STOCHASTIC DESCENT ANALYSIS OF REPRESENTATION LEARNING ALGORITHMS

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ABSTRACT

Although stochastic approximation learning methods have been widely used in the machine learning literature for over 50 years, formal theoretical analyses of specific machine learning algorithms are less common because stochastic approximation theorems typically possess assumptions which are difficult to communicate and verify. This paper presents a new stochastic approximation theorem for state-dependent noise with easily verifiable assumptions applicable to the analysis and design of important deep learning algorithms including: adaptive learning, contrastive divergence learning, stochastic descent expectation maximization, active learning, and adaptive control.

The Representation Learning Problem is concerned with identifying an alternative representation of a data set which facilitates both learning and generalization performance (see Bengio, Courville, and Vincent, 2013, for a recent review). For example, contrastive divergence type learning algorithms have been developed for learning joint distributions of large numbers of observable and hidden variables (e.g., Younes, 1999; Hinton et al., 2006; Tieleman, 2008; Swersky et al., 2010; Salakhutdinov and Hinton, 2012). Active learning algorithms (e.g., Sutton and Barto, 1998; Feldman and Balcanand, 2013) have also been developed for learning in statistical environments whose characteristics are shaped by the learning machine’s actions.

Stochastic approximation theorems have played a vital role in increasing our understanding of a wide variety of machine learning algorithms since the early beginnings of machine learning (e.g., Amari, 1967; Duda and Hart, 1973) and continue to do so (e.g., White, 1989; Jaakkola et al., 1994; Jani et al., 2000; Golden, 1996; Bottou, 2004; Paik et al., 2006; Swersky et al., 2010; Mohri et al., 2012). All of the above published papers consider the analysis of learning machines which update their parameter estimates at iteration \( t \), \( \hat{\theta}(t) \), to obtain a revised parameter estimate at iteration \( t + 1 \), \( \hat{\theta}(t + 1) \) according to formulas of the form:

\[
\hat{\theta}(t + 1) = \hat{\theta}(t) + \gamma_t \mathbf{d}(\tilde{x}(t), \hat{\theta}(t))
\]

where the search direction function \( \mathbf{d} : \mathbb{R}^d \times \mathbb{R}^q \rightarrow \mathbb{R}^q \) specifies how the learning machine will update its parameters, a particular realization of the random vector \( \tilde{x}(t) \) is a training stimulus which may possibly be corrupted with an additional internal or external noise source, and \( \gamma_t \) is called the stepsize or learning rate of the learning machine. More generally, with some additional conditions, an algorithm of the form of (1) is called a stochastic approximation algorithm.

Assume the stochastic sequence of \( d \)-dimensional random vectors \( \tilde{x}^1, \tilde{x}^2, \ldots \) modeling the training stimuli are independent and identically distributed with common Data Generating Process (DGP) probability density \( p_{\text{o}} : \mathbb{R}^d \rightarrow [0, \infty) \). In other words, each time the learning machine updates its parameters, the likelihood of observing a particular training stimulus \( x(t) \) at iteration \( t \) is given by \( p_{\text{o}} \). The goal of an adaptive learning machine is to estimate (learn) the global minimizer, \( \theta^* \in \mathbb{R}^d \), of a smooth risk function \( \ell : \mathbb{R}^q \rightarrow \mathbb{R} \) which specifies the learning machine’s optimal behavior. In addition, let a smooth function \( c \) be defined such that \( c(x, \theta) \) is the penalty or "loss" incurred by the
learning machine for choosing parameter value $\theta$ for training stimulus $x$ where $x \in \mathcal{R}^d$. The risk function $\ell$ is defined such that for all $\theta \in \mathcal{R}^d$:

$$
\ell(\theta) = \int c(x, \theta)p_o(x)d\nu(x).
$$

Note that the notation $\int c(x, \theta)p_o(x)d\nu(x)$ may be interpreted as $\sum c(x, \theta)p_o(x)$ when $p_o$ is a probability mass function for a discrete random vector and $\int c(x, \theta)p_o(x)dx$ when $p_o$ is an (absolutely continuous) probability density function for a continuous random vector. The notation $\int c(x, \theta)p_o(x)d\nu(x)$ also is used to specify the appropriate combination of sums and Riemann integrals for situations where the random training stimulus $\tilde{x}$ includes both discrete and absolutely continuous random variables. Thus, the goal of learning is to minimize the expected loss (or equivalently risk) associated with choosing a parameter value with respect to a particular statistical environment characterized by the DGP density $p_o$.

Many previously published machine learning papers which consider parameter update equations of the form of (1) in the field of machine learning (e.g., White, 1989; Bottou, 1991; Golden, 1996; Bottou, 1998) assume that the risk function has the form of (2). That is, that at each parameter update, the training stimulus is sampled from the statistical environment using the probability density $p_o$. This assumption, unfortunately, is not directly relevant to many important problems in the areas of: (i) contrastive divergence learning, (ii) learning in the presence of missing data or latent variables, and (iii) active learning and adaptive control. Such problems typically require that the training stimulus is sampled from a statistical environment which is specified by the current parameter estimates so that rather than sampling from the density $p_o$, one samples from the density $p_x(\cdot|\theta)$ where $\theta$ is the current knowledge state of the learning machine. Thus, rather than using the risk function in (2), the active learning environment assumption uses the risk function:

$$
\ell(\theta) = \int c(x, \theta)p_x(x|\theta)d\nu(x).
$$

The majority of stochastic approximation papers in the field of machine learning which provide explicit convergence theorems (e.g., White, 1989; Bottou, 1991; Golden, 1996; Bottou, 1998) do not explicitly assume that the probability distribution of a training stimulus changes as the parameter estimates change.

