On a path integral representation of the Nekrasov instanton partition function and its Nekrasov–Shatashvili limit

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Abstract

In this work we study the Nekrasov–Shatashvili limit of the Nekrasov instanton partition function of Yang–Mills field theories with $\mathcal{N} = 2$ supersymmetry and gauge group $SU(N)$. The theories are coupled with fundamental matter. A path integral expression of the full instanton partition function is derived. It is checked that in the Nekrasov–Shatashvili (thermodynamic) limit the action of the field theory obtained in this way reproduces exactly the equation of motion used in the saddle-point calculations.

1 Introduction

In this letter we derive a path integral expression of the Nekrasov multi-instanton partition function $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ that takes into account the instantonic sector of gauge field theories with $\mathcal{N} = 2$ supersymmetry and gauge group $SU(N)$. Here $q$ denotes an effective scale, while $\epsilon_1$ and $\epsilon_2$ are the

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deformation parameters of the $\Omega$–background in which the $N = 2$ gauge theory has been embedded before applying the localization procedure. More details on the derivation of $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ can be found in Refs. [1, 2, 3, 4]. The Nekrasov multi–instanton partition function plays a crucial role in the so–called AGT-W [5, 6, 7] and Bethe/gauge correspondences [8, 9, 10].

To provide a representation of $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ in the form of the partition function of scalar fields we apply techniques borrowed from the theory of matrix models [11]. Usually the multi–instanton partition function is given in the form of a sum over $k$ variables $\phi_I$, $I = 1, \ldots, k$, called eigenvalues, where the integer $k$ ranges from zero to infinity. In the path integral formulation, one scalar field is related to the density of eigenvalues and the second is a lagrange multiplier. Let us note that matrix models methods have already been applied in the past to investigate the instantonic partition function in different contexts, see for instance [12, 13, 14, 15, 16, 17, 18, 19, 20], but up to now not to the problem of deriving a path integral expression of $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$. Starting from this expression we are able to prove in a simple and rigorous way that in the Nekrasov–Shatashvili limit $\epsilon_2 \to 0$, performed by keeping $\epsilon_1$ constant, the full multi–instanton partition function reduces to its saddle point approximation given in [21]. This limit is relevant in a wide set of applications, see for example [8, 21, 22, 23, 24, 25, 26, 27]. With the series representation of $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ the computation of the Nekrasov–Shatashvili limit is particularly cumbersome, because it requires to extract from $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ the contribution coming from the tree diagrams, which leads to rather complicated calculations. Within the path integral representation the limiting procedure becomes straightforward since it is performed directly inside the action of the theory.

2 Path integral formulation

The starting point is the instanton partition function of a $N = 2$ gauge field theory with gauge group $SU(N)$ and matter in the fundamental representation $[2, 21]$:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = 1 + \sum_{k=1}^{\infty} \frac{q^k}{k!} \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \int_{\mathbb{R}^k} \prod_{l=1}^{k} \frac{d\phi_l}{2\pi i} \prod_{l \neq J = 1}^{k} D(\phi_l - \phi_J) \prod_{l=1}^{k} Q(\phi_l),$$

(1)
where
\[ D(z) = \frac{z(z + \epsilon_1 + \epsilon_2)}{(z + \epsilon_1)(z + \epsilon_2)}, \quad Q(z) = \frac{M(z)}{P(z + \epsilon_1 + \epsilon_2)P(z)} \]

and
\[ M(z) = \prod_{r=1}^{N_f}(z + m_r), \quad P(z) = \prod_{l=1}^{N}(z - a_l). \]

Here the \( a_l \)'s, \( l = 1, \ldots, N \), are the vacuum expectation values of the adjoint scalar field in the \( SU(N) \) vector multiplet, while the \( m_r \)'s parametrize the masses of the fundamental matter. \( N_f \) can be indentified with the number of flavors of the theory, though a more precise description of its meaning can be found in [1]. Let us note that the integrals over the real line \( \mathbb{R} \) in Eq. (1) require some form of regularization and should be intended as integrals over a closed contour. The details are explained in [1] and, for generic matrix models, in [11].

