TWISTING LEMMA FOR $\Lambda$-ADIC MODULES

SOHAN GHOSH, SOMNATH JHA, SUDHANSHU SHEKHAR

ABSTRACT. A classical twisting lemma says that given a finitely generated torsion module $M$ over the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$ with $\Gamma \cong \mathbb{Z}_p$, $\exists$ a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ such that, the $\Gamma^n$-Euler characteristic of the twist $M(\theta)$ is finite for every $n$. This twisting lemma has been generalized for the Iwasawa algebra of a general compact $p$-adic Lie group $G$. In this article, we consider a further generalization of the twisting lemma to $T[[G]]$ modules, where $G$ is a compact $p$-adic Lie group and $T$ is a finite extension of $\mathbb{Z}_p[[X]]$. Such modules naturally occur in Hida theory. We also indicate arithmetic application by considering the twisted Euler Characteristic of the big Selmer (respectively fine Selmer) group of a $\Lambda$-adic form over a $p$-adic Lie extension.

INTRODUCTION

In this article, we discuss some topics in non-commutative Iwasawa theory for modules over $T[[G]]$: the completed group ring of a compact $p$-adic Lie group $G$ with coefficients in the ring $T$, where $T$ is a finite extension of $\mathbb{Z}_p[[X]]$. We fix an odd prime $p$ throughout. Let $B$ be any commutative, complete, noetherian local domain of characteristic 0 with finite residue field of characteristic $p$. For a profinite group $G$, recall the Iwasawa algebra of $G$ over $B$ is defined as $B[[G]] := \lim_{\leftarrow} B[G/U]$, where $U$ varies over open normal subgroups of $G$ and the inverse limit is taken with respect to the canonical projection maps.

Let $G$ be a compact $p$-adic Lie group with a closed normal subgroup $H$ such that $\Gamma := G/H \cong \mathbb{Z}_p$. For a left $O[[G]]$ module $M$ and a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$, denote by $M(\theta)$ the $O[[G]]$-module $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\theta)$ with diagonal $G$-action. We will assume throughout that $G$ has no $p$-torsion element. We will denote by $O$, the ring of integers of a finite extension of $\mathbb{Q}_p$. Recall that for a compact $p$-adic Lie group $G$ with no element of order $p$, the Iwasawa algebra $O[[G]]$ has finite global dimension [Br, La].

Definition. Let $G$ be a compact $p$-adic Lie group without any element of order $p$. For a finitely generated $O[[G]]$-module $M$, we say that the $G$-Euler characteristic of $M$ exists if the homology groups $H_i(G, M)$ are all finite and we define it as

$$\chi(G, M) := \prod_{i \geq 0} (\#H_i(G, M))^{(-1)^i}. $$

Given a $O[[G]]$-module $M$, $\chi(G, M)$ is an invariant attached to $M$. Natural example of $O[[G]]$ modules in arithmetic comes from Selmer group attached to

AMS Subject Classification: 11R23, 14H52
Keywords and phrases: Iwasawa theory, Selmer groups, $\Lambda$-adic form, $G$-Euler characteristic.
a motive over $p$-adic Lie extension of a number field, with $G$ being the corresponding Galois group. Euler characteristic of Selmer group naturally carries arithmetical information. For any number field $K$, let $K_{cyc}$ denote the cyclotomic $\mathbb{Z}_p$-extension and set $\Gamma = \text{Gal}(K_{cyc}/K)$. Let $E/\mathbb{Q}$ be an elliptic curve with good, ordinary reduction at $p$ and let $X(E/\mathbb{Q}_{cyc})$ denote the dual $p\infty$-Selmer group of $E$ over $\mathbb{Q}_{cyc}$. Then it is known that, under suitable condition, the $p$-adic valuation of $\chi(\Gamma, X(E/\mathbb{Q}_{cyc}))$ is related to the $p$-adic valuation of the special value $\frac{L_E(1)}{\Omega_E} (\# E(\mathbb{F}_p)(p))^2$ of the complex $L$-function of $E$ over $\mathbb{Q}$. (see [JS, Introduction])

On the other hand, consider a $p$-adic Lie extension $K \subset K_{cyc} \subset K_\infty$ of a number field $K$ which is unramified outside a finitely many places of $K$ and set $G = \text{Gal}(K_\infty/K)$ and $H = \text{Gal}(K_\infty/K_{cyc})$. Then $\Gamma := G/H \cong \mathbb{Z}_p$. For such a general $p$-adic Lie extension $K_\infty/K$, the twisted Euler characteristic of $\mathbb{Z}_p[[G]]$ modules $M$, such that $M/M(p)$ is finitely generated over $\mathbb{Z}_p[[H]]$, has been discussed in [CFKSV, Theorem 3.6], [JS]. Also for an elliptic curve $E/\mathbb{Q}$, the conjectural relation between $\chi(G, X(E/K_\infty))(\theta)$ and twisted $L$-values are studied (cf. [CFKSV, Theorem 3.6], [JS]).

For $G = \Gamma$, following classical twisting lemma is well known in Iwasawa theory and can be found in the works of Greenberg [Gr] and Perrin-Riou [Pr]: For any finitely generated torsion $O[[\Gamma]]$-module $M$, there exists a continuous character $\theta : \Gamma \to \mathbb{Z}_p^{\times}$ such that the largest $\Gamma^{\theta}$-covariant quotient $H_0(\Gamma^{\theta^n}, M(\theta)) = (M(\theta))_{\Gamma^{\theta^n}}$ is finite for every $n \in \mathbb{N}$. Note that $H_0(\Gamma^{\theta^n}, M(\theta))$ is finite if and only if $\chi(\Gamma^{\theta^n}, M(\theta))$ is finite.

Let $R$ be a ring and $M$ be a left $R$ module. Define, $M(p) := \bigcup_{r \geq 1} M[p^r]$, where $M[p^r]$ is the set of $p^r$ torsion points of $M$. Unless stated otherwise, when we consider a module $M$ over a ring $R$, we mean $M$ is a left module over $R$.

The twisting Lemma in the non-commutative setting was established in [JOZ]:

**Theorem (JOZ).** Let $G$ be a compact $p$-adic Lie group and $H$ be a closed normal subgroup of $G$ such that $\Gamma := G/H \cong \mathbb{Z}_p$. Let $M$ be an $O[[G]]$ module which is finitely generated over $O[[H]]$. Then, there exist a continuous character $\theta : \Gamma \to \mathbb{Z}_p^{\times}$ such that $M(\theta)_{U} = H_{0}(U,M(\theta))$ is finite for every open normal subgroup $U$ of $G$.

Note that for a general compact $p$-adic Lie group $G$ and an $O[[G]]$ module $M$, $H_{0}(G,M)$ is finite does not necessarily imply $\chi(G,M)$ exists (i.e. finite) [JS, see Remark 1.5]. In [JS], Theorem (JOZ) was extended to the following result on the twisted Euler characteristic.

**Theorem (JS).** Let $G$ be a compact $p$-adic Lie group without any element of order $p$ and $H$ be a closed normal subgroup of $H$ such that $\Gamma := G/H \cong \mathbb{Z}_p$. Let $M$ be a finitely generated $O[[G]]$ module such that $M/M(p)$ is a finitely generated $O[[H]]$ module. Then there exist a continuous character $\theta : \Gamma \to \mathbb{Z}_p^{\times}$ such that $\chi(U,M(\theta))$ exists for every open normal subgroup $U$ of $G$.

