LEFT INVARIANT CONTACT STRUCTURES ON LIE GROUPS

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Abstract

A result from Gromov ensures the existence of a contact structure on any connected non-compact odd dimensional Lie group. But in general such structures are not invariant under left translations. The problem of finding which Lie groups admit a left invariant contact structure (contact Lie groups), is then still wide open. We perform a ‘contactization’ method to construct, in every odd dimension, many contact Lie groups with a discrete centre and discuss some applications and consequences of such a construction. We give classification results in low dimensions. In any dimension $7$, there are infinitely many locally non-isomorphic solvable contact Lie groups. We also classify contact Lie groups having a prescribed Riemannian or semi-Riemannian structure and derive obstructions results.

1 Introduction-Summary

A contact form on a manifold $M^{2n+1}$ is a differential 1-form such that $(\omega_f)^n \wedge \theta \neq 0$ pointwise over $M$. The kernel $\ker f = \{0\}$ then defines a maximally non-integrable smooth field of tangent hyperplanes on $M^{2n+1}$. A fundamental question about contact structures is their existence on a given manifold. Every closed oriented 3-manifold admits a contact structure (J. Martinet 1971, see also [40]). The question remains open in higher dimensions, some answers have been obtained using surgery-like tools mainly (see e.g. [41], [17], ...)

According to M. Gromov [23], there is a contact structure on every odd dimensional connected non-compact Lie group. Still, in general, such contact structures are not invariant under left translations (left invariant) of the Lie group. Furthermore, the methods used by Gromov in his proofs do not, a priori, involve any kind of invariance.

The aim of this paper is the study of these Lie groups having a left invariant contact form, also termed contact Lie groups, in the sequel. Beyond the geometric interest, contact Lie groups appear in a natural way in all areas using contact Geometry or Topology (for these areas, see e.g. [2], [5], [24], [29], ... and excellent review-like sources by Lutz [32] and Geiges [16]).

The question whether symplectic compact manifolds with a boundary of contact type, admit a connected boundary, as it is the case for compact complex manifolds with strictly pseudo-convex boundary, was raised up by E. Calabi. In [15], Geiges uses some 3-dimensional contact Lie groups to build up counterexamples to such a question. The constructions in [15] can be generalised in any odd dimension to unimodular contact Lie groups admitting a lattice.

While Lie groups with left invariant symplectic structures are widely studied by a great number of authors (amongst which A. Lichnerowicz; E.B. Vinberg; I.I. Pjateckiï-Šapiro; S. G. Gindikin; A. Medina; Ph. Revoy; M. Goze, J. Dorfmeister; K. Nakajima; etc.), contact Lie groups still remain quite unexplored. So far, the main known examples of contact Lie groups in dimension $> 3$, have a (non-discrete) centre of dimension 1. Among other results in [20], the authors solved the existence question for left invariant contact forms on filiform Lie groups (i.e. with a nilpotent Lie algebra $\mathfrak{g}$ whose nilindex equals $\dim(\mathfrak{g}) - 1$), and classify all contact structures in such Lie groups.

Some earlier results of the present work presented in [11] have been applied by D. Iglesias and J. C. Marrero in [29], to get some of their nice results about generalized Lie bialgebras and Jacobi Structures.
In Section 3 we give a construction result that allows to get many contact Lie groups (and especially those with a discrete center) in any odd dimension (Theorem 1). We discuss some applications such as the construction of some special principle fibre bundles (Remark 4), the construction of contact Lie groups with left invariant Einstein metrics (Theorem 6), give several special examples (Corollaries 2, 3, 4), and extend such a result to all Lie groups with bi-invariant Riemannian or semi-Riemannian structure (Theorem 7). For the existence of left invariant contact structures on odd dimensional Lie groups G, the non-degeneracy of the Killing form (Theorem 5 of [6]) and the dimension of the centre of G (it is readily checked that the center should have dimension 1) are the main obstructions so far known to the author.

In Section 3 using some known results from Riemannian Geometry, we classify contact Lie groups (via their Lie algebras) having some prescribed Riemannian or semi-Riemannian structure, give properties and derive some obstructions to the existence of left invariant contact structures on Lie groups as well. For the present purposes, we only need to use the presence of left invariant contact and some given Riemannian structures on the same Lie group. The actual behaviour of such structures with respect to one another as in [5], will be studied in a subsequent work [13]. A Riemannian or semi-Riemannian structure in a Lie group is said to be bi-invariant if it is invariant under both left and right translations. The Killing forms of semi-simple Lie groups are examples of such bi-invariant structures.

In Theorem 5 of [6], W.M. Boothby and H.C. Wang proved, by generalising a result from J.W. Gray [21], that the only semi-simple Lie groups that carry a left invariant contact structure are those which are locally isomorphic to \( SL(2) \) or to \( SO(3) \). We extend such a result to all Lie groups with bi-invariant Riemannian or semi-Riemannian structures (Theorem 3).

In his main result of [4] (see also [5]), D.E. Blair proved that a flat Riemannian metric in a contact manifold \( M \) of dimension 5, cannot be a contact metric structure (see "Some preliminaries and notations" for the definition). We prove that in the case of contact Lie groups of dimension 5, there is no flat left invariant Riemannian metric at all, even if such a metric has nothing to do with the given contact structure (Theorem 5).

We give a characterisation of contact Lie groups which have a left invariant Riemannian metric of negative sectional curvature (Proposition 5). We also show that if \( \dim(G) = 5 \), there is no left invariant contact structure in any of the following cases: (a) \( G \) has the property that every left invariant metric has a sectional curvature of constant sign (Proposition 4), (b) \( G \) is a negatively curved 2-step solvable Lie group (Corollary 4), (c) \( G \) has a left invariant Riemannian metric with negative sectional curvature, such that the Levi-Civita connection \( \nabla \) and the curvature tensor \( R \) satisfy \( \nabla R = 0 \) (Corollary 3). Proposition 5 proves that there is no left invariant K-contact structure (if \( \dim(G) > 3 \)) whose underlying Riemannian metric has a Ricci curvature of constant sign. In particular, there is no K-contact-Einstein, a fortiori no Sasaki-Einstein, left invariant structures on Lie groups of dimension 5.

Section 3 is devoted to the classification problem in dimensions 7. We also exhibit an infinite family of non-isomorphic contact Lie algebras in dimension 7 and hence in any dimension \( 2n + 1 = 7 \).

**Some preliminaries and notations.** Throughout this paper, \( \langle \cdot , \cdot \rangle \) always stands for the duality pairing between a vector space and its dual, unless otherwise stated. Let \( G \) be a Lie group, its unit, and \( G \) its Lie algebra identified with the tangent space \( T_eG \) to \( G \) at \( e \). If \( x \in G \), let \( x^t \) stand for the left invariant vector field on \( G \) with value \( x = x^t \) at \( e \). If \( G \) has dimension \( 2n + 1 \), a left invariant differential 1-form \( \omega \) on \( G \) is a contact form if its de Rham differential \( \partial^+ \) caps up, together with \( \omega \), to a volume form \( \omega^+ \). This is equivalent to \( \partial^+ \) a volume form in \( G \), where \( \omega = \partial^+ \) and \( \omega \otimes (x^t; y) = \omega((x^t; y)) \). In this case \( \langle \omega; \omega \rangle \) (resp. \( \langle \omega; 0 \rangle \)) is termed a contact Lie group (resp. algebra). The Reeb vector field is the unique vector field \( \dot{x} \) satisfying \( \partial^+ \dot{x} = 0 \), \( 8x^0 \) and \( \dot{x}(\partial^+) = 1 \). From now on, we will also usually write \( \partial^+ \) instead of \( \partial^+ \). Every 3-dimensional
nonabelian Lie group is a contact Lie group, except the one (unique, up to a local isomorphism) all of whose left invariant Riemannian metrics have sectional curvature of constant sign (Proposition 6). Every Heisenberg Lie group $H_{2n+1}$ is a contact Lie group.

A contact metric structure on a contact manifold $(\mathcal{M};\alpha)$ is given by a Riemannian metric $g$ and a field $\xi$ of endomorphisms of its tangent bundle such that for all vector fields $X;Y$,

$$\xi(X;Y) = g(\xi X;Y) \quad \text{and} \quad g(X;\xi Y) = g(\xi X;Y)$$

(1)

(see e.g. [5]). If in addition the Reeb vector field is a Killing vector field (ie, generates a group of isometries) with respect to $g$, then $(\mathcal{M};\xi)$ is termed a K-contact contact Lie group.

**Lemma 1.** (Lemma 5.2.0.1 of [11]). If $(\mathcal{M};\alpha)$ is a contact form in a Lie algebra $\mathcal{G}$, with Reeb vector field, then its kernel (nullspace) $\ker(\alpha)$ is not a Lie subalgebra of $\mathcal{G}$, whereas the radical (nullspace) $\text{Rad}(\ker(\alpha)) = \mathbb{R}$ of $\ker(\alpha)$ is a reductive subalgebra of $\mathcal{G}$.

