We discuss topological theories, arising from the general $\mathcal{N} = 2$ twisted gauge theories. We initiate a program of their study in the Gromov-Witten paradigm. We re-examine the low-energy effective abelian theory in the presence of sources and study the mixing between the various $p$-observables. We present the twisted superfield formalism which makes duality transformations transparent. We propose a scheme which uniquely fixes all the contact terms. We derive a formula for the correlation functions of $p$-observables on the manifolds of generalized simple type for $0 \leq p \leq 4$ and on some manifolds with $b_2^+ = 1$. We study the theories with matter and explore the properties of universal instanton. We also discuss the compactifications of higher dimensional theories. Some relations to sigma models of type A and B are pointed out and exploited.
1. Introduction and summary

1.1. Motivations

The study of topological theories has several important reasons even in the era of string duality. The first reason has to do with the sensitivity of certain topological correlation functions to the geometry of the space of vacua of the physical theory. This may provide a check on many conjectures on non-perturbative behavior of supersymmetric field theories in various dimensions. The second reason is the desire to understand the proper structure of the space-time in four dimensional field theories in the following sense. The prescription of evaluation of correlations functions in topological theory relies on a particular way the moduli space of distinct points on the space-time manifold is compactified. In most two dimensional theories this compactification is that of Deligne-Mumford. The proper analogue of this construction in four dimensions is unknown\footnote{There exists an analogue of Deligne-Mumford compactification of the space of configurations of points on complex surfaces}. In order to get a hint of how this space might look like we propose to study the contact terms in the topological gauge theory. We expect that the algebraic structures arising in the course of the study would generalize the associativity (WDVV) equations and Whitham hierarchies in two dimensions.

One of the sources of the duality revolution is the solution of N. Seiberg and E. Witten\cite{1} of $\mathcal{N}=2$ supersymmetric Yang-Mills theory which has been tested in many indirect ways but never directly. The solution, among other things, predicts the formula for the effective coupling constant of the low-energy theory as a function of the order parameter $u$ (for concreteness we talk about $SU(2)$ theory):

$$\tau(u) = \frac{2i}{\pi} \log \left( \frac{u}{\Lambda^2} \right) + \sum_{n=1}^{\infty} \tau_n \left( \frac{\Lambda^2}{u} \right)^{2n}$$

(1.1)

The coefficients $\tau_n$ are claimed to be instanton corrections. The direct test of (1.1) would involve integration over the moduli space of instantons on $\mathbb{R}^4$ of certain form. This integration can be shown to localize onto the space of point-like instantons\cite{2} and has a potential divergence in it related to the fact that the space of point-like instantons is non-compact (it is a resolution of singularities of $S^k\mathbb{R}^4$). Although this difficulty seems to be avoidable by appropriate regularization\cite{2} so far no substantial success has been achieved on this route (nevertheless, see the related discussion in \cite{3}). Instead, suppose that one may calculate

\footnote{There exists an analogue of Deligne-Mumford compactification of the space of configurations of points on complex surfaces}
exactly some correlation functions in the theory without using the form of the effective action (we shall make this statement precise later) and compare it to the computation using the techniques of effective actions. Then, if the correlators “probe” the moduli space of vacua sufficiently well then the function $\tau(u)$ can be reconstructed. The tricky point of this idea is that the exactly computable quantities are the correlation functions in the topological (twisted) theory. The latter has more freedom in the choice of action then the original physical theory. In particular, one may take the limit of zero coupling in the non-abelian theory while keeping the representatives of the observables being such that the resulting finite-dimensional counting problem has a well-defined formulation\(^2\). In the context of gauge theory such a counting problem is equivalent to the way S. Donaldson formulated his invariants in the language of problems of “generic position” (for the gauge groups $SU(2)$ and $SO(3)$). The insight coming from the equivalence of Donaldson theory and twisted version of $\mathcal{N} = 2$ super-Yang-Mills theory is that the same problem may be addressed in the infrared limit. It seems that the well-defined ultraviolet problem in the field-theoretic formulation involves the definition of the compactified moduli space of distinct points on the space-time manifold $\Sigma$. The infrared theory contains the information about this compactification in the contact terms between the observables. In the context of gauge theory these contact terms can be studied using severe constraints of modular invariance and ghost number anomalies.

1.2. Extract of Gromov-Witten theory

The story above has a counterpart in the context of two dimensional field theories. Namely, let us consider the conformal theory which is type A topological sigma model in the limit of infinite volume of the target space. One can show that such a limit exists by passing to the first order action of a curved $\beta - \gamma$ system perturbed by marginal (but not exactly marginal) operator proportional to inverse metric. The unperturbed system is naturally an interacting\(^3\) conformal theory. Such a conformal model could be consistently coupled to topological gravity to produce Gromov-Witten (GW) invariants\(^4\). Recall, that GW invariants are the maps from the $n$-th tensor power of the space of zero-observables to the cohomologies of Deligne-Mumford compactification $\bar{M}_{g,n}$ of the

\(^2\) It is important to understand that the theory is non-trivial even at zero coupling precisely due to the existence of instantons

\(^3\) i.e. nonlinear
moduli space of complex structures of Riemann surfaces with \( n \) distinct marked points. The correlators of \( p \)-observables can be interpreted as the integrals of the GW invariants over some cycles in \( \bar{M}_{g,n} \), corresponding to the fixed complex structure of the curve \( \Sigma \) without marked points.

The important and rather subtle point is that the compactification of the certain spaces is necessary for producing field theoretic interpretation of the enumerative problems while in most cases the problems themselves are formulated without this complication.

The construction of Gromov-Witten classes is possible if the following triple of compact spaces and maps between them is presented \[5\] \[6\]. The first space is the compactification of the space of \( n \) points on the worldsheet: \( \bar{M}_{n,\Sigma} \). The second space is the compactified moduli space \( \mathcal{V}_{\Sigma,n;T} \) of the pairs \((\Sigma, x_1, \ldots, x_n; f)\) where \( f \) is the holomorphic map (instanton) of the worldsheet \( \Sigma \) to the target space \( T \). The non-trivial part of the compactification is that it possesses two maps:

\[
\begin{array}{ccc}
\bar{M}_{n,\Sigma} & \xrightarrow{p} & \mathcal{V}_{\Sigma,n;T} \\
\; & \searrow \ev & \; \\
\; & T^{\times n} & \; \\
\end{array}
\]

where \( T \) is the target space and the restriction of the map \( \ev = \times_i \ev_i \) onto the internal part of \( \mathcal{V} \) is simply the evaluation of \( f \) at the points \( x_i \): \( \ev_i = f(x_i) \), while that of \( p \) is the projection \( p(\Sigma, x_1, \ldots, x_n; f) = (\Sigma, x_1, \ldots, x_n) \). Then the construction of the classes proceeds as follows: take the cohomology elements \( g_1, \ldots, g_n \in H^*(T) \). Then the corresponding Gromov-Witten class is

\[
\mathcal{I}(g_1 \otimes \ldots \otimes g_n) = p_*(\otimes_i \ev^*_i g_i) \in H^*(\bar{M}_{n,\Sigma})
\]

It is desirable that the four dimensional gauge theories also may be treated within similar paradigm. The first problem is to find the analogue of \( \bar{M}_{n,\Sigma} \). In the case where the instantons can be described as holomorphic bundles the natural candidate is the resolution of diagonals constructed by W. Fulton and R. MacPherson \[7\]. Unfortunately this appeals to the choice of complex structure on \( \Sigma \) and hence violates the Lorentz invariance.

\[4\] More precisely, let \( m \in \bar{M}_{g,0} \) be the point and \( \pi : \bar{M}_{g,n} \to \bar{M}_{g,0} \) be the forgetful map. Then the cycles lie in \( \pi^{-1}(m) \).
1.3. Summary of the results

This paper makes the first steps in the direction of GW program. Specifically we discuss the geometry of the twisted abelian gauge theories and present the superfield formalism which makes duality transformations transparent (chapter 3). We study the deformations of the theory. It turns out that the proper formulation of the deformation problem is equivalent to the studies of the deformations of $\Gamma$-invariant Lagrangian submanifolds in a complex vector symplectic space, where $\Gamma$ is a certain discrete subgroup of linear symplectic group.

We propose the extension of these results to the case of supermanifolds. In other words, we include all observables into consideration. In particular, we derive the formula for the contact terms between the $p_1$- and $p_2$-observables for $p_1 + p_2 \geq 4$ (otherwise the contact term vanishes). We propose the conjecture of universality of contact terms (see chapter 2) and test it in some examples. The validity of this conjecture is a strong hint about the topology of the compactified configuration space $\bar{M}_{n,\Sigma}$. In the chapter 4 we apply these results to the computations of generalized Donaldson invariants - the correlation functions of $p$-observables of the functions $(\text{Tr}\phi^2)^r$ (Donaldson studied the case $r = 1$). The subtleties of this definition are discussed in the chapter 5. We compute all topological correlation functions in the theory on some manifolds with $b^+_2 = 1$ thus generalizing the results of [8][9]. They are not exactly topological invariants. The reason is that sometimes the enumerative problem of Donaldson’s is not well-behaved under the variations of parameters, such as the metric. This occurs precisely when $b^+_2 = 1$ due to the abelian instantons. The jumps of the correlations functions/Donaldson invariants are under control [8][10][9]. We use these formulae to test certain conjectures which formed the ground for the analysis of the chapter 2, 3.

Then we go on and discuss more general theories. We explain the meaning of the theories with matter, and the compactified higher dimensional theories, both microscopically (i.e. from the point of view of intersection theory on instanton moduli spaces) in chapter 5 and macroscopically in chapter 6. In particular, we show that the theory with massive matter can be interpreted as the integration of the equivariant Euler class of the Dirac index bundle over instanton moduli space. The higher dimensional theories are interpreted as the integration of $\hat{A}$-genus and elliptic genus respectively. We discuss the $A, D, E$ singularities

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5 They are honest solutions of instanton equations with abelian gauge group.
on the Coulomb branches and their contribution to the correlation functions using equiv-
ariant cohomology with respect to the enhanced global symmetry groups. In the chapter
7 we discuss the two dimensional analogues of several constructions which we considered
in the gauge theory case. We present the proper analogues of the theories with matter
in the type A sigma model framework. We explain the relations of gauge theories (both
two and four-dimensional) to the type B sigma models. The formulae we obtain have very
transparent meaning in some cases. This is the limit of the metric on the manifold Σ where
it looks like a two dimensional surface. For example, for \( \mathbb{P}^1 \times \mathbb{P}^1 \) one may take one of
the spheres much smaller then the other. In this limit the theory becomes equivalent to a
kind of type B two dimensional sigma model with Landau-Ginzburg superpotential. The
target space of the sigma model is a disjoint union of the spaces \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \), where
\( \Lambda \) is the weight lattice of \( G \), and \( M_\lambda \) is the moduli space of curves, corresponding to the
Seiberg-Witten theory with the group \( G \) together with the choice of cycle in their first
homology, proportional to \( \lambda \). The superpotential can be written universally for all theories
with arbitrary gauge groups:

\[
W = \sum_{i=1}^{r} \lambda_i a^i + f(u^k)
\]  

(1.4)

where \( u^k \) are the invariants (casimirs) of the group \( G \), \( f \) is some holomorphic function
 corresponding to the 2-observable, \( \lambda_i \) are the components of the weight vector, \( a^i \) are the
central charges in the \( \mathcal{N} = 2 \) algebra in four dimensions, viewed as functions of \( u^k \). We
compare two dimensional Yang-Mills theory and Landau-Ginzburg model. We explain the
absence of the contact terms in the two dimensional Yang-Mills theory as a result of its
hidden higher dimensional nature.

Due to the length of the paper we present a short overview of explicit formulae and
the relevant notations in the chapter 2. The material of the chapters 3 and 4 has been
reported at the Cargese conference on “Strings, Branes and Duality” in June 1997.

The recent paper \[9\] has some overlap with these chapters. The beautiful paper \[9\]
treated extensively the Donaldson invariants for the \( SO(3) \) group for various \( w_2(E) \). The
reader should consult it for the detailed formulae for Donaldson invariants of manifolds
with \( b_2^+ = 1 \). Also, some formulae for the theories with matter are presented there.\[6\]

\[6\] G. Moore has informed us that he and M. Mariño had some progress in extending the results
of \[9\] to the case of higher rank groups.
2. Answers for Correlators and Principles of Their Computation in the Pure Gauge Theory

The objective of this section is to formulate certain principles of calculations of the correlation functions of various observables in topological gauge theory.

We start with presentation of the explicit formulae for the correlators in the case of pure Donaldson theory. We compute the correlation functions of $p$-observables $O_r^{(p)}(p = 0, 1, 2, 3, 4)$ of operators $\left(-\frac{\text{Tr}\phi^2}{4\pi^2}\right)^r$ ($r = 1, \ldots$) in $SU(2)$ twisted gauge theory on a oriented closed four-manifold $\Sigma$.

2.1. The setup

The metric $g$ allows to split the bundle of the real two-forms $\Omega^2(\Sigma)$ into the sum $\Omega^2 = \Omega^{2,+} \oplus \Omega^{2,-}$ of the self-dual and anti-self-dual forms. This decomposition descends to cohomology: $H^2(\Sigma; \mathbb{R}) = H^{2,+} \oplus H^{2,-}$. Let $b_p = \dim H^p(\Sigma; \mathbb{R})$, $b^{\pm}_2 = \dim H^{2,\pm}$.

Let $e_\alpha$ be the base in the cohomology group of $\Sigma$: $e_\alpha \in H^*(\Sigma; \mathbb{C})$. Let $d_\alpha$ denote the degree of $e_\alpha$.

The breakthrough in the physical approach to the Donaldson theory came from the realization of the fact that the essential contribution to the correlation functions in the twisted low-energy effective theory comes from the singularities in the space of vacua (the “u-plane”) where extra massless particles appear. In the paper [11] the following equations were proposed:

$$F_{ij}^+ = -\frac{i}{2} \hat{M} \Gamma_{ij} M$$

$$\sum_i \Gamma^i D_i M = 0$$

(2.1)

where $M$ is the section of the $Spin^c$ bundle $S^+ \otimes L$, $\hat{M}$ is the section of $S^+ \otimes L^{-1}$, $F$ is the curvature of the connection in the line bundle $L^2$, $\Gamma^i$ are the Clifford matrices, and $\Gamma_{ij} = \frac{1}{2} \{\Gamma_i, \Gamma_j\}$. Let $\ell \in H^2(\Sigma, \mathbb{Z})$ be the class such that $\ell \equiv w_2(\Sigma)\mod 2$. Let $\mathcal{M}(\ell)$ be the moduli space of the solutions of the equations (2.1) with $c_1(L^2) = \ell$. Its dimension can be computed by virtue of the index theorem [11]:

$$d_\ell = \dim \mathcal{M}(\ell) = \frac{(\ell, \ell) - 2\chi - 3\sigma}{4}$$

(2.2)

7 Here $L^2$ is the ordinary complex line bundle. For two $Spin^c$ structures $S_+ \otimes L$ and $S_+ \otimes L'$ the ratio $L' \otimes L^{-1}$ is also the ordinary line bundle. Notice that $c_1(L^2) \equiv w_2(\Sigma)\mod 2$. 

6
where $\chi$ and $\sigma$ denote the Euler characteristics and the signature of the manifold $\Sigma$ respectively, $(\ell, \ell)$ is the intersection pairing in $H^*(\Sigma)$. Fix a point $P \in \Sigma$. One may also form the framed moduli space $\mathcal{M}(\ell, P)$. It is the space of solutions of the equations (2.1) modulo gauge transformations equal to identity at $P$. The space $\mathcal{M}(\ell, P)$ is the $U(1)$ bundle over $\mathcal{M}(\ell)$. Let $c_1$ be its first Chern class. It does not depend on $P$. Define the Seiberg-Witten invariant corresponding to the $Spin^c$ structure $\ell$ the integral [9]:

$$SW(\ell) = \int_{\mathcal{M}(\ell)} c_1^d$$

(2.3)

It vanishes if the dimension $d_\ell$ is odd. The manifold $\Sigma$ is of generalized simple type if there is a finite number $l$ such that $SW(\ell)$ vanishes for $|(\ell, \ell)| > l$.

2.2. The explicit formulae

First we present the answers for the correlators. The relevant notations are explained shortly. The conjectures which form the ground for our computations are formulated afterwards.

The expressions for correlators involve various derivatives of the following master function $F(a; \{t_r\})$. Let $(a(u), a_D(u))$ be the functions, defined as follows. Both $a$ and $a_D$ are the solutions of the second-order differential equation:

$$\left[ (1 - u^2) \frac{d^2}{du^2} + \frac{1}{4} \right] \left( \begin{array}{c} a(u) \\ a_D(u) \end{array} \right) = 0 \quad (2.4)$$

with the asymptotics at $u \to \infty$:

$$a(u) \sim \sqrt{\frac{u}{2}} + \ldots, \quad a_D \sim -\frac{2a}{\pi \ell} \log(u) + \ldots \quad (2.5)$$

Then $F(a; 0)$ is defined as the solution to the equation:

$$dF = a_D(u) da(u)$$

Next we define the function $H(a, a_D)$ which obeys the following three properties:

1. **Normalization**: 

$$H(a(u), a_D(u)) = -\frac{u}{2} \quad (2.6)$$

2. **Homogeneity.** For $\mu \neq 0$:

$$H(\mu a, \mu a_D) = \mu^2 H(a, a_D) \quad (2.7)$$
3. **Modular invariance.** For any \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma \subset SL_2(\mathbb{Z}) \):

\[
H(\gamma a_D + \delta a, \alpha a_D + \beta a) = H(a, a_D),
\]  
(2.8)

The function \( F(a; t_r) \) is the (formal) solution to the following system of partial differential equations:

\[
\frac{\partial F}{\partial t_r} = -H^r(a, \frac{\partial F}{\partial a})
\]  
(2.9)

The master function \( F \) can be analytically continued by allowing the parameters \( t_r \) to take values in any (super)commutative algebra \( V \). In our case the algebra \( V \) is that of cohomologies of \( \Sigma \):

\[
V = H^*(\Sigma, \mathbb{C}).
\]

Now, \( F \) solves the system:

\[
\frac{\partial F}{\partial T_\alpha^r} = -e_\alpha H^r(a, \frac{\partial F}{\partial a})
\]  
(2.10)

Consider the formal seria:

\[
\begin{pmatrix} a(u; \{ t_r \}) \\ a_D(u; \{ t_r \}) \end{pmatrix} = e^{\frac{\lambda(u, t)}{2}} \begin{pmatrix} a(ue^{-\frac{\lambda(u, t)}{4}}) \\ a_D(ue^{-\frac{\lambda(u, t)}{4}}) \end{pmatrix}
\]  
(2.11)

where

\[
\lambda(u, t) = \sum_{r=1}^{\infty} r t_r u^{r-1}
\]  
(2.12)

Define \( a_{1,2} \) as follows:

\[
a_1 = a_D, \quad a_2 = -a_D + 2a
\]

Together with \( F(a, t) \) we also need its Legendre transforms:

\[
F_1(a_1, t) = -aa_D + F(a), \quad F_2(a_2, t) = -3aa_D + 3a^2 - \frac{1}{2}a_D^2 + F(a)
\]  
(2.13)

The last notation is the following. Let \( \psi \) denote the odd (fermionic) variable with values in \( H^1(\Sigma; \mathbb{R}) \). Let \([d\psi]\) be the fermionic measure on \( H^1(\Sigma; \mathbb{R}) \).

**Claim 1.** for the manifolds \( \Sigma \) with \( b_2^+ > 1 \):

\[
\mathcal{Z}(T^{α, r}) = \langle \exp \left( T^{α, r} \int_{\Sigma} e_\alpha O_r^{(1-d_α)} \right) \rangle =
\sum_{i=1}^{2} \int_{a_i=0}^{\infty} du \left[ \frac{du}{da_i} \psi \right] a_i^{\frac{α+1}{4}}(u^2 - 1)^{\frac{b_2}{2}} \left( \frac{du}{da_i} \right)^{\frac{b_2}{2}} \sum_{\ell \in H^2(\Sigma, \mathbb{Z})} SW(\ell) e^{\frac{\pi}{4} \int_{\Sigma} F_i(a_i + \psi + \ell, T)}
\]  
(2.14)
Here we substitute $a_i = a_i(u,t)$.

As an example of the correlator on the manifold $\Sigma$ with $b_2^+ = 1$ we consider the case of $\Sigma_0^g = \mathbb{P}^1 \times C_g$, where $C_g$ is the genus $g$ Riemann surface and that of $\Sigma_0^g$, which is $\Sigma_0^g$ blown up at $l$ points in generic position (for $l \leq 8$). The surfaces $\Sigma_{0}^{g}$ are called Del Pezzo surfaces. We study them in the limit where the sizes of the two-sphere and all exceptional divisors are small compared to that of $C_g$.

**Claim 2.** the answer for the correlator (2.14) is given by the contour integral

$$\sum_{N \in \mathbb{Z}} \oint B^l(u) \frac{(du)^2[d\tilde{\psi}]}{dW_N} e^{\frac{i}{\pi} \int_{\Sigma^+} \mathcal{F}(a+\psi;T)}, \quad W_N = Na(u) - \int_{C_g} \mathcal{F}(a+\psi;T)$$

(2.15)

where $\tilde{\psi} = \frac{du}{da} \psi$ and $B^l(u)$ is the blowup factor which is equal to

$$B^l(u; t) = \prod_{i=1}^l \frac{\theta_{00}(\tau(u), \frac{1}{\pi} \int_{e_i} \mathcal{F}(a; t))}{\theta_{00}(\tau(u), 0)}$$

with $e_i$ denoting the exceptional divisors (the definition of $\theta_{00}, \theta_{01}$ etc. is given below).

