Topological classes of rotating black holes

Di Wu

School of Physics and Astronomy, China West Normal University, Nanchong, Sichuan 637002, People’s Republic of China
(Dated: January 18, 2023)

In this paper, we investigate the topological numbers for the singly rotating Kerr black holes in all dimensions and the four-dimensional Kerr-Newman black hole. We show that for uncharged black holes, the rotation parameter has a significant effect on the topological number, and for rotating black holes, the dimension of spacetimes has a remarkable effect on the topological number too. The remaining part of this paper is organized as follows. In Sec. II, we first give a brief review of the topological approach, and then investigate the topological number of the four-dimensional Kerr black hole. In Sec. III, we extend the discussion in Sec. II to the cases of the \(d\)-dimensional singly rotating Kerr black holes. In Sec. IV, we turn to discuss the topological number of the four-dimensional Kerr-Newman black hole. Finally, we present our conclusions in Sec. V.

I. INTRODUCTION

As one of the most remarkable and fascinating objects in nature, the black hole has always been a subject of extensive theoretical and observational studies. On the observational side, recent years have witnessed remarkable successes in “seeing” the shadow of a black hole via the event horizon telescope (EHT) [1, 2] and “hearing” gravitational waves from black hole coalescence [3, 4]. On the theoretical side, the study of the topology of black holes has very recently shed new light on the nature of gravity by means of light rings [5–8], timelike circular orbit [9], and thermodynamic properties [10–19].

In particular, in Ref. [10], the black holes have been treated as topological thermodynamic defects by using the generalized off-shell free energy, and thus black hole solutions are divided into three different topological classes according to their different topological numbers, which shed new light on the fundamental nature of quantum gravity. Because of its simplicity and easy maneuverability of the procedure, the topological method proposed in Ref. [10] soon attracted a great deal of attention and was then successfully applied to investigate the topological numbers of some known black hole solutions [15, 17, 19], i.e., the Schwarzschild-AdS black hole solutions [15], the static black hole solutions in Lovelock gravity [16], the static Gauss-Bonnet-AdS black hole solutions [17], and the static black hole solution in nonlinear electrodynamics [19]. However, all of the above-mentioned progresses [10, 15–17, 19] are only restricted to the static cases, leaving the topological numbers of the rotating black holes unexplored. Up to date, the astronomical observation of black holes are basically rotating black holes, so it is very important and necessary to study the topological number of rotating black holes, which also provides the motivation of this paper.

In this paper, we investigate the topological numbers for the singly rotating Kerr black holes in arbitrary dimensions and the four-dimensional Kerr-Newman black hole. We show that for uncharged black holes, the rotation parameter has a significant effect on the topological number, and for rotating black holes, the dimension of spacetimes has a remarkable effect on the topological number too. The remaining part of this paper is organized as follows. In Sec. II, we first give a brief review of the topological approach, and then investigate the topological number of the four-dimensional Kerr black hole.

II. FOUR-DIMENSIONAL KERR BLACK HOLE

We first investigate the effect of the rotation parameter on the topological number of black holes. We begin with by considering the four-dimensional Kerr black hole case [20], whose metric in the Boyer-Lindquist coordinates has the form

\[
d s^2 = -\frac{\Delta_r}{\Sigma} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} [adt - (r^2 + a^2) d\phi]^2
\]

where

\[
\Delta_r = r^2 + a^2 - 2mr, \quad \Sigma = r^2 + a^2 \cos^2 \theta,
\]

in which \(m\) and \(a\) are the mass and the rotation parameters of the black hole. The thermodynamic quantities associated with the above solution (1) can be computed via the standard method and have the following exquisite expressions:

\[
M = m, \quad J = ma, \quad \Omega = \frac{a}{r_h^2 + a^2},
\]

\[
S = \pi (r_h^2 + a^2), \quad T = \frac{r_h^2 - a^2}{4\pi r_h (r_h^2 + a^2)},
\]

where \(r_h = m \pm \sqrt{m^2 - a^2}\) are the locations of the event and Cauchy horizons.

With the expressions of the thermodynamical quantities in hand, we are now ready to investigate the topological number...
of the four-dimensional Kerr black hole. As shown in Ref. [10], one can first introduce the generalized off-shell free energy as

$$ \mathcal{F} = M - \frac{S}{\tau}, $$ (3)

for a black hole system with mass $M$ and entropy $S$, where $\tau$ is an extra variable that can be considered as the inverse temperature of the cavity surrounding the black hole. Only when $\tau = 1/T$, the generalized free energy becomes on-shell.