Moreover, it is shown that for a given $M$, there is a countable subset $S_M$ of all continuous characters from $\Gamma$ to $\mathbb{Z}_p^{\times}$, such that for any choice of a continuous $\theta : \Gamma \to \mathbb{Z}_p^{\times}$ outside $S_M$, Theorem (JOZ) and Theorem (JS) hold.
Congruence of modular forms is an important topic in number theory and it naturally leads to the study of modules over two variable Iwasawa algebra with coefficients in certain universal ordinary deformation rings (cf. [Hi], [Wi]). Examples of these rings includes finite extensions of $\mathbb{Z}_p[[X]]$. Thus it is natural to ask for a generalization of twisting Lemma for modules over $T[[G]]$, with $T$ finite over $\mathbb{Z}_p[[X]]$. The main result of the article is the following:

**Theorem 0.1.** Let $G$ be a compact $p$-adic Lie group without any element of order $p$ and $H$ be a closed normal subgroup of $G$ with $\Gamma := G/H \cong \mathbb{Z}_p$. Let $T$ be a commutative, complete local domain which is finite over $\mathbb{Z}_p[[X]]$. Let $M$ be a finitely generated $T[[G]]$ module such that $M/M(p)$ is finitely generated over $T[[H]]$. Then $\exists$ a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ and a countable set $C_{M,\theta}$ of height 1 prime ideals of $T$ such that if we choose any height 1 prime ideal $Q \notin C_{M,\theta}$, then $\chi(U, M_{QM}(\theta))$ is finite for every open normal subgroup $U$ of $G$.

**Definition 0.2.** Define $S := \{\theta|\theta$ is a continuous character from $\Gamma$ to $\mathbb{Z}_p^\times\}$ and set $C = C^T := \{Q : Q$ is a height 1 prime ideal of $T, Q \nmid (p)\}$.

**Remark 0.3.** In the proof of Theorem 0.1, we will show slightly stronger result than stated in the theorem. We will show there exists a countable subset $S_M$ of $S$ such that for any choice of $\theta \in S \setminus S_M$, Theorem 0.1 holds.

**Remark 0.4.** In the above theorems, we consider finitely generated $B[[G]]$ modules $M$ such that $M/M(p)$ is finitely generated over $B[[H]]$. This comes from the formulation of non-commutative $GL_2$ Iwasawa main conjecture in [CFKSV], where it is suggested, many arithmetic modules, including the dual Selmer group of an elliptic curve over a $p$-adic Lie extension would satisfy this condition.

**Remark 0.5.** Let us keep the setting and hypotheses of Theorem 0.1. Also for simplicity take $T = O[[X]]$. Then, from Theorem 0.1, one may naturally ask if the following statements are true:

Statement 1: For all but finitely many height 1 prime ideals in $O[[X]]$, $(M_{QM})_U = H_0(U, M_{QM})$ is finite, for every open normal subgroup $U$ of $G$.

Statement 2: There exists a continuous character $\theta : \Gamma \to \mathbb{Z}_p^\times$ and a finite set $C'$ of $C$ such that if we choose and fix any $Q \in C \setminus C'$ then $(M_{QM}(\theta))_U$ is finite for every open normal subgroup $U$ of $G$.

Statement 3: There exists a countable subset $S'$ of $S$ and a countable subset $C'$ of $C$ such that for any choice of $\theta \in S \setminus S'$ and for any choice of a height 1 prime $Q \in C \setminus C'$, $(M_{QM}(\theta))_U$ is finite for every open normal subgroup $U$ of $G$.

Statement 4: There exists a countable set $S' \times C' \subset S \times C$, such that for any choice of $(\theta, Q) \notin S' \times C'$, $(M_{QM}(\theta))_U$ is finite, for every open normal subgroup $U$ of $G$.

In Examples 1 to 4 we show that all of the above 4 statements are false.

**Example 1.** In this example, we consider the statement 1 in Remark 0.5. Let $G = \Gamma = \langle \gamma \rangle$, $H = \{1\}$ and $O = \mathbb{Z}_p$. Let $M = \frac{\mathbb{Z}_p[[X]][[T]]}{(X - T)}$, where $1 + T$ corresponds to $\gamma$. Let $Q \neq (p)$ be a height 1 prime ideal of $\mathbb{Z}_p[[X]]$. Write $Q = (g(X))$, where $(g(X))$ is an irreducible Weierstrass polynomial. Then
shows that for Example 3 is false. Example 1 is false. For a continuous we see that statement 4 in holds true.

Note for $C$ be in $M$ Then

Recall that

For each $(\theta)$ for $S$ subsets of $S$.

Therefore from the proof of Example 3.

Then let us choose a $\theta$ for $\theta$ is infinite. Hence for every $S$, $\exists$ an integer $n_0$ such that $(M_{QM})_{\Gamma^p,n}$ is infinite. Thus Statement 1 in Remark 0.5 is false.

**Example 2.** Let $G,H,O$ and $M$ be the same as in Example 1. For a continuous character $\theta : \Gamma \longrightarrow \mathbb{Z}_p^*$, we can write $\theta(\gamma) = 1 + p\lambda$ for some (fixed) $\lambda \in \mathbb{Z}_p$.

Then $M(\theta) = \frac{\mathbb{Z}_p[[X]][[T]]}{(1 + p\lambda)(T + 1) - (X + 1)}$. Now, if $Q \neq (p)$ is a prime ideal of height 1 in $\mathbb{Z}_p[[X]]$, then

$$\left(\frac{M}{QM}(\theta)\right)_{\Gamma^p,n} \cong \frac{\mathbb{Z}_p[[X]][[T]]}{(1 + p\lambda)(T + 1) - (X + 1), (X + 1)^{p^n} - 1, Q}$$

is finite if and only if $(Q, (X + 1)^{p^n} - (1 + p\lambda)^{p^n})$ has height 2. Let

$S^\theta_{\Gamma^p,n} := \{Q \in \mathbb{Z}_p[[X]] : Q \neq (p) \text{ and } (M_{QM}(\theta))_{\Gamma^p,n} \text{ is infinite}\}$.

Then $S^\theta_{\Gamma^p,n} = \{(g(X)) \mid g(X) \text{ is an irreducible divisor of } (X + 1)^{p^n} - (1 + p\lambda)^{p^n}\}$. A similar argument used in Example 1 shows that for $m > n$, $S^\theta_{\Gamma^p,m} \subseteq S^\theta_{\Gamma^p,n}$ and $S := \bigcup_{n \geq 1} S^\theta_{\Gamma^p,n}$ is an infinite set. Now, for each $Q \in S$, $\exists$ an integer $n_0$ such that $(M_{QM}(\theta))_{\Gamma^p,n_0}$ is infinite. This shows that statement 2 in remark 0.5 is false.

**Example 3.** Let $G,H,O$ and $M$ be as in Examples 1 and 2. Recall the sets $S$ and $C$ defined in definition 0.2. If possible, assume that Statement 3 in Remark 0.5 is true. Then, there exists a countable subset $S'$ of $S$ and $C'$ of $C$ such that for $\theta \in S \setminus S'$ and $Q \in C \setminus C'$, $H_0((\Gamma^p), (M_{QM}(\theta)))$ is finite for every $n \geq 0$.

For every $\theta \in S$, we can write $\theta(\gamma) = 1 + p\lambda_{\theta}$ for some (fixed) $\lambda = \lambda_{\theta} \in \mathbb{Z}_p$.

Then $(M_{QM}(\theta))_{\Gamma^p,n} \cong \frac{\mathbb{Z}_p[[X]]}{((X + 1)^{p^n} - (1 + p\lambda)^{p^n}, Q)}$. Observe that if $Q = (X - p\lambda)$ then $(M_{QM}(\theta))_{\Gamma^p,n}$ is infinite for every $n \in \mathbb{N}$.

