Hydroelastic Oscillations of a Circular Plate, Resting on Winkler Foundation

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Abstract. The forced hydroelastic oscillations of a circular plate resting on elastic foundation are investigated. The oscillations are caused by a stamp vibration under interaction with a plate through a thin layer of viscous incompressible liquid. The axis-symmetric problem for the regime of the steady-state harmonic oscillations is considered. On the basis of hydroelasticity problem solution the laws of plate deflection and pressure in the liquid are found. The functions of the amplitudes deflection distribution and liquid pressure along the plate are constructed. The presented mathematical model provides for investigating viscous liquid layer interaction dynamics with a circular plate resting on an elastic foundation. The above-mentioned model makes it possible to define the plate oscillations resonance frequencies and the corresponding amplitudes of deflection and liquid pressure, as well. Keywords – circular plate; hydroelastic oscillations; vibration stamp; viscous liquid; Winkler foundation

1. Introduction

The investigations of the plates and supporting elastic foundations interaction are of theoretical, as well as of practical interest. For example, the review [1], devoted to this subject matter, considers the items of elastic foundations development models, as well as various approaches to analytical and numerical investigation of beams and plates with elastic foundations interaction. Reference [2] studies free and forced oscillations of a circular plate, the plate resting on Winkler foundation. The investigation is carried out in axis-symmetric setting for the cases of a simply supported edge and a clamped one. Reference [3] studies stability and oscillations of circular orthotropic plates of variable thickness resting on Winkler foundation. The cases of the simply supported and clamped edge of the plates are considered. Reference [4] deals with investigation of oscillations of a thin circular plate with elastic fixation along the contour, the plate resting on Winkler elastic foundation. Reference [5] studies free oscillations of thin circular plate on Winkler foundation. The axis-symmetric problem under the condition of foundation rigidity change in radial direction is investigated in the above-mentioned reference. References [6, 7] consider oscillations and stability of multi-layered circular plates on elastic foundation under the impact of local and distributed loadings of various nature. The Winkler and Pasternak models are used for foundation reactions modelling. On the other hand, the investigations of hydroelastic interaction of circular plates with an ideal liquid are of
importance, too. For example, reference [8] is one of the first publications on investigation of natural oscillation of a circular plate interacting with an ideal liquid. The study of a circular plate vibration on a free surface of an ideal incompressible liquid is made in [9]. The problem is considered for the case, where liquid volume is limited by a rigid bottom and cylinder surface. Reference [10] studies the oscillations of a circular plate, plunged into an ideal incompressible liquid with a free surface in rigid cylinder. Mathematical modeling and a series of experimental investigations are carried out by the authors of this paper. The damping qualities of the liquid, conditioned by its viscosity are not considered in the references mentioned above. The problem of vibrating discs interaction with viscous incompressible liquid layer between them is solved in [11]. The present paper studies the case, where the discs are considered to be rigid and the case with one of the discs being elastic. The analogous problem in a two-dimensional setting for two plates of finite sizes is solved in [12]. The investigation of a beam hydroelastic oscillations in a viscous liquid flow with an application to piezoelectric elements, aimed at obtaining energy from the flow, is made in [13]. The problem of bending hydroelastic oscillations of the plate forming the narrow channel wall under the impact of viscous liquid pulsating layer is solved in [14]. The forced hydroelastic oscillations of a three-layered circular plate, interacting with viscous incompressible liquid layer under channel foundation vibration, are investigated in [15].

However, evaluating foundation elasticity impact on hydroelastic oscillations is of theoretical and practical interest. Membrane oscillations on Winkler elastic foundation, the membrane situated on the reservoir bottom full of ideal incompressible liquid with a free surface, are investigated in [16]. References [17–19] consider hydroelastic oscillations of rectangular plates, resting on Pasternak foundation and interacting with an ideal incompressible liquid with a free surface. The investigation of rectangular plate, resting on Winkler elastic foundation and interacting with a pulsating layer of viscous incompressible liquid is carried out in [20–23]. Nevertheless, the case of hydroelastic oscillations of a circular plate on an elastic foundation is not considered in the investigations mentioned.

