Electroweak effects in the $B^0 - \bar{B}^0$ mixing

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Abstract

We compute analytically the complete electroweak two-loop corrections to the $B^0 - \bar{B}^0$ mixing. These corrections fix the normalization of the electroweak coupling employed in the extraction of $|V_{td}|$ and reduce the theoretical uncertainty due to higher order electroweak effects from several percent to a few parts in a thousand. If the LO result is expressed in terms of $G_\mu$ or of the $\overline{\text{MS}}$ coupling $\hat{g}(M_Z)$, the two-loop corrections are $O(1\%)$, the exact value depending on the mass of the Higgs boson. We discuss in detail the renormalization procedure and the scheme and scale dependence, and provide practical formulas for the numerical implementation of our results. We also consider the heavy top mass expansion and show that in the case at hand it converges very slowly.

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1 Introduction

The $B^0 - \bar{B}^0$ system offers rich possibilities for studies of CP violation and the quark mixing structure of the Standard Model (see [1] for a recent review). The physics of this system is well described by $\mathcal{H}_{\Delta B=2}^{\text{eff}}$, the effective low-energy Hamiltonian for the $B^0 \leftrightarrow \bar{B}^0$ transition, and the most important observable directly linked to $\mathcal{H}_{\Delta B=2}^{\text{eff}}$ is $\Delta M_{B^0}$, the mass difference between the heavy and the light mass eigenstates in the $B^0 - \bar{B}^0$ system. Theoretically, this quantity is given by

$$\Delta M_{B^0} = \frac{1}{m_B} |\langle \bar{B}^0 | \mathcal{H}_{\Delta B=2}^{\text{eff}} | B^0 \rangle|,$$

(1)

and the experimentally measured value is [2]

$$\Delta M_{B^0} = (0.46 \pm 0.02) \times 10^{12} \text{ s}^{-1}.$$ 

(2)

If all other ingredients in the evaluation of the r.h.s. of Eq. (1) are sufficiently well known, one can extract from this measurement the absolute value of the Cabibbo-Kobayashi-Maskawa (CKM) parameter $V_{td}$, which plays an important role in the standard analysis of the unitary triangle [1]. This example illustrates the phenomenological relevance of a precise determination of $\mathcal{H}_{\Delta B=2}^{\text{eff}}$ and its matrix element, especially if one takes into account the experimental accuracy already achieved in the measurement of $\Delta M_{B^0}$.

The two main steps for such a determination are (i) the calculation of $\mathcal{H}_{\Delta B=2}^{\text{eff}}$ (or more precisely: the corresponding Wilson coefficient) in perturbation theory – in particular, this calculation yields the complete dependence of $\mathcal{H}_{\Delta B=2}^{\text{eff}}$ on all the heavy degrees of freedom, and (ii) the evaluation of the remaining low-energy matrix element on the lattice or by some other non-perturbative method. Presently, this second step causes the largest theoretical uncertainty in the determination of $\langle \mathcal{H}_{\Delta B=2}^{\text{eff}} \rangle$, about $\pm 20\%$ (dominated by systematics of the lattice calculation, i.e. extrapolation to the continuum [3,4]). There is, however, well-founded hope that with increasing computer power or by new developments in lattice theory, this uncertainty may be significantly reduced in the near future.

The perturbative calculation, on the other hand, has been by now performed at the level of next-to-leading order QCD [5]. This analysis includes the matching at $O(\alpha_s)$ and the $O(\alpha_s^2)$ anomalous dimension of the relevant four-quark operator; the achieved accuracy is better than $\pm 1\%$ [6].

In this paper, we focus on a different part of the perturbative analysis which has not yet been considered in the literature: the two-loop electroweak correction to the effective Hamiltonian $\mathcal{H}_{\Delta B=2}^{\text{eff}}$. In our opinion, at least three reasons suggest such a calculation:

- **Reduction of scheme dependence.** The leading order (LO) result (Fig. 1a shows one of the relevant diagrams) is proportional to $g^4$, where $g$ is the weak coupling. However,
Figure 1: (a) shows one of the box diagrams contributing to the $B^0 \rightarrow \bar{B}^0$ transition at leading order, whereas (b) depicts an example of an electroweak two-loop correction to this process which may be seizable in the large $M_t$ limit.

at LO no renormalization prescription for $g$ is needed, so one can use the numerical value of $g$ in any renormalization scheme. For example, the difference between $g^4$ at the scale $M_Z$ calculated from $\sin^2 \theta_{w}^{\text{eff}} = 0.23155$ and $\alpha(M_Z) = (128.9)^{-1}$, or from the relation $g^2 = 8M_w^2G_\mu/\sqrt{2}$ amounts to about 2.5%. As noted in [8], such ambiguity reflects the uncertainty of the LO result due to the uncalculated electroweak two-loop correction, and it already exceeds the existing perturbative QCD uncertainty mentioned above. In a similar way, if we express the LO result in terms of the \overline{MS} coupling $\hat{g}(\mu)$, and change the scale $\mu$ between $M_W/2$ and $2M_W$, we obtain a 5% variation, a normalization ambiguity which is almost completely removed by the consideration of two-loop electroweak effects.

- **Possible large corrections due to a heavy top.** The coupling of the Higgs particle to the top quark is proportional to $g M_t/M_w$. Since $g M_t/M_w \approx g_s$ at the scale $M_Z$, from diagrams like the one in Fig. 1b one can expect large contributions of roughly the same order of magnitude as the QCD corrections.

- **Performing a complete two-loop electroweak calculation.** Despite recent efforts, there exist very few nearly complete electroweak calculations at the two-loop level [10], and many available results rely on heavy mass expansions [11, 14], which are known to work very well in specific examples [14]. In other cases (e.g. the important two-loop QED corrections to muon decay [16] and some electroweak corrections to $B \rightarrow X_s\gamma$ computed in [17]), only gauge-invariant subsets of diagrams have been considered. Similarly to the case of the anomalous magnetic moment of the muon.
considered in [10], the mixing $B^0 - \bar{B}^0$ is a low-energy process which first occurs at one-loop level. Its two-loop electroweak corrections therefore require only one-loop renormalization and the relevant two-loop integrals can be expressed in terms of elementary functions, because the external momenta can be generally neglected. As will be illustrated in the following, this allows one to compute analytically the complete electroweak effects in a relatively simple way. The calculation presented here has therefore some interest in itself, independent of phenomenological applications. In particular, it exhibits the complete Higgs mass dependence and enables one to assess the validity of the heavy mass expansion in the case of electroweak boxes. Interestingly, our results show a very slow convergence of the heavy top mass expansion in the case at hand.

In addition to the strong motivation provided by the above three points, it should also be mentioned that our results represent an important subset of the electroweak corrections to the $K^0 - \bar{K}^0$ mixing, whose treatment is more involved and will not be considered in the present work.

This paper is organized as follows: in Sec. 2 we outline the strategy we have devised and the general features of the calculation; in Sec. 3 we describe the renormalization procedure in detail, giving explicit expressions for the renormalization constants we have used. Finally, Sec. 4 is devoted to a discussion of our results and of their main consequences, which are summarized in the Conclusions.

2 Outline of the calculation

2.1 The effective Hamiltonian

The effective Hamiltonian $H_{\Delta B^0=2}^{\text{eff}}$ for the $B^0 \leftrightarrow \bar{B}^0$ transition can schematically be written as

$$H_{\Delta B^0=2}^{\text{eff}} = \frac{1}{16\pi^2} \frac{g^4 \lambda_t^2}{32 M_W^2} C_{BB}(\{M\}, \mu) \hat{Q}_{LL} + \text{H.c.}, \quad (3)$$

where $\lambda_t = V_{tb}^* V_{td}$ denotes the CKM factor, $\hat{Q}_{LL} = \bar{b} \gamma^\mu (1 - \gamma_5) d \otimes \bar{b} \gamma^\mu (1 - \gamma_5) d$ is the relevant four-quark operator, and $C_{BB}(\{M\}, \mu)$ is the corresponding Wilson coefficient, to be calculated within perturbation theory. The latter contains all information on the heavy degrees of freedom, generically indicated by $\{M\}$, and depends additionally on the scale $\mu$ at which the operator $\hat{Q}_{LL}$ is renormalized. In principle, this scale can be chosen arbitrarily — physical quantities do not depend on it and therefore the $\mu$-dependence of the Wilson coefficient will cancel against an analogous $\mu$-dependence of the matrix element.
\(\langle \hat{Q}_{LL} \rangle\) order by order in perturbation theory. It is physically natural, however, to choose \(\mu\) of the order of \(m_b\), since the \(b\) mass is clearly the relevant scale for the evaluation of the matrix element \(\langle \hat{Q}_{LL} \rangle\).

