ON ASYMPTOTICALLY SHARP BI-LIPSCHITZ INEQUALITIES
OF QUASICONFORMAL MAPPINGS SATISFYING INHOMOGENEOUS POLYHARMONIC EQUATIONS

SHAOLIN CHEN AND DAVID KALAJ

Abstract. Suppose that \( f \) is a \( K \)-quasiconformal ((\( K, K' \))-quasiconformal resp.) self-mapping of the unit disk \( \mathbb{D} \), which satisfies the following: (1) the inhomogeneous polyharmonic equation \( \Delta^n f = \Delta(\Delta^{n-1} f) = \varphi_n (\varphi_n \in C(\mathbb{D})) \), (2) the boundary conditions \( \Delta^{n-1} f|_{\mathbb{T}} = \varphi_{n-1}, \ldots, \Delta^1 f|_{\mathbb{T}} = \varphi_1 (\varphi_j \in C(\mathbb{T}) \) for \( j \in \{1, \ldots, n-1\} \) and \( \mathbb{T} \) denotes the unit circle), and (3) \( f(0) = 0 \), where \( n \geq 2 \) is an integer and \( K \geq 1 \) (\( K' \geq 0 \) resp.). The main aim of this paper is to prove that \( f \) is Lipschitz continuous, and, further, it is bi-Lipschitz continuous when \( \|\varphi_j\|_{\infty} \) are small enough for \( j \in \{1, \ldots, n\} \). Moreover, the estimates are asymptotically sharp as \( K \to 1 \) (\( K' \to 0 \) resp.) and \( \|\varphi_j\|_{\infty} \to 0 \) for \( j \in \{1, \ldots, n\} \), and thus, such a mapping \( f \) behaves almost like a rotation for sufficiently small \( K \) (\( K' \) resp.) and \( \|\varphi_j\|_{\infty} \) for \( j \in \{1, \ldots, n\} \).

1. Preliminaries and main results

Let \( \mathbb{C} \cong \mathbb{R}^2 \) be the complex plane. For \( a \in \mathbb{C} \) and \( r > 0 \), let \( \mathbb{D}(a, r) = \{ z : |z - a| < r \} \), the open disk with center \( a \) and radius \( r \). For convenience, we use \( \mathbb{D}_r \) to denote \( \mathbb{D}(0, r) \), and \( \mathbb{D} \) the open unit disk \( \mathbb{D}_1 \). Let \( \mathbb{T} \) be the unit circle, i.e., the boundary \( \partial \mathbb{D} \) of \( \mathbb{D} \) and \( \overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T} \). Also, we denote by \( C^m(D) \) the set of all complex-valued \( m \)-times continuously differentiable functions from \( D \) into \( \mathbb{C} \), where \( D \) is a subset of \( \mathbb{C} \) and \( m \in \{0, 1, 2, \ldots\} \). In particular, let \( C(D) := C^0(D) \), the set of all continuous functions in \( D \).

For a real \( 2 \times 2 \) matrix \( A \), we use the matrix norm
\[
\|A\| = \sup\{|Az| : |z| = 1\}
\]
and the matrix function
\[
\lambda(A) = \inf\{|Az| : |z| = 1\}.
\]

For \( z = x + iy \in \mathbb{C} \), the formal derivative of a complex-valued function \( f = u + iv \) is given by
\[
D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},
\]

2000 Mathematics Subject Classification. Primary: 30C62; Secondary: 31A05.

Key words and phrases. Bi-Lipschitz continuity, Quasiconformal mapping, Polyharmonic equation.
where $x, y \in \mathbb{R}$, and $u, v$ are real-valued functions with partial derivatives. Then,
\[
\|Df\| = |f_z| + |f_{\overline{z}}| \quad \text{and} \quad \lambda(Df) = |f_z| - |f_{\overline{z}}|,
\]
where
\[
f_z = \frac{\partial f}{\partial z} = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(f_x + if_y).
\]
Moreover, we use
\[
J_f := \det Df = |f_z|^2 - |f_{\overline{z}}|^2
\]
to denote the Jacobian of $f$.

1.1. Bi-Lipschitz continuity of $K$-quasiconformal self-mappings of $\mathbb{D}$ satisfying the inhomogeneous polyharmonic equation. A sense-preserving homeomorphism $f$ from a domain $\Omega$ onto $\Omega'$, contained in the Sobolev class $W^{1,2}_{loc}(\Omega)$, is said to be a $K$-quasiconformal mapping if, for $z \in \Omega$,
\[
\|Df(z)\|^2 \leq K|\det Df(z)|, \quad \text{i.e.,} \quad \|Df(z)\| \leq K\lambda(Df(z)),
\]
where $K \geq 1$ (cf. [1, 31]).

Given a subset $\Omega$ of $\mathbb{C}$, a function $\psi : \Omega \rightarrow \mathbb{C}$ is said to be bi-Lipschitz if there is a constant $L \geq 1$ such that for all $z_1, z_2 \in \Omega$,
\[
(1.1) \quad \frac{1}{L}|z_1 - z_2| \leq |\psi(z_1) - \psi(z_2)| \leq L|z_1 - z_2|.
\]

In particular, $\psi$ is called Lipschitz if the inequality on the right of (1.1) holds, and $\psi$ is said to be co-Lipschitz if it satisfies the inequality on the left of (1.1).

It is clear that any sense-preserving bi-Lipschitz mapping is quasiconformal mapping (cf. [2]). But quasiconformal mappings are not necessarily bi-Lipschitz, not even Lipschitz (see the Example 1.1).

Example 1.1. Let
\[
f(z) = \begin{cases} 
z \log^\alpha \left(\frac{e}{|z|^2}\right), & z \in \mathbb{D} \setminus \{0\}, \\
0, & z = 0,
\end{cases}
\]
where $\alpha \in (0, 1/2)$ is a constant. Then $f$ is a quasiconformal self-homeomorphism of $\mathbb{D}$. However, $f$ is not Lipschitz at the origin (cf. [29]).

Example 1.2. The mapping $f(z) = z \log(|z|^2)$ is bi-harmonic (i.e., $\Delta(\Delta f) = 0$) in $\mathbb{D}_{e^{-2}} \setminus \{0\}$ and quasiconformal in $\mathbb{D}_{e^{-2}}$ but is not Lipschitz in any neighborhood of $z = 0$. The mapping $f$ is not bi-harmonic in 0, since $\Delta f(z) = 1/\overline{z}$. The mapping $f(z) = z \log(|z|^2)$ is bi-harmonic in $\mathbb{D}(ri, r)$ for small enough positive number $r$, and maps $\mathbb{D}(ri, r)$ onto a convex Jordan domain $\Omega$ with $C^2$ boundary. Thus the bi-harmonic mapping $h(z) = f(r(z + i))$ maps $\mathbb{D}$ quasiconformally onto the Jordan domain $\Omega$ with $\partial \Omega \in C^2$, but it is not Lipschitz. So the Lipschitz continuity fails if
we drop the condition $\Delta f$ is continuous up to the boundary. To prove that $\partial \Omega \in C^2$ we observe first that $\partial \Omega$ is rectifiable. Namely by direct computation, we have

$$|\partial \Omega| = \int_0^{2\pi} \left| \frac{\partial}{\partial t} h(e^{it}) \right| dt$$

$$= \int_0^{2\pi} \pi r \sqrt{1 + (1 + \log [2r^2(1 + \sin t)])^2 - 2 \sin t - 2 \sin t \log [2r^2(1 + \sin t)]} dt$$

$$< \infty$$

and

$$\frac{\partial}{\partial \Omega} h(e^{it}) \left[ \frac{\partial}{\partial \Omega} h(e^{it}) \right] = \frac{1 + ie^{it}(1 + \log[r(1 + \sin t)])}{|1 + ie^{it}(1 + \log[r(1 + \sin t)])|}.$$ 

Since

$$\frac{\partial}{\partial \Omega} h(e^{it}) = e^{i\varphi(s(t))},$$

where

$$s(t) = \int_0^t \left| \frac{\partial}{\partial \tau} h(e^{i\tau}) \right| d\tau$$

is the natural parameter, and since the limit of left-hand side in (1.2) tends to $-1$, it follows that $\varphi$ is continuous in $s = 0$, and therefore the function $s \to h(t(s))$ is $C^1$. To show that the curve is $C^2$, we find the curvature of $\partial \Omega$ at 0. Namely if $x(t) = \text{Re}(h(e^{it}))$ and $y(t) = \text{Im}(h(e^{it}))$, then the curvature

$$\kappa(t) = \frac{|\ddot{x} \dot{y} - \ddot{y} \dot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$ 

Then it can be proved that $\lim_{t \to 0} \kappa(t) = 0$. Thus $\kappa$ is continuous in $\partial \Omega$ which means that the curve is $C^2$.

From the Example 1.2, we conclude that bi-harmonic quasiconformal mappings between smooth domains are not necessarily Lipschitz. Hence there is a classical problem which is as follows.

**Question 1.3.** For $K \geq 1$ ($K' \geq 0$ resp.), what conditions do $K$-quasiconformal mappings ($K, K'$)-quasiconformal mappings resp.) satisfy to be bi-Lipschitz continuous? Furthermore, can you find the sharp bi-Lipschitz constants of the bi-Lipschitz continuous mappings?

Recently, this problem has been attracted much attention. For example, the Lipschitz continuity of harmonic quasiconformal mappings has been discussed by many authors (see [22, 24, 30, 36, 40, 42]). The Lipschitz continuity of $(K, K')$-quasiconformal harmonic mappings (see Section 1.2) has also been investigated in [7, 26, 48]. On the discussion of the related topic, we refer to [12, 13, 18, 21, 27, 35, 42, 45, 46] and the related references therein. On the study of the Lipschitz characteristic of quasiconformal mappings satisfying certain elliptic PDEs, see [1, 6, 23, 25, 28, 29].

The main aim of this paper is to investigate the Problem 1.3 for $K$-quasiconformal ($K, K'$)-quasiconformal resp.) self-mappings of $\mathbb{D}$ satisfying the inhomogeneous
polyharmonic equations. In order to state our main results, we need to recall some basic definitions and some results which motivate the present work.

For $z, \zeta \in \mathbb{D}$ with $z \neq \zeta$, let

$$G(z, \zeta) = \frac{1}{2\pi} \log \left| \frac{1 - z\zeta}{z - \zeta} \right| \quad \text{and} \quad P(z, e^{it}) = \frac{1}{2\pi} \frac{1 - |z|^2}{|1 - z e^{-it}|^2}$$

be the Green function and (harmonic) Poisson kernel, respectively, where $t \in [0, 2\pi]$.

Let $\varphi_n \in \mathcal{C}(\mathbb{T})$ and $f \in \mathcal{C}^{2n}(\mathbb{D})$, where $n \geq 2$ is an integer. Of particular interest for our investigation is the following inhomogeneous polyharmonic equation (or $n$-harmonic equation):

$$\Delta^n f = \Delta(\Delta^{n-1} f) = \varphi_n \text{ in } \mathbb{D}$$

with the following associated Dirichlet boundary value condition:

$$\Delta^{n-1} f|_T = \varphi_{n-1}, \ldots, \Delta^1 f|_T = \varphi_1, \Delta^0 f|_T = \varphi_0,$$

where $\Delta^0 f := f$,

$$\Delta^1 f := \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4f,$$

stands for the Laplacian of $f$, and $\varphi_k \in \mathcal{C}(\mathbb{T})$ for $k \in \{0, 1, \ldots, n-1\}$.

By the iterated Poly-Cauchy integral operators (cf. [3]), we see that all solutions to the equation (1.3) satisfying (1.4) are given by

$$f(z) = P[\varphi_0](z) + \sum_{k=1}^{n} (-1)^k G_k[\varphi_k](z), \ z \in \mathbb{D},$$

where

$$P[\varphi_0](z) = \int_0^{2\pi} P(z, e^{it})\varphi_0(e^{it})dt,$$

$$G_k[\varphi_k](z) = \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} G(z, \xi_1) \cdots G(\xi_{k-1}, \xi_k) \times \left( \int_0^{2\pi} P(\xi_k, e^{it})\varphi_k(e^{it})dt \right) d\sigma(\xi_k) \cdots d\sigma(\xi_1)$$

for $k \in \{1, \ldots, n-1\}$, and

$$G_n[\varphi_n](z) = \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} G(z, \zeta_1) \cdots G(\zeta_{n-2}, \zeta_{n-1}) \times \left( \int_{\mathbb{D}} G(\zeta_{n-1}, \zeta_n)\varphi_n(\zeta_n)d\sigma(\zeta_n) \right) d\sigma(\zeta_{n-1}) \cdots d\sigma(\zeta_1).$$

Here $d\sigma$ is the Lebesgue area measure in $\mathbb{D}$.

The behavior of solutions to the polyharmonic equations with the different boundary value conditions has attracted much attention of many authors (cf. [4, 11, 14, ...]}
where the notation \( Q \) following result.

\[ N \]

and different from the Poisson equations. In this sense, Theorem 1.4 is a generalization of Theorem A. Suppose that \( f \) is a \( K \)-quasiconformal self-mapping of \( \mathbb{D} \) satisfying the inhomogeneous polyharmonic equation (1.3) with the boundary condition (1.4).

Our result is as follows.

**Theorem 1.4.** Let \( \varphi_n \in C(\mathbb{D}) \) and \( \varphi_k \in C(\mathbb{T}) \), and let \( K \geq 1 \) be a constant, where \( n \geq 2 \) and \( k \in \{1, \ldots, n-1\} \). Suppose that \( f \) is a \( K \)-quasiconformal self-mapping of \( \mathbb{D} \) satisfying the inhomogeneous polyharmonic equation (1.3) with \( \Delta^{n-1} f|_T = \varphi_{n-1}, \ldots, \Delta^1 f|_T = \varphi_1 \) and \( f(0) = 0 \). Then, there are nonnegative constants \( M_j(K) \) and \( N_j(K, \varphi_1, \ldots, \varphi_n) \) \((j \in \{1, 2\})\) with

\[
\lim_{K \to 1} M_j(K) = 1 \quad \text{and} \quad \lim_{\|\varphi_1\|_{\infty} \to 0, \ldots, \|\varphi_n\|_{\infty} \to 0} N_j(K, \varphi_1, \ldots, \varphi_n) = 0
\]
such that for all \( z_1, z_2 \in \mathbb{D} \),

\[
(M_1(K) - N_1(K, \varphi_1, \ldots, \varphi_n))|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq (M_2(K) + N_2(K, \varphi_1, \ldots, \varphi_n))|z_1 - z_2|,
\]

where \( \|\varphi_n\|_{\infty} = \sup_{z \in \mathbb{D}} |\varphi_n(z)| \) and \( \|\varphi_k\|_{\infty} = \sup_{\zeta \in \mathbb{T}} |\varphi_k(\zeta)| \) for \( k \in \{1, \ldots, n-1\} \).

**Remark 1.5.** For some given functions \( g_1 \) and \( g_2 \) defined in \( \mathbb{D} \), let

\[
\mathcal{P} \mathcal{Y}(g_1) = \{ f : \Delta^n f = \Delta(\Delta^{n-1} f) = g_1 \text{ in } \mathbb{D} \} \quad \text{and} \quad \mathcal{P} \mathcal{Y}(g_2) = \{ f : \Delta f = g_2 \text{ in } \mathbb{D} \},
\]

where \( n \geq 2 \). Then \( \mathcal{P}(0) \subset \mathcal{P} \mathcal{Y}(0) \). Hence, the polyharmonic equations is essentially different from the Poisson equations. In this sense, Theorem 1.4 is a generalization of [28, Theorem 1.2] and [40, Theorem 3.3].

