The cause of universality in growth fluctuations

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Abstract

Phenomena as diverse as breeding bird populations, the size of U.S. firms, money invested in mutual funds, the GDP of individual countries and the scientific output of universities all show unusual but remarkably similar growth fluctuations. The fluctuations display characteristic features, including double exponential scaling in the body of the distribution and power law scaling of the standard deviation as a function of size. To explain this we propose a remarkably simple additive replication model: At each step each individual is replaced by a new number of individuals drawn from the same replication distribution. If the replication distribution is sufficiently heavy tailed then the growth fluctuations are Levy distributed. We analyze the data from bird populations, firms, and mutual funds and show that our predictions match the data well, in several respects: Our theory results in a much better collapse of the individual distributions onto a single curve and also correctly predicts the scaling of the standard deviation with size. To illustrate how this can emerge from a collective microscopic dynamics we propose a model based on stochastic influence dynamics over a scale-free contact network and show that it produces results similar to those observed. We also extend the model to deal with correlations between individual elements. Our main conclusion is that the universality of growth fluctuations is driven by the additivity of growth processes and the action of the generalized central limit theorem.
I. INTRODUCTION

Recent research has revealed surprising properties in the fluctuations in the size of entities such as breeding bird populations along given migration routes [1], U.S. firm size [2–7], money invested in mutual funds [8], GDP [9–11], scientific output of universities [12], and many other phenomena [13–17]. This is illustrated in Figures 1 and 2. The first unusual property is in the logarithmic annual growth rates $g_t$, defined as $g_t = \log(\frac{N_{t+1}}{N_t} + 1)$, where $N_t$ is the size in year $t$. As seen in the top panel of Figure 1, all of the data sets show a similar double exponential scaling in the body of the distribution, indicating heavy tails. The second surprising feature is the power law scaling of the standard deviation $\sigma$ with size, as illustrated in Figure 2. In each case the standard deviation scales as $\sigma \sim N^{-\beta}$ with $\beta \approx 0.3$.

These results are viewed as interesting because they suggest a non-trivial collective phenomena with universal properties. If the individual elements fluctuate independently, then (with a caveat we will state shortly) the standard deviation of the growth rates scales as a function of size with an exponent $\beta = 1/2$, whereas if the individual elements of the population move in tandem the standard deviation scales with $\beta = 0$, i.e. it is independent of size. The fact that we instead observe a power law with an intermediate exponent $0 < \beta < 1/2$ suggests that the individual elements neither change independently nor in tandem. Instead it suggests some form of nontrivial long-range coupling. Why should phenomena as diverse as breeding bird populations and firm size show such similar behavior? There is a substantial body of previous work attempting to explain individual phenomena, such as firm size or GDP [18–32]. However none of these theories has the generality to explain how this behavior could occur so widely.

The caveat in the above reasoning is the assumption that the fluctuations of the individual elements are well-behaved, in the sense that they are not too heavy-tailed. As we show in a moment, if the growth fluctuations of the individual elements are sufficiently heavy-tailed then the fluctuations of the population are also heavy tailed, even if there are no collective dynamics. Under the simple additive replication model that we propose the fluctuations in size are Levy distributed in the large $N$ limit. This predicts a scaling exponent $0 < \beta < 1/2$ and the shape parameter of the Levy distribution predicts the value of $\beta$. We show here that this model provides an excellent fit to the data.
TABLE I. The parameter values for fitting the data with a Levy distribution

| year         | $\alpha$ | $\kappa$ | $c$   | $\mu$  |
|--------------|----------|----------|-------|--------|
| NABB         | 1.40     | 0.81     | 0.156 | -0.037 |
| Mutual funds | 1.48     | 0.3      | 0.111 | -0.015 |
| Firms        | 1.53     | 0.80     | 0.16  | -0.05  |

In the first part of this paper we develop the additive replication model and show that it gives a good fit to the data. Our analysis in the first part is predicated on the existence of a heavy-tailed replication distribution. In the second part of the paper we present one possible explanation for the heavy-tailed replication distribution in terms of stochastic influence dynamics on a scale-free contact network, and argue that such an explanation could apply to any of the diverse settings in which these scaling phenomena have occurred. This influence dynamics is an example of “nontrivial” collective dynamics. Thus, the process that generates the heavy tails in individual fluctuations may come from nontrivial collective dynamics even though the replication model does not depend on this.

