Lattice calculation of the lowest-order hadronic contribution to the muon anomalous magnetic moment

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I present quenched domain wall fermion and 2+1 flavor improved Kogut-Susskind fermion calculations of the hadronic vacuum polarization which are used to calculate the $\mathcal{O}(\alpha^2)$ hadronic contribution to the anomalous magnetic moment of the muon. Together with previous quenched calculations, the new results confirm that in the quenched theory the hadronic contribution is significantly smaller ($\sim 30\%$) than the value obtained from the total cross section of $e^+ e^-$ annihilation to hadrons. The 2+1 flavor results show an increasing contribution to $g - 2$ as the quark mass is reduced.

1. Introduction

The anomalous magnetic moment of the muon ($g-2$) has been measured to incredible accuracy at Brookhaven National Lab’s E861 experiment [1] and has also been calculated precisely in the Standard Model using the dispersion relation for the magnetic moment of the muon. Together with previous quenched calculations, the new results confirm that in the quenched theory the hadronic contribution is significantly smaller ($\sim 30\%$) than the value obtained from the total cross section of $e^+ e^-$ annihilation to hadrons. The 2+1 flavor results show an increasing contribution to $g - 2$ as the quark mass is reduced.

**magnetic current**

$$
\int d^4x \, J^\mu(x)J^{\nu}(y)e^{iq(x-y)} =
(q^2 \gamma^{\mu\nu} - q^\mu q^\nu)\Pi(q^2),
$$

where $q$ is photon momentum. See [4,6] for details of the lattice calculation. The central idea is to calculate $\Pi(q^2)$ on the lattice, and fit it to obtain a continuous function that can be reliably integrated (numerically) with a known function from continuum perturbation theory from $q^2 = 0$ up to a cut-off $\sim (1/a)^2$, the rest of the integral being done using perturbation theory. For the quenched case, the (continuum) form of $\Pi(q^2)$ is known and provides an ansatz for the fit [8].

$$
\Pi(q^2) = \frac{f_V^2}{q^2 + m_V^2} + C \ln (a^2(q^2 + \mu^2)),
$$

where the first term comes from the vector meson bound state (delta function) and the second from a two particle continuum (cut, $q^2 \geq \mu^2$). Note that $f_V$ and $m_V$ can be extracted in the usual way from the zero momentum correlator as was done in [6] which leads to a more accurate fit, or treated as free parameters (here, I treat $f_V$ as a free parameter and take $m_V$ from [7]). This ansatz fits the data well and leads to (statistically) accurate results that are extrapolated to $q^2 = 0$, which is important because the low $q^2$ region dominates the one-loop integral that gives the hadronic contribution. For the 2+1 flavor case there is, of course, no such physical ansatz, excluding the experimental one which is, after all, the thing we’re trying to compute in the first place. In this case I try several forms: the pole fit just described,
a simple polynomial, a log for the low $q^2$ region, and combinations of these.

2. Results and Discussion

The quenched calculations with domain wall fermions were done with inverse spacing $a^{-1} \approx 1.3$ and 2 GeV on $16^3 \times 32$ lattices, and $m_{\text{val}} = 0.02$ and 0.04. $\Pi(q^2)$ is shown in Figure 1 and values of $a_\mu = (g-2)/2$ for a single quark with unit charge are summarized in Table 1. After including the u, d, and s quark charges, the total contribution for three degenerate quarks is consistent with previous quenched results[4,6], confirming that the hadronic contribution to $g-2$ is significantly less ($\sim 30\%$) than in the real world[2]. Thus it is probably not worthwhile to pursue further quenched calculations designed to eliminate systematic errors due to finite volume, non-zero lattice spacing, and unphysically large quark mass (which has been done to some extent in [4,6]).

![Figure 1. The hadronic vacuum polarization for quenched domain wall fermions, $a^{-1} = 1.3$ GeV, $m_{\text{val}} = 0.02$ (circles) and 0.04 (squares). The line shows the result from continuum 3-loop perturbation theory [8] for comparison.](image)

Instead, I have started a calculation on the 2+1 flavor lattices from the MILC collaboration that were generated using $a^2$–tad fermions (here I omit the Naik term for the valence quarks[9]). Initial results were presented last year [5], and currently $\Pi(q^2)$ is being calculated on new $40^3 \times 96$ lattices with $m_l = 0.0031$. Of course, these are very aggressive parameters, so the lattice generation is somewhat slow. At the time of the meeting some 84 configurations existed on which $\Pi(q^2)$ was calculated from a point-split current from a single site (and its nearest neighbors). These were separated by six trajectories and so are probably not independent. To improve statistics, I have begun calculating on a time-slice that is one-half the lattice size distant from the first. In addition the MILC collaboration plans to at least triple the length of the evolution ($\sim 3000$ trajectories).

In Figure 2 $\Pi(q^2)$ is shown for $m_{\text{val}} = m_l = 0.0031$ ($40^3 \times 96$) and 0.0062 ($28^3 \times 96$), or $0.1$ and $0.2$ times the strange quark mass, respectively. Note that the large volumes and time sizes used here lead to very small values of $q^2$ which is quite important. The data show a slight increase as $q^2 \to 0$ as $m_l$ decreases by a factor of two. While it appears small, a slight increase in this region changes the contribution to $g-2$ significantly, so the fit in this region must be quite accurate and precise. In Figure 3 I show covariant polynomial fits to $\Pi(q^2)$ which tend to under-predict the data as $q^2 \to 0$ (the $\chi^2$ of the cubic and quartic fits is acceptable, though). Pole fits like Eq. 2 do a bit worse, and a log fit does about the same. It seems that a better fit ansatz is needed.

Chiral perturbation theory offers the means to understand the mass dependence and the small $q^2$ behavior of $\Pi(q^2)$. This result does not appear in the literature, to the best of my knowledge, and even if it did, to be effective in this case, it would probably have to be augmented with so-called staggered chiral perturbation theory ($S\chi$PT) [9] to obtain accurate results. Indeed, recently the MILC collaboration has performed a fit to all of their 2+1 flavor data for the pseudo-scalar decay constant using $S\chi$PT[10]. The combination of a large data set and $S\chi$PT allowed for a $\sim 1$-2% determination of $f_K/f\pi$ which is quite remarkable. Since the vacuum polarization can be measured accurately on these same lattices, perhaps a similar fit will prove to be as precise here as there. I
am now considering such an analysis for $\Pi(q^2)$.

### Table 1

The magnetic anomaly $a_\mu$ for a single flavor with unit charge from quenched domain wall fermions.

| $a^{-1}$ | $m_{\text{val}}$ | configs | $(\alpha^2) a_\mu^{\text{val}}$ |
|----------|------------------|---------|----------------------------------|
| 1.31     | 0.02             | 337     | $750(35) \times 10^{-10}$       |
| 1.31     | 0.04             | 337     | $669(23) \times 10^{-10}$       |
| 1.98     | 0.02             | 299     | $730(51) \times 10^{-10}$       |

Figure 2. Same as Figure 1 except for 2+1 flavor improved Kogut-Susskind fermions, $a^{-1} \approx 2.3$ GeV, $m_\pi = 0.031$, $m_{\text{val}} = m_\pi = 0.0031$ (open circles) and 0.0062 (filled squares).

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