Statistically anisotropic curvature perturbation generated during the waterfall

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ABSTRACT: If the waterfall field of hybrid inflation couples to a $U(1)$ gauge field, the waterfall can generate a statistically anisotropic contribution to the curvature perturbation. We investigate this possibility, generalising in several directions the seminal work of Yokoyama and Soda. The statistical anisotropy of the bispectrum could be detectable by PLANCK even if the statistical anisotropy of the spectrum is too small to detect.

KEYWORDS: Primordial curvature perturbation.
1. Introduction

Although there is so far no evidence for statistical anisotropy of the primordial curvature perturbation $\zeta$, mechanisms have been proposed for generating it. Most of them invoke a vector field.

One mechanism takes the vector field to be homogeneous during inflation, but causes significant anisotropy in the expansion [1] (for a recent review of this approach see [2]). Then the perturbations of scalar fields generated from the vacuum fluctuation will be statistically anisotropic, and so too will be $\zeta$ on the usual assumption that it originates from one or more of these perturbations.

We here invoke a different mechanism [3, 4] (for the most recent paper on this approach see [5]) #1 Taking the inflationary expansion to be practically isotropic, this

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#1 The use of a vector field to generate a contribution to $\zeta$ was first mooted in [6].
mechanism generates a perturbation of the vector field from the vacuum fluctuation, which in turn generates a contribution to $\zeta$. Results using this mechanism are at best approximate, because the unperturbed part of the vector field will cause at some level anisotropic expansion which generates additional statistical anisotropy through the first mechanism, but existing calculations ignore that effect and we will do the same.

We work with the setup of [3]. The vector field is a $U(1)$ gauge field coupled to the waterfall field of hybrid inflation.\(^*\)\(^2\) The dominant contribution to $\zeta$ is supposed to come from the perturbation of the inflaton field. But the perturbation of the gauge field is supposed to generate an additional contribution during the waterfall that ends inflation. The waterfall is taken to be practically instantaneous. We extend the original treatment of the scenario in several respects. First, we do not assume the inflaton potential $V = V_0 + m^2 \phi^2 / 2$ (which is ruled out by observation). Second, we do not assume that the perturbation of the gauge field is exactly scale-invariant. Third, we take into account the time-dependence of the gauge field.

We take for granted the main ideas of modern cosmology described for instance in [8], and use the notation and definitions of [4, 8]. The unperturbed universe has the line element

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^idx^j.$$  

(1.1)

For any smooth rotationally invariant quantity $g(x, t)$, uniquely defined during some era, we can choose a slicing (fixed $t$) and threading (fixed $x$) and then write

$$g(x, t) = g(t) + \delta g(x, t).$$

(1.2)

Going to a different slicing with a time displacement $\delta t(x, t)$, we have to first order

$$[\tilde{\delta}g(x, \tilde{t})] - \delta g(x, t) = g(x, t) - g(x, \tilde{t}) \simeq -\dot{g}(t)\delta t(x, t).$$

(1.3)

We will invoke this ‘gauge transformation’ without comment. In most cases $g$ is homogeneous on one of the slicings.

2. The curvature perturbation $\zeta$

2.1 Definition and $\delta N$ formula

To define $\zeta$ one smoothes the metric on a super-horizon scale, and adopts the comoving threading and the slicing of uniform energy density $\rho$. Then [3]

$$\zeta(x, t) \equiv \delta[\ln a(x, t)] = \delta[\ln (a(x, t)/a(t))] \equiv \delta N(x, t),$$

(2.1)

where $a(x, t)$ is the locally defined scale factor (such that a comoving volume element is proportional to $a^3(x, t)$) and $a(t)$ is its unperturbed value. The number of e-folds

\(^*\)The study of this setup with non-Abelian gauge fields can be found in [6].
of expansion $N(x, t, t_\ast)$ starts from a slice at time $t_\ast$ on which $a$ is unperturbed (‘flat slice’) and ends on a uniform $\rho$ slice at time $t$. Since the expansion between two flat slices is uniform, $\delta N$ is independent of $t_\ast$. The change in $\zeta$ between two times is

$$\zeta(x, t_2) - \zeta(x, t_1) = \delta N(x, t_1, t_2), \quad (2.2)$$

where now both the initial and final slices have uniform $\rho$.

By virtue of the smoothing, the energy conservation equation is valid locally:

$$\dot{\rho}(t) = 3 \frac{\partial a(x, t)}{\partial t} (\rho(t) + P(x, t)). \quad (2.3)$$

In consequence, $\dot{\zeta} = 0$ during an era when $P(\rho)$ is a unique function. The success of the BBN calculation shows that $P = \rho/3$ to high accuracy when cosmological scales start to enter the horizon. Then $\zeta$ has a time-independent value $\zeta(x)$ that is strongly constrained by observation. Within observational errors it is gaussian and statistically isotropic. Its spectrum is nearly independent of $k$, with

$$P_\zeta(k) \simeq (5 \times 10^{-5})^2. \quad (2.4)$$

For the reduced bispectrum $f_{NL}$, current observation give $|f_{NL}| \lesssim 100$ and barring a detection PLANCK will give $|f_{NL}| \lesssim 10$. For $f_{NL}$ to ever be observable we need $|f_{NL}| \gtrsim 1$.

We will work to first order in $\zeta$, so that

$$\zeta(x, t) = H(t) \delta t_{t\rho}, \quad (2.5)$$

where $\delta t_{t\rho}$ is the time displacement from the flat slice to the the uniform-$\rho$ slice. A second-order calculation of $\zeta$ is needed only to treat very small non-gaussianity corresponding to $|f_{NL}| \lesssim 1$.

