EXISTENCE AND STABILITY OF SYMMETRIC PERIODIC SIMULTANEOUS BINARY COLLISION ORBITS IN THE PLANAR PAIRWISE SYMMETRIC FOUR-BODY PROBLEM

LENNARD F. BAKKER, TIANCHENG OUYANG, DUOKUI YAN, AND SKYLER SIMMONS

ABSTRACT. We prove the analytic existence of a symmetric periodic simultaneous binary collision orbit in a regularized planar pairwise symmetric equal mass four-body problem. We provide some analytic and numerical evidence for this periodic orbit to be linearly stable. We then use a continuation method to numerically find symmetric periodic simultaneous binary collision orbits in a regularized planar pairwise symmetric 1, m, 1, m four-body problem for \( m \) between 0 and 1. We numerically investigate the linear stability of these periodic orbits through long-term integration of the regularized equations, showing that linear stability occurs when \( 0.538 \leq m \leq 1 \), and instability occurs when \( 0 < m \leq 0.537 \) with spectral stability for \( m \approx 0.537 \).

1. INTRODUCTION

In the \( N \)-body problem, linearly stable periodic orbits trap around themselves bounded, non-chaotic motion of the \( N \) masses [4]. Some of the known examples of linearly stable periodic orbits in the three-body problem are the elliptic Lagrangian triangular periodic orbits for certain values of eccentricity and the three masses [9],[15], and the Montgomery-Chenciner figure-eight periodic orbit for equal masses [3],[11],[16],[7],[8]. Other examples of linearly stable periodic orbits in the three or four-body problem involve binary collisions (BC) and/or simultaneous binary collisions (SBC). The regularization of these kinds of singularities is achieved by a generalized Levi-Civita type transformation and an appropriate scaling of time, as adapted from Aarseth and Zare [1] to the particular problem (also see [13]).

Schubart [21] numerically discovered a singular symmetric periodic orbit in the collinear three-body equal mass problem. In this orbit, the inner mass alternates between binary collisions with the two outer masses. Hénon [5] extended Schubart’s numerical investigations to the case of unequal masses. Only recently did Venturelli [24] and Moeckel [10] prove the analytic existence of the Schubart orbit when the outer masses are equal and the inner mass is arbitrary. The linear stability of the Schubart orbit was determined numerically by Hietarinta and Mikkola [6] revealing that linear stability occurs only for certain choices of the three masses. Numerically, non-Schubart-like linearly stable periodic orbits in the collinear three-body problem were found by Saito and Tanikawa for certain choices of the three masses [17],[18].

Sweatman [22], [23] and Sekiguchi and Tanikawa [19] numerically found a symmetric Schubart-like orbit in the symmetric collinear four-body problem with masses 1, \( m \), \( m \), and 1. This Schubart-like periodic orbit alternates between simultaneous

2000 Mathematics Subject Classification. Primary: 70F10, 70H12, 70H14; Secondary: 70F16, 70H33.

Key words and phrases. N-Body Problem, Regularization, Periodic Orbits, Stability.
binary collisions of the two outer pairs of masses and binary collisions of the inner two masses. Ouyang and Yan [14] proved analytically the existence and symmetry of this orbit. In the regularized setting, this periodic orbit has a symmetry group isomorphic to $D_2$, of which both of the generators are time-reversing symmetries. (Here the dihedral group $D_k$ is the group of symmetries of the regular $k$-gon.) Sweatman [23] numerically showed that the Schubart-like orbit is linearly stable when $0 < m < 2.83$ or $m > 35.4$, and is otherwise unstable. This linear stability was confirmed in [2] using Robert’s symmetry reduction technique [16].

Ouyang, Yan, and Simmons [12] numerically found and analytically proved the existence and symmetry of a singular symmetric periodic orbit in the fully symmetric planar four-body problem with equal masses. (In the fully symmetric planar four-body equal mass problem, the position of one mass determines the positions of the other three masses.) In this orbit, the four masses alternate between different simultaneous binary collisions. In the regularized setting, this periodic orbit has a symmetry group isomorphic to $D_4$, of which one of the two generators is a time-reversing symmetry, while the other generator is a time-preserving symmetry.

In this paper we consider the existence and linear stability of time-reversible periodic simultaneous binary collision orbits in the planar pairwise symmetric four-body problem. The positions of the four bodies in the plane are $(x_1, x_2), (x_3, x_4), (-x_1, -x_2), \text{ and } (-x_3, -x_4)$, where the corresponding masses are $1, m, 1, m$ with $0 < m \leq 1$. With $t$ as the time variable and $\dot{} = d/dt$, the momenta for the four masses are $(\omega_1, \omega_2) = 2(\dot{x}_1, \dot{x}_2), (\omega_3, \omega_4) = 2m(\dot{x}_3, \dot{x}_4), -(\omega_1, \omega_2),$ and $-(\omega_3, \omega_4)$. The Hamiltonian for the pairwise symmetric planar four-body problem is $H = K - U$, where

$$K = \frac{1}{4} [\omega_1^2 + \omega_2^2] + \frac{1}{4m} [\omega_3^2 + \omega_4^2],$$

and

$$U = \frac{1}{2\sqrt{x_1^2 + x_2^2}} + \frac{2m}{\sqrt{(x_3 - x_1)^2 + (x_4 - x_2)^2}} + \frac{2m}{\sqrt{(x_1 + x_3)^2 + (x_2 + x_4)^2}} + \frac{m^2}{2\sqrt{x_3^2 + x_4^2}}.$$  

The angular momentum for the pairwise symmetric planar four-body problem is

$$A = x_1\omega_2 - x_2\omega_1 + x_3\omega_4 - x_4\omega_3.$$  

The center of mass is fixed at the origin, and the linear momentum is zero. With $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ for $I$ the $4 \times 4$ identity matrix, the vector field for the pairwise symmetric planar four-body problem is $J\nabla H$, i.e., the Hamiltonian system of equations with Hamiltonian $H$ are $\dot{x}_i = \partial H/\partial \omega_i, \dot{\omega}_i = -\partial H/\partial x_i, i = 1, 2, 3, 4$.

The initial conditions for the orbits of interest has the first body of mass $1$ located on the positive horizontal axis with its momentum perpendicular to the horizontal axis, and the first body of mass $m$ located on the positive vertical axis with its momentum perpendicular to the vertical axis. Specifically, at $t = 0$ we have

$$x_1 > 0, x_2 = 0, x_3 = 0, x_4 > 0, \text{ with } x_4 \leq x_1,$$

$$\omega_1 = 0, \omega_2 > 0, \omega_3 > 0, \omega_4 = 0, \text{ with } \omega_2 \leq \omega_3,$$
Figure 1. The symmetric periodic simultaneous binary collision orbit in the planar pairwise symmetric four-body problem for $m = 1$ (left) and $m = 0.539$ (right). The two red curves are those traced out by $\pm (x_1(t), x_2(t))$, and the two blue curves are those traced out by $\pm (x_3(t), x_4(t))$.

at which $H$ is defined. The first objective is to find, for $0 < m \leq 1$, values of $x_1, x_4, \omega_2, \omega_3$ at $t = 0$ such that (i) $x_3 - x_1 = 0$ and $x_4 - x_2 = 0$ with $x_1^2 + x_2^2 \neq 0$ at some $t = t_0 > 0$, (ii) $x_1 + x_3 = 0$ and $x_2 + x_4 = 0$ with $x_1^2 + x_2^2 \neq 0$ at some $t = t_1 > t_0$, (iii) the orbit extends to a symmetric periodic orbit, and (iv) the periodic orbit avoids all the other kinds of collisions. Such an orbit experiences a simultaneous binary collision in the first and third quadrant at $t = t_0$, and then another simultaneous binary collision in the second and fourth quadrants at $t = t_1$, before returning to its initial conditions at some $t = t_2 > t_1$. The presence of collisions along the orbit necessarily imposes zero angular momentum on the orbit, thus requiring that $x_1\omega_2 - x_4\omega_3 = 0$ at $t = 0$. Examples of these symmetric periodic simultaneous binary collision orbits in the planar pairwise symmetric four body problem $m, 1, m$ problem are illustrated in Figure 1 for $m = 1$ and $m = 0.539$. The second objective is to investigate the linear stability of the symmetric periodic simultaneous binary collision orbits as $m$ varies over interval $(0, 1]$.

The regularization of the simultaneous binary collisions, as described by (i) and (ii) above, in the Hamiltonian system of equations with Hamiltonian $H$ plays a key role in achieving the two objectives. Section 2 details this regularization which consists of two canonical transformations followed by a scaling of time $t = \theta(s)$ with $s$ as the regularizing time variable, producing a new Hamiltonian $\hat{\Gamma}$ in the extended phase space. Section 3 describes a scaling of orbits of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ which shows that any such periodic solution always belongs to a one-parameter family of periodic solutions for which the linear stability is the same for all periodic solutions in the family. Section 4 describes the symmetries of the Hamiltonian $\hat{\Gamma}$ which are used to construct periodic solutions with a $D_4$ symmetry group generated by a time-reversing symmetry and a time-preserving symmetry.

In Sections 5 and 6, we prove the analytic existence of a periodic simultaneous binary collision orbit $\gamma(s)$, with a $D_4$ symmetry group, for the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ with $m = 1$, and investigate its linear stability. The proof extends the analytic existence of a symmetric periodic simultaneous binary collision orbit in the fully symmetric planar four-body equal mass problem, as found in [12], to the planar pairwise symmetric four-body equal mass problem.
The symmetric periodic simultaneous binary collision orbit in the fully symmetric four-body equal mass problem is known to be linearly stable [2], which provides some analytic evidence for the linear stability of \( \gamma(s) \). We give numerical evidence that supports the linear stability of \( \gamma(s) \).

In Section 7, we numerically continue the symmetric periodic simultaneous binary collision orbit \( \gamma(s) \) from \( m = 1 \) to a symmetric periodic simultaneous binary collision orbit \( \gamma(s;m) \) for \( m < 1 \), and then investigate the linear stability of \( \gamma(s;m) \) as \( m \) varies in the interval \((0,1]\). We use trigonometric polynomials as approximations of the periodic orbits (cf. [20]). The numerical algorithm for continuation starts with a trigonometric polynomial approximation of \( \gamma(s;1) = \gamma(s) \) that is used as a initial guess for a trigonometric polynomial approximation of \( \gamma(s;0.99) \), where the coefficients of the trigonometric polynomial are optimized through a variational approach. This process is repeated, using the optimized approximation of \( \gamma(s;0.99) \) as the initial guess for \( \gamma(s;0.98) \), etc., until an optimized approximation of \( \gamma(s;0.01) \) is obtained. Numerical integrations of the Hamiltonian system of equations with Hamiltonian \( \tilde{\Gamma} \) show that \( \gamma(s;m) \) is linearly stable when \( 0.54 \leq m \leq 1 \), and is unstable when \( 0 < m \leq 0.53 \). We then conduct a refined continuation for values of \( m \) between 0.53 and 0.54 at 0.001 increments. Numerical integration of \( \gamma(s;m) \) for these values of \( m \) reveals that linear stability appears to hold for \( \gamma(s;m) \) when \( 0.538 \leq m \leq 1 \) and that instability appears to hold when \( 0 < m \leq 0.537 \) with spectral stability for \( m \approx 0.537 \).

2. Regularization

We adapt the regularization of Aarseth and Zare [1] to the planar pairwise symmetric four-body problem to regularize simultaneous binary collisions as described in the first objective. The first canonical transformation is

\[
(x_1, x_2, x_3, x_4, \omega_1, \omega_2, \omega_3, \omega_4) \rightarrow (g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4)
\]
determined by the generating function

\[
F_1(x_1, x_2, x_3, x_4, h_1, h_2, h_3, h_4) = h_1(x_1-x_3)+h_2(x_2-x_4)+h_3(x_1+x_3)+h_4(x_2+x_4).
\]

So the first canonical transformation is

\[
\begin{align*}
\omega_1 &= \frac{\partial F_1}{\partial x_1} = h_1 + h_3, & \omega_2 &= \frac{\partial F_1}{\partial x_2} = h_2 + h_4, \\
\omega_3 &= \frac{\partial F_1}{\partial x_3} = -h_1 + h_3, & \omega_4 &= \frac{\partial F_1}{\partial x_4} = -h_2 + h_4, \\
g_1 &= \frac{\partial F_1}{\partial h_1} = x_1 - x_3, & g_2 &= \frac{\partial F_1}{\partial h_2} = x_2 - x_4, \\
g_3 &= \frac{\partial F_1}{\partial h_3} = x_1 + x_3, & g_4 &= \frac{\partial F_1}{\partial h_4} = x_2 + x_4.
\end{align*}
\]

Here

\[
\begin{align*}
x_1 &= \frac{g_1 + g_3}{2}, & x_2 &= \frac{g_2 + g_4}{2}, & x_3 &= \frac{g_3 - g_1}{2}, & x_4 &= \frac{g_4 - g_2}{2}, \\
\end{align*}
\]

and

\[
\begin{align*}
h_1 &= \frac{\omega_1 - \omega_3}{2}, & h_2 &= \frac{\omega_2 - \omega_4}{2}, & h_3 &= \frac{\omega_1 + \omega_3}{2}, & h_4 &= \frac{\omega_2 + \omega_4}{2}.
\end{align*}
\]

So

\[
\begin{align*}
x_1^2 + x_2^2 &= \frac{(g_1 + g_3)^2 + (g_2 + g_4)^2}{4}, & x_3^2 + x_4^2 &= \frac{(g_3 - g_1)^2 + (g_4 - g_2)^2}{4}.
\end{align*}
\]
The new Hamiltonian is $\tilde{H} = \tilde{K} - \tilde{U}$, where
\[
\tilde{K} = \frac{(h_1 + h_3)^2 + (h_2 + h_4)^2}{4} + \frac{(h_3 - h_1)^2 + (h_4 - h_2)^2}{4m},
\]
and
\[
\tilde{U} = \frac{1}{\sqrt{(g_1 + g_3)^2 + (g_2 + g_4)^2}} + \frac{2m}{\sqrt{g_1^2 + g_2^2}} + \frac{2m}{\sqrt{(g_1 - g_3)^2 + (g_2 - g_4)^2}}.
\]

The second canonical transformation is
\[
(g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4) \rightarrow (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)
\]
determined by the generating function
\[
F_2(h_1, h_2, h_3, h_4, u_1, u_2, u_3, u_4) = \sum_{j=1}^{4} h_j f_j(u_1, u_2, u_3, u_4),
\]
where
\[
f_1 = u_1^2 - u_2^2, \quad f_2 = 2u_1u_2, \quad f_3 = u_3^2 - u_4^2, \quad f_4 = 2u_3u_4.
\]

So the second canonical transformation is
\[
g_1 = \frac{\partial F_2}{\partial h_1} = u_1^2 - u_2^2, \quad g_2 = \frac{\partial F_2}{\partial h_2} = 2u_1u_2,
\]
\[
g_3 = \frac{\partial F_2}{\partial h_3} = u_3^2 - u_4^2, \quad g_4 = \frac{\partial F_2}{\partial h_4} = 2u_3u_4,
\]
\[
v_1 = \frac{\partial F_2}{\partial u_1} = 2h_1u_1 + 2h_2u_2, \quad v_2 = \frac{\partial F_2}{\partial u_2} = -2h_1u_2 + 2h_2u_1,
\]
\[
v_3 = \frac{\partial F_2}{\partial u_3} = 2h_3u_3 + 2h_4u_4, \quad v_4 = \frac{\partial F_2}{\partial u_4} = -2h_3u_4 + 2h_4u_3.
\]

Here
\[
g_1^2 + g_2^2 = (u_1^2 - u_2^2)^2 + 4u_1^2u_2^2 = (u_1^2 + u_2^2)^2,
\]
and
\[
g_3^2 + g_4^2 = (u_3^2 - u_4^2)^2 + 4u_3^2u_4^2 = (u_3^2 + u_4^2)^2.
\]

Also, solving $g_2 = 2u_1u_2$ for $u_2$ and substituting this into $g_1 = u_1^2 - u_2^2$ gives
\[
u_1^4 - g_1u_1^2 - \frac{g_2^2}{4} = 0.
\]

Solving this quadratic in $u_1^2$ and noting that $u_1^2 \geq 0$ gives
\[
u_1^2 = \frac{g_1 + \sqrt{g_1^2 + g_2^2}}{2}.
\]

Substituting this into $g_1 = u_1^2 - u_2^2$ and solving for $u_2^2$ gives
\[
u_2^2 = \frac{-g_1 + \sqrt{g_1^2 + g_2^2}}{2}.
\]

Similarly
\[
u_3^2 = \frac{g_3 + \sqrt{g_3^2 + g_4^2}}{2} \quad \text{and} \quad u_4^2 = \frac{-g_3 + \sqrt{g_3^2 + g_4^2}}{2}.
\]

The equations in $v_1, v_2, v_3, v_4$ are linear in $h_1, h_2, h_3, h_4$:
\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} -u_1 & u_2 \\ -u_2 & u_1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = 2 \begin{bmatrix} u_3 & u_4 \\ -u_4 & u_3 \end{bmatrix} \begin{bmatrix} h_3 \\ h_4 \end{bmatrix}.
\]
Solving these equations for \( h_1, h_2, h_3, h_4 \) gives
\[
\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{2(u_1^2 + u_2^2)} \begin{bmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{2(u_3^2 + u_4^2)} \begin{bmatrix} v_1 u_1 - v_2 u_2 \\ v_1 u_2 + v_2 u_1 \end{bmatrix},
\]
and
\[
\begin{bmatrix} h_3 \\ h_4 \end{bmatrix} = \frac{1}{2(u_3^2 + u_4^2)} \begin{bmatrix} u_3 & -u_4 \\ u_4 & u_3 \end{bmatrix} \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = \frac{1}{2(u_3^2 + u_4^2)} \begin{bmatrix} v_3 u_3 - v_4 u_4 \\ v_3 u_4 + v_4 u_3 \end{bmatrix}.
\]

The new Hamiltonian is \( \hat{H} = \hat{K} - \hat{U} \), where
\[
\hat{K} = \frac{1}{16} \left( 1 + \frac{m}{M_5 + M_6} \right) \frac{(v_1^2 + v_2^2)(u_3^2 + u_4^2) + (v_3^2 + v_4^2)(u_1^2 + u_2^2)}{(u_1^2 + u_2^2)(u_3^2 + u_4^2)}
\]
\[
+ \frac{1}{8} \left( 1 - \frac{m}{M_5 + M_6} \right) \frac{(v_3 u_3 - v_4 u_4)(v_1 u_1 - v_2 u_2) + (v_3 u_4 + v_4 u_3)(v_1 u_2 + v_2 u_1)}{(u_1^2 + u_2^2)(u_3^2 + u_4^2)},
\]
and
\[
\hat{U} = \frac{1}{\sqrt{(u_1^2 - u_2^2 + u_3^2 - u_4^2)^2 + (2u_1 u_2 + 2u_3 u_4)^2}} + \frac{2m}{u_1^2 + u_2^2} + \frac{2m}{u_3^2 + u_4^2}
\]
\[
+ \frac{m^2}{\sqrt{(u_1^2 - u_2^2 + u_3^2 - u_4^2)^2 + (2u_1 u_2 - 2u_3 u_4)^2}}.
\]

Introduce a new time variable \( s \) by the regularizing change of time
\[
\frac{dt}{ds} = (u_1^2 + u_2^2)(u_3^2 + u_4^2).
\]

To simplify notation, set
\[
M_1 = v_1 u_1 - v_2 u_2, \quad M_2 = v_1 u_2 + v_2 u_1, \\
M_3 = v_3 u_3 - v_4 u_4, \quad M_4 = v_3 u_4 + v_4 u_3, \\
M_5 = u_1^2 - u_2^2 + u_3^2 - u_4^2, \quad M_6 = 2u_1 u_2 + 2u_3 u_4, \\
M_7 = u_1^2 - u_2^2 - u_3^2 + u_4^2, \quad M_8 = 2u_1 u_2 - 2u_3 u_4.
\]

The Hamiltonian in the extended phase space with coordinates \( u_1, u_2, u_3, u_4, \hat{E}, v_1, v_2, v_3, v_4, t \) is
\[
\hat{\Gamma} = \frac{dt}{ds} (\hat{H} - \hat{E}) = \frac{1}{16} \left( 1 + \frac{m}{M_5 + M_6} \right) \left( (v_1^2 + v_2^2)(u_3^2 + u_4^2) + (v_3^2 + v_4^2)(u_1^2 + u_2^2) \right)
\]
\[
+ \frac{1}{8} \left( 1 - \frac{m}{M_5 + M_6} \right) (M_3 M_1 + M_4 M_2)
\]
\[
- \frac{(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{\sqrt{M_5^2 + M_6^2}} - 2m(u_1^2 + u_2^2 + u_3^2 + u_4^2)
\]
\[
- \frac{m^2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{\sqrt{M_7^2 + M_8^2}} - \hat{E}(u_1^2 + u_2^2)(u_3^2 + u_4^2).
\]
With $' = \frac{d}{ds}$, the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ is:

\[ u_1' = \frac{\partial \hat{\Gamma}}{\partial u_1} = \frac{1}{8} \left( 1 + \frac{1}{m} \right) v_1 (u_3^2 + u_4^2) + \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_3 u_1 + M_4 u_2), \]

\[ u_2' = \frac{\partial \hat{\Gamma}}{\partial u_2} = \frac{1}{8} \left( 1 + \frac{1}{m} \right) v_2 (u_3^2 + u_4^2) + \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_3 (-u_2) + M_4 u_1), \]

\[ u_3' = \frac{\partial \hat{\Gamma}}{\partial u_3} = \frac{1}{8} \left( 1 + \frac{1}{m} \right) v_3 (u_1^2 + u_2^2) + \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_1 u_3 + M_2 u_4), \]

\[ u_4' = \frac{\partial \hat{\Gamma}}{\partial u_4} = \frac{1}{8} \left( 1 + \frac{1}{m} \right) v_4 (u_1^2 + u_2^2) + \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_1 (-u_4) + M_2 u_3), \]

and

\[ v_1' = - \frac{\partial \hat{\Gamma}}{\partial v_1} = - \frac{1}{8} \left( 1 + \frac{1}{m} \right) u_1 (v_3^2 + v_4^2) - \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_3 v_1 + M_4 v_2), \]

\[ + \frac{2u_1(u_3^2 + u_4^2)}{\sqrt{M_5^2 + M_6^2}} - \frac{2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{[M_5^2 + M_6^2]^{3/2}}, \]

\[ + \frac{2m^2 u_1(u_3^2 + u_4^2)}{\sqrt{M_7^2 + M_8^2}} - \frac{2m^2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{[M_7^2 + M_8^2]^{3/2}} + 4mu_1 + 2\hat{E}u_1(u_3^2 + u_4^2), \]

\[ v_2' = - \frac{\partial \hat{\Gamma}}{\partial v_2} = - \frac{1}{8} \left( 1 + \frac{1}{m} \right) u_2 (v_3^2 + v_4^2) - \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_3 (-v_2) + M_4 v_1), \]

\[ + \frac{2u_2(u_3^2 + u_4^2)}{\sqrt{M_5^2 + M_6^2}} - \frac{2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{[M_5^2 + M_6^2]^{3/2}}, \]

\[ + \frac{2m^2 u_2(u_3^2 + u_4^2)}{\sqrt{M_7^2 + M_8^2}} - \frac{2m^2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{[M_7^2 + M_8^2]^{3/2}} + 4mu_2 + 2\hat{E}u_2(u_3^2 + u_4^2), \]

\[ v_3' = - \frac{\partial \hat{\Gamma}}{\partial v_3} = - \frac{1}{8} \left( 1 + \frac{1}{m} \right) u_3 (v_1^2 + v_2^2) - \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_1 v_3 + M_2 v_4), \]

\[ + \frac{2u_3(u_1^2 + u_2^2)}{\sqrt{M_5^2 + M_6^2}} - \frac{2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{[M_5^2 + M_6^2]^{3/2}}, \]

\[ + \frac{2m^2 u_3(u_1^2 + u_2^2)}{\sqrt{M_7^2 + M_8^2}} - \frac{2m^2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{[M_7^2 + M_8^2]^{3/2}} + 4mu_3 + 2\hat{E}u_3(u_1^2 + u_2^2), \]

\[ v_4' = - \frac{\partial \hat{\Gamma}}{\partial v_4} = - \frac{1}{8} \left( 1 + \frac{1}{m} \right) u_4 (v_1^2 + v_2^2) - \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_1 (-v_4) + M_2 v_3), \]

\[ + \frac{2u_4(u_1^2 + u_2^2)}{\sqrt{M_5^2 + M_6^2}} - \frac{2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{[M_5^2 + M_6^2]^{3/2}}, \]

\[ + \frac{2m^2 u_4(u_1^2 + u_2^2)}{\sqrt{M_7^2 + M_8^2}} - \frac{2m^2(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{[M_7^2 + M_8^2]^{3/2}} + 4mu_4 + 2\hat{E}u_4(u_1^2 + u_2^2), \]
along with the auxiliary equations,
\[ \dot{E}' = \frac{\partial \hat{\Gamma}}{\partial t} = 0, \quad t' = -\frac{\partial \hat{\Gamma}}{\partial E} = (u_1^2 + u_2^2)(u_3^2 + u_4^2). \]

