1. Introduction

1.1. Let $\mathcal{G}$ be a noncompact connected real semi-simple Lie group with trivial center and with no nontrivial compact connected normal subgroups, and $\mathfrak{g}$ be its Lie algebra. The group $\text{Aut}(\mathcal{G})$ ($=\text{Aut}(\mathfrak{g})$) of automorphisms of $\mathcal{G}$ is a Lie group with finitely many connected components, and $\overline{\mathcal{G}}$ is its identity component. We will denote the identity component of $\text{Aut}(\mathcal{G})$ in the Zariski-topology by $\text{Int}(\mathcal{G})$. Let $X$ be the symmetric space of $\mathcal{G}$ ($X$ is the space of maximal compact subgroups of $\mathcal{G}$), and $X_u$ be the compact dual of $X$. There is a natural identification of the group of isometries of $X$ with $\text{Aut}(\mathcal{G})$. We assume in this paper that $X$ (and hence $X_u$) is hermitian. Then every holomorphic automorphism of $X$ is an isometry. The group $\text{Hol}(X)$ of holomorphic automorphisms of $X$ is a subgroup of finite index of the group $\text{Aut}(\mathcal{G})$ of isometries, and it is known (see [Ta], the remark in §5) that $\text{Hol}(X) \cap \text{Int}(\mathcal{G}) = \mathcal{G}$.

1.2. We will say that the quotient $X/\Pi$ of $X$ by a torsion-free cocompact discrete subgroup $\Pi$ of $\overline{\mathcal{G}}$ is a fake compact hermitian symmetric space if its Betti numbers are same as that of $X_u$; $X/\Pi$ is an irreducible arithmetic fake compact hermitian symmetric space if, further, $\Pi$ is irreducible (i.e., no subgroup of $\Pi$ of finite index is a direct product of two infinite normal subgroups) and it is an arithmetic subgroup of $\overline{\mathcal{G}}$. Any such space can be endowed with the structure of a smooth complex projective variety. Several such spaces have been constructed in our two earlier papers [PY1] and [PY2]. In [PY1] we have given a classification of “fake projective planes”, the first of which was constructed by David Mumford in [Mu] using $p$-adic uniformization. In [PY2] we have constructed four arithmetic fake $\mathbb{P}^1_{\mathbb{C}}$, four arithmetic fake Grassmannians $\text{Gr}_{2,5}$, and five irreducible arithmetic fake $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$. All these are Shimura varieties.

We note that if $\mathcal{G}$ contains an irreducible arithmetic subgroup, then the simple factors of its complexification are isomorphic to each other, see [Ma], Corollary 4.5 in Ch. IX. Also, if the real rank of $\overline{\mathcal{G}}$ is at least 2, then by Margulis’ arithmeticity theorem ([Ma], Ch. IX), any irreducible discrete cocompact subgroup of $\overline{\mathcal{G}}$ (in fact, any irreducible lattice) is arithmetic.

If $\Pi$ is a torsion-free cocompact discrete subgroup of $\overline{\mathcal{G}}$, then there is a natural embedding of $H^*(X_u, \mathbb{C})$ in $H^*(X/\Pi, \mathbb{C})$, see [B], 3.1 and 10.2, and hence $X/\Pi$ is a fake compact hermitian symmetric space if and only if the natural homomorphism $H^*(X_u, \mathbb{C}) \to H^*(X/\Pi, \mathbb{C})$ is an isomorphism.
Let $\mathcal{G}$, $X$ and $X_u$ be as above, and let $\Pi$ be a torsion-free cocompact discrete subgroup of $\mathcal{G}$. Let $Z = X/\Pi$. If $Z$ is a fake compact hermitian symmetric space, then the Euler-Poincaré characteristic $\chi(Z)$ of $Z = X/\Pi$, and so the Euler-Poincaré characteristic $\chi(\Pi)$ of $\Pi$ equals $\chi(X_u)$. As $X$ has been assumed to be hermitian, the Euler-Poincaré characteristic of $X_u$ is positive. On the other hand, it follows from Hirzebruch proportionality principle, see [S], Proposition 23, that the Euler-Poincaré characteristic of $X/\Pi$ is positive if and only if the complex dimension of $X$ is even.

Using the results of [BP], we can easily conclude that there are only finitely many irreducible arithmetic fake compact hermitian symmetric spaces. It is of interest to determine them all.

1.4. Hermitian symmetric spaces have been classified by Élie Cartan; see [H], Ch. IX. We recall that the irreducible hermitian symmetric spaces are the symmetric spaces of Lie groups $\text{SU}(n + 1 - m, m)$, $\text{SO}(2, 2n - 1)$, $\text{Sp}(2n)$, $\text{SO}(2, 2n - 2)$, $\text{SO}^*(2n)$, an absolutely simple real Lie group of type $E_6$ with Tits index $2E_6^{16}$, and an absolutely simple real Lie group of type $E_7$ with Tits index $E_7^{28}$ respectively (for Tits indices see Table II in [Ti1]). The complex dimensions of these spaces are $(n + 1 - m)m$, $2n - 1$, $n(n + 1)/2$, $2n - 2$, $n(n - 1)/2$, 16 and 27 respectively. The Lie groups listed above are of type $A_n$, $B_n$, $C_n$, $D_n$, $D_n$, $E_6$ and $E_7$ respectively. We will say that an irreducible symmetric space is one of these types if it is the symmetric space of a simple Lie group of that type, and say that a locally hermitian symmetric space is of one of these types if its simply connected cover is a product of irreducible hermitian symmetric spaces of that type.

The following is the main theorem of this paper.

**Theorem.** There does not exist an irreducible arithmetic fake compact hermitian symmetric space of type $B_n$, $C_n$, $E_6$ or $E_7$.

In the following paragraph we will explain the strategy of the proof, and fix notation which will be used throughout the paper.

