COMBINATORIAL OPTIMIZATION PROBLEMS WITH INTERACTION COSTS: COMPLEXITY AND SOLVABLE CASES

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ABSTRACT. We introduce and study the combinatorial optimization problem with interaction costs (COPIC). COPIC is the problem of finding two combinatorial structures, one from each of two given families, such that the sum of their independent linear costs and the interaction costs between elements of the two selected structures is minimized. COPIC generalizes the quadratic assignment problem and many other well studied combinatorial optimization problems, and hence covers many real world applications. We show how various topics from different areas in the literature can be formulated as special cases of COPIC. The main contributions of this paper are results on the computational complexity and approximability of COPIC for different families of combinatorial structures (e.g. spanning trees, paths, matroids), and special structures of the interaction costs. More specifically, we analyze the complexity if the interaction cost matrix is parameterized by its rank and if it is a diagonal matrix. Also, we determine the structure of the intersection cost matrix, such that COPIC is equivalent to independently solving linear optimization problems for the two given families of combinatorial structures.

1. INTRODUCTION

Let a family $\mathcal{F}_1$ of subset of $[m] = \{1, 2, \ldots, m\}$, and a family $\mathcal{F}_2$ of subsets of $[n] = \{1, 2, \ldots, n\}$ represent feasible solutions. We assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ have a compact representation of size polynomial in $m$ and $n$, respectively, although the number of feasible solutions in each family could be of size exponential in $m$ or $n$. For each element $i \in [m]$ a linear cost $c_i$ is given. Also, for each element $j \in [n]$ a linear cost $d_j$ is given. In addition, for any $(i, j) \in [m] \times [n]$ their interaction cost $q_{ij}$ is given. Then the combinatorial optimization problem with interaction costs (COPIC) is the problem of finding $S_1 \in \mathcal{F}_1$ and $S_2 \in \mathcal{F}_2$ such that

$$f(S_1, S_2) = \sum_{i \in S_1} \sum_{j \in S_2} q_{ij} + \sum_{i \in S_1} c_i + \sum_{j \in S_2} d_j$$

is minimized. We denote an instance of this problem by COPIC($\mathcal{F}_1, \mathcal{F}_2, Q, c, d$), where $Q = (q_{ij})$ is the interaction cost matrix and $c = (c_i), d = (d_j)$ are linear cost vectors of the instance. This generalizes the classical linear cost combinatorial optimization problem, where for a given family $\mathcal{F}$ of subsets of $[n]$, and cost vector $w \in \mathbb{R}^n$ one tries to find a set $S \in \mathcal{F}$ minimizing

$$\sum_{i \in S} w_i.$$
We denote an instance of this problem by \( \text{LCOP}(\mathcal{F}, w) \).

COPIC generalizes many well studied combinatorial optimization problems. For example, when \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are respectively the family of perfect matchings in bipartite graphs \( G_1 \) and \( G_2 \) with respective edge sets \([m]\) and \([n]\), then COPIC reduces to the bilinear assignment problem (BAP) \cite{22}. BAP is a generalization of the well studied quadratic assignment problem \cite{18} and the three-dimensional assignment problem \cite{51} and hence COPIC generalizes these problems as well. When \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) contain all subsets of \([m]\) and \([n]\) respectively, COPIC reduces to the bipartite unconstrained quadratic programming problem \cite{23,47,35,39} studied in the literature by various authors and under different names. Also, when \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are feasible solutions of generalized upper bound constraints on \( m \) and \( n \) variables, respectively, COPIC reduces to the bipartite quadratic assignment problem and its variations \cite{21,48}. Most quadratic combinatorial optimization problems can also be viewed as special cases of COPIC, including the quadratic minimum spanning tree problem \cite{4}, quadratic set covering problem \cite{5}, quadratic travelling salesman problem \cite{37}, etc. Thus all the applications studied in the context of these special cases are applications of COPIC as well. COPIC is a special case of bilinear integer programs \cite{12,24,30} when \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) can be represented by polyhedral sets. To further motivate the study of COPIC, let us consider the following illustration.

A spanning tree of a graph needs to be constructed as a backbone network. To construct a link of the tree, many different tasks need to be completed, such as digging, building conduits, laying fiber cables, lighting dark fiber etc. Each of the tasks needs to be assigned to different contractors and for each link in a graph the costs vary by quotes from different contractors. We want to assign the tasks to contractors and choose an appropriate tree topology so that the overall construction cost is minimized. This optimization problem can be formulated as a COPIC where feasible solution sets \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) correspond to spanning trees and assignments of tasks to contractors, respectively.

In this paper we investigate various theoretical properties of COPIC. To understand the impact of interaction costs in combinatorial optimization we will analyze special cases of the interaction cost matrix \( Q \) for representative well-studied sets of feasible solutions. Among others, the classes of interaction cost matrices \( Q \) that we will be focused on in this paper include matrices of fixed rank, and diagonal matrices. In the literature many quadratic-like optimization problems have been investigated in the context of fixed rank or low rank cost matrices, for example see \cite{3,11,47,53}. Further, the importance of investigating COPIC with diagonal matrices is illustrated by its direct connections to problems of disjointness of combinatorial structures \cite{50,32,53,28}, packing, covering and partitioning problems \cite{7}, as well as to problems of congestion games \cite{11,54}. In this paper we also pose the problem of identifying cost structures of COPIC instances that can be reduced to an instance with no interaction costs. These instances are called linearizable instances \cite{17,38,40,20,22}. We suggest an approach of identifying such instances for COPIC with specific feasible solution structures along with a characterization of linearizable instances.

The aforementioned topics are investigated on COPIC’s with representative well-studied sets of feasible solutions \( \mathcal{F}_1, \mathcal{F}_2 \). To make easy future references to different sets of feasible solutions we introduce shorthand notations. We denote by \( 2^{[n]} = \{ S : S \subseteq [n] \} \) the unconstrained solution set. Given a matroid \( \mathcal{M} \) we denote by \( \mathcal{B}(\mathcal{M}) \) the set of bases of \( \mathcal{M} \). We denote by \( \mathcal{U}_n^k \) the uniform matroid, whose base set \( \mathcal{B}(\mathcal{U}_n^k) \) is the set of all \( k \)-sets of \([n]\). Given
a graph $G$, $\mathcal{M}(G)$ is the graphic matroid of $G$, whose base set $\mathcal{B}(\mathcal{M}(G))$ is the set of all spanning trees of $G$ (or spanning forests if $G$ is not connected). The set of all maximum matchings of $G$ is denoted by $\mathcal{P}(G)$. Given two terminals $s,t \in V(G)$ the set of all $s$-$t$-paths in $G$ is denoted by $\mathcal{P}(s,t)$. If $G$ is a directed graph $\mathcal{P}(s,t)$ is the set of all directed $s$-$t$-paths in $G$. The set of all cuts in $G$ is denoted by $\mathcal{CUT}(G)$ and $\mathcal{CUT}(s,t)$ is the set of all $s$-$t$-cuts in $G$.

Using these definitions, for example, the bipartite unconstrained quadratic programming problem [47] is denoted by COPIC($2^{[m]}$, $2^{[n]}$, $Q$, $c$, $d$).

The structure of this paper is as follows. We begin by discussing the complexity of COPIC with no significant constraints on the cost structure in Section 2. Section 3 investigates COPIC’s where interaction cost matrix $Q$ is of fixed rank. Using the methods from parametric optimization we show that in the case when one of the solution sets is unconstrained, i.e. $\mathcal{F}_1 = 2^{[n]}$ or $\mathcal{F}_2 = 2^{[m]}$, and linear cost optimization over the other solution set can be done in polynomial time, the problem becomes polynomially solvable. Further, we show that approximability may be achieved in the case of $Q$ with fixed rank. We also show that if the number of breakpoints of multi-parametric linear optimization over both sets of feasible solutions is polynomially bounded and if $Q$ has fixed rank, then COPIC can be solved in polynomial time. Section 4 investigates COPIC’s where interaction cost matrix $Q$ is diagonal. That is, there is a one-to-one relation between ground elements of $\mathcal{F}_1$ and $\mathcal{F}_2$ and the interaction costs appear only between the pairs of the relation. The complexity of COPIC with various well-knows feasible structures (matroids, paths, matchings, cuts, etc.) in the context of diagonal matrix $Q$ are considered, and their relationship to some existing results in the literature is presented. Characterization of linearizable instances is investigated in Section 5. The paper is concluded with Section 6 where we summarize the results and suggest some problems for future work.

2. General complexity

Being a generalization of many hard combinatorial optimization problems, the general COPIC is NP-hard. Moreover, even for the “simple” case with no constraints on the feasible solutions it results in the bipartite unconstrained quadratic programming problem which is NP-hard [47]. COPIC($2^{[m]}$, $2^{[n]}$, $Q$, $c$, $d$) can easily be embedded into a COPIC for most sets of feasible solutions $\mathcal{F}_1$ and $\mathcal{F}_2$, which implies again NP-hardness. However, COPIC($2^{[m]}$, $2^{[n]}$, $Q$, $c$, $d$) is known to be solvable in polynomial time if $Q \leq 0$ and if $Q, c, d \geq 0$ (see Punnen et al. [47]). This is not true anymore if $\mathcal{F}_1, \mathcal{F}_2$ are bases of a uniform matroid, for which we obtain the following hardness result.

**Theorem 1.** COPIC($\mathcal{B}(U_m^{k_1})$, $\mathcal{B}(U_n^{k_2})$, $Q$, $0$, $0$) is strongly NP-hard even if $Q \geq 0$.

**Proof.** We give a reduction from a strongly NP-hard version of the cardinality constrained directed minimum cut problem.

