GEOMETRIC REGULARITY CRITERIA FOR INCOMPRESSIBLE
NAVIER–STOKES EQUATIONS WITH NAVIER BOUNDARY CONDITIONS
I: SPECIAL DOMAINS

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Abstract. We study the regularity criteria for weak solutions to the 3D incompressible Navier–Stokes equations in terms of the geometry of vortex structures, taking into account the boundary effects. A boundary regularity theorem is proved on regular domains with a class of oblique derivative boundary conditions, providing that the vorticity of the fluid is coherently aligned. In particular, we establish the boundary regularity on round balls, half-spaces and right circular cylindrical ducts, subject to the classical Navier and kinematic boundary conditions.

1. Introduction and Statement of Main Results

We are concerned with the regularity of weak solutions to the 3-dimensional incompressible Navier–Stokes equations on a regular domain \( \Omega \subset \mathbb{R}^3 \):

\[
\begin{align*}
\partial_t u + \text{div} (u \otimes u) - \nu \Delta u + \nabla p &= 0 \quad \text{in } [0, T^*] \times \Omega, \\
\text{div} u &= 0 \quad \text{in } [0, T^*] \times \Omega, \\
\mathbf{u}|_{t=0} &= \mathbf{u}_0 \quad \text{on } \{0\} \times \Omega.
\end{align*}
\]

The fluid boundary \( \partial \Omega =: \Sigma \) is a regular surface. Here \( u : \Omega \to \mathbb{R}^3 \) is the velocity, \( p : \Omega \to \mathbb{R} \) is the pressure, and \( \nu > 0 \) is the viscosity of the fluid. We study the regularity criteria up to the boundary under geometric assumptions on the alignment of vortex structures of the fluid. Eqs. (1.1)-(1.3) are complemented by the classical Navier and kinematic boundary conditions.

Let \( T \in \mathfrak{gl}(3; \mathbb{R}) \) be the Cauchy stress tensor of the fluid in \( \Omega \) (\( \mathfrak{gl}(3, \mathbb{R}) \) is the space of \( 3 \times 3 \) real matrices), defined by

\[
T_{ij} := \nu (\nabla_i u^j + \nabla_j u^i)
\]

for \( i, j \in \{1, 2, 3\} \). (1.4)

Its contraction with the normal vectorfield on \( \Sigma \), known as the Cauchy stress vector \( \mathbf{t} \in \Gamma(T\Sigma) \), describes the stress on the boundary contributed by the fluid from the normal direction:

\[
\mathbf{t}^i := \sum_{j=1}^{3} T_{ij} n^j \quad \text{for } i \in \{1, 2, 3\}.
\]

The Navier boundary condition, first proposed by Navier [31] in 1816, requires the tangential component of the Cauchy stress vector to be proportional to the tangential component of the velocity on \( \Sigma \):

\[
\beta u \cdot \tau + \mathbf{t} \cdot \tau = 0 \quad \text{for each } \tau \in \Gamma(T\Sigma) \text{ on } [0, T^*] \times \Sigma,
\]

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where the constant $\beta > 0$ is known as the slip length of the fluid. Throughout the paper we write $\Gamma(T\Sigma)$ for the space of tangential vectorfields on $\Sigma$. We moreover impose the kinematic or impenetrable boundary condition:

$$u \cdot n = 0 \quad \text{on } [0, T^*] \times \Sigma,$$

(1.7)

where $n$ is the outward unit normal vector field along $\Sigma$. The above choices for domains $\Omega$ and boundary conditions all have physical relevance.

Throughout we say $u$ is a weak solution to the Navier–Stokes equations (1.1)(1.2) if $u \in L^\infty(0, T^*; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0, T^*; H^1(\Omega; \mathbb{R}^3))$ satisfies the equations in the sense of distributions and, additionally, the energy inequality holds:

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u(t, x)|^2 \, dx + \nu \int_\Omega |\nabla u(t, x)|^2 \, dx - c \int_\Sigma |u(t, y)|^2 \, dH^2(y) \leq 0 \quad \text{for each } t \in [0, T^*],$$

(1.8)

where $c$ is a constant depending only on $\Omega$ and $\nu$. The energy inequality was proposed in the classical works by Leray ([25]) and Hopf ([23]) on Eqs. (1.1) (1.2) in $\Omega = \mathbb{R}^3$, where $c = 0$. Here the $c$ term is introduced to account for the boundary conditions; we shall give a justification in Lemma 3.4 in Sect. 3 below.

A weak solution $u$ is said to be a strong solution if it further satisfies

$$\nabla u \in L^\infty(0, T^*; L^2(\Omega; gl(3, \mathbb{R}))) \cap L^2(0, T^*; H^1(\Omega; gl(3, \mathbb{R}))).$$

The above definitions for weak and strong solutions are also adapted for more general boundary conditions in this paper, e.g., the oblique derivative boundary condition (2.2) and the Navier and kinematic boundary conditions (1.6)(1.7).

The main result of the paper is as follows:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^3$ be one of the following domains: a round ball, a half-space, or a right circular cylindrical duct. Assume that the initial data satisfies $u_0 \in H^1(\Omega; \mathbb{R}^3)$ and $\text{div} \, u_0 = 0$. Let $u$ be a weak solution to the Navier–Stokes equations (1.1)(1.2)(1.3) with the Navier and kinematic boundary conditions (1.6)(1.7). Define the turning angle $\theta$ of vorticity $\omega = \nabla \times u$ by

$$\theta(t; x, y) := \angle(\omega(t, x), \omega(t, y)).$$

(1.9)

If the vorticity directions are coherently aligned, i.e., there is a constant $\rho > 0$ so that

$$|\sin \theta(t; x, y)| \leq \rho \sqrt{|x - y|} \quad \text{for all } x, y \in \Omega, \ t < T^*,$$

(1.10)

then $u$ is the strong solution on $[0, T^*]$.

**Remark 1.2.** For $y \in \Sigma$, $\omega(t, y)$ is understood in the sense of trace.
The regularity theory for the incompressible Navier–Stokes equations has long been a central topic in PDE and mathematical hydrodynamics; cf. Constantin–Foias [13], Fefferman [16], Lemarié-Rieusset [24], Temam [37], Seregin [33] and many references cited therein. One major problem in the regularity theory is the regularity criteria for weak solutions: under what conditions can a weak solution be the strong solution. In [12] Constantin and Fefferman first proposed the following geometric regularity criterion: For a weak solution to the Navier–Stokes equations on the whole space $\Omega = \mathbb{R}^3$, if there are constants $\rho, \Lambda > 0$ such that

$$\left| \sin \theta(t; x, y) \right| \mathbb{1}_{\{ |\omega(t,x)| \geq \Lambda, |\omega(t,y)| \geq \Lambda \}} \leq \rho |x - y| \quad \text{for all } x, y \in \mathbb{R}^3, t < T^*, \quad (1.11)$$

then the weak solution is indeed strong ($\theta$ is the turning angle of vorticity as in Theorem 1.1). This result suggests that, if the vortex lines of the fluid are coherently aligned, i.e., without sharp turnings before time $T^*$, then the weak solutions cannot blow up by $T^*$. It opens up the ways for many subsequent works on regularity conditions in terms of the geometry of vortex structures; see Beirão da Veiga–Berselli [5, 6], Beirão da Veiga [7], Chae [9], Li [26], Giga–Miura [19], Grujić [21], Grujić–Ruzmaikina [22], Vasseur [38] and many others. Let us remark that, in [5], Beirão da Veiga–Berselli improved the right-hand side of (1.11) to $\rho \sqrt{|x - y|}$.

In line with the above results, Theorem 1.1 proposes a geometric regularity condition for the weak solutions to the Navier–Stokes equations. As the main new feature of our work, we investigate the regularity theory up to the boundary. Such “geometric boundary regularity conditions” have been studied for only one special boundary condition proposed by Solonnikov–Šcadilov in [36] (also see Xiao–Xin [39]), which agrees with the Navier and kinematic boundary conditions (1.6)(1.7) if and only if $\Omega = \mathbb{R}^3_+$:

$$u \cdot n = 0, \quad \omega \times n = 0 \quad \text{on } [0, T^*] \times \Sigma; \quad (1.12)$$

see Beirão da Veiga [7] for the case of $\Omega = \mathbb{R}^3_+$ and Beirão da Veiga–Berselli [6] for the case of general bounded $C^{3,\alpha}$-domains $\Omega \subset \mathbb{R}^3$. It is crucial to note that, in the latter case, Eq. (1.12) no longer agrees with the Navier and kinematic boundary conditions. Therefore, our work is the first in the literature to prove the geometric boundary regularity under physical (Navier and kinematic) boundary conditions on certain curvilinear domains.

Let us comment on the Navier and kinematic boundary conditions (1.6)(1.7). The kinematic boundary condition requires the fluid motion on the boundary $\Sigma$ to be tangential. That is, $\Sigma$ is impermeable. The Navier boundary condition further describes the tangential motion of the fluid: the velocity is proportional to the tangential component of the Cauchy stress vector $\mathbf{t}$. It was proposed by Navier [31] in 1816 to resolve the incompatibility between the theoretical predictions from the Dirichlet boundary condition ($u = 0$ on $\Sigma$) and the experimental data, and was later investigated by Maxwell (30) in 1879 for rarefied gases. In recent years, the Navier boundary condition has been extensively studied in fluid models when the curvature effect of the boundary becomes considerable. In particular, free capillary boundaries, perforated boundaries or the presence of an exterior electric field may lead to such situations for high-Reynold number flows; cf. Achdou–Pironneau–Valentin [3], Bänsch [11], Einzel–Panzer–Liu [14] and many others for physical and numerical studies, and cf. Berselli–Spirtito [8], Chen–Qian [10], Iftimie–Raugel–Sell [17], Jäger–Mikelić [18], Masmoudi–Rousset [28], Neustupa–Penel [32], Xiao–Xin [39] and the references cited therein for mathematical analyses of the Navier boundary condition.
Our strategy for proving Theorem 1.1 is as follows. First we notice that, by energy estimates (Sect. 3), it suffices to control the vortex stretching term:

\[
\text{[Stretch]} := \left| \int_\Omega S\!u(t,x) : \omega(t,x) \otimes \omega(t,x) \, dx \right|,
\]

where \( S\!u \) is the rate-of-strain tensor, i.e., the symmetrised gradient of \( u \):

\[
S\!u := \frac{\nabla u + \nabla^\top u}{2} : [0, T^*] \times \Omega \to \mathfrak{gl}(3, \mathbb{R}).
\]

Following [12], we represent \( S\!u \) by a singular integral of \( \omega \). We first localise to coordinate charts on \( \Omega \) (cf. Grujić [21]). In the interior charts, the integral kernel looks like the corresponding kernel on \( \mathbb{R}^3 \), whose estimates are obtained by Constantin–Fefferman in [12]. In each boundary chart, thanks to the results by Solonnikov [34, 35], there exists one single Green’s matrix for Laplacian, which can be explicitly constructed by transforming to the model problem — the Poisson equation with oblique derivative boundary conditions on the half space \( \mathbb{R}^3_+ \). With suitable bounds for the term \([\text{Stretch}]\) at hand (these estimates occupy the major part of the paper; see Sect. 4), we can conclude using interpolation and the Grönwall’s lemma. In this process, new estimates are needed to control the boundary terms and the non-trivial geometry of \( \Omega \).

We mention again that the approach in this paper applies to more general boundary conditions than those in Theorem 1.1:

1. The energy estimates in Sect. 3 below are valid for Navier and kinematic boundary on arbitrary regular embedded surfaces in \( \mathbb{R}^3 \);
2. The potential estimates is applicable to the diagonal oblique derivative boundary conditions with constant coefficients (see Sect. 4).

In both (1) and (2), we do not need to impose any restriction on the specific geometry of \( \Omega \).

The remaining parts of the paper is organised as follows:

In Sect. 2 we present Solonnikov’s theory on the Green’s matrices for certain elliptic PDE systems. Next, in Sect. 3 we collect the energy estimates for the Navier–Stokes system (1.1)–(1.2)–(1.6)–(1.7). In Sect. 4 we prove the boundary regularity theorem for the Navier–Stokes equations under the general diagonal oblique derivative conditions. This is achieved by potential estimates based on the theory outlined in Sect. 2. Finally, in Sect. 5 we deduce Theorem 1.1 for the Navier and kinematic boundary conditions as an instance of the results in Sect. 4.

2. Green’s Matrices

In this section we summarise the theory of Green’s matrices for a general family of boundary value problems for the diagonal elliptic PDE systems. It is the foundation of the subsequent developments. For the convenience of exposition we focus only on \( 3 \times 3 \) elliptic systems, although the general theory applies to \( N \times M \) elliptic systems for arbitrary \( N, M \geq 2 \).

2.1. Regular Oblique Derivative Boundary Conditions. Consider the system with the homogeneous boundary conditions:

\[
-\Delta u = f := \nabla \times \omega \quad \text{in} \ [0, T^*] \times \Omega,
\]

\[
(Nu)^i = a^{(i)} u^i + \sum_{j=1}^3 b^{(i)}_{j} \nabla_j u^i = 0 \quad \text{on} \ [0, T^*] \times \partial \Omega \quad \text{for each} \ i = 1, 2, 3,
\]

\[
(Nu)^i = a^{(i)} u^i + \sum_{j=1}^3 b^{(i)}_{j} \nabla_j u^i = 0 \quad \text{on} \ [0, T^*] \times \partial \Omega \quad \text{for each} \ i = 1, 2, 3,
\]

\[
(Nu)^i = a^{(i)} u^i + \sum_{j=1}^3 b^{(i)}_{j} \nabla_j u^i = 0 \quad \text{on} \ [0, T^*] \times \partial \Omega \quad \text{for each} \ i = 1, 2, 3,
\]
where, without loss of generality, we assume
\[ a^{(i)} \leq 0, \quad \sum_{j=1}^{3} \left[ b_{j}^{(i)} \right]^2 = 1 \] (2.3)
for each \( i = 1, 2, 3 \) and \( Nu = \{(Nu)^{i}\}_{i=1}^{3} \), in some local coordinates \( \{x^1, x^2, x^3\} \) near a point \( p \in \Sigma := \partial \Omega \).

The key assumption above is that the boundary conditions (2.2) are diagonal: in suitable coordinates it is decoupled into three scalar equations in \( u_1, u_2 \) and \( u_3 \). This ensures that the Green’s matrices for Eqs. (2.1)(2.2), constructed by Solonnikov ([34, 35]; see below for details), are diagonal. Also, in order to write down the explicit expressions for the Green’s matrices, we require that \( a^{(i)}, b^{(i)} \) are constants for each \( i \in \{1, 2, 3\} \).

Our goal is to represent \( u \) in terms of \( \omega \); in the case of \( \Omega = \mathbb{R}^3 \) and no boundary conditions other than suitable decay at infinity, the above system is solved by the convolution \( u = K_{bs} * \omega \), where \( K_{bs} \) is the classical Biot–Savart kernel.

The system (2.1)(2.2) is known as an oblique derivative problem for the Poisson equation.

