Electric fields at finite temperature

A. D. Bermúdez Manjarres, N. G. Kelkar and M. Nowakowski

Departamento de Física
Universidad de los Andes
Cra. 1E No. 18A-10 Bogotá, Colombia

Abstract

Partial differential equations for the electric potential at finite temperature, taking into account the thermal Euler-Heisenberg contribution to the electromagnetic Lagrangian are derived. This complete temperature dependence introduces quantum corrections to several well known equations such as the Thomas-Fermi and the Poisson-Boltzmann equation. Our unified approach allows at the same time to derive other similar equations which take into account the effect of the surrounding heat bath on electric fields. We vary our approach by considering a neutral plasma as well as the screening caused by electrons only. The effects of changing the statistics from Fermi-Dirac to the Tsallis statistics and including the presence of a magnetic field are also investigated. Some useful applications of the above formalism are presented.

PACS numbers: 11.10.Wx,12.20.Ps,03.50.De,26.20.-f

*Electronic address: ad.bermudez168@uniandes.edu.co
†Electronic address: nkelkar@uniandes.edu.co
‡Electronic address: mnowakos@uniandes.edu.co
I. INTRODUCTION

A class of nonlinear Poisson equations of the form $\nabla^2 \Phi = F(\Phi, T, r)$ (with $F$ a function) which take into account the effects (like the temperature dependence $T$) of the surrounding matter on the electric potential $\Phi$ play an important role in many branches of physics. We mention here the Thomas-Fermi equation which finds its applications in atomic physics [1], astrophysics [2] and solid states physics [3] and Poisson-Boltzmann equation applied in plasma physics [4] and solutions [5]. The derivation of these equations is seemingly unrelated and yet, as shown in this work, they are based on one and the same principle. Feynman, Metropolis and Teller [6] have have postulated a self-consistent Poisson-like equation of the form $\nabla^2 \Phi = \int d^3 p F_D(\Phi, T, p)$, where $F_D$ stands for the Fermi-Dirac distribution, from which the Thomas-Fermi, Poisson-Boltzmann and other similar equations can be derived. We use this unifying principle to calculate quantum corrections to these nonlinear Poisson equations. These corrections arise when we use the Quantum Electrodynamics (QED) at finite temperature to calculate the first quantum corrections to the classical electrodynamics known as the Euler-Heisenberg theory (in our case at finite $T$). The Euler-Heisenberg theory at finite $T$ brings yet another temperature dependence of the electric potential. To be specific, the QED effective Lagrangian in the presence of a thermal bath and arbitrary slowly varying electromagnetic field can be written as

$$\mathcal{L} = \frac{E^2 - B^2}{8\pi} + \mathcal{L}^0_{EH}(\mathbf{E}, \mathbf{B}) + \mathcal{L}^T_{EH}(\mathbf{E}, \mathbf{B}; T),$$

where $\mathcal{L}^0_{EH}(\mathbf{E}, \mathbf{B})$ is the zero temperature effective Lagrangian of QED [7–9] giving rise to new effects like vacuum birefringence [10–12], vacuum dichroism [15], corrections to the Lorentz force [16], corrections to the field and energy of point charges [13, 14] among others (see [17, 18] for comprehensive reviews); and $\mathcal{L}^T_{EH}(\mathbf{E}, \mathbf{B}; T)$ is the contribution from the thermal bath to the effective Lagrangian [19]. The Lagrangian [1] gives rise to modifications of the Maxwell’s equations that can be used to study electromagnetic phenomena that occur beyond the classical electrodynamics [20].

The initial investigation about the finite temperature effective Lagrangian was done by Dittrich [21]. Further developments were made in [22–29]. In particular, Ref [27] was the first one to show that, at temperatures below $m_e$ (mass of the electron), the two loop contributions dominate over the one loop term. A review and an expanded bibliography can be found in [19].
Among the applications of the finite temperature Lagrangian, we can find the study of thermally induced photon splitting\cite{30, 31}, thermally induced pair production \cite{32–34}, and the velocity shift of light in thermalized media \cite{35–37} (see \cite{19} for more references).

Classical (or semi-classical) methods have been developed for the study of matter at laboratory conditions or plasmas in stars, supernovas, and even the electron-positron plasma at an early stage of the big-bang \cite{38}. We refine these methods by including the effects of the QED effective Lagrangian (1). We do so by implementing the effects of the Euler-Heisenberg theory via the modified Gauss law in the Poisson-like equations. The set of the latter encompasses known equations (like the Thomas-Fermi or Poisson-Boltzmann, now equipped with quantum corrections) as well as new equations which will be derived in this work.

The paper is organized as follows. In the next section we present the low temperature and high temperature approximation for the Euler-Heisenberg effective Lagrangian and we will discuss the way of incorporating the effective Lagrangian into the equations of classical electrodynamics.

In section III we calculate the correction to the electrostatic potential of point-like and extended charged objects when the charge density is given. An explicit solution of an electric field at finite temperature due to the Euler-Heisenberg theory is given. This solution neglects the fact that the particles surrounding the charge whose potential we wish to calculate can also be in a heat bath. However, the solution is part of a more general treatment where it appears in the boundary conditions.

In section IV we focus on the temperature dependent charge densities. We shall write the charge density with two separated terms as $\rho = \rho_c + \rho_m$ where $\rho_c$ is the density of the object whose effective electrostatic potential we want to compute and $\rho_m$ is the charge density of the surrounding media. This results in the Feynman-Metropolis-Teller equation. We discuss several limiting cases of this master equation treating the degenerate and non-degenerate cases and carefully distinguishing between the relativistic and non-relativistic situation and the high and low temperature cases. Taking into account the corrections from the Euler-Heisenberg theory we derive several nonlinear Poisson-like equations at finite temperature.

Section V is devoted to the “relatives” of the Thomas-Fermi equation, namely equations derived under a change of assumptions. In the first case we change the Fermi-Dirac distribution for the Tsallis statistics and the second case considers a Thomas-Fermi equation in
the presence of a magnetic field.

In the section VI we discuss two possible applications, one connected with tunneling in
the presence of a surrounding heat bath and the second one treating an electron-positron
neutral plasma.

In the last section we draw our conclusions.

II. EULER HEISENBERG LAGRANGIAN FOR LOW AND HIGH TEMPERATURE

The full expression of both the zero temperature and the thermal Euler-Heisenberg La-
grangian is very complex. In this work we shall concentrate on some special cases where the
effective Lagrangian can be approximated by more manageable expressions. First, we shall
deal only with electromagnetic fields that are weak compared to the so called critical field
\( B_c = \frac{e^2}{m_e} \). Secondly, all the fields are considered to be slowly varying compared to the
scales of the problem, i.e., the fields obey \( |\partial_a F_{\mu\nu}| / |F_{\mu\nu}|^2 \ll |2F_{\mu\nu}|^{1/2}, m_e, T \) and \( \eta^{1/3} \), where
\( \eta \) is the particle density. Finally, for the thermal Lagrangian, we will only consider the two
limiting cases of temperature much bigger or much smaller than the electron mass.

With the above restrictions the zero temperature Lagrangian can be written as \[7–9\],
\[
L_{EH}^0 = a \left( 4F^2 + 7G^2 \right),
\]
(2)

where
\[
a = \frac{e^4}{360\pi^2 m_e^4},
\]
(3)

and the two relativistic invariants of the electromagnetic fields are given by
\[
F = -\frac{E^2 - B^2}{2},
\]
(4)
\[
G = E \cdot B.
\]
(5)

As mentioned in the introduction, for temperatures below the electron mass \( T \ll m_e \),
the dominant contribution in the thermal Lagrangian comes from the two loop term \[27\]. To
quadratic order in the field invariants, the weak field expansion for the two loop Lagrangian is \[27\],
\[ \mathcal{L}_{EH}^{T}(T \ll m_e) = b (\mathcal{F} + \mathcal{E}) - c \mathcal{F} (\mathcal{F} + \mathcal{E}) + k (2\mathcal{F}^2 + 6\mathcal{F} \mathcal{E} + 3\mathcal{E}^2 - G^2). \] (6)

The coefficients appearing in (6) are

\[ b = \frac{44\alpha^2 \pi^2 T^4}{2025 m_e^4}, \] (7)

\[ c = \frac{2^6 \times 37 \alpha^3 \pi^3 T^4}{3^4 \times 5^2 \times 7 m_e^8}, \] (8)

\[ k = \frac{2^{13} \alpha^3 \pi^5 T^6}{3^6 \times 5 \times 7^2 m_e^{10}}, \] (9)

and \( \mathcal{E} \) is a term involving the relative velocity of the thermal bath. We will work only in the reference frame where the bath is at rest, and in that special frame we have \( \mathcal{E} = E^2 \).