To explain the near scale-invariance of the observed $P_\zeta(k)$, it is usually supposed that $N(x, t)$ is determined by the values of one or more fields $\phi_i(x, t)$, evaluated during inflation at an epoch $t_\ast$ when relevant scales have left the horizon:

$$N(x, t) = N(\phi_1^*(x), \phi_2^*(x), \cdots, t). \quad (2.6)$$

The fields are defined on a flat slice and denoting their values by $\phi_i^*$ we write

$$\phi_i^*(x) = \phi_i^* + \delta \phi_i^*(x) \quad (2.7)$$

and

$$\zeta(x, t) = \sum N_i \delta \phi_i^*(x) + \frac{1}{2} \sum_{ij} N_{ij} \delta \phi_i^*(x) \delta \phi_j^*(x) + \cdots, \quad (2.8)$$

where a subscript $i$ denotes $\delta/\phi_i^*$ evaluated at $\phi_i^*(x) = \phi_i^*$. The $\phi_i$ are usually taken to be scalar fields, but it has been proposed that some or all of them may be components of a vector field.
On each scale $k$, the field perturbations are generated from the vacuum fluctuation at horizon exit. Ignoring scales leaving the horizon after $t_*$ Eq. (2.8) defines a classical quantity $\zeta$, which is independent of the choice of $t_*$. In general it depends on $t$, settling down to the observed quantity $\zeta(x)$ by some time $t_f$.

Cosmological scales have a fairly narrow range $\Delta k \sim 15$ or so. Choosing $t_*$ as the epoch when the shortest scale leaves the horizon, scalar fields with the canonical kinetic term, that satisfy the slow-roll approximation, have a nearly Gaussian uncorrelated perturbations with spectrum $P_{\delta\phi} \simeq (H/2\pi)^2$. To have the observed nearly gaussian $\zeta(x)$ Eq. (2.8) has to be dominated by one or more linear terms (at least when $t = t_f$). Keeping only linear terms,

$$P_\zeta(x, t) \simeq \sum N_i^2 P_{\delta\phi_i^*} + \cdots,$$

where the terms exhibited correspond to scalar fields, and the dots indicate vector field contributions. The contribution of the latter is positive like the rest [4]. The non-linear terms may give non-gaussianity that can be observed in the future.

### 2.2 Slow-roll inflation

Inflation corresponds to $\epsilon_H < 1$ where $\epsilon_H \equiv -\dot{H}/H^2$. During inflation each coordinate wavenumber $k$ (scale) leaves the horizon when $k = aH$, and we are interested only in the era of inflation after horizon exit for the biggest observable scale $k \sim a_0H_0$. Here $H \equiv \dot{a}(t)/a(t)$, $k$ is the coordinate wavenumber and the subscript 0 denotes the present. We need $\epsilon_H \ll 1$ at least while cosmological scales leave the horizon to generate the nearly scale-invariant $P_\zeta(k)$.

We are interested in single-field slow-roll inflation. Here, the only field with significant variation during inflation is the inflaton. Its unperturbed value $\phi(t)$ satisfies the slow-roll approximation.

$$3H\dot{\phi} \simeq -V'(\phi),$$

$$\epsilon \equiv \frac{1}{2}M_P^2(V'/V)^2 \simeq \epsilon_H \ll 1$$

$$\rho(t) = 3M_P^2H^2 \simeq V(\phi).$$

The perturbation $\delta\phi_*$ can be removed by a shift $\delta t_*(x, t)$, which means that it generates a time-independent contribution to $\zeta_\phi$. Since the $f_{NL}$ generated by $\zeta_\phi$ is negligible [12], we can work to first order in $\delta\phi_*$,

$$\zeta_\phi(x) = -(H/\dot{\phi})\delta\phi_*(x).$$

Choosing $t_*$ as the epoch of horizon exit, we find the spectral index is given by

$$P_{\zeta_\phi}(k) \simeq \frac{1}{2\epsilon M_P^2} \left( \frac{H}{2\pi} \right)^2$$

$$n_{\phi}(k) - 1 \equiv dP_{\zeta_\phi}/d\ln k = 2\eta - 6\epsilon,$$
where $\eta \equiv M^2_P V''/V$ with $|\eta| \ll 1$, and the right hand sides are evaluated at the epoch of horizon exit $aH = k$.

Although it is not our central concern, we mention at this point the case of multi-field slow-roll inflation, where two or more fields vary significantly. Taking $\phi$ to be the field pointing along the trajectory at horizon exit, Eq. (2.14) still applies to that case.

If $\zeta$ depends only on the part of the action that we are considering, $\zeta_\phi(k)$ can be identified with the observed quantity $\zeta(k)$. More generally we have

$$\mathcal{P}_{\zeta_\phi}(k) \lesssim \mathcal{P}(k) \simeq (5 \times 10^{-5})^2.$$ \hspace{1cm} (2.16)

This inequality is important for two reasons. First, it makes the tensor fraction $r \leq 16 \epsilon$. Then the slow roll approximation gives what has been called the Lyth bound, on the variation $\Delta \phi$ of the inflaton field after the observable universe leaves the horizon. Without any assumption about the function $\epsilon(\phi)$ after the first few $e$-folds, one finds \cite{13}

$$10^{-1} \left( \frac{\Delta \phi}{M_P} \right)^2 \gtrsim 16 \epsilon \geq r.$$ \hspace{1cm} (2.17)

If $\epsilon(\phi(t))$ increases with time, $10^{-1}$ in the above expression is replaced \cite{14} by 0.0003. According to these results, an observable $r$ cannot be obtained with $\Delta \phi \ll M_P$ (small-field model).

The other use of the inequality is for curvaton-type models, where $\zeta$ is generated almost entirely by the perturbation of some field that has a negligible effect during inflation. Then $\mathcal{P}_{\zeta_\phi}$ will be negligible compared with $\mathcal{P}_\zeta$, and $r$ will be negligible compared with $16 \epsilon$ so that it is unobservable.

All of this assumes slow-roll inflation, in which it is assumed that there is no time-dependent field except the slowly-rolling inflaton fields. If there is such a field the shift in the initial time generated by $\delta \phi_*$ will be accompanied by a shift in the value of that field, which could allow $\zeta_\phi$ to be time-dependent and avoid the inequality (2.16).

The possibility of avoiding this inequality was mooted in \cite{13,16} but they did not find a mechanism. One can easily avoid the inequality by abandoning the canonical kinetic term for the inflaton \cite{17} and we are for the first time pointing to a possible mechanism with the canonical kinetic term.

