COMPACT MODULI OF K3 SURFACES

VALERY ALEXEEV AND PHILIP ENGEL

ABSTRACT. We construct geometric compactifications of the moduli space $F_{2d}$ of polarized K3 surfaces, in any degree $2d$. Our construction is via KSBA theory, by considering canonical choices of divisor $R \in |nL|$ on each polarized K3 surface $(X, L) \in F_{2d}$. The main new notion is that of a recognizable divisor $R$, a choice which can be consistently extended to all central fibers of Kulikov models. We prove that any choice of recognizable divisor leads to a semitoroidal compactification of the period space, at least up to normalization. Finally, we prove that the rational curve divisor is recognizable for all degrees.

CONTENTS

1. Introduction 2
   Relation to earlier work 3
   Summary of contents 4

2. Moduli of K3 surfaces 4
   2A. Analytic moduli 4
   2B. Quasipolarized moduli 5
   2C. ADE moduli 6

3. One-parameter degenerations 6
   3A. Kulikov models 6
   3B. Nef, divisor, and stable models 7
   3C. Topology of Kulikov models 8

4. The period map 9
   4A. The period of a Kulikov surface 9
   4B. Markings of Kulikov surfaces 10
   4C. Partial markings of K3 surfaces 11

5. Compactifications of arithmetic quotients 12
   5A. Baily-Borel compactification 12
   5B. Toroidal compactification 13
   5C. Semitoroidal compactification 14

6. Moduli of stable slc pairs 16
   6A. Canonical choices of polarizing divisor 16
   6B. Compact moduli of stable pairs 16

7. $\lambda$-families 17
   7A. Deformation spaces of Kulikov models 18
   7B. The gluing and period complexes 19
   7C. The global Friedman-Scattone theorem 21
   7D. Type II $\lambda$-families 23
   7E. Quasipolarized $\lambda$-families 24

8. Recognizable divisors 25

9. Main theorem for recognizable divisors 29

Date: March 31, 2023.
1. **Introduction**

Let $F_{2d}$ be the coarse moduli space of complex K3 surfaces $X$ having ADE singularities with an ample line bundle $L$ of degree $L^2 = 2d$. A well known corollary of the Torelli theorem [PSS71] is that $F_{2d} = \mathbb{D}/\Gamma$ is the quotient of a 19-dimensional symmetric type IV domain $\mathbb{D}$ by an arithmetic group $\Gamma \subset O(2,19)$. In this capacity, $F_{2d}$ admits a Baily-Borel compactification $F_{BB}^{2d}$ [BB66] and infinitely many toroidal compactifications $F_{2d}$ [AMRT75]. An admissible fan $\mathfrak{F}$ consists of polyhedral decompositions of the positive cones of finitely many hyperbolic signature lattices (Def. 5.9). Looijenga [Loo03] simultaneously generalized the Baily-Borel and toroidal compactifications to the semi-toroidal compactifications, where $\mathfrak{F}$ may only be locally rational polyhedral (Def. 5.11).

Toroidal compactifications enjoy a number of geometric properties by virtue of the fact that they are analytically-locally modeled at the boundary by finite quotients of toric varieties. But there are infinitely many, with seemingly no one being distinguished. An old and deep question is whether any toroidal, or semitoroidal, compactifications can be understood as moduli spaces parameterizing geometric objects—some generalized “stable” K3 surfaces, similar to the Deligne-Mumford’s compactifications $\overline{M}_{g,n}$.

For the moduli space $A_g$ of principally polarized abelian varieties (PPAVs) the answer is yes by [Ale02]. A PPAV $(A,\lambda) \in A_g$ determines uniquely an abelian torsor $A \rightarrow X$ together with a theta divisor $\Theta \subset X$. For pairs $(X,\Theta)$ or, even better $(X,\epsilon\Theta)$, there is a generalization of $\overline{M}_{g,n}$. This moduli space is projective, and gives a geometrically meaningful compactification $\overline{A}_g^{\Theta}$. Furthermore, the normalization of $\overline{A}_g^{\Theta}$ is the toroidal compactification $\overline{A}_g^{\mathfrak{F}}$ associated to the 2nd Voronoi fan, and so admits a purely period-theoretic definition. Thus, among the infinitely many toroidal compactifications, this one has a clear geometric meaning.

To extend this construction to polarized K3 surfaces $(X,L)$, first one needs a canonical choice of polarizing divisor (Def. 6.1), an effective divisor $R \in |nL|$ in a fixed multiple of the polarization. Given this choice, the general theory produces a modular compactification $F_{2d}^R$ (Def. 6.11) via slc stable pairs $(X,\epsilon R)$, see [KXX20, Kol23], [AET19, Sec. 3]. The divisor is needed because for the general theory to work, the divisor $K_X + \epsilon R$ must be ample. One can work with all divisors in $\mathcal{L}$, without making a canonical choice, e.g. [Laz16], but that gives a larger moduli space $P_{2d}$ of dimension $20 + d$.

At least two canonical choices for ample divisors on polarized K3 surfaces have been identified. About 15 years ago, Sean Keel proposed to consider, for a general polarized K3 surface $(X,L)$, the sum $R^{\text{rc}} = \sum C_i$ of rational curves in $|L|$. We call this the rational curve divisor (Def. 10.3). One has $R^{\text{rc}} \in |n_dL|$, where the multiple $n_d$ is given by the Yau-Zaslow formula. For instance $n_1 = 324$, $n_2 = 3200$, etc. The second choice, suggested to the authors by Claire Voisin, is called the flex divisor $R^{\text{flex}}$ [AE21]. It generalizes to all degrees the fixed locus $R \in |3L|$ of the involution on a K3 surface of degree 2.
By the Kulikov-Persson-Pinkham theorem [Kul77, PP81], any one-parameter degeneration of K3 surfaces $X \to (C, 0)$ admits a Kulikov model: a $K$-trivial model with smooth total space and reduced normal crossings central fiber $X_0$ (Def. 3.1). The key notion of this paper is a recognizable divisor (Def. 6.2). Heuristically, it is a canonical choice of polarizing divisor which can be extended to any such $X_0$. More precisely: Given any Kulikov surface $X_0$ appearing as a one-parameter degeneration of polarized K3 surfaces $(X_t, L_t)$, the limit of the canonically chosen divisors $R_t \subset X_t$, $R_t \in [nL_t]$ is a unique curve $R_0 = \lim_{t \to 0} R_t \subset X_0$. Such a limit $R_0$ exists on any fixed Kulikov model, but recognizability additionally states that $R_0$ is independent of how $X_0$ gets smoothed.

Our two main theorems are:

**Theorem 1** (Thm. 9.1). Suppose that $R$ is a recognizable choice of polarizing divisor. There is a unique semifan $\tilde{R}$ for which $F_{2d}^R$ is the normalization of $F_{2d}^R$.

**Theorem 2** (Thm. 10.11). The rational curve divisor $R^{rc}$ is recognizable for $F_{2d}$.

Theorem 1 holds more generally for moduli of lattice-polarized K3 surfaces. Combined, Theorems 1 and 2 give an affirmative answer to the existence of a compactification of K3 moduli which is simultaneously geometric and period-theoretic:

**Corollary 3.** For all degrees $2d$, there is a KSBA compactification of $F_{2d}$ by slc stable pairs, whose normalization is semitoroidal.

Theorem 1 is proven as follows: Kulikov models $X \to (C, 0)$ with a given Picard-Lefschetz transformation, encoded by a lattice vector $\lambda$, can be packaged into a 19-dimensional family of polarized surfaces $\mathcal{X} \to S_\lambda$ we call $\lambda$-families (Def. 7.14). The general fiber is smooth, and the discriminant $\Delta_\lambda \subset S_\lambda$ is a smooth divisor, isomorphic to $(\mathbb{C}^*)^{18}$ for Type III Kulikov models. The discriminant parameterizes the equisingular, quasipolarized, smoothable deformations of $X_0$.

Recognizability implies that the divisor $R$ extends over the boundary $\Delta_\lambda$ to give a family of pairs $(\mathcal{X}, \mathcal{R}) \to S_\lambda$ (Prop. 8.1). We modify $\mathcal{X}$ until $\mathcal{R} \subset \mathcal{X}$ is relatively nef and contains no singular strata of any fiber (Prop. 8.8, Sec. 7C). Taking the relative canonical model shows that all degenerations with a given Picard-Lefschetz transformation limit to stable pairs $(X_0, \epsilon R_0)$ of a fixed combinatorial type (Cor. 8.13). This fact, together with an argument involving resolution of indeterminacy (Lem. 9.18) and quasi-affineness of the strata of the KSBA compactification (Thm. 9.16), imply that there is a toroidal compactification dominating the normalization of the KSBA compactification. The proof concludes with a new characterization: semitoroidal compactifications are exactly those normal compactifications of $F_{2d}$ that are dominated by a toroidal compactification, and dominating Baily-Borel (Thm. 5.14).

Theorem 2 is proven by borrowing ideas from Gromov-Witten theory and degenerations of stable maps. $R^{rc}$ is recognizable because any limit $R_{t,0} \subset X_0$ of a family of rational curves $R_{t,1} \subset R_{t}^{rc} \subset X_t$ in a Kulikov model $X \to (C, 0)$ enjoys a geometric property which ensures its rigidity: $R_{t,0}$ is the image of an admissible stable genus zero map $f : T \to X_0$. Using $K$-triviality of $X_0$ and adjunction, we prove that $f(T)$ is locally constant on the Kontsevich space of admissible stable maps (Lem. 10.14).

**Relation to earlier work.** The notion of a recognizable divisor presented here arose from generalizing certain specific examples, for moduli of degree 2 [AET19] and elliptic K3 surfaces [ABE22]. In both of these papers, Kulikov models with divisor are constructed explicitly for all possible $\lambda$, providing the necessary input for the general theory to work.

Every degree 2 K3 surface $(X, L)$ with ADE singularities admits an involution, and the fixed locus $R \in [3L]$ can be taken as a canonical choice of polarizing divisor. Every elliptic K3 surface admits the polarizing divisor $R := s + \sum_{i=1}^{24} f_i \in [s + 24f]$ formed from the section, plus the sum of the singular fibers counted with appropriate multiplicity.

Using the theory of integral-affine structures on the two-sphere $S^2$ [Eng18, EF21], one can, in both of these cases, explicitly construct a family of Kulikov surfaces $X_0 \to \Delta_\lambda$ which smooths to...
a \( \lambda \)-family \( (\mathcal{X}, \mathcal{R}) \to S_\lambda \) with \( \mathcal{R} \) relatively big and nef, and not containing strata of any fiber. This is achieved by building a continuously varying family of “polarized integral-affine spheres” \( (\mathcal{B}, \mathcal{R}_{\text{trop}}) \to C_\delta^+ \) over a cone associated to an appropriate hyperbolic lattice (Def. 5.8). The cone \( C_\delta^+ \) contains all possible \( \lambda \). Once triangulated, the integral-affine sphere \( \mathcal{B}_\lambda = \Gamma(\mathcal{X}_0) \) can be identified with the dual complex of \( \mathcal{X}_0 \) for some Kulikov model \( \mathcal{X} \to (\mathcal{C}, 0) \) with monodromy \( \lambda \). The tropical divisor \( \mathcal{R}_{\text{trop}, \lambda} \subset \mathcal{B}_\lambda \) describes the dual complex \( \Gamma(\mathcal{R}_0) \) of \( \mathcal{R}_0 \subset \mathcal{X}_0 \).

The upshot of these constructions is that recognizability is verified explicitly, and all degenerations with a fixed Picard-Lefschetz transformation admit a stable model of a fixed combinatorial type. Furthermore, the family \( (\mathcal{B}, \mathcal{R}_{\text{trop}}) \to C_\delta^+ \) determines which monodromy invariants \( \lambda \) give rise to degenerations into a specified stratum of slc stable pairs—it is those \( \lambda \) on which the family \( (\mathcal{B}, \mathcal{R}_{\text{trop}}) \) is combinatorially constant. In the above examples, these loci of combinatorial constancy in \( C_\delta^+ \) are the cones of a semifan \( \mathfrak{F}_R \). The semifan of Theorem 1 is exactly this one. In fact, we prove here that for any recognizable divisor, the cones of \( \mathfrak{F}_R \) are the loci on which a well-defined “stratum” function \( \mathfrak{S}: C_\delta^+ \to \{ \text{slc strata of } \mathcal{F}_{2d} \} \) is constant (Thm. 9.3).

For degree 2 K3 surfaces with ramification divisor, \( \mathfrak{F}_R \) is a semifan but not a fan. It is a coarsening of the Coxeter fan. For elliptic K3 surfaces with divisor \( s + \sum_{i=1}^{24} f_i \) it is a fan which refines the maximal cone of the Coxeter fan into 9 subcones. Theorems 1 and 2 imply the existence of a semifan \( \mathfrak{F}_{\text{eq}} \) for all degrees 2d. But unlike for \( A_d \) and the 2nd Voronoi fan, or the above two examples, we have no explicit description, because the structure of a hypothetical “tropical K3-rational curves pair” \( (\mathcal{B}, \mathcal{R}_{\text{trop}}^+) \) is unknown. Such a description is an interesting open question. Integral-affine structures make no appearance in this paper because we do not explicitly construct \( \mathfrak{F}_R \) for any given \( R \)—we prove a general existence result.

After this work appeared, the authors, with Han [AEH21], proved recognizability for fixed curves of non-symplectic automorphisms. Explicit semifans \( \mathfrak{F}_R \) for the fixed divisor \( R \) were given in [AE22], for moduli spaces of K3 surfaces with nonsymplectic involution.

**Summary of contents.** In Section 2, we recall different notions of moduli of K3 surfaces, such as smooth analytic, \( M \)-quasipolarized, and polarized with ADE singularities. In Section 3 we study one-parameter degenerations: Kulikov models, as well as nef, divisor, and stable models, by adding nef line bundles, effective nef divisors, and by taking the canonical models of the latter, respectively. In Section 4 we define the periods of Kulikov surfaces. These sections compile known results about K3 moduli, giving a unified treatment of Type II and III degenerations.

Section 5 recalls the combinatorially defined, period-theoretic compactifications of arithmetic quotients: Baily-Borel, toroidal, and semitoroidal. A major result is Theorem 5.14 which states that, for Type IV arithmetic quotients, a semitoroidal compactification is precisely a normal compactification which is sandwiched between the Baily-Borel and some toroidal compactification. Section 6 discusses the compactifications of \( F_{2d} \) via stable pairs, associated to a canonical choice of divisor \( R \). Here, we introduce the critical notion of recognizable divisors (Def. 6.2).

In Section 7, we construct the \( \lambda \)-families which appear in the proof of Theorem 1. A new result (Theorem 7.19) globalizes the main result of Friedman-Scattone [FS86]: Any two \( \lambda \)-families with the same values of \( \lambda^2 \) and imprimitivity of \( \lambda \) are connected by a series of birational modifications falling into three special forms. In Section 8, we prove the main properties of recognizable divisors with respect to \( \lambda \)-families. Theorem 8.11 summarizes equivalent formulations of recognizability.

Sections 9 and 10 contain the proofs of Theorems 1 and 2, respectively.

2. **Moduli of K3 surfaces**

2A. **Analytic moduli.** We begin by setting notation and reviewing fundamental results about K3 surfaces. For general references, see [Huy16] or [ast85].

**Definition 2.1.** A K3 surface \( X \) is a compact, complex surface with \( h^1(X) = 0 \) and \( K_X = \mathcal{O}_X \).

**Definition 2.2.** Let \( L_{K3} := H_{3,19} = H^{\oplus 3} \oplus E_8^{\oplus 2} \) be a fixed copy of the unique even unimodular lattice of signature \( (3,19) \).
Endowed with the cup product, $H^2(X,\mathbb{Z})$ is isometric to $L_{K3}$ for any K3 surface $X$. Let $\mathcal{K}_X \subset H^{1,1}(X,\mathbb{R})$ denote the Kähler cone of $X$. It is a fundamental chamber for the group $W_X = \langle r_\beta \rangle \subset O(H^2(X,\mathbb{Z}))$ generated by reflections in the roots $\beta \in \text{NS}(X)$, $\beta^2 = -2$ acting on the positive cone of $H^{1,1}(X,\mathbb{R})$.

**Theorem 2.3** ([PS71, LP81]). Two K3 surfaces $X$, $X'$ are isomorphic if and only if they are Hodge-isometric: there is an isometry $i : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ for which $i(H^{2,0}(X')) = H^{2,0}(X)$. Furthermore, $i = f^*$ for a unique isomorphism $f : X \to X'$ if and only if $i(K_X) = K_X$.

Note that $\pm 1$ and $g \in W_X$ act by Hodge isometries on $H^2(X,\mathbb{Z})$. For any Hodge isometry $i$ between $X'$ and $X$, there is a unique sign and unique element $g \in W_X$ such that $\pm g \circ i(K_{X'}) = K_X$. Thus, the group of Hodge isometries of $X$ fits into a split exact sequence of groups

$$0 \to \{ \pm 1 \} \times W_X \to \text{HodgeIsom}(X) \to \text{Aut}(X) \to 0.$$

**Definition 2.4.** Let $\pi : X \to S$ be a family of smooth analytic K3 surfaces over an analytic space $S$. A *marking* is an isometry of local systems $\sigma : R^2\pi_*\mathbb{Z} \to L_{K3}$.

**Definition 2.5.** The *period domain* of analytic K3 surfaces is

$$\mathbb{D} := \mathbb{P}\{ x \in L_{K3} \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \overline{x} > 0 \}.$$

It is an analytic open subset of a 20-dimensional quadric in $\mathbb{P}^{21}$. Let $(X \to S, \sigma)$ be a marked family of K3 surfaces. The *period map* $P : S \to \mathbb{D}$ is defined by $s \mapsto \sigma(H^{2,0}(X_s))$.

By [ast85, Exp. XIII], there is a non-Hausdorff complex manifold $\mathcal{M}$ of dimension 20, forming a fine moduli space of marked K3 surfaces, together with a period map $P : \mathcal{M} \to \mathbb{D}$. For $x \in \mathbb{D}$ a period, let $W_x$ be the group generated by reflections in roots of $x^\perp \cap L_{K3}$. Then $P^{-1}(x)$ is a torsor over $\{ \pm 1 \} \times W_x$ with action given by $(X, \sigma) \mapsto (X, g \circ \sigma)$.

**2B. Quasipolarized moduli.** We now give analogous definitions to Section 2A in the polarized case. The standard reference is [Dol96]. However, Thm. 3.1 in *ibid* is incorrect. We modify the definition in a way that this theorem remains true.

Let $J : M \to L_{K3}$ be a primitive hyperbolic sublattice of signature $(1, r - 1)$ with $r \leq 20$. A vector $h \in M \otimes \mathbb{R}$ is very irrational if $h \notin M' \otimes \mathbb{R}$ for any proper sublattice $M' \subset M$. We will henceforth fix one such, of positive norm $h^2 > 0$.

**Definition 2.6.** An *$M$-quasipolarized K3 surface* $(X, j)$ is a K3 surface $X$, and a primitive lattice embedding $j : M \to \text{NS}(X)$ for which $j(h) \in \overline{\mathcal{K}_X}$ is big and nef. Two such $(X, j)$, $(X', j')$ are isomorphic if there is an isomorphism $f : X \to X'$ of K3 surfaces for which $j = f^* \circ j'$.

**Definition 2.7.** A marking of $(X, j)$ is an isometry $\sigma : H^2(X,\mathbb{Z}) \to L_{K3}$ for which $J = \sigma \circ j$.

The $M$-quasipolarized period domain is

$$\mathbb{D}_M := \mathbb{P}\{ x \in M^\perp \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \overline{x} > 0 \}.$$

Define the Weyl group of a point $x \in \mathbb{D}_M$ to be $W_x(M^\perp) := \langle r_\beta : \beta \in x^\perp \cap M^\perp \rangle$. Note that now $W_x(M^\perp)$ is finite since $x^\perp \cap M^\perp$ is negative-definite.

**Theorem 2.8.** There is a non-Hausdorff complex manifold $\mathcal{M}_M \subset \mathcal{M}$ of dimension $20 - r$, admitting a universal family of marked $M$-quasipolarized K3 surfaces. The fiber $P^{-1}(x)$ of the period map $P : \mathcal{M}_M \to \mathbb{D}_M$ is a torsor over $W_x(M^\perp)$.

**Proof.** The proof follows that of [Dol96, Thm. 3.1], which now works because of the modified Definition 2.6 for an $M$-quasipolarization. \(\square\)

Let $\mathcal{F}_M^q$ denote the moduli stack of (unmarked) $M$-quasipolarized K3 surfaces.

**Corollary 2.9.** There is an isomorphism of stacks $\mathcal{F}_M^q = [\mathcal{M}_M : \Gamma]$ where

$$\Gamma := \{ \gamma \in O(L_{K3}) : \gamma|_M = \text{id}_M \}$$

is the group of changes-of-marking. The quotiented period map $\mathcal{M}_M/\Gamma \to \mathbb{D}_M/\Gamma$ is a bijection.
2C. **ADE moduli.** We now modify the above moduli problems to produce a Hausdorff moduli space. Theorem 2.11 below is well-known.

**Definition 2.10.** An $M$-polarized K3 surface $(\overline{X}, j)$ is a surface $\overline{X}$ with at worst rational double point (ADE) singularities whose minimal resolution $X \to \overline{X}$ is a smooth K3 surface, together with an isometric embedding $j : M \to \text{Pic}(\overline{X})$ for which $j(h)$ is ample.

**Theorem 2.11.** The coarse moduli space of $M$-polarized K3 surfaces is $F_M = \mathbb{D}_M/\Gamma$. The moduli stack $\mathcal{F}_M$ is the separated quotient of the stack $[\mathcal{M}_M : \Gamma]$.

**Remark 2.12.** The stack $\mathcal{F}_M$ and the quotient stack $[\mathbb{D}_M : \Gamma]$ are not equal. In the latter stack, the inertia group at $x \in \mathbb{D}_M$ is the stabilizer $\Gamma_x$. In the former stack, the Torelli Theorem 2.3 implies the inertia group is rather $\Gamma_x/W_x(M^\perp) = \text{Aut}(\overline{X}, j)$.

Consider an open neighborhood $U_x \ni x$ in $\mathbb{D}_M$ preserved by $\Gamma_x$. First, quotient $U_x$ by $W_x(M^\perp)$. Since $W_x(M^\perp) \subset \Gamma_x$ is normal, the quotient group acts on the coarse space $U_x/W_x(M^\perp)$, which is a smooth complex manifold. The stack quotient $[U_x/W_x(M^\perp) : \Gamma_x/W_x(M^\perp)]$ defines orbifold charts for the smooth DM stack $\mathcal{F}_M$.

3. **One-parameter degenerations**

3A. **Kulikov models.** We now examine degenerations of K3 surfaces over a curve. Let $(C, 0)$ denote the analytic germ of a smooth curve at a point $0 \in C$ and let $C^* = C \setminus 0$. Let $X^* \to C^*$ be a family of smooth analytic K3 surfaces.

**Definition 3.1.** A Kulikov model $X \to (C, 0)$ is an extension of $X^* \to C^*$ for which $X$ is smooth, $K_X \sim_C 0$, and $X_0$ has reduced normal crossings with all components Kähler. We say $X$ is Type I, II, or III, respectively, depending on whether $X_0$ is smooth, has double curves but no triple points, or has triple points, respectively.

A key result is the theorem of Kulikov [Kul77] and Persson-Pinkham [PP81]:

**Theorem 3.2.** Let $Y^* \to C^*$ be a family of analytic K3 surfaces admitting an extension $Y \to (C, 0)$ for which every component of $Y_0$ is Kähler. There is a base change $(C', 0) \to (C, 0)$ and a sequence of bimeromorphic modifications $Y' \to X$ of the pullback, such that $X$ is a Kulikov model.

Assume for notational convenience that the strata of $X_0$ are globally normal crossings. Let $V_i \subset X_0$ denote the irreducible components, $D_{ij} = V_i \cap V_j$ and $T_{ijk} = V_i \cap V_j \cap V_k$ the double curves and triple points, respectively. By convention, we write $D_{ij} \subset V_i$ and $D_{ji} \subset V_j$.

**Proposition 3.3.** Let $X \to (C, 0)$ be a Kulikov model. Let $D_i = \sum_j D_{ij}$ be the part of the double locus of $X_0$ lying on $V_i$. Then:

1. $D_i \in |-K_{X_0}|$ is an anticanonical cycle of rational curves in Type III, and an elliptic curve or the disjoint union of two elliptic curves in Type II.
2. $D_{ij}^2 + D_{ji}^2 = -2 + 2g$ where $g$ is the arithmetic genus of $D_{ij}$ in $X_0$.
3. The dual complex $\Gamma(X_0)$ is a triangulation of $S^2$ in Type III, and a segment decomposed into subsegments in Type II.

**Definition 3.4.** A reduced normal crossings surface $X_0$ satisfying (1), (2), (3) is a Kulikov surface.

There is a converse to Proposition 3.3 due to Friedman [Fri83b]:

**Theorem 3.5.** Let $X_0$ be a Kulikov surface. Then, $X_0$ deforms to a smooth K3 surface if and only if it satisfies an additional property called $d$-semistability:

$$\mathcal{E}xt^1(\Omega_{X_0}, \mathcal{O}_{X_0}) \cong \mathcal{O}_{(X_0)_{\text{sing}}}.$$
Definition 3.6. An anticanonical pair or simply pair \((V, D)\) is a smooth surface \(V\) with a reduced, at worst nodal, anticanonical divisor \(D \in |−K_V|\). A toric pair \((\overline{V}, \overline{D})\) is a smooth toric surface \(\overline{V}\) with \(\overline{D} \in |−K_{\overline{V}}|\) the toric boundary.

