THE DIRICHLET PROBLEM FOR A CLASS OF DEGENERATE FULLY NONLINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS WITH MEAN CONCAVE BOUNDARY

RI-RONG YUAN

Abstract. This article studies the Dirichlet problem for a class of degenerate fully nonlinear elliptic equations on Riemannian manifolds with mean concave boundary in the sense that the mean curvature of the boundary is nonpositive. The proof is primarily based on a quantitative boundary estimate. Also, we obtain analogous results in complex variables. In Appendix, the subsolutions are also constructed on certain topologically product manifolds.

Mathematics Subject Classification (2010): 35J70, 58J05, 35B45.

Keywords: Riemannian manifolds with mean concave boundary, Dirichlet problem, Degenerate fully nonlinear elliptic equation, Quantitative boundary estimate.

1. Introduction

Let $(M, g)$ be an $n$-dimensional $(n \geq 3)$ compact Riemannian manifold with mean concave boundary $\partial M$ and the Levi-Civita connection $\nabla, \bar{M} := M \cup \partial M$. Let $A$ be a smoothly symmetric $(0,2)$-type tensor, $\eta = \eta_{ij} dx^i \otimes dx^j$ be a smooth $(1,2)$-type tensor with $\eta_{ij} = \eta_{ji}$, $Z(du) = \text{tr}_g(W(du))g - W(du)$, where $(W(du))_{ij} = \sum_{k=1}^{n} u_k \eta_{ij}^k$. A further technical hypothesis is needed: For each $p_0 \in \partial M$, under local coordinates with $g_{ij}(p_0) = \delta_{ij}$, $\sum_{i \neq k} \eta_{ik}^k(p_0) = 0$ for each $k$. Typical examples are $\eta \equiv 0$ and more general $\eta_{ik}^k = \delta_{ik} \zeta_i + \delta_{jk} \zeta_i$ for a smooth $(0,1)$ tensor $\zeta = \zeta_i dx^i$.

In this paper, we propose a new hypothesis on the boundary in an attempt to establish a quantitative boundary estimate and solve the following fully nonlinear elliptic equation possibly with degenerate right-hand side

(1.1) \[ f(\lambda(U[u])) = \psi \text{ in } M, \]

(1.2) \[ u = \varphi \text{ on } \partial M, \]

where $\lambda(U[u])$ denote the eigenvalues of $U[u] = A + (\Delta u)g - \nabla^2 u + Z(du)$ with respect to $g$, $\nabla^2 u$ is real Hessian with $\nabla^2 u(X,Y) = \nabla_{XY} u := YXu - (\nabla_Y X)u$ for $X,Y \in TM$, $\Delta = \text{tr}_g \nabla^2$ is the Laplacian operator. Moreover, $\varphi$ and $\psi$ are sufficiently smooth functions with $w|_{\partial M} = \varphi$ for some admissible function $w \in C^2(\bar{M})$ defined as $\lambda(U[w]) \in \Gamma$, also with $\sup_M \psi < \sup_{\Gamma} f$ that is automatically satisfied if there exists an admissible subsolution satisfying (1.10) below. As in [11], $\Gamma$ is an open symmetric
and convex cone with vertex at the origin, $\Gamma_n \subseteq \Gamma \subseteq \Gamma_1$ and boundary $\partial \Gamma \neq \emptyset$, on which $f$ is a smooth and symmetric function satisfying

\begin{equation}
(1.3) \quad f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \text{ in } \Gamma, \ 1 \leq i \leq n,
\end{equation}

\begin{equation}
(1.4) \quad f \text{ is concave in } \Gamma.
\end{equation}

where $\Gamma_1 := \{ \lambda \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i > 0 \}$ and $\Gamma_n := \{ \lambda \in \mathbb{R}^n : \text{ each } \lambda_i > 0 \}$. In order to study equation (1.1) within the framework of elliptic equations, we shall seek solutions in the class of $C^2$-admissible functions.

The study of such equations goes back at least to [1, 25], and since then it has been carried widely out in numerous literature (cf. [10, 12, 13, 14, 5, 30, 35, 17, 39, 41, 43]) on Monge-Ampère equation for $(n-1)$-plurisubharmonic functions from non-Kähler geometry. Specifically, due to the interests and problems from differential geometry and PDEs (cf. [20] and references therein), it would be important to understand the solutions whenever the right-hand side is degenerate

\[ \inf_M \psi = \sup_{\partial \Gamma} f, \]

where $\sup_{\partial \Gamma} f := \sup_{\lambda_0 \in \partial \Gamma} \lim_{\lambda \to \lambda_0} \sup_{\lambda \to \lambda_0} f(\lambda)$. There are some papers should be mentioned (cf. [2, 19, 21, 26, 40], see also [20] for some relative open problems), which concern degenerate real Monge-Ampère equation and more general degenerate fully nonlinear elliptic equations on $\Omega \subset \mathbb{R}^n$. However, it is still less known for general cases of both equations and background spaces. On the background space being a curved Riemannian manifold, with the assumption that the boundary is concave and there is a strictly admissible subsolution, the author [43, 44] solved the Dirichlet problem for the following degenerate fully nonlinear elliptic equations (see Theorem 2.7 below, while the papers primarily treat equations on complex manifolds)

\begin{equation}
(1.5) \quad F(g[u]) := f(\lambda(g[u])) = \psi,
\end{equation}

in which $g[u] = \chi + \nabla^2 u + W(du)$, $\chi$ is a smoothly symmetric $(0,2)$ tensor, and $f$ further satisfies

\begin{equation}
(1.6) \quad \text{For each } \sigma < \sup_{\Gamma} f \text{ and } \lambda \in \Gamma, \lim_{t \to +\infty} f(t\lambda) > \sigma,
\end{equation}

\*We call $\partial M$ is concave, if the second fundamental form with respect to $-\nu$, denoted by $\Pi_{\partial M}$, is negative semidefinite, where and hereafter $\nu$ is the unit inner normal vector along boundary.

Here are examples of such spaces: Let $M = X \times [0,1]$ ($\partial X = \emptyset$) equip with $g = e^\phi g_X + dx^n \otimes dx^n$ where $\phi$ is a smooth function (on $M$) with $\nabla \phi|_{\partial M} \geq 0$, while on such warped product spaces subsolutions for certain equations are constructed in Appendix A.
or equivalently (according to the Lemma 3.4 of [44])

\[(1.7) \sum_{i=1}^{n} f_i(\lambda)\mu_i > 0, \forall \lambda, \mu \in \Gamma,\]

which is satisfied by the functions with \(\Gamma = \Gamma_n\), and also by homogenous functions of degree one with \(f|_\Gamma > 0\).

In the following theorem we show an interesting approach to degenerate equations of the form (1.1), when \((M, g)\) supposes mean concave boundary which includes among others Riemannian manifold whose boundary is a minimal hypersurface. Such a hypothesis is reasonable according to significant progress on Yau’s conjecture (cf. [24, 29, 34] and references therein).

**Theorem 1.1.** Let \((M, g)\) be a compact Riemannian manifold with smooth mean concave boundary. In addition to (1.3), (1.4), (1.6) and \(f \in C^\infty(\Gamma) \cap C(\overline{\Gamma})\), we suppose \(\varphi \in C^{2,1}(\partial M)\), \(\psi \in C^{1,1}(\overline{M})\), \(\inf_M \psi = \sup_{\partial \Gamma} f\). Then Dirichlet problem (1.1)-(1.2) admits a (weak) solution \(u \in C^{1,1}(\overline{M})\) with \(\lambda(U[u]) \in \Gamma\) and \(\Delta u \in L^\infty(\overline{M})\), provided that there is a strictly admissible subsolution \(u \in C^{2,1}(\overline{M})\) obeying

\[(1.8) f(\lambda(U[u])) \geq \psi + \delta_0 \text{ in } \overline{M}, \ u = \varphi \text{ on } \partial M,\]

for some \(\delta_0 > 0\). Moreover, if the mean curvature of boundary, say \(H_{\partial M}\), is strictly negative then the above statement still holds for \(\partial M \in C^{2,1}\).

