\textbf{$L^p$-parabolic regularity and non-degenerate Ornstein-Uhlenbeck type operators}

Enrico Priola

Dipartimento di Matematica “Giuseppe Peano”,
Università di Torino,
via Carlo Alberto 10, 10123 Torino (Italy)
enrico.priola@unito.it

\begin{abstract}
We prove $L^p$-parabolic a-priori estimates for
\[ \partial_t u + \sum_{i,j=1}^d c_{ij}(t) \partial_{x_i x_j}^2 u = f \]
on $\mathbb{R}^{d+1}$ when the coefficients $c_{ij}$ are locally bounded functions on $\mathbb{R}$. We slightly generalize the usual parabolicity assumption and show that still $L^p$-estimates hold for the second spatial derivatives of $u$. We also investigate the dependence of the constant appearing in such estimates from the parabolicity constant. Finally we extend our estimates to parabolic equations involving non-degenerate Ornstein-Uhlenbeck type operators.

\textbf{Keywords} Parabolic equations, a-priori $L^p$-estimates, Ornstein-Uhlenbeck operators

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\end{abstract}

\section{Introduction and basic notations}

In this paper we deal with global a-priori $L^p$-estimates for solutions $u$ to second order parabolic equations like

\[ u_t(t,x) + \sum_{i,j=1}^d c_{ij}(t) u_{x_i x_j}(t,x) = f(t,x), \quad (t,x) \in \mathbb{R}^{d+1}, \quad (1.1) \]

$d \geq 1$, with locally bounded coefficients $c_{ij}(t)$. Here $u_t$ and $u_{x_i x_j}$ denote respectively the first partial derivative with respect to $t$ and the second partial derivative with respect to $x_i$ and $x_j$. We slightly generalize the usual parabolicity assumption and show that still $L^p$-estimates hold for the second spatial derivatives of $u$. We also investigate the dependence of the constant appearing in such estimates from the symmetric $d \times d$-matrix $c(t) = (c_{ij}(t))_{i,j=1,...,d}$. In the final section we treat more general equations involving Ornstein-Uhlenbeck type operators and show that the previous a-priori estimates are still true.

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The $L^p$-estimates we are interested in are the following: for any $p \in (1, \infty)$, there exists $\tilde{M} > 0$ such that, for any $u \in C_0^\infty(\mathbb{R}^{d+1})$ which solves (1.1), we have
\[
\|u_{i,j}\|_{L^p(\mathbb{R}^{d+1})} \leq \tilde{M} \|f\|_{L^p(\mathbb{R}^{d+1})}, \quad i, j = 1, \ldots, d,
\]
where the $L^p$-spaces are considered with respect to the $d + 1$-dimensional Lebesgue measure. Usually, in the literature such a-priori estimates are stated requiring that there exists $\lambda$ and $\Lambda > 0$ such that
\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^d c_{ij}(t) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad t \in \mathbb{R}, \xi \in \mathbb{R}^d,
\]
where $|\xi|^2 = \sum_{i=1}^d \xi_i^2$. We refer to Chapter 4 in [16], Appendix in [23], Section VII.3 in [17], which also assumes that $c_{ij}$ are uniformly continuous, and Chapter 4 in [15]. The proofs are based on parabolic extensions of the Calderon-Zygmund theory for singular integrals (cf. [8] and [11]). This theory was originally used to prove a-priori Sobolev estimates for the Laplace equation (see [5]). In the above mentioned references, it is stated that $\tilde{M}$ depends not only on $d$, $p$, $\lambda$ (the parabolicity constant) but also on $\Lambda$. An attempt to determine the explicit dependence of $\tilde{M}$ from $\lambda$ and $\Lambda$ has been done in Theorem A.2.4 of [23] finding a quite complicate constant.

The fact that $\tilde{M}$ is actually independent of $\Lambda$ is mentioned in Remark 2.5 of [13]. This property follows from a general result given in Theorem 2.2 of [13]. Once this independence from $\Lambda$ is proved one can use a rescaling argument (cf. Corollary 2.4) to show that we have
\[
\tilde{M} = \frac{M_0}{\lambda}, \tag{1.4}
\]
for a suitable positive constant $M_0$ depending only on $d$ and $p$.

In Theorem 2.3 and Corollary 2.4 we generalize the parabolicity condition by requiring that the symmetric $d \times d$ matrix $c(t) = (c_{ij}(t))$ is non-negative definite, for any $t \in \mathbb{R}$, and, moreover, that there exists and integer $p_0$, $1 \leq p_0 \leq d$, and $\tilde{\lambda} \in (0, \infty)$ such that
\[
\lambda \sum_{j=1}^{p_0} \xi_j^2 \leq \sum_{i,j=1}^d c_{ij}(t) \xi_i \xi_j, \quad t \in \mathbb{R}, \xi \in \mathbb{R}^d \tag{1.5}
\]
(cf. Hypothesis H in Section 2). We show that (1.5) is enough to get estimates like (1.2) for $i, j = 1, \ldots, p_0$, with a constant $\tilde{M}$ as in (1.4) (now $M_0$ depends on $p, d$ and $p_0$). An example in which (1.5) holds is
\[
u(t,x,y) + u_{tx}(t,x,y) + tu_{xy}(t,x,y) + t^2u_{yy}(t,x,y) = f(t,x,y), \tag{1.6}
\]
$(t,x,y) \in \mathbb{R}^3$ (see Example 2.5). In this case we have an a-priori estimates for $\|u_{tx}\|_{L^p}$.

We will first provide a purely analytic proof of Theorem 2.3 in the case of $L^2$-estimates. This is based on Fourier transform techniques. Then we provide the proof for the general case $1 < p < \infty$ in Section 2.2. This proof is inspired by the one of Theorem 2.2 in [13] and requires the concept of stochastic integral with respect to the Wiener process. In Section 2.2.1 we recall basic properties of the stochastic integral. It is not clear how to prove Theorem 2.3 for $p \neq 2$ in a purely analytic way. One possibility could be to
follow step by step the proof given in Appendix of [23] trying to improve the constants appearing in the various estimates.

In Section 3 we will extend our estimates to more general equations like

$$u_t(t,x) + \sum_{i,j=1}^d c_{ij}(t) u_{x_i x_j}(t,x) + \sum_{i,j=1}^d a_{ij} x_i u_x(t,x) = f(t,x), \quad (1.7)$$

where $A = (a_{ij})$ is a given real $d \times d$-matrix. If $(1.5)$ holds with $p_0 = d$ then we show that estimate $(1.9)$ is still true with $M_0 = M_0(d,p,T,A) > 0$ for any solution $u \in C^p_0((-T,T) \times \mathbb{R}^d)$ of $(1.7)$ (see Theorem 3.1 for a more general statement).

