ON THE STABILITY OF PULLED BACK PARABOLIC VECTOR BUNDLES

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Abstract. Take an irreducible smooth projective curve $X$ defined over an algebraically closed field of characteristic zero, and fix finitely many distinct point $D = \{x_1, \cdots, x_n\}$ of it; for each point $x \in D$ fix a positive integer $N_x$. Take a nonconstant map $f : Y \rightarrow X$ from an irreducible smooth projective curve. We construct a natural subbundle $F \subset f_*\mathcal{O}_Y$ using $(D, \{N_x\}_{x \in D})$. Let $E_*$ be a stable parabolic vector bundle whose parabolic weights at each $x \in D$ are integral multiples of $\frac{1}{N_x}$. We prove that the pullback $f^*E_*$ is also parabolic stable, if rank($F$) = 1.

1. Introduction

We begin by recalling the main result of [BP]. Let $f : Y \rightarrow X$ be a surjective separable morphism between irreducible smooth projective curves defined over an algebraically closed field $k$. It is called genuinely ramified if the rank of the maximal semistable subbundle $F \subset f_*\mathcal{O}_Y$ is one. The main result of [BP] says that the pullback $f^*E$ of every stable vector bundle $E$ on $X$ is also stable, provided $f$ is genuinely ramified.

Our aim here is to prove an analogue of it for parabolic vector bundles, but under an extra assumption that the characteristic of the base field $k$ is zero.

Let $X$ be an irreducible smooth projective curve defined over an algebraically closed field $k$ of characteristic zero. Fix finitely many points $D = \{x_1, \cdots, x_n\} \subset X$, and for each $x \in D$ fix a positive integer $N_x$. We consider the category of parabolic vector bundles $E_*$ on $X$ with parabolic divisor $D$ such that all the parabolic weights of $E_*$ at any $x \in D$ are integral multiples of $\frac{1}{N_x}$.

Let $f : Y \rightarrow X$ be a nonconstant morphism from an irreducible smooth projective curve $Y$. Using the above data $(D, \{N_x\}_{x \in D})$ we construct a natural subbundle

$$F \subset f_*\mathcal{O}_Y.$$ (1.1)

This $F$ is an analogue of the above mentioned maximal semistable subbundle $F \subset f_*\mathcal{O}_Y$ in the context of parabolic bundles. It should be clarified that $f_*\mathcal{O}_Y$ is parabolic polystable (see Proposition 2.1), so $F$ is not related to the Harder–Narasimhan filtration or the socle filtration of $f_*\mathcal{O}_Y$.

We prove the following (see Theorem 5.1):

Theorem 1.1. Let

$$f : Y \rightarrow X$$

Mathematics Subject Classification. 14H30, 14H60, 14E20.
Key words and phrases. Parabolic bundle, stability, orbifold, canonical subsheaf.
be a nonconstant map between irreducible smooth projective curves defined over an algebraically closed field $k$ of characteristic zero. Take any stable parabolic vector bundle $E_*$ on $X$ with parabolic structure over $D$ such that all the parabolic weights of $E_*$ at each point $x \in D$ are integral multiples of $\frac{1}{N_x}$. If the rank of the vector bundle $\mathcal{F} \to X$ in (1.1) is one, then the parabolic vector bundle $f^*E_*$ on $Y$ is also stable.

We also prove the following converse of Theorem 1.1 (see Lemma 4.3):

**Lemma 1.2.** Let $f : Y \to X$ be a nonconstant map between irreducible smooth projective curves defined over an algebraically closed field $k$ of characteristic zero. Assume that the rank of the holomorphic vector bundle $\mathcal{F} \to X$ in (1.1) is at least two. Then there is a stable parabolic vector bundle $E_*$ on $X$ with parabolic structure over $D$ such that

1. all the parabolic weights of $E_*$ at each $x \in D$ are integral multiples of $\frac{1}{N_x}$, and
2. the parabolic vector bundle $f^*E_*$ on $Y$ is not stable.

Theorem 1.1 is first proved for $k = \mathbb{C}$; see Theorem 4.2. Then theorem 1.1 is deduced using Theorem 4.2.

2. **Direct image and parabolic structure**

Let $k$ be an algebraically closed field of characteristic zero.

Let $X$ be an irreducible smooth projective curve defined over $k$. Fix a nonempty finite subset $D := \{x_1, \ldots, x_n\} \subset X$.

The reduced effective divisor $x_1 + \ldots + x_n$ on $X$ will also be denoted by $D$. A quasiparabolic structure on a vector bundle $E$ on $X$ is a filtration of subspaces of the fiber $E_{x_i}$ of $E$ over $x_i$

$$E_{x_i} = E_{i,1} \supset E_{i,2} \supset \cdots \supset E_{i,l_i} \supset E_{i,l_i+1} = 0$$

for every $1 \leq i \leq n$. A parabolic structure on $E$ is a quasiparabolic structure as above together with a string of rational numbers

$$0 \leq \alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,l_i} < 1$$

for every $1 \leq i \leq n$. The above number $\alpha_{i,j}$ is called the parabolic weight of the subspace $E_{i,j}$. (See [MS], [MY], [Bh], [IIS], [In].)

A parabolic vector bundle is a vector bundle $E$ equipped with a parabolic structure $(\{E_{i,j}\}, \{\alpha_{i,j}\})$. For notational convenience, $(E, (\{E_{i,j}\}, \{\alpha_{i,j}\}))$ will be denoted by $E_*$. The divisor $D$ is known as the parabolic divisor of $E_*$. The parabolic degree of $E_*$ is defined to be

$$\text{par-deg}(E_*) := \deg(E) + \sum_{i=1}^n \sum_{j=1}^{l_i} \alpha_{i,j} \cdot \dim(E_{i,j}/E_{i,j+1}),$$
and \( \frac{\text{par-deg}(E_\ast)}{\text{rank}(E)} \in \mathbb{Q} \) is denoted by \( \text{par-}\mu(E_\ast) \).

Let \( F \subset E \) be a subbundle. Then a parabolic structure on \( E \) produces a parabolic structure on \( F \). The parabolic divisor for the induced parabolic structure on \( F \) is \( D \) itself. A subspace \( 0 \neq V \subset F_{x_i} \) appears in the quasiparabolic filtration of \( F_{x_i} \) if

\[
F_{x_i} \cap E_{i,j+1} \neq V = F_{x_i} \cap E_{i,j}
\]

for some \( 1 \leq j \leq l_i \). The parabolic weight of such a subspace \( V \) is \( \alpha_{i,j} \). The vector bundle \( F \) with the induced parabolic structure will be denoted by \( F_\ast \).

