Abstract. A continuum limit of the Toda lattice field theory, called the SDiff(2) Toda equation, is shown to have a Lax formalism and an infinite hierarchy of higher flows. The Lax formalism is very similar to the case of the self-dual vacuum Einstein equation and its hyper-Kähler version, however now based upon a symplectic structure on a cylinder $S^1 \times \mathbb{R}$. An analogue of the Toda lattice tau function is introduced. The existence of hidden SDiff(2) symmetries are derived from a Riemann-Hilbert problem in the SDiff(2) group. Symmetries of the tau function turn out to have commutator anomalies, hence give a representation of a central extension of the SDiff(2) algebra.
1. Introduction

Two dimensional Toda fields on an infinite chain, \( \phi_i, \ i \in \mathbb{Z} \), are described by the nonlinear field equation

\[
\partial_z \partial_{\bar{z}} \phi_i + \exp(\phi_{i+1} - \phi_i) - \exp(\phi_i - \phi_{i-1}) = 0
\]  

(1)

or, equivalently, for \( \varphi_i = \phi_i - \phi_{i-1} \) by

\[
\partial_z \partial_{\bar{z}} \varphi_i + \exp \varphi_{i+1} + \exp \varphi_{i-1} - 2 \exp \varphi_i = 0.
\]

(2)

In continuum limit as lattice spacing tends to 0, \( \phi_i \) and \( \varphi_i \) will become three dimensional fields \( \phi = \phi(z, \bar{z}, s) \) and \( \varphi = \partial \phi / \partial s \) with an extra space variable \( s \in \mathbb{R} \). The equation of motions are then given by

\[
\partial_z \partial_{\bar{z}} \phi + \partial_s \exp \partial_s \phi = 0
\]

(3)

and

\[
\partial_z \partial_{\bar{z}} \varphi + \partial_s^2 \exp \varphi = 0.
\]

(4)

From this observation, Saveliev and Vershik [1] introduced the notion of Cartan operators and corresponding (continual) Lie algebras, and applied them to solving the above three dimensional equations. Bakas [2] and Park [3] gave another interpretation of these equations in the language of extended conformal symmetries (area-preserving diffeomorphisms, \( w_\infty \) algebras, etc.). Since the group of area-preserving (i.e., symplectic) diffeomorphisms, \( \text{SDiff}(2) \), plays a key role in the latter point of view, let us call the equation of motion of the \( \phi \) and \( \varphi \) fields the \( \text{SDiff}(2) \) Toda equation. The original Toda equation, in this sense, may be called the \( \text{GL}(\infty) \) Toda equation.

In fact, there are further two sources of the \( \text{SDiff}(2) \) Toda equation. One is discovered by relativists, Boyer and Finley [4] and Gegenberg and Das [5], in the study of \( \mathcal{H} \)-spaces (heavens) with a rotational Killing symmetry. Another source is Einstein-Weyl
geometry studied by twistor people [6] [7] [8] [9] in the context of curved minitwistor spaces.

The contents of this letter are influenced by all these approaches, but mostly based upon the method of Kyoto group [10] [11] originally developed for soliton equations. A direct prototype is the theory of the Toda lattice (TL) hierarchy [12] [13] that was proposed as a Toda lattice version of the KP hierarchy. Our aim is to give a similar framework to study the SDiff(2) Toda equation.

We first show a Lax formalism along with a set of higher commuting flows (hierarchy), secondly, introduce an analogue of the tau functions of the TL hierarchy, and finally, with the aid of a Riemann-Hilbert problem in the SDiff(2) group, construct SDiff(2) symmetries of the hierarchy explicitly. A remarkable result is that these symmetries exhibit commutator anomalies at the level of the tau function, hence the true symmetry algebra should be a central extension of the original SDiff(2) algebra. Apart from the treatment of the tau function, technical details are mostly the same as in the case of the self-dual vacuum Einstein equation [14]. Unlike that case, however, the spectral parameter $\lambda$ now plays the role of a true variable: $\lambda$ and $s$ form a coordinate system on a cylinder $S^1 \times \mathbf{R}$ on which we consider a symplectic structure and associated symplectic diffeomorphisms.