An interesting case of $(1.7)$ is when $c(t)$ is constant, i.e., $c(t) = Q$, $t \in \mathbb{R}$. Then equation $(1.7)$ becomes

$$u_t + \mathcal{A} u = f,$$

where $\mathcal{A}$ is the Ornstein-Uhlenbeck operator, i.e.,

$$\mathcal{A} v(x) = \text{Tr}(QD^2 v(x)) + \langle Ax, Dv(x) \rangle, \quad x \in \mathbb{R}^d, \quad v \in C^p_0(\mathbb{R}^d). \quad (1.8)$$

The operator $\mathcal{A}$ and its parabolic counterpart $\mathcal{L} = \mathcal{A} - \partial_t$, which is also called Kolmogorov-Fokker-Planck operator, have recently received much attention (see, for instance, [3], [4], [6], [7], [9], [18], [19], [22] and the references therein). The operator $\mathcal{A}$ is the generator of the Ornstein-Uhlenbeck process which solves a linear stochastic differential equation (SDE) describing the random motion of a particle in a fluid (see [20]). Several interpretations in physics and finance for $\mathcal{A}$ and $\mathcal{L}$ are explained in the survey [21].

From the a-priori estimates for the parabolic equation $(1.7)$ one can deduce elliptic estimates like

$$\|v_{x_i}\|_{L^p(\mathbb{R}^d)} \leq C_1 \left( \|\mathcal{A} v\|_{L^p(\mathbb{R}^d)} + \|v\|_{L^p(\mathbb{R}^d)} \right), \quad (1.9)$$

with $C_1 = \frac{M_0(d,p,A)}{\lambda}$, assuming that $\mathcal{A}$ is non-degenerate (i.e., $Q$ is positive definite; see Corollary 3.4). Similar estimates have been already obtained in [19]. Here we can show in addition the precise dependence of the constant $C_1$ from the matrix $Q$.

More generally, estimates like $(1.9)$ hold for possibly degenerate hypoelliptic Ornstein-Uhlenbeck operators $\mathcal{A}$ (see [3]); a typical example in $\mathbb{R}^2$ is $\mathcal{A} v = q v_{xx} + x v_y$ with $q > 0$ (cf. Example 3.10). In this case we have

$$\|v_{x_i}\|_{L^p(\mathbb{R}^2)} \leq C_1 \left( \|q v_{xx} + x v_y\|_{L^p(\mathbb{R}^2)} + \|v\|_{L^p(\mathbb{R}^2)} \right). \quad (1.10)$$

Estimates as $(1.10)$ have been deduced in [3] by corresponding parabolic estimates for $\mathcal{A} - \partial_t$, using that such operator is left invariant with respect to a suitable Lie group structure on $\mathbb{R}^{d+1}$ (see [13]). We also mention [4] which contains a generalization of [3] when $Q$ may also depend on $x$ and [22] where the results in [3] are used to study well-posedness of related SDEs. Finally, we point out that in the degenerate hypoelliptic case considered in [3] it is not clear how to prove the precise dependence of the constant appearing in the a-priori $L^p$-estimates from the matrix $Q$.

We denote by $|\cdot|$ the usual euclidean norm in any $\mathbb{R}^k$, $k \geq 1$. Moreover, $\langle \cdot, \cdot \rangle$ indicates the usual inner product in $\mathbb{R}^k$.

We denote by $L^p(\mathbb{R}^k)$, $k \geq 1$, $1 < p < \infty$ the usual Banach spaces of measurable real functions $f$ such that $|f|^p$ is integrable on $\mathbb{R}^k$ with respect to
the Lebesgue measure. The space of all $L^p$-functions $f : \mathbb{R}^k \to \mathbb{R}^l$ with $j > 1$ is indicated with $L^p(\mathbb{R}^k; \mathbb{R}^l)$. Let $H$ be an open set in $\mathbb{R}^k$; $C_0^\infty (H)$ stands for the vector space of all real $C^\infty$-functions $f : H \to \mathbb{R}$ which have compact support.

Let $d \geq 1$. Given a regular function $u : \mathbb{R}^{d+1} \to \mathbb{R}$, we denote by $D_w^2 u(t, x)$ the $d \times d$ Hessian matrix of $u$ with respect to the spatial variables at $(t, x) \in \mathbb{R}^{d+1}$, i.e., $D_w^2 u(t, x) = (u_{x_{ij}}(t, x))_{i,j=1,..,d}$. Similarly we define the gradient $D_t u(t, x) \in \mathbb{R}^d$ with respect to the spatial variables.

Given a real $k \times k$ matrix $A$, $\|A\|$ denotes its operator norm and $Tr(A)$ its trace.

Let us recall the notion of Gaussian measure (see, for instance, Section 1.2 in [2] or Chapter 1 in [7] for more details). Let $d \geq 1$. Given a symmetric non-negative definite $d \times d$ matrix $Q$, the symmetric Gaussian measure $N(0, Q)$ is the unique Borel probability measure on $\mathbb{R}^d$ such that its Fourier transform is

$$
\int_{\mathbb{R}^d} e^{i(x, \xi)} N(0, Q)(dx) = e^{-\frac{1}{2} \langle \xi, Q \xi \rangle}, \quad \xi \in \mathbb{R}^d; \tag{1.11}
$$

$N(0, Q)$ is the Gaussian measure with mean 0 and covariance matrix $2Q$. If in addition $Q$ is positive definite than $N(0, Q)$ has the following density $f$ with respect to the $d$-dimensional Lebesgue measure

$$
f(x) = \frac{1}{\sqrt{(4\pi)^d \det(Q)}} e^{-\frac{1}{2} \langle Q^{-1}x, x \rangle}, \quad x \in \mathbb{R}^d. \tag{1.12}
$$

Given two Borel probability measures $\mu_1$ and $\mu_2$ on $\mathbb{R}^d$ the convolution $\mu_1 * \mu_2$ is the Borel probability measure denoted as

$$
\mu_1 * \mu_2(B) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_B(x+y) \mu_1(dx) \mu_2(dy) = \int_{\mathbb{R}^d} \mu_1(dx) \int_{\mathbb{R}^d} 1_B(x+y) \mu_2(dy),
$$

for any Borel set $B \subset \mathbb{R}^d$. Here $1_B$ is the indicator function of $B$ (i.e., $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ if $x \not\in B$). It can be easily verified that

$$
N(0, Q) * N(0, R) = N(0, Q + R), \tag{1.13}
$$

where $Q + R$ is the sum of the two symmetric non-negative definite matrices $Q$ and $R$.

### 2 A-priori $L^p$-estimates

In this section we consider parabolic equations like \([1.1]\).