The parabolic bundle \( E_\ast \) is called stable (respectively, semistable) if

\[
\text{par-}\mu(F_\ast) < \text{par-}\mu(E_\ast) \quad \text{(respectively, } \text{par-}\mu(F_\ast) \leq \text{par-}\mu(E_\ast))
\]

for every subbundle \( 0 \neq F \subset E \). The parabolic bundle \( E_\ast \) is called polystable if the following two conditions hold:

- \( E_\ast \) is parabolic semistable, and
- \( E_\ast \) is a direct sum of stable parabolic bundles.

Let \( Y \) be an irreducible smooth projective curve and

\[
f : Y \longrightarrow X
\]

a nonconstant morphism. Let

\[
D_f := \{p_1, \ldots, p_m\} \subset X
\]

be the subset over which the map \( f \) is ramified.

**Proposition 2.1.** The direct image \( f_\ast\mathcal{O}_Y \) has a parabolic structure whose parabolic divisor is \( D_f \) defined in (2.2). This parabolic bundle given by \( f_\ast\mathcal{O}_Y \) is parabolic polystable of parabolic degree zero.

**Proof.** We use a local model of the map \( f \) to describe the parabolic structure on \( f_\ast\mathcal{O}_Y \). Take

- \( Y \) to be an open subset of \( U \subset \mathbb{A}_k^1 \) containing 0,
- \( X \) to be the image of \( U \) under the map \( \mathbb{A}_k^1 \longrightarrow \mathbb{A}_k^1 \) defined by \( z \longmapsto z^d \), where \( d \) is a positive integer, and
- \( f \) to be the map \( z \longmapsto z^d \).

Then the quasiparabolic filtration of the fiber \( (f_\ast\mathcal{O}_U)_0 = (f_\ast\mathcal{O}_U)_{p_1} \) over \( 0 = p_1 \) is given by the image of the fibers of the filtration of subsheaves

\[
f_\ast\mathcal{O}_U \supset f_\ast(\mathcal{O}_U(-p_1)) \supset f_\ast(\mathcal{O}_U(-2p_1)) \supset \cdots \supset f_\ast(\mathcal{O}_U(-(d-1)p_1)) \supset f_\ast(\mathcal{O}_U(-dp_1)),
\]

and the parabolic weight of the image of the fiber \( (f_\ast(\mathcal{O}_U(-kp_1)))_{p_1} \) in \( (f_\ast\mathcal{O}_U)_{p_1} \) is \( \frac{k}{d} \). Note that the image of the fiber \( (f_\ast(\mathcal{O}_U(-dp_1)))_{p_1} \) in \( (f_\ast\mathcal{O}_U)_{p_1} \) is zero, because by the projection formula we have \( f_\ast(\mathcal{O}_U(-dp_1)) = (f_\ast\mathcal{O}_U) \otimes \mathcal{O}_{f(U)}(-p_1) \).

Now consider the map \( f \) in (2.1). For each \( z_i \in D_f \) (see (2.2)), let \( \{y_i^1, \ldots, y_i^{b_i}\} \) be the reduced inverse image \( f^{-1}(z_i)_{\text{red}} \). For \( 1 \leq j \leq b_i \), let \( U^i_j \) be the formal completion of \( y_i^j \).
in \( Y \). The restriction of \( f \) to \( U_i^j \) will be denoted by \( f_i^j \). Now we have

\[
(f_\ast \mathcal{O}_Y)_{z_i} = \bigoplus_{j=1}^{b_i} \left( (f_i^j)_\ast \mathcal{O}_{U_i^j} \right)_{z_i}.
\]  

(2.3)

Each direct summand \( \left( (f_i^j)_\ast \mathcal{O}_{U_i^j} \right)_{z_i} \) of the fiber \((f_\ast \mathcal{O}_Y)_{z_i}\) in (2.3) has a parabolic structure which is described above. The parabolic structure on \((f_\ast \mathcal{O}_X)_{z_i}\) is given by the direct sum of the parabolic structures on the direct summands in (2.3).

We will give an alternative description of the parabolic structure on \( f_\ast \mathcal{O}_Y \). Let \( Z \) be an irreducible smooth projective curve and

\[
\phi : Z \longrightarrow Y
\]

(2.4)
a nonconstant morphism, such that \( f \circ \phi : Z \longrightarrow X \) is a (ramified) Galois covering. Let \( \Gamma = \text{Gal}(f \circ \phi) \subset \text{Aut}(Z) \) be the Galois group.

Let \( k[\Gamma] \) denote the algebra of functions on the finite group \( \Gamma \). The left-translation action of \( \Gamma \) on itself produces an action of \( \Gamma \) on \( k[\Gamma] \). On the other hand, the group \( \Gamma \subset \text{Aut}(Z) \) has a tautological action on \( Z \). Consider the diagonal action of \( \Gamma \) on \( Z \times k[\Gamma] \). This action makes the trivial vector bundle 

\[
Z \times k[\Gamma] \longrightarrow Z
\]

(2.5)
a \( \Gamma \)-equivariant vector bundle on \( Z \). Using the natural correspondence between equivariant bundles and parabolic bundles (see [Bis1], [Bo1], [Bo2]), this \( \Gamma \)-equivariant vector bundle \( Z \times k[\Gamma] \) on \( Z \) produces a parabolic vector bundle on \( Z/\Gamma = X \). This parabolic vector bundle on \( X \) will be denoted by \( W_\ast \). The parabolic divisor \( \text{par-div}(W_\ast) \) for \( W_\ast \) is the subset of \( X \) over which the map \( f \circ \phi \) is ramified, where \( \phi \) is the map in (2.4). Note that \( D_f \subset \text{par-div}(W_\ast) \), and \( \text{par-div}(W_\ast) \) may be larger than \( D_f \).

The vector bundle underlying the parabolic vector bundle \( W_\ast \) is \( (f \circ \phi)_\ast \mathcal{O}_Z \) [Bis2], [Par]. On the other hand, we have

\[
f_\ast \mathcal{O}_Y \subset (f \circ \phi)_\ast \mathcal{O}_Z.
\]

In fact, \( f_\ast \mathcal{O}_Y \) is a subbundle of \((f \circ \phi)_\ast \mathcal{O}_Z\).

