Renormalisation Group Flows
and Conserved Vector Currents

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Abstract

Irreversibility of RG flows in two dimensions is shown using conserved vector currents. Out of a conserved vector current, a quantity decreasing along the RG flow is built up such that it is stationary at fixed points where it coincides with the constant coefficient of the two current correlation function. For Wess-Zumino-Novikov-Witten models this constant coefficient is the level of their associated affine Lie algebra. Extensions to higher dimensions using the spectral decomposition of the two current correlation function are studied.

1.- Introduction

Zamolodchikov has established the irreversibility of Renormalisation Group flows in two dimensions, through his celebrated $c$-theorem [1]. The theorem is proved by constructing a quantity that is monotonically decreasing along the RG flow and is stationary at fixed points where it coincides with the central charge of the corresponding Conformal Field Theory. Three are the basic ingredients to the demonstration: Lorentz invariance, unitarity and existence and conservation all along the RG flow of the stress tensor. The difference between the UV and IR central charges is shown to be computable from the correlation function of two traces of the stress tensor using a very simple sum rule [2]. An alternative proof of the $c$-theorem which uses the spectral decomposition of the correlation function of precisely two stress tensors has been given in references [3] and [4]. This is a very convenient picture to understand the $c$-theorem in terms of loss of degrees of freedom. In effect, the central charge is a measure of the degrees of freedom in the theory. When the UV CFT is relevantly perturbed, some will become massive and decouple in the IR limit. This feature is explicitly seen on the spectral representation of the two stress tensor correlation function. Recall
that the stress tensor is the quantity to which all degrees of freedom couple. In unitary theories, where all degrees of freedom add positively to their counting, the central charge will effectively decrease. By integrating from the spectral density those degrees of freedom that have become massive, one can quantify how many are lost along the RG flow. In this manner a number of sum rules can be build up, Cardy’s [2] among them. There have been several attempts to enlarge the scope of the c-theorem beyond two dimensions, though no general conclusive result has been attained so far [5] [4] [6].

The purpose of this paper is to elaborate on a similar result for two dimensional theories with conserved vector currents [3]. In effect, one can mimic both Zamolodchikov’s and spectral proof of the c-theorem substituting the stress tensor by a conserved vector current. One constructs in this manner a quantity monotonically decreasing along the RG flow and stationary at fixed points where it coincides with the constant coefficient of the two current correlation function. Whenever vector currents satisfy a Kac-Moody algebra this constant coefficient corresponds to the level of its central extension [7]. In this sense, we shall colloquially talk about the k-theorem. Originally, Friedan proved the k-theorem using the spectral representation technique [3]. The basic conditions required to formulate and prove it are mostly the same one needs for the c-theorem, namely, the existence of a conserved vector current along the RG flow, Lorentz invariance and unitarity. Let us insist that the only requirement made on the current is its conservation. One should not be mislead by the particular case of fermion theories where the two point function of the vector current encodes information about the chiral anomaly. The k-theorem is a general statement on conserved currents and, as such, has no relation with the chiral anomaly.

Just like the c-theorem, the k-theorem portrays the loss along the RG of the degrees of freedom coupling to the vector current. These are accounted at CFT by the constant coefficient of the two current correlation function. In contradistinction to the c-theorem case, it may happen that none of the degrees of freedom coupling to the current becomes massive under a certain relevant perturbation. Then, the constant coefficient remains the same all along the flow. The quantification of the change in the constant coefficient is achieved through a number of sum rules, just like in the c-theorem case.

In order to enlarge the scope of the theorem to higher dimensions, the spectral proof seems to provide a natural framework. This is because the formalism is set up with no reference to the dimension of space time. Besides, the k-theorem case looks a priori simpler than the c-theorem one. In effect, when looking for an an extension of the spectral proof for the c-theorem, one has to consider two spin structures for the decomposition of the two stress tensor correlation function while in two dimensions, only one structure is present. Then, the information encoded in the central charge is split into the coefficients of each structure and gets so mistified. For conserved vector currents, only one structure occurs in two as well as in more dimensions. This is because the two possible possible structures one can build for the two point function with one momentum and the metric tensor are constrained to one by the current Ward Identity. Although this is a simplification with respect to the c-theorem shall see that it does not keep up expectations.

The paper unfolds as follows. In section 2, the irreversibility of RG flows for two dimensional theories with conserved vector currents is proved using techniques similar to Zamolodchikov’s in his original proof of the c-theorem. From that, the equivalent of the “Cardy sum rule” is derived. In section 3, the spectral decomposition of the two current correlation function is studied and the original proof by Friedan [3] of the k-theorem for two dimensional theories is reproduced. The
previous sum rule will then be recovered in this framework, and new sum rules will be obtained. The features described in sections 2 and 3 are then illustrated in section 4 by a simple example, two dimensional free massive fermions. Finally, the naive extension of the $k$-theorem using the spectral representation is considered in section 5, computing explicitly as a particular case the spectral density of gauge currents for massive bosons and fermions.