2. Stetting the problem

Let us consider the forced oscillations of a circular plate 1, resting on Winkler foundation (see fig. 1). The plate oscillations are caused by its interaction with a vibrating circular stamp 2 through viscous liquid 3. We will consider the axis-symmetric problem. We bind cylindrical coordinate system $\text{O}_r\varphi z$ with the medium surface center of the plate in an unperturbed state. Let us consider that the stamp movement takes place according to a set harmonic law in a vertical plane. The plate has thickness $h_0$, radius $R$. It is rested on Winkler foundation and clamped at the edge. The viscous liquid fills fully a narrow channel, formed by the plate and stamp, the liquid layer medium thickness in the channel being $\delta_0$. We think that $\delta_0<<R$ and the stamp oscillations amplitude are considerably less than $\delta_0$. The liquid on the plate edge freely leaks into the same liquid with a constant pressure $p_0$. The presence of viscous liquid between the plate and the stamp leads to quick going out of transitional processes. Therefore, the initial processes impact can be excluded from the very beginning of the investigation, according [24]. Thus, further we will study the forced steady-state harmonic oscillations.

Figure 1. The circular plate 1, resting on Winkler foundation and interacting with vibrating circular stamp 2 through viscous liquid 3.
We will present the stamp movement law in the form of:

\[ z = z_m f(\omega t), \quad f(\omega t) = \sin \omega t, \]  

where \( z_m \) is the stamp oscillations amplitude, \( \omega \) is the frequency, \( t \) is the time.

The liquid movement in a narrow split between the stamp and the plate can be considered to be creeping in [25]. In this case viscous liquid dynamics equations take the form of:

\[
\begin{align*}
\frac{1}{\rho} \frac{\partial p}{\partial r} &= \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} - \frac{u_r}{r} \right), \\
\frac{1}{\rho} \frac{\partial p}{\partial z} &= \nu \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} \right), \\
\frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{\partial u_z}{\partial z} &= 0,
\end{align*}
\]  

where \( r, z \) are cylindrical coordinates; \( u_r, u_z \) are liquid velocity vector projections on a coordinate axis; \( p \) is the pressure; \( \rho \) is density of liquid; \( \nu \) is the kinematic viscosity coefficient of liquid.

The boundary conditions of equation (2) are the ones of liquid movements velocity and a plate velocity shift coincidence:

\[
\begin{align*}
&u_r = \frac{\partial u}{\partial t}, \quad u_z = \frac{\partial w}{\partial t} \text{ at } z = h_0/2 + w, \\
&u_r = 0, \quad u_z = 0 \text{ at } z = h_0/2 + \delta_0 + z_m f(\omega t),
\end{align*}
\]

and also the conditions for the pressure limit on symmetry axis, as well as the ones for the pressure on edge:

\[ r \frac{\partial p}{\partial r} = 0 \text{ at } r = 0, \quad p = p_0 \text{ at } r = R. \]  

Here \( u, w \) are the plate radial shifts an deflection.

The equations of the circular plate oscillations resting on Winkler foundation can be written down, as [1, 6]:

\[
D \left( \frac{\partial^4 w}{\partial r^4} + \frac{2}{r} \frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^4} \frac{\partial w}{\partial r} \right) + \kappa w + \rho_0 h_0 \frac{\partial^2 w}{\partial r^2} = q_z,
\]

where \( D \) is the flexural rigidity of the plate, \( h_0 \) is the plate thickness, \( \rho_0 \) is the plate material density, \( \kappa \) is the foundation modulus, \( q_z = -p + 2\mu \frac{\partial^2 w}{\partial r^2} \) is the normal stress in a liquid layer.

