ORIENTABLE QUADRATIC EQUATIONS IN FREE METABELIAN GROUPS

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Abstract. We prove that the Diophantine problem for orientable quadratic equations in free metabelian groups is decidable and furthermore, \textbf{NP}-complete. In the case when the number of variables in the equation is bounded, the problem is decidable in polynomial time.

1. Introduction

Let $G$ be a group, $X$ a set of variables, and $F_X$ the free group on $X$. An \textit{equation} in $G$ is a formal equality $W = 1$ where $W \in G \ast F_X$. A \textit{solution} of an equation $W = 1$ is a homomorphism $\alpha : G \ast F_X \rightarrow G$ such that $\alpha(W) = 1$ and $\alpha(g) = g$ for all $g \in G$.

\textbf{Diophantine problem DP in a group} $G$ for a class of equations $C$ is an algorithmic question to decide if a given equation $W = 1$ in $C$ has a solution, or not.

1.1. \textbf{Quadratic equations in a group} $G$. A word $W \in G \ast F_X$ and an equation $W = 1$ in $G$ are \textit{quadratic} if every variable $x \in X$ occurring in $W$ occurs exactly twice (as $x$ or $x^{-1}$). The Diophantine problem for quadratic equations in $G$ is quite general, for example it naturally contains the word problem and the conjugacy problem for $G$. The problem received a lot of attention recently and was investigated for several classes of groups. In particular, it was shown in [10] that solving quadratic equations is \textbf{NP}-complete for free non-abelian groups. Later the same result was generalized in [11] to the class of non-cyclic hyperbolic torsion-free groups. Quadratic equations are solvable in the first Grigorchuk group [13]. Recently Roman’kov showed in [22] that quadratic equations are unsolvable in a nilpotent group of class 2.

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In general, quadratic equations are more approachable than general ones. Typically, they can be treated by a specialized method. For instance, solving quadratic equations in free groups is relatively easy, it essentially reduces to enumeration of the minimal automorphic orbit of the equation (see [3]), while solving general equations require very sophisticated tools [15, 4]. Similarly, the main result of this paper demonstrates the difference between quadratic and general equations in free metabelian groups. The former are proved to be solvable in \( \text{NP} \), while the latter are known to be unsolvable.

A quadratic equation \( W = 1 \) in \( G \) is called orientable if each variable \( x \) occurring in \( W \) occurs once as \( x \) and once as \( x^{-1} \), and non-orientable otherwise. Two special forms of orientable and non-orientable equations are called standard quadratic equations:

1. \[ [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] c_1 z_2^{-1} c_2 z_2 \cdots z_m^{-1} c_m z_m = 1 \quad (g \geq 0, m \geq 0), \]
2. \[ x_1^2 x_2^2 \cdots x_g^2 c_1 z_2^{-1} c_2 z_2 \cdots z_m^{-1} c_m z_m = 1 \quad (g > 0, m \geq 0), \]

where \( x_i, y_i, z_i \) are variables and \( c_i \in G \) are coefficients. We assume that the coefficient part \( c_1 z_2^{-1} c_2 z_2 \cdots z_m^{-1} c_m z_m \) is void in the case \( m = 0 \). The choice of terminology comes the following well-known fact (whose proof essentially repeats the proof of the classification theorem for compact surfaces with boundary, see for example a classical topology textbook [21, Sections 38–40]; a detailed argument can be found for example in [6]). See also [3, Theorem 3.2].

**Proposition 1.** Let \( W \in G * F_X \) be a quadratic word in \( G \). If \( W \) is orientable (non-orientable resp.), then there exists an automorphism \( \phi \) of \( G * F_X \) with \( \phi|_G = \text{id} \) such that \( \phi(W) \) is of the form (1) (form (2) resp.). Furthermore, the automorphism \( \phi \) (written as a composition of some elementary automorphisms) and the word \( \phi(W) \) can be computed in time bounded by \( O(|W|^2) \).

Proposition 1 allows us to restrict our attention to standard quadratic equations. In what follows, we prefer to consider standard orientable quadratic equations written in a more symmetric form with an extra variable \( z_1 \):

3. \[ [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] = z_1 c_1 z_1^{-1} z_2 c_2 z_2^{-1} \cdots z_m c_m z_m^{-1} \]

which is obviously equivalent to (1).

1.2. **Main results.** In this paper we consider orientable quadratic equations in the free metabelian group \( M_n \) of rank \( n \geq 2 \).
Theorem 2. The Diophantine problem for orientable quadratic equations in $M_n$ is solvable.

Theorem 3. The Diophantine problem for orientable quadratic equations in $M_n$ is NP-complete.

Theorem 4. For fixed $g$ and $m$, the Diophantine problem for orientable quadratic equations in $M_n$ can be solved by a polynomial time algorithm.

In fact, the proof of Theorem 2 in Section 5 yields a slightly stronger result. It works for the uniform version of the Diophantine problem in free metabelian groups.

Uniform Diophantine problem for a class of groups $G$ and a class of equations $C$. Given a group $G \in G$ and an equation $W = 1$ from $C$ with coefficients from $G$ decide if $W = 1$ has a solution, or not.

Theorem 5. The uniform Diophantine problem for orientable quadratic equations in free metabelian groups is solvable and, furthermore, NP-complete.

1.3. Known facts on equations in free metabelian groups. In general, for every $n \geq 2$ the Diophantine problem for equations of the form $W(x_1, \ldots, x_k) = c$ in $M_n$ is undecidable. By [20], there exist a word $W = W(x_1, \ldots, x_k)$ and an element $u \in M_n$ such that the set of integers $\{ l \in \mathbb{Z} | W = u^l \text{ has a solution} \}$ is non-recursive. In particular, this implies undecidability of the general Diophantine problem for $M_n$. In [14] the authors proved that the problem is solvable and, moreover, is NP-complete for the special class of quadratic equations in $M_n$ of the form (1) with $g = 0$.

It is known that metabelian groups have finite commutator width. This fact was first established by Rhemtulla in [19] who proved that every finitely generated solvable of class $\leq 3$ group has a finite commutator width. The precise value of the commutator width of $M_n$ is $n$. This is a consequence of a more general result by Akhavan-Malayeri and Rhemtulla [1] and for $n \geq 3$ follows also from several earlier results, see section 7.1 in [21]. In connection with our Theorem 2 this means that for an equation of the form (1) in $M_n$ we can always assume that $g \leq n$.

2. Preliminaries

2.1. Elements of $M_n$ as 1-cycles. We fix a set $\{a_1, a_2, \ldots, a_n\}$ of free generators for $M_n$. By $A_n$ we denote the free abelian group $M_n/M'_n$ and by $x \mapsto \bar{x}$ the canonical epimorphism $M_n \to A_n$. In particular, $\{\bar{a}_1, \ldots, \bar{a}_n\}$ is a basis for $A_n$. We use additive notation for $A_n$. The
Cayley graph of $A_n$ with respect to the generating set $\{a_1, \ldots, a_n\}$ is denoted $\Gamma_n$. We identify vertices of $\Gamma_n$ with elements of $A_n$ and assume that directed edges of $\Gamma_n$ are labeled with letters $a_i^{\pm 1}$.

A word $w \in F_n$ determines a unique edge path $p_w$ in $\Gamma_n$ labeled by $w$ which starts at 1 (the vertex corresponding to the identity of $A_n$). It defines a 1-chain $\sigma(w)$ in $\Gamma_n$ which is the algebraic sum of all edges traversed by $p_w$; the sum of two mutually inverse directed edges is defined to be 0. It is well known that the mapping $w \mapsto \sigma(w)$ induces a well-defined injective map of $M_n$ to the group $C_1(\Gamma_n)$ of 1-chains of $\Gamma_n$ over $\mathbb{Z}$; that is, two words $u$ and $w$ define the same element of $M_n$ if and only $\sigma(u) = \sigma(w)$ (see [5, 26, 16]). For $g \in M_n$, we use the same notation $\sigma(g)$ for the image of $g$ under the induced map. It is an easy exercise to check that for any $g, h \in M_n$,

$$\sigma(gh) = \sigma(g) + \bar{g}\sigma(h),$$

where $\bar{g}\sigma(h)$ is obtained by shifting $\sigma(h)$ by $\bar{g}$ (via the action of $A_n$ on $\Gamma_n$). In particular, we have $\sigma(g^{-1}) = -\bar{g}\sigma(g)$ and the action of $A_n$ on $C_1(\Gamma_n)$ agrees with conjugation in $M_n$:

$$\sigma(ghg^{-1}) = \bar{g}\sigma(h).$$

In additive notation for $A_n$, we have

$$(g_1 + g_2)\sigma(h) = g_1(\bar{g}_2\sigma(h)),$$

and, obviously, for the boundary $\partial_1\sigma(g)$ we have

$$\partial_1\sigma(g) = \bar{g} - 1.$$

This implies that $g \in M'_n$ if and only if $\sigma(g)$ is a 1-cycle. Since $\sigma(gh) = \sigma(g) + \sigma(h)$ if $g \in M'_n$, $\sigma$ induces an isomorphism between $M'_n$ and the group $Z_1(\Gamma_n)$ of 1-cycles of $\Gamma_n$.