II. THE ADDITIVE REPLICATION MODEL

We assume an additive replication process: At each time step each individual element is replaced by $k$ new elements drawn at random from a replication distribution $p(k)$, where $0 \leq k < \infty$. An individual element could be a bird, a sale by a given firm, or the holdings of a given investor in a mutual fund. By definition the number of elements $N_{t+1}$ on the next time step is

$$N_{t+1} = \sum_{j=1}^{N_t} k_{jt},$$

where $k_{jt}$ is the number of new elements replacing element $j$ at time $t$. The growth $G_t$ is given by

$$G_t = \frac{N_{t+1} - N_t}{N_t} = \frac{\sum_{j=1}^{N_t} k_{jt}}{N_t} - 1.\tag{2}$$

The simplest version of our model assumes that draws from the replication distribution $p(k)$ are independent; we later relax this assumption to allow for correlations.

Why might such a model be justified? First note that additivity of the elements is automatic, since by definition the size is the sum of the number of elements. The assumption
FIG. 1. An illustration of how our theory reveals the underlying regularity in the distribution of growth fluctuations of highly diverse phenomena. The three data sets studied here are North American Breeding Birds (○), US firm sales (□) and US equity mutual funds (◇). The data is the same in all three panels, the only change is the presentation. A: The traditional view. Histograms of the logarithmic growth rates are plotted on semi-log scale, normalized such that the mean vanishes \( \text{E}[g] = 0 \) and the variance is unity \( \text{Var}[g] = 1 \). The collapse is good for the body of the distribution, revealing double-exponential scaling, but poor in the tails, where the three data sets look quite different. B: Comparison to a Levy distribution. The cumulative distribution \( P(G > X) \) of relative growth rates for the three data sets are compared to fits to the Levy distributions predicted by our theory (solid curves) and plotted on double logarithmic scale (for positive \( X \) only). See table 2 for parameter values. C: Superior collapse onto a single curve when the data is scaled as predicted by our theory. The empirical values of the relative growth \( G \) (rather than the logarithmic growth rate \( g \)) are normalized so they all have a scale parameter approximately one, as described in the text. In order to compare to the top panel, we plot the logarithmic growth \( g \) and compare to a Levy distribution (solid curve). This gives a better collapse of the data which works in the tails as well as the body.
that each element replicates itself in the next year amounts to a persistence assumption, i.e. that the number of elements in one year is linearly related to the number in the previous year, with each element influencing the next year independently of the others. We also assume uniformity by letting all elements have the same replication distribution \( p(k) \). For the case of firms, for example, each sale in year \( t \) can be viewed as replicating itself in year \( t+1 \). This is plausible if the typical customer remains faithful to the same firm, normally continuing to buy the product from the same company, but occasionally changing to buy more or less of the product. For migrating birds this is plausible if the number of birds taking a given route in a given year is related to the number taking it last year, either because of the survival probability of individual birds or flocks of birds, or because individual birds influence other birds to take a given migration route.

III. PREDICTIONS OF THE MODEL

Given that the size \( N_t \) at time \( t \) is known and the drawings from \( p(k) \) are independent, the growth rate \( G_t \) is a sum of \( N_t \) I.I.D. random variables. Under the generalized central limit theorem \([33, 34]\), in the large \( N_t \) limit the growth \( P_G \) converges to a Levy skew alpha-stable distribution

\[
P_G(G_t|N_t) = N_t^{\frac{1-\alpha}{\alpha}} L_\kappa^\alpha(G_t N_t^{-\frac{1-\alpha}{\alpha}}; c, \mu).
\]

(3)

\( 0 < \alpha \leq 2 \) is the shape parameter, \( -1 \leq \kappa \leq 1 \) is the asymmetry parameter, \( \mu \) is the shift parameter and \( c \) is a scale parameter.

The normal distribution is a special case corresponding to \( \alpha = 2 \). This occurs if the second moment of \( p(k) \) is finite. However, if the second moment diverges according to extreme value theory, under conditions that are usually satisfied, it is possible to write \( p(k) \sim k^{-\gamma} \) for large \( k \). When \( 1 < \gamma < 3 \) the Levy distribution has heavy tails that asymptotically scale as a power law with \( P(G > x) \sim x^{-\alpha} \), where \( \alpha = \gamma - 1 \).

