Poincaré-Snyder Relativity with Quantization

Otto C. W. Kong and Hung-Yi Lee

Department of Physics and Center for Mathematics and Theoretical Physics,
National Central University, Chung-li, TAIWAN 32054.

Abstract

Based on a linear realization formulation of a quantum relativity, — the proposed relativity for quantum ‘space-time’, we introduce the new Poincaré-Snyder relativity and Snyder relativity as relativities in between the latter and the well known Galilean and Einstein cases. We discuss how the Poincaré-Snyder relativity may provide a stronger framework for the description of the usual (Einstein) relativistic quantum mechanics and beyond with particular focus first on a geometric quantization picture through the $U(1)$ central extension of the relativity group, which had been establish to work well for the Galilean case but not for the Einstein case.

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I. INTRODUCTION

Possible relativities as described by relativity symmetries beyond the Lorentz or Poincaré group of Einstein relativity have been catching quite some interest recently (see for example Refs. [1, 2]). Since the pioneering work of Snyder [3], symmetry deformation, mostly considered as required to implement an invariant quantum scale, has been a main key for the direction of theoretical pursuit. That gives the idea of a quantum relativity. On the other hand, if one does believe that the entity we used to know as space-time does have a different structure at the true microscopic/quantum level that can plausibly be described directly, such a ‘quantum space-time’ will have its own relativity. The relativity symmetry deformations could be nicely formulated as Lie algebra stabilizations [4]. Following the line of thinking, we implemented in Ref. [5] a linear realization perspective to arrive at the ‘quantum space-time’ description with the quantum relativity symmetry as the starting point. Lorentz or Poincaré symmetry (of Einstein relativity) can be considered exactly a result of the stabilization of the Galilean relativity symmetry. The linear realization scheme in that setting is nothing other than the Minkowski space-time picture. Such a mathematically conservative approach, however, leads to a very radical physics perspective that at the quantum level space-time is to be described as part of something bigger [5]. The latter as the arena for the description of the new fundamental physics is called the quantum world in Ref. [6]. It is to be identified, mathematically, as the coset space $SO(2, 4)/SO(2, 3)$ [7], or the hypersurface $\eta_{MN} X^M X^N = 1$ [$\eta_{MN} = (-1, 1, 1, 1, 1, -1)$], \(^1\) within the 6D classical geometry with $X^\mu$ ($\mu = 0$ to 3) being space-time coordinates while $X^4$ and $X^5$ being non-space-time ones. The ‘time’ of Minkowski space-time is not just an extra spatial dimension. Its nature is dictated, from the symmetry stabilization perspective, by the physics of having the invariant speed of light $c$. The other two new coordinates in our ‘quantum space-time’ picture are likewise dictated to be neither space nor time [5, 6]. We reproduce in table 1 the suggested physics of the stabilizations/deformations involved in our stabilizations and extensions by translations (of the corresponding arenas for the linear realizations) sequence

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\(^1\) Note that we have flipped the metric sign convention adopted in our earlier publications [3–7]; from now on, the time-like (space-like) geometric signature is -1 (+1)
arriving at the $SO(2, 4)$ quantum relativity as illustrated by

$$
ISO(3) \rightarrow SO(1, 3) \iff ISO(1, 3) \\
\rightarrow SO(1, 4) \iff ISO(1, 4) \rightarrow SO(2, 4)
$$

Like $X^0 = ct$, we have $X^4 = \kappa c\sigma$ and $X^5 = \ell \rho$ with however, $\sigma$ having the physics dimension of time/mass (and a space-like geometric signature) and $\rho$ a pure number. Understanding the physics role of the two extra coordinates $\sigma$ and $\rho$ of the quantum world is considered crucial for any attempt to formulate the dynamics. Here in this letter, we report a way to approach the challenge — analyzing the physics of what we called the Poincaré-Snyder relativity.

We will explain first, in the next section, structure of the Poincaré-Snyder relativity, with symmetry denoted by $G(1, 3)$. In short, it is mathematically sort of a ‘Galilie group’ for 4D space-time. The analog of time $t$ as an external evolution parameter for Galilean dynamics is here given by $\sigma$. Recall that $\sigma$ has a space-like geometric signature but the physics dimension of time/mass. We are inspired to consider the new relativity by our studies on the quantum relativity. The ultimate goal is to analyze and formulate physics directly for the intrinsically quantum arena — the quantum world (see discussions in Refs. [6, 7]).

To better prepare ourselves for the formidable challenge, we want to take a step backward and study the relativity(ies) with symmetry between the Einstein and our quantum case. From the latter perspective, the Poincaré-Snyder relativity is the first step beyond Einstein relativity. Its physics setting should be not much different from the latter. It has, however, a mathematical structure very similar to the $G(3)$ Galilean case. The latter suggests similar mathematics in the formulation of the admissible dynamics for the two cases, both at the classical and quantum level. The Poincaré-Snyder relativity mechanics may hence be a much more familiar object. We must warn the readers that the physics interpretations of the similar mathematics are expected to be quite nontrivial and un-conventional though.

The Poincaré-Snyder relativity is still a relativity on 4D Minkowski space-time, only with an extra kind of momentum dependent reference frame transformations admitted. These momentum boosts are independent of the usual velocity boosts, but reduce to the latter when $\sigma = \tau/m$, the Einstein proper time over the standard (fixed) particle rest mass. Just as Galilean velocity boosts are transformations on 3D space dependent on an external parameter time, the momentum boosts enforce the independent $\sigma$-coordinate external to
4D space-time. The ‘dynamic’ formulation naturally suggests taking $\sigma$ as a sort of ‘evolution’ parameter. We called that $\sigma$-dynamics or $\sigma$-evolution, withholding the exact physics interpretation. Within the Einstein framework, $\sigma$-evolution looks like proper time evolution, and as such have actually been used quite a lot in the literature to describe Einstein relativistic dynamics, both classical and quantum\[8, 9\]. This letter is the first step to take a second look at the kind of studies, focusing on the difference and superiority of the new Poincaré-Snyder perspective. In particular, we will present in section 3 results of the picture of quantization as $U(1)$ central extension \[10\]. Notice that unlike the quantum relativity, and possibly the Snyder relativity obtained from the stabilization of the Poincaré-Snyder relativity, the construction of the $G(1, 3)$ symmetry involves none of the quantum physics inspired deformations with quantum scale(s) as deformation parameters. Hence, there is no reason at all to expect the relativity to be in any sense quantum. It looks only like a different perspective to look at classical physics on Minkowski space-time; and as such should be liable to quantization. Results of section 3 actually lay further justification to that a posteriori. Of course the ultimate justification for the $G(1, 3)$ approach from the theoretical side has to come from the relativities beyond. Or there is the possibility of seeing experimental evidence for Poincaré-Snyder relativity or Snyder relativity through careful studies of the $\sigma$-dynamics and its physics interpretations.

