Nonlinear Time-Delay Observer-Based Control to Estimate Vehicle States: Lateral Vehicle Model

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ABSTRACT This paper deals with the state estimation and control problem for nonlinear lateral vehicle dynamics with time delays. First, a novel time-varying delay vehicle model described as a Takagi-Sugeno fuzzy model is presented. In particular, it is considered that the lateral force contains an air resistance term which is assumed to be a quadratic function of the lateral vehicle velocity. A time-varying delay has been included in the vehicle states by a simple formula in order to capture brake actuation aspects or other practical aspects that may generate a delayed response, while the nonlinear part of the vehicle model is described as a Lipschitz function. A Takagi-Sugeno time-delay observer-based control that satisfies the Lipschitz condition is proposed to get closed-loop stability conditions. These results generalize existing ones in the literature on lateral dynamics control. Additionally, we provide a new methodology for the controller and observer gains design that can be cast as linear matrix inequality constraints. Finally, we illustrate our results with numerical examples, which also reveal the negative effect of not considering the presence of delays in the controller design.

INDEX TERMS Lateral vehicle model, time-varying delay, observer-based control, T-S fuzzy model, LMIs.

I. INTRODUCTION

Every year around the world, over a million people are killed and millions are injured in car accidents. This can be seen as a huge road war in some countries, since many people risk their lives every day. Researchers have tried to tackle the challenge of road traffic safety from different angles (see for example [1] and the references therein). The study and development of vehicular systems play a significant role in overcoming this challenge, and the automotive engineering industry and research laboratories in this field have constantly been striving to make vehicles comfortable, efficient and safe. In this regard, this work delves in the topic of observer-based control systems for lateral vehicle dynamics, and in particular, expanding the analysis to consider time-varying delays with nonlinear vehicle dynamics.

The research on lateral vehicle dynamics and their control design has received a great deal of attention in the recent decade [2], [3], [4], [5]. Model predictive control (MPC) schemes have been used to track a lateral vehicle desired path [6], [7], [8]. In [9], a state-feedback control is investigated for the lateral motion of an intelligent vehicle. A great deal of work has been done to estimate and control lateral vehicle systems. The active yaw stabilizer (AYS) system was proposed in [2] to improve vehicle lateral stability control. In [3], the authors used the steer-by-wire (SBW) system to improve vehicle safety, maneuverability and steering flexibility. In [4], the authors investigated two-dimensional state portrait analysis to enable the control designer to test the stability and trajectories of the vehicle system. In addition, many approaches and methodologies addressing vehicle systems’ problems are discussed in [4] and the references therein. In [10], the authors used a fuzzy control method for vehicle systems to increase stability and reduce the rollover risk when...
considering the lateral forces’ nonlinearity using the Takagi-Sugeno (T-S) representation.

It is well known that many physical, industrial, and engineering systems are nonlinear. In this context, the T-S fuzzy approach has become essential for dealing with complex nonlinear systems, since fuzzy models can adequately describe several natural systems. These systems are competent to describe a large class of nonlinear systems using fuzzy IF-THEN rules, which are described in the form of local linear or affine models connected by fuzzy membership functions [10], [11]. Recently, fuzzy observer-based control has attracted the attention of several researchers due to the importance of unmeasured internal states of real systems. Several studies have dealt with fuzzy observer design problems for continuous systems [10], [11], [12]. In [10], the authors used a fuzzy observer-based control to evaluate the vehicle states, while in [11], the observer-based control strategy was used to control T-S fuzzy descriptor systems with time-varying delays and a $H_{\infty}$ performance index.

Researchers have also studied the delay-free lateral vehicle dynamics, and several mathematical models have been suggested [13], [14], [15]. However, most of these models are derived by using assumptions and approximations, which can turn into the loss of critical information, especially when dealing with safety related behaviors of nonlinear models. In particular, the delay information could be regarded as an essential component of an accurate model for lateral vehicle dynamics, when considering the effect that irregular or asymmetrical road conditions could have on the steering. This work introduces a time-varying delay function that aims to recover information that would be otherwise lost when applying approximations to deal with delays in the real model. We believe that this can help to get a new time-delay model that may perform better for observer-based control applications.

Time delays have not attracted much interest from researchers in the field of lateral vehicular motion. However, it is well known that delay occurs in most natural systems and is part of many processes. For example, delays may occur as a result of the transmission of information between various parts of a system and because of limited information processing capabilities [11], [16], [17]. Furthermore, delays can not be ignored, as they can cause instability and system performance degradation [18], [19], [20]. Several works have been reported that deal with a time-varying delay system, see for comparison purposes [11], [16], [17] and the references therein.

The first contribution of the present work is to propose a novel lateral vehicle model by introducing a time-varying delay in its states. The proposed model is more general than those in the literature [10], [21], which can be recovered as the time-varying delay tends to zero. Also, the lateral forces have been represented by the Takagi-Sugeno (T-S) model, which globalizes the lateral forces given in other works such as [10], [21], and [11]. The nonlinear part not introduced in the T-S model is considered to be a Lipschitz function. The second contribution is the proposal of a T-S observer-based control to estimate the yaw rate and side-slip angles, together with a new methodology for simultaneously designing the observer and controller gains based on LMI conditions.

The outline of this paper is as follows: Section II presents the notation and nomenclature used throughout the paper. In Section III, we present the lateral vehicle model under study. In contrast, Section IV presents the problem formulation and the main contributions of this work regarding observer-based control design. Section V presents simulation results and their analysis, illustrating the theoretical results and the design methodology for the proposed observer-based control scheme. Finally, conclusions and future work are placed in VII.

II. NOTATION AND NOMENCLATURE

We use $\mathbb{R}^n$ to denote the n-dimensional Euclidean space, and $\mathbb{R}^{n \times n}$ to denote the set of all $n \times n$ real matrices. For a matrix $X \in \mathbb{R}^{n \times n}$, $X^{-1}$ and $X^T$ will denote the inverse and the transpose of $X$, respectively. Asterisks (*) will denote the symmetric part of a symmetric matrix. The notation $X > 0$ means that the matrix $X$ is real symmetric positive definite. We denote by $I$ ($I_n$) the identity matrix of adequate order ($n$) and by $0_{n \times m}$ the $n \times m$ null matrix. We define $\text{sym}(A) = A + A^T$. Finally, $\text{diag}(...)$ denotes a block-diagonal matrix.

NOMENCLATURE

(See Fig. 1):

- $F_x$ Is a longitudinal force.
- $F_y$ Is a lateral force.
- $M_z$ Is the yaw moment, pointing in the positive $z$ direction.
- $V_x(t)$ Is the yaw moment, pointing in the positive $z$ direction.
- $V_y(t)$ Is the yaw rate.
- $I_z$ Is yaw moment of inertia.
- $F_{sf}$ Is the front tire force.
- $F_{sr}$ Is the rear tire force.
- $a_1$ Is the distance from the center of gravity to the front tires.
- $a_2$ Is the distance from the center of gravity to the rear tires.
- $\delta(t)$ Is the front wheel steering angle.
- $\alpha_f(t)$ Is the front tire side-slip angle between wheel $x$-axis and the velocity $v$.
- $\alpha_r(t)$ Is the rear tire side-slip angle between wheel $x$-axis and the velocity $v$.
- $C_f$ Is the cornering stiffness of the front tires.
- $C_r$ Is the cornering stiffness of the rear tires.
- $\beta_f(t)$ Is the front side-slip angle of wheel vehicle.
- $\beta_r(t)$ Is the rear side-slip angle of wheel vehicle.
- $\beta(t)$ Is the vehicle side-slip angle.