2.- RG flows in two dimensions and conserved vector currents

Let us start by giving a proof of the irreversibility of RG flows for theories with conserved vector currents in two dimensions which mimics Zamolodchikov’s proof of the $c$-theorem [1], since this is the historical standard and the way of reasoning the reader might be more familiar with.

Consider a two-dimensional Quantum Field Theory which is Euclidean invariant and reflection positive (reflection positivity amounts to the Euclidean equivalent of unitarity at the level of Green functions). Let $J_\alpha(x)$ be a vector conserved current in the theory, at classical and quantum level. Let us introduce complex coordinates, $z = x_0 + ix_1$, $\bar{z} = x_0 - ix_1$ and the following notation,

$$
J(z) = J_z(z, \bar{z}),
$$
$$
\bar{J}(z) = J_{\bar{z}}(z, \bar{z}).
$$

Consider the two current correlation function. Euclidean invariance arguments and the absence of anomalous dimensions for a conserved current allow us to write

$$
\langle J(z) J(w) \rangle = \frac{R(\tau)}{(z-w)^2},
$$
$$
\frac{1}{2} \langle J(z) \bar{J}(w) + \bar{J}(z) J(w) \rangle = \frac{S(\tau)}{(z-w)(\bar{z}-\bar{w})},
$$

where $\tau = \ln(z-w)(\bar{z}-\bar{w})\Lambda^2$, and $\Lambda$ is the renormalisation scale of the theory. For reflection-positive theories, $R(\tau)$ is a positive function while $S(\tau)$ is negative. AT CFT, $S(\tau) = 0$ and $R(\tau)$ becomes a constant, which we shall denote by $K_{\text{CFT}}$. In the case of a non-abelian conserved gauge current, it is related to the level of the central extension of the corresponding affine Lie algebra [7].

Among the two current Ward Identities of $J_\alpha$, we have

$$
\langle (\partial_\tau J(z) + \partial_{\bar{z}} \bar{J}(z)) J(w) \rangle = 0,
$$
$$
\langle J(z) (\partial_{\bar{z}} J(w) + \partial_{\bar{w}} \bar{J}(w)) \rangle = 0.
$$

Using definitions (2.1) and subtracting the two equations in (2.2) we obtain

$$
\dot{R}(\tau) + \dot{S}(\tau) = S(\tau),
$$

where $\dot{R} = \frac{d}{d\tau} R$. From that, a quantity decreasing along the RG flow can immediately be defined, namely,

$$
K(\tau) = 2(R(\tau) + S(\tau)),
$$
$$
\dot{K}(\tau) = 2S(\tau) \leq 0.
$$
At a fixed point, since $S(\tau) = 0$, $K(\tau)$ coincides with the constant coefficient of the two point function and is stationary. If we fix the value of $\tau$ (say, to $\tau_0$), $K(\tau = \tau_0)$ will depend only on the coupling constants of the theory, $g = \{g_i\}$. Then, the RG flow is given by

$$\beta_i(g) \frac{\partial}{\partial g_i} K(g) = \frac{1}{2} (z - w)(\overline{z} - \overline{w}) \langle J(z)\overline{J}(w) + \overline{J}(z)J(w) \rangle |_{\tau = \tau_0}. \quad (2.5)$$

A sum rule à la Cardy [2] can immediately be derived from the previous result. In effect, we can rewrite

$$\dot{K} = \frac{d}{d\tau} K = r^2 \frac{d}{dr^2} K, \quad (2.6)$$

where $r^2 = (z - w)(\overline{z} - \overline{w})$. The total change in the coefficient $K_{CFT} = K(\tau)|_{CFT}$ is given by

$$\Delta K_{CFT} \equiv K_{UV} - K_{IR} = K(\tau = -\infty) - K(\tau = \infty) = K(r^2 = 0) - K(r^2 = \infty) = - \int_0^\infty dr^2 \frac{d}{dr^2} K, \quad (2.7)$$

which, in the light of equation (2.6), becomes

$$\Delta K_{CFT} = -\frac{2}{\pi} \int d^2 x \frac{1}{2} \langle J(z)\overline{J}(w) + \overline{J}(z)J(w) \rangle. \quad (2.8)$$

We have so established the irreversibility of the RG flow for theories with conserved vector currents, using arguments similar to Zamolodchikov’s in his original proof of the $c$-theorem.

### 3. k-theorem and spectral representation

**3.1 - Spectral decomposition of a two current correlation function**

Just like the $c$-theorem, the $k$-theorem also admits a proof using the spectral decomposition of a two point function [3] [4]. Actually, this is the original way the theorem was demonstrated in reference [3]. Before going on with the proof, we shall study some necessary properties of the correlation function of two conserved vector currents.