2.2. Elliptic Systems and the Existence of Green’s Matrices. In this subsection we outline the theory for the elliptic systems of the Petrovsky type developed by Solonnikov [34, 35]. Our use of Solonnikov’s theory is motivated by [6] by Beirão da Veiga–Berselli; also see Proposition 2.2 in Temam [37].

We consider a \( 3 \times 3 \) linear PDE system
\[ \mathcal{L}_x u := \sum_{j=1}^{3} l_{ij}(x, \nabla) u^j = f_i \quad \text{in } \Omega \subset \mathbb{R}^3 \] (2.4)
which is elliptic in the sense of the ADN theory due to Agmon–Douglis–Nirenberg ([12]). Here \( i, j \in \{1, 2, 3\}, u = (u^1, u^2, u^3), f = (f_1, f_2, f_3) : \Omega \rightarrow \mathbb{R}^3 \) are vectorfields/one-forms on \( \Omega \), and \( \{l_{ij}\} \) is a \( 3 \times 3 \) matrix of differential operators. A family of weights \( \{s_1, s_2, s_3; t_1, t_2, t_3\} \subset \mathbb{Z} \) is associated to the system (2.4), such that
\[ s_i \leq 0 \text{ for each } i, \quad \text{the order of } l_{ij} \leq \max\{0, s_i + t_j\}. \] (2.5)
Then, we set \( l'_{ij}(x, \nabla) \) to be the principal part of \( l_{ij} \), namely the sum of all terms in \( l_{ij}(x, \nabla) \) of order \((s_i + t_j)\), and consider the characteristic matrix \( \{l'_{ij}(x, \xi)\}_{1 \leq i,j \leq 3} \). Then, (2.4) is elliptic if and only if \( s_i, t_j \) satisfying (2.5) exist for every \( x \in \Omega \), and that
\[
\det \left\{ l'_{ij}(x, \xi) \right\} \neq 0 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{2.6}
\]

Now we consider the boundary conditions imposed to the system (2.4). Throughout, \( \Sigma := \partial \Omega \) is a \( C^2 \) surface, and we use \( p \) to denote a typical boundary point on \( \Sigma \). A generic (linear) boundary condition is of the form
\[
\sum_{j=1}^{3} B_{hj}(p, \nabla) u_j(p) = \phi_h(p) \quad \text{on } \Sigma \text{ for } h = 1, 2, \ldots, m, \tag{2.7}
\]
where
\[
m := \frac{1}{2} \deg \det \left\{ l'_{ij}(p, \xi) \right\} > 0, \tag{2.8}
\]
for which the determinant (as in Eq. (2.6)) is viewed as a polynomial in \( \xi \). Similarly, viewing \( B_{hj}(p, \xi) \) as a \( C \)-coefficient polynomial in \( \xi \) (depending on \( p \)), we consider another set of weights \( \{r_1, r_2, \ldots, r_m\} \subset \mathbb{Z} \) such that
\[
\deg \left\{ B_{hj}(p, \xi) \right\} \leq \max \{r_h + t_j, 0\} \tag{2.9}
\]
with \( t_j \) given as above. For any \( p \in \Sigma \), consider \( \Xi \in T_p \Sigma \setminus \{0\} \) and
\[
\tau^+_{h}(p, \Xi) := \text{roots in } \tau \text{ with positive imaginary part of } L_{h}(p, \Xi + \tau \eta) = 0, \quad (2.10)
\]
\[
M^{+}(p, \Xi, \tau) := \prod_{h=1}^{m} \left( \tau - \tau^+_{h}(p, \Xi) \right). \tag{2.11}
\]
We also write \( \{B'_{hj}\} \) for the principal part of \( B_{hj} \), and view \( M^{+}(p, \Xi, \tau) \) as a polynomial in \( \tau \). The boundary condition (2.7) is said to be complementing to the elliptic system (2.4) if, for every \( p \in \Sigma \) and every \( \Xi \in T_p \Sigma \setminus \{0\} \), there exist \( \{r_h\}_{h=1,2,\ldots,m} \) everywhere satisfying (2.9), and
\[
\sum_{h=1}^{m} C_h \sum_{j=1}^{3} B'_{hj} \left\{ \text{adjoint matrix of } L'_{ij}(p, \Xi + \tau \eta) \right\} \equiv 0 \pmod{M^{+}} \iff C_h = 0 \text{ for all } h. \tag{2.12}
\]

All the classical boundary conditions (Dirichlet, Neumann, regular oblique derivative etc., homogeneous or inhomogeneous) are known to be complementing to the Poisson equation.

**Definition 2.2.** Consider the elliptic PDE system (2.4) (2.7) with complementing boundary conditions in the ADN sense, and with weights \( \{s_i, t_j, r_h\} \) as above. If one can choose \( s_i = 0 \) and \( r_h < 0 \) for all \( i \in \{1, 2, 3\} \) and \( h \in \{1, \ldots, m\} \), then (2.4) is said to be of the Petrovsky type.

**Lemma 2.3.** The system (2.1) (2.2) is of the Petrovsky type.

**Proof.** In this case we have \( L = -\Delta \) and \( l'_{ij}(x, \xi) = (\xi^1)^2(\xi^2)^2(\xi^3)^2 \), hence \( m = 3 \). Using \( N = \{B_{hj}\}_{1 \leq h, j \leq 3} \) in (2.2), we can pick \( s_1 = s_2 = s_3 = 0 \), \( t_1 = t_2 = t_3 = 2 \) and \( r_1 = r_2 = r_3 = -1 \). \( \square \)

Therefore, in view of Solonnikov’s theory on the existence of Green’s matrices for Petrovsky-type elliptic systems (cf. p126, [55] and p606, [4]), we may deduce:

**Lemma 2.4.** A matrix field \( \{G_{ij}\}_{1 \leq i, j \leq 3} : \Omega \times \Omega \to \mathbb{R} \) exists for the system (2.1) (2.2) such that
\[
u^i(x) = \sum_{j=1}^{3} \int_{\Omega} G_{ij}(x, y) f^j(y) \, dy \quad \text{for each } i = 1, 2, 3. \tag{2.13}
\]
Moreover, we have the decomposition \( \mathcal{G} = \mathcal{G}^{\text{good}} + \mathcal{G}^{\text{bad}} \), where

\[
\exists C_{\text{bad}} > 0 : \left| \nabla_x^\alpha \nabla_y^\beta \mathcal{G}^{\text{bad}}(x, y) \right| \leq \frac{C_{\text{bad}}}{|x - y|^{|\alpha|+|\beta|+1}} \quad \text{for all } x \neq y \in \Omega, \tag{2.14}
\]

and

\[
\exists C_{\text{good}} > 0, \delta > 0 : \left| \nabla_x^\alpha \nabla_y^\beta \mathcal{G}^{\text{good}}(x, y) \right| \leq \frac{C_{\text{good}}}{|x - y|^{|\alpha|+|\beta|+1-\delta}} \quad \text{for all } x \neq y \in \Omega, \tag{2.15}
\]

for any multi-indices \( \alpha, \beta \in \mathbb{N}^N \). For \( \Sigma = \partial \Omega \) sufficiently regular, one can take \( \delta > 1/2 \).

In the lemma above, \( \mathcal{G} \) is known as the Green’s matrix for the oblique derivative boundary value problem for the Poisson equation \((2.1)(2.2)\). The crucial point is that the solution can be represented by one single matrix. Moreover, under our assumption that the boundary conditions are diagonal (decoupled), we know that \( \mathcal{G} \) is a diagonal matrix, namely

\[
\mathcal{G}_{ij}(x, y) = g(x, y) \delta_{ij} \tag{2.16}
\]

for a scalar function \( g : \Omega \times \Omega \to \mathbb{R} \). In this case

\[
u_i(x) = \int_{\Omega} g(x, y) f^i(y) \, dy,
\]

so we can carry out potential estimates for the corresponding scalar functions. Thus we can resort to well-developed theories in PDE; cf. Gilbarg–Trudinger [20].

### 2.3. Diagonality of the Boundary Conditions.

Now, let us discuss the system \((2.1)(2.2)\) on the half space \( \mathbb{R}^3_+ := \{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^3 > 0 \} \), namely

\[
\begin{align*}
-\Delta u &= f \quad \text{in } \mathbb{R}^3_+, \tag{2.17} \\
\mathcal{N} u^i &= a^{(i)} u^i + \sum_{j=1}^3 b^{(i)}_{j} \nabla_j u^i = 0 \quad \text{on } \{x^3 = 0\} \text{ for each } i = 1, 2, 3, \tag{2.18}
\end{align*}
\]

where \( a^{(i)}, b^{(i)} \) are constants. In Sect. 4 below we shall localise \((2.1)(2.2)\) so that, in each chart near the boundary \( \Sigma \), the system “looks like” the above model system \((2.17)(2.18) \) (i.e., modulo certain linear transforms which can be nicely controlled).

For each \( y \in \mathbb{R}^3_+ \) let us write:

\[
y = (y^1, y^3) \quad \text{where } y' = (y^1, y^2), \quad y^* := (y', -y^3).
\]

That is, \( y^* \) is the reflected point (the “virtual charge”) across the boundary \( \{x^3 = 0\} \). We use \( \langle \cdot, \cdot \rangle \) to denote the Euclidean inner product. Also, for \( x, y \in \mathbb{R}^3 \) we write

\[
\Gamma(x, y) := \frac{1}{|x - y|}, \tag{2.19}
\]

namely the fundamental solution to the Laplace equation in \( \mathbb{R}^3 \) (up to a multiplicative constant). One also denotes by

\[
\xi := \frac{x - y^*}{|x - y^*|} \quad \text{for } x, y \in \mathbb{R}^3_+. \tag{2.20}
\]

Then, following Sect. 6.7 in Gilbarg–Trudinger [20], the Green’s matrix \( \{\mathcal{G}_{ij}\} \) for the model problem \((2.17)(2.18)\) takes the following explicit form:

\[
\mathcal{G}_{ij}(x, y) = \frac{\delta_{ij}}{4\pi} \left\{ \Gamma(x - y) - \Gamma(x - y^*) - \frac{2b^{(i)}_j}{3|x - y^*|} \Omega^{(i)}(x, y^*) \right\}, \tag{2.21}
\]
where for each $i = 1, 2, 3$,

$$
\Theta^{(i)}(x, y^*) := \int_0^\infty \left\{ e^{a^{(i)}|x-y^*|s} \frac{\xi_s + b_3^{(i)}s}{[1 + 2(b^{(i)}, \xi)s + s^2]^{3/2}} \right\} \, ds.
$$

(2.22)

In fact, later (in Lemma 4.9) we shall check that $\Theta^{(i)}$ is smooth in $(x, y)$.

The Green’s matrix in Eq. (2.21) reduces to that for the Dirichlet boundary condition $(u|^{\Sigma} = 0)$ when the third term on the right-hand side is not present. (Recall that $b_3^{(i)} \neq 0$ for some $i$ if the oblique boundary condition is regular). The above representation formulae (2.21) (2.22) are the starting point of our subsequent estimates.

3. Basic Energy Estimates

Here we collect some energy estimates for the Navier–Stokes Eqs. (1.1) (1.2), subject to the Navier and kinematic boundary conditions (1.6) (1.7); cf. e.g., [10] by Chen–Qian and many other references.

Let us first fix some notations: for $a, b \in \mathbb{R}^3$, we write

$$
a \otimes b = \{a \otimes b\}_{ij} \in \mathfrak{gl}(3, \mathbb{R}), \quad (a \otimes b)_{ij} := a^i b^j \text{ for } i, j \in \{1, 2, 3\};
$$

and for $A, B \in \mathfrak{gl}(3, \mathbb{R})$, write

$$
A : B := \text{Trace} (AB), \quad |A| := \sqrt{\sum_{i,j=1}^3 |A_{ij}|^2}.
$$

We also need the following geometric quantities:

$$
\Pi := -\nabla n : \Gamma(T^*\Sigma) \times \Gamma(T\Sigma) \to \Gamma(T\Sigma^\perp)
$$

(3.1)

is the second fundamental form of $\Sigma$, and

$$
H_\Sigma := \text{Trace} (\Pi)
$$

(3.2)

is the mean curvature of $\Sigma$. The metric on $\Sigma$ (with respect to which we are taking the trace) is the pullback of the Euclidean metric via the inclusion $\Sigma \hookrightarrow \mathbb{R}^3$. We use $\Gamma(T^*\Sigma)$ to denote the space of vectorfields tangential to $\Sigma$, and $\mathcal{H}^2$ to denote the 2-dimensional Hausdorff measure on $\Sigma$. Whenever the estimates are kinematic, i.e., valid pointwise in time, we suppress the variable $t$ for notational convenience.

To begin with, we take the gradient of Eq. (1.1) and anti-symmetrise it. This yields the vorticity equation:

$$
\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + Su \cdot \omega
$$

(3.3)

in $[0, T^*] \times \Omega$. In the sequel, for $\omega$ and $u$ to satisfy the Navier boundary condition (1.6), we understand (1.6) in the sense of trace. In particular, let us impose the following, which shall be taken as part of the definition for the weak solutions to the system (1.1) (1.2) (1.6) (1.7).

Assumption 3.1. Both the tangential and the normal traces of $\omega$ on $\Sigma = \partial \Omega$ exist. The incompressibility condition $\nabla \cdot u = 0$ holds on $\Sigma$, also in the sense of trace.

Let us establish several energy estimates for the strong solution. First, note that the $L^2$-norm of $\nabla u$ can be bounded by the $L^2$-norms of $u$ and $\omega = \nabla \times u$.
Lemma 3.2. Let $u$ be the strong solution to Eqs. (1.1) (1.2) (1.6) (1.7) on $[0, T^*] \times \Omega$. Then, for all $t \in [0, T^*]$, 
\[
\int_{\Omega} |\nabla u|^2 \, dx \leq \|\Pi\|_{L^2(\Sigma)} \int_{\Omega} |u|^2 \, dx + \int_{\Omega} |\omega|^2 \, dx. \tag{3.4}
\]
Proof. See (3.4), p.728 in [10]. □

The next result concerns the growth of enstrophy, namely the square of the $L^2$-norm of vorticity. One may compare it with Lemma 2.6 in [6] (recall that the vorticity stretching term [Stretch] is defined in Eq. (1.13)):

Lemma 3.3. Let $u$ be the strong solution to Eqs. (1.1) (1.2) (1.6) (1.7) on $[0, T^*] \times \Omega$. Then there exists a constant $c_0$ depending only on $\beta, \nu$ and $\|\Pi\|_{C^1(\Sigma)}$ such that for each $t \in [0, T^*]$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\omega|^2 \, dx + \nu \int_{\Omega} |\nabla \omega|^2 \, dx - c_0 \int_{\Sigma} (|\nabla u|^2 + |u|^2) \, d\mathcal{H}^2 \leq [\text{Stretch}]. \tag{3.5}
\]
Proof. We divide our arguments into four steps.

