Event-triggered stabilization over digital channels of linear systems with disturbances*

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Abstract

In the same way that subsequent pauses in spoken language are used to convey information, it is also possible to transmit information in communication systems not only by message content (data payload), but also with its timing. This paper presents an event-triggering strategy that utilizes timing information by transmitting in a state-dependent fashion. We consider the stabilization of a continuous-time, time-invariant, linear plant over a digital communication channel with bounded delay and in the presence of bounded plant disturbances. We propose an encoding-decoding scheme that guarantees a sufficient information transmission rate for stabilization of the plant. In addition, we determine a lower bound on the information transmission rate of the sensor which is necessary for stabilization. We show that for small values of the delay, the timing information implicit in the triggering events is enough to stabilize the plant with any positive information transmission rate. In contrast, when the delay increases beyond a critical threshold, the timing information alone is not enough to stabilize the plant and the data payload transmission rate begins to increase. Finally, large values of the delay require transmission rates higher than what prescribed by the classic data-rate theorem. Our results also provide a novel encoding-decoding scheme for complex systems, which can be readily applied to diagonalizable multivariate systems with complex eigenvalues. The identified rates depend on parameters in the event-triggered law and plant dynamics, and the bounds on the plant disturbances and channel delays. The technical treatment does not rely upon any a priori probabilistic distribution of plant disturbances, initial condition, or delay. Our theoretical results are also validated by numerical simulation.

Key words: Control under communication constraints; event-triggered control; networked control systems; quantized control; feedback stabilization with delay.

1 Introduction

Networked control systems (NCS) [Bemporad et al., 2010], where the feedback loop is closed over a communication channel, are a fundamental component of cyber-physical systems (CPS) [Kim and Kumar, 2012]. In this context, data-rate theorems state that the minimum communication rate to achieve stabilization is equal to the entropy rate of the plant, expressed by the sum of the unstable modes in nats (one nat corresponds to 1/ \ln 2 bits.) Key contributions by Tatikonda and Mitter [2004a] and Nair and Evans [2004] consider a “bit-pipe” communication channel, capable of noiseless transmission of a finite number of bits per unit time evolution of the plant. Extensions to noisy communication channels are considered in [Tatikonda and Mitter, 2004b, Sahai and Mitter, 2006, Matveev and Savkin, 2009]. Stabilization over time-varying bit-pipe channels, including the erasure channel as a special case, are studied by Minero et al. [2009, 2013]. Additional formulations include stabilization of switched linear systems [Liberzon, 2014], uncertain systems [Ishii, 2010], multiplicative noise [Ding et al., 2016], optimal control [Tatikonda et al., 2004, Kostina and Hassibi, 2016, Khina et al., 2017], and stabilization using event-triggered strategies [Tallapragada and Cortés, 2016, Pearson et al., 2017, Ling, 2017, Linsenmayer et al., 2017, Tallapragada et al., 2018].

While the majority of communication systems transmit information by adjusting the signal amplitude, it is also possible to communicate information by adjusting the transmission time of a symbol [Anantharam and Verdu, 1996, Rose and Mian, 2016]. In a general framework, Khojasteh et al. [2018a] study the fundamental limitations of using timing information for stabilization and show that it is possible to stabilize a plant using inherent information in the timing of the transmissions. In fact, it was shown earlier that event-triggering control techniques encode information in the timing in a state-dependent fashion. In this context, Kofman and Braslavsky [2006] have shown that in the absence of delay in the communication process and without plant disturbances and assuming the controller has knowledge of the triggering strategy, one can stabilize the plant with any positive data payload transmission rate. Building upon this observation, our previous work [Khojasteh et al., 2016] considers scalar plants.
without disturbances and quantifies the information contained in the timing of the triggering events as a function of the delay in the communication channel. For small values of the delay, we show that stability can be achieved with any positive information transmission rate (the rate at which sensor transmits data payload). However, as the delay increases to values larger than a critical threshold, the timing information contained in the triggering action itself may not be enough to stabilize the plant and the information transmission rate must be increased. The work in [Khojasteh et al., 2016] also extends the treatment to the vector case, but the analysis is limited to plants with only real eigenvalues of the open-loop gain matrix. Furthermore, the required exponential convergence guarantees lead to a mismatch between sensor and controller about the possible values of the state estimation error, which requires an additional layer of complexity in the sensor’s transmission policy of the event-triggered control design. In contrast, in this work we consider the weaker stability notion of practically stable solutions, requiring the state to be bounded at all times beyond a fixed horizon, allowing us to simplify the treatment and design a simpler event-triggered control strategy. In addition, the literature has not considered to what extent the implicit timing information in the triggering events is still useful in the presence of plant disturbances. Beyond the uncertainty due to the unknown delay in communication, disturbances add an additional degree of uncertainty to the state estimation process and therefore their effect should be properly accounted for. With this in mind, we study the stabilization of a linear, time-invariant plant subject to bounded disturbance over a communication channel with bounded delay.

Our contributions are threefold. First, for scalar real plants with disturbances, we derive a sufficient condition on the information transmission rate. Specifically, we design an encoding-decoding scheme that, together with the proposed event-triggering strategy, rules out Zeno behavior and ensures that the state remains bounded over time. We show that for small values of the delay, our event-triggering strategy achieves stabilization using only implicit timing information and transmitting data payload at a rate arbitrarily close to zero. On the other hand, since larger values of the delay imply that information has been excessively aged by delay and corrupted by the disturbance, as the delay becomes larger, increasingly higher communication rates are required. Our second contribution pertains to the generalization of the sufficient condition to complex plants with complex open-loop gain subject to disturbances. This result sets the basis for the generalization of event-triggered control strategies that meet the bounds on the information transmission rate for the stabilization of vector systems under disturbances and with any real open-loop gain matrix (with complex eigenvalues). Our final contribution consists of deriving for scalar real plants a necessary condition on the information transmission rate, assuming that at each triggering time, the sensor transmits the smallest possible packet size to achieve the triggering goal for all realizations of the delay and plant disturbance.

Notation: Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ represent the set of real, complex, and natural numbers, respectively. We let $|\cdot|$ and $\|\cdot\|$ denote absolute value and complex norm, respectively. Let $\log$ and $\ln$ represent base 2 and natural logarithms, respectively. For a function $f : \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$, we let $f(t^+) = \lim_{s \to t^+} f(s)$ denote the right-hand limit of $f$ at $t$. In addition, $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denotes the nearest integer less (resp. greater) than or equal to $x$. We denote the modulo function by $\mod(x, y)$, representing the remainder after division of $x$ by $y$. The function $\sign(x)$ denotes the sign of $x$. Any $Q \in \mathbb{C}$ can be written as $Q = \Re(Q) + i \Im(Q) = \|Q\| e^{i\phi_Q}$, and for any $y \in \mathbb{R}$, we have $\|y Q^k\| = e^{\Re(Q) y}$. Finally, $\tr(A)$ denotes the trace of matrix $A$, and $m$ denotes the Lebesgue measure.

2 Problem formulation

We consider a networked control system described by a plant-sensor-channel-controller tuple, cf. in Figure 1. The plant is described by a scalar, continuous-time, linear time-invariant model,

$$\dot{x} = Ax(t) + Bu(t) + w(t), \quad (1)$$

where $x(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$ for $t \in [0, \infty)$ are the plant state and control input, respectively, and $w(t) \in \mathbb{R}$ represents the plant disturbance. The latter is upper bounded as:

$$|w(t)| \leq M, \quad (2)$$

where $M$ is a nonnegative real number. In (1), $A$ is a positive real number (i.e., the plant is unstable), $B \in \mathbb{R}$ and the initial condition $x(0)$ is bounded. We consider the following notion of stability.

**Definition 1** The plant (1) is practically stable if for any $x(0)$ in a closed interval, there exists an increasing function $\alpha(M)$, with $0 \leq \alpha(0) < \epsilon$ for any $\epsilon > 0$, such that for all $\Psi > \alpha(M)$, there exists $T$ such that $|x(t)| \leq \Psi$ for all $t \geq T$.

Our objective is to ensure the dynamics (1) is practically stable given the limitations posed by the system model described in Figure 1. In particular, we assume the sensor measurements are exact and there is no delay in the control action, which is executed with infinite precision. However, measurements are transmitted from sensor to controller over a communication channel subject to a finite data rate and bounded unknown delay, as described later in this paper. We denote by $\{t_k^s\}_{k \in \mathbb{N}}$ the sequence of times when the sensor transmits a packet of length $g(t_k^s)$ bits that contains a quantized version of the state. We let the $k^{th}$ triggering interval as $\Delta_k = t_{k+1}^s - t_k^s$. The packets are delivered to the controller without error and entirely but with unknown upper bounded delay. Let $\{t_k^c\}_{k \in \mathbb{N}}$ be the sequence of times when the controller receives the packets transmitted at times $\{t_k^s\}_{k \in \mathbb{N}}$. We assume that the communication delays $\Delta_k = t_k^c - t_k^s$, for all $k \in \mathbb{N}$, satisfy

$$\Delta_k \leq \gamma, \quad (3)$$

where $\gamma$ is a non-negative real number. From this point on, when referring to a generic triggering or reception time, for convenience we skip the super-script $k$ in $t_k^s$ and $t_k^c$, and the sub-script $k$ in $\Delta_k$. 
We do not consider delays, plant disturbances, and initial condition to be chosen from any specific distribution. Therefore, our results are valid for any arbitrary delay, plant disturbances, and initial condition with finite supports.