Hence the parabolic structure of \( W_\ast \) induces a parabolic structure on \( f_\ast \mathcal{O}_Y \). Let

\[
(f_\ast \mathcal{O}_Y)_\ast \longrightarrow X
\]

denote the parabolic vector bundle with parabolic structure on \( f_\ast \mathcal{O}_Y \) induced by \( W_\ast \).

The parabolic structure on \((f_\ast \mathcal{O}_Y)_\ast \) over the complement \( \text{par-div}(W_\ast) \setminus D_f \) is the trivial one, meaning \((f_\ast \mathcal{O}_Y)_\ast \) does not have any nonzero parabolic weight on the points of \( \text{par-div}(W_\ast) \setminus D_f \).

It is straightforward to check that \((f_\ast \mathcal{O}_Y)_\ast \) coincides with the parabolic bundle given by the parabolic structure on \( f_\ast \mathcal{O}_Y \) constructed earlier. In particular, the above parabolic structure on \((f_\ast \mathcal{O}_Y)_\ast \) does not depend on the choice of the pair \((Z, \phi)\).

Since the vector bundle underlying the \( \Gamma \)-equivariant vector bundle in (2.5) is polystable (it is in fact trivial), the corresponding parabolic bundle \((f_\ast \mathcal{O}_Y)_\ast \) is polystable [BBN].
p. 350–351, Theorem 4.3]. Since the degree of the $\Gamma$–equivariant vector bundle in $2.5$ is zero, it follows that the parabolic degree of $(f_*O_Y)_*$ is zero [BisI, p. 318, (3.12)].

We refer the reader to [Yo] for the definition of parabolic dual of parabolic vector bundles.

Lemma 2.2. The parabolic dual of the parabolic vector bundle $(f_*O_Y)_*$, constructed in the proof of Proposition 2.1, is $(f_*O_Y)_*$ itself.

Proof. If $E_*$ is the parabolic vector bundle corresponding to an equivariant bundle $V$, then the parabolic vector bundle corresponding to the equivariant bundle $V^*$ is the parabolic dual $E^*_*$ of $E_* [BBN]$. Let

$$k[\Gamma] \otimes k[\Gamma] \longrightarrow k$$

be the pairing defined by

$$(\sum_{\gamma \in \Gamma} a_\gamma \gamma, \sum_{\gamma \in \Gamma} b_\gamma \gamma) \longmapsto \sum_{\gamma \in \Gamma} a_\gamma b_\gamma,$$

where $a_\gamma, b_\gamma \in k$. Consider the $\Gamma$–equivariant vector bundle $Z \times k[\Gamma] \longrightarrow Z$ in $2.5$. The paring in $2.6$ defines a homomorphism of coherent sheaves

$$(Z \times k[\Gamma]) \otimes (Z \times k[\Gamma]) \longrightarrow O_Z$$

which is fiberwise nondegenerate. The resulting isomorphism of vector bundles

$$Z \times k[\Gamma] \cong Z \times k[\Gamma]^* = (Z \times k[\Gamma]^*)^*$$

is in fact $\Gamma$–equivariant. Therefore, we conclude that the parabolic dual of $(f_*O_Y)_*$ is $(f_*O_Y)_*$ itself. \qed

3. Construction of a parabolic subbundle

Let $V_* = (V, \{\{V_{ij}\}, \{\alpha_{i,j}\}\})$ be a semistable parabolic bundle on $X$ with parabolic divisor

$$D := \{t_1, \cdots, t_r\} \subset X.$$

For any subbundle $F \subset V$, the parabolic vector bundle defined by $F$ equipped with the parabolic structure induced by $V_*$ will be denoted by $F_*$. For each parabolic point $t \in D$, we fix an integer $N_t \geq 1$. Assume that there is a subbundle

$$F \subset V$$

satisfying the following two conditions:

1. All the parabolic weights of $F_*$ at every $t \in D$ are integral multiples of $\frac{1}{N_t}$. (If $N_t = 1$, then $F_*$ does not have any nonzero parabolic weight at $t$.)
2. $\text{par}-\mu(F_*) = \text{par}-\mu(V_*)$. 


Lemma 3.1. There is a unique maximal subbundle 
\[ \mathcal{F} \subset V \]

satisfying the following two conditions:

1. All the parabolic weights of \( \mathcal{F}_* \) at every \( t \in \mathbb{D} \) are integral multiples of \( \frac{1}{N_t} \), and
2. \( \text{par-} \mu(\mathcal{F}_*) = \text{par-} \mu(V_*) \).

Proof. Let \( F^1 \) and \( F^2 \) be two subbundles of \( V \) such that for \( 1 \leq j \leq 2 \),

1. all the parabolic weights of \( F^j_* \) at every \( t \in \mathbb{D} \) are integral multiples of \( \frac{1}{N_t} \), and
2. \( \text{par-} \mu(F^j_*) = \text{par-} \mu(V_*) \).

Since \( V_* \) is parabolic semistable, and \( \text{par-} \mu(F^j_*) = \text{par-} \mu(V_*) \), it follows immediately that 
\( F^j_* \) is semistable for \( j = 1, 2 \). Consider the subsheaf \( F^1 + F^2 \subset V \) equipped with the 
parabolic structure induced by the parabolic structure of \( V_* \); the resulting parabolic bundle 
will be denoted by \( (F^1 + F^2)_* \). So
\[ \text{par-} \mu((F^1 + F^2)_*) \leq \text{par-} \mu(V_*) \] (3.1)
because \( V_* \) is parabolic semistable. On the other hand, \( (F^1 + F^2)_* \) is a quotient of the 
direct sum \( F^1_* \oplus F^2_* \), and \( F^1_* \oplus F^2_* \) is parabolic semistable with
\[ \text{par-} \mu((F^1 \oplus F^2)_*) = \text{par-} \mu(F^1_*) = \text{par-} \mu(F^2_*) = \text{par-} \mu(V_*) \].