Consequently, we have a corollary.

**Corollary 1.2.** Let \((M, g)\) be a closed connected Riemannian manifold, \(\Sigma \subset M\) be a closed connected smoothly minimal hypersurface in \((M, g)\) such that \(M \setminus \Sigma = M_1 \cup M_2\), \(M_1 \cap M_2 = \emptyset\) and \(\partial M_1 = \partial M_2 = \Sigma\). Assume (1.3), (1.4), (1.6), \(\lambda(A) \in \Gamma\) and \(f \in C^\infty(\Gamma) \cap C(\overline{\Gamma})\) with \(f|_{\partial \Gamma} = 0\). Then the following conclusions are true:

- For each smooth function \(\psi\) with \(e^\psi \leq f(\lambda(A))\), there is a unique function \(u\) that is smooth and admissible when restricted to \(M \setminus \Sigma\), to satisfy \(f(\lambda(U[u])) = e^\psi\) in \(M \setminus \Sigma\).
- There is \(u\) which is \(C^{1,1}\) when restricted to \(M \setminus \Sigma\), to satisfy \(f(\lambda(U[u])) = 0\) in \(M \setminus \Sigma\).

In the both two cases, \(u|_{\Sigma} = 0\).

In order to solve the degenerate equation, we approximate it by a sequence of non-degenerate equations with

\[(1.9) \delta_{\psi, f} := \inf_M \psi - \sup_{\partial \Gamma} f > 0.\]

Besides, the crucial ingredient is to establish a quantitative boundary estimate for such non-degenerate equations which says that second order estimate at the boundary can be bounded from above by a constant depending not on \((\delta_{\psi, f})^{-1}\). More precisely,

\[\text{We say } C \text{ is independent of } (\delta_{\psi, f})^{-1} \text{ if it remains uniformly bounded as } \delta_{\psi, f} \to 0.\]
Theorem 1.3. Let \((M, g)\) be a compact Riemannian manifold with smooth mean concave boundary, \(\psi \in C^1(M)\), \(\varphi \in C^3(\partial M)\), and we assume (1.3), (1.4) and (1.9). Suppose there is an admissible subsolution \(\underline{u} \in C^2(\bar{M})\) satisfying
\[
(1.10) \quad f(\lambda(U(\underline{u}))) \geq \psi \text{ in } M, \text{ and } \underline{u} = \varphi \text{ on } \partial M.
\]
Then for any admissible solution \(u \in C^3(M) \cap C^2(\bar{M})\) to (1.11)-(1.2), there exists a uniformly positive constant depending only on \(\varphi|_{C^3(M)}\), \(\psi|_{C^1(\bar{M})}\), \(\underline{u}|_{C^2(\bar{M})}\), \(\partial M\) up to its second derivatives and other known data (but not on \((\delta, f)^{-1}\)) such that
\[
(1.11) \quad \sup_{\partial M} \Delta u \leq C(1 + \sup_M |\nabla u|^2).
\]
Moreover, if the boundary data \(\varphi\) is a constant then \(C\) depends only on \(\psi|_{C^1(\bar{M})}\), \(\underline{u}|_{C^2(\bar{M})}\), \(\partial M\) up to its second derivatives and other known data.

The hypothesis of mean concave boundary is only used to derive such a quantitative boundary estimate that is primarily used to deal with degenerate equations on such Riemannian manifolds and further weaken the regularity assumptions on the boundary and boundary data as well. More precisely, it only requires \(\varphi, \partial M \in C^{2,1}\) in Theorem 1.1 while such regularity assumptions on boundary and boundary data are impossible for homogeneous real Monge-Ampère equation on certain bounded domains \(\Omega \subset \mathbb{R}^n\), as the counterexamples in [32] show that the hypotheses of \(C^{3,1}\) regularity on boundary and boundary data are optimal for \(C^{1,1}\) regularity of weak solution to homogeneous real Monge-Ampère equation on \(\Omega\). On the other hand, as shown by counterexamples in [32], the solutions to Dirichlet problem for (nondegenerate) real Monge-Ampère equation on \(\Omega \subset \mathbb{R}^2\) may fail to \(C^2\) if either boundary or boundary data is only in class of \(C^{2,1}\). While for the case that boundary data is a constant, we can further weaken the regularity assumption on \(\partial M\) and prove the following theorem as a result.

Theorem 1.4. Let \((M, g)\) be a compact Riemannian manifold with \(C^{2,\beta}\)-smooth strictly mean concave boundary of \(H_{\partial M} < 0\), \(0 < \beta < 1\). Suppose, in addition to (1.3), (1.4), (1.6), (1.9), \(\varphi = 0\) and \(\psi \in C^2(M)\), that Dirichlet problem (1.1)-
(1.2) with homogeneous boundary data has a \(C^{2,\beta}\)-smooth admissible subsolution with \(\nabla \varphi \underline{u}|_{\partial M} \leq 0\) or \(\nabla \varphi \underline{u}|_{\partial M} \geq 0\). Then the Dirichlet problem admits a unique \(C^{2,\alpha}\)-admissible solution for some \(0 < \alpha \leq \beta\). In particular, if \(\lambda(A) \in \Gamma\) and \(f(\lambda(A)) \geq \psi\), then \(\underline{u} = 0\).

Let \(f \in C^\infty(\Gamma) \cap C(\Gamma)\). With (1.9), \(\psi \in C^2(M)\) and the existence of admissible subsolution replaced by \(\inf_M \psi = \sup_{\partial \Gamma} f\), \(\psi \in C^{1,1}(M)\) and (1.8), respectively, we have the existence of \(C^{1,1}\) weak solution to the Dirichlet problem for degenerate equations.

Theorem 1.4 is a complement of relative results of [33] in complex variables. In the proof, follows an idea of [44], based on \(u\) we construct domains and apply them to approximate the original Dirichlet problem. A result of [33] is also needed. Moreover, according to a result in [31], \(\alpha = \beta\) if \(\beta < \gamma\) for a universal constant \(0 < \gamma < 1\).
The paper is organized as follows. In Section 2 we mainly derive the quantitative
boundary estimate and then apply it to complete the proof of main theorem. In
Section 3 we summarize the analogous result in the counterpart of complex variables.
In Appendix A under certain assumptions, we can construct strictly admissible subsolutions
for equations on certain topologically product spaces.

2. Proof of main results

2.1. Sketch of proof of Theorem 1.3  First of all, we see that equation (1.1) is of
the form (1.5) with
\[ g[u] = \chi + \nabla^2 u + W(du), \]
where \( \chi = \frac{1}{n-1} (\text{tr}_g A) g - A \). Then, in main equation (1.1), \( U[u] = (\text{tr}_g g[u]) g - g[u] \).