On the level set \( \hat{\Gamma} = 0 \), the value of the Hamiltonian \( \hat{H} \) (i.e., the energy) along solutions of the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) is \( E \). Independent of the values of \( E \) and \( \hat{\Gamma} \), the angular momentum \( A = x_1 \omega_2 - x_2 \omega_1 + x_3 \omega_4 - x_4 \omega_3 \) in the coordinates \( u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 \) simplifies to
\[ A = \frac{1}{2} [-v_1 u_2 + v_2 u_1 - v_3 u_4 + v_4 u_3]. \]

On the level set \( \hat{\Gamma} = 0 \), two simultaneous binary collisions in the planar pairwise symmetric four-body problem have been regularized in the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \). The simultaneous binary collision \( x_3 - x_1 = 0 \) and \( x_4 - x_2 = 0 \) with \( x_1^2 + x_2^2 \neq 0 \) corresponds to \( u_1^2 + u_2^2 = 0 \) with \( u_3^2 + u_4^2 \neq 0 \). These imply that \( M_2^6 + M_2^6 = 4(x_1^2 + x_2^2) \neq 0 \) and \( M_2^6 + M_2^6 = 4(x_1^2 + x_2^2) \neq 0 \). From \( \hat{\Gamma} = 0 \) it follows that
\[ v_1^2 + v_2^2 = \frac{32 m^2}{m + 1}. \]

Similarly, the simultaneous binary collision \( x_3 + x_1 = 0 \) and \( x_4 + x_2 = 0 \) with \( x_1^2 + x_2^2 \neq 0 \) corresponds to \( u_3^2 + u_4^2 = 0 \) with \( u_1^2 + u_2^2 \neq 0 \), and hence that \( M_2^6 + M_2^6 = 4(x_1^2 + x_2^2) \neq 0 \), \( M_2^6 + M_2^6 = 4(x_1^2 + x_2^2) \neq 0 \), and, from \( \hat{\Gamma} = 0 \), that
\[ v_3^2 + v_4^2 = \frac{32 m^2}{m + 1}. \]

On the level set \( \hat{\Gamma} = 0 \), the other singularities of the planar pairwise symmetric four-body problem have not been regularized in the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \). The binary collision \( x_1 = 0, x_2 = 0 \) with \( x_3^2 + x_4^2 \neq 0 \) corresponds to \( M_2^6 + M_6^2 = 0 \) and \( M_2^6 + M_6^2 = 0 \) with \( u_1^2 + u_2^2 \neq 0 \) and \( u_3^2 + u_4^2 \neq 0 \). The binary collision \( x_3 = 0, x_4 = 0 \) with \( x_1^2 + x_2^2 \neq 0 \) corresponds to \( M_2^6 + M_6^2 = 0 \) and \( M_2^6 + M_6^2 = 0 \) with \( u_3^2 + u_4^2 \neq 0 \) and \( u_1^2 + u_2^2 \neq 0 \). Because of the pairwise symmetry, there are no triple collisions. Total collapse \( x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0 \) corresponds to \( u_1 = 0, u_2 = 0, u_3 = 0, and u_4 = 0 \). A solution of the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) is called nonsingular if it avoids the unregularized binary collisions and total collapse singularities.

We establish next the correspondence between the original coordinates and the regularized coordinates for the initial conditions given in the Introduction.

**Lemma 2.1.** The conditions (at \( t = 0 \))
\[ x_1 > 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 > 0, \quad \text{with} \quad x_4 \leq x_1, \]
\[ \omega_1 = 0, \quad \omega_2 > 0, \quad \omega_3 > 0, \quad \omega_4 = 0, \quad \text{with} \quad \omega_2 \leq \omega_3, \]
respond to the conditions (at \( s = 0 \))
\[ u_3 = \pm u_1, \quad u_4 = \mp u_2, \quad \text{with} \quad u_1 u_2 < 0, \quad |u_2| \leq (\sqrt{2} - 1)|u_1|, \]
\[ v_3 = \mp v_1, \quad v_4 = \pm v_2, \quad \text{with} \quad 0 < v_1 u_2 + v_2 u_1 \leq v_2 u_2 - v_1 u_1. \]
Proof. Suppose that $x_2 = 0$, $x_3 = 0$, $\omega_1 = 0$, $\omega_4 = 0$, and that $x_1$, $x_4$, $\omega_2$, and $\omega_3$ are positive with $x_4 \leq x_1$ and $\omega_2 \leq \omega_3$. The first canonical transformation implies that

$$g_1 = g_3 > 0, \quad g_2 = -g_4 < 0, \quad \text{with } |g_2| \leq g_1,$$

$$h_1 = -h_3 < 0, \quad h_2 = h_4 > 0, \quad \text{with } h_2 \leq |h_1|.$$ 

The second canonical transformation can be rendered in the following complex notation identities:

$$g_1 + ig_2 = u_1^2 - u_2^2 + 2iu_1u_2 = (u_1 + iu_2)^2,$$

$$g_3 - ig_4 = u_3^2 - u_4^2 - 2iu_3u_4 = (u_3 - iu_4)^2,$$

$$-h_1 + ih_2 = \frac{v_2u_2 - v_1u_1}{2(u_1^2 + u_2^2)} + i\frac{v_1u_2 + v_2u_1}{2(u_1^2 + u_2^2)}$$

$$= (-v_1 + iv_2)(u_1 - iu_2) = -v_1 + iv_2 \frac{2}{2(u_1 + iu_2)},$$

and

$$h_3 + ih_4 = \frac{v_3u_3 - v_4u_4}{2(u_3^2 + u_4^2)} + i\frac{v_3u_4 + v_4u_3}{2(u_3^2 + u_4^2)}$$

$$= (v_3 + iv_4)(u_3 + iu_4) = v_3 + iv_4 \frac{2}{2(u_3 - iu_4)}.$$ 

The two identities relating $g_1, g_2, g_3, g_4$ with $u_1, u_2, u_3, u_4$ imply that

$$(u_1 + iu_2)^2 = g_1 + ig_2 = g_3 - ig_4 = (u_3 - iu_4)^2.$$ 

Thus $u_1 + iu_2 = \pm (u_3 - iu_4)$, and so

$$u_3 = \pm u_1, \quad u_4 = \mp u_2.$$ 

Now $g_2 = 2u_1u_2$ and $g_2 < 0$ imply that

$$u_1u_2 < 0.$$ 

For a complex number $z$, let $\text{arg}(z)$ denote the argument of $z$, i.e., the angle $z$ makes with the positive horizontal axis, modulo $2\pi$. Since $g_1 > 0$, $g_2 < 0$, and $|g_2| \leq g_1$, then $\text{arg}(g_1 + ig_2) \in [7\pi/4, 2\pi)$. Since $(u_1 + iu_2)^2 = g_1 + ig_2$, it follows that $\text{arg}(u_1 + iu_2) \in [7\pi/8, \pi) \cup [15\pi/8, 2\pi)$. This implies that

$$|u_2| \leq \tan(\pi/8)|u_1|.$$ 

It is easily shown that $\tan(\pi/8) = \sqrt{2} - 1.$

The equalities $h_1 = -h_3$ and $h_2 = h_4$ imply that $-h_1 + ih_2 = h_3 + ih_4$. By the identities relating $h_1, h_2, h_3, h_4$ and $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4$, we have that

$$-v_1 + iv_2 = \frac{v_3 + iv_4}{2(u_3 - iu_4)}.$$ 

Since $u_1 + iu_2 = \pm (u_3 - iu_4)$, then $-v_1 + iv_2 = \pm (v_3 + iv_4)$, and so

$$v_3 = \mp v_1, \quad v_4 = \pm v_2.$$ 

Since $h_2 \leq |h_1|$, $h_1 < 0$, and $h_2 > 0$, the second canonical transformation implies

$$0 < \frac{v_1u_2 + v_2u_1}{2(u_1^2 + u_2^2)} = h_2 \leq -h_1 = \frac{v_1u_2 - v_1u_2}{2(u_1^2 + u_2^2)}.$$
From this it follows
\[ 0 < v_1 u_2 + v_2 u_1 \leq v_2 u_2 - v_1 u_1. \]

Now suppose that \( u_3 = \pm u_1, u_4 = \mp u_2, u_1 u_2 < 0, |u_2| \leq (\sqrt{2} - 1)|u_1|, v_3 = \mp v_1, v_4 = \pm v_2, \) and \( 0 < v_1 u_2 + v_2 u_1 \leq v_2 u_2 - v_1 u_1. \) The second canonical transformation implies that
\[ g_1 = u_1^2 - u_2^2, \quad g_2 = 2u_1 u_2 < 0, \quad g_3 = u_1^2 - u_2^2, \quad g_4 = -2u_1 u_2 > 0. \]

The first canonical transformation implies
\[ x_1 = u_1^2 - u_2^2 > 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = -2u_1 u_2 > 0. \]

In complex notation, \( x_1 - i x_2 = (u_1 + i u_2)^2. \) Then \( |u_2| \leq (\sqrt{2} - 1)|u_1| \) implies that \( |x_4| \leq \tan(\pi/4)|x_1| = |x_1|, \) i.e.,
\[ x_4 \leq x_1. \]

The equalities \( u_3 = \pm u_1, u_4 = \mp u_2, v_3 = \mp v_1, \) and \( v_4 = \pm v_2 \) imply that \( u_1^2 + u_2^2 = u_3^2 + u_4^2, \) and
\[ v_3 u_4 + v_4 u_3 = v_1 u_2 + v_2 u_1, \quad v_3 u_3 - v_4 u_4 = -v_1 u_1 + v_2 u_2. \]

Thus the second canonical transformation implies
\[ -h_1 = \frac{-v_1 u_1 + v_2 u_2}{2(u_1^2 + u_2^2)} = \frac{v_3 u_3 - v_4 u_4}{2(u_3^2 + u_4^2)} = h_3, \]
\[ h_2 = \frac{v_1 u_2 + v_2 u_1}{2(u_1^2 + u_2^2)} = \frac{v_3 u_4 + v_4 u_3}{2(u_3^2 + u_4^2)} = h_4. \]

Furthermore, the inequalities \( 0 < v_1 u_2 + v_2 u_1 \leq v_2 u_2 - v_1 u_1 \) imply that
\[ -h_1 \geq h_2 > 0. \]

The first canonical transformation now implies that
\[ \omega_1 = h_1 + h_3 = 0, \quad \omega_2 = h_2 + h_4 > 0, \quad \omega_3 = -h_1 + h_3 > 0, \quad \omega_4 = -h_2 + h_4 = 0 \]
with \( \omega_3 = -2h_1 \geq 2h_2 = \omega_2. \)

3. A Scaling of Periodic Orbits and Linear Stability

A certain scaling of solutions of the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) produces more solutions. When applied to a periodic solution, this scaling leads to a one-parameter family of periodic solutions. The proof of the following result is a straight-forward verification.

**Lemma 3.1.** If \( \gamma(s) = (u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s)) \) is a periodic solution of the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) on the level set \( \hat{\Gamma} = 0 \) with period \( T \) and energy \( E \), then for every \( \epsilon > 0 \), the function
\[ \gamma_\epsilon(s) = (\epsilon u_1(\epsilon s), \epsilon u_2(\epsilon s), \epsilon u_3(\epsilon s), \epsilon u_4(\epsilon s), \epsilon v_1(\epsilon s), \epsilon v_2(\epsilon s), \epsilon v_3(\epsilon s), \epsilon v_4(\epsilon s)) \]
is a periodic solution of the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) on the level set \( \hat{\Gamma} = 0 \) with period \( T_\epsilon = \epsilon^{-1}T \) and energy \( \hat{E}_\epsilon = \epsilon^{-2}E \).

The linear stability of a periodic orbit of the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) is determined by the linearization of the equations along the periodic orbit. By Lemma 3.1, a periodic orbit \( \gamma(s) \) on the level set \( \Gamma = 0 \) with period \( T \) and energy \( \hat{E} \) embeds into a one-parameter family \( \gamma_\epsilon(s) \) of periodic orbits on the level set \( \hat{\Gamma} = 0 \) with period \( T_\epsilon \) and energy \( \hat{E}_\epsilon \). The linearization of the
Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ along the periodic orbit $\gamma_\epsilon(s)$ is

$$X' = J\nabla^2\hat{\Gamma}(\gamma_\epsilon(s))X$$

where $\nabla^2\hat{\Gamma}$ is the matrix of second-order partials of $\hat{\Gamma}$. Let $X_\epsilon(s)$ be the solution of the linearization of the equations along $\gamma_\epsilon(s)$ that satisfies $X_\epsilon(0) = I$ (the $8 \times 8$ identity matrix). The monodromy matrix for $\gamma_\epsilon(s)$ is $X_\epsilon(T_\epsilon)$, and the eigenvalues of $X_\epsilon(T_\epsilon)$ are the characteristic multipliers of $\gamma_\epsilon(s)$. A characteristic multiplier $\lambda$ of $\gamma_\epsilon(s)$ is defective if its geometric multiplicity is smaller than its algebraic multiplicity, i.e., its generalized eigenspace $\cup_{j \geq 1} \ker(X_\epsilon(T_\epsilon) - \lambda I)^j$ is not the same as its eigenspace $\ker(X_\epsilon(T_\epsilon) - \lambda I)$.

**Lemma 3.2.** If $\gamma(s)$ is a periodic orbit of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$, then for each $\epsilon > 0$, the periodic orbit $\gamma_\epsilon(s)$ has 1 as a defective characteristic multiplier with algebraic multiplicity at least two.

**Proof.** By Lemma 3.1, $\gamma_\epsilon(s)$ is a one-parameter family of periodic orbits on the level set $\hat{\Gamma} = 0$ with period $T_\epsilon = \epsilon^{-1}T$ and energy $\hat{E}_\epsilon = \epsilon^{-2}\hat{E}$, where $\gamma_1(s) = \gamma(s)$ and $T$ is the period and $\hat{E}$ is the energy of $\gamma(s)$. For each $\epsilon > 0$, the periodic solution $\gamma_\epsilon(s)$ satisfies $\gamma_\epsilon'(s) = J\nabla\hat{\Gamma}(\gamma_\epsilon(s))$, and so

$$\left(\gamma_\epsilon'(s)\right)' = J\nabla^2\hat{\Gamma}(\gamma_\epsilon(s))\gamma_\epsilon''(s),$$

and

$$\left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)'(s) = J\nabla^2\hat{\Gamma}(\gamma_\epsilon(s))\left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(s).$$

Since $X_\epsilon$ is a fundamental matrix solution that satisfies $X_\epsilon(0) = I$, it follows that

$$\gamma_\epsilon'(s) = X_\epsilon(s)\gamma_\epsilon'(0) \quad \text{and} \quad \frac{\partial}{\partial \epsilon}\gamma_\epsilon(s) = X_\epsilon(s)\left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(0).$$

The $T_\epsilon$-periodicity of $\gamma_\epsilon(s)$ implies the $T_\epsilon$-periodicity of $\gamma_\epsilon'(s)$, and so

$$\gamma_\epsilon'(0) = \gamma_\epsilon'(T_\epsilon) = X_\epsilon(T_\epsilon)\gamma_\epsilon'(0)$$

where $\gamma_\epsilon'(0) \neq 0$. Thus 1 is an characteristic multiplier of $\gamma_\epsilon(s)$. Since $X_\epsilon(T_\epsilon)$ is symplectic, the algebraic multiplicity of this characteristic multiplier is at least two. Now

$$\gamma_\epsilon'(s) = (\epsilon^2u_1'(\epsilon s), \ldots, \epsilon^2u_4'(\epsilon s), \epsilon v'(\epsilon s), \ldots, \epsilon v_4'(\epsilon s))$$

and

$$\left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(s) = (u_1(\epsilon s) + \epsilon su_1'(\epsilon s), \ldots, u_4(\epsilon s) + \epsilon su_4'(\epsilon s), sv_1'(\epsilon s), \ldots, sv_4'(\epsilon s)).$$

The $T_\epsilon$-periodicity of $\gamma_\epsilon(s)$ and $\gamma_\epsilon'(s)$ implies that

$$\left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(T_\epsilon) = \left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(0) + \epsilon^{-1}T_\epsilon\gamma_\epsilon'(0).$$

Thus

$$X_\epsilon(T_\epsilon)\left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(0) = \left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(T_\epsilon) = \left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(0) + \epsilon^{-1}T_\epsilon\gamma_\epsilon'(0).$$

Consequently,

$$(X_\epsilon(T_\epsilon) - I)\left(\frac{\partial}{\partial \epsilon}\gamma_\epsilon\right)(0) = \epsilon^{-1}T_\epsilon\gamma_\epsilon'(0) \neq 0$$
and
\[(X_\epsilon(T_\epsilon) - I)^2 \left( \frac{\partial}{\partial \epsilon} \gamma_\epsilon \right)(0) = 0.\]

This shows that
\[\bigcup_{j \geq 1} \ker(X_\epsilon(T_\epsilon) - I)^j \neq \ker(X_\epsilon(T_\epsilon) - I),\]
i.e., that 1 is a defective characteristic multiplier of \(\gamma_\epsilon(s)\) for each \(\epsilon > 0\).

For each \(\epsilon > 0\), the periodic orbit \(\gamma_\epsilon(s)\) is spectrally stable if all of its characteristic multipliers have modulus one. By Lemma 3.2, the periodic orbit \(\gamma_\epsilon(s)\) has 1 as a defective characteristic multiplier with algebraic multiplicity at least two, and so the monodromy matrix \(X_\epsilon(T_\epsilon)\) it not semisimple. However, as shown in the proof of Lemma 3.2, the two-dimensional subspace
\[U_1 = \text{Span} \left( \gamma'_\epsilon(0), \left( \frac{\partial}{\partial \epsilon} \gamma_\epsilon \right)(0) \right)\]
is \(X_\epsilon(T_\epsilon)\)-invariant. The periodic orbit \(\gamma_\epsilon(s)\) is said to be linearly stable if it is spectrally stable and there exists a 6-dimensional \(X_\epsilon(T_\epsilon)\)-invariant subspace \(U_2\) such that \(U_1 + U_2 = \mathbb{R}^8\) and \(X_\epsilon(T_\epsilon)\) restricted to \(U_2\) is semisimple.

**Theorem 3.3.** Suppose \(\gamma(s)\) is a periodic orbit of the Hamiltonian system of equations with Hamiltonian \(\hat{\Gamma}\) on the level set \(\hat{\Gamma} = 0\). Then \(\gamma_\epsilon(s)\) is spectrally (linearly) stable for some \(\epsilon > 0\) if and only if \(\gamma_\epsilon(s)\) is spectrally (linearly) stable for all \(\epsilon > 0\).

**Proof.** Let \(\gamma(s)\) be a periodic orbit with period \(T\) and energy \(\hat{E}\). For each \(\epsilon > 0\), the periodic orbit \(\gamma_\epsilon(s)\) has period \(T_\epsilon = \epsilon^{-1} T\) and energy \(\hat{E}_\epsilon = \epsilon^{-2} \hat{E}\). Note that \(\gamma_1(s) = \gamma(s), T_1 = T, \) and \(\hat{E}_1 = \hat{E}\).

It suffices to show that \(X_\epsilon(T_\epsilon)\) is similar to \(X_1(T_1)\) for all \(\epsilon > 0\). The fundamental matrix \(X_\epsilon(s)\) satisfies \(X'_\epsilon(s) = \nabla^2 \hat{\Gamma}(\gamma_\epsilon(s))X_\epsilon(s), X_\epsilon(0) = I\). Since the matrix \(\nabla^2 \hat{\Gamma}\) is symmetric, there are \(4 \times 4\) matrix functions \(A,B,D\) that satisfy
\[\nabla^2 \hat{\Gamma} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}\]
with \(A^T = A\) and \(D^T = D\). Here
\[A = \left[ \frac{\partial^2 \hat{\Gamma}}{\partial u_i \partial u_j} \right]_{i,j=1,\ldots,4}, \quad B = \left[ \frac{\partial^2 \hat{\Gamma}}{\partial u_i \partial v_j} \right]_{i,j=1,\ldots,4}, \quad D = \left[ \frac{\partial^2 \hat{\Gamma}}{\partial v_i \partial v_j} \right]_{i,j=1,\ldots,4}.
\]

From these, it is straightforward to show that
\[A(\gamma_\epsilon(s)) = A(\gamma_1(\epsilon s)), \quad B(\gamma_\epsilon(s)) = \epsilon B(\gamma_1(\epsilon s)), \quad D(\gamma_\epsilon(s)) = \epsilon^2 D(\gamma_1(\epsilon s)),\]
where the value of energy on the left of each of these is \(\hat{E}_\epsilon\) and on the right is \(\hat{E}_1\).