III. VEHICLE MODEL

This section describes the vehicular model to be studied.
A. INITIAL VEHICLE MODEL

The bicycle model (Fig. 1) describes the essential nonlinear lateral dynamics of the vehicle. They balance complexity and computation efficiency. The first contribution of this work is to propose a new vehicle model that describes the relationship between the relevant variables of the lateral dynamics, for example, the yaw rate and the side-slip angle (see Fig. 1).

**FIGURE 1. Vehicle model.**

The bicycle vehicle model (single-track model) was invented in 1940 and is still widely used today [22], [23], [24]. Several significant omissions and simplifications are included in the bicycle model, which effectively reduces the degree of freedom of the model without affecting the vehicle’s dynamic behavior in the linear range. The bicycle model can be converted to a computer simulation model and can be used to quickly and easily analyze vehicle behavior [25], [26]. Based on the assumptions and simplifications considered in [25], [26], the geometric relationships for the bicycle model can be developed as shown in Fig. 1.

By using Newton-Euler motion equations for a rigid vehicle in the body coordinate frame connected to the vehicle at its mass center, the following equations are obtained [27]:

\[
\begin{align*}
F_x &= m\dot{V}_x(t) - m\psi(t)V_y(t), \\
F_y &= m\dot{V}_y(t) + m\psi(t)V_x(t), \\
M_z &= \ddot{\psi}(t)I_z,
\end{align*}
\]

where \( m \) is the mass of the vehicle, and \( M_z \) is the yaw moment, pointing in the positive \( z \) direction. We can now consider the four-wheel vehicle whose longitudinal and lateral forces are projected on the body coordinate as follows

\[
\begin{align*}
F_x &= 2F_{sx}\cos\delta(t) - 2F_{sy}\sin\delta(t) + 2F_{sr}, \\
F_y &= 2F_{sx}\sin\delta(t) + 2F_{sy}\cos\delta(t) + 2F_{sr}, \\
M_z &= 2a_1F_{sf}\cos\delta(t) + 2a_2F_{sr}\sin\delta(t) - 2a_2F_{yr}.
\end{align*}
\]

The nonlinear model given in (2) has been obtained by assuming that the left and right wheels have the same longitudinal and lateral force. Moreover, in [27], [28], and [21], the authors considered the steer-angle \( \delta(t) \) to be small, to obtain linear models for the yaw moment, and the longitudinal and lateral forces. Based on this approximation, large amount of work has been done, in order to deal with the challenges arising from nonlinear terms in the lateral vehicle dynamics.

Considering small steer-angle \( \delta(t) \) approximations, the above nonlinear forces in (2) can be simplified as (see for instance [27], [28] and [21]):

\[
\begin{align*}
F_x &= 2F_{sf} + 2F_{sr}, \\
F_y &= 2F_{sy}, \\
M_z &= 2a_1F_{sf} - 2a_2F_{sy}.
\end{align*}
\]

In [21] and [10], the authors have considered that the lateral forces depend on the front and rear side-slip angles \( \alpha_f(t) \) and \( \alpha_r(t) \), and the relationship between these forces and the angles \( \alpha_f(t) \) and \( \alpha_r(t) \) to be linear. By assuming such approximations, these relationships can be expressed as

\[
\begin{align*}
F_{sf} &= -C_{a_f}\alpha_f(t), \\
F_{sr} &= -C_{a_r}\alpha_r(t).
\end{align*}
\]

Notice that the relationships given in (4) are valid for bounded side-slip angles, which implies that they are applicable when the vehicle velocity is small. Thus, the conditions in (4) are valid in the linear region (steady-state), which makes dynamic relationships much more complex. Furthermore, we can see that the lateral forces suggested in (4) tend to be constant for all vehicle wheels. This is not generally true since the lateral force generated by each wheel is different from the ones produced by the other wheels. In this context, we consider the nonlinearities of the lateral forces by using the Takagi–Sugeno (T-S) representation, and we recall the lateral forces proposed in [10] and [29] described as follows

\[
\begin{align*}
\text{if} \ | \alpha_f(t) | \leq M_1 \text{ then } F_{sf} &= -C_{a_f}\alpha_f(t); \\
& \quad F_{yr} = -C_{a_r}\alpha_r(t), \\
\text{if} \ | \alpha_f(t) | \leq M_2 \text{ then } F_{sf} &= -C_{a_f}\alpha_f(t); \\
& \quad F_{yr} = -C_{a_r}\alpha_r(t),
\end{align*}
\]

where \( C_{a_f} \) and \( C_{a_r} \) are the fuzzy sets for slip angles. Using the membership function, the overall cornering forces (5) are described as following

\[
\begin{align*}
F_{sf} &= -\sum_{i=1}^{2} \theta_i(|\alpha_f(t)|)C_{a_f}\alpha_f(t), \\
F_{yr} &= -\sum_{i=1}^{2} \theta_i(|\alpha_f(t)|)C_{a_r}\alpha_r(t),
\end{align*}
\]

where the membership functions \( \theta_i, \quad i = 1, 2 \) satisfy the following conditions

\[
\sum_{i=1}^{2} \theta_i(|\alpha_f(t)|) = 1, \quad 0 \leq \theta_i(|\alpha_f(t)|) \leq 1, \quad i = 1, 2.
\]

and are given by the following expression

\[
\theta_i(|\alpha_f(t)|) = \frac{\omega_i(|\alpha_f(t)|)}{\sum_{i=1}^{2} \omega_i(|\alpha_f(t)|)}.
\]
where \( \omega_1(\alpha_f(t)) \) and \( \omega_2(\alpha_f(t)) \) are membership functions given by

\[
\omega_k(\alpha_f(t)) = \frac{1}{1 + \left|\frac{\alpha_f(t) - c_k}{d_k}\right|^n},
\]

(9)

**Remark 1:** To reduce the number of membership functions, the authors in [10] have considered \( \alpha_f(t) \) and \( \alpha_r(t) \) in the same fuzzy set. That is why the rules are only made for \( \alpha_f(t) \), which in turn allows us to reduce the number of parameters to identify [30].

**B. EXTERNAL FORCES**

When the vehicles move and their velocity increases, it appears that external forces try to resist their movement, which may impact the protection requirements of the vehicle as a safety measure. Among these external forces we find the intensity of the wind, which is evident when the vehicle’s velocity increases or when the weather changes. Assuming that the lateral force comprises an air resistance term, that is assumed to be a quadratic function of the lateral velocity of the vehicle \( V_\ell(t) \), and using the T-S model given in (6), the lateral forces in (4) can be written as follows.

\[
F_{yf} = -\sum_{i=1}^{2} \theta_i(\alpha_f(t))C_{af}\alpha_f(t) - C_{Af}V_\ell^2(t),
\]

\[
F_{yr} = -\sum_{i=1}^{2} \theta_i(\alpha_r(t))C_{ar}\alpha_r(t) - C_{Ar}V_\ell^2(t).
\]

(10)

where \( C_{Af} \) and \( C_{Ar} \) are the front and rear air resistance coefficient, respectively.