Let $\overline{K}_{m,n}$ be a digraph with vertex sets $[m]$ and $[n]$ and arcs $(i, j)$ for each $i \in [m]$ and $j \in [n]$. The $k$-card min directed cut problem asks for a minimum cost directed cut $\delta^+(S) = \{(i, j) : i \in S, j \notin S\}$ such that $|\delta^+(S)| = k$. Using similar arguments as in [13] one can show that this directed version of the minimum cut problem is strongly NP-hard. Now we show how this problem can be solved in polynomial time, assuming a polynomial time algorithm for COPIC($\mathcal{B}(U_m^{k_1})$, $\mathcal{B}(U_n^{k_2})$, $Q$, $0$, $0$) exists.
For each \( k_1 = 1, 2, \ldots, m \) check if \( \frac{k}{k_1} \) is an integer. If so set \( k_2 = \frac{k}{k_1} \) and solve the instance COPIC(\( \mathcal{B}(U_m^{k_1}), \mathcal{B}(U_m^{k_2}), Q, 0, 0 \)), obtaining solution sets \( S_1, S_2 \). Note that \( |S_1||S_2| = k \), i.e. it corresponds to exactly \( k \) edges. We can define an equivalent directed cut \( \delta^+(S) \) by setting

\[
S = S_1 \cup ([n] \setminus S_2).
\]

This way the directed cuts \( \delta^+(S) \) are in one to one correspondence with solutions of COPIC. Doing this for all possible pairs \((k_1, k_2)\), we can obtain all possible \( k \)-cuts as feasible solutions of instances of COPIC(\( \mathcal{B}(U_m^{k_1}), \mathcal{B}(U_m^{k_2}), Q, 0, 0 \)). Taking the minimum found via all such COPIC problems solves the \( k \)-card directed min cut problem in the given bipartite digraph. \( \square \)

Theorem \( \Box \) can be used to show that COPIC(\( \mathcal{F}_1, \mathcal{F}_2, Q, 0, 0 \)) is NP-hard already for \( Q \geq 0 \) for most sets of feasible solutions \( \mathcal{F}_1, \mathcal{F}_2 \), since in many cases cardinality constraints can be easily encoded in more complicated sets of feasible solutions.

On the positive side, if we fix one of the two solutions, e.g. \( S_1 \in \mathcal{F}_1 \), then finding the corresponding optimal solution \( S_2 \in \mathcal{F}_2 \) reduces to solving LCOP(\( \mathcal{F}_2, h \)), where

\[
h_j := \sum_{i \in S_1} q_{ij} + d_j \quad \text{for} \quad j \in [n].
\]

This implies that if the cardinality of one set of feasible solutions, say \( \mathcal{F}_1 \), is polynomially bounded in the size of the input, then we can solve COPIC by solving linear instances LCOP(\( \mathcal{F}_2, h \)) (where \( h \) is defined by \( (2) \)) for all \( S_1 \in \mathcal{F}_1 \).

**Theorem 2.** If \( m = O(\log n) \) and LCOP(\( \mathcal{F}_2, h \)) can be solved in polynomial time for any cost vector \( h \in \mathbb{R}^n \), then COPIC(\( \mathcal{F}_1, \mathcal{F}_2, Q, c, d \)) can be solved in polynomial time.

### 3. The Interaction Matrix with Fixed Rank

In this section we investigate the behavior of COPIC in terms of complexity and approximability when the rank of the interaction costs matrix \( Q \) is fixed. In the literature, many optimization problems have been investigated in the context of fixed rank or low rank cost matrices. This also includes problems with quadratic-like objective functions. For example, the Koopmans-Beckmann QAP \([11]\), the unconstrained zero-one quadratic maximization problem \([3]\), bilinear programming problems \([55]\), the bipartite unconstrained quadratic programming problem \([17]\), among others.

Let \( \text{rk}(Q) \) denotes the rank of a matrix \( Q \). Then \( \text{rk}(Q) \) is at most \( r \), if and only if there exist vectors \( a_p = (a_1^{(p)}, a_2^{(p)}, \ldots, a_m^{(p)}) \in \mathbb{R}^m \) and \( b_p = (b_1^{(p)}, b_2^{(p)}, \ldots, b_n^{(p)}) \in \mathbb{R}^n \) for \( p = 1, 2, \ldots, r \), such that

\[
Q = \sum_{p=1}^{r} a_p b_p^T.
\]

We say that \( (3) \) is a factored form of \( Q \). Then COPIC(\( \mathcal{F}_1, \mathcal{F}_2, Q, c, d \)), where \( Q \) is of fixed rank \( r \), becomes minimizing

\[
f(S_1, S_2) = \sum_{p=1}^{r} \left( \sum_{i \in S_1} a_i^{(p)} \sum_{j \in S_2} b_j^{(p)} \right) + \sum_{i \in S_1} c_i + \sum_{j \in S_2} d_j,
\]

such that \( S_1 \in \mathcal{F}_1, S_2 \in \mathcal{F}_2 \).
In the following, we show that if \( \mathcal{F}_1(= 2^{[m]}) \) is unrestricted, i.e. the set of all subsets of \([m]\), then we can generalize the results of Punnen et al. \[47\] to solve the problem. Using methods of multi-parametric optimization we also demonstrate how to tackle more-general problems where both sets of feasible solutions are constrained, if their parametric complexity is bounded.

These results are obtained using methods from binary and linear optimization. To apply these techniques we will formulate our problem in terms of binary variables. We achieve this in a straightforward way, by introducing variables \( x \in \{0,1\}^m, y \in \{0,1\}^n \) in one to one correspondence with a solution \( S_1, S_2 \), such that \( x_i = 1 \) iff \( i \in S_1 \), and \( y_j = 1 \) iff \( j \in S_2 \). The vector \( x \) and \( y \) are respectively called the incidence vectors of \( S_1 \) and \( S_2 \). Thus the family of feasible solutions can be represented in terms of the incidence vectors, i.e. \( \mathcal{F}_1 = \{ x \in \{0,1\}^m : S_1 \in \mathcal{F}_1 \text{ and } (x_j = 1 \iff j \in S_1) \} \) and \( \mathcal{F}_2 = \{ y \in \{0,1\}^n : S_2 \in \mathcal{F}_2 \text{ and } (y_j = 1 \iff j \in S_2) \} \). Now, rank \( r \) COPIC can be formulated as the binary optimization problem:

\[
\begin{align*}
\min_{x,y} & \sum_{p=1}^{r} (a_p^T x)(b_p^T y) + c^T x + d^T y \\
\text{s.t.} & \quad x \in \mathcal{F}_1 \\
& \quad y \in \mathcal{F}_2
\end{align*}
\]

3.1. One-sided unconstrained fixed rank COPIC. In this section we consider the case where \( \mathcal{F}_1 = \{0,1\}^m \). Observe that COPIC is equivalent to the following linear relaxation of the constraint \( x \in \{0,1\}^m \).

\[
\begin{align*}
\min_{x,y} & \sum_{p=1}^{r} (a_p^T x)(b_p^T y) + c^T x + d^T y \\
\text{s.t.} & \quad x \in [0,1]^m \\
& \quad y \in \mathcal{F}_2
\end{align*}
\]

To solve this problem, consider the multi-parametric linear program (MLP)

\[
h_1(\lambda) := \min c^T x \\
\text{s.t. } a_p^T x = \lambda_p \text{ for } p = 1, 2, \ldots, r \\
\quad x \in [0,1]^m,
\]

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{R}^r \). Then \( h_1(\lambda) \) is a piecewise linear convex function \[33\]. A basis structure for MLP is a partition \( (B,L,U) \) of \([m]\), such that \(|B| = r\). With each basic feasible solution of MLP we associate a basis structure \( (B,L,U) \), where \( L \) is the index set of nonbasic variables at the lower bound 0, \( U \) is the index set of nonbasic variables at the upper bound 1 and \( B \) is the index set of basic variables. Given a dual feasible basis structure \( (B,L,U) \), the set of values \( \lambda \in \mathbb{R}^r \) for which the corresponding basic solution is optimal is called the characteristic region of \( (B,L,U) \). Since \( h_1(\lambda) \) is piecewise linear convex, \( h_1(\lambda) \) is linear if \( \lambda \) is restricted to a characteristic region associated with a dual feasible basic structure \( (B,L,U) \). We call the extreme points of the characteristic regions of \( (B,L,U) \) as breakpoints and denote the set of these breakpoints by \( B_1 \) and define \( x(\lambda) \) as the optimal basic feasible solution of \( h_1(\lambda) \) at each \( \lambda \in B_1 \). By the results of Punnen et al. \[47\] Theorem 3 we know
that $x(\lambda) \in \{0,1\}^m$. Let $y(\lambda) \in \mathcal{F}_2$ be an optimal solution to our instance of COPIC when $x$ is fixed at $x(\lambda)$. In this case COPIC reduces to

$$\min \left( \sum_{p=1}^r (a^T_p x(\lambda)) b_p^T + d^T \right) y$$

s.t. $y \in \mathcal{F}_2$

which is an instance of LCOP($\mathcal{F}_2, f$), with $f = \sum_{p=1}^r (a^T_p x(\lambda)) b_p + d$. This allows us to calculate $y(\lambda)$ in $O(\max\{rmn, T(\mathcal{F}_2)\})$ time, using an $T(\mathcal{F}_2)$-time algorithm for LCOP($\mathcal{F}_2, f$), for each $\lambda \in B_1$.

**Theorem 3.** There exists an optimal solution to COPIC($2^{[m]}, \mathcal{F}_2, Q, c, d$) with $\text{rk}(Q) = r$ amongst the solutions $\{(x(\lambda), y(\lambda)) : \lambda \in B_1\}$.

**Proof.** Rank $r$ COPIC is equivalent to solving the bilinear program

$$\min \sum_{p=1}^r \lambda_p (b^T_p y) + c^T x + d^T y$$

s.t. $a^T_p x = \lambda_p$, $p = 1, 2, \ldots, r$

$x \in [0,1]^m$, $y \in \mathcal{F}_2$, $\lambda \in \mathbb{R}_r$.

Let $h(\lambda)$ be the optimal value if $\lambda$ is fixed, then we can decompose $h(\lambda)$ into $h(\lambda) = h_1(\lambda) + h_2(\lambda)$, where

$$h_2(\lambda) = \min \sum_{p=1}^r \lambda_p (b^T_p y) + d^T y$$

s.t. $y \in \mathcal{F}_2$.

