Relativistic breather-like solitary waves with linear polarization in cold plasmas

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Linearly polarized solitary waves, arising from the interaction of an intense laser pulse with a plasma, are investigated. New localized structures, in the form of exact, nonlinear solutions of the one-dimensional Maxwell-fluid model for a cold plasma with fixed ions are presented. Unlike stationary circularly polarized solitary waves, the linear polarization gives rise to a breather-like behavior and a periodic exchange of electromagnetic energy and electron kinetic energy at twice the frequency of the wave. A numerical method based on a finite-differences scheme allows to compute a branch of solutions within the frequency range \( \Omega_{\text{min}} < \Omega < \omega_{\text{pe}} \), where \( \omega_{\text{pe}} \) and \( \Omega_{\text{min}} \) are the electron plasma frequency and the frequency value for which the plasma density vanishes locally, respectively. A detailed description of the spatio-temporal structure of the waves and their main properties as a function of \( \Omega \) are presented. Small amplitude oscillations appearing in the tail of the solitary waves, a consequence of the linear polarization and harmonic excitation, are explained with the aid of the Akhiezer-Polovin system. Direct numerical simulations of the Maxwell-fluid model show that these solitary waves propagate without change for a long time.

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I. INTRODUCTION

Electromagnetic relativistic solitary waves, commonly named relativistic solitons, are self-trapped localized structures that are excited during the interaction of a high-intensity laser with a plasma. They typically consist on a region of rarified plasma where a high amplitude electromagnetic wave, with frequency below the laser frequency, is trapped. A solitary wave can be classified according to its polarization (linear or circular), its group velocity (standing or moving waves) and also its state (isolated structure or embedded in long laser pulses). Since part of the laser energy is dedicated to excite them, they may affect to certain laser applications like inertial confinement fusion, particle acceleration, plasma lens for ultra-intense laser focusing and high-brightness X- and gamma rays generation [1]. They also offer an excellent opportunity to confront theoretical and experimental works and enhance our knowledge on non-linear waves in plasmas.

This great variety of solitary waves has been the subject of an important research activity on plasma physics, including analytical work, numerical simulations and laboratory experiments. Pioneer works [2–4] were followed by an intense activity mainly in the framework of the relativistic fluid model. Exact 1-dimensional circularly polarized solitary waves solutions, including isolated [5–7] and embedded in long laser pulses [7–9] waves, were found. Single-hump and multi-hump solitons are possible. The analysis of linearly polarized waves, however, is more difficult mainly due to the high harmonic generation. Some one-dimensional solutions have been presented in the weak amplitude limit [10, 11] and in the framework of the Akhiezer-Polovin system [12]. Exact 2-dimensional linearly polarized solitary wave solutions of the fluid model were also found [13]. Solitary waves were observed in particle-in-cell (PIC) codes [14–16] and their footprints were detected in laboratory experiments [17–21].

In the theoretical works, the analysis was mostly focused on the spatio-temporal structures of the plasma and electromagnetic fields and the relationship between the amplitude of the electromagnetic wave and its frequency. A central role in the analysis was also played by the organization of the solitons in parameter space, i.e., in the velocity and frequency plane. Some solitary waves are organized in branches: given a value of the group velocity, the soliton exists for a single frequency value. Examples of this type are the circularly polarized moving solitons discovered in Ref. 6. On the other hand, other types of solitons, like the circularly polarized standing soliton found in Ref. 5 or the embedded solitons of Ref. 8 and [9], have a continuous spectrum. In this case, for a given velocity, solitons exist for any frequency within a certain range. The discrete or continuous character of the spectrum can be anticipated by using arguments from the dynamical system theory, including the dimension of the system, the characteristics of the stable and unstable manifolds of the fixed point where the orbit connects, and the Hamiltonian and/or reversible character of the system [8].

A common feature to all the previously cited 1-dimensional solitons is the stationary character of the soliton profile in a frame moving with its group velocity.
Under the traveling wave ansatz, the amplitude of the solutions just depends on the coordinate $\xi = x - Vt$. However, in recent fluid simulations aimed at the interaction of an ultrashort laser pulse with an overdense plasma in the relativistic transparency regime, solitonic structures with a breather-like behavior were observed [22]. Energy exchange between the soliton and the plasma occurred at twice the laser frequency. These periodic oscillations in time of the soliton amplitude is a novel aspect in one-dimensional fluid theory but a well-known feature for linearly s-polarized 2-dimensional solitons, as shown by past particle-in-cell simulation [15] and more recent solutions found in the fluid model [8]. An oscillatory behavior of 1-dimensional linearly polarized solitary waves was also observed in PIC simulations of laser-plasma interactions [23].