The kind of physics picture we have in mind behind our work and the earlier papers \[5, 6\] is not quite like any of the familiar old pictures, and admittedly not yet fully conceptualized. The research program is a very ambitious one, aiming at dynamics beyond any existing framework. We find the need to take the most conservative strategy, trying to commit to the minimal conceptual physics picture on any particular new aspects arising from the formulation before one can be sure that it is the right way to look at it. We try to learn from the mathematics and logics of the basic formulation what it can offer. One will see that such a conservative strategy can still bring out quite some interesting results presented in this letter and in another accompanying long paper \[11\].

Readers be very cautious. This letter can be read without detailed reading of the earlier papers, but what motivates a particular definition or approach would then be difficult to appreciate. With or without reference to Refs.\[5, 6\], it is important for readers to read first what we presented as it is, without assuming a perspective from any other theory standard or less conventional. This is especially true with the very similar looking structure from the
line of work on covariant formulation of Einstein relativistic dynamics. We highlight here a few crucial points to bear in mind. The momentum boosts are newly introduced transformations with physical meaning still to be clarified (see for some discussions). That comes with a modified or sort of generalized definition of energy-momentum, as \( \sigma \) derivatives. The \( \sigma \) parameter is of central importance, with physics content still to be fully understood. It is definitely not a measure of time. We will introduce the dynamics of the Poincaré-Snyder relativity as a formal \( \sigma \)-evolution, more or less duplicating mathematically the time dynamics of the Galilean case. In fact, a main aim of the studies is to learn about how to think about the physics of the \( \sigma \) parameter. Comparing any expression here with similar ones having essentially a time evolution perspective from the physics point of view leads only to confusion. In fact, in the long paper to follow, we will illustrate the right way to really look at the time evolution results our \( \sigma \)-evolution formulation provides in a more fundamental setting — that of canonical classical mechanics. For instance, we have derived directly from the formulation an interesting solution of particle-antiparticle annihilation, which is considered to be a nontrivial success of our approach.

II. POINCARÉ-SNYDER RELATIVITY

The Poincaré-Snyder relativity is a relativity on 4D Minkowski space-time, with \( \sigma \) as an external parameter. It has otherwise a structure mimic that of the Galilean case on 3D space. The complete Galilei group has rotations, translations, as well as velocity boosts as symmetry transformations on 3D space together with an external time parameter. Comparing the first two columns of table 1, we can see that the implementation of Poincaré symmetry stabilization through the invariant (quantum) energy-momentum scale requires considering a new kind of momentum boosts, as independent from the velocity boosts. The usual relation between momentum and velocity has to be relaxed to hold only at the Einstein limit. The Poincaré-Snyder relativity is then just the relativity with Poincaré symmetry extended by such momentum boosts before the deformation, \( i.e. \) at the unconstrained commuting limit. The relativity may hence provide a window for us to understand the \( \sigma \) parameter in the most familiar context. Interestingly enough, we came to realize that parameter(s) of quite close a nature to that of \( \sigma \) had been used quite a lot, to different extents, in the regime of (Einstein) relativistic quantum mechanics in somewhat ambiguous ways. Adopting the
perspective of the Poincaré-Snyder relativity we propose here actually helps to put many of such earlier attempts on solid theoretical footings. From the perspective alone, that comes at a great cost though — a new definition of energy-momentum as $\sigma$ derivatives with a physics picture still to be fully understood [5].

Let us start by a clear illustration of the structure of our Poincaré-Snyder relativity. Following Ref. [12] (see Fig. 10.6 for an illustration), we can describe the Poincaré group and the Galilei group as sequential contractions from $SO(1,4)$

$$SO(1,4) \rightarrow ISO(1,3) \rightarrow G(3).$$

The first step is the well known Inönü-Wigner contraction, a reverse of the symmetry stabilization. A further, similar, contraction gives the Galilei group with commuting translations as well as commuting velocity boosts. We are more interested in the other contraction sequence, from the symmetry of our quantum relativity. That is the sequence

$$SO(2,4) \rightarrow ISO(1,4) \rightarrow G(1,3),$$

giving the newly named Snyder and Poincaré-Snyder relativities. In table 2, we list the full set of generators for the symmetry groups of all the five relativities. ² Wherever there is a change of notation from the $J_{MN}$ generator(s) moving across a row, a contraction is involved. $J_{MN}$ here denotes the 15 generators of the the $so(2,4)$ algebra, satisfying

$$[J_{RS},J_{MN}] = - \left( \eta_{SM}J_{RN} - \eta_{RM}J_{SN} + \eta_{RN}J_{SM} - \eta_{SN}J_{RM} \right),$$

(1)

where again $\eta_{MN} = (-1, +1, +1, +1, +1, -1)$ with the indices go from 0 to 5; we use also $\eta_{AB}$ to denote the 0 to 4 part; others indices follow the common notation. Note that the $J_{\mu\nu}$’s are the (subset of) Lorentz transformation generators, etc. All $P_A$’s denote (generators for) the coordinate translations on the corresponding arena for the linear realizations — $M^5$, the Minkowski space-time $M^4$, and the 3D space. The $K_i$’s are Galilean velocity boosts, and $K'_\mu$’s the new commuting momentum boosts. We have the standard structure

$$[J_{AB},P_C] = - (\eta_{AC}P_B - \eta_{BC}P_A),$$

(2)