Hence we have
\[ \text{par-} \mu((F^1 + F^2)_*) \geq \text{par-} \mu(V_*) \).
Combining this with (3.1) we conclude that
\[ \text{par-} \mu((F^1 + F^2)_*) = \text{par-} \mu(V_*) \] (3.2)

We will show that \( V/(F^1 + F^2) \) is torsionfree. To prove this, if \( T_0 \) is the torsion part 
of \( V/(F^1 + F^2) \), consider \( \mathbb{S} = q_0^{-1}(T_0) \), where \( q_0 : V \rightarrow V/(F^1 + F^2) \) is the quotient 
map. Let \( \mathbb{S}_* \) denote the parabolic vector bundle given by \( \mathbb{S} \) equipped with the parabolic 
structure induced by the parabolic structure of \( V_* \). If \( T_0 \neq 0 \), then
\[ \text{par-} \mu(\mathbb{S}_*) > \text{par-} \mu((F^1 + F^2)_*) = \text{par-} \mu(V_*) \] (see (3.2)). But this contradicts the given condition that \( V_* \) is parabolic semistable. Therefore, we conclude that \( V/(F^1 + F^2) \) is torsionfree. In other words,
\[ F^3 := F^1 + F^2 \subset V \]
is a subbundle.

Consider the parabolic vector bundle \( F^3_* \) defined by \( F^3 \) equipped with the parabolic 
structure induced by the parabolic structure of \( V_* \). Recall that for \( 1 \leq j \leq 2 \), all the 
parabolic weights of \( F^j_* \) at every \( t \in \mathbb{D} \) are integral multiples of \( \frac{1}{N_t} \). This immediately 
implies that all the parabolic weights of \( F^3_* \) at each \( t \in \mathbb{D} \) are also integral multiples of \( \frac{1}{N_t} \).

In (3.2) we have seen that \( \text{par-} \mu(F^3_*) = \text{par-} \mu(V_*) \).
Now take $F$ to be the coherent subsheaf of $V$ generated by all subbundles $F \subset V$ such that

1. all the parabolic weights of $F_*$ at each $t \in \mathbb{D}$ are integral multiples of $\frac{1}{N_t}$, and
2. $\text{par-}\mu(F_*) = \text{par-}\mu(V_*)$.

From the above observations on $F^3$ it follows immediately that this coherent subsheaf $F \subset V$ satisfies all the conditions in the statement of the lemma. \hfill \Box

As in (2.1), take any irreducible smooth projective curve $Y$ together with a nonconstant morphism $f : Y \to X$. As in (2.2), $D_f := \{p_1, \cdots, p_m\} \subset X$ denotes the subset over which $f$ is ramified. Fix a divisor

$$D := \{x_1, \cdots, x_n\} \subset X.$$ (3.3)

Also, fix an integer

$$N_x \geq 1$$ (3.4)

for each point $x \in D$.

**Proposition 3.2.** Let $(f_*\mathcal{O}_Y)_*$ be the parabolic bundle defined by $f_*\mathcal{O}_Y$ equipped with the natural parabolic structure (see Proposition 2.1). Then there is a unique maximal subbundle $F \subset f_*\mathcal{O}_Y$ satisfying the following three conditions:

1. For any $x \in D \cap D_f$, all the parabolic weights of $F_*$ at $x$ are integral multiples of $\frac{1}{N_x}$ (see (3.4)),
2. $F_*$ does not have any nonzero parabolic weight over any point of $D_f \setminus (D \cap D_f)$, and
3. $\text{par-deg}(F_*) = 0$.

**Proof.** In Lemma 3.1, set

- $V_* = (f_*\mathcal{O}_Y)_*$, (so the parabolic divisor $\mathbb{D}$ in Lemma 3.1 is now $D_f$),
- $N_x = 1$ if $x \in D_f \setminus (D \cap D_f)$ (see (3.3) for $D$), and
- $N_x = N_x$ (see (3.4)) if $x \in D \cap D_f$.

Recall from Proposition 2.1 that $\text{par-deg}((f_*\mathcal{O}_Y)_*) = 0$. So we have

$$\text{par-}\mu((f_*\mathcal{O}_Y)_*) = 0.$$ 

Therefore, in view of Lemma 3.1 it suffices to show that there is a subbundle $F \subset f_*\mathcal{O}_Y$ satisfying the following two conditions:

1. All the parabolic weights of $F_*$ at each $x \in D_f$ are integral multiples of $\frac{1}{N_x}$, and
2. $\text{par-deg}(F_*) = 0$. 


Since
\[ H^0(Y, \text{Hom}(O_Y, O_Y)) = H^0(Y, \text{Hom}(f^*O_X, O_Y)) = H^0(X, \text{Hom}(O_X, f_*O_Y)) \]
(see [Ha, p. 110]), the identity map of \( O_Y \) produces a nonzero homomorphism
\[ O_X \hookrightarrow f_*O_Y. \]
This coherent subsheaf is actually a subbundle. Indeed, this follows immediately from the fact that for any \( \psi \in H^0(U, O_U) \), where \( U \subset X \) is a Zariski open subset, the section \( \psi \circ f \in H^0(f^{-1}(U), O_{f^{-1}(U)}) \) has the property that if \( \psi(x) \neq 0 \), then \( \psi \circ f \) does not vanish on any point of \( f^{-1}(x) \).

From the construction of the parabolic structure on \( f_*O_Y \) in Proposition 2.1 it follows immediately that the induced parabolic weight on \( O_X \) at any \( x \in D_f \) is zero. Consequently, \( F = O_X \) satisfies the above two conditions. This proves the proposition. \( \square \)

Equip the curve \( X \) with the following orbifold structure: For each point \( x \in D \) the inertia group is \( \mathbb{Z}/N_x\mathbb{Z} \), where \( N_x \) is the integer in (3.4). The curve \( X \) equipped with this orbifold structure will be denoted by \( \mathcal{X} \). An \( \acute{e} \text{tale covering} \)
\[ \varphi : Z \longrightarrow \mathcal{X} \quad (3.5) \]
is an irreducible smooth projective curve \( Z \) together with a nonconstant morphism
\[ \varphi_0 : Z \longrightarrow X \quad (3.6) \]
such that the following conditions hold:

- the map \( \varphi_0 \) is unramified over \( X \setminus D = X \setminus \{x_1, \ldots, x_n\} \), and
- for every \( x \in D \), the order of ramification of \( \varphi_0 \) at each \( z \in \varphi_0^{-1}(x) \) is a divisor of \( N_x \).

An \( \acute{e} \text{tale covering} \) \( \varphi \) of \( \mathcal{X} \) will be called nontrivial if \( \text{degree}(\varphi_0) \geq 2 \).

