SHARP LOWER BOUNDS FOR MOMENTS OF $\zeta'(\rho)$

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Abstract. We study the $2k$-th discrete moment of the derivative of the Riemann zeta-function at nontrivial zeros to establish sharp lower bounds for all real $k \geq 0$ under the Riemann hypothesis (RH).

Mathematics Subject Classification (2010): 11M06, 11M26

Keywords: lower bounds, moments, nontrivial zeros, Riemann zeta-function

1. Introduction

Various types of moments of the Riemann zeta function $\zeta(s)$ have been extensively studied in the literature. In this paper, we are interested in the $2k$-th discrete moment of the derivative of $\zeta(s)$ at nontrivial zeros denoted by

$$J_k(T) := \frac{1}{N(T)} \sum_{0 < \Im(\rho) \leq T} |\zeta'(\rho)|^{2k},$$

where we write $\rho$ for the nontrivial zeros of $\zeta(s)$ and

$$N(T) = \sum_{0 < \Im(\rho) \leq T} 1.$$

In [5], S. M. Gonek initiated the study on $J_k(T)$ to show that under the truth of the Riemann hypothesis (RH), one has asymptotically

$$J_1(T) \sim \frac{1}{12} (\log T)^3.$$

Regarding the order of magnitude for $J_k(T)$, S. M. Gonek [6] and D. Hejhal [10] independently conjectured that for any real $k$,

$$(1.1) \quad J_k(T) \asymp (\log T)^{k(k+2)}.$$

A precisely asymptotic formula is further conjectured by C. P. Hughes, J. P. Keating and N. O’Connell in [11], building on connections with the random matrix theory. The evidence from the random matrix side also suggests that (1.1) may not be valid for $k \leq -3/2$.

A proof of (1.1) for the case of $k = 2$ is given by N. Ng [19] assuming RH. Under RH and the additional assumption that the zeros of $\zeta(s)$ are simple, S. M. Gonek [6] obtained a lower bound for $J_{-1}(T)$ of the conjectured order of magnitude. An explicit estimation for the constant involved is further given by M. B. Milinovich and N. Ng [15]. In [16], M. B. Milinovich and N. Ng also proved that for all natural number $k$,

$$(1.2) \quad J_k(T) \gg_k (\log T)^{k(k+2)}.$$

On the other hand, M. B. Milinovich [13] proved that under RH, for any natural number $k$ and any $\varepsilon > 0$,

$$(1.3) \quad J_k(T) \ll_{k, \varepsilon} (\log T)^{k(k+2)+\varepsilon}.$$

In [12], S. Kirila obtained sharp upper bounds for $J_k(T)$ for all real $k > 0$ under RH. In particular, this implies the validity of (1.3) for natural numbers $k$ without the extra $\varepsilon$ power. Together with (1.2), we see that (1.1) is valid for all natural numbers $k$.

The approach taken by M. B. Milinovich and N. Ng to obtain their result concerning (1.2) follows from a simple and powerful method developed by Z. Rudnick and K. Soundararajan in [22,23] for establishing lower bounds for moments of families of $L$-functions, while the approach taken by S. Kirila follows from a method of K. Soundararajan [24] together with its refinement by A. J. Harper [8] for establishing upper bounds for moments of families of $L$-functions.

There are now a few more approaches towards establishing sharp bounds for moments of $L$-functions, notably the upper bounds principal of M. Radziwill and K. Soundararajan [21] and the lower bounds principal of W. Heap and K. Soundararajan [9]. One then expects to apply these approaches to obtain sharp bounds concerning $J_k(T)$. In fact, it is
pointed out in [12] that one should be able to establish sharp lower bounds for all real \( k > 0 \) using the approaches in [20][21]. The aim of this paper is to achieve this and our main result is as follows.

**Theorem 1.1.** Assuming RH. For large \( T \) and any \( k \geq 0 \), we have

\[
J_k(T) \gg_k (\log T)^{k(k+2)}. 
\]

We shall instead apply the lower bounds principal of W. Heap and K. Soundararajan [9] in the proof of Theorem 1.1. The proof also uses the arguments by A. J. Harper in [8] and by S. Kirila in [12]. Combining Theorem 1.1 and the above mentioned result of S. Kirila in [12], we immediately obtain the following result concerning the order of magnitude of \( J_k(T) \).

**Corollary 1.2.** Assuming RH. For large \( T \), the estimation given in (1.1) is valid for all real \( k \geq 0 \).

2. Preliminaries

We now introduce some notations and auxiliary results used in the paper. We assume the truth of RH throughout so that we may write each nontrivial zero \( \rho \) of \( \zeta(s) \) as \( \rho = \frac{1}{2} + i\gamma \), where we denote \( \gamma \in \mathbb{R} \) for the imaginary part of \( \rho \). We denote \( \omega(n) \) for the number of distinct prime factors of \( n \) and \( \Omega(n) \) for the number of prime powers dividing \( n \). We note the following estimation (see [17] Theorem 2.10]) for \( \omega(n) \),

\[
(2.1) \quad \omega(n) \leq \frac{\log n}{\log \log n} (1 + O\left(\frac{1}{\log \log n}\right)), \quad n \geq 3.
\]

We also define \( \Lambda_j(n) \) for all integers \( j \geq 0 \) to be the coefficient of \( n^{-s} \) in the Dirichlet series expansion of \((-1)^j \frac{\zeta^{(j)}(s)}{\zeta(s)}\). In particular, we have \( \Lambda_1(n) = \Lambda(n) \), the usual von Mangoldt function. We extend the definition of \( \Lambda \) to all real numbers \( x \) by defining \( \Lambda(x) = 0 \) when \( x \) is not an integer and we note the following uniform version of Landau’s formula [13], originally proved by S. M. Gonek [7].

**Lemma 2.1.** Assume RH. Then we have for \( T \) large and any positive integers \( a, b \),

\[
(2.2) \quad \sum_{T < \gamma \leq 2T} \frac{(a/b)^i\gamma}{\sqrt{a/b}} = \begin{cases} 
T \Lambda(a/b), & a = b, \\
\frac{T}{2\pi} \sqrt{a/b} + O\left(\sqrt{ab}(\log T)^2\right), & a > b, \\
\frac{T}{2\pi} \sqrt{b/a} + O\left(\sqrt{ab}(\log T)^2\right), & b > a,
\end{cases}
\]

where we denote \( N(T, 2T) \) for the number of nontrivial zeros \( \rho \) such that \( T < \Im(\rho) \leq 2T \).

The cases when \( a \neq b \) of Lemma 2.1 are given in [12] Lemma 5.1] while the case \( a = b \) of Lemma 2.1 is trivial. Recall that the Riemann–von Mangoldt formula asserts (see [2] Chapter 15]) that

\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).
\]

It follows from this and the relation \( N(T, 2T) = N(2T) - N(T) \) that

\[
(2.3) \quad N(T, 2T) \ll T \log T.
\]

We reserve the letter \( p \) for a prime number in this paper and we note that \( \Lambda_2(n) \) is supported on integers \( n \) with \( \omega(n) \leq 2 \) satisfying for primes \( p, q, p \neq q \) and positive integers \( i, j \),

\[
(2.4) \quad \Lambda_2(p^i) \ll i(\log p)^2, \quad \Lambda_2(p^i q^j) \ll (\log p)(\log q).
\]

We recall the following well-known Mertens’ formula (see [18] Theorem 2.7]).

**Lemma 2.2.** Let \( x \geq 2 \). We have, for some constant \( b \),

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + b + O\left(\frac{1}{\log x}\right).
\]

We end this section by including a mean value theorem given in [16] Lemma 4.1 concerning integrals over Dirichlet polynomials.
Lemma 2.3. Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of complex numbers. Let \( T_1 \) and \( T_2 \) be positive real numbers and \( g(t) \) be a real-valued function that is continuously differentiable on the interval \([T_1, T_2] \). Then

\[
\int_{T_1}^{T_2} g(t) \left( \sum_{n=1}^{\infty} a_n t^{-it} \right) \left( \sum_{n=1}^{\infty} b_n t^{it} \right) dt \\
= \int_{T_1}^{T_2} g(t) dt \sum_{n=1}^{\infty} a_n b_n + O \left( \left( |g(T_1)| + |g(T_2)| + \int_{T_1}^{T_2} |g'(t)| dt \right) \left( \sum_{n=1}^{\infty} n |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} n |b_n|^2 \right)^{1/2} \right).
\]

3. Proof of Theorem 1.1

3.1. The lower bound principle. We assume that \( T \) is a large number throughout the proof. As the case \( k = 0 \) is trivial, we only consider the case \( k > 0 \) in the proof. Moreover, we note that in the rest of the paper, the explicit constants involved in estimations using \( \ll \) or the big-\( O \) notations depend on \( k \) only and are uniform with respect to \( \rho \). We further make the convention that an empty product is defined to be 1.