² In a forthcoming paper, we will present more details of the mathematics and plausible physics pictures of relativity symmetries from the contraction schemes based on the kind of perspective [13].
\[ [J_{\mu\nu}, K^{(\prime)}_{\lambda}] = - (\eta_{\nu\lambda} K^{(\prime)}_{\mu} - \eta_{\mu\lambda} K^{(\prime)}_{\nu}) , \] (3)

where the latter applies to the two different types of boosts \( K'_\mu \) and \( K_i \)'s. Translations and the boosts (not including the so-called Lorentz boosts which are space-time rotations) always from a commuting set. The external time evolution \( H \) commutes with all others with the only exception given by

\[ [K_i, H] = P_i . \] (4)

The latter is actually the same commutation relation as that of the generators \( J_{0i} \) and \( P_0 \) before the \( ISO(1,3) \to G(3) \) symmetry contraction. In fact, no commutation relation between time translation and the other generators is changed in the contraction. We choose to use \( H \) instead of \( P_0 \) to denote time translation for the Galilean case only to highlight time being a parameter external to the geometric realization arena of 3D space. Similarly the external \( \sigma \)-translation \( H' \) has the only non-zero commutators

\[ [K'_\mu, H'] = P_\mu , \] (5)

equal to that of the corresponding ones between \( J_{4\mu} \) and \( P_4 \).

Note that the translations for Einstein and Galilean relativities are listed in the rows of the \( J_{4\mu} \)'s and \( K'_\mu \)'s for the other three cases, rather than with the other \( P_\mu \) translations. That is done to emphasize that all the 10 generators of the Poincaré or Galilei group, like the case of the \( J_{\mu\nu} \) and \( K'_\mu \) subset for the \( G(1,3) \) group of the Poincaré-Snyder relativity, can be obtained through proper contractions of an \( SO(1,4) \) symmetry. This is more in correspondence with the stabilization sequence presentation in our earlier publications [5, 6], in which we make no clear distinction nor simultaneous treatment of boosts and translations. The \( J_{\mu\nu} \) and \( K'_\mu \) set indeed gives an algebra isomorphic to that of the \( J_{\mu\nu} \) and \( P_\mu \) set. However, one can also keep all the \( P_i \)'s and \( P_0 \) on the same role. That corresponds to seeing the last two groups as from the equally valid contraction sequence [12]

\[ SO(2,3) \to ISO(1,3) \to G(3) , \]

perhaps more adapted to tracing back their relations to the full \( SO(2,4) \) symmetry. We are interested here mostly in illustrating the structure of the \( G(1,3) \) symmetry for the Poincaré-Snyder relativity.
As suggested by the notation, the $G(1, 3)$ symmetry has a very similar structure to that of the Galilean $G(3)$, hence similar mathematical properties. The latter may imply similar properties, at the level of mathematical formulations, when applied to describe physics. Two main features are considered specially interesting. Firstly, taking away only $H$ from the set of generators of $G(3)$, the rest generates a subgroup, likewise for taking away $H'$ in the case of $G(1, 3)$. This is not the case for the set of $ISO(1, 3)$ generators with $P_0$, for example. Particle dynamics under Poincaré symmetry has a no-interaction theorem \cite{14, 15}. The latter can be interpreted as a consequence of different subgroup structure, as compared to that of the $G(3)$ symmetry. All generators besides the Hamiltonian $H$ for the symmetry stay as kinematical ones, which has to generate a subgroup. The generators besides the Hamiltonian fail to do the same for Poincaré symmetry, leaving rather the three admissible forms of Hamiltonian dynamics as noted by Dirac\cite{16} (see also Ref.\cite{15}) as the alternatives. The generators besides the $H'$ Hamiltonian of the $G(1, 3)$ symmetry can be taken as all being kinematical. Using $G(1, 3)$ symmetry to describe ‘relativistic dynamics’, or rather the dynamics of $\sigma$-evolution on Minkowski space-time, would admit direct description of interactions as in the Galilean case. It will be interesting to see if we can learn something about ‘relativistic dynamics’ from such an approach (see Ref.\cite{11}). Next, we turn to a feature that we want to focus on here.

The $G(1, 3)$ group, like $G(3)$, admits a non-trivial $U(1)$ central extension. Projective group representations required to describe quantum mechanics are simply unitary representations of the $U(1)$ central extension \cite{10}. Hence, the $G(1, 3)$ may be a better candidate than the $ISO(1, 3)$ for the description of ‘relativistic quantum mechanics’ as a quantization of ‘relativistic mechanics’.