Thus
\[J \nabla^2 \hat{\Gamma}(\gamma_\epsilon(s)) = \begin{bmatrix} \epsilon B^T(\gamma_1(\epsilon s)) & \epsilon^2 D(\gamma_1(\epsilon s)) \\ -A(\gamma_1(\epsilon s)) & -\epsilon B(\gamma_1(\epsilon s)) \end{bmatrix}.
\]

For \(I\) the \(4 \times 4\) identity matrix, define the nonsingular matrix
\[Y_\epsilon = \begin{bmatrix} \epsilon^{-1/2} I & 0 \\ 0 & \epsilon^{1/2} I \end{bmatrix}.
\]

Then
\[\epsilon Y_\epsilon^{-1} J \nabla^2 \hat{\Gamma}(\gamma_1(\epsilon s)) Y_\epsilon = \epsilon \begin{bmatrix} B^T(\gamma_1(\epsilon s)) & \epsilon D(\gamma_1(\epsilon s)) \\ -\epsilon^{-1} A(\gamma_1(\epsilon s)) & -B(\gamma_1(\epsilon s)) \end{bmatrix} = J \nabla^2 \hat{\Gamma}(\gamma_\epsilon(s)).\]
Thus
\[
(Y^{-1}X_1(\epsilon s)Y_{\epsilon})' = \epsilon Y^{-1}J\nabla^2\hat{\Gamma}(\gamma(\epsilon s))X_1(\epsilon s)Y_{\epsilon} \\
= \epsilon Y^{-1}J\nabla^2\hat{\Gamma}(\gamma(\epsilon s))YY^{-1}X_1(\epsilon s)Y_{\epsilon} \\
= \epsilon \nabla^2\hat{\Gamma}(\gamma(\epsilon s))Y^{-1}X_1(\epsilon s)Y_{\epsilon}.
\]
Since \(X_\epsilon(s)\) and \(Y^{-1}X_1(\epsilon s)Y_{\epsilon}\) both evaluate to the \(8 \times 8\) identity matrix at \(s = 0\), uniqueness of solutions implies that \(X_\epsilon(s) = Y^{-1}X_1(\epsilon s)Y_{\epsilon}\) for all \(s\). In particular,
\[
X_\epsilon(T_\epsilon) = Y^{-1}X_1(\epsilon T_\epsilon)Y_{\epsilon} = Y^{-1}X_1(T_1)Y_{\epsilon}.
\]
Therefore, \(X_\epsilon(T_\epsilon)\) and \(X_1(T_1)\) are similar. \(\square\)

4. Symmetries

The Hamiltonian system of equations with Hamiltonian \(\hat{\Gamma}\) has a group of symmetries isomorphic to the dihedral group \(D_4 = (a, b : a^2 = b^4 = (ab)^2 = e)\). With
\[
F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]
define the matrices
\[
S_F = \begin{bmatrix} 0 & F & 0 & 0 \\ -F & 0 & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & -F & 0 \end{bmatrix}, \quad S_G = \begin{bmatrix} -G & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & -G \end{bmatrix}.
\]
These matrices satisfy \(S_F^2 = \hat{1}, S_F^4 = I, S_G^2 = I,\) and \((S_F S_G)^2 = \hat{1}\). Fixing the value of \(\hat{E}\), these matrices satisfy \(\hat{\Gamma} \circ S_F = \hat{\Gamma}\) and \(\hat{\Gamma} \circ S_G = \hat{\Gamma}\), and so \(S_F\) and \(S_G\) are the generators of the \(D_4\)-symmetry group for \(\hat{\Gamma}\). If
\[
\gamma(s) = (u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s))
\]
is a solution of the Hamiltonian system of equations with Hamiltonian \(\hat{\Gamma}\), then \(S_F \gamma(s), S_F^2 \gamma(s),\) and \(S_G \gamma(-s)\) are also solutions of the Hamiltonian system of equations with Hamiltonian \(\hat{\Gamma}\). This means that \(S_F\) is a time-preserving symmetry and that \(S_G\) is a time-reversing symmetry.

**Lemma 4.1.** If for some \(s_0 > 0\) there is a nonsingular solution \(\gamma(s), s \in [0, s_0]\), of the Hamiltonian system of equations with Hamiltonian \(\hat{\Gamma}\) such that for constants \(\zeta_1 \neq 0, \zeta_2 \neq 0, \rho_1 \neq 0,\) and \(\rho_2 \neq 0\) there holds
\[
\begin{align*}
&u_1(0) = \zeta_1, \quad u_2(0) = \zeta_2, \quad u_3(0) = \zeta_1, \quad u_4(0) = -\zeta_2, \\
&v_1(0) = \rho_1, \quad v_2(0) = \rho_2, \quad v_3(0) = -\rho_1, \quad v_4(0) = \rho_2,
\end{align*}
\]
and
\[
\begin{align*}
&u_1(s_0) = 0, \quad u_2(s_0) = 0, \quad u_3(s_0) \neq 0, \quad u_4(s_0) \neq 0, \\
&v_1(s_0) \neq 0, \quad v_2(s_0) \neq 0, \quad v_3(s_0) = 0, \quad v_4(s_0) = 0,
\end{align*}
\]
then \(\gamma(s)\) extends to a periodic orbit with period \(8s_0\) and a symmetry group isomorphic to \(D_4\) such that
\[
\begin{align*}
&u_1(3s_0) \neq 0, \quad u_2(3s_0) \neq 0, \quad u_3(3s_0) = 0, \quad u_4(3s_0) = 0, \\
&v_1(3s_0) = 0, \quad v_2(3s_0) = 0, \quad v_3(3s_0) \neq 0, \quad v_4(3s_0) \neq 0,
\end{align*}
\]
Proof. The curve $S_G\gamma(2s_0 - s)$, $s \in [s_0, 2s_0]$, is a nonsingular solution of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ whose conditions at $s = s_0$ are

$S_G\gamma(2s_0 - s_0) = S_G\gamma(s_0) = (0, 0, u_3(s_0), u_4(s_0), v_1(s_0), v_2(s_0), 0, 0) = \gamma(s_0)$.

By uniqueness of solutions, $S_G\gamma(2s_0 - s)$, $s \in [s_0, 2s_0]$, is the extension of $\gamma(s)$, $s \in [0, s_0]$, to $[s_0, 2s_0]$, i.e., for $s \in [s_0, 2s_0]$

$u_1(s) = -u_1(2s_0 - s), \quad u_2(s) = -u_2(2s_0 - s), \quad u_3(s) = u_3(2s_0 - s), \quad u_4(s) = u_4(2s_0 - s), \quad v_1(s) = v_1(2s_0 - s), \quad v_2(s) = v_2(2s_0 - s), \quad v_3(s) = -v_3(2s_0 - s), \quad v_4(s) = -v_4(2s_0 - s)$.

The conditions of this extension at $s = 2s_0$ are

$S_G\gamma(2s_0 - 2s_0) = S_G\gamma(0) = (-\zeta_1, -\zeta_2, -\zeta_1, \zeta_2, -\rho_1, -\rho_2, \rho_1, -\rho_2)$.

The curve $S_F\gamma(s - 2s_0)$, $s \in [2s_0, 4s_0]$, is a nonsingular solution of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$. Its conditions at $s = 2s_0$ are

$S_F\gamma(2s_0 - 2s_0) = S_F\gamma(0) = (-\zeta_1, -\zeta_2, -\zeta_1, -\zeta_2, \rho_1, \rho_2, \rho_1, -\rho_2)$.

By uniqueness of solutions, $S_F\gamma(s - 2s_0)$, $s \in [2s_0, 4s_0]$, is the extension of $\gamma(s)$, $s \in [0, 2s_0]$, to $[2s_0, 4s_0]$, i.e., for $s \in [2s_0, 4s_0]$

$u_1(s) = -u_3(s - 2s_0), \quad u_2(s) = u_4(s - 2s_0), \quad u_3(s) = u_1(s - 2s_0), \quad u_4(s) = -u_2(s - 2s_0), \quad v_1(s) = -v_3(s - 2s_0), \quad v_2(s) = v_4(s - 2s_0), \quad v_3(s) = v_1(s - 2s_0), \quad v_4(s) = -v_2(s - 2s_0)$.

The conditions of this extension at $s = 4s_0$ are

$S_F\gamma(4s_0 - 2s_0) = S_F\gamma(2s_0) = S_F S_G\gamma(0)$

$= (-\zeta_1, -\zeta_2, -\zeta_1, -\zeta_2, -\rho_1, -\rho_2, \rho_1, -\rho_2)$.

The curve $S_F^2\gamma(s - 4s_0)$, $s \in [4s_0, 8s_0]$, is a nonsingular solution of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$. Its conditions at $s = 4s_0$ are

$S_F^2(4s_0 - 4s_0) = S_F^2\gamma(0) = (-\zeta_1, -\zeta_2, -\zeta_1, \zeta_2, -\rho_1, -\rho_2, \rho_1, -\rho_2)$. 
By uniqueness of solutions, $S^2_F \gamma(s - 4s_0)$, $s \in [4s_0, 8s_0]$, is an extension of $\gamma(s)$, $s \in [0, 4s_0]$, to $[4s_0, 8s_0]$, i.e., for $s \in [4s_0, 8s_0]$, 

\[
\begin{align*}
    u_1(s) &= -u_1(s - 4s_0), \quad u_2(s) = -u_2(s - 4s_0), \\
    u_3(s) &= -u_3(s - 4s_0), \quad u_4(s) = -u_4(s - 4s_0), \\
    v_1(s) &= -v_1(s - 4s_0), \quad v_2(s) = -v_2(s - 4s_0), \\
    v_3(s) &= -v_3(s - 4s_0), \quad v_4(s) = -v_4(s - 4s_0).
\end{align*}
\]

The conditions of this extension at $s = 8s_0$ are 

\[
S^2_F \gamma(8s_0 - 4s_0) = S^2_F \gamma(4s_0) = S^2_F S_F \gamma(2s_0)
\]

\[
= (\zeta_1, \zeta_2, \zeta_1, -\zeta_2, \rho_1, \rho_2, -\rho_1, \rho_2) = \gamma(0).
\]

Thus the overall extension of $\gamma(s)$ from $[0, s_0]$ to $[0, 8s_0]$ is nonsingular and periodic with period $8s_0$. By the construction of this extension, 

\[
\gamma(2s_0 - s) = S_F \gamma(s) \quad \text{and} \quad \gamma(s + 2s_0) = S_F \gamma(s)
\]

for all $s$, and so the periodic extension of $\gamma(s)$ has a symmetry group isomorphic to $D_4$.

Simultaneous binary collision occur at $s = s_0$ by hypothesis, at $s = 3s_0$ where 

\[
\gamma(3s_0) = S_F \gamma(s_0) = (-u_3(0), u_4(0), 0, 0, 0, v_1(0), -v_2(0)),
\]

at $s = 5s_0$ where 

\[
\gamma(5s_0) = S^2_F \gamma(s_0) = (0, 0, -u_3(0), -u_4(0), -v_1(0), -v_2(0), 0, 0),
\]

and at $s = 7s_0$ where 

\[
\gamma(7s_0) = S^2_F \gamma(3s_0) = (u_3(0), -u_4(0), 0, 0, 0, -v_1(0), v_2(0)).
\]

Therefore, the symmetric periodic extension of $\gamma(s)$ has four distinct simultaneous binary collisions between $s = 0$ and $s = 8s_0$. $\square$

5. **Analytic Existence in the Equal Mass Case**

When $m = 1$, there is an additional symmetry in the positions of the four masses that reduces the planar pairwise symmetric four-body equal mass problem to the fully symmetric planar four-body equal mass problem. We exploit this reduction to prove the existence of a symmetric periodic simultaneous binary collision orbit in the equal mass case.

The additional symmetry is the Ansatz, 

\[
x_4 = x_1, \quad x_3 = x_2, \quad \text{with } |x_2| \leq x_1.
\]

From this it follows that 

\[
\omega_4 = \omega_1, \quad \omega_3 = \omega_2 \quad x_1 - x_2 \geq 0, \quad x_1 + x_2 \geq 0.
\]

From the first canonical transformation, we have 

\[
\begin{align*}
    g_1 &= x_1 - x_2, \quad \quad \quad g_2 = x_2 - x_1, \\
    g_3 &= x_1 + x_2, \quad \quad \quad g_4 = x_1 + x_2, \\
    h_1 &= \frac{\omega_1 - \omega_2}{2}, \quad \quad \quad h_2 = \frac{\omega_2 - \omega_1}{2}, \\
    h_3 &= \frac{\omega_1 + \omega_2}{2}, \quad \quad \quad h_4 = \frac{\omega_1 + \omega_2}{2}.
\end{align*}
\]
From the second canonical transformation, we have

\[
\begin{align*}
    u_1^2 &= \frac{1 + \sqrt{2}}{2} (x_1 - x_2), & u_2^2 &= \frac{-1 + \sqrt{2}}{2} (x_1 - x_2), \\
    u_3^2 &= \frac{1 + \sqrt{2}}{2} (x_1 + x_2), & u_4^2 &= \frac{-1 + \sqrt{2}}{2} (x_1 + x_2).
\end{align*}
\]

Thus

\[
\begin{align*}
    \frac{2u_1^2}{1 + \sqrt{2}} &= x_1 - x_2 = \frac{2u_3^2}{-1 + \sqrt{2}}, & \frac{2u_2^2}{1 + \sqrt{2}} &= x_1 + x_2 = \frac{2u_4^2}{-1 + \sqrt{2}}.
\end{align*}
\]

Since \(2u_1u_2 = g_2 = x_2 - x_1 \leq 0\) and \(2u_3u_4 = g_4 = x_1 + x_2 \geq 0\), it follows that

\[
\begin{align*}
    u_2 &= \sqrt{-\frac{1 + \sqrt{2}}{1 + \sqrt{2}}} u_1 = -((\sqrt{2} - 1)u_1), \\
    u_4 &= \sqrt{-\frac{1 + \sqrt{2}}{1 + \sqrt{2}}} u_3 = ((\sqrt{2} - 1)u_3).
\end{align*}
\]

From the second canonical transformation, we have

\[
\begin{align*}
    v_1 &= \sqrt{2} (\omega_1 - \omega_2) u_1, & v_2 &= -(2 - \sqrt{2}) (\omega_1 - \omega_2) u_1, \\
    v_3 &= \sqrt{2} (\omega_1 + \omega_2) u_3, & v_4 &= (2 - \sqrt{2}) (\omega_1 + \omega_2) u_3.
\end{align*}
\]

These imply that

\[
\begin{align*}
    \frac{v_1}{\sqrt{2}} &= (\omega_1 - \omega_2) u_1 = \frac{v_2}{-2 - \sqrt{2}}, & \frac{v_3}{\sqrt{2}} &= (\omega_1 + \omega_2) u_3 = \frac{v_4}{2 - \sqrt{2}},
\end{align*}
\]

and thus

\[
\begin{align*}
    v_2 &= -\frac{2 - \sqrt{2}}{\sqrt{2}} v_1 = -((\sqrt{2} - 1)v_1), & v_4 &= \frac{2 - \sqrt{2}}{\sqrt{2}} v_3 = ((\sqrt{2} - 1)v_3).
\end{align*}
\]

Substitution into the Hamiltonian system of equations with Hamiltonian \(\hat{\Gamma}\) (and with \(m = 1\)) gives

\[
\begin{align*}
    u_1' &= \frac{4 - 2\sqrt{2}}{4} v_1 u_3, & u_2' &= -\frac{(\sqrt{2} - 1)(4 - 2\sqrt{2})}{4} v_1 u_3, \\
    u_3' &= \frac{4 - 2\sqrt{2}}{4} v_3 u_1, & u_4' &= \frac{(\sqrt{2} - 1)(4 - 2\sqrt{2})}{4} v_3 u_1.
\end{align*}
\]
\[\dot{E'} = 0, \text{ and} \]
\[v'_1 = -\frac{(4 - 2\sqrt{2})u_1v_1^2}{4} + 4u_1 + \frac{4u_1^5}{u_1^4 + u_1^3} - \frac{4u_1^5}{(u_1^4 + u_1^3)^{3/2}} + 2(4 - 2\sqrt{2})\dot{E}u_1u_2, \]
\[v'_2 = -\left(\sqrt{2} - 1\right) \left[ -\frac{(4 - 2\sqrt{2})u_1v_1^2}{4} + 4u_1 + \frac{4u_1^5}{u_1^4 + u_1^3} - \frac{4u_1^5}{(u_1^4 + u_1^3)^{3/2}} \right], \]
\[v'_3 = -\frac{(4 - 2\sqrt{2})u_3v_3^2}{4} + 4u_3 + \frac{4u_1^5}{u_1^4 + u_1^3} - \frac{4u_1^5}{(u_1^4 + u_1^3)^{3/2}} + 2(4 - 2\sqrt{2})\dot{E}u_1u_3, \]
\[v'_4 = \left(\sqrt{2} - 1\right) \left[ -\frac{(4 - 2\sqrt{2})u_3v_3^2}{4} + 4u_3 + \frac{4u_1^5}{u_1^4 + u_1^3} - \frac{4u_1^5}{(u_1^4 + u_1^3)^{3/2}} \right], \]
\[t' = (4 - 2\sqrt{2})^2u_1u_3^2. \]

Because \(u_2 = -(\sqrt{2} - 1)u_1, u_4 = (\sqrt{2} - 1)u_3, v_2 = -(\sqrt{2} - 1)v_1, \) and \(v_4 = (\sqrt{2} - 1)v_3,\) the equations in \(u'_2, u'_4, v'_2, \) and \(v'_4\) duplicate those in \(u'_1, u'_3, v'_1, \) and \(v'_3.\) The Ansatz \(x_4 = x_1, x_3 = x_2\) with \(|x_2| \leq x_1,\) therefore leads to the reduced system of equations,

\[u'_1 = \frac{4 - 2\sqrt{2}}{4}v_1u_3^2, \]
\[u'_3 = \frac{4 - 2\sqrt{2}}{4}v_3u_1^2, \]
\[\dot{E'} = 0, \]
\[v'_1 = -\frac{(4 - 2\sqrt{2})u_1v_1^2}{4} + 4u_1 + \frac{4u_1^5}{u_1^4 + u_1^3} - \frac{4u_1^5}{(u_1^4 + u_1^3)^{3/2}} + 2(4 - 2\sqrt{2})\dot{E}u_1u_2, \]
\[v'_3 = -\frac{(4 - 2\sqrt{2})u_3v_3^2}{4} + 4u_3 + \frac{4u_1^5}{u_1^4 + u_1^3} - \frac{4u_1^5}{(u_1^4 + u_1^3)^{3/2}} + 2(4 - 2\sqrt{2})\dot{E}u_1u_3, \]
\[t' = (4 - 2\sqrt{2})^2u_1u_3^2. \]

Scale the value of \(\dot{E} \) by

\[\dot{E} = \frac{\dot{E}}{4 - 2\sqrt{2}}, \]

and define

\[\dot{\Gamma} = \frac{4 - 2\sqrt{2}}{8}(v_1u_3^2 + v_3u_1^2) - 2(u_1^4 + u_3^4) - \frac{2u_1^4u_3^4}{\sqrt{u_1^4 + u_3^4}} + (4 - 2\sqrt{2})^2\dot{E}u_1u_3^2. \]

It is straight-forward to check that the reduced system of equations satisfies

\[u'_i = \frac{\partial \dot{\Gamma}}{\partial v_i}, \quad v'_i = -\frac{\partial \dot{\Gamma}}{\partial u_i}, \quad i = 1, 2, \]
and
\[ \dot{E}' = \frac{\partial \tilde{\Gamma}}{\partial t}, \quad t' = -\frac{\partial \tilde{\Gamma}}{\partial E}. \]
Thus the system of reduced equations is Hamiltonian.

We will simplify the Hamiltonian \( \tilde{\Gamma} \) by a linear symplectic transformation with a multiplier \( \mu \neq 1 \). Define new coordinates \( (Q_1, Q_2, E, P_1, P_2, \tau) \) by
\[
\begin{align*}
\frac{u_1}{2^{1/4}} &= \frac{Q_1}{2^{1/4}}, & v_1 &= \frac{P_1}{2\sqrt{2} - 1}, \\
\frac{u_3}{2^{1/4}} &= \frac{Q_2}{2}, & v_3 &= \frac{P_2}{2\sqrt{2} - 1}, \\
\dot{E} &= \frac{2E}{(4 - 2\sqrt{2})^2}, & t &= 2^{3/4}(\sqrt{2} - 1)^{3/2} \tau.
\end{align*}
\]
This is a linear symplectic change of coordinates with multiplier
\[ \mu = \frac{1}{2^{5/4}\sqrt{2} - 1}. \]
Under this linear symplectic transformation and the accompanying scaling \( s = s/\mu \) of the independent variable \( s \), the Hamiltonian \( \tilde{\Gamma} \) becomes
\[ \Gamma = \frac{1}{16} (P_1^2 Q_2^2 + P_2^2 Q_1^7) - \sqrt{2}(Q_1^3 + Q_2^2) - \frac{\sqrt{2}Q_1^5 Q_2^3}{\sqrt{Q_1^3 + Q_2^2}} - EQ_1^2 Q_2^2. \]
The reduced system of equations is the Hamiltonian system of equations with Hamiltonian \( \Gamma \),
\[
\begin{align*}
\frac{dQ_1}{d\sigma} &= \frac{1}{8} P_1 Q_2, \\
\frac{dQ_2}{d\sigma} &= \frac{1}{8} P_2 Q_1, \\
\frac{dP_1}{d\sigma} &= -\frac{1}{8} P_2^2 Q_1 + 2\sqrt{2} Q_1 + \frac{2\sqrt{2} Q_1 Q_2^2}{\sqrt{Q_1^3 + Q_2^2}} - \frac{2\sqrt{2} Q_1^5 Q_2^3}{(Q_1^3 + Q_2^2)^{3/2}} + 2EQ_1 Q_2^2, \\
\frac{dP_2}{d\sigma} &= -\frac{1}{8} P_1^2 Q_2 + 2\sqrt{2} Q_2 + \frac{2\sqrt{2} Q_2 Q_1^2}{\sqrt{Q_1^3 + Q_2^2}} - \frac{2\sqrt{2} Q_1^5 Q_2^3}{(Q_1^3 + Q_2^2)^{3/2}} + 2EQ_2 Q_1^2,
\end{align*}
\]
along with the auxiliary equations,
\[ \frac{dE}{d\sigma} = 0, \quad \frac{d\tau}{d\sigma} = Q_1^2 Q_2^2. \]

The function \( \Gamma \) is a regularized Hamiltonian for the fully symmetric planar four-body equal mass problem with the bodies located at \((x_1, x_2), (x_2, x_1), (-x_1, -x_2),\) and \((-x_2, -x_1)\) (see [12]). On the level set \( \Gamma = 0 \), the solutions have energy \( E \). One regularized simultaneous binary collision occurs when \( Q_1 = 0 \) and \( Q_2 \neq 0 \), for which \( \Gamma = 0 \) implies \( P_1^2 = 16\sqrt{2} \), and for which the transformation between \( Q_1, Q_2 \) and \( x_1, x_2 \) implies \( x_1 - x_2 = 0 \) and \( x_1 + x_2 \neq 0 \). The other regularized simultaneous binary collision occurs when \( Q_1 \neq 0 \) and \( Q_2 = 0 \), for which \( \Gamma = 0 \) implies \( P_2^2 = 16\sqrt{2} \), and for which the transformation between \( Q_1, Q_2 \) and \( x_1, x_2 \) implies \( x_1 - x_2 \neq 0 \) and \( x_1 + x_2 = 0 \). Total collapse occurs when \( Q_1 = 0 \) and \( Q_2 = 0 \), and is the only singularity in \( \Gamma \) that is not regularized. A solution \( Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma), \sigma \in [0, \sigma_0] \), for \( \sigma_0 > 0 \), of the Hamiltonian system equations
with Hamiltonian $\Gamma$ is nonsingular if it avoids total collapse, i.e., $Q_1^4 + Q_4^4 \neq 0$ for all $\sigma \in [0, \sigma_0]$.