We follow the ideas of A. J. Harper in [8] and the notations of S. Kirila in [12] to define for a large number \( M \) depending on \( k \) only,

\[
\alpha_0 = 0, \quad \alpha_j = \frac{20^{j-1}}{(\log \log T)^2} \quad \forall j \geq 1, \quad J = I_{k,T} = 1 + \max \{ j : \alpha_j \leq 10^{-M} \}.
\]

It follows from the above notations and Lemma 2.2 that we have for \( 1 \leq j \leq J - 1 \) and \( T \) large enough,

\[
(3.1) \quad \sum_{\tau^* < \rho \leq \tau^{*+1}} \frac{1}{\rho} = \log \alpha_{j+1} - \log \alpha_j + o(1) = \log 20 + o(1) \leq 10.
\]

We denote for any real number \( \ell \) and any \( x \in \mathbb{R} \),

\[
E_\ell(x) = \sum_{j=0}^{[\ell]} \frac{x^j}{j!}.
\]

We then define for any real number \( \alpha \) and any \( 1 \leq j \leq J \),

\[
\mathcal{P}_j(s, \alpha) = \sum_{p \in I_j} \frac{1}{p^s}, \quad \mathcal{N}_j(s, \alpha) = E_{e^{2k\alpha_j^{-3/4}}} \left( \alpha \mathcal{P}_j(s) \right), \quad \mathcal{N}(s, \alpha) = \prod_{j=1}^{J} \mathcal{N}_j(s, \alpha).
\]

Denote \( g(n) \) for the multiplicative function given on prime powers by \( g(p^r) = 1/r! \) and define functions \( b_j(n), 1 \leq j \leq J \) such that \( b_j(n) = 0 \) or 1 and that \( b_j(n) = 1 \) only when \( n \) is composed of at most \( \lfloor e^{2k\alpha_j^{-3/4}} \rfloor \) primes, all from the interval \( I_j \). We then have

\[
\mathcal{N}_j(s, \alpha) = \sum_{n_j} \alpha^{\Omega(n_j)} g(n_j) b_j(n_j) \frac{1}{n_j^s}, \quad 1 \leq j \leq J.
\]

Note that each \( \mathcal{N}_j(s, \alpha) \) is a short Dirichlet polynomial of length at most \( T^{\alpha_j [e^{2k\alpha_j^{-3/4}}]} \). By taking \( T \) large enough, we notice that

\[
\sum_{j=1}^{J} \alpha_j [e^{2k\alpha_j^{-3/4}}] \leq 40 e^2 k 10^{-M/4}.
\]

It follows that \( \mathcal{N}(s, \alpha) \) is also a short Dirichlet polynomial of length at most \( T^{40 e^2 k 10^{-M/4}} \).

Moreover, we write for simplicity that

\[
(3.2) \quad \mathcal{N}(s, \alpha) = \sum_n a_\alpha(n) \frac{n^s}{n^*}.
\]

We note that \( a_\alpha(n) \neq 0 \) only when \( n = \prod_{1 \leq j \leq J} n_j \), in which case we have

\[
a_\alpha(n) = \prod_{n_j} \alpha^{\Omega(n_j)} g(n_j) b_j(n_j).
\]

Taking note also the estimation that for all integers \( i \geq 0 \),

\[
\frac{\alpha^i}{i!} \ll e^{|\alpha|},
\]
we conclude from above discussions that for all $n$,  
\begin{equation}  
a_n(n) \leq e^{\epsilon |\omega(n)|}, \quad a_k(n) = 0, \quad \text{when } n > T^{40e^{2/k}10^{-M/4}}.  
\end{equation}  

We apply the above estimations and (2.1) in (3.2) to see that for $\Re(s) \geq -1/\log T$ and $T$ large enough,  
\begin{equation}  
|N(s, \alpha)| \ll e^{\epsilon \log T \log \log T (1+O(1/\log T)) T^{40e^{2/k}10^{-M/4}(1+1/\log T)}.  
\end{equation}  

In the proof of Theorem 1.1 we need the following bounds concerning expressions involving with various $N(s, \alpha)$.

**Lemma 3.2.** With the notations as above, we have for $0 < k \leq 1/2$ and $1 \leq j \leq J$,  
\begin{equation}  
|N_j(s, k)|^{2/k} |N_j(s, k - 1)|^2 \leq |N_j(s, k)|^2 \left(1 + e^{-e^2k\alpha_j^{-3/4}}\right)^{2/k+2} \left(1 - e^{-e^2k\alpha_j^{-3/4}}\right)^{-2} + |Q_j(s, k)|^{2r_k}.  
\end{equation}  

We also have for $k > 1/2$ and $1 \leq j \leq J$,  
\begin{equation}  
|N_j(s, k - 1)|/|N_j(s, k)|^{2/k} \leq |N_j(s, k)|^2 \left(1 + e^{-e^2k\alpha_j^{-3/4}}\right)^{2/k} \left(1 - e^{-e^2k\alpha_j^{-3/4}}\right)^{-2} + |Q_j(s, k)|^{2r_k}.  
\end{equation}

Here the implied constants in (3.5) and (3.6) are absolute, and we define  
\begin{equation}  
Q_j(s, k) = \left(\frac{64 \max(2, k + 3/2)P_j(s)}{e^{2k\alpha_j^{-3/4}}}\right) |e^{2k\alpha_j^{-3/4}}|,  
\end{equation}  

with $r_k = 2 + [1/k]$ for $0 < k \leq 1/2$ and $r_k = 1 + [2k/(2k - 1)]$ for $k > 1/2$.

**Proof.** As in the proof of Lemma 3.4, we have for $|z| \leq aK/20$ with $0 < a \leq 2$,  
\begin{equation}  
\left|\sum_{r=0}^{K} \frac{e^{zr}}{r!} - e^z\right| \leq \frac{|z|^K}{K!} \leq \left(\frac{ae}{20}\right)^K.  
\end{equation}

By taking $z = \alpha P_j(s), K = [e^{2k\alpha_j^{-3/4}}]$ and $a = \min(|\alpha|, 2)$ in (3.7), we see that when $|P_j(s)| \leq [e^{2k\alpha_j^{-3/4}}]/(20(1 + |\alpha|))$,  
\begin{equation}  
N_j(s, \alpha) \leq \exp(\alpha P_j(s)) \left(1 + \exp(|\alpha P_j(s)|) \left(\frac{ae}{20}\right)^{e^{2k\alpha_j^{-3/4}}}\right) \leq \exp(\alpha P_j(s)) \left(1 + e^{-e^2k\alpha_j^{-3/4}}\right).  
\end{equation}

Similarly, we have  
\begin{equation}  
N_j(s, \alpha) \geq \exp(\alpha P_j(s)) \left(1 - \exp(|\alpha P_j(s)|) \left(\frac{ae}{20}\right)^{e^{2k\alpha_j^{-3/4}}}\right) \geq \exp(\alpha P_j(s)) \left(1 - e^{-e^2k\alpha_j^{-3/4}}\right).  
\end{equation}

We apply the above estimation to $N_j(s, k), N_j(s, k - 1)$ to see that when $0 < k \leq 1/2$ and $|P_j(s)| \leq [e^{2k\alpha_j^{-3/4}}]/60$, we have  
\begin{equation}  
|N_j(s, k)|^{2/k} |N_j(s, k - 1)|^2 \leq \exp(2kR\alpha P_j(s)) \left(1 + e^{-e^2k\alpha_j^{-3/4}}\right)^{2/k+2} \left(1 - e^{-e^2k\alpha_j^{-3/4}}\right)^{-2}.  
\end{equation}

The above arguments also imply that when $k > 1/2$ and $|P_j(s)| \leq [e^{2k\alpha_j^{-3/4}}]/60$, then  
\begin{equation}  
|N_j(s, 2k - 1)|^{2/k} \leq |N_j(s, k)|^2 \left(1 + e^{-e^2k\alpha_j^{-3/4}}\right)^{2/k} \left(1 - e^{-e^2k\alpha_j^{-3/4}}\right)^{-2}.  
\end{equation}