So rank $r$ COPIC can be reduced to solving

$$\min_{\lambda \in \mathbb{R}_r} h(\lambda).$$

We already argued above that $h_1(\lambda)$ is a piecewise linear convex function in $\lambda$. Using the fact that $h_2(\lambda)$ is the pointwise minimum of linear functions, we obtain that $h_2(\lambda)$ is a piecewise linear concave function in $\lambda$ [12]. This implies that $h_1(\lambda)$ is linear, if $\lambda$ is restricted to any characteristic region of $h_1(\lambda)$ and thus $h(\lambda)$ is concave on each of these regions. This implies that the minimum of $h(\lambda)$ is attained at a breakpoint of $h_1(\lambda)$, which implies the result since $B_1$ is defined as the set of these breakpoints. □

Analogously to Punnen et al. [47], we can use Theorem 3 to solve rank $r$ COPIC using the following approach.

1. Compute the set $\bar{S}$ of all optimal basic feasible solutions corresponding to the extreme points of the characteristic region of a dual feasible basis structure $(\mathcal{E}, \mathcal{L}, \mathcal{U})$ of $h_1(\lambda)$.
2. For each $x \in \bar{S}$ compute the best $y \in \mathcal{F}_2$ by solving LCOP($\mathcal{F}_2, f$), with $f = \sum_{p=1}^r (a^T_p x)b_p + d$.
3. Output the best pair $(x, y)$ with minimum total cost found in the last step.
By the arguments above it follows that this algorithm finds an optimal solution. There are \( \binom{m}{r} \) choices for \( \mathcal{B} \) and each of them gives a unique allocation of non-basic variables to \( \mathcal{L} \) and \( \mathcal{U} \) (uniqueness following from non-degeneracy which can be achieved by appropriate perturbation of the cost vector). The basis inverse can be obtained in \( O(r^3) \) time and given this inverse \( \mathcal{L} \) and \( \mathcal{U} \) can be identified in \( O(mr^3) \) time, such that \( (\mathcal{B},\mathcal{L},\mathcal{U}) \) is dual feasible. This implies that the set of dual feasible basis structures is bounded by \( \binom{m}{r} \) and can be calculated in \( O((\binom{m}{r})(r^3 + mr^2)) \) time. By [47, Theorem 3], we know that the number of extreme points associated with \( (\mathcal{B},\mathcal{L},\mathcal{U}) \) is bounded by \( 2^r \) and how to calculate the optimal solution of \( h_1(\lambda) \) for \( \lambda \) fixed at these extreme points without explicitly calculating \( \lambda \). This allows us to compute \( \tilde{S} \) in \( O((\binom{m}{r})2^mm) \) time. Fixing \( x \in \tilde{S} \), the best corresponding solution \( y \) can be computed in \( O(\max\{mn,TCOP(\mathcal{F}_2)\}) \) time. Summarizing this gives the following result.

**Theorem 4.** If \( \text{rk}(Q) = r \) and there is a \( T(\mathcal{F}_2) \)-time algorithm for LCOP(\( \mathcal{F}_2, f \)) for every \( f \in \mathbb{R}^n \), then COPIC(\( 2^{[m]} \), \( \mathcal{F}_2, Q, c, d \)) can be solved in \( O((\binom{m}{r})2^r \max\{mn, TCOP(\mathcal{F}_2)\}) \) time.

**Remark.** An identical approach works for sets of feasible solutions \( \mathcal{F}_1 \), for which we can solve the linear cost minimization problem, extended by a constant number of side constraints of the form \( a_p^T x = \lambda_p \) and the number of breakpoints (in \( \lambda \)) is polynomially bounded. But this does not help for most non-continuous problems, because already for the bases of a uniform matroid this corresponds to a partition problem.

We can now use Theorem 3 to obtain approximation algorithms for rank \( r \) COPIC based on approximation algorithms for the linear problem with feasible solutions in \( \mathcal{F}_2 \).

**Theorem 5.** COPIC(\( 2^{[m]} \), \( \mathcal{F}_2, Q, c, d \)) such that LCOP(\( \mathcal{F}_2, f \)) admits a \( T(\mathcal{F}_2) \) time \( \alpha \)-approximation algorithm for arbitrary \( f \in \mathbb{R}^n \), has a \( O((\binom{m}{r})2^r \max\{mn, TCOP(\mathcal{F}_2)\}) \) time \( \alpha \)-approximation algorithm.

**Proof.** By Theorem 3 there exists an optimal solution

\[
(x^*, y^*) = (x(\lambda^*), y(\lambda^*)) \in \{(x(\lambda), y(\lambda)) : \lambda \in B_1\}.
\]

By the method above we will in some iteration find \( x^* \) as one of the extreme points of a characteristic region of \( h_1(\lambda) \). Then calculating \( y^* \) is equivalent to solving LCOP(\( \mathcal{F}_2, f \)) with \( f = \sum_{p=1}^{r} (a_p^T x^*) b_p + d \). Instead of solving this problem to optimality we can use our \( \alpha \)-approximation algorithm and obtain a solution \( \tilde{y} \in \mathcal{F}_2 \) such that

\[
\tilde{h}_2 := \sum_{p=1}^{r} (a_p^T x^*) (b_p^T \tilde{y}) + d^T \tilde{y} \leq \alpha h_2(\lambda^*).
\]

Altogether for our found solution \( (x^*, \tilde{y}) \) we obtain a bound on the objective value given by

\[
h_1(\lambda^*) + \tilde{h}_2 \leq h_1(\lambda^*) + \alpha h_2(\lambda^*) \leq \alpha h(\lambda^*).
\]

For the more general case of rank \( r \) COPIC, where both \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are constrained, we can still obtain a FPTAS based on the results of Mittal and Schulz [44], for a restricted class of objective functions.
Theorem 6 (Mittal and Schulz [44]). Consider the separable bi-linear programming problem

\[ \min \sum_{p=1}^{r} (a_p^T x)(b_p^T y) + c^T x + d^T y \]
\[ \text{s.t. } x \in P_1 \]
\[ y \in P_2 \]

where \( P_1, P_2 \) are polytopes, completely given in terms of linear inequalities or by a polynomial time separation oracle, for fixed \( r \). Then the problem admits a FPTAS giving a solution that is an extreme point of \( P_1, P_2 \), if \( c^T x > 0, d^T y > 0 \) and \( a_p^T x > 0, b_p^T y > 0 \) for \( p = 1, 2, \ldots, r \) over the polytopes \( P_1, P_2 \).

This result directly implies a FPTAS for COPIC(\( F_1, F_2, Q, c, d \)), if \( \text{rk}(Q) = r \) and the sets \( F_1, F_2 \) can be represented as polytopes of polynomial size or polytopes with a polynomial time separation oracle. This is for instance the case for matroid constraints. See [44] for a detailed description of the FPTAS.

3.2. General fixed rank COPIC via multi-parametric optimization. To solve fixed rank COPIC when both sets of feasible solutions \( F_1 \) and \( F_2 \) are constrained, we again apply methods from parametric optimization. Since in many cases additional linear constraints of the form \( a_p^T x = \lambda_p \) imply NP-hardness, we cannot follow an identical approach as above. Instead, we analyze and solve multi-parametric objective versions for both sets of feasible solutions directly. Given linear cost vectors \( a_1, a_2, \ldots, a_r \in \mathbb{R}^n \) and \( c \in \mathbb{R}^n \) in addition to a set of feasible solutions \( F \subseteq \{0,1\}^n \), the problem of finding optimal solutions to

\[ \min \sum_{p=1}^{r} \mu_p(a_p^T x) + c^T x \]
\[ \text{s.t. } x \in F \]

for all possible values of \( \mu \in \mathbb{R}^r \) is called multi-parametric linear optimization over \( F \). In this section the number of vectors \( a \) will always be fixed to \( r \). For every fixed \( \mu \in \mathbb{R}^r \) this is equivalent to solving an instance of LCOP(\( F, h \)) for \( h = \sum_{p=1}^{r} \mu_p a_p + c \). We denote this problem by MPLCOP(\( F, a, c \))(\( \mu \)).

It is well known that MPLCOP(\( F, a, c \))(\( \mu \)) is a piecewise-linear concave function in \( \mu \) on \( \mathbb{R}^r \). For such a function the parameter space \( \mathbb{R}^r \) can be partitioned into regions \( M_1, M_2, \ldots, M_l \), such that in each of these regions the optimal objective value is linear in \( \mu \) and for each \( i = 1, 2, \ldots, l \) there exists a solution \( x_i \in F \) that achieves this value on the whole region \( M_i \). The smallest needed number \( l \) of such regions is called the parametric complexity of MPLCOP(\( F, a, c \)). Böckler and Mutzel [3] showed that there is an output-sensitive algorithm for MPLCOP(\( F, a, c \)) to obtain all the solutions \( x_1, x_2, \ldots, x_l \) with running time \( O(\text{poly}(n, m, r^r)) \), if LCOP(\( F, h \)) can be solved in polynomial time.
Given an instance of fixed rank COPIC

\[
\min \sum_{p=1}^{r} (a_p^T x)(b_p^T y) + c^T x + d^T y
\]

s.t. \( x \in F_1 \)
\( y \in F_2 \)

and its optimal solution \((x^*, y^*) \in F_1 \times F_2\), we observe that \(x^*\) is an optimal solution to MPLCOP\((F_1', a, c)(\mu^*)\) for \(\mu^*_p = b_p^T y^*\) and \(y^*\) is an optimal solution to MPLCOP\((F_2', b, d)(\lambda^*)\) for \(\lambda^*_p = a_p^T x^*\). This yields the following approach for solving such instances of COPIC:

1. Obtain optimal solutions \(x_1, x_2, \ldots, x_{l_1}\) for all possible parameter values \(\mu\) of MPLCOP\((F_1', a, c)(\mu)\) and \(y_1, y_2, \ldots, y_{l_2}\) for all possible parameter values \(\lambda\) of MPLCOP\((F_2', b, d)\).
2. Calculate their corresponding parameter values \(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l_1)}\) and \(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(l_1)}\) as \(\lambda^{(i)}_p = a_p^T x_i\) and \(\mu^{(j)}_p = b_p^T y_j\).
3. For each pair \((x_i, y_j)\) check if \(x_i\) is optimal for LCOP\((F_1', a, c)(\mu^{(j)})\) and \(y_j\) is optimal for LCOP\((F_2', b, d)(\lambda^{(i)})\).
4. Among all the pairs that fulfill conditions in (3), take the one with minimum objective value for our instance of COPIC.