A symplectic Lie group $(\mathcal{G};\omega)$ is a Lie group $\mathcal{G}$ together with a left invariant symplectic form $\omega$ (see [9], [10], [30], [31], ...). It is well known that a symplectic Lie group carries a left invariant flat affine structure (see e.g. [9], [10], [12]). But this is no longer true for contact Lie groups such as $SU(2)$, $\mathbb{R}^n \circ SL(n;\mathbb{R})$ and even for nilpotent ones, as shown by the example of Y. Benoist in 1992.

The ‘Classical’ Contactization is obtained as follows. From a symplectic Lie algebra $(\mathcal{H};\omega)$, perform the central extension $\mathcal{G} = \mathcal{H} \circ \mathbb{R}$, using the 2-cocycle $\omega$. Then $\mathcal{G}$ is a contact Lie algebra with center $\mathbb{Z}(\mathcal{G}) = \mathbb{R}$. The converse is easy to see as stated below.

**Lemma 2.** (Lemma 5.2.0.3 of [11]). A contact Lie algebra with nontrivial center is a central extension $\mathcal{H} \circ \mathbb{R}$ of a symplectic Lie algebra $(\mathcal{G};\omega)$ using the non-degenerate 2-cocycle $\omega$.

If $\mathcal{G}$ is the differential $\mathcal{G}^+ = \mathcal{G}^+$ of a left invariant differential 1-form $\omega$, then $(\mathcal{G};\omega)$ (resp. $(\mathcal{G};\omega)$) is an exact symplectic (or a Frobenius) Lie group (resp. Lie algebra).

A Lie algebra $\mathcal{G}$ is said to be decomposable, if it is a direct sum $\mathcal{G} = \mathcal{A}_1 \oplus \mathcal{A}_2$ of two ideals $\mathcal{A}_1$ and $\mathcal{A}_2$. It is readily checked that a decomposable Lie algebra $\mathcal{G} = \mathcal{A}_1 \oplus \mathcal{A}_2$ is contact if and only if $\mathcal{A}_1$ is contact and $\mathcal{A}_2$ exact symplectic or vice versa. Exact symplectic Lie algebras of dimension $6$ are all well known, a list of those in dimension $4$ is quoted e.g. in [11]. A particular family of Frobenius Lie algebras, the so-called j-algebras, plays a central role in the study of the homogeneous Kähler Manifolds and in particular homogeneous bounded domains [14]. If $n$ invertible matrices act on the space $M_{n,p}$ of $n \times p$ matrices by ordinary left multiplication of matrices. If $p$ divides $n$, the resulting semi-direct product $M_{n,p} \circ GL(n;\mathbb{R})$ is a (non-solvable) Frobenius Lie group with Lie algebra $M_{n,p} \circ GL(n;\mathbb{R})$ [38]. In particular, if $p = 1$ the group $Aff(\mathbb{R}^n)$ of affine motions of $\mathbb{R}^n$ is a Frobenius Lie group (see also [7]). In [19], one can find infinite $(n-1)$-parameter families of nonisomorphic solvable exact symplectic Lie algebras (in dimension $2n+2$), obtained as 1-dimensional extensions of the Heisenberg Lie algebras.

## 2 Construction of contact Lie groups.

The construction and classification of contact manifolds is a basic problem in differential topology (see e.g. Weinstein [41]). The main purpose here, is to perform a contactization method to construct contact Lie groups, from exact symplectic Lie groups. In particular, we obtain contact Lie groups with discrete center, while the classical contactization gives only those contact Lie groups with a 1-dimensional center. The inverse process of building exact symplectic Lie groups from contact Lie groups, arises also naturally. We will work locally, i.e at the Lie algebra level, the results for Lie groups are obtained by left-translating those structures about the corresponding Lie groups. Given an exact symplectic Lie algebra $(\mathcal{H};\omega)$, we will find all contact Lie algebras $(\mathcal{G};\omega)$ containing $\mathcal{H}$ as a codimension 1 subalgebra such
that \( i = \text{dom} : H \) is the natural inclusion. We first solve the following embedding problem for Lie algebras: given a Lie algebra \( H \), find all Lie algebras \( G \) containing \( H \) as a codimension 1 subalgebra. Choose a line \( \mathbb{R} e_o \) complementary to \( H \) so that, as a vector space, \( G \) can be written as \( G = H \oplus \mathbb{R} e_o \).

In the following lemma, \( i \) stands for the (Chevalley-Eilenberg) coboundary operator associated to the adjoint action of Lie algebras. In particular, if \( \text{dom} \) is a linear transformation on \( H \), then \( 2 \) \( H \text{dom} (x^2 H; H) \) is given by \( (x; y) = (x; y) + [x; e_o] \). The corresponding left invariant vector field \( x^e_o \) along \( x^e_o \) satisfies \( \text{dom} (x^e_o) = 1 \).

**Lemma 3.** A Lie algebra \( G \supseteq H \) \( \cap e_o \) containing \( H \) as a codimension 1 subalgebra consists of a couple \( (H; \mathfrak{g}) \) such that \( 2 \) \( H \) \( H \) \( 2 \) \( H \) is again viewed as a linear subalgebra. Choose a line \( \mathbb{R} e_o \) complementary to \( H \). There exists \( \mathfrak{g} \) \( 2 \) \( \mathfrak{e} \mathfrak{n} \mathfrak{d} (H) \) \( H \) such that the Lie bracket reads as in (3). The Jacobi identity gives the result. Conversely, it is obvious that a couple \( \mathfrak{g} \) \( 2 \) \( \mathfrak{e} \mathfrak{n} \mathfrak{d} (H) \) satisfying the conditions in lemma, defines a Lie algebra structure, with Lie bracket as in (3).

Now, for an exact symplectic Lie algebra \( (H; \mathfrak{g}) \) containing \( H \) as a codimension 1 subalgebra such that \( i = \text{dom} : H \) is the natural inclusion. Set \( ! = \mathfrak{g} \) and consider the vector space isomorphism \( \mathfrak{q} : H \to H \), \( \mathfrak{q}(\mathfrak{g}) = \mathfrak{g}(\mathfrak{g}) \). There exists a unique vector \( x_o \) in \( H \) such that \( \mathfrak{q}(\mathfrak{g}(\mathfrak{g})) = \mathfrak{g} \). The corresponding left invariant vector field \( x^e_o \) in any symplectic Lie group \( (H; \mathfrak{g}) \) with Lie algebra \( H \), is a Liouville vector field, i.e the Lie derivative \( L_{x^e_o} \) along \( x^e_o \) satisfies \( L_{x^e_o} = 1 \).

**Theorem 1.** Let \( (H; \mathfrak{g}) \) be an exact symplectic Lie algebra and \( x_o \) \( H \) such that \( ! (x_o; \mathfrak{g}) = \mathfrak{g} \). The Lie algebras \( G = H \oplus \mathbb{R} e_o \) of lemma 3, which admit a contact form \( s = \mathfrak{e} e_o \), correspond to the couples \( (H; \mathfrak{g}) \) \( \mathfrak{e} \mathfrak{n} \mathfrak{d} (H) \) \( H \) satisfying, for some \( s \) \( 2 \) \( R \):

\[
(\mathfrak{g}(\mathfrak{g})) + s(1 + f(\mathfrak{g}(\mathfrak{g}))) \neq 0
\]

Here \( e_o \) \( G \) satisfies \( \mathfrak{g} e_o \) and \( e_o H \) = 1 and \( e_o H \) = 0.

Remark 1. 1. Theorem 1 essentially says that, if \( f(\mathfrak{g}(\mathfrak{g})) \neq 1 \) or if \( x_o \) and \( \mathfrak{g}(\mathfrak{g}) \) are not \( ! \)-orthogonal (or equivalently \( \mathfrak{g}(\mathfrak{g}) \) is not in the kernel of \( f \), then every Lie group \( G \) whose Lie algebra is obtained from \( (H; \mathfrak{g}) = \mathfrak{g}(\mathfrak{g}) \) as in Lemma 3 is a contact Lie group. Furthermore, \( G \) contains a connected exact symplectic codimension 1 subgroup \( i : (G; \mathfrak{g}) \) \( ! \mathfrak{g} \) and \( \text{Lie}(G) = H \).

2. If in Lemma 3 we choose \( H \) to be a symplectic Lie group (which needs not be exact, here) then we exhaust the list of all \( 2n \) \( 1 \)-dimensional Lie algebras admitting a solution of the Classical Yang-Baxter Equation of (maximal) rank \( 2n \) (see e.g. [12]).