Considering the relationship between the lateral velocity \( V_\ell(t) \), the vehicle side-slip angle \( \beta(t) \) and the longitudinal velocity \( V_x(t) \), that is \( \beta(t) = \frac{V_\ell(t)}{V_x(t)} \), (10) can be rewritten as

\[
F_{yf} = -\sum_{i=1}^{2} \theta_i(\alpha_f(t))C_{af}\alpha_f(t) - V_x^2(t)C_{Af}\beta^2(t),
\]

\[
F_{yr} = -\sum_{i=1}^{2} \theta_i(\alpha_r(t))C_{ar}\alpha_r(t) - V_x^2(t)C_{Ar}\beta^2(t).
\]

(11)

**Remark 2:** The vehicle model’s complexity depends on the model of the cornering forces \( F_{yf} \) and \( F_{yr} \). In the literature, several formulations have been proposed to define these forces. Moreover, these forces are given as a function of the tire slip angles using nonlinear relationships, as in [10], or as the form in (4). Furthermore, note that when \( C_{Af} = C_{Ar} = 0 \), (11) recovers the results presented in [10] and [29], which means that the front and rear tire forces proposed in this work are more general than the existing ones in the literature.

To obtain the coefficients defined in the membership parameters of the T–S model, we use the method cited in [10]. The obtained coefficients are described in Table 1.

**C. TIME DELAYS**

As discussed in [10], [21], [28] and [11], and the references therein, many works have considered both side-slip angles \( \beta_f(t) \) and \( \beta_r(t) \), under the assumption that they are small. The relationship between them and the front and rear tire side-slip angles, \( \alpha_f(t) \) and \( \alpha_r(t) \), are given as follows

\[
\alpha_f(t) = \beta_f(t) - \delta(t) = \beta(t) + \frac{a_1}{V_x} \psi(t) - \delta(t),
\]

\[
\alpha_r(t) = \beta_r(t) = \beta(t) - \frac{a_2}{V_x} \psi(t).
\]

(12)

Note that the approximation used to represent both side-slip angles is not unique and various approximations have been used in the literature. The assumption is unrealistic despite the representation used for the front and rear tire side-slip angles. In a practical setting, and when the vehicle’s slip angle changes, the force system on the planar vehicle model can not respond immediately, which could translate into the presence of delays. Delays exist in many practical dynamical systems but are usually disregarded when designing controllers and observers. However, especially in safety-critical applications, they can be a source of performance and safety loss for the overall system, if they are not adequately addressed when synthesizing a controller or observer. Furthermore, since the vehicle should be safe for humans, it is essential to detect any possible problems that may arise due to its dynamical properties. For example, when a vehicle enters a turn, the lateral forces between the wheels and the roadway are proportional to the side-slip angle of the wheels. For a front-steered vehicle, the side-slip angle on the unsteered rear wheels takes a moment to build up, delaying the application of cornering forces to the rear wheels [25]. To improve the deficiencies of driver assistance systems, drawbacks in each system must be reduced or compensated. One such weakness is the delay required to generate adequate braking torque on a wheel brake [25]. This time delay of several tenths of a second is caused in part by the following factors [25]:

- Elasticities in the brake lines;
- The compressibility of the brake pads;
- Elasticities in the brake calipers.

Certain environmental conditions, such as heavy moisture and road salt build-up on the brake discs, can increase the time delay in the braking system.

Under the assumption that the equations of motion of the tire are stationary or at least quasi-stationary, we have that the introduced parameters, such as the values of circumferential slip, slip angle \( \beta(t) \), camber angle, as well as the tire forces \( F_x, F_y \), and tire torques \( M_z \), remain constant over time or fluctuate very slowly [26]. These requirements are absent or insufficiently fulfilled to justify this assumption. In these

**TABLE 1. Parameters of the membership functions.**

| Parameters | Units |
|------------|-------|
| \( c_1 \) | 0.2844 |
| \( c_2 \) | 0.1647 |
| \( b_1 \) | 1.7009 |
| \( b_2 \) | 12.0064 |
| \( d_1 \) | 0.0785 |
| \( d_2 \) | 0.1126 |
| \( C_{af,1} \) | 55234 |
| \( C_{af,2} \) | 15544 |
| \( C_{ar,2} \) | 49200 |
| \( \frac{N}{m} \) | |

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| \( C_{ar,2} \) | 49200 |
| \( \frac{N}{m} \) | |
cases, the longitudinal slip and the side-slip angle ($\beta(t)$) change very fast with time, implying that the tire forces ($F_x, F_y$) and torques ($M_z$) can only build up with a time delay [26]. To consider the effect of the potential time-delay on the side slip angles, we consider the following representation of the front and rear tire side slip angles of the vehicle:

$$\alpha_f(t) = l_1 \beta(t) + (1 - l_1) \beta(t - h(t)) + \frac{a_1}{V_x} \dot{\psi}(t) - \delta(t),$$

$$\alpha_r(t) = l_2 \beta(t) + (1 - l_2) \beta(t - h(t)) - \frac{a_2}{V_x} \dot{\psi}(t),$$

where $l_1$ and $l_2$ are delay parameters satisfying

$$0 \leq l_1 \leq 1 \quad \text{and} \quad 0 \leq l_2 \leq 1$$

and $h(t)$ is a time-varying delay satisfying the following constraint

$$0 \leq h_1 \leq h(t) \leq h_2, \quad \dot{h}(t) \leq \mu,$$  

where $h_1$ and $h_2$ are the lower and upper bound of the time-varying delay, and $\mu$ is the upper bound of the time-varying delay derivative. Based on the above discussion, it can be concluded that the steering angles can be described by a delay while the yaw rate cannot.

**Remark 3:** The idea behind considering different delay parameters $l_1$ and $l_2$, as opposed to being equal, is that the front and rear side slip angles may not be affected by delays equally. Take for instance when a vehicle begins to navigate on irregular terrain (change of road conditions), or if the wheels are of different characteristics, or under an emergency situation (punctured wheel) or similar cases. Therefore, the proposed model is more general than the existing ones in the literature as it considers a potential delay parameter for the front side slip angle that may differ from the delay parameter for the rear side slip angle.

**Remark 4:** Notice that the time-delay tire side-slip angles given in (13) are more general than those provided in [21] and [10]. Thus, if $l_1 = l_2 = 1$ or $h(t) = 0$ we obtain the delay-free tire side-slip angles as indicated in [21] and [10] and otherwise the tire side-slip angles are totally affected by delay. Moreover, the delay parameters $l_1$ and $l_2$ are the tuning parameters that help us introduce a delay in side-slip and yaw rate angles and allow us to control the percentage of the delay in the vehicle model.

A simple differential equation cannot successfully model many complex systems. For many systems, the future evolution of the state variables $x(t)$ depends not only on their current value of the state and its derivatives but also on their past values [31]. Furthermore, the delay is not just a mathematical concept; more significantly, it is embedded in many natural systems, such as biochemical reactions, industrial processes and mechatronic motions. Delay, as a universal part of many processes, is a phenomenon that undoubtedly deteriorates the quality of the control, causing oscillations and instability. Therefore, the stability and control analysis of such a system are of theoretical and practical importance. Considering a time-delay in the lateral vehicle model may allow us more information about the system’s behavior, especially when these delays may be exacerbated in the event of an emergency or a change in road conditions.

Furthermore, the idea behind using (13) is to reconstruct the information that could be eliminated from the system when using the linear approximation in (3), which could affect the system’s performance. Thus, the delayed part has been identified as global information, which is ignored when approximations are used.

It is assumed that the vehicle body moves at a constant forward speed, which means that $V_x = 0$ and the first line in (1) becomes an independent algebraic equation. Therefore, the longitudinal force does not affect vehicle stability, and the integration into the system equation is unnecessary.

