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On the eigenfrequencies of elastic shear waves propagating in an inhomogeneous layer

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Abstract. In this work, we consider the problem of eigenfrequencies of elastic shear waves propagating in a layer whose Young’s modulus and density are functions of the longitudinal coordinate. Taking into account the material inhomogeneity makes the problem of the eigenfrequencies of the waves propagating in the layer more complicated. In this paper, the problem of pure shear is considered. To solve the problem, we use an integral formula which allows us to represent the general solution of the original equation with variable coefficients in terms of the general solution of the accompanying equation with constant coefficients.

1. Formulation of the problem
Suppose that an elastic layer occupies the area \(0 \leq x \leq a, \ 0 \leq y \leq b, \ |z| < \infty\), for which the equation of motion of an elastic inhomogeneous medium in pure shear (antiplane problem) has the form [1, 2]

\[
\frac{\partial}{\partial x} \left[ G(x) \frac{\partial W}{\partial x} \right] + G(x) \frac{\partial^2 W}{\partial y^2} = \rho(x) \frac{\partial^2 W}{\partial t^2},
\]

(1)

where \(G(x)\) is the shear modulus, \(\rho(x)\) is the density of the medium material, which are variable, and \(W = W(x, y, t)\) is the elastic displacement.

The boundary conditions are

\[ W = 0 \ \text{at} \ y = 0, b. \]

(2)

The solution of (1) satisfying boundary conditions (2) is represented as

\[ W(x, y, t) = \sum_{n=1}^{\infty} F_n(x, t) \sin(\lambda_n y), \quad \lambda_n = \frac{\pi n}{b}. \]

(3)

Substituting (3) into (1) and equating the corresponding coefficients of the series, we obtain

\[
\frac{\partial}{\partial x} \left[ G(x) \frac{\partial F_n(x, t)}{\partial x} \right] - \lambda_n^2 G(x) F_n(x, t) = \rho(x) \frac{\partial^2 F_n(x, t)}{\partial t^2}.
\]

(4)

The following three cases of boundary conditions at \(x = 0, a\) are considered:

1) \(W = 0\) at \(x = 0, a\). Then

\[ F_n(0, t) = 0, \quad F_n(a, t) = 0, \quad n = 1, 2, \ldots . \]

(5)
(2) $W = 0$ at $x = 0$ and $\sigma_{13} = 0$ at $x = a$. Then

$$F_n(0, t) = 0, \quad \left. \frac{\partial F_n(x, t)}{\partial x} \right|_{x=a} = 0, \quad n = 1, 2, \ldots .$$  \hfill (6)

(3) $\sigma_{13} = 0$ at $x = 0, a$. Then

$$\left. \frac{\partial F_n(x, t)}{\partial x} \right|_{x=0, a} = 0, \quad n = 1, 2, \ldots .$$  \hfill (7)

Taking into account the inhomogeneity of the material makes the problem of the eigenfrequencies of the waves propagating in the layer more complicated.

2. Representation of the general solution of the original equation through the general solution of the accompanying equation

Let us average the boundary-value problem for equation (4). In [3] the averaging process for nonperiodic inhomogeneity is described in detail. On the same interval $0 \leq x \leq a$, we consider equation (4) with parameters $G_0 = \text{const}$ and $\rho_0 = \text{const}$:

$$\frac{\partial}{\partial x} \left[ G_0 \frac{\partial P_n(x, t)}{\partial x} \right] - \lambda_n^2 G_0 P_n(x, t) = \rho_0 \frac{\partial^2 P_n(x, t)}{\partial t^2}. \hfill (8)$$

This case corresponds to a homogeneous layer. It is assumed that $P_n(x, t)$ satisfies the same boundary conditions as $F_n(x, t)$. Formula (8) is the accompanying problem. The coefficients of equation (8) of the accompanying problem can be any positive constants, but it is expedient to associate them with the characteristics of the original problem [4]. Suppose that

$$G_0 = \frac{1}{\langle 1/G(x) \rangle}, \quad \rho_0 = \langle \rho(x) \rangle. \hfill (9)$$

By averaging we mean the expression for the solution of original problem (4) in terms of the solution of accompanying problem (8).