Consider again a two-dimensional Quantum Field Theory which is Lorentz invariant and unitary. Let $J_\alpha(x)$ be a conserved current in the theory, at classical and quantum level. Let us study the two current correlation function. By inserting a resolution of the identity made out of representations of the Poincaré group, we obtain the spectral decomposition of $\langle J_\alpha(x)J_\beta(0) \rangle$ (see, for instance, [8]),

$$\langle J_\alpha(x)J_\beta(0) \rangle = (\partial_\alpha \partial_\beta - \eta_{\alpha\beta}\Box) \frac{1}{\pi} \int_0^\infty d\mu \ k(\mu) \Delta(x, \mu), \quad (3.1)$$

being $\Delta(x, \mu)$ the free propagator for a spinless particle of mass $\mu$, namely, in two dimensions,

$$\Delta(x, \mu) = \frac{1}{2\pi} K_0(\mu |x|). \quad (3.2)$$
The $K_n$ symbol stands, as usual, for the modified Bessel function of order $n$. The spectral density $k(\mu)d\mu$ measures the density of degrees of freedom coupling to the current at distance $\mu^{-1}$. It is made out of objects living in the Hilbert space of the theory, being therefore well defined.

In complex coordinates, with the conventions established in the previous section, equation (3.1) becomes

$$\langle J(z)J(w) \rangle = \frac{1}{8\pi^2} \int_0^\infty d\mu \, k(\mu) \frac{\mu^2 K_2(\mu |x|)}{p^2 + \mu^2} + \frac{1}{4\pi} \delta^2(x) \int_0^\infty d\mu \, k(\mu).$$

To determine the spectral density, one can proceed in several ways, from using form factors to direct calculation [9]. However, equation (3.1) written in momentum space can be recast into a dispersion relation, which provides a very convenient way to compute $k(\mu)$. In effect, in Euclidean space, we have,

$$\Pi_{\alpha\beta}(p) = \left(p_\alpha p_\beta - \delta_{\alpha\beta} p^2\right) \frac{1}{\pi} \int_0^\infty d\mu \, k(\mu) \frac{1}{p^2 + \mu^2}. \quad (3.4)$$

In the following, $\Pi_{\alpha\beta}(x)$ shall denote the two current correlation function $\langle J_\alpha(x)J_\beta(0) \rangle$. If we take the trace over both sides of equation (3.4), we have

$$\delta^{\alpha\beta} \Pi_{\alpha\beta}(p) = \frac{1}{\pi} \int_0^\infty d\mu \, k(\mu) \frac{\mu^2}{p^2 + \mu^2} + \text{constant},$$

which can be interpreted as a dispersion relation. Therefore, $k(\mu)$ is related to the imaginary part of the trace of $\Pi_{\alpha\beta},$

$$k(\mu) = \frac{2}{\mu} \text{Im} \delta^{\alpha\beta} \Pi_{\alpha\beta}(p^2 = -\mu^2). \quad (3.6)$$

This provides a suitable formula to evaluate the spectral density.

Yet preserving full generality, one can make some statements regarding the functional form of the spectral density [3] [4]. One has to start by looking at CFT, where no scales are present. By power counting arguments, it can be established that only two behaviours are allowed,

(i) $k_{\text{CFT}}(\mu) = k_0 \delta(\mu),$

(ii) $k_{\text{CFT}}(\mu) = \frac{k_0}{\mu},$

being $k_0$ a constant related to the constant coefficient of the two current correlation function. The case (ii) raises non-existing IR singularities which leaves form (i) as the correct one. Then, $\Pi_{\alpha\beta}(x)$ at CFT is

$$\Pi_{\alpha\beta}(x)|_{\text{CFT}} = \frac{k_0}{2\pi^2} \left(2x_\alpha x_\beta - \delta_{\alpha\beta}\right) \frac{1}{x^2}, \quad (3.7)$$

in agreement with reference [11]. For a general theory, the spectral function should then have the form,

$$k(\mu) = k_0 \delta(\mu) + k_1(\mu, \Lambda), \quad (3.8)$$

being $\Lambda$ the renormalisation scale in the theory and $k_1(\mu, \Lambda)$ a smooth function, non singular when $\mu \to 0$. The delta term will account for the contribution of the massless degrees of freedom coupling to the current while $k_1(\mu, \Lambda)$ comes from the intermediate states of mass $\mu > 0$. The role of unitarity is to ensure the positivity of $k_0$ and $k_1(\mu, \Lambda).$
3.2 - Spectral proof of the $k$-theorem