Including next-to-leading order (NLO) QCD corrections as well as two-loop electroweak corrections, the Wilson coefficient \(C_{BB}\) assumes the form

\[
C_{BB}(\{M\}, \mu_b) = \bar{\eta}_{2B} \left\{ S_0(w_t) + \delta S_{ew} \right\},
\]

where the Inami-Lim function \(w_t = M_W^2/M_t^2\)

\[
S_0(w_t) = \frac{1 - 11w_t + 4w_t^2}{4w_t(w_t - 1)^2} + \frac{3 \ln w_t}{2(w_t - 1)^3},
\]

represents the leading order result [18]. The factor \(\bar{\eta}_{2B}\) contains the complete NLO QCD corrections including the running of the Wilson coefficient down to the scale \(\mu = m_b\) [6]; \(\bar{\eta}_{2B}\) is scheme dependent and its scheme dependence is canceled by analogous terms in the matrix element (see discussion in [6]). In the NDR scheme and using an \(\overline{\text{MS}}\) top mass evaluated at \(M_t\) (see Sec. 3.1), \(\bar{\eta}_{2B} \approx 0.85\). By \(\delta S_{ew}\), instead, we denote the electroweak two-loop correction of order \(g^2\). The purpose of this paper is to calculate this quantity. For a better understanding of this calculation, and in order to introduce some relevant notation, we will first briefly recall some important features of the leading order (one-loop) calculation.

### 2.2 Summary of one-loop results

An important attribute of Wilson coefficients in general is their independence of the infrared region of the theory, i.e. “soft” physics. In particular, this means that one can calculate them using arbitrary kinematic configurations of the external particles (on-shell as well as off-shell), and arbitrary values for the light quark masses. In the case at hand, all quarks but the top can be considered light, and only the top has been integrated out in order to obtain the effective theory of Eq. (3). This freedom in the choice of the “soft” parameters crucially simplifies the calculation: in the following we will always adopt the simplest approach and work with zero external momenta and zero masses of the light quarks (except where these masses are needed as infrared regulator).

Keeping this in mind, the one-loop amplitude for the process \(\bar{b} + d \to b + \bar{d}\) with vanishing external momenta and all light quark masses set to zero can be written as

\[
\mathcal{M}_{1\text{loop}} = \frac{-i}{16\pi^2} \frac{g^4}{8M_W^2} \sum_{i,j} \lambda_i \lambda_j S^{(i,j)} \langle \hat{Q}_{LL} \rangle
\]
where $\lambda_i = V_{it}^2 V_{id}$ and the sum is over the leading order box diagrams with internal quarks $i$ and $j$ ($i, j = u, c, t$), calculated in $n$ dimensions. $S^{(i,j)}$ is the general box function given e.g. in Eq. (2.2) of [3]. Using $m_u = m_c = 0$ and the unitarity of the CKM matrix, one obtains

$$M_{1\text{loop}} = \frac{-i}{16\pi^2} \frac{g_4^4}{8M_W^2} \left[ S^{(t,t)} - 2 S^{(t,c)} + S^{(c,c)} \right] Q_{LL}. \quad (7)$$

The survival of the sole $\lambda_t$ term in Eq. (7) is a consequence of the hard (power-like) GIM cancellation mechanism which takes place in the case of electroweak boxes. For later convenience, we present here the explicit result for the function

$$S(w_t, \bar{\mu}^2/M_t^2) \equiv S^{(t,t)} - 2 S^{(t,c)} + S^{(c,c)} = S_0(w_t) + \epsilon S_1(w_t, \bar{\mu}^2/M_t^2) + O(\epsilon^2) \quad (8)$$

where $\epsilon = 2 - n/2$ and we have included the $O(\epsilon)$ terms: $S_0(w_t)$ has already been given in Eq. (3), and the function $S_1$ reads

$$S_1(w_t, \bar{\mu}^2/M_t^2) = \frac{3 - 33w_t - 4w_t^2}{8w_t(w_t - 1)^2} + \frac{2 + 15w_t}{4(w_t - 1)^3} \ln w_t - \frac{3\ln^2 w_t}{4(w_t - 1)^3} + S_0(w_t) \ln \frac{\bar{\mu}^2}{M_t^2}, \quad (9)$$

with $\bar{\mu}^2 = 4\pi e^{-\gamma_E} \mu^2$.

Before proceeding further, some notation should be introduced. The electroweak coupling is generically denoted by $g$, the sine and cosine of the Weinberg angle by $s$ and $c$, respectively. At the order we are working at, however, it is at some point necessary to specify the renormalization schemes (to be discussed in the following) in which these quantities are defined. Specifically, a hat ($\hat{g}$, $\hat{s}$, $\hat{c}$) will always denote $\overline{\text{MS}}$ scheme quantities and the subscript $W$ ($g_W$, $s_W$, $c_W$) the electroweak on-shell scheme. Furthermore, throughout the paper the following short-hand forms for the ratios of masses are used:

$$w_t = \frac{M_W^2}{M_t^2}, \quad z_t = \frac{M_Z^2}{M_t^2}, \quad h_t = \frac{M_H^2}{M_t^2}, \quad (10)$$

where $M_W$, $M_Z$, $M_t$ and $M_H$ are understood as on-shell masses.

### 2.3 The matching procedure at two loops

We are now ready to describe the strategy we have followed to perform our calculation. As a first step, we compute the complete electroweak corrections to the amplitude, setting all external masses and momenta and the internal light quark masses to zero. After renormalization, these corrections are of course ultraviolet finite, but some of the diagrams
containing the photon need the introduction of an infrared regulator. Following the QCD analysis of [5], we choose it to be a common internal light quark mass. The renormalized two-loop amplitude can then be expressed as

$$M_{2\text{loop}} = -\frac{i}{(16\pi^2)^2} \frac{g^6 \lambda^2}{8 M_W^2} \Delta^{(2)} Q_{LL},$$

(11)

where $\Delta^{(2)}$ is given by the sum of (i) the unrenormalized one-particle irreducible two-loop diagrams shown in Fig. 2a-h, (ii) the various counterterm contributions $\Delta^c_i$ (to be discussed in Sec. 3), (iii) a contribution $\Delta_{DT}$ from the one-particle reducible diagrams depicted in Fig. 2i:

$$\Delta^{(2)} = \Delta^{\text{unren}}_{2\text{loop}} + \Delta^c_{M_t} + \Delta^c_{M_W} + \Delta^c_g + \Delta^c_\psi + \Delta^c_{\text{Tad}} + \Delta_{DT}.$$  

(12)

The last contribution, $\Delta_{DT}$, can be gleaned from [18]; it originates from a finite but gauge-dependent subset of two-loop diagrams and was already considered in [5]. We will discuss it in detail in Sec. 2.5.

As a second step, we consider the effective theory of Eq. (3), in which all degrees of freedom of the Standard Model (SM) with $M \geq M_W$ are decoupled, and perform the matching between the full theory and this effective theory at first order in QED. The amplitude in the effective theory is

$$M_{\text{eff}} = -\frac{i}{(16\pi^2)^2} \frac{g^6 \lambda^2}{8 M_W^2} \Delta_{\text{eff}}^{QED} (\mu) Q_{LL},$$

(13)

where

$$\Delta_{\text{eff}}^{QED} (\mu) = \frac{2}{3} s^2 S_0(w_t) \left( \ln q_t - \ln \frac{\mu^2}{M_t^2} - 2 \right).$$

(14)

Here, $q_t = m_q^2 / M_t^2$ ($m_q$ denotes the generic mass of the internal light quarks) plays the role of an infrared regulator. The electroweak correction to the relevant Wilson coefficient, proportional to the difference $\Delta^{(2)} - \Delta_{\text{eff}}^{QED} (\mu)$, is free from infrared divergences, as the dependence on $q_t$ cancels against analogous terms in $\Delta^{(2)}$. The Wilson coefficient, however, depends crucially on the scale $\mu$ at which the operator $\hat{Q}_{LL}$ is renormalized. It is clear that our analysis also requires the knowledge of QED corrected matrix elements at the same scale $\mu$. At least in principle, they could be provided by dedicated lattice calculations, but from a practical point of view it is likely that, since the QED corrections turn out to be very small, their impact would be irrelevant.