Now let us choose a $\theta \in S \setminus S'$. Then by our assumption in Statement 3, $(M_{QM}(\theta))_{\Gamma^p,n}$ is finite for every $Q \in C \setminus C'$. Thus the height 1 prime $(X - p\lambda)$ must be in $C'$. Hence for every $\theta \in S \setminus S'$, we get an element $Q_{\theta} = (X + 1 - \theta(\gamma)) \in C'$. Note for $\theta_1, \theta_2 \in S \setminus S'$ with $\theta_1 \neq \theta_2$, $Q_{\theta_1} \neq Q_{\theta_2}$. Since $S \setminus S'$ is uncountable, the set $C'$ is also uncountable. This is a contradiction.

**Example 4.** Recall $S$ and $C$ be from Definition 0.2. Let $S'$ and $C'$ be countable subsets of $S$ and $C$ respectively. Note that $(S \setminus S') \times (C \setminus C') \subseteq ((S \times C) \setminus (S' \times C'))$.

Therefore from the proof of Example 3 we see that statement 4 in 0.5 is false.

However, the following variant of the Theorem 0.1 holds true.
Proposition 0.6. Let \(G, H, \mathcal{T}\) and \(M\) be as in Theorem 0.1. Let \(C'\) be a (finite or a) countable subset of \(C\). Then there exists a countable set \(S_M\) of \(S\), such that for any choice of \(\theta \in S \setminus S_M\) and for any choice of \(Q \in C'\), \(\chi(U, \frac{M}{Q_M} (\theta))\) exists, for every open normal subgroup \(U\) of \(G\).

Proof. Let us enumerate \(C' = \{Q_1, Q_2, \cdots\}\). For each \(Q_i\), note that \(\frac{\mathcal{T}}{Q_i}[[G]] \cong O_i[[G]]\) and \(\frac{\mathcal{T}}{Q_i}[[H]] \cong O_i[[H]]\), where \(O_i\) is the ring of integers of a finite extension \(L^i\) of \(\mathbb{Q}_p\). Then from Theorem(JS), there exists a countable subset \(S_{Q_i,M}\) of all the continuous character from \(\Gamma\) to \(\mathbb{Z}_p^\times\), such that for any choice of \(\theta \notin S_{Q_i,M}\), \(\chi(U, \frac{M}{Q_M} (\theta))\) exists for every open normal subgroup \(U\) of \(G\). Now choose and fix any \(\theta : \Gamma \rightarrow \mathbb{Z}_p^\times\) outside the countable set \(S_M := \bigcup_{i=1}^k S_{Q_i,M}\). Then it follows that for any choice of \(Q_i \in C'\), \(\chi(U, \frac{M}{Q_M} (\theta))\) exists for every open normal subgroup \(U\) of \(G\). \(\square\)

Let \(\mu_\infty\) denote the group of \(p\) power roots of unity. Define a homomorphism \(\nu_{k,\zeta} : \mathbb{Z}_p[[X]] \rightarrow \hat{\mathbb{Q}}_p^\times\) by \(\nu_{k,\zeta}(1 + X) = (1 + p)^k\), where \(\zeta \in \mu_\infty\) and \(k \in \mathbb{N}\). Define \(A_{\text{arith}}(\mathbb{Z}_p[[X]]) := \{Q|Q\text{ is kernel of }\nu_{k,\zeta}\text{ for some }k \geq 1\text{ and }\zeta \in \mu_\infty\}\). Clearly, elements of \(A_{\text{arith}}(\mathbb{Z}_p[[X]])\) are height 1 prime ideals of \(\mathbb{Z}_p[[X]]\). Similarly, we define \(A_{\text{arith}}(\mathcal{T})\) as set of those height 1 prime ideals of \(\mathcal{T}\) which divides some height 1 prime ideal in \(A_{\text{arith}}(\mathbb{Z}_p[[X]])\). The elements of \(A_{\text{arith}}(\mathcal{T})\) are called the arithmetic primes or classical primes of \(\mathcal{T}\) [see [Wi]]. As an immediate corollary of Proposition 0.6, we deduce the following:

Corollary 0.7. Let us keep the hypotheses and setting of Proposition 0.6. Then, there exist a countable subset \(S_M\) of \(S\), such that for any choice of \(\theta \in S \setminus S_M\) and for any \(Q \in A_{\text{arith}}(\mathcal{T})\), \(\chi(U, \frac{M}{Q_M} (\theta))\) exists for every open normal subgroup \(U\) of \(G\).

Remark 0.8 (Significance in Arithmetic). Consider a \(p\)-adic Lie extension \(K \subset K_{\text{cycl}} \subset K_\infty\) of number fields such that \(K_\infty\) is unramified outside finitely many places of \(K\) and \(G = \text{Gal}(K_\infty/K)\) has no element of order \(p\). Put \(H = \text{Gal}(K_\infty/K_{\text{cycl}})\). Let \(f\) be \(p\)-ordinary newform and assume that the dual Selmer group \(X(L_f/K_\infty)\) of \(f\) over \(K_\infty\) [JO, Definition 1.11] satisfies the property that \(\frac{X(L_f/K_\infty)}{X(L_f/K_{\text{cycl}})[p]}\) is finitely generated over \(O_f[[H]]\). Here \(L_f\) is a lattice of the \(p\)-Galois representation associated to \(f\) and \(O_f\) is the ring of integers of the \(p\)-adic field defined by the Fourier coefficients of \(f\). Then Theorem(JOZ) along with an appropriate control theorem (see [JO, Theorem 0.1]) shows \(\exists \theta : \Gamma \rightarrow \mathbb{Z}_p^\times\) such that the twisted Selmer group \(X(L_f(\theta)/K_U)\) corresponding to \(V_f(\theta)\) (see [JO, Definition 1.11]) is finite for every finite extension \(K_U = K_{\infty}^U\) of \(K_\infty\) contained in \(K_\infty\).

This finiteness of \(X(L_f(\theta)/K_U)\) for a fixed \(\theta\), where \(K_U\) varies over every intermediate finite extension of \(K\) inside \(K_{\infty}\), can be useful in various arithmetical situations; for example, finiteness of \(X(L_f(\theta)/K_U)\) is crucial in the proof of algebraic functional equation for \(X(L_f/K_\infty)\) (see [JO, §5, Theorem 0.3]). Theorem(JS) was also used in the proof in [JO]. Indeed, this is a generalization of the fact that classical twisting lemma is crucial in Greenberg’s [Gr, Theorem 2] and Perrin-Riou’s [Pr, Theorem 4.2.1] proof of algebraic functional equation.
Now, our Corollary 0.7 in particular shows that \( \exists \theta \in S \) such that for each \( Q \in A_{\text{arith}}(T) \), \( \left( \frac{f}{Q} \right)_{M} \) is finite for every \( U \). Let \( F \) be a \( \Lambda \)-adic newform with universal ordinary Hecke algebra \( T = T_{F} \), finite over \( \mathbb{Z}_{p}[[X]] \) and let \( f_{Q} \) denote the \( p \)-ordinary, \( p \)-stabilized newform obtained as a specialization at the arithmetic point \( Q \). Take \( M \) to be the ‘big’ Selmer group \( \mathcal{X}(L_{F}/K_{\infty}) \) of \( F \) over \( K_{\infty} \) (see \S 2). Assume \( \frac{x_{i}(L_{F}/K_{\infty})(p)}{\mathcal{X}(L_{F}/K_{\infty})(p)} \) is a finitely generated \( T_{F}[[H]] \) module. Then under an appropriate control theorem, we can deduce the following: For any chosen \( \theta \in S \backslash S_{\mathcal{X}(L_{F}/K_{\infty})} \), the twisted Selmer groups \( X(L_{Q}(\theta)/K_{U}) \) of the congruent family of cuspforms \( f_{Q}, Q \in A_{\text{arith}}(T) \) are all simultaneously finite when \( K_{U} \) varies over every finite extension of \( K \) inside \( K_{\infty} \). This can be used to establish an algebraic functional equation for the ‘big’ Selmer group \( \mathcal{X}(L_{F}/K_{\infty}) \) of \( F \) over \( K_{\infty} \).

