Crossed Threshold Resummation

George Sterman\textsuperscript{1}, Werner Vogelsang\textsuperscript{2}

\textsuperscript{1}C.N. Yang Institute for Theoretical Physics, Stony Brook University Stony Brook, New York 11794 – 3840, U.S.A.
\textsuperscript{2}Physics Department, Brookhaven National Laboratory, Upton, New York 11973, U.S.A.

(Dated: May 10, 2018)

We show that certain general properties of threshold and joint resummations in Drell-Yan cross sections hold as well for their crossed analogs in semi-inclusive deep-inelastic scattering and double-inclusive leptonic annihilation. We show that all plus-distribution corrections near threshold show the same structure, and are determined to all logarithmic order by two anomalous dimensions, one of which is a generalization of the $D$-term previously derived in Drell-Yan. We also discuss the possibility of universality in power corrections implied by the resummation.

I. INTRODUCTION

Resummations organize sets of potentially large higher-order contributions to perturbative series. In hard inclusive and semi-inclusive cross sections, such corrections typically arise when restrictions on phase space result in incomplete cancellations between partonic emission and virtual corrections. The realization of a cross section that respects unitarity may then require the resummation of a portion of the infinite series, involving unlimited numbers of partons.

For hadron-hadron scattering processes, both transverse momentum and threshold resummations have been studied extensively, especially for differential and inclusive cross sections for electroweak vector and heavy scalar (Higgs) production. In these cases, the technique of joint resummation provides a unification of the two methods in situations where both may be important. In addition, transverse momentum resummation has also found applications in crossed processes involving the action of electromagnetic currents, especially doubly-inclusive hadron production in leptonic annihilation and single-inclusive hadron production in deep-inelastic scattering.

In this short paper, we show how in these cases there are natural crossed analogs of threshold and joint resummations for Drell-Yan (DY) cross sections. Extensions of this type for threshold resummation were previously discussed in Refs. \textsuperscript{[1,2,3,4]}. Here we will introduce specific new observables in single-inclusive deep-inelastic scattering (SIDIS) and double-inclusive leptonic annihilation (DIA). These are related to the doubly-differential distributions already studied at next-to-leading logarithm in \textsuperscript{[5,6,7]}. We will see that the singly-differential threshold-resummed cross sections in this set share a transparent structure that extends to all logarithms.

These considerations also have implications for non-perturbative power corrections in these cross sections, through the running of the QCD coupling \textsuperscript{[11,12]}. In principle, perturbation theory applies to all infrared safe observables, which depend only on large dimensional scales, $Q$, up to corrections that are formally suppressed by powers of $Q$. There is a close relationship between resummed perturbative predictions and power corrections, however, because the perturbative series does not converge. In our analysis of observables related by crossing we will propose a universal structure for both resummed logarithmic and power corrections. Such universality conjectures on power corrections have previously been tested extensively in average and differential jet event shapes in $e^+e^-$ annihilation cross sections, with phenomenological success.

In Sec. \textsuperscript{II} we introduce the cross sections in DY, DIA, and SIDIS that we will consider in this paper. Section \textsuperscript{III} analyzes their underlying partonic cross sections near threshold, and in Sections \textsuperscript{IV} and \textsuperscript{V} we investigate the corresponding threshold resummation. In Sec. \textsuperscript{VI} we discuss the universality properties of the cross sections, and we extend our results to the case of joint resummation.

II. CROSSED THRESHOLD VARIABLES

We consider processes that are characterized by the lowest-order (LO) partonic reaction $q\bar{q} \to \gamma^*$ or one of its crossed versions. These are the Drell-Yan (DY) process $h_1h_2 \to \ell^+\ell^-X$, “semi-inclusive” deeply-inelastic scattering (SIDIS) $\ell h_1 \to \ell h_2 X$, and “double-inclusive leptonic annihilation” (DIA), $\ell^+\ell^- \to h_1h_2 X$, where $h_{1,2}$ denote hadrons. For each of these processes, we are interested in the cross section with the least restrictive kinematics, and the crossed versions of the variable that controls threshold logarithms for DY cross sections.

In the DY process, the most inclusive observable is the usual “total” cross section, differential only in the variable

$$\tau_{DY} \equiv \frac{Q^2}{(P_1+P_2)^2},$$

(1)

where $Q$ is the invariant mass of the lepton pair, and $P_{1,2}$ are the momenta of the initial hadrons, so that the denominator is the hadronic center of mass energy squared. Cross sections for SIDIS and DIA have customarily been considered differential in two light-cone scaling variables, one associated with each of the two hadrons \textsuperscript{[2,12,13]}. For example, in the case of SIDIS, one usually employs the Bjorken variable $x \equiv Q^2/2P_1 \cdot q$ (with $q$ the momentum of the virtual photon; $Q^2 \equiv -q^2$) and the “fragmentation” variable $z \equiv P_1 \cdot P_2/P_1 \cdot q$ and studies the cross

...
section differential in both of these. This treatment is also followed in experimental studies. In contrast to this we define, in analogy with the DY process, the “τ-variable”

\[ \tau_{\text{SIDIS}} \equiv xz = \left( \frac{Q^2}{2P_1 \cdot q} \right) \left( \frac{P_1 \cdot P_2}{P_1 \cdot q} \right), \]

and consider the cross section differential in \( \tau_{\text{SIDIS}} \). We are not aware of a discussion of this cross section in the earlier literature. As we shall show below, this cross section has remarkable similarities with the DY cross section. Given the experimental studies of SIDIS in terms of \( x \) and \( z \), it should actually be relatively straightforward to perform measurements of the cross section differential in the variable \( \tau_{\text{SIDIS}} \) as well.