The following result is from [12]. The proof of it is a consequence of four equal mass bodies starting at $(x_1, x_2), (x_2, x_1), (−x_1, −x_2),$ and $(−x_2, −x_1)$ with $x_1 = 1$ and $x_2 = 0$, and with the momenta $(0, \vartheta), (\vartheta, 0), (0, −\vartheta),$ and $(−\vartheta, 0)$ for any $\vartheta > 0$, always having a simultaneous binary collision on the line $x_2 = x_1$ at a time $t_0 > 0$ continuously depending on $\vartheta$, such that the cluster velocity $\dot{x}_1(t_0) + \dot{x}_2(t_0)$ is a continuous function of $\vartheta$.

**Lemma 5.1.** There exists $\vartheta > 0$, $\sigma_0 > 0$, and a nonsingular solution $Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma), \sigma \in [0, \sigma_0]$, of the Hamiltonian system of equations with Hamiltonian $\Gamma$ on the level set $\hat{\Gamma} = 0$ such that

$$Q_1(0) = 1, \quad Q_2(0) = 1, \quad P_1(0) = −\vartheta, \quad P_2(0) = \vartheta,$$

$$E = \frac{\vartheta^2 − 16\sqrt{2} − 8}{8} < 0, \quad \tau(\sigma) = \int_0^\sigma Q_1^2(y)Q_2^2(y) \, dy,$$

and

$$Q_1(\sigma_0) = 0, \quad Q_2(\sigma_0) > 0, \quad P_1(\sigma_0) = −4(2^{1/4}), \quad P_2(\sigma_0) = 0.$$  

This Lemma gives the existence of a solution of a boundary value problem for the Hamiltonian system of equations with Hamiltonian $\Gamma$. It is this solution whose symmetric extension gives a symmetric periodic simultaneous binary collision orbit in the planar pairwise symmetric four-body equal mass problem.

**Theorem 5.2.** Fix $m = 1$. For each $\hat{E} < 0$, there exists a time-reversible periodic regularized simultaneous binary collision orbit

$$\gamma(s) = (u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s))$$

with period $T > 0$, angular momentum $A = 0$, and a symmetry group isomorphic to $D_4$, for the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$ such that distinct regularized simultaneous binary collisions occur at $s = T/8, 3T/8, 5T/8, 7T/8$. This periodic orbit corresponds to a symmetric periodic singular orbit

$$(x_1(t), x_2(t), x_3(t), x_4(t), \omega_1(t), \omega_2(t), \omega_3(t), \omega_4(t))$$

with energy $\hat{E}$ for the planar pairwise symmetric four-body equal mass problem where for all $t$,

$$x_4(t) = x_1(t), \quad x_3(t) = x_2(t), \quad |x_2(t)| \leq x_1(t), \quad \omega_4(t) = \omega_1(t), \quad \omega_3(t) = \omega_2(t),$$

with initial conditions

$$x_1(0) > 0, \quad x_2(0) = 0, \quad \omega_1(0) = 0, \quad \omega_2(0) > 0,$$

and period

$$R = \int_0^{T/2} (u_1^2(s) + u_2^2(s))(u_3^2(s) + u_4^2(s)) \, ds,$$

where for $t \in [0, R]$, the only singularities are two distinct simultaneous binary collisions occurring at $t = R/4$ and $t = 3R/4$. 
Proof. Let $Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma)$, $\sigma \in [0,\sigma_0]$, be the nonsingular solution of the Hamiltonian system of equations with Hamiltonian $\Gamma$ on the level set $\Gamma = 0$, whose existence and properties are given in Lemma 5.1. Using the scaling $\sigma = s/\mu$, set $s_0 = \mu \sigma_0$. By the linear symplectic transformation with multiplier $\mu$, we have

$$u_1(s) = \frac{Q_1(s/\mu)}{2^{1/4}}, \quad u_3(s) = \frac{Q_2(s/\mu)}{2^{1/4}}, \quad v_1(s) = \frac{P_1(s/\mu)}{2\sqrt{\sqrt{2} - 1}}, \quad v_3(s) = \frac{P_2(s/\mu)}{2\sqrt{\sqrt{2} - 1}}.$$  

Since $u_2 = -(\sqrt{2} - 1)u_1$, $u_4 = (\sqrt{2} - 1)u_3$, $v_2 = -(\sqrt{2} - 1)v_1$, and $v_4 = (\sqrt{2} - 1)v_3$, we have

$$u_2(s) = -\frac{(\sqrt{2} - 1)Q_1(s/\mu)}{2^{1/4}}, \quad u_4(s) = \frac{(\sqrt{2} - 1)Q_2(s/\mu)}{2^{1/4}},$$

$$v_2(s) = -\frac{\sqrt{2} - 1P_1(s/\mu)}{2}, \quad v_4(s) = \frac{\sqrt{2} - 1P_2(s/\mu)}{2}.$$  

Set $\gamma(s) = (u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s))$, $s \in [0, s_0]$. From $E = (\vartheta^2 - 16\sqrt{2} - 8)/\theta < 0$ and $\dot{E} = (4 - 2\sqrt{2})\dot{E}$ and $\dot{E} = 2E/(4 - 2\sqrt{2})^2$, we get

$$\dot{E} = \frac{(2 + \sqrt{2})\vartheta^2}{16} - 3 - \frac{5\sqrt{2}}{2} < 0.$$  

With this value of $\dot{E}$, it follows that the value of $\dot{\Gamma}$ at $\gamma(0)$ is 0. Set

$$\zeta_1 = \frac{1}{2^{1/4}} > 0, \quad \zeta_2 = -\frac{(-\sqrt{2} - 1)}{2^{1/4}} < 0,$$

$$\rho_1 = \frac{-\theta}{2\sqrt{\sqrt{2} - 1}} < 0, \quad \rho_2 = \frac{\theta\sqrt{\sqrt{2} - 1}}{2} > 0.$$  

With $Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma)$, $\sigma \in [0,\sigma_0]$, being a nonsingular solution of the Hamiltonian system of equations with Hamiltonian $\Gamma$, we have $Q_1^4(\sigma) + Q_2^4(\sigma) \neq 0$ for all $\sigma \in [0,\sigma_0]$. From this it follows for all $s \in [0, s_0]$ that

$$M_5 = u_1^2(s) - u_2^2(s) + u_3^2(s) - u_4^2(s) = (2 - \sqrt{2})[Q_1^2(s/\mu) + Q_2^2(s/\mu)] \neq 0,$$

and

$$M_8 = 2u_1(s)u_2(s) - 2u_3(s)u_4(s) = -(2 - \sqrt{2})[Q_1^2(s/\mu) + Q_2^2(s/\mu)] \neq 0.$$  

These imply that $M_5^2 + M_6^2 \neq 0$ and $M_7^2 + M_8^2 \neq 0$ for all $s \in [0, s_0]$. Thus the function $\gamma(s)$ is a nonsingular solution of the Hamiltonian system of equations with Hamiltonian $\dot{\Gamma}$ on the level set $\dot{\Gamma} = 0$ that satisfies

$$u_1(0) = \zeta_1, \quad u_2(0) = \zeta_2, \quad u_3(0) = \zeta_1, \quad u_4(0) = -\zeta_2,$$

$$v_1(0) = \rho_1, \quad v_2(0) = \rho_2, \quad v_3(0) = -\rho_1, \quad v_4(0) = \rho_2,$$

$$u_1(s_0) = 0, \quad u_2(s_0) = 0, \quad u_3(s_0) > 0, \quad u_4(s_0) > 0,$$

$$v_1(s_0) < 0, \quad v_2(s_0) > 0, \quad v_3(s_0) = 0, \quad v_4(s_0) = 0.$$  

By Lemma 4.1, the solution $\gamma(s)$ extends to a $T = 8s_0$ periodic solution, call it $\gamma(s)$, with a $D_4$ symmetry group generated by the symmetries $S_F$ and $S_G$, and four
distinct regularized simultaneous binary collisions at \( s = s_0, 3s_0, 5s_0, 7s_0, \) for which
\[
\begin{align*}
    u_1^2(s_0) + u_2^2(s_0) &= 0, \quad u_3^2(s_0) + u_4^2(s_0) \neq 0, \\
    u_1^2(3s_0) + u_2^2(3s_0) &= 0, \quad u_3^2(3s_0) + u_4^2(3s_0) \neq 0, \\
    u_1^2(5s_0) + u_2^2(5s_0) &= 0, \quad u_3^2(5s_0) + u_4^2(5s_0) \neq 0, \\
    u_1^2(7s_0) + u_2^2(7s_0) &= 0, \quad u_3^2(7s_0) + u_4^2(7s_0) = 0.
\end{align*}
\]
Since \( Q_1^k(\sigma) + Q_2^k(\sigma) \neq 0 \) for all \( \sigma \in [0, \sigma_0], \) it follows that
\[
(u_1^2(s) + u_2^2(s))(u_3^2(s) + u_4^2(s)) \neq 0 \quad \text{for} \quad s \in [0, T] \setminus \{s_0, 3s_0, 5s_0, 7s_0\}. 
\]
The regularizing change of time
\[
\frac{dt}{ds} = (u_1^2(s) + u_2^2(s))(u_3^2(s) + u_4^2(s)) 
\]
defines \( t \) an invertible differentiable function of \( s, \) i.e., \( t = \theta(s) \) with \( \theta(0) = 0 \) and \( \theta'(s) = 0 \) when \( s = (2k + 1)s_0 \) for \( k \in \mathbb{Z}. \) The symmetry \( S_F \) satisfies \( S_F^{-1} \gamma(s) = \gamma(s + 2s_0) \) and \( -\gamma(s) = S_F^2 \gamma(s) = \gamma(s + 4s_0). \) The symmetry \( S_G \) satisfies \( S_G \gamma(s) = \gamma(2s_0 - s), \) and so \( \gamma(s) \) has a time-reversing symmetry. The angular momentum of \( \gamma(s) \) at \( s = 0 \) is
\[
    A = \frac{1}{2} \left[ -v_1 u_2 + v_2 u_1 - v_3 u_4 + v_4 u_3 \right] = \rho_2 \zeta_1 - \rho_1 \zeta_2 = 0.
\]
The extended solution \( \gamma(s) \) gives a singular symmetric solution
\[
z(t) = (x_1(t), x_2(t), x_3(t), x_4(t), \omega_1(t), \omega_2(t), \omega_3(t), \omega_4(t)),
\]
of the planar pairwise symmetric four-body equal mass problem. Under the Ansatz, the components of \( z(t) \) satisfy \( x_4(t) = x_1(t), x_3(t) = x_2(t), |x_2(t)| \leq x_1(t), \omega_4(t) = \omega_1(t), \omega_3(t) = \omega_2(t), \) where
\[
\begin{align*}
x_1(t) &= \frac{u_1^2(s) - u_2^2(s) + u_3^2(s) + u_4^2(s)}{2}, \\
x_2(t) &= u_1(s)u_2(s) + u_3(s)u_4(s), \\
\omega_1(t) &= \frac{v_1(s)u_1(s) - v_2(s)u_2(s)}{2(u_1^2(s) + u_2^2(s))} + \frac{v_3(s)u_3(s) - v_4(s)u_4(s)}{2(u_3^2(s) + u_4^2(s))}, \\
\omega_2(t) &= \frac{v_1(s)u_2(s) + v_2(s)u_1(s)}{2(u_1^2(s) + u_2^2(s))} + \frac{v_3(s)u_4(s) + v_4(s)u_3(s)}{2(u_3^2(s) + u_4^2(s))},
\end{align*}
\]
for \( s = \theta^{-1}(t). \) The components of the extended solution \( \gamma(s) \) satisfy \( u_3(0) = u_1(0), \)
\( u_4(0) = -u_2(0), u_1(0)u_2(0) < 0, |u_2(0)| = (\sqrt{2} - 1)|u_1(0)|, v_3(0) = -v_1(0), v_4(0) = v_2(0), \)
\( v_1(0)u_2(0) + v_2(0)u_1(0) = \rho_1 \zeta_2 + \rho_2 \zeta_1 = \frac{\sqrt{2} - 1}{2^{1/4}} \theta > 0, \)
and
\[
v_2(0)u_2(0) - v_1(0)u_1(0) = \rho_2 \zeta_2 - \rho_1 \zeta_1 = \frac{\sqrt{2} - 1}{2^{1/4}} \theta.
\]
From Lemma 2.1, it follows that \( x_1(0) > 0, x_2(0) = 0, \omega_1(0) = 0, \) and \( \omega_2(0) > 0. \) Set \( R = \theta(T/2). \) Since \( \gamma(4s_0) = -\gamma(0), \) it follows that \( x_1(R) = x_1(0), x_2(R) = x_2(0), \)
\( \omega_1(R) = \omega_1(0), \) and \( \omega_2(R) = \omega_2(0). \) Thus the singular symmetric solution \( z(t) \) has period \( R. \) By the construction of the extension of \( \gamma(s) \) given in Lemma 4.1, there holds
\[
\int_{k = s_0}^{(k+1)s_0} (u_1^2(s) + u_2^2(s))(u_3^2(s) + u_4^2(s)) \, ds = \int_0^{s_0} (u_1^2(s) + u_2^2(s))(u_3^2(s) + u_4^2(s)) \, ds
\]
for all \( k = 1, \ldots, 7 \). This implies that \( R/4 = \theta((k+1)s_0) - \theta(ks_0) \) for all \( k = 0, 1, \ldots, 7 \). The first regularized simultaneous binary collision for \( \gamma(s) \) occurs at \( s = s_0 \), and this corresponds to \( t = \theta(s_0) = R/4 \). The next regularized simultaneous binary collision for \( \gamma(s) \) occurs at \( s = 3s_0 \), and this corresponds to

\[
    t = \theta(3s_0) = (\theta(3s_0) - \theta(2s_0)) + (\theta(2s_0) - \theta(s_0)) + \theta(s_0) = \frac{R}{4} + \frac{R}{4} + \frac{R}{4} = \frac{3R}{4}.
\]

Similarly, the regularized simultaneous binary collisions for \( \gamma(s) \) occurring at \( s = 5s_0, 7s_0 \) correspond to \( t = 5R/4, 7R/4 \). Hence, for \( t \in [0, R] \), the periodic solution \( z(t) \) has simultaneous binary collisions as its only singularities, and these occur at \( t = R/4, 3R/4 \).

For a fixed but arbitrary \( \epsilon > 0 \), the value of \( e^{-2\hat{E}} \) is a fixed but arbitrary negative real number. By Lemma 3.1, the scaled extended solution

\[
    \gamma_\epsilon(s) = (\epsilon u_1(\epsilon s), \epsilon u_2(\epsilon s), \epsilon u_3(\epsilon s), \epsilon u_4(\epsilon s), v_1(\epsilon s), v_2(\epsilon s), v_3(\epsilon s), v_4(\epsilon s))
\]

is a periodic solution for the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) on the level set \( \hat{\Gamma} = 0 \), having period \( \epsilon^{-1}T \) and energy \( \epsilon^{-2}\hat{E} < 0 \). The extended solution \( \gamma_\epsilon(s) \) has four distinct regularized simultaneous binary collision occurring at \( s = (2k-1)\epsilon^{-1}s_0 \) for \( k = 1, 2, 3, 4 \). The regularizing change of time now defines \( t \) as an invertible differentiable function of \( s \) by

\[
    t = \int_0^s \epsilon^4(u_1^2(ey) + u_2^2(ey))(u_3^2(ey) + u_4^2(ey)) \, dy = \int_0^{\epsilon s} \epsilon^3(u_1^2(y) + u_2^2(y))(u_3^2(y) + u_4^2(y)) \, dy = \epsilon^3 \theta(\epsilon s).
\]

The extended solution \( \gamma_\epsilon(s) \) defines a singular symmetric solution

\[
    z_\epsilon(t) = (x_1^\epsilon(t), x_2^\epsilon(t), x_3^\epsilon(t), x_4^\epsilon(t), \omega_1^\epsilon(t), \omega_2^\epsilon(t), \omega_3^\epsilon(t), \omega_4^\epsilon(t))
\]

of the planar pairwise symmetric four-body equal mass problem, with \( x_4^\epsilon(t) = x_1^\epsilon(t) \), \( x_3^\epsilon(t) = x_2^\epsilon(t) \), \( |x_2^\epsilon(t)| \leq x_1^\epsilon(t) \), \( \omega_1^\epsilon(t) = \omega_1^\epsilon(t) \), \( \omega_3^\epsilon(t) = \omega_3^\epsilon(t) \), where

\[
    x_1^\epsilon(t) = \frac{c^2}{2} \left[ u_1^2(\epsilon s) - u_2^2(\epsilon s) + u_3^2(\epsilon s) + u_4^2(\epsilon s) \right],
\]

\[
    x_2^\epsilon(t) = c^2[ u_1(\epsilon s) u_2(\epsilon s) + u_3(\epsilon s) u_4(\epsilon s)],
\]

\[
    \omega_1^\epsilon(t) = \frac{v_1(\epsilon s) u_1(\epsilon s) - v_2(\epsilon s) u_2(\epsilon s)}{2c(u_1^2(\epsilon s) + u_2^2(\epsilon s))} + \frac{v_3(\epsilon s) u_3(\epsilon s) - v_4(\epsilon s) u_4(\epsilon s)}{2c(u_3^2(\epsilon s) + u_4^2(\epsilon s))},
\]

\[
    \omega_3^\epsilon(t) = \frac{v_1(\epsilon s) u_2(\epsilon s) + v_2(\epsilon s) u_1(\epsilon s)}{2c(u_1^2(\epsilon s) + u_2^2(\epsilon s))} + \frac{v_3(\epsilon s) u_4(\epsilon s) + v_4(\epsilon s) u_3(\epsilon s)}{2c(u_3^2(\epsilon s) + u_4^2(\epsilon s))},
\]

for \( \epsilon s = \theta^{-1}(\theta^{-3}t) \). The period of \( z_\epsilon(t) \) is \( \epsilon^3 R = \epsilon^3 \theta(T/2) \). The argument for the case of \( \epsilon = 1 \) now applies to \( \epsilon \neq 1 \) with \( \gamma(s) \) replaced by \( \gamma_\epsilon(s) \), \( s_0 \) replaced by \( \epsilon s_0 \), \( z(t) \) replaced by \( z_\epsilon(t) \), \( T \) replaced by \( \epsilon^{-1}T \), \( \hat{E} \) replaced by \( \epsilon^{-2}\hat{E} \), and \( R \) replaced by \( \epsilon^3 R \). This gives the existence of a symmetric periodic simultaneous binary collision orbit \( z_\epsilon(t) \) of the planar pairwise symmetric four-body equal mass problem for every negative value of energy. \( \square \)
6. Numerical Estimates in the Equal Mass Case

In the equal mass case, there is by Theorem 5.2 a time-reversible periodic orbit $\gamma(s)$ for the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$ with period $T$. The components $u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s)$ of $\gamma(s)$ satisfy $u_1(s) = -(\sqrt{2} - 1)u_1(s), u_2(s) = (\sqrt{2} - 1)u_3(s), v_2(s) = -(\sqrt{2} - 1)v_1(s)$, and $v_4(s) = (\sqrt{2} - 1)v_3(s)$. The $D_4$ symmetry group of $\gamma(s)$ is generated by $S_F\gamma(s) = \gamma(s + T/4)$ and $S_G\gamma(s) = \gamma(T/4 - s)$. Under the linear symplectic transformation with multiplier $\mu$, this gives a periodic orbit $Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma)$ of the Hamiltonian system of equations with Hamiltonian $\Gamma$ on the level set $\Gamma = 0$, which by Lemma 5.1 satisfies $Q_1(0) = 1, Q_2(0) = 1, P_1(0) = -\vartheta, P_2(0) = \vartheta, Q_1(\sigma_0) = 0, Q_2(\sigma_0) > 0, P_1(\sigma_0) = -4(2^{1/4}),$ and $P_2(\sigma_0) = 0$ for some $\sigma_0 > 0$ and $\vartheta > 0$. In [2], we numerically estimated

$$\sigma_0 = 1.62047369909693, \quad \vartheta = 2.57486992651942.$$ 

The period of this periodic orbit is $8\sigma_0 \approx 12.96378959$ and its energy is $E \approx -2.999682732$. From the linear symplectic transformation with multiplier $\mu$ and the relations among the components of $\gamma(s)$, we have for the values of components of $\gamma(0)$ the exact

$$u_1(0) = u_3(0) = 2^{-1/4}, \quad -u_2(0) = u_4(0) = (\sqrt{2} - 1)2^{-1/4},$$

and the estimates

$$v_1(0) = -v_3(0) = -\frac{\vartheta}{2\sqrt{2} - 1} \approx -2.000382939,$$

$$v_2(0) = v_4(0) = \frac{\vartheta\sqrt{2} - 1}{2} \approx 0.8285857433.$$ 

Since $\sigma = s/\mu$, the period of $\gamma(s)$ is $T \approx 8.469003682$. Since $\hat{E} = E/(2 - \sqrt{2})$, the value of the energy for $\gamma(s)$ is $\hat{E} \approx -5.120778733$.