The structure of the rest of the article is as follows: In \S 1, we prove Theorem 0.1. We indicate some more arithmetic applications of Theorem 0.1 and Corollary 0.7 in \S 2.

1. Proof of Theorem 0.1

In this section, we will prove Theorem 0.1. This proof is divided several remarks, lemmas and propositions. We begin with various reductions steps.

**Remark 1.1.** Let \( G, H, T \) be as in Theorem 0.1. Let \( N \) be a finitely generated \( T[[G]] \) module. Then consider the module \( N(p) \). Note that \( T[[G]] \) is noetherian and hence \( N(p) = N[p^{r}] \) for some \( r \in \mathbb{N} \). Thus from the short exact sequence \( 0 \rightarrow N[p^{r}] \rightarrow N \rightarrow \frac{N}{N[p^{r}]} \rightarrow 0 \), we get another right exact sequence

\[
\frac{N[p^{r}]}{QN[p^{r}]} \rightarrow \frac{N}{QN} \rightarrow \frac{\frac{N}{N[p^{r}]}}{\frac{N}{N[p^{r}][pr]}} \rightarrow 0,
\]

for any \( Q \in C \). Now let \( I_{N} \) be the image of \( \frac{N[p^{r}]}{QN[p^{r}]} \) in \( \frac{N}{QN} \). Then \( I_{N} \) is a finitely generated \( p^{r} \)-torsion \( T[F][G] \) module and hence \( \chi(U, I_{N}) \) exists for every \( U \) \[Ho, Proposition 1.6\]. Thus, \( \chi(U, \frac{N}{QN}) \) is finite if and only if \( \chi(U, \frac{N}{QN[p^{r}]}) \) is finite.

We can apply this observation for \( N = M(\theta) \), with \( M, \theta \) as in Theorem 0.1 and without any loss of generality, we may assume that \( M \) in Theorem 0.1 is a finitely generated \( T[[H]] \) module.

Next we consider finitely generated \( T[[G]] \) modules which are \( T \)-torsion.

**Proposition 1.2.** Let \( G, H \) be as in Theorem 0.1. Let \( N \) be a finitely generated \( T[[G]] \) module such that \( N \) is also \( T \)-torsion. Also assume \( T \) is a regular (local) ring. Then there exists a finite subset \( C_{N} \) of \( C \) such that for any \( Q \in C \backslash C_{N}, \chi(U, \frac{N}{QN}) \) exists for every open normal subgroup \( U \) of \( G \).

**Proof.** By Remark 1.1, we may assume \( N \) is \( p \)-torsion free. Let \( N \) be \( \langle x_{1}, x_{2}, ..., x_{n} \rangle \) as a finitely generated \( T[[G]] \) module. As \( N \) in \( T \)-torsion, \( \exists x_{i} \in T \) such that \( r_{i}x_{i} = 0 \). Set \( r := r_{1}r_{2}...r_{n} \) and notice that \( r \) being an element of \( T \) commutes with the elements of \( T[[G]] \). Thus we get that \( rN = 0 \). As \( T \) is a UFD, let \( r = p_{1}^{n_{1}}p_{2}^{n_{2}}...p_{t}^{n_{t}} \) be the unique factorization of \( r \) where \( p_{i}, 1 \leq i \leq t \) are height 1 primes in \( T \). Set \( C_{N} := \{ p_{1}, p_{2}, ..., p_{t} \} \). Then for any \( Q \in C \backslash C_{N}, \frac{N}{QN[p^{r}]}, \frac{N}{QN[p^{r}][pr]} \rightarrow 0, \frac{N}{N[p^{r}]} \rightarrow 0 \).
$\frac{N}{QN}$ is annihilated by the height 2 ideal $(r, Q)$. Now $T$ is a regular local ring of dimension 2. Thus for some $n_0 \in \mathbb{N}$, $p^{n_0} \in (r, Q)$. This shows $p^{n_0} \frac{N}{QN} = 0$. Hence for any choice of $Q \in C \setminus C_N$, $\chi(U, \frac{N}{QN})$ exists for every $U$ [Ho, Prop. 1.6].

**Corollary 1.3.** Let us keep the setting of Proposition 1.2. Then in the proof of Theorem 0.1, we may assume that $M$ is $T$-torsion free.

**Proof.** Take any $\theta \in S$ and write $N = M(\theta)$. Let $N_T$ denote the $T$-torsion submodule of $N$. Consider the exact sequence $0 \to N_T \to N \to N/N_T \to 0$. Then for any $Q \in C$, we get the induced exact sequence

$$0 \to \frac{N_T}{QN_T} \to \frac{N}{QN} \to \frac{N}{QN_T} \to 0.$$ 

By Proposition 1.2, there exists a finite subset $C_N = C_{M, \theta}$ of $C$ such that for any $Q \in C \setminus C_N$, $\chi(U, \frac{N}{QN})$ exists for every $U$. Thus the statement of Theorem 0.1 holds for $M$ if and only if it holds for $M/M_T$. □

Recall the following well known result, which can be conveniently found in [SS, Lemma 6.15].

**Lemma 1.4.** Let $T$ be a complete noetherian local domain of characteristic 0 with finite residue field of characteristic $p$ and let $T$ be finite over $\mathbb{Z}_p[[X]]$. Let $Q$ be a height 1 prime ideal in $T$. Put $q = Q \cap \mathbb{Z}_p[[X]]$. Let $Q = Q_1, Q_2, \ldots, Q_d$ be the height 1 prime ideals in $T$ lying above $q$. If $Q$ is unramified in $T$, then the kernel and cokernel of the natural map $T/qT \to \bigoplus_{1 \leq i \leq d} T/Q_iT$ are finite of $p$-power order. □

Using Lemma 1.4, we further reduce Theorem 0.1 to the case $T = \mathbb{Z}_p[[X]]$.

**Corollary 1.5.** Let us keep the hypotheses and setting of Theorem 0.1. Then in Theorem 0.1, without any loss of generality, we may assume $T = \mathbb{Z}_p[[X]]$.

**Proof.** Since the Kähler differential $\Omega_T/\mathbb{Z}_p[[X]]$ is supported outside finitely many height 1 prime ideals in $\mathbb{Z}_p[[X]]$, it follows that only finitely many height 1 primes of $\mathbb{Z}_p[[X]]$ can ramify in $T$. Let $C_1^{\mathbb{Z}_p[[X]]}$ be the finite set of height 1 prime ideals in $\mathbb{Z}_p[[X]]$ that ramify in $T$. Let $q \in C_1^{\mathbb{Z}_p[[X]]} \setminus C_{1}^{\mathbb{Z}_p[[X]]}$ and let $Q = Q_1, Q_2, \ldots, Q_d$ be the height 1 primes in $T$ dividing $q$. Recall $M$ in Theorem 0.1 is a finitely generated $T[\mathbb{G}]$ module. By Lemma 1.4, for any $\theta \in S$, there exists an exact sequence

$$0 \to K_q^\theta \to \frac{M(\theta)}{qM(\theta)} \to \bigoplus_{1 \leq i \leq d} \frac{M(\theta)}{Q_iM(\theta)} \to CK_q^\theta \to 0, \tag{1}$$

where $K_q^\theta$ and $CK_q^\theta$ are finitely generated $p$-power torsion $T/\mathbb{Z}_p[\mathbb{G}]$ modules. In particular, $K_q^\theta$ and $CK_q^\theta$ are finitely generated $p$-power torsion $\mathbb{Z}_p[\mathbb{G}]$ modules.