The considerations about the functional form of $k(\mu)$ are the first step towards the spectral proof of the $k$-theorem. In order to finish the demonstration after reference [3] we shall study the short and long distance behaviour of the two current correlation function. At short distances, when $x \to 0$, the two current correlation function (3.1) takes the form of that at CFT,

$$x \to 0 \Rightarrow \Pi_{\alpha\beta}(x) \to \frac{1}{2\pi^2} \left( 2 \frac{x_\alpha x_\beta}{x^2} - \delta_{\alpha\beta} \right) \frac{1}{x^2} \int_0^\infty d\mu \ k(\mu). \quad (3.9)$$

This expression recovers a more familiar aspect in complex coordinates,

$$x \to 0 \Rightarrow \begin{cases} \langle J(z)J(w) \rangle \to \frac{1}{\pi^2} \frac{1}{z^2} \int_0^\infty d\mu \ k(\mu), \\ \frac{1}{2} \langle J(z)\overline{J}(w) + \overline{J}(z)J(w) \rangle \to 0. \end{cases} \quad (3.10)$$

Identically, in the long distance limit, $x \to \infty$, $\Pi_{\alpha\beta}$ has the CFT form. This time,

$$x \to \infty \Rightarrow \begin{cases} \langle J(z)J(w) \rangle \to \frac{1}{\pi^2} \frac{1}{z^2} \lim_{\epsilon \to 0} \int_0^\epsilon d\mu \ k(\mu), \\ \frac{1}{2} \langle J(z)\overline{J}(w) + \overline{J}(z)J(w) \rangle \to 0. \end{cases} \quad (3.11)$$

We can identify the UV and IR coefficients of the two point function,

$$k_{UV} = \int_0^\infty d\mu \ k(\mu), \quad k_{IR} = \lim_{\epsilon \to 0} \int_0^\epsilon d\mu \ k(\mu), \quad (3.12)$$

From the general functional form for $k(\mu)$ (3.8), we get

$$k_{UV} = \int_0^\infty d\mu \ k(\mu) = k_{IR} + \int_0^\infty d\mu \ k_1(\mu, \Lambda). \quad (3.13)$$

For unitary theories, by positivity of the spectral density, we have

$$k_{UV} \geq k_{IR}. \quad (3.14)$$

We have thus shown that the coefficient of the two current correlation function at CFT decreases along the RG flow. The RG flow will thus be irreversible. The amount the coefficient decreases is given by the spectral sum rule

$$\Delta k = k_{UV} - k_{IR} = \int_0^\infty d\mu \ k_1(\mu, \Lambda). \quad (3.15)$$

This sum rule deserves some considerations. Being $J_\alpha(x)$ a conserved current, the spectral density $k(\mu)d\mu$ gets no renormalisation. Therefore, a change of scale is absorbed as

$$k(\mu)d\mu = k(\lambda\mu)\lambda d\mu \quad (3.16)$$
and the sum rule

$$
\Delta k = \int_0^\infty d\mu \, k_1(\mu, \Lambda) = \int_0^\infty \lambda d\mu \, k_1(\lambda\mu, \Lambda) = \int_0^\infty d\mu \, k_1(\mu, \frac{\Lambda}{\lambda})
$$

(3.17)
can be computed at any point along the RG flow, that is, at any $\Lambda$. The UV limit corresponds to $\lambda \to \infty$, which is equivalent to setting to zero the scale $\Lambda$. In this limit the spectral function must vanish for all $\mu \neq 0$ but still have a finite integral, which means that it becomes a representation of a Dirac $\delta$-function. This is the rigorous expression of the picture one draws for the behaviour of the spectral density along the RG flow: when we perturb the initial CFT some degrees of freedom coupling to the current become massive eventually decoupling in the IR limit. If the relevant perturbation does not succeed to turn massive any of the degrees of freedom in the theory, the spectral density will maintain the original UV delta form, and we will recover the equal sign in equation (3.14).

With these tools one can build a function which decreases along the RG flow and which is stationary at fixed points in order to contact with the proof in the previous section. For this purpose, we just need a positive smooth function $f(\mu)$ such that $f(0) = 1$, $f$ decreases exponentially for large $\mu$ and its derivative is negative. From the spectral density we have

$$
k(\Lambda) = \int_0^\infty d\mu \, k(\mu)f(\mu) = k_{IR} + \int_0^\infty d\mu \, k_1(\mu, \Lambda)f(\mu),
$$

$$
\Rightarrow \quad \Lambda \frac{d}{d\Lambda} k(\Lambda) \leq 0.
$$

Since the full dependence of $k(\Lambda)$ in $\Lambda$ will be given by its dependence in the couplings of the theory, the previous equation becomes equivalent to equation (2.5). The function $K(g)$ in the previous section would correspond to a particular choice of the function $f(\mu)$.