1.5. Let $\mathcal{G}$, $X$, $X_u$ be as above. $X$ will be assumed to be a hermitian symmetric space of one of the following types $B_n$, $C_n$, $E_6$ and $E_7$. Assume, if possible, that $\mathcal{G}$ contains a cocompact irreducible arithmetic subgroup $\Pi$ whose orbifold Euler-Poincaré characteristic $\chi(\Pi)$ equals $\chi(X_u)$. Then there exist a totally real number field $k$, a connected adjoint absolutely simple algebraic $k$-group $\mathcal{G}$ of same type as $X$, $\mathcal{G}$ of $k$-rank 0 (by Godement criterion since $\Pi$ is cocompact), real places $v_1, \cdots, v_r$ of $k$ such that $\mathcal{G}(k_v)$ is compact for every real place $v$ different from $v_1, \cdots, v_r$. $\mathcal{G}$ is isomorphic to $\prod_{v=1}^r \mathcal{G}(k_v)^0$ (and will be identified with it), and $\Pi$ is commensurable with an arithmetic subgroup of $\mathcal{G}(k)$. Let $\pi : G \to \mathcal{G}$ be the simply connected covering of $\mathcal{G}$ defined over $k$. The kernel of the isogeny $\pi$ is the center $C$ of the simply connected $k$-group $G$. If $G$ is of type $E_6$, then it is an “outer $k$-form” of a split group (i.e., it is of type $2E_6$) since $X$ is hermitian. In this case let $\ell$ be the quadratic extension of $k$ over which $G$ is an “inner $k$-form”. Then $\ell$ is totally complex.
Description of $C$: For a positive integer $s$, let $\mu_s$ be the kernel of the endomorphism $x \mapsto x^s$ of $GL_1$. Then if $G$ is of type $2E_6$, its center $C$ is $k$-isomorphic to the kernel of the norm map $N_{\ell/k}$ from the algebraic group $R_{\ell/k}(\mu_3)$, obtained from $\mu_3$ with Weil’s restriction of scalars, to $\mu_3$. If $G$ is of type $B_n$, $C_n$ or $E_7$, then $C$ is isomorphic to $\mu_2$.

It is known, and easy to see using the above description of $C$, that for any real place $v$ of $k$, the order of the kernel of the induced homomorphism $G(k_v) \to \overline{G}(k_v)$ is $2$ if $G$ is not of type $2E_6$, and of order $3$ if it is of type $2E_6$. Moreover, as $G(k_v)$ is connected, $\pi(G(k_v)) = \overline{G}(k_v)^0$. Let $\mathcal{G} = \prod_{v \mid k} G(k_v)$, and let $\tilde{\Pi}$ be the inverse image of $\Pi$ in $\mathcal{G}$. Then the kernel of the homomorphism $\pi : \mathcal{G} \to \overline{\mathcal{G}}$ is of order $s^r$, and hence the orbifold Euler-Poincaré characteristic $\chi(\Pi)$ of $\tilde{\Pi}$ equals $\chi(\Pi)/s^r = \chi(X_u)/s^r$, where, here and in the sequel, $s = 2$ if $G$ is not of type $E_6$, and $s = 3$ if $G$ is of type $E_6$. Now let $\Gamma$ be a maximal discrete subgroup of $\mathcal{G}$ containing $\tilde{\Pi}$. Then the orbifold Euler-Poincaré characteristic $\chi(\Gamma)$ of $\Gamma$ is a submultiple of $\chi(\Pi) = \chi(X_u)/s^r$. Using the volume formula of $[P]$, and some number theoretic estimates, we will prove that $\mathcal{G}$ does not contain such a subgroup.

2. Preliminaries

2.1. We will use the notations introduced in 1.5. Thus $k$ will be a totally real number field, $G$ an absolutely simple simply connected algebraic $k$-group (of one of the following four types: $B_n$, $C_n$, $2E_6$, and $E_7$), $C$ its center, $\mathcal{G} = \prod_{v \mid k} G(k_v)$. We will think of $G(k)$ as a subgroup of $\mathcal{G}$ in terms of its diagonal embedding. $\Gamma$ is a maximal arithmetic subgroup of $\mathcal{G}$ (arithmetic with respect to the $k$-structure on $G$) whose orbifold Euler-Poincaré characteristic is a submultiple of $\chi(X_u)/s^r$. Then $\Lambda := \Gamma \cap G(k)$ is a “principal” arithmetic subgroup, i.e., for every nonarchimedean place $v$ of $k$, the closure $P_v$ of $\Lambda$ in $G(k_v)$ is a parahoric subgroup, $\Lambda = G(k) \cap \prod_{v \mid k} P_v$, and $\Gamma$ is the normalizer of $\Lambda$ in $\mathcal{G}$, see Proposition 1.4(iv) of [BP]. Let the “type” $\Theta_v$ of $P_v$ be as in [BP], 2.2, and $\Xi_{\Theta_v}$ be as in 2.8 there. If either $P_v$ is hyperspecial, or $G$ is of type $2E_6$ and it does not split over $k_v$, then $\Xi_{\Theta_v}$ is trivial. The order of $\Xi_{\Theta_v}$ is always a divisor of $s$.

In terms of the normalized Haar-measure $\mu$ on $\mathcal{G} = \prod_{v \mid k} G(k_v)$ used in $[P]$ and [BP], and to be used in this paper, $\chi(\Gamma) = \chi(X_u)\mu(\mathcal{G}/\Gamma)$ (see [BP], 4.2). Thus the condition that $\chi(\Gamma)$ is a submultiple of $\chi(X_u)/s^r$ is equivalent to the condition that $\mu(\mathcal{G}/\Gamma)$ is a submultiple of $1/s^r$. We will prove that there does not exist such a $\Gamma$.

A comprehensive survey of the basic notions and the main results of the Bruhat-Tits theory of reductive groups over nonarchimedean local fields is given in [Ti2].

2.2. All unexplained notations are as in [BP] and [P]. Thus for a number field $K$, $D_K$ will denote the absolute value of its discriminant, $h_K$ its class number, i.e., the order of its class group $Cl(K)$. We will denote by $h_{K,s}$ the order of the subgroup of $Cl(K)$ consisting of the elements of order dividing $s$, where, as in 1.5, $s = 2$ if $G$ is not of type $E_6$, and $s = 3$ if $G$ is of type $E_6$. Then $h_{K,s} \leq h_K$. We will denote by $U_K$
the multiplicative-group of units of $K$, and by $K_s$ the subgroup of $K^\times$ consisting of the elements $x$ such that for every normalized valuation $v$ of $K$, $v(x) \in s\mathbb{Z}$.

$V_f$ (resp. $V_\infty$) will denote the set of nonarchimedean (resp. archimedean) places of $k$. As $k$ admits at least $r$ distinct real places, see 1.5, $d := [k : \mathbb{Q}] \geq r$. For $v \in V_f$, $q_v$ will denote the cardinality of the residue field $\mathfrak{v}_v$ of $k_v$.