We always assume that the coefficients $c_{ij}(t)$ of the symmetric $d \times d$ matrix $c(t)$ appearing in \([1.1]\) are (Borel) measurable and locally bounded on $\mathbb{R}$ and, moreover, that $\langle c(t) \xi, \xi \rangle \geq 0$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^d$. Moreover, we will consider the symmetric non-negative $d \times d$ matrix

$$
C_{sr} = \int_t^s c(t) dt, \quad s \leq r, \quad s, r \in \mathbb{R}. \tag{2.1}
$$

We start with a simple representation formula for solutions to equation \([1.1]\). This formula is usually obtained assuming that $c(t)$ is uniformly positive. However there are no difficulties to prove it even in the present case when $c(t)$ is only non-negative definite.
Proposition 2.1. Let $u \in C^\infty_0(\mathbb{R}^{d+1})$ be a solution to (1.1). Then we have, for $(s,x) \in \mathbb{R}^{d+1}$,

$$u(s,x) = -\int_s^\infty dr \int_{\mathbb{R}^d} f(r,x+y)N(0,C_r)(dy).$$

(2.2)

Proof. Let us denote by $\hat{u}(t,\cdot)$ the Fourier transform of $u(t,\cdot)$ in the space variable $x$. Applying such partial Fourier transform to both sides of (1.1) we obtain

$$\hat{u}_t(s,\xi) - \sum_{i,j=1}^d c_{ij}(s)\xi_i\xi_j \hat{u}(s,\xi) = \hat{f}(s,\xi),$$

i.e., we have

$$\hat{u}(s,\xi) = -\int_s^\infty e^{-(C_r,\xi,\xi)} \hat{f}(r,\xi)dr, \quad (s,\xi) \in \mathbb{R}^{d+1}.$$  

(2.3)

It follows that

$$\hat{u}(s,\xi) = -\int_s^\infty \left( \int_{\mathbb{R}^d} e^{i(s,\xi)\frac{d}{dx}} N(0,C_r)(dx) \right) \hat{f}(r,\xi)dr.$$ 

By some straightforward computations, using also the uniqueness property of the Fourier transform, we get (2.2).

Alternatively, starting from (2.3) one can directly follow the computations of pages 48 in [15] and obtain (2.2). These computations use that there exists $\epsilon > 0$ such that $\langle c(t)\xi,\xi \rangle \geq \epsilon |\xi|^2$, $\xi \in \mathbb{R}^d$. We write, for $\epsilon > 0$, using the Laplace operator,

$$u_t(t,x) + \sum_{i,j=1}^d c_{ij}(t)u_{x_ix_j}(t,x) + \epsilon \triangle u(t,x) = f(t,x) + \epsilon \triangle u(t,x),$$

$(t,x) \in \mathbb{R}^{d+1}$; since $c(t) + \epsilon I$ is uniformly positive, following [15] we find

$$u(s,x) = -\int_s^\infty dr \int_{\mathbb{R}^d} f(r,y+x)N(0,C_r + \epsilon(r-s)I)(dy)$$

$$-\epsilon \int_s^\infty dr \int_{\mathbb{R}^d} \triangle u(r,y+x)N(0,C_r + \epsilon(r-s)I)(dy).$$

Using also (1.13) we get

$$u(s,x) = -\int_s^\infty dr \int_{\mathbb{R}^d} N(0,(r-s)I)(dz) \int_{\mathbb{R}^d} f(r,x+y + \sqrt{\epsilon}z)N(0,C_r)(dy)$$

$$-\epsilon \int_s^\infty dr \int_{\mathbb{R}^d} N(0,(r-s)I)(dz) \int_{\mathbb{R}^d} \triangle u(r,x+y + \sqrt{\epsilon}z)N(0,C_r)(dy).$$

Now we can pass to the limit as $\epsilon \to 0^+$ by the Lebesgue theorem and get (2.2).

The next assumption is a slight generalization of the usual parabolicity condition which corresponds to the case $p_0 = d$ (see also Remark 2.6).

Hypothesis 1. The coefficients $c_{ij}$ are locally bounded on $\mathbb{R}$ and the matrix $c(t) = (c_{ij}(t))$ is symmetric non-negative definite, $t \in \mathbb{R}$. In addition, there exists an integer $p_0$, $1 \leq p_0 \leq d$, and $\lambda \in (0,\infty)$ such that

$$\langle c(t)\xi,\xi \rangle = \sum_{i,j=1}^d c_{ij}(t)\xi_i\xi_j \geq \lambda \sum_{j=1}^{p_0} \xi_j^2, \quad t \in \mathbb{R}, \xi \in \mathbb{R}^d.$$  

(2.4)
A possible generalization of this hypothesis is given in Remark 2.6. Note that if we introduce the orthogonal projection

\[ I_0 : \mathbb{R}^d \to F_{p_0}, \]

where \( F_{p_0} \) is the subspace generated by \( \{e_1, \ldots, e_{p_0}\} \) (here \( \{e_i\}_{i=1}^d \) denotes the canonical basis in \( \mathbb{R}^d \)) then \( (2.6) \) can be rewritten as

\[ (c(t) \xi, \xi) \geq \lambda I_0 \xi \xi, \quad t \in \mathbb{R}, \quad \xi \in \mathbb{R}^d. \]  

**Lemma 2.2.** Let \( g : \mathbb{R}^{d+1} \to \mathbb{R} \) be Borel, bounded, with compact support and such that \( g(t, \cdot) \in C_0^\infty(\mathbb{R}^d), t \in \mathbb{R}. \) Fix \( i, j \in \{1, \ldots, p_0\} \) and consider

\[ w_{ij}(s, x) = - \int_s^\infty dr \int_{\mathbb{R}^d} g_{s, x}(r, x+y)N(0, I_0(r-s))(dy), \quad (s, x) \in \mathbb{R}^{d+1}, \]

where \( I_0 \) is defined in \( (2.3) \). For any \( p \in (1, \infty) \), there exists \( M_0 = M_0(d, p, p_0) > 0 \), such that

\[ \|w_{ij}\|_{L^p(\mathbb{R}^{d+1})} \leq M_0 \|g\|_{L^p(\mathbb{R}^{d+1})}. \]  

**Proof.** If \( p_0 = d \) the estimate is classical. In such case we are dealing with the heat equation

\[ \partial_t u + \Delta u = g \]

on \( \mathbb{R}^{d+1} \) and \( w_{ij} \) coincides with the second partial derivative with respect to \( x_i \) and \( x_j \) of the heat potential applied to \( g \) (see, for instance, page 288 in [16] or Appendix in [23]). If \( p_0 < d \) we write \( x = (x', x'') \) for \( x \in \mathbb{R}^d \), where \( x' \in \mathbb{R}^{p_0} \) and \( x'' \in \mathbb{R}^{d-p_0} \). We get

\[ w_{ij}(s, x', x'') = - \int_s^\infty dr \int_{\mathbb{R}^{p_0}} g_{s, x'}(r, x'+y', x'')N(0, I_{p_0}(r-s))(dy), \]

where \( I_{p_0} \) is the identity matrix in \( \mathbb{R}^{p_0} \). Let us fix \( x'' \in \mathbb{R}^{d-p_0} \) and consider the function \( l(t, x') = g(t, x', x'') \) defined on \( \mathbb{R} \times \mathbb{R}^{p_0} \). By classical estimates for the heat equation \( \partial_t u + \Delta u = f \) on \( \mathbb{R}^{p_0+1} \) we obtain

\[ \int_{\mathbb{R}^{p_0+1}} |w_{ij}(s, x', x'')|^p dx' \leq M_0^p \int_{\mathbb{R}^{p_0+1}} |g(s, x', x'')|^p dx'. \]

Integrating with respect to \( x'' \) we get the assertion.

