Derivation of a matrix-valued Boltzmann equation for the Hubbard model

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Abstract
For the spin-1/2 Fermi–Hubbard model, we derive the kinetic equation valid for weak interactions by using a time-dependent perturbation expansion up to second order. In recent theoretical and numerical studies, the kinetic equation has merely been stated without further details. In this contribution, we provide the required background material.

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(Some figures may appear in colour only in the online journal)

1. Introduction and main result
In its original form, the Hubbard model is a simplified description of electrons in a solid, for which the lattice periodic potential is treated in the tight binding approximation, and the interaction between electrons is reduced to an on-site potential. One particular realization would be graphene, which has stimulated a lot of activity \cite{1, 2}. In graphene, the C atoms form a sheet arranged as a honeycomb lattice resulting in a two-band Hubbard model. The energy bands exhibit conical intersections which are at the core of interesting dynamical behavior. The Fermi–Hubbard model has also been realized in a very different context as an accurate description of the motion of cold atoms in an optical lattice \cite{3}. Thereby one has at their disposal new possibilities and methods to study the dynamical properties of the Hubbard model.

In our contribution, we will derive the kinetic equation for the Hubbard model which will be an accurate description for sufficiently small interactions. Let us first recall the structure of
the Hubbard model. For simplicity, we restrict ourselves to the lattice $\mathbb{Z}^d$ (single band), but generalizations are easily implemented. The basic object is thus a spin-1/2 Fermi field, $a_\sigma(x), x \in \mathbb{Z}^d, \sigma \in \{\uparrow, \downarrow\}$, with an anticommutation relation
\[ [a_\sigma(x), a_\tau(y)] = \delta_{xy} \delta_{\sigma\tau}, \]
and
\[ [a_\sigma(x), a_\tau(y)] = 0, \]

\[ [a_\sigma(x), a_\sigma(y)] = 0, \]

where $[A, B] = AB + BA$ and $A^*$ denotes the adjoint operator to $A$. The Hubbard Hamiltonian reads
\[ H = \sum_{x,y \in \mathbb{Z}^d} a(x - y) a(x)^* \cdot a(y) + \lambda \frac{1}{2} \sum_{x,y \in \mathbb{Z}^d} V(x - y)(a(x)^* \cdot a(x))(a(y)^* \cdot a(y)) \]
with $a(x)^* \cdot a(y) = \sum_{\sigma \in \{\uparrow, \downarrow\}} a_\sigma(x)^* a_\sigma(y)$. $\alpha$ is the hopping amplitude, with the properties $\alpha(x) = \bar{\sigma}(x), \alpha(x) = \alpha(-x)$. The particles interact through the weak pair potential $\lambda V$, $V : \mathbb{Z}^d \to \mathbb{R}, V(x) = V(-x)$, $V$ is assumed to decay fast enough to be absolutely summable and $0 < \lambda \ll 1$. A particular case of interest is the on-site, $\delta$-potential $V(x) = \delta_{x0}$. Our notation emphasizes the invariance under global spin rotations.

For the Fourier transformation, we use the convention
\[ \hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{-2\pi i k \cdot x} \]
and correspondingly for operator-valued functions; for instance, $[\hat{a}_\sigma(k), \hat{a}_\tau(\tilde{k})] = \delta_{\sigma\tau} \delta(k - \tilde{k})$. Then the first Brillouin zone is the set $B^d = [-\frac{1}{4}, \frac{1}{4})^d$ with periodic boundary conditions. The dispersion relation is $\omega(k) = \hat{\sigma}(k)$, and in Fourier space $H$ can be written as
\[ H = H_0 + \lambda H_1 = \int_{B^d} dk \omega(k) \hat{\sigma}(k) \cdot \hat{\sigma}(k) + \lambda \frac{1}{2} \int_{B^d} dk_1234 (\delta(k_1 - k_2 + k_3 - k_4)) \]
\[ \times \hat{V}(k_1 - k_2)(\hat{\sigma}(k_1)^* \cdot \hat{\sigma}(k_2))(\hat{\sigma}(k_3)^* \cdot \hat{\sigma}(k_4)), \]

where $dk_1234 = dk_1 dk_2 dk_3 dk_4$.

In the spatially homogeneous case, the central quantity is the time-dependent average Wigner matrix $W$ defined by
\[ \langle \hat{a}_\sigma(k, t)^* \hat{a}_\tau(\tilde{k}, t) \rangle = \delta(k - \tilde{k})W_{\sigma\tau}(k, t), \]
which for times up to order $\lambda^{-2}$ will satisfy in approximation a kinetic equation. In (7), $\langle \cdot \rangle$ denotes the average over the initial state and the operators are computed in the Heisenberg picture, $A(t) = e^{iHt} A e^{-iHt}$.

Our goal is to derive the Boltzmann-type equation for the Hamiltonian (6). At first sight this may look like a problem treated in textbooks. However, to the best of our knowledge, the standard discussion assumes that initially $W_{\sigma\tau}(k) = 2 \rho_{\sigma\tau} W_0(k)$, a property which is preserved by the kinetic equation. One obtains then a coupled set of equations for $W_1$ and $W_1'$, which have the same structure as a classical kinetic equation for a classical two-component system. Physically, there is no compelling reason to have $W$ diagonal. In fact, interesting aspects of the spin dynamics may be lost. In our contribution, we will treat general initial Wigner matrices. This may look like an easy exercise—now there will simply be a coupled set of equations for $W_{1\uparrow}, W_{1\downarrow}, W_{2\uparrow}$. However, already at second order, the number of terms steeply increases. Even more importantly, one loses sight of any comprehensible structure. A more optimal strategy is to regard $W(k, t)$ as a $2 \times 2$ matrix and to completely avoid the representation in a specific
The effective Hamiltonian is given by

\[ \lambda^{-1} \int_{\mathcal{W}} d\hat{k} \hat{V}(\hat{k} - \hat{k}) W(\hat{k}, t), \]

which generates fast oscillations on the kinetic timescale. Of course, such a unitary evolution does not produce any entropy.

As our main result, we obtain a kinetic equation valid for the kinetic timescale, where \( t \) is replaced by \( \lambda^{-2} t \). It is an evolution equation for the 2 \times 2-matrix-valued Wigner function of the form

\[ \frac{\partial}{\partial t} W(t) = C[W(t)], \quad C[W] = C_c[W] + C_d[W], \]

which has to be supplemented with some initial condition \( W(k, 0) = W^{(0)}(k) \). The conservative term, \( C_c \), has the form

\[ C_c[W(t)](k) = -i[H_{\text{eff}}(W(t))(k), W(k, t)] - \lambda^{-1} i[R(W(t))(k), W(k, t)]. \]

Here the effective Hamiltonian is given by

\[ H_{\text{eff}}[W_1] = \int_{\mathcal{W}} d\epsilon \left( \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \mathcal{P} \left( \frac{1}{\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4} \right) \right) \times (\hat{V}(k_2 - k_3)\hat{V}(k_2 - k_4)A_{X,c}[W_1] + \hat{V}(k_2 - k_4)^2A_{tr,c}[W_1]), \]

where

\[ A_{X,c}[W_1] = W_2 W_4 - W_2 W_3 - W_3 W_2 + W_2 \]

and

\[ A_{tr,c}[W_1] = (\text{tr}[W_2] - \text{tr}[W_4])W_3, \]

with \( d\epsilon = d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 \). We have introduced the shorthand notations \( W_j = W(k_j, t) \), \( \epsilon_j = \epsilon(k_j) \), \( H_{\text{eff},1} = H_{\text{eff}}(k_1, t) \). Since \( W \) is 2 \times 2-matrix-valued function, \( \text{tr} \cdot \) is the trace in spin space. Finally, \( \mathcal{P} \) denotes a principal value integral: the notation \( \mathcal{P}(1/f(k)) \) means that for small \( \epsilon > 0 \), we first integrate over \( k \) with \( |f(k)| > \epsilon \), and the result is then given by the \( \epsilon \to 0 \) limit of these integrals. Since the \( k_3 \), \( k_4 \) integration can be interchanged, \( H_{\text{eff}} = H_{\text{eff}}^* \), as it should be. The second summand in (10) is linear in \( W \) and reads

\[ R[W](k) = \int_{\mathcal{W}} d\hat{k} \hat{V}(\hat{k} - \hat{k}) W(\hat{k}). \]