On the other hand, when $|P_j(s)| \geq [e^{2k\alpha_j^{-3/4}}]/60$, we have that  
\begin{equation}  
N_j(s, \alpha) \leq \sum_{r=0}^{\lfloor e^{2k\alpha_j^{-3/4}}\rfloor} \frac{|\alpha P_j(s)|^r}{r!} \leq |\lfloor |\alpha| + 1\rfloor P_j(s)| [e^{2k\alpha_j^{-3/4}}] \sum_{r=0}^{\lfloor e^{2k\alpha_j^{-3/4}}\rfloor} \frac{60 [e^{2k\alpha_j^{-3/4}}] - r}{r!} \leq \left(\frac{64(|\alpha| + 1)|P_j(s)|}{[e^{2k\alpha_j^{-3/4}}]}\right) [e^{2k\alpha_j^{-3/4}}].  
\end{equation}

We then set $\alpha = 1$ in the last expression in (3.10) to deduce that when $0 < k \leq 1/2$ and $|P_j(s)| \geq [e^{2k\alpha_j^{-3/4}}]/60$,  
\begin{equation}  
|N_j(s, k)|^{2/k} |N_j(s, k - 1)|^2 \leq |Q_j(s, k)|^{2r_k}.  
\end{equation}
Moreover, we set $\alpha = k$ in the last expression in (3.10) to deduce that when $k > 1/2$ and $|P_j(s)| \geq \left( e^{2}k^{3/4}/(10(1+|a|)) \right)$,

\[(3.12) \quad \left| N_j(s,k-1)N_j(s,k) \right|^{2k} \leq |Q_j(s,k)|^{2k}. \]

The assertion of the lemma now follows from (3.8), (3.9), (3.11) and (3.12). □

Next, we state the needed lower bounds principle of W. Heap and K. Soundararajan in [9] for our situation.

**Lemma 3.3.** With notations as above. For $0 < k \leq 1/2$, we have

\[
\sum_{0<\gamma \leq T} \zeta'(\rho)N(\rho, k-1)N(\gamma, k) \ll \left( \sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2k} \right)^{1/2} \left( \sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2} |N(\rho, k-1)|^{2} \right)^{(1-k)/2} \quad \text{(3.13)}
\]

\[
\quad \times \left( \sum_{0<\gamma \leq T} \prod_{t=1}^{\mathcal{J}} \left( |N_t(\rho, k)|^{2} + |Q_t(\rho, k)|^{2r_k} \right) \right)^{k/2}.
\]

For $k > 1/2$, we have

\[
(3.14) \quad \sum_{0<\gamma \leq T} \zeta'(\rho)N(\rho, k-1)N(\gamma, k) \leq \left( \sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2k} \right)^{1/2} \left( \sum_{0<\gamma \leq T} \prod_{t=1}^{\mathcal{J}} \left( |N_t(\rho, k)|^{2} + |Q_t(\rho, k)|^{2r_k} \right) \right)^{2k \cdot 1/k - 1} \quad \text{(3.14)}
\]

The implied constants in (3.13) and (3.14) depend on $k$ only.

**Proof.** We assume $0 < k \leq 1/2$ first and apply Hölder’s inequality to see that the left side of (3.13) is

\[
(3.15) \quad \leq \left( \sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2k} \right)^{1/2} \left( \sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2} |N(\rho, k-1)|^{2} \right)^{(1-k)/2} \left( \sum_{0<\gamma \leq T} |N(\rho, k)|^{2/k} |N(\rho, k-1)|^{2} \right)^{k/2}.
\]

We apply the estimation in (3.10) in the last sum of (3.15) above and note that, upon applying the estimation that $1 - e^{-u} \geq e^{-u/4}$ for $u > 0$, we have

\[
\prod_{1\leq j \leq \mathcal{J}} \left( 1 + e^{-e^{2k^{3/4}}/2} \right)^{2k+2} \left( 1 - e^{-e^{2k^{3/4}}/2} \right)^{-2} \leq \prod_{1\leq j \leq \mathcal{J}} \left( 1 - e^{-e^{2k^{3/4}}/2} \right)^{-2} \leq \prod_{1\leq j \leq \mathcal{J}} e^{2k^{3/4}/(e^{2})} < \infty.
\]

This leads to the estimation given in (3.13).

It remains to consider the case $k > 1/2$ and we apply Hölder’s inequality again to see that the left side of (3.14) is

\[
(3.16) \quad \leq \left( \sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2k} \right)^{1/2} \left( \sum_{0<\gamma \leq T} |N(\rho, k-1)N(\rho, k)|^{2k} \right)^{2k \cdot 1/k - 1}.
\]

We apply the estimation in (3.10) in the last sum of (3.16) above and note that the product

\[
\prod_{1\leq j \leq \mathcal{J}} \left( 1 + e^{-e^{2k^{3/4}}/2} \right)^{2k+2} \left( 1 - e^{-e^{2k^{3/4}}/2} \right)^{-2} < \infty.
\]

This leads to the estimation given in (3.14) and this completes the proof. □

It follows from the above lemma and the observation that $\gamma = 1 - \rho$ that in order to establish Theorem 1.1 it suffices to prove the following three propositions.

**Proposition 3.4.** With notations as above, we have for $k > 0$,

\[
(3.17) \quad \sum_{0<\gamma \leq T} \zeta'(\rho)N(\rho, k-1)N(1-\rho, k) \gg T(\log T)^{k^2+2}.
\]

**Proposition 3.5.** With notations as above, we have for $k > 0$,

\[
\sum_{0<\gamma \leq T} \prod_{j=1}^{\mathcal{J}} \left( |N_j(\rho, k)|^{2} + |Q_j(\rho, k)|^{2r_k} \right) \ll T(\log T)^{k^2+1}.
\]

**Proposition 3.6.** With notations as above, we have for $0 < k \leq 1/2$,

\[
\sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2} |N(\rho, k-1)|^{2} \ll T(\log T)^{k^2+3}.
\]

We shall prove the above propositions in the rest of the paper.
3.7. Proof of Proposition 3.4. The proof follows largely the arguments in Section 5 of [16]. We begin by recalling a few results on the Riemann zeta function $\zeta(s)$. Notice that $\zeta(s)$ satisfies the functional equation (see [17, Corollary 10.4]):

\begin{equation}
\zeta(s) = \chi(s)\zeta(1 - s),
\end{equation}

where

\begin{equation}
\chi(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2).
\end{equation}

Logarithmically differentiating the functional equation above implies that

\begin{equation}
\frac{\zeta'(s)}{\zeta(s)} = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(s).
\end{equation}

Moreover, we have (see [16, (8)]) uniformly for $-1 \leq \sigma \leq 2$ and $|t| \geq 1$,

\begin{equation}
\frac{\chi'}{\chi}(\sigma + it) = \frac{\chi'}{\chi}(1 - \sigma - it) = -\log\left(\frac{|t|}{2\pi}\right) + O\left(\frac{1}{|t|}\right).
\end{equation}

Note that it follows from [24, p. 108] that for every $t \geq 2$ and all nontrivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$, there exists a number $T$ satisfying $t \leq T \leq t + 1$ such that

\begin{equation}
\frac{\chi'}{\zeta}(\sigma + it) \ll (\log T)^2, \quad \text{for} \quad -1 \leq \sigma \leq 2 \quad \text{and} \quad |\gamma - T| \gg (\log T)^{-1}.
\end{equation}

Note also that by differentiating the functional equation (3.18) above, we have

\begin{equation}
\zeta'(s) = -\chi(s) \left(\zeta'(1 - s) - \frac{\chi'}{\chi}(s)\zeta(1 - s)\right).
\end{equation}

We deduce from this that if we denote the left side expression in (3.17) by $S$, then

\begin{equation}
S = -\sum_{0 < \gamma \leq T} \chi(\rho)\zeta'(1 - \rho)N(\rho, k - 1)N'(1 - \rho, k) = \frac{1}{2\pi i} \int_{C} \frac{\zeta'}{\zeta}(1 - s)\chi(s)\zeta'(1 - s)N(s, k - 1)N(1 - s, k) ds,
\end{equation}

where $C$ is the positively oriented rectangle with vertices at $1 - \kappa + i, \kappa + i, \kappa + iT$, and $1 - \kappa + iT$, and $\kappa = 1 + (\log T)^{-1}$. Here we may choose $T$ to satisfy the conditions in (3.21). We then apply (3.4), (3.21) and the estimations

\begin{equation}
\chi(s) \ll T^{1/2 - \sigma}, \quad \zeta'(1 - s) \ll T^{\sigma/2 + \epsilon}
\end{equation}

to see that the integral is bounded by $O(T^{1-\epsilon})$ on the horizontal edges of the contour.