1. First, multiplying $\omega$ to the vorticity equation (3.3), we get
\[
\partial_t(|\omega|^2) + u \cdot \nabla(|\omega|^2) - \nu \Delta(|\omega|^2) + 2 \nu |\nabla \omega|^2 = 2 \mathcal{S} u : (\omega \otimes \omega). \tag{3.6}
\]
By Eq. (1.2) and the divergence theorem,
\[
\int_{\Omega} u \cdot \nabla(|\omega|^2) \, dx = \int_{\Sigma} |\omega|^2 u \cdot n \, d\mathcal{H}^2,
\]
which vanishes due to the kinematic boundary condition (1.7). On the other hand, by the divergence theorem again, we have
\[
\int_{\Omega} \Delta(|\omega|^2) \, dx = \int_{\Sigma} \partial_n |\omega|^2 \, d\mathcal{H}^2 = 2 \int_{\Sigma} \omega \cdot \partial_n \, d\mathcal{H}^2,
\]
where $\partial_n := n \cdot \nabla$. In view of Eq. (3.6) and the triangle inequality, it remains to establish
\[
\left| \int_{\Sigma} \omega \cdot \partial_n \, d\mathcal{H}^2 \right| \leq \frac{\nu}{2} \int_{\Omega} |\nabla \omega|^2 \, dx + c_0 \int_{\Sigma} (|\nabla u|^2 + |u|^2) \, d\mathcal{H}^2. \tag{3.8}
\]

2. To deal with the last term in (3.7), one utilises the Navier boundary condition (1.6). Take an arbitrary orthonormal frame $\{\partial/\partial x^i\}$ on $\mathbb{R}^3$, and assume $\tau = \partial/\partial x^k$ is a tangential vectorfield to $\Sigma$; then
\[
0 = \beta u^k + \frac{3}{i=1} \nu(\nabla_i u^k + \nabla_k u^i) n^i
= \beta u^k + \frac{3}{i=1} \left( \nu(-\nabla_k u^i + \nabla_i u^k) n^i + 2 \nu (\nabla_k u^i) n^i \right)
= \beta u^k + \nu \sum_{i,j=1}^3 \epsilon^{ikl} \omega^l + 2 \nu \nabla_k (\sum_{i=1}^3 u^i n^i) - 2 \nu \sum_{i=1}^3 u^i \nabla_k n^i
\]
for any $i \in \{1, 2, 3\}$. Thanks to $u \cdot n = 0$ and the definition of the second fundamental form, we obtain an equivalent formulation of the Navier boundary condition as follows:
\[
0 = \beta u \cdot \tau + \nu (\omega \times n) \cdot \tau - 2 \nu \Pi(u, \tau) \quad \text{on } \Sigma \text{ for each } \tau \in \Gamma(T\Sigma). \tag{3.9}
\]
Moreover, if we decompose $\omega$ into tangential and normal components:

$$\omega := \omega^\parallel + \omega^\perp$$

for $\omega^\parallel \in \Gamma(T\Sigma)$, $\omega^\perp \in \Gamma(T\Sigma^\perp)$, then $\omega^\parallel$ can be pointwise controlled by $u$ and the geometry of $\Sigma$:

$$|\omega^\parallel| \leq \left( \beta \nu^{-1} + 2\|\Pi\|_{L^\infty(\Sigma)} \right) |u|.$$  

(3.11)

3. Now let us estimate

$$2 \int_\Sigma \omega \cdot \frac{\partial \omega^\parallel}{\partial n} d\mathcal{H}^2 = 2 \int_\Sigma \left\{ \omega^\parallel \cdot \frac{\partial \omega^\parallel}{\partial n} + \omega^\perp \cdot \frac{\partial \omega^\perp}{\partial n} + \omega^\parallel \cdot \frac{\partial \omega^\perp}{\partial n} + \omega^\perp \cdot \frac{\partial \omega^\parallel}{\partial n} \right\} d\mathcal{H}^2$$

(3.12)

in Eq. (3.7). For the first two terms, let us use Eq. (3.9) to derive that

$$\nabla \omega^\parallel = L(\nabla u, \nabla \Pi u, \Pi \nabla u),$$

(3.13)

where the schematic tensor $L(X_1, X_2, \ldots)$ denotes a linear combination of $X_1, X_2, \ldots$ with coefficients depending only on $\beta, \nu$, and $X \ast Y$ denotes a generic quadratic term in $X, Y$ with constant coefficients. Thus, we have the pointwise estimate

$$|\nabla \omega^\parallel| \leq C(|\nabla u| + |u|),$$

(3.14)

where $C$ depends only on $\|\Pi\|_{C^1}, \beta$ and $\nu$. We can bound

$$\left| \int_\Sigma \left\{ \omega^\parallel \cdot \frac{\partial \omega^\parallel}{\partial n} + \omega^\perp \cdot \frac{\partial \omega^\parallel}{\partial n} \right\} d\mathcal{H}^2 \right| \leq C \int_\Sigma \left( |\omega^\parallel| \nabla u | + |\omega^\parallel| u \right) d\mathcal{H}^2 \leq C \left\{ 2 \int_\Sigma |\nabla u|^2 d\mathcal{H}^2 + \int_\Sigma |u|^2 d\mathcal{H}^2 \right\},$$

(3.15)

using Eq. (3.14) and Cauchy–Schwarz, with the constant $C = C(\|\Pi\|_{C^1}, \beta, \nu, \Sigma)$.

For the third term, notice that

$$\int_\Sigma \omega^\parallel \cdot \frac{\partial \omega^\perp}{\partial n} d\mathcal{H}^2 = -\int_\Sigma \omega^\perp \cdot \frac{\partial \omega^\parallel}{\partial n} d\mathcal{H}^2 + \int_\Sigma n \cdot \nabla (\omega^\parallel \cdot \omega^\perp) d\mathcal{H}^2 = -\int_\Sigma \omega^\perp \cdot \frac{\partial \omega^\parallel}{\partial n} d\mathcal{H}^2.$$  

Just as above, we get

$$\left| \int_\Sigma \omega^\parallel \cdot \frac{\partial \omega^\perp}{\partial n} d\mathcal{H}^2 \right| \leq C \int_\Sigma \left( |\nabla u|^2 + |u|^2 \right) d\mathcal{H}^2.$$  

(3.16)

4. To control the remaining term $\int_\Sigma \omega^\perp \cdot (\partial \omega^\parallel / \partial n) d\mathcal{H}^2$, let us first establish a simple claim: For any vertical vector field $\eta \in \Gamma(T\Sigma^\perp)$, there holds

$$\frac{1}{2} n \cdot \nabla (|\eta|^2) - (\nabla \cdot \eta)(\eta \cdot n) = H_\Sigma (|\eta|^2).$$

(3.17)

Indeed, write $\eta = \phi n$ for some scalar function $\phi : \Sigma \to \mathbb{R}$. Then

$$\frac{1}{2} n \cdot \nabla (|\eta|^2) - (\nabla \cdot \eta)(\eta \cdot n) = \sum_{i,j=1}^3 \left( \eta^i n^i \nabla_i \eta^j - (\nabla_i \eta^j)(\eta^i n^j) \right) = \phi^2 \sum_{i,j=1}^3 (n^i n^i \nabla_i \eta^j - \nabla_i \eta^j) = H_\Sigma \phi^2,$$

where the last equality follows from $|n| = 1$ and the definition of mean curvature. As a side remark, this claim gives a geometric interpretation to the boundary term in the case of the “slip-type” boundary condition $\omega \times n = 0$ as in Lemma 2.6, [3].
In the above claim let us take $\eta = \omega^\perp$. Thanks to the incompressibility of $\omega$, we have $\nabla \cdot \omega = -\nabla \cdot \omega^\perp$; thus,
\[
\omega^\perp \cdot \frac{\partial \omega^\perp}{\partial n} = -(\nabla \cdot \omega^\perp) |\omega^\perp| + H_{\Sigma} |\omega^\perp|^2.
\] (3.18)

Therefore, using Eq. (3.14) again and arguing as in (3.15), one obtains
\[
\left| \int_{\Sigma} \omega^\perp \cdot \frac{\partial \omega^\perp}{\partial n} \, d\mathcal{H}^2 \right| \leq C \int_{\Sigma} (|\nabla u|^2 + |u|^2) \, d\mathcal{H}^2.
\] (3.19)

Finally, we put together Eqs. (3.12), (3.15), (3.16), (3.19) to complete the proof. \hfill \Box

The lemma below justifies the energy inequality (1.8) in the definition of weak solutions:

**Lemma 3.4.** Let $u$ be the strong solution to Eqs. (1.1), (1.2), (1.6), (1.7) on $[0, T] \times \Omega$. There exists a constant $c_1 > 0$ depending only on $\beta, \nu$ and $\|\Sigma\|_{L^\infty(\Omega)}$ such that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 \, dx + \nu \int_{\Omega} |\nabla u|^2 \, dx - c_1 \int_{\Sigma} |u|^2 \, d\mathcal{H}^2 \leq 0
\] (3.20)
for each $t \in [0, T^*]$.

**Proof.** This follows from standard energy estimates. Multiplying $u$ to the Navier–Stokes equations (1.1), (1.2) and integration by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 \, dx + \nu \int_{\Omega} |\nabla u|^2 \, dx - \nu \int_{\Sigma} u \cdot \frac{\partial u}{\partial n} \, d\mathcal{H}^2 = 0.
\] (3.21)

To estimate the last term, let $\{\partial/\partial x^i\}_{i=1}^3$ be an arbitrary local orthonormal frame on $\mathbb{R}^3$; then
\[
u \int_{\Sigma} u \cdot \frac{\partial u}{\partial n} \, d\mathcal{H}^2 = \sum_{i,j=1}^3 u^i n^j \nabla_j u^i = \sum_{i,j=1}^3 \left( u^i n^j (\nabla_j u^i - \nabla_i u^j) + u^i n^j \nabla_i u^j \right)
\]
\[
= \sum_{i,j,k=1}^3 \left( \epsilon^{kji} u^i n^k \omega^j + u^i \nabla_i (u^j n^j) - u^j \nabla_i u^i \right)
\]
\[
= u \cdot (\omega \times n) + u \cdot \nabla (u \cdot n) + II(u, u).
\]
In view of the incompressibility of $u$ and that $\omega \times n = \omega^\parallel \times n$, we have
\[
\left| \int_{\Sigma} u \cdot \frac{\partial u}{\partial n} \, d\mathcal{H}^2 \right| \leq \int_{\Sigma} \left( |u||\omega^\parallel| + ||II||_{L^\infty(\Sigma)} |u|^2 \right) \, d\mathcal{H}^2.
\] (3.22)

But $|\omega^\parallel|$ can be estimated by $|u|$ as in Eq. (3.11); by (3.21), we may thus take
\[
c_1 := \beta + 3\nu ||II||_{L^\infty(\Sigma)}
\]
to complete the proof. \hfill \Box

Several bounds can be deduced immediately from Lemmas 3.2, 3.3 and 3.4. First, by the trace inequality
\[
\frac{1}{2} \int_{\Sigma} |u|^2 \, d\mathcal{H}^2 \leq \frac{\nu}{2} \int_{\Omega} |\nabla u|^2 \, dx + c_2 \int_{\Omega} |u|^2 \, dx,
\]
Lemma 3.4 implies
\[
\frac{d}{dt} \int_{\Omega} |u|^2 \, dx + \nu \int_{\Omega} |\nabla u|^2 \, dx \leq 2c_2 \int_{\Omega} |u|^2 \, dx,
\] (3.23)
where $c_2$ depends on $c_1, \Omega$ and $\nu^{-1}$. Then, thanks to Grönwall’s lemma, one has
\begin{equation}
\|u(t, \cdot)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} e^{c_2 t} \quad \text{for each } t \in [0, T_*].
\end{equation}
(3.24)
Thus,
\begin{equation}
\|\nabla u(t, \cdot)\|_{L^2(\Omega)} \leq \sqrt{2c_2/\nu} \|u_0\|_{L^2(\Omega)} e^{c_2 t} \quad \text{for each } t \in [0, T_*].
\end{equation}
(3.25)
Next, applying again the trace inequality to Lemma 3.3 yields that, for any given $\delta > 0$,
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\omega|^2 \, dx + \frac{\nu}{2} \int_{\Omega} |\nabla \omega|^2 \, dx \
\leq \text{[Stretch]} + \delta \int_{\Omega} |\nabla \nabla u|^2 \, dx + c_3 \int_{\Omega} |\nabla \omega|^2 \, dx + c_3 \int_{\Omega} |\omega|^2 \, dx + c_3 \int_{\Omega} |u|^2 \, dx.
\end{equation}
Here $c_3$ depends on $c_0$ and $\Omega$. By (3.24) (3.25), the last three terms on the right-hand side are bounded by a constant $c_4 = C(c_3, c_2, \nu, \|u_0\|_{L^2(\Omega)}, T_*, \delta)$. Moreover, we have
\begin{proposition}
Let $u$ be the strong solution to Eqs. (1.1) (1.2) (1.6) (1.7) on $[0, T_*] \times \Omega$. There exists $c_5$ depending only on $\Omega$ such that
\begin{equation}
\int_{\Omega} |\nabla \nabla u|^2 \, dx \leq c_5 \left( \int_{\Omega} |\nabla \omega|^2 \, dx + \int_{\Omega} |\omega|^2 \, dx + \int_{\Omega} |u|^2 \, dx \right).
\end{equation}
(3.26)
\end{proposition}

Proof. See Theorem 3.3, p.729 in [10].

