BOUNDDED COHOMOLOGY AND NON-UNIFORM PERFECTION OF MAPPING CLASS GROUPS

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ABSTRACT. Using the existence of certain symplectic submanifolds in symplectic 4-manifolds, we prove an estimate from above for the number of singular fibers with separating vanishing cycles in minimal Lefschetz fibrations over surfaces of positive genus. This estimate is then used to deduce that mapping class groups are not uniformly perfect, and that the map from their second bounded cohomology to ordinary cohomology is not injective.

1. Introduction

It is well-known that the mapping class group of a closed oriented surface $F$ of genus $\geq 3$ is perfect. In this paper we shall prove that it is not uniformly perfect, meaning that there is no number $N$ such that every element of the group is a product of at most $N$ commutators. By a result of Bavard [1], this statement implies the non-injectivity of the map from the second bounded cohomology of the mapping class group to its ordinary cohomology. This non-injectivity and the non-perfection of the mapping class groups confirm two conjectures of Morita [16].

We shall also prove the above statement about the bounded cohomology for some non-perfect subgroups of mapping class groups, namely for the Torelli groups and for the hyperelliptic mapping class groups.

As in [11], we use results of Taubes [19] on the Seiberg–Witten invariants of 4–dimensional symplectic manifolds to study the mapping class groups of surfaces. We generalise the main argument of [11] from smooth surface bundles to Lefschetz fibrations. The result is an estimate from above for the number of separating singular fibers in Lefschetz fibrations over aspherical surfaces, in terms of the base and fiber genus (and of the number of nonseparating singular fibers). The existence of such an estimate implies that mapping class groups are not uniformly perfect, even if one does not know the exact shape of the estimate. Every concrete estimate, however, quantifies the failure of uniformity of perfection; see Theorem 2 below.

Date: March 28, 2022; MSC 2000: 57R17, 57R57, 20F12.
Support from the Deutsche Forschungsgemeinschaft is gratefully acknowledged.
In this section we prove the main technical result about Lefschetz fibrations and conclude that mapping class groups are not uniformly perfect. For definitions and further information on smooth Lefschetz fibrations we refer to [6]. We assume throughout that Lefschetz fibrations have at most one critical point in each fiber, and we call the corresponding singular fiber separating or nonseparating according to whether the vanishing cycle is separating or nonseparating.

**Theorem 1.** Let $X$ be a connected smooth closed oriented 4-manifold and $f: X \to B$ a relatively minimal Lefschetz fibration with fiber genus $h \geq 2$ and base genus $g \geq 1$ having $s$ separating and $n$ nonseparating singular fibers. Then

$$s \leq 6(3h-1)(g-1) + 5n .$$

**Proof.** By a result of Gompf, see [6], $X$ is a symplectic manifold, and the fibers are symplectic submanifolds. As has been observed before [12, 18], the assumption of relative minimality here implies that $X$ is minimal and not ruled, because any pseudo-holomorphic sphere in $X$ would have to be contained in a fiber because $g \geq 1$. Thus Liu’s extension [13] of Taubes’s results [19] implies $K^2 \geq 0$, which we can write as

$$b_2^+(X) \geq \frac{1}{5}(b_2^-(X) + 4b_1(X) - 4) .$$

Every reducible singular fiber contains a curve of negative selfintersection, and all of these are linearly independent in homology and independent of the class of a smooth fiber, which has selfintersection zero. Therefore $b_2^-(X) \geq s + 1$. Substituting this in (3) and using $b_1(X) \geq 2g \geq 2$, we obtain

$$b_2^+(X) \geq \frac{1}{5}s .$$

As the claim (1) is trivial for $s = 0$, we may assume $s \geq 1$, and therefore $b_2^+(X) \geq 2$.

The Euler characteristic of $X$ is

$$\chi(X) = 4(g-1)(h-1) + s + n .$$

We estimate the signature of $X$ using Novikov additivity by decomposing the fibration into two pieces. Let $D \subseteq B$ be an embedded 2-disk containing all the critical values of $f$. If $X_1 = f^{-1}(D)$, then a result of Ozbagci [17] gives

$$\sigma(X_1) \leq n - s .$$

Let $X_2$ denote the restriction of $X$ to $B \setminus D$. As $B \setminus D$ can be decomposed into $2g-1$ pairs of pants, and the signature of $X_2$ over each of them is given by the Meyer cocycle [17] and therefore bounded by $2h$, we conclude

$$\sigma(X_2) \leq 2h(2g - 1) .$$
Combining (4) and (5), we obtain
\[ \sigma(X) \leq 2h(2g - 1) + n - s. \]

Passing to finite covers of \( B \) and applying (6) to the pulled-back fibrations, we finally have
\[ \sigma(X) \leq 2h(2g - 2) + n - s. \]

As \( b_2^+(X) \geq 2 \), a result of Taubes [19] ensures that the canonical class \( K \) of \( X \) is represented by a symplectically embedded surface \( \Sigma \subset X \). It may be disconnected, but because \( X \) is minimal, \( \Sigma \) has no spherical component. In the argument below we will tacitly assume that it is connected. In the general case the same argument works by summing over the components.