Next, we deduce from (3.22) and the functional equation (3.18) that

\begin{equation}
\chi(s)\zeta'(1 - s) = -\zeta'(s) + \frac{\chi'}{\chi}(s)\zeta(s).
\end{equation}

Combining this with (3.19), we see that the integral on the right edge of the contour equals

\begin{equation}
S_R = \frac{1}{2\pi i} \int_{(\kappa + iT} \left(\frac{\chi'}{\chi}(s)\zeta(s) - 2\frac{\chi'}{\chi}(s)\zeta'(s) + \frac{\zeta'}{\zeta}(s)\zeta'(s)\right) N(s, k - 1)N(1 - s, k) ds.
\end{equation}

Also, the integral on the left edge of the contour equals

\begin{equation}
S_L = \frac{1}{2\pi i} \int_{(1 - \kappa + iT} \frac{\zeta'}{\zeta}(1 - s)\chi(s)\zeta'(1 - s)N(s, k - 1)N(1 - s, k) ds.
\end{equation}

we make a change of variable $s \to 1 - s$ to see that $S_L = -T_I$, where

\begin{equation}
I_L = \frac{1}{2\pi i} \int_{(\kappa + iT} \chi(1 - s)\frac{\zeta'}{\zeta}(s)\zeta'(1 - s)N(1 - s, k - 1)N(s, k) ds.
\end{equation}

We then conclude that

\begin{equation}
S = S_R - T_I + O(T^{1-\epsilon}).
\end{equation}

We now apply (3.4), (3.20) and the bounds (see [17, Corollary 1.17, Theorem 6.7]) that when $s = \kappa + it$,

\begin{equation}
\zeta(s) \ll (1 + (|t| + 4)^{1-\kappa}) \min\left(\frac{1}{|\kappa - 1|}, \log(|t| + 4)\right) + \frac{1}{s - 1} + O(1),
\end{equation}

\begin{equation}
\frac{\zeta'}{\zeta}(s) \ll \log(|t| + 4) - \frac{1}{s - 1} + O(1),
\end{equation}

and we conclude that

\begin{equation}
S = S_R - T_I + O(T^{1-\epsilon}).
\end{equation}
to see that

$$S_R = \frac{1}{2\pi i} \int_{-\infty+iT}^{\infty+iT} \left( \log^2 \left( \frac{t}{2\pi} \right) \zeta'(s) + 2 \log \left( \frac{t}{2\pi} \right) \zeta'(s) + \frac{\zeta'}{\zeta}(s) \zeta'(s) \right) \mathcal{N}(s, k-1)\mathcal{N}(1-s, k) \, ds + O(T^{1-\varepsilon}).$$

To evaluate $S_R$, we define the Dirichlet convolution $f \ast g$ for two arithmetic functions $f(k), g(k)$ by

$$f \ast g(k) = \sum_{m=k} f(m)g(n).$$

We then denote the integral given in (3.23) as a sum of three terms: $S_{R,1}, S_{R,2}$ and $S_{R,3}$, where

$$S_{R,1} = \frac{1}{2\pi i} \int_{-\infty+iT}^{\infty+iT} \log^2 \left( \frac{t}{2\pi} \right) \zeta(s)\mathcal{N}(s, k-1)\mathcal{N}(1-s, k) \, ds,$$

$$S_{R,2} = \frac{2}{2\pi i} \int_{-\infty+iT}^{\infty+iT} \log \left( \frac{t}{2\pi} \right) \zeta'(s)\mathcal{N}(s, k-1)\mathcal{N}(1-s, k) \, ds,$$

$$S_{R,3} = \frac{1}{2\pi i} \int_{-\infty+iT}^{\infty+iT} \frac{\zeta'}{\zeta}(s)\zeta'(s)\mathcal{N}(s, k-1)\mathcal{N}(1-s, k) \, ds.$$

We use the notation given in (3.24) and apply Lemma 2.3 to evaluate $S_{R,1}$ to obtain that

$$S_{R,1} = \left( \frac{1}{2\pi} \right)^T \left( \log^2 \frac{t}{2\pi} \right) dt \sum_n \frac{1 \ast a_{k-1}(n) \cdot a_k(n)}{n}$$

$$+ \left( \log^2 T + \int_1^T |(\log^2 \frac{t}{2\pi})| dt \right) \left( \sum_{n=1}^\infty \frac{(1 \ast a_{k-1})(n)^2}{n^{2\kappa-1}} \right)^{\frac{1}{2}} \left( \sum_{n=1}^\infty \frac{a_k(n)^2}{n^{1-2\kappa}} \right)^{\frac{1}{2}}.$$

Similarly, we have

$$S_{R,2} = -2 \left( \frac{1}{2\pi} \right)^T \left( \log^2 \frac{t}{2\pi} \right) dt \sum_n \frac{\log \ast a_{k-1}(n) \cdot a_k(n)}{n}$$

$$+ \left( \log T + \int_1^T |(\log \frac{t}{2\pi})| dt \right) \left( \sum_{n=1}^\infty \frac{(\log \ast a_{k-1})(n)^2}{n^{2\kappa-1}} \right)^{\frac{1}{2}} \left( \sum_{n=1}^\infty \frac{a_k(n)^2}{n^{1-2\kappa}} \right)^{\frac{1}{2}}.$$

Also,

$$S_{R,3} = \left( \frac{1}{2\pi} \right)^T \sum_n \frac{(\Lambda \ast \log) \ast a_{k-1}(n) \cdot a_k(n)}{n}$$

$$+ \left( \sum_{n=1}^\infty \frac{(\Lambda \ast \log) \ast a_{k-1}(n)^2}{n^{2\kappa-1}} \right)^{\frac{1}{2}} \left( \sum_{n=1}^\infty \frac{a_k(n)^2}{n^{1-2\kappa}} \right)^{\frac{1}{2}}.$$

We now apply the estimations given in (3.24) to see that for $T$ large enough,

$$\sum_{n=1}^\infty \frac{a_k(n)^2}{n^{1-2\kappa}} \ll e^{4\kappa \log T / \log \log T} \sum_{n \leq T^{4\kappa^{2}x_{10}^{-M/4}}} \frac{1}{n^{1-2\kappa}} \ll T^{1-\varepsilon}.$$

Moreover, using the estimation $(\Lambda \ast \log)(n) \leq \log n \sum_{d|n} \Lambda(d) = \log^2 n$, we see that

$$1 \ast a_{k-1}(n) \leq \sum_{n \leq T^{4\kappa^{2}x_{10}^{-M/4}}} a_{k-1}(n) \leq T^{1/2-\varepsilon},$$

$$\log \ast a_{k-1}(n) \leq \log n \sum_{n \leq T^{4\kappa^{2}x_{10}^{-M/4}}} a_{k-1}(n) \leq T^{1/2-\varepsilon} \log n,$$

$$\Lambda \ast \log \ast a_{k-1}(n) \leq \log^2 n \sum_{n \leq T^{4\kappa^{2}x_{10}^{-M/4}}} a_{k-1}(n) \leq T^{1/2-\varepsilon} \log^2 n.$$

It follows that

$$\sum_{n=1}^\infty \frac{1 \ast a_{k-1}(n)^2 + (\log \ast a_{k-1}(n)^2 + (\Lambda \ast \log) \ast a_{k-1}(n)^2}{n^{2\kappa-1}} \ll T^{-2\varepsilon} \sum_{n=1}^\infty \frac{\log^4 n}{n^{2\kappa-1}} \ll T^{1-\varepsilon},$$
where the last estimation above follows from the bound that \((\text{see (16)})\) uniformly for \(\sigma > 1\) and any integer \(i \geq 0\),
\[
(3.25) \quad \sum_{n=1}^{\infty} \frac{\log^i n}{n^\sigma} \ll \frac{1}{\sigma^{i+1}}.
\]

We apply the above estimations in the evaluations of \(S_{R,1}, S_{R,2}\) and \(S_{R,3}\) to see that the contributions from the error terms can be ignored. Furthermore, in the evaluation of \(S_{R,1}\), we see that, for a monic polynomial \(Q_2\) of degree 2,
\[
\frac{1}{2\pi} \int_{1}^{T} \left( \log^2 \frac{t}{2\pi} \right) dt = \frac{T}{2\pi} Q_2(\mathcal{L}) + O(1),
\]
where we denote \(\mathcal{L} = \log(T/(2\pi))\).