If $L$ is a subgroup of $A_n$, then by $\tau_L(w)$ we denote the projection of $\sigma(w)$ in the quotient $\Gamma_n/L$:

$$\tau_L : M_n \xrightarrow{\sigma} C_1(\Gamma_n) \rightarrow C_1(\Gamma_n/L).$$

Now, with a collection of elements $h_1, \ldots, h_r \in M_n$ we associate the subgroup $L = \langle h_1, \ldots, h_r \rangle$ of $A_n$ and two subgroups of $M_n$:

$$H'_L = \langle [a_i, a_j]^g, h_k \rangle, \text{ where } 1 \leq i, j \leq n, \ 1 \leq k \leq r, \ g \in M_n \rangle,$$

$$H_L = \left\langle \begin{array}{c} [a_i, a_j]^g, h_k, \ \text{ where } 1 \leq i, j \leq n, \ 1 \leq k \leq r, \ g \in M_n \\ [h_i, h_j], \ \text{ where } 1 \leq i, j \leq r \end{array} \right\rangle.$$

The following proposition describes elements of $M_n$ vanishing under $\tau_L$.

**Proposition 6** (See [14, Proposition 2.2]). $\ker(\tau_L) = H_L$. $\square$
Remark. The technique of representing elements in $M_n$ as 1-chains in the Cayley graph of $\mathbb{Z}^n$ is not new and was introduced several times under different names, see for example [16, 26]. Equivalently, we could use the language of Fox derivatives: it can be easily checked that for any word $w$ representing an element $g \in M_n$

$$\sigma(g) = \sum_i \frac{\partial w}{\partial a_i} \sigma(a_i)$$

where $\frac{\partial w}{\partial a_i}$ is the Fox derivative of $w$ with respect to a generator $a_i$ of $M_n$ and images $\sigma(a_i)$ of generators $a_i$ form a generating set for $C_1(\Gamma_n)$ as an $\mathbb{Z}M_n$-module.

2.2. External square. As a convenient tool, we use the following well known construction. Let $X$ be an $R$-module over a commutative ring $R$ with identity. The external square $\Lambda^2(X)$ of $X$ defined as the skew-symmetric quotient of the tensor square $X \otimes X$, i.e. the quotient over the submodule generated by all elements $x \otimes y + y \otimes x$ and $x \otimes x$. By construction, the map $(x, y) \mapsto x \otimes y$ induces the skew-symmetric bilinear map $X \times X \to \Lambda^2(X)$, the wedge product $x \wedge y$. If $X$ is a free $R$-module with basis $B = \{b_1, \ldots, b_k\}$ then $\Lambda^2(X)$ is a free $R$-module with basis $B^\wedge_2 = \{b_i \wedge b_j \mid 1 \leq i < j \leq k\}$.

We consider $\Lambda^2(A_n)$ as the external square of the $\mathbb{Z}$-module $A_n$. It is a free abelian group with basis $\{\bar{a}_i \wedge \bar{a}_j \mid 1 \leq i < j \leq n\}$ and hence is isomorphic to the second factor $M'_n/[M'_n, M_n]$ of the upper central series of $M_n$ where $\bar{a}_i \wedge \bar{a}_j$ maps to the image of the basis commutator $[a_i, a_j]$. Thus, we have an epimorphism $\phi : M'_n \to \Lambda^2(A_n)$ given by

$$\phi([a_i, a_j]^g) = \bar{a}_i \wedge \bar{a}_j$$

for any $g \in M_n$.

Since metabelian groups satisfy the identities

$$(uv, x) = [u, x][v, x], \quad [x, uv] = [x, u][x, v],$$

together with linearity of the wedge product we obtain

$$\phi([g, h]) = \bar{g} \wedge \bar{h}.$$

If $\{b_1, b_2, \ldots, b_n\}$ is a basis for $A_n$ and $u, v \in A_n$ are expressed as

$$u = \sum_{i=1}^n t_i b_i \quad \text{and} \quad v = \sum_{i=1}^n s_i b_i,$$

then the definition implies

$$u \wedge v = \sum_{1 \leq i < j \leq n} (t_i s_j - t_j s_i) b_i \wedge b_j.$$
Lemma 7. In the notation of Proposition 6, \( \ker(\phi) \cap \ker(\tau_L) = H'_L \).

Proof. The definition of \( \phi \) implies \( H'_L \subseteq \ker \phi \). Due to identities (3), we may assume without loss of generality that \( \{\bar{h}_1, \ldots, \bar{h}_r\} \) is a basis for \( L \). Then \( H_L \) is generated by \( H'_L \) and commutators \( [h_i, h_j], 1 \leq i < j \leq r \). Now observe that the images \( \{\bar{h}_i \wedge \bar{h}_j \mid 1 \leq i < j \leq r\} \) of these commutators in \( \Lambda^2(A_n) \) are a basis for a free abelian subgroup (to see this, for example, we can enlarge the basis \( \{\bar{h}_i\} \) to obtain a basis for \( \mathbb{Q}^n \cong A_n \otimes \mathbb{Q} \); then \( \{\bar{h}_i \wedge \bar{h}_j\} \) is a part of a basis for \( \Lambda^2(\mathbb{Q}^n) \)). This implies that \( H_L \cap \ker \phi \) contains no elements outside \( H'_L \). \( \square \)

3. Abelian reduction

Let \( Y = (u_1, v_1, \ldots, u_g, v_g, w_1, \ldots, w_m) \) be a tuple of elements of \( M_n \). In this section we give a necessary and sufficient condition for \( Y \) to be a solution of the equation (3), i.e., to satisfy

\[
[u_1, v_1] \cdots [u_g, v_g] = w_1 c_1 w_1^{-1} \cdots w_m c_m w_m^{-1}.
\]

The condition is formulated purely in terms of the image of \( Y \) in \( A_n \).

Assume that \( Y \) is a solution of (3). Note that the left-hand side of (3) belongs to \( M'_n \) and, hence, \( c_1 c_2 \cdots c_m \in M'_n \). Define a subgroup \( L \) in \( A_n \):

\[(7) \quad L = \langle \bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2, \ldots, \bar{u}_g, \bar{v}_g, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m \rangle.\]

Applying \( \tau_L \) to the both sides of (3) we get

\[
0 = \tau_L (w_1^{-1} c_1 w_1 \cdots w_m^{-1} c_m w_m) = \sum_{i=1}^{m} \bar{w}_i \tau_L (c_i).
\]

Applying \( \phi \) to the left-hand side and to the right-hand side of (3) we get

\[
\phi([u_1, v_1][u_2, v_2] \cdots [u_g, v_g]) = \bar{u}_1 \wedge \bar{v}_1 + \cdots + \bar{u}_g \wedge \bar{v}_g
\]

and

\[
\phi(w_1^{-1} c_1 w_1 \cdots w_m^{-1} c_m w_m) = \phi([w_1, c_1^{-1}] \cdots [w_m, c_m^{-1}]) + \phi(c_1 c_2 \cdots c_m)
\]

\[
= \bar{c}_1 \wedge \bar{w}_1 + \cdots + \bar{c}_m \wedge \bar{w}_m + \phi(c_1 c_2 \cdots c_m).
\]

That gives a necessary condition for \( Y \) to be a solution of (3). Below we prove that it is also sufficient.