### 3. The model

#### 3.1 Hybrid inflation

The relevant part of the action is

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M^2_P R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{1}{4} f^2(\phi) F_{\mu\nu} F^{\mu\nu} - V \right],$$ \hspace{1cm} (3.1)
with \( F_{\mu \nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu \) and \( B_\mu \) a \( U(1) \) gauge field. Following [1] we use the gauge with \( B_0 = \partial_i B_i = 0 \), and work with \( A_i \equiv B_i / a \) which is the field defined with respect to the locally orthonormal basis (as opposed to \( B_i \) which is defined with respect to the coordinate basis). The raised component is \( A^i = A_i \) (as opposed to \( B^i = B_i / a^2 \)). We also define \( W \equiv f A \), which would be the canonically normalized field if \( f \) were constant. To fix the normalization of \( f \), we set \( f = 1 \) at a time \( t_w \) just before the waterfall begins.

Before the waterfall the potential is

\[
V(\phi, \chi, A) = V_0 + V(\phi) + \frac{1}{2} m^2(\phi, A) \chi^2 + \frac{1}{4} \lambda \chi^4 + \frac{1}{2} \mu^2 A^2 \tag{3.2}
\]

\[
m^2(\phi, A) \equiv h^2 A^2 + g^2 \phi^2 - m^2. \tag{3.3}
\]

The waterfall field \( \chi \) is supposed to be the radial part of a complex field which is charged under the \( U(1) \), generating the first term of Eq. (3.3).

This is the usual hybrid inflation potential [18, 19] except for the presence of \( A \). We assume that the values of the parameters and fields give what has been called standard hybrid inflation [20]. At each location, the waterfall begins when \( m^2(\phi(x, t), A(x, t)) \) falls to zero. Before it begins, the waterfall field \( \chi \) vanishes up to a vacuum fluctuation which is set to zero, and we have slow-roll inflation with

\[
V = V_0 + V(\phi) \simeq V_0. \tag{3.4}
\]

Cosmological scales are supposed to leave the horizon before the waterfall begins. We will take \( H \) to be constant which is typically a good approximation.

### 3.2 Field equations

During the waterfall, \( \chi \) moves to it’s vev and then inflation ends. We will assume that the duration of the waterfall is so short that it can be taken to occur on a practically unique slice of spacetime. The evolution of \( \phi \) and \( A \) is therefore required only before the waterfall begins.

To work out the field equations, previous authors have taken \( f(\phi) \) to be a function of time with \( f \propto a^\alpha(t) \), and have taken spacetime to be unperturbed. Then the action (3.1) gives for the unperturbed fields

\[
\ddot{\phi}(t) + 3H \dot{\phi}(t) + V'(\phi(t)) = 0 \tag{3.5}
\]

\[
\ddot{W}(t) + 3H \dot{W}(t) + \mu^2 W(t) = 0, \tag{3.6}
\]

where

\[
\mu^2 \equiv H^2(2 + \alpha)(1 - \alpha). \tag{3.7}
\]

By virtue of the flatness conditions on the potential (\( \epsilon \ll 1 \) and \( |\eta| \ll 1 \)), the first expression is expected to give the slow-roll approximation (2.10) more or less
independently of the initial condition. Similarly, the second equation is expected to give the slow-roll approximation

\[ 3H \dot{W} \simeq -\mu^2 W \]  

if \(|\mu|^2 \ll H^2\) which we assume.

In terms of \(W\), the coupling \(h^2 A^2 \chi^2\) becomes \(\tilde{h}^2 W^2 \chi^2\), where \(\tilde{h} \equiv hf\). Before horizon exit on cosmological scales we are taking \(W\) to be a practically free field corresponding to \(\tilde{h} \ll 1\). With \(\alpha \simeq 1\) this would give at \(t = t_w\) a tiny coupling \(h \ll e^{-Nk}\) which would have practically no effect. We therefore assume \(\alpha \simeq -2\).

For the first order perturbations, \(f \propto a^\alpha\) gives

\[ \delta \ddot{\phi}_k(t) + 3H \delta \dot{\phi}_k(t) + \left((k/a)^2 + V''(\phi(t))\right) \delta \phi_k = 0 \]  

\[ \delta \ddot{W}_k(t) + 3H \delta \dot{W}_k(t) + \left((k/a)^2 + \mu^2\right) \delta W_k = 0. \]  

(3.9)  

(3.10)

Since \(\phi\) and \(W\) are slowly varying, this flat spacetime calculation is expected to hold in the perturbed universe on the flat slicing. It is expected because \([8, 21]\) the effect of the metric perturbation (back-reaction) on Eq. (3.10) is proportional to the small quantity \(\dot{W}\).

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Since \(f(\phi)\), the choice \(f \propto a^\alpha\) corresponds to

\[ f \propto \exp \left(\alpha \int_{\phi}^{\phi_0} \frac{1}{\sqrt{2\epsilon(\phi)}M_P^{-1}} d\phi\right). \]  

(3.11)

This gives the perturbation

\[ \delta f/f = \frac{\alpha}{\sqrt{2\epsilon M_P}} \delta \phi. \]  

(3.12)

Since \(f\) is a function of \(\phi\), the term \(\propto f F_{\mu \nu} F^{\mu \nu}\) in the action couples \(\phi\) and \(W\) so that the right hand sides of Eqs. (2.10), (3.8), (3.9) and (3.10) are nonzero. We calculate them in the Appendix, and show that they are negligible if

\[ \frac{\rho_B}{\epsilon \rho} = \frac{1}{2} \frac{\dot{W}^2}{\epsilon \rho} \simeq \frac{1}{2} \frac{\mu^2 W^2}{\epsilon V} \simeq \frac{1}{6} \frac{W^2}{\epsilon M_P^2} \ll 1, \]  

(3.13)

where \(\rho_W\) is the energy density of \(W\). We will assume this condition. Note that it implies \(\rho_W \ll \rho\), which is anyway needed because we are taking the expansion of the universe to be isotropic. From Eq. (2.10), the condition is guaranteed if \(W/H \lesssim 10^5\).