\footnote{In calculating Eq. (14) we have employed the projection method described in the next subsection, thus conforming to the same choice of evanescent operators as in [R]. A detailed discussion of the definition of evanescent operators can be found in [27, 28].}
We now consider the evolution of the Wilson coefficient down to the bottom mass scale. At lowest order in $\alpha_{QED}$, this amounts simply to setting $\mu = \mu_b$, with $\mu_b \approx m_b$ in Eq. (13). By adapting the available QCD calculations \cite{5} for the NLO anomalous dimension of the operator $\hat{Q}_{LL}$ to the case of QED, the evolution of the Wilson coefficient down to $\mu_b$ could in principle be performed consistently at NLO. However, as the QED coupling is very small and the QED logarithms have a negligible effect on the result, this simple approach (setting $\mu = \mu_b$) seems to us sufficient. Indeed, we notice that the second term in the parenthesis on the r.h.s. of Eq. (14), corresponding to the leading logarithmic QED correction, provides only a $-0.3\%$ correction to the one-loop result in the case $\mu_b = 4.8$ GeV.

Clearly, QCD corrections modify the evolution of the Wilson coefficient if the renormalization group equation is solved keeping both QCD and QED effects into account. Using $\alpha \ll \alpha_s$, the formalism is well known \cite{19,20}, and one finds that the second logarithmic term in Eq. (14) should be multiplied by the QCD screening factor $\eta'$, given by

$$\eta' = \frac{4\pi}{\beta_0} \left( \frac{1}{\alpha_s(\mu_b)} - \frac{1}{\alpha_s(M_t)} \right) \left( \frac{\alpha_s(M_t)}{\alpha_s(\mu_b)} \right)^{\frac{7\pi}{2\beta_0}} \left( \ln \frac{\mu_b^2}{M_t^2} \right)^{-1}; \tag{15}$$

the rest of the Wilson coefficient undergoes the usual NLO QCD evolution summarized by $\tilde{\eta}_{2B}$. Using $\beta_0 = \frac{33}{3}$ and the QCD anomalous dimension $\gamma_0 = 4$ \cite{6}, one obtains the numerical value $\eta' \approx 0.9$. This modest reduction of an already small effect can be approximately taken into account by noting that $\eta' \approx \tilde{\eta}_{2B}$ and including the QED logarithm $\ln \mu_b^2/M_t^2$ (without the screening factor) inside the curly parenthesis of Eq. (4). As a consequence of this discussion, we identify the electroweak correction $\Delta_{QED}$ of Eq. (4) with the Wilson coefficient evaluated at a scale $\mu_b$, which for definiteness we set equal to $m_b$:

$$\delta S_{ew} = \frac{g^2}{16\pi^2} \left[ \Delta^{(2)} - \Delta^{QED}_{\text{eff}}(\mu = m_b) \right]. \tag{16}$$

### 2.4 Calculation of the two-loop diagrams

As far as the two-loop diagrams contributing to $\Delta^{(2)}$ in Eq. (16) are concerned, we need to consider all the topologies shown in Fig. 2, which correspond to a very large number of diagrams. We have therefore decided to use the MATHEMATICA \cite{21} package FeynArts 1.2 \cite{22} to generate automatically all the two-loop amplitudes. Dirac algebra operations, reduction to scalar integrals, and substitution of the scalar integrals have all been performed independently in two different ways involving various combinations of the MATHEMATICA packages Tracer \cite{23} and ProcessDiagram \cite{24}, and of FORM \cite{25}. The calculation has been performed in the 't Hooft-Feynman gauge, but we have checked the
Figure 2: The 26 two-loop topologies contributing to the $B^0 \to \bar{B}^0$ transition at $O(g^6)$. 
\( \xi_Z \) independence. As we neglect all external momenta, the complete result for the un-renormalized two-loop amplitudes can be written in terms of two-loop vacuum integrals, which admit a relatively compact representation in terms of logarithms and dilogarithms of the internal masses (see e.g. [26]). The final expression for the two-loop diagrams, however, is very long, due to the presence of four different heavy masses. Hence, we have decided to provide only approximate formulas that allow the reader to reproduce our numerical results with high accuracy. On the other hand, we feel that it may be more useful to explain in some detail the renormalization procedure that we have adopted. This is done in Sec. 3, where we stress the importance of keeping it as simple as possible in order to reduce the number of terms we have to deal with. All partial and final results are available in full analytic form from the authors. The complete analytic result has also allowed us to perform various heavy mass expansions, which will be presented in Sec. 4.

The computation of the contributions of individual two-loop diagrams to the Wilson coefficient of the operator \( \hat{Q}_{LL} \) can be considerably simplified by the implementation of a suitable projection procedure. In our case, such a projection can be achieved by forming appropriate traces (see e.g. [28] for a nice explanation of the method): writing a generic two-loop amplitude as

\[
\mathcal{M} = \bar{b} D_1 d \otimes \bar{b} D_2 d, \tag{17}
\]

where \( D_1 \) and \( D_2 \) are arbitrary Dirac structures with saturated Lorentz indices (for instance \( D_1 \otimes D_2 = \gamma^\nu \gamma^\lambda \gamma^\rho (1 - \gamma_5) \otimes \gamma_\mu \gamma^\lambda \gamma^\rho (1 - \gamma_5) \)), the projection on the operator \( Q_{LL} \) reads

\[
\mathcal{M} \to -\frac{1}{256} \left( 1 + \frac{3}{2} \epsilon \right) \text{tr} \left[ \gamma^\mu (1 + \gamma_5) D_1 \gamma_\mu (1 + \gamma_5) D_2 \right] Q_{LL}, \tag{18}
\]

where terms up to \( O(\epsilon) \) have been included. Furthermore, since \( Q_{LL} \) represents the only operator contributing to the \( B^0 - \bar{B}^0 \) mixing at the order we are working at, Eq. (18) is in fact an identity.

A few additional comments are now in order:

- We work in the framework of Naive Dimensional Regularization, a choice justified by the absence of any \( \gamma_5 \) ambiguity in the two-loop graphs we have considered. In particular, no closed fermion loop appears, apart from the ones in the self-energy insertions of the topologies in Fig. 2a, which pose no problem.

- The number of diagrams is significantly reduced by the following simple observation: for a given topology, one can consider subsets of diagrams characterized by the number of light quark propagators they contain and by the lines to which the light quarks are assigned. Since all light quarks are treated as massless, diagrams within such subsets differ solely by the CKM factor, and can be grouped together, reducing the total number of diagrams to be actually computed. Using the unitarity of the
CKM matrix, the overall CKM factor turns out to be always proportional to $\lambda^2$, as expected from the above-mentioned hard GIM cancellation.

- We perform the calculation in the electroweak scheme of Ref. [29], which uses $\overline{\text{MS}}$ couplings and on-shell masses as basic parameters. The transition to other popular schemes amounts to a finite renormalization of the electroweak coupling and is therefore straightforward: we discuss it in detail in the next section. The residual $\overline{\text{MS}}$ scale and scheme dependence allows us to gauge the remaining ambiguity due to the truncation of the perturbative series, as discussed in Sec. 4.

2.5 Double triangle diagrams

As mentioned above, the full electroweak two-loop correction receives also a contribution $\Delta_{DT}$ from the one-particle reducible topologies depicted in Fig. 2i. This contribution is finite but gauge dependent. The two penguin diagrams may be connected by $\gamma$, $Z$, $H$, or $\phi^0$, but only the $\bar{b}dZ$ vertex contributes to $\Delta_{DT}$ for vanishing external momenta and masses. Denoting this vertex by $\Gamma_{\bar{b}dZ}^\mu$, one finds in the ‘t Hooft-Feynman gauge [18]

$$i\Gamma_{\bar{b}dZ}^\mu = \frac{g^3\lambda_t}{16\pi^2 c} C(w_t) \bar{b}\gamma^\mu(1 - \gamma_5)d,$$

(19)

with

$$C(w_t) = \frac{6w_t - 1}{8w_t(w_t - 1)} - \frac{3 + 2w_t}{8(w_t - 1)^2} \ln w_t.$$

(20)

This result already includes the on-shell wave-function renormalization of the external legs. For the “double triangle” contribution $\Delta_{DT}$ we therefore obtain

$$\Delta_{DT} = 16 C(w_t)^2.$$

(21)

Numerically, $g^2/(16\pi^2) \times (\Delta_{DT}/S_0) \approx +1.1\%$.