Proof of Theorem 1 Let's identify the dual space \( H \) of \( H \) with the annihilator \( \mathfrak{a} e_o ^0 \) of \( e_o \) in \( G \), the space of linear forms on \( G \) which vanish on \( e_o \). So, \( e_o \) is an element of \( \mathfrak{a} e_o ^0 \). Denote \( e_o \) the element of the annihilator \( \mathfrak{a} e_o ^0 \) of \( H \) such that \( e_o \) has value 1 at \( e_o \). The exact symplectic form \( ! (\mathfrak{g}(\mathfrak{g}) = \mathfrak{g}(\mathfrak{g}) = < \mathfrak{g}; e_o > \mathfrak{g}(\mathfrak{g}) ) > e_o \) on \( H \) is again viewed as a linear 2-form on \( G \) with radical \( \mathbb{R} e_o \). Now for \( s \) \( 2 \) \( R \), let's compute the differential \( \mathfrak{g} e_o \) of \( s = \mathfrak{e} e_o \). Let \( x; y \) be in the subalgebra \( H \) of
First, let $x; y$ equals $(x; y)$ and from (5) it follows
\[ 0 \leq e_0 = < ; ; x; e_0 > = < ; x + f(x); e_0 > = \langle \epsilon(x); e_0 \rangle \]. The expression of $e_0$ then reads $e_0 = \langle \epsilon(x); e_0 \rangle$. On the other hand, bearing in mind that $e_0$ vanishes on $H$, one has $\langle e_0; x; y; H \rangle = 0$ and $\langle e_0; x; e_0; x \rangle = \langle e_0; x + f(x); e_0 \rangle = f(x)$, that is $\langle e_0; f^* \rangle = e_0$. Finally, $e_0$ equals $(\epsilon(x)) (x; e_0)$ and caps up as
\[ (\epsilon(x)) = n! n! n! n! (\epsilon(x)) (x; e_0); The linear $(2n + 1)$-form on $G$ we are looking for is
\[ (\epsilon(x)) = n! n! n! (\epsilon(x)) (x; e_0). \]
We now need to find necessary and sufficient conditions for this latter to be nonzero i.e to be a volume form. To do so in a simple way, let’s express it in terms of a well-chosen decomposition of $G$. Let $x_0 H$ such that $\varphi(x_0) = \{ x : H ! H \}$ be the isomorphism $x \mapsto \varphi(x) = \epsilon(x)$. Consider an $x_0$ in $H$ satisfying $(x_0; x_0) = 1$ and set $q(x_0; x_0)$. Then we get $H = (R_{x_0} R_{x_0}) R_{x_0}$, where $R_{x_0} R_{x_0}$ is the orthogonal of the 2-space $R_{x_0} R_{x_0}$ with respect to the symplectic form $\sigma$. On $H$. We can then write $1 = \sigma / \sigma$ here $1 / \sigma$ is the restriction of $1 / \sigma$ to $R_{x_0} R_{x_0}$. Then it follows $1 / \sigma = (1 / \sigma) + s f / \sigma$ and $\epsilon_0 = n! n! n! \sigma (\epsilon(x)) (x; e_0)$. Obviously $(1 / \sigma) + s f / \sigma$ has a nonzero component along $R$ relative to the decomposition $H = R R q(R_{x_0} R_{x_0})$. This is equivalent to $< x + \epsilon(x); e_0 > = \varphi(x_0; \varphi(x_0); s f / \sigma) \in 0$. Example 1. The special affine group $R^2 \circ S(2)$ is a contact Lie group. Its Lie algebra $G$ has a basis $(e_1; e_2; x; y; H; \sigma)$ and Lie bracket $[ ; ] = e_1, [ ; e_2] = e_2, [ ; H; e_2] = e_2, [ ; e_2; e_2] = e_2, [ ; y; x] = 2x, [ ; y; y] = 2y$. Set $e_2 = x, e_3 = y, e_4 = H$. Now $G$ is obtained from the exact symplectic subalgebra $(\text{span} (e_1; e_2; e_3; e_4); \sigma = \sigma(e_1; e_1))$ using Theorem 1 where $x_0 = e_0, f = 2e_0, e_0 = e_0, e_1 = e_2, e_2 = 0, e_3 = e_4, e_4 = 0$ and contact form $\sigma = x_{12} + x_{13} + s x_{14} + 2 R f$. Here is an immediate simple consequence of Theorem 1.

Theorem 2. If a Lie group $G$ contains an exact symplectic Lie group $(G; \theta^+)$, as a codimension 1 distinguished Lie subgroup, then $G$ has a family of left invariant contact forms $\alpha$ satisfying $\alpha = \alpha^+ = \alpha^+$, where $\alpha^+ = \alpha^+$ is the inclusion. Conversely, if $(G; \theta^+)$ is an exact symplectic Lie group, there is a connected exact symplectic Lie group $(G; \theta^+) (\text{symplecto-})$ isomorphic to $(G; \theta^+)$ and a Lie group $G$ of discrete centre, containing $H^0$ as a codimension 1 distinguished Lie subgroup, $G$ admits a family of left invariant contact forms $\alpha$ with $\alpha = 0^\alpha$. In particular, if one can embed an exact symplectic Lie group as a distinguished codimension 1 subgroup of a Lie group $G$, then $G$ is a contact Lie group.

Remark 2. Theorem 2 allows, in particular, to construct contact Lie groups as follows. Let $K = R$ or $S^1$ act on an exact symplectic Lie group $(G; \theta^+)$ by automorphisms $(t, s)$, $t 2 K$ of $G$ which preserve $\theta^+$. The semi-direct product $G = G_0 K$ is a contact Lie group, with $\alpha = \alpha^+$ and $\sigma$ is in some open $I R$. Recall that such an action is Hamiltonian with a (Marsden-Weinstein) moment $J : G_1 ! R$.

Example 2. Let $(G_3; \theta^+)$ be the exact symplectic Lie group $G_3 = R^4$ with product $(x_1; x_2; x_3; x_4) (x_1^0; x_2^0; x_3^0; x_4^0) = (x_1 + e^{x_1} x_2^0; x_3 + e^{x_3} x_1^0; x_4 = e^{x_4}) (x_1; x_2; x_3; x_4) = (e^{x_1}; e^{x_2}; e^{x_3}; e^{x_4})$. Each $t$ is an automorphism of the Lie group $G_3$ which preserves $\theta^+$. The map $J : G_3 ! R, (x_1; x_2; x_3; x_4) \mapsto e^{x_1} x_2$ is a moment of this action. The resulting semi-direct product $G = G_3 o R$ is a contact Lie group, with $\alpha = \alpha^+ + \sigma t = e^{x_1} (x_1; x_2; x_3; x_4) + \sigma t, s 2 R$. Actually, the Lie algebra $G$ of $G$ is obtained, using
Theorem 1 from the exact symplectic Lie algebra \( \langle H; \{ \cdot ; \cdot \} \rangle \): \( \{ e_1; e_2 \} = e_3 \), \( \{ e_4; e_1 \} = e_5 \), \( \{ e_5; e_3 \} = e_6 \), with the following setting! \( \Psi = \{ e_3 \} \), \( \phi = e_4 \), \( \xi = 0 \) and \( \{ e_1 \} = e_5 \), \( \{ e_2 \} = e_6 \), \( \{ e_3 \} = (e_4) = 0 \).

Recall that, the opposite Lie algebra \( G^{op} \) of \( G \) is defined by the Lie bracket \( \{ \cdot ; \cdot \} \) opposite to \( \{ ; \cdot \} \) on the vector space underlying \( G \). That is \( \{ \cdot ; \cdot \} \). Remark that \( \langle G; \{ ; \cdot \} \rangle \) is contact if and only if \( G^{op}; \{ \cdot ; \cdot \} \) is a contact Lie algebra.

**Corollary 1.** Let \( \mathbb{V} \) be a vector space of dimension \( n \geq 2 \) and \( \mathbb{W} \) be a subspace of dimension \( p \leq 1 \). If \( p \) divides \( n \), then the space \( G \) of all endomorphisms of \( \mathbb{V} \) preserving \( \mathbb{W} \) and whose restrictions to \( \mathbb{W} \) are homotheties, is a contact Lie algebra and \( G^{op} \) contains a codimension 1 Lie ideal isomorphic to the exact symplectic Lie algebra \( M_{n,p} \circ GL(n; \mathbb{R}) \).

**Proof of the corollary** Suppose \( p \) divides \( n \), so that \( M_{n,p} \circ GL(n; \mathbb{R}) \) is exact symplectic. To show that \( G^{op} \) contains \( M_{n,p} \circ GL(n; \mathbb{R}) \) with the Lie algebra \( H \) of \( (n + p) \times (n + p) \) matrices all of whose entries, on the last \( p \) rows, are zero. Now, by the transpose \( H \) of matrices, the opposite \( G^{op} \) of \( G \) is isomorphic to the Lie algebra \( G^{op} \) of matrices of the form

\[
\begin{pmatrix}
A_{nn} & A_{np} \\
0 & I_p
\end{pmatrix}
\]

where \( A_{nn} \) (resp. \( A_{np} \)) is an \( n \times n \) (resp. \( n \times p \)) matrix and \( I_p \) the identity map of \( \mathbb{W} \).

So \( G^{op} \) contains \( M_{n,p} \circ GL(n; \mathbb{R}) \) as a codimension 1 ideal, as \( M_{n,p} \circ GL(n; \mathbb{R}) \) contains its derived ideal \( [G, G^{op}] \). From Theorem 2 above, \( G^{op} \) is a contact Lie algebra, so is \( G \).

When \( p = 1 \), considering again the opposite Lie algebra \( G^{op} \), it follows.