Proposition 8. Let

\[
Y = (\bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2, \ldots, \bar{u}_g, \bar{v}_g, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)
\]
be a tuple of elements of \( A_n \). Let \( L \) be the subgroup of \( A_n \) generated by all \( \bar{u}_i, \bar{v}_i \) and \( \bar{c}_i \). Then \( \bar{Y} \) can be lifted to a solution of the equation (3) if and only if the following conditions are satisfied:

\[
\sum_{i=1}^{m} \bar{w}_i \tau_L(c_i) = 0, \\
\sum_{i=1}^{m} \bar{u}_i \wedge \bar{v}_i = \sum_{i=1}^{m} \bar{c}_i \wedge \bar{w}_i + \phi(c_1c_2 \ldots c_m).
\]

**Proof.** We need to prove the ‘if’ part. Assume that \( \bar{Y} \) satisfies the conditions (8) and (9). Let \( Y = (u_1, v_1, u_2, v_2, \ldots, u_g, v_g, w_1, w_2, \ldots, w_m) \) be any lift of \( \bar{Y} \) in \( M_n \). Define

\[
W = [x_1, y_1] \ldots [x_g, y_g]z_1c_1^{-1}z_1^{-1} \ldots z_mc_m^{-1}z_m^{-1},
\]

and associate with \( Y \) an element

\[
W(Y) = [u_1, v_1] \ldots [u_g, v_g]w_1c_1^{-1}w_1^{-1} \ldots w_mc_m^{-1}w_m^{-1},
\]

obtained by substituting the elements of \( Y \) into \( W \). Conditions (8) and (9) with Lemma 7 imply that

\[
W(Y) \in \ker(\phi) \cap \ker(\tau_L) = H'_L.
\]

Now, it is sufficient to prove that the set

\[
\{W(Y) \mid Y \text{ is a lift of } \bar{Y}\}
\]

is a coset of \( H'_L \).

Recall that \( H'_L \) is generated by the elements of the form \([a_i, a_j]^g, x]\) where \( g \in M_n \) and \( x \) belongs to the set \( \{u_i, v_j, c_k\} \). Fix an occurrence of \( z_kc_k^{-1}z_k^{-1} \) in \( W \):

\[
W = Uz_kc_k^{-1}z_k^{-1}V.
\]

Multiplication \( w_k \mapsto w_k[a_i, a_j]^g \) of \( w_k \) by a generator of \( M'_n \) results in multiplication:

\[
W(Y) \mapsto [[a_i, a_j]^gU(Y)w_k, c_k]W(Y),
\]

of the value of \( W \). Since \([a_i, a_j]^g\) depends only on \( \bar{g} \), the factor

\[
[[a_i, a_j]^gU(X)w_k, c_k]
\]

runs over all generators of \( H'_L \) of the form \([a_i, a_j]^g, c_k\]. In a similar way, multiplication of \( u_k \) and \( v_k \) by generators of \( M'_n \) produces generators of \( H'_L \) of the form \([a_i, a_j]^g, v_k] \) and \([a_i, a_j]^g, u_k] \) respectively. This finishes the proof.
We now examine conditions (8) and (9). Observe that (8) depends only on images of the elements $\bar{w}_i$ in $A_n/L$. Changing $\bar{w}_i$’s while keeping their images in $A_n/L$ fixed adds to the right-hand side of (9) an element of the subgroup $K$ of $\Lambda^2(A_n)$ defined by

$$K = \langle \bar{c}_i \wedge g \text{ for all } i \text{ and } g \in L \rangle.$$ 

Furthermore, for any $k \in K$ we can find appropriate $\bar{w}_i$’s that add $k$ to the right-hand side of (9). Hence, (9) can be replaced with the following condition:

$$(10) \quad g \sum_{i=1}^{g} \bar{u}_i \wedge \bar{v}_i = \sum_{i=1}^{m} \bar{c}_i \wedge \bar{w}_i + \phi(c_1c_2 \ldots c_m) \mod K.$$ 

Note that $K$ is the kernel of the natural epimorphism $\Lambda^2(L) \to \Lambda^2(L/Q)$, where $Q = \langle \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m \rangle$.

3.1. Restricting to finitely many $w_i$’s. Here we provide an effective bound on the size of elements $\bar{w}_i$.

**Proposition 9.** An equation (3) has a solution if and only if there exists a tuple $\bar{Y}$ of elements $\bar{u}_i$, $\bar{v}_i$ and $\bar{w}_i$ satisfying (8), (10) and

$$(11) \quad |\bar{w}_i| \leq \sum_{j=1}^{m} |c_j|, \quad i = 1, 2, \ldots, m.$$ 

**Proof.** We need to prove the ‘only if’ part. Let $\bar{Y}$ be a tuple satisfying (8) and (10). Below we prove that $\bar{Y}$ can be modified to satisfy (8), (10), and (11). The argument essentially repeats the proof of [14, Proposition 2.6].

Denote $I = \{1, \ldots, m\}$ and for each $i = 1, \ldots, m$ define

$$\tau_i = \bar{w}_i \tau_L(c_i).$$

It follows from (8) that $\sum_{i=1}^{m} \tau_i = 0$. Given a 1-chain $\rho \in C_1(\Gamma_n/L)$, define $\text{supp}(\rho)$ to be the set of edges in $\Gamma_n/L$ that occur in $\rho$ with a non-zero coefficient. We call a non-empty subset $J \subseteq I$ a cluster if $\text{supp}(\tau_i) \cap \text{supp}(\tau_j) = \emptyset$ for any $i \in J$ and $j \notin J$. It follows from the definition that if $J_1$ and $J_2$ are clusters and $J_1 \cap J_2 \neq \emptyset$, then $J_1 \cap J_2$ is a cluster. Hence, $I$ can be partitioned into a finite disjoint union of minimal clusters.

We introduce an integer-valued distance function $d(e, f)$ on the set $E(\Gamma_n/L)$ of edges of $\Gamma_n/L$. By definition, the distance between edges $e$ and $f$ is the distance between their midpoints in the graph $\Gamma_n/L$ (where all edges are assumed to have length 1). For example, $d(e, f) = 1$ if and only if $e$ and $f$ are distinct and have a common vertex. The following statements follow directly from the definition of a cluster:
(i) If \( J \) is a cluster, then \( \sum_{i \in J} \tau_i = 0. \)

(ii) If \( J \) is a minimal cluster, then:

\[
\text{diam} \left( \bigcup_{i \in J} \text{supp}(\tau_i) \right) \leq \sum_{i \in J} |c_i|.
\]

Note that the composition \( M_n \xrightarrow{\tau} C_1(\Gamma_n/L) \to C_1(\Gamma_n/A_n) \simeq A_n \) gives the canonical epimorphism \( g \mapsto \bar{g} \). Hence, (i) implies that

\[
\prod_{i \in J} w_i c_i w_i^{-1} = \sum_{i \in J} \bar{c}_i = 0.
\]

Now consider an arbitrary minimal cluster \( J = \{ j_1, j_2, \ldots, j_k \} \). By (ii), there exist \( r_2, r_3, \ldots, r_k \in L \) such that

\[
|\bar{w}_{j_t} - \bar{w}_{j_1} - r_t| \leq \sum_{i \in J} |c_i|, \quad t = 2, 3, \ldots, k.
\]

Define new values of \( \bar{w}_i \) for \( i \in J \) by

\[
\bar{w}_{j_1}^* = 0, \quad \bar{w}_{j_t}^* = \bar{w}_{j_t} - \bar{w}_{j_1} - r_t, \quad t = 2, 3, \ldots, k.
\]

Then

\[
\sum_{i \in J} \bar{w}_i^* \tau_L(c_i) = \bar{w}_{j_1}^{-1} \sum_{i \in J} \tau_i = 0
\]

and

\[
\sum_{i \in J} \bar{c}_i \wedge \bar{w}_i^* = \sum_{i \in J} \bar{c}_i \wedge \bar{w}_i - \left( \sum_{i \in J} \bar{c}_i \right) \wedge \bar{w}_{j_1} \quad \text{(mod } K) \]

\[
= \sum_{i \in J} \bar{c}_i \wedge \bar{w}_i.
\]

Therefore, replacing \( \bar{w}_i \) with \( \bar{w}_i^* \) in \( \bar{X} \) for each \( i \in J \) we preserve conditions (8) and (10). Performing the procedure for each minimal cluster \( J \) we achieve the required bound \( |\bar{w}_i^*| \leq \sum_{i=1}^m |c_i| \) for each \( i \in I \). \( \Box \)

Enumerating (finitely many) values of \( \bar{w}_i \)'s satisfying condition (11) we eliminate \( \bar{w}_i \)'s from (8) and (10), and further reduce the equation (3).

