BRANCHED SL(r, C)-OPERS

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Abstract. Branched projective structures were introduced by Mandelbaum [22], [23], and opers were introduced by Beilinson and Drinfeld [2], [3]. We define the branched analog of SL(r, C)-opers and investigate their properties. For the usual SL(r, C)-opers, the underlying holomorphic vector bundle is actually determined uniquely up to tensoring with a holomorphic line bundle of order r. For the branched SL(r, C)-opers, the underlying holomorphic vector bundle depends more intricately on the oper. While the holomorphic connection for a branched SL(r, C)-oper is nonsingular, given a branched SL(r, C)-oper, we associate to it a certain holomorphic vector bundle equipped with a logarithmic connection. This holomorphic vector bundle in question supporting a logarithmic connection does not depend on the branched oper. We characterize the branched SL(r, C)-opers in terms of the logarithmic connections on this fixed holomorphic vector bundle.

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1. Introduction

Opers were introduced by Beilinson and Drinfeld \[2, 3\]. For this they were motivated by the earlier works of Drinfeld and Sokolov \[10, 11\]. Given a semisimple complex Lie group $G$, a $G$–oper on a compact Riemann surface $X$ consists of

- a holomorphic principal $G$–bundle $P$ on $X$ equipped with a holomorphic connection $\nabla$, and
- a holomorphic reduction of the structure group of $P$ to a Borel subgroup $B$ of $G$,

such that the reduction to $B$ satisfies an analogue of the Griffiths transversality condition with respect to this connection $\nabla$, and moreover the second fundamental form of the above reduction of structure group to $B$, for the connection $\nabla$, satisfies certain nondegeneracy conditions. Opers are useful in diverse topics, for example in geometric Langlands correspondence, nonabelian Hodge theory, some branches of mathematical physics, differential equations et cetera; see \[4, 12, 18, 17, 13, 14, 15, 21, 24, 7\] and references therein.

An SL(2, $\mathbb{C}$)–oper on a compact Riemann surface $X$ corresponds to a complex projective structure on $X$ together with a theta characteristic on $X$. We recall that a complex projective structure on $X$ is given by a covering of $X$ by holomorphic coordinate charts such that all the transition functions are restrictions of Möbius transformations \[20\]. During the early seventies, Mandelbaum introduced, and studied, the notion of a branched complex projective structure \[22, 23\]. Examples of branched complex projective structures are provided by pulling back the usual complex projective structures through a holomorphic ramified covering map; see \[6\] for a recent study of branched complex projective structures. For higher dimensional complex manifolds a branched analog of the more general notion of holomorphic Cartan geometries was introduced in \[5\].

Our aim here is to introduce, and investigate, the branched analog of SL($r, \mathbb{C}$)–opers.

Let $X$ be a compact connected Riemann surface, and let $S \subset X$ be a finite subset. The holomorphic cotangent bundle and the structure sheaf of $X$ are denoted by $K_X$ respectively $\mathcal{O}_X$.

A branched SL($r, \mathbb{C}$)–oper on $X$, with branching over $S$, is given by a triple of the form $(V, \mathcal{F}, D)$, where

- $V$ is a rank $r$ holomorphic vector bundle on $X$ such that $\bigwedge^r V = \mathcal{O}_X$,
- $\mathcal{F}$ is a filtration of holomorphic subbundles

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = V \quad (1.1)$$

with $\text{rank}(F_i) = i$, for all $1 \leq i \leq r$, and
- $D$ is a holomorphic connection on $V$ such that $D(F_i) \subset F_{i+1} \otimes K_X$, for all $1 \leq i \leq r - 1$, and $D$ induces the trivial connection on $\bigwedge^r V$,

such that for all $1 \leq i \leq r - 1$, the second fundamental form of $F_i$ vanishes exactly on the reduced divisor $S$ (see Definition \[21\]).
We recall that for a usual SL($r, \mathbb{C}$)-oper, the second fundamental forms of the filtration $\mathcal{F}$ are required to be nonzero everywhere.

For usual SL($r, \mathbb{C}$)-opers, i.e., when $S = 0$, the underlying holomorphic vector bundle $V$ supporting the holomorphic connection is very special. To explain this, note that any holomorphic line bundle $\xi$ on $X$ of order $r$ has a tautological holomorphic connection $D_\xi$ which is uniquely determined by the condition that the connection on $\xi^{\otimes r} = \mathcal{O}_X$ induced by $D_\xi$ has trivial monodromy. For any usual SL($r, \mathbb{C}$)-oper $(V, F, D)$, note that

$$(V \otimes \xi, F \otimes \xi, D \otimes \text{Id}_\xi + \text{Id}_V \otimes D_\xi)$$

is also a usual SL($r, \mathbb{C}$)-oper. If we fix a usual SL($r, \mathbb{C}$)-oper $(V_0, F_0, D_0)$ on $X$, then for any usual SL($r, \mathbb{C}$)-oper $(V', F', D')$ on $X$, there is a holomorphic line bundle $\xi$ on $X$ of order $r$ such that $V' = V_0 \otimes \xi$. Moreover, we have

$$F' = F_0 \otimes \xi;$$

so the filtration for a usual SL($r, \mathbb{C}$)-oper also does not depend on the SL($r, \mathbb{C}$)-oper. In the case of usual SL($r, \mathbb{C}$)-opers, the underlying holomorphic vector bundle $V$ in (1.1) is a jet bundle, and the filtration $\mathcal{F}$ is the natural filtration of a jet bundle.

However, for branched SL($r, \mathbb{C}$)-opers the underlying vector bundle $V$ supporting the holomorphic connection depends more intimately on the oper, although each graded piece $F_{i+1}/F_i$ for the filtration $\mathcal{F}$ in (1.1) is actually independent of the branched SL($r, \mathbb{C}$)-oper. More precisely, $F_{i+1}/F_i$ is independent of the branched SL($r, \mathbb{C}$)-oper if $r$ is odd, and when $r$ is even, the quotients $F_{i+1}/F_i$ are determined by the theta characteristic on $X$ associated to the branched oper.

Denote the above quotient line bundle $F_r/F_{r-1}$ by $Q$. We prove that a branched SL($r, \mathbb{C}$)-oper produces a logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$, where $J^{r-1}(Q)$ is the $(r-1)$-th jet bundle of $Q$; see Proposition 3.3. Given the logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ associated to a branched SL($r, \mathbb{C}$)-oper, the branched SL($r, \mathbb{C}$)-oper can actually be recovered using the Hecke transformations on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ and the induced logarithmic connection on the Hecke transformations. The residues of these logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$, and also the local properties of the logarithmic connection, are studied here. Based on these studies, we characterize all logarithmic connections on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ that arise from branched SL($r, \mathbb{C}$)-opers; see Theorem 6.1.

The branched SL($r, \mathbb{C}$)-opers are closely related with the SL($r, \mathbb{C}$)-opers with regular singularity introduced by Beilinson and Drinfeld; this is explained in Section 6.1, which was kindly communicated to us by Edward Frenkel.

Once the branched SL($r, \mathbb{C}$)-opers have been defined, it is straightforward to define the branched orthogonal and branched symplectic opers. We have omitted this exercise.

2. Branched SL($r$)-opers

As before, $X$ is a compact connected Riemann surface of genus $g$. The holomorphic tangent bundle of $X$ is denoted by $TX$. 
Let $V$ be a holomorphic vector bundle on $X$. A logarithmic connection on $V$ singular over a reduced effective divisor $S$ is a holomorphic differential operator
\[ D : V \longrightarrow V \otimes K_X \otimes \mathcal{O}_X(S) \]
satisfying the Leibniz identity which states that
\[ D(fs) = fD(s) + s \otimes df \quad (2.1) \]
for any locally defined holomorphic function $f$ on $X$ and any locally defined holomorphic section $s$ of $V$. If $S$ is the zero divisor, then $D$ is called a holomorphic connection [1]. Any logarithmic connection on $V$ is integrable (same as flat) because $\Omega^2_X = 0$. A criterion of Weil and Atiyah says that $V$ admits a holomorphic connection if and only if the degree of every indecomposable component of $V$ is zero [1, p. 203, Theorem 10], [25]. In particular, if $V$ is indecomposable, and degree($V$) = 0, then $V$ admits a holomorphic connection.

Let $D$ be a logarithmic connection on $V$. For a holomorphic subbundle $F \subset V$, consider the following composition of homomorphisms
\[ F \hookrightarrow V \xrightarrow{D} V \otimes K_X \otimes \mathcal{O}_X(S) \xrightarrow{q_0 \otimes \text{Id}_{K_X} \otimes \mathcal{O}_X(S)} (V/F) \otimes K_X \otimes \mathcal{O}_X(S), \quad (2.2) \]
where $q_0 : V \longrightarrow V/F$ is the natural quotient map. From (2.1) it follows that this composition of homomorphisms is actually $\mathcal{O}_X$–linear, and hence it produces a holomorphic section
\[ \text{SF}(D, F) \in H^0(X, \text{Hom}(F, V/F) \otimes K_X \otimes \mathcal{O}_X(S)) = H^0(X, F^* \otimes (V/F) \otimes K_X \otimes \mathcal{O}_X(S)), \quad (2.3) \]
which is known as the second fundamental form of $F$ for the logarithmic connection $D$.

If $D$ is a holomorphic connection on $V$, then the composition of homomorphisms in (2.2) simplifies as
\[ F \hookrightarrow V \xrightarrow{D} V \otimes K_X \xrightarrow{q_0 \otimes \text{Id}_{K_X}} (V/F) \otimes K_X, \quad (2.4) \]
and it gives the following analogue of (2.3)
\[ \text{SF}(D, F) \in H^0(X, \text{Hom}(F, V/F) \otimes K_X) = H^0(X, F^* \otimes K_X \otimes (V/F)), \quad (2.5) \]
which is the second fundamental form of $F$ for the holomorphic connection $D$.

Assume that the rank $r$ of $V$ is at least two, and $D$ is, as before, a holomorphic connection on $V$. Let
\[ 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = V \]
be a filtration of holomorphic subbundles such that for all $1 \leq i \leq r-1$,
\begin{itemize}
  \item rank($F_i$) = $i$, and
  \item $D(F_i) \subset F_{i+1} \otimes K_X$.
\end{itemize}

Consequently, the second fundamental form $\text{SF}(D, F_i)$ in (2.3) satisfies the following condition:
\[ \text{SF}(D, F_i)(F_i) \subset (F_{i+1}/F_i) \otimes K_X \subset (V/F_i) \otimes K_X \]
for all $1 \leq i \leq r-1$. Therefore, $\text{SF}(D, F_i)$ produces a holomorphic homomorphism
\[ \text{SF}(D, i) : F_i/F_{i-1} \longrightarrow (F_{i+1}/F_i) \otimes K_X \quad (2.6) \]
for every $1 \leq i \leq r - 1$.

Assume that there is holomorphic line bundle $L_0$ on $X$ such that the line bundle $\text{Hom}(F_i/F_{i-1}, (F_{i+1}/F_i) \otimes K_X) = (F_{i+1}/F_i) \otimes (F_i/F_{i-1})^* \otimes K_X$ is holomorphically isomorphic to $L_0$ for all $1 \leq i \leq r - 1$. This implies that

$$F_{i+1}/F_i = (F_i/F_{i-1}) \otimes (L_0 \otimes TX) = (F_i/F_{i-1}) \otimes L,$$

where $L := L_0 \otimes TX$. Then we have

$$\det V := \bigwedge^r V = \bigotimes_{j=1}^r F_j/F_{j-1} = F_1^\otimes r \otimes L^\otimes \frac{(r-1)}{2} = (F_r/F_{r-1})^\otimes r \otimes \left(L^\otimes \frac{(r-1)}{2}\right)^*.$$ 

(2.8)

Fix a reduced effective divisor

$$S := \sum_{k=1}^d x_k$$

on $X$ of degree $d \geq 0$; so $\{x_1, \cdots, x_d\}$ are distinct points of $X$. We will introduce branched $SL(r, \mathbb{C})$–opers on $X$ with branching over $S$.

When $r$ is an even integer, we assume that $d$ is also even. If $r$ is an even integer, fix a holomorphic line bundle $\mathcal{L}$ on $X$ of degree $1 + \frac{d}{2} - g \in \mathbb{Z}$ such that

$$\mathcal{L}^\otimes 2 = TX \otimes \mathcal{O}_X(S);$$

(2.10)

also, fix a holomorphic isomorphism of $\mathcal{L}^\otimes 2$ with $TX \otimes \mathcal{O}_X(S)$. Note that if $S$ is the zero divisor, then the dual $\mathcal{L}^*$ is a theta characteristic on $X$.

**Definition 2.1.** A branched $SL(r, \mathbb{C})$–oper over $X$ with branching over $S$ is a triple

$$(V, \mathcal{F}, D),$$

where

- $V$ is a rank $r$ holomorphic vector bundle on $X$ such that $\bigwedge^r V = \mathcal{O}_X$,
- $\mathcal{F}$ is a filtration of holomorphic subbundles

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = V$$

(2.11)

with rank$(F_i) = i$, for all $1 \leq i \leq r$, and
- $D$ is a holomorphic connection on $V$,

such that the following five conditions hold:

1. if $r$ is even, then $F_1 = \left(\mathcal{L}^\otimes (r-1)\right)^*$, where $\mathcal{L}$ is a line bundle as in (2.10), and if $r$ is odd, then $F_1 = \left((TX \otimes \mathcal{O}_X(S))^\otimes \frac{(r-1)}{2}\right)^*$,
2. $F_{j+1}/F_j \simeq F_1 \otimes (TX \otimes \mathcal{O}_X(S))^\otimes j$, for all $0 \leq j \leq r - 1$,
3. the connection on $\bigwedge^r V = \mathcal{O}_X$ induced by the connection $D$ on $V$ coincides with the trivial connection given by the de Rham differential $d$ on $\mathcal{O}_X$,
4. $D(F_i) \subset F_{i+1} \otimes K_X$, for all $1 \leq i \leq r - 1$, and
the holomorphic connection on the dual vector bundle $V^*$, actually independent of the branched \( \text{SL}(r, \mathbb{C}) \text{–oper} \).

Definition 2.1. Using the above observation we conclude that these two spaces are actually two different spaces of branched \( \text{SL}(r, \mathbb{C}) \text{–oper} \).

