REGULARITY OF FREE BOUNDARY
FOR THE MONGE-AMPERÈ OBSTACLE PROBLEM

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Abstract. In this paper, we prove the regularity of the free boundary in the Monge-Ampère obstacle problem \( \det D^2 v = f(y) \chi_{\{v>0\}} \). By duality, the regularity of the free boundary is equivalent to that of the asymptotic cone of the solution to the singular Monge-Ampère equation \( \det D^2 u = 1/f(Du) + \delta_0 \) at the origin. We first establish an asymptotic estimate for the solution \( u \) near the singular point, then use a partial Legendre transform to change the Monge-Ampère equation to a singular, fully nonlinear elliptic equation, and establish the regularity of solutions to the singular elliptic equation.

1. Introduction

In this paper we study the Monge-Ampère obstacle problem

\[
\det D^2 v = f \chi_{\{v>0\}} \quad \text{in } \Omega, \\
v = v_0 \quad \text{on } \partial \Omega,
\]

where \( \Omega \) is a bounded domain in the Euclidean space \( \mathbb{R}^n \), \( f, v_0 \) are positive functions on \( \Omega \), and \( \chi \) is the characteristic function. Denote \( \Gamma := \partial \{ v = 0 \} \) the free boundary. The main objective of the paper is to prove the regularity of the free boundary \( \Gamma \).

Problem (1.1) is the Monge-Ampère counterpart of the classical free boundary problem

\[
\Delta v = f \chi_{\{v>0\}} \quad \text{in } \Omega, \\
v = v_0 \quad \text{on } \partial \Omega.
\]

A central issue for the prototypal obstacle problem (1.2) is the regularity of the free boundary \( \Gamma = \partial \{ v = 0 \} \). In a seminal work [3], Caffarelli proved that the free boundary is \( C^1 \) smooth at regular points; and hence is \( C^\infty \) smooth and analytic [28]. Since then the regularity of free boundary problems has been extensively studied.

Regularity of free boundary problems associated with the Monge-Ampère equation has also been studied in a number of papers [7, 9, 16, 17, 30, 31, 35]. See also [10, 11, 19] for...
a related free boundary problem with the Gauss curvature flow. In [35], Savin studied problem (1.1) and proved that the free boundary $\Gamma$ is uniformly convex and $C^{1,1}$ smooth. He also pointed out two other interpretations of the obstacle problem (1.1), as a model in the optimal transportation with a Dirac measure and in the Monge-Ampère equation with a cone singularity. In dimensions two, Galvez, Jiménez and Mira [16] proved that the free boundary $\Gamma$ is $C^\infty$ smooth and analytic, if $f$ is respectively smooth and analytic. As pointed out in [16], some of the arguments in [16] are specific of the two-dimensional case, since they rely on complex analysis and surface theory. Also in dimension two, Daskalopoulos and Lee [11] obtained the regularity of the free boundary in the Gauss curvature flow. For fully nonlinear, uniformly elliptic equations, the corresponding results can be found in [29, 38, 39]. An open problem is the regularity of the free boundary problem (1.1) in high dimensions. In this paper we resolve the problem completely.

**Theorem 1.1.** Let $v$ be a generalized solution to the obstacle problem (1.1), in the sense of Aleksandrov. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$, and $f, v_0 > 0$. Then the free boundary $\Gamma$ is smooth if $f$ is smooth; and $\Gamma$ is analytic if $f$ is analytic.

Monge-Ampère type equations are different from uniformly elliptic equations in many ways, and the techniques for uniformly elliptic equations [3, 6, 29, 38, 39] do not apply to the Monge-Ampère obstacle problem (1.1). A key property for uniformly elliptic equations is the growth estimate $\sup_{B_r(0)} v \approx r^2$ (assuming that 0 is a point on the free boundary). By this property, the blow-up profile at 0 is either of the form $\frac{1}{2} \max\{x \cdot e, 0\}^2$ for a vector $e$, or of the form $\frac{1}{2} x \cdot Ax$ for a matrix $A$. In the former case, 0 is a regular point and the free boundary is smooth at the point. But for the Monge-Ampère equation (1.1), we have instead the estimate $\sup_{B_r(0)} v \approx r^{1+\frac{1}{n}}$. It implies that equation (1.1) is both singular and degenerate near the free boundary. Therefore to prove the regularity of the free boundary, one needs a completely different approach.

Our new approach to the problem involves a series of transforms. We first make the Legendre transform (1.3) to obtain the equation (1.4) with point singularity. By estimate (1.9) we then make the change (1.8) to get equation (1.10) in the polar coordinates. Regularity of the free boundary $\Gamma$ is thus reduced to that for equation (1.10). To establish the a priori estimate for (1.10), we again make a partial Legendre transform (1.13) to obtain a fully nonlinear, singular elliptic equation (1.14). The linearized equation of (1.14) is a degenerate equation of Keldysh type after a change of variables. By the Hölder regularity of the linearized equation of (1.14), we can prove the blow-up limits for equation (1.14) is a quadratic function (Theorem 1.3). Technically the most difficult part in the
approach is to prove the uniqueness of the blow-up limits, namely all blow-up sequences around a point converge to the same limit, from which the \( C^2 \) regularity of solutions follows. We will prove the uniqueness of the blow-up limits by consecutively using the maximum principle in an infinite sequence of domains. See also Remark 4.4.

The Monge-Ampère equation has a very useful property, namely the duality. Let \( v \) be the solution to the obstacle problem (1.1), as in Theorem 1.1. Then \( v \) is convex and the set \( \{v = 0\} \) is a convex sub-set of \( \Omega \). We may assume that the set \( \{v = 0\} \) has positive measure, otherwise the free boundary does not exist [35]. Let \( u \) be the Legendre transform of \( v \), given by

\[
(1.3) \quad u(x) = \sup\{x \cdot y - v(y) : y \in \Omega\}, \quad x \in \Omega^* =: D_y v(\Omega).
\]

Then \( u \) is a generalized solution to the following Monge-Ampère equation with point singularity,

\[
(1.4) \quad \det D^2 u = g(Du) + c^* \delta_0 \quad \text{in} \quad \Omega^*,
\]

where \( g(Du(x)) = \frac{1}{f(y)} \) at \( y = Du(x) \), and \( c^* = |\{v = 0\}| \) is a constant. There is no loss of generality in assuming that \( c^* = 1 \). By a translation of the coordinates we assume that the origin is an interior point of the convex set \( \{v = 0\} \). Then \( u(0) = 0 \) and \( u(y) > 0 \ \forall \ y \neq 0 \).

Let \( \phi \) be the tangential cone of \( u \) at 0, namely it is a homogeneous function of degree one defined in \( \mathbb{R}^n \) and satisfying

\[
(1.5) \quad u(x) \geq \phi(x) \quad \forall \ x \in \Omega^*,
\]

\[
(1.5) \quad u(x) - \phi(x) = o(|x|) \quad \text{as} \quad x \to 0,
\]

such that \( \partial \phi\{0\} = \partial u\{0\} \). Given a convex function \( w \), we denote by \( \partial w\{p\} \) the subdifferential of \( w \) at \( p \),

\[
\partial w\{p\} = \{\xi \in \mathbb{R}^n : w(x) \geq \xi \cdot (x - p) + w(p) \ \forall \ x \ \text{near} \ p\}.
\]

By duality,

\[
\partial \phi\{0\} = \{v = 0\}.
\]

Hence the section \( S_{1,\phi} =: \{x \in \mathbb{R}^n : \phi(x) < 1\} \) is the polar body of \( \{v = 0\} \), i.e.,

\[
(1.6) \quad S_{1,\phi} = \{x \in \mathbb{R}^n : x \cdot y < 1 \ \forall \ y \in \{v = 0\}\}.
\]

Denote \( L = \partial S_{1,\phi} \). Hence \( L \) is \( C^k \) smooth (\( k \geq 2 \)) and uniformly convex if and only if the free boundary \( \Gamma \) is. It was proved in [35] that \( \Gamma \) is uniformly convex and \( C^{1,1} \) smooth for \( f \equiv 1 \). In the two dimensional case, it was proved in [16] that the curve \( L \) is smooth and analytic. Moreover, if \( u \) is an entire solution to (1.4) with \( g \equiv 1 \), namely \( u \) is defined in
the whole space \( \mathbb{R}^n \), then \( u \) must be rotationally symmetric after an affine transform of coordinates \(^{25}\)[26]. In this paper, we prove

**Theorem 1.2.** Let \( u \) be a strictly convex solution of \(^{(1.4)}\). Then in the spherical coordinates \((\theta,r)\), \( u \) is smooth as a function of \( \theta \) and \( r \) if \( f \) is smooth, and we have the Taylor expansion

\[
(1.7) \quad u(\theta,r) = r^k \sum_{i=0}^{k} \phi_i(\theta)r^i + O(r^{1+k+1})
\]

for any integer \( k \geq 0 \), where \( \phi_i \) are smooth functions of \( \theta \). Moreover, if \( f \) is analytic, then \( u \) is analytic in \( \theta \).

By the condition \( v_0 > 0 \) on \( \partial \Omega \), we know that \( u \) is strictly convex in \( \Omega^* \) \(^{[5]}\). Theorem 1.1 follows from Theorem 1.2 by the duality between the free boundary \( \Gamma \) and the section \( L \). The Taylor expansion \(^{(1.7)}\) reveals a special geometric profile of the solution \( u \) at the origin, namely \( \frac{u(\theta,r)}{r} \) is a smooth function of \( \theta \) and \( r^n \).

To prove Theorem 1.2, our goal is to prove that \( u/r \), as a function of \( \theta, r \), is smooth. Therefore we will make the change

\[
(1.8) \quad \zeta(\theta,r) = \frac{u(\theta,r)}{r}, \quad s = r^\frac{n}{2}.
\]

The proof of Theorems 1.2 can be divided into three steps.

**Step 1:** We first prove that the function \( w = u - \phi \) satisfies the growth condition near 0,

\[
(1.9) \quad C_1|x|^{n+1} \leq w(x) \leq C_2|x|^{n+1}
\]

for two positive constants \( C_2 \geq C_1 > 0 \). Estimates \(^{(1.9)}\) are built on Pogorelov and Savin’s interior second derivative estimates. \(^{(1.9)}\) is a key estimate in this paper. It implies that equation \(^{(1.10)}\) below is uniformly elliptic.

**Step 2:** We next prove that the free boundary is \( C^2 \) smooth. We express the solution \( u \) in the spherical coordinates \((\theta,r)\), and make the changes \(^{(1.8)}\). Then in an orthonormal frame \( \theta \) on the unit sphere \( S^{n-1} \), \( \zeta \) satisfies the equation

\[
(1.10) \quad \text{det} \begin{pmatrix}
(\frac{n}{2})^2 \zeta_{ss} + \frac{n(n+2)}{4} \frac{\zeta_s}{s} & \frac{n}{2} \zeta_{s\theta_1} & \cdots & \frac{n}{2} \zeta_{s\theta_{n-1}} \\
\frac{n}{2} \zeta_{s\theta_1} & \zeta_{\theta_1\theta_1} + \zeta + \frac{n}{2} s \zeta_s & \cdots & \zeta_{\theta_1\theta_{n-1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{n}{2} \zeta_{s\theta_{n-1}} & \zeta_{\theta_{n-1}\theta_{n-1}} & \cdots & \zeta_{\theta_{n-1}\theta_{n-1}} + \zeta + \frac{n}{2} s \zeta_s
\end{pmatrix} = g.
\]

This is a fully nonlinear singular elliptic equation. By \(^{(1.9)}\) we have

\[
(1.11) \quad C_1 s^2 \leq \zeta(\theta,s) - \frac{\phi}{r} \leq C_2 s^2.
\]
From (1.11), we deduce that equation (1.10) is uniformly elliptic. Moreover, the regularity of the free boundary is equivalent to that of the function $\zeta(\theta, 0)$.

Therefore the main task of the paper is to establish the regularity, at the boundary $\{s = 0\}$, for the fully nonlinear, singular elliptic equation (1.10). A key step is to obtain the $C^2$ regularity of the solution $\zeta$. Recall that for the classical obstacle problem (1.2), the key estimate is the $C^1$ regularity, proved in [3].

The proof of the $C^2$ regularity of $\zeta$ will be carried out as follows. We first use a blow-up argument to simplify equation (1.10) to the following equation

$$\det \begin{pmatrix} \psi_{x_n x_n} + \frac{n+2}{n} \frac{\psi_{x_n}}{x_n} & \psi_{x_n x_1} & \cdots & \psi_{x_n x_{n-1}} \\ \psi_{x_1 x_n} & \psi_{x_1 x_1} & \cdots & \psi_{x_1 x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{x_{n-1} x_n} & \psi_{x_{n-1} x_{n-1}} & \cdots & \psi_{x_{n-1} x_{n-1}} \end{pmatrix} = 1 \quad \text{in} \quad \mathbb{R}^n_+,$$

where $\mathbb{R}^n_+ = \mathbb{R}^n \cap \{x_n > 0\}$. The RHS of (1.12) is a positive constant, which we assume is one.

We then make a partial Legendre transform for $\psi$ [32], i.e.,

$$y_n = x_n,$$

(1.13)

$$y' = D_{x'} \psi,$$

$$\psi^* = x' \cdot D_{x'} \psi - \psi,$$

where $x' = (x_1, \cdots, x_{n-1})$. Then $\psi^*$ satisfies

$$\psi^{**}_{y_n y_n} + \frac{n+2}{n} \frac{\psi^{**}_{y_n}}{y_n} + \det D^2_{y'} \psi^* = 0 \quad \text{in} \quad \mathbb{R}^n_+.$$

A nice feature of equation (1.14) is that the singular part $\psi^*_{y_n y_n}$ is separate from the nonlinear part of the equation, which enables us to prove the $C^{2,\alpha}$ regularity for equation (1.14). Hence by a scaling argument, we obtain the following Bernstein theorem.

**Theorem 1.3.** Let $\psi \in C^{1,1}(\mathbb{R}^n_+)$ be a solution to (1.12). Assume that $\psi_{x_n}(x', 0) = 0 \ \forall \ x' \in \mathbb{R}^{n-1}$, and equation (1.12) is uniformly elliptic. Then

$$\psi(x) = q(x') + ax_n^2$$

(1.15)

where $a$ is a positive constant and $q$ is a quadratic polynomial.

To prove the $C^2$-continuity of $\zeta(\theta, s)$ at $\mathbb{S}^{n-1} \times \{s = 0\}$, we use a blow-up argument. By Theorem 1.3, the limit of a blow-up sequence is a quadratic polynomial of the form (1.15). The most delicate part of the paper is to prove the uniqueness of the limit, namely the limit is independent of the choice of the blow-up sequences. The uniqueness of the
limit implies that the second derivatives $D^2 \zeta$ are continuous. Our proof of the uniqueness is by employing the maximum principle consecutively in an infinite sequence of domains.

**Step 3:** We differentiate equation (1.10) to obtain a linearized equation, of which a prototype is of the form

\[(1.16) \quad \Delta_x u + u_{nn} + b\frac{u_n}{x_n} = f \quad \text{in} \quad \mathbb{R}^n_+,
\]

where $b > 1$, $u_n = u_{x_n}$. Equation (1.16) is a singular elliptic equation of Keldysh type. In [22, 23], Horiuchi introduced the Green function for (1.16) and proved the $C^{2,\alpha}$ estimate for (1.16). In this paper, we use his Green function to establish a weighted $W^{2,p}$ estimate for equation (1.16), extending the classical $W^{2,p}$ estimate for the Poisson equation.

We then use the freezing coefficient method and the weighted $W^{2,p}$ estimate to obtain the $C^{2,\alpha}$ regularity of $\zeta$ in $\theta$. By the weighted $W^{2,p}$ estimate and the bootstrap technique, we show that the solution $\zeta \in C^\infty$ up to the boundary.

Finally we prove that the free boundary is analytic. The analyticity of solutions has been extensively studied in literature [15, 33, 28]. A simple proof was found in [27, 2]. To prove the analyticity of our free boundary, we need to prove the analyticity of $\zeta$ in $\theta$. We will adopt the method in [27, 2].

The paper is organized as follows. In Section 2 we establish the asymptotic estimate (1.9), which is the first step of our proof. The second step of the proof consists of Sections 3 and 4. In Section 3 we prove the Hölder continuity for the singular term $\frac{\psi_n}{y_n}$ in equation (1.14), from which we obtain the Bernstein Theorem (1.3). In Section 4, we use the Bernstein Theorem and construct auxiliary functions to prove the $C^2$-regularity of the free boundary. Step 3 of the proof consists of Sections 5 and 6. In Section 5 we establish a weighted $W^{2,p}$-estimate for (1.16), and use it to prove the $C^\infty$ regularity of the free boundary. The analyticity of the free boundary will be proved in Section 6.

2. Asymptotic behaviour at the singularity

Let $u$ be a generalized solution to

\[(2.1) \quad \det D^2 u = g(Du) + \delta_0 \quad \text{in} \quad B_1(0).
\]

Assume that $u(0) = 0$, $u \geq 0$ and $u > 0$ on $\partial B_1(0)$. Assume also that $g$ is a smooth and positive function. There is no loss of generality in assuming that the unit ball $B_1(0) \subset \Omega^*$. By extending $v$ to $\mathbb{R}^n$ such that $v$ is smooth and uniformly convex away from the free boundary $\Gamma$, we may also assume that $\Gamma \subset B_1(0) \subset \Omega$. Therefore in the following we
will consider problems (1.1) and (1.4) in $B_1(0)$. By the regularity of the Monge-Ampère equation, we may also assume that $|D^4u| \leq M_0, |D^4v| \leq M_0$ near $\partial B_1(0)$. In the following we will use $\Omega$ to denote a general bounded convex domain.

Let
\begin{equation}
(2.2)
  u = w + \phi
\end{equation}

where $\phi$ is the tangent cone of $u$ at $0$, defined in (1.5). Then
\begin{equation}
(2.3)
  w(x) \geq 0, \quad w(x) = o(|x|) \quad \text{as} \quad x \to 0.
\end{equation}

Let $\Gamma$ be the free boundary of the obstacle problem (1.1). In [35], Savin proved that $\Gamma$ is uniformly convex and $C^{1,1}$ smooth for the case $f \equiv 1$. In this section, we show that his argument also applies to general function $f(x)$ provided $f(x)$ is positive and smooth, and obtain the estimate (1.9).

Lemma 2.1. (Pogorelov’s estimate) Let $\psi \in C^4(\Omega)$ be a convex solution to
\begin{equation}
(2.4)
  \det D^2\psi = h(x, \psi, D\psi) \quad \text{in} \quad \Omega,
  \quad \psi = 0 \quad \text{on} \quad \partial \Omega.
\end{equation}

Then there exists a constant $C > 0$, depending only on $n, \sup_{\Omega}(|\psi| + |D\psi|)$, and $\|\log h\|_{C^2}$, such that
\begin{equation}
(2.5)
  [-\psi(x)] |D^2\psi(x)| \leq C \quad \forall \quad x \in \Omega.
\end{equation}

The proof for Pogorelov’s estimate can be found in several papers. See for instance [18, 34]. Here we omit the proof.

Corollary 2.1. Let $u$ be the strictly convex solution to (2.1). We have the estimate
\begin{equation}
(2.6)
  |x||D^2u(x)| \leq C \quad \forall \quad x \in B_1 \setminus \{0\}.
\end{equation}

where $C$ depends only on $n, M_0, \sup_{B_1}(|u| + |Du|)$, and $\|\log g\|_{C^2}$.