Next, we evaluate \(I_1\) using the notation given in (3.2) to see that
\[
(3.26) \quad S_R = \frac{T}{2\pi} \sum_{n,m} \frac{a_{k-1}(m) a_k(n)}{m n} (Q_2(\mathcal{L}) - 2Q_1(\mathcal{L}) \log n) + ((\Lambda * \log)(n)) + O(T^{1-\varepsilon}).
\]

We treat the integrals in the expressions for \(S_{R,2}\) and \(S_{R,3}\) similarly to arrive that, for monic polynomials \(Q_i, i = 1, 2\) of degree \(i\),
\[
(3.27) \quad I_L = \sum_{n} \sum_{m} \frac{a_{k-1}(m) a_k(n)}{m} \sum_k (\Lambda * \log)(k) \left( \frac{1}{2\pi i} \int_{\kappa+i}^{\kappa+iT} \chi(1-s) \left( \frac{n k}{m} \right)^{-s} ds \right).
\]

To evaluate the integral above, we need the following result from [16, Lemma 5.2].

**Lemma 3.8.** Let \(r, \kappa_0 > 0\). We have uniformly for \(\kappa_0 \leq \kappa \leq 2\) that
\[
\frac{1}{2\pi i} \int_{\kappa+i}^{\kappa+iT} \chi(1-s) js ds = I_{[0,T/2\pi]}(r) e(r) + O \left( r^{\kappa} \left( T^{\kappa-1/2} + \frac{T^{\kappa+1/2}}{|T - 2\pi r| + T^{1/2}} \right) \right),
\]
where we write \(e(x) = e^{2\pi i x}\) and we denote \(I_{[0,T/2\pi]}(r)\) for the indicator function of the interval \([0, T/2\pi]\), namely, \(I_{[0,T/2\pi]}(r) = 1\) if \(r \in [0, T/2\pi]\) and \(I_{[0,T/2\pi]}(r) = 0\) otherwise.

We apply Lemma 3.8 to see that the contribution from the error term in (3.27) is
\[
\ll \sum_m \sum_n a_{k-1}(m) a_k(n) \sum_k (\Lambda * \log)(k) \left( \frac{n k}{m} \right)^{-\kappa} T^{\kappa-1/2} + \frac{T^{\kappa+1/2}}{|T - 2\pi \left( \frac{n k}{m} \right) | + T^{1/2}}.
\]

Using (3.25), we see that
\[
\sum_{k=1}^{\infty} \frac{(\Lambda * \log)(k)}{k^\kappa} \leq \sum_{k=1}^{\infty} \frac{\log^2(k)}{k^\kappa} \ll \frac{1}{(\kappa - 1)^3}.
\]

We deduce from this and (3.25) that
\[
\ll \sum_m \sum_n a_{k-1}(m) a_k(n) \sum_k (\Lambda * \log)(k) \left( \frac{n k}{m} \right)^{-\kappa} T^{\kappa-1/2} = O(T^{1-\varepsilon}).
\]

We now use the ideas in the proof of [1, Lemma 2] to estimate
\[
\sum_m \sum_n a_{k-1}(m) a_k(n) \sum_k (\Lambda * \log)(k) \left( \frac{n k}{m} \right)^{-\kappa} \left( \frac{T^{\kappa+1/2}}{|T - 2\pi \left( \frac{n k}{m} \right) | + T^{1/2}} \right).
\]

We break up the sum into three parts. The terms with \(|T - 2\pi \left( \frac{n k}{m} \right) | > \frac{T}{2} \) contribute \(O(T^{1-\varepsilon})\) as our discussions above. The terms \(T^{1/2} \leq |T - 2\pi \left( \frac{n k}{m} \right) | \leq \frac{T}{2} \) are further divided into cases that \(T^{1/2} \leq T - 2\pi \left( \frac{n k}{m} \right) \leq \frac{T}{2} \) and \(T^{1/2} \leq 2\pi \left( \frac{n k}{m} \right) - T \leq \frac{T}{2} \). Without loss of generality, we consider the case that \(T^{1/2} \leq 2\pi \left( \frac{n k}{m} \right) - T \leq \frac{T}{2} \). This implies that \(T + T^{1/2} \leq 2\pi \left( \frac{n k}{m} \right) \leq T + \frac{T}{2} \). We then split the sum into \(\ll \log T\) sums of the shape
\[
T + P \leq 2\pi \left( \frac{n k}{m} \right) \leq T + 2P
\]
where $T^{\frac{1}{2}} \ll P \ll T$. The above implies that

\[(nk)^{-\kappa} \ll (mT)^{-\kappa}.\]

Moreover, the sum over $n, k$ ranges over an interval of length $\ll mP$, Thus, the contribution form the corresponding terms (using $a_k(n) \ll T^\kappa$, $(A + \log)(k) \ll \log^2(m(T + P)) \ll T^\kappa$, $1/([T - 2\pi (\frac{nk}{m})] + T^{1/2}) \ll 1/P$) is

\[T^\kappa \sum_m \sum_n \frac{a_k(n)}{m} mP(mT)^{-\kappa} \frac{T^{\kappa + 1/2}}{P} \ll T^{1-\varepsilon}.\]

Lastly, we consider the contributions from the terms $|T - 2\pi (\frac{nk}{m})| \leq T^{\frac{1}{2}}$ by noticing that in this case the estimation (3.28) is still valid. Moreover, the sum over $n, k$ ranges over an interval of length $\ll mT^{\frac{1}{2}}$, Thus, the contribution form the corresponding terms is

\[\ll T^\kappa \sum_m \sum_n \frac{a_k(n)}{m} mT^{\frac{1}{2}} (mT)^{-\kappa} \frac{T^{\kappa + 1/2}}{T^{\frac{1}{2}}} \ll T^{1-\varepsilon}.\]

We then conclude from the above discussions that

\[I_L = \sum_m \sum_n \frac{a_k(n)}{m} m \sum_{1 \leq k \leq \frac{mk}{m}} e\left(-\frac{kn}{m}\right) + O(T^{1-\varepsilon}).\]

We now proceed as in [16, p. 3212-3213] to see that, for a monic polynomial $P_2$ of degree 2,

\[I_L = \frac{T}{4\pi} \sum_{n,m} \frac{a_k(n)}{mn} P_2(\log \frac{mT}{2\pi n}) + O(T^{1-\varepsilon}) + O(\sum_{n,m} \frac{a_k(n)}{mn}) (m, n) (\Lambda^2(\frac{m}{m}, n) + \Lambda(\frac{m}{m}, n) \log T).\]

To estimate the last error term in (3.29), we write $d = (m, n), m = dL, n = dN$ and use the easily checked property that $a_{k-1}(mn) \leq a_k(m)a_{k-1}(n)$ to see that it is

\[\ll \sum_{d, N} \frac{a_k(d)}{dN} \sum_{L} \frac{a_{k-1}(L)}{L} (L^2 + \log L).\]

Using (2.4) and the observation from (3.3) that $a_k(L)$ is bounded when $L$ is supported on integers $L$ with $\omega(L) \leq 2$ and $a_{k-1}(L) \neq 0$ only for $L \leq T^{40\omega^2 k^2}10^{-M/4}$, we proceed as in [16, p. 3213] to see that

\[\sum_{L} \frac{a_{k-1}(L)}{L} (\Lambda^2(L) + \Lambda(L) \log T) \ll 10^{-M/4} \log^2 T.\]

We conclude from (3.28), (3.29) and the above estimation that

\[S \geq T \sum_{n,m} \frac{a_k(n)}{mn} \left(Q_2(L) - 2Q_1(L) \log n - \frac{1}{2} P_2(L - \log n) + ((A + \log)(n))
+ O(10^{-M/4} T \log^2 T \sum_{n,m} \frac{a_k(m)}{mn}) + O(T^{1-\varepsilon}).\]

We now take $M$ large enough and argue as in [16, p. 3214] to see that

\[S \gg T \log^2 T \sum_{n,m} \frac{a_k(m)}{mn}.\]