3.3 Spectrum of \(W\)

The evolution equation for \(W(x, t)\) is the same as that of a free scalar field with mass-squared \(\mu^2\), and we are assuming \(|\mu|^2 \ll H^2\). Treating \(\delta W_k\) as an operator and assuming the vacuum state well before horizon exit gives the approximately scale-independent vacuum expectation value

\[ \frac{k^3}{2\pi^2} \langle \delta W_k^i(t) \delta W_k^j(t) \rangle = \left(\delta^{ij} - \hat{k}^i \hat{k}^j\right) \delta^3(k + k') \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{a(t)H}\right)^{n_{vec} - 1} \]  

(3.14)  

\[ n_{vec} - 1 = 2\mu^2/3H^2, \]  

(3.15)
where hats denote unit vectors. According to Eq. (3.8), $\delta W_k$ has constant phase which means that it can be treated as a classical quantity with this correlator.

We are interested in $t = t_w$, when

$$\left( \frac{k}{a(t_w)H} \right)^{n_{\text{vec}} - 1} = e^{-N_k(n_{\text{vec}} - 1)} \equiv e^x. \quad (3.16)$$

Also, we are interested in cosmological scales, which have a range $\Delta N_k \sim 15$ and a typical central value $N_k \sim 50$.

The decomposition

$$W(x, t) = W(t) + \delta W(x, t) \quad (3.17)$$

is made in some box of coordinate size $L$ around the observable universe, with $W(t)$ the average within the box. After smoothing on a cosmological scale $k$, the spatial average of $(\delta W)^2$ (evaluated within a region not many orders of magnitude bigger than the observable universe) is of order $\ln(kL)H$ and we assume that the box is not too big, $\ln(kL)$ roughly of order 1. Guided by these results, we assume $W(t) \gg H$, which is reasonable because $W^2(t)$ at a typical position is expected to be at least of order the mean square of $(\delta W)^2$ evaluated within a much larger box \[22\].

### 4. Including the waterfall contribution

#### 4.1 End-of-inflation formula

At an epoch $t_+$ just after inflation ends,

$$\zeta(x, t_+) = \zeta_+(x) \equiv \zeta_\phi(x) + \zeta_w(x), \quad (4.1)$$

where $\zeta_w$ is the waterfall contribution.

To calculate $\zeta_w$, we suppose that the waterfall happens very quickly so that it can be regarded as taking place on a single spacetime slice. Then \[20, 23\]

$$\zeta_w(x) = H \delta t_{12}(x) = H \left[ \frac{\delta \rho_w(x)}{\dot{\rho}(t_w)} - \frac{\delta \rho_w(x)}{\dot{\rho}(t_+)} \right] \approx H \frac{\delta \rho_w(x)}{\dot{\rho}(t_w)} \approx H \delta t_{\rho w}. \quad (4.2)$$

In this equation, $\delta t_{12}(x)$ is the proper time elapsing between a uniform-$\rho$ slice at time $t_w$ just before the waterfall, and a uniform-$\rho$ slice at time $t_+$ just after the waterfall. Because $|\dot{\rho}|$ is much smaller during inflation than afterwards, $\delta t_{12}$ is practically the same as $\delta t_{\rho w}$, the displacement from the initial uniform-$\rho$ slice to the waterfall slice. Using Eq. (2.3), we see that

$$\zeta(x, t_+) = H \delta t(x), \quad (4.3)$$

where $\delta t$ is the displacement from the flat slice at $t_w$ to the waterfall slice.

This end-of-inflation formula actually holds if the waterfall slice is replaced by any sufficiently brief transition from inflation to non-inflation. In \[20\] it is invoked
for the transition beginning during the waterfall, at the epoch when the evolution of $\chi$ becomes non-linear. We are here applying it to the entire waterfall. It was first given \[23\] with $A$ in Eq. (3.3) replaced by a scalar field. In \[23\] the slope of the potential in the $A$ direction was assumed to be negligible corresponding to single-field hybrid inflation, and the same assumption was made in several later papers \[24\]. The assumption was relaxed in \[25, 26, 27\], corresponding to what has been called \[26\] multi-brid inflation. Following \[3\] we are here taking $A$ to be the magnitude of $U^1(1)$ gauge field. One can also replace $A$ by a non-Abelian gauge field \[7, 28, 29\].

Before continuing, let us ask what is required to make the waterfall sufficiently brief. Just to have the error $|\Delta \zeta_w| \ll |\zeta_w|$, one presumably needs $\Delta t(x) \ll |\delta t_{12}(x)|$, which is equivalent to \[20\].

\[ H \Delta t \ll P_{\zeta}^{1/2} \leq P_{\zeta}^{1/2} = 5 \times 10^{-5}. \] (4.4)

During the waterfall $m^2(\phi, A)$ goes from 0 to $-m^2$ of the waterfall, with $m^2 \gtrsim H^2$. We therefore expect $\Delta t$ to be at least of order $1/m$, and we need $m/H \ll \sqrt{M_P/H}$ so that $\lambda \ll 1 \ [23]$. Hence Eq. (4.4) requires an inflation scale $H/M_P \ll 10^9$ GeV or $V^{1/4} \ll 10^{14}$ GeV. This rules out GUT hybrid inflation ($V^{1/4} \sim 10^{15}$ or so) but easily allows inflation at the scale of supersymmetry breaking ($H$ of order the gravitino mass $\lesssim 10^5$ GeV or so).

A stronger requirement might be needed to justify a calculation of the non-gaussianity parameter $f_{NL}$, because it refers to the non-gaussian part of $\zeta$ that is only of order $P_{\zeta}^{1/2}f_{NL} \lesssim 10^{-3}$. A reasonable estimate for $|\Delta \zeta_w/\zeta_w|$ might be $H \Delta t/P_{\zeta}^{1/2}$. Then, in the worst case that $\Delta \zeta_w$ is completely non-gaussian, one would require the very low inflation scale $H/M_P \ll 10^{-18}f_{NL}^2$. We proceed on the assumption that the inflation scale is sufficiently small.