Now, as explained in Remark 1.1, $\chi(U, K_q^\theta)$ and $\chi(U, CK_q^\theta)$ always exist for any open normal subgroup $U$ of $G$.

Now assume Theorem 0.1 holds for $T = \mathbb{Z}_p[[X]]$. Then there exists a countable subset $S_M$ of $S$ and for any $\theta \in S \setminus S_M$ there exists a countable subset $C_{M, \theta}$ of $C_1^{\mathbb{Z}_p[[X]]}$ such that the following holds: For any $\theta \in S \setminus S_M$ and for any $Q \in C_1^{\mathbb{Z}_p[[X]]} \setminus C_{M, \theta}$, the $U$-Euler characteristic $\chi \left(U, \frac{M}{qM(\theta)} \right)$ exists for every
open normal subgroup $U$ of $G$. Note $C^Z_{p,[X]} \cup C^1_{p,[X]}$ is countable and hence $C^0_{M,\theta} := \{ Q \in C^T : Q \cap Z_p[[X]] \in C_{p,[X]}^Z \cup C_{p,[X]}^1 \}$ is a countable subset of $C^T$. Now from (1), for any $\theta \in S \setminus S_M$ and for any $Q \in C \setminus C^0_{M,\theta}, \chi(U, M, \theta)_{U}$ is finite for every $U$. Thus Theorem 0.1 holds for a general $T$.

Remark 1.6. As explained in [JS, Remark 2.3], we may also assume without any loss of generality, that $G$ is a compact, pro-$p$, $p$-adic Lie group without any element of order $p$. It should be noted [JS, Remark 2.3] uses fundamental work of Lazard [La].

Lemma 1.7. Let $G, H$ be in the setting of Theorem 0.1 and assume $T = Z_p[[X]]$. Let $M$ be a finitely generated $Z_p[[X]][G]$ module which is finitely generated over $Z_p[[X]][[H]]$. Also assume $\exists \ a$ height 1 prime ideal $Q_0$ in $Z_p[[X]]$ with $Q_0 \neq (p)$, such that $\#(\frac{M}{QM})_U$ is finite for every open normal subgroup $U$ of $G$. Then for all but countably many height 1 prime ideals $Q$ in $Z_p[[X]], \#(\frac{M}{QM})_U$ is finite for every open normal subgroup $U$ of $G$.

Proof. Let $U$ be an open normal subgroup of $G$. As $[G : U] < \infty$, $M_U$ is a finitely generated $Z_p[[X]]$ module. Now, from the structure theorem for finitely generated $Z_p[[X]]$ modules, we know that there exists a $Z_p[[X]]$ module homomorphism:

$$M_U \rightarrow Z_p[[X]]^{r_U} \oplus \bigoplus_{i=1}^{s_U} Z_p[[X]]/(Q_{i,U})^n$$

with finite kernel and cokernel, where $r_U, n_i, s_U \in \mathbb{N} \cup \{0\}$ and $Q_{i,U}$'s are height 1 prime ideals in $Z_p[[X]]$. Note that $(\frac{M}{QM})_U \cong \frac{M_U}{QM_U}$. Hence using the fact that the cardinality of $(\frac{M}{QM})_U$ is finite, we deduce from (2) that $r_U = 0$. So, if we choose $Q \in C$ such that $Q \notin C_{M,U} := \bigcup_{1 \leq i \leq s_U} Q_{i,U}$, then $\#H_0(U, \frac{M}{QM})$ is finite.

Since $G$ is a profinite group, it has countable base at identity. Thus we can take the set $U$ of open normal subgroups $U$ of $G$ to be countable. Then for any $Q \in C$ which does not belong to the countable set $C_M := \bigcup_{U \in \mathcal{U}} C_{M,U}$, we get $\#(\frac{M}{QM})_U$ is finite for every $U$. This completes the proof of the lemma.

We next prove the following proposition:

Proposition 1.8. Let $G, H$ be in the setting of Theorem 0.1. Let $M$ be a finitely generated $Z_p[[X]][G]$ module which is finitely generated over $Z_p[[X]][[H]]$. Then there exists a countable subset $S_M$ of $S$ and further, for each $\theta \in S \setminus S_M$, there exists a countable subset $C_{M,\theta}$ of $C$ such the the following holds: If we choose any $\theta \in S \setminus S_M$ and any $Q \in C \setminus C_{M,\theta}$, then $\#(\frac{M}{QM})_U$ is finite for every open normal subgroup $U$ of $G$.

Proof. Note that $Z_p[[X]][[G]] \cong Z_p[[G_1 \times G]]$, where $G_1 \cong Z_p$. Let $U$ be any open normal subgroup of $G$. Then $U' := G_1 \times U$ is an open normal subgroup of $G_1 \times G$ that maps onto $U$ under the projection map $\pi : G_1 \times G \rightarrow G$. Now by Theorem ([JOZ]), there is a countable subset $S_M$ of $S$, such that for any $\theta \in S \setminus S_M$, we have $(M(\theta))_{U'} = (M(\theta))_{G_1 \times U}$ is finite for every $U$. Therefore, $(M(\theta))_{U'} = (M(\theta))_{U'/U} \cong \frac{M(\theta)}{U}/(X)$ is finite for every $U$.
So writing \( N = M(\theta) \) and \( Q_0 = (X) \), we have \( (\overline{Q}_0 \cap Q_i) \) is finite for every \( U \) open normal in \( G \). Now applying Lemma 1.7, there is a countable subset \( C_{M,\theta} \subseteq C \) such that for any \( Q \in C \setminus C_{M,\theta} \), we have \( (\overline{Q}_0 \cap Q_i) \) is finite for every open normal subgroup \( U \).

The next lemma is easy to prove and used later.

**Lemma 1.9.** Let \( G, H \) be as in Theorem 0.1. Let \( G_1 \) be a \( p \)-adic Lie group isomorphic to \( \mathbb{Z}_p \). Let \( K \) be an open subgroup of \( G_1 \times G \) such that \( G_1 \times H \subseteq K \). Then \( K = G_1 \times G^0 \), where \( G^0 \) is an open subgroup of \( G \).

Using Lemma 1.9, we deduce the following result.

**Lemma 1.10.** Let \( G \) be a compact, pro-\( p \), \( p \)-adic Lie group without any \( p \)-torsion element. Let \( H \) be a closed normal subgroup of \( G \) with \( \Gamma := G/H \cong \mathbb{Z}_p \). Let \( M \) be a finitely generated \( \mathbb{Z}_p[[X]][[G]] \) module which is also finitely generated over \( \mathbb{Z}_p[[X]][[H]] \). Then, there exists an open subgroup \( G^0 \) of \( G \) with \( H \subseteq G^0 \) and a resolution

\[
0 \to N_k \to N_{k-1} \to \cdots \to N_1 \to M \to 0
\]

of \( M \) by finitely generated \( \mathbb{Z}_p[[X]][[G^0]] \) modules \( N_i \), \( 1 \leq i \leq k \) such that each \( N_i \) is a free \( \mathbb{Z}_p[[X]][[H]] \) module of finite rank.