In Ref. [10], a vector $\phi$ is established as

$$ \phi = \left( \frac{\partial \mathcal{F}}{\partial r_h}, -\cot \Theta \csc \Theta \right), $$ (4)

where the parameter $0 \leq \Theta \leq \pi$ is introduced for convenience and intuition. The component $\phi^\Theta$ is divergent at $\Theta = 0, \pi$, and the direction of the vector points outward there. Furthermore, as discussed in Ref. [17], the extremal points of the generalized free energy landscape exactly correspond to the on-shell black holes, so the zero point of the component $\phi^h$ exactly meets the black hole solution. The component $\phi^\Theta = 0$ will yield $\Theta = \pi/2$.

According to Duan’s $\phi$-mapping topological current theory, a topological current can be defined as [21–23]

$$ j^\mu = \frac{1}{2\pi} e^{\mu\nu\rho} \varepsilon_{ijk} \partial_i n^a \partial^r n^b, \quad \mu, \nu, \rho = 0, 1, 2, $$ (5)

where $\partial_i = \partial / \partial x^i$ and $x^i = (\tau, r_h, \Theta)$. The unit vector $n$ reads as $n = (n^\tau, n^r, n^\Theta)$, where $n^\tau = \phi^h / ||\phi||$ and $n^r = \phi^h / ||\phi||$. It is a simple matter to prove that the above topological current (5) is conserved, thus one can easily derive $\partial_\mu j^\mu = 0$. It is further shown that the topological current $j^\mu$ is a $\delta$-function of the field configuration [7, 22, 23]

$$ j^\mu = \delta^2(\phi) J^\mu \left( \frac{\phi}{x} \right), $$ (6)

where the three dimensional Jacobian $J^\mu (\phi / x)$ is defined as:

$$ e^{\mu\nu\rho} j^\mu (\phi / x) = e^{\mu\nu\rho} \partial_\nu n^a \partial_\rho n^b. $$

It is easy to see that $j^h$ equals to zero except when $\phi^h(x_i) = 0$, and one can derive the expressions of the topological number $W$ as:

$$ W = \int_S \delta^2(\phi) J^\mu \left( \frac{\phi}{x} \right), $$ (7)

where $\beta_i$ is the positive Hopf index counting the number of the loops of the vector $\phi^h$ in the $\phi$-space when $x^h$ are around the zero point $z_i$, $\eta_i = \text{sign}(J^0 (\phi / x)_{z_i}) = \pm 1$ is the Brouwer degree, and $w_i$ is the winding number for the $i$th zero point of $\phi$ that is contained in $S$. It is worth to note that if two loops $\Sigma_1$ and $\Sigma_2$ surround the same zero point of $\phi$, then they have the same winding number. On the other hand, if there is no zero point in the enclosed region, then one can have $W = 0$.

From the results already given above in Eq. (2), one can easily obtain the generalised free energy

$$ \mathcal{F} = \frac{r_h^2 + a^2}{2r_h} - \frac{\pi (r_h^2 + a^2)}{\tau}, $$ (8)

of the four-dimensional Kerr black hole. Then the components of the vector $\phi$ can be computed as

$$ \phi^h = \frac{1}{2} - \frac{a^2}{2r_h^2} - \frac{2\pi r_h}{\tau}, $$ (9)

$$ \phi^\Theta = -\cot \Theta \csc \Theta. $$ (10)

By solving the equation $\phi^h = 0$, one can obtain its solution that is depicted by a curve on the $r_h - \tau$ plane. For the four-dimensional Kerr black hole, one can get

$$ \tau = \frac{4\pi r_h^3}{r_h^2 - a^2}. $$ (11)