Combining (1), (10) and (13), and assuming the yaw rate $\dot{\psi}(t)$ as the output, the modelled vehicle system can be represented as follows:

$$\dot{x}(t) = \sum_{i=1}^{n} \theta_i(\alpha_f(t)) [A_i x(t) + A_{di} x(t - h(t))] + B_i u(t) + f(x(t)),\]

$$y(t) = \sum_{i=1}^{n} \theta_i(\alpha_f(t)) C_i x(t),$$

where $x^T(t) = [x_1(t), x_2(t), \ldots, x_n(t)] = [\beta(t), \dot{\psi}(t)]$ is the state vector and $\delta(t) = u(t)$ is the control input,

$$A_i = \begin{bmatrix} -\pi_1^1 & -(1 + \pi_2^1) \\ -\pi_3^1 & -\pi_4^1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 2 \frac{C_{a1}}{m V_x} \\ \frac{2 a_1 C_{a1}}{I_z} \end{bmatrix},$$

$$f(x(t)) = \begin{bmatrix} -\chi_1 x_1^2(t) \\ -\chi_3 x_3^2(t) \end{bmatrix}, \quad A_{di} = \begin{bmatrix} -\pi_5^1 \\ -\pi_6^1 \end{bmatrix},$$

$$C_i = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

with

$$\pi_1^1 = 2 \frac{l_1 C_{a1} + l_2 C_{a2}}{m V_x}, \quad \pi_2^1 = 2 \frac{a_1 C_{a1} - a_2 C_{a2}}{m V_x^2},$$

$$\pi_3^1 = 2 \frac{a_1 l_1 C_{a1} - a_2 l_2 C_{a2}}{I_z},$$

$$\pi_4^1 = 2 \frac{a_1^2 C_{a1} - a_2^2 C_{a2}}{I_z V_x},$$

$$\pi_5^1 = 2 \frac{(1 - l_1) C_{a1} - (1 - l_2) C_{a2}}{m V_x},$$

$$\pi_6^1 = 2 \frac{a_1 (1 - l_1) C_{a1} - a_2 (1 - l_2) C_{a2}}{I_z V_x}.$$
with (13) are reduced to those in [10], [21], and [11] when \( h(t) = 0 \) and \( C_Af = Ca_r = 0 \) or when \( l_l = l_2 = 1 \) and \( C_Af = Ca_r = 0 \), which shows that the current model is more general.

The proposed vehicle system model in (16) is a nonlinear time-varying delay system, which is more general than some systems found in the literature. Moreover, since the nonlinear parts are of the kind \( x_i^2 \), they can be regarded as Lipschitz functions, at least locally, by considering that the operating range of \( x_1 \) is bounded. Because of that, in this work, it is assumed that the nonlinear function \( f(x(t)) \) satisfies the one-sided Lipschitz condition in [32], that is,

\[
< f(x(t)), x(t) > \leq \rho_1 \| x(t) \|^2, \quad \rho_1 \in \mathbb{R}, \tag{18}
\]

and also, verifies the quadratic inner-bounded condition given by

\[
\| f(x(t)) \|^2 \leq \rho_2 \| x(t) \|^2 + \rho_3 < f(x(t)), x(t) > . \tag{19}
\]

**IV. OBSERVER-BASED CONTROL**

As in many practical systems, the vehicle system states are not necessarily fully observable due to sensor and technology limitations or other practical reasons. Because of this, observer-based control is very relevant, as it seems to be well-suited for estimating the vehicle states from the measured vehicle output states. In this context, we will present the main results of this work by formulating an observer-based control problem for the proposed vehicle dynamics, followed by an LMI based approach for its design.

**A. PROBLEM FORMULATION**

In order to estimate the vehicle system states in (16), consider the following observer which has the same structure as the vehicle system:

\[
\dot{x}(t) = A_l x(t) + B_l e(t) + A_d x(t - h(t)) + f(x(t)),
\]

\[
\dot{e}(t) = A_L e(t) + A_d e(t - h(t)) + \Delta f(x(t), \bar{x}(t)). \tag{21}
\]

In order to achieve this, let us define the augmented error system as follows

\[
\dot{x}(t) = A_k x(t) - B_k e(t) + A_d x(t - h(t)) + f(x(t)),
\]

\[
\dot{e}(t) = A_L e(t) + A_d e(t - h(t)) + \Delta f(x(t), \bar{x}(t)). \tag{21}
\]

where

\[
A_k = \sum_{i=1}^{2} \sum_{j=1}^{2} \theta_i(|a_j(t)|) \theta_j(|a_j(t)|) (A_i + B_i K_j),
\]

\[
A_L = \sum_{i=1}^{2} \sum_{j=1}^{2} \theta_i(|a_j(t)|) \theta_j(|a_j(t)|) (A_i - L_i C_j),
\]

\[
B_k = \sum_{i=1}^{2} \sum_{j=1}^{2} \theta_i(|a_j(t)|) \theta_j(|a_j(t)|) (B_i K_j),
\]

\[
\Delta f(\bar{x}(t), x(t)) = f(x(t)) - f(\bar{x}(t)). \tag{22}
\]

Moreover, it is assumed that \( \Delta f(x(t)) \) verifies the one-sided Lipschitz and quadratic inner-bounded conditions described in the following conditions, for given scalars \( \vartheta_i \in \mathbb{R}, i = 1, 2, 3 \).

\[
< \Delta f(\bar{x}(t), x(t)), e(t) > \leq \vartheta_1 \| e(t) \|^2
\]

\[
\| \Delta f(\bar{x}(t), x(t)) \|^2 \leq \vartheta_2 \| e(t) \|^2 + \vartheta_3 < \Delta f(\bar{x}(t), x(t)), e(t) > . \tag{23}
\]

One can see that the error goes to zero when the time-delay system (21) is asymptotically stable. The notation in (22) is used to help us in the development of our results.

Most of the works in the literature that study time-delay systems use the boundedness of integral terms, since these terms appear in the time-derivative of a Lyapunov-Krasovskii functional. In recent years, a significant reduction in the conservatism of stability results has been achieved due to the boundedness of these terms and the use of Wirtinger’s inequality (Lemma 1). Next, we present two technical lemmas that are helpful when deriving our results.

**Lemma 1 [16]:** For any matrix \( R > 0 \), and a differentiable signal \( x \in [h_1, h_2] \to \mathbb{R}^n \), the following inequality holds:

\[
- \int_{h_1}^{h_2} \dot{x}(s) R \dot{x}(s) ds \leq \frac{1}{h_2 - h_1} \xi^T \Phi(R) \xi, \tag{24}
\]

where \( \xi = [x^T(h_2) x^T(h_1)] \frac{1}{h_2 - h_1} \int_{h_1}^{h_2} x^T(s) ds \) and

\[
\Phi(R) = \begin{bmatrix}
-4R & -2R & 6R \\
* & -4R & 6R \\
* & * & -12R
\end{bmatrix}. \tag{25}
\]

**Lemma 2 [33]:** For matrices \( V, L, Z > 0 \) with appropriate dimensions, the following inequality holds:

\[
VL + L^T V^T \leq V Z V^T + L^T (e Z)^{-1} \mathbf{L}. \tag{26}
\]

**B. TIME-DELAY OBSERVER-BASED CONTROL DESIGN**

Now, we consider a time-delay observer-based control to estimate the vehicle model states developed in the above section and to ensure that the closed-loop system is asymptotically stable. Additionally, we consider that the controller gains are...
known and, based on the Lyapunov-Krasovskii functional, we present the condition that ensures the stability of system (21). First we have the following result.