In the sequel we also consider the differential operator \( L \)

\[ Lu(t, x) = \sum_{i,j=1}^d c_{ij}(t) u_{x_i x_j}(t, x), \quad (t, x) \in \mathbb{R}^{d+1}, \quad u \in C_0^\infty(\mathbb{R}^{d+1}). \]  

The next regularity result when \( p_0 = d \) follows by a general result given in Theorem 2.2 of [13] (cf. Remark 2.5 in [14]).

In the next two sections we provide the proof. First we give a direct and self-contained proof in the case \( p = 2 \) by Fourier transform techniques (see Section 2.1). Then in Section 2.2 we consider the general case. The proof for \( 1 < p < \infty \) is inspired by the one of Theorem 2.2 in [13] and uses also a probabilistic argument. This argument is used to “decompose” a suitable Gaussian measure in order to apply successfully the Fubini theorem (cf. 2.17 and 2.18).

We stress again that in the case of \( d = p_0 \), usually, the next result is stated under the stronger assumption that \( (2.4) \) holds with \( \lambda = 1 \) and also that \( c_{ij} \) are bounded, i.e., assuming (1.3) with \( \lambda = 1 \) and \( \Lambda \geq 1 \) (see, for instance, Appendix in [23] and [16]).
Proof. Now the matrix $A$ is a Borel set of Lebesgue measure zero. Moreover, if (2.11) holds then by
Applying Theorem 2.3 to $H_0$ we may assume that
Corollary 2.4. Assume Hypothesis 1. Then, for any $u \in C_0^\infty(\mathbb{R}_{d+1}^d)$, $i, j = 1, \ldots, p_0$, we have
Indeed following the proof of Theorem 2.3 it is clear that assertion (2.9)

**Remark 2.6.** One can easily generalize Hypothesis 1 as follows:
the coefficients $c_{ij}$ are locally bounded on $\mathbb{R}$ and, moreover, there exists an orthogonal projection $I_0 : \mathbb{R}^d \to \mathbb{R}^d$ and $\lambda > 0$ such that, for any $t \in \mathbb{R}$, a.e.,

\[
\langle e(t) \xi, \xi \rangle \geq \lambda |I_0 \xi|^2, \quad \xi \in \mathbb{R}^d.
\]

Theorem 2.3 and Corollary 2.4 continue to hold under this assumption.

Under hypothesis (2.11) assertion (2.9) in Theorem 2.3 becomes

\[
\| \langle \mathcal{L}_0^2 u(\cdot) \rangle h, k \|_{L^p(\mathbb{R}^{d+1})} \leq M_0 \| u_t + Lu \|_{L^p(\mathbb{R}^{d+1})},
\]

where $h, k \in I_0(\mathbb{R}^d)$. 

\[\text{Theorem 2.3. Assume Hypothesis 1 with } \lambda = 1 \text{ in (2.4). Then, for } p \in (1, \infty), \text{ there exists a constant } M_0 = M_0(d, p, p_0) \text{ such that, for any } u \in C_0^\infty(\mathbb{R}_{d+1}^d), \text{ we have}
\]

\[
\|u_{x_i x_j}\|_{L^p(\mathbb{R}^{d+1})} \leq M_0 \| u_t + Lu \|_{L^p(\mathbb{R}^{d+1})}.
\]

As a consequence of the previous result we obtain

Corollary 2.4. Assume Hypothesis 1. Then, for any $u \in C_0^\infty(\mathbb{R}_{d+1}^d)$, $p \in (1, \infty)$, $i, j = 1, \ldots, p_0$, we have (see (2.8))

\[
\|u_{x_i x_j}\|_{L^p(\mathbb{R}^{d+1})} \leq \frac{M_0}{\lambda} \| u_t + Lu \|_{L^p(\mathbb{R}^{d+1})},
\]

where $M_0 = M_0(d, p, p_0)$ is the same constant appearing in (2.9).

Proof. Let us define $w(t, y) = u(t, \sqrt{\lambda} y)$. Set $f = u_t + Lu$; since $u(t, x) = w(t, \sqrt{\lambda} x)$, we find

\[
f(t, \sqrt{\lambda} y) = w_t(t, y) + \frac{1}{\lambda} Lw(t, y)
\]

Now the matrix $\left(\frac{1}{\lambda} c_{ij}\right)$ satisfies

\[
\frac{1}{\lambda} \sum_{j=1}^p c_{ij}(t) \xi_j \xi_j \geq \sum_{j=1}^p \xi_j^2, \quad t \in \mathbb{R}, \quad \xi \in \mathbb{R}.
\]

Applying Theorem 2.3 to $w$ we find

\[
\|w_{x_i x_j}\|_{L^p} \leq M_0 \lambda^{-1} \frac{\lambda}{\lambda} \| f \|_{L^p}
\]

and so

\[
\lambda^{1-\frac{\lambda}{\lambda}} \|u_{x_i x_j}\|_{L^p} \leq M_0 \lambda^{-1} \frac{\lambda}{\lambda} \| f \|_{L^p}
\]

which is the assertion.

**Examples 2.5.** The equation (1.6) verifies the assumptions of Corollary 2.4 with $p_0 = 1$ and $\lambda = 3/4$ since

\[
\sum_{i,j=1}^2 c_{ij}(t) \xi_i \xi_j = \xi_1^2 + t \xi_1 \xi_2 + t^2 \xi_2^2 \geq \frac{3}{4} \xi_1^2, \quad (t, \xi_1, \xi_2) \in \mathbb{R}^3.
\]

Hence there exists $M_0 > 0$ such that if $u \in C_0^\infty(\mathbb{R}^3)$ solves (1.6) then

\[
\|u_{xx}\|_{L^p(\mathbb{R}^3)} \leq \frac{M_0}{\lambda} \| f \|_{L^p(\mathbb{R}^3)}.
\]
2.1 Proof of Theorem 2.3 when $p = 2$

This proof is inspired by the one of Lemma A.2.2 in [23]. Note that such lemma has $p_0 = d$ and, moreover, it assumes the stronger condition (1.3). In Lemma A.2.2 the constant $M_0$ appearing in (2.9) is $2\sqrt{A}$.