3 Renormalization

In renormalizing the two-loop amplitude our aim is to attain the maximal simplicity. We avoid all wave function renormalization of the internal lines, and choose a particularly simple procedure for the unphysical sector. All masses are defined on-shell and for the electroweak coupling we use the $\overline{\text{MS}}$ scheme [29], although we explain in detail the connection to other schemes.
In the following we give explicit expressions for the various counterterms. They are written in terms of logarithms and of a single function \( B_0(x, y) \), which is defined through the one-loop integral

\[
-\text{Re} \left( q^2 e^{\gamma} \right) \epsilon \int \frac{d^nk}{\pi^{n/2}} \frac{i}{[k^2 - m_1^2][(k - q)^2 - m_2^2]} = \frac{1}{\epsilon} + B_0 \left( \frac{m_1^2}{q^2}, \frac{m_2^2}{q^2} \right) + O(\epsilon),
\]

whose analytic form is well known. The \( O(\epsilon) \) part of the counterterms is not needed. To the reader’s convenience we report here the explicit expression of \( B_0 \) for the three special cases that are needed in our calculation:

\[
\begin{align*}
B_0(1, x) &= 2 - \frac{x}{2} \ln x - \frac{1}{2} a(x), \\
B_0(0, x) &= 2 - x \ln x - (1-x) \ln |1-x|, \\
B_0(1, 0) &= 2,
\end{align*}
\]

where the function \( a(x) \) is given by

\[
a(x) = \begin{cases} \\
2\sqrt{4 - x^2} \arctan \sqrt{4/x - 1}, & 0 < x \leq 4, \\
\sqrt{x^2 - 4} x \ln \frac{1 - \sqrt{1 - 4/x}}{1 + \sqrt{1 - 4/x}}, & x > 4.
\end{cases}
\]

### 3.1 Top Mass Counterterm

The complete on-shell top mass counterterm is gauge-independent only after the inclusion of tadpoles, whose explicit expression can be found, for instance, in [30]. Here we report only the two-point function contribution to \( \delta M_t \) in the ’t Hooft-Feynman gauge. It can be written as

\[
\frac{\delta M_t}{M_t} = \frac{g^2}{16\pi^2} \left( \frac{\tilde{\mu}^2}{M_t^2} \right) \epsilon \left\{ -\frac{3}{8\epsilon} \left[ \frac{1}{w_t} + \frac{1}{c^2} \left( 1 - \frac{14}{9} s^2 \right) \right] + F_t + O(\epsilon) \right\},
\]

where \( F_t \) is given by

\[
\begin{align*}
F_t &= \frac{1}{w_t} \left[ \frac{4 - h_t}{16} a(h_t) + \frac{h_t - 7}{8} + \frac{6 - h_t}{16} h_t \ln h_t \right] \\
&+ \frac{1}{4} + \frac{16}{9} s^2 - \frac{1}{8} \left( 1 + \frac{1}{w_t} - 2w_t \right) B_0(0, w_t) - \frac{2w_t + 1}{8} (1 - \ln w_t) \\
&+ \frac{1}{c^2} \left[ \left( \frac{1}{8} + \left( \frac{1}{8} - \frac{s^2}{3} + \frac{4}{9} s^4 \right) z_t \right) \left( \ln z_t - 1 + \frac{1}{z_t} \right) \\
&+ \left( \frac{z_t - 1}{8} + (2 + z_t) \left( \frac{4}{9} s^2 - \frac{1}{3} \right) s^2 \right) B_0(1, z_t) + \frac{1}{8} + \frac{s^2}{3} - \frac{4}{9} s^4 \right].
\end{align*}
\]
Replacing $M_t \rightarrow M_t - \delta M_t$ in the LO result (7), one finds the following top counterterm contribution to be added to $\Delta_{2\text{loop}}^{\text{unren}}$:

$$\Delta_{M_t}^c = 2 w_t \left( \frac{\bar{\mu}^2}{M_t^2} \right)^\epsilon \left\{ -\frac{3}{8\epsilon} \left[ \frac{1}{w_t} + \frac{1}{c^2} \left( 1 - \frac{14}{9} s^2 \right) \right] \bar{S}' + S_0' F_t \right\},$$

(25)

with $\bar{S}' \equiv \partial \bar{S}(w_t, \bar{\mu}^2/M_w^2)/\partial w_t$, $S_0' \equiv \partial S_0/\partial w_t$, the function $S_0(w_t)$ as given in Eq. (5), and $\bar{S}$ defined by

$$\bar{S}(w_t, \bar{\mu}^2/M_w^2) = S(w_t, w_t \bar{\mu}^2/M_w^2),$$

where $S(w_t, \bar{\mu}^2/M_w^2)$ has been given in Eq. (8).

The top quark mass is therefore renormalized on-shell for what concerns electroweak effects\(^3\), while $\bar{\eta}_{2B} \approx 0.85$ in Eq. (4) implies the use of a $\overline{\text{MS}}$ mass $\overline{M}_t(M_t)$ as far as QCD effects are considered, in accordance to the standard convention $\overline{\text{MS}}$. Using the LO QCD relation between on-shell and the $\overline{\text{MS}}$ mass and the pole mass value $M_t = 174$ GeV $\overline{\text{MS}}$, we find $\overline{M}_t(M_t) = 166$ GeV, which will be our input in the following and will be denoted for simplicity just by $\overline{M}_t$.

### 3.2 Renormalization of the unphysical sector

Before considering the renormalization of the $W$ mass it is necessary to explain the procedure we have followed for the unphysical scalars. As is well known, the renormalization of the unphysical sector is not independent from the way the physical sector is treated. Indeed, the renormalization procedure must respect the Slavnov-Taylor Identities (STI) which are induced by the local gauge invariance of the original Lagrangian before spontaneous symmetry breaking. According to the organization of the calculation, it is possible to use different procedures that respect the STI’s and are particularly convenient in order, for example, to minimize the number of counterterms to be considered. Of course, physical amplitudes are independent of the chosen procedure, and this can be used as an additional check of the calculation. For the problem at hand, the discussion can be kept at the one-loop level.

If we split the unrenormalized $W$ polarization tensor into transverse and longitudinal parts

$$\Pi_{wW}^{\mu\nu}(q) = \left( g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2} \right) A_{wW}(q^2) + \frac{q^{\mu}q^{\nu}}{q^2} \Pi_{wW}^L(q^2),$$

(26)

\(^3\)A $\overline{\text{MS}}$ top mass renormalization in the electroweak sector would make the renormalized top mass dependent on the precise value of the Higgs mass, which is unknown; we do not consider it here.
and we denote by $\Pi_{WW}$ the two-point function for the mixing between the $W$ and its pseudo-Goldstone boson, $\phi$, and by $\Pi_{\phi\phi}$ the self-energy of the latter, we obtain the following STI in a general $R_\xi$ gauge (see for ex. \cite{31,32})

$$q^2 \left( \Pi_{WW}(q^2) + 2M_W \Pi_{W\phi}(q^2) \right) + M_W^2 \Pi_{\phi\phi}(q^2) + M_W^2 T = 0,$$

where $T$ represents the tadpole contribution. In particular, at $q^2 = 0$, the first two terms vanish, and the STI implies the cancelation between $\Pi_{\phi\phi}(0)$ and the tadpole contribution. This uncovers the connection between the renormalization of the Goldstone boson mass and the one of the tadpole. As anticipated, the counterterm contributions to the various terms in Eq. (27) must also respect the STI. In practice, the usual tadpole renormalization that minimizes the effective potential and consists in removing all tadpole graphs implies the subtraction of $\Pi_{\phi\phi}(0)$ from the two-point function of the pseudo-Goldstone boson $^4$. As mentioned above, we adopt the physical mass to define the masses of the vector bosons. It is therefore convenient to renormalize the longitudinal component of the two-point function of the $W$ in the same way as the transverse, using $\delta M_W^2 = \text{Re} A_{WW}(M_W^2)$. For the other two-point functions different options are possible, which all respect the STI, and are equivalent at the level of physical amplitudes. They correspond to different ways of renormalizing the gauge-fixing parameters.

One possibility consists in assigning no counterterm to the $W - \phi$ transition; in the 't Hooft-Feynman gauge this corresponds to renormalizing the masses of the vector boson and of the associated scalar boson in the same way (bare gauge fixing). Of course, the mass of the scalar boson will still need a supplementary subtraction at $q^2 = 0$, corresponding to the tadpole contribution. This choice clearly verifies Eq. (27) at $q^2 = M_W^2$, leaving room for a further arbitrary wave-function renormalization, which we avoid altogether, as the $W$ boson appears only inside the loops. Because of its simplicity, this is our preferred option: it amounts to fixing $\delta M_\phi^2 = \delta M_W^2 + T$ and is the closest to the naive parameter renormalization.

Another possibility, for example, would imply the subtraction of the first two terms of the Taylor expansion around $q^2 = M_W^2$ of the individual two-point functions in the external momentum, and it would obviously respect the identities, as the unrenormalized self-energies do. A counterterm for the $W - \phi$ transition is now involved. We have explicitly verified that all the renormalization options that satisfy the STI are equivalent at the level of physical amplitudes.