The following is the so-called Mori’s Theorem (cf. [9, 28, 38]). We refer to [10, 34] for some analogical results of Theorem A in the higher dimensional case.

**Theorem A.** Suppose that \( f \) is a \( K \)-quasiconformal self-mapping of \( \mathbb{D} \) with \( f(0) = 0 \). Then, there exists a constant \( Q(K) \), satisfying the condition \( Q(K) \to 1 \) as \( K \to 1 \), such that

\[
|f(z_2) - f(z_1)| \leq Q(K)|z_2 - z_1|^\frac{1}{K},
\]

where the notation \( Q(K) \) means that the constant \( Q \) depends only on \( K \).

We remark that in [43] it is proved

\[
1 \leq Q(K) \leq 16^{1 - \frac{1}{K}} \min \left\{ \left( \frac{23}{8} \right)^{1 - \frac{1}{K}}, (1 + 2^{3 - 2K})^\frac{1}{K} \right\}.
\]

As a direct consequence of Claim 3.8 in the proof of Theorem 1.4, we have the following result.
Corollary 1.6. Under the assumptions of Theorem 1.4, if, further,

\[
\frac{(Q(K))^{-2K}K^{-2}}{2\pi} \int_0^{2\pi} |e^{it} - e^{i\theta}|^{2K-2} dt > \left( \frac{7}{6} + \frac{1}{2K^2} \right) \|\varphi_1\|_{\infty} + \sum_{j=2}^{n} \left( \frac{47}{240} + \frac{1}{16K^2} \right) \|\varphi_j\|_{\infty} \left( \frac{3}{16} \right)^{j-2},
\]

then \( f \) is co-Lipschitz continuous, and so, it is bi-Lipschitz continuous, where \( Q(K) \) is the same as in Theorem A.

By (1.8) and [28, Formula 3.27], we see that

\[
K^{-2}(Q(K))^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{it} - e^{i\theta}|^{2K-2} dt = \frac{2^{2K-2}\Gamma(K - \frac{1}{2})}{\sqrt{\pi}K^2(K - 1)\Gamma(K - 1)(Q(K))^{2K}} \geq \frac{1}{K^2(Q(K))^{2K}} \geq \frac{1}{K^{246K^{-2}}},
\]

which gives the following result, where \( \Gamma \) is the Gamma function.

Corollary 1.7. Under the assumptions of Theorem 1.4, if, further,

\[
\frac{1}{K^{246K^{-2}}} > \left( \frac{7}{6} + \frac{1}{2K^2} \right) \|\varphi_1\|_{\infty} + \sum_{j=2}^{n} \left( \frac{47}{240} + \frac{1}{16K^2} \right) \|\varphi_j\|_{\infty} \left( \frac{3}{16} \right)^{j-2},
\]

then \( f \) is co-Lipschitz continuous, and so, it is bi-Lipschitz continuous.

By the discussions in Step 3 of the proof of Theorem 1.4 in Section 3 or by Corollary 1.6, we see that the co-Lipschitz continuity coefficient

\[
M_1(K) - N_1(K, \varphi_1, \ldots, \varphi_n)
\]

is positive for small enough norms \( \|\varphi_k\|_{\infty} \), where \( k \in \{1, \ldots, n\} \). The following example (Example 1.8) shows that the condition for \( f \) to be co-Lipschitz continuous cannot be replaced by the one that \( \varphi_k \) are arbitrary, where \( k \in \{1, \ldots, n\} \).

Example 1.8. For \( z \in \overline{D} \), let

\[
\varphi_n(z) = \beta \left( \prod_{j=1}^{n}(\tau - 2j + 4) \right) \left( \prod_{j=1}^{n}(\tau - 2j + 2) \right) z^{\frac{n}{2} - n + 1} \overline{z}^{-\frac{n}{2}},
\]

where \( \tau > 2n - 1 \) and \( \beta \) are constants with \( n \geq 2 \) and \( |\beta| = 1 \). Suppose that \( f \) satisfies the following polyharmonic equation

(1.9) \[ \Delta^n f(z) = \Delta(\Delta^{n-1} f(z)) = \varphi_n(z), \quad z \in D, \]

with the following associated Dirichlet boundary value condition:

\[ \Delta^k f(\xi) = \varphi_k(\xi) \text{ and } f(\xi) = \varphi_0(\xi), \quad \xi \in T, \]
where \( \varphi_0(\xi) = \beta \xi \), and for \( k \in \{1, \ldots, n-1\} \),

\[
\varphi_k(\xi) = \beta \left( \prod_{j=1}^{k} (\tau - 2j + 4) \right) \left( \prod_{j=1}^{k} (\tau - 2j + 2) \right) \xi.
\]

It follows from (1.5) that

\[
f(z) = \beta |z|^\tau z, \quad z \in \mathbb{D},
\]

is the solution to (1.9). Obviously, \( f \) is a \( K \)-quasiconformal self-mapping of \( \mathbb{D} \) with \( f(0) = 0 \) and \( K = 1 + \tau \). Furthermore,

\[
\|\varphi_n\|_{\infty} = \left( \prod_{j=1}^{n} (\tau - 2j + 4) \right) \left( \prod_{j=1}^{n} (\tau - 2j + 2) \right),
\]

and for \( k \in \{1, \ldots, n-1\} \),

\[
\|\varphi_k\|_{\infty} = \left( \prod_{j=1}^{k} (\tau - 2j + 4) \right) \left( \prod_{j=1}^{k} (\tau - 2j + 2) \right) \quad \text{and} \quad \|\varphi_0\|_{\infty} = 1.
\]

However, \( f \) is not co-Lipschitz continuous because

\[
\lambda(Df(0)) = |f_z(0)| - |f_\bar{z}(0)| = 0.
\]

By applying Corollary 1.7, we illustrate the possibility of \( f \) to be bi-Lipschitz continuous by the following example.

**Example 1.9.** Suppose that \( f \) satisfies the following bi-harmonic equation

\[
\Delta(\Delta f(z)) = -\frac{16}{15}, \quad z \in \mathbb{D},
\]

with the following associated Dirichlet boundary value condition:

\[
\Delta f(\xi) = -\frac{1}{5} \quad \text{and} \quad f(\xi) = \xi, \quad \xi \in \mathbb{T}.
\]

By (1.5), we see that

\[
f(z) = z + \frac{1}{60}(|z|^2 - |z|^4), \quad z \in \mathbb{D},
\]

is the solution to (1.10). It is not difficult to know that \( f \) is a \( K \)-quasiconformal self-mapping of \( \mathbb{D} \) with

\[
K = \max_{z \in \mathbb{T}} \left\{ \frac{1 + \overline{z}(1 - 2|z|^2)M}{1 + z(1 - 2|z|^2)M} \left\| \frac{Mz(1 - 2|z|^2)}{Mz(1 - 2|z|^2) - |Mz(1 - 2|z|^2)|} \right\|_{\infty} \right\} = \frac{30}{29},
\]

where \( M = \frac{1}{60} \). Since elementary computations lead to

\[
\frac{1}{K^246^2(K-1)} = \frac{29^2}{30^246^2} > 0.717, \quad \left( \frac{7}{6} + \frac{1}{2K^2} \right) \|\varphi_1\|_{\infty} < 0.326
\]

and

\[
\left( \frac{47}{240} + \frac{1}{16K^2} \right) \|\varphi_2\|_{\infty} < 0.271,
\]

\( \|\varphi_1\|_{\infty} < 0.326 \) and \( \|\varphi_2\|_{\infty} < 0.271 \).
we see that
\[
\frac{1}{K^246^{2K-2}} \left( \frac{7}{6} + \frac{1}{2K^2} \right) \|\varphi_1\|_\infty + \left( \frac{47}{240} + \frac{1}{16K^2} \right) \|\varphi_2\|_\infty,
\]
where \(\|\varphi_1\|_\infty = \frac{1}{6}\) and \(\|\varphi_2\|_\infty = \frac{16}{7}\). Now, it follows from Corollary 1.7 that \(f\) is co-Lipschitz continuous, and so, it is bi-Lipschitz continuous.

1.2. **Bi-Lipschitz continuity of \((K, K')\)-quasiconformal self-mappings of \(\mathbb{D}\) satisfying the inhomogeneous polyharmonic equation.** A sense-preserving homeomorphism

\[f : \Omega_1 \to \Omega_2,\]

where \(\Omega_1\) and \(\Omega_2\) are subdomains of \(\mathbb{C}\), is said to be a \((K, K')\)-quasiconformal mapping if \(f\) is absolutely continuous on lines in \(\Omega_1\), and there are constants \(K \geq 1\) and \(K' \geq 0\) such that

\[\|D_f(z)\|^2 \leq KJ_f(z) + K', \quad z \in \Omega_1.\]

In particular, if \(K' = 0\), then \(f\) is a \(K\)-quasiconformal mapping (cf. [26]). The second aim of this paper is to study the the asymptotically sharp bi-Lipschitz inequalities of \((K, K')\)-quasiconformal self-mapping of \(\mathbb{D}\) satisfying the inhomogeneous polyharmonic equation (1.3) with the boundary condition (1.4). It is read as follows.

**Theorem 1.10.** Let \(\varphi_0\) be a sense-preserving homeomorphism of \(\mathbb{T}\) onto itself. For \(n \geq 2\) and \(k \in \{1, \ldots, n-1\}\), let \(\varphi_n \in \mathcal{C}(\overline{\mathbb{D}})\) and \(\varphi_k \in \mathcal{C}(\mathbb{T})\), and let \(K \geq 1\) and \(K' \geq 0\) be constants. Suppose that \(f\) is a \((K, K')\)-quasiconformal self-mapping of \(\overline{\mathbb{D}}\) satisfying the inhomogeneous polyharmonic equation (1.3) with the Dirichlet boundary value condition (1.4).

(a) If \(|P(\varphi_0)(0)| + \sqrt{K'} + 2K \left( \frac{1}{3} \|\varphi_1\|_\infty + \frac{1}{15} \sum_{k=2}^n \left( \frac{3}{16} \right)^{k-2} \|\varphi_k\|_\infty \right) < \frac{2}{\pi}\), then \(f\) is bi-Lipschitz continuous in \(\mathbb{D}\).

(b) If \(|P(\varphi_0)(0)| = 0\) and \(\sqrt{K'} + 2K \left( \frac{1}{3} \|\varphi_1\|_\infty + \frac{1}{15} \sum_{k=2}^n \left( \frac{3}{16} \right)^{k-2} \|\varphi_k\|_\infty \right) < \frac{2}{\pi}\), then, there are nonnegative constants \(M_j(K, K')\) and \(N_j(K, \varphi_1, \ldots, \varphi_n)\) \((j \in \{3, 4\})\) with

\[
\lim_{K \to 1, K' \to 0} M_j(K, K') = 1, \quad \lim_{\|\varphi_1\|_\infty \to 0, \ldots, \|\varphi_n\|_\infty \to 0} N_j(K, \varphi_1, \ldots, \varphi_n) = 0
\]

and \(M_4(K, K') - N_4(K, \varphi_1, \ldots, \varphi_n) > 0\) such that for all \(z_1, z_2 \in \mathbb{D}\),

\[
(M_4(K, K') - N_4(K, \varphi_1, \ldots, \varphi_n)) |z_1 - z_2| \leq |f(z_1) - f(z_2)|
\]

\[
\leq \left( M_3(K, K') + N_3(K, \varphi_1, \ldots, \varphi_n) \right) |z_1 - z_2|.
\]

We remark that Theorem 1.10 is a generalization of [40, Theorem 3.3]. Moreover, if \(n = 2\), then Theorem 1.10 is also an improvement of [48, Theorem 1.2].

1.3. **Lipschitz continuity with respect to some certain boundary conditions.** We recall that the (periodic) *Hilbert transformation* of a \(2\pi\)-periodic function \(\Psi \in L^1(\mathbb{T})\) is defined by

\[
H(\Psi)(\theta) = -\frac{1}{\pi} \int_0^\pi \frac{\Psi(\theta + t) - \Psi(\theta + t)}{2 \tan(t/2)} dt.
\]
It is well known that the Lipschitz continuity of $\varphi$ in $T$ is not enough to guarantee that its harmonic extension $P[\varphi]$ is also Lipschitz continuous. In fact, $P[\varphi]$ is Lipschitz continuous if and only if the Hilbert transform of $d\varphi(e^{i\theta})/d\theta \in L^\infty(T)$ (cf. [5, 49]). The last aim of this paper is to investigate the Lipschitz continuity of solutions to the inhomogeneous polyharmonic equation (1.3) satisfying some certain boundary conditions.

Theorem 1.11. For $n \geq 2$ and $k \in \{1, \ldots, n-1\}$, let $\varphi_n \in C(D)$ and $\varphi_k \in C(T)$, and let $\varphi_0 \in C(T)$ be differentiable. Suppose that $f$ is a solution to the inhomogeneous polyharmonic equation (1.3) satisfying $\Delta^{n-1}f|_T = \varphi_{n-1}, \ldots, \Delta^1f|_T = \varphi_1, \Delta^0f|_T = \varphi_0$. Then $f$ is Lipschitz continuous in $D$ if and only if the Hilbert transform of $d\varphi_0(e^{i\theta})/d\theta \in L^\infty(T)$.

We will prove several auxiliary results in Section 2. The proof of Theorem 1.4 will be presented in Section 3. Theorems 1.10 and 1.11 will be proved in Sections 4 and 5, respectively.

2. Some auxiliary results

In this section, we shall prove several lemmas which will be used later on. The first lemma is as follows.

Lemma 2.1. Let $G$ be the Green function. Then, for $z \in D$,

\begin{equation}
\int_{D} |G(z, \zeta)|d\sigma(\zeta) = \frac{1 - |z|^2}{4}
\end{equation}

and

\begin{equation}
\int_{D} (1 - |\zeta|^2)|G(z, \zeta)|d\sigma(\zeta) = \frac{(1 - |z|^2)(3 - |z|^2)}{16} \leq \frac{3(1 - |z|^2)}{16}.
\end{equation}

Theorem B. (cf. [33]) For $z \in D$, we have

\begin{equation}
\frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\theta}{|1 - z e^{i\theta}|^{2\alpha}} = \sum_{k=0}^{\infty} \left( \frac{\Gamma(k + \alpha)}{k! \Gamma(\alpha)} \right)^2 |z|^{2k},
\end{equation}

where $\alpha > 0$.