**Claim 3.** The formulae (2.14) are easily modified to cover the case of the $SU(2)$ theory on spin manifold $\Sigma$ (we relax this requirement in the sequel) with $N_f \leq 3$ massless multiplets (its mathematical meaning is explained in chapter 5). The modification is as follows. The “functions” $a(u)$ and $a_D(u)$ obey the equations:

$$\left[ -\Delta \frac{d^2}{du^2} + \frac{f}{4} \right] \begin{pmatrix} a \\ a_D \end{pmatrix} = 0$$

(2.16)

where

$$\begin{array}{c|c|c|c|c}
N_f & \Delta & f & u & 0 \\
1 & 1-u^3 & u & 0 & 1 \\
2 & (u^2-1)^2 & u^2-1 & 0 & 2 \\
3 & (u + \frac{2}{3})^3 (\frac{1}{3} - u) & (u + \frac{2}{3})^2 & 0 & 3 \\
\end{array}$$

(2.17)

The solutions to (2.16) are fixed by their asymptotics at $u \to \infty$:

$$a(u) \sim \frac{1}{2} \sqrt{2u} + \ldots, \quad a_D(u) \sim \frac{N_f-4}{2\pi i} a(u) \log u + \ldots$$

(2.18)

The factor $(u^2-1)^{\frac{2}{3}}$ is replaced by $\Delta^{\frac{2}{3}}$. Notice that in the case $N_f = 3$ one has to make a shift $u \mapsto u - \frac{2}{3}$ while comparing to the formulae of [1][1][1] (the formalism of [1][1] allowed for an arbitrary shift in $u$). The rest of the formalism is unchanged.

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8 We set the scales of [1] to: $\Lambda_3 = 8, \Lambda_2^2 = 8, \Lambda_1^6 = -\frac{256}{27}$.
2.3. Geometry of the abelian $\mathcal{N} = 2$ theory

Here discuss the geometric representation of the data entering the construction of the low-energy effective abelian theory. Although the various parts of it are well-known [1][2], we present our reformulation since in this setting the deformation problem can be clearly stated.

**Embedded vacua**

Consider the complex symplectic vector space $\mathbb{C}^{2r}$ with the symplectic form $\omega = \sum_{i=1}^{r} da_i \wedge da_i^D$. Let $\theta = \sum_{i=1}^{r} a_i da_i^D \equiv (a^D, da)$. Let $\Gamma$ be a subgroup of $Sp(2r, \mathbb{Z})$. Let $\mathcal{L}$ be a $\Gamma$-invariant Lagrangian submanifold in $\mathbb{C}^{2r}$. By definition, the restriction of $\omega$ on $\mathcal{L}$ vanishes, hence

$$\theta|_{\mathcal{L}} = dF, \quad F: \mathcal{L} \to \mathbb{C}$$  \hspace{1cm} (2.19)

The function $F$ is called the generating function of the Lagrangian submanifold. It transforms under the action of element $g \in \Gamma$ as follows:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad g^*F(x) = F(x) + (Ba, Ca_D) + \frac{1}{2}(Ba, Da) + \frac{1}{2}(Aa_D, Ca_D) + c(g)$$  \hspace{1cm} (2.20)

where $c(g)$ is a certain cocycle: $[c(g)] \in H^1(\Gamma, \mathbb{C})$. If the cocycle $c(g)$ is trivial then one can solve (2.20) as follows:

$$F = \frac{1}{2}(a, a_D) + \frac{u}{\pi i}$$  \hspace{1cm} (2.21)

where $u$ is some $\Gamma$-invariant function on $\mathcal{L}$. This property of the prepotential $F$ has been observed by several authors (cf. [13][14][15]). It reflects the scaling properties of $F$. To see this one inserts into (2.21) that $a_D = \partial F/\partial a$ and use the equation (2.9) for the evolution generated by $u$ provided that the extension of $u$ to $\mathbb{C}^{2r}$ is known. We claim that the $\Gamma$-invariant Lagrangian submanifold determines an effective abelian $\mathcal{N} = 2$ gauge theory, whose duality group is precisely $\Gamma$.

**A note on definitions.** There is one confusing point which must be kept in mind. The name *generating function* for the prepotential $F$ which comes from symplectic geometry may be confusing with the notion of generating function for correlators. In fact, in two dimensional topological field theories the same letter $F$ goes under the name of prepotential and the generating function of correlators. Moreover, sometimes $F$ is the prepotential of the effective four dimensional supersymmetric field theory. Throughout this paper we use

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9. It is globally well-defined on $\mathcal{L}$ if $\mathcal{L}$ is simply-connected.
the letter $F$ only for the prepotential and when we call it generating function it is only the generating function of Lagrangian variety which is meant by the name. The generating functions of the correlators will be called by $Z, Z, \ldots$.

**Deformations of the theory**

The symplectomorphisms of $\mathfrak{C}^{2r}$ map $\mathcal{L}$ to another Lagrangian submanifolds. The symplectomorphisms in the component of identity are generated by the (time-dependent) Hamiltonians $H(a, a_D)$. The generating function $\mathcal{F}$ changes according to the Hamilton-Jacobi equation:

$$\frac{\partial \mathcal{F}}{\partial t} = -H(a, \frac{\partial \mathcal{F}}{\partial a}, t) \quad (2.22)$$

where we view $\mathcal{F}$ as a function of $a$. This description is local. The flows which preserve the property of $\Gamma$-invariance are generated by $\Gamma$-invariant Hamiltonians. Let us denote the space of all $\Gamma$-invariant holomorphic functions on $\mathfrak{C}^{2r}$ by $\mathcal{C}$. The Hamiltonian flows in $\mathfrak{C}^{2r}$ which do not change $\mathcal{L}$ are generated by the Hamiltonians obeying

$$\tau_{ij} \frac{\partial H}{\partial a_D, i} = \frac{\partial H}{\partial a^j} \quad (2.23)$$

where

$$\tau_{ij} = \frac{\partial \mathcal{F}}{\partial a^i \partial a^j} = \frac{\partial a_{D,j}}{\partial a^i} \quad (2.24)$$

The space of such Hamiltonians is denoted as $\mathcal{C}_L$. The quotient $\mathcal{W}_L = \mathcal{C}/\mathcal{C}_L$ can be identified with the space of $\Gamma$-invariant functions on $\mathcal{L}$. In general there is no canonical way of extending the $\Gamma$-invariant function on $\mathcal{L}$ to the $\Gamma$-invariant function on $\mathfrak{C}^{2r}$. Thus, we have two potential difficulties: 1) the Hamiltonians may be time-dependent and 2) even if they are time-independent there are many ways to extend a given function $u \in \mathcal{C}$ to the whole $\mathfrak{C}^{2r}$.

Let us dispose of the problems 1) and 2). In general, the family $H_k(a, a_D, t)$ may be used in defining a consistent system of the type $(2.22)$ if and only if:

$$\frac{\partial H_k}{\partial t} - \frac{\partial H_l}{\partial t} = \{H_l, H_k\} = 0 \quad (2.25)$$

We propose to impose the extra condition that the Hamiltonians are actually time-independent: $\frac{\partial}{\partial t} H_k = 0$. This is the principle of background independence. Clearly, relaxing the background independence principle allows to add to $\mathcal{F}(a, t)$ an arbitrary $\Gamma$-invariant function on $\mathcal{L}$ at any order in $t$. This freedom would spoil the uniqueness of the contact terms which we discuss shortly.
The problem 2) we shall solve in the important Rank one case. Let \( r = 1 \). The group \( \mathbb{C}^* \) acts in \( \mathbb{C}^2 \) in a standard way. This action commutes with that of \( \Gamma \). Suppose that the basis of \( \Gamma \)-invariant functions on \( L \) is chosen. As \( L \) is one-dimensional it is sufficient to choose one function. We call it \( u \). Then any other admissible function is a rational function of \( u \). Fix an integer \( p \).

The next principle is called “Homogeneity”: The function \( u \) extends to a function \( \Gamma \)-invariant \( H(a,a_D) \) on \( \mathbb{C}^2 \) with the following properties:

\[
H(\mu a, \mu a_D) = \mu^d H(a, a_D) \\
H(a,a_D)|_L = u
\]  

(2.26)

We will see that in asymptotically free theories \( d = 2 \).

Once we have fixed a continuation of one function the extension of the rest is unique once the two principles are at their power. Indeed, since the Hamiltonians must Poisson-commute the only possibility is that the higher Hamiltonians are the functions of the Hamiltonian corresponding to \( u \). Homogeneity fixes this function uniquely. In particular, for a polynomial \( P \) the function \( P(u) \) extends to \( P(H(a,a_D)) \). The deformation problem is well-posed now [16]. We have to solve the equation of motion

\[
\dot{a} = \frac{\partial H}{\partial a_D}, \quad \dot{a}_D = -\frac{\partial H}{\partial a}
\]

(2.27)

with the initial condition \((a(0),a_D(0)) \in L\). Then

\[
\mathcal{F}(a,t) = \mathcal{F}(\tilde{a},0) + \int_0^t (a_D(t')\dot{a}(t') - H(a(t'),a_D(t'))) \, dt'
\]

(2.28)

where the trajectory \((a(t'),a_D(t'))\) is such that

\[
a(0) = \tilde{a}, \quad a(t) = a
\]

(2.29)

We introduce the set of “times” \( t_1, t_2, \ldots \):

\[
\frac{\partial \mathcal{F}}{\partial t^k} = -H^k(a, \frac{\partial \mathcal{F}}{\partial a})
\]

(2.30)

In applications it is sometimes useful to have an expansion of the form:

\[
\mathcal{F}(a,t) = \mathcal{F}_0(a) + \sum_{k>0} t^k \mathcal{F}_k + \sum_{k,l>0} \frac{1}{2} t^k t^l \mathcal{F}_{k,l} + \ldots
\]

(2.31)
The equations (2.30) allow to calculate all the coefficients $F_{k_1,\ldots,k_p}$ term by term. In particular, the value of $F_k$ depends only on the restriction of the function $H^k$ to $L$. The next term $F_{k,l}$ which we call a pair contact term depends on the first jets of $H^k$ and $H^l$, etc. It is easy to calculate:

$$F_{k,l} = klu^{k+l-2} \frac{\partial H}{\partial a} \frac{du}{da}$$

where the last factor is the derivative of the function $u$ along $L$. The quasihomogeneity of $H$ allows to calculate (2.32) quite explicitly:

$$F_{k,l} = klu^{k+l-2} \left( \frac{du}{da} - a \frac{du}{da} \right)$$

**Higher rank.** In the case of higher rank we do not know how to extend the functions $u_1, \ldots, u_r$. However, we can solve the equations for the evolution with respect to the Hamiltonian $u_1$. It is done again by the homogeneity principles. Let $d_k$ be the degree of the coordinate $u_k$ with respect to the $U(1)$ ghost number symmetry. For the low-energy effective theory corresponding to the ultra-violet gauge theory with simple group $G$ the degrees of the basic invariant polynomials $u_k$ are $\deg u_k = d_k = m_k + 1$ where $m_k$ are the corresponding exponents of the Weyl group $W$ of $G$. Thus, the ghost number of $u_k$ is equal to $2m_k + 2$. In particular, for $G = SU(r + 1)$ one has: $d_k = k + 1$. Then:

$$a^i(t; \{u^k\}) = a_{D_i}(t; \{u^k\}) = e^{\frac{2\pi i d_k}{h^\vee}} \left( \frac{du}{da} - a \frac{du}{da} \right)$$

where $h^\vee$ is the dual Coxeter number. For $G = SU(r + 1)$, $h^\vee = r + 1$. Here the functions $a^i, a_{D,i}, i = 1, \ldots, r$ are the central charges of $N = 2$ susy algebra. They are given by the periods of a certain meromorphic differential $\eta$ on the Riemann surface $C_{u_1,\ldots,u_r}$ of genus $r$. For the group $SU(r + 1)$ the curve is:

$$z + \frac{1}{4z} = x^{r+1} + \sum_{k=1}^{r} u_k x^{r+1-k}$$

and the differential

$$\eta = x \frac{dz}{z}$$

One can choose a set of $A$ and $B$ cycles in $H_1(C_{\vec{u}}; \mathbb{Z})$ which form a base with canonical intersections. Then

$$a^i = \oint_{A_i} \eta, \quad a_{D,i} = \oint_{B_i} \eta, \quad A_i \cap B_j = \delta^i_j$$
We can also write down a formula for the pair contact term between the 2-observables $O^{(2)}_{u_k}$ and $O^{(2)}_{u_l}$ in the first non-vanishing order. It is:

$$F_{u_k, u_l} = \frac{\partial u_k \partial u_l \partial \log \Theta}{\partial a^i \partial a^j \partial \tau_{ij}} \quad (2.37)$$

where

$$\Theta = \sum_{\lambda \in \Lambda} \exp \left(2\pi i \langle \lambda, \tau \lambda \rangle + \pi i \langle \lambda, \rho \rangle \right) \quad (2.38)$$

with $\Lambda$ being the set of weights and $\langle , \rangle$ the restriction of the Killing form on the Cartan subalgebra of $\mathfrak{g}$. The matrix $\tau_{ij}$ is the period matrix of $C_{\vec{u}}$ in the given basis of $A$ and $B$ cycles. The vector $\rho$ equals the half the sum of positive roots.

**Observables as nilpotent deformations.** The advantage of (2.31) is the possibility to have nilpotent parameters $t^k$. In the context of gauge theory these parameters are the elements of the cohomology of the space-time manifold $\Sigma$. The function $F$ takes now values in the graded (super)commutative algebra $\mathcal{V} = H^*(\Sigma, \mathfrak{g})$. Let $e_\alpha$ be the base of $\mathcal{V}$. The product of the classes $a$ and $b$ is denoted as $a \cdot b$. The times are denoted as $T^{k,\alpha}$, where $k = 1, 2, \ldots$ and $\alpha$ runs over the base of $\mathcal{V}$. The corresponding $\mathcal{V}$-valued Hamiltonians $H_{k,\alpha}$ are of the form $H_k(a, a_D)e_\alpha$ with $e_\alpha \in \mathcal{V}$. We may immediately apply (2.31) and (2.33) to get:

$$\frac{\partial F(a, a_D)}{\partial T^{k,\alpha}} = -H_k(a, a_D)e_\alpha$$

$$F_{(k,\alpha), (l,\beta)} = kD^{k+l-2} \left( \frac{du - b_{da}}{a_{D}da - b_{da_D}} \right) e_\alpha \cdot e_\beta \quad (2.39)$$

The meaning of the nilpotent deformations along $H^*(\Sigma)$ is clear. They simply correspond to the observables added to the action. The $p$-observable corresponds to $H^4-p(\Sigma)$. The meaning of (2.39) is simply the **pair contact term** between $p$ and $q$-observables, which is $4-p-q$-observable.

The fact that the form of the contact terms is essentially the same for all $p$ and $q$ is formulated as the following **universality principle**:  

**The contact term between the observables $O^{(p)}_{\phi_1}$ and $O^{(q)}_{\phi_2}$ is equal to $O^{(p+q-4)}_{C(\phi_1, \phi_2)}$, with some universal (i.e. independent of $p$ and $q$) $C(\phi_1, \phi_2)$.**

The principle of holomorphic integration is rather methodological. It states that in order to figure out the structure of the contact terms and to compute the correlation functions it is sufficient to look at the expression for the effective action (with all observables included) and evaluate it on the harmonic forms only. Also, one needs only the
minimal number of fields entering the twisted $\mathcal{N} = 2$ multiplet (see below). It is analogous to the methods used in [14]. This principle does not explain why the contribution of the moduli space of vacua vanishes for the manifolds $\Sigma$ with $b_2^+ > 1$. Nevertheless it proves to be useful in the course of getting the right expressions for the observables. To make the statement properly we need to reinspect the twisted supersymmetry transformations. The following chapter is devoted to this question.

3. Geometry of twisted supersymmetry.

We start with reminding the structure of the twisted vector multiplet, $Q$-symmetry and observables. We discuss the subtleties concerning preserving duality invariance and present the formulae for the modular covariant $Q$-operator in the effective low-energy abelian gauge theory. We discuss $Q$-invariant observables in the ultraviolet non-abelian theory and their low-energy abelian counterparts. We show that the proposed picture of $\Gamma$-invariant Lagrangian submanifolds of $\mathbb{C}^{2r}$ naturally arises in the attempt to define the low-energy effective measure.

3.1. Microscopic pure gauge theory. $Q$ operator and observables.

The field content of the twisted $\mathcal{N} = 2$ vector multiplet is the gauge field $A = A_\mu dx^\mu$, the complex scalar $\phi$ and its conjugate $\bar{\phi}$ and the fermions: the one-form $\psi$, the scalar $\eta$ and the self-dual two-form $\chi$. All fields take values in the adjoint representation of the gauge group. The $Q$-transformation has the form;

\begin{align*}
Q\phi &= 0, & Q\bar{\phi} &= \eta, & Q\eta &= [\phi, \bar{\phi}] \\
Q\chi &= H, & QH &= [\phi, \chi] \\
QA &= \psi, & Q\psi &= dA\phi
\end{align*}

(3.1)

Here $H$ is an auxiliary bosonic self-dual two-form field with values in the adjoint representation. The operator $Q$ squares to the gauge transformation generated by $\phi$.

The theory has a set of natural observables. Start with invariant polynomial $\mathcal{P}$ on the algebra $\mathfrak{g}$. Let $C^k, k = 0, \ldots 4$ be the closed $k$-cycles in the space-time manifold $\Sigma$. 

Their homology cycles are denoted as \( [C^k] \in H_k(\Sigma; \Psi) \). The observables form the descent sequence:

\[
O^{(0)} = \mathcal{P}(\phi) \quad \{Q, O^{(0)}\} = 0
\]

\[
dO^{(0)} = \{Q, O^{(1)}\} \quad (O^{(1)}, [C^1]) = \int_{C^1} O^{(1)} = \int_{C^1} \frac{\partial P}{\partial \phi^a} \psi^a
\]

\[
dO^{(1)} = \{Q, O^{(2)}\} \quad (O^{(2)}, [C^2]) = \int_{C^2} O^{(2)} = \int_{C^2} \frac{\partial P}{\partial \phi^a} F^a + \frac{1}{2} \frac{\partial^2 P}{\partial \phi^a \partial \phi^b} \psi^a \wedge \psi^b
\]

and so on.

In particular, the top degree observable has the form:

\[
O^{(4)} = \mathcal{P}^2 \quad \{Q, O^{(4)}\} = 0
\]

\[
dO^{(4)} = \{Q, O^{(5)}\} \quad (O^{(5)}, [C^3]) = \int_{C^3} O^{(5)} = \int_{C^3} \frac{\partial^2 P}{\partial \phi^a \partial \phi^b} F^a + \frac{1}{2} \frac{\partial^3 P}{\partial \phi^a \partial \phi^b \partial \phi^c} F^a \psi^b \psi^c + \frac{1}{4} \frac{\partial^4 P}{\partial \phi^a \partial \phi^b \partial \phi^c \partial \phi^d} \psi^a \psi^b \psi^c \psi^d
\]

It enters the Seiberg-Witten low-energy effective action, where all the fields are specialized to be abelian. In general, the whole action \( S \) equals the sum of the 4-observable, constructed out of the prepotential \( \mathcal{F} \) and the \( Q \)-exact term:

\[
S = O^{(4)}_\mathcal{F} + \{Q, R\}. \quad (3.3)
\]

The standard choice is

\[
\mathcal{F} = \left( \frac{i\theta}{8\pi^2} + \frac{1}{e^2} \right) \text{Tr}^2, \quad R = \frac{1}{e^2} \text{Tr} \left( \alpha \chi F^+ - \beta \chi H + \gamma D_A \bar{\phi} \ast \psi + \delta \eta \ast [\phi, \bar{\phi}] \right) \quad (3.4)
\]

for \( \alpha = \beta = \gamma = \delta = 1 \).

The gauge invariant observables annihilated by \( Q \) form a special class of operators. Their correlation functions do not change under a small variation of metric on the four-manifold \( \Sigma \). Let us denote

\[
\mathcal{V} = \oplus_{p=0}^4 H^p(\Sigma; \Psi) \quad (3.5)
\]

In the Donaldson theory \( (G = SU(2) \text{ or } G = SO(3)) \) one’s aim is to compute:

\[
\langle \exp((O_u^{(2)}, w) + \lambda O_u^{(0)}) \rangle \quad (3.6)
\]

where \( w \in H^2(\Sigma; \mathbb{R}) \), \( O_u^{(0)} = \text{Tr}\phi^2 \),

\[
(O_u^{(2)}, w) = -\frac{1}{4\pi^2} \int_{\Sigma} \text{Tr}(\phi F + \frac{1}{2} \psi^2) \wedge w \quad (3.7)
\]
From the point of view of topological theory the full set of correlators of interest is given by the generating function of correlators of all \( p \)-observables:

\[
Z(\{T^k, \alpha\}) = \langle e^{T^k, \alpha (O^{4-d\alpha}, \epsilon)} \rangle \quad (3.8)
\]

**Choices.** In principle the dependence on the coefficients \( (\alpha, \beta, \gamma, \delta) \) which were introduced in (3.4) might be non-trivial. Nevertheless the following general remarks are in order. There exist the following non-anomalous symmetries of the measure:

\[
(\chi, H) \mapsto t_1(\chi, H) \\
(\eta, \tilde{\phi}) \mapsto t_2(\eta, \tilde{\phi})
\]

Hence the correlation functions depend only on the orbit of \( (\alpha, \beta, \gamma, \delta) \) under the action of the group of rescalings:

\[
(\alpha, \beta, \gamma, \delta) \mapsto (t_1^{-1} \alpha, t_1^{-2} \beta, t_2^{-1} \gamma, t_2^{-2} \delta)
\]

(3.10)

One momentarily notices that the points \((1, 1, 1, 1)\) and \((1, 0, 1, 0)\) belong to the different orbits. The orbit of \((1, 1, 1, 1)\) corresponds to the \( \mathcal{N} = 2 \) twisted theory, while \((1, 0, 1, 0)\) has a transparent meaning from the point of view of Donaldson theory. Indeed, if \( \beta = \delta = 0 \) then the action (3.3) yields the Gaussian integral which localizes onto the space of solutions of the following system:

\[
\begin{align*}
F_A^+ &= 0 \\
d_A^+ \psi &= 0, \quad d_A^* \psi &= 0 \\
\phi &= \frac{1}{\Delta_A} [\psi, \ast \psi]
\end{align*}
\]

(3.11)

where \( \Delta_A = d_A^* d_A \) is the gauge-covariant Laplacian on the scalars. The rest of the fields is integrated out. We assume that the gauge -fields are irreducible, i.e. there is no normalizable solutions to the equation \( d_A \phi = 0 \).