\[ Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \]

\[ = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \int_{\mathbb{R}} \prod_{l=1}^{k} \frac{d\phi_l}{2\pi i} e^{H_k}, \]

where
\[ H_k = \sum_{l \neq j=1}^{k} \log(D(\phi_l - \phi_j)) + \sum_{l=1}^{k} \log(Q(\phi_l)). \]

After introducing the density of eigenvalues
\[ \rho(\phi) = \sum_{l=1}^{k} \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_l) \]

it is possible to express \( Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \) as follows [11]:

\[ Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \]

\[ = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \]

\[ \times \int_{\mathbb{R}} \prod_{l=1}^{k} \frac{d\phi_l}{2\pi i} \int D\rho(\phi) \delta \left( \rho(\phi) - \sum_{l=1}^{k} \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_l) \right) \]

\[ \times \left[ \int_{-\infty}^{\infty} d\phi d\phi' \left( \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \right)^2 \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') + \int_{-\infty}^{\infty} d\phi \left( \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \right) \rho(\phi) \log(Q(\phi)) \right] . \]

In the above equation the principal value prescription of the integrals over the variable \( \phi \) is necessary in order to avoid the singularities of \( \log(D(\phi - \phi')) \)
when $\phi = \phi'$. This is the analog in the continuous case of the condition $I \neq J$ in the double discrete sum appearing in $H_k$. After introducing the Fourier representation of the Dirac delta function appearing in the above equation:

$$
\delta \left( \rho(\phi) - \sum_{l=1}^{k} \frac{\epsilon_l \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_l) \right) = 
\int D\lambda \exp \left[ i \int_{-\infty}^{+\infty} d\phi \lambda(\phi) \left( \rho(\phi) - \sum_{l=1}^{k} \frac{\epsilon_l \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_l) \right) \right]
$$

we obtain:

$$
Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \int D\rho D\lambda \left( \int_{\mathbb{R}} \frac{d\phi}{2\pi i} \frac{(\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} \exp \left( -\frac{i q \epsilon_2}{1 + \epsilon_2} \lambda(\phi) \right) \right)^k \times \exp \left[ \int_{-\infty}^{+\infty} d\phi d\phi' \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^2 \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') 
+ \int_{-\infty}^{+\infty} d\phi \left( i \lambda(\phi) \rho(\phi) + \frac{(\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} \rho(\phi) \log(Q(\phi)) \right) \right]
$$

(9)

Let us note that the contour integration over $\mathbb{R}$ is now restricted to the integral of the quantity $\exp(-i q \lambda(\phi))$. As mentioned in [11], the choice of the contour is not important when performing integrals over densities like $\rho(\phi)$ or $\lambda(\phi)$, so that it is possible to replace in Eq. (9) the contour integration with the integration over the whole real line, i.e.: $\int_{\mathbb{R}} \rightarrow \int_{-\infty}^{+\infty}$.

The summation over the index $k$ in Eq. (9) can be easily performed and gives as a result:

$$
Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = 
\int D\rho D\lambda \exp \left[ \int_{-\infty}^{+\infty} d\phi d\phi' \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^2 \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') 
+ \int_{-\infty}^{+\infty} d\phi \left[ i \rho(\phi) \lambda(\phi) + \frac{(\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} \rho(\phi) \log(Q(\phi)) \right] \right].
$$

(10)

For future purposes, it will be convenient to perform in Eq. (10) the following shift of the auxiliary field $\lambda(\phi)$:

$$
\lambda(\phi) = \lambda'(\phi) - i \frac{(\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} \log(q).
$$

(11)
After the above shift, the instanton partition function $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ takes the form:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = \int D\rho D\lambda' \exp \left\{ \int_{-\infty}^{+\infty} d\phi d\phi' \left[ \rho(\phi) \log \left( D(\phi - \phi') \right) \rho(\phi') + \int_{-\infty}^{+\infty} d\phi \left[ i\rho(\phi) \lambda'(\phi) + \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right) \left( \rho(\phi) \log(q Q(\phi)) + e^{-i\epsilon_1 \epsilon_2} \lambda'(\phi) \right) \right] \right\}.$$  \hfill (12)

This is the desired path integral expression of the instanton partition function.