Finally, we recall that the renormalization of the three point function $d ud \phi$ is fixed by the Ward identity that links the Yukawa coupling counterterm to the gauge coupling and fermion mass renormalization.

$^4$This point is nicely explained in Taylor’s book \cite{33}, see section 14.6.
3.3 W mass counterterm

Following the strategy outlined in the preceding subsection, we use the same counterterm to renormalize the W and the pseudo-Goldstone boson mass terms. In the 't Hooft-Feynman gauge, one has for the two-point contributions to the W mass counterterm (see e.g. [34]):

$$\frac{\delta M_W^2}{M_W^2} = \text{Re}A_{ww}(M_W^2) = \frac{g^2}{16\pi^2} \left( \frac{\bar{\mu}^2}{M_t^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} \left[ \frac{3}{2 w_t} - \frac{1}{c^2} + \frac{7}{6} \right] + F_W + O(\epsilon) \right\}, \quad (28)$$

where the finite part $F_W$ is given by

$$F_W = \left( \frac{1}{2 w_t^2} + \frac{1}{2 w_t} - 1 \right) B_0 \left( 0, \frac{1}{w_t} \right) - \left( \frac{h_t^2}{12 w_t^2} - \frac{h_t}{3 w_t} + 1 \right) B_0 \left( 1, \frac{h_t}{w_t} \right) + \left( 4 c^2 + \frac{17}{3} - \frac{4}{3 c^2} - \frac{1}{12 c^4} \right) B_0 \left( 1, \frac{z_t}{w_t} \right) - \left( 4 - \frac{5}{6 c^2} - \frac{1}{6 c^4} \right) \ln c$$

$$- \left[ \frac{1}{w_t^2} \left( \frac{1}{2} - \frac{h_t^2}{12} \right) + \frac{1}{w_t} \left( \frac{1}{2} + \frac{h_t}{4} \right) + \frac{7}{6} - \frac{1}{c^2} \right] \ln w_t - \left( \frac{h_t^2}{12 w_t^2} - \frac{h_t}{4 w_t} \right) \ln h_t$$

$$- 8 c^2 - \frac{1}{w_t^2} \left( \frac{1}{2} - \frac{h_t^2}{12} \right) - \frac{h_t}{6 w_t} + \frac{53}{18} + \frac{1}{2 c^2} + \frac{1}{12 c^4}.$$ 

Replacing $M_w \rightarrow M_w - \delta M_w$ in the LO result Eq. (7), one obtains the W mass counterterm contribution

$$\Delta^c_{M_w} = \left( \frac{\bar{\mu}^2}{M_t^2} \right)^\epsilon \left\{ \frac{1}{\epsilon} \left[ \frac{3}{2 w_t} - \frac{1}{c^2} + \frac{7}{6} \right] (S - w_t S') + (S_0 - w_t S_0') F_W \right\}. \quad (29)$$

Here again the notations $S' \equiv \partial S(w_t, \bar{\mu}^2/M_t^2) / \partial w_t$ and $S_0' \equiv \partial S_0 / \partial w_t$ are understood, and the functions $S_0(w_t), S(w_t, \bar{\mu}^2/M_t^2)$ are given in Eqs. (5) and (8). We treat the W mass as a fundamental input parameter, whose value is taken directly from the experiment: $M_w = 80.385 \pm 0.065$ [33]. Although the determination of $M_w$ from $M_Z, G_\mu$, and $\alpha$ is presently more precise than the experimental one after including all theoretical uncertainties [32], this is certainly sufficient for our purposes.

3.4 Tadpole contribution

As mentioned above, the renormalization of the pseudo-Goldstone boson mass term can be performed (in the 't Hooft-Feynman gauge) in the same way as the one of the W mass, apart from an additional tadpole contribution, whose physical origin has been already explained. In practice, this means that the one-loop diagrams containing scalars induce
a further counterterm contribution that can be easily calculated from the complete one-loop tadpole (see for instance Eq. (12) of [30]) and the one-loop amplitudes involving only scalars. Here we just give the final result for this additional contribution, again in the \'t Hooft-Feynman gauge:

\[ \Delta^c_{Tad} = \left( \frac{\bar{\mu}^2}{M_t^2} \right) \epsilon \left[ E_{Tad} \left( \frac{T_0}{\epsilon} + T_1 \right) + F_{Tad} T_0 + O(\epsilon) \right], \]  \hspace{1cm} (30)

with

\[
T_0 = \frac{-1 + 19w_t}{4(w_t - 1)^3} + \frac{1 - 6w_t - 4w_t^2}{2(w_t - 1)^4} \ln w_t,
\]

\[
T_1 = \frac{-13 + 43w_t}{8(w_t - 1)^3} + \frac{4 - 19w_t}{4(w_t - 1)^4} \ln w_t - \frac{1 - 6w_t - 4w_t^2}{4(w_t - 1)^4} \ln^2 w_t + T_0 \ln \frac{\bar{\mu}^2}{M_t^2},
\]

\[
E_{Tad} = -\frac{3}{w_t^2} + \frac{3h_t^2}{8w_t^2} + \frac{h_t}{4w_t} + \frac{h_t}{8c^2w_t} + \frac{3}{2} + \frac{3}{4c^4},
\]

\[
F_{Tad} = E_{Tad} - 1 - \frac{1}{2c^4} - \frac{3h_t^2}{8w_t^2} \ln h_t - \left( \frac{3}{2} + \frac{h_t}{4w_t} \right) \ln w_t - \left( \frac{3}{4c^4} + \frac{h_t}{8c^2w_t} \right) \ln z_t.
\]

3.5 Coupling counterterm

The SU(2) coupling counterterm in the \overline{MS} scheme is (up to $\ln 4\pi$ and $\gamma_E$ constants)

\[
\frac{\delta \hat{g}}{\hat{g}} = \frac{\hat{g}^2}{16\pi^2} \frac{19}{12\epsilon}.
\]

Performing the renormalization $g_0 = \hat{g} - \delta \hat{g}$, we obtain the following contribution to $\Delta^{(2)}$:

\[
\Delta^c_g = -\frac{19}{3\epsilon} S(w_t, \mu^2/M_t^2).
\]  \hspace{1cm} (31)

In the $\overline{MS}$ scheme $\hat{g}(M_Z)$ can be calculated from $\hat{\alpha}(M_Z) = (128.1)^{-1}$ and $\hat{s}^2(M_Z) \approx \sin^2 \theta_{\text{eff}}^\text{lep} - 1 \times 10^{-4} = 0.23145$ [33], obtaining $\hat{g}^2(M_Z) = 0.423842$; this is the value that will be used in the following numerical calculations.

Let us now consider what happens in different renormalization schemes. In any scheme $X$ but $\overline{MS}$ the counterterm would have a finite part $F^X_g$, namely

\[
\frac{\delta g^X}{g} = \frac{g^2}{16\pi^2} \frac{19}{12\epsilon} + F^X_g,
\]

so that the difference between the result in a scheme $X$ and the result in the $\overline{MS}$ scheme at the scale $\mu_g = M_Z$ is just

\[
\delta S_X = -4 S_0(w_t) F^X_g,
\]  \hspace{1cm} (32)

16
which should be added to $\delta g_{ew}^{\text{MS}}$, also evaluated at $\mu_g = M_Z$. It is well-known that the use of the electromagnetic fine structure constant $\alpha(0) = 1/137.036$ to normalize the electroweak coupling would introduce large and uncompensated mass singularities in the two-loop result. Therefore, the coupling $g$ must be normalized at short distances. This is the case, for instance, if we use $G_\mu$ and $M_W$: $g^2 = 4 \sqrt{2} G_\mu M_W^2$. Then we find

$$F_{OS}^g = \frac{1}{2} \left( \Delta \hat{r}_W + \frac{2\delta e}{e} \left| \frac{\mu}{M_Z} \right| \right), \quad (33)$$

where $\Delta \hat{r}_W$ is a function of the top, Higgs, $W$ and $Z$ masses, as well as of the couplings, which summarizes the electroweak corrections to the muon decay in the $\overline{\text{MS}}$ scheme and is given explicitly in [29]. The second term in parenthesis, instead, represents the finite part of the conventional electric charge counterterm in $\overline{\text{MS}}$, evaluated at the $\overline{\text{MS}}$ scale $\mu_g$. $F_{OS}^g$ depends mildly (logarithmically) on $M_\mu$ and $M_t$; numerically it is very small, as can be seen comparing $g^2$ obtained from $G_\mu M_W^2$ and the $\overline{\text{MS}}$ value of $\hat{g}^2(M_Z)$: they differ by half a percent.