Therefore, choosing $\delta := \nu/(4c_3)$ and invoking once more (3.24) (3.25), one may conclude:

\begin{theorem}[Energy Estimate]
Let $u$ be the strong solution to Eqs. (1.1) (1.2) (1.6) (1.7) on $[0, T_*] \times \Omega$. There is a constant $M$ depending on $\Omega, \beta_1, \|\Pi\|_{L^1(\Sigma)}, \nu, \|u_0\|_{L^2(\Omega)}$ and $T_*$, such that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\omega|^2 \, dx + \frac{\nu}{4} \int_{\Omega} |\nabla \omega|^2 \, dx \leq \text{[Stretch]} + M.
\end{equation}
(3.27)
The vorticity stretching term $\text{[Stretch]}$ is defined in Eq. (1.13). Moreover, the supremum of $\|u(t, \cdot)\|_{W^{1,2}(\Omega)}$ is bounded on $[0, T_*]$ by (3.23) (3.24).
\end{theorem}

4. THE GEOMETRIC BOUNDARY REGULARITY CRITERION

In this section we prove Theorem 4.1 below. It is a generalisation of Theorem 3.6 (see Sect. 3), with the more general diagonal oblique derivative boundary condition (2.2) considered on arbitrary regular curvilinear domains rather than the Navier and kinematic boundary conditions (1.6) (1.7) on round balls, half-spaces and right cylinders.

\begin{theorem}
Let $\Omega \subset \mathbb{R}^3$ be a regular domain. Let $u$ be a weak solution to the Navier–Stokes equations (1.1) (1.2) on $[0, T_*] \times \Omega$ with the regular oblique derivative boundary condition (2.2).
Assume that the energy estimate in Theorem 3.6 is valid for strong solutions. Then, under the assumptions of Theorem 4.1, i.e., if the vorticity turning angle $\theta$ satisfies
\begin{equation}
|\sin \theta(t; x, y)| \leq \rho \sqrt{|x - y|} \quad \text{for all } t \in [0, T_*], x, y \in \Omega,
\end{equation}
(4.1)
for some $\rho > 0$, then $u$ is strong on $[0, T_*] \times \Omega$.
\end{theorem}

To prove Theorem 4.1 in Sect. 4.1 we first localise the problem to small coordinates charts in the interior or near the boundary. The key is to estimate the vortex stretching term $\text{[Stretch]}$, which is carried out in Sects. 4.2–4.10. Finally, we conclude the proof in Sect. 4.11, thanks to the preliminary energy estimates obtained in Sect. 3.
The general strategy for the proof is based on the continuation argument as follows. By the definition of weak/strong solutions, we know that
\[
\limsup_{t \to T} \int_{\Omega} |\omega(t)|^2 \, dx = \infty,
\]
is a breakdown criterion for the strong solution. That is, a weak solution \(u\) on \([0,T]\) cannot be strong beyond the time \(T\) if the above quantity blows up. Therefore, we assume that \(u\) is a strong solution on \([0,T]\) for some \(T \leq T^*\), and that \(T\) is the maximal lifespan of \(u\). Utilising the energy estimate in Theorem 3.6 and the bound for [Stretch] in the current section, we prove that such a blowup shall not occur. Thus, \(u\) is strong on \([0, T + \delta]\) for some \(\delta > 0\), which contradicts the maximality of \(T\). Therefore, \(u\) is strong all the way up to \(T^*\).

4.1. Localisation. We adopt Solonnikov’s method of localisation in the construction of Green’s matrices; see p.150, [34] and p.609, [6]. For the convenience of the readers, let us briefly summarise the construction in four steps below:

1. There exists a finite family of open cover for \(\overline{\Omega}\), written as
   \[
   \{U_a\}_{a \in \mathcal{I}} \cup \{U_b\}_{b \in \mathcal{B}},
   \]
   where \(U_a \cap \Sigma = \emptyset\) for each \(a \in \mathcal{I}\), and \(U_b \cap \Sigma \neq \emptyset\) for each \(b \in \mathcal{B}\). Each \(U_a\) is known as an interior chart, and each \(U_b\) as a boundary chart.

2. Each interior chart is a cube: there exists \(d_1 > 0\) (independent of \(a \in \mathcal{I}\)) such that
   \[
   U_a = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : |x^i - x^i_a| \leq d_1\}\quad \text{for some } x_a \in \mathbb{R}^3,
   \]
   which also satisfies
   \[
   \text{dist} \,(U_a, \Sigma) \geq d_1.
   \]

3. In each boundary chart \(U_b\), we can find a boundary point \(x_b \in \Sigma\), a local Euclidean coordinate system \(\{z^1_b, z^2_b, z^3_b\}\), and a \(C^2\) map \(F_b : [0,d_2]^2 \to \mathbb{R}\) such that
   \[
   |z^1_b|, |z^2_b| \leq d_2, \quad 0 \leq z^3_b - F_b(z^1_b, z^2_b) \leq 2d_2
   \]
   for some constant \(d_2 > 0\) independent of \(b \in \mathcal{B}\), and that the portion of the boundary \(\Sigma \cap U_b\) in \(z_b\) coordinates is the graph of \(F_b\).

4. Let \(\{\chi_a\}_{a \in \mathcal{I}} \cup \{\chi_b\}_{b \in \mathcal{B}}\) be a \(C^\infty\) partition of unity subordinate to the cover in Step 1. That is, \(0 \leq \chi_c \leq 1\), \(\chi_c \in C^\infty(\overline{\Omega})\), \(\sum_{c \in \mathcal{I} \cup \mathcal{B}} \chi_c(x) = 1\) for each \(x \in \overline{\Omega}\) and \(\text{spt} \,(\chi_c) \subseteq U_c\) for each \(c \in \mathcal{I}\) or \(\mathcal{B}\).

With the help of the above steps, we can now localise the Green’s matrices. Indeed, in Step 3 above let us further introduce the notations:
\[
z_b := \mathcal{O}_b(x - x_b) \quad \text{for } \mathcal{O}_b \in SO(3),
\]
\[
(z')_b^1, (z')_b^2, (z')_b^3 := (z^1_b, z^2_b, z^3_b - F_b(z^1_b, z^2_b)) \equiv \bar{F}_b(z_b)
\]
and
\[
T_b(x) := \bar{F}_b \circ \mathcal{O}_b(x - x_b).
\]
That is, \(\mathcal{O}_b\) is the rotation of Euclidean coordinates, and \(\bar{F}_b \in C^2(U_b; [0, d_2]^2 \times [0, 2d_2])\) is the boundary straightening map, which satisfies
\[
T_b(\Sigma \cap U_b) \subset \{(z')^3_b = 0\}.
\]
Then, setting
\[ d_3 := \min\{d_1, d_2\}, \tag{4.11} \]
we can compute \( u(x) \) from the following integral formula (comparing with Eq. (29), [6], p.610):

**Lemma 4.2.** Let \( u \) be the strong solution to Eqs. (2.1)(2.2). Fix a cut-off function \( \zeta \in C_c^\infty(\mathbb{R}) \) such that \( 0 \leq \zeta \leq 1 \), \( \zeta \) is non-increasing on \( \mathbb{R} \), \( \zeta \equiv 1 \) on \([0,1/4] \), \( \zeta \equiv 0 \) on \([3/4,\infty) \) and \( \|\zeta\|_{C^6(\mathbb{R})} \leq 4 \). Then we have

\[
\begin{align*}
u^i(x) & = \sum_{j=1}^{3} \sum_{a \in \mathcal{A}} \int_{\Omega} \chi_a(y) \left\{ \frac{\delta_{ij}}{4\pi|x-y|} (\nabla \cdot \omega)^j(y) \right\} \zeta \left( \frac{|x-y|}{d_3} \right) dy \\
& \quad + \sum_{j=1}^{3} \sum_{b \in \mathcal{B}} \int_{\Omega} \chi_b(y) \delta_{ij} \left\{ \frac{1}{|T_{b,x} - T_{b,y}|} - \frac{1}{|T_{b,x} - (T_{b,y})^*|} \right\} \\
& \quad \times \left( 1 + \frac{2b_3^{(i)}}{3} \Theta^{(i)}(T_{b,x},(T_{b,y})^*) \right) (\nabla \cdot \omega)^j(y) \zeta \left( \frac{|T_{b,x} - T_{b,y}|}{d_3} \right) dy \\
& \quad + \sum_{j=1}^{3} \int_{\Omega} G_{\text{good}}^{ij}(x,y)(\nabla \cdot \omega)^j(y) dy \\
= & : J_1(x) + J_2(x) + J_3(x), \tag{4.12}
\end{align*}
\]

where \( G_{\text{good}}^{ij} \) satisfies the estimate in Eq. (2.15), and \( \Theta^{(i)} \) is given by Eq. (2.22).

**Proof.** As Eqs. (2.1)(2.2) are an elliptic PDE system of Petrovsky type, by Lemma 2.4 we can find a single Green’s matrix \( G \) such that

\[
u^i(x) = \sum_{a \in \mathcal{A}} \sum_{j=1}^{3} \int_{\Omega} G_{ij}(x,y) \chi_a(y)(\nabla \cdot \omega)^j(y) dy + \sum_{b \in \mathcal{B}} \sum_{j=1}^{3} \int_{\Omega} G_{ij}(x,y) \chi_b(y)(\nabla \cdot \omega)^j(y) dy
\]

\[
= u^i_{\text{int}}(x) + u^i_{\text{bdry}}(x), \tag{4.13}
\]

where \( \{\chi_a\}_{a \in \mathcal{A}} \uplus \{\chi_b\}_{b \in \mathcal{B}} \) is the aforementioned partition-of-unity.

For \( u^i_{\text{int}}(x) \), let us decompose each of its summands as

\[
u^i_{\text{int, near}}(x) + u^i_{\text{int, far}}(x) = \int_{\Omega} G_{ij}(x,y) \chi_a(y)(\nabla \cdot \omega)^j(y) \left\{ \zeta \left( \frac{|x-y|}{d_3} \right) \right\} dy \\
+ \int_{\Omega} G_{ij}(x,y) \chi_a(y)(\nabla \cdot \omega)^j(y) \left\{ 1 - \zeta \left( \frac{|x-y|}{d_3} \right) \right\} dy. \tag{4.14}
\]

The non-zero contribution to \( u^i_{\text{int, near}}(x) \) comes from \( \{y \in U_a : |y-x| \leq 3d_3/4\} \), which is uniformly away from the boundary \( \Sigma \). Thus

\[
G_{ij}(x,y)1_{\{y \in U_a : |y-x| \leq 3d_3/4\}} \equiv \frac{\delta_{ij}}{4\pi} \Gamma(x,y) + G_{ij}^{\text{good}}(x,y), \tag{4.15}
\]

where the leading term \( \frac{\delta_{ij}}{4\pi} \Gamma(x,y) \) is the Green’s matrix on \( \mathbb{R}^3 \), and the error term \( G_{ij}^{\text{good}} \) satisfies (2.15) (the explicit form of \( G_{ij}^{\text{good}} \) may differ from line to line, though). On the other hand, the non-zero contribution to \( u^i_{\text{int, far}}(x) \) comes only from \( \{y \in U_a : |y-x| > d_3/4\} \), but the Green’s matrix \( G_{ij} \) is smooth away from the diagonal \( \{x = y\} \subset \mathbb{R}^3 \times \mathbb{R}^3 \). That is,

\[
G_{ij}(x,y)1_{\{y \in U_a : |y-x| > d_3/4\}} = G_{ij}^{\text{good}}(x,y). \tag{4.16}
\]

For the boundary term \( u^i_{\text{bdry}}(x) \), we apply the boundary-straightening map \( T_b \) in each boundary chart; cf. Eq. (4.12). Indeed, for each \( x \in U_b, b \in \mathcal{B} \), arguments analogous to those for
the \( u_{\text{int}}^i(x) \) term show that

\[
    u_{\text{dry}}^i(x) = \sum_{b \in B} \sum_{j=1}^{3} \int_{\Omega} G_{ij}(x, y) \chi_b(y)(\nabla \times \omega)^j(y) \zeta \left( \frac{|x - y|}{d_3} \right) \, dy + \sum_{j=1}^{3} \int_{\Omega} G_{ij}^\text{good}(x, y)(\nabla \times \omega)^j(y) \, dy. \tag{4.17}
\]

We further claim that

\[
    \int_{\Omega} G_{ij}(x, y) \chi_b(y)(\nabla \times \omega)^j(y) \zeta \left( \frac{|x - y|}{d_3} \right) \, dy
    = \int_{\Omega} G_{ij}(T_b x, T_b y) \chi_b(y)(\nabla \times \omega)^j(y) \zeta \left( \frac{|T_b x - T_b y|}{d_3} \right) \, dy + \int_{\Omega} G_{ij}^\text{good}(x, y)(\nabla \times \omega)^j(y). \tag{4.18}
\]

Indeed, by the definition of \( T_b \) we have

\[
    \nabla T_b(x) = \begin{bmatrix} 1 & 0 & 0 \\ -\nabla_1 F_b & 1 & 0 \end{bmatrix} \cdot \mathcal{O}_b(x - x_b), \quad \text{where } \mathcal{O}_b \in O(3). \tag{4.19}
\]

So \( \det(\nabla T_b) = 1 \); thus \( \nabla T_b(\cdot) \in O(3) \) modulo a translation in \( \mathbb{R}^3 \). It means that the boundary-straightening map \( T_b \) is almost a Euclidean isometry. Now, Taylor expansion gives us

\[
    |T_b x - T_b y| = |x - y| + o(|x - y|) \quad \tag{4.20}
\]

and

\[
    |G_{ij}(T_b x, T_b y) - G_{ij}(x, y)| = o(|x - y|), \tag{4.21}
\]

where \( o \) is the usual “small-o” notation in the limit of \( |x - y| \to 0 \). These higher order terms contribute to \( G^\text{good} \), as they cancel the singularities in the denominator of \( G_{ij} \); see Lemma 2.4. Therefore, the claim (4.18) follows.

Finally, in the boundary chart \( U_b \), the boundary condition pulled back by \( T_b \) is in the form of (2.18), which is the oblique derivative boundary condition on the half-space. Thus, choosing the local coordinate frame \( \{ x^1, x^2, x^3 \} \) such that \( \partial / \partial x^3 = n \), we have

\[
    G_{ij}(x, y) \mathbb{1}_{\{(x, y) \in U_b \times U_b\}} = \frac{\delta_{ij}}{4\pi} \left\{ \Gamma(T_b x - T_b y) - \Gamma(T_b x - (T_b y)^*) - \frac{2b^{(i)}}{3} \Theta^{(i)}(T_b x, (T_b y)^*) \right\} \tag{4.22}
\]

by (2.21) (also see Sect. 6.7 in Gilbarg–Trudinger [20]). Eqs. (4.15) (4.16) (4.17) (4.18) and (4.22) together complete the proof.

In the proof we have deduced the following identity:

\[
    \{ \nabla T_b \}^i_j(x) \equiv \nabla_i (T_b x)^j = \sum_{k=1}^{3} \mathcal{O}^k_j \left( \delta^i_k - \delta^i_k \nabla_k F_b \right)(x - x_b) \quad \text{for each } i, j \in \{1, 2, 3\}, \tag{4.23}
\]

where \( \mathcal{O}^k_j \in O(3) \); see Eq. (4.19). It will be repeatedly used in the subsequent development.

4.2. Potential Estimates for the Vortex Stretching Term. In the following nine subsections we shall estimate the term

\[
    \text{[Stretch]} := \left| \int_{\Omega} S u(t, x) : \omega(t, x) \otimes \omega(t, x) \, dx \right|
\]
using the representation formula for \( u \) in Lemma \[4.2\] recall that \( Su = (\nabla u + \nabla^\top u)/2 \). To this end, we first need the expressions for \( \nabla J_i : \omega \otimes \omega, \ i = 1, 2, 3 \). The major novelty and difficulty of the current work comes from the \( J_2 \) term, due to the non-triviality of the boundary conditions.

Before further development, let us introduce a notation used throughout the paper:

\[
\hat{a} := \frac{a}{|a|} \quad \text{for } a \in \mathbb{R}^3.
\]

Also, in what follows let us write \( \nabla_{y,j} = \nabla_j \) for \( \partial/\partial y^j \), and \( \nabla_{x,k} \) for \( \partial/\partial x^k \). Furthermore, \( \epsilon^{kij} \) denotes the Levi-Civita tensor which equals to 1 if \((kij)\) is an even permutation of \((123)\), to \(-1\) if \((kij)\) is an odd permutation of \((123)\), and to \(0\) if there are repeated indices in \( \{k, l, j\} \).