For the genus of \( \Sigma \) we have the adjunction formula \( g(\Sigma) = 1 + K^2 = 1 + 2\chi(X) + 3\sigma(X) \). Using (3) and (7) we obtain:
\[ g(\Sigma) - 1 \leq 2(10h - 4)(g - 1) + 5n - s. \]

The fibration \( f \) induces a smooth map \( \Sigma \to B \) of degree \( d \) equal to the algebraic intersection number of \( \Sigma \) with a fiber. This is calculated from the adjunction formula applied to a smooth fiber \( F \):
\[ d = \Sigma \cdot F = K \cdot F = 2h - 2. \]

Now Kneser’s inequality \( g(\Sigma) - 1 \geq |d|(g(B) - 1) \) gives:
\[ g(\Sigma) - 1 \geq 2(h - 1)(g - 1), \]
which together with (8) completes the proof of (1).

The following consequence of Theorem 1 makes precise the failure of uniformity of perfection of mapping class groups. It also applies to the genus 2 case, where the mapping class group is not perfect.

**Theorem 2.** Let \( a \) be a nontrivial separating simple closed curve on a surface \( F \) of genus \( h \geq 2 \), and let \( t_a \) be the corresponding Dehn twist. Suppose that \( t_a^k \) with \( k > 0 \) can be written as a product of \( N \) commutators. Then
\[ N \geq 1 + \frac{k}{6(3h - 1)}. \]

**Proof.** We consider a Lefschetz fibration over the 2-disk \( D \) with precisely \( k \) singular fibers, such that with respect to a basepoint on the boundary of \( D \) the vanishing cycles of all the singular fibers can be identified with \( a \). Then the monodromy of the fibration around the boundary of \( D \) is \( t_a^k \).

If this can be expressed as a product of \( N \) commutators, then we can find a smooth surface bundle with fiber \( F \) over a surface of genus \( N \) with one boundary component and the same restriction to the boundary. Let \( X \) be the Lefschetz fibration over the closed surface \( B \) of genus \( N \) obtained by gluing together the two fibrations along their common boundary.

By construction, no fiber contains a sphere, so \( X \) is relatively minimal. Thus we can apply Theorem 1 to conclude \( k \leq 6(3h - 1)(N - 1) \) as \( n = 0 \) in this case. \( \square \)
Remark 3. Theorem 2 implies in particular that no \( t^k_a \) equals a single commutator. For \( k = 1 \) this was previously proved in \([10]\). The proof there depends on a result of \([18]\) whose proof is not correct as written, but can by salvaged in the case needed for \([10]\), see the Erratum to \([18]\).

Remark 4. It is clear that the number of factors needed to express \( t^k_a \) as a product of commutators grows at most linearly with \( k \). Thus Theorem 2 settles Problem 2.13 (D) in Kirby’s list \([9]\) qualitatively.

Corollary 5. Let \( \Gamma_{k,h,r} \) be the mapping class group of genus \( h \) with respect to \( r \) marked points and \( k \) boundary components (fixed pointwise). The group \( \Gamma_{k,h,r} \) is not uniformly perfect for \( h \geq 2 \).

Proof. If \( k \geq 1 \), then we have a surjective homomorphism
\[
\Gamma_{k,h,r} \longrightarrow \Gamma_{k-1,h,r+1}
\]
given by collapsing a boundary component to a point. We also have surjective forgetful homomorphisms
\[
\Gamma_{k,h,r} \longrightarrow \Gamma_{k,h,r-1},
\]
so it is enough to prove the claim in the case \( r = k = 0 \). But this case is immediate from Theorem 2.

3. BOUNDED COHOMOLOGY AND COMMUTATOR LENGTHS

In this section we relate Theorems 1 and 2 to the second bounded cohomology.

Let \( G \) be a group, and \([G,G]\) its commutator subgroup. For an element \( g \in [G,G] \), the minimal number \( c_G(g) \) of factors in an expression of \( g \) as a product commutators is called the commutator length of \( g \). The limit
\[
||g||_{G} = \lim_{n \to \infty} \frac{c_G(g^n)}{n}
\]
is called the stable commutator length of \( g \). This is related to the second bounded cohomology of \( G \) by the following result:

**Theorem 6.** (Bavard \([1]\)) The map \( H^2_b(G) \to H^2(G) \) is injective if and only if the stable commutator length \( ||g||_{G} \) vanishes identically on \([G,G]\).