We will consider later also another possibility to define $g$, namely through the relation $g^2 = 4\pi \hat{\alpha}(M_Z)/s_W^2$. Here we have adopted the notation $s_W^2 = 1 - M_W^2/M_Z^2$ for the on-shell definition of the weak mixing angle [30] and $\hat{\alpha}(M_Z)$ stands for the $\overline{\text{MS}}$ running electromagnetic coupling at the scale $M_Z$. In this case

$$F_{OS}^g = \frac{1}{2} \frac{c_W^2}{s_W^2} \Delta \hat{\rho}, \quad (34)$$

where $\Delta \hat{\rho}$ is another function of the heavy masses given in Ref. [29]. Unlike Eq. (33), however, $F_{OS}^g$ is very sensitive to $M_t$, due to its quadratic dependence enhanced by the factor $c_W^2/s_W^2$. Indeed, the $\overline{\text{MS}}$ and on-shell definitions of $\sin^2 \theta_W$, and so the corresponding $g^2$, differ by almost 4%! We can therefore anticipate that this normalization choice will introduce large two-loop corrections when compared to the $\overline{\text{MS}}$ scheme. Moreover, as the origin of the large counterterm contributions has nothing to do with the process at hand, we can also expect this choice to give unnaturally large two-loop corrections.

### 3.6 Wave function counterterm

The left-handed quark fields of the down type (i.e. the external fields) are renormalized according to

$$\langle d^L \rangle_0 = \left[ (Z_\text{d}^L)^{1/2} \right]_{ij} d^L_i, \quad (35)$$

To keep the notation simple, we use here the $\overline{\text{MS}}$ running electromagnetic coupling $\hat{\alpha}(M_Z) \simeq (128.1)^{-1}$ instead of the conventional $\alpha(M_Z) \simeq (128.9)^{-1}$ [3], which is defined by subtraction of the renormalized light fermion photon correlators only, and is more generally used in the context of the on-shell scheme. The difference is not negligible, and must be taken into account in the two-loop corrections.
and the (matrix-valued) wave-function renormalization constant $Z_L^d = 1 + \delta Z_L^d$ is determined as described by Aoki et al. [31]. Denoting the sum of all one-loop diagrams contributing to the transition $d_i \rightarrow d_j$ with external momentum $p$ by $i\Sigma_{ij}(p)$, and writing $(L, R = (1 \mp \gamma_5)/2)$

$$
\Sigma_{ij}(p) = \Sigma_{ij}^L(p^2)\phi L + \Sigma_{ij}^R(p^2)\phi R + \Sigma_{ij}^S(p^2)(m_iL + m_jR),
$$

(36)

this prescription implies in the limiting case of vanishing masses of the down-type quarks the customary relation

$$(\delta Z_L^d)_{ij} = -\Sigma_{ij}^L(0).$$

(37)

In this equation we have considered only the hermitian part of the wave function renormalization constant. For what concerns the antihermitian part of $\delta Z_L^d$, it is not defined in the limit of vanishing quark masses that we have adopted from the beginning. One can use the renormalization prescription of Denner and Sack [37] for the CKM matrix elements and remove it completely. In the case of the up quarks, the top mass cannot be set to zero, but the antihermitian part of the wave function renormalization vanishes because of the GIM mechanism.

By calculating the relevant diagrams, we obtain in this way

$$(\delta Z_L^d)_{ij} = \frac{g^2}{16\pi^2} \left( \frac{\mu^2}{m_t^2} \right)^\epsilon \left( \delta_{ij}A + V_{ti}V_{tj}B \right),$$

(38)

where

$$A = \frac{1}{c^2} \left[ -\frac{1}{\epsilon} \left( \frac{1}{36} + \frac{13}{18}c^2 \right) + \frac{1}{72} + \frac{1}{12}c^2 + \frac{5}{18}c^4 + \left( \frac{1}{36} + \frac{1}{9}c^2 + \frac{1}{9}c^4 \right) \ln z_t \right] + \frac{1}{2} \ln w_t$$

$$+ \frac{1}{9} s^2 \ln q_t, \quad (39)$$

$$B = -\frac{1}{\epsilon} \frac{1}{4w_t} - \frac{3(1 + w_t)}{8w_t(1 - w_t)} + \frac{w_t - 4}{4(1 - w_t)^2} \ln w_t. \quad (40)$$

The corresponding counterterm contribution $\Delta c_\psi$ is obtained as follows: the amputated amplitude $M_{1\text{loop}}^{(k)}$ for the transition $\bar{k} + d \rightarrow b + \bar{d}$ $(k = d, s, b)$ is in the notation of Sec. 2.2 given by

$$M_{1\text{loop}}^{(k)} = \frac{-i}{16\pi^2} \frac{g^4}{8M_W^2} \sum_{ij} \lambda_i^{(k)} \lambda_j S^{(i,j)} Q_{LL},$$

(41)

with $\lambda_i^{(k)} = V_{ti}^*V_{td}$. Using the unitarity of the CKM matrix, this simplifies to

$$M_{1\text{loop}}^{(k)} = \frac{-i}{16\pi^2} \frac{g^4}{8M_W^2} \left\{ \lambda_t^{(k)} \lambda_l \left[ S^{(t,t)} - 2S^{(t,c)} + S^{(c,c)} \right] + \delta_{kd} \lambda_l \left[ S^{(t,c)} - S^{(c,c)} \right] \right\}. \quad (42)$$
The second term contributes only for $k \equiv d$, and it originates in the unitarity relation $\lambda^{(k)}_t + \lambda^{(k)}_c + \lambda^{(k)}_u = \delta_{kd}$. The counterterm contribution due to the wave function renormalization of the external $b$-field is then

$$\Delta^{c,1}_\psi = \frac{1}{2} \sum_k \left[ \delta Z L^*_d \right]_{bk} M^{(k)}_{\text{loop}}.$$  (43)

Combining (38) and (42), one thus obtains

$$\Delta^{c,1}_\psi = \frac{1}{2} \left( \frac{\bar{\mu}^2}{m^2_t} \right)^\epsilon \left[ (A + B) S(w_t, \bar{\mu}^2/M^2_t) + B U(w_t, \bar{\mu}^2/M^2_t), \right]$$  (44)

with $S(w_t, \bar{\mu}^2/M^2_t)$ as given in (8), and

$$U(w_t, \bar{\mu}^2/M^2_t) \equiv \mathcal{S}^{(t,c)} - \mathcal{S}^{(c,c)} = \left( \frac{1}{w_t - 1} - \frac{w_t \ln w_t}{(w_t - 1)^2} \right) \left( 1 + \epsilon \ln \frac{\bar{\mu}^2}{M^2_t} \right) + \frac{1}{2(w_t - 1)} + \frac{2 - w_t}{2(w_t - 1)^2} \ln w_t + \frac{w_t \ln^2 w_t}{2(w_t - 1)^2} + O(\epsilon^2).$$

An analogous reasoning holds for the wave function renormalization of the other three external fields, and the final counterterm contribution is simply

$$\Delta^c_\psi = 4 \Delta^{c,1}_\psi.$$  (45)

4 Results and discussion

Combining the counterterm contributions of Eqs. (25), (29–31), (45) with Eq. (21) and with the unrenormalized two-loop contributions yields our final result in the $\overline{\text{MS}}$ scheme. The size of the effect can be seen in Fig. 3, where the following input values have been used [3]: $M_W = 80.385 \text{ GeV}$, $\mu_g = M_Z = 91.187 \text{ GeV}$, $M_t = 166 \text{ GeV}$, $\mu_b = 4.8 \text{ GeV}$, $\hat{s}^2(M_Z) = 0.23145$. These are the values we will use throughout this section. The electroweak correction is less than 1% of the LO result for any value of the Higgs mass below 1 TeV, and is particularly small for a light Higgs, namely in the region preferred by recent global fits [3]. The logarithmic asymptotic dependence for large Higgs masses is already evident around 1 TeV. In Sec. 4.3 we will present a simple approximate formula that reproduces the complete result very accurately. Before, however, we consider the heavy mass expansions of the full result and study the residual scheme and scale dependence.
4.1 Heavy mass expansions

We first consider the heavy Higgs expansion. In the case of a very heavy Higgs boson, $M_H \gg M_W, M_t$, we have verified that the leading behaviour is logarithmic in $M_H$. In particular, the MS scheme result reads

$$
\delta S_{ew}^{hh} = \frac{g^2}{16\pi^2} \frac{1}{w_t^2} \left[ \ln^2 h_t - \ln h_t \right] + \frac{8 w_t^3 - 27 w_t^2 + 30 w_t - 5}{48(w_t - 1)^2} + \frac{w_t^2 (3w_t - 5) \ln w_t}{16 (w_t - 1)^3} \ln h_t + \ldots \right]
$$

where the ellipses stand for constants or terms suppressed by inverse powers of the Higgs mass. Of course, in another scheme $X$ the Higgs dependence contained in $\delta S_X$ will change the $\ln h_t$ term in the above expression. The absence of a leading quadratic Higgs behaviour in the complete result can be explained in the context of an effective theory approach where the Higgs boson is integrated out \[38\].

