Cohomology of Pure Braid Groups of exceptional cases

Simona Settepanella

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ABSTRACT. Consider the ring $R := \mathbb{Q}[\tau, \tau^{-1}]$ of Laurent polynomials in the variable $\tau$. The Artin’s Pure Braid Groups (or Generalized Pure Braid Groups) act over $R$, where the action of every standard generator is the multiplication by $\tau$. In this paper we consider the cohomology of these groups with coefficients in the module $R$ (it is well known that such cohomology is strictly related to the untwisted integral cohomology of the Milnor fibration naturally associated to the reflection arrangement). We compute this cohomology for the cases $I_2(m), H_3, H_4, F_4$ and $A_n$ with $1 \leq n \leq 7$.

1 Introduction

Let $(W, S)$ be a finite Coxeter system realized as a reflection group in $\mathbb{R}^n$, $A(W)$ the arrangement in $\mathbb{C}^n$ obtained by complexifying the reflection hyperplanes of $W$. Let

$$Y(W) = Y(A(W)) = \mathbb{C}^n \setminus \bigcup_{H \in A(W)} H.$$ 

be the complement to the arrangement, then $W$ acts freely on $Y(W)$ and the fundamental group $G_W$ of the orbit space $Y(W)/W$ is the so called Artin group associated to $W$ (see [3]). Likewise the fundamental group $P_W$ of $Y(W)$ is the Pure Artin group or the pure braid group of the series $W$. It
is well known ([4]) that these spaces $Y(W) (Y(W)/W)$ are of type $K(\pi, 1)$, so their cohomologies equal that of $P_W (G_W)$.

The integer cohomology of $Y(W)$ is well known (see [4], [13], [2], [10]) and so is the integer cohomology of the Artin groups associated to finite Coxeter groups (see [18], [11], [15]).

Let $R = \mathbb{Q}[\tau, \tau^{-1}]$ be the ring of rational Laurent polynomials. To $R$ can be given a structure of module over the Artin group $G_W$, where standard generators of $G_W$ act as $\tau$-multiplication.

In [6] and [7] the authors compute the cohomology of all Artin groups associated to finite Coxeter groups with coefficients in the previous module.

In a similar way we define a $P_W$-module $R_\tau$, where standard generators of $P_W$ act over the ring $R$ as $\tau$-multiplication.

Equivalently, one defines an abelian local system (also called $R_\tau$) over $Y(W)$ with fiber $R$ and local monodromy around each hyperplane given by $\tau$-multiplication (for local systems on $Y(W)$ see [12], [14]).

It is known that the cohomologies of $Y(W)$ for the series $A_n$, $B_n$ and $D_n$ stabilize, with respect to the natural inclusion, at a well known number of copies of the trivial $R$-module $\mathbb{Q}$ ([17]).

In this paper we give that cohomology for the finite Coxeter groups $H_3$, $H_4$, $F_4$ and $A_n$ with $n = 7$ (for $n \leq 5$ see [5] and for $n = 6$ [8]). We did these computations using an algorithm based on the Salvetti’s complex.

Moreover, using results of G. Denham [9] and H. Barcelo [1], we create algorithms to compute the cohomologies of the the flag complex $Fl_4(W)$ defined by Schechtman V. and Varchenko A. in [16] (see also [9]). More precisely, these algorithms allow to perform calculations for finite Coxeter groups $H_3, H_4, F_4$ and $A_n$ with $n \leq 7$.

These computations are quite interesting and it’s also very interesting to compare them. In fact this comparison supports the conjecture that the integral cohomology of the Milnor Fibre associated to the reflection arrangements is torsion-free (see [9] and [8]).
2 Cohomology of pure braid groups $H_3$, $H_4$, $F_4$ and $A_7$.

In this section we give results obtained by direct computations.