4.3. Estimates for \( J_2 \): Preliminaries. Let us first integrate by parts to re-write the \( J_2 \) term. It suffices to bound \( J_2 \) in each fixed \( U_b \) for \( b \in \mathcal{B} \). With a slight abuse of notations, let us denote

\[
J^b_2(x) := \sum_j \int_\Omega \left\{ \chi_b(y) G_{ij}(x, y) (\nabla \times \omega)^j(y) \zeta_j \left( \frac{|T_b x - T_b y|}{d_3} \right) \right\} dy,
\]

where

\[
G_{ij}(T_b x, T_b y) = \delta_{ij} \left\{ \frac{1}{|T_b x - T_b y|} - \frac{1}{|T_b x - (T_b y)|} \left( 1 + \frac{2b^{(i)}_{(j)}}{3} \Theta^{(i)}(T_b x, (T_b y)^*) \right) \right\},
\]

and \( d_4 \) is chosen to be the minimum of \( d_3/2 \) and the maximal width of the tubular neighbourhood of \( \Sigma = \partial \Omega \) such that the nearest point projection onto \( \Sigma \) is a bi-Lipschitz homotopy retract.

For notational convenience, we fix \( b \in \mathcal{B} \) and drop the subscripts \( b \) from now on.

Recall that

\[
(\nabla \times \omega)(y) = \sum_{k,l,j=1}^3 \epsilon^{kij} \nabla_k \omega^l \frac{\partial}{\partial y^j}, \quad (\omega \times n)(y) = \sum_{k,l,j=1}^3 \epsilon^{kij} \omega^l \frac{\partial}{\partial y^j},
\]

where \( \epsilon^{kij} \) is the Levi-Civita symbol and \( \partial/\partial y^j \) is the unit vectorfield in \( j \)-th coordinate. Thus, integrating by parts and utilising the Stokes’ thorem, we obtain

\[
J^i_2(x) = -\sum_{k,j,l} \epsilon^{kij} \int_{\Omega} \chi(y) \zeta_j \left( \frac{|T x - T y|}{d_4} \right) \cdot \nabla_k \left( G_{ij}(T x, T y) \right) \omega^l(y) dy
\]

\[
- \sum_{k,j,l} \epsilon^{kij} \int_{\Omega} \nabla_k \left( \chi(y) \zeta_j \left( \frac{|T x - T y|}{d_4} \right) \right) G_{ij}(T x, T y) \omega^l(y) dy
\]

\[
- \sum_j \int_{\Sigma = \partial \Omega} \chi(y) \zeta_j \left( \frac{|T x - T y|}{d_4} \right) G_{ij}(T x, T y)(\omega \times n)^j(y) d\mathcal{H}^2(y)
\]

\[
=: J^i_{21}(x) + J^i_{22}(x) + J^i_{23}(x).
\]

Here \( \nabla_k = \partial/\partial y^k \) and \( \mathcal{H}^2 \) is the 2-dimensional Hausdorff measure on \( \Sigma \) induced by \( \Sigma \hookrightarrow \mathbb{R}^3 \).

In the next six subsections, we estimate the terms \( J_{2j}, \ j = 1, 2, 3 \) one by one.

4.4. Decomposition of \( J_{21} \) into Three Terms. Let us introduce the symbol

\[
\sigma_j := \begin{cases} 
1 & \text{if } i = 1 \text{ or } 2, \\
-1 & \text{if } i = 3,
\end{cases}
\]

and adopt the convention \( \gamma, \eta \in \{1, 2\}; \ i, j, k, l, p, q \ldots \in \{1, 2, 3\} \). Then, \( J_{21} \) can be further decomposed into three terms:
Lemma 4.3. $J_{21}$ can be written as follows:

$$[J_{21}(x)]^i = \sum_{kli} \frac{e^{ki}}{4\pi} \left\{ \int_{\Omega} \chi(y) \left( \frac{|T_x-T_y|}{d_4} \right)^2 \left[ -2 \frac{(T_x-T_y)^p}{|T_x-T_y|^3} \left( \mathcal{O}_p^k - \delta_3^k \nabla_T \mathcal{F}(y-x_b) \mathcal{O}_p^j \right) 
\right. \\
+ 2 \frac{(T_x-(Ty)^*)^p}{|T_x-(Ty)^*|^3} \delta_p \left( \mathcal{O}_p^k - \delta_3^k \nabla_T \mathcal{F}(y-x_b) \mathcal{O}_p^j \right) 
\left( 1 + \frac{2h(i)}{3} \Theta(i) (T_x,(Ty)^*) \right) \\
+ \frac{2h(i)}{3|T_x-(Ty)^*|} \nabla_k \left[ \Theta(i) (T_x,(Ty)^*) \right] \right] d\gamma \right\} dy
=: [J_{211}(x)]^i + [J_{212}(x)]^i + [J_{213}(x)]^i. \tag{4.28}$$

From now on, $\mathcal{F} := \mathcal{F}_b$ as in Step 3, Sect. 4.1; $x_b$ is the centre of the boundary chart $U_b$.

**Proof.** It follows from a direct computation for $\nabla_k \mathcal{G}_{ij}$. Note that

$$\nabla_k \left( \frac{1}{|T_x-T_y|^3} \right) = \sum_{p=1}^3 \frac{-2(T_x-T_y)^p \nabla_p (Ty)^k}{|T_x-T_y|^3},$$

where $\nabla_p (Ty)^k = \sum_q (\nabla \mathcal{F})_q p \nabla_q y^k = (\nabla \mathcal{F})^k_p$. Thus,

$$\nabla_k \left( \frac{1}{|T_x-T_y|^3} \right) = \sum_{p=1}^3 \sum_{\gamma=1}^2 \frac{(T_x-T_y)^p}{|T_x-T_y|^3} \left( \mathcal{O}_p^k - \delta_3^k \nabla_T \mathcal{F}(y-x_b) \mathcal{O}_p^j \right). \tag{4.29}$$

Analogously, we have

$$\nabla_k \left( \frac{1}{|T_x-(Ty)^*|^3} \right) = \sum_{p=1}^3 \sum_{\gamma=1}^2 \frac{(T_x-(Ty)^* p)^p}{|T_x-(Ty)^*|^3} \left( \mathcal{O}_p^k - \delta_3^k \nabla_T \mathcal{F}(y-x_b) \mathcal{O}_p^j \right). \tag{4.30}$$

Hence, the assertion follows from the explicit formula for $\mathcal{G}_{ij}$ in Eq. (2.21). \qed

In what follows we compute the vortex stretching terms involving $J_{21k}$, $k = 1, 2, 3$.

4.5. **Estimates for $J_{211}$**. For this term, one has

$$\nabla_j [J_{211}]^i (x) = \sum_{kli} \sum_{qny} \frac{e^{ki}}{2\pi} \left\{ \int_{\Omega} \frac{2}{d_4} \chi(y) \zeta \left( \frac{|T_x-T_y|}{d_4} \right)^2 \frac{(T_x-T_y)^q}{|T_x-T_y|^3} \left( \mathcal{O}_q^j - \delta_3^j \nabla_T \mathcal{F}(x-x_b) \mathcal{O}_q^y \right) \times \right. \\
\times \left( \frac{T_x-T_y)^p}{|T_x-T_y|^3} \left( \mathcal{O}_p^k - \delta_3^k \nabla_T \mathcal{F}(y-x_b) \mathcal{O}_p^j \right) \right] d\gamma \right\} dy
=: K_1(x) + K_2(x). \tag{4.31}$$

In the sequel let us simply the notations by setting

$$\Xi(z)^j := \mathcal{O}_j^q - \sum_{\gamma=1}^2 \delta_{3\gamma} \nabla_T \mathcal{F}(z-x_c) \mathcal{O}_q^\gamma \quad \text{for } z \in U_c, c \in I \cup \mathcal{B}. \tag{4.32}$$

Then, as $\Omega$ is a $C^2$ bounded domain,

$$\|\Xi\|_{C^0(U_c)} \leq 2 + \|\mathcal{F}\|_{\text{Lip}(U_c)} =: C_1. \tag{4.33}$$
As a result, since \( \| \zeta \|_{C^0(\mathbb{R})} \leq 4 \) and \( T \) is almost an isometry (see Eq. (4.23)), we can bound
\[
|K_1(x)| \leq C_2 \int_{\Omega} \frac{\omega(y)}{|x - y|^2} \, dy \quad \text{for } x \in U_b,
\]
where the constant \( C_2 = C(\| F \|_{\text{Lip}(U_b)}, 1/d_4) \). The same bound remains valid with the indices \( i, j \) interchanged. For the \( K_2 \) term, one observes that
\[
\nabla_{x,j} \frac{(Tx - Ty)^p}{|Tx - Ty|^3} = \frac{\nabla_j(Tx)^p}{|Tx - Ty|^3} - 6 \sum_q \frac{(Tx - Ty)^p(Tx - Ty)^q \nabla_j(Tx)^q}{|Tx - Ty|^5}
\]
\[= \frac{\Xi^j_1(x)}{|Tx - Ty|^3} - 6 \sum_q \frac{(Tx - Ty)^p(Tx - Ty)^q \Xi^j_q(x)}{|Tx - Ty|^5}.
\]

Hence, the symmetric gradient of \( J_{211} \) equals to
\[
\frac{1}{2} \left( \nabla_j[J_{211}] + \nabla_i[J_{211}] \right)(x)
\]
\[= \sum_{klpq} \frac{\epsilon^{kli}}{4\pi} \int_{\Omega} \chi(y) \zeta \left( \frac{|Tx - Ty|}{d_4} \right) \omega^j(y) \Xi^k_p(y) \left[ \frac{\Xi^j_1(x)}{|Tx - Ty|^3} - 6 \frac{(Tx - Ty)^p(Tx - Ty)^q \Xi^j_q(x)}{|Tx - Ty|^5} \right] dy
\]
\[+ \sum_{klpq} \frac{\epsilon^{klij}}{4\pi} \int_{\Omega} \chi(y) \zeta \left( \frac{|Tx - Ty|}{d_4} \right) \omega^j(y) \Xi^k_p(y) \left[ \frac{\Xi^j_1(x)}{|Tx - Ty|^3} - 6 \frac{(Tx - Ty)^p(Tx - Ty)^q \Xi^j_q(x)}{|Tx - Ty|^5} \right] dy
\]
\[+ K_3(x),
\]
where \( K_3 \) has the same bound (4.34) as for \( K_1 \). The first terms in the second and third lines above have nice cancellation properties, thanks to the following observation:

**Lemma 4.4.** For some \( C_3 = C(\| \nabla^2 F \|_{C^0(U_b)}), \) there holds
\[
\sum_{ij} \epsilon^{klij} (\Xi^k_p(y) \Xi^j_1(x) + \Xi^k_p(y) \Xi^j_q(x)) \leq C_3 |x - y|
\]
for \( x, y \) sufficiently close in \( U_b \).

**Proof.** Using \( O^{-1} = O^\top \) and the definition of \( \Xi \) in (4.32), we have
\[
\Xi^k_p(y) \Xi^j_1(x) = \delta^k_{ij} + \sum_{\gamma, \eta=1} \delta^k_{ij} \delta^\gamma_3 \delta^\eta_3 \nabla_\gamma F(y - x) \nabla_\eta F(x - y - x_b)
\]
\[= \left( \delta^k_3 \nabla_k F(x - x_b) + \delta^k_3 \nabla_i F(y - x_b) \right)
\]
\[= \delta^k_{ij} + \sum_{\gamma, \eta=1} \delta^k_{ij} \delta^\gamma_3 \delta^\eta_3 \nabla_\gamma F(y - x) \nabla_\eta F(x - y - x_b)
\]
\[= \left( \delta^k_3 \nabla_k F(x - x_b) + \delta^k_3 \nabla_i F(y - x_b) \right) + \delta^k_3 \left( \nabla_i F(x - x_b) - \nabla_i F(y - x_b) \right).
\]

The first three terms on the right-hand side are symmetric in \( i \) and \( k \); hence, multiplying with \( \epsilon^{klij} \) and symmetrising over \( i, j \) yield zero. For the last term, one may use the definition of \( T \) and Taylor expansion to deduce
\[
|\delta^k_3 \left( \nabla_i F(x - x_b) - \nabla_i F(y - x_b) \right)| \leq C_3 |Tx - Ty| = C_4 |x - y| \quad \text{for } x, y \in U_b.
\]
Hence the assertion follows. □
The above lemma implies that
\[
\left| \sum_{klpq} \frac{\epsilon_{kl}}{4\pi} \int_{\Omega} \chi(y) \left( \frac{|Tx - Ty|}{d_4} \right)^i \omega_j(y) \frac{\Xi_k(y)}{|Tx - Ty|^3} \right| \\
+ \sum_{klpq} \frac{\epsilon_{kl}}{4\pi} \int_{\Omega} \chi(y) \left( \frac{|Tx - Ty|}{d_4} \right)^j \omega_k(y) \frac{\Xi_l(y)}{|Tx - Ty|^3} \right| \\
\leq C_2 \int_{\Omega} \frac{|\omega(y)|}{|x - y|^2} \, dy, 
\]
which is the same bound as for $K_1, K_3$. For the remaining terms (denoted by $\mathcal{R}$) in Eq. (4.36), let us introduce the short-hand notation
\[
\Psi^2(x, y) := \Xi(y) \cdot (Tx - Ty), \\
\Psi^\times (x, y) := \Xi(x) \cdot (Tx - Ty).
\]
Thus,
\[
\mathcal{R} \equiv \sum_{klpq} \frac{\epsilon_{kl}}{4\pi} \int_{\Omega} \chi(y) \left( \frac{|Tx - Ty|}{d_4} \right)^i \omega_j(y) \frac{\Xi_k(y)}{|Tx - Ty|^3} \right] \\
- 6 \frac{(Tx - Ty)^p(Tx - Ty)^q \Xi_j(y)}{|Tx - Ty|^5} \right) \, dy \\
+ \sum_{klpq} \frac{\epsilon_{kl}}{4\pi} \int_{\Omega} \chi(y) \left( \frac{|Tx - Ty|}{d_4} \right)^j \omega_k(y) \frac{\Xi_l(y)}{|Tx - Ty|^3} \right] \\
- 6 \frac{(Tx - Ty)^p(Tx - Ty)^q \Xi_j(y)}{|Tx - Ty|^5} \right) \, dy \\
= - \frac{3}{2\pi} \sum_{kl} \int_{\Omega} \chi(y) \left( \frac{|Tx - Ty|}{d_4} \right)^i \left( \frac{\epsilon_{kl} \Phi^2(y)}{|Tx - Ty|^5} \right) \, dy \\
= - \frac{3}{2\pi} \int_{\Omega} \chi(y) \left( \frac{|Tx - Ty|}{d_4} \right)^i \left( \frac{\Phi^2 \omega(y) \otimes \Phi^2 + \Phi^\times \otimes \Phi^3 \omega(y)}{|Tx - Ty|^5} \right) \, dy. 
\]
Here and throughout, the notation for tensor product is understood as follows:
\[
\{a \times b \otimes c\}^{ij} := (a \times b)^i c^j \quad \text{for } a, b, c \in \mathbb{R}^3, i, j \in \{1, 2, 3\}. 
\]
We further notice that
\[
[a \times b \otimes c + b \otimes c \times a] : (d \otimes d) = 2(c, d) \det(a, b, d) \quad \text{for } a, b, c, d \in \mathbb{R}^3, 
\]
where $\det(a, b, d)$ is the determinant of the $3 \times 3$ matrix with columns $a, b$ and $d$ in order. Hence, in view of Eqs. (4.36) (4.40) (4.33) and (4.44) and Lemma 4.4 one obtains
\[
\left| \int_{\Omega} \nabla J_{211}(x) + \nabla^\top J_{211}(x) \right| \omega(x) \, dx \\
\leq \frac{3}{\pi} \int_{\Omega} \int_{\Omega} \chi(y) \left( \frac{|Tx - Ty|}{d_4} \right)^i \left( \Phi^2(x, y), \omega(x) \right) \cdot \det \left( \Phi^2(x, y), \omega(y), \omega(x) \right) \, dy \, dx + K_4, 
\]
where, for some $C_5 = C(\|\mathcal{F}\|_{C^2(\Omega)}^1, 1/d_4)$, there holds
\[
|K_4| \leq C_5 \int_{\Omega} |\omega(x)|^2 \int_{U_6} \frac{|\omega(y)|}{|x - y|^2} \, dy \, dx. 
\]
It remains to bound $K_5$. The key is to explore the geometric meaning of the determinant, as in Constantin–Fefferman [12] and Constantin [11]. This is achieved by the following lemmas.
Lemma 4.5. The determinant term in (4.45) satisfies
\[
\left| \det \left( \Psi^t(x, y), \omega(y), \omega(x) \right) \right|_{|Tx - Ty|} \simeq \left\{ \left| \det \left( \frac{x - y}{|x - y|}, \omega(y), \omega(x) \right) \right| + \left| \det \left( \frac{\mathcal{R}_x(y; x - y)}{|x - y|}, \omega(x), \omega(y) \right) \right| \right\}.
\]