**Theorem 1:** Let us consider the system (16), with conditions (18), (19), and (23) satisfied, and the proposed observer in (20) with given gains \( K_i \) and \( L_i, i = 1, 2 \). The augmented time-varying delay error system in (21) is asymptotically stable, with given gains \( K_i \) and \( L_i \), if for given positive scalars \( \gamma_1, \gamma_2 \), there exist positive definite matrices \( \gamma \), \( N_r, r = 1, 2, 3 \), \( \text{diag}(P_{e11}, P_{e22}, P_{e33}), R_1, R_2, T_1, T_2, \text{diag}(P_{e11}, P_{e22}, P_{e33}) \), and positive scalars \( \gamma_3 \) and \( \gamma_4 \) such that the following conditions hold

\[
\Omega_{ij} < 0, \quad i, j = 1, 2, \quad (27)
\]

where

\[
\Omega_{ij} = \left[ \begin{array}{cc}
\Omega_{ij} & \Omega_{ij} \\
* & \Omega_{ij}
\end{array} \right] + \left[ \begin{array}{c}
\Gamma_{ij}^T
\end{array} \right] \left( h_i^2 R_1 + h_i^2 R_2 \right) \left[ \begin{array}{c}
\Gamma_{ij}^T
\end{array} \right]^T,
\]

with \( h_{12}, \bar{\Omega}_e, \Omega_x, \) and \( \Omega_{ij} \), as shown in the equation at the bottom of the page.

\[
\Omega_{x11} = \text{sym}(P_{x11}A_{1x}^T) + Q_1 + 2\sigma_1 I - 4R_1,
\]

\[
\Omega_{x15} = h_1P_{x22} + 6R_1,
\]

\[
\Omega_{x17} = P_{x11} + 2\sigma_2 I,
\]

\[
\Omega_{x22} = -(Q_1 - Q_2) - 4R_1 - 4R_2,
\]

\[
\Omega_{x25} = -h_1P_{x22} + 6R_1,
\]

\[
\Omega_{x26} = h_1P_{x33} + 6R_2,
\]

\[
\Omega_{x33} = -(1 - \mu)(Q_2 - Q_3),
\]

\[
\Omega_{x44} = -Q_3 - 4R_2,
\]

\[
\Omega_{x46} = -h_1P_{x33} + 6R_2,
\]

\[
\sigma_1 = \gamma_1\rho_1 + \gamma_2\rho_2, \quad \sigma_2 = \gamma_2\rho_3 - \gamma_1,
\]

\[
\Omega_{e11} = \text{sym}(P_{e11}A_{1e}^T) + N_1 + 2\sigma_3 I - 4T_1,
\]

\[
\Omega_{e15} = h_1P_{e22} + 6T_1,
\]

\[
\Omega_{e17} = P_{e11} + 2\sigma_4 I,
\]

\[
\Omega_{e22} = -(N_1 - N_2) - 4T_1 - 4T_2,
\]

\[
\Omega_{e25} = -h_1P_{e22} + 6T_1,
\]

\[
\Omega_{e26} = h_1P_{e33} + 6T_2,
\]

\[
\Omega_{e33} = -(1 - \mu)(N_2 - N_3),
\]

\[
\Omega_{e44} = -N_3 - 4T_2,
\]

\[
\Omega_{e46} = -h_1P_{e33} + 6T_2,
\]

\[
\sigma_3 = \gamma_3\theta_1 + \gamma_4\theta_2, \quad \sigma_4 = \gamma_4\theta_3 - \gamma_3,
\]

\[
\Omega_{xe} = \left[ \begin{array}{c}
-P_{x11}B_d^T \quad 0_{n \times 6n}
\end{array} \right],
\]

\[
\Gamma_{ij} = \left[ \begin{array}{c}
A_d^T \quad 0_{n \times 3n} \quad I_n
\end{array} \right],
\]

\[
\Pi_{ij} = \left[ \begin{array}{c}
A_d^T \quad 0_{n \times 3n} \quad I_n
\end{array} \right].
\]

**Proof:** See Appendix A.

Theorem 1 was obtained by using Wirtinger’s inequality (Lemma 1) [16], which has been shown to reduce conservatism on the related bounds when compared to using Jensen’s inequality.

Notice that Theorem 1 provides a delay-dependent stability analysis of the lateral vehicle described in (1). Moreover, the conditions presented in Theorem 1 are linear for given controller gains \( K_i \) and \( L_i \). Thus, they can be solved in a straightforward manner using the standard LMI-Toolbox solver [34].

If the controller gains \( K_i \) and \( L_i \) are variable, we have that condition (27) is not an LMI, since nonlinear terms will appear. Therefore, more work is required to solve the nonlinearity problem in Theorem 1. The following theorem presents sufficient conditions to design both gains \( K_i \) and \( L_i \) simultaneously, in order to ensure the asymptotic stability of the closed-loop system (21).
Theorem 2: System (16) is asymptotically stabilizable by (20) if for given positive scalars \( \gamma_1, \gamma_2 \), there exist positive definite matrices \( \tilde{Q}_x, N_x, r = 1, 2, 3, \) \( \text{diag}(X_{11}, \tilde{P}_{e11}, \tilde{P}_{e33}), \) \( \text{diag}(P_{e11}, P_{e22}, P_{e33}), \) \( \tilde{R}_1, \tilde{R}_2, T_1, T_2, \) matrices \( Y_i, G_i \), and positive scalars \( \gamma_3 \) and \( \gamma_4 \) such that the following conditions hold

\[
\Xi_{ij} < 0, \quad i = 1, 2, \quad (29)
\]
\[
\Xi_{ij} + \Xi_{ji} < 0, \quad 1 \leq i < j \leq 2, \quad (30)
\]

where \( h_{12}, \Xi_{ij}, \tilde{\Xi}_{ij}(0), \tilde{\Omega}_{ij} \), and \( \Omega_{ij} \), as shown in the equation at the bottom of the page.

\[
\tilde{x}_1 = T_1 - 2P_{e11},
\]
\[
\tilde{x}_2 = T_2 - 2P_{e11},
\]
\[
\tilde{x}_1 = \tilde{R}_1 - 2X_{11},
\]
\[
\tilde{x}_2 = \tilde{R}_2 - 2X_{11},
\]
\[
\tilde{\Omega}_{ij}^{(x11)} = \text{sym}(A_iX_{11} + B_iY_j) + \tilde{Q}_1 + 2\sigma_1 X_{11}X_{11} - 4\tilde{R}_1,
\]
\[
\tilde{\Omega}_{ij}^{(x15)} = h_1\tilde{P}_{e22} + 6\tilde{R}_1,
\]
\[
\tilde{\Omega}_{ij}^{(x17)} = X_{11} + 2\sigma_2 X_{11},
\]
\[
\tilde{\Omega}_{ij}^{(x22)} = -(\tilde{Q}_1 - \tilde{Q}_2) - 4\tilde{R}_1 - 4\tilde{R}_2,
\]
\[
\tilde{\Omega}_{ij}^{(x25)} = -h_1\tilde{P}_{e22} + 6\tilde{R}_1,
\]
\[
\tilde{\Omega}_{ij}^{(x26)} = h_1\tilde{P}_{e33} + 6\tilde{R}_2,
\]
\[
\tilde{\Omega}_{ij}^{(x33)} = -(1 - \mu)(\tilde{Q}_2 - \tilde{Q}_1),
\]
\[
\tilde{\Omega}_{ij}^{(x44)} = -\tilde{Q}_3 - 4\tilde{R}_2,
\]
\[
\tilde{\Omega}_{ij}^{(x46)} = -h_1\tilde{P}_{e33} + 6\tilde{R}_2,
\]
\[
\sigma_1 = \gamma_1 \rho_1 + \gamma_2 \rho_2,
\]
\[
\sigma_2 = \gamma_2 \rho_3 - \gamma_1,
\]
\[
\Omega_{ij}^{(e11)} = \text{sym}(P_{e11}A_i - G_iC_j) + N_1 + 2\sigma_3 I - 4T_1,
\]
\[
\Omega_{ij}^{(e15)} = \text{sym}(P_{e11}A_i - G_iC_j) + N_1 + 2\sigma_3 I - 4T_1,
\]
\[
\tilde{\Gamma}_{ij}^{(1)} = \left[ A_iX_{11} + B_iY_j \quad 0 \quad A_{di}X_{11} \quad 0_{n\times3n} \quad I_n \right]^T,
\]
\[
L = \begin{bmatrix} 0_{n\times7n} & I & 0_{n\times4n} \end{bmatrix}.
\]

Moreover, if the above conditions are satisfied, the parameters of the observer-based controller are obtained as follows

\[
K_i = Y_iX_{11}^{-1}, \quad L_i = P_{e11}^{-1} G_i.
\]

Proof: The proof can be found in Appendix B.

