Numerical model of the structural element complex geometry with a coating

S N Yakupov, H G Kiyamov and N M Yakupov
Institute of Mechanics and Engineering – Subdivision of the Federal State Budgetary Institution of Science “Kazan Scientific Center of the Russian Academy of Sciences” (IME – Subdivision of FIC KazanSC of RAS), 420111, Kazan
E-mail: tamas_86@mail.ru

Abstract. A model of the composition of complex geometry based on three-dimensional elements with a cubic approximation of the desired variables for the structural element and two-dimensional elements with a cubic approximation of the desired variables for the coating is described. An example of calculating the stress-strain state of a plate with thin steel coatings is considered.

1. Introduction
About the known methods of calculation. Among the methods for calculating shells of complex geometry, the finite element method (FEM) is widely used [1–8]. An effective method is the spline version of the FEM (SV FEM-2) with bicubic approximation of the desired variables [8–10], which is developed for calculating the VAT of structural elements in a three-dimensional formulation (SV FEM-3) [11, 12].

In this paper, we propose the development of the idea of a spline version of the FEM-a numerical model of the system "structural element + coating" of complex geometry based on three-dimensional elements with a cubic approximation of the desired variables in all three directions for the structural element and two-dimensional elements with a cubic approximation of the desired variables for the coating. The numerical model allows us to take into account the specific properties of the coating, which differ from the properties of the main structural element.

The element model. We consider a structural element of complex geometry, on the upper face of which a coating is applied (figure 1).

Figure 1. Coated construction element.
The structural element is modeled in a three-dimensional setting, and the coating is modeled in a two-dimensional one. The radii-vectors of the structural element \( \vec{r}_{el} \) and the cover \( \vec{r}_{pk} \) are set by the coordinates of the rectangular parallelepiped \( t^1, t^2, t^3 \) and the rectangle \( t^1 \) and \( t^2 \), respectively:

\[
\vec{r}_{el} = \vec{r}(t^1, t^2, t^3), \quad \vec{r}_{pk} = \vec{r}(t^1, t^2).
\] (1)

Differentiating the relations (1) by the corresponding coordinates, the coordinate vectors, the components of the metric tensor and its discriminant are determined. The Christoffel symbols for the structural element and the coating are calculated using the formulas given in [8–12]. In the derivation of the equilibrium equations, geometric and physical relations similar to [8–12] are used.

2. Representation of the desired unknowns

The region of the parallelepiped describing the construction element is divided into finite elements and the solution \( u, v, \) and \( w \) in each of them is represented as an interpolation Hermitian cubic spline of three variables [11, 12]:

\[
\begin{align*}
\psi_1(t^1) \times \psi_2(t^2) \times \psi_3(t^3) \\
\psi_1(t^1) \times \psi_2(t^2) \times \psi_3(t^3) \\
\psi_1(t^1) \times \psi_2(t^2) \times \psi_3(t^3)
\end{align*} \otimes F_u
\]

where \( \psi_1, \psi_2, \psi_3 \) – are vectors of coordinate functions along the coordinate lines \( t^1, t^2, t^3 \); \( F_u, F_v, F_w \) – are three-dimensional matrices of the components of the desired unknowns: \( u, v, w \); the first: \( u^{00}, u^{010}, u^{100}, v^{100}, v^{001}, w^{100}, w^{001}, w^{010}, w^{010} \); the second: \( u^{110}, u^{101}, v^{110}, v^{101}, v^{011}, w^{110}, w^{101}, w^{011} \); and the third derivatives: \( u^{111}, v^{111}, w^{111} \) at the corresponding coordinates; \( \otimes \) - the symbol of the product of the corresponding components of the matrices, followed by their summation. In each grid node of the region, we have 24 unknowns: 8 for group \( u \), 8 for group \( v \), and 8 for group \( w \).

The finite element grid of the median surface of the cover coincides with the grid of the upper face of the parallelepiped and the solution in each of the rectangles is represented as an interpolation Hermitian cubic spline of two variables:

\[
\begin{align*}
\psi_1(t^1) \times \psi_2(t^2) \\
\psi_1(t^1) \times \psi_2(t^2) \\
\psi_1(t^1) \times \psi_2(t^2)
\end{align*} \otimes F_{w1}
\]

where \( F_U, F_V, F_W \) – are two-dimensional matrices of the components of the desired unknowns: \( u, v, w \); the first: \( u^{10}, u^{01}, v^{10}, v^{01}, w^{10}, w^{01} \) and the second: \( u^{11}, v^{11}, w^{11} \), derived from the corresponding coordinates.