**III. QUANTIZATION AS $U(1)$ CENTRAL EXTENSION**

Aldaya and de Azcárraga \cite{17} presented a particularly nice approach to geometric quantization in which the quantum dynamical description of the system can be obtained with the symmetry group as the basic starting point (see also Ref.\cite{10}). The approach looks especially relevant to our case in which we have a new relativity symmetry in search of a clear understanding of the physics involved. In fact, while the approach gives an elegant presentation for the quantization of the Galilean system, its application to the case of Einstein relativity is less than equally appealing. For the former case, the group to be considered is a $U(1)$ central
extension of the symmetry for the corresponding classical system — $G(3)$. The essentially unique nontrivial central extension is depicted by the modified commutator $[K_i, P_j] = m\delta_{ij}\Xi$, where $\Xi$ is a central charge ($m$ the particle mass). The $ISO(1,3)$ symmetry, however, admits no such nontrivial central extension (for an explicit discussion on admissible central charges for both cases, see Ref. [15]). It is easy to see from our discussion above of the $G(1,3)$ symmetry for the Poincaré-Snyder case that it has a structure mostly parallel to that of the Galilean $G(3)$. Hence, we should have the same nontrivial $U(1)$ central extension available for the implementation of such a quantization scheme. Indeed, when we set out to perform the analysis, we realized that the work, at least most of the mathematics, has actually been done [18] under a different premise. Confronted with the difficulty on applying the elegant quantization scheme to (Einstein) relativistic dynamics, Aldaya and de Azcárraga chose to put the Poincaré symmetry into a mathematical framework that make the scheme applicable — essentially taking it to $G(1,3)$. They basically considered promoting the proper time to an ‘absolute time’ parameter for the formulation. The latter was used more like as a mathematical trick with any independent physics meaning not explicitly addressed. The physics results are discussed only after a symmetry reduction back to the Einstein setting has been implemented (see also Ref. [19]). We choose here to follow mostly the approach of Ref. [17] and present first the quantization results in the language of our Poincaré-Snyder relativity formulation. After that we will discuss the very important difference in physics premise and interpretation we introduce here. We discuss why the Poincaré-Snyder relativity perspective is considered to provide a plausibly more interesting framework for the bold attempt at the group quantization formulation of the ‘quantum relativistic system’. Our approach may also provides an interesting way to avoid the many ‘uncomfortable’ features well appreciated in the usual (Einstein) relativistic quantum mechanics, which otherwise need to be resolved in the quantum field theory framework.

The standard action of $G(1,3)$ on the Minkowski spacetime $(x^\mu)$ with the extra, external, parameter $\sigma$ is given by

$$
\begin{align*}
x'{}^\mu &= \Lambda^\mu{}_\nu x^\nu + p^\mu \sigma + A^\mu, \\
\sigma' &= \sigma + b.
\end{align*}
$$

An element of our extended $G(1,3)$ group may be written as $g = (b, A^\mu, p^\mu, \Lambda^\nu{}_{\nu'}, e^{i\theta})$, with
group product rule given by
\[ A''{}^\mu = \Lambda''{}^\nu A''{}^\nu + p''{}^\mu b + A''{}^\mu , \quad b'' = b' + b , \]
\[ p'' = \Lambda''{}^\mu p{}^\nu + p''{}^\mu , \quad \Lambda''{}^\mu{}^\nu = \Lambda''{}^\mu{}^\rho \Lambda''{}^\rho{}^\nu , \quad (7) \]
and the nontrivial \( U(1) \) extension of given by
\[ \theta'' = \theta' + \theta + z \left[ A'{}^\mu \Lambda'{}^\nu p{}^\nu + b(p'{}^\mu \Lambda'{}^\nu P{}^\nu + \frac{1}{2} p'{}^\mu p'{}^\nu) \right] . \quad (8) \]
The last term is the cocycle the exact choice of which is arbitrary up to a coboundary [10]; \( z \) corresponds to a value of the central charge is taken as an arbitrary constant at this point.

The right-invariant vector fields are given by
\[ X_R^b = \frac{\partial}{\partial b} , \quad X_R^{A{}^\mu} = \frac{\partial}{\partial A{}^\mu} + z p{}^\mu \frac{i \zeta}{\hbar} \frac{\partial}{\partial \zeta} , \]
\[ X_R^{p{}^\mu} = b \frac{\partial}{\partial A{}^\mu} + \frac{\partial}{\partial p{}^\mu} + z b p{}^\mu \frac{i \zeta}{\hbar} \frac{\partial}{\partial \zeta} , \]
\[ X_R^{\omega{}^\mu\nu} = \tilde{X}_R^{\omega{}^\mu\nu} + A{}^\nu \frac{\partial}{\partial A{}^\mu} - A{}^\mu \frac{\partial}{\partial A{}^\nu} + p{}^\nu \frac{\partial}{\partial p{}^\mu} - p{}^\mu \frac{\partial}{\partial p{}^\nu} , \]
\[ X_R^\zeta = \frac{i \zeta}{\hbar} \frac{\partial}{\partial \zeta} ; \quad (9) \]
where we skip the details of \( \tilde{X}_R^{\omega{}^\mu\nu} \), the invariant vector field for the \( SO(1, 3) \) subgroup [with \( \Lambda(\omega) = e^{\frac{i}{2} \omega} J^{}_{\mu\nu} \cdot J^{}_{\mu\nu} \)] leaving it to be given explicitly in the appendix. Note that \( \zeta = \exp\left(\frac{i}{\hbar} \theta\right) \) with \( \frac{i \zeta}{\hbar} \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial \theta} \) locally. The left-invariant vector fields are given by
\[ X_L^b = \frac{\partial}{\partial b} + p{}^\mu \frac{\partial}{\partial A{}^\mu} + \frac{z}{2} p{}^\mu p{}^\nu \frac{i \zeta}{\hbar} \frac{\partial}{\partial \zeta} , \]
\[ X_L^{A{}^\mu} = \frac{\partial}{\partial A{}^\nu} \Lambda{}^\nu{}^\mu , \quad X_L^{p{}^\mu} = \frac{\partial}{\partial p{}^\nu} \Lambda{}^\nu{}^\mu + z A{}^\nu \Lambda{}^\nu{}^\mu \frac{i \zeta}{\hbar} \frac{\partial}{\partial \zeta} , \]
\[ X_L^{\omega{}^\mu\nu} = \tilde{X}_L^{\omega{}^\mu\nu} , \quad X_L^\zeta = \frac{i \zeta}{\hbar} \frac{\partial}{\partial \zeta} ; \quad (10) \]
again explicit expression for the \( SO(1, 3) \) vector field \( \tilde{X}_L^{\omega{}^\mu\nu} \) is left to the appendix. We have the quantization form given by the left-invariant 1-forms conjugate to \( X_L^\zeta \), the vertical 1-form; explicitly
\[ \Theta = -z A{}^\nu \Lambda{}^\nu{}^\mu d p{}^\mu - \frac{z}{2} p{}^\mu p{}^\nu d b + \frac{\hbar d \zeta}{i \zeta} . \quad (11) \]
The characteristic module is defined through the conditions \( i_X \Theta = 0 \) and \( i_X d \Theta = 0 \), characterizing the differential system given by the vector field \( X_L^b \). We have the equations of
motion given by

\begin{align*}
\frac{db}{d\sigma} &= 1, & \frac{dA^\mu}{d\sigma} &= p^\mu, \\
\frac{dp^\mu}{d\sigma} &= 0, & \frac{d\Lambda^\mu_{\nu}}{d\sigma} &= 0 \quad \left( \frac{d\omega^{\mu\nu}}{d\sigma} = 0 \right), \\
\frac{d\zeta}{d\sigma} &= \frac{z}{2} p^\mu p^\mu i\zeta \bar{h} .
\end{align*}