Remark 2.2. Note that the above condition that $F_{j+1}/F_j = F_1 \otimes (TX \otimes \mathcal{O}_X(S)) \otimes \mathcal{O}_X(S)$ implies that $(F_{i+1}/F_i) = (F_i/F_{i-1}) \otimes TX \otimes \mathcal{O}_X(S)$. The fifth condition that $SF(D, i)$ is the natural inclusion map coincides with the condition that $SF(D, i)$ is given by tensoring with the section of $\mathcal{O}_X(S)$ defined by the constant function 1 on $X$.

Remark 2.3. Comparing (2.12) and (2.7) we conclude that in Definition 2.1, the holomorphic line bundle $\mathcal{O}_X(S)$ plays the role of $L_0$ in (2.7). So in Definition 2.1, the holomorphic line bundle $\mathcal{O}_X(S) \otimes TX$ plays the role of $L$ in (2.8). The second condition in Definition 2.1 that $F_{j+1}/F_j \simeq F_1 \otimes (TX \otimes \mathcal{O}_X(S)) \otimes \mathcal{O}_X(S)$ is obtained from (2.7), and the first condition in Definition 2.1 is motivated by (2.8), because $\bigwedge V = \mathcal{O}_X$.

Remark 2.4. Any two choices of the holomorphic line bundle $\mathcal{L}$ in (2.10) differ by tensoring with a line bundle on $X$ of order two. Any holomorphic line bundles $\xi$ of order two on $X$ has a unique holomorphic connection $D_\xi$ satisfying the following condition: Any holomorphic isomorphism between $\xi \otimes 2$ and $\mathcal{O}_X$ takes the connection $D_\xi \otimes \text{Id}_\xi + \text{Id}_\xi \otimes D_\xi$ on $\xi \otimes 2$ to the connection on $\mathcal{O}_X$ given by the de Rham differential $d$. If $W$ is a holomorphic vector bundle on $X$ equipped with a holomorphic connection $D_W$, then $D_\xi$ and $D_W$ together produce a holomorphic connection $D_W \otimes \text{Id}_\xi + \text{Id}_W \otimes D_\xi$ on $W \otimes \xi$. When $r$ is even, for any two choices of the holomorphic line bundle $\mathcal{L}$ in (2.10), a priori, we have two different spaces of branched $\text{SL}(r, \mathbb{C})$–opers over $X$, with branching over $S$, given by Definition 2.1. Using the above observation we conclude that these two spaces are actually canonically identified.

When $S$ in (2.9) is the zero divisor, a branched $\text{SL}(r, \mathbb{C})$–oper is a usual $\text{SL}(r, \mathbb{C})$–oper [2], [3]. In that special (unbranched) case, the underlying holomorphic vector bundle $V$ of an $\text{SL}(r, \mathbb{C})$–oper does not depend on the oper in the following sense: The vector bundle $V$ is the jet bundle $\mathcal{J}^{r-1}(\mathcal{L}^{(r-1)} \otimes \xi)$, where $\mathcal{L}$ is a fixed holomorphic line bundle on $X$ such that $\mathcal{L}^2 = TX$ while $\xi$ is a holomorphic line bundle on $X$ of order $r$. Contrary to the unbranched case, when degree($S$) = $d > 0$, the underlying holomorphic vector bundle $V$ of a $\text{SL}(r, \mathbb{C})$–oper with branching over $S$ generally depends on the branched oper. Nevertheless the successive quotients for the filtration $\mathcal{F}$ of $V$ are clearly independent of the branched $\text{SL}(r, \mathbb{C})$–oper. In Section 3 we will show that a branched $\text{SL}(r, \mathbb{C})$–oper $(V, F, D)$ with branching over $S$ gives rise to a logarithmic connection on a certain holomorphic vector bundle over $X$ obtained by performing Hecke transformation on $V$ over the points of $S$. The holomorphic vector bundle in question turns out to be actually independent of the branched $\text{SL}(r, \mathbb{C})$–oper.

Take a branched $\text{SL}(r, \mathbb{C})$–oper $(V, F, D)$ over $X$ with branching over $S$. Let $D^*$ be the holomorphic connection on the dual vector bundle $V^*$ induced by the connection $D$.
The filtration $\mathcal{F}$ of $V$ in (2.11) produces a filtration of $V^*$ by holomorphic subbundles as follows: Consider the dual homomorphisms of the inclusion maps in (2.11)

$$V^* = F^*_r \rightarrow F^*_{r-1} \rightarrow \cdots \rightarrow F^*_2 \rightarrow F^*_1 \rightarrow F^*_0 = 0;$$

the subbundles of $V^*$ giving the filtration are the kernels of the above projections $V^* \rightarrow F^*_i$. This filtration by holomorphic subbundles of $V^*$ will be denoted by $\mathcal{F}^*$.

Lemma 2.5. The above triple $(V^*, \mathcal{F}^*, D^*)$ is a branched $\text{SL}(r, \mathbb{C})$–oper over $X$ with branching over $S$.

Proof. As in (2.4), take a holomorphic subbundle $F \subset V$. So we have the holomorphic subbundle $(V/F)^* \subset V^*$. Note that the quotient bundle $V^*/((V/F)^*)$ is identified with $F^*$. Let

$$\text{SF}(D^*, (V/F)^*) \in H^0(X, \text{Hom}((V/F)^*, V^*/((V/F)^*)) \otimes K_X)$$

$$= H^0(X, \text{Hom}((V/F)^*, F^*) \otimes K_X) = H^0(X, (V/F) \otimes F^* \otimes K_X)$$

be the second fundamental form of $(V/F)^* \subset V^*$ for the connection $D^*$ on $V^*$ (see (2.5)). Then it is straightforward to check that

$$\text{SF}(D^*, (V/F)^*) = \text{SF}(D, F), \quad (2.13)$$

where $\text{SF}(D, F)$ is constructed in (2.5). The lemma is a straightforward consequence of (2.13). $\square$

2.1. Examples. We now give some examples of branched $\text{SL}(r, \mathbb{C})$–oper.

Let

$$\varpi : X \rightarrow Y$$

be a nonconstant holomorphic map between compact connected Riemann surfaces such that all the branch points of $\varpi$ have branch number 1, and let $(V, \mathcal{F}, D)$ be a usual $\text{SL}(r, \mathbb{C})$–oper over $Y$. Then the pullback $(\varpi^*V, \varpi^*\mathcal{F}, \varpi^*D)$ is a branched $\text{SL}(r, \mathbb{C})$–oper over $X$. The branching divisor for $(\varpi^*V, \varpi^*\mathcal{F}, \varpi^*D)$ coincides with the branching divisor for the map $\varpi$.

Let $\varpi : X \rightarrow \mathbb{C}P^1$ be any nonconstant holomorphic map such that all the branch points of $\varpi$ have branch number 1. Then pulling back the standard $\text{SL}(2, \mathbb{C})$–oper on $\mathbb{C}P^1$ by $\varpi$ we get a branched $\text{SL}(2, \mathbb{C})$–oper over $X$.

More generally, let $V$ be a rank two holomorphic vector bundle on $X$ with $\bigwedge^2 V = \mathcal{O}_X$, and let $D$ be a holomorphic connection on $V$ such that the induced holomorphic connection on $\bigwedge^2 V$ coincides with the holomorphic connection on $\mathcal{O}_X$ given by the de Rham differential $d$. Let $L \subset V$ be a holomorphic line subbundle satisfying the following two conditions:

1. the second fundamental form $\text{SF}(D, L)$ is not identically zero, and
2. all the zeros of $\text{SF}(D, L)$ are of order one.

Then $(V, L, D)$ is a branched $\text{SL}(2, \mathbb{C})$–oper on $X$. The branching divisor for $(V, L, D)$ coincides with the vanishing divisor of $\text{SF}(D, L)$. The previous example, given by the pull-back of the standard $\text{SL}(2, \mathbb{C})$–oper on $\mathbb{C}P^1$, actually corresponds to the special case.
where \( V = \varpi^* \mathcal{O}_{\mathbf{CP}^1}^{\oplus 2} = \mathcal{O}_{X}^{\oplus 2} \), \( D \) is the trivial connection on it, and \( L \) is the pull-back, by \( \varpi \), of the tautological line subbundle of \( \mathcal{O}_{\mathbf{CP}^1}^{\oplus 2} \).

Let \((V, L, D)\) be a branched \( \text{SL}(2; \mathbb{C}) \)-oper over \( X \). Then the holomorphic line subbundle \( L \subset V \) produces a filtration of holomorphic subbundles \( \{F_{i}\}_{i=1}^{r} \) of the symmetric product \( \text{Sym}^{r-1}(V) \) whose \( i \)-th term \( F_{i} \) is the image of \( L^{\otimes (r-i)} \otimes V^{\otimes (i-1)} \) in \( \text{Sym}^{r-1}(V) \). The triple \((\text{Sym}^{r-1}(V), \{F_{i}\}_{i=1}^{r}, \tilde{D})\), where \( \tilde{D} \) is the holomorphic connection on \( \text{Sym}^{r-1}(V) \) induced by \( D \), is a branched \( \text{SL}(r; \mathbb{C}) \)-oper over \( X \). Its branching divisor coincides with that of \((V, L, D)\).

3. Logarithmic connection from branched \( \text{SL}(r; \mathbb{C}) \)-opers

3.1. Homomorphism to a jet bundle. As in Definition \( 2.1 \), let

\[(V, \mathcal{F}, D)\]  

be a branched \( \text{SL}(r; \mathbb{C}) \)-oper over \( X \), with branching over \( S \), where \( \mathcal{F} \) stands for the filtration in \((2.11)\) of holomorphic subbundles of \( V \).

In this subsection we will construct another filtered holomorphic vector bundle from the branched \( \text{SL}(r; \mathbb{C}) \)-oper \((V, \mathcal{F}, D)\) in \((3.1)\).

For notational convenience, let

\[Q := F_{r}/F_{r-1} = F_{1} \otimes (TX \otimes \mathcal{O}_{X}(S))^{\otimes (r-1)}\]  

(3.2)

denote the quotient line bundle in \((2.11)\); the second statement in Definition \( 2.1 \) gives the above isomorphism. We will construct a holomorphic homomorphism

\[\Phi : V \rightarrow J^{r-1}(Q)\]  

(3.3)

to the \((r-1)\)-th order jet bundle of the line bundle \( Q \) in \((3.2)\); see \[20, \ 6\] for jet bundles.

Let

\[q : V \rightarrow Q = V/F_{r-1}\]  

(3.4)

be the natural quotient map. Take any \( x \in X \) and also take any \( v \in V_{x} \). Consider a simply connected open neighborhood \( x \in U \subset X \) of \( x \), and denote by

\[\tilde{v} \in H^{0}(U, V)\]

the unique flat section of \( V|_{U} \), for the connection \( D \), such that \( \tilde{v}(x) = v \). Restricting the section \( q(\tilde{v}) \in H^{0}(U, Q) \) to the \((r-1)\)-th order infinitesimal neighborhood of \( x \), where \( q \) is the quotient map in \((3.4)\), we get an element

\[\tilde{v}' := q(\tilde{v})|_{rx} \in J^{r-1}(Q)_{x},\]

where \( rx \) is the nonreduced divisor with multiplicity \( r \). The map \( \Phi \) in \((3.3)\) sends \( v \) to this element \( \tilde{v}' \in J^{r-1}(Q)_{x} \) constructed from \( v \) using the connection \( D \). The homomorphism \( \Phi \) is evidently holomorphic.

We have the natural short exact sequence of jet bundles

\[0 \rightarrow Q \otimes K_{X}^{\otimes k} \rightarrow J^{k}(Q) \rightarrow J^{k-1}(Q) \rightarrow 0\]  

(3.5)

for all \( k \geq 1 \). For \( 0 \leq j \leq r \), let \( H_{j} \) be the kernel of the projection \( J^{r-1}(Q) \rightarrow J^{r-1-j}(Q) \) obtained by iterating the projection in \((3.5)\); we use the convention that
\[ J^k(W) = 0 \text{ if } k < 0. \] So \( \text{rank}(H_j) = j \), and we have the short exact sequence of holomorphic vector bundles
\[
0 \longrightarrow H_j \longrightarrow J^{r-1}(Q) \longrightarrow J^{r-1-j}(Q) \longrightarrow 0 \quad (3.6)
\]
on \( X \). From (3.6) it follows that the quotient \( H_j/H_{j-1} \) coincides with the kernel of the projection \( J^{r-j}(Q) \longrightarrow J^{r-j-1}(Q) \). Therefore, from (3.5) it follows that
\[
H_j/H_{j-1} = Q \otimes K_X^{\otimes(r-j)} \quad (3.7)
\]
for all \( 1 \leq j \leq r \).

Let
\[
X_0 := X \setminus \{ x_1, \ldots, x_d \} = X \setminus S \subset X \quad (3.8)
\]
be the complement of \( S \) in (2.9).

The triple \((V, F, D)\) in (3.1) defines a usual \( \text{SL}(r, \mathbb{C})\)-oper over the open subset \( X_0 \subset X \) in (3.3). The following proposition is well-known for the usual \( \text{SL}(r, \mathbb{C})\)-opers.

**Proposition 3.1.** The restriction \( \Phi|_{X_0} \) of the homomorphism in (3.3) to \( X_0 \) is an isomorphism \( V|_{X_0} \sim J^{r-1}(Q)|_{X_0} \).

For all \( 0 \leq j \leq r \), the homomorphism \( \Phi|_{X_0} \) takes the subbundle \( F_j|_{X_0} \) (defined in (2.11)) isomorphically to the subbundle \( H_j|_{X_0} \) (constructed in (3.6)).

From Proposition 3.1 it follows that when \( S \) is the zero divisor, then
\[
F_r = V = J^{r-1}(Q). \]

The following is a consequence of Proposition 3.1.

**Corollary 3.2.** The homomorphism of \( \mathcal{O}_X \)-modules \( \Phi \) in (3.3) is injective. It satisfies the condition
\[
\Phi(F_j) \subset H_j
\]
for all \( 0 \leq j \leq r \), where \( F_j \) and \( H_j \) are defined in (2.11) and (3.6) respectively.