Proof. Consider a point $x_0 \in B_1(0) \setminus \{0\}$ near the origin. Choose the coordinates such that $x_0 = |x_0|e_n$, where $e_n = (0, \cdots, 0, 1)$. Subtracting a linear function, one can assume
\[ \phi(te_n) = 0 \quad \text{for} \quad t > 0, \quad \text{and} \quad \phi(x) \geq 0 \quad \text{for} \quad x \in B_1. \]

Denote $\Omega_{\varepsilon_0} = \{ x \in B_1 \mid u(x) < \varepsilon_0 x_n \}$, for some small but fixed $\varepsilon_0 > 0$. Applying Lemma 2.1 to $u - \varepsilon_0 x_n$ in $\Omega_{\varepsilon_0}$, we obtain
\[ (\varepsilon_0|x_0| - u(x_0))|D^2u(x_0)| \leq C \quad \forall \quad x_0 \in \Omega_{\varepsilon_0}. \]

Note that $u(x_0) = o(|x_0|)$. We obtain (2.6). \qed
Corollary 2.2. Let \( \phi \) be the tangential convex cone of \( u \) at 0. Then there holds

\[
|\mathbf{x}| |D^2 \phi(x)| \leq C \quad \forall \mathbf{x} \neq 0 \in \mathbb{R}^n,
\]

for the same constant \( C \) in (2.4).

Proof. Consider a point \( x_0 \in B_1(0) \setminus \{0\} \) near the origin. Denote \( t = |x_0| \in (0,1) \). Make the dilation \( X = \mathbf{x}/t \) and \( U_t(X) = u(x)/t \), and choose the coordinates such that \( X_0 =: x_0/t = e_n \). Then \( \phi \) is also the tangential convex cone of \( U_t \) at 0. By Corollary 2.1, \( U_t \) satisfies

\[
|X||D^2 U_t(X)| = |\mathbf{x}||D^2 u(x)| \leq C \quad \forall \mathbf{x} \neq 0.
\]

Since \( U_t(X) \to \phi(X) \) locally uniformly in \( \mathbb{R}^n \) as \( t \to 0 \), we obtain (2.7). \( \square \)

Estimates (2.6), (2.7) are due to Savin [35]. By duality, (2.7) implies that the free boundary \( \Gamma \) is strictly convex. In fact, for any point \( p \in \Gamma \), we can choose the coordinates such that \( p = 0, \Gamma \subset \{x_n \geq 0\} \), and locally \( \Gamma \) is given by \( x_n = \eta(x') \). Then (2.7) implies that \( \eta(x') \geq |x'|^2/C \). By (2.6) and (2.7), we also have

\[
|x||D^2 w(x)| \leq C \quad \forall \mathbf{x} \in B_1 \setminus \{0\}.
\]

where \( w \) is the function given in (2.2).

Lemma 2.2. (Savin’s estimate) Consider the Monge-Ampère obstacle problem

\[
det D^2 v = f(x,v)\chi_{\{v>0\}} \quad \text{in } \Omega,
\]

\[
v = v_0 > 0 \quad \text{on } \partial \Omega.
\]

Assume that \( f \) is non-decreasing in \( v \) and \( 0 < \lambda \leq f \leq \Lambda < +\infty \). Then there exists a constant \( C \), depending only on \( n, \lambda, \Lambda, \|D_x \log f\|_{L^\infty}, \|D_x^2 \log f\|_{L^\infty} \) and the strict convexity of \( \Gamma \), such that

\[
\kappa_i \leq C \quad \forall \, i = 1, \ldots, n-1,
\]

where \( (\kappa_1, \ldots, \kappa_{n-1}) \) are the principal curvatures of \( \partial \{v = h\} \), for \( h > 0 \) small.

Proof. The following proof is due to Savin’s [35], where he considered the case \( f = f(v) \). His proof also applies to the case \( f = f(x,v) \), so we will just sketch the proof.

1). Let \( p_0 \) be a point on the free boundary \( \Gamma \). We choose the coordinates such that \( x_n \) is the outer normal of \( \{v = 0\} \) at \( p_0 \), and express the graph of \( v \) by a function \( \mathbf{v} \) in the form \( x_n = -\mathbf{v}(x_1, \ldots, x_{n-1}, x_{n+1}) \). Then

\[
\det D^2 \mathbf{v} \left(1 + |D \mathbf{v}|^2\right)^{(n+2)/2} = K = \frac{\det D^2 v}{(1 + |Dv|^2)^{(n+2)/2}} = \frac{f(x,v)}{(1 + |Dv|^2)^{(n+2)/2}},
\]
where $K$ is the Gauss curvature of the graph of $v$. Hence $v$ satisfies the equation

$$\det D^2v = f(x,v)\left(1 + |Dv|^2\right)^{(n+2)/2} \left(1 + |Dv|^2\right)^{(n+2)/2}.$$  

From the relation $x_{n+1} = v(x_1, \cdots, x_{n-1}, -v(x_1, \cdots, x_{n-1}, x_{n+1}))$, we have

$$v_1 - v_n v_1 = 0, \quad \cdots, \quad v_{n-1} - v_n v_{n-1} = 0, \quad v_n v_{n+1} = -1.$$  

Hence $Dv = \left(\frac{-v_1}{v_{n+1}}, \cdots, \frac{-v_{n-1}}{v_{n+1}}, \frac{-1}{v_{n+1}}\right)$, and $\left(1 + |Dv|^2\right)^{(n+2)/2} = |v_{n+1}|^{n+2}$. The above equation becomes

$$(2.11) \quad \det D^2v = \tilde{f}(x,v)|v_{n+1}|^{n+2},$$

where $\tilde{f}(x,v) = f(x_1, \cdots, x_{n-1}, -v, x_{n+1})$. The variables of $v$ are $x_1, \cdots, x_{n-1}, x_{n+1}$.

By a translation of the coordinates, assume that $p_0 = ae_n$ for a small constant $a > 0$. By the strict convexity of $\Gamma$, there exists a small constant $\delta_0 > 0$ such that $v(x_1, \cdots, x_{n-1}, x_{n+1})$ is a graph in $(-2\delta_0, 2\delta_0)^{n-1} \times (0, 2\delta_0)$, $v_1$ is bounded in $(-\delta_0, \delta_0)^{n-1} \times (0, \delta_0)$, $v(0) < 0$, and $v > 0$ on $\partial((-\delta_0, \delta_0)^{n-1}) \times [0, \delta_0]$.

2). To prove (2.10), it suffices to prove that $v_{11}$ is bounded near $p_0$. The same argument applies to $v_{ii}$ for $2 \leq i \leq n - 1$. Let

$$G = -(v)\eta\left(\frac{1}{\sqrt{2}}v^2\right)v_{11}.$$  

be the auxiliary function of Pogorelov. Assume that $G$ attains its maximum at an interior point $x_0 \in \{v < 0\} \cap \{(-\delta_0, \delta_0)^{n-1} \times (0, \delta_0)\}$. We make the coordinate transform

$$(2.12) \quad y_1 = x_1 + \frac{v_{11}(x_0)}{v_{11}(x_0)}x_i, \quad y_i = x_i \ (i = 2, \cdots, n - 1), \quad \text{and} \quad y_n = x_{n+1},$$

such that $D^2v$ is diagonal at $x_0$. This coordinate transform does not change the value of $v_1, v_{11}$. Then at $x_0$, we have

$$(\log G)_i = 0 \quad \text{and} \quad \sum_{i=1}^{n}v^{ii}(\log G)_{ii} \leq 0.$$  

Carrying out the calculation in [35] and choosing $\eta(t) = e^{\alpha t}$ for some $\alpha > 0$ small, we obtain $G \leq C$.

3). In step 2), we assume that $x_0$ is an interior point. Next we use approximation [35] to rule out the case when $x_0 \in \{x_{n+1} = 0\}$ is a boundary point. Consider

$$\det D^2v = f_\varepsilon(x,v) \quad \text{in} \ \Omega,$$

$$v = v_0 > 0 \quad \text{on} \ \partial \Omega$$

$$f_\varepsilon(x,v) = (1 + |Dv|^2)^{-(n+2)/2}.$$
where \( f_\varepsilon(x,v) = a_\varepsilon(v)f(x,v) \). We choose smooth functions \( a_\varepsilon \) such that

\[
a_\varepsilon(t) = \begin{cases} 
  \varepsilon & \text{when } t < -\varepsilon, \\
  1 & \text{when } t > 0.
\end{cases}
\]

Let \( v_\varepsilon \) be the solution to (2.13). By assumption, \( f(x,v) \) is non-decreasing in \( v \), there is a unique solution to the obstacle problem. Hence \( v_\varepsilon \) converges to the unique solution \( v \) as \( \varepsilon \to 0 \).

Define \( v_\varepsilon \) similarly as above, such that \( x_{n+1} = v_\varepsilon(x_1, \ldots, x_{n-1}, -v_\varepsilon) \). We then apply the above argument to \( v_\varepsilon \) and obtain an upper bound for \( (v_\varepsilon)_{11} \) near the origin, and the upper bound is independent of \( \varepsilon \). Sending \( \varepsilon \to 0 \), we obtain (2.10).

**Corollary 2.3.** Let \( u \) be the solution to (2.1). We have the estimate

\[
|x| |\partial^2_x u(x)| \geq C_1 \quad \forall \ 0 \neq x \in B_1(0), \ |\xi| = 1
\]

where the constant \( C_1 \) depends only on \( n, M_0, \sup_{B_1} (|u| + |Du|) \), and \( \| \log g \|_{C^{1,1}} \).

**Proof.** Denote by \( \ell_x \) the tangent plane of \( u \) at \( x \). For any given point \( x_0 \neq 0 \) in \( B_1(0) \) near the origin, let \( h = \ell_{x_0}(0) \) \((h < 0)\) and \( \phi_h(x) \) be the convex cone given by

\[
\phi_h(x) = \sup \{ L(x) : L \text{ is affine function, } L(\cdot) < u(\cdot) \text{ in } B_1(0) \text{ and } L(0) = h \}.
\]

Let \( v \) be Legendre transform of \( u \), i.e. \( v = x \cdot Du - u \). Then \( v \) is a solution to (1.1). By the duality between \( u \) and \( v \), we have \( \partial \phi_h \{0\} = \{v \leq -h\} \). So (2.10) implies that

\[
|x| |\partial^2_x \phi_h(x)| \geq C_1 \quad \forall \ x \neq 0, \ |\xi| = 1.
\]

It is easy to see that (2.15) implies (2.14) at point \( x_0 \).

Sending \( h \to 0 \), from (2.15) we also obtain

**Corollary 2.4.** Let \( \phi \) be the tangential cone of \( u \) at 0. Then

\[
|x| |\partial^2_\xi \phi(x)| \geq C_1 \quad \forall \ 0 \neq x \in B_1(0), \ |\xi| = 1.
\]

We denote \( a \approx b \) if two quantities \( a \) and \( b \) are positive and there is a constant \( C \) under control such that \( \frac{a}{b} + \frac{b}{a} \leq C \). Given two convex domains \( A \) and \( B \), we denote \( A \sim B \) if \( C^{-1}(A - p) \subset B - q \subset C(A - p) \), where \( p, q \) are the geometric centres of \( A \) and \( B \), respectively.

**Corollary 2.5.** Let \( u \) be the solution to (2.1). Let \( \lambda_1(x) \geq \cdots \geq \lambda_n(x) \) be the eigenvalues of \( D^2 u \) at \( x \neq 0 \). Then \( \lambda_1(x) \approx \cdots \approx \lambda_{n-1}(x) \approx |x|^{-1} \) and \( \lambda_n(x) \approx |x|^{n-1} \).
Proof. By (2.6) and (2.14), we have $\lambda_1 \approx \cdots \approx \lambda_{n-1} \approx |x|^{-1}$. By virtue of the equation (2.1) we then have $\lambda_n \approx |x|^{-1}$.

□

Lemma 2.3. Let $w$ be the function given in (2.2). There exist two constants $C_1, C_2 > 0$, depending only on $n, M_0, \sup_{B_1}(|u| + |Du|)$, and $\|\log g\|_{C^{1,1}}$, such that

\begin{equation}
C_1|x|^{n+1} \leq w(x) \leq C_2|x|^{n+1} \quad \forall \ x \in B_1(0).
\end{equation}

Proof. For any given point $x_0 \in B_1(0) \{0\}$ near the origin, by a rotation of the coordinates we assume that $x_0$ is on the positive $x_1$-axis. Subtracting a linear function we assume

\begin{equation}
\phi(te_1) = 0 \quad \forall \ t \geq 0,
\end{equation}

\begin{equation}
\phi(x) \geq 0 \quad \forall \ x \in B_1(0),
\end{equation}

where $e_1 = (1, 0, \cdots, 0)$. To prove the first inequality of (2.17), it suffices to prove

$u(te_1) \geq Ct^{n+1}$ for $t > 0$ small.

By Corollary 2.5 we have

$u_{11}(se_1) \geq Cs^{n-1}.$

Hence

$u_1(te_1) = \int_0^t u_{11}(se_1)ds \geq Ct^n,$

and so we have

$u(te_1) = \int_0^t u_1(se_1)ds \geq Ct^{n+1}.$

Next we prove the second inequality of (2.17). Similarly as in Corollary 2.3, denote by $\ell_x$ the tangent plane of $u$ at $x$. For any given point $te_1, t > 0$, let $\phi_t(x)$ be the convex cone, given by

$\phi_t(x) = \sup\{L(x): L \text{ is affine function, } L(\cdot) < u(\cdot) \text{ in } B_1(0) \text{ and } L(0) = \ell_{te_1}(0)\}.$

Then from the proof of Corollary 2.3, we have

$\partial_{x_i}^2 \phi_t(te_1) = 0, \quad \partial_{x_i}^2 \phi_t(te_1) \geq \frac{C}{t}, \quad i = 2, \cdots, n.$

Let $\xi$ be a unit eigenvector of the least eigenvalue of $D^2u(te_1)$. We claim

\begin{equation}
|\langle \xi, e_1 \rangle| \geq 1 - Ct^n
\end{equation}

for a sufficiently large constant $C_1$. Indeed, let $\xi = ae_1 + b\eta$, where $\eta \perp e_1$ and $|\eta| = 1$. If (2.19) is not true, then $|b| \geq (C_1t^n)^{1/2}$ for $t$ small enough. Since $u$ and $\phi_t$ are tangent at $te_1$ and $u \geq \phi_t$, and since $\phi_t(te_1)$ is linear in $t$, we have

$t^{n-1} \approx \partial_\xi^2 u(te_1) \geq \partial_\xi^2 \phi_t(te_1) = b^2 \partial_\eta^2 \phi_t(te_1) \geq C_1 t^{n-1}$
which is impossible if we choose $C_1$ large enough. This proves the claim.

By (2.19), we infer that
\begin{equation}
\partial^2_{x_1} u(te_1) \leq C_2 t^{n-1}.
\end{equation}
Indeed, from (2.19), we have $a \approx 1$ and $|b| \leq C t^{\frac{n}{2}}$. By the convexity of $u$, we have
\begin{equation}
t^{n-1} \approx \partial^2_x u(te_1) = a^2 \partial^2_{x_1} u(te_1) + 2ab \partial_{x_1} \eta u(te_1) + b^2 \partial^2_\eta u(te_1)
\end{equation}
if $\partial^2_{x_1} u(te_1) \geq C_2 t^{n-1}$ for a sufficiently large $C_2$, which is a contradiction. Hence
\begin{equation}
u_1(te_1) = \int_0^t u_{11}(se_1) ds \leq C_3 t^n
\end{equation}
and so we have
\begin{equation}
u(te_1) = \int_0^t u_1(se_1) ds \leq C_4 t^{n+1}.
\end{equation}
This finishes the proof. \hfill \Box

We express equation (2.1) in the spherical coordinates $(\theta, r)$, where $r = |x|$ and $\theta = (\theta_1, \cdots, \theta_{n-1})$ is an orthonormal frame on $\mathbb{S}^{n-1}$.

**Lemma 2.4.** In the spherical coordinate $(\theta, r)$, one has
\begin{align}
|\partial^k_\theta w(p)| &\leq c|p|^{n+1-k}, \quad k = 0, \cdots, n+1, \\
|\partial^k_r w(p)| &\leq c|p|, \\
|\partial_{r\theta} w(p)| &\leq c|p|^\frac{n}{2},
\end{align}
for any point $p \neq 0$ near the origin.

**Proof.** As in Lemma 2.3, we may assume (2.18) holds. Denote $G_\varepsilon = \{x \in B_1(0) : u(x) < \varepsilon x_1\}$, where $\varepsilon > 0$ is a small constant. By the strict convexity of $u$, we have $G_\varepsilon \subset B_1$.

For any point $x = (x_1, \bar{x}) \in G_\varepsilon$, where $\bar{x} = (x_2, \cdots, x_n)$, by (2.16) we have
\begin{equation}
u(x) \geq \phi(x) \geq c_0 \frac{|\bar{x}|^2}{x_1}.
\end{equation}
Hence
\begin{equation}G_\varepsilon \subset \{x \in B_1(0) : |\bar{x}| \leq c_1 \varepsilon \frac{1}{x_1}\}.
\end{equation}
Denote
\[ s_\varepsilon = \sup \{ s : s \varepsilon_1 \in G_\varepsilon \}, \]
\[ t_\varepsilon = \sup \{ x_1 : x \in G_\varepsilon \}. \]
(2.25)

By Lemma 2.3, we have \( t_\varepsilon \geq s_\varepsilon \approx \varepsilon^{\frac{1}{n}}. \)

By the definition of \( t_\varepsilon \) in (2.25), and the strict convexity of \( u \), there exists a unique \( \tilde{x}_\varepsilon \) such that \( (t_\varepsilon, \tilde{x}_\varepsilon) \in \partial G_\varepsilon \). Then by Lemma 2.3
\[ \varepsilon t_\varepsilon = u(t_\varepsilon, \tilde{x}_\varepsilon) \geq \phi(t_\varepsilon, \tilde{x}_\varepsilon) + C(t_\varepsilon^2 + |\tilde{x}_\varepsilon|^2)^{\frac{n+1}{2}} \geq Ct_\varepsilon^{n+1}, \]
which implies \( t_\varepsilon \leq C\varepsilon^{\frac{1}{n}}. \) Hence \( t_\varepsilon \approx s_\varepsilon \approx \varepsilon^{\frac{1}{n}}. \)

By (2.17), we also have
\[ \inf_{G_\varepsilon}(u - \varepsilon x_1) = \inf_{G_\varepsilon}(\phi + w - \varepsilon x_1) \approx -\varepsilon s_\varepsilon. \]

Let \( A_\varepsilon = G_\varepsilon \cap \{ x_1 = \beta s_\varepsilon \} \), where \( \beta > 0 \) is a small constant. By Corollary 2.4, we have \( A_\varepsilon \subset \{ |\tilde{x}| < C\varepsilon^{1/2} s_\varepsilon \} \cap \{ x_1 = \beta s_\varepsilon \} \), where \( \tilde{x} = (x_2, \ldots, x_n) \). For a point \( (x_1, \tilde{x}) \in \partial A_\varepsilon \), by Corollary 2.2 and Lemma 2.3, we have
\[ u(x_1, \tilde{x}) \leq \phi(x_1, \tilde{x}) + C(|x_1|^2 + |\tilde{x}|^2)^{\frac{n+1}{2}} \leq C\frac{|\tilde{x}|^2}{x_1} + 2C\beta^{n+1}s_\varepsilon^{n+1}. \]
The left hand side \( u(x_1, \tilde{x}) = \varepsilon x_1 = \beta \varepsilon s_\varepsilon \) since \( (x_1, \tilde{x}) \in \partial A_\varepsilon \). Choosing a small \( \beta \ll 1 \) such that \( \beta^{n+1}s_\varepsilon^{n+1} \approx \beta^{n+1}\varepsilon s_\varepsilon \ll \beta \varepsilon s_\varepsilon \), we obtain \( |\tilde{x}| \geq c\varepsilon^{1/2}s_\varepsilon \), namely \( \{ |\tilde{x}| < c\varepsilon^{1/2}s_\varepsilon \} \cap \{ x_1 = \beta s_\varepsilon \} \subset A_\varepsilon \).