The additive replication process theory predicts power law behavior for \( \sigma(N) \) and predicts its scaling exponent based on the growth distribution. If \( \gamma > 3 \) the growth rate distribution converges to a normal with \( \beta = 1/2 \). However, when \( \alpha = \gamma + 1 < 2 \), using standard

\[\text{1 Under extreme value theory there are distributions for which there is no convergent behavior; the power law assumes convergence.}\]

\[\text{2 For } \gamma = 3 \text{ and } \gamma = 2 \text{ there are logarithmic corrections to the results.}\]
results in extreme value theory \[33, 34\] the standard deviation scales as a power law with size, \( \sigma_G \sim N_t^{-\beta} \), where

\[
\beta = (\gamma - 2)/(\gamma - 1). \tag{4}
\]

IV. TESTING THE PREDICTIONS

To test the prediction that the data is Levy-distributed, in the central panel of Figure 1 we compare each of our three data sets to Levy distributions. The three data sets are (1) the number of birds of a given species observed along a given migration route, (2) the size of a firm as represented by its sales, and (3) the size of a U.S. mutual fund. The data shown in the middle panel of Figure 1 are exactly the same as in the upper panel, except that we plot the growth fluctuations \( G \) rather than their logarithmic counterpart \( g \), we plot a cumulative distribution rather than a histogram, and we graph the data on double logarithmic scale. The fits are all good.

Because we are lucky enough that the shape parameter \( \alpha \) and the asymmetry parameter \( \kappa \) are similar in all three data sets, we can collapse them onto a single curve. This is done by transforming all the data sets to the same scale in \( G \) by dividing by an empirically computed scale factor equal to the 0.75 quantile minus the 0.25 quantile (we do it this way rather than dividing by the standard deviation because the standard deviation does not exist). It is important that this normalization is done in terms of \( G \), in contrast to the standard method which normalizes the logarithmic growth \( g \). The standard method, illustrated in the top panel, produces a collapse for the body of the distribution, but there is no collapse for the tails – mutual funds have very heavy tails while the breeding birds closely follow the exponential even for large values of \( g \). In contrast, the collapse using \( G \) as suggested by our theory, illustrated in the bottom panel, works for both the body and the tails.

To test the prediction of the power law scaling of the standard deviation with size we estimated \( \gamma \) from the data shown in Figure 1 and \( \beta \) from the data in Figure 2. We then make a prediction \( \hat{\beta} \) for each data set using Eq. 4 and the estimated value of \( \gamma \) for each data set. The results given in Table 1 are in good statistical agreement in every case. (See Materials and Methods.)
FIG. 2. Illustration of the non-trivial scaling of the standard deviation $\sigma$ as a function of size $N$. The straight lines on double logarithmic scale indicate power law scaling. Same symbols as in Figure 1. The standard deviation is computed by binning the data into bins of exponentially increasing size and computing the sample standard deviation in each bin. For clarity the breeding bird population is shifted by a factor of 10 and the mutual fund data set by a factor of $10^{-1}$. The empirical data are compared to lines of slopes $-0.303$, $-0.308$ and $-0.309$ respectively.

TABLE II. A demonstration that the Levy distribution makes a good prediction of the scaling of the standard deviation as a function of size. The measured value of $\gamma$ based on the center panel of Figure 1 is used to make a prediction, $\hat{\beta}$, of the exponent of the scaling of the standard deviation. This is in good statistical agreement with $\beta$, the measured value. NABB stands for North American Breeding Birds.

| year    | $\beta$    | $\hat{\beta}$ | $\gamma$ |
|---------|-------------|----------------|----------|
| NABB    | $0.30 \pm 0.07$ | $0.29 \pm 0.03$ | $2.40 \pm 0.06$ |
| Mutual funds | $0.29 \pm 0.03$ | $0.32 \pm 0.04$ | $2.48 \pm 0.08$ |
| Firms   | $0.31 \pm 0.07$ | $0.35 \pm 0.03$ | $2.53 \pm 0.07$ |

V. WHY IS THE REPLICATION DISTRIBUTION HEAVY-TAILED?

Part of the original motivation for the interest in the non-normal properties and power law scalings of the growth fluctuations is the possibility that they illustrate an interesting collective growth phenomenon with universal applicability ranging from biology to economics. Our explanation so far seems to suggest the opposite: In our additive replication model each
element acts independently of the others. As long as the replicating distribution is heavy tailed the scaling properties illustrated in Figures 1 and 2 will be observed, even without any collective interactions.

