Maximizing the Mutual Information of Multi-Antenna Links Under an Interfered Receiver Power Constraint

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Abstract

Single-user multiple-input / multiple-output (SU-MIMO) communication systems have been successfully used over the years and have provided a significant increase on a wireless link’s capacity by enabling the transmission of multiple data streams. Assuming channel knowledge at the transmitter, the maximization of the mutual information of a MIMO link is achieved by finding the optimal power allocation under a given sum-power constraint, which is in turn obtained by the water-filling (WF) algorithm. However, in spectrum sharing setups, such as Licensed Shared Access (LSA), where a primary link (PL) and a secondary link (SL) coexist, the power transmitted by the SL transmitter may induce harmful interference to the PL receiver. While such co-existing links have been considered extensively in various spectrum sharing setups, the mutual information of the SL under a constraint on the interference it may cause to the PL receiver has, quite astonishingly, not been evaluated so far. In this paper, we solve this problem, find its unique optimal solution and provide the power allocation policy and corresponding precoding solution that achieves the optimal capacity under the imposed constraint. The performance of the optimal solution and the penalty due to the interference constraint are evaluated over some indicative Rayleigh fading channel conditions and interference thresholds. We believe that the obtained results are of general nature and that they may apply, beyond spectrum sharing, to a variety of applications that admit a similar setup.

1 Introduction

Nowadays, wireless network operators are struggling with the rapidly growing volume of the mobile data traffic, [1], [2]. The exponential increase of
the mobile traffic is expected to continue in an even more abrupt pace in the 5th generation (5G) era. Given the shortage and the consequent high acquisition cost of the spectral resources, several techniques that improve the spectral efficiency (SE) of cellular systems have been proposed.

Spectrum sharing methods constitute a representative example. Initially, the community was focused on opportunistic spectrum access (OSA), where cognitive radio (CR) nodes sense the spectrum and transmit over detected idle channels, [3]. However, the lack of provision of any quality-of-service (QoS) guarantees for the CR secondary systems (SS) and the inability to ensure interference-free operation for the legacy primary systems (PS) prohibited the utilization of this paradigm in practice. Recently, a new approach that addresses these issues, called Licensed Shared Access (LSA), has emerged. LSA is based on commonly agreed spectrum usage rules between the interested parties and the use of a shared database that stores spectrum occupancy information [4]. Lately, a dynamic LSA variant that combines spectrum sensing with conventional LSA in order to enable more aggressive spectrum sharing has been proposed, [5].

Multiple-input multiple-output (MIMO) communication methods enhance also the capacity of a system by enabling the transmission of multiple data streams destined to a single user or a group of active users on the same time-frequency resource, [6]. The capacity of open-loop (OL) and closed-loop (CL) single-user MIMO (SU-MIMO) systems has been derived in the middle 1990’s, [7], [8]. In the latter case, it has been proven that the use of singular value decomposition (SVD) based pre- and post-coding is required, in order to reach the capacity. Then, the mutual information maximization problem reduces to one of finding the optimal power allocation under a sum-power constraint. The optimal power levels are obtained via the water-filling (WF) algorithm, see [9]. A WF power allocation solution also maximizes the mutual information in CL SU-MIMO systems operating under interference, assuming that the TX knows perfectly the interference plus additive white Gaussian noise (AWGN) covariance matrix, see [10], as well as the sum-rate (SR) throughput in multi-user MIMO (MU-MIMO) setups, [6], and cooperative multi-cell MIMO (Co-MC-MIMO) paradigms, [11], for given linear precoding and user selection schemes.

The lesson learned from dynamic LSA and Co-MC-MIMO is that the cooperation between the interested parties allows for efficient non-orthogonal spectrum sharing. This can be achieved via the use of beamforming / precoding, power allocation or/and power control techniques at the SS. The objective is to maximize the mutual information / SR throughput of the SS under some given power constraint, while at the same time ensuring that the interference incurred at the PS does not exceed some predefined threshold, see [12]. While this problem has been studied for various multi-user and multi-cell setups in [12], [11], the use case of SU-MIMO links has not been considered under this context in the literature, to the best of our knowledge,
despite its significance.