Finally, for DIA, we follow to define \( x \equiv 2P_1 \cdot q/Q^2 \), and \( z \equiv P_1 \cdot P_2/P_1 \cdot q \) and, following the same logic as above, we shall consider the cross section differential in

\[ \tau_{\text{DIA}} \equiv xz = \left( \frac{2P_1 \cdot q}{Q^2} \right) \left( \frac{P_1 \cdot P_2}{P_1 \cdot q} \right) = \left( \frac{P_1 + P_2}{Q^2} \right), \]

where for the last equality we have neglected the masses of the produced hadrons to introduce their pair invariant mass \((P_1 + P_2)^2\).

Each of the cross sections discussed above is given by a convolution of parton distribution functions \( f_i^h(\xi, \mu) \) in hadron \( h \) and/or hadron fragmentation functions \( D_i^h(\xi, \mu) \) with partonic hard-scattering functions, where \( i \) runs over quarks, antiquarks, and gluons, and \( \mu \) is a factorization scale. We shall collectively refer to the parton distributions and fragmentation functions as \( F_i^h(\xi, \mu) \). The functions depend on the variable \( \xi \), which for the parton distributions is the light-cone momentum fraction of the initial hadron momentum taken by the parton, while for the fragmentation functions it is the momentum fraction that the produced hadron takes from the parent parton. Denoting parton momenta by \( p \) and hadron momenta by \( P \), we have \( p = \xi P \) for the initial partons and \( p = P/\xi \) for the final-state ones. The partonic hard-scattering functions will be denoted by \( \omega_i^j \), where again \( i, j \) run over parton types, and \( A = DY, \) SIDIS, DIA. The \( \omega_i^j \) are perturbative; they begin at lowest order with the simple processes \( q\bar{q} \rightarrow \gamma^* \) (or crossed). For each of the processes we consider, we introduce a partonic τ-variable in terms of the corresponding partonic momenta:

\[ \hat{\tau}_{\text{DY}} \equiv \frac{Q^2}{(P_1 + P_2)^2} = \frac{\tau_{\text{DY}}}{\xi_1 \xi_2}, \]
\[ \hat{\tau}_{\text{SIDIS}} \equiv \frac{Q^2}{2P_1 \cdot q} \left( \frac{P_1 \cdot P_2}{P_1 \cdot q} \right) = \frac{\tau_{\text{SIDIS}}}{\xi_1 \xi_2}, \]
\[ \hat{\tau}_{\text{DIA}} \equiv \frac{2P_1 \cdot q}{Q^2} \left( \frac{P_1 \cdot P_2}{P_1 \cdot q} \right) = \frac{\tau_{\text{DIA}}}{\xi_1 \xi_2}. \]

Apart from dependence on the strong coupling \( \alpha_s(\mu) \) and on the ratio \( Q/\mu \), where \( \mu \) is the renormalization/factorization scale, the \( \omega_i^j \) will be functions only of the variable \( \hat{\tau}_A \). The LO partonic cross section is in each case proportional to \( \delta(1 - \hat{\tau}_A) \). From now on, we will choose \( \mu = Q \) throughout.

In the following, we shall mostly be interested in the behavior of the partonic cross sections at large \( \hat{\tau}_A \), \( \hat{\tau}_A \rightarrow 1 \). Since the parton distributions and fragmentation functions are steeply falling functions of the \( \xi \), they emphasize the region \( \xi_1 \xi_2 \sim \hat{\tau}_A \), and therefore \( \hat{\tau}_A \sim 1 \). For \( \hat{\tau}_A \rightarrow 1 \), the partonic cross sections at higher orders in perturbation theory develop large logarithmic corrections. For example, for the Drell-Yan process, the emission of a single gluon in the process \( q\bar{q} \rightarrow \gamma^* \) gives rise to a leading term of the form \( \alpha_s \ln (1 - \hat{\tau}_{\text{DY}})/(1 - \hat{\tau}_{\text{DY}}) \), where the plus-distribution makes the cross section integrable at \( \hat{\tau}_{\text{DY}} = 1 \) in the standard way. At yet higher orders in \( \alpha_s \), one finds leading terms of the form \( \alpha_s^k \left[ \ln (1 - \hat{\tau}_{\text{DY}})/(1 - \hat{\tau}_{\text{DY}}) \right]^+ \), plus subleading terms that are down by one or more powers of the logarithm.

