When Risks and Uncertainties Collide: Mathematical Finance for Arbitrage Markets in a Quantum Mechanical View

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Abstract

Geometric Arbitrage Theory reformulates a generic asset model possibly allowing for arbitrage by packaging all assets and their forwards dynamics into a stochastic principal fibre bundle, with a connection whose parallel transport encodes discounting and portfolio rebalancing, and whose curvature measures, in this geometric language, the "instantaneous arbitrage capability" generated by the market itself. The asset and market portfolio dynamics have a quantum mechanical description, which is constructed by quantizing the deterministic version of the stochastic Lagrangian system describing a market allowing for arbitrage. Results, obtained by solving explicitly the Schrödinger equations by means of spectral decomposition of the Hamilton operator, coincides with those obtained by solving the stochastic Euler Lagrange equations derived by a variational principle and providing therefore consistency. Arbitrage bubbles are computed.
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1 Introduction

This paper further develops a conceptual structure - called Geometric Arbitrage Theory - to link arbitrage modeling in generic markets with quantum mechanics.

GAT rephrases classical stochastic finance in stochastic differential geometric terms in order to characterize arbitrage. The main idea of the GAT approach consists of modeling markets made of basic financial instruments together with their term structures as principal fibre bundles. Financial features of this market - like no arbitrage and equilibrium - are then characterized in terms of standard differential geometric constructions - like curvature - associated to a natural connection in this fibre bundle. Principal fibre bundle theory has been heavily exploited in theoretical physics as the language in which laws of nature can be best formulated by providing an invariant framework to describe physical systems and their dynamics. These ideas can be carried over to mathematical finance and economics. A market is a financial-economic system that can be described by an appropriate principle fibre bundle. A principle like the invariance of market laws under change of numéraire can be seen then as gauge invariance. Concepts like No-Free-Lunch-with-Vanishing-Risk (NFLVR) and No-Unbounded-Profit-with-Bounded-Risk (NUPBR) have a geometric characterization, which can be represented in exact mathematical terms either by means of stochastic differential geometry or via quantum mechanics.

The fact that gauge theories are the natural language to describe economics was first proposed by Malaney and Weinstein in the context of the economic index problem ([Ma96], [We06]). Ilinski (see [Il00] and [Il01]) and Young ([Yo99]) proposed to view arbitrage as the curvature of a gauge connection, in analogy to some physical theories. Independently, Cliff and Speed ([SmSp98]) further developed Flesaker and Hughston seminal work ([FHu96]) and utilized techniques from differential geometry to reduce the complexity of asset models before stochastic modeling.

This paper is structured as follows. Section 2 reviews classical stochastic finance and Geometric Arbitrage Theory. Arbitrage is seen as curvature of a principal fibre bundle representing the market which defines the quantity of arbitrage associated to it. Proofs are omitted and can be found in [Fa15] and in [FaTa19], where Geometric Arbitrage Theory has been given a rigorous mathematical foundation utilizing the formal background of stochastic differential geometry as in Schwartz ([Schw80]), Elworthy ([El82]), Eméry([Em89]), Hackenbroch and Thalmaier ([HaTh94]), Stroock ([St00]) and Hsu ([Hs02]).

Section 3 describes the intertwined dynamics of assets, term structures and market portfolio as constrained Lagrange system deriving it from a stochastic variational principle whose Lagrange function measures the arbitrage quantity allowed by the market. This constrained Lagrange system and its stochastic Euler-Lagrange equation is equivalent to a constrained Hamilton system, obtained by Legendre transform, with its stochastic Hamilton equations. These stochastic Hamilton system is, on
its turn, equivalent to a quantum mechanical system, obtained by quantizing the deterministic version of the Hamilton system. This is shown in Section 4 where we reformulate mathematical finance in terms of quantum mechanics. The Schrödinger equation describes then both the asset and market portfolio dynamics, which can be explicitly computed once the spectrum of the Hamilton operator, a second order non-elliptic self adjoint pseudodifferential operator is known. The expected values of future asset values and market portfolio nominals are identical to those computed with the stochastic Euler Lagrange equation, demonstrating the consistency of the quantum mechanical approach.

In Section 5 we explicitly compute the market dynamics of the solution of the minimal arbitrage problem and the of the corresponding bubbles for base assets and derivatives of European style. In section 6 we show how Feynman’s path integral can be utilized to solve the Schrödinger equation and compute the arbitrage market dynamics. The appeal of this approach is that it can be directly implemented as a numerical algorithm in a simulation procedure. Appendix A reviews Nelson’s stochastic derivatives. Section 7 concludes.

2 Geometric Arbitrage Theory Background

In this section we explain the main concepts of Geometric Arbitrage Theory introduced in [Fa15], to which we refer for proofs and examples.

2.1 The Classical Market Model

In this subsection we will summarize the classical set up, which will be rephrased in section (2.4) in differential geometric terms. We basically follow [HuKe04] and the ultimate reference [DeSc08].

We assume continuous time trading and that the set of trading dates is \([0, +\infty]\). This assumption is general enough to embed the cases of finite and infinite discrete times as well as the one with a finite horizon in continuous time. Note that while it is true that in the real world trading occurs at discrete times only, these are not known a priori and can be virtually any points in the time continuum. This motivates the technical effort of continuous time stochastic finance.

The uncertainty is modelled by a filtered probability space \((\Omega, \mathcal{A}, \mathbb{P})\), where \(\mathbb{P}\) is the statistical (physical) probability measure, \(\mathcal{A} = \{\mathcal{A}_t\}_{t \in [0, +\infty]}\) an increasing family of sub-\(\sigma\)-algebras of \(\mathcal{A}_\infty\) and \((\Omega, \mathcal{A}_\infty, \mathbb{P})\) is a probability space. The filtration \(\mathcal{A}\) is assumed to satisfy the usual conditions, that is

- right continuity: \(\mathcal{A}_t = \bigcap_{s > t} \mathcal{A}_s\) for all \(t \in [0, +\infty]\).
- \(\mathcal{A}_0\) contains all null sets of \(\mathcal{A}_\infty\).
The market consists of finitely many assets indexed by \( j = 1, \ldots, N \), whose nominal prices are given by the vector valued semimartingale \( S : [0, +\infty[ \times \Omega \to \mathbb{R}^N \) denoted by \( (S_t)_{t \in [0, +\infty[} \) adapted to the filtration \( \mathcal{A} \). The stochastic process \( (S^j_t)_{t \in [0, +\infty[} \) describes the price at time \( t \) of the \( j \)th asset in terms of unit of cash at time \( t = 0 \). More precisely, we assume the existence of a 0th asset, the cash, a strictly positive semimartingale, which evolves according to \( S^0_t = \exp\left(\int_0^t d\rho_u^0\right) \), where the predictable semimartingale \( (\rho^0_t)_{t \in [0, +\infty[} \) represents the continuous interest rate provided by the cash account: one always knows in advance what the interest rate on the own bank account is, but this can change from time to time. The cash account is therefore considered the locally risk less asset in contrast to the other assets, the risky ones. In the following we will mainly utilize discounted prices, defined as \( \hat{S}^j_t := S^j_t / S^0_t \), representing the asset prices in terms of current unit of cash.

We remark that there is no need to assume that asset prices are positive. But, there must be at least one strictly positive asset, in our case the cash. If we want to renormalize the prices by choosing another asset instead of the cash as reference, i.e. by making it to our numéraire, then this asset must have a strictly positive price process. More precisely, a generic numéraire is an asset, whose nominal price is represented by a strictly positive stochastic process \( (B_t)_{t \in [0, +\infty[} \), and which is a portfolio of the original assets \( j = 0, 1, 2, \ldots, N \). The discounted prices of the original assets are then represented in terms of the numéraire by the semimartingales \( \hat{S}^j_t := S^j_t / B_t \).

We assume that there are no transaction costs and that short sales are allowed. Remark that the absence of transaction costs can be a serious limitation for a realistic model. The filtration \( \mathcal{A} \) is not necessarily generated by the price process \( (S_t)_{t \in [0, +\infty[} \); other sources of information than prices are allowed. All agents have access to the same information structure, that is to the filtration \( \mathcal{A} \).

An admissible strategy \( x = (x_t)_{t \in [0, +\infty[} \) is a predictable semimartingale for which the Itô integral \( \int_0^t x \cdot S \) is almost surely \( t \)-uniformly bounded from below.

**Definition 1 (Arbitrage).** Let \( T \leq +\infty \), the process \( (S_t)_{[0, +\infty[} \) be a semimartingale and \( (x_t)_{t \in [0, +\infty[} \) and admissible strategy. We denote by \( (x \cdot S)_T := \lim_{t \to T} \int_0^t x_u \cdot S_u \) if such limit exits, and by \( K_0 \) the subset of \( L^0(\Omega, \mathcal{A}_T, P) \) containing all such \( (x \cdot S)_T \). Then, we define

- \( C_0 := K_0 - L^0_+(\Omega, \mathcal{A}_T, P) \).
- \( C := C_0 \cap L^\infty_+(\Omega, \mathcal{A}_T, P) \).
- \( \bar{C} \): the closure of \( C \) in \( L^\infty \) with respect to the norm topology.
- \( \mathcal{X}^Vu_T := \{(x \cdot S)_T \mid (x \cdot S)_0 = V_0, x \text{ admissible}\} \).

We say that \( S \) satisfies
• **(NA), no arbitrage**, if and only if \( C \cap L^\infty(\Omega, \mathcal{A}_T, P) = \{0\} \).

• **(NFLVR), no-free-lunch-with-vanishing-risk**, if and only if \( \bar{C} \cap L^\infty(\Omega, \mathcal{A}_T, P) = \{0\} \).

• **(NUPBR), no-unbounded-profit-with-bounded-risk**, if and only if \( X V_0^T \) is bounded in \( L^0 \) for some \( V_0 > 0 \).

\[ \text{Theorem 2 (First fundamental theorem of asset pricing). The market } (S, \mathcal{A}) \text{ satisfies the (NFLVR) condition if and only if there exists an equivalent local martingale measure } P^*. \]

\[ \text{Remark 3. In the first fundamental theorem of asset pricing we just assumed that the price process } S \text{ is locally bounded. If } S \text{ is bounded, then (NFLVR) is equivalent to the existence of a martingale measure. But without this additional assumption (NFLVR) only implies the existence of a local martingale measure, i.e. a local martingale which is not a martingale. This distinction is important, because the difference between a security price process being a strict local martingale versus a martingale under a probability } P^* \text{ relates to the existence of asset price bubbles.} \]

\[ \text{Definition 4 (Complete market). The market } (S, \mathcal{A}) \text{ is complete on } [0, T] \text{ if for all contingent claims } C \in L_+(P^*, \mathcal{A}_T) := \{ C : \Omega \to [0, +\infty] | C \text{ is } \mathcal{A}_T \text{ measurable and } E^{P^*}_0 [\|C\|] < +\infty \} \text{ there exists an admissible self-financing strategy } x \text{ such that } C = (x \cdot S)_T. \]

\[ \text{Theorem 5 (Second fundamental theorem of asset pricing). Given } (S, \mathcal{A}) \text{ satisfies the (NFLVR) condition, the market is complete on } [0, T] \text{ if and only if the equivalent local martingale } P^* \text{ is unique.} \]

\[ \text{Definition 6 (Dominance). The } j \text{-th security } S^j = (S^j_t)_{t \in [0, T]} \text{ is undominated on } [0, T] \text{ if there is no admissible strategy } (x_t)_{t \in [0, T]} \text{ such that} \]

\[ S^j_0 + (x \cdot S)_T \geq S^j_T \quad \text{a.s. and} \quad P[S^j_0 + (x \cdot S)_T > S^j_T] > 0. \]  

(1)

\[ \text{We say that } S \text{ satisfies the no dominance condition } (ND) \text{ on } [0, T] \text{ if and only if each } S^j, j = 0, 1, \ldots, N \text{ is undominated on } [0, T]. \]

\[ \text{Definition 7 (Economy). An economy consists of a market given by } (S, \mathcal{A}) \text{ and a finite number of investors } k = 1, \ldots, K \text{ characterized by their beliefs, information, preferences and endowment. Moreover, there is a single consumption good that is perishable. The price of the consumption good in units of the cash account is the denoted by } \Psi = (\Psi_t)_{t \in [0, T]}. \text{ We assume that } \Psi \text{ is strictly positive.} \]

\[ \text{The } k \text{-th investor is characterized by the following quantities:} \]
**Beliefs and information:** $\{P_k, A\}$. We assume that the investor’s beliefs $P_k$ are equivalent to $P$. All investors have the same information filtration $A$.

**Utility function:** $U_k : [0, T] \times [0, +\infty] \to \mathbb{R}$ and $\mu$ a probability measure on $[0, T]$ with $\mu(\{T\}) > 0$ such that for every $t$ in the support of $\mu$, the function $U_k(t, \cdot)$ is strictly increasing. We also assume $\lim_{v \to +\infty} U_k(T, v) = +\infty$. The utility that agent $k$ derives from consuming $c_t \mu(dt)$ at each time $t \leq T$ is

$$U_k(c) = \mathbb{E}_k^0 \int_0^T U_k(t, c_t) \mu(dt),$$

where $\mathbb{E}_k$ is the expectation with respect to $P_k$. Since $\mu(\{T\}) > 0$ the utility is strictly increasing in the final consumption $c_T$.

**Initial wealth:** $v_k$. Given a trading strategy $x = (x^1, \cdots, x^N)$, the investor will be required to choose his initial holding $x^0_0$ in the cash account such that

$$v_k = x^0_0 + \sum_{j=1}^N x^j_0 S^j_0.$$  

**Stochastic endowment stream:** $\varepsilon^k_t$, $t < T$ of the commodity. This means that the investors receive $c_t \mu(dt)$ units of the commodity at time $t \leq T$. The cumulative endowment of the $k$-th investor, in units of the cash account, is given by

$$E^k_t := \int_0^t \Psi s \varepsilon^k_s \mu(ds).$$

**Definition 8 (Consumption plan and strategy).** A pair $(c^k_t, x^k_t)_{t \in [0, T]}$ is called admissible if $(c^k_t, x^k_t)_{t \in [0, T]}$ is progressively measurable with respect to the filtration $A$, $(x^k_t)_{t \in [0, T]}$ is admissible in the usual sense, and it generates a wealth process $V^k = (V^k_t)_{t \in [0, T]}$ with non-negative terminal wealth $V^k_T \geq 0$.

**Definition 9 (Equilibrium).** Given an economy $\{\{P_k\}_{k=1}^{K}, A_t, \varepsilon_k, \varepsilon_k^k, U_k\}_{t \in [0, T]}$, a consumption good price index $\Psi$, financial assets $S = [S^0, S^1, \ldots, S^N]^T$, and investor consumption-investment plans $(c^k; \hat{x}^k)$ for $k = 1, \ldots, K$, the pair $(\Psi, S)$ is an equilibrium price process if for all $t \leq T$ $P$-a.s.