2.3. For a parahoric subgroup $P_v$ of $G(k_v)$, we define $e(P_v)$ and $e'(P_v)$ by the following formulae (cf. Theorem 3.7 of [P]):

\[ e(P_v) = \frac{q_v((\dim \mathfrak{M}_v+\dim \pi_f)/2)}{\# \mathfrak{M}_v(\mathfrak{f}_v)}. \]

\[ e'(P_v) = e(P_v) \cdot \frac{\# \mathfrak{M}_v(\mathfrak{f}_v)}{q_v^{\dim \mathfrak{M}_v}} = q_v(\dim \mathfrak{M}_v-\dim \pi_f)/2 \cdot \frac{\# \mathfrak{M}_v(\mathfrak{f}_v)}{\# \mathfrak{M}_v(\mathfrak{f}_v)}. \]

2.4. Let $m_1, \ldots, m_n$ ($m_1 < \cdots < m_n$), where $n$ is the absolute rank of $G$, be the exponents of the Weyl group of $G$. For type $B_n$ and $C_n$, $m_j = 2j - 1$; for type $E_6$, the exponents are 1, 4, 5, 7, 8 and 11; and for type $E_7$, the exponents are 1, 5, 7, 9, 11, 13 and 17. Then

- if either $G$ is not of type $^2E_6$, or $v$ splits in $\ell$,

\[ e'(P_v) = e(P_v) \prod_{j=1}^{n} (1 - \frac{1}{q_v^{m_j}+1}); \]

- if $G$ is of type $^2E_6$ and $v$ does not split in $\ell$,

\[ e'(P_v) = e(P_v)(1 - \frac{1}{q_v^0})(1 + \frac{1}{q_v^6})(1 - \frac{1}{q_v^8})(1 + \frac{1}{q_v^8})(1 - \frac{1}{q_v^{12}}), \]

or

\[ e'(P_v) = e(P_v)(1 - \frac{1}{q_v^0})(1 - \frac{1}{q_v^6})(1 - \frac{1}{q_v^8})(1 - \frac{1}{q_v^{12}}) \]

according as $v$ does not or does ramify in $\ell$.

2.5. It is obvious that $e'(P_v) < e(P_v)$. It is not difficult to check using (2) that for all $v \in V_f$, and an arbitrary parahoric subgroup $P_v$ of $G(k_v)$, $e'(P_v)$ is an integer.

2.6. Now we will use the volume formula of [P] to write down the precise value of $\mu(G/\Lambda)$. As the Tamagawa number $\tau_k(G)$ of $G$ equals 1, Theorem 3.7 of [P] (recalled in 3.7 of [BP]), for $S = V_\infty$, provides us the following if $G$ is not of type $^2E_6$,

\[ \mu(G/\Lambda) = D_k^{\frac{1}{2}(\dim G)} \left( \prod_{j=1}^{n} \frac{m_j!}{(2\pi)^{m_j+1}} \right)^d \mathcal{E}, \]

and if $G$ is of type $^2E_6$,

\[ \mu(G/\Lambda) = (D_k D_\ell)^{13} \left( \frac{4!5!7!8!11!11!}{(2\pi)^{42}} \right)^d \mathcal{E}, \]

where $n$ is the absolute rank of $G$, and $\mathcal{E} = \prod_{v \in V_f} e(P_v)$, with $e(P_v)$ as in 2.1.
2.7. Let $\zeta_k$ be the Dedekind zeta-function of $k$, and $L_{\ell|k}$ be the Hecke $L$-function associated to the quadratic Dirichlet character of $\ell/k$. Then

$$\zeta_k(a) = \prod_{v \in V_f} (1 - \frac{1}{q_v^a})^{-1};$$

$$L_{\ell|k}(a) = \prod' (1 - \frac{1}{q_v^a})^{-1} \prod'' (1 + \frac{1}{q_v^a})^{-1},$$

where $\prod'$ is the product over the nonarchimedean places $v$ of $k$ which split in $\ell$, and $\prod''$ is the product over all the other nonarchimedean places $v$ which do not ramify in $\ell$. Hence the Euler product $E$ appearing in (3) can be rewritten as

$$E = \prod_{v \in V_f} e'(P_v) \prod_{j=1}^n \zeta_k(m_j + 1),$$

and the one appearing in (4) can be rewritten as

$$E = \prod_{v \in V_f} e'(P_v) \cdot \zeta_k(2)L_{\ell|k}(5)\zeta_k(6)L_{\ell|k}(8)L_{\ell|k}(9)\zeta_k(12).$$

2.8. If $G$ is not of type $2E_6$, let $T$ be the set of all $v \in V_f$ such that $P_v$ is not a hyperspecial parahoric subgroup of $G(k_v)$, and $T'$ be the empty set. If $G$ is of type $2E_6$, let $T$ be the set of all $v \in V_f$ which splits in $\ell$ and $P_v$ is not a hyperspecial parahoric subgroup, and let $T'$ be the set of $v (\in V_f)$ which does not split in $\ell$, and either $P_v$ is not a hyperspecial parahoric subgroup of $G(k_v)$ but $v$ is unramified over $\ell$, or $v$ is ramified in $\ell$ and $P_v$ is not a special parahoric subgroup. Then for all nonarchimedean $v \notin T$, $\Xi_{\Theta_v}$ is trivial; if $v \notin T \cup T'$, $e'(P_v) = 1$, and $e'(P_v) > s$ if $v \in T$. Therefore, $1 < e'(P_v) < e(P_v)$, and $E = \prod_{v \in V_f} e'(P_v) > s^{#T}$. Hence, if $G$ is not of type $2E_6$,

$$\mu(G/\Lambda) > D_k^{\frac{1}{2}} \dim G \left( \prod_{j=1}^n \frac{m_j!}{(2\pi)^{m_j+1}} \right)^{d^{#T}},$$

and if $G$ is of type $2E_6$,

$$\mu(G/\Lambda) > (D_kD_\ell)^{\frac{13}{2}} \left( \frac{4!5!7!8!11!}{(2\pi)^{42}} \right)^{d^{#T}}.$$

2.9. If $G$ is not of type $2E_6$, let

$$R = D_k^{\frac{1}{2}} \dim G \left( \prod_{j=1}^n \frac{m_j!}{(2\pi)^{m_j+1}} \right)^{d} \prod_{j=1}^n \zeta_k(m_j + 1),$$

and if $G$ is of type $2E_6$, let

$$R = (D_kD_\ell)^{\frac{13}{2}} \left( \frac{4!5!7!8!11!}{(2\pi)^{42}} \right)^{d} \zeta_k(2)L_{\ell|k}(5)\zeta_k(6)L_{\ell|k}(8)L_{\ell|k}(9)\zeta_k(12).$$
Then
\[
(11) \quad \mu(G/\Lambda) = \mathbb{R} \prod_{v \in T \cup \tilde{T}} e'(P_v).
\]

As \(e'(P_v)\) is an integer for all \(v\) (see 2.5), we conclude that \(\mu(G/\Lambda)\) is an integral multiple of \(\mathbb{R}\).