**Corollary 2.** 1) The subgroup of \( GL(n; \mathbb{R}) \), \( n \geq 2 \), that globally preserves a hyperplane of \( \mathbb{R}^n \) is a contact Lie group which contains the group \( Aff(\mathbb{R}^n) \) of affine diffeomorphisms of \( \mathbb{R}^n \), as a distinguished subgroup of codimension 1, where \( GL(n; \mathbb{R}) \) stands for the group of linear diffeomorphisms of \( \mathbb{R}^n \).

2) Let \( \nu \) be a non-zero vector in \( \mathbb{R}^n \). The Lie subgroup of \( GL(n; \mathbb{R}) \) consisting of all linear diffeomorphisms of \( \mathbb{R}^n \) with common eigenvector \( \nu \), is a contact Lie group.

Theorem 1 allows, starting from exact symplectic Lie algebras, to get all contact Lie algebras (and hence Lie groups) containing a codimension 1 subalgebra which has an exact symplectic form. Now naturally considering the inverse process of building exact symplectic Lie groups from contact Lie groups \( G \), we get the following.

**Proposition 1.** Let \( \langle G; \{ ; \cdot \} \rangle \) be a contact Lie algebra with Reeb vector \( \{ e_0 \} \). Then with the same notations as in Theorem 1 for every \( \{ ; \cdot \} \) \( 2 \) \( \mathbb{R} \) satisfying \( 2 \) and for every \( s \) \( 2 \) \( \mathbb{R} \) satisfying \( \{ s \cdot \} \) \( + \) \( \{ s \cdot \} \) \( \neq 0 \), the Lie algebra \( G = G \) \( \mathbb{R} e_0 \) obtained from Lemma 3 using \( \{ ; \cdot \} \) \( + \) \( \{ s \cdot \} \), has exact symplectic forms \( \{ s \} = \{ e_0 \} \) with \( s \) \( \neq 0 \)

**Remark 3.** Proposition 1 allows to get all exact symplectic Lie algebras containing \( G \) as codimension 1 subalgebra transverse to their Liouville vector \( e_0 \). Such a construction is not always possible, for example starting with \( G \) if \( H = GL(n; \mathbb{R}) = \mathbb{R} e_0 \) and all derivations are inner. However, it also allows one to construct contact Lie algebras without using, a priori, results on exact symplectic Lie algebras. One applies Proposition 1 to a contact Lie algebra \( G \) by adding a line \( \mathbb{R} e_0 \) to get \( G \) and then applies Theorem 1 to \( G \) to get contact Lie algebras containing \( G \) as a codimension 2 contact Lie subalgebra.

As a corollary we have

**Proposition 2.** The special affine group \( \mathbb{R}^n \circ SL(n; \mathbb{R}) \) of affine motions whose linear part has determinant 1, is a contact Lie group.

As a proof of Proposition 2 we can also write the Lie algebra of \( \mathbb{R}^n \circ SL(n; \mathbb{R}) \) as a subalgebra transverse to the Liouville vector \( e_0 \) of \( aff(\mathbb{R}^n) \) consisting of the diagonal \( n \times n \) matrix \( e_0 = diag(1; 2; \ldots; n) \) in the canonical basis \( \{ e_1; \ldots; e_n \} \) of \( \mathbb{R}^n \). Indeed, if we write elements of \( H = aff(\mathbb{R}^n) \) as \( (\cdot ; M) \), where \( \cdot \) \( 2 \) \( \mathbb{R}^n \) and \( M \) is an \( n \times n \) matrix, every \( H \) can be written in a unique way as \( (\cdot ; M) = g \) \( (\cdot ; trace M = 0) \) for some \( g \) \( 2 \) \( \mathbb{R}^n \) and some \( n \times n \) matrix \( M \). Now taking
is a symplectic form on $\mathbb{M}$.

Let $m$ be a dimension.

Remark 4. Let $\mathfrak{g}$ be a Lie algebra of dimension $2n + 1$. As in Lemma 3, suppose $G = \mathfrak{g}$ and $\mathfrak{e}_0$ is a Lie algebra containing $\mathfrak{g}$ as a codimension 1 subalgebra. Let $e_0$ be in the dual $G$ of $\mathfrak{g}$ such that $\langle e_0; G \rangle = 0$ and $\langle e_0; e_0 \rangle = 1$ and denote by $(\ker e_0)$ the annihilator of $R$ in $\mathfrak{g}$. Then $G$ splits as $G = (\ker e_0) \oplus R \oplus e_0$.

Let $s = e_0 + e_0$ and denote the restriction of $\theta$ to $G$. We have $\theta_s = 0$ and $\theta = (\tau(\cdot) + s\theta)^\wedge e_0$ and $\theta = (\tau(\cdot) + s\theta)^\wedge e_0$ is a volume form if and only if $\tau(\cdot) + s\theta$ has a nonzero component along $R$ or equivalently if $\tau(\cdot) + s\theta > \theta 0$.

Let $G_1$ be a Lie group, $G_1$ its Lie algebra, $\mathfrak{h}^1(G_1; \mathbb{R})$ the space of left invariant closed forms on $G_1$. Taking $= 0$ in Theorem 1 we can easily deduce

**Remark 4.** Let $(G_1; \theta^+)$ be a connected and simply connected exact symplectic Lie group with Lie algebra $G_1$ and $x_0$ the Liouville vector as above. There is a correspondence between the open subset of $\mathfrak{h}^1(G_1; \mathbb{R})$ consisting of those $f$ satisfying $f(x_0) \in 1$ and the principal fibre bundles $p : G_2 ! G_1 = G_2 = H$ such that (a) the structural group $H$ is 1-dimensional, (b) the total space is a simply connected contact Lie group $G_2$ connected with $\theta^+$, the projection $p$ is a Lie group homomorphism, (c) and which admit a Lie group homomorphism $S$ as a section such that $S^+ = \theta^+$.

Notice that $H^1(G_2; \mathbb{R}) \not\in 0$, as $G_1$ has a left invariant locally flat affine structure induced by $\theta + \theta^+$ (27). This is no longer true in the contact case as $H^1(\mathfrak{so}(3); \mathbb{R}) = H^1(\mathbb{R}^n \circ \mathfrak{sl}(n); \mathbb{R}) = 0$.

### 3 Invariant Contact and (semi-)Riemannian Geometry

Here we consider contact Lie groups $G$ which display an additional structure, namely a left invariant Riemannian or Semi-Riemannian metric with specific properties such as being bi-invariant, flat, negatively or non-negatively curved, Einstein, etc. This can be motivated in the one hand by the fact that the relationship between the contact and the algebraic structures of Lie groups does not, a priori, show to be strong enough to ensure certain general consequences or to affect certain invariants of Lie groups. In the other hand, this section can be very useful for Riemannian or Sub-Riemannian (and CR) Geometry, Control Theory, Vision Models, ...

#### 3.1 Contact Lie groups with a bi-invariant (semi-) Riemannian metric

Our aim in this subsection is to extend a result on semi-simple contact Lie groups due to Boothby and Wang (Theorem 5 of [6]) to all Lie groups with a bi-invariant Riemannian or semi-Riemannian metric. A semi-Riemannian metric is a smooth field of bilinear symmetric non-degenerate real-valued forms.

In Theorem 5 of [6], Boothby and Wang showed, by generalising a result from J.W. Gray [21], that the only contact Lie groups that are semi-simple are those locally isomorphic to $SL(2; \mathbb{R})$ or to $SU(2)$. Actually, semi-simple Lie groups, with their Killing form, are a small part of the much wider family of Lie groups with a Riemannian or semi-Riemannian metric which is bi-invariant, i.e invariant under both left and right translations. For a connected Lie group, the above property is equivalent to the existence of a symmetric bilinear non-degenerate scalar form $\theta$ in its Lie algebra $\mathfrak{g}$, such that the adjoint representation lies in the Lie algebra $O(\mathfrak{g}; 2)$ of infinitesimal isometries. Such Lie groups and their Lie algebras are called orthogonal (see e.g. [34]). This is, for instance, the case of reductive Lie groups and Lie algebras (e.g. the Lie algebra of all linear transformations of a finite dimensional vector space), the so-called oscillator groups with their bi-invariant Lorentzian metrics (see [35]), the cotangent bundle of any Lie group (with its natural Lie group structure) and in general any element of the large and
interesting family of the so-called Drinfeld doubles or Manin algebras which appear as one of the key tools for the study of the so-called Poisson-Lie groups and corresponding quantum analogs, Hamiltonian systems (see V.G. Drinfeld 3), etc. It is then natural to interest ourselves in the existence of left invariant contact structures on such Lie groups. Here is our main result.

**Theorem 3.** Let $G$ be a Lie group. Suppose (i) $G$ admits a bi-invariant Riemannian or semi-Riemannian metric and (ii) $G$ admits a left invariant contact structure. Then $G$ is locally isomorphic to $SL(2;R)$ or to $SU(2)$.

Unlike the contact Lie groups, there is a great deal of symplectic Lie groups $G$ which also have bi-invariant Riemannian or semi-Riemannian metrics. The underlying symplectic form is related to the bi-invariant metric by a nonsingular derivation of the Lie algebra $\mathfrak{g}$, hence $G$ must be nilpotent.