Theorem 2 provides a new design methodology for both controller and observer gains simultaneously. The design methodology has been developed without structural restrictions on the Lyapunov matrices, which reduces conservatism. The obtained controller and observer gains ensure that the closed-loop vehicle system is asymptotically stable for all admissible delays.

Remark 7: To get less restrictive conditions for the control and observer gains design, a new matrix variable is introduced in Theorem 2 in order to separate the dynamical system matrices from the Lyapunov matrix. This delivers the condition (31), which is not linear for a given scalar \( \epsilon \). Several
researchers have explored how to choose this scalar, and several methods have been developed to find the optimal scalar, namely \textit{fminsearch}. To overcome the complexity of choosing \( \epsilon \), we implement the inequality \( X_{11} > \delta I \) by means of the inequality \( \frac{1}{\epsilon} \leq - (2 - \epsilon) \). These inequalities help us deal with the nonlinearity set out in (51) and allow us to obtain the LMI conditions (29) and (30).

V. SIMULATION RESULTS

To illustrate the efficacy of the proposed methodology to obtain the gains of the fuzzy observer-based control to estimate the vehicle states, we use the vehicle parameters [10], [30] given in Table 2 and the membership parameters [10] given in Table 1.

| TABLE 2. Vehicle parameters. |
|---|
| Parameter | Value | Unit |
| \( V_x \) | 80 | m/s \(^2\) |
| \( l_x \) | 2988 | kg m \(^2\) |
| \( a_1 \) | 1.18 | m |
| \( a_2 \) | 1.77 | m |
| \( m \) | 1832 | kg |

First, we will demonstrate the impact of the function \( f(x(t)) \) and the time-varying delay on the side-slip and yaw rate angles by perturbing the vehicle model with the delay \( h(t) \) with different delay parameters \( l_1 \neq l_2 \) and with \( l_1 = l_2 \). Consider an input signal \( \delta(t) = 0.3 \), \( C_{yf} = C_{yr} = 0.5 \), \( 0.1 \leq h(t) \leq 0.7 \), \( \mu = 0.3 \) and the following initial condition \( x(0) = [0 \ 0]^T \). The side-slip angles are plotted in Fig. 2 for \( l_1 \neq 1 \) or \( l_2 \neq 1 \) (delayed side-slip angle) and for \( l_1 = l_2 = 1 \) (delay-free side-slip angle). The yaw rate angles are plotted in Fig. 3 for \( l_1 \neq 1 \) or \( l_2 \neq 1 \) (delayed yaw rate angle) and for \( l_1 = l_2 = 1 \) (delay-free yaw rate angle).

According to Fig. 2 and Fig. 3, the time-varying delay and nonlinear dynamics have a significant impact on the side-slip and yaw-rate angles, as large oscillations occur when the time-varying delay becomes meaningful. Furthermore, it can be seen that when the delay is high in the vehicle states (side-slip and yaw-rate angles), it can cause destabilization and performance degradation. Additionally, when \( l_1 = l_2 = 1 \), which means that the system states are delay-free, one can see the effect of the nonlinearity \( f(x(t)) \) on the vehicle states. This demonstrates the significance of including the possible appearance of a delay in the vehicle model states.

We now consider the delay parameters as \( l_1 = 0.5 \) and \( l_2 = 0.7 \). The time-varying delay \( h(t) \) as \( 0.1 \leq h(t) \leq 0.6 \) which means that \( h_1 = 0.1 \) and \( h_2 = 0.6 \) and \( \mu = 0.3 \). Applying Theorem 2, the observer and controller gains obtained for \( \gamma_1 = 2 \), \( \rho_1 = -0.9 \), \( \gamma_2 = 5 \), \( \rho_2 = 0.7 \), \( \rho_3 = -0.15 \), \( \delta_1 = 0.09 \), \( \delta_2 = 0.9 \), \( \delta_3 = 0.05 \) are given by

\[
\begin{bmatrix}
K_1 \\
K_2 \\
L_1 \\
L_2
\end{bmatrix} = 
\begin{bmatrix}
-0.3876 & -0.1719 \\
-0.1279 & 0.4935 \\
0.6128 & 12.8226 \\
0.8630 & 7.8630
\end{bmatrix}.
\]

(32)

To demonstrate the effectiveness of the proposed approach, we will disturb the vehicle system with time-varying delay, and we will use the observer-based control scheme with gains (32) to estimate the vehicle states \( \beta(t) \) and \( \psi(t) \). The vehicle side-slip \( \beta(t) \) and yaw rate \( \psi(t) \) angles over time are displayed in Fig. 4 and Fig. 5, respectively. Fig. 4 shows the trajectory of the side-slip angle \( \beta(t) \) and its estimate \( \tilde{\beta}(t) \), while Fig. 5 shows the trajectory of the yaw rate angle \( \psi(t) \) and its estimate \( \tilde{\psi}(t) \). These simulations were obtained by using the initial conditions \( x(0) = 0 \) and \( \tilde{x}(0) = [-0.1 \pi \ 0.3 \pi]^T \). The time-varying delay is chosen as \( h(t) = 0.1 + 0.5 \sin 0.6 t \).

From Fig. 4 and Fig. 5, we can see that the estimation of the vehicle’s state with a small-amplitude converges to the system states after 1 second for the side-slip angle and before 1 second for the yaw rate angle. This estimation is inaccurate before 1 second. This can occur for several reasons, for instance, the initial conditions, the time-varying delay, the nonlinear function \( f(x(t)) \), and the dynamics not included in the mathematical model. As a result, the estimation becomes significant and successful after 1 second for side-slip and yaw rate angles. Moreover, the effect of the delay can be seen in the trajectory of the vehicle states where the oscillation has...
Finally, the system states are asymptotically stable using the controller gains (32).

In the case where $l_1 = 1$ and $l_2 = 1$, which means that $A_{di} = 0$, $i = 1, 2$ and by applying Theorem 2, the observer-based controller gains for (20) are given by

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -0.3099 & -0.1520 \\ -0.1553 & -0.3771 \end{bmatrix} & \begin{bmatrix} 0.4002 \\ 0.2835 \end{bmatrix} \\ \begin{bmatrix} 10.4119 \\ 6.3246 \end{bmatrix} \end{bmatrix}.$$  

(33)

Fig 6 and Fig 7 display the vehicle states for the delay-free model and for the standard observer-based control structure with $l_1 = 1$ and $l_2 = 1$. The simulation was performed with the controller gains (33) and the initial conditions

$$x(0) = 0 \quad \text{and} \quad \bar{x}(0) = [-0.1\pi, 0.3\pi]^T.$$

For comparison purposes, in Fig 6 and Fig 7, we can see that the vehicle states and their estimates converge to zero, implying that they are asymptotically stable when using the delay-free observer.