The first equation implies that \( A \) is an anti-self-dual gauge field. The second equation implies that \( \psi \) is the tangent vector to the moduli space \( \mathcal{M} \) of the ASD gauge fields. The third equation means that \( \phi \) is certain \( \mathfrak{g} \)-valued two-form on \( \mathcal{M} \). More thorough study (which involves gauge fixing and Faddeev-Popov ghosts \( c \)) identifies \( \phi + \psi + F \) with the curvature of the universal connection \( A + c \) in the universal bundle \( \mathcal{E} \) over \( \mathcal{M} \times \Sigma \).

The solutions to (3.11) when substituted into (3.2) produce representatives of cohomology classes of the moduli space \( \mathcal{M} \). The correlation functions are the integrals of the
cohomology classes over the fundamental cycle of the compactified moduli space $\tilde{M}$. It is non-trivial to check that the representative constructed with the help of (3.11) actually determines a cohomology class of $\tilde{M}$. For the observables constructed out of $\text{Tr}\phi^2$ it has been done by indirect methods (cf. [18]). For the higher observables $\text{Tr}\phi^k$ it is not known to authors.

“Quantum multiplication.” Classically, in $SU(2)$ theory one certainly has the equality:

$$\text{Tr}\phi^{2r} = 2^{1-r} (\text{Tr}\phi^2)^r$$

(3.12)

When the expressions (3.11) are substituted in (3.12) one gets the equality of the representatives of the cohomology classes of $\left[\text{Tr}\phi^{2r}\right]$ and $2^{1-r} \left[\text{Tr}\phi^2\right]^r$ on $\mathcal{M}$. However the divergencies which appear near the boundary $\tilde{M}\setminus M$ of $\mathcal{M}$ may spoil the validity of (3.12) in $H^*(\tilde{M})$. Thus we expect that a priori the observables constructed out of $\text{Tr}\phi^{2r}$ may differ from the cohomology class of $2^{1-r} \left[\text{Tr}\phi^2\right]^r$ where the power is understood as a multiplication in the cohomology of $\tilde{M}$. We return to this subtlety in the chapter 5.

3.2. Macroscopic theory. Electric-magnetic duality.

We shall make use of the low-energy effective theory, whose action on $\mathbb{R}^4$ has been computed in [1], and certain aspects of it for the general four-manifold $\Sigma$ have been worked out in [19] and also recently in [9].

The low-energy theory contains $r \mathcal{N} = 2$ vector multiplets, which are defined up to $\Gamma$-transformation, where $\Gamma$ is a subgroup of $Sp_{2r}(\mathbb{Z})$, e.g $\Gamma(2)$ or $\Gamma_0(4)$ for $r = 1$. Let us denote the scalar components of the multiplet, which are monodromy invariant (up to a sign) at $u^k = \infty$ by $a^i$. Then the $S$-dual ones will be denoted as $a^i_D$. The low-energy effective couplings are denoted as:

$$\tau_{ij}(a) = \left(\frac{4\pi i}{g_{eff}^2} + \frac{\theta_{eff}}{2\pi}\right)_{ij} = \frac{\partial a_{i,D}}{\partial a^j} = \tau_{ij,1} + i\tau_{ij,2} \equiv \Re\tau_{ij} + i\Im\tau_{ij}$$

(3.13)

Our problem here is to write down the $Q$ transformations for the fields entering the twisted vector multiplet. Naively we might simply adopt the transformations (3.1) from the previous section. The trouble is that they are valid only for the theory with field independent coupling constant. It is known from the studies of two dimensional topological strings that the deformation of the theory by the observable of top degree (two in two dimensional case) changes the transformations of the fields.
Let us present the correct transformations and then discuss them:

\[ Q\psi^i = da^i \quad Qa^i = 0 \]
\[ Q(3\tau\chi)_i = (3\tau H)_i \quad Q(3\tau H)_i = 0 \]
\[ Q\bar{a}^i = \eta^i \quad Q\eta^i = 0 \]  

(3.14)

We introduce a notation, inspired by the analogy with type B sigma model:

\[ \theta_i = 3\tau_{ij}\chi^j, \quad \bar{F}_i = 3\tau_{ij}H^j \]  

(3.15)

We also need the transformations for the gauge field. We shall write them a bit later.

Notice that the transformations (3.14) are consistent with the following action of the modular group on fields:

\[ \psi^i \mapsto \psi_{i,D} = \tau_{ij}\psi^j \]
\[ a^i \mapsto a_{i,D} \quad \tau_{ij} \mapsto \tau_{ij,D} = -(\tau^{-1})^{ij} \]
\[ \chi^j \mapsto \chi_{i,D} = \bar{\tau}_{ij}\chi^j \quad \eta^i \mapsto \eta_{i,D} = \bar{\tau}_{ij}\eta^j \quad H^i \mapsto H_{i,D} = \bar{\tau}_{ij}H^j \]  

(3.16)

The \( Q \)-transformations of the gauge fields also must be consistent with the electromagnetic duality. It is not clear a priori that such a \( Q \)-action exists, since the duality is a non-local operation on the fields, while the \( Q \) is a local one. One may try to imitate the twisted version of the supersymmetric duality transformation presented in [1]. We do it in the next subsection, while here simply present the result. Introduce two more fields: the two-form \( F^i \) and one-form \( A_{i,D} \) with the following \( Q \)-transformations:

\[ QA_{i,D} = \tau_{ij}\psi^j \quad QF^i = d\psi^i \]  

(3.17)

Then one has two options. Either \( \tilde{A}_{i,D} = (A_{i,D}, \psi_{i,D}, a_{i,D}, \bar{a}_{i,D}, \chi_{i,D}, H_{i,D}; \eta_{i,D}) \) is treated as a twisted multiplet with the standard \( Q \)-action, or \textsuperscript{“on-shell”} \( F^i = dA^i \) and then \( \tilde{A}^i = (A^i, \psi^i, a^i, \bar{a}^i, \chi^i, \eta^i, H^i) \) form the twisted multiplet. The passage from \( \tilde{A}^i \) to \( \tilde{A}_{i,D} \) can be done by a Gaussian integration and is discussed in some details in the next two sections.
3.3. Holomorphic approach.

Here we state the last principle of the computations. This principle allows one to check the modular invariance of the measure in a relatively simple setting, where all the $Q$-exact (see below) terms are “thrown out” and one works with (formal) contour integrals. Most of the constructions can be done working with the “holomorphic” fields $a, \psi, A$ only. The only trouble with such a prescription is the absence of Laplacians and non-definiteness of the topological term $F \wedge F$. The first problem is avoided in certain cases by working with cohomology (with harmonic forms) while the second may be treated via analytic continuation. In any case such an approach is useful in getting the right structures. Once it is done one may introduce the quartet of the fields $\bar{a}, \eta, \chi, H$ and justify the constructions by working with the standard positive-definite actions.

3.4. “Off-shell” holomorphic formulation

Consider the short superfield:

$$\mathcal{A}_{i,D} = a_{i,D} + \psi_{i,D} + F_{i,D}$$

(3.18)

where $dF_{i,D} = 0$. The operator $Q$ acts as follows:

$$Qa_{i,D} = 0, \quad Q\psi_{i,D} = da_{i,D}, \quad QF_{D} = d\psi_{i,D}$$

(3.19)

We impose (by hands) the condition that $F_{i,D}$ represents the integral cohomology class of the space-time manifold $\Sigma$. Thus,

$$\mathcal{A}_{i,D} \in \Omega^0(\Sigma)_B \oplus \Omega^1(\Sigma)_F \oplus \Omega^2(\Sigma)_B$$

Here $\Omega^2(\Sigma)_B$ is the space of closed two-forms with periods in $2\pi i \mathbb{Z}$. The indices $B, F$ denote the bosonic and fermionic fields respectively. The superfield $\mathcal{A}_{D,i}$ obeys the condition $(Q - d)\mathcal{A}_{D,i} = 0$. One may also fulfill the condition of $Q - d$-closedness by introducing a complete set of $p$-forms which we call the long superfield:

$$\mathcal{A}^i = a^i + \psi^i + F^i + \rho^i + D^i$$

$$0 \quad 1 \quad 2 \quad 3 \quad 4$$

(3.20)

$$\mathcal{A}^i = \sum_{p=0}^{4} \mathcal{A}^{i,p} \in V = \oplus_{p=0}^{4} \Omega^p(\Sigma)$$
and $Q$ acts as follows:

$$Q A^{i,p} = dA^{i,p-1}$$

(3.21)

Let $F_D$ be a holomorphic function on $C^r$. The “action”

$$S = \int_\Sigma F_D(A_D)$$

(3.22)

is clearly $Q$-invariant. The long superfield $A$ allows reparameterizations:

$$A^i \mapsto \tilde{A}^i(A)$$

(3.23)

induced by the holomorphic maps $a^i \mapsto \tilde{a}^i(a^k)$. Let $L \subset C^{2r}$ be a $\Gamma$-invariant Lagrangian subvariety. Let $u^k$, $k = 1, \ldots, r$ be the generators of the ring $\mathcal{W}_L$ of globally defined $\Gamma$-invariant holomorphic functions on $L$. Extend them to the long superfields $U^k$, $k = 1, \ldots, r$.

Define the measure

$$[DU] = \prod_{k=1}^{r} du^k d\psi^k u dF^k u d\rho^k u dD^k u$$

(3.24)

where $(Q - d) (u^k + \psi^k u + F^k u + \rho^k u + D^k u) = 0$.

The duality transformation proceeds as follows: introduce both $A_{i,D}$ and $A^i$ and consider the action

$$\pi i S' = \int_\Sigma A^i A_{i,D} - F(A^i)$$

$$= \int_\Sigma F^i F_{i,D} + \rho^i \psi_{i,D} + D^i (a_{i,D} - \partial_i F) - \frac{1}{2} \partial^2_{ij} F(F^i F^j + 2\rho^i \psi^j)$$

$$- \frac{1}{2} \partial^3_{ijk} F F^i \psi^j \psi^k - \frac{1}{24} \partial^4_{ijkl} F \psi^i \psi^j \psi^k \psi^l$$

(3.25)

Let us consider the following (formal) path integral:

$$\int \mathcal{D}A^i \mathcal{D}A_{i,D} e^{-S}.$$  

(3.26)

The measure $\mathcal{D}A_{i,D}$ is defined canonically. The dependence of the measure on $A^i$ on the choice of the measure $du^1 \wedge \ldots du^r$ is completely parallel to anomaly in Type B sigma models in two dimensions [17][20]. The integral over $A_{i,D}$ (together with summation over the fluxes of $F_{i,D}$) forces $D^i, \rho^i$ to vanish, while $F^i$ becomes a curvature of a connection $A^i$. We hope that the reader will not confuse the fields $D^i, \rho^i$ etc. entering $A^i$ with the fields
$D^i_u, \rho^i_u \ldots$ which enter $U^i$. Of course, there is a simple formula which expresses $D^i_u, \rho^i_u \ldots$ in terms of $D^i, \rho^i, \ldots$. As a result one gets a measure

$$\operatorname{Det}_{ij} \left( \frac{\partial u^i}{\partial a^j} \right)^\frac{\chi}{2} \prod_{k=1}^{r} da^k d\psi^k dF^k$$

On the other hand, performing the integral over $U$ gives us:

$$a_{i,D} = \frac{\partial F}{\partial a^i}, \quad \psi^i = (\tau^{-1})^{ij} \psi_{j,D}$$

$$F^i = (\tau^{-1})^{ij} \left( F_{j,D} - \frac{1}{2} (\tau^{-1})^{lm} (\tau^{-1})^{kp} (\partial_{lkj} F) \psi_{m,D} \psi_{p,D} \right) \quad (3.27)$$

with

$$\tau_{ij} = \frac{\partial^2 F}{\partial a^i \partial a^j}$$

The determinants in this case are slightly more involved:

$$\operatorname{Det}_{ij} \left( \frac{\partial u^i}{\partial a^j} \right)^\frac{\chi}{2} (\operatorname{Det} \tau)^{-\operatorname{dim}\Omega^0 + \operatorname{dim}\Omega^1 - \frac{1}{2} \operatorname{dim}\Omega^2} =$$

$$(\operatorname{Det} \tau)^{-\frac{\chi}{2}} \operatorname{Det}_{ij} \left( \frac{\partial u^i}{\partial a^j} \right)^\frac{\chi}{2} =$$

$$\operatorname{Det}_{ij} \left( \frac{\partial u^i}{\partial a_{D,j}} \right)^\frac{\chi}{2} \quad (3.28)$$

Without the factor $\operatorname{Det}_{ij} \left( \frac{\partial u^i}{\partial a^j} \right)^\frac{\chi}{2}$ the duality transformation would be anomalous. This anomaly was already observed in [19] (for $r = 1$). The “action” (3.25) turns into (3.22) with the substitution of $F$ by $F_D$, which is the Legendre transform of $F$.

### 3.5. Justification of the action

In this section we pass from the holomorphic approach to the “harmonic” one. We introduce the regulator fields $H, \chi, \eta, \bar{a}$ and make the manipulations of the previous section well-defined. We first write down the expression for four-observable:

$$\mathcal{O}^{(4)} = \frac{1}{2} \tau F \wedge F + \frac{1}{2} \frac{\partial \tau}{\partial a} F \psi^2 + \frac{1}{24} \frac{\partial^2 \tau}{\partial a^2} \psi^4 + F F_D \quad (3.29)$$
where we write \( F_D = dA_D \) in order to stress the fact that \( F_D \) may be closed, but not exact form with integral periods. Now let us add to (3.29) a \( Q \)-exact term, which would enforce electric-magnetic duality.\(^{10}\)

\[
L = \frac{i}{4} \mathcal{O}^{(4)} + \{Q, R_0\} \quad R_0 = \tau_2 (\chi(F^+ - H) + d\bar{a} \ast \psi) + \frac{1}{2} \frac{d\tau_2}{da} \psi^2 \chi + \frac{1}{6} \frac{d\tau_2}{d\bar{a}} \chi^3
\]

(3.30)

Expanding \( \{Q, \ldots \} \) out we get:

\[
L = \frac{i}{8} \tau F^2 + FF_D + \tau_2 (H(F^+ - H) + da \ast d\bar{a}) + \\
+ \tau_2 (\chi(d\psi)^+ + \eta d^* \psi) + \\
+ \frac{i}{8} \frac{d\tau}{da} F \psi^2 + \frac{d\tau}{da} \chi(da \wedge \psi) + H \left( \frac{d\tau_2}{da} \left( \frac{1}{2} \chi^2 + \chi \eta \right) + \frac{1}{2} \frac{d\tau_2}{da} \psi^2 \right) \\
+ \frac{i}{96} \frac{d^2 \tau}{da^2} \psi^4 - \frac{1}{2} \frac{d \log \tau_2}{da} \frac{d\tau_2}{da} \chi \eta \psi^2 - \frac{1}{12} \frac{d^2(\tau_2^{-2})}{da^2} \eta(\tau_2 \chi)^3
\]

(3.31)

Performing gaussian integration over \( H \) we get:

\[
H = \frac{1}{2} F^+ + \frac{1}{\tau_2} \left( \frac{d\tau_2}{da} \left( \frac{1}{2} \chi^2 \right)^+ + \chi \eta \right) + \frac{d\tau_2}{da} \left( \psi^2 \right)^+
\]

and

\[
-i\mathcal{L} = \frac{1}{2} (\tau(F^-)^2 - \bar{\tau}(F^+)^2) + \tau_2 (\chi(d\psi)^+ + \eta d^* \psi + da \ast d\bar{a}) \\
+ \frac{1}{2} \frac{d\tau}{da} F(\psi^2)^- + \frac{d\tau}{da} \chi(da \wedge \psi) + FF_D + \\
+ F^+ \frac{d\tau_2}{d\bar{a}} \left( \frac{1}{2} \chi^2 \right)^+ + \chi \eta \right) + \ldots
\]

(3.32)

where \( \ldots \) denote the quartic fermionic terms.

\textbf{Duality}

The duality is manifested in two ways one may treat the fields \( F \) and \( A_D \). The first option is to integrate \( A_D \) out and to sum over all line bundles thus ensuring that \( F \) is a closed two-form with integral periods. Then \( F = \tau A \), where \( A \) is a connection in some line bundle over \( X \). The \( Q \) transformation for \( A \) can be read off from (3.17): \( QA = \psi \).

The second option is to integrate out \( F \). We get:

\[
F^- = -\frac{1}{\tau} \left( F^+_D + \frac{d\tau_2}{da} (\psi^2)^- \right)
\]

\[
F^+ = -\frac{1}{\tau} \left( F^-_D + \frac{d\tau_2}{da} \eta \chi \right)
\]

\(^{10}\) We should warn the reader that we merely point out the appearing structures without exact coefficients.
Both the action (3.32) and the $Q$ transformations become identical to those we get in the previous case with the replacements of all the fields by their duals. In particular the transformation law for $A_D$ is $QA_D = \psi_D$ as expected.

As an illustration of how this works let us look at the quartic fermionic terms, involving $\psi^4$:

\[ \xi_{\psi^4} = \frac{1}{24} \frac{d^2 \tau}{da^2} \psi^4 - \frac{1}{\tau_2} \left( \frac{d\tau_2}{da} (\psi^2)^+ \right)^2 \]  

(3.34)

As a result of integration over $F$ this term changes by a piece $\delta \xi_{\psi^4} = -\frac{1}{\tau} \left( \frac{d\tau}{da} (\psi^2)^- \right)^2$. The sum $\xi_{\psi^4} + \delta \xi_{\psi^4}$ is equal to (3.34) after the substitution $a \rightarrow a_D$ etc.

4. Correlation functions

In the next sections we explain the origin of the principles and derive the formulas for the correlators. We also test them in certain simple cases, where alternative calculations are available.

4.1. Observables and Contact terms: First appearance

We discuss the issue of observables first in holomorphic setting. Namely, we study formally the deformations of four observable by adding $p$-observables. We immediately face the necessity of introduction of the contact terms in order to preserve modular invariance. We then prove that the deformations along the nilpotent directions, studied in the previous section contain those terms. In the next section we shall supply the expressions for the deformed action, containing the anti-holomorphic fields.

The first question is to determine the appropriate deformation space. Since one may add all $p$-observables to the action it is natural to assume that the tangent space to the space of deformations is the space $V_{\mathcal{L},\Sigma}$ of $\Gamma$-invariant $V_{\Sigma} \equiv H^*(\Sigma, \mathbb{C})$-valued holomorphic functions $f$ on $\mathcal{L}$.

Given such a function $f(a)$ one may analytically continue it to a holomorphic function on $\mathcal{V}$ ($d$-exact terms do not matter) and form the action:

\[ S = \int_{\Sigma} -\mathcal{A} A_D + \mathcal{F}(A) \]  

(4.1)

The total deformation can be expanded in a series:

\[ \mathcal{F} = \mathcal{F}_0 + \sum t^{(p)} \mathcal{F}_{(p)} + \sum t^{(p_1)} t^{(p_2)} \mathcal{F}_{(p_1 p_2)} + \ldots, \quad f = t^{(p)} \mathcal{F}_{(p)} \]  

(4.2)
One may get a set of recursion relations between the terms $F_{(p_1...p_k)}$ by examining the duality properties of (4.2).