Using the functional equations
\[
\zeta_k(2a) = D_k^{\frac{1}{2} - 2a} \left(\frac{-1)^a 2^{2a-1} \pi^{2a}}{(2a-1)!}\right)^d \zeta_k(1 - 2a),
\]
and
\[
L_{\ell|k}(2a + 1) = \left(D_{\ell} / D_k\right)^{2a + \frac{1}{2}} \left(\frac{-1)^a 2^{2a} \pi^{2a+1}}{(2a)!}\right)^d L_{\ell|k}(-2a),
\]
for every positive integer \(a\), and the fact that \(\dim G = n + 2 \sum m_j\), we find that if \(G\) is not of type \(2E_6\),
\[
(12) \quad \mathbb{R} = 2^{-dn} \prod_{j=1}^n \zeta_k(-m_j),
\]
and if \(G\) is of type \(2E_6\),
\[
(13) \quad \mathbb{R} = 2^{-6d} \zeta_k(-1) L_{\ell|k}(-4) \zeta_k(-5) \zeta_k(-7) L_{\ell|k}(-8) \zeta_k(-11).
\]

2.10. As \(\chi(\Lambda) = \chi(X_u) \mu(G/\Lambda)\) ([BP], 4.2), we have the following
\[
(14) \quad \chi(\Gamma) = \frac{\chi(\Lambda)}{[\Gamma : \Lambda]} = \frac{\chi(X_u) \mu(G/\Lambda)}{[\Gamma : \Lambda]}.
\]
Proposition 2.9 of [BP] applied to \(G' = G\) and \(\Gamma' = \Gamma\) implies that \([\Gamma : \Lambda]\) is a power of the prime number \(s\). Now since \(\mu(G/\Lambda)\) is an integral multiple of \(\mathbb{R}\), we conclude from (10) that if \(\chi(\Gamma)\) is a submultiple of \(\chi(X_u)\), then the numerator of the rational number \(\mathbb{R}\) is a power of \(s\). We state this as the following proposition.

**Proposition 1.** If the orbifold Euler-Poincaré characteristic of \(\Gamma\) is a submultiple of \(\chi(X_u)\), then the numerator of the rational number \(\mathbb{R}\) is a power of \(s\).

2.11. In this paragraph we assume that \(G\) is not of type \(2E_6\). Then \(C \cong \mu_2\), and the Galois cohomology \(H^1(k, C) \cong k^\times / k^{\times 2}\). The order of the first term of the short exact sequence of Proposition 2.9 of [BP], for \(G' = G\) and \(S = V_\infty\), is \(2^{r-1}\). From the proof of Proposition 0.12 of [BP], we easily conclude that \(# k_2 / k^{\times 2} \leq h_{k,2} 2^d\).

Now we can adapt the argument used to prove Proposition 5.1, and the argument in 5.5, of [BP], for \(S = V_\infty\) and \(G' = G\), to derive the following bound:
\[
(15) \quad [\Gamma : \Lambda] \leq 2^{d+r-1+\# h_{k,2}}.
\]

2.12. We shall assume now that \(G\) is of type \(2E_6\). As the norm map \(N_{\ell/k} : \mu_3(\ell) \rightarrow \mu_3(k)\) is onto, the Galois cohomology group \(H^1(k, C)\) is isomorphic to the kernel of the homomorphism \((k^\times / k^{\times 3}) \rightarrow (\ell^\times / \ell^{\times 3})\) induced by the norm map. We shall denote this kernel by \((\ell^\times / \ell^{\times 3})^\bullet\).
By Dirichlet’s unit theorem, \( U_k \cong \{ \pm 1 \} \times \mathbb{Z}^{d-1} \), and \( U_\ell \cong \mu(\ell) \times \mathbb{Z}^{d-1} \), where \( \mu(\ell) \) is the finite cyclic group of roots of unity in \( \ell \). Hence, \( U_k/U_k^3 \cong \mathbb{Z}/3\mathbb{Z} \), and \( U_\ell/U_\ell^3 \cong \mu(\ell)^3 \times \mathbb{Z}/3\mathbb{Z}^{d-1} \), where \( \mu(\ell)^3 \) is the group of cube-roots of unity in \( \ell \). Now we observe that \( N_{\ell/k}(U_\ell) \supset N_{\ell/k}(U_k) = U_k^2 \), which implies that the homomorphism \( U_\ell/U_\ell^3 \to U_k/U_k^3 \), induced by the norm map, is onto. Therefore, the order of the kernel \( (U_\ell/U_\ell^3) \) of this homomorphism equals \( \# \mu(\ell)^3 \).

The short exact sequence (4) in the proof of Proposition 0.12 of [BP] gives us the following exact sequence:

\[
1 \to (U_\ell/U_\ell^3) \to (\ell^x/\ell^{x^3}) \to (\mathbb{P} \cap \mathbb{J}^3)/\mathbb{P}^3,
\]

where \( (\ell^x/\ell^{x^3}) = (\ell^x/\ell^{x^3}) \cap (\ell^x/\ell^{x^3}) \), \( \mathbb{P} \) is the group of all fractional principal ideals of \( \ell \), and \( \mathbb{J} \) the group of all fractional ideals (we use multiplicative notation for the group operation in both \( \mathbb{J} \) and \( \mathbb{P} \)). Since the order of the last group of the above exact sequence is \( h_{\ell,3} \), see (5) in the proof of Proposition 0.12 of [BP], we conclude that

\[
\#(\ell^x/\ell^{x^3}) \leq \# \mu(\ell)^3 \cdot h_{\ell,3}.
\]

Now we note that the order of the first term of the short exact sequence of Proposition 2.9 of [BP], for \( G' = G \) and \( S = V_\infty \), is \( 3^r/\# \mu(\ell)^3 \).

Using the above observations, together with Proposition 2.9 and Lemma 5.4 of [BP], and a close look at the arguments in 5.3 and 5.5 of [BP] for \( S = V_\infty \) and \( G \) as above, we can derive the following upper bound:

\[
[\Gamma : \Lambda] \leq 3^r + \# \mathbb{J} h_{\ell,3}.
\]

2.13. Since \( \mu(G/\Gamma) = \mu(G/\Lambda)/[\Gamma : \Lambda] \) is a submultiple of \( 1/s^r \) (see 2.1), we conclude that \( \mu(G/\Lambda) \leq [\Gamma : \Lambda]/s^r \). From the bound for \( [\Gamma : \Lambda] \) derived in 2.11 and 2.12 we obtain that if \( G \) is not of type \( 2E_6 \), then

\[
\mu(G/\Lambda) \leq 2^{d-1+\# \mathbb{J}} h_{k,2},
\]

and if \( G \) is of type \( 2E_6 \),

\[
\mu(G/\Lambda) \leq 3^{\# \mathbb{J}} h_{k,3}.
\]

Now combining these with (7) and (8) respectively, we obtain

\[
D_k^\dim G < 2^{d-1} h_{k,2} \left( \prod_{j=1}^{n} \frac{2\pi^{m_j+1}}{m_j!} \right)^d,
\]

if \( G \) is not of type \( 2E_6 \), and

\[
(D_k D_\ell)^{13} < h_{k,3} \left( \frac{2\pi^{42}}{4!\cdot 5!\cdot 7!\cdot 11!} \right)^d
\]

if \( G \) is of type \( 2E_6 \).
3. Discriminant bounds

We will recall discriminant bounds required in later discussions. We define $M_r(d) = \min_K D_1^{1/d}$, where the minimum is taken over all totally real number fields $K$ of degree $d$. Similarly, we define $M_c(d) = \min_K D_1^{1/d}$, by taking the minimum over all totally complex number fields $K$ of degree $d$.