3.3 - Sum rules

Sum rules are used to evaluate quantitatively the change in $k_0$ along the flow. We have just seen how the spectral decomposition formulation of the $k$-theorem provides immediately a manner to compute such change, $\Delta k = k_{UV} - k_{IR}$, namely,

$$
\Delta k = \int_0^\infty d\mu \, k_1(\mu, \Lambda) = \lim_{\varepsilon \to 0} \int_0^\infty d\mu \, k(\mu).
$$

(3.19)

Besides, this spectral proof allows to recover the sum rule (2.8) of the previous section. We just have to use the decomposition (3.3) to see that

$$
\int d^2x \, \frac{1}{2} \langle J(z)\overline{J}(0) + \overline{J}(z)J(0) \rangle = -\frac{\pi}{4} \int_0^\infty d\mu \, k(\mu)
$$

$$
+ \text{contribution of the contact term.}
$$

(3.20)

Neglecting the contribution of the contact term, we recover equation (2.8),

$$
\Delta k = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty d\mu \, k(\mu) = -\lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} d^2x \, \frac{2}{\pi} \langle J(z)\overline{J}(0) + \overline{J}(z)J(0) \rangle.
$$

(3.21)
Recall that the coefficients $k$ and $K_{CFT}$ are related by a factor 4.

Similarly, a third sum rule can be found using $\langle J(z)J(0) \rangle$,

$$\Delta k = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} d\mu k(\mu) = -\frac{4}{\pi} \lim_{\epsilon \to 0} \int |x| > \epsilon d^2x \frac{\partial}{\partial z} \langle J(z)J(0) \rangle. \quad (3.22)$$

Thus, given a theory, we are armed with several tools to compute the change in the constant coefficient of the two point function.

4.- Example: fermions in two dimensions

In order to illustrate the features appearing in the previous sections, we shall study the case of two dimensional free fermions.

4.1 - The Abelian gauge current

Let $\psi(x)$ describe a free two-dimensional Dirac fermion of mass $m$. The theory has a well known U(1) symmetry whose conserved current is

$$J_\alpha(x) =: \bar{\psi}\gamma_\alpha \psi : (x) \quad (4.1)$$

The two current correlation function, $\Pi_{\alpha\beta}$, happens to be the one-loop contribution to the vacuum polarisation of the photon in the Schwinger model and encodes the chiral anomaly. If the mass is set to zero, we recover a Conformal Field Theory. Then,

$$\Pi_{\alpha\beta}(x)|_{m=0} = \frac{1}{2\pi^2} \left( \frac{2x_\alpha x_\beta}{x^2} - \delta_{\alpha\beta} \right) \frac{1}{x^2}, \quad (4.2)$$

which is the standard form for a two vector conserved current correlation function in CFT. This implies that $k_0 = 1$. Actually, the normalisation of equation (3.1) was devised to fit this result. When we switch on the massive perturbation, the theory flows towards the trivial fixed point where no degree of freedom couples to the current and, thus $k_{IR} = 0$. So, since $k_{UV} = 1$, we expect $\Delta k = 1$. In order to check this prediction, we need the massive version of $\Pi_{\alpha\beta}$. In momentum space, we have

$$\Pi_{\alpha\beta}(p) = \left( \delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2} \right) \left[ \frac{m^2}{\pi} \int_0^1 dt \frac{1}{-t^2p^2 + tp^2 + m^2} - \frac{1}{\pi} \right]. \quad (4.3)$$

From this equation, we can immediately check, using the sum rule (3.21), that, in effect,

$$\Delta k = 1. \quad (4.4)$$

We can also use equation (4.3), together with the dispersion relation (3.6), to evaluate the spectral density of $\Pi_{\alpha\beta}$. We find

$$k(\mu, m) = \frac{4m^2}{\mu^2 \sqrt{\mu^2 - 4m^2}} \theta(\mu^2 - 4m^2), \quad (4.5)$$
being $\theta(x)$ the step function. This result coincides with the computation in reference [10]. Note the two particle production threshold appearing. By the field content of the current, when introducing a resolution of the identity made out of Poincaré representations, one can see that only two particle states will saturate the correlation function. Precisely, Poincaré invariance will not allow any two particle state below the mass gap, so the threshold is a feature that the correct solution should exhibit. With equation (4.5), we can immediately check that

$$\Delta k = \int_0^\infty d\mu \ k(\mu, m) = 1. \quad (4.6)$$

In the limit $m \to \infty$, $k(\mu, m)$ tends to zero as expected. On the other end, in the limit $m \to 0$, $k(\mu, m)$ reproduces the behaviour of a Dirac delta function, with coefficient 1, recovering so the CFT behaviour. This fact can be checked by integrating $k(\mu, m)$ with a test function $f(\mu)$. In the integral, one must rescale the integration variable $\mu$ by $m$ in order to expand $f$ in Taylor series.