The boundary conditions of (5) take the form of:

\[ r \frac{\partial w}{\partial r} = 0 \text{ at } r = 0, \quad w = \frac{\partial^2 w}{\partial r^2} = 0 \text{ at } r = R. \]

3. The Theory

Let us introduce dimensionless variables and small parameters into the consideration:

\[
\begin{align*}
\xi &= r/R, \quad \zeta = (z - 0.5h_0)/\delta_0, \quad w = w_m W, \quad u = u_m U, \\
\eta &= \frac{z_m}{\delta_0}, \quad \nu = \frac{\omega}{\delta_0} U_z, \quad p = p_0 + P \nu \frac{\delta_0}{\delta_0} \psi^{-2}/\delta_0, \\
\tau &= \omega t, \quad \lambda = \frac{z_m}{\delta_0} << 1, \quad \psi = \delta_0/R << 1.
\end{align*}
\]
While taking into account (7), hydromechanics problem in a dimensionless state in a zero approximation in \( \psi \) will be written down as:

\[
\frac{\partial P}{\partial \xi} = \frac{\partial^2 U_\xi}{\partial \zeta^2}, \quad \frac{\partial P}{\partial \zeta} = 0, \quad \frac{\partial U_\xi}{\partial \zeta} + \frac{1}{\zeta} U_\xi + \frac{\partial U_\xi}{\partial \xi} = 0 \tag{8}
\]

With boundary conditions (with accuracy up to \( \psi \) and \( \lambda \)):

\[
U_\xi = 0, U_\zeta = df(r)/d\tau \text{ at } \zeta = 1,
\]
\[
U_\xi = 0, U_\zeta = (w_m/z_m)\partial W/\partial \tau \text{ at } \zeta = 0, \tag{9}
\]
\[
P = 0 \text{ at } \zeta = 1, \xi \partial P/\partial \xi = 0 \text{ at } \zeta = 0.
\]

Normal stress in the liquid with consideration of variables (7) in zero approximation in \( \psi \) takes the form of

\[
q_{zz} = -p_0 - \rho v \varepsilon_m \omega \delta \psi \frac{\partial \psi}{\partial \xi} P. \tag{10}
\]

Further, by solving (8) with (9), we get:

\[
U_\xi = \frac{\zeta^2 - \zeta}{2} \frac{\partial P}{\partial \xi}, \quad U_\zeta = \frac{\partial W}{\partial \tau} - \frac{2}{12} \left[ \frac{1}{\zeta} \frac{\partial P}{\partial \xi} + \frac{\partial^2 P}{\partial \xi^2} \right]. \tag{11}
\]

\[
P = 3(\xi^2 - 1) \frac{df}{d\tau} + 12 \frac{w_m}{z_m} \left[ \frac{\xi}{0} \frac{\partial W}{\partial \tau} d\xi \right]. \tag{12}
\]

We present the shape of the plate deflection as the series in eigenfunctions of Sturm-Loiville problem:

\[
W = \sum_{k=1}^{\infty} \left( R_k^0 + R_k(r) \right) \left[ J_0(\beta_k \xi)/J_0(\beta_k) - I_0(\beta_k \xi)/I_0(\beta_k) \right], \tag{13}
\]

here \( R_k(r) \) is the time harmonic function, \( R_k^0 \) is the constant, \( J_0 \) is the Bessel function of the first kind; \( I_0 \) is the modified Bessel function of the first kind; \( \beta_k \) is the root of transcendental equation \( I_1(\beta_k)/I_0(\beta_k) = -J_1(\beta_k)/J_0(\beta_k) \) (\( k = 1, 2, \ldots \)), \( J_1(\beta_k), I_1(\beta_k) \) are Bessel function and modified Bessel function of the first kind.