To emphasize the implications of (3), consider the risk function $\ell(\theta)$ in (2) for a passive statistical environment specified by:

$$
\nabla_{\theta}\ell(\theta) = \int \nabla_{\theta}c(x, \theta)p_o(x)d\nu(x),
$$

This formula for the gradient provided in (4) is not correct for an active learning environment where the risk function is given by (3). The gradient of (3) for an active learning environment is given instead by the formula:

$$
\nabla_{\theta}\ell(\theta) = \int \nabla_{\theta}c(x, \theta)p_x(x|\theta)d\nu(x) + \int c(x, \theta)\nabla_{\theta}p_x(x|\theta)d\nu(x).
$$

Sunehag et al. (2009) provides a theorem which implicitly considers a risk function such as (3) but does not present this function in an explicit manner. Furthermore, Sunehag et al. (2009) discusses the relevance of (3) for handling the case of active learning but does not emphasize or mention that this approach is also applicable to handling missing data, latent (hidden) variables, and adaptive control problems. A key assumption of the Sunehag et al. (2009) convergence theorem is that the risk function has a single minimizer which is not always the case in many machine learning applications.

Indeed, the assumption of a unique minimizer of the risk function in (2) or that the risk function is strictly convex is a common assumption of stochastic approximation theorems in the machine learning literature (e.g., Sunehag et al., 2009; Younes, 1999) yet these assumptions are not appropriate for classes of important machine learning problems. Note, however, other stochastic approximation theorems published in the machine learning literature (e.g., White, 1989; Bottou, 1991; Golden, 1996; Mohri et al., 2012) can handle these more general cases but they assume a risk function of the form of (2).
Assumptions used to identify the assumptions of convergence theorems for equations of the form of (1) are usually very difficult to check for someone who is not a researcher in the field of stochastic approximation theory. For example, one assumption common to many published stochastic approximation theorems (e.g., Bottou, 1991; Sunehag et al., 2009; Mohri et al., 2012) and which is representative of the cryptic nature of such assumptions is for all positive $\epsilon$:

$$\inf_{|\theta - \theta^*| > \epsilon} \{[\theta - \theta^*]^T \nabla \theta \ell(\theta) \} > 0.$$  \hfill (6)

The assumption in (6) is representative of the types of assumptions commonly encountered in stochastic approximation theorems. A sufficient but not necessary condition for (6) to hold is that the objective function $\ell$ is strictly convex. However, many important problems in learning in machine learning involve objective functions which are not strictly convex. For example, a logistic regression model where the Hessian is positive semidefinite has a negative log-likelihood risk function which is convex but not strictly convex. More complicated multi-layer network architectures may have multiple minimizers and multiple saddle points.

Thus, assumptions such as (6) cannot simply be assumed to hold. Such assumptions are generally not satisfied for a large class of important machine learning algorithms. In order to investigate the asymptotic behavior of a learning algorithm of the form of (1), it is necessary to check that assumptions such as (6) hold. For many practitioners in the field of machine learning whose specialty is not stochastic approximation learning algorithms, simply checking that an assumption such as (6) holds might require considerable time and effort. And, if the assumption does not hold, additional time and effort is required to figure out if it is possible to modify the algorithm or theorem to obtain a convergence analysis.

The conclusions of several stochastic approximation theorems in the machine learning literature also could be improved. In many (but not all) cases, the few additional arguments and assumptions required to obtain more relevant conclusions may be “obvious” to an expert in the field of stochastic approximation theory but virtually impossible for the typical practitioner in the field of machine learning without training in mathematical statistics to confidently construct. For example, theorems which explicitly state a conclusion such that $\Theta(t)$ converges to a set of critical points with probability one are more desirable (e.g., see White, 1989; Golden, 1996; for theorems of this type) than theorems which state that the gradient of the risk function converges to zero with probability one (e.g., see Bottou, 1991; Sunehag et al., 2009; Mohri et al., 2012; for theorems of this type).

Furthermore, most of the stochastic approximation theorems published in the machine learning literature focus more upon stochastic gradient methods (e.g., see Bottou, 1991, for a good review) rather than variable metric stochastic approximation methods which include discussions of Quasi-Newton stochastic descent algorithms such as Levenberg-Marquardt and Broyden-Fletcher-Goldfarb-Shanno accelerated learning algorithms. Sunehag et al. (2009) analyzes a class of variable metric stochastic approximation algorithms that assume a unique minimizer for risk functions of the form of (3) and concludes by stating that the gradient of the risk function converges to zero with probability one. White (1989) and Golden (1996) provide explicit theorems guaranteeing convergence of a large class of variable metric methods to the set of critical points of the risk function. Jani, Dowling, Golden, and Wang (2000) and Paik, Golden, Torlak, and Dowling (2006) used the methods of Golden (1996) to derive variable metric stochastic approximation algorithms based upon the one-step Broyden-Fletcher-Goldfarb-Shanno (BFGS) approach.

And finally, it is assumed that the conditional density $p(x|\theta)$ is a Radon-Nikodým density. This assumption permits $p(x|\theta)$ to specify the joint distribution of a collection of random variables which consists of both discrete random variables and continuous random variables. In the special case, where $x$ is a discrete random variable, then $p(x|\theta)$ is a probability mass function. Generative models involving both discrete and continuous random variables play an important role in modern theories of machine learning in applications since training stimuli, in practice, often include a mixture of both discrete and continuous random variables.