**Securities markets clear:**

$$\sum_{k=1}^K \hat{x}^k_t j = \alpha^j \quad (j = 0, 1, \ldots, N),$$

where $\alpha^j$ is the aggregate net supply of the $j$-th security. It is assumed that each $\alpha^j$ is non-random and constant over time, with $\alpha^0 = 0$ and $\alpha^j > 0$ for $j = 1, \ldots, N$. 
Commodity markets clear:

\[
\sum_{k=1}^{K} \hat{c}_k^k = \sum_{k=1}^{K} \epsilon_k^k.
\]  

(6)

Investors’ choices are optimal: \((\hat{c}_k^k, \hat{x}_k^k)\) solves the k-th investor’s utility maximization problem

\[
u_k(x) := \sup \{ U_k(c) \mid c \text{ admissible consumption plan}, x^k = x \},
\]  

and the optimal value is finite.

Definition 10 (Efficiency). A market model given by \(S\) is called efficient on \([0, T]\) with respect to \((\mathcal{A}_t)_{t \in [0, T]}\), i.e. (E), if there exists a consumption good price index \(\Psi\) and an economy \((\{P_k\}_{k=1}^{K}, \mathcal{A}_t)_{t \in [0, T]}\{\epsilon_k\}_{k=1}^{K}, \{U_k\}_{k=1}^{K})\), for which \((\Psi, S)\) is an equilibrium price process on \([0, T]\).

In [JaLa12] we find the proof of

Theorem 11 (Third fundamental theorem of asset pricing, characterization of efficiency). Let \((S, \mathcal{A})\) be a market. The following statements are equivalent

(i) \((E)\): \((S, \mathcal{A})\) is efficient in \([0, T]\).

(ii) \((S, \mathcal{A})\) satisfies both (NFLVR) and (ND) on \([0, T]\).

(iii) \((EMM)\): There exists a probability \(P^*\) equivalent to \(P\), such that \(S\) is a \((P^*, \mathcal{A})\) martingale on \([0, T]\).

2.2 Geometric Reformulation of the Market Model: Primitives

We are going to introduce a more general representation of the market model introduced in section 2.1, which better suits to the arbitrage modeling task.

Definition 12. A gauge is an ordered pair of two \(\mathcal{A}\)-adapted real valued semimartingales \((D, P)\), where \(D = (D_t)_{t \geq 0} : [0, +\infty) \times \Omega \to \mathbb{R}\) is called deflator and \(P = (P_{t,s})_{t,s} : \mathcal{T} \times \Omega \to \mathbb{R}\), which is called term structure, is considered as a stochastic process with respect to the time \(t\), termed valuation date and \(\mathcal{T} := \{(t, s) \in [0, +\infty)^2 \mid s \geq t\}\). The parameter \(s \geq t\) is referred as maturity date. The following properties must be satisfied a.s. for all \(t, s\) such that \(s \geq t \geq 0\):

(i) \(P_{t,s} > 0\),

(ii) \(P_{t,t} = 1\).
Remark 13. Deflators and term structures can be considered outside the context of fixed income. An arbitrary financial instrument is mapped to a gauge \((D, P)\) with the following economic interpretation:

- **Deflator:** \(D_t\) is the value of the financial instrument at time \(t\) expressed in terms of some numéraire. If we choose the cash account, the 0-th asset as numéraire, then we can set \(D^j_t := \frac{\hat{S}^j_t}{S_0^j} (j = 1, \ldots, N)\).

- **Term structure:** \(P_{t,s}\) is the value at time \(t\) (expressed in units of deflator at time \(t\)) of a synthetic zero coupon bond with maturity \(s\) delivering one unit of financial instrument at time \(s\). It represents a term structure of forward prices with respect to the chosen numéraire.

We point out that there is no unique choice for deflators and term structures describing an asset model. For example, if a set of deflators qualifies, then we can multiply every deflator by the same positive semimartingale to obtain another suitable set of deflators. Of course term structures have to be modified accordingly. The term "deflator" is clearly inspired by actuarial mathematics. In the present context it refers to a nominal asset value up division by a strictly positive semimartingale (which can be the state price deflator if this exists and it is made to the numéraire). There is no need to assume that a deflator is a positive process. However, if we want to make an asset to our numéraire, then we have to make sure that the corresponding deflator is a strictly positive stochastic process.

### 2.3 Geometric Reformulation of the Market Model: Portfolios

We want now to introduce transforms of deflators and term structures in order to group gauges containing the same (or less) stochastic information. That for, we will consider deterministic linear combinations of assets modelled by the same gauge (e.g. zero bonds of the same credit quality with different maturities).

**Definition 14.** Let \(\pi : [0, +\infty[ \longrightarrow \mathbb{R}\) be a deterministic cashflow intensity (possibly generalized) function. It induces a gauge transform \((D, P) \mapsto (D, P)^\pi := (D^\pi, P^\pi)\) by the formulae

\[
D^\pi_t := D_t \int_0^{+\infty} dh \pi_h P_{t,t+h} \quad P^\pi_{t,s} := \frac{\int_0^{+\infty} dh \pi_h P_{t,s+h}}{\int_0^{+\infty} dh \pi_h P_{t,t+h}}.
\]

**Proposition 15.** Gauge transforms induced by cashflow vectors have the following property:

\[
((D, P)^\pi)^\nu = ((D, P)^\nu)^\pi = (D, P)^{\pi \ast \nu},
\]

9
where $\ast$ denotes the convolution product of two cashflow vectors or intensities respectively:

$$(\pi \ast \nu)_t := \int_0^t dh \pi_h \nu_{t-h}. \quad (10)$$

The convolution of two non-invertible gauge transform is non-invertible. The convolution of a non-invertible with an invertible gauge transform is non-invertible.

**Definition 16.** The term structure can be written as a functional of the instantaneous forward rate $f$ defined as

$$f_{t,s} := -\frac{\partial}{\partial s} \log P_{t,s}, \quad P_{t,s} = \exp \left( -\int_t^s dh f_{t,h} \right). \quad (11)$$

and

$$r_t := \lim_{s \to t^+} f_{t,s} \quad (12)$$

is termed short rate.

**Remark 17.** Since $(P_{t,s})_{t,s}$ is a $t$-stochastic process (semimartingale) depending on a parameter $s \geq t$, the $s$-derivative can be defined deterministically, and the expressions above make sense pathwise in a both classical and generalized sense. In a generalized sense we will always have a $D'$ derivative for any $\omega \in \Omega$; this corresponds to a classic $s$-continuous derivative if $P_{t,s}(\omega)$ is a $C^1$-function of $s$ for any fixed $t \geq 0$ and $\omega \in \Omega$.

**Remark 18.** The special choice of vanishing interest rate $r \equiv 0$ or flat term structure $P \equiv 1$ for all assets corresponds to the classical model, where only asset prices and their dynamics are relevant.

### 2.4 Arbitrage Theory in a Differential Geometric Framework

Now we are in the position to rephrase the asset model presented in subsection 2.1 in terms of a natural geometric language. Given $N$ base assets we want to construct a portfolio theory and study arbitrage and thus we cannot a priori assume the existence of a risk neutral measure or of a state price deflator. In terms of differential geometry, we will adopt the mathematician’s and not the physicist’s approach. The market model is seen as a principal fibre bundle of the (deflator, term structure) pairs, discounting and foreign exchange as a parallel transport, numéraire as global section of the gauge bundle, arbitrage as curvature. The no-free-lunch-with-vanishing-risk condition is proved to be equivalent to a zero curvature condition.
2.4.1 Market Model as Principal Fibre Bundle

Let us consider - in continuous time- a market with \( N \) assets and a numéraire. A general portfolio at time \( t \) is described by the vector of nominals \( x \in \mathcal{X} \), for an open set \( \mathcal{X} \subset \mathbb{R}^N \). Following Definition 12, the asset model induces for \( j = 1, \ldots, N \) the gauge

\[
(D^j, P^j) = (\{D^j_t\}_{t \in [0, +\infty]}, \{P^j_t\}_{s \geq t}),
\]

where \( D^j \) denotes the deflator and \( P^j \) the term structure. This can be written as

\[
P^j_{t,s} = \exp \left( -\int_t^s f^j_{t,u} du \right),
\]

where \( f^j \) is the instantaneous forward rate process for the \( j \)-th asset and the corresponding short rate is given by \( r^j_t := \lim_{u \to 0^+} f^j_{t,u} \). For a portfolio with nominals \( x \in \mathcal{X} \subset \mathbb{R}^N \) we define

\[
D^x_t := \sum_{j=1}^N x_j D^j_t, \quad f^{x}_{t,u} := \sum_{j=1}^N \frac{x_j D^j_t}{\sum_{j=1}^N x_j D^j_t} f^j_{t,u}, \quad P^x_{t,s} := \exp \left( -\int_t^s f^x_{t,u} du \right),
\]

The short rate writes

\[
r^x_t := \lim_{u \to 0^+} f^x_{t,u} = \sum_{j=1}^N \frac{x_j D^j_t}{\sum_{j=1}^N x_j D^j_t} r^j_t.
\]

The image space of all possible strategies reads

\[
M := \{ (t, x) \in [0, +\infty[ \times \mathcal{X} \}.
\]

In subsection 2.3 cashflow intensities and the corresponding gauge transforms were introduced. They have the structure of an Abelian semigroup

\[
H := \mathcal{E}'([0, +\infty[, \mathbb{R}) = \{ F \in \mathcal{D}'([0, +\infty]) \mid \text{supp}(F) \subset [0, +\infty[ \text{ is compact} \},
\]

where the semigroup operation on distributions with compact support is the convolution (see [H603], Chapter IV), which extends the convolution of regular functions as defined by formula (10).

**Definition 19.** The **Market Fibre Bundle** is defined as the fibre bundle of gauges

\[
\mathcal{B} := \{(D^x_t, P^x_{t,s})^\pi \mid (t, x) \in M, \pi \in G \}.
\]
The cashflow intensities defining invertible transforms constitute an Abelian group

\[ G := \{ \pi \in H \mid \text{it exists } \nu \in H \text{ such that } \pi \ast \nu = \delta \} \subset \mathcal{E}'([0, +\infty[; \mathbb{R}). \]  

(20)

From Proposition 15 we obtain

**Theorem 20.** The market fibre bundle \( \mathcal{B} \) has the structure of a \( G \)-principal fibre bundle given by the action

\[ \mathcal{B} \times G \longrightarrow \mathcal{B} \]

\[ ((D, P), \pi) \mapsto (D, P^\pi) = (D^{\pi}, P^{\pi}) \]

(21)

The group \( G \) acts freely and differentiably on \( \mathcal{B} \) to the right.

### 2.4.2 Stochastic Parallel Transport

Let us consider the projection of \( \mathcal{B} \) onto \( M \)

\[ p : \mathcal{B} \cong M \times G \longrightarrow M \]  

\[ (t, x, g) \mapsto (t, x) \]  

(22)

and its differential map at \( (t, x, g) \in \mathcal{B} \) denoted by \( T_{(t,x,g)}p \), see for example, Definition 0.2.5 in \([Bl81]\)

\[ T_{(t,x,g)}p : T_{(t,x,g)}\mathcal{B} \longrightarrow T_{(t,x)}M. \]

(23)

The vertical directions are

\[ \mathcal{V}_{(t,x,g)}\mathcal{B} := \ker (T_{(t,x,g)}p) \cong \mathbb{R}^{[0, +\infty[}, \]  

(24)

and the horizontal ones are

\[ \mathcal{H}_{(t,x,g)}\mathcal{B} \cong \mathbb{R}^{N+1}. \]

(25)

An Ehresmann connection on \( \mathcal{B} \) is a projection \( TB \rightarrow VB \). More precisely, the vertical projection must have the form

\[ \Pi^v_{(t,x,g)} : T_{(t,x,g)}\mathcal{B} \longrightarrow \mathcal{V}_{(t,x,g)}\mathcal{B} \]

\[ (\delta x, \delta t, \delta g) \mapsto (0, 0, \delta g + \Gamma(t, x, g)(\delta x, \delta t)), \]

(26)
and the horizontal one must read

\[ \Pi_{(t,x,g)}^h : T_{(t,x,g)}B \rightarrow \mathcal{H}_{(t,x,g)}B \]

\[ (\delta x, \delta t, \delta g) \mapsto (\delta x, \delta t, -\Gamma(t,x,g).\delta x), \quad (27) \]

such that

\[ \Pi^v + \Pi^h = 1_B. \quad (28) \]

Stochastic parallel transport on a principal fibre bundle along a semimartingale is a well defined construction (cf. [HaTh94], Chapter 7.4 and [Hs02] Chapter 2.3 for the frame bundle case) in terms of Stratonovich integral. Existence and uniqueness can be proved analogously to the deterministic case by formally substituting the deterministic time derivative \( \frac{d}{dt} \) with the stochastic one \( \mathcal{D} \) corresponding to the Stratonovich integral.

Following Ilinski’s idea ([Il01]), we motivate the choice of a particular connection by the fact that it allows to encode foreign exchange and discounting as parallel transport.

**Theorem 21.** With the choice of connection

\[ \chi(t,x,g).(\delta x, \delta t) := \left( \frac{D\delta x}{D^r_t} - r^r_t \right) g, \quad (29) \]

the parallel transport in \( B \) has the following financial interpretations:

- Parallel transport along the nominal directions (x-lines) corresponds to a multiplication by an exchange rate.
- Parallel transport along the time direction (t-line) corresponds to a division by a stochastic discount factor.

Recall that time derivatives needed to define the parallel transport along the time lines have to be understood in Stratonovich’s sense. We see that the bundle is trivial, because it has a global trivialization, but the connection is not trivial.

**Remark 22.** An Ehresmann connection on \( B \) is called **principal Ehresmann connection** if and only if the decomposition \( T_{(t,x,g)}B = \mathcal{V}_{(t,x,g)}B \oplus \mathcal{H}_{(t,x,g)}B \) is invariant under the action of \( G \). Equivalently, the corresponding connection 1-form \( \chi \) must be smooth with respect to \( x, t \) and \( g \) and \( G \)-invariant, which
is the case, since, for arbitrary \((t,x,g) \in \mathcal{B}\) and \(a \in G\)
\[
(R_{a*})\chi(x,t,g).(\delta x, \delta t) = \frac{d}{ds}|_{s=0} g \exp \left( s \left( \frac{D^\delta x}{D^t} - r^\tau \delta t \right) g \right) \cdot a
\]
\[
= \frac{d}{ds}|_{s=0} g \cdot a \exp \left( s \left( \frac{D^\delta x}{D^t} - r^\tau \delta t \right) g \right)
\]
\[
= \chi(x,t,g \cdot a).(\delta x, \delta t),
\]
where \(R_a\) denotes the (right) action of \(a \in G\) and \(R_{a*}\) is the differential of the mapping \(R_a : G \to G\).

2.4.3 Nelson \(\mathcal{D}\) Differentiable Market Model

We continue to reformulate the classic asset model introduced in subsection 2.1 in terms of stochastic differential geometry. We refer to Appendix A for a short background in stochastic derivatives.

**Definition 23.** A **Nelson \(\mathcal{D}\) differentiable market model** for \(N\) assets is described by \(N\) gauges which are Nelson \(\mathcal{D}\) differentiable with respect to the time variable. More exactly, for all \(t \in [0, +\infty[\) and \(s \geq t\) there is an open time interval \(I \ni t\) such that for the deflators \(D_t := [D^1_t, \ldots, D^N_t]^\dagger\) and the term structures \(P_{t,s} := [P^1_{t,s}, \ldots, P^N_{t,s}]^\dagger\), the latter seen as processes in \(t\) and parameter \(s\), there exist a \(\mathcal{D}\) \(t\)-derivative. The short rates are defined by \(r_t := \lim_{s \to t-} \frac{D}{ds} \log P_{ts}\).