The precise values of $M_r(d)$, $M_c(d)$ for low values of $d$ are given in the following table (cf. [N]).

| $d$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $M_r(d)$ | 5 | 49 | 725 | 14641 | 300125 | 20134393 | 282300416 |
| $M_c(d)$ | 3 | 117 | 9747 | 1257728 |

The following proposition can be proved in the same way as Proposition 2 in [PY2] has been proved.

**Proposition 2.** Let $k$ and $\ell$ be a totally real number field and a totally imaginary number field of degree $d$ and $2d$ respectively.

$$\forall d \geq 2 \quad D_1^{1/d} > 2.23 \quad 3.65 \quad 5.18 \quad 6.8 \quad 8.18 \quad 11.05 \quad 11.38$$

$$D_\ell^{1/2d} > 1.73 \quad 3.28 \quad 4.62 \quad 5.78$$

4. $G$ of type $B_n$ or $C_n$

4.1. In this section we assume that $G$ is either of type $B_n$ or $C_n$. Then its dimension is $n(2n+1)$. The $j$-th exponent $m_j = 2j - 1$, and the complex dimension of the symmetric space $X$ of $G = \prod_{j=1}^r G_{k(v_j)}$ is $r(2n-1)$ if $G$ is of type $B_n$, and is $rn(n+1)/2$ if $G$ is of type $C_n$. From (19) we obtain

$$D_k^{1/d} < f_1(n, d, h_k, 2) := \left[ \frac{1}{2} \prod_{j=1}^n (\frac{2\pi}{2j-1}) \right]^d \cdot \frac{h_{k, 2}}{2}^{2/dn(2n+1)}.$$

According to the Brauer-Siegel Theorem, for a totally real number field $k$, and all real $\delta > 0$,

$$h_k R_k \leq 2^{-1-d} \delta (1 + \delta) \Gamma((1 + \delta)/2)^d (\pi^{-d} D_k)^{(1+d)/2} \zeta_k(1 + \delta),$$

where $R_k$ is the regulator of $k$. Now from (21) we get the following bound:

$$D_k^{1/d} < f_2(n, d, R_k, \delta)$$

$$:= \left[ \frac{\Gamma((1 + \delta)/2) \zeta(1 + \delta) \prod_{j=1}^n (2\pi)^{2j} \zeta_k^{(\delta(1 + \delta)) R_k}}{(2j-1)!} \right]^{1/d} \cdot 2^{1/2n(2n+1)} \cdot (\pi^{(1+\delta)/2} D_k^{(1+d)/2}),$$

since $\zeta_k(1 + \delta) < \zeta(1 + \delta)$, where $\zeta = \zeta_Q$. Using the lower bound $R_k \geq 0.04 e^{0.46d}$, for a totally real number field $k$, due to R. Zimmert [Z], we get

$$D_k^{1/d} < f_3(n, d, \delta)$$

$$:= \left[ \frac{\Gamma((1 + \delta)/2) \zeta(1 + \delta) \prod_{j=1}^n (2\pi)^{2j} \zeta_k^{(\delta(1 + \delta)) R_k}}{(2j-1)!} \right]^{1/d} \cdot 2^{1/2n(2n+1)} \cdot (\pi^{(1+\delta)/2} D_k^{(1+d)/2}),$$

since $\zeta_k(1 + \delta) < \zeta(1 + \delta)$, where $\zeta = \zeta_Q$. Using the lower bound $R_k \geq 0.04 e^{0.46d}$, for a totally real number field $k$, due to R. Zimmert [Z], we get
From the bounds provided by the above table and the properties of established in the preceding paragraph, that 
\[ \delta \in \mathbb{R} \]
we infer that separately.

see that if for a given \( f \) that unless increases. Now we observe that for \( n \)
and if \( f \) and if \( n \), then \( f_3(n + 1, d, \delta) < f_3(n, d, \delta) \), and if \( f_3(n, d, \delta) < 1 \), then \( f_3(n + 1, d, \delta) < 1 \). In particular, if for given \( d \), and \( \delta \in [0.04, 9] \), \( f_3(8, d, \delta) < c \), with \( c \geq 1 \), then \( f_3(n, d', \delta) < c \) for all \( n \geq 8 \) and \( d' \geq d \).

We obtain by a direct computation the following upper bound for the value of \( f_3(n, 2, 3) \) for \( 6 \leq n \leq 14 \).

| \( n \) | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 |
|-----|----|----|----|----|----|---|---|---|---|
| \( f_3(n, 2, 3) \) | < 1 | 1.1 | 1.2 | 1.3 | 1.4 | 1.6 | 1.8 | 2.1 | 2.4 |

From the bounds provided by the above table and the properties of \( f_3 \) we conclude that \( f_3(n, d, 3) < 2.1 \) for all \( n \geq 7 \), and \( d \geq 2 \), and we conclude from Proposition 2 that unless \( k = \mathbb{Q} \) (i.e., \( d = 1 \)), \( n \leq 6 \).

We assert now that \( n \leq 13 \). To prove this, we can assume, in view of the result established in the preceding paragraph, that \( k = \mathbb{Q} \). By a direct computation we see that \( f_1(14, 1, 1) < 1 \). Hence, \( f_1(n, 1, 1) < 1 \) for all \( n \geq 14 \). As \( D_\mathbb{Q} = 1 \), from bound (21) we conclude that \( n \leq 13 \).

We will now assume that \( d \geq 2 \) and consider each of the possible cases \( 2 \leq n \leq 6 \) separately.

- **\( n = 6 \):** As \( D_k^{1/d} < f_3(6, 3, 2.1) < 2.5 \), from Proposition 2 we conclude that if \( n = 6 \), \( d < 3 \). If \( d = 2 \), \( D_k^{1/d} < f_3(6, 2, 1.54) < 2.36 \). Therefore, \( D_k < 6 \), which implies that \( k = \mathbb{Q}(\sqrt{5}) \) is the only possibility.