The dissipative part of the collision term, \( C_d \), is given by

\[ C_d[W_1] = \pi \int_{\mathcal{W}} d\epsilon \left( \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \mathcal{P} \left( \frac{1}{\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4} \right) \right) \times (\hat{V}(k_2 - k_3)\hat{V}(k_2 - k_4)A_{\text{quad},d}[W_1] + \hat{V}(k_2 - k_4)^2A_{tr,d}[W_1]), \]

where

\[ A_{\text{quad},d}[W_1] = \tilde{W}_4 W_2 \tilde{W}_3 W_1 - \tilde{W}_4 \tilde{W}_2 W_3 \tilde{W}_1 - \tilde{W}_1 W_3 \tilde{W}_2 W_4 + W_1 \tilde{W}_3 \tilde{W}_2 \tilde{W}_4 \]

and

\[ A_{tr,d}[W_1] = (\tilde{W}_1 W_3 + \tilde{W}_3 W_1)\text{tr}[\tilde{W}_2 W_4] - (W_1 \tilde{W}_3 + \tilde{W}_3 W_1)\text{tr}[\tilde{W}_2 W_4] \]

with \( \tilde{W} = 1_{2\times2} - W \). Note that in the commuting scalar case \( H_{\text{eff}} \) and \( R \) have no effect.
2. Expansion in $\lambda$

We assume that the initial state, $\langle \cdot \rangle$, denotes the average over the initial state which is gauge invariant, invariant under translations and quasi-free. The state is then completely determined by the two-point function:

$$\langle \hat{a}_i(k)^* \hat{a}_\tau(\tilde{k}) \rangle = \delta(k - \tilde{k}) W_{\tau\tau}(k, 0), \quad \sigma, \tau \in \{\uparrow, \downarrow\}. \tag{19}$$

Averages of the form $\langle (a^*)^m a^n \rangle$ vanish unless $m = n$, and all other moments are determined by the Wick pairing rule. The state at time $t$ will still be gauge invariant and invariant under translations. But quasi-freeness will not be preserved. The basic tenet of kinetic theory is that for small $\lambda$ and times of order $\lambda^{-2}$ the quasi-free property is approximately maintained. However, the initial $W(0)$ will have evolved to $W(t)$ (on the timescale $\lambda^{-2}$). To determine the collision operator $C$ of (9), one has to study the increment $W(t + dt) - W(t)$. Since $W(t)$ is approximately quasi-free by assumption, we might as well set $t = 0$ and then evaluate the collision operator at $W(dt)$. (This is a version of the much debated repeated random phase approximation.) $dt$ is long on the microscopic scale, but short on the kinetic scale.

More formally, we expand the true two-point function $W_{\sigma\tau}$, defined by the relation

$$\delta(k - \tilde{k}) W_{\sigma\tau}(k, t)_{\tau\tau} = \langle \hat{a}_\sigma(k, t)^* \hat{a}_\tau(\tilde{k}, t) \rangle,$$

for fixed $t$ up to order $\lambda^2$ as

$$W_{\sigma\tau}(k, t) = W^{(0)}(k) + \lambda W^{(1)}(k, t) + \lambda^2 W^{(2)}(k, t) + O(\lambda^3). \tag{20}$$

The collision operator will then be extracted from $W^{(2)}$, see section 5. But the main effort is to properly organize the expansion. To avoid a specific spin basis, we choose arbitrary vectors $f, g \in \mathbb{C}^2$ and consider $\langle f, W_{\sigma}(k, t) g \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product (anti-linear on the left) in spin space. It is advantageous to introduce the vector-valued operators

$$\hat{a}_i(k)^* = \begin{pmatrix} \bar{f}_i, \bar{a}_i(k)^* \\ \bar{f}_i, \bar{a}_i(k)^* \end{pmatrix} \quad \text{and} \quad \hat{a}_i(k) = \begin{pmatrix} f_i, \hat{a}_i(k) \\ g_i, \hat{a}_i(k) \end{pmatrix}, \tag{21}$$

where $\bar{f}$ denotes complex conjugate and $f_\sigma, g_\sigma, \sigma \in \{\uparrow, \downarrow\}$ denote the components of $f$ and $g$. We will also use the following operations mapping two 2-vector-valued operators into a scalar-valued one:

$$v \odot w = \sum_{\sigma, \tau \in \{\uparrow, \downarrow\}} v_{\sigma} w_{\tau} \quad \text{and} \quad v \cdot w = \sum_{\sigma \in \{\uparrow, \downarrow\}} v_{\sigma} w_{\sigma}. \tag{22}$$

For instance, then $\langle \hat{a}_i(k, t)^* \odot \hat{a}_\tau(\tilde{k}, t) \rangle = \delta(k - \tilde{k}) \langle f, W_{i\tau}(k, t) g \rangle$.

Let us first compute the time derivative of the basic 2-vector-valued operator

$$\frac{d}{dt} \hat{a}_g(k, t)^g = i[H, \hat{a}_g(k, t)^g] = i[H_0, \hat{a}_g(k)^g](t) + \lambda \hat{a}_g(k)^g(t). \tag{23}$$

where $\#$ denotes either nothing or an adjoint (annihilation or creation operator). For the quadratic $H_0$, it follows directly from the commutation relations that

$$[H_0, \hat{a}_g(k)] = \int_{\mathbb{R}^d} dk' \omega(k') [\hat{a}(k')^* \cdot \hat{a}(k'), \hat{a}_g(k)] = -\omega(k) \hat{a}_g(k) \tag{24}$$

In the special case of an on-site potential, one finds $\hat{V} = 1$ and hence $R[W](k) = \int_{\mathbb{R}^d} dk W(k)$. $R[W]$ does not depend on $k$ and, by conservation of spin, also not on $t$. We denote the resulting matrix by $R$, which still depends on the initial condition. Since $R$ is constant, it can be removed explicitly through the unitary transformation

$$W(k, t) \mapsto e^{i\kappa - i\kappa R} W(k, t) e^{-i\kappa - i\kappa R}. \tag{18}$$

For a general potential, one has to solve the full nonlinear equation including the $R[W(t)]$ term.
With this notation we have

\[ [H_0, \hat{a}_g(k)^*] = -[H_0, \hat{a}_g(k)]^* = \omega(k) \cdot \hat{a}_g(k)^*. \]  

(25)

For \( H_1 \), we use

\[ [H_1, \hat{a}_g(k)] = \frac{1}{2} \int (d^3 \nu) \delta(k) \hat{V}(k_2 - k_3) [\hat{a}(k_3)^* \cdot \hat{a}(k_2)] (\hat{a}(k_3)^* \cdot \hat{a}(k_4)) \cdot \hat{a}_g(k)], \]