**Corollary 10.** An equation (3) can be effectively reduced to a disjunction of finitely many systems of the form

\[
\left\{ \bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2, \ldots, \bar{u}_g, \bar{v}_g, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m \right\} = L,
\]

\[
\pi_L(\delta) = 0,
\]

\[
\bar{u}_1 \wedge \bar{v}_1 + \cdots + \bar{u}_g \wedge \bar{v}_g = h \mod K,
\]

(12)
with constant entries \( \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m \in A_n \), \( \delta \in C_1(\Gamma_n) \), \( h \in \Lambda^2(A_n) \) and unknowns \( \bar{u}_i, \bar{v}_i \in A_n \) and \( L \leq A_n \).

Proof. By Proposition 9, equation (3) is effectively reduced to the disjunction of systems (8) & (10) for finitely many values of \( \bar{w}_i \)'s. Observe that the left-hand side of (8) can be rewritten as \( \tau_L(d) \), where

\[
d = w_1^{-1}c_1w_1 \cdots w_m^{-1}c_mw_m.
\]

By definition, \( \tau_L(d) = \pi_L(\sigma(d)) \), where \( \pi_L : C_1(\Gamma_n) \to C_1(\Gamma_n/L) \) is the quotient map. The value of \( \sigma(d) \) depends only on \( \bar{w}_i \) and \( c_i \) and can be expressed as

\[
\sigma(d) = \sum_i \bar{w}_i \sigma(c_i).
\]

Then equation (8) can be written as \( \pi_L(\delta) = 0 \), where \( \delta = \sigma(d) \), and equation (10) can be written as

\[
\bar{u}_1 \wedge \bar{v}_1 + \cdots + \bar{u}_g \wedge \bar{v}_g = h \mod K,
\]

where \( h = \phi(c_1c_2 \cdots c_m) + \sum_{i=1}^m c_i \wedge \bar{w}_i \). \( \square \)

3.2. Symplectic transformations. Recall that \( \bar{u}_i \) and \( \bar{v}_i \) in (12) are images in \( A_n \) of the variables in the commutator part of the initial equation (3). The stabilizer of the conjugacy class of the product of commutators in the free group, modulo inner automorphisms, is isomorphic to the mapping class group of the closed orientable surface of genus \( g \). This group acts on the set of abelianized tuples \( (\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_g, \bar{v}_g) \) as the symplectic group \( \text{Sp}(2g, \mathbb{Z}) \), see for example [8, Theorem 6.4]. In particular, the set of tuples \( \{\bar{u}_i, \bar{v}_i\} \) satisfying (12) is invariant under symplectic transformations. It is well known that the complete set of symplectic transformations over \( \mathbb{Z} \) is generated by the following transformations:

- (S1) Transposition: \( (\bar{u}_i \to \bar{v}_j, \bar{v}_i \to \bar{u}_j, \bar{u}_j \to \bar{u}_i, \bar{v}_j \to \bar{v}_i) \).
- (S2) \( \text{SL}_2(\mathbb{Z}) \) on a pair: \( (\bar{u}_i \to \bar{u}_i + t\bar{v}_i) \) and \( (\bar{v}_i \to \bar{v}_i + t\bar{u}_i) \).
- (S3) Mixing pairs: \( (\bar{u}_i \to \bar{u}_i + t\bar{u}_j, \bar{v}_j \to \bar{v}_j + t\bar{v}_i), i \neq j \).
- (S4) Mixing pairs II: \( (\bar{u}_i \to \bar{u}_i + t\bar{v}_j, \bar{u}_j \to \bar{u}_j + t\bar{v}_i), i \neq j \).

(deducible from (S1), (S2), and (S3)).

Remark. Formally speaking, we only need the fact that transformations (S1)–(S4) preserve the value of \( \bar{u}_1 \wedge \bar{v}_1 + \cdots + \bar{u}_g \wedge \bar{v}_g \) and the subgroup generated by the elements \( \bar{u}_i \) and \( \bar{v}_i \). This can be seen directly from the definition.

Lemma 11. Let \( T = (\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_g, \bar{v}_g) \) be a tuple of \( 2g \) elements of \( A_n \) and \( \{b_1, b_2, \ldots, b_r\} \) a basis of the subgroup generated by \( T \). Then
$T$ can be transformed by a symplectic transformation to a tuple $T' = (\bar{u}_1', \bar{u}_2', \ldots, \bar{u}_g', \bar{v}_g')$ satisfying the following conditions:

$$u'_1 = b_1 \quad \text{and} \quad \langle \bar{v}_1', \bar{u}_2', \bar{v}_2', \ldots, \bar{u}_g', \bar{v}_g' \rangle = \langle b_2, \ldots, b_r \rangle.$$  

**Proof.** Denote $\langle b_2, \ldots, b_r \rangle$ by $P$. Using (S2) we can act on each pair $(\bar{u}_i, \bar{v}_i)$ as $SL_2(\mathbb{Z})$. In particular, we can implement the Euclidean algorithm on the $b_1$-components of $\bar{u}_i$ and $\bar{v}_i$, and transform $T$ so that $v_i \in P$ for each $i$. Next, using (S1) and (S3) we can act as $SL_n(\mathbb{Z})$ on $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ and change $T$ so that

$$\bar{u}_1 = b_1 + h \quad \text{and} \quad h, \bar{v}_1, \bar{u}_2, \ldots, \bar{v}_g \in P.$$  

The elements $\bar{v}_1, \bar{u}_2, \bar{v}_2, \ldots, \bar{v}_g$ generate $P$. In particular, we have:

$$h = s_1 \bar{v}_1 + s_2 \bar{u}_1 + s_3 \bar{v}_2 + \cdots + s_{2g-1} \bar{v}.$$  

for some coefficients $s_1, \ldots, s_{2g-1}$. Using (S2) we make $s_1 = 0$ and then using (S3) and (S4) with $i = 1$ we eliminate the entire $h$ from $\bar{u}_1$. \hfill $\Box$

### 3.3. Restricting to finitely many subgroups $L$

We now show that we can restrict to an effective finite set of possibilities for $L$ in (12).

The idea can be roughly described as follows: if $L$ is “too stretched” in some direction, then elements $\bar{u}_i$ and $\bar{v}_i$ in (12) (and the depending subgroup $L$) can be changed making the dimension of $L$ smaller. To control “the stretching factor” we use Hermit’s reduced basis for $L$ which is, in some sense, “close to orthogonal” as much as possible.

We use the standard notations $\|x\|$ for the norm of a vector $x$ in a Euclidean normed space $\mathbb{R}^r$ and $x \cdot y$ for the scalar product of $x$ and $y$. If $E = \{e_1, e_2, \ldots, e_r\}$ is a canonical orthonormal basis for $\mathbb{R}^r$, then we view $\Lambda^2(\mathbb{R}^r)$ as a Euclidean normed space with orthonormal basis $E^\wedge^2 = \{e_i \wedge e_j \mid 1 \leq i < j \leq r\}$.

**Remark 12.** If $E' = \{e'_1, e'_2, \ldots, e'_r\}$ is another orthonormal basis for $\mathbb{R}^r$ then $E'^\wedge^2$ is an orthonormal basis for $\Lambda^2(\mathbb{R}^r)$. This can be seen using the fact that the orthogonal group $O(r)$ is generated by rotations in the coordinate planes $S(e_i, e_j)$ and a single reflection $e_1 \mapsto -e_1$. Then checking orthonormality of $E'^\wedge^2$ is reduced to the case of dimension $r \leq 4$ which can be done by direct computation.