As we shall discuss below, the logarithms arising at \( \hat{\tau}_A \rightarrow 1 \) have a universal structure in the three cross sections we are discussing in this paper. They are associated with emission of relatively soft gluons into the final state. The analysis of the large corrections is most conveniently performed in a reference frame where the energy of the emitted soft gluons is the only relevant quantity. For the Drell-Yan process, this approach has been followed in earlier treatments, where the resummation of the large logarithms to all orders of perturbation theory was derived. The frame chosen here is the center-of-mass frame of the initial hadrons. The large corrections then arise when the partons have “just enough” energy to produce the final state. For this reason, the logarithms are also referred to as threshold logarithms, and their resummation as threshold resummation.

Restricting ourselves to the terms in the partonic cross sections that dominate at large \( \hat{\tau}_A \), we find the following generic structure for our three cross sections of interest:

\[ \frac{d\sigma_A(\tau_A)}{d\tau_A} = \sigma_A^0 \sum_{i = q, \bar{q}} \int_{\tau_{\text{DY}}}^{1} \frac{d\xi_1}{\xi_1} f_{i}^{h_1}(\xi_1, Q) \times \int_{\tau_{\text{DY}}/\xi_1}^{1} \frac{d\xi_2}{\xi_2} f_{i}^{h_2}(\xi_2, Q) \omega_{i}^{\delta}(\tau_A, \alpha_s(\xi_1, Q)), \]

where for \( A = DY, \) \( f_{i}^{h_1} \equiv f_{i}^{h_1}, f_{i}^{h_2} \equiv f_{i}^{h_2}, \) for \( A = SIDIS, \) \( f_{i}^{h_1} \equiv f_{i}^{h_1}, f_{i}^{h_2} \equiv D_{i}^{h_2}, \) and for \( A = DIA, \) \( f_{i}^{h_1} \equiv D_{i}^{h_1}, f_{i}^{h_2} \equiv D_{i}^{h_2}. \) The normalization \( \sigma_A^0 \) is specific to each process; it may depend on additional variables such as lepton scattering angles (the cross section may also be differential in these, in addition to \( \tau_A \)). It is chosen in such a way that each of the \( \omega_{i}^{\delta} \) begins with \( \delta(1 - \hat{\tau}_A) \) at LO. Note that the sum in Eq. (5) only runs over quarks and antiquarks and is diagonal in flavor. This is because soft gluon emission gives rise to threshold logarithms only in the process \( q\bar{q} \rightarrow \gamma^* \) (or crossed). Partonic channels with an initial gluon, or with a gluon fragmenting into the observed hadron, are suppressed near threshold. For this reason, we keep only a single partonic index on \( \omega_A. \)
Also note that in general there could be two forms of the term in Eq. 3 for a given process, depending on whether the virtual photon carries transverse or longitudinal polarization. For example, in DIS, there are contributions involving the structure functions $F_1$ and $F_L$. However, it turns out that near partonic threshold the longitudinal polarization component is suppressed, so that there is only a single structure like Eq. 4 for each process.

In the following, it will be convenient to introduce Mellin moments of the cross sections in Eq. 5 in $\tau_A$. Defining for any function $f(x)$ the moments $\tilde{f}(N) \equiv \int_0^1 dx x^{-N-1} f(x)$, we find:

$$\tilde{\sigma}_A(N) = \sigma_A^0 \sum_{i=q,\bar{q}} \tilde{F}_i^{h_1}(N,Q) \tilde{F}_i^{h_2}(N,Q,\omega_A(N,\alpha_s(Q))).$$

(6)

Small $1 - \tilde{\tau}_A$ corresponds to large $N$. The inverse transformation reads:

$$\frac{d\sigma_A}{d\tau_A} = \sigma_A^0 \sum_{i=q,\bar{q}} \int_C \frac{dN}{2\pi i} \tau_A^{-N} \tilde{F}_i^{h_1}(N,Q) \tilde{F}_i^{h_2}(N,Q) \times \tilde{\omega}_A(N,\alpha_s(Q)),$$

(7)

where $C$ denotes a contour in complex-$N$ space. In moment space, the partonic hard-scattering functions have the perturbative expansion

$$\tilde{\omega}_A^{(1)}(N,\alpha_s(Q)) = 1 + \frac{\alpha_s(Q)}{2\pi} \tilde{\omega}_A^{(1)}(N) + O(\alpha_s(Q)^2).$$

(8)

The explicit forms of the coefficients $\tilde{\omega}_A^{(1)}(N)$ may be obtained from results in the literature [12, 15, 16]. One finds for large $N$:

$$\tilde{\omega}_A^{(1)}(N) = 4 \ln^2(N) - 2 + \frac{\pi^2}{3} + O(1/N)$$

$$= \tilde{\omega}_A^{(1)}(N),$$

$$\tilde{\omega}_A^{(1)}(N) = \tilde{\omega}_A^{(1)}(N) - \pi^2 + O(1/N),$$

(9)

where $\tilde{N} \equiv N e^{\gamma_E}$ with $\gamma_E$ the Euler constant. The logarithmic term is the moment-space equivalent of the threshold logarithm $\ln(1-\tilde{\tau}_A)/(1-\tilde{\tau}_A)$ in $\tilde{\tau}_A$ space mentioned above. As we anticipated, this term is universal in all $\tilde{\omega}_A^{(1)}(N)$. The $N$-independent pieces partly result from virtual corrections, which explains the difference $\sim \pi^2$ between the time-like DY and DIA processes and the space-like SIDIS. We will show below that the close relationship between the $\tilde{\omega}_A^{(1)}$ extends to all orders.