If we are interested in the leading heavy-top corrections, $O(g^2 M_t^4/M_W^4)$, we may work in the framework of a Yukawa Lagrangian where the heavy fermions couple only to the Higgs boson and to the longitudinal components of the gauge bosons, a situation which corresponds to the gaugeless limit of the SM \[14\]. In the covariant gauges, it suffices to consider only the diagrams that involve exchanges of scalar bosons. Even in the case of a heavy Higgs boson, in which the scalar coupling $\lambda \sim g^2 M_H^2/M_W^2$ cannot be neglected with respect to the Yukawa coupling of the top, only a few tens of two-loop diagrams...
contribute. Notice that in this framework the mass of the Higgs boson is completely arbitrary.

Computing the various counterterms from the scalar one-loop diagrams only (which corresponds to the heavy top limit of our renormalization constants), adding these to the sum of the scalar two-loop diagrams, and expanding in small $w_t$, we get

$$
\delta S_{\text{lead}}^{\text{ew}} = \frac{g^2 \phi}{16\pi^2 w_t^2} \left[ -44 + 112 h_t - 66h_t^2 + 14h_t^3 - h_t^4 \right] \phi \left( \frac{h_t}{4} \right) + \frac{52 - 16 h_t - 9 h_t^2}{32} \\
- \left( h_t - 1 \right)^2 \left( 4 - 14 h_t + h_t^2 \right) \text{Li}_2(1 - h_t) - \frac{22h_t^2 - 20h_t^3 + h_t^4}{64} \ln^2 h_t \\
- \frac{h_t \left( 14 + 9 h_t \right)}{32} \ln h_t - \frac{8 + 22h_t^2 - 20h_t^3 + h_t^4}{192} \pi^2 + \frac{h_t - 4}{32} a(h_t) \right].
$$

(47)

Here we have indicated the dilogarithmic function as $\text{Li}_2(x) = -\int_0^x dt \frac{\ln(1 - t)}{t}$, and introduced

$$
\phi(z) = \begin{cases} 
4 \sqrt{1 - z} \text{Cl}_2(2 \arcsin \sqrt{z}), & 0 < z \leq 1, \\
\frac{4}{\lambda} \left[ -4\text{Li}_2\left( \frac{1 - \lambda}{2} \right) + 2 \ln^2\left( \frac{1 - \lambda}{2} \right) - \ln^2(4z) + \frac{\pi^2}{3} \right], & z > 1,
\end{cases}
$$

(48)

where $\lambda = \sqrt{1 - \frac{2}{z}}$ and $\text{Cl}_2(x) = \text{Im} \text{Li}_2(e^{ix})$ is the Clausen function. It is a welcome check of our result that Eq. (47) coincides with the limit $w_t \to 0$ of the complete result, which is scheme and scale independent. The limit for $M_H \to 0$ of the square parenthesis in Eq. (47) is $\frac{13}{8} - \frac{\pi^2}{12}$. One can also verify that the heavy Higgs limit of Eq. (47) coincides with the heavy top limit of Eq. (46).

Numerically, $\delta S_{\text{lead}}^{\text{ew}} / S_0 = 1.12\%$ (for $M_H = 100$ GeV), which should be compared to 0.18% obtained using the full result. Clearly, the leading term of a heavy top expansion does not approximate well the complete result. This is not surprising since, for the measured value of the top quark mass, the heavy top expansion of the LO result,

$$
S_0(w_t) = \frac{1}{4} w_t - \frac{9}{4} - \frac{3}{2} \ln w_t - w_t \left( \frac{15}{4} + \frac{9}{2} \ln w_t \right) + O(w_t^2)
$$

(49)

must be pushed up to the fourth term in order to reach a 10% accuracy. An example of slow convergence of the heavy top expansion can also be found in the $O(\alpha_s\alpha)$ corrections to the $Z \to \bar{b}b$ decay amplitude [39], although in that case the leading top contribution dominates, because of large cancellations among the subleading terms.

It is therefore interesting to investigate further the convergence of the heavy top expansion in a case where the complete two-loop contributions are available in analytical form. We have expanded our complete $\overline{\text{MS}}$ result in inverse powers of the heavy top mass,
Figure 4: (a) Renormalized box contributions to $\delta S_{\text{ew}}$ in the $\overline{\text{MS}}$ scheme, normalized to the LO result, as a function of the Higgs mass. The solid line represents the complete result, the dotted line the leading top approximation, and the dashed (dashed-dotted) line the heavy (light) Higgs top expansions up to $O(1/M_t)$. Inputs as in the text. (b) The same, but for $M_t = 350\,\text{GeV}$, demonstrating the much better convergence of the heavy top expansions in this case.

up to the third term, namely excluding only contributions which are formally $O(M_W/M_t)$.

With respect to the Higgs boson mass, we either consider it much smaller than the top mass, or comparable, i.e. heavy compared to $M_W$. Consequently, we obtain two expansions, one valid for small $M_H$, the other for large $M_H$. In Fig. 4, we compare them with the leading top result of Eq. (47) and with the complete result; we display the various results as functions of $M_H$ for the two cases $\overline{M}_t = 166\,\text{GeV}$ and $\overline{M}_t = 350\,\text{GeV}$, respectively. We show only the contributions of the renormalized box diagrams, i.e. we leave aside the double triangle contribution $\Delta_{DT}$; the latter does not depend on $M_H$ and its top expansion converges also very slowly, as can be seen from its explicit expression of Eq. (21). We observe that in both cases the light and heavy Higgs expansions match quite well around $M_H \approx 60$ and 100 GeV (similar to what happens in [12]), but start to converge to the complete result only for a very heavy top, with $\overline{M}_t = 350\,\text{GeV}$. For even heavier top masses, the expansion up to $O(1/M_t)$ terms provides an excellent approximation.

The slow convergence of the heavy top expansion of the two-loop contributions may be related to the slow convergence of the expansion of the LO result. It is therefore quite different from what happens in the case of the calculation of precision observables [12-14], where the top contributions originate from two-point functions. The heavy top mass expansion for the latter seems to work remarkably well up to two-loop [15] and has been checked even at the three-loop level in the case of mixed $O(\alpha\alpha_s^2)$ corrections [14].
4.2 Scale and scheme dependence

In the $\overline{\text{MS}}$ scheme, we have so far fixed $\mu_g = M_Z$. However, by varying $\mu_g$ around $M_W$, we can compare the dependence on the renormalization scale $\mu_g$ of the LO result with the one of the sum of one and two-loop results. We recall that the LO result is expected to exhibit a strong scale dependence because of the factor $\hat{g}^4(\mu_g)$. Our two-loop correction cancels the leading logarithmic dependence on $\mu_g$, and the remaining much weaker scale dependence can be interpreted as an indication of the importance of higher order effects. This is illustrated in Fig. 5, where we show that in varying the scale $\mu_g$ between $M_W/2$ and $2M_W$ the combination $\hat{g}^4 S_0(w_t)$ varies by about 5%, while the combination $\hat{g}^4 (S_0(w_t) + \delta S_{ew})$ undergoes a maximum variation of less than 0.1%. This dramatic reduction of the scale dependence demonstrates the importance of a complete two-loop electroweak analysis in order to reduce the normalization ambiguities of the LO result.