A strategy is a curve \(\gamma : I \to \mathfrak{X}\) in the portfolio space parameterized by the time. This means that the allocation at time \(t\) is given by the vector of nominals \(x_t := \gamma(t)\). We denote by \(\bar{\gamma}\) the lift of \(\gamma\) to \(M\), that is \(\bar{\gamma}(t) := (\gamma(t), t)\). A strategy is said to be **closed** if it represented by a closed curve. A **\(\mathcal{D}\)-admissible strategy** is predictable and \(\mathcal{D}\)-differentiable.

In general the allocation can depend on the state of the nature i.e. \(x_t = x_t(\omega)\) for \(\omega \in \Omega\).

**Proposition 24.** A \(\mathcal{D}\)-admissible strategy is self-financing if and only if

\[
\mathcal{D}(x_t \cdot D_t) = x_t \cdot DD_t - \frac{1}{2} D_x \langle x, D \rangle_t \quad \text{or} \quad D x_t \cdot D_t = -\frac{1}{2} D_x \langle x, D \rangle_t,
\]

almost surely.

For the reminder of this paper unless otherwise stated we will deal only with \(\mathcal{D}\) differentiable market models, \(\mathcal{D}\) differentiable strategies, and, when necessary, with \(\mathcal{D}\) differentiable state price deflators. All Itô processes are \(\mathcal{D}\) differentiable, so that the class of considered admissible strategies is very large.
2.4.4 Arbitrage as Curvature

The Lie algebra of $G$ is

$$\mathfrak{g} = \mathbb{R}^{[0, +\infty]}$$

and therefore commutative. The $\mathfrak{g}$-valued connection 1-form writes as

$$\chi(t, x, g)(\delta x, \delta t) = \left( \frac{D_t^\delta x}{D_t^t} - r_t^\delta \delta t \right) g,$$

or as a linear combination of basis differential forms as

$$\chi(t, x, g) = \left( \frac{1}{D_t^t} \sum_{j=1}^N D_t^j dx_j - r_t^\delta dt \right) g. \quad (33)$$

The $\mathfrak{g}$-valued curvature 2-form is defined as

$$R := d\chi + [\chi, \chi],$$

meaning by this, that for all $(t, x, g) \in \mathcal{B}$ and for all $\xi, \eta \in T_{(t, x)} M$

$$R(t, x, g)(\xi, \eta) := d\chi(t, x, g)(\xi, \eta) + [\chi(t, x, g)(\xi), \chi(t, x, g)(\eta)]. \quad (35)$$

Remark that, being the Lie algebra commutative, the Lie bracket $[\cdot, \cdot]$ vanishes. After some calculations we obtain

$$R(t, x, g) = \frac{g}{D_t^t} \sum_{j=1}^N D_t^j \left( r_t^\delta + \mathcal{D} \log(D_t^x) - r_t^j - \mathcal{D} \log(D_t^j) \right) dx_j \wedge dt, \quad (36)$$

summarized as

**Proposition 25 (Curvature Formula).** Let $R$ be the curvature. Then, the following quality holds:

$$R(t, x, g) = g dt \wedge dx \left[ \mathcal{D} \log(D_t^x) + r_t^\delta \right]. \quad (37)$$

We can prove following results which characterizes arbitrage as curvature.

**Theorem 26 (No Arbitrage).** The following assertions are equivalent:

(i) The market model satisfies the no-free-lunch-with-vanishing-risk condition.

(ii) There exists a positive semimartingale $\beta = (\beta_t)_{t \geq 0}$ such that deflators and short rates satisfy for
all portfolio nominals and all times the condition

\[ r_t^x = -D \log(\beta_t D_t^x). \]  

(iii) There exists a positive semimartingale \( \beta = (\beta_t)_{t \geq 0} \) such that deflators and term structures satisfy for all portfolio nominals and all times the condition

\[ P_{t,s}^x = \frac{E_t[\beta_s D_s^x]}{\beta_t D_t^x}. \]  

This motivates the following definition.

**Definition 27.** The market model satisfies the zero curvature (ZC) if and only if the curvature vanishes a.s.

The different arbitrage concepts are related in the following logical representation, which is a compact representation of results of Delbaen-Schachermeyer ([DeSc94]), Kabanov ([Ka97]), Jarrow-Larsson ([JaLa12]) and Farinelli-Takada ([FaTa19]):

**Theorem 28.**

\[
\begin{array}{c}
(EMM) \Leftrightarrow (E) \Leftrightarrow \left\{ (NFLVR) \Leftrightarrow (NUPBR) \Leftrightarrow (EUM) \Leftrightarrow (ZC) \right. \\
(NA) \\
(ND)
\end{array}
\]  

2.5 Cashflows as Sections of the Associated Vector Bundle

By choosing the fiber \( V := \mathbb{R}^{[0, +\infty]} \) and the representation \( \rho : G \to \text{GL}(V) \) induced by the gauge transform definition, and therefore satisfying the homomorphism relation \( \rho(g_1 * g_2) = \rho(g_1)\rho(g_2) \), we obtain the associated vector bundle \( V \). Its sections represent cashflow streams - expressed in terms of the deflators - generated by portfolios of the base assets. If \( v = (v_t^x)_{(t,x) \in M} \) is the deterministic cashflow stream, then its value at time \( t \) is equal to

- the deterministic quantity \( v_t^x \), if the value is measured in terms of the deflator \( D_t^x \),
- the stochastic quantity \( v_t^x D_t^x \), if the value is measured in terms of the numéraire (e.g. the cash account for the choice \( D_t^j := \hat{S}_t^j \) for all \( j = 1, \ldots, N \)).

In the general theory of principal fibre bundles, gauge transforms are bundle automorphisms preserving the group action and equal to the identity on the base space. Gauge transforms of \( B \) are naturally
isomorphic to the sections of the bundle \( \mathcal{B} \) (See Theorem 3.2.2 in [Bl81]). Since \( G \) is Abelian, right multiplications are gauge transforms. Hence, there is a bijective correspondence between gauge transforms and cashflow intensities admitting an inverse. This justifies the terminology introduced in Definition 14.

2.6 The Connection Laplacian Associated to the Market Model

This subsection summarizes definitions and results introduced in [FaTa19Bis]. The connection \( \chi \) on the market principal fibre bundle \( \mathcal{B} \) induces a covariant differentiation \( \nabla^\mathcal{V} \) on the associated vector bundle \( \mathcal{V} \), with the same interpretation for the corresponding parallel transport as that in Theorem 21, i.e. portfolio rebalancing along the asset nominal dimensions and discounting along the time dimension. More exactly, we have

**Proposition 29.** Let us extend the coordinate vector \( x \in \mathbb{R}^N \) with a 0th component given by the time \( t \). Let \( X = \sum_{j=0}^{N} X_j \frac{\partial}{\partial x_j} \) be a vector field over \( M \) and \( f = (f_s)_s \) a section of the cashflow bundle \( \mathcal{V} \). Then

\[
\nabla^\mathcal{V}_X f_t = \sum_{j=0}^{N} \left( \frac{\partial f_t}{\partial x_j} + K_j f_t \right) X_j, \tag{41}
\]

where

\[
K_0(x) = -r^x_t \\
K_j(x) = \frac{D^x_j}{D^x_t} \quad (1 \leq j \leq N). \tag{42}
\]

**Proposition 30.** The curvature of the connection \( \nabla^\mathcal{V} \) is

\[
R^\mathcal{V}(X, Y) := \nabla^\mathcal{V}_X \nabla^\mathcal{V}_Y - \nabla^\mathcal{V}_Y \nabla^\mathcal{V}_X - \nabla^\mathcal{V}_{[X, Y]} = [p] \circ (R(X^*, Y^*, e) * \cdot) \circ [p]^{-1}, \tag{43}
\]

where \( R \) is the curvature on the principal fibre bundle \( \mathcal{B} \), \( X^*, Y^* \in T_p \mathcal{B} \) the horizontal lifts of \( X, Y \in T(t, x)M \) and

\[
[p] : \mathcal{V} \mapsto \mathcal{V}_{(t, x)} := \mathcal{B}_{(t, x)} \times_G \mathcal{V} \\
v \mapsto [p](v) = [p, v] \tag{44}
\]

is the fibre isomorphism between \( \mathcal{B} \) and \( \mathcal{V} \). In particular the curvature on the principal fibre bundle vanishes if and only if the curvature on the associated vector bundle vanishes.

We now continue by introducing the connection Laplacian on an appropriate Hilbert space.
Definition 31. The space of the sections of the cashflow bundle can be made into a scalar product space by introducing, for stochastic sections \( f = f(x,t,\omega) = (f_s(t,x,\omega))_{s \in [0,\infty[} \) and \( g = g(t,x,\omega) = (g_s(t,x,\omega))_{s \in [0,\infty[} \)

\[
(f,g) := \int_{\Omega} dP \int_{\mathbb{R}^N} \int_0^{+\infty} dt \, (f,g)(t,x,\omega) = E_0 \left[ (f,g)_{L^2(M,\mathbb{R}[0,\infty[)} \right] = (f,g)_{L^2(\Omega,\mathbb{R}[0,\infty[)}
\]

where

\[
(f,g)(x,t,\omega) := \int_0^{+\infty} ds f_s(t,x,\omega) g_s(t,x,\omega).
\]

The Hilbert space of integrable sections reads

\[
\mathcal{H} := L^2(\Omega,\mathbb{R},dP) = \left\{ f = f(t,x,\omega) = (f_s(t,x,\omega))_{s \in [0,\infty[} \mid (f,f)_{L^2(\Omega,\mathbb{R},dP)} < +\infty \right\}.
\]

When considering the connection Laplacian, there are two standard choices for a local elliptic boundary condition which guarantees selfadjointness:

- **Dirichlet boundary condition:**
  \[
  B_D(f) := f|_{\partial M}.
  \]

- **Neumann boundary condition:**
  \[
  B_N(f) := \langle \nabla_V f \rangle|_{\partial M},
  \]

  where \( \nu \) denotes the normal unit vector field to \( \partial M \).

By considering the \( \omega \) as a parameter dependence we can apply a standard result functional analysis to obtain

**Proposition 32.** The connection Laplacian \( \Delta^V := \nabla^V \ast \nabla^V \) with domain of definition given by the Neumann boundary condition

\[
\text{dom} \left( \Delta_{B_N}^V \right) := \left\{ f \in \mathcal{H} \mid f(\omega,\cdot,\cdot) \in H^2(M,\mathbb{R}[0,\infty[), B_N(f(\omega,\cdot,\cdot)) = 0 \ \forall \ \omega \in \Omega \right\}
\]

is a selfadjoint operator on \( \mathcal{H} \). Its spectrum consists in the disjoint union of discrete spectrum (eigenvalues) and continuous spectrum (approximate eigenvalues) lies in \([0,\infty[\):

\[
\text{spec} \left( \Delta_{B_N}^V \right) = \text{spec}_d \left( \Delta_{B_N}^V \right) \cup \text{spec}_c \left( \Delta_{B_N}^V \right).
\]
If \( M \) is compact, for example by setting \( M := [0, T] \times X, X \subset \mathbb{R}^N \) compact and \( T < +\infty \), then the continuous spectrum is empty and the eigenvalues can be ordered in a monotone increasing sequence converging to \(+\infty\).

**Remark 33.** When we choose \( M = [0, T] \times X \) with \( T < +\infty \), we have to adapt the construction of the principal fibre bundle and the associated vector bundle accordingly. Note that the structure group of \( B \) and its Lie-Algebra remain \( G \) and \( \mathbb{R}^{[0, +\infty]} \), respectively, and the fibre of \( V \) is still \( \mathbb{R}^{[0, +\infty]} \). Only the integration over the time dimension in the base space \( M \) is performed till \( T \).

**Remark 34.** For a fixed \( \omega \in \Omega \) the domain of definition of \( \Delta^V_{B_N} \) is a subset of the Sobolev space \( H^2(M, \mathbb{R}) \). If \( M \) is compact, then the eigenvectors of \( \Delta^V_{B_N} \) lie in \( C^\infty(M, \mathbb{R}) \) and satisfy the Neumann boundary condition. Proposition 32 follows from standard elliptic spectral theory by means of an integration over \( \Omega \).

The spectrum of the connection Laplacian under the Neumann boundary condition contains information about arbitrage possibilities in the market. More exactly,

**Theorem 35.** The market model satisfies the (NFLVR) condition if and only if \( 0 \in \text{spec}_d(\Delta^V_{B_N}) \). The harmonic sections parametrize the Radon-Nykodim derivative for the change of measure from the statistical to the risk neutral measures.

**Remark 36.** Note that if \( f = f(\omega, t, x) \equiv f(\omega) \), and at least one of the components of \( r \) or \( D \) does not vanish, then \( f = 0, 0 \notin \text{spec}(\Delta^V_{B_N}) \), confirming and extending Remark ??.

**Remark 37.** Any harmonic \( f = f_t(x) \) defines a risk neutral measure by means of the Radon-Nykodim derivative

\[
\frac{dP^*}{dP} = \frac{\beta_t}{\beta_0} = \frac{D^*_t f_0(x)}{D^*_0 f_t(x)}, \quad (51)
\]

which does not depend on \( x \).

From formula (51) we derive

**Corollary 38.** The market model is complete if and only if \( 0 \in \text{spec}(\Delta^V)_{B_N} \) is an eigenvalue with simple multiplicity.

**Remark 39.** The situation for the Dirichlet boundary condition is similar. The proposition and remark analogous to Proposition 32 and Remark 34 hold true. But because of the unique continuation property for elliptic operators \( 0 \) never lies in \( \text{spec}_d(\Delta^V_{B_D}) \), wether the (NFLVR) property is satisfied or not.

### 2.7 Arbitrage Bubbles

This subsection summarizes definitions and results introduced in [FaTa19Bis].
Definition 40 (Spectral Lower Bound). The highest spectral lower bound of the connection Laplacian on the cash flow bundle $\mathcal{V}$ is given by

$$
\lambda_0 := \inf_{\varphi \in C^\infty(M, \mathcal{V})} \frac{(\nabla^V \varphi \cdot \nabla^V \varphi)_H}{(\varphi \cdot \varphi)_H}
$$

(52)

and it is assumed on the subspace

$$
E_{\lambda_0} := \{ \varphi \mid \varphi \in C^\infty(M, \mathcal{V}) \cap \mathcal{H}, B_N(\varphi) = 0, (\nabla^V \varphi \cdot \nabla^V \varphi)_H \geq \lambda_0 (\varphi \cdot \varphi)_H \}.
$$

(53)

The space

$$
K_{\lambda_0} := \{ \varphi \in E_{\lambda_0} \mid \varphi \geq 0 \mathbb{E}[\varphi] = 1 \}
$$

(54)

contains all candidates for the Radon-Nikodym derivative

$$
\frac{dP^*}{dP} = \varphi,
$$

(55)

for a probability measure $P^*$ absolutely continuous with respect to the statistical measure $P$.