Let $b_s(t)$ be the number of bits transmitted in the data payload by the sensor up to time $t$. The information transmission rate is

$$ R_s = \limsup_{t \to \infty} \frac{b_s(t)}{t}. $$

In addition to the data payload, the reception time of the packets carries information. Consequently, let $b_c(t)$ be the amount of information measured in bits included in data payload and timing information received at the controller until time $t$. The information access rate is

$$ R_c = \limsup_{t \to \infty} \frac{b_c(t)}{t}. $$

At the controller, the estimated state is represented by $\hat{x}$ and evolves during the inter-reception times as

$$ \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad t \in [t^k_c, t^{k+1}_c], $$

starting from $\hat{x}(t^{k+1}_c)$, which represents the state estimate of the controller with the information received up to time $t^k_c$ (The exact way to construct $\hat{x}(t^{k+1}_c)$ is explained later in this section). We also set $\hat{x}(0) = \hat{x}_0$. We assume that the sensor knows $\hat{x}_0$ and has knowledge of the times the actuator performs the control action. This is to ensure that the sensor can also compute $\hat{x}(t)$ for all time $t$. In practice, this corresponds to assuming an instantaneous acknowledgment from the actuator to the sensor via the control input, as discussed in Sahai and Mitter [2006], Ling [2018]. To obtain such causal knowledge, one can monitor the output of the actuator provided that the control input changes at each reception time. In case the sensor has only access to the plant state, one can use a narrowband signal in the control input to excite a specific frequency of the state, that can signal the time at which the control action has been applied. The state estimation error is

$$ z(t) = x(t) - \hat{x}(t), $$

where $z(0) = x(0) - \hat{x}_0$. We rely on this error to determine when a triggering event occurs in our controller design.

According to the data-rate theorem, if $R_c < A/\ln 2$, the value of the state in (1) becomes unbounded as $t \to \infty$ (the result for plants evolving in continuous time stated in [Hespanha et al., 2002, Theorem 1] does not consider disturbances, but can readily be generalized to account for them), and (1) is not practically stable. In our discussion below, the data-rate theorem serves as a baseline for comparison with our results on the information transmission rate $R_s$ to better understand the amount of timing information contained in event-triggered control designs in presence of unknown communication delays.

### 3 Event-triggered control design

In general, the transmission times should be determined so that the plant (1) is practically stable. Our approach to ensure this is through event-triggered control. Consider the following class of triggers: for $J \in \mathbb{R}$ positive, the sensor sends a message to the controller at $t^{k+1}_c$ if

$$ |z(t^{k+1}_c)| = J, $$

provided $t^k_c < t^{k+1}_c$ for $k \in \mathbb{N}$ and $t^1_c \geq 0$. In this case, a new transmission happens only after the previous packet has been received by the controller. If the controller knows this criterion and the triggering time $t_s$, it can compute $x(t_s) = \pm J + \hat{x}(t_s)$. Therefore, at every triggering event, the transmission of only a single bit in data payload (to determine the sign) is required to compute the exact value of $x(t_s)$, which can then be used for stabilization.

However, due to the unknown bounded delay in the communication channel, the controller does not have perfect knowledge of the time $t_s$, and as a consequence may not be able to compute the exact value of $x(t_s)$ by receiving a single bit. To address this, let $\tilde{z}(t_s)$ be an estimated version of $z(t_s)$ reconstructed by the controller knowing $|z(t_s)| = J$, the bound (3) on the delay and the packet received through the communication channel. With this information, the controller updates the state estimate using the jump strategy,

$$ \hat{x}(t^+_c) = \tilde{z}(t_c) + \hat{x}(t_c). $$

Note that $|z(t^+_c)| = |x(t_s) - \tilde{z}(t^+_c)| = |z(t_s) - \tilde{z}(t_s)|$. In addition, the packet size $g(t_s)$ is calculated at the sensor to ensure that

$$ |z(t^+_c)| = |z(t_c) - \tilde{z}(t_c)| \leq J, $$

is satisfied for all $t_c \in [t_s, t_s + \gamma]$.

We next show that, if (8) holds at each reception time $\{t^k_c\}_{k \in \mathbb{N}}$, then the plant (1) is practically stable. Consequently, finding a sufficient condition on the transmission rate to guarantee (1) is practically stable reduces to finding conditions to achieve (8) for all reception times.

**Lemma 2** Consider the plant-sensor-channel-controller model with plant dynamics (1), estimator dynamics (4), triggering strategy (6), and jump strategy (7). Assume the controller has enough information about $x(0)$ such that $|z(0)| < J$ and the packet size is large enough to ensure (8) for all reception times $\{t^k_c\}_{k \in \mathbb{N}}$. Then for all time $t$,

$$ |z(t)| \leq Je^{At} + \frac{M}{A} (e^{At} - 1). $$

**PROOF.** At the reception time $z(t^+_c)$ satisfies (8), hence using triggering rule (6) we deduce $|z(t)| \leq J$ for all $t \in [t^k_c, t^{k+1}_c]$. Noting that $J$ is smaller than the upper bound in (9), it remains to prove (9) for all time $t \in [t^k_c, t^{k+1}_c]$. From (5), we have

$$ z(t) = A z(t) + w(t) $$

during inter-reception time intervals, therefore

$$ z(t) = e^{A(t-c_c)} z(t_c) + \int_{t_c}^t e^{A(t-c_c - \tau)} w(\tau) d\tau. $$

When a triggering occurs $|z(t_s)| = J$, hence from (3) the ab-
solute value of the first addend in (10) is upper bounded as 
\[ | \int_{t_s}^{t_c} e^{A(t_c-\tau)}w(\tau) d\tau | \leq M \int_{t_s}^{t_c} |e^{A(t_c-\tau)}| d\tau = M \left( e^{A(t_c-t_s)} - 1 \right), \]
consequently, the result follows. \( \square \)

Next, using Lemma 2, we prove that if (8) holds for all reception times \( \{ t_k^k \} \), then we can ensure (1) is practically stable.

**Proposition 3** Under the assumptions of Lemma 2, using the control rule \( u(t) = -K \dot{x}(t) \) the system is practically stable, provided \( A - BK < 0 \).

**PROOF.** By letting \( u(t) = -K(x(t) - z(t)) \), we rewrite (1) as 
\[ \dot{x}(t) = (A - BK)x(t) + BKz(t) + w(t). \]

Consequently, we have
\[
\begin{align*}
x(t) &= e^{(A-BK)t}x(0) \\
&+ e^{(A-BK)t} \int_0^t e^{-(A-BK)\tau}(BKz(\tau) + w(\tau)) d\tau.
\end{align*}
\]

Since \( |z(t)| \leq J \) we deduce \( |x(t)| \leq L := \max \{|\hat{x}_0 - J|, |\hat{x}_0 + J|\} \). Using the upper bounds in (2) and (9), and \( |x(0)| \leq L \), we obtain
\[
|z(t)| \leq e^{(A-BK)t}L + e^{(A-BK)t} \int_0^t e^{-(A-BK)\tau}(BK|z(\tau)| + M) d\tau \\
\leq e^{(A-BK)t}L \left( BK + 1 \right) + \frac{M}{A - BK} \left( e^{A\gamma} - 1 \right) \frac{M}{A - BK} \left( e^{A\gamma} - 1 \right) + M \\
(1 - e^{(A-BK)t}).
\]

Since \( A - BK \) is negative we deduce,
\[
\limsup_{t \to \infty} |x(t)| \leq \frac{BK (Je^{A\gamma} + \frac{M}{A} (e^{A\gamma} - 1)) + M}{A - BK}.
\]
The result now follows with the choice \( x(0) \in [\hat{x}_0 - J, \hat{x}_0 + J] \) and
\[
\Psi > \alpha(M) = \frac{BK (Je^{A\gamma} + \frac{M}{A} (e^{A\gamma} - 1)) + M}{-(A - BK)}\]
since \( \alpha(0) \to 0 \) as \( J \) tends to zero. \( \square \)

We conclude this section by showing that using a proper packet size \( g(t_s) \) our event-triggered control design does not suffer from “Zeno behavior”, i.e., an infinite amount of triggering events in a finite time interval. To show this, let \( 0 < \rho_0 < 1 \) be a design parameter, and select the packet size \( g(t_s) \) at the sensor to ensure that
\[
|z(t_c^+)| = |z(t_c) - z(t_c)| \leq \rho_0 J.
\]
Clearly, having (12) at each reception time implies (8). The following result shows that given (12) the time between consecutive triggers is uniformly lower bounded.