**Proof.** Since \( \Phi|_{X_0} \) is injective it follows immediately that \( \Phi \) is injective. Since
\[
\Phi|_{X_0}(F_j|_{X_0}) \subset H_j|_{X_0}
\]
(see Proposition 3.1), and \( X_0 \) is a dense subset of \( X \), we conclude that \( \Phi(F_j) \subset H_j \). \( \square \)

From Corollary 3.2 it follows that \( \Phi \) produces a grading preserving holomorphic homomorphism from the graded vector bundle \( \bigoplus_{j=1}^r F_j/F_{j-1} \) in (2.11) to the graded vector bundle \( \bigoplus_{j=1}^r H_j/H_{j-1} \) in (3.6). For any \( 1 \leq j \leq r \), let
\[
\Phi_j : F_j/F_{j-1} \longrightarrow H_j/H_{j-1} = Q \otimes K_X^{\otimes(r-j)} \quad (3.9)
\]
be the homomorphism induced by \( \Phi \); see (3.7) for the isomorphism in (3.9).

For any \( 1 \leq j \leq r \), note that \( Q \otimes K_X^{\otimes(r-j)} \otimes \mathcal{O}_X(-(r-j)S) \) is actually a subsheaf of \( Q \otimes K_X^{\otimes(r-j)} \) because \( S \) is an effective divisor. The following proposition identifies this subsheaf \( Q \otimes K_X^{\otimes(r-j)} \otimes \mathcal{O}_X(-(r-j)S) \) with the image of the homomorphism \( \Phi_j \).
Proposition 3.3. For each $1 \leq j \leq r$, the image of the homomorphism $\Phi_j$ in (3.9) is the subsheaf

$$Q \otimes K_X^{(r-j)} \otimes \mathcal{O}_X(-(r-j)S) \subset Q \otimes K_X^{(r-j)} = H_j/H_{j-1}.$$  

Proof. Recall the homomorphisms $\text{SF}(D, i)$ in statement (5) of Definition 2.1. Take any $1 \leq k \leq r - 1$. Consider the composition of homomorphisms

$$(\text{SF}(D, r - 1) \otimes \text{Id}_{K_X^{(r-1-k)}}) \circ \cdots \circ (\text{SF}(D, k+1) \otimes \text{Id}_{K_X}) \circ \text{SF}(D, k) : F_k/F_{k-1} \rightarrow (F_r/F_{r-1}) \otimes K_X^{(r-k)} = Q \otimes K_X^{(r-k)} = (F_k/F_{k-1}) \otimes \mathcal{O}_X((r-k)S);$$

(3.10)

note that since $(F_{i+1}/F_i) \otimes K_X = (F_i/F_{i-1}) \otimes \mathcal{O}_X(S)$ for all $1 \leq i \leq r - 1$ (see Remark 2.2), it follows that $(F_r/F_{r-1}) \otimes K_X^{(r-k)} = (F_k/F_{k-1}) \otimes \mathcal{O}_X((r-k)S)$. From statement (5) in Definition 2.1, we know that the composition of homomorphisms in (3.10) coincides with the homomorphism map

$$F_k/F_{k-1} \hookrightarrow (F_k/F_{k-1}) \otimes \mathcal{O}_X((r-k)S).$$

In other words, if $1 \in H^0(X, \mathcal{O}_X((r-k)S))$ is the section given by the constant function $1$ on $X$, then the composition of homomorphisms in (3.10) coincides with the homomorphism

$$\text{Id}_{F_k/F_{k-1}} \otimes 1 : F_k/F_{k-1} \rightarrow (F_k/F_{k-1}) \otimes \mathcal{O}_X((r-k)S).$$

On the other hand, we have $Q \otimes K_X^{(r-k)} = H_k/H_{k-1}$ (see (3.7)). Therefore, the image of the composition of homomorphisms in (3.10) is the subsheaf

$$(H_k/H_{k-1}) \otimes \mathcal{O}_X(-(r-k)S) = Q \otimes K_X^{(r-k)} \otimes \mathcal{O}_X(-(r-k)S) \subset Q \otimes K_X^{(r-k)}.$$

This completes the proof. \hfill \Box

For $0 \leq j \leq r$, define

$$\widehat{H}_j := H_j \otimes \mathcal{O}_X(-(r-1)S) \subset J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S),$$

where $H_j$ is constructed in (3.6). So the filtration $\{H_j\}_{j=0}^r$ of $J^{r-1}(Q)$ produces a filtration of holomorphic subbundles

$$0 = \widehat{H}_0 \subset \widehat{H}_1 \subset \cdots \subset \widehat{H}_{r-1} \subset \widehat{H}_r = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$$

(3.11)

of $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$.

The next result is deduced using Proposition 3.3 and Corollary 3.2.

Corollary 3.4. The natural inclusion map $I : J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \hookrightarrow J^{r-1}(Q)$ factors through $\Phi$ in (3.3), meaning there is a unique homomorphism

$$i : J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \rightarrow V$$

such that $I = \Phi \circ i$. For all $0 \leq j \leq r$,

$$i(\widehat{H}_j) = F_j,$$

where $\widehat{H}_j$ and $F_j$ are as in (3.11) and (2.11), and moreover, $\Phi(F_j) = H_j.$
Proof. By Proposition 3.3, the image of the injective homomorphism \( \Phi \) in (3.9) is the subsheaf
\[
Q \otimes K_X^{(r-j)} \otimes \mathcal{O}_X(-(r-j)S) \subset Q \otimes K_X^{(r-j)} = H_j/H_{j-1}.
\]
Consequently, we have the following inclusions:
\[
\tilde{H}_j/\tilde{H}_{j-1} = (H_j/H_{j-1}) \otimes \mathcal{O}_X(-(r-1)S) \hookrightarrow F_j/F_{j-1}
\]
\[
= (H_j/H_{j-1}) \otimes \mathcal{O}_X(-(r-j)S) \hookrightarrow H_j/H_{j-1},
\]
where the isomorphism \( F_j/F_{j-1} = (H_j/H_{j-1}) \otimes \mathcal{O}_X(-(r-j)S) \) is given by \( \Phi_j \). In view of Corollary 3.2, from these inclusion maps we conclude that the homomorphism \( \Phi \) in (3.3) satisfies the following:
\[
\tilde{H}_r := J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \hookrightarrow \Phi(V) \subset J^{-1}(Q).
\]
Since the coherent analytic sheaf \( V \) is identified with the coherent analytic sheaf \( \Phi(V) \) using the isomorphism \( \Phi \) between them, the homomorphism \( \iota' \) produces the homomorphism \( i \) in the statement of the corollary, in other words, \( i \) is defined by the equality \( \iota' = \Phi \circ i \).

The map \( \iota' \) in (3.13) evidently takes \( \tilde{H}_j \) in (3.11) into \( \Phi(F_j) \). Also, the inclusion map \( \Phi(V) \hookrightarrow J^{-1}(Q) \) in (3.13) takes \( \Phi(F_j) \) into the subbundle \( \tilde{H}_j \subset J^{-1}(Q) \) in (3.6).

In view of Corollary 3.4, the filtration of holomorphic subbundles in (3.11) fits in the following filtration of coherent analytic subsheaves:
\[
0 = \tilde{H}_0 \subset \tilde{H}_1 \subset \cdots \subset \tilde{H}_{r-1} \subset \tilde{H}_r = J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \subset V. \tag{3.14}
\]

3.2. A logarithmic connection. Take a branched SL\((r, \mathbb{C})\)-oper \( (V, \mathcal{F}, D) \) as in (3.1). The following proposition shows that the holomorphic connection \( D \) produces a logarithmic connection on the holomorphic vector bundle \( J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \).

Proposition 3.5. The holomorphic connection \( D : V \to V \otimes K_X \) in (3.1) sends the subsheaf \( \tilde{H}_r := J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \subset V \) in (3.14) to the subsheaf
\[
(J^{-1}(Q) \otimes \mathcal{O}_X(-(r-2)S) \otimes K_X) \bigcap (V \otimes K_X) \subset V \otimes K_X.
\]
In other words, \( D \) produces a logarithmic connection, singular over \( S \), on the holomorphic vector bundle \( J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \).

Proof. Recall that \( V|_{X_0} = J^{-1}(Q)|_{X_0} = (J^{-1}(Q) \otimes \mathcal{O}_X(-(r-2)S))|_{X_0} \), where \( X_0 \) is the open subset in (3.8). So we need to investigate the connection \( D \) only around the points of \( S \). Take any point
\[
x' \in S.
\]
Fix a holomorphic splitting of the filtration \( \{F_i\}_{i=0} \) of \( V \) in (2.11) over a sufficiently small analytic neighborhood \( U \) of \( x' \); this subset \( U \) is chosen such that \( U \cap S = x' \). So we have a holomorphic isomorphism
\[
V|_U \sim \bigoplus_{i=1}^r (F_i/F_{i-1})|_U =: \bigoplus_{i=1}^r \mathcal{F}_i \tag{3.15}
\]
using the notation \( \mathcal{F}_i := (F_i/F_{i-1})|_{U} \); it should be clarified that there is no natural isomorphism. We would express the holomorphic connection \( D|_U \) in (3.1) in terms of the
decomposition in (3.15). From the fifth condition in Definition 2.1 it follows that $D|_U$ has the following expression in terms of the decomposition in (3.15):

$$
D|_U = \begin{pmatrix}
D_1 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \cdots & \alpha_{1,r-2} & \alpha_{1,r-1} & \alpha_{1,r} \\
\gamma_1 & D_2 & \alpha_{2,3} & \alpha_{2,4} & \cdots & \alpha_{2,r-2} & \alpha_{2,r-1} & \alpha_{2,r} \\
0 & \gamma_2 & D_3 & \alpha_{3,4} & \cdots & \alpha_{3,r-2} & \alpha_{3,r-1} & \alpha_{3,r} \\
0 & 0 & \gamma_3 & D_4 & \cdots & \alpha_{4,r-2} & \alpha_{4,r-1} & \alpha_{4,r} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & D_{r-2} & \alpha_{r-2,r-1} & \alpha_{r-2,r} \\
0 & 0 & 0 & 0 & \cdots & D_{r-1} & \alpha_{r-1,r-1} & \alpha_{r-1,r} \\
0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{r-1} & D_r
\end{pmatrix}
$$

(3.16)

where $D_i$ (the entry at the $i \times i$-th position of the matrix) is a holomorphic connection on $\mathcal{F}_i$, and $\gamma_i$ (the entry at the $(i+1) \times i$-th position) is a section

$$
\gamma_i \in H^0(U, \text{Hom}(\mathcal{F}_i, \mathcal{F}_{i+1}) \otimes (K_X|_U) \otimes \mathcal{O}_U(-x')) ;
$$

(3.17)

in other words, $\gamma_i$ is a holomorphic homomorphism $\mathcal{F}_i \rightarrow \mathcal{F}_{i+1} \otimes (K_X|_U)$ that vanishes at the point $x' \in U$. For any $j > i$, we have

$$
\alpha_{i,j} \in H^0(U, \text{Hom}(\mathcal{F}_j, \mathcal{F}_i) \otimes (K_X|_U)) .
$$

The entry at the $i \times j$-th position of the matrix in (3.16) is zero if $i > j + 1$. It may be mentioned that by choosing the splitting in (3.15) carefully it is possible to make $\alpha_{i,j} = 0$ for all $j > i$; but this will not be needed here.

The decomposition of $V|_U$ in (3.15) produces a holomorphic decomposition of the vector bundle $(J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))|_U$. To see this, consider the intersection of coherent analytic subsheaves of $V|_U$

$$
\mathcal{G}_i := \mathcal{F}_i \cap (J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))|_U = \mathcal{F}_i \cap (J^{-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x')) \subset V|_U ;
$$

(3.18)

note that both $\mathcal{F}_i$ and $J^{-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x')$ are subsheaves of $V|_U$ (see Corollary 3.3 and (3.15)). Then we have a holomorphic decomposition of $J^{-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x')$

$$
J^{-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x') = \bigoplus_{i=1}^{r} \mathcal{G}_i
$$

(3.19)

into a direct sum of holomorphic line bundles on $U$. Indeed, the natural homomorphism of coherent analytic sheaves

$$
\bigoplus_{i=1}^{r} \mathcal{G}_i \rightarrow J^{-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x')
$$

is clearly surjective; it is also injective because it is injective over the open subset $U \setminus \{x'\} \subset U$ and the coherent analytic sheaf $\bigoplus_{i=1}^{r} \mathcal{G}_i$ is torsionfree. From (3.12) it follows immediately that

- $\mathcal{G}_i = \mathcal{F}_i \otimes \mathcal{O}_U(-(i-1)x')$, and
- the inclusion map $\mathcal{G}_i \hookrightarrow \mathcal{F}_i$ (see (3.18)) coincides with the natural inclusion map

$$
\mathcal{F}_i \otimes \mathcal{O}_U(-(i-1)x')) \hookrightarrow \mathcal{F}_i .
$$
Now it is straight-forward to check that the connection operator $D\big|_U$ in (3.16) on $\bigoplus_{i=1}^r \mathcal{F}_i$ produces a holomorphic differential operator

$$\bigoplus_{i=1}^r \mathcal{G}_i \rightarrow \left( \bigoplus_{i=1}^r \mathcal{G}_i \right) \otimes (K_X|_U) \otimes \mathcal{O}_U(x') .$$

To see this, first note that the section $\gamma_i$ in (3.16) produces a holomorphic homomorphism

$$\mathcal{G}_i \rightarrow \mathcal{G}_{i+1} \otimes (K_X|_U)$$

because the homomorphism $\gamma_i \in H^0(U, \text{Hom}(\mathcal{F}_i, \mathcal{F}_{i+1} \otimes (K_X|_U)))$ in (3.17) vanishes at the point $x'$. Secondly, for any $j > i$, the section $\alpha_{ij}$ in (3.16) produces a section

$$\tilde{\alpha}_{ij} \in H^0(U, \text{Hom}(\mathcal{G}_j, \mathcal{G}_i \otimes (K_X|_U))) \otimes \mathcal{O}_U(-(j - i)x')),$$

in particular, $\tilde{\alpha}_{ij} \in H^0(U, \text{Hom}(\mathcal{G}_j, \mathcal{G}_i \otimes (K_X|_U)))$. Thirdly, the connection operator

$$D_i : \mathcal{F}_i \rightarrow \mathcal{F}_i \otimes (K_X|_U)$$

in (3.16) produces a first order holomorphic differential operator

$$\mathcal{G}_i \rightarrow \mathcal{G}_i \otimes (K_X|_U) \otimes \mathcal{O}_U(x')$$

that satisfies the Leibniz identity, because $D_i$ itself satisfies the Leibniz identity.