Theorem 2, that is the first fundamental theorem of asset pricing can be reformulated as

Proposition 41. The market model satisfies the (NFLVR) condition if and only if $\lambda_0 = 0$. Any probability measure defined by (55) with $\varphi \in K_0$ is a risk neutral measure, that is $(D_t)_{t \in [0,T]}$ is a vector valued martingale with respect to $P^*$, i.e.

$$
\mathbb{E}^*_t[D_s] = D_t \quad \text{for all } s \geq t \text{ in } [0,T].
$$

(56)

The market is complete if and only if $\lambda_0 = 0$ and $\dim E_0 = 1$.

For arbitrage markets we have that $\lambda_0 > 0$ and there exists no risk neutral probability measures. Nevertheless it is possible to define a fundamental value, however not in a unique way.

Definition 42 (Basic Assets' Arbitrage Fundamental Prices and Bubbles). Let $(C_t)_{t \in [0,T]}$ the $\mathbb{R}^N$ cash flow stream stochastic process associated to the $N$ assets of the market model with given spectral lower bound $\lambda_0$ and Radon-Nikodym subspace $K_{\lambda_0}$. For a given choice of $\varphi \in K_{\lambda_0}$ the approximated fundamental value of the assets with stochastic $\mathbb{R}^N$-valued price process $(S_t)_{t \in [0,T]}$ is defined as

$$
S_t^{*, \varphi} := \mathbb{E}_t \left[ \varphi \left( \int_t^T dC_u \exp \left( - \int_t^u r_s^0 ds \right) + S_T \exp \left( - \int_t^T r_s^0 ds \right) 1_{\{\tau < +\infty\}} \right) 1_{\{t < \tau\}} \right],
$$

(57)
where \( \tau \) denotes the maturity time of all risky assets in the market model, and, the approximated bubble is defined as

\[
B_t^\tau := S_t - S_t^\tau.
\]

The fundamental price vector for the assets and their asset bubble prices are defined as

\[
S_t^* := S_t^* \varphi_0 \\
B_t := B_t^{\varphi_0} \\
\varphi_0 := \arg \min_{\varphi \in K_{\lambda_0}} \mathbb{E}_0 \left[ \int_0^T ds |B_t^\varphi|^2 \right].
\]

The probability measure \( P^* \) with Radon-Nikodym derivative

\[
\frac{dP^*}{dP} = \varphi_0
\]

is termed \textit{minimal arbitrage measure}.

**Proposition 43.** The assets’ fundamental values can be expressed as conditional expectation with respect to the minimal arbitrage measure as

\[
S_t^* := \mathbb{E}_t^* \left[ \int_t^\tau dC_u \exp \left( -\int_t^u r^0_s ds \right) + S_T \exp \left( -\int_t^\tau r^0_s ds \right) \right] 1_{\{\tau < +\infty\}} 1_{\{t < \tau\}}.
\]

Formula (61) can be reformulated in terms of the curvature, by means of which we can extend Jarrow-Protter-Shimbo’s results in [JPS10] to the following bubble decomposition and classifications theorems proved in [FaTa19Bis].

**Theorem 44 (Bubble decomposition and types).** Let \( T = +\infty \) and \( \tau \) denote the maturity time of all risky assets in the market model. \( S_t \) admits a unique (up to \( P \)-evanescent set) decomposition

\[
S_t = \tilde{S}_t + B_t,
\]

where \( B = (B_t)_{t \in [0,T]} \) is a càdlàg process satisfying for all \( j = 1, \ldots, N \)

\[
B_t^j = S_t^j - \mathbb{E}_t^j \left[ \int_t^\tau dC_u^j \exp \left( -\int_t^u ds r^0_s \right) + \exp \left( \int_t^\tau ds r^0_s \right) S_T^{j1_{\{\tau < +\infty\}}} \right] 1_{\{t < \tau\}},
\]

into a sum of fundamental and bubble values.

If there exists a non-trivial bubble \( B_t^j \) in an asset’s price for \( j = 1, \ldots, N \), then, there exists a
probability measure $P^*$ equivalent to $P$, for which we have three and only three possibilities:

**Type 1:** $B^i_t$ is local super- or submartingale with respect to both $P$ and $P^*$, if $P[\tau = +\infty] > 0$.

**Type 2:** $B^i_t$ is local super- or submartingale with respect to both $P$ and $P^*$, but not uniformly integrable super- or submartingale, if $B^i_t$ is unbounded but with $P[\tau < +\infty] = 1$.

**Type 3:** $B^i_t$ is a strict local super- or sub-$P$- and $P^*$-martingale, if $\tau$ is a bounded stopping time.

Next we analyze the situation for derivatives.

**Definition 45 (Contingent Claim’s Arbitrage Fundamental Price and Bubble).** Let us consider in the context of Definition (42) a European option given by the contingent claim with a unique payoff $G(S_T)$ at time $T$ for an appropriate real valued function $G$ of $N$ real variables. The contingent claim fundamental price and its corresponding arbitrage bubble is defined in the case of base assets paying no dividends as

$$V_t^*(G) := E_t \left[ \varphi_0 \exp \left( - \int_t^T r^0_s ds \right) G(S_T) 1_{\{T<+\infty\}} \right] 1_{\{t<T\}} =$$

$$= \mathbb{E}^* \left[ \exp \left( - \int_t^T r^0_s ds \right) G(S_T) 1_{\{T<+\infty\}} \right] 1_{\{t<T\}}$$

$$B_t(G) := V_t(G) - V_t^*(G),$$

where $\varphi_0$ is the minimizer for the basic assets bubbled defined in (59), $P^*$ the minimal arbitrage measure and $(V_t(G))_{t \in [0,T]}$ is the price process of the European option.

In the case of base assets paying dividends the definition becomes

$$V_t^*(G) := E_t \left[ \varphi_0 \exp \left( - \int_t^T r^0_s ds \right) G(S_T \exp \left( \frac{C_t}{S_T} (T-t) \right)) \right] 1_{\{T<+\infty\}} 1_{\{t<T\}} =$$

$$= \mathbb{E}^* \left[ \exp \left( - \int_t^T r^0_s ds \right) G(S_T \exp \left( \frac{C_t}{S_T} (T-t) \right)) \right] 1_{\{T<+\infty\}} 1_{\{t<T\}}$$

$$B_t(H) := V_t(G) - V_t^*(G),$$

where $\frac{C_t}{S_t}$ is the instantaneous dividend rate for the $j$-th asset.

**Remark 46.** If the market is complete, then $\lambda_0 = 0$ and $K_{\lambda_0} = \{\varphi_0\}$, where $\varphi_0$ is the Radon-Nykodim derivative of the unique risk neutral probability measure with respect to the statistical probability measure. The definitions in (42) and in (44) coincide for the complete market with the definitions of fundamental value and asset bubble price for both base asset and contingent claim introduced by Jarrow, Protter and Shimbo in [JPS10], proving that they are a natural extension to markets allowing for arbitrage opportunities.
eside the put-call parity for European options in the case of markets allowing for arbitrage one can prove following result as well.

**Corollary 47.** The bubble discounted values for the base assets in Definition 42) and for the contingent claim on the base assets paying dividends in Definition 44

\[
\hat{B}_t := \exp \left( -\int_0^t r_s^0 ds \right) B_t \quad \hat{B}(G)_t := \exp \left( -\int_0^t r_s^0 ds \right) B(G)_t
\]

(66)

satisfy the equalities

\[
\begin{align*}
\hat{B}_t &= D_t - \left( E_t^* \left[ D_t^j 1_{\{\tau<+\infty\}} \right] + E_t^* \left[ \hat{C}_t^j 1_{\{\tau<+\infty\}} \right] - \hat{C}_t^j \right) 1_{\{t<\tau\}} \\
\hat{B}_t(G) &= \hat{V}_t(G) - E_t^* \left[ \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T-t) \right) \right) 1_{\{T<+\infty\}} \right] 1_{\{t<T\}}.
\end{align*}
\]

(67)

where

\[
\begin{align*}
\hat{C}_t^j &= \exp \left( -\int_0^t r_s^0 ds \right) C_t^j \quad \hat{G} := \exp \left( -\int_0^T r_s^0 ds \right) G \quad \hat{V}_t(G) := \exp \left( -\int_0^t r_s^0 ds \right) V_t(G)
\end{align*}
\]

(68)

are the discounted cashflow for the j-th asset, the discounted contingent claim payoff, and the discounted value of the derivative.

### 3 Asset and Market Portfolio Dynamics as a Constrained Lagrangian System

In [Fa15] the minimal arbitrage principle, stating that asset dynamics and market portfolio choose the path guaranteeing the minimization of arbitrage, was encoded as the Hamilton principle under constraints for a Lagrangian measuring the arbitrage. Then, the SDE describing asset deflators, term structures and market portfolio were derived by means of a stochasticization procedure of the Euler-Lagrange equations following a technique developed by Cresson and Darses ([CrDa07] who follow previous works of Yasue ([Ya81]) and Nelson ([Ne01]). Since we need this set up to proceed with its quantization, we briefly summarize it here below.

**Definition 48.** Let \( \gamma \) be the market \( \mathcal{D} \)-admissible strategy, and \( \delta \gamma, \delta D, \delta r \) be perturbations of the market strategy, deflators’ and short rates’ dynamics. The variation of \( (\gamma, D, r) \) with respect to the given perturbations is the following one parameter family:

\[
\epsilon \mapsto (\gamma^\epsilon, D^\epsilon, r^\epsilon) := (\gamma, D, r) + \epsilon(\delta \gamma, \delta D, \delta r).
\]

(69)
Thereby, the parameter $\epsilon$ belongs to some open neighborhood of $0 \in \mathbb{R}$. The arbitrage action with respect to a positive semimartingale $\beta$ can be consistently defined, assuming that the time is the arc length parameter of $\gamma$, by

$$A^\beta(\gamma; D, r) := \int_0^1 dt x_t' \{ D \log(\beta_t D_t x_t) + r_t D_t x_t \}$$

and the first variation of the arbitrage action as

$$\delta A^\beta(\gamma; D, r) := \frac{d}{d\epsilon} A^\beta(\gamma'; D', r') |_{\epsilon=0}.$$  

This leads to the following

**Definition 49.** Let us introduce the notation $q := (x, D, r)$ and $q' := (x', D', r')$ for two vectors in $\mathbb{R}^{3N}$. The Lagrangian (or Lagrange function) is defined as

$$L(q, q') := L(x, D, r, x', D', r') := |x'| \frac{x \cdot (D' + rD)}{x \cdot D}.$$  

The self-financing constraint is defined as

$$C(q, q') := L(x, D, r, x', D', r') := x' \cdot D.$$  

**Remark 50.** In the deterministic definition (72) there is no contribution from the quadratic covariation $\langle \cdot, \cdot \rangle$ in (70).

**Lemma 51.** The arbitrage action for a self-financing strategy $\gamma$ is the integral of the Lagrange function along the $D$-admissible strategy:

$$A^\beta(\gamma; D, r) = \int_\gamma dt L(q_t, q'_t) + \log \frac{\beta_1}{\beta_0} = \int_\gamma dt L(x_t, D_t, r_t, x'_t, D'_t, r'_t) + \log \frac{\beta_1}{\beta_0}.$$  

A fundamental result of classical mechanics allows to compute the extrema of the arbitrage action in the deterministic case as the solution of a system of ordinary differential equations.

**Theorem 52 (Hamilton Principle).** Let us denote the derivative with respect to time as $\frac{d}{dt} := t$ and assume that all quantities observed are deterministic. The local extrema of the arbitrage action satisfy
the Lagrange equations under the self-financing constraint

\[
\begin{cases}
\delta A(\gamma; D, r) = 0 \text{ for all } (\delta\gamma, \delta D, \delta r) \\
such \text{ that } x^\epsilon_t \cdot D^\epsilon_t = 0 \text{ for all } \epsilon
\end{cases}
\]

\[
\begin{align*}
\frac{d}{dt} \frac{\partial L_\lambda}{\partial q}(q_t, dq_t) - \frac{\partial L_\lambda}{\partial q'}(q_t, dq_t') &= 0 \\
C(q_t, q_t') &= 0
\end{align*}
\]

(75)

where \( \lambda \in \mathbb{R} \) denotes the self-financing constraint Lagrange multiplier and \( L_\lambda := L - \lambda C \).

**Definition 53.** Let \( L = L(q,q') \) be the Lagrange function of a deterministic Lagrangian system with the non-holonomic constraint \( C(q,q') = 0 \). Setting \( L_\lambda := L - \lambda C \) for the constraint Lagrange multipliers the dynamics is given by the extended Euler-Lagrange equations

\[
\begin{align*}
(EL) \quad \begin{cases}
\frac{d}{dt} \frac{\partial L_\lambda}{\partial q}(q_t, dq_t) - \frac{\partial L_\lambda}{\partial q'}(q_t, dq_t') &= 0 \\
C(q_t, q_t') &= 0
\end{cases}
\end{align*}
\]

(76)

meaning by this that the deterministic solution \( q = q_t \) and \( \lambda \in \mathbb{R} \) satisfy the constraint and

\[
\frac{d}{dt} \frac{\partial L_\lambda}{\partial q'}(q_t, dq_t') - \frac{\partial L_\lambda}{\partial q}(q_t, dq_t) = 0.
\]

(77)

The formal **stochastic embedding of the Euler-Lagrange equations** is obtained by the formal substitution

\[
S : \frac{d}{dt} \mapsto D,
\]

(78)

and allowing the coordinates of the tangent bundle to be stochastic

\[
\begin{align*}
(SEL) \quad \begin{cases}
D\frac{\partial L_\lambda}{\partial q'}(q_t, dq_t') - \frac{\partial L_\lambda}{\partial q}(q_t, dq_t) &= 0 \\
C(q_t, q_t') &= 0
\end{cases}
\end{align*}
\]

(79)

meaning by this that the stochastic solution \( Q = Q_t \) and the random variable \( \lambda \) satisfy the constraint and

\[
\begin{align*}
\left\{ \begin{array}{l}
D\frac{\partial L_\lambda}{\partial q'}(Q_t, DQ_t') - \frac{\partial L_\lambda}{\partial q}(Q_t, DQ_t) = 0 \\
C(Q_t, DQ_t) = 0
\end{array} \right.
\end{align*}
\]

(80)

**Definition 54.** Let \( L = L(q,q') \) be the Lagrange function of a deterministic Lagrangian system on a time interval \( I \) with constraint \( C = 0 \). Set

\[
\Xi := \left\{ Q \in C^1(I) \mid E \left[ \int_I |L_\lambda(Q_t, DQ_t)|dt \right] < +\infty \right\}.
\]

(81)
The action functional associated to $L_\lambda$ defined by

$$F : \Xi \rightarrow \mathbb{R}$$
$$Q \mapsto E\left[\int L_\lambda(Q_t, DQ_t) dt\right]$$

is called stochastic analogue of the classic action under the constraint $C = 0$.

For a sufficiently smooth extended Lagrangian $L_\lambda$ a necessary and sufficient condition for a stochastic process to be a critical point of the action functional $F$ is the fulfillment of the stochastic Euler-Lagrange equations (SEL), as it can be seen in Theorem 7.1 page 54 in [CrDa07]. Moreover we have the following

**Lemma 55 (Coherence).** The following diagram commutes

$$\begin{array}{ccc}
L_\lambda(q_t, q'_t) & \xrightarrow{S} & L_\lambda(Q_t, DQ_t) \\
\downarrow \text{Critical Action Principle} & & \downarrow \text{Stochastic Critical Action Principle} \\
(EL) & \xrightarrow{S} & (SEL)
\end{array}$$

### 4 Quantum Mechanical Reformulation of Mathematical Finance

In this section we show how the assets’ and market portfolio’s dynamics is the solution of the Schrödinger equation for the quantum mechanical system obtained by means of the quantization of the deterministic constrained Hamilton system equivalent to the Lagrange one introduced in Section 3. As a general background to the mathematics of quantum mechanics we refer to [Ta08] and [Ha13].