Now we make the coordinate change \( x \rightarrow y = T(x) \), given by
\[ y_1 = \frac{x_1}{s_\varepsilon}, \quad y_k = \frac{x_k}{\varepsilon^{\frac{n}{2}}s_\varepsilon} \quad (k = 2, \ldots, n), \]
(2.26)

We have shown that \( A_\varepsilon \sim \{ |\tilde{x}| < \varepsilon^{1/2}s_\varepsilon \} \) (as a convex domain in \( \mathbb{R}^{n-1} \)). Hence \( T(G_\varepsilon) \) has a good shape, namely, \( T(G_\varepsilon) \sim B_1(0) \). Let \( \tilde{u}(y) = \frac{u(x) - \varepsilon x_1}{\varepsilon s_\varepsilon} \). Then \( \tilde{u} \) satisfies
\[ \det D^2\tilde{u} = c_\varepsilon \tilde{g}(D\tilde{u}) \quad \text{in} \quad T(G_\varepsilon), \]
(2.27)
\[ \tilde{u} = 0 \quad \text{on} \quad \partial T(G_\varepsilon), \]
where \( c_\varepsilon = \varepsilon^{-1}s_\varepsilon^n \approx 1, \tilde{g}(D\tilde{u}) = g(\varepsilon \tilde{u}_{y_1} + \varepsilon, \varepsilon^{\frac{1}{2}}\tilde{u}_y) \). Hence by the regularity theory for the Monge-Ampère equation, \( \tilde{u} \) is smooth in \( T(G_\varepsilon) \), and we have the estimate
\[ \| D^k \tilde{u} \|_{s_{h_0/2, a}} \leq C_k, \quad k \geq 2 \]
(2.28)
where \( s_{h_0/2, a} = \{ \tilde{u} < -h_0/2 \}, h_0 = -\tilde{u}(\frac{1}{2}, 0) \approx 1. \) Restricting to the \( x_1 \)-axis, we obtain
\[ \| D^k \tilde{u} \|_{s_{h_0/2, a} \cap \{|\tilde{x}| = 0\}} = \| D^k \tilde{u} \|_{s_{h_0/2, a} \cap \{|\tilde{x}| = 0\}} \leq C_k, \quad k \geq 2, \]
where \( \tilde{w}(y) = \frac{w(x)}{\varepsilon^k} \). Scaling back to the original coordinates, we obtain

\[
|D^k_x w(p)| \leq C_k t^{n+1-k}, \quad k \geq 2,
\]

where \( p = te_1 \) with \( t \approx t\varepsilon \). We obtain the first estimate in (2.22). The second and third estimates in (2.22), for which \( k = 2 \), also follows from (2.28) by rescaling. \( \square \)

3. Bernstein theorem for a singular Monge-Ampère equation

In this section we prove a Bernstein theorem for the singular Monge-Ampère type equation in half space,

\[
\det \begin{pmatrix}
    \psi_{x_n x_n} + b \frac{\psi_{x_n}}{x_n} & \psi_{x_n x_1} & \cdots & \psi_{x_n x_{n-1}} \\
    \psi_{x_n x_1} & \psi_{x_1 x_1} & \cdots & \psi_{x_1 x_{n-1}} \\
    \vdots & \vdots & \ddots & \vdots \\
    \psi_{x_n x_{n-1}} & \psi_{x_1 x_{n-1}} & \cdots & \psi_{x_{n-1} x_{n-1}}
\end{pmatrix} = 1 \quad \text{in} \quad \mathbb{R}^n_+ = \mathbb{R}^n \cap \{x_n > 0\}.
\]

Equation (3.1) is the limit of equation (1.10) in a blow-up argument. We have the following Bernstein theorem.

**Theorem 3.1.** Let \( \psi \in C^{1,1}(\overline{\mathbb{R}^n_+}) \) be a solution to (3.1) with constant \( b > -1 \). Assume that \( D\psi(0) = 0 \), \( \psi_{x_n}(x',0) = 0 \quad \forall \ x' \in \mathbb{R}^{n-1} \), and equation (3.1) is uniformly elliptic. Then \( \psi \) is a quadratic polynomial of the form

\[
\psi(x) = \frac{1}{2} \sum_{i,j=1}^{n-1} c_{ij} x_i x_j + \frac{1}{2} c_{nn} x_n^2
\]

where \( \{c_{ij}\}_{i,j=1}^{n-1} \) is positive definite and \( c_{nn} > 0 \).

In Theorem 3.1 we denote \( \overline{\mathbb{R}^n_+} = \mathbb{R}^n_+ \cup \{x_n = 0\} \). To prove the Bernstein theorem, we make use of the Hölder continuity for the following degenerate elliptic equation.

\[
\partial_n(x_n \partial_n u) + \sum_{i,j=1}^{n-1} \partial_i(a_{ij}(x) \partial_j u) + \sum_{i=1}^{n} b_i(x) \partial_i u = f(x) \quad \text{in} \quad \mathbb{R}^n_+.
\]

We assume that the coefficients \( a_{ij} \) and \( b_i \) satisfy the following conditions.

(i) \( a_{ij} \) are measurable and satisfy

\[
C_s^{-1} |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq C_s |\xi|^2 \quad \forall \ \xi \in \mathbb{R}^{n-1},
\]

where \( C_s \) is a positive constant.

(ii) \( b_1 = \cdots = b_{n-1} = 0 \) and \( b_n \) is a positive constant.
By a change of variables, a model of equation (3.3) is

\begin{equation}
\Delta u + \frac{b}{x_n}u_{x_n} = f(x) \quad \text{in } \mathbb{R}^n_+.
\end{equation}

This is the classical Keldysh equation.

Denote \( B_R^+(0) = B_R(0) \cap \{x_n > 0\} \). We have the following Hölder continuity.

\textbf{Proposition 3.1.} Let \( u \in C^2(B_1^+) \cap L^\infty(B_1^+) \) be a solution to (3.3). Assume conditions (i), (ii), and \( f \in L^q(B_1^+) \) for some \( q > (n + 1)/2 \). Then \( u \) is continuous up to \( x_n = 0 \), and there exists \( \alpha \in (0, 1) \) such that

\begin{equation}
|u(x) - u(\tilde{x})| \leq C \left( \sup_{B_1^+} |u| + \|f\|_{L^q(B_1^+)} \right) |x - \tilde{x}|^\alpha \quad \forall \ x, \tilde{x} \in B_1^+/2,
\end{equation}

where \( \alpha \) and \( C \) are positive constants depending only on \( n, b, q, C_* \).

The Hölder continuity of solutions for degenerate elliptic equations has been studied by many authors. For proofs of Proposition 3.1 we refer the readers to [13, 20].

In Theorem 1.11 of [13], the authors proved the Hölder continuity for weak solutions to the variational equation (1.19) in [13], where the bilinear \( a(u, v) \) was given in (1.13) and the coefficients satisfy Assumption 1.11 of the paper. In Section 5 of [20], the authors studied equation (3.3) and proved the Hölder continuity in dimension two, with application to a geometric problem in \( \mathbb{R}^3 \), but the proof in [20] is valid in all dimensions. In fact, the proofs in [13] and [20] are similar, both of them use the Nash-Moser iteration.

In [13, 20] the coefficients \( b_i \) can be more general, here we assume condition (ii), which suffices for our purpose. In Proposition 3.1 we also assume that \( u \in C^2(B_1^+) \cap L^\infty(B_1^+) \), which is stronger than the assumption that \( u \) is a weak solution in [13, 20].

To apply Proposition 3.1 to the singular Monge-Ampère equation (3.1), we make a partial Legendre transform [32], to change equation (3.1) to the form (3.3).

Let

\begin{equation}
y_n = x_n, \\
y' = D_{x'} \psi, \\
\psi^* = x' \cdot D_{x'} \psi - \psi.
\end{equation}
Then by direct computation, we have
\[
\frac{\partial y_n}{\partial x_n} = 1, \quad \frac{\partial y_n}{\partial x'} = 0, \\
\frac{\partial y'}{\partial x_n} = D_{x'}^n \psi_{x'n}, \quad \frac{\partial y'}{\partial x'} = D_{x'} \psi,
\]
and
\[
\frac{\partial x_n}{\partial y_n} = 1, \quad \frac{\partial x_n}{\partial y'} = 0, \\
\frac{\partial x'}{\partial y'} = (D_{x'}^2 \psi)^{-1}, \\
\frac{\partial x_i}{\partial y_n} = \Psi_{in} \left( \det(D_{x'}^2 \psi) \right)^{-1}, \quad i = 1, \ldots, n - 1
\]
where \( \{ \Psi_{ij} \} \) is the cofactor matrix of \( D_{x'}^2 \psi \). Hence \( \psi^* \) satisfies
\[
\psi_{yn}^* = -\psi_{xn}, \quad D_{y'}^n \psi^* = x', \\
D_{y'}^2 \psi^* = (D_{x'}^2 \psi)^{-1},
\]
and
\[
\psi_{ynyn}^* = -\frac{\partial \psi_{xn}}{\partial y_n} \\
= -\psi_{xnyn} - \sum_{i=1}^{n-1} \frac{\partial \psi_{xn}}{\partial x_i} \frac{\partial x_i}{\partial y_n} \\
= -\psi_{xnyn} - \sum_{i=1}^{n-1} \psi_{xiyn} \Psi_{in} \left( \det(D_{x'}^2 \psi) \right)^{-1}.
\]
We obtain
\[
-\frac{\psi_{ynyn}^* + \frac{\psi_{yn}}{y_n} \Psi_{ynyn}}{\det D_{y'}^2 \psi^*} = \psi_{x'xn} \Psi_{in} + \left( \psi_{x'xn} + \frac{\psi_{xn}}{x_n} \right) \det D_{x'}^2 \psi = 1.
\]
Hence \( \psi^* \) satisfies
\[
(3.7) \quad \psi_{ynyn}^* + \frac{\psi_{yn}}{y_n} + \det D_{y'}^2 \psi^* = 0 \quad \text{in} \quad \mathbb{R}^n_+.
\]

In equation (3.7), the singular term \( \frac{\psi_{yn}}{y_n} \) is separate from the nonlinear part \( \det D_{y'}^2 \psi^* \). This is a very helpful property. Moreover, the Monge-Ampère operator \( \det D_{y'}^2 \psi^* \) is of divergence form. Hence equation (3.7) is of the same form as (3.4). Moreover, we assume that \( \psi \in C^{1,1} \) such that (3.1) is uniformly elliptic. We have the following key estimate.

**Lemma 3.1.** Let \( \psi^* \in C^{1,1}(\mathbb{R}^n_+) \) be a solution to (3.7) with \( b > -1 \). Assume \( \psi_{yn}^* (y', 0) = 0 \ \forall \ y' \in \mathbb{R}^{n-1} \), and \( D_{y'}^2 \psi^* \) is positive definite. Then \( \frac{\psi_{yn}}{y_n} \in C^\alpha(\mathbb{R}^n_+) \) for some \( \alpha \in (0, 1) \),
and we have the estimate

\[(3.8) \quad \left\| \frac{\psi^*_y}{y_n} \right\|_{C^\alpha([R^{n-1} \times [0,1])} \leq C \]

for a constant \(C\) depending only on \(b, n, \|D^2_0 \psi^*\|_{L^\infty(R^n_+)}\) and \(\|(D^2_0 \psi^*)^{-1}\|_{L^\infty(R^n_+)}\).

**Proof.** Let \(z_n = \frac{1}{2}y_n^2, \ z' = y'.\) Then equation \((3.7)\) is changed to

\[z_n \psi^*_{x_n z_n} + \frac{b+1}{2} \psi^*_n + \det D^2_\psi^* = 0 \ in \ \mathbb{R}^n_+.
\]

Denote \(\Psi = \psi^*_n.\) Differentiating the above equation in \(z_n\) gives

\[\partial_{z_n}(z_n \Psi_{z_n}) + \sum_{i,j=1}^{n-1} \partial_{z_i}(a_{ij} \Psi_{z_j}) + \frac{b+1}{2} \Psi_{z_n} = 0 \ in \ \mathbb{R}^n_+.
\]

Here \(\{a_{ij}\}_{i,j=1}^{n-1}\) is the the cofactor matrix of \(D^2_\psi^*\). By assumption, \(D^2_\psi^*\) is positive definite. Hence \(\lambda I \leq \{a_{ij}\} \leq \Lambda I\) for two positive constants \(\lambda, \Lambda\) depending only on \(\|D^2_\psi^*\|_{L^\infty(R^n_+)}\) and \(\|(D^2_\psi^*)^{-1}\|_{L^\infty(R^n_+)}\). Moreover.

\[(3.9) \quad \Psi(z) = \psi^*_n = \frac{2 \psi^*_y}{y_n} = 2 \int_0^1 \psi^*_y(y_n, ty_n) dt \in L^\infty(R^n_+).
\]

Therefore all the conditions in Proposition \(3.1\) are satisfied.

By Proposition \(3.1\) we obtain the H"{o}lder continuity of \(\Psi.\) By \(3.9\), \(\Psi(z) = \frac{2 \psi^*_y}{y_n}.\) Hence we obtain \((3.8).\)

**Lemma 3.2.** Let \(\psi \in C^{1,1}(\overline{R^n_+})\) be a solution to \((3.1)\) with \(b > -1.\) Assume \(\psi_{x_n}(x', 0) = 0 \ \forall \ x' \in R^{n-1},\) and equation \((3.1)\) is uniformly elliptic. Then \(\psi \in C^{2,\alpha}(\overline{R^n_+})\) for some \(\alpha \in (0, 1).\)

**Proof.** Let \(\psi^*\) be the partial Legendre transform of \(\psi.\) Then \(\psi^*\) satisfies equation \((3.7)\) and the assumptions of Lemma \(3.1.\) Hence by Lemma \(3.1\) \(\frac{\psi^*_y}{y_n} \in C^{\alpha}(\overline{R^n_+}).\) Recall that

\[\frac{\psi^*_y}{x_n} = -\frac{\psi^*_y}{y_n}.\]

We therefore have

\[\left| \frac{\psi^*_y(x)}{x_n} - \frac{\psi^*_y(\tilde{x})}{x_n} \right| = \left| \frac{\psi^*_y}{y_n} - \frac{\psi^*_y}{\tilde{y}_n} \right| \leq C|y - \tilde{y}|^{\alpha}.
\]

By the partial Legendre transform, \(y_n = x_n, \ y' = D_x \psi.\) It follows that

\[|y' - \tilde{y}'| = |D_x \psi(x) - D_x \psi(\tilde{x})| \leq \|D^2 \psi\|_{L^\infty(R^n_+)}|x - \tilde{x}|.
\]

Hence \(\psi^*_y \in C^{\alpha}(\overline{R^n_+})\) and we have the estimate

\[(3.10) \quad \left\| \frac{\psi^*_y}{x_n} \right\|_{C^\alpha([R^{n-1} \times [0,1])} \leq C
\]
for a constant \( C \) depending only on \( b, n, \) and \( \| D^2 \psi \|_{L^\infty(\mathbb{R}^n)} \).

We make an even extension of \( \psi(x) \) with respect to the variable \( x_n \) and still denote it by \( \psi(x) \). Regard \( \psi_{x_n} \) as a known function, which is Hölder continuous. Then we can write equation (3.1) in the form
\[
F(x, D^2 \psi) = 1.
\]
By our assumption, \( F \) is fully nonlinear, uniformly elliptic, and is \( C^\alpha \) smooth in \( x \). Since \( F_{1n} \) is concave in \( D^2 \psi \), by the \( C^{2,\alpha} \) regularity \([4]\), we also conclude that \( \psi \in C^{2,\alpha}(\mathbb{R}^n) \). \( \square \)

In Lemma 3.2 we assume that \( \psi \) is \( C^{1,1} \) and the equation (3.1) is uniformly elliptic. Without these conditions, Lemma 3.2 does not hold. Here is an example.

**Example 3.1.** The function
\[
u(x,y) = \frac{1}{2} x^2 + \frac{|y|^{1+\varepsilon}}{(1+\varepsilon)\varepsilon},
\]
is strictly convex and satisfies the equation
\[
(u_{xx} - 1 + |y|^{1-\varepsilon})u_{yy} - u_{xy}^2 = 1.
\]
But \( u \) is not \( C^{1,1} \) smooth.

With the aid of Lemma 3.2 we can now prove Theorem 3.1.

**Proof of Theorem 3.1:** Let \( \psi \) be the solution in Theorem 3.1. Let
\[
\psi^m(x) = \frac{\psi(mx)}{m^2}, \quad m = 1, 2, \ldots
\]
be a blow-down sequence of \( \psi \). Since (3.1) is uniformly elliptic for \( \psi \), it is also uniformly elliptic for \( \psi^m \) with the same ellipticity constants. The uniform ellipticity implies that there is a constant \( \hat{C} > 0 \), independent of \( m \), such that
\[
\hat{C}^{-1} \mathcal{I} \leq \mathcal{M}_{\psi^m} \leq \hat{C} \mathcal{I},
\]
where \( \mathcal{I} \) is the unit matrix and \( \mathcal{M}_{\psi} \) denotes matrix in equation (3.1). Hence the first entry in the matrix \( \mathcal{M}_{\psi^m} \) satisfies
\[
\psi^m_{x_nx_n} + b \frac{\psi^m_{x_n}}{x_n} = \hat{f},
\]
for a function \( \hat{f} \) satisfying \( \hat{C}^{-1} \leq \hat{f} \leq \hat{C} \). We can solve the above equation, regarding it as an ode with variable \( x_n \),
\[
\psi^m(\cdot, x_n) = \psi^m(\cdot, 0) + \int_0^{x_n} r^{-b} \int_0^r s^b \hat{f}(\cdot, s)ds.
\]
In (3.14) we have used the initial condition \(\psi_m^x(x', 0) = 0\). Note that (3.13) implies that \(\psi_m^x(x', 0) = O(|x'|^2)\). Hence from (3.14) we have \(\psi_m(x) = O(|x|^2)\) near 0.

Hence by the assumptions in Theorem 3.1, \(\psi^m\) satisfies the conditions in Lemma 3.2, uniformly in \(m\). Therefore, by Lemma 3.2 we have

\[
|D^2\psi(x) - D^2\psi(0)| = \lim_{m \to +\infty} |D^2\psi^m\left(\frac{x}{m}\right) - D^2\psi^m(0)| = 0
\]

for any given point \(x \in \mathbb{R}^n_+\). That is, \(D^2\psi(x) = D^2\psi(0) \forall x \in \mathbb{R}^n_+\). Hence \(\psi\) is a quadratic polynomial. By the assumption \(\psi^x_n(x', 0) = 0 \forall x' \in \mathbb{R}^{n-1}\), we have \(c_m = 0\) in the polynomial (3.2).

The following example shows that the Bernstein theorem 3.1 is not unconditionally true.

**Example 3.2.** Let

\[
\psi(x) = \frac{1}{2}(x_2^2 + \cdots + x_{n-1}^2) + \frac{1}{2}x_1^2x_n^{b-1} + \frac{x_n^{3-b}}{2(3-b)},
\]

where \(b > 1\). By direct computation, \(\psi\) satisfies equation (3.1).