Proof of Lemma 2.1. We first prove (2.1). Let

\begin{equation}
w = \frac{z - \zeta}{1 - \overline{z}\zeta} = re^{it},
\end{equation}

where $r \in [0, 1)$ and $t \in [0, 2\pi]$. Since Theorem B implies

\begin{equation}
\frac{1}{2\pi} \int_{0}^{2\pi} \frac{dt}{|1 - z e^{i\theta}|^{2\alpha}} = \sum_{j=0}^{\infty} (j + 1)^2 |z|^{2j} r^{2j},
\end{equation}

by (2.3), we obtain

\begin{equation}
\int_{D} |G(z, \zeta)|d\sigma(\zeta) = \frac{1}{2\pi} \int_{D} \left( \log \frac{1}{|w|} \right) \frac{(1 - |z|^2)^2}{|1 - \overline{z}w|^4}d\sigma(w) = \frac{1 - |z|^2}{4}.
\end{equation}
Now we show that (2.2). For \( z \in \mathbb{D} \), let
\[
I_1(z) = \int_{\mathbb{D}} (1 - |\zeta|^2)|G(z, \zeta)|d\sigma(\zeta).
\]

By Theorem B, we have
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - zre^{it}|^6} = \sum_{j=0}^{\infty} \frac{(j + 1)^2(j + 2)^2 |z|^{2j}r^{2j}}{4},
\]
which, together with
\[
1 - |\zeta|^2 = (1 - |w|^2)(1 - |z|^2)
\]
implies that
\[
I_1(z) = \frac{1}{2\pi} \int_{\mathbb{D}} \frac{(1 - |w|^2)(1 - |z|^2)^3 \log \frac{1}{|w|}}{|1 - \zeta w|^6} d\sigma(w)
\]
\[
= (1 - |z|^2)^3 \int_0^1 (1 - r^2)r \log \frac{1}{r} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - zre^{it}|^6} \right) dr
\]
\[
= \frac{(1 - |z|^2)(3 - |z|^2)}{16}.
\]
The proof of this lemma is complete. \( \square \)

**Lemma 2.2.** For \( z \in \mathbb{D} \),
\[
(2.5) \quad \frac{1}{2\pi} \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^2}{|1 - z\zeta||z - \zeta|} d\sigma(\zeta) \leq \frac{4(2 - |z|^2)}{15}.
\]
In particular, the inequality (2.5) is sharp at \( z = 0 \).

**Proof.** Let
\[
\eta = \frac{z - \zeta}{1 - \zeta} = \rho e^{i\theta}.
\]
Then
\[
1 - \zeta = \frac{1 - |z|^2}{1 - \zeta \eta} \quad \text{and} \quad 1 - |\zeta|^2 = \frac{(1 - |\eta|^2)(1 - |z|^2)}{|1 - \zeta \eta|^2}.
\]
By Theorem B, we obtain
\[
(2.7) \quad \frac{1}{2\pi} \int_{\mathbb{D}} \frac{(1 - |\eta|^2)^2}{|\eta||1 - \zeta \eta|^6} d\sigma(\eta) = \int_0^1 (1 - \rho^2)^2 \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - \zeta \rho e^{i\theta}|^6} \right) d\rho
\]
\[
= \sum_{j=0}^{\infty} \frac{(j + 1)^2(j + 2)^2}{4} |z|^{2j} \int_0^1 (1 - \rho^2)^2 \rho^{2j} d\rho
\]
\[
= \sum_{j=0}^{\infty} \frac{(j + 1)^2(j + 2)^2}{(2j + 1)(2j + 3)(j + \frac{5}{2})} |z|^{2j}.
\]
By computation, we have

\[(2.8) \quad \frac{(j+1)^2(j+2)^2}{(2j+1)(2j+3)(j+\frac{5}{2})} \leq \frac{4}{15}(j+2).\]

It follows from (2.6), (2.7) and (2.8) that

\[
\frac{1}{2\pi} \int_{D} \frac{(1-|z|^2)^2}{|1-z\zeta||z-\zeta|} d\sigma(\zeta) = \frac{(1-|\eta|^2)^2}{2\pi} \int_{D} \frac{1}{|1-\tau\eta|^6} d\sigma(\eta)
\]
\[
= (1-|\eta|^2)^2 \sum_{j=0}^{\infty} \frac{(j+1)^2(j+2)^2}{(2j+1)(2j+3)(j+\frac{5}{2})}|z|^{2j}
\]
\[
\leq \frac{4(1-|\eta|^2)^2}{15} \sum_{j=0}^{\infty} (j+2)|z|^{2j}
\]
\[
= \frac{4(2-|z|^2)}{15}.
\]

The proof of this lemma is finished. \(\Box\)

**Lemma 2.3.** Let \(P\) be the Poisson kernel and \(\theta \in [0, 2\pi]\). Then

\[
\int_{D} P(\zeta, e^{i\theta})(1-|\zeta|^2) d\sigma(\zeta) = \frac{1}{4}.
\]

**Proof.** Let \(\zeta = \rho e^{i\tau}\). By Theorem B, we have

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \frac{dt}{|1-\rho e^{it} e^{-i\theta}|^2} = \frac{1}{1-\rho^2},
\]
which gives that

\[
\int_{D} P(\zeta, e^{i\theta})(1-|\zeta|^2) d\sigma(\zeta) = \int_{0}^{1} (1-\rho^2)^2 \rho \left[ \frac{1}{2\pi} \int_{0}^{2\pi} \frac{dt}{|1-\rho e^{it} e^{-i\theta}|^2} \right] d\rho
\]
\[
= \frac{1}{4}.\]

\(\Box\)

**Lemma 2.4.** Suppose that \(\varphi_k \in C(\mathbb{T})\) and \(G_k[\varphi_k]\) are defined in (1.6), where \(k \in \{1, \ldots, n-1\}\) and \(n \geq 2\). Then, the following statements hold:

1. For \(z \in D\),

\[
\max \left\{ \left| \frac{\partial}{\partial z} G_k[\varphi_k](z) \right|, \left| \frac{\partial}{\partial \bar{z}} G_k[\varphi_k](z) \right| \right\} \leq \nu_k(z),
\]

where
\[ \nu_k(z) = \begin{cases} 
\frac{1}{3} \| \varphi_1 \|_\infty, & \text{if } k = 1, \\
\| \varphi_k \|_\infty \left( \frac{3}{16} \right)^{k-2} \frac{(2 - |z|^2)}{30}, & \text{if } 2 \leq k \leq n - 1.
\end{cases} \]

(2) Both \( \frac{\partial}{\partial z} G_k[\varphi_k] \) and \( \frac{\partial}{\partial z} G_k[\varphi_k] \) have continuous extensions to the boundary, and further, for \( \theta \in [0, 2\pi] \),

\[
\max \left\{ \left| \frac{\partial}{\partial z} G_k[\varphi_k](e^{i\theta}) \right|, \left| \frac{\partial}{\partial z} G_k[\varphi_k](e^{i\theta}) \right| \right\} \leq \nu_k^*(e^{i\theta}),
\]

where

\[
\nu_k^*(e^{i\theta}) = \begin{cases} 
\frac{1}{4} \| \varphi_1 \|_\infty, & \text{if } k = 1, \\
\frac{1}{32} \left( \frac{3}{16} \right)^{k-2} \| \varphi_k \|_\infty, & \text{if } 2 \leq k \leq n - 1.
\end{cases}
\]

**Proof.** In order to prove the first statement of this Lemma, we only need to prove the following inequality

\[
\left| \frac{\partial}{\partial z} G_k[\varphi_k](z) \right| \leq \nu_k(z)
\]

because the proof of the other one is similar, where \( \nu_k \) is defined in the first statement of this Lemma. For this, let

\[
I_2(z) = \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \left| \frac{\partial}{\partial z} G(z, \xi_1) \right| \cdots \left| G(\xi_{k-1}, \xi_k) \right| \times \left| \int_0^{2\pi} P(\xi_k, e^{it}) \varphi_k(e^{it}) dt \right| d\sigma(\xi_k) \cdots d\sigma(\xi_1).
\]

**Case 1.** \( k = 1 \).

Then, by [28, Lemma 2.7], we have

\[
I_2(z) = \int_{\mathbb{D}} \left| \frac{\partial}{\partial z} G(z, \xi_1) \right| \left| \int_0^{2\pi} P(\xi_1, e^{it}) \varphi_1(e^{it}) dt \right| d\sigma(\xi_1) \leq \frac{1}{3} \| \varphi_1 \|_\infty,
\]

which, together with [28, Proposition 2.4] (see also [44]), gives that

\[
\left| \frac{\partial}{\partial z} G_1[\varphi_1](z) \right| = \left| \int_{\mathbb{D}} \frac{\partial}{\partial z} G(z, \xi_1) \left( \int_0^{2\pi} P(\xi_1, e^{it}) \varphi_1(e^{it}) dt \right) d\sigma(\xi_1) \right| \leq I_2(z) \leq \frac{1}{3} \| \varphi_1 \|_\infty.
\]

**Case 2.** \( 2 \leq k = n - 1 \).

Let

\[
A_1 = \int_{\mathbb{D}} \int_{\mathbb{D}} |G(\xi_{k-2}, \xi_{k-1})||G(\xi_k, \xi_k)| \left| \int_0^{2\pi} P(\xi_k, e^{it}) \varphi_k(e^{it}) dt \right| d\sigma(\xi_k)d\sigma(\xi_{k-1}).
\]

It follows from Lemma 2.1 that
\[ A_1 \leq \| \varphi_k \|_\infty \int_{\mathbb{D}} \int_{\mathbb{D}} |G(\xi_{k-2}, \xi_{k-1})| |G(\xi_{k-1}, \xi_k)| d\sigma(\xi_k) d\sigma(\xi_{k-1}) \]
\[ = \frac{\| \varphi_k \|_\infty}{4} \int_{\mathbb{D}} |G(\xi_{k-2}, \xi_{k-1})|(1 - |\xi_{k-1}|^2) d\sigma(\xi_{k-1}) \]
\[ \leq \frac{3(1 - |\xi_{k-2}|^2)}{64} \| \varphi_k \|_\infty, \]
which, together with Lemma 2.2, implies that
\[ I_2(z) \leq \frac{3\| \varphi_k \|_\infty}{64} \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \left| \frac{\partial}{\partial z} G(z, \xi_1) \right| |G(\xi_1, \xi_2)| \cdots |G(\xi_{k-3}, \xi_{k-2})|(1 - |\xi_{k-2}|^2) d\sigma(\xi_{k-2}) \cdots d\sigma(\xi_1) \]
\[ \leq \frac{\| \varphi_k \|_\infty}{4} \left( \frac{3}{16} \right)^{k-2} \int_{\mathbb{D}} (1 - |\xi_1|^2) \left| \frac{\partial}{\partial z} G(z, \xi_1) \right| d\sigma(\xi_1) \]
\[ = \frac{\| \varphi_k \|_\infty}{16\pi} \left( \frac{3}{16} \right)^{k-2} \int_{\mathbb{D}} \frac{(1 - |\xi_1|^2)^2}{|1 - z\xi_1||z - \xi_1|} d\sigma(\xi_1) \]
\[ \leq \| \varphi_k \|_\infty \left( \frac{3}{16} \right)^{k-2} \frac{(2 - |z|^2)^3}{30}. \]

By (2.9) and [28, Proposition 2.4] (see also [44]), we conclude that
\[
\left| \frac{\partial}{\partial z} G_k[\varphi_k](z) \right| = \left| \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \frac{\partial}{\partial z} G(z, \xi_1)G(\xi_1, \xi_2) \cdots G(\xi_{k-1}, \xi_k) \right. \]
\[
\left. \times \left( \int_0^{2\pi} P(\xi_k, e^{it})|\varphi_k(e^{it})| dt \right) d\sigma(\xi_k) \cdots d\sigma(\xi_1) \right| \]
\[ \leq I_2(z) \leq \| \varphi_k \|_\infty \left( \frac{3}{16} \right)^{k-2} \frac{(2 - |z|^2)^3}{30}. \]

Now we prove the second statement of this Lemma. In order to show this statement, we use the Vitali theorem (see [16, Theorem 26.C]) which asserts that if \( \Omega \) is a measurable space with finite measure \( \mu \) and that \( F_n : \Omega \to \mathbb{C} \) is a sequence of functions such that
\[
\lim_{n \to \infty} F_n(x) = F(x) \quad \text{a.e. and} \quad \sup_{n \geq 1} \int_{\Omega} |F_n|^q d\mu < \infty \text{ for some } q > 1,
\]
then
\[
\lim_{n \to \infty} \int_{\Omega} F_n d\mu = \int_{\Omega} F d\mu.
\]
Let
\[ A_2 = \int_{\mathbb{D}} \left( \left| \frac{\partial}{\partial z} G(z, \xi_1) \right|^{\frac{3}{2}} \right) \int_{\mathbb{D}} \left( G(\xi_1, \xi_2) \cdots \int_{\mathbb{D}} \left( G(\xi_{k-1}, \xi_k) \right. \left. \times \left( \int_0^{2\pi} P(\xi_k, e^{it}) d\sigma(\xi_k) \right) \right) \right) \right) d\sigma(\xi_2) \cdots d\sigma(\xi_1), \]

where \( k \in \{1, \ldots, n-1\} \). In order to estimate \( A_2 \), we let

\[ (2.10) \quad \eta_1 = \frac{z - \xi_1}{1 - \overline{\xi_1}} = r_1 e^{i\theta_1}, \]

where \( r_1 \in [0, 1) \) and \( t \in [0, 2\pi] \). Since

\[ \int_0^1 r_1^{2j - \frac{1}{2}} (1 - r_1^2)\frac{3}{2} dr_1 = \frac{3}{(2j + \frac{5}{2})(2j + \frac{3}{2})} \int_0^1 \frac{x^{2j + \frac{3}{2}}}{(1 - x^2)^{\frac{3}{2}}} dx \]

\[ = \frac{3}{(2j + \frac{5}{2})(2j + \frac{3}{2})} \int_0^{\frac{\pi}{2}} \sin^{2j + \frac{1}{2}} t dt \]

\[ \leq \frac{3}{(2j + \frac{5}{2})(2j + \frac{3}{2})} \int_0^{\frac{\pi}{2}} \sin^{2j + 2} t dt \]

\[ = \frac{3\pi}{2(2j + \frac{5}{2})(2j + \frac{3}{2})} \cdot \frac{(2j + 1)!!}{(2j + 2)!!}, \]

we see that

\[ \sum_{j=0}^{\infty} (j + 1)^2 |z|^{2j} \int_0^1 r_1^{2j - \frac{1}{2}} (1 - r_1^2)\frac{3}{2} dr_1 \leq \frac{6\pi}{5} \sum_{j=0}^{\infty} \frac{(2j + 1)!!}{(2j + 2)!!} |z|^{2j} \]

\[ < \frac{6\pi}{5} \left( 1 + \sum_{j=1}^{\infty} \frac{(2j - 1)!!}{(2j)!!} |z|^{2j} \right) \]

\[ = \frac{6\pi}{5} \frac{1}{(1 - |z|^2)^{\frac{1}{2}}}, \]

which, together with (2.10) and Theorem B, yield that

\[ (2.11) \quad \int_{\mathbb{D}} \left| \frac{\partial}{\partial z} G(z, \xi_1) \right|^{\frac{3}{2}} d\sigma(\xi_1) = \frac{1}{2\pi} \int_{\mathbb{D}} \left( \frac{1 - |\eta_1|^2}{|\eta_1|^2 |1 - \overline{\eta_1}|^4} \right) d\sigma(\eta_1) \]

\[ = (1 - |z|^2)^{\frac{1}{2}} \sum_{j=0}^{\infty} (j + 1)^2 |z|^{2j} \]

\[ \times \int_0^1 r_1^{2j - \frac{1}{2}} (1 - r_1^2)\frac{3}{2} dr_1 < \frac{6\pi}{5}. \]

It follows from (2.1) and (2.11) that
\[ A_2 \leq \| \varphi_k \|_{L^\infty} \frac{3}{16} \int_{\partial D} \left( \left| \frac{\partial}{\partial z} G(z, \xi) \right| \right)^{\frac{3}{2}} \times \left( \int_{\partial D} \left( |G(\xi_1, \xi_2)| \cdots \int_{\partial D} |G(\xi_{k-1}, \xi_k)| d\sigma(\xi_k) \right) \cdots d\sigma(\xi_2) \right)^{\frac{3}{2}} d\sigma(\xi_1) \leq \| \varphi_k \|_{L^\infty} \frac{3}{16} \int_{\partial D} \left| \frac{\partial}{\partial z} G(z, \xi_1) \right|^{\frac{3}{2}} d\sigma(\xi_1) \leq \frac{5\| \varphi_k \|_{L^\infty}}{6}\pi \left( \frac{1}{8} \right)^{k-1} < \infty. \]

Therefore, by the Vitali theorem, we conclude that \( \frac{\partial}{\partial z} G_k[\varphi_k] \) has continuous extension to the boundary, and further, by [28, Lemma 2.7],

\[ \left| \frac{\partial}{\partial z} G_1[\varphi_1](e^{i\theta}) \right| \leq \frac{1}{4}\| \varphi_1 \|_{L^\infty}, \]

where \( \theta \in [0, 2\pi] \).