Consequently, $D$ sends the subsheaf $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S) \subset V$ in (3.14) to the subsheaf

$$(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 2)S) \otimes K_X) \cap (V \otimes K_X) \subset V \otimes K_X .$$

Since

$$J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 2)S) \otimes K_X = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S) \otimes K_X \otimes \mathcal{O}_X(S) ,$$

this implies that $D$ produces a logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S)$. \hfill \Box

**Remark 3.6.** Consider the holomorphic connection $D_i$ in (3.16) on the holomorphic vector bundle $\mathcal{F}_i$ on $U$. The connection on $\mathcal{G}_i := \mathcal{F}_i \cap (J^{r-1}(Q|_U) \otimes \mathcal{O}_U(-(r - 1)x'))$ induced by $D_i$ is singular at $x'$ if $i \geq 2$. Therefore, the singular locus of the logarithmic connections on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S)$ constructed in Proposition 3.5 is exactly $S$.

Take any branched SL$(r, \mathbb{C})$–oper $(V, \mathcal{F}, D)$ as in (3.1). Let $\mathcal{D}$ denote the logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S)$ constructed in Proposition 3.5, from $(V, \mathcal{F}, D)$. The following is a straight-forward consequence of the construction of $\mathcal{D}$.

**Corollary 3.7.**

1. The logarithmic connection $\mathcal{D}$ on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S)$ constructed in Proposition 3.5 satisfies the condition

$$\mathcal{D}(\tilde{H}_i \cap \tilde{H}_{i+1} \otimes K_X \otimes \mathcal{O}_X(S))$$

for all $1 \leq i \leq r - 1$, where $\{\tilde{H}_{ij}\}_{j=0}^r$ is the filtration of $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S)$ in (3.11).
(2) The second fundamental form of the subbundle
\[ \hat{H}_i|_{X_0} \subset (J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))|_{X_0} \]
for the holomorphic connection \( D|_{X_0} \) is nowhere zero on \( X_0 \) for all \( 1 \leq i \leq r-1 \), where \( X_0 \) is the open subset in (3.8).

Proof. Over the open subset \( X_0 \) in (3.8), we have
\[ J^{r-1}(Q)|_{X_0} = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)|_{X_0} = V|_{X_0} \]
(see Proposition 3.1). This isomorphism takes \( F_j|_{X_0} \subset V|_{X_0} \) isomorphically to \( H_j|_{X_0} = \hat{H}_j|_{X_0} \) (see Proposition 3.1). It also takes the holomorphic connection \( D|_{X_0} \to D|_{X_0} \); this is an immediate consequence of the construction of \( \hat{D} \). Since \( X_0 \) is dense in \( X \), the first statement now follows from the facts that \( D(F_i) \subset F_{i+1} \otimes K_X \) for all \( 1 \leq i \leq r-1 \) (see the fourth statement in Definition 2.1), while the second statement follows from the fact that the second fundamental form of the subbundle \( F_i \subset V \), for the holomorphic connection \( D \), is nowhere zero on \( X_0 \) (see the fifth statement in Definition 2.1). \( \square \)

For any \( 1 \leq i \leq r-1 \), let
\[ SF(D, \hat{H}_i) \in H^0(X, \text{Hom}(\hat{H}_i, \hat{H}_{r}/\hat{H}_i) \otimes K_X \otimes \mathcal{O}_X(S)) \]
be the second fundamental form of the subbundle \( \hat{H}_i \) in (3.11) for the logarithmic connection \( \mathcal{D} \) on \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) = \hat{H}_r \) constructed in Proposition 3.5; see (2.3) for the second fundamental form. From Corollary 3.7(1) we know that \( SF(D, \hat{H}_i) \) lies in the image of the natural inclusion map
\[ H^0(X, \text{Hom}(\hat{H}_i, \hat{H}_{i+1}/\hat{H}_i) \otimes K_X \otimes \mathcal{O}_X(S)) \hookrightarrow H^0(X, \text{Hom}(\hat{H}_i, \hat{H}_{r}/\hat{H}_i) \otimes K_X \otimes \mathcal{O}_X(S)) \]
given by the inclusion map \( \hat{H}_{i+1} \hookrightarrow \hat{H}_r \). So we get a homomorphism
\[ SF(D, i) \in H^0(X, \text{Hom}(\hat{H}_i/\hat{H}_{i-1}, \hat{H}_{r}/\hat{H}_i) \otimes K_X \otimes \mathcal{O}_X(S)) \]  
(3.21)
for every \( 1 \leq i \leq r-1 \) given by these \( SF(D, \hat{H}_i) \).

For the filtration of \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \), from (3.12) and (3.7) it follows immediately that
\[ \hat{H}_j/\hat{H}_{j-1} = Q \otimes K_X^{\otimes (r-j)} \otimes \mathcal{O}_X(-(r-1)S) \]  
(3.22)
for all \( 1 \leq j \leq r \). Therefore, \( SF(D, i) \) in (3.21) is a holomorphic section
\[ SF(D, i) \in H^0(X, \mathcal{O}_X(S)) \]  
(3.23)
for every \( 1 \leq i \leq r-1 \).

The following lemma is a refinement of Corollary 3.7(2).

Lemma 3.8. The holomorphic section \( SF(D, i) \) in (3.23) coincides with the section of \( \mathcal{O}_X(S) \) given by the constant function 1 on \( X \).

Proof. In the proof of Corollary 3.7 it was observed that
\[ (J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S), \{ \hat{H}_j \}_{i=1}^r, \mathcal{D})|_{X_0} = (V, \{ F_i \}_{i=1}^r, D)|_{X_0}, \]
where $X_0$ the open subset $X_0$ in (3.3). Therefore, the lemma follows from the fifth statement in Definition 2.1 and Remark 2.2.

Using (3.5) it is deduced that

$$\det J^{-1}(Q) := \bigwedge_r J^{-1}(Q) = Q^\otimes r \otimes K_X^{\otimes r(r-1)/2}.$$ 

Therefore, from (3.2) it follows that

$$\det J^{-1}(Q) = F_1^\otimes (TX)^{\otimes r(r-1)/2} \otimes \mathcal{O}_X(r(r-1)S).$$

Now the expression of $F_1$ in the first statement in Definition 2.1 gives that

$$\det J^{-1}(Q) = \mathcal{O}_X\left(\frac{r(r-1)}{2}S\right).$$

Hence we have the following:

$$\det(J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) = \mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right). \quad (3.24)$$

Take any branched $SL(r, \mathbb{C})$–oper $(V, F, D)$ as in (3.1). Let $D$ denote the logarithmic connection on $J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ constructed in Proposition 3.5 from $(V, F, D)$.

**Lemma 3.9.** The logarithmic connection on $\det(J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ induced by the logarithmic connection $D$ on $J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ coincides with the logarithmic connection on $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ given by the de Rham differential (it sends $f$ to $df$, in particular, constant functions are covariant constant), once $\det(J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ is identified with $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ using (3.24).

**Proof.** Consider the homomorphism $i$ in Corollary 3.4. Let

$$\bigwedge^r i : \det(J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) = \mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right) \rightarrow \det V = \mathcal{O}_X$$

be the corresponding homomorphism of the top exterior products. This homomorphism $\bigwedge^r i$ evidently coincides with the natural inclusion map of $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ into $\mathcal{O}_X$. Now, since $D$ induces the connection on $\det V$ given by the de Rham differential (see the third statement in Definition 2.1), and $D$ restricts to $\mathcal{O}_X(J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$, the lemma follows.

In Section 6 we will determine all the logarithmic connections on the rank $r$ vector bundle $J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ that arise from branched $SL(r, \mathbb{C})$–opers on $X$ with branching over $S$.

4. **Residues**

Take a point $y \in S$. The fiber of $K_X \otimes \mathcal{O}_X(S)$ over $y$ is identified with $\mathbb{C}$ by the Poincaré adjunction formula [19, p. 146]. To explain this isomorphism

$$(K_X \otimes \mathcal{O}_X(S))_y \sim \mathbb{C}, \quad (4.1)$$
let $z$ be a holomorphic coordinate function on $X$ defined on an analytic open neighborhood of $y$ such that $z(y) = 0$. Then we have an isomorphism $\mathbb{C} \rightarrow (K_X \otimes \mathcal{O}_X(S))_y$ that sends any $c \in \mathbb{C}$ to $c \cdot \frac{dz}{z}(y) \in (K_X \otimes \mathcal{O}_X(S))_y$. It is straightforward to check that this map $\mathbb{C} \rightarrow (K_X \otimes \mathcal{O}_X(S))_y$ is independent of the choice of the holomorphic coordinate function $z$.

Let $D_W : W \rightarrow W \otimes K_X \otimes \mathcal{O}_X(S)$ be a logarithmic connection on a holomorphic vector bundle $W$ on $X$. Consider the composition of homomorphisms

$$W \xrightarrow{D_W} W \otimes K_X \otimes \mathcal{O}_X(S) \rightarrow (W \otimes K_X \otimes \mathcal{O}_X(S))_y = W_y.$$ 

This composition of homomorphisms is evidently $\mathcal{O}_X$–linear and therefore is given by an endomorphism

$$\text{Res}(D_W, y) : W_y \rightarrow W_y;$$

called the \textit{residue} of $D_W$ at $y$.

If $r = \text{rank}(W)$ and $\lambda_1, \ldots, \lambda_r$ are the generalized eigenvalues, with multiplicity, of $\text{Res}(D_W, y)$, then the generalized eigenvalues of the local monodromy of $D_W$ around $y$ are

$$\exp(-2\pi\sqrt{-1}\lambda_1), \exp(-2\pi\sqrt{-1}\lambda_1), \ldots, \exp(-2\pi\sqrt{-1}\lambda_r);$$

see [9].

For convenience we use the notation

$$E := \hat{H}_r = J^r-1(Q) \otimes \mathcal{O}_X(-(r-1)S)$$

(see (3.11)).

Take a branched $\text{SL}(r, \mathbb{C})$–oper $(V, \mathcal{F}, D)$ as in (3.1). As before, let $\mathcal{D}$ denote the logarithmic connection on $E$ (see (4.2)) constructed in Proposition 3.5 from $(V, \mathcal{F}, D)$. Recall from Remark 3.6 that the singular locus of $\mathcal{D}$ is exactly $S$. The following lemma describes the residues of $\mathcal{D}$.

**Lemma 4.1.** For any point $x' \in S$, let

$$\text{Res}(\mathcal{D}, x') \in \text{End}(E_{x'})$$

be the residue of $\mathcal{D}$ at $x'$. Then the eigenvalues of $\text{Res}(\mathcal{D}, x')$ are the integers $\{0, 1, \ldots, r-2, r-1\}$, and the multiplicity of each of them is one. For any $0 \leq i \leq r-1$, the eigenspace of $\text{Res}(\mathcal{D}, x')$ for the eigenvalue $i$ is contained in the subspace

$$(\hat{H}_{i+1})_{x'} \subset E_{x'}$$

(see (3.11) and (4.2)).

**Proof.** Let $L$ be a holomorphic line bundle on the open neighborhood $U \subset X$ of $x'$ (as in (3.15)) equipped with a holomorphic connection $D_L$. Then for any integer $k$, the differential operator $D_L|_{U \setminus \{x'\}}$ on $L|_{U \setminus \{x'\}} = (L \otimes \mathcal{O}_U(kx'))|_{U \setminus \{x'\}}$ extends to a logarithmic connection on $L \otimes \mathcal{O}_U(kx')$ over $U$. In fact it coincides with the logarithmic connection on $L \otimes \mathcal{O}_U(kx')$ given by the holomorphic connection $D_L$ on $L$ and the logarithmic connection on $\mathcal{O}_U(kx')$ defined by the de Rham differential $d$. The residue, at $x'$, of this logarithmic connection on $L \otimes \mathcal{O}_U(kx')$ is $-k$; indeed, this follows immediately from the fact that the
residue of the logarithmic connection on $\mathcal{O}_U(kx')$ defined by the de Rham differential $d$ is $-k$. In view of this, the lemma follows from the expression of $D|_U$ in (3.16) in terms of the direct sum of line bundles $\mathcal{G}_i$ in (3.19). Recall the observation in the proof of Proposition 3.5 that the section $\gamma_i$ in (3.17) produces a holomorphic homomorphism $\mathcal{G}_i \to \mathcal{G}_{i+1} \otimes (K_X|_U)$.

Therefore, $\gamma_i$ does not contribute to the residue $\text{Res}(D, x')$. Since $\tilde{\alpha}_{i,j}$ in (3.20) does not have any pole as a homomorphism from $\mathcal{G}_j$ to $\mathcal{G}_i \otimes (K_X|_U)$, it also does not contribute to the residue $\text{Res}(D, x')$.

Therefore, the residue of $D$ at $x'$ is given by the residues on the logarithmic connections on $\mathcal{G}_j$, $1 \leq j \leq r$, induced by the holomorphic connections $D_j$ in (3.16). From the above observation on the residue of the logarithmic connection on $L \otimes \mathcal{O}_U(kx')$ we know that the residues on this logarithmic connection $\mathcal{G}_j$ is $j - 1$. This proves the lemma.

We will end this section on residues by noting some general properties of it.

Let $W \to X$ be a holomorphic vector bundle, and let $D_W$ be a logarithmic connection on $W$, singular at $y \in X$. Assume that the residue $\text{Res}(D_W, y)$ of $D_W$ at $y$ is semisimple, meaning $\text{Res}(D_W, y)$ is diagonalizable. Let $\lambda_1, \cdots, \lambda_b$ be the eigenvalues of $\text{Res}(D_W, y)$ (they need not be of multiplicity one). For any $1 \leq i \leq b$, let

$$W^i_y \subset W_y$$

be the eigenspace of $\text{Res}(D_W, y) \in \text{End}(W_y)$ for the eigenvalue $\lambda_i$. For any given $1 \leq k < b$, consider the following natural homomorphisms

$$W \to W_y \to W_y / \left( \bigoplus_{i=1}^k W^i_y \right);$$

both are natural quotient maps. The kernel of this composition of homomorphisms will be denoted by $\tilde{W}$. So $\tilde{W}$ is a torsion-free coherent analytic sheaf on $X$ that fits in the following short exact sequence of coherent analytic sheaves on $X$:

$$0 \to \tilde{W} \to W \to W_y / \left( \bigoplus_{i=1}^k W^i_y \right) \to 0. \quad (4.3)$$

The following is straight-forward to check.