We start from (2.3) with

\[ f = u_t + Lu. \]

Recall that for $g : \mathbb{R}^{d+1} \to \mathbb{R}$, $\hat{g}(s, \xi)$ denotes the Fourier transform of $g(s, \cdot)$ with respect to the $s$-variable ($s \in \mathbb{R}$, $\xi \in \mathbb{R}^d$) assuming that $g(s, \cdot) \in L^1(\mathbb{R}^d)$. Let us fix $s \in \mathbb{R}$. Let $i, j = 1, \ldots, p_0$. We easily compute the Fourier transform of $u_{t,x_j}(s, \cdot)$ (the matrix $C_\nu$ is defined in (2.1)):

\[ \hat{u}_{t,x_j}(s, \xi) = -\xi_j^* \xi_j \hat{u}(s, \xi) = \xi_j \xi j \int_{\mathbb{R}^d} e^{-\langle C_\nu \xi, \xi \rangle} \hat{f}(r, \xi) dr, \quad \xi \in \mathbb{R}^d. \]

Since $|I_0^s|^2 = \sum_{i=1}^{p_0} |\xi_i|^2$, we get

\[ 2|\hat{u}_{t,x_j}(s, \xi)| \leq |I_0^s|^2 \int_{\mathbb{R}^d} e^{-\langle (C_\nu \xi, \xi) - (C_\nu \xi, \xi) \rangle} \hat{f}(r, \xi) dr = G_\xi(s). \]

Now we fix $\xi \in \mathbb{R}^d$, such that $|I_0^s| \neq 0$, and define

\[ g_\xi(r) = \langle C_\nu \xi, \xi \rangle = \int_{\mathbb{R}^d} \langle c(p) \xi, \xi \rangle dp, \quad r \in \mathbb{R}. \]

Changing variable $t = g_\xi(r)$, we get

\[ G_\xi(s) = |I_0^s|^2 \int_{g_\xi(s)} e^{\langle g_\xi(s), t \rangle} \hat{f}(g_\xi^{-1}(t), \xi) \frac{1}{\langle c(g_\xi^{-1}(t)) \xi, \xi \rangle} dt. \]

Let us introduce $\varphi(t) = e^t \cdot 1_{(-\infty,0)}(t)$, $t \in \mathbb{R}$, and

\[ F_\xi(t) = |I_0^s|^2 |\hat{f}(g_\xi^{-1}(t), \xi)\rangle \frac{1}{\langle c(g_\xi^{-1}(t)) \xi, \xi \rangle}. \]

Using the standard convolution for real functions defined on $\mathbb{R}$ we find

\[ G_\xi(s) = (\varphi * F_\xi)(g_\xi(s)). \]

Therefore (recall (2.6) with $\lambda = 1$)

\[ \|G_\xi\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \|\varphi * F_\xi\|^2(t) \frac{1}{\langle c(g_\xi^{-1}(t)) \xi, \xi \rangle} dt \leq \frac{1}{|I_0^s|^2} \|\varphi \ast F_\xi\|_{L^2(\mathbb{R})}^2, \tag{2.12} \]

which implies $\|G_\xi\|_{L^2(\mathbb{R})} \leq \frac{1}{|I_0^s|^2} \|\varphi \ast F_\xi\|_{L^2(\mathbb{R})}$. On the other hand, using the Young inequality, we find, for any $\xi \in \mathbb{R}^d$ with $|I_0^s| \neq 0$,

\[
\begin{align*}
\|\varphi \ast F_\xi\|_{L^2(\mathbb{R})} &\leq \|\varphi\|_{L^1(\mathbb{R})} \|F_\xi\|_{L^2(\mathbb{R})} = \|F_\xi\|_{L^2(\mathbb{R})} \\
&= |I_0^s|^2 \left( \int_{\mathbb{R}} |\hat{f}(g_\xi^{-1}(t), \xi)|^2 \frac{1}{\langle c(g_\xi^{-1}(t)) \xi, \xi \rangle^2} dt \right)^{1/2} \\
&= |I_0^s|^2 \left( \int_{\mathbb{R}} |\hat{f}(r, \xi)|^2 \left( \frac{1}{\langle c(r) \xi, \xi \rangle} \right)^2 \langle c(r) \xi, \xi \rangle dr \right)^{1/2} \\
&\leq \frac{|I_0^s|^2}{|I_0^s|} \left( \int_{\mathbb{R}} |\hat{f}(r, \xi)|^2 \right)^{1/2} = |I_0^s| \cdot \|f(\cdot, \xi)\|_{L^2(\mathbb{R})}. \nonumber
\end{align*}
\]
Using also (2.12) we obtain, for any \( \xi \in \mathbb{R}^d \), \( |t_0 \xi| \neq 0 \),
\[
2\|\hat{u}_{kx_k}(\xi)\|_{L^2(\mathbb{R})} \leq \|G_x\|_{L^2(\mathbb{R})} \leq \|\hat{f}(\xi)\|_{L^2(\mathbb{R})}.
\]
From the previous inequality, integrating with respect to \( \xi \) over \( \mathbb{R}^d \) we find
\[
4 \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\hat{u}_{kx_k}(s, \xi)|^2 d\xi ds \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\hat{f}(s, \xi)|^2 d\xi ds.
\]
By using the Plancherel theorem in \( L^2(\mathbb{R}^d) \) we easily obtain (2.9) with \( M_0 = 1/2 \). The proof is complete.

### 2.2 Proof of Theorem 2.3 when \( 1 < p < \infty \)

The proof uses the concept of stochastic integral in a crucial point (see (2.17) and (2.18)). Before starting the proof we collect some basic properties of the stochastic integral with respect to the Wiener process which are needed (see, for instance, Chapter 4 in [1] or Section 4.3 in [23] for more details).

#### 2.2.1 The stochastic integral

Let \( W = (W_t)_{t \geq 0} \) be a standard \( d \)-dimensional Wiener process defined on a probability space \( (\Omega, \mathcal{F}, P) \). Denote by \( E \) the expectation with respect to \( P \).

Consider a function \( F \in L^2([a, b]; \mathbb{R}^d \otimes \mathbb{R}^d) \) (here \( 0 \leq a \leq b \) and \( \mathbb{R}^d \otimes \mathbb{R}^d \) denotes the space of all real \( d \times d \)-matrices).