In the high temperature limit \( (T \gg m_e) \) the one loop correction is the dominating term and the thermal correction takes the form \[ \mathcal{L}_{EH}^{T}(T \gg m_e) = -\frac{2\alpha}{3\pi} \mathcal{F} \ln \left(\frac{T}{m_e}\right) + \frac{\alpha}{6\pi} \mathcal{E} - \mathcal{L}_{EH}^0. \] (10)

It can be seen from the above equation (10) that, for temperatures above the electron mass, the thermal bath cancels the vacuum polarization effects from the zero temperature Euler-Heisenberg Lagrangian.

In this paper, the interest in the effective Lagrangian comes from the fact that it can be related to a modification of the Maxwell’s equations at the purely classical level.

Faraday’s and magnetic Gauss’s laws remain unchanged

\[ \nabla \cdot \mathbf{B} = 0, \] (11)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \] (12)

We see from (11) and (12) that electromagnetic potentials are still defined as in classical electrodynamics i.e, \( \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \) and \( \mathbf{B} = \nabla \times \mathbf{A} \).

The electric Gauss’s and Ampere-Maxwell’s law are modified by the use of the effective Lagrangian. They now resembles the the form of the Maxwell’s equation in matter \[ 39 \], namely,

\[ \nabla \cdot \mathbf{D} = 4\pi \rho, \] (13)

\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \] (14)
where $\rho$ is the charge density and the auxiliary fields $D$ and $H$ are given by,

\begin{align}
D &= E + 4\pi \frac{\partial L_{EH}}{\partial E}, \\
H &= B - 4\pi \frac{\partial L_{EH}}{\partial B}.
\end{align}

(15)  

(16)

In classical electrodynamics the Gauss law $\nabla \cdot E = \rho$ or Laplace equation $-\nabla^2 \phi = \rho$ is solved in order to find the field created by a given charge density distribution. Here we shall tackle the problem of finding the effective electric field of a spherical charge distribution produced by the modified Gauss Law \[13\].

III. TEMPERATURE INDEPENDENT CHARGE DENSITY

To begin with, we shall derive the fields in the limit of low and high temperatures for the case of temperature independent charge densities.

A. Low temperature

For pure electric field ($B = 0$) and in the plasma rest frame, the effective Lagrangian takes the form

\[ L_{Maxwell} + L_{EH}^0 + L_{EH}^T = \left( \frac{1}{8\pi} + \frac{b}{2} \right) E^2 + (a + \frac{k}{2} - \frac{c}{4}) E^4. \]

(17)

With this Lagrangian the Gauss law reads

\[ \nabla \cdot (A(T)E^2E + B(T)E) = 4\pi \rho(r) \]

(18)

with

\[ A(T) = 16\pi \left( a + \frac{k}{2} - \frac{c}{4} \right), \]

(19)

\[ B(T) = 1 + 4\pi b. \]

(20)

The rest of this sections follows closely the works \[13, 14\]. In general, The charge distribution can be written as

\[ 4\pi \rho(r) = \nabla \cdot E_c \]

(21)

where $E_c$ is the field that would be produced by $\rho$ in Maxwell's theory.
The field \( E \) given by \((18)\) has to approach \( E_c \) in the limit that the Euler-Heisenberg coefficients vanish. We can then write from \((18)\) and \((21)\) the following algebraic equation

\[
A(T)E^3 + B(T)E = E_c. \tag{22}
\]

This equation is a cubic equation which has only one real solution and is given by Cardano’s formula,

\[
E = \sqrt[3]{\frac{E_c}{A(T)}} + \sqrt[3]{\frac{E_c^2}{A^2(T)} + \frac{B^3(T)}{A^3(T)}} + \sqrt[3]{\frac{E_c}{A(T)} - \sqrt{\frac{E_c^2}{A^2(T)} + \frac{B^3(T)}{A^3(T)}}}. \tag{23}
\]

Let’s note that at large distances, the behaviour of \( E \) is

\[
E \sim \frac{e}{B(T)r^2} \tag{24}
\]

while for short distances it is

\[
E \sim \left[ \frac{e}{A(T)r^2} \right]^{1/3}. \tag{25}
\]

In particular, we will later need the form of the potential at short distances. The behaviour of the potential for small \( r \) follows from \((25)\) to be

\[
\phi \sim -\frac{1}{3} \left[ \frac{e}{A(T)} \right]^{1/3} r^{1/3} + \phi(0) \tag{26}
\]

where \( \phi(0) \) is a positive constant given by

\[
\phi(0) = -\int_0^\infty E dr. \tag{27}
\]

Equation \((23)\) would be a complete result if we could neglect screening effects and neglect the temperature dependence of the surrounding matter. Nevertheless the result \((23)\) is important for later, more exhaustive considerations when we will take into account the temperature dependence of the matter. Then the boundary condition of the resulting partial differential equations will be formulated with the help of \((23)\), i.e., at small distances the potential should follow \((26)\).

B. High Temperature

The high temperature case is mathematically easier.

\[
\nabla \cdot E_c = \rho(r) = \nabla \cdot D = \nabla \cdot (E + 4\pi P) = \left[ \frac{8\alpha}{3} \ln \left( \frac{T}{m_e} \right) + \frac{4\alpha}{3} + 1 \right] E
\]
The solution for $E$ is trivial if $E_c$ is known and is given by

$$E = \frac{1}{\frac{8\alpha}{3} \ln \left( \frac{T}{m_e} \right) + \frac{4\alpha}{3} + 1} E_c$$

IV. TEMPERATURE DEPENDENT CHARGE DENSITY

We will work out the effective electric field of a point charge that is submerged in a neutral plasma consisting of electrons and positive charges which can consist of protons or positrons.

As is customary, statistical methods are needed when the charge density depends on temperature. The density of fermions is governed by the Fermi-Dirac distribution and is given by

$$\eta(r, T) = 4(2\pi)^4 \int_0^\infty \frac{(p^2/h^3)}{e^{\beta(K + q\phi + \mu)} + 1} dp,$$

where $\mu$ stands for the chemical potential, $\hbar$ denotes the Planck’s constant, we use $c = 1$ and $K$ is the kinetic energy

$$K = \begin{cases} \frac{p^2}{2m} & \text{non-relativistic}, \\ \sqrt{p^2 + m^2} & \text{relativistic}, \\ p & \text{ultra-relativistic}. \end{cases}$$

The electron charge density is then given by $-e\eta_e(r, T)$ and an analogous analysis holds for the proton or positron charge density.

The equation for the effective field at $r$ created by a charge distribution is the modified Poisson’s equation

$$\nabla \cdot \left( E + 4\pi \frac{\partial L_{EH}}{\partial E} \right) = \rho(r, T)_{\text{total}},$$

where the term $\rho(r, T)_{\text{total}}$ means the density taking into account all the electric charges and it depends on the system considered. For example, consider the densities given by

$$\begin{align*}
\rho(r, T)_{\text{total}} &= -\rho_e(r, T), \\
\rho(r, T)_{\text{total}} &= \rho(r) - e\eta_e(r, T), \\
\rho(r, T)_{\text{total}} &= \rho(r) + 4\pi e\eta_p(r, T) - 4\pi e\eta_e(r, T).
\end{align*}$$

The above densities stand respectively for (a) a cloud of electron, (b) a given particle with charge density $\rho(r)$ that is surrounded by a cloud of electrons and (c) a charge $\rho(r)$
surrounded by a cloud of electrons and protons. We always consider the charges to have spherical symmetry. Neglecting the Euler-Heisenberg contribution in (30) and introducing an electric potential $\phi$, the resulting equation

$$\nabla^2 \phi = \rho(r, T)_{\text{total}},$$

is referred to as the Feynman-Metropolis-Teller (FMT) equation.