The topologically trivial deformations of \(X_0\) consist of deforming the moduli of the pairs \((V_i, D_i)\), and regrading the double curves by an element of \(\mathbb{C}^*\) (in Type III) or by a translation of the elliptic double curve \(E\) (in Type II). Only some of these regulings are smoothable by Theorem 3.5.

Definition 3.7. The charge of an anticanonical pair \((V, D)\) is \(\chi_{\text{top}}(V \setminus D)\). If \(D = \sum D_j\),

\[
Q(V, D) := \begin{cases} 
12 + \sum (-3 - D_j^2) & \text{if } D \text{ is nodal with at least two components,} \\
11 - D^2 & \text{if } D \text{ is nodal and irreducible,} \\
12\chi(\mathcal{O}_V) - D^2 & \text{if } D \text{ is smooth.}
\end{cases}
\]

Proposition 3.8 (Conservation of Charge, [FM83a, Prop. 3.7]). Let \(X = \bigcup(V_i, D_i)\) be a Kulikov surface. Then \(\sum Q(V_i, D_i) = 24\).

Definition 3.9. A corner blow-up of \((V, D)\) is the blow-up at a node of \(D\). An internal blow-up is the blow-up at a smooth point of \(D\).

Both the corner and internal blow-ups \(\overline{V} \to V\) are naturally anticanonical pairs \((\overline{V}, \overline{D})\). For a corner blow-up, \(\overline{D}\) is the reduced inverse image of \(D\). For an internal blow-up, \(\overline{D}\) is the strict transform of \(D\). The formula for charge easily implies \(Q(\overline{V}, \overline{D}) = Q(V, D)\) for a corner blowup, while \(Q(\overline{V}, \overline{D}) = Q(V, D) + 1\) for an internal blow-up.

Any toric pair satisfies \(Q(\overline{V}, \overline{D}) = 0\). When \(V\) is rational and \(D\) has nodes, as is the case for any component in Type III, [GHK15, Prop. 1.3] proves the existence of a diagram

\[
(V, D) \xleftarrow{f} (\overline{V}, \overline{D}) \xrightarrow{g} (\overline{V}, \overline{D})
\]

where \(f\) is a sequence of corner blow-ups, \(g\) is a sequence of internal blow-downs, and \((\overline{V}, \overline{D})\) is a toric pair. We call this data a toric model of \((V, D)\). By the existence of a toric model, \(Q(V, D) \geq 0\) for all \((V, D)\) in Type III, with \(Q(V, D) = 0\) if and only if \((V, D)\) is toric. So the conservation-of-charge formula (3.8) says that \(X_0\) is “24 steps from being toric.”

When \((V, D)\) is an elliptic ruled component of a Type II Kulikov surface, we have \(Q(V, D) = 0\) if and only if \(V \cong \mathbb{P}_E(\mathcal{O} \oplus L)\) with \(D\) the disjoint union of the zero and infinity sections. Otherwise \(Q(V, D)\) measures the number of steps from being a smooth \(\mathbb{P}^1\)-bundle over an elliptic curve \(E\).

Finally, we discuss base change, following [Fri83a]. Consider an order \(k\) base change \(X' \to X\) of a Kulikov model along a branched cover \((C', 0) \to (C, 0)\). Let \(t\) be an analytic coordinate on \((C, 0)\).

The smoothing of \(X_0\) is locally \(xy = t\) or \(xyz = t\) near a double curve or triple point, respectively.

So the base change is locally \(x'y = t^k\) or \(x'yz = t^k\). There is a locally toric, SNC resolution \(X[k] \to X'\) near the singular locus of \(X_0\) corresponding to the standard order \(k\) subdivision of the simplices of the dual complex \(\Gamma(X_0)\). Each triangle decomposes into \(k^2\) triangles, and each segment into \(k\) subsegments.

All components of \(X_0[k]\) not appearing in \(X_0\) satisfy \(Q = 0\).

3B. Nef, divisor, and stable models. We now describe some additional structures on a Kulikov model in the presence of a quasipolarization.

Definition 3.10. Let \(L^*\) be a line bundle on \(X^*\), relatively nef and big over \(C^*\). A relatively nef extension \(L\) to a Kulikov model \((X, R)\) over \(C\) is called a nef model.

Definition 3.11. Let \(R^* \subset X^*\) be the vanishing locus of a section of \(L^*\) as above, containing no vertical components. A divisor model is an extension \(R \subset X\) to a relatively nef divisor \(R \in |L|\) for which \(R_0\) contains no strata of \(X_0\).

Definition 3.12. The stable model of \((X^*, R^*)\) is

\[
(\overline{X}, \overline{R}) := \text{Proj}_C \bigoplus_{n \geq 0} H^0(X, \mathcal{O}(nR))
\]

for some divisor model. It is unique and independent of the choice of divisor model \((X, R)\) by the theory of canonical models, since for \(0 < \epsilon \ll 1\) the pair \((\overline{X}, \overline{X}_0 + \epsilon\overline{R})\) has log canonical singularities and the divisor \(K_{\overline{X}} + \epsilon\overline{R}\) is relatively ample.
By adjunction, the central fiber \((X_0, \epsilon R_0)\) has semi log canonical (slc) singularities and the divisor \(K_{X_0} + \epsilon R_0\) is ample.

The existence of a nef model is due to Shepherd-Barron [SB83], and the existence of a divisor model is proved in [Laz16, Thm. 2.11], [AET19, Thm. 3.12].

Now suppose one starts with a family \((X, \tilde{R}) \to C^*\) of K3 surfaces with ADE singularities. After a finite base change it admits a simultaneous resolution of singularities \(f: X^* \to X^*\). Let \(R^* = f^*(\tilde{R})\). After a further finite base change, by the above we get a divisor model, whose stable model \((X, \tilde{R})\) is the stable extension of \((X^*, \tilde{R}^*)\) over \(C\). It is unique and stable under base changes by a standard argument, see e.g. [Kol23, Thm. 2.47].

3C. Topology of Kulikov models. The primary reference for this section is [FS86]. Let \(X \to (C, 0)\) be a Kulikov model. For convenience, denote integral singular cohomology by \(H^*\). Let \(T: H^2(X_\epsilon) \to H^2(X_\epsilon)\) be the Picard-Lefschetz transformation along an oriented simple loop in \(C^*\) encircling \(0\). Since \(X_0\) is reduced normal crossings, \(T\) is unipotent. Let \(N := \log T\) be its logarithm.

**Theorem 3.13 ([FS86, Fri84]).** Let \(X \to (C, 0)\) be a Kulikov model. We have that

- if \(X\) is Type I, then \(N = 0\),
- if \(X\) is Type II, then \(N^2 = 0\) but \(N \neq 0\),
- if \(X\) is Type III, then \(N^3 = 0\) but \(N^2 \neq 0\).

Furthermore, \(N\) is integral, and of the form \(Nx = (x \cdot \lambda)\delta - (x \cdot \delta)\lambda\) for \(\delta \in H^2(X_\epsilon)\) a primitive isotropic vector, and \(\lambda \in \delta^\perp/\delta\) satisfying \(\lambda^2 = \#\) triple points of \(X_0\). When \(\lambda^2 = 0\), its imprimitivity is the number of double curves of \(X_0\).

Thus, the Types I, II, III of Kulikov model are distinguished by the behavior of the monodromy invariant \(\lambda\): either \(\lambda = 0\), \(\lambda^2 = 0\) but \(\lambda \neq 0\), or \(\lambda^2 \neq 0\) respectively.

**Remark 3.14.** If \(X^* \to C^*\) admits a quasipolarization \(M \hookrightarrow \text{Pic}(X^*)\) then \(T \in O(H^2(X_\epsilon))\) lies in the subgroup \(\Gamma\) fixing \(M\). In particular, \(\delta \in M^\perp\) and we can consider the lattice of monodromy invariants \(\lambda \in \delta^\perp/\delta\) as valued in a subquotient of \(M^\perp\).

**Definition 3.15.** Let \(I \subset H^2(X_\epsilon)\) denote the primitive isotropic lattice \(\mathbb{Z}\delta\) in Type III or the saturation of \(\mathbb{Z}\delta \oplus \mathbb{Z}\lambda\) in Type II.

As a simple normal crossings degeneration, there is a deformation-retraction \(c: X \times [0,1] \to X_0\) called the Clemens collapse [Cle69]. So we have \(H^*(X_0) = H^*(X)\). In particular, the map \(c^*_i: H^*(X_0) \to H^*(X_\epsilon)\) coincides with restriction from \(X\) to \(X_\epsilon\).

The integral cohomology of a Type III Kulikov surface \(X_0\) is computed in [FS86, Sec. 1] by the Mayer-Vietoris spectral sequence, associated to the exact sequence of sheaves

\[
0 \to \mathbb{Z}_{X_0} \to \bigoplus_i \mathbb{Z}_{V_i} \to \bigoplus_{i,j} \mathbb{Z}_{D_{ij}} \to \bigoplus_{i,j,k} \mathbb{Z}_{T_{ijk}} \to 0.
\]

It follows that there is an exact sequence

\[
0 \to \mathbb{Z} \to H^2(X_0) \to \tilde{\Lambda} \to 0,
\]

where \(\tilde{\Lambda}\) are collections of classes \((\alpha_i)\) for which \(n_{ij} = \alpha_i \cdot D_{ij} = \alpha_j \cdot D_{ji}\) for all double curves.

**Definition 3.16.** The numerically Cartier classes on a Kulikov model \(X_0\)

\[
\tilde{\Lambda} = \tilde{\Lambda}(X_0) := \ker \left( \bigoplus_i H^2(V_i) \to \bigoplus_{i,j} H^2(D_{ij}) \right)
\]

are collections of classes \((\alpha_i)\) for which \(n_{ij} = \alpha_i \cdot D_{ij} = \alpha_j \cdot D_{ji}\) for all double curves.

The lefthand term \(\mathbb{Z}\) arises in the spectral sequence from the second simplicial cohomology \(H^2(\Gamma(X_0))\) of the dual complex. Choosing an orientation on \(\Gamma(X_0)\) gives a generator \(1 \in \mathbb{Z}\) which satisfies \(c^*_i(1) = \delta\).

Mayer-Vietoris for a Type II Kulikov surface \(X_0\) implies that there is an analogous exact sequence \(0 \to \mathbb{Z}^2 \to H^2(X_0) \to \tilde{\Lambda}(X_0) \to 0\) with the \(\mathbb{Z}^2\) arising from \(H^1(D_{i,i+1})\) for some double curve \(D_{i,i+1}\). Here the image \(c^*_i(\mathbb{Z}^2)\) is identified with the rank two lattice \(I\).
Definition 3.17. Let $I_0$ denote the sublattice $\mathbb{Z} \cong \text{H}^2(\Gamma(X_0)) \subset \text{H}^2(X_0)$ in Type III or $\mathbb{Z}^2 \cong \text{H}^1(D_{i,i+1}) \subset \text{H}^2(X_0)$ in Type II arising from Mayer-Vietoris.

So for both Type II and III, $c_1^i(I_0) = I$ and $\text{H}^2(X_0)/I_0 = \bar{\Lambda}$.

Definition 3.18. Define the intersection form $\text{H}^2(X_0) \rightarrow \text{H}^2(X_0)$.

Definition 3.19. Define $\xi_i := c_1(\mathcal{O}_X(V_i))|_{X_0}$ to be the image of $\xi_i$. Then $\xi_i = \sum_j(D_{ji} - D_{ij})$ and $\sum_i \xi_i = 0$. Define $\Xi := \mathbb{Z}\text{-span}\{\xi_i\} \subset \bar{\Lambda}$ and declare $\Lambda := \bar{\Lambda}/\Xi$.

It is easy to check directly from property (2) of Proposition 3.3 that $\Xi \subset \bar{\Lambda}$ is contained in the null sublattice of the intersection form.

Proposition 3.20. The map $c_1^i : \text{H}^2(X_0) \rightarrow \text{H}^2(X_0)$ induces a surjection $\bar{\Lambda} \rightarrow \{\delta, \lambda\}^\perp/I$ sending $\Xi$ to zero, which thus descends to $\Lambda$. Furthermore, $\Xi = \bar{\Lambda}^{\text{null}}$ is the null sublattice. Hence $\Xi$ is saturated, $\Lambda$ is torsion-free and the induced map $\Lambda \rightarrow \{\delta, \lambda\}^\perp/I$ is an isometry of lattices.

Proof. [FS86, 4.13] gives an exact sequence

$$0 \rightarrow \hat{\Xi} \rightarrow \text{H}^2(X_0) \xrightarrow{c_1^i} \ker(N) = \{\delta, \lambda\}^\perp \rightarrow 0$$

where $\hat{\Xi} := \mathbb{Z}\text{-span}\{\xi_i\}$. Noting that $c_1^i(I_0) = I$, we can quotient the second and third factors in the above exact sequence to get an exact sequence

$$0 \rightarrow \Xi \rightarrow \bar{\Lambda} \rightarrow \{\delta, \lambda\}^\perp/I \rightarrow 0.$$

Since the third term is torsion-free, the kernel $\Xi$ must be the saturated. It is exactly the null lattice because the target $\{\delta, \lambda\}^\perp/I$ is nondegenerate and $c_1^i$ preserves the intersection form. \qed

4. The period map

4A. The period of a Kulikov surface. Let $X_0$ be a Kulikov surface, not necessarily $d$-semistable. The period map is a homomorphism $\psi$ from $\bar{\Lambda}(X_0)$ (see Def. 3.16) to $\mathbb{C}^*$ in Type III or the elliptic double curve $E$ in Type II, which measures the obstruction to a class being represented by a Cartier divisor. First, we consider the Type III case.

A resolution of the sheaf of non-vanishing holomorphic functions is given by

$$1 \rightarrow \mathcal{O}_{X_0} \rightarrow \bigoplus_i \mathcal{O}_{V_i} \rightarrow \bigoplus_{i \leq j} \mathcal{O}_{D_{ij}} \rightarrow \bigoplus_{i \leq j \leq k} \mathcal{O}_{T_{ijk}} \rightarrow 1.$$ 

Computing $\text{Pic}(X_0) = \text{H}^1(X_0, \mathcal{O}_{X_0})$ via the Mayer-Vietoris spectral sequence [FS86, Sec. 3] shows that $\text{Pic}(X_0)$ is the kernel of a homomorphism

$$\ker \left( \bigoplus_i \text{Pic}(V_i) \rightarrow \bigoplus_{i \leq j} \text{Pic}(D_{ij}) \right) \rightarrow \text{H}^2(\Gamma(X_0), \mathbb{C}^*) \cong \mathbb{C}^*$$

where the latter space is identified with $\mathbb{C}^*$ by choosing an orientation on the dual complex. Note that since $V_i$ and $D_{ij}$ are rational, we have $\text{Pic}(V_i) = \text{H}^2(V_i)$ and $\text{Pic}(D_{ij}) = \text{H}^2(D_{ij})$ so the first term is nothing more than the lattice $\bar{\Lambda}$ of (3.16).

Definition 4.1. The period point of a Type III Kulikov surface $X_0$ is $\bar{\psi}_{X_0} \in \text{Hom}(\bar{\Lambda}, \mathbb{C}^*)$.

Construction 4.2. Unwinding the maps in the spectral sequence, one can explicitly construct the homomorphism $\bar{\psi}_{X_0}$. Let $\alpha = (\alpha_i) \in \bar{\Lambda}$ be a numerically Cartier divisor. Then $\alpha_i$ determines a unique line bundle $L_i \in \text{Pic}(V_i)$ for all $i$. We have

$$L_i|_{D_{ij}} \cong L_j|_{D_{ij}} \cong \mathcal{O}_{T_{ij}}(n_{ij})$$

so we can extend a line bundle $L_i \rightarrow V_i$ by $L_j \rightarrow V_j$ to a line bundle on $V_i \cup V_j$. We may continue successively until only one component $V_1$ remains. The result is a line bundle $L \rightarrow X_0 \setminus V_1$ and we may consider the line bundle

$$L|_{D_1} \otimes L^{-1}|_{D_1} =: L_\alpha \in \text{Pic}^0(D_1).$$
We have \( \text{Pic}^0(D_1) = \mathbb{C}^* \) because the cycle \( D_1 \) is oriented by the choice of orientation on the dual complex \( \Gamma(X_0) \). So \( \alpha \) determines a period \( \tilde{\psi}_{X_0}(\alpha) = L_\alpha \in \mathbb{C}^* \). It is independent of the choice of component \( V_i \) and clearly obstructs \( \alpha \) being represented by a Cartier divisor.

**Construction 4.3.** In analogy to Construction 4.2, we now construct a period map \( \tilde{\psi}_{X_0} : \tilde{\Lambda} \to E \) in Type II. Orient the segment \( \Gamma(X_0) \) so that \( X_0 = V_0 \cup \cdots \cup V_k \) with indices increasing with respect to the orientation. Let \( \alpha = (\alpha_i) \in \tilde{\Lambda} \). Then \( \alpha_0 \in H^2(V_0) \) and \( \alpha_k \in H^2(V_k) \) define line bundles \( L_0 \) and \( L_k \) because the end surfaces are rational elliptic. On the other hand, the lifts of an element \( \alpha_i \in H^2(V_i) \), \( i \neq 0, k \) to an element \( L_i \in \text{Pic}(V_i) \) form a torsor over \( E = \text{Pic}^0(D_{i,i+1}) \). So there is a unique lift \( L_1 \) of \( \alpha_1 \) for which \( L_0|_{D_{01}} \cong L_1|_{D_{10}} \). Take this lift to extend \( L_0 \) to \( V_0 \cup V_1 \) by \( L_1 \). Continuing inductively gives a unique line bundle \( L \to X_0 \backslash V_k \). Then define

\[
\tilde{\psi}_{X_0}(\alpha) := L|_{D_{k-1,k}} \otimes L_{k-1}^{-1}|_{D_{k,k-1}} \in \text{Pic}^0(D_{k-1,k}) = E.
\]

The period map can also be defined from the exponential long exact sequence

\[
\cdots \to H^1(X_0) \to H^1(X_0, \mathcal{O}) \to \text{Pic}(X_0) \to H^2(X_0, \mathcal{O}) \xrightarrow{\psi} H^2(X_0, \mathcal{O}^*) \to \cdots.
\]

Note that \( H^2(X_0, \mathcal{O}) = H^0(X_0, \omega_{X_0})^* \cong \mathbb{C} \) is one-dimensional. Quotienting by the image of \( I_0 \subset H^2(X_0) \), we reproduce the period homomorphism

\[
\tilde{\psi}_{X_0} : \tilde{\Lambda} \to \mathbb{C}/\Psi(I_0).
\]

In Type III, we have \( \mathbb{C}/\Psi(I_0) \cong \mathbb{C}^* \) while in Type II we have \( \mathbb{C}/\Psi(I_0) \cong E \) for an elliptic curve \( E \). In both cases, \( \text{Pic}(X_0) \) is the kernel because \( H^1(X_0, \mathcal{O}) = 0 \).

**Proposition 4.4.** The surface \( X_0 \) is smoothable if and only if the period point \( \tilde{\psi}_{X_0} \in \text{Hom}(\tilde{\Lambda}, \mathbb{C}^*) \) or \( \text{Hom}(\Lambda, E) \) descends to a period point \( \psi_{X_0} \in \text{Hom}(\Lambda, \mathbb{C}^*) \) or \( \text{Hom}(\Lambda, E) \).

**Proof.** By Theorem 3.5, \( X_0 \) is smoothable if and only if it is \( d \)-semistable. But \( X_0 \) is \( d \)-semistable if and only if \( \tilde{\psi}_{X_0}(\xi_i) = 1 \) for all \( i \), i.e. \( \tilde{\psi}_{X_0} \) descends to \( \Lambda = \tilde{\Lambda}/\mathbb{Z} \).

### 4B. Markings of Kulikov surfaces

In this section, we define the analogues of markings for Kulikov surfaces \( X_0 \) to properly formulate results on the period map. Let \( \Lambda_0 = \Lambda_0(t,k) \) denote a model for \( \{\delta, \lambda\}^t/I \) in \( L_{K3} \). It depends only on the even integer \( \lambda^2 = t \) giving the number of triple points of \( X_0 \) and the imprimitivity \( k \in \delta^t/\delta \). We suppress \( (t,k) \) in the notation.

**Definition 4.5.** Let \( X_0 \) a \( d \)-semistable Kulikov surface. A marking \( (\sigma, b) \) consists of:

1. An isometry \( \sigma : \Lambda(X_0) \to \Lambda_0 \) (see Def. 3.16) and
2. An ordered basis \( b \) of \( I_0 \subset H^2(X_0) \) (see Def. 3.17).

The notion of a marking naturally extends to equisingular families \( \mathcal{X} \to S \) of Kulikov surfaces using local systems. We can now define the period map:

**Definition 4.6.** Let \( (\mathcal{X} \to S, \sigma) \) be a family of marked \( d \)-semistable Kulikov surfaces. The period map is defined by

\[
S \to \text{Hom}(\Lambda_0, \mathbb{C}^*) \text{ or } \text{Hom}(\Lambda_0, \tilde{E}), \quad s \mapsto B \circ \Psi_s \circ \sigma^{-1}.
\]

Here \( \Psi_s \) comes from the exponential exact sequence as in Section 4A, \( \tilde{E} \to \mathbb{C} \setminus \mathbb{R} \) is the universal marked elliptic curve \( \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} \tau \to \{ \tau \in \mathbb{C} \setminus \mathbb{R} \} \), and \( B \) is the quotient map

\[
B : H^2(X_0, \mathcal{O}_{X_0}) \to \mathbb{C}/\mathbb{Z} = \mathbb{C}^* \text{ or } B : H^2(X_0, \mathcal{O}_{X_0}) \to \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} \tau
\]

induced by the ordered basis \( b \) of \( I_0 \) (the first element \( b_1 \) of the basis is sent to \( 1 \in \mathbb{C} \)).

**Remark 4.7.** In the Type II case, we could also require that \( b \) is an oriented basis, in the sense that \( \tau \in \mathbb{H} \). Then the period map can be defined with target \( \tilde{E}^+ := \tilde{E}|_{\mathbb{R}} \) instead.
4C. Partial markings of K3 surfaces. Let \( X \to (C,0) \) be a Kulikov model. We determine what information a marking of \( X_0 \) induces on the general fiber \( X_t = Y \). In this subsection, we denote an analytic K3 surface by \( X \) to distinguish it from the Kulikov model \( X \).

**Definition 4.8.** A partial marking of \( Y \) is a distinguished primitive isotropic class \( \delta \in H^2(Y) \), a distinguished vector \( \lambda \in \delta^\perp/\delta \) of non-negative norm, and an isometry \( \sigma : \{ \delta, \lambda \}^\perp/I \to \Lambda_0 \). We say the partial marking is Type II or III depending on whether \( \lambda^2 = 0 \) or \( \lambda^2 > 0 \), respectively.

**Proposition 4.9.** Let \( X \to (C,0) \) be a Kulikov model. A partial marking of \( X_t \) whose distinguished classes \( \delta, \lambda \) are the monodromy invariants determines uniquely a marking of \( X_0 \).

**Proof.** Proposition 3.20 gives an isometry \( c_\ast^\delta : \Lambda(X_0) \to \{ \delta, \lambda \}^\perp/I \). So a partial marking of \( X_t \) induces an isometry \( \Lambda(X_0) \to \Lambda_0 \) by composing with \( c_\ast^\delta \). The class \( \delta \) (and \( \lambda \) in Type II) determines a basis of \( I_0 \subset H^2(X_0) \) via \( c_\ast^\delta \).

**Definition 4.10.** The parabolic stabilizer of an isotropic lattice \( I \subset L_{K3} \) fits into an exact sequence \( 0 \to U_I \to \text{Stab}_{O(L_{K3})}(I) \to \Gamma_I \to 0 \) where \( U_I \) is the unipotent radical: the normal subgroup acting trivially on the graded pieces \( I \) and \( I^\perp/I \). In Type III, \( U_I \) is isomorphic to the additive group \( \text{Hom}(I^\perp/I, I) \). In Type II, \( U_I \) is a central \( \mathbb{Z} \)-extension of \( \text{Hom}(I^\perp/I, I) \). The quotient has the structure \( \Gamma_I \cong O(I^\perp/I) \times GL(I) \). These exact sequences play an important role in the toroidal compactifications (Sec. 5B).

**Definition 4.11.** A partially marked K3 surface \( (Y, \sigma) \) is admissible if \( [\Omega] : I \otimes \mathbb{R} \to \mathbb{C} \) sending \( i \mapsto [\Omega] \cdot i \) is injective for any non-zero two-form \( \Omega \) on \( Y \). Similarly, define
\[
\mathbb{D}^I := \{ x \in \mathbb{D} \mid I \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{C} \text{ is injective} \}.
\]

Note that \( \mathbb{D}^I \subset \mathbb{D} \) is an open subset. The period maps described in Sec. 4A can be understood as Carlson’s extension class [Car85] for the limit mixed Hodge structure of surfaces, admitting a period map to \( D \).

**Proposition 4.12.** There is a fine moduli space of admissible, partially marked, analytic K3 surfaces, admitting a period map to \( \mathbb{D}^I/U_I \).

**Proof.** The partial markings of a K3 surface \( Y \) are identified with \( U_I \)-orbits of the set of markings of \( Y \). The fine moduli space \( \mathcal{M} \) of marked analytic K3 surfaces admits a period map \( P : \mathcal{M} \to \mathbb{D} \), and if the partial marking associated to a marked K3 surface \( (Y, \sigma) \) is admissible, then its image under the period map lies in \( \mathbb{D}^I \). The action of \( U_I \) by post-composition on \( \sigma \) is free on \( P^{-1}(\mathbb{D}^I) \) (as it is free on \( \mathbb{D} \)). The quotient is a non-Hausdorff complex manifold. By the Torelli theorem, the universal family descends to a universal family of partially marked K3 surfaces.