Lastly, we apply arguments used in [4, Section 4] to estimate the sums above to arrive that

\[S \gg T \log T \ll^{2+}.\]

This completes the proof of the proposition.
3.9. Proof of Proposition 3.5. Upon dividing the range of $\gamma$ into dyadic blocks and replacing $T$ by $2T$, we see that it suffices to show for large $T$,

$$
\sum_{T < \gamma \leq 2T} \prod_{j=1}^{\mathcal{J}} (|W_j(\rho, k)|^2 + |Q_j(\rho, k)|^{2r_k}) \ll T^{(\log T)^{k+1}}.
$$

We apply Lemma 2.1 to evaluate the left side expression above. In this process, we may ignore the contributions from the error terms in (2.30), using arguments similar to our treatments on various error terms in the proof of Proposition 3.4. Moreover, we may also ignore the contributions from the main terms in (2.2) from the cases $a \neq b$, since these terms are negative and we are seeking for an upper bound here. Thus, only the diagonal terms in the left side expression of (3.30) survive. Now applying (2.3), we conclude that

$$
\sum_{T < \gamma \leq 2T} \prod_{j=1}^{\mathcal{J}} (|W_j(\rho, k)|^2 + |Q_j(\rho, k)|^{2r_k}) \\
\ll T^{(\log T)} \prod_{j=1}^{\mathcal{J}} \left( \sum_{n_j} \frac{k^{2\Omega(n_j)}}{n_j g^2(n_j)} b_j(n_j) \right) \left( \frac{64 \max(2, k + 3/2)}{e^2 k^{3/4}} \right)^{2r_k} \left( \frac{(r_k e^{2k\alpha^{-3/4}})!^2}{\prod_{p|n_j} p \cdot e^{2k\alpha^{-3/4}}} \right) \sum_{n_j g^2(n_j)} \frac{1}{n_j g^2(n_j)}.
$$

The product above is analogous to the product in (4.6). Using similar estimations, we see that it is $\ll (\log T)^k$. This implies (3.30) and completes the proof of the proposition.

3.10. Proof of Proposition 3.6. Again by dividing the range of $\gamma$ into dyadic blocks and replacing $T$ by $2T$, we see that it suffices to show for large $T$,

$$
\sum_{T < \gamma \leq 2T} |\zeta'(\rho)|^2 |\mathcal{N}(\rho, k-1)|^2 \ll T^{(\log T)^{k+3}}.
$$

We split the interval $(0, T^{\alpha'})$ into disjoint subintervals $I_j = (T^{\alpha_i-1}, T^{\alpha_i})$ for $1 \leq j \leq \mathcal{J}$ and define

$$
w_j(n) = \frac{\Lambda_{\mathcal{L}}(n)}{n^{1/(\alpha_j \log T) - \log n}} \log(T^{\alpha_j}/n) \log T^{\alpha_j},
$$

where

$$
\Lambda_{\mathcal{L}}(n) = \begin{cases} 
\Lambda(n), & \text{if } n = p \text{ or if } n = p^2 \text{ and } n \leq \mathcal{L}, \\
0, & \text{otherwise}.
\end{cases}
$$

For $1 \leq l \leq j \leq \mathcal{J}$, we define

$$
G_{l,j} = \Re \sum_{n \in I_l} w_j(n) n^{-it}.
$$

Note that our definition of $G_{l,j}$ is slightly different from that in (12), due to our definition on $\mathcal{L}$. However, we notice the bounds

$$
w_j(p) \leq 1, \quad w_j(p^2) \leq \frac{1}{2},
$$

so that it follows from the above and Lemma 2.3 that

$$
\sum_{\mathcal{L} \leq p \leq \log T} \frac{w_j(p^2)}{p} = O(1).
$$

We then deduce from (12) (4.1) the following upper bound for $\log |\zeta'(\rho)|$, which says that for any $1 \leq j \leq \mathcal{J}$,

$$
\log |\zeta'(\rho)| \ll \sum_{l=1}^{j} G_{l,j} + \log \log T + \alpha_j^{-1} + O(1).
$$

We also define the following sets:

$$
S(0) = \{ T \leq \gamma \leq 2T : |G_{1,l}(\gamma)| > \alpha_-^{-3/4} \text{ for some } 1 \leq l \leq \mathcal{J} \},
$$

$$
S(j) = \{ T \leq \gamma \leq 2T : |G_{m,l}(\gamma)| \leq \alpha_-^{-3/4} \forall 1 \leq m \leq j, \forall m \leq l \leq \mathcal{J}, \text{ but } |G_{j+1,l}(\gamma)| > \alpha_+^{-3/4} \text{ for some } j+1 \leq l \leq \mathcal{J}, \} \quad 1 \leq j \leq \mathcal{J},
$$

$$
S(\mathcal{J}) = \{ T \leq \gamma \leq 2T : |G_{m,l}(\gamma)| \leq \alpha_-^{-3/4} \forall 1 \leq m \leq \mathcal{J} \}.
$$
so that

$$\{ T < \gamma \leq 2T \} = \bigcup_{j=0}^{J} S(j).$$

It follows that

$$\sum_{T < \gamma \leq 2T} |\zeta'(\rho)|^2 |N(\rho, k - 1)|^2 = \sum_{j=0}^{J} \sum_{\gamma \in S(j)} |\zeta'(\rho)|^2 |N(\rho, k - 1)|^2.$$

We note from [12, Lemma 5.5] and (2.3) that we have

$$\text{meas}(S(0)) \ll (\log T)^{e^{(\log \log T)^2/10}}.$$

We then deduce via the Cauchy-Schwarz inequality that

$$\sum_{\gamma \in S(0)} |\zeta'(\rho)|^2 |N(\rho, k - 1)|^2 \leq \left( \text{meas}(S(0)) \right)^{1/4} \left( \sum_{T < \gamma \leq 2T} |\zeta'(\rho)|^8 \right)^{1/4} \left( \sum_{T < \gamma \leq 2T} |N(\rho, k - 1)|^4 \right)^{1/2}.$$

Similar to the proof of Proposition 3.5, we have that

$$\sum_{T < \gamma \leq 2T} |N(\rho, k - 1)|^4 \ll T(\log T)^{(2k-2)^2+1}.$$

Also, applying (1.3) with \( k = 4, \varepsilon = 1 \), we see that

$$\sum_{T < \gamma \leq 2T} |\zeta'(\rho)|^8 \leq T(\log T)^{25}.$$

We use the bounds given in (3.34), (3.37) and (3.38) in (3.36) to conclude that

$$\sum_{\gamma \in S(0)} |\zeta'(\rho)|^2 |N(\rho, k - 1)|^2 \ll (\log T)^{k^2+3}.$$

The above estimation implies that it remains to consider the cases \( j \geq 1 \) in (3.34). Without loss of generality, we may assume that \( 1 \leq j \leq J - 1 \) here. When \( \gamma \in S(j) \), we deduce from (3.33) that

$$\sum_{\gamma \in S(j)} |\zeta'(\rho)|^2 |N(\rho, k - 1)|^2 \ll (\log T)^2 \exp \left( \frac{2}{\alpha_j} \right) \sum_{\gamma \in S(j)} \exp \left( 2 \sum_{l=1}^{j} G_{l,j}(\gamma) \right) |N(\rho, k - 1)|^2.$$

As we have \( G_{i,j} \leq \alpha_{i}^{-3/4} \) when \( \gamma \in S(j) \), we apply [12, Lemma 5.2] to see that

$$\exp \left( 2 \sum_{l=1}^{j} G_{l,j}(\gamma) \right) \ll \prod_{l=1}^{j} E_{e^{2\alpha_{i}^{-3/4}}(G_{i,j}(\gamma))}.$$