Here, recall the notation \( x - y := (x - y)/|x - y| \); also, we write \( A \simeq B \) to mean that \( C^{-1}A \leq B \leq CA \) for a universal constant \( C \).

Proof. We make a detailed analysis of the term \( \Psi^t \). By Taylor expansion and \( \mathcal{O}^{-1} = \mathcal{O^T} \) one may deduce
\[
|\Psi^t(x, y)|^t = \sum_j \Xi_j^t(y) (Tx - Ty)^j
\]
\[
\simeq \sum_{j,k\eta\gamma} \left( \mathcal{O}_j^t + \delta_3 \nabla, \mathcal{F}(y) \mathcal{O}_j^t \right) \left( \mathcal{O}_k^t + \delta_3 \nabla, \mathcal{F}(y) \mathcal{O}_k^t \right) (x - y)^k + \mathcal{O}(|x - y|)
\]
\[
= (x^i - y^i) + \delta_3 \left\{ \nabla_1 \mathcal{F}(y)(x^1 - y^1) + \nabla_2 \mathcal{F}(y)(x^2 - y^2) + (\nabla \mathcal{F}(y))^2(x^3 - y^3) \right\}
\]
\[
+ \nabla_1 \mathcal{F}(y)(x^3 - y^3) + \mathcal{O}(|x - y|);
\]
Equivalently,
\[
\Psi^t(x, y) = (x - y) + \begin{bmatrix}
\nabla_1 \mathcal{F}(y)(x^3 - y^3) \\
\nabla_2 \mathcal{F}(y)(x^3 - y^3) \\
\nabla_1 \mathcal{F}(y)(x^1 - y^1) + \nabla_2 \mathcal{F}(y)(x^2 - y^2) + (\nabla \mathcal{F}(y))^2(x^3 - y^3)
\end{bmatrix}
\]
\[
=: (x - y) + \mathcal{R}_x(y; x - y).
\]
On the other hand, by shrinking \( d_4 > 0 \) if necessary, we conclude from Eq. (4.19) that
\[
\frac{1}{2} |x - y| \leq |Tx - Ty| \leq 2 |x - y|.
\]
Hence the assertion follows.

By analogous arguments, we have

Lemma 4.6. There holds
\[
\frac{|\Psi^t(x, y)|}{|Tx - Ty|} \simeq 1 + \frac{|\mathcal{R}_x(y; x - y)|}{|x - y|}.
\]

Proof. A computation similar to (4.49) gives us
\[
\Psi^t(x, y) = (x - y) + \begin{bmatrix}
\nabla_1 \mathcal{F}(x)(x^3 - y^3) \\
\nabla_2 \mathcal{F}(x)(x^3 - y^3) \\
\nabla_1 \mathcal{F}(x)(x^1 - y^1) + \nabla_2 \mathcal{F}(x)(x^2 - y^2) + (\nabla \mathcal{F}(x))^2(x^3 - y^3)
\end{bmatrix}
\]
The assertion follows immediately from Eq. (4.51).

Now, utilising the crucial geometric observation by Constantin [11] and Constantin–Fefferman [12], we can finalise the estimate for $K_5$. This is the first place where we need the geometric condition in the hypotheses of Theorem [11].

**Lemma 4.7.** Under the assumption of Theorem [11], i.e., if \( \theta(x,y) := \angle(\hat{\omega}(x),\hat{\omega}(y)) \) satisfies \( |\sin \theta(x,y)| \leq C_6 \sqrt{|x-y|} \) for a universal constant $C_6 > 0$, then we can find $C_7 = C(C_6, \|\mathcal{F}\|_{C^1(\Omega)})$ such that

\[
|K_5| \leq C_7 \int_\Omega |\omega(x)|^2 \int_{U_h} \frac{|\omega(y)|}{|x-y|^{5/2}} \, dy \, dx.
\]

**Proof.** In view of Lemmas 4.5 and 4.6, substituting Eqs. (4.48)–(4.52) into (4.45), we have:

\[
|K_5| \approx \int_\Omega |\omega(x)|^2 \int_{U_h} \left\{ \frac{|T_x y|}{d_4} \right\} \frac{|\omega(y)|}{|x-y|} \left( 1 + \frac{\mathcal{R}_\mathcal{F}(x; x-y)}{|x-y|} \right) \times
\]

\[
\times \left\{ \left| \det \left( \frac{\omega(x)}{|\omega(x)|} \right) \right| + \left| \det \left( \frac{\mathcal{R}_\mathcal{F}(x; x-y)}{|x-y|} \right) \right| \right\} \, dy \, dx.
\]

Now we invoke the geometric observation by Constantin [11] and Constantin–Fefferman [12] (also see Beirão da Veiga–Berselli [1] and the references cited therein): Consider the expression

\[
\det \left( \hat{a}, \hat{\omega}(x), \hat{\omega}(y) \right)
\]

for any unit vector $\hat{a} \in \mathbb{R}^3$. It is the volume of the parallelepiped spanned by the sides $\hat{a}$, $\hat{\omega}(x)$ and $\hat{\omega}(y)$, hence equals to

\[
\det \left( \hat{a}, \hat{\omega}(x), pr_{\hat{\omega}(x)^\perp} \hat{\omega}(y) \right).
\]

Here $pr_{\hat{\omega}(x)^\perp}(\cdot)$ denotes the orthogonal projection onto the subspace perpendicular to $\hat{\omega}(x)$. Moreover, as $|\hat{\omega}(y)| = 1$, one has

\[
\left| \det \left( \hat{a}, \hat{\omega}(x), pr_{\hat{\omega}(x)^\perp} \hat{\omega}(y) \right) \right| \leq \left| \det pr_{\hat{\omega}(x)^\perp} \hat{\omega}(y) \right| \leq \left| \sin \theta(x,y) \right|.
\]

Finally, it is clear that

\[
\frac{|\mathcal{R}_\mathcal{F}(\bullet; x-y)|}{|x-y|} \leq \sqrt{3} \left\| \mathcal{F} \right\|_{C^0(\Gamma)}.
\]

Therefore, we complete the proof in view of (4.55) and by considering $\hat{a} = \hat{x-y}$ in (4.56). \( \square \)

We conclude this subsection with the following bound for the contribution of $J_{211}$ to the vortex stretching term:

**Proposition 4.8.** Under the assumption of Theorem [11]

\[
\left| \int_\Omega \frac{\nabla J_{211}(x)}{2} : \omega(x) \otimes \omega(x) \, dx \right| \leq C_8 \left( \int_\Omega |\omega(x)|^2 \int_{U_h} \frac{|\omega(y)|}{|x-y|} \, dy \, dx + \int_\Omega |\omega(x)|^2 \int_{U_h} \frac{|\omega(y)|}{|x-y|^{5/2}} \, dy \, dx \right)
\]

where $C_8 = C(\left\| \mathcal{F} \right\|_{C^2(\Omega)}, 1/d_4)$.

**Proof.** Immediate from Lemma 4.7 and Eqs. (4.46), (4.46). \( \square \)
4.6. Estimates for $J_{213}$. This is a good term, due to the decay properties of the kernel $\Theta^{(i)}$. Recall from (4.26) that

$$[J_{213}(x)]^i = \sum_{kl} \frac{e^{ikl}b^{(i)}_k}{6\pi} \int_{\Omega} \chi(y) \zeta \left( \frac{|Tx - Ty|}{d_4} \right) \nabla_k \left( \Theta^{(i)}(Tx, (Ty)^*) \right) \omega'(y) dy,$$

where the $\Theta^{(i)}$ term is given by (4.6.2):

$$\Theta^{(i)}(Tx, (Ty)^*) = \int_0^\infty e^{a^{(i)}|Tx - (Ty)^*|s} \frac{\xi^3 + b^{(i)}_3 s}{[1 + 2(b^{(i)}, \xi) s + s^2]^{3/2}} ds,$$

and

$$\xi = \frac{Tx - (Ty)^*}{|Tx - (Ty)^*|}.$$

**Lemma 4.9.** For the regular oblique derivative boundary condition (2.2), i.e., if for each $i \in \{1, 2, 3\}$ one has

$$b^{(i)}_3 > 0, \quad n = \frac{\partial}{\partial x^3} \text{ on } \Sigma, \quad (4.59)$$

$\Theta^{(i)}(Tx, (Ty)^*)$ is a smooth function in $x$ and $y$.

**Proof.** First of all, note that $|Tx - (Ty)^*| \neq 0$ from the definition of the boundary-straightening map $T = T_0$; hence $\xi$ is well defined and smooth for all $x$ and $y$, so are

$$\nabla_{x,j}|Tx - (Ty)^*| = 2 \frac{(Tx - (Ty)^*)^k [\Xi^j_k(x) + o(|x - x_b|)]}{|Tx - (Ty)^*|} \quad (4.60)$$

and

$$\nabla_{x,j}\xi^k = \frac{\Xi^j_k(x) + o(|x - x_b|)}{|Tx - Ty|}$$

$$- \frac{2(Tx - (Ty)^*)^k(Tx - (Ty)^*)^l [\Xi^j_l(x) + o(|x - x_b|)]}{|Tx - (Ty)^*|^3} \quad (4.61).$$

Next, we can compute

$$\nabla_{x,j}\Theta^{(i)}(Tx, (Ty)^*) = \int_0^\infty e^{a^{(i)}|Tx - (Ty)^*|s} \frac{\xi^3 + b^{(i)}_3 s}{[1 + 2(b^{(i)}, \xi) s + s^2]^{3/2}} \nabla_{x,j}|Tx - (Ty)^*| ds$$

$$+ \int_0^\infty e^{a^{(i)}|Tx - (Ty)^*|s} \frac{\nabla_{x,j}\xi^3}{[1 + 2(b^{(i)}, \xi) s + s^2]^{3/2}} ds$$

$$- \int_0^\infty e^{a^{(i)}|Tx - (Ty)^*|s} \frac{(\xi^3 + b^{(i)}_3 s)(b^{(i)}_{ij}, \nabla_{x,j}\xi^k)}{[1 + 2(b^{(i)}, \xi) s + s^2]^{3/2}} ds, \quad (4.62)$$

and by a simple induction, for any multi-index $\alpha \in \mathbb{N}^3$ we have

$$\nabla^\alpha_y \Theta^{(i)}(Tx, (Ty)^*) = \int_0^\infty e^{a^{(i)}|Tx - (Ty)^*|s} \mathcal{P}_\alpha(x, y, s) ds, \quad (4.63)$$

where $\mathcal{P}_\alpha(x, y, s)$ is a linear combination of polynomials in $s$. The coefficients of such polynomials are products of components of $\xi$, $\nabla_x|Tx - (Ty)^*|$, $\nabla_x\xi$ and $(1 + 2(b^{(i)}, \xi) s + s^2)^k$ for $k \leq -3/2$. In view of (4.60), (4.61) and the assumptions $a^{(i)} \leq 0$, $b^{(i)}_3 > 0$, $|b^{(i)}| = 1$ for the regular oblique derivative condition, the integral (4.12) converges for any multi-index $\alpha$, and is continuous in the $x$-variable. Finally, the derivatives $\nabla^\alpha_y \Theta^{(i)}(Tx, (Ty)^*)$ differs from (4.12) only by multiplications of the constant matrix $M = \text{diag } (1, 1, -1)$. Hence the assertion follows. \[\square\]
As a consequence, the gradient of $J_{213}$:

$$
\nabla_j[J_{213}(x)]^i = \sum_{kl} \frac{e^{kili_b}}{6d_4\pi} \int_\Omega \chi(y)\zeta'\left(\frac{|T_x - T_y|}{d_4}\right) \times
$$

$$
\times \frac{(\nabla_{x,j}|T_x - (Ty)^*|\nabla_{y,k}[(\Theta^{(i)}(T_x,(Ty)^*))^l]}{|T_x - (Ty)^*|} dy
$$

$$
+ \sum_{kl} \frac{e^{kili_b}}{6\pi} \int_\Omega \chi(y)\zeta\left(\frac{|T_x - T_y|}{d_4}\right) \frac{(\nabla_{x,j}\nabla_{y,k}[(\Theta^{(i)}(T_x,(Ty)^*))^l]}{|T_x - (Ty)^*|} dy
$$

$$
+ \sum_{kl} \frac{e^{kili_b}}{6\pi} \int_\Omega \chi(y)\zeta\left(\frac{|T_x - T_y|}{d_4}\right) \frac{(\nabla_{y,k}[(\Theta^{(i)}(T_x,(Ty)^*))^l]}{|T_x - (Ty)^*|} (\nabla_{x,j}|T_x - T_y|)^l} dy
$$

(4.64)

satisfies good bounds (so does its symmetrisation), because

$$
|\nabla|T_x - (Ty)^*|| \leq C_{14} = C(\|F\|_{C^1(U_b)})
$$

and $\Theta^{(i)}(T_x,(Ty)^*) \in C^\infty$ by Eq. (4.60) and Lemma 4.9. More precisely,

**Proposition 4.10.** Under the assumption of Theorem 4.7, we have

$$
\left| \int_\Omega \nabla J_{213}(x) + \nabla^T J_{213}(x) : \omega(x) \otimes \omega(x) dx \right| \leq C_{15} \int_\Omega |\omega(x)|^2 \int_\Omega \frac{|\omega(y)|}{|x - y|} dy dx,
$$

where $C_{15} = C(\|F\|_{C^1(U)}, 1/d_4, b^{(i)}, a^{(i)})$.