**Proposition 4.13.** In Type III, there is an open embedding \( \mathbb{D}^I/U_I \hookrightarrow I^\perp/I \otimes \mathbb{C}^* \) into a 20-dimensional algebraic torus. In Type II, there is an open embedding \( \mathbb{D}^I/U_I \hookrightarrow A_I \) where \( A_I \to I^\perp/I \otimes \mathbb{E} \) is a punctured holomorphic disk bundle.

**Sketch.** Though the period domain \( \mathbb{D} \) of analytic K3s is not Hermitian symmetric, these embeddings are defined in exactly the same way as the “torus embeddings” of the unipotent quotients of Hermitian symmetric domains [AMRT75]. In Type III, one realizes \( \mathbb{D}^I \) as a tube domain inside \( \mathbb{C}^{20} \). The translation group \( U_I = \text{Hom}(I^\perp/I, I) \) acts by translations by \( \mathbb{Z}^{20} \) on \( \mathbb{C}^{20} \) and so the quotient \( \mathbb{D}^I/U_I \) embeds into \( (\mathbb{C}^*)^{20} \).

In Type II, \( \mathbb{D}^I \) is contained in an upper half-plane bundle, fibered over the total space of a \( C^{18} \)-bundle over \( C \setminus \mathbb{R} \). The central \( \mathbb{Z} \) acts on the upper-half plane bundle by fiberwise translation. Quotienting gives a punctured holomorphic disk bundle over the \( C^{18} \)-bundle. Then \( \text{Hom}(I^\perp/I, I) \) further acts on the \( C^{18} \)-fiber over \( \tau \in C \setminus \mathbb{R} \) by translation by \( (\mathbb{Z} \oplus \mathbb{Z}\tau)^{18} \). So the quotient \( \mathbb{D}^I/U_I \) embeds into a punctured holomorphic disk bundle \( A_I \to I^\perp/I \otimes \mathbb{E} \).

The unipotent quotient of \( \mathbb{D}_M \) embeds into \( \mathbb{D}^I/U_I \) for all \( M \). Let \( \mathbb{D}(I) := \mathbb{D}^I/U_I \).
Definition 4.14. Define an enlargement $D(I) \hookrightarrow D(I)^\lambda$ as follows: In Type III, it is the closure of $D(I)$ in the toric variety $T_\lambda$ extending the torus $I^\perp/I \otimes \mathbb{C}^*$ whose fan consists of the unique ray $\mathbb{R}_{\geq 0}\lambda$. In Type II, it is the holomorphic disk bundle $\tilde{A}_I$ extending the punctured disk bundle $A_I$.

In Type III, the boundary divisor in $D(I)^\lambda$ is isomorphic to $\delta^\perp/\{\delta, \lambda\}^{sat} \otimes \mathbb{C}^*$. Since $\delta^\perp/\delta$ is unimodular, this torus can be identified with $\text{Hom}(\Lambda, \mathbb{C}^*)$. Similarly, the added boundary divisor in Type II is naturally isomorphic to the base $I^\perp/I \otimes \tilde{E}$ of the disk bundle, which is identified with $\text{Hom}(\Lambda, \tilde{E})$ again because $I^\perp/I$ is unimodular.

Definition 4.15. Let $\mathcal{X} \to S$ be a family of $d$-semistable Kulikov surfaces of Types I + II or I + III over a smooth base $S$. Suppose furthermore that the discriminant locus $\Delta \subset S$ is a smooth divisor. A mixed marking $\sigma$ is a partial marking of the family $\mathcal{X}^* \to S\setminus \Delta$ of smooth fibers together with a compatible (Prop. 4.9) marking of the equisingular family $\mathcal{X}_0 \to \Delta$ of Kulikov surfaces.

Theorem 4.16. Let $(\mathcal{X} \to S, \sigma)$ be a mixed marked family of admissible surfaces as in Definition 4.15. The period map $\psi: S \setminus \Delta \to D(I)$ extends to a morphism

$$\overline{\psi}: S \to D(I)^\lambda$$

sending the discriminant $\Delta$ to the boundary divisor $\text{Hom}(\Lambda, \mathbb{C}^*)$ or $\text{Hom}(\Lambda, \tilde{E})$. Furthermore, $\overline{\psi}|_\Delta$ is the period map for the family of marked Kulikov surfaces $\mathcal{X}_0 \to \Delta$, as in Definition 4.6.

Proof. This theorem is essentially the same as [FS86, Thm. 5.3]. The primary tool is the nilpotent orbit theorem [Sch73, Thm. 4.9].

For an $M$-quasipolarized $d$-semistable Kulikov surface $X_0$, we fix an embedding $M \to \ker(\psi|_{X_0}) \subset \Lambda(X_0)$. In the $M$-quasipolarized case, the period point $\text{Hom}(\Lambda, \mathbb{C}^* \text{ or } E)$ descends to a period point in $\text{Hom}(\Lambda/M, \mathbb{C}^* \text{ or } E)$ which we will also denote $\psi|_{X_0}$ by abuse of notation. More precisely, a primitive sublattice $M \subset \Lambda \oplus I$ is the same as a not necessarily primitive sublattice $M \subset \Lambda$ plus a homomorphism $\Psi: Tors(\Lambda/M) \to \mathbb{C}^* \text{ or } E$. The period point belongs to the coset of $\text{Hom}(\Lambda/M^{sat}, \mathbb{C}^* \text{ or } E)$ of points with $\psi|_{X_0}$ which is impossible since $M^{\perp}$ has signature $(2, 20 - r)$.

The moduli space of partially marked, $M$-quasipolarized K3 surfaces admits a period map to the torsion translate of a subtorus $D_M(I) := D_M/U_I \cap \Gamma \subset D(I)$ which is generically one-to-one. A mixed marked $M$-quasipolarized family admits a period map to the toroidal extension

$$D_M(I)^\lambda := \overline{D_M(I)} \subset D(I)^\lambda.$$

Notation 4.18. We henceforth write $I^\perp_M$ simply as $I^\perp$ whenever it is clear from context that we are working with $M$-quasipolarized surfaces.

5. Compactifications of arithmetic quotients

5A. Baily-Borel compactification. By Theorem 2.11, the coarse space of $M$-polarized (ADE) K3 surfaces $F_M$ is the quotient of the period domain

$$D_M := \mathbb{P}\{x \in M^{\perp} \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \pi > 0\}$$

by the arithmetic group $\Gamma$. In this capacity, the space $F_M = D_M/\Gamma$ has a Baily-Borel compactification [BB66], which we now describe.
Remark 5.1. Note that $\mathbb{D}_M = \mathbb{D}_M^1 \cup \mathbb{D}_M^2$ has two connected components and so $F_M$ may have either 1 or 2 connected components, depending on whether or not $\Gamma$ contains an element interchanging the two components. To simplify (but abuse) notation, we refer to $\mathbb{D}_M^1$ and its stabilizer $\Gamma^1 \subset \Gamma$ as $\mathbb{D}_M$ and $\Gamma$, respectively.

Definition 5.2. The **compact dual** is $\mathbb{D}_M^c := \mathbb{P}\{x \in M^+ \otimes \mathbb{C} \mid x \cdot x = 0\}$. It is the compact hermitian symmetric domain containing $\mathbb{D}_M$ as an open subdomain.

Definition 5.3. A **boundary component** of $\mathbb{D}_M$ is a maximal connected complex submanifold of the boundary $\partial \mathbb{D}_M \subset \mathbb{D}_M^c$. The rational boundary components $B_I$ are in bijection with primitive isotropic lattices $I \subset M^+$ via

$$B_I = \{x \in \partial \mathbb{D}_M \mid \text{span}\{\text{Re} \, x, \text{Im} \, x\} = I \otimes \mathbb{R}\}.$$  

We have $B_I \cong \mathbb{H}$ when $\text{rk} \, I = 2$. We have $B_I \cong \{\text{pt}\}$ when $\text{rk} \, I = 1$. We call these **Type II and III** boundary components, respectively. The **rational closure** of $\mathbb{D}_M$ is $\mathbb{D}_M^+ := \mathbb{D}_M \cup_{\Gamma} B_I \subset \mathbb{D}_M^c$ topologized at the boundary points using horoballs as a neighborhood base.

**Theorem 5.4** ([BB66]). The quotient $\overline{F}^\text{BB}_M := \mathbb{D}_M^+ / \Gamma$ is compact and has the structure of a projective variety, and the projective coordinate ring is the ring of modular forms for $\Gamma$.

The image of a boundary component $B_I$ in $\overline{F}^\text{BB}_M$ is isomorphic to $B_I / \text{Stab}_\Gamma(I)$ and so is either a point when $\text{rk} \, I = 1$ or a modular curve when $\text{rk} \, I = 2$.

**Definition 5.5.** The 0- and 1-cusps of $\overline{F}^\text{BB}_M$ are the zero- and one-dimensional boundary components, respectively. They are, respectively, in bijection with $\Gamma$-orbits of rank 1 and 2 primitive isotropic lattices $I \subset M^+$.

**Proposition 5.6.** Let $X \to (C, 0)$ be an $M$-quasipolarized Kulikov model. The extension of the period map $C^* \to F_M$ to the Baily-Borel compactification sends 0 into the cusp associated to the monodromy lattice $I$. In Type II, the $j$-invariant $j(D_{i,i+1})$ of a double curve agrees with the $j$-invariant $j : B_I / \text{Stab}_I(I) \to \mathbb{H} / \text{SL}_2(\mathbb{Z}) = \mathbb{A}_1^1$ of the corresponding image point.

**Proof.** This well-known fact follows directly from the asymptotics of the period map and the nilpotent orbit theorem, as in Theorem 4.16. $\square$

**5B. Toroidal compactification.** The original source on this subject is [AMRT75]. The reference [Nam80] in the case of Siegel space $\mathbb{D} = \mathbb{H}_g$ is particularly clear. The following well-known theorem is key to constructing toroidal compactifications:

**Theorem 5.7.** Let $B_I$ be a rational boundary component of $\mathbb{D}_M$. There exists a horball neighborhood $\overline{N}_I \supset B_I$ preserved by $\text{Stab}_I(I)$ and an embedding

$$N_I / \text{Stab}_I(I) \hookrightarrow \mathbb{D}_M / \Gamma$$

where $N_I = \overline{N}_I \setminus B_I$.

So a punctured neighborhood of a Baily-Borel cusp can be constructed locally as a quotient by the parabolic stabilizer $\text{Stab}_I(I)$. Let $0 \to U_I \to \text{Stab}_I(I) \to \Gamma_I \to 0$ be the exact sequence associated to the unipotent radical $U_I$ (4.10). Then $N_I / U_I \hookrightarrow \mathbb{D}_M(I) = \mathbb{D}_M / U_I$ has an open embedding into the unipotent quotient. The Levi group $\Gamma_I$ has a residual action on both.

**Definition 5.8.** Let $I = \mathbb{Z} \delta$ be a rank 1 isotropic lattice. Let $C_{\delta} \subset \delta^+ / \delta \otimes \mathbb{R}$ denote a connected component of the positive norm vectors and let $C_{\delta}^+$ be its rational closure: the union of $C_{\delta}$ with all rational rays on its boundary.

**Definition 5.9.** A $\Gamma$-admissible collection of fans $\mathfrak{F}_{\delta}$ (or for short, fan) is, for each $I = \mathbb{Z} \delta$, a fan $\mathfrak{F}_{\delta}$ with support $C_{\delta}^+$, such that the collection $\{\mathfrak{F}_{\delta}\}$ is $\Gamma$-invariant, with finitely many orbits of cones.

By “fan” $\mathfrak{F}_{\delta}$ we mean a decomposition into rational polyhedral cones, closed under taking faces and intersections, and locally finite in the positive cone $C_{\delta}$. Infinitely many cones meet at rational rays on the boundary of $C_{\delta}^+$. Recall, when $\text{rk} \, I = 1$, there is a “torus embedding” $\mathbb{D}_M(I) \hookrightarrow \delta^+ / \delta \otimes \mathbb{C}^*$ (4.13).
Construction 5.10. The toroidal compactification $F^\tilde{\mathcal{S}}_M$ associated to a fan $\tilde{\mathcal{S}}$ is built as follows: Take the closure $N_I/U_I \hookrightarrow \overline{N_I/U_I} \subset X(\tilde{\mathcal{S}}_{\delta})$ in the toric variety containing $\delta^+ / \delta \otimes \mathbb{C}^*$ associated to the fan $\tilde{\mathcal{S}}_{\delta}$. Then quotient by $\Gamma_I$ to get an analytic space $V_I := (N_I/U_I) / \Gamma_I$. This is possible by $\Gamma$-invariance of $\tilde{\mathcal{S}}$. Note that $V_I$ contains an open subset $(N_I/U_I) / \Gamma_I = N_I / \text{Stab}_I(I) \hookrightarrow \mathbb{D}_M / \Gamma_I$.

Define the Type III extension to be the gluing of $F_M = \mathbb{D}_M / \Gamma$ to $V_I$ along this open set, ranging over all $\Gamma$-orbits of rank 1 isotropic $I$.

If $I = \mathbb{Z} \delta \oplus \mathbb{Z} \lambda$ is isotropic of rank 2, take the closure $\overline{N_I/U_I} \subset \mathbb{D}_M(I)^{\lambda}$ in the projective line bundle over $I^+ / I \otimes \mathbb{E}^+$ (4.13) and define $V_I$ as above. The Type II extension is the gluing of $F_M$ with $V_I$ along their common open subset $(N_I/U_I) / \Gamma_I = N_I / \text{Stab}_I(I)$.

The toroidal compactification $F^\tilde{\mathcal{S}}_M := F_M \cup \bigcup V_I$ is the gluing of the Type II and III extensions.

Let $\delta \in I = \mathbb{Z} \delta \oplus \mathbb{Z} \lambda$. The analytic structure where the corresponding Type III and II loci meet is described by the Mumford construction [Mum72] applied to a periodic, rational polyhedral tiling $\tilde{\mathcal{S}}_{\delta,I}$ of $I^+ / I \otimes \mathbb{R}$. The polyhedral tiles are defined as follows: Quotient the cones of $\tilde{\mathcal{S}}_{\delta}$ passing through $\mathbb{R}^+ \lambda \subset C^+_\delta$, viewing $I^+ / I$ as the subquotient $\lambda^+ / \lambda$ of $\delta^+ / \delta$. Geometrically, $I^+ / I \otimes \mathbb{R}$ is identified with a small horosphere through $\lambda$ (minus $\lambda$) in the hyperbolic space $\mathbb{P}C^\delta$. The projectivized cones of $\tilde{\mathcal{S}}_{\delta}$ decompose this horosphere in a $\text{Stab}_{\Gamma_\delta}(\lambda)$-invariant manner.

5C. Semitoroidal compactification. The papers [Loo85, Loo03] are the only references for this section. Semitoroidal compactifications are determined combinatorially, unify the toroidal and Baily-Borel compactifications, and form the smallest class of compactifications closed under taking normal images of toroidal compactifications (proven in Theorem 5.14 below).

The combinatorial input is similar to toroidal compactifications, with two differences:

Definition 5.11 ([Loo03, Def. 6.1]). A semifan $\tilde{\mathcal{S}}$ requires the same data as a fan (Def. 5.9), but we allow the cones in $\tilde{\mathcal{S}}_{\delta}$ to be only locally polyhedral in $C^\delta$.

We additionally require “compatibility” at each 1-cusp: Let $\delta \in I$ be a primitive integral vector in a rank 2 isotropic lattice and let $\tilde{\mathcal{S}}_{\delta,I}$ denote the corresponding polyhedral tiling of $I^+ / I \otimes \mathbb{R}$. The tiles of $\tilde{\mathcal{S}}_{\delta,I}$ are of the form $B \times (H_{I,\delta} \otimes \mathbb{R})$ for bounded polytopes $B$ and $H_{I,\delta} \subset I^+ / I$ a primitive sublattice. We require that $H_{I,\delta} = H_I$ is independent of choice of $\delta$.

Example 5.12. Any fan is a semifan. The tiles of $\tilde{\mathcal{S}}_{\delta,I}$ are bounded polytopes, so $H_I = \{0\}$ for all $I$. At the other extreme, the semifan $\tilde{\mathcal{S}}$ for which $\tilde{\mathcal{S}}_{\delta} = \{C^\delta_{\delta}\}$ is locally finite and the compatibility condition holds: $H_{I,\delta} = I^+ / I$ for all $\delta \in I$. The resulting compactification is $F^\tilde{\mathcal{S}}_M = F^\text{BB}_M$.

We now compile the key results we need about semitoroidal compactifications:

Theorem 5.13. There is a normal compactification $F^\mathcal{S}_M$ whose boundary strata are in bijection with $\Gamma$-orbits of cones of $\mathcal{S}$ [Loo03, Thm. 6.7]. The stratum $\text{Str}_{\sigma}$ corresponding to a cone $\sigma \subset \mathcal{S}_{\delta}$ is finite quotient of $\delta^+ / \{\delta, \sigma\} \otimes \mathbb{C}^*$ in Type III, and a finite quotient of $I^+ / \{I, H_I\} \otimes \mathbb{E}$ in Type II [Loo03, p. 552]. For any semifan $\mathcal{S}$ which refines $\tilde{\mathcal{S}}$, there is a morphism $F^\mathcal{S}_M \to F^\tilde{\mathcal{S}}_M$ mapping strata to strata [Loo03, Lem. 6.6]. Given an inclusion of cones $\sigma \subset \sigma_{\delta}$ the map of corresponding strata is induced by the natural quotient map on tori.

Unlike for fans, a Type III cone $\sigma$ of a semifan may have an infinite stabilizer $\text{Stab}_{\Gamma_\delta}(\sigma)$. Still, the corresponding stratum is a finite quotient of a torus.

A simple way to visualize a semifan or fan $\tilde{\mathcal{S}}$ is as follows: For each $\Gamma$-orbit of isotropic vector $\delta$, associate a cusped, real-hyperbolic orbifold $M_{\delta} := \mathbb{H}^{19-rk M} / \Gamma_{\delta}$ where $\mathbb{H}^{19-rk M} = PC_{\delta}$ is real-hyperbolic space. The cusps of $M_{\delta}$ correspond to $\Gamma_{\delta}$-orbits of isotropic rays in $C^+_\delta$. A semifan $\tilde{\mathcal{S}}_{\delta}$ gives rise to a finite decomposition of $M_{\delta}$ (for all $\delta$) into metrically convex, rational, locally polyhedral cells, compatible with the hyperbolic cusps. For a fan, these cells are polyhedra, while for a semifan, they may have nontrivial topology.

14
We now prove a key theorem characterizing semitoroidal compactifications:

**Theorem 5.14.** Let $F$ be a Type IV arithmetic quotient and let $\overline{F}$ be a normal compactification of $F$. The following are equivalent:

1. $\overline{F}$ sits between some toroidal and the Baily-Borel compactification: $F^\delta \rightarrow \overline{F} \rightarrow \overline{F}^{BB}$.
2. There exists a semifan $\mathfrak{F}$ for which $\overline{F} = F^\mathfrak{F}$.

**Proof.** The implication (2) $\Rightarrow$ (1) follows from refining $\mathfrak{F}$ to some fan $\mathfrak{G}$.

Now we prove (1) $\Rightarrow$ (2). Define an equivalence relation $\sigma_1 \sim \sigma_2$ on maximal cones of $\mathfrak{G}$ generated by: $\sigma_1 \sim \sigma_2$ if $\sigma_1$ and $\sigma_2$ share a codimension one face $\tau$ such that the corresponding 1-dimensional boundary stratum $\text{Str}_\tau$ is contracted by $m$. Our strategy is to show that the curves contracted by any birational morphism $m: F^\delta \rightarrow \overline{F}$ over $F^{BB}$ are algebraically equivalent to a union of 1-dimensional torus orbits. So the “toroidally definable” equivalence relation $\sim$ captures everything one needs to know about the contracting morphism $m$.

Define a decomposition of $C^+_\delta$ into a collection of maximal dimensional sets

$$[\sigma_0] := \bigcup_{\sigma \sim \sigma_0} \sigma.$$

We claim that the $[\sigma_0]$ form the maximal cones of some semifan $\mathfrak{F}$. The $\Gamma$-invariance is automatic, so it suffices to show that $[\sigma_0]$ satisfy the semifan axioms, including the compatibility condition (Def. 5.11) over the 1-cusps.

Begin with a Type III cone $\tau \in \Theta_\delta$. The stratum $\overline{\text{Str}}_{\tau} \subset F^\delta$ is the $\text{Stab}_\tau(\tau)$-quotient of the toric variety $X(\Theta_\delta/\tau)$ associated to the quotient fan. Consider the Stein factorization $X(\Theta_\delta/\tau) \rightarrow Z_{\tau} \rightarrow m(\text{Str}_\tau)$. It is proved in [BMSZ18, Lem. 2.3.4] that the target of a morphism from a proper toric variety to a normal variety $(\overline{Z}, \tau)$ is independent of $I, \delta$. The stratum $\text{Str}_\tau$ on maximal cones induces a polyhedral decomposition of $X(\Theta_\delta/\tau)$, which is the quotient of $X(\Theta_\delta/\tau)$ by the cone $\tau$. So the equivalence relation $\sim$ on the maximal cones containing $\tau$ is induced by the morphism of fans corresponding to the toric morphism $X(\Theta_\delta/\tau) \rightarrow Z_{\tau}$. Thus the cones $[\sigma_0]$ locally form a fan in a tubular neighborhood of $\tau \subset C^+_\delta$. In particular, they are locally polyhedral and convex at their boundary. So the $[\sigma_0]$ define a semifan within $C_\delta$.

Next, we examine the Type II locus. Since $\overline{F}$ has a morphism to $\overline{F}^{BB}$, we conclude that $m$ induces a fiberwise morphism over the modular curve 1-cusp. Let $j$ be a point in the 1-cusp of $\overline{F}^{BB}$. The fiber over $j$ in any toroidal compactification is a finite quotient of $I_{+}/I \otimes E_j$ so there is a morphism $I_{+}/I \otimes E_j \rightarrow F^j_{\tau}$ induced by $m$. In analogy with the Type III case, take the Stein factorization of this morphism $I_{+}/I \otimes E_j \rightarrow Z_j \rightarrow F^j_{\tau}$. Since the normal image of an abelian variety is an abelian variety, this map is the quotient by a sub-abelian variety $H_j \otimes E_j$.

The contracted curves are generated, up to algebraic equivalence, by $h \otimes E_j$ for $h \in H_j$. Taking the limit

$$j \rightarrow 0 \text{-cusp of } \overline{F}^{BB} \text{ associated to } \delta,$$

the elliptic curve $h \otimes E_j$ breaks in the Type II locus to a cycle of rational curves, according to the Mumford degeneration discussed after Construction 5.10.

Applying the torus action to the cycle of rational curves $\lim_j h \otimes E_j$ we can break it further into a cycle of contracted 1-dimensional boundary strata connecting 0-dimensional boundary strata of Type III. So the equivalence relation $\sim$ on maximal cones induces a polyhedral decomposition of $I_{+}/I \otimes \mathbb{R}$ whose tiles are fixed under translation by $h$, and in turn by $H_j$. Conversely, consider an $h \in I_{+}/I$ fixing all tiles in the polyhedral decomposition $H_{1, \delta}$ of $I_{+}/I$ induced by the cones $[\sigma_0]$. This $h$ corresponds to a contracted cycle of rational curves, which deforms to a contracted elliptic curve $h \otimes E_j$. Hence $h \in H_j$. Thus $H_j = H_{1, \delta}$ is independent of $\delta$. So there exists a semifan $\mathfrak{F}$ whose maximal cones are $[\sigma_0]$.

Since $\mathfrak{F}$ is a coarsening of $\mathfrak{G}$, there is a morphism $F^\mathfrak{G} \rightarrow F^\mathfrak{F}$ and the above arguments prove that the curves contracted by $n$ are exactly those contracted by $m$. We conclude by Zariski’s main theorem that $\overline{F} = F^\mathfrak{F}$, because $\overline{F}$ is normal. \qed
6. Moduli of stable slc pairs

6A. Canonical choices of polarizing divisor. Let \( \mathcal{F}^q_M \) be the moduli stack of \( M \)-quasipolarized K3 surfaces. Fix a class \( L \in M \), not necessarily primitive, which defines a relatively big and nef line bundle \( \mathcal{L} \rightarrow \mathcal{X} \rightarrow \mathcal{F}^q_M \) on the universal family, canonical up to twisting by line bundles pulled back from \( \mathcal{F}^q_M \). Since \( \mathcal{L} \) is big and nef on every fiber, \( h^i(\mathcal{X}_s, \mathcal{L}_s) = 0 \) for \( i > 0 \) for all \( s \in \mathcal{F}^q_M \). By Cohomology and Base Change, the pushforward of \( \mathcal{L} \) from the universal family defines a vector bundle of rank \( 2 + \frac{1}{2}L^2 \) on \( \mathcal{F}^q_M \), canonical up to twisting by line bundles, cf. [Kol23, Cor. 2.69]. Let \( \mathbb{P}_L \) denote its projectivization, a \( \mathbb{P}^g \)-bundle over the stack, where \( g = d + 1 \).

**Definition 6.1.** A canonical choice of polarizing divisor is a rational section \( R \) of the projective bundle \( \mathbb{P}_L \). Alternatively, it is an ample divisor \( R \) on the generic K3 surface.