**Proposition 56.** The Hamilton function $H$ defined as Legendre transform of the Lagrangian $L$ is

$$H(p, q, t) := (p \cdot q' - L(q, q', t))|_{p} = \frac{\partial L}{\partial q'} = \frac{x \cdot p_{D}}{|x|^2}((x \cdot D)|p_x| - x \cdot (rD)),$$

where $q = (x, D, r)$ and $p = (p_x, p_D, p_r)$.

The constraint $C$ is equivalent to

$$\bar{C}(q, q') := \frac{x \cdot (D' + (rD))}{x \cdot D} \frac{x' \cdot D}{|x'|}$$

which reads

$$E(p, q) := \bar{C}(q, q')|_{p} = \frac{\partial L}{\partial q'} = D \cdot p_x.$$
Proof. It follows directly by inserting
\[\begin{align*}
p_x &= \frac{\partial L}{\partial x} = \frac{x \cdot (D' + (rD))}{x \cdot D} x' \\
p_D &= \frac{\partial L}{\partial D} = \frac{|x'|}{x} x' \\
p_r &= \frac{\partial L}{\partial r} = 0
\end{align*}\]
into equations (72) and (75).

Proposition 57. The selfadjoint Hamilton operator obtained by the standard quantization procedure
\[q \rightarrow q \text{ (multiplication operator)}\]
\[p \rightarrow \frac{1}{i} \frac{\partial}{\partial q} \text{ (differential operator)}\]
is
\[H := \frac{1}{2} \left( H \left( q, \frac{1}{i} \frac{\partial}{\partial q} \right) + H \left( q, \frac{1}{i} \frac{\partial}{\partial q} \right)^* \right) = \frac{1}{2} \left[ \left( \frac{x}{|x|^2} \cdot \frac{1}{i} \frac{\partial}{\partial D} \right) \left( \pm \left| \frac{\partial}{\partial x} \right| (x \cdot D) - x \cdot (rD) \right) + \left( \pm (x \cdot D) \left| \frac{\partial}{\partial x} \right| - x \cdot (rD) \left( \frac{x}{|x|^2} \cdot \frac{1}{i} \frac{\partial}{\partial D} \right) \right) \right]
\]
with domain of definition
\[\text{dom}(H) := \{ \varphi \in L^2(\mathfrak{X} \times \mathbb{R}^{2N}, \mathbb{C}, d^{2N}q) \mid H\varphi \in L^2(\mathfrak{X} \times \mathbb{R}^{2N}, \mathbb{C}, d^{2N}q), \varphi|_{\partial \mathfrak{X} \times \mathbb{R}^{2N}} = 0, E\varphi = 0 \},\]
where \(E\) is the quantization of the constraint, defined as
\[E := E \left( q, \frac{1}{i} \frac{\partial}{\partial q} \right) = D \cdot \frac{1}{i} \frac{\partial}{\partial x}\]

Proof. The quantization procedure of constrained Lagrangian systems, explained in [Di64], [Kl01] and [FaJa88], is directly applied here.

Remark 58. The Hamilton operator is a second order pseudodifferential operator with leading symbol
given by

\[ \sigma_L(H)(x, D, r)(\xi_x, \xi_D, \xi_r) = \pm \left( \frac{x}{|x|^2} \cdot \xi_D \right) (|\xi_x|(x \cdot D)), \tag{92} \]

where \( \xi_x, \xi_D, \xi_r \in \mathbb{R}^N \). We notice that \( \sigma_L(H)(x, D, r) \) is not injective and thus \( H \) cannot be elliptic. Therefore, we cannot infer that \( H \) has a pure point spectrum, even when we restrict support of the functions in its domain of definition on a bounded region of \( \mathbb{R}^N \). Note that the operator \( H \) as true pseudodifferential operator is not local but only pseudolocal.

**Proposition 59.**

\[ (\text{NUPBR}) \iff H = 0. \tag{93} \]

**Proof.**

\( \Rightarrow \): The (NUPBR) condition is equivalent with the existence of a positive semimartingale \((\beta_t)_{t \in [0, T]}\) such that for all \( x \in \mathcal{X} \) and all \( t \in [0, T] \)

\[ \mathcal{D} \log(\beta_t D_t^x) + r_t^x = 0. \tag{94} \]

Therefore, on the optimal path \( q = q_t = (x_t, D_t, r_t) \) the equations

\[
\begin{align*}
L(q, q') &:= |x'| \frac{x(D' + rD)}{|x'|} = 0 \\
\tilde{C}(q, q') &:= \frac{x(D' + rD)}{|x'|} x' D = 0
\end{align*}
\]  

must hold. Hence, the Hamilton function becomes

\[ H(p, q, t) := (p \cdot q' - L(q, q', t)) \Big|_{pc = \frac{\delta}{\delta p}} = 0, \tag{96} \]

and the Hamilton operator vanishes, i.e. \( H = 0 \).

\( \Leftarrow \): The converse is obtained by going back one step after the other because of the equivalence of the different statements.

\[ \square \]

**Theorem 60.** The asset and market portfolio dynamics is given by the solution of the Schrödinger equation

\[ \begin{cases}
\frac{i}{\Delta t} \psi(q, t) = H \psi(q, t) \\
\psi(q, 0) = \psi_0(q),
\end{cases} \tag{97} \]

where \( \psi_0 \) is the initial state satisfying \( \mathcal{C}\psi_0 = 0 \) and \( \int_{\mathcal{X} \times \mathbb{R}^N} dq^3 |\psi_0(q)|^2 = 1 \).
The solution is given by

$$\psi(q,t) = e^{iHt}\psi_0.$$  \hfill (98)

where $\{e^{iHt}\}_{t \geq 0}$ is the strong continuous, unitary one parameter group associated to the selfadjoint $H$ by Stone’s theorem.

**Remark 61.** This is the quantum mechanical formulation of the constrained stochastic Lagrangian system described by the SDE (80). The interpretation of $|\psi(q,t)|^2$ is the probability density at time $t$ for the coordinates $q$:

$$P[q_t \in Q] = \int_Q dq^{3N} |\psi(q,t)|^2.$$  \hfill (99)

Therefore, if we have a random variable $a_t = a(p,q,t)$, by mean of its quantization

$$A := a\left(\frac{1}{i} \frac{\partial}{\partial q}, q, t\right)$$  \hfill (100)

we can compute its expectation by means of both the **Schrödinger and Heisenberg representation**

$$\mathbb{E}_0[a_t] = (A\psi, \psi) = \int_{\mathbb{R}^{2N}} dq^{3N} A\psi(q,t)\bar{\psi}(q,t) = \int_{\mathbb{R}^{2N}} dq^{3N} A_t\psi(q,0)\bar{\psi}(q,0),$$  \hfill (101)

where the time dependent operator $A_t$, the **Heisenberg representation** of the operator $A$ is defined as

$$A_t := e^{iHt}Ae^{-iHt}.$$  \hfill (102)

Higher moments of random variables $a_t$, like any measurable functions $f(a_t)$ of them can be computed by means of this technique as

$$\mathbb{E}_0[f(a_t)] = (f(A)\psi, \psi) = \int_{\mathbb{R}^{2N}} dq^{3N} f(A_t)\psi(q,0)\bar{\psi}(q,0),$$  \hfill (103)

transforming the problem in one of operator calculus. Similarly, for stochastic dependencies between two processes $(a_t)_{t \in [0,T]}$ and $(b_t)_{t \in [0,T]}$ we have for $t \leq t_1, t_2$

$$\mathbb{E}_t[f(a_t^1, g(b_{t_2}))] = (f(A)^1 g(B)\psi_t, \psi_t) = \int_{\mathbb{R}^{2N}} dq^{3N} f(A_{t_1}^1)g(B_{t_2}\psi(q,t)\bar{\psi}(q,t) = $$

$$= \left(\exp(+iH(t_1 - t))f(A)^1\exp(+iH(t_2 - t_1))g(B)\exp(-iH(t_2 - t))\psi_0, \psi_0\right),$$  \hfill (104)

where $f, g$ are real matrix valued measurable functions of a real variable.

**Theorem 62 (Ehrenfest).** The time derivative of the expectation of a selfadjoint operator $A$ is given
\[ \frac{d}{dt}(A\psi, \psi) = \frac{1}{i} ([A, H]\psi, \psi) + \left( \frac{\partial A}{\partial t} \psi, \psi \right). \] (105)

By applying Ehrenfest’s theorem, which is proven in [Ha13], to the operator $E$ and noting that $[H, E] \neq 0$, we cannot conclude that the self-financing $E\psi(q, t) \equiv 0$ condition is satisfied for all times. To avoid this problem we follow [FaJa88] by solving the constraint
\[ D \cdot p_x = 0 \] (106)

with respect to $p_x^N$ to obtain
\[ p_x^N = -\frac{1}{D^N} \sum_{j=1}^{N-1} D^j p_x^j. \] (107)

When this expression is inserted into the Hamilton function (84), its quantization is equivalent to the substitution
\[ \left| \frac{\partial}{\partial x} \right| = \sqrt{-\sum_{j=1}^{N-1} \left(1 + \left(\frac{D^j}{D^N}\right)^2\right) \frac{\partial^2}{\partial x^2}} \] (108)

into the expression (89) for the Hamilton operator, in whose domain of definition we can drop the condition $E\varphi = 0$. Hence, with this substitution we made sure that the solution of the Schrödinger equation (98) fulfills the self-financing condition.

**Proposition 63.** If $\lambda$ is an eigenvalue of $H$ with eigenvector $\psi_0 \in \text{dom}(H)$, then the dynamics of the expected values of market portfolio, asset values and term structures given initial state $\psi_0$ is constant.

**Proof.** The solution of the Schrödinger equation (98) reads
\[ \psi(q, t) = e^{i\lambda t} \psi_0 \] (109)

and the expectation for any operator $A$ not explicitly depending on the time $t$ is
\[ (A\psi(\cdot, t), \psi(\cdot, t)) = (Ae^{i\lambda t} \psi_0, e^{i\lambda t} \psi_0) \equiv (A\psi_0, \psi_0) \] (110)

Therefore,
\[ \mathbb{E}_0[x_t] = (x\psi, \psi) \equiv \mathbb{E}_0[x_0], \quad \mathbb{E}_0[D_t] = (D\psi, \psi) \equiv \mathbb{E}_0[D_0], \quad \mathbb{E}_0[r_t] = (r\psi, \psi) \equiv \mathbb{E}_0[r_0], \] (111)

and the proof is completed.
Therefore, in view of Ehrenfest’s theorem the computation of the spectrum of the Hamilton operator $H$ is the key to the computation of the expectations of the market portfolio and the asset dynamics. First we analyze the case of bounded domains.

**Theorem 64.** Let

(a) the Hamilton operator be defined on a domain which is bounded in $x \in X$ with the Dirichlet boundary condition on $\partial X$, bounded in $D \in D$ with the Neumann boundary condition on $\partial D$ and unbounded in $r$: 

$$\text{dom}(H) := \left\{ \varphi \in L^2(X \times D \times \mathbb{R}^N, \mathbb{C}, d^Nq) \mid H\varphi \in L^2(X \times D \times \mathbb{R}^N, \mathbb{C}, d^Nq), \varphi \mid_{\partial X \times D \times \mathbb{R}^N} = 0, \left( \frac{\partial \varphi}{\partial \nu} \right)_{x \times \partial D \times \mathbb{R}^N} = 0 \right\}. \tag{112}$$

(b) $\{(\alpha_i, \lambda_i^\alpha)\}_{i \geq 0}$ be a spectral decomposition of $\left| \frac{\partial}{\partial x} \right|$, where the eigenvectors $\alpha_i$ satisfy the Dirichlet boundary condition on $\partial X$, meaning by this for all $i \geq 0$

$$\frac{1}{i} \frac{\partial \alpha_i}{\partial x} = \lambda_i^\alpha \alpha_i e. \tag{113}$$

(c) $\{(\beta_j, \lambda_j^\beta)\}_{j \geq 0}$ be a spectral decomposition of the $\mathbb{R}^N$-valued operator $\frac{1}{i} \frac{\partial}{\partial D}$, where the eigenvectors $\beta_j$ satisfy the Neumann boundary condition on $\partial D$, meaning by this for all $j \geq 0$

$$\frac{1}{i} \frac{\partial \beta_j}{\partial D} = \lambda_j^\beta \beta_j e. \tag{114}$$

(d) $\{\gamma_k\}_{k \geq 0}$ be a basis of $L^2(\mathbb{R}^N, \mathbb{C}, d^N r)$,

Then:

(a) $\{\varphi_{i,j,k} := \alpha_i \beta_j \gamma_k\}_{i,j,k \geq 0}$ is a o.n.b of $\text{dom}(H)$,

(b) the images of the basis $\{\varphi_{i,j,k}\}_{i,j,k \geq 0}$ are

$$H\varphi_{i,j,k} = \lambda_j^\beta \alpha_i(x) \left( \frac{x}{|x|^2} \cdot e \right) \beta_j(D) \left[ -\lambda_i^\alpha(e \cdot D) + \lambda_i^\alpha(x \cdot D) \right] \gamma_k(r). \tag{115}$$

(c) the real number

$$\lambda_{i,j} := \frac{\lambda_i^\alpha \lambda_j^\beta}{2} \int_X d^N x \int_D d^N D \left[ (x \cdot D) - \lambda_i^\alpha(e \cdot D) \right] \left( \frac{x}{|x|^2} \cdot e \right) |\alpha_i(x)|^2 |\beta_j(D)|^2 \tag{116}$$
is eigenvalue of $H$ for the eigenvectors $\{\alpha_i \beta_j \gamma_k\}_{k \geq 0}$. In particular, this gives a spectral decomposition of $H$ and all eigenspaces are infinite dimensional, and

$$\text{spec}_d(H) = \{\lambda_{i,j}\}_{i,j \geq 0} \quad \text{spec}_c(H) = \emptyset.$$  \hfill (117)

Note that 0 is always an eigenvalue.

(d) the spectrum of $\mid \frac{\partial}{\partial x} \mid$ satisfying the self-financing condition and the the Dirichlet boundary condition is obtained as the spectrum of $\mid \frac{\partial}{\partial x} \mid$ in one dimension lower, satisfying the Dirichlet boundary condition. More precisely:

$$\text{spec} \left( \mid \frac{\partial}{\partial x} \mid_{BD} \right) = \{0 < \lambda_1^{\alpha,N} < \lambda_2^{\alpha,N} \leq \lambda_3^{\alpha,N} \ldots \}$$

$$\text{spec} \left( \sqrt{\sum_{i=1}^{N-1} \left( 1 + \left( \frac{D_i}{DN} \right)^2 \frac{\partial^2}{\partial x_i^2} \right)_{BD} } \right) = \{0 < \lambda_1^{\alpha,N-1} < \lambda_2^{\alpha,N-1} \leq \lambda_3^{\alpha,N-1} \ldots \}$$ \hfill (118)

and $\lambda_i^{\alpha,N} \uparrow +\infty$ for $i \to +\infty$ for all dimensions $N \geq 1$.