(26)

with \( k = k_1 - k_2 + k_3 - k_4 \). Using the commutators

\[ [(\hat{a}(k_1)^* \cdot \hat{a}(k_2)) \cdot \hat{a}(k_3)^* \cdot \hat{a}(k_4)), \hat{a}_g(k)] = -\delta(k_1 - k) \hat{a}_g(k_2) (\hat{a}(k_3)^* \cdot \hat{a}(k_4)) \]

\[ -\delta(k_3 - k) \hat{a}_g(k_4) (\hat{a}(k_1)^* \cdot \hat{a}(k_2)) + \delta(k_1 - k_4) \delta(k_3 - k) \hat{a}_g(k_2), \]

(27)

we obtain

\[ \frac{d}{dt} \hat{a}_g(k_1, t) = i[H, \hat{a}_g(k_1, t)] = -i \omega(k_1) \hat{a}_g(k_1, t) + \frac{i}{2} V(0) \hat{a}_g(k_1, t) \]

\[ -i\lambda \int (d^3 \nu) \delta(k) \hat{V}(k_2 - k_3) \cdot \hat{a}_g(k_2, t) (\hat{a}(k_3)^* \cdot \hat{a}(k_4)). \]

(28)

To proceed further we need convenient shorthand notations. With the notation \( k_{1234} = (k_1, k_2, k_3, k_4) \) and for the complex-valued functions \( h \), we set

\[ A[h, a, b, c](k_1, t) = \int (d^3 \nu) \delta(k) h(k_{1234}, t) \hat{V}(k_3 - k_4)a(k_2, t)(b(k_3, t) \cdot c(k_4, t)), \]

(29)

\[ A_4[H, a, b, c](k_1, t) = \int (d^3 \nu) \delta(k) H(k_{1234}, t) \hat{V}(k_2 - k_3)(a(k_2, t) \cdot b(k_3, t)) c(k_4, t), \]

(30)

where \( a, b \) and \( c \) are two-component vector-valued operators as in (21). Then \( A \) is again a vector-valued operator and it holds

\[ (A[h, a, b, c](k, t))^* = A_4[H, c^*, b, a^*](k, t). \]

(31)

With this notation we have

\[ A[\text{id}, \hat{a}_g, a^*, \hat{a}](k_1, t) = \int (d^3 \nu) \delta(k) \hat{V}(k_3 - k_4) \hat{a}_g(k_2, t) (\hat{a}(k_3)^* \cdot \hat{a}(k_4)), \]

(32)

\[ A_a[\text{id}, \hat{a}_g, a^*, \hat{a}^a](k_1, t) = \int (d^3 \nu) \delta(k) \hat{V}(k_2 - k_3) (\hat{a}(k_2, t)^* \cdot \hat{a}(k_3))^* \hat{a}(k_4, t)^*, \]

(33)

where ‘id’ is the identity function. The evolution equation (28) is then

\[ \frac{d}{dt} \hat{a}_g(k, t) = -i \left( \omega(k) - \frac{1}{2} \lambda V(0) \right) \hat{a}_g(k, t) - i\lambda A[\text{id}, \hat{a}_g, a^*, \hat{a}](k, t) \]

(34)

and correspondingly for the creation operator

\[ \left( \frac{d}{dt} \hat{a}_i(k, t) \right)^* = \frac{d}{dt} \hat{a}_i(k, t)^* = i \left( \omega(k) - \frac{1}{2} \lambda V(0) \right) \hat{a}_i(k, t)^* + i\lambda A_a[\text{id}, \hat{a}_i, a^*, \hat{a}^a](k, t). \]

(35)

The linear part can be removed through defining

\[ a_g(k, t) = e^{i(\omega(k) - \frac{1}{2} \lambda V(0)) t} \hat{a}_g(k, t), \]

(36)

where \( a \) always acts in Fourier space. Clearly,

\[ a^a_g(k, t) = (a_g(k, t))^* = e^{-i(\omega(k) - \frac{1}{2} \lambda V(0)) t} \hat{a}_g(k, t)^*, \]

(37)
and for the correlation it still holds that
\[ \langle a_\lambda^*(k, t) \otimes a_\lambda(k, t) \rangle = \langle \hat{a}_\lambda(k, t)^* \otimes \hat{a}_\lambda(\bar{k}, t) \rangle. \]  
(38)

Introducing the further shorthand
\[ \omega_{abcd} = \omega(k_a) - \omega(k_b) + \omega(k_c) - \omega(k_d), \]
(39)
one finally arrives at
\[ \frac{d}{dt} a_\lambda(k_1, t) = -i \lambda \mathcal{A}[e^{i \lambda_{abcd} t}, a_\lambda, a^*, a](k_1, t) \]
(40)
and for the adjoint
\[ \frac{d}{dt} a_\lambda^*(k_1, t) = i \lambda \mathcal{A}[e^{-i \lambda_{abcd} t}, a^*, a, a_\lambda^*](k_1, t). \]
(41)

By the fundamental theorem of calculus
\[ \int_0^t \frac{d}{ds} a_\lambda^*(k, s) = a_\lambda^*(k, t) - a_\lambda^*(k, 0), \]
(42)
which implies
\[ a_\lambda(k_1, t) = a_\lambda(k_1, 0) - i \lambda \int_0^t \frac{d}{ds} \mathcal{A}[e^{i \lambda_{abcd} s}, a_\lambda, a^*, a](k_1, s). \]
(43)

Iterating (40) twice up to second order of the Dyson expansion, with an error of order \( \lambda^3 \),
\[ \frac{d}{dt} a_\lambda(k_1, t) = -i \lambda \mathcal{A}[e^{i \lambda_{abcd} t}, a_\lambda, a^*, a](k_1, 0) \]
\[ - \lambda^2 \int_0^t \frac{d}{ds} \mathcal{A}[e^{i \lambda_{abcd} s}, \mathcal{A}[e^{i \lambda_{abc} t}, \hat{a}_\lambda, a^*, a, a]](k_1, s) \]
\[ + \lambda^2 \int_0^t \frac{d}{ds} \mathcal{A}[e^{i \lambda_{abcd} s}, \hat{a}_\lambda, \mathcal{A}[e^{i \lambda_{abc} t}, a^*, a, a^*]](k_1, s) \]
\[ - \lambda^2 \int_0^t \frac{d}{ds} \mathcal{A}[e^{i \lambda_{abcd} s}, \hat{a}_\lambda, a^*, a^*, a^*]](k_1, s) + O(\lambda^3) \]
\[ = \lambda \frac{d}{dt} a_\lambda^{(1)}(k_1, t) + \lambda^2 \frac{d^2}{dt^2} a_\lambda^{(2)}(k_1, t) + O(\lambda^3). \]
(44)

Hence, for fixed \( t \), as an expansion in \( \lambda \),
\[ a_\lambda(k, t) = a_\lambda^{(0)}(k, t) + \lambda a_\lambda^{(1)}(k, t) + \lambda^2 a_\lambda^{(2)}(k, t) + O(\lambda^3), \]
(45)
where \( a_\lambda^{(0)}(k, t) = a_\lambda^{(0)}(k, 0) = \hat{a}_\lambda(k) \). A corresponding expression is satisfied by \( a_\lambda^*(k, t) \).