Let \( U \) be the regular locus of this rational section. The key definition of the paper is:

**Definition 6.2.** A canonical choice of polarizing divisor \( R \) is recognizable for \( F_M \) if every \( M \)-quasipolarized Kulikov surface \( X_0 \) of Type I, II, or III contains a divisor \( R_0 \subset X_0 \) which, for any \( M \)-quasipolarized smoothing \( X \rightarrow (C, 0) \) with \( C^* \subset U \), has the property that \( R_0 \) is the flat limit of \( R_t \subset X_t, t \neq 0 \), up to the action of \( \text{Aut}^0(X_0) \).

Here \( \text{Aut}^0(X_0) \) is the connected component of the identity of the automorphism group, which is always trivial in Type III, and is isomorphic to \( (C^*)^{k-1} \) where \( k-1 \) is the number of intermediate elliptic ruled components, in Type II. We use the term “smoothing” to mean specifically a Kulikov model \( X \rightarrow (C, 0) \). Roughly, Definition 6.2 amounts to saying that the canonical choice \( R \) can also be made on any Kulikov surface, including smooth K3s, at least up to \( \text{Aut}^0(X_0) \).

**Proposition 6.3.** Let \( (\mathcal{X}^*, \mathcal{R}^*) \rightarrow U \) be the universal family of pairs. If \( R \) is recognizable, it extends to a flat family of pairs \( (\mathcal{X}, \mathcal{R}) \rightarrow \mathcal{F}^q_M \). That is, \( R \) defines a regular section of \( \mathbb{P}_L \rightarrow \mathcal{F}^q_M \).

**Proof.** Let \( 0 \in \mathcal{F}^q_M \) be in the complement of \( U \). Choose any curve \( C \subset \mathcal{F}^q_M \) containing \( 0 \) for which \( C^* = C \setminus 0 \subset U \). Then \( X \rightarrow C \) is a Type I Kulikov model, for which \( X_0 \) is a smooth \( M \)-quasipolarized K3 surface. By assumption, there is a divisor \( R_0 \in |L| \) on \( X_0 \) which is the flat limit of the curves \( R_t \) for \( t \neq 0 \). We may extend the section \( \mathcal{R}^*: (\mathcal{X}, \mathcal{R}) \rightarrow \mathbb{P}_L \rightarrow \mathcal{F}^q_M \), so this extension is algebraic when restricted to any curve in \( U \cup \{0\} \). Since \( \mathcal{F}^q_M \) is connected, we conclude that the rational section \( \mathcal{R}^* \) extends over \( 0 \).

This proposition only concerns Type I Kulikov models. The properties of recognizability in Types II and III is discussed in Section 8.

6B. Compact moduli of stable pairs. We refer the reader to [Kol23] for a definitive account. An slc (or KSBA) stable pair \( (X, B = \sum b_i B_i) \) consists of a projective variety and a \( \mathbb{Q} \)-divisor which has semi log canonical (slc) singularities such that the divisor \( K_X + B \) is ample. A particular case is a log Calabi-Yau pair \( (X, \Delta + \epsilon R) \) such that \( \Delta \) is reduced and log canonical, \( 0 < \epsilon \ll 1, K_X + \Delta \sim_{\mathbb{Q}} 0 \) and \( R \) is ample, not containing any log centers of \( \Delta \). In our notations, \( R \) is a polarizing divisor. By [KK20], in any dimension the irreducible components of the moduli of log Calabi-Yau pairs with a polarizing divisor are projective. Sections 6.4 and 8.3 of [Kol23] are closely related to our setup.

The situation for K3 surfaces (note \( \Delta = 0 \)) is easier because if \( (X_0, \epsilon R_0) \) is the stable limit of a one-parameter family of K3 pairs \( (X_t, \epsilon R_t) \) then the divisor \( R_0 \) is, perhaps surprisingly, Cartier and not merely \( \mathbb{Q} \)-Cartier. Indeed, the pair \( (X_0, \epsilon R_0) \) is the central fiber of the stable model of a divisor model we defined and discussed in Section 3B. We state the main theorem for the moduli functor we need in this paper. The details are given in [AET19, Sec. 3].

**Definition 6.4.** For a fixed degree \( e \in \mathbb{N} \) and fixed rational number \( 0 < \epsilon \leq 1 \), a stable K-trivial pair of type \( (e, \epsilon) \) is a pair \( (X, \epsilon R) \) such that

1. \( X \) is a Gorenstein surface with \( \omega_X \simeq \mathcal{O}_X \),
2. The divisor \( R \) is an effective, ample Cartier divisor of degree \( R^2 = e \).
(3) The pair \((X, \epsilon R)\) has semi log canonical singularities.

**Definition 6.5.** A family of stable K-trivial pairs of type \((e, \epsilon)\) is a flat morphism \(f: (X, \epsilon R) \to S\) such that \(\omega_{X/S} \simeq O_X\) locally on \(S\), the divisor \(R\) is a relative Cartier divisor, such that every fiber is a stable K-trivial pair of type \((e, \epsilon)\).

By [AET19, Lem. 3.6], for a fixed degree \(e\) there exists an \(\epsilon_0(e) > 0\) such that for any \(0 < \epsilon \leq \epsilon_0\) the moduli stacks \(M^\text{slc}(e, \epsilon_0)\) and \(M^\text{slc}(e, \epsilon)\) coincide.

**Definition 6.6.** A family of stable K-trivial pairs of degree \(e\) is a family of type \((e, \epsilon_0)\), with \(\epsilon_0(e)\) chosen as above. We will denote the corresponding moduli functor by \(M^\text{slc}_e\). For a scheme \(S\), \(M^\text{slc}_e(S) = \{\text{families of type } (e, \epsilon_0(e)) \text{ over } S\}\), with the equivalence relation being \(S\)-isomorphisms of the family \(\mathcal{X} \to S\) preserving \(R\).

**Proposition 6.7 ([AET19, Prop. 3.8]).** \(M^\text{slc}_e\) is a Deligne-Mumford stack of stable K-trivial pairs.

We denote the coarse moduli space by \(\overline{M}^\text{slc}_e\).

**Definition 6.8.** Let \(N \in \mathbb{N}\). The moduli stack \(\mathcal{P}_{N,2d}\) parameterizes proper flat families of pairs \((X, R)\) such that \((X, L)\) is a polarized K3 surface with ADE singularities and a primitive ample line bundle \(L\), \(L^2 = 2d\), and \(R \in |NL|\) is an arbitrary divisor. One has \(R^2 = 2dN^2\). In particular, one defines \(\mathcal{P}_{2d} := \mathcal{P}_{1,2d}\).

If we take \(\epsilon_0(e)\) as above then the pair \((X, \epsilon_0 R)\) is stable. Obviously, the stack \(\mathcal{P}_{N,2d}\) is fibered over the stack \(F_{2d}\) with fibers isomorphic to \(\mathbb{P}^{2dN^2+1}\). The automorphism groups of stable pairs are finite, and it is easy to see that \(\mathcal{P}_{N,2d}\) is coarsely represented by a scheme \(P_{N,2d}\).

**Definition 6.9.** One defines \(\overline{P}_{N,2d}\) (resp. \(\overline{P}_{N,2d}^\text{rig}\)) to be the closure of the coarse moduli space \(P_{N,2d}\) (resp. stack \(\mathcal{P}_{N,2d}\)) in \(M^\text{slc}_e\) (resp. \(M^\text{slc}^\text{rig}_e\)) for \(e = 2dN^2\).

For K3 surfaces polarized by a lattice \(M \subset \text{Pic} X\), choose a primitive vector \(L \in M\) with \(L^2 > 0\). Then the substack \(F_M \subset F_{2d}\) parameterizing \(M\)-polarized K3 surfaces inside of \(2L\)-polarized K3 surfaces has dimension \(20 - \text{rank } M\). A canonical choice \(R\) of polarizing divisor over a Zariski open subset \(U \subset F_M\) (or equivalently \(F_M^\text{rig}\) as in Def. 6.1) defines an embedding of \(U \subset \mathcal{P}_{N,2d}\) if \(R \in |NL|\).

**Remark 6.10.** We should choose \(U\) to avoid the non-separated locus of \(F_M^\text{rig}\) to ensure that \(U \subset \mathcal{P}_{N,2d}\). This embedding exists on the stack level. For instance, on the stack \(F_M^\text{rig}\) of degree 2 K3 surfaces, there is a nontrivial generic inertia group \(\mathbb{Z}_2\). Then \(R\) must be preserved by the involution, and defines an embedding of stacks \(U \subset \mathcal{P}_{N,2d}\).

**Definition 6.11.** Let \(\overline{F}^R_M\) denote the closure of \(U\) in \(\overline{P}_{N,2d}\) and let \(\overline{F}^R_M\) be its coarse space.

**Proposition 6.12.** If \(R\) is recognizable, \(\overline{F}^R_M\) contains \(F_M\) as an open substack.

**Proof.** By Proposition 6.3, the choice of divisor \(R\) extends to all of \(F_M^\text{rig}\) when \(R\) is recognizable. Taking the relative stable model of the universal family of pairs \((\mathcal{X}, R) \to F_M^\text{rig}\) gives a classifying morphism \(F_M^\text{rig} \to \overline{F}^R_M\) which necessarily factors through the separated quotient \(F_M\).

**Theorem 6.13 ([AET19, Thm. 3.11]).** \(\overline{P}_{N,2d}\) and thus also \(\overline{F}^R_M\) are projective.

### 7. \(\lambda\)-families

The goal of this section is to construct “\(\lambda\)-families” of Kulikov models, both unpolarized and \(M\)-quasipolarized, of a fixed combinatorial type, and to describe the birational modifications which relate them. These are families of Kulikov models, which complete any one-parameter degeneration with monodromy invariant \(\lambda\) and play a critical role in the main theorem of [FS86]: two Kulikov models with the same \(\lambda\) are related by Atiyah flops and topologically trivial deformations.

Some improvements on loc.cit. are made: We construct families for which the boundary period mapping is an isomorphism onto the period torus \(\text{Hom}(\Lambda, \mathbb{C}^* \text{ or } \mathcal{E})\), as opposed to simply an
isogeny. Also, we globalize the main theorem of [FS86]: two \( \lambda \)-families are related by certain
global birational modifications (Thm. 7.19, Thm. 7.28). These global modifications are key to
proving that different formulations of recognizability are equivalent (Sec. 8).

Unlike Kulikov models, which depend on continuous parameters, the \( \lambda \)-families depend only on
combinatorial parameters, and thus are countable in number. Similar families of Kulikov surfaces
previously appear in work of Olsson [Ols04]. See Remark 7.35 for a comparison with our version.

7A. Deformation spaces of Kulikov models. We recall the description of the universal de-
formation of a \( d \)-semistable Kulikov surface \( X_0 \) given in [Fri83b, Thm. 5.10], when \( (X_0)_{\text{sing}} \) is
connected. The deformation space \( S \cup T \) has two smooth components. The component \( S \) is
smooth and 20-dimensional, with a smooth, divisorial discriminant locus \( \Delta \). The general fiber over
\( s \in S \) is a smooth K3 surface. The other component \( T \) has large dimension \( \text{rk} \, \Lambda(X_0) \), and consists of
the topologically trivial deformations of \( X_0 \). These result from deforming the gluings of double
curves or the moduli of anticanonical pairs \( (V_i, D_i) \) and are generally not \( d \)-semistable. \( \Delta = S \cap T \)
consists of the \( d \)-semistable, topologically trivial deformations of \( X_0 \). The universal family \( \mathcal{X} \to S \)
is topologically a product \( \mathcal{X} \approx_{\text{diff}} \Delta \times X \) with a fixed Kulikov model \( X \to (C, 0) \), and has smooth
total space. In particular, \( \mathcal{X} \to S \) admits a mixed marking (4.15) over a contractible \( S \).

As for deformations of smooth K3 surfaces, the local period map on \( S \) is understood:

**Theorem 7.1.** Let \( X_0 \) be a \( d \)-semistable Kulikov surface. Suppose \( t > 0 \), or \( t = 0 \), \( k = 1 \) (Sec. 4B).
The period map \( S \to \mathbb{D}(I)^\lambda \) is an order \( k \) cyclic cover, branched along the boundary divisor.

*Proof.* In Type III, this is [FS86, Thm. 5.3]. The Type II case is similar [Fri84]. \( \square \)

By Theorem 7.1 we can ensure that \( \mathcal{X} \to S \) is universal at all \( s \in S \): A topologically trivial
family \( \mathcal{X}_0 \to \Delta \) of \( d \)-semistable Kulikov surfaces is a fiberwise universal deformation if and only if
the period map to the boundary divisor \( \text{Hom}(\Lambda, \mathbb{C}^* \text{ or } \tilde{E}) \subset \mathbb{D}(I)^\lambda \) is a local isomorphism.

**Remark 7.2.** In the remaining case \( t = 0 \), \( k > 1 \) the singular locus of \( X_0 \) is disconnected, making
it possible to independently smooth each double curve. The \( d \)-semistable deformations of \( X_0 \) have
dimension \( 19 + k \) and fiber over \( \mathbb{C}^k \), with each coordinate hyperplane parameterizing deformations
which do not smooth a given double curve of \( X_0 \).

In this case, we define \( S \) as the inverse image of the line \( \mathbb{C}(1, \ldots, 1) \subset \mathbb{C}^k \). It gives a slice
transverse to the natural action of \( \text{Aut}^0(X_0) \cong (\mathbb{C}^*)^{k-1} \). The discriminant locus \( \Delta \subset S \) is the
inverse image of \( 0 \in \mathbb{C}^k \) and is still the universal \( d \)-semistable topologically trivial deformation,
while the general fiber is a K3 surface that simultaneously smooths all \( k \) double curves.

**Proposition 7.3.** Let \( \mathcal{X}_0 \to \Delta_\lambda \) be a topologically trivial family of marked Kulikov surfaces for
which the period map \( \Delta_\lambda \to \text{Hom}(\Lambda, \mathbb{C}^* \text{ or } \tilde{E}) \subset \mathbb{D}(I)^\lambda \) is an isomorphism. There is a smoothing

\[
\begin{array}{ccc}
\mathcal{X}_0 & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\Delta_\lambda & \longrightarrow & S_\lambda
\end{array}
\]

for which the mixed period map \( S_\lambda \to \mathbb{D}(I)^\lambda \) defines an order \( k \) cyclic branched cover to an open
neighborhood of the boundary divisor. The analytic germ of the family along \( \Delta_\lambda \subset S_\lambda \) is unique.

*Sketch.* The construction parallels that of [ast85, Exp. XIII]. When \( t > 0 \), or \( t = 0 \) and \( k = 1 \), we
 glue together the 20-dimensional bases of everywhere-universal deformations of the fibers \( X_0 \subset \mathcal{X}_0 \). With the mixed markings, these bases either glue uniquely (when \( k = 1 \)) or uniquely up to the
order \( k \) cyclic action permuting the sheets of the period mapping (Thm. 7.1). Taking care to
ensure that the glued base is Hausdorff, the resulting family \( \mathcal{X} \to S_\lambda \) smooths \( \mathcal{X}_0 \to \Delta_\lambda \) and the
germ is unique by local universality. The \( t = 0 \), \( k > 1 \) case is proven in the same way, by instead
 glueing the slices \( S \) of the \( \text{Aut}^0(X_0) \) action, see Remark 7.2. \( \square \)
7B. The gluing and period complexes. We now explicitly construct families of Kulikov surfaces $X_0 \rightarrow \Delta_\lambda$ satisfying the hypotheses of Proposition 7.3, developing ideas in [FS86, Sec. 4]. We assume here that $X_0 = \bigcup (V_i, D_i)$ is Type III. For notational convenience, we drop the index $i$ when analyzing an individual component.

Each component $(V, D)$ admits a toric model $(\tilde{V}, \tilde{D}) \xrightarrow{\mathcal{L}} (\tilde{V}, \tilde{D})$ where $f$ is a sequence of corner blow-ups and $g$ is a sequence of internal blow-ups (3.9). Note that $f$ has no moduli whereas $g$ can be varied by moving the non-nodal points blown up on the $\overline{D}_j$. Note that unless $(V, D)$ is itself toric, the toric model is non-unique.

**Definition 7.4.** An ordered toric model of $(V, D)$ is an orientation of the cycle $D$ and a toric model $f, g$ as above, together with a factorization $g = \tau_Q \circ \cdots \circ \tau_1$ into internal blow-ups. Here $Q = Q(V, D)$ is the charge (3.7). An ordered toric model of $X_0$ is an orientation of $\Gamma(X_0)$, a toric model of each component $(V_i, D_i)$, and a total ordering of the 24 internal blow-ups.

An ordered toric model of $X_0$ orients each cycle $D_i \subset V_i$ and thus gives a way to label the nodes on the component $D_{ij} \subset D_i$ as 0 and $\infty$. But on the double curve $D_{ji} \subset D_j$ the corresponding nodes have the opposite label, so the non-nodal points of $D_{ij}$ and $D_{ji}$ are inverse torsors for $\mathbb{C}^*$. 

**Construction 7.5.** Fix an ordered toric model of $X_0$ and fix copies of the toric surfaces $\tilde{(V_i, \overline{D}_i)}$.

For a given toric surface $(\tilde{V}, \overline{D})$, construct a family

\[(\tilde{V}, \overline{D}) \xrightarrow{\tau_Q} \cdots \xrightarrow{\tau_1} (\tilde{V}, \overline{D}) \times (\mathbb{C}^*)^Q\]

of anticanonical pairs over $(\mathbb{C}^*)^Q$ by freely varying the points blown up by $\tau_k$. There exists a simultaneous contraction $(\tilde{V}, \overline{D}) \rightarrow (\mathbb{C}^*)^Q$ which contracts the corner blowdowns of $f$ fiberwise. So we have families $(V_i, D_i) \rightarrow (\mathbb{C}^*)^Q_i$ for all $i$.

Now, for each $i$, choose some fiber $(V_i, D_i)$ of this family and glue $D_{ij} \subset D_i$ to $D_{ji} \subset D_j$ by a map identifying the appropriate nodes of $D_i$ and $D_j$. The set of such gluings is a torsor over $\mathbb{C}^*$. Varying all such gluings, we get a family of Kulikov surfaces

\[\mathcal{X}_0^{\text{gig}} \rightarrow \prod_i (\mathbb{C}^*)^Q_i \times (\mathbb{C}^*)^E = (\mathbb{C}^*)^{24+E}\]

whose fibers are not necessarily $d$-semistable. Here $E$ is the number of double curves on $X_0$. We call this the gigantic gluing family of Kulikov surfaces associated to the ordered toric model of $X_0$. It is globally topologically trivial by construction.

Choose an origin of the open torus orbit in the fixed toric surface $(\tilde{V}_i, \overline{D}_i)$. This choice defines a distinguished origin point of any toric boundary component and thus defines an isomorphism of the $\mathbb{C}^*$-torsor associated to any internal blow-up or any edge-gluing with $\mathbb{C}^*$. So such a choice identifies the base of the gigantic gluing family with $\text{Hom}(\mathcal{G}_0, \mathbb{C}^*)$ where

\[\mathcal{G}_0 := \bigoplus_{k=1}^{24} \mathbb{Z}E_{ijk} \oplus \bigoplus_{i<j} \mathbb{Z}D_{ij}\]

is a free $\mathbb{Z}$-module encoding the blow-up points of the $\tau_k$ and the gluing maps. Here the index $ijk$ indicates that $E_{ijk}$ meets the component $D_{ij}$ and is the $k$th internal blow-up in the ordered toric model. Note that $D_{ij}$ range only over the curves corresponding to actual double curves appearing in $X_0$ and not to boundary components blown down in $(\tilde{V}_i, \overline{D}_i)$.

Consider now automorphisms. Define $\mathcal{G}_1 := \bigoplus M_i$ with each $M_i \cong \mathbb{Z}^2$ the character lattice of the toric surface $(\tilde{V}_i, \overline{D}_i)$. The set of choices of origin points in the open torus orbit of $(\tilde{V}_i, \overline{D}_i)$ is naturally a torsor over $\text{Hom}(\mathcal{G}_1, \mathbb{C}^*)$. Fixing the family $\mathcal{X}_0^{\text{gig}}$ but varying the chosen origin point defines an equivariant action of $\text{Hom}(\mathcal{G}_1, \mathbb{C}^*)$ on $\mathcal{X}_0^{\text{gig}} \rightarrow \text{Hom}(\mathcal{G}_0, \mathbb{C}^*)$ by isomorphisms. On the base, this action is determined by a map of $\mathbb{Z}$-modules $\mathcal{G}_0 \rightarrow \mathcal{G}_1$.

**Definition 7.6.** The gluing complex $\mathcal{G}$ is the two-step complex $\mathcal{G}_0 \xrightarrow{\partial_\mathcal{G}} \mathcal{G}_1$.

We describe the map $\partial_\mathcal{G}$ explicitly. An orientation of the cycle $D_i$ (and thus of $\overline{D}_i$) gives a canonical identification $w_i : M_i \rightarrow N_i$ sending $v \rightarrow \det(v, -)$. 

**Proposition 7.7.** We have
\[ \partial \mathcal{G}(E_{ijk}) = w_i^{-1}(v_{ij}), \quad \partial \mathcal{G}(D_{ij}) = w_i^{-1}(v_{ij}) + w_j^{-1}(v_{ji}). \]
Here \( v_{ij} \in N_i \) in the cocharacter lattice of \((V_i, \mathcal{D}_i)\) is the primitive integral vector in the fan of \((V_i, \mathcal{D}_i)\) corresponding to the component \( \mathcal{D}_{ij} \).

**Proof.** The action of a change-of-origin \( c_i \in \text{Hom}(M_i, \mathbb{C}^*) \) on the induced origin point of \( \mathcal{D}_{ij} \) is given by \( c_i(w_i^{-1}(v_{ij})) \). This factor scales either the gluing parameter between \( D_{ij} \) and \( D_{ji} \) or the position of any blow-up \( E_{ijk} \) on the edge \( \mathcal{D}_{ij} \).

Given Proposition 7.7, it is convenient to identify \( \mathcal{G}_i \cong \bigoplus_i N_i \) using the isomorphisms \( w_i^{-1} \) on each summand, so that \( \partial \mathcal{G}(E_{ijk}) = v_{ij} \) and \( \partial \mathcal{G}(D_{ij}) = v_{ij} + v_{ji} \).

**Definition 7.8.** The period complex \( \mathcal{P} \) of a Kulikov model is the two-step complex \( \mathcal{P}_0 \xrightarrow{\partial_2} \mathcal{P}_1 \) where \( \mathcal{P}_0 = \bigoplus_i H^2(V_i), \mathcal{P}_1 = \bigoplus_{i,j} H^2(D_{ij}), \) and \( \partial_2 \) is the signed restriction map with respect to an orientation of the edges of \( \Gamma(X_0) \).

**Theorem 7.9.** Let \( X_0 \) be a Type III Kulikov surface with an ordered toric model. The gluing and period complexes are quasi-isomorphic as complexes of \( \mathbb{Z} \)-modules. In particular \( H^{0}(\mathcal{G}) = H^0(\mathcal{P}) = \bar{\Lambda} \) and \( K := H^1(\mathcal{G}) = H^1(\mathcal{P}) \).

**Proof.** We first record some exact sequences of Picard groups arising from the basic results on smooth projective toric surfaces:

**Lemma 7.10.** Write \( (V_i, D_i = \sum_j D_{ij}) \) as \( (V, D = \sum D_j) \). For each component, one has the following exact sequences:

\[
\begin{align*}
0 & \rightarrow \text{Pic} V \rightarrow \bigoplus \mathbb{Z} \mathcal{D}_j \rightarrow N \rightarrow 0, \quad \mathcal{L} \mapsto \sum (\mathcal{L} \cdot \mathcal{D}_j) \mathcal{D}_j \\
0 & \rightarrow \text{Pic} V \rightarrow \bigoplus \mathbb{Z} \mathcal{D}_j \oplus \mathbb{Z} E_{jk} \rightarrow N \rightarrow 0, \quad \mathcal{L} \mapsto \sum (\mathcal{L} \cdot \mathcal{D}_j) \mathcal{D}_j + \sum (\mathcal{L} \cdot E_{jk}) E_{jk} \\
0 & \rightarrow \text{Pic} V \rightarrow \bigoplus \mathbb{Z} \mathcal{D}_j \oplus \mathbb{Z} E_{jk} \rightarrow N \rightarrow 0, \quad L \mapsto \sum (L \cdot \mathcal{D}_j) \mathcal{D}_j + \sum (L \cdot E_{jk}) E_{jk}
\end{align*}
\]

where

1. \( \mathcal{L}, \mathcal{L}, \mathcal{L} \) are the line bundles on \( V, \tilde{V}, \tilde{V} \), and \( \mathcal{L} = g_\ast \tilde{L} \).
2. In the last line we take \( \tilde{L} = f^\ast L \).
3. In the last line the sum goes only over \( \mathcal{D}_j \) such that \( D_j = f_\ast \tilde{D}_j \neq 0 \).
4. \( \mathcal{D}_j \mapsto v_j \) and \( E_{jk} \mapsto v_j \).

Notationally reincorporating the dependence on \( i \), and summing these exact sequences, we get a short exact sequence of two-term complexes:

\[
\begin{array}{cccccc}
0 & \rightarrow & \bigoplus \text{Pic} V_i & \rightarrow & \bigoplus_{i \leq j} \mathbb{Z} D_{ij} & \rightarrow & 0 \\
0 & \rightarrow & \bigoplus_{i,j} \mathbb{Z} D_{ij} \oplus \bigoplus_{k=1}^{24} \mathbb{Z} E_{ijk} & \rightarrow & \bigoplus_{i \leq j} \mathbb{Z} D_{ij} \oplus N_i & \rightarrow & 0 \\
0 & \rightarrow & \bigoplus N_i & \rightarrow & \bigoplus N_i & \rightarrow & 0
\end{array}
\]

Note that:

1. The first column is a direct sum of sequences from the previous lemma.
2. In the first line \( \partial_2 : L_i \mapsto \pm \sum (L_i \cdot D_{ij}) D_{ij} \) is the signed restriction map.
(3) In the second column, the first map sends \( D_{ij} \mapsto D_{ij} + v_{ij} \).
(4) In the second line, \( \overline{D}_{ij} \mapsto D_{ij} + v_{ij} \) and \( \overline{D}_{ji} \mapsto -D_{ij} + v_{ji} \) if the corresponding edge is oriented from \( i \) to \( j \). Also, for all \( i \) and \( j \), \( E_{ij} \mapsto v_{ij} \).