The novel contribution of this paper is to provide a new stochastic approximation theorem which is general enough to handle situations involving both the passive risk function in (2) and the active risk function in (3). The assumption that a unique minimizer exists or that the objective function is strictly convex is not required. The theorem also handles the stochastic gradient descent case as well as the variable metric case as important special cases. Moreover, the statement and proof of the theorem are specifically designed to provide relatively easily verifiable assumptions and inter-
prettable conclusions so that the theorem can be applied by practitioners in the field. And finally, the framework is directly applicable to joint distributions of both discrete and absolutely continuous random variables. To the author’s knowledge, a single stochastic approximation theorem with all five of these features does not exist in either the machine learning literature or even the stochastic approximation literature.

1 The New Stochastic Approximation Theorem

The new state-dependent stochastic approximation theorem which minimizes the active learning risk function in (3) as well as the passive learning risk function in (2) is similar to theorems described by Gu and Kong (1998) and Delyon et al. (1999) (also see Kushner, 1984; Benveniste et al., 1990; Theorem 2; Kushner, 2010). However, the assumptions, conclusions, and proof of this theorem are specifically designed to be easily understood by researchers outside the field of stochastic approximation theory. The accessibility of these theoretical results is fundamentally important for the development of the field of machine learning to ensure that such results are correctly applied in specific applications.

The theorem is also sufficiently general to handle not only the case of stochastic gradient descent but also cases involving variable metric descent methods in situations where one or more multiple minimizers of the risk function may exist and the risk function is not necessarily strictly convex.

A crucial assumption of the following theorem is that the expected value of $\bar{d}(\tilde{x}(t), \theta), \bar{d}$, is continuous in $\theta$. Given that $g(\theta)^T \bar{d}(\theta) \to 0$ with probability one, the assumption $g(\theta)^T \bar{d}(\theta)$ is continuous in $\theta$ is used to establish in Step 6 of the Proof in Section 3 that

$$\hat{\theta}(t) \to \{\theta : g(\theta)^T \bar{d}(\theta) = 0\}.$$  

(7)

Showing that $d(\tilde{x}(t), \theta)$ is continuous in $\theta$, is not sufficient to obtain the convergence result in (7) for the case where $p_x$ is functionally dependent upon $\theta$. For example, if $\tilde{x}(t)$ is a discrete random vector on a finite sample space $S$, so that

$$\bar{d} = \sum_S d(x, \theta)p_x(x|\theta),$$

then a sufficient (but not necessary) condition for $\bar{d}$ to be continuous is that both $d(x, \theta)$ and $p_x(x|\theta)$ are continuous in $\theta$.

The terminology that a function $d : \mathcal{R}^d \times \mathcal{R}^q \to \mathcal{R}^q$ is bounded means that for all $(x, \theta) \in \mathcal{R}^d \times \mathcal{R}^q$ there exists a finite number $K$ such that: $|d| \leq K$.

**Theorem 1.1** (State-Dependent Stochastic Approximation Theorem). Let $\ell : \mathcal{R}^q \to \mathcal{R}$ be a twice continuously differentiable function with a lower bound. Let $g \equiv \nabla \ell$. Let $H \equiv \nabla^2 \ell$. Assume that there exists a finite number $K$ for all $\theta \in \mathcal{R}^q$ so that $|\tilde{x}_\theta| \leq K$ with probability one where $\tilde{x}_\theta$ has Radon-Nikodym density $p_x(x|\theta) : \mathcal{R}^d \to [0, \infty)$. Let $d : \mathcal{R}^d \times \mathcal{R}^q \to \mathcal{R}^q$ be a function piecewise continuous on a finite partition in its first argument and continuous in its second argument. Let $d : \mathcal{R}^q \to \mathcal{R}^q$ be defined such that for all $\theta \in \mathcal{R}^q$:

$$\bar{d}(\theta) = \int d(x, \theta)p_x(x|\theta)d\mu(x)$$

when it exists. Assume $d$ and $p_x$ are defined such that $\bar{d}$ is continuous on $\mathcal{R}^q$. In addition, assume that for all $\theta \in \mathcal{R}^q$:

$$g(\theta)^T \overline{d}(\theta) \leq 0.$$  

(8)

Let $\gamma_0, \gamma_1, \gamma_2, \ldots$ be a sequence of positive real numbers such that:

$$\sum_{t=0}^{\infty} \gamma_t^2 < \infty$$  

(9)

and

$$\sum_{t=0}^{\infty} \gamma_t = \infty.$$  

(10)
Let \( \tilde{\theta}(0) \) be a \( q \)-dimensional bounded random vector. Let \( \tilde{\theta}(1), \tilde{\theta}(2), \ldots \) be a sequence of \( q \)-dimensional random vectors defined such that for \( t = 0, 1, 2, \ldots \):

\[
\tilde{\theta}(t + 1) = \tilde{\theta}(t) + \gamma_t \mathbf{d}(\tilde{x}_\theta(t), \tilde{\theta}(t))
\]  

(11)

where \( \tilde{x}_\theta(t) \) is a \( d \)-dimensional random vector with conditional probability density \( p_x(\cdot | \theta) \) for each \( \theta \in \mathbb{R}^q \).

Assume, in addition, that either of the following two conditions holds:

- **A1.** both \( \mathbf{d} \) and \( \nabla^2 \ell \) are bounded functions, or
- **A2.** \( \tilde{\theta}(1), \tilde{\theta}(2), \ldots \) converges to a bounded set with probability one.