Iterating it further yields the formal expansion
\[ \frac{d}{dt} (a_\lambda^*(k, t) \otimes a_\lambda(k, t)) = \sum_{n=0}^\infty \lambda^n \sum_{m=0}^n \frac{d^n}{dt^n} (a_\lambda^*(k, t)^{(m)} \otimes a_\lambda(k, t)^{(n-m)}). \]
(46)

Therefore, \( W_\lambda(k, t) \) can be written as
\[ \delta(k - \bar{k}) \langle \mathcal{W}_\lambda(k, t) \rangle = \langle a_\lambda^*(k, 0) \otimes a_\lambda(k, 0) \rangle \]
\[ + \sum_{n=1}^\infty \lambda^n \int_0^t ds \sum_{m=0}^n \frac{d^n}{ds^n} (a_\lambda^*(k, s)^{(m)} \otimes a_\lambda(k, s)^{(n-m)}) \]
\[ = \delta(k - \bar{k}) \langle \mathcal{W}_\lambda^{(0)}(k, t) \rangle + \delta(k - \bar{k}) \sum_{n=1}^\infty \lambda^n \langle \mathcal{W}_\lambda^{(n)}(k, t) \rangle. \]
(47)

The zeroth-order term of equation (47) reads
\[ \delta(k - \bar{k}) \langle \mathcal{W}_\lambda^{(0)}(k) \rangle = \langle a_\lambda^*(k, 0) \otimes a_\lambda(k, 0) \rangle = \langle \hat{a}_\lambda(k) \otimes \hat{a}_\lambda(\bar{k}) \rangle. \]
(48)

In the next two sections we determine the terms of first and second orders.
3. First-order terms

Let us consider the $W^{(1)}(k, t)$-term of equation (47). Its structure will be easier to capture once we represent the various summands as Feynman diagrams. The first-order terms are determined by

$$\delta(k_1 - k_5) \langle f, W^{(1)}(k_1, t) g \rangle = i \int_0^t ds \langle A_s [e^{-i\omega_{1234}s}, a^*, a, a^*_5](k_1) \odot a_g(k_5, s) \rangle$$

$$- i \int_0^t ds \langle a^*_5(k_1, s)^{(0)} \odot A[e^{i\omega_{5234}s}, a_g, a^*, a](k_5) \rangle$$

$$= i \int_0^t ds \int (\mathcal{T}_d)^3 dk_{234} \delta(k) \hat{V}(k_2 - k_3) \times e^{-i\omega_{1234}s} (\hat{a}(k_2)^* \cdot \hat{a}(k_3)) (\hat{a}(k_4)^* \odot \hat{a}_g(k_5)))$$

$$- i \int_0^t ds \int (\mathcal{T}_d)^3 dk_{234} \delta(k) \hat{V}(k_3 - k_4) \times e^{i\omega_{5234}s} (\hat{a}_g(k_1)^* \odot \hat{a}(k_3)) (\hat{a}_g(k_3)^* \cdot \hat{a}(k_4)).$$

(49)

The first term is represented by the left graph in figure 1. Let us first explain the structure of the graph. Each graph consists of the following symbols: vertices, edges and time slices. The time direction points from bottom to top. The $n$th-order terms have $n$ vertices, and so the first-order terms have only a single vertex. The vertex represents the interaction of particles. The edges are labeled by oriented momentum variables $k_i$. If the earlier of the endpoints is a creation operator, the arrow points in the time direction, and if it is an annihilation operator, the arrow points opposite to the time direction. Then, by the definition of $\mathcal{A}$, at every vertex there are two ingoing and two outgoing arrows.

To reconstruct the corresponding integral from a given graph, one needs to iteratively add the following five operations for each vertex.

(i) An integration of a time variable $s$ from zero to the end of the time slice after the vertex. In figure 1, this amounts to using the time integral $\int_0^t ds$.

(ii) The integration over the momentum variables can be read of as follows: one needs to add $\int (\mathcal{T}_d)^3 dk_{ij}$, where $k_i$, $k_j$ and $k_l$ label the three ‘earlier’ edges.
(iii) A product of four phase factors $e^{\pm i\omega(k)\lambda}$, one for each arrow attached to the vertex, where $\omega$ denotes the time integration variable of the vertex. A negative sign is chosen if the arrow points in the time direction, and a positive sign if it points against the time direction.

(iv) A $\delta$-function ensuring the momentum conservation, in which a positive sign is used if the corresponding arrow points to the vertex, and a negative sign if the arrow points away from the vertex.

(v) A factor `$\pm i$' with a positive sign if the single later edge points away from the vertex, and a negative sign if it points towards the vertex.

Finally, an average $\langle \cdot \rangle$ over the initial state needs to be taken of the product of creation and annihilation operators at the bottom of the graph. Each case of $(\hat{a}(k)\ast \hat{a}(k))$ apart from those labeled with an index $f$ or $g$ in figure 1 also represents a multiplication of $\tilde{V}(k_i - k_f)$ in the integral representation of a given graph. By construction, if one starts to count the direction of the arrows from left to right in any of the time slices, they always start with an up arrow and alternate from left to right in up–down combinations. This results in an alternating sequence of creation and annihilation operators at the bottom of the graph. The Wick-pairings `$\cup$' shown under the graph follow from averaging this alternating sequence over the initial quasi-free state. The average has a particularly simple form for the alternating order of creation and annihilation operators: it can then be computed according to the Wick rule

$$\langle \hat{a}_1^* \cdots \hat{a}_n^* \hat{a}_1 \cdots \hat{a}_n \rangle = \text{det}(K(i, j))_{1 \leq i, j \leq n}, \tag{50}$$

where

$$K(i, j) = \begin{cases} \langle \hat{a}_i^* \hat{a}_j \rangle, & \text{if } k \leq l, \\ -\langle \hat{a}_j \hat{a}_i^* \rangle, & \text{if } k > l. \end{cases} \tag{51}$$

For instance, the expectation value $\langle \cdot \rangle$ over the initial state in the first term in (49) can be expressed as

$$\langle (\hat{a}(k)\ast \hat{a}(k)) (\hat{a}(k)\ast \hat{a}_g(k)) \rangle = \sum_{\sigma, \tau \in \{\uparrow, \downarrow\}} \bar{t}_g \bar{t}_\tau \text{det} \begin{bmatrix} \langle \hat{a}_5 \hat{a}_1^* \rangle & \langle \hat{a}_5 \hat{a}_2 \rangle & \langle \hat{a}_5 \hat{a}_3 \rangle & \langle \hat{a}_5 \hat{a}_4 \rangle \\ -\langle \hat{a}_5 \hat{a}_4 \rangle & \langle \hat{a}_5 \hat{a}_3^* \rangle & \langle \hat{a}_5 \hat{a}_2^* \rangle & \langle \hat{a}_5 \hat{a}_1 \rangle \\ \langle \hat{a}_4 \hat{a}_1 \rangle & \langle \hat{a}_4 \hat{a}_2^* \rangle & \langle \hat{a}_4 \hat{a}_3^* \rangle & \langle \hat{a}_4 \hat{a}_5 \rangle \\ -\langle \hat{a}_4 \hat{a}_5 \rangle & \langle \hat{a}_4 \hat{a}_3 \rangle & \langle \hat{a}_4 \hat{a}_2 \rangle & \langle \hat{a}_4 \hat{a}_1^* \rangle \end{bmatrix}. \tag{52}$$