These observations in both fluid [22] and PIC simulations [23] motivate our work, which shows that these coherent structures are exact linearly polarized solitary wave solutions of the relativistic fluid model. Following the findings of Ref. [22], we did not restrict the analysis to stationary solutions, but let them oscillate periodically in time. From a numerical point of view, this a challenging problem because one needs to work with a system of partial differential equations (instead of ordinary differential equations). Section II introduces the relevant equations of the model and presents the numerical method used to compute the solitary waves. Results on standing solitary waves are presented in Sec. III, including their spatio-temporal structure, a simple analysis based on energy conservation and the dependence of their main properties as a function of the frequency. The stability is explored in Sec. IV with the aid of full nonstationary fluid simulations. The conclusions are summarized in Sec. V.

II. PHYSICAL AND NUMERICAL MODEL

A. The relativistic fluid model

We consider a plasma consisting of electrons and immobile ions. For convenience, length, time, velocity, momentum, vector and scalar potentials and density are normalized over $c/\omega_{pe}$, $\omega_{pe}^{-1}$, $e$, $m_ec$, $m_ec^2/e$ and $n_0$, respectively. Here $n_0$, $\omega_{pe} = \sqrt{4\pi n_0 e^2/m_e}$, $m_e$ and $c$ are the unperturbed plasma density, the electron plasma frequency, the electron mass and the speed of light. Maxwell (in the Coulomb gauge) and plasma equations then read

$$\Delta A - \frac{\partial^2 A}{\partial t^2} - \frac{\partial}{\partial t} \nabla \phi = n v$$ (1a)

$$\Delta \phi = n - 1$$ (1b)

$$\frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0$$ (1c)

$$\frac{\partial P}{\partial t} - v \times (\nabla \times P) = \nabla (\phi - \gamma)$$ (1d)

where $A$ and $\phi$ are the vector and scalar potentials, $P = p - A$, $\gamma = \sqrt{1 + |p|^2}$ and $p$ and $v = p/\gamma$ are the electron momentum and velocity, respectively.

Here we restrict the analysis to one dimensional waves ($\partial_y = \partial_z = 0$) propagating along the $x$ direction. Coulomb gauge and the transverse components of Eq. (1d) give $A_x = 0$ and $P_y = P_z = 0$, respectively. Using these results in Sys. Eq. (1), one finds

$$\frac{\partial^2 \phi}{\partial t \partial x} + \frac{n}{\gamma} p_x = 0$$ (2a)

$$\frac{\partial A_y}{\partial x^2} - \frac{\partial^2 A_z}{\partial t^2} - \frac{n}{\gamma} A_y = 0$$ (2b)

$$n = 1 + \frac{\partial^2 \phi}{\partial x^2}$$ (2c)

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} \left( \frac{n p_x}{\gamma} \right) = 0$$ (2d)

$$\frac{\partial p_x}{\partial t} = \frac{\partial}{\partial x} (\phi - \gamma)$$ (2e)

B. The Akhiezer-Polovin system

Before we discuss solitary waves solutions of Sys. (2), we first review some concepts of the Akhiezer-Polovin set of equations [2]. This system, which is a subset of Sys. Eq. (2), is obtained by assuming that all the variables depend on the coordinate $\xi = x - Vt$, with $V \equiv \omega/k > 1$ the normalized phase velocity. The result is a set of two ordinary differential equations that are simpler than Sys. (2). The purpose of this short analysis is twofold. On one hand, analytical tools from the theory of dynamical system show that linearly polarized and isolated solitary waves depending only on $\xi$ are not possible. This property suggests that one may look for solitary waves depending separately on $x$ and $t$. This procedure is followed in Sec. III. On the other hand, the dispersion relation of the Akhiezer-Polovin system will help us to understand the results of Sec. III; small oscillations appearing in the tail of our solitary waves are not a numerical artefact but a physical consequence of the linear polarization.