**Theorem 3.3.** Consider the map \( f : Y \longrightarrow X \), and the corresponding subbundle
\[ F \subset f_*O_Y \]
in Proposition 3.2. Then the following two statements are equivalent:

1. There is a nontrivial \( \acute{e} \text{tale covering} \)
\[ \varphi : Z \longrightarrow \mathcal{X} \]
(see (3.5) and (3.6)) and a morphism \( \beta : Y \longrightarrow Z \), such that \( \varphi_0 \circ \beta = f \).
2. The rank of \( F \) is bigger than one.

**Proof.** First assume that there is a nontrivial \( \acute{e} \text{tale covering} \)
\[ \varphi : Z \longrightarrow \mathcal{X}, \]
and a morphism \( \beta : Y \longrightarrow Z \), such that \( \varphi_0 \circ \beta = f \). Consider the subbundle
\[ (\varphi_0)_*O_Z \subset f_*O_Y. \quad (3.7) \]
The parabolic structure on \((\varphi_0)_*\mathcal{O}_Z\) constructed in Proposition 2.1 coincides with the one induced by the parabolic structure of \(f_*\mathcal{O}_Y\) on the subbundle in (3.7). The parabolic bundle defined by this parabolic structure on \((\varphi_0)_*\mathcal{O}_Z\) will be denoted by \(((\varphi_0)_*\mathcal{O}_Z)_*\).

Using the given condition that \(\varphi\) is an étale covering of \(X\) it is straight-forward to verify that for every point \(x \in D\), all the parabolic weights of \(((\varphi_0)_*\mathcal{O}_Z)_*\) at \(x\) are integral multiples of \(\frac{1}{N_x}\). Since \(\varphi_0\) is unramified over the complement \(X \setminus D\), the parabolic bundle \(((\varphi_0)_*\mathcal{O}_Z)_*\) does not have any nonzero parabolic weights on \(X \setminus D\). Also, from Proposition 2.1 we know that \(\text{par-deg}(((\varphi_0)_*\mathcal{O}_Z)_*) = 0\).

In view of these, from the uniqueness property of \(F\) in Proposition 3.2 we know that \((\varphi_0)_*\mathcal{O}_Z \subset F\).

Since \(\text{deg}(\varphi_0) \geq 2\), from (3.8) we conclude that \(\text{rank}(F) \geq 2\).

To prove the converse, assume that \(\text{rank}(F) \geq 2\).

As before, \(F_*\) denotes the parabolic bundle defined by \(F\) equipped with the parabolic structure induced by the parabolic structure of \(f_*\mathcal{O}_Y\).

The algebra structure of \(\mathcal{O}_Y\) produces an algebra structure

\[\Phi : (f_*\mathcal{O}_Y)_* \otimes (f_*\mathcal{O}_Y)_* \longrightarrow (f_*\mathcal{O}_Y)_*\]  (3.10)

(see [Yo], [BBN] for the tensor product of parabolic vector bundles).

Since the parabolic weights of \(F_*\) at every \(x \in D\) are integral multiples of \(\frac{1}{N_x}\), we conclude that the parabolic weights of \(F_* \otimes F_*\) at \(x \in D\) are also integral multiples of \(\frac{1}{N_x}\). From the given condition that \(\text{par-deg}(F_*) = 0\) it follows immediately that \(\text{par-deg}(F_* \otimes F_*) = 0\).

We note that \(F_* \otimes F_*\) is parabolic semistable because \(F_*\) is so [BBN] p. 346, Proposition 3.2]. Let \(\Phi(F_* \otimes F_*)_*\) be the parabolic vector bundle defined by \(\Phi(F_* \otimes F_*)\) equipped with the induced parabolic structure, where \(\Phi\) is the homomorphism in (3.10). Since \(\Phi(F_* \otimes F_*)_* \subset (f_*\mathcal{O}_Y)_*\) is a quotient parabolic bundle of the semistable parabolic bundle \(F_* \otimes F_*\), we have

\[\text{par-}\mu(\Phi(F_* \otimes F_*)_*) \geq \text{par-}\mu(F_* \otimes F_*) = 0\.

On the other hand,

\[\text{par-}\mu(\Phi(F_* \otimes F_*)_*) \leq \text{par-}\mu((f_*\mathcal{O}_Y)_*) = 0\],

because \((f_*\mathcal{O}_Y)_*\) is polystable. Combining these, we have

\[\text{par-}\mu(\Phi(F_* \otimes F_*)_*) = 0\.

Since \(\Phi(F_* \otimes F_*)_*\) is a quotient of \(F_* \otimes F_*\), and all the parabolic weights of \(F_* \otimes F_*\) at every \(x \in D\) are integral multiples of \(\frac{1}{N_x}\), it follows that all the parabolic weights of \(\Phi(F_* \otimes F_*)_*\) at every \(x \in D\) are also integral multiples of \(\frac{1}{N_x}\).
Therefore, from the uniqueness property of $\mathcal{F}$ we conclude that
\[ \Phi(\mathcal{F}_* \otimes \mathcal{F}_*^*) \subset \mathcal{F}_*. \] (3.11)

From (3.11) it follows that there is a unique étale covering
\[ \varphi : Z \rightarrow \mathcal{X}, \] (3.12)
where $\mathcal{X}$ is the orbifold in (3.5), and a morphism $\beta : Y \rightarrow Z$, such that following two hold:

1. $\varphi_0 \circ \beta = f$, and
2. the two subsheaves $(\varphi_0)_* \mathcal{O}_Z$ and $\mathcal{F}$ of $f_* \mathcal{O}_Y$ coincide.

From (3.9) and the above statement (2) we know that the étale covering $\varphi$ in (3.12) is nontrivial. This completes the proof. \[ \square \]

Consider the set-up of Theorem 3.3. Let
\[ Y' := Y \setminus f^{-1}(D) \subset Y \]
be the complement. Let
\[ f' := f|_{Y'} : Y' \rightarrow \mathcal{X} \] (3.13)
be the restriction of $f$ to $Y'$. The étale fundamental groups of $Y'$ and $\mathcal{X}$ will be denoted by $\pi_1(Y')$ and $\pi_1(\mathcal{X})$ respectively. The following two statements are evidently equivalent:

1. There is a nontrivial étale covering
   \[ \varphi : Z \rightarrow \mathcal{X} \]
   (see (3.5) and (3.6)) and a morphism $\beta : Y \rightarrow Z$, such that $\varphi_0 \circ \beta = f$.
2. The homomorphism of étale fundamental groups
   \[ (f'_\text{et})_* : \pi_1(Y') \rightarrow \pi_1(\mathcal{X}) \] (3.14)
   induced by $f'$ in (3.13) is not surjective.