For \( 2 \leq k = n - 1 \), by Lemmas 2.1 and 2.3, we have

\[ \left| \frac{\partial}{\partial z} G_k[\varphi_k](e^{i\theta}) \right| = \left| \int_{\partial D} \cdots \int_{\partial D} \frac{\partial}{\partial z} G(e^{i\theta}, \xi_1)G(\xi_1, \xi_2) \cdots G(\xi_{k-1}, \xi_k) \right. \times \left( \int_0^{2\pi} P(\xi_k, e^{i\theta})\varphi_k(e^{i\theta}) dt \right) d\sigma(\xi_k) \cdots d\sigma(\xi_1) \leq \frac{1}{4} \left( \frac{3}{16} \right)^{k-2} \| \varphi_k \|_{L^\infty} \int_{\partial D} \left| \frac{\partial}{\partial z} G(e^{i\theta}, \xi_1) \right| (1 - |\xi_1|^2) d\sigma(\xi_1) = \frac{1}{32} \left( \frac{3}{16} \right)^{k-2} \| \varphi_k \|_{L^\infty}. \]

Similarly, we can show that \( \frac{\partial}{\partial \xi} G_k[\varphi_k] \) has continuous extension to the boundary, and for \( \theta \in [0, 2\pi] \),

\[ \left| \frac{\partial}{\partial \xi} G_k[\varphi_k](e^{i\theta}) \right| \leq \nu_k^*(e^{i\theta}), \]

where \( k \in \{1, \ldots, n - 1\} \). The proof of this lemma is complete. \( \square \)

**Lemma 2.5.** Suppose \( \varphi_n \in C(\overline{D}) \) and \( G_n[\varphi_n] \) is defined in (1.7). Then, the following statements hold:

(1) For \( z \in D \),

\[ \max \left\{ \left| \frac{\partial}{\partial z} G_n[\varphi_n](z) \right|, \left| \frac{\partial}{\partial \xi} G_n[\varphi_n](z) \right| \right\} \leq \| \varphi_n \|_{L^\infty} \left( \frac{3}{16} \right)^{n-2} \frac{(2 - |z|^2)}{30}. \]

(2) Both \( \frac{\partial}{\partial z} G_n[\varphi_n] \) and \( \frac{\partial}{\partial \xi} G_n[\varphi_n] \) have continuous extensions to the boundary, and further, for \( \theta \in [0, 2\pi] \),
max \left\{ \frac{\partial}{\partial z} G_n[\varphi_n](e^{i\theta}), \frac{\partial}{\partial \bar{z}} G_n[\varphi_n](e^{i\theta}) \right\} \leq \frac{1}{32} \left( \frac{3}{16} \right)^{n-2} \| \varphi_n \|_{\infty}.

Proof. To prove the first statement, we only need to prove the inequality:

\[ \frac{\partial}{\partial z} G_n[\varphi_n](z) \leq \| \varphi_n \|_{\infty} \left( \frac{3}{16} \right)^{n-2} \frac{(2 - |z|^2)}{30}, \quad z \in \mathbb{D}, \]

because the proof to the other one is similar. For this, let

\[ I_3(z) = \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \left| \frac{\partial}{\partial z} G(z, \zeta_1) \right| |G(\zeta_1, \zeta_2)| \cdots |G(\zeta_{n-2}, \zeta_{n-1})| \times \left| \int_{\mathbb{D}} G(\zeta_{n-1}, \zeta_n) \varphi_n(\zeta_n) d\sigma(\zeta_n) \right| d\sigma(\zeta_{n-1}) \cdots d\sigma(\zeta_1). \]

By calculation, we have

\[ (2.12) \quad \left| \int_{\mathbb{D}} G(\zeta_{n-1}, \zeta_n) \varphi_n(\zeta_n) d\sigma(\zeta_n) \right| \leq \| \varphi_n \|_{\infty} \int_{\mathbb{D}} |G(\zeta_{n-1}, \zeta_n)| d\sigma(\zeta_n) \]

\[ = \| \varphi_n \|_{\infty} (1 - |\zeta_{n-1}|^2) \frac{4}{3}. \]

By (2.12), Lemmas 2.1 and 2.2, we see that

\[ I_3(z) \leq \frac{\| \varphi_n \|_{\infty}^2}{4} \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \left| \frac{\partial}{\partial z} G(z, \zeta_1) \right| |G(\zeta_1, \zeta_2)| \cdots |G(\zeta_{n-2}, \zeta_{n-1})| \times (1 - |\zeta_{n-1}|^2) d\sigma(\zeta_{n-1}) \cdots d\sigma(\zeta_1) \]

\[ \leq \frac{\| \varphi_n \|_{\infty}^2}{16\pi} \left( \frac{3}{16} \right)^{n-2} \int_{\mathbb{D}} \frac{(1 - |\zeta_n|^2)^2}{|1 - \zeta_n|\bar{\zeta}_1|z - \zeta_1|} d\sigma(\zeta_1) \]

\[ \leq \| \varphi_n \|_{\infty} \left( \frac{3}{16} \right)^{n-2} \frac{(2 - |z|^2)}{30}, \]

which, together with [28, Proposition 2.4] (see also [44]), yields that

\[ (2.13) \quad \frac{\partial}{\partial z} G_n[\varphi_n](z) = \left| \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \frac{\partial}{\partial z} G(z, \zeta_1) \cdots G(\zeta_{n-2}, \zeta_{n-1}) \right| \times \left| \int_{\mathbb{D}} G(\zeta_{n-1}, \zeta_n) \varphi_n(\zeta_n) d\sigma(\zeta_n) \right| d\sigma(\zeta_{n-1}) \cdots d\sigma(\zeta_1) \]

\[ \leq I_3 \leq \| \varphi_n \|_{\infty} \left( \frac{3}{16} \right)^{n-2} \frac{(2 - |z|^2)}{30}. \]

Next, we prove the second part of this Lemma. Set
\[ A_3 = \int_D \left( \left| \frac{\partial}{\partial z} G(z, \zeta_1) \right| \right)^{\frac{3}{2}} \int_D \left( G(\zeta_1, \zeta_2) \cdots \int_D \left( G(\zeta_{n-2}, \zeta_{n-1}) \times \left( \int_D G(\zeta_{n-1}, \zeta_n) \varphi_n(\zeta_n) d\sigma(\zeta_n) \right) \right) \right) d\sigma(\zeta_{n-1}) \cdots d\sigma(\zeta_2) \right) d\sigma(\zeta_1). \]

Then by (2.11) and Lemma 2.1 (2.1), we get
\[ A_3 \leq \frac{\|\varphi_n\|_{\infty}^{\frac{3}{8}}}{8} \int_D \left( \left| \frac{\partial}{\partial z} G(z, \zeta_1) \right| \right)^{\frac{3}{2}} \int_D \left( G(\zeta_1, \zeta_2) \cdots \int_D \left( G(\zeta_{n-2}, \zeta_{n-1}) \times (1 - |\zeta_n|^2) \right) d\sigma(\zeta_{n-1}) \cdots d\sigma(\zeta_2) \right) d\sigma(\zeta_1) \]
\[ \leq \frac{6\pi}{5} \left( \frac{1}{8} \right)^{n-1} \|\varphi_n\|_{\infty}^{\frac{3}{8}} < \infty. \]

Hence, by the Vitali theorem, we see that $\frac{\partial}{\partial z} G_n[\varphi_n]$ has continuous extension to the boundary, and further, by Lemmas 2.1 and 2.3, we have
\[ \left| \frac{\partial}{\partial z} G_n[\varphi_n](e^{i\theta}) \right| = \left| \int_D \cdots \int_D \frac{\partial}{\partial z} G(e^{i\theta}, \zeta_1)G(\zeta_1, \zeta_2) \cdots G(\zeta_{n-1}, \zeta_n) \times \left( \int_D G(\zeta_{n-1}, \zeta_n) \varphi_n(\zeta_n) d\sigma(\zeta_n) \right) d\sigma(\zeta_{n-1}) \cdots d\sigma(\zeta_1) \right| \]
\[ \leq \frac{1}{4} \left( \frac{3}{16} \right)^{n-2} \|\varphi_n\|_{\infty} \int_D \left| \frac{\partial}{\partial z} G(e^{i\theta}, \zeta_1) \right| (1 - |\zeta_1|^2) d\sigma(\zeta_1) \]
\[ = \frac{1}{32} \left( \frac{3}{16} \right)^{n-2} \|\varphi_n\|_{\infty}. \]

where $\theta \in [0, 2\pi]$.

Similarly, we can prove that $\frac{\partial}{\partial z} G_n[\varphi_n]$ has continuous extension to the boundary, and for $\theta \in [0, 2\pi]$,
\[ \left| \frac{\partial}{\partial z} G_n[\varphi_n](e^{i\theta}) \right| \leq \frac{\|\varphi_n\|_{\infty} \left( \frac{3}{16} \right)^{n-2}}{32}. \]

The proof of this lemma is finished. $\square$

**Lemma 2.6.** For $\varphi_k \in C(\mathbb{T})$ and $\varphi_n \in C(\overline{\mathbb{D}})$, suppose that $f$ is a sense-preserving homeomorphism from $\overline{\mathbb{D}}$ onto itself satisfying (1.3) and the boundary conditions $\Delta^{n-1} f|_{\mathbb{T}} = \varphi_{n-1}$, ..., $\Delta^1 f|_{\mathbb{T}} = \varphi_1$, and suppose that $f$ is Lipschitz continuous.
in \( \mathbb{D} \), where \( n \geq 2 \) and \( k \in \{1,\ldots,n-1\} \). Then, for almost every \( e^{i\theta} \in \mathbb{T} \), the following limits exist:

\[
D_f(e^{i\theta}) := \lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_f(z) \quad \text{and} \quad J_f(e^{i\theta}) := \lim_{z \to e^{i\theta}, z \in \mathbb{D}} J_f(z).
\]

Further, we have

\[
J_f(e^{i\theta}) \leq \frac{\gamma'(\theta)}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt + \frac{\gamma'(\theta)\|\varphi_1\|_{\infty}}{2} + \frac{1}{16} k^{-2}
\]

and

\[
J_f(e^{i\theta}) \geq \frac{\gamma'(\theta)}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt - \frac{\gamma'(\theta)\|\varphi_1\|_{\infty}}{2} - \frac{1}{16} k^{-2},
\]

where \( f(e^{i\theta}) = e^{\gamma(\theta)} \) and \( \gamma(\theta) \) is a real-valued function in \([0,2\pi]\).

Before the proof of Lemma 2.6, let us recall the following result (cf. [28, Lemma 2.1]).

**Theorem C.** Suppose that \( f \) is a harmonic mapping defined in \( \mathbb{D} \) and its formal derivative \( D_f \) is bounded in \( \mathbb{D} \) (or equivalently, according to Rademacher’s theorem, suppose that \( f \) itself is Lipschitz continuous in \( \mathbb{D} \)). Then, there exists a mapping \( A \in L_{\infty}(\mathbb{T}) \) such that for any \( e^{i\theta} \in \mathbb{D} \),

\[
D_f(e^{i\theta}) = P[A](e^{i\theta}).
\]

Moreover, the function \( F(e^{i\theta}) := f(e^{i\theta}) \) is differentiable almost everywhere in \([0,2\pi]\) and

\[
A(e^{i\theta})i e^{i\theta} = \frac{\partial}{\partial \theta} F(e^{i\theta}).
\]

**Proof of Lemma 2.6.** We first prove the existence of the two limits in (2.14). By Lemmas 2.4 and 2.5, we get that for any \( e^{i\theta} \in \mathbb{D} \),

\[
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} G_k[\varphi_k](z) = D_f(e^{i\theta}),
\]

where \( k \in \{1,2,\ldots,n\} \).

Again, by Lemmas 2.4 and 2.5, we know that \( \|D_{G_k[\varphi_k]}\| \) is bounded, which implies the Lipschitz continuity of \( G_k[\varphi_k] \) in \( \mathbb{D} \). Since \( f \) is Lipschitz continuous in \( \mathbb{D} \), we see that \( \|D_f\| \) is bounded in \( \mathbb{D} \). Thus, it follows from (1.5) that

\[
P[\varphi_0] = f - \sum_{k=1}^{n} (-1)^k G_k[\varphi_k]
\]
is also Lipschitz continuous in $\mathbb{D}$, where $\varphi_0 = f|_T$. Now, we conclude from Theorem C that for almost every $e^{i\theta} \in T$,

$$
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_{P[\varphi_0]}(z)
$$

does exist, which, together with (1.5) and (2.17), guarantees that for almost every $\theta \in [0, 2\pi]$,

$$
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_f(z)
$$

also exists.

Since

$$
J_f(z) = \det D_f(z),
$$

obviously, we see that

$$
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} J_f(z)
$$

exists for almost every $\theta \in [0, 2\pi]$.

Next, we demonstrate the estimates in (2.15) and (2.16). For convenience, in the rest of the proof of the lemma, let

$$
D_f(e^{i\theta}) = \lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_f(z) \quad \text{and} \quad J_f(e^{i\theta}) = \lim_{z \to e^{i\theta}, z \in \mathbb{D}} J_f(z).
$$

By Lebesgue Dominated Convergence Theorem, the boundedness of $\|D_f\|$, and by letting $z = re^{i\theta} \in \mathbb{D}$, we see that for any fixed $\theta_0 \in [0, 2\pi]$,

$$(2.18) \quad f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}) = \lim_{r \to 1^-} \int_{\theta_0}^{\theta} \frac{\partial}{\partial t} f(re^{it})dt + f(e^{i\theta_0})$$

$$= \int_{\theta_0}^{\theta} \lim_{r \to 1^-} \left[ ir(f_z(re^{it})e^{it} - f_x(re^{it})e^{-it})\right] dt + f(e^{i\theta_0}),$$

which implies that $f(e^{i\theta})$ is absolutely continuous. Let $\gamma(\theta)$ be a real-valued function in $[0, 2\pi]$ such that

$$e^{i\gamma(\theta)} = f(e^{i\theta}).$$

Then,

$$(2.19) \quad f'(e^{i\theta}) = i\gamma'(\theta)e^{i\gamma(\theta)}$$

holds almost everywhere in $[0, 2\pi]$.