We then deduce from the description on \( S(j) \) that when \( j \geq 1 \),

$$\sum_{\gamma \in S(j)} |\zeta'(\rho)|^2 |N(\rho, k - 1)|^2$$

$$\ll (\log T)^2 \exp \left( \frac{2}{\alpha_j} \right) \sum_{m=j+1}^{T} \sum_{\gamma \in S(j)} \exp \left( \sum_{l=1}^{j} G_{l,j}(\gamma) \right) |N(\rho, k - 1)|^2 \left( \alpha_{j+1}^{3/4} G_{j+1,m}(\gamma) \right)^{2[1/(10\alpha_{j+1})]}$$

$$\ll (\log T)^2 \exp \left( \frac{2}{\alpha_j} \right) \sum_{m=j+1}^{T} \sum_{\gamma \in S(j)} \prod_{l=1}^{j} E_{e^{2\alpha_{i}^{-3/4}}(G_{i,j}(\gamma))} |E_{e^{2\alpha_{i}^{-3/4}}((k-1)P_{j+1}(\rho))}|$$

$$\times |E_{e^{2\alpha_{i}^{-3/4}}((k-1)P_{j+1}(\rho))}|^{2[1/(10\alpha_{j+1})]} \prod_{n=j+2}^{T} |E_{e^{2\alpha_{n}^{-3/4}}((2k-2)P_{n}(\rho))}|.$$

SHARP LOWER BOUNDS FOR MOMENTS OF \( \zeta'(\rho) \)
Note that
\[
\sum_{\gamma \in S_j} \prod_{i=1}^{J} E_{e^{2k\alpha_i^{-1/4}}}^{j} (G_{i,j} (\gamma)) \left| E_{e^{2k\alpha_i^{-1/4}}}^{j} ((k - 1) \mathcal{P}_l (\rho))^2 \right|
\times |E_{e^{2k\alpha_i^{-1/4}}}^{j} ((k - 1) \mathcal{P}_l (\rho))^2 (\alpha_{t+1}^{-4/3} G_{j+1, m}(\gamma))^{2[1/(10\alpha_{t+1})]} | \prod_{n=j+2}^{J} |E_{e^{2k\alpha_i^{-1/4}}}^{j} ((2k - 2) \mathcal{P}_n (\rho))^2 |
\leq \sum_{T < \gamma \leq 2T} \prod_{l=1}^{J} E_{e^{2k\alpha_i^{-1/4}}}^{j} (G_{l,j} (\gamma)) \left| E_{e^{2k\alpha_i^{-1/4}}}^{j} ((k - 1) \mathcal{P}_l (\rho))^2 \right|
\times |E_{e^{2k\alpha_i^{-1/4}}}^{j} ((k - 1) \mathcal{P}_l (\rho))^2 (\alpha_{t+1}^{-4/3} G_{j+1, m}(\gamma))^{2[1/(10\alpha_{t+1})]} | \prod_{n=j+2}^{J} |E_{e^{2k\alpha_i^{-1/4}}}^{j} ((2k - 2) \mathcal{P}_n (\rho))^2 |.
\]

(3.41)

We shall apply Lemma [2.1] to evaluate the last sum above. As in the case for the proof of Proposition 3.4, we may only focus on the main term in the process. To facilitate our evaluation of the last sum above, we follow the treatments in [12] by introducing a sequence of independent random variables \{X_p\} such that each \(X_p\) is uniformly distributed on the upper half plane in the complex plane. We also define
\[
X_n = X_{h_1} \cdots X_{h_r}
\]
for \(n = p_{h_1} \cdots p_{h_r}\) so that \(X_n\) is a random completely multiplicative function. We then define random models \(G_{l,j}(X)\) for \(1 \leq l \leq j \leq J\) by
\[
G_{l,j}(X) = \Re \sum_{n \in I_l} \frac{w_j(n)}{\sqrt{n}} X_n, \quad \mathcal{P}_j(X) = \sum_{p \in I_j} \frac{1}{\sqrt{p}} X_p.
\]

Similar to [12] Lemma 5.3, we have under RH and other than a negligible error term, for \(1 \leq l \leq j \leq J\) and non-negative integers \(n_{l,1}, n_{l,2}, n_{l,3}\) satisfying \(\max(n_{l,1}/2, n_{l,2}, n_{l,3}) \leq \max([e^{2k\alpha_l^{-1/4}}, 2[1/(10\alpha_{l+1})]])\),
\[
\sum_{T < \gamma \leq 2T} \prod_{l=1}^{J} G_{l,j}^{n_{l,1}}(\gamma) \mathcal{P}_l^{n_{l,2}}(\rho)^{\mathcal{P}_l^{n_{l,3}}(\rho)} \leq N(T, 2T) \mathbb{E} \left( \prod_{l=1}^{J} G_{l,j}^{n_{l,1}}(X) \mathcal{P}_l^{n_{l,2}}(X)^{\mathcal{P}_l^{n_{l,3}}(X)} \right).
\]

We now proceed as in the proof of [12] Lemma 5.5 to see that, under RH and other than a negligible error term, the last sum in (3.41) is
\[
\ll \left( \alpha_{t+1}^{-4/3} \right)^{2[1/(10\alpha_{l+1})]} N(T, 2T) \mathbb{E} \left( G_{j+1, m}(X)^{2[1/(10\alpha_{l+1})]} \exp \left( 2 \sum_{l=1}^{J} G_{l,j}(X) + 2(k - 1) \sum_{n=1}^{J} \Re \mathcal{P}_n (X) \right) \right) \]
\[
= \left( \alpha_{t+1}^{-4/3} \right)^{2[1/(10\alpha_{l+1})]} N(T, 2T) \mathbb{E} \left( \exp \left( 2G_{l,j}(X) + 2(k - 1) \Re \mathcal{P}_l (X) \right) \right) \prod_{l=2}^{J} \mathbb{E} \left( \exp \left( 2G_{l,j}(X) + 2(k - 1) \Re \mathcal{P}_l (X) \right) \right) \]
\[
\times \left( \alpha_{l+1}^{-4/3} \right)^{2[1/(10\alpha_{l+1})]} \exp \left( 2(k - 1) \Re \mathcal{P}_{j+1}(X) \right) \right) \times \prod_{l=j+2}^{J} \mathbb{E} \left( \exp \left( 2(k - 1) \Re \mathcal{P}_l (X) \right) \right).
\]

Similar to the evaluation done on [12] p. 492, we see that
\[
\prod_{l=2}^{J} \mathbb{E} \left( \exp \left( 2G_{l,j}(X) + 2(k - 1) \Re \mathcal{P}_l (X) \right) \right) = \prod_{l=2}^{J} \prod_{p \in I_l} I_0 \left( \frac{2w_j(p)}{\sqrt{p}} + \frac{2(k - 1)}{\sqrt{p}} \right),
\]
\[
\prod_{l=j+2}^{J} \mathbb{E} \left( \exp \left( 2(k - 1) \Re \mathcal{P}_l (X) \right) \right) = \prod_{l=j+2}^{J} \prod_{p \in I_l} I_0 \left( 2(k - 1) \frac{1}{\sqrt{p}} \right),
\]

(3.42)

where \(I_0(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{(n!)^2} \) is the modified Bessel function of the first kind.
Next, using the arguments similar to those in [12 (6.3)], we have that
\[
\mathbb{E}\left( \exp \left( 2G_{1,j}(X) + 2(k-1)\Re P_1(X) \right) \right) = \prod_{p \in I_1, p > \log T} I_0 \left( \frac{2w_j(p)}{\sqrt{p}} + \frac{2(k-1)}{\sqrt{p}} \right) \times \mathbb{E}\left( \exp \left( \sum_{p \leq \log T} \frac{2w_j(p)}{\sqrt{p}} \right) \Re P_1 + \sum_{p \leq \log T} \frac{4w_j(p^2)}{\sqrt{p}} \Re X_1^2 - \sum_{p \leq \log T} \frac{4w_j(p^2)}{\sqrt{p}} \right) \).
\]

We apply the bounds given in [332] and proceed as in [12, p. 492-493] to deduce that the last expression above is
\[
\leq \prod_{p \in I_1} I_0 \left( \frac{2w_j(p)}{\sqrt{p}} + \frac{2(k-1)}{\sqrt{p}} \right).
\]

Furthermore, we notice that
\[
\mathbb{E}\left( G_{j+1,m}(X)^{2[1/(10\alpha_j+1)]} \right) \exp \left( 2(k-1)\Re P_{j+1}(X) \right) = \sum_{n=0}^{\infty} \mathbb{E}\left( G_{j+1,m}(X)^{2[1/(10\alpha_j+1)]} \right) \frac{(2(k-1)\Re P_{j+1}(X))^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \mathbb{E}\left( G_{j+1,m}(X)^{2[1/(10\alpha_j+1)]} \right) \frac{(2(k-1)\Re P_{j+1}(X))^{2n}}{(2n)!}
\]
\[
\leq \sum_{n=0}^{\infty} \left( \frac{2\Re P_{j+1}(X)^{2[1/(10\alpha_j+1)]} + 2n}{(2n)!} \right),
\]

where the last estimation above follows from the observation that, upon using [332] and noting that \(0 < k \leq 1/2\), we have for any integer \(n \geq 0\),
\[
\mathbb{E}\left( G_{j+1,m}(X)^{2[1/(10\alpha_j+1)]} \right) \frac{(2(k-1)\Re P_{j+1}(X))^{2n}}{(2n)!} \leq \mathbb{E}\left( \frac{(2\Re P_{j+1}(X))^{2[1/(10\alpha_j+1)]} + 2n}{(2n)!} \right).
\]