Write the deformed $F$ as a sum: $F = \sum_{p=0}^{4} F_p$, $F_p \in H^p(\Sigma)$. The action density $F(A) - AA_D$ equals $(\partial = \partial/\partial a)$:

$$
S = a_D D + \psi_D \rho + F_D F + \\
+ \partial F_0 D + \frac{1}{2} \partial^2 F_0 (F^2 + 2\psi \rho) + \frac{1}{2} \partial^3 F_0 F \psi^2 + \frac{1}{24} \partial^4 F_0 \psi^4 \\
+ \partial F_1 \rho + \partial^2 F_1 F \psi + \frac{1}{6} \partial^3 F_1 \psi^3 + \\
+ \partial F_2 F + \frac{1}{2} \partial^2 F_2 \psi^2 + \\
+ \partial F_3 \psi + \\
+ F_4
$$

(4.3)

If we integrate out $a_D, \psi_D, F_D$ first, then we get the standard action in the presence of observables. If $D, \rho, F$ are integrated out then the rest of the fields assumes the values:

$$
a_D = \partial F_0 \\
\psi_D = \partial^2 F_0 \psi + \partial F_1 \\
F_D = \partial^2 F_0 F + \frac{1}{2} \partial^3 F_0 \psi^2 + \partial^2 F_1 \psi + \partial F_2
$$

and the new action is equal to $(\tau = \partial^2 F_0)$:

$$
S_D = - \int_{\Sigma} \frac{1}{2\tau} (F_D - \frac{1}{2} \partial^3 F_0 \psi^2 - \partial^2 F_1 \psi - \partial F_2)^2 \\
+ \frac{1}{24} \partial^4 F_0 \psi^4 + \frac{1}{6} \partial^3 F_1 \psi^3 + \frac{1}{2} \partial^2 F_2 \psi^2 + \partial F_3 \psi + F_4
$$

(4.5)

We have already taken care of the determinants by the virtue of the factor $(\frac{du}{da})^{\frac{c}{2}}$ (cf. (3.28)). The new action can be interpreted as the old one written in new coordinates $a_D = \partial F_0, \psi_D = \tau \psi$ and with the new function $F_0^D$ such that $\frac{\partial^2 F_0^D}{\partial a_D^2} = -\frac{1}{\tau}$ iff the following equations are obeyed:

$$
F_p(a_D) - F_p(a) = \sum_{l=2}^{4} \frac{(-1)^{l-1}}{l!} \sum_{i_1+...+i_l=p} \frac{\partial^{l-2}}{\partial a_D^{l-2}} \frac{1}{\tau} \partial F_{i_1} ... \partial F_{i_l}
$$

(4.6)

modulo terms which do not affect the observables, e.g. $F_3 \sim F_3 + \text{const}$. Some of the equations (4.6) can be solved explicitly. In particular, if all $F_l$ for $l < p$ vanish then $F_p$
must be modular invariant: $\mathcal{F}_p = g_p(u)$. Also, if $\mathcal{F}_1 = 0$, then $\mathcal{F}_2$ and $\mathcal{F}_3$ are modular invariant $\mathcal{F}_{2,3} = g_{2,3}(u)$, while

$$\mathcal{F}_4 = g_4(u) - (\partial g_2(u))^2 \partial_\tau \log \theta(\tau)$$ (4.7)

Here $g_p(u)$ are modular invariant functions,

$$\theta(\tau) = \theta_{00}(\tau, 0),$$

$$\theta_{\alpha\beta}(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (\tau n + \beta/2)^2 + 2(n + \beta/2)z}, \quad \alpha, \beta = 0, 1$$ (4.8)

Although the solution of the equations (4.6) seems to be a complicated problem in general the formalism of Hamilton-Jacobi equations (2.22) automatically takes care of it since the modular invariance is build in it. Indeed, if we forget about the nilpotent terms in $f$ then the condition on $\mathcal{F}_0$ is simply the requirement that it defines a new $\Gamma$-invariant Lagrangian submanifold $\mathcal{L}$. The observables modify the equation of the Lagrangian submanifold to (4.4). One may think of (4.4) as of some kind of supermanifold in the space with coordinates $a, a_D, \psi, \psi_D, F, F_D$. Our super-Hamilton-Jacobi equations (2.10) are simply the generalizations of the standard canonical formalism for such supermanifolds.

The general problem is to understand how to extend a function $f$ on $\mathcal{L}$, perhaps with values in $\mathcal{V}_\Sigma$ to a globally well-defined Hamiltonian on $\mathcal{C}^{2r}$. Our proposal is to use the homogeneity properties of the asymptotically free theories. In particular, we extend the function $u$ to the homogeneous function of degree 2. Of course, in the case of rank higher then one this condition does not fix the continuation uniquely. But it is quite satisfying to see that in the case $r = 1$ this principle leads to the sensible results consistent with ghost number anomalies.

4.2. The tests of the universality and homogeneity

In this section we test the proposed principles.

**Ghost number anomaly**

The first test deals with the deformation of the microscopic theory by the four-observable constructed out of $-\frac{1}{8\pi} \text{Tr} \phi^2$. Consider the $SU(2)$ theory with $N_f$ massless hypermulitplets in the fundamental representation. The deformation amounts to the multiplication of the $k$-instanton contribution to any correlation function by the factor $e^{2\pi i tk}$.
This can be accomplished by multiplying every operator $O$ of the ghost charge $\Delta O$ by the factor $\exp\left(\frac{\pi it}{4 - N_f} \Delta O\right)$ and the whole correlation function by the factor
\[
\exp\left(\frac{3(\chi + \sigma) + N_f \sigma}{2(4 - N_f)} \pi it\right)
\]
(neglecting for the moment the subtleties related to the choice of $Spin^c$ structure, see below). Since the effective measure contains a factor $(\frac{du}{da})^2$, the $\chi$-dependent part of (4.9) can be absorbed into the redefinition of $a$:
\[
a \mapsto e^{\frac{2\pi it}{4 - N_f}} a
\]
(one should remember that the measure has an anomaly under the transformation (4.10) which is proportional to $\frac{\chi + \sigma}{2}$). The remaining part $\frac{2 + N_f}{4(4 - N_f)}$ is absorbed in the redefinition of the $\sigma$-dependent factor in the measure: $\Delta \hat{\sigma}$, since $\Delta$ is the polynomial in $u$ of degree $N_f + 2$. Nevertheless, as $\Delta$ does not coincide with $u^{N_f + 2}$ the last rescaling maps $\Delta(u)$ to another polynomial in $u$. To get around this point we “undo” the transformation on $u$ inside the integral using the invariance of integral under the changes of variables. As a result the function $a(u)$ has changed:
\[
a(u) \mapsto e^{\frac{2\pi it}{4 - N_f}} a(e^{-\frac{4\pi it}{4 - N_f}} u)
\]
Also, all the $p$-observables which involve $a$ or its derivatives have changed. We claim that (4.11) is the result of the evolution with respect to the degree $d = 2$ homogeneous Hamiltonian $H(a, a_D)$, whose restriction on $\mathcal{L}$ is $u$. Indeed, since the Hamiltonian is the conserved quantity in the course of evolution the following identity holds:
\[
u = H(a(u), a_D(u)) = H\left(e^{\frac{2\pi it}{4 - N_f}} a(e^{-\frac{4\pi it}{4 - N_f}} u), e^{\frac{2\pi it}{4 - N_f}} a_D(e^{-\frac{4\pi it}{4 - N_f}} u)\right)
\]
To check that the parameter $t$ actually coincides with the evolution time (it could have been some function of the latter) it is sufficient to compare the first derivatives. Let the Hamiltonian $H$ be the degree $d = 2$ homogeneous extension of the function $\gamma \cdot u$ where $\gamma$ is some constant. As we explain momentarily the equations of motion (2.27) yield:
\[
\frac{da(u,t)}{dt} = \frac{\partial H}{\partial a_D} = \gamma \frac{a - 2u \frac{da}{du}}{W(u)}
\]
\[\text{footnote}{\text{The factor is motivated by the ghost number anomaly } 2(4 - N_f)k - \frac{3}{2} \chi - \frac{N_f + 3}{2} \sigma \text{ in the case of trivial } Spin^c \text{ structure, see below}} \]
\[\text{footnote}{\text{which can be obtained by rescaling } \Lambda_{N_f}} \]
where
\[ W(u) = a \frac{da_D}{du} - a_D \frac{da}{du} \] (4.14)

The equations (4.13)(4.14) follow from the two conditions (2.26):
\[ \frac{da}{du} \frac{\partial H}{\partial a} + \frac{da_D}{du} \frac{\partial H}{\partial a_D} = \gamma \]
\[ a \frac{\partial H}{\partial a} + a_D \frac{\partial H}{\partial a_D} = 2\gamma u \] (4.15)

where all derivatives are restricted onto \( \mathcal{L} \). So it is enough to find \( \gamma \) such that
\[ W(u) = \gamma \frac{4 - N_f}{2\pi i} \]

Since in all cases with \( N_f < 4 \) both \( a \) and \( a_D \) are the solutions of the corresponding Picard-Fuchs equations of the form:
\[ \frac{d^2}{du^2} \begin{pmatrix} a \\ a_D \end{pmatrix} = f(u) \begin{pmatrix} a \\ a_D \end{pmatrix} / \Delta(u) \]

for appropriate \( f(u), \Delta(u) \) the quantity \( W(u) \) being the Wronskian is actually \( u \)-independent. To calculate its value it is sufficient to check the large \( u \) asymptotics in the normalization \( a(u) \sim \frac{1}{2} \sqrt{2u} + \ldots, a_D(u) \sim \frac{N_f - 4}{2\pi i} a(u) \log u + \ldots \). Using the formula (4.4) of the second paper in [1] we get that \( \gamma = -\frac{1}{2} \). Thus the evolution with respect to the Hamiltonian \( -\frac{1}{2} u \) leads to the expected dependence on the instanton charge.

Contact terms of two-observables.

The paper [8] computes the Donaldson invariants of \( \mathbb{IP}^1 \times \mathbb{IP}^1 \) and other rational surfaces. It is possible to extract the expression for the contact term between the two 2-observables constructed out of \( u \):
\[ C_{2,2}(u, u) = \left( \frac{du}{da} \right)^2 G(\tau), \] (4.16)

where
\[ G(\tau) = -\frac{1}{24} \left( E_2(\tau) - 8u \left( \frac{da}{du} \right)^2 \right) \] (4.17)

We use the formulae:
\[ a = \frac{2E_2 + \theta_{01}^4 + \theta_{00}^4}{6\theta_{00}\theta_{01}} \]
\[ \frac{da}{du} = \frac{\theta_{00}\theta_{01}}{2} \] (4.18)
\[ u = \left( \frac{\theta_{00}^4 + \theta_{01}^4}{2\theta_{00}^2\theta_{01}^2} \right) \]
It is easy to check that our expression

\[ \frac{1}{4} \left( a \frac{du}{da} - 2u \right) \]  

(4.19)

coincides with (4.16) for the pure SU(2) theory. It is also easy to check that for the theory with \( N_f \) flavors our proposal (2.39) yields for \( p = 2 \):

\[ C_{2,2}(u, u) = \frac{1}{4 - N_f} \left( a \frac{du}{da} - 2u \right) \]  

(4.20)

which again coincides with

\[ -\frac{1}{24} E_2(\tau) \left( \frac{du}{da} \right)^2 + \frac{u}{3} \]

of [9] (provided that one performs the important shift \( u \mapsto u + \frac{2}{3} \) which is not detected just by the modular invariance and asymptotics at infinity requirements of [9]).

**Derivation from blow up.** In order to derive the contact term between the two two-observables \( \mathcal{O}_u^{(2)} \) in the theory with massless quarks we perform the blow up of the manifold \( \Sigma \) at the point of the intersection of the two-surfaces \( C_1 \) and \( C_2 \). The next arguments are presented for the case of the gauge group \( G = SU(N) \) but are easily generalized for any simple-laced gauge group. We assume that the intersection is transverse and that it contributes +1 to the intersection index \( C_1 \cap C_2 \). The blow up can be achieved by gluing to \( \Sigma \) a copy of the projective plane \( \mathbb{CP}^2 \) with the orientation opposite to the one induced from the complex structure: \( \tilde{\Sigma} = \Sigma \# \mathbb{CP}^2 \). The homology lattice \( H_*(\tilde{\Sigma}) \) of \( \tilde{\Sigma} \) is that of \( \Sigma \) plus a factor of \( \mathbb{Z} \). The intersection form is simply

\[(\cdot, \cdot)_{\tilde{\Sigma}} = (\cdot, \cdot)_{\Sigma} \oplus (-1)\]

as the exceptional divisor \( e \) (the two-sphere inside \( \mathbb{CP}^2 \)) has self-intersection \(-1\). Under the isomorphism \( H_*(\tilde{\Sigma}) = H_*(\Sigma) \oplus \mathbb{Z} \) the inverse images of the cycles in \( \Sigma \) belong to the component \( H_*(\Sigma) \) of \( H_*(\tilde{\Sigma}) \). We shall denote them by the same letters as the cycles in \( \Sigma \). To derive the contact term we compare the correlation functions

\[ \langle \int_{C_1} \mathcal{O}_u^{(2)} \int_{C_2} \mathcal{O}_u^{(2)} \cdots \rangle_{\Sigma} \]  

(4.21)

and

\[ \langle \int_{C_1} \mathcal{O}_u^{(2)} \int_{C_2} \mathcal{O}_u^{(2)} \cdots \rangle_{\Sigma} \]  

(4.22)
where the cycles $\tilde{C}_1, \tilde{C}_2 \in H_*(\tilde{\Sigma})$ do not intersect each other in the vicinity of $p$ and are
given by the formulae:

$$\tilde{C}_k = C_k - e, \quad \#\tilde{C}_1 \cap \tilde{C}_2 = \#C_1 \cap C_2 - 1$$ (4.23)

First of all:

$$\langle \int_{C_1} \mathcal{O}_u^{(2)} \int_{C_2} \mathcal{O}_u^{(2)} \ldots \rangle_{\Sigma} = \langle \int_{\tilde{C}_1} \mathcal{O}_u^{(2)} \int_{\tilde{C}_2} \mathcal{O}_u^{(2)} \ldots \rangle_{\tilde{\Sigma}}$$ (4.24)

This follows from the topological invariance of the correlators: the point of blow up can be
taken far away from the intersection point of $C_1$ and $C_2$. The very existence of the blowup
of the zero size should not affect the correlation functions of the operators which do not
contain integration over the glued $\mathbb{P}^2$. The next step is the exploring of the structure
of the moduli space of instantons on $\tilde{\Sigma}$. Let us take the metric on $\tilde{\Sigma}$ to be such that the
 glued $\mathbb{P}^2$ is very far away from the rest. By conformal transformation we may view the
manifold $\tilde{\Sigma}$ as the bouquet of $\Sigma$ and $\mathbb{P}^2$ glued at the point $p$. This picture is similar to
the familiar representation of the moduli space of holomorphic bundles on two dimensional
surface (see, e.g. [21]). The instanton charge may be divided between $\Sigma$ and $\mathbb{P}^2$. We
represent the moduli space of instantons on $\tilde{\Sigma}$ as the quotient of the disjoint union of the
products of the framed moduli spaces of instantons on $\Sigma$ and $\mathbb{P}^2$:

$$\mathcal{M}_{k,\tilde{\Sigma}} = \left( \coprod_{l=0}^{k} \mathcal{M}_{l,\Sigma; P} \times \mathcal{M}_{k-l,\mathbb{P}^2; P} \right) / G$$ (4.25)

where $P$ denotes the point where the allowed gauge transformations must be equal to one.
The dimensions of the framed moduli spaces for the group $G = SU(N)$ are equal to:

$$\dim \mathcal{M}_{l,\Sigma; P} = 4Nl - \frac{N^2 - 1}{2} (\chi + \sigma - 2)$$
$$\dim \mathcal{M}_{k-l,\mathbb{P}^2; P} = 4N(k-l)$$ (4.26)

The correlation functions in the theory are the integrals of certain cohomology classes over
the moduli spaces of instantons. To get the different correlation function the observable
associated with the cycles in $\mathbb{P}^2$ must have the ghost charge at least $4N$. One and two
2-observables constructed out of $u^k$ for any $k = 1, \ldots, N-1$ have the total ghost charge
less or equal to $4N - 4$. Therefore, the correlation function does not change if we replace

---

13 The framed moduli space is the quotient of the space of solutions of instanton equations by
the group of punctured gauge transformations
$C_k$ by $\tilde{C}_k$ (see [22][23] for the mathematical proof of this result for $G = SU(2)$).

The net result of our manipulations is the replacement of the intersecting cycles on the manifold $\Sigma$ by the non-intersecting cycles on the manifold $\tilde{\Sigma}$. Physically the crucial fact is that under the blowup of a point $P$ a new two-cycle $e$ appears and it leads to the possibility for the gauge field to have a flux through it. In the low-energy effective theory the insertion of the new two-cycle must be reflected in the new factor in the Maxwell partition function, which is the sum over all line bundles on $\mathbb{P}^2$ in the presence of two 2-observables $O^{(2)}_{u_k}$ and $O^{(2)}_{u_l}$ integrated over $e$. This partition function is simply:

$$\frac{\partial u_k}{\partial a^i} \frac{\partial u_l}{\partial a^j} \frac{\partial}{\partial \tau_{ij}} \log \Theta(\tau)$$

(4.27)

In the rank one case it reduces to

$$\left( \frac{du}{da} \right)^2 \frac{\partial}{\partial \tau} \log \theta(\tau)$$

(4.28)

after the shift $\tau \rightarrow \tau + 1$. The origin of the sign $(-1)^{(\lambda, \rho)}$ in $\Theta(\tau)$ is the same curious minus sign of [19]. It can be derived by examining the sign of fermion determinants by means of index theorem if the fact that $c_1(\mathbb{P}^2) = 3$ is taken into account. By the way, the appearance of this minus sign is quite analogous to the shift $\lambda \rightarrow \lambda + \rho$ in two dimensional gauge theories [24][25][26].

One may check that this expression is again equivalent to (4.16). The reasoning above does not immediately carry over to the case of arbitrary 2-observables constructed out of $P_1(u)$ and $P_2(u)$. However, assume the validity of the “Künneth” formula inside the correlator:

$$\langle \int_C O^{(2)}_{P(u)} \ldots \rangle = \langle O^{(0)}_{dP(u)/du} \int_C O^{(2)}_{u} \ldots \rangle + \langle \sum_{i,j} \eta^{ij}_C \int_{L_i} O^{(1)}_{dP(u)/du} \int_{L_j} O^{(1)}_{u} \ldots \rangle$$

with $\Delta(C) = [pt] \otimes C \oplus C \otimes [pt] \oplus \eta^{ij}_C L_i \otimes L_j$, $L_i \in H_1(\Sigma)$. Hence we reduce this case to the one already studied.

Finally, notice that the argument with the counting of the dimensions of framed moduli space suggests that the contact term between two 2-observables $O^{(2)}_{u}$ in the $SU(2)$

\footnote{For general simple groups the number $4N$ is replaced by $4h^\vee$ where $h^\vee$ is the dual Coxeter number, while the maximal ghost charge of two 2-observables is $2(2h - 2)$ since the highest degree of the invariant polynomial on $\mathfrak{g}$ is $h$. For simply-laced groups $h^\vee = h$.}
theory with $N_f = 1$ coincides with (4.28). However, two 2-observables may saturate the dimension bound 4 in the theory with $N_f = 2$. Indeed, the formula (4.20) differs from (4.28) in this case by a one-instanton contribution. We hope to elaborate further on this subject elsewhere.

Blowup formula. If the 2-observable $\mathcal{O}^{(2)}_g$ corresponding to the exceptional divisor is inserted in the correlation functions then the Maxwell partition function with the contact term included changes by a factor:

$$\frac{\theta_{00}(\tau, \frac{dg}{d\tau})}{\theta_{00}(\tau, 0)} e^{-\left(\frac{dg}{d\tau}\right)^2 \frac{1}{2} \log \theta_{00}(\tau, 0)}$$ (4.29)

This explains the origin of the blowup factor in the formula (2.15). It is also equivalent to the formula of R. Fintushel and R. Stern [23]. In the case of the theory with matter one must take into account the curious minus sign of [19] and the change in the gravitational renormalization factor.

Comments on the last derivation of the contact term. As an attempt to formulate the appropriate analogue of Gromov-Witten paradigm we propose to think of the computation of the correlation function as of the two step procedure. Let $g_1, \ldots, g_k$ be the zero-observables. First of all, the topological field theory prepares the cohomology class

$$\mathcal{I}_k(g_1 \otimes \ldots \otimes g_k) \in H^*(\bar{M}_k, \Sigma)$$ (4.30)

where $\bar{M}_k, \Sigma$ is the suitable compactification of the moduli space of $k$ distinct points on $\Sigma$ (an analogue of Deligne-Mumford compactification).\footnote{As an example one may keep in mind Fulton-MacPherson’s space, which is the resolution of diagonals in $\Sigma^k$ for complex $\Sigma$}

Next, the actual correlation function is defined as long as the cycle $\Theta_C \in H_*(\bar{M}_k, \Sigma)$ is chosen:

$$\langle \int_{C_1^{(l_1)}} (g_1)^{(l_1)} \cdots \int_{C_k^{(l_k)}} (g_k)^{(l_k)} \rangle = \int_{\Theta_C} \mathcal{I}_k(g_1 \otimes \ldots \otimes g_k)$$ (4.31)

Here $\Theta_C$ is the cycle in $\bar{M}_k, \Sigma$ constructed as follows. Let $\pi : \bar{M}_k, \Sigma \to \Sigma^k$ be the tautological map. Let $\Delta$ be the big diagonal where at least for one pair $i \neq j$, $x_i = x_j$. The preimage
of the cycle $C_1 \times \ldots \times C_k - \Delta$ is a locally finite chain in $\bar{M}_{k,\Sigma}$. Its closure in $\bar{M}_{k,\Sigma}$ is the cycle $\Theta_{\bar{C}}$.

Now suppose that all $C_\alpha$’s have dimensions zero and two. It means that the representative in $H_*(\bar{M}_{k,\Sigma})$ of $\Theta_{\bar{C}}$ can be chosen in such a way that its projection $\pi(\Theta_{\bar{C}})$ intersects the big diagonal $\Delta$ only at the stratum of the lowest codimension (this is a fancy way of saying that the cycles $C_\alpha$ can be chosen either intersecting each other transversely or not intersecting at all).

Hence we must understand the topology of $\bar{M}_{k,\Sigma}$ in the vicinity of these strata. In the example of [7] it is the blowup of the diagonal $\Delta \subset \Sigma \times \Sigma$. Since the blowup can be performed in the real category it is plausible that the proper moduli space $\bar{M}_{k,\Sigma}$ does look like a blowup of the diagonal near this region. Fix one point $P \in \Sigma$. Its preimage under the forgetful map $\bar{M}_{2,\Sigma} \to \bar{M}_{1,\Sigma} = \Sigma$ is the manifold $\Sigma$ blown up at the point $P$. If we interpret the fiber of the forgetful map over the point $P$ as the “effective space-time manifold in the presence of the observable inserted at the point $P$ (it may be a density of some operator)” then the appearance of the gauge theory on the blowup becomes natural. However, we don’t know whether these ideas can be easily generalized to the theories with matter.

Loosely speaking, one may say that while the origin of the non-perturbative corrections to the effective coupling comes from the point-like instantons, the contact terms between non-local operators come from the point-like monopoles created at the intersection points.

We shall discuss the compatibility of the contact terms in the four dimensional theory and the contact terms in the two dimensional type B theory in the corresponding chapter 7.