2. LAX FORMALISM – HEURISTIC CONSIDERATION

A “zero-curvature representation” of the SDiff(2) Toda equation is given by

$$\partial_z \mathcal{B} - \partial_{\bar{z}} \bar{\mathcal{B}} + \{ \mathcal{B}, \bar{\mathcal{B}} \} = 0,$$

where $\{ \ , \ \}$ stands for the Poisson bracket

$$\{ F, G \} \overset{\text{def}}{=} \lambda (\partial_\lambda F)(\partial_s G) - \lambda (\partial_\lambda G)(\partial_s F)$$

and $\mathcal{B}$ and $\bar{\mathcal{B}}$ are given by

$$\begin{align*}
\mathcal{B} &= \lambda \exp[(-\alpha + \frac{1}{2})\partial_s \phi] + (\alpha + \frac{1}{2})\partial_z \phi, \\
\bar{\mathcal{B}} &= \lambda^{-1} \exp[(\alpha + \frac{1}{2})\partial_s \phi] + (\alpha - \frac{1}{2})\partial_z \phi.
\end{align*}$$
Here $\alpha$ is a gauge parameter that also arises in the TL hierarchy [13]; we mostly consider the $\alpha = 1/2$ gauge. The Poisson bracket is obviously related with area-preserving diffeomorphisms on a cylinder. This zero-curvature equation is already proposed by Kashaev et al. [15].

We note that similar zero-curvature representations take place in the case of the self-dual vacuum Einstein equation [14]. The only difference is that $\lambda$ is no more a parameter (spectral parameter) but a true variable. Bearing this difference in mind, we now modify the framework developed therein so as to fit into the present setting. We first introduce an exterior differential 2-form as

$$\omega = \frac{d\lambda}{\lambda} \wedge ds + dB \wedge dz + \bar{B} \wedge d\bar{z},$$

(8)

where $d$ here (and from now on) stands for total differentiation with respect to the space-time variables $z, \bar{z}, s$ and the spectral variable $\lambda$. This 2-form is obviously a closed form,

$$d\omega = 0,$$

(9)

and satisfies the algebraic relation

$$\omega \wedge \omega = 0.$$  

(10)

The latter is equivalent to zero-curvature equation (5). These two relations imply the existence of two functions $P$ and $Q$ that give a pair of “Darboux coordinates” as

$$\omega = \frac{dP}{P} \wedge dQ.$$  

(11)

(Note that the Poisson bracket \{ , \} is twisted by an extra factor $\lambda$. The presence of the denominator $P$ is due to this fact.) From (11) one can deduce the Lax equations

$$\partial z P = \{ B, P \}, \quad \partial \bar{z} P = \{ \bar{B}, P \},$$

$$\partial z Q = \{ B, Q \}, \quad \partial \bar{z} Q = \{ \bar{B}, Q \},$$

(12)
and the canonical Poisson relation

\[ \{P, Q\} = P. \] (13)

Conversely, from (12) and (13) one can go back to (11); they are equivalent. In the case of the self-dual vacuum Einstein equations [14], Lax equations like (12) are also interpreted as a “linear system.” This double nature seems to be a common characteristic of this kind of nonlinear systems (nonlinear graviton equations).

In a generic case (see Section 4), one can find two distinct pairs of Darboux coordinates, \((\mathcal{L}, \mathcal{M})\) and \((\hat{\mathcal{L}}, \hat{\mathcal{M}})\), with the following analyticity properties on the complex \(\lambda\) plane.

i) these four functions are holomorphic functions of \(\lambda\) in a neighborhood of a circle \(\Gamma\) with center at the origin \(\lambda = 0\).

ii) \(\mathcal{L}\) and \(\mathcal{M}\) can be analytically extended outside \(\Gamma\) up to the point \(\lambda = \infty\) where they have first order poles.

iii) \(\hat{\mathcal{L}}\) and \(\hat{\mathcal{M}}\) can be analytically extended inside \(\Gamma\).

More specifically, one may select them to have Laurent expansion as

\[
\mathcal{L} = \lambda + \sum_{n \leq 0} u_n \lambda^n, \quad \mathcal{M} = z \mathcal{L} + s + \sum_{n \leq -1} v_n \mathcal{L}^n, \\
\hat{\mathcal{L}} = \sum_{n \geq 1} \hat{u}_n \lambda^n, \quad \hat{\mathcal{M}} = -\hat{z} \hat{\mathcal{L}}^{-1} + s + \sum_{n \geq 1} \hat{v}_n \hat{\mathcal{L}}^n. \] (14)