It is well known that the classical Bernstein theorem for the Monge-Ampère equation was proved by Jörgens for dimension \(n = 2\), Calabi for \(3 \leq n \leq 5\), and Pogorelov for all dimensions [34]. In [24], a Bernstein type theorem on a different singular Monge-Ampère type equation in half space was proved.

**Example 3.3 ([24]).** Let \(u\) be a convex solution to

\[
\det D^2u = \left(\frac{ux_n}{x_n}\right)^{n+2} \quad \text{in} \quad \mathbb{R}^n_+,
\]

\[
u(x', 0) = \frac{1}{2}|x'|^2.
\]

Then either \(u = \frac{1}{2}|x|^2\), or \(u(x) = \frac{1}{2}|x'|^2\).

In contrast to the above example, it is also interesting to mention the following counter-example by Savin [37].

**Example 3.4 ([37]).** Let

\[
u(x) = \frac{x_1^2}{2(1+x_n)} + \frac{1}{2}(x_2^2 + \cdots + x_n^2) + \frac{1}{6}x_n^3.
\]

Then \(u\) satisfies

\[
\det D^2u = 1 \quad \text{in} \quad \mathbb{R}^n_+,
\]

\[
u(x', 0) = \frac{1}{2}|x'|^2.
\]
4. $C^2$ regularity of free boundary

In this section, we prove the $C^2$ regularity of the free boundary, which is equivalent to the $C^2$ regularity of the tangent cone of the following singular Monge-Ampère equation

\[(4.1) \det D^2 u = g(Du) + \delta_0 \quad \text{in} \quad B_1(0).\]

Assume that $g$ is a smooth function and satisfies

\[(4.2) 0 < \lambda \leq g \leq \Lambda < +\infty\]

for positive constants $\Lambda \geq \lambda > 0$. Let $u$ be a strictly convex solution to (4.1). Assume that $u(0) = 0$, $u \geq 0$ and $u > 0$ on $\partial B_1(0)$.

**Theorem 4.1.** Let $\phi$ be the tangential cone of $u$ at $0$. Then the section $S_{1,\phi} = \{x \in \mathbb{R}^n : \phi(x) < 1\}$ is uniformly convex and $C^2$ smooth provided $g$ is $C^2$ smooth and satisfies (4.2).

We have shown in §2 that $S_{1,\phi}$ is uniformly convex and $C^{1,1}$ smooth. In this section we raise the regularity of $S_{1,\phi}$ from $C^{1,1}$ to $C^2$. This is the most delicate part in the proof of Theorem 1.1. Our proof uses a blow-up argument, and uses the maximum principle in an infinite sequence of specially chosen domains.

Denote

\[(4.3) \quad \zeta = \frac{u}{r},\]

where $r = |x| \in (0, 1]$. We can extend $\zeta$ continuously to $r = 0$. By (2.17), $\zeta(\theta, 0) = \phi(\theta, r)/r$. To prove Theorem 4.1, we will prove that $\zeta(\theta, r) \in C^2(\mathbb{S}^{n-1} \times [0, 1])$, namely the second derivatives of $\zeta$ can continuously extend to $r = 0$ (Theorem 4.2). First we derive the equation for $\zeta$ in $(\theta, r)$.

**Lemma 4.1.** Let $\theta$ be an orthonormal frame on $\mathbb{S}^{n-1}$. Then $\zeta$ satisfies the Monge-Ampère type equation

\[(4.4) \quad \det \begin{pmatrix} \frac{\zeta_{rr}}{r^{n-2}} + \frac{2\zeta_r}{r^{n-1}} & \frac{\zeta_{r\theta_1}}{r^{n-1}} & \cdots & \frac{\zeta_{r\theta_{n-1}}}{r^{n-1}} \\ \frac{\zeta_{r\theta_1}}{r^{n-1}} & \frac{\zeta_{\theta_1\theta_1} + \zeta + r\zeta_r}{r^{n-2}} & \cdots & \frac{\zeta_{\theta_1\theta_{n-1}}}{r^{n-2}} \\ \vdots & \cdots & \cdots & \cdots \\ \frac{\zeta_{r\theta_{n-1}}}{r^{n-2}} & \frac{\zeta_{\theta_1\theta_{n-1}}}{r^{n-2}} & \cdots & \frac{\zeta_{\theta_{n-1}\theta_{n-1}} + \zeta + r\zeta_r}{r^{n-2}} \end{pmatrix} = \bar{g},\]

where $\bar{g} =: g(Du)$ is a smooth function of $\zeta, r\zeta_r, \zeta_\theta$. The subscript $\theta$ means covariant derivatives on the sphere $\mathbb{S}^{n-1}$. 
Proof. For any given point \((\theta, r) \in S^{n-1} \times (0, 1]\), there is no loss in assuming that \(\theta = 0\) and \(r = 1\). To derive equation (4.4), we use the spherical polar coordinates

\[
\begin{aligned}
x_1 &= r \sin \theta_1, \\
x_2 &= r \cos \theta_1 \sin \theta_2, \\
&\quad \ddots \\
x_{n-1} &= r \cos \theta_1 \cdots \cos \theta_{n-2} \sin \theta_{n-1}, \\
x_n &= r \cos \theta_1 \cdots \cos \theta_{n-1}.
\end{aligned}
\]

(4.5)

In the local coordinates (4.5), we have

\[
\begin{bmatrix}
\frac{\partial \theta_1}{\partial x_1} & \cdots & \frac{\partial \theta_{n-1}}{\partial x_1} & \frac{\partial r}{\partial x_1} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial \theta_1}{\partial x_n} & \cdots & \frac{\partial \theta_{n-1}}{\partial x_n} & \frac{\partial r}{\partial x_n}
\end{bmatrix} = I \text{ at } \theta = 0.
\]

By direct computation, we also have, at \(\theta = 0\),

\[
\frac{\partial^2 \theta_\alpha}{\partial x_i \partial x_j} = \begin{cases} 
-\delta_{\alpha j} & \text{if } 1 \leq j \leq n - 1, i = n, \\
-\delta_{\alpha i} & \text{if } 1 \leq i \leq n - 1, j = n, \\
0 & \text{otherwise}.
\end{cases}
\]

\[
\frac{\partial^2 r}{\partial x_i \partial x_j} = \begin{cases} 
1 & \text{if } i = j \leq n - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Rescaling back to \(r \in (0, 1]\), we obtain

\[
D^2_{x_i x_j} u = \begin{cases} 
\frac{1}{r} u_{\theta_i \theta_j} + \frac{u_{rr}}{r} \delta_{ij} & \text{if } 1 \leq i, j \leq n - 1, \\
\frac{u_{r \theta_i}}{r} - \frac{u_{\theta_i}}{r} & \text{if } 1 \leq i \leq n - 1, j = n, \\
u_{rr} & \text{if } i = j = n.
\end{cases}
\]

(4.6)

By the change \(u = r\zeta\), we then obtain

\[
D^2 u = \begin{pmatrix}
\zeta_{rr} + 2\zeta_r & \zeta_r \theta_1 & \cdots & \zeta_r \theta_{n-1} \\
\zeta_r \theta_1 & \frac{1}{r} \zeta_{\theta_1 \theta_1} + \zeta_r + \frac{1}{r} \zeta & \cdots & \frac{1}{r} \zeta_{\theta_1 \theta_{n-1}} \\
\cdots & \cdots & \cdots & \cdots \\
\zeta_r \theta_{n-1} & \frac{r}{r} \zeta_{\theta_{n-1} \theta_{n-1}} & \cdots & \frac{1}{r} \zeta_{\theta_{n-1} \theta_{n-1}} + \zeta_r + \frac{1}{r} \zeta
\end{pmatrix}.
\]

Hence we obtain (4.4) at \(\theta = 0\). As the Monge-Ampère equation is invariant under rotation of the coordinates, we see that (4.4) holds at a general point \((\theta, r) \in S^{n-1} \times (0, 1]\).

To see that \(\tilde{g} := g(Du)\) is a smooth function of \(\zeta, r\zeta, \zeta_\theta\), we just need to compute

\[
u_{x_i} = (r\zeta)_{x_i} = (r\zeta) \frac{\partial r}{\partial x_i} + r\zeta \frac{\partial \theta_i}{\partial x_i} = \begin{cases} 
\zeta_i & \text{if } i < n, \\
\zeta + r\zeta & \text{if } i = n,
\end{cases} \text{ at } \theta = 0.
\]

\(\square\)
Note that the local coordinates $\theta$ in (4.5) is not orthonormal on $\mathbb{S}^{n-1}$, but it is second order close to an orthonormal frame at $\theta = 0$. Hence we can use it to calculate the equation at $\theta = 0$. Here we say a coordinate system $\alpha = \alpha(\theta)$ is second order close to $\theta$ if
\[
\frac{d\alpha}{d\theta} = \mathcal{I} + O(|\theta|^2).
\]
By our notation, $\zeta = \phi(\theta) + \frac{w}{r}$ (here we denote $\phi(\theta) = \phi(\theta, 1)$ for the function $\phi$ in (2.2)). Hence
\[
(4.8)
\]
\[
\begin{align*}
\zeta_r &= \frac{w_r}{r} - \frac{w}{r^2}, \\
\zeta_{rr} &= \frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3}, \\
\zeta_{r\theta} &= \frac{w_{r\theta}}{r} - \frac{w_\theta}{r^2}, \\
\zeta_{\theta\theta} &= \phi_{\theta\theta} + \frac{w_{\theta\theta}}{r}.
\end{align*}
\]
By Lemma 2.4, all the entries in (4.4) are uniformly bounded. To make (4.4) uniformly elliptic, let
\[
s = r^{\frac{n}{2}}.
\]
Then
\[
(4.9)
\]
\[
\begin{align*}
\zeta_s &= 2n r^{-\frac{n+2}{2}} \zeta_r, \\
\zeta_{s\theta} &= 2n r^{-\frac{n+2}{2}} \zeta_{r\theta}, \\
\zeta_{ss} &= \frac{4\zeta_{rr}}{n^2 r^{n-2}} - \frac{2(n - 2)\zeta_r}{n^2 r^{n-1}}.
\end{align*}
\]
Hence by (4.8), (4.9), and Lemma 2.4 we have

**Corollary 4.1.** As a function of $\theta$ and $s$, $\zeta$ satisfies $\zeta_s(\theta, 0) = 0$ and $\zeta \in C^{1,1}(\mathbb{S}^{n-1} \times [0, 1))$.

By (4.9), equation (4.4) changes to
\[
(4.10)
\]
\[
\begin{pmatrix}
\frac{1}{2} \zeta_{ss} + \frac{n(n+2)}{4} \zeta_s & \frac{n}{2} \zeta_{s\theta_1} & \cdots & \frac{n}{2} \zeta_{s\theta_{n-1}} \\
\frac{n}{2} \zeta_{\theta_1 \theta_1} & \zeta_{\theta_1 \theta_1} + \zeta + \frac{n}{2} s \zeta_s & \cdots & \zeta_{\theta_1 \theta_{n-1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{n}{2} \zeta_{\theta_{n-1} \theta_{n-1}} & \zeta_{\theta_{n-1} \theta_{n-1}} & \cdots & \zeta_{\theta_{n-1} \theta_{n-1}} + \zeta + \frac{n}{2} s \zeta_s
\end{pmatrix} = \bar{g}.
\]
After the change $s = r^{\frac{n}{2}}$, we have $r\zeta_r = \frac{n}{2} s \zeta_s$. Hence by (4.7), $\bar{g}$ is still a smooth function, and $\bar{g}$ is analytic if $g$ is.

In the following we regard $\zeta$ as a function of $\theta$ and $s$, except otherwise specified.

**Lemma 4.2.** Equation (4.10) is uniformly elliptic in $\theta$ and $s$. 
Proof. As shown in Corollary 4.1, $\zeta$ is $C^{1,1}$ in $(\theta, s)$. Note that all the entries of the matrix in (4.10) are uniformly bounded. By our assumption, the function $\bar{g} > 0$. Hence the eigenvalues of the matrix are uniformly bounded and positive. Hence equation (4.10) is uniformly elliptic. $\square$

**Theorem 4.2.** Let $\zeta(\theta, s) \in C^{1,1}(S^{n-1} \times [0, 1))$ be a solution to (4.10) and satisfy $\zeta_s(\theta, 0) = 0$. Assume that $\bar{g}$ is positive and $C^2$ smooth. Then $\zeta(\theta, s) \in C^2(S^{n-1} \times [0, 1))$.

Once Theorem 4.2 is proved, Theorem 4.1 follows by our definition of $\zeta$.

To prove Theorem 4.2, one may wish to apply the partial Legendre transform (3.6) to (4.10). But due to the term $s\zeta_s$, equation (4.10) becomes very complicated after the change (3.6). In the following we will use a blow-up argument to prove Theorem 4.2. By the $C^2$ regularity of $\zeta$ for $s > 0$, it suffices to prove it at $S^{n-1} \times \{s = 0\}$. By Theorem 3.1 a blow-up sequence converges to a quadratic polynomial $\psi$ of the form (3.2), in which the mixed derivatives $\psi_{x_i x_n}$ vanish for $i < n$. It implies that all the blow-up sequences sub-converge to the same limit and so the mixed derivatives $\zeta_{\theta \theta}$ are continuous at $s = 0$. For other second derivatives, we need to prove that all the blow-up sequences at a given point converge to the same quadratic polynomial, namely the limit is independent of the blow-up sequences. It implies not only the existence but also the continuity of the second derivatives. For clarity we divide the proof into three lemmas.

For any given point $\theta_0 \in S^{n-1}$, by a rotation of the coordinate system, we assume $\theta_0 = 0$. By subtracting a linear function, there is no loss of generality in assuming that $\zeta(0) = D\zeta(0) = 0$ in the following argument.

**Lemma 4.3.** Let $\zeta$ be as in Theorem 4.2. Then $\zeta_{s\theta} \in C(S^{n-1} \times [0, 1))$.

**Proof.** We only need to prove that

(4.11) \[ \lim_{s \to 0^+} \zeta_{s\theta}(\theta, s) = 0. \]

If (4.11) is not true, there exists a sub-sequence $(\theta^k, s^k) \to (0, 0)$ such that

(4.12) \[ \lim_{k \to +\infty} |\zeta_{s\theta}(\theta^k, s^k)| \geq \varepsilon_0 \]

for a constant $\varepsilon_0 > 0$. In this case, we make the coordinate transform

(4.13) \[ \theta = \lambda_k \varphi + \theta^k, \]
\[ s = \lambda_k \tau, \]

where $\lambda_k \to 0$ as $k \to +\infty$.
where $\lambda_k = s^k$, and set
\begin{equation}
\zeta^k(\varphi, \tau) = \frac{\zeta(\theta, s) - \zeta(\theta^k, 0) - D_\theta \zeta(\theta^k, 0) \cdot (\theta - \theta^k)}{\lambda_k^2}. \tag{4.14}
\end{equation}

Note that (4.13) means a dilation and a rotation of the coordinates. We can regard $\varphi$ as an orthonormal frame on $\lambda_k^{-1}S^{n-1}$. Then by the $C^{1,1}$ regularity of $\zeta$, we have
\[|\zeta^k(\varphi, \tau)| \leq C(\tau^2 + |\varphi|^2)\]
for a constant $C > 0$ independent of $k$. Moreover, $\zeta^k$ satisfies the equation
\begin{equation}
\det\begin{pmatrix}
\left(\frac{n}{2}\right)^2 \zeta_{\tau\tau} + \frac{n(n+2)}{4} \frac{\zeta_k^k}{\tau} & \frac{n}{2} \zeta_{\tau\varphi_1}^k & \cdots & \frac{n}{2} \zeta_{\tau\varphi_{n-1}}^k \\
\frac{n}{2} \zeta_{\varphi_1\varphi_1}^k & \zeta_{\varphi_1\varphi_1}^k + h^k(\varphi, \tau) & \cdots & \zeta_{\varphi_1\varphi_{n-1}}^k \\
\cdots & \cdots & \cdots & \cdots \\
\frac{n}{2} \zeta_{\varphi_{n-1}\varphi_{n-1}}^k & \zeta_{\varphi_{n-1}\varphi_{n-1}}^k & \cdots & \zeta_{\varphi_{n-1}\varphi_{n-1}}^k + h^k(\varphi, \tau)
\end{pmatrix} = \bar{g}^k
\end{equation}
where $h^k = \lambda_k^2(\zeta_k + \frac{n}{2} \tau \zeta_\tau) + \zeta(\theta^k, 0) + \lambda_k D_\theta \zeta(\theta^k, 0) \cdot \varphi$, and $\bar{g}^k$ is a smooth function converging to a positive constant.

Write (4.15) as a general fully nonlinear elliptic equation of the form
\begin{equation}
F_k(\varphi, \tau, \zeta^k, D\zeta^k, D^2\zeta^k) = 0. \tag{4.16}
\end{equation}
By Lemma 4.2 $F_k$ is uniformly elliptic. Moreover, $F_k$ is concave with respect to its variables $D^2\zeta^k$ and is $C^{1,1}$ smooth in all arguments in $\tau > 0$. Hence by Evans-Krylov’s interior regularity theory [18], we have
\begin{equation}
\|\zeta^k\|_{C^{3,\alpha}(\Omega)} \leq C_\Omega \quad \forall \ \Omega \subset \subset \mathbb{R}^n_+,
\end{equation}
where the constant $C_\Omega$ is independent of $k$. By passing to a subsequence, we have
\begin{equation}
\zeta^k(\varphi, \tau) \to \tilde{\zeta}(\varphi, \tau) \quad \text{in} \quad C^{3,\alpha}_{loc}(\mathbb{R}^n_+) \cap C^{1,1-\varepsilon}_{loc}(\mathbb{R}^n_+),
\end{equation}
for a function $\tilde{\zeta} \in C^{3,\alpha}_{loc}(\mathbb{R}^n_+) \cap C^{1,1}_{loc}(\mathbb{R}^n_+)$, where $\varepsilon > 0$ is any small constant, and $\mathbb{R}^n_+ = \mathbb{R}^n \cup \{x_n = 0\}$. Moreover, the $C^{1,1}$ norm of $\tilde{\zeta}$ is independent of the choice of the blow-up sequences, it depends only on the $C^{1,1}$ norm of $\zeta$. Hence $\tilde{\zeta}$ satisfies equation (3.1) with $b = \frac{n+2}{n}$ under the coordinates $x' = \varphi$, $x_n = \tau$. By Theorem 3.1 $\tilde{\zeta}$ is a quadratic polynomial of the form
\begin{equation}
\tilde{\zeta}(\varphi, \tau) = \frac{1}{2} c_{00} \tau^2 + \frac{1}{2} \sum_{i,j=1}^{n-1} c_{ij} \varphi_i \varphi_j.
\end{equation}
Note that in (4.19), the mixed derivatives $\tilde{\xi}_{\tau\varphi}(0', 1) = 0$. By the interior regularity for (4.16), it implies that
\begin{equation}
\lim_{k \to +\infty} \zeta_{s\theta}(\theta^k, s^k) = \lim_{k \to +\infty} \zeta^k_{\tau\varphi}(0', 1) = \tilde{\xi}_{\tau\varphi}(0', 1) = 0.
\end{equation}
By the interior regularity (4.17), we see that (4.20) is in contradiction with (4.12). This completes the proof. □