4.3. Observables and contact terms: second act

The next issue is to find the expressions for the observables consistent with the duality in the presence of $Q$-exact regulators. In this section we consider the non-trivial case of the two-observable. Here we also meet the difficulty which we mentioned at the end of the discussion of microscopic theory. Two-cycles on a four-manifold typically intersect each other. Therefore we expect to have contact terms associated with any such intersection.

The obvious problem is that the 2-observable, being holomorphic in $a$ (and $\tau$) under the duality transformation can only by multiplied by $\tau$. Let $w$ represents the cohomology
class Poincare dual to $C^2$. It can be decomposed as $w_+ + w_-$ - the sum of the self-dual and anti-selfdual parts, thereby

$$(O^2_u, w) \sim \int F^+ \wedge v_+ + F^- \wedge v_- = 2\pi (m_+, v_+) + 2\pi (m_-, v_-)$$

where $v = \frac{du}{da} w$. The equations (3.33) suggest that under the duality the curvature $F$ “transforms” as:

$$F^+_D \sim \bar{\tau}F^+, \quad F^-_D \sim \tau F^-$$

thereby establishing a contradiction between the modular invariance of $(O^2_u, w)$ and its holomorphy.

The idea is to examine the supersymmetry transformations (3.1). On the equations of motion $H \sim F^+$ and therefore the problematic part of the 2-observable is $Q$-exact on shell. It appears reasonable to eliminate this part of the observable by adding something $Q$-exact to the action. To do it carefully we write everything with the help of the auxiliary field $H$ and eliminate it only when the final expression for the action including the observables is constructed.

**Modified action.**

We deform $L_0$ by adding to it the naive expression for two-observable:

$$S_0 = L_0 + \int \Sigma w_\alpha \wedge \left( \frac{dg^\alpha}{da} F + \frac{1}{2} \frac{d^2 g^\alpha}{da^2} \psi \psi \right)$$

(4.32)

We also add to (3.31) a $Q$-exact term:

$$S = S_0 + \{Q, R_{w,g} \}$$

(4.33)

with $R_{w,g} = R_0 - 2 \frac{dg^\alpha}{da} (\chi_+, w\alpha,+)$. Computing $\{Q, \ldots \}$ and integrating out $H$ we get:

(i) a factor $(2\tau_2)^{1/2} b^+_2$ (this is regularization independent part of the determinant - see [19] for thorough discussion);

(ii) effective $w$-dependent interaction:

$$\frac{dg^\alpha}{da} (w\alpha, -, F) + \frac{dg^\alpha}{da} (w\alpha, +, \frac{1}{\tau_2} \frac{d \log \tau_2}{da} \eta \chi) + \frac{1}{\tau_2} (w\alpha, +, \frac{dg^\alpha}{da}, w\beta, +, \frac{dg^\beta}{da})$$

(4.34)

**Contact term again.** The next question is whether the modification of the naive observable by the $Q$-exact term provides the desired modular invariance. To check this we
must compare two results: an integration over \( F \) with fixed \( A_D \) or integration over \( A_D \) (with summation over all line bundles understood) with fixed \( F \) which turns out to be \( F = dA \).

The first integration yields:

\[
F^- = -\frac{1}{\tau} \left( F_D^- + \frac{d\tau}{da}(\psi^2)^- + w_{\alpha,-} \frac{dg^\alpha}{da} \right) \\
F^+ = -\frac{1}{\bar{\tau}} \left( F_D^+ + \frac{d\tau_2}{da} \eta \chi \right) 
\]

with the resulting action for \( A_D \) and the rest of the fields equal to the sum of (3.32) and

\[
\frac{dg^\alpha}{da}(w_{\alpha,-}, F) + \frac{dg^\alpha}{da}(w_{\alpha,+}, \frac{d\log \tau_2}{da} \eta \chi) + \frac{1}{\tau_2}(w_{\alpha,+} \frac{dg^\alpha}{da}, w_{\beta,+} \frac{dg^\beta}{da}) + \\
\frac{1}{\tau}(w_{\alpha} \frac{dg^\alpha}{da}, \beta \frac{dg^\beta}{da}) 
\]

On the other hand, the integration over \( A_D \) would simply produce (4.34) as \( w \)-dependent piece of the resulting action.

The difference is therefore (we temporarily do not discuss the prefactors which come from the evaluations of determinants):

\[
\frac{1}{\tau}(w_{\alpha}, w_{\beta}) \frac{dg^\alpha}{da} \frac{dg^\beta}{da} 
\]

This “anomaly” clearly has to do with the intersections of the two-cycles, since it is proportional to \((w_{\alpha}, w_{\beta})\). This is precisely the extra terms which we saw in (4.5) and whose cancelation was encoded in the equations (4.6). As we mentioned in the previous section, when the cycles \( C^k \) intersect the truly \( Q \)-closed observables in the effective theory are modified by the additions of contact terms, attributed to the intersections of the cycles. In our case the contact term can be guessed from the requirement that it must be holomorphic, almost modular invariant and vanish in the perturbative regime \( u \to \infty \) where everything can be calculated using Feynmann diagrams\(^{16}\).

These conditions yield the operator (up to modular invariant operator \( O'_{\alpha \beta} \)):

\[
O_{\alpha \beta} = \frac{dg^\alpha}{du} \frac{dg^\beta}{du} G(u, \tau) 
\]

\(^{16}\) We did perform this computation
which is to be inserted at the point \( p_{\alpha\beta} = C_\alpha \cap C_\beta \). Here:

\[
G(u, \tau) = -\frac{1}{24} \left( E_2(\tau) \left( \frac{du}{da} \right)^2 - 8u \right) = \frac{1}{\pi i} \left( \frac{du}{da} \right)^2 \partial_\tau \log \theta(\tau) \quad (4.39)
\]

The first form of the contact term was presented in [9]. The second form is valid in the pure Donaldson theory and its rôle is already explained in the section devoted to the blowups. The third form \( (4.19) \) was discussed in the previous section. In order to normalize \( G(u, \tau) \) properly we use the fact that as \( u \to \infty, \tau \to i\infty \) and \( a \sim \frac{1}{\sqrt{2}} \sqrt{u} \).

### 4.4. Macroscopic measure

As the last check of our assertions let us try to calculate the \( u \)-plane integral, contributing to

\[
\langle e^{pO_u(0)} \rangle \quad (4.40)
\]

We denote the contribution of the \( u \)-plane to this correlator by \( Z_f \). We set \( b_2^+ = 1 \).

The measure which we get has the form:

\[
d\mu = (\tau)^{-b_2^+/2} D_a D\bar{a} D\chi D\eta e^{S+b(u)\chi(X)+c(u)\sigma(X)+pu} \quad (4.41)
\]

(to compare with [8] notice that the powers of \( \tau^2 \) coming from the kinetic term of the scalars \( a, \bar{a} \) are cancelled by the similar terms coming from the kinetic term of \( \eta, \psi \) by supersymmetry).

Here \( b(u), c(u) \) are the gravitational renormalization coefficients computed for the low-energy \( SU(2) \) theory in [19]:

\[
e^{b(u)\chi+c(u)\sigma} = \left( (u^2 - 1) \frac{d\tau}{du} \right)^{\frac{\chi}{\pi}} (u^2 - 1)^{\frac{\sigma}{\pi}} \quad (4.42)
\]

We can rewrite it using the formulae of [8]:

\[
\theta(\tau)^{-\chi} \left( \frac{(du)^3}{(da)^2 d\tau} \right)^{\frac{\chi+\sigma}{\pi}} \quad (4.43)
\]

In the form \( (4.43) \) the gravitational renormalization factor is not valid in arbitrary abelian twisted \( \mathcal{N} = 2 \) theory, but has more transparent physical meaning in the Seiberg-Witten theory. The modular non-invariant part \( \theta^{-\chi} \) has the following origin. Suppose one performs a blowup of the manifold \( \Sigma \) at some point \( p \), i.e. glues \( \mathbb{P}^2 \) to it. If the size
of the glued piece is small and no operators are inserted at the point of the blowup then their correlations functions must be unchanged. On the other hand the partition function of Maxwell theory gets extra factor $\theta(\tau)$ from summing over the magnetic fluxes through the two-sphere created in the blowup. The Euler characteristics $\chi$ increases by one and therefore the factor $\theta^{-\chi}$ cancels the extra piece of the Maxwell partition function.

The factor $\left(\frac{(du)^3}{(da)^2d\tau}\right)^{\frac{\chi+\sigma}{4}}$ is modular invariant. It accounts for the $U(1)_R$ anomaly in the perturbative $u \to \infty$ regime. It follows easily from the Picard-Fuchs equations (2.16) (cf. [14] [17] for the case of pure $N=2$ theory) that the factor $\left(\frac{(du)^3}{(da)^2d\tau}\right)^{\frac{\chi+\sigma}{4}}$ equals $\left(\frac{\Delta}{f}\right)^{\frac{\chi+\sigma}{4}}$.

There is another gravitational correction to the effective action, described in [19], namely, if the manifold $X$ is not spin, then there is a term:

$\left(-1\right)^{(w_2(X),m)}$

We can get rid of it in the course of the study of $SU(2)$ theory by the shift $\tau \to \tau + 1$ thanks to Wu formula

$(m,m) \equiv (w_2(X),m)_{\mod 2}$

for any $m \in H^2(X;\mathbb{Z})$ (see the discussion under the formula (4.28)).

Putting all things together we arrive at the following measure on the $u$-plane:

$$d\mu = d\mu_0\Phi(u,w^2)$$

$$d\mu_0 = \frac{D\alpha D\alpha D\hat{\chi}}{4\pi i/2/\theta(\tau)} \left(\frac{(du)^3}{(da)^2d\tau}\right)^{\frac{\chi+\sigma}{4}} \frac{d\tau_2}{d\alpha} e^{\pi(v_+,v_+)/2\tau_2} + pu \Theta(\tau, w) \quad (4.44)$$

$$\Theta(\tau, w) = \sum_{m \in \Lambda} \left(m_+ + \frac{v_+}{2i\tau_2}\right) e^{\left(v_- + i\tau_2\right)(m, m) - \frac{\tau_1}{2}(m, m)}$$

Here

$$\Phi(u,w^2) = e^{2\pi i G(\tau,u)(w,w)} \quad (4.45)$$

Sample computation on $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. The cohomology lattice of $\Sigma$ is two dimensional: $H^2 = \mathbb{Z}e_b \oplus \mathbb{Z}e_f$ and the self(anti-self)dual components of the element $m = m_1e_b \oplus m_2e_f$ are:

$$m_\pm = \frac{m_1}{2R} \pm m_2R \quad (4.46)$$
Here $2R$ is the ratio of the areas of the base $\mathbb{P}^1$ and fiber $\mathbb{P}^1$. The Kähler form decomposes as:

$$\omega = t \left( \sqrt{2R} e_b \oplus \frac{1}{\sqrt{2R}} e_f \right)$$

(4.47)

where $t$ measures the overall size of the manifold, while $R$ corresponds to its “shape”\[17\]. The computations of the correlation functions of zero- and two-observables can be performed in the large $t$ limit, where all fields can be replaced by the zero modes, i.e. by the harmonic forms. In particular we may drop all the fields $\psi$. Also, the curvature of the gauge field $F$ equals $2\pi$ times the harmonic representative of the integral cohomology element $m$.

Collecting all the terms in the measure we arrive at the formula:

$$d\mu = \frac{d\tau \, d\bar{\tau}}{\tau_2^{1/2} \theta(\tau)^4} \left( \frac{(du)^3}{(da)^2 d\tau} \right) e^{\frac{\pi v_+^2}{\tau_2} + \frac{4\pi^2 w_1 w_2 G(\tau, u) + pu}{2\tau_2} \Theta(\tau, w)}$$

(4.48)

where the function $\Theta(\tau, w)$ is given by:

$$\sum_{m_1, m_2 \in \mathbb{Z}} \left( \frac{m_1}{2R} + m_2 R + \frac{v_+}{2\tau_2} \right) e\left[ -v_-(\frac{m_1}{2R} - m_2 R) - \tau_1 m_1 m_2 + i\tau_2 \left( \frac{m_1^2}{4R^2} + m_2^2 R^2 \right) \right]$$

(4.49)

Here

$$v_\pm = \frac{v_1}{2R} \pm v_2 R$$

and $v_1 = \frac{dH}{da} w_1$, $v_2 = \frac{dV}{da} w_2$ for the holomorphic functions $H$ and $V$ corresponding to the “horizontal” (base $\mathbb{P}^1$) and “vertical” (fiber $\mathbb{P}^1$) two-observables. It is convenient to perform Poisson resummation in $m_2$ (or $m_1$). Denoting the dual variable by the same letter and omitting the unimportant overall factors we get:

$$d\mu = \frac{du \wedge d\bar{\tau}}{R^2 \tau_2^2} e^{v_1 v_2 \hat{G} + pu} \frac{du}{da} \sum_{m_1, m_2} (m_2 - m_1 \tau - 2v_2 R^2) \times$$

$$\times e^{\left[ \frac{i}{4\tau_2} |m_2 - m_1 \tau|^2 + \frac{i v_-}{2\tau_2 R} \right]}$$

(4.50)

where

$$\hat{G} = -\frac{\pi^2}{6} \left( \hat{E}_2 - 8u \left( \frac{da}{du} \right)^2 \right), \quad \hat{E}_2 = E_2 - \frac{3}{\pi \tau_2}$$

\[17\] The strange notation $2R$ is motivated by the analogy of the integral we are computing to the one-loop computation of the partition function of the string on a circle of radius $R$ \[27\]\[28\].

38
The sum over \((m_1, m_2)\) is performed as follows. First, the contribution of \(m_1 = m_2 = 0\) equals

\[
\bar{\partial} \left( \frac{du}{w_1} e^{v_1 v_2 \hat{G}} \right) \tag{4.51}
\]

Now assume \(m_1^2 + m_2^2 \neq 0\). Let \(N\) denotes the maximal common divisor of \(m_1\) and \(m_2\) is both of them are not zero. Otherwise \(N\) denotes the one of numbers \(m_1\) and \(m_2\) which does not vanish. We fix its sign by the requirement \(N < 0\) if \(m_1 > 0\). Define the element of the quotient

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \mathbb{Z} \begin{pmatrix} c & d \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})/\mathbb{Z}
\]

by the conditions: \(m_1 = -Nc, m_2 = Nd, c > 0\). The sum over the rest of \(m_1\) and \(m_2\) is performed in two steps. First we sum over \(g\) by unfolding the integration region and replacing it by the strip \(\mathcal{H}/\mathbb{Z}\). The integrand for given \(N\) equals:

\[
\bar{\partial} \left( \frac{(du)^2}{Nd a + w_1 dH} e^{v_1 v_2 G + p u} \mathbf{e}^{\left[ \frac{i}{4 R^2 \tau_2} (N + v_1)(N - 2 v_2 R^2) \right]} \right) \tag{4.52}
\]

The final step is the integration by parts. The boundary of the integration region has various parts. In the limit \(R \to 0\) the only surviving contribution is that of \(\tau_2 \to \infty\). One may drop the non-holomorphic part of the \(\hat{G}\) function there. To prove this one integrates along the contour \(\tau_2 = \text{const}\) first (integration across the strip of \(14\)) and then takes the limit \(\tau_2 \to \infty\). The integration across the strip simply picks up a residue.

One should understand that the integral we were evaluating is not absolutely converging. The last steps of its computation involved a particular prescription. In fact, what is crucial is the choice of the particular coordinate near the cusp which determines the sequence of contours. Our choice \(\tau_2 = \text{const}\) is dictated by the metric on the Coulomb branch \(\tau_2 \text{d} \bar{a}\) but there may be other choices as well. Their study is important in the problem of deformation of theory \(29\).

Thus, we arrive at the expression:

\[
\sum_{N \in \mathbb{Z}} \frac{(du)^2}{Nd a + w_1 dH} e^{v_1 v_2 G + p u} \tag{4.53}
\]

The series in \(N\) may be summed up to the \(\text{cotan}\) function \(8\) but we shall keep it in this form as it is this form which reveals the similarity to the Landau-Ginzburg model of chapter 7.

\[\text{18 One must average over the quotient } SL_2(\mathbb{Z})/\Gamma^0(4) \text{ but we skip this step as it is straightforward in this case.}\]
4.5. Contour integral representation of the correlators.

To finish with the derivations of the formulae for the correlation functions of all \( p \)-observables announced in the beginning of the paper we need a contour integral representation for the contribution of singular points on the \( u \)-plane to the correlator.

Introduce the following Seiberg-Witten function -

\[
SW(T) = \sum_{\ell} SW(\ell) e^{\mathcal{F}(a + \psi + \ell; T)} \tag{4.54}
\]

Here \( SW(\ell) \) is the Seiberg-Witten invariant defined in the section 2. Then the contribution of the singularity where \( a = 0 \) on the \( u \)-plane to the correlation function can be expressed as the residue:

\[
\langle \ldots \rangle = \int [d\psi] \oint \frac{da}{a} a^{\frac{\chi + \sigma}{8}} \Delta^a_s \left( \frac{du}{da} \right)^\frac{\chi}{2} SW(T) \tag{4.55}
\]

To prove this one notices that the effective couplings and the gravitational renormalization function change upon including the massless field of the hypermultiplet. This change is taken into account by the multiplication of the effective measure by the factor:

\[
a^{\ell, \ell - \sigma} = a^{\frac{\chi + \sigma}{8} \cdot \frac{df}{d\ell}} \tag{4.56}
\]

In the chapter 6 we discuss this issue in more details. The rest is done by the remark that \( a + \psi + \ell \) is the decomposition of the universal curvature form over the product \( \mathcal{M}(\ell) \times \Sigma \), therefore the integral over \( \mathcal{M}(\ell) \) is computed by picking out of the effective measure including the factor \( (4.56) \) the term proportional to \( a^{\frac{df}{d\ell}} \) and replacing it by \( SW(\ell) \). This is what \( (4.55) \) does\(^{\text{19}} \). The formula similar to \( (4.55) \) (for all higher times set to zero and only with 2-observables) is written in \[9\] where it was derived by the analysis of the jumps of the Coulomb branch contribution to the correlation function on the manifolds with \( b^+_2 = 1 \) and \( b^-_2 \geq 9 \). These jumps must be cancelled by the similar jumps of the contributions of the monopole and dyon points, anticipated in \[11\]. The comparison gives the formula of the form similar to \( (4.55) \) which by universality extends to other cases \((b^+_2 > 1)\) as well.

\(^{19}\) One does not have to worry about the higher observables since in this case the moduli space \( \mathcal{M}(\ell) \) is compact and the K"unneth formula works, see below
5. Generalizations I. Theories with matter and higher dimensional theories.

The pure $\mathcal{N} = 2$ super-Yang-Mills theory has been shown \cite{4,20} to be equivalent (after appropriate twisting) to the theory, computing the Donaldson invariants of a four-fold. Donaldson theory studies the intersections of certain homology classes of the moduli space of instantons.

The idea of physical approach is the following. The twisted theory, when formulated on a general four-fold has an unbroken scalar supercharge $Q$ which squares to zero on the gauge-invariant observables. The action of the theory is $Q$-closed, and its metric-dependent part is $Q$-exact. Therefore, the correlation functions of $Q$-closed operators are metric-independent and define the invariants of a smooth structure of manifold (for a review, see \cite{30}). Now, the important piece of the action to look at is

$$\frac{1}{e^2} \|F^+\|^2 + \ldots$$

where $\ldots$ denote fermionic part, making the whole thing $Q$-exact. By taking the limit $e^2 \to 0$ one realizes that the $Q$-closed observables correlators are nothing but the integrals over instanton moduli spaces $\mathcal{M}_k$. This argument shows that in any $\mathcal{N} = 2$ twisted gauge theory the correlators of $Q$-closed operators are certain integrals over $\mathcal{M}_k$.

5.1. Observables and K"unneth formula

The $Q$-cohomology in the sector of zero-observables is provided by the space of all gauge invariant functions of $\phi$, i.e. the functions of $u \sim \text{Tr}\phi^2$. Since the field $\phi$ is the two-by-two traceless matrix all the higher casimirs $\text{Tr}\phi^{2r}$ are up to a multiple the powers of $u$. Let us introduce the notation:

$$Ch_r = \text{Tr} \left( \frac{\phi}{2\pi i} \right)^r$$

Then the property of $\phi$ to be the traceless two-by-two matrix translates to the equation:

$$Ch_{2r+1} = 0, \quad Ch_{2r} = 2^{1-r} Ch_r^r, \quad r = 0, 1, \ldots$$

These equations are the classical relations in the cohomology of $BG$ for $G = SU(2)$. In the quantum theory the relations (5.2) might be modified. To explain what we mean let us discuss the meaning of the observables (5.1) in the gauge theory.
One usually assumes the existence of the universal instanton $\mathcal{E}$ over the product $\tilde{\mathcal{M}}_k \times \Sigma$ of the compactified moduli space of instantons and the space-time manifold $\Sigma$. The object $\mathcal{E}$ is the bundle over the interior $\mathcal{M}_k$ of $\tilde{\mathcal{M}}_k$ corresponding to the ordinary gauge connections. We assume that there are no reducible connections. The trouble hides at the “boundary” of $\tilde{\mathcal{M}}_k$, where the point-like instantons are situated. K. Uhlenbeck’s compactification \cite{31} which simply adds the point-like instantons does not allow for the universal instanton to exist as a bundle over the totality of $\mathcal{M}_k \times \Sigma$. Nevertheless, the bundle $\mathcal{E}$ does exists over the complement $\mathcal{M}_k^c$ to the submanifold $S$ of codimension 8 in $\tilde{\mathcal{M}}_k \times \Sigma$. The submanifold $S$ is the union of the product $\mathcal{M}_{k-1} \times (\Delta \subset \Sigma \times \Sigma)$ and higher codimension strata of Uhlenbeck’s compactification. The complement to this submanifold is the space of pairs $(A_k, x)$ where $A_k$ is the instanton connection of charge $k$ and $x \in \Sigma$ or $(A_{k-1}, y, x)$ where $A_{k-1}$ is the instanton connection of charge $k-1$ and $x \neq y \in \Sigma$. Here $y$ is the center of the point-like instanton. The fact that the running point $x$ does not hit the center of the point-like instanton is the origin of the existence of the continuation of the universal instanton. The existence of $\mathcal{E}$ over $\mathcal{M}_k^c$ allows to define the following element $\Theta$ of $H^4(\tilde{\mathcal{M}}_k \times \Sigma)$. Its integral over a four-cycle represented by the simplices, which do not

\[ \sum_{p=0}^{4} Ch_{r}^{(p)} \otimes e_p \in H^*(\mathcal{M} \times \Sigma) \] \hfill (5.3)

is nothing but the class $Ch_r(\mathcal{E})$ (the usage of complex language is not important here). As it was already noticed in the physical context in \cite{32} and subsequently widely used in the studies of $D$-branes the relations (5.2) which are certainly obeyed for a rank two bundle with trivial determinant may be violated if the object $\mathcal{E}$ is a sheaf. The simplest type of such violation is the relation between $Ch_4$ and $\frac{1}{2} Ch_2^2$: from the sequence one gets the relation between the Chern characters.

\[ Ch_4 = \frac{1}{2} Ch_2^2 + \delta_1 \] \hfill (5.4)

where $\delta_1$ is the Poincare dual to the submanifold $\mathcal{M}_{k-1} \times (\Delta \subset \Sigma \times \Sigma) \subset \mathcal{M}_k$ of the point-like instantons of charge 1.

\[ 42 \]
intersect $S$ (such a representative exists for dimensional reasons) is given by the integral of the second Chern class of $\mathcal{E}$. In other words, the imbedding $\mathcal{M}^o_k \rightarrow \bar{\mathcal{M}}_k \times \Sigma$ induces the isomorphism of the fourth cohomology and the class $\Theta$ is the class whose pull-back coincides with $c_2(\mathcal{E}) \in H^4(\mathcal{M}^o_\Sigma)$.