**Lemma 4.2.** The logarithmic connection $D_W : W \to W \otimes K_X \otimes \mathcal{O}_X(y)$ sends the subsheaf $\tilde{W} \to W$ to $\tilde{W} \otimes K_X \otimes \mathcal{O}_X(y)$. Hence $D_W$ induces a logarithmic connection on $\tilde{W}$.

The logarithmic connection on $\tilde{W}$ induced by $D_W$ will be denoted by $\tilde{D}$.

We will now describe the residue of $\tilde{D}$ at the singular point $y$.

Let

$$0 \to \text{kernel}(\phi(y)) \to \tilde{W}_y \to W_y \to \text{cokernel}(\phi(y)) = W_y / \left( \bigoplus_{i=1}^k W^i_y \right) \to 0 \quad (4.4)$$
be the exact sequence of vector spaces obtained by restricting, to the point \( y \), the short exact sequence of coherent analytic sheaves in (4.3). Note that the homomorphism of fibers of vector bundles corresponding to an injective homomorphism of coherent analytic sheaves need not be injective, so \( \text{kernel}(\phi(y)) \) may be nonzero.

We will now show that there is a canonical isomorphism

\[
\left( \bigoplus_{i=k+1}^{b} W^i_y \right) \otimes (K_X)_{y} \xrightarrow{\sim} \text{kernel}(\phi(y)). \tag{4.5}
\]

To prove (4.5), take any \( w \in \left( \bigoplus_{i=k+1}^{b} W^i_y \right) \otimes (K_X)_{y} \). Using the isomorphism \((K_X)_{y} = O_Y(\cdot)_{y}\) (see (4.1)), we have

\[
w \in \left( \bigoplus_{i=k+1}^{b} W^i_y \right) \otimes O_Y(-y)_{y} = \bigoplus_{i=k+1}^{b} (W^i \otimes O_Y(-y))_{y}.
\]

Now take a holomorphic section defined on an analytic open neighborhood \( U \subset X \)

\[
s \in H^0(U, (W|_U) \otimes O_U(-y))
\]
such that \( s(y) = w \). We note that \( q_1(s) = 0 \), where \( q_1 \) is the projection in (4.3). Therefore, from (4.3) it follows that \( s \) is the image of a holomorphic section of \( \widetilde{W}|_U \) under the homomorphism \( \phi \) in (4.3). Let

\[
\tilde{s} \in H^0(U, \widetilde{W}|_U)
\]
be the unique holomorphic section such that \( \phi(\tilde{s}) = s \). It can be shown that the evaluation

\[
\tilde{s}(y) \in \widetilde{W}_y
\]
is independent of the choice of the above section \( s \in H^0(U, (W|_U) \otimes O_U(-y)) \) satisfying \( s(y) = w \). Indeed, for another holomorphic section

\[
t \in H^0(U, (W|_U) \otimes O_U(-y))
\]
with \( t(y) = w \), we have

\[
s - t \in H^0(U, (W|_U) \otimes O_U(-2y)),
\]
and hence

\[
\tilde{s} - \tilde{t} \in H^0(U, (\widetilde{W}|_U) \otimes O_U(-y)), \tag{4.6}
\]
where \( \tilde{t} \in H^0(U, \widetilde{W}|_U) \) is the unique section for which \( \phi(\tilde{t}) = t \). From (4.6) it follows immediately that \( \tilde{s}(y) = \tilde{t}(y) \), and hence \( \tilde{s}(y) \in \widetilde{W}_y \) is independent of the choice of the section \( s \in H^0(U, (W|_U) \otimes O_U(-y)) \) satisfying \( s(y) = w \).

For the homomorphism \( \phi(y) \) in (4.4) we have

\[
\phi(y)(\tilde{s}(y)) = 0,
\]
because \( \phi(\tilde{s}) = s \in H^0(U, (W|_U) \otimes O_U(-y)) \). Therefore, from (4.4) we conclude that

\[
\tilde{s}(y) \in \text{kernel}(\phi(y)).
\]
The isomorphism in (4.5) sends any
\[ w \in \bigoplus_{i=k+1}^{b} W^i_y \otimes (K_X)_y \]
to \( \tilde{s}(y) \in \text{kernel}(\phi(y)) \) constructed above from it.

The following lemma is a straight-forward consequence of the construction of residue of a logarithmic connection.

**Lemma 4.3.** The residue \( \text{Res}(\tilde{D}, y) \) of the logarithmic connection \( \tilde{D} \) on \( \tilde{W} \) (see Lemma 4.2) has the following properties:

1. The eigenvalues of \( \text{Res}(\tilde{D}, y) \) are \( \{ \lambda_i \}_{i=1}^{k} \cup \{ \lambda_i + 1 \}_{i=k+1}^{b} \).
2. For any \( k + 1 \leq i \leq b \), the eigenspace of \( \text{Res}(\tilde{D}, y) \) for the eigenvalue \( \lambda_i + 1 \) is the subspace \( W^i_y \otimes (K_X)_y \subset \tilde{W}_y \) (see (4.5) and (4.4)).
3. For any \( 1 \leq i \leq k \), the eigenspace of \( \text{Res}(\tilde{D}, y) \) for the eigenvalue \( \lambda_i \) is taken isomorphically to the eigenspace \( W^i_y \) by the homomorphism \( \phi(y) \) in (4.4).

## 5. Local Monodromy

A logarithmic connection has local monodromy around a singular point of the connection. The two holomorphic vector bundles \( V \) and
\[ E := J^{r-1}(Q) \otimes O_X(-(r-1)S) \] (5.1)
are identified over \( X_0 \) (3.8) (see Corollary 3.4), and this identification takes the holomorphic connection \( D|_{X_0} \) on \( E|_{X_0} \) in Lemma 4.1 to the holomorphic connection \( D|_{X_0} \) on \( V|_{X_0} \) in (3.1). Now \( D|_{X_0} \) does not have local monodromy around any point \( x' \in S \) because \( D \) is a holomorphic connection on \( V \). Hence the logarithmic connection \( D \) has trivial local monodromy around every point of \( S \), as well. In this section we will reformulate this condition of vanishing of local monodromies.

Let
\[ D : E \longrightarrow E \otimes K_X \otimes O_X(S) \] (5.2)
be a logarithmic connection on \( E \) (see (5.1)) singular over \( S \) that satisfies the following four conditions:

1. \( D(\tilde{H}_i) \subset \tilde{H}_{i+1} \otimes K_X \otimes O_X(S) \), for all \( 1 \leq i \leq r-1 \), where \( \{ \tilde{H}_i \}_{i=0}^{r} \) is the filtration of \( J^{r-1}(Q) \otimes O_X(-(r-1)S) \) in (3.11). This implies that the second fundamental forms of the subbundles \( \tilde{H}_i \) produce a section
\[ SF(D, i) \in H^0(X, O_X(S)) \]
as in (3.23) for every \( 1 \leq i \leq r-1 \).
2. For all \( 1 \leq i \leq r-1 \), the above section \( SF(D, i) \) coincides with the section of \( O_X(S) \) given by the constant function 1 on \( X \).
3. For every \( x' \in S \), the eigenvalues of \( \text{Res}(D, x') \) are the integers \( \{0, 1, \cdots, r - 2, r - 1\} \). Note that the multiplicity of each of the eigenvalues is one.
(4) For all $0 \leq i \leq r - 1$ and every $x' \in S$, the eigenspace of $\text{Res}(\mathcal{D}, x')$ for the eigenvalue $i$ is contained in the subspace

$$(\hat{H}_{i+1})_{x'} \subset E_{x'}$$

(see (3.11) and (4.2)).

In other words, $\mathcal{D}$ shares all the properties of $\mathcal{D}$ stated in Lemma 4.1, Corollary 3.7 and Lemma 3.8.

**Remark 5.1.** Note that we did not impose the condition that all the local monodromies of $\mathcal{D}$ are trivial, despite the fact that $\mathcal{D}$ enjoys this property (this was shown above).

Let

$$L_{x'}(i) \subset E_{x'}$$

be the eigenline for the eigenvalue $0 \leq i \leq r - 1$ of the residue $\text{Res}(\mathcal{D}, x')$ at $x' \in S$. So we have a decomposition of the fiber $E_{x'}$

$$E_{x'} = \bigoplus_{i=0}^{r-1} L_{x'}(i),$$

(5.4)

where $L_{x'}(i)$ is the subspace in (5.3). The above condition that the eigenspace of $\text{Res}(\mathcal{D}, x')$ for the eigenvalue $i$ is contained in the subspace $(\hat{H}_{i+1})_{x'} \subset E_{x'}$ implies that

$$(\hat{H}_{j+1})_{x'} = \bigoplus_{i=0}^{j} L_{x'}(i)$$

for all $0 \leq j \leq r - 1$; so we have

$$L_{x'}(j) = (\hat{H}_{j+1})_{x'}/(\hat{H}_j)_{x'}.$$  (5.5)

We recall from (3.22) that for the filtration of $E$ given in (3.11),

$$\hat{H}_j/\hat{H}_{j-1} = Q \otimes K_X^{\otimes (r-j)} \otimes \mathcal{O}_X(-(r-1)S)$$

for all $1 \leq j \leq r$.

Take any point $x' \in S$, and also take any integer $2 \leq j \leq r$. We will construct, from $\mathcal{D}$, an element

$$M_j(\mathcal{D}, x') \in \text{Hom}((\hat{H}_j/\hat{H}_{j-1})_{x'} \otimes (T_{x'}X)^{\otimes (j-1)}, \hat{H}_{j-1}/\hat{H}_{j-2})_{x'} \otimes (T_{x'}X)^{\otimes (j-2)}) \quad (5.6)$$

$$= \text{Hom}((\hat{H}_j/\hat{H}_{j-1})_{x'}, (\hat{H}_{j-1}/\hat{H}_{j-2})_{x'} \otimes (K_X)_{x'}) = (K_X^{\otimes 2})_{x'},$$

where $T_{x'}X$ is the holomorphic tangent space to $X$ at $x'$; see (3.22) for the equality

$$\text{Hom}((\hat{H}_j/\hat{H}_{j-1})_{x'}, (\hat{H}_{j-1}/\hat{H}_{j-2})_{x'}) = (K_X)_{x'}$$

in (5.6).

To construct $M_j(\mathcal{D}, x')$, first note that the fiber $\mathcal{O}_X((j-1)S)_{x'}$ is identified with $(T_{x'}X)^{\otimes (j-1)}$ using the Poincaré adjunction formula (see (4.1)). This produces an isomorphism

$$\eta : (\hat{H}_j/\hat{H}_{j-1})_{x'} \otimes \mathcal{O}_X((j-1)S)_{x'} \sim (\hat{H}_j/\hat{H}_{j-1})_{x'} \otimes (T_{x'}X)^{\otimes (j-1)}$$  (5.7)
Take any 
\[ w \in (\tilde{\mathcal{H}}_j/\tilde{\mathcal{H}}_{j-1})_{x'} \otimes (T_{x'}X)^{\otimes (j-1)}. \] 
(5.8)

Using (5.5) and \( \eta \) (constructed in (5.7)) we have 
\[ (\tilde{\mathcal{H}}_j/\tilde{\mathcal{H}}_{j-1})_{x'} \otimes (T_{x'}X)^{\otimes (j-1)} = L_{x'}(j-1) \otimes \mathcal{O}_X((j-1)x')_{x'}. \]

Let 
\[ w' \in L_{x'}(j-1) \otimes \mathcal{O}_X((j-1)x')_{x'} \]
be the element that corresponds to \( w \) (see (5.8)) by this isomorphism.

Now we choose a holomorphic section 
\[ \tilde{w} \in H^0(U, (E|_U) \otimes \mathcal{O}_U((j-1)x')), \] 
(5.9)
defined on some sufficiently small analytic neighborhood \( U \subset X \) of \( x' \), such that 
\[ \tilde{w}(x') = w'. \]

Let \( \hat{\mathbb{D}} \) be the logarithmic connection on \( (E|_U) \otimes \mathcal{O}_U((j-1)x') \) induced by the logarithmic connection \( \mathbb{D}|_U \) on \( E|_U \) and the logarithmic connection on \( \mathcal{O}_U((j-1)x') \) given by the de Rham differential \( d \). Consider the residue \( \text{Res}(\hat{\mathbb{D}}, x') \in \text{End}(E_{x'} \otimes \mathcal{O}_U((j-1)x')_{x'}) \) of the logarithmic connection \( \hat{\mathbb{D}} \) at the point \( x' \). It can be shown that the line 
\[ L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'} \subset E_{x'} \otimes \mathcal{O}_U((j-1)x')_{x'} \]
(see (5.4)) is contained in the eigenspace of \( \text{Res}(\hat{\mathbb{D}}, x') \) for the eigenvalue 0. Indeed, \( \text{Res}(\hat{\mathbb{D}}, x') \) acts on \( L_{x'}(j-1) \) as multiplication by \( j-1 \) and the logarithmic connection on \( \mathcal{O}_U((j-1)x') \) given by the de Rham differential \( d \) has the property that its residue at \( x' \) is \( 1 - j \). Hence \( L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'} \) is contained in the eigenspace of \( \text{Res}(\hat{\mathbb{D}}, x') \) for the eigenvalue 0. Actually, \( L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'} \) is the eigenspace of \( \text{Res}(\hat{\mathbb{D}}, x') \) for the eigenvalue 0.

Since the section \( \tilde{w} \) in (5.9) satisfies the condition \( \tilde{w}(x') \in L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'} \), from the above property of \( \text{Res}(\hat{\mathbb{D}}, x') \) that \( L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'} \) is contained in the eigenspace of \( \text{Res}(\hat{\mathbb{D}}, x') \) for the eigenvalue 0 it follows that 
\[ \hat{\mathbb{D}}(\tilde{w}) \in H^0(U, (E|_U) \otimes K_U \otimes \mathcal{O}_U((j-1)x')), \]
(5.10)
where \( K_U := K_X|_U \).