Taking linear polarization, $A = a(\xi) u_y$, Eq. (2) becomes

$$\frac{d}{d\xi} \left[ -V n + \frac{n p_x}{\gamma} \right] = 0 \quad \rightarrow \quad -V n + \frac{n p_x}{\gamma} = -V$$ (3a)
\[ \frac{d}{d \xi} [V p_x + \phi - \gamma] = 0 \quad \rightarrow \quad V p_x + \phi - \gamma = -\Gamma \] (3b)

\[ \frac{d^2 \phi}{d \xi^2} = n - 1 \] (3c)

\[ (V^2 - 1) \frac{d^2 a}{d \xi^2} + \frac{n}{\gamma} a = 0 \] (3d)

where we assumed that at certain position, \( \xi_0 \), one has

\[ a(\xi_0) = a_0, \quad \phi(\xi_0) = 0, \quad p_x(\xi_0) = 0, \quad n(\xi_0) = 1 \] (4)

and we introduced the constant \( \Gamma \equiv \sqrt{1 + a_0^2} \). Defining \( \psi \equiv \Gamma + \phi \) and \( R \equiv \sqrt{\psi^2 - (1 - V^2)(1 + a^2)} \), equations Eq. (3a), Eq. (3b) and \( \gamma = \sqrt{1 + a^2 + R^2} \) give

\[ p_x = (V \psi - R)/(1 - V^2), \quad n = V (\psi - VR)/R(1 - V^2) \] and \( \gamma = (\psi - VR)/(1 - V^2) \). The substitution of these results in Eqs. (3c) and Eq. (3d) yields [2, 24]

\[ \frac{d^2 a}{d \xi^2} = -\frac{V}{V^2 - 1} R \] (5a)

\[ \frac{d^2 \phi}{d \xi^2} = -\frac{V}{V^2 - 1} \left( \frac{\psi - 1}{R} \right) \] (5b)

The above discussion rules out the existence of homoclinic orbits to \( Q_0 \). From a physical point of view they would represent isolated solitary waves, with linear polarization, and stationary in a frame of reference moving with \( V \). However, as previously discussed, one could also try to look for homoclinic orbits connecting to a periodic solution as \( \xi \to \pm \infty \). These structures, that were constructed using Poincaré analysis in Ref. [12], are solitary waves embedded in long laser pulses. We remark that, since we used Eq. (4) to derive Eq. (5), only orbits of Sys. (5) passing through the surface \( (a, \dot{a}, \phi, \dot{\phi}) = (a_0, \dot{a}, 0, \dot{\phi}) \) are relevant from a physical point of view. Other orbits would be a solution of Eq. (5) but not of the fluid model Eq. (1).

### C. Locating generalized solitary waves

A new class of solitary waves is found if we let the solutions vary with time in the \( \xi \)-frame. Physically, such a broader model is necessary to take into account the appearance of new frequencies in the solution. It requires a more general mathematical framework than the Akhiezer-Polovin system. For this purpose, scaled spatial and temporal variables \( X = x/L \) and \( \tau = t/T \) are introduced, with \( T \) and \( L \) the temporal and spatial periods of the solution. System (2) then reads

\[ \frac{1}{L^2} \frac{\partial^2 A_{y,z}}{\partial X^2} - \frac{1}{T^2} \frac{\partial^2 A_{y,z}}{\partial \tau^2} - \frac{n}{\gamma} A_{y,z} = 0 \] (8a)

\[ \frac{1}{T^2} \frac{\partial^2 p_x}{\partial \tau^2} + \frac{1}{T L} \frac{\partial^2 \gamma}{\partial X \partial \tau} + \frac{n}{\gamma} p_x = 0 \] (8b)

\[ n = 1 + \frac{1}{T L} \frac{\partial^2 p_x}{\partial X \partial \tau} + \frac{1}{L^2} \frac{\partial^2 \gamma}{\partial X^2} \] (9a)
where
\[ \gamma = \sqrt{1 + p_y^2 + A_y^2 + A_z^2} \]  

(9b)

and Eq. (2e) was used to eliminate \( \phi \). We remark that \( T \) and \( L \) appear now as free parameters in Eq. (8). For convenience, the solitary wave frequency \( \Omega = 2\pi/T \) (instead of \( T \)) will be used as a bifurcation parameter to present our results. As explained below, \( L \) is found with the aid of a phase condition for each \( \Omega \) value.