Using equation (30) may prove to be too difficult as stated. It is convenient to look for special cases where the density (28) can be reduced to a simpler expression. The way to simplify (28) depends on the relation between the temperature and the chemical potential in a given situation.

At this point it is important to make some clarifications about what we mean by the temperature regime. The problem at hand has two natural scales of temperatures and they enter in different ways in the modified Poisson’s equation (30). First, we can compare $T$ with $m_e$, and as we have seen this affects the form of the effective Lagrangian e.g, for $T \gg m_e$ we use (10) in the left hand side of (30) and for $T \ll m_e$ we use the Lagrangian (6) instead. On the other hand, for the right hand side of (28) the temperature has to be compared with the chemical potential $\mu$ in order to know what simplification can be done to the charge density. We have a degenerate system when $T \ll \mu$ and a dilute system when $T \gg \mu$, both with different approximate expansions (the form of the kinetic energy also affects the approximation of (28)).

In the appendix we summarize different ways to approximate equation (28).

A. Low temperature

We deal first with the systems whose temperatures are below $m_e$ and are composed of electrons and protons at equilibrium. The equation of interest is

$$\nabla \cdot (A(T)E^2E + B(T)E) = 4\pi \rho(r, T)_{\text{total}},$$

Since the charge density is related to the electric potential, it is better to work with the potential instead of the electric field. To accomplish the above we first mention that the usual relation between the electric field and the potential is maintained i.e, $E = -\nabla \phi$. Then, it is just a matter of replacing the potential for electric field into (35). Taking into
account the spherical symmetry of the problem, the differential equation for the potential of a charge distribution can be written as

\[-(\nabla^2 + \mathcal{D}_{EH}^2) \phi = 4\pi \rho(r, T)_{\text{total}},\]  

(36)

where the non-linear differential operator \(\mathcal{D}_{EH}^2\) stands for

\[\mathcal{D}_{EH}^2 = A(T) \frac{1}{r^2} \left( \frac{d}{dr} \right)^2 \left( r^2 \frac{d}{dr} \right) + A(T) \frac{d^2}{dr^2} + 4\pi b \nabla^2 \Phi,\]  

(37)

with \(A(T)\) and \(B(T)\) given by (19) and (20). Let’s note that \(-\mathcal{D}^2 \to 0\) when \(\alpha \to 0\).

1. Degenerate matter

Before we embark on the derivation of the Thomas-Fermi like differential equations we note that for astrophysical applications a rough distinction between degenerate and non-degenerate matter can be made according to the mass density \(\rho\). For non-degenerate matter we should have

\[\rho \leq 8.49 \times 10^{-17} \left( \frac{T^3}{1K} \right) \frac{\text{g}}{\text{cm}^3},\]  

(38)

whereas the degenerate matter should satisfy

\[\rho \geq 4.2 \times 10^{-10} \left( \frac{T^3}{1K} \right) \frac{\text{g}}{\text{cm}^3}.\]  

(39)

We now present the differential equations for the potentials in the case where the temperature is low compared to the chemical potential.

For the degenerate case we can use the particle density (109) (given in the appendix) to write the non-relativistic equation as

\[-(\nabla^2 + \mathcal{D}_{EH}^2) \phi = \rho(r)\]

\[+ \frac{4(2\pi)^5 e^2 (2m_e T)^{3/2}}{\hbar^3} \left( \frac{2}{3} \left( \frac{\mu_p - e\phi}{T} \right)^{3/2} + \frac{\pi^2}{12} \left( \frac{\mu_p - e\phi}{T} \right)^{-1/2} \right),\]

\[+ \frac{4(2\pi)^5 e^2 (2m_e T)^{3/2}}{\hbar^3} \left( \frac{2}{3} \left( \frac{\mu_e + e\phi}{T} \right)^{3/2} + \frac{\pi^2}{12} \left( \frac{\mu_e + e\phi}{T} \right)^{-1/2} \right).\]  

(40)

Using (110) (from the appendix) the ultra-relativistic equation is

\[-(\nabla^2 + \mathcal{D}_{EH}^2) \phi = \rho(r)\]

\[+ \frac{8(2\pi)^5 e}{\hbar^3} \left( \mu_p - e\phi \right)^3 \left( 1 + \frac{3}{4} \pi^2 \left( \frac{T}{\mu_p - e\phi} \right)^2 \right),\]

\[+ \frac{8(2\pi)^5 e}{\hbar^3} \left( \mu_e + e\phi \right)^3 \left( 1 + \frac{3}{4} \pi^2 \left( \frac{T}{e\phi + \mu_e} \right)^2 \right).\]  

(41)
Equations (40) and (41) are the low T Euler-Heisenberg generalization to the non-relativistic and ultra-relativistic Thomas-Fermi equations with the first thermal correction and a contribution from positive charges. As already mentioned in section III we can drop the source $\rho$ from these equations (and the other equations derived below) by incorporating it into the boundary conditions. This is to say, we demand that at small distances the potential behaves as (26) which amounts to saying that the effect of matter is negligible. In the standard Thomas-Fermi-like equations for spherical charge distributions and without the Euler-Heisenberg corrections this is equivalent to demanding that the Coulomb law be valid at short distances.

For the sake of comparison we now present the standard Thomas-Fermi equations for the non-relativistic and the ultra-relativistic cases. Considering a point charge $Z_1 e$ surrounded by a cloud of electrons, the non-relativistic Thomas-Fermi equation with the first thermal term is [41]

$$\nabla^2 \phi = \frac{8e(2\pi)^5(2m_e)^{3/2}}{3\hbar^3} \left(\mu_e + e\phi\right)^{3/2} \left(1 + \frac{\pi^2}{8} \left(\frac{\mu_e + e\phi}{T}\right)^{-2}\right). \tag{42}$$

The ultra-relativistic Thomas-Fermi equation reads

$$\nabla^2 \phi = \frac{4(2\pi)^4 e}{\hbar^3} \left(\mu + e\phi\right)^3 \left(1 + \frac{3}{4} \pi^2 \left(\frac{T}{\mu + e\phi}\right)^2\right). \tag{43}$$

It is possible to rewrite the Thomas-Fermi equation (42) using the following change of variables

$$\mu_e + e\phi = \frac{Z_1 e^2}{r}\psi(s), \tag{44}$$

$$r = Cs, \tag{45}$$

$$C = \frac{1}{2} \left(\frac{3}{8(2\pi)^5}\right)^{2/3} Z_1^{-1/3} a_0, \tag{46}$$

where $a_0 = \hbar^2/e^2 m_e$ is the Bohr radius and $s$ is a dimensionless quantity.

After using (44), (45) and (46), the Thomas-Fermi equation (42) becomes

$$\frac{d^2\psi(s)}{ds^2} = \psi(s)^{3/2} \left(1 + \frac{\gamma T^2 s^2}{\psi^2(s)}\right), \tag{47}$$

where $\gamma = \frac{s^2}{8} \left(\frac{3}{8(2\pi)^5}\right)^{2/3} Z_1^{-1/3} a_0^2 / c^4$. 