Therefore, Theorem 3.3 gives the following:

**Corollary 3.4.** Consider the map $f : Y \rightarrow X$, and the corresponding subbundle
\[ \mathcal{F} \subset f_* \mathcal{O}_Y \]
in Proposition 3.2. Then the following two statements are equivalent:

1. The homomorphism of étale fundamental groups $(f'_\text{et})_*$ in (3.14) is not surjective.
2. The rank of $\mathcal{F}$ is bigger than one.

4. **Complex curves and pullback of stable parabolic bundles**

Throughout this section we assume that $k = \mathbb{C}$. The topological fundamental group of any complex manifold or orbifold $\mathcal{N}$ will be denoted by $\pi^t_1(\mathcal{N})$; this is to distinguish it from the étale fundamental group of $\mathcal{N}$. 
4.1. **Homomorphism of topological fundamental groups.** As before, \( f : Y \rightarrow X \) is a nonconstant holomorphic map between irreducible complex projective curves; the map \( f \) is ramified exactly over
\[
D_f := \{ p_1, \ldots, p_m \} \subset X.
\]
Fix an integer \( N_x \geq 1 \) for each \( x \in D \), and the resulting orbifold is denoted by \( \mathcal{X} \). The curve \( Y' \) and the map \( f' \) are both as in (3.13).

**Proposition 4.1.** The following two statements are equivalent:

1. the homomorphism of topological fundamental groups
   \[
   f'_* : \pi_1^t(Y') \rightarrow \pi_1^t(\mathcal{X})
   \]  
   induced by \( f' \) in (3.13) is surjective.

2. The rank of the holomorphic vector bundle \( \mathcal{F} \rightarrow X \) in Proposition 3.2 is one.

**Proof.** First assume that the homomorphism
\[
f'_* : \pi_1^t(Y') \rightarrow \pi_1^t(\mathcal{X})
\]  
is surjective. The group \( \pi_1(Y') \) (respectively, \( \pi_1(\mathcal{X}) \)) is the profinite completion of \( \pi_1^t(Y') \) (respectively, \( \pi_1^t(\mathcal{X}) \)). Therefore, from the surjectivity of the above homomorphism \( f'_* \) it follows immediately that the homomorphism \( (f'_{\text{et}})_* \) in (3.14) is surjective. Now Corollary 3.4 says that
\[
\text{rank}(\mathcal{F}) = 1,
\]
where \( \mathcal{F} \rightarrow X \) is the holomorphic vector bundle in Proposition 3.2.

To prove the converse, assume that
\[
\text{rank}(\mathcal{F}) = 1.
\]  
In view of (4.2), from Corollary 3.4 it follows that the homomorphism \( (f'_{\text{et}})_* \) in (3.14) is surjective. From the surjectivity of \( (f'_{\text{et}})_* \), it can be deduced that the homomorphism of topological fundamental groups
\[
f'_* : \pi_1^t(Y') \rightarrow \pi_1^t(\mathcal{X})
\]  
induced by \( f' \) in (3.13) is surjective. To see this, first note that \( \pi_1(\mathcal{X}) \) is residually finite and \( \pi_1^t(Y') \) is finitely generated as they are both surface groups. Now a result of Peter Scott, [Sc, p. 555, Theorem 3.3], says that for any finitely generated subgroup \( H \) of \( \pi_1^t(\mathcal{X}) \), and any \( t \in \pi_1^t(\mathcal{X}) \setminus H \), there is a finite index subgroup
\[
\tilde{H} \subset \pi_1^t(\mathcal{X})
\]  
such that
\[
t \notin \tilde{H} \supset H
\]  
(see [Pat] p. 2892 Theorem 1.2 for an effective version of the theorem of Scott). Applying this to the image \( f'_*(\pi_1^t(Y')) \subset \pi_1^t(\mathcal{X}) \) we conclude that if \( f'_* \) is not surjective then the image \( f'_*(\pi_1^t(Y')) \subset \pi_1^t(\mathcal{X}) \) is contained in a proper subgroup
\[
\Gamma \subsetneq \pi_1^t(\mathcal{X})
\]  
of finite index.
Consider the finite étale covering
\[ \varphi : Z \rightarrow X \]
given by the subgroup \( \Gamma \) in (4.4). Since \( f'_* (\pi_1(Y')) \subset \Gamma \), and a morphism \( \beta : Y \rightarrow Z \), such that \( \varphi_0 \circ \beta = f \). But this implies that
\[ (f'_*) (\pi_1(Y')) \subset \pi_1(Z) \subset \pi_1(X), \]
where \( (f'_*) \) is the homomorphism in (3.14). But this contradicts the fact that the homomorphism \( (f'_*) \) is surjective. Therefore, the homomorphism \( f_*' \) in (4.3) is surjective. \( \square \)

4.2. Pullback of parabolic bundles. Let \( E_* = (E, \{ E_{i,j} \}, \{ \alpha_{i,j} \}) \) be a parabolic vector bundle with parabolic divisor \( D = \{ x_1, \cdots, x_n \} \). Take a nonconstant holomorphic map
\[ f : Y \rightarrow X \]
from an irreducible complex projective curve \( Y \). Then, using \( f \), the parabolic bundle \( E_* \) pulls back to a parabolic bundle \( f^*E_* \) on \( Y \). We will briefly recall the construction of the parabolic bundle \( f^*E_* \).

We first consider the case where \( \text{rank}(E) = 1 \). So for each \( x_i \in D \) the parabolic weight of \( E_* \) is \( \alpha_{i,1} = \alpha_i \). The parabolic divisor for \( f^*E_* \) is the reduced effective divisor \( f^{-1}(D)_{\text{red}} \). For \( 1 \leq i \leq n \), let
\[ f^{-1}(x_i) = \{ y_{i,1}, \cdots, y_{i,b_i} \} \subset Y \]
be the inverse image, and let \( m_{i,j} \) be the multiplicity of \( f \) at \( y_{i,j} \) for every \( 1 \leq j \leq b_i \). For any \( \lambda \in \mathbb{Q} \), let \( \lfloor \lambda \rfloor \) be the integral part of \( \lambda \), so \( 0 \leq \lambda - \lfloor \lambda \rfloor < 1 \).