We consider the case of steady-state harmonic oscillations. Suppose that

$$F_n(x, t) = f_n(x) e^{i\omega_n t}, \quad P_n(x, t) = p_n(x) e^{i\omega_n t}. \hfill (10)$$

Substituting (10) into (4)–(7), we obtain

$$\frac{\partial}{\partial x} \left[ G(x) \frac{df_n(x)}{dx} \right] + \left[ \omega_n^2 \rho(x) - \lambda_n^2 G(x) \right] f_n(x) = 0$$  \hfill (11)

with the boundary conditions

$$f_n(0) = 0, \quad f_n(a) = 0, \quad n = 1, 2, \ldots \quad (\text{case 1}),$$  \hfill (12)

$$f_n(0) = 0, \quad \left. \frac{df_n(x)}{dx} \right|_{x=a} = 0, \quad n = 1, 2, \ldots \quad (\text{case 2}),$$  \hfill (13)

$$\left. \frac{df_n(x)}{dx} \right|_{x=0,a} = 0, \quad n = 1, 2, \ldots \quad (\text{case 3}).$$  \hfill (14)

Substituting (10) into (8), we obtain

$$G_0 \frac{d^2 p_n(x)}{dx^2} + (\omega_n^2 \rho_0 - \lambda_n^2 G_0) p_n(x) = 0. \hfill (15)$$
The boundary conditions for \( p_n(x) \) are the same as for \( f_n(x) \):
\[
\begin{align*}
    p_n(0) &= 0, \quad p_n(a) = 0, \quad n = 1, 2, \ldots \quad \text{(case 1)}, \\
    p_n(0) &= 0, \quad \left. \frac{dp_n(x)}{dx} \right|_{x=a} = 0, \quad n = 1, 2, \ldots \quad \text{(case 2)}, \\
    \left. \frac{dp_n(x)}{dx} \right|_{x=0,a} &= 0, \quad n = 1, 2, \ldots \quad \text{(case 3)}.
\end{align*}
\]

The general solution of the accompanying equation (15) has the form
\[
    p_n(x) = Ae^{i\lambda_n \sqrt{\eta_n^2 - \omega^2}} + Be^{-i\lambda_n \sqrt{\eta_n^2 - \omega^2}},
\]
where \( \eta_n \) is a dimensionless characteristic, \( V_{\Phi_n} \) are the phase velocities of the corresponding modes, and \( c \) is the velocity of the elastic shear wave propagation
\[
    \eta_n = \frac{V_{\Phi_n}}{c}, \quad V_{\Phi_n} = \frac{\omega_n}{\lambda_n}, \quad c = \sqrt{\frac{G_0}{\rho_0}}.
\]

The integral formula that permits determining the solution of the original equation (11) in terms of the solution of the accompanying equation (15) has the form
\[
    f_n(x) = p_n(x) + \int_0^a \frac{\partial K_n(s,x)}{\partial s} \tilde{G}(s) \frac{dp_n(s)}{ds} \, ds - \int_0^a \left[ \omega_n^2 \tilde{\rho}(s) - \lambda_n^2 \tilde{G}(s) \right] p_n(s) K_n(s,x) \, ds,
\]
where
\[
    \tilde{G}(s) = G_0 - G(s), \quad \tilde{\rho}(s) = \rho_0 - \rho(s),
\]
and \( K_n(s,x) \) is the continuous fundamental solution of (11) [5], which satisfies the equation
\[
    \frac{\partial}{\partial x} \left[ G(x) \frac{\partial K_n(x,s)}{\partial x} \right] + \left[ \omega_n^2 \tilde{\rho}(x) - \lambda_n^2 G(x) \right] K_n(x,s) = -\delta(x-s),
\]
where \( x, s \in [0, a] \).