As a direct corollary of Theorem 3 we have

**Theorem 4.** Suppose a Lie algebra $G$ splits as a direct sum $G = G_1 \oplus G_2$ of two ideals $G_1$ and $G_2$, where $G_1$ is an orthogonal Lie algebra. Then $G$ carries a contact form if and only if $G_1$ is $so(2)$ or $sl(2)$ and $G_2$ is an exact symplectic Lie algebra.

Theorem 4 implies in particular that if a Lie algebra $G$ is a direct sum of its Levi (semi-simple) subalgebra $G_1$ and its radical (maximal solvable ideal) $G_2$, then $G$ carries a contact form if and only if its Levi component is 3-dimensional and its radical is an exact symplectic Lie algebra. This is a simple way to construct many non-solvable contact Lie algebras in any dimension $2n + 1$, where $n \geq 1$. Recall that the situation is different in the symplectic case. A symplectic Lie group whose Lie algebra splits as a direct sum of its Levi subalgebra and its radical, must be solvable as shown in Theorem 10 of 8.

As we need a local isomorphism for the proof of Theorem 3 we can work with Lie algebras.

Our following lemma is central in the proof of Theorem 3.

**Lemma 4.** If an orthogonal Lie algebra $(G; b)$ has a contact form, then $G$ equals its derived ideal $G = [G; G]$. Furthermore, there exists $x \in G$ such that as a vector space $G = \ker(ad_x)$ and $\ker(ad_x)$ is of dimension 1, hence $\ker(ad_x) = R \cdot x$.

*Proof.* Let $G$ be a Lie algebra, and $b$ a (possibly non-definite) scalar product on it. For $x \in G$, denote by $(x)$ the element of $G$ defined by $(x)_y = b(x; y)$ for all $y \in G$, where $< ; >$ is the duality pairing between $G$ and $G$. Then $(G; b)$ is an orthogonal Lie algebra if and only if its adjoint and co-adjoint representations are isomorphic via the linear map $G \to G$ (see e.g. (34)). Suppose $(x)$ is a contact form on $G$. There exists $x \in G$ such that $(x) = x$. The differential of $(x)$ is $\theta(y) = \theta(y) > = b(x; y) = b(y; x)$. This implies in particular that the radical (nullspace) $R \cdot ad(\theta)$ equals the kernel $\ker(ad_x)$ of $ad_x$. As $(x)$ is a contact form, the vector space underlying $G$ splits as $G = R \cdot ad(\theta) \oplus \ker(\theta)$ and $\dim(R \cdot ad(\theta)) = 1$, that is $\ker(ad_x) = R \cdot x$. It then follows that $\dim(\ker(ad_x)) = \dim G = 1$. As $ad_x$ is an infinitesimal isometry of $\theta$ then $\ker(ad_x)$ is a subspace of the $b$-orthogonal $(R \cdot x)^2$ of $R \cdot x$ and finally $\ker(ad_x) = (R \cdot x)^2$. We have proved that $G = \ker(ad_x) \oplus \ker(ad_x)$ and $\ker(ad_x) = R \cdot x$. On the other hand, as $\ker(\theta)$ is not a Lie subalgebra of $G$ (see Lemma 1), there exist $x; y \in G$ such that $(x)_y$ is not in $\ker(\theta)$, and has the form $(x)_y = ax + (x')$, where $a \in R \cdot 0$ and $x' \in G$. But then $x = \frac{1}{a} (x)_y = (x')$ is in the derived ideal $[G; G]$ of $G$ and consequently we have $G = [G; G]$.

*Proof of Theorem 3.* Let $G = S \oplus R$ be the Levi decomposition of $G$, where $S$ is the Levi (semi-simple) subalgebra and $R$ is the maximal solvable ideal of $G$. The inequality $\dim(S) \leq 3$ follows from Lemma 4 as $S$ is non-trivial. We are now going to show that $G$ is semi-simple.

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3 V.G. Drinfeld, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations. Dokl. Akad. Nauk SSSR 268 (1983), no. 2, 285-287.
Lemma 5. [34] A subspace \( J \) of an orthogonal Lie algebra \( \mathbb{G} \) is an ideal of \( \mathbb{G} \) if and only if the centraliser \( Z_{\mathbb{G}}(J) = \{ x \in \mathbb{G} : [x,y] = 0 \text{ for all } y \in J \} \) is contained in \( J \).

Lemma 5 ensures that \( Z_{\mathbb{G}}(\mathbb{R}) \) contains \( \mathbb{R}^2 \) and hence \( \dim(Z_{\mathbb{G}}(\mathbb{R})) = \dim(\mathbb{G}) \). If the element \( x \) of Lemma 4 was in \( \mathbb{R} \), then \( Z_{\mathbb{G}}(\mathbb{R}) = \ker(ad_x) \) would contain \( \mathbb{R}^2 \), contradicting Lemma 4. Suppose the restriction \( \mathbb{R} \) of \( ad_x \) to \( \mathbb{R} \) is not injective. There exists \( y_0 \neq 0 \) in the intersection of \( \mathbb{R} \) and \( \ker(ad_x) \). As \( x \) is not in \( \mathbb{R} \), there exists at least two linearly independent elements \( x, y_0 \) in \( \ker(ad_x) \), which again contradicts Lemma 4. So \( \mathbb{R} \) is injective and the image \( \Im(ad_x) = (\mathbb{R}x)^2 \) of \( ad_x \) then contains \( \mathbb{R} = \mathbb{R} \). Now the closures of \( \mathbb{R} \times \mathbb{R} \) in \( G \) imply that \( x \) commutes with every element of \( \mathbb{R} \) and hence this latter is a subset of \( \ker(ad_x) \). We conclude that \( \mathbb{R} \) is zero, as it is contained in both \( \Im(ad_x) \) and \( \ker(ad_x) \). So \( \mathbb{G} \) is semi-simple. But Theorem 5 of [6] asserts that the only semi-simple Lie algebras with a contact structure are \( \mathfrak{sl}(2;\mathbb{R}) \) and \( \mathfrak{so}(3) \).

3.2 Flat Riemannian metrics in Contact Lie Groups

In his main result of [4] (see also [5]), Blair proved that a contact manifold of dimension 5 does not admit a flat contact metric, i.e., a metric satisfying the condition (11) whose sectional curvature vanishes. Below, we prove that in the case of contact Lie groups of dimension 5, there is no flat left invariant metric at all, even if such a metric has nothing to do with the given contact structure.

Theorem 5. Let \( G \) be a Lie group of dimension 5. Suppose \( G \) admits a left invariant contact structure. Then, there is no flat left invariant Riemannian metric on \( G \).

The following complete classification of contact Lie groups which carry a flat left invariant metric is a direct consequence of Theorem 5.

Corollary 3. A contact Lie group admits a flat left invariant Riemannian metric if and only it is locally isomorphic to the group \( E(2) \) \( = \mathbb{R}^2 \circ O(2) \) of rigid motions of the Euclidean 2-space.

Unlike contact Lie groups which cannot display flat left invariant metrics in dimension \( > 3 \) (Theorem 5), we have again a different scenario for symplectic Lie groups. At each even dimension there are several non-isomorphic symplectic Lie groups with some flat left invariant metric (see Theorem 2 of Lichnerowicz [30], Theorem 2.2 of [9]).

Proof of Theorem 5. Let \( G \) be a connected Lie group of dimension \( m \), with a left invariant Riemannian metric \( < ; > \). Then \( < ; > \) is flat if and only if its Levi-Civita connection \( \bar{\nabla} \) is a left-invariant \( \mathfrak{g} \)-valued linear connection on \( \mathfrak{g} \). This allows Milnor (Theorem 1.5 of [36]) to establish that \( (G, < ; >) \) is flat if and only if every \( G \)-invariant form is flat. It follows from the above that \( (G, < ; >) \) is flat if and only if it is isomorphic to \( E(2) \).

Proof of Theorem 5. Let \( G \) be a connected Lie group of dimension \( m \), with a left invariant Riemannian metric \( < ; > \). Then \( < ; > \) is flat if and only if its Levi-Civita connection \( \bar{\nabla} \) is a left-invariant \( \mathfrak{g} \)-valued linear connection on \( \mathfrak{g} \). This allows Milnor (Theorem 1.5 of [36]) to establish that \( (G, < ; >) \) is flat if and only if it is isomorphic to \( E(2) \).