Let us note that we could have arrived to Eq. (12) starting from Eq. (7) and performing there the substitution:

$$q^k = \exp \left[ \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \sum_{I=1}^{k} \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \log(q) \right].$$ \hfill (13)

This replacement obviously does not change Eq. (7) because it is easy to prove the following sequence of identities:

$$q^k = e^{k \log(q)} = e^{\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \sum_{I=1}^{k} \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \log(q)}. \hfill (14)$$

Using the condition (6) that is imposed by the Dirac delta function appearing in Eq. (7), we may finally write:

$$q^k = \exp \left[ \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \int_{-\infty}^{+\infty} d\phi \rho(\phi) \log(q) \right].$$ \hfill (15)

In this way Eq. (7) becomes:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \times \int_{\mathbb{R}} \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \int D\rho(\phi) \rho(\phi) \delta\left( \rho(\phi) - \sum_{I=1}^{k} \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_I) \right) \times e^{\int_{-\infty}^{+\infty} d\phi d\phi' \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^2 \rho(\phi) \log(D(\phi - \phi')) \rho(\phi')} + \int_{-\infty}^{+\infty} d\phi \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \rho(\phi) \log(q Q(\phi)) \right)}.$$  \hfill (16)
After exploiting the Fourier representation of the Dirac delta function given in Eq. (8), it is possible to obtain Eq. (12) directly from (16) without performing the shift (11). As a matter of fact, the right hand side of Eq. (15) provides exactly the term proportional to $\log(q)$ that is present in (16) but is apparently missing in the action of Eq. (10) and has been recovered only after the shift (11).

In order to pass to the limit $\epsilon_2 \to 0$ in Eq. (12), the following two formulas will be useful:

\[
\int_{-\infty}^{+\infty} d\phi d\phi' \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') = \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} d\phi' \rho(\phi) [\log(D(\phi - \phi')) - \log(D(\phi' - \phi))] \rho(\phi')
\]

(17)

and

\[
\lim_{\epsilon_2 \to 0} \left[ \frac{\log(D(\phi' - \phi')) + \log(D(\phi - \phi'))}{\epsilon_2} \right] = \frac{2\epsilon_1}{\epsilon_1^2 - (\phi - \phi')^2}.
\]

(18)

With the help of equations (17) and (18) it is easy to prove that, when $\epsilon_2 \sim 0$, the expression of $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ given in Eq. (12) may be approximated up to an irrelevant constant as shown below:

\[
Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \sim \int \mathcal{D}\rho(\phi) \exp \left\{ \frac{1}{\epsilon_2} \left[ \frac{1}{2} \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} d\phi' \rho(\phi) G(\phi - \phi') \rho(\phi') + \int_{-\infty}^{+\infty} d\phi \rho(\phi) \log(qQ_0(\phi)) \right] \right\},
\]

(19)

where

\[
G(\phi - \phi') = \frac{2\epsilon_1}{\epsilon_1^2 - (\phi - \phi')^2}.
\]

(20)

Let us notice that the dependence on the auxiliary field $\lambda(\phi)$ disappeared after the limit $\epsilon_2 \to 0$. The above equation is in agreement with the saddle–point approximation that have been derived in the literature, see for example Ref. [21]. In order to derive Eq. (19) we have used the fact that:

\[
\int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} d\phi' \rho(\phi) G(\phi - \phi') \rho(\phi') = \frac{1}{2} \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} d\phi' \rho(\phi) G(\phi - \phi') \rho(\phi').
\]

(21)
3 Conclusions

In this work a path integral formulation of the full Nekrasov instanton partition function has been derived using the methods of matrix models, see Eq. (12). As a byproduct, it has been shown in a rigorous and simple way that in the semi-classical limit, valid when $\epsilon^2$ is very small, the leading order contribution in the partition function is provided by Eq. (19), in agreement with the argument of [2] and statistical mechanics.

In the future it is planned to study the path integral appearing in Eq. (12) using the method of Ref. [28] that allows to perform analytic calculations in the case of theories with nonpolynomial and complex potentials.

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