Then as \( t \to \infty \), \( \tilde{\theta}(t) \to \mathcal{H} \) with probability one where

\[
\mathcal{H} = \left\{ \theta : g(\theta)^T \mathbf{\bar{d}}(\theta) = 0 \right\}.
\]  

(12)

The condition that there exists a finite number \( K \) for all \( \theta \in \mathbb{R}^q \) so that \( |\tilde{x}_\theta| \leq K \) with probability one is satisfied, for example, if \( \tilde{x}_\theta \in X \) where \( X \) is either a finite subset of \( \mathbb{R}^d \) or a bounded subset of \( \mathbb{R}^d \). In other words, if \( \tilde{x}_\theta \) is a discrete random vector taking on values in a finite sample space, then this condition is automatically satisfied.

Assumption A1 of the Stochastic Approximation Theorem requires that the descent search direction function \( \mathbf{d} \) is bounded. This constraint can be usually satisfied by an appropriate choice of \( \mathbf{d} \). However, Assumption A1 also requires that the Hessian of \( \ell \) is bounded. The Hessian of \( \ell \) is only bounded for some types of risk functions commonly encountered in machine learning applications. For example, consider a logistic regression modeling problem where the training stimuli are sampled using the State-Dependent Stochastic Approximation Theorem presented in Section 2.

In this section, we discuss several examples of adaptive learning algorithms which can be analyzed using the State-Dependent Stochastic Approximation Theorem presented in Section 2.

### 2 Adaptive Algorithms for Representation Learning

**2.1 Adaptive Learning in Passive Statistical Environments**

Stochastic approximation algorithms provide a methodology for the analysis and design of adaptive learning machines. A stochastic approximation algorithm is defined by beginning with an initial guess for the parameter values of the learning machine denoted by \( \tilde{\theta}(0) \) and then updating that initial guess to obtain a refined estimate called \( \tilde{\theta}(1) \), more specifically the process of iterated updates is defined by:

\[
\tilde{\theta}(k + 1) = \tilde{\theta}(k) + (\gamma_k/m) \sum_{j=1}^{m} \mathbf{d}(\tilde{x}_j^k, \tilde{\theta}(k))
\]  

(13)

where it is assumed that the mini-batch of \( m \) observations \( \tilde{x}_1^k, \ldots, \tilde{x}_m^k \) are independent and identically distributed with common density \( p_x \). The mini-batch size is denoted by \( m \) which is a positive integer whose value can be as small as one. Because \( p_x \) is not functionally dependent upon the current state of the learning machine, this is an example of learning in a passive statistical environment.
The function $\mathbf{d}$ is called the \textit{search direction function} which attempts to use the current guess for the parameter values $\hat{\theta}(k)$ and the current observation (or equivalently training stimulus) $\mathbf{x}^j$ to calculate the change to the current parameter estimate which is given by the second term on the right-hand side of (13). The magnitude of this change is governed by the strictly positive \textit{step size} parameter $\gamma_k$. In order for the stochastic approximation algorithm to converge to an appropriate solution, both the search direction function and step size sequence must be appropriately chosen.

In order to determine if $\mathbf{d}$ is an appropriate search direction function, one typically uses $\mathbf{d}$ to compute the \textit{expected search direction}

$$\mathbf{d}(\theta) \equiv \int \mathbf{d}(\mathbf{x}, \theta)p_o(\mathbf{x})d\nu(\mathbf{x})$$

(14)

and then one checks if $\mathbf{d}$ satisfies the \textit{downhill condition}

$$\frac{d\ell}{d\theta}(\mathbf{d}(\theta)) \leq 0.$$ (15)

Note that the downhill condition in (15) is a commonly used condition for ensuring that the deterministic gradient descent algorithm defined by:

$$\theta(k + 1) = \theta(k) + \gamma_k \mathbf{d}$$ (16)

converges where the sequence of positive step sizes $\gamma_1, \gamma_2, \ldots$ are appropriately chosen. To summarize, the stochastic search direction $\mathbf{d}(\mathbf{x}, \theta)$ is chosen such that, on the average, the search direction $\mathbf{d}(\theta)$ is downhill by satisfying the relation in (15). As the mini-batch size $m$ increases, the actual search direction will tend to converge to the expected search direction when an appropriate law of large numbers holds. However, it is not necessary that $m$ increase or take on a large value to establish convergence of the algorithm in (13).

Selecting the search direction in an appropriate manner, however, is not sufficient to ensure convergence of the stochastic descent algorithm in (13). The sequence of positive step sizes must also be appropriately chosen. One common choice for the step size sequence $\gamma_1, \gamma_2, \ldots$ is to use a ”search and converge” approach. In this type of approach, the step size is initially held constant or even increased but then eventually decreased at an appropriate rate. For example, Darken et al. (1992) proposed that:

$$\gamma_k = \frac{\gamma_0}{(k/\tau) + (k/\tau)^2 + (k/\tau) + 1}$$ (17)

where $\gamma_0$ is the initial positive step size and $k < \tau$ specifies the ”search” time period where the stepsize is relatively constant, while $k >> \tau$ corresponds to the ”converge” time period where the stepsize $\gamma_k$ tends to decrease for $k = 0, 1, 2, \ldots$.