To guarantee that this method finds the optimal solution \((x^*, y^*)\) the two given instances of MPLCOP must be non-degenerate. This can be guaranteed by appropriate perturbations of the cost vectors. Based on the algorithm of Böckler and Mutzel [9] we obtain the following result.

**Theorem 7.** Let \(l_1, l_2\) be the parametric complexity of MPLCOP\((F_1, a, c)\), MPLCOP\((F_2', b, d)\) respectively, and \(\text{rk}(Q) = r\) is a constant. If both LCOP\((F_1, h)\) and LCOP\((F_2, h)\) can be solved in polynomial time for arbitrary linear cost vectors \(h\), then COPIC\((F_1, F_2, Q, c, d)\) can be solved in \(O(\text{poly}(n, m, l_1, l_2))\) time.

If \(\mathcal{F}\) is the set of bases of a matroid, Ganley et al. [34] showed that the parametric complexity of MPLCOP\((\mathcal{F}, a, c)\) for arbitrary \(a, c\) over the whole parameter region \(\mathbb{R}^r\) is polynomially bounded, if \(r\) is fixed.

**Theorem 8** (Ganley et al. [34]). If \(\mathcal{F} = \mathcal{B}(\mathcal{M})\) is the set of bases of a matroid \(\mathcal{M}\) with \(n\) elements, the parametric complexity of MPLCOP\((\mathcal{F}, a, c)\) for arbitrary \(a\) and \(c\) is bounded by \(O(n^{2r-2})\).

This implies a polynomial time algorithm for COPIC\((\mathcal{B}(\mathcal{M}_1), \mathcal{B}(\mathcal{M}_2), Q, c, d)\) for arbitrary matroids \(\mathcal{M}_1, \mathcal{M}_2\) and fixed rank matrix \(Q\). For rank 1 problems Eppstein [26] gives stronger bounds for the parametric complexity. It is conjectured that for higher rank matrices and bases of matroids as feasible solution, stronger bounds than the one given in Theorem 8 can be achieved. For other types of feasible solutions, like paths or bipartite matchings, this approach does not yield polynomial time algorithms.

Another set of feasible solutions for which this approach yields a polynomial time algorithm are global cuts in a graph. Karger [40] currently gives the best bound for the parametric complexity of cuts and obtains several other related sets of feasible solutions with similar polynomial bounds. In this case we are even able to bound the number of distinct cuts that
can become optimal, instead of just the parametric complexity, so degeneracy is not even an
issue here.

**Theorem 9** (Karger [40]). If \( \mathcal{F} = \text{CUT}(G) \) the number of cuts that can become optimal in
MPLCOP(\( \mathcal{F}, a, c \)) for arbitrary \( a \) and \( c \) over all choices of \( \mu \) is bounded by \( O(n^{r+1}) \).

Already for the case \( r = 1 \) subexponential lower bounds for the parametric complexity
of MPLCOP(\( \mathcal{F}, a, c \)) for paths (\( \mathcal{F} = \text{P}_{x,t}(G) \)) and matchings (\( \mathcal{F} = \text{PM}(G) \)) in a graph
\( G \) are known (Gusfield [36], Carstensen [16]). However, in the setting of smoothed analysis
Brunsch and Röglin [14] showed that the parametric complexity of MPLCOP(\( \mathcal{F}, a, c \)) is
bounded by \( O(n^{2r} \phi^r) \) for every perturbation parameter \( \phi \geq 1 \), all costs \( a, c \) and arbitrary
sets of feasible solutions \( \mathcal{F} \).

4. **Diagonal interaction matrix**

In this section we analyze the special case of COPIC, referred to as diagonal COPIC,
where for a given vector \( a \in \mathbb{R}^n \) the matrix \( Q = (q_{ij}) \) is given as the diagonal \( n \times n \) matrix
\[
q_{ij} = \begin{cases} 
  a_i & \text{if } i = j \\
  0 & \text{otherwise}
\end{cases}
\]
This results in finding solutions \( S_1 \in \mathcal{F}_1 \subseteq \{0,1\}^n \) and \( S_2 \in \mathcal{F}_2 \subseteq \{0,1\}^n \) that minimize the
objective function
\[
f(S_1, S_2) = \sum_{i \in S_1 \cap S_2} a_i + \sum_{i \in S_1} c_i + \sum_{j \in S_2} d_j.
\]
Such instances are denoted by COPIC(\( \mathcal{F}_1, \mathcal{F}_2, \text{diag}(a), c, d \)).

Already this very restricted version of COPIC includes many well-studied problems of
combinatorial optimization. For example, problems that ask for two disjoint combinatorial
structures among an element set can all be handled by solving COPIC with identity interaction
matrix \( Q = I \) and \( c = d = 0 \). This includes the disjoint spanning tree problem [50],
disjoint matroid base problem [32], disjoint path problems [53, 28], disjoint matchings problem [31]
and many others. Bernth and Kirly [7] analyzed the computational complexity of many combinations of
different packing, covering and partitioning problems on graphs and
matroids. It is easy to model all of these problems as instances of diagonal COPIC. The
hardness results for packing problems in this paper directly imply NP-hardness results for
diagonal COPIC with \( Q = I \) and \( c = d = 0 \) for several classes of problems. In this section we
further investigate complexity of diagonal COPIC. Some results investigated in this section
are summarized in Table 1.

4.1. **Unconstrained feasible sets.** We start by considering diagonal COPIC with unconstrained feasible sets.

**Theorem 10.** COPIC(\( 2^n, 2^n, \text{diag}(a), c, d \)) can be solved in linear time.

**Proof.** For each \( e \in [n] \) independently we have four different choices:

- \( e \notin S_1 \cup S_2 \): this contributes 0 to \( f(S_1, S_2) \)
- \( e \in S_1, i \notin S_2 \): this contributes \( c_e \) to \( f(S_1, S_2) \)
- \( e \notin S_1, i \in S_2 \): this contributes \( d_e \) to \( f(S_1, S_2) \)
- \( e \in S_1 \cap S_2 \): this contributes \( a_e + c_e + d_e \) to \( f(S_1, S_2) \)
So for each $e \in [n]$ we can independently find $\min\{0, c_e, d_e, a_e + c_e + d_e\}$ and select the corresponding solution accordingly. This can be done in constant time for each $e \in [n]$, so the overall running time is $O(n)$.

The result of Theorem 10 can be generalized. Using a straightforward dynamic programming approach, COPIC($\mathcal{F}_1, \mathcal{F}_2, Q, c, d$) with matrix $Q$ of bandwidth $O(\log n)$ can be solved in polynomial time. This result is presented as Theorem 2.11 in the PhD thesis of Sripratak [52].

**Theorem 11.** COPIC($\mathcal{F}, 2^{[n]}$, diag$(a, c, d)$) can be solved by solving LCOP($\mathcal{F}, f$), where $f_i = \min\{c_i + d_i + a_i, c_i\} - \min\{d_i, 0\}$ for each $i \in [n]$.

**Proof.** For each $i \in [n]$ we can determine independently if it should be included in $S_2$, given that it is included in $S_1$ or not. The cost for an element $i \in [n]$ is therefore uniquely determined as $f_1(i) = \min\{c_i + d_i + a_i, c_i\}$, if $i \in S_1$ and as $f_2(i) = \min\{d_i, 0\}$ if $i \notin S_1$. So we can determine an optimal solution $S_1$ by solving the minimization problem over $\mathcal{F}$ for the linear cost function $f_1 - f_2$. The corresponding corresponding optimal $S_2$ can be easily obtained in $O(n)$ time. \hfill \Box

### 4.2. Uniform and Partition Matroids

In the following two subsections we investigate diagonal COPIC where $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond to bases of different types of matroids. For bases of uniform and partition matroids, which are defined by standard cardinality constraints, the main insight is that we can solve our problem in polynomial time using matching algorithms.

Given a graph $G = (V, E)$ and a function $b: V \to 2^N$ an edge set $M \subseteq E$ is a $b$-factor, if $|M \cap \delta(v)| \in b(v)$ for each $v \in V$. If $b(v) = \{k\}$ for some integer $k \in \mathbb{N}$ we simply write $b(v) = k$. Given an additional cost function $c: E \to \mathbb{R}$ a minimum cost $b$-factor can be found in polynomial time, if all the $b$-values $b(v)$ are sequences of consecutive integers $[b_1; b_2] = \{b_1, b_1 + 1, \ldots, b_2\}$ (see [43, Section 10.2]).

**Theorem 12.** COPIC($\mathcal{B}(U_n^{k_1}), \mathcal{B}(U_n^{k_2}), \text{diag}(a, c, d)$) can be solved in polynomial time.

**Proof.** We create an equivalent instance of the minimum cost $b$-factor problem on a graph $G$ (see Figure 1). To achieve this, we introduce two special vertices $x$ and $y$ with $b(x) = k_1$ and $b(y) = k_2$ and another $3n$ vertices $i_x, i_y$ and $i_m$ for $i = 1, 2, \ldots, n$, i.e., for each element of the ground set of the two matroids. We set $b(i_x) = b(i_y) = 1$ and $b(i_m) = \{0, 1\}$. The $k_1$ vertices matched with $x$ and $k_2$ vertices matched with $y$ correspond to the sets $S_1$ and $S_2$, respectively.
We introduce edges \( \{x, i_x\} \) with cost \( c_i + \frac{a_i}{2} \) and \( \{y, i_y\} \) with cost \( d_i + \frac{a_i}{2} \). We also connect \( \{i_x, i_y\} \) with edges of cost 0 and \( \{i_x, i_m\} \) and \( \{i_x, i_m\} \) both with cost \(-\frac{a_i}{2}\).