In this work, we compute the mutual information for the link of the secondary user (SL) under an interfered receiver power constraint, which is imposed by the link of the primary user (PL). Moreover, we provide a power allocation algorithm and also the precoding technique that achieves the optimal capacity. The performance of the system is evaluated for various interference thresholds and an insight on the effect of this parameter is provided.

We should note that the problem under study finds a wide range of applications. For instance, due to the enormous backhaul capacity requirements of 5G systems, the use of SU-MIMO in terrestrial point-to-point (PTP) backhaul links and the spectrum sharing between terrestrial and satellite backhaul links has been proposed [13].

2 Signal Model and Problem Formulation

In this paper, we consider a MIMO setup in which two different entities share the same resources, such as in a typical spectrum sharing setup. The system architecture is depicted in Figure 1. Thus, having a Primary Link (PL), TX_p − RX_p and a Secondary Link, TX_s − RX_s, we model the signals from each link, by taking into account the possible interference that is caused from the cross channels. Our goal is to find the mutual information of the SL, under an additional interference constraint, which is imposed by the PS and provides QoS guarantees to its receiver. As in standard MIMO setups, we assume that each transmitter / receiver is equipped with an antenna array of multiple antennas. Let the PL consist of k elements for the transmitter and ℓ for the receiver, while the SL consists of m elements for the transmitter and n for its receiver. The received signals at both SL and PL are modeled as:

\[ y_s = H_s s + H_{ps} x + \eta, \]  
\[ y_p = H_p x + H_{sp} s + v, \]

respectively\[1\]. The transmitted signals for the SL and the PL are denoted as \( s \in \mathbb{C}^m \) and \( x \in \mathbb{C}^k \), respectively, and are zero mean complex Gaussian (uncorrelated); the channel gain from the \( i \)-th transmitter to the \( j \)-th receiver element is denoted as \( h_{ij} \), thus, for the channels of each link of Figure 1 we have: \( H_s \in \mathbb{C}^{n \times m}, H_p \in \mathbb{C}^{\ell \times k}, H_{sp} \in \mathbb{C}^{\ell \times m} \) and \( H_{ps} \in \mathbb{C}^{n \times k} \) and are assumed fixed and frequency flat. We have also considered that \( v \sim \mathcal{CN}(0, I_\ell) \) and \( \eta \sim \mathcal{CN}(0, I_n) \) are additive white circularly complex Gaussian noise processes. Both the signals and the noise are assumed uncorrelated with each other.

\[^1\text{The dependency of the signals and random variables over time are omitted for simplicity.}\]
Let $z = H_{ps}x + \eta$. According to (1), it can be readily seen that the covariance matrix of the signal received by the SL is:

$$R_{ys} := E\left\{y_s y_s^\dagger\right\} = H_s R_s H_s^\dagger + R_z,$$

where $R_s$ is the covariance matrix of the SL’s transmitted signal; $R_z$ is the covariance matrix of the vector $z$, i.e., $R_z = H_{ps} R_s H_{ps}^\dagger + I_n$, and $R_s$ is the covariance matrix of the primary’s signal, which we assume known.

The maximum mutual information of the SL (disregarding any constraint on the interference caused to RX$_p$), see [14], is given by:

$$I(y_s; z) = \log_2 \det (\pi e R_{ys}) - \log_2 \det (\pi e R_z)$$

for $n \geq m$. At this point we consider the eigendecomposition of matrix $H_s^\dagger R_z^{-1} H_s$, i.e.,

$$H_s^\dagger R_z^{-1} H_s = U \Lambda U^\dagger,$$

where $U$ is a unitary matrix and $\Lambda$ is the diagonal matrix with the eigenvalues. Therefore, by imposing the SL’s transmitted signal to be of the form $s = Us_w$, where $s_w$ is spatially white, leads to $R_s = E(s s^\dagger) = U D U^\dagger$, where $D = E(s_w s_w^\dagger)$ is diagonal. Thus, (3) is simplified to:

$$I(y_s; z) = \log_2 \det (I_m + \Lambda D).$$

Hence, the standard mutual information maximization task for the SL’s transmitted signal is given by:

$$\max_D \log_2 \det (I_m + \Lambda D)$$

subject to

$$D \succeq 0,$$

$$\text{tr}(D) \leq 1,$$
where without loss of generality (avoiding an equivalent normalization) we have considered that the maximum transmission power of the SL’s MIMO antenna array is 1. The optimization task in (6a)-(6c) obtains the standard water-filling solution\(^2\) which is given by:

\[ d_i = (\rho - \lambda_i^{-1})^+, \ i = 1, \ldots, m, \]  

(7)

where \(\rho\) is the water-level chosen to satisfy the power constraint with equality, i.e., \(\sum_{i=1}^{m} d_i = 1\).