Appropriate choice of the search direction and step size sequence are the essential ingredients for guaranteeing that the stochastic sequence $\hat{\theta}(0), \hat{\theta}(1), \ldots$ will converge with probability one to the set of points where \( \frac{d\ell}{d\theta}(\mathbf{d}(\theta)) = 0 \) whenever it does converge. In the special case where the search direction

$$\mathbf{d} = -\frac{d\ell}{d\theta},$$

the stochastic descent algorithm in (13) is called a \textit{stochastic gradient descent algorithm}. When it converges, a stochastic gradient descent algorithm converges to the set of points where \( \frac{d\ell}{d\theta} = 0 \) which corresponds to the set of critical points of $\ell$. Variable metric search directions for accelerating convergence of adaptive learning (e.g., Jani et al., 2000; Paik et al., 2006; Sunehag et al., 2009) such as Quasi-Newton methods and Broyden-Fletcher-Goldfarb-Shanno (BFGS) methods can also be analyzed using the Theorem presented in Section 1.

2.2 Normalization Constants and Contrastive Divergence

Maximum likelihood estimation is a method for computing the parameter estimates that maximize the likelihood of the observed data or equivalently minimize the cross-entropy between the researcher’s model and the empirical distribution of the observed data. For example, suppose that the observed data is a collection of $n$ $d$-dimensional vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, which are presumed to be a particular realization of a sequence of independent and identically distributed random vectors with
common density \( p_o : \mathcal{R}^d \to (0, \infty) \). Then the method of maximum likelihood estimation corresponds to finding the parameter vector \( \hat{\theta} \) that is a global minimizer of

\[
\ell_n(\theta) = -\frac{1}{n} \sum_{i=1}^{n} \log p(x_i|\theta)
\]

on \( \Theta \). In addition, as \( n \to \infty \), \( \hat{\theta} \to \theta^* \) with probability one where \( \theta^* \) is a particular global minimizer of

\[
\ell(\theta) = -\int p_o(x) \log p(x|\theta) d\nu(x)
\]

under appropriate regularity conditions.

Let \( V : \mathcal{R}^d \times \mathcal{R}^m \to \mathcal{R} \). Let \( \Theta \) be a closed and bounded subset of \( \mathcal{R}^d \). Assume for each \( \theta \in \Theta \) that the probability density of \( \tilde{x} \) is a Gibbs density \( p(\cdot|\theta) : \mathcal{R}^d \to (0, \infty) \) defined such that

\[
p(x|\theta) = \left[Z(\theta)\right]^{-1} \exp(-V(x;\theta))
\]

where the normalization constant \( Z(\theta) \) is defined as:

\[
Z(\theta) = \int \exp(-V(y;\theta)) d\nu(y).
\]

The derivative of \( \ell_n \) in (18) is given by the formula:

\[
\frac{d\ell_n}{d\theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{d\ell_n,i}{d\theta}
\]

where

\[
\frac{d\ell_n,i}{d\theta} = \frac{dV(x_i,\theta)}{d\theta} - \int \frac{dV(y;\theta)}{d\theta} p(y|\theta) d\nu(y).
\]

Even though \( \tilde{d} \) satisfies (15) with \( d = -d\ell_n/d\theta \). Equation (22) can not, however, be immediately used to derive a stochastic gradient descent algorithm that minimizes \( \ell \) for the following reasons. The first term on the right-hand side of (23) is usually relatively easy to evaluate. On the other hand, the second term on the right-hand side of (23) is usually very difficult to evaluate because it involves a computationally intractable multidimensional integration.

Let \( \tilde{y}^1, \ldots, \tilde{y}^m \) be a sequence of \( m \) possibly correlated distributed random vectors with a common mean whose joint density is \( p(y^1, \ldots, y^m | \theta) \) for a given \( \theta \). To obtain a computationally practical method of evaluating the second term on the right-hand side of (23), note that the expected value of

\[
\frac{1}{m} \sum_{j=1}^{m} \frac{dV(\tilde{y}^j;\theta)}{d\theta}
\]

is

\[
\int \frac{dV(y;\theta)}{d\theta} p(y|\theta) d\nu(y)
\]

which corresponds to the second term on the right-hand side of (23).

Substituting the Monte Carlo approximation in (24) for the multidimensional integral in (23) and then using the resulting approximate derivative as a stochastic search direction for a stochastic approximation algorithm defined by:

\[
\tilde{\theta}(k+1) = \tilde{\theta}(k) - \gamma_k \frac{dV(\tilde{x}(k);\tilde{\theta}(k))}{d\theta} + \left(\gamma_k/m\right) \sum_{j=1}^{m} \frac{dV(\tilde{y}^j;\tilde{\theta}(k))}{d\theta}
\]

where the mini-batch \( \tilde{y}^1, \ldots, \tilde{y}^m \) is a collection of \( m \) possibly highly correlated observations with joint density \( p(y^1, \ldots, y^m | \theta(k)) \) for the \( k \)th iteration of (26). It is assumed that mini-batches are independent and identically distributed with common density \( p(y^1, \ldots, y^m | \theta(k)) \). Equation (26) is an example of contrastive divergence type learning algorithm which can be interpreted as a stochastic approximation algorithm. The mini-batch size \( m \) can be a fixed integer (e.g., \( m = 3 \) or \( m = 100 \)) or
\( m \) can be varied (e.g., initially \( m \) is chosen to be small and then gradually increased to some finite positive integer during the learning process).

Note that the statistical environment used to generate the data for the stochastic approximation algorithm in (26) is not a passive statistical environment since the parameters of the learning machine are updated at learning trial \( k \) not only by the observation \( \tilde{x}(k) \) but also by the observations \( \tilde{y}_1, \ldots, \tilde{y}_m \) whose joint distribution is functionally dependent upon the current parameter estimates \( \hat{\theta}(k) \). Thus, contrastive-divergence algorithms of this type can be analyzed using the Theorem presented in Section 1.