11
Another customary way to rewrite equation (42) is by using the change of variables [6],

\[
\frac{\mu_e + e\phi}{T} = \frac{\psi}{s},
\]

\[
C' = \sqrt{\frac{\hbar^3}{8e^2(2\pi)^5(2m_eT)^{1/2}}}. 
\]

In this case \( \psi \) obeys

\[
\frac{d^2\psi(s)}{ds^2} = \frac{\psi(s)^{3/2}}{s} \left( 1 + \frac{\pi^2s^2}{8\psi(s)^2} \right). 
\]

With the definitions (44), (45), and (46), we can write the low T Euler-Heisenberg generalization to the Thomas-Fermi equation (47) in the form

\[
\frac{d^2\psi(s)}{ds^2} + \frac{1}{C^4} \mathcal{D}_{EH}^2 \psi(s) = \frac{\psi(s)^{3/2}}{\sqrt{s}} \left( 1 + \frac{\gamma T^2 s^2}{\psi(s)^2} \right), 
\]

where \( \mathcal{D}_{EH}^2 \) in terms of \( \psi \) and \( s \) reads

\[
\mathcal{D}_{EH}^2 = A(T) \left[ \frac{1}{s} \frac{d}{ds} - \frac{\bullet}{s^2} \right]^2 
\times \left[ 2\frac{d^2}{ds^2} - \frac{2}{s} \frac{d}{ds} + 2 \frac{\bullet}{s^2} \right].
\]

The equation (40) that considers both a cloud of negative and positive charges can also be rewritten using the change of variables (44) and (45), with the result

\[
\frac{d^2\psi(s)}{ds^2} + \frac{1}{C^4} \mathcal{D}_{EH}^2 \psi(s) = -s\rho(Cs) 
\times \left( \frac{e^2\psi(s)}{Cs} + \mu_+ - \mu_e \right)^{3/2} \left( 1 + \frac{\pi^2T^2 s^2}{8} \left( \frac{e^2\psi(s)}{Cs} + \mu_+ - \mu_e \right)^{-2} \right). 
\]

We can see that the whole effect of the low T Euler-Heisenberg is contained in the term \( \frac{1}{C^4} \mathcal{D}_{EH}^2 \psi \).

A similar treatment can be given for the ultra-relativistic equations. The change of variables is the same as (44) and (45) but with \( C = \sqrt{\frac{\hbar^3}{4(2\pi)^4 e}} \) (though in this case \( s \) is not dimensionless). With this change of variables equation (43) becomes

\[
\frac{d^2\psi(s)}{ds^2} = \frac{\psi(s)^3}{s^2} \left( 1 + \frac{3\pi^2T^2 s^2}{4\psi(s)^2} \right). 
\]
The Euler-Heisenberg generalization to (54) now reads
\[ \frac{d^2 \psi(s)}{ds^2} + \frac{1}{C^4} \mathcal{D}^2_{EH} \psi(s) = \frac{\psi(s)^3}{s^2} \left( 1 + \frac{3\pi^2 T^2 s^2}{4\psi(s)^2} \right). \] (55)

With the inclusion of the test charge and a cloud of positive charges, the equation (41) becomes
\[ \frac{d^2 \psi(s)}{ds^2} + \frac{1}{C^4} \mathcal{D}^2_{EH} \psi(s) = -s \rho(Cs) + \frac{\psi(s)^3}{s^2} \left( 1 + \frac{3\pi^2}{4} \frac{s^2}{\psi(s)^2} \right) - \frac{1}{s^2} \left( \frac{e^2 \psi(s)}{s} + \mu_+ - \mu_- \right)^3 \left( 1 + \frac{3}{4\pi^2 T^2} \left( \frac{e^2 \psi(s)}{s} + \mu_+ - \mu_- \right)^2 \right). \] (56)

In the standard Thomas-Fermi theory, the function \( \psi(r) \) has to obey the following boundary conditions
\[ \psi(0) = 1, \] (57)
\[ \psi(\infty) = 0, \] (58)
where we have ignored the size of the charged object. Condition (57) ensures that we recover Coulomb electrostatic energy at short distances. Condition (58) ensures the right behaviour at large distances. However, for the Euler-Heisenberg generalization of Thomas-Fermi equations, the potential has to reduce to its Euler-Heisenberg form (26) at short distances. Therefore, for small \( r \), \( \psi \) has to behave like
\[ \psi(s) \sim -\frac{1}{3e} \left[ \frac{e}{A(T)} \right]^{1/3} s^{7/3} + \phi(0) \frac{s^2}{e}. \] (59)

2. Dilute matter

For dilute matter, the Maxwell’s-Boltzmann distribution (107) given in the appendix, can be used to write the density of particles:
\[ n_e = n_{e0} e^{\frac{\phi}{T}}, \] (60)
\[ n_p = n_{p0} e^{-\frac{\phi}{T}}, \] (61)
where \( n_{e0} \) and \( n_{i0} \) are the concentrations of electrons and protons respectively.

Assuming the concentration of negative and positive charges to be equal to \( n_0 \), we can write the equation
\[ - (\nabla^2 + \mathcal{D}^2_{EH}) \phi = \rho(r) - en_0 e^{\frac{\phi}{T}} + en_0 e^{-\frac{\phi}{T}}. \] (62)
Equation (62) is a generalization of the Poisson-Boltzmann equation.

For regions where the perturbed potential obeys \( \phi \ll T \), the exponentials in (62) can be expanded and keeping only the first term, we get,

\[
-\left( \nabla^2 + \mathcal{D}_{EH}^2 \right) \phi + \kappa^2 \phi = \rho(r)
\]

(63)

with, \( \kappa \), the Debye parameter, given by

\[
\kappa^2 = \frac{2e^2n_0}{T}.
\]

(64)

Equation (63) is a generalization of the linearized version of the Poisson-Boltzmann or Debye-Hückel equation [42]. Without considering the Euler-Heisenberg term, the Debye-Hückel equation is

\[
- \nabla^2 \phi + \kappa^2 \phi = \rho(r).
\]

(65)

For for point charges \( \rho(r) = q\delta(r) \), equation (65) has the solution

\[
\phi = \frac{q}{r} e^{-\kappa r}.
\]

(66)

The screening effect is evident.

**B. High Temperature**

1. **Dilute matter**

In the high temperature regime the particles move ultra-relativistically with kinetic energy \( E = p \) and for the non-degenerate case, charge densities are given by the distribution (107). We assume the electric interaction between charges to be small as compared to the temperature so that we can write the differential equation for the screened potential for a point-charge as

\[
\left[ 8\alpha \ln \left( \frac{T}{m_e} \right) + \frac{4\alpha}{3} + 1 \right] \nabla^2 \phi = q\delta(r) + n_0 e^{-\frac{e\phi}{T}} - n_0 e^{\frac{e\phi}{T}}
\]

\[
\approx q\delta(r) - \kappa^2 \phi.
\]

(67)

(68)

Equation (68) can easily be rewritten as

\[
- \nabla^2 \phi + \kappa_{EH}^2 \phi = q_{EH}(T)\delta(r),
\]

(69)
where we have defined

\[ \kappa_{EH}^2 = \frac{\kappa^2}{\frac{8\alpha}{3} \ln \left( \frac{T}{m_e} \right) + \frac{4\alpha}{3} + 1}, \]  
(70)

\[ q_{EH}(T) = \frac{q}{\frac{8\alpha}{3} \ln \left( \frac{T}{m_e} \right) + \frac{4\alpha}{3} + 1}. \]  
(71)

We see that for dilute gases at temperatures above the electron mass, the effects of the Euler-Heisenberg Lagrangian are the renormalization of the Debye parameter (70) and the electric charge (71).

The solution of (69) is given by [42]

\[ \phi = \frac{q_{EH}(T)}{r} e^{-\kappa_{EH} \tau}. \]  
(72)

This represents an analytical solution of a Poisson-Boltzmann problem with Euler-Heisenberg corrections.

V. RELATED EQUATIONS

Having discussed the Thomas-Fermi and other equations at different temperatures and densities, we now consider the variants for different statistics and the equations obtained in the presence of magnetic fields.

A. Tsallis Statistics

Tsallis statistics have been with us for about 30 years now [43]. It has been applied to physical situations like Euler turbulence [44], gravitating systems [45], ferrofluid-like systems [46] and neutron stars [47], among others. Recently, it has been suggested that the Tsallis statistics could eventually explain the Lithium anomaly of early nucleosynthesis [48].

In the Tsallis statistics for Fermi particles the occupation number is given by

\[ \eta = 4(2\pi)^4 \int_0^\infty \frac{(p^2/\hbar^2) dp}{e_q(\beta (K + q\phi - \mu) + 1)} \]  
(73)

where \( q \) is a real number and

\[ e_q(x) = \left[ 1 + (1 - q)x \right]^{1/q}. \]  
(74)
is a generalization of the standard exponential function, which is recovered in the limit $q \to 1$.