**Proposition 13** (Hermite’s reduced basis, see for example [17] Chapter 2). Let $L$ be a discrete subgroup of $\mathbb{R}^n$, $\dim_{\mathbb{Z}}(L) = r$. Then there exists a basis $(b_1, \ldots, b_r)$ for $L$ satisfying the following. Define

$$L_i = \langle b_{i+1}, b_{i+2}, \ldots, b_r \rangle, \quad \forall i = 0, \ldots, r.$$  

Let $b_1^*, b_2^*, \ldots, b_r^*$ be obtained from $b_1, b_2, \ldots, b_r$ by the Gram–Schmidt orthogonalization process starting from $b_r$; that is, $b_i^*$ is the orthogonal
projection of \( b_i \) to the orthogonal complement of \( L_i \). Then

\[
||b_i^*|| \geq \frac{\sqrt{3}}{2} ||b_{i+1}^*||, \quad \forall \ i = 1, \ldots, r - 1, 
\]

and \( |b_i \cdot b_j^*| \leq \frac{1}{2} ||b_j^*||^2 \) for \( i < j \) or, equivalently,

\[
||\text{proj}_{b_j^*}(b_i)|| \leq \frac{||b_j^*||}{2}. \tag{14}
\]

If \( V \) is a nontrivial discrete subgroup of \( \mathbb{R}^n \), then by \( \text{Vol}(V) \) we denote the Euclidean volume of the parallelepiped spanned by a free abelian basis for \( V \). Recall that \( \text{Vol}(V)^2 = \det(AA^T) \geq 1 \), where \( A \) is the \( r \times n \) matrix of the generators of \( V \). Therefore, \( \text{Vol}(V) \geq 1 \) for any non-trivial \( V \subseteq \mathbb{Z}^n \).

Suppose that \( (b_1, \ldots, b_r) \) is a basis satisfying the conclusion of Proposition 13. Since \( \text{Vol}(L) = ||b_1^*|| ||b_2^*|| \ldots ||b_r^*|| \), it follows from (13) that

\[
||b_1^*|| \geq \omega^\frac{r-i}{2} (\text{Vol}(L))^{\frac{1}{r}}, \quad \text{where} \ \omega = \frac{\sqrt{3}}{2}. \tag{15}
\]

Next, from (14) for \( j = i + 1, \ldots, r \) we obtain

\[
||b_i||^2 \leq \sum_{j=i}^r ||\text{proj}_{b_j^*}(b_i)||^2 
\]

\[
\leq ||b_i^*||^2 + \frac{1}{4} ||b_{i+1}^*||^2 + \cdots + \frac{1}{4} ||b_r^*||^2 
\]

\[
\leq ||b_i^*||^2 \left( 1 + \frac{1}{4\omega^2} + \frac{1}{4\omega^4} + \cdots + \frac{1}{4\omega^{2r-2i}} \right) 
\]

\[
< 2\omega^{2i-2r} ||b_i^*||^2. 
\]

Therefore,

\[
||b_i|| < 2\omega^{r-i} ||b_i^*|| < 2\omega^{1-r} ||b_i^*||, \quad \forall \ i = 1, \ldots, r. \tag{16}
\]

In particular,

\[
\sin \angle(b_i, L_i) = \frac{||b_i^*||}{||b_i||} > \frac{1}{2} \omega^{r-i}, \quad \forall \ i = 1, \ldots, r, \tag{17}
\]

where \( \angle(x, y) \) denotes the Euclidean angle between \( x \) and \( y \).

In the case when \( L \) is a subgroup of \( \mathbb{Z}^n \), we have \( ||b_i|| \geq 1 \) for each \( i \) and, hence, \( ||b_i^*|| > \frac{1}{2} \omega^{r-1} \) by the first inequality (16). Then from \( \text{Vol}(L) = ||b_1^*|| ||b_2^*|| \ldots ||b_r^*|| \) and \( ||b_i|| < 2\omega^{r-i} ||b_i^*|| \) we get an upper bound on \( ||b_i|| \):

\[
||b_i|| < 2^r \omega^{-\frac{r(r-1)}{2}} \text{Vol}(L), \quad \forall \ i = 1, \ldots, r. \tag{18}
\]
Proposition 14. Assume that system (12) has a solution. Then (12) has a solution \((\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_g, \bar{v}_g)\) such that the subgroup

\[ L = \langle \bar{u}_1, \bar{v}_1, \ldots, \bar{u}_g, \bar{v}_g, \bar{c}_1, \ldots, \bar{c}_m \rangle \]

has a basis \(\{f_1, f_2, \ldots, f_k\}\) satisfying

\[ ||f_i|| \leq n 2^n \omega^{-\frac{\alpha(n-1)}{2}} D^n + 2 \omega^{1-n} ||h|| \left( \max_i ||\bar{c}_i|| \right)^n, \quad \forall i = 1, \ldots, k, \]

where \(D = \max_i ||\bar{c}_i|| + \text{diam} \supp(\delta)\).

Proof. Consider a solution \((\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_g, \bar{v}_g)\) of (12). We will show that if \(L\) is, in a certain sense, “sufficiently large” then the solution \((\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_g, \bar{v}_g)\) can be changed to decrease the dimension of \(L\). As above, let \(Q = \langle \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m \rangle\). We proceed in two steps.

Step 1. Define a (finite) subset \(\Delta\) of \(A_n\):

\[ \Delta = \{ g \in A_n \mid gs = t \text{ for some } s, t \in \supp(\delta) \}. \]

Let \(L_0\) be the minimal direct summand of \(L\) containing both \(Q\) and \(\Delta \cap L\). Since \(\pi_L(\delta) = 0\), the equality \(\pi_{L_0}(\delta) = 0\) holds by construction. Let \(U\) be a maximal set of linearly independent elements of \(\{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m\} \cup (\Delta \cap L)\). Then \(\langle U \rangle\) has finite index in \(\langle Q \cup (\Delta \cap L) \rangle\) and, hence, in \(L_0\). Since \(||u|| \leq D\) for each \(u \in U\), we have

\[ \text{Vol}(L_0) \leq \text{Vol}(\langle U \rangle) \leq D^n. \]

By Proposition 13 and (18) there exists a basis \(\{f_1, f_2, \ldots, f_q\}\) for \(L_0\) satisfying

\[ ||f_i|| \leq 2^n \omega^{-\frac{\alpha}{2}} D^n, \quad \forall i = 1, \ldots, q. \]

Step 2. Here we perform the reduction step if \(L\) is “sufficiently large”. We consider the free abelian group \(A_n\) with basis \((\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)\) as canonically embedded in \(\mathbb{R}^n\). For \(X \subseteq \mathbb{R}^n\), denote by \(S(X)\) the linear \(\mathbb{R}\)-subspace spanned by \(X\). Let \(L_0^\perp\) be the orthogonal complement of \(S(L_0)\) and let \(\pi : \mathbb{R}^n \to L_0^\perp\) denote the orthogonal projection.

Choose a basis \((b_1, \ldots, b_r)\) for \(\pi(L)\) as in Proposition 13 and let vectors \(b_1^*, b_2^*, \ldots, b_r^*\) be obtained from \(b_1, b_2, \ldots, b_r\) by the Gram–Schmidt orthogonalization process starting from \(b_r\). By (10),

\[ ||b_i|| < 2 \omega^{1-r} ||b_i^*||, \quad \forall i = 1, \ldots, r. \]

For each \(i\), choose \(d_i \in L\) such that \(b_i = \pi(d_i)\). We may assume that \(d_i - b_i\) is a real linear combination of vectors in the basis \(\{f_1, f_2, \ldots, f_q\}\)
for $L_0$ with coefficients in the interval $[0, 1)$ (otherwise we add an appropriate integral linear combination of $f_1, \ldots, f_q$ to $d_i$). Hence,

$$\|d_i\| < 2\omega^{1-r}\|b_1^*\| + \sum_{i=1}^{q} \|f_i\|, \quad \forall i = 1, \ldots, r.$$  

(21)

By construction, the tuple $(d_1, \ldots, d_r)$ is a basis of a free abelian subgroup of $L$ and $L = L_0 \oplus \langle d_1, \ldots, d_r \rangle$. We prove that if $\|b_1^*\|$ is sufficiently large, then there is a solution of (12) with the subgroup $L' = \langle L_0, d_2, \ldots, d_r \rangle$, instead of $L$. Observe that $d_1$ and $b_1$ have the same orthogonal projection $b_1^*$ to the orthogonal complement of $S(L_0)$. By Lemma [11] we can assume that $\bar{u}_1 \in d_1 + L'$, then $\bar{u}_1, \bar{v}_2, \bar{v}_2, \ldots, \bar{v}_2 g \in L'$.