Fig. 1 shows the zero points of the component $\phi^h$ for the four-dimensional Kerr black hole with $a = r_0$, where $r_0$ is an arbitrary length scale set by the size of a cavity enclosing the black hole. For large $\tau$, such as $\tau = \tau_2$, there are one and two intersection points for the Schwarzschild and Kerr black holes, respectively. The intersection points exactly satisfy the condition $\tau = 1/T$, and therefore represent the on-shell black hole solutions with the characteristic temperature $T = 1/\tau$. Similar to the RN black hole [10], but in contrast to the Schwarzschild black hole, the two intersection points for the Kerr black hole can coincide each other when $\tau = \tau_2$, and then vanish when $\tau < \tau_c$. Especially, at the point $\tau_c = 6\sqrt{3} \pi a$, one can easily find $(d^2 \tau / d r_h^2) = 6\sqrt{3} \pi / a > 0$ for the four-dimensional Kerr black hole. This indicates that $\tau_c$ is a generation point, which can also be seen straightforward from Fig. 1. Furthermore, the generation point $\tau_c$ divides the Kerr black hole into the upper and lower branches with the winding number $w = -1$ and $w = 1$, respectively. Thus one obtains the topological number $W = 0$ for the four-dimensional Kerr black hole. Alternatively, one can also figure out the topological number for the four-dimensional Kerr black hole by plotting the unit vector field $n$ for arbitrarily chosen typical values.
FIG. 2. The red arrows represent the unit vector field \( \mathbf{n} \) on a portion of the \( r_h - \Theta \) plane. The unit vector field is plotted for the four-dimensional Kerr black hole with \( \tau / r_0 = 34.48 \) and \( a / r_0 = 1 \). The zero points (ZPs) marked with blue dots are at \( (r_h / r_0, \Theta) = (1.46, \pi / 2) \), and \( (2.15, \pi / 2) \), for ZP1 and ZP2, respectively. Due to the two winding numbers \( w_1 = 1 \) and \( w_2 = -1 \), the topological number is: \( W = w_1 + w_2 = 0 \).

(Note that \( \tau \) must be larger than \( \tau_c \)), for example, \( \tau / r_0 = 34.48 \) and \( a / r_0 = 1 \) in Fig. 2 where we find two zero points: ZP1 at \( r_h = 1.46 r_0 \) and ZP2 at \( r_h = 2.15 r_0 \), with the winding numbers \( w_1 = 1 \), \( w_2 = -1 \), respectively. So one can get the topological number \( W = w_1 + w_2 = 0 \) for the four-dimensional Kerr black hole. According to the classification proposal of black hole solutions, which is based upon their different topological numbers [10], the Kerr black hole and RN black hole are the same kind of black hole solutions since both of their topological numbers are equal to zero. In addition, since the topological number of the Schwarzschild black hole is -1, while that of Kerr black hole is 0, it implies that the rotation parameter has an important effect on the topological number for the uncharged black hole.

III. HIGHER DIMENSIONAL SINGLEY ROTATING KERR BLACK HOLES

In this section, we will extend the above discussion to the cases of a rotating black hole in higher dimensions by considering the \( d \)-dimensional singly rotating Kerr black holes. For the singly rotating Kerr black holes in arbitrary dimensions, the metric has the form [24, 25]

\[
d^2 = -\frac{\Delta_r}{\Sigma} \left( dt - a \sin^2 \theta d\phi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} \left[ a dt - (r^2 + a^2) d\phi \right]^2 + r^2 \cos^2 \theta d\Omega_{d-4}^2,
\]

where \( d\Omega_d \) represents the line element of a \( d \)-dimensional unit sphere, and

\[
\Delta_r = r^2 + a^2 - 2mr^{5-d}, \quad \Sigma = r^2 + a^2 \cos^2 \theta.
\]

The thermodynamic quantities are [25]

\[
M = \frac{d-2}{8\pi} \omega_{d-2} m, \quad J = \frac{\omega_{d-2} m a}{4\pi},
\]

\[
\Omega = \frac{a}{r_h^2 + a^2}, \quad S = \frac{\omega_{d-2}}{4} (r_h^2 + a^2) r_h^{d-4},
\]

\[
T = \frac{r_h}{2\pi} \left( \frac{1}{r_h^2 + a^2} + \frac{d-3}{2r_h^2} \right) - \frac{1}{2\pi r_h},
\]

where \( \omega_{d-2} = 2\pi^{(d-1)/2} / [\Gamma((d-1)/2)] \), and the horizon radius \( r_h \) of the black hole is determined by the equation: \( \Delta_r = 0 \).