For simplicity we denote \( g = g[u] \) and \( g = g[u] \) for \( u \) and \( u \), respectively. One
denotes \( \lambda(g[u]) = (\lambda_1, \ldots, \lambda_n) \) and \( \lambda(U[u]) = (\mu_1, \ldots, \mu_n)Q^{-1} \),
where \( Q = (q_{ij}) \) and \( q_{ij} = 1 - \delta_{ij} \) (\( Q \) is symmetric). Here \( Q^{-1} \) is well defined, since
\( \det Q = (-1)^{n-1}(n-1) \neq 0 \). Then \( \tilde{\Gamma} \) is also an open symmetric convex cone of \( \mathbb{R}^n \),
and we thus define \( \tilde{f} : \tilde{\Gamma} \to \mathbb{R} \) by \( f(\mu) = \tilde{f}(\lambda) \). So equation (1.1) is
\[ \tilde{f}(\lambda(g[u])) = \psi. \]
That is, equation (1.1) is of the form (1.5). We can verify that if \( f \) satisfies (1.3),
(1.4) and (1.6) in \( \Gamma \), then so does \( \tilde{f} \) in \( \tilde{\Gamma} \).

The quantitative boundary estimate follows immediately from Propositions 2.1 and
2.2.

Proposition 2.1. Let \( (M, g) \) be a compact Riemannian manifold with mean concave
boundary. Suppose, in addition to (1.3), (1.4), (1.9), that for \( \psi \in C^0(\bar{M}) \) and \( \varphi \in C^2(\partial M) \) there is a \( C^2 \)-admissible subsolution obeying (1.10). Let \( u \in C^3(M) \cap C^2(\bar{M}) \)
be an admissible solution to Dirichlet problem (1.1)–(1.2). Fix \( x_0 \in \partial M \). Then there
is a uniformly positive constant \( C \) depending only on \( |u|_{C^0(\bar{M})}, |u|_{C^2(\bar{M})}, \partial M \)
up to second order derivatives and other known data (but neither on \( \sup_M |\nabla u| \) nor on
(\( \delta_{\psi, f} \)) such that

\[ \text{tr}_g(g)(x_0) \leq C \left( 1 + \sum_{\alpha=1}^{n-1} |g(e_\alpha, \nu)(x_0)|^2 \right), \]
where \( e_\alpha, e_\beta \in T_{\partial M, x_0} \) (\( \alpha, \beta = 1, \ldots, n-1 \)) with \( g(e_\alpha, e_\beta)(x_0) = \delta_{\alpha\beta} \).
This proposition is a crucial ingredient for studying degenerate equation (1.1) and for weakening the regularity assumptions on boundary and boundary data as well. The proposition of this type is first proved by the author [43] for equation (3.1) (with \( \eta^1,0 = 0 \)) on compact Hermitian manifolds with Levi flat boundary that is later extended to more general case that \( \partial M \) is pseudoconcave in [44]. In final section of the same paper [44], it was extended to more general fully nonlinear elliptic equations than Monge-Ampère equation for \((n - 1)\)-plurisubharmonic functions associated to Gauduchon’s conjecture on complex manifolds with holomorphically flat boundary. Clearly, when \( \partial M \) is concave the proof presented there automatically works for Dirichlet problem (1.5) and (1.2) in real variables, while for Dirichlet problem (1.1)–(1.2) we assume in Proposition 2.1 that \( \partial M \) is only mean concave that is much more general than the condition of concave for \( n \geq 3 \). Moreover, with replaced \( H_{\partial M} \leq 0 \) by the condition that the Levi form of \( \partial M \), denoted by \( L_{\partial M} \), has nonpositive trace, we have a similar proposition in complex variables.

According to Proposition 2.1, the other issue for our goal is to prove that the bounds for mixed derivatives at the boundary can be controlled linearly by \( L^\infty \)-norm of gradient term. That is

**Proposition 2.2.** Let \((M, g)\) be a compact Riemannian manifold with boundary (but without restriction to second fundamental form of \( \partial M \)), and \( \psi \in C^1(\bar{M}) \), \( \varphi \in C^3(\partial M) \). In addition to (1.3), (1.4) and (1.9), we suppose Dirichlet problem (1.5) and (1.2) has a \( C^2 \) subsolution \( \lambda(g[u]) \in \Gamma \) in \( M \), \( u|_{\partial M} = \varphi \).

Then each solution \( u \in C^3(M) \cap C^2(\bar{M}) \) to Dirichlet problem (1.5) and (1.2) with \( \lambda(g[u]) \in \Gamma \) satisfies

\[
|\nabla^2 u(X, \nu)| \leq C(1 + \sup_M |\nabla u|), \quad \forall X \in T_{\partial M} \text{ with } |X| = 1,
\]

where \( C \) depends on \( |\varphi|_{C^1(\bar{M})}, |\psi|_{C^1(\bar{M})}, |u|_{C^2(\bar{M})}, \partial M \) up to its second derivatives and other known data (but neither on \( \sup_M |\nabla u| \) nor on \( (\delta_{\psi,f})^{-1} \)).

**2.2. Bounds for \( \sup_M |u| \) and \( \sup_{\partial M} |\nabla u| \).** Let \( x_0 \in \partial M \), \( \sigma \) be the distance function to boundary, and \( \rho \) be the distance function to the given point \( x_0 \), and \( \Omega_\delta = \{ x \in M : \rho(x) < \delta \} \).

We construct the supersolution \( w \in C^2(\bar{M}) \) by solving \( \text{tr}_g(g[w]) = 0 \) in \( M \), \( w = \varphi \) on \( \partial M \). The existence follows from the theory of PDEs, and the maximum principle implies \( w \leq u \leq w \) in \( M \), and then yields, on \( \partial M \),

\[
\nabla_{e_\alpha} u = \nabla_{e_\alpha} u, \quad \nabla_{\nu} u \leq \nabla_{\nu} w \leq \nabla_{\nu} w,
\]

\[
\nabla^2 (u - u)(e_\alpha, e_\beta) = -\nabla_{\nu}(u - u)\Pi_{\partial M}(e_\alpha, e_\beta), \quad \forall e_\alpha, e_\beta \in T_{\partial M}.
\]

Thus

\[
\sup_M |u| + \sup_{\partial M} |\nabla u| \leq C,
\]
\begin{equation}
\sup_{\partial M} |\nabla^2 u(e_\alpha, e_\beta)| \leq C, \forall e_\alpha, e_\beta \in T_{\partial M}, |e_\alpha| = |e_\beta| = 1.
\end{equation}

2.3. **Proof of Proposition 2.1.** To complete the proof, we need the following lemma which follows from an idea and deformation argument from the Lemma 3.1 in [43] (or equivalently Lemma 2.2 of [44]).

**Lemma 2.3.** Let $H$ be an $n \times n$ symmetric matrix of the form
\[
\begin{pmatrix}
  a_1 + d_1 & a_2 & \cdots & a_n \\
  a_2 & a_1 + d_2 & \cdots & \\
  & \ddots & \ddots & \cdots \\
  a_n & \cdots & a_2 & a_1 + d_{n-1} \\
\end{pmatrix}
\]

with $d_1, \ldots, d_n, a_1, \ldots, a_{n-1}$ fixed, and with $a$ variable. Denote $\lambda_1, \ldots, \lambda_n$ by the eigenvalues of $H$. Let $\epsilon > 0$ be a fixed constant. Suppose that the parameter $a$ in $H$ satisfies the quadratic growth condition
\[
a \geq \frac{2n-3}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n-1) \sum_{i=1}^{n} |d_i| + \frac{(n-2)\epsilon}{2n-3}.
\]

Then the eigenvalues (possibly with an order) behavior like
\[
|a + d_\alpha - \lambda_\alpha| < \epsilon, \forall 1 \leq \alpha \leq n-1, \quad |d_n - \lambda_n| < (n-1)\epsilon.
\]

