mathbb{N} \). \( \Box \)

The next lemma shows how to combine blocks which correspond to different progressions.

**Lemma 8.** Let \( \gamma > 0 \), \( S \subseteq \mathbb{T} \) with \( |S| > 0 \), and \( B_1, B_2 \subseteq \mathbb{N} \) finite subsets s.t \( \gamma \) is a lower Riesz bound (in \( L^2(S) \)) for \( E(B_j) \), \( j = 1, 2 \). Then for any \( 0 < \gamma' < \gamma \) there exists \( M \in \mathbb{Z} \) s.t. the system \( E(B_1 \cup (M + B_2)) \) has \( \gamma' \) as a lower Riesz bound.

**Proof.** Denote \( P_j(t) = \sum_{\lambda \in B_j} c(\lambda) e^{i\lambda t}, j = 1, 2 \). Notice that for sufficiently large \( M = M(S) \), the polynomials \( P_1 \) and \( e^{iMt} \cdot P_2 \) are "almost orthogonal" on \( S \), meaning
\[
\int_S \left| P_1(t) + e^{iMt} \cdot P_2(t) \right|^2 dt \geq \frac{1}{2\pi} \left\| P_1 \right\|^2_{L^2(S)} + \frac{1}{2\pi} \left\| P_2 \right\|^2_{L^2(S)} + o(1),
\]
where the last term is uniform w.r. to all polynomials having \( \|P\|_{L^2(T)} = 1 \). \( \Box \)
Now we are ready to finish the proof of Theorem 2. Given $S$ take $\gamma$ from Lemma 6 and denote by $N$ the set of all natural numbers $n$ for which $\gamma$ is a lower Riesz bound (in $L^2(S)$) for $E(B_n)$. Define

$$\Lambda = \bigcup_{n \in N} (M_n + B_n).$$

Due to Lemma 7 we can define subsequently for every $n \in N$, an integer $M_n$ s.t. for any partial union

$$\Lambda(N) = \bigcup_{n \in N, n < N} (M_n + B_n), N \in N$$

the corresponding exponential system $E(\Lambda(N))$ has lower Riesz bound $\frac{\gamma}{2} \cdot (1 + \frac{1}{N})$, so we get that $E(\Lambda)$ is a RS in $L^2(S)$.

5. Proof of Theorem 3

In order to obtain $\Lambda$ which satisfies property $(i')$ we will require the following result.

**Theorem A.** ([1], Thm. 13.12) Let $d(n)$ denote the number of divisors of an integer $n$. Then $d(n) = o(n^\varepsilon)$ for every $\varepsilon > 0$.

The next lemma demonstrates how the contribution of blocks can be controlled even when they are not disjoint.

**Lemma 9.** Let $\{a(n)\}_{n \in \mathbb{N}}$ a sequence of non-negative numbers s.t. $\sum_{n=1}^{\infty} a(n) \leq 1$. Then for every $\alpha > 1$ there exist $\varepsilon(\alpha) > 0$ and $\nu(\alpha) \in \mathbb{N}$ s.t. for every $N \geq \nu(\alpha)$ one can find an integer $\ell_{\alpha,N} < N^\alpha$ satisfying

$$\sum_{n=1}^{N} a(n \cdot \ell_{\alpha,N}) < \frac{1}{N^{1+\varepsilon(\alpha)}}.\tag{4}$$

**Proof.** Fix $\alpha > 1$ and apply Theorem A with $\varepsilon$ small enough, depending on $\alpha$, to be chosen later. We get the inequality $d(k) < k^\varepsilon$ holds for every $k \geq \nu(\alpha)$. Fix $N \geq \nu(\alpha)$,
and notice that for every $L \in \mathbb{N}$

$$\sum_{\ell=1}^{L} \sum_{n=1}^{N} a(n \cdot \ell) \leq \sum_{k=1}^{L \cdot N} d(k) \cdot a(k) < (L \cdot N)^\varepsilon.$$ 