The commutativity of the diagram follows from \( \overline{D}_i = f^*_i(\overline{D}_i) - \sum_{j,k} E_{ijk} \). Since the last complex is acyclic, the complex \( \mathcal{P} \) is quasi-isomorphic to the second complex \( \mathcal{G} \). There also is a quasi-isomorphism \( \mathcal{G} \to \mathcal{G} \). On \( \mathcal{G}_0 \) it maps \( \overline{D}_{ij} \to \overline{D}_{ij} + \overline{D}_{ji} \), and \( E_{ijk} \to E_{ijk} \) and on \( \mathcal{G}_1 \) it is \( (0, id) \). \( \square \)

**Proposition 7.11.** The gigantic gluing family \( X^\text{gig}_0 \to \text{Hom}(\mathcal{G}_0, \mathbb{C}^*) \) descends along the canonical surjection \( \text{Hom}(\mathcal{G}_0, \mathbb{C}^*) \to \text{Hom}(H^0(\mathcal{G}), \mathbb{C}^*) \). Furthermore, the isomorphism \( \text{Hom}(H^0(\mathcal{G}), \mathbb{C}^*) = \text{Hom}(\mathcal{\tilde{A}}, \mathbb{C}^*) \) induced by Theorem 7.9 is the period map of the descended family.

**Proof.** As noted, \( \text{Hom}(\mathcal{G}_1, \mathbb{C}^*) \) acts by automorphisms on \( X^\text{gig}_0 \). The action is free and the quotient can be constructed, for instance, by restricting \( X^\text{gig}_0 \) to a subtorus of \( \text{Hom}(\mathcal{G}_0, \mathbb{C}^*) \) which intersects each orbit of \( \text{Hom}(\mathcal{G}_1, \mathbb{C}^*) \) exactly once.

For the second statement, it suffices to show that the action of relinking on periods is described by the isomorphism \( H^0(\mathcal{G}) \to \tilde{\mathcal{A}} \). By Construction 4.2, relinking \( D_{ij} \) or moving the blow-up point of \( E_{ijk} \) by \( c \in \mathbb{C}^* \) can be computed by gluing in \( V_i \) as the last component. The action is

\[
\psi_{X_0}(\gamma) \mapsto e^{\gamma D_{ij}} \psi_{X_0}(\gamma) \quad \text{and} \quad \psi_{X_0}(\gamma) \mapsto e^{\gamma E_{ijk}} \psi_{X_0}(\gamma) \quad \text{for} \quad \gamma \in \tilde{\mathcal{A}}.
\]

This exactly corresponds to the first chain map \( \mathcal{P}_0 \to \tilde{\mathcal{G}}_0 \) in Theorem 7.9. Thus the map \( \mathcal{P}_0 \to \tilde{\mathcal{G}}_0 \) in Lemma 7.10 is the natural one, \( H^0(\mathcal{P}) = H^0(\tilde{\mathcal{G}}) \) canonically, and the proposition follows. \( \square \)

**Definition 7.12.** Let \( X_0 \) be a Type III Kulikov surface with ordered toric model. Define the big gluing family to be the descended family \( X^\text{big}_0 \to \text{Hom}(\tilde{\mathcal{A}}, \mathbb{C}^*) \) from Proposition 7.11, for which the period map is an isomorphism.

Now observe that \( \text{Hom}(\mathcal{\tilde{A}}, \mathbb{C}^*) \) is the subtorus of \( \text{Hom}(\tilde{\mathcal{A}}, \mathbb{C}^*) \) corresponding to the \( d \)-semistable Kulikov surfaces. So we define:

**Definition 7.13.** The gluing family associated to the ordered toric model of \( X_0 \) is the restriction of \( X^\text{big}_0 \) to the subtorus \( \mathcal{X}_0 \to \text{Hom}(\mathcal{\tilde{A}}, \mathbb{C}^*) = \Delta_\lambda \). The period map is an isomorphism.

**Definition 7.14.** The \( \lambda \)-family of a Type III Kulikov surface \( X_0 \) with ordered toric model is the unique germ \( \mathcal{X} \to S_\lambda \) of the universal smoothing (Prop. 7.3) of the gluing family \( X_0 \to \Delta_\lambda \).

We delay the construction of \( \lambda \)-families in Type II, as some complications arise from the non-existence of toric models.

7C. The global Friedman-Scattone theorem. We now discuss birational modifications. Let \( X \to (\mathbb{C}, 0) \) be a Kulikov model of Type I, II, or III.

**Definition 7.15.** An \((M0)\), \((M1)\), or \((M2)\) modification of \( X \) is the flop along a curve \( E \cong \mathbb{P}^1 \) in the central fiber \( X_0 \). The cases are distinguished by when \( E \cap (X_0)_{\text{sing}} = \emptyset \), when \( E \cap (X_0)_{\text{sing}} = \{\text{pt}\} \), or when \( E \subset (X_0)_{\text{sing}} \), respectively.

We describe the effect of each modification on the central fiber \( X_0 \):

(M0) flops a smooth \((-2)\)-curve in \( X_0 \) which does not deform to the general fiber. It leaves the isomorphism type of \( X_0 \) invariant.
(M1) flops an internal exceptional \((-1)\)-curve \( E \) on a component \( V_i \subset X_0 \). The effect on the central fiber is to contract \( E \subset V_i \) and blow up the intersection point \( E \cap D_{ij} \) on \( V_j \).
(M2) flops a double curve \( D_{ij} \) which is exceptional on both components on which it lies. The effect on \( X_0 \) is to contract \( D_{ij} \) on both \( V_i \), \( V_j \) and to make corner blow-ups on the two remaining components \( V_i, V_j \) which \( E \) intersects.

**Notation 7.16.** In the book [FM83b], M0, M1, M2 modifications are called Type 0, 1, 2 modifications, but we find this to conflict with the already existing usage of the word “Type.”
Definition 7.17. Let $\mathcal{X} \rightarrow S_\lambda$ be the $\lambda$-family associated to some ordered toric model of $X_0$. Let $B \subset S_\lambda$ be a smooth divisor and let $\mathcal{E} \rightarrow B$ be a smooth $\mathbb{P}^1$-fibration for which the normal bundle to $\mathcal{E}$ restricts to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on every fiber. We call the relative flop along $\mathcal{E}$:

(GM0) if $B$ is the closure in $S_\lambda$ of a Noether-Lefschetz divisor of K3 surfaces with a $(-2)$-curve, and $\mathcal{E}$ is the family of $(-2)$-curves.

(GM1) if $B = \Delta_\lambda$ is the discriminant, and $\mathcal{E}$ is a family of internal exceptional curves meeting a relative double curve $D_{ij}$.

(GM2) $B = \Delta_\lambda$ is the discriminant, and $\mathcal{E}$ is a family of relative double curves $D_{ij}$ which is, on each fiber, exceptional on both components.

In all three cases, the divisor $B \subset S_\lambda$ is smooth. Indeed in the GM1, GM2 cases, $B$ is the (smooth) discriminant divisor $\Delta$, and in the GM0 case it is the closure of Noether-Lefschetz divisor, a hypersurface subtorus, in the divisorial toroidal extension $S_\lambda$.

The relative flop $\mathcal{X} \dashrightarrow \mathcal{X}'$ along $\mathcal{E}$ exists. Indeed, let $\mathcal{X} \rightarrow \mathcal{X}$ be the blowup along $\mathcal{E}$. The exceptional divisor $\tilde{\mathcal{E}}$ is a $\mathbb{P}^1$-fibration over $B$ and $\tilde{\mathcal{E}} \cdot t = -1$ for the lines of either ruling. By [Nak71] there exists the contraction $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$ along the second ruling so that $\tilde{\mathcal{X}}$ is the blowup along $\mathcal{E}' \subset \mathcal{X}'$, $\mathbb{P}^1$-fibration over $B$.

Example 7.18. Fix a Kulikov surface $X_0 = \bigcup_{i} V_i$ in the $\lambda$-family $\mathcal{X} \rightarrow S_\lambda$ and consider a boundary divisor $\tilde{D}_{ij} \subset \tilde{V}_i$ of some ordered toric model which receives two internal blow-ups $E_1$ and $E_2$. On the sub locus of $\Delta_\lambda$ where the two blow-up points coincide, the first $(-1)$-curve $E_1$ breaks into the union of the $(-1)$-curve $E_2$ and a $(-2)$-curve with class $E_1 - E_2$. But the second exceptional curve $E_2$ never breaks, and thus satisfies the conditions of Definition 7.17(GM1).

Theorem 7.19. Any two $\lambda$-families $\mathcal{X} \rightarrow S_\lambda$ and $\mathcal{X}' \rightarrow S_\lambda$ with the same $(k, t)$ are related by a series of GM0, GM1, and GM2 modifications.

Proof. Choose an arc $(C, 0)$ intersecting $\Delta_\lambda \subset S_\lambda$ transversely, mapping $C^*$ generically into a locus of K3 surfaces with Picard group $\mathbb{Z}L$, $L^2 = 2d$. Consider the two Kulikov models $X \rightarrow (C, 0)$ and $X' \rightarrow (C, 0)$. Then the punctured families $X^*, (X')^* \rightarrow (C, 0)$ are isomorphic as families of (partially marked) K3 surfaces, and admit polarizations. So by [SB83, Cor. 3.1] there exists a sequence of M1 and M2 modifications $X \dashrightarrow X'$ connecting them. Requiring $\ker \psi_{X_0} = \mathbb{Z}L$, no modification of $X$ supports a $(-2)$-curve, eliminating the need for M0 modifications.

We now seek to globalize these modifications to a sequence of GM0, GM1, GM2 modifications. There is no obstruction to globalizing an M2 modification to GM2, since the relative double curve $D_{ij}$ never breaks. For GM0 and GM1 modifications, it suffices to work component-wise.

Lemma 7.20. Let $(\mathcal{V}, D) \rightarrow (C^*)^Q$ and $(\mathcal{V}', D') \rightarrow (C^*)^Q$ be families of anticanonical pairs (see 7.5) associated to two ordered toric models of a given pair $(\mathcal{V}, D)$. Then, there is an isomorphism in $S_Q$ of the bases and a sequence of GM0 modifications connecting $\mathcal{V}$ and $\mathcal{V}'$.

Here $S_Q$ is the signed symmetric group, acting on $(C^*)^Q$ by permuting and inverting coordinates.

Proof. First, note that M0 and GM0 modifications also make sense for anticanonical pairs, by flopping $(-2)$-curves in the complement $V \setminus D$. Since we will only be making birational modifications in the complement of the anticanonical cycle, and $V \setminus D = \tilde{V} \setminus D$, we may as well assume that $(\mathcal{V}, D) = (\tilde{\mathcal{V}}, \tilde{D})$ i.e. there are no corner blow-ups in the toric model.

Fix $(\mathcal{V}, D)$ very general, in the sense that it has no $(-2)$-curves disjoint from $D$. An ordered toric model is given by an ordered collection $(E_1, \ldots, E_Q)$ of $Q$ disjoint internal exceptional curves. It follows from a theorem of Blanc [Bla13, Thm. 1] describing the birational automorphism group of $((C^*)^2, \frac{dx}{d_1} \wedge \frac{dy}{d_2})$—see [HK20, Prop. 3.27] for the interpretation we employ—that any two such tuples are related by a series of two moves:
Definition 7.21. An elementary mutation replaces the first exceptional curve \((E_1, E_2, \ldots, E_Q) \mapsto (E'_1, E'_2, \ldots, E'_Q)\) where \(E_1 + E'_1\) is the pullback of a fiber of a toric ruling on \((\mathcal{V}, \mathcal{D})\).

Definition 7.22. An order switch sends \((E_1, \ldots, E_i, E_{i+1}, \ldots, E_Q) \mapsto (E_1, \ldots, E_{i+1}, E_i, \ldots, E_Q)\).

An elementary mutation of the ordered toric model gives rise to an isomorphism \((\mathcal{V}, \mathcal{D}) \mapsto (\mathcal{V}', \mathcal{D}')\) of the corresponding families: The construction of the family by successive blow-ups

\[(\mathcal{V}, \mathcal{D}) \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_1} (\mathcal{V}_1, \mathcal{D}_1) \xrightarrow{\tau_0} (\mathcal{V}_0, \mathcal{D}_0) = (\mathcal{V}, \mathcal{D}) \times (\mathbb{C}^*)^Q\]

is unaltered when the blow-up \(\tau_1\) is replaced with the blow-up \(\tau'_1\). The bases \(B \cong (\mathbb{C}^*)^Q \cong B'\) of the two families of varying blow-ups \(g = \tau_Q \circ \cdots \circ \tau_2 \circ \tau_1\) and \(g' = \tau_Q' \circ \cdots \circ \tau_2 \circ \tau'_1\) can thus be canonically identified. But with respect to this canonical identification, the point blown up by \(\tau'_1\) lives in the inverse \(\mathbb{C}^*\)-torsor to the point blown up by \(\tau_1\). We require the coordinates on \(B' \cong (\mathbb{C}^*)^Q\) to be compatible with the orientation, so we must invert the first coordinate.

An order switch gives an isomorphism of the families whenever \(E_i\) and \(E_{i+1}\) meet distinct components—\(\tau_i\) and \(\tau_{i+1}\) commute. But when \(E_i\) and \(E_{i+1}\) meet the same component, the families are only canonically isomorphic over the locus where \(E_i\) and \(E_{i+1}\) meet distinct points. Then \((\mathcal{V}_{i+1}, \mathcal{D}_{i+1})\) and the family \((\mathcal{V}_{i+1}', \mathcal{D}'_{i+1})\) constructed with the reverse ordering are related by a flop along the relative \((−2)\)-curve \(\mathcal{E}_{i+1} - \mathcal{E}_i \subset \mathcal{V}_{i+1}\)|\(_L\) fibering over the locus \(L\) where the blow-up points coincide. The remaining blow-ups \(\tau_j\) for \(j > i + 1\) do not interfere with the flop because \(\mathcal{E}_{i+1} - \mathcal{E}_i\) is disjoint from the boundary. The order switch permutes two \(\mathbb{C}^*\) coordinates of \(B\).

Thus, we can connect any two families \((\mathcal{V}, \mathcal{D})\) and \((\mathcal{V}', \mathcal{D}')\) by a series of isomorphisms and GM0 modifications. The sequence of elementary mutations and order switches connecting \((E_1, \ldots, E_Q)\) to \((E'_1, \ldots, E'_Q)\) induces an isomorphism \(B \to B'\) valued in \(S_Q\). \(\square\)

By Lemma 7.20, two families \((\mathcal{V}_i, \mathcal{D}_i)\) of anticanonical pairs associated to ordered toric models of a given component \((\mathcal{V}_i, \mathcal{D}_i) \subset X_0\) are connected by GM0 modifications. To globalize an M1 modification along \(E \subset \mathcal{V}_i\) we apply Lemma 7.20 to find a sequence of isomorphisms and GM0 modifications until \(E = E_Q\) is the last exceptional curve in the ordered toric model. Then \(\mathcal{E}_Q\) never breaks in the gluing family \(X_0\) as it is the last blow-up performed on any given fiber. So \(\mathcal{E}_Q\) can be flopped in \(X\). The GM0 modifications on the discriminant family \(X_0\) extend to the smoothing \(X\) because the relative \((−2)\)-curve \(\mathcal{E}_{i+1} - \mathcal{E}_i\) deforms over the Noether-Lefschetz divisor \(B \subset \lambda_\mathcal{S}\).

This proves that there exists a series of GM0, GM1, GM2 modifications of \(X \to \lambda_\mathcal{S}\) to a new \(\lambda\)-family \(X' \to \lambda_\mathcal{S}\) for which the sequence of modifications restricts to the given sequence of birational modifications \(X \dasharrow X'\). Again applying Lemma 7.20, perform a sequence of GM0 modifications until the ordered toric models defining \(X''\) and \(X'\) are the same. The theorem follows. \(\square\)

7D. Type II \(\lambda\)-families. We construct topologically trivial families of Type II Kulikov surfaces for which the period map is an isomorphism. It is simplest to construct a family for a single combinatorial type with \((k, t) = (k, 0)\), then just apply GM0 and GM1 modifications to it.

Proposition 7.23. For each \(k\), there exists a family of Type II Kulikov models \(X_0 \to \text{Hom}(\Lambda, \hat{E})\) for which the period map is the identity.

Proof. It suffices restrict to the \(k = 1\) as we may otherwise insert \(k - 1\) intermediate components which are \(\mathbb{P}^1\)-bundles over elliptic curves.

Let \(\hat{D} \subset \mathbb{P}^2\) be an arbitrary smooth cubic. Take 18 points \(p_1, \ldots, p_9, q_1, \ldots, q_9 \in \hat{D}\) satisfying the single condition \(\Omega_{\hat{D}}(0) \cong \Omega_{\hat{D}}(\sum p_i + \sum q_i)\). Let \(D_1\) denote the strict transform of \(\hat{D}\) in \(V_1 := Bl_{p_1, \ldots, p_9}\mathbb{P}^2\) and let \(D_2\) denote the strict transform of \(\hat{D}\) in \(V_2 := Bl_{q_1, \ldots, q_9}\mathbb{P}^2\). Then \(X_0 := (V_1, D_1) \cup (V_2, D_2)\) is a d-semistable Type II Kulikov surface, even when \(D_1\) and \(D_2\) are glued via an arbitrary translation. This construction produces a \(1 + (18 - 1) + 1 = 19\)-dimensional space of Kulikov surfaces. Respectively, the parameters are the \(\bar{j}\)-invariant of \(\hat{D}\), the 18 points \(p_i, q_i\) subject to the single condition, and the translation to glue by.

There is a projective linear automorphism acting by translation on \(\hat{D}\) and sending one 9-tuple to another \((p_1, \ldots, p_9) \mapsto (p_1', \ldots, p_9')\) if and only if \(p'_i - p_i\) are all equal to a fixed element of \(\text{Pic}^0(\hat{D})[3] \cong \mathbb{Z}_3^3\). Thus the family of Kulikov surfaces \(\tilde{X}_0 \to \hat{S}\) gotten by varying the data of \(\hat{D}, p_i, q_i\),
noether-lexschetz locus $D \to S$ An Definition 7.26. Quasipolarized modifications of the one in Corollary 7.24. The period map is an order $k$ branched cover of a tubular neighborhood of the boundary divisor of $\mathbb{D}(I)^{\lambda}$.

Definition 7.27. A Type II $\lambda$-family is a family of surfaces which arises from a series of GM0, GM1 modifications of the family $X \to S_\lambda$ in Corollary 7.24.

Using techniques of Theorem 7.19, replacing toric models with minimal models, we can construct a Type II $\lambda$-family for any fixed combinatorial type of surface $X_0$ via a series of GM0, GM1 modifications of the one in Corollary 7.24.

7. Quasipolarized $\lambda$-families.

Definition 7.26. An $M$-quasipolarized $\lambda$-family is the restriction of a $\lambda$-family $X \to S_\lambda$ to the Noether-Lefschetz locus $\mathbb{D}_M(I)^{\lambda} \cap S_\lambda \subset \mathbb{D}(I)^{\lambda}$, such that the embedding $M \to \text{Pic}(X_\lambda)$ induced by the marking defines an $M$-quasipolarization on a generic fiber $X_\lambda$. Notation 7.27. When the context is clear, we reuse symbols $S_\lambda$, $S_\lambda$ and $X_\lambda$, $X_0$ for the intersections $\mathbb{D}_M(I)^{\lambda} \cap S_\lambda$, $\mathbb{D}_M(I)^{\lambda} \cap S_\lambda$ and the restrictions of the unpolarized $\lambda$-families $X$, $X_0$ to them.

The elements $L \in M$ extend to line bundles $L \to X$ which are unique up to twisting by the relative components $O_X(V_i)$ and line bundles pulled back from the base $S_\lambda$.

Theorem 7.28. Any two $M$-quasipolarized $\lambda$-families are related by a series of GM0, GM1, GM2 modifications.

Proof. By Theorem 7.19, the two unpolarized $\lambda$-families from which they are restricted (see Def. 7.26) are related by GM0, GM1, GM2 modifications. These modifications specialize to birational modifications of the restricted family in all cases, except for a GM0 modification associated to a $(-2)$-curve $\beta \in M$. But in this case, the two restricted families are isomorphic before and after the modification so we simply replace the GM0 modification with this isomorphism. □

We now define analogues of nef and divisor models.

Definition 7.29. A nef $\lambda$-family is an $M$-quasipolarized $\lambda$-family $X \to S_\lambda$ together with an extension of $L \in M$ to a relatively big and nef line bundle $L \to X$.

Definition 7.30. A divisor $\lambda$-family $(X, R) \to S_\lambda$ is an $M$-quasipolarized nef $\lambda$-family and a relatively big and nef divisor $R \in |L|$ which contains no stratum of any fiber.

Proposition 7.31. Given a nef model $L \to X$ of a Type III $M$-quasipolarized Kulikov model $X \to (C, 0)$, there is an ordered toric model of $X_0$ for which $L$ defines a nef $\lambda$-family $L \to X \to S_\lambda$.

Proof. Write $L|_{X_0} = (L_i)$ with each $L_i \in \text{Pic}(V_i)$. Note that $L_i$ is nef for all $i$ and at least one $L_i$ is big. It follows from [EF21, Prop. 1.5] that there exists a toric model of $V_i$ for which

$$f_i^*L_i = \sum a_{ij} \tilde{D}_{ij} + \sum b_{ijk}E_{ijk},$$

with $a_{ij}, b_{ijk} \geq 0$. We order this toric model so that $b_{ijk} > b_{ijk}$ implies that $E_{ijk}$ is blown up after $E_{ijk}$. Then $L_i$ defines a relatively nef line bundle on the family $(V_i, D_i)$ because the only irreducible curves which $L_i$ could possibly intersect negatively are $(-2)$-curves of the form $\beta = (f_i)_*(E_{ijk} - E_{ijk})$ but

$$L_i \cdot \beta = f_i^*L_i \cdot (E_{ijk} - E_{ijk}) = b_{ijk} - b_{ijk} > 0.$$ 

Definition 7.32. An element $(\alpha_i) \in \tilde{\Lambda}$ is numerically nef if $\alpha_i$ is the class of a nef line bundle on each component $V_i$. 

24
We have that $L_i$ defines a numerically nef class on every fiber of the unpolarized gluing family over $\text{Hom}(\Lambda, C^\ast)$ or $\tilde{E}$. On the sublocus of the discriminant $\Delta_\lambda$ where $(L_i)$ actually defines a Cartier divisor, in particular over the locus where $\psi_{X_0}(M) = 1$, we get a relatively big and nef line bundle $L_0 \to X_0$. On the smoothing $\mathcal{X} \to S_\lambda$, the line bundle $L_0$ extends to a relatively big and nef line bundle $\mathcal{L}$, because big and nefness is an open condition. □

**Proposition 7.33.** Given a nef model $L \to X$ of a Type II $M$-quasipolarized Kulikov model $X \to (C, 0)$, there is a nef $\lambda$-family $\mathcal{L} \to \mathcal{X} \to S_\lambda$ extending it.

**Sketch.** The proof is roughly the same as Proposition 7.31, the key point being to order the $k\lambda$ multiplicities $k$ of the arc $C$ in $X$. This follows from recognizability because $X_0$ is of Type II.

**Definition 7.34.** A stable $\lambda$-family $(\mathcal{X}, \epsilon R) \to S_\lambda$ is defined as $\text{Proj}_{S_\lambda} \oplus_{n \geq 0} \pi_\ast \mathcal{O}(nR)$ for a divisor $\mathcal{L}$-family.

Cohomology and Base Change theorem [Har77, III.12.11] implies that the fibers of a stable $\lambda$-family are stable pairs $(\mathcal{X}, \epsilon R)$.

**Remark 7.35.** Olsson defined a moduli space closely related to $\lambda$-families in [Ols04]. The functor is defined by families of Kulikov surfaces together with a line bundle $L$ extending a polarization, such that $L^n$ for some $n > 0$ gives a morphism fiberwise contracting only finitely many curves. (Olsson uses the language of stacks and log schemes, so this description is approximate, see [Ols04] for complete details.) Our $\lambda$-families are different in a number of ways: our primary focus is a divisor $R$, and the corresponding nef line bundle $L^n = \mathcal{O}(nR)$ usually contracts irreducible components of the fibers.

**8. Recognizable divisors**

When a canonical choice of polarizing divisor (6.1) is recognizable (6.2), Proposition 6.3 allows us to extended $R^\ast$ to the whole quasipolarized moduli space $\mathcal{F}_M^q$. We now generalize this to $\lambda$-families $\mathcal{X} \to S_\lambda$. Recall that $\text{Aut}^0(X_0)$ is non-trivial only when $t = 0, k > 1$, i.e. $X_0$ is of Type II with intermediate elliptic ruled components. This case for $\lambda$ has a number of subtleties not present in the general case, and we delay its treatment to Proposition 8.10.