The two Wick pairings shown in figure 1 represent the two different pairings in equation (52). Since for instance, $(\hat{a}_5(k_5)\ast \hat{a}_a(k_4)) = \delta(k_5 - k_a)W(k_5)\delta_{\sigma\sigma}$, the left diagram yields

$$\int_0^T ds \langle \hat{a}_5^*(k_1, s)^{(1)} \ast \hat{a}_a(k_5, s)^{(0)} \rangle = it \delta(k_1 - k_5) \int_0^\infty dk_2 \langle \hat{V}(0) \text{tr}[\hat{W}_2][f, W_1 g] \rangle \tag{53}$$

$$+ \hat{V}(k_1 - k_2)(f, \tilde{W}_2 W_1 g),$$

where $\hat{a}(k, t) = \frac{\partial}{\partial t} \hat{a}(k, t)$. The contribution of the right diagram in figure 1 can also be computed directly by taking an adjoint of the result above, yielding

$$\int_0^T ds \langle \hat{a}_5^*(k_1, s)^{(0)} \ast \hat{a}_a(k_5, s)^{(1)} \rangle = -it \delta(k_1 - k_5) \int_0^\infty dk_2 \langle \hat{V}(0) \text{tr}[\hat{W}_2][f, W_1 g] \rangle \tag{54}$$

$$+ \hat{V}(k_1 - k_2)(f, \tilde{W}_2 W_1 g).$$

Thus, the first-order term is given by

$$W^{(1)}(k_1, t) = -it [R(W), W_1], \quad R(W) = \int_0^\infty dk \hat{V}(k_1 - k) W(k) \in C_{2\times2}. \tag{55}$$
Figure 2. The diagrams of the \((1', 1)\)-terms.

All four diagrams in figure 1 have an interaction with zero momentum transfer (for instance, using the top left pairing leads to \(k_4 = k_1\)). Such diagrams will also appear in the second order and we call them zero momentum transfer diagrams.

4. Second-order terms

We next consider the second-order term which we decompose into a sum of four terms, obtained by evaluating the time derivative in the equality

\[
\delta(k - \tilde{k})(f, W^{(2)}(k, t)g) = \int_0^t ds \sum_{m=0}^2 \frac{d}{ds} \langle a^*_\sigma(k, s)^{(m)} \odot a_\tau(\tilde{k}, s)^{(2-m)} \rangle.
\]

\((1', 1)\)-term

In the previous section, we have already shown that

\[
\int_0^t ds \langle \hat{a}^*_\sigma(k_1, s)^{(1)} \odot a_\tau(k_5, s)^{(1)} \rangle = \int_0^t ds_1 \int_0^{2s_1} \frac{d}{ds} \langle A_s[\epsilon^{-i\omega_{23} s_2}, a^*, a, a^*_\mu_2](k_1)
\odot A[\epsilon^{i\omega_{23} s_1}, a_\tau, a^*, a](k_3) \rangle
\]

which can be represented by the Feynman diagram of figure 2.

In order to evaluate the diagram, we start with

\[
\langle \hat{a}(k_2)^* \cdot \hat{a}(k_3) \rangle (\hat{a}(k_4)^* \odot \hat{a}_\tau(k_5)) (\hat{a}(k_7)^* \cdot \hat{a}(k_8)) \rangle

= \sum_{\sigma, \tau, \mu_1, \mu_2} \overline{t}_{\sigma \tau \mu_1 \mu_2} \langle \hat{a}_{\mu_1}(k_2)^* \hat{a}_{\mu_1}(k_3) \hat{a}_\tau(k_4)^* \hat{a}_\tau(k_5) \hat{a}_\tau(k_7)^* \hat{a}_{\mu_2}(k_8) \rangle.
\]

(58)
Using
\[
\langle \hat{a}_s(i_1)^* \hat{a}_t(j_1) \hat{a}_s(i_2)^* \hat{a}_t(j_2) \hat{a}_s(i_3)^* \hat{a}_t(j_3) \rangle \\
= \det \begin{bmatrix}
\langle \hat{a}_s(i_1)^* \hat{a}_t(j_1) \rangle & \langle \hat{a}_s(i_1)^* \hat{a}_t(j_2) \rangle & \langle \hat{a}_s(i_1)^* \hat{a}_t(j_3) \rangle \\
\langle \hat{a}_s(i_1) \hat{a}_s(i_2)^* \rangle & \langle \hat{a}_s(i_2)^* \hat{a}_t(j_2) \rangle & \langle \hat{a}_s(i_2)^* \hat{a}_t(j_3) \rangle \\
\langle \hat{a}_s(j_1) \hat{a}_s(i_3)^* \rangle & \langle \hat{a}_s(j_2) \hat{a}_s(i_3)^* \rangle & \langle \hat{a}_s(j_3) \hat{a}_s(i_3)^* \rangle \\
\end{bmatrix} ,
\]
(59)
on one arrives at
\[
\langle (\hat{a}\langle k_2 \rangle^* \cdot \hat{a}(k_3)) (\hat{a}^\dagger(k_4)^* \circ \hat{a}_g(k_5)) (\hat{a}^\dagger(k_7)^* \cdot \hat{a}(k_8)) \rangle \\
= \delta(k_5 - k_7) \delta(k_4 - k_6) \delta(k_2 - k_8) \langle f, \mathcal{W}_s W_2 \rangle \text{g} \\
- \delta(k_2 - k_6) \delta(k_4 - k_6) \delta(k_3 - k_7) \langle f, \mathcal{W}_s W_2 \rangle \text{g} \\
+ \delta(k_2 - k_6) \delta(k_4 - k_6) \delta(k_3 - k_7) \langle f, \mathcal{W}_s W_2 \rangle \text{g} \\
+ \delta(k_6 - k_8) \delta(k_3 - k_4) \delta(k_2 - k_7) \langle f, \mathcal{W}_s W_2 \rangle \text{g} \\
+ \delta(k_5 - k_8) \delta(k_3 - k_4) \delta(k_2 - k_6) \langle f, \mathcal{W}_s W_2 \rangle \text{g}.
\]
(60)
Using this formula in (57) yields the following expression for the (1’, 1)-term:
\[
\int_0^t ds \langle \hat{a}^\dagger(k_1, s)^{(1)} \circ \hat{a}_g(k_5, s)^{(1)} \rangle = \delta(k_1 - k_5) \frac{1}{2} \mathcal{T}^2 \langle f, \mathcal{Z}[W_1^{(1)}] \rangle \\
+ \delta(k_1 - k_5) \int_0^t ds_2 \int_0^{s_2} ds_1 \int_{(\mathcal{T}^1)} d\mathcal{W}_{234} \delta(k) e^{-i021234(k_2 - s_1)} \langle f, \mathcal{D}[W_{234}] \rangle.
\]
(61)
Here
\[
\mathcal{D}[W_{234}] = \tilde{\mathcal{V}}(k_2 - k_3)^2 \mathcal{W}_2 \mathcal{W}_2 - \tilde{\mathcal{V}}(k_2 - k_3) \mathcal{V}(k_3 - k_4) \mathcal{W}_2 \tilde{\mathcal{W}}_2,
\]
(62)
and it results from the first two terms in equation (60). The remaining four terms all lead to a diagram with a zero momentum transfer and summing up their contribution yields
\[
\mathcal{Z}[W_1^{(1)}] = \tilde{\mathcal{V}}(0) [W_{1}, R[\tilde{\mathcal{W}}_1]] \text{tr}[R] + R[\tilde{\mathcal{W}}_1] W_1 R[\tilde{\mathcal{W}}_1] + \tilde{\mathcal{V}}(0)^2 \text{tr}[R] \text{tr}[R].
\]
(63)