**Lemma 4** Consider the plant-sensor-channel-controller model with plant dynamics (1), estimator dynamics (4), triggering strategy (6), and jump strategy (7). Assume the controller has enough information about \( x(0) \) such that \( |z(0)| < J \) and the packet size is large enough to ensure (12) for all \( k \in \mathbb{N} \). Then
\[
t_k^{k+1} - t_k^k \geq \frac{1}{A} \ln \left( \frac{J + M}{\rho_0 J + M} \right).
\]

**PROOF.** By considering two successive triggering times \( t_k^k \) and \( t_k^{k+1} \) and the reception time \( t_{k+1}^k \), from (6) it follows \( t_k^k \leq t_{k+1}^k \leq t_k^{k+1} \). From (5), we have \( \dot{z}(t) = A\gamma z(t) + w(t) \), consequently using the definition of the triggering time \( t_k^{k+1} \) it follows
\[
|z(t_{k+1}^k + t_{k+1}^k) - e^{A(t_{k+1}^k - t_k^k)}| + | \int_{t_k^k}^{t_{k+1}^k} e^{A(t_{k+1}^k - \tau)}w(\tau) d\tau | \geq J.
\]
Using (12) and (11), we have
\[
\rho_0 Je^{A(t_{k+1}^k - t_k^k)} + M \left( e^{A(t_{k+1}^k - t_k^k)} - 1 \right) \geq J,
\]
which is equivalent to \( t_k^{k+1} - t_k^k \geq \frac{1}{A} \ln \left( \frac{J + M}{\rho_0 J + M} \right) \). The result follows from using \( t_k^k \leq t_{k+1}^k \) in this inequality. \( \square \)

Given the uniform lower bound on the inter-event time obtained in Lemma 4, we deduce that the event-triggered control design does not exhibit Zeno behavior. The frequency with which transmission events are triggered is captured by the triggering rate
\[
R_{tr} = \limsup_{N \to \infty} \frac{N}{N} \sum_{k=1}^N \Delta t_k.
\]

Using Lemma 4, we deduce that the triggering rate (13) is uniformly upper bounded under the event-triggered control design, i.e., for all initial conditions, possible delay and plant noise values, we have
\[
R_{tr} \leq \frac{A}{\ln \left( \frac{J + M}{\rho_0 J + M} \right)}.
\]

**4 Sufficient and necessary conditions on the information transmission rate**

Here, we derive sufficient and necessary conditions on the information transmission rate to ensure (1) is practically stable. Our
approach to derive them revolves around the characterization of the transmission rate required to ensure that (8) holds at all reception times. Section 4.1 introduces a quantization policy that, together with the event-triggered scheme discussed above, provides a complete control design to guarantee (1) is practically stable and rule out Zeno behavior by ensuring (12). Section 4.2 presents lower bounds on the packet size and triggering rate required to guarantee (1) is practically stable, leading to our bound on the necessary information transmission rate. We conclude the section by comparing the sufficient and necessary bounds, and discussing the gap between them.

4.1 A sufficient information transmission rate

To finalize the design in Section 3, here we specify a quantization policy and determine the resulting error as a function of the number of bits transmitted. This allows us to determine the packet size that ensures (12) (and consequently (8)) holds, thereby leading to a complete control design which ensures (1) is practically stable and rules out Zeno behavior. In turn, this also yields a sufficient condition on the information transmission rate.

In our sufficient design the controller estimates \( z(t_s) \) as:

\[
\hat{z}(t_s) = \operatorname{sign}(z(t_s)) Je^{A(t_s - q(t_s))}.
\]

(15)

According to (6), at every triggering event, the sensor encodes \( t_s \) and transmits a packet \( p(t_s) \). The packet \( p(t_s) \) consists of \( g(t_s) \) bits of information and is generated according to the following quantization policy. The first bit \( p(t_s)[1] \) denotes the sign of \( z(t_s) \). As shown in Figure 2, the reception time \( t_e \) provides information to the controller that \( t_s \) could fall anywhere between \( t_e - \gamma \) and \( t_e \). Let \( b > 1 \). To determine the time interval of the triggering event, we break the positive time line into intervals of length \( b \gamma \). Consequently, \( t_s \) falls into \([j \gamma, (j+1)\gamma]\) or \([(j+1)\gamma, (j+2)\gamma]\), with \( j \) being a natural number. We use the second bit of the packet to determine the correct interval of \( t_s \). This bit is zero if the nearest integer less than or equal to the beginning number of the interval is an even number and is 1 otherwise. This can be written mathematically as \( p(t_s)[2] = \operatorname{mod}(\lfloor \frac{t_s}{b \gamma} \rfloor, 2) \). For the remaining bits of the packet, the encoder breaks the interval containing \( t_s \) into \( 2^q(t_s) - 2 \) equal sub-intervals. Once the packet is complete, it is transmitted to the controller, where it is decoded and the center point of the smallest sub-interval is selected as the best estimate of \( t_s \). Therefore, we have

\[
|t_s - q(t_s)| \leq \frac{b \gamma}{2^q(t_s) - 1}.
\]

(16)

We have employed this quantization policy in our previous work [Khojasteh et al., 2016] and analyzed its behavior in the case with no disturbances. Here, we extend our analysis to scenarios with both unknown delays and plant disturbances.

We start by showing that under the proposed encoding-decoding scheme, if the sensor has a causal knowledge of the delay in the communication channel, it can compute the state estimated by the controller.

**Proposition 5** Under the assumptions of Lemma 4, using the estimation (15) and the quantization policy described in Figure 2, if the sensor has causal knowledge of delay in the communication channel, then the sensor can calculate \( \hat{x}(t) \) for all time \( t \).

**PROOF.** The proof is based on induction. Knowing that \( \hat{x}(0) = \hat{x}_0 \) sensor can construct the value of \( \hat{x}(t) \) for \( t \in [0, t^1] \) according to (4). Note that we are using the proposed quantizer in Figure 2, hence given \( t^1_s, q(t^1_s) \) gets identified deterministically. Consequently, given \( t^1_s \) and using (15), the sensor constructs the value of \( z(t^1_s) \) and it determines the value of \( x(t^1_s) \).

Now assuming that the sensor is aware of the value of \( \hat{x}(t^k_s) \) we will prove that the sensor can find the value of \( \hat{x}(t^k_s) \) too. Since the sensor is aware of the \( \hat{x}(t^k_s) \) and it knows that \( \hat{x}(t) \) evolves according to (4) for \( t \in [t^k_s, t^{k+1}] \), starting from \( \hat{x}(t^k_s) \) sensor can calculate all the values of \( \hat{x}(t) \) until \( t^{k+1} \). Using our proposed quantizer and given \( t^{k+1}_s, q(t^{k+1}_s) \) can be identified deterministically, therefore by knowing the value of \( (k+1)th \) delay the sensor can calculate the value of \( \hat{x}(t_s) \) from (15). Then using the jump strategy (7) it can calculate \( \hat{x}(t^k_s) \). So the result follows.

Our next result bounds the difference \( |t_s - q(t_s)| \) between the triggering time and its quantized version so that (12) holds at all reception times.

**Lemma 6** Under the assumptions of Lemma 4, using the estimation (15) and the quantization policy described in Figure 2, if

\[
|t_s - q(t_s)| \leq \frac{1}{A} \ln(1 + \frac{p_0 - \frac{M}{A\rho_0}}{e^{A\gamma} - 1})
\]

then (12) holds for all reception times \( \{t^k_s\}_{k \in \mathbb{N}} \), provided \( J > \frac{M}{A\rho_0}(e^{A\gamma} - 1) \).

**PROOF.** Using (10), (15), and the triangular inequality, we deduce

\[
|z(t_e) - \hat{z}(t_e)| \leq
\]
By applying the bounds (3) and (11) on first and second addend respectively it follows

\[
|z(t_c) - \bar{z}(t_c)| \leq |J e^{A(t_c - q(t_c))}| + \frac{M}{A} (e^{A\gamma} - 1).
\]

Therefore, ensuring (12) reduce to

\[
|1 - e^{A(t_c - q(t_c))}| \leq \eta,
\]

where \(\eta = e^{-A\gamma}(\rho_0 - \frac{M}{AJ}(e^{A\gamma} - 1))\). Since \(J > \frac{M}{A\rho_0}(e^{A\gamma} - 1)\), we have \(0 \leq \eta < 1\). Consequently, using (17), we deduce

\[
\frac{\ln(1 - \eta)}{A} \leq t_s - q(t_s) \leq \frac{\ln(\eta + 1)}{A}
\]

It follows that to satisfy (12) for all delay values, requiring

\[
|t_s - q(t_s)| \leq \min \left\{ \frac{\ln(1 - \eta)}{A}, \frac{\ln(\eta + 1)}{A} \right\}
\]

suffices, and the result now follows. \(\square\)

Building on our results above, the following result provides a lower bound on the packet size so that (12) is ensured at all reception times.

**Theorem 7** Under the assumptions of Lemma 4, there exists a quantization policy that achieves (12) for all reception times \(\{t^{k}_c\}_{k \in \mathbb{N}}\) with any packet size

\[
g(t^{k}_s) \geq \max \left\{ 0, 1 + \log \frac{Ab\gamma}{\ln(1 + \frac{\rho_0 - (M/J)(e^{A\gamma} - 1)}{e^{A\gamma}})} \right\},
\]

where \(b > 1\) and \(J > \frac{M}{A\rho_0}(e^{A\gamma} - 1)\).