From Eq. (13), one can get the generalized free energy

\[
\mathcal{F} = \frac{M}{T} - S \tau = \frac{\omega_{d-2} (d-2)}{16\pi r_h^{d-4}} \left( r_h^2 + a^2 \right) - \frac{\omega_{d-2} (r_h^2 + a^2) r_h^{d-4}}{4\pi}. \tag{14}
\]

The components of the vector \( \phi \) can be calculated as

\[
\phi^\tau = \frac{\omega_{d-2} t^{d-6}}{16\pi r_h} \left( (d-2) \left[ (d-3) \tau - 4\pi r_h \right] r_h^2 + a^2 \left[ (d-2) (d-5) - 4(d-4) \pi r_h \right] \right), \tag{15}
\]

\[
\phi^\Theta = -\cot \Theta \csc \Theta. \tag{16}
\]

By solving the equation \( \phi^\tau = 0 \), one can obtain

\[
\tau = \frac{4\pi [(d-2) r_h^2 + (d-4) a^2 r_h]}{(d-2) [(d-3) r_h^2 + (d-5) a^2]} \tag{17}
\]

as the zero point of the vector field \( \phi \).

As some examples, we show the zero points of the component \( \phi^\tau \) for five- to nine-dimensional singly rotating Kerr black holes with \( a = r_0 \) in Fig. 3, and the unit vector field \( n \) for five- to seven-dimensional singly rotating Kerr black holes in Fig. 4 with \( a = r_0 \), \( \tau = 2\pi r_0 \) (\( d = 5 \)), \( 3\pi r_0 / 2 \) (\( d = 6 \)), and \( 16\pi r_0 / 15 \) (\( d = 7 \)), respectively. It is worthwhile to note that
the unit vector field $n$ for eight- and nine-dimensional singly rotating Kerr black holes is similar to that for six- and seven-dimensional singly rotating Kerr black holes. The only difference is that the typical values are $\tau = 5\pi/6$ ($d = 8$) and $\tau = 24\pi/35$ ($d = 9$), but here we are not going to plot them again.

From Figs. 3 and 4, one can obtain the topological numbers of the higher dimensional singly rotating Kerr black holes. For instance, the topological number of the five-dimensional singly rotating Kerr black hole is $W = 1 - 1 = 0$, which is the same one as that of the four-dimensional Kerr black hole. However, via comparing Figs. 1-4, the topological numbers of all $d \geq 6$ singly rotating Kerr black holes are found to be equal to -1, which are different from the four- and five-dimensional Kerr black holes. To summarize, according to the topological approach proposed in Ref. [10], we find that the $d \geq 6$ singly rotating Kerr black holes and the $d = 4, 5$ singly rotating Kerr black holes belongs to two different topological classes. Therefore, the dimension of spacetimes has an important effect on the topological number of the rotating black holes, which are also reported in Ref. [16] for the higher-dimensional static uncharged black holes in Lovelock gravity theory.

### IV. FOUR-DIMENSIONAL KERR-NEWMAN BLACK HOLE

Finally, we want to explore the effect of the electric charge parameter on the topological number of the four-dimensional rotating black holes in the pure Einstein-Maxwell gravity theory. So in this section, we turn to investigate the topological number of the four-dimensional Kerr-Newman black hole [26, 27], whose metric and Abelian gauge potential are

$$
 ds^2 = \frac{\Delta_r}{\Sigma} \left( dt - a \sin^2 \theta d\phi \right)^2 - \frac{\Sigma}{\Delta_r} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} \left[ a dt - (r^2 + a^2) d\phi \right]^2 ,
$$

$$
 A = \frac{qr}{\Sigma} (dt - a \sin^2 \theta d\phi) ,
$$

where

$$
 \Delta_r = r^2 + a^2 - 2mr + q^2 , \quad \Sigma = r^2 + a^2 \cos^2 \theta .
$$

In the above, $m$ is the mass parameter, $a$ and $q$ are the rotation and electric charge parameters, respectively.

The thermodynamic quantities can be evaluated via the standard method as follows:

$$
 M = m , \quad J = ma , \quad \Omega = \frac{a}{r_h^2 + a^2} ,
$$

$$
 Q = q , \quad \Phi = \frac{qr}{r_h^2 + a^2} ,
$$

$$
 S = \pi (r_h^2 + a^2) , \quad T = \frac{r_h^2 - a^2 - q^2}{4\pi r_h (r_h^2 + a^2)} ,
$$

where $r_h = m \pm \sqrt{m^2 - a^2 - q^2}$ are the locations of the outer and inner horizons.
Using the results given by Eq. (20), one can directly obtain the generalized free energy of the Kerr-Newman black hole as

\[ \mathcal{F} = M - \frac{S}{\tau} = \frac{r_h^2 + a^2 + Q^2}{2r_h} - \frac{\pi(r_h^3 + a^2)}{\tau}, \quad (21) \]

and the components of the vector \( \phi \) can be computed as

\[ \phi^n = 1 - \frac{r_h^2 + a^2 + Q^2}{2r_h} - \frac{2\pi r_h}{\tau}, \quad (22) \]

\[ \phi^\Theta = -\cot \Theta \csc \Theta. \quad (23) \]

One can also get

\[ \tau = \frac{4\pi r_h^3}{r_h^2 - a^2 - Q^2}, \quad (24) \]

as the zero point of the vector field \( \phi \).