Substituting expression (21) into equation (11), we see that it is exactly satisfied if equations (15) and (23) are true.

Substituting (19) into (21), we obtain the general solution of the original equation
\[
    f_n(x) = A \left\{ e^{\beta_n(\omega) x + \beta_n(\omega)} \int_0^a \frac{\partial K_n(s,x)}{\partial s} \tilde{G}(s)e^{\beta_n(\omega)s} \, ds - \int_0^a K_n(s,x)[\omega_n^2 \tilde{\rho}(s) - \lambda_n^2 \tilde{G}(s)]e^{\beta_n(\omega)s} \, ds \right\} \\
    + B \left\{ e^{-\beta_n(\omega) x - \beta_n(\omega)} \int_0^a \frac{\partial K_n(s,x)}{\partial s} \tilde{G}(s)e^{-\beta_n(\omega)s} \, ds - \int_0^a K_n(s,x)[\omega_n^2 \tilde{\rho}(s) - \lambda_n^2 \tilde{G}(s)]e^{-\beta_n(\omega)s} \, ds \right\},
\]
where
\[
    \beta_n(\omega) = i\alpha_n(\omega), \quad \alpha_n(\omega) = \lambda_n \sqrt{\eta_n^2(\omega) - 1}.
\]

3. **Fundamental solution of the original equation**

To find the Green’s function satisfying equation (23), we use the method of successive approximations. For the zero approximation, we have
\[
    \frac{\partial}{\partial x} \left[ G(x) \frac{\partial K_{00}(x,s)}{\partial x} \right] = -\delta(x-s).
\]
For the next approximations, we obtain the recurrent equations

$$\frac{\partial}{\partial x} \left[ G(x) \frac{\partial K_{nn}(x, s)}{\partial x} \right] = -[\omega_n^2 \rho(x) - \lambda_n^2 G(x)] K_{nn-1}(x, s) - \delta(x - s), \quad m \geq 1. \quad (27)$$

From (26) we obtain

$$K_{n0}(x, s) = - \int_0^x \frac{h(z - s)}{G(z)} dz = \begin{cases} 0, & x < s, \\ - \int_s^x \frac{dz}{G(z)}, & x \geq s. \end{cases} \quad (28)$$

From (27) we have

$$K_{nm}(x, s) = - \int_0^x \frac{dx_1}{G(x_1)} \int_0^{x_1} [\omega_n^2 \rho(x_2) - \lambda_n^2 G(x_2)] K_{nm-1}(x_2, s) dx_2 - \int_0^x \frac{h(x_1 - s)}{G(x_1)} dx_1, \quad m \geq 1. \quad (29)$$

Hence, if \( \lim_{m \to \infty} K_{nm}(x, s) \) converges, then

$$K_n(x, s) = \lim_{m \to \infty} K_{nm}(x, s). \quad (30)$$

Substituting (30) into (24) and satisfying boundary conditions (12), (13) or (14), we obtain the corresponding equations for the eigenfrequencies \( \omega_n \).

4. Frequency equations of the zero approximation

Considering only the zero approximation for Green’s function, we obtain

$$K_n(x, s) \approx - \int_s^x \frac{dz}{G(z)}, \quad x \geq s, \quad K_n(x, s) = 0, \quad x < s. \quad (31)$$

For the derivatives, we have

$$\frac{\partial K_n(s, x)}{\partial s} = \frac{1}{G(s)}, \quad \frac{\partial K_n(s, x)}{\partial x} = - \frac{1}{G(s)}, \quad \frac{\partial^2 K_n(s, x)}{\partial s \partial x} = 0 \quad (32)$$

and

$$\frac{\partial f_n(x)}{\partial x} = A \left\{ \beta_n(\omega) \left[ e^{\beta_n(\omega)x} + \int_0^x \frac{\partial^2 K_n(s, x)}{\partial s \partial x} G(s) e^{\beta_n(\omega)s} ds \right] - \int_0^x \frac{\partial K_n(s, x)}{\partial x} [\omega_n^2 \rho(s)] \right\}$$