(12)

Identifying the integration parameter as \( \sigma \) gives

\begin{align*}
b &= \sigma , & A^\mu &= p^\mu \sigma + K'^\mu_{\nu} \Lambda^\nu_{\mu} , & p^\mu &= P^\mu , \\
\zeta &= \zeta_0 \exp \left( \frac{iz}{2\hbar} p^\mu p^\mu \sigma \right) .
\end{align*}

(13)

Naturally, \( A^\mu \) is to be identified as \( x^\mu \) giving \( p^\mu \) as \( \frac{dx^\mu}{d\sigma} \), showing consistence with our original introduction of the momentum boosts (see table 1) as the extra symmetry transformations to supplement \( SO(1,3) \) and hence getting to \( G(1,3) \). We have constants of motion, \( K'^\mu_{\nu} \Lambda^\nu_{\mu} \), \( P^\mu \), and \( \zeta_0 \) which parametrize the quantum manifold. Passing to the latter, \( \Theta \) takes the form

\[ \Theta_P = -z K'^\mu dP_\mu + \frac{\hbar d\zeta_0}{i\zeta_0} . \]

(14)

The symplectic 2-form is given by \( \omega = d\Theta \). Taking \( z = 1 \), \( H' = \frac{1}{2} p^\mu p_\mu \), and \( K'^\mu = A^\nu \Lambda^\nu_{\mu} \) we have

\[ \omega = d\Theta_P = -dK'^\mu \wedge dp_\mu , \]

(15)

where we have taken \( z = 1 \) corresponding to \( H' = \frac{1}{2} p^\mu p_\mu \), which gives the right form for the classical \( \sigma \)-Hamiltonian \[11\]. The expression suggested the identification of \((K'^\mu, b)\) as particle ‘position’ variables \((x^\mu, \sigma)\) and \(H'\) as the \( \sigma \)-Hamiltonian generating ‘evolution’ in the absolute parameter \( \sigma \). The prequantum operator associated with a real function \( f \) on the classical phase space acting on wavefunction \( \psi \) is given by

\[ \hat{f} \psi \equiv -i\hbar \hat{X} f \cdot \psi = -i\hbar X_f \cdot \psi + \left[ f - \Theta(X_f) \right] \psi , \]

(16)

where \( i_{X_f} \omega = -df \). In particular,

\begin{align*}
\hat{K}'^\mu &= i\hbar \frac{\partial}{\partial P_\mu} , & \hat{P}_\mu &= -i\hbar \frac{\partial}{\partial K'^\mu} + P_\mu , \\
\hat{\sigma} &= i\hbar \frac{\partial}{\partial H' \sigma} , & \hat{H}' &= i\hbar \frac{\partial}{\partial \sigma} ,
\end{align*}

(17)
where an extra negative sign is adopted in $\hat{H}'$. The operators $K'_{\mu}$ and $P_{\mu}$ can also be obtained from $X_{A \mu}$ and $X_{\mu \nu}$. The full polarization subalgebra can be taken as spanned by \( \{ X_{b}, X_{A \mu}, X_{\omega \mu \nu} \} \), giving the momentum space wavefunction $\phi(P_{\mu})$, i.e. the wavefunction as dependent only on $P_{\mu}$ but not $K'_{\mu}$. We have then simply $\hat{P}_{\mu} \phi(P_{\mu}) = P_{\mu} \phi(P_{\mu})$. $X_{b}$ generates the Euler-Lagrange equation

\[
i \hbar \partial_{\sigma} \psi + \frac{\hbar^{2}}{2} \partial_{\mu} \partial^{\mu} \psi = 0 \tag{18}
\]

for the Fourier transform $\psi$ of ‘momentum’ space wavefunction $\phi$. Note that $\hat{H}'$ and $\hat{P}_{\mu}$ constitute a complete set of commuting observables for the configuration space wavefunction. The equation expresses an operator form of $H' = \frac{1}{2} p^{\mu} p_{\mu}$ with $\hat{P}_{\mu}$ reduced to just $-i \hbar \frac{\partial}{\partial K'_{\mu}}$, i.e. $-i \hbar \frac{\partial}{\partial x'_{\mu}}$. Eq. (18) is of the same form as the so-called (Lorentz) covariant Schrödinger equation studied in the literature[9], except with the parameter $\sigma$ replacing the proper time $\tau$ (or rather $\tau/m$). The equation, with again essentially the proper time as evolution parameter, is also what is obtained in Ref.[18]. One can see that the rest mass, or $m^{2}/2$ to be exact, of an Einstein particle is just the $\hat{H}'$ eigenvalue. Without considering the spin degree of freedom, the usual (Einstein) relativistic quantum mechanics corresponds to the $\sigma$ independent eigenvalue equation for $\hat{H}'$, obtainable from Eq. (18) separation of the $\sigma$ variable from the $x^{\mu}$ variables. The eigenvalue equation is the Klein-Gordon equation. Recall that for an Einstein particle, i.e. taking the Poincaré-Snyder free particle to the Einstein relativistic limit, we have $\sigma \rightarrow \tau/m$[5, 6].