Similar to the procedure adopted in the last two sections, we show the zero points of the component \( \phi^n \) with \( a = r_0 \) and \( Q = r_0 \) in Fig. 5, and the unit vector field \( n \) with \( \tau = 50r_0 \), \( a = r_0 \), and \( Q = r_0 \) in Fig. 6. These two figures allow us to determine that the topological number of the four-dimensional Kerr-Newman black hole is \( W = 0 \), which is the same one as found for the four-dimensional Kerr black hole in Sec. II and the five-dimensional singly rotating Kerr black hole in Sec. III. In addition, this also seems to imply that the electric charge parameter has no effect on the topological number of rotating black holes. However, this conclusion needs to be further tested by investigating the topological numbers of many other rotating charged black holes.

**V. CONCLUSIONS**

In this paper, we have investigated the topological numbers of the singly rotating Kerr black holes in arbitrary dimensions and the four-dimensional Kerr-Newman black hole. Combined ours with those in Ref. [10], Table I summarizes some interesting results. The \( d \geq 6 \) singly rotating Kerr black holes and the Schwarzschild black hole belong to the first kind of topological classes since their topological number \( W = -1 \); the RN black hole, the Kerr-Newman black hole, and the \( d = 4, 5 \) singly rotating Kerr black holes belong to the second kind of topological classes due to their topological number \( W = 0 \); the RN-AdS black hole belongs to the third kind of topological classes since its topological number equals to 1.

In this work, we have arrived at two particularly exciting consequences: (i) the existence of the rotation parameter has an important effect on the topological number of the uncharged black hole, and (ii) the dimension of spacetimes has an important effect on the topological numbers of the rotating black holes. In addition, the fact that the topological numbers of the four-dimensional Kerr and Kerr-Newman black holes are identical seems to suggest that the electric charge parameter has no effect on the topological number of the rotating black holes. Our current work also supports the conjecture proposed in Ref. [10] that all black hole solutions should be classified into three different topological classes, at least in the pure Einstein-Maxwell gravity theory.

There are two further promising topics to be pursued in the future. One intriguing topic is to extend the present work to the more general cases with a nonzero cosmological constant.
and the black hole solutions in the supergravity and modified gravity theories. Another interesting object is to investigate the topological number of the black hole solutions with the planar [28], toroidal [29], and hyperbolic [30], as well as spindle [31–35] horizons to explore whether there is a relation between the topological number and the horizon topology of the black holes. We hope to report these related progress along these two directions soon.

ACKNOWLEDGMENTS

We thank Prof. Shao-Wen Wei for very helpful discussions. This work is supported by the National Natural Science Foundation of China (NSFC) under Grant No. 12205243, No. 11675130, by the Natural Science Foundation of Sichuan Province under Grant No. 2023NSFSC1347, and by the Doctoral Research Initiation Project of China West Normal University under Grant No. 21E028.