**Proof of Proposition 2.1.** Inspired by an idea from [44], we give the proof based on the structure of (1.1).

Given $x_0 \in \partial M$. We choose local coordinates $x = (x_1, \cdots, x_n)$ centered at $x_0$, such that at $x_0$, $g_{ij} = \delta_{ij}$ and $(g_{\alpha\beta})$ is diagonal, $\frac{\partial}{\partial x_\alpha}$ is the unit inner normal vector. In what follows all the discussions will be given at $x_0$, and the Greek letters $\alpha, \beta$ range from 1 to $n - 1$. Let’s denote
\[
A(R) = \begin{pmatrix}
  R - g_{11} & & -g_{1n} \\
  & \ddots & \vdots \\
  -g_{n1} & \cdots & R - g_{n(n-1)(n-1)} - g_{(n-1)n} \\
\end{pmatrix},
\]

\[
A(R) = \begin{pmatrix}
  R - g_{11} & & -g_{1n} \\
  & \ddots & \vdots \\
  -g_{n1} & \cdots & R - g_{n(n-1)(n-1)} - g_{(n-1)n} \sum_{\alpha=1}^{n-1} g_{\alpha\alpha} \\
\end{pmatrix}.
\]
and
\[
B(R) = \begin{pmatrix}
R - \mathbf{g}_{11} & -\mathbf{g}_{12} & \cdots & -\mathbf{g}_{1(n-1)} & -\mathbf{g}_{1n} \\
-\mathbf{g}_{21} & R - \mathbf{g}_{22} & \cdots & -\mathbf{g}_{2(n-1)} & -\mathbf{g}_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\mathbf{g}_{(n-1)1} & -\mathbf{g}_{(n-1)2} & \cdots & R - \mathbf{g}_{(n-1)(n-1)} & -\mathbf{g}_{(n-1)n} \\
-\mathbf{g}_{n1} & -\mathbf{g}_{n2} & \cdots & -\mathbf{g}_{n(n-1)} & \sum_{\alpha=1}^{n-1} \mathbf{g}_{\alpha\alpha}
\end{pmatrix}.
\]

In particular, \( A(\text{tr}_g(\mathbf{g})) = (U_{ij}([u])) \) and \( B(\text{tr}_g(\mathbf{g})) = (U_{ij}([u])) \). Lemma 2.3 implies
\[
(R - \lambda^r_1, \ldots, R - \lambda^r_{n-1}, \sum_{\alpha=1}^{n-1} \mathbf{g}_{\alpha\alpha}) \in \Gamma \text{ for } R \gg 1,
\]
here we use the openness of \( \Gamma \).

The ellipticity and concavity of the equation, couple with Lemma 6.2 in [1], yield that
\[
F(A) - F(B) \geq F^{ij}(A)(a_{ij} - b_{ij})
\]
for the symmetric matrices \( A = \{a_{ij}\} \) and \( B = \{b_{ij}\} \) with \( \lambda(A), \lambda(B) \in \Gamma \). One thus obtains that there is \( R_1 > 0 \) depending only on \( \mathbf{g} \) such that
\[
f(R_1 - \lambda^r_1, \ldots, R_1 - \lambda^r_{n-1}, \sum_{\alpha=1}^{n-1} \mathbf{g}_{\alpha\alpha}) \geq F(\lambda(B(\text{tr}_g(\mathbf{g})))) \geq \psi,
\]
\((R_1 - \lambda^r_1, \ldots, R_1 - \lambda^r_{n-1}, \sum_{\alpha=1}^{n-1} \mathbf{g}_{\alpha\alpha}) \in \Gamma \), here we use the fact that if \( A \) is diagonal then so is \( F^{ij}(A) \). Therefore, there exist two uniformly positive constants \( \varepsilon_0, R_0 \) depending on \( \mathbf{g} \) and \( f \), such that
\[
(2.7) \quad f(R_0 - \lambda^r_1 - \varepsilon_0, \ldots, R_0 - \lambda^r_{n-1} - \varepsilon_0, \sum_{\alpha=1}^{n-1} \mathbf{g}_{\alpha\alpha} - \varepsilon_0) \geq \psi
\]
and \((R_0 - \lambda^r_1 - \varepsilon_0, \ldots, R_0 - \lambda^r_{n-1} - \varepsilon_0, \sum_{\alpha=1}^{n-1} \mathbf{g}_{\alpha\alpha} - \varepsilon_0) \in \Gamma \). Here (1.3), (1.4) and the openness of \( \Gamma \) are needed.

Next, we apply Lemma 2.3 to matrix \( A(R) \). Let
\[
R_c = 2\frac{(n-1)(2n-3)}{\varepsilon_0} \sum_{\alpha=1}^{n-1} |\mathbf{g}_{\alpha\alpha}|^2 + (n-1) \sum_{\alpha=1}^{n-1} (|\mathbf{g}_{\alpha\alpha}| + |\mathbf{g}_{\alpha\alpha}|) + \sum_{\alpha=1}^{n-1} |\lambda_\alpha'| + R_0 + \varepsilon_0,
\]
where \( \varepsilon_0 \) and \( R_0 \) are fixed constants so that (2.1) holds. It follows from Lemma 2.3 that the eigenvalues of \( A(R_c) \) (possibly with an order) shall behavior like
\[
(2.8) \quad \lambda(A(R_c)) \in \left(R_c - \mathbf{g}_{11} - \frac{\varepsilon_0}{2}, \ldots, R_c - \mathbf{g}_{(n-1)(n-1)} - \frac{\varepsilon_0}{2}, \sum_{\alpha=1}^{n-1} \mathbf{g}_{\alpha\alpha} - \frac{\varepsilon_0}{2}\right) + \Gamma_n \subset \Gamma.
\]
It follows from (2.4), the technical hypothesis
$$\sum_{i \neq k} \eta_i^k(x_0) = 0,$$
and \(H_{\partial M} \leq 0\) that
$$\sum_{\alpha=1}^{n-1} g_{\alpha\alpha} \geq \sum_{\alpha=1}^{n-1} g_{\alpha\alpha}.\) Thus
\[(2.9)\]
$$A(R) \geq A(R).$$
This is the only place where we use the mean concavity of boundary and technical hypothesis on \(\eta\) as well. Thus
$$\text{tr}_g(g) < R_c \leq C(1 + \sum_{\alpha=1}^{n-1} |g_{\alpha\alpha}|^2),$$
since (1.3), (2.7), (2.8) and (2.9) imply
\[(2.10)\]
$$f(\lambda(U[u])) = \psi(x, u, \nabla u)$$
(without assumptions on precise dependences of \(u\) and \(\nabla u\) on \(\psi\)), if we further assume
\[(2.11)\]
$$\lim_{t \to +\infty} f(\lambda_1 + t, \cdots, \lambda_{n-1} + t, \lambda_n) = \sup_{\Gamma} f, \; \forall \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma,$$
including \(f(\lambda) = \sum_{i=1}^n \log \lambda_i\) corresponding to Monge-Ampère type equation. Therefore, (2.1) holds for equation (2.10) with (1.2) when \(f\) satisfies (2.11) and the right-hand side depends on \(u\) and \(\nabla u\) as well.