Since

$$J_f(re^{i\theta}) = |f_z(re^{i\theta})|^2 - |f_x(re^{i\theta})|^2 = -\Re \left( \frac{\partial f}{\partial r} i \frac{\partial f}{\partial \theta} \right),$$

we infer from (2.19) that

$$(2.20) \quad J_f(e^{i\theta}) = \lim_{r \to 1^-} J_f(re^{i\theta}) = -\lim_{r \to 1^-} \Re \left( \frac{\partial f}{\partial r} i \frac{\partial f}{\partial \theta} \right) = \chi_0 - \sum_{k=1}^{n} (-1)^k \chi_k,$$

where

$$\chi_0 = \lim_{r \to 1^-} \Re \left( \frac{f(e^{i\theta}) - P[\varphi_0](re^{i\theta})}{1 - r} \cdot \gamma'(\theta) f(e^{i\theta}) \right).$$
and

$$\chi_k = \lim_{r \to 1^-} \text{Re} \left( \frac{G_k[\varphi_k](re^{i\theta})}{1 - r} \cdot \gamma'(\theta)f(e^{i\theta}) \right).$$

Now, we are going to prove (2.15) and (2.16) by estimating the quantities $|\chi_0|$ and $|\chi_k|$, respectively, where $k \in \{1, 2, \ldots, n\}$. We start with the estimate of $|\chi_0|$. Since

$$\text{Re}(f(e^{i\theta}), f(e^{i\theta}) - f(e^{i\theta})) = \text{Re}[f(e^{i\theta})(f(e^{i\theta}) - f(e^{i\theta}))] = \frac{1}{2}|f(e^{i\theta}) - f(e^{i\theta})|^2$$

and

$$\chi_0 = \lim_{r \to 1^-} \text{Re} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + r}{1 - r e^{i(\theta - t)}} \langle \gamma'(\theta)f(e^{i\theta}), f(e^{i\theta}) - f(e^{i\theta}) \rangle \, dt \right),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product, it follows that

$$\chi_0 = \gamma'(\theta) \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}) - f(e^{i\theta})|^2 \, dt.$$

Next, we estimate $|\chi_k|$ for $k \in \{1, 2, \ldots, n\}$. Since

$$\lim_{z \to e^{i\theta}, z \in \mathbb{D}} \frac{G(z, \xi_1)}{1 - |z|} = \lim_{z \to e^{i\theta}, z \in \mathbb{D}} \frac{G(z, \xi_1) - G(e^{i\theta}, \xi_1)}{1 - |z|} = P(\xi_1, e^{i\theta}),$$

we deduce that

$$\lim_{z \to e^{i\theta}, z \in \mathbb{D}} \frac{G_k[\varphi_k](z)}{1 - |z|} = \lim_{z \to e^{i\theta}, z \in \mathbb{D}} \left[ P(\xi_1, e^{i\theta}) \frac{G(z, \xi_1)}{1 - |z|} \cdots \frac{G(z, \xi_k)}{1 - |z|} \right] \times \left[ \int_0^{2\pi} P(\xi_k, e^{i\theta}) \varphi_k(e^{i\theta}) \, dt \right] \, d\sigma(\xi_k) \cdots d\sigma(\xi_1)

= \int_\mathbb{D} \cdots \int_\mathbb{D} P(\xi_1, e^{i\theta}) \frac{G(\xi_1, \xi_2)}{1 - |z|} \cdots \frac{G(\xi_k, \xi_{k-1})}{1 - |z|} \times \left[ \int_0^{2\pi} P(\xi_k, e^{i\theta}) \varphi_k(e^{i\theta}) \, dt \right] \, d\sigma(\xi_k) \cdots d\sigma(\xi_1),$$

where $k \in \{1, 2, \ldots, n - 1\}$.

**Case 1.** $2 \leq k \leq n - 1$.

It follows from (2.23), Lemmas 2.1 and 2.3 that

$$|\chi_k| \leq \gamma'(\theta) \|\varphi_k\| \int_\mathbb{D} \cdots \int_\mathbb{D} P(\xi_1, e^{i\theta}) |G(\xi_1, \xi_2)| \cdots |G(\xi_{k-1}, \xi_k)| \, d\sigma(\xi_k) \cdots d\sigma(\xi_1)

\leq \frac{\gamma'(\theta) \|\varphi_k\|}{4} \left( \frac{3}{16} \right)^{k-2} \int_\mathbb{D} P(\xi_1, e^{i\theta}) (1 - |\xi_1|^2) \, d\sigma(\xi_1)

= \frac{\gamma'(\theta) \|\varphi_k\|}{16} \left( \frac{3}{16} \right)^{k-2}.$$

**Case 2.** $k = 1$. 


By (2.23), we get

\[(2.25) \quad |\chi_1| \leq \gamma'(\theta)\|\varphi_1\|_{\infty} \int_{\mathbb{D}} P(\xi_1, e^{i\theta}) d\sigma(\xi_1) = \frac{\gamma'(\theta)\|\varphi_1\|_{\infty}}{2}.
\]

At last, we estimate \(|\chi_n|\). By (2.22), Lemmas 2.1 and 2.3, we obtain

\[(2.26) \quad |\chi_n| \leq \gamma'(\theta) \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} P(\zeta_1, e^{i\theta}) |G(\zeta_1, \zeta_2)| \cdots |G(\zeta_{n-2}, \zeta_{n-1})| \times \left| \int_{\mathbb{D}} G(\zeta_{n-1}, \zeta_n) \varphi_n(\zeta_n) d\sigma(\zeta_n) \right| d\sigma(\zeta_{n-1}) \cdots d\sigma(\zeta_1)
\]

\[\leq \gamma'(\theta)\|\varphi_n\|_{\infty} \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} P(\zeta_1, e^{i\theta}) |G(\zeta_1, \zeta_2)| \cdots |G(\zeta_{n-1}, \zeta_n)| d\sigma(\zeta_n) \cdots d\sigma(\zeta_1)
\]

\[\leq \frac{\gamma'(\theta)\|\varphi_n\|_{\infty}}{4} \left( \frac{3}{16} \right)^{n-2} \int_{\mathbb{D}} P(\zeta_1, e^{i\theta})(1 - |\zeta_1|^2) d\sigma(\zeta_1)
\]

\[= \frac{\gamma'(\theta)\|\varphi_n\|_{\infty}}{16} \left( \frac{3}{16} \right)^{n-2}.
\]

Hence, (2.15) and (2.16) follow from the inequalities (2.21), (2.24) and (2.26) along with the following chain of inequalities:

\[|\chi_0| - \sum_{k=1}^{n} |\chi_k| \leq J_f(e^{i\theta}) \leq \sum_{k=0}^{n} |\chi_k|.
\]

The proof of the lemma is complete. \(\square\)

**Theorem D.** ([25, Theorem 3.4]) Suppose that \(f\) is a quasiconformal \(C^2\) diffeomorphism from the plane domain \(\Omega\) with \(C^{1,\alpha}\) compact boundary onto the plane domain \(\Omega^*\) with \(C^{2,\alpha}\) compact boundary. If there exist constants \(a_1\) and \(b_1\) such that

\[|\Delta f(z)| \leq a_1\|Df(z)\|^2 + b_1
\]

in \(\Omega\), then \(f\) has bounded partial derivatives. In particular, it is a Lipschitz mapping in \(\Omega\).

**Theorem E.** ([41, Theorem 2.2]) Given \(K \geq 1\), let \(f\) be a \(K\)-quasiconformal and harmonic self-mapping of \(\mathbb{D}\) satisfying \(f(0) = 0\). Then, for \(z \in \mathbb{D}\),

\[|f_z(z)| \geq \frac{K + 1}{2K} \max \left\{ \frac{2}{\pi}, L_K \right\},
\]

where \(L_K\) is defined in [41, Lemma 1.4] and \(\lim_{K \to 1} L_K = 1\).
3. THE PROOF OF THEOREM 1.4

The purpose of this section is to prove Theorem 1.4. The proof consists of three steps. In the first step, the Lipschitz continuity of the mappings \( f \) is proved, the co-Lipschitz continuity of \( f \) is demonstrated in the second step, and in the third step, the Lipschitz and co-Lipschitz continuity coefficients obtained in the first two steps are shown to have bounds with the forms as required in Theorem 1.4.

**Step 3.1.** The asymptotically sharp Lipschitz inequality of \( f \).

We start the discussions of this step with the following claim.

**Claim 3.1.** The limits

\[
\lim_{z \to \xi \in T, z \in D} D_f(z) \quad \text{and} \quad \lim_{z \to \xi \in T, z \in D} J_f(z)
\]

exist almost everywhere in \( T \).

We are going to verify the existence of these two limits by applying Theorem D and Lemma 2.6. For this, we need to get an upper bound of \( |\Delta f| \) as stated in (3.1) and (3.2) below, and we will divide it into two cases to estimate.

**Case 1.** \( n = 2 \).

By the formula (1.3) in [28] (see also [20, pp. 118-120]), we have that for \( z \in D \),

\[
\Delta f(z) = P[\varphi_1](z) - \int_D G(z, \zeta) \varphi_2(\zeta) d\sigma(\zeta).
\]

It follows from Lemma 2.1 (2.1) that

\[
|\Delta f(z)| \leq |P[\varphi_1](z)| + \|\varphi_2\|_\infty \int_D |G(z, \zeta)| d\sigma(\zeta) \leq \|\varphi_1\|_\infty + \frac{\|\varphi_2\|_\infty}{4}.
\]

**Case 2.** \( n \geq 3 \).

Since

\[
\Delta^{n-1}(\Delta f) = \varphi_n \quad \text{in} \quad D,
\]

and

\[
\Delta^{n-2}(\Delta f)|_T = \varphi_{n-1}, \ldots, \Delta f|_T = \varphi_1,
\]

by (1.5), we see that, for \( z \in D \),

\[
\Delta f(z) = P[\varphi_1](z) + \sum_{j=1}^{n-1} (-1)^j G_j[\varphi_{j+1}](z),
\]

where

\[
G_k[\varphi_{k+1}](z) = \int_D \cdots \int_D G(z, \xi_1) \cdots G(\xi_{k-1}, \xi_k) \times \left( \int_0^{2\pi} P(\xi_k, e^{it}) \varphi_{k+1}(e^{it}) dt \right) d\sigma(\xi_k) \cdots d\sigma(\xi_1)
\]

for \( k \in \{1, \ldots, n-2\} \), and
\[ G_{n-1}[\varphi_n](z) = \int_D \cdots \int_D G(z, \zeta_1) \cdots G(\zeta_{n-3}, \zeta_{n-2}) \]
\[ \times \left( \int_D G(\zeta_{n-2}, \zeta_{n-1}) \varphi_n(\zeta_{n-1}) d\sigma(\zeta_{n-1}) \right) d\sigma(\zeta_{n-2}) \cdots d\sigma(\zeta_1). \]

By Lemmas 2.1 and , for \( z \in \mathbb{D} \), we obtain that
\[
|G_k[\varphi_{k+1}](z)| \leq \|\varphi_{k+1}\|_{\infty} \int_D \cdots \int_D |G(z, \xi_1)| \cdots |G(\xi_{k-1}, \xi_k)| d\sigma(\xi_k) \cdots d\sigma(\xi_1)
\]
\[
\leq \|\varphi_{k+1}\|_{\infty} \left( \frac{3}{16} \right)^{k-1} (1 - |z|^2)
\]
for \( k \in \{1, \ldots, n - 2\} \), and
\[
|G_{n-1}[\varphi_n](z)| = \|\varphi_n\|_{\infty} \int_D \cdots \int_D |G(z, \zeta_1)| \cdots |G(\zeta_{n-3}, \zeta_{n-2})| d\sigma(\zeta_{n-2}) \cdots d\sigma(\zeta_1)
\]
\[
\leq \|\varphi_n\|_{\infty} \left( \frac{3}{16} \right)^{n-2} (1 - |z|^2)
\]
which give that
\[
|\Delta f(z)| = |P[\varphi_1](z)| + \sum_{j=1}^{n-1} |G_j[\varphi_{j+1}](z)|
\]
\[
\leq \|\varphi_1\|_{\infty} + \sum_{j=1}^{n-1} \|\varphi_{j+1}\|_{\infty} \left( \frac{3}{16} \right)^{j-1} < \infty.
\]

Since \( f \) is a \( K \)-quasiconformal self-mapping of \( \mathbb{D} \), we see that \( f \) can be extended to the homeomorphism of \( \overline{\mathbb{D}} \) onto itself. Now, the existence of the limits
\[ D_f(\xi) = \lim_{z \to \xi \in \mathbb{T}, z \in \mathbb{D}} D_f(z) \]
and \( J_f(\xi) = \lim_{z \to \xi \in \mathbb{T}, z \in \mathbb{D}} J_f(z) \)
almost everywhere in \( \mathbb{T} \) follows from (3.2), Theorem D and Lemma 2.6.

For convenience, in the following, let
\[ C_2(K, \varphi_1, \ldots, \varphi_n) = \sup_{z \in \mathbb{D}} \|D_f(z)\|. \]

Since for almost all \( z_1 \) and \( z_2 \in \mathbb{D} \),
\[
|f(z_1) - f(z_2)| = \left| \int_{[z_1, z_2]} f_z dz + f_\xi d\xi \right| \leq C_2(K, \varphi_1, \ldots, \varphi_n)|z_1 - z_2|,
\]
we see that, to prove the Lipschitz continuity of $f$ and investigate the behavior of the Lipschitz coefficient, it suffices to estimate the quantity $C_2(K, \varphi_1, \cdots, \varphi_n)$. To reach this goal, we first show that the quantity $C_2(K, \varphi_1, \cdots, \varphi_n)$ satisfies an inequality which is stated in the following claim.

**Claim 3.2.** $C_2(K, \varphi_1, \cdots, \varphi_n) \leq \left( C_2(K, \varphi_1, \cdots, \varphi_n) \right)^{1-\frac{1}{K}} \mu_1 + \mu_2$, where

$$
\mu_1 = \frac{K(Q(K))^{\frac{1}{K}+1}}{2\pi} \int_0^{2\pi} |1 - e^{it}|^{-1+\frac{1}{K} \pi^2} dt,
$$

$Q(K)$ is from Theorem A, $\mu_2 = \mu_3 + \mu_4$,

$$
\mu_3 = \frac{K\|\varphi_1\|_{\infty}}{2} + K \sum_{k=2}^{n} \frac{\|\varphi_k\|_{\infty}}{16} \left( \frac{3}{16} \right)^{k-2},
$$

and

$$
\mu_4 = \frac{7}{6} \|\varphi_1\|_{\infty} + \sum_{k=2}^{n} \frac{47\|\varphi_k\|_{\infty}}{240} \left( \frac{3}{16} \right)^{k-2}.
$$

To prove the claim, we need the following preparation. Firstly, we prove that for almost every $\theta \in [0, 2\pi]$,

$$
\|D_f(e^{i\theta})\| \leq \frac{K}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt + \mu_3.
$$

(3.4)

For $\theta \in [0, 2\pi]$, let $\varphi_0(e^{i\theta}) = f(e^{i\theta}) = e^{i\gamma(\theta)}$.