- **\( n = 5 \):** As \( D_k^{1/d} < f_3(5, 3, 1.8) < 2.9 \), from Proposition 2 we conclude that if \( n = 5 \), \( d < 3 \). If \( d = 2 \), \( D_k^{1/d} < f_3(5, 2, 1.3) < 2.9 \). Therefore, \( D_k < 9 \). So there are two possible real quadratic fields \( k \), their discriminants are 5 and 8. Both the fields have class number 1, and we use the bound (21) to obtain \( D_k^{1/d} < f_1(5, 2, 1) < 2.73 \). So only \( D_k = 5 \) can occur.

- **\( n = 4 \):** As \( D_k^{1/d} < f_3(4, 3, 1.3) < 3.61 \), from Proposition 2 we conclude that if \( n = 4 \), \( d < 3 \). Let us assume that \( d = 2 \). Then since \( D_k^{1/d} < f_3(4, 2, 1.1) < 3.76 \), \( D_k < 15 \) and so the possible values of \( D_k \) are 5, 8, 12 or 13. The quadratic fields with these \( D_k \) have class number 1. Now from bound (21) we obtain \( D_k^{1/d} < f_1(4, 2, 1) < 3.4 \). Hence, \( D_k < 12 \), and only \( D_k = 5, 8 \) can occur.

- **\( n = 3 \):** As \( D_k^{1/d} < f_3(3, 4, 1.3) < 5.1 \), from Proposition 2 we conclude that if \( n = 3 \), \( d < 4 \). Now since \( D_k^{1/d} < f_3(3, 3, 1.13) < 5.21 \), if \( d = 3 = n \), \( D_k < 142 \) from which we infer that \( D_k = 49 \) or 81. Since \( D_k^{1/d} < f_3(3, 2, 0.8) < 5.6 \), if \( d = 2 \) (and \( n = 3 \)), \( D_k < 32 \), and in this case the possible values of \( D_k \) are 5, 8, 12, 13, 17, 21, 24, 28 or 29. The quadratic fields with these discriminants have class number 1, and we
use bound (21) to obtain $D_k^{1/d} < f_1(3, 2, 1) < 4.52$. Hence, $D_k < 21$ and only $D_k = 5, 8, 12, 13, 17$ can occur.

- $n = 2$: As $D_k^{1/d} < f_3(2, 7, 1.1) < 9$, Proposition 2 implies that $d \leq 6$.

  - $n = 2$ and $d = 6$: As $D_k^{1/d} < f_3(2, 6, 1) < 9$, $D_k < 531441$. One can check from the table in [1] that $h_k = 1$ for all the five number fields satisfying this bound. We now use bound (21) to obtain $D_k^{1/d} < f_1(2, 6, 1) < 7.2$, which contradicts Proposition 2.

  - $n = 2$ and $d = 5$: As $D_k^{1/d} < f_3(2, 5, 1) < 9.3$, $D_k < 69569$. Again, one can check from the table in [1] that there are five such number fields and the class number of each of them is 1. Now we use bound (21) to obtain $D_k^{1/d} < f_1(2, 5, 1) < 7.1$. Hence, $D_k < 18043$. From [1] we find that $D_k = 14641$ is the only possibility.

  - $n = 2$ and $d = 4$: As $D_k^{1/d} < f_3(2, 4, 0.92) < 9.74$, $D_k < 9000$. According to [1], there are 45 totally real quartic number fields with discriminant < 9000, all of them have class number 1. We use bound (21) to obtain $D_k^{1/d} < f_1(2, 4, 1) < 7.037$. Hence, $D_k < 2453$. We find from [1] that there are eight totally real quartic number fields $k$ with $D_k < 2453$. Their discriminants are

  $$725, 1125, 1600, 1957, 2000, 2048, 2225, 2304.$$ 

  - $n = 2$ and $d = 3$: As $D_k^{1/d} < f_3(2, 3, 0.78) < 10.5$, $D_k < 1158$. From table B.4 of [C] we find that there are altogether 31 totally real cubics satisfying this discriminant bound. Each of these fields has class number 1. We use bound (21) to obtain $D_k^{1/d} < f_1(2, 3, 1) < 6.96$, which implies that $D_k < 338$. There are eight real cubic number fields satisfying this bound. The values of $D_k$ are

  $$49, 81, 148, 169, 229, 257, 316, 321.$$ 

  - $n = 2$ and $d = 2$: As $D_k^{1/d} < f_3(2, 2, 0.52) < 12$, $D_k < 144$. From table B.2 of totally real quadratic number fields given in [C], we check that the class number of all these fields are bounded from above by 2. Hence, $D_k^{1/d} < f_1(2, 2, 2) < 7.285$. So $D_k < 53$. Among the real quadratic fields with $D_k < 53$, there is only one field whose class number is 2, it is the field with $D_k = 40$. All the rest have class number 1, and from bound (21) we conclude that $D_k^{1/d} < f_1(2, 2, 1) < 6.8$, i.e., $D_k < 47$. Therefore, the following is the list of the possible values of $D_k$:

  $$5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44.$$ 

  To summarize, for $G$ of type $B_n$ or $C_n$, the possible $n$, $d$ and $D_k$ are given in the following table.
We will show that none of the possibilities listed in the above table actually give rise to an arithmetic fake compact hermitian symmetric space of type $B_n$ or $C_n$. For this we recall first of all that $G$, and so also $G$, are anisotropic over $k$ (1.5).

Now we observe that if $G$ is a group of type $B_n$ ($n \geq 2$), then it is $k$-isotropic if and only if it is isotropic at all real places of $k$ (this is an immediate consequence of the classical Hasse principle for quadratic forms which says that a quadratic form over $k$ is isotropic if and only if it is isotropic at every place of $k$, and the well-known fact that a quadratic form of dimension $> 4$ is isotropic at every nonarchimedean place). Also, a $k$-group of type $C_n$ ($n \geq 2$) is isotropic if it is isotropic at all the real places of $k$ (this is known, and follows, for example, from Proposition 7.1 of [PR]). These results imply that if $d = 1$, i.e., if $k = \mathbb{Q}$, then $G$ is isotropic, and so $k = \mathbb{Q}$ is not possible.

Now let us take up the case where $d = 2$, i.e., $k$ is a real quadratic field, and $n = 2, 5$ or 6. Then for any real place $v$ of $k$ where $G$ is isotropic, the complex dimension of the symmetric space of $G(k_v)$ is odd (recall from 1.4 that the complex dimension of the symmetric space of $G(k_v)$ is $2n - 1$ if $G$ is of type $B_n$, and it is $n(n + 1)/2$ if $G$ is of type $C_n$). But as the complex dimension of the hermitian symmetric space $X$ is even (since the orbifold Euler-Poincaré characteristic of $\Gamma$ is positive, see 1.3), we conclude that $G$ must be isotropic at both the real places of $k$ (note that $G$ is anisotropic at a place $v$ of $k$ if and only if $G(k_v)$ is compact). From this observation we conclude that $G$ is $k$-isotropic also in case $d = 2$, and $n = 2, 5$ or 6. Therefore these possibilities do not occur.