**Proposition 8.1.** Let $\mathcal{X} \to S_\lambda$ be an $M$-quasipolarized $\lambda$-family. If $R$ is recognizable, then the Zariski closure of $R^\ast$ is a flat family of curves in $\mathcal{X}$. Conversely, if the canonical choice of divisor $R$ extends to a flat family of divisors $\mathcal{R}^\ast$ on $\mathcal{F}_M^q$, then the existence of a further flat extension of $\mathcal{R}^\ast$ over any $\lambda$-family $\mathcal{X}$ implies $R$ is recognizable.

**Proof.** Note that $\mathcal{R}^\ast$ extends to a flat family of curves in $\mathcal{X}$ if and only if the Zariski closure $\mathcal{R} := \mathcal{R}^\ast \subset \mathcal{X}$ defines a relative curve, even over the discriminant $\Delta_\lambda$. Equivalently, $\mathcal{R}$ contains no component of any singular fiber $X_0$. By recognizability, there is a “candidate curve” $R_0 \subset X_0$ which enjoys the following property: if we take any curve $(C, 0)$ transverse to $\Delta_\lambda$ at 0, then the Zariski closure of $\mathcal{R}^\ast|_{C^\ast} \subset \mathcal{X}|_{C}$ intersects $X_0$ at $R_0$. We say that $R_0$ is the flat limit of $\mathcal{R}^\ast$ along the arc $C$. This follows from recognizability because $\mathcal{X}|_{C}$ is Kulikov.

More generally, suppose that $(C, 0)$ is an arc passing through 0 which has intersection multiplicity $k$ with $\Delta_\lambda$. This arc defines a degenerating family $X \to (C^\ast, 0)$ with monodromy invariant $k\lambda$. Letting $t$ be a local parameter at 0, the analytic equation of the smoothing is of the form $xy = t^k$ and $x^2 y = t^k$ near the double curves and triple points of $X_0$.

Such a family admits a standard resolution (Sec. 3A) to a new Kulikov model $X[k] \to (C^\ast, 0)$ whose dual complex $\Gamma(X_0[k])$ is gotten by subdividing the triangles and segments of $\Gamma(X_0)$ into $k^2$ triangles and $k$ segments. Then $X[k]$ defines a map $(C^\ast, 0) \to S_{k\lambda}$ which is transverse to $\Delta_{k\lambda}$. Here the Kulikov surfaces over the discriminant have the same combinatorial type as $X_0[k]$. The boundary divisors $\Delta_{k\lambda} = \Delta_\lambda$ are naturally isomorphic and the arcs $(C^\ast, 0)$ in both $S_\lambda$ and $S_{k\lambda}$ limit to the same point under this isomorphism.

25
Then $X_0[k]$ contains a distinguished curve $R_0[k]$ which is the flat limit of the canonically chosen divisors over any arc transverse to $\Delta_\lambda$. So the image $\overline{R}_0[k]$ under the morphism $X_0[k] \to X_0$ is equal to the flat limit of the restriction of $\mathcal{R}^*$ to any arc with tangency $k$ to $\Delta_\lambda$. So the flat limit of $\mathcal{R}^*$ over any arc $(C, 0)$ not fully contained in $\Delta_\lambda$ lies in the countable union of curves $\bigcup_{k \geq 1} \overline{R}_0[k] \subset X_0$.  

Supposing for the sake of contradiction $\mathcal{R} \cap X_0$ contained a component $V_i$, there would be some point $p \in \mathcal{R} \cap V_i$ avoiding the above countable union. Choose some irreducible curve contained in $\mathcal{R}$ passing through $p$ whose projection is not contained in $\Delta_\lambda$. Taking the image in $S_1$, gives an arc $C$ passing through 0, possibly singular, which intersects $\Delta_\lambda$ with some finite multiplicity $k$ for which the restriction $\mathcal{R}^*|_C$, contains $p$ in its Zariski closure. Contradiction.

To prove the converse is easy: Every $M$-quasipolarized smoothing of $X_0$ corresponds to a transverse arc $(C, 0)$ in the base of the $\lambda$-family $\mathcal{X} \to S_\lambda$ and so the flat extension $\mathcal{R} \cap X_0$ defines a curve $R_0$ satisfying the recognizability property.

Intuitively, recognizability implies that the limits of canonically chosen curves over arcs $(C, 0)$ approaching the discriminant with tangency $k$ are rigid, for all $k$. On the other hand, if the closure of $\mathcal{R}^*$ contained a surface in $X_0$, there would have to be some finite tangency order $k$ for which these limit curves moved.

**Remark 8.2.** Proposition 8.1 implies that any of the images $\overline{R}_0[k]$ must in fact equal $R_0$. In particular, the divisor $R_0 \subset X_0$ is compatible with base change plus standard resolution.

**Definition 8.3.** We say that $R$ is (resp. weakly) $\lambda$-recognizable if $\mathcal{R}^*$ extends to a flat family of curves in $\mathcal{X} \to S_\lambda$ for any (resp. some) ordered toric model of any (resp. some) Kulikov model with monodromy invariant $\lambda$.

**Remark 8.4.** The existence of an extension of $\mathcal{R}^*$ to $\mathcal{F}_M^3$ can be considered as $\lambda$-recognizability in the $\lambda = 0$ case. Then Proposition 8.1 states that $R$ is recognizable if and only if it is $\lambda$-recognizable for all possible $\lambda$, including $\lambda = 0$.

We now show equivalence with weak recognizability:

**Proposition 8.5.** $R$ is $\lambda$-recognizable if and only if it is weakly $\lambda$-recognizable.

**Proof.** $\lambda$-recognizability clearly implies weak $\lambda$-recognizability. To show the converse, apply Theorem 7.28: There exists a sequence of GM0, GM1, GM2 modifications connecting any two $\lambda$-families. The condition that the closure of $\mathcal{R}^*$ in a $\lambda$-family $\mathcal{X} \to S_\lambda$ contain no fiber component is a property invariant under all three types of modifications, because the center of any such modification contains no fiber component. Hence weak $\lambda$-recognizability implies $\lambda$-recognizability.

Proposition 8.5 shows that recognizability can be certified by finding some $\lambda$-family $\mathcal{X}$ for which $\mathcal{R}^*$ extends, for all $\lambda$. The following is a key statement:

**Proposition 8.6.** Suppose $R$ is recognizable, and let $X \to (C, 0)$ be a Kulikov model for which $R_0$ contains no strata of $X_0$. Then all fibers of the flat extension $(\mathcal{X}, \mathcal{R}) \to S_\lambda$ (Prop. 8.1) enjoy the same property: $\mathcal{R} \cap X_p$ contains no strata of $X_p$.

**Proof.** We show the Type III case; Type II works the same but easier. Assume the opposite: for some $p \in \Delta_\lambda$ the divisor $\mathcal{R} \cap X_p$ contains a triple point. Following the argument in [AET19, Claim 3.13], there is an order $k$ base change and (possibly non-standard) simultaneous toric resolution producing a $k\lambda$-family $\mathcal{X}' \to S_{k\lambda}$ for which the closure of $\mathcal{R}'|_C$, in $\mathcal{X}'|_{(C,p)}$, contains no strata. Here $(C, p)$ is an arc intersecting $\Delta_{k\lambda}$ transversely at $p$ and $\mathcal{R}' \subset \mathcal{X}'$ extends (Prop. 8.1) the canonical choice of polarizing divisor. The discriminant family $\mathcal{X}'_0 \to \Delta_{k\lambda}$ is topologically trivial. The divisor $\mathcal{R}'$ intersects some irreducible component $V'_p \subset X'_p$ lying over the triple points of $X_p$ but it is disjoint from the corresponding component $V'_0 \subset X'_0$. The divisor $\mathcal{R}'$ is a section of a line bundle $\mathcal{L}' = \mathcal{O}_{\mathcal{X}'}(\mathcal{R}')$. It restricts to line bundles $L'_p$ resp. $L'_0$ on $V'_p$ resp. $V'_0$, with $R'_p \in |L'_p|$ and $R'_0 \in |L'_0|$.
But since $L_0'$ restricts to the trivial bundle on $V_0'$, the topological triviality implies that $L_0'$ is the trivial bundle on $V_0'$. So $R_p'$ contains $V_p'$ if it intersects it. Contradiction. An alternative contradiction avoiding reference to the line bundles is that $R_0' \subset X_0'$ is a flat family of curves intersecting $V_0'$ but not intersecting the corresponding component for a generic nearby fiber. This would only possible if $R_p'$ contained a triple point, which is does not.

Next, we study when we have the freedom to multiply $\lambda$ by an integer:

**Proposition 8.7.** Suppose $R$ is $m\lambda$-recognizable and the fibers of the flat extension $R \subset \mathcal{X} \rightarrow S_{m\lambda}$ contain no strata of any fiber. Then $R$ is $n\lambda$-recognizable for all $n$. Conversely, if $R$ is $n\lambda$-recognizable for all $n \in \mathbb{N}$, then there is an $m \in \mathbb{N}$ for which the flat extension $R \subset \mathcal{X}$ contains no strata of fibers.

**Proof.** First we prove the forward direction, i.e. we have a flat family of curves $\mathcal{X} \rightarrow \mathcal{N}$ with strata of fibers, let $R$ be a flat extension. Consider the standard resolution $\mathcal{X}[k]$ of the global base change. On any fiber, the map $u : X_0[k] \rightarrow X_0$ satisfies the property that the inverse image $R_0[k] := u^{-1}(R_0)$ is still a divisor. This divisor certifies recognizability for $X_0[k]$. This would be false if $R_0$ contained a singular stratum of $X_0$, as then $u^{-1}(R_0)$ would contain a component.

Hence $R$ is weakly $n\lambda$-recognizable for all $m \mid n$. By Proposition 8.5, we conclude that $R$ is $n\lambda$-recognizable whenever $m \mid n$. So consider the case $m \nmid n$. Supposing $R$ were not $n\lambda$-recognizable, the limiting divisor $R_0$ would vary depending on the chosen arc $(C, 0) \rightarrow S_{n\lambda}$. But taking a standard resolution and base change of order $r$, we would conclude that $R$ is not $rn\lambda$-recognizable for an $r \in \mathbb{N}$ as the base-changed arcs would also produce different limiting divisors. Taking $r = m$ gives a contradiction.

The reverse direction follows from the existence of divisor models: There exists some Kulikov model $X \rightarrow (C, 0)$ with monodromy $m\lambda$ for which the limit $R_0$ contains no strata of $X_0$. Taking a $\lambda$-family, Proposition 8.6 shows we get a flat extension $R \subset \mathcal{X}$ containing no strata of fibers.

**Proposition 8.8.** $R$ is $n\lambda$-recognizable for all $n \in \mathbb{N}$ if and only if there exists a divisor $m\lambda$-family $(\mathcal{X}, R) \rightarrow S_{m\lambda}$ for some $m \in \mathbb{N}$.

**Proof.** The existence of a divisor $m\lambda$-family $(\mathcal{X}, R)$ implies that $R$ is (weakly) $m\lambda$-recognizable with $R$ containing no strata of fibers, so Proposition 8.7 implies that $R$ is $n\lambda$-recognizable for all $n \in \mathbb{N}$. Conversely, choose a divisor model $(X, R) \rightarrow C$ with monodromy invariant $m\lambda$. Then Proposition 7.31 (or Proposition 7.33 for Type II) implies that we may choose an ordered toric model of $X_0$ for which the line bundle $\mathcal{O}_{X_0}(R_0)$ extends to a relatively big and nef line bundle $\mathcal{L} \rightarrow \mathcal{X}$ on the corresponding $\lambda$-family. By recognizability and Proposition 8.6, the closure $\overline{\mathcal{R}} = \overline{\mathcal{L}}^\vee$ is a section of $\mathcal{L}$ which doesn’t contain strata. We conclude that $(\mathcal{X}, \mathcal{R})$ it is a divisor $m\lambda$-family.

We also show equivalence with a weaker condition:

**Proposition 8.9.** Let $X \rightarrow (C, 0) \times B$ be a family of Kulikov models over a curve $B$ for which the discriminant family $X_0 = X_0 \times B$ is constant, and the restriction of $X$ to $(C, 0) \times \{b_0\}$ gives a divisor model. Then $R$ is recognizable if and only if $R_{0, b} := \lim_{t \rightarrow 0} R_{t, b}$ is independent of $b$, i.e. $R_{0, b} = R_{0, b_0} \subset X_0$ for any such $X \rightarrow (C, 0) \times B$.

**Proof.** Certainly if $R$ is recognizable, then $R_{0, b}$ will equal the divisor $R_0 \subset X_0$ certifying recognizability for any $b$. Conversely, suppose $R$ is not recognizable. Following the proof of Proposition 8.1, there must be a one-parameter family of Kulikov models $Y \rightarrow (C, 0) \times B$ for which $R_{0, b}$ varies. It remains to show that we may assume these Kulikov models are divisor models. To do so, we perform a series of GM0, GM1, GM2 modifications (possibly after a global base change and standard resolution) until the restriction of the modified family $Y'$ to a fixed arc $(C', 0) \times \{b_0\}$ is a divisor model. These modifications do not affect the triviality of the discriminant family $X_0' = X_0' \times B$ and the limit curves $R_{0, b}$ still vary on $X_0'$ because they cover some component.

**Proposition 8.10.** Suppose that $t = 0$ and $k > 1$. That is, $X_0$ is a Type II Kulikov surface with intermediate elliptic ruled components. Then, there exist $\lambda$-families $\mathcal{X} \rightarrow S_{\lambda}$ for which Propositions 8.1, 8.5, 8.6, 8.7, 8.8, 8.9 hold.

27
Proof. Recall that the smoothing component of such a Kulikov surface $X_0$ has dimension $19+k$ and is fibered over $\mathbb{C}^k$, with the $k$th coordinate axis corresponding to the deformations which smooth the $k$th double curve. Imposing an $M$-quasipolarization reduces the dimension to $19+k-rkM$. Given any smooth arc $(C, 0) \hookrightarrow (\mathbb{C}^{19+k-rkM}, 0) =: (S, 0)$ whose tangent direction $T_0C$ is transverse to all the coordinate axes under the projection to $(\mathbb{C}^k, 0)$, the restriction of the universal family to $(C, 0)$ is a Kulikov model, simultaneously smoothing all of the double curves.

The closure $\mathcal{R} = \overline{\mathcal{R}}$ over the full $(19+k-rkM)$-dimensional smoothing component of such an $X_0$ could contain an entire intermediate elliptic ruled component. In fact, this does occur: Applying $g \in \text{Aut}^0(X_0) \cong (\mathbb{C}^*)^{k-1}$ to the arc $(C, 0)$ in the deformation space will translate the flat limit $R_0 \subset X_0$ by $g$. But a recognizable divisor $R_0$ need not be $\text{Aut}^0(X_0)$-invariant, see the $A_{17}$ case in [AET19, Construction 9.27].

Fixing one arc $(C, 0) \hookrightarrow (S, 0)$ gives a flat limit $R_0 \subset X_0$ and assuming $R$ is recognizable, the flat limit $R_0'$ along any other arc $(C', 0) \hookrightarrow (S, 0)$ differs from $R_0$ by an element $g \in \text{Aut}^0(X_0)$, i.e. $g(R_0) = R_0$. But then, the flat limit along $g^*(C', 0)$ equals $R_0$. So for any arc transverse to the coordinate axes of $(\mathbb{C}^k, 0)$, there is a representative of its $\text{Aut}^0(X_0)$-orbit for which the flat limit is equal to $R_0$. Thus, there exists a slice of the $\text{Aut}^0(X_0)$-action on $(S, 0)$ for which the flat limit along the slice is always $R_0$.

This procedure can be performed analytically-locally along the fibers over the equisingular locus $\Delta \subset S$. We call such a slice well-chosen. Summarizing, a well-chosen slice gives a local $\lambda$-family over an open set $U \subset S_{\lambda}$ around $0 \in \Delta_{\lambda}$ for which $\mathcal{R}^*$ extends to a flat family of divisors $\mathcal{R}$.

Now consider a collection $\{U_i\}$ of well-chosen slices for which $U_i \cap \Delta$ cover the equisingular deformation space $\Delta_{\lambda}$. On the double overlaps $U_i \cap U_j$ these well-chosen slices are isomorphic, by a unique isomorphism preserving the mixed marking, because the isomorphisms on the smooth smooth fibers are unique (Prop. 4.12). Thus, when $R$ is recognizable, we can glue to form a $\lambda$-family $(\mathcal{X}, \mathcal{R}) \to S_{\lambda}$ on which $\mathcal{R}$ extends to a flat family of divisors. The arguments of the above propositions apply verbatim to such a well-chosen slice. \qed

We summarize the results proven above:

**Theorem 8.11.** Let $R$ be a canonical choice of polarizing divisor, defining a divisor $\mathcal{R}^*$ on the universal $K3$ surface over a Zariski open subset $U \subset \mathcal{F}_M^3$. Then the following are equivalent:

1. Any one-parameter deformation of a divisor model $(X, R) \to (C, 0)$ keeping $X_0$ constant in moduli gives rise to a constant limiting curve $R_0$, up to $\text{Aut}^0(X_0)$.
2. $R$ is recognizable.
3. For all primitive isotropic $\delta$ and all $\lambda \in C^*_\delta \cap \delta^\perp / \delta$, there is some $\lambda$-family for which $\mathcal{R}^*$ extends a flat divisor $\mathcal{R} \subset \mathcal{X}$.
4. $\mathcal{R}^*$ extends to a flat divisor $\mathcal{R} \subset \mathcal{X}$ in every $\lambda$-family.
5. For every projective class $[\lambda]$, there exists some $k \in \mathbb{N}$ for which $\mathcal{R}^*$ extends to a divisor $\lambda$-family $(\mathcal{X}, \mathcal{R}) \to S_{\lambda}$.

If $t = 0$, $k > 1$, the above equivalences hold when the $\lambda$-family is a well-chosen slice.

**Proof.** Note that we are allowing the case $\lambda = 0$, which in conditions (3), (4), (5) amounts to saying that $\mathcal{R}^*$ extends to a section of the projective bundle $\mathbb{P}_\mathcal{L} \to \mathcal{F}_M^3$. Then (2) $\iff$ (4) by Proposition 8.1, (3) $\iff$ (4) by Proposition 8.5, and (4) $\iff$ (5) by Proposition 8.8. Finally, (1) $\iff$ (2) by Proposition 8.9. \qed

The conditions in Theorem 8.11 are roughly in increasing order of strength. As such, we use condition (5) in the proof of Theorem 1, but use condition (1) in the proof of Theorem 2.

**Definition 8.12.** Let $(\mathcal{X}, \epsilon \mathcal{R}) = \bigcup_i (V_i, \mathcal{D}_i, \epsilon \mathcal{R}_i)$ be a stable degeneration of $K3$ pairs. The slc combinatorial type is the data of:

1. The deformation types of the quasipolarized minimal resolutions $(V_i, D_i, L_i)$ of each component, where $L_i = \mathcal{O}_{V_i}(R_i)$, and
2. the combinatorics $\Gamma(\mathcal{X})$ of the singular strata.
Corollary 8.13. Suppose \( R \) is recognizable and let \( (\overline{X}^*,\epsilon\overline{\mathcal{R}}^*) \rightarrow \mathcal{C}^*, \epsilon \ll 1 \) be a family of stable K3 pairs over a punctured curve \( \mathcal{C}^* = \mathcal{C} \setminus 0 \). The slc combinatorial type of the unique stable limit \( (\overline{X}_0,\epsilon\overline{\mathcal{R}}_0) \) depends only on the projective class \([\lambda]\) of the monodromy invariant.

Proof. Consider the divisor \( \lambda \)-family as in Theorem 8.11(5). The family of canonical models \((\overline{X},\epsilon\overline{\mathcal{R}})\), where \( \overline{\mathcal{X}} = \text{Proj} \oplus_{n \geq 0} \pi_*\mathcal{O}_\mathcal{X}(n\mathcal{R}) \), and \( \overline{\mathcal{R}} = \text{im} \mathcal{R} \), is the corresponding family of stable slc pairs. Every one-parameter degeneration with monodromy invariant \( \lambda \) has a unique limit in this family. The combinatorial type of the discriminant family \((\mathcal{X}_0,\mathcal{R}_0)\) is fixed, with the line bundles \( L_0 = \mathcal{O}_{\mathcal{X}_0}(\mathcal{R}_0) \) on every fiber identified by the Gauss-Manin connection because \( \mathcal{X}_0 \) is topologically trivial. Since the contraction \( \mathcal{X}_0 \rightarrow \overline{\mathcal{X}}_0 \) is defined only by the line bundle \( L_0 \), the combinatorial type of the stable models is also fixed. \( \square \)

9. Main theorem for recognizable divisors

9A. Proof of Theorem 1. We have proven in Corollary 8.13 that whenever \( R \) is recognizable, the slc combinatorial type of an \( M \)-polarized degeneration depends only on the projective class \([\lambda]\) of the monodromy invariant. This is the key input which recognizability gives us: from here we have an essentially birational-geometric argument to show that the KSBA compactifications associated to recognizable divisors are (up to normalization) semitoroidal.

Theorem 9.1. If \( R \) is recognizable, there exists a unique semifan \( \mathfrak{S}_R \) for which \( \mathcal{F}_M^{\mathfrak{S}_R} \rightarrow \mathcal{F}_M^R \) is the normalization.

Proof. Recall that \( \mathcal{F}_M^R \) is, by Definition 6.11, the coarse space of the closure \( \mathcal{P}_{N,2d} \) of the stack of pairs parameterized by \( U \subseteq \mathcal{F}_M \).

We define the interior of \( \mathcal{F}_M^R \) to be the locus in this closure parameterizing \( M \)-polarized ADE K3 surface pairs \((\overline{X},\epsilon\overline{\mathcal{R}})\). Proposition 6.12 implies that this locus is isomorphic to \( \mathcal{F}_M \).

Let \( \mathfrak{S} \) be some regular fan (cones are standard affine) and let \( u: \mathcal{F}_M^{\mathfrak{S}} \rightarrow \mathcal{F}_M^R \) be the birational map which is isomorphism on the interiors. Let \( \sigma = \text{span}\{\lambda_1,\ldots,\lambda_d\} \) be a Type III standard affine cone of \( \mathfrak{S} \) of maximal dimension. Associated to this cone is an analytic, finite morphism from a tubular neighborhood \( N(\sigma) \) of the toric boundary of

\[ X(\sigma) = \mathbb{C}^d = \mathbb{C}\lambda_1 \oplus \cdots \oplus \mathbb{C}\lambda_d \]

to a neighborhood of the boundary strata of \( \mathcal{F}_M^{\mathfrak{S}} \) containing the 0-dimensional stratum associated \( \sigma \). The finiteness arises from quotienting by the \( \text{Stab}_{\mathcal{T}_\sigma}(\sigma) \) action on this toric chart.

Let \( u(\sigma): N(\sigma) \rightarrow \mathcal{F}_M^R \) denote the corresponding meromorphic map. Consider an arc germ \( (C,0) \subset (\mathbb{C}^d,0) \) with \( C^* \subset (\mathbb{C}^*)^d \) contained in the open torus orbit. Since \( \mathcal{F}_M^R \) is proper, \( u(\sigma) \) extends uniquely over \( C^* \) to the origin 0. By Corollary 8.13, the combinatorial type of the stable model depends only on the orders \( r_i \) of tangency of \((C,0)\) with the coordinate hyperplanes of \( \mathbb{C}^d \), since this determines the monodromy invariant of \((C,0)\) to be \( \lambda = r_1\lambda_1 + \cdots + r_d\lambda_d \).

The meromorphic map \( u(\sigma): N(\sigma) \rightarrow \mathcal{F}_M^R \) thus satisfies the following conditions:

1. There is a stratification (by slc combinatorial type) of \( \mathcal{F}_M^R \) for which the extension of \( u(\sigma) \) over any arc \((C,0)\) with fixed tangency orders \( r_i \) to the coordinate hyperplanes of \( \mathbb{C}^d \) lies in a fixed slc stratum.

2. The indeterminacy locus lies in the coordinate hyperplanes, which map by \( u(\sigma) \) into the union of Type III slc strata.

No Type III slc stratum contains a complete curve by Corollary 9.17. We conclude by Lemma 9.18 that there exists a toric blow-up of \( X(\sigma) \) eliminating the indeterminacy of \( u(\sigma) \). Further refining, we may assume this toric blow-up is given by a \( \text{Stab}_{\mathcal{T}_\sigma}(\sigma) \)-invariant fan. Thus, we may refine \( \mathfrak{S} \) so that \( u \) defines a morphism over the refinement of \( \sigma \). Applying this argument to all \( \Gamma \)-orbits of maximal cones \( \sigma \in \mathfrak{S} \), we may as well have assumed that \( u: \overline{\mathcal{F}_M^{\mathfrak{S}}} \rightarrow \mathcal{F}_M^R \) has no indeterminacy over the Type III extension of \( \mathcal{F}_M \).
In fact, there is no indeterminacy in the Type II ($\lambda^2 = 0$) locus either: By Theorem 8.11, there is a divisor $\lambda$-family $(\mathcal{X}, \mathcal{R}) \to S_\lambda$. Consider the resulting stable $\lambda$-family $(\mathcal{X}, \mathcal{R}) \to S_\lambda$. The base $S_\lambda$ is an order $k$ branched cover of a tubular neighborhood of the boundary divisor in the unipotent quotient $\mathbb{D}_M(I)$. There is a natural quotient map $v: S_\lambda \to F_M^R$ by the action of $\Gamma_k$.

The classifying morphism $S_\lambda \to F_M^R$ for the stable $\lambda$-family must factor through $v$ because the fibers of $v$ not lying in the boundary give isomorphic ADE K3 surfaces with divisor. Ranging over all $I = \mathbb{Z} \oplus \mathbb{Z}_k$, the maps $v$ surject onto the Type II locus, so $u$ extends to a morphism over the Type II extension of $F_M$.