Figure 2 illustrates the graphs of the components of the scaled periodic orbit $\gamma_s(s)$ for $\epsilon > 2$. Readily observable in these graphs are the symmetries $S_F\gamma_s(s) = \gamma_s(s + T/4)$ and $S_G\gamma_s(s) = \gamma_s(T/4 - s)$. Figure 1 illustrates the curves in the physical plane that the four equal masses follow in the symmetric simultaneously binary collision orbit $z(t) = (x_1(t), x_2(t), x_3(t), x_4(t), \omega(t), \omega_2(t), \omega_3(t), \omega_4(t))$ of the planar pairwise symmetric four-body problem, corresponding to $\gamma_s(s)$ for an $\epsilon > 0$. The initial conditions for $z(t)$ are

$$x_1(0) = x_4(0) = 1, \quad x_2(0) = x_3(0) = 0,$$

$$\omega_1(0) = \omega_4(0) = 0, \quad \omega_2(0) = \omega_3(0) \approx 1.287434964.$$ 

The value of $\epsilon = 1/\sqrt{2 - \sqrt{2}}$ here for the scaling is determined by the relation $x_1(t) = \epsilon^2(u_1^2(\epsilon s) - u_2^2(\epsilon s))$ coming from the regularization transformation applied to the scaled periodic solution $\gamma_s(s)$, together with the initial condition $x_1(0) = 1$, where $t = 0$ corresponds to $s = 0$. The value of the Hamiltonian $H$ along $z(t)$ is $\epsilon^{-2}\hat{E} \approx -2.999682732$.

Some analytic evidence for the linear stability of $\gamma(s)$ is provided by an investigation of the linear stability of the periodic orbit $Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma)$ of the Hamiltonian system of equations with Hamiltonian $\Gamma$, corresponding to $\gamma_1(s) = \gamma(s)$. In [2], we showed that the periodic orbit $Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma)$ is linearly stable for the Hamiltonian system of equations with Hamiltonian $\Gamma$. We
did this by applying the symmetry reduction technique of Roberts [16] to compute its two nontrivial characteristic multipliers to be a complex conjugate pair lying on the unit circle close to but not equal to $-1$. (Of course, the trivial characteristic multiplier for this periodic orbit is 1 whose algebraic multiplicity is two). This means that $\gamma(s)$ is linearly stable among all solutions of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$ whose components satisfy $u_2 = - (\sqrt{2} - 1) u_1$, $u_4 = (\sqrt{2} - 1) u_3$, $v_2 = -(\sqrt{2} - 1) v_1$, and $v_4 = (\sqrt{2} - 1) v_3$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The $u_1$, $u_2$, $u_3$, $u_4$, $v_1$, $v_2$, $v_3$, $v_4$ coordinates of the symmetric periodic simultaneous binary collision orbit $\gamma_\epsilon(s)$ for $m = 1$ and an $\epsilon > 2$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{A long-term numerical integration of the symmetric periodic simultaneous binary collision orbit $\gamma_\epsilon(s)$ for $m = 1$ and an $\epsilon > 2$.}
\end{figure}

Numerical evidence for the linear stability of the periodic orbit $\gamma(s)$ is provided by a long-term numerical integration of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$. We used the Runge-Kutta-Fehlberg algorithm, with a time-step of $0.001$, for the numerical integration. By Theorem 3.3, the linear stability of $\gamma_\epsilon(s)$ is the same for all $\epsilon > 0$. Figure 3 illustrates the components of $\gamma_\epsilon(s)$, for the same $\epsilon > 2$ as in Figure 2, over the time interval $[0, 100]$, where $100$ is approximately $26T_\epsilon$. We also performed a longer term numerical integration of $\gamma_\epsilon(s)$ for the value of $\epsilon$ that satisfies $T_\epsilon = 2\pi$ (see next Section). From these numerical integrations it appears that $\gamma_\epsilon(s)$ is linearly stable for some value of $\epsilon$, and hence for all $\epsilon > 0$. Further numerical evidence of the linear stability of the periodic orbit $\gamma(s)$ is provided by a long-term numerical integration of the orbit of four equal masses starting at the initial conditions determined by $z(0)$. We did this, not for the Hamiltonian system of equations with Hamiltonian $H$, but for the three-dimensional four-body equal mass problem. We carried out the numerical integration using Euler’s method combined with a subroutine that gives an elastic bounce at binary collisions. None of the symmetry properties of the singular periodic simultaneous binary collision orbit $z(t)$, nor the pairwise symmetry of the four masses, were coded into the algorithm. Because the motion of the four masses is planar, the numerical integration produces an orbit of the planar four-body equal mass problem. Any instability in this planar orbit should be amplified by Euler’s method and become apparent in a timely manner. The numerical integration from $t = 0$ to $t = 100000$, with a time-step of $0.001$, shows the orbit remains bounded, with the four masses tracing out planar curves close to those illustrated in Figure 1, while retaining its regular pattern of alternating simultaneous binary collisions.

7. **Numerical Estimates in the Unequal Mass Case**

We numerically continue to $0 < m < 1$ the time-reversible periodic regularized simultaneous binary collision orbit for the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$, whose analytic existence is given by Theorem 5.2 for $m = 1$. We also investigate through long-term integrations the values of $0 < m < 1$ for which the continued time-reversible periodic regularized simultaneous binary collision orbits are linearly stable. For $m = 1$, we assume by Lemma 3.1 that the time-reversible periodic regularized simultaneous binary collision orbit $\gamma(s;1)$ with the $D_4$ symmetry has period $T = 2\pi$ and energy $\hat{E} \approx -2.818584789$. Recall that the $D_4$ symmetry group of $\gamma(s;1)$ is generated by $S_F\gamma(s;1) = \gamma(s+\pi/2;1)$ and $S_G\gamma(s;1) = \gamma(\pi/2-s;1)$. We assume without loss of generality by Lemma 3.1 and Lemma 4.1, that the continued time-reversible periodic regularized simultaneous binary collision orbit $\gamma(s;m)$ for $0 < m < 1$, has period $T = 2\pi$, energy $\hat{E}(m)$, and a $D_4$ symmetry group generated by $S_F\gamma(s;m) = \gamma(s+\pi/2;m)$ and $S_G\gamma(s;m) = \gamma(\pi/2-s;m)$. We assume that $\hat{E}(m)$ is a continuous function of $0 < m \leq 1$, with $\hat{E}(1) = \hat{E}$. We shift the regularized time variable $s$ to $s + \pi/4$, so that for $\gamma(s;m) = (u_1(s;m), u_2(s;m), u_3(s;m), u_4(s;m), v_1(s;m), v_2(s;m), v_3(s;m), v_4(s;m))$, the value $s = 0$ now corresponds to the first simultaneous binary collision, i.e., $u_1(0;m) = 0$, $u_2(0;m) = 0$, $u_3(0;m) \neq 0$, $u_4(0;m) \neq 0$, $v_1(0;m) \neq 0$, $v_2(0;m) \neq 0$, $v_3(0;m) = 0$, and $v_4(0;m) = 0$. We approximate the continued time-reversible
periodic orbits $\gamma(s; m)$, $0 < m \leq 1$, by the trigonometric polynomials,

$$
\begin{align*}
  u_1(s; m) &= \sum_{i=1}^{n} a_i \sin((2i-1)s), & u_2(s; m) &= -\sum_{i=1}^{n} b_i \sin((2i-1)s), \\
  u_3(s; m) &= u_1(s - \pi/2; m), & u_4(s; m) &= u_2(s + \pi/2; m), \\
  v_1(s; m) &= \sum_{i=1}^{n} c_i \cos((2i-1)s), & v_2(s; m) &= -\sum_{i=1}^{n} d_i \cos((2i-1)s), \\
  v_3(s; m) &= v_1(s - \pi/2; m), & v_4(s; m) &= v_2(s + \pi/2; m), 
\end{align*}
$$

for a positive integer $n$ and constants $a_i$, $b_i$, $c_i$, $d_i$, $i = 1, \ldots, n$, that are assumed to be continuous functions of $m$. The presence of odd positive integer frequencies $2i-1$ in these trigonometric polynomials is to ensure that the period functions defined by them have the $D_4$ symmetry group generated by $S_F$ and $S_G$. So in particular, the periodic orbits in the continuation are time-reversible for all $0 < m < 1$. An numerical estimate of the periodic solution $\gamma(s; m)$ is found through the variational approach of minimizing the functional

$$
L = \int_0^{2\pi} \| \gamma'(s; m) - J \nabla \Gamma(\gamma(s; m)) \| \, ds
$$

over the space $\mathbb{R}^{4n}$ of coefficients $a_i$, $b_i$, $c_i$, $d_i$, $i = 1, \ldots, n$, for an appropriate choice of $n$. The use of trigonometric polynomials for numerically approximating periodic solutions is a classic approach (see, for example, Simó [20]).

The numerical algorithm for finding a trigonometric polynomial that approximates the periodic solution $\gamma(s; m)$, $0 < m < 1$, proceeds in two steps. First, we consider a guess for the set of values for $a_i$, $b_i$, $c_i$, $d_i$, $i = 1, \ldots, n$, as well as a guess for $\hat{E}(m)$. Starting with a reasonably low number of terms, $n = 5$, we let a numerical minimization algorithm (in this case, MATLAB’s `fminunc`) find the minimum of $L$ near the starting guess. Then we add an additional non-zero term to each of the trigonometric polynomials, and the minimizing solution from the previous iteration is used as a starting guess for the next iteration. This process continues until we reach $n = 10$. Second, since the Hamiltonian equations with Hamiltonian $\hat{\Gamma}$ requires a specified value of $\hat{E}(m)$ to compute $L$, we need to make sure that we are getting a good estimate of $\hat{E}(m)$. We evaluate $\hat{\Gamma}$ at the point $\gamma(\pi/4; m)$, away from simultaneous binary collisions. If this value is not sufficiently close to 0 (within about $5 \times 10^{-10}$ of 0), we adjust $\hat{E}(m)$ using the bisection method in a small interval about the initial guess of $\hat{E}(m)$ until $\hat{\Gamma}(\gamma(\pi/4; m))$ is sufficiently close to 0, re-minimizing the trigonometric polynomial approximation of $\gamma(s; m)$ for each new choice of $\hat{E}(m)$.

This numerical method only works well if we have a good initial guesses for $a_i$, $b_i$, $c_i$, $d_i$, $i = 1, \ldots, 5$, and $\hat{E}(m)$ for some value of $m$. This we have when $m = 1$. We use our estimate of $\gamma(s; 1)$ and $\hat{E}(1)$ to provide the initial guesses for $a_i$, $b_i$, $c_i$, $d_i$, $i = 1, \ldots, 5$, and $\hat{E}(m)$ for $m = 0.99$. The numerical algorithm produces a trigonometric polynomial approximation of $\gamma(s; 0.99)$ and an estimate of $\hat{E}(0.99)$, which then provide the initial guesses for $a_i$, $b_i$, $c_i$, $d_i$, $i = 1, \ldots, 5$, and $\hat{E}(m)$ for $m = 0.98$. We continue decreasing $m$ by 0.01 and using the numerical algorithm until we reach $m = 0.01$. In Figure 4, we plot the graphs of the numerical estimates of $u_3(0; m)$, $u_4(0; m)$, $v_1(0; m)$, and $v_2(0; m)$. Notice that $v_1^2(0; m) + v_2^2(0; m) \approx$
Figure 4. The estimated values of $u_3(0; m)$ (blue), $u_4(0; m)$ (green), $v_1(0; m)$ (red), and $v_2(0; m)$ (light blue) for the nonzero components of $\gamma(0; m)$ over $0 < m \leq 1$.

Figure 5. The energy $\hat{E}(m)$ of the periodic orbits $\gamma(s; m)$ for $0 < m \leq 1$.

$32m^2/(m + 1)$, as is expected from the regularization of the simultaneous binary collisions. In Figure 5, we plot the graph of the numerical estimate of $\hat{E}(m)$. In Figure 6, we graph of value of $\hat{\Gamma}(\gamma(\pi/4; m))$ over $0 < m \leq 1$. 
We use the optimized trigonometric approximations of $\gamma(s; m)$, $0 < m < 1$, to estimate $\gamma(-\pi/4; m)$, which are the conditions corresponding to $s = 0$ before the shift of $s$. For each $0 < m < 1$, the conditions $\gamma(-\pi/4; m)$ satisfy Lemma 2.1, and so they correspond to the initial conditions of interest as given in the Introduction.

We also use the optimized trigonometric approximation of $\gamma(s; m)$ to estimate $\gamma(0; m)$. Using the estimated value of $\hat{E}(m)$, for values of $m$ between 0.01 and 1 separated by 0.01 increments, we integrated the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ using the Runge-Kutta-Fehlberg algorithm, with a time-step of 0.001, starting at $\gamma(0; m)$. By the time $s$ reached $40\pi$, i.e., twenty periods, the periodic orbits were easily recognized as unstable for $m \leq 0.53$. On the other hand, with $s$ reaching values much larger than $40\pi$, the periodic orbits appear to be linearly stable for $m \geq 0.54$.

Following these observations, we refine the numerical continuation to approximate the periodic orbits $\gamma(s; m)$ and the energy $\hat{E}(m)$ for $m = 0.531, 0.532, \ldots, 0.539$ using the numerical algorithm described previously. Then the Hamiltonian systems of equations with Hamiltonian $\hat{\Gamma}$ were integrated by the Runge-Kutta-Fehlberg algorithm, with a time-step of 0.001, for each of the initial conditions $\gamma(0; m)$ with $m = 0.531, 0.532, \ldots, 0.539$, until the norm of the orbit, $\|\gamma(s; m)\|$, exceeded 100 for some $s > 0$. By the time $s$ reached $10000\pi$, i.e., 5000 periods, the only orbits whose norms were still below 100 were those with $0.535 \leq m \leq 0.539$. The norm of the orbit for $m = 0.535$ exceeded 100 after $s$ reached $15688\pi$, i.e., 7844 periods. The integrations for the orbits with masses $0.536 \leq m \leq 0.539$ continued until the value of $s$ was approximately $201864\pi$, i.e., 100,932 periods, at which point the integrations were halted.

The graphs of the numerical integration of the components of $\gamma(s; 0.536)$ are illustrated in Figure 7, where $s = 0$ on the horizontal axis corresponds to $s \approx
Figure 7. Graphs of the components of $\gamma(s; m)$ when $m = 0.536$, where $s = 0$ on the horizontal axis corresponds to $s \approx 201864\pi$.

Figure 8. The motion of the four masses $\pm(x_1(t), x_2(t))$ and $\pm(x_3(t), x_4(t))$ when $m = 0.536$ for large $t$, corresponding to the numerical integration of $\gamma(s; 0.536)$ with $s \approx 201864\pi$.

201864$\pi$. These graphs clearly show that the numerical integration of $\gamma(s; 0.536)$ has ceased to have a period of $2\pi$. Illustrated in Figure 8 is the motion of the four pairwise symmetric masses corresponding to the numerical integration of the components of $\gamma(s; 0.536)$ as shown in Figure 7. The rotational drift from the initial conditions and the irregularity of this motion indicates that there is instability near the periodic orbit $\gamma(s; 0.536)$.

The graphs of the numerical integration of the components of $\gamma(s; 0.537)$ are illustrated in Figure 9, where $s = 0$ on the horizontal axis corresponds to $s \approx 201864\pi$. From these graphs it appears that the numerical integration has retained the periodicity of $2\pi$. Illustrated in Figure 10 is the motion of the four pairwise symmetric masses corresponding to the numerical integration of the components of $\gamma(s; 0.537)$ as shown in Figure 9. The rotational drift from the initial conditions indicates that there is an instability near the periodic orbit $\gamma(s; 0.537)$. The regularity of the motion in Figure 10 indicates that $m = 0.537$ is close to the value of $m$ at which the $\gamma(s; m)$ is spectrally stable but not linearly stable.

The graphs of the numerical integrations of the components of $\gamma(s; 0.538)$ are illustrated in Figure 11, where $s = 0$ on the horizontal axis corresponds to $s \approx$
Figure 9. Graphs of the components of $\gamma(s; m)$ when $m = 0.537$, where $s = 0$ on the horizontal axis corresponds to $s \approx 201864\pi$. From these graphs it appears that the numerical integration of $\gamma(s; 0.538)$ has retained the periodicity of $2\pi$. Illustrated in Figure 12 is the motion of the four pairwise symmetric masses corresponding to the numerical integration of the components of $\gamma(s; 0.538)$ as shown in Figure 11. The lack of rotation and the presence of a thin annulus-like region traced out by the motion of four pairwise symmetric masses indicates that $\gamma(s; 0.538)$ is linearly stable.

The graphs of the numerical integrations of the components of $\gamma(s; 0.539)$ are illustrated in Figure 13, where $s = 0$ on the horizontal axis corresponds to $s \approx 201864\pi$. From these graphs it appears that the numerical integration of $\gamma(s; 0.539)$ has retained the periodicity of $2\pi$. Illustrated in Figure 1 is the motion of the four pairwise symmetric masses corresponding to the numerical integrations of the components of $\gamma(s; 0.539)$ as shown in Figure 13. The lack of rotation and the regularity of the motion indicates that $\gamma(s; 0.539)$ is linearly stable.

This numerical investigation shows that $\gamma(s; 1)$ continues as a one-parameter family $\gamma(s; m)$, $0 < m < 1$, of symmetric periodic simultaneous binary collision orbits, all with period $2\pi$ and varying energy $\hat{E}(m)$. It further shows that $\gamma(s; m)$
Figure 11. Graphs of the components of $\gamma(s; m)$ when $m = 0.538$, where $s = 0$ on the horizontal axis corresponds to $s \approx 201864\pi$.

Figure 12. The motion of the four masses $\pm(x_1(t), x_2(t))$ and $\pm(x_3(t), x_4(t))$ when $m = 0.538$ for large $t$, corresponding to the numerical integration of $\gamma(s; 0.538)$ with $s \approx 201864\pi$.

Figure 13. Graphs of the components of $\gamma(s; m)$ when $m = 0.539$, where $s = 0$ on the horizontal axis corresponds to $s \approx 201864\pi$.

is linearly stable when $0.538 \leq m \leq 1$ and is unstable when $0 \leq m \leq 0.537$, with spectral stability occurring without linear stability for a value of $m$ close to 0.537.
References

[1] Aarseth, S.J., and Zare, K., A Regularization of the Three-Body Problem, Celest. Mech. 10 (1974), 185-205.

[2] Bakker, L.F., Ouyang, T., Yan, D., Simmons, S.C., and Roberts, G.E., Linear Stability for Some Symmetric Periodic Simultaneous Binary Collision Orbits in the Four-Body Problem, submitted to Celest. Mech. Dynam. Astron., preprint posted on arxiv, 2009.

[3] Chenciner, A., and Montgomery, R., A remarkable periodic solution of the three-body problem in the case of equal masses, Ann. of Math. 152 (2000), 881-901.

[4] Contopoulos, G., Order and Chaos in Dynamical Astronomy, Springer-Verlag, New York, 2002.

[5] Hénon, M., Stability of interplay orbits, Cel. Mech., 15 (1977), 243-261.

[6] Hietarinta, J., and Mikkola, S., Chaos in the one-dimensional gravitational three-body problem, Chaos 3 (2) (1993), 183-203.

[7] Hu, X., and Sun, S., Morse Index and Stability of Elliptic Lagrangian Solutions in the Planar 3-body Problem, preprint, 2008.

[8] Hu, X., and Sun, S., Index Stability of Symmetric periodic Orbits in Hamiltonian Systems with Applications to Figure-Eight Orbit, Commun. Math. Phys., 290 (2009), 737-777.

[9] Meyer, K.R., and Schmidt, D.S., Elliptic relative equilibria in the N-body problem, J. Diff. Eqn. 214 (2005), 256-298.

[10] Moelckel, R., A Topological Existence Proof for the Schubart Orbits in the Collinear Three-Body Problem, Dis. Con. Dyn. Syst. Series B, Vol. 10, No. 2 & 3 (2008), 609-620.

[11] Moore, C., Braids in classical dynamics, Phys. Rev. Lett. 70 (24) (1993), 3675-3679.

[12] Ouyang, T., Simmons, S.C., and Yan, D., Periodic Solutions with Singularities in Two Dimensions in the n-body Problem, to appear in Rocky Mountain Journal.

[13] Ouyang, T., Xie, Z., Regularization of Simultaneous Binary Collisions and Solutions with Singularities in the Collinear Four-Body Problem, Dis. Con. Dyn. Sys. Vol. 24, No. 3 (2009), 909-932.

[14] Ouyang, T. and Yan D., Periodic Solutions with Alternating Singularities in the Collinear Four-body Problem, submitted to Celest. Mech. Dynam. Astron., preprint posted on arxiv, 2008.

[15] Roberts, G.E., Linear Stability of the Elliptic Lagrangian Triangle Solutions in the Three-body Problem, J. Diff. Eqn. 182 (2002), 191-218.

[16] Roberts, G.E., Linear Stability analysis of the figure-eight orbit in the three-body problem, Ergod. Th. & Dynam. Sys. 27 (2007), 1947-1963.

[17] Saito, M.M. and Tanikawa, K., The rectilinear three-body problem using symbol sequence I: Role of triple collisions, Celest. Mech. Dynam. Astron. 98 (2007), 95-120.

[18] Saito, M.M. and Tanikawa, K., The rectilinear three-body problem using symbol sequence II: Role of periodic orbits, posted on arxiv.

[19] Sekiguchi, M, and Tanikawa, K., On the Symmetric Collinear Four-Body Problem, Publ. Astron. Soc. Japan 56 (2004), 235-251.

[20] Simó, C., New families of solutions in the N-body problem, Progress in Mathematics Vol. 201, Birkhäuser (2001), 101-115.

[21] Schubart, J., Numerische Aufsuchung periodischer Lösungen im Dreikörperproblem, Astronomische Nachrichten, 283 (1956), 17-22.

[22] Sweatman, W.L., Symmetrical one-dimensional four-body problem, Celest. Mech. Dynam. Astron. 82 (2002), 179-201.

[23] Sweatman, W.L., A Family of Symmetrical Schubart-Like Interplay Orbits and their Stability in the One-Dimensional Four-Body Problem, Celest. Mech. Dynam. Astron. 94(1) (2006), 37-65.

[24] Venturelli, A., A Variational Proof of the Existence of Von Schubart’s Orbit, Dis. Con. Dyn. Syst. Series B, Vol. 10, No. 2&3 (2008), 699-717.

Department of Mathematics, Brigham Young University, Provo, UT USA 84602
E-mail address, Lennard F. Bakker: bakker@math.byu.edu
E-mail address, Tiancheng Ouyang: ouyang@math.byu.edu
E-mail address, Skyler Simmons: xinkaisen@yahoo.com
PERIODIC SBC ORBITS IN THE PLANAR PAIRWISE SYMMETRIC PROBLEM

Chern Institute of Mathematics, Nankai University, Tianjin 300071, P.R.China
E-mail address, Duokui Yan: duokuiyan@gmail.com
EXISTENCE AND STABILITY OF SYMMETRIC PERIODIC SIMULTANEOUS BINARY COLLISION ORBITS IN THE PLANAR PAIRWISE SYMMETRIC FOUR-BODY PROBLEM

LENNARD F. BAKKER, TIANCHENG OUYANG, DUOKUI YAN, AND SKYLER SIMMONS

Abstract. We extend our previous analytic existence of a symmetric periodic simultaneous binary collision orbit in a regularized fully symmetric equal mass four-body problem to the analytic existence of a symmetric periodic simultaneous binary collision orbit in a regularized planar pairwise symmetric equal mass four-body problem. We then use a continuation method to numerically find symmetric periodic simultaneous binary collision orbits in a regularized planar pairwise symmetric 1, m, 1, m four-body problem for m between 0 and 1. Numerical estimates of the characteristic multipliers show that these periodic orbits are linearly stable when 0.54 ≤ m ≤ 1, and are linearly unstable when 0 < m ≤ 0.53.