Our algorithm solves Eq. (8) as follows. The computational box \( (0 < X < 1) \times (0 < \tau < 1) \) is discretized with \( N_x \times N_y \) uniformly distributed points. Here \( N_x \) and \( N_y \) are odd numbers. A state vector \( \mathbf{x}_s = [A_y \ A_z \ p_x \ L] \) of dimension \( 3 \times N_x \times N_y + 1 \) is constructed. It contains the unknowns of the problem, i.e., the values of \( A_y, A_z, \) and \( p_x \) at the grid points and the length of the computational domain. Differential operators in Eq. (8) are substituted by second order finite difference formulas. At the border of the box one may take into account the phase conditions of the problem. Here we are interested in periodic in time solutions \( A_{y,z}(X, \tau + T) = A_{y,z}(X, \tau) \) and \( p_x(X, \tau + T) = p_x(X, \tau) \) and two types of spatial boundary conditions: (i) vanishing \( A_{y,z}(0, \tau) = A_{y,z}(L, \tau) = p_x(0, \tau) = p_x(L, \tau) = 0 \) and (ii) periodic \( A_{y,z}(X, \tau) = A_{y,z}(X + L, \tau) \) and \( p_x(X, \tau) = p_x(X + L, \tau) \). The use of finite differences transforms Eq. (8) into a system of \( 3N_x \times N_y \) nonlinear algebraic equations.

The algorithm is closed by taking into account that the problem is invariant under temporal and spatial translations. Its solution is not unique, but rather is a continuous family. In order to remove this arbitrary phase and restrict the solution to a unique member of the family, a phase condition was added. We found that

\[ \frac{\partial^2 A_y}{\partial X \partial \tau} |_{X=0.5, \tau=0.5} = 0 \]  

(10)

is satisfactory for this purpose. Equation Eq. (10) was also approximated by the corresponding finite-difference formula.

The set of \( 3N_x \times N_y + 1 \) algebraic equations,

\[ \mathbf{F}(\mathbf{x}_s) = 0, \]  

(11)

was solved with a Newton-Raphson method. The speed of the algorithm was enhanced by computing analytically the Jacobian, \( \mathbf{J} \), of \( \mathbf{F}(\mathbf{x}_s) \) and using parallel computations to find the LU decomposition of \( \mathbf{J} \) for each iteration. The tolerance of our solutions, \( Error = \max(\mathbf{F}(\mathbf{x}_s)) \), was less than \( 10^{-8} \). The initial guess used to initialize the Newton method depends on the solution under consideration, as discussed shortly.

The code was validated with the circularly polarized solitary waves obtained by Esirkepov et al [5]. For these calculations we set \( N_x = 1000 \) and \( N_y = 60 \). The comparison between the analytical formula given in Ref. [5] and the result of our code is shown in Fig. 1. At each \( \Omega \), we used as initial guess for the Newton method the solution obtained at the previous \( \Omega \) value in the branch. For the first frequency, \( \Omega = 0.996 \), the analytical formula was used. The calculations were carried out with periodic and vanishing boundary conditions in space. Both schemes yielded the same results, that are in good agreement with the analytical formulae. For this solution one has \( |A| \rightarrow 0 \) as \( \Omega \rightarrow \omega_{pe} \), and the branch extends until \( \Omega = \sqrt{2/3}\omega_{pe} \), where the minimum density vanishes.

III. BREATHER-LIKE SOLITARY WAVES

A. Branch of solutions

We now focus on linearly polarized solitary waves and set \( A_z = 0 \) in Eqs. Eq. (8a) and Eq. (8b). The number of unknowns in Sys. Eq. (11) is equal to \( 2N_xN_y + 1 \), thus reducing the computational load as compared with the circularly polarized case. The guess to initialize the Newton algorithm was taken from Ref. [11] that presented an analytical formula for linearly polarized solitary waves. This solution is not exact but only valid for frequency close to \( \omega_{pe} \), where the amplitude of the wave vanishes. In this limit, the authors derived a nonlinear Schrödinger equation with local and non-local cubic nonlinearities and found a standing electromagnetic solitary wave. Fortunately, this solution is close enough to our breather-like wave. Here close means that our Newton method started with this wave converges to a solution during the iterative process. Once a breather-like solution is known for a \( \Omega \) value close to \( \omega_{pe} \), the branch of solution is continued by using \( \Omega \) as a bifurcation parameter. For each parameter value the spatio-temporal structure of the solitary
wave was obtained. Typical resolutions in the calculations were \( N_x = 1001 \) and \( N_x = 101 \) and the length of the computational box dynamically changed during the calculation within the range \( 100 < L < 200 \). To check the integrity of the solutions, the same calculations but with different resolutions were also carried out.