Since the Type II and III extensions of $F_M$ cover all of $F_M^{\delta}$, we conclude that there is a morphism $F_M^{\delta} \to F_M^R$—on the intersection of the closure of the Type II locus with the Type III locus, it is a morphism as opposed to just a set-theoretic map because $F_M^{\delta}$ is normal.

By Lemma 9.19, we also have a morphism $(F_M^R)^\gamma \to F_M^{BB}$. So by Theorem 5.14, the normalization of $F_M^R$ is semitoroidal for a unique semifan $\mathfrak{F}_R$.

\textbf{Corollary 9.2.} Suppose $R$ is recognizable. The normalization map $F_M^{\delta} \to F_M^R$ sends semitoroidal strata to slc strata.

\textbf{Proof.} Let $\sigma \in \mathfrak{F}_R$ be any cone and choose $\lambda$ in the relative interior $\text{int}(\sigma)$. By Corollary 8.13, the stable limit of any degeneration with monodromy invariant $\lambda$ lies in a fixed slc stratum. Since the natural map $\delta^+/\{\delta, \lambda\} \to \delta^+/\{\delta, \sigma\}$ is surjective, every point in $\text{Str}_\sigma$ is the limit of some arc with monodromy invariant $\lambda$. So the combinatorial type of the slc stable model at any point in $\text{Str}_\sigma$ is the same.

Corollary 9.2 implies that there is a well-defined function

$$
\text{S}: \{\text{cones of } \mathfrak{F}_R \text{ mod } \Gamma\} \to \left\{\text{combinatorial types of slc strata which appear in } F_M^R\right\}.
$$

Note that $\text{S}$ may not be injective. For instance, $\text{S}(\sigma) = \text{S}(\tau)$ if the corresponding strata are unglued by normalizing. By abuse, let $\text{S}(\lambda) := \text{S}(\sigma)$ where $\lambda \in \text{int}(\sigma)$.

\textbf{Theorem 9.3.} Let $R$ be a recognizable divisor for $F_M$. Let $D$ be the decomposition of monodromy invariants into loci $\{\lambda \in \coprod I_s C^+_\delta \cap \delta^+/\delta \mid \text{S}(\lambda) \text{ is constant}\}$. Then maximal cones of $\mathfrak{F}_R$ and $D$ are the same.

A maximal cone of $D$ is a top-dimensional, convex cone in $C^+_\delta$ whose integral interior points lie in a single element of $D$, and which is maximal for this property.

\textbf{Proof.} $\text{S}$ is constant on cones of $\mathfrak{F}_R$ by Corollary 9.2, so it suffices to show that $\text{S}$ cannot take the same value on two maximal dimensional cones $\sigma_1, \sigma_2 \in \mathfrak{F}_R$ and a codimension 1 face $\tau \subset \sigma_1 \cap \sigma_2$ they share. If this were the case, the closed boundary stratum $\text{Str}_\tau$ would map to a fixed slc stratum $\text{S}(\sigma_1) = \text{S}(\sigma_2) = \text{S}(\tau)$. But the Type III slc strata contain no complete curve by Corollary 9.17. So $\text{Str}_\tau$ would be contracted to a point, contradicting finiteness of the normalization $F_M^{\delta} \to F_M^R$.

Theorem 9.3 gives a method to compute the semifan $\mathfrak{F}_R$. Up to taking faces, its cones are sets of monodromy invariants $\lambda$ which produce a fixed combinatorial slc type. This is how $\mathfrak{F}_R$ was computed in Examples 9.20, 9.21 below.

The semifan $\mathfrak{F}_R$ is also functorial under restriction to Type IV subdomains of $F_M$, i.e. Noether-Lefschetz loci. Let $M \subset M' \subset L_{K3}$ be primitive hyperbolic sublattices. Then there is a natural map of moduli stacks $F_{M'}^R \to F_{M}^R$ sending $(X,j) \mapsto (X,j)|_{M'}$. Let $L \subset M$.

\textbf{Proposition 9.4.} Suppose $R \in |L|$ is recognizable for $F_{M'}^R$. Then its restriction to $F_{M'}^R$ is also recognizable. Furthermore, $\mathfrak{F}_R(M')$ is the restriction of the semifan $\mathfrak{F}_R(M)$ to the appropriate linear subspaces of $C^+_\delta \subset \delta^+/\delta$.  


More precisely, if $\delta \in M' = M$ is an isotropic vector corresponding to some 0-cusp of $F_{M'}$, we restrict the decomposition $R_{\delta}(M)$ to the subspace $\delta_{M',\delta}/\delta$.

**Proof.** Proposition 9.4 follows from the fact that any $M'$-quasipolarized Kulikov model is also $M$-quasipolarized, plus the functoriality of the stable pair and semitoroidal constructions under restriction to Noether-Lefschetz subdomains. \[\Box\]

9B. **Moduli of anticanonical pairs.** We prove here that Type III slc strata contain no complete curve by considering the periods of anticanonical pairs. A useful general reference is [Fri15].

**Definition 9.5.** Let $(V, D)$ be an anticanonical pair with $D = D_1 + \cdots + D_n$ an oriented, labeled cycle of rational curves. Define $\Lambda_{(V, D)} := \{D_1, \ldots, D_n\} \subseteq H^2(V)$ and define the period point $\psi(\Lambda_{(V, D)}) \in \text{Hom}(\Lambda_{(V, D)}, \mathbb{C}^*)$ to be the restriction map $\gamma \mapsto \gamma|_D \in \text{Pic}^0(D) = \mathbb{C}^*$.

**Definition 9.6 ([Fri15, Def. 5.4]).** The generic ample cone $A_{\text{gen}} \subseteq H^2(V)$ is the ample cone of a very general topologically trivial deformation of $(V, D)$.

It suffices to take a deformation for which $\text{ker}(\psi(\Lambda_{(V, D)})) = 0$. This is possible because there is a local universal deformation $(V, D) \rightarrow S$ of pairs for which the assignment $s \mapsto \psi(V_s, D_s)$ is an isomorphism to an open subset of $\text{Hom}(\Lambda_{(V, D)}, \mathbb{C}^*)$.

**Definition 9.7 ([Fri15, Def. 6.5]).** A Looijenga root $\beta \in \Lambda_{(V, D)}$ is a class of square $\beta^2 = -2$ which represents a smooth $(-2)$-curve on some topologically trivial deformation of $(V, D)$, and for which $\psi(\Lambda_{(V, D)})(\beta) = 1$.

Reflections in Looijenga roots act on $A_{\text{gen}}$. The ample cone $A$ of $(V, D)$ is a fundamental chamber for the action of the group $W_{(V, D)} := \langle r_\beta : \beta \text{ a Looijenga root} \rangle$ on $A_{\text{gen}}$. We can now recall the Torelli theorem for anticanonical pairs:

**Theorem 9.8 ([Fri15, Thm. 8.7]).** Two pairs $(V, D)$ and $(V', D')$ (with oriented, labeled cycle) are isomorphic if and only if there exists an isometry $\phi : H^2(V) \rightarrow H^2(V')$ for which $\phi(D_j) = D'_j$, $\phi(A_{\text{gen}}) = A'_{\text{gen}}$, and $\psi(V, D) = \psi(V', D') \circ \phi$. Furthermore, $\phi = f^*$ is induced by an isomorphism $f : (V', D') \rightarrow (V, D)$ if and only if $\phi(A) = \phi(A')$. This isomorphism is unique up to the action of continuous automorphisms $\text{Aut}(V, D)$.

So the analogue of the Torelli Theorem 2.3 holds nearly verbatim, replacing $C$ with $A_{\text{gen}}$ (which is notably not the positive cone), $K$ with $A$ (which is the Kähler cone), and $W_X$ with $W_{(V, D)}$.

**Definition 9.9.** Fix a reference lattice $L_{(V, D)}$ isomorphic to $H^2(V)$. Fix classes $D_j^0 \in L_{(V, D)}$ and fix a cone $A_{\text{gen}}^0 \subseteq L_{(V, D)} \otimes \mathbb{R}$. A marking of $(V, D)$ is an isometry $\sigma : H^2(V) \rightarrow L_{(V, D)}$ sending $\sigma(D_j) = (D_j)^0$ and $\sigma(A_{\text{gen}}) = A_{\text{gen}}^0$. Let $\Gamma_{(V, D)} \subseteq O(L_{(V, D)})$ be the subgroup fixing all this data.

**Theorem 9.10 ([Fri15, Thm. 8.13]).** Assume $\text{Aut}(V, D)$ is trivial. There is a fine moduli space $\mathcal{M}_{(V, D)}$ of marked anticanonical pairs deformation-equivalent to $(V, D)$. It has a period map

$$\mathcal{M}_{(V, D)} \rightarrow \text{Hom}(L_{(V, D)}, \mathbb{C}^*)$$

which is generically one-to-one, and whose fibers are torsors over a group isomorphic to $W_{(V, D)}$ with the action on a fiber given by $(X, \sigma) \mapsto (X, g \circ \sigma)$.

When $\text{Aut}(V, D)$ is non-trivial, there is still a space $\mathcal{M}(V, D)$ admitting a family which defines at every point a universal deformation, and for which every isomorphism type is represented, but it is not a fine moduli space.

**Definition 9.11.** A quasi-polarized triple $(V, D, L)$ is an anticanonical pair $(V, D)$ and a big and nef line bundle $L \in \text{Pic}(V)$. A polarized ADE triple is an image $(\mathcal{V}, \mathcal{D}, \epsilon)$ of such under the linear system $\phi_{[nL]}$, $n \gg 0$ (we must add the condition that $\psi_{(V, D)}(L) = 1$ when $L \in \Lambda$). A divisor triple $(V, D, R)$ is the extra data of an element $R \in |L|$ such that $R$ contains no nodes of $D$. A stable triple $(\mathcal{V}, \mathcal{D}, \epsilon R)$ is an image of a divisor triple $(V, D, \epsilon R)$ under $\phi_{[nR]}$, $n \gg 0$. 

31
The map $(V, D) \to (\overline{V}, \overline{D})$ contracts the components of $D$ for which $L \cdot D_j = 0$, together with some negative-definite ADE configuration of $(-2)$-curves whose classes lie in $\Lambda_{(V, D)}$.

**Theorem 9.12.** The coarse moduli space of polarized ADE triples $F_{(V, D, \Gamma)}$ of a fixed deformation type is the quotient of $\text{Hom}(L_{(V, D)}, \mathbb{C}^*)$ by the finite group $\Gamma_{(V, D, L)} := \text{Stab}_{(V, D)}(L)$.

**Proof.** The result is analogous to Theorem 2.11. If $L \notin \Lambda$, take the sublocus $\mathcal{M}_{(V, D, L)} \subset \mathcal{M}_{(V, D)}$ where $L$ defines a big and nef divisor—this surjects onto the period torus with fibers a torsor over the reflection subgroup $W_{(V, D, L)} := \text{Stab}_{(V, D)}(L)$. When $L \in \Lambda$, we restrict to the sublocus $\psi_{(V, D)}(L) = 1$. Now take the relative linear system of $nL$, which simultaneously contracts the ADE configuration in $\Lambda_{(V, D)} \cap L^-$ and some components of $D$.

The fibers of the period map $\mathcal{M}_{(V, D, L)} \to \text{Hom}(L_{(V, D)}, \mathbb{C}^*)$ (or $\text{Hom}(L_{(V, D)}/ZL, \mathbb{C}^*)$ when $L \in \Lambda$) are identified with distinct resolutions of the contraction, and the moduli functor factors through the separated quotient of $\mathcal{M}_{(V, D, L)}$. Since we have included $L$ as part of the data, our change-of-markings in $\Gamma_{(V, D)}$ must preserve $L$. The result follows.

We can even identify (when $\text{Aut}^0(V, D)$ is trivial) the moduli stack as the separated quotient of $[\mathcal{M}_{(V, D, L)} : \Gamma_{(V, D, L)}]$. Like in the K3 case (Rem. 2.12), its only difference with the quotient stack $[\text{Hom}(L_{(V, D)}, \mathbb{C}^*) : \Gamma_{(V, D, L)}]$ is that the inertia groups are locally quotiented by $W_{(V, D, L)}$. □

Let $F_{(V, D, \overline{D}, \overline{R})}$ denote the coarse moduli space of stable triples $(V, D, cR)$ with a fixed deformation type of minimal resolution. Here $c$ is a fixed small number.

**Lemma 9.13.** $F_{(V, D, \overline{D}, \overline{R})}$ is a (possibly non-flat) family of affine varieties over the coarse moduli space $F_{(V, D, \Gamma)}$.

**Proof.** On a given polarized ADE triple $(V, D, \Gamma)$, we may choose $\overline{R} \in |\Gamma|$ arbitrarily, subject to the condition that $\overline{R}$ not contain any nodes of $\overline{D}$. This condition is either not satisfied by any element of $|L|$, or is the complement of a non-zero number of hyperplanes, corresponding to sections which go through some node. Thus, the set of choices of $\overline{R}$ on a fixed ADE triple forms an affine variety.

The automorphism group of $(V, D, \Gamma)$ acts on the set of such choices $\overline{R}$. So when this automorphism group is finite, the choices form an affine variety. If the automorphism group contains a continuous part of dimension 1 or 2, we may rigidify by requiring $\overline{R}$ to go through 1 or 2 generically chosen points of $\overline{V} \setminus \overline{D}$. Then, the coarse moduli space is a finite image of the rigidified moduli space, which is again affine by the reasoning of the first paragraph. □

**Corollary 9.14.** The coarse moduli space $F_{(V, D, \overline{D}, \overline{R})}$ contains no complete curves.

**Proof.** This follows immediately from Lemma 9.13 and $F_{(V, D, \Gamma)}$ being affine. □

Let $F_{(\overline{X}, \overline{R})}$ denote the coarse moduli space of stable slc pairs of a fixed combinatorial type, as in Definition 8.12.

**Remark 9.15.** Semi log canonical singularities are seminormal. The seminormality implies that the scheme-theoretic structure of a 0-stratum of $\overline{X}$ is unique, since $\overline{X}$ is the direct limit of the diagram of strata, partially ordered by inclusion. So moduli is uniquely determined by the moduli of components and gluings of double curves.

**Theorem 9.16.** Let $F_{(\overline{X}, \overline{R})}$ be a coarse moduli space of glued seminormal stable pairs containing a Type III stable pair degeneration of K3 surfaces. Then $F_{(\overline{X}, \overline{R})}$ contains no complete curve.

**Proof.** We can construct the coarse moduli space as follows: First, take the product of the coarse moduli spaces of each component $\prod_i F_{(\overline{V}_i, \overline{D}_i, \overline{R}_i)}$. Let $\{\mu_{ij}\} \subset \mathbb{C}^*$ be the (possibly empty, but always finite) set of gluings of $\overline{D}_{ij}$ to $\overline{D}_{ji}$ which identify the nodes of $\overline{D}_i$ and $\overline{D}_j$ and for which $\overline{R}_i \cap \overline{D}_{ij} = \overline{R}_j \cap \overline{D}_{ji}$. The space $G_{(\overline{X}, \overline{R})}$ of such glued pairs is $\prod_i F_{(\overline{V}_i, \overline{D}_i, \overline{R}_i)} \times \prod_{i,j} \{\mu_{ij}\}$ which has a finite map to $\prod_i F_{(\overline{V}_i, \overline{D}_i, \overline{R}_i)}$. So by Corollary 9.14, $G_{(\overline{X}, \overline{R})}$ contains no complete curves. □
The space $G_{(X, R)}$ parameterizes seminormal pairs $(X, \epsilon R)$ together with a combinatorial labeling of the dual complex $\Gamma(X)$. Consider the finite group of combinatorial self-maps of $\Gamma(X)$ preserving the combinatorial types of all stable triples. The coarse moduli space $F_{(X, R)}$ is the quotient of $G_{(X, R)}$ by this finite group. Since $G_{(X, R)}$ contains no complete curve, neither does $F_{(X, R)}$.

**Corollary 9.17.** No Type III stratum of $F_M^{R}$ contains a complete curve.

**Proof.** A Type III stratum of $F_M^{R}$ is a sublocus of the coarse moduli space of pairs $(X, \epsilon R)$ as in Theorem 9.16. The corollary follows. \qed

9C. Other lemmas. We prove the remaining lemmas used in Theorem 9.1. Let $(C^n, B)$ denote the analytic germ of $B := \{x_1 \cdots x_n = 0\}$, the union of the coordinate hyperplanes, in $C^n$.

**Lemma 9.18.** Let $V$ be an analytic variety stratified by sub-varieties $V_i$. Consider a meromorphic map $\phi: (C^n, B) \rightarrow V$ with locus of indeterminacy contained in $B$. Assume that the image of the indeterminacy locus is contained in $\bigcup_{i \in I} V_i$ and that no $V_i$ for $i \in I$ contains a complete curve.

Assume that for any arc germ $f: (C, 0) \rightarrow (C^n, 0)$ with $f(C \setminus 0) \subset (C^*)^n$, there exists an extension $g: (C, 0) \rightarrow V$ of $\phi \circ f$. Moreover, assume that for any such $f$, the stratum $V_i \ni g(0)$ depends only on the orders of tangency of $C$ to the coordinate hyperplanes.

Then the indeterminacy of $\phi$ can be resolved by toric blow-ups.

**Proof.** Fix the standard torus action of $T = (C^*)^n$ on $C^n$. By Hironaka (see Wlodarczyk [Wlo09] for a careful treatment of analytic spaces), there exists a sequence of blowups at smooth centers in the indeterminacy loci that resolves $\phi$. Let $H$ be the first center which is not $T$-invariant.

Let $O$ be the largest $T$-orbit with $O \cap H \neq \emptyset$. By restricting to an open subset, we can assume that $H \subseteq O$. Consider the toric cross-sections normal to $O$. These cross-sections satisfy the conditions of the Lemma, and so applying an inductive hypothesis in $n$, we resolve the indeterminacy of $\phi$ generically along $O$, by a series of toric blowups.

So we get a rational map $\phi': (X', B') \rightarrow V$ from a toric variety, a torus orbit $O'$, and a nontoric center of indeterminacy $H' \subset O'$ such that $\phi'$ is regular on an open set $U \subset X'$ intersecting $O'$. Then $\phi'(U \cap O')$ is contained in a single stratum $V_i$. The stratum containing the limit of an arc in $X'$ again depends only on the orders of tangency with the components of $B'$.

Let $X' \leftarrow Z \rightarrow V$ be a resolution of singularities. Then there exists $p \in O' \cap H'$ such that for the fiber $Z_p$ of $Z \rightarrow X'$ the morphism $Z_p \rightarrow V$ is non-constant. Since $Z_p$ is proper and the strata $V_i$ contain no complete curve, there exist two arcs with $f_1(0) = f_2(0) = p$ and with $g_1(0), g_2(0)$ lying in different strata $V_i$. But shifts of these arcs by the torus action have the same tangency conditions with the coordinate hyperplanes and satisfy $f(0) \in U$. So for them $g(0)$ lie in the same stratum of $V$. Contradiction. \qed

**Lemma 9.19.** There is a morphism $(F_M^{R})^c \rightarrow F_M^{RB}$ for any canonical choice of polarizing divisor $R$ (recognizable or not).

This is proved in [AET19, Thm. 3.15] and amounts to the observation that in Type II, the $j$-invariant of the corresponding point in the 1-cusp of $F_M^{RB}$ can be recovered from the stable slc pair $(X, \epsilon R)$. Indeed, either $X$ is nonnormal and every connected component of the double locus is an elliptic curve $E$ with this $j$-invariant, or $X$ has an elliptic singularity corresponding to $E$.

9D. Examples. Previously known examples of recognizable divisors come from [AET19], [ABE22].

**Example 9.20** (Degree 2 K3s). Let $(X, L)$ be a quasipolarized K3 surface of degree $L^2 = 2$. Let $\overline{X}$ denote the corresponding polarized ADE K3 surface. Then $\phi_{\vert L} \circ \phi$ defines a branched double cover $\overline{X} \rightarrow \mathbb{P}^2$ or $\overline{X} \rightarrow \mathbb{P}_4^2$ (the contraction of the Hirzebruch surface $F_4$ along its negative section). Define $R \in |3L|$ to be the pullback of the ramification locus $\overline{R} \subset \overline{X}$.

There is only one $\Gamma$-orbit of primitive isotropic vector $\delta \in L^\perp$, and so a semitoroidal compactification is determined by a single $\Gamma_\delta$-invariant semifan $\overline{\delta} = \overline{\delta}_3$ in the positive cone of $C_\delta^+$. Then
[AET19] verifies Theorem 8.11(5) directly, by constructing for each monodromy invariant \( \lambda \), a divisor \( \lambda \)-family with monodromy invariant in the projective class \([\lambda]\).

These divisor models are constructed by ensuring the involution on the general fiber \( X_t \) of Kulikov model \( X \to (C, 0) \) extends to the central fiber \( X_0 \). Then the fixed locus \( R_0 \subset X_0 \) is the canonical choice of divisor certifying recognizability. The resulting semifan \( \mathfrak{F}_R \) is not a fan. The lattice \( \delta^+ / \delta \) is a hyperbolic root lattice with a finite covolume Coxeter chamber \( R \). There is an infinite subgroup \( W \subset \Gamma_4 \) for which \( \mathfrak{F}_R \) is the \( \Gamma_4 \)-orbit of a single chamber \( \Sigma := W \cdot \mathfrak{F} \).

Thus, Theorem 9.1 cannot be strengthened by replacing “semifan” with “fan.”

**Example 9.21.** (Elliptic K3s). Let \((X, j)\) be an \( H \)-quasipolarized K3 surface, i.e. an elliptic K3 surface, with fiber class \( f \) and section \( s \) (so \( h \) lies in cone spanned by \( f, s + 2f \)). Let \( R = s + m \sum f_i \) be the section plus the sum of the singular fibers, with multiplicity, and weighted by \( m \). Here \( L = s + 2mf \in H \) is the relevant big and nef class. \( F^R_H \) is the same for all \( m > \frac{1}{2} \).

As in the previous example, [ABE22] find Kulikov models \( X \to (C, 0) \) for any monodromy invariant which preserve the existing structures on the general fiber: \( X_0 \) admits a fibration by genus 1 curves \( \pi_0 : X_0 \to B_0 \) over a chain of rational curves \( B_0 \), with a section \( s_0 \). Finitely many fibers \( \pi_{i,0} = \pi_0^{-1}(b_i) \) not contained in the double locus of \( X_0 \) have more nodes than all analytically nearby fibers. Counting \( f_{i,0} \) with the correct multiplicity, the recognizable divisor on \( X_0 \) is

\[
R_0 = s_0 + m \sum f_{i,0}.
\]

There is a unique \( \Gamma \)-orbit of primitive isotropic \( \delta \in H^+ = H_{2,18} \) and \( \delta^+ / \delta = H_{1,17} \) is a hyperbolic root lattice with a finite covolume Coxeter chamber \( \mathcal{R} \). Then \( \mathfrak{F}_R \) is the \( \Gamma_4 \)-orbit of a subdivision of \( \mathcal{R} \) into 9 subchambers. So \( \mathfrak{F}_R \) is a fan.

**Example 9.22.** Any choice of divisor \( R \) when \( \dim D_M = 1 \) is recognizable: There exists a divisor model in the neighborhood of any point \( p \in F_M \) in the unique toroidal compactification, which also equals the Baily-Borel compactification.

**Remark 9.23.** The necessity of normalizing \( F^R_M \) to get a semitoroidal compactification is apparent in both Examples 9.20, 9.21. [AET19, ABE22] compute the normalization map explicitly.

10. **The Rational Curve Divisor**

Our goal is to now make a canonical choice of divisor for \( F_{2d} \) for any \( d > 0 \), then prove its recognizability. From this, we can conclude Corollary 3: there are KSBA compactifications of \( F_{2d} \) whose normalizations are semitoroidal, for all degrees. Our divisor is roughly the sum of all rational curves in \( |L| \). Its recognizability is proven below, by showing that the image of a predeformable, stable, genus zero map to any Kulikov surface is rigid.

10A. **Definition of \( R^R \).** Consider the moduli space \( \mathcal{F}_{2d} \) of quasipolarized K3 surfaces of degree \( 2d \). Let \((X, L) \in \mathcal{F}_{2d} \).

**Definition 10.1.** We say that \( G \in |L| \) is a rational curve if the normalization of every irreducible component of \( G \) is \( \mathbb{P}^1 \).

**Theorem 10.2** (Yau-Zaslow formula [YZ96, Bea99, Che99, Che02]). There is a Zariski open subset \( U \subset \mathcal{F}_{2d} \) for which any rational curve \( G \in |L| \) for \((X, L) \in U \) is irreducible, nodal, and for which the number of such rational curves is exactly

\[
n_d := [q^d] \frac{1}{q} \prod_{k \geq 1} \left(1 - q^k\right)^{24} = [q^d] \frac{1}{\Delta(q)}
\]

where \( \Delta(q) \) is the modular discriminant, and \( [q^d] \) denotes the \( q^d \)-coefficient.

The integer \( n_d \) is the number of 24-colored partitions of \( d + 1 \).

**Definition 10.3.** The rational curve divisor is the canonical choice of polarizing divisor \( R^R := \sum_{G \in |L| \text{ rational}} G \in |n_d L| \) defined over the open subset \( U \subset \mathcal{F}_{2d} \).
We now outline an alternative definition using Gromov-Witten invariants.

**Definition 10.4.** Let $X$ be a smooth complete variety and let $\beta \in H_2(X, \mathbb{Z})$. The Kontsevich space $M_\beta(X)$ is the moduli space of stable maps $f : T \to X$ from a genus $g$ nodal curve $T$, for which $f_*[T] = \beta$.