The non-vanishing terms in the expansion when $m \to 0$ should be kept to compare the result with that of the action of a delta function. The spectral decomposition is, thus, as announced in section 3, a representation of the Dirac delta function when the scale of the theory is removed.

4.2 - Non abelian gauge currents

Consider now a theory of two-dimensional free fermions, $\psi_i$, of equal mass $m$, with a non-abelian symmetry group, SU(N) for instance. Let $\Psi$ denote the $N$-plet of fermions and $t^a$ the generators of the algebra of the symmetry group in the representation in which the fermions live. Then the non-abelian conserved current is

$$J^a_\alpha(x) =: \Psi \bar{t}^a \gamma_\alpha \Psi : (x) \quad (4.7)$$

The two current correlation function is rapidly related to the Dirac fermion one by the formula

$$\Pi^{ab}_{\alpha\beta}(x) = \langle J^a_\alpha(x)J^b_\beta(0) \rangle = C_A \delta^{ab} \Pi^{\alpha\beta}(x), \quad (4.8)$$

where $\text{Tr} t^a t^b = C_A \delta^{ab}$. From this, we deduce that $k_0 = C_A$ when $m = 0$ and therefore, when the mass perturbation is switched on, $\Delta k = C_A$. We can change the normalisation of the generators,

$$\hat{J}^a_\alpha(x) =: \overline{\Psi} \hat{t}^a \gamma_\alpha \Psi : (x),$$

$$\text{Tr} \hat{t}^a \hat{t}^b = \delta^{ab},$$

in order to have $k_0 = 1$. If we now consider $n$ copies of the theory, this is equivalent to a Wess-Zumino-Novikov-Witten model of level $k = n$ [12]. When we add a mass to one of the copies, the total theory will change from $k_{UV} = n$ to $k_{IR} = n - 1$, $k$ decreasing along the RG flow. This is a particular case of theory with an affine Lie algebra associated in which we check the decreasing of the level of the algebra along the flow.
5.- Extension to higher dimensions

5.1 - Spectral decomposition in more than two dimensions

The extension to more than two dimensions of the previous results on the irreversibility of RG flows for theories with vector conserved currents would be very interesting. To achieve this goal, the spectral representation approach seems most promising because its essential set up is made irrespective of the dimension of space time. On the other hand, the proof à la Zamolodchikov is very much constrained by to the two dimensional formalism. Besides, the vector current case looks a priori simpler than the c-theorem one. In effect, the spectral decomposition of the two stress tensors correlation function has two spin structures [4], while the two current correlation function has only one. For this reason we shall study the extension of the k-theorem to higher dimensions through this approach.

We consider this time a d-dimensional Quantum Field Theory which is Lorentz invariant and unitary. We denote by $J_\alpha(x)$ the required conserved current at classical and quantum level. The analysis of the spectral decomposition of $\langle J_\alpha(x) J_\beta(0) \rangle$ is essentially the same than the two dimensional case. We have

$$
\Pi_{\alpha\beta}(x) = \langle J_\alpha(x) J_\beta(0) \rangle = (\partial_\alpha \partial_\beta - \eta_{\alpha\beta} \Box) \int_0^\infty d\mu \, k(\mu) \Delta(x,\mu),
$$

(5.1)

being $\Delta(x,\mu)$ the d-dimensional free propagator for a spinless particle of mass $\mu$,

$$
\Delta(x,\mu) = \frac{1}{2\pi} \left( \frac{\mu}{2\pi x} \right)^{d/2} K_{d/2}(\mu \vert x \vert).
$$

(5.2)

By rewriting equation (5.1) in Euclidean momentum space and taking the trace over space time indices, we obtain, just like in the two-dimensional case, a dispersion relation. It relates the spectral density $k(\mu)$ with the imaginary part of the correlation function,

$$
k(\mu) = \frac{2}{(d-1)\pi \mu} \text{Im} \delta^{\alpha\beta} \Pi_{\alpha\beta}(p^2 = -\mu^2).
$$

(5.3)

For unitary theories, $k(\mu)$ is positive, as in the two dimensional case.

In more than two dimensions it becomes harder to put forward a general statement on the form of $k(\mu)$, in the line of equation (3.8). If we start by the form at CFT, it is well known that, in any dimension, Conformal Invariance completely constrains the form of two point functions (see, for instance [13] and [11]). In the case of conserved vector currents, this compulsory behaviour is

$$
\langle J_\alpha(x) J_\beta(0) \rangle = \frac{A}{|x|^{2(d-1)}} \left( \frac{x_\alpha x_\beta}{x^2} - \delta_{\alpha\beta} \right),
$$

(5.4)

where $A$ is a constant. Regarding the functional form of $k(\mu)$, which must reproduce equation (5.4), dimensional analysis allows only two possibilities,