2.4. **Proof of Proposition 2.2.** The proof of Proposition 2.2 is almost parallel to that of Proposition 4.2 in [43] as well as of Proposition 4.7 in [44] for the equations on Hermitian manifolds on which the boundary is assumed to be *holomorphically flat*. In contrast with the rigidity of complex structure of \(M\) and the CR structure on \(\partial M\) as well, the differential structure is soft enough to allow one to flatten boundary and impose (2.3) without any restriction to the second fundamental form of boundary. For completeness we present the proof here. Given \(x_0 \in \partial M\) one has local coordinates
\[(2.12)\]
$$x = (x_1, \cdots, x_n)$$

\[\text{The admissible subsolution satisfying (1.10) is clearly a } C\text{-subsolution for equation (1.1).}\]
with origin at $x_0$ such that $\partial M$ is locally given by $x_n = 0$. Moreover, we can assume $g_{ij}(x_0) = \delta_{ij}$.

We will carry out the computations in such local coordinates, and we set $\nabla_i = \nabla_{\partial \over \partial x_i}$, $\nabla_{ij} = \partial^2_{x_i \partial x_j} - \Gamma^l_{ij} \partial_{x_l}$, with a similar convention for higher derivatives, where $\Gamma^k_{ij}$ are the Christoffel symbols defined by $\nabla_{\partial \over \partial x_i} \partial_{x_j} = \Gamma^k_{ji} \partial_{x_k}$. Under Levi-Civita connection, $\Gamma^l_{ij} = \Gamma^l_{ji}$. By direct computation one has

\begin{equation}
\nabla_{ij} \nabla_k u = {\partial^3 u \over \partial x_i \partial x_j \partial x_k} - \Gamma^l_{ij} {\partial^2 u \over \partial x_x \partial x_l},
\end{equation}

\begin{equation}
\nabla_{ijk} u = {\partial^3 u \over \partial x_i \partial x_j \partial x_k} - \Gamma^l_{ij} {\partial^2 u \over \partial x_x \partial x_l} - \partial \Gamma^l_{ij} \partial_{x_l} u + \Gamma^l_{ik} \nabla_i u + \Gamma^l_{kj} \nabla_i u.
\end{equation}

Under local coordinates (2.12), we take the tangential operator on boundary as

\begin{equation}
\mathcal{T} = \pm \partial_{x_{\alpha}}, \quad 1 \leq \alpha \leq n - 1.
\end{equation}

Let $\mathcal{L}$ be the linearized operator at $u$ of equation (1.5) which is given by

$$\mathcal{L} v = F^{ij} \nabla_{ij} v + F^{ij} \eta^{\ell}_{ij} \nabla_{\ell} v,$$

for $v \in C^2(M)$, where $F^{ij} = \partial F_{\alpha} \over \partial a_{ij}(g[u])$.

First of all, we have the following lemma.

**Lemma 2.5.** Let $u \in C^3(M) \cap C^1(\bar{M})$ be an admissible solution to equation (1.5). There is a positive constant $C$ depending only on $|\varphi|_{C^3(\bar{M})}$, $|\chi|_{C^1(\bar{M})}$, $\psi_{C^1(\bar{M})}$ and other known data (but not on $(\delta_{\psi,f})^{-1}$) such that

\begin{equation}
|\mathcal{L}(\mathcal{T}(u - \varphi))| \leq C \left(1 + (1 + \sup_M |\nabla u|) \sum_{i=1}^{n} f_i + \sum_{i=1}^{n} f_i |\lambda| \right), \quad \text{in } \Omega_{\delta}
\end{equation}

for some small $\delta > 0$. Moreover, if the boundary data $\varphi$ is a constant, then $C$ depends only on $|\chi|_{C^1(M)}$, $|\psi|_{C^1(M)}$ and other known data.

**Proof.** Differentiating equation (1.5) one has

$$F^{ij}(\nabla_{ij} u + \chi_{ij} u + \eta^l_{ij} \nabla_l u + \eta^l_{ij} \nabla_{\ell} u) = \nabla_k \psi.$$

Combining with (2.13) and (2.14) one derives (2.16). \qed

**Construction of barriers and completion of proof of Proposition 2.2.** The proposition can be proved by constructing barrier functions similar to that used in [43, 44, 45] in complex variables, and the construction of this type of barriers goes back at least to [22, 18, 11]. Let’s take

$$\tilde{\psi} = A_1 \sqrt{b_1(u - u)} - A_2 \sqrt{b_1 \rho^2} + A_3 \sqrt{b_1(N \sigma^2 - t \sigma)} + {1 \over \sqrt{b_1}} \sum_{\tau < n} |\nabla_{\tau}(u - \varphi)|^2 + \mathcal{T}(u - \varphi),$$

where $\mathcal{T} = \pm \partial_{x_{\alpha}}$, $1 \leq \alpha \leq n - 1$.
where \( b_1 = 1 + \sup_{M} |\nabla(u - \varphi)|^2 \). In particular, if \( \varphi \) is a constant function, then
\[
\widetilde{\Psi} = A_1 \sqrt{b_1(\varphi - u)} - A_2 \sqrt{b_1 \rho^2} + A_3 \sqrt{b_1} (N\sigma^2 - t\sigma) + \frac{1}{\sqrt{b_1}} \sum_{\tau < n} |\nabla_{\tau} u|^2 + T u.
\]

Let \( \delta > 0 \) and \( t > 0 \) be sufficiently small such that \( N\delta - t \leq 0 \) (where \( N \) is a positive constant sufficiently large to be determined later), \( \sigma \) is \( C^2 \) and
\[
\frac{1}{2} \leq |\nabla \sigma| \leq 2, \quad |L\sigma| \leq C_2 \sum_{i=1}^{n} f_i, \quad |L\rho^2| \leq C_2 \sum_{i=1}^{n} f_i, \quad \text{in } \Omega_{\delta}.
\]

Furthermore, we can choose \( \delta \) and \( t \) small enough such that \( |2N\delta - t| \) and \( t \) are both small.

By straightforward calculation and \( |a - b|^2 \geq \frac{1}{2} |a|^2 - |b|^2 \), one has
\[
L(\sum_{\tau < n} |\nabla_{\tau}(u - \varphi)|^2) \geq \sum_{\tau < n} F^{ij}_{\tau i \tau j} - C_1 \sqrt{b_1} \sum_{i=1}^{n} f_i |\lambda_i| - C_1 \sqrt{b_1} \sum_{i=1}^{n} f_i - C_1 \sqrt{b_1}.
\]

By Proposition 2.19 in [13], there is an index \( r \) such that
\[
\sum_{\tau < n} F^{ij}_{\tau i \tau j} \geq \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2.
\]

By (2.17), (2.18) and Lemma 2.5 we therefore arrive at the following key inequality
\[
L(\widetilde{\Psi}) \geq A_1 \sqrt{b_1} L(\varphi - u) + \frac{1}{2} \sqrt{b_1} \sum_{i \neq r} f_i \lambda_i^2 + A_3 \sqrt{b_1} L(N\sigma^2 - t\sigma) - C_1 - C_1 \sum_{i=1}^{n} f_i |\lambda_i| - (A_2 C_2 + C_1) \sqrt{b_1} \sum_{i=1}^{n} f_i.
\]

Same as the discussion presented in proof of Proposition 4.7 of [44] and Proposition 4.2 in [43] as well, we can prove
\[
L \widetilde{\Psi} \geq 0, \quad \text{in } \Omega_{\delta}
\]
for \( 0 < \delta \ll 1 \), if we appropriately choose \( A_1 \gg A_2 \gg A_3 > 1, N > 1 \) and \( 0 < t \ll 1 \).