Figure 2 shows the maximum of the vector potential (blue thick line) and the minimum density (dashed red line) values of the solitary waves versus the frequency \( \Omega \). The behavior is qualitatively similar to the circularly polarized case (see Fig. 1): (i) solitary waves exist in a frequency range \( \Omega_{\text{min}} < \Omega < \omega_{pe} \), (ii) \( \max(|A|) \to 0 \), \( \min(n) \to 1 \), and the width increases as \( \Omega \to \omega_{pe} \), and (iii) the maximum of the amplitude increases and \( \min(n) \to 0 \) as the frequency approaches \( \Omega_{\text{min}} \approx 0.694 \). This can also be seen in Fig. 6. A comparison of Fig. 1 and 2 shows that, for a given frequency, linearly polarized waves have greater amplitude than circularly polarized waves. They also exist in a broader frequency range because \( \Omega_{\text{min}} = 0.694 < \sqrt{2/3} \approx 0.817 \).

**Figure 2.** Maximum amplitude (solid blue line) and minimum density (dashed red line) values versus the normalized frequency for linearly polarized solitary waves.

### B. Spatio-temporal structure of the waves

Figure 3 shows the spatio-temporal evolution of a solitary wave with \( \Omega = 0.9 \omega_{pe} \). Similarly to circularly polarized solitons, there is a rarified plasma region and an electromagnetic wave oscillating inside [see panels (a) and (c)]. However, linearly polarized waves present two distinguishing features: (i) the plasma cavity has a time modulation [see panel (c)] and (ii) longitudinal electron momentum \( p_x \), electron density and \( \gamma \) factor oscillates twice faster than the vector potential. From Eq. (2e), we observe that electron oscillations come from the actions of the electrostatic force due to the charge separation and the ponderomotive force. The momentum in panel (b) oscillates twice faster because of the second-harmonic oscillating component of the ponderomotive force. The modulation in time of the plasma cavity is correlated with the behavior of the vector potential: as shown by panels (a) and (c), the lowest plasma density inside the cavity is reached when the electromagnetic wave vanishes.

A first sight to the maps of Fig. 3 may suggest that all plasma and electromagnetic perturbations vanish far-away from the localized structure \( (A_y \to 0, p_x \to 0 \) and \( n \to 1 \) as \( X \to \pm \infty \)). However, this is not the case and, unlike circularly polarized waves that decay exponentially, perturbations exist at the tails of a linearly polarized solitary wave. This is shown in Fig. 4, which display the spatial structure of the wave at certain instant. Low amplitude oscillations in the vector potential are evident in the solitary wave’s tails.

Previous results are in agreement with the analysis of the Akhiezer-Polovin system, which is a particular case included in Sys. Eq. (8). In the soliton’s tail \( A_y \) and \( \phi \) are very small, thus indicating that the dynamics occurs close to the Akhiezer-Polovin equilibrium point \( Q_0 \equiv (a, a, \phi, \psi) = (0, 0, 0, 0) \). Small oscillations appear because, as shown in Sec. II B, \( Q_0 \) is a center and orbits cannot connect exponentially to it. Oscillations following the dispersion relation given by Eq. (7) may occur. This statement is confirmed by looking closer to the small oscillations in the tail of a wave. Figure 5 shows the tail of a solitary wave of \( \Omega = 0.9 \omega_{pe} \). From panels (a) and (b), which show the spatial structure at \( \tau = 0.5 \) and the temporal behavior at \( x = 0 \) of the vector potential, we find the frequency \( \omega \approx 2.67 \omega_{pe} \), the wavenumber \( k \approx 2.51c/\omega_{pe} \), and \( a_0 \approx 5 \times 10^{-5} \). The phase velocity of these oscillations in the tail is \( V = \omega/k \approx 1.06 \), close to one. Using \( k = 2.51c/\omega_{pe} \) in Eq. (7) yields \( \omega \approx 2.71 \omega_{pe} \), which is in good agreement with \( \omega \approx 2.67 \omega_{pe} \). As expected, the longitudinal variable \( p_x \) has an amplitude of the order of \( \sim a_0^2 \) and oscillates with frequency \( 2\omega \) [see panel (c)].