Below we prove that the inequality

$$\|b_1^*\| > \|h\| \text{Vol}(Q),$$  

(22)

implies that $\bar{v}_1 \in Q$. On the way to contrary assume that $\bar{v}_1 \notin Q$. Choose an orthonormal basis $(e_1, e_2, \ldots, e_n)$ for $\mathbb{R}^n$ compatible with the following chain of linear subspaces of $\mathbb{R}^n$:

$$S(Q) \subseteq S(\langle Q, \bar{v}_1 \rangle) \subseteq S(L') \subseteq S(L),$$

that is,

$$S(Q) = S(e_1, \ldots, e_{p-1}),$$

$$S(\langle Q, \bar{v}_1 \rangle) = S(e_1, \ldots, e_p),$$

$$S(L') = S(e_1, \ldots, e_{r-1}),$$

$$S(L) = S(e_1, \ldots, e_r).$$

We have

$$\bar{u}_1 \in \pm \|b_1^*\| e_r + S(e_1, \ldots, e_{r-1}) \quad \text{and} \quad \bar{v}_1 \in \alpha e_p + S(e_1, \ldots, e_{p-1}),$$

for some $\alpha \neq 0$. Note that $\text{Vol}(\langle Q, \bar{v}_1 \rangle) = |\alpha| \text{Vol}(Q)$ and hence $|\alpha| \geq (\text{Vol}(Q))^{-1}$. Project both sides of the equality

$$\bar{u}_1 \wedge \bar{v}_1 + \cdots + \bar{u}_g \wedge \bar{v}_g = h \text{ mod } K,$$

onto the basis bivector $e_p \wedge e_r$ of $\Lambda^2(\mathbb{R}^n)$. The projection of the right-hand side is bounded above by $\|h\|$ (because $K$ is projected onto 0). The projection of each $\bar{u}_i \wedge \bar{v}_i$ for $i = 2, \ldots, g$ is 0 because $\bar{u}_i, \bar{v}_i \in L'$. The projection of $\bar{u}_1 \wedge \bar{v}_1$ is $\pm \|b_1^*\| |\alpha|$ that has the absolute value greater than $\|h\|$ by (22). This gives a contradiction. Thus, $\bar{v}_1 \in Q$.

Observe that $\bar{v}_1 \in Q$ implies that $L \wedge \bar{v}_1 \subseteq K$ and, hence, the tuple

$$(\bar{u}_1 - d_1, \bar{v}_1, \bar{u}_2, \bar{v}_2, \ldots, \bar{u}_g, \bar{v}_g),$$
is a solution of (12) with the subgroup $L'$ instead of $L$, where $\dim(L') < \dim(L)$. Since the reduction is possible under the assumption (22) we can assume that (22) does not hold, i.e.:

$$\|b_1^*\| \leq ||h|| \text{ Vol}(Q).$$

For the required basis $\{f_1, f_2, \ldots, f_k\}$ of $L$ we then take

$$\{f_1, f_2, \ldots, f_q, d_1, d_2, \ldots, d_r\}.$$

Inequalities (20), (21) and (23) imply the required bound (19). □

Define a (finite) set $L$ of subgroups of $A_n$:

$$L = \{L = \langle f_1, \ldots, f_k \rangle \mid f_1, \ldots, f_k \text{ satisfy } (19), \pi_L(\delta) = 0, \text{ and } c_1, \ldots, c_m \in L.\}.$$

Clearly, the set $L$ can be effectively computed for a given equation (12). The next statement is an immediate corollary of Proposition 14.

**Corollary 15.** A system (12) is solvable if and only if the following system in unknowns $\bar{u}_i, \bar{v}_i \in A_n$ is solvable for some $L \in L$:

$$\begin{cases}
\langle \bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2, \ldots, \bar{u}_g, \bar{v}_g, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m \rangle = L, \\
\bar{u}_1 \wedge \bar{v}_1 + \cdots + \bar{u}_g \wedge \bar{v}_g = h \mod K.
\end{cases}$$

Let $R = L/Q$. As observed above, the subgroup $K$ of $\Lambda^2(L)$ is the kernel of the natural epimorphism $\Lambda^2(L) \to \Lambda^2(R)$. Passing to images of $\bar{u}_i, \bar{v}_i$ in $R$ and to the image of $h$ in $\Lambda^2(R)$ (and, with a slight abuse, coming back to notations $u_i, v_i$ and $h$) we rewrite the system (24) as follows:

$$\begin{cases}
\langle u_1, v_1, u_2, v_2, \ldots, u_g, v_g \rangle = R, \\
u_1 \wedge v_1 + \cdots + u_g \wedge v_g = h.
\end{cases}$$

Thus, the Diophantine problem for the equation (3) is reduced to the following problem: *Given a finitely generated abelian group $R$, an element $h \in \Lambda^2(R)$ and a number $g$, determine if there exist elements $u_i, v_i \in R$ satisfying (25).*

4. Solution of the reduced problem

Here we prove that the reduced problem is algorithmically decidable.

**Proposition 16.** There is an algorithm that determines if the system (25) has a solution $(u_1, v_1, \ldots, u_g, v_g)$ or not.

We prove Proposition 16 for free abelian $R$ in Section 4.1. The general case is considered in Section 4.2.
4.1. The case of free abelian $R$. Suppose that $R$ is a free abelian group, equipped with a fixed basis $E = \{e_1, e_2, \ldots, e_r\}$, and naturally embedded in the normed space $\mathbb{R}^n$ with the Euclidean norm $|| \cdot ||$. We view $\Lambda^2(R)$ as embedded in $\Lambda^2(\mathbb{R}^r)$; on $\Lambda^2(\mathbb{R}^r)$ we have an induced norm defined by setting $E^\wedge 2 = \{e_i \wedge e_j \mid 1 \leq i < j \leq r\}$ to be an orthonormal basis. Below we find an effective bound on the sizes of elements $u_i$ and $v_i$ satisfying \[(25)\].

Lemma 17. Let $0 \neq b \in R^r$ and $L$ be a subspace of $R^r$ with $\dim(L) \geq 2$. Then the angle between subspaces $\Lambda^2(L)$ and $b \wedge L$ of $\Lambda^2(\mathbb{R}^r)$ is equal to the angle between $b$ and $L$.

Proof. If $b \in L$, then both angles are 0. Hence, we can assume that $b \not\in L$ and, consequently, $\dim(L) < r$. By Remark 12 we can assume that the basis $E$ is chosen in a “nice” way with respect to $b$ and $L$, e.g.: 

$$L = S(e_1, e_2, \ldots, e_k) \quad \text{and} \quad b = ||b||\left(e_1 \cos \alpha + e_r \sin \alpha\right),$$

where $k < r$ and $\alpha = \angle(b, L)$. Note that $(b \wedge L) \cap \Lambda^2(L) = 0$. If $S_1$ and $S_2$ are two subspaces of $\mathbb{R}^m$ with $S_1 \cap S_2 = 0$, then the angle between $S_1$ and $S_2$ is the minimal possible angle between a vector $x \in S_1$ and its orthogonal projection onto $S_2$. In our case, $S_1 = b \wedge L$, $S_2 = \Lambda^2(L)$ and $x = b \wedge y$ for some $y \in L$. Up to further change of $E$ we can assume that $y \in S(e_1, e_2)$. Then an easy computation shows that this angle is precisely $\alpha$. \hfill $\square$

Proposition 18. Assume that $R$ is a free abelian group of rank $r$. Then any solution $(u_1, v_1, \ldots, u_g, v_g)$ of the system \[(25)\] can be transformed by symplectic transformations into a new solution $(u_1^*, v_1^*, \ldots, u_g^*, v_g^*)$ such that for each $i$:

\[(26)\] 

$$||u_i^*||, ||v_i^*|| < 2^r (||h|| + 1).$$

Proof. We consider $R$ as embedded in the Euclidean space $\mathbb{R}^r$. For any $x, y \in \mathbb{R}^r$ we have:

\[(27)\] 

$$||x \wedge y|| = \sin \angle(x, y) \cdot ||x|| \cdot ||y||.$$

Step 1: By induction on $g$ we prove the following. If a tuple $(u_1, v_1, \ldots, u_g, v_g) \in \mathbb{R}^{2g}$ satisfies:

$$u_1 \wedge v_1 + \cdots + u_g \wedge v_g = h,$$

then it can be transformed by symplectic transformations to the form:

$$(u_1, v_1, \ldots, u_k, v_k, u_{k+1}, 0, u_{k+2}, 0, \ldots, u_g, 0),$$

where $||u_i||, ||v_i|| \leq 4^r \omega^{-2(r-1)}||h||$ for each $i = 1, \ldots, k$. 