The boundary is locally given by \( x_n = 0 \) in local coordinates (2.12), we know that \( T(u - \varphi) = 0 \) and \( \nabla_{\tau}(u - \varphi) = 0 \) on \( \partial M \cap \overline{\Omega}_{\delta}, \forall 1 \leq \tau < n \). Thus
\[
\widetilde{\Psi} = A_1 \sqrt{b_1} (\varphi - u) - A_2 \sqrt{b_1} \rho^2 + A_3 \sqrt{b_1} (N\sigma^2 - t\sigma)
+ \frac{1}{\sqrt{b_1}} \sum_{\tau < n} |\nabla_{\tau}(u - \varphi)|^2 + T(u - \varphi) \leq 0, \quad \text{on } \partial M \cap \overline{\Omega}_{\delta}.
\]

On the other hand, \( \rho = \delta \) and \( u - u \leq 0 \) on \( M \cap \partial \Omega_{\delta} \). Hence, if \( A_2 \gg 1 \) then \( \widetilde{\Psi} \leq 0 \) on \( M \cap \partial \Omega_{\delta} \), where we use again \( N\delta - t \leq 0 \). Therefore \( \widetilde{\Psi} \leq 0 \) in \( \Omega_{\delta} \) by applying
maximum principle. Together with \( \Psi(0) = 0 \), one has \( \nabla_{\nu} \Psi(0) \leq 0 \). Thus

\[
\nabla_{\nu} T(u - \varphi)(0) \leq -A_1 \sqrt{b_1} \nabla_{\nu} (u - u)(0) + A_2 \sqrt{b_1} \nabla_{\nu} (\rho^2)(0)
- A_3 \sqrt{b_1} \nabla_{\nu} (N\sigma^2 - t\sigma)(0)
- \frac{2}{\sqrt{b_1}} \sum_{\tau < n} (\nabla_{\nu} (u - \varphi) \nabla_{\nu} (\nabla_{\tau}(u - \varphi)))(0)
\leq C(1 + \sup_{\partial M} |\nabla(u - \varphi)|)(1 + \sup_M |\nabla u|) \leq C'(1 + \sup_M |\nabla u|).
\]

Here we use (2.5). Therefore

\[
\nabla_{\nu} T(u) = \pm \nabla_{\alpha n} u \leq C(1 + \sup_M |\nabla u|), \text{ at } x_0,
\]

where \( C \) depends only on \( |\varphi|_{C^0(M)}, |u|_{C^2(M)}, |\psi|_{C^1(M)} \) and other known data (but not on \( \sup_M |\nabla u| \)). Moreover, the constant \( C \) in (2.20) does not depend on \((\delta_{\psi,f})^{-1}\).

Remark 2.6. To assure \( \varphi \in C^3(\partial M) \) and the computations (near the boundary) in the proof of Proposition 2.2 (and so of Theorem 1.3) make sense, the regularity of boundary is assumed to be of class \( C^3 \); while it is subtle if boundary data is constant.

2.5. Completion of the proof of Theorem 1.1. In analogy with the Theorem 4.9 proved in [44], if there is an admissible subsolution \( u \in C^2(\bar{M}) \), then for any admissible solution \( u \in C^4(M) \cap C^2(\bar{M}) \) to Dirichlet problem (1.3) and (1.2), there is a uniformly positive constant \( C \) depending only on \( |u|_{C^0(\bar{M})}, |u|_{C^2(\bar{M})}, |\chi|_{C^2(\bar{M})}, |\psi|_{C^2(\bar{M})} \) and other known data (but not on \( (\delta_{\psi,f})^{-1} \)) such that

\[
\sup_M \Delta u \leq C(1 + \sup_M |\nabla u|^2 + \sup_{\partial M} |\Delta u|).
\]

Comparing to that of equation (3.1) involving gradient terms in complex variables studied in [44], the proof of (2.21) is much more simple in real variables. We hence only summarize it but omit the details. The estimation (2.21) for \( \eta = 0 \) in real variables was mentioned in Section 8 of [35] where Székelyhidi assumes the existence of \( C \)-subsolution, and was also proved in the Theorem 1.6 of [14] for Dirichlet problem for more general equations with removing (1.6). With (1.11), (2.21) and \( \text{tr}_g(U[u]) > 0 \) at hand, we establish

\[
\sup_M \Delta u \leq C(1 + \sup_M |\nabla u|^2)
\]

and then apply it to prove gradient bound by using a blow-up argument proposed by [3] and [23, 4] for complex Monge-Ampère equation and complex \( k \)-Hessian equation respectively, and further by Székelyhidi [35] for general complex fully nonlinear elliptic equations. We also refer the reader to Section 8 of [35] for brief discussion on Liouville type theorem and blow-up argument in real case. Evans-Krylov theorem [7, 28] and Schauder theory give \( C^{2,\alpha} \) and higher order regularity.
Indeed, Guan [13] and Guan-Jiao [15] proved a second estimate for a class of fully nonlinear elliptic equations possibly with gradient terms on Riemannian manifolds, however, the bound does not tell one how it precisely depends on the gradient bound which is not enough to apply blow-up argument. Our quantitative boundary estimate enables us to to achieve this goal. Also, for the proof of gradient estimate for Hessian type equations (η = 0) on curved Riemannian manifolds, without using second estimate, please refer for instance to [30, 41, 13] and progress made by Guan (see Theorem 1.6 in current version of [14]).

It would be worthwhile to note that, besides with giving this interesting and different approach to gradient estimate, what the major role of our quantitative boundary estimate (1.4) plays is allowing one to deal with degenerate equations and to impose some regularity assumptions on boundary and boundary data as well, since a priori estimates up to second order among others boundary estimate for second derivatives are all independent of (δω, f)−1. We thus apply approximation to study degenerate equations and finally derive Theorem 1.1 as a complement of the following theorem.

(See the final section of [43] for η = 0 and Remark 7.5 in first version of [44] for the announcement on general η).

**Theorem 2.7** ([13][14]). Let (M, g) be a compact Riemannian manifold with smooth concave boundary. Let ηij = δikζj + δjkζi. In addition to (1.3), (1.4), (1.6), ϕ ∈ C2,1(∂M), ψ ∈ C1,1(M), f ∈ C∞(Γ) ∩ C(Γ) and infM ψ = sup∂M f, we further assume that there is a strictly admissible subsolution u ∈ C2,1(M) with

\[ f(λ(g[u])) ≥ ψ + δ_0 \]

in M, u|∂M = ϕ. Then (1.5) admits a C1,1 (weak) solution u with u|∂M = ϕ, λ(g[u]) ∈ Γ and ∆u ∈ L∞(M).

3. The Dirichlet problem for degenerate equations on certain Kähler manifolds

By assuming that (M, ω) is a compact Kähler manifold with nonnegative orthogonal bisectional curvature and the existence of subsolutions, the author proved (2.21) and gradient estimate in [44], and solved

\[ f(λ(g[u])) = ψ, \ g[u] = χ + \sqrt{−1}\partial\bar{\partial}u + \sqrt{−1}\partial u ∧ η^{1,0} + \sqrt{−1}η^{1,0} ∧ \bar{\partial}u, \]

with possibly degenerate right hand side when ∂M is pseudoconcave of L∂M ≤ 0. Here χ is a smooth real (1, 1)-form and η^{1,0} = η dz\overline{z}\ is a smooth (1, 0)-form.

In this section, with replaced L∂M ≤ 0 by trω′(L∂M) ≤ 0 where ω′ = ω|T∂M\cap JT∂M, we study the following equation possibly with degenerate right hand side

\[ f(λ(W[u])) = ψ in M, \]

where W[u] = χ + (∆u)ω − \sqrt{−1}\partial\bar{\partial}u + Z(∂u, \bar{\partial}u), χ is a smooth real (1, 1)-form, Z(∂u, \bar{\partial}u) = trω(\sqrt{−1}\partial u ∧ η^{1,0} + \sqrt{−1}η^{1,0} ∧ \bar{\partial}u)ω − (√−1∂u ∧ η^{1,0} + √−1η^{1,0} ∧ \bar{\partial}u).