There is a subtle point here, however. Our discussion so far leaves open the question of why the replication distribution might be heavy-tailed. Based on the limited data that is currently available there are many possible explanations – it is not possible to choose one over another. One can postulate mechanisms that involve no collective behavior at all, for example, if individual birds had huge variations in the number of surviving offspring. (This might be plausible for mosquitos but does not seem plausible for birds). One can also postulate mechanisms that involve collective behavior, as we do in the next section.

VI. THE CONTACT NETWORK EXPLANATION FOR HEAVY TAILS

In this section we present a plausible explanation for power law tails of \( p(k) \) in terms of random influence on a scale-free contact network. This example nicely illustrates how the heavy tails of the individual replication distribution \( p(k) \) can be caused by a collective phenomenon.

Assume a contact network \[35\] where each node represents individuals. They are connected by an edge if they influence each other. For simplicity assume that influence is bi-directional and equal, i.e. that the edges are undirected and unweighted. Let individual \( i \) be connected to \( d_i \) other individuals, where \( d_i \in \{1, \ldots, M\} \) is the degree of the node. The degree distribution \( D(d) \) is the probability that a randomly selected node has degree \( d \).

Let each individual belong to one of \( \Gamma \) groups. For example, belonging to group \( a \in \{1, \ldots, \Gamma\} \) can represent a consumer owning a product of firm \( a \), an investor with money in mutual fund \( a \), or a bird of a given species taking migration route \( a \). The groups are the same as the populations discussed earlier, i.e. \( N_t^a \) is the size of group \( a \) at time \( t \). The dynamics are epidemiological in the sense that an individual will stay in her group unless her contacts influence her to switch. The switching is stochastic: An individual in group \( a \) with a contact in group \( b \) will switch to group \( b \) with a rate \( \rho_{ab} \). Furthermore, the switching rate is linearly proportional to the number of contacts in that group, i.e. if an individual belonging to group \( a \) has \( n \) contacts in group \( b \), she will switch with a rate \( n\rho_{ab} \). As an example, the individual in the center of the graph in Fig \( 4 \) has a degree \( d = 8 \) and belongs
FIG. 3. Here we show an example of a simple network. Each node represents an individual and each edge represents a contact between them. The labels represent the group the individual belongs to.

to group $a$. She will switch to group $b$ with a rate $4\rho_{ab}$, to group $c$ with a rate $2\rho_{ac}$ and to group $d$ with a rate $\rho_{ad}$.

For example consider firm sales. If a given consumer likes the product of a given firm, she might influence her friends to buy more, and if she doesn’t like it, she might influence them to buy less. Thus each sale in a given year influences the sales in the following year. A similar explanation applies to mutual funds, under the assumption that each investor influences her friends, or it applies to birds, under the assumption that each bird influences other birds that it comes into contact with.

We now show how the contact network gives rise to an additive replication model. To calculate $N_{t+1}^a$ consider each of the $N_t^a$ individuals in group $a$ one at a time. Individual $j$ in group $a$ replicates if she remains in the group, and/or if one or more of her contacts that belong to other groups join group $a$. She fails to replicate if she leaves the group and also fails to influence anyone else to join. Let the resulting number of individuals that replace individual $j$ be $k_{jt}$. This implies

$$N_{t+1}^a = \sum_{j \in \text{Group } a} N_t^a k_{jt},$$

which is identical to Eq. 1 except for the group label (which was previously implicit).

The replication factor $k_{jt}$ is a random number with values in the range $k_i \in [0, d_j]$. Given the stochastic nature of the influence process we approximate $k_{jt}$ as a Poisson random

3 It has recently been shown that influence in flocking pigeons is hierarchical. 36, 37.

4 This approximation is valid for random networks, which have a local tree-like structure 35.
FIG. 4. A demonstration that influence dynamics on a scale-free contact network give rise to the Levy behavior predicted by the additive replication model. The influence model was simulated for $10^3$ groups on a network of $10^6$ nodes, an average degree $\langle d \rangle = 10$ and a power law degree distribution $\mathcal{D}(d) \sim d^{-\gamma}$ with $\gamma = 2.2$. The cumulative growth rate distribution $P(G' > G)$ is in good agreement with the predicted Levy distribution $G$.

Inset: the fluctuations are compared to a line of slope $\beta = -0.1667$, illustrating the expected power law scaling.