### 4.2 Waterfall contribution in our model

Without at first specifying the nature of $A$, we now calculate $\delta t$. The fields on the waterfall slice have values given by $m^2(\phi_w(x), A_w(x)) = 0$. It was noted in \[7\] that the time dependence of a waterfall slice could be important. Thus let us define a ‘time-dependent waterfall slice’ $\phi_w(x, t)$ by

\[ m^2(\phi_w(x, t), A(x, t)) = 0. \] (4.5)

(If this equation has more than one solution $\phi_w(x, t)$ we choose one of them.) If $\dot{\phi_w}$ is negligible, the waterfall occurs when $\phi(x, t)$ falls to the practically time-independent waterfall slice $\phi_w(x)$. If instead $\dot{\phi}$ is negligible, the waterfall occurs when the time-dependent waterfall slice $\phi_w(x, t)$ meets the practically time-independent field value

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#3The second inequality allows for contributions to $\zeta(x)$ that might be generated after inflation by fields different from $\phi$ and $A$, and assumes that the contribution of the latter undergoes no further change.
\( \phi(x) \). Since we deal with hybrid inflation \( \dot{\phi} \) is negative, but \( \dot{\phi}_w \) might have either sign. If it is also negative we need \( |\dot{\phi}_w| < |\dot{\phi}| \) or inflation will never end.

If \( \delta t(x) \) is the displacement from the flat slice at \( t_w \) to the waterfall slice, this gives to first order in \( \delta t \)

\[
\begin{align*}
\phi(x, t_w + \delta t(x)) &= \phi(t_w) + \delta \phi(x, t_w) + \dot{\phi}(t_w)\delta t(x) \\
\phi_w(x, t_w + \delta t(x)) &= \phi_w(t_w) + \delta \phi_w(x, t_w) + \dot{\phi}_w(t_w)\delta t(x),
\end{align*}
\]

where \( \delta \phi_w \) is defined on the flat slice. Setting \( \delta t = 0 \) gives \( \phi(t_w) = \phi_w(t_w) \). Evaluating \( \delta t \) we have

\[
\zeta(x, t_+) = H\delta t(x) = H \frac{\delta \phi_w(x, t_w) - \delta \phi(x, t_w)}{\dot{\phi}(t_w) - \dot{\phi}_w(t_w)}.
\]

Now we invoke Eq. (3.3). Discounting the strong cancellation \( m^2 \simeq h^2 A^2 \) it gives

\[
\dot{\phi}_w(x, t) = \frac{1}{g} \left( m^2 - h^2 A^2(x, t) \right)^{1/2} \simeq \frac{m}{g} \left( 1 - \frac{m^2}{2mg} \right). \tag{4.9}
\]

Using Eqs. (2.13), (3.8), (3.12) and (4.9) we get#4

\[
\zeta(x, t_+) = \zeta_\phi(x) \left( 1 + \frac{\mu^2}{H^2} \left( \frac{2XA^2}{1 + 2XA^2} \right) \right) + \frac{\hat{\zeta}_w}{1 + 2XA^2}, \tag{4.10}
\]

where \( \zeta_\phi(x) \) is defined by Eq. (2.13) and

\[
\hat{\zeta}_w = -X \left( W(t_w) \cdot \delta W(x, t_w) + (\delta W(x, t_w))^2 \right), \tag{4.11}
\]

\[
X \equiv h^2/\sqrt{2\pi mg}. \tag{4.12}
\]

Previous authors except [30] ignored the time-dependence of \( A \), which means that they implicitly set \( \dot{\phi}_w(t_w) = 0 \) to obtain \( \zeta_w = \hat{\zeta}_w \).#5

Taking \( t_* = t_w \), the second term of Eq. (4.10) is the contribution of \( \delta W_* \) to \( \zeta \), which means that the first term is the contribution of \( \delta \phi_* \). It differs slightly from the result found earlier in Eq. (2.13), but the difference is not significant; to calculate the contribution of \( \delta \phi_* \) taking account of \( W \), one would have to include the anisotropy of the expansion of the universe caused by \( W \), which is beyond the scope of our investigation. Therefore, at the level of our calculation there is no change in the usual assumption that \( \zeta_\phi(x, t) \) is constant.

The formalism that we have given involves \( \delta \phi_w \), which is a function of \( \delta A \) and hence of both \( \delta \phi \) and \( \delta W \). A more direct approach is to use \( \dot{\phi}_w(t) \) defined by

\[
m^2 \left( \dot{\phi}_w, (W/f(\dot{\phi}_w))^2 \right). \tag{4.13}
\]

Then \( \dot{\phi}_w \) is a function only of \( W \), and

\[
\frac{\dot{\phi}_w}{\dot{\phi}} = -\frac{\mu^2}{H^2} \frac{2XA^2}{1 + 2XA^2}, \tag{4.13}
\]

leading directly to Eq. (4.10).

---

#4Terms involving a product of \( \delta \phi \) with itself or \( \delta W \) are dropped because they are negligible.

#5The spectrum of the waterfall contribution found in [30] is negligible (smaller than ours by a factor \( (1 + 2XA^2)^2 e^{-2N_*} \)). We have not been able to follow this calculation, which is not from first principles because \( A \) is treated as a scalar.
4.3 Anisotropic spectrum and bispectrum

Since $W \gg H$, the linear term of Eq. (4.11) dominates, leading to \[3, 4\]

$$P_\zeta(k) = P_{\zeta}^{\text{iso}} \left[ 1 - \beta \left( \hat{\mathbf{A}} \cdot \mathbf{k} \right)^2 \right]$$  \hspace{1cm} (4.14)

$$P_{\zeta}^{\text{iso}} = \frac{P_{\zeta}(1 + \beta)}{1 + 2XH^2},$$ \hspace{1cm} (4.15)

where

$$\beta = \frac{h^4 A^2(t_w)}{m^2 g^2} e^x.$$ \hspace{1cm} (4.16)

Current observation requires $\beta \lesssim 10^{-1}$, and barring a detection PLANCK will give $\beta \lesssim 10^{-2}$ \[31\]. Using Eqs. (2.4) and (2.14), the observed value of $P_\zeta$ requires

$$\frac{1}{2\epsilon M_p^2} \left( \frac{H}{2\pi} \right)^2 \simeq (5 \times 10^{-5})^2 (1 + 2XA^2)^2.$$ \hspace{1cm} (4.17)