Choose a basis $B = \langle b_1, b_2, \ldots, b_r \rangle$ for $\langle u_1, v_1, \ldots, u_g, v_g \rangle$ by Lemma 13. Using Lemma 11 we may assume that $u_1 = b_1$ and:

$$\langle v_1, u_2, \ldots, v_g \rangle = \langle b_2, \ldots, b_r \rangle = L_1.$$ 

If $v_1 = 0$, then using transformations (S1) we put the pair $(u_1, 0)$ to the end as $(u_g, 0)$ and use induction. Assume that $v_1 \neq 0$. Let $\alpha$ be the Euclidean angle between $u_1 \wedge v_1$ and $h - u_1 \wedge v_1$. We have:

$$||u_1 \wedge v_1||, ||h - u_1 \wedge v_1|| \leq \frac{||h||}{\sin \alpha}.$$ 

Since $u_1 \wedge v_1 \in u_1 \wedge L_1$ and $h - u_1 \wedge v_1 \in \Lambda^2(L_1)$, by Lemma 17 and inequality (17) we have:

$$\sin \alpha \geq \sin \angle(u_1 \wedge L_1, \Lambda^2(L_1)) \geq \frac{1}{2} \omega^{r-1},$$

and, hence:

$$||u_1 \wedge v_1||, ||h - u_1 \wedge v_1|| \leq 2\omega^{1-r}||h||.$$ 

By (27), since $||u_1||, ||v_1|| \geq 1$, we have:

$$||u_1||, ||v_1|| \leq 4\omega^{2-2r}||h||.$$ 

Then we use the inductive hypothesis for $(u_2, v_2, \ldots, u_g, v_g)$ with $h - u_1 \wedge v_1$ instead of $h$.

**Step 2.** Now assume that we are given a tuple $(u_1, v_1, \ldots, u_g, v_g)$ satisfying (25). Let $S_1 = S(u_1, v_1, \ldots, u_k, v_k)$ and let $S_2$ be the orthogonal complement of $S_1$ in $R$. Denote $\pi_i : R \to S_i$ ($i = 1, 2$) the orthogonal projection map. Using symplectic transformations (S3) we can implement the action of $SL_{g-k}$ on the tuple $(u_{k+1}, u_{k+2}, \ldots, u_g)$. Hence, by Lemma 13 we can change $u_i$ so that for some $t \leq g$ the projections $\pi_2(u_i)$ ($i = k + 1, \ldots, t$) form Hermit’s reduced basis for $S_2$ and $\pi_2(u_i) = 0$ for $i > t$. Let $u^*_t$ ($i = k + 1, \ldots, t$) denote the vector obtained from $\pi_2(u_i)$ by the Gram–Schmidt orthogonalization process starting from $\pi_2(u_t)$. Since $u_1, v_1, \ldots, u_g, v_g$ generate $R$ we have:

$$||u^*_t|| = \frac{\text{Vol}(S_1, u_{k+1}, \ldots, u_t)}{\text{Vol}(S_1, u_{k+2}, \ldots, u_t)} \leq 1.$$ 

Then by (16):

$$||\pi_2(u_i)|| < 2\omega^{1-r}, \quad i = k + 1, \ldots, g.$$ 

It remains to bound the other projections $\pi_1(u_i)$ for $i \geq k + 1$. Using transformations (S3) we can add to $u_i$ for $i \geq k + 1$ any integer linear
combinations of vectors $u_i$ with $i \leq k$. Hence, we can change each $u_i$ for $i \geq k + 1$ so that:

$$||\pi_1(u_i)|| < \sum_{i=1}^{k}||u_i|| < r \cdot \omega^{-2r(r-1)}||h||, \quad i = k + 1, \ldots, g.$$ 

It can be easily checked that the bound obtained for $||u_i||$ and $||v_i||$ is less than the right-hand side of (26).

Proposition 18 immediately implies Proposition 16 for a free abelian group $R$.

4.2. The case of a general abelian $R$. Assume that $R$ is a general finitely generated abelian group. To fix a way of presenting the input data, we assume that the quotient $R = L/Q$ is given by specifying subgroups $L$ and $Q$ of $A_n$, i.e. we are given a set $\{c_1, c_2, \ldots, c_m\}$ of generators for $Q$ and a set $\{e_1, e_2, \ldots, e_s\}$ of generators for $L$ over $Q$. The element $h \in \Lambda^2(R)$ is given by a tuple $(h_{ij})_{1 \leq i < j \leq s}$ of (non-uniquely) defined integers so that $h$ is the canonical image of the linear combination $\sum_{i,j} h_{ij} (\bar{a}_i \wedge \bar{a}_j)$. The algorithm goes as follows.

**Step 1.** Represent $R$ in the form

$$R = R_{\text{free}} \oplus R_{\text{tor}},$$

where $R_{\text{free}} \simeq \mathbb{Z}^r$ and $R_{\text{tor}}$ is a finite abelian group. From the computational point of view, this means that we find a generating set $\{b_1, b_2, \ldots, b_q\}$ of $R$ such that $B = (b_1, b_2, \ldots, b_r)$ is a basis for $R_{\text{free}}$ and $R_{\text{tor}}$ is the direct sum of finite cyclic subgroups $\langle b_i \rangle$ of orders $d_i$ for $r + 1 \leq i \leq q$.

**Step 2.** By Proposition 18 any solution $(u_1, v_1, \ldots, u_g, v_g)$ of (25) can be modified in such a way that its coefficient matrix $T = (t_{ij})_{ij}$ in the generators $b_j$ satisfies:

$$|t_{ij}| \leq 2^{r^2} (||h||_{B^{\wedge 2}} + 1) \quad \text{for } j \leq r,$$

where $||h||_{B^{\wedge 2}}$ is the norm of $h$ written in the basis $B^{\wedge 2} = \{b_i \wedge b_j \mid 1 \leq i < j \leq r\}$. Since $\langle b_j \rangle$ has finite order $d_j$ for $j > r$, we obtain also:

$$|t_{ij}| < d_j \quad \text{for } j > r.$$ 

This gives a finite search space for a solution of (25). Checking each equality in (25) is obviously algorithmic. This finishes the proof of Proposition 16 and Theorem 2.
5. Complexity analysis

According to the main result of [14], the Diophantine problem for orientable quadratic equations is \( \text{NP} \)-hard. In this section we prove the upper bound for Theorem 3 and Theorem 4: the problem belongs to \( \text{NP} \) and if the rank \( n \) of the free metabelian group \( M_n \) and the number of variables in the equation are fixed, then there exists a polynomial-time algorithm.

We rely on existence of polynomial-time algorithms [9] solving two principal problems of integer linear algebra: reduction of an integer matrix to the Hermite and the Smith normal forms. More precisely, there exist the following algorithms working in time bounded by a polynomial on the size of the input (see [25]):

**Extended Hermite normal form algorithm:** Given an integer \((p \times q)\)-matrix \( M \), the algorithm computes the Hermite normal form \( M^\dagger \) of \( M \) and a matrix \( U \in \text{GL}(p, \mathbb{Z}) \) such that \( M^\dagger = UM \). Absolute values of entries of \( U \) and \( M^\dagger \) are bounded by \( r ||M||^r \), where \( r = \max(p, q) \) and \( ||M|| \) is the largest absolute value of an entry in \( M \).

**Extended Smith normal form algorithm:** Given an integer \((p \times q)\)-matrix \( M \), the algorithm computes the Smith normal form \( M^\ddagger \) of \( M \) and matrices \( U \in \text{GL}(p, \mathbb{Z}) \) and \( V \in \text{GL}(q, \mathbb{Z}) \) such that \( M^\ddagger = UMV \). Absolute values of entries of \( U \) and \( V \) are bounded by \( r^{4r+5} ||M||^{4r+1} \).

As a consequence, we obtain a polynomial-time algorithm for the following version of the membership problem for finitely generated free abelian groups: given a finite set \( S \) of elements of a finitely generated free abelian group \( A \) and an element \( x \in A \), determine if \( x \) belongs to the subgroup \( \langle S \rangle \). In particular, given two elements \( x, y \in A \) we can determine in polynomial time whether \( x \) and \( y \) belong to the same coset modulo \( S \). It is assumed here that elements of \( A \) are presented as vectors of integers with entries written in the binary form.

Note that the diagonal entries of the Smith normal form of a matrix \( M \) are equal to the greatest common divisor of certain sets of minors of \( M \). In particular, if \( d \) is such an entry, then by Hadamard’s inequality we have \(|d| \leq r^{r/2} ||M||^r \).