Then, by (2.18), we see that $f(e^{i\theta})$ is absolutely continuous. It follows that

$$
\gamma'(\theta)e^{i\gamma(\theta)} = \frac{d}{d\theta} f(e^{i\theta}) = \lim_{r \to 1^-} \frac{\partial}{\partial \theta} f(re^{i\theta}) = \lim_{r \to 1^-} \left[ i r \left( f_z(re^{i\theta})e^{i\theta} - f_{\overline{z}}(re^{i\theta})e^{-i\theta} \right) \right],
$$

which implies

$$
\frac{1}{K} \|D_f(e^{i\theta})\| \leq \lim_{r \to 1^-} \lambda(D_f(re^{i\theta})) \leq \gamma'(\theta) \leq \lim_{r \to 1^-} \|D_f(re^{i\theta})\| = \|D_f(e^{i\theta})\|
$$

almost everywhere in $[0, 2\pi]$, where $r \in [0, 1]$.

Since the existence of the two limits

$$
D_f(e^{i\theta}) = \lim_{z \to e^{i\theta}} D_f(z) \text{ and } J_f(e^{i\theta}) = \lim_{z \to e^{i\theta}} J_f(z)
$$

almost everywhere in $[0, 2\pi]$ guarantees that

$$
\|D_f(e^{i\theta})\|^2 \leq KJ_f(e^{i\theta}),
$$

we deduce from (2.15) and (3.5) that

$$
\|D_f(e^{i\theta})\|^2 \leq K\|D_f(e^{i\theta})\| \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt + \frac{\mu_3}{K} \right\},
$$

from which the inequality (3.4) follows.

Secondly, we show that for any $\epsilon > 0$, there exists $\theta_{\epsilon} \in [0, 2\pi]$ such that

$$
C_2(K, \varphi_1, \cdots, \varphi_n) \leq (1 + \epsilon)\|D_f(e^{i\theta_{\epsilon}})\| + \mu_4.
$$

(3.6)
For the proof, let $t \in [0, 2\pi]$, and let

$$H_t(z) = \frac{\partial}{\partial z} P[\varphi_0](z) + e^{it} \frac{\partial}{\partial \zeta} P[\varphi_0](z)$$

in $\mathbb{D}$.

Since $P[\varphi_0] = f - \sum_{k=1}^{n} (-1)^k G_k[\varphi_k]$ is harmonic, we see that $H_t$ is analytic in $\mathbb{D}$, and thus,

$$|H_t(z)| \leq \text{esssup}_{\theta \in [0, 2\pi]} |H_t(e^{i\theta})| \leq \text{esssup}_{\theta \in [0, 2\pi]} \|D_P[\varphi_0](e^{i\theta})\|.$$  

Then, the facts

$$\|D_P[\varphi_0](z)\| = \max_{t \in [0, 2\pi]} |H_t(z)| \leq \text{esssup}_{\theta \in [0, 2\pi]} \|D_P[\varphi_0](e^{i\theta})\|$$

and

$$\|D_P[\varphi_0](z)\| = \left| \frac{\partial f}{\partial z} - \sum_{k=1}^{n} (-1)^k \frac{\partial}{\partial z} G_k[\varphi_k] \right| + \left| \frac{\partial f}{\partial \zeta} - \sum_{k=1}^{n} (-1)^k \frac{\partial}{\partial \zeta} G_k[\varphi_k] \right|$$

ensure

$$\|D_P[\varphi_0](z)\| \leq \text{esssup}_{\theta \in [0, 2\pi]} \|D_f(e^{i\theta})\| + \sum_{k=1}^{n} \text{esssup}_{\theta \in [0, 2\pi]} \|D_{G_k[\varphi_k]}(e^{i\theta})\|,$$

which, together with Lemmas 2.4 and 2.5, guarantees that for all $z \in \mathbb{D},$

$$\|D_f(z)\| \leq \|D_P[\varphi_0](z)\| + \sum_{k=1}^{n} \|D_{G_k[\varphi_k]}(z)\| \leq \text{esssup}_{\theta \in [0, 2\pi]} \|D_f(e^{i\theta})\| + \mu_4,$$

from which the inequality (3.6) follows.

Let

$$\nu = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta_1})|^2}{|e^{it} - e^{i\theta_1}|^2} dt.$$  

Finally, we need the following estimate of $\nu_2$:

$$\nu_2 \leq \frac{(C_2(K, \varphi_1, \cdots, \varphi_n))^{1+\frac{1}{\kappa}} (Q(K))^{\frac{1}{\kappa}+1}}{2\pi} \int_{0}^{2\pi} |e^{it} - e^{i\theta_1}|^{-1 + \frac{1}{\kappa}} dt.$$  

Since it follows from (3.3) that for almost all $\theta_1, \theta_2 \in [0, 2\pi],$

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq C_2(K, \varphi_1, \cdots, \varphi_n) |e^{i\theta_1} - e^{i\theta_2}|,$$

we infer that

$$\nu \leq \frac{(C_2(K, \varphi_1, \cdots, \varphi_n))^{1+\frac{1}{\kappa}}}{2\pi} \int_{0}^{2\pi} |e^{it} - e^{i\theta_1}|^{-1 + \frac{1}{\kappa}} \frac{|f(e^{it}) - f(e^{i\theta_1})|^{1+\frac{1}{\kappa}}}{|e^{it} - e^{i\theta_1}|^{\frac{1}{\kappa}} dt},$$

from which, together with Theorem A, the inequality (3.7) follows.

Now, we are ready to finish the proof of the claim. It follows from (3.6) that

$$C_2(K, \varphi_1, \cdots, \varphi_n) \leq (1 + \epsilon)\|D_f(e^{i\theta_1})\| + \mu_4,$$
and so, (3.4) and (3.7) give

\[(3.9) \quad C_2(K, \varphi_1, \cdots, \varphi_n) \leq \left( C_2(K, \varphi_1, \cdots, \varphi_n) \right)^{1-\frac{1}{K}} \mu_1(1 + \epsilon) + \mu_3(1 + \epsilon) + \mu_4. \]

Moreover, by [21, Lemma 1.6], we know that

\[
\int_0^{2\pi} |e^{it} - e^{i\theta}|^{-1 + \frac{1}{K}} dt < \infty,
\]

which shows \( \mu_1 < \infty \).

By letting \( \epsilon \to 0^+ \), we get from (3.9) that

\[
C_2(K, \varphi_1, \cdots, \varphi_n) \leq \left( C_2(K, \varphi_1, \cdots, \varphi_n) \right)^{1-\frac{1}{K}} \mu_1 + \mu_2,
\]

as required.

The following is a lower bound for \( C_2(K, \varphi_1, \cdots, \varphi_n) \).

**Claim 3.3.** \( C_2(K, \varphi_1, \cdots, \varphi_n) \geq 1 \).

Since

\[
\int_0^{2\pi} \gamma'(\theta) d\theta = \gamma(2\pi) - \gamma(0) = 2\pi,
\]

we conclude that

\[
\text{esssup}_{\theta \in [0, 2\pi]} \lim_{t \to \theta} \left| f(e^{i\theta}) - f(e^{it}) \right| = \text{esssup}_{\theta \in [0, 2\pi]} \gamma'(\theta) \geq 1.
\]

Then, it follows from (3.8) and the following fact

\[
\text{esssup}_{\theta \in [0, 2\pi]} \lim_{t \to \theta} \left| f(e^{i\theta}) - f(e^{it}) \right| \leq \text{esssup}_{0 \leq \theta \neq t \leq 2\pi} \left| f(e^{i\theta}) - f(e^{it}) \right| \frac{e^{i\theta} - e^{it}}{e^{i\theta} - e^{it}}
\]

that

\[
C_2(K, \varphi_1, \cdots, \varphi_n) \geq 1.
\]

Hence, the claim is true.

An upper bound of \( C_2(K, \varphi_1, \cdots, \varphi_n) \) is established in the following claim.

**Claim 3.4.** If \( \frac{(K-1)}{K} \mu_1 < 1 \), then

\[
C_2(K, \varphi_1, \cdots, \varphi_n) \leq \mu_5,
\]

where \( \mu_5 = \frac{\mu_1 + \mu_2}{K (1 - \mu_1 (1 - \frac{1}{K})))} \).

The proof of this claim easily follows from [28, Lemma 2.9].

Now, we are ready to finish the discussions in this step. By Claims 3.2 and 3.3, we obtain

\[
1 \leq C_2(K, \varphi_1, \cdots, \varphi_n) \leq \mu_6,
\]

where \( \mu_6 = (\mu_1 + \mu_2)^K \).

By letting

\[
C_3 = \begin{cases} 
\mu_6, & \text{if } \frac{(K-1)}{K} \mu_1 \geq 1, \\
\min\{\mu_5, \mu_6\}, & \text{if } \frac{(K-1)}{K} \mu_1 < 1,
\end{cases}
\]

we have

\[
C_3 \leq \mu_6.
\]
we infer that

\[1 < C_2(K, \varphi_1, \ldots, \varphi_n) \leq C_3.\]

Then, the Lipschitz continuity of \( f \) follows from these estimates of \( C_2(K, \varphi_1, \ldots, \varphi_n) \).

**Step 3.2.** *The asymptotically sharp co-Lipschitz inequality of \( f \).*

We begin the discussions of this step with some preparation which consists of the following two claims.

**Claim 3.5.** \( \lambda(D_{P_0}(e^{i\theta})) \geq \frac{\mu_7}{K} - \frac{1}{K^2} \mu_8 \) almost everywhere on \( \theta \in [0, 2\pi] \), where

\[
\mu_7 = \max\{\mu_7', \mu_7''\}, \quad \mu_7' = (Q(K))^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{it} - e^{i\theta}|^{2K-2} |2K|^{-2} dt,
\]

\[
\mu_7'' = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{\|\varphi_k\|_\infty}{8} \left(\frac{3}{16}\right)^{k-1},
\]

and

\[
\mu_8 = \frac{\|\varphi_1\|_\infty}{2} + \sum_{k=2}^{\infty} \frac{\|\varphi_k\|_\infty}{16} \left(\frac{3}{16}\right)^{k-2}.
\]

By (3.5), we have

\[
\frac{\gamma'(\theta)}{K} \leq \frac{\|D_f(e^{i\theta})\|}{K} \leq \lambda(D_f(e^{i\theta})) \leq \lambda(D_{P_0}(e^{i\theta})) + \sum_{k=1}^{n} \|D_{G_k}(e^{i\theta})\|,
\]

which, together with Lemmas 2.4 and 2.5, implies

\[
\lambda(D_{P_0}(e^{i\theta})) \geq \frac{\gamma'(\theta)}{K} - \frac{1}{K} \sum_{k=1}^{n} \|D_{G_k}(e^{i\theta})\| \geq \frac{\gamma'(\theta)}{K} - \mu_8.
\]

Then, we know from (3.13) that, to prove the claim, it suffices to show that

\[
K\gamma'(\theta) \geq \mu_7.
\]

Again, it follows from (3.5) that

\[
\frac{J_f(e^{i\theta})}{\gamma'(\theta)} \leq \frac{J_f(e^{i\theta})}{\lambda(D_f(e^{i\theta}))} \leq K \lambda(D_f(e^{i\theta})) \leq K\gamma'(\theta),
\]

and thus, (2.16) gives

\[
K\gamma'(\theta) \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt - \mu_8.
\]

This implies that, to prove (3.14), we only need to verify the validity of the following inequality:

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt \geq \mu_7.
\]
We now prove this inequality. On the one hand, since \( f^{-1} \) is a \( K \)-quasiconformal mapping, it follows from Theorem A that for any \( z_1, z_2 \in \mathbb{D} \),
\[
(Q(K))^{-K} |z_1 - z_2|^K \leq |f(z_1) - f(z_2)|,
\]
which implies
\[
(3.16) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt \geq \mu'_T.
\]

On the other hand, since \( f(0) = 0 \), we see from
\[
|G_k[\varphi_k](0)| = \|\varphi_k\|\infty \int_\mathbb{D} \cdots \int_\mathbb{D} |G(0, \zeta_1)| \cdots |G(\xi_{k-1}, \xi_k)| d\sigma(\xi_k) \cdots d\sigma(\xi_1)
\leq \frac{\|\varphi_k\|\infty}{4} \left( \frac{3}{16} \right)^{k-1}
\]
and
\[
|G_n[\varphi_n](0)| = \frac{\|\varphi_n\|\infty}{4} \int_\mathbb{D} \cdots \int_\mathbb{D} |G(0, \zeta_1)| \cdots |G(\xi_{n-2}, \zeta_{n-1})| \times (1 - |\zeta_{n-1}|^2) d\sigma(\zeta_{n-1}) \cdots d\sigma(\xi_1)
\leq \frac{\|\varphi_n\|\infty}{4} \left( \frac{3}{16} \right)^{n-1}
\]
that
\[
(3.17) \quad |P[\varphi_0](0)| \leq \sum_{k=1}^n |G_k[\varphi_k](0)| \leq \sum_{k=1}^n \frac{\|\varphi_k\|\infty}{4} \left( \frac{3}{16} \right)^{k-1}.
\]

Then, we infer from (3.17) and the following fact:
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt \geq \frac{1}{4\pi} \int_0^{2\pi} \left[ 1 - \text{Re}(f(e^{it})\overline{f(e^{i\theta})}) \right] dt \geq \frac{1 - |P[\varphi_0](0)|}{2}
\]
that
\[
(3.18) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt \geq \mu''_T.
\]

Obviously, the inequality (3.15) follows from (3.16) and (3.18), and so, the claim is proved.

**Claim 3.6.** For \( z \in \mathbb{D} \), \( \lambda(D_{P[\varphi_0]}(z)) \geq \frac{\mu''_T}{K^2} - \left( 1 + \frac{1}{K^2} \right) \mu_8. \)

By the Choquet-Radó-Kneser theorem (see [8]), we see that \( P[\varphi_0] \) is a sense-preserving harmonic diffeomorphism of \( \mathbb{D} \) onto itself. It follows from Lewy’s theorem (cf.[32]) and [19, Inequality (17)] that

\[
(3.19) \quad \inf_{z \in \mathbb{D}} \left| \frac{\partial}{\partial z} P[\varphi_0](z) \right| > 0.
\]
Hence, for $z \in \mathbb{D}$, we can let

$$p_1(z) = \frac{\partial}{\partial z} P[\varphi_0](z) \quad \text{and} \quad p_2(z) = \left( \frac{\mu_7}{K^2} - \mu_8 \right) \frac{1}{\partial P[\varphi_0](z)},$$

and let

$$q_\vartheta(z) = p_1(z) + e^{i\vartheta} p_2(z),$$

where $\vartheta \in [0, 2\pi]$. Since $P[\varphi_0]$ is a sense-preserving harmonic diffeomorphism of $\mathbb{D}$, by (3.19), we see that

$$\sup_{z \in \mathbb{D}} |q_\vartheta(z)| < +\infty.$$  

By Claim 3.5, we have

$$|q_\vartheta(e^{i\vartheta})| \leq |p_1(e^{i\vartheta})| + |p_2(e^{i\vartheta})| = \left| \frac{\partial}{\partial z} P[\varphi_0](e^{i\vartheta}) \right| + \frac{\mu_7}{K^2} - \left( 1 + \frac{1}{K^2} \right) \mu_8 \left| \frac{\partial}{\partial z} P[\varphi_0](e^{i\vartheta}) \right| \leq 1$$

almost everywhere on $\vartheta \in [0, 2\pi]$.