1. Introduction

In the N-body problem, linearly stable periodic orbits may trap around themselves bounded, non-chaotic motion of the N masses [5]. Some of the known examples of linearly stable periodic orbits in the three-body problem are the elliptic Lagrangian triangular periodic orbits for certain values of eccentricity and the three masses [12],[18], and the Montgomery-Chenciner figure-eight periodic orbit for equal masses [4],[14],[19],[8],[9]. Other examples of linearly stable periodic orbits in the three or four-body problem involve binary collisions (BC) and/or simultaneous binary collisions (SBCs). The regularization of these kinds of singularities is achieved by a generalized Levi-Civita type transformation and an appropriate scaling of time, as adapted from Aarseth and Zare [1] to the particular problem (also see [10],[16]).

Schubart [26] numerically discovered a singular symmetric periodic orbit in the collinear three-body equal mass problem. In this orbit, the inner mass alternates between binary collisions with the two outer masses. Hénon [6] extended Schubart’s numerical investigations to the case of unequal masses. Only recently did Venturelli [29] and Moeckel [13] prove the analytic existence of the Schubart orbit when the outer masses are equal and the inner mass is arbitrary. The linear stability of the Schubart orbit was determined numerically by Hietarinta and Mikkola [7] revealing that linear stability occurs only for certain choices of the three masses. Numerically, non-Schubart-like linearly stable periodic orbits in the collinear three-body problem were found by Saito and Tanikawa for certain choices of the masses [20],[21],[22].

Sweatman [27], [28] and Sekiguchi and Tanikawa [24] numerically found a symmetric Schubart-like orbit in the symmetric collinear four-body problem with masses 1, m, m, and 1. This Schubart-like periodic orbit alternates between SBCs of the
two outer pairs of masses and binary collisions of the inner two masses. Ouyang and Yan [17] proved analytically the existence and symmetry of this orbit. In the regularized setting, this periodic orbit has a symmetry group isomorphic to $D_2$, of which both of the generators are time-reversing symmetries. (Here the dihedral group $D_n$ is the group of symmetries of the regular $n$-gon.) Sweatman [28] numerically showed that the Schubart-like orbit is linearly stable when $0 < m < 2.83$ or $m > 35.4$, and is otherwise unstable. This linear stability was confirmed [3] using Robert’s symmetry reduction technique [19].

Ouyang, Yan, and Simmons [15] numerically found and analytically proved the existence and symmetries of a singular symmetric periodic orbit in the fully symmetric planar four-body problem with equal masses. (In the fully symmetric planar four-body equal mass problem, the position of one mass determines the positions of the other three masses.) In this orbit, the four masses alternate between different SBC’s. In the regularized setting, this periodic orbit has a symmetry group isomorphic to $D_4$, of which one of the two generators is a time-reversing symmetry, while the other generator is a time-preserving symmetry. By using Roberts’ symmetry reduction method, we showed [3] that this symmetric periodic simultaneous binary collision orbit is linearly stable.

In this paper we extend the analytic existence of a symmetric periodic SBC orbit in the fully symmetric planar four-body equal mass problem [15] to the analytic existence of a symmetric periodic SBC orbit in the planar pairwise symmetric four-body problem, or PPS4BP for short. The positions of the four pairwise symmetric bodies in the plane are $(x_1, x_2), (x_3, x_4), (-x_1, -x_2), and (-x_3, -x_4)$, where the corresponding masses are $1, m, 1, m$ with $0 < m \leq 1$. With $t$ as the time variable and $\dot{} = d/dt$, the momenta for the four masses are $(\omega_1, \omega_2) = 2(\dot{x_1}, \dot{x_2}), (\omega_3, \omega_4) = 2m(\dot{x_3}, \dot{x_4}), -(\omega_1, \omega_2), and -(\omega_3, \omega_4)$. The Hamiltonian for the PPS4BP is $H = K - U$, where

$$K = \frac{1}{4}[\omega_1^2 + \omega_2^2] + \frac{1}{4m}[\omega_3^2 + \omega_4^2],$$

and

$$U = \frac{1}{2\sqrt{x_1^2 + x_2^2}} + \frac{2m}{\sqrt{(x_3 - x_1)^2 + (x_4 - x_2)^2}} + \frac{2m}{\sqrt{(x_1 + x_3)^2 + (x_2 + x_4)^2}} + \frac{m^2}{2\sqrt{x_3^2 + x_4^2}}.$$

The angular momentum for the PPS4BP is

$$A = x_1\omega_2 - x_2\omega_1 + x_3\omega_4 - x_4\omega_3.$$

The center of mass is fixed at the origin, and the linear momentum is zero. With

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

for $I$ the $4 \times 4$ identity matrix, the vector field for the PPS4BP is $J\nabla H$, i.e., the Hamiltonian system of equations with Hamiltonian $H$ are $\dot{x}_i = \partial H/\partial \omega_i$, $\dot{\omega}_i = -\partial H/\partial x_i$, $i = 1, 2, 3, 4$. The PPS4BP presented here is the Caledonian symmetric four-body problem [23] with non-collinear initial positions.

The initial conditions for the orbits of interest has the first body of mass 1 located on the positive horizontal axis with its momentum perpendicular to the horizontal
Figure 1. The symmetric periodic SBC orbit in the PPS4BP for $m = 1$ (left) and $m = 0.539$ (right). The two red curves are those traced out by $\pm(x_1(t), x_2(t))$, and the two blue curves are those traced out by $\pm(x_3(t), x_4(t))$.

axis, and the first body of mass $m$ located on the positive vertical axis with its momentum perpendicular to the vertical axis. Specifically, at $t = 0$ we have

\[
x_1 > 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 > 0, \quad x_2^2 + x_3^2 \neq 0, \quad x_1 > x_3, \quad \omega_1 = 0, \quad \omega_2 > 0, \quad \omega_3 > 0, \quad \omega_4 = 0, \quad \omega_2 \leq \omega_3,
\]

at which $H$ is defined. The first objective is to find, for $0 < m \leq 1$, values of $x_1, x_4, \omega_2, \omega_3$ at $t = 0$ such that (i) $x_3 - x_1 = 0$ and $x_4 - x_2 = 0$ with $x_1^2 + x_2^2 \neq 0$ at some $t = t_0 > 0$, (ii) $x_1 + x_3 = 0$ and $x_2 + x_4 = 0$ with $x_1^2 + x_2^2 \neq 0$ at some $t = t_1 > t_0$, (iii) the orbit extends to a symmetric periodic orbit, and (iv) the periodic orbit avoids all the other kinds of collisions. Such an orbit experiences a SBC in the first and third quadrant at $t = t_0$, and then another SBC in the second and fourth quadrants at $t = t_1$, before returning to its initial conditions at some $t = t_2 > t_1$. The presence of collisions along the orbit necessarily imposes zero angular momentum on the orbit, thus requiring that $x_1 \omega_2 - x_4 \omega_3 = 0$ at $t = 0$.

Examples of these symmetric periodic SBC orbits in the PPS4BP with masses 1, $m$, 1, $m$ are illustrated in Figure 1 for $m = 1$ and $m = 0.539$. The second objective is to numerically investigate the linear stability of the symmetric periodic SBC orbits as $m$ varies over interval $(0, 1]$.

The regularization of the SBCs, as described by (i) and (ii) above, in the Hamiltonian system of equations with Hamiltonian $H$ plays a key role in achieving the two objectives. Section 2 details this regularization which consists of two canonical transformations followed by a scaling of time $t = \theta(s)$ with $s$ as the regularizing time variable, producing a new Hamiltonian $\tilde{\Gamma}$ for the PPS4BP in extended phase space. Section 3 describes a scaling of orbits of the Hamiltonian system of equations with Hamiltonian $\tilde{\Gamma}$ which shows that any such periodic solution always belongs to a one-parameter family of periodic solutions for which the linear stability is the same for all periodic solutions in the family. Section 4 describes the symmetries of the Hamiltonian $\tilde{\Gamma}$ which are used to construct periodic solutions with a $D_4$ symmetry group generated by a time-reversing symmetry and a time-preserving symmetry.
In Sections 5 and 6, we prove the analytic existence of a periodic SBC orbit \( \gamma(s) \), with a \( D_4 \) symmetry group, for the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) with \( m = 1 \), and numerically investigate its stability. The proof extends the analytic existence of a symmetric periodic SBC orbit in the fully symmetric planar four-body equal mass problem, as found in [15], to the PPS4BP with equal masses. Our numerical estimates of the characteristic multipliers show that this periodic orbit is linearly stable.

In Section 7, we numerically continue the linearly stable symmetric periodic SBC orbit \( \gamma(s) \) from \( m = 1 \) to a symmetric periodic SBC orbit \( \gamma(s; m) \) for \( m < 1 \), and then investigate the linear stability of \( \gamma(s; m) \) as \( m \) varies in the interval \((0, 1]\). We use trigonometric polynomials as approximations of the periodic orbits (cf. [25]). The numerical algorithm for continuation starts with a trigonometric polynomial approximation of \( \gamma(s; 1) = \gamma(s) \) that is used as an initial guess for a trigonometric polynomial approximation of \( \gamma(s; 0.99) \), where the coefficients of the trigonometric polynomial are optimized through a variational approach. This process is repeated, using the optimized approximation of \( \gamma(s; 0.99) \) as the initial guess for \( \gamma(s; 0.98) \), etc., until an optimized approximation of \( \gamma(s; 0.01) \) is obtained. Numerical estimates of the characteristic multipliers of \( \gamma(s; m) \) show it is linearly stable when \( 0.54 \leq m \leq 1 \) and unstable when \( 0 < m < 0.53 \). We have analyzed the linear stability of these orbits for \( 0 < m \leq 1 \), especially for \( 0.53 < m < 0.54 \) with increments of 0.001, using Roberts’ symmetry reduction method, the results of which will appear in a subsequent paper [2].

2. Regularization

We adapt the regularization of Aarseth and Zare [1] to the PPS4BP to regularize SBCs as described in the first objective. This regularization differs from the one used in the Caledonian symmetric four-body problem [23] in that we only regularize the SBCs as described in the first objective. This regularization differs from the one used in the Caledonian symmetric four-body problem [23] in that we only regularize the SBCs as described in the first objective. This regularization differs from the one used in the Caledonian symmetric four-body problem [23] in that we only regularize

The first canonical transformation in our regularization is

\[
(x_1, x_2, x_3, x_4, \omega_1, \omega_2, \omega_3, \omega_4) \rightarrow (g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4)
\]
determined by the generating function

\[
F_1(x_1, x_2, x_3, x_4, h_1, h_2, h_3, h_4) = h_1(x_1 - x_3) + h_2(x_2 - x_4) + h_3(x_1 + x_3) + h_4(x_2 + x_4).
\]

So the first canonical transformation is determined by

\[
\omega_i = \frac{\partial F_1}{\partial x_i}, \quad g_i = \frac{\partial F_1}{\partial \omega_i}, \quad i = 1, 2, 3, 4.
\]

The new Hamiltonian is \( \hat{H} = \hat{K} - \hat{U} \), where

\[
\hat{K} = \frac{(h_1 + h_3)^2 + (h_2 + h_4)^2}{4} + \frac{(h_3 - h_1)^2 + (h_4 - h_2)^2}{4m},
\]

and

\[
\hat{U} = \frac{1}{\sqrt{(g_1 + g_3)^2 + (g_2 + g_4)^2}} + 2m \frac{1}{\sqrt{g_1^2 + g_2^2}} + 2m \frac{1}{\sqrt{g_3^2 + g_4^2}} + \frac{m^2}{\sqrt{(g_1 - g_3)^2 + (g_2 - g_4)^2}}.
\]

The second canonical transformation in our regularization,

\[
(g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4) \rightarrow (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)
\]
is determined by the generating function

\[ F_2(h_1, h_2, h_3, h_4, u_1, u_2, u_3, u_4) = - \sum_{j=1}^{4} h_j f_j (u_1, u_2, u_3, u_4), \]

where

\[ f_1 = u_1^2 - u_2^2, \quad f_2 = 2u_1u_2, \quad f_3 = u_3^2 - u_4^2, \quad f_4 = 2u_3u_4. \]

So the second canonical transformation is determined by

\[ g_i = -\frac{\partial F_2}{\partial h_i}, \quad v_i = -\frac{\partial F_2}{\partial u_i}, \quad i = 1, 2, 3, 4. \]

The new Hamiltonian is \( \hat{H} = \hat{K} - \hat{U} \), where

\[
\begin{align*}
\hat{K} &= \frac{1}{16} \left( 1 + \frac{1}{m} \right) \left[ \frac{(v_1^2 + v_2^2)(u_1^2 + u_2^2) + (v_3^2 + v_4^2)(u_3^2 + u_4^2)}{(u_1^2 + u_2^2)(u_3^2 + u_4^2)} \right] \\
&\quad + \frac{1}{8} \left( 1 - \frac{1}{m} \right) \frac{(v_3 u_3 - v_4 u_4)(v_1 u_1 - v_2 u_2) + (v_3 u_4 + v_4 u_3)(v_1 u_2 + v_2 u_1)}{(u_1^2 + u_2^2)(u_3^2 + u_4^2)},
\end{align*}
\]

and

\[
\hat{U} = \frac{1}{\sqrt{(u_1^2 - u_2^2 + u_3^2 - u_4^2)^2 + (2u_1 u_2 + 2u_3 u_4)^2}} \frac{2m}{u_1^2 + u_2^2} + \frac{2m}{u_3^2 + u_4^2} + \frac{m^2}{\sqrt{(u_1^2 - u_2^2 + u_3^2 - u_4^2)^2 + (2u_1 u_2 - 2u_3 u_4)^2}}.
\]

We introduce a new time variable \( s \) by the regularizing change of time

\[ \frac{dt}{ds} = (u_1^2 + u_2^2)(u_3^2 + u_4^2). \]

To simplify notation, we set

\[
\begin{align*}
M_1 &= v_1 u_1 - v_2 u_2, & M_2 &= v_1 u_2 + v_2 u_1, \\
M_3 &= v_3 u_3 - v_4 u_4, & M_4 &= v_3 u_4 + v_4 u_3, \\
M_5 &= u_1^2 - u_2^2 + u_3^2 - u_4^2, & M_6 &= 2u_1 u_2 + 2u_3 u_4, \\
M_7 &= u_1^2 - u_2^2 - u_3^2 + u_4^2, & M_8 &= 2u_1 u_2 - 2u_3 u_4.
\end{align*}
\]

The Hamiltonian in the extended phase space with coordinates \( u_1, \ u_2, \ u_3, \ u_4, \ \hat{E}, \ v_1, \ v_2, \ v_3, \ v_4, \ t \) is

\[
\dot{\hat{\Gamma}} = \frac{dt}{ds} (\hat{H} - \hat{E}) = \frac{1}{16} \left( 1 + \frac{1}{m} \right) \left( (v_1^2 + v_2^2)(u_1^2 + u_2^2) + (v_3^2 + v_4^2)(u_3^2 + u_4^2) \right) \\
&\quad + \frac{1}{8} \left( 1 - \frac{1}{m} \right) (M_3 M_1 + M_4 M_2) \\
&\quad - \frac{(u_1^2 + u_2^2)(u_3^2 + u_4^2)}{\sqrt{M_5^2 + M_6^2}} - 2m (u_1^2 + u_2^2 + u_3^2 + u_4^2) \\
&\quad - \frac{m^2(u_2^2 + u_4^2)(u_3^2 + u_4^2)}{\sqrt{M_7^2 + M_8^2}} - \hat{E}(u_1^2 + u_2^2)(u_3^2 + u_4^2).
\]

With \( ' = d/ds \), the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) is

\[ u'_i = \frac{\partial \hat{\Gamma}}{\partial v_i}, \quad v'_i = -\frac{\partial \hat{\Gamma}}{\partial u_i}, \quad i = 1, 2, 3, 4, \]
along with the auxiliary equations,

\[
\dot{E}' = \frac{\partial \hat{\Gamma}}{\partial t} = 0, \quad t' = -\frac{\partial \hat{\Gamma}}{\partial E} = (u^2_1 + \ldots s=0)u^3 = \pm u^1, u^4 = \mp u^2, \text{ with } u^1u^2 < 0, |u^2| \leq (\sqrt{2} - 1)|u^1|,
\]

\[
v^3 = \mp v^1, v^4 = \pm v^2, \text{ with } 0 < v^1u^2 + v^2u^1 \leq v^2u^2 - v^1u^1.
\]

Lemma 2.1. The conditions (at \( t = 0 \))

\[
x_1 > 0, x_2 = 0, x_3 = 0, x_4 > 0, \text{ with } x_4 \leq x_1,
\]

\[
\omega_1 = 0, \omega_2 > 0, \omega_3 > 0, \omega_4 = 0, \text{ with } \omega_2 \leq \omega_3,
\]

Correspond to the conditions (at \( s = 0 \))

\[
u^3 = \pm u^1, u^4 = \mp u^2, \text{ with } u^1u^2 < 0, |u^2| \leq (\sqrt{2} - 1)|u^1|,
\]

\[
v^3 = \mp v^1, v^4 = \pm v^2, \text{ with } 0 < v^1u^2 + v^2u^1 \leq v^2u^2 - v^1u^1.
\]
3. A Scaling of Periodic Orbits and Linear Stability

A certain scaling of solutions of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ produces more solutions. When applied to a periodic solution, this scaling leads to a one-parameter family of periodic solutions. The proof of the following result is a straightforward verification.

**Lemma 3.1.** If $\gamma(s) = (u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s))$ is a periodic solution of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$ with period $T$ and energy $\hat{E}$, then for every $\epsilon > 0$, the function

$$
\gamma_{\epsilon}(s) = (\epsilon u_1(\epsilon s), \epsilon u_2(\epsilon s), \epsilon u_3(\epsilon s), \epsilon u_4(\epsilon s), \epsilon v_1(\epsilon s), \epsilon v_2(\epsilon s), \epsilon v_3(\epsilon s), \epsilon v_4(\epsilon s))
$$

is a periodic solution of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$ with period $\epsilon T$ and energy $\hat{E}_\epsilon = \epsilon^{-2}\hat{E}$.

The linear stability of a periodic orbit of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ is determined by the linearization of the equations along the periodic orbit. By Lemma 3.1, a periodic orbit $\gamma(s)$ on the level set $\hat{\Gamma} = 0$ with period $T$ and energy $\hat{E}$ embeds into a one-parameter family $\gamma_{\epsilon}(s)$ of periodic orbits on the level set $\hat{\Gamma} = 0$ with period $T_\epsilon$ and energy $\hat{E}_\epsilon$. The linearization of the Hamiltonian system of equations (3) with Hamiltonian $\hat{\Gamma}$ along the periodic orbit $\gamma_{\epsilon}(s)$ is

$$
X' = J\nabla^2\hat{\Gamma}(\gamma_{\epsilon}(s))X
$$

where $\nabla^2\hat{\Gamma}$ is the matrix of second-order partials of $\hat{\Gamma}$. Let $X_\epsilon(s)$ be the solution of the linearization of the equations along $\gamma_{\epsilon}(s)$ that satisfies $X_\epsilon(0) = I$ (the $8 \times 8$ identity matrix). The monodromy matrix for $\gamma_{\epsilon}(s)$ is $X_\epsilon(T_\epsilon)$, and the eigenvalues of $X_\epsilon(T_\epsilon)$ are the characteristic multipliers of $\gamma_{\epsilon}(s)$. A characteristic multiplier $\lambda$ of $\gamma_{\epsilon}(s)$ is defective if its geometric multiplicity is smaller than its algebraic multiplicity, i.e., its generalized eigenspace $\cup_{j \geq 1} \ker(X_\epsilon(T_\epsilon) - \lambda I)^j$ is not the same as its eigenspace $\ker(X_\epsilon(T_\epsilon) - \lambda I)$. Proving the following result is routine (see [11]).

**Lemma 3.2.** If $\gamma(s)$ is a periodic orbit of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$, then for each $\epsilon > 0$, the periodic orbit $\gamma_{\epsilon}(s)$ has 1 as a defective characteristic multiplier with algebraic multiplicity at least two.

For each $\epsilon > 0$, the periodic orbit $\gamma_{\epsilon}(s)$ is spectrally stable if all of its characteristic multipliers have modulus one. By Lemma 3.2, the periodic orbit $\gamma_{\epsilon}(s)$ has 1 as a defective characteristic multiplier with algebraic multiplicity at least two, and so the monodromy matrix $X_\epsilon(T_\epsilon)$ it not semisimple. However, the two-dimensional subspace

$$
U_1 = \text{Span} \left( \gamma_{\epsilon}'(0), \frac{\partial}{\partial \epsilon} \gamma_{\epsilon}(0) \right)
$$

is $X_\epsilon(T_\epsilon)$-invariant. The periodic orbit $\gamma_{\epsilon}(s)$ is said to be linearly stable if it is spectrally stable and there exists a 6-dimensional $X_\epsilon(T_\epsilon)$-invariant subspace $U_2$ such that $U_1 + U_2 = \mathbb{R}^8$ and $X_\epsilon(T_\epsilon)$ restricted to $U_2$ is semisimple. A proof of the following is routine.

**Theorem 3.3.** Suppose $\gamma(s)$ is a periodic orbit of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ on the level set $\hat{\Gamma} = 0$. Then $\gamma_{\epsilon}(s)$ is spectrally (linearly) stable for some $\epsilon > 0$ if and only if $\gamma_{\epsilon}(s)$ is spectrally (linearly) stable for all $\epsilon > 0$. 

4. Symmetries

The Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ has a group of symmetries isomorphic to the dihedral group $D_4 = \langle a, b : a^2 = b^4 = (ab)^2 = e \rangle$. With

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

define the matrices

$$S_F = \begin{bmatrix} 0 & F & 0 & 0 \\ -F & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & -F & 0 \end{bmatrix}, \quad S_G = \begin{bmatrix} -G & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & -G \end{bmatrix}. $$

These matrices satisfy $S_F^2 = -I$, $S_F^4 = I$, $S_G^2 = I$, and $(S_F S_G)^2 = I$. Fixing the value of $\hat{E}$, these matrices satisfy $\hat{\Gamma} \circ S_F = \hat{\Gamma}$ and $\hat{\Gamma} \circ S_G = \hat{\Gamma}$, and so $S_F$ and $S_G$ are the generators of the $D_4$-symmetry group for $\hat{\Gamma}$. If

$$\gamma(s) = (u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s))$$

is a solution of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$, then $S_F \gamma(s)$, $S_F^2 \gamma(s)$, and $S_G \gamma(-s)$ are also solutions of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$. This means that $S_F$ is a time-preserving symmetry and that $S_G$ is a time-reversing symmetry. The proof of the following is routine.