**4.7. Estimates for $J_{212}$**. The computation for $J_{212}$ is similar to that for $J_{211}$ in Sect. 4.5. Recall from Sect. 4.4:

$$
[J_{212}(x)]^i = \sum_{klp} e^{kili_b} \int_\Omega \chi(y)\zeta\left(\frac{|T_x - T_y|}{d_4}\right) \frac{(T_x - (Ty)^*)^p}{|T_x - (Ty)^*|} \sigma_p \Xi^k_p(y) dy
$$

for $x \in U_b$. Then,

$$
\nabla_j[J_{212}]^i(x) = \sum_{klpq} \frac{e^{kili_b}}{2\pi} \left\{ \int_\Omega \frac{2}{d_4} \chi(y)\zeta'\left(\frac{|T_x - T_y|}{d_4}\right) \frac{(T_x - (Ty)^*)^p \sigma_p}{|T_x - (Ty)^*|^3} \Xi^k_p(y) \right\}
$$

$$
\times \left\{ \frac{(T_x - (Ty)^*)^p \sigma_p}{|T_x - (Ty)^*|^3} \Xi^k_p(y) \omega^l(y) dy + \int_\Omega \chi(y)\zeta\left(\frac{|T_x - T_y|}{d_4}\right) \nabla_{x,j} \left[ \frac{(T_x - (Ty)^*)^p \sigma_p}{|T_x - (Ty)^*|^3} \Xi^k_p(y) \right] \omega^l(y) dy + \int_\Omega \chi(y)\zeta\left(\frac{|T_x - T_y|}{d_4}\right) \frac{(T_x - (Ty)^*)^p}{|T_x - (Ty)^*|} \sigma_p \Xi^k_p(y) \right\}
$$

by a direct computation. Using similar arguments as for $J_{211}$ (in particular, Lemma 4.4) and the smoothness of the $\Theta$-kernel (Lemma 4.3), we can deduce

$$
\left| \frac{1}{2} \int_\Omega \left( \nabla_J[J_{212}]^i(x) + \nabla_i[J_{212}]^i(x) \right) : \omega(x) \otimes \omega(x) dx \right| \leq K_6 + K_7.
$$

(4.66)

In the above, the “nice” term is $K_6$: we can control

$$
K_6 \leq C_9 \int_\Omega |\omega(x)|^2 \int_{U_b} \frac{|\omega(y)|}{|x - y|}^2 dy dx
$$

(4.67)

for some constant $C_9$ depends only on $\|F\|_{C^2(U)}$. 
The “bad” term in Eq. (4.66) equals to

\[ K_7 = \frac{C_{10}}{2} \sum_{ijklpq} \int \int \chi(y) \zeta \left( \frac{|Tx - Ty|}{d_4} \right) \times \frac{k^{ij}(Tx - (Ty)^*)^p(Tx - (Ty)^*)^q \omega^j(y) \sigma_k \Xi^l(y) \Xi^q(y) \omega^i(x) \omega^j(x)}{|Tx - (Ty)^*|^5} \, dy \, dx, \]

where \( C_{10} \) is a universal constant. In the above these symbols are introduced:

\[ \tilde{\Psi}^i = \tilde{\Psi}^i(x, y) := M \Xi(y) \cdot (Tx - (Ty)^*), \]

\[ \tilde{\Psi}^o = \tilde{\Psi}^o(x, y) := \Xi(x) \cdot (Tx - (Ty)^*), \]

\( z^* \) denotes the reflection of \( z \in \mathbb{R}^3_+ \) across the boundary as usual, as well as

\[ M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \]

Thus, using the geometric observation in [12, 11], we find:

\[ K_7 = C_{10} \sum_{ijklpq} \int \int \chi(y) \zeta \left( \frac{|Tx - Ty|}{d_4} \right) \left| \tilde{\Psi}^i, \omega(x) \right| \left| \tilde{\Psi}^o, \omega(y), \omega(x) \right| \frac{dy \, dx}{|Tx - (Ty)^*|^5}, \]

which is analogous to \( K_5 \) in Eq. (4.45) in Sect. 4.5.

However, it is clear that

\[ \frac{|\tilde{\Psi}^o|}{|Tx - (Ty)^*|} \leq |\Xi(x)| \leq C_{11} = C(||\mathcal{F}||_{C^1(U_b)}); \]

in addition, assuming the hypothesis in Theorem 1.1 one obtains

\[ \left| \det \left( \frac{\tilde{\Psi}^i}{|Tx - (Ty)^*|}, \tilde{\omega}(y), \tilde{\omega}(x) \right) \right| \leq C_{12} \sqrt{|x - y|} \]

for \( C_{12} = C(||\mathcal{F}||_{C^1(U_b)}). \) Indeed, one easily bounds

\[ \frac{|\tilde{\Psi}^i|}{|Tx - (Ty)^*|} \leq |M\mathcal{O}| + |M\mathcal{O}\nabla \mathcal{F}|, \]

where both \( M, \mathcal{O} \) are orthogonal matrices, in view of (4.32).

Putting together the estimates in Eqs. (4.66), (4.67), (4.68), (4.69) and (4.70), we can deduce:

**Proposition 4.11.** **Under the assumption of Theorem 1.1,** we have

\[ \left| \int \nabla J_{212}(x) + \nabla^T J_{212}(x) \right| = \omega(x) \otimes \omega(x) \, dx \]

\[ \leq C_{13} \left\{ \int \Omega |\omega(x)|^2 \int_{U_b} \frac{|\omega(y)|}{|x - y|^2} \, dy \, dx + \int \Omega |\omega(x)|^2 \int_{U_b} \frac{|\omega(y)|}{|x - y|^5} \, dy \, dx \right\} \]

where \( C_{13} = C(||\mathcal{F}||_{C^1(U)}, 1/d_4). \)

### 4.8. Estimates for \( J_{22}. \)

Next, \( J_{22} \) is also a good term (recall from Eq. (4.26)):

\[ [J_{22}(x)]^i = -\sum_{kij} \epsilon^{kli} \int \nabla_k \chi(y) \zeta \left( \frac{|Tx - Ty|}{d_4} \right) \mathcal{G}_{ij}(Tx, Ty) \omega^j(y) \, dy. \]
Clearly, by the definition of \( \chi \) and \( \zeta \),
\[
\left| \nabla_k \left[ \chi(y)\zeta \left( \frac{|Tx - Ty|}{d_4} \right) \right] \right| \leq C_{16} = C(\|\mathcal{F}\|_{C^1(\Omega)}, 1/d_4).
\]
In addition, in light of Lemma 4.9
\[
G_{ij}(Tx, Ty) = \frac{\delta_{ij}}{4\pi} \left\{ \frac{1}{|Tx - Ty|} - \frac{1}{|Tx - (Ty)^*|} \right\} \left( 1 + \frac{2b_{ij}(i)}{3} \Theta(i)(Tx, (Ty)^*) \right)
\]
has a singularity of order \(-1\), i.e.,
\[
|G_{ij}(Tx, Ty)| \leq C_{17} \frac{1}{|x - y|} \tag{4.72}
\]
for some constant \( C_{17} = C(\|\mathcal{F}\|_{C^1(\Omega)}, b^{(i)}, a^{(i)}) \). Therefore, we may easily deduce

**Proposition 4.12.** Under the assumption of Theorem 1.1, we have
\[
\left| \int_{\Omega} \nabla J_{22}(x) + \nabla^T J_{22}(x) : \omega(x) \otimes \omega(x) \, dx \right| \leq C_{18} \int_{\Omega} \|\omega(x)\|^2 \int_{\Omega} \frac{|\omega(y)|}{|x - y|^2} \, dy \, dx, \tag{4.73}
\]
where \( C_{18} = C(\|\mathcal{F}\|_{C^1(\Omega)}, 1/d_4, b^{(i)}, a^{(i)}) \).

Again, in Proposition 4.12 we do not need the hypothesis on vorticity direction alignment.

4.9. **Estimates for** \( J_{23} \): **the Boundary Term.** One of the new features of this work is the analysis of the boundary term, reproduced below from Eq. (4.26):
\[
[J_{23}(x)]^i = -\sum_j \int_\Sigma \chi(y)\zeta \left( \frac{|Tx - Ty|}{d_4} \right) G_{ij}(Tx, Ty)(\omega \times n)^j \, d\mathcal{H}^2(y).
\]
In the literature the geometric regularity conditions for the weak solutions to the Navier–Stokes equations are usually studied on the whole space \( \mathbb{R}^3 \), i.e., in the absence of physical boundaries of the fluid domain. In Beirão da Veiga–Berselli [6] and Beirão da Veiga [7] the boundary conditions were first considered. Therein the slip-type condition
\[
\omega \times n = 0 \quad \text{on} \ [0, T^*] \times \Sigma \tag{4.74}
\]
was imposed (which were first studied by Solonnikov–Šćadilov [36]), so that the boundary term vanishes: \( J_{23} \equiv 0 \). It is a very strong condition on the geometry of the vortex structure \( \Sigma \), which entails the vorticity to be perpendicular to the boundary.

In our current work the condition (4.74) is not required. We shall establish:

**Proposition 4.13.** Under the assumption of Theorem 1.1, we have
\[
\left| \int_{\Omega} \nabla J_{23}(x) + \nabla^T J_{23}(x) : \omega(x) \otimes \omega(x) \, dx \right| \leq \epsilon \|\nabla \omega\|_{L^2(\Omega)}^2 + \frac{C_{19}}{\epsilon} \|u\|^2_{H^1(\Omega)} \|\omega\|^2_{L^2(\Omega)} \tag{4.75}
\]
for any \( \epsilon > 0 \). Here \( C_{19} \) depends on \( \|\mathcal{F}\|_{C^1}, 1/d_4, b^{(i)}, a^{(i)} \) and the geometry of \( \Omega \) and \( \Sigma \).

**Proof.** Using integration by parts, we have
\[
\int_{\Omega} \nabla J_{23}(x) + \nabla^T J_{23}(x) : \omega(x) \otimes \omega(x) \, dx = \int_{\Sigma} \left\langle J_{23}, \omega \right\rangle \langle \omega, n \rangle \, d\mathcal{H}^2 - \int_{\Omega} \left\langle J_{23}, \omega \cdot \nabla \omega \right\rangle \, dx. \tag{4.76}
\]
By the definition of $J_{23}$ in Eq. (4.26) and the pointwise bound on the tangential part of $\omega$ in Eq. (3.11), one easily deduces

$$|J_{23}(x)| \leq C_{20} \int_\Sigma \frac{|u(y)|}{|x - y|} \, d\mathcal{H}^2(y),$$

where $C_{20} = C(\|F\|_{C^1}, 1/d_4, b^{(i)}, a^{(i)})$. Thus, by Hölder’s inequality we get

$$|J_{23}(x)| \leq C_{21} \left( \int_\Sigma |u|^p \, d\mathcal{H}^2 \right)^{\frac{2}{p}} \text{ for any } p > 2,$$

(4.77)

with $C_{21}$ depending on $C_{20}, \Sigma$ and $p$. Thanks to the Hölder and trace inequalities, we may continue to estimate Eq. (4.76) by

$$\left| \int_\Omega \frac{\nabla J_{23}(x) + \nabla^\top J_{23}(x)}{2} : \omega(x) \otimes \omega(x) \, dx \right| \leq C_{22} \|u\|_{W^{s+1/p,p}(\Omega)} \|\nabla \omega\|_{L^2(\Omega)} \|\omega\|_{L^2(\Omega)}^2$$

(4.78)

for any $p > 2, s > 0$; $C_{22}$ depends on $C_{21}$ and geometric constants from the trace inequalities. Therefore, utilising Cauchy–Schwarz and choosing suitable $s, p$ so that $H^1(\Omega) \hookrightarrow W^{s+1/p,p}(\Omega)$ continuously, one may deduce

$$\left| \int_\Omega \frac{\nabla J_{23}(x) + \nabla^\top J_{23}(x)}{2} : \omega(x) \otimes \omega(x) \, dx \right| \leq \epsilon \|\nabla \omega\|_{L^2(\Omega)}^2 + \frac{C_{23}}{\epsilon} \|u\|_{H^1(\Omega)}^2 \|\omega\|_{L^2(\Omega)}^2$$

(4.79)

where $C_{23} = C(C_{22}, \Omega)$. Hence the assertion follows.

4.10. Estimates for $J_1$, $J_3$. As $J_1$ (reproduced below) only involves the interior charts

$$J_1(x) = \sum_{j=1}^3 \sum_{\alpha=1}^3 \int_\Omega \chi_\alpha(y) \left\{ \frac{\delta_{ij}}{4\pi|x - y|} (\nabla \times \omega)^j(y) \right\} \zeta \left( \left| \frac{x - y}{d_3} \right| \right) \, dy,$$

its contribution to $[Stretch]$ can be estimated as in the pioneering works by Constantin–Fefferman [12] and Beirão da Veiga–Berselli [3].

**Proposition 4.14.** Under the assumption of Theorem [14] there is a constant $C_{25} = C(\|F\|_{C^2(\Omega)})$ such that

$$\left| \int_\Omega \frac{\nabla J_1(x) + \nabla^\top J_1(x)}{2} : \omega(x) \otimes \omega(x) \, dx \right| \leq C_{25} \int_\Omega |\omega(x)|^2 \int_\Omega \frac{|\omega(y)|}{|x - y|^{5/2}} \, dy \, dx.$$

(4.80)

For $J_3$, Solonnikov [35] (also see p.610 and Appendix B, p.626 in Beirão da Veiga–Berselli [6], and Lemma [2.3] in this paper) showed that, for $C^{3,\alpha}$-boundary $\Sigma$, the good part of the kernel $G^{\text{good}}$ satisfies

$$\left| \frac{\partial^{\alpha}}{\partial x^\alpha} \frac{\partial^{\beta}}{\partial y^\beta} G^{\text{good}}(x,y) \right| \leq \frac{C_{\text{good}}}{|x - y|^{\alpha + |\beta| + 1 - \delta}}$$

for all $x \neq y$ in $\Omega$ with $\delta > 1/2$.