An interesting equation of this type is a Monge-Ampère equation for (n − 1)-plurisubharmonic functions that is connected to the Gauduchon conjecture on closed
astheno-Kähler manifolds first studied by Jost-Yau [27] (i.e. \( \overline{\partial}(\omega^{n-2}) = 0 \)), and complex surfaces are all astheno-Kähler). On such manifolds the Gauduchon conjecture was proved by Cherrier [4] for \( n = 2 \) and later by Tosatti-Weinkove [38] for higher dimensions, while for the Gauduchon conjecture on arbitrary closed Hermitian manifolds without carrying astheno-Kähler metric, the corresponding equation is much more hard to handle. For more references on Form-type Calabi-Yau equation and Monge-Ampère equation for \((n-1)\)-PSH functions, please refer to [8, 9, 37, 36, 32, 16], and also to [44] for Dirichlet problem on compact Hermitian manifolds with holomorphic flat boundary.

The following theorem concludes the existence of weak solutions to Dirichlet problems of degenerate equation (3.2) on certain Kähler manifolds.

**Theorem 3.1.** Let \((M, \omega)\) be a compact Kähler manifold with nonnegative orthogonal bisectional curvature and with smooth boundary of \( \text{tr} \omega'(\mathcal{L}_{\partial M}) \leq 0 \). In addition to (1.3), (1.4) and (1.6), we assume \( \varphi \in C^{2,1}(\partial M) \), \( \psi \in C^{1,1}(M) \), \( f \in C^\infty(\Gamma) \cap C(\overline{\Gamma}) \) and \( \inf_M \psi = \sup_{\partial M} f \). Then equation (3.2) supposes a (weak) solution \( u \in C^{1,\alpha}(\overline{M}) \) \( (\forall 0 < \alpha < 1) \) with \( u|_{\partial M} = \varphi \), \( \lambda(W[u]) \in \Gamma \) and \( \Delta u \in L^\infty(M) \), provided that there is a strictly admissible subsolution \( u \in C^{2,1}(\overline{M}) \). Moreover, if \( \text{tr} \omega'(\mathcal{L}_{\partial M}) < 0 \) then the above statement is still true when we assume \( \partial M \in C^{2,1} \).

**Appendix A. Construction of subsolutions**

In what follows we assume \( \eta = 0 \) and \( \eta^{1,0} = 0 \) for simplicity. The existence of subsolutions is required according to above theorems. Inspired by an idea of [44], on certain topologically product spaces, we can construct strictly admissible subsolutions with \( \nabla \nu u|_{\partial M} < 0 \) for some equations (1.1) and more general (1.5).

**Real variables:**
- **Case I:** \( (M, g) \) is a warped product space \( (X \times (0,1), e^t g_X + dx^n \otimes dx^n) \) for \( \varrho \in C^\infty(\overline{M}) \), \( (\overline{M} = X \times [0,1]) \).
  - For equation (1.5): Suppose there is an admissible function \( w \) with \( \lambda(g[w]) \in \Gamma \) such that
    \[
    \lim_{t \to +\infty} f(\lambda(g[w] + tdx^n \otimes dx^n)) > \psi \text{ in } \overline{M}, \quad w = \varphi \text{ on } \partial M
    \]
    which is automatically satisfied if \( f \) further obeys the unbounded condition
    \[
    (A.1) \quad \lim_{t \to +\infty} f(\lambda_1, \ldots, \lambda_n + t) = \sup_{\Gamma} f, \quad \forall \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma.
    \]
    Then we can construct subsolutions for (1.5) on such warped product spaces, and the subsolution is given by \( \underline{u} = w + A(x_n^2 - x_n) \) for \( A \gg 1 \).
  - For equation (1.1): Similarly, on such warped product spaces, we can construct subsolutions, if
    \[
    \lim_{t \to +\infty} f(\lambda(g[w] + tg_X)) > \psi \text{ in } \overline{M}, \quad w = \varphi \text{ on } \partial M
    \]
holds for an admissible function $w$ with $\lambda(U[w]) \in \Gamma$.

- **Case II**: $(M, g) = (X \times \Omega, g)$ is a product of $(n - k)$-dimensional closed Riemannian manifold $(X, g_X)$ with a bounded smooth domain $\Omega \subset \mathbb{R}^k$, $2 \leq k \leq n$. Also, we denote $g_\Omega = \sum_{j=n-k+1}^n dx^j \otimes dx^j$ and

$$\Gamma_\mathbb{R}^1 = \{ c \in \mathbb{R} : (t, \cdots, t, c) \in \Gamma \text{ for } t \gg 1 \}.$$ 

In particular, if $c > 0$ then $c \in \Gamma_\mathbb{R}^1$.

- For equation (1.5): Suppose furthermore that $\Omega$ is a strictly convex domain. If there is an admissible function $w$ such that

$$\lim_{t \to +\infty} f(g[w] + tg_\Omega)) > \psi \text{ in } \bar{M}, \ w = \varphi \text{ on } \partial M,$$

then the subsolution is given by $u = w + Ah$ for large $A$, where $h$ is a smooth strictly convex function with $h|_{\partial \Omega} = 0$.

- For equation (1.1): We assume $H_{g\Omega} \in \Gamma_\mathbb{R}^1$, $g = e^\rho g_X + g_\Omega$, $\varphi \in C^\infty(M)$, and $f$ further satisfies (1.6).

A somewhat interesting fact is that under such assumptions, we can construct subsolutions with arbitrary $\varphi \in C^2(\partial M)$, if $f$ further satisfies (1.6).

**Complex variables**: $M = X \times N$ is a product of a closed complex manifold $(X, \omega_X)$ of complex dimension $(n - k)$ with a compact complex manifold $(N, \omega_N)$ of complex dimension $k$ with boundary. When $N$ is a compact Riemannian surface with boundary, i.e. $k = 1$, this case was already considered by the author in [44].

Suppose now that $N = \Omega \subset \mathbb{C}^k$ is a bounded smooth domain, as above $2 \leq k \leq n$. Let $\omega_\Omega = \sqrt{-1} \sum_{j=n-k+1}^n dz^j \wedge d\bar{z}^j$ denote the standard metric of $\mathbb{C}^k$.

Similar to Riemannian case, we can construct strictly admissible subsolutions. More precisely,

- For (3.1): Let $\Omega$ be a strictly pseudoconvex domain, $h$ be a smooth strictly pseudoconvex function with $h|_{\partial \Omega} = 0$. The subsolution is given by $u = w + Ah$ for large $A$, provided there is an admissible function $w$ such that

$$\lim_{t \to +\infty} f(g[w] + tg_\Omega)) > \psi \text{ in } \bar{M}, \ w = \varphi \text{ on } \partial M.$$ 

- For (3.2): We assume $\text{tr}(\mathcal{L}_{g\Omega}) \in \Gamma_\mathbb{R}^1$, $\omega = e^\rho \omega_X + e^\rho \omega_\Omega$, for $\varphi \in C^\infty(M)$. Similarity, we can construct subsolutions with arbitrary $\varphi \in C^2(\partial M)$, if $f$ further satisfies (1.6).
References