The amplitude of the oscillations in the tail increases as the frequency of the solitary waves decreases. This is evident in Figure 6, which shows the vector potential and the density of a solitary wave with frequency \( \Omega = 0.694 \omega_{pe} \) at \( \tau = 0.5 \). The oscillations have \( \omega \approx 2.05 \omega_{pe} \), \( k \approx 1.795 \omega_{pe}/c \). The normalized amplitude \( a_0 \approx 0.06 \) and the phase velocity \( V \approx 1.144 \). These values are in agreement with the dispersion relation Eq. (7).

### C. Energy balance

Linearly polarized solitary waves are localized structures where electromagnetic energy and electron kinetic energy are exchanged periodically. This is an important difference compared to the circularly polarized waves, that exhibit a stationary character. Energy evolution is here analyzed using Sys. Eq. (2). It conserves the sum of
the three normalized energies, $E = E_l + E_p + E_e$, where

$$E_l = \frac{1}{2} \int \left[ \left( \frac{\partial A_y}{\partial t} \right)^2 + \left( \frac{\partial A_y}{\partial x} \right)^2 \right] dx \quad (12)$$

$$E_p = \frac{1}{2} \int \left( \frac{\partial \phi}{\partial x} \right)^2 dx \quad (13)$$

$$E_e = \int (\gamma - 1) n \, dx. \quad (14)$$

Here $E_l$, $E_p$ and $E_e$ are the energy of the electromagnetic wave, the energy of the electrostatic plasma wave and the kinetic energy of the electrons, respectively.

As observed in Ref. [22], linearly polarized breather-like solitary waves exhibit a periodic exchange of energy. This feature is shown in Fig. 7, which displays the time evolution of the three energy contributions for a solitary wave with $\Omega = 0.8\omega_{pe}$. Each period, the energy exchange is repeated twice; the two minimums in the electron kinetic energy coincide with the two maxima in the electromagnetic and plasma wave energies. The electrostatic energy, albeit being the smallest contribution, never vanishes, thus helping to maintain the plasma cavity.

IV. DIRECT NUMERICAL SIMULATION OF THE SOLITARY WAVES

Having discussed a new family of breather like solitary wave solutions in the previous section, we now turn to direct fluid-Maxwell simulations of solitary waves of this family, in order to validate the numerical procedure that we used to detect these solutions. Moreover, since we integrate the fluid-Maxwell equations for several periods of the solitary wave, we expect that if these waves are prone to instabilities, these will be triggered within the simulation time. A more rigorous stability study is beyond the scope of this work. The simulations involve numerically solving the full relativistic fluid-Maxwell model [Sys. (2)]. We have used two different codes for this purpose, a finite difference code and a pseudospectral one.

As all the time evolution equations we wish to solve here are either in continuity or convective form, it was very convenient to employ the subroutine package LCPFCT for the finite difference solver. LCPFCT is a freely available subroutine package developed at Naval Research Laboratory (NRL), USA. These subroutines are based on the principle of flux-corrected transport [27]. Periodic boundary conditions are implemented for the simulations presented here and Courant stability condition is used to calculate appropriate value of integration time step for ensuring the numerical stability. Profiles of fluid variables are specified at the grid centers whereas the interface values of flow variables is used. The flux-correction method has been quite successful in solving fluid flow problems and it ensures density positivity as well as numerical accuracy.

The pseudospectral code has been presented in Ref. [28] and uses Fourier space discretization of the partial derivative with respect to space and an adaptive, fourth order Runge-Kutta scheme for time stepping.

The initial conditions for the numerical simulations are
chosen in accordance with the numerical solutions of Sec. III and are then evolved with above discussed method. In order to explore the full branch presented in Fig. 2, we ran several simulations. Each one was initialized with a solitary wave of a given frequency. For instance, Fig. 8 shows a fluid-Maxwell simulation initialized with a solitary wave of frequency $\Omega = 0.75\omega_{pe}$, using the finite difference solver. In all cases, using either the pseudospectral or the finite difference code, the structures remain unchanged for several plasma periods and they do not seem to be prone to any instability. The amplitude and the oscillation frequency of the waves during the simulations coincide with the values expected from the analysis of Sec. III. This confirms the integrity of the method.