It is easy to see that there is a one-to-one mapping between feasible solutions of the given diagonal COPIC and this instance of the \( b \)-factor problem, and moreover, the corresponding costs are the same. Any feasible \( b \)-factor \( M \) must contain exactly \( k_1 \) edges of the form \( \{x, i_x\} \) and \( k_2 \) edges of the form \( \{y, i_y\} \). These can be identified with the solution sets \( S_1 \) and \( S_2 \) for our diagonal COPIC. Given any such partial \( b \)-factor there exists exactly one completion to a feasible \( b \)-factor, using additional edges inside the triangles \( i_x, i_y, i_m \) for each \( i \in [n] \), according to the following four cases. We can also directly observe that the cost of the enforced \( b \)-factor and the solution \( S_1, S_2 \) is the same.

1. \( i \notin S_1, i \notin S_2 \): Both \( i_x \) and \( i_y \) are unmatched. The only way to match both is by using the single edge \( \{i_x, i_y\} \) and leaving \( i_m \) unmatched, which is feasible since \( 0 \in b(i_m) \).
   The contribution to the total cost is 0.

2. \( i \in S_1, i \notin S_2 \): In this case \( i_x \) is already matched but \( i_y \) is still unmatched. The only feasible way to match \( i_y \) is using the edge \( \{i_y, i_m\} \), which contributes \( c_i \) to the cost.

3. \( i \notin S_1, i \in S_2 \): This case is symmetric to case (2). The cost contribution is \( d_i \).

4. \( i \in S_1, i \in S_2 \): In this case \( i_x \) and \( i_y \) are both already matched and \( i_m \) cannot be matched anymore. We get a cost contribution of \( a_i + c_i + d_i \).

\[ \square \]

Given a partition \( S_1, S_2, \ldots, S_t \) of the ground set \( E \) and integers \( g_1, g_2, \ldots, g_t \), such that \( 0 \leq g_i \leq |B_i| \) for all \( i = 1, 2, \ldots, t \), the set \( \{X \subseteq E : |X \cap S_i| = g_i \text{ for all } i = 1, 2, \ldots, t\} \) forms the collection of all bases of a partition matroid.

**Corollary 13.** If \( \mathcal{M}_1, \mathcal{M}_2 \) are partition matroids, \( \text{COPIC}(\mathcal{B}(\mathcal{M}_1), \mathcal{B}(\mathcal{M}_2), \text{diag}(a, c, d)) \) can be solved in polynomial time.

**Proof.** To prove this theorem, we use the approach based on matchings as in the proof of Theorem 12 with minor modifications. Instead of special vertices \( x \) and \( y \), we introduce \( x \)-and \( y \)-vertices for each set in the partition of the ground set and connect these vertices only to the \( i_x, i_y \)-vertices that are in the corresponding set of the partition. The equivalence of this construction can be shown analogously. \[ \square \]
One can even further generalize the concept of partition matroids. Given a partition $S_1, S_2, \ldots, S_t$ of the element set $E$ and integers $f_1, f_2, \ldots, f_t, g_1, g_2, \ldots, g_t, k$, such that $0 \leq f_i \leq g_i \leq |S_i|$ for all $i = 1, 2, \ldots, t$ and $\sum_{i=1}^{t} f_i \leq k \leq \sum_{i=1}^{t} g_i$. The set $\{X \subseteq E : |X| = k \text{ and } f_i \leq |X \cap S_i| \leq g_i \forall i = 1, 2, \ldots, t\}$ is the set of bases of a generalized partition matroid [29]. A similar approach based on matchings still applies, if $F_1, F_2$ are sets of bases of a generalized partition matroids.

The combination of bases of a uniform matroid with other sets of feasible solutions in diagonal COPIC is similar to different versions of linear problems with capacity side constraints. The following result is an example that can be derived using methods for the well-studied constrained shortest path problem with uniform edge weights, for which dynamic programming can be used to solve the problem in polynomial time (see Dumitrescu and Boland [24] for a review).

**Theorem 14.** COPIC($\mathcal{B}(U_m^k), \mathcal{P}_{s,t}(G), \text{diag}(a), 0, d$) can be solved in polynomial time, if $a \geq 0$ and $d \geq 0$.

### 4.3. Matroid bases as feasible sets

Another problem of great interest is the case when $F_1, F_2$ are sets of spanning trees of a graph, especially if the underlying graphs are isomorphic. A generalization of this problem is the case when $F_i = \mathcal{B}(\mathcal{M}_i)$ are the sets of bases of (not necessarily isomorphic) matroids $\mathcal{M}_1, \mathcal{M}_2$. In this section we assume familiarity with matroids and refer the reader to Oxley [45] for further definitions, results and notations.

We will first focus on the case without linear costs, i.e. $c \equiv d \equiv 0$. So the problem we are interested in is, given a ground set $E = [n]$ and a cost vector $a \in \mathbb{R}^n$, to minimize the objective function

$$f(B_1, B_2) = \sum_{i \in B_1 \cap B_2} a_i$$

under the restrictions that $B_1 \in \mathcal{B}(\mathcal{M}_1), B_2 \in \mathcal{B}(\mathcal{M}_2)$ for two given matroids $\mathcal{M}_1, \mathcal{M}_2$ over the ground set $E$.

#### 4.3.1. Minimum cardinality base intersection

If the cost vector $a \equiv 1$ (together with $c \equiv d \equiv 0$) this gives the problem of minimizing the size of the intersection of the two matroid bases $B_1$ and $B_2$. It contains as a special case the disjoint matroid base problem for two given matroids, since there exist two disjoint bases if and only if the optimal solution has objective value 0.

Gabow and Westermann [32] showed that the disjoint matroid base problem can be efficiently solved under the assumption that there exist efficient oracles to solve the static-base circuit problem. This means that for both matroids $\mathcal{M}_i$, $i = 1, 2$, independent set $S$ and element $e \notin S$, we can efficiently decide if $S \cup \{e\}$ is independent in $\mathcal{M}_i$, and if not, output all elements in $C(e, S)$, the unique cycle contained in $S \cup \{e\}$ of the matroid $\mathcal{M}_i$.

#### 4.3.2. Minimum cost base intersection

The more general case, where for each element $i \in B_1 \cap B_2$ we pay a non-negative cost $a_i \geq 0$ was already studied in the algorithmic game theory literature. It is equivalent to computing the socially optimal state of a two player matroid congestion game. Ackermann et al. [1] show that this problem can be solved in polynomial time for an arbitrary number of players using the same approach that was used by Werneck et al. [54] to calculate the socially optimal state in spanning tree congestion games.
To keep this work self contained we give a summary of their algorithm using the notation of diagonal COPIC. We reduce the problem to an equivalent instance of the minimum cost disjoint base problem, for which we can guarantee the existence of two disjoint bases.

The idea of the construction is to double all elements of $E$. The new ground set of elements is denoted by $E' = E_1 \cup E_2$, where $E_1, E_2$ are two disjoint copies of the original ground set $E$. For $i \in E$ we write $i_1$ for the copy of $i$ inside $E_1$ and $i_2$ for its copy in $E_2$. We set $a_{i_1} = a_i$ and $a_{i_2} := 0$ for each $i \in E$ and introduce two new matroids $\mathcal{M}_1', \mathcal{M}_2'$, each with $E'$ as their ground set. The independent sets of $\mathcal{M}_j'$ are all sets $S' \subseteq E'$ that do not contain both $i_1$ and $i_2$ for any $i \in E$ and where $\{i \in E: i_1 \in S' \text{ or } i_2 \in S'\}$ is independent in $\mathcal{M}_j$, for $j = 1, 2$.

Given two disjoint bases $B_1'$ of $\mathcal{M}_1'$ and $B_2'$ of $\mathcal{M}_2'$, they induce, not necessarily disjoint, bases $B_1, B_2$ of $\mathcal{M}_1, \mathcal{M}_2$. For every element $i \in B_1 \cap B_2$ we know that both $i_1$ and $i_2$ were used in $B_1'$ and $B_2'$. So for this element the cost $a_i$ is payed in the disjoint base problem. For all other elements $i_2$ is used, since $0 = a_{i_2} \leq a_{i_1}$.

Efficient methods for solving the minimum cost disjoint base problem for general matroids obtained by Gabow and Westermann [32] can be used to solve our transformed minimum cost base intersection instance.

The case of arbitrary real costs $a_e \in \mathbb{R}$ can also be handled. This is not included in the algorithmic game theory literature, since in that context a positive impact of congestion (i.e. $a_i < 0$) does not make sense.

First, we find a set $B \in \mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{M}_2)$ of minimum cost and we contract this set. For all edges $e \in E \setminus B$ with $a_e < 0$ it holds that $B + e \notin \mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{M}_2)$, or we could improve the solution, so these elements can never be in the intersection of a feasible solution together with $B$. Hence we can run the algorithm from above on the remaining instance. The optimality of this approach follows from the following lemma.

**Lemma 15.** Let $B$ be an element of $\mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{M}_2)$ with minimum cost $a(B) := \sum_{i \in B} a_i$, and $B_1, B_2$ be two bases. Then $B_1, B_2$ can be transformed into two new bases $\tilde{B}_1, \tilde{B}_2$ such that $B \subseteq \tilde{B}_1 \cap \tilde{B}_2$ and $a(\tilde{B}_1 \cap \tilde{B}_2) \leq a(B_1 \cap B_2)$.

**Proof.** Let $e \in B \setminus (B_1 \cap B_2)$. There are three different cases on how to add $e$ to the intersection.