III. PHASE SPACE NEAR PARTONIC THRESHOLD

The one-loop double logarithmic structure in the moment variable $N$ exhibited in the previous subsection can be generalized and resummed to all orders in each of the DY, SIDIS and DIA processes. The same reasoning will enable us to exhibit jointly resummed cross sections that organize the singular behavior in both $N$ and the impact parameter $b$ conjugate to the transverse momentum of soft gluon radiation. The key to these results is an analysis of the phase space available to partonic radiation near threshold in each case. We will find that, by a suitable choice of frame, all logarithmic behavior in the partonic variables $\tau_A$, $A = DY$, SIDIS or DIA, arises from a region where there is a limitation on the total energy of partonic radiation. The powerful consequences of this restriction were explored in detail for Drell-Yan processes in Ref. [5]. We derive the analogous results for SIDIS and DIA in the following section. Here, we review the DY phase space and provide its extensions to the other cases.

For Drell-Yan processes $h_1(P_1)h_2(P_2) \rightarrow \ell^+\ell^- X$, we choose the overall partonic rest frame, with

$$(p_1 + p_2)^\mu = \sqrt{s}\delta_{\mu 0}.$$  

(10)

The relation between the observed vector boson momentum $q$ and the momentum $k$ of unobserved radiation is

$$p_1 + p_2 = q + k.$$  

(11)

In the rest frame, we then evaluate the partonic $\tilde{\tau}_A$ variable and find

$$\tilde{\tau}_{DY} = \frac{(p_1 + p_2 - k)^2}{s} = 1 - \frac{2k_0}{Q} + O[(1 - \tilde{\tau}_{DY})^2].$$  

(12)

Thus, the difference between $\tilde{\tau}_{DY}$ and unity is twice the total energy of partonic radiation divided by $Q$, up to corrections that vanish as a power for $\tilde{\tau}_{DY} \rightarrow 1$, and which therefore do not affect logarithmic behavior in the moments.

To construct a similar analysis for the SIDIS process $\ell h_1(P_1) \rightarrow \ell h_2(P_2) X$, we choose a partonic Breit frame where

$$p_1 = (p_1^+,0^+,0_T),$$

$$q = (-q^+,Q^2/2q^+,0_T),$$

$$p_1^+ = q^-.\quad (13)$$

As before, we define $k^\mu$ as the momentum of all unobserved radiation,

$$p_1 + q = p_2 + k.$$  

(14)

We then have $k_T = -p_{2,T}$ for the transverse components. A brief calculation gives

$$\tilde{\tau}_{SIDIS} = \left(1 - \frac{k_T^2}{2p_1^+ q^-} - k^+_{1\ell} / p_1^+ \right) \left(1 - k^-_{1\ell} / q^- \right)$$

$$\sim 1 - \frac{2k_0}{Q} + O[(1 - \tilde{\tau}_{SIDIS})^2],$$  

(15)

a result precisely analogous to [12], up to nonsingular corrections.
Finally, for DIA $\ell^+\ell^- \to h_1(P_1)h_2(P_2)X$, we use the overall rest frame, with
\[ q^\mu = Q\delta_{\mu 0}. \] (16)
The relation between the observed momenta and inclusive radiation in this case is
\[ q = p_1 + p_2 + k, \] (17)
and we again find:
\[ \hat{\tau}_{\text{DIA}} = \frac{(q - k)^2}{2p_1 \cdot q} = 1 - 2k_0^2/Q + O\left[(1 - \hat{\tau}_{\text{DIA}})^2\right]. \] (18)
Thus, in each case the available phase space is most easily characterized by a limitation on the total energy of gluon radiation.

IV. THRESHOLD RESUMMATION
Near partonic threshold, in each of the cross sections discussed above singular behavior appears as plus-distributions, up to $\ln^{2n-1}(1 - \hat{\tau}_A)/(1 - \hat{\tau}_A)_+$ in momentum space and $\ln^{2n} N$ in Mellin-moment space at $n$th order in $\alpha_s$, with $A = \text{DY, SIDIS, DIA}$. Using the results of the previous section, we can write a universal form for threshold resummation in these processes.

As discussed in some detail in Ref. [8], all singular corrections in the region near partonic threshold for the Drell-Yan process can be factorized into parton distribution functions in convolution with an inclusive cross section for the production of soft radiation with total momentum $k$. In the appropriate frame, corrections to this factorization are suppressed by powers of the energy, $k_0/Q$ to the hard-scattering scale $Q$, in momentum space, and by powers of the moment variable $N$ in the transform space.

Because we have established that the threshold regions in SIDIS and DIA are also characterized by small energy of soft gluon radiation $k_0$, essentially identical arguments apply in these cases as well. The only significant difference is that, as in Eq. (5), for SIDIS one of the parton distributions is replaced in the convolution by a fragmentation function, while for DIA both are replaced.