The input data for our problem consists of the number \( g \) and the coefficients \( c_1, \ldots, c_m \) written as words in the alphabet \( \{a_1^\pm1, \ldots, a_n^\pm1\} \). As observed in Introduction, we can assume that \( g \leq n \), i.e. if the group \( M_n \) is fixed then \( g \) is bounded by a constant. For the size of the input we take \( N = \sum_{i=1}^m |c_i| + m + n \).

We analyze computational complexity of each step of the algorithm and prove existence of an appropriate \( \text{NP} \)-certificate. For fixed \( m \)
and $n$ we show that the step can be performed in polynomial time. By writing $x \leq O(F(N, m, n))$ we mean that $x \leq c \cdot F(N, m, n)$ for some constant $c$ independent of $N$, $m$ and $n$.

Restricting to finitely many $w_i$’s. Proposition [9] reduces the problem to checking existence of a solution of finitely many systems (8) & (10) over all possible choices of tuples $\overline{W} = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_m) \in (A_n)^m$ satisfying $|\overline{w}_i| \leq \sum_j |c_j|$. The record size of $W$ is bounded by $O(N^2)$, so we can consider $\overline{W}$ as a part of an NP-certificate for solvability of the initial system (3) and, thus, reduce the problem to a single choice of $\overline{W}$. The size of the search space for $\overline{W}$ is bounded by $O((2N+1)^{mn})$ which is polynomial in $N$ when $m$ and $n$ are fixed.

For each $\overline{W}$ we reduce the problem to a system of the form (12). It can be easily seen from the proof of Corollary [10] that for constants $\delta \in C_1(\Gamma_n)$ and $h = \sum_{1 \leq i < j \leq n} h_{ij}(\overline{a}_i \wedge \overline{a}_j)$ occurring in (12) we have:

\begin{align}
(28) \quad \text{diam supp}(\delta) \leq 2N \quad \text{and} \quad \max_{i,j} |h_{ij}| \leq 2N^2.
\end{align}

Restricting to finitely many subgroups $L$. The next step reduces the system (12) to finitely many systems (24). Each system (24) is defined by a choice of a subgroup $L$ generated by a basis $U = \{f_1, \ldots, f_k\}$ satisfying inequality (19) and the equality $\pi_L(\delta) = 0$. The right-hand side of (19) is bounded by $O(C^{n^2}N^{n+1})$ for some constant $C > 1$. Hence, the number of possible choices for $U$ is bounded by $O(C^{n^3}N^{n(n+1)})$. For each choice of $U$, the equality $\pi_L(\delta) = 0$ and the membership $c_i \in L$ can be checked in polynomial time using the algorithm for the membership problem in finitely generated free abelian groups. Thus, we can include a generating set $U$ for $L$ into our NP-certificate and reduce the problem to a single choice of $L$. For a fixed $n$ the set $L$ can be computed in polynomial time.

As a final step of the reduction, for a particular choice of a generating set for $L$ we rewrite the system (24) in the form (25). The input data for (25) is already presented in the form described in Section 4.2.

Solution of the reduced problem. We follow the procedure described in Section 4.2 which consists of two steps: at Step 1 we obtain a normalized basis for the abelian group $R = L/Q$ and at Step 2 we give a bound on the solution in terms of the new basis. We show that Step 1 can be performed in polynomial time and bound at Step 2 gives a solution of the size bounded by a polynomial in $N$.

The input data for (25) consists of three matrices: a matrix $K_0$ expressing the generators for $Q$ in the basis $B_0 = \{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n\}$, a matrix $M_0$ expressing the generators for $L$ in $B_0$ and a skew-symmetric matrix $H_0$ expressing $h$ in the basis $B_0^{\wedge 2}$ of $\Lambda^2(A_n)$. We have upper
bounds:
$$||K_0|| \leq N, \quad ||M_0|| \leq C^{n^2} N^{n+1}, \quad ||H_0|| \leq 2N^2.$$  

The computation consists of the following steps.

**Transferring the data to a basis of $L$.** We reduce the matrix $M_0$ to a Hermite normal form $M_1$ producing a basis $B_1$ for $L$. According to the above bound we can assume that:
$$||M_1|| \leq n^n ||M_0||^n.$$  

We express the generators for $Q$ in the new basis $B_1$. For the new matrix $K_1$ of coordinates of generators of $Q$ we have:
$$K_1 = K_0^* M_1^{*-1},$$
where $K_0^*$ and $M_1^*$ are certain submatrices of $K_0$ and $M_1$. This gives the bound:
$$||K_1|| \leq n ||K_0|| \cdot ||M_1^{*-1}|| < n ||K_0|| \cdot n^{n/2} ||M_1||^n.$$  

In a similar way, for the new matrix $H_1$ of coordinates of $h$ in the basis $B_1^\wedge 2$ we have:
$$H_1 = (M_1^{*-1})^T H_0^* M_1^{*-1},$$
and, hence:
$$||H_1|| < n^2 ||H_0|| \cdot n^n ||M_1||^{2n}.$$  

**Representing the subgroup $R = L/Q$ as $R_{\text{free}} \oplus R_{\text{tor}}$.** This requires reducing the matrix $K_1$ to the Smith normal form $K_2$. We have:
$$K_2 = U K_1 V$$
for some unimodular matrices $U$ and $V$ satisfying
$$||U||, ||V|| \leq n^{4n+5} ||K_1||^{4n+1}.$$  

We obtain a basis $B_2 = \{b_1, b_2, \ldots, b_q\}$ of $R$ such that $R_{\text{free}} = \langle b_1, b_2, \ldots, b_r \rangle$ and $R_{\text{tor}} = \mathbb{Z}_{d_{r+1}} \oplus \cdots \oplus \mathbb{Z}_{d_q}$ where $\mathbb{Z}_{d_i} = \langle b_i \rangle$ for $i > r$. The numbers $d_i$ are non-unit diagonal elements of $K_2$ and we have
$$d_i < n^{n/2} ||K_1||^n, \quad i = r + 1, \ldots, q.$$  

We express the canonical image of $h$ in the new basis $B_2^\wedge 2$ obtaining a matrix $H_2$ computed as
$$H_2 = (V^{*-1})^T H_1^* V^{*-1}$$
This gives the bound
$$||H_2|| < n^2 ||H_1|| \cdot n^n ||V||^{2n}.$$
The final solution bound. According to Step 2 of the procedure in [1, 2] if the system (25) is solvable then it has a solution \((u_1, v_1, \ldots, u_g, v_g)\) represented by a matrix \(T\) in the basis \(B_2\) satisfying

\[
||T|| \leq \max \left\{ 2^{n^2}(||H_2|| + 1), \max_i |d_i| \right\}
\]

We observe that all numerical upper bounds in our computation are of the form \(O(2^{f(n)} N^{g(n)})\) where \(g(n)\) and \(f(n)\) are bounded by polynomials on \(n\). This gives a similar bound on \(||T||\). This implies that the record size of \(T\) is bounded by a polynomial on the input size of the problem, and hence the problem belongs to \(\text{NP}\). If \(n\) is fixed, we obtain a search space for \(T\) of a polynomial size. The proof is completed.

6. Open problems

We left the case of non-orientable quadratic equations open. We think that methods of the current paper can be appropriately modified to treat that case as well.

As mentioned in Introduction, the general Diophantine problem in free metabelian groups is undecidable. It would be interesting to investigate some specific (non-quadratic) classes of equations. For instance, Baumslag, Mahler, and Lyndon in [2, 12] study equations of the form \(x^n y^m z^l = 1\). Also, Roman’kov in [23] investigates solvability of regular one-variable equations in the class of metabelian groups.

Spherical equations are a straightforward generalization of conjugacy equations. It is shown in [18] that the conjugacy problem in finitely generated metabelian groups is decidable. It is currently unknown if a similar result holds for spherical quadratic equations. It is also interesting to investigate the Diophantine problem for quadratic equations in the whole class of finitely generated metabelian groups and in specific groups such as Baumslag–Solitar groups.

Another possible generalization of our work is a study of groups of type \(F/N’\) where \(F\) is a free group and \(N \trianglelefteq F\). Recall that the power problem in a group \(G\) is to determine if \(u \in \langle v \rangle\) for given \(u, v \in G\). It is shown in [7] that the conjugacy problem in \(F/N’\) is decidable if and only if the power problem in \(F/N\) is decidable. It would be interesting to generalize that result at least to spherical quadratic equations.

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