By applying Proposition 59 to Theorem 64 (c) we obtain the following

**Corollary 65.** Under assumption (a) we have the equivalence

$$\forall i \geq 0, \forall j \geq 1 \quad \int_{\mathcal{X}} d^N x \int_{\mathcal{D}} d^N D \left[ \left( (x \cdot D) - \lambda_i^{\alpha}(e \cdot D) \right) \left( \frac{x}{(x \cdot e)} \cdot e \right) \right] |\alpha_i(x)|^2 |\beta_j(D)|^2 = 0.$$ \hfill (119)

**Corollary 66.** Under the same assumptions as in Theorem 64 with $\mathcal{X} = \prod_{i=1}^{N} [0, A_i]$ and $\mathcal{D} = \prod_{i=1}^{N} [0, B_i]$ the following statements hold true:

(e) the functions $\{\alpha_i\}_{i \geq 0}$ can be rearranged as $\{\alpha_I\}_{I \in \mathbb{N}^N}$, the Dirichlet eigenvectors are

$$\alpha_I(x) = \exp \left( -\pi \sum_{i=1}^{N} \frac{x_i}{A_i} \right) \sqrt{\prod_{i=1}^{N} A_i},$$ \hfill (120)

and the Dirichlet eigenvalues are

$$\lambda_i^{\alpha,N} = \sqrt{\sum_{i=1}^{N} \frac{x_i^2}{A_i^2} I_i^2}.$$ \hfill (121)
(f) the functions \( \{ \beta_j \}_{j \geq 0} \) can be rearranged as \( \{ \beta_j \}_{j \in \mathbb{Z}^N} \), the Neumann eigenvectors are
\[
\beta_J(D) = \frac{\exp \left( -i\pi \sum_{l=1}^{N} \frac{J_l}{B_l} D_l \right)}{\sqrt{\prod_{l=1}^{N} B_l}}.
\]
and the Neumann eigenvalues are
\[
\lambda_J^\beta = \prod_{l=1}^{N} \text{sgn}(J_l) \sqrt{\sum_{l=1}^{N} \frac{\pi^2 J_l^2}{B_l^2}}.
\]

(g) The eigenvalues of \( H \) can be written as
\[
\lambda_{I,J} := \frac{\lambda_I^\alpha \lambda_J^\beta}{2 \prod_{l=1}^{N} A_l B_l} \int_X d^N x \int_D d^N D \left[ ((x \cdot D) - \lambda_I^\alpha (e \cdot D)) \left( \frac{x}{|x|^2} \cdot e \right) \right].
\]
Expression (124) can be computed in a closed form, which reads for
\( N = 1 \):
\[
\lambda_{I,J} = 0 \quad \text{for all} \quad I, J \in \mathbb{Z} \quad (\text{i.e.} \ H = 0),
\]
This means that a market model with cash and just one risky assets must always satisfy the (NUPBR) condition.

\( N = 2 \):
\[
\lambda_{I,J} = \frac{\text{sgn}(J_1) \text{sgn}(J_2) B_2 \pi^2 \sqrt{\frac{J_1^2}{B_1} + \frac{J_2^2}{B_2}} |I_1|}{48 A_1^2} \cdot \left\{ -2A_1 B_1^2 + B_2^2 \right\} \left[ -2A_1^3 + 2A_1^2 \sqrt{A_1^2 + A_2^2} + A_1 A_2 \sqrt{A_1^2 + A_2^2} + 2A_2^2 (A_2 - A_1) \right] +
-2A_1 A_2^3 B_1^2 \arctanh \left( \frac{A_1}{\sqrt{A_1^2 + A_2^2}} \right) + 2B_2^2 (\log(A_2) - \log \left( A_1 + \sqrt{A_1^2 + A_2^2} \right)) +
-3B_1 (B_1 + B_2) \pi |I_1| \left[ 2A_1 \sqrt{A_1^2 + A_2^2} + 2A_2 \sqrt{A_1^2 + A_2^2} +

\vphantom{\int} -2A_2^2 \left( 1 + \log(A_2) - \log \left( A_1 + \sqrt{A_1^2 + A_2^2} \right) \right) +

\vphantom{\int} + A_1^2 \left( -2 + \log \left( 1 + \frac{2A_2 (A_2 + \sqrt{A_1^2 + A_2^2})}{A_1^2} \right) \right) +

\vphantom{\int} + A_1^2 (2B_1^2 - B_2^2) \log \left( 1 + \frac{2A_2 (A_2 + \sqrt{A_1^2 + A_2^2})}{A_1^2} \right) \right\}.
\]
Explicit values in closed form for $N \geq 3$ are available but require more complicated analytic expressions. By applying Proposition 59 to Corollary 66 we obtain the following

**Corollary 67.** Under the same assumptions as Corollary 66 an equivalent condition to the (NUPBR) is

$$\{(A, B) \in \bigcap_{I, J \in \mathbb{Z}^N} \left\{ (A, B) \in [0, +\infty)^{2N} \mid \lambda_{I,J}(A, B) = 0 \right\} \},$$

which means that the domain upper bounds for the market portfolio nominals and asset values must belong to the intersection of an infinite number of hypersurfaces in the $(A, B)$-space.

**Proof of Theorem 64.** We insert the separation Ansatz

$$\varphi(x, D, r) = \alpha(x)\beta(D)\gamma(r)$$

into the eigenvalue equation

$$H\varphi = \lambda\varphi,$$

and obtain, after having noted that $\gamma$ can be chosen arbitrarily in $L^2(\mathbb{R}^N, d^N r)$,

$$H(\alpha\beta) = \lambda(\alpha\beta).$$

Inserting the additional Ansatz for $\alpha$ satisfying the Dirichlet boundary condition on $\partial \mathfrak{X}$, and for $\beta$ satisfying the Neumann boundary condition on $\partial \mathfrak{D}$,

$$\frac{1}{r} \frac{\partial}{\partial x} \alpha = \lambda^\alpha \alpha \quad \text{and} \quad \frac{1}{r} \frac{\partial}{\partial D} \beta = \lambda^\beta \beta e$$

into equation (130), we obtain the compatibility condition

$$\lambda^\beta \left( \frac{x}{|x|^2} \cdot e \right) \left[ -\lambda_1^\alpha (e \cdot D) + \frac{\lambda^\alpha (x \cdot D)}{2} - x \cdot (r D) \right] = \lambda,$$

which must hold true for all $x, D$ and $r$. Thereby we have utilized the property

$$\left| \frac{\partial}{\partial x} \right| (D \cdot x)u(x) = - \left( D \cdot \frac{\partial}{\partial x} \right) \left| \frac{\partial}{\partial x} \right| u(x),$$

which follows from the following computation, which utilizes the fact that $\left| \frac{\partial}{\partial x} \right|$ is a pseudodifferential...
operator:
\[
\frac{\partial}{\partial x} (D \cdot x) u(x) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} d^N \xi \, e^{i x \cdot \xi} \xi |(D \cdot x) u(x) = (2\pi)^{-\frac{N}{2}} \sum_{j=1}^{N} D^j \int_{\mathbb{R}^N} d^N \xi \, e^{i x \cdot \xi} \frac{1}{i} \frac{\partial \hat{u}}{\partial x_j} (\xi) = -(2\pi)^{-\frac{N}{2}} \sum_{j=1}^{N} D^j \int_{\mathbb{R}^N} d^N \xi \xi_j e^{i x \cdot \xi} \hat{u}(\xi) = -(2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} d^N \xi \, (D \cdot \xi) |u(x) e^{i x \cdot \xi} \hat{u}(\xi) = - \left( D \cdot \frac{\partial}{\partial x} \right) | \frac{\partial}{\partial x} | u(x),
\]

where
\[
\hat{u}(\xi) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} d^N x \, e^{-i x \cdot \xi} u(x)
\]
denotes the Fourier transform of a \( u \in L^2(\mathbb{R}^N, d^N x) \).

Therefore, \( \lambda^\beta = \lambda = 0 \) and \( 0 \in \text{spec}_d(H) \), and equation (115) for a fix choice of \( \alpha, \beta \) and \( \gamma \) can be obtained by inserting (132) into (130):
\[
H(\alpha \beta \gamma) = \lambda^\beta \alpha(x) \left( \frac{x}{|x|^2} \cdot e \right) \beta(D) \left[ \frac{-\lambda_i^\alpha (e \cdot D) + \lambda_i^\alpha (x \cdot D)}{2} - x \cdot (rD) \right] \gamma(r).
\]

By introducing spectral decomposition of the selfadjoint elliptic operators on compact domains \( \frac{1}{i} \frac{\partial}{\partial x} \) and \( \frac{1}{i} \frac{\partial}{\partial D} \), that is \( (\alpha_i, \lambda_i^\alpha)_{i \geq 0} \), and \( (\beta_j, \lambda_j^\beta)_{j \geq 0} \) such that
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial}{\partial x} | \alpha_i = | \lambda_i^\alpha | \alpha_i \\
\alpha_i |_{\partial x} = 0
\end{array} \right. \\
\left\{ \begin{array}{l}
\frac{1}{i} \frac{\partial}{\partial D} = \lambda_j^\beta \beta_j e \\
\left( \frac{\beta_j}{D} \cdot \nu \right) |_{\partial D} = 0,
\end{array} \right.
\end{aligned}
\]

where \( \{ \alpha_i \}_{i \geq 0} \) is an o.n.b of \( L^2(\mathbb{R}^N, d^N x) \) and \( \{ \beta_j \}_{j \geq 0} \) is an o.n.b of \( L^2(\mathbb{D}, C, d^N D) \), and considering an arbitrary o.n.b \( \{ \gamma_k \}_{k \geq 0} \) of \( L^2(\mathbb{R}^N, d^N r) \), we obtain an o.n.b of \( \text{dom}(H) \) with the choice \( (\alpha_i \beta_j \gamma_k)_{i,j,k \geq 0} \) and (a) is proved. Inserting \( \{ \alpha_i \beta_j \gamma_k \}_{i,j,k \geq 0} \) into (136) proves (b).

If we insert the Fourier decomposition
\[
\varphi = \sum_{i,j,k \geq 0} \epsilon_{i,j,k} \alpha_i \beta_j \gamma_k
\]
into the eigenvalue equation
\[
H \varphi = \lambda \varphi,
\]

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we obtain
\[
\sum_{i,j,k \geq 0} c_{i,j,k} \left[ \lambda^2_j \left( \frac{x}{|x|^2} \cdot e \right) \left( -\lambda^2_i (e \cdot D) + \lambda^0_i (x \cdot D) \right) - x \cdot (rD) \right] - \lambda \alpha_i (x) \beta_j (D) \gamma_k (r) = 0. \quad (140)
\]

If we choose in (140) just one Fourier coefficient \( c_{i,j,k} \neq 0 \), and compute we have
\[
\left[ \lambda^2_j \left( \frac{x}{|x|^2} \cdot e \right) \left( -\lambda^2_i (e \cdot D) + \lambda^0_i (x \cdot D) \right) - x \cdot (rD) \right] - \lambda \alpha_i (x) \beta_j (D) \gamma_k (r) = 0, \quad (141)
\]
where we can compute the scalar product of both sides with \( \alpha_i \beta_j \gamma_k \) in \( L^2 (X \times D \times \mathbb{R}^N, d^3 q) \), obtaining
\[
\lambda = \lambda^2_j \left( \left( \frac{x}{|x|^2} \cdot e \right) \left( -\lambda^2_i (e \cdot D) + \lambda^0_i (x \cdot D) \right) - x \cdot (rD) \right) \alpha_i \beta_j \gamma_k, \quad (142)
\]
The factor \(-x \cdot (rD)\) gives no contribution to \( \lambda \), because for fixed \( x \) and \( D \) it is an odd function of \( r \). Therefore,
\[
\lambda = \lambda^2_j \left( \left( \frac{x}{|x|^2} \cdot e \right) \left( -\lambda^2_i (e \cdot D) + \lambda^0_i (x \cdot D) \right) - x \cdot (rD) \right) \alpha_i \beta_j \gamma_k, \quad (143)
\]
Since \( \{ \gamma_k \}_{k \geq 0} \) is an o.n.B of \( L^2 (\mathbb{R}^N, d^N r) \), we obtain (e).

With the \( x \) coordinate transform for fixed \( D \)
\[
\bar{x}_j := \left( 1 + \left( \frac{D_j}{D^N} \right) \right)^{-\frac{1}{2}} x_j \quad (1 \leq j \leq N - 1)
\]
we see that
\[
\sqrt{- \sum_{l=1}^{N-1} \left( 1 + \left( \frac{D_l}{D^N} \right) \right)^2 \frac{\partial^2}{\partial x_l^2}} = \sqrt{- \sum_{l=1}^{N-1} \frac{\partial^2}{\partial x_l^2}}, \quad (145)
\]
and, thus (d) follows for the operators acting on \( L^2 (X, C, d^N x) \) functions satisfying both Dirichlet boundary and self-financing conditions.

Now we can compute explicitly eigenvectors and eigenvalues for the operators \( \frac{\partial}{\partial x_l} \) and \( \frac{1}{2} \frac{\partial^2}{\partial x_l^2} \). A given (possibly unbounded) selfadjoint operator \( A \) on a Hilbert space \( \mathcal{H} \) can be represented by means of its spectral decomposition as
\[
A \chi = \int_{\mathbb{R}} \lambda dP_{\lambda} \chi, \quad \text{for any } \chi \in \text{dom}(A), \quad (146)
\]
where \( \{P_{\lambda}\}_{\lambda \in \mathbb{R}} \) is the projection valued measure for \( A \). The Riemann-Stieltjes integral converges in the strong sense. Given a real valued measurable function \( f \) with domain of definition included in \( \text{spec}(A) \),
the operator $f(A)$ is defined as
\[ f(A)\eta := \int_{\mathbb{R}} f(\lambda) dP_\lambda \eta, \quad \text{for any } \eta \in \text{dom}(A) \]
\[ \text{dom}(f(A)) := \left\{ \eta \in \mathcal{H} \mid \int_{\mathbb{R}} |f(\lambda)|^2 d(P_\lambda \eta, \eta) < +\infty \right\}, \quad (147) \]
is selfadjoint, has spectrum $\text{spec}(f(A)) = f(\text{spec}(A))$, and the same the projection valued measure $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ as $A$. We apply this fact to compute the a spectral decomposition of the two operators in (137) utilizing the spectral decomposition of the Laplacian on functions on a bounded domain. To do so, we observe that
\[ \left| \frac{\partial}{\partial x} \right| = \sqrt{-\Delta_{\mathbb{R}^N}} \]
\[ \left( \frac{1}{i} \frac{d}{dD_l} \right)^2 = -\frac{d^2}{dD_l^2} \quad (148) \]
On bounded domains with the Dirichlet, and, respectively Neumann boundary condition we obtain a spectral decomposition as
\[ \left| \frac{\partial}{\partial x} \right| \alpha_i = \sqrt{\mu_i^2} \alpha_i \]
\[ \left( \frac{1}{i} \frac{d}{dD_l} \right) \beta_j = \pm \sqrt{\mu_j^2} \beta_j, \quad (149) \]
where $(\alpha_i, \mu_i)_{i \geq 0}$ is a spectral decomposition of $-\Delta_{\mathbb{R}^N}$, and $(\beta_j, \mu_j^2)_{j \geq 0}$ is a spectral decomposition of $-\frac{d^2}{dD_l^2}$.