**Proof.** Note that \( \mathbb{Z}_p[[X]][[G]] \cong \mathbb{Z}_p[[G_1 \times G]] \), where \( G_1 \cong \mathbb{Z}_p \). Also \( \frac{G_1 \times G}{G/H} \cong \Gamma \). Using [JS, Lemma 2.4], \( \exists \) an open subgroup \( G^{00} \) of \( G_1 \times G \) such that \( G_1 \times H \subset G^{00} \) and a resolution

\[
0 \to N_k \to N_{k-1} \to \cdots \to N_1 \to M \to 0
\]

of \( M \) by finitely generated \( \mathbb{Z}_p[[G^{00}]] \) modules \( N_i \), \( 1 \leq i \leq k \) such that each \( N_i \) is a free \( \mathbb{Z}_p[[G_1 \times H]] \) module of finite rank. By Lemma 1.9, \( G^{00} = G_1 \times G^0 \), where \( G^0 \) is an open subgroup of \( G \) containing \( H \).

Next we calculate the Euler characteristic of a free \( \mathbb{Z}_p[[X]][[H]] \) module. Recall, for any \( Q \in C_{\mathbb{Z}_p[[X]]} \), \( \frac{\mathbb{Z}_p[[X]]}{Q} \cong O_Q \), the ring of integer of certain finite extension of \( \mathbb{Q}_p \).

**Proposition 1.11.** Let \( G \) be a compact, pro-\( p \), \( p \)-adic Lie group without any \( p \)-torsion element. Let \( H \) be a closed normal subgroup of \( G \) with \( \Gamma := G/H \cong \mathbb{Z}_p \). Let \( N \) be a finitely generated \( \mathbb{Z}_p[[X]][[G]] \) module which is also a finitely generated free \( \mathbb{Z}_p[[X]][[H]] \) module of rank \( d \). Then there exists a countable subset \( S_N \) of \( S \) and for any \( \theta \in S \setminus S_N \) there exists a countable subset \( C_{N,\theta} \) of \( C \) such that the following holds: For any \( \theta \in S \setminus S_N \) and for any \( Q \in C \setminus C_{N,\theta} \), the U-Euler characteristic \( \chi \left( U, \frac{N}{Q} \right) \) exists for every open normal subgroup \( U \) of \( G \). Moreover, \( \chi \left( U, \frac{N}{Q} \right) = \# \left( \frac{N}{Q} \right) \).

**Proof.** By Proposition 1.8, \( \exists \) a countable subset \( S_N \) of \( S \) and for any \( \theta \in S \setminus S_N \) there exists a countable subset \( C_{N,\theta} \) of \( C \) such that the following holds: For any \( \theta \in S \setminus S_N \) and for any \( Q \in C \setminus C_{N,\theta} \), \( H_0 \left( U, \frac{N}{Q} \right) \) is finite for every open normal subgroup \( U \) of \( G \). Now as \( \frac{N}{Q} \) is a free \( O_Q[[H]] \) module of finite rank, the result follows directly from [JS, Proposition 2.7].
**Proof of Theorem 0.1:** First of all, using Corollary 1.5, we can assume without any loss of generality, that $T = \mathbb{Z}_p[[X]]$. Next, by Remark 1.1, we can assume $M$ is a finitely generated $\mathbb{Z}_p[[X]][[H]]$ module. Further, as $\mathbb{Z}_p[[X]]$ is regular, we can assume using Corollary 1.3 that $M$ is $T$-torsion free. Moreover, following Remark 1.6, we will assume that $G$ is a compact, pro-$p$, $p$-adic Lie group without any element of order $p$.

By Lemma 1.10, there exists an open normal subgroup $G^0$ of $G$ with $H \subset G^0$ and a resolution

$$0 \rightarrow N_k \xrightarrow{f_k} N_{k-1} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} N_1 \xrightarrow{f_1} M \rightarrow 0 \quad (3)$$

of $M$ by finitely generated $\mathbb{Z}_p[[X]][[G^0]]$ modules $N_i$, $1 \leq i \leq k$ such that each $N_i$ is a free $\mathbb{Z}_p[[H]]$ module of finite rank.

Set $T^0 := G^0/H$. Then as explained in the proof of [JS, Theorem 1.2], for any given $\theta \in S$, $Q \in C$ and any open normal subgroup $U$ of $G$, if $\chi(U \cap G^0, \frac{M}{QM}(\theta))$ is finite, then we can deduce $\chi(U, \frac{M}{QM}(\theta))$ is also finite. Thus, for the rest of the proof, we will only discuss the finiteness of $\chi(U \cap G^0, \frac{M}{QM}(\theta))$; and further, to ease the burden of notation, will write $G^0 = G$ and $T^0 = T$ for the rest of the proof.

We will proceed by induction on $k$ in (3). For $k = 1$, $M/QM$ is a free $\mathbb{Z}_p[[X]][[H]] \cong O_Q[[H]]$ module of finite rank, for any $Q \in C$. Hence by Proposition 1.11, there exists a countable subset $S_M$ of $S$ and a countable subset $C_{M, \theta}$ of $C$, such that the following holds: For any $\theta \in S \setminus S_M$ and for any $Q \in C \setminus C_{M, \theta}$, $\chi(U, \frac{N}{QN}(\theta))$ exists for every open normal subgroup $U$ of $G$.

Next, pick any $Q \in C$. Then $\mathbb{Z}_p[[X]]$ being a regular local ring, we get $Q = (q)$ and hence $M[q] = 0$ as $M$ is $T$-torsion free. Thus (3) gives rise to another exact sequence of $O_Q[[G]]$ modules

$$0 \rightarrow \text{Img}(f_2)/Q\text{Img}(f_2) \rightarrow N_1/QN_1 \xrightarrow{f_1} M/QM \rightarrow 0. \quad (4)$$

By induction, there exists a countable subset $S_2$ of $S$ and a $C_{2, \theta}$ of $C$, such that the following holds: For any $\theta \in S \setminus S_2$ and for any $Q \in C \setminus C_{2, \theta}$, $\chi(U, \frac{\text{Img}(f_2)}{Q\text{Img}(f_2)}(\theta))$ exists for every open normal subgroup $U$ of $G$. Similarly, $N_1/QN_1$ is a free $O_Q[[H]]$ module of finite rank and hence there exists a countable subset $S_1$ of $S$ and a countable subset $C_{1, \theta}$ of $C$, such that the following holds: For any $\theta \in S \setminus S_1$ and for any $Q \in C \setminus C_{1, \theta}$, $\chi(U, \frac{N_1}{QN_1}(\theta))$ exists for every open normal subgroup $U$ of $G$. Define $S_M := S \cup S_2$, and for any $\theta \in S \setminus S_M$, set $C_{M, \theta} = C_{1, \theta} \cup C_{2, \theta}$. Then from (4), for any $\theta \in S \setminus S_M$ and any $Q \in C \setminus C_{M, \theta}$, $\chi(U, \frac{M}{QM}(\theta))$ is finite for every $U$. This completes the proof of Theorem 0.1.  

**2. Arithmetic Applications**

In addition to Remark 0.8, we indicate some more arithmetic applications of our results.
2.1. fine Selmer group: Let $K \subset K_{\text{cyc}} \subset K_\infty$ be a $p$-adic Lie extension of a number field $K$, unramified outside finitely many primes of $K$. We continue to assume $G = \text{Gal}(K_\infty/K)$ has no element of order $p$. Let $X(L_f/K_\infty)$ denote the dual Selmer group [JO, Definition 1.11] of a $p$-ordinary newform $f$, where $L_f$ is a lattice of Deligne’s Galois representation associated to the modular form $f$. The dual fine Selmer group $Y(L_f/K_\infty)$ of a newform $f$ over $K_\infty$, is a quotient of $X(L_f/K_\infty)$ (see [Jh, §2]) and has interesting arithmetic properties. Following a conjecture of Coates-Sujatha in [CS], it is believed that $Y(L_f/K_\infty)$ is a finitely generated $\mathbb{Z}_p$ module, for any $f$. For a general $p$-adic Lie extension $K_\infty/K$, it is not easy to determine if the Euler characteristic $\chi(G, Y(L_f/K_\infty))$ exists (even if we assume $Y(L_f/K)$ is finite) and not much is known in the literature. Thus it is reasonable to discuss, if at least after a twist, the Euler characteristic of the fine Selmer group exists and also make sense to consider the Euler characteristic of a twist of the fine Selmer groups in a congruent family.