The proof is a direct consequence of (16) and Lemma 6. The combination of the upper bound (14) obtained for the triggering rate and Theorem 7 yields a sufficient bound on the information transmission rate. To sum it up, we conclude that any information transmission rate

\[
R_s \geq \frac{A}{\ln(\frac{J + \frac{M}{AJ}}{\rho_0 + \frac{M}{AJ}})} \max \left\{ 0, 1 + \log \frac{Ab\gamma}{\ln(1 + \frac{\rho_0 - (M/J)(e^{A\gamma} - 1)}{e^{A\gamma}})} \right\},
\]

is sufficient to ensure (12) and, as a consequence (8), for all reception times \(\{t^{k}_c\}_{k \in \mathbb{N}}\). Therefore, from Proposition 3, the bound (18) is sufficient to ensure the plant (1) is practically stable.

### 4.2 A necessary information transmission rate

Here, we present a necessary condition on the information transmission rate required for stabilizing the plant (1) under the class of event-triggering strategies described in Section 3. In Section 4.1, to derive a sufficient bound that guarantees (1) is practically stable, our focus has been on identifying a quantization policy that could handle any realization of initial condition, delay, and disturbance. Instead, the treatment in this section switches gears to focus on any given quantization policy, for which we identify at least a realization of initial condition, delay, and disturbance that requires the necessary bound on the information transmission rate.

We start our discussion by making the following observation about the condition (8). If this condition is not satisfied at an arbitrary reception time \(t^{k}_s\), i.e., \(z(t^{k}_s) > J\) and \(w(t) > 0\) for all \(t > t^{k}_s\), then the state estimation error grows exponentially with time. In this case, (1) is not practically stable. As a consequence, our worst-case analysis providing a necessary condition for (1) to be practically stable reduces to identifying a necessary condition for having (8) at all reception times \(\{t^{k}_c\}_{k \in \mathbb{N}}\).

Our approach to find a lower bound on the information transmission rate \(R_s\) consists of finding lower bounds on the packet size \(g(t_{s})\) and the triggering rate \(\hat{R}_{tr}\). The combination of both bounds yields then a characterization of the necessary rate. We start by finding a lower bound on the number of bits transmitted at each triggering event. Using (10), we define the uncertainty set of the sensor about the estimation error at the controller \(z(t_{c})\), given \(t_{s}\) as follows

\[
\Omega(z(t_{c})|t_{s}) = \{ y : y = \pm Je^{A(t_{c} - t_{s})} + \int_{t_{s}}^{t_{c}} e^{A(t_{c} - \tau)} w(\tau) d\tau, t_{c} \in [t_{s}, t_{s} + \gamma], |w(\tau)| \leq M \text{ for } \tau \in [t_{s}, t_{c}] \}.
\]

Additionally, we define the uncertainty of the controller about \(z(t_{c})\), given \(t_{c}\), as follows

\[
\Omega(z(t_{c})|t_{c}) = \{ y : y = \pm Je^{A(t_{c} - t_{s})} + \int_{t_{s}}^{t_{c}} e^{A(t_{c} - \tau)} w(\tau) d\tau, t_{c} \in [t_{c} - \gamma, t_{c}], |w(\tau)| \leq M \text{ for } \tau \in [t_{c}, t_{c}] \}.
\]

We next show the relationship between these uncertainty sets.

**Lemma 8** Assume the plant-sensor-channel-controller model described in Section 2, with plant dynamics (1), estimator dynamics (4), triggering strategy (6), and jump strategy (7). Moreover, assume \(M \leq AJ\). Then

\[
\Omega(z(t_{c})|t_{s}) = \Omega(z(t_{c})|t_{s}) \cap \left[ - Je^{A\gamma} - (M/A)(e^{A\gamma} - 1), -J \right] \cup \left[ J, Je^{A\gamma} + (M/A)(e^{A\gamma} - 1) \right],
\]

and \(m(\Omega(z(t_{c})|t_{s})) = m(\Omega(z(t_{c})|t_{s})) = 2\frac{M}{AJ}(e^{A\gamma} - 1)\).

**Proof.** Due to symmetry, we can clearly say \(\Omega(z(t_{s})|t_{c})\) is the same as \(\Omega(z(t_{s})|t_{c})\). Using (10) we characterize the set \(\Omega(z(t_{c})|t_{s})\) as follows. The analyses for the case where \(z(t_{s}) = J\) and \(z(t_{s}) = -J\) are similar. Accordingly, without loss of generality, we assume \(z(t_{s}) = J\). Clearly, \(z(t_{c})\) takes its largest value when \(t_{c} = t_{s} + \gamma\) and \(w(\tau) = M\) for \(\tau \in [t_{s}, t_{c}]\),
which is equal to
\[ z(t_c) = Je^{A\gamma} + (M/A)(e^{A\gamma} - 1). \]

On the other hand, finding the smallest value of \( z(t_c) \) is more challenging. First, when \( t_c = t_s \) we have
\[ z(t_c) = J. \]  \hspace{1cm} (19)

Second, by setting \( w(\tau) = -M \) for \( \tau \in [t_s, t_c] \) and \( t_c = t_s + \Delta \), we have
\[ z(t_c) = Je^{A\Delta} - (M/A)(e^{A\Delta} - 1). \]  \hspace{1cm} (20)

Taking the derivative of (20) with respect to \( \Delta \) results in
\[ \frac{dz(t_c)}{d\Delta} = Ae^{A\Delta} - Me^{A\Delta} = e^{A\Delta}(AJ - M). \]  \hspace{1cm} (21)

If \( M \leq AJ \) and the derivative in (21) is non-negative, \( z(t_c) \) in (20) would be a non decreasing function of \( \Delta \). Hence, the smallest value of \( z(t_c) \) in (20) occurs for \( \Delta = 0 \) which is equal to the value of \( z(t_c) \) in (19). Hence, when \( M \leq AJ \) and \( z(t_s) = J \) it follows \( \Omega(z(t_s)|t_s) = [J, Je^{A\gamma} + (M/A)(e^{A\gamma} - 1)] \). By symmetry similar argument is valid for \( z(t_s) = -J \), thus when \( M \leq AJ \) we deduce
\[ m(\Omega(z(t_c)|t_s)) = 2(M/A + J)(e^{A\gamma} - 1). \]

To ensure (8) for all reception times, we calculate a lower bound on the number of bits to be transmitted to ensure the sensor uncertainty set \( \Omega(z(t_c)|t_s) \) is covered by quantization cells of measure \( 2J \). Moreover, \( g(t_s) \) is the packet size, hence it is non-negative, and we have
\[ g(t_s) \geq \max \left\{ 0, \log \left( \frac{M}{AJ} + 1 \right)(e^{A\gamma} - 1) \right\}. \]  \hspace{1cm} (22)

**PROOF.** To ensure (8) for all reception times, we calculate a lower bound on the number of bits to be transmitted to ensure the sensor uncertainty set \( \Omega(z(t_c)|t_s) \) is covered by quantization cells of measure \( 2J \). Moreover, \( g(t_s) \) is the packet size, hence it is non-negative, and we have
\[ g(t_s) \geq \max \left\{ 0, \log \left( \frac{m(\Omega(z(t_c)|t_s))}{m(B(J))} \right) \right\}, \]
\[ \text{where } B(J) \text{ denotes a ball centered at } 0 \text{ of radius } J. \]  \hspace{1cm} By Lemma 8 it follows
\[ \log \left( \frac{m(\Omega(z(t_c)|t_s))}{m(B(J))} \right) \geq \log \frac{2(M/A + J)(e^{A\gamma} - 1)}{2J}. \]  \hspace{1cm} \( \square \)

Having found a lower bound on the packet size, our next step is to determine a lower bound on the triggering rate. To obtain this, we note that it is required to limit the collection of permissible quantization policies. This is because, without such a restriction, a packet may carry an unbounded amount of information, which can bring the state estimation error arbitrarily close to zero at all reception times and for all delay and disturbance values, resulting in a triggering rate which is arbitrarily close to zero. Since ensuring (8) at each reception time is equivalent to dividing the uncertainty set at the controller \( \Omega(z(t_c)|t_c) \) by quantization cells of measure of at most \( 2J \), our approach is to restrict the class of quantization policies to those that use the minimum possible number of bits to ensure (8).

**Assumption 10** We assume at each triggering time the sensor transmits the smallest possible packet size (data payload) to ensure (8) at each reception time for all initial conditions and all possible realizations of the delay and plant disturbance. Moreover, to simplify our analysis in the encoding-decoding scheme, we choose the center of each quantization cell as \( z(t_c) \).

Based on this assumption, the sensor brings the uncertainty about \( z(t_c) \) at the controller down to a quantization cell of measure at most \( 2J \), using the smallest possible packet size. The following result shows that, for this class of quantization policies, there exists a delay realization such that the sensor can only shrink the estimation error for the controller to at most half of the largest value of \( J \) dictated by (8).