**Lemma 4.1.** If for some $s_0 > 0$ there is a nonsingular solution $\gamma(s)$, $s \in [0, s_0]$, of the Hamiltonian system of equations with Hamiltonian $\hat{\Gamma}$ such that for constants $\zeta_1 \neq 0$, $\zeta_2 \neq 0$, $\rho_1 \neq 0$, and $\rho_2 \neq 0$ there holds

$$u_1(0) = \zeta_1, \ u_2(0) = \zeta_2, \ u_3(0) = \zeta_1, \ u_4(0) = -\zeta_2,$$

$$v_1(0) = \rho_1, \ v_2(0) = \rho_2, \ v_3(0) = -\rho_1, \ v_4(0) = \rho_2,$$

and

$$u_1(s_0) = 0, \ u_2(s_0) = 0, \ u_3(s_0) \neq 0, \ u_4(s_0) \neq 0,$$

$$v_1(s_0) \neq 0, \ v_2(s_0) \neq 0, \ v_3(s_0) = 0, \ v_4(s_0) = 0,$$

then $\gamma(s)$ extends to a periodic orbit with period $8s_0$ and a symmetry group isomorphic to $D_4$ such that

$$u_1(3s_0) \neq 0, \ u_2(3s_0) \neq 0, \ u_3(3s_0) = 0, \ u_4(3s_0) = 0,$$

$$v_1(3s_0) = 0, \ v_2(3s_0) = 0, \ v_3(3s_0) \neq 0, \ v_4(3s_0) \neq 0,$$

and

$$u_1(5s_0) = 0, \ u_2(5s_0) = 0, \ u_3(5s_0) \neq 0, \ u_4(5s_0) \neq 0,$$

$$v_1(5s_0) \neq 0, \ v_2(5s_0) \neq 0, \ v_3(5s_0) = 0, \ v_4(5s_0) = 0,$$

and

$$u_1(7s_0) \neq 0, \ u_2(7s_0) \neq 0, \ u_3(7s_0) = 0, \ u_4(7s_0) = 0,$$

$$v_1(7s_0) = 0, \ v_2(7s_0) = 0, \ v_3(7s_0) \neq 0, \ v_4(7s_0) \neq 0.$$
5. Analytic Existence in the Equal Mass Case

When \( m = 1 \), there is an additional symmetry in the positions of the four masses that reduces the PPS4BP with equal masses to the fully symmetric planar four-body equal mass problem. We exploit this reduction to prove the existence of a symmetric periodic simultaneous binary collision orbit in the equal mass case.

The additional symmetry is the Ansatz, \( x_4 = x_1, x_3 = x_2 \), with \( |x_2| \leq x_1 \). From this it follows that \( \omega_4 = \omega_1, \omega_3 = \omega_2, x_1 - x_2 \geq 0, x_1 + x_2 \geq 0 \). From the canonical transformations (1) and (2), we have

\[
\frac{2u_1^2 + u_2^2}{1 + \sqrt{2}} = x_1 - x_2 = \frac{2u_2^2}{1 + \sqrt{2}}, \quad \frac{2u_3^2}{1 + \sqrt{2}} = x_1 + x_2 = \frac{2u_1^2}{1 + \sqrt{2}}.
\]

Since \( 2u_1 u_2 = g_2 = x_2 - x_1 \leq 0 \) and \( 2u_3 u_4 = g_4 = x_1 + x_2 \geq 0 \), it follows that

\[
(7) \quad u_2 = -(\sqrt{2} - 1)u_1, \quad u_4 = (\sqrt{2} - 1)u_3.
\]

From the second canonical transformation (2), we have

\[
v_1 = \sqrt{2}(\omega_1 - \omega_2)u_1, \quad v_2 = -(2 - \sqrt{2})(\omega_1 - \omega_2)u_1, \quad v_3 = \sqrt{2}(\omega_1 + \omega_2)u_3, \quad v_4 = (2 - \sqrt{2})(\omega_1 + \omega_2)u_3.
\]

These imply that

\[(8) \quad v_2 = -(\sqrt{2} - 1)v_1, \quad v_4 = (\sqrt{2} - 1)v_3.\]

Substitution into the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) (and with \( m = 1 \)) gives

\[
\begin{align*}
u_1' &= \frac{4 - 2\sqrt{2}}{4} v_1 u_3^2, & u_2' &= -\frac{(\sqrt{2} - 1)(4 - 2\sqrt{2})}{4} v_1 u_3^2, \\
u_3' &= \frac{4 - 2\sqrt{2}}{4} v_3 u_1^2, & u_4' &= \frac{(\sqrt{2} - 1)(4 - 2\sqrt{2})}{4} v_3 u_1^2,
\end{align*}
\]

\( \dot{E}' = 0 \), and

\[
\begin{align*}
v_1' &= -(\frac{4 - 2\sqrt{2}}{4} u_1 v_3^2 + 4u_1 + \frac{4u_1 u_3^2}{\sqrt{u_1^4 + u_3^4}} - \frac{4u_1^2 u_3^2}{(u_1^4 + u_3^4)^{3/2}} + 2(4 - 2\sqrt{2})\dot{E}u_1 u_3^2, \\
v_2' &= -(\sqrt{2} - 1) \left[ -\frac{(4 - 2\sqrt{2}) u_1 v_3^2}{4} + 4u_1 + \frac{4u_1 u_3^2}{\sqrt{u_1^4 + u_3^4}} - \frac{4u_1^2 u_3^2}{(u_1^4 + u_3^4)^{3/2}} + 2(4 - 2\sqrt{2})\dot{E}u_1 u_3^2\right], \\
v_3' &= -(\frac{4 - 2\sqrt{2}}{4} u_3 v_1^2 + 4u_3 + \frac{4u_1 u_3^2}{\sqrt{u_1^4 + u_3^4}} - \frac{4u_1^2 u_3^2}{(u_1^4 + u_3^4)^{3/2}} + 2(4 - 2\sqrt{2})\dot{E}u_1 u_3^2, \\
v_4' &= (\sqrt{2} - 1) \left[ -\frac{(4 - 2\sqrt{2}) u_3 v_1^2}{4} + 4u_3 + \frac{4u_1^2 u_3^2}{\sqrt{u_1^4 + u_3^4}} - \frac{4u_1^2 u_3^2}{(u_1^4 + u_3^4)^{3/2}} + 2(4 - 2\sqrt{2})\dot{E}u_1 u_3^2\right], \\
t' &= (4 - 2\sqrt{2})^2 u_1^2 u_3^2.
\end{align*}
\]
Because of Equations (7) and (8), the equations in \( u'_2, u'_4, v'_2, \) and \( v'_4 \) duplicate those in \( u'_1, u'_3, v'_1, \) and \( v'_3 \). The Ansatz \( x_4 = x_1, \ x_3 = x_2 \) with \(|x_2| \leq x_1\), therefore leads to the reduced system of equations,

\[
\begin{align*}
u'_2 &= \frac{4 - 2\sqrt{2}}{4} v_1 u_3^2, \\
u'_3 &= \frac{4 - 2\sqrt{2}}{4} v_3 u_2^2, \\
\dot{E}' &= 0, \\
v'_1 &= -\frac{(4 - 2\sqrt{2}) u_1 v_2^2}{4} + 4u_1 + \frac{4u_1 u_3^2}{u_1^2 + u_3^2} - \frac{4u_1^5 u_3^2}{(u_1^4 + u_3^4)^{3/2}} + 2(4 - 2\sqrt{2}) \dot{E} u_1 u_3^2, \\
v'_3 &= -\frac{(4 - 2\sqrt{2}) u_3 v_1^2}{4} + 4u_3 + \frac{4u_2^7 u_3}{u_1^2 + u_3^2} - \frac{4u_2^7 u_3^5}{(u_1^4 + u_3^4)^{3/2}} + 2(4 - 2\sqrt{2}) \dot{E} u_2^7 u_3, \\
t' &= (4 - 2\sqrt{2})^2 u_1^2 u_3^2.
\end{align*}
\]

Scale the value of \( \hat{E} \) by

\[
\hat{E} = \frac{\dot{E}}{4 - 2\sqrt{2}},
\]

and define

\[
\hat{\Gamma} = \frac{4 - 2\sqrt{2}}{8} (v_1^2 u_3^2 + v_3^2 u_1^2) - 2(u_1^2 + u_3^2) - \frac{2u_1^7 u_3^2}{u_1^4 + u_3^4} - (4 - 2\sqrt{2})^2 \dot{E} u_1^2 u_3^2.
\]

It is straightforward to check that the reduced system of equations satisfies

\[
\begin{align*}
u'_i &= \frac{\partial \hat{\Gamma}}{\partial v_i}, \\
v'_i &= -\frac{\partial \hat{\Gamma}}{\partial u_i}, \quad i = 1, 2, \\
\dot{E}' &= \frac{\partial \hat{\Gamma}}{\partial \dot{\theta}}, \\
t' &= -\frac{\partial \hat{\Gamma}}{\partial \dot{E}}.
\end{align*}
\]

Thus the system of reduced equations is Hamiltonian.

We will simplify the Hamiltonian \( \hat{\Gamma} \) by a linear symplectic transformation with a multiplier \( \mu \neq 1 \). Define new coordinates \((Q_1, Q_2, E, P_1, P_2, \tau)\) by

\[
\begin{align*}
u_1 &= \frac{Q_1}{2^{1/4}}, \\
u_3 &= \frac{Q_2}{2^{1/4}}, \\
\dot{E} &= \frac{2E}{(4 - 2\sqrt{2})^2}, \\
t &= 2^{3/4} (\sqrt{2} - 1)^{3/2} \tau.
\end{align*}
\]

This is a linear symplectic change of coordinates with multiplier

\[
\mu = \frac{1}{2^{5/4} \sqrt{\sqrt{2} - 1}}.
\]

Under this linear symplectic transformation and the accompanying scaling \( \sigma = s/\mu \) of the independent variable \( s \), the Hamiltonian \( \hat{\Gamma} \) becomes

\[
\hat{\Gamma} = \frac{1}{16} (P_1^2 Q_2^2 + P_2^2 Q_1^2) - \sqrt{2}(Q_1^2 + Q_2^2) - \frac{\sqrt{2} Q_1 Q_2^3}{\sqrt{Q_1^2 + Q_2^2}} - EQ_1 Q_2^2.
\]
The reduced system of equations is the Hamiltonian system of equations with Hamiltonian \( \Gamma \),
\[
\frac{dQ_i}{d\sigma} = \frac{\partial \Gamma}{\partial P_i}, \quad \frac{dP_i}{d\sigma} = -\frac{\partial \Gamma}{\partial Q_i}, \quad i = 1, 2, \quad \frac{dE}{d\sigma} = 0, \quad \frac{d\tau}{d\sigma} = Q_1^2Q_2^2.
\]

The function \( \Gamma \) is a regularized Hamiltonian for the fully symmetric planar four-body equal mass problem with the bodies located at \((x_1, x_2), (x_2, x_1), (-x_1, -x_2),\) and \((-x_2, -x_1)\) (see [15]). On the level set \( \Gamma = 0 \), the solutions have energy \( E \).

One regularized simultaneous binary collision occurs when \( Q_1 = 0 \) and \( Q_2 \neq 0 \), for which \( \Gamma = 0 \) implies \( P_1^2 = 16\sqrt{2} \), and for which the transformation between \( Q_1, Q_2 \) and \( x_1, x_2 \) implies \( x_1 - x_2 = 0 \) and \( x_1 + x_2 \neq 0 \). The other regularized simultaneous binary collision occurs when \( Q_1 \neq 0 \) and \( Q_2 = 0 \), for which \( \Gamma = 0 \) implies \( P_2^2 = 16\sqrt{2} \), and for which the transformation between \( Q_1, Q_2 \) and \( x_1, x_2 \) implies \( x_1 - x_2 \neq 0 \) and \( x_1 + x_2 = 0 \). Total collapse occurs when \( Q_1 = 0 \) and \( Q_2 = 0 \), and is the only singularity in \( \Gamma \) that is not regularized. A solution \( Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma), \sigma \in [0, \sigma_0] \), for \( \sigma_0 > 0 \), of the Hamiltonian system equations with Hamiltonian \( \Gamma \) is non-singular if it avoids total collapse, i.e., \( Q_1^4 + Q_2^4 \neq 0 \) for all \( \sigma \in [0, \sigma_0] \).

The following result is from [15]. The proof of it is a consequence of four equal mass bodies starting at \((x_1, x_2), (x_2, x_1), (-x_1, -x_2),\) and \((-x_2, -x_1)\) with \( x_1 = 1 \) and \( x_2 = 0 \), and with the momenta \((0, \vartheta), (\vartheta, 0), (0, -\vartheta),\) and \((-\vartheta, 0)\) for any \( \vartheta > 0 \), always having a simultaneous binary collision on the line \( x_2 = x_1 \) at a time \( t_0 > 0 \) continuously depending on \( \vartheta \), such that the cluster velocity \( \dot{x}_1(t_0) + \dot{x}_2(t_0) \) is a continuous function of \( \vartheta \).

**Lemma 5.1.** There exists \( \vartheta > 0, \sigma_0 > 0, \) and a non-singular solution \( Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma), \sigma \in [0, \sigma_0] \), of the Hamiltonian system of equations with Hamiltonian \( \Gamma \) on the level set \( \Gamma = 0 \) such that
\[
Q_1(0) = 1, \quad Q_2(0) = 1, \quad P_1(0) = -\vartheta, \quad P_2(0) = \vartheta,
\]
\[
E = \frac{\vartheta^2 - 16\sqrt{2} - 8}{8} < 0, \quad \tau(\sigma) = \int_0^\sigma Q_1^2(y)Q_2^2(y) \, dy,
\]
and
\[
Q_1(\sigma_0) = 0, \quad Q_2(\sigma_0) > 0, \quad P_1(\sigma_0) = -4(2^{1/4}), \quad P_2(\sigma_0) = 0.
\]

This Lemma gives the existence of a solution of a boundary value problem for the Hamiltonian system of equations with Hamiltonian \( \Gamma \). It is this solution whose symmetric extension gives a symmetric periodic SBC orbit in the PPS4BP with equal masses.

**Theorem 5.2.** Fix \( m = 1 \). For each \( \dot{E} < 0 \), there exists a time-reversible periodic regularized SBC orbit
\[
\gamma(s) = (u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s))
\]
with period \( T > 0 \), angular momentum \( A = 0 \), and a symmetry group isomorphic to \( D_4 \), for the Hamiltonian system of equations with Hamiltonian \( \Gamma \) on the level set \( \Gamma = 0 \) such that distinct regularized SBCs occur at \( s = T/8, 3T/8, 5T/8, 7T/8 \). This periodic orbit corresponds to a symmetric periodic singular orbit
\[
(x_1(t), x_2(t), x_3(t), x_4(t), \omega_1(t), \omega_2(t), \omega_3(t), \omega_4(t))
\]
with energy $\hat{E}$ for the PPS4BP with equal masses $m = 1$ where for all $t$,
\[ x_4(t) = x_1(t), \ x_3(t) = x_2(t), \ |x_2(t)| \leq x_1(t), \ \omega_4(t) = \omega_1(t), \ \omega_3(t) = \omega_2(t), \]
with initial conditions
\[ x_1(0) > 0, \ x_2(0) = 0, \ \omega_1(0) = 0, \ \omega_2(0) > 0, \]
and period
\[ R = \int_{0}^{\pi/2} (u_1^2(s) + u_2^2(s)) \left(u_3^2(s) + u_4^2(s)\right) \, ds, \]
where for $t \in [0, R]$, the only singularities are two distinct SBC's occurring at $t = R/4$ and $t = 3R/4$.

**Proof.** By Lemma 5.1, let $Q_1(\sigma)$, $Q_2(\sigma)$, $P_1(\sigma)$, $P_2(\sigma)$, $\sigma \in [0, \sigma_0]$, be the nonsingular solution of the Hamiltonian system of equations (14) with Hamiltonian $\Gamma$ on the level set $\Gamma = 0$, whose properties are given in (15), (16), and (17). Using the scaling $\sigma = s/\mu$, set $s_0 = \mu \sigma_0$. By the linear symplectic transformation given in (10), (11), and (12) with multiplier $\mu$ (as given in (13)), we have
\[ u_1(s) = {Q_1(s/\mu) \over 2^{1/4}}, \quad u_3(s) = {Q_2(s/\mu) \over 2^{1/4}}, \quad v_1(s) = {P_1(s/\mu) \over 2\sqrt{2} - 1}, \quad v_3(s) = {P_2(s/\mu) \over 2\sqrt{2} - 1}. \]
By Equations (7) and (8), we have
\[ u_2(s) = -({2} - 1)Q_1(s/\mu) \over 2^{1/4}, \quad u_4(s) = ({2} - 1)Q_2(s/\mu) \over 2^{1/4}, \]
\[ v_2(s) = -\sqrt{2} - 1P_1(s/\mu), \quad v_4(s) = \sqrt{2} - 1P_2(s/\mu). \]
Set $\gamma(s) = (u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s))$, $s \in [0, s_0]$. From the Equations (9), (12), (16) in $E$, $\hat{E}$, $\hat{E}$, and $\hat{\vartheta}$, we obtain
\[ \hat{E} = {2 + \sqrt{2}\vartheta^2 \over 16} - 3 - {5\sqrt{2} \over 2} < 0. \]
With this value of $\hat{E}$, it follows that the value of $\hat{\Gamma}$ at $\gamma(0)$ is 0. Set
\[ \zeta_1 = {1 \over 2^{1/4}} > 0, \quad \zeta_2 = -({2} - 1) \over 2^{1/4} < 0, \]
\[ \rho_1 = -{\vartheta \over 2\sqrt{2} - 1} < 0, \quad \rho_2 = {\vartheta \sqrt{2} - 1 \over 2} > 0. \]
With $Q_1(\sigma)$, $Q_2(\sigma)$, $P_1(\sigma)$, $P_2(\sigma)$, $\sigma \in [0, \sigma_0]$, being a nonsingular solution of the Hamiltonian system of equations with Hamiltonian $\Gamma$, we have $Q_1^2(\sigma) + Q_2^2(\sigma) \neq 0$ for all $\sigma \in [0, \sigma_0]$. From this it follows for all $s \in [0, s_0]$ that
\[ M_5 = u_1^2(s) - u_2^2(s) + u_3^2(s) - u_4^2(s) = (2 - \sqrt{2})[Q_1^2(s/\mu) + Q_2^2(s/\mu)] \neq 0, \]
and
\[ M_8 = 2u_1(s)u_2(s) - 2u_3(s)u_4(s) = -(2 - \sqrt{2})[Q_1^2(s/\mu) + Q_2^2(s/\mu)] \neq 0. \]
These imply that $M_5^2 + M_6^2 \neq 0$ and $M_2^2 + M_8^2 \neq 0$ for all $s \in [0, s_0]$. Thus the function $\gamma(s)$ is a nonsingular solution of the Hamiltonian system of equations (3) with Hamiltonian $\Gamma$ on the level set $\Gamma = 0$ that satisfies
\[ u_1(0) = \zeta_1, \quad u_2(0) = \zeta_2, \quad u_3(0) = \zeta_1, \quad u_4(0) = -\zeta_2. \]
By Lemma 4.1, the solution $\gamma(s)$ extends to a $T = 8s_0$ periodic solution, call it $\gamma(s)$, with a $D_4$ symmetry group generated by the symmetries $S_F$ and $S_G$, and four distinct regularized simultaneous binary collisions at $s = s_0, 3s_0, 5s_0, 7s_0$, for which

\begin{align*}
u_1(s_0) = 0, & \quad u_2(s_0) = 0, \quad u_3(s_0) > 0, \quad u_4(s_0) > 0, \\
v_1(s_0) < 0, & \quad v_2(s_0) > 0, \quad v_3(s_0) = 0, \quad v_4(s_0) = 0.
\end{align*}

Since $Q^1_F(\sigma) + Q^1_G(\sigma) \neq 0$ for all $\sigma \in [0, \sigma_0]$, it follows that $(u_2(s) + u_4(s))(u_2(s) + u_4(s)) \neq 0$ for $s \in [0, T]$ except $s = s_0, 3s_0, 5s_0, 7s_0$. The regularizing change of time (4),

$$\frac{dt}{ds} = (u_2(s) + u_4(s))(u_2(s) + u_4(s)),$$

defines $t$ as an invertible differentiable function of $s$, i.e., $t = \theta(s)$ with $\theta(0) = 0$ and $\theta'(s) = 0$ when $s = (2k + 1)s_0$ for $k \in \mathbb{Z}$. The symmetry $S_F$ satisfies $S_F\gamma(s) = \gamma(s + 2s_0)$ and $-\gamma(s) = S^2_F\gamma(s) = \gamma(s + 4s_0)$. The symmetry $S_G$ satisfies $S_G\gamma(s) = \gamma(2s_0 - s)$, and so $\gamma(s)$ has a time-reversing symmetry. The angular momentum (5) for $\gamma(s)$ at $s = 0$ is

$$A = \frac{1}{2}[-v_1u_2 + v_2u_1 - v_3u_4 + v_4u_3] = \rho_2\zeta_1 - \rho_1\zeta_2 = 0.$$

The extended solution $\gamma(s)$ gives a singular symmetric solution

$$z(t) = (x_1(t), x_2(t), x_3(t), x_4(t), \omega_1(t), \omega_2(t), \omega_3(t), \omega_4(t)),$$

of the PPS4BP with $m = 1$. Under the Ansätz, the components of $z(t)$ satisfies $x_4(t) = x_1(t)$, $x_3(t) = x_2(t)$, $|x_2(t)| \leq x_1(t)$, $\omega_4(t) = \omega_1(t)$, $\omega_3(t) = \omega_2(t)$, where

\begin{align*}
x_1(t) &= \frac{u_2(s) - u_4(s) + u_2(s) - u_4(s)}{2} \\
x_2(t) &= u_1(s)u_2(s) + u_3(s)u_4(s), \\
\omega_1(t) &= \frac{v_1(s)v_3(s) - v_2(s)v_4(s)}{2(u_2(s) + u_4(s))} + \frac{v_3(s)v_4(s)}{2(u_2(s) + u_4(s))}, \\
\omega_2(t) &= \frac{v_1(s)v_3(s) - v_2(s)v_4(s)}{2(u_2(s) + u_4(s))} + \frac{v_3(s)v_4(s)}{2(u_2(s) + u_4(s))},
\end{align*}

for $s = \theta^{-1}(t)$. The components of the extended solution $\gamma(s)$ satisfy $u_3(0) = u_1(0)$, $u_4(0) = u_2(0)$, $u_1(0)u_2(0) < 0$, $|u_2(t)| = (\sqrt{2} - 1)|u_1(0)|$, $v_3(0) = -v_1(0)$, $v_4(0) = v_2(0)$,

$$v_1(0)u_2(0) + v_2(0)u_1(0) = \rho_1\zeta_2 + \rho_2\zeta_1 = \frac{\sqrt{2} - 1}{2^{1/4}} \theta > 0,$$

and

$$v_2(0)u_2(0) - v_1(0)u_1(0) = \rho_2\zeta_2 - \rho_1\zeta_1 = \frac{\sqrt{2} - 1}{2^{1/4}} \theta.$$

From Lemma 2.1, it follows that $x_1(0) > 0$, $x_2(0) = 0$, $\omega_1(0) = 0$, and $\omega_2(0) > 0$. Set $R = \theta(T/2)$. Since $\gamma(4s_0) = -\gamma(0)$, it follows that $x_1(R) = x_1(0)$, $x_2(R) = x_2(0)$,
\( \omega_1(R) = \omega_1(0) \), and \( \omega_2(R) = \omega_2(0) \). Thus the singular symmetric solution \( z(t) \) has period \( R \). By the construction of the extension of \( \gamma(s) \) given in Lemma 4.1, there holds

\[
\int_{s_0}^{(k+1)s_0} (u_1^2(s) + u_2^2(s))(u_3^2(s) + u_4^2(s)) \, ds = \int_0^{s_0} (u_1^2(s) + u_2^2(s))(u_3^2(s) + u_4^2(s)) \, ds
\]

for all \( k = 1, \ldots, 7 \). This implies that \( R/4 = \theta((k+1)s_0) - \theta(ks_0) \) for all \( k = 0, 1, \ldots, 7 \). The first regularized SBC for \( \gamma(s) \) occurs at \( s = s_0 \), and this corresponds to \( t = \theta(s_0) = R/4 \). The next regularized SBC for \( \gamma(s) \) occurs at \( s = 3s_0 \), and this corresponds to

\[
t = \theta(3s_0) = (\theta(3s_0) - \theta(2s_0)) + (\theta(2s_0) - \theta(s_0)) + \theta(s_0) = \frac{R}{4} + \frac{R}{4} + \frac{R}{4} = \frac{3R}{4}.
\]

Similarly, the regularized SBCs for \( \gamma(s) \) occurring at \( s = 5s_0, 7s_0 \) correspond to \( t = 5R/4, 7R/4 \). Hence, for \( t \in [0, R] \), the periodic solution \( z(t) \) has SBCs as its only singularities, and these occur at \( t = R/4, 3R/4 \).