IV. DISCUSSIONS

One can see from the above that Poincaré-Snyder relativity provides a very nice mathematical framework to formulate the the covariant quantum mechanics except with the Lorentz invariant evolution parameter $\tau$ replaced by the truly independent variable $\sigma$ as suggested from the quantum relativity framework. The introduction of an extra evolution parameter in the beautiful quantization scheme of Ref.[18] and the various early discussions of the covariant Schrödinger before the 50’s[9] as sort of a mathematical tool is now dictated by the quantum relativity picture. It remains a challenge to interpret the $\sigma$ dependent mechanics at both the quantum and the classical level beyond the case of an Einstein particle. We emphasize again that $\sigma$ is not a kind of time parameter. The key lesson from our perspective is that one has to go beyond thinking about the ‘evolution’ parameter as
essentially time, which confines all earlier literature. In a somewhat different background setting, a first discussion of the physics of the $\sigma$ coordinate has been given in Ref. [5]. The most important point to note is that the framework actually redefine energy-momentum through $p^\mu = \frac{dx^\mu}{d\sigma}$, making it not equal to the Einstein form of $m \frac{dx^\mu}{d\tau}$. In general, for the quantum relativity or the Poincaré-Snyder relativity, particle rest mass becomes a reference frame dependent quantity. A momentum boost transformation changes the value of $m$ as the magnitude of the energy-momentum four-vector. In Poincaré-Snyder relativity, the vector transforms by simple addition like the Galilean velocity. This is the new and most fundamental feature offered by our framework. A related aspect is the lost of the Einstein rest mass as an intrinsic or fundamental character of a particle. Here, it is only the magnitude of the particle energy-momentum four-vector which can be modified by interaction. The feature is illustrated to be useful, or even necessary, in describing some interesting physics scenario like particle-antiparticle annihilation [11].

In Galilean relativity, the kinetic energy of a particle ($\propto v^2$) is both reference frame dependent and interaction dependent hence time dependent. Similarly, the (expectation) value of the $\sigma$-Hamiltonian is, in general, $\sigma$ dependent. To put it another way, the $\sigma$ dependent covariant Schrödinger equation is to be given by

$$i\hbar \partial_\sigma \psi - \hat{H}' \psi = 0 \quad (19)$$

where $\sigma$-Hamiltonian $\hat{H}'$ operator should be given by $\frac{1}{2} \hat{P}_\mu \hat{P}_\mu + \hat{V}'$ with $V'$ depicting an ‘interaction potential’ under the $\sigma$-evolution. The value for the magnitude of the energy-momentum four-vector would hence change with $\sigma$. Such a picture is fully collaborated by a classical canonical Hamiltonian picture [11].

It is interested to note that in various studies of the essentially $\tau$-parametrized covariant Schrödinger equation there had been discussions of notion of mass indefiniteness [8, 9]. Naively, Eq.(18) admits mixture of eigenstates of different $m^2$ value. Some author actually went so far as to absorb $m$ into the evolution parameter and made it $\tau/m$, which is indeed close to our $\sigma$. An example of an explicit physics considerations of statistical nature, for example, was offered by Hostler [20]. In our opinion, Feynman was the one that went beyond everybody, in his works on quantum electrodynamics. Not only did he rewrote the Klein-Gordon equation in the form of Eq.(18) with $u \equiv \tau/m$ in the place of $\sigma$ [21], he actually considered the case of $dt/du < 0$ hence taking the ‘evolution’ somewhat beyond a physical
time variable. That was actually behind the antiparticle picture in Feynman diagrams used in the standard quantum field theory [22]. There was no indication, however, that the Feynman $u$ parameter was taken to have any independent physical meaning. Our framework discussed above certainly asks for the $\sigma$ parameter to be taken as a truly important physics parameter beyond the $\tau/m$ limit. And one should take special caution against thinking about it too much as a quantity analogous to any ‘time’ variable. For example, Ref.[6] illustrates it has a close connection to a scaling parameter in the full quantum relativity. The challenge to fully appreciate the $\sigma$ variable is beyond us here, but sure a main target of the research program.

In this letter, we introduce the Poincaré-Snyder relativity and Snyder relativity as relativities in between the well known Galilean and Einstein cases and the quantum relativity — the relativity for ‘quantum space-time’. We illustrate, using the symmetry group geometric quantization framework, how the Poincaré-Snyder relativity may be providing a stronger framework for the description of the usual relativistic quantum mechanics, from the perspective of the which the formulation under Einstein relativity is sort of an incomplete description. The extra ‘evolution’ parameter $\sigma$ have been actually used in various limiting form by earlier authors. Our Poincaré-Snyder relativity provides a formulation for thinking about the $\sigma$ variable in more serious manner, on a similar footing as the space and time variables. We will report further on the physics of $\sigma$-evolutions in future publications.

V. APPENDIX : INVARIANT VECTOR FIELDS ON $SO(1,3)$ GROUP MANIFOLD

We give in this appendix some details of our results on invariant vector fields on the $SO(1,3)$ group manifold. Starting with a generic group element $\Lambda(\omega) = e^{\frac{i}{2} \omega_{\mu\nu} J_{\mu\nu}}$ in terms of the generators $J_{\mu\nu}$ in the standard form, we rewrite it in terms of two commuting sets of generators for separate $SU(2)$ groups as $\Lambda(\omega) = e^{-i(\omega_+^+ J_+^+)} e^{-i(\omega_-^- J_-^-)}$, where $J_\pm^\pm = \frac{1}{2} \left( \frac{1}{2} \epsilon_{i j k} J_{jk} \pm i J_{0i} \right)$ respecting $[J_i^\pm, J_j^\pm] = i \epsilon_{ij}^k J_k^\pm$, and $\omega_{\pm}^i = \frac{1}{2} \epsilon_{i j k} \omega^{j k} \mp i \omega_0^i \equiv \theta^i \mp i \eta^i$. The group product may be written as

$$\Lambda(\omega'') = \Lambda(\omega') \Lambda(\omega) = e^{-i(\omega'_+^+ J_+^+)} e^{-i(\omega'_-^- J_-^-)} e^{-i(\omega_+^+ J_+^+)} e^{-i(\omega_-^- J_-^-)} e^{-i(\omega'_-^- J_-^-)} \ e^{-i(\omega'_+^+ J_+^+)} .$$