(i) $k_{\text{CFT}}(\mu) = k_0 \mu^{d-2} \delta(\mu),$

(ii) $k_{\text{CFT}}(\mu) = k_0 \mu^{d-3},$

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with, $k_0$, of course, a constant. In contradistinction to two dimensions, case (ii) does not arise IR singularities, and it is perfectly well defined. Actually, if we plug case (i) into formula (5.1), the two current correlation function vanishes (which implies that $A = 0$). However, form (i) allows the splitting of a generic spectral density into

$$k(\mu) = k_0 \mu^{d-2} \delta(\mu) + k_1(\mu, \Lambda),$$

being $k_1(\mu, \Lambda)$ smooth in the limit $\mu \to 0$. In this way, the contribution from massless degrees of freedom is separated from the contribution of massive ones, just like in the two dimensional analysis. On the other hand, case (ii) delivers the general result (5.4). The constants $k_0$ and $A$ are then related by the following equation,

$$k_0 = \frac{2^{4-d} \pi^{d/2}}{\Gamma(d) \Gamma(\frac{d}{2} - 1)} A,$$  \hspace{1cm} (5.5)

where $\Gamma(t)$ denotes the Euler $\Gamma$-function. The splitting of $k(\mu)$ makes no sense any more since for such kind of currents, massless degrees of freedom couple to all distances $\mu^{-1}$ and thus they are hard to separate from massive ones.

5.2 - Examples : free bosons and fermions

To show how the spectral decomposition works in more than two dimensions, we shall consider two very simple examples. These are free bosons and fermions of mass $m$, with a U(1) symmetry.

Let us start by studying a free complex boson $\varphi(x)$ in $d$ dimensions. The theory has a U(1) conserved current

$$J_{\alpha}^{\text{bos}}(x) = i : \varphi^* \partial_\alpha \varphi : (x).$$  \hspace{1cm} (5.6)

A first test consists in checking the CFT behaviour of the two current correlation function for the $m = 0$ case. One recovers the form (5.4) with

$$A^{\text{bos}} = \frac{2}{(d - 2) S_d^2},$$  \hspace{1cm} (5.7)

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$. Since the value of $A$ is non-vanishing, we expect that the spectral density will follow the power law behaviour rather than the delta one. When switching on the massive perturbation, the computation of the two current correlation function in Euclidean momentum space delivers

$$\Pi^{\text{bos}}_{\alpha \beta}(p) = (p_\alpha p_\beta - \delta_{\alpha \beta} p^2) \frac{\Gamma \left(2 - \frac{d}{2} \right)}{(4\pi)^{d/2}} \int_0^1 dt \frac{(1 - 2t)^2}{(t(1 - t)p^2 + m^2)^{d - \frac{d}{2}}},$$  \hspace{1cm} (5.8)

Using the dispersion relation (5.3), we retrieve the spectral density of the two current correlation function. We find

$$k^{\text{bos}}(\mu) = \frac{1}{2(4\pi)^{d-1}} \frac{1}{\Gamma \left(\frac{d+1}{2} \right)} \left(\frac{\mu}{2}\right)^{d-3} \left(1 - \frac{4m^2}{\mu^2}\right)^{\frac{d-1}{2}} \theta(\mu^2 - 4m^2).$$  \hspace{1cm} (5.9)
We see the two particle production threshold appearing as a healthy sign of the result. Besides, if we take the limit \( m \to 0 \), we recover the expected power law behaviour and the coefficient \( k_0 \) matches the conformal result (5.7). Another interesting limit is the \( d \to 2 \) one. In effect, we see that the power law behaviour still holds when we further take \( m \to 0 \). This is due to the IR troubles of the free massless boson in two dimensions. The generalisation of these results to the non-abelian case is straightforward. If \( t^a \) are the generators of the non-abelian symmetry group of a free boson theory, with the normalisation,

\[
J_a^a(x) =: \varphi \partial_\alpha \varphi : (x),
\]

\[
\text{Tr} (t^at^b) = -N_\varphi \delta^{ab},
\]

we have,

\[
\langle J_\alpha^a(x)J_\beta^b(0) \rangle = \frac{1}{2} N_\varphi \delta^{ab} \Pi_{\alpha\beta}^{\text{bos}}(x).
\]

In the massless limit, we recover the results from reference [11].