For a fixed but arbitrary \( \epsilon > 0 \), the value of \( \epsilon^{-2} \hat{E} \) is a fixed but arbitrary negative real number. By Lemma 3.1, the scaled extended solution

\[
\gamma_\epsilon(s) = (\epsilon u_1(\epsilon s), \epsilon u_2(\epsilon s), \epsilon u_3(\epsilon s), \epsilon u_4(\epsilon s), v_1(\epsilon s), v_2(\epsilon s), v_3(\epsilon s), v_4(\epsilon s))
\]

is a periodic solution for the Hamiltonian system of equations (3) with Hamiltonian \( \hat{\Gamma} \) on the level set \( \hat{\Gamma} = 0 \), having period \( \epsilon^{-1} T \) and energy \( \epsilon^{-2} \hat{E} < 0 \). By an argument similar to above, \( \gamma_\epsilon(s) \) satisfies the required conditions.

### 6. Numerical Estimates in the Equal Mass Case

In the equal mass case, there is by Theorem 5.2 a time-reversible periodic orbit \( \gamma(s) \) for the Hamiltonian system of equations with Hamiltonian \( \hat{\Gamma} \) on the level set \( \hat{\Gamma} = 0 \) with period \( T \). The components \( u_1(s), u_2(s), u_3(s), u_4(s), v_1(s), v_2(s), v_3(s), v_4(s) \) of \( \gamma(s) \) satisfy Equations (7) and (8). The \( D_4 \) symmetry group of \( \gamma(s) \) is generated by \( S_2 \gamma(s) = \gamma(s + T/4) \) and \( S_2 \gamma(s) = \gamma(T/4 - s) \). Under the linear symplectic transformation (10), (11), and (12) with multiplier \( \mu \) (as given by (13)), this gives a periodic orbit \( Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma) \) of the Hamiltonian system of equations (14) with Hamiltonian \( \Gamma \) on the level set \( \Gamma = 0 \), which by Lemma 5.1 satisfies \( Q_1(0) = 1, Q_2(0) = 1, P_1(0) = -\vartheta, P_2(0) = \vartheta, Q_1(\sigma_0) = 0, Q_2(\sigma_0) > 0, P_1(\sigma_0) = -4(2^{1/4}), P_2(\sigma_0) = 0 \) for some \( \sigma_0 > 0 \) and \( \vartheta > 0 \). In [3], we numerically estimated

\[
\sigma_0 = 1.62047369909693, \quad \vartheta = 2.57486992651942.
\]

The period of this periodic orbit is \( 8\sigma_0 \approx 12.96378959 \) and its energy is \( E \approx -2.999682732 \). From the linear symplectic transformation (10), (11), and (12), with multiplier \( \mu \) and from Equations (7) and (8), we have for the values of components of \( \gamma(0) \) the exact

\[
u_1(0) = u_3(0) = 2^{-1/4}, \quad -u_2(0) = u_4(0) = (\sqrt{2} - 1)2^{-1/4},
\]

and the estimates

\[
v_1(0) = -v_3(0) = -\frac{\vartheta}{2\sqrt{\sqrt{2} - 1}} \approx -2.000382939,
\]

\[
v_2(0) = v_4(0) = \frac{\vartheta\sqrt{\sqrt{2} - 1}}{2} \approx 0.8285857433.
\]
Since \( \sigma = s/\mu \), the period of \( \gamma(s) \) is \( T \approx 8.469003682 \). From Equation (9), the value of the energy for \( \gamma(s) \) is \( \hat{E} \approx -5.120778733 \).

The components of the scaled periodic orbit \( \gamma_{\epsilon}(s) \) for \( \epsilon > 1 \), shown in Figure 2, satisfy the symmetries

\[
S_F \gamma_{\epsilon}(s) = \gamma_{\epsilon}(s + T_{\epsilon}/4) \quad \text{and} \quad S_G \gamma_{\epsilon}(s) = \gamma_{\epsilon}(T_{\epsilon}/4 - s).
\]

We choose \( \epsilon \) so that \( T_{\epsilon} = 2\pi \), and check the linear stability of \( \gamma_{\epsilon}(s) \) (see Theorem 3.3). Using a Runge-Kutta order 4 algorithm with a fixed time step of \( 2\pi/50000 \), we computed \( X_{\epsilon}(2\pi) \). Two of the eigenvalues of \( X_{\epsilon}(2\pi) \) are 1 by Theorem 3.2. Numerical estimates of the remaining eigenvalues of \( X_{\epsilon}(2\pi) \) are

\[
-0.9888731375 \pm 0.1487612779i,
-0.9973584383 \pm 0.07263708002i,
0.9999060579 \pm 0.01370676220i,
\]

that have modulus one. Thus numerically, the periodic orbits \( \gamma_{\epsilon}(s) \) are linearly stable for all \( \epsilon > 0 \). The first complex conjugate pair of eigenvalues for \( X_{\epsilon}(2\pi) \) matches the complex conjugate pair of characteristic multipliers for the periodic orbit \( Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma) \) of the Hamiltonian system of equations (14) with Hamiltonian \( \Gamma \), corresponding to \( \gamma_1(s) = \gamma(s) \), where we computed [3] the real part of the complex conjugate pair of modulus one to be \(-0.9888840619\). Because, by a lengthy computation, \( J \nabla^2 \hat{\Gamma}(\gamma_{\epsilon}(s)) \) is not block diagonal, the last two complex conjugate pairs of eigenvalues of \( X_{\epsilon}(2\pi) \) are not repeats of the characteristic multipliers of the periodic orbit \( Q_1(\sigma), Q_2(\sigma), P_1(\sigma), P_2(\sigma) \).

Figure 1 illustrates the curves in the physical plane that the four equal masses follow in the linearly stable SBC orbit \( z(t) = (x_1(t), x_2(t), x_3(t), x_4(t), \omega(t), \omega_2(t), \omega_3(t), \omega_4(t)) \) of the PPS4BP, corresponding to \( \gamma_{\epsilon}(s) \) with \( \epsilon = 1/(2 - \sqrt{2}) \). The initial conditions for \( z(t) \) are

\[
x_1(0) = x_4(0) = 1, \quad x_2(0) = x_3(0) = 0,
\omega_1(0) = \omega_4(0) = 0, \quad \omega_2(0) = \omega_3(0) \approx 1.287434964.
\]
The value of $\epsilon = 1/\sqrt{2} - \sqrt{2}$ here for the scaling is determined by the equation $x_1(0) = L^2(u_2^2(0) - u_2^2(0))$ coming from Lemma 2.1 and the canonical transformations (1) and (2) applied to the scaled periodic solution $\gamma(s)$, together with the initial condition $x_1(0) = 1$, where $t = 0$ corresponds to $s = 0$. The value of the Hamiltonian $H$ along $z(t)$ is $\epsilon^{-2}\hat{E} \approx -2.999682732$.

7. Numerical Estimates in the Unequal Mass Case

We numerically continue to $0 < m < 1$ the time-reversible periodic regularized simultaneous binary collision orbit for the Hamiltonian system of equations with Hamiltonian $\Gamma$ on the level set $\Gamma = 0$. The analytic existence of this orbit is given by Theorem 5.2 for $m = 1$. We also numerically estimate the monodromy matrices and their eigenvalues for these continued time-reversible periodic regularized simultaneous binary collision orbits for $0 < m < 1$. For $m = 1$, we assume by Lemma 3.1 that the time-reversible periodic regularized simultaneous binary collision orbit $\gamma(s;1)$ with the $D_4$ symmetry has period $T = 2\pi$ and energy $\hat{E} \approx -2.818548479$. Recall that the $D_4$ symmetry group of $\gamma(s;1)$ is generated by $S_F \gamma(s;1) = \gamma(s + \pi/2;1)$ and $S_G \gamma(s;1) = \gamma(s + \pi/2;1)$. We assume without loss of generality by Lemma 3.1 and Lemma 4.1, that the continued time-reversible periodic regularized simultaneous binary collision orbit $\gamma(s;m)$ for $0 < m < 1$, has period $T = 2\pi$, energy $\hat{E}(m)$, and a $D_4$ symmetry group generated by $S_F \gamma(s;m) = \gamma(s + \pi/2;m)$ and $S_G \gamma(s;m) = \gamma(s + \pi/2;m)$. We assume $\hat{E}(m)$ depends continuously on $0 < m \leq 1$, with $\hat{E}(1) = \hat{E}$. We shift the regularized time variable $s$ to $s + \pi/4$, so that for

$$\gamma(s;m) = (u_1(s;m), u_2(s;m), u_3(s;m), u_4(s;m), v_1(s;m), v_2(s;m), v_3(s;m), v_4(s;m)),$$

the value $s = 0$ now corresponds to the first simultaneous binary collision, i.e., $u_1(0;m) = 0$, $u_2(0;m) = 0$, $u_3(0;m) \neq 0$, $u_4(0;m) \neq 0$, $v_1(0;m) \neq 0$, $v_2(0;m) \neq 0$, $v_3(0;m) = 0$, and $v_4(0;m) = 0$. We approximate the continued time-reversible periodic orbits $\gamma(s;m)$, $0 < m \leq 1$, by the trigonometric polynomials,

$$u_1(s;m) = \sum_{i=1}^{n} a_i \sin((2i - 1)m), \quad u_2(s;m) = -\sum_{i=1}^{n} b_i \sin((2i - 1)m),$$

$$u_3(s;m) = u_1(s - \pi/2;m), \quad u_4(s;m) = u_2(s + \pi/2;m),$$

$$v_1(s;m) = \sum_{i=1}^{n} c_i \cos((2i - 1)m), \quad v_2(s;m) = -\sum_{i=1}^{n} d_i \cos((2i - 1)m),$$

$$v_3(s;m) = v_1(s - \pi/2;m), \quad v_4(s;m) = v_2(s + \pi/2;m),$$

for a positive integer $n$ and constants $a_i$, $b_i$, $c_i$, $d_i$, $i = 1, \ldots, n$, that are assumed to be continuous functions of $m$. The presence of odd positive integer frequencies $2i - 1$ in these trigonometric polynomials is to ensure that the period functions defined by them have the $D_4$ symmetry group generated by $S_F$ and $S_G$. So in particular, the periodic orbits in the continuation are time-reversible for all $0 < m < 1$. An numerical estimate of the periodic solution $\gamma(s;m)$ is found through the variational approach of minimizing the functional

$$L = \int_0^{2\pi} \|\gamma'(s;m) - J\nabla\hat{\Gamma}(\gamma(s;m))\| \, ds.$$
over the space $\mathbb{R}^{4n}$ of coefficients $a_i, b_i, c_i, d_i, i = 1, \ldots, n$, for an appropriate choice of $n$. The use of trigonometric polynomials for numerically approximating periodic solutions is a classic approach (see, for example, Simó [25]).

The numerical algorithm for finding a trigonometric polynomial that approximates the periodic solution $\gamma(s; m), 0 < m < 1,$ proceeds in two steps. First, we consider a guess for the set of values for $a_i, b_i, c_i, d_i, i = 1, \ldots, n$, as well as a guess for $\hat{E}(m)$. Starting with a reasonably low number of terms, $n = 5$, we let a numerical minimization algorithm (in this case, MATLAB’s `fminunc`) find the minimum of $L$ near the starting guess. Then we add an additional non-zero term to each of the trigonometric polynomials, and the minimizing solution from the previous iteration is used as a starting guess for the next iteration. This process continues until we reach $n = 10$. Second, since the Hamiltonian equations with Hamiltonian $\hat{\Gamma}$ requires a specified value of $\hat{E}(m)$ to compute $\gamma(s; m)$, we need to make certain that we are getting a good estimate of $\hat{E}(m)$. We evaluate $\hat{\Gamma}$ at the point $\gamma(\pi/4; m)$, away from simultaneous binary collisions. If this value is not sufficiently close to 0 (within about $5 \times 10^{-10}$ of 0), we adjust $\hat{E}(m)$ using the bisection method in a small interval about the initial guess of $\hat{E}(m)$ until $\hat{\Gamma}(\gamma(\pi/4; m))$ is sufficiently close to 0, re-minimizing the trigonometric polynomial approximation of $\gamma(s; m)$ for each new choice of $\hat{E}(m)$.

This numerical method only works well if we have a good initial guesses for $a_i, b_i, c_i, d_i, i = 1, \ldots, 5$, and $\hat{E}(m)$ for some value of $m$. This we have when $m = 1$. We use our estimate of $\gamma(s; 1)$ and $\hat{E}(1)$ to provide the initial guesses for $a_i, b_i, c_i, d_i, i = 1, \ldots, 5$, and $\hat{E}(m)$ for $m = 0.99$. The numerical algorithm produces a trigonometric polynomial approximation of $\gamma(s; 0.99)$ and an estimate of $\hat{E}(0.99)$, which then provide the initial guesses for $a_i, b_i, c_i, d_i, i = 1, \ldots, 5$, and $\hat{E}(m)$ for

Figure 3. The estimated values of $u_3(0; m), u_4(0; m), v_1(0; m),$ and $v_2(0; m)$ for the nonzero components of $\gamma(0; m)$ for $0 < m \leq 1$. 

[Diagram of the estimated values of $u_3(0; m), u_4(0; m), v_1(0; m),$ and $v_2(0; m)$ for the nonzero components of $\gamma(0; m)$ for $0 < m \leq 1$.]


m = 0.98. We continue decreasing m by 0.01 and using the numerical algorithm until we reach m = 0.01. In Figure 3, we plot the graphs of the numerical estimates of $u_3(0;m)$, $u_4(0;m)$, $v_1(0;m)$, and $v_2(0;m)$. Notice that $v_1(0;m)$ and $v_2(0;m)$ satisfy Equation (6), as is expected from the regularization of the simultaneous binary collisions. In Figure 4, we plot the graph of the numerical estimate of $\hat{E}(m)$.

In Figure 5, we graph of value of $\hat{\Gamma}(\gamma(\pi/4;m))$ for the optimized trigonometric polynomial approximation of $\gamma(s;m)$ over $0 < m \leq 1$.

We use the optimized trigonometric approximations of $\gamma(s;m)$, $0 < m < 1$, to estimate $\gamma(-\pi/4;m)$, which are the conditions corresponding to $s = 0$ before the shift of $s$. For each $0 < m < 1$, the conditions $\gamma(-\pi/4;m)$ satisfy Lemma 2.1, and so they corresponds to the initial conditions of interest as given in the Introduction.

We numerically estimate the monodromy matrices $X(s;m)$ and their eigenvalues for $\gamma(s;m)$ with $0 < m \leq 1$. We used a Runge-Kutta order 4 algorithm coded in
MATLAB with a fixed time step of $2\pi/50000$ at increments of 0.01 in $m$. In Figure 6 we graph the maximum modulus of the eight eigenvalues of $X(s; m)$ over $0.2 < m \leq 1$. The maximum modulus of the eigenvalues of $X(s; m)$ for $0 < m \leq 0.2$ becomes large as $m \to 0$, and are therefore not included in Figure 6. The maximum modulus for the eight eigenvalues shows that $\gamma(s; m)$ is linearly unstable for $0 < m \leq 0.53$. The error in the numerical estimation of the repeated eigenvalue 1 of $X(s; m)$ is large enough to account for the graph in Figure 6 not lying at 1 over $0.54 \leq m \leq 1$. However, for $0.54 \leq m \leq 1$, the other six eigenvalues come in three distinct complex conjugate pairs that are each of modulus one and located away from ±1. This shows that $\gamma(s; m)$ is linearly stable for $0.54 \leq m \leq 1$. 

Acknowledgements. We thank the referees for their valuable comments that improved the paper.

References

[1] Aarseth, S.J., and Zare, K., A Regularization of the Three-Body Problem, Celest. Mech. 10, 185-205 (1974)

[2] Bakker, L.F., Simmons, S.C., and Mancuso, S., Linear Stability Analysis of Symmetric Periodic Simultaneous Binary Collision Orbits in the Planar Pairwise Symmetric Four-Body Problem, submitted to Celest. Mech. Dynam. Astron., and posted on ArXiv.

[3] Bakker, L.F., Ouyang, T., Yan, D., Simmons, S.C., and Roberts, G.E., Linear Stability for Some Symmetric Periodic Simultaneous Binary Collision Orbits in the Four-Body Problem, Celest. Mech. Dynam. Astron. 108, 147-164 (2010)

[4] Chenciner, A., and Montgomery, R., A remarkable periodic solution of the three-body problem in the case of equal masses, Ann. of Math. 152, 881-901 (2000)

[5] Contopoulos, G., Order and Chaos in Dynamical Astronomy, Springer-Verlag, New York, 2002

[6] Hénon, M., Stability of interplay orbits, Cel. Mech., 15, 243-261 (1977)
[7] Hietarinta, J., and Mikkola, S., *Chaos in the one-dimensional gravitational three-body problem*, Chaos 3, 183-203 (1993)
[8] Hu, X., and Sun, S., *Morse Index and Stability of Elliptic Lagrangian Solutions in the Planar 3-body Problem*, Adv. Math. 223, 98-119 (2010)
[9] Hu, X., and Sun, S., *Index Stability of Symmetric periodic Orbits in Hamiltonian Systems with Applications to Figure-Eight Orbit*, Commun. Math. Phys., 290, 737-777 (2009)
[10] Martinez, R., Simó, C., *Simultaneous binary collisions in the planar four-body problem*, Nonlinearity 12, 903-930, (1999)
[11] Meyer, K.R., and Hall, G.R., *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, Springer-Verlag, New York, 1992
[12] Meyer, K.R., and Schmidt, D.S., *Elliptic relative equilibria in the N-body problem*, J. Diff. Eqn. 214, 256-298 (2005)
[13] Moeckel, R., *A Topological Existence Proof for the Schubart Orbits in the Collinear Three-Body Problem*, Dis. Con. Dyn. Syst. Series B, 10, 609-620 (2008)
[14] Moore, C., *Braids in classical dynamics*, Phys. Rev. Lett. 70, 3675-3679 (1993)
[15] Ouyang, T., Simmons, S.C., and Yan, D., *Periodic Solutions with Singularities in Two Dimensions in the n-body Problem*, to appear in Rocky Mountain Journal
[16] Ouyang, T., Xie, Z., *Regularization of Simultaneous Binary Collisions and Solutions with Singularities in the Collinear Four-Body Problem*, Dis. Con. Dyn. Sys. 24, 909-932 (2009)
[17] Roberts, G.E., *Linear Stability of the Elliptic Lagrangian Triangle Solutions in the Three-Body Problem*, J. Diff. Eqn. 202, 191-218 (2002)
[18] Roberts, G.E., *Linear Stability analysis of the figure-eight orbit in the three-body problem*, Ergod. Th. & Dynam. Sys. 27, 1947-1963 (2007)
[19] Saito, M.M. and Tanikawa, K., *The rectilinear three-body problem using symbol sequence I: Role of triple collisions*, Celest. Mech. Dynam. Astron. 98, 95-120 (2007)
[20] Saito, M.M. and Tanikawa, K., *The rectilinear three-body problem using symbol sequence II: Role of periodic orbits*, Celest. Mech. Dynam. Astron. 103, 191-207 (2009)
[21] Saito, M.M., and Tanikawa, K., *Non-schubart periodic orbits in the rectilinear three-body problem*, Celest. Mech. Dynam. Astron. 107, 397-407 (2010)
[22] Sivasankaran, A., Steves, B.A., and Sweatman, W.L., *A global regularisation for integrating the Caledonian symmetric four-body problem*, Celest. Mech. Dynam. Astron. 107, 157-168 (2010)
[23] Sekiguchi, M. and Tanikawa, K., *On the Symmetric Collinear Four-Body Problem*, Publ. Astron. Soc. Japan 56, 235-251 (2004)
[24] Simó, C., *New families of solutions in the N-body problem*, Progress in Mathematics Vol. 20, Birkhäuser, 101-115, 2001.
[25] Schubart, J., *Numerische Aufsuchung periodischer Lösungen im Dreikörperproblem*, Astronomische Nachrichten, 283, 17-22 (1956)
[26] Sweatman, W.L., *Symmetrical one-dimensional four-body problem*, Celest. Mech. Dynam. Astron. 82, 179-201 (2002).
[27] Sweatman, W.L., *A Family of Symmetrical Schubart-Like Interplay Orbits and their Stability in the One-Dimensional Four-Body Problem*, Celest. Mech. Dynam. Astron. 94, 37-65, (2006)
[28] Venturelli, A., *A Variational Proof of the Existence of Von Schubart’s Orbit*, Dis. Con. Dyn. Syst. Series B, 10, 699-717 (2008)