To summarize, we expect that the correlation functions of $Ch_2^{(p)}$ differ from those of $2^{1-r}(Ch^r_2)^{(p)}$. To test that the correlation functions of the latter are computed correctly we check the validity of “Künneth” formula:

$$\langle \int_C \mathcal{O}^{(dC)}_{P_1 P_2 \cdots} \rangle = \langle \sum_{A,B \in H_*(\Sigma)} \eta^{AB}_C \int_A \mathcal{O}^{dA}_{P_1} \int_B \mathcal{O}^{dB}_{P_2} \cdots \rangle$$ (5.5)

where the structure constants $\eta^{AB}_C$ are the coefficients of the decomposition of the diagonally embedded class $C \in H_{dC}(\Sigma)$ into $H_*(\Sigma \times \Sigma)$ (hence the name of the formula):

$$\Delta(C) = \eta^{AB}_C A \otimes B, \ A \in H_{dA}(\Sigma), \ etc.$$  

Here $P_{1,2}$ denote the polynomials in $u$. The formula (5.5) has to hold if the construction of the descendants goes via decomposing the class

$$P_1(Ch_2)P_2(Ch_2) \in H^*(\mathcal{M} \times \Sigma) \approx H^*(\mathcal{M}) \otimes H^*(\Sigma)$$

The formula (5.5) is not guaranteed to hold true in any topological field theory. For example, it is not valid in two dimensional type A sigma model. The interested reader may try to compute a few correlators of $(\omega^2)^{(2)}$ in CP$^2$ sigma model using (5.5) and compare them with the highly non-trivial results of [3].

Check on $\mathbb{P}^1 \times \mathbb{P}^1$. We check (5.5) in the case of $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. We have $H^*(\Sigma) = \mathbb{C}\langle 1, e_b, e_f, e_\Sigma \rangle$ with the multiplication law:

$$e_\alpha \cdot 1 = e_\alpha, \ e_b \cdot e_f = e_f \cdot e_b = e_\Sigma, \ e_\alpha^2 = 0, e_\alpha \neq 1$$

By generalizing (4.53) to the case of two-observables constructed out the polynomials $H(u) = \sum_k H_k u^k$ (corresponding to the large $\mathbb{P}^1$) and $V(u) = \sum_k V_k u^k$ (corresponding to the small $\mathbb{P}^1$) we get the formula (the times $T^{k,1}$ are implicitly included):

$$Z = \langle e^{\int_\Sigma e_f \wedge \mathcal{O}^{(2)}_H + e_b \wedge \mathcal{O}^{(2)}_V + e_\Sigma \wedge \mathcal{O}^{(0)}_H} \rangle = \sum_{N \in \mathbb{Z}} \int \frac{(du)^2}{N da + dH} e^{\Pi(u) + \frac{\mu'}{4}(a_{\Sigma}^\mu - 2u)}$$ (5.6)
For the polynomial $P = \sum_k P_k u^k$ we denote by $\nabla_{P,\alpha}$ the differential operator $\sum_k P_k \frac{\partial}{\partial T_k^\alpha}$. The formula (5.3) is equivalent to the identity:

$$\nabla_{P_1,1} P_{2,1} Z = \left[ \nabla_{P_1,1} \nabla_{P_2,e_S} + \nabla_{P_1,e_b} \nabla_{P_2,e_f} + (1 \leftrightarrow 2) \right] Z$$  \hspace{0.5cm} (5.7)

The left hand side of (5.7) equals:

$$\sum_{N \in \mathbb{Z}} \oint \frac{(du)^2}{Nd a + dH} e^{\Pi(u) + \frac{V'H'}{4} (a \frac{da}{du} - 2u)} \times \left[ \frac{V'H'(P_1P_2)'}{4} \frac{\partial a}{\partial \tau_1} \frac{du}{da} - \frac{\partial a}{\partial \tau_1} \frac{(P_1P_2)'}{a} \left( \frac{du}{da} \frac{V'H'}{4} + \frac{Nd a}{Nd a + dH} \right) \right]$$  \hspace{0.5cm} (5.8)

where

$$\frac{\partial a}{\partial \tau_1} = \frac{1}{4} \left( a - 2u \frac{da}{du} \right)$$

The difference

$$\left[ \nabla_{P_1,1} P_{2,1} - \nabla_{P_1,1} \nabla_{P_2,e_S} - \nabla_{P_2,1} \nabla_{P_1,e_S} \right] Z$$

as it follows immediately from (5.8) is equal to:

$$-2 \sum_{N \in \mathbb{Z}} \oint \frac{(du)^2}{Nd a + dH} e^{\Pi(u) + \frac{V'H'}{4} (a \frac{da}{du} - 2u)} \frac{\partial a}{\partial \tau_1} \frac{du}{da} \left[ \frac{P_1'P_2'}{a} \frac{du}{da} + \frac{Nd a}{Nd a + dH} \right]$$  \hspace{0.5cm} (5.9)

which coincides with $\left[ \nabla_{P_1,e_b} \nabla_{P_2,e_f} + \nabla_{P_1,e_f} \nabla_{P_2,e_b} \right] Z$ (the terms in the square brackets in (5.5) come from the term $V'H'$ in the exponential and $Nd a + dH$ in the denominator of (5.5) respectively).

Hence we conclude that in the infrared theory in the presented formalism the observables $(u^r)^{(p)}$ correspond to $(Ch_2^r)^{(p)}$ and a priori not to $(Ch_{2r})^{(p)}$.

5.2. Theory with matter as integration of the equivariant Euler class

To get a hint on what the correlation functions of the descendents of $Ch_r$ might be let us study the theory with fundamental matter, e.g. $N_f$ hypermultiplets in fundamental representation. Its interpretation is the following.

Let $M_k$ be the uncompactified moduli space of instantons on $\Sigma$. We again assume that there are no reducible connections. Let us pick for any point $m \in M_k$ its representative gauge field $A_m$. Let $\mathcal{E}$ be the rank two complex bundle over $M_k \times \Sigma$ with connection $\mathcal{A}$, whose $\Sigma$ components coincide with $A_m$ at $\{m\} \times \Sigma$. This connection exists if the only
non-trivial cohomology group in Atiyah-Hitchin-Singer complex is $H^1 \cong T_m M_k$. In fact, if the component of the connection $\mathcal{A}$ along $\mathcal{M}_k$ is denoted as $c$, then the following equations hold:

$$QA_m = dm^i a_i + dA_m c$$
$$\psi = \psi^i a_i \quad (dA_m^+ \oplus dA_m^*) a_i = 0$$
$$\phi = Qc + \frac{1}{2}[c, c]$$
$$Q \equiv dm^i \frac{\partial}{\partial m^i}$$

(5.10)

Let $\mathcal{E}_m$ be the restriction of $\mathcal{E}$ on $\{m\} \times \Sigma$. We have: $c_1(\mathcal{E}_m) = 0$, $ch_2 = c_2(\mathcal{E}_m) = -k$.

Let $S^+ \otimes \mathcal{F}$ be a $Spin^c$ bundle over $\Sigma$, i.e. rank two complex bundle $S^+ \otimes \mathcal{F}$ whose projectivization coincides with the projectivization of the spinor bundle $\mathbb{P}(S^+)$ (the latter exists even if $S^+$ doesn’t). Strictly speaking the line bundle $\mathcal{F}$ may not exist. However its square $\mathcal{F}^2 = L_c$ is the honest line bundle $L_c$ on $\Sigma$, whose first Chern class is a lift of $w_2(\Sigma)$:

$$c_1(L_c) \equiv w_2(\Sigma) \mod 2$$

Going from one $Spin^c$ structure to another one is equivalent to multiplication of $L_c$ by a square of a line bundle. Similarly one defines $S^- \otimes \mathcal{F}$.

There exists a Dirac-like operator $D_{A_m}$ which maps the sections of $S^+ \otimes \mathcal{F} \otimes \mathcal{E}_m$ to the sections of $S^- \otimes \mathcal{F} \otimes \mathcal{E}_m$. Its index bundle $E_k$, $(E_k)_m = \text{Ind}D_{A_m} = \text{Ker}D_{A_m} - \text{Coker}D_{A_m}$ defines an element of the $K$-group of $\mathcal{M}_k$. The rank of $E_k$ is given by the index theorem:

$$\text{rk} E_k = \int_{\Sigma} Ch(\mathcal{E}_m)Ch(\mathcal{F})\hat{A}_\Sigma = -2k + 4(ch_2(\mathcal{F}) - \frac{\sigma}{8})$$

(5.11)

(since for $k \gg 0$ this quantity is negative it is $-E_k$ who has positive dimension). The reason for the coefficients 2 and 4 in (5.11) is the fact that the bundle $\mathcal{F}$ viewed as a real bundle has the rank two while the bundle $\mathcal{E}_m$ has rank four. The trouble is that when we approach the point-like instanton the rank of $-E_k$ drops by two. The idea is to extend $-E_k$ to the compactification stratum $\mathcal{M}_{k-1} \times \Sigma$ corresponding to the point-like instantons of charge one as follows. Physically the extra two zero modes of the Dirac operator are localized at the point $P$ where the instanton is going to shrink to zero and become singular in the limit. The rest of the zero-modes can be mapped (by a gauge transformation) to the honest zero modes of the instanton of charge $k - 1$ (which exists by Uhlenbeck’s theorem [31]).
Thus we expect that the zero modes bundle over the stratum \( \mathcal{M}_{k-1} \times \Sigma \) of codimension 4 can be decomposed as \(-E_{k-1} \oplus \mathcal{T}_\Sigma\) where \( \mathcal{T}_\Sigma \) is some rank two vector bundle over \( \Sigma \), whose existence follows from the excision principle \[13\]. In fact, let \( \tilde{\mathcal{M}}_k \) be the compactification of the moduli space of instantons by adding the ideal instantons:

\[
\tilde{\mathcal{M}}_k = \mathcal{M}_k \cup \mathcal{M}_{k-1} \times \Sigma \cup \ldots \cup \mathcal{M}_{k-l} \times S^l \Sigma \ldots
\]

Let \( \Delta_l \subset \Sigma \times S^l \Sigma \) be the subspace, consisting of the pairs \((x, I = \sum_i \nu_i[x_i])\), s.t. \( x = x_i \) for some \( i \). Then it is possible to extend the bundle \( \mathcal{E}_k \) to the complement in \( \tilde{\mathcal{M}}_k \times \Sigma \) to the submanifold \( \tilde{\Delta}_l = \mathcal{M}_{k-1} \times \Delta_1 \cup \ldots \cup \mathcal{M}_{k-l} \times \Delta_l \cup \ldots \):

\[
\tilde{\mathcal{E}}_k \\
\downarrow
\\
\mathcal{M}_k \times \Sigma \setminus \tilde{\Delta}_l
\]

It seems that the index bundle of the Dirac operator coupled to \( \tilde{\mathcal{E}}_k \) actually extends to the bundle \( \tilde{E}_k \) over the whole \( \tilde{\mathcal{M}}_k \). Then the Chern classes of the bundle \( \tilde{E}_k \) are the cohomology classes of \( \mathcal{M}_k \). Let us denote them as: \( \sigma_l = c_l(\tilde{E}_k) \). The theory above easily generalizes to the case of \( N_f > 1 \) copies of \( \mathcal{F} \). More precisely, let \( S^+ \otimes \mathcal{F}' \) be a \( Spin^c \) bundle and \( F \) a \( Spin(2N_f) \) bundle over \( \Sigma \). Their tensor product we denote as \( S^+ \otimes \mathcal{F} \) and symbolically \( \mathcal{F} = \mathcal{F}' \otimes F \). These “equations” are useful when computing Chern classes.

The bundle \( E_k = \text{Ind}D_{\mathcal{E} \otimes \mathcal{F}} \) has the rank

\[
-2N_f k - \frac{\sigma}{2} + 4 \int_\Sigma \text{ch}_2(\mathcal{F})
\]

The bundle \( \mathcal{F} \) is acted on by the global group \( Spin(2N_f) \). Let us denote the scalar generator of the equivariant cohomology of \( Spin(2N_f) \) by \( \bar{m} = (m_1, \ldots, m_{N_f}) \). It belongs to the Cartan subalgebra of \( Spin(2N_f) \) which is \( N_f \) dimensional. The bundle \( E_k \) is also \( Spin(2N_f) \) equivariant and one may compute its equivariant Chern character using index theorem (we do it in the next section).

We claim that the correlation functions in the theory with massive matter are given by

\[
\langle \mathcal{O}_1 \ldots \mathcal{O}_p \rangle_{N_f, \bar{m}} = \sum_k \Lambda_{N_f}^{(4-N_f)k} \left( \prod_{i=1}^{N_f} m_i \right)^{\text{rk} \tilde{E}_k} \int_{\tilde{\mathcal{M}}_k} \left( \prod_{i=1}^{N_f} \left( \sum_l \frac{\sigma_l}{m_i^l} \right) \right) \omega_1 \wedge \ldots \wedge \omega_p
\]

with \( \omega_k \) being the ordinary Donaldson classes.
In other words, the inclusion of the matter amounts to inserting into integral over $\mathcal{M}_k$ of the equivariant Euler class of the Dirac index bundle.

Therefore the correlation functions in the theory with massive matter are the generating functions for the intersection numbers of the standard Donaldson observables and the Poincare duals to the Chern classes of the various vector bundles on $\mathcal{M}_k$.

The twisted theory with matter has been considered by several authors [33][34][9]. The relation of the studies of non-abelian monopoles [33][34] to our approach is simply the localization with respect to the global group $Spin(2N_f)$. Consideration of the general $Spin(2N_f)$ bundles is the generalization of the approach of [34] where several $Spin^c$ structures were studied (in our language it corresponds to the global group broken to its maximal torus).

5.3. Subtleties of the theories with matter

The construction with general bundle $\mathcal{F}$ may seem to be too abstract. There are at least five reasons why these theories may (and should!) be studied.

1. If $\Sigma$ is not spin for $N_f > 1$ one has to study various options: the $Spin^c$ structures for different quarks may be chosen differently. Clearly this is just the study of the bundles with the structure group being the maximal torus of $Spin(2N_f)$. The twisted theory depends on the choice of $Spin^c$ structure. If the gauge group contained a $U(1)$ factor, which acts non-trivially on $\det(\mathcal{E})$ then we would sum over all such line bundles and henceforth over all $Spin^c$ structures. This is the case of $U(1)$ theory [11]. However in the $SU(2)$ case this is not true. Therefore, the theories with matter are labelled by some extra discrete data, such as a choice of $\mathcal{F}$.

2. The second occasion where the different $\mathcal{F}'s$ may appear is the study of $D3$ probes in $F$-theory compactifications. The group $Spin(2N_f)$ is the gauge group of the $D7$ background branes. The additional branes may create a non-trivial instanton background for the global group on the $D3$ probe.

3. The third reason is the possibility to learn about the topology of the universal “bundle” and the “quantum cohomology” of the classifying space $BG$. Indeed, one may compute the Chern character of $\bar{E}_k$ and express it through the characteristic classes of the would-be-universal instanton. Given the knowledge of the correlation functions in the theory with matter (see [9] for the first steps in this direction) one may deduce the intersection numbers of the Chern classes of $\bar{E}_k$ using (5.13). The standard formulae, relating the Chern character of $E_k$ and topology of $\mathcal{E}$ to the Euler class of $E_k$ become
the perturbative expressions for the correlation function in the physical language. The subtle issues of the compactification of the moduli space, extension of $E_k$ to the compactification strata and defining the Euler class are resolved by summing up the “instanton corrections” to the perturbative answer. This point is completely analogous to the examples, studied in [35][36][37] so we just sketch the line of arguments. The family index theorem gives

$$Ch(E_k) = \int_\Sigma Ch(E)Ch(F)\,\hat{A}_\Sigma =$$

$$\int_\Sigma \text{Tr}(e^{\frac{1}{2\pi i}(\phi+\psi+F)}) (2N_f + c_1(F) + ch_2(F)) \,\hat{A}_\Sigma$$

(5.14)

where we have used the standard decomposition of the curvature of the universal instanton [20]. To convert the Chern character into the equivariant Euler class we use the following trick:

$$\log Eu_m = \sum_{l,i} \log(x_l + m_i) = \lim_{s \to 0} \frac{1}{\Gamma(-s)} \left( \frac{1}{s} \int_0^\infty \frac{dt}{t^{1+s}} \sum_l \sum_i e^{t(x_l + m_i)} - A \frac{s}{s} \right)$$

(5.15)

where $\frac{A}{s}$ is the divergent part, contributing to the perturbative beta function. The sum $\sum e^{t(x+m)}$ is obtained from (5.14) by multiplying $\phi$ by $t$, $\psi$ by $t^{1/2}$ and by shifting $\phi$ by $m$. One should also remember that (5.14) gives the Chern character of the alternated sum of the bundles:

$$H^0 - H^1 + H^2$$

while we need the expression for the Euler class of $H^1$ (if all other cohomology groups vanish). This is another way of saying that we need the bundle $-E_k$ rather then $E_k$. So we change the sign in (5.14) and get:

$$\log Eu_m = \int_\Sigma O^{(4)}_f - \frac{\sigma}{8} O^{(0)}_h$$

(5.16)

with

$$f = \frac{1}{2} \sum_{k=1}^{N_f} \text{Tr} \left( \frac{\phi + m_k}{2\pi i} \right)^2 \log(\phi + m_k) \sim \sum_{n=1}^{\infty} \sum_{i=1}^{N_f} \left( \frac{2\pi i}{m_i} \right)^{2n} \frac{Ch_{2n+2}}{n(n+1)(2n+1)}$$

$$h = -\text{Tr} \left[ \log(m_k + \phi) \right]$$

$$A = \int_\Sigma O^{(4)}_{f_a} - \frac{\sigma}{4} \quad f_a = 2\text{Tr} \phi^2$$

(5.17)

21 Riemann-Roch-Grothendieck would give the same answer with the replacement $F \to F \otimes K^{\frac{1}{2}}_\Sigma$
4. The forth reason to study more general $\mathcal{F}$’s is a possibility to have vectormultiplets charged under $G$. We write a more general formula, which would contain the contributions both from the hypermultiplets and vectormultiplets:

$$
\langle \omega_1 \ldots \omega_l \rangle_{V,H} = \sum_k \int_{\mathcal{M}_k} \omega_1 \wedge \ldots \wedge \omega_l \frac{\prod_{i=1}^{N_h} \prod_{l_i} (x_{l_i} + m_i)}{\prod_{j=1}^{N_v} \prod_{l_j} (x_{l_j} + m_j)}
$$

(5.18)

where the notations are self-explanatory.

The generalization of the theory containing both hyper- and vectormultiplets charged under the gauge group is the following. Let $\mathcal{F}$ be any bundle over $\Sigma$ and $R$ some representation of $G$. It may be the case that the complex of bundles (sheaf)

$$
\Omega^0 \otimes R(\mathcal{E}) \otimes \mathcal{F} \rightarrow \Omega^1 \otimes R(\mathcal{E}) \otimes \mathcal{F} \rightarrow \Omega^{2,+} \otimes R(\mathcal{E}) \otimes \mathcal{F}
$$

has non-trivial cohomology in various dimensions. Denote the cohomology groups as $\mathcal{E}^i_{R,\mathcal{F}} = H^i(\Sigma, \mathcal{E} \otimes \mathcal{F})$. The ranks of the bundles $\mathcal{E}^i_{R,\mathcal{F}}$ over $\mathcal{M}_k$ are locally constant and one may define the ratio:

$$
\frac{\text{Eu}_m(\mathcal{E}^1_{R,\mathcal{F}})}{\text{Eu}_m(\mathcal{E}^0_{R,\mathcal{F}}) \text{Eu}_m(\mathcal{E}^2_{R,\mathcal{F}})}
$$

(5.19)

The hope is that the ratio (5.19) defines a cohomology class of the compactification $\mathcal{M}_k$.