Dimensionless pressure (12) with consideration the plate deflection (13), takes the form:

\[
P = 3(\xi^2 - 1) \frac{df}{d\tau} + \sum_{k=1}^{\infty} \frac{12 w_m}{z_m} \frac{dR_k}{d\tau} \left[ \frac{J_0(\beta_k \xi)}{J_0(\beta_k)} + \frac{I_0(\beta_k \xi)}{I_0(\beta_k)} - 2 \right]. \tag{14}
\]

By substituting (14) in (10) and expanding in the series in eigenfunctions of the Sturm-Loiville problem, we get:

\[
q_{zz} = -p_0 \sum_{k=1}^{\infty} 2 \frac{J_1(\beta_k)}{\beta_k} \left[ J_0(\beta_k \xi)/J_0(\beta_k) - I_0(\beta_k \xi)/I_0(\beta_k) \right] - \\
- \rho v w_m \omega \delta \psi \frac{\partial \psi}{\partial \xi} \sum_{k=1}^{\infty} \frac{1}{\beta_k} \left[ J_1(\beta_k \xi)/J_0(\beta_k) - I_1(\beta_k \xi)/I_0(\beta_k) \right]. \tag{15}
\]
Further, we consider that \( k = 1, 2, \ldots, n \), \( i = 1, 2, \ldots, n \). By substituting (13), (15) in (5) and equalizing the coefficients at identical eigenfunctions of the Sturm-Loiville problem in the obtained equations, we define the expressions for \( R_k^0 \):

\[
R_k^0 = -p_0 \frac{2}{w_m \beta_k (\kappa + D \beta_k^4 / R^4)} \frac{J_0(\beta_k)}{J_0(\beta_k)},
\]

and the system of linear ordinary differential equations in time to define \( R(t) \):

\[
w_m \left( \frac{D \beta_k^4}{R^4} + \kappa \right) R_k(t) + \rho \beta_k w_m \omega^2 \frac{d^2 R_k(t)}{d t^2} =
\]

\[
= \frac{\rho \nu m \omega}{\delta \eta^2} \left[ \frac{12}{\beta_i^4} \frac{d R_k(t)}{d t} + \frac{w_m}{\beta_i^4} \left( \frac{12}{\beta_i^4} \frac{d R_k(t)}{d t} - \frac{4}{\beta_i^4} \frac{J_1(\beta_i)}{J_0(\beta_i)} \right) + \frac{w_m}{z_m} \sum_{i=1}^{n} \left( \frac{12}{\beta_i^4} \right) \frac{d R_k(t)}{d t} \right],
\]

The inverse transformation in time gives us the solution for \( R(t) \):
By substituting (19) in (14) with taking into account (7), the pressure in liquid layer between the plates, stamp movement, takes the form of:

\[ P = \frac{w_m}{\beta_k^2} \left[ J_1^2(\beta_k) \right] dR_k(\tau) \]

Finally, with consideration of (16), (19) the law of the plate hydroelastic oscillations caused by the stamp movement, takes the form of:

\[ w_m \left( \frac{D \beta_k^4}{R^4} + \kappa \right) R_k(\tau) + \rho \omega R_m \omega^2 \frac{d^2 R_k(\tau)}{d\tau^2} = - \frac{\rho \omega R_m \omega^2}{\beta_k^2} \left[ \frac{df(\tau)}{d\tau} \right] R_k(\tau) - \frac{\rho \omega R_m \omega^2}{\beta_k^0} \left[ \frac{df(\tau)}{d\tau} \right] R_k(\tau)
\]

The solution of equation (18) for the regime of the steady-state harmonic oscillations takes the form:

\[ w_m R_k = -z_m \left( \frac{K_i^1 K_i^w \omega^2}{(\bar{D}_1 - \rho \omega^3 \omega^2)^2 + (K_i^w \omega^2)^2} f(\tau) + \frac{K_i^1 \omega (\bar{D}_1 - \rho \omega^3 \omega^2)}{(\bar{D}_1 - \rho \omega^3 \omega^2)^2 + (K_i^w \omega^2)^2} \frac{df(\tau)}{d\tau} \right) \]

\[ K_i^1 = \frac{\rho \omega}{\beta_k^2} \left[ \frac{J_1^2(\beta_k)}{J_0(\beta_k)} \left[ \frac{J_1(\beta_k)}{J_0(\beta_k)} \right] \cdot \frac{4 J_1(\beta_k)}{J_0(\beta_k)} \right] \]

