Modeling a reverse taper pile for frost heave conditions

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Abstract. The research addresses the issue mathematical modeling of effective pile for construction in cold regions on the base of existing experimental studies. The problem of piles uplift by frost heave is critical and results in damage to overlying structures. Previous research and practice has shown that the existing methods of frost heave protection involving pile length extension lead to piles cost increase. It is assumed that a novel approach to the piles design is needed. The paper proposes directional reducing the influence of heaving soil on the reverse taper pile due to pile design features without its length extension. Positive effects which arise in this connection have been studied. The devised approach was based on mathematical modeling techniques. Heaving soil-pile interaction model has been developed. The study conducts an investigation of the pile frost heaving phenomenon that occurs in the annular space surrounding the pile. Particular attention has been given to working out pile geometry equations. It is concluded that the pile geometrical features developed by the devised mathematical modeling techniques allow to effectively reducing the frost heave influence. We can foresee that the research findings and the approach implementation will entail the reduction of construction costs in cold environments.

1. Introduction
Frost heave causes great damage to buildings and structures on pile foundation in cold regions [1]. A number of frost heave protection techniques have been developed: engineering amelioration, thermal, chemical, and structural protection techniques. All of them use a lot of additional design as well as materials or require length extension of piles that drive up the cost of construction and operation of structures. The research highlights the need for methods based on improving the effectiveness of structures without additional elements and which are therefore cost-neutral. This has led researchers for instance Yushkov and Repetskii (2014) [2], Domaschuk (1984) [3], Stelzer and Andersland (2007) [4], Huang and Sheng (2018) [5] to investigate the issue of piles design improvement. Domaschuk (1984) [3] conducted experimental and theoretical study of the opposition of pipe piles with expanded bases to frost heave forces. Stelzer and Andersland (2007) [4] conducted experimental study of the model steel piles with controlled lugs geometry on pile surface. This provides data on displacement mechanisms acting at the pile surface-frozen sand interface. Huang and Sheng (2018) [5] have considered the belled piles embedded in seasonally frozen soil. Experimental results showed that vertical displacements due to tangential frost heave forces of belled piles were less than the displacements of pile with uniform cross section. Kulchitskiy and Khamidullin (1976) [6] have noted high resistance of diamond shaped piles to tangential frost heave forces. Yushkov and Repetskii (2014) have shown experimentally an increased load capacity of pile with reverse surface inclination.
when it is uplifted due to tangential frost heave forces. Ponomarev and Bartholomey conducted studies of conical piles under various conditions [7-8].

Frost heave of soil is a complex phenomenon. A number of scientists have attempted to model these effects in order to facilitate the explanation of them: Dalmatov (1966) [9], Volokhov (1997-2017) [10-12], Kudryavtsev (2018) [13], Korshunov, Churkin and Nevzorov (2014) [14], Ulitskii (2015) [15], Ponomarev (2018) [16], Dagli (2018) [17], Ghoreishian and Grimstad (2017) [18], Peppin and Style (2013) [19], Zhang and Michalowski (2015) [20], Lebeau and Konrad (2012) [21]. Researchers have developed different models of soil parameters under freezing. Kudryavtsev (2018) [13] have shown the linear depth temperature distribution function. Golli (1999) [22] has used assumption of exponential distribution function temperature and frost heave normal stresses in depth. Volokhov, Nikitin and Lavrov (2017) [12] have reported that the boundary layer adjacent to the pile surface plays an important role in developing frost heave tangential stresses. Steiner, Vardon and Broere (2018) [23] investigated the influence of freezing rate on the shear strength and the cryogenic structure of clay soil.

Dalmatov (1966) [9], Golli 1999 [22] have formulated modeling conditions and similarity criteria of frost heave phenomenon. Grechishchev 1980 [24] has highlighted that mathematical modeling techniques are most applicable for the study of cryogenic heaving. Thus there are a number of remarkable models which allow to describing the frost heave phenomenon and a heaving soil-foundation interaction. However the models do not enable influence on these processes, establish no relationship between stress-related deformed states of heaving soil and the foundation design.