Note that \(\mathcal{M}\) and \(\hat{\mathcal{M}}\) are Laurent expanded in \(\mathcal{L}\) and \(\hat{\mathcal{L}}\) rather than in \(\lambda\); coefficients thus defined will be used for defining a tau function. A few Laurent coefficients are directly related with \(\phi\) as

\[
\partial_z \phi = u_0, \\
\partial_z \phi = \hat{u}_2 \hat{u}_1^{-2}, \\
\partial_s \phi = -\log \hat{u}_1 \] (15)
One can thus single out two particular pairs of Darboux coordinates \((L, M)\) and \((\hat{L}, \hat{M})\) for the 2-form \(\omega\),
\[
\omega = \frac{dL}{L} \wedge dM = \frac{d\hat{L}}{\hat{L}} \wedge d\hat{M}.
\]
(16)

Of course they both can also be characterized by a Lax system and a canonical Poisson relation like (12) and (13).

3. Hierarchy and tau function

The notion of the SDiff(2) Toda hierarchy is a straightforward extension of the above Lax formalism. We now start from two pairs of Laurent series \((L, M)\) and \((\hat{L}, \hat{M})\) of the form

\[
L = \lambda + \sum_{n \leq 0} u_n \lambda^n, \quad M = \sum_{n \geq 1} n z_n L^n + s + \sum_{n \leq -1} v_n L^n,
\]
\[
\hat{L} = \sum_{n \geq 1} \hat{u}_n \lambda^n, \quad \hat{M} = -\sum_{n \geq 1} n \hat{z}_n \hat{L}^{-n} + s + \sum_{n \geq 1} \hat{v}_n \hat{L}^n,
\]
(17)

where \(z_n, \hat{z}_n, n = 1, 2, \ldots\), now supply an infinite number of independent variables along with \(s\) and \(\lambda\). The hierarchy consists of the Lax equations

\[
\partial_{z_n} K = \{B_n, K\}, \quad \partial_{\hat{z}_n} K = \{\hat{B}_n, K\}
\]
(18)

for \(K = L, M, \hat{L}, \hat{M}\) and the canonical Poisson relations

\[
\{L, M\} = L, \quad \{\hat{L}, \hat{M}\} = \hat{L}.
\]
(19)

Here \(B_n, \hat{B}_n\) are given by

\[
B_n \overset{\text{def}}{=} (L^n)_{\geq 0}, \quad \hat{B}_n \overset{\text{def}}{=} (\hat{L}^{-n})_{\leq -1},
\]
(20)

where \((\quad)\ldots\) stands for extracting powers of \(\lambda\) with exponents running over the range shown in the suffix. These equations can be recast into a compact form as

\[
\omega = \frac{dL}{L} \wedge dM = \frac{d\hat{L}}{\hat{L}} \wedge d\hat{M}.
\]
(21)
where the 2-form $\omega$ is now given by

$$\omega = \frac{d\lambda}{\lambda} \wedge ds + \sum_{n \geq 1} dB_n \wedge dz_n + \sum_{n \geq 1} d\hat{B}_n \wedge d\hat{z}_n,$$

(22)

and, accordingly, satisfies the equations

$$d\omega = 0, \quad \omega \wedge \omega = 0.$$  
(23)

The second relation of (23) is equivalent to the zero-curvature equations

$$\partial_{z_n} B_m - \partial_{z_m} B_n + \{B_m, B_n\} = 0,$$
$$\partial_{\hat{z}_n} \hat{B}_m - \partial_{\hat{z}_m} \hat{B}_n + \{\hat{B}_m, \hat{B}_n\} = 0,$$
$$\partial_{\hat{z}_n} B_m - \partial_{z_m} \hat{B}_n + \{B_m, \hat{B}_n\} = 0.$$  
(24)

The original SDiff(2) Toda equation is included in the $(z_1, \hat{z}_1)$ sector with $z_1 = z$, $\bar{z} = \hat{z}_1$, $B = B_1$ and $\hat{B} = \hat{B}_1$. We note that Lax equations like (18) containing a Poisson bracket are already studied to some extent by Golenisheva-Kutuzova and Reiman [16] by means of the coadjoint orbit method.