**Proof of Corollary 66.** For simple domains as cuboids, $\mathcal{X} = \prod_{i=1}^{N} [0, A_i]$ and $\mathcal{D} = \prod_{i=1}^{N} [0, B_i]$ the computation of the spectrum of the Laplacian can be performed by separation of variables, reducing the problem to the solution of second order ODE with constant coefficients, leading to (c) and (f). The formula for the eigenvalues in (g) follows from (c) once inserted (e) and (f). The explicit expression (126) has been computed with the symbolic computation software Mathematica.

**Remark 68.** Note that the eigenspaces of $H$ for all eigenvalues are infinite dimensional, which is line with the fact that $H$ is not elliptic.

**Corollary 69.** If the Hamilton operator is defined on the whole $\mathbb{R}^N$:
\[ \text{dom}(H) := \left\{ \varphi \in L^2(\mathbb{R}^3, \mathbb{C}, d^3q) \mid H\varphi \in L^2(\mathbb{R}^3, \mathbb{C}, d^3q) \right\}, \quad (150) \]
or on an unbounded domain in \(x\) and \(r\) but bounded in \(D\), \(\mathbb{X} \times \mathbb{D} \times \mathbb{R}^N\)

\[
\text{dom}(H) := \left\{ \varphi \in L^2(\mathbb{X} \times \mathbb{D} \times \mathbb{R}^N, \mathbb{C}, d^3N) \left| H\varphi \in L^2(\mathbb{X} \times \mathbb{D} \times \mathbb{R}^N, \mathbb{C}, d^3N), \right. \right. \\
\left. \left. \left. \left( \frac{\partial \varphi}{\partial D}, \nu \right) \right|_{\mathbb{X} \times \partial \mathbb{D} \times \mathbb{R}^N} = 0 \right\},
\]

(151)

then there are two cases:

\[
\begin{align*}
H \neq 0 &\Rightarrow \text{spec}_d(H) = \emptyset, \text{spec}_c(H) = \mathbb{R} \\
H = 0 &\Rightarrow \text{spec}_d(H) = \{0\}, E_0 = \mathcal{H}, \text{spec}_c(H) = \emptyset.
\end{align*}
\]

(152)

**Proof.** It follows from a limit argument of Theorem 64, by noting the eigenvalues in the discrete spectrum for the bounded domain clusters to an element of the continuous spectrum for the unbounded domain, when the diameter of the bounded domain tends to infinity. The limits of sequences of eigenvectors is no longer in the Hilbert space of the \(L^2(\mathbb{R}^3N, \mathbb{C}, d^3Nq)\) functions but lies in the space of tempered distributions \(S'(\mathbb{R}^3N, \mathbb{C})\). These limits are approximate eigenvectors for the rigged Hilbert space \(L^2 \subset S'\). The special case \(H = 0\) follows from the application of Proposition 59.

\[\square\]

Now we can apply Ehrenfest’s theorem to the operators \(x, D, r\) we obtain the following

**Proposition 70.** The dynamics of the expected values of market portfolio, asset values and term structures in the bounded case (112) is given by

\[
\begin{align*}
\mathbb{E}_0[x_t] &= \mathbb{E}_0[x_1] = \frac{1}{2}[A_1, A_2, \ldots, A_N]^\dagger \\
\mathbb{E}_0[D_t] &= \mathbb{E}_0[D_1] = \frac{1}{2}[B_1, B_2, \ldots, B_N]^\dagger \\
\mathbb{E}_0[r_t] &= \mathbb{E}_0[r_1].
\end{align*}
\]

(153)

for a unit norm initial state \(\|\psi_0\|_{L^2(\mathbb{X} \times \mathbb{D} \times \mathbb{R}^N, \mathbb{C}, d^3Nq)} = 1\).

**Proof.** If the initial state \(\psi_0\) is an eigenvector of the Hamilton operator \(H\), then the result follows from Proposition 63. If this is not the case, by Theorem 62 we obtain, after inserting \([r, H] = 0\) into equation (105) we obtain

\[
\frac{d}{dt}\mathbb{E}_0[r_t] = 0,
\]

(154)

from which the last equation of (153) follows. Since \([x, H]\) and \([D, H]\) do not vanish, a direct computation
of \( \mathbb{E}_0[x_t] \) and \( \mathbb{E}_0[D_t] \) is better than applying Ehrenfest's Theorem:

\[
\mathbb{E}_0[x_t] = \langle x e^{iHt} 0, e^{iHt} 0 \rangle = \sum_{i,j,k \geq 0} |c_{i,j,k}|^2 \langle x \alpha_i \beta_j \gamma_k, e^{i\alpha_i \beta_j \gamma_k} \rangle = \sum_{i,j,k \geq 0} |c_{i,j,k}|^2 \langle x \alpha_i \beta_j \gamma_k, \alpha_i \beta_j \gamma_k \rangle = \sum_{i,j,k \geq 0} |c_{i,j,k}|^2 \int \prod_{l=1}^N [0, A_l] dx |\alpha_i|^2 = \frac{1}{2} \langle \psi_0 | x_1, A_1, \ldots, A_N \rangle = \frac{1}{2} \langle A_1, A_2, \ldots, A_N \rangle,
\]

and, analogously,

\[
\mathbb{E}_0[D_t] = \frac{1}{2} \langle B_1, B_2, \ldots, B_N \rangle.
\]

The expressions

\[
c_{i,j,k} := \langle \psi_0, \alpha_i \beta_j \gamma_k \rangle_{L^2(\mathbb{R}^d \times \mathbb{R}^N, C, d^N q)}
\]

are the Fourier coefficients of the unit norm initial state \( \psi_0 \).

An similar computation leads to the probability distribution of the asset values and the market portfolio, which turns out to be the multivariate uniform distribution in the asset nominals and asset values, for arbitrary distributions of the term structures associated to the assets.

**Proposition 71.**

\[
P[q_t \leq (x^0, D^0, r^0)] = \left( \prod_{l=1}^N \frac{x^0_l}{A_l} \right) \left( \prod_{l=1}^N \frac{D^0_l}{B_l} \right) P[r_0 \leq r^0].
\]

**Remark 72.** Note that formulae (153) coincide with those in ([Fa19]), where the stochastic Lagrange equations (80) have been explicitly solved. These demonstrates the consistency and compatibility of the quantum mechanical reformulation to mathematical finance.

**Theorem 73 (Heisenberg’s uncertainty relation).** Let \( A \) and \( B \) two selfadjoint operators on \( \mathcal{H} \). The variance of the corresponding observables in the state \( \varphi \in \text{dom}(A) \cap \text{dom}(B) \) is

\[
\sigma^2_{\varphi}(A) := ||A\phi - ||A\phi||^2||^2 \quad \sigma^2_{\varphi}(B) := ||B\phi - ||B\phi||^2||^2,
\]

where \( \| \cdot \| \) and \( (\cdot, \cdot) \) are the norm and the scalar product in \( \mathcal{H} = L^2(\mathbb{R}^d \times \mathbb{R}^N, C, d^N q) \). Then,

\[
\sigma^2_{\varphi}(A)\sigma^2_{\varphi}(B) \geq \frac{1}{4} ||[A, B]\varphi||^2.
\]
The proof of Theorem 73 can be found f.i. in [Ta08] or in [Ha13]. By applying Heisenberg’s uncertainty relation to the quantum mechanical representation of our market model we obtain the following

**Proposition 74.** The dynamics of the volatilities of market portfolio and asset values satisfies the inequalities

\[
\begin{align*}
\text{Var}_0 \left(x_t^j \right) & \cdot \text{Var}_0 \left( \frac{x_t \cdot (DD_t + r_t D_t) \ D x_t^j}{x_t \cdot D_t |Dx_t|} \right) \geq \frac{1}{4} \\
\text{Var}_0 \left(D_t^j \right) & \cdot \text{Var}_0 \left( \frac{|Dx_t| D x_t^j}{x_t \cdot D_t |Dx_t|} \right) \geq \frac{1}{4},
\end{align*}
\]

for all indices \( j = 1, \ldots, N \).

**Proof.** For \( q = (x, D) \) we choose \( A := q^i \) and \( B := \frac{1}{x} \frac{\partial}{\partial q^i} \), obtaining by Theorem 73, since \( [A, B] = i \delta^{i,j} \), \( \varphi_t = e^{iH} \varphi_0 \) and \( ||\varphi_t||^2 = 1 \),

\[
\sigma_{\varphi_t}^2 (q^i) \sigma_{\varphi_t}^2 \left( \frac{1}{x} \frac{\partial}{\partial q^i} \right) \geq \frac{1}{4},
\]

(162)

Now we can identify

\[
\begin{align*}
\sigma_{\varphi_t}^2 (q^j) & = \text{Var}_0 (q_t^j) \\
\sigma_{\varphi_t}^2 \left( \frac{1}{x} \frac{\partial}{\partial q^j} \right) & = \text{Var}_0 (p_t^j),
\end{align*}
\]

(163)

and (161) follows after inserting the first two equations of (87))

\[
\begin{align*}
p_x & = \frac{\partial L}{\partial x} = \frac{x \cdot (D' + (rD))}{x \cdot D} \frac{x'}{|x'|} \\
p_D & = \frac{\partial L}{\partial D} = \frac{|x'|}{x \cdot D} x.
\end{align*}
\]

(164)

The proof is completed.

\[ \Box \]

**Remark 75.** Note that we can apply Theorem 73 to \( q = r \), but, since \( p_r = 0 \) as computed in the third equation (87), we cannot identify \( \text{Var}(p_r^j) = 0 \) with \( \sigma_{\varphi_t}^2 \left( \frac{1}{x} \frac{\partial}{\partial r^j} \right) \neq 0 \).

## 5 Asset Bubble in Arbitrage Markets

We would like to transpose the results of Corollary 47 about the valuation of bubbles for the base assets and their European style derivatives into the quantum mechanical context developed so far and apply
the mathematical machinery to explicitly compute the bubble values specifically using the technique proposed in Remark 61. As a matter of fact we obtain

**Theorem 76.** The following statements hold true for any market model with $T \leq +\infty$ allowing for arbitrage:

(a) Market portfolio, asset values and term structures solving the minimal arbitrage problem, i.e. the stochastic Lagrange system (80) are serially independent, more exactly

$$((x_t, D_t, r_t))_{t \in [0,T]} \text{ is an i.i.d. process.}$$ (165)

In particular, conditional and total expectations of asset values, nominals and term structures are constant over time. For all $s > t > 0$:

$$\begin{align*}
\mathbb{E}_0[x_t] &\equiv \mathbb{E}_0[x_1] & \mathbb{E}_0[D_t] &\equiv \mathbb{E}_0[D_1] & \mathbb{E}_0[r_t] &\equiv \mathbb{E}_0[r_1] \\
\mathbb{E}_t[x_s] &\equiv \mathbb{E}_0[x_1] & \mathbb{E}_t[D_s] &\equiv \mathbb{E}_0[D_1] & \mathbb{E}_t[r_s] &\equiv \mathbb{E}_0[r_1].
\end{align*}$$ (166)

The volatilities satisfy

$$\text{Var}_0 \left( x_t^2 \right) \text{ Var}_0 (r_t^2) \geq \frac{1}{4}$$ (167)

(b) Expectation and variance of the bubble discounted value for the $j$-th asset read

$$\begin{align*}
\mathbb{E}_0[\tilde{B}_t^j] &= \mathbb{E}_0[D_t^j] - \mathbb{E}_0^*[D_t^j] \\
\text{Var}_0(\tilde{B}_t^j) &= \text{Var}_0(D_t^j) + \text{Var}_0^*(D_t^j). 
\end{align*}$$ (168)

(c) Expectation and variance of the bubble discounted value for the contingent claim $G(S_T)$ on the base assets reads

$$\begin{align*}
\mathbb{E}_0[\tilde{B}_t(G)] &= \mathbb{E}_0[\tilde{V}_t(G)] - \mathbb{E}_0^* \left[ \tilde{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] \\
\text{Var}_0(\tilde{B}_t(G)) &= \text{Var}_0(\tilde{V}_t(G)) + \text{Var}_0^* \left( \tilde{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right). 
\end{align*}$$ (169)

Proof. (a): The serial independence follows from formula (104) with the choice

$$A = B := q \quad \text{i.e. the multiplication operator,}$$ (170)
and noting that for the operator

$$X_{t_1, t_2}^{f,g} := \exp(+iH(t_1 - t)f(A)^\dagger \exp(+iH(t_2 - t_1)g(B) \exp(-iH(t_2 - t)) \tag{171}$$

applied to any element of the basis of $\mathcal{H}$ from Theorem 64 gives

$$X_{t_1, t_2}^{f,g} \varphi_{i,j,k} =$$

$$= \exp(+i\Lambda_{i,j}(q)(t_1 - t)f(q)^\dagger \exp(+i\Lambda_{i,j}(q)(t_2 - t_1)g(q) \exp(-i\Lambda_{i,j}(q)(t_2 - t) = \tag{172}$$

where

$$\Lambda_{i,j}(x, D, r) := \frac{\beta_j}{|x|^2} \cdot e \left( \frac{-\lambda_i^2(e \cdot D) + \lambda_i(x \cdot D)}{2} - x \cdot (rD) \right). \tag{173}$$

We therefore obtain by inserting into formula (104)

$$E_t [f(q_{t_1})^\dagger g(q_{t_2})] = 0 \tag{174}$$

for $t_1 \neq t_2$, $t \leq t_1, t_2$, and all measurable $f$ and $g$, from which we infer the serial independence of $(q_t)_{t \in [0,T]}$.