For soft emission, the hard scattering functions $\tilde{\omega}_A^i$ may be evaluated in the eikonal approximation for the quarks and/or antiquarks involved in the hard scattering. The fermions are then represented by recoilless color sources, characterized by velocities $\beta$ and $\beta'$ in opposite directions along the axis defined by the relevant center-of-mass frames. In these frames, the $\tilde{\omega}_A^i$ are invariant under both boosts and rescalings of the eikonal velocities $\beta, \beta'$. In moment space, the eikonal hard scattering functions exponentiate [9, 17].

\[ \tilde{\omega}_A^i(N, Q) = \exp \left\{ \int_0^{Q^2 - k_0^2} dk^2 \frac{W^i_{\lambda}(k_T^2, k_T^2 + k^2, \mu^2, \alpha_s(\mu, \varepsilon), \varepsilon)}{2\pi^2} \right\} \times \left[ \frac{2N \sqrt{k_T^2 + k^2}}{Q^2} \ln \frac{Q^2}{k_T^2 + k^2} \right] + \frac{2}{(k_T^2)^{1-\varepsilon}} \ln N A_i(\alpha_s(k_T, \varepsilon)) \right\}, \] (19)
where we have chosen dimensional regularization with $d = 4 - 2\varepsilon$ dimensions, in order to make all contributions to the exponent individually finite. We have suppressed the finite dependence on $\varepsilon$ in the function $\tilde{\omega}_A^i$. All dynamical information in Eq. (19) is contained in the “web” functions $W^i_{\lambda}(k_T^2, k_T^2 + k^2, \mu^2, \alpha_s(\mu, \varepsilon), \varepsilon)$. The web functions possess the same boost-invariance and scaling properties as the full eikonal cross sections. As such, they can depend only on the boost invariant quantities $k^2$ and $k_T^2$, the squared total invariant mass and transverse momentum of radiation. In the frame we have chosen, $k^2 + k_T^2 = k \cdot \beta k \cdot \beta'/(|\beta' - \beta|)$. The web functions may also depend on the boost-invariant signs: $\text{sgn}(\beta' \cdot k)$, which in general leads to differences between the web functions for the various processes at a non-leading level. An example at one loop is the $\pi^2$ term in Eq. (19).

Webs can be defined graphically in terms of sums of diagrams that are irreducible by cutting two eikonal lines, and the web functions are the result of summing over all final-state cuts of these diagrams, integrating over phase space at fixed total final-state momentum, $k$. The web functions for the three cross sections under consideration are all related simply by crossings of the eikonal lines.

The boost invariance of the webs allows us to integrate over one light-cone component of $k$, resulting in the Bessel function $K_0$ in Eq. (19), with the specific boost-invariant momentum dependence shown. Corrections to (19) are exponentially suppressed in $N$ for all contributions with radiation.

The logarithmic term in the square brackets accounts for virtual corrections, that is, final states without radiation. Only these contributions are sensitive to the upper limits of the $k$ integrals for large $N$. We will see that in (19) the virtual corrections are defined to set $\tilde{\omega}_A^i$ to zero.
at \( N = 0 \). This condition, of course, does not affect the logarithmic large-\( N \) behavior in moment space, or the singular \( \tau_n \) behavior in momentum-space convolutions.

The final term in (19) represents the subtraction of collinear singularities. In this term, the anomalous dimension \( \alpha_i(\alpha_s) \) is the same coefficient that appears in the single-log term in the moments of the \( i \rightarrow \bar{i} \) splitting function, \( P_{i\bar{i}}(\alpha_s, N) = -\alpha_i(\alpha_s) \ln N + \ldots \). \( \alpha_i(\alpha_s) \) is the same for parton distributions and for fragmentation functions because it is determined by the elastic form factor. \( i \) and \( \bar{i} \), which are both either a quark or an antiquark, have the same subtraction of collinear singularities, hence the factor 2 in this term of Eq. (19).

Finally, we recall the very useful property that the web functions are invariant under variations of the renormalization scale \( \mu \):

\[
\frac{\partial}{\partial \ln \mu^2} \ln A \left( k_T^2, k_T^2 + \mu^2, \mu, \epsilon \right) = 0. \tag{20}
\]

This enables us to shift the scale of the running coupling with the integration over momentum, a feature that we will exploit in the next section.

In the first term of this expression, all \( k_T \)-dependence is in the web function at fixed \( u \), and it is natural to define an anomalous dimension that is a function of \( u \) only:

\[
\rho_A^i(\alpha_s(u, \epsilon), \epsilon) \equiv \int_0^{u^2} \frac{dk_T^2}{k_T^2} (k_T^2)^{-\epsilon} \ln N \mathcal{A} \left( k_T^2, u^2, \alpha_s(u, \epsilon) \right) \times W_A \left( k_T^2, u^2, \alpha_s(u, \epsilon) \right). \tag{23}
\]

The function \( \rho_A^i \) defined in this fashion is dimensionless, so that the overall factor \( u^{-2} \) carries all dimensional information. Note that in the subtraction term of Eq. (22) the power is also \( u^{-2} \) with no dependence on \( \epsilon \). These integrals are defined by reexpanding the coupling \( \alpha_s(u, \epsilon) \) in terms of the coupling at a fixed scale, for instance \( \alpha_s(Q) \). As long as \( \epsilon < 0 \), the running coupling vanishes for \( u^2 \rightarrow 0 \) order-by-order in this expansion, and the integrals all exist. Our ignorance of the true behavior of \( \alpha_s(u, \epsilon) \), however, may be considered a signal of power corrections \( \alpha^{(2)} \), which we discuss briefly below.