**Lemma 4.4.** Let \( \zeta \) be as in Theorem 4.2. Then \( \zeta_{ss} \in C(S^{n-1} \times [0,1]) \).

**Proof.** To prove the continuity of \( \zeta_{ss} \) on \( S^{n-1} \times \{s = 0\} \), we need to prove the limit
\[
\lim_{s \to 0^+, \theta \to \theta} \zeta_{ss}(\theta, s) = c_{0}.
\]
By the proof of Lemma 4.3, we have
\[
(4.21) \quad \zeta^k(\varphi, \tau) \to \frac{1}{2} c_{00} \tau^2 + \frac{1}{2} \sum_{i,j=1}^{n-1} c_{ij} \varphi_i \varphi_j \quad \text{in} \quad C^3_{\text{loc}}(\mathbb{R}^n_+) \cap C^{1,1-\varepsilon}_{\text{loc}}(\mathbb{R}^n_+),
\]
by passing to a subsequence if necessary. To prove Lemma 4.4, we need to prove that
\[
(4.22) \quad \lim_{s \to 0^+, \theta \to \theta} \zeta_{ss}(\theta, s) = c_{00}.
\]
By the convergence (4.21) and the interior regularity of equation (4.15), we can choose a subsequence, such that
\[
(4.23) \quad \left\| \zeta_{ss}(\varphi, \tau) - c_{00} \right\|_{L^\infty(Q_k)} \leq \frac{1}{2k} \quad \text{in} \quad Q_k =: \{(\varphi, \tau) : |\varphi| \leq 1, \frac{1}{k} \leq \tau \leq 1\}.
\]
Scaling back to \( \zeta(\theta, s) \), we obtain
\[
(4.24) \quad \left\| \zeta_s(\theta, s) - c_{00} \right\|_{L^\infty(\Sigma_k)} \leq \frac{1}{2k} \quad \text{in} \quad \Sigma_k =: \{ (\theta, s) : |\theta| \leq \lambda_k, \frac{\lambda_k}{k} \leq s \leq \lambda_k \}.
\]
Let \( t = \frac{s^2}{4} \). The above estimate implies that
\[
(4.25) \quad \left\| \zeta_t - 2c_{00} \right\|_{L^\infty(\Sigma_k)} \leq \frac{1}{k} \quad \text{in} \quad \Sigma_k =: \{ (\theta, t) : |\theta| \leq \lambda_k, \frac{\lambda_k^2}{4k^2} \leq t \leq \frac{\lambda_k^2}{4} \}.
\]
Moreover, equation (4.10) changes to
\[
(4.26) \quad \det \begin{pmatrix}
    t \zeta_{tt} + \frac{n+1}{n} \zeta_t & \zeta_{t\theta_1} & \cdots & \zeta_{t\theta_{n-1}} \\
    t \zeta_{\theta_1} & \zeta_{\theta_1 \theta_1} + \zeta + t \zeta_t & \cdots & \zeta_{\theta_1 \theta_{n-1}} \\
    \cdots & \cdots & \cdots & \cdots \\
    t \zeta_{\theta_{n-1}} & \zeta_{\theta_{n-1} \theta_1} & \cdots & \zeta_{\theta_{n-1} \theta_{n-1}} + \zeta + t \zeta_t
\end{pmatrix} = \frac{4\tilde{g}}{n^2}.
\]
Note that in the matrix (4.26), there is a coefficient \( t \) in the first column. This is due to \( \zeta_{\theta \theta} = t^{1/2} \zeta_{\theta t} \). After the change \( t = \frac{s^2}{4} \), we have \( s \zeta_s = 2t \zeta_t \). Hence by (4.7), \( \tilde{g} \) is still a positive and smooth function of its arguments.
Writing equation (4.26) as \( \log \det W_{ij} = \log \frac{4g}{n^2} \) and differentiating in \( t \), we have

\[
W^{ij} \partial_t W_{ij} = \partial_t \log \frac{4\bar{g}}{n^2},
\]

where \( \{W_{ij}\} \) is the inverse of \( \{W_{ij}\} \). Dividing the above equation by \( W^{11} \), and denoting \( V = \zeta_t \), we can write the above equation as

\[
\mathcal{L}(V) =: tV_{tt} + \frac{2n+1}{n} V_t + \sum_{i,j=1}^{n-1} a^{ij} V_{\theta_i \theta_j} + \sum_{i,j=1}^{n-1} (\zeta_{\theta_i} b^{ij} + b^j) V_{\theta_j} = h
\]

(4.27)

where \( a^{ij}, \tilde{a}^{ij}, b^{ij}, b^j \) and \( h \) are continuous functions of the elements in the matrix in (4.26), namely \( t, \zeta, \zeta_t, \zeta_{\theta_i}, t_{\zeta_{t_0}}, t_{\zeta_{\theta_i}} \). From the assumptions in Theorem 4.2, all the elements are uniformly bounded, namely

\[
|\zeta_t| + |t\zeta_{tt}| + |t\zeta_{\theta_i} \zeta_{\theta_j}| + |D^2_{\theta} \zeta| \leq C \quad \forall (\theta, t) \in \mathbb{S}^{n-1} \times [0, \frac{1}{16}],
\]

(4.28)

It implies that \( a^{ij}, \tilde{a}^{ij}, b^{ij}, b^j, h \) are uniformly bounded.

Let \( \eta \) be a cut-off function of \( \theta \) such that

\[
\eta(\theta) \equiv 1 \quad \text{when } |\theta| \leq \frac{1}{2}, \quad \eta \equiv 0 \quad \text{when } |\theta| > 1, \quad \text{and } 0 \leq \eta \leq 1.
\]

Denote \( \eta_k(\theta) = \eta\left(\frac{\theta}{\lambda_k}\right) \) and \( V_k = \eta_k V \). Then \( V_k \) satisfies

\[
\mathcal{L}(V_k) = \eta_k h - [\eta_k, \mathcal{L}] V =: h_k,
\]

where \([\eta_k, \mathcal{L}] V = \eta_k \mathcal{L} V - \mathcal{L}(\eta_k V)\). By (4.28), we have

\[
|h_k| \leq C(1 + \lambda_k^{-2} + \lambda_k^{-1} t^{-\frac{3}{2}}) \leq C\lambda_k^{-1} t^{-\frac{3}{2}} \quad \text{when } 0 < t < \lambda_k^2,
\]

(4.29)

where \( C \) is a positive constant independent of \( k \).

Denote \( \delta_{k,-1} = 1, \delta_{k,0} = \frac{1}{4k^2} \). Let

\[
\delta_{k,l+1} = (\delta_{k,l})^{1 + \frac{1}{4}}, \quad \varepsilon_{k,l} = C_1 \delta_{k,l}^{\frac{1}{4} + \frac{1}{4}} \lambda_k^{-\frac{1}{2}}, \quad l = 0, 1, 2, \ldots,
\]

(4.30)

where \( C_1 \) is a fixed large constant.

Claim: For any given \( m \geq 1 \), we have

\[
|V_k - 2\eta_k c_{00}| \leq \frac{1}{k} + C_1 \sum_{l=0}^{m-1} \delta_{k,l}^{\frac{1}{4}}, \quad \text{when } |\theta| \leq \lambda_k, \delta_{k,m} \lambda_k^2 \leq t \leq \delta_{k,m-1} \lambda_k^2.
\]

(4.31)

We prove (4.31) by induction. First note that by (4.26),

\[
|V_k - 2\eta_k c_{00}| \leq \frac{1}{k} \quad \text{when } |\theta| \leq \lambda_k, \delta_{k,m} \lambda_k^2 \leq t \leq \delta_{k,m-1} \lambda_k^2,
\]

(4.32)
which is exactly (4.31) for $m = 0$. Assuming that (4.31) holds for $m$, we prove that it holds for $m + 1$. We introduce the auxiliary functions

$$
(4.32) \quad \sigma_{k,m}^\pm(\theta,t) = 2\eta_k c_{00} \pm \left(\frac{1}{k} + C_1 \sum_{l=0}^{m-1} \delta_{k,l}^\mp + \varepsilon_{k,m} t^{-\frac{1}{n}}\right).
$$

By our choice of $\eta_k$ and (4.29), we have

$$
(4.33) \quad \mathcal{L}(V_k - \sigma_{k,m}^+) = h_k - 2c_{00} \mathcal{L}(\eta_k) + \frac{\varepsilon_{k,m} t^{-1-\frac{1}{n}}}{n} \geq \frac{t^{-\frac{1}{n}}}{n} (\varepsilon_{k,m} t^{-\frac{1}{2} - \frac{1}{n}} - C\lambda_k^{-1})
$$

if the constant $C_1$ in (4.30) is chosen large. By our induction assumptions, we have

$$
V_k - \sigma_{k,m}^+ \leq 0 \quad \text{if} \quad |\theta| \leq \lambda_k, \quad t = \delta_{k,m} \lambda_k^2,
$$

$$
V_k - \sigma_{k,m}^+ = -\left(\frac{1}{k} + C_1 \sum_{l=0}^{m-1} \delta_{k,l}^+ + \varepsilon_{k,m} t^{-\frac{1}{n}}\right) < 0 \quad \text{if} \quad |\theta| = \lambda_k,
$$

$$
\limsup_{t \to 0^+} (V_k - \sigma_{k,m}^+) < 0.
$$

By the maximum principle, it follows that

$$
V_k - \sigma_{k,m}^+ \leq 0 \quad \text{if} \quad |\theta| \leq \lambda_k, \quad 0 < t \leq \delta_{k,m} \lambda_k^2.
$$

Similarly, we have

$$
V_k - \sigma_{k,m}^- \geq 0 \quad \text{if} \quad |\theta| \leq \lambda_k, \quad 0 < t \leq \delta_{k,m} \lambda_k^2.
$$

Therefore we obtain

$$
|V_k - 2c_{00} \eta_k| \leq \frac{1}{k} + C_1 \sum_{l=0}^{m-1} \delta_{k,l}^+ + \varepsilon_{k,m} t^{-\frac{1}{n}}
$$

$$
\leq \frac{1}{k} + C_1 \sum_{l=0}^{m-1} \delta_{k,l}^+ + \varepsilon_{k,m} \delta_{k,m+1} \lambda_k^{-\frac{1}{n}}
$$

$$
\leq \frac{1}{k} + C_1 \sum_{l=0}^{m-1} \delta_{k,l}^+ \quad \text{if} \quad |\theta| \leq \lambda_k, \quad \delta_{k,m+1} \lambda_k^2 \leq t \leq \delta_{k,m} \lambda_k^2.
$$

The claim (4.31) is proved.

For any point $(\hat{\theta}, \hat{t})$ near $(0,0)$ with $\hat{t} > 0$, we can choose $k > 0$ such that

$$
(\hat{\theta}, \hat{t}) \in \{(\theta,t) : |\theta| \leq \frac{\lambda_k}{2}, 0 < t \leq \frac{\lambda_k^2}{4}\}.
$$
We then choose $m \geq 0$ such that $\delta_{k,m+1}^2 \leq \hat{t} \leq \delta_{k,m}^2$. Hence we have

$$|V_k - 2\eta_k c_{00}| \leq \frac{1}{k} + C_1 \sum_{l=0}^{m} \frac{1}{\delta_{k,l}} \leq \frac{1}{k} + C_1 \sum_{l=0}^{\infty} \left(\frac{1}{4k^2}\right)^{(1+n/4)/4} \leq \frac{C_1}{\sqrt{k}} \text{ at } (\hat{t}, \hat{t}).$$

(4.34)

Because $(\hat{t}, \hat{t})$ is an arbitrary point near $(0, 0)$ with $\hat{t} > 0$. Hence from (4.34) we conclude that (recall that $V = \zeta_t = 2\zeta_s(\theta,s)$)

$$\lim_{\theta \to 0, s \to 0^+} \zeta_s(\theta,s) = \frac{1}{2} \lim_{\theta \to 0, s \to 0^+} V(\theta,s) = c_{00}.$$

(4.35)

The convergence (4.35) implies that the constant $c_{00}$ in the blow-up limit (4.19) is independent of the choice of the blow-up sequence. Hence by the blow-up argument in the proof of Lemma 4.3, we infer that

$$\lim_{\theta \to 0, s \to 0^+} \zeta_{ss}(\theta,s) = c_{00}.$$

(4.36)

By the convergence (4.36), we can define $\zeta_{ss}$ on $S^{n-1} \times \{s = 0\}$ as the limit $\lim_{s \to 0^+} \zeta_{ss}(\theta,s)$. The above proof also implies that $\zeta_{ss}$ is continuous on $\{s = 0\}$. For if not, let us assume that $\zeta_{ss}$ is discontinuous at $(\hat{\theta}, \hat{t}) = (0, 0)$. Then there exist two sequences $(\theta^k, s^k_1) \to 0$ and $(\theta^k, s^k_2) \to 0$ such that $\zeta_{ss}(\theta^k, s^k_1)$ and $\zeta_{ss}(\theta^k, s^k_2)$ converge to different limits, which is in contradiction with (4.36). This completes the proof.

**Remark 4.1.** In the first paragraph of the above proof, we must choose the sequence $(\theta^k, s^k)$ on the line $\{\theta = 0\}$, namely we must assume $\theta^k = 0$. Otherwise if $\frac{\theta^k}{s^k}$ is too large, by the changes (4.13), (4.14), and the definition of the cube $Q_k$, the set $\Sigma_k$ and $\hat{\Sigma}_k$ may not intersect with the line $\{\theta = 0\}$. In this case, we can only obtain the convergence $\zeta_{ss}(\theta,s)$ for $\theta, s$ near $(\theta^k, s^k)$, but not the full convergence (4.35) and (4.36).

**Lemma 4.5.** Let $\zeta$ be as in Theorem 4.2. Then $D^2_\theta \zeta \in C(S^{n-1} \times [0,1))$.

**Proof.** To prove the continuity of $D^2_\theta \zeta$ on $S^{n-1} \times \{s = 0\}$, we need to prove the limit $\lim_{s \to 0^+, \theta \to 0} D^2_\theta \zeta(\theta,s)$ exists. By Lemma 4.2, $D^2_\theta \zeta(\theta,s)$ is uniformly bounded. Hence there is a sub-sequence $s^k \to 0^+$ such that $D^2_\theta \zeta(0, s^k)$ is convergent. We introduce the coordinates $(\varphi, \tau)$ and function $\zeta^k$ as in (4.13) and (4.14), with $\theta^k = 0$. 


Let $\mathcal{B}$ denote the set of all convergent blow up sequences $\{\zeta^k\}$ given by (4.14) (with $\theta^k \equiv 0$). For any fixed unit vector $\gamma \in \mathbb{R}^{n-1}$, define
\begin{equation}
(4.37)
c_{\gamma\gamma} = \inf \lim_{k \to +\infty} \zeta^k(0', 1).
\end{equation}

where $\zeta^k_{\gamma\gamma} = \zeta^k_{\theta, \theta_j} \gamma_i \gamma_j$. By a diagonal process, we can extract a subsequence in $\mathcal{B}$, which for simplicity we still denote as $\{\zeta^k\}$, such that
\begin{equation}
(4.38)
c_{\gamma\gamma} = \lim_{k \to +\infty} \zeta^k_{\gamma\gamma}(0', 1).
\end{equation}

We claim
\begin{equation}
(4.39)
\lim_{\theta \to 0, s \to 0^+} \zeta^k_{\gamma\gamma}(\theta, s) \leq c_{\gamma\gamma}.
\end{equation}

Indeed, by the convergence (4.21) and the interior regularity of equation (4.15), similarly to (4.23) we can pass to a subsequence such that
\begin{equation}
\| \zeta^k_{\gamma\gamma}(\varphi, \tau) - c_{\gamma\gamma} \|_{L^\infty(Q_k)} \leq \frac{1}{k} \text{ in } Q_k.
\end{equation}

Scaling back to $\zeta(\theta, s)$, this implies
\begin{equation}
(4.40)
\| \zeta_{\gamma\gamma}(\theta, s) - c_{\gamma\gamma} \|_{L^\infty(\Sigma_k)} \leq \frac{1}{k} \text{ in } \Sigma_k,
\end{equation}

Here the domains $Q_k, \Sigma_k$ are the same as in (4.23) and (4.24).

To simplify the notation, let us denote the matrix in (4.26) as $R = (r_{ij})_{i,j=1}^n$, and rewrite equation (4.10) as
\begin{equation}
(4.41)
\mathcal{F}(r_{ij}) =: \log(\det R) = \log \bar{g}.
\end{equation}

Then $\mathcal{F}$ is concave in its variables $r_{ij}$. Differentiating (4.41) in direction $\gamma$ twice and by the concavity, we have
\begin{equation*}
\mathcal{F}_{r_{ij}, r_{ij, \gamma\gamma}} \geq (\log \bar{g})_{\gamma\gamma}.
\end{equation*}

Denote $V = \zeta_{\gamma\gamma}$. Similarly to (4.27), one obtains
\begin{equation}
(4.42)
\mathcal{L}(V) := V_{ss} + \frac{n + 2 V_s}{n} \frac{V_s}{s} + \sum_{i,j=1}^{n-1} a^{ij} V_{\theta_i \theta_j} + \sum_{i=1}^{n-1} a^i V_{\theta_i s} + \sum_{i=1}^{n-1} b^i V_i + b^0 V_s \geq h,
\end{equation}

where $a^{ij}, a^i, b^i, b^0, h$ are continuous functions of $\theta, s, \zeta, D\zeta, D^2\zeta, \frac{\zeta}{s}$. Here $D\zeta$ and $D^2\zeta$ denotes derivatives of $\zeta$ with respect to both $s$ and $\theta$. Note that the operator in (4.42) is different from that in (4.27) but we use the same notation $\mathcal{L}$. When differentiating (4.41) in $\gamma$ twice to obtain (4.42), we need to exchange the derivatives in $\gamma$ and $\theta$. By the Ricci identity, it arises some second derivatives of $\zeta$. They are all merged to the RHS $h$. 

We now introduce the auxiliary function $\sigma_{k,m}^{\pm}$ as in (4.32), and apply the same argument thereafter to obtain (4.39).

**Remark 4.2.** In the proof of Lemma 4.4, we differentiate equation (4.26) in $t$ once to obtain equation (4.27). Here we differentiate equation (4.26) in $\theta$ twice and use the concavity of $F$ to obtain the inequality (4.42). Hence by the auxiliary function $\sigma^+$, we obtain the inequality (4.39). We cannot obtain the reverse of the inequality by the auxiliary function $\sigma^-$.

**Remark 4.3.** Combining the definition of $c_{\gamma \gamma}$ in (4.37) and the estimate (4.39), we obtain (4.43)

$$\lim_{s \to 0^+} \zeta_{\gamma \gamma}(0, s) = c_{\gamma \gamma}.$$  

Note that the convergence in (4.43) is on the line $\theta = 0$.

To prove the convergence $\lim_{\theta \to 0, s \to 0^+} \zeta_{\gamma \gamma}(\theta, s) = c_{\gamma \gamma}$, we make use of the equation (4.10). By Lemmas 4.3 and 4.4, and noting that $s \zeta_s = o(1)$ near $s = 0$, we can write (4.10) as

$$\det(D^2_\theta \zeta + \zeta I) = \frac{2\bar{g}(0)}{n(n+1)c_{00}} + o(1) \text{ for } (\theta, s) \text{ near } (0', 0).$$  

The left hand side is the standard Monge-Ampère operator on the sphere $S^{n-1}$. For a $k \times k$ positive definite matrix $W$, its determinant $\det W$ can be written as

$$\det W = \min_{(\nu_1, \ldots, \nu_k) \in SO(k)} \prod_{i=1}^k \nu_i^T W \nu_i$$  

where $SO(k)$ is the set of orthogonal matrices and $\nu_i$ are column vectors of the matrix.

Consider a blow up sequence $\{\zeta^k\} \in B$. By the convergence (4.21) and the interior regularity we may assume that, after passing to a subsequence,

$$\lim_{k \to \infty, (\theta, s) \in \Sigma_k} (D^2_\theta \zeta + \zeta I) \to A$$  

for a positive definite matrix $A$, where $\Sigma_k$ is given in (4.21).

