Mutual Conversions of Physical and Mathematical Quantities in Measurement

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Abstract. Metrology is known to be a science of measurement and its application. Measurement is carried out by indicating measuring instruments, transducers, material measures, comparators and other measuring devices. Quality of many modern measuring devices is based often on characteristics of analog integrated circuits. Mathematical modelling of measurement (including the evaluation of a measurement uncertainty) is very important. Measuring devices and modelling can be described by the conversion of input quantities to an output quantity. All possible quantities can be divided into two parts: physical and mathematical quantities. General approach to the evaluation of uncertainty for the all kinds of conversions is offered in this paper.

1. Introduction

Basic and general concepts and associated terms of metrology are well known [1]. But some private definitions and recommendations are very different in papers. For example, an absolute error of a material measure is defined as difference between the nominal value of this measure and the true value of a quantity reproduced by this measure [2]:

$$\Delta = x_{nom} - y_{tr}$$ (1)

Here and below a conventionally true value can be used instead of the true value [1, 2.12]. The opposite definition of the absolute error is used, in instance, for resistors: difference between the true value of a quantity reproduced by a measure and the nominal value of this measure:

$$\Delta = y_{tr} - x_{nom}$$ (2)

Such difference of the definitions is not important if a tolerance is given by “±” form. The situation changes dramatically if the temperature coefficient of a material measure is found from experiments in order to exclude a systematic error. The mistake in the sign of the error can decrease accuracy instead of improvement.

The absolute input and output errors of a transducer [1] are

$$\Delta_{in} = \varphi_{nom}(y_{tr}) - x_{tr}$$ (3)
$$\Delta_{out} = y_{tr} - f_{nom}(x_{tr})$$ (4)

where $x_{tr}$ and $y_{tr}$ – the true values of input quantities of the transducer, $f_{nom}(x_{tr})$ and $\varphi_{nom}(y_{tr})$ – the nominal (ideal) conversion and invers functions of the transducer.

The most nominal conversion and reverse functions are linear and can be described by one parameter – sensitivity [1, 4.12]. The measurent error of an ideal measuring instrument, the input and output errors of an ideal transducer are equal to zero.

Integrated circuits (IC) are described by different models [3]. The ideal IC is described by ideal values for the parameters of the model [3, III-1]. The chois of “important” parameters for the ideal IC is not usually based on a mathematical base but rather on a “good sence”.

The goal of this paper is to give General Theory of Conversion (GTC) for all devices used in measurement including measuring instruments, transducers, analog integrated circuits etc. The theory is based on divisions of all quantities in a measurement into two parts: physical and mathematical quantities. Then all possible actions in a measurement can be divided into one of four conversions.

2. Two quantities and three conversions for measuring devices
In accordance with dichotomous divisions in logic, we offer to divide all quantities used in measurement into two parts: physical quantities and mathematical quantities.

The value of a physical quantity depends on many influence quantities. It has a definitional uncertainty [1, 2.27] due to three reasons: it is not all influence quantities are taken into account; any influence quantity cannot be described by one value, a physical quantity is a function of time even for the constant values of all influence quantities. Any physical quantity has many true values in accordance with values of influence quantities and time moments. For example, electrical resistance of any object depends on temperature, current, time. The mathematical quantity has a definite value or can be presented by the value with any necessary accuracy. All results of measurements, codes in electronic devices, nominal values of material measures, mathematical constants (e, \( \pi \) etc.) are considered in this paper as mathematical values. All measuring devices including analog integrated circuits can be considered as converters of an input physical quantity \( x_{ph} \) into an output physical quantity \( y_{ph} \) or a mathematical quantity \( y_{mat} \) as a converter of an input mathematical quantity \( x_{mat} \) into an output physical quantity \( y_{ph} \):

\[
\begin{align*}
 y_{ph} & = f(x_{ph}) \\ 
 y_{mat} & = f(x_{mat}) \\ 
 y_{ph} & = f(x_{mat})
\end{align*}
\]
where \( R_S = 0.5 \cdot (R_S^+ + R_S^-) \), \( \Delta R_S = R_S^+ - R_S^- \), \( i_{in,cm} \) – common mode input current of the OA, \( i_{in,dif} \) – differential current of the OA.