Let

$$E = \{ \vartheta \in [0, 2\pi] : \lim_{z \to e^{i\vartheta}} q_\vartheta(z) \text{ exists} \}.$$

Then the measure of the set $[0, 2\pi] \setminus E$ is zero. Hence, for $r \in [0, 1)$, we have

$$|q_\vartheta(rz)| \leq \int_0^{2\pi} P(z, e^{i\vartheta})|q_\vartheta(re^{i\vartheta})|d\vartheta$$

$$\leq \int_E P(z, e^{i\vartheta})|q_\vartheta(re^{i\vartheta})|d\vartheta + \int_{[0, 2\pi] \setminus E} P(z, e^{i\vartheta})|q_\vartheta(re^{i\vartheta})|d\vartheta$$

which, together with (3.20), (3.21) and the Lebesgue Dominated Convergence Theorem, implies

$$|q_\vartheta(z)| \leq \int_E P(z, e^{i\vartheta})|q_\vartheta(e^{i\vartheta})|d\vartheta \leq 1,$$

where $z \in \mathbb{D}$. It follows from (3.22) and the arbitrariness of $\vartheta \in [0, 2\pi]$ that, for $z \in \mathbb{D}$,

$$|p_1(z)| + |p_2(z)| \leq 1,$$

from which the claim follows.

Now, we are ready to finish the proof of the co-Lipschitz continuity of $f$. Since

$$\lambda(Df(z)) \geq \lambda(DP[\varphi_0](z)) - \sum_{k=1}^{n} \|DG_{\varphi_k}(z)\|,$$

we see from Claim 3.6, Lemmas 2.4 and 2.5 that

$$\lambda(Df(z)) \geq C_1(K, \varphi_1, \ldots, \varphi_n),$$
where

\[
C_1(K, \varphi_1, \ldots, \varphi_n) = \frac{\mu_7}{K^2} - \left(1 + \frac{1}{K^2}\right)\mu_8 - \frac{2}{3}\|\varphi_1\|_\infty - \sum_{k=2}^n \frac{2\|\varphi_k\|_\infty}{15} \left(\frac{3}{16}\right)^{k-2}.
\]

And, we know from (3.11) and (3.12) that $C_1(K, \varphi_1, \ldots, \varphi_n) > 0$ for small enough $\|\varphi_k\|_\infty$, where $k \in \{1, 2, \ldots, n\}$. Since for all $z_1, z_2 \in \mathbb{D}$,

\[
|f(z_1) - f(z_2)| \geq \int_{[z_1, z_2]} \lambda(D_f(z))|dz| \geq C_1(K, \varphi_1, \ldots, \varphi_n)|z_1 - z_2|,
\]

we conclude that $f$ is co-Lipschitz continuous.

**Step 3.3. Bounds of the Lipschitz continuity coefficients $C_1(K, \varphi_1, \ldots, \varphi_n)$ and $C_2(K, \varphi_1, \ldots, \varphi_n)$.**

The discussions of this step consists of the following two claims.

**Claim 3.7.** There are constants $M_2(K)$ and $N_2(K, \varphi_1, \ldots, \varphi_n)$ such that

1. $C_2(K, \varphi_1, \ldots, \varphi_n) \leq M_2(K) + N_2(K, \varphi_1, \ldots, \varphi_n)$;
2. $\lim_{K \to 1} M_2(K) = 1$, and
3. $\lim_{\|\varphi_1\|_\infty \to 0, \ldots, \|\varphi_n\|_\infty \to 0} N_2(K, \varphi_1, \ldots, \varphi_n) = 0$.

From (3.10), we see that

\[
1 \leq C_2(K, \varphi_1, \ldots, \varphi_n) \leq C_3,
\]

where

\[
C_3 = \begin{cases} 
(\mu_1 + \mu_2)^K, & \text{if } \frac{(K-1)}{K} \mu_1 \geq 1, \\
\min \left\{ (\mu_1 + \mu_2)^K, \frac{\mu_1 + \mu_2}{1 - \mu_1 \left(1 - \frac{1}{K}\right)} \right\}, & \text{if } \frac{(K-1)}{K} \mu_1 < 1.
\end{cases}
\]

Then, we have

\[
C_3 = \begin{cases} 
M_1^*, & \text{if } \frac{(K-1)}{K} \mu_1 \geq 1, \\
\min \{ M_1^*, M_2^* \}, & \text{if } \frac{(K-1)}{K} \mu_1 < 1.
\end{cases}
\]

where $M_1^* = M_2'(K) + N_2'(K, \varphi_1, \ldots, \varphi_n)$, $M_2^* = M_2''(K) + N_2''(K, \varphi_1, \ldots, \varphi_n)$, $M_2(K) = \mu_1^K$, $M_2'(K) = \frac{\mu_2}{K-\mu_1(K-1)}$, $N_2(K, \varphi_1, \ldots, \varphi_n) = (\mu_1 + \mu_2)^K - \mu_1^K$, and

\[
N_2''(K, \varphi_1, \ldots, \varphi_n) = \frac{\mu_2}{1 - \mu_1 \left(1 - \frac{1}{K}\right)}.
\]

Let

\[
M_2(K) = \begin{cases} 
M_2'(K), & \text{if } (1 - K^{-1}) \mu_1 \geq 1, \\
M_2''(K), & \text{if } (1 - K^{-1}) \mu_1 < 1 \text{ and } M_1^* \geq M_2^*, \\
M_2(K), & \text{if } (1 - K^{-1}) \mu_1 < 1 \text{ and } M_1^* \leq M_2^*.
\end{cases}
\]
Bi-Lipschitz continuity of quasiconformal self-mappings of the unit disk

and

\[ N_2(K, \varphi_1, \ldots, \varphi_n) = \begin{cases} 
N_2'(K, \varphi_1, \ldots, \varphi_n), & \text{if } (1 - K^{-1}) \mu_1 \geq 1, \\
N_2''(K, \varphi_1, \ldots, \varphi_n), & \text{if } (1 - K^{-1}) \mu_1 < 1 \text{ and } M^*_1 \geq M^*_2, \\
N_2(K, \varphi_1, \ldots, \varphi_n), & \text{if } (1 - K^{-1}) \mu_1 < 1 \text{ and } M^*_1 \leq M^*_2.
\]

It follows from the facts

\[ \lim_{K \to 1} M_2(K) = 1 \quad \text{and} \quad \lim_{\|\varphi_1\|_\infty \to 0, \ldots, \|\varphi_n\|_\infty \to 0} N_2(K, \varphi_1, \ldots, \varphi_n) = 0 \]

that these two constants are what we need, and so, the claim is proved.

Claim 3.8. There are constants \( M_1(K) \) and \( N_1(K, \varphi_1, \ldots, \varphi_n) \) such that

1. \( C_1(K, \varphi_1, \ldots, \varphi_n) \geq M_1(K) - N_1(K, \varphi_1, \ldots, \varphi_n) \);
2. \( \lim_{K \to 1} M_1(K) = 1 \), and
3. \( \lim_{\|\varphi_1\|_\infty \to 0, \ldots, \|\varphi_n\|_\infty \to 0} N_1(K, \varphi_1, \ldots, \varphi_n) = 0 \).

By (3.24), we have

\[ C_1(K, \varphi_1, \ldots, \varphi_n) \geq M_1(K) - N_1(K, \varphi_1, \ldots, \varphi_n), \]

where

\[ M_1(K) = K^{-2}(Q(K))^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{it} - e^{i\theta}|^{2K-2} dt \]

and

\[ N_1(K, \varphi_1, \ldots, \varphi_n) = \left( \frac{7}{6} + \frac{1}{2K^2} \right) \|\varphi_1\|_\infty \]

\[ + \sum_{j=2}^n \left( \frac{47}{240} + \frac{1}{16K^2} \right) \|\varphi_j\|_\infty \left( \frac{3}{16} \right)^{j-2}. \]

The following facts

\[ \lim_{K \to 1} M_1(K) = 1 \quad \text{and} \quad \lim_{\|\varphi_1\|_\infty \to 0, \ldots, \|\varphi_n\|_\infty \to 0} N_1(K, \varphi_1, \ldots, \varphi_n) = 0 \]

show that these two constants are what we want, and thus, the claim is true.

Now, by the discussions of Steps 3.1 \sim 3.3, we see that the theorem is proved. \( \square \)

4. The proof of Theorem 1.10

We first prove part (a).

Step 4.1. The co-Lipschitz continuity of \( f \).

Now we begin to prove the co-Lipschitz continuity of \( f \). Since \( f \) is a \((K, K')\)-quasiconformal mapping, by [26, Lemma 4.2], we see that, for \( z \in \mathbb{D} \),

\[ \|D_f(z)\| \leq K\lambda(D_f(z)) + \sqrt{K'}, \]

which implies that
onto itself. Then, by [47, Lemma 2.1], we obtain Rado-Kneser theorem (see [8]), we see that

\[ |f_\varphi(z)| \leq \frac{(K - 1)}{K + 1} |f_\varphi(z)| + \frac{\sqrt{K^7}}{K + 1}. \]

By (1.5), we have

\[ |f_\varphi(z)| = \left| P[\varphi_0]z(z) + \sum_{k=1}^{n} (-1)^k G_k[\varphi_k]z(z) \right| \geq |P[\varphi_0]z(z)| - \sum_{k=1}^{n} |G_k[\varphi_k]z(z)| \]

and

\[ |f_\varphi(z)| = \left| P[\varphi_0]z(z) + \sum_{k=1}^{n} (-1)^k G_k[\varphi_k]z(z) \right| \leq |P[\varphi_0]z(z)| + \sum_{k=1}^{n} |G_k[\varphi_k]z(z)|, \]

which, together with (4.1), yield that

\[ |P[\varphi_0]z(z)| \leq \frac{(K - 1)}{K + 1} |P[\varphi_0]z(z)| + \Lambda(z), \]

where

\[ \Lambda(z) = \frac{(K - 1)}{K + 1} \sum_{k=1}^{n} |G_k[\varphi_k]z(z)| + \sum_{k=1}^{n} |G_k[\varphi_k]z(z)| + \frac{\sqrt{K^7}}{1 + K}. \]

By Lemmas 2.4 and 2.5, we have

\[ \Lambda(z) \leq \frac{2K}{K + 1} H(\varphi_1, \cdots, \varphi_n) + \frac{\sqrt{K^7}}{1 + K}, \]

where

\[ H(\varphi_1, \cdots, \varphi_n) = \left( \frac{1}{3} \|\varphi_1\|_\infty + \frac{1}{15} \sum_{k=2}^{n} \left( \frac{3}{16} \|\varphi_k\|_\infty \right)^{k-2} \right). \]

Since \( \varphi_0 \) is a sense-preserving homeomorphic self-mapping of \( \mathbb{T} \), by the Choquet-Rado-Kneser theorem (see [8]), we see that \( P[\varphi_0] \) is a harmonic diffeomorphism of \( \mathbb{D} \) onto itself. Then, by [47, Lemma 2.1], we obtain

\[ \frac{1}{\pi - \frac{|P[\varphi_0](0)|}{2}} \geq 1. \]

It follows from (4.2), (4.3) and (4.4) that

\[ |P[\varphi_0]z(z)| \leq \frac{(K - 1)}{K + 1} |P[\varphi_0]z(z)| + \frac{2K}{K + 1} H(\varphi_1, \cdots, \varphi_n) + \frac{\sqrt{K^7}}{K + 1} \frac{|P[\varphi_0]z(z)|}{\left( \frac{1}{\pi - \frac{|P[\varphi_0](0)|}{2}} \right)}, \]

which, together with the assumptions, gives that

\[ \frac{|P[\varphi_0]z(z)|}{|P[\varphi_0]z(z)|} \leq \frac{K - 1}{K + 1} + \frac{2K H(\varphi_1, \cdots, \varphi_n) + \sqrt{K^7}}{(K + 1) \left( \frac{1}{\pi - \frac{|P[\varphi_0](0)|}{2}} \right)} < 1. \]

Then \( P[\varphi_0] \) is a \( K^* \)-quasiconformal mapping in \( \mathbb{D} \), where
\[(4.6)\quad K^* = \frac{K \left( \frac{2}{\pi} - |P[\varphi_0](0)| \right) + 2KH(\varphi_1, \ldots, \varphi_n) + \sqrt{K'}}{\frac{2}{\pi} - |P[\varphi_0](0)| - 2KH(\varphi_1, \ldots, \varphi_n) - \sqrt{K'}}.\]

Hence, by [47, Lemma 2.4], we have
\[(4.7)\quad |P[\varphi_0]z(z)| \geq \frac{1 + K^*}{2K^*} \left( \frac{2}{\pi} - |P[\varphi_0](0)| \right),\]

which, together with (4.1), yields that
\[(4.8)\quad \lambda(D_f(z)) \geq \frac{2}{K+1} |f_z(z)| - \frac{\sqrt{K'}}{K+1}
\]
\[\geq \frac{2}{K+1} \left( |P[\varphi_0]z(z)| - H(\varphi_1, \ldots, \varphi_n) - \frac{\sqrt{K'}}{K+1} \right)
\geq \frac{(1 + K^*)}{K^*(K+1)} \left( \frac{2}{\pi} - |P[\varphi_0](0)| \right) - \frac{2H(\varphi_1, \ldots, \varphi_n) + \sqrt{K'}}{K+1}.
\]

Claim 4.1. \[\frac{(1 + K^*)}{K^*(K+1)} \left( \frac{2}{\pi} - |P[\varphi_0](0)| \right) - \frac{2H(\varphi_1, \ldots, \varphi_n) + \sqrt{K'}}{K+1} > 0.\]

Now we prove this Claim. Let \(B = 2/\pi - |P[\varphi_0](0)|\). By the assumptions, we have

\[(4.9)\quad 2KH(\varphi_1, \ldots, \varphi_n) + \sqrt{K'} < B.\]

It follows from (4.6) and (4.9) that
\[\frac{K^* + 1}{K^*(K+1)} B = \frac{B^2}{KB + 2KH(\varphi_1, \ldots, \varphi_n) + \sqrt{K'}} > \frac{B^2}{KB + B}
= \frac{B}{K+1} \geq \frac{2KH(\varphi_1, \ldots, \varphi_n) + \sqrt{K'}}{K+1}
\geq \frac{2H(\varphi_1, \ldots, \varphi_n) + \sqrt{K'}}{K+1},\]

which implies that the Claim 4.1 is true. Since for all \(z_1, z_2 \in \mathbb{D},\)

\[|f(z_1) - f(z_2)| \geq \int_{[z_1,z_2]} \lambda(D_f(z)) |dz|
\geq \left( \frac{(1 + K^*)}{K^*(K+1)} B - \frac{2H(\varphi_1, \ldots, \varphi_n) + \sqrt{K'}}{K+1} \right) |z_1 - z_2|,
\]
we conclude that \(f\) is also co-Lipschitz continuous.
Step 4.2. The Lipschitz continuity of $f$.