A numerical simulation verifies these results. We simulated a network of $10^6$ nodes with a power law tailed degree distribution $\mathcal{D}(d) \sim d^{-\gamma}$ with $\gamma = 2.2$ and average degree $\langle d \rangle = 10$. The average number of individuals and the average growth rate of a group can be approximated using a mean field approach. The mean field growth rates are given by

$$\frac{\partial N_a}{\partial t} = \langle d \rangle M \theta_a (1 - \theta_a) \sum_{b=1}^{r} (\rho_{ab} - \rho_{ba}) \text{[40]}.$$ 

and $\theta_a = \langle d \rangle^{-1} \sum_{d'} d' f_a^d \mathcal{D}(d')$, where $f_a^d$ is the fraction of individual elements with degree $d$ that belong to group $a$. We know of no analytic method to compute the growth fluctuations.
The dynamics were simulated for $10^3$ groups with a homogeneous switching rate $\rho_{ab} = \rho$. As expected the growth rates have a Levy distribution $P(G) \sim G^{-\gamma}$ as shown in Figure 4. The fitted parameter values are $\alpha = 1.2$, $\kappa = 0.25$, $c = 0.09$ and $\mu = -0.17$. The fitted value of the fluctuation scaling $\beta = 0.14 \pm 0.03$, shown in the inset of Figure 4, is in agreement with the predicted value of $\beta = (\gamma - 2)/(\gamma - 1) = 1/6$.

VII. CORRELATIONS AND FINITE SIZE EFFECTS

So far we have assumed that the growth process for individual elements is uncorrelated, i.e. that the draws from $p(k)$ are I.I.D. Sufficiently strong correlations can change the results substantially. There can be correlations among the individual elements or correlations in time. For example, suppose some groups are intrinsically more or less popular than others. For example, the popularity of a city might depend on its economy and living conditions. This can be modeled by assuming that the replication of individual $j$ in group $i$ is given by a random variable $\hat{k}_{jt}^i$ which is the sum of a random variable that depends on the individual and one that is common for the group, i.e. $\hat{k}_{jt}^i = k_{jt} + \zeta_{it}$. We can then write the replication model in the form

$$N_{t+1} = \sum_{i=1, j=1}^{\Gamma, N_i} k_{jt} + \zeta_{it}. \quad (7)$$

As shown in the supplementary materials, for small sizes the individual fluctuations $k_{jt}$ dominate, so that there is a power law scaling of $\sigma$, but for larger sizes the group fluctuations $\zeta_{it}$ dominate, and $\sigma$ becomes constant (i.e. $\beta = 0$). This is indeed what we observe for cities 6.

We have also assumed in our analysis that the number of elements is infinite, i.e. that there is no upper limit on the replication factors. For finite systems the growth of one group is at the expense of another. This can induce correlations which affect both the growth rate distribution and the fluctuation scaling. Nevertheless, as our simulation shows, under appropriate circumstances the theory can still describe finite systems to a very good approximation. A more detailed discussion is provided in the supplementary materials.

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6 Note that the nature of the scalings for cities is controversial and strongly depends on how a city is defined - our results are in agreement with those who claim the scaling is not very good [41]. Rather than using the census definitions, Rozenfeld et al [17] use a clustering algorithm for defining cities and then the fluctuation scaling (without the group correlations) seems to hold.
VIII. DISCUSSION

The explanation that we offer here is widely applicable and very robust. The idea that a larger entity can be decomposed into a sum of smaller elements, and that the smaller elements can be modeled as if they replicate, is quite generic. As discussed in the previous section this can be broken if the growth of the elements is too correlated. Our explanation for the heavy tailed growth rate distributions and fluctuation scaling requires that the replication distribution \( p(k) \) is heavy tailed. The key thing we have shown is that when this occurs, the generalized central limit theorem dictates that the growth distribution \( P_G \) will be Levy, which in turn dictates the power law size dependence of the standard deviation, \( \sigma(N) \).

The previous models which are closest to ours are the model of firm size of Wyart and Bouchaud [29] and the model of GDP due to Gabaix [31]. Both of these models assume that the size distribution \( P(N_t) \) has power law tails and that firms grow via multiplicative fluctuations. They each suggested (without any testing) that additivity might lead to Levy distributions for their specific phenomena (GDP or firm size). This is in contrast to our model, which requires neither the assumption of power tails for size nor multiplicative growth. This is a critical point because the size of mutual funds does not obey a power law distribution [8], which rules out both the the Wyart and Bouchaud and Gabaix models as general explanations. We are apparently the first to realize that these diverse phenomena all obey Levy distributions, and that this explains the power law scaling of \( \sigma(N) \).