Let \( (\pi_n) \) be any sequence of partitions of \([a, b]\) such that \( |\pi_n| \to 0 \) as \( n \to \infty \) (given a partition \( \pi = \{t_0 = a, ..., t_N = b\} \) we set \( |\pi| = \sup_{t_{j+1} \in \pi} |t_{j+1} - t_j| \)). One defines the stochastic integral \( \int_a^b F(s)dW_s \) as the limit in \( L^2(\Omega; P; \mathbb{R}^d) \) of
\[
J_n = \sum_{i_0, i_1, ..., i_N \in \pi_n} F(t_{i_1}) (W_{t_{i_1+1}} - W_{t_{i_1}})
\]
as \( n \to \infty \) (recall that the previous formula means
\[
J_n(\omega) = \sum_{i_0, i_1, ..., i_N \in \pi_n} F(t_{i_1}) (W_{t_{i_1+1}}(\omega) - W_{t_{i_1}}(\omega))
\]
for any \( \omega \in \Omega \). One can prove that the previous limit is independent of the choice of \( (\pi_n) \). Moreover, we have, \( P \)-a.s.,
\[
\int_a^b F(s)dW_s = \int_0^b F(s)dW_s - \int_0^a F(s)dW_s.
\] (2.13)

Set \( \Gamma_{ab} = \int_a^b F(s)F^*(s)ds \) where \( F^*(s) \) denotes the adjoint matrix of \( F(s) \). Clearly, \( \Gamma_{ab} \) is a \( d \times d \) symmetric non-negative definite matrix. Moreover, we have (see, for instance, page 77 in [1])
\[
E\left[e^{i\langle \xi, \Gamma_{ab} \xi \rangle/2}\right] = \int_{\Omega} e^{i\langle \xi, \Gamma_{ab} \xi \rangle} \mathbb{P}(d\omega) \tag{2.14}
\]
\[
= \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} N(0, \Gamma_{ab})(dx) = e^{-\langle \xi, \xi \rangle/2}, \quad \xi \in \mathbb{R}^d.
\]

Formula (2.14) is equivalent to require that for any Borel and bounded \( f : \mathbb{R}^d \to \mathbb{R} \),
\[
E \left[ f(\sqrt{2} \int_a^b F(s)dW_s) \right] = \int_{\mathbb{R}^d} f(y) N(0, \Gamma_{ab})(dy). \tag{2.15}
\]
Equivalently, one can say that the law (or image measure) of \( \sqrt{2} \int_a^b F(s)dW_s \) is \( N(0, \Gamma_{ab}) \).
2.2.2 Proof of the theorem

It is convenient to suppose that \( u(t, \cdot) = 0 \) if \( t \leq 0 \) so that \( u \in C^0_\infty([0, \infty) \times \mathbb{R}^d) \).

Indeed if \( u(t, \cdot) = 0 \), \( t \leq T \), for some \( T \in \mathbb{R} \), then we can introduce \( v(t, x) = u(t + T, x) \) which belongs to \( u \in C^0_\infty([0, \infty) \times \mathbb{R}^d) \); from the a-priori estimate for \( v_{u,x} \) it follows (2.9) since \( \|v_{u,x}\|_{L^p(\mathbb{R}^{d+1})} = \|u_{u,x}\|_{L^p(\mathbb{R}^{d+1})} \).

We know that, for \( s \geq 0 \), \( x \in \mathbb{R}^d \),

\[
\begin{align*}
  u(s, x) &= -\int_s^\infty dr \int_{\mathbb{R}^d} f(r, x + y) N(0, C_{sr})(dy),
\end{align*}
\]

where \( f = u_t + Lu \) is bounded, with compact support on \( \mathbb{R}^{d+1} \) and such that \( f(t, \cdot) \in C^0_\infty(\mathbb{R}^d) \), \( t \geq 0 \). Let us fix \( i, j \in \{1, \ldots, p_0\} \).

Differentiating under the integral sign it is not difficult to prove that

\[
\begin{align*}
  u_{u,xj}(s, x) &= -\int_s^\infty dr \int_{\mathbb{R}^d} f_{u,xj}(r, x + y) N(0, C_{sr})(dy).
\end{align*}
\]

Let us fix \( s \) and \( r, 0 \leq s \leq r \), and consider

\[
C_{sr} = A_{sr} + (r-s)I_0, \quad \text{where} \quad A_{sr} = \int_s^r (c(t) - I_0) dt.
\]

By (1.13) we know that \( N(0, C_{sr}) = N(0, A_{sr}) + N(0, (r-s)I_0) \) and so

\[
\begin{align*}
  \int_{\mathbb{R}^d} f_{u,xj}(r, x + y) N(0, C_{sr})(dy)
  &= \int_{\mathbb{R}^d} N(0, A_{sr})(dz) \int_{\mathbb{R}^d} f_{u,xj}(r, x + y + z) N(0, (r-s)I_0)(dy).
\end{align*}
\]

Now we introduce a standard \( d \)-dimensional Wiener process \( W = (W_t)_{t \geq 0} \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) (see Section 2.2.1). Consider the symmetric \( d \times d \) square root \( \sqrt{c(t) - I_0} \) of \( c(t) - I_0 \) and define the stochastic integral

\[
\Lambda_{sr} = \sqrt{2} \int_s^r \sqrt{c(t) - I_0} dW_t.
\]

By (1.13) we know that

\[
\Lambda_{sr} = b_r - b_s, \quad \text{where} \quad b_t = \sqrt{2} \int_0^t \sqrt{c(p) - I_0} dW_p,
\]

\( t \geq 0 \), and \( b_t = 0 \) if \( t \leq 0 \). Moreover (cf. (2.13)) for any Borel and bounded \( g: \mathbb{R}^d \to \mathbb{R} \),

\[
\mathbb{E}[g(b_r - b_s)] = \int_{\Omega} g(b_r(\omega) - b_s(\omega)) P(d\omega) = \int_{\mathbb{R}^d} g(y) N(0, A_{sr})(dy).
\]

Using this fact and the Fubini theorem we get from (2.16)

\[
\begin{align*}
  \int_{\mathbb{R}^d} f_{u,xj}(r, x + y) N(0, C_{sr})(dy)
  &= \mathbb{E} \left[ \int_{\mathbb{R}^d} f_{u,xj}(r, x + y + \Lambda_{sr}) N(0, (r-s)I_0)(dy) \right]
  = \mathbb{E} \left[ \int_{\mathbb{R}^d} f_{u,xj}(r, x + y + b_r - b_s) N(0, (r-s)I_0)(dy) \right].
\end{align*}
\]

Therefore we find

\[
\begin{align*}
  u_{u,xj}(s, x) &= -\mathbb{E} \left[ \int_s^\infty dr \int_{\mathbb{R}^d} f_{u,xj}(r, x + y + b_r - b_s) N(0, (r-s)I_0)(dy) \right].
\end{align*}
\]
Now we estimate the $L^p$-norm of $u_{ixj}$. To simplify the notation in the sequel we set $N(0, (r-s)I_0) = \mu_{sr}$. Using the Jensen inequality and the Fubini theorem we get
\[
\int_{R^d_+} ds \int_{R^d} |u_{ixj}(s,x)|^p dx
= \int_{R^d_+} ds \int_{R^d} \left[ \int_0^\infty dr \int_{R^d} f_{xj}(r,x+y+b_r) \mu_{sr}(dy) \right]^p dx
\leq \mathbb{E} \left[ \int_{R^d_+} ds \int_{R^d} \left[ \int_0^\infty dr \int_{R^d} f_{xj}(r,x+y+b_r) \mu_{sr}(dy) \right]^p dx \right].
\]