The Lipschitz continuity of $f$ easily follows from (3.1), (3.2) and Theorem D. Next, we prove part (b).

Step 4.3. The asymptotically sharp Lipschitz inequality of $f$.

Since $P[\varphi_0]$ is a $K^*$-quasiconformal mapping of $\mathbb{D}$ onto itself with $P[\varphi_0](0) = 0$, by [40, Theorem 3.3], we see that, for all $z_1, z_2 \in \mathbb{D}$,

\begin{equation}
|P[\varphi_0](z_1) - P[\varphi_0](z_2)| \leq (K^*)^{3K^*+125(K^*-1/K^*)/2}|z_1 - z_2|.
\end{equation}

By Lemmas 2.4 and 2.5, we obtain that, for all $z_1, z_2 \in \mathbb{D}$,

\begin{equation}
|G_1[\varphi_1](z_1) - G_1[\varphi_1](z_2)| \leq \frac{2}{3}\|\varphi_1\|_\infty|z_1 - z_2|
\end{equation}

and

\begin{equation}
|G_k[\varphi_k](z_1) - G_k[\varphi_k](z_2)| \leq \frac{2}{15}\left(\frac{3}{16}\right)^{k-1}\|\varphi_k\|_\infty|z_1 - z_2|,
\end{equation}

where $k \in \{2, \ldots, n\}$. It follows from (1.5), (4.10), (4.11) and (4.12) that, for all $z_1, z_2 \in \mathbb{D}$,

\begin{equation}
|f(z_1) - f(z_2)| \leq |P[\varphi_0](z_1) - P[\varphi_0](z_2)| + \sum_{k=1}^{n} |G_k[\varphi_k](z_1) - G_k[\varphi_k](z_2)|
\end{equation}

\begin{equation}
\leq \left(M_3(K, K') + N_3(\varphi_1, \ldots, \varphi_n)\right)|z_1 - z_2|,
\end{equation}

where $M_3(K, K') = (K^*)^{3K^*+125(K^*-1/K^*)/2}$ and

\begin{equation}
N_3(\varphi_1, \ldots, \varphi_n) = \frac{2}{3}\|\varphi_1\|_\infty + \sum_{k=2}^{n} \frac{2}{15}\left(\frac{3}{16}\right)^{k-1}\|\varphi_k\|_\infty.
\end{equation}

It is easy to know that

\begin{equation}
\lim_{K \to 1, K' \to 0} M_3(K, K') = 1 \quad \text{and} \quad \lim_{\|\varphi_1\|_\infty \to 0, \ldots, \|\varphi_n\|_\infty \to 0} N_3(\varphi_1, \ldots, \varphi_n) = 0.
\end{equation}

Step 4.4. The asymptotically sharp co-Lipschitz inequality of $f$.

Let

\begin{equation}
M_4(K, K') = \frac{(1 + K^*)}{K^*(1 + K)} \max \left\{\frac{2}{\pi}, L_{K^*}\right\} - \sqrt{K'} \frac{K}{K + 1}
\end{equation}

and

\begin{equation}
N_4(\varphi_1, \ldots, \varphi_n) = \frac{2\mathcal{H}(\varphi_1, \ldots, \varphi_n)}{K + 1},
\end{equation}

where $K^*$ is defined in (4.6) and $L_{K^*}$ is a positive constant satisfying

\begin{equation}
\lim_{K \to 1, K' \to 0} L_{K^*} = 1.
\end{equation}
Then
\[ \lim_{K \to 1, K' \to 0} M_4(K, K') = 1 \text{ and } \lim_{\|\varphi_1\|_\infty \to 0, \ldots, \|\varphi_n\|_\infty \to 0} N_4(\varphi_1, \ldots, \varphi_n) = 0. \]

It follows from the Claim 4.1 that
\[ (4.13) \quad M_4(K, K') - N_4(\varphi_1, \ldots, \varphi_n) > 0. \]

Then, by Theorem E, we have
\[ (4.14) \quad |P[\varphi_0]_z(z)| \geq \frac{K^* + 1}{2K^*} \max \left\{ \frac{2}{\pi}, L_{K^*} \right\}, \quad z \in \mathbb{D}, \]

which, together with (4.1) and (4.13), yields that, for all \( z_1, z_2 \in \mathbb{D} \),
\[ |f(z_1) - f(z_2)| \geq \int_{[z_1, z_2]} |D_f(z)| |dz| \geq \int_{[z_1, z_2]} \left( \frac{2}{K + 1} |f_z(z)| - \frac{\sqrt{K'}}{K + 1} \right) |dz| \]
\[ \geq \int_{[z_1, z_2]} \left( \frac{2}{K + 1} \left( |P[\varphi_0]_z(z)| - \mathcal{H}(\varphi_1, \ldots, \varphi_n) \right) - \frac{\sqrt{K'}}{K + 1} \right) |dz| \]
\[ \geq (M_4(K, K') - N_4(\varphi_1, \ldots, \varphi_n)) |z_1 - z_2|. \]

Therefore, \( f \) is co-Lipschitz continuous in \( \mathbb{D} \). The proof of this theorem is complete. \( \square \)

5. THE PROOF OF THEOREM 1.11

By (1.5), we have
\[ f(z) = P[\varphi_0](z) + \sum_{k=1}^{n} (-1)^k G_k[\varphi_k](z), \quad z \in \mathbb{D}. \]

For \( k \in \{1, 2, \ldots, n\} \), it follows from Lemmas 2.4 and 2.5 that \( G_k[\varphi_k] \) are Lipschitz continuous in \( \mathbb{D} \). Since \( P[\varphi_0] \) is Lipschitz continuous in \( \mathbb{D} \) if and only if the Hilbert transform of \( d\varphi_0(e^{i\theta})/d\theta \in L^\infty(\mathbb{T}) \) (see the part 1.2), together with the Lipschitz continuity of \( G_k[\varphi_k] \), we conclude that \( f \) is Lipschitz continuous in \( \mathbb{D} \) if and only if the Hilbert transform of \( d\varphi_0(e^{i\theta})/d\theta \in L^\infty(\mathbb{T}) \), where \( k \in \{1, 2, \ldots, n\} \). The proof of this proposition is complete. \( \square \)

Acknowledgements: This research was partly supported by the exchange project for the third regular session of the China-Montenegro Committee for Cooperation in Science and Technology (No. 3-13), the Hunan Provincial Education Department Outstanding Youth Project (No. 18B365), the Science and Technology Plan Project of Hengyang City (No. 2018KJ125), the National Natural Science Foundation of China (No. 11571216), the Science and Technology Plan Project of Hunan Province (No. 2016TP1020), the Science and Technology Plan Project of Hengyang City (No. 2017KJ183), and the Application-Oriented Characterized Disciplines, Double First-Class University Project of Hunan Province (Xiangjiaotong [2018]469).


REFERENCES

1. K. Astala, T. Iwaniec and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, in: Princeton Mathematical Series, vol. 48, Princeton University Press, Princeton, NJ, 2009, p. xviii+677.

2. K. Astala, T. Iwaniec, I. Prause and E. Saksman, Bilipschitz and quasiconformal rotation, stretching and multifractal spectra, *Publ. Math. Inst. Hautes Études Sci.*, 121 (2015), 113–154.

3. H. Begehr, *Complex Analytic Methods for Partial Differential Equation. An Introductory Text*. World Scientific, Singapore (1994).

4. A. Borichev and H. Hedenmalm, Weighted integrability of polyharmonic functions, *Adv. Math.*, 264 (2014), 464–505.

5. Sh. Chen, P. Li, and X. Wang, Schwarz-type Lemma, Landau-type Theorem, and Lipschitz-type space of solutions to inhomogeneous biharmonic equations, *J. Geom. Anal.*, (2018), https://doi.org/10.1007/s12220-018-0083-6.

6. Sh. Chen and X. Wang, Bi-Lipschitz characteristic of quasiconformal self-mappings of the unit disk satisfying bi-harmonic equation, Submitted.

7. J. Chen, P. Li, S. K. Sahoo and X. Wang, On the Lipschitz continuity of certain quasiregular mappings between smooth Jordan domain, *Israel J. Math.*, 220 (2017), 453–478.

8. G. Choquet, Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques, *Bull. Sci. Math.*, 69 (1945), 156–165.

9. G. Cui and Zh. Li, A note on mori’s theorem of K-quasiconformal mappings, *Acta Math. Sinica (N. S.)*, 9 (1993), 55–62.

10. R. Fehlmann and M. Vuorinen, Mori’s theorem for n-dimensional quasiconformal mappings, *Ann. Acad. Sci. Fenn. Math.*, 13 (1988), 111–124.

11. F. Gazzola, H.-C. Grunau and G. Sweer, *Polyharmonic boundary value problems*, Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, 2010. xviii+423 pp.

12. S. I. Goldberg and T. Ishihara, Harmonic quasiconformal mappings of Riemannian manifolds, *Bull. Amer. Math. Soc.*, 80 (1974), 562–566.

13. S. I. Goldberg and T. Ishihara, Harmonic quasiconformal mappings of Riemannian manifolds, *Amer. J. Math.*, 98 (1976), 225–240.

14. H. -Ch. Grunau and F. Robert, Positivity and almost positivity of biharmonic Green’s functions under Dirichlet boundary conditions, *Arch. Rational Mech. Anal.*, 195 (2010), 865–898.

15. H. -Ch. Grunau and G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, *Math. Ann.*, 307 (1997), 589–626.

16. P. R. Halmos, Measure theory, D. Van Nostrand Company, Inc., New York, N. Y., 1950.

17. W. K. Hayman and B. Korenblum, Representation and uniqueness theorems for polyharmonic functions, *J. Anal. Math.*, 60 (1993), 113–133.

18. W. Hengartner and G. Schober, Harmonic mappings with given dilatation, *J. London Math. Soc.*, 33 (1986), 473–483.

19. E. Heinz, On one-to-one harmonic mappings, *Pacific J. Math.*, 9 (1959), 101–105.

20. L. Hörmander, *Notions of convexity*, Progress in Mathematics, Vol. 127, Birkhäuser Boston Inc, Boston 1994.

21. D. Kalaj, On harmonic quasiconformal self-mappings of the unit ball, *Ann. Acad. Sci. Fenn. Math.*, 33 (2008), 261–271.

22. D. Kalaj, Quasiconformal and harmonic mappings between Jordan domains, *Math. Z.*, 260 (2008), 237–252.

23. D. Kalaj, On the quasiconformal self-mappings of the unit ball satisfying the Poisson differential equation, *Ann. Acad. Sci. Fenn. Math.*, 36 (2011), 177–194.

24. D. Kalaj and M. Mateljević, Inner estimate and quasiconformal harmonic maps between smooth domains, *J. Anal. Math.*, 100 (2006), 117–132.
25. D. Kalaj and M. Mateljević, On certain nonlinear elliptic PDE and quasiconformal maps between Euclidean surfaces, Potential Anal., 34 (2011), 13–22.
26. D. Kalaj and M. Mateljević, \((K, K')\)-quasiconformal harmonic mappings, Potential Anal., 36 (2012), 117–135.
27. D. Kalaj and M. Pavlović, Boundary correspondence under harmonic quasiconformal diffeomorphisms of a half-plane, Ann. Acad. Sci. Fenn. Math. 30 (2005), 159–165.
28. D. Kalaj and M. Pavlović, On quasiconformal self-mappings of the unit disk satisfying Poisson’s equation, Trans. Amer. Math. Soc., 363 (2011), 4043–4061.
29. D. Kalaj and E. Saksman, Quasiconformal maps with controlled Laplacian, J. Anal. Math., 137 (2019), 251–268.
30. M. Knežević and M. Mateljević, On the quasi-isometries of harmonic quasiconformal mappings, J. Math. Anal. Appl., 334 (2007), 404–413.
31. O. Lehto and K. I. Virtanen, Quasiconformal mappings in the plane, Springer Verlag, 1973.
32. H. Lewy, On the non-vanishing of Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc., 42 (1936), 689–692.
33. P. Li and S. Ponnusamy, Representation formula and bi-Lipschitz continuity of solutions to inhomogeneous bi-harmonic Dirichlet problems in the unit disk, J. Math. Anal. Appl., 456 (2017), 1150–1175.
34. O. Martio and R. Näkki, Boundary Hölder continuity and quasiconformal mappings, J. London Math. Soc., 44 (1991), 339–350.
35. O. Martio, On harmonic quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A I, 425 (1968), 3–10.
36. M. Mateljević and M. Vuorinen, On harmonic quasiconformal quasi-isometries, J. Inequal. Appl., 2010, Art. ID 178732.
37. S. Mayboroda and V. Maz’ya, Regularity of solutions to the polyharmonic equation in general domains, Invent. Math., 196 (2014), 1–68.
38. A. Monr, On an absolute constant in the theory of quasiconformal mappings, J. Math. Soc. Japan, 8 (1956), 156–166.
39. A. Olofsson, Differential operators for a scale of Poisson type kernels in the unit disc, J. Anal. Math., 123 (2014), 227–249.
40. M. Part silky and K. Sakam, On bi-Lipschitz type inequalities for quasiconformal harmonic mappings, Ann. Acad. Sci. Fenn. Math., 32 (2007), 579–594.
41. M. Part silky and K. Sakam, On an asymptotically sharp variant of Heinz’s inequality, Ann. Acad. Sci. Fenn. Math., 30 (2005), 167–182.
42. M. Pavlović, Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disk, Ann. Acad. Sci. Fenn. Math., 27 (2002), 365–372.
43. S. Qiu, On Mori’s theorem in quasiconformal theory, Acta Math. Sinica (N.S.) 13 (1997), 35–44.
44. E. Talvila, Necessary and sufficient conditions for differentiating under the integral sign, Amer. Math. Monthly, 108 (2001), 544–548.
45. Luen-Fai Tam and Tom Y.-H. Wan, Quasiconformal harmonic diffeomorphism and the universal Teichmüller space, J. Differential Geom., 42 (1995), 368–410.
46. Luen-Fai Tam and Tom Y.-H. Wan, On quasiconformal harmonic maps, Pacific J. Math., 182 (1998), 359–383.
47. D. Zhong and H. Tu, Heinz type inequalities for mappings satisfying Poisson’s equation, Indag. Math., 30 (2019), 98–105.
48. D. Zhong, Y. Zhou and W. Yuan, Heinz-type inequality and bi-Lipschitz continuity for quasiconformal mappings satisfying inhomogeneous biharmonic equations, Math. Meth. Appl. Sci., 2019, 1–8, Doi: 10.1002/mma.5611.
49. A. Zygmund, Trigonometrical series, Chelsea Publishing Co., 2nd edition, New York, 1952.
Sh. Chen, College of Mathematics and Statistics, Hengyang Normal University, Hengyang, Hunan 421008, People’s Republic of China.
E-mail address: mathechen@126.com

D. Kalaj, Faculty of Natural Sciences and Mathematics, University of Montenegro, Cetinjski put b. b. 81000 Podgorica, Montenegro.
E-mail address: davidk@ac.me; didekalaj@gmail.com