**Lemma 11** Let
\[ \beta = \frac{1}{A} \ln \left( 1 + \frac{2}{1 + \frac{M}{AJ}} \right). \]  \hspace{1cm} (23)

Under the assumptions of Lemma 8, for all the quantization policies ensuring (8) at all reception times \( \{t^k_c\}_{k \in \mathbb{N}} \) with Assumption 10 in place, there exists a delay realization \( \{\Delta_k \leq \beta\}_{k \in \mathbb{N}} \), initial condition, and plant disturbance such that
\[ |z(t^k_c) - z(t^k_c) - \bar{z}(t^k_c)| \geq \frac{J}{2}. \]  \hspace{1cm} (24)

**PROOF.** Without loss of generality assume that \( z(t_s) = J \) throughout this proof. We also consider the realization of \( w(t) = M \) for all time \( t \). We first show \( \beta \) is the time needed for the state estimation error to grow from \( z(t_s) \) to \( z(t_s) + 2J \). From (10), we deduce at delay \( \beta \) we have
\[ z(t_c) = e^{A\beta}J + \frac{M}{A}(e^{A\beta} - 1). \]  \hspace{1cm} (25)

By combining (25), (23), and \( z(t_s) = J \) it follows \( z(t_c) = z(t_s) + 2J \). Hence, the value of \( z(t_c) \) sweeps an area of measure \( 2J \) when the delay takes values in \( [0, \beta] \).

We continue by distinguishing between two classes of quantization cells. We call a quantization cell perfect, if its measure is equal to \( 2J \), and when the measure of a quantization cell is less than \( 2J \) we call it defective. Using these definitions we now prove the occurrence of (24) with delay of at most \( \beta \), in three different cases. First, when \( z(t_s) \) is in a perfect cell, clearly for a delay of at most \( \beta \) we have \( |z(t^k_c) - \bar{z}(t^k_c)| \geq J \), and (24) follows. Second, when \( z(t_s) \) is in a defective cell which is adjacent to a perfect cell, for a delay of at most \( \beta \) we are on the boundary of the adjacent perfect cell and it follows \( |z(t^k_c) - \bar{z}(t^k_c)| \geq J \) and (24) follows. It remains to check the assertion when \( z(t_s) \) is in a defective quantization cell which is adjacent to another defective quantization cell. Due to the restriction on the quantization policies as in...
Assumption 10, the sensor transmits the minimum required bits to divide the uncertainty set at the controller to quantization cell of measure of at most $2J$. If the measure of union of two adjacent cells is at most $2J$, these two balls could be replaced by one quantization cell to reduce the number of quantization cells. As a consequence, under Assumption 10, the measure of union of two adjacent quantization cells is greater than $2J$. Assume the defective quantization cell that contain $z(t_s)$ is of the measure $\mu_1$ and the measure of the adjacent defective cell is $\mu_2$. As a result, we have $\mu_1 + \mu_2 > 2J$. Therefore, at least one of the $\mu_1$ or $\mu_2$ is at least $J$, thus with a delay of at most $\beta$, we have $|z(t^k_s) - \tilde{z}(t^k_s)| \geq J/2$, and (24) follows. \hfill \square

Our next result uses Lemma 11 to establish a lower bound on the triggering rate that is valid for all quantization policies that use the minimum required packet size according to Assumption 10.

**Lemma 12** Under the assumptions of Lemma 8, for all the quantization policies which ensure (8) at all reception times \(\{t^k_s\}_{k \in \mathbb{N}}\) with Assumption 10 in place, there exists a delay realization \(\{\Delta_k\}_{k \in \mathbb{N}}\), a disturbance realization, and an initial condition such that

\[
R_{tr} \geq \left( \frac{1}{A} \ln \left( 1 + \frac{2}{1 + \frac{J}{2J + M}} \right) \right)^{-1}.
\]

**Proof.** Using the definition of the triggering time (6), (24), $t^k_c = t^k_s + \Delta_k$, and (10), we have

\[
\frac{1}{2} J e^{\frac{A \Delta_k}{2} - J - \Delta_k} + \frac{M}{A} \left( e^{A(t^k_s + 1 - t^k_s + \Delta_k) - 1} - 1 \right) \leq J.
\]

Hence we need to have

\[
e^{A(t^k_s + 1 - t^k_s + \Delta_k)} \leq \frac{J + M}{2J + M}.
\]

By Lemma 11, (24) occurs for all $k \in \mathbb{N}$ when $\Delta_k \leq \beta$, and from (26) it follows

\[
e^{A(t^k_s + 1 - t^k_s)} \leq e^{-\frac{A\beta}{2J + M}} \cdot \frac{J + M}{2J + M}.
\]

Therefore, from (23) we deduce

\[
t^k_{s+1} - t^k_s \leq \frac{1}{A} \ln \left( 1 + \frac{2}{1 + \frac{J}{2J + M}} \right) \cdot \frac{J + M}{2J + M}.
\]

The result follows by substituting (27) into (13). \hfill \square

**Remark 13** Lemma 12 establishes a lower bound on the triggering rate that is valid for all quantization policies that use the minimum required packet size according to Assumption 10. Note that, for this class of quantization policies, there might exist a realization that results in unbounded triggering rates (in fact, a quantization policy that makes $|z(t^k_s)| = J$ with a delay realization of $\Delta_k = 0$ for all $k \geq k_0$, for some $k_0$, would result in Zeno behavior and $R_{tr} = \infty$). This is not in contradiction with Lemma 4, which led to the upper bound (13) on the triggering rate, since this result relies on (12) instead of (8). \hfill \bullet

The following result states a lower bound on the information transmission rate and follows from Lemmas 9 and 12.

**Theorem 14** Under the assumptions of Lemma 8, for all the quantization policies which ensure (8) at all reception times \(\{t^k_c\}_{k \in \mathbb{N}}\) with Assumption 10 in place, there exists a delay realization \(\{\Delta_k\}_{k \in \mathbb{N}}\), a disturbance realization, and an initial condition such that

\[
R_s \geq \max \left\{ 0, \log \left( \frac{J + M}{J + M - 1} e^{A\gamma - 1} \right) \right\},
\]

\[
\frac{1}{A} \ln \left( \frac{J + M}{J + M - 1} \right) = \frac{J + M}{M}.
\]

Note that the necessary condition is valid even if the sensor does not have knowledge of $\dot{x}(t)$ at all times. Figure 3 compares our bounds on the sufficient (18) and necessary (28) information transmission rates for (1) to become practically stable. One can observe a gap between them, which we attribute to the fact that, while the necessary condition employs quantization policies with the minimum possible packet size according to Assumption 10, the encoding-decoding scheme proposed in the sufficient design does not generally satisfy this assumption.

**Remark 15** By setting $M = 0$ in (28), the necessary condition to guarantee (1) is practically stable, reduces to $R_s \geq \frac{1}{A} \max \left\{ 0, \log (e^{A\gamma} - 1) \right\}$. The critical value on the delay $\gamma_s = \ln 2/A$ is the threshold distinguishing whether this lower bound is zero or strictly positive. A similar phase transition behavior is observed in Khojasteh et al. [2016] in the absence of disturbances with a different triggering strategy and packet size constraint. Furthermore, for the value $\gamma = \ln(1 + 2\log 6)/A$ of delay upper bound, the result returns the rate dictated by the data-rate theorem. \hfill \bullet
5 Extension to complex linear systems

In this section, we generalize our investigation to complex linear plants with disturbances. The results presented here can be readily applied to multivariate linear plants with disturbance and diagonalizable open loop-gain matrix (possibly, with complex eigenvalues). This corresponds to handling the $n$-dimensional real plant as $n$ scalar (and possibly complex) plants, and derive a sufficient condition for them. We consider a plant, sensor, communication channel, and controller described by the following continuous linear time-invariant system

$$\dot{x} = Ax(t) + Bu(t) + w(t),$$

(29)

where the plant state $x(t)$ and control input $u(t)$ are complex numbers for $t \in [0, \infty)$. Here $w(t) \in \mathbb{C}$ represents a plant disturbance, which is upper bounded as $\|w(t)\| \leq M$, with $M \in \mathbb{R}$ nonnegative. Here, $A \in \mathbb{C}$ with $\text{Re}(A) \geq 0$ (since we are only interested in unstable plants), and $B \in \mathbb{C}$. The model for the communication channel is the same as in Section 2, with unknown delay upper bounded by (3).

5.1 Data-rate theorem for complex linear system

To establish a baseline against which we can compare the bounds on the information transmission rate, we start by stating a generalization of the classical data-rate theorem for the complex plant (29).