To show the efficiency of the control structures, $A_d \neq 0$ with the controller gains in (32), and $A_d = 0$ (standard control structure) with the controller gains in (33), we perturb the vehicle model with a time-varying delay and use the two control structures to estimate the vehicle states. The initial conditions used for the simulation are $x(0) = 0$ and $\bar{x}(0) = [-0.1\pi, 0.3\pi]^T$, and the time-varying delay considered is $h(t) = 0.1 + 0.5 \cdot |\sin 0.6t|$. The vehicle side-slip and yaw rate angles and their estimates are displayed in Fig. 8 and Fig 9, respectively.
It is evident from Fig. 8 that the standard control scheme \( A_d = 0 \) does not estimate the vehicle side-slip angle \( \beta(t) \) when the time-varying delay occurs. While the proposed control scheme \( A_d \neq 0 \) provides a good estimation of the vehicle state. This illustrates the efficacy of the proposed control scheme in the presence of time-varying delays and nonlinear dynamics.

![Graph showing yaw rate angle and its estimate](image)

**FIGURE 9.** Yaw rate angle and its estimate.

Fig. 9 shows that the proposed control scheme \( (A_d \neq 0) \) is successful in estimating the vehicle states while the standard control scheme \( A_d = 0 \) offers defective estimation of the yaw rate \( \dot{\psi}(t) \). We can infer from these results that the efficiency given by the proposed control structure is better than that provided by the traditional control scheme. This has been demonstrated by comparing the state estimation of the standard control scheme and the proposed scheme. We can also see that the gains obtained can control the vehicle system. Finally, these results highlight the importance of considering the presence of a delay in such systems.

**VI. DISCUSSION**

The lateral vehicle model, widely used in vehicle dynamics, offers a detailed representation of chassis motion [10], [11], [21], [22], [24]. In particular, [23] had proposed an extended representation of the vehicle model considering roll, which yields better results. Note that, in comparison to the proposed lateral vehicle model, the aforementioned model is a special case when \( l_1 = l_2 = 1 \) (or \( h(t) = 0 \)). Since the lateral vehicle models found in the literature are delay-free [10], [11], [21], [22], [24], [35], a direct comparison with the proposed time delay model may be unfair. Because of that, a general comparison in terms of time-varying delay may be more significant, and it is done in this paper, where a delay-free lateral model and a time-varying delay lateral vehicle model have been compared in Section V.

**VII. CONCLUSION**

In this work, we developed a time-delay vehicle model using a simple formula that helps us introduce a time-varying delay in its states and considering that the lateral vehicle’s air resistance can be represented as a Lipschitz function. The global nonlinear vehicle model was represented by a fuzzy T-S model with a time-varying delay and stability conditions were obtained based on an appropriate Lyapunov-Krasovskii functional in combination with Wirtinger’s inequality.

The vehicle side-slip and yaw rate angles were estimated using a T-S observer-based control with a time-varying delay. The observer and controller gains have been designed simultaneously using a new design methodology formulated in terms of linear matrix inequalities. Numerical simulations showed a clear benefit of considering a model with delays as the control scheme without it underperforms when it is used to control the delayed system. In contrast, the proposed approach can estimate the vehicle states better and stabilize its dynamics. Future work could focus on full vehicle models with time-varying parameters and delays.

**CONFLICT OF INTEREST**

The authors declare that they have no conflict of interest.

**APPENDIX A**

**PROOF OF THEOREM 1**

Let us consider the following Lyapunov-Krasovskii functional

\[
V(x(t), e(t)) = V_1(x(t)) + V_2(e(t)),
\]

where

\[
V_1(x(t)) = \eta^T(t)P_a\eta_1(x(t)) + \int_{t-h_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-h_2}^{t} x^T(s)Q_2x(s)ds + h_1\int_{-h_1}^{0} \int_{t+\lambda}^{t} \dot{x}(s)R_1x(s)dsd\lambda,
\]

\[
V_2(e(t)) = \eta^T(t)P_d\eta_2(e(t)) + \int_{t-h_1}^{t} e^T(s)N_1e(s)ds + \int_{t-h_2}^{t} e^T(s)N_2e(s)ds + h_1\int_{-h_1}^{0} \int_{t+\lambda}^{t} \dot{\hat{e}}(s)T_1\dot{\hat{e}}(s)dsd\lambda,
\]

with

\[
h_{12} = h_2 - h_1,
\]

\[
\eta_0^T = [\otimes(t) \in \mathbb{R}^n] \in \mathbb{R}^{m}, \quad \otimes(t) = [\otimes_1(t), \otimes_2(t), \dot{\otimes}_3(t)],
\]

\[
P_d = \text{diag}(P_{11}, P_{22}, P_{33}), \quad \otimes = [x, e].
\]
Taking the derivative of $V_1(x(t))$ along the trajectory of $x(t)$, we obtain
\[
\dot{V}_1(x(t)) \leq \eta_1^T(t)P_\delta \dot{h}_1(t) + x^T(t)Q_1 x(t)
- x^T(t - h_1)(Q_1 - Q_2)x(t - h_1)
- (1 - \mu)x^T(t - h(t))(Q_2 - Q_3)x(t - h(t))
- x^T(t - h_2)Q_3 x(t - h_2)
+ \dot{x}^T(t)(h_1^T R_1 + h_2^T R_2)\dot{x}(t)
- h_1 \int_{t - h_1}^t \dot{x}^T(s) R_1 \dot{x}(s)ds
- h_2 \int_{t - h_2}^{t - h_1} \dot{x}^T(s) R_2 \dot{x}(s)ds.
\] (37)

Using Lemma 1 for the last two integral terms, one can obtain
\[
- h_1 \int_{t - h_1}^t \dot{x}^T(s) R_1 \dot{x}(s)ds \leq \xi^T \Phi(R_1) \xi,
- h_2 \int_{t - h_2}^{t - h_1} \dot{x}^T(s) R_2 \dot{x}(s)ds \leq \xi^T \Phi(R_2) \xi.
\] (38)

Condition (18) implies that there exists a $\gamma_1 > 0$ such that
\[
2\gamma_1(\rho_2 x^T(t)x(t) + \rho_3 f^T(x(t))x(t) - f^T(x(t))f(x(t))) \geq 0.
\] (40)

Substituting (38) into (37) in combination with (39) and (40), we obtain
\[
\dot{V}_1(x(t)) \leq \left[\begin{array}{c}
\xi(x(t))
\end{array}\right]^T \left[\begin{array}{c}
\xi(x(t))
\end{array}\right]
+ \Pi_1^T(h_1^T R_1 + h_2^T R_2) \Pi_1 \left[\begin{array}{c}
\xi(x(t))
\end{array}\right],
\] (41)

where
\[
\xi^T = [x^T(t)x^T(t - h_1)x^T(t - h(t))x^T(t - h_2)] x^T(t - h_2)
\times \varphi^T \varphi^T f^T(x(t))],
\varphi_1^T = \left(\frac{1}{h_1} \int_{t - h_1}^t x(s)ds\right)^T,
\varphi_2^T = \left(\frac{1}{h_2} \int_{t - h_2}^{t - h_1} x(s)ds\right)^T.
\]