However, in the presence of the PL, i.e., \(TX_p - RX_p\), computing the optimum value for the transmission power in (7) does not take into account the interference that will be caused to the PL. In order to avoid causing excessive interference to the PL’ receiver, \(RX_p\), an additional constraint should be satisfied, which can be expressed in view of (2) as:

\[ \text{tr} \left( H_{sp}R_sH_{sp}^\dagger \right) = \text{tr} \left( \tilde{H}_{sp}D\tilde{H}_{sp}^\dagger \right) \leq P_I, \]  

(8)

where \(\tilde{H}_{sp} = H_{sp}U\) and \(P_I > 0\) is the maximum value of interference that is tolerable to the PL receiver due to \(TX_s\). Thus, our goal now is to find a solution for (6a)-(6c) under the additional constraint in (8).

3 Derivation of the Solution and algorithm

By considering only the positive eigenvalues of \(\Lambda\), i.e., \(\text{diag}(\lambda_1, \ldots, \lambda_r)\), where \(r\) is the rank of the decomposed matrix \(\Lambda\), the cost function in (6a) and (6b) can be further simplified and thus the new optimization power allocation water-filling with the interference constraint task is now formulated as:

\[ \begin{align*}
\text{minimize} & \quad -\sum_{i=1}^{r} \log_2 (1 + \lambda_i d_i) \\
\text{subject to} & \quad d_i \geq 0, \ i = 1, \ldots, r, \\
& \quad \sum_{i=1}^{r} d_i \leq 1, \\
& \quad \sum_{i=1}^{r} \alpha_i d_i \leq P_I,
\end{align*} \]  

(9a)-(9d)

where \(\alpha_i = \left\| \tilde{h}_i \right\|_2^2, \ i = 1, \ldots, r\) is the squared norm of the column vectors of matrix \(\tilde{H}_{sp}\). The objective function in (9a) is convex and the constraints in (9b)-(9d) define a polyhedron, as demonstrated in Figure 2 for \(r = 2\). Thus, the optimization task we are attempting to solve is convex and hence it attains a unique minimum.

\(^2\)The task can be equivalently transformed to a convex optimization one (since the cost function is concave), thus a unique solution exists.
3.1 Optimality Conditions

For the solution of this convex optimization task we use the Karush-Kuhn-Tucker (KKT) conditions (also known as optimality conditions), see \([15, 16]\). In order to maximize the capacity, (9c) should be satisfied with equality. Let \(\nu\) denote the Lagrange multiplier corresponding to the constraint of (9c), \(\mu\) the Lagrange multiplier corresponding to the constraint of (9d) and \(\xi_1, \ldots, \xi_r\) denote the Lagrange multipliers corresponding to the constraints that force the powers to be positive. The Lagrangian form for the solution of the optimization task (9a)-(9d) is:

\[
L(d_i; \nu, \mu, \xi_i) = -\sum_{i=1}^{r} \log_2 \left(1 + \lambda_i d_i \right) + \nu \left( \sum_{i=1}^{r} d_i - 1 \right) + \\
+ \mu \left( \sum_{i=1}^{r} \alpha_i d_i - P_I \right) - \sum_{i=1}^{r} \xi_i d_i, \tag{10}
\]

where \(\nu, \mu, \xi_i\), \(i = 1, \ldots, r\) are the Lagrange multipliers associated to the constraints. Hence, at the minimum we imply that (9b), (9d) hold, as well as:

\[
\xi_i \geq 0, \text{ for all } i = 1, \ldots, r \tag{11a}
\]
\[
\xi_i d_i = 0, \text{ } i = 1, \ldots, r \tag{11b}
\]
\[
\sum_{i=1}^{r} d_i = 1, \tag{11c}
\]
\[
\mu \geq 0 \tag{11d}
\]
\[
\mu \left( \sum_{i=1}^{r} \alpha_i d_i - P_I \right) = 0 \tag{11e}
\]
\[
\frac{\lambda_i \log_2 e}{(1 + \lambda_i d_i)} + \xi_i = \nu + \mu \alpha_i, \text{ } i = 1, \ldots, r \tag{11f}
\]
By observing (11c) we first notice that $\nu + \mu \alpha_i > 0$, since $\lambda_i > 0$.

3.2 Solution

Next, we provide the solution to the power allocation task under the interference constraint.

**Restriction 1**: From (11b), it is observed that if $d_i > 0$, then $\xi_i = 0$. Thus, according to (11d) and (11f) we have the restriction that $\log_2 e/\left(\lambda_i^{-1} + d_i\right) - \nu \geq 0$, which leads to $\lambda_i^{-1} < \log_2 e/\nu$.

**Restriction 2**: If $\lambda_i^{-1} \geq \log_2 e/\nu$, then from the derived inequality (Restriction 1) we obtain that $d_i = 0$.

Thus, the derived solution of the first stage is given by \[ \rho = \log_2 e/\nu \] and $m = r$. Next, we should differentiate between the two following cases in the power allocation:

- **Case 1**: If $\sum_{i=1}^{r} a_i \left(\frac{\log_2 e}{\nu} - \frac{1}{\lambda_i}\right)^+ \leq P_I$, then the power allocation $d_i = \left(\frac{\log_2 e}{\nu} - \frac{1}{\lambda_i}\right)^+$, $i = 1, \ldots, r$ is a valid solution that satisfies all the KKT conditions. It should also be noted that in this case $\mu = 0$.

- **Case 2**: If $\sum_{i=1}^{r} a_i \left(\frac{\log_2 e}{\nu} - \frac{1}{\lambda_i}\right)^+ > P_I$, then $\mu > 0$ and thus according to (11e) we have $\sum_{i=1}^{r} a_i d_i = P_I$. Thus, two options exist:

  - (a) : If $\lambda_i^{-1} \geq \log_2 e/(\nu + \mu \alpha_i)$, then $d_i = 0$. This is proved by contradiction, since if we assume $d_i > 0$ it would lead to $\xi_i = 0$ and thus $\lambda_i^{-1} < \log_2 e/(\nu + \mu \alpha_i)$.

  - (b) : If $\lambda_i^{-1} < \log_2 e/(\nu + \mu \alpha_i)$, then $\xi_i = 0$. This can also be proved by contradiction. Let’s instead assume that $\xi_i > 0$. Thus, from (11b) $d_i = 0$ and according to (11f) leads to $\lambda_i^{-1} > \log_2 e/(\nu + \mu \alpha_i)$, which is a contradiction.

Summarizing Case 2, the solution of the second equality is given by:

$$d_i = \left(\frac{\log_2 e}{\nu + \mu \alpha_i} - \frac{1}{\lambda_i}\right)^+, \quad i = 1, \ldots, r,$$

(12)

where $\mu$ is obtained from

$$\sum_{i=1}^{r} a_i \left(\frac{\log_2 e}{\nu + \mu \alpha_i} - \frac{1}{\lambda_i}\right)^+ = P_I.$$  

(13)

It should be noted that the solution to the above equality cannot be obtained in closed form; however, it can be solved iteratively. Existence and uniqueness of the solution is proved in Section 3.3.

According to the aforementioned analysis, we provide the following expression for the solution of our convex optimization task.
Theorem 1  The solution to the optimization task \((9a)-(9d)\) is:

\[
d_i = \begin{cases} 
\left(\frac{\log_2 e}{\nu} - \frac{1}{\lambda_i}\right)^+, & \text{if } \sum_{i=1}^r \alpha_i \left(\frac{\log_2 e}{\nu} - \frac{1}{\lambda_i}\right)^+ \leq P_I, \\
\left(\frac{\log_2 e}{\nu + \mu \alpha_i} - \frac{1}{\lambda_i}\right)^+, & \text{otherwise}
\end{cases}
\]  \hspace{1cm} (14)

where the Lagrange multipliers are obtained from a two stage procedure. First, \(\nu\) is obtained by solving \((11c)\) and, if required, \(\mu\) is obtained by solving \((13)\) with the \(\nu\) that is obtained from the first stage.