(1, 1’)-term

A similar discussion applies to
\[
\int_0^t ds \langle \hat{a}^\dagger(k_1, s)^{(1)} \circ \hat{a}_g(k_5, s)^{(1)} \rangle = \int_0^t ds_2 \int_0^{s_2} ds_1 \langle A_s[e^{-i01234k_5}, a, a_1^*](k_1) \\
\circ A[e^{i01234k_5}, a, a^*, a](k_5) \rangle,
\]
(64)
which can also be computed by taking the adjoint of the (1’, 1)-term. This shows that
\[
\int_0^t ds \langle \hat{a}^\dagger(k_1, s)^{(1)} \circ \hat{a}_g(k_5, s)^{(1)} \rangle = \delta(k_1 - k_5) \frac{1}{2} \mathcal{T}^2 \langle f, \mathcal{Z}[W_1^{(1)}] \rangle \\
+ \delta(k_1 - k_5) \int_0^t ds_2 \int_0^{s_2} ds_1 \int_{(\mathcal{T}^1)} d\mathcal{W}_{234} \delta(k) e^{-i021234(k_2 - s_1)} \langle f, \mathcal{D}[W_{234}] \rangle,
\]
(65)
where \(\mathcal{Z}[W_1^{(1)}] = (\mathcal{Z}[W_1^{(1)}]^*) = \mathcal{Z}[W_1^{(1)}]\) and
\[
\mathcal{D}[W_{234}] = \tilde{\mathcal{V}}(k_2 - k_3)^2 \mathcal{W}_2 \mathcal{W}_2 - \tilde{\mathcal{V}}(k_2 - k_3) \mathcal{V}(k_3 - k_4) \mathcal{W}_2 \tilde{\mathcal{W}}_2,
\]
(66)
such that it holds \(\mathcal{D}[W_{234}] = \mathcal{D}[W_{234}]\) by interchanging \(k_2 \leftrightarrow k_4\) for the second term.
Figure 3. Graphs related to the three terms in (67): (a) first term, (b) second term and (c) third term.

(2, 0)-term

The (2, 0)-term is given by the following expression:

\[
\int_0^t ds \langle \hat{a}^*_7(k_1, s) \odot \hat{a}_5(k_5, s) \rangle
\]

\[
= - \int_0^t ds_2 \int_0^{s_2} ds_1 \langle \hat{A}_s[e^{-i\omega_1 s_2 s} A_s[e^{-i\omega_2 s}, a^*, a, a^*], a, a^*](k_1) \odot a_5(k_5) \rangle
\]

\[
+ \int_0^t ds_2 \int_0^{s_2} ds_1 \langle \hat{A}_s[e^{-i\omega_1 s_2 s}, a^*, A[e^{-i\omega_3 s], a, a^*], a^*](k_1) \odot a_5(k_5) \rangle
\]

\[
- \int_0^t ds_2 \int_0^{s_2} ds_1 \langle \hat{A}_s[e^{-i\omega_1 s_2 s}, a^*, a, A[e^{-i\omega_4 s}, a, a^*], a^*](k_1) \odot a_5(k_5) \rangle.
\]

The associated graphs are shown in figure 3.

To evaluate the contribution of the pairings to the first term in equation (67), we use

\[
\langle (\hat{a}(k_6) \cdot \hat{a}(k_7)) (\hat{a}(k_8) \cdot \hat{a}(k_9)) (\hat{a}(k_1) \cdot \hat{a}(k_2)) \rangle
\]

\[
= \delta(k_7 - k_4) \delta(k_8 - k_3) \delta(k_6 - k_5) \langle f, \hat{W}_4 \hat{W}_5 \text{tr}[W_5]g \rangle
\]

\[
- \delta(k_6 - k_3) \delta(k_8 - k_5) \delta(k_7 - k_1) \langle f, \hat{W}_4 \hat{W}_5 \hat{W}_3 \hat{W}_5 g \rangle
\]

+ zero momentum transfer diagrams.

The contribution to the second term in equation (67) can be computed using

\[
\langle (\hat{a}(k_2) \cdot \hat{a}(k_3)) (\hat{a}(k_8) \cdot \hat{a}(k_9)) (\hat{a}(k_1) \cdot \hat{a}(k_2)) \rangle
\]

\[
= \delta(k_8 - k_4) \delta(k_7 - k_3) \delta(k_6 - k_2) \langle f, \hat{W}_4 \hat{W}_5 \text{tr}[W_2]g \rangle
\]

\[
- \delta(k_2 - k_3) \delta(k_7 - k_5) \delta(k_6 - k_4) \langle f, \hat{W}_4 \hat{W}_5 \hat{W}_2 \hat{W}_5 g \rangle
\]

+ zero momentum transfer diagrams.
and the contribution to the third term in equation (67) by
\[
\langle (\hat{a}(k_2)^* \cdot \hat{a}(k_3)) (\hat{a}(k_6)^* \cdot \hat{a}(k_7)) (\hat{a}(k_8)^* \cdot \hat{a}(k_9)) \rangle
\]
\[
= \delta(k_8 - k_5) \delta(k_3 - k_6) \delta(k_2 - k_7) \langle f, W_5 W_2 W_5 \rangle
\]
\[
- \delta(k_2 - k_7) \delta(k_6 - k_5) \delta(k_3 - k_8) \langle f, \bar{W}_5 W_2 W_5 \rangle
\]
\[
+ \text{zero momentum transfer diagrams.}
\] (70)

With the definitions
\[
B[W]_{1234} = \bar{V}(k_2 - k_5) \bar{V}(k_3 - k_4) \left( \bar{W}_5 W_2 W_1 + \bar{W}_5 W_3 W_1 - \bar{W}_4 W_1 \right)
\]
\[
+ \hat{V}(k_2 - k_3)^2 \left( \bar{W}_4 W_1 tr[W_2] - \bar{W}_4 W_1 tr[W_5] - W_1 tr[W_5] \right)
\] (71)

and
\[
Z[W]_{1}^{(20)} = -\hat{V}(0) W_1 tr[R] tr[R] - R[\bar{W}] W_1 W_1 - \hat{V}(0) R[\bar{W}] W_1 W_1 tr[R]
\]
\[
- \hat{V}(0) R[\bar{W}] W_1 W_1 tr[R],
\] (72)

we obtain
\[
\int_0^t ds \langle \hat{a}_5^+(k_1, t)^{(2)} \circ \hat{a}_5(k_5, t)^{(0)} \rangle = \delta(k_1 - k_5) \frac{1}{2} \langle f, Z[W]_{1}^{(20)} \rangle
\]
\[
+ \delta(k_1 - k_5) \int_0^t ds_1 \int_0^{s_1} ds_2 \int_{(70)} d^3k_{234} \delta(k) e^{i\omega_{1234}(s_2 - s_1)} \langle f, B[W]_{1234} \rangle.
\] (73)

(0, 2)-term

The (0, 2)-term is given by the following expression:
\[
\int_0^t ds \langle a_5^+(k_1, s)^{(0)} \circ \hat{a}_5(k_5, s)^{(2)} \rangle
\]
\[
= - \int_0^t ds_2 \int_0^{s_2} ds_1 \langle a_5^+(k_1) \circ A[e^{i\omega_{1234} s_2}, A[e^{i\omega_{1234} s_1}, A_5, a^+, a^+, a_5, a_5]] \rangle
\]
\[
+ \int_0^t ds_2 \int_0^{s_2} ds_1 \langle a_5^+(k_1) \circ A[e^{i\omega_{1234} s_2}, A_5, A[e^{i\omega_{1234} s_1}, a^+, a^+, a_5, a_5]] \rangle
\]
\[
- \int_0^t ds_2 \int_0^{s_2} ds_1 \langle a_5^+(k_1) \circ A[e^{i\omega_{1234} s_2}, A_5, A[e^{i\omega_{1234} s_1}, a^+, a^+, a_5, a_5]] \rangle. \]
(74)