We now show that four important potentials are hidden in the above equations. The first one is the $\phi$ field itself. This is characterized as

$$d\phi = \sum_{n \geq 1} \text{res} (L^n d \log \lambda) dz_n - \sum_{n \geq 1} \text{res} (\hat{L}^{-n} d \log \lambda) d\hat{z}_n - \log \hat{u}_1 ds,$$

(25)

where “res” denotes the formal residue operator

$$\text{res} \sum a_n \lambda^n d\lambda \overset{\text{def}}{=} a_{-1}.$$  
(26)

The construction of the second and third potentials are inspired by Krichever's work [17] on a SDiff(2) version of the KP hierarchy. These potentials, $S$ and $\hat{S}$, are defined as

$$dS = M d \log L + \log \lambda ds + \sum_{n \geq 1} B_n dz_n + \sum_{n \geq 1} \hat{B}_n d\hat{z}_n,$$

(27)

$$d\hat{S} = \hat{M} d \log \hat{L} + \log \lambda ds + \sum_{n \geq 1} B_n dz_n + \sum_{n \geq 1} \hat{B}_n d\hat{z}_n.$$  
(28)
Actually, $S$ and $\hat{S}$ mostly play rather technical roles, however it seems likely that they also have some deep meaning. Comparing with the Laurent expansion of $M$ and $\hat{M}$, one can see that $S$ and $\hat{S}$ have Laurent expansion of the following form.

$$S = \sum_{n \geq 1} z_n L^n + s \log L + \sum_{n \leq -1} v_n L^n / n, \quad (29)$$

$$\hat{S} = \sum_{n \geq 1} \hat{z}_n \hat{L}^{-n} + s \log \hat{L} + \phi + \sum_{n \geq 1} \hat{v}_n \hat{L}^n / n. \quad (30)$$

This expansion is reminiscent of the Fourier expansion of free bosons in the KP hierarchy [11]. Remarkably, $\phi$ now arise as a “zero-mode” and $s$ may be interpreted as its “conjugate variable.” Finally, the tau function $\tau$ can be defined as

$$d \log \tau = \sum_{n \geq 1} v_{-n} dz_n - \sum_{n \geq 1} \hat{v}_n d\hat{z}_n + \phi ds. \quad (31)$$

**Theorem 1.** The right hand side of the above equations defining $\phi, S, \hat{S}$ and $\tau$ are all closed differential forms.

One should note here that all the ingredients of the $\text{SDiff}(2)$ Toda hierarchy, $L$, $M$, $\hat{L}$, $\hat{M}$ and $\phi$, can be reproduced from the tau function. Indeed, $\phi$ and the Laurent coefficients $v_n$ and $\hat{v}_n$ of $M$ and $\hat{M}$ are now given by a derivative of $\log \tau$, see (31). The Laurent coefficients of $L$ and $\hat{L}$, as (25) shows, can be written in terms of derivatives of $\phi$, and $\phi$ is already a derivative of $\log \tau$. The whole hierarchy can thus, in principle, be rewritten into a system of nonlinear differential equations with just a single unknown function, $\tau$. In the case of the KP and TL hierarchy [11] [12], these equations take the form of Hirota’s bilinear equations. We do not yet know if such a beautiful structure persists in the $\text{SDiff}(2)$ Toda hierarchy.

4. **RIEMANN-HILBERT PROBLEM AND $\text{SDiff}(2)$ SYMMETRIES**

The nonlinear graviton construction of Penrose [18] can be reformulated in the present setting as follows. As basic relation (21) shows, the two pairs of “Darboux coordinates”
(\mathcal{L}, \mathcal{M}) and (\hat{\mathcal{L}}, \hat{\mathcal{M}}) should be connected by a SDiff(2) group element, i.e., an area-preserving diffeomorphism. This means that there are two functions \( f = f(\lambda, s) \) and \( g = g(\lambda, s) \) such that

\[
f(\mathcal{L}, \mathcal{M}) = \hat{\mathcal{L}}, \quad g(\mathcal{L}, \mathcal{M}) = \hat{\mathcal{M}}
\]

(32)

and

\[
\{f, g\} = f.
\]

(33)

More precisely, we have to impose some analyticity conditions to \( \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}} \) and \( \hat{\mathcal{M}} \) so that functional relations (32) are meaningful. It is customary to assume that the Laurent series giving \( \mathcal{L} \) etc. have a common domain of convergence, say a neighborhood of a circle \( \Gamma \) as mentioned in Section 2. On the complex \( \lambda \) plane, then, \( \mathcal{L} \) and \( \mathcal{M} \) can be analytically extended outside \( \Gamma \) whereas \( \hat{\mathcal{L}} \) and \( \hat{\mathcal{M}} \) inside \( \Gamma \). The “patching functions” \( f \) and \( g \) will then become holomorphic functions in a neighborhood of \( \Gamma \). (We do not specify the domain of the \( s \) plane where they are to be defined; a rigorous formulation should be given in the language of sheaf cohomology [19].) Conversely, given a pair of patching functions satisfying (33), one may ask if functional equations (32) for \( \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}} \) and \( \hat{\mathcal{M}} \) have a solution. This is a kind of Riemann-Hilbert problem, now referring to the SDiff(2) group. As Penrose [18] first observed, this kind of Riemann-Hilbert problem has a unique solution as far as the area-preserving diffeomorphism \((f, g)\) is sufficiently close the identity map.