[1] L. Caffarelli, L. Nirenberg, and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations, III: Functions of eigenvalues of the Hessians, Acta Math. 155 (1985), 261–301.
[2] L. Caffarelli, L. Nirenberg, and J. Spruck, The Dirichlet problem for the degenerate Monge-Ampère equation, Rev. Mat. Iberoamericana 2 (1986), 19–27.
[3] X.-X. Chen, The space of Kähler metrics, J. Differential Geom. 56 (2000), 189–234.
[4] P. Cherrier, Équations de Monge-Ampère sur les variétés hermitiennes compactes, Bull. Sci. Math. 111 (1987), 343–385.
[5] J.-C. Chu, and H.-M. Jiao, Curvature estimates for a class of Hessian type equations, arXiv:2004.05463.
[6] S. Dinew, and S. Kołodziej, Liouville and Calabi-Yau type theorems for complex Hessian equations, Amer. J. Math. 139 (2017), 403–415.
[7] L. C. Evans, Classical solutions of fully nonlinear convex, second order elliptic equations, Comm. Pure Appl. Math. 35 (1982) 333-363.
[8] J.-X. Fu, Z.-Z. Wang, and D.-M. Wu, Form-type Calabi-Yau equations, Math. Res. Lett. 17 (2010), 887–903.
[9] J.-X. Fu, Z.-Z. Wang, and D.-M. Wu, Form-type equations on Kähler manifolds of nonnegative orthogonal bisectional curvature, Calc. Var. Partial Differential Equations 52 (2015), 327–344.
[10] B. Guan, The Dirichlet problem for a class of fully nonlinear elliptic equations, Comm. Partial Differential Equations 19 (1994), 399–416.
[11] B. Guan, The Dirichlet problem for complex Monge-Ampère equations and regularity of the pluri-complex Green function, Comm. Anal. Geom. 6 (1998), 687–703.
[12] B. Guan, The Dirichlet problem for Hessian equations on Riemannian manifolds, Calc. Var. Partial Differential Equations 8 (1999), 45–69.
[13] B. Guan, Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, Duke Math. J. 163 (2014), 1491–1524.
[14] B. Guan, The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds, preprint, arXiv:1403.2133v1.
[15] B. Guan, and H.-M. Jiao, Second order estimates for Hessian type fully nonlinear elliptic equations on Riemannian manifolds, Calc. Var. Partial Differential Equations 54 (2015), 2693–2712.
[16] B. Guan, and X.-L. Nie, Fully nonlinear elliptic equations with gradient terms on Hermitian manifolds, preprint.
[17] B. Guan, C.-H. Qiu, and R.-R. Yuan, Fully nonlinear elliptic equations for conformal deformation of Chern-Ricci forms, Adv. Math. 343 (2019), 538–566.
[18] B. Guan, and J. Spruck, Boundary-value problems on $\mathbb{S}^n$ for surfaces of constant Gauss curvature, Ann. of Math. 138 (1993), 601–624.
[19] P.-F. Guan, $C^2$ a priori estimates for degenerate Monge-Ampère equations, Duke Math. J. 86 (1997), 323–346.
[20] P.-F. Guan, Nonlinear degenerate elliptic equations, Proc. ICCM 2001. Edited by C. S. Lin, L. Yang and S. T. Yau, International Press, (2004), 257–266.
[21] P.-F. Guan, N. Trudinger, and X.-J. Wang, On the Dirichlet problem for degenerate Monge-Ampère equations, Acta Math. 182 (1999), 87–104.
[22] D. Hoffman, H. Rosenberg, and J. Spruck, Boundary value problems for surfaces of constant Gauss curvature, Comm. Pure Appl. Math. 45 (1992), 1051–1062.
[23] Z.-L. Hou, X.-N. Ma, and D.-M. Wu, A second order estimate for complex Hessian equations on a compact Kähler manifold, Math. Res. Lett. 17 (2010), 547–561.
[24] K. Irie, F. Marques, and A. Neves, *Density of minimal hypersurfaces for generic metrics*, Ann. of Math. **187** (2018), 963–972.

[25] N. M. Ivochkina, *The integral method of barrier functions and the Dirichlet problem for equations with operators of the Monge-Ampère type*, (Russian) Mat. Sb. (N.S.) **112** (1980), 193–206; English transl.: Math. USSR Sb. **40** (1981), 179–192.

[26] N. M. Ivochkina, N. Trudinger, and X.-J. Wang, *The Dirichlet Problem for degenerate Hessian equations*, Comm. Partial Differential Equations **29** (2004), 219–235.

[27] J. Jost, and S.-T. Yau, *A nonlinear elliptic systems for maps from Hermitian to Riemannian manifolds and rigidity theorem in Hermitian geometry*, Acta Math. **170** (1993), 221–254.

[28] N. V. Krylov, *Boundedly nonhomogeneous elliptic and parabolic equations in a domain*, Izvestia Math. Ser. **47** (1983), 75–108.

[29] Yangyang Li, *Existence of infinitely many minimal hypersurfaces in higher-dimensional closed manifolds with generic metrics*, preprint, arXiv:1901.08440.

[30] Y.-Y. Li, *Some existence results of fully nonlinear elliptic equations of Monge-Ampère type*, Comm. Pure Appl. Math. **43** (1990), 233–271.

[31] Y.-Y. Li, and K. Zhang, *Boundary pointwise $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity for fully nonlinear elliptic equations*, preprint, arXiv:1901.06060 to appear in JDE.

[32] D. Popovici, *Aeppli cohomology classes associated with Gauduchon metrics on compact complex manifolds*, Bull. Soc. Math. France **143** (2015), 763–800.

[33] L. Silvestre, and B. Sirakov, *Boundary regularity for viscosity solutions of fully nonlinear elliptic equations*, Comm. Partial Differential Equations **39** (2014), 1694–1717.

[34] A. Song, *Existence of infinitely many minimal hypersurfaces in closed manifolds*, preprint, arXiv:1806.08816.

[35] G. Székelyhidi, *Fully non-linear elliptic equations on compact Hermitian manifolds*, J. Differential Geom. **109** (2018), 337–378.

[36] G. Székelyhidi, V. Tosatti, and B. Weinkove, *Gauduchon metrics with prescribed volume form*, Acta Math. **219** (2017), 181–211.

[37] V. Tosatti, and B. Weinkove, *The Monge-Ampère equation for $(n-1)$-plurisubharmonic functions on a compact Kähler manifold*, J. Amer. Math. Soc. **30** (2017), 311–346.

[38] V. Tosatti, and B. Weinkove, *Hermitian metrics, $(n-1,n-1)$ forms and Monge-Ampère equations*, J. Reine Angew. Math. **755** (2019), 67–101.

[39] N. Trudinger, *On the Dirichlet problem for Hessian equations*, Acta Math. **175** (1995), 151–164.

[40] N. Trudinger, and J. Urbas, *On second derivative estimates for equations of Monge-Ampère type*, Bull. Aust. Math. Soc. **30** (1984), 321–334.

[41] J. Urbas, *Hessian equations on compact Riemannian manifolds*, in Nonlinear problems in mathematical physics and related topics, II, vol. 2 of Int. Math. Ser. (N. Y.), Kluwer/Plenum, New York, 2002, pp. 367–377.

[42] X.-J. Wang, *Regularity for Monge-Ampère equation near the boundary*, Analysis **16** (1996), 101–107.

[43] R.-R. Yuan, *Regularity of fully non-linear elliptic equations on Hermitian manifolds*, preprint.

[44] R.-R. Yuan, *Regularity of fully non-linear elliptic equations on Hermitian manifolds. II*, preprint, arXiv:2001.09238.

[45] R.-R. Yuan, *Regularity of fully non-linear elliptic equations on Kähler cones*, preprint.

E-mail address: rirongyuan@stu.xmu.edu.cn