**Theorem 16** Consider the plant-sensor-channel-controller model with plant dynamics (29). If $x(t)$ remains bounded as $t \to \infty$, then

$$R_c \geq \frac{2 \text{Re}(A)}{\ln 2}.$$

**Proof.** It is enough to prove the assertion when $w(t) = 0$. By rewriting (29) when $w(t) = 0$ we have

$$
\begin{align*}
\text{Re}(x) + i\text{Im}(x) &= \text{Re}(A)\text{Re}(x) - \text{Im}(A)\text{Im}(x) + i(\text{Re}(A)\text{Im}(x) + \text{Im}(A)\text{Re}(x)) \\
\text{Im}(x) &= \text{Im}(A)\text{Re}(x) + \text{Re}(A)\text{Im}(x),
\end{align*}
$$

which is equivalent to

$$
\begin{bmatrix}
\text{Re}(x) \\
\text{Im}(x)
\end{bmatrix} =
\begin{bmatrix}
\text{Re}(A) & -\text{Im}(A) \\
\text{Im}(A) & \text{Re}(A)
\end{bmatrix}
\begin{bmatrix}
\text{Re}(x)(t) \\
\text{Im}(x)(t)
\end{bmatrix}.
$$

Since $\|x\| = \sqrt{\text{Re}(x)^2 + \text{Im}(x)^2}$, if $\text{Re}(x)$ or $\text{Im}(x)$ becomes unbounded, $\|x\|$ becomes unbounded. Consequently, using [Hespanha et al., 2002, Theorem 1], we need to have

$$R_c \geq \text{tr} \left( \begin{bmatrix}
\text{Re}(A) & -\text{Im}(A) \\
\text{Im}(A) & \text{Re}(A)
\end{bmatrix} \right) / \ln 2,$$

and the result follows. $\Box$

5.2 Event-triggered control for complex linear system

The state estimate $\hat{x}$ evolves according to the dynamics (4) along the inter-reception time intervals starting from $\hat{x}(t^+_k)$ with initial condition $\hat{x}(0) = \hat{x}_0$. We use the state estimation error defined as (5) with initial condition $z(0) = x(0) - \hat{x}_0$. A triggering event happens at $t^+_k$ if

$$\|z(t^+_k)\| = J,$$

(30)

provided $t^+_k \leq t^{k+1}_s$ for natural number $k$ and $t^{1}_s \geq 0$, and the triggering radius $J$ is a positive real number. At triggering time, the packet $p(t_s)$ of size $g(t_s)$ is transmitted from the sensor to the controller. The packet $p(t_s)$ consists of a quantized version of the phase of $z(t_s)$, denoted $\phi_q(z(t_s))$, and a quantized version of the triggering time $t_s$. By (30), we have

$$z(t_s) = Je^{i\phi_q(z(t_s))}.$$

We construct a quantized version, denoted $q(z(t_s))$, of $z(t_s)$ at the controller as follows

$$q(z(t_s)) = Je^{i\phi_q(z(t_s))}.$$

Additionally, using the bound (3) and the packet at the controller, the quantized version of $t_s$ is reconstructed and denoted by $q(t_s)$. Consequently, at the controller, $z(t_c)$ can be estimated as follows

$$\bar{z}(t_c) = e^{A(t_c-q(t_s))}q(z(t_s)),$$

(31)

We use the jump strategy (7) to update the value of $\hat{x}(t^+_k)$. Hence, $\|z(t^+_k)\| = \|z(t_c) - \bar{z}(t_c)\|$ holds. At the sensor, the packet size $g(t_s)$ is chosen to be large enough such that the following equation for all $t \in [t_s, t_s + \gamma]$ is satisfied

$$\|z(t^+_k)\| = \|z(t_c) - \bar{z}(t_c)\| \leq \rho_0J,$$

(32)

where $0 < \rho_0 < 1$ is a design parameter. A typical realization of $\hat{z}(t)$ under the proposed event-triggering strategy before and after one triggering is represented in Figure 4. The notion of practically stable remains the same as in Definition 1 by replacing absolute value with complex norm.

**Remark 17** Similar to Proposition 3, one can show that if (32) occurs at all reception times and $(A, B)$ is a stabilizable pair, then under the control rule $u(t) = -K\hat{x}(t)$, the plant (29) is practically stable, provided the real part of $A - BK$ is negative. As a consequence of this observation, our analysis focuses on ensuring (32) at each reception time. The lower bound on the inter-event time of Lemma 4 and the upper bound on triggering rate (14) also holds, where in both cases one needs to replace $A$ by $\text{Re}(A)$ for the complex plant.

5.3 A sufficient information transmission rate

In this section, we design a quantization policy that, utilizing the event-triggered controller of Section 5.2, ensures the plant (29) is practically stable. We rely on this design to establish a sufficient bound on the information transmission rate.
the packet, the encoder breaks the interval containing $q$ to append a quantized version of triggering time $t$. Consequently, similar to Proposition 5, if the sensor has a causal knowledge of the delay in the communication channel, it can calculate the state estimation $\hat{x}(t)$ for all time $t$.

5.3.2 A sufficient packet size

Here we show that with a sufficiently large packet size, we can achieve (32) at all reception times $\{t_{c,k}\}_{k \in \mathbb{N}}$ using the quantization policy designed in Section 5.3.1.

Theorem 18 Consider the plant-sensor-channel-controller model with plant dynamics (29), estimator dynamics (4), triggering strategy (30), and jump strategy (7). If the controller has enough information about $x(0)$ such that state estimation error satisfies $\|z(0)\| < J$, then the quantization policy designed above achieves (32) for all reception times $\{t_{c,k}\}_{k \in \mathbb{N}}$ with any packet size lower bounded by

$$g(t_s) \geq \frac{\gamma}{\Delta}$$

$$\max \left\{ 0, \lambda + \log \frac{\text{Re}(A)b_{\gamma}}{\ln \left( \frac{1 + e^{-\text{Re}(A)\gamma}}{\text{Re}(A)\gamma} \left( e^{\text{Re}(A)\gamma} - 1 \right) \right)} \right\} ,$$

provided $\cos \left( \text{Im}(A) (t_s - q(t_s)) \right) = 1 - \zeta, \ b > 1$, $\rho_0 \geq \frac{M}{\text{Re}(A)J} \left( e^{\text{Re}(A)\gamma} - 1 \right) + e^{\text{Re}(A)\gamma} \left( 2 \sin(\pi/2^{\lambda+1}) + \sqrt{2\zeta} \right),$ $J \geq \frac{M}{\text{Re}(A)\chi} \left( e^{\text{Re}(A)\gamma} - 1 \right),$ $\sqrt{2\zeta} e^{\text{Re}(A)\gamma} \leq \chi'$, $\lambda > \log \left( \frac{\pi}{\arcsin \frac{\chi - \chi'}{2\text{Re}(A)\gamma}} \right) - 1$, where $0 < \chi + \chi' < 1$.

**PROOF.** In our design, the controller estimates $z(t_c)$ as in (31), and the encoding-decoding scheme is as depicted in Figures 2 and 5. Using (10), (31), and the triangle inequality, it follows

$$\|z(t_c) - \tilde{z}(t_c)\| \leq \|e^{A(t_c - t)} z(t_s) - e^{A(t_c - q(t_s))} q(z(t_s))\| + \left\| \int_{t_s}^{t_c} e^{A(t_c - \tau)} w(\tau) d\tau \right\| .$$

$$\|z(t_c) - \tilde{z}(t_c)\| \leq \|e^{A(t_c - t)} z(t_s) - e^{A(t_c - q(t_s))} q(z(t_s))\| + \left\| \int_{t_s}^{t_c} e^{A(t_c - \tau)} w(\tau) d\tau \right\| .$$
Similarly to (11), since \( \|w(t)\| \leq M \), the second summand in (37) is upper bounded as
\[
\left\| \int_{t_s}^{t_e} e^{A(t_e - \tau)} w(\tau) d\tau \right\| \leq \frac{M}{\Re(A)} \left( e^{\Re(A)\gamma} - 1 \right). \tag{38}
\]
To find a proper upper bound on the first summand in (37), assuming \( q(z(t_s)) = z(t_s) - v_1 \) and \( q(t_s) = t_s - v_2 \), we have
\[
\left\| e^{A(t_e - t_s)} (z(t_s) - e^{A v_2} (z(t_s) - v_1)) \right\| \leq e^{\Re(A)\gamma} \left( J \|1 - e^{A v_2}\| + e^{\Re(A)v_2} \|v_1\| \right). \tag{39}
\]
Next, we find an upper bound of \( \|v_1\| \). Since the sensor devotes \( \lambda \) bits to transmit a quantized version of the phase of \( z(t_s) \) to the controller, we have the upper bound (33) on the difference of the phases of \( z(t_s) \) and \( q(z(t_s)) \). Also, over \([\frac{-\pi}{2}, \frac{\pi}{2}]\), the cosine function is concave, with global maximum at 0. Hence, as depicted in Figure 5, from the law of cosines, we have
\[
\|v_1\| = \|z(t_s) - q(z(t_s))\| \leq 2J \sin(\pi/2^{\lambda+1}).
\]
Combining this with (39), the first summand in (37) is upper bounded by
\[
J e^{\Re(A)\gamma} \left( \|1 - e^{A v_2}\| + 2e^{\Re(A)v_2} \sin(\pi/2^{\lambda+1}) \right).
\]
Note that \( \|1 - e^{A v_2}\|^2 = (1 - e^{\Re(A)v_2})^2 + 2e^{\Re(A)v_2} \zeta \), where \( \cos(\Im(A)v_2) = 1 - \zeta \), and 0 \( \leq \zeta \leq 2 \). Thus, the first summand in (37) is upper bounded by
\[
J e^{\Re(A)\gamma} \left( \|1 - e^{\Re(A)v_2}\| + \sqrt{2e^{\Re(A)v_2} \zeta} + 2e^{\Re(A)v_2} \sin(\pi/2^{\lambda+1}) \right).
\]
For any positive real number \( \epsilon \) we know \( \epsilon + 1/\epsilon \geq 2 \), hence, \( e^{\Re(A)v_2} - 1 \geq 1 - e^{-\Re(A)v_2} \). Therefore, for the rest of the proof, and without loss of generality, we assume \( v_2 \geq 0 \), and the first summand in (37) is upper bounded by
\[
J e^{\Re(A)\gamma} \left( e^{\Re(A)v_2} - 1 + \sqrt{2 \zeta} e^{\Re(A)v_2} + 2e^{\Re(A)v_2} \sin(\pi/2^{\lambda+1}) \right). \tag{40}
\]
Combining (37), (38), and (40) we deduce
\[
e^{\Re(A)v_2} \leq \frac{1 + e^{-\Re(A)\gamma} \left( \rho_0 - \frac{M}{\Re(A)J} \left( e^{\Re(A)\gamma} - 1 \right) \right)}{2 \sin(\pi/2^{\lambda+1}) + 1 + \sqrt{2 \zeta}}, \tag{41}
\]
which suffices to ensure (32). Recalling \( v_2 = t_s - q(t_s) \), using (34) and by setting
\[
\frac{b \gamma}{2 \rho(t_s) - \lambda} \leq \frac{1}{\Re(A)} \ln \left( \frac{1 + e^{-\Re(A)\gamma} \left( \rho_0 - \frac{M}{\Re(A)J} \left( e^{\Re(A)\gamma} - 1 \right) \right)}{2 \sin(\pi/2^{\lambda+1}) + 1 + \sqrt{2 \zeta}} \right),
\]
(41) is ensured. Consequently, the packet size in (35) is sufficient to ensure (32) for all reception times. However, (41) is well defined only when the upper bound in (41) is at least one, namely
\[
e^{\Re(A)\gamma} \left( \rho_0 - \frac{M}{\Re(A)J} \left( e^{\Re(A)\gamma} - 1 \right) \right) \geq 2 \sin(\pi/2^{\lambda+1}) + \sqrt{2 \zeta},
\]
which holds because of (36a). Moreover, the design parameter \( \rho_0 \) in (32) should be in the open interval \((0, 1)\). Therefore, the lower bound in (36a) should be smaller than 1, namely
\[
\frac{M}{\Re(A)J} \left( e^{\Re(A)\gamma} - 1 \right) + e^{\Re(A)\gamma} (2 \sin(\pi/2^{\lambda+1}) + \sqrt{2 \zeta}) < 1.
\]
The result now follows by noting that (36b), (36c) and (36d) ensure this inequality holds. \( \square \)