Following the discussion of Sec. 3 on the normalization of the electroweak coupling $g$, we can also compare the LO result $g^4 S_0(w_t)$ and the electroweak corrected $g^4 (S_0(w_t) + \delta S_{ew})$ in different renormalization schemes. Table 1 summarizes the scheme dependence before and after the inclusion of our two-loop calculation by comparing the three representative cases introduced in Sec. 3: $\overline{\text{MS}}$ scheme at the scale $M_Z$, $g$ expressed in terms of $G_\mu M_W^2$, $g$ expressed in terms of the on-shell sine $s_{ew}^2$ and of $\hat{\alpha}(M_Z)$. The input values that we use are to a good approximation compatible with the precise calculation summarized by...
Table 1: Scheme dependence: comparison between LO and electroweak corrected results for $M_H = 100\text{GeV}$.

| scheme | $g^4 S_0$ | $g^4 (S_0 + \delta S_{\text{ew}})$ |
|--------|-----------|-------------------------------|
| $\overline{\text{MS}}$ | 0.4270 | 0.4277 |
| $G_\mu M_W^2$ | 0.4320 | 0.4276 |
| OS | 0.4603 | 0.4270 |

the numerical formulas provided in Ref. 35 in the case $M_H = 100\text{GeV}$ and $M_t = 174\text{GeV}$ (equivalent to $\overline{M}_t = 166\text{GeV}$). For our purposes it is therefore sufficient to evaluate numerically the extra terms $\delta S_{OS}$ and $\delta S_{G_\mu}$ inverting the definitions of $\Delta\hat{\rho}$ and $\Delta\hat{\tau}_w$: one finds $F_{g OS}^{\hat{G}_\mu} = (1 - s_W^2/\hat{s}^2(M_Z))/2$ and $F_{g G_\mu}^{\hat{G}_\mu} = 1/2 (1 - \pi\hat{\alpha}(M_Z)/\sqrt{2} \hat{s}^2(M_Z) G_\mu M_W^2)$, from which $\delta S_{OS}$ and $\delta S_{G_\mu}$ can be computed using also $G_\mu = 1.16639 \times 10^{-5}\text{GeV}^{-2}$. Here we have implicitly assumed that the theoretical determination of the relations among the electroweak couplings is ideally accurate; although this is obviously not the case, the theoretical errors involved are of the order of a few parts in $10^4$ [13] and can be neglected at this stage.

As could be expected after the remarks made in Sec. 3 about the scheme dependence, the difference between $\overline{\text{MS}}$ scheme and $G_\mu$ normalization is only 1.1% at LO, while the one between the $\overline{\text{MS}}$ and on-shell schemes is unnaturally very large for a purely electroweak correction at low-energies, almost 8%. After the inclusion of our two-loop corrections, the situation changes drastically. The maximum differences are now about 0.2%, which agrees with a rough estimate of the residual uncertainty, obtained by squaring the relative difference between $s_W^2$ and $\hat{s}^2(M_Z)$, that gives $\approx 0.002$.

Because the on-shell scheme induces large and theoretically well-controlled radiative corrections $\delta S_{\text{ew}}$, however, it cannot be used to estimate higher order effects when the couplings are normalized in a different way. We have therefore extended the scheme comparison of Table 1 to different values of $M_H$, limiting ourselves to the case where the result is expressed in terms of $\overline{\text{MS}}$ couplings or of $G_\mu$. To this end, one needs to recalculate $\hat{s}^2(M_Z)$ for each $M_H$ from $G_\mu$ and $M_W$, and to use the complete expression for $F_{g G_\mu}^{\hat{G}_\mu}$. We find excellent agreement between the two schemes: the residual ambiguity is always well below 0.1%.
4.3 Approximate formulae

In this subsection we present some very simple approximate formulae that reproduce with excellent accuracy the result of our calculation when the LO result is normalized to the MS coupling \( \hat{g}(\mu_g) \) and to \( G_\mu M_w^2 \). Unlike the original very long analytic formulae, they are suitable for a simple implementation in numerical analyses.

In the MS scheme, we can rewrite \( \delta S_{\text{ew}} \) isolating the dependence of the complete result on the two scales \( \mu_g \), at which the coupling \( \hat{g} \) is normalized, and \( \mu_b \approx m_b \), at which the Wilson coefficient of the underlying effective theory is evaluated (\( C(w_t) \) is the penguin function given in Eq. (20)):

\[
\delta S_{\text{ew}}^{\text{MS}} = \frac{\hat{g}^2}{16\pi^2} \left[ S_0(w_t) \left( \frac{19}{3} \ln \frac{\mu_g^2}{M_t^2} + \frac{2}{3} s^2 \ln \frac{\mu_b^2}{M_t^2} \right) + A(M_H, \overline{M}_t) + 16 C(w_t)^2 \right].
\] (50)

The following simple approximate formula reproduces with high accuracy the analytic result for \( A(M_H, \overline{M}_t) \), which cannot be reduced to a compact form:

\[
A(M_H, \overline{M}_t) = 12.20 + 52.74 \left( \frac{\overline{M}_t}{166} - 1 \right) - 0.977 \ln \frac{M_H}{100} + 1.709 \ln^2 \frac{M_H}{100},
\] (51)

where we have expressed the top and Higgs masses in GeV. For \( 80 < M_H < 650 \) GeV, \( \overline{M}_t \) within 1\( \sigma \) from the present experimental value, \( \overline{M}_t = 166 \pm 5 \) GeV, and using \( s^2(M_Z) = 0.23145 \) and \( M_w = 80.385 \) GeV, Eq. (51) does not deviate from the analytic result by more than 1\% which induces in the complete result a very small \( O(10^{-4}) \) relative error.

In the case the LO result is expressed in terms of \( G_\mu M_w^2 \), only the \( \mu_b \) scale dependence is left and the Higgs and top dependence is different due to the additional term \( \delta S_{G_\mu} \), as shown in Fig. 6. The result can then be written as

\[
\delta S_{\text{ew}}^{G_\mu} = \frac{\sqrt{2} G_\mu M_w^2}{4\pi^2} \left[ \frac{2}{3} \sqrt{\frac{\hat{g}(M_Z)\pi}{2 G_\mu M_w^2}} S_0(w_t) \ln \frac{\mu_b^2}{M_t^2} + B(M_H, \overline{M}_t) + 16 C(w_t)^2 \right],
\] (52)

where

\[
B(M_H, \overline{M}_t) = -15.49 - 31.65 \left( \frac{\overline{M}_t}{166} - 1 \right) - 2.296 \ln \frac{M_H}{100} + 1.868 \ln^2 \frac{M_H}{100},
\] (53)

and again \( \overline{M}_t \) and \( M_H \) are expressed in GeV. In the same ranges of \( \overline{M}_t \) and \( M_H \) that we considered for the function \( A(M_H, \overline{M}_t) \), \( B(M_H, \overline{M}_t) \) approximates the complete result with analogous accuracy.
Figure 6: The electroweak correction $\delta S_{\text{ew}}$ when the LO result is expressed in terms of $G_\mu$, normalized to the LO result, as a function of the Higgs mass. Inputs as in Fig. 3.

5 Conclusions

In this paper we have computed the complete electroweak effects in the $B^0 - \bar{B}^0$ mixing in the SM. We have used an effective theory approach in treating the QED corrections and neglected QED effects in the matrix elements. The calculation has been performed analytically, and the results expressed for arbitrary Higgs mass in terms of the very simple approximate formulae of Eqs. (50–53). The discussion of Sec. 4 can be summarized in the following way:

1. Unless the renormalization procedure introduces unnaturally large correction unrelated to the process, the two-loop electroweak corrections are small, $O(1\%)$, for any realistic value of the Higgs mass. This is of the same order of magnitude of the NLO QCD corrections. If one uses an on-shell definition of the electroweak coupling, however, they reach almost 8% for a light Higgs boson.

2. The electroweak scheme and scale dependence of the result, which we have shown to be quite large at LO, is consistently reduced to the permille level. If $G_\mu M_W^2$ or $\overline{\text{MS}}$ couplings are used to normalize the result, the ambiguity is below 0.1%, much less than the present perturbative QCD error.

3. The heavy top expansion converges very slowly in the case of the $B^0 - \bar{B}^0$ mixing.
This was known at LO and we have verified the same pattern at the two-loop level. The fact that the leading heavy top limit does not approximate well the complete result suggests that it cannot be used as a reliable estimate of the full electroweak effects in cases where already at LO it does not work properly. In particular, this refers to some weak decays for which the leading heavy top corrections have recently been computed: see [40] (for the decay $B \rightarrow X_s \gamma$) and [9] (for a class of rare decays including $K \rightarrow \pi\nu\bar{\nu}$).

If one expresses the LO result in terms of $G_\mu$ and $M_W$, as it is customarily done [6], the correction as a function of the Higgs mass is shown in Fig. 6 and the effect on the extraction of $|V_{td}|$ from $\Delta M_{B_d}$ is tiny. Indeed, $|V_{td}| \sim (S_0 + \delta S_{ew})^{-1/2}$, so the electroweak corrections induce approximately a $+0.5\%$ change in the extracted value of $|V_{td}|$, for values of the Higgs mass below 400 GeV.

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