It follows that there exists an integer $0 < \ell < L$ s.t.

$$\sum_{n=1}^{N} a(n \cdot \ell) < \frac{(L \cdot N)^\varepsilon}{L} = \frac{N^\varepsilon}{L^{1-\varepsilon}}.$$ 

In order to get (4) we ask

$$\frac{N^\varepsilon}{L^{1-\varepsilon}} < \frac{1}{N^{1+\varepsilon}},$$

which yields

$$N^{\frac{1+2\varepsilon}{1-\varepsilon}} < L.$$ 

Therefore, choosing $\varepsilon = \varepsilon(\alpha)$ sufficiently small we see that $L$ may be chosen to be smaller than $N^\alpha$. 

Setting

$$B_{\alpha,N} := \{\ell_{\alpha,N}, 2\ell_{\alpha,N}, \ldots, N \cdot \ell_{\alpha,N}\},$$

we get

**Corollary 10.** For every $\alpha > 1$ and $N \geq \nu(\alpha)$

(5) $$\sum_{\substack{\lambda, \mu \in B_{\alpha,N} \\ \mu < \lambda}} a(\lambda - \mu) < \frac{1}{N^{\varepsilon(\alpha)}}.$$ 

The proof is identical to that of Corollary 6.

We combine our estimates.

**Lemma 11.** Given $S \subset \mathbb{T}$ of positive measure, there exist a constant $\gamma = \gamma(S) > 0$ s.t. for every $\alpha > 1$ there exists $N(\alpha) \in \mathbb{N}$ for which the following holds: For every integer $N \geq N(\alpha)$ one can find $\ell_{\alpha,N} \in \mathbb{N}$ satisfying $\ell_{\alpha,N} < N^\alpha$ and $\gamma$ is a lower Riesz bound (in $L^2(S)$) for $E(B_{\alpha,N})$. 

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Proof. Let $S \subset T$, with $|S| > 0$. We fix $\alpha > 1$ and apply corollary 10 to the sequence
\[
\{a(n)\}_{n \in \mathbb{N}} := \left\{\left|\sum_{n \in N} \hat{S}(n) \right|^2 \right\}_{n \in \mathbb{N}},
\]
we get $\varepsilon(\alpha)$ and for every $N \geq \nu(\alpha)$ a positive integer $\ell_{\alpha,N} < N^\alpha$ satisfying (5). Proceeding as in the proof of Lemma 7 we get
\[
\sum_{\lambda \in B_{\alpha,N}} c(\lambda) e^{i\lambda t} \geq \left(\frac{|S|}{N^{\alpha}/2} \right) \sum_{\lambda \in B_{\alpha,N}} |c(\lambda)|^2 \geq \frac{|S|}{2} \sum_{\lambda \in B_{\alpha,N}} |c(\lambda)|^2,
\]
where last inequality holds for all $N \geq N(\alpha)$. □

For the last step of the proof we will use a diagonal process.

Given $S$ find $\gamma$ using Lemma 11. This provides, for every $\alpha > 1$ and every $N \geq N(\alpha)$, a block $B_{\alpha,N}$ s.t. $\gamma$ is a lower Riesz bound (in $L^2(S)$) for $E(B_{\alpha,N})$. Let $\alpha_k \to 1$ be a decreasing sequence. Define
\[
\Lambda = \bigcup_{k \in \mathbb{N}} \bigcup_{N=N(\alpha_k)}^{N(\alpha_k+1)-1} (M_N + B_{\alpha_k,N})
\]
Again, by Lemma 7, we can make sure any partial union has lower Riesz bound not smaller than $\frac{\gamma}{2}$, and so $E(\Lambda)$ is a RS in $L^2(S)$.

It follows directly from the construction that for every $N \in \mathbb{N} \Lambda$ contains an arithmetic progression of length $N$ and step $\ell < C(\alpha) \cdot N^\alpha$, for any $\alpha > 1$, as required.

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