Based on the analysis of existing experimental studies, we came to the conclusion that the design of a pile with a reverse surface incline ensures its stability in the frozen soil under frost heave. Considering the extensive experience of mathematical modeling of frost heaving of the soil and its interaction with structures, given in numerous works of researchers, we believe that there is a need for further development of the models. The reason for this is that there are no analytical dependencies for determining the geometry of piles with inverse surface slope and no existing mathematical models can allow to varying the geometric parameters of the pile, ensuring its stability in the ground, depending on the stress-strain state of the freezing heaving soil and structural features of the structure.

The aim of the research is to develop the mathematical model of a special pile with reverse surface inclination at the top of the pile in order to provide resistance to the frost heave forces without pile length increase. The model will allow not only to predict pile behavior, but also to control it. It will enable a direct influence upon the construction behavior under the conditions of frost heave when the pile geometry changes. The related series of objectives include: to work out analytic models of piles with the reverse surface inclination at the top taking into account the soil-pile interaction physical model; to model the frost heave phenomenon; to establish the relation between the pile geometry and stress strain state of heaving soil around the pile; to develop a heaving soil and the pile interaction mathematical model.

2. Experimental
Experimental studies of the work of piles with a reverse surface gradient under conditions of frost heaving of the soil were carried out by Kulchitskiy and Khamidullin (1976) [6] on the example of diamond-shaped piles. The diamond-shaped piles showed higher stability in heaving soil than uniform cross-section piles (constant dimension pile), due to a decrease in the tangential frost heave forces of the soil, which is reflected in table 1. This decrease in heaving forces was achieved due to the reverse slope of the surface of the upper part of the pile.

The analytical solutions obtained in this work take into account the经验of diamond-shaped piles research and are based on experimental studies of double-cone piles conducted since 2003 till 2008 under the guidance of Yushkov and Repetskii [2]. The soil conditions of experiments were follows: organic clay with high plasticity, firm, organic clay with low plasticity, firm-stiff and soft-firm clay. The results of the experiments showed an increased bearing capacity of the biconical pile when being lifted by the tangential forces of frost heaving due to the operation of the upper reverse
cone, which makes it possible to reduce the tangential forces of frost heave. As a result, there was a smaller rise of biconical piles in heaving soil compared to a diamond-shaped pile and pile of constant cross section, as shown in table 1.

**Table 1.** The results of experimental studies of piles with the surface reverse inclination of Kulchitskiy and Khamidullin (1976) [6], Yushkov and Repetskii (2014) [2]

| Type of pile                | Vertical movements of piles for the autumn-winter season, mm |
|-----------------------------|-------------------------------------------------------------|
|                             | Pile length 3m | Pile length 6m | Pile length 8m |
| Diamond-shaped piles        | 15             | 5              | 8              |
| Biconical piles             | 13             | -              | -              |
| Pile of constant cross-section | 36             | 20             | 28; 41.4       |

The above studies present the high-quality work of the upper reverse cone of the pile in the heaving soil for a limited number of geometric parameters of the piles. The present paper develops the concept of reducing the forces of heaving due to pile geometry. It follows the abovementioned experimental studies and carries out analytical studies on the mathematical formalization of pile operation in heaving soil. These developments allowed the experimental results to be extended to other possible geometrical parameters of piles with an upper reverse cone and the entire range of soils subject to frost heaving.

3. **Theory**

The mathematical soil-pile interaction model of piles is the system of equations which describes pile behavior in soil under freezing. The model reflects the relation between pile geometry and forces which affect the pile under given soil and climatic conditions. The model made possible to transform the forces which act on the pile in frost heave soil by the pile geometry change. In total the model provided the forces equilibrium and stability of the pile in the frost soil and as a consequence preventing of pile uplift.

4. **Results**

4.1. **Analytic models of reverse taper pile**

Generation of the pile analytic models was based on the study of forces which occur due to pile geometry change and can neutralize frost heave tangential forces partly or fully. The pile geometry change involves creating surface reverse inclination at the top part of the pile. A particular case of this is a pile with a reverse taper at the top. This paper considers a pile which is referred to as a reverse taper pile. That is a vertical cylindrical pile with the top part in the form of a reverse frustum taper.

Tretyakova and Yushkov (2017) [25].