Now in the last line of the previous formula we change variable in the integral over $R^d$ with respect to the $x$-variable; we obtain
\[
\int_{R^d_+} ds \int_{R^d} |u_{ixj}(s,x)|^p dx \leq \mathbb{E} \left[ \int_{R^d_+} ds \int_{R^d} \int_0^\infty dr \int_{R^d} f_{xj}(r,u_{xj}+y+b_r) \mu_{sr}(dy) dx \right].
\]

To estimate the last term we fix $\omega \in \Omega$ and consider the function $g_\omega(t,x) = f(t,x + b_t(\omega))$, $(t,x) \in R^{d+1}$. The function $g_\omega$ is bounded, with compact support on $R^{d+1}$ and such that $g_\omega(t, \cdot) \in C_0^1(R^d), t \in R$.

By Lemma 2.2 we know that there exists $M_0 = M_0(d,p,p_0) > 0$ such that, for any $\omega \in \Omega$,
\[
\int_{R^d_+} ds \int_{R^d} \int_0^\infty dr \int_{R^d} f_{xj}(r,z+y+b_r(\omega)) \mu_{sr}(dy) dx \leq M_0^p \|g_\omega\|^p_{L^p}.
\]

Using also (2.20) we find
\[
\int_{R^d_+} ds \int_{R^d} |u_{ixj}(s,x)|^p dx \leq M_0^p \mathbb{E} \left[ \int_{R^d} \int_{R^d} |g_\omega(s,x)|^p dx \right]
= M_0^p \mathbb{E} \left[ \int_{R^d} \int_{R^d} \int_{R^d} |f(s,x+b_s)|^p dx \right]
= M_0^p \int_{R^d} \int_{R^d} |f(s,z)|^p dz.
\]

The proof is complete.

3 $L^p$-estimates involving Ornstein-Uhlenbeck operators

Let $A = (a_{ij})$ be a given real $d \times d$-matrix. We consider the following Ornstein-Uhlenbeck type operator
\[
L_0 u(t,x) = \sum_{i,j=1}^d c_{ij}(t)u_{ij}(t,x) + \sum_{i,j=1}^d a_{ij}x_ju_i(t,x)
= \text{Tr}(c(t)D_z^2 u(t,x)) + \langle Ax, D_z u(t,x) \rangle,
\]
$(t,x) \in R^{d+1}, u \in C_0^1(R^{d+1})$. This is a kind of perturbation of $L$ given in (2.8) by the first order term $\langle Ax, D_z u(t,x) \rangle$ which has linear coefficients.
We will extend Corollary 2.4 to cover the parabolic equation

$$u_t(t,x) + L_0u(t,x) = f(t,x)$$

(3.1)
on $\mathbb{R}^{d+1}$. We will assume Hypothesis 1 and also

Hypothesis 2. Let $p_0$ as in Hypothesis 1. Define $F_{p_0} \subset \mathbb{R}^{p_0}$ as the linear subspace generated by $\{e_1, \ldots, e_{p_0}\}$. Let $F_{p_0}$ be the linear subspace generated by $\{e_{p_0+1}, \ldots, e_d\}$ if $p_0 < d$ (when $p_0 = d$, $F_{p_0} = \{0\}$). We suppose that

$$A(F_{p_0}) \subset F_{p_0}, \quad A(F_{p_0}) \subset F_{p_0}. \quad (3.2)$$

Recall that given a $d \times d$-matrix $B$, $\|B\|$ and $Tr(B)$ denote, respectively, the operator norm and the trace of $B$. In the next result we will use that there exists $\omega > 0$ and $\eta > 0$ such that

$$\|e^{tA}\| \leq \eta e^{\omega t}, \quad t \in \mathbb{R}, \quad (3.3)$$

where $e^{tA}$ is the exponential matrix of $tA$. Note that the constant $M_0$ below is the same given in (2.9).

**Theorem 3.1.** Assume Hypotheses 1 and 2. Let $T > 0$ and set $S_T = (-T, T) \times \mathbb{R}^d$. Suppose that $u \in C_0^\infty(S_T)$.

For any $p \in (1, \infty)$, $i, j = 1, \ldots, p_0$,

$$\|u_{x_i x_j}\|_{L^p(\mathbb{R}^{d+1})} \leq \frac{M_1(T)}{\lambda} \|u_t + L_0u\|_{L^p(\mathbb{R}^{d+1})}; \quad (3.4)$$

with $M_1(T) = c(d)p_0\eta^4 e^{4\omega T} + \frac{\|c\|_{C_0^\infty(S_T)}}{\|c\|_{C_0^\infty(S_T)}} |Tr(A)|$.

**Proof.** We fix $T > 0$ and use a change of variable similar to that used in page 100 of [5]. Define $v(t,y) = u(t, e^{tA}y)$, $(t,y) \in \mathbb{R}^{d+1}$.

We have $v \in C_0^\infty(\mathbb{R}^{d+1})$, $u(t,x) = v(t, e^{-tA}x)$ and

$$u_t(t,x) + L_0u(t,x) = v_t(t,x) - \langle D_y v(t, e^{-tA}x), Ae^{-tA}x \rangle + Tr(e^{-tA}c(t)e^{-tA}D_y^2 v(t, e^{-tA}x))$$

$$+ \langle D_y v(t, e^{-tA}x), Ae^{-tA}x \rangle + Tr(e^{-tA}c(t)e^{-tA}D_y^2 v(t, e^{-tA}x)).$$

It follows that

$$u_t(t, e^{tA}y) + L_0u(t, e^{tA}y) = v_t(t,y) + Tr(e^{-tA}c(t)e^{-tA}D_y^2 v(t,y)). \quad (3.5)$$

Now we have to check Hypothesis 1. We first define $c_0(t), t \in \mathbb{R}$,

$$c_0(t) = e^{-tA}c(t)e^{-tA}, \quad t \in [-T, T], \quad (3.6)$$

$$c_0(t) = e^{-tA}c(T)e^{-tA}, \quad t > T, \quad c_0(t) = e^{tA}c(-T)e^{tA}, \quad t \leq -T.$$