[1] The Event Horizon Telescope Collaboration, First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole, Astrophys. J. Lett. 875, L1 (2019).
[2] The Event Horizon Telescope Collaboration, First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way, Astrophys. J. Lett. 930, L12 (2022).
[3] B.P. Abbott et al. (LIGO Scientific and Virgo Collaborations), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116, 061102 (2016).
[4] B.P. Abbott et al. (LIGO Scientific and Virgo Collaborations), GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. 116, 241103 (2016).
[5] P.V.P. Cunha, E. Berti, and C.A.R. Herdeiro, Light Ring Stability in Ultra-Compact Objects, Phys. Rev. Lett. 119, 251102 (2017).
[6] P.V.P. Cunha and C.A.R. Herdeiro, Stationary Black Holes and Light Rings, Phys. Rev. Lett. 124, 181101 (2020).
[7] S.-W. Wei, Topological charge and black hole photon spheres, Phys. Rev. D 102, 064039 (2020).
[8] M. Guo and S. Gao, Universal properties of light rings for stationary axisymmetric spacetimes, Phys. Rev. D 103, 104031 (2021).
[9] S.-W. Wei and Y.-X. Liu, Topology of equatorial timelike circular orbits around stationary black holes, arXiv: 2207.08397.
[10] S.-W. Wei, Y.-X. Liu, and R.B. Mann, Black Hole Solutions as Topological Thermodynamic Defects, Phys. Rev. Lett. 129, 191101 (2022).
[11] S.-W. Wei and Y.-X. Liu, Topology of black hole thermodynamics, Phys. Rev. D 105, 104003 (2022).
[12] P.K. Yerra and C. Bhamidipati, Topology of black hole thermodynamics in Gauss-Bonnet gravity, Phys. Rev. D 105, 104053 (2022).
[13] M.B. Ahmed, D. Kubiznak, and R. B. Mann, Vortex/antivortex pair creation in black hole thermodynamics, arXiv: 2207.02147.
[14] P.K. Yerra and C. Bhamidipati, Topology of Born-Infeld AdS black holes in 4D novel Einstein-Gauss-Bonnet gravity, Phys. Lett. B 835, 137591 (2022).
[15] P.K. Yerra, C. Bhamidipati, and S. Mukherji, Topology of critical points and Hawking-Page transition, Phys. Rev. D 106, 064059 (2022).
[16] N.-C. Bai, L. Li, and J. Tao, Topology of black hole thermodynamics in Lovelock gravity, arXiv: 2208.10177.
[17] C.H. Liu and J. Wang, The topological nature of the Gauss-Bonnet black hole in AdS space, arXiv: 2211.05524.
[18] Z.-Y. Fan, Topological interpretation for phase transitions of black holes, arXiv: 2211.12957.
[19] C.X. Fang, J. Jiang, and M. Zhang, Revisiting thermodynamic topologies of black holes, arXiv: 2211.15534.
[20] R.P. Kerr, Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics, Phys. Rev. Lett. 11, 237 (1963).
[21] Y.-S. Duan and M.-L. Ge, SU (2) gauge theory and electrodynamics of N moving magnetic monopoles, Sci. Sin. 9, 1072 (1979).
[22] Y.-S. Duan, S. Li, and G.-H. Yang, The bifurcation theory of the Gauss-Bonnet-Chern topological current and Morse function, Nucl. Phys. B514, 705 (1998).
[23] L.-B. Fu, Y.-S. Duan, and H. Zhang, Evolution of the Chern-Simons vortices, Phys. Rev. D 61, 045004 (2000).
[24] R.C. Myers and M.J. Perry, Black holes in higher dimensional space-times, Ann. Phys. (N.Y.) 172, 304 (1986).
[25] S.-W. Wei, P. Cheng, and Y.-X. Liu, Analytical and exact critical phenomena of d-dimensional singly spinning Kerr-AdS black holes, Phys. Rev. D 93, 084015 (2016).
[26] E.T. Newman and A.I. Janis, Note on the Kerr spinning particle metric, J. Math. Phys. (N.Y.) 6, 915 (1965).
[27] E.T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating, charged mass, J. Math. Phys. (N.Y.) 6, 918 (1965).
[28] R.G. Cai and Y.Z. Zhang, Black plane solutions in four-dimensional space-times, Phys. Rev. D 54, 4891 (1996).
[29] D.R. Brill and J. Louko, Thermodynamics of (3+1)-dimensional black holes with toroidal or higher genus horizons, Phys. Rev. D 56, 3600 (1997).
[30] Y. Chen, Y.K. Lim, and E. Teo, Deformed hyperbolic black holes, Phys. Rev. D 92, 044058 (2015).
[31] D. Klemm, Four-dimensional black holes with unusual horizons, Phys. Rev. D 89, 084007 (2014).
[32] R.A. Hennigar, R.B. Mann, and D. Kubizňák, Entropy Inequality Violations from Ultraspinning Black Holes, Phys. Rev. Lett. 115, 031101 (2015).
[33] D. Wu, P. Wu, H. Yu, and S.-Q. Wu, Are ultraspinning Kerr-Sen-AdS4 black holes always superentropic?, Phys. Rev. D 102, 044007 (2020).
[34] D. Wu, S.-Q. Wu, P. Wu, and H. Yu, Aspects of the dyonic Kerr-Sen-AdS4 black hole and its ultraspinning version, Phys. Rev. D 103, 044014 (2021).
[35] D. Wu and S.-Q. Wu, Ultraspinning Chow’s black holes in six-dimensional gauged supergravity and their thermodynamical properties, J. High Energy Phys. 11 (2021) 031.