Having this correspondence between a solution of the SDiff(2) Toda hierarchy and a pair of patching functions, we can apply the previous method for the self-dual vacuum Einstein equation [14] to the present setting with slightest modifications. According to a general prescription presented therein, we now consider left and right translations caused by two Hamiltonian vector fields

\[
\{F, \cdot\} = \lambda(\partial_\lambda F)\partial_s - \lambda(\partial_s F)\partial_\lambda,
\]

\[
\{\hat{F}, \cdot\} = \lambda(\partial_\lambda \hat{F})\partial_s - \lambda(\partial_s \hat{F})\partial_\lambda
\]

(34)
as
\[(f, g) \to \exp \epsilon \{\hat{F}, \cdot \} \circ (f, g) \circ \exp(-\epsilon \{F, \cdot \}), \quad (35)\]
where \(\epsilon\) is an infinitesimal parameter; “\(\circ\)” stands for composition of diffeomorphisms; 
\(F = F(\lambda, s)\) and \(\hat{F} = \hat{F}(\lambda, s)\) are arbitrary functions with the same analyticity as the patching functions. This should give rise to an infinitesimal variation of the corresponding solution of the Riemann-Hilbert problem. From the construction, one can expect a chiral structure in these symmetries, i.e., left and right components give rise to two distinct symmetries that commute each other. Expanded to the first order of \(\epsilon\), \(L, M, \hat{L}\) and \(\hat{M}\) will transform as
\[
L \to L + \epsilon \delta L + O(\epsilon^2), \quad M \to M + \epsilon \delta M + O(\epsilon^2),
\]
\[
\hat{L} \to \hat{L} + \epsilon \delta \hat{L} + O(\epsilon^2), \quad \hat{M} \to \hat{M} + \epsilon \delta \hat{M} + O(\epsilon^2). \quad (36)
\]
The coefficients of \(\epsilon\) thus define a linear operator \(\delta = \delta_{F, \hat{F}}\) that gives an infinitesimal symmetry of the SDiff(2) Toda hierarchy. From a differential-algebraic point of view [20], \(\delta_{F, \hat{F}}\) is an operator that acts on \(L, M, \hat{L}\) and \(\hat{M}\) as an abstract derivation. By definition, this is an inner symmetry,
\[
\delta_{F, \hat{F}} \lambda = \delta_{F, \hat{F}} s = \delta_{F, \hat{F}} z_n = \delta_{F, \hat{F}} \hat{z}_n = 0. \quad (37)
\]
In fact, we have the following very explicit result.

**Theorem 2.** The infinitesimal symmetries of \(L, M, \hat{L}, \hat{M}\) are given by
\[
\delta_{F, \hat{F}} K = \{F(L, M)_{\leq -1} - \hat{F}(\hat{L}, \hat{M})_{\leq -1}, K\} \quad \text{for } K = L, M,
\]
\[
\delta_{F, \hat{F}} K = \{\hat{F}(\hat{L}, \hat{M})_{\geq 0} - F(L, M)_{\geq 0}, K\} \quad \text{for } K = \hat{L}, \hat{M}. \quad (38)
\]
We now show that these infinitesimal symmetries can be extended to the potentials that we have introduced in the previous section. A priori, this is by no means obvious, because the potentials are defined only up to an integration constant, hence one has to show a way to fix this ambiguity in the course of transformation. We first show a result on \(\phi\).
Theorem 3. An infinitesimal transformation of \( \phi \) consistent with the previous one for \( \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}} \) and \( \hat{\mathcal{M}} \) is given by

\[
\delta_{\mathcal{F}, \hat{\mathcal{F}}} \phi = - \text{res} \ F(\mathcal{L}, \mathcal{M})d\log \lambda + \text{res} \ \hat{\mathcal{F}}(\hat{\mathcal{L}}, \hat{\mathcal{M}})d\log \hat{\lambda}.
\]

(39)

“Consistency” in the statement of the theorem means that the infinitesimal symmetry retains the basic relation, (25), connecting \( \phi \) with other quantities. This result also shows how \( S \) and \( \hat{S} \) should be transformed. Finally, we have the following result for the tau function.

Theorem 4. An infinitesimal transformation of \( \tau \) consistent with the previous one for \( \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}} \) and \( \phi \) is given by

\[
\delta_{\mathcal{F}, \hat{\mathcal{F}}} \log \tau = - \text{res} \ F^s(\mathcal{L}, \mathcal{M})d\lambda \log \mathcal{L} + \text{res} \ \hat{F}^s(\hat{\mathcal{L}}, \hat{\mathcal{M}})d\lambda \log \hat{\mathcal{L}},
\]

where \( d\lambda \) stands for total differentiation with respect to \( \lambda \), and \( F^s = F^s(\lambda, s) \) and \( \hat{F}^s = \hat{F}^s(\lambda, s) \) are given by

\[
F^s(\lambda, s) = \text{def} \int_0^s F(\lambda, \sigma)d\sigma, \quad \hat{F}^s(\lambda, s) = \text{def} \int_0^s \hat{F}(\lambda, \sigma)d\sigma.
\]

(41)

Commutation relations for these infinitesimal symmetries can also be calculated.

Theorem 5. The infinitesimal symmetries \( \delta_{\mathcal{F}, \hat{\mathcal{F}}} \) on \( \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}} \) and \( \phi \) respect the \( \text{SDiff}(2) \) structure in the sense that the commutation relations

\[
\left[ \delta_{\mathcal{F}_1, \hat{\mathcal{F}}_1}, \delta_{\mathcal{F}_2, \hat{\mathcal{F}}_2} \right] K = \delta_{\{\mathcal{F}_1, \mathcal{F}_2\}, \{\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2\}} K
\]

are satisfied for \( K = \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}}, \phi \).

This result is not very surprising, because we already know of a similar result in the case of the self-dual vacuum Einstein equations [14]. For several reasons, \( \phi \) may be considered an analogue of the Plebanski (first and second) key functions, and it is shown therein that infinitesimal symmetries of the above type can be extended to the key functions without any anomaly. Anomalous commutation relations, however, take place at the level of the tau function.
Theorem 6. For the tau function,
\[
\delta_{F_1, \hat{F}_1, \delta_{F_2, \hat{F}_2}} \log \tau = \delta_{\{F_1, F_2\}, \{\hat{F}_1, \hat{F}_2\}} \log \tau + c(F_1, F_2) + \hat{c}(\hat{F}_1, \hat{F}_2),
\]
where \( c \) and \( \hat{c} \) are cocycles of the SDiff(2) algebra given by
\[
c(F_1, F_2) \overset{\text{def}}{=} - \text{res} \int F_2(\lambda, 0) dF_1(\lambda, 0),
\]
\[
\hat{c}(\hat{F}_1, \hat{F}_2) \overset{\text{def}}{=} \text{res} \int \hat{F}_2(\lambda, 0) d\hat{F}_1(\lambda, 0).
\]

We are thus naturally led to a central extension of the SDiff(2) algebra (or, more precisely, the direct sum of two copies of the SDiff(2) algebra). The cocycles are of Kac-Moody type and give rise to a U(1) current algebra in the “spin 1” sector of SDiff(2).

It is amusing to compare the above result with physicists’ calculation of cocycles for SDiff(2) algebras on various surfaces [21] [22] [23] [24]; they observed that there are exactly \( 2g \) linearly independent cocycles on a genus \( g \) surface. Since the cylinder \( S^1 \times \mathbb{R} \) may be thought of as a genus \( g = 1/2 \) surface, our result seems to fit well into physicists’ observation.

5. Conclusion

We have thus introduced the SDiff(2) Toda hierarchy and shown that it shares a number of remarkable characteristics with the ordinary KP/TL hierarchies. In particular, SDiff(2) symmetries of the tau function exhibit commutator anomalies, hence the tau function requires a central extension of SDiff(2) as a true symmetry algebra. On the other hand, the Lax formalism and related technical details such as the Riemann-Hilbert problem are obviously of the same type as the self-dual vacuum Einstein equations and its hyper-Kähler version. We thereby expect that the SDiff(2) Toda equation (hierarchy) will be a nice laboratory for an attempt to unify two apparently distinct families of nonlinear “integrable” systems, soliton equations and nonlinear graviton equations, on an equal footing.

One can deduce a similar conclusion for Krichever’s SDiff(2) version of the KP hierarchy [17]. This issue will be reported elsewhere.
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