5. The proper understanding of these issues would allow to check independently the solutions of Seiberg and Witten [1] of the $\mathcal{N} = 2$ $SU(2)$ theories with matter in the fundamental and adjoint representations. The theory with adjoint matter can be treated similarly. One just plugs into (5.19) the representation $R = \text{ad}(E)$.

5.4. Higher dimensional theories

The toroidal compactifications of the higher dimensional theories are the particular examples of this construction [35]. In these cases an infinite set of vector multiplets forms the tower of Kaluza-Klein states. For example, a five dimensional theory on a circle can be considered as a deformation of the four dimensional theory, the radius $R$ of the fifth dimension being the deformation parameter. The analogue of (5.13) looks in this case as:

$$
\langle O_1 \ldots O_p \rangle = \sum_k \int_{\mathcal{M}_k} \omega_1 \wedge \ldots \wedge \omega_p \wedge \hat{A}(\mathcal{M}_k)
$$

(5.20)
To make contact with (5.18) we expand $\hat{A}$-genus as:

$$
\hat{A}(\mathcal{M}_k) = \prod_{j=1}^{\infty} \prod_{l} \frac{1}{x_l + \frac{2\pi ij}{R}}
$$

(5.21)

Finally, one may replace the circle by a two-torus $T^2$ and study the compactification on $T^2$ of the six dimensional $\mathcal{N} = 1$ theory. Then (5.20) generalizes to

$$
\langle \mathcal{O}_1 \ldots \mathcal{O}_l \rangle = \int_{\mathcal{M}_k} \omega_1 \wedge \ldots \omega_l \wedge E_{q,y}(\mathcal{M}_k)
$$

(5.22)

where $E_{q,y}(\mathcal{M}_k)$ is the combination of the characteristic classes of $\mathcal{M}_k$ entering its elliptic genus, $q = e^{2\pi i \tau}$ is the modular parameter of the two torus, $y$ is a $U(1)$ Wilson loop, measuring the fermion number.

The formula (5.18) is useful in the study of the breaking of the gauge group $G$ to its subgroup $H$ by the vacuum expectation value of the adjoint scalar $\phi$ which commutes with $H$. In this case the supersymmetric configurations are the solutions to the equations:

$$
F^+_A = 0, \quad A \in \mathfrak{h}, \quad d_A \phi = 0
$$

The moduli space $\mathcal{M}_H$ of irreducible solutions of these equations are the instantons of the group $H$. The standard localization arguments show that the instanton measure is modified by the equivariant Euler class of the normal bundle to $\mathcal{M}_H$ viewed as a submanifold in $\mathcal{M}_G$. Since the masses of the vector multiplets are determined by the eigenvalues of $\phi$ the localization formulae of [38] gives precisely (5.18).

**Four dimensional Verlinde formula.** Consider the five dimensional example of the previous section. We may think of it as of the deformation of the four dimensional theory by adding an infinite number of four-observables to the action [35] [36]. In fact,

$$
\log \hat{A}(\mathcal{M}_k) = \int_{\Sigma} \mathcal{O}^{(4)} - \frac{\chi + \sigma}{4} \mathcal{O}^{(0)}
$$

(5.23)

with

$$
f = \frac{1}{R^2} \text{Tr} \left( \text{Li}_3(e^{R\phi}) + \frac{\phi^2}{2} \log(-R^2 e^{3\phi^2}) \right) =
$$

$$
= \sum_{k=1}^{\infty} \frac{B_{2k} R^{2k} Ch_{2k+2}}{k(2k + 2)!}
$$

(5.24)

$$
h = \sum_{k=1}^{\infty} \frac{B_{2k} R^{2k} Ch_{2k}}{k(2k)!}
$$
with $B_{2k}$ being the Bernoulli numbers: $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}$ etc. As explained in [32], the correlation function of the exponential of two-observable of $\text{Tr} \phi^2$ in this theory gives rise to the four-dimensional analogue of Verlinde formula:

$$\int_{\mathcal{M}_k} e^{c_1(L)} \hat{A}(\mathcal{M}_k)$$

(5.25)

where $L$ is a line bundle over $\bar{\mathcal{M}}_k$. From the field theory point of view (5.25) is the expectation value of Chern-Simons observable [32] [35] [36] [37]. One may, of course, study the simpler observables - the descendants of the Wilson loops. These correspond to the computations of the integrals like:

$$\langle \exp \left( \int_C O^{(2)}_{\text{Tr}g} + O^{(0)}_{\text{Tr}g}(1 - \text{genus}(C)) \right) \rangle = \int_{\mathcal{M}_k} e^{Ch(E)} \hat{A}(\mathcal{M}_k)$$

(5.26)

where

$$g = P \exp \int (A_5 + i\varphi) dx^5$$

$A_5$ is the fifth-component of the gauge field, $\varphi$ is the scalar in the $\mathcal{N} = 1$ $d = 5$ multiplet, the bundle $E_C$ is the Dirac index bundle of the restriction of the Dirac operator coupled to the universal instanton to the two dimensional surface $C \subset \Sigma$.

### 6. Generalizations II. Macroscopic theories

In this chapter we briefly discuss the subtleties of the low-energy effective description of the theories considered in the previous chapter. For simplicity we study the rank one case only. Thus the Lagrangian manifold $\mathcal{L}$ is one-dimensional and is described by the one-dimensional family of elliptic curves:

$$y^2 = 4x^3 - g_2(u)x - g_3(u)$$

(6.1)

with the holomorphic symplectic form on the total space of fibration:

$$\omega = \frac{dx \wedge du}{y}$$

(6.2)

---

22 If the manifold $\Sigma$ is not spin one replaces $L$ by $L \otimes K^\frac{1}{2}$, where $K$ is the canonical bundle of some almost complex structure on $\Sigma$. This replacement changes $\hat{A} \to Td$
There is a classification of possible singularities of elliptic fibrations due to Kodaira. Its physical interpretation has been discussed in [39]. We are mainly interested in the singularities of $A_k$ and $D_k$ type, which correspond to the appearance of massless particles.

In either $A$ or $D$ case the coupling constant (elliptic modulus) exhibits a logarithmic singularity (we denote by $z$ a local coordinate near the singularity, the point of singularity being $z = 0$):

$$
A_k \text{ case : } \tau(z) = (k + 1) \frac{\log(z)}{2\pi i} + \ldots, \quad M = \begin{pmatrix} 1 & k + 1 \\ 0 & 1 \end{pmatrix} \\
D_k \text{ case : } \tau(z) = (k - 4) \frac{\log(z)}{2\pi i} + \ldots, \quad M = \begin{pmatrix} -1 & 4 - k \\ 0 & -1 \end{pmatrix}
$$

(6.3)

The $A_k$ singularity corresponds to the $U(1)$ theory with $k + 1$ massless hypermultiplets of charge 1, while $D_k$ corresponds to the $SU(2)$ gauge theory with $k$ massless hypermultiplets in the fundamental representation.

The issue is to find the proper analogue of the numbers $SW(\ell)$ which correspond to the $A_0$ case. Of course, for the $A_k$ and $D_k$ case we know the answer. One simply should study the moduli spaces of the generalized monopole equations and their intersection theory produces the analogues of the Seiberg-Witten invariants. This approach looks like a difficult problem, though. However, there is a trick which reduces everything to the already studied $A_0$ case.

Consider the $A_k$ case first. The monopole equations have the schematic form:

$$
F_{ij}^+ = -\frac{i}{2} \sum_{\alpha=1}^{k+1} \tilde{M}_{ij} \Gamma_{ij} M^\alpha \\
\sum_i \Gamma^i D_i M^\alpha = 0
$$

(6.4)

where the index $\alpha$ runs from 1 through $k + 1$. Let $\mathcal{M}_{A_k}(\ell)$ be the moduli space of solutions of (6.4) for $[F] = 2\pi i \ell \in H^2(\Sigma, 2\pi i \mathbb{Z})$. Clearly it is acted on by the global symmetry group $SU(k + 1)$. The cohomology of $\mathcal{M}_{A_k}$ can be studied with the help of equivariant cohomology of $\mathcal{M}_{A_k}$ with respect to the action of $SU(k + 1)$. In particular, the integration over $\mathcal{M}_{A_k}$ of $SU(k + 1)$-invariant classes is reduced to the fixed points of the action of the generic element of the maximal torus of $SU(k + 1)$. These are, in turn, the solutions to the $A_0$ equations, which are obtained from (6.4) by setting $M^\alpha = 0$ for $\alpha \neq \beta$. The contribution of the fixed point set $\mathcal{M}_{A_0}^\beta$ is the integral over $\mathcal{M}(\ell)$ of a ratio of the evaluation of the corresponding equivariant class at $\mathcal{M}_{A_0}^\beta(\ell)$ and the equivariant Euler class of the
normal bundle to $\mathcal{M}_{A_0}^\beta(\ell)$ in $\mathcal{M}_{A_k}(\ell)$. To be specific consider the first Chern class $c_1(\mathcal{L})$ of the line bundle $\mathcal{L}$ associated to the $U(1)$ bundle $\mathcal{M}_{A_k}(\ell, P) \to \mathcal{M}_{A_k}(\ell)$ where $\mathcal{M}_{A_k}(\ell, P)$ is the framed moduli space. The class $c_1(\mathcal{L})$ is $SU(k+1)$-invariant. In fact, its representative can be rather explicitly written:

$$c_1(\mathcal{L}) = \frac{1}{-\Delta + \bar{M}_\gamma M^\gamma}(\delta \bar{M}_\alpha \wedge \delta M^\alpha)$$

(6.5)

where $\Delta$ is the scalar Laplacian, all functions are evaluated at the point $P$ and again we are not careful about the exact coefficients. Let diag($m_1, \ldots, m_{k+1}$) be the scalar generator of the $SU(k+1)$ equivariant cohomology, i.e. $m_1 + \ldots + m_{k+1} = 0$. The equivariant extension of (6.5) is given by the form:

$$c_1(\mathcal{L}) + \sum_{\alpha=1}^{k+1} m_\alpha H^\alpha_\alpha$$

(6.6)

where

$$H^\beta_\alpha = \frac{1}{-\Delta + \sum_\gamma \bar{M}_\gamma M^\gamma} \bar{M}_\alpha M^\beta(P)$$

(6.7)

The collection of the Hamiltonians $H^\beta_\alpha$ forms the moment map for the $SU(k+1)$ action on $\mathcal{M}_{A_k}(\ell)$ if (6.5) is treated as a symplectic form. In order to check that the form (6.6) is equivariantly closed one must keep in mind that the variations $\delta A$ and $\delta M^\beta$ of the solutions to (6.4) obey in addition the gauge fixing condition:

$$d^* \delta A + \delta \bar{M}_\gamma M^\gamma - \bar{M}_\gamma \delta M^\gamma = 0$$

(6.8)

(which was used in deriving (6.5)). In particular, one gets that the infinitesimal global rotation generated by $H^\beta_\alpha$ goes together with the compensating gauge transformation:

$$\Delta M^\gamma = \delta_\alpha^\gamma M^\beta - M^\gamma \frac{1}{-\Delta + \bar{M}_\gamma M^\gamma} \bar{M}_\alpha M^\beta$$

The fixed points of the flow generated by $H^\alpha_\alpha$ are easily seen from this to have $M^\gamma = 0$ for $\alpha \neq \gamma$. The value of the Hamiltonians $H^\beta_\gamma$ is computed from (6.7):

$$H^\beta_\gamma |_{\mathcal{M}_{A_0}^\alpha} = \delta_\gamma^\alpha \delta^\beta_\alpha$$

(6.9)

\[23\] The operator $(-\Delta + \bar{M} M)$ has positive spectrum and is invertible iff there are no abelian instantons
These remarks are useful for computing the equivariant Euler class of the normal bundle $\mathcal{N}$ to $\mathcal{M}_A^{\beta}$ in $\mathcal{M}_A$. It is easy to see that the normal bundle is spanned by the solutions to the equation

$$D_A M^\gamma = 0$$

with $\gamma \neq \beta$. There are $\frac{k}{4}((\ell, \ell) - \sigma)$ solutions to this equation. We must understand the topology of $\mathcal{N}$. Clearly,

$$\mathcal{N} = \mathcal{L}|_{\mathcal{M}_A} \otimes \mathcal{Q}_\beta^{((\ell, \ell) - \sigma)}$$

as the vector bundle over $\mathcal{M}_A$. As the equivariant bundle it decomposes further with the result:

$$\text{Eu}_m(\mathcal{N}) = \prod_{\beta \neq \alpha} (c_1(\mathcal{L}) + m_\alpha - m_\beta)^{\frac{1}{2}((\ell, \ell) - \sigma)}$$

Hence

$$\int_{\mathcal{M}_A^{(\ell)}} e^{c_1(\mathcal{L}) + \sum_\alpha m_\alpha H_\alpha^m} = \sum_\alpha \int_{\mathcal{M}_A} \frac{e^{c_1(\mathcal{L}) + m_\alpha}}{\prod_{\beta \neq \alpha} (c_1(\mathcal{L}) + m_\alpha - m_\beta)^{\frac{1}{2}((\ell, \ell) - \sigma)}}$$

Finally, the right hand side of (6.12) is simplified using the notation $SW(\ell)$ introduced in the chapter 2 as follows:

$$SW(\ell) \int e^a da P(a) \frac{1}{\ell^{(\ell, \ell) - \sigma}} \sum_{a=1}^{k+1} (a - m_\alpha)^{\frac{k+1}{2} - 1}$$

where $P(a) = \prod_\alpha (a - m_\alpha)$. The limit $m \to 0$ exists and it implies that the integral of $e^{c_1(\mathcal{L})}$ over $\mathcal{M}_A^{(\ell)}$ is equal to $(k + 1)$ times $SW(\ell)$.

**Apologies.** Of course, the trick with the equivariant cohomology $H^*_\text{SU}(k+1)(\mathcal{M}_A)$ is nothing but the deformation of the theory by giving bare masses to the hypermultiplets. It serves as a further justification of the analysis which led to (1.53).

**Higher critical points.** It is straightforward to generalize this idea to the case of $A,D,E$ singularities. In either case one has a global symmetry group $G$ of the type $A,D,E$ and the corresponding moduli space $\mathcal{M}_G$ (which remains a mystery for the $E_k$ case) is acted on by $H$. The study of $H^*_G(\mathcal{M}_G)$ is done by the localization techniques. It is equivalent to the unfolding of the singularity by turning on the masses. In the $D_k$ cases the global group is $SO(2k)$. By turning on $k$ masses one gets various critical points, some of which are “mutually non-local”. The analysis of our paper carries over to these cases. It would be interesting to see whether one can learn anything about the spaces $\mathcal{M}_{E_k}$ using our techniques [40]. We hope to return to this subject elsewhere.
Nontrivial bundles $\mathcal{F}$. It is interesting to generalize the formulae for the correlation functions in the theories with matter to the case of non-trivial bundles $\mathcal{F}$, say with $c_2(\mathcal{F}) \neq 0$ (for $N_f > 1$). Our proposal is the following. The formula (4.54) involves the function $\mathcal{F}$. It is the function of the masses $m_i$ and can be expanded in $m_i$ as a series. Moreover, the coefficients of the expansion are the functions which are invariant under the action of the Weyl group of $Spin(2N_f)$. Write

$$\mathcal{F} = \sum_l \mathcal{F}_l p_l(m)$$

where $p_l$ run through a basis in the space of Weyl invariant functions on the Cartan subalgebra of $Spin(2N_f)$. These functions are in one-to-one correspondence with the universal characteristic classes of the $Spin(2N_f)$ bundles. Replace the function $p_l(m)$ by the corresponding Pontryagin class. We get another function $\tilde{\mathcal{F}}$ with values in $H^*(\Sigma)$. Substitute it into (4.54) and compute the contour integral (4.55). We conjecture that this procedure yields the correct answer. The motivation is clear - the masses and the curvature of the bundle $\mathcal{F}$ are related by the $Q$-symmetry. Our formalism allows the analytic continuation in the fermionic directions.

Chern–Simons observable. This is the issue of the low-energy expression for the observable in the five dimensional theory compactified on a circle containing Chern-Simons functional. To explain what the problem is let us recall the expression for the two-observable in the holomorphic approach:

$$\mathcal{O}_u^{(2)} = \frac{du}{da} F + \frac{1}{2} \frac{d^2 u}{da^2} \psi^2$$

(6.14)

The difficulty with Chern-Simons terms is that they are not the descendants of the gauge-invariant functionals and moreover give rise to the gauge-invariant observables only in the exponential with quantized coefficients. We try:

$$e^{\int_{C \times S^1} CS(A) + \ldots} \sim e^{\int_C z F + \frac{1}{2} \frac{4\pi}{a} \psi^2}$$

(6.15)

Here $z$ is a section of a certain local system over the Coulomb branch of the theory. The expression (6.15) allows for $z$ to be defined up to the shifts (which are remnants of the five-dimensional gauge invariance):

$$z \rightarrow z + n, \quad n \in \mathbb{Z}$$

(6.16)
The question is whether the modular invariance can be preserved. In fact, under the holomorphic modular transformation the factor
\[ e^{\frac{z^2}{2}} \#(C \cap C) \]
appears. The contact term \( z^2 E_2 + \ldots \) will not work because it violates the gauge invariance (6.16). But the following contact term works:
\[ e^{\frac{z^2}{2}} \#(C \cap C) \mapsto \left( \frac{\theta_{00}(\tau, z)}{\theta_{00}(\tau)} \right)^\#(C \cap C) \]
At the moment we do not know how to complete this program by including the full set of fields \( \chi, \eta \) etc.

7. Two dimensional analogies

7.1. Reflections on type A sigma models

The analogue of the theory with matter in two dimensions exists. Consider the type A sigma model with target space \( B \). Let \( E \) be rank \( r \) complex hermitian vector bundle over \( B \). The group \( G = U(r) \) acts on \( E \), preserving the hermitian metric on the fibers. The sigma model with the target \( E \) is probably ill-defined since the space is non-compact. Nevertheless, consider the following equivariant model. First we present its mathematical definition and then discuss its physical realization. Let \( \mathcal{M}_{\beta;n} \) denote the moduli space of degree \( \beta \in H_2(B; \mathbb{Z}) \) stable (pseudo)-holomorphic maps of the \( n \)-punctured worldsheet \( \Sigma \) to \( B \). There are the bundles (sheaves, complexes of sheaves, ...) \( \mathcal{E}^i (i = 0, 1) \) over \( \mathcal{M}_{\beta;n} \) which are defined as follows. The fiber \( \mathcal{E}^i_f \) over the map \( f \) is the \( i \)'th cohomology group of the pullback of \( E \):
\[ \mathcal{E}^i_f = H^i(\Sigma, f^* E) \quad (7.1) \]
More invariantly the sheaves \( \mathcal{E}^i \) are defined as \( \mathcal{E}^i = R^i(f_{gt_{n+1}})_* ev_{n+1}^* E \), where \( ev_{n+1} : \mathcal{M}_{\beta;n+1} \to B \) is the evaluation at the \( n + 1 \)'st point and \( f_{gt_{n+1}} : \mathcal{M}_{\beta;n+1} \to \mathcal{M}_{\beta;n} \) is the map which forgets the position of the \( n + 1 \)'st point on \( \Sigma \). The group \( G \) acts on \( \mathcal{E}^i \) naturally. Let \( E_{um}(\mathcal{E}^i) \) denote the \( G \)-equivariant Euler classes of the bundles \( \mathcal{E}^i \). Let \( \omega_1, \ldots, \omega_n \) be the natural cohomology classes of \( \mathcal{M}_{\beta;n} \), which were defined in the sigma model with the
target $B$. For $\phi_1, \ldots, \phi_n \in H^*(B)$ we write $\omega_i = \text{ev}_i^* \phi_i$. We define the correlation function in the sigma model with the target $E$ the following integral:

$$\langle \int_{\Sigma} \phi_1^{(2)} \cdots \int_{\Sigma} \phi_n^{(2)} \rangle_{E,m,\beta} = \int_{\mathcal{M}_{\beta,n}} \frac{E_{u_m}(\mathcal{E}^1)}{E_{u_m}(\mathcal{E}^0)} \omega_1 \wedge \ldots \omega_n$$

(7.2)

Remarks.

1. The definition (7.2) is motivated by the constructions of [41, 42] and [43, 44].

2. If $\mathcal{E}^1 = 0$ then the formula (7.2) can be interpreted as a fixed locus contribution to the correlation function in the theory on the target space $X$, which is $G$-space and the fixed locus contains $B$. In this case the bundle $E$ may be interpreted as a normal bundle to $B \subset X$, while the bundle $\mathcal{E}^0$ is the normal bundle to the submanifold $\mathcal{M}_d$ in the moduli space of the stable maps $\Sigma \to X$.

3. If $\mathcal{E}^0 = 0$ then the formula (7.2) can be interpreted in the limit $m = 0$ as a computation of the number of the curves in $E$ in the generic almost complex structure, whose homology class is determined by $\beta$.

In any case one may think of the set of correlation functions (7.2) as of the “theory” interpolating between the sigma model on $B$ and the sigma model on $E$. In particular, by choosing the bundle $E$ in such a way, that $c_1(E) = -c_1(B)$ one gets a non-compact Calabi-Yau manifold. In this case the correlation functions (7.2) are the “regularized” correlators in the theory on $B$, where the regularization is achieved by embedding the theory into the ultra-violet finite (= conformal) theory on Calabi-Yau. The phase diagram and the beta function of the two dimensional theory can be recovered from the computation of the powers of $m$ entering the correlation function.

The leading power of $m$ is given by the index formula:

$$\text{rk}\mathcal{E}^0 - \text{rk}\mathcal{E}^1 = \int_{\beta} c_1(E) + \frac{\chi r}{2}$$

(7.3)

where $\chi$ is the Euler characteristics of $\Sigma$. Thus,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_l \rangle_{E,m,\beta} \sim_{m \to \infty} m^{d+\chi r/2} \langle \mathcal{O}_1 \cdots \mathcal{O}_l \rangle_{B,\beta}$$

(7.4)

If $\mathcal{E}^0 = 0$, then the correlation function (7.2) is the polynomial in $m$ of the degree $\text{rk}\mathcal{E}^1$. If $\mathcal{E}^1 = 0$ then the correlator (7.2) is the polynomial in $m^{-1}$ of the degree $\text{rk}\mathcal{E}^0$.