Combining the bound on the triggering rate from Remark 17 with Theorem 18, it follows that any information transmission rate
\[
R_s \geq \frac{\Re(A)}{\ln \left( \frac{J + \frac{M}{\Re(A)J}}{\rho_0 J + \frac{M}{\Re(A)J}} \right)} \bar{g}, \tag{42}
\]
achieves (32) for all reception times \( \{t_k\}_{k \in N} \), and is therefore, sufficient to ensure (29) is practically stable. Figure 6 shows the sufficient information transmission rate in (42) as a function of the upper bound \( \gamma \) on the channel delay. One can observe that for small values of the delay, the sufficient information transmission rate is smaller than the rate required by the extension of the data-rate result in Theorem 16, and as the delay upper bound \( \gamma \) increases, the sufficient information transmission rate increases accordingly.

Fig. 6. Sufficient information transmission rate (42) as a function of channel delay upper bound \( \gamma \). We assume \( A = 1 + i \), \( B = 0.5 \), \( M = 0.1 \), \( \rho_0 = 0.9 \) and \( b = 1.0001 \). Also \( \lambda = \log \left( \pi/2 \arcsin \left( \frac{7}{8} \right) e^{\Re(A)\gamma} \right) \) and \( J = \frac{\lambda b}{\Re(A)} \left( e^{\Re(A)\gamma} - 1 \right) + 0.002 \). In this example, the rate dictated by data-rate theorem (Theorem 16) is \( 2 \Re(A) / \ln 2 = 2.885 \).
6 Simulation

This section presents simulation results validating the proposed event-triggered control scheme for real-valued plants. The interested reader can find simulations for a complex-valued plant in [Khojasteh et al., 2018b]. While our analysis is for continuous-time plants, the simulations are performed in discrete time with a small sampling time δ'. As a consequence, the minimum upper bound for the channel delay is equal to two sampling times in the digital environment (this is because a delay of at most one sampling time might occur from the time that the sensor took a sample from the plant state and another delay of at most one sampling time might occur from the time that the packet is received to the time the control input is applied to the plant). The packet size for the simulation has two differences from the lower bound provided in Theorem 7. Since the packet size should be an integer, we use the ceiling operator, and because we should have at least one bit, we set the minimum size of the packet to one.

We consider a linearized version of the two-dimensional problem of balancing an inverted pendulum mounted on a cart, where the motion of the pendulum is constrained in a plane and its position can be measured by an angle θ, cf. Figure 7. The inverted pendulum has mass $m_1$, length $l$, and moment of inertia $I$. Also, the pendulum is mounted on top of a cart of mass $m_2$, constrained to move in $y$ direction. The nonlinear equations governing the motion of the cart and pendulum are

$$(m_1 + m_2)\ddot{y} + m_1 l \dot{\theta}\cos\theta - m_1 l \dot{\theta}^2 \sin\theta = F$$

$$(I + m_1 l^2)\ddot{\theta} + m_1 g_0 l \sin\theta = -m_1 l \dot{\theta} \cos\theta$$

where $\nu$ is the damping coefficient between the pendulum and the cart and $g_0$ is the gravitational acceleration. We define $\theta = \pi$ as the equilibrium position of the pendulum and $\phi$ as small deviations from $\theta$. We derive the linearized equations of motion using small angle approximation, noting that this linearization is only valid for sufficiently small values of the delay upper bound $\gamma$. Define the state variable $s = [y, \dot{y}, \theta, \dot{\theta}]^T$, where $y$ and $\dot{y}$ are the position and velocity of the cart respectively. Assuming $m_1 = 0.2$ kg, $m_2 = 0.5$ kg, $\nu = 0.1$ N/m/s, $l = 0.3$ m, $I = 0.006$ kg/m$^2$, one can write the evolution of $s$ in time as

$$\dot{s} = As(t) + Bu(t) + w(t),$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6730 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1800 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1.8180 \\ 0 \\ 4.5450 \end{bmatrix}$$

In addition, we add the plant noise $w(t) \in \mathbb{R}^4$ to the linearized plant model, and we assume that all of its elements are upper bounded by $M$. A simple feedback control law can be derived for (43) as $u = -Ks$, where $K = [-1.00 - 2.04 20.36 3.93]$. is chosen such that $A - BK$ is Hurwitz.

The eigenvalues of the open-loop gain of the plant $A$ are $e = [0 - 5.6041 - 0.1428 5.5651]$. Hence, three out of the four modes of the plant are stable and they need no actuation. Also, the open-loop gain of the plant $A$ is diagonalizable (all eigenvalues of $A$ are distinct). As a result, diagonalization of the matrix $A$, allows us to apply Theorem 7 to the unstable mode of the plant, and consequently stabilize the whole plant.

Using the eigenvector matrix $P$, we diagonalize the plant to obtain

$$\dot{s} = \hat{A}s(t) + \hat{B}u(t) + \hat{w}(t)$$

(44)

where

$$\hat{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5.6041 & 0 & 0 \\ 0 & 0 & -0.1428 & 0 \\ 0 & 0 & 0 & 5.5651 \end{bmatrix}, \hat{B} = \begin{bmatrix} 10.0000 \\ -2.3865 \\ 10.0979 \\ 2.2513 \end{bmatrix}$$

For the first three coordinates of the diagonalized plant in (44), which are already stable, the state estimation $\hat{s}$ at the controller simply constructs as follows:

$$\dot{\hat{s}} = \hat{A}\hat{s}(t) + \hat{B}u(t)$$

starting from $\hat{s}(0)$. In addition, using the problem formulation in Section 2, the estimated state for the unstable mode $\hat{s}_4$ evolves during the inter-reception times as

$$\dot{\hat{s}}_4(t) = 5.5651\hat{s}_4(t) + 2.2513\hat{\omega}(t), \quad t \in [t^k_{c}, t^{k+1}_{c}],$$

(45)

starting from $\hat{s}_4(t^{k+1}_{c})$ and $\hat{s}_4(0)$. Also, a triggering occurs when

$$|\hat{\omega}_4(t)| = |\hat{s}_4(t) - \hat{s}_4(t)| = J,$$

where $|\hat{\omega}_4(t)|$ is the state estimation error for the unstable mode, and assuming the previous packet is already delivered to the controller. In the simulation environment, since the sampling time is small, a triggering happens as soon as $|\hat{s}_4(t)|$ is equal or greater than $J$ and the previous packet has been received by the controller. Let $\lambda_4 = 5.5651$ be the eigenvalue corresponding to the unstable
mode. Using Theorem 7, we choose
\[
J = \frac{M}{\lambda_4 \rho_0} (e^{\lambda_4 \gamma} - 1) + 0.005, \\
\]
and the size of the packet for all \( t_s \) to be
\[
g(t_s) = \max \left\{ 1, \left[ 1 + \log \frac{Ab \gamma}{\ln(1 + e^{(M/A \lambda)(e^{\lambda_4 \gamma} - 1)})} \right] \right\},
\]
where \( b = 1.0001 \) and \( \rho_0 = 0.9 \).