$$- \lambda_n^2 G(s) e^{\beta_n(\omega)x} - \int_0^x \frac{\partial^2 K_n(s, x)}{\partial s \partial x} G(s) e^{-\beta_n(\omega)s} ds \right\} - B \left\{ \beta_n(\omega) \left[ e^{-\beta_n(\omega)x} + \int_0^x \frac{\partial^2 K_n(s, x)}{\partial s \partial x} G(s) e^{-\beta_n(\omega)s} ds \right]$$

$$+ \int_0^x \frac{\partial K_n(s, x)}{\partial x} [\omega_n^2 \rho(s) - \lambda_n^2 G(s)] e^{-\beta_n(\omega)s} ds \right\}. \quad (33)$$

Satisfying boundary conditions (12)–(14), we obtain

$$\begin{cases} A + B = 0, \\ A \left\{ e^{\alpha \beta_n(\omega)} + \int_0^a [\beta_n(\omega)Q(s) + R_n(s)L(s)] e^{\beta_n(\omega)s} ds \right\}$$

$$+ B \left\{ e^{-\alpha \beta_n(\omega)} - \int_0^a [\beta_n(\omega)Q(s) - R_n(s)L(s)] e^{-\beta_n(\omega)s} ds \right\} = 0, \quad (34)$$

\[ \text{[4.1088/1742-6596/991/1/012041]} \]
\begin{align}
A + B &= 0, \\
A \left[ \beta_n(\omega)e^{a\beta_n(\omega)} + \frac{1}{G(a)} \int_0^a R_n(s)e^{a\beta_n(\omega)s} \, ds \right] \\
- B \left[ \beta_n(\omega)e^{-a\beta_n(\omega)} - \frac{1}{G(a)} \int_0^a R_n(s)e^{-a\beta_n(\omega)s} \, ds \right] &= 0, \\
\beta_n(\omega)(A - B) &= 0, \\
A \left[ \beta_n(\omega)e^{a\beta_n(\omega)} + \frac{1}{G(a)} \int_0^a R_n(s)e^{a\beta_n(\omega)s} \, ds \right] \\
- B \left[ \beta_n(\omega)e^{-a\beta_n(\omega)} - \frac{1}{G(a)} \int_0^a R_n(s)e^{-a\beta_n(\omega)s} \, ds \right] &= 0,
\end{align}

where

\[ Q(s) = \frac{G_0}{G(s)} - 1, \quad R_n(s) = \omega_n^2[\rho_0 - \rho(s)] - \lambda_n^2[G_0 - G(s)], \quad L(s) = \int_s^a \frac{dz}{G(z)}. \]

Equating the determinants of systems (34)–(36) to zero, we obtain the following approximate frequency equations:

- for clamped ends,

\[ \sin[a\alpha_n(\omega)] + \alpha_n(\omega) \int_0^a Q(s) \cos s\alpha_n(\omega) \, ds + \int_0^a R_n(s)L(s)\sin[s\alpha_n(\omega)] \, ds = 0, \]  

- for the clamped end \( x = 0 \) and the free end \( x = a, \)

\[ G(a)\alpha_n(\omega)\cos[a\alpha_n(\omega)] + \int_0^a R_n(s)\sin[s\alpha_n(\omega)] \, ds = 0, \]

- and for free ends,

\[ \alpha_n(\omega)\left\{ G(a)\alpha_n(\omega)\sin[a\alpha_n(\omega)] - \int_0^a R_n(s)\cos[s\alpha_n(\omega)] \, ds \right\} = 0. \]

Specific examples of inhomogeneities will be considered in a subsequent paper.

**Conclusions**

Taking into account the material inhomogeneity makes the problem of the eigenfrequencies of the waves propagating in the layer more complicated. The above-mentioned integral formula allows us to solve the problem of pure shear by representing the general solution of the original equation with variable coefficients in terms of the general solution of the accompanying equation with constant coefficients.

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