1. $e \notin B_1 \cup B_2$: In this case we have $f_i \in C_i(e, B_i) \setminus B$ for both $i = 1, 2$. By modifying the bases to $\tilde{B}_i = B_i + e - f_i$ we get that

   $a(\tilde{B}_1 \cap \tilde{B}_2) = a(B_1 \cap B_2) + a_e - \begin{cases} a_f & f_1 = f_2 \\ 0 & f_1 \neq f_2 \end{cases}$

2. $e \in B_1, e \notin B_2$: In this case we have $f \in C_2(e, B_2) \setminus B$ and we can modify $\tilde{B}_2 = B_2 + e - f$. this gives a modified cost of

   $a(\tilde{B}_1 \cap \tilde{B}_2) = a(B_1 \cap B_2) + a_e - \begin{cases} a_f & f \in B_1 \\ 0 & f \notin B_1 \end{cases}$

3. $e \notin B_1, e \in B_2$: symmetric to case (2).

We apply these steps iteratively until $B$ is contained in the intersection. We know that the sum of costs of the elements $e \in \tilde{B}_1 \cap \tilde{B}_2$ with $a_e \leq 0$ must now be smaller than before,
since \( B \) is minimum. We never added any element \( e \) to \( \tilde{B}_1 \cap \tilde{B}_2 \) with \( a_e > 0 \). This implies that \( a(\tilde{B}_1 \cap \tilde{B}_2) \leq a(B_1 \cap B_2) \).

The approach above gives us the following result.

**Theorem 16.** COPIC(\( \mathcal{B}(\mathcal{M}_1), \mathcal{B}(\mathcal{M}_2), \text{diag}(a), 0, 0 \)) can be solved in polynomial time, for any two matroids \( \mathcal{M}_1, \mathcal{M}_2 \) and cost vector \( a \in \mathbb{R}^n \).

4.3.3. The case \( a \geq 0, c \equiv d \). This case can be solved analogously to the case without linear costs. We create two identical helper matroids \( \mathcal{M}_1', \mathcal{M}_2' \), with the only difference that we set the costs of the elements to \( a_e + c_e \) and \( c_e \). Since \( a_e \geq 0 \), it follows that the algorithm will prefer the copy of cost \( c_e \) if it takes only one of the two elements into the solution. This again implies that we obtain a one to one correspondence of solutions as in the discussion above.

**Theorem 17.** COPIC(\( \mathcal{B}(\mathcal{M}_1), \mathcal{B}(\mathcal{M}_2), \text{diag}(a), c, c \)) can be solved in polynomial time, for any two matroids \( \mathcal{M}_1, \mathcal{M}_2 \) and cost vectors \( a \in \mathbb{R}^n \geq 0, c \in \mathbb{R}^n \).

It remains an interesting open question whether we can also solve the case with arbitrary costs \( a \in \mathbb{R}^n \) and the case with non-equal linear costs \( c \neq d \) in polynomial time, as it is possible for uniform and partition matroids.

4.4. **Pairs of paths.** In this section we analyze the special case when \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) correspond to the set of \( s_1-t_1 \)- and \( s_2-t_2 \)-paths in a graph. We will again look at the case where the graphs corresponding to \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are identical. One must also make sure that there do not exist negative circles in the graph, else already optimizing over a linear cost function without interaction costs is NP-hard. To simplify the exposition we will focus on \( \mathcal{Q}, c, d \geq 0 \). Table 2 is a summary of the results in this subsection. It is important to differentiate between directed and undirected graphs, which is clear in the light of Proposition 18 and the known complexity results of the edge-disjoint paths problem.

**Proposition 18.** Given a graph \( G \), COPIC(\( \mathcal{P}_{s_1,t_1}(G), \mathcal{P}_{s_2,t_2}(G), \text{diag}(a), 0, 0 \)) with \( a > 0 \) has a solution with objective value 0, if and only if there exist two edge-disjoint paths \( s_1-t_1 \)-paths in \( G \).

| directedness | terminals | cost restrictions | complexity |
|--------------|-----------|-------------------|------------|
| directed     | arbitrary | \( Q = I, c = d = 0 \) | NP-hard    |
| directed     | common    | \( Q = \text{diag}(\infty) \) | NP-hard    |
| undirected   | arbitrary | \( Q = \text{diag}(\infty), d = 0 \) | NP-hard    |
| undirected   | arbitrary | \( c = d = 0 \) | open       |
| undirected   | common    | \( Q = \text{diag}(\infty) \) | NP-hard    |
| both         | common    | \( c = d \) | P          |

**Table 2.** Summary of the results for diagonal COPIC with paths as feasible solutions

It is well known that the edge-disjoint paths problem is polynomial time solvable for every constant number of paths in undirected graphs [49], but NP-hard already for 2 paths in directed graphs [27]. This immediately yields the following result.
Corollary 19. Given a directed graph $G$, $\text{COPIC}(\mathcal{P}_{s_1,t_1}(G), \mathcal{P}_{s_2,t_2}(G), \text{diag}(a), 0, 0)$ is NP-hard, even for $a \equiv 1$.

We use the following results obtained by Eilam-Tzoreff [25] to further classify the complexity of our problem.

Theorem 20 (Eilam-Tzoreff [25]). The undirected edge-disjoint two shortest paths problem is polynomial time solvable, even in the weighted case. On the other hand, the undirected two edge-disjoint one shortest paths problem is NP-hard.

It is important to note that in the results of Eilam-Tzoreff, a shortest path always means a shortest path in the original graph, not a shortest path after removing the edges of the other disjoint path. This is the reason why using Theorem 20 we cannot conclude that COPIC($\mathcal{P}_{s_1,t_1}(G), \mathcal{P}_{s_2,t_2}(G), \text{diag}(\infty), c, c$) is polynomial time solvable, since in our model we cannot enforce two shortest paths of the original graph. If $c = d = 1$ and $Q = \text{diag}(\infty)$ Björklund and Husfeldt [8] showed in 2014 how to solve the problem using a polynomial time Monte Carlo algorithm. The existence of a deterministic polynomial time algorithm is still unknown and a long-standing open problem.

Nevertheless, it is possible to use the hardness results of Eilam-Tzoreff [25] to show that for general costs $c, d \geq 0$ the problem is NP-hard.

Corollary 21. Given an undirected graph $G$, $\text{COPIC}(\mathcal{P}_{s_1,t_1}(G), \mathcal{P}_{s_2,t_2}(G), \text{diag}(\infty), c, 0)$ is NP-hard for $c \geq 0$.

Proof. Using a polynomial time algorithm for COPIC we can determine, if the two edge-disjoint one shortest paths problem has a solution. Just run the algorithm and check if the objective value equals the length of a shortest $s_1$-$t_1$-path in the given graph. □

This covers the case if $s_1 \neq s_2$ and $t_1 \neq t_2$. From the edge-disjoint path literature we know that the problem becomes easier, if one assumes a common source $s$ and a common sink $t$ for all the paths. We can classify the complexity of this case for our problem, using the following results.

Theorem 22. Given a graph or digraph $G$, $\text{COPIC}(\mathcal{P}_{s,t}(G), \mathcal{P}_{s,t}(G), \text{diag}(a), c, c)$ is solvable in polynomial time, for cost vectors $a, c \geq 0$.

Proof. We reduce to a minimum cost flow problem. Set $b(s) = 2$ and $b(t) = -2$ and double each edge/arc $e \in E$ to two versions $e_1, e_2$ with $\tilde{c}_{e_1} = c_e$ and $\tilde{c}_{e_2} = a_e + c_e$. Now a minimum cost flow in this network will be integral and can be decomposed into two path flows, each sending one unit from $s$ to $t$. The cost of the flow corresponds to the cost of these two paths in our problem. □

Theorem 23. Given a graph or digraph $G$, $\text{COPIC}(\mathcal{P}_{s,t}(G), \mathcal{P}_{s,t}(G), \text{diag}(\infty), c, d)$ is NP-hard.

Proof. For digraphs the statement follows from a reduction from directed two disjoint paths. Given such an instance we introduce the new terminals $s$ and $t$ and add arcs $(s, s_1), (s, s_2), (t_1, t), (t_2, t)$. We use $Q = \text{diag}(\infty)$ and as linear costs $c_{(s,s_1)} = c_{(t_1,t)} = d_{(s,s_2)} = d_{(t_2,t)} = 0$ and $c_{(s,s_2)} = c_{(t_2,t)} = d_{(s,s_1)} = d_{(t_1,t)} = \infty$ and $c_e = d_e = 0$ for all other edges. This enforces that paths $S_i$ are $s_i$-$t_i$-paths and the diagonal matrix with infinite entries ensures disjointness.
In the undirected case we apply the same construction as above but using the undirected two edge-disjoint one shortest paths problem. To solve the decision problem analyzed by Eilam-Tzoreff [25], we create COPIC with \( c_e = 1 \) and \( d_e = 0 \) for all the edges in the original network to enforce that \( S_1 \) is a shortest path. After finding a finite cost solution to this problem we check if the length of \( S_1 \) is equal to the length of a shortest \( s_1-t_1 \)-path in \( G \). □

5. Linearizable instances

In this section we explore for which cost matrices COPIC leads to an equivalent problem where there is essentially no interaction between two structures of COPIC.

More precisely, we say that an interaction cost matrix \( Q \) of a COPIC is linearizable, if there exist vectors \( a = (a_i) \) and \( b = (b_i) \) such that for all \( S_1 \in F_1 \) and \( S_2 \in F_2 \)

\[
\sum_{i \in S_1} \sum_{j \in S_2} q_{ij} = \sum_{i \in S_1} a_i + \sum_{j \in S_2} b_j
\]

holds. In that case we say that the pair of vectors \( a \) and \( b \) together is a linearization of \( Q \).

Note that for an instance COPIC(\( F_1, F_2, Q, c, d \)), \( f(S_1, S_2) = \sum_{i \in S_1} \bar{a}_i + \sum_{j \in S_2} \bar{b}_j \) for some \( \bar{a} = (\bar{a}_i), \bar{b} = (\bar{b}_i) \) and all \( S_1 \in F_1, S_2 \in F_2 \), if and only if \( Q \) is linearizable. Hence, we extend our notion of linearizability and say that an instance COPIC(\( F_1, F_2, Q, c, d \)) is linearizable if and only if \( Q \) is linearizable. Our aim is to characterize all linearizable instances of COPIC, with respect to given solution sets \( F_1 \) and \( F_2 \).