Now the proof of Theorem 2 implies that the stable commutator length of the Dehn twist along a separating simple closed curve is bounded below by \( \frac{1}{6(3h-1)} \). More generally, if \( g \in \Gamma_{k,h,r} \) is a product of \( s \) separating Dehn twists, not necessarily along the same curve, then we obtain
\[
||g||_{\Gamma_{k,h,r}} \geq \frac{s}{6(3h-1)}.
\]

Thus Theorem 3 implies:

**Corollary 7.** The map \( H^2_b(\Gamma_{k,h,r}) \to H^2(\Gamma_{k,h,r}) \) is not injective for all \( h \geq 2 \) and \( k,r \geq 0 \).
As Theorem 3 is not limited to perfect groups, we can also deal with some subgroups of mapping class groups.

3.1. Hyperelliptic mapping class groups. Let $\Delta_h$ be the hyperelliptic mapping class group of genus $h \geq 2$. It is known that $H^2(\Delta_h, \mathbb{R}) = 0$, cf. [5, 8], so that the statement analogous to Corollary 7 is just:

**Corollary 8.** The bounded cohomology $H^2_b(\Delta_h)$ is non-trivial.

**Proof.** From the presentation of $\Delta_h$ due to Birman-Hilden [2], it follows that the Abelianisation of $\Delta_h$ is a finite cyclic group of order $4(2h + 1)$ if $h$ is odd, and of order $2(2h + 1)$ if $h$ is even.

Let $a$ be a non-trivial separating simple closed curve on the surface of genus $h$ which is invariant under the hyperelliptic involution. Then $t_a^{4(2h+1)} \in [\Delta_h, \Delta_h]$, and using (11) we obtain:

$$||t_a^{4(2h+1)}||_{\Delta_h} \geq ||t_a^{4(2h+1)}||_{\Gamma_h} \geq \frac{4(2h + 1)}{6(3h - 1)} > 0.$$ 

Thus the claim follows from Theorem 6. \qed

3.2. Torelli groups. Let $T_h$ be the Torelli group of a closed oriented surface of genus $h \geq 2$ defined by

$$1 \rightarrow T_h \rightarrow \Gamma_h \xrightarrow{\phi} Sp(2h, \mathbb{Z}) \rightarrow 1,$$

where $\phi$ denotes the representation of the mapping class group on homology.

**Corollary 9.** The map $H^2_b(T_h) \to H^2(T_h)$ is not injective.

**Proof.** For $h = 2$ this follows from the result of Mess [14] that $T_2$ is a free group (on infinitely many generators).

Johnson [7] proved that for $h \geq 3$ the commutator subgroup of the Torelli group is:

$$[T_h, T_h] = T_h^2 \cap K_h,$$

where $T_h^2$ is the subgroup generated by all squares of elements of $T_h$, and $K_h$ is the subgroup generated by the Dehn twists along separating simple closed curves.

Thus, if $t_a$ is a separating Dehn twist, then $t_a^2 \in [T_h, T_h]$. Using (11), we have

$$||t_a^2||_{T_h} \geq ||t_a^2||_{\Gamma_h} \geq \frac{2}{6(3h - 1)} > 0,$$

and so the claim follows from Theorem 6. \qed

**Remark 10.** Here we have used the bound (11) on the stable commutator length in $\Gamma_h$ obtained from Theorem 4 which is certainly not optimal for $T_h$. Examining the proof of Theorem 4, we see that for Lefschetz fibrations whose monodromy is in the Torelli group, we have $\sigma(X) \leq n - s$ instead of (5), so that (6) is replaced by

$$s \leq 6(h - 1)(g - 1) + 5n.$$
With this we obtain

\[ ||t^a||_{\mathcal{T}_h} \geq \frac{2}{6(h - 1)}. \]

Remark 11. It has been pointed out to us by M. Burger that the kernel of \( H^2_b(\Gamma_h) \to H^2(\Gamma_h) \), which is non-trivial by Corollary 7, injects into the kernel of \( H^2_b(T_h) \to H^2(T_h) \) under the restriction map. This follows from the exact sequence in bounded cohomology associated to an extension of the form (12), see [3], together with the injectivity in degree 2 of the map from the bounded to the ordinary cohomology of \( \text{Sp}(2h, \mathbb{Z}) \) proved in [4].

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