Let us consider the first formula of (161) of Proposition 74, and note that

$$\frac{\mathcal{D} x_i^j}{|\mathcal{D} x_i|} = \frac{\mathcal{D} x_i^j}{\sqrt{|\mathcal{D} x_i|^2 + |\mathcal{D} x_i|^2}} = \frac{\mathcal{D} x_i^j}{|\mathcal{D} x_i|^2} = \text{sgn} \left( \mathcal{D} x_i^j \right), \tag{175}$$

if $|\mathcal{D} x_i|^2 - |\mathcal{D} x_i|^2 = 0$. All Nelson time derivatives of $(q_t)_{t \in [0,T]}$ vanish because of the serial independence, and, after inserting $\mathcal{D} x_t = 0$ and (175) into (161), we obtain

$$\text{Var}_0 \left( x_i^j \right) \text{Var}_0 \left( \frac{x_t \cdot (r_t D_t)}{x_t \cdot D_t} \right) \geq \frac{1}{4}, \tag{176}$$

in which we insert

$$\frac{x_t \cdot (r_t D_t)}{x_t \cdot D_t} = \frac{\sum_{j=1}^N x_i^j r_t^j D_t^i}{\sum_{j=1}^N x_i^j D_t^i} = \sum_{j=1}^N \frac{x_i^j D_t^i}{\sum_{j=1}^N x_i^j D_t^i} r_t^j = r_t^x, \tag{177}$$

leading to equation (167).
(b): Follows from (c) noting that, with the choice \( G(S) := S^j \),

\[
E_t^* \left[ \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] = E_t[\varphi_0 D_T^j]
\]

\[
E_0 \left[ E_t^* \left[ \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] \right] = E_0[\varphi_0 D_T^j] = E_0^*[D_T^j] = E_0^*[D_T^j]
\]

\[
\text{Var}_0 \left( E_t^* \left[ \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] \right) = \text{Var}_0(\varphi_0 D_T^j) = \text{Var}_0^*(D_T^j) = \text{Var}_0^*(D_T^j).
\]

(178)

(c): The second equation of (67) in Corollary 47 becomes when \( \tau = T < +\infty \)

\[
\widehat{B}_t(G) = \widehat{V}_t(G) - E_t^* \left[ \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] = \widehat{V}_t(G) - E_t \left[ \varphi_0 \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right].
\]

(179)

By applying the absolute expectation operator \( E_0[\cdot] \) on both side of (179) we obtain

\[
E_0[\widehat{B}_t(G)] = E_0[\widehat{V}_t(G)] - E_0 \left[ E_t \left[ \varphi_0 \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] \right] =
\]

\[
= E_0[\widehat{V}_t(G)] - E_0 \left[ \varphi_0 \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] =
\]

\[
= E_0[\widehat{V}_t(G)] - E_0^* \left[ \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right].
\]

(180)

By applying the absolute variance operator \( \text{Var}_0(\cdot) \) on both side of (179) we obtain

\[
\text{Var}_0(\widehat{B}_t(G)) =
\]

\[
= E_0[\widehat{B}_t(G)]^2 - E_0[\widehat{B}_t(G)]^2 =
\]

\[
= E_0[\widehat{V}_t(G)]^2 + E_0 \left[ \varphi_0^2 \hat{G}^2 \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] +
\]

\[
- 2E_0 \left[ \varphi_0 \widehat{V}_t(G) \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] +
\]

\[
- E_0[\widehat{V}_t(G)]^2 + E_0 \left[ \varphi_0 \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right]^2 +
\]

\[
- 2E_0 \left[ \varphi_0 \widehat{V}_t(G) \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] =
\]

\[
= \text{Var}_0(\widehat{V}_t(G)) - \text{Var}_0 \left( E_t \left[ \varphi_0 \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right] \right) =
\]

\[
= \text{Var}_0(\widehat{V}_t(G)) - \text{Var}_0 \left( \varphi_0 \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right) =
\]

\[
= \text{Var}_0(\widehat{V}_t(G)) + \text{Var}_0^* \left( \hat{G} \left( S_T \exp \left( \frac{C_T}{S_T} (T - t) \right) \right) \right).\]

(181)
6 The (Numerical) Solution of the Schrödinger Equation via Feynman Integrals

6.1 From the Stochastic Euler-Lagrangian Equations to Schrödinger’s Equation: Nelson’s method

Following chapter 14 of [Ne85] we consider diffusions on $N$-dimensional Riemannian manifold satisfying the SDE

$$d\xi_t = b(t,\xi_t)dt + \sigma(\xi_t)dW_t,$$  \hspace{1cm} (182)

where $(W_t)_{t \geq 0}$ is a $K$-dimensional Brownian motion, and

$$b : [0, +\infty[ \times \mathbb{R}^N \rightarrow \mathbb{R}^N \quad \text{and} \quad \sigma : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times K}$$  \hspace{1cm} (183)

are vector and matrix valued functions with appropriate regularity. We assume that

$$\sigma^2(q)(q', q') := q'\sigma(q)\sigma^\dagger(q)q' = \sum_{j=1}^N q'^j q'_j$$  \hspace{1cm} (184)

defines a Riemannian metric, and introduce the notation $v^j := \sum_{i=1}^N (\sigma\sigma^\dagger)_{j,i} v_i$.

We consider a Lagrangian on $M$ given as

$$L(q, q', t) := \sum_{j=1}^N \left[ \frac{1}{2} q'^j q'_j - \Phi(q) + A_j(q)q'^j \right],$$  \hspace{1cm} (185)

for given potentials $\Phi$ and $A$. For the diffusion (182) the Guerra-Morato Lagrangian writes

$$L_+(\zeta, t) := \sum_{j=1}^N \left[ \frac{1}{2} b^j(t, \zeta)b_j(t, \zeta) + \frac{1}{2} \nabla_j b^j(t, \zeta) - \Phi(\zeta) + A_j(\zeta)b_j(t, \zeta) + \frac{1}{2} \nabla_j A^j(\zeta) \right]$$  \hspace{1cm} (186)

We define

$$R(t, q) := \frac{1}{2} \log \rho(t, q),$$  \hspace{1cm} (187)

where $\rho$ is the density of the process $(\xi_t)_{t \geq 0}$, and

$$S(t, q) := \mathbb{E} \left[ \int_0^t L_+(\xi_s, s)ds \bigg| \xi_t = q \right]$$  \hspace{1cm} (188)
Hamilton’s principle for the Guerra-Morato Lagrangian implies that

\[
\left( \frac{\partial}{\partial t} + \sum_{j=1}^{N} b^j \nabla_j + \frac{1}{2} \Delta \right) S = \sum_{j=1}^{N} \left[ \frac{1}{2} b^j b_j + \frac{1}{2} \nabla_j b^j - \Phi + A_j b^j + \frac{1}{2} \nabla_j b_j \right],
\]

(189)

which, since

\[b^j = \nabla_j S - A^j + \nabla_j R,\]

(190)

becomes the Hamilton-Jacobi equation

\[
\frac{\partial S}{\partial t} + \frac{1}{2} \sum_{j=1}^{N} \left( \nabla^j S - A^j \right) \left( \nabla_j S - A_j \right) + \Phi - \frac{1}{2} \sum_{j=1}^{N} \nabla^j R \nabla_j R - \frac{1}{2} \Delta R = 0.
\]

(191)

The continuity equation

\[
\frac{\partial \rho}{\partial t} = - \sum_{j=1}^{N} \nabla_j (v^j \rho),
\]

(192)

where

\[v^j = \frac{1}{2} (b^j + b^j^*), \quad b^j^* = b^j - \nabla^j \log \rho,\]

(193)

becomes

\[
\frac{\partial R}{\partial t} + \sum_{j=1}^{N} \left( \nabla_j R \right) \left( \nabla^j S - A^j \right) + \frac{1}{2} \Delta S - \frac{1}{2} \nabla_j A^j = 0.
\]

(194)

The non linear Hamilton-Jacobi and continuity PDE lead to the linear Schrödinger equation

\[
\frac{i \partial \psi}{\partial t} = \left[ \frac{1}{2} \sum_{j=1}^{N} \left( \frac{1}{i} \nabla^j - A^j \right) \left( \frac{1}{i} \nabla_j - A_j \right) + \Phi \right] \psi,
\]

(195)

for the Schrödinger operator \(H\), if we define the probability amplitude

\[\psi(q,t) := e^{iR(q,t) + iS(q,t)}.
\]

(196)

Note that

\[\rho(q,t) = |\psi(q,t)|^2.
\]

(197)
6.2 Solution to Schrödinger’s Equation via Feynman’s Path Integral

The Hamilton function is the Legendre transformation of the Lagrangian:

\[
H(p,q,t) := \left( \sum_{j=1}^{N} p_j^i q_j^i - L(q,q',t) \right) \bigg|_{p_i = \frac{\partial L}{\partial q_j}} = \frac{1}{2} \sum_{j=1}^{N} (p_j - A_j') (p_j - A_j) + \Phi,
\]

and the Schrödinger operator is obtained by the quantization

\[
q \to q \quad \text{(Multiplication operator)} \quad p \to \frac{1}{i} \nabla \quad \text{(Differential operator)}.
\]

The solution of the Schrödinger initial value problem

\[
\begin{aligned}
   i \frac{\partial \psi}{\partial t} &= H \psi \\
   \psi(q,0) &= \psi_0(q),
\end{aligned}
\]

can be obtained as the convolution of the initial condition with Feynman’s path integral:

\[
\psi(y,t) = \int \psi_0(q) \left( \int_{q(0)=q}^{q(t)=y} \exp \left( i \int_{0}^{t} L(u(s), u'(s), s) \, ds \right) Du \right) dq,
\]

An approximation of Feynman’s path integral can be obtained by averaging over a number of possible paths. If the original Lagrangian problem has to fulfill some constraints, these can be enforced in the choice of the paths to be averaged over in the integral.

6.3 Application to Geometric Arbitrage Theory

The GAT Lagrangian reads

\[
L(q,q',t) := |x'| \frac{x \cdot (D' + r D)}{x \cdot D},
\]

for \( q := (x, D, r) \in \mathbb{R}^{3N} \), where \( x, D \) and \( r \) represent portfolio nominals, deflators and short rates. The portfolios under consideration have to satisfy the self-financing condition

\[
x' \cdot D = 0.
\]
Let us assume that the diffusions can be written separately as

\[
\begin{align*}
\frac{dx_t}{dt} &= b_x(t, x_t) dt + \sigma_x(x_t) dW_t \\
\frac{dD_t}{dt} &= b_D(t, D_t) dt + \sigma_D(D_t) dW_t \\
\frac{dr_t}{dt} &= b_r(t, r_t) dt + \sigma_r(r_t) dW_t,
\end{align*}
\]

(204)

where

\[
\begin{align*}
b_x, b_D, b_r : [0, +\infty) \times \mathbb{R}^N &\to \mathbb{R}^N \\
\sigma_x, \sigma_D, \sigma_r : \mathbb{R}^N &\to \mathbb{R}^{N \times K}
\end{align*}
\]

(205)

are vector and matrix valued functions with appropriate regularity.

The GAT Lagrangian can be written in the form (185)

\[
L(q, q', t) = \left( \frac{1}{2} \sum_{j=1}^{3N} q'^j q_j - \frac{1}{2} \right) + \Phi(q) + \sum_{j=1}^{3N} A_j(q) q'^j,
\]

(206)

if we add the additional constraint

\[
\sum_{j=1}^{3N} q'^j q_j \equiv 1
\]

(207)

and set

\[
\Phi(q) := -\frac{x \cdot (rD)}{x \cdot D} - \frac{1}{2}, \quad A^D_j(q) := -\frac{\sigma^{-2}_{j,i} x_i}{x \cdot D}, \\
A^r_j(q) := 0, \quad A^q_j(q) := 0.
\]

(208)

Therefore, the solution of Schrödinger’s initial value problem (200) reads

\[
\psi(y, t) = \int \psi_0(q) \left( \int_{q(0)=y}^{q(t)=y} \exp \left( i \int_0^t \frac{x_s \cdot (D'_s + r_s D_s)}{x_s \cdot D_s} ds \right) Du \right) dq,
\]

(209)

where the Feynman integration is over all paths satisfying the constraints

\[
\sum_{j=1}^{3N} q'^j q_j = \sum_{i,j=1}^N \left[ (\sigma^2)^2_{j,i}(x) x_i x'_j + (\sigma^D)^2_{j,i}(D) D'_i D'_j + (\sigma^r)^2(r) x'_i x'_j \right] \equiv 1
\]

(210)

\[
x' \cdot D \equiv 0.
\]

The first constraints is satisfied by all path where time is the arc length parameter.
7 Conclusion

By introducing an appropriate stochastic differential geometric formalism, the classical theory of stochastic finance can be embedded into a conceptual framework called Geometric Arbitrage Theory, where the market is modelled with a principal fibre bundle with a connection and arbitrage corresponds to its curvature. The asset and market dynamics have a Lagrangian, an Hamiltonian and a quantum mechanical formulation, the latter in terms of Schrödinger equation which can be solved explicitly by means of the spectral decomposition of the Hamiltonian operator. We compute the arbitrage dynamics for the assets and the market portfolio assuming the minimization of arbitrage as a natural principle governing the interactions among market participants. We explicitly compute asset values, term structures, market portfolio nominals and the implied asset bubbles.

A Derivatives of Stochastic Processes

In stochastic differential geometry one would like to lift the constructions of stochastic analysis from open subsets of \( \mathbb{R}^N \) to \( N \) dimensional differentiable manifolds. To that aim, chart invariant definitions are needed and hence a stochastic calculus satisfying the usual chain rule and not Itô’s Lemma is required, (cf. [HaTh94], Chapter 7, and the remark in Chapter 4 at the beginning of page 200). That is why we will be mainly concerned in this paper by stochastic integrals and derivatives meant in Stratonovich’s sense and not in Itô’s.

Definition 77. Let \( I \) be a real interval and \( Q = (Q_t)_{t \in I} \) be a vector valued stochastic process on the probability space \( (\Omega, \mathcal{A}, P) \). The process \( Q \) determines three families of \( \sigma \)-subalgebras of the \( \sigma \)-algebra \( \mathcal{A} \):

(i) "Past" \( \mathcal{P}_t \), generated by the preimages of Borel sets in \( \mathbb{R}^N \) by all mappings \( Q_s : \Omega \to \mathbb{R}^N \) for \( 0 < s < t \).

(ii) "Future" \( \mathcal{F}_t \), generated by the preimages of Borel sets in \( \mathbb{R}^N \) by all mappings \( Q_s : \Omega \to \mathbb{R}^N \) for \( 0 < t < s \).

(iii) "Present" \( \mathcal{N}_t \), generated by the preimages of Borel sets in \( \mathbb{R}^N \) by the mapping \( Q_s : \Omega \to \mathbb{R}^N \).

Let \( Q = (Q_t)_{t \in I} \) be continuous. Assuming that the following limits exist, Nelson’s stochastic deriva-
tives are defined as

\[ DQ_t := \lim_{h \to 0^+} \mathbb{E} \left[ \frac{Q_{t+h} - Q_t}{h} \bigg| \mathcal{P}_t \right] : \text{forward derivative}, \]

\[ D_*Q_t := \lim_{h \to 0^+} \mathbb{E} \left[ \frac{Q_t - Q_{t-h}}{h} \bigg| \mathcal{F}_t \right] : \text{backward derivative}, \]

\[ DQ_t := \frac{DQ_t + D_*Q_t}{2} : \text{mean derivative}. \] (211)

Let \( S^1(I) \) the set of all processes \( Q \) such that \( t \mapsto Q_t \), \( t \mapsto DQ_t \) and \( t \mapsto D_*Q_t \) are continuous mappings from \( I \) to \( L^2(\Omega, \mathcal{A}) \). Let \( C^1(I) \) the completion of \( S^1(I) \) with respect to the norm

\[
\|Q\| := \sup_{t \in I} \left( \|Q_t\|_{L^2(\Omega, \mathcal{A})} + \|DQ_t\|_{L^2(\Omega, \mathcal{A})} + \|D_*Q_t\|_{L^2(\Omega, \mathcal{A})} \right). \] (212)

**Remark 78.** The stochastic derivatives \( D, D_* \) and \( D \) correspond to Itô’s, to the anticipative and, respectively, to Stratonovich’s integral (cf. [Gl11]). The process space \( C^1(I) \) contains all Itô processes. If \( Q \) is a Markov process, then the sigma algebras \( \mathcal{P}_t \) (”past”) and \( \mathcal{F}_t \) (”future”) in the definitions of forward and backward derivatives can be substituted by the sigma algebra \( \mathcal{N}_t \) (”present”), see Chapter 6.1 and 8.1 in ([Gl11]).

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