(20)
The left invariant vector fields for the $SU(2)$ group $X^\omega_{\omega_\pm}$'s can be computed directly. We list the result here dropping the + or - sign:

$$X^\omega_{\omega} = \frac{\omega}{2} \left( \frac{\partial}{\partial \omega} \right) + \frac{2}{\sqrt{2}} \left( \frac{\partial}{\partial \omega_1} \right) + \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \omega_2} \right) - \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \omega_3} \right).$$

We have then from the relations $\bar{X}_{\theta_i} = X^\omega_{\omega_i} + X^\omega_{\omega_i}$ and $\bar{X}_{\eta_i} = -i \left( X^\omega_{\omega_i} - X^\omega_{\omega_i} \right)$

$$\bar{X}_{\theta_i} = A \frac{\partial}{\partial \theta_i} + B \frac{\partial}{\partial \eta_i} + \frac{1}{2} \left( C\theta_i - D\eta_i \right) \left( \theta^k \frac{\partial}{\partial \theta^k} + \eta^k \frac{\partial}{\partial \eta^k} \right) + \frac{1}{2} \left( C\eta + D\theta_i \right) \left( \theta^k \frac{\partial}{\partial \eta^k} - \eta^k \frac{\partial}{\partial \theta^k} \right)$$

$$- \frac{1}{2} \eta_i \left( \theta^k \frac{\partial}{\partial \theta^k} + \eta^k \frac{\partial}{\partial \eta^k} \right),$$

$$\bar{X}_{\eta_i} = A \frac{\partial}{\partial \eta_i} - B \frac{\partial}{\partial \theta_i} - \frac{1}{2} \left( C\eta_i + D\theta_i \right) \left( \theta^k \frac{\partial}{\partial \theta^k} + \eta^k \frac{\partial}{\partial \eta^k} \right) + \frac{1}{2} \left( C\theta_i - D\eta_i \right) \left( \theta^k \frac{\partial}{\partial \eta^k} - \eta^k \frac{\partial}{\partial \theta^k} \right)$$

$$- \frac{1}{2} \eta_i \left( \theta^k \frac{\partial}{\partial \eta^k} - \eta^k \frac{\partial}{\partial \theta^k} \right).$$

[we mark $SO(1, 3)$ vector fields with $\bar{X}_{\omega_i}$ instead of just $X_{\omega_i}$, following the main text], where

$$A(\omega) = \frac{\alpha \sin \alpha + \beta \sinh \beta}{\cosh \beta - \cos \alpha}, \quad B(\omega) = \frac{1}{2} \beta \sin \alpha - \alpha \sinh \beta,$$

$$C(\omega) = \frac{2(\alpha^2 - \beta^2)(\cos \alpha - \cosh \beta) + (\alpha^2 + \beta^2)(\alpha \sin \alpha - \beta \sinh \beta)}{(\alpha^2 + \beta^2)^2(\cos \alpha - \cosh \beta)},$$

$$D(\omega) = \frac{4\alpha\beta(\cos \alpha - \cosh \beta) + (\alpha^2 + \beta^2)(\beta \sin \alpha + \alpha \sinh \beta)}{(\alpha^2 + \beta^2)^2(\cos \alpha - \cosh \beta)}.$$

The above expressions can be written in the recombined form as

$$\bar{X}_{\omega_{\mu}^\nu} = \frac{1}{2} A(\omega) \frac{\partial}{\partial \omega_{\mu}^\nu} - \frac{1}{2} B(\omega) \frac{\partial}{\partial \omega_{\sigma}^\nu} + \frac{1}{2} C_{\mu \nu}(\omega) \frac{\partial}{\partial \omega_{\mu}^\sigma} - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \omega_{\omega \rho \sigma} \frac{\partial}{\partial \omega_{\mu}^\sigma}$$

$$- \frac{1}{2} \epsilon_{\mu \nu \sigma \rho} \omega_{\omega \rho \sigma} \frac{\partial}{\partial \omega_{\mu}^\sigma} \left( \eta_{\mu} \frac{\partial}{\partial \omega_{\sigma}} + \eta_{\sigma} \frac{\partial}{\partial \omega_{\mu}} \right),$$

where

$$C_{\mu \nu}(\omega) = \frac{1}{2} \left[ C(\omega) \omega_{\mu}^\nu - \frac{1}{2} D(\omega) \epsilon_{\mu \nu \rho \sigma} \omega_{\omega \rho \sigma} \right].$$

Similarly, we have

$$\bar{X}_{\omega_{\mu}^\nu} = \frac{1}{2} A(\omega) \frac{\partial}{\partial \omega_{\mu}^\nu} - \frac{1}{2} B(\omega) \frac{\partial}{\partial \omega_{\sigma}^\nu} + \frac{1}{2} C_{\mu \nu}(\omega) \frac{\partial}{\partial \omega_{\mu}^\sigma} - \frac{1}{2} \epsilon_{\mu \nu \sigma \rho} \omega_{\omega \rho \sigma} \frac{\partial}{\partial \omega_{\mu}^\sigma}$$

$$- \frac{1}{2} \epsilon_{\mu \nu \sigma \rho} \omega_{\omega \rho \sigma} \frac{\partial}{\partial \omega_{\mu}^\sigma} \left( \eta_{\mu} \frac{\partial}{\partial \omega_{\sigma}} + \eta_{\sigma} \frac{\partial}{\partial \omega_{\mu}} \right).$$
The left- and right-invariant vector fields for the Lorentz group are available in the literature, for example in Ref. [19], typically not in the form directly with \( \omega^{\mu\nu} \) as coordinate parameters. Indeed, we do not find even expression (21) in the literature. For instance, vector fields are given in Ref. [19] with a prior splitting between the Lorentz boosts and rotations, each in terms of parameters that constitute a 3D vector. We find the form as presented here more appealing, and hope that it can be generalized to \( SO(p, q) \). The vector fields actually have little role to play in the quantization procedure as shown above. As for the expressions of the other \( G(1, 3) \) invariant vector fields, we use the Lorentz transformation matrix \( \Lambda^e_\mu \) instead. We consider the expressions illustrating more transparent physics.