It should be noted that for the first case the value \(\rho = \log_2 e/\nu\) can be interpreted as the standard water level of the water-filling method. However, for the second case, the initial water level violates the second condition, i.e., \((9d)\) and the initial water level is penalized by the term \(\mu \alpha_i\), which is different for each channel, since it depends on \(\alpha_i\)'s. Moreover, it can be readily seen that, for the new power level and the \(\nu\) obtained at the first stage, \(\sum_{i=1}^r d_i < 1\), for any \(\mu > 0\).

3.3 Algorithm

The established iterative scheme for the power allocation task under PL receiver interference constraint is presented in Algorithm 1. It should be noted that this is a generic method, whose standard WF algorithmic part is only a special case. Thus, the case of greater interesting, is when the interference constraint is not satisfied, i.e., the bottom case of \((14)\).

At the first stage, the algorithm computes a \(\nu\), which is related to a specific water level, according to the standard WF solution. At the second stage, a decision is taken; the derived solution can either satisfy the interference power constraint or not. In the latter case, given the \(\nu\) that is already computed, the algorithm computes a \(\mu > 0\) from the solution of \((13)\), which is equivalent to obtaining the root of the following function:

\[
g_p(\mu) := \sum_{i=1}^{r-p+1} \frac{\alpha_i}{\nu + \mu \alpha_i} - \gamma_p,
\]  \hspace{1cm} (15)

for \(p = 1, \ldots, r\), where \(\gamma_p\) is given in the 5-th row of Algorithm 1. At this point one should notice that the function \(g_p\) is strictly decreasing for \(\mu \geq 0\). Moreover, \(g_p(0) > 0\) (Case 2 of Section 3.2) and \(\lim_{\mu \to \infty} g_p(\mu) = -\gamma_p < 0\). Thus, \(g_p(\mu) = 0\) has a unique solution for every \(\nu\) obtained from the first stage of the algorithm, which can be derived via an iterative method, such as the bisection or the Newton’s method.

3.4 Precoding Technique and Power Allocation Strategy

We should note that, along with matrix \(U\), which is obtained from the eigendecomposition in (4), we also know how to precode in order to achieve the capacity computed by Algorithm 1.
Algorithm 1 Interference Constraint Water-Filling

1: procedure ICWF(λᵢ, αᵢ, Pᵢ)
2:  \[ dᵢ = \left( \log_{2} e - \frac{1}{\lambdaᵢ} \right) \]
3:  where \( \nu \) is obtained from
4:  \[ rᵢ = \frac{1}{dᵢ} \]
5:  \[ \text{if } \sumᵢₐᵢdᵢ > Pᵢ \text{ then } p \leftarrow 1 \]
6:  \[ \text{while } p \leq r \text{ do} \]
7:  \[ \gamma_p = \left( Pᵢ + \sumᵢ₋p+1 \left( \frac{αᵢ}{λᵢ} \right) \right) / \log_{2} e \]
8:  \[ \text{Find } μ \text{ as the solution of } g_p(μ) = 0 \text{ in (15)} \]
9:  \[ \text{Compute: } dᵢ = \left( \log_{2} e - \frac{1}{μ \lambdaᵢ} \right) \]
10:  \[ p \leftarrow p + 1 \]
11: \text{Output: } dᵢ, \text{ for } i = 1, \ldots, r

4 Experimental Evaluation

For the evaluation of the derived power allocation technique, we perform the following experiment. We consider two 3×3 MIMO links, one for the primary and another one for the secondary user \((n, m, k, ℓ = 3)\) and attempt to maximize the SL’s capacity. The channels between the direct and the cross links are assumed Rayleigh fading. For each value of interference constraint, \(Pᵢ\), we perform 10000 Monte Carlo (independent) runs and average the maximum achieved capacity. Due to the chosen normalization, we have considered \(P = 1\); however, if the sum-power constraint was chosen equal to \(P \neq 1\) one should measure the capacity for different values of the ratio \(Pᵢ/P\).