Let $T = T_F$ be the quotient of the universal ordinary Hecke algebra that corresponds to an ordinary $\Lambda$-adic newform $F$ (see [Hi]). The algebra $T_F$ is a local domain and is finite flat over $\mathbb{Z}_p[[X]]$. By a celebrated result of Hida ([Hi], [Wi]), $\exists$ a `large’ continuous irreducible representation $\rho_F : G_\mathbb{Q} \rightarrow \text{Aut}_{T_F}(L_F)$, where $L_F$ is a finitely generated, torsion-free module of generic rank 2 over $T_F$. We assume that the residual representation $\bar{\rho}_F$ associated to $\rho_F$ (see [Hi]) is absolutely irreducible. For each $Q \in \mathcal{A}_{\text{arith}}(T_F)$, $\exists$ a $p$-ordinary, $p$-stabilized newform $f_Q$, such that the quotient $L_F/QL_F$ is isomorphic to $L_Q$, the unique lattice of Deligne’s Galois representation associated to the cuspspace $f_Q$.

Under appropriate assumption, one defines the dual Selmer group $\mathcal{X}(L_F/K_\infty)$ (respectively the dual fine Selmer group $\mathcal{Y}(L_F/K_\infty)$) of the $\Lambda$-adic newform $F$ [Jh, §2]. Under suitable hypotheses, the fine Selmer group $\mathcal{Y}(L_F/K_\infty)_{\mathbb{Z}_p}$ is a finitely generated $T_F[[H]]$ module and similarly $\mathcal{Y}(L_F/K_\infty)_{\mathbb{Q}}/\mathcal{Y}(L_F/K_\infty)_{\mathbb{Z}_p}$ is a finitely generated $O_Q[[H]]$ module ([Jh, §2]). Then applying Corollary 0.7, we deduce the following:

**Corollary 2.1.** We keep the setting described above. There exists a countable subset $S_{\mathcal{Y}(L_F/K_\infty)}$ of $S$ such that for any chosen $\theta \in S \setminus S_{\mathcal{Y}(L_F/K_\infty)}$ and for every choice of $Q \in \mathcal{A}_{\text{arith}}(T_F)$, $\chi(U, \mathcal{Y}(L_F/K_\infty)_{\mathbb{Q}}(\mathcal{Y}(L_F/K_\infty)_{\mathbb{Z}_p}(\theta))$ is finite for every open normal subgroup $U$ of $G$.

Now consider a special case $K = \mathbb{Q}(\mu_p)$, $p \geq 3$ and the ‘false Tate curve’ extension $K_\infty = \mathbb{Q}(\mu_{p^n}, m^1/p^n)$, where $m$ is any $p$-power free integer and assume that $T_F = O[[X]]$. Then it is shown in [Jh, Theorem 4 & Remark 11] that the kernel and the cokernel of the natural map $\mathcal{Y}(L_F/K_\infty)_{\mathbb{Q}}/\mathcal{Y}(L_F/K_\infty)_{\mathbb{Z}_p} \rightarrow Y(L_Q/K_\infty)$ are finitely generated $O_Q$ modules. Also assume for some $Q_0 \in \mathcal{A}_{\text{arith}}(T_F)$, $Y(L_{Q_0}/\mathbb{Q}(\mu_{p^n}))$ is a finitely generated $O_{Q_0}$ module. Then $\mathcal{Y}(L_F/K_\infty)$ and $Y(L_Q/K_\infty)$ are finitely generated modules respectively over $T_F[[H]]$ and $O_Q[[H]]$, for every $Q \in \mathcal{A}_{\text{arith}}(T_F)$ [Jh, §2].

Then from the proof of Proposition 0.6 and Corollary 2.1, we obtain the following result on the fine Selmer groups $Y(L_Q/K_\infty)$ of the congruent family of cuspforms $f_Q$, where $Q \in \mathcal{A}_{\text{arith}}(T_F)$. 

11
Theorem 2.2. Let $K_{\infty}/Q(\mu_p)$ be the false Tate curve extension. Let $F$ be a $\Lambda$-adic newform. Assume that $T_F \cong O[[X]]$ is a power series ring and $\exists Q_0 \in A_{\text{arith}}(T_F)$ such that $Y(L_{Q_0}/Q(\mu_p))$ is a finitely generated $O_{Q_0}$ module. 

Then there exists a countable subset $S_{Y(L_F/K_{\infty})}$ of $S$, such that for any chosen $\theta \in S \setminus S_{Y(L_F/K_{\infty})}$ and for any choice of $Q \in A_{\text{arith}}(T_F)$, the $U$-Euler characteristic $\chi(U, Y(L_Q/K_{\infty})/(\theta))$ exists, for every open normal subgroup $U$ of $G$. \hfill $\square$

**Remark 2.3.** A result similar to Corollary 2.1 and Theorem 2.2 respectively for the usual ‘big’ dual Selmer group $X(L_F/K_{\infty})$ (see [Jh, $\S$3]) and for the Selmer group $X(L_Q/K_{\infty})$ of $f_{Q}$ for $Q \in A_{\text{arith}}(T_F)$ can also be obtained.

2.2. $\mathbb{Z}_p[[H]]$ free Selmer group: Let $E$ be the elliptic curve of conductor 121 defined by the equation $y^2 + y = x^3 - x^2 - 887x - 10143$. Let us consider a particular false Tate curve extension by taking $p = 5$ and $m = 11$. Then $G = \text{Gal}(K_{\infty}/K) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. Let $X(E/K_{\infty})$ (respectively $X(E/K_{\text{cyc}})$) denote the dual of the $5\text{-}S$-Selmer group of $E$ over $K_{\infty}$ (respectively $K_{\text{cyc}}$). Then using the fact that $X(E/K_{\text{cyc}})$ has $\mu$-invariant zero together with [HV, Theorem 3.1], $\exists$ an injective $\mathbb{Z}_5[[H]]$ module homomorphism $f : X(E/K_{\infty}) \to (\mathbb{Z}_5[[H]])^t$, for some integer $t$. In fact, using $E(\mathbb{F}_5)[5] = 0$, from the proof of [HV, Theorem 3.1] we can show that $f$ is an isomorphism. Let $Q_0 \in A_{\text{arith}}(T_F)$ be such that the weight 2 specialization $f_{Q_0}$ corresponds to $E$. Since $E$ has CM, in this case $T \cong \mathbb{Z}_5[[X]]$. Then following an argument similar to [Sh, page 412], we can deduce that the natural map $X(L_F/K_{\infty})/Q_0 \to X(E/K_{\infty})$ is an isomorphism. Now it is well known that $X(L_F/K_{\infty})$ has no non-trivial pseudounit $T[[G]]$ sub-module. In particular, $X(L_F/K_{\infty})[Q_0] = 0$. Since $X(E/K_{\infty})$ is a free $\mathbb{Z}_5[[H]]$-module of finite rank, by Nakayama’s lemma $X(L_F/K_{\infty})$ is also a free $T[[H]]$-module of finite rank. This also implies that $X(L_F/K_{\infty})/Q \cong X(L_Q/K_{\infty})$ is free $O_{Q_0}[[H]]$-module of finite rank for every $Q \in A_{\text{arith}}(T_F)$. We notice that as the rank of $E(\mathbb{Q}) = 1$, the $U$-Euler characteristics of $X(E/K_{\infty})$ does not exist for any open normal subgroup $U$ of $G$.