Necessary and sufficient conditions for an ideal amplifier with any source resistances, in accordance with (10), are

\[
v_{in,dif} = 0, \quad i_{in,cm} = 0, \quad i_{in,dif} = 0
\]  

(11)

In practice, the members of (11) are not equal to zero due to many influence quantities. The most important quantities are usually: output voltage \( v_{out} \), input common mode voltage \( v_{in,cm} \), power supply voltage, output current \( i_{out} \), ambient temperature. In linear approximation, for low frequency, we can find, in accordance with (11), three dc parameters and 15 influence coefficients in order to evaluate systematic static errors of a conversion by the OA. Let us find only \( v_{in,dif} \) and \( i_{in,dif} \) as functions of only two influence quantities \( v_{out} \) and \( i_{out} \) in order to compare the results found by the GTS and results found due to “common sense” in [3, III-1]:

\[
v_{in,dif} = v_{out} + \frac{v_{cm}}{CMRR} + \frac{i_{out} \cdot R_{out}}{A}
\]

(12)

\[
i_{dif} = \frac{v_{out}}{0.5 \cdot A \cdot R_{in}} + \frac{d i_{in,dif}}{dv_{cm}} \cdot v_{cm} + \frac{i_{out} \cdot R_{out}}{0.5 \cdot A \cdot R_{in}}
\]

(13)

where parameters of the OA: \( A \) – open-loop gain, \( CMRR \) – common-mode rejection ratio, \( R_{out} \) – output impedance, \( R_{in} \) – input impedance. All mentioned parameters are well-known and are usually specified. The parameter \( d i_{in,dif} / dv_{cm} \) is found due to the GTS and is not specified in all the books known to the authors of this paper. The conditions \( A = \infty \) and \( CMRR = \infty \) are necessary both for known theory [3] and the GTS. The conditions \( R_{in} = \infty \) and \( R_{out} = 0 \) [3, III-1] are not necessary if \( A = \infty \). Even more, for high frequency, the input impedance is designed comparatively low to decrease influence of an input capacitance (THS3001, eg). The second term of (13) can be very important at low frequency if \( R_S \) is very high and \( \Delta R_S \) is low.

For modulator of delta-sigma ADC, the standard deviation of quantization noise \( \sigma \) is supposed by many authors equal to \( V_{ref}/\sqrt{3} \), where \( V_{ref} \) is a reference voltage [5]. According to the GTC, the standard deviation of quantization noise depends very much on input signal. The results were found by analytical modelling and confirmed by simulation and physical experiments [6]. If the input voltage \( V_{in} \) is constant then \( \sigma = \sqrt{V_{ref}^2 - V_{in}^2} \).

### 3. Conversion of mathematical quantities to mathematical quantities

A typical example of such a conversion is rounding a numerical value. An input mathematical quantity \( x_{mat} (\pi, \sqrt{3}, \text{etc.}) \) is transformed to an output mathematical quantity \( y_{mat} \) after rounding. The sensitivities of the nominal conversion and inverse functions are equal to one. According to (3) and (4), \( \delta_{IN} = \delta_{OUT} = \Delta = y_{mat} - x_{mat} \). Relative error is approximately \( \delta = (\Delta / y_{mat}) \cdot 100\% \). For the final result of measurement uncertainty, we recommend to use always two significant digits. The worst case \( (x_{mat} = 1.0, y_{mat} = 1.0) \) gives \( \delta = -5\% \). This number can be used as a standard to consider any error as negligible with regard to other. Some authors advice to use only one significant digit if the first digit is 3 or more. We think that it is a bad advice because relative error for \( x_{mat} = 3.499, y_{mat} = 3.499 \) is approximately 16 %, much more than the negligible level of 5 % offered before.

The evaluation of uncertainty is a very important case of conversions from mathematical quantities (input uncertainties) to the output mathematical quantity (expanded uncertainty of measurement, in instance). It is an example of mathematical modelling [7]. Application of the GTC can be used here to find errors of modelling for different conditions [7]. Let us consider only one example. If input uncertainties referred to output are represented by bounds \( \pm a_i \), a rectangular distribution law is usually used [4]. Then extended uncertainty Type B with a coverage probability \( P = 0.95 \) is
We recommend to use also other method of modelling – the worst-case one with \( P = 1 \) [8,9]:

\[
U_{0.95} = \frac{2}{\sqrt{3}} \sqrt{\sum_{i=1}^{n} a_i^2}
\]  

(14)

Let us compare (14) and (15) for the case when one of input uncertainties dominates. Then we can find an absurd result \( U_{0.95} \geq U_1 \). The rectangular law is often unphysical [4, 4.3.9]. It is more realistic to use trapezoidal distribution with a standard uncertainty in the range \( a_i/\sqrt{6} \leq s_i \leq a_i/\sqrt{3} \) [4]. We recommend to use approximately the average value \( s_i = a_i/2 \). Then

\[
U_{0.95} = \sqrt{\sum_{i=1}^{n} a_i^2}
\]  

(16)

Now the uncertainty for \( P = 1 \) is always more than the uncertainty for \( P = 0.95 \).

4. Conclusions

Application of the GTC allowed us to find the true definition of a material measure error; to give the mathematical definition of an ideal operational amplifier and to predict the important parameter which is not specified now; to define a more accurate standard deviation of the quantisation noise for a delta-sigma ADC; to show necessity to use always two significant digits for the final measurement uncertainty; to offer the more realistic evaluation of measurement uncertainty with a coverage probability \( P = 0.95 \).

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