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version with some different background discussions.

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3 The actual vectorial parameter for the the rotation part for example is essentially a sine function of the \(|\omega/2|\). In particular, the explicit form as given there has an extra factor of 2 coming up in the commutator, which is a result of a 1/2 factor when each of the parameters is matched to \( \omega^{ij} \) in the infinitesimal limit effectively rescaling the generators.
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TABLE I: Physics of the Relativity Deformations Summarized: The first column shows the familiar Galilean to Einstein case. Having an invariant speed $c$ change the velocity from a unconstrained three-vector $v^i$ to a constrained four-vector $u^\mu$, matched to a 4D arena with an extra coordinate $ct$. Mathematically, the zero commutator between Galilean boosts is deformed to a rotation. The latter two columns similarly summarized the physics of the two further steps towards the quantum relativity of Ref. [6]. For example (2nd column), having an invariant magnitude $\kappa c$ for $p^\mu = \frac{dx^\mu}{d\sigma}$ change the unconstrained four-vector to a constrained five-vector $\pi_A$, to be matched to a 5D arena with an extra coordinate $\kappa c \sigma$. The orginally zero commutator between two momentum boosts, to be described before imposing the constraint as $\Delta x^\mu = p^\mu \sigma$ on 4D space-time, is deformed to a Lorentz transformation. The introduction of the $\sigma$ parameter/coordinate and the momentum boosts before the deformation are the key features behind the Poincaré-Snyder relativity introduced in this letter.

| $\Delta x^i(t) = v^i \cdot t$ | $\Delta x^\mu(\sigma) = p^\mu \cdot \sigma$ | $\Delta x^A(\rho) = z^A \cdot \rho$ |
|-----------------|------------------|-----------------|
| $|v^i| \leq c$ | $\sqrt{-\eta_{\mu\nu} p^\mu p^\nu} \leq \kappa c$ | $|z^A| \leq \ell$ |
| $\eta_{ij} v^i v^j = c^2 \left(1 - \frac{1}{\gamma^2}\right)$ | $\eta_{\mu\nu} p^\mu p^\nu = -\kappa^2 c^2 \left(1 - \frac{1}{\Gamma^2}\right)$ | $\eta_{AB} z^A z^B = \ell^2 \left(1 + \frac{1}{\kappa^2}\right)$ |
| $M_{io} \equiv K_i \sim P_i$ | $J_{i\mu} \equiv O_{i\mu} \sim P_\mu$ | $J_{iA} \equiv O'_{iA} \sim P_A$ |
| $[K_i, K_j] \longrightarrow - M_{ij}$ | $[O_{i\mu}, O_{j\nu}] \longrightarrow M_{\mu\nu}$ | $[O'_{iA}, O'_{jA}] \longrightarrow J_{AB}$ |
| $\bar{\pi}^4 = \frac{2}{c}(c, v^i)$ | $\bar{\pi}^5 = \frac{r}{\kappa^2}(p^\mu, \kappa c)$ | $\bar{X}^6 = \frac{\ell^2}{r^2}(z^A, \ell)$ |
| $-\eta_{\mu\nu} u^\mu u^\nu = 1$ | $\eta_{AB} \pi^A \pi^B = 1$ | $\eta_{MN} X^M X^N = 1$ |

$R^3 \rightarrow SO(1, 3)/SO(3)$ $M^4 \rightarrow SO(1, 4)/SO(1, 3)$ $M^5 \rightarrow SO(2, 4)/SO(2, 3)$
TABLE II: The various relativities – matching the generators: The table matches out the generators for the various relativity symmetries from a pure mathematical point of view. Note as algebras, the mathematical structures of translations (denoted by $P$) or the boosts (denoted by $K$ and $K'$ – the so-called Lorentz boosts not included as they are really space-time rotations) in relation to rotations $J_{\mu}$ are the same. Algebraically, translation and boost generators are distinguished only by the commutation with the Hamiltonian ($H$ and $H'$). Successive contractions retrieve $G(1,3)$ and $ISO(1,4)$ from $SO(2,4)$, similar to the more familiar $G(3)$ and $ISO(1,3)$ from $SO(1,4)$. In the physics picture under discussion, however, $SO(1,4)$ part of our so-called Snyder relativity $ISO(1,4)$ is different from the usual de-Sitter $SO(1,4)$ contracting to $ISO(1,3)$. We consider simply keeping only the $P_{\mu}$ and $J_{\mu\nu}$ generators to reduce from our Poincar’e-Snyder $G(1,3)$ to the Einstein $ISO(1,3)$.

| Relativity Symmetry | Quantum | Snyder | Poincaré-Snyder | Einstein | Galilean |
|---------------------|---------|--------|----------------|----------|----------|
|                     | $SO(2,4)$ | $ISO(1,4)$ | $G(1,3)$ | $ISO(1,3)$ | $G(3)$ |
| Arena               | ‘AdS$_5$’ | $M^5$ | $M^4$ (with $\sigma$) | $M^4$ | $\mathbb{R}^3$ (with $t$) |
| SO(1,4) part        | $J_{ij}$ | $J_{ij}$ | $J_{ij}$ | $J_{ij}$ | $J_{ij}$ |
|                     | $J_{i0}$ | $J_{i0}$ | $J_{i0}$ | $J_{i0}$ | $K_i$ |
|                     | $J_{0i}$ | $J_{0i}$ | $K_0'$ | $P_0$ | $H$ |
|                     | $J_{4i}$ | $J_{4i}$ | $K_4'$ | $P_i$ | $P_i$ |
|                     | $J_{50}$ | $P_0$ | $P_0$ | | |
|                     | $J_{5i}$ | $P_i$ | $P_i$ | | |
|                     | $J_{54}$ | $P_4$ | $H'$ | | |