Linearizable instances have been studied by various authors for the case of quadratic assignment problem [17, 38, 46], quadratic spanning tree problem [20] and bilinear assignment problem [22]. Here we generalize the ideas from [22] and suggest an approach for finding a characterization of linearizable instances of COPIC’s.

An interaction cost matrix \( Q \) of a COPIC has constant objective property with respect to \( F_1 \) if for every \( j \in [n] \) there exist a constant \( K_j^{(1)} \), so that

\[
\sum_{i \in S_1} q_{ij} = K_j^{(1)} \quad \text{for all } S_1 \in F_1.
\]

Similarly, \( Q \) has constant objective property with respect to \( F_2 \) if for every \( i \in [m] \) there exist a constant \( K_i^{(2)} \), so that

\[
\sum_{j \in S_2} q_{ij} = K_i^{(2)} \quad \text{for all } S_2 \in F_2.
\]

For \( F_i, i = 1, 2 \), let \( \text{CVP}_i(F_i) \) be the vector space of all matrices with constant objective property with respect to \( F_i \).

Combinatorial optimization problems with constant objective property have been studied by various authors [6, 13, 19, 41].

Let \( \text{CVP}_1(F_1) + \text{CVP}_2(F_2) \) be the vector space of all interaction matrices \( Q = (q_{ij}) \) of COPIC, such that \( q_{ij} = a_{ij} + b_{ij} \) \( \forall i, j \), for some \( A = (a_{ij}) \in \text{CVP}_1(F_1) \) and \( B = (b_{ij}) \in \text{CVP}_2(F_2) \).

**Lemma 24** (Sufficient conditions). If the interaction cost matrix \( Q \) of COPIC(\( F_1, F_2, Q, c, d \)) is an element of \( \text{CVP}_1(F_1) + \text{CVP}_2(F_2) \), then \( Q \) is linearizable.
Proof. Let $Q$ be of the form $Q = E + F$, where $E = (e_{ij}) \in \text{CVP}_1(F_1)$ and $F = (f_{ij}) \in \text{CVP}_2(F_2)$. Then

$$
\sum_{i \in S_1, j \in S_2} q_{ij} = \sum_{i \in S_1, j \in S_2} (e_{ij} + f_{ij})
= \sum_{j \in S_2} \left( \sum_{i \in S_1} e_{ij} \right) + \sum_{i \in S_1} \left( \sum_{j \in S_2} f_{ij} \right)
= \sum_{j \in S_2} K_j^{(1)} + \sum_{i \in S_1} K_i^{(2)}.
$$

Hence $Q$ is linearizable, and $a = (a_i), b = (b_j)$ with $a_i = K_i^{(2)}, b_j = K_j^{(1)}$ is a linearization of $Q$.

Now we show that the opposite direction is also true, provided some additional conditions are satisfied. In fact, these additional conditions are satisfied for many well studied combinatorial optimization problems.

**Lemma 25** (Necessary conditions). Let $F_1 \subseteq 2^{|m|}$ and $F_2 \subseteq 2^{|n|}$ be such that:

(i) There exist an $m$ vector $a = (a_i)$, an $n$ vector $b = (b_j)$ and two non-zero constants $K_a, K_b$, such that

$$
\sum_{i \in S_1} a_i = K_a \quad \forall S_1 \in F_1 \quad \text{and} \quad \sum_{j \in S_2} b_j = K_b \quad \forall S_2 \in F_2.
$$

(ii) If an $m \times n$ matrix $\bar{Q} = (\bar{q}_{ij})$ is such that $\sum_{i \in S_1} \sum_{j \in S_2} \bar{q}_{ij} = 0$ for all $S_1 \in F_1, S_2 \in F_2$, then $\bar{Q} \in \text{CVP}_1(F_1) + \text{CVP}_2(F_2)$.

If COPIC($F_1, F_2, Q, c, d$) is linearizable, then $Q \in \text{CVP}_1(F_1) + \text{CVP}_2(F_2)$.

Proof. Assume that the conditions (i) and (ii) of Lemma 25 are satisfied, and that $Q$ is linearizable. We will show that $Q \in \text{CVP}_1(F_1) + \text{CVP}_2(F_2)$ by reconstructing the proof of Lemma 24 in reverse direction.

Since $Q$ is linearizable, there exist $a = (a_i)$ and $b = (b_j)$ such that

$$
\sum_{i \in S_1, j \in S_2} q_{ij} = \sum_{i \in S_1} a_i + \sum_{j \in S_2} b_j \quad \forall S_1 \in F_1, S_2 \in F_2.
$$

(5)

Note that from (i) it follows that there exist matrices $\hat{E} = (\hat{e}_{ij}) \in \text{CVP}_1(F_1)$ and $\hat{F} = (\hat{f}_{ij}) \in \text{CVP}_2(F_2)$ such that

$$
\sum_{j \in S_2} \hat{f}_{ij} = a_i \quad \forall S_2 \in F_2, \ i \in M,
$$

(6)

$$
\sum_{i \in S_1} \hat{e}_{ij} = b_j \quad \forall S_1 \in F_1, \ j \in N.
$$

(7)
Proof. We present a complete proof for (i) of COPIC and perfect matchings of a complete bipartite graph. Theorem 26.

We consider unconstrained solution sets, i.e., when \( Q \) is not satisfied for unconstrained solution sets, i.e., when \( Q = (E + \hat{F}) + (F + \hat{E}) \) is satisfied, although feasible solutions are of different cardinality. Condition (i) is not satisfied for unconstrained solution sets, i.e., when \( \mathcal{F}_1 (\mathcal{F}_2) \) is linearizable if and only if there are some \( a \) and \( b \) such that \( q_{ij} = a_i + b_j \).

Now, from (ii) it follows that \( Q = (E + \hat{F}) = E + F \) for some \( E \in \text{CVP}_1 (\mathcal{F}_1) \), \( F \in \text{CVP}_2 (\mathcal{F}_2) \), and hence, \( Q = (E + \hat{F}) + (F + \hat{E}) \in \text{CVP}_1 (\mathcal{F}_1) + \text{CVP}_2 (\mathcal{F}_2) \). □

From Lemma 24 and Lemma 25 it follows that \( \text{CVP}_1 (\mathcal{F}_1) + \text{CVP}_2 (\mathcal{F}_2) \) is the set of all linearizable matrices, provided that the corresponding COPIC satisfies properties (i) and (ii) of Lemma 25.

In most cases, property (i) is straightforward to check. For example, it is true for all COPIC’s for which elements of \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are of fixed cardinality. If \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are \( s \)-\( t \) paths in a graph, then again property (i) is satisfied, although feasible solutions are of different cardinality. Condition (i) is not satisfied for unconstrained solution sets, i.e., when \( \mathcal{F}_1 (\mathcal{F}_2) \) is \( 2^{[m]} \) (\( 2^{[n]} \)).

Now we show how Lemma 24 and Lemma 25 can be used to characterize linearizable instances for some specific COPIC’s. In particular, we consider unconstrained solution sets \( 2^{[m]} \), bases of the uniform matroids \( \mathcal{B}(U_{m}^n) \), spanning trees of a complete graph \( \mathcal{B}(\mathcal{M}(K_m)) \) and perfect matchings of a complete bipartite graph \( \mathcal{P}\mathcal{M}(K_m m) \). For the case of \( \mathcal{P}\mathcal{M}(K_m m) \) the set \([m] \times [m]\) will be our set of edges of the perfect bipartite graph \( K_{m m} \). Hence, in the case of COPIC(\( \mathcal{F}_1, \mathcal{F}_2, Q, c, d \)) where \( \mathcal{F}_i = \mathcal{P}\mathcal{M}(K_m m) \), the dimensions (number of indices) of the cost arrays \( Q \) and \( c \) or \( d \) is increased by one, however our lemmas and \( \text{CVP}_i (\mathcal{P}\mathcal{M}(K_m m)) \) remain to be well defined.

**Theorem 26.**

(i) COPIC(\( \mathcal{P}\mathcal{M}(K_m m), \mathcal{P}\mathcal{M}(K_n n), Q, c, d \)) is linearizable if and only if there are some arrays \( A, B, C, D \) such that \( q_{ijk} = a_{ij} + b_{ij} + e_{ik} + d_{jk} \).

(ii) COPIC(\( \mathcal{B}(\mathcal{M}(K_m)), \mathcal{B}(\mathcal{M}(K_n)), Q, c, d \)) is linearizable if and only if there are some vectors \( a, b \) such that \( q_{ij} = a_i + b_j \).

(iii) COPIC(\( \mathcal{B}(U_{m}^1), \mathcal{B}(U_{n}^2), Q, c, d \)) is linearizable if and only if there are some vectors \( a, b \) such that \( q_{ij} = a_i + b_j \).

(iv) COPIC(\( \mathcal{P}\mathcal{M}(K_m m), \mathcal{B}(\mathcal{M}(K_n)), Q, c, d \)) is linearizable if and only if there are some arrays \( A, B, C \) such that \( q_{ijk} = a_{ij} + b_{ik} + c_{jk} \).

(v) COPIC(\( \mathcal{B}(\mathcal{M}(K_m)), \mathcal{B}(U_{n}^1), Q, c, d \)) is linearizable if and only if there are some vectors \( a, b \) such that \( q_{ij} = a_i + b_j \).

(vi) COPIC(\( \mathcal{P}\mathcal{M}(K_m m), \mathcal{B}(U_{n}^1), Q, c, d \)) is linearizable if and only if there are some arrays \( A, B, C \) such that \( q_{ijk} = a_{ij} + b_{ik} + c_{jk} \).