Since $v \in C_0^\infty(S_T)$ we have on $\mathbb{R}^{d+1}$

$$v_t(t,y) + Tr(e^{-tA}c(t)e^{-tA}D_y^2 v(t,y)) = v_t(t,y) + Tr(c_0(t)D_y^2 v(t,y))$$

and so it is enough to check that $c_0(t)$ verifies (3.6). Moreover, by (3.6) it is enough to verify (2.6) for $t \in [-T, T]$. We have

$$\langle c_0(t)\xi, \xi \rangle = \langle c(t)e^{-tA}\xi, e^{-tA}\xi \rangle \geq \frac{\lambda}{4}(e^{-tA})^2 \xi|^2.$$
By (3.2) we deduce that $F_{p_0}$ and $F^{p_0}$ are both invariant for $A^*$. It follows easily that

$$I_0 e^{\alpha A^*} \xi = e^{\alpha A^*} I_0 \xi, \quad \xi \in \mathbb{R}^d, \quad s \in \mathbb{R}. \tag{3.7}$$

Using this fact we find for $t \in [-T, T]$, $\xi \in \mathbb{R}^d$,

$$|I_0 \xi|^2 = |I_0 e^{\alpha A^*} e^{-\alpha A^*} \xi|^2 = |e^{\alpha A^*} I_0 e^{-\alpha A^*} \xi|^2 \leq \eta^2 e^{2T\omega} |I_0 e^{-\alpha A^*} \xi|^2$$

and so

$$\lambda |I_0 \xi|^2 \leq \lambda \eta^2 e^{2T\omega} |I_0 e^{-\alpha A^*} \xi|^2 \leq \eta^2 e^{2T\omega} \langle c_0(t) \xi, \xi \rangle,$$

which implies $\lambda \eta^2 e^{-2T\omega} |I_0 \xi|^2 \leq \langle c_0(t) \xi, \xi \rangle$. By Corollary 2.4 and (3.5) we get, for any $i, j = 1, \ldots, p_0$,

$$\|v_{i,j}\|_{L^p} = \|\langle D_y^2 v(t,y) e_i, e_j \rangle\|_{L^p} \leq \frac{M_0 \eta^2 e^{2T\omega}}{\lambda} \|v_t + \text{Tr}(c_0(t) D^2_Y v)\|_{L^p} \tag{3.8}$$

and so $I_0 D_y^2 v(t,y) I_0 = \langle e^{A^*} I_0 D^2_x u(t, e^{A^*} y) e^A e_i, e_j \rangle$ and so $I_0 D_y^2 v(t,y) I_0 = \langle e^{A^*} I_0 D^2_x u(t, e^{A^*} y) I_0 e^A, t \in \mathbb{R}, y \in \mathbb{R}^d$. Indicating by $\mathbb{R}^{p_0} \otimes \mathbb{R}^{p_0}$ the space of all real $p_0 \times p_0$-matrices, we find

$$\|I_0 D_y^2 v I_0\|_{L^p(\mathbb{R}^{d+1}; \mathbb{R}^{p_0} \otimes \mathbb{R}^{p_0})} \geq e^{-\frac{2T}{\text{Tr}(A)}} \|e^{A^*} I_0 D^2_x u I_0 e^A\|_{L^p(\mathbb{R}^{d+1}; \mathbb{R}^{p_0} \otimes \mathbb{R}^{p_0})}.$$

Since, for $(t,x) \in \mathbb{R}^{d+1}$,

$$\|I_0 D_y^2 u(t,x) I_0\| \leq \eta^2 e^{2T\omega} \|e^{A^*} I_0 D_y^2 u(t,x) I_0 e^A\|$$

by (3.8) we deduce

$$\|I_0 D_y^2 u I_0\|_{L^p(\mathbb{R}^{d+1}; \mathbb{R}^{p_0} \otimes \mathbb{R}^{p_0})} \leq \eta^2 e^{2T\omega} \|e^{A^*} I_0 D_y^2 u(t,x) I_0 e^A\|\|u_t + L_0 u\|_{L^p}$$

which gives (3.4). The proof is complete.

**Examples 3.2.** The equation

$$u_t(t,x,y) + (1 + e^d) u_{xx}(t,x,y) + u_{xy}(t,x,y) + t^2 u_{yy}(t,x,y) + y u_y(t,x,y) = f(t,x,y), \tag{3.9}$$

$(t,x,y) \in \mathbb{R}^3$, verifying the assumptions of Theorem 3.1 with $p_0 = 1$ and so estimate (3.4) holds for $u_{xx}$.

**Remark 3.3.** Assumption (3.2) does not hold for the degenerate hypoelliptic operators considered in [3]. To see this let us consider the following classical example of hypoelliptic operator (cf. [10] and [12])

$$u_t(t,x,y) + u_{xx}(t,x,y) + x u_y(t,x,y) = f(t,x,y), \tag{3.10}$$

$(t,x,y) \in \mathbb{R}^3$. In this case $p_0 = 1$ and $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It is clear that (3.2) does not hold in this case. Indeed we can not recover the $L^p$-estimates in [3].
As an application of the previous theorem we obtain elliptic estimates for non-degenerate Ornstein-Uhlenbeck operators $\mathcal{A}$. These estimates have been first proved in [19]. Differently with respect to [19] in the next result we can show the explicit dependence of the constant $C_1$ in (3.13) from the ellipticity constant $\lambda$.

Let
$$\mathcal{A}u(x) = \text{Tr}(QD^2u(x)) + \langle Ax, Du(x) \rangle,$$  \hspace{1cm} (3.11)

$x \in \mathbb{R}^d$, $u \in C^\infty_0(\mathbb{R}^d)$, where $A$ is a $d \times d$ matrix and $Q$ is a symmetric positive [342] define $d \times d$-matrix such that

$$\langle Q\xi, \xi \rangle \geq \lambda |\xi|^2, \quad \xi \in \mathbb{R}^d,$$  \hspace{1cm} (3.12)

for some $\lambda > 0$.

**Corollary 3.4.** Let us consider (3.11) under assumption (3.12). For any $w \in C^\infty_0(\mathbb{R}^d)$, $p \in (1, \infty)$, $i, j = 1, \ldots, d$, we have (the constant $M_1(1)$ is given in (3.4))

$$\|w_{x_i x_j}\|_{L^p(\mathbb{R}^d)} \leq \frac{c(p)M_1(1)}{\lambda} \left(\|w\|_{L^p(\mathbb{R}^d)} + \|w\|_{L^p(\mathbb{R}^d)}\right).$$  \hspace{1cm} (3.13)

**Proof.** We will deduce (3.13) from (3.4) in $S_1 = (-1, 1) \times \mathbb{R}^d$ with $p_0 = d$.

Let $\psi \in C^\infty_0(-1,1)$ with $\int_{-1}^1 \psi(t)\,dt > 0$. We define, similarly to Section 1.3 of [3],

$$u(t,x) = \psi(t)w(x).$$

Since $u_t + L_0u = \psi'(t)w(x) + \psi(t)\mathcal{A}w(x)$, applying (3.4) to $u$ we easily get (3.13). $\blacksquare$

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