2.3 Missing Data, Hidden Variables, and Expectation Maximization

In this section, the problem of "hidden variables" is considered. The presence of hidden variables is a characteristic feature of deep learning architectures. Suppose the \( d \)-dimensional random vector \( \tilde{x} \) could be partitioned such that \( \tilde{x} = [\tilde{v}, \tilde{h}] \) where \( \tilde{v} \) is the observable component of \( \tilde{x} \) and \( \tilde{h} \) is the unobservable component whose probability distribution is functionally dependent upon a realization of \( \tilde{v} \). The elements of \( \tilde{v} \) correspond to the "visible random variables" while the elements of \( \tilde{h} \) correspond to the "hidden random variables" or the "missing data".

The missing data negative log-likelihood analogous to the complete data negative log-likelihood in (18) is given by the formula:

\[
\ell_n(\theta) = -\frac{1}{n} \sum_{i=1}^{n} \log p(v_i | \theta)
\]  

(27)

which can be rewritten in terms of the joint density \( p(v_i, h_i | \theta) \) as:

\[
\ell_n(\theta) = -\frac{1}{n} \sum_{i=1}^{n} \log \left[ \int p(v_i, h_i | \theta) d\nu(h_i) \right].
\]  

(28)

Now take the derivative of (28) under the assumption that the interchange of derivative and integral operators is permissible to obtain:

\[
\frac{d\ell_n}{d\theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{d\ell_{i,n}}{d\theta}
\]  

(29)

where

\[
\frac{d\ell_{i,n}}{d\theta} = -\int \frac{1}{p(v_i | \theta)} \frac{dp(v_i, h_i | \theta)}{d\theta} d\nu(h_i).
\]  

(30)

The derivative in the integrand of (30) is obtained using the identity (e.g., see Louis, 1982; McLachlan and Thriyambakam, 1996):

\[
\frac{dp(v_i, h_i | \theta)}{d\theta} = \frac{d\log p(v_i, h_i | \theta)}{d\theta} p(v_i, h_i | \theta).
\]  

(31)

Substitution of (31) into (30) gives:

\[
\int \frac{d\log p(v_i, h_i | \theta)}{d\theta} p(h_i | v_i, \theta) d\nu(h_i)
\]

which is then approximated using a Monte Carlo approximation using the formula:

\[
\frac{1}{m} \sum_{j=1}^{m} \frac{d\log [p(v_i, h^j | \theta)]}{d\theta}
\]  

(32)

where the stochastic imputation \( h^j \) is a realization of \( \tilde{h}^j \) whose distribution is specified by the conditional density \( p(h^j | v, \theta) \) for a given realization \( v \) and parameter vector \( \theta \).

The final stochastic descent expectation maximization algorithm is then defined by constructing a stochastic gradient descent algorithm by defining the stochastic search direction as negative one
multiplied by the derivative in (30) and then replacing the integral in (30) with the Monte Carlo approximation in (32) to obtain:

\[ \tilde{\theta}(k + 1) = \tilde{\theta}(k) - (\gamma_k / m) \sum_{j=1}^{m} \frac{d \log p(\tilde{v}_j, \tilde{h} | \theta)}{d \theta} \]

where the mini-batch \( \tilde{h}_1, \ldots, \tilde{h}_m \) at the kth learning trial is generated by first sampling a realization \( v_j \) from the environment and then sampling \( m \) times from \( p(h | v_j, \theta(k)) \) using the sampled value \( v_j \) and the current parameter estimates \( \theta(k) \) at the kth learning trial. Thus, the new stochastic approximation theorem provides a method for analyzing the asymptotic behavior of the stochastic descent expectation-maximization algorithm.

Note that \( m \) can be chosen equal to 1 or any positive integer. In the case where \( m = \infty \), then the resulting algorithm approximates the deterministic Generalized Expectation Maximization (GEM) algorithm (see McLachlan and Thriyambakam, 1996, for a formal definition of a GEM algorithm) in which the learning machine uses its current probabilistic model to compute the expected downhill search direction, takes a downhill step, updates its current probabilistic model, and then repeats this process in an iterative manner.

2.4 Active Learning Machines

Another aspect of deep learning is the ability of a learning machine to learn in environments whose statistical characteristics are molded by its actions. In other words, an optimal "deep representation" depends not only upon the learning machine’s sensory representation of its statistical environment but also upon the effects of a learning machine’s interactions upon its environment as well. In this section, one particular implementation of a learning machine capable of active learning in statistical environments is discussed.

Suppose that a learning machine experiences a collection of "episodes". The episodes \( \tilde{u}(0), \tilde{u}(1), \ldots \) are assumed to be independent and identically distributed. In addition, the tth episode \( u(t) \) is defined such that \( u(t) \equiv [s_o(t), s_F(t + 1)] \) where \( s_o(t) \) is called the initial state of episode \( u(t) \) and \( s_F(t) \) is called the final state of episode \( u(t) \). The probability density of \( \tilde{u}_k \) when the learning machine is a “passive learner” is specified by the density \( p_o(u) = p_o(s_o, s_F) \) where \( p_o(u) \) specifies the likelihood that \( u \) is observed by the learning machine in its statistical environment.