There are many possible explanations that could generate a heavy tailed replication distribution \( p(k) \). Here we proposed an influence process on a scale free contact network as a possible example. This mechanism is quite general and relies on the assumption that an individual element’s actions are affected by those of its contacts. Scale free networks are surprisingly ubiquitous and the existence of social, information and biological networks with power law tails with \( 2 < \gamma < 3 \) is well documented [38, 39], and suggests that the assumption that the degree distribution \( \mathcal{D}(d) \) and hence the replication distribution \( p(k) \) are heavy-tailed is plausible.

The influence model shows that the question of whether the interesting scaling properties of these systems should be regarded as “interesting collective dynamics” can be subtle. On one hand the additive replication model suppresses this – any possibility for collective action is swept into the individual replication process. On the other hand, the influence
model shows that the heavy tails may nonetheless come from a collective interaction. More detailed data is needed to make this distinction.

Our model shows that, whenever its assumptions are satisfied, one should expect universal behavior as dictated by the central limit theorem: The growth fluctuations should be Levy distributed (with the normal distribution as a special case). Our model does not suggest that the tail parameter should be universal, though of course this could be possible for other reasons. Based on our model there is no reason to expect that the value of the exponent \( \alpha \) (or equivalently \( \gamma \) or \( \beta \)) will not depend on factors that vary from example to example. Thus the growth process is universal in one sense but not in another.

IX. MATERIALS AND METHODS

1. North american breeding birds dataset

We use the the North American breeding bird survey, which contains 42 yearly observations for over 600 species along more than 3,000 observation routes. For each route the number of birds from each species is quoted for each year in the period 1966-2007. For each year in the data set, from 1966 to 2007, we computed the yearly growth with respect to each species in each route. The data set can be found online at ftp://ftpext.usgs.gov/pub/er/md/laurel/BBS/DataFiles/

2. US public firms dataset

We use the 2008 COMPUSTAT dataset containing information on all US public firms. As the size of a firm we use the dollar amount of sales. Growth is given by the 3 year growth in sales.

3. US equity mutual fund dataset

We use the Center for Research in Security Prices (CRSP) mutual fund database, restricted to equity mutual funds existing in the years 1997 to 2007. An equity fund is one with at least 80% of its portfolio in stocks. As the size of the Mutual fund we use the total
net assets value (TNA) in real US dollars as reported monthly. Growth in the mutual fund industry, measured by change in TNA, is comprised of two sources: growth due to the funds performance and growth due to flux of money from investors, i.e. mutual funds can grow in size if their assets increase in value or due to new money coming in from investors. We define the relative growth in the size of a fund at time $t$ as

$$G_{TNA}(t) = \frac{TNA_{t+1}}{TNA_{t}} - 1$$

and decompose it as follows;

$$G_{TNA}(t) = r_t + G_t,$$  \hspace{1cm} (8)

where $r_t$ is the fund’s return, quoted monthly in the database, and $G_t$ is the growth due to investors. For our purposes here we only consider $G_t$, the growth due to investors.

**A. Empirical fitting procedures**

The empirical investigation is conducted as follows: We first estimate the fluctuation scaling exponent $\beta$. The relative growth rate distribution $G = N_{t+1}/N_{t} - 1$ is binned into 10 exponentially spaced bins according to size $N_t$. For each bin $i$, the sample estimate of the variance of the growth rates $\sigma_i^2$ is estimated in the usual way. Then the logarithm of the measured variances are regressed on the logarithm of the average size $\bar{N}_i$

$$\log(\sigma) = \beta \log(N) + \sigma_1$$

such that the slope is the ordinary least squares (OLS) estimator of $\beta$.

To estimate the tail exponent we normalize the growth rate $G$ such that it has zero mean and we divide by the 0.75 quartile - the 0.25 quartile. We estimate tail exponents using the technique described in Clauset et al [42]. The method used uses the following modified Kolmogorov-Smirnoff statistic

$$KS = \max_{x > x_{\text{min}}} \frac{|s(x) - p(x)|}{\sqrt{p(x)(1 - p(x))}},$$

where $s$ is the empirical cumulative distribution and $p$ is the hypothesized cumulative distribution. Using the maximum likelihood estimator (MLE) of the tail exponent $\gamma$ we can predict the fluctuation scaling exponent $\hat{\beta}$ using Eq. 4. and compare to the measured OLS estimator of $\beta$. 

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