Denote by $\varphi_i$ the cyclotomic polynomial having as roots the primitive $i$-roots of 1 and let

$$\{\varphi_i\} := \mathbb{Q}[\tau, \tau^{-1}]/(\varphi_i) = \mathbb{Q}[\tau]/(\varphi_i)$$

be the cyclotomic field of $i$-roots of 1, thought as $R$-module.

|       | $H_3$          | $H_4$          | $F_4$          |
|-------|----------------|----------------|----------------|
| $H^0$ | 0              | 0              | 0              |
| $H^1$ | $\{\varphi_1\}$ | $\{\varphi_1\}$ | $\{\varphi_1\}$ |
| $H^2$ | $\{\varphi_1\}^{14}$ | $\{\varphi_1\}^{59}$ | $\{\varphi_1\}^{23}$ |
| $H^3$ | $\{\varphi_1\}^{45} \oplus \bigoplus_{i \mid 15, i \neq 1} \{\varphi_i\}^{32}$ | $\{\varphi_1\}^{1079} \oplus \{\varphi_3\}$ | $\{\varphi_1\}^{167} \oplus \{\varphi_3\}^8$ |
| $H^4$ | 0              | $\{\varphi_1\}^{6061} \oplus \bigoplus_{i \mid 60, i \neq 1, 3} \{\varphi_i\}^{5040}$ | $\{\varphi_1\}^{385} \oplus \{\varphi_3\}^{232}$ $\oplus \bigoplus_{i \mid 24, i \neq 1, 3} \{\varphi_i\}^{240}$ |

Table 1: Cohomologies of Pure Braid groups for exceptional cases.
|     | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|-----|------|------|------|------|------|------|------|
| $H^0$ | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| $H^1$ | $\{\varphi_1\}$ | $\{\varphi_1\}$ | $\{\varphi_1\}$ | $\{\varphi_1\}$ | $\{\varphi_1\}$ | $\{\varphi_1\}$ | $\{\varphi_1\}$ |
| $H^2$ | $\{\varphi_1\}^2$ | $\{\varphi_1\}^5 \oplus \{\varphi_3\}$ | $\{\varphi_1\}^9$ | $\{\varphi_1\}^{14}$ | $\{\varphi_1\}^{20}$ | $\{\varphi_1\}^{27}$ |
| $H^3$ | $\{\varphi_1\}^6 \oplus \{\varphi_3\}$ | $\{\varphi_1\}^{26} \oplus \{\varphi_2\}^2$ | $\{\varphi_1\}^{71} \oplus \{\varphi_3\}$ | $\{\varphi_1\}^{155} \oplus \{\varphi_3\}$ | $\{\varphi_1\}^{295}$ |
| $H^4$ | $\{\varphi_1\}^{24} \oplus \{\varphi_2\}^8$ | $\{\varphi_1\}^{154} \oplus \{\varphi_3\}^{14}$ | $\{\varphi_1\}^{580}$ | $\{\varphi_1\}^{1665}$ |
| $H^5$ | $\{\varphi_1\}^{120} \oplus \{\varphi_3\}^{37}$ | $\{\varphi_1\}^{121}$ | $\{\varphi_1\}^{1044}$ | $\{\varphi_1\}^{5104}$ |
| $H^6$ | $\{\varphi_1\}^{720}$ | $\{\varphi_1\}^{8028}$ | $\{\varphi_1\}^{222}$ | $\{\varphi_2\}^{140}$ |
| $H^7$ | $\{\varphi_1\}^{5040}$ | $\{\varphi_1\}^{858}$ | $\{\varphi_7\}^{840}$ | $\{\varphi_7\}^{120}$ |

Table 2: Cohomologies of Pure Braid Groups for $1 \leq n \leq 7$. 
3 Cohomology of \((Fl_*(W), \delta)\) for exceptional cases

Recall, briefly, that a flag in a graded poset \(P\) is a tuple of elements \((X_0, \cdots, X_k)\) of \(P\) where each \(X_i\) has rank \(i\) and \(X_i < X_{i+1}\) for \(0 \leq i < k\) (the order given by the reverse inclusion). The flag complex \(Fl_p\) in the intersection lattice of an arrangement \(A\), is defined to be the free abelian group of flags of length \(p + 1\), modulo the relations:

\[
\sum_{Y: X_{i-1} < Y < X_{i+1}} (X_0, \cdots, X_{i-1}, Y, X_{i+1}, \cdots, X_p) = 0,
\]

where \(i \geq 1\).