Observe that for any positive integer \(m\),
\[
\mathbb{E}[1/(\Re P_1)^m] = \left\{ \begin{array}{ll} 2^m & \text{if } m = 2h, \\ 0 & \text{if } m \text{ is odd.} \end{array} \right.
\]

It follows from this that if we denote \( \{p | p \in I_{j+1} \} = \{p_1, \cdots, p_s\} \), then we have for any positive integer \(m\),
\[
\mathbb{E}\left( (2\Re P_{j+1}(X))^{2m} \right) = \sum_{m_1, m_2, \cdots, m_s \geq 0} \binom{2m}{2m_1, \cdots, 2m_s} \prod_{i=1}^{s} \binom{2m_i}{m_i} \mathbb{E}[\Re P_1^{2m_i} (\Re P_1)^{2m_i}]
\]
\[
= 2^{-2m} \sum_{m_1, m_2, \cdots, m_s \geq 0} \binom{2m}{2m_1, \cdots, 2m_s} \prod_{i=1}^{s} \binom{2m_i}{m_i} \left( \frac{4}{p_i} \right)^{m_i}
\]
\[
\leq \frac{(2m)!}{2^{2m} m!} \left( \sum_{p \in I_{j+1}} \frac{4}{p} \right)^m.
\]

We apply the above estimation to the last expression in (3.34) to deduce that
\[
\mathbb{E}\left( G_{j+1,m}(X)^{2[1/(10\alpha_j+1)]} \right) \frac{(2(k-1)\Re P_{j+1}(X))^{2n}}{(2n)!} \leq \frac{(2[1/(10\alpha_j+1)] + 2n)!}{2^{2[1/(10\alpha_j+1)] + 2n} (2n)!} \left( \sum_{p \in I_{j+1}} \frac{4}{p} \right)^{[1/10\alpha_j+1] + n}.
\]

Using the bounds
\[
\left( \frac{n}{e} \right)^n \leq n! \leq n \left( \frac{n}{e} \right)^n,
\]

(3.45) and (3.46) hold.
we see that the last expression in (3.45) is
$$
(3.47) \quad \ll \left( \frac{2[1/(10\alpha_{j+1})] + 2n}{(2n)!} \right) \left( \frac{1/(10\alpha_{j+1})}{e} \right) \sum_{T^{\alpha_j} < p \leq T^{\alpha_{j+1}}} \frac{4}{p} \left[ 1/(10\alpha_{j+1}) \right] + n.
$$

Notice that when \( n \geq 2[1/(10\alpha_{j+1})] \), the expression above is
$$
\leq \frac{3n}{(2n)!} \left( \frac{3n}{2e} \right) \sum_{T^{\alpha_j} < p \leq T^{\alpha_{j+1}}} \frac{4}{p} \leq 3n \left( \frac{2n}{e} \right)^{3n/2},
$$
where the last estimation above follows from (3.50) and (2.3). As the sum of the last term above over \( n \) is convergent, we see that the contribution of these terms to the last sum of (3.43) is \( O(1) \) and may be ignored.

Now for \( \frac{1}{4}[1/(10\alpha_{j+1})] \leq n < 2[1/(10\alpha_{j+1})] \), we apply (3.32) to see that the expression in (3.41) is
$$
\ll \frac{6[1/(10\alpha_{j+1})]}{(2n)!} \left( \frac{6[1/(10\alpha_{j+1})]}{e} \right) \sum_{T^{\alpha_j} < p \leq T^{\alpha_{j+1}}} \frac{4}{p} \left[ 1/(10\alpha_{j+1}) \right]^n.
$$

Applying (3.40), again we see that the sum of the last term above over \( n \) is convergent so that the contribution of these terms to the last sum of (3.43) is
$$
(3.48) \quad \ll \left( \frac{120[1/(10\alpha_{j+1})]}{4e} \right)^{1/(10\alpha_{j+1})}.
$$

Lastly, for \( n < \frac{1}{4}[1/(10\alpha_{j+1})] \), we apply (3.31) and (3.40) one more time to see that the expression in (3.47) is
$$
\leq \frac{5[1/(10\alpha_{j+1})]/2}{(2n)!} \left( \frac{5[1/(10\alpha_{j+1})]}{4e} \right) \sum_{T^{\alpha_j} < p \leq T^{\alpha_{j+1}}} \frac{4}{p} \left[ 1/(10\alpha_{j+1}) \right]^{1/4}
\leq \frac{5[1/(10\alpha_{j+1})]/2}{(2n)!} \left( \frac{200[1/(10\alpha_{j+1})]}{4e} \right)^{5[1/(10\alpha_{j+1})]/4}.
$$

Upon summing over \( n \), we see that the contribution of these terms to the last sum of (3.43) is
$$
(3.49) \quad \ll \left( \frac{200[1/(10\alpha_{j+1})]}{4e} \right)^{5[1/(10\alpha_{j+1})]/4}.
$$

We apply the bounds (3.48) and (3.49) in (3.43) to conclude that
$$
\mathbb{E} \left( G_{j+1,m}(X)^2 [1/(10\alpha_{j+1})] \exp \left( 2(k - 1) \Re P_{j+1}(X) \right) \right) \ll \left( \frac{120[1/(10\alpha_{j+1})]}{e} \right)^{5[1/(10\alpha_{j+1})]/4}.
$$

We combine (3.42), (3.48) and the last estimation above to see that, via using (3.32), the last expression in (3.34) is
$$
\ll \left( \frac{3}{\alpha_{j+1}} \right)^{2[1/(10\alpha_{j+1})]} \left( \frac{120[1/(10\alpha_{j+1})]}{e} \right)^{5[1/(10\alpha_{j+1})]/4} \left( N(T, 2T) \prod_{l=1}^\mathcal{J} \prod_{l=1}^\mathcal{J} I_0 \left( \frac{2k(p)}{\sqrt{p}} + \frac{2(k - 1)}{\sqrt{p}} \right) \right)
\ll \left( \frac{1}{\alpha_{j+1}} \right)^{-[1/(10\alpha_{j+1})]/8} \left( N(T, 2T) \prod_{l=1}^\mathcal{J} \prod_{l=1}^\mathcal{J} I_0 \left( \frac{2k}{\sqrt{p}} \right) \right).
$$

Using further the estimation that \( I_0(2x) \leq e^{x^2} \), we deduce from the above that the last expression in (3.44) is
$$
(3.50) \quad \ll \left( \frac{1}{\alpha_{j+1}} \right)^{-[1/(10\alpha_{j+1})]/8} \left( N(T, 2T) \exp \left( \sum_{p \leq T^{\alpha_j}} \frac{k^2}{p} \right) \right) \ll \left( \frac{1}{\alpha_{j+1}} \right)^{-[1/(10\alpha_{j+1})]/8} T(\log T)^{k^2+1},
$$
where the last estimation above follows from (3.50) and (2.3).
As $20/\alpha_{j+1} = 1/\alpha_j$, we conclude from (3.31), (3.30), (3.11), (3.10) and (3.8) and that
\[
\sum_{T<\gamma \leq 2T} |\zeta'(\rho)|^2 |N(\rho, k-1)|^2 \leq \sum_{j=1}^{J} \sum_{\gamma \in S(j)} |\zeta'(\rho)|^2 |N(\rho, k-1)|^2 + T(\log T)^{k^2+3}
\]
\[
\leq T(\log T)^{k^2+3} \sum_{j=1}^{J} \left( \frac{1}{\alpha_{j+1}} \right)^{\left[ 1/(10\alpha_{j+1}) \right]/10}.
\]
As the sum of the right side expression over $j$ converges, we see that the above estimation implies (3.31) and this completes the proof of Proposition 3.6.

**Acknowledgments.** P. G. is supported in part by NSFC grant 11871082.

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