In terms of \( \rho_A^i \) the eikonal hard-scattering function simplifies to

\[
\ln \tilde{\omega}_A^i(N, Q) = \int_0^{Q^2} du^2 \left\{ \int_0^{u^2} \frac{dk_T^2}{k_T^2} (k_T^2)^{-\epsilon} \ln N \mathcal{A} \left( k_T^2, u^2, \alpha_s(u, \epsilon) \right) \right. \\
\times \left. \left[ K_0 \left( \frac{2Nu}{Q} \right) - \ln \frac{Q}{u} \right] + \frac{2}{u^2} \ln N \mathcal{A} \right( \alpha_s(u, \epsilon) \right) \right\}. \tag{22}
\]

In this section we will provide a new analysis of the exponents of the resummed cross sections in moment space, Eq. (19), that will for the first time relate the exponent in terms of webs to the anomalous dimensions that have customarily been introduced to generate nonleading logarithms. These have variously been denoted by \( g_3(\alpha_s) \) and \( D(\alpha_s) \), and are process dependent. We will see that for the set of cross sections discussed here, these anomalous dimensions are very closely related.

In the double integral of Eq. (19), we change variables from \( k^2 \) to

\[
u^2 \equiv k^2 + k_T^2, \tag{21}\]

and in the integral over the term proportional to \( A_i \) we separate the angular integration and relabel the radial variable \( k_T^2 \) as \( u^2 \). Using the renormalization scale invariance of the web functions, Eq. (20), we choose \( \mu = u \) and write the logarithm of the eikonal cross section as

\[
\ln \tilde{\omega}_A^i(N, Q) \tag{24}
\]

The double-logarithmic structure of the exponential is manifest in this form, with additional logarithms associated only with the running of the QCD coupling. Both \( u \) integrals show a (collinear) divergence at \( u^2 = 0 \). Factorization theorems require that these divergences cancel between the eikonal cross section and the collinear subtraction (the \( A \) term). This implies that the function \( \rho_A^i \) is given, up to terms that vanish at \( \epsilon = 0 \), by the universal \textquotedblleft cusped\textquotedblright anomalous dimensions \( A_i \):

\[
\rho_A^i(\alpha_s(u, \epsilon), \epsilon) = 2A_i(\alpha_s(u, \epsilon)) + F_A^i(\alpha_s(u, \epsilon), \epsilon), \tag{25}
\]

where \( F_A^i \) is a function that vanishes at \( \epsilon = 0 \). However, its integral over \( u \) in Eq. (24) need not vanish.
We can isolate the contribution of $F_A^i$ to the hard-scattering function in (24) by an integration by parts. For this purpose, we introduce a new function, $D_A^i$, defined by

$$
\int_0^Q \frac{d^2u}{u^2} F_A^i(\alpha_s(u, \varepsilon), \varepsilon) = D_A^i(\alpha_s(Q, \varepsilon), \varepsilon).
$$

The function $D_A^i(\alpha_s, \varepsilon)$ does not necessarily vanish at $\varepsilon = 0$, and is related to $F_A^i$ by

$$
F_A^i(\alpha_s(u, \varepsilon), \varepsilon) = \frac{\partial}{\partial \ln u^2} D_A^i(\alpha_s(u, \varepsilon), \varepsilon).
$$

For the logarithm of the eikonal hard-scattering function, we now have, after an integration by parts, a universal behavior. For the second term in Eq. (28), the second term in Eq. (20) vanishes rapidly, leaving a $\ln(\bar{N}/Nu)$.

Here we have set $\varepsilon = 0$ on both sides of the equation, because the integrals on the right are now both finite in this limit. The first $(A)$ term generates the leading double logarithms, because for $u \gg Q/N$ the Bessel function $K_0(2Nu/Q)$ vanishes rapidly, leaving a $\ln(\bar{N}/Nu)$ behavior. For the second $(D_A^i)$ term, the derivative of the expression in square brackets is again power-suppressed when $u \ll Q/\bar{N}$, but lacks the logarithmic enhancement for larger $u$, where it behaves simply as $1/u^2$.

$$
\frac{\partial}{\partial u^2} \left[ K_0 \left( \frac{2Nu}{Q} \right) - \ln \frac{Q}{u} \right] 
\sim \frac{N^2}{Q^2} \left( \ln \frac{\bar{N}u}{Q} - \frac{1}{2} \right), \quad \frac{\bar{N}u}{Q} \ll 1,
$$

$$
\sim \frac{1}{2u^2}, \quad \frac{Nv}{Q} \gg 1.
$$

Because the function $D_A^i$ receives no contributions from a single-gluon final state, the second term in Eq. (28) begins at next-to-next-to-leading logarithm in the moment variable. This term generalizes the $D$-terms found in Drell-Yan threshold resummation [3, 4, 19]. We note again that this result is found by consistently using the coupling in $4 - 2\varepsilon$ dimensions, with $\varepsilon < 0$. The final expression, however, is finite for $\varepsilon \to 0$, as it must be in the short-distance function $\tilde{\omega}_A^i(N, Q)$.