Proof of Theorem 5. Let \( G \) be a connected Lie group of dimension \( m \), with a left invariant Riemannian metric \( < ; > \). Then \( < ; > \) is flat if and only if its Levi-Civita connection \( \bar{\nabla} \) is a left-invariant \( \mathfrak{g} \)-valued linear connection on \( \mathfrak{g} \). This allows Milnor (Theorem 1.5 of [36]) to establish that \( (G, < ; >) \) is flat if and only if it is isomorphic to \( E(2) \).
for each \( i = 1; \cdots; p \), the \( 2(p + j) \)-form \( \Omega^0 \) is identically zero, if \( j \neq 1 \). But obviously we have \( p = m \in (p_1,p_2) \). Thus as \( p_1 + p_2 = 2n + 1 \), the non-vanishing condition on \( \Omega^0 \) imposes that \( \dim N = p \) and either \( p_1 = p_2 + 1 = n + 1 \) or \( p_1 = p_2 = 1 = n \). Hence the dimension of the abelian subalgebra \( \mathfrak{a}(2) \) of \( \mathfrak{p}(2) \) satisfies \( \dim ( \mathfrak{a}(2) ) = p_1 + 1 \). But the maximal abelian subalgebras of \( \mathfrak{O}(p_1) \) are conjugate to the Lie algebra of a maximal torus of the compact Lie group \( SO(p_1) \) (real special orthogonal group of degree \( p_1 \)). It is well known that the dimension of maximal tori in \( SO(p_1) \) equals \( \frac{p_1}{2} \) if \( p_1 \) is even, and \( \frac{p_1 - 1}{2} \) if \( p_1 \) is odd. This is incompatible with the inequality \( \dim ( \mathfrak{a}(2) ) = p_1 + 1 \), unless \( p_1 = 2 \) and \( p_2 = 1 \), hence \( \dim G = 3 \).

\[ \square \]

### 3.3 Contact Lie Groups with a Riemannian metric of negative curvature

This subsection is devoted to the study of contact Lie groups (resp. algebras) having a left invariant Riemannian metric of negative sectional curvature. Nevertheless, the main result outlined here characterises the more general case of solvable contact Lie algebras whose derived ideal has codimension 1. For the negative sectional curvature case, see Remark 5. See also Corollary 4 for some obstructions in the locally symmetric and in the 2-step solvable cases.

**Proposition 3.** (1) If the derived ideal \( N \) of a solvable contact Lie algebra \( G \) has codimension 1 in \( G \), then the following hold. (a) The center \( Z(\mathfrak{N}) \) of \( N \) has dimension \( \dim Z(\mathfrak{N}) = 2 \). If moreover \( \dim Z(\mathfrak{N}) = 2 \), then there exists \( e \in G \), such that \( Z(\mathfrak{N}) \) is not an eigenspace of \( \text{ad}_e \).

(b) There is a linear form on \( N \) with \( \langle \cdot \rangle^n \wedge \mathfrak{g} = 0 \), where \( \dim (\mathfrak{G}) = 2n + 1 \).

(2) If a Lie algebra \( G \) has a codimension 1 abelian subalgebra, then \( G \) has neither a contact form nor an exact symplectic form if \( \dim G = 4 \).

**Proof.** (1) Let \( \dim (\mathfrak{G}) = 2n + 1 \). Write \( G \) as the direct sum of vector spaces \( G = \mathfrak{g} \oplus \mathfrak{n} \), where \( \mathfrak{n} \) is the derived ideal \( \mathfrak{g} \oplus \mathfrak{g} = \mathfrak{n} \). Let \( e \) be the unique linear form on \( G \) satisfying \( e(\mathfrak{g}) = 1 \) and \( e(\mathfrak{n}) = 0 \). Any \( \in G \) can be written as \( + t e \), where \( t = \langle \cdot \rangle(\mathfrak{g}) \) and the restrictions to \( \mathfrak{n} \) of \( \cdot \mathfrak{g} \) and \( \text{ad}_e \), respectively. The formula \( \langle \cdot \rangle^n \wedge \mathfrak{g} = n! \wedge \mathfrak{g} \), implies in particular that if \( \mathfrak{g} \) is a contact form, then \( e \) must have rank 2 \( \dim (\mathfrak{N}) = 1 \) and satisfies \( \wedge \mathfrak{g} \wedge \mathfrak{g} = 0 \). Hence its radical (nullspace) \( \text{Rad}(\cdot) \) must have dimension \( \dim (\mathfrak{N}) = 1 \). As it is contained in \( \text{Rad}(\cdot) \), then \( \text{ad}(\mathfrak{g}) = 0 \) and \( \text{ad}_{\mathfrak{g}}(\mathfrak{g}) = \mathfrak{g} \) and \( \text{ad}_{\mathfrak{g}}(\mathfrak{g}) = \mathfrak{g} \). In the other hand, if \( \dim (\mathfrak{N}) = 2 \) and there was \( \mathfrak{g} \) such that \( \text{ad}_{\mathfrak{g}}(\mathfrak{g}) = \mathfrak{g} \) for all \( \mathfrak{g} \), then \( \text{Rad}(\cdot) \) would coincide on \( \mathfrak{g} \), and \( \mathfrak{g} \) would vanish identically, - 8 \( \mathfrak{g} \).

(2) Suppose a Lie algebra \( G \) contains a codimension 1 abelian subalgebra \( \mathfrak{V} \) and let \( \mathfrak{V} \) be a complement of \( \mathfrak{V} \) in \( G \). There are \( 2 \mathfrak{e} \mathfrak{g}(\mathfrak{g}) \), \( f \in G \), such that the Lie bracket of \( G \) reads: \( \langle \cdot \rangle f = 0 \) and \( \langle \cdot \rangle f \mathfrak{g}(\mathfrak{g}) = \mathfrak{g} + f(\mathfrak{g}) \mathfrak{g}(\mathfrak{g}) \). So, with the notations as above, every form \( \eta \) \( \mathfrak{g}(\mathfrak{g}) \) satisfies \( \mathfrak{g}(\mathfrak{g}) = \mathfrak{g} + f(\mathfrak{g}) \mathfrak{g}(\mathfrak{g}) \) and \( \langle \cdot \rangle^n \wedge \mathfrak{g} \) would vanish identically, - 8 \( \mathfrak{g} \).

To fix ideas, here are two typical examples of \( G \) for Proposition 3(1). (i) From a nilpotent symplectic Lie algebra \( (\mathfrak{n},\mathfrak{l},\mathfrak{o}) \), perform the central extension \( N = \mathfrak{n} \oplus \mathfrak{l} \), using \( \mathfrak{V} \), to get a nilpotent contact Lie algebra with center \( \mathfrak{V} \mathfrak{l} = \mathfrak{g} \). Let a 1-dimensional Lie algebra \( \mathfrak{g} \) act on \( N \) by a nilpotent derivation \( D_1 \) with \( D_1(\mathfrak{n}) = \mathfrak{g} \) and \( D_1(\mathfrak{g}) = 0 \). We set \( N = \mathfrak{n} \mathfrak{g} \mathfrak{g} \mathfrak{g} \) so that if \( x \in N \) then \( \mathfrak{g}(\mathfrak{g}) = \mathfrak{g} \) and \( \mathfrak{g}(\mathfrak{g}) = \mathfrak{g} \). Now we have \( \mathfrak{g} \mathfrak{g}(\mathfrak{g}) = \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \) if \( D_1 = \text{ad}_{\mathfrak{g}}(\mathfrak{g}) \). So, with the notations as above, every form \( \eta \) satisfies \( \eta = \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \) and \( \langle \cdot \rangle^n \wedge \mathfrak{g} \) would vanish identically, - 8 \( \mathfrak{g} \).

(ii) Another example is the direct sum \( N = N = N \) of two nilpotent contact Lie algebras \( N \) and \( N \). For instance, two Heisenberg Lie algebras \( H_{2p+1} \) and \( H_{2q+1} \), thus \( Z(\mathfrak{N}) \) \( \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \) and \( \dim Z(\mathfrak{N}) = 2q+1 \). There are only two \( N \) for Proposition 3(1) if \( \dim (\mathfrak{N}) = 4 \), namely \( N = N \), with a basis \( (\mathfrak{e}_1) \) and \( \mathfrak{g}(\mathfrak{g}) = \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \). With bracket \( [\mathfrak{e}_1] = \mathfrak{e}_1 \), \( [\mathfrak{e}_2] = \mathfrak{e}_2 \), \( [\mathfrak{e}_3] = \mathfrak{e}_3 \), \( [\mathfrak{e}_4] = \mathfrak{e}_4 \), and \( Z(\mathfrak{N}) = \mathfrak{e}_1 \mathfrak{e}_2 \mathfrak{e}_3 \mathfrak{e}_4 \). For example in Subsection 4, the Lie algebra number 4 is obtained from \( N_{2p} \) and has a metric with negative sectional curvature when \( p > 0 \), \( q > 0 \) and \( q = 6p + 1 \). (Example 3). Likewise, the Lie algebra number 15 is obtained from \( N_{1;2} \) and has a metric with negative sectional curvature when \( p > 0 \).
Example 3. Let $\mathbb{R}$ act on the closed connected subgroup $G_1 := f = \begin{pmatrix} 0 & 1 & x_1 & x_3 & 0 & 1 \\ 0 & 1 & x_2 & 0 & C & \psi_1 x_4 \\ 0 & 0 & 1 & 0 & A & x_1 \\ 0 & 0 & 0 & e^{x_5} & x_6 & x_7 \\ 0 & 0 & 0 & 0 & e^{x_8} & x_9 \\ 0 & 0 & 0 & 0 & 0 & e^{x_{10}} \end{pmatrix}$ of $GL(4; \mathbb{C})$ by $(x_5) = (x_1 e^{x_5}; x_2 e^{x_5}; x_3 e^{(l+p)x_5}; x_4 e^{x_5})$, where is written as $(x_1; x_2; x_3; x_4)$ for simplicity. If $q \in 1 + p$, the semi-direct product $G = G_1 \circ R = R^3 S^1 R$ is a contact Lie group and has a left invariant Riemannian metric of negative sectional curvature if moreover $pr_q > 0$. Recall that $G_1$ is the nilpotent Lie group used by E. Abbena to model the Kodaira-Thurston Manifold as a nilmanifold, which is symplectic but not Kählerian. It might be interesting to work out the behaviour of the extensions to $G$ of the Abbena metric and its relationships with the contact structure.