The holomorphic line bundle on \( Y \) underlying the parabolic line bundle \( f^*E_* \) is
\[ F := (f^*E) \otimes \mathcal{O}_Y (\sum_{i=1}^n \sum_{j=1}^{b_i} [m_{i,j} \alpha_i] \cdot y_{i,j}), \]
and the parabolic weight of \( F_{y_{i,j}} \) is \( m_{i,j} \alpha_i - [m_{i,j} \alpha_i] \). Note that
\[ \text{par-deg}(f^*E_*) = \text{degree}(f) \cdot \text{par-deg}(E_*). \tag{4.5} \]

Any parabolic vector bundle \( E_* \) can locally be expressed as a direct sum of parabolic line bundles. In other words, \( X \) can be covered by Zariski open subsets \( U_1, \cdots, U_m \) such that \( E_*|_{U_j} \) is a direct sum of parabolic line bundles on \( U_j \) for all \( 1 \leq j \leq m \). We have described above the pullback of parabolic line bundles. The pullback of a direct sum of parabolic line bundles is the direct sum of the pulled back parabolic line bundles. Using the decomposition of \( E_*|_{U_j} \) into a direct sum of parabolic line bundles we now have a description of the parabolic pullback \( f^*E_* \). From (4.5) it follows that
\[ \text{par-deg}(f^*E_*) = \text{degree}(f) \cdot \text{par-deg}(E_*), \tag{4.6} \]
for any parabolic vector bundle \( E_* \).

For each point \( x \in D \) fix an integer \( N_x \geq 1 \).
Theorem 4.2. Take any stable parabolic vector bundle \( E_* \) on \( X \) with parabolic structure over \( D \) such that all the parabolic weights of \( E_* \) at each \( x \in D \) are integral multiples of \( \frac{1}{N_x} \). If the rank of the holomorphic vector bundle \( \mathcal{F} \to X \) in Proposition 3.2 is one, then the parabolic vector bundle \( f^*E_* \) on \( Y \) is also stable.

Proof. Assume that
\[
\text{rank}(\mathcal{F}) = 1. \tag{4.7}
\]

Let \( E_* \) be a stable parabolic vector bundle of rank \( r \) on \( X \) with parabolic structure over \( D \) such that all the parabolic weights of \( E_* \) at each \( x \in D \) are integral multiples of \( \frac{1}{N_x} \). Consider the parabolic principal \( \text{PGL}(r, \mathbb{C}) \)-bundle \( \mathbb{P}(E_*) \) defined by \( E_* \); see [BBN] for parabolic principal bundles. Since \( E_* \) is stable, we know that the parabolic principal \( \text{PGL}(r, \mathbb{C}) \)-bundle \( \mathbb{P}(E_*) \) is given by an irreducible homomorphism
\[
\rho : \pi_1^t(X) \to \text{PU}(r) \tag{4.8}
\]
[MS], [Biq]. Let \( \mathbb{P}(f^*E_*) \) denote the parabolic principal \( \text{PGL}(r, \mathbb{C}) \)-bundle on \( Y \) defined by the parabolic vector bundle \( f^*E_* \). Since \( \mathbb{P}(E_*) \) is given by the homomorphism \( \rho \) in (4.8), we conclude that the parabolic principal \( \text{PGL}(r, \mathbb{C}) \)-bundle \( \mathbb{P}(f^*E_*) \) is given by the homomorphism
\[
\rho \circ f'_*: \pi_1^t(Y') \to \text{PU}(r), \tag{4.9}
\]
where \( f'_* \) is the homomorphism in (4.1).

From (4.7) and Proposition 4.1 we know that the homomorphism \( f'_* \) in (4.9) is surjective. Therefore, from the property of the homomorphism \( \rho \) in (4.8) that it is irreducible we conclude that the homomorphism \( \rho \circ f'_* \) in (4.9) is also irreducible. Since the parabolic principal \( \text{PGL}(r, \mathbb{C}) \)-bundle \( \mathbb{P}(f^*E_*) \) is given by the irreducible projective unitary representation \( \rho \circ f'_* \) in (4.9), we now conclude that the parabolic vector bundle \( f^*E_* \) is stable [MS], [Biq].

The following lemma is a converse of Theorem 4.2.

Lemma 4.3. Assume that the rank of the holomorphic vector bundle \( \mathcal{F} \to X \) in Proposition 3.2 is at least two. Then there is a stable parabolic vector bundle \( E_* \) on \( X \) with parabolic structure over \( D \) such that

1. all the parabolic weights of \( E_* \) at each \( x \in D \) are integral multiples of \( \frac{1}{N_x} \), and
2. the parabolic vector bundle \( f^*E_* \) on \( Y \) is not stable.

Proof. Since \( \text{rank}(\mathcal{F}) > 1 \), from Proposition 4.1 we know that the homomorphism of topological fundamental groups
\[
f'_*: \pi_1^t(Y') \to \pi_1^t(X)
\]
(see (4.1)) induced by \( f' \) in (3.13) is not surjective. Fix an irreducible representation
\[
\rho : \pi_1^t(X) \to U(r),
\]
for some \( r \geq 2 \), such that the composition of homomorphisms
\[
\rho \circ f'_*: \pi_1^t(Y') \to U(r)
\]
is not irreducible; such a $\rho$ exists because $f'_*\rho$ is not surjective.

Let $E_*$ be the parabolic vector bundle of rank $r$ on $X$, with parabolic structure over $D$, given by $\rho$. We note that

1. all the parabolic weights of $E_*$ at each $x \in D$ are integral multiples of $\frac{1}{N_x}$, and
2. the parabolic vector bundle $E_*$ is stable, because $\rho$ is irreducible [MS].

Since $\rho \circ f'_* : \pi_1^t(Y') \longrightarrow U(r)$ is not irreducible, it follows that the parabolic vector bundle $f^*E_*$ on $Y$ is not stable. \hfill \Box

5. Algebraically closed fields of characteristic zero

Now let $k$ be any algebraically closed fields of characteristic zero. As in Section 3, $X$ is an irreducible smooth projective curve defined over $k$, and $D = \{x_1, \ldots, x_n\} \subset X$ is a finite subset.

For each point $x \in D$ fix an integer $N_x \geq 1$. Consider the holomorphic vector bundle $F \longrightarrow X$ in Proposition 3.2.