We see explicitly from (28) that the expression in Eq. (19) vanishes for $N \to 0$, which can be thought of as a normalization condition for virtual corrections. This confirms that the resummed short-distance function has been constructed to consist of plus distributions only. It may, however, be corrected by $N$-independent constants [18].

VI. DISCUSSION

We have shown that the full formalism for threshold resummation can be extended from the Drell-Yan cross section to its crossed relatives in single-inclusive deep-inelastic scattering and double-inclusive annihilation. The hard-scattering functions for all of these processes can now be expressed in terms of exponentiated integrals of eikonal web functions, related by crossing eikonal lines. We have also shown how process-dependent nonleading logarithms arise naturally in the eikonal formalism and that, within this set of diagrams, the relevant $(D_A^i)$ anomalous dimensions are all closely related. These results extend the set of observables that can be analyzed to all orders and logarithms in terms of a limited set of anomalous dimensions, and should facilitate the investigation of the relationship between perturbative and nonperturbative dynamics in these and related hard-scattering processes. Here we shall make a few observations on possible applications.

A. Joint resummation

Following the analysis of Ref. [4], the threshold resummed hard scattering functions above can all be extended to joint resummation. At leading power in $N$, the double transform to Mellin moment and impact parameter space in joint resummation, is derived by simply inserting the factor $\exp[b \cdot k_T]$ in the integral over final state momenta in Eq. (19):

$$
\ln \tilde{\omega}_A^i(N, Q, b) = 2 \int_0^Q \frac{d^2u}{u^2} \left\{ \int_{k_T^2 \leq u^2} \frac{d^2 \cdot k_T}{2\pi^{1-\varepsilon}/\Gamma(1-\varepsilon)} W_A^i(k_T^2, u^2, u, \alpha_s(u, \varepsilon), \varepsilon) \right. 
\times \left[ e^{b \cdot k_T} K_0 \left( \frac{2Nu}{Q} \right) - \ln \frac{Q}{u} \right] + \frac{2}{u^2} \ln \bar{N} A_i(\alpha_s(u, \varepsilon)) \right\}.
$$

That is, at leading power of $N$, all logarithms in $b$ are controlled by the same web functions as in threshold resummation. The extension to $b$-dependence at nonlead-
In the resummed expressions of Eq. (28), as in resummations of many other physically-relevant quantities, perturbative non-convergence arises from the integral over a momentum scale $u$ that can be identified with the argument of the strong coupling, $\alpha_s(u)$. In cases where $u = 0$ is the endpoint of the integral, re-expanding the running coupling $\alpha_s(u)$ in terms of any coupling at fixed scale, say, $\alpha_s(Q)$, leads to integrals that are finite for infrared-safe quantities, but which grow factorially with $Q$ in the large $Q$ limit. A conjecture of universality is natural when the function $f(\alpha_s) = f(\beta_0)$ is the same for different processes. Expanding the $K_0$ function, we find for each of these observables an expansion in even powers (only) of $N/Q$ and $bQ$.

Many investigations of power corrections have been carried out in this fashion \cite{10, 11, 20}, normally on the basis of cross sections that have been resummed to next-to-leading logarithm. It is interesting, therefore, to make a similar analysis in these cases, where the resummation has been carried out in principle for all logarithms. This reasoning would seem to imply identical power corrections from the $A_i$ term in Eq. (28), which generates leading and next-to-leading logarithms in perturbation theory, since this anomalous dimension is the same in each of the cross sections. The anomalous dimensions $D_{iA}$, on the other hand, may vary between these processes, although only through the dependence of the web functions $W_{iA}$ on the signs of the invariants $\beta \cdot k$, $\beta' \cdot k$, as mentioned above. It will clearly be of interest to investigate further the possible influence of the $D_i$ terms on power corrections in these cross sections, where the perturbative structure is relatively simple.

### C. Phenomenological applications and generalizations

We close with the observation that the cross sections resummed in this paper can in principle be compared directly with data from deep-inelastic and leptonic annihilation processes over a wide range of energy scales. It may also be possible to generalize this analysis to a number of observables in hadron-hadron scattering, for example dihadron cross sections. We leave these applications and generalizations to future work.

### Acknowledgements

The work of G.S. was supported in part by the National Science Foundation, grants PHY-0354776, and PHY-0354822. W.V. is grateful to RIKEN, BNL and the U.S. Department of Energy (contract number DE-AC02-98CH10886) for providing the facilities essential for the completion of part of this work.