As a direct consequence of Proposition 3, we have the following.

Corollary 4. If $\dim (G) \neq 5$, then $G$ is not a contact Lie group, in any the following cases.
1. $G$ is a negatively curved locally symmetric Lie group, i.e has a left invariant Riemannian metric with negative sectional curvature, such that the Levi-Civita connection $\nabla$ and the curvature tensor $\nabla^2$ satisfy $\nabla R = 0$.
2. $G$ is a negatively curved 2-step solvable Lie group.

Proof. (1). From Proposition 3 of [28], there exists a vector $e$ in the Lie algebra $\mathfrak{g}$ of $G$ such that $\mathfrak{g}$ splits as a direct sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2$, where $\mathfrak{r} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ is a 2-step nilpotent ideal, with derived ideals $(\mathfrak{r}; \mathfrak{g}) = N$ and $(\mathfrak{n}; N) = \mathfrak{a}_2$. It follows that $\mathfrak{a}_2 = Z \mathfrak{n}$. But from [28], $\dim (\mathfrak{a}_2) = 0; 1; 3$, or 7. So $G$ has no contact form. The part (2) also follows from Proposition 3 and Heintze’s main result [28], as the derived ideal of the Lie algebra of $G$ must have codimension 1 and is abelian.

Remark 5. Proposition 3 also characterizes contact Lie groups with a left invariant Riemannian metric of negative sectional curvature. Their Lie algebras are solvable with a codimension 1 derived ideal ([28]).

Proposition 4. If a Lie group $G$ has the property that for every left invariant Riemannian metric, the sectional curvature has a constant sign, then $G$ does not carry any left invariant contact (or exact symplectic) structure. Moreover, such a Lie group is unique, up to a local isomorphism, in any dimension.

As a byproduct, the uniqueness result must have another interest (independant from Contact Geometry) in the framework of Riemannian Geometry (compare with [36], [37]).

Proof of Proposition 4 From theorem 2.5 of Milnor [36] (see also [37]), the Lie bracket $[x; y]$ is always equal to a linear combination of $x$ and $y$, for all $x, y$ in the Lie algebra $G$ of such a Lie group. There exists a well defined real-valued linear map $l$ on $G$ such that $[x; y] = l(y)x - l(x)y$.

Now identifying the kernel of $l$ with $R^n$ and choosing a vector $e_1$ satisfying $l(e_1) = 1$, allows us to see that all such Lie algebras are actually isomorphic to the sum $R^n \oplus e_1$ of a codimension 1 abelian ideal $R^n$ and a complementary $R e_1$, where the restriction of $ad_{e_1}$ to $R^n$ is opposite the identity mapping $ad_{e_1}$ and $n + 1 = \dim (G)$. So any linear form on $G$, has differential $\theta = ^\wedge 1$. Hence we have $\theta ^\wedge = 0$ and $\theta = 0$, $\theta^P = 0, 8, 2$.

3.4 Left invariant Einstein metrics on contact Lie groups

As well known, a connected Lie group $G$ must be compact with finite fundamental group, if some of its left invariant metrics has positive Ricci curvature (see e.g theorem 2.2. of [36]). Thus, Theorem 3 ensures that the only Einstein contact Lie groups with a positive Ricci curvature are those locally isomorphic to $SU(2)$. In the other hand, a contact metric structure in a $(2n + 1)$-dimensional manifold, is K-contact.
if only if the Ricci curvature on the direction of the Reeb vector field is equal to 2\(n\) (see Blair\(^5\)). A direct consequence of this,

**Proposition 5.** There is no left invariant K-contact structure on Lie groups of dimension > 3 whose underlying Riemannian metric has a Ricci curvature of constant sign. In particular, there is no K-contact-Einstein, and a fortiori no Sasaki-Einstein, left invariant structures on Lie groups of dimension 5.

**Remark 6.** Nevertheless, there are contact Lie groups with a left invariant Riemannian metric of nonnegative Ricci curvature, this is the case for any 7 dimensional Lie group with Lie algebra \(R^4 \circ \text{so}(3)\) in the Subsection 4.3. As a Lie group with a left invariant Riemannian metric of nonnegative Ricci curvature must be unimodular, then from J. Hano (see also \[8\]) it is solvable if it admits a left invariant symplectic structure. In this case the metric must be flat (see Lichnerowicz\[31\]).

Recall that an Einstein metric on a solvable Lie algebra is standard if the orthogonal complement of the derived ideal is an abelian subalgebra (see e.g. \[26\]).

**Theorem 6.** Suppose \(\mathfrak{h} \circ \mathfrak{g}\) is an exact symplectic solvable Lie algebra that carries a standard Einstein metric. Let \(\mathfrak{a}\) be the orthogonal complement of the derived ideal \([\mathfrak{h} ; \mathfrak{h}\) with respect to the Einstein metric. Then for any symmetric derivation \(D : 2D \in [\mathfrak{h}] \) \(f\) commuting with \(ad_a\), for all \(a \in \mathfrak{a}\), the semidirect product Lie algebra \(G = H \circ RD\) is a contact Lie algebra endowed with an Einstein metric.

**Proof.** From Theorem \[2\] if \(G\) is a semidirect product of \(\mathfrak{h} \circ \mathfrak{g}\) and a derivation \(D\) of \(H\), then \(G\) carries a 1-parameter family of contact structures \(t)_{t \subset T} \) satisfying \(i_t = \), where \(i : H \to G\) is the natural inclusion and \(T\) is an open nonempty subset of \(R\). In the other hand, from a result of Heber in \[26\], any semidirect product of a standard Einstein Lie algebra \(H\) by a symmetric non-trivial derivation commuting with \(ad_a\) for all \(a \in \mathfrak{a}\), is again a standard Einstein Lie algebra.

Theorem \[6\] gives several such examples using in particular \(j\)-algebras from \[14, 18, 39\], ...

### 4 On the classification problem in low dimensions.

#### 4.1 Contact Lie algebras of dimension 3

Let \(R\) act on the abelian Lie algebra \(R^2\) via a linear map \(D\) and let \(R^2 \circ RD\) be the resulting semi-direct product. Denote \(D_0\) an endomorphism of \(R^2\) with no real eigenvalue. It is straightforward to check the

**Proposition 6.** Every 3 dimensional nonabelian Lie algebra has a contact form, except \(R^2 \circ R^2\). Furthermore, apart from \(\text{so}(3;R)\) and \(R^2 \circ RD\), every 3 dimensional contact Lie algebra can be built up by Theorem \[1\] from the Lie algebra \(\text{aff}(R)\) of affine transformations of \(R\).

The Lie algebra \(\text{so}(3;R)\) contains no subalgebra of codimension 1, so it cannot be constructed by Theorem \[1\]. As far as \(R^2 \circ RD\) is concerned, it contains no nonabelian codimensional 1 subalgebra, so it doesn’t contain \(\text{aff}(R)\). Recall that the simplest exact symplectic Lie algebra is \(\text{aff}(R)\). It has a basis \((e_1, e_2)\) with Lie bracket \([e_1 ; e_2]\) = \(e_2\). If \(\mathfrak{g} \circ \mathfrak{g}\) \(\neq 0\) \(e_2 = e_1 \circ e_2\), then \(x_0 = e_1\). For example \(\mathfrak{s}(2;R)\) is obtained using \(f = e_1\), \((e_1) = 0\), \((e_2) = 2e_1\).
4.2 Contact Lie algebras of dimension 5

A decomposable (direct sum of two ideals) 5-dimensional contact Lie algebra is either (a) the direct sum $G = \text{aff}(\mathbb{R})$ A where A is any 3-dimensional Lie algebra different from $R^2 \circ R \mathbb{C}_{x_2}$, or else (b) the direct sum of an exact symplectic 4-dimensional Lie algebra and the line $R$.

**Theorem 7.** (1) A 5-dimensional non-solvable Lie group $G$ is a contact Lie group if and only if its Lie algebra is one of the following: (i) decomposable: $\text{aff}(\mathbb{R}) \oplus \mathfrak{sl}(2)$, $\text{aff}(\mathbb{R}) \oplus \mathfrak{so}(3)$, (ii) nondecomposable: $R^2 \circ \mathfrak{sl}(2)$.

(2) Let $G$ be a 5-dimensional non-decomposable solvable Lie algebra with trivial centre $Z(G) = 0$.

(i) If the derived ideal $[G;G]$ has dimension 3 and is nonabelian, then $G$ is a contact Lie algebra.