The corresponding graphs are computed similar to the (2, 0)-term and can also be obtained by reflecting each of the three graphs in figure 3 at the vertical green (grey) line through \(\circ\) and interchanging \(f\) and \(g\). By defining
\[
B[W]_{1234} = \bar{V}(k_2 - k_5) \bar{V}(k_3 - k_4) (W_1 W_2 W_3 + W_1 W_5 W_2 - W_1 W_5 W_3)
\]
\[
+ \hat{V}(k_2 - k_3)^2 (W_1 W_2 tr[W_4] - W_1 tr[W_5 W_3] - W_1 tr[W_5 W_3]),
\] (75)

one can show that the value of the integral in (73) does not change if \(B[W]_{1234}\) is replaced there by \(B[W]_{1234}^\star\). The contribution of the zero momentum transfer graphs is given by
\[
Z[W]_{1}^{(20)} = -\hat{V}(0) W_1 tr[R] tr[R] - W_1 R[\bar{W}] W_1 W_1 - \hat{V}(0) W_1 R[\bar{W}] W_1 W_1 tr[R]
\]
\[
- \hat{V}(0) W_1 R[\bar{W}] W_1 W_1 tr[R],
\] (76)

It holds that \((Z[W]_{1}^{(20)})^\star = Z[W]_{1}^{(02)}\), and we finally obtain
\[
\int_0^t ds \langle a_5^+(k_1, s)^{(0)} \circ \hat{a}_5(k_5, s)^{(2)} \rangle = \delta(k_1 - k_5) \frac{1}{2} \langle f, Z[W]_{1}^{(02)} \rangle + \delta(k_1 - k_5)
\]
\[
\times \int_0^t ds_2 \int_0^{s_2} ds_1 \int_{(70)} d^3k_{234} \delta(k) e^{i\omega_{1234}(s_2 - s_1)} \langle f, B[W]_{1234} \rangle.
\] (77)
5. The limit $\lambda \to 0, t = O(\lambda^{-2})$

Before we consider the limit $\lambda \to 0$, we summarize all second-order diagrams.

Defining

$$\mathcal{A}[W] = \mathcal{D}[W] + \mathcal{B}[W],$$

and using the identity

$$- [\mathcal{R}[W], [\mathcal{R}[W], W_j]] = \mathcal{Z}[W]^{(1)} + \mathcal{Z}[W]^{(11)} + \mathcal{Z}[W]^{(20)} + \mathcal{Z}[W]^{(02)},$$

we thus find that

$$\int_0^t ds \frac{d}{ds} \sum_{m=0}^2 \langle a_s^m(k_1, s) (m) \rangle \circ \alpha_s(k_5, s)^{(2-m)} \rangle = -\delta(k_1 - k_5) \frac{1}{2} t^2 \langle f, [\mathcal{R}[W], [\mathcal{R}[W], W_j]]g \rangle$$

$$+ \delta(k_1 - k_5) \int_0^t ds_1 \int_0^{s_1} ds_2 \int_{(T_x^y)^3} d_{k_{234}} \delta(k_1) e^{i_{1234}(\omega_{k_1} - \omega)} \langle f, \mathcal{A}[W]_{1234} g \rangle$$

$$+ \delta(k_1 - k_5) \int_0^t ds_1 \int_0^{s_1} ds_2 \int_{(T_x^y)^3} d_{k_{234}} \delta(k_2) e^{-i_{1234}(\omega_{k_2} - \omega)} \langle f, \mathcal{A}[W]_{1234}^* g \rangle.$$ (81)

Hence the second-order term $W_{(1)}^{(2)}$ is given by

$$W_{(1)}^{(2)}(k_1, t) = W_{x}^{(2)}(k_1, t) + W_{c}^{(2)}(k_1, t),$$ (82)

where

$$W_{x}^{(2)}(k_1, t) = -\frac{1}{2} t^2 \langle f, [\mathcal{R}[W], [\mathcal{R}[W], W_j]] g \rangle$$ (83)

and

$$W_{c}^{(2)}(k_1, t) = \int_0^t ds_1 \int_0^{s_1} ds_2 \int_{(T_x^y)^3} d_{k_{234}} \delta(k_1) e^{i_{1234}(\omega_{k_1} - \omega)} \mathcal{A}[W]_{1234}$$

$$+ e^{-i_{1234}(\omega_{k_1} - \omega)} \mathcal{A}[W]_{1234}^*.$$ (84)

The collision operator is determined by taking at second order the limit $\lambda \to 0$ and simultaneous long times $\lambda^{-2} t$ with $t$ of order 1. More explicitly,

$$t \mathcal{C}[W^{(0)}](k) = \lim_{\lambda \to 0} \lambda^2 W_{c}^{(2)}(k, \lambda^{-2} t),$$ (85)

where $W_{c}^{(2)}$ is defined in (84). To evaluate the limit, we make use of

$$\lim_{\lambda \to 0} \lambda^2 \int_0^t ds_1 \int_0^{s_1} ds_2 e^{i_{1234}(\omega_{k_1} - \omega)} = t \int_0^s ds e^{i_{1234} \omega t}$$

$$= t \left( \pm i \mathcal{P} \left( \frac{1}{\omega_{1234}} \right) + \pi \delta(\omega_{1234}) \right),$$ (86)

where $\mathcal{P}$ denotes the principal value integral, as defined in section 1. This yields

$$\lim_{\lambda \to 0} \lambda^2 W_{c}^{(2)}(k, \lambda^{-2} t) = t \pi \int_{(T_x^y)^3} d_{k_{234}} \delta(k) \delta(\omega_{1234}) \langle f, \mathcal{A}[W]_{1234} + \mathcal{A}[W]_{1234}^* \rangle g \rangle$$

$$+ t i \int_{(T_x^y)^3} d_{k_{234}} \delta(k) \mathcal{P} \left( \frac{1}{\omega_{1234}} \right) \langle f, \mathcal{A}[W]_{1234} - \mathcal{A}[W]_{1234}^* \rangle g \rangle. \quad (87)$$

This agrees with the result stated in the introduction.

We note that in case $\mathcal{W}_{e}(k) = \delta_{e, 0} \mathcal{W}_{e}(k)$, the term containing the principal part vanishes. The effective Hamiltonian results from the twofold degeneracy of the unperturbed $H_0$.  

13
6. Conclusions

The kinetic equation for the Hubbard model has two novel features. Firstly, the Boltzmann f-function becomes in a natural way the $2 \times 2$ matrix-valued function. Furthermore, besides the conventional collision term, there appears a conservative, Vlasov-type term with an effective Hamiltonian depending itself on $W(t)$.

Of course, the next goal would be to arrive at predictions based on kinetic theory. In [5, 6], we studied the one-dimensional model by numerically integrating the Boltzmann equation. In the spatially homogeneous case, we find exponential convergence to the steady state and the related entropy increase. However, the family of stationary solutions depends on the precise form of the dispersion relation $\omega$ and is related to the integrability of the Hubbard chain. In the more mathematical investigation [7], we examine the role of the effective Hamiltonian for $d \geq 3$. Because of the principal part, this term is in fact fairly singular and may induce rapid oscillations in the solution $W(k, t)$. In [5–7], we use an on-site interaction, which simplifies somewhat the structure of the kinetic equation, as explained in the appendix.

From a theoretical perspective, one might wonder about the structure of the higher order diagrams. For example, at order $\lambda^4$ one expects four types of diagrams:

(i) those vanishing in the kinetic limit,

(ii) non-vanishing and summing up to the term

$$\frac{1}{2} \lambda^4 t^2 C[C[W]],$$  \hspace{1cm} (88)

(iii) zero momentum transfer diagrams summing up to

$$\frac{1}{4!} \lambda^4 t^4 [R[W], [R[W], [R[W], [R[W], W]]]].$$  \hspace{1cm} (89)

(iv) the cross-terms of types (ii) and (iii).