The decomposition of \( E_{x'} \) in (5.4) gives a decomposition 
\[ E_{x'} \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} = \bigoplus_{i=0}^{r-1} L_{x'}(i) \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'}. \]
(5.11)

Let 
\[ \beta_{j-2}(w) \in L_{x'}(j-2) \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} \]
(5.12)
be the component of \( \hat{\mathbb{D}}(\tilde{w})(x') \in (E \otimes K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} \) (see (5.10)) in 
\[ L_{x'}(j-2) \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} \subset E_{x'} \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'}. \]
with respect to the decomposition in (5.11). Since \( L_x' (j - 2) = (\hat{H}_{j-1})_x' / (\hat{H}_{j-2})_x' \) (see 5.5), and \( \mathcal{O}_U ((j - 1)x')_x' = (T_x' X) \otimes (j - 1) \) (see (1.1)), the element \( \beta_{j-2} (w) \) in (5.12) is also an element

\[
\beta_{j-2} (w) \in (\hat{H}_{j-1} / \hat{H}_{j-2})_x' \otimes (T_x' X) \otimes (j - 2).
\]

(5.13)

The map \( M_j (\mathbb{D}, x') \) in (5.6) sends the element \( w \) in (5.8) to \( \beta_{j-2} (w) \) constructed in (5.13).

But we need to show that this map is well-defined in the sense that \( \beta_{j-2} (w) \) depends only on \( w \), in other words, \( \beta_{j-2} (w) \) is independent of the choice of the section \( \tilde{w} \) in (5.9). The following lemma shows that \( \beta_{j-2} (w) \) depends only on \( w \).

**Lemma 5.2.** The element \( \beta_{j-2} (w) \) constructed in (5.13) does not depend on the choice of the section \( \tilde{w} \) in (5.9).

**Proof.** We may replace \( \tilde{w} \) in (5.9) by \( \tilde{w} + t \), where

\[
t \in H^0 (U, (E|_U) \otimes \mathcal{O}_U ((j - 1)x'))
\]

with \( t(x') = 0 \), where \( E \) is defined in (4.2). Let

\[
\beta_{j-2}' (w) \in (\hat{H}_{j-1} / \hat{H}_{j-2})_x' \otimes (T_x' X) \otimes (j - 2)
\]

be the element constructed as in (5.13) after substituting \( \tilde{w} + t \) in place of \( \tilde{w} \) in the construction of \( \beta_{j-2} (w) \). To prove the lemma we need to show that

\[
\beta_{j-2}' (w) = \beta_{j-2} (w).
\]

(5.14)

To prove (5.14), first note that

\[
t \in H^0 (U, (E|_U) \otimes \mathcal{O}_U ((j - 2)x')) \subset H^0 (U, (E|_U) \otimes \mathcal{O}_U ((j - 1)x'))
\]

(5.15)

because of the given condition that \( t(x') = 0 \).

Let \( \hat{D}_1 \) be the logarithmic connection on \( (E|_U) \otimes \mathcal{O}_U ((j - 2)x') \) given by the logarithmic connection \( \hat{D}|_U \) on \( E|_U \) and the logarithmic connection on \( \mathcal{O}_U ((j - 2)x') \) given by the de Rham differential \( d \). We note that \( \hat{D}_1 \) is simply the restriction of the logarithmic connection \( \hat{D} \) to the subsheaf

\[
(E|_U) \otimes \mathcal{O}_U ((j - 2)x') \subset (E|_U) \otimes \mathcal{O}_U ((j - 1)x').
\]

From (5.15) we have

\[
\hat{D}_1 (t) \in H^0 (U, (E|_U) \otimes K_U \otimes \mathcal{O}_U ((j - 1)x')).
\]

Using the isomorphism in (4.1), the evaluation, at \( x' \), of this section \( \hat{D}_1 (t) \) is considered as an element of

\[
\hat{D}_1 (t)(x') \in E_{x'} \otimes (K_U \otimes \mathcal{O}_U ((j - 1)x'))_{x'} = E_{x'} \otimes \mathcal{O}_U ((j - 2)x')_{x'}.
\]

(5.16)

The decomposition in (5.4) produces a decomposition

\[
E_{x'} \otimes \mathcal{O}_U ((j - 2)x')_{x'} = \bigoplus_{i=0}^{r-1} L_{x'} (i) \otimes \mathcal{O}_U ((j - 2)x')_{x'}.
\]

(5.17)
The residue of the logarithmic connection $\hat{\mathbb{D}}$ at $x'\prime$

$$\text{Res}(\hat{\mathbb{D}}, x') \in \text{End}(E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'})$$

preserves the decomposition in (5.17). Moreover, $\text{Res}(\hat{\mathbb{D}}, x')$ acts on the subspace

$$L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'} \subset E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'}$$

in (5.17) as multiplication by $i - j + 2$. Indeed, the residue $\text{Res}(\mathbb{D}, x')$ acts on $L_{x'}(i)$ as multiplication by $i$ (see (5.3)), and the residue, at $x'$, of the logarithmic connection on $\mathcal{O}_U((j-2)x')$ given by the de Rham differential $d$ is $2 - j$. Consequently, $\text{Res}(\hat{\mathbb{D}}, x')$ acts on $L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'}$ as multiplication by $i - j + 2$. This implies that

$$\text{Res}(\hat{\mathbb{D}}, x')(E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'}) \subset \bigoplus_{i \in \{0, \ldots, r-1\}\setminus\{j-2\}} L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'}, \quad (5.18)$$

and $\text{kernel}(\text{Res}(\hat{\mathbb{D}}, x')) = L_{x'}(j-2) \otimes \mathcal{O}_U((j-2)x')_{x'}$.

On the other hand, the evaluation $\hat{\mathbb{D}}_1(t)(x') \in E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'}$ in (5.18) satisfies the identity

$$\hat{\mathbb{D}}_1(t)(x') = \text{Res}(\hat{\mathbb{D}}, x')(t)(x').$$

Therefore, from (5.18) it follows that

$$\hat{\mathbb{D}}_1(t)(x') \in \bigoplus_{i \in \{0, \ldots, r-1\}\setminus\{j-2\}} L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'}. \quad (5.19)$$

Recall that $\beta_{j-2}(w)$ in (5.12) is the component of $\mathbb{D}(\hat{w})(x')$ in

$$L_{x'}(j-2) \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} = L_{x'}(j-2) \otimes \mathcal{O}_U((j-2)x')_{x'}$$

with respect to the decomposition in (5.11). Therefore, from (5.19) it follows immediately that the equality in (5.14) holds. As noted before, (5.14) completes the proof. \qed

The map $M_j(\mathbb{D}, x')$ in (5.6) is defined by sending any element $w$ as in (5.8) to $\beta_{j-2}(w)$ constructed in (5.13) from $w$. Lemma 5.2 ensures that it is well-defined.

**Remark 5.3.** Let $D'$ be a logarithmic connection singular at a point $x$, such that residue at $x$ is semisimple. If $\lambda$ is an eigenvalue of the local monodromy of $D'$ around $x$, then $\lambda = \exp(2\pi \sqrt{-1}b)$, where $b$ is an eigenvalue of the residue $\text{Res}(D', x)$ [9]. For any point $x' \in S$, the eigenvalues of the residue $\text{Res}(\mathbb{D}, x')$ of the connection $\mathbb{D}$ in (5.2) are integers. Therefore, we conclude that 1 is the only eigenvalue of the local monodromy of $\mathbb{D}$ around the point $x'$. In other words, the local monodromy of $\mathbb{D}$ around $x'$ is a unipotent automorphism. This local monodromy is given by

$$(M_2(\mathbb{D}, x'), M_3(\mathbb{D}, x'), \cdots, M_r(\mathbb{D}, x')) \in ((K_{x}^{\otimes 2})_{x'})^{\otimes (r-1)},$$

where the elements $M_j(\mathbb{D}, x')$ are constructed in (5.6). The elements $M_j(\mathbb{D}, x')$ will be studied in the next section.
6. Characterizing the logarithmic connections

As in (5.2), let
\[ \mathbb{D} : E := J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \rightarrow J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-2)S) \otimes K_X \] (6.1)
satisfying the following four conditions:

1. \( \mathbb{D}(\hat{H}_i) \subset \hat{H}_{i+1} \otimes K_X \otimes \mathcal{O}_X(S) \), for all \( 1 \leq i \leq r-1 \), where \( \{\hat{H}_i\}_{i=0}^r \) is the filtration of \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \) in (3.11). This implies that the second fundamental forms of the subbundles \( \hat{H}_i \) produce a section
\[ \text{SF}(\mathbb{D}, i) \in H^0(X, \mathcal{O}_X(S)) \] (6.2)
as in (3.23) for every \( 1 \leq i \leq r-1 \).

2. For all \( 1 \leq i \leq r-1 \), the section \( \text{SF}(\mathbb{D}, i) \) in (6.2) coincides with the section of \( \mathcal{O}_X(S) \) given by the constant function 1 on \( X \).

3. For every \( x' \in S \), the eigenvalues of \( \text{Res}(\mathbb{D}, x') \) are the integers
\[ \{0, 1, \ldots, r-2, r-1\}, \] (6.3)
with the multiplicity of each of them being one.

4. For all \( 0 \leq i \leq r-1 \) and every \( x' \in S \), the eigenspace of \( \text{Res}(\mathbb{D}, x') \) for the eigenvalue \( i \) is contained in the subspace \( (\hat{H}_{i+1})_{x'} \subset E_{x'} \) in (4.2).

**Theorem 6.1.** There is a branched \( \text{SL}(r, \mathbb{C}) \)-oper
\[ (V, \mathcal{F}, D) \]
such that the logarithmic connection \( \mathbb{D} \) in (6.1) coincides with the logarithmic connection on \( E \) associated to \( (V, \mathcal{F}, D) \) by Proposition 3.3 if and only if the following two conditions hold:

1. The logarithmic connection on \( \text{det}(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) \) induced by the logarithmic connection \( \mathbb{D} \) on \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \) coincides with the logarithmic connection on \( \mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right) \) given by the de Rham differential \( d \), once \( \text{det}(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) \) is identified with \( \mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right) \) using (3.24).

2. \( M_j(\mathbb{D}, x') = 0 \), for all \( 2 \leq j \leq r \) and every \( x' \in S \), where \( M_j(\mathbb{D}, x') \) are constructed in (5.6).

**Proof.** If there is a branched \( \text{SL}(r, \mathbb{C}) \)-oper \( (V, \mathcal{F}, D) \) such that \( \mathbb{D} \) coincides with the logarithmic connection on \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \) associated to \( (V, \mathcal{F}, D) \) by Proposition 3.3, then from Lemma 3.9 we know that the logarithmic connection on \( \text{det}(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) \) induced by \( \mathbb{D} \) on \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \) coincides with the logarithmic connection on \( \mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right) \) given by the de Rham differential \( d \), once \( \text{det}(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) \) is identified with \( \mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right) \) using (3.24).

Therefore, we assume that the logarithmic connection on \( \text{det}(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) \) induced by \( \mathbb{D} \) on \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \) coincides with the logarithmic connection on...
\[ \mathcal{O}_X \left( -\frac{r(r-1)}{2} S \right) \] given by the de Rham differential \( d \), after \( \det(J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) \) is identified with \( \mathcal{O}_X \left( -\frac{r(r-1)}{2} S \right) \) using (3.24).

To prove the theorem we need to show the following: There is a branched \( \text{SL}(r, \mathbb{C}) \)-oper \((V, \mathcal{F}, D)\) such that \( \mathbb{D} \) coincides with the logarithmic connection on \( E = J^{-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \) associated to \((V, \mathcal{F}, D)\) by Proposition 3.5 if and only if \( M_j(\mathbb{D}, x') = 0 \) for all \( 2 \leq j \leq r \).

In Proposition 3.5 we constructed a logarithmic connection on \( E \) from a branched \( \text{SL}(r, \mathbb{C}) \)-oper. The above statement will be proved by establishing an inverse of this construction in Proposition 3.5.

Consider the holomorphic vector bundle
\[ \mathcal{W} := J^{-1}(Q) = E \otimes \mathcal{O}_X((r-1)S) \] (6.4) on \( X \) (see (4.2)). The logarithmic connection \( \mathbb{D} \) on \( E \) and the logarithmic connection on \( \mathcal{O}_X((r-1)S) \) given by the de Rham differential \( d \) together produce a logarithmic connection
\[ \widehat{\mathbb{D}} : \mathcal{W} \longrightarrow \mathcal{W} \otimes K_X \otimes \mathcal{O}_X(S) \] (6.5) on the holomorphic vector bundle \( \mathcal{W} \) in (6.4). At any point \( x' \in S \), the residue of the logarithmic connection on \( \mathcal{O}_X((r-1)S) \) given by the de Rham differential \( d \) is \( 1-r \). On the other hand, the eigenvalues of \( \text{Res}(\mathbb{D}, x') \) are given to be \( \{0, 1, \cdots, r-2, r-1\} \) (see (6.3)). Therefore, the eigenvalues of the residue \( \text{Res}(\widehat{\mathbb{D}}, x') \) of \( \mathbb{D} \) at \( x' \) are \( \{1-r, 2-r, \cdots, -1, 0\} \). We note that the multiplicity of every eigenvalue of \( \text{Res}(\mathbb{D}, x') \) is one.

For each \( x' \in S \), let
\[ \ell_{x'}(0) \subset \mathcal{W}_{x'} \] be the eigenspace, for the eigenvalue 0, of \( \text{Res}(\mathbb{D}, x') \); so \( \ell_{x'}(0) \) is a line in \( \mathcal{W}_{x'} \). Let \( \mathcal{W}_1 \) be the holomorphic vector bundle of rank \( r \) on \( X \) defined by the following short exact sequence of coherent analytic sheaves on \( X \):
\[ 0 \longrightarrow \mathcal{W}_1 \longrightarrow \mathcal{W} \longrightarrow \bigoplus_{x' \in S} \mathcal{W}_{x'}/\ell_{x'}(0) \longrightarrow 0. \] (6.6)

From Lemma 4.2 we know that the logarithmic connection \( \mathbb{D} : \mathcal{W} \longrightarrow \mathcal{W} \otimes K_X \otimes \mathcal{O}_X(S) \) preserves the subsheaf \( \mathcal{W}_1 \) in (6.6). Let
\[ \mathbb{D}_1 : \mathcal{W}_1 \longrightarrow \mathcal{W}_1 \otimes K_X \otimes \mathcal{O}_X(S) \] be the logarithmic connection on \( \mathcal{W}_1 \) induced by \( \mathbb{D} \). Since the eigenvalues of \( \text{Res}(\mathbb{D}, x') \) are \( \{0, -1, \cdots, 2-r, 1-r\} \), and \( \ell_{x'}(0) \) is the eigenspace of \( \text{Res}(\mathbb{D}, x') \) for the eigenvalue 0, from Lemma 4.3 it follows that the eigenvalues of \( \text{Res}(\mathbb{D}_1, x') \) are \( \{0, -1, \cdots, 2-r\} \). The eigenvalue 0 of \( \text{Res}(\mathbb{D}_1, x') \) has multiplicity two, while the rest of the eigenvalues of \( \text{Res}(\mathbb{D}_1, x') \) are of multiplicity one.