Write $M = X(L_F/K_{\infty})$. Then applying Theorem 0.1, we get a countable subset $S_M$ of $S$ and for each $\theta \in S \setminus S_M$, $\exists$ a countable subset $C_{M, \theta}$ of $C$ such that the following holds: For every $\theta \in S \setminus S_M$ and for any $Q \in C_{M, \theta}$, the $U$-Euler characteristic $\chi(U, M_{Q_M}(\theta))$ exists for every $U$ and is given by $\chi(U, M_{Q_M}(\theta)) = \#(M_{Q_M}(\theta))_U$.

We continue to write $M = X(L_F/K_{\infty})$. Then by Corollary 0.7, we deduce $\exists$ a countable subset $S_M$ of $S$ such that for any chosen $\theta \in S \setminus S_M$ and for every $Q \in A_{\text{arith}}(T_F)$, the $U$-Euler characteristic $\chi(U, M_{Q_M}(\theta))$ exists for every $U$ and is given by $\chi(U, M_{Q_M}(\theta)) = \#(M_{Q_M}(\theta))_U = \#(X(L_Q/K_{\infty})(\theta))_U = \#(X(E/K_{\infty})(\theta))_U$. In particular, for every $Q \in A_{\text{arith}}(T_F)$, there exists a countable subset $S_M$ of $S$ such that for any chosen $\theta \in S \setminus S_M$, $\chi(U, X(L_Q/K_{\infty})(\theta))$ is finite and $\chi(U, X(L_Q/K_{\infty})(\theta))$ is finite and $\#(X(E/K_{\infty})(\theta))_U$, for every open normal subgroup $U$ of $G$.

**Remark 2.4.** Let $E/K$ be an elliptic with good, ordinary reduction at primes of $K$ dividing $p$. Let $p$-cohomological dimension $cd_p(G) = \text{Gal}(K_{\infty}/K)$ be $\geq 3$. Also assume (i) for a prime $v$ of $K$ dividing $p$, $cd_p(G_v) < cd_p(G)$ and (ii) for a prime $v$ of $K$ not dividing $p$, that either ramifies in $K_{\infty}$ or is a bad prime of $E$, $cd_p(G_v) \geq 2$. Further assume, $X(E/K_{\text{cyc}})$ is a finitely generated $\mathbb{Z}_p$ module. Then by [OV, Proposition 5.2], the $\mathbb{Z}_p[[G]]$ projective dimension of $X(E/K_{\infty})$ is
\[ cd_p(G) - 1 \text{ if } E(K)[p] \neq 0 \text{ and } = cd_p(G) - 2 \text{ if } E(K)[p] = 0. \] Now, as \( X(E/K_{\text{cyc}}) \) is a finitely generated \( \mathbb{Z}_p \) module, applying a control theorem, we can deduce \( X(E/K_{\infty}) \) is a finitely generated \( \mathbb{Z}_p[[H]] \) module. Since \( cd_p(H) = cd_p(G) - 1, \mathbb{Z}_p[[H]] \) projective dimension of \( X(E/K_{\infty}) \) is \( cd_p(H) - 1 \) if \( E(K)[p] \neq 0 \) and \( = cd_p(H) - 2 \text{ if } E(K)[p] = 0. \) In particular, if \( \dim H \geq 3 \), then \( X(E/K_{\infty}) \) cannot be a free \( \mathbb{Z}_p[[H]] \) module. Nevertheless, \( \exists \) an open subgroup \( G^0 \) of \( G \) containing \( H \) and a \( \mathbb{Z}_p[[G^0]] \) resolution of \( X(E/K_{\infty}) \) of length \( k \) by finitely generated \( \mathbb{Z}_p[[H]] \) modules [JS, Lemma 2.4]. Moreover, the length of the resolution is given by \( k = cd_p(H) - 1 \) if \( E(K)[p] \neq 0 \) and \( k = cd_p(H) - 2 \) if \( E(K)[p] = 0. \) A similar assertion holds for the ‘big’ Selmer group \( \mathcal{X}(T_{\infty}/K_{\infty}) \). (See Lemma 1.10).

Acknowledgement

S. Jha acknowledges the support of SERB MATRICS grant and SERB ECR grant. S. Shekhar is supported by DST INSPIRE faculty award grant. We thank Tadashi Ochiai for some discussions.

References

[Br] A. Brumer, Pseudocompact Algebras, Profinite Groups and Class Formations, Journal of Algebra 4 (1966) 442-470.
[CS] J. Coates and R. Sujatha, Fine Selmer group of elliptic curves over p-adic Lie extensions, Math. Ann. 331 (4) (2005), 809-839.
[CFKSV] J. H. Coates, T. Fukaya, K. Kato, R. Sujatha, O. Venjakob, GL2 main conjecture of elliptic curves without complex multiplication, Publ. Math. IHES 101 (2005), 163-208.
[Gr] R. Greenberg, Iwasawa theory for p-adic representations, in Algebraic number theory, Adv. Stud. Pure Math. 17 (1989) 97-137.
[HV] Y. Hachimori and O. Venjakob, Completely Faithful Selmer Groups over Kummer Extensions, Documenta Math. Extra Volume Kato, (2003), 443-478.
[Hi] H. Hida, Galois representations into \( \text{GL}_2(\mathbb{Z}_p[[X]]) \) attached to ordinary cusp forms, Invent. Math. 85 (3) (1986), 545-613.
[Ho] S. Howson, Euler characteristics as invariants of Iwasawa modules, Proc. London Math. Soc. 85 (3) (2002), no. 3, 634-658.
[Jh] S. Jha, Fine Selmer group of Hida deformations over non-commutative p-adic Lie extensions, Asian J. Math., 16(2)(2012), 353-365.
[JO] S. Jha and T. Ochiai, Control theorem and functional equation of Selmer groups over p-adic Lie extensions, preprint, arXiv:1910.05454.
[JS] S. Jha, and S. Shekhar, Non-commutative twisted Euler characteristic, Münster J. Math. 11 (1) (2018) , 1-12.
[JOZ] S. Jha, T. Ochiai and G. Zábrádi, On twists of modules over Non-Commutative Iwasawa algebras, Algebra & Number Theory, 10(3) 2016, 685-694.
[La] M. Lazard, Groupes analytiques p-adiques, IHES Publ. Math. 26 (1965), 5-219.
[OV] Y. Ochi and O. Venjakob, On the ranks of Iwasawa modules over p-adic Lie extensions, Math. Proc. Camb. Phil. Soc. 135 (2003) 25-43.
[Pr] B. Perrin-Riou, Groupes de Selmer et accouplements: cas particulier des courbes elliptiques, Documenta Math. Extra Vol. Kato (2003), 725-760.
[Sh] S. Shekhar, Euler characteristic of \( \Lambda \)-adic forms over Kummer extensions, Int. J. Number Theory, 10 (2) (2014), 401-419.
[SS] R. Sujatha and S. Shekhar, On the Structure of Selmer Groups of \( \Lambda \)-Adic Deformations over p-Adic Lie Extensions, Documenta Math. 17 (2012) 573-606.
[Wi] A.Wiles, The Iwasawa conjecture for totally real fields, Ann. of Math. (2) 131 (3) (1990), 493-540.

Department of Mathematics and Statistics, IIT Kanpur, Kanpur 208016, India

E-mail address: gsohan@iitk.ac.in, jhasom@iitk.ac.in, sudhansh@iitk.ac.in