The time derivative of $V_2(e(t))$ along the trajectory of $e(t)$ can be obtained by following a similar approach as in the derivative for $V_1(x(t))$:
\[
\dot{V}_2(e(t)) \leq \eta_2^T(t)P_\delta \dot{h}_2(e(t) + e^T(t)N_1 e(t)
- e^T(t - h_1)(N_1 - N_2)e(t - h_1)
- (1 - \mu)e^T(t - h(t))(N_2 - N_3)e(t - h(t))
- e^T(t - h_2)N_3 e(t - h_2)
+ \dot{e}^T(t)(h_1^T T_1 + h_2^T T_2)\dot{e}(t)
- h_1 \int_{t - h_1}^t \dot{e}^T(s) T_1 \dot{e}(s)ds
- h_2 \int_{t - h_2}^{t - h_1} \dot{e}^T(s) T_2 \dot{e}(s)ds.
\] (42)

As in (18) and (19), condition (23) implies that there exist $\gamma_3 > 0$ and $\gamma_4 > 0$ such that
\[
0 \leq 2\gamma_3(\vartheta_1 e^T(t)e(t) - \Delta f^T(s(t), x(t))e(t)),
0 \leq 2\gamma_4(\vartheta_2 e^T(t)e(t) + \vartheta_3 \Delta f^T(s(t), x(t))e(t)
- \Delta f^T(s(t), x(t))\Delta f^T(s(t), x(t))).
\] (43)

Combining (43) with (42) and using Lemma 1, we obtain
\[
\dot{V}_2(e(t)) \leq \zeta^T \left[\begin{array}{c}
\zeta(e(t))
\end{array}\right]
+ \Pi_2^T(h_1^T T_1 + h_2^T T_2) \Pi_2 \zeta(e(t)),
\] (44)

where
\[
\zeta^T = [e^T(t)e^T(t - h_1)e^T(t - h(t))e^T(t - h_2) \varphi_3 \varphi_4]
\times \Delta f^T(s(t), x(t)),
\varphi_3^T = \left(\frac{1}{h_1} \int_{t - h_1}^t e(s)ds\right)^T,
\varphi_4^T = \left(\frac{1}{h_2} \int_{t - h_2}^{t - h_1} e(s)ds\right)^T.
\]

The time derivative of (34) along the trajectory of (21) is obtained as follows
\[
\dot{V}(x(t), e(t)) = \dot{V}_1(x(t)) + \dot{V}_2(e(t)) \leq \xi^T \xi e(t),
\] (45)

where $\xi^T = [\xi^T \xi^T]$. By considering the augmented system (21) with (22) and known $K_i$ and $L_i$, $\Omega$ will be given by
\[
\Omega = \sum_{i=1}^2 \sum_{j=1}^2 \theta_i(\alpha_j(t))\theta_j(\alpha_j(t)) \Omega_{ij}.
\] (46)

Note that considering the conditions in (7) and (8), if Theorem 1 holds, then $\Omega < 0$, which implies that $\dot{V}(x(t), e(t)) < 0$. Thus, the closed-loop system (21) with known $K_i$ and $L_i$ is asymptotically stable, which completes the proof.

**APPENDIX B**

**PROOF OF THEOREM 2**

By using the Schur complement, $\Omega$ can be rewritten as in (47), shown at the top of the next page, with
\[
\Xi(R_1^{-1}R_1) = -P_{11}R_1^{-1}P_{11}, \Xi(R_2^{-1}) = -P_{12} R_2^{-1} P_{12},
\]
\[
\Xi(T_1^{-1}) = -P_{11} T_1^{-1} P_{11}, \Xi(T_2^{-1}) = -P_{12} T_2^{-1} P_{12}.
\]

Furthermore, $\Omega_{xx}$ and $\Gamma_2^T$ can be rewritten as
\[
\Omega_{xx} = [B_k 0_{n \times 6} \ 0_{6 \times n} - I_n] \Gamma_2^T = [B_k 0_{6 \times n} \ 0_{n \times 6} I_n] \Gamma_2^T.
\] (48)

with $\Sigma_1 = \text{diag}[P_{11}, P_{11}, P_{11}, P_{11}, P_{11}, P_{11}, P_{11}, P_{11}, L_n]$.

To obtain the observer and controller gains, we define
\[
X_{11} = P_{11}^{-1}, \Sigma = \text{diag}[\Sigma_1^{-1}, I_n, I_n, I_n, P_{11}^{-1}, P_{11}^{-1}].
\]

Then, pre and post multiplying $\hat{\Omega}$ in (47) by $\Sigma^T$ and its transpose respectively, we obtain
\[
\hat{\Omega} = \Sigma^T \Omega \Sigma = \hat{\Omega}(0) + VL + L^T V^T,
\] (49)

where, as shown in the equation at the top of the next page, with
\[
\Gamma_1^T = \Sigma_1^{-1} \Gamma_1^T, \quad \hat{\Omega}_i = \Sigma_1^{-1} \Omega_i \Sigma_1^{-1}.
\]
Using Lemma 2, we obtain
\[
\tilde{\Omega} \leq \tilde{\Omega}(0) + V(\epsilon X_{11}) V^T + L^T(\epsilon X_{11})^{-1} L. \tag{50}
\]
Using the Schur complement Lemma, we have
\[
\tilde{\Omega} \leq \begin{bmatrix} \tilde{\Omega}(0) & VX_{11} & L^T \\ * & -\frac{1}{\epsilon}X_{11} & 0 \\ * & * & -\epsilon X_{11} \end{bmatrix}. \tag{51}
\]
For a positive definite matrix, the following inequalities hold
\[
\begin{align*}
\Xi(X_{11}R_{1}^{-1}X_{11}) & \leq \tilde{R}_{1} - 2X_{11}, \\
\Xi(T_{1}^{-1}) & \leq T_{1} - 2P_{e11}, \\
\Xi(X_{11}R_{2}^{-1}X_{11}) & \leq \tilde{R}_{2} - 2X_{11}, \\
\Xi(T_{2}^{-1}) & \leq T_{2} - 2P_{e11}.
\end{align*} \tag{52}
\]
For a positive scalar \( \epsilon \), we have \(-\frac{1}{\epsilon} \leq -(2-\epsilon) \). By assuming \( X_{11} > \delta_{1}I \), we obtain
\[
-\frac{1}{\epsilon}X_{11} \leq -(2\delta_{1} - \epsilon \delta_{1})I, \quad -\epsilon X_{11} \leq -\delta_{2}I. \tag{53}
\]
Define the following change of coordinates
\[
\begin{align*}
\tilde{P}_{x_{22}} &= X_{11}P_{x_{22}}X_{11}, \\
\tilde{P}_{x_{33}} &= X_{11}P_{x_{33}}X_{11}, \\
\tilde{R}_{\Theta} &= X_{11}R_{\Theta}X_{11}, \quad \Theta = \{1, 2\}, \\
\tilde{Q}_{\Theta} &= X_{11}Q_{\Theta}X_{11}, \quad \Theta = \{1, 2, 3\}.
\end{align*} \tag{54}
\]
Substituting (53) and (52) from the right hand side of (51) and using (22) with the relaxation scheme in [36], we obtain
\[
\Xi = \sum_{i=1}^{2} \theta_{i}^2(\alpha(t) | i)\Xi_{ij} + \sum_{i=1}^{2} \sum_{i<j} \theta_{i}^2(\alpha(t) | i)\theta_{j}^2(\alpha(t) | j)(\Xi_{ij} + \Xi_{ji}) \tag{55}
\]
Thus, one can see that if conditions (29) and (30) are satisfied, then (55) holds as well, implying that \( \Omega < 0 \), which completes the proof. \( \square \)

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