For each \( x' \in S \), let
\[ H_1(x') \subset (\mathcal{W}_1)_{x'} \] be the eigenspace of \( \text{Res}(\mathbb{D}_1, x') \in \text{End}((\mathcal{W}_1)_{x'}) \) for the eigenvalue 0. As noted above, we have \( \dim H_1(x') = 2 \). Imitating (6.6) we define \( \mathcal{W}_2 \). More precisely, let \( \mathcal{W}_2 \) be the
holomorphic vector bundle of rank \( r \) on \( X \) defined by the following short exact sequence of coherent analytic sheaves on \( X \):

\[
0 \longrightarrow \mathcal{W}_2 \longrightarrow \mathcal{W}_1 \longrightarrow \bigoplus_{x' \in S} (\mathcal{W}_1)_{x'}/H_1(x') \longrightarrow 0.
\]  

(6.7)

From Lemma 4.2 we know that the logarithmic connection \( \mathbb{D}_1 : \mathcal{W}_1 \longrightarrow \mathcal{W}_1 \otimes K_X \otimes \mathcal{O}_X(S) \) preserves the subsheaf \( \mathcal{W}_2 \) (6.7). Let

\[
\mathbb{D}_2 : \mathcal{W}_2 \longrightarrow \mathcal{W}_2 \otimes K_X \otimes \mathcal{O}_X(S)
\]

be the logarithmic connection on \( \mathcal{W}_2 \) induced by \( \mathbb{D}_1 \). Since the eigenvalues of \( \text{Res}(\mathbb{D}_1, x') \) are \( \{0, -1, \cdots, 2-r\} \), from Lemma 1.3 it follows that the eigenvalues of \( \text{Res}(\mathbb{D}_2, x') \) are \( \{0, -1, \cdots, 3-r\} \). The eigenvalue 0 of \( \text{Res}(\mathbb{D}_2, x') \) has multiplicity three.

For each \( x' \in S \), let \( H_2(x') \subset (\mathcal{W}_2)_{x'} \) be the eigenspace of \( \text{Res}(\mathbb{D}_2, x') \) for the eigenvalue 0. Define the holomorphic vector bundle \( \mathcal{W}_3 \) by the short exact sequence of coherent analytic sheaves

\[
0 \longrightarrow \mathcal{W}_3 \longrightarrow \mathcal{W}_2 \longrightarrow \bigoplus_{x' \in S} (\mathcal{W}_2)_{x'}/H_2(x') \longrightarrow 0.
\]

We now proceed inductively. To explain this, for \( 2 \leq j \leq r - 2 \), suppose that we have constructed a holomorphic vector bundle \( \mathcal{W}_j \), and a logarithmic connection \( \mathbb{D}_j \) on it, such that the following conditions hold:

- For each \( x' \in S \), the eigenvalues of \( \text{Res}(\mathbb{D}_j, x') \) are \( \{0, -1, \cdots, j+1-r\} \).
- The multiplicity of the eigenvalue zero of \( \text{Res}(\mathbb{D}_j, x') \) is \( j+1 \).

Let

\[
H_j(x') \subset (\mathcal{W}_j)_{x'}
\]

be the eigenspace of \( \text{Res}(\mathbb{D}_j, x') \) for the eigenvalue 0. Then define the holomorphic vector bundle \( \mathcal{W}_{j+1} \) by the short exact sequence of coherent analytic sheaves

\[
0 \longrightarrow \mathcal{W}_{j+1} \longrightarrow \mathcal{W}_j \longrightarrow \bigoplus_{x' \in S} (\mathcal{W}_j)_{x'}/H_j(x') \longrightarrow 0.
\]  

(6.8)

From Lemma 4.2 we know that the logarithmic connection \( \mathbb{D}_j \) on \( \mathcal{W}_j \) preserves the subsheaf \( \mathcal{W}_{j+1} \subset \mathcal{W}_j \) in (6.8); the logarithmic connection on \( \mathcal{W}_{j+1} \) induced by \( \mathbb{D}_j \) is denoted by \( \mathbb{D}_{j+1} \). Since the eigenvalues of \( \text{Res}(\mathbb{D}_j, x') \) are \( \{0, -1, \cdots, j+1-r\} \), from Lemma 4.3 it follows that the eigenvalues of \( \text{Res}(\mathbb{D}_{j+1}, x') \) are \( \{0, -1, \cdots, j+2-r\} \). The eigenvalue 0 of \( \text{Res}(\mathbb{D}_{j+1}, x') \) has multiplicity \( j+2 \).

Proceeding inductively, we finally obtain the following:

(1) a holomorphic vector bundle \( \mathcal{W}_{r-1} \) on \( X \) of rank \( r \), and
(2) a logarithmic connection \( \mathbb{D}_{r-1} \) on \( \mathcal{W}_{r-1} \) whose singular locus is contained in \( S \), and for each point \( x' \in S \), the residue

\[
\text{Res}(\mathbb{D}_{r-1}, x') \in \text{End}((\mathcal{W}_{r-1})_{x'})
\]  

(6.9)

is nilpotent (meaning, zero is the only eigenvalue of it).

The next step in the proof of the theorem is to prove the following proposition.
Proposition 6.2. Take any point \( x' \in S \). Then

\[ \text{Res}(\mathbb{D}_{r-1}, x') = 0 \]  

(see (6.9)) if and only if \( M_j(D, x') = 0 \), for all \( 2 \leq j \leq r \), where \( M_j(D, x') \) are constructed in (5.6).

Proof. Consider the holomorphic vector bundle \( \mathcal{W} \) in (6.4). Let

\[ \varphi : \mathcal{W}_{r-1} \longrightarrow \mathcal{W} \]  

(6.10)

be the following composition of homomorphisms

\[ \mathcal{W}_{r-1} \longrightarrow \mathcal{W}_{r-2} \longrightarrow \cdots \longrightarrow \mathcal{W}_2 \longrightarrow \mathcal{W}_1 \longrightarrow \mathcal{W}; \]

see (6.8), (6.7) and (6.6) for the above homomorphisms. We note that the homomorphism \( \varphi \) in (6.10) is an isomorphism over the open subset \( X_0 \) in (3.8). We will now show that any holomorphic subbundle \( \mathcal{V} \subset \mathcal{W} \) produces a holomorphic subbundle of \( \mathcal{W}_{r-1} \). To prove this, let

\[ \mathcal{V}' \subset \mathcal{W}_{r-1} \]

be the coherent analytic subsheaf uniquely defined by the following condition: A holomorphic section \( \sigma \in H^0(U, \mathcal{W}_{r-1}) \) over some analytic open subset \( U \subset X \) is a section of \( \mathcal{V}' \) if and only if the restriction of \( \varphi(\sigma) \) to the complement \( U \setminus (U \cap S) \) is a section of the subbundle \( \mathcal{V} \subset \mathcal{W} \). It is straightforward to check that \( \mathcal{V}' \) is a holomorphic subbundle of \( \mathcal{W}_{r-1} \).

For \( 0 \leq j \leq r \), consider the holomorphic subbundle

\[ H_j \subset J^{r-1}(Q) = \mathcal{W} \]  

(see (3.6) and (6.4)). Let

\[ \mathcal{E}_j \subset \mathcal{W}_{r-1} \]

be the holomorphic subbundle corresponding to \( H_j \). So we have the filtration of holomorphic subbundles

\[ 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \mathcal{W}_{r-1} \]  

(6.11)

of \( \mathcal{W}_{r-1} \). Note that we have

\[ \varphi(\mathcal{E}_j) \subset H_j \]  

(6.12)

for all \( 0 \leq j \leq r \), where \( \varphi \) is the homomorphism in (6.10). Therefore, \( \varphi \) produces a homomorphism

\[ \varphi_j : \mathcal{E}_j/\mathcal{E}_{j-1} \longrightarrow H_j/H_{j-1} = Q \otimes K_X^{\otimes(r-j)} \]  

(6.13)

for all \( 1 \leq j \leq r \); see (3.7) for the isomorphism in (6.13).

Recall that the logarithmic connection \( \mathbb{D} \) on \( E = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \) satisfies the following condition: For any \( x' \in S \) and any \( 0 \leq i \leq r-1 \), the eigenspace of \( \text{Res}(\mathbb{D}, x') \) for the eigenvalue \( i \) is contained in the subspace \( (\mathcal{H}_{i+1})_{x'} \subset \mathcal{E}_{x'} \) (see (3.11) and (4.2)). This condition implies that the logarithmic connection \( \widehat{\mathbb{D}} \) on \( \mathcal{W} \) in (6.5) has the following property: For any \( x' \in S \) and any \( 0 \leq i \leq r-1 \), the eigenspace of \( \text{Res}(\widehat{\mathbb{D}}, x') \) for the

eigenvalue $i - r + 1$ is contained in the subspace $(H_{i+1})_{x'} \subset \mathcal{W}_{x'}$ in (3.6). Using this and (6.12) it follows that $\text{Res}(\mathcal{D}_{r-1}, x')$ in (6.9) preserves the filtration of subspaces

$$0 = (\mathcal{E}_0)_{x'} \subset (\mathcal{E}_1)_{x'} \subset (\mathcal{E}_2)_{x'} \subset \cdots \subset (\mathcal{E}_r)_{x'} = (\mathcal{W}_{r-1})_{x'}$$

obtained from (6.11). Since $\text{Res}(\mathcal{D}_{r-1}, x')$ is a nilpotent endomorphism, and it preserves the filtration in (6.14), we conclude that

$$\text{Res}(\mathcal{D}_{r-1}, x')((\mathcal{E}_i)_{x'}) \subset (\mathcal{E}_{i-1})_{x'}$$

for all $1 \leq i \leq r$. It also follows from the construction of $\mathcal{W}_{r-1}$ that

$$\text{Res}(\mathcal{D}_{r-1}, x')((\mathcal{E}_i)_{x'}) \cap (\mathcal{E}_{i-2})_{x'} = 0$$

for all $2 \leq i \leq r$.

The above observations on $\text{Res}(\mathcal{D}_{r-1}, x')$ combine together to give the following:

For every $x \in S$, the residue $\text{Res}(\mathcal{D}_{r-1}, x')$ gives an element

$$\mathbb{R}(\mathcal{D}_{r-1}, x') \in \bigoplus_{i=1}^{r-1} \text{Hom}(\mathcal{E}_{i+1}/\mathcal{E}_i, \mathcal{E}_i/\mathcal{E}_{i-1})_{x'}.$$  

(6.15)

This $\mathbb{R}(\mathcal{D}_{r-1}, x')$ has the property that $\text{Res}(\mathcal{D}_{r-1}, x') = 0$ if and only if $\mathbb{R}(\mathcal{D}_{r-1}, x') = 0$.

From (6.13) and the construction of $\mathcal{W}_{r-1}$ it follows that

$$\mathcal{E}_i/\mathcal{E}_{i-1} = (H_i/H_{i-1}) \otimes \mathcal{O}_X(-(r-i)S) = Q \otimes K_X^{(r-i)} \otimes \mathcal{O}_X(-(r-i)S)$$

for all $1 \leq i \leq r$. From this we conclude that

$$\mathcal{E}_{i+1}/\mathcal{E}_i = (\mathcal{E}_i/\mathcal{E}_{i-1}) \otimes TX \otimes \mathcal{O}_X(S).$$

(6.16)

The isomorphism in (6.16) implies that for any $x' \in S$ and all $1 \leq i \leq k - 1$, we have

$$\text{Hom}(\mathcal{E}_{i+1}/\mathcal{E}_i, \mathcal{E}_i/\mathcal{E}_{i-1})_{x'} = (K_X \otimes \mathcal{O}_X(-S))_{x'} = (K_X^{(r-i)})_{x'}.$$  

(see (4.11) for the last isomorphism). Therefore, $\mathbb{R}(\mathcal{D}_{r-1}, x')$ in (6.15) can be considered as an element

$$\mathbb{R}(\mathcal{D}_{r-1}, x') \in \bigoplus_{i=1}^{r-1} (K_X^{(r-i)})_{x'} = ((K_X^{(r-1)})_{x'})^\oplus.$$  

(6.17)

For any $1 \leq i \leq r - 1$, the element of $(K_X^{(r-1)})_{x'}$ in the $i$--th component of $\mathbb{R}(\mathcal{D}_{r-1}, x')$, with respect to the above decomposition, coincides with $M_{i+1}(D, x')$ constructed in (5.6).

It was noted earlier that $\text{Res}(\mathcal{D}_{r-1}, x') = 0$ if and only if $\mathbb{R}(\mathcal{D}_{r-1}, x') = 0$. Therefore, the proof of the proposition is complete. $\Box$

Continuing with the proof of Theorem 6.1 first assume that there is a branched $\text{SL}(r, \mathbb{C})$--operator $(V, \mathcal{F}, D)$ such that $\mathcal{D}$ coincides with the logarithmic connection on $E = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ associated to $(V, \mathcal{F}, D)$ by Proposition 3.5.5.

It is straightforward to check that the construction of the triple $(\mathcal{W}_{r-1}, \{\mathcal{E}_i\}_{i=0}^r, \mathcal{D}_{r-1})$ from $(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S), \mathcal{D})$ is the inverse of the construction of $(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S), \mathcal{D})$ from $(V, \mathcal{F}, D)$. More precisely, $(\mathcal{W}_{r-1}, \{\mathcal{E}_i\}_{i=0}^r, \mathcal{D}_{r-1})$ coincides with $(V, \mathcal{F}, D)$. In particular, $\mathcal{D}_{r-1}$ is a holomorphic connection on $\mathcal{W}_{r-1}$, as $D$ is a holomorphic
connection on $V$. Now from Proposition 6.2 we conclude that $M_j(\mathcal{D}, x') = 0$ for all $2 \leq j \leq r$ and every $x' \in S$.