Density (73) has been used in literature to form a non-extensive generalization of the Thomas-Fermi equations; in the nonrelatistic case by [49], and in the relativistic case by [50].

The relativistic Poisson equation reads

$$\nabla^2 \phi = \frac{e m_e^3}{3 \pi^2 h^2} \left[ \frac{(\mu + m_e + \phi)}{m_e^2} - 1 \right]^{3/2} \times \left\{ 1 + \frac{3 T I_1^{(q)}}{m_e} \left[ \frac{(\mu + m_e + \phi)}{m_e^2} - 1 \right]^{-1} + \frac{3 T I_2^{(q)}}{m_e} \left[ \frac{(\mu + m_e + \phi)}{m_e^2} - 1 \right]^{-2} + \ldots \right\},$$

(75)

where the q-generalized Fermi-Dirac integral $I_k^{(q)}$ is defined by

$$I_k^{(q)} = q \int_{-\infty}^{\infty} \frac{z^n}{1 + [1 + (q - 1) z]^{1/(q-1)}} dz.$$  

(76)

Numerical evaluation of (76) for different q can be found in [49, 51].

With the following change of variables

$$\mu_e + e \phi = \frac{Z_1 e^2}{r} \psi(s),$$

(77)

$$r = Cs,$$  

(78)

$$C = \left( \frac{9 \pi^2}{128} \right)^{1/3} Z_1^{-1/3} a_0,$$  

(79)

equation (75) transform into the non-extensive relativistic generalization of the Thomas-Fermi equation

$$\frac{d^2 \psi}{ds^2} = \frac{\psi^{3/2}}{\sqrt{s}} \left( 1 + \gamma T^2 s^2 \right) \times \left\{ 1 + \chi_1 \frac{T s}{\psi} \left[ 1 + \gamma \frac{s}{\psi} \right]^{-1} + \chi_2 \frac{T^2 s^2}{\psi^2} \left[ 1 + \gamma \frac{s}{\psi} \right]^{-2} + \ldots \right\},$$

(80)

where

$$\gamma = \left[ \frac{4 Z_1^2 \gamma}{3 \pi} \right]^{2/3} \frac{e^4}{h^2}, \quad \chi_1 = \frac{3 C}{2 e^2 Z_1} I_1^{(q)}, \quad \chi_2 = \frac{3 C^2}{8 e^4 Z_1^2} I_2^{(q)}.$$  

(81)

In the limit where the relativistic contribution is neglected ($\gamma \to 0$), equation (80) reduces to the following non-relativistic expression (originally obtained in [49])

$$\frac{d^2 \psi}{ds^2} = \frac{\psi^{3/2}}{\sqrt{s}} \left[ 1 + \chi_1 \frac{T s}{\psi} + \chi_2 \frac{T^2 s^2}{\psi^2} \right].$$

(82)
B. Thomas-Fermi equations in presence of magnetic fields

In the context of nuclear astrophysics, there are cases where the process of interest occurs in presence of magnetic fields. When the magnetic field is intense enough, the quantum nature of the motion of the charged particle can not be ignored.

The first investigation of the modification of the Thomas-Fermi equation due to a magnetic field was done in [52]. Further developments were done in [53–55]. We follow the procedure of [56], where the discretization of the transverse motion into Landau levels is taken into account. The motion of electrons perpendicular to the magnetic field is quantized into the discrete Landau Levels $\nu B$, with $\nu = 0, 1, 2, \ldots$. The degeneracy of the levels, per unit area, is $\frac{B}{2\pi}$ for $\nu = 0$, but, due to the electron spin, the degeneracy is twice as high for the higher $\nu$. Along the direction of the field the motion is not quantized, and the degeneracy of states is $D(\varepsilon) = \varepsilon^{-1/2}/(2^{1/2}\pi)$, where $\varepsilon$ is the energy of the translational motion.

Taking the above into consideration, it follows that the density of electrons at temperature $T$ and electrical potential $-e\phi$ is given by

$$
\eta = \frac{B}{2\pi} \frac{1}{2^{1/2}\pi} \int_0^{\infty} \left[ \int_0^{\infty} \frac{\varepsilon^{-1/2}}{e^{(\varepsilon-\mu-e\phi)/T} + 1} d\varepsilon + 2 \sum_{\nu=1}^{\infty} \int_0^{\infty} \frac{\varepsilon^{-1/2}}{e^{(\varepsilon+\nu B-\mu-e\phi)/T} + 1} d\varepsilon \right]
$$

$$
= \frac{BT^{1/2}}{2^{3/2}\pi^2} \left[ I_{-1/2} \left( \frac{\mu + e\phi}{T} \right) + 2 \sum_{\nu=1}^{\infty} I_{-1/2} \left( \frac{\mu + e\phi - \nu B}{T} \right) \right] \tag{83}
$$

where the Fermi-Dirac integral for $k > -1$ is defined by

$$
I_k(x) = \int_0^{\infty} \frac{y^k}{e^{y-x} + 1} dy. \tag{84}
$$

Combining (83) with the Poisson’s equation yields

$$
\nabla^2 \phi = 4 \frac{BT^{1/2}}{2^{3/2}\pi} \left[ I_{-1/2} \left( \frac{\mu + e\phi}{T} \right) + 2 \sum_{\nu=1}^{\infty} I_{-1/2} \left( \frac{\mu + e\phi - \nu B}{T} \right) \right]. \tag{85}
$$

In the case where only the lowest Landau level is taken into account, equation (78) reduces to the one that can be found in [53, 55], namely,

$$
\nabla^2 \phi = 4 \frac{BT^{1/2}}{2^{3/2}\pi} I_{-1/2} \left( \frac{\mu + e\phi}{T} \right). \tag{86}
$$
Using the relation $\frac{d}{dx} I_k(x) = k I_{k-1}(x)$ we can obtain a low $T$ expression for (86). Indeed, for low $T$ we can write

$$I_{-1/2}(x) = 2 \frac{d}{dx} I_{1/2}(x) \approx 2 \frac{d}{dx} \left[ \frac{2}{3} x^{3/2} \left\{ 1 + \frac{3}{8x^2} \right\} \right]$$

$$= 2x^{1/2} \left\{ 1 + \frac{3}{8x^2} \right\} - \frac{3}{2x^{3/2}}. \quad (87)$$

Then, at low $T$, equation (86) can be expanded as

$$\nabla^2 \phi = 4 \frac{B}{2^{3/2} \pi} \left[ 2(\mu + e\phi)^{1/2} \left\{ 1 + \frac{3T^2}{8(\mu + e\phi)^2} \right\} - \frac{3T^2}{2(\mu + e\phi)^{3/2}} \right]. \quad (88)$$

To calculate the low-T Euler-Heisenberg correction to equation (86) we have to consider the Lagrangian (6), this time taking into account a magnetic term of the form $B = \hat{B} \hat{k}$ in the electromagnetic invariants (4) and (5). With the magnetic terms included, the Gauss’s law now reads

$$\nabla \cdot (A(T)E^2E + B(T)E - E_z B^2 \hat{k}) = 4\pi \rho, \quad (89)$$

where

$$B(T) = B(T) + 4\pi (4k - 6c)B^2. \quad (90)$$

From the form of the Gauss’s law (89), the modified Poisson’s equation

$$\nabla^2 \phi + \mathcal{D}^2_{EH-B} \phi = 4 \frac{BT^{1/2}}{2^{3/2} \pi} I_{-1/2} \left( \frac{\mu + e\phi}{T} \right), \quad (91)$$

with

$$\mathcal{D}^2_{EH-B} \bullet = A(T)(\nabla \bullet)^2 \nabla^2 \bullet + 2A(T) (\nabla \bullet) \cdot [(\nabla \bullet) \cdot \nabla] \nabla \bullet + 4\pi (b+4kB^2 - 6cB^2) \nabla^2 \bullet + kB^2 \frac{d^2 \bullet}{dz^2}. \quad (92)$$

We can see that, due to existence of the magnetic field, the operator $\mathcal{D}^2_{EH-B}$ is not spherically symmetric.