Let $O$ be an orthogonal matrix such that $O^T AO$ is diagonal. Then we have

$$\det(D^2_\theta \zeta + \zeta I) = \det(O^T (D^2_\theta \zeta + \zeta I) O)$$  

$$= \prod_{i=1}^{n-1} (\zeta_{\gamma(i)\gamma(i)} + \zeta) + o(1) \text{ in } \Sigma_k$$  

where $\{\gamma^{(1)}, \ldots, \gamma^{(n-1)}\}$ is an orthonormal basis of $\mathbb{R}^{n-1}$. By (4.39) it then follows

$$\det A = \lim_{\Sigma_k \ni (\theta, s) \to (0', 0)} \prod_{i=1}^{n-1} (\zeta_{\gamma(i)\gamma(i)} + \zeta)$$  

$$\leq \prod_{i=1}^{n-1} (c_{\gamma(i)\gamma(i)} + \zeta(0)).$$
By the definition of \( c_{\gamma \gamma} \) in (4.37) and the convergence (4.46), we have
\[
\prod_{i=1}^{n-1} (c_{\gamma(i)\gamma(i)} + \zeta(0)) \leq \lim_{\Sigma_{x} \in \{(\theta,s) \to (0',0)} \prod_{i=1}^{n-1} (\zeta_{\gamma(i)\gamma(i)} + \zeta) = \det A.
\]

Combining the above two inequalities, we obtain
(4.48)
\[
\prod_{i=1}^{n-1} (c_{\gamma(i)\gamma(i)} + \zeta(0)) = \det A = \frac{2\bar{g}(0)}{n(n+1)c_{00}}.
\]

We are ready to conclude the continuity of \( D_{\theta}^2 \zeta \). Indeed, by (4.44) and (4.45), we have
(4.49)
\[
\frac{2\bar{g}(0)}{n(n+1)c_{00}} = \lim_{\theta \to 0, s \to 0^+} \det(D_{\theta}^2 \zeta + \zeta I) \leq \lim_{\theta \to 0, s \to 0^+} \prod_{i=1}^{n-1} (\zeta_{\gamma(i)\gamma(i)} + \zeta).
\]

By (4.39) and (4.48), the RHS of (4.49)
(4.50)
\[
\leq \prod_{i=1}^{n-1} (c_{\gamma(i)\gamma(i)} + \zeta(0)) = \det A.
\]

Therefore the inequalities in (4.49) and (4.50) becomes equalities. Choosing the coordinates such that \( \gamma^{(1)}, \ldots, \gamma^{(n-1)} \) are the axial directions. As the inequality in (4.49) becomes equality, we see that
\[
\lim_{\theta \to 0, s \to 0^+} D_{\theta\theta}^2 \zeta(\theta, s) = 0 \quad \text{for all } i \neq j.
\]

By (4.39) and since the inequality (4.50) becomes equality, we infer that
(4.51)
\[
\lim_{\theta \to 0, s \to 0^+} \zeta_{\gamma(i)\gamma(i)} = c_{\gamma(i)\gamma(i)}.
\]

This completes the proof. \( \square \)

**Remark 4.4.** In §3 and §4, we employ the blow-up argument to prove the continuity of the second derivatives. The blow-up technique has been frequently used in geometric and analysis problems. It usually contains two steps, one is to classify the blow-up limits and the other is to show that the limit is independent of choice of the blow-up sequences. In the first step, one proves that a blow-up sequence at a fixed point converges along a subsequence to a nice limit. In the second step, one needs to prove that all blow-up sequences around a given point converges to the same limit. The second step is usually rather difficult. See [3, 6] for the classical obstacle problem (1.2). There are many examples, for which one can prove the first step but the second one becomes extremely complicated, such as singularity profiles for the Ricci flow and the mean curvature flow, and the \( C^1 \) regularity of infinity harmonic functions. For infinity harmonic functions, the blow-up at a fixed point is an affine function [12], but the \( C^1 \) regularity in high dimensions is still open.
5. Higher regularity of free boundary

In this section we first establish a weighted $W^{2,p}$ estimate for a linear singular elliptic equation of Keldysh type. Then we prove the higher regularity of the free boundary.

5.1. Linear singular elliptic equations of Keldysh type. First we consider the Neumann problem of the singular elliptic equation with constant coefficients:

$$
L_b(u) = -\Delta u - b \frac{\partial u}{\partial x_n} = f \quad \text{in} \quad \mathbb{R}^n_+,
$$

$$
\left. u \right|_{x_n(x')} = 0.
$$

We assume that $b > 0$ is a constant, $f \in L^\infty(\mathbb{R}^n_+)$. Horiuchi [22, 23] introduced the Green function for (5.1) and proved a weighted Schauder type estimate. In [22], he found the following representation formula for the solution to (5.1),

$$
u(x) = \int_{\mathbb{R}^n_+} K_b(x,y) y^n f(y) dy =: T_b(f)(x),
$$

where $K_b(x,y)$ is the Green function, given by

$$
K_b(x,y) = D_b \int_0^1 \left( |x - y|^2 (1 - \tau) + |x - y^*|^2 \tau \right)^{-\frac{n-2+b}{2}} \frac{\tau (1 - \tau)}{(1 - \tau)^{\frac{b}{2} - 1}} d\tau,
$$

$$
D_b = 2^{b-2} \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+b-2}{2}\right)}{\Gamma\left(\frac{b}{2}\right)}, \quad y^* = (y_1, \cdots, y_{n-1}, -y_n).
$$

Moreover, for $b > 0$, the following estimates hold,

$$
|\partial^{\gamma_1}_{x} \partial^{\gamma_2}_{y} K_b(x,y)| \leq \left\{ \begin{array}{ll}
C |x - y^*|^{-b} \ln \left( 2 + \frac{|x - y^*|}{|x - y|} \right) & \text{if } n = 2 \text{ and } |\gamma_1| = |\gamma_2| = 0, \\
C |x - y^*|^{-b} |x - y|^{2-n-|\gamma_1|-|\gamma_2|}, & \text{otherwise}
\end{array} \right.
$$

for any indexes $\gamma_1, \gamma_2 \in \mathbb{N}^n$, where $C$ depends on $n, b, \gamma_1, \gamma_2$ (see Lemma 5-3 in [22]).

Horiuchi [22] proved that the function $u = T_b(f)$, given in (5.2), is a solution to (5.1). By the representation formula (5.2), he also proved the following $C^{2,\alpha}$ estimate by careful computation as in [18].

**Theorem 5.1.** [Theorem 2-1, [22]] Assume that $f \in C^\alpha(B^+_1)$, supp $f \subset B_1$, and $b > 0$. Then

$$
\sum_{i,j=1}^n \| \partial_{ij} T_b(f)(x) \|_{C^\alpha(B^+_1)} + \| \frac{\partial_n T_b(f)(x)}{x_n} \|_{C^\alpha(B^+_1)} \leq C \| f \|_{C^\alpha(B^+_1)},
$$

where $\alpha \in (0,1)$, and $C$ depends only on $n, b, \alpha$. 
Since \( u = T_b(f) \) and \( f \) has compact support, from (5.5) we also have

\[
\|u\|_{C^{2,\alpha}(\mathbb{R}^n_+)} + \left\| \frac{u_n}{x_n} \right\|_{C^{\alpha}(\mathbb{R}^n_+)} \leq C \|f\|_{C^{\alpha}(\mathbb{R}^n_+)}.
\]

Denote

\[
\|f\|_{L^p_{\mu_b}(\mathbb{R}^n_+)} = \left( \int_{\mathbb{R}^n_+} |f|^p(x) \mu_b(x) \, dx \right)^{1/p},
\]

\[
\|f\|_{W^{k,p}_{\mu_b}(\mathbb{R}^n_+)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p_{\mu_b}(\mathbb{R}^n_+)}.\]

We have the following weighted \( W^{2,p} \) estimate.

**Theorem 5.2.** Let \( u = T_b(f) \) be given by (5.2). Assume that \( f \) has compact support. Then for any \( p > 1 \) and \( b > 1 \), we have the estimate

\[
\sum_{i,j=1}^n \left\| u_{x_i x_j} \right\|_{L^p_{\mu_b}(\mathbb{R}^n_+)} + \left\| \frac{u_n}{x_n} \right\|_{L^p_{\mu_b}(\mathbb{R}^n_+)} \leq C \|f\|_{L^p_{\mu_b}(\mathbb{R}^n_+)}
\]

for a constant \( C > 0 \) depending only on \( p, n, b \).

Note that (5.7) is invariant under dilation of coordinates. Hence to prove Theorem 5.2 we may assume that \( \text{supp}\, f \subset B_1(0) \). By Theorem 5.2 we have

**Corollary 5.1.** Let \( u = T_b(f) \) be given by (5.2). Assume that \( \text{supp}\, f \subset B_1(0) \). Then for any \( p > \frac{n+b}{n-2+b} \) and \( b > 1 \), we have the estimate

\[
\|u\|_{W^{2,p}_{\mu_b}(\mathbb{R}^n_+)} + \left\| \frac{u_n}{x_n} \right\|_{L^p_{\mu_b}(\mathbb{R}^n_+)} \leq C \|f\|_{L^p_{\mu_b}(\mathbb{R}^n_+)},
\]

where the constant \( C \) depends on \( p, n, b \).

**Remark 5.1.** (i) The condition \( b > 1 \) is used only in (5.31), otherwise it suffices to assume \( b > 0 \). The condition \( p > \frac{n+b}{n-2+b} \) in Corollary 5.1 is for the estimate \( \|u\|_{L^p_{\mu_b}} \).

(ii) A different version of the \( C^{2,\alpha} \) estimate in Theorem 5.1 was also obtained in [21]. The paper [21] has also established a weighted \( W^{2,p} \) estimate by the method of Fourier transformation and oscillatory integral method [21, Theorem 1.2]. However, the coefficient \( a > \frac{3}{2} \) is needed in [21] which corresponds to \( b > 2 \) in our present case. Hence we can’t apply [21] to the problem in this paper.

In the following we prove Theorem 5.2. Recall that the \( W^{2,p} \) estimate for the Laplace equation can be obtained by the Calderon-Zygmund decomposition and the Marcinkiewicz interpolation [18]. Our proof will adopt a similar idea. But due to the singular term \( \frac{u_n}{x_n} \), we need to introduce a new variable in the proof (see function \( \tilde{u} \) in (5.29)). We include the details of the proof for convenience of the reader.
First, we notice the following asymptotic estimate for $T_b(f)$ near $\infty$.

**Lemma 5.1.** Let $u = T_b f$ be given by (5.2). Assume that $f \in L^\infty(\mathbb{R}_+^n)$, suppf $\subset B_1(0)$, and $b > 0$. Then we have the following asymptotic estimate,

$$
|D^2_x u(x)| \leq \frac{C \|f\|_{L^1_b(\mathbb{R}_+^n)}}{|x|^{n-2+b+|\gamma|}} \quad \forall |x| > 2,
$$

where the constant $C$ depends only on $n, \gamma, b$.

**Proof.** From (5.4), one has

$$
|D^2_x u(x)| = \left| \int_{\mathbb{R}_+^n} \partial^2_x K_b(x, y) y^b f(y) dy \right|
$$

$$
\leq C(\gamma, b) \int_{\mathbb{R}_+^n} |x-y|^{|\gamma|} |x-y|^{2-n-|\gamma|} f(y) y^b dy
$$

$$
\leq \frac{C(\gamma, b) \|f\|_{L^1_b(\mathbb{R}_+^n)}}{|x|^{n-2+b+|\gamma|}}
$$

provided $|x| > 2$. \hfill \Box

**Lemma 5.2.** Assume that $f \in L^\infty(\mathbb{R}_+^n)$, suppf $\subset B_1$, and $b > 0$. Then $u = T_b f$ satisfies the weighted $W^{2,2}$ estimate

$$
\|D^2 u\|_{L^2_b(\mathbb{R}_+^n)}^2 + b \left( \|u_n\|_{L^2_b(\mathbb{R}_+^n)}^2 \right) = \|f\|_{L^2_b(\mathbb{R}_+^n)}^2.
$$

**Proof.** Since $u$ satisfies equation (5.1), we have

$$
\int_{\mathbb{R}_+^n} \left( \Delta u + b \frac{u_n}{x_n} \right)^2 x^b_n dx = \int_{\mathbb{R}_+^n} f^2 x^b_n dx.
$$

By Lemma 5.1, we can carry out integration by parts and obtain

$$
\int_{\mathbb{R}_+^n} (\Delta_x u)^2 x^b_n dx = \sum_{i,j=1}^{n-1} \int_{\mathbb{R}_+^n} u^2_{x_i x_j} x^b_n dx,
$$

$$
2 \int_{\mathbb{R}_+^n} u_{i} u_{n n} x^b_n dx = -2 \int_{\mathbb{R}_+^n} u_{i} u_{n i} x^b_n dx
$$

$$
= 2 \int_{\mathbb{R}_+^n} u^2_{n i} x^b_n dx + 2b \int_{\mathbb{R}_+^n} u_{i} u_{n i} x^{n-1} dx, \quad i = 1, \ldots, n-1,
$$

and

$$
2b \int_{\mathbb{R}_+^n} u_{i i} u_{n i} x^{n-1} dx = -2b \int_{\mathbb{R}_+^n} u_{i} u_{n i} x^{n-1} dx, \quad i = 1, \ldots, n-1,
$$

$$
2b \int_{\mathbb{R}_+^n} u_{n n} u_{n i} x^{n-1} dx = -b(b-1) \int_{\mathbb{R}_+^n} u^2_{n} x^{n-2} dx.
$$
Summing up, and noticing that the two integrals \( 2b \int_{\mathbb{R}^n_+} u_iu_{ni}x_n^{b-1} \) are cancelled each other, and the last integral \(-b(b-1) \int_{\mathbb{R}^n_+} u_n^2x_n^{b-2} \) is partly cancelled by the left hand side of (5.12), we obtain (5.11).

By Lemma 5.2, we see that \( \partial_{ij}T_b, \frac{\partial T_b}{\partial x_\alpha} \) are bounded linear operators from \( L^2_{\mu_b}(\mathbb{R}^n_+) \) to \( L^2_{\mu_b}(\mathbb{R}^n_+) \). Next we show that they are bounded linear operators from \( L^1_{\mu_b}(\mathbb{R}^n_+) \) to \( L^1_{\mu_b}(\mathbb{R}^n_+) \), where

\[
L^1_{\mu_b}(\mathbb{R}^n_+) = \{ v \text{ is measurable : } \mu_b\{ x \in \mathbb{R}^n_+ : |v(x)| > \alpha \} \leq C\alpha^{-1} \forall \alpha > 0 \},
\]

and the measure \( \mu_b \) is defined as \( \mu_b(E) = \int_E x_n^b \) for any measurable set \( E \).

As a first step, we give the following weighted Calderon-Zygmund decomposition in \( \mathbb{R}^n_+ \).

**Lemma 5.3.** Suppose \( f \in L^1_{\mu_b}(\mathbb{R}^n_+) \) and \( b > 0 \). Then for any given constant \( \alpha > 0 \), there is a sequence of disjoint cubes \( \{Q_k\}_{k=1}^{\infty} \) and a decomposition

\[
f = g + h = g + \sum_{k \geq 1} h_k
\]
such that

(i) \( |g(x)| \leq c\alpha \) for \( \mu_b \)-a.e. \( x \in \mathbb{R}^n_+ \), and

\[
\int_{\mathbb{R}^n_+} |g(x)| \, d\mu_b \leq c \int_{\mathbb{R}^n_+} |f(x)| \, d\mu_b.
\]

(ii) For each \( k \geq 1 \), \( h_k \) is supported in \( Q_k \) and satisfies

\[
\int_{\mathbb{R}^n_+} |h_k| \, d\mu_b \leq c\alpha \mu_b(Q_k) \quad \text{and} \quad \int_{\mathbb{R}^n_+} h_k \, d\mu_b = 0.
\]

(iii) \( \sum_k \mu_b(Q_k) \leq \frac{c}{\alpha} \int_{\mathbb{R}^n_+} |f| \, d\mu_b. \)

Here the constant \( c \) depends only on \( n \) and \( b \).

**Proof.** The Calderon-Zygmund decomposition for the Lebesgue measure is well-known [4, 18]. Here we replace the Lebesgue measure by the measure \( \mu_b \) but the idea is the same. Consider a partition of \( \mathbb{R}^n_+ \) by cubes \( \mathcal{K}_0 = \{Q\} \) whose side length \( d \) is chosen such that

\[
\alpha > \frac{1}{\mu_b([0,d]^n)} \int_{\mathbb{R}^n_+} |f| \, d\mu_b.
\]

Then for any cube \( Q \in \mathcal{K}_0 \), we have \( |f|_Q \leq \alpha \), where \( |f|_Q = \frac{1}{\mu_b(Q)} \int_Q |f| \, d\mu_b. \)

To obtain the sequence of disjoint cubes \( \{Q_k\}_{k=1}^{\infty} \) stated in the lemma, we first consider an arbitrary cube \( Q \in \mathcal{K}_0 \), by bisecting the edges of \( Q \), we subdivide \( Q \) into \( 2^n \) congruent
sub-cubes \( \{Q'_i\}_{i=1}^{2^n} \) with disjoint interiors. For any sub-cube \( Q'_i \), we have either \( |f|_{Q'_i} \geq \alpha \) or \( |f|_{Q'_i} < \alpha \). Denote by \( \mathcal{K}_1 \) the set of all sub-cubes of side-length \( d/2 \).

(a) If \( |f|_{Q'_i} \geq \alpha \), set

\[
h_{Q'_i} = \chi_{Q'_i}(f - f_{Q'_i}) \quad \text{and} \quad g_{Q'_i} = \chi_{Q'_i}f_{Q'_i},
\]

where \( \chi \) is the characteristic function, \( f_{Q'_i} = \frac{1}{\mu_b(Q'_i)} \int_{Q'_i} f \, d\mu_b \) is the \( \mu_b \)-average of \( f \) in \( Q'_i \). In this case we will not divide \( Q'_i \) any more, and this \( Q'_i \) will be counted as an element in the family \( \{Q_k\} \).

(b) If \( |f|_{Q'_i} < \alpha \), we divide \( Q'_i \) into \( 2^n \) equal sub-cubes \( \{Q''_{i,j}\} \) as above, and denote by \( \mathcal{K}_2 \) the set of all the sub-cubes of side-length \( d/4 \). For any sub-cube \( Q''_{i,j} \in \mathcal{K}_1 \), if \( |f|_{Q''_{i,j}} \geq \alpha \), we define the functions \( h_{Q''_{i,j}} \) and \( g_{Q''_{i,j}} \) as in (a) above, and count \( Q''_{i,j} \) as an element in the family \( \{Q_k\} \). If \( |f|_{Q''_{i,j}} < \alpha \), we divide \( Q''_{i,j} \) into \( 2^n \) equal sub-cubes again.