Physical realization. To realize the correlation functions (7.2) physically we start with the set of fields $x^\mu, \rho^\mu, \psi_\zeta, \psi^\zeta$ of the type A sigma model with the target space $B$
and add the fields $X^\mu, \epsilon^\mu, \Psi^i, \bar{\Psi}^i$ which represent the maps to $E$. Also we introduce the auxiliary fields $p^i, p^i_\bar{z}$ and $P^i, P^i_\bar{z}$ (which are usually omitted). The $Q$-operator representing the equivariant derivative with respect to $G$ reads as follows:

$$
\begin{align*}
Qx^\mu &= \rho^\mu & Q\rho^\mu &= 0 \\
Q\psi^i &= p^i & Qp^i_\bar{z} &= 0 \\
QX^M &= \epsilon^M & Q\epsilon^M &= \phi^A T^M_{A,N} X^N \\
Q\Psi^I &= P^I & QP^I_\bar{z} &= \phi^A T^I_{A,J} \Psi^J
\end{align*}
$$

(7.5)

modulo the terms involving various connections which can be trivially reconstructed on the general covariance principles. The action of the theory is given by the sum of the pull-back of the equivariant extension of the symplectic form $\omega_E$ on $E$ and the standard $Q$-exact terms:

$$
S = \int_{\Sigma} \omega_{\mu\nu} dx^\mu \wedge dx^\nu + \int_{\Sigma} \{Q, \epsilon^M X^N \omega_{MN} d^2 z\} + {}^* \{Q, \int_{\Sigma} \Psi^I_\bar{z} (\partial_\bar{z} X^I - P^I_\bar{z}) + \psi^i_\bar{z} (\partial_\bar{z} x^i - p^i_\bar{z}) + c.c.\}
$$

(7.6)

The formula (7.2) arises as a standard result of the expansion around the zero locus of the $Q$-transformation.

Concluding this section let us make the following remarks. Historically the first supersymmetric models which were used to compute certain topological invariants of manifolds or operators were precisely of this type. Namely, the relevant $Q$-transformation squares to the global symmetry transformation [44][45]. Of course, the mathematical apparatus of equivariant cohomology corresponds precisely to that kind of constructions [38]. Recently the aspects of such models and their relation to the twisted $\mathcal{N} = 2$ algebras with central extension was pointed out in [33]. The compactifications of higher dimensional theories on tori were also interpreted along these lines [35][36][37].

The models (7.5)(7.6) are useful in understanding the local mirror symmetry [46][47].

### 7.2. Comparison with topological type B sigma model

**Supersymmetry algebra.** Let us look once again at the transformations (3.14)(3.17). Introduce the auxiliary fields:

$$
\theta_i = \Im \tau_{ij} \chi^j, \quad \bar{F}_i = \Im \tau_{ij} H^j
$$

(7.7)
Also, change the notations $\psi^i \rightarrow \rho^i$, $a^i \rightarrow X^i$, $\bar{a}^i \rightarrow \bar{X}^i$.

Then the algebra (3.14) (3.17) assumes the form:

$$
Q F^i = d\rho^i, \quad Q \rho^i = dX^i \\
Q \theta_i = \bar{F}_i \quad Q \bar{X}^i = \eta^i
$$

(7.8)

and $Q^2 = 0$. Except for the piece with $A_D$ (where $Q^2$ does not vanish) this algebra is identical to the algebra of sigma model of type B [20] (before eliminating the auxiliary fields $F, \bar{F}$).

This analogy will turn out to be fruitful in the next sections.

**Target space.**

The target space of the sigma model which is relevant for four dimensional gauge theory can be worked out as follows. Suppose the space-time manifold $\Sigma$ contains a two-cycle with zero self-intersection. Then $\Sigma$ may be represented as a fibration over a two-dimensional base $B$. Let us study the limit of the metric on $\Sigma$ in which the size of the generic fiber $F$ of the fibration is much smaller then that of $B$. In this limit the gauge degrees of freedom along the fiber $F$ are almost frozen.

Let us study this effective two dimensional theory. For simplicity we consider the pure Yang-Mills case only. The generalization involving matter fields or more complicated backgrounds are straightforward. The theory turns out to be equivalent to Landau-Ginzburg model formulated on a target space $\mathcal{M}$ defined below.

The target space $\mathcal{M}$ is the moduli space of the following triples $(E, C, P)$, where $E$ is elliptic curve, $C$ is a one-cycle, $C \in H_1(E, \mathbb{Z})$ and $P$ is a cyclic subgroup of $E$ of the order 4.

There exists bundle $S \rightarrow \mathcal{M}$, whose fiber at the triple $(E, C, P)$ is the curve $E$ itself. The total space $S$ is endowed with holomorphic symplectic form $\omega$.

The model has superpotential $W$, whose differential is defined canonically by the formula:

$$
dW_{(E, C, P)} = \int_C \omega
$$

(7.9)

The space $\mathcal{M}$ is not connected. Its connected components are labelled by an integer $N \in \mathbb{Z}$: $\mathcal{M} = \coprod_N \mathcal{M}_N$. The component $\mathcal{M}_N$ consists of the triples $(E, C, P)$ where $C$ is $N$ times the primitive element of $H_1(E, \mathbb{Z})$. All components $\mathcal{M}_N$ with $N \neq 0$ are isomorphic to each other and to the strip $\mathcal{H}/4\mathbb{Z} = \{ \tau \mid \tau_2 > 0 \}/(\tau \sim \tau + 4)$. The component $\mathcal{M}_0$ is isomorphic to $\mathcal{H}/\Gamma_0^0(4)$.
The holomorphic functions on $M$ all come from the functions $f(u)$, where $u$ is the canonically defined function on $M$, whose description follows. The curve $E$ with the subgroup $P$ is identified with the double cover of $\mathbb{P}^1$ which can be written as follows:

$$y^2 = x(x^2 - ux + \frac{1}{4})$$  \hspace{1cm} (7.10)

The subgroup $P$ is generated by the points $x = \frac{1}{2}, y = \pm \frac{1}{2}\sqrt{1-u}$.

The space $S$ also splits as $S = \prod_N S_N$, with $S_0$ being the surface in $\mathbb{C}^3$:

$$y^2 = x(x^2 - ux + \frac{1}{4})$$  \hspace{1cm} (7.11)

The symplectic form $\omega_0$ on $S_0$ can be written explicitly as follows:

$$\omega_0 = \frac{dx \wedge dy}{x^2}$$  \hspace{1cm} (7.12)

There exists a covering $p_f : M \to M_0$, defined by forgetting $C$. It extends to the map of the spaces $S \to S_0$. The form $\omega$ on $S$ is the pullback of $\omega_0$ with respect to $p_f$.

The superpotential $W$ is defined in the coordinates $u$ by the integral:

$$W = Na(u) \equiv N \int_{-1}^{+1} \frac{ydx}{x^2}$$  \hspace{1cm} (7.13)

It vanishes on $M_0$.

The data defining the Landau-Ginzburg model contains the Kähler metric on $M$, the choice of distinguished coordinates near infinities of $M$ along with the superpotential $W$. In our case the metric is fixed by special geometry:

$$g_{u\bar{u}}dud\bar{u} = \tau_2dad\bar{a}$$  \hspace{1cm} (7.14)

The distinguished coordinates are exactly the special coordinates (each cusp goes together with a set of canonical special coordinates corresponding to the vanishing cycles of the fiber). The distinguished coordinates allow to define the principal value integrals over $M$. We have met this necessity while computing Coulomb branch integrals above.

Now we claim that the gauge theory on $\Sigma$, whose intersection form contains a factor

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$  \hspace{1cm} (7.15)

is equivalent in certain chamber to the type B topological sigma model with the worldsheet $B$ and the target space $M$. The surface $B$ in question may be singular. It is a representative
of a class \( x \in H_2(\Sigma) \), whose self-intersection zero. This class together with another null class \( x', (x')^2 = 0 \) span the two dimensional lattice \( H \). Hence \( x \cdot x' = 1 \). We call \( x' \) the class of a fiber.

The chamber we mentioned corresponds to the limit of the metric on \( \Sigma \) in which the size of \( B \) is much larger then that of the rest of \( H^2(\Sigma) \). In computing a particular combination of observables of finite ghost number one needs some particular inequality to be fulfilled. The ratio of the classes must be larger then a certain number depending on the net ghost number of the observables.

The fluxes of the gauge field strength span the lattice \( H \oplus \Gamma \), where \( \Gamma \) denotes the rest of \( H^2(\Sigma, \mathbb{Z}) \). Let us replace the summation over the fluxes through \( B \) by the integral over the auxiliary field \( F \) with the constraint \( \int_B F \in 2\pi i \mathbb{Z} \) imposed by means of the delta function:

\[
\sum_{m \in \mathbb{Z}} e^{m \int_B F} \tag{7.16}
\]

The part of the effective action, containing \( F \) is simply:

\[
\int_B F(m + n\tau + \frac{dh}{da}) + \{Q, \ldots\} \tag{7.17}
\]

where \( h \) is the function of \( u \), whose 2-observable is to be integrated over \( B \). The number \( n \) is the flux through the cycle, which intersects \( B \) (since the intersection form is \( H \oplus \Gamma \) there is only one such cycle). From the point of view of the theory on \( B \) the field \( F \) is nothing but the auxiliary field of the type \( B \) sigma model, while \( m, n \) are some extra labels. If \( m = n = 0 \) we get a sigma model with the target being the \( u \)-plane with the superpotential \( W_0 = h \). If \( m^2 + n^2 \neq 0 \) then summation over all pairs \((m, n)\) with a given maximal common divisor \( N \) is equivalent to unfolding the quotient by the group \( \Gamma^0(4) \). Thus we get a target space \( \mathcal{M}_N \) and the superpotential \( W_N = Na + h \). In the absence of the observable associated with \( h \) we get the announced superpotential \((7.13)\). The rest of \( \Gamma \) is encoded in the peculiar zero-observables which are to be inserted at some points of \( B \). The generalization to the case of higher rank groups \( G \) is straightforward and yields the formula \((7.9)\) for the superpotential.

We conclude that the origin of the disconnectness of the target space is the presence of the discrete degrees of freedom in the gauge theory.

Given the identification of target spaces one may wonder about the geometrical meaning of the various observables of the gauge theory in terms of sigma model. Recall that in
the type $\mathcal{B}$ sigma model with the compact target space $X$ the $Q$-invariant observables are in one-to-one correspondence with the elements of the cohomology groups:

$$\mathcal{H} = \bigoplus_{p,q} H^p(X, \Lambda^q T_X) \quad (7.18)$$

In our case the target space is non-compact. This is not a completely unknown case since the Landau-Ginzburg models are usually formulated on the non-compact spaces and the appropriate cohomology theory is provided by the cohomology of the operator

$$Q = \bar{\partial} + \partial W$$

while the rest of the $(7.18)$ are the deformations of the operator $Q$ (in particular $H^0(\Lambda^0 T_X)$ deforms $W$ itself, $H^1(\Lambda^1 T_X)$ deforms the complex structure on $X$, $H^0(\Lambda^2 T_X)$ yields the deformation quantization of $X$ [48] and so on).

In the gauge theory on $\Sigma$ the observables correspond to the (co-)homology of $\Sigma$ while in the sigma model they come from the (co-)homology of $B$. For simplicity assume that the manifold $\Sigma$ projects to $B$. The fibers over the different points on $B$ need not to be the same. The projection $p : \Sigma \to B$ induces a pushforward map in cohomology $p_* : H^*(\Sigma) \to H^*(B)$. The imbedding $i : B \to \Sigma$ yields another map $i^* : H^*(\Sigma) \to H^*(B)$. The net effect of these fancy operations is simply the correspondence $p_*$ between the 4, 3, 2 observables in gauge theory and one set of 2, 1, 0 observables in sigma model (we call them vertical observables) and the correspondence $i^*$ between 2, 1, 0 observables in gauge theory and another set of 2, 1, 0 observables in sigma model (which we call horizontal).

The horizontal 0 -observable is simply the holomorphic function $h$ of $u$. The standard descend applied to the function $h$ produces 1 and 2 -observables which turn out to be horizontal 1 and 2 observables.

The vertical 0-observable inserted at the point $P \in B$ is the result of integration of 2-observable $O_v^{(2)}$ of gauge theory along the fiber $p^{-1}(P)$. Here $v$ denotes a holomorphic function of $u$ whose second descend gives rise to $O_v^{(2)}$. It can be expressed in terms of the standard fields of the sigma model (modulo irrelevant terms):

$$\frac{1}{\tau_2} \frac{dv}{da} \frac{d\bar{\tau}}{da} \theta \eta \quad (7.19)$$

where $\theta$ is the integral of the two-form field $\tau_2 \chi$ of the gauge theory along the fiber. The structure $\eta \theta$ corresponds to the elements of $H^1(\Lambda^1 T_X)$ (cf. the indices in $(7.8)$). Thus
the vertical zero-observable is the Beltrami differential responsible for the deformations of the complex structure of the target space. Its descendants produce the deformation of the theory.

This identification allows to perform one extra check of the formula for the contact terms by comparing it with the known expressions for the contact terms

\[ C(\phi_1, \phi_2) = G \frac{1}{Q} \left[ 1 - \Pi \right] (\phi_1 \phi_2) \]

where \( \Pi \) is the projector onto the “harmonic” representative. in the Landau-Ginzburg theory \[49\]. Unfortunately in this case it is not really an independent test, since we must rely on the requirement of modular invariance (it enters the definition of the target space).

**Contact terms.** It turns out that the analogy is not as simple as it may seem from the previous arguments. The subtle point is the behavior of the observables under the deformations of the theory.

The two-dimensional theory is deformed by adding two-observables to the action. These observables come from \( p \)-observables in four dimensional theory, integrated over \( p - 2 \)-cycles in the fiber. The zero-observables come from \( p \)-observables in four dimensions, integrated along \( p \)-dimensional cycles in the fiber. Now, if \( p + q < 4 \) then in four dimensions the corresponding cycles do not intersect each other and no contact terms appear. The distinction between different types of observables is not clear to the two-dimensional observer. Therefore the sigma model must be treated not as the conventional type \( B \) model (although we don’t know how to deal with the conventional sigma model on the non-compact target space in the absence of sufficient superpotential) but rather as a secretly four dimensional theory. The structure of contact terms mimics the geometry and the topology of the compactified space of distinct points on \( \Sigma \).

Nevertheless the analogy with two dimensional model suggests the principle of universality of contact terms.

Before stating it in the full generality let us consider the abstract two dimensional topological theory, whose correlation functions are computed as the integrals of Gromov-Witten invariants. Consider the correlator of 0 and 2-observables on the two-torus \( E \):

\[ \langle \mathcal{O}_{\phi_1}^{(0)} \int_E \mathcal{O}_{\phi_2}^{(2)} \rangle \]  

(7.20)

Consider also the correlator of two one-observables:

\[ \langle \int_A \mathcal{O}_{\phi_1}^{(1)} \int_B \mathcal{O}_{\phi_2}^{(1)} \rangle \]  

(7.21)
where $A$ and $B$ are the basic elements of $H_1(E)$. These correlators are equal as both can be represented as the integrals of

$$GW_{1,2}(\phi_1 \otimes \phi_2)$$

over the generic fiber of the projection $\tilde{M}_{1,2} \to \tilde{M}_{1,1}$.

Now to be specific let consider the example of Landau-Ginzburg theory, where it is believed that

$$\langle O^{(0)}(0) \phi_1 \int_E O^{(2)}(0) \phi_2 \rangle_W = \frac{d}{dt} \bigg|_{t=0} \langle O^{(0)}(0) \phi_1 \rangle_W + t \phi_2 + \langle O^{(0)}(0) C W;2+0(\phi_1,\phi_2) \rangle_W \tag{7.22}$$

with

$$C_{W;2+0}(\phi_1,\phi_2)$$

being the contact term between the zero-observable $\phi_1$ and the two-observable $\phi_2$. One may rewrite (7.22) explicitly as:

$$\oint -d\phi_1 d\phi_2 + (dX)^2 W'' C_{W;2+0}(\phi_1,\phi_2) \frac{dW}{dW} \tag{7.23}$$

On the other hand, the correlator (7.21) also has the contour integral representation:

$$\oint -d\phi_1 d\phi_2 + (dX)^2 W'' C_{W;1+1}(\phi_1,\phi_2) \frac{dW}{dW} \tag{7.24}$$

Here the first term appears as a result of saturating the fermionic zero modes in the one-observables $\partial \phi_1 \psi$, while the second is the contact term. Comparing the expressions (7.24) and (7.23) we arrive at the conclusion: “$1+1 = 2+0$”, or more drastically that the contact term is universal.

Flow to two-dimensional Yang-Mills theory. The trick with replacing the summation over the magnetic fluxes by an integral over the constrained field $F$ has been widely used in the context of two-dimensional gauge theories (see, for example [50]). The fact that this established an equivalence with the type $B$ sigma model has not been really appreciated, though (the papers [51][52] point out the relation to Landau-Ginzburg theory). Now we can easily formulate the sigma model description of the two dimensional topological Yang-Mills theory. It is again a sigma model with non-compact disconnected target space $\mathcal{M}$, with the components labelled by the dominant weights $\lambda$ of the group $G$ (cf. [24]). The component $\mathcal{M}_\lambda$ with $\lambda \neq 0$ is isomorphic to the Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ of the complexified
Lie algebra $\mathfrak{g}_C$: $\mathcal{M}_\lambda \approx \mathfrak{t}_C$. The component $\mathcal{M}_0$ is the set of semi-simple conjugacy classes in $\mathfrak{g}_C$: $\mathcal{M}_0 \approx \mathfrak{t}_C/W$. The model has the superpotential, which is equal to

$$W_\lambda = \langle \lambda, X \rangle, \quad X \in \mathcal{M}_\lambda$$  \hspace{1cm} (7.25)

Upon the deformation by the two-observable constructed out of the gauge-invariant function $F$ the superpotential changes to

$$W_\lambda = \langle \lambda, X \rangle + F(X)$$  \hspace{1cm} (7.26)

To complete the description of the model one needs the expression for the holomorphic volume form $dQ$ on $\mathcal{M}$. It is induced from the Haar measure on $G$. The latter yields $W$-invariant holomorphic volume form on $\mathfrak{t}_C$.

As a check of these assertions one may re-derive E. Witten’s results [24] by examining the handle gluing operator:

$$H = \text{Det} \frac{\partial^2 W_\lambda}{\partial X^i \partial X^j} \left( \frac{dX^1 \wedge \ldots \wedge dX^r}{dQ} \right)^2$$  \hspace{1cm} (7.27)

It is interesting to note that the results for the computations of the Donaldson invariants on the manifold $\mathbb{P}^1 \times B^2$ where $B$ is the Riemann surface are equal to those of the two dimensional gauge theory on $B$ in the limit $\Lambda \to 0$. Of course this is an expected result since in this limit the four dimensional instantons are forced to have vanishing instanton charge and become simply flat connections on $B$. In this sense the four dimensional perspective provides one more “deformation” of the intersection theory on the moduli space of flat connections on $B$. It is interesting to see whether it coincides with the quantum cohomology\textsuperscript{25}

7.3. Sigma model on $BG$?

The diagram (1.2) seems to exist with the target space $T$ being $BG$ - the classifying space of the (finite-dimensional) gauge group. This idea is based on two arguments. The first argument compares the space $B^*$ of gauge equivalence classes of irreducible connections with the classifying space of the group of the gauge transformations. The latter is homotopy\textsuperscript{24}

\textsuperscript{24} This case wasn’t studied in [53] precisely because of the difficulties of the Donaldson theory in the $b_2^+ = 1$ case

\textsuperscript{25} To compare it with the results of [53] one needs to pass through an infinite number of walls

65
equivalent to the space of maps of $\Sigma$ into $BG$. The idea that the gauge theory shares many similar properties with sigma model is by no means new, but here we suggest more concrete realization of this idea at least in the topological setting.

The standard obstructions in defining the sensible (topological) sigma model with the worldsheet of dimension higher then two are that there are no natural first order conditions to impose on the map which would yield a finite-dimensional moduli space. However, in the context of the gauge theory there exists such a condition - the instanton equation. The instantons form a submanifold $\mathcal{M}^+$ in the infinite-dimensional manifold $B^*$. Now, given a homotopy

$$h : B^* \to Map(\Sigma, BG)$$

one obtains the “submanifold of special maps ” $h(\mathcal{M}^+) \subset Map(\Sigma, BG)$. All the computations of the correlation functions in the topological gauge theory can be translated into the sigma model language. It would be interesting to pursue this idea futher.

To conclude we restate one of the puzzles which we tried to solve in this paper. There are at least three definitions of the higher characteristic classes of the universal instanton over $\mathcal{M} \times \Sigma$. One is the continuation of the harmonic representative $\text{Tr} (\Delta^{-1}_A [\psi, \star \psi])^{2r}$ given by the ultraviolet gauge theory to the compactification of the moduli space of instantons. The second uses the construction of the universal sheaf (in the complex situation). The last simply takes the four dimensional class $\Theta$ and raises it to the $r$’th power. We have seen that the infrared theory is best suited for the last definition. Nevertheless, the study of the theory with matter suggests that the first two definitions are also relevant. It would be highly desirable to understand the relation between these three and apply it to the theories, which are not well-studied yet, such as the compactifications of five- and six-dimensional theories and $F$-theory vacua. It would also be interesting to compare the formalism of the Hamilton -Jacobi equations which we used in deriving the contact terms to the Whitham hierarchies of [57].

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