On the other hand, define the probability density of \( \tilde{u} \) when the learning machine is an "active learner". In this case, the probability that the learning machine selects action \( a_j \) given the current state of the environment \( s_o \) and the learning machine’s current state of knowledge \( \theta \) is expressed by the conditional probability mass function \( p(a_j | s_o, \theta), j = 1, \ldots, J \). The statistical environment of the learning machine is characterized by the probability density, \( p_o(s_o) \) specifying the likelihood of a given initial state of an episode and the conditional density \( p_o(s_F | a_j, s_o) \) which specifies the likelihood of a final state of an episode \( s_F \) given the learning machine’s action \( a_j \) and the initial state of the episode \( s_o \).

Thus, the probability distribution of an episode \( \tilde{u}(t) \) is specified by the density

\[ p(u(\theta) = p_o(s_o)p(s_F | s_o, \theta) \]

where

\[ p(s_F | s_o, \theta) = \sum_{j=1}^{J} p_o(s_F | a_j, s_o)p(a_j | s_o, \theta). \]

Let \( c(u, \theta) \) specify the cost incurred by the learning machine when episode \( u \) is encountered in its environment for a particular state of knowledge \( \theta \). Notice that the cost \( c(u, \theta) \) is functionally dependent upon \( \theta \) as well as \( u \) allowing for the possibility of a learning machine with an "adaptive critic" (e.g., Sutton and Barton, 1998). One possible goal of an adaptive learning machine in an active statistical environment is to minimize the objective function \( \ell \) defined by the formula:

\[ \ell(\theta) = \int c(u, \theta)p(u(\theta))du \]  (33)
where $p(\cdot|\theta)$ is a Gibbs density for each $\theta \in \mathcal{R}^q$.

Now take the derivative of (33), interchange the integral and derivative operators, and use a Monte Carlo approximation for the integral in (33) similar to the approximations in (24) and (32). The resulting derivative can then be used to construct the stochastic gradient descent algorithm:

$$
\hat{\theta}(k + 1) = \hat{\theta}(k) - \gamma_k \frac{d\bar{c}(\bar{u}(k), \theta)}{d\theta} - \gamma_k c(u, \hat{\theta}(k)) \frac{d\log p(u(k)|\hat{\theta}(k))}{d\theta}
$$

(34)

where the probability distribution of $\bar{u}(k)$ at learning trial $k$ is specified by the conditional probability density $p(u(k)|\hat{\theta}(k))$. Note that the identity

$$
\frac{dp(u|\theta)}{d\theta} = p(u|\theta) \frac{d\log p(u|\theta)}{d\theta}
$$

was used to derive equation (33) (e.g., see Louis, 1982; McLachlan and Thriambakam, 1996).

Note that in this example, the action $a_j$ is functionally dependent upon the initial state of the episode and the parameters of the learning machine. Since the initial state of the episode is independently and identically distributed, one could argue that this example of active learning is an example of learning in an “open-loop” system. However, this approach is easily generalizable and the Theorem presented in Section 1 is still applicable to a problem which has more “closed-loop” characteristics. In particular, define an episode $u(t)$ such that:

$$
u(t) \equiv [s_0(t), a_0(t), s_1(t), a_1(t), s_2(t)].
$$

(35)

That is, an episode is chosen at random by identifying an initial state $s_0(t)$, then the learning machine chooses an action $a_0(t)$ at random using $s_0(t)$ and its current parameter estimates $\hat{\theta}(t)$ which influences its statistical environment and generates the random state $s_1(t)$. Next, the random state $s_1(t)$ and $\hat{\theta}(t)$ are used to generate at random the next action $a_1(t)$ which results in a final state $s_2(t)$. At this time, the parameter update equation in (11) is applied to generate $\hat{\theta}(t + 1)$. This same methodology can clearly be extended to situations involving episodes of longer time periods.

The key assumption here is that the episodes are independent and identically distributed (i.i.d.) for each parameter vector $\theta$ in the parameter space. Learning can take place as the learning machine is interacting with its environment provided that the episodes are sampled such that they are not overlapping in time and can be effectively modeled as i.i.d.. This type of strategy may be interpreted as Besag’s (1974) coding assumption for the special case of sequences of random vectors. This approach to adaptive control in an active learning environment has been demonstrated using a toy lunar lander problem within the MATLAB environment. Please visit the website:

www.utdallas.edu/~golden/lunarlander

to download some unpublished preliminary notes, preliminary results, and preliminary software associated with the toy lunar lander problem.

3 PROOF OF THE NEW STOCHASTIC APPROXIMATION THEOREM

In this section, the proof of the stochastic approximation theorem for state-dependent learning is provided which minimizes the active environment risk function in (3) as well as the passive environment risk function in (2). The proof of the theorem is based upon a combination of arguments by Blum (1954) and the Appendix of Benveniste et al. (1990) and the well-known Robbins-Siegmund Lemma (Robbins and Siegmund, 1971) (e.g., see Benveniste et al., 1987, p. 344 or Mohri et al, 2012 for relevant reviews of this Lemma).

The terminology that a stochastic sequence $\tilde{x}(0), \tilde{x}(1), \ldots$ is uniformly bounded means that there exists a finite number $K$ such that for all $t$: $|\tilde{x}(t)| \leq K$ with probability one.