Repeating the above procedure indefinitely, we obtain a sequence of disjoint cubes \( \{Q_k\} \) such that \( |f|_{Q_k} \geq \alpha \) for all \( k \geq 1 \). Now we define

\[
h_i = h_{Q_k} \quad i = 1, 2, \ldots ,
\]

\[
(5.17)
g(x) = \begin{cases} 
 f_{Q_k} & \text{if } x \in Q_k, \quad k = 1, 2, \ldots , \\
 f(x) & \text{if } x \in \mathbb{R}_+^n \setminus (\bigcup_{k=1}^\infty Q_k).
\end{cases}
\]

For any cube \( Q_k \) in the sequence, let \( \tilde{Q}_k \) be its predecessor, namely \( Q_k \) is one of the \( 2^n \) sub-cubes obtained from \( \tilde{Q}_k \). By the above decomposition, it is easy to see

\[
(5.18) \quad |f|_{Q_k} \leq \frac{\mu_b(\tilde{Q}_k)}{\mu_b(Q_k)}|f|_{\tilde{Q}_k} \leq 2^{n+2b+1}|f|_{\tilde{Q}_k} \leq 2^{n+2b+1}\alpha.
\]

One can verify that \( (5.14) - (5.16) \) hold with \( c = 2^{n+2b+1} \). To finish the proof, it remains to show that \( |g(x)| \leq C\alpha \quad \forall \ x \in \mathbb{R}_+^n \). Since the set \( \bigcup_k \partial Q_k \) has measure zero, there is no loss of generality in assuming that \( x \) is a Lebesgue point and \( x \notin \bigcup_k \partial Q_k \). Then in the procedure described in (a) and (b) above, eventually there will be a cube \( Q_k \) such that \( x \in Q_k \) and \( |f|_{Q_k} \geq \alpha \) if \( g(x) > \alpha \). Hence by our definition of \( g \) in \( (5.17) \), we have \( |g(x)| = |f_{Q_k}| \leq C\alpha \) for \( x \in Q_k \) by \( (5.18) \). This completes the proof. \( \square \)

Next we show that \( \partial_{x_i}\partial_{x_j} T_b \) and \( \partial_{x_i}\partial_{x_i} T_b \) are bounded linear operators from \( L^1_{\mu_b}(\mathbb{R}_+^n) \) to \( L^1_{\mu_b, \omega}(\mathbb{R}_+^n) \), for any \( b > 0 \).
Lemma 5.4. Assume \( f \in L^{1}_{\mu_b}(\mathbb{R}^n_+) \) and \( \text{supp} \, f \subset B_1(0) \). Then for any constant \( \alpha > 0 \), \( u = T_b(f) \) satisfies
\[
\mu_b \{ x \in \mathbb{R}^n_+ : |u(x)| > \alpha \} \leq C \alpha^{-1} \| f \|_{L^{1}_{\mu_b}(\mathbb{R}^n_+)},
\]
(5.19)
\[
\mu_b \{ x \in \mathbb{R}^n_+ : \frac{|u_n(x)|}{x_n} > \alpha \} \leq C \alpha^{-1} \| f \|_{L^{1}_{\mu_b}(\mathbb{R}^n_+)},
\]
(5.20)
for a positive constant \( C \) depending only on \( n, b \).

Proof. Let \( f = g + h = g + \sum_k h_k \) be the decomposition given in Lemma 5.3. Since \( T_b \) is a linear operator, we have
\[
\mu_b \{ x \in \mathbb{R}^n_+ : |\partial_{x,i,j} T_b f(x)| > \alpha \}
\leq \mu_b \{ x \in \mathbb{R}^n_+ : |\partial_{x,i,j} T_b g(x)| > \frac{\alpha}{2} \} + \mu_b \{ x \in \mathbb{R}^n_+ : |\partial_{x,i,j} T_b h(x)| > \frac{\alpha}{2} \}.
\]
(5.21)
By Lemma 5.3 we have \( |g| \leq C\alpha \), and \( g \) is compactly supported since \( f \) is. By Lemma 5.2 one has
\[
\| \partial_{x,i,j} T_b g(x) \|_{L^{2}_{\mu_b}(\mathbb{R}^n_+)}^2 \leq \| g \|_{L^{2}_{\mu_b}(\mathbb{R}^n_+)}^2 \leq C \alpha \| g \|_{L^{1}_{\mu_b}(\mathbb{R}^n_+)} \leq C \alpha \| f \|_{L^{1}_{\mu_b}(\mathbb{R}^n_+)}.
\]
This implies
\[
\mu_b \{ x \in \mathbb{R}^n_+ : |\partial_{x,i,j} T_b g(x)| > \frac{\alpha}{2} \} \leq C \alpha^{-1} \| f \|_{L^{1}_{\mu_b}(\mathbb{R}^n_+)}. \]
(5.22)
Now for any cube \( Q_k \) in Lemma 5.3 let \( Q_k^* = (q + 2(Q_k - q)) \cap \mathbb{R}^n_+ \), where \( q \) is the center of \( Q_k \). Then for any \( x \in (Q_k^*)^c \), there holds
\[
|\partial_{x,i,j} T b h_k(x)| = \left| \int_{\mathbb{R}^n_+} \partial_{x,i,j} K_b(x,y) h_k(y) d\mu_b(y) \right|
\leq \int_{Q_k} \left| \partial_{x,i,j} K_b(x,y) - \partial_{x,i,j} K_b(x,q) \right| h_k(y) d\mu_b(y)
\leq C \int_{Q_k} \frac{|y_n - q_n| h_k(y)}{|x - y|^n} d\mu_b(y),
\]
(5.23)
where in the second equality, we used (ii) in Lemma 5.3. By (5.23) we then have
\[
\int_{(Q_k^*)^c} |\partial_{x,i,j} T b h_k(x)| d\mu_b(x)
\leq C \int_{Q_k} |h_k(y)| d\mu_b(y) \int_{(Q_k^*)^c} \frac{|y_n - q_n| d\mu_b(x)}{|x - y|^n}
\leq C \int_{Q_k} |h_k(y)| d\mu_b(y) \leq C \alpha \mu_b(Q_k),
\]
(5.24)
where $y^*$ is reflection of $y$ in $\{x_n = 0\}$. Then
\[
\mu_b(\{x \in \mathbb{R}^n_+ : |\partial_{x,x_j} Th(x)| > \frac{\alpha}{2}\})
\leq \sum_k \mu_b(Q_k^x) + \frac{2}{\alpha} \sum_k \int_{(Q_k^x)^c} |(\partial_{x,x_j} Th_k)(x)|d\mu_b(x) \leq \frac{C\|f\|L_{\mu_b}(\mathbb{R}^n_+)}{\alpha}.
\]
We obtain the first inequality in (5.19).

For the second inequality in (5.19), recall that $T_b f$ satisfies equation (5.1). Hence by (5.1), we infer that $\frac{\partial_T x_n}{x_n}$ is also a bounded linear operator from $L^1_{\mu_b}(\mathbb{R}^n_+)$ to $L^1_{\mu_b,\omega}(\mathbb{R}^n_+)$. \[\square\]

With the above preparation, we are ready to prove Theorem 5.2

**Proof of Theorem 5.2.** By Lemma 5.2 and Lemma 5.4, the operator $\partial_{x,x_j} T_b$ satisfies
\[
\|\partial_{x,x_j} T_b(f)\|L_{\mu_b}(\mathbb{R}^n_+) \leq C_{n,b}\|f\|L_{\mu_b}(\mathbb{R}^n_+),
\]
\[
\|\partial_{x,x_j} T_b(f)\|L_{\mu_b,\omega}(\mathbb{R}^n_+) \leq C_{n,b}\|f\|L_{\mu_b}(\mathbb{R}^n_+)
\]
for any $b > 0$, where
\[
\|v\|L_{\mu_b,\omega}(\mathbb{R}^n_+) = \sup_{\alpha > 0} \alpha \mu_b\{x \in \mathbb{R}^n_+ : |v(x)| > \alpha\}.
\]
Hence by the Marcinkiewicz Interpolation [Theorem 1.3.1, [1]], we infer that
\[
\|\partial_{x,x_j} T_b(f)\|L_{\mu_b}(\mathbb{R}^n_+) \leq C_{n,b,p}\|f\|L_{\mu_b}(\mathbb{R}^n_+)
\]
for any $p \in (1,2]$, and for all $1 \leq i, j \leq n$. In applying the Marcinkiewicz Interpolation, we regard $\partial_{x,x_j} T_b$ as the mapping in [1].

When $p > 2$, let $p' = \frac{p}{p-1} < 2$. For any $f, g \in C^\infty_c(\mathbb{R}^n_+)$, noting that the kernel $K_b(x,y)$ in (5.3) is symmetric in $x'$ and $y'$, we have
\[
\int_{\mathbb{R}^n_+} g \partial_{ij} T_b(f)d\mu_b(x) = \int_{\mathbb{R}^n_+} f \partial_{ij} T_b(g)d\mu_b(x) \quad \forall 1 \leq i, j \leq n - 1.
\]
Here we denote by $C^\infty_c(\mathbb{R}^n_+)$ smooth functions in $\mathbb{R}^n_+$ with compact support. Hence we have
\[
\|\partial_{ij} T_b(g)\|L_{\mu_b}(\mathbb{R}^n_+) = \sup_{\|f\|L_{\mu_b}(\mathbb{R}^n_+) = 1} \int_{\mathbb{R}^n_+} f \partial_{ij} T_b(g)d\mu_b(x)
\leq \sup_{\|f\|L_{\mu_b}(\mathbb{R}^n_+) = 1} \|g\|L_{\mu_b}(\mathbb{R}^n_+) \|\partial_{ij} T_b(f)\|L_{\mu_b}(\mathbb{R}^n_+) \leq C_{p'}\|g\|L_{\mu_b}(\mathbb{R}^n_+) \quad \forall 1 \leq i, j \leq n - 1.
\]
By approximation, we see that (5.28) holds for non-smooth function $g$. 
To prove the estimate (5.28) for $\partial_n T_b(g)$, $i = 1, \cdots, n$, we need to introduce a new function $\tilde{u}$ defined in $\mathbb{R}^{n+1}$ (raising the dimension to $n+1$). Denote $u(x) = T_b(g)$ and

\begin{equation}
\tilde{u}(x, x_{n+1}) = u\left(x', \sqrt{x_n^2 + 2x_{n+1}}\right) = u(x', r),
\end{equation}

where $(x, x_{n+1}) \in \mathbb{R}^{n+1}$, $r = \sqrt{x_n^2 + x_{n+1}^2}$. By direct computation, we have

\begin{equation}
\begin{aligned}
\tilde{u}_x(x', x, x_{n+1}) &= u_x(x', r), \quad i = 1, \cdots, n-1, \\
\frac{\tilde{u}_x}{x_n} &= \frac{u_r}{r}, \\
\frac{\tilde{u}_x}{x_{n+1}} &= \frac{u_r}{r},
\end{aligned}
\end{equation}

Hence $\tilde{u}$ satisfies

\begin{equation}
\begin{cases}
\Delta_{x,x_{n+1}} \tilde{u} + \frac{b-1}{x_{n+1}} \tilde{u}_{n+1} = \tilde{g} & \text{in } \mathbb{R}^{n+1}, \\
\tilde{u}_{n+1}(x, 0) = 0 & \forall \ x \in \mathbb{R}^n,
\end{cases}
\end{equation}

where $\tilde{g}(x, x_{n+1}) = g(x', r)$, $\tilde{u}_{n+1} := \tilde{u}_{x_{n+1}}$. The boundary condition $\tilde{u}_{n+1}(x, 0) = 0$ is due to that $u$ is even in $x_{n+1}$. For equation (5.31), we need to assume that $b > 1$ for the $W^{2,p}$ estimate.

As $\tilde{u}$ and $T_{b-1}(\tilde{g})$ satisfy the same equation (5.31), by the Neumann boundary condition and the decay at $\infty$, we infer that $\tilde{u} = T_{b-1}(\tilde{g})$. Hence for $b > 1$ and $i = 1, \cdots, n$, similarly to estimate (5.28), we have

\begin{equation}
\int_{\mathbb{R}^{n+1}_+} |\tilde{u}_{n+1}|^p x_{n+1}^{b-1} dxdx_{n+1} \leq C_{p,b} \int_{\mathbb{R}^{n+1}_+} |\tilde{g}|^p x_{n+1}^{b-1} dxdx_{n+1}
\end{equation}

\begin{equation}
= \tilde{C}_{p,b} \int_{\mathbb{R}^n_+} |g|^p x_n^b dx.
\end{equation}

For the equality above, we use the polar coordinates for the variables $x_n, x_{n+1}$, i.e. $x_n = r \cos \theta$, $x_{n+1} = r \sin \theta$, $\theta \in [0, \pi]$. Notice that

\begin{align*}
\int_{\mathbb{R}^{n+1}_+} |\tilde{u}_{n+1}|^p x_{n+1}^{b-1} dxdx_{n+1} \\
= \int_{\mathbb{R}^{n+1}_+} |u_r| \frac{x_n}{r} |^p x_{n+1}^{b-1} dxdx_{n+1} \\
= \int_{\mathbb{R}^n_+} |u_r|^p r^{b} d^r \cdot \int_0^\pi |\cos \theta|^{p-b} d\theta \\
= \tilde{C}_{p,b} \int_{\mathbb{R}^n} |u_{n+1}|^p x_n^b dx.
\end{align*}
We obtain
\begin{equation}
\|\partial_{ij} T_b g\|_{L_p^b(\mathbb{R}^n_+)} \leq C_{b,n,p} \|g\|_{L_p^b(\mathbb{R}^n_+)}, \quad p \in (1, +\infty)
\end{equation}
for all $1 \leq i, j \leq n$.

By equation (5.1) and estimate (5.33), we also have
\begin{equation}
\|u_n\|_{L_p^b(\mathbb{R}^n_+)} \leq C \|f\|_{L_p^b(\mathbb{R}^n_+)},
\end{equation}
This completes the proof of Theorem 5.2.

Proof of Corollary 5.1. By Theorem 5.2 and the interpolation inequality, it suffices to prove
\begin{equation}
\|u\|_{L_p^b(\mathbb{R}^n_+)} \leq C \|f\|_{L_p^b(\mathbb{R}^n_+)}.\tag{5.35}
\end{equation}
By a dilation of the coordinates, we assume that supp $f \subset B_1(0)$. By Lemma 5.1, we have
\begin{equation}
|u(x)| \leq \frac{C \|f\|_{L_p^b(\mathbb{R}^n_+)}}{|x|^{n-2+b}} \leq \frac{C \|f\|_{L_p^b(\mathbb{R}^n_+)}}{|x|^{n-2+b}}, \quad |x| \geq 2.
\end{equation}
It follows that
\begin{equation}
\int_{\mathbb{R}^n_+ \backslash B_2} |u|^p d\mu_b \leq C_p \|f\|_{L_p^b(\mathbb{R}^n_+)}^p \int_{\mathbb{R}^n_+ \backslash B_2} \frac{x_n^b}{|x|^{p(n-2+b)}} dx \leq C_1 C_p \|f\|_{L_p^b(\mathbb{R}^n_+)}^p
\end{equation}
provided $p(n - 2 + b) > n + b$.

Let $\xi \in C^\infty(\mathbb{R}^n)$ be a cut-off function satisfying $0 \leq \xi \leq 1$ in $\mathbb{R}^n$, $\xi = 1$ in $B_2$, and $\xi = 0$ outside $B_4$. Then by Poincare’s inequality, we have
\begin{equation}
\int_{\mathbb{R}^n_+} |\xi u|^p d\mu_b \leq C \int_{\mathbb{R}^n_+} |D^2(\xi u)|^p d\mu_b \\
\leq C \int_{B_4 \cap \mathbb{R}^n_+} (|D^2 u|^p + |D^2 \xi|^p |u|^p) d\mu_b \\
\leq C \|f\|_{L_p^b(\mathbb{R}^n_+)}^p.
\end{equation}
Combining the above two estimates yields (5.35). \hfill \Box

We have obtained the $C^{2,\alpha}$ and $W^{2,p}$ estimates for the special solution $u = T_b(f)$. Next we prove that these two a priori estimates hold for any other solutions to (5.1). First we prove
Lemma 5.5. Let $u \in W^{2,p}_{\text{loc}}(\mathbb{R}_+^n) \cap C^1(\mathbb{R}_+^n)$ be a solution to
\begin{equation}
-\Delta u - b \frac{u_{x_n}}{x_n} = f \quad \text{in} \quad \mathbb{R}_+^n.
\end{equation}
Assume $b > 1$, $p > n + b$, and $f \in L^p_{\mu_b}(B_1^n)$. Then $u$ satisfies
\begin{equation}
u_{x_n} = 0 \quad \text{on} \quad x_n = 0.
\end{equation}

Proof. Let $\eta(\tau) \in C^\infty_c(\mathbb{R})$ be a cut-off function which satisfies $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_1(0)$, and supp $\eta \subset B_2(0)$. Let $\eta_\varepsilon(x) = \eta\left(\frac{|x|}{\varepsilon}\right)$ for a small constant $\varepsilon > 0$, and let $v = \eta_\varepsilon u$. Then $v$ solves the following equation
\begin{equation}
\mathcal{R}(v) =: \Delta v + b \frac{v_{x_n}}{x_n} = \eta_\varepsilon f + 2\partial_i u \partial_i \eta_\varepsilon + u \Delta \eta_\varepsilon + b u \frac{\partial_n \eta_\varepsilon}{x_n} =: \hat{f}.
\end{equation}
Denote $W = T_b(\hat{f})$, where $T_b(\cdot)$ is the integral operator given by (5.2). Denote $H(x_n) = x_n^{1-b}$. Then $\mathcal{R}(H) = 0$ and $H \to \infty$ as $x_n \to 0^+$. Hence
\begin{equation}
\mathcal{R}(v - W + \varepsilon H) = 0 \quad \text{in} \quad \mathbb{R}_+^n.
\end{equation}
By Lemma 5.1, $|W(x)| \leq C/|x|^{n-2+b} = o(H)$ as $|x| \to \infty$. Hence
\begin{equation}
\lim_{x \to \infty} (v - W + \varepsilon H) \geq 0
\end{equation}
for any given small $\varepsilon > 0$. By Theorem 5.2, $W \in W^{2,p}_{\mu_b}(\mathbb{R}_+^n)$. Hence $W \in L^\infty_{\text{loc}}(\mathbb{R}_+^n)$ by the Sobolev embedding [21, Lemma B.3]. We obtain
\begin{equation}
\lim_{x_n \to 0^+} (v - W + \varepsilon H) \geq 0.
\end{equation}
Moreover, since $W \in W^{2,p}_{\mu_b}(\mathbb{R}_+^n)$ for $p > n$ and $v \in C^2(B_1^n)$, we can apply Aleksandrov’s maximum principle for strong solution [18] and obtain $v - W \geq -\varepsilon H$. Letting $\varepsilon \to 0$, we obtain $v \geq W$. Similarly, we have $W \geq v$. This implies
\begin{equation}
\eta_\varepsilon u = v = W = T_b(\hat{f})
\end{equation}
and so (5.40) is proved. \qed

The condition $u \in C^1(\mathbb{R}_+^n)$ in Lemma 5.5 is such that $u_{x_n}$ exists on $x_n = 0$. If $p > n + b$ and $u \in W^{2,p}_{\text{loc}}(\mathbb{R}_+^n)$, by the Sobolev embedding we have $u \in C^1(\mathbb{R}_+^n)$. For the proof of (5.41), it suffices to assume that $u \in W^{2,p}_{\text{loc}}(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$. When $b \geq 1$ is an integer, $u_{x_n} + \frac{b}{x_n} u_{x_n}$ is actually the Laplacian operator for rotationally symmetric functions in $\mathbb{R}^{1+b}$. Lemma 5.5 means for bounded solutions, the singularity at $x_n = 0$ is removable.
By (5.41), we see that the $C^{2,\alpha}$ and $W^{2,p}$ estimates in Theorems 5.1 and 5.2 hold for any solutions $u \in W_{\text{loc}}^{2,p}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$. Moreover, the estimates can be extended easily to linear singular elliptic equations of Keldysh type with variable coefficients,

$$ \sum_{i,j=1}^n a^{ij}_i \partial_{ij} u + \sum_{i=1}^{n-1} b^i \partial_i u + \frac{b^n}{x_n} \partial_n u + cu = f \quad \text{in } B_1^+. $$

Assume that

$$ 0 < \lambda I \leq (a^{ij})_{i,j=1}^n \leq \Lambda I < +\infty \quad \text{in } B_1^+, $$

$$ \frac{b^p}{a^{nn}} = b > 1 \quad \text{is a constant,} $$

$$ |c| + \sum_{i=1}^n |b^i| \leq \Lambda \quad \text{in } B_1^+, $$

for two positive constants $\lambda, \Lambda$. Then by the freezing coefficient method, we have the following a priori estimates.