Including the second term of Eq. (4.11) we find \[3, 4\]

$$f_{\text{NL}} = f_{\text{NL}}^{\text{iso}} \left( 1 + f_{\text{ani}}(k_1, k_2, k_3) \right)$$ \hspace{1cm} (4.18)

where

$$f_{\text{ani}} = -\frac{(\hat{\mathbf{A}} \cdot \mathbf{k}_1)^2 - (\hat{\mathbf{A}} \cdot \mathbf{k}_2)^2 + (\mathbf{k}_1 \cdot \mathbf{k}_2)(\hat{\mathbf{A}} \cdot \mathbf{k}_1)(\hat{\mathbf{A}} \cdot \mathbf{k}_2)}{\sum k_i^3/k_3^3} + 2 \text{ perms.}$$ \hspace{1cm} (4.19)

(with $k_1 + k_2 + k_3 = 0$) and

$$\frac{6}{5} f_{\text{NL}} = \frac{1 + 2XA^2}{XA^2} \beta^2 \simeq 3 \left( 100\beta \right)^{3/2} \frac{H}{A(t_w)} 1 + 2XA^2 e^{x/2}.$$ \hspace{1cm} (4.20)

(For the final expression we used Eqs. (2.14), (4.16), and (4.17).) The last two factors, omitted in the original calculation \[3\], allow PLANCK to detect $f_{\text{NL}}^{\text{iso}}$ even if it does not detect $\beta$.

This calculation ignores the time-dependence of $\epsilon$. Allowing time-dependence for $\epsilon$ would multiply $\zeta_w$ by a factor $[\epsilon(t_{\cos})/\epsilon(t_w)]^{1/2}$, where $t_{\cos}$ is the epoch of horizon exit for a typical cosmological scale. The factor might be significantly different from 1, but there is little point in including it because its effect is indistinguishable from the effect of the tilt factor $e^x$.

5. Conclusion

If the waterfall of hybrid inflation is sufficiently brief, it takes place on a practically unique slice of spacetime. Then the waterfall slice contributes to the curvature
perturbation \( \zeta \), if its location depends on some field \( A \) different from both the inflaton and the waterfall field.

We generalised in several directions the model of Yokoyama and Soda [3], that takes \( A \) to be a \( U(1) \) gauge field. The model makes \( \zeta \) statistically anisotropic, and we find that the prediction for \( f_{\text{NL}} \) could be verified by PLANCK, even if the prediction for the anisotropy of \( P_\zeta \) is too small to be detected.

The weak point of the model is the special form Eq. (3.11), that is required to get the gauge kinetic function \( f(\phi) \propto a^{-2} \). We are not aware of any well-motivated hybrid inflaton potential that would lead to a well-motivated \( f(\phi) \). This is in contrast with the case of non-hybrid inflation [32], where one can take \( f(\phi) \) and \( V(\phi) \) to have exponentially increasing behaviour that might be reasonable in string theory [33], and which could correspond to an attractor (late-time limit) [34].

In the course of our investigation we noticed that the presence, during slow-roll inflation, of a time-dependent field different from the inflaton might allow a significant decrease in the spectrum of the curvature perturbation after horizon exit. That does not however happen in our case.

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A. Equations of Motion for \( \phi(x, t) \) and \( W(x, t) \)

Extremizing the action in Eq. (3.1) with respect to fields \( \phi, B_\mu \) and their derivatives we obtain field equations

\[
[\partial_\mu + \partial_\mu \ln \sqrt{-g}] \partial^\mu \phi + V'(\phi) + \frac{1}{2} f f'(\phi) F_{\mu\nu} F^{\mu\nu} = 0; \quad (A.1)
\]

\[
[\partial_\mu + \partial_\mu \ln \sqrt{-g}] f F^{\mu\nu} = 0, \quad (A.2)
\]

where \( g \equiv \det(g_{\mu\nu}) \) and \( f_{,\phi} \equiv \partial f / \partial \phi \). Choosing the temporal gauge \( B_0 = 0 \) and a line element of the unperturbed universe in Eq. (1.1), one finds equations of motion for the fields \( \phi(x, t) \) and \( B(x, t) \)

\[
\ddot{\phi} + 3H \dot{\phi} - a^{-2} \nabla^2 \phi + V'(\phi) = -\frac{1}{2} f(\phi) f'(\phi) F_{\mu\nu} F^{\mu\nu}, \quad (A.3)
\]

\[
\ddot{B}_i + \left( H + 2 \frac{\dot{f}}{f} \right) \dot{B}_i - a^{-2} \nabla^2 B_i = a^{-2} \partial_j f \partial_j B_i, \quad (A.4)
\]
Recasting the above equations in terms of $W \equiv fB/a$ and dropping gradient terms, one arrives at equations of motion for homogeneous fields $\phi(t)$ and $W(t)$

$$\ddot{\phi} + 3H\dot{\phi} + V' = \frac{f'(\phi)}{f} \left[ \dot{W} + \left( H - \frac{j}{f} \right) W \right]^2,$$

(A.5)

$$\ddot{W} + 3H\dot{W} + \left( 2H^2 - H\frac{j}{f} - \frac{j^2}{f} \right) W = 0,$$

(A.6)

where we also used $\dot{H} \simeq 0$.