In the procedure above there is a subtlety that we have to mention. When substituting into the electromagnetic invariants we have considered the magnetic field to be of the form $B = B\hat{k}$. However, an external magnetic field can induce the electric charges to produce a magnetic field of their own \[57, 58\]. So, in reality, the Gauss’s law has to take into account this induced field as well. However, we have ignored the induced field since it will be much smaller than the original external one.
VI. APPLICATIONS

In this section we remind the reader of some applications. We will explicitly examine the details of an electric potential in a neutral electron-positron plasma under conditions encountered in the beginning of the universe. Secondly, we will recall how screening of charges affects the alpha decay.

A. Ultra relativistic degenerate electron-positron gas

The electron-positron plasma at an early stage of the Big-Bang presents a situation where the thermal Euler-Heisenberg Lagrangian might prove of great relevance. It is believed that the early pre-stellar period of the evolution of the Universe was dominated by electrons and positrons having ultra relativistic temperatures [59]. In the time between $10^{-6}s$ and $10s$ after the big bang, the universe reached temperatures between $10^9K$ and $10^{13}K$ and was composed mainly of electrons, positrons, and photons in thermodynamic equilibrium. Furthermore, statistical mechanics states that for an electron-positron plasma in an electrostatic field which is in equilibrium, the chemical potential of the positrons and electrons must be the same in magnitude at all points [60, 61].

Furthermore, in thermodynamic equilibrium the mean particle numbers will change via the creation and annihilation processes, therefore the total density $\eta_- - \eta_+$ will remain a constant. The total charge density was calculated in [38] and can be written as

$$e\eta_- - e\eta_+ = \frac{(e\phi + \mu)}{3\hbar^3} \left[ T^2 + \frac{(e\phi + \mu)^2}{\pi^2} \right].$$

(93)

With the charge density (93) and the Euler-Heisenberg contribution we can write for the potential the following equation

$$\nabla^2 \phi = \frac{4\pi e}{3\hbar^3} \left[ T^2 + \frac{(e\phi + \mu)^2}{\pi^2 T^2} \right].$$

(94)

With the change of variable $\Phi = \frac{e\phi + \mu}{T}$, the equation (94) can be written as

$$\nabla^2 \Phi = \phi \frac{\Phi}{r_{EH}^2} \left[ 1 + \Phi^2 \right],$$

(95)

where

$$r_{EH}^2 = \frac{3/4\pi}{\frac{8\alpha}{3} \ln \left( \frac{T}{m_e} \right) + \frac{4\alpha}{3} + 1}.$$  

(96)
B. Effect on tunneling probability

The original Thomas-Fermi equation was derived for bound electrons. The derivation presented in this work shows that it is equally valid if the screening happens in a gas of free electrons. We will use the Thomas-Fermi equation (47) in its simplest form, i.e., without the term proportional to \( T^2 \) and without Euler-Heisenberg corrections. It is evident that in equation (47) the length scale is given by the atomic Bohr radius whereas the important quantities entering the tunneling probability of an alpha particle have to do with the much smaller nuclear scale. Following [62, 63] one can expand the solution \( \psi \) of the Thomas-Fermi equation which simplifies the calculations. According to (44) we can write the interaction potential between two positive charges (characterized by \( Z_1 \) and \( Z_2 \)) as

\[
V(r) = \frac{Z_1 Z_2 \alpha}{r} \psi(s) \tag{97}
\]

where \( r = Cs \) as used before. We will look for solutions of \( \psi \) which at the lowest order behave linearly, i.e.,

\[
\psi_i(s) \simeq 1 - d_i s . \tag{98}
\]

One such solution with a linear behaviour at the origin, which is one of the first attempts to derive a semi-analytical solution of the Thomas-Fermi equation, is given by [64] with \( d_0 = 1.588558 \). Other semi-analytical solutions [65–69] have been attempted and we list below some of them in the order in which they are cited:

\[
\begin{align*}
\psi_1(s) &= (1 + \eta \sqrt{s})e^{-\eta \sqrt{s}} \simeq 1 - d_1 s, \quad d_1 = 3.6229 \\
\psi_2(s) &= (a_0 e^{-\alpha x} + b_0 e^{-\beta x})^2 \simeq 1 - d_2 x, \quad d_2 = 1.2357 \\
\psi_3(s) &= (a e^{-\alpha x} + b e^{-\beta x} + c e^{-\gamma x})^2 \simeq 1 - d_3 x, \quad d_3 = 1.4042 \\
\psi_4(s) &= (1 + A \sqrt{x} + B x e^{-D \sqrt{x}})^2 e^{-2A \sqrt{x}} \simeq 1 - d_4 s, \quad d_4 = 1.45612 \\
\psi_5(s) &= \frac{1}{(1 + A_0 x)^2} \simeq 1 - d_5 s, \quad d_5 = 0.9615
\end{align*}
\]

The potential for the alpha tunneling is in the first approximation given by a potential well modeling the nuclear interaction plus the Coulomb or the modified Coulomb potential given in (97). In the semiclassical JWKB approximation, the tunneling probability is simply given by [62],

\[
P \propto e^{-\gamma}
\]

\[
\gamma(E, r_1) = 2\sqrt{2mI(E, r_1)} = 2\sqrt{2m} \int_{r_1}^{r_2} \sqrt{[V(r) - E]}dr \tag{100}
\]
where \( m \) is the reduced mass, \( r_1 \) the first turning point given in our simple model by the radius of the nucleus and \( r_2 \) the second turning point determined by \( V(r_2) = E \), with \( E \) being the energy of the tunneling particle. In passing we note that we have omitted some other approximate solutions which exist in the literature [71–73].

The integral \( I \) with the Coulomb potential can be solved analytically to be [62],

\[
I(E, r_1) = 2\sqrt{2m} \frac{Z_1 Z_2 \alpha}{\sqrt{E}} \left[ \cos^{-1}(x^{1/2}) - x^{1/2}(1 - x)^{1/2} \right]
\]

(101)

with \( x = (Er_1)/(Z_1 Z_2 \alpha) \). Since the modification of the electromagnetic interaction brought by the Thomas-Fermi equation can be approximated by \( 1 - \delta s \), the correction to the potential is simply a constant. The integral for the modified Coulomb problem is then \( I(E^*, r_1) \) with \( E^* = E + Z_1 Z_2 \delta s \). Correspondingly, we have \( \gamma^* = \gamma(E^*, r_1) \). We have chosen the few examples (with experimental \( Q \)-values [74] denoted above as \( E \)) with some of them being the same as in [62]. The nuclear radii are taken from [75]. In table 1 we summarize the effects in the form of the ratio of half-lives, \( \tau/\tau^* \) for the decays, \(^{106}\text{Te} \rightarrow ^{4}\text{He} + ^{102}\text{Sn} \), \(^{148}\text{Sm} \rightarrow ^{2}\text{He} + ^{144}\text{Nd} \), \(^{222}\text{Rn} \rightarrow ^{4}\text{He} + ^{218}\text{Po} \), and \(^{240}\text{Cm} \rightarrow ^{4}\text{He} + ^{236}\text{Pu} \). Though the exact values of half-lives (and hence also the screening effects) are sensitive to the \( Q \)-values [76], the increase in the half-life due to screening seems to be quite sizable in some of the cases considered. The results prompt us to consider a more sophisticated calculation, with the following points in future: (i) Inclusion of the \( T^2 \) term in the Thomas-Fermi equation for different gas temperatures, (ii) including the Euler-Heisenberg corrections and (iii) improving the nuclear model such that the first turning point is also sensitive to the nuclear potential.