Theorem 5.1. Let

$$f : Y \longrightarrow X$$

be a nonconstant map from an irreducible smooth projective curve $Y$. Take any stable parabolic vector bundle $E_*$ on $X$ with parabolic structure over $D$ such that all the parabolic weights of $E_*$ at each point $x \in D$ are integral multiples of $\frac{1}{N_x}$. If the rank of the vector bundle $F \longrightarrow X$ in Proposition 3.2 is one, then the parabolic vector bundle $f^*E_*$ on $Y$ is also stable.

Proof. Assume that

$$\text{rank}(F) = 1. \quad (5.1)$$

Take any stable parabolic vector bundle $E_*$ on $X$ with parabolic structure over $D$ such that all the parabolic weights of $E_*$ at each $x \in D$ are integral multiples of $\frac{1}{N_x}$. We need to show that the parabolic vector bundle $f^*E_*$ on $Y$ is stable.

Let $k_0 \subset k$ be an algebraically closed field of characteristic of finite transcende nce degree over $\mathbb{Q}$ such that $X, Y, D, f$ and $E_*$ are defined over $k_0$. Fix an embedding of $k_0$ in $\mathbb{C}$. Let

$$X_\mathbb{C} := X \times_{k_0} \mathbb{C}, \quad Y_\mathbb{C} := Y \times_{k_0} \mathbb{C}, \quad f_\mathbb{C} := f \times_{k_0} \mathbb{C} \quad \text{and} \quad E^\mathbb{C}_* := E_* \otimes_{k_0} \mathbb{C} \quad (5.2)$$

be the base changes to $\mathbb{C}$ of $X, Y, f$ and $E_*$ respectively. We need to show that the parabolic vector bundle $f^*E_*$ on $Y$ is stable.

Let $\mathcal{X}_\mathbb{C} := \mathcal{X} \times_{k_0} \mathbb{C}, \quad \mathcal{Y}'_\mathbb{C} := Y' \times_{k_0} \mathbb{C}$ and $f'_\mathbb{C} := f' \times_{k_0} \mathbb{C}$

where $Y'$ and $f'$ are as in (3.13), be the base changes to $\mathbb{C}$ of $\mathcal{X}, \mathcal{Y}'$ and $f'$ respectively.

We need the following lemma.

Lemma 5.2. The parabolic vector bundle $E^\mathbb{C}_* = E_* \otimes_{k_0} \mathbb{C}$ in (5.2) is stable.
Proof of Lemma 5.2. An equivariant vector bundle is equivariantly semistable if the underlying vector bundle is semistable, because the Harder–Narasimhan filtration of an equivariant bundle is preserved by the action of the group. It is known that the property of semistability of a vector bundle is preserved under field extensions (see [HL, p. 18, Corollary 1.3.8]). Now using the correspondence between the parabolic bundles and the equivariant bundles we conclude that for a semistable parabolic bundle $V_*$ on $X$ the parabolic bundle $V_* \times_{k_0} \mathbb{C}$ on $X_{\mathbb{C}}$ is also semistable. Therefore, the given condition that the parabolic bundle $E_*$ is semistable implies that the parabolic bundle $E_*^\mathbb{C}$ is also semistable. Since the unique maximal polystable parabolic subbundle of the semistable parabolic bundle $E_*^\mathbb{C}$ (it is also known as the socle of $E_*^\mathbb{C}$ (see [HL, p. 23, Lemma 1.5.5]) is defined over $k_0$, and $E_*$ is polystable, we conclude that the parabolic bundle $E_*^\mathbb{C}$ is polystable (see [HL, p. 24, Corollary 1.5.11]).

For a parabolic vector bundle $F_*$, the sheaf of quasiparabolic structure preserving endomorphisms of the underlying vector bundle $F$ will be denoted by $\text{End}_P(F_*)$. A polystable parabolic vector bundle $F_*$ is stable if and only if the space of global sections of $\text{End}_P(F_*)$ is the base field. Since

$$\text{End}_P(E_*^\mathbb{C}) = \text{End}_P(E_*) \otimes_{k_0} \mathbb{C},$$

and $E_*$ is stable, we have

$$H^0(X_{\mathbb{C}}, \text{End}_P(E_*^\mathbb{C})) = H^0(X, \text{End}_P(E_*)) \otimes_{k_0} \mathbb{C} = \mathbb{C}.$$  

This implies that the polystable parabolic bundle $E_*^\mathbb{C}$ is stable. □

Continuing with the proof of Theorem 5.1 from Corollary 3.4 and (5.1) we know that the homomorphism of étale fundamental groups $(f'_\mathbb{C})_*$ in (3.14) is surjective. This implies that the homomorphism of étale fundamental groups

$$(f'_\mathbb{C})_* : \pi_1(Y'_{\mathbb{C}}) \longrightarrow \pi_1(X_{\mathbb{C}})$$

induced by $f'_\mathbb{C}$ in (5.3) is surjective; both $Y'_{\mathbb{C}}$ and $X$ are defined in (5.3). Hence from Theorem 4.2 and Lemma 5.2 we conclude that the parabolic vector bundle $f'_\mathbb{C}E_*^\mathbb{C}$ on $Y_{\mathbb{C}}$ is stable, where $f_\mathbb{C}$ is the map in (5.2). This implies that the parabolic vector bundle $f^*E$ is stable. □

Remark 5.3. We note that the main result of [BKP] implies that if the base field is algebraically closed of characteristic 0 then the conclusion to Theorem 5.1 holds under the strict condition that for every $x \in X$ the number $N_x$ is coprime to the ramification indices of $f$ at points above $x$. Theorem 5.1 is more general than this as illustrated by Example 5.4.

Example 5.4. Let $X = Y = \mathbb{P}^1_\mathbb{C}$ and $f : Y \longrightarrow X$ be the cyclic covering of degree 6 ramified at 0 and $\infty$. Let $N_0 = 2$ and $N_\infty = 3$; then the map $\pi_1(Y \setminus \{0, \infty\}) \longrightarrow \pi_1(X)$ is surjective, hence the rank of $\mathcal{F}$ is one so Theorem 5.1 applies. Though the hypothesis of [BKP] Theorem 5.1 does not hold in this example.
Acknowledgements

We are very grateful to the referee for the question on the proof of Proposition 4.1. We thank Mahan Mj for pointing out [Sc] and [Pat]. The first-named author thanks Université Côte d’Azur for hospitality while parts of the work were carried out. He is partially supported by a J. C. Bose Fellowship.

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