To prove the converse, assume that

$$M_j(\mathcal{D}, x') = 0$$

for all $2 \leq j \leq r$ and every $x' \in S$, where $M_j(\mathcal{D}, x')$ are constructed in (5.6). We will show that there is a branched $\text{SL}(r, \mathbb{C})$–oper $(V, \mathcal{F}, D)$ such that $\mathcal{D}$ coincides with the logarithmic connection on $E = J^{r-1}(Q) \otimes O_X(-(r-1)S)$ associated to $(V, \mathcal{F}, D)$ by Proposition 3.5.

Since (6.17) holds, from Proposition 6.2 we know that the logarithmic connection $\mathcal{D}_{r-1}$ on $\mathcal{W}_{r-1}$ is actually a holomorphic connection. Consider the filtration $\{E_i\}_{i=0}^r$ of $\mathcal{W}_{r-1}$ in (6.11). It can be shown that

$$\mathcal{D}_{r-1}(E_i) \subset E_{i+1} \otimes K_X$$

for all $0 \leq i \leq r - 1$. Indeed, we have

$$(\mathcal{W}_{r-1}, \{E_i\}_{i=0}^r, \mathcal{D}_{r-1})|_{X_0} = (J^{r-1}(Q) \otimes O_X(-(r-1)S), \{\hat{H}_i\}_{i=0}^r, \mathcal{D})$$

over the nonempty open subset $X_0$ in (3.3); the filtration $\{\hat{H}_i\}_{i=0}^r$ is constructed in (3.11). So (6.18) follows from the given condition that

$$\mathcal{D}(\hat{H}_i) \subset \hat{H}_{i+1} \otimes K_X \otimes O_X(S)$$

for all $0 \leq i \leq r - 1$.

In view of (6.18), the second fundamental form of $E_i \subset \mathcal{W}_{r-1}$ for the holomorphic connection $\mathcal{D}_{r-1}$ produces a homomorphism

$$\Psi_i \in H^0(X, \text{Hom}(E_i/E_{i-1}, E_{i+1}/E_i) \otimes K_X)$$

for every $1 \leq i \leq r - 1$. Now using the isomorphism in (6.16) we conclude that

$$\Psi_i \in H^0(X, O_X(S)).$$

(6.20)

Recall the given condition that for all $1 \leq i \leq r - 1$, the section $\text{SF}(\mathcal{D}, i)$ in (6.2) is given by the constant function $1$ on $X$. Therefore, from the isomorphism in (6.19) we conclude that the section $\Psi_i$ in (6.20) coincides with the section of $O_X(S)$ given by the constant function $1$ on $X$. From this it follows that

$$(\mathcal{W}_{r-1}, \{E_i\}_{i=0}^r, \mathcal{D}_{r-1})$$

is a branched $\text{SL}(r, \mathbb{C})$–oper. The logarithmic connection on $J^{r-1}(Q) \otimes O_X(-(r-1)S)$ that corresponds to the branched $\text{SL}(r, \mathbb{C})$–oper $(\mathcal{W}_{r-1}, \{E_i\}_{i=0}^r, \mathcal{D}_{r-1})$ by Proposition 3.5 coincides with $\mathcal{D}$, because the construction $(\mathcal{W}_{r-1}, \{E_i\}_{i=0}^r, \mathcal{D}_{r-1})$ from $\mathcal{D}$ is the inverse of the construction in Proposition 3.5. This completes the proof of Theorem 6.1. □

Let

$$\mathcal{D} : E := J^{r-1}(Q) \otimes O_X(-(r-1)S) \rightarrow J^{r-1}(Q) \otimes O_X(-(r-2)S) \otimes K_X$$

be a logarithmic connection satisfying the following five conditions:
(1) The logarithmic connection on \( \det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S)) \) induced by the logarithmic connection \( \mathbb{D} \) on \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S) \) coincides with the logarithmic connection on \( \mathcal{O}_X \left( -\frac{r(r-1)}{2}S \right) \) given by the de Rham differential \( d \), once \( \det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S)) \) is identified with \( \mathcal{O}_X \left( -\frac{r(r-1)}{2}S \right) \) using \((3.24)\).  

(2) \( \mathbb{D}(\hat{H}_i) \subset \hat{H}_{i+1} \otimes K_X \otimes \mathcal{O}_X(S) \), for all \( 1 \leq i \leq r-1 \), where \( \{\hat{H}_i\}_{i=0}^r \) is the filtration of \( J^{r-1}(Q) \otimes \mathcal{O}_X(-(r - 1)S) \) in \((3.11)\). This implies that the second fundamental forms of the subbundles \( \hat{H}_i \) produce a section  

\[
\text{SF}(\mathbb{D}, i) \in H^0(X, \mathcal{O}_X(S))
\]  

as in \((3.23)\) for every \( 1 \leq i \leq r-1 \).  

(3) For all \( 1 \leq i \leq r-1 \), the section \( \text{SF}(\mathbb{D}, i) \) in \((6.22)\) coincides with the section of \( \mathcal{O}_X(S) \) given by the constant function \( 1 \) on \( X \).  

(4) For every \( x' \in S \), the eigenvalues of \( \text{Res}(\mathbb{D}, x') \) are the integers \( \{0, 1, \cdots, r - 2, r - 1\} \), with the multiplicity of each of them being one.  

(5) For all \( 0 \leq i \leq r-1 \) and every \( x' \in S \), the eigenspace of \( \text{Res}(\mathbb{D}, x') \) for the eigenvalue \( i \) is contained in the subspace \( (\hat{H}_{i+1})_{x'} \subset E_{x'} \) in \((1.2)\).  

In the proof of Theorem \(6.1\) we constructed the following:  

(1) a holomorphic vector bundle \( \mathcal{W}_{r-1} \) on \( X \) of rank \( r \), and  

(2) a logarithmic connection \( \mathbb{D}_{r-1} \) on \( \mathcal{W}_{r-1} \) whose singular locus is contained in \( S \), and for each point \( x' \in S \), the residue  

\[
\text{Res}(\mathbb{D}_{r-1}, x') \in \text{End}((\mathcal{W}_{r-1})_{x'})
\]

is nilpotent (see \((6.9)\)).  

If the residue \( \text{Res}(\nabla, y) \) of a logarithmic connection \( \nabla \) at a point \( y \) is nilpotent, then the local monodromy of \( \nabla \) around \( y \) is trivial if and only if \( \text{Res}(\nabla, y) = 0 \). Therefore, Proposition \(6.2\) has the following immediate corollary:  

**Corollary 6.3.** The connection \( \mathbb{D}_{r-1} \) on \( \mathcal{W}_{r-1} \) is nonsingular (meaning it is a holomorphic connection) if and only if the local monodromy of \( \mathbb{D}_{r-1} \) around each point of \( S \) is trivial.

In view of Corollary \(6.3\), from the last part of the proof of Theorem \(6.1\) (the part after Proposition \(6.2\)) we have the following:  

**Proposition 6.4.** There is a branched \( \text{SL}(r, \mathbb{C}) \)-oper \( (V, \mathcal{F}, D) \) such that the logarithmic connection \( \mathbb{D} \) in \((6.21)\) coincides with the logarithmic connection on \( E \) associated to \( (V, \mathcal{F}, D) \) by Proposition \(3.7\) if and only if the local monodromy of \( \mathbb{D} \) around each point \( S \) is trivial.

6.1. \( \text{SL}(r, \mathbb{C}) \)-opers with regular singularity and branched \( \text{SL}(r, \mathbb{C}) \)-opers. This subsection was communicated to us by Edward Frenkel.  

In this subsection we relate the branched \( \text{SL}(r, \mathbb{C}) \)-opers with the \( \text{SL}(r, \mathbb{C}) \)-opers with regular singularity and trivial monodromy introduced by Frenkel and Gaitsgory in [16 Section 2.9].
Let $G$ be a connected simple affine algebraic group over $\mathbb{C}$, and let $X$ be a smooth projective algebraic curve over $\mathbb{C}$. As before, $S$ is a finite subset of $X$. Beilinson and Drinfeld have introduced in [2, Section 4] and [3, Section 3.8] the notion of a $G$–oper on $X$ with regular singularity at $S$. We recall that it is an ordinary $G$–oper on the complement of $S$ in $X$, whose restriction to the formal punctured disc $D^\times_{x_k}$ at each point $x_k \in S$ is a $G$–oper on the punctured disc with regular singularity; the details are in [3, Section 3.8.8] (see also [16, Section 2.4]). Further, Beilinson and Drinfeld have defined the residue of such an oper. This residue is a point of the geometric invariant theoretic quotient $\mathfrak{g}/G \simeq \mathfrak{h}/W$, where $\mathfrak{g}$ is the Lie algebra of $G$, $\mathfrak{h} \subset \mathfrak{g}$ is its Cartan subalgebra, and $W$ is the corresponding Weyl group $N_G(T)/T$.

Denote by $\varpi$ the natural projection $\mathfrak{h} \to \mathfrak{h}/W$. For a dominant integral coweight $\check{\lambda} \in \mathfrak{h}$, denote by

$$\text{Op}_{\mathfrak{g}}^{\text{RS}, \varpi(-\check{\lambda} - \check{\rho})}$$

the space of $G$–opers with regular singularity on a punctured disc with fixed residue $\varpi(-\check{\lambda} - \check{\rho})$.

Frenkel and Gaitsgory have defined in [16, Section 2.9], following Beilinson and Drinfeld (unpublished), a subspace

$$\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}} \subset \text{Op}_{\mathfrak{g}}^{\text{RS}, \varpi(-\check{\lambda} - \check{\rho})}$$

(6.23)

consisting of all those $G$–opers whose monodromy around the origin is trivial. So, in particular, $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ is naturally identified with the space of regular $G$–opers on the disc. It may be mentioned that opers of this kind play an important rôle in the geometric Langlands correspondence (see [17] and [15, Section 9.6]).

The following proposition is due to E. Frenkel.

**Proposition 6.5.** There is a natural isomorphism between the space of all branched $\text{SL}(r, \mathbb{C})$–opers on $X$ with branching over $S$ and the space of all $\text{SL}(r, \mathbb{C})$–opers on $X$ with regular singularity at $S$ satisfying the following two conditions:

1. the restriction to the punctured formal disc around each point $x_k \in S$ has residue $\varpi(-2\check{\rho})$, and
2. this oper belongs to the subspace $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ in (6.23).

The proof of Proposition 6.5 follows directly by comparing Definition 2.1 and Theorem 6.1 with the definition of the subspace $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ in [16].

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**References**

[1] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* 85 (1957), 181–207.

[2] A. Beilinson and V. G. Drinfeld, Opers, arXiv:0501398.
[3] A. Beilinson and V. G. Drinfeld, Quantization of Hitchin’s integrable system and Hecke eigen-sheaves, (1991).

[4] D. Ben-Zvi and E. Frenkel, Spectral curves, opers and integrable systems, Publ. Math. Inst. Hautes Études Sci. 94 (2001), 87–159.

[5] I. Biswas and S. Dumitrescu, Branched holomorphic Cartan geometries and Calabi-Yau manifolds, Int. Math. Res. Not. 23 (2019), 7428–7458.

[6] I. Biswas and S. Dumitrescu, Branched projective structures, branched SO(3, C)-opers and logarithmic connections on jet bundle, Geom. Dedicata 215 (2021), 191–227.

[7] I. Biswas, L. P. Schaposnik and M. Yang, Generalized B-opers. Symmetry Integrability Geom. Methods Appl. 16 (2020), Article 041.

[8] B. Collier and A. Sanders, (G,P)-opers and global Slodowy slices, Adv. Math. 377 (2021), Paper No. 107490, 43 pp.

[9] P. Deligne, Équations différentielles à points singuliers réguliers, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.

[10] V. G. Drinfeld and V. V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, Current problems in mathematics, Vol. 24, 81–180, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[11] V. G. Drinfeld and V. V. Sokolov, Equations of Korteweg-de Vries type, and simple Lie algebras, Dokl. Akad. Nauk SSSR 258 (1981), 11–16.

[12] O. Dumitrescu, L. Fredrickson, G. Kydonakis, R. Mazzeo, M. Mulase and A. Neitzke, From the Hitchin section to opers through nonabelian Hodge, J. Differential Geom. 117 (2021), 223–253.

[13] E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, Second edition, Mathematical Surveys and Monographs, 88, American Mathematical Society, Providence, RI, 2004.

[14] E. Frenkel, Gaudin model and opers, Infinite dimensional algebras and quantum integrable systems, 1–58, Progr. Math., 237, Birkhäuser, Basel, 2005.

[15] E. Frenkel, Lectures on the Langlands program and conformal field theory, Frontiers in number theory, physics, and geometry. II, 387–533, Springer, Berlin, 2007.

[16] E. Frenkel and D. Gaitsgory, Local geometric Langlands correspondence and affine Kac-Moody algebras, Algebraic geometry and number theory, 69–260, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006.

[17] E. Frenkel and D. Gaitsgory, Weyl modules and opers without monodromy, Arithmetic and geometry around quantization, 101–121, Progr. Math., 279, Birkhäuser Boston, Boston, MA, 2010.

[18] E. Frenkel and C. TELEMAN, Geometric Langlands correspondence near opers, J. Ramanujan Math. Soc. 28 (2013), 123–147.

[19] P. Griffiths and J. Harris, Principles of algebraic geometry, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.

[20] R. C. Gunning, On uniformization of complex manifolds: the role of connections, Princeton Univ. Press, 1978.

[21] P. Koroteev, D. S. Sage and A. M. Zeitlin, (SL(N), q)-Opers, the q-Langlands correspondence, and quantum/classical duality, Comm. Math. Phys. 381 (2021), 641–672.

[22] R. Mandelbaum, Branched structures on Riemann surfaces, Trans. Amer. Math. Soc. 163 (1972), 261–275.

[23] R. Mandelbaum, Branched structures and affine and projective bundles on Riemann surfaces, Trans. Amer. Math. Soc. 183 (1973), 37–58.

[24] D. Masoero and A. Raimondo, Opers for higher states of quantum KdV models, Comm. Math. Phys. 378 (2020), 1–74.

[25] A. Weil, Généralisation des fonctions abéliennes, Jour. Math. Pures Appl. 17 (1938), 47–87.
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