Soil mass around the reverse taper pile is regarded as linearly elastic half-space with rigid inclusion in the form of a special pile working as a friction-bearing pile. The investigated freezing zone is located within the half-space. The lower boundary of the freezing zone is movable. That movable freezing front shifts from ground surface to seasonal frost level and is an interphase boundary. The following processes occur in the freezing zone of soil: crystallization of water together with 9 percent volumetric gain; adding of water sucked up from thawed layer to unfrozen films on mineral particles (soil swelling type) Amount of the frozen and unfrozen water in the soil interstices increases due to these processes.

Internal stresses in soil occur when the amount of the frozen and unfrozen water exceeds the volume of the soil interstices. This stresses give rise to displacement of the soil interstices walls, which in its turn, leads to the increase in the soil volume. The constraining of the soil volume increase by the pile surface brings into existence the frost heave normal stresses. The normal stresses apply a
compressive effect on the pile and may trigger the tangential stresses of shear in the frozen soil with respect to the pile.

Adfreeze cohesion of frozen soil with the pile disrupts when flexural deformations and shear displacements of pile and frozen soil do not coincide. In the result tangential frost heave forces occur. Thus these forces are determined by soil shear stress with respect to the pile under freezing. The tangential forces of frost heave lead to the pile uplift.

The normal forces of frost heave act perpendicular to the pile surface. If there is a reverse taper of the pile, vertical downward component of the frost heave normal forces emerges. This component of frost heave normal forces is created due to the reverse taper of the pile surface and partly compensates upward-directed tangential frost heave forces which lead to the pile uplift. So this component of the frost heave normal forces resists the pile uplift by tangential forces of frost heave.

Kinetic interaction of heaving soil and reverse taper pile has allowed to proceed from stress fields to forces in the frozen soil layer around the pile and to give the analytic models as the base of mathematical model of this process. Two the analytic models of the pile are shown in figure 1.

![Figure 1](image)

**Figure 1.** The analytic models of the reverse taper pile: (a) if the frost depth position was located in the reverse taper zone of the pile; (b) if the frost depth position was located in the cylindrical part zone of the pile.

One model for the frost depth position in the reverse taper zone and other model for the frost depth position below that zone. The models represent following components of stress strain state the pile:

- The frost heave tangential force on the part of the pile with a reverse taper ($T_1$);
- The frost heave tangential force on the part of the pile with a constant cross-section ($T_2$);
- The frost heave normal force directed perpendicular to the pile lateral surface located at frozen soil ($S_1$);
- The friction resistance force on the lateral surface of the pile part with a reverse taper located in thawed soil ($F_1$);
- The friction resistance force at the lateral surface of the pile part with a constant cross-section located in thawed soil ($F_2$).

The following simplifications are adopted in figure 1:
- the external permanent load and proper weight of the pile ($P$);
- the coordinate of seasonal frost depth ($z$);
- the coordinate of base the taper pile part ($z_1$);
- the coordinate of base the cylindrical pile part ($z_2$).
4.2. Simulation of frost heave phenomenon

Frost heave forces which affect the reverse taper pile during each design-basis time interval of cold season can be represented in the form of an integral. The integral depicts the range of forces acting during the design-basis time interval but it is used to calculate the single value of the pile taper angle. As a result the pile can be overloaded with frost heave forces at some instant of the design-basis time interval.

Following the above reasoning, it is admitted that although theoretical model has the form of an integral, a different model for application in actual practice is needed. Such a model was developed to employ the worst combination of loads instead of the integral forces. The combination involved the maximum tangential force of frost heave (uplift force) and the corresponding normal force (hold-in force). That provided for a certain strength reserve while considering frost heave phenomenon which does not lend itself to a precise definition.

The objective to simulate the frost heave phenomenon resolves itself into finding the seasonal frost depth, evaluating the temperature distribution and temperature gradient in borehole, calculating the frost heave forces which act during the design-basis time intervals, and detecting the worst combination of these forces.

The following assumptions were accepted in the model: the soil along the pile is homogeneous; frost soil deformation modulus along the pile is permanent; temperature varies linearly in the depth of frozen soil layer.

The frost depth position is found with the aid of the Stefan condition. Considering that all phase transitions occur at the temperature \( t(z, \theta) = \text{const} = 0 \) and the temperature in the frozen zone linearly vary with the depth, the Stefan condition can be written in