**Theorem 5.3.** Let $u \in W_{\text{loc}}^{2,p}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ be a solution to (5.42). Assume that $a^{ij} \in C(B_1^+)$ satisfy conditions (5.43), and $f \in L^p_{\mu_b}(B_1^+)$ for some $p > n + b$. Then $u \in W^{2,p}_{\mu_b}(B_{1/2}^+)$ and satisfies the estimate

$$ \|u\|_{W^{2,p}_{\mu_b}(B_{1/2}^+)} + \|\frac{u_n}{x_n}\|_{L^p_{\mu_b}(B_{1/2}^+)} \leq C \left( \|f\|_{L^p_{\mu_b}(B_1^+)} + \|u\|_{L^p_{\mu_b}(B_1^+)} \right), $$

where $C > 0$ depends only on $p, n, \lambda, \Lambda$ and the modulus of continuity of $a^{ij}$.

**Theorem 5.4.** Let $u \in C^2(B_1^+) \cap C^{1,\alpha}(\overline{B_1^+})$ be a solution to (5.42). Assume that $a^{ij}, b^n, c, f \in C^{\alpha}(\overline{B_1^+})$ and condition (5.43) holds. Then $u \in C^{2,\alpha}(B_{1/2}^+)$ and we have the estimate

$$ \|u\|_{C^{2,\alpha}(B_{1/2}^+)} + \|\frac{u_n}{x_n}\|_{C^{\alpha}(B_{1/2}^+)} \leq C \left( \|f\|_{C^{\alpha}(B_1^+)} + \|u\|_{C^{\alpha}(B_1^+)} \right), $$

for a constant $C > 0$ depending only on $n, \alpha, \lambda, \Lambda$ and $\|a^{ij}\|_{C^{\alpha}(\overline{B_1^+})}, \|b^i\|_{C^{\alpha}(\overline{B_1^+})}, |c|_{C^{\alpha}(\overline{B_1^+})}$.  

**5.2. Smoothness of free boundary.** By the $C^{2,\alpha}$ and $W^{2,p}$ estimates in Section 5.1, we can prove the higher regularity of the tangent cone of the solution $u$ to (1.1). As before we assume that $u(0) = 0$ and $u(x) > 0 \forall x \neq 0$.

**Theorem 5.5.** Let $\phi$ be the tangential cone of $u$ at 0. Then the section $S_{\phi} = \{x \in \mathbb{R}^n : \phi(x) < 1\}$ is uniformly convex and $C^\infty$ smooth if $g$ is positive and $C^\infty$ smooth.

To prove Theorem 5.5 by the definition of $\zeta$ in (4.3), it suffices to prove that $\zeta(\theta, 0) \in C^\infty(S^{n-1})$. We have the following stronger result, which implies Theorems 5.5 and 1.2.
Theorem 5.6. Let $\zeta(\theta, s) \in C^2(S^{n-1} \times [0, 1))$ be a solution to (4.10). Assume that $\zeta_s(\theta, 0) = 0$ and $\bar{g}$ is positive and smooth. Then $\zeta(\theta, s) \in C^\infty(S^{n-1} \times [0, 1))$.

Proof. Differentiating (4.10) with respect to $\theta_k$, one gets

$$(5.46) \quad L(V) = V_{ss} + \frac{n + 2}{n} \frac{V_s}{s} + \sum_{i,j=1}^{n-1} a^{ij} V_{\theta_i \theta_j} + \sum_{i=1}^{n-1} a^{i\alpha} V_{\theta_i \alpha} = h$$

where $V = \zeta_{\theta_k}$. To apply the a priori estimate in §5.1 to (5.46), we express equation (5.46) in a local coordinates on $S^{n-1}$. By Theorem 4.2, $a^{ij}$ and $h$ are continuous in $\theta, s$. By Lemma 4.2 the operator $L$ is uniformly elliptic. Hence all the assumptions in Theorem 5.3 are fulfilled for $V$. Letting $p > n + b$ and by the Sobolev embedding, $W^{1,p} \to C^\alpha$ for some $\alpha > 0$ (Lemma B.3, [21]), we have $V = \zeta_{\theta_k} \in C^{1,\alpha}(S^{n-1} \times [0, 1))$.

Write equation (4.10) in the form

$$(5.47) \quad \zeta_{ss} + \frac{n + 2}{n} \zeta_s = \tilde{f} \in C^\alpha(S^{n-1} \times [0, 1]),$$

where $\tilde{f}$ is a smooth function of $\theta, s, \zeta, D\zeta, D\zeta$. Hence $\tilde{f}$ is Hölder continuous in $s, \theta$. The solution to (5.47) is given by

$$(5.48) \quad \zeta(\theta, s) = \zeta(\theta, 0) + \int_0^s r^{-\frac{n+2}{n}} \int_0^r \lambda^{\frac{n+2}{n}} \tilde{f}(\theta, \lambda) d\lambda.$$

Hence we have

$$(5.49) \quad \zeta_s(\theta, s) = s^{-\frac{n+2}{n}} \int_0^s \lambda^{\frac{n+2}{n}} \tilde{f}(\theta, \lambda) d\lambda,$$

$$\zeta_{ss}(\theta, s) = -\frac{n + 2}{n} s^{-\frac{n+2}{n}-1} \int_0^s \lambda^{\frac{n+2}{n}} \tilde{f}(\theta, \lambda) d\lambda + \tilde{f} \in C^\alpha(S^{n-1} \times [0, 1]).$$

This implies $\zeta \in C^{2,\alpha}(S^{n-1} \times [0, 1))$. Hence the coefficients $a^{ij}, h \in C^\alpha(S^{n-1} \times [0, 1))$.

By the Hölder continuity of the coefficients, we can obtain the $C^{2,\alpha}$ regularity of $D^k_\theta \zeta$ for all $k \geq 1$. Indeed, differentiating equation (5.46) in $\theta$, and by Theorem 5.4, we infer that $\zeta_{\theta_k} \in C^{2,\alpha}(S^{n-1} \times [0, 1))$. Differentiating equation (5.46) in $\theta$ again, and also by Theorem 5.4, we have $D^2_\theta \zeta \in C^{2,\alpha}(S^{n-1} \times [0, 1))$. Repeating the argument we obtain $D^k_\theta \zeta \in C^{2,\alpha}(S^{n-1} \times [0, 1))$ for all integers $k \geq 0$.

To prove the higher order regularity of $\zeta$ in $s$ and the expansion (1.7), let $t = \frac{s^2}{2}$. Then equation (4.10) changes to equation (4.20). Differentiating (4.20) in $t$ and letting $V = \zeta_t$, one gets

$$(5.50) \quad L(V) = tV_{tt} + \frac{2n + 1}{n} V_t + \sum_{i,j=1}^{n-1} a^{ij} V_{\theta_i \theta_j} + t \sum_{i=1}^{n-1} \tilde{a}^{i\alpha} V_{\theta_i \alpha} = h$$
where \( a^{ij}, \tilde{a}^{ij}, h \) are smooth as functions of \( \theta, t, \zeta, D_{\theta t}^2 \zeta, D_{\theta}^2 \zeta, \zeta_{\theta}, t_{\zeta t} \). Note that the operator \( L \) in (5.50) is different from that in (4.27). We have moved some terms to the right hand side.

By the regularity \( \zeta(\theta, s), \zeta_{\theta}(\theta, s) \in C^{2, \alpha}(S^{n-1} \times [0, 1]) \) and the relation \( t = \frac{s^2}{4} \), we have \( D_{\theta}^2 \zeta, \zeta_{\theta}, t_{\zeta t} \in C^{0}(S^{n-1} \times [0, 1]) \) in variables \( \theta, t \). Hence, \( a^{ij}, \tilde{a}^{ij}, h \in C^{\alpha} \) in variables \( \theta, t \).

To apply Theorem 5.4 to equation (5.50), we need to change the Hölder norm from the variable \( s \) to \( t = s^2/4 \). Hence we denote

\[
\| f \|_{\tilde{C}^\alpha(B_t^1)} = \| f \|_{L^\infty(B_t^1)} + \sup_{x \neq y \in B_t^1} \frac{|f(x) - f(y)|}{|x' - y'|^2 + (\sqrt{x_n} - \sqrt{y_n})^2}^{\frac{\alpha}{2}},
\]

\[
\| f \|_{\tilde{C}^{k, \alpha}(B_t^1)} = \| f \|_{C^k(B_t^1)} + \sup_{x \neq y \in B_t^1, |\beta| = k} \frac{|D^k f(x) - D^k f(y)|}{|x' - y'|^2 + (\sqrt{x_n} - \sqrt{y_n})^2}^{\frac{\alpha}{2}}.
\]

Then estimate (5.45) can be reiterated as

\[
\| tu_{tt} \|_{\tilde{C}^\alpha(B_{t}^{1/2})} + \| t^{\frac{1}{2}} D_{\theta} u_t \|_{\tilde{C}^\alpha(B_{t}^{1/2})} + \| D_{\theta}^2 u \|_{\tilde{C}^\alpha(B_{t}^{1/2})}
\]

\[
+ \| u \|_{\tilde{C}^{1, \alpha}(B_{t}^{1/2})} \leq C \left( \| f \|_{\tilde{C}^\alpha(B_t^1)} + \| u \|_{\tilde{C}^\alpha(B_t^1)} \right).
\]

Applying (5.51) to equation (5.50), we obtain \( t \partial_\theta^2 \zeta, \partial_\theta^2 \partial_\theta \zeta, t^{1/2} \partial_\theta^2 \partial_\theta \zeta \in \tilde{C}^\alpha(S^{n-1} \times [0, 1]) \), and \( \zeta \in \tilde{C}^{1, \alpha}(S^{n-1} \times [0, 1]) \). Differentiating (5.50) in \( t \) repeatedly and using estimate (5.51), we obtain \( \zeta \in C^{\infty}(S^{n-1} \times [0, 1]) \).

In the above we have shown that \( \zeta \) is smooth in \( t \). Recall that \( t = \frac{s^2}{4} = \frac{r^2}{4} \), and \( \zeta = \frac{r}{4} \). Hence we obtain the Taylor expansion (11.7). This completes the proof of Theorem 5.6 and also that of Theorem (1.2). □

6. Analyticity of the free boundary

In this section, we prove the analyticity of the free boundary \( \Gamma \). Let \( u \) be the solution to (4.1). Let \( \zeta = \frac{r}{4} \) and \( s = \frac{r^{n/2}}{4} \) as in §4, so that \( \zeta \) satisfies equation (4.10). Then the analyticity of the free boundary is equivalent to showing \( \zeta(\theta, 0) \in C^\omega(S^{n-1}) \). Here \( C^\omega \) denotes the set of analytic functions. As before we assume that \( u(0) = 0 \) and \( u(x) > 0 \) \( \forall x \neq 0 \). We have the following result.

**Theorem 6.1.** Let \( \zeta(\theta, s) \) be a solution to (4.10). Assume that \( \bar{g} \) is positive and analytic. Then \( \zeta(\cdot, s) \in C^\omega(S^{n-1}) \), for any \( s \in [0, 1] \).
To prove the analyticity of a function $u$, one needs to control the growth rate of its derivatives. That is, for any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we need to prove
\begin{equation}
|\partial^\alpha u| \leq CA^{\lvert \alpha \rvert} \alpha!
\end{equation}
for sufficiently large constants $C, A$, independent of $\alpha$, where $\lvert \alpha \rvert = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

We will use the local coordinate system (4.5) in a neighbourhood of $\theta = 0$. For simplicity of notations, we use $x' = (x_1, \cdots, x_{n-1})$ to denote $\theta = (\theta_1, \cdots, \theta_{n-1})$, and use $x_n$ to denote $s$. Then in the coordinates (4.5), equation (4.10) can be written in the form
\begin{equation}
\zeta_{nn} + b\zeta_n + F(x, \zeta, \zeta_x, \zeta_{xx}) = 0, \quad (x', x_n) \in Q_{r_0},
\end{equation}
where $b = \frac{n+2}{n}$, $Q_{r_0} = B'_{r_0}(0) \times [0, 1]$, and $B'_{r_0}(0) = \{|x'| < r_0\}$ is a ball in $\mathbb{R}^{n-1}$.

The function $F(x, z, p, r)$ is defined for $x \in Q_{r_0}$, $z \in \mathbb{R}^1$, $p \in \mathbb{R}^n$, and $r = (r_{ij}) \in \mathbb{S}^{n \times n}$ but is independent of $r_{nn}$. As a function, $F$ is analytic in its arguments. By Lemma 4.2, equation (6.2) is uniformly elliptic.

The analyticity of solutions to uniformly elliptic equations has been studied by many people [15, 33]. A simple proof for the linear elliptic equation was found in [27], and it was extended to nonlinear elliptic equation in [2]. Here we adopt the proof from [27, 2]. By [15, 33], $\zeta(x)$ is analytic when $x_n > 0$. Here we show that $\zeta(x)$ is analytic in $x'$ when $x_n = 0$.

In [27], Kato demonstrated his idea by considering the equation
\begin{equation}
\Delta u = u^2 \quad \text{in} \quad \Omega,
\end{equation}
where $\Omega$ is a domain in $\mathbb{R}^n$. Instead of (6.1), Kato’s strategy is to establish the estimate
\begin{equation}
\|\rho|\alpha|\partial^\alpha u\|_{H^m(B_{r_0})} \leq CA^{\lvert \alpha \rvert}|\alpha|!,
\end{equation}
where $\rho$ is a cut-off function such that $\rho = 1$ in $B_{r_0/2}$ and $\rho = 0$ outside $B_{r_0}$. He chooses $m = \left[\frac{n}{2}\right] + 1$ such that $\|u\|_{L^\infty(B_{r_0})} \leq C_{r_0} \|u\|_{H^m(B_{r_0})}$. Hence (6.4) implies (6.1). In [2], Blatt extended the estimate (6.4) to the general fully nonlinear, uniformly elliptic equation
\begin{equation}
\Phi(x, u(x), Du(x), D^2u(x)) = 0 \quad \text{in} \quad \Omega.
\end{equation}

The proof in [27] is rather simple, one can easily see that the norm $H^m(B_{r_0})$ in [27, 2] can be replaced by the Hölder space $C^\delta(B_{r_0})$, using the $C^{2,\delta}$ estimate (Schauder estimate) for the Laplace equation.
To apply the argument in [27, 2] to our equation (6.2), we use the Hölder norm $\| \cdot \|_{C^8(B_{r_0})}$ instead of the norm $\| \cdot \|_{H^6(B_{r_0})}$, and use the $C^{2,\delta}$ estimate (Theorem 5.4). As our equation contains the singular term $\frac{\rho}{\rho_n}$, we cannot obtain the analyticity of $\zeta$ on $x_n$ (near $x_n = 0$) by their simple proof, but we can obtain the analyticity of $\zeta$ on $x'$, namely

$$\| \rho^{|\alpha|} \partial_x^\alpha \zeta \|_{C^8(B_{r_0})} \leq CA|\alpha|!$$

for all multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$.  

**Proof of Theorem 6.1**. As the proof is similar to that in [27, 2], we sketch the main idea only. Let

$$L[\phi] := \phi_{nn} + b\frac{\phi_n}{x_n} + \sum_{i+j\leq 2n} a_{ij}(x)\phi_{ij} + \sum_{i=1}^n b_i(x)\phi_i + c(x)\phi$$

be the linearized operator of (6.2), where $a_{ij}, b_i, c$ are functions of $x, \zeta, \partial_i \zeta$ ($1 \leq i \leq n$) and $\partial^2 \zeta$ ($i+j < 2n$), and $b = \frac{n+2}{n}$. From the proof of Lemma 4.3 we have $a_{in}(0) = 0$ for $i \leq n - 1$. Denote

$$L_0[\phi] := \phi_{nn} + b\frac{\phi_n}{x_n} + \sum_{i,j < n} a_{ij}(0)\phi_{ij}.$$ 

Let $\rho = \rho(|x'|)$ be a cut-off function of $x'$, such that $\rho(x') = 1$ when $|x'| < r_0/2$ and $\rho(x) = 0$ when $|x'| > r_0$. For any multi-indices $\alpha \in \mathbb{R}^{n-1}$ and $\beta \in \mathbb{R}^n$ with $|\alpha| = N-1 \geq 1$ and $|\beta| = 2$, as in [27, 2], we compute

$$\| \rho^{N-1} \partial^\alpha x_{\beta} \phi \| \leq \| \partial^\alpha x_{\beta} [\rho^{N-1} \partial^\alpha x_{\beta} u] \| + \| \partial^\alpha x_{\beta} \rho^{N-1} \partial^\alpha x_{\beta} u \| \leq C\| L_0 [\rho^{N-1} \partial^\alpha x_{\beta} u] \| + \| \partial^\alpha x_{\beta} \rho^{N-1} \partial^\alpha x_{\beta} u \| + CA^{N+1}(N+1)!$$

(6.8) 

$$= C\| L_0 [\rho^{N-1} \partial^\alpha x_{\beta} u] \| + \| \partial^\alpha x_{\beta} \rho^{N-1} \partial^\alpha x_{\beta} u \| + CA^{N+1}(N+1)!$$

Here $\| \cdot \| := \| \cdot \|_{C^4(\overline{B}_{3r_0} \times [0, r_0])}$, $B'_{r_0} = \{ x' \in \mathbb{R}^{n-1} : |x'| < r_0 \}$, and

$$[\partial^\alpha x_{\beta} \rho^{N-1} \partial^\alpha x_{\beta} u] = \rho^{N-1} \partial^\alpha x_{\beta} \partial^\alpha x_{\beta} u - \partial^\alpha x_{\beta} [\rho^{N-1} \partial^\alpha x_{\beta} u].$$

In the second inequality of (6.8), we use the $C^{2,\delta}$ estimate (Theorem 5.4) in the form

$$\| u \|_{C^{2,\delta}(\overline{B}_{2r_0} \times [0, 2r_0])} \leq C \left( \| f \|_{C^0(\overline{B}_{2r_0} \times [0, 2r_0])} + \| u \|_{C^0(\overline{B}_{2r_0} \times [0, 2r_0])} \right)$$

(6.9) 

But the cut-off function $\rho$ is supported in $B'_{r_0}$ and $u$ is analytic in the interior. Hence when applying (6.9) to (6.8), the domain $\overline{B}_{2r_0} \times [0, 2r_0]$ on the RHS of (6.9) can be replaced by $\overline{B}_{r_0} \times [0, r_0]$ plus an additional term $CA^{N+1}(N+1)!$. 
The estimation for $I_1, I_2, I_3$ is the same as in \cite{27} and is omitted here. Note that by (6.8) and iteration, we not only obtain (6.6), but also

\begin{equation}
\|\rho^{[\alpha]} \partial_x^2 \partial_x^{-2} \zeta\|_{C^5(B_{r_0})} \leq CA^{[\alpha]} |\alpha|!.
\end{equation}

(6.10) is needed in the estimation of $I_1, I_2, I_3$ when differentiating equation (6.7).

\[\square\]

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