Proof. The proof of this theorem is a variation of the analysis of Blum (1954; also see Appendix of Benveniste et al., 1987). Let $\tilde{g}_t \equiv g(\tilde{\theta}(t))$. Let $\tilde{d}_t \equiv d(\tilde{x}(t), \tilde{\theta}(t))$. Let $\tilde{\ell}_t \equiv \ell(\tilde{\theta}(t))$. 
Step 1: Expand objective function using a second-order mean value expansion. Expand \( \ell \) about \( \tilde{\theta}(t) \) and evaluate at \( \theta(t+1) \) using the mean value theorem for random vectors (Jennrich, 1969, Lemma 3) to obtain:
\[
\ell_{t+1} = \ell_t + \gamma^2 \left[ \theta(t+1) - \tilde{\theta}(t) \right] + \gamma^2 R_t
\]
with
\[
R_t \equiv \frac{1}{2} d_t^T H(\zeta_t) d_t
\]
where the random variable \( \zeta_t \) can be defined using Lemma 3 of Jennrich (1969) as a point on the chord connecting \( \tilde{\theta}(t) \) and \( \theta(t+1) \). Substituting the relation
\[
\gamma_t d_t = \theta(t+1) - \theta(t)
\]
into (36) gives:
\[
\ell_{t+1} = \ell_t + \gamma_t g^T d_t + \gamma^2 R_t
\]

Step 2: Identify conditions required for the remainder term of the expansion to be bounded. If Assumption \( A_1 \) holds so that both \( d \) and \( H \) are bounded functions, then there exists a finite number \( R_{\max} \) such that for all \( t = 0, 1, 2, \ldots \):
\[
|R_t| < R_{\max}. \tag{39}
\]
If Assumption \( A_2 \) holds with respect to a bounded set \( S \), this implies that for a given positive \( \epsilon \), there exists a finite positive integer \( T_\epsilon \) such that for all \( t > T_\epsilon : |\theta(t) - y| < \epsilon \) for some \( y \in S \) with probability one. Thus, with probability one, the stochastic sequence \( \{\theta(t)\} \) is uniformly bounded by a constant equal to the sum of \( \epsilon \), an element in the closure of \( S \) with the largest magnitude, and the largest element of the finite set \( \{\theta(0), \ldots, \theta(T_\epsilon)\} \). Since \( \{\theta(t)\} \) is uniformly bounded with probability one, and \( d \) and \( H \) are continuous functions of \( \theta(t) \) and \( d \) is piecewise continuous on the \( \theta \)-dependent uniformly bounded stochastic sequence \( \{\tilde{x}_\theta(t)\} \) for all \( \theta \in \mathcal{R}^q \), it follows that there exists a finite number \( R_{\max} \) such that for all \( t = 0, 1, 2, \ldots \) : \( |\tilde{R}_t| < R_{\max} \) with probability one.

Step 3: Show asymptotic average decrease in objective function value. Taking the conditional expectation of both sides of (38) with respect to the conditional density \( p_x \) and evaluating at \( \theta(t) \) and \( \gamma_t \) yields:
\[
E \{ \ell(\theta(t+1))|\theta(t) \} = \ell_t + \gamma_t g(\theta(t))^T d(\theta(t)) + \gamma^2 E \{ R_t|\theta(t) \}. \tag{40}
\]

Now use the relation that \( |\tilde{R}_t| < R_{\max} \) with probability one from Step 2 to obtain:
\[
E \{ \ell(\theta(t+1))|\theta(t) \} \leq \ell_t + \gamma_t g(\theta(t))^T d(\theta(t)) + R_{\max} \gamma^2. \tag{41}
\]

Step 4: Apply the Robbins-Siegmund Lemma. The Robbins-Siegmund Lemma (Robbins and Siegmund, 1971; also see Beneveniste et al., 1987, p. 344 or Mohri et al., 2012 for relevant reviews) asserts that since \( \ell \) has a lower bound, and additionally (8), (9), (10), and (41) hold then: (1) \( \ell_t \) converges with probability one as \( t \to \infty \) to a random variable \( \ell^* \), and (2)\[
- \sum_{t=0}^{\infty} \gamma_t g(\theta(t))^T d(\theta(t)) < \infty \]
with probability one.

Step 5: Show the state sequence converges to a random vector. A proof by contradiction is used to demonstrate \( \{\tilde{\theta}(t)\} \) converges w.p.1 to a random vector \( \tilde{\theta}^* \). Assume \( \{\tilde{\theta}(t)\} \) does not converge to \( \tilde{\theta}^* \) as \( t \to \infty \). This means there exists a subsequence of \( \{\tilde{\theta}(t)\} \) which does not converge to \( \tilde{\theta}^* \) w.p.1 as \( t \to \infty \). Since \( \ell \) is continuous, this implies that there exists a subsequence of \( \{\ell(\tilde{\theta}(t))\} \) which does not converge to \( \ell^* \). But this contradicts the conclusion in Step 4 that every subsequence of \( \{\ell(\tilde{\theta}(t))\} \) converges to the random variable \( \ell^* \) with probability one.

Step 6: Show the state sequence converges to \( \mathcal{H} \). Equation (42) obtained in Step 4 in conjunction with the relation \( \sum \gamma_t = \infty \) in (10) implies that a subsequence of \( g(\theta(t))^T d(\theta(t)) \to 0 \) as \( t \to \infty \) with probability one. Since \( g^T d \) is continuous, this implies that a subsequence of \( \{\tilde{\theta}(t)\} \) converges to \( \mathcal{H} \) with probability one. Since \( \{\theta(t)\} \) (and thus every subsequence of \( \{\theta(t)\} \) converges w.p.1 to the same random vector by the results of Step 5, it follows that \( \{\tilde{\theta}(t)\} \) converges with probability one to \( \mathcal{H} \).
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