Now we will rule out the case where $n = 2, d = 5$, and $D_k = 14641$. In this case, $k = \mathbb{Q}[x]/(x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1)$. The class number of $k$ is 1. It is easy to see that the cardinality $q_v$ of the residue field of $k$ at any nonarchimedean place $v$ is at least 4.

Since $\mu(G/\Gamma)$ is a submultiple of $1/2^r$ (2.1), and $[\Gamma : \Lambda] = 2^m$, where $m \leq 4 + r + \#\mathcal{J}$, see (15), using (3) we obtain

$$
1 \geq 2^r \frac{\mu(G/\Gamma)}{[\Gamma : \Lambda]} > \frac{2^r}{[\Gamma : \Lambda]} \left( \frac{6D_k}{(2\pi)^6} \right)^5 \mathcal{E} > 2^{-4-\#\mathcal{J}} \left( \frac{87846}{(2\pi)^6} \right)^5 \prod_{v \in \mathcal{J}} e(P_v),
$$
The group $G$ is a simply connected $k$-anisotropic group of type $C_2$. Such a group is described in terms of a quaternion division algebra $D$, with center $k$ (see [Ti1]). Since the symmetric space $X$ is a complex analytic space of even complex dimension, and $d = 5$, we conclude that $G$ is anisotropic at an odd number of real places of $k$. Hence the quaternion division algebra $D$ ramifies at an odd number of real places of $k$. Since a quaternion division algebra ramifies at an even number of places, we infer that there is at least one nonarchimedean place where $D$ ramifies. At such a place, $G$ is of rank 1, so such a place lies in $\mathcal{T}$. This shows that $\mathcal{T}$ is nonempty. For $v \in \mathcal{T}$, $e(P_v)$ equals either $q_v^6/(q_v^4 - 1)$, or $q_v^6/(q_v + 1)(q_v^2 - 1)$, or $q_v^6/(q_v^2 - 1)$. All these numbers are larger than $q_v^2$. Now as $q_v \geq 4$, and $\mathcal{T}$ is nonempty, from (24) we conclude that

$$1 \geq 2^r \mu(G/\Gamma) > 2^{-4-\#\mathcal{T}} \left(\frac{87846}{(2\pi)^6}\right)^5 16^{\#\mathcal{T}} > \frac{1}{2} \left(\frac{87846}{(2\pi)^6}\right)^5 > 1,$$

which is absurd.

4.4. To rule out the remaining cases listed in the table in 4.2, we compute the value of $R$ ($R$ as in (12)) in each case. These values are given below. It turns out that in none of the remaining cases the numerator of $R$ is a power of 2 and Proposition 1 then eliminates these cases.

| $n$ | $d$ | $k$ | $D_k$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ | $\zeta_k(-5)$ | $\zeta_k(-7)$ | $R$                     |
|-----|-----|-----|-------|-------------|-------------|-------------|-------------|--------------------------|
| 4   | 2   | $x^2 - 5$ | 5     | 1/30        | 1/60        | 67/630      | 361/120     | 24187/34836480000       |
| 4   | 2   | $x^2 - 2$ | 8     | 1/12        | 11/120      | 361/252     | 24611/240   | 97730281/22295347200    |

| $n$ | $d$ | $k$ | $D_k$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ | $\zeta_k(-5)$ | $R$                     |
|-----|-----|-----|-------|-------------|-------------|-------------|--------------------------|
| 3   | 3   | $x^3 - x^2 - 2x + 1$ | 49    | -1/21      | 79/210      | -7393/63    | 584047/142248960       |
| 3   | 3   | $x^3 - 3x - 1$ | 81    | -1/9       | 199/90      | -50353/27   | 10020247/11197440      |
| 3   | 2   | $x^2 - 17$ | 17    | 1/3        | 41/30       | 5791/63     | 237431/362880         |
| 3   | 2   | $x^2 - 13$ | 13    | 1/6        | 29/60       | 33463/1638  | 970427/37739520       |
| 3   | 2   | $x^2 - 3$  | 12    | 1/6        | 23/60       | 1681/126    | 38663/2903040         |
| 3   | 2   | $x^2 - 2$  | 8     | 1/12       | 11/120      | 361/252     | 3971/23224320         |
| 3   | 2   | $x^2 - 5$  | 5     | 1/30       | 1/60        | 67/630      | 67/72576000          |
The Brauer-Siegel Theorem for the totally complex number field \( \ell \) asserts that for all real \( \delta > 0 \),
\[
(25) \quad h_\ell R_\ell \leq w_\ell \delta(1 + \delta)\Gamma(1 + \delta)^d ((2\pi)^{-2d} D_\ell)^{(1+\delta)/2} \zeta(1 + \delta),
\]
where \( R_\ell \) is the regulator of \( \ell \) and \( w_\ell \) is the number of roots of unity contained in \( \ell \). Using this, from bound (20) we obtain
\[
(D_k D_\ell)^{13} < h_\ell A^d \leq \frac{\delta(1 + \delta)A^d\Gamma(1 + \delta)^dD_\ell^{(1+\delta)/2}\zeta(1 + \delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.
\]

Hence,
\[
D_k^{13} D_\ell^{13-\frac{1+\delta}{d}} < \frac{\delta(1 + \delta)A^d\Gamma(1 + \delta)^d\zeta(1 + \delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.
\]

As \( D_k^2 \leq D_\ell \), and \( \zeta(1 + \delta) \leq \zeta(1 + \delta)^{2d} \), we conclude that
\[
D_k^{38-\delta} < \frac{\delta(1 + \delta)A^d\Gamma(1 + \delta)^d\zeta(1 + \delta)^{2d}}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.
\]

5. **G of type \( ^2E_6 \)**

5.1. In this section \( G \) is of type \( ^2E_6 \). Its dimension is 78 and the complex dimension of the symmetric space of \( G = \prod_{j=1}^{\ell} G(k_{v_j}) \) is 16\( r \). Let
\[
A = \frac{(2\pi)^{42}}{4!5!7!8!11!}.
\]

The Brauer-Siegel Theorem for the totally complex number field \( \ell \) asserts that for all real \( \delta > 0 \),
\[
(25) \quad h_\ell R_\ell \leq w_\ell \delta(1 + \delta)\Gamma(1 + \delta)^d ((2\pi)^{-2d} D_\ell)^{(1+\delta)/2} \zeta(1 + \delta),
\]
where \( R_\ell \) is the regulator of \( \ell \) and \( w_\ell \) is the number of roots of unity contained in \( \ell \). Using this, from bound (20) we obtain
\[
(D_k D_\ell)^{13} < h_\ell A^d \leq \frac{\delta(1 + \delta)A^d\Gamma(1 + \delta)^dD_\ell^{(1+\delta)/2}\zeta(1 + \delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.
\]