A set of two simulations are carried out as follows. In simulation (a) the plant disturbance is upper bounded by \( M = 0.05 \) and channel delay is upper bounded by the two sampling time \( 2\delta \). In simulation (b), the plant disturbance is upper bounded by \( M = 0.05 \) and channel delay is upper bounded by \( \gamma = 0.1 \). Each row in Figure 8 presents a different simulation. The first column shows the triggering function for \( \bar{s}_4 \) in (44) and the absolute value of the state estimation error for the unstable coordinate, that is, \( |\bar{s}_4(t)| = |\bar{s}_4(t) - s(t)| \). As soon as the absolute value of this error is equal or greater than the triggering function, the sensor transmits a packet, and the jumping strategy adjusts \( \bar{s}_4 \) at the reception time to ensure the plant is practically stable. Note that the amount this error exceeds the triggering function depends on the random channel delay upper bounded by \( \gamma \). Since \( \gamma \) in simulation (b) is larger than in simulation (a), the absolute value of the state estimation error grows beyond the triggering function depending on the random delay in the communication channel. The second column of Figure 8 presents the evolution of the unstable state in (44) and its estimation in (45). The last column in Figure 8 represents the evolution of all the actual states of the linearized plant in (43) in time. In the second and third columns, as expected, when \( \gamma \) increases, the controller performance deteriorate significantly. However, all the states of the plant remain bounded and the plant is practically stable.

Finally, Figure 9 presents the simulation of information transmission rate versus the delay upper bound \( \gamma \) in the communication channel for stabilizing the linearized model of the inverted pendulum. It can be seen that for small \( \gamma \), the plant is practically stable with an information transmission rate smaller than the one prescribed by the data-rate theorem.

7 Conclusions

We have presented an event-triggered control scheme for the stabilization of noisy, scalar real and complex, continuous, linear time-invariant systems over a communication channel subject to random bounded delay. We have developed an algorithm for encoding-decoding the quantized version of the estimated state, leading to the characterization of a sufficient transmission rate for stabilizing these systems. We also identified a necessary condition on the transmission rate for the real system. Future work will study the identification of necessary conditions on the transmission rate in complex systems, developing an event-triggering design for a vector system with real and complex eigenvalues based on the complex system design, and experimentation of the proposed control strategies on real systems.

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References

Venkat Anantharam and Sergio Verdu. Bits through queues. IEEE Transactions on Information Theory, 42(1):4–18, 1996.
Alberto Bemporad, Maurice Heemels, and Mikael Johansson. Networked control systems, volume 406. Springer, 2010.
Jian Ding, Yuval Peres, Gireeja Ranade, and Alex Zhai. When multiplicative noise stymies control. arXiv preprint arXiv:1612.03259, 2016.
Joao P Hespanha, Antonio Ortega, and Lavanya Vasudevan. Towards the control of linear systems with minimum bit-rate. In Proc. 15th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS), 2002.
Hideaki Ishii. Feedback control over limited capacity channels. In Alberto Bemporad, Maurice Heemels, and Mikael Johansson, editors, Networked Control Systems, pages 255–291. Springer London, London, 2010.
Anatoly Khina, Yorie Nakahira, Yu Su, and Babak Hassibi. Algorithms for optimal control with fixed-rate feedback. In 2017 IEEE 56th Annual Conference on Decision and Control (CDC), pages 6015–6020, Dec 2017.
Mohammad Javad Khojasteh, Pavan Kumar Tallapragada, Jorge Cortés, and Massimo Franceschetti. The value of timing information in event-triggered control. arXiv preprint arXiv:1609.09594, 2016.
Mohammad Javad Khojasteh, Massimo Franceschetti, and Gireeja Ranade. Stabilizing a linear system using phone calls. arXiv preprint arXiv:1804.00351, 2018a.
Mohammad Javad Khojasteh, Mojtaba Hedayatpour, Jorge Cortés, and Massimo Franceschetti. Event-triggering stabilization of complex linear systems with disturbances over digital channels. In 2018 IEEE 57th Annual Conference on Decision and Control (CDC), Dec 2018b.
Mohammad Javad Khojasteh, Mojtaba Hedayatpour, Jorge Cortés, and Massimo Franceschetti. Event-triggered stabilization of disturbed linear systems over digital channels. In 2018 52nd Annual Conference on Information Sciences and Systems (CISS), pages 1–6, March 2018c.
Kyoung-Dae Kim and Panganamala R Kumar. Cyber–physical systems: A perspective at the centennial. Proceedings of the IEEE, 100(Special Centennial Issue):1287–1308, 2012.
Ernesto Kofman and Julio H Braslavsky. Level crossing sampling in feedback stabilization under data-rate constraints. In 45st IEEE Conference on Decision and Control (CDC), pages 4423–4428. IEEE, 2006.
Victoria Kostina and Babak Hassibi. Rate-cost tradeoffs in control. In Communication, Control, and Computing (Allerton), 2016 54th Annual Allerton Conference on, pages 1157–1164. IEEE, 2016.
Daniel Liberzon. Finite data-rate feedback stabilization of switched and hybrid linear systems. Automatica, 50(2):409–420, 2014.
Qiang Ling. Bit rate conditions to stabilize a continuous-time scalar linear system based on event triggering. IEEE Transactions on Automatic Control, 62(8):4093–4100, 2017.
Qiang Ling. Bit-rate conditions to stabilize a continuous-time linear system with feedback dropouts. IEEE Transactions on Automatic Control, 63(7):2176–2183, July 2018.
Steffen Linsenmayer, Rainer Blind, and Frank Allgöwer. Delay-dependent data rate bounds for containability of scalar systems. IFAC-PapersOnLine, 50(1):7875–7880, 2017.
Alexey S Matveev and Andrey V Savkin. Estimation and control over communication networks. Springer Science & Business Media, 2009.
Paolo Minero, Massimo Franceschetti, Subhrakanti Dey, and Girish N Nair. Data rate theorem for stabilization over time-varying feedback channels. IEEE Transactions on Automatic Control, 54(2):243, 2009.
Simulation (a): $M = 0.05$, $\gamma = 0.01$ sec, $g(t_s) = 1$ bit

Simulation (b): $M = 0.05$, $\gamma = 0.1$ sec, $g(t_s) = 4$ bits

Fig. 8. Simulation results: The following simulation parameters are chosen for the system: simulation time $T = 5$ seconds, sampling time $\delta' = 0.005$ seconds, $\tilde{s}(0) = P^{-1}[0, 0, 0, 0.1001]^T$, and $\hat{s}(0) = P^{-1}[0, 0, 0, 0.10]^T$. The first column represents the evolution of the absolute value of state estimation error for the unstable mode of the plant in (44). The second column represents the evolution of the unstable state in (44), and its estimate in (45). Finally, and the last column represents the evolution of all the actual states of the plant given in (43) in time.

Fig. 9. Information transmission rate in simulations compared to the data-rate theorem. Note that the rate calculated from simulations does not start at $\gamma = 0$ because the minimum channel delay upper bound is equal to two sampling time (0.005 seconds in this example). $M$ is chosen to be 0.2 in these simulations, and simulation time is $T = 5$ seconds.

Pavankumar Tallapragada, Massimo Franceschetti, and Jorge Cortés. Event-triggered second-moment stabilization of linear systems under packet drops. *IEEE Transactions on Automatic Control*, 63(8):2374–2388, 2018.

Sekhar Tatikonda and Sanjoy Mitter. Control under communication constraints. *IEEE Transactions on Automatic Control*, 49(7):1056–1068, 2004a.

Sekhar Tatikonda and Sanjoy Mitter. Control over noisy channels. *IEEE transactions on Automatic Control*, 49(7):1196–1201, 2004b.

Sekhar Tatikonda, Anant Sahai, and Sanjoy Mitter. Stochastic linear control over a communication channel. *IEEE transactions on Automatic Control*, 49(9):1549–1561, 2004.

Paolo Minero, Lorenzo Coviello, and Massimo Franceschetti. Stabilization over Markov feedback channels: the general case. *IEEE Transactions on Automatic Control*, 58(2):349–362, 2013.

Girish N Nair and Robin J Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM Journal on Control and Optimization*, 43(2):413–436, 2004.

Justin Pearson, Joao P Hespanha, and Daniel Liberzon. Control with minimal cost-per-symbol encoding and quasi-optimality of event-based encoders. *IEEE Transactions on Automatic Control*, 62(5):2286–2301, 2017.

Christopher Rose and I Saira Mian. Inscribed matter communication: Part I. *IEEE Transactions on Molecular, Biological and Multi-Scale Communications*, 2(2):209–227, 2016.

Anant Sahai and Sanjoy Mitter. The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link, Part I: Scalar systems. *IEEE Transactions on Information Theory*, 52(8):3369–3395, 2006.

Pavankumar Tallapragada and Jorge Cortés. Event-triggered stabilization of linear systems under bounded bit rates. *IEEE Transactions on Automatic Control*, 61(6):1575–1589, 2016.