(ii) If $[G;G]$ has dimension 4, then $G$ is contact if and only if either (a) $\dim(Z([G;G])) = 1$ or else (b) $\dim(Z([G;G])) = 2$ and there is a 2-dimensional nonsolvable Lie algebra has trivial centre if and only if it is contact.

**Proof of Theorem 7.** (1) The Lie algebras $\text{aff}(\mathbb{R}) \oplus \mathfrak{sl}(2)$, $\text{aff}(\mathbb{R}) \oplus \mathfrak{so}(3)$ are contact Lie algebras, as they are direct sums of a contact and an exact symplectic Lie algebras. For $R^2 \circ \mathfrak{sl}(2)$, see Example 1.

Conversely, suppose $G$ is contact, nonsolvable and $\dim(G) = 5$. From Theorem 3, $G$ splits as (Levi decomposition) $G = R \circ S$, where $S$ is either $\mathfrak{so}(3)$ or $\mathfrak{sl}(2;R)$ and $R$ is either the abelian algebra $R^2$ or the nonnilpotent one $\text{aff}(\mathbb{R})$. The semidirect product $R \circ S$ is given by a representation of $S$ by derivations of $R$, which is either trivial or faithful, as $S$ is simple. But as a subalgebra of the space $\mathfrak{gl}(R^2)$ of linear maps of $R^2$, the space $\mathfrak{der}(R)$ of derivations of $R$ does not contain a copy of $\mathfrak{so}(3)$. Hence, only the trivial representation occurs when $S = \mathfrak{so}(3)$ and as the center satisfies $\dim(Z(G)) = 1$, we necessarily have $R = \text{aff}(\mathbb{R})$ and $G = \mathfrak{so}(3) \oplus \text{aff}(\mathbb{R})$. Now for $S = \mathfrak{sl}(2;R)$, either $G$ is the direct sum $\text{aff}(\mathbb{R}) \oplus \mathfrak{sl}(2;R)$ or the semidirect product $R^2 \circ \mathfrak{sl}(2;R)$, where $\mathfrak{sl}(2;R)$ acts in the natural way (matrix multiplication) on $R^2$. This last claim is due to the fact that all representations $\mathfrak{sl}(2;R)$ of $G \mathfrak{sl}(2;R)$ are conjugate and given by inner automorphisms of $\mathfrak{sl}(2;R)$.

(2) Now suppose $G$ is solvable, nondecomposable with trivial center. (i) If $\dim([G;G]) = 3$, then $[G;G]$ is either the Heisenberg Lie algebra $H_3$ or the abelian Lie algebra $R^3$. If $[G;G] = H_3$, since the center $Z(G)$ is trivial, there exists $y \in G$ such that the restriction of $ad_y$ to the center of $H_3$ is not trivial. So the (codimension 1) ideal of $G$ spanned by $H_3$ and $R y$ is an exact symplectic Lie algebra. By Theorem 2, $G$ is a contact Lie algebra. (ii) The case $\dim([G;G]) = 4$ is obtained by a direct calculation using Proposition 3 and $N_{1,2}, N_{2,2}$ for $N \Rightarrow [G;G]$.

The following is a direct consequence of Theorem 7.

**Corollary 5.** A 5-dimensional nonsolvable and nonsemisimple Lie algebra is a contact Lie algebra if and only if its centre is trivial.

**Proof.** A 5-dimensional nonsolvable and nonsemisimple Lie algebra has trivial centre if and only if it is one of the following $\text{aff}(\mathbb{R}) \oplus \mathfrak{sl}(2)$, $\text{aff}(\mathbb{R}) \oplus \mathfrak{so}(3)$ or $R^2 \circ \mathfrak{sl}(2)$.

A list of solvable contact Lie algebras in dimension 5.

Applying the above results to the list of 5-dimensional Lie algebras quoted from 3 together with some direct extra calculations, we get the following list of all 5-dimensional nondecomposable solvable contact Lie algebras, each case along with an example of a contact form $\omega$. Only nonvanishing Lie brackets are listed in a basis $\{e_1; \cdots; e_5\}$ with dual $\{e^1; \cdots; e^5\}$. The parameters $p; q$ are in $R$. Assuming the list from 3 is complete, then together with the decomposable and the nonsolvable ones (Theorem 7), we get a complete classification of all contact Lie algebras of dimension 5. In 25, among other results, the author gives a method of constructing 5-dimensional compact contact manifolds which are not covered by the Boothby-Wang fibration method. The reader can also see 17 for the topology of contact 5-manifolds $M$ with $\pi_1(M) = \mathbb{Z}_2$. As a byproduct, the list below also allows to get contact solvmanifolds modeled on 5-dimensional Lie groups.

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13
| Step | Equation/solution |
|------|-------------------|
| 20.  | \[ e_1 = e_2, \quad [e_1, e_2, e_3] = e_1, \quad [e_1, e_2, e_3] = e_2. \]  |
| 21.  | \[ [e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_3] = e_3 = e_2. \]  |
| 22.  | \[ e_1 = e_2 = e_3 = e_4, \quad [e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_3] = e_3 = e_2. \]  |
| 23.  | \[ e_1 = e_2, \quad [e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_3] = e_3 = e_2 + e_3. \]  |
| 24.  | \[ e_1 = e_2, \quad [e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_3] = e_2, \quad [e_1, e_2, e_3] = e_3 = e_2 + e_3. \]  |
4.3 Contact Lie algebras of dimension 7.

We quote below an infinite family $G_c$, $c \in \mathbb{R}$, of 7-dimensional nilpotent contact Lie algebras with $e_0 = e_2$. According to [33], if $c \neq 0$ then $G_c$ and $G_d$ are not isomorphic. Hence in any dimension $2n + 1 > 7$, on can again obtain infinite families of contact Lie algebras as the direct sum of $G_c$ and exact symplectic Lie algebras.

1. $[e_1; e_4] = e_1$, $[e_2; e_3] = e_7$, $[e_3; e_4] = e_7$, $[e_1; e_2] = e_4 + te_5$, $[e_1; e_3] = e_6$, $[e_2; e_1] = e_5$. $\Rightarrow e_7$.

The nonsolvable case. Using the same arguments as in Subsection 4.2, a nonsolvable contact Lie algebra of dimension 7 is either the direct sum $R \oplus S$ of an exact symplectic (solvable) Lie algebra $R$ of dimension 4 and $S = \mathfrak{s}l(2;\mathbb{R})$ or $\mathfrak{s}o(3)$; or the semi-direct product $R \rtimes S$ where $S = \mathfrak{s}l(2;\mathbb{R})$ or $\mathfrak{s}o(3)$ acts faithfully on the 4-dimensional solvable Lie algebra $R$, by derivations. The following examples are non-decomposable.

2. $R^4 \circ \mathfrak{s}l(2;\mathbb{R})$: $[e_1; e_2] = 2e_2$, $[e_1; e_3] = 2e_3$, $[e_1; e_4] = e_1$, $[e_1; e_5] = 3e_4$, $[e_1; e_6] = 3e_5$, $[e_2; e_3] = e_6$, $[e_3; e_4] = e_2$, $[e_3; e_5] = 2e_2$, $[e_4; e_5] = 2e_6$, $[e_4; e_6] = e_5$, $[e_5; e_6] = 3e_7$.

3. $R^4 \circ \mathfrak{s}l(2;\mathbb{R})$: $[e_1; e_2] = 2e_2$, $[e_1; e_3] = 2e_3$, $[e_2; e_3] = e_1$, $[e_1; e_4] = e_4$, $[e_2; e_1] = e_5$, $[e_3; e_4] = e_6$, $[e_2; e_4] = e_7$, $[e_3; e_5] = e_7$.

4. $[e_1; e_2] = e_1$, $[e_2; e_3] = e_2$, $[e_3; e_1] = e_3$, $[e_1; e_3] = e_6$, $[e_2; e_4] = e_5$, $[e_3; e_4] = e_5$, $[e_2; e_3] = e_4$, $[e_1; e_4] = e_4$, $[e_3; e_5] = e_6$, $[e_1; e_7] = e_6$, $[e_1; e_2] = e_7$, $[e_2; e_1] = e_7$, $[e_3; e_4] = e_7$, $[e_3; e_5] = e_7$, $[e_4; e_5] = e_7$, $[e_4; e_6] = e_7$.

5. $R^4 \circ \mathfrak{s}o(3)$: $[e_1; e_2] = e_1$, $[e_1; e_3] = e_3$, $[e_2; e_1] = e_2$, $[e_2; e_4] = \frac{1}{2}e_7$, $[e_2; e_5] = \frac{1}{2}e_7$, $[e_3; e_4] = \frac{1}{2}e_6$, $[e_3; e_5] = \frac{1}{2}e_6$, $[e_3; e_4] = \frac{1}{2}e_5$, $[e_3; e_5] = \frac{1}{2}e_5$, $[e_4; e_5] = \frac{1}{2}e_4$, $[e_4; e_7] = \frac{1}{2}e_5$, $[e_5; e_7] = \frac{1}{2}e_5$, with at least 4 independent contact forms $e_1$, $e_2$, $e_3$, $e_4$. This latter Lie algebra has interesting structures [13].

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