In passing we note that the Gamow factor \( e^{-\gamma} \) appears also in stellar reaction rates \( R \) defined by

\[
R \propto \int_0^\infty e^{-\gamma} S(E) e^{-E/kT} dE
\]

(102)

where \( S(E) \) is the astrophysical S-factor [77], which is sometimes approximated by a constant. It would be interesting to study the screening effects in the reaction rates (which eventually affect the abundance of elements) in the fusion reactions in stars within a more refined model as mentioned above.
TABLE I: The effect of electron gas on alpha tunneling. \( \tau^* \) is the half-life of the decaying nucleus within the electron medium.

| \( d_i \) | \( \tau/\tau^* \) | \( \tau/\tau^* \) | \( \tau/\tau^* \) | \( \tau/\tau^* \) |
|---------|----------------|----------------|----------------|----------------|
| \(^{106}\text{Te}\) | 0.962 | 1.123 | 1.776 | 1.302 | 1.324 |
| \(^{148}\text{Sm}\) | 1.236 | 1.161 | 2.090 | 1.404 | 1.434 |
| \(^{222}\text{Rn}\) | 1.404 | 1.185 | 2.309 | 1.471 | 1.506 |
| \(^{240}\text{Cm}\) | 1.456 | 1.192 | 2.381 | 1.492 | 1.529 |
| \(^{222}\text{Rn}\) | 1.589 | 1.211 | 2.575 | 1.547 | 1.589 |
| \(^{240}\text{Cm}\) | 3.630 | 1.547 | 8.476 | 2.693 | 2.859 |

VII. CONCLUSIONS

The effect of surrounding matter at finite temperature on the electric potential of an object is encoded in the Feynman-Metropolis-Teller equation (34). From this equation, various equations can be derived imposing different conditions on the matter. Among the well-known equations which emerge are the Thomas-Fermi and Poisson-Boltzmann equations. Other, new equations like the relativistic Thomas-Fermi equation have been derived in the present work. We have stressed the importance and the universal applicability of these equations. Therefore, it appears timely to consider quantum corrections to these equations. We have calculated these corrections using the Euler-Heisenberg theory at finite temperature. For non-degenerate matter and high temperature analytical solutions have been presented. Although our emphasis was on the derivations of these equations we have touched upon two examples where it can be applied. One example concerns the electron-positron neutral plasma under the Big-Bang conditions in the early universe. The other was a reminder of the state of art of screening charges in astrophysics and its effect on alpha tunneling. The size of the effect makes us think that a more detailed investigation including temperature effects and the quantum corrections is in order. This will be attempted in a future publication. As we already mentioned the applicability of the equations resulting from the Feynman-Metropolis-Teller is manifold and not limited to the examples we presented here. Apart from atomic physics [78], plasma physics [79] and biological applications [80], one can
also find Thomas-Fermi like equations in gravitational physics \[81\]. Future projects could probe into such equations replacing the Fermi-Dirac distribution by the corresponding Bose-Einstein for bosons. Regarding the novel aspects where Thomas-Fermi equations could be used we mention graphene where the electrons are treated relativistically \[82\].

With the inclusion of the quantum corrections we obtain a complete picture of the electric fields at finite temperature from which the electromagnetic force can be easily calculated. Forces at finite temperature, of a different nature than the electromagnetic one, can, in general, be treated within quantum field theory at finite temperature (see \[83\] for an example).

**Appendix: Expansions for the charge density**

We review the form of the particle density for the limiting cases of both non-relativistic and ultra relativistic particles. The special case of ultra relativistic electron-positron plasma is shown at the end.

The quantity of interest is

\[
\eta(r, T) = 2\left(\frac{2\pi}{\hbar}\right)^3 \int_0^\infty \frac{dp}{e^{\beta(K+q\phi+\mu)} + 1},
\]  

(103)

where \(K\) is the kinetic energy of the particles.

For high temperatures the +1 in the denominator of (103) can be ignored. Under this consideration of non-degeneracy, the equation (103) simplifies to

\[
\eta(r, T) \approx 2\left(\frac{2\pi}{\hbar}\right)^3 e^{-\beta(q\phi+\mu)} \int_0^\infty p^2 e^{-\beta K} dp.
\]

(104)

Equation (103) can be simplified further by taking into account the normalization condition

\[
N = \int \eta(r, T) dV = 2\left(\frac{2\pi}{\hbar}\right)^3 e^{-\beta\mu} \int e^{-\beta(K+q\phi)} dp dV,
\]  

(105)

where \(N\) is the total number of particles. From the above we can write for the chemical potential

\[
e^{-\beta\mu} = \frac{N}{2\left(\frac{2\pi}{\hbar}\right)^3 \int e^{-\beta(K+q\phi)} dp dV}.
\]

(106)
Replacing (106) into (104) we get

$$\eta(r, T) = Ne^{\alpha} - q\phi \beta \int e^{-\beta(K+q\phi)}dKd\nu \approx \left(\frac{N}{V}\right) e^{-q\phi \beta}.$$  \hspace{1cm} \text{(107)}$$

In the last step of (107) we have made the final approximation $\int e^{-q\phi \beta} d\nu \approx V$, the total volume. The justification is based on the assumption that for high $T$ the exponential $e^{-q\phi \beta}$ will be small for almost all the volume considered.

The approximation for the degenerate case involves a Sommerfeld’s expansion in power series of $\frac{\mu + q\phi}{T}$ for the equation

$$\eta(r, T) \approx 4(2\pi)^4 \int_0^\infty \frac{p^2 dp}{e^{\beta(K+q\phi-\mu)} + 1}.$$  \hspace{1cm} \text{(108)}$$

The first two terms for the non-relativistic cases read \[6, 84\]

$$\eta(r, T) \approx \frac{2(2\pi)^4(2mT)^{3/2}}{h^3} \left(\frac{2}{3} \left(\frac{q\phi - \mu}{T}\right)^{3/2} + \frac{\pi^2}{12} \left(\frac{q\phi - \mu}{T}\right)^{-1/2}\right).$$  \hspace{1cm} \text{(109)}$$

The ultra relativistic expansion is given by \[84\]

$$\eta(r, T) \approx \frac{4(2\pi)^4}{h^3} (q\phi - \mu)^3 \left[1 + \frac{3}{4} \pi^2 \left(\frac{T}{q\phi - \mu}\right)^2\right].$$  \hspace{1cm} \text{(110)}$$

A special case is the electron-positron plasma \[38\]. Due to the relation between their chemical potentials, the exact total charge density $e\eta_- - e\eta_+$ can be written without any simplifying assumption as

$$e\eta_- - e\eta_+ = \frac{4(2\pi)^4}{h^3} \int_0^\infty dp p^2 \left[\frac{1}{e^{\beta(p-e\phi-\mu)} + 1} - \frac{1}{e^{\beta(p+e\phi+\mu)} + 1}\right]$$

$$= (2\pi)^3 \left(\frac{e\phi + \mu}{3h^3}\right) \left[\frac{1}{T^2} + \frac{(e\phi + \mu)^2}{\pi^2}\right].$$  \hspace{1cm} \text{(111)}$$

[1] See S. Flügge, Practical Quantum Mechanics, Springer, Berlin 1971; J. C. Slater, Quantum Theory of Atoms, McGraw-Hill, New York 1960; L. Spruch, Rev. Mod. Phys. 63 (1991) 151.

[2] S. L. Shapiro and S. A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, Wiley, New York 1983.
[3] J. C. Slater Rev. Mod. Phys. 6 (1934) 209; J. C. Slater and H. M. Krutter, Phys. Rev. 47 (1935) 559; A. Meyer and W. H. Young, J. Phys. C: Metal Phys, Suppl. 3 (1970) S348.

[4] R. Ying and G. Kalman, Phys. Rev. A40, 3927 (1989); M. Akbari-Moghanjoughi, Physics of Plasma 21, 102702 (2014).

[5] M. Z. Bazant, M. S. Kilic and Ajdari, Advances in Colloid and Interface Science, 152, 458 (2009).

[6] R. P. Feynman, N. Metropolis, and E. Teller, Phys. Rev. 75, 1561 (1949).

[7] W. Heisenberg and H. Euler, Z. Phys. 98 (1936), 714.

[8] V. Weisskopf, Mat.-Fis. Med. Dan. Vidensk. Selsk. 14 (1936), 6.

[9] J. Schwinger, Phys. Rev. 82 (1951), 664.

[10] S. Adler, Ann. Phys. N.Y. 67 (1971), 599.