\begin{thebibliography}{99}
\item [1] W. Giele et al., QCD/SM working group: Summary report, contributed to Workshop on Physics at TeV Colliders, Les Houches, France, 21 May - 1 Jun 2001; arXiv:hep-ph/0204316, Chapter 3.
\item [2] Y. L. Dokshitzer, D. Diakonov and S. I. Trojan, Phys. Lett. B 79, 269 (1978); G. Parisi and R. Petronzio, Nucl. Phys. B 154, 427 (1979); G. Altarelli, R.K. Ellis, M. Greco and G. Martinelli, Nucl. Phys. B 246, 12 (1984); J.C. Collins, D.E. Soper and G. Sterman, Nucl. Phys. B 250, 199 (1985).
\item [3] G. Sterman, Nucl. Phys. B 281, 310 (1987).
\item [4] S. Catani and L. Trentadue, Nucl. Phys. B 327, 323 (1989); Nucl. Phys. B 353, 183 (1991).
\item [5] E. Laenen, G. Sterman, and W. Vogelsang, Phys. Rev. Lett. 84, 4296 (2000) arXiv:hep-ph/0002078; Phys. Rev. D 63, 114018 (2001) arXiv:hep-ph/0010080.
\item [6] A. Kulesza, G. Sterman and W. Vogelsang, Phys. Rev. D 66, 014011 (2002) arXiv:hep-ph/0202251; Phys. Rev. D 69, 014012 (2004) arXiv:hep-ph/0309264.
\item [7] A. Banfi and E. Laenen, Phys. Rev. D 71, 034003 (2005) arXiv:hep-ph/0411241.
\item [8] J.C. Collins and D.E. Soper, Nucl. Phys. B 193, 381 (1981); E. B 213, 545 (1983); Nucl. Phys. B 197, 446 (1982); R. Meng, F. I. Ohness and D. E. Soper, Phys. Rev. D 54, 1919 (1996) arXiv:hep-ph/9511311; P. Nadolsky, D. R. Stump and C. P. Yuan, Phys. Rev. D 61, 014003 (2000) [Erratum-ibid. D 64, 059903 (2001)] arXiv:hep-ph/9906280; Phys. Rev. D 64, 114011 (2001) arXiv:hep-ph/0012261; Phys. Lett. B 515, 175 (2001) arXiv:hep-ph/0012262; S. D. Ellis, D. G. Richards and W. J. Stirling, Phys. Lett. B 136, 99 (1984); D. de Florian and M. Grazzini, Nucl. Phys. B 704, 387 (2005) arXiv:hep-ph/0407241; D. Boer, Nucl. Phys. B 603, 195 (2001) arXiv:hep-ph/0102071; A. Idilbi, X. d. Ji, J. P. Ma and F. Yuan, Phys. Rev. D 70, 074021 (2004) arXiv:hep-ph/0406302; Y. Koike, J. Nagnashima and W. Vogelsang, Nucl. Phys. B 744, 59 (2006) arXiv:hep-ph/0602188.
\item [9] M. Cacciari and S. Catani, Nucl. Phys. B 617, 253 (2001) arXiv:hep-ph/0107138.
\item [10] M. Beneke and V.M. Braun, in the Boris Ioffe Festschrift at the frontier of particle physics / handbook of QCD, ed. M. Shifman (World Scientific, Singapore, 2001) vol. 3, p. 1719 arXiv:hep-ph/0010208.
\item [11] M. Dasgupta and G. P. Salam, J. Phys. G 30, R143 (2004) arXiv:hep-ph/0312283.
\end{thebibliography}
[12] G. Altarelli, R. K. Ellis, G. Martinelli and S. Y. Pi, Nucl. Phys. B 160, 301 (1979).
[13] D. de Florian and L. Vanni, Phys. Lett. B 578, 139 (2004) [arXiv:hep-ph/0310196].
[14] see, for example: A. Airapetian et al. [HERMES Collaboration], Eur. Phys. J. C 21, 599 (2001) [arXiv:hep-ex/0104004], and references therein.
[15] G. Altarelli, R. K. Ellis and G. Martinelli, Nucl. Phys. B 157, 461 (1979).
[16] D. de Florian, M. Stratmann and W. Vogelsang, Phys. Rev. D 57, 5811 (1998) [arXiv:hep-ph/9711387].
[17] G. Sterman, in AIP Conference Proceedings Tallahassee, Perturbative Quantum Chromodynamics, eds. D. W. Duke, J. F. Owens, New York, 1981, p. 22; J. G. M. Gatheral, Phys. Lett. B 133, 90 (1983); J. Frenkel and J. C. Taylor, Nucl. Phys. B 246, 231 (1984); C. F. Berger, arXiv:hep-ph/0305076.
[18] T. O. Eynck, E. Laenen and L. Magnea, JHEP 0306, 057 (2003) [arXiv:hep-ph/0305179].
[19] A. Vogt, Phys. Lett. B 497, 228 (2001) [arXiv:hep-ph/0010146]; S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B 726, 317 (2005) [arXiv:hep-ph/0506288]; V. Ravindran, arXiv:hep-ph/0512249; V. Ravindran, arXiv:hep-ph/0603041.
[20] G. P. Korchemsky and S. Tafat, JHEP 0010, 010 (2000) [arXiv:hep-ph/0007005]; E. Gardi and J. Rathsman, Nucl. Phys. B 609, 123 (2001) [arXiv:hep-ph/0103217]; E. Gardi and J. Rathsman, Nucl. Phys. B 638, 243 (2002) [arXiv:hep-ph/0201019]; E. Gardi and L. Magnea, JHEP 0308, 030 (2003) [arXiv:hep-ph/0306094]; C. F. Berger and L. Magnea, Phys. Rev. D 70, 094010 (2004) [arXiv:hep-ph/0407024].