Decomposing the field $W(x, t)$ as in Eq. (3.17) and similarly the field $\phi(x, t)$, we find equations of motion for perturbations $\delta \phi(x, t)$ and $\delta W(x, t)$ from Eqs. (A.4) and (A.4). Keeping only the first order terms and switching to the Fourier space they become

$$\ddot{\delta \phi} + 3H\dot{\delta \phi} + (k/a)^2 \delta \phi + V''\delta \phi =$$

$$= 2\frac{f'}{f} \left[ \dot{W} + \left( H - \frac{j}{f} \right) W \right] \left[ \dot{\delta W} + \left( H - \frac{j}{f} \right) \delta W - \delta \left( \frac{j}{f} \right) W \right],$$

(A.7)

$$\delta \ddot{W} + 3H\delta \dot{W} + \left( 2H^2 - H\frac{j}{f} - \frac{j^2}{f} \right) \delta W - a^{-2}\nabla^2 \delta W =$$

$$= \left[ H\delta \left( \frac{j}{f} \right) + \delta \left( \frac{j}{f} \right) + \frac{f'}{f} \left( \frac{k}{a} \right)^2 \delta \phi \right] W.$$  

(A.8)

For exponentially varying gauge kinetic function $f$ in Eq. (3.11) the above expressions become

$$\ddot{\delta \phi} + 3H\dot{\delta \phi} + (k/a)^2 \delta \phi + V''\delta \phi =$$

$$= \frac{2\alpha}{\sqrt{2\epsilon M_P}} \left[ \dot{W} + H (1 - \alpha) W \right] \left[ \dot{\delta W} + H (1 - \alpha) \delta W + \alpha \frac{H}{\phi} W \delta \phi \right],$$

(A.9)

$$\delta \ddot{W} + 3H\delta \dot{W} + \mu^2 \delta W + \left( \frac{k}{a} \right)^2 \delta W =$$

$$= \frac{\alpha W}{\sqrt{2\epsilon M_P}} \left[ \ddot{\delta \phi} + H (1 + 2\alpha) \dot{\delta \phi} + \left( \frac{k}{a} \right)^2 \delta \phi \right],$$

(A.10)

where $\mu^2 = (2 + \alpha)(1 - \alpha)H^2$ and $\alpha \simeq -2$.

The energy density of the vector field in Eq. (3.11) is given by $\rho_B(x, t) = -f^2 F_{\mu\nu} F^{\mu\nu}/4$. From this it is easy to see that the background value of $\rho_B(x, t)$ is given by

$$\rho_B(t) = \frac{1}{2} f^2 \left( \frac{\dot{B}}{a} \right)^2 = \frac{1}{2} \left[ \dot{W} + \left( H - \frac{j}{f} \right) W \right]^2 \simeq \frac{1}{2} H^2 W^2.$$

(A.11)
The right hand side of Eq. (A.5) is negligible if \( \rho_B \) satisfies Eq. (3.13). We now show that the same is true of the right hand sides of Eqs. (A.9) and (A.10). At the epoch \( k \sim aH \), the terms on the left hand sides are of order \( H^3 \) and Eq. (3.13) ensures that the right hand sides are indeed much smaller. At the epoch \( aH/k = \exp(N_k(t)) \gg 1 \), the first term of each left hand side is negligible. The other two terms are of order \( |\eta| \equiv |V''|/3H^2 \) for Eq. (A.9) and of order \( |\eta_W| \equiv |\mu|^2/3H^2 \) for Eq. (A.10). Eq. (3.13) ensures that the right hand side of Eq. (A.9) is negligible, and it ensures that the right hand side of Eq. (A.10) is negligible if also \( |\eta_W| \gg 10^{-5} \). But the latter condition is irrelevant, because its violation makes the time-dependence of \( W \) (coming then from the right hand side) negligible.

References

[1] S. Kanno, M. Kimura, J. Soda and S. Yokoyama, “Anisotropic Inflation from Vector Impurity,” JCAP 0808 (2008) 034.

[2] J. Soda, “Statistical Anisotropy from Anisotropic Inflation,” arXiv:1201.6434 [hep-th].

[3] S. Yokoyama and J. Soda, “Primordial statistical anisotropy generated at the end of inflation,” JCAP 0808 (2008) 005.

[4] K. Dimopoulos, M. Karciauskas, D. H. Lyth and Y. Rodriguez, “Statistical anisotropy of the curvature perturbation from vector field perturbations,” JCAP 0905 (2009) 013.

[5] K. Dimopoulos and M. Karciauskas, “Parity Violating Statistical Anisotropy,” arXiv:1203.0230 [hep-ph].

[6] K. Dimopoulos, “Can a vector field be responsible for the curvature perturbation in the Universe?,“ Phys. Rev. D 74 (2006) 083502.

[7] M. Karciauskas, “The Primordial Curvature Perturbation from Vector Fields of General non-Abelian Groups,” JCAP 1201, 014 (2012).

[8] D. H. Lyth and A. R. Liddle, The primordial density perturbation, Cambridge University Press, 2009; http://astronomy.sussex.ac.uk/ andrewl/PDP/errata.pdf; http://astronomy.sussex.ac.uk/ andrewl/PDP/extensions.pdf.

[9] A. A. Starobinsky, “Multicomponent de Sitter (inflationary) stages and the generation of perturbations“, Pisma Zh. Eksp. Teor. Fiz. 42 (1985) 124 [JETP Lett. 42 (1985) 152]; M. Sasaki and E. D. Stewart, “A General Analytic Formula For The Spectral Index Of The Density Perturbations Produced During Inflation,” Prog. Theor. Phys. 95 (1996) 71; D. H. Lyth, K. A. Malik, and M. Sasaki, “A general proof of the conservation of the curvature perturbation,” JCAP 0505
(2005) 004; D. H. Lyth and Y. Rodriguez, “The inflationary prediction for primordial non-gaussianity,” Phys. Rev. Lett. 95 (2005) 121302.

[10] E. Komatsu, N. Afshordi, N. Bartolo, D. Baumann, J. R. Bond, E. I. Buchbinder, C. T. Byrnes and X. Chen et al., “Non-Gaussianity as a Probe of the Physics of the Primordial Universe and the Astrophysics of the Low Redshift Universe,” arXiv:0902.4759 [astro-ph.CO].

[11] D. H. Lyth and Y. Rodriguez, “The Inflationary prediction for primordial non-Gaussianity,” Phys. Rev. Lett. 95, 121302 (2005).

[12] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP 0305, 013 (2003).

[13] D. H. Lyth, ‘What would we learn by detecting a gravitational wave signal in the cosmic microwave background anisotropy?,” Phys. Rev. Lett. 78, 1861 (1997).

[14] L. Boubekeur and D. H. Lyth, “Hilltop inflation,” JCAP 0507, 010 (2005).