A set of \(p\) independent hyperplanes \(H_1, \cdots, H_p\) determines a flag with \(X_i = H_1 \cap \cdots \cap H_i\) for \(0 \leq i \leq p\) (see [16]). We will denote this flag by \(\Phi = \lambda(H_1, \cdots, H_p)\). Accordingly, set \(\varepsilon(H, \Phi) = (-1)^{i-1}\) and let define \(\delta : Fl_p \to Fl_{p-1}\) as follows. Let \(\Phi = (X_0, \cdots, X_p)\) be a flag and \(H\) an hyperplane such that \(H \leq X_p\). \(D(\Phi, H)\) will denote the set of all \((p - 1)\)-flags

\[
\Psi = \lambda(H_1, \cdots, H_{i-1}, H_{i+1}, \cdots, H_p),
\]

such that \(\Phi = \lambda(H_1, \cdots, H_i = H, \cdots, H_p)\).

Now, if \(a : A \to \mathbb{Z}_+\) is a weight function given by the multiplicities of the hyperplanes in an unreduced arrangement, we set (see [16])

\[
\delta \Phi = \sum_{H \leq X_p} \sum_{\Psi \in D(\Phi, H)} \varepsilon(H, \Phi)a(H)\Psi.
\]

In the following table we give the cohomology of \((Fl_*(W), \delta)\) for reflection arrangements given by the finite Coxeter groups \(H_3, H_4\) and \(F_4\).
Table 3: Cohomologies of flag complex for some exceptional cases.

| $H^3$ | $H^4$ | $F_4$ |
|-------|-------|-------|
| $H^0$ | 0     | 0     |
| $H^1$ | 0     | 0     |
| $H^2$ | 0     | 0     |
| $H^3$ | $\oplus_{i|15,i\neq 1} (\mathbb{Z}/i\mathbb{Z})^{32}$ | $\mathbb{Z}/3\mathbb{Z}$ | $(\mathbb{Z}/3\mathbb{Z})^8$ |
| $H^4$ | $\oplus_{i|60,i\neq 1,3} (\mathbb{Z}/i\mathbb{Z})^{5039}$ | $(\mathbb{Z}/3\mathbb{Z})^{232}$ | $\oplus_{i|24,i\neq 1,3} (\mathbb{Z}/i\mathbb{Z})^{240}$ |

4 A filtration for the complex $(Fl_p(A_n), \delta)$

Recall that a NBC-basis for $A_p(A_n) = A_p(A(A_n))$ is given by standard $p$-tuples

$$\{ (H_{i_1,j_1}, \ldots, H_{i_p,j_p}) \}_{1 \leq i_1 < \cdots < i_p \leq n}.$$  

and that $\{ \lambda(H_{i_1,j_1}, \ldots, H_{i_p,j_p}) \}_{1 \leq i_1 < \cdots < i_p \leq n}$ is a basis for $Fl_p(A_n)$ (see [16]).

In order to construct a filtration for the complex $(Fl_p(A_n), \delta)$ we need some notations and definitions.

We consider the set $[n+1] := \{ 1, \ldots, n+1 \}$, to each $H_{i,j}$ we associate the pair $(i, j)$; then to each $p$-tuple $H_{(i,t)_{1 \leq t \leq p}} := (H_{i_1,j_1}, \ldots, H_{i_p,j_p})$ corresponds a graph

$$G_{(i,t)_{1 \leq t \leq p}} := G(H_{(i,t)_{1 \leq t \leq p}}).$$

For each point $h \in [n+1]$ of this graph we consider the set of vertices which
are in the same connected component of $G_{(i,t),j} \leq t \leq p$ as $h$; precisely:

$$c_h(G_{(i,t),j} \leq t \leq p) := \{i \in [n+1] \mid \exists k_0 = i, \ldots, k_r = h \in [n+1] \text{ s.t. } H_{\sigma(k_q,k_{q+1})} \in \mathcal{H}_{(i,t),j} \leq t \leq p \forall \ 0 \leq q \leq r - 1\}$$

where

$$\sigma(i,j) = \begin{cases} (i,j) & \text{if } i < j \\ (j,i) & \text{if } i > j \end{cases}$$

and

$$\mathcal{H}_{(i,t),j} \leq t \leq p) := \{H_{i,j}, H_{i,j+1}, \ldots, H_{i,p} \}.$$

We set

$$l_h(G_{(i,t),j} \leq t \leq p) := \#c_h(G_{(i,t),j} \leq t \leq p)$$

its length.