The structure of higher order terms has already been investigated by van Hove [8] for quantum fluids and by Prigogine [9] for classical anharmonic crystals. In fact, the diagrams become rather intricate, iterated oscillatory integrals and their asymptotic is difficult to handle; see [10–13] for more recent related work.

From the rigorous perspective, the best understood model seems to be the weakly nonlinear Schrödinger equation on a lattice with the on-site interaction [14]. In this work, equilibrium time correlations are studied in the regime of small coupling and the limit equation is a version of the Boltzmann equation linearized at equilibrium. The $R$ matrix becomes just a number. Still it gives rise to rapid oscillations on the kinetic scale, and the analysis indicates that higher order terms will have a more complicated structure than anticipated in (88) and (89). In particular, the higher order zero momentum transfer diagrams generate terms diverging on the kinetic timescale. Therefore, the strategy is to first subtract the rapid oscillations related to $R$ and to show that thereby, in a certain sense, the sum of all zero momentum transfer diagrams cancels each other with a sufficiently high precision. On a technical level, this separation is achieved by the pair truncation, as explained in section 3 of [14]. After pair truncation, the diagrams are indeed separated into (i) and (ii) and follow the anticipated pattern, i.e. at order $\lambda^{2n}$ the diagrams of (ii) sum up to

$$\frac{1}{n!} (\lambda^2 t)^n C^{(n)}[W],$$ \hspace{1cm} $n$-fold iteration.  \hspace{1cm} (90)

One might hope that a similar type of analysis can be achieved for the Hubbard model, but this will be a task for the future.
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Appendix A. The collision operator

For the special case of an on-site interaction, we show that the kinetic equation (9) agrees with the one in [5–7]. Since $\bar{V} = 1$, the starting point is

$$C_1(W(t))(k) = -i[H_{\text{eff}}(k,t), W(k,t)] \quad (A.1)$$

with

$$H_{\text{eff},1} = \int_{\mathbb{R}^3} d \mathbf{k}_{234} \delta(\mathbf{k}) \mathcal{P}\left(\frac{1}{\omega}\right) (W_2 W_4 - W_3 W_5 - \text{tr}[W_4]W_2 + \text{tr}[W_5]W_2 + W_3) \quad (A.2)$$

and

$$C_2[W_1] = \pi \int_{\mathbb{R}^3} d \mathbf{k}_{234} \delta(\mathbf{k}) \delta(\omega) (\mathcal{A}[W]_{1234} + \mathcal{A}[W]_{1234}''), \quad (A.3)$$

where

$$\mathcal{A}[W]_{1234} = -W_4 \tilde{W}_3 W_2 + W_4 \text{tr}[\tilde{W}_3 W_2] - [\tilde{W}_4 W_2 - \tilde{W}_4 W_3 - \tilde{W}_3 W_2] + \tilde{W}_3 \text{tr}[W_4] - \tilde{W}_4 \text{tr}[W_3] + \text{tr}[W_3 W_2] W_1. \quad (A.4)$$

There are many possible representations of the collision operator. The goal here is to show that the form derived in section 5 agrees with the simpler expressions given in equations (A.12) and (A.13). To achieve the representation of the dissipative part given in (15), we consider $\mathcal{A}[W]_{1234} + \mathcal{A}[W]_{1234}''$ and add the zero term:

$$0 = W_1 W_2 [W_4 W_3] W_2 - W_1 W_2 W_3 W_2 W_4 + \text{tr}[W_4 W_5] W_3 W_1 - \text{tr}[W_4 W_3 W_2 W_1] + W_1 W_2 W_3 W_2 - W_1 W_2 W_3 W_2 W_4 + W_1 W_2 W_3 W_2 W_4 - W_1 W_2 W_3 W_2 W_4.$$

Then we use the symmetry $k_2 \leftrightarrow k_4$ to replace $\mathcal{A}[W]_{1234} + \mathcal{A}[W]_{1234}''$ in the first integral in (87) by

$$\tilde{W}_1 W_4 \mathcal{J}[\tilde{W}_3 W_2] + \mathcal{J}[W_2 \tilde{W}_3] W_4 \tilde{W}_1 - W_1 \tilde{W}_4 \mathcal{J}[W_3 \tilde{W}_2] - \mathcal{J}[\tilde{W}_2 W_3] \tilde{W}_4 W_1, \quad (A.6)$$

where $\mathcal{J}[W] = 1_{\mathbb{C}^2} \text{tr}[W] - W$. We again make a change of variables $k_2 \leftrightarrow k_3$, which implies that $\delta(k_1 + k_2 - k_3 - k_4) \rightarrow \delta(k_1 + k_2 - k_3 - k_4)$, and results in the integrand

$$\tilde{W}_1 W_2 \mathcal{J}[\tilde{W}_3 W_4] + \mathcal{J}[W_4 \tilde{W}_3] W_2 \tilde{W}_1 - W_1 \tilde{W}_4 \mathcal{J}[W_3 \tilde{W}_4] - \mathcal{J}[\tilde{W}_4 W_3] \tilde{W}_2 W_1. \quad (A.7)$$

The conservative part can be written as a commutator

$$\mathcal{A}[W]_{1234} = [W_4 W_3 + W_4 \text{tr}[W_2] + W_3 W_2 - W_4 W_2 - W_4 \text{tr}[W_3], W_1]. \quad (A.8)$$
In the second integral in (87) we then exchange $k_2 \leftrightarrow k_3$, leading to $\delta(k_1 - k_2 + k_3 - k_4)$ and afterwards make use of $k_3 \leftrightarrow k_4$ resulting in the integrand

$$[-W_2 + W_3 W_1 + W_4 W_2 - W_5 W_3 + W_4 \text{tr}[W_5] - W_5 \text{tr}[W_4], W_1].$$

(A.9)

Hence the second term in (87) is equal to $-i\langle f, [H_{\text{eff}}[W], W_1]\rangle$ with $H_{\text{eff}}[W]$ defined by (11). On the other hand, using the symmetry of the delta function under $k_3 \leftrightarrow k_4$, we can conclude that replacing (A.9) in the integrand by

$$\frac{1}{2}[W_2, J][W_5 W_4] + J[W_4 W_2] W_3 + W_5 J[W_2 W_4] + J[W_2 W_3 W_5, W_1]$$

(A.10)

yields the same result.

Therefore, in summary, the second-order results are compatible with defining

$$C[W](k, t) = C_0[W](k, t) + C_4[W](k, t),$$

(A.11)

where

$$C_4[W](k, t) = -i[H_{\text{eff}}(k, t), W(k, t)]$$

(A.12)

and $H_{\text{eff}}$ is defined either by (11) or by

$$H_{\text{eff},1} = -\frac{1}{2} \int d\kappa_2 d\kappa_3 d\kappa_4 \delta(k_1 - k_2 + k_3 - k_4) \mathcal{P} \left(\frac{1}{\omega_1 + \omega_2 - \omega_3 - \omega_4}\right)$$

$$\times (W_2 J[W_2 W_4] + J[W_4 W_2] W_3 + W_5 J[W_2 W_4] + J[W_2 W_3 W_5, W_1],$$

(A.13)

$\omega_i = \omega(k_i), i \in \{1, 2, 3, 4\}$, etc. (The latter form was used as the starting point in [7].)

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