[15] N. Bartolo, E. W. Kolb and A. Riotto, “Post-inflation increase of the cosmological tensor-to-scalar perturbation ratio,” Mod. Phys. Lett. A 20, 3077 (2005).

[16] M. S. Sloth, “Suppressing super-horizon curvature perturbations?,” Mod. Phys. Lett. A 21, 961 (2006).

[17] V. F. Mukhanov and A. Vikman, “Enhancing the tensor-to-scalar ratio in simple inflation,” JCAP 0602, 004 (2006).

[18] A. D. Linde, “Axions in inflationary cosmology,” Phys. Lett. B 259 (1991) 38.

[19] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, “False vacuum inflation with Einstein gravity,” Phys. Rev. D 49, 6410 (1994).

[20] D. H. Lyth, “The hybrid inflation waterfall and the primordial curvature perturbation,” arXiv:1201.4312 [astro-ph.CO].

[21] A. R. Liddle and D. H. Lyth, Cosmological Inflation and Large-Scale Structure, Cambridge University Press, 2000.

[22] D. H. Lyth, “The curvature perturbation in a box,” JCAP 0712 (2007) 016.

[23] F. Bernardeau and J. P. Uzan, “Inflationary models inducing non-gaussian metric fluctuations,” Phys. Rev. D 67 (2003) 121301; F. Bernardeau, L. Kofman and J. P. Uzan, “Modulated fluctuations from hybrid inflation,” Phys. Rev. D 70 (2004) 083004; D. H. Lyth, “Generating the curvature perturbation at the end of inflation,” JCAP 0511, 006 (2005).

[24] L. Alabidi and D. Lyth, “Curvature perturbation from symmetry breaking the end of inflation,” JCAP 0608 (2006) 006; D. H. Lyth and A. Riotto, “Generating the
Curvature Perturbation at the End of Inflation in String Theory, "Phys. Rev. Lett. 97 (2006) 121301; M. P. Salem, "On the generation of density perturbations at the end of inflation," Phys. Rev. D 72 (2005) 123516; L. Leblond and S. Shandera, "Cosmology of the Tachyon in Brane Inflation," JCAP 0701 (2007) 009; B. Dutta, L. Leblond and J. Kumar, "Tachyon Mediated Non-Gaussianity," Phys. Rev. D 78 (2008) 083522; C. -M. Lin, "Large non-Gaussianity generated at the end of Extended D-term Hybrid Inflation," arXiv:0908.4168 [hep-ph]; T. Suyama, T. Takahashi, M. Yamaguchi and S. Yokoyama, "On Classification of Models of Large Local-Type Non-Gaussianity," JCAP 1012, 030 (2010).

[25] F. Vernizzi and D. Wands, "Non-gaussianities in two-field inflation," JCAP 0605 (2006) 019.

[26] M. Sasaki, "Multi-brid inflation and non-Gaussianity," Prog. Theor. Phys. 120, 159 (2008).

[27] A. Naruko and M. Sasaki, "Large non-Gaussianity from multi-brid inflation," Prog. Theor. Phys. 121 (2009) 193; H. -Y. Chen, J. -O. Gong and G. Shiu, "Systematics of multi-field effects at the end of warped brane inflation," JHEP 0809 (2008) 011; H. -Y. Chen, J. -O. Gong and G. Shiu, "Systematics of multi-field effects at the end of warped brane inflation," JHEP 0809 (2008) 011; C. T. Byrnes, "Large non-Gaussianity from two-component hybrid inflation," JCAP 0902 (2009) 017; L. Alabidi, K. Malik, C. T. Byrnes and K. -Y. Choi, "How the curvaton scenario, modulated reheating and an inhomogeneous end of inflation are related," JCAP 1011 (2010) 037;

[28] N. Bartolo, E. Dimastrogiovanni, S. Matarrese and A. Riotto, "Anisotropic bispectrum of curvature perturbations from primordial non-Abelian vector fields," JCAP 0910, 015 (2009); N. Bartolo, E. Dimastrogiovanni, S. Matarrese and A. Riotto, "Anisotropic Trispectrum of Curvature Perturbations Induced by Primordial Non-Abelian Vector Fields," JCAP 0911, 028 (2009).

[29] M. Karciaskas, "Generating $\zeta$ with non-Abelian Vector Fields," arXiv:1202.3133 [astro-ph.CO].

[30] R. Emami and H. Firouzjahi, "Issues on Generating Primordial Anisotropies at the End of Inflation," JCAP 1201, 022 (2012).

[31] A. R. Pullen and M. Kamionkowski, "Cosmic Microwave Background Statistics for a Direction-Dependent Primordial Power Spectrum," Phys. Rev. D 76 (2007) 103529.

[32] B. Ratra, "Cosmological 'seed' magnetic field from inflation," Astrophys. J. 391 (1992) L1.

[33] J. Martin and J. 'i. Yokoyama, "Generation of Large-Scale Magnetic Fields in Single-Field Inflation," JCAP 0801 (2008) 025 [arXiv:0711.4307 [astro-ph]].
[34] M. -a. Watanabe, S. Kanno and J. Soda, “Inflationary Universe with Anisotropic Hair,” Phys. Rev. Lett. 102 (2009) 191302 [arXiv:0902.2833 [hep-th]]. S. Kanno, J. Soda and M. -a. Watanabe, “Anisotropic Power-law Inflation,” JCAP 1012 (2010) 024 [arXiv:1010.5307 [hep-th]]. R. Emami, H. Firouzjahi, S. M. Sadegh Movahed and M. Zarei, “Anisotropic Inflation from Charged Scalar Fields,” JCAP 1102 (2011) 005 [arXiv:1010.5495 [astro-ph.CO]]. J. M. Wagstaff and K. Dimopoulos, “Particle Production of Vector Fields: Scale Invariance is Attractive,” Phys. Rev. D 83 (2011) 023523 [arXiv:1011.2517 [hep-ph]].

[35] K. Dimopoulos, M. Karciauskas and J. M. Wagstaff, “Vector Curvaton with varying Kinetic Function,” Phys. Rev. D 81 (2010) 023522 [arXiv:0907.1838 [hep-ph]].