These definitions can be extended to a flag $\Phi = \lambda(H_{i,j})$. In this case we will denote:

$$c_h(\Phi) := c_h(G_{(i,t),j} \leq t \leq p)$$

$$l_h(\Phi) := l_h(G_{(i,t),j} \leq t \leq p)$$

Now let us consider the subcomplexes:

$$G^k_{n+1} := \{\lambda(\mathcal{H}_{(i,t),j} \leq t \leq p) \in Fl_*(A_n) \mid l_1(G_{(i,t),j} \leq t \leq p) \leq k\},$$

clearly the boundary map preserves $G^k_{n+1}$. Let $F^k_{n+1}$ the cokernel of the natural inclusion in $Fl_*(A_n)$ endowed with the induced boundary.

Notice that the map $\delta$ defined on a flag $\Phi = \lambda(\mathcal{H}_{(i,t),j} \leq t \leq p)$ depends on the set $D(\Phi, H_{(i,j)})$ for $H_{(i,j)} \in \{\mathcal{H}_{(i,t),j} \leq t \leq p\}$ (see [11]). In particular it is easy to see that if $\Psi = \lambda(\mathcal{H}_{(i',j')} \leq t \leq p - 1)$ then for all $(i', j')$ s.t. $H_{(i',j')} \in \mathcal{H}_{(i,t),j} \leq t \leq p$ and $i', j' \notin c_t(\Phi)$, $H_{(i',j')} \in \{\mathcal{H}_{(i,t),j} \leq t \leq p - 1\}$.

From the previous remark it follows that if $\Phi = \lambda(\mathcal{H}) \in (F^k_{n+1})_*$ is a flag with $l_1(\Phi) = k$ then exists a flag $\Phi' = \lambda(\mathcal{H}') \in Fl_{*-k}(A_n)$ s.t.

$$\{\mathcal{H}'\} \cup \{\mathcal{H}_{(i,j)}\}_{i,j \in c_1(\Phi)} = \{\mathcal{H}\}$$
and
\[ \delta_s \Phi = (H_{i,j})_{i,j \in c_1(\Phi)} \delta_s - (k-1) \Phi' \]
where the product is defined pointwise for all \( \overline{H} \in \{H_{i,j}\}_{i,j \in c_1(\Phi)} \) as follows:
\[
\overline{H} \cdot \delta \Phi' = \overline{H} \cdot \sum_{H \leq X_p} \sum_{\Psi' \in D(\Phi', H)} \varepsilon(H, \Phi') a(H) \Psi' := \sum_{H \leq X_p} \sum_{\Psi' \in D(\Phi', H)} \varepsilon(H, \Phi') a(H) \overline{H} . \Psi'
\]
and
\[
\overline{H} . \Psi' = \overline{H} . \lambda(H_1, \cdots, H_p) := \lambda(H_1, \cdots H_{i-1}, \overline{H}, H_i, \cdots, H_p)
\]
with \( H_{i-1} < \overline{H} < H_i \).

In other words for a flag \( \Phi \) in \( F_{k,n}^{l+1} \), \( l_1(\Phi) = k \), the boundary map does not change components with indices in \( c_1(\Phi) \).

Then, if we define a map of complexes
\[
i_n := i : Fl_*(A_{n-1}) \longrightarrow Fl_*(A_n),
i((h_1, \cdots, h_k-1), \Phi) = i((h_1, \cdots, h_{k-1}). \lambda(H_{(i_t,j_t)_{1 \leq t \leq p}})) = \lambda(H_{(i_t,j_t)_{1 \leq t \leq p}})
\]
the cokernel of the map \( i \) is the complex \( F_{n+1}^2 \) and we can iterate this construction considering the map
\[
i_n[1] := i : (n)^1 Fl_*(A_{n-2})[1] \longrightarrow F_{n+1}^2,
i((i_1, \cdots, i_{k-1}), \Phi) = i((i_1, \cdots, i_{k-1}). \lambda(H_{(i_t,j_t)_{1 \leq t \leq p}})) = \lambda(H_{(i_t,j_t)_{1 \leq t \leq p}})
\]
where \( (n)^1 = \{(i) \mid 2 \leq i \leq n\} \), \( (i_0', j_0') = (1, j) \) and 
\( (i_t', j_t') = (2, \cdots, \tilde{j}, \cdots, n+1)_{(i_t,j_t)} \), i.e. the indexes in the positions \( (i_t,j_t) \) in the \( (n-1) \)-tuple \( (2, \cdots, \tilde{j}, \cdots, n+1) \).