Hence,
\[
D_k^{13} D_\ell^{13-\frac{1+\delta}{d}} < \frac{\delta(1 + \delta)A^d\Gamma(1 + \delta)^d\zeta(1 + \delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.
\]

As \( D_k^2 \leq D_\ell \), and \( \zeta(1 + \delta) \leq \zeta(1 + \delta)^{2d} \), we conclude that
\[
D_k^{38-\delta} < \frac{\delta(1 + \delta)A^d\Gamma(1 + \delta)^d\zeta(1 + \delta)^{2d}}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.
\]
Therefore,

\[ D_k^{1/d} < \left[ \left\{ A \frac{\Gamma(1 + \delta) \zeta(1 + \delta)^2}{(2\pi)^{1+\delta}} \right\} \cdot \left\{ \frac{\delta(1 + \delta)}{R/\ell w} \right\}^{1/d} \right]^{1/(38-\delta)}. \]

Using the lower bound \( R/\ell \geq 0.02w e^{0.1d} \) due to R. Zimmert [Z], we obtain from this the following

\[ (26) \quad D_k^{1/d} < f(d, \delta) := \left[ \left\{ A \frac{\Gamma(1 + \delta) \zeta(1 + \delta)^2}{(2\pi)^{1+\delta} e^{0.1}} \right\} \cdot \left\{ 50\delta(1 + \delta) \right\}^{1/d} \right]^{1/(38-\delta)}. \]

From bound (20) we also obtain,

\[ (27) \quad D/\ell^2 < \left[ A^{d} \frac{h/\ell}{D^{39}} \right]^{1/13}. \]

Furthermore, using (25) and Zimmert's bound \( R/\ell \geq 0.02w e^{0.1d} \), we get from this that

\[ (28) \quad D/\ell^2 < p(d, D, \delta) := \left[ \left\{ A \frac{\Gamma(1 + \delta) \zeta(1 + \delta)^2}{(2\pi)^{1+\delta} e^{0.1}} \right\} \cdot \left\{ 50\delta(1 + \delta) \right\}^{1/d} \right]^{2d/(25-\delta)}. \]

5.2. For a fixed \( \delta \), \( f(d, \delta) \) clearly decreases as \( d \) increases. By a direct computation we find that \( f(3, 2) < 2.3 \), and hence for all \( d \geq 3 \),

\[ D_k^{1/d} < f(d, 2) \leq f(3, 2) < 2.3. \]

But according to Proposition 2, for totally real number fields of degree \( d \geq 3 \), \( D_k^{1/d} > 3.65 \), so we conclude that \( d \leq 2 \).

Assume now that \( d = 2 \). Then \( D_k^{1/2} \leq f(2, 1.94) < 2.4 \). Therefore \( D_k < 6 \) and hence \( D_k = 5 \). It follows from bound (28) with \( \delta = 1.9 \) that \( D/\ell^2 < p(2, 5, 1.9) < 1.4 \). Hence \( D/\ell^2 = 1 \) and \( D/\ell = 25 \), which contradicts the bound given by Proposition 2. Hence, \( d = 1 \) and \( k = Q \).

It is known, and follows, for example, from Proposition 7.1 of [PR], that a \( Q \)-group \( G \) of type \( 2E_6 \), which at the unique real place of \( Q \) is the outer form of rank 2 (this is the form \( 2E_{6,2}^{16'} \) which gives rise to a hermitian symmetric space), is isotropic over \( Q \). This contradicts the fact that \( G \) is anisotropic over \( Q (1.5) \), and hence we conclude that groups of type \( 2E_6 \) do not give rise to arithmetic fake compact hermitian symmetric spaces.

6. \( G \) of type \( E_7 \)

6.1. In this section \( G \) is assumed to be of type \( E_7 \). Its dimension is 133, the exponents are 1, 5, 7, 9, 11, 13 and 17. The dimension of the symmetric space \( X \) of \( G = \prod_{j=1}^{r} G(k_{v_j}) \) is \( 27r \).
Let
\[ B = \prod_{j=1}^{7} \frac{(2\pi)^{m_j+1}}{m_j!}. \]

From (19) we obtain the following:
\[ D_k^{1/d} < \left[ 2B(h_{k,2}/2)^{1/d} \right]^{2/133}. \]

Using the Brauer-Siegel Theorem for totally real number fields (see 4.1), and the obvious bound \( \zeta_k(1 + \delta) \leq \zeta(1 + \delta)^d \), we obtain
\[ D_k^{1/d} < \left[ \left\{ B \frac{\Gamma((1 + \delta)/2)\zeta(1 + \delta)}{\pi(1 + \delta)/2} \right\} \cdot \left\{ \frac{\delta(1 + \delta)}{R_k} \right\}^{1/d} \right]^{2/(132-\delta)}. \]

Now using the lower bound \( R_k \geq 0.04 e^{0.46d} \) due to R. Zimmert [Z] again, we get
\[ D_k^{1/d} < \phi(d, \delta) := \left[ \left\{ B \frac{\Gamma((1 + \delta)/2)\zeta(1 + \delta)}{\pi(1 + \delta)/2} e^{0.46} \right\} \cdot \left\{ 25\delta(1 + \delta) \right\}^{1/d} \right]^{2/(132-\delta)}. \]

6.2. For a fixed \( \delta \geq 0.04 \), \( \phi(d, \delta) \) clearly decreases as \( d \) increases. By a direct computation we see that \( \phi(2, 4) < 2 \), and hence for all totally real number field \( k \) of degree \( d \geq 2 \),
\[ D_k^{1/d} < \phi(d, 4) \leq \phi(2, 4) < 2. \]

From this bound and Proposition 2 we conclude that \( d \) can only be 1, i.e., \( k = \mathbb{Q} \). But then \( r = 1 \) and the complex dimension of the associated symmetric space \( X \) is 27. Then the Euler-Poincaré characteristic of any quotient of \( X \) by a cocompact torsion-free discrete subgroup of \( \overline{G} \) is negative (1.3), and hence it cannot be a fake compact hermitian symmetric space. Another way to eliminate this case is to observe that an absolutely simple \( \mathbb{Q} \)-group of type \( E_7 \) is isotropic if it is isotropic over \( \mathbb{R} \) (this result follows, for example, from Proposition 7.1 of [PR]).

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**University of Michigan, Ann Arbor, MI 48109**
e-mail: gprasad@umich.edu

**Purdue University, West Lafayette, IN 47907**
e-mail: yeung@math.purdue.edu