The second example is a free \( d \)-dimensional fermion \( \psi(x) \) of mass \( m \). The U(1) conserved current is

\[
J_\alpha(x) =: \overline{\psi} \gamma_\alpha \psi : (x).
\]

Again we start by checking the CFT behaviour in the \( m = 0 \) case. We have

\[
A^{\text{fer}} = \frac{2\sqrt{\pi}}{\sqrt{d}}.
\]

Then,

\[
\Pi^{\text{fer}}_{\alpha\beta}(p) = -2(p_\alpha p_\beta - \delta_{\alpha\beta}p^2) \frac{\Gamma \left( 2 - \frac{d}{2} \right)}{2(2\pi)^{\frac{d}{2}}} \int_0^1 dt \frac{t(1-t)}{(t(1-t)p^2 + m^2)^{\frac{d}{2}-\frac{3}{2}}},
\]

and

\[
k^{\text{fer}}(\mu) = \frac{\sqrt{\pi}}{(2\pi)^{\frac{d}{2}} \Gamma \left( \frac{d+1}{2} \right)} \left( \frac{\mu}{2} \right)^{d-3} \left( 1 - \frac{4m^2}{\mu^2} \right)^{\frac{d-3}{2}} \left[ 1 - \frac{1}{d-1} \left( 1 - \frac{4m^2}{\mu^2} \right) \right] \theta(\mu^2 - 4m^2).
\]

Just like for the boson case, the two particle production threshold is found. The limit \( m \to 0 \) also delivers the expected power law behaviour with a coefficient compatible with (5.13). For the \( d \to 2 \) limit, we recover equation (4.5), and the following limit \( m \to 0 \) gives the correct delta result. Curiously, we can invert the order of the two limits (first \( m \to 0 \) and then \( d \to 2 \)), and see how the delta behaviour is surprisingly recovered from a clever appearing of \((d-2)\) factors. Regarding the non-abelian generalisation, we just take

\[
J_\alpha^a(x) =: \overline{\psi} i t^u_\alpha \gamma_\alpha \psi_j : (x),
\]

\[
\text{Tr} (t^a t^b) = -N_\psi \delta^{ab},
\]

to get

\[
\langle J_\alpha^a(x)J_\beta^b(0) \rangle = N_\psi \delta^{ab} \Pi^{\text{fermion}}_{\alpha\beta}(x),
\]

which is again consistent with the results from reference [11].
5.3 - Extension of the $k$-theorem to higher dimensions

The two possible behaviours for the spectral density $k(\mu)$ described in paragraph 5.1 are the same that exhibit, respectively, the spin zero and the spin two spectral densities, when we decompose the correlation function of two stress tensors for $d > 2$. Therefore, we shall briefly sketch the discussion in references [4] and [6] about whether a quantity decreasing along the RG flow exists.

In case (i), such a quantity can be built up by following closely the two dimensional argument. This can be done because the contribution of massless degrees of freedom is clearly separated from that of the massive ones. However, at present, it is not known whether the coefficient $k_0$ can be defined always and uniquely for any CFT, like $c$ or $k$ are in two dimensions. Then, we cannot speak about irreversibility of the RG flow since a theory could come back to herself in a RG loop, with a different value of $k_0$. The only existing approach defines $k_0$ using a limiting procedure away from criticality, which is only consistent if the space of theories is a manifold. In this case, we can rigorously speak about irreversibility of the RG flow. Therefore, we are in the same situation than in the $c$-theorem with the spin zero spectral density.

In case (ii), on the other hand, the constant $k_0$ is well defined at the Conformal point, by the two point function, following formula (5.5). However, a quantity decreasing along the RG flow cannot immediately be built since, despite $k(\mu)$ is positive, information about the positivity of the derivative of $k(\mu)$ would be needed. Such information requires dynamical considerations, spoiling a general statement about the RG flow. This is precisely the same problem encountered in the analysis of the spin two spectral density of the two stress tensor correlation function. In the case of the examples in paragraph 5.2, both theories flow from the gaussian Conformal point $m = 0$ towards the trivial fixed point. This means that $k_0$ goes from the free theory value towards 0, effectively decreasing along the flow. However, this is only a very simple example, and does not allow to draw any conclusion for more complicated flows in interacting theories.

Summing up, we have shown the irreversibility of RG flows in two dimensions for theories with vector conserved currents. The naive extension of the spectral version of the $k$-theorem encounters the same difficulties than its corresponding version of the $c$-theorem, in spite of the a priori simplification drawn by the only spin structure of the decomposition of the correlation function of two conserved currents. From here, one can look for particular applications of the two dimensional result to a number of conserved currents. Regarding the extension to higher dimensions, the study of particular flows in interactive theories might shed some light on the behaviour of the derivative of $k(\mu)$ in case (ii), just to see if a theorem is ruled out by a counterexample or it is verified in some specific cases. However, these extensions are beyond the scope of this introductory paper.

Acknowledgements

This work has been partially supported by the Ministerio de Educaci´on y Ciencia through an FPI grant, CICYT under contract AEN 90-0033 and NATO under contract CRG-910890. I would like to acknowledge the warm hospitality of the Lawrence Berkeley Laboratory where part of this work was developped. I thank very specially Jos´e I. Latorre for introducing me to the subject and sharing his insight. I am indebted to O.Alvarez, P.E.Haagensen and J.Soto for discussions, and A.Cappelli for his comments on the manuscript.
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