(4.81)

In fact, the range of $\delta$ depends only on the regularity of the solution to the elliptic system (2.1)(2.2); as a consequence of the standard Schauder theory, this in turn depends only on the regularity of $\Omega$. Thanks to Eq. (4.81), a direct computation give us:

**Proposition 4.15.** Under the assumption of Theorem [14] there is a constant $C_{26}$ such that

$$\left| \int_\Omega \frac{\nabla J_3(x) + \nabla^\top J_3(x)}{2} : \omega(x) \otimes \omega(x) \, dx \right| \leq C_{26} \int_\Omega |\omega(x)|^2 \int_\Omega \frac{|\omega(y)|}{|x - y|^{5/2}} \, dy \, dx.$$

(4.82)

Here $C_{26}$ depends only on the regularity of $\Omega$. 26
The estimation for $J_3$ is the only place where we possibly need higher regularity of the domain $\Omega$ than $C^2$. In the case of the slip-type boundary conditions (1.12), it is shown in [6] that $\Omega \in C^{3,\alpha}$ is enough. In our case of the general diagonal oblique derivative conditions (2.2) $\Omega \in C^{3,\alpha}$ will also suffice, in view of the Schauder theory for the oblique derivative problem; cf. Gilbarg–Trudinger, Chapter 6 [20]. This is true when the coefficients of the boundary conditions $(a^{(i)}, b^{(i)})$ are constant.

4.11. Completion of the Proof of Theorem 4.1. We are now at the stage of proving Theorem 4.1. Let us first recall the Hardy–Littlewood–Sobolev interpolation inequality (e.g., see p.106, Lieb–Loss [27]):

**Lemma 4.16** (Hardy–Littlewood–Sobolev). Let $1 < p, r < \infty$ and $0 < \lambda < n$ satisfy $1/p + \lambda/n + 1/r = 2$. Let $f \in L^p(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. Then there exists $K = C(n, \lambda, p)$ such that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\lambda} \, dx \, dy \right| \leq K \|f\|_{L^p(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}.$$  

**Proof of Theorem 4.1**. First of all, in view of the localisation procedure in Sect. 4.1, it suffices to prove the result on each local chart. Thus, without loss of generality, let us assume $\Omega$ to be bounded in $\mathbb{R}^3$. The unbounded case can be deduced via a partition-of-unity argument.

The proof follows from the continuation argument outline at the beginning of this section. Suppose there were a time $T \in [0, T^*)$ such that the weak solution $u$ is strong on $[0, T]$, but cannot be continued as a strong solution past $T$. We shall establish

$$\limsup_{t \nearrow T} \int_\Omega |\omega(t)|^2 \, dx < \infty$$  

(4.83)

for any such given $T$. It shows that $u$ can be extended to a strong solution to $[0, T+\delta]$ for some $\delta > 0$. This contradicts the maximality of $T$. Hence, $u$ is strong on $[0, T^*)$.

To this end, by collecting the estimates in Subsections 4.2–4.10 (in particular, Propositions 4.8, 4.11, 4.10, 4.12, 4.13 and 4.14) and recalling Eq. (4.12) in Lemma 4.2 let us first bound

$$[\text{Stretch}] = 2 \left| \int_\Omega \nabla u : \omega \otimes \omega \, dx \right|$$

$$\leq C_{27} \left( \int_\Omega |\omega(x)|^2 \int_\Omega \frac{|\omega(y)|}{|x-y|^{5/2}} \, dy \, dx \right) + \left\{ \frac{\epsilon}{2} \|\nabla \omega\|^2_{L^2(\Omega)} + \frac{C_{27}}{\epsilon} \|u\|_{H^1(\Omega)}^2 \|\omega\|^2_{L^2(\Omega)} \right\}$$

(4.84)

where $C_{27} = C(\Omega, b^{(i)}, a^{(i)})$; note that $d_4$ depends only on the geometry of $\Omega$ and the partition-of-unity, so we do not write it explicitly here; $\epsilon$ is an arbitrary positive number.

Let us first control the bulk term $I_\Omega$. By Lemma 4.16 above, one obtains

$$I_\Omega \leq C_{28} \left( \int_\Omega |\omega|^3 \, dx \right)^{2/3} \left( \int_\Omega |\omega|^2 \, dx \right)^{1/3},$$

where $C_{28}$ equals the product of $C_{27}$ and the constant in the Hardy–Littlewood–Sobolev inequality. In addition, thanks to the interpolation inequality, there holds

$$\left( \int_\Omega |\omega|^3 \, dx \right)^{2/3} \leq \left( \int_\Omega |\omega|^2 \, dx \right) \left( \int_\Omega |\omega|^6 \, dx \right)^{1/6}.$$
and, by the Sobolev inequality,
\[ \|\omega\|_{L^6(\Omega)} \leq C_{29}\left(\|\omega\|_{H^1(\Omega)}\right) \]
where \(C_{29}\) depends only on the geometry of \(\Omega\). Thus, by Young’s inequality we conclude:
\[ I_\Omega \leq \frac{\epsilon}{2} \int_\Omega |\nabla \omega|^2 \, dx + C_{30}\left(\int_\Omega |\omega|^2 \, dx\right)^2 + \frac{\epsilon}{2} \int_\Omega |\omega|^2 \, dx, \]
with any \(\epsilon > 0\) and \(C_{30} = C(\epsilon, \Omega, \beta, a)\).

For the boundary term \(I_\Sigma\), we may easily deduce the same bound:
\[ I_\Omega \leq \frac{\epsilon}{2} \int_\Omega |\nabla \omega|^2 \, dx + C_{31}\left(\int_\Omega |\omega|^2 \, dx\right)^2 + \frac{\epsilon}{2} \int_\Omega |\omega|^2 \, dx, \]
with \(C_{31} = C(\epsilon, \Omega, \beta, a)\).

Putting together the estimates (4.85)-(4.86), one obtains
\[ [\text{Stretch}] \leq \epsilon \int_\Omega |\nabla \omega|^2 \, dx + C_{32}\left(\int_\Omega |\omega|^2 \, dx\right)^2 + C_{33}\left(\int_\Omega |\omega|^2 \, dx\right), \]
where the constant \(C_{32} := C_{30} + C_{31}\).

Now, in view of the differential inequality for the enstrophy \(\mathcal{E}_t\), by choosing \(\epsilon = \nu/16\) in Eq. (4.87) we may deduce
\[ \frac{d}{dt}\left(\int_\Omega |\omega|^2 \, dx\right) + \nu \int_\Omega |\nabla \omega|^2 \, dx \leq C_{33}\left(\int_\Omega |\omega|^2 \, dx\right)^2 + \nu \int_\Omega |\omega|^2 \, dx + M. \]

The constant \(C_{33}\) depends on \(\Omega, \nu, \beta, a\), the initial energy \(\|u_0\|_{L^2(\Omega)}\) and \(M\). Moreover, \(M = M(\Omega, \beta, \|\Pi\|_{C^1(\Sigma)}, \nu, \|u_0\|_{L^2(\Omega)}, T^*)\) as in Theorem 5.10.

Thus, by Grönwall’s lemma,
\[ \int_\Omega |\omega(T)|^2 \, dx \leq \left(\int_\Omega |\omega(0)|^2 \, dx\right) \exp\left\{C_{33}\int_0^T \int_\Omega |\omega(t, x)|^2 \, dx \, dt\right\} + \nu \int_0^T \exp\left\{C_{33}\int_s^T \int_\Omega |\omega(t, x)|^2 \, dx \, dt\right\} \, ds. \]

But, in view of Lemma 3.2, the control on \(\int_0^T \int_\Omega |\omega|^2 \, dx \, dt\) is equivalent to that on \(\int_0^T \int_\Omega |\nabla u|^2 \, dx \, dt\), which is bounded by the energy inequality (4.28). Hence \(\lim_{T \to T^*} \sup_t \int_\Omega |\omega(t)|^2 \, dx < \infty\). Thanks to Lemma 4.2, we have \(\nabla u \in L^\infty(0, T; L^2(\Omega; \mathfrak{gl}(3, \mathbb{R})))\). Then, substituting this back into Eq. (4.88) and invoking Lemma 3.5 we may infer that \(\nabla u \in L^2(0, T; H^1(\Omega; \mathfrak{gl}(3, \mathbb{R})))\). Therefore, \(u\) can be continued as a strong solution past the time \(T\). This contradicts the blowup at \(T\).

The proof is now complete. \(\square\)

At the end of this section, we mention the following result \(\text{à la} \) Constantin–Fefferman [12], which can be proved by a slight modification of the arguments in Sect. 4:

**Corollary 4.17.** Let \(\Omega \subset \mathbb{R}^3\) be a regular domain. Let \(u\) be a weak solution to the Navier-Stokes equations (1.1)-(1.2) on \([0, T_\star]\times\Omega\) with the regular oblique derivative boundary condition (2.2). Assume that the energy estimate in Theorem 5.10 is valid for the strong solution. Then, if there are constants \(\rho, \Lambda > 0\) such that the vorticity turning angle \(\theta\) satisfies the following condition:
\[ |\sin \theta(t; x, y)| \mathbb{1}_{\{|\omega(t, x)| \geq \Lambda, |\omega(t, y)| \geq \Lambda\}} \leq \rho \sqrt{|x - y|} \quad \text{for all } t \in [0, T_\star[, \ x, y \in \Omega, \]

(4.90)
then $u$ is also strong on $[0,T^*]\times\Omega$.

5. Geometric Boundary Regularity Criterion on Special Domains

In Sect. 4 we proved the estimates for the system \((2.1)\) under the diagonal regular oblique derivative boundary condition \((2.2)\) with constant coefficients. Now let us specialise to the regularity problem for the incompressible Navier–Stokes equations under Navier and kinematic boundary conditions on $B^3,\mathbb{R}_+^3$ and $B^2\times\mathbb{R}$.

Our crucial observation is that the Navier and kinematic boundary conditions, in suitable local coordinate frames, can be cast into the form of Eq. \((2.2)\). The special geometry of the designated domains ensures that the boundary conditions in Eq. \((2.2)\) have constant coefficients, thanks to the differential geometry of surfaces. Thus, Theorem 1.1 follows from Theorem 4.1.

**Proof of Theorem 1.1.** Let us establish the following claim: Given each boundary point $p \in \Sigma$, we can find a local coordinate chart $U \subset \mathbb{R}^3$ containing $p$ and an orthonormal frame $\{\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3\}$ on $U$ with $\{\partial/\partial x^1, \partial/\partial x^2\}$ spanning $\Gamma(TU)$ and $\partial/\partial x^3 = n$, in which the boundary conditions \((1.6)-(1.7)\) take the form of Eq. \((2.2)\) (reproduced below):
\[
a^{(i)}u^i + \sum_{j=1}^3 b^{(i)}_{j}\nabla_j u^i = 0 \quad \text{on } [0,T^*]\times\Sigma
\]
for each $i \in \{1, 2, 3\}$.

To see this, we take $\{\partial/\partial x^1, \partial/\partial x^2\} \subset \Gamma(T\Sigma)$ to be the principal direction fields: that is, we require that the second fundamental form $\Pi = -\nabla n : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ to be diagonalised with respect to this basis. Such coordinate frames always exist, as $\Pi$ is a self-adjoint operator on each $T_p\Sigma$. Then, the Navier boundary condition \((1.6)\) can be rewritten as follows:
\[
0 = \beta u^i + \nu(\nabla_k u^i + \nabla_i u^k)n^k
= \beta u^i + \nu n \cdot \nabla u^i + \nu \nabla_i (u \cdot n) - \sum_{k=1}^3 \nu u^k \nabla_i n^k
\quad \text{for } i \in \{1, 2\}. \quad (5.1)
\]

In regards to the kinematic boundary condition \((1.7)\), the third term on the second line above vanishes. Moreover, the fourth term equals
\[
\sum_{k=1}^3 \nu \Pi_{ik} u^k = \nu \kappa_i u^i,
\]
where $\kappa_i$ is the $i$-th principal curvature, namely the eigenvalue of $\Pi$ that corresponds to the eigenvector $\partial/\partial x^i$. Thus, taking $n = \partial/\partial x^3 \in \Gamma(TM^\perp)$, we may conclude that \((1.6)-(1.7)\) are equivalent to the following system of boundary conditions:
\[
(\beta + \nu \kappa_1)u^1 + \nu \nabla_3 u^1 = 0, \quad (5.2)
(\beta + \nu \kappa_2)u^2 + \nu \nabla_3 u^2 = 0, \quad (5.3)
u^3 = 0 \quad \text{on } [0,T^*]\times\Sigma. \quad (5.4)
\]
Now, let us set (up to normalisations)
\[ a^{(i)} = -\beta - \nu \kappa_i, \quad b_1^{(i)} = b_2^{(i)} = 0, \quad b_3^{(i)} = -\nu \quad \text{for } i \in \{1, 2\} \]
and
\[ a^{(3)} = -1, \quad b_3^{(3)} = 0 \quad \text{for any } j \in \{1, 2, 3\} \]
to recover Eq. (2.2), namely the oblique derivative boundary condition. Due to the presence of viscosity (and the choice \( \mathbf{n} = \partial / \partial x^3 \)), the oblique condition is regular.

For \( \Sigma = \) round spheres, 2-planes and right circular cylinder surfaces, both the mean curvature and the Gauss curvature of the surface are constant, hence \( \kappa_1 \) and \( \kappa_2 \) are constant on \( \Sigma \). In fact, by elementary differential geometry of surfaces, these are the only embedded/immersed surfaces in \( \mathbb{R}^3 \) with constant principal curvatures; see Montiel–Ros [29]. Therefore, in these cases the Navier and kinematic boundary conditions (1.6)(1.7) can be recast to the homogeneous diagonal oblique boundary derivative conditions with constant coefficients, i.e., Eq. (2.2). Hence, thanks to Theorem 4.1 the proof is now complete. □

Using the proof above, we can deduce the following result from Corollary 4.17:

**Corollary 5.1.** Let \( \Omega \subset \mathbb{R}^3 \) be one of the following domains: a round ball, a half-space, or a right circular cylindrical duct. Let \( u \) be a weak solution to the Navier–Stokes equations (1.1)(1.2)(1.3) with the Navier and kinematic boundary conditions (1.6)(1.7). Suppose that the vorticity \( \omega = \nabla \times u \) is coherently aligned, i.e., there are constants \( \rho, \Lambda > 0 \) so that
\[
| \sin \theta(t; x, y) | |_{| \omega(t, x) | \geq \Lambda, | \omega(t, y) | \geq \Lambda} \leq \rho \sqrt{|x - y|} \quad \text{for all } x, y \in \Omega, t < T^*. \tag{5.5}
\]
Then \( u \) is strong on \([0, T^*]\).

It is an interesting problem to study the geometric regularity criteria for weak solutions to the Navier–Stokes equations in general regular domains in \( \mathbb{R}^3 \) under the Navier and kinematic conditions (1.6)(1.7). In full generality, one has difficulty finding nice local frames in which Eqs. (1.6)(1.7) can be transformed to constant-coefficient diagonal oblique derivative boundary conditions. Thus, to analyse the boundary conditions (1.6)(1.7) on general embedded surfaces in \( \mathbb{R}^3 \) calls for new ideas. Also, our current analyses are based on the regular oblique derivative boundary condition. It does not include the Dirichlet/full-slip boundary conditions, which are of great importance in mathematical hydrodynamics. We shall investigate some of these issues in forthcoming works.

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