In Figure 3a, we have evaluated the maximum capacity for various values of interference constraint \(Pᵢ\). The solid line corresponds to the power allocation under the PL interference constraint, which is achieved via Algorithm 1 and the dashed one to the power allocation without the interference constraint \((Pᵢ = ∞)\). Moreover, in Figure 3b, we have computed the percentage of capacity loss that is caused by the interference constraint imposed by the primary system. It is clear that a tighter constraint translates to a greater penalty.

Finally, in Figure 4, we present the empirical cumulative distribution function’s (CDF’s) for capacities achieved with various interference levels \(Pᵢ\)’s. The dashed line corresponds to the unconstrained power allocation. It is observed that the CDF’s for small \(Pᵢ\) are far from the ideal case of the unconstrained task.

5 Summary and Conclusions

In this work, a power allocation strategy and its precoding technique has been proposed for the spectrum sharing scenario of a primary and a secondary user MIMO link. The problem is formulated and solved via convex
optimization techniques. The derived algorithm maximizes the mutual information, while it satisfies the interference constraint. The evaluation of the method is performed over a set of experiments, where it is shown that the level of the interference constraint determines the loss on the SL’s capacity. Due to its large potential in various applications, we believe that this study plays an important role in the adoption of spectrum sharing techniques in next generation communications networks.
Figure 4: CDF’s of a $3 \times 3$ MIMO link for different capacity values, which correspond to interference levels $P_I$. The dashed line corresponds to the case where no interference constraint exists.

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References

[1] “Cisco Visual Networking Index: Global Mobile Data Traffic Forecast Update 2014-2019,” Cisco, White Paper, February 2015.

[2] Ericsson, “Mobility Report,” Tech. Rep., June 2016.

[3] A. Khattab, D. Perkins, and M. A. Bayoumi, “Opportunistic spectrum access: From theory to practice,” IEEE Vehicular Technology Magazine, vol. 7, no. 2, pp. 62–68, 2012.

[4] Nokia and Qualcomm, “Authorised Shared Access (ASA) - An Evolutionary Spectrum Authorisation Scheme For Sustainable Economic Growth And Consumer Benefit,” Input Document FM(11)116, 72nd Meeting of the WG FM, May 2011.

[5] A. Morgado et al., “Dynamic LSA for 5G networks: The ADEL perspective,” in 24th European Conference on Networks and Communication (EuCNC 2015), Paris, France, July 2015, pp. 190–194.

[6] H. Huang, C. B. Papadias, and S. Venkatesan, MIMO Communication for Cellular Networks. Springer Science & Business Media, 2011.
[7] G. J. Foschini, “Layered Space-Time Architecture for Wireless Communication in Fading Environments When Using Multi-Element Antennas,” Bell Labs, Tech. Rep., 1996.

[8] E. Telatar, “Capacity of Multi-Antenna Gaussian Channels,” *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585–596, November 1999.

[9] R. G. Gallager, *Information theory and reliable communication*. Springer, 1968, vol. 2.

[10] D.-S. Shiu, G. J. Foschini, M. J. Gans, and J. M. Kahn, “Fading correlation and its effect on the capacity of multielement antenna systems,” *IEEE Transactions on communications*, vol. 48, no. 3, pp. 502–513, 2000.

[11] E. Bjornson and E. Jorswieck, “Optimal resource allocation in coordinated multi-cell systems,” *Foundations and Trends in Communications and Information Theory*, vol. 9, no. 2-3, pp. 113–381, January 2013.

[12] Advanced Dynamic Spectrum 5G mobile networks Employing Licensed shared access. [Online]. Available: http://www.fp7-adel.eu/

[13] Shared Access Terrestrial-Satellite Backhaul Network enabled by Smart Antennas. [Online]. Available: sansa-h2020.eu

[14] A. Paulraj, R. Nabar, and D. Gore, *Introduction to space-time wireless communications*. Cambridge university press, 2003.

[15] S. Theodoridis, *Machine learning: a Bayesian and optimization perspective*. Academic Press, 2015.

[16] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.