[11] Z. Bialynicka-Birula and I. Bialynicki-Birula, Phys. Rev. D 2 (1970), 2341.

[12] S. I. Kruglov, Phys. Rev. D 75, (2007), 117301

[13] C. V. Costa, D. M. Gitman and A. E. Shabad, Phys. Scripta 90, 074012 (2015)

[14] S. I. Kruglov, Mod. Phys. Lett. A, 32, 1750092 (2017)

[15] J.J. Klein and B.P. Nigam, Phys. Rev. 136 (1964), B1540.

[16] L. Labun and J. Rafelski, Acta Phys. Pol. B 43, 2237 (2012).

[17] R Battesti and C Rizzo 2013 Rep. Prog. Phys. 76 016401.

[18] M. Marklund and J. Lundin, Eur. Phys. J. D 55 (2009) 319.

[19] W. Dittrich and H. Gies, “Probing the Quantum Vacuum: Perturbative Effective Action Approach in Quantum Electrodynamics and Its Applications”, Springer-Verlag-NY (2000).

[20] A. D. Bermudez Manjarres and M. Nowakowski, Phys. Rev. A (2017) (in press).

[21] W. Dittrich, Phys. Rev. D 19, 2385 (1979).

[22] P.H. Cox, W.S. Hellman and A. Yildiz, Ann. Phys. 154, 211 (1984).

[23] M. Loewe and J.C. Rojas, Phys. Rev. D 46, 2689 (1992).

[24] A.K. Ganguly, P.K. Kaw and J.C. Parikh, Phys. Rev. C 51, 2091 (1995).

[25] I.A. Shovkovy, Phys. Lett. B 441, 313 (1998).

[26] A. Chodos, K. Everding and D.A. Owen, Phys. Rev. D 42, 2881 (1990).

[27] H. Gies, Phys. Rev. D 61, 085021 (2000).

[28] F.T. Brandt, J. Frenkel and J.C. Taylor, Phys. Rev. D 50, 4110 (1994).

[29] F.T. Brandt and J. Frenkel, Phys. Rev. Lett. 74, 1705 (1995).
[30] P. Elmfors and B.-S. Skagerstam, Phys. Lett. B 427, 197 (1998).
[31] V. Ch. Zhukovsky, T.L. Shoniya and P.A. Eminov, Zh. Eksp. Teor. Fiz. 107, 299 (1995); J. Exp. Theor. Phys. 80, 158 (1995).
[32] P. H. Cox, W.S. Hellman and A. Yildiz, Ann. Phys. 154, 211 (1984).
[33] J. Hallin and P. Liljenberg, Phys. Rev. D 46, 2689 (1992).
[34] A.K. Ganguly, P.K. Kaw and J.C. Parikh, Phys. Rev. C 51, 2091 (1995).
[35] W. Dittrich and H. Gies, Phys. Rev. D 58, 025004 (1998).
[36] J.L. Latorre, P. Pascual and R. Tarrach, Nucl. Phys. B 437, 60 (1995).
[37] H. Gies, Phys. Rev. D 60, 105033 (1999).
[38] N. L. Tsintsadze, A. Rasheed, H. A. Shah, and G. Murtaza, Phys. Plasmas 16, 112307 (2009).
[39] B.V. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, "Quantum Electrodynamics", Butterworth Heinemann, Oxford, 1999.
[40] C. J. Hansen, S. D. Kawaler Stellar interiors-Physical Principles, Structure and Evolution, Springer, New York 1994.
[41] R.E. Marshak, H. Bethe, Astrophys. J. 91 (1940) 239.
[42] A. Piel, "Plasma Physics: An Introduction to Laboratory, Space, and Fusion Plasmas", Springer-Verlag, Berlin Heidelberg (2010), p. 35.
[43] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[44] B.M. Boghosian, Phys. Rev. E 53 (1996) 4754.
[45] A.R. Plastino, A. Plastino, Phys. Lett. A 174 (1993) 384.
[46] P. Jund, S.G. Kim, C. Tsallis, Phys. Rev. B 52 (1995) 50.
[47] D. Menezes, A. Deppman, E. Megías, L. Castro, Eur. Phys. J. A 51 (2015) 155.
[48] S.Q. Hou, J.J. He, A. Parikh, D. Kahl, C.A. Bertulani, T. Kajino, G.J. Mathews, G. Zhao, Astrophys. J. 834 (2017) 165.
[49] E. Martinenko, B. K. Shivamoggi Phys. Rev. A 69, 052504 (2004).
[50] K. Ourabah, M. Tribeche, Physica A 393 (2014) 470-474.
[51] D.F. Torres, U. Tiranaki, Physica A 261 (1998) 499.
[52] B.B. Kadomtsev, Soviet Phys. JETP, 31 (1970), p. 945.
[53] D.H. Constantinescu and G. Moruzzi, Phys. Rev. D 18 (1978) 1820.
[54] S.H. Hill, P.J. Grout and N.H. March, J. Phys. B18 (1985) 4665.
[55] B.K. Shivamoggi, P.P.J.M. Schram, Physica A 215 (1995) 387-39.
[56] A. Thorolfsson, Ö. Rögnvaldsson, J. Yngvason. Astrophys. J. 502 847 (1998).

[57] D.M Gitman and A.E. Shabad, Phys.Rev.D 86, 125028 (2012).

[58] T.C. Adorno, D.M. Gitman, A.E. Shabad, Eur. Phys. J. C 74, 2838 (2014); Phys. Rev. D 89, 047504 (2014).

[59] W. Misner, K. S. Thorne, and J. A. Wheeler, “Gravitation”, Freeman, San Francisco, 1980, p. 764.

[60] L. D. Landau and E. M. Lifshitz, “Statistical Physics”, Butterworth-Heinemann, Oxford, 1998, p. 315.

[61] Ya. B. Zel’dovich and Y. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena Academic, New York (1966), p. 222.

[62] V. Erma, Phys. Rev. 105 (1957) 1784.

[63] A. Jain and V. K. Tewary, Zetschrift für Astrphysics, 54 (1962) 107.

[64] E. B. Baker, Quart. Appl. Math. 36 (1930) 630.

[65] E. Roberts, Phys. Rev. 170 (1968) 8.

[66] P. Csavinsky, Phys. Rev. A8 (1973) 1688.

[67] R. N. Kesarwani and Y. P. Varshni, Phys. Rev. A23 (1981) 991.

[68] M. Oulne, Int. Rev. Phys. 6 (2010) 349, ibid Applied Mathematics and Computation 228 (2011) 303.

[69] L. Bougoffa and R. Rach, Rom. J. Phys. 60 (2015) 1032.

[70] C. Tsallis, “Introduction to nonextensive statistical mechanics: approaching a complex world”, Springer-Verlag, New York (2009).

[71] M. Desaix, D. Anderson and M. Lisak, Eur. J. Phys. 24 (2004) 699.

[72] M. Wu, Phys. Rev A26 (1982) 57.

[73] S. Esposito, Am J. Phys. 70 (2002) 852.

[74] Q-values obtained from the database at http://www.nndc.bnl.gov/qcalc/.

[75] I. Angeli and K. P. Marinova, Atomic Data and Nuclear Data Tables 99 (2013) 69.

[76] N. G. Kelkar and M. Nowakowski, J. Phys. G 43 (2016) 105102.

[77] C. Iliadis, Nuclear Physics of Stars, Wiley-VCH 2007.

[78] E. H. Lieb and B. Simon, Advances in Mathematics, 23 (1977) 22.

[79] D. Michta, F. Graziani, and M. Bonitz, Contrib. Plasma Phys. 55, 437 (2015).

[80] Fogolari F, Brigo A, Molinari H, J. Mol. Recognit. (2002), 15 377-392.
[81] Bilić N and Viollier R D 1999 Eur. Phys. J.C 11 173.

[82] A. H. Castro Neto at al., Rev. Mod. Phys. 81 (2009) 109.

[83] F. Ferrer, J.A. Grifols and M. Nowakowski, Phys. Rev. D61 (2000) 057304.

[84] John P. Cox, “Principles of stellar structure Volume II: Applications to stars”, Routledge (1968), p. 801.