Anisotropic Lavine’s Formula and Symmetrised Time Delay in Scattering Theory

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Abstract We consider, in quantum scattering theory, symmetrised time delay defined in terms of sojourn times in arbitrary spatial regions symmetric with respect to the origin. For potentials decaying more rapidly than $|x|^{-4}$ at infinity, we show the existence of symmetrised time delay, and prove that it satisfies an anisotropic version of Lavine’s formula. The importance of an anisotropic dilations-type operator is revealed in our study.

Keywords Time delay · Lavine’s formula · Scattering theory

Mathematics Subject Classifications (2000) 46N50 · 81Q10 · 35J10

1 Introduction and Main Results

It is known for quite some time that the definition of time delay (in terms of sojourn times) in scattering theory has to be symmetrised in the case of multichannel-type scattering processes (see e.g. [3, 4, 12, 13, 21, 22]). More recently [6] it has been shown that symmetrised time delay does exist, in two-body scattering processes, for arbitrary dilated spatial regions symmetric with respect to the origin (the usual time delay does exist only for spherical spatial regions [20]). This leads to a generalised formula for time delay, which reduces to the usual one in the case of spherical spatial regions. The aim of the present paper is to provide a reasonable interpretation of this formula...
for potential scattering by proving its identity with an anisotropic version of Lavine’s formula [11].

Let us recall the definition of symmetrised time delay for a two-body scattering process in $\mathbb{R}^d$, $d \geq 1$. Consider a bounded open set $\Sigma$ in $\mathbb{R}^d$ containing the origin and the dilated spatial regions $\Sigma_r := \{rx \mid x \in \Sigma\}$, $r > 0$. Let $H_0 := -\frac{1}{2} \Delta$ be the kinetic energy operator in $\mathcal{H} := L^2(\mathbb{R}^d)$ (endowed with the norm $\| \cdot \|$ and scalar product $\langle \cdot, \cdot \rangle$). Let $H$ be a selfadjoint perturbation of $H_0$ such that the wave operators $W_\pm := \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$ exist and are complete (so that the scattering operator $S := W_+ W_-$ is unitary). Then one defines for some states $\phi \in \mathcal{H}$ and $r > 0$ two sojourn times, namely:

$$T_0^r(\phi) := \int_{-\infty}^{\infty} dt \int_{x \in \Sigma_r} d^d x \left| (e^{-itH_0} \phi)(x) \right|^2$$

and

$$T_r(\phi) := \int_{-\infty}^{\infty} dt \int_{x \in \Sigma_r} d^d x \left| (e^{-itH} W_- \phi)(x) \right|^2.$$ 

If the state $\phi$ is normalized to one the first number is interpreted as the time spent by the freely evolving state $e^{-itH_0} \phi$ inside the set $\Sigma_r$, whereas the second one is interpreted as the time spent by the associated scattering state $e^{-itH} W_- \phi$ within the same region. The usual time delay of the scattering process for $\Sigma_r$ with incoming state $\phi$ is defined as

$$\tau_{in}^r(\phi) := T_r(\phi) - T_0^r(\phi),$$

and the corresponding symmetrised time delay for $\Sigma_r$ is given by

$$\tau_r(\phi) := T_r(\phi) - \frac{1}{2} [T_0^r(\phi) + T_0^r(S\phi)].$$

If $\Sigma$ is spherical and some abstract assumptions are verified, the limits of $\tau_{in}^r(\phi)$ and $\tau_r(\phi)$ as $r \to \infty$ exist and satisfy [6, Sec. 4.3]

$$\lim_{r \to \infty} \tau_r(\phi) = \lim_{r \to \infty} \tau_{in}^r(\phi) = -\frac{1}{2} \langle H_0^{-1/2} \phi, S^*[D, S]H_0^{-1/2} \phi \rangle, \quad (1.1)$$

where $D$ is the generator of dilations. If $\Sigma$ is not spherical the limit of $\tau_{in}^r(\phi)$ as $r \to \infty$ does not exist anymore [20], but the limit of $\tau_r(\phi)$ as $r \to \infty$ still exists, provided that $\Sigma$ is symmetric with respect to the origin [6, Rem. 4.8].

In this paper we study $\tau_r(\phi)$ in the setting of potential scattering. For potentials decaying more rapidly than $|x|^{-4}$ at infinity, we prove the existence of $\lim_{r \to \infty} \tau_r(\phi)$ by using the results of [6]. In a first step we show that the limit satisfies the equation

$$\lim_{r \to \infty} \tau_r(\phi) = -\langle f(H_0)^{-1/2} \phi, S^*[D_\Sigma, S] f(H_0)^{-1/2} \phi \rangle, \quad (1.2)$$

where $f$ is a real symbol of degree 1 and $D_\Sigma \equiv D_\Sigma(f)$ is an operator acting as an anisotropic generator of dilations. Then we prove that formula (1.2) can be rewritten as an anisotropic Lavine’s formula. Namely, one has (see Theorem