**Proof.** We present a complete proof for (iv), and indicate how other statements can be shown analogously.
In the case of COPIC($\mathcal{PM}(K_{m,m}), \mathcal{B}(\mathcal{M}(K_n)), Q, c, d$), the interaction costs are represented in a three-dimensional array $Q$, since for convenience we represent the cost vector of $\mathcal{F}_1 = \mathcal{PM}(K_{m,m})$ in two indices. It is well known that a linear assignment problem instance $R = (r_{ij})$ has the constant objective property if and only if $r_{ij} = s_i + t_j$, for some vectors $s$ and $t$. Hence CVP$_1(\mathcal{PM}(K_{m,m})) = \{A = (a_{ijk}) : a_{ijk} = b_{ik} + c_{jk} \text{ for some } B = (b_{ij}), C = (c_{ij})\}$. A spanning tree problem on a complete graph has the constant objective property if and only if the cost vector is constant, therefore CVP$_2(\mathcal{B}(\mathcal{M}(K_n))) = \{A = (a_{ijk}) : a_{ijk} = b_{ij} \text{ for some } B = (b_{ij})\}$. Hence, $Q$ is an element of CVP$_1(\mathcal{PM}(K_{m,m})) + \text{CVP}_2(\mathcal{B}(\mathcal{M}(K_n)))$ if and only if there are some $A, B$ and $C$ such that

$$q_{ijk} = a_{ij} + b_{ik} + c_{jk}. \quad (10)$$

Lemma 24 tells us that (10) is a sufficient condition for $Q$ to be linearizable. To show that it is also a necessary condition, we just need to show that properties $(i)$ and $(ii)$ of Lemma 25 are true for COPIC($\mathcal{PM}(K_{m,m}), \mathcal{B}(\mathcal{M}(K_n)), Q, c, d$). $(i)$ is obviously true, hence it remains to show that if $Q$ is such that

$$\sum_{(i,j) \in S_1} \sum_{k \in S_2} q_{ijk} = 0 \quad \forall S_1 \in \mathcal{PM}(K_{m,m}), \ S_2 \in \mathcal{B}(\mathcal{M}(K_n)),$$

then $Q \in \text{CVP}_1(\mathcal{PM}(K_{m,m})) + \text{CVP}_2(\mathcal{B}(\mathcal{M}(K_n)))$.

Let $i, j \in \{2, 3, \ldots, m\}$ be fixed, and let $S'_{PM}, S''_{PM} \in \mathcal{PM}(K_{m,m})$ be such that $S'_{PM} \setminus S''_{PM} = \{(1, 1), (i, j)\}$ and $S''_{PM} \setminus S'_{PM} = \{(1, j), (i, 1)\}$. Further, let $k \in \{2, 3, \ldots, n\}$ be fixed, and $S'_{ST}, S''_{ST} \in \mathcal{B}(\mathcal{M}(K_n))$ be such that $S'_{ST} \setminus S''_{ST} = \{1\}$ and $S''_{ST} \setminus S'_{ST} = \{k\}$. Note that such $S'_{PM}, S''_{PM}, S'_{ST}, S''_{ST}$ exist for all $i, j \in \{2, 3, \ldots, m\}, k \in \{2, 3, \ldots, n\}$.

Let us assume that $Q$ satisfies property $(ii)$ of Lemma 25. Then, in particular, we have that

$$\sum_{(i,j) \in S'_{PM}} \sum_{k \in S'_{ST}} q_{ijk} + \sum_{(i,j) \in S''_{PM}} \sum_{k \in S''_{ST}} q_{ijk} = \sum_{(i,j) \in S'_{PM}} \sum_{k \in S'_{ST}} q_{ijk} + \sum_{(i,j) \in S''_{PM}} \sum_{k \in S''_{ST}} q_{ijk}, \quad (11)$$

which, after cancellations, gives us

$$q_{111} + q_{ij1} + q_{ijk} + q_{11k} = q_{111} + q_{ijk} + q_{1j1} + q_{111} \quad (12)$$

for all $i, j \in \{2, 3, \ldots, m\}, k \in \{2, 3, \ldots, n\}$. Note that (12) holds true even if $i, j$ or $k$ is equal to 1, since in that case everything cancels out. Therefore, $q_{ijk}$ can be expressed as

$$q_{ijk} = a_{ij} + b_{ik} + c_{jk} \quad \forall i, j \in [m], \forall k \in [n], \quad (13)$$

where

$$a_{ij} := q_{ij1} - \frac{1}{2}q_{1j1} - \frac{1}{2}q_{111} + \frac{1}{3}q_{111},$$

$$b_{ik} := q_{i1k} - \frac{1}{2}q_{11k} - \frac{1}{2}q_{111} + \frac{1}{3}q_{111},$$

$$c_{jk} := q_{ijk} - \frac{1}{2}q_{11k} - \frac{1}{2}q_{1j1} + \frac{1}{3}q_{111},$$

i.e., $Q \in \text{CVP}_1(\mathcal{PM}(K_{m,m})) + \text{CVP}_2(\mathcal{B}(\mathcal{M}(K_n)))$. That proves statement $(iv)$ of the theorem.

Statements $(i)$ and $(ii)$ of the theorem can be proved by considering equation (11) with two pairs of $S'_{PM}, S''_{PM}$ for the case of COPIC($\mathcal{PM}(K_{m,m}), \mathcal{PM}(K_{n,n}), Q, c, d$), and two
pairs of $S'_{ST}, S''_{ST}$ for the case of COPIC($\mathcal{B}(\mathcal{M}(K_m)), \mathcal{B}(\mathcal{M}(K_n)), Q, c, d$). Using analogous approach, the remaining statements of the theorem can be shown. □

As we mentioned before, property (i) of Lemma 25 does not hold for unconstrained solution set $2^{|m|}$ ($2^{|n|}$), nevertheless, it is not hard to show that CVP$_1(\mathcal{F}_1) +$ CVP$_2(\mathcal{F}_2)$ characterizes all linearizable matrices even if $\mathcal{F}_1 = 2^{|m|}$ or $\mathcal{F}_2 = 2^{|n|}$.

**Theorem 27.** COPIC($\mathcal{F}_1, \mathcal{F}_2, Q, c, d$) with $\mathcal{F}_1 = 2^{|m|} (\mathcal{F}_2 = 2^{|n|})$ is linearizable if and only if $Q \in$ CVP$_2(\mathcal{F}_2) (Q \in$ CVP$_1(\mathcal{F}_1))$.

**Proof.** Assume that $\mathcal{F}_1 = 2^{|m|}$. Note that CVP$_1(2^{|m|})$ contains only the $m \times n$ zero matrix, hence Lemma 24 implies that elements of CVP$_2(\mathcal{F}_2)$ are linearizable.

Now let us assume that $Q$ is linearizable and not an element of CVP$_2(\mathcal{F}_2)$. Then there must exist some $i' \in [m]$ and $S_2, S'_2 \subseteq \mathcal{F}_2$ such that $\sum_{j \in S_2} q_{i'j} \neq \sum_{j \in S'_2} q_{i'j}$. Let $a = (a_i)$ and $b = (b_i)$ be a linearization of $Q$. Since $\{i'\} \in 2^{|m|}$, we have that

$$\sum_{j \in S_2} q_{i'j} = \sum_{i \in \{i'\}} \sum_{j \in S_2} q_{ij} = a_{i'} + \sum_{j \in S_2} b_j,$$

$$\sum_{j \in S'_2} q_{i'j} = \sum_{i \in \{i'\}} \sum_{j \in S'_2} q_{ij} = a_{i'} + \sum_{j \in S'_2} b_j.$$  

Hence, $\sum_{j \in S_2} b_j \neq \sum_{j \in S'_2} b_j$. However, since $\emptyset \in 2^{|m|}$ we have

$$0 = \sum_{i \in \emptyset} \sum_{j \in S_2} q_{ij} = \sum_{j \in S_2} b_j \quad \text{and} \quad 0 = \sum_{i \in \emptyset} \sum_{j \in S'_2} q_{ij} = \sum_{j \in S'_2} b_j$$

which implies that $\sum_{j \in S_2} b_j = \sum_{j \in S'_2} b_j$, a contradiction. □

**6. Conclusion**

We introduced a general model to study combinatorial optimization problems with interaction costs and showed that many classical hard combinatorial optimization problems are special cases. In many cases, interaction costs can be identified as the origin of the hardness of these problems. Therefore we considered special structures of interaction costs, and their impact on the computational complexity of the underlying combinatorial optimization problems. We presented a general approach based on multi-parametric programming to solve instances parametrized with the rank of the interaction cost matrix $Q$. Complementary to that, we analyzed problems with diagonal interaction cost matrix $Q$, which can be used to enforce disjointness constraints. Even for this special type of interaction costs, we can show that for many common sets of feasible solutions, that have no matroid structure, COPIC becomes NP-hard. We also identified conditions on the interaction costs so that COPIC can be reduced to an equivalent instance with no interaction costs.

To further characterize how interaction costs impact the computational complexity of different combinatorial optimization problems, the following questions could be addressed.

(1) Are the polynomially solvable cases of COPIC where matrix $Q$ has fixed rank $r$ \(W[1]-hard\)?

(2) For cases of COPIC with diagonal matrix that can be efficiently solved, analyze the parameterized complexity with respect to the bandwidth of $Q$.  


(3) Can COPIC($B(U^k_m), P_{s,t}(G), \text{diag}(a, c, d)$) be solved in polynomial time, if $a \geq 0, c \geq 0$ and $d \geq 0$?

(4) Is COPIC($B(M_1), B(M_2), \text{diag}(a, c, d)$) solvable in polynomial time, without any restrictions on $M_1, M_2, a, c$ and $d$?

For the case of diagonal COPIC it would be interesting to study further types of sets of feasible solutions. For example the matching-cut problem analyzed by Bonsma [10] can be also formulated as a special case of diagonal COPIC, so analyzing graph cuts as feasible sets in diagonal COPIC is an interesting candidate for further research.

Additionally, understanding the influence of interaction costs with other special matrix structures, besides fixed rank and diagonal matrices, to the computational complexity of combinatorial optimization problems would be of interest.

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