The Kontsevich space is a proper Deligne-Mumford stack. For a surface, $H_2(X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ are canonically identified by Poincaré duality, so we make no distinction. We will take $L = \beta$.

There is a virtual fundamental cycle $[M_\beta(X)]^{\text{vir}} \in A_{\text{exp.dim}}(M_\beta(X))$ where $\text{exp.dim} = (\dim X - 3)(1 - g) + c_1(T_X) \cdot \beta$ is the expected dimension of the moduli space $[BF97]$. In particular, for stable genus 0 maps to a K3 surface, $\text{exp.dim} = -1$ so $[M_0(X,L)]^{\text{vir}} = 0$. Geometrically, this can be explained by the fact that GW invariants are deformation-invariant, but that a generic deformation of $X$ has no nontrivial line bundles, so $\beta$ cannot represent an algebraic curve.

For polarized K3 surfaces $(X, L)$, there is a reduced virtual fundamental cycle $[M_0(X,L)]^{\text{vir,red}} \in A_0(M_0(X,L))$, see $[KT14]$. Roughly, it is built to be invariant only under the deformations of $X$ which stay in $\mathcal{F}_{2d}$. This decreases dimension of the obstruction space by one, increasing the expected dimension by one.

**Lemma 10.5.** Let $(T, f)$ be a stable map $f : T \to X$ with $T$ a nodal curve of arithmetic genus 0 and $X$ a smooth K3 surface. Under any deformation of the stable map $(T, f) \in M_0(X, L)$, the image divisor $f_*T$ is constant.

**Proof.** If the image divisor $f_*T$ moves under a deformation of $(T, f)$, a restriction of $f$ gives a dominant map $S = B \times \mathbb{P}^1 \to X$, for some (possibly incomplete) curve $B$. Letting $\text{Ram} \subset S$ be the ramification divisor of the map $S \to X$, the Riemann-Hurwitz formula gives $K_S = \text{Ram}$. This contradicts the adjunction formula, because restricting to a general fiber $F = \{b\} \times \mathbb{P}^1$ gives $-2 = 2g(F) - 2 = F \cdot (F + \text{Ram}) = F \cdot \text{Ram} \geq 0$. \hfill $\square$

Replicating this argument for a Kulikov surface is the key to proving that $R^{rc}$ is recognizable.

**Definition 10.6.** Let $G \in |L|$ be a rational curve. Define $M_G(X)$ to be the union of the connected components of $M_0(X, L)$ for which $f_*T = G$. This is well-defined because $f_*T$ is constant on any connected component by Lemma 10.5. Define

$$n_G := \deg_{M_G(X)}[M_0(X,L)]^{\text{vir,red}} \in \mathbb{Q}.$$

This quantity a priori only lies in $\mathbb{Q}$ because of stack-theoretic issues.

**Proposition 10.7.** For any smooth quasi-polarized K3 surface $(X, L)$, we have

$$R^{rc} = \sum_{G \in |L|} \text{rational } n_G G.$$

Furthermore, $n_G$ is a non-negative integer for all rational curves $G \in |L|$.

**Proof.** Chen’s theorem $[Che02]$ implies that for a sufficiently general $(X, L) \in U$, we have $n_G = 1$ for all rational curves $G$. Fix an $(X_0, L_0) \in \mathcal{F}_{2d}$ not in $U$ and consider a 1-parameter deformation $(\mathcal{X}, \mathcal{L}) \to (C, 0)$ over an analytic disc $C$ for which $X_t \in U$ for all $t \in C^*$. Consider the moduli space of relative stable maps $M_0(\mathcal{X}, \beta)$, where $\beta$ is the class of $L_0$ pushed forward to $\mathcal{X}$.

There is a proper morphism $M_0(\mathcal{X}, \beta) \to C$ sending a curve to the fiber it is supported on, but the fibers of this family are in general poorly behaved. For instance, the dimension can and often does suddenly jump at $t = 0$. But by assumption, the fiber over any point $t \in C^*$ is a reduced zero-dimensional scheme consisting of exactly $n_d$ points. The proposition follows if we can prove that the scheme-theoretic intersection $\overline{M_0(\mathcal{X}^*, \beta)} \cap M_0(X_0, L_0)$ represents the reduced virtual fundamental class, in homology.

The constancy of the reduced Gromov-Witten invariants $n_d = \deg_{M_0(X_t, \beta)}[M_0(X_t, \beta)]^{\text{vir,red}}$ as one varies $t$ follows from the existence of a relative perfect obstruction theory $[BF97, \text{Sec. 7}]$, $[KT14, \text{Rem. 3.1}]$. Without going into the details, this is a perfect two-term complex with a morphism to the relative cotangent complex, satisfying various axioms.
Now let \( W \subset M_0(\mathcal{X}, \beta) \) be a connected component. The restriction of the axioms of a (relative) perfect obstruction theory still hold under restricting this two-term complex to \( W \). Hence the constancy of reduced GW invariants still holds, i.e. \( \deg_W [M_0(X_0, L_0)]^{\text{vir, red}} \) will equal the number of sheaves of \( M_0(X_t, L_t) \) whose closures over \( t = 0 \) lie in \( W \). This implies the first statement.

Summing these integrals over the components \( W \) for which the image curve \( f_* T = G \), we also see that \( n_G \) is a non-negative integer. \( \square \)

**Remark 10.8.** A priori, the contribution \( n_G \) could equal zero. Perhaps no genus 0 stable map with image \( G \) deforms to the general fiber \( X_t \). Notably, this cannot occur when there is a component \( W \subset M_G(X) \) of dimension \( \dim W = 0 \), see [Huy16, Ch. 13.2.3] and references therein.

Proposition 10.7 provides us with a definition of \( R^{rc} \) on all \( (X, L) \in F_{2d}^q \).

**Definition 10.9.** A quasipolarized K3 surface \((X, L)\) of degree \(2d\) is **unigonal** if it is elliptic, with section and fiber classes \(s, f\) and \( L = s + (d + 1)f\).

As a Noether-Lefschetz locus of Picard rank 2, the unigonal locus forms a divisor in \( F_{2d}^q \) isomorphic to the moduli space \( F_H^q \) of elliptic K3s.

**Proposition 10.10.** On a unigonal K3 surface \((X, L)\), the rational curve divisor is

\[
R^{rc} := n_d(s + \frac{d+1}{2d} \sum f_i)
\]

where \(f_i\) are the 24 singular fibers in \(|f|\), counted with multiplicity.

**Proof.** The proposition follows immediately from Proposition 10.7 and the main result of [BL00], though historically [Che02] relies on [BL00]. \( \square \)

10B. Proof of Theorem 2. We now prove our second main result:

**Theorem 10.11.** The rational curve divisor \( R^{rc} \) is recognizable for \( F_{2d} \) for all \( d > 0 \).

**Proof.** Take a divisor model \((X, R) \to (C, 0)\). We verify Theorem 8.11(1) by showing that the limiting curve \( R_0 \subset X_0 \) satisfies some geometric property ensuring its rigidity on \( X_0 \) even as we deform the smoothing of \( X_0 \). Take a base change and standard resolution of \((X, R) \to (C, 0)\), so that the irreducible components \( G_i \) of \( R_t \) are not permuted by monodromy. Then, Lemmas 10.12 and 10.14 imply that the limit of any individual rational curve \( G_t \) is rigid. \( \square \)

**Lemma 10.12.** Let \( X \to (C, 0) \) be a Kulikov model and let \( G \subset X \) be a flat family of curves for which \( G_0 \) is an irreducible rational curve for \( t \neq 0 \), and \( G_0 \) contains no strata. Then, after a finite base change and resolution of \( X \to (C, 0) \), there is a stable map \( f : T \to X_0 \) from a nodal, genus 0 curve (a tree of \( \mathbb{P}^1 \)'s) for which \( f_* T = G_0 \) and \((T, f)\) is predeformable (see Def. 10.13).

**Definition 10.13.** We say that \((T, f)\) is predeformable [L01, Def. 2.5] if no component of \( T \) is contracted into the double locus, and for each node \( p \in T \) with \( f(p) \in D_{ij} \), the two arcs \((T_k, p)\), \((T_t, p)\) with \( f(T_k) \subset V_i \) and \( f(T_t) \subset V_j \) satisfy

the tangency order of \( f(T_k, p) \) to \( D_{ij} \) = the tangency order of \( f(T_t, p) \) to \( D_{ji} \).

**Proof of Lemma 10.12.** Because \( G_0 \) contains no strata, it maps into the complement of the triple points of \( X_0 \), i.e. the union of the non-singular locus and the double locus. Then, the result follows from the properness over \((C, 0)\) of the space of predeformable stable maps [L01, Thm. 3.10] to varieties with only double crossings. \( \square \)

**Lemma 10.14.** Let \( f : T \to X_0 \times B \) be a family of stable maps over a local curve \( B \), such that \( T_b \) is a tree of \( \mathbb{P}^1 \)'s of fixed combinatorial type for all \( b \in B \), and for which \((T_b, f_b)\) is predeformable. Then the image curves \( f_* (T_b) = G_0 \) are constant.

**Proof.** Let \( T = \cup T_k \) be the components of \( T \). We have \( T_k \cong \mathbb{P}^1 \times B \). Let \( N_{k\ell} = T_k \cap T_\ell \) be the relative nodes over \( B \). We label the vertices \( \Gamma(T)^{(0)} \) of the dual complex \( \Gamma(T) \) as follows:

(V0) \( T_k \) is contracted to a point inside a component.

(V1b) \( T_k \) is contracted along multisections to a curve inside a component.
As discussed in Section 5A, they are in bijection with the $\Gamma$-orbits of primitive isotropic lattices $\Gamma(T)$.

These are the only possibilities, by noting that $T_k \to B$ is proper and that the image of $f$ contains no triple points. Next, we label the edges $\Gamma(T)[1]$ of the dual complex $\Gamma(T)$ as follows:

- (v0) $N_{v0}$ maps to a point in the interior of a component.
- (v1) $N_{v1}$ maps to a point in the interior of a component.
- (d0) $N_{d0}$ maps to a point in a double curve.
- (d1) $N_{d1}$ maps to a curve in a double curve.

Table 1 records the allowable adjacencies for the labeled dual complex $\Gamma(T)$, which can be verified from predeformability by straightforward geometric arguments.

Let $\Gamma \subseteq \Gamma(T)$ be a maximal subtree consisting of only V2-vertices and d1-edges. Let $T_1 \subseteq T$ be the sub-family of curves with dual complex $\Gamma$. Consider the restricted family $f_1 : T_1 \to X_0 \times B$.

The fibers $T_{V_i}$ may only fail to map in a predeformable way to $X_0$ at the leaves of $\Gamma$ which are not leaves of $\Gamma(T)$. Consider the edges emanating from such a V2 leaf which are connected to the rest of $\Gamma(T)$. Disconnecting $\Gamma(T)$ at a v-edge does not interfere with the condition of being predeformable, so consider only the d-edges. By Table 1 and maximality of $\Gamma$, such a V2 leaf of $\Gamma$ must connect by a d0-edge.

So fix one V2 leaf of $\Gamma$, associated to a component $T_k \cong \mathbb{P}^1 \times B \subset T_1$ attached to the rest of $\Gamma(T)$ by d0-edges. The further restriction $f_k : T_k \to V_i$ is now a map of smooth surfaces.

Each outgoing d0-edge corresponds to a relative node $N_{d0}$ of $T$ which maps under $f_k$ to a single point $p_{d0} \in D_{ij}$. There is at most one remaining relative node $N \subset T_k$ which attaches $T_k$ to the rest of $T_1$ and for which $f(N)$ is a curve in one boundary component of $V_i$. Make an interior blow-up $\tilde{V}_i \to V_i$ at each fixed attaching point $p_{d0}$. Taking the strict transforms of the images of fibers of $T_k \to B$, we can lift $f_k$ to a map $\tilde{f}_k : T_k \to \tilde{V}_i$. If $\tilde{f}_k$ still sends any $N_{d0}$ to a point in the new anticanonical boundary $\tilde{D}_{ij}$, we continue to blow up at the fixed attaching points, until the lifted map satisfies the property $\tilde{f}_k^{-1}(\tilde{D}_i) = N$.

Since both $N$ and $\tilde{D}_i$ are divisors with coefficient 1, we have by Riemann-Hurwitz that

$$\omega_{T_k}(N) = \tilde{f}_k^*(\omega_{\tilde{V}_i}(\tilde{D}_i)) \otimes \mathcal{O}(\text{Ram}) \otimes \mathcal{O}(\sum a_i E_i).$$

Here $\text{Ram} \subset T_k$ is the interior ramification divisor, i.e. the ramification away from the boundary $\tilde{D}_i$, and $E_i \subset T_k$ are the contracted curves of $\tilde{f}_k$. Note that $a_i \geq 0$ because $\tilde{V}_i$ is smooth. Since $\omega_{\tilde{V}_i}(\tilde{D}_i) = \mathcal{O}$ we conclude $\omega_{T_k}(N)$ is effective, implying $-1 = \omega_{T_k}(N) \cdot \mathbb{P}^1 \geq 0$. Contradiction. $\square$

In analogy with Proposition 10.7, Lemma 10.14 allows us to define $R_{\text{rec}}$ for Kulikov surfaces inherently, in terms of logarithmic Gromov-Witten invariants [Che14, AC14, AMW14].

### 10C. The rational curve semifan

We first give some general results concerning the Baily-Borel compactification of $F_{2d}$, following [Sca87].

The number of 0-cusps of $F_{2d}^{BB}$ is exactly $\left\lfloor \frac{N + 2}{2} \right\rfloor$ where $d = N^2 d_0$ for a square-free integer $d_0$.

As discussed in Section 5A, they are in bijection with the $\Gamma$-orbits of primitive isotropic lattices $I = \mathbb{Z} \delta$ in the lattice

$$L_{2d} := \langle -2d \rangle \oplus H^{\oplus 2} \oplus E^{\oplus 2}_8 = v^\perp \subset L_{K3}.$$

\[
\begin{array}{|c|c|c|c|}
\hline
 & V0 & V1b & V1f & V2 \\
\hline
V0 & v0 & v0 & - & v0 \\
V1b & v0/v1/d0 & v1 & v0/v1/d0 & \\
V1f & v1 & v1 & & \\
V2 & & & all & \\
\hline
\end{array}
\]

**Table 1. Allowable adjacencies for the labeled dual complex $\Gamma(T)$.**
The $\Gamma$-orbit of a generator $\delta \in I$ is determined by the following invariant: $\delta^* = \frac{\delta}{p^*(\delta)} \in \Delta_{2d} := \frac{L_{2d}^*}{L_{2d}}$, where $p^*$ is by definition the imprimitivity in $L_{2d}^*$, then $\delta^*$ is an isotropic vector for the quadratic form on $\Delta_{2d}$ valued in $\frac{1}{2d}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Identifying the source $\Delta_{2d} = \mathbb{Z}/2d\mathbb{Z}$ and the target with $\mathbb{Z}/4d\mathbb{Z}$, the quadratic form is given by $x \mapsto x^2$. We must have $\delta^* = 2qNd_0 \in \mathbb{Z}/2d\mathbb{Z}$ for $q \in \mathbb{Z}/N\mathbb{Z}$. So $I = \mathbb{Z}\delta$ is determined by $\{\pm q\}$ and we have

$$\delta^0 / \delta = (\frac{\delta_{2d}}{p^*(\delta)}) \oplus H \oplus E_8^{\oplus 2}.$$

A semitoroidal compactification of $F_{2d}$ is determined by a collection of $\Gamma_\delta$-invariant semifans $\tilde{S}_\delta$ decomposing the rational closures $C^\perp_\delta$ of the positive cones of each lattice $\delta^0 / \delta$ as above, as one ranges over the $\lfloor \frac{N + 2}{2} \rfloor$ possible values of $\{\pm \delta^0\}$. By Theorems 1 and 2, we may define:

**Definition 10.15.** Let $\tilde{S}^{\text{rc}}$ be the semifan for which $\nu: \tilde{F}^{\text{rc}}_{2d} \to \tilde{F}^{\text{rc}}_{2d}$ is the normalization map.

Some facts about the combinatorics of $\tilde{S}^{\text{rc}}$ can be deduced from Proposition 10.10 and [ABE22], by restricting to the locus of elliptic K3 surfaces.

**Theorem 10.16.** Consider a cone $C^\perp_\delta$ with invariant $p^*(\delta) = 1$, that is, where $\delta$ is primitive in $L_{2d}^*$ (all such $\delta$ are equivalent under $\Gamma$). The restriction of $\tilde{S}^{\text{rc}}$ to $\langle -2d \rangle^\perp = H \oplus E_8^{\oplus 2}$, or any $\Gamma_\delta$-orbit of it, is a fan. Furthermore, $\langle -2d \rangle^\perp \cap C^\perp_\delta$ is a union of cones of $\tilde{S}^{\text{rc}}_\delta$.

We call such hyperplanes unigonal. The last statement in the theorem implies that $\tilde{S}^{\text{rc}}$ refines the unigonal hyperplane arrangement in $C^\perp_\delta$.

**Proof.** Suppose $(X, L)$ is in the unigonal locus, so that $L = s + (d + 1)f$. The inclusion $\mathbb{Z}L \hookrightarrow H$ induces an inclusion of moduli spaces $F_H \to F_{2d}$. So the restriction of $R^{\text{rc}}$ to $F_H$ is recognizable. Suppose that $\delta \in H^\perp$ is primitive isotropic. There is a unique isometry orbit of such and $\delta$ is primitive in $(L^\perp)^*$. So $\delta^0 / \delta$ includes into the lattice $\delta^0 / \delta^\perp$ corresponding an isotropic vector with invariant $\delta^* = 0$. Concretely, it is the summand inclusion $H \oplus E_8^{\oplus 2} \hookrightarrow \langle -2d \rangle \oplus H \oplus E_8^{\oplus 2}$.

By Proposition 9.4, the restriction of $\tilde{S}^{\text{rc}}$ to $H \oplus E_8^{\oplus 2}$ is the semifan $\tilde{S}^{\text{rc}}_H$ whose corresponding semitoroidal compactification normalizes $\tilde{F}^{\text{rc}}_H$. By Proposition 10.10, the rational curve divisor, as in Definition 10.3, when extended to the unigonal locus, is a multiple of $R = s + m \sum f_i$ for $m = \frac{d + 1}{24}$. [ABE22, Thm. 1.2] gives an explicit description of the fan $\tilde{S}^{\text{rc}}_H$ modulo the following caveat: The divisor models described in [ABE22, Sec. 7B] require a threshold value of $m > \frac{1}{3}$ for the divisors $R_0 \subset X_0$ constructed therein to be nef. The threshold is achieved when $\Gamma(X_0)$ has a so-called $X_3$ end singularity. So it is automatic that $\mathcal{loc}$.cit. describes the restriction of $\tilde{S}^{\text{rc}}$ to the unigonal hyperplane when $d > 7$. For $m \leq \frac{1}{4}$ or $d \leq 7$, the stable models only differ from those in $\mathcal{loc}$.cit. in a minor way—one might contract the section on one or both end surfaces. But the stratum function $\mathcal{S}$ has the same level sets and so $\tilde{S}^{\text{rc}}_H$ is the same (Prop. 9.3).

The fan $\tilde{S}^{\text{rc}}_H$ consists of six orbits of maximal cones [ABE22, Sec. 4C]. To prove the final statement of the theorem, we must show that all six of the 18-dimensional cones $\sigma_H \in \tilde{S}^{\text{rc}}_H$ are themselves cones of $\tilde{S}^{\text{rc}}$ and not simply slices of the interior of some 19-dimensional cone $\sigma \in \tilde{S}^{\text{rc}}$.

A maximal cone $\sigma_H$ corresponds to a 0-stratum of the stable pair compactification of elliptic K3s and hence to unique Type III elliptic stable K3 pair $(X_0, R_0)$. If $(X_0, R_0)$ deforms out of the unigonal locus as rational curve K3 pair, keeping the combinatorial type constant, then $\sigma_H$ must be a cone of $\tilde{S}^{\text{rc}}$. But if the elliptic stable K3 pair $(X_0, R_0)$ is, as a rational curve K3 pair, rigid in its combinatorial type, then $\sigma_H$ must be the slice of a larger dimensional cone of $\tilde{S}^{\text{rc}}$.

Let $(X_0, R_0)$ be a Type III divisor model whose stable model is $(\overline{X}_0, \overline{R}_0)$, see [ABE22, Sec. 7A] for an explicit description. Let $L_0 = \mathcal{O}_{X_0}(R_0)$. We can deform $(X_0, L_0)$ to a non-elliptic, $d$-semistable Kulikov model $(X'_0, L'_0)$ by regrading double curves so that $\psi_{X'_0}(f) \neq 1$. Concretely, comparing to [ABE22, Def. 7.10], it corresponds to when a connected chain of fibers of vertical rulings fails to glue to a closed cycle, destroying the elliptic fibration and the Cartierness of $f$.

Since $R^{\text{rc}}$ is recognizable, the rational curve divisor on such a deformed Kulikov model is necessarily a deformation of the curve $n_4(s + \frac{d + 1}{24} \sum f_{i,0})$ (see 9.21) living in the linear system $|L_0|$. 

38
So the resulting stable model has the same combinatorial type as the elliptic one
\[
(X_0', \epsilon R_0) = \bigcup_{i=1}^r (V_i, D_i, \epsilon R_i)
\]
for \(r = 18, 19, 20\) depending on the cone \(\sigma_H\).

In the elliptic case, the intermediate components \(V_i\) for \(i \neq 1, r\) are the result of gluing two sections of \(\mathbb{P}^1 \times \mathbb{P}^1\) via the isomorphism provided by the vertical fibration. But when we deform \(X_0\) out of the elliptic locus to the Kulikov surface \(X'_0\), the surface \(V_i\) also deforms: The gluing map between the two sections includes a shift exactly equal to \(\psi_{X'_0}(f)\).

Hence \((X_0, \epsilon R_0)\) is not rigid in \(\mathcal{F}_{2d}^{rc}\) within its slc combinatorial type. Even forgetting the divisor, the underlying surface \(X_0\) is not rigid. We conclude that \(\sigma_H\) is a cone of \(\mathfrak{S}^{rc}\). \(\square\)

**Remark 10.17.** The results of this section hold for the *imprimitive rational curve divisor*
\[
R^{rc}(m) = \sum_{G \in \{mL\} \text{ rational}} n_G G
\]
where the coefficients \(n_G\) are defined using reduced GW invariants as in Definition 10.6. A naive version of Chen’s theorem (that generically all rational curves are nodal) is false: For instance one can take \(mG\) for \(G \in \{L\}\) rational. It is not clear whether one can recover Chen’s theorem by subtracting out these and other obvious non-reduced and non-irreducible contributions to get a divisor \(R^{rc}_{prim}(m)\). Regardless, the above serves as a definition of \(R^{rc}(m)\) and produces a canonical choice of polarizing divisor. The proof of recognizability, Theorem 10.11 applies verbatim because the normalization of any irreducible component of \(R^{rc}(m)\) is \(\mathbb{P}^1\).

Thus, there are semifans \(\mathfrak{S}^{rc}(m)\) for all \(m \geq 1\) which give the normalization of the KSBA compactification associated to \(R^{rc}(m)\).

**References**

[ABE22] Valery Alexeev, Adrian Brunyate, and Philip Engel, *Compactifications of moduli of elliptic K3 surfaces: Stable pair and toroidal*, Geom. Topol. **26** (2022), no. 8, 3525–3588.

[AC14] Dan Abramovich and Qile Chen, *Stable logarithmic maps to Deligne-Faltings pairs II*, Asian J. Math. **18** (2014), no. 3, 465–488.

[AET19] Valery Alexeev, Philip Engel, and Changho Han, *Complete moduli of K3 surfaces with a nonsymplectic automorphism*, Trans. Amer. Math. Soc. (to appear) (2021), arXiv:2110.13834.

[AET22] Valery Alexeev, Philip Engel, and Changho Han, *Mirror symmetric compactifications of moduli spaces of K3 surfaces with a nonsymplectic involution*, arXiv:2208.10983 (2022).

[AEH21] Valery Alexeev, Philip Engel, and Changho Han, *Complete moduli of K3 surfaces with a nonsymplectic automorphism*, Trans. Amer. Math. Soc., to appear (2021), arXiv:2010.15929.

[Ale06] Valery Alexeev, *Log canonical singularities and complete moduli of stable pairs*, arXiv alg-geom/9608013 (1996), 1–13.

[Ale02] Valery Alexeev, *Complete moduli in the presence of semialabel group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708.

[Ale06] Valery Alexeev, *Higher-dimensional analogues of stable curves*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 515–536.

[AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, Brookline, Mass., 1975, Lie Groups: History, Frontiers and Applications, Vol. IV.

[AMW14] Dan Abramovich, Steffen Marcus, and Jonathan Wise, *Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenerations*, Ann. Inst. Fourier (Grenoble) **64** (2014), no. 4, 1611–1667.

[ast85] Géométrie des surfaces K3: modules et périodes, Société Mathématique de France, Paris, 1985, Papers from the seminar held in Palaiseau, October 1981–January 1982, Astérisque No. 126 (1985).

[BB66] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.

[Bea99] Arnaud Beauville, *Counting rational curves on K3 surfaces*, Duke Math. J. **97** (1999), no. 1, 99–108.

[BF97] Kai Behrend and Barbara Fantechi, *The intrinsic normal cone*, Inventiones mathematicae **128** (1997), no. 1, 45–88.

[BL00] Jim Bryan and Naichung Conan Leung, *The enumerative geometry of K3 surfaces and modular forms*, J. Amer. Math. Soc. **13** (2000), no. 2, 371–410.