3. Equations of equilibrium

The resolution relations are obtained from the Lagrange variational equation:

\[
\delta W - \delta A = 0,
\]

where \( \delta W \) – is the variation of the strain energy, \( \delta A \) – is the variation of the work of the forces acting on the structural element.

Substituting the variations of displacements and deformations for the structural elements and the coating in equation (4), taking into account the independence of the nodal displacements and their derivatives, after a series of transformations, a system of algebraic equations of the form is obtained.

\[
[ A ] \{ U \} = \{ R \},
\]

where \( [A] \) – is the stiffness matrix of the tape structure system, \( \{U\} \) – is the vector of unknowns, and, \( \{R\} \) – is the load vector.
The total stiffness matrix \([A]\) of the "structural element + coating" system is assembled from the stiffness matrices of each three-dimensional finite element, to which the elements of the coating stiffness matrix are summed in the corresponding cells. The structure of the integrals is the same. The integration uses the Gaussian quadrature formula according to the eight-node scheme. All calculations are performed with double precision.

4. Example of the calculation

We consider a plate with a length of \(L = 600\) cm, a width of \(B = 120\) cm, a thickness of \(H_p = 20\) cm, and an elastic modulus of \(E = 20000\) MPa. On the lower and upper surfaces there are thin metal coatings with a thickness of \(H_n = 0.08\) cm, the elastic modulus of which is \(E = 200000\) MPa. Dividing the plate into finite elements: 10 elements in length, 4 elements in width, 1 element in thickness. Figure 2 shows the layout of the plate in the plan and shows the four mounting points of the coated plate on the lower surface, in which \(u = v = w = 0\). A distributed load \(q = -0.02\) MPa acts on the upper surface of the plate.

![Figure 2. The grid for splitting the plate and its attachment points.](image)

The changes in the normal longitudinal stresses \(\sigma_x\) on the surfaces \(z = -9.5\) cm (a) and \(z = 10\) cm (b) are shown for the coated plate in Figure 3 and for the uncoated plate in Figure 4.

![Figure 3. Stresses \(\sigma_x\) on the surfaces of the coated plate \(z = -9.5\) cm (a) and \(z = 10\) cm (b).](image)

As can be seen from Figure 3 and Figure 4, the presence of rigid coatings increases the rigidity of the plate and significantly redistributes the stresses.

![Figure 4. Stresses \(\sigma_x\) on uncoated plate surfaces \(z = -9.5\) cm (a) and \(z = 10\) cm (b).](image)

5. Conclusion

To assess the load-bearing capacity of the "structural element + coating" system with a rigid coating, it is necessary to study the structure as a composite structure.
The paper describes a model of the composition of complex geometry based on three-dimensional elements with a cubic approximation of the desired variables for the structural element and two-dimensional elements with a cubic approximation of the desired variables for the coating, which allows us to study the structural element with a rigid coating. As an illustration, an example of calculating the stress-strain state of a plate with thin steel coatings is considered.

References
[1] Montemor M F 2014 Surface & Coatings Technology 258 17–37
[2] Yakupov S N and Yakupov N M 2017 Journal of Physics: Conference series 857
[3] Yakupov S N 2010 Mechanics of Composite Materials and Structures 16 436–44
[4] Ahmad S, Irons B M and Zienkiewicz O C 1970 Int. J. Num. Meth. Eng. 3 419–51
[5] Argyris J H, Fried I and Scarff D W 1968 The Aeronautical Journal 692 701–9
[6] Gallagher R H 1974 Buckling of structures, Symposium Cambridge 40–51
[7] Golovanov A I, Pesoshin A V and Tyuleneva O N 2005 Modern finite element models and methods for studies of thin-walled structures 442
[8] Yakupov N M 2020 The mechanics of thin-walled structures: history, diagnostics, treatment 159
[9] Yakupov N M 1984 Research on the theory of shells 4–17
[10] Kornishin M S and Yakupov N M 1987 Applied mechanics 3 38–44
[11] Kornishin M S and Yakupov N M 1989 Applied mechanics 8 53–60
[12] Yakupov N M, Kiyamov H G, Yakupov S N and Kiyamov I H 2011 Mechanics of composite materials and structures 1 145–54