Figure 4. Spatial structure of a linearly polarized soliton with $\Omega = 0.9\omega_{pe}$ at certain instant. Panels (a) to (d) show $A_y$, $p_x$, $n$ and $\gamma$, respectively. The vector potential exhibits small amplitude oscillations at the solitary wave tail [see also panel (a) in Fig. 5].

Figure 5. Vector potential at the tail of a solitary wave with $\Omega = 0.9\omega_{pe}$. Panel (a) shows the spatial structure at $\tau = 0.5$ and panel (b) the temporal behavior at $x = 0$.

Figure 6. Vector potential and density at at $\tau = 0.5$ for a solitary wave of $\Omega = 0.694\omega_{pe}$.
Figure 7. Evolution of the energy of the electromagnetic wave ($E_l$), the energy of the electrostatic plasma wave ($E_p$) and the kinetic energy of the electrons ($E_e$) for a solitary wave with frequency $\Omega = 0.8\omega_{pe}$. The black line represents the sum of the three terms.

ology and correctness of the solutions presented in this work. As the solitary waves appear stable they are ideal candidates for further research, for example, for the investigation of the mutual collisions among two or more standing structures as has been studied for the circular polarization case in Ref. [28, 29].

V. CONCLUSIONS

We have presented novel exact solutions of the relativistic Maxwell-fluid model in a cold plasma with fixed ions. One-dimensional solitary waves with linear polarization and a breather-like behavior were computed by using a numerical method based on a finite-difference scheme. The spatio-temporal structure of the electromagnetic and plasma fields was presented as well as the main properties of the waves as a function of their frequency $\Omega$. The solitary waves exist in the frequency range $\Omega_{\text{min}} \equiv 0.694 < \Omega/\omega_{pe} < 1$, with $\Omega_{\text{min}}$ the frequency value where the minimum of the plasma density vanishes. An analysis based on the different energy contributions showed that these localized structures consist of a plasma cavity where a periodic exchange of energy at frequency $2\Omega$ occurs between an electromagnetic wave and the plasma electrons. The small amplitude oscillations appearing in the tails of the solitary wave, a consequence of the linear polarization, was explained using the Akhiezer-Polovin system, which is a particular case of our model.

Unlike circularly polarized waves, which are stationary, linearly polarized waves exhibit a breather like behavior. Longitudinal variables, like the electron momentum, oscillate with the frequency $2\Omega$; electrons are moved inward and outward from the plasma cavity twice per period. Besides this fundamental difference, other features of the solitary waves are shared by both types of polarization. For instance, the amplitude vanishes (is enhanced) and the width increases (decreases) as the frequency approaches to $\omega_{pe}$ ($\Omega_{\text{min}}$). For a given frequency, the amplitude of linearly polarized waves is greater than for circular polarization. The frequency range of existence is also broader.

Past fluid [22] and PIC [23] simulations demonstrated that breather-like solitary waves are easily excited during the interaction of a linearly polarized high-intensity laser with a plasma. Our fluid simulations, initialized with a single solitary wave, now reveal that these coherent structures persist for long time. Since part of the electromagnetic energy is trapped inside the wave cavity, our numerical simulations suggest that solitary waves may play an important role in several applications like fast ignition in inertial confinement fusion. A good understanding of some aspects, like the excitation process or solitary wave interactions, are relevant for these applications. In this respect, the exact solitary wave solutions here presented are useful. For instance, fluid or particle-in-cell codes can be prepared to analyze several scenarios. An example of this technique applied to relativistic solitons was given in Ref. [30], where stability, collisions, electromagnetic bursts and post-soliton evolution of s-polarized 2-dimensional solitary waves were analyzed.

The present analysis can be extended to find exact moving solitary waves solutions with linear polarization in the relativistic fluid model. The small amplitude limit of such solutions was presented in Ref. [10], thus suggesting that large amplitude waves may also exist. The results of this work, that would complete our knowledge on the organization of linearly polarized solitary waves, will be presented elsewhere.

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Figure 8. Fluid-Maxwell simulation initialized with a solitary wave of frequency $\Omega = 0.75\omega_{pe}$.

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