Notice that a chain of degree \( k \) in \( Fl_*(A_{n-2}) \) maps to one of degree \( k + 1 \), for this we prefer to shift the complex \( Fl_*(A_{n-2}) \) by 1.

Again the cokernel of \( i_n[1] \) is the complex \( F_{n+1}^3 \).

We continue in this way getting maps
\[
i_n[k-1] := i : (n)^{k-1} Fl_*(A_{n-k})[k-1] \longrightarrow F_{n+1}^k,
i((h_1, \cdots, h_{k-1}), \Phi) = i((h_1, \cdots, h_{k-1}). \lambda(H_{(h_t,j_t)_{1 \leq t \leq p}})) = \lambda(H_{(h_t,j_t)_{1 \leq t \leq p}})
\]
where $h_0 = 1$, $\sigma$ is the map $f$

$$(n)^{k-1} = \{(h_1, \cdots, h_{k-1}) \mid 2 \leq h_i \leq n + 1 \text{ and } h_i \neq h_j \text{ if } i \neq j\}$$

and $(i^*_t, j^*_t) = (2, \cdots, \tilde{h}_i, \cdots, n + 1)_{(i_t, j_t)}$, i.e. the indexes in the positions $(i_t, j_t)$ in the $(n - (k - 1))$-tuple $(2, \cdots, \tilde{h}_i, \cdots, n + 1)$.

Each $i_n[k]$ gives rise to the exact sequence of complexes

$$0 \rightarrow (n)^{k}Fl_*(\mathcal{A}_{n-k-1})[k] \rightarrow F_{n+1}^{k+1} \rightarrow F_{n+1}^{k+2} \rightarrow 0. \quad (4)$$

The exact sequences $4$ give rise to long exact sequences in homology.

With this filtration we are able to compute the (co)-homology groups of $Fl_{\mathcal{A}_n}$ for $n \leq 7$ (see table 4).

We can remark how, for these cases, the cohomology of the Flag Complex is 0 when the cohomology of the complement to the arrangement is a trivial $\mathbb{Z}$-module. An interesting question could be to verify if this is always true. In this case we could extend the “stability” theorem proved in [17] to the Cohomology of the Flag Complex.
| $H^0$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     | 0     | 0     | 0     |

| $H^1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     | 0     | 0     | 0     |

| $H^2$ | $Z/3Z$ | $Z/3Z$ | $(Z/2Z)^2$ | $Z/3Z$ | $Z/3Z$ |
|-------|--------|--------|------------|--------|--------|
|       |        |        | $\oplus_{i\neq 1} (Z/iZ)^2$ |        |        |

| $H^3$ | $Z/3Z$ | $(Z/2Z)^2$ | $Z/3Z$ | $Z/3Z$ |
|-------|--------|------------|--------|--------|
|       |        | $\oplus_{i\neq 1,2} (Z/iZ)^2$ |        |        |

| $H^4$ | $(Z/2Z)^8$ | $(Z/3Z)^{14}$ | $(Z/3Z)^{30}$ |
|-------|------------|---------------|---------------|
|       | $\oplus_{i\neq 1,2} (Z/iZ)^6$ | $\oplus (Z/5Z)^6$ |               |

| $H^5$ | $(Z/3Z)^{37}$ | $(Z/3Z)^{121}$ | $(Z/2Z)^2$ |
|-------|---------------|---------------|------------|
|       | $\oplus (Z/5Z)^{30}$ | $\oplus (Z/15Z)^{24}$ |               |

| $H^6$ | $(Z/3Z)^{222}$ | $(Z/2Z)^{140}$ | $(Z/7Z)^{120}$ |
|-------|---------------|---------------|---------------|
|       | $\oplus_{i\neq 1,3} (Z/iZ)^{120}$ |               |               |

| $H^7$ | $(Z/2Z)^{858}$ | $(Z/7Z)^{840}$ |
|-------|---------------|---------------|
|       | $\oplus_{i\neq 1,2,7} (Z/iZ)^{720}$ |               |
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