Finally, with consideration of (16), (19) the law of the plate hydroelastic oscillations caused by the stamp movement, takes the form of:

\[ w = -\frac{p_0}{\beta_k(\kappa + D \beta_k^4/R^4)} \left[ \frac{2 J_1(\beta_k)}{J_0(\beta_k)} \left[ \frac{J_0(\beta_k, \omega)}{J_0(\beta_k)} \right] - \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} \right] - z_m A(\xi, \omega) \sin(\alpha + \phi(\xi, \omega)), \]

\[ A(\xi, \omega) = \sqrt{C^2 + B^2}, \quad \phi(\xi, \omega) = \arctan(B/C), \]

\[ C = \frac{K_i^1 K_i^w \omega^2}{(\bar{D}_1 - \rho \omega^3 \omega^2)^2 + (K_i^w \omega^2)^2} \left[ \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} - \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} \right], \quad B = \frac{K_i^1 \omega (\bar{D}_1 - \rho \omega^3 \omega^2)}{(\bar{D}_1 - \rho \omega^3 \omega^2)^2 + (K_i^w \omega^2)^2} \left[ \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} - \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} \right] \]

By substituting (19) in (14) with taking into account (7), the pressure in liquid layer between the plates and stamp can be written down as:

\[ p = p_0 + z_m \Pi(\xi, \omega) \sin(\alpha + \phi_p(\xi, \omega)), \]

\[ \Pi(\xi, \omega) = \sqrt{S^2 + Q^2}, \quad \phi_p(\xi, \omega) = \arctan(Q/S), \]

\[ Q = \left( \frac{\xi^2}{4} - 1 \right) \beta_k^2 K_i^1 \omega - \frac{\rho v}{\beta_k^2} \left[ \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} \right] - \frac{1}{2} \left[ \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} \right] B, \quad S = \frac{1}{2} \left[ \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} \right] B, \quad \frac{1}{2} \left[ \frac{J_0(\beta_k, \omega)}{I_0(\beta_k)} \right] B \]
4. Summary and Conclusion

The investigation, carried out above, allows making the following conclusions. The mathematical model, allowing to investigate a circular plate hydroelastic oscillations, caused by stamp vibration, is carried out. The model takes into account the impact of Winkler foundation with a plate resting on it and the presence of viscous liquid layer between the plate and stamp. The expressions for the plate deflection and pressure in the liquid in the case of preserving one member of the series in (13) are obtained. However, it is possible to take into account the subsequent members of the series within the framework of the proposed model. To do this, it is necessary to use expression (16) and to define $R_i(\tau), i = 1,2,\ldots,n$ out of (17), and then to write down the corresponding expressions for the plate deflection and the pressure. According to (20) the amplitude of dynamic deflection is defined by the function $A(\xi, \omega)$. This function must be considered as frequency dependent function of deflection amplitudes distribution along the channel. The amplitude of liquid dynamic pressure in the channel is defined by the function $\Pi(\xi, \omega)$. The functions $\varphi(\xi, \omega), \varphi_p(\xi, \omega)$ are frequency dependent functions of phase shift distribution of the plate deflection and the pressure along the channel, accordingly. The investigation of the functions, mentioned above, makes it possible to study dynamic processes in the considered oscillation system. The analysis of the obtained expressions for the plate deflection (20) and pressure (21) allows to make a conclusion, that the foundation rigidity influences the plate static deflection and dynamic deflection amplitude and pressure in the liquid, as well. The obtained solution provides for transition from Winkler foundation to the plate with a clamped edge. To do this, it is necessary to equal the foundation modulus to zero.

Thus, the obtained results can be used for mathematical modeling and analysis of hydroelastic oscillations of circular plates, resting on elastic foundations. The presented model can be used to develop the methods of nondestructive testing of elastic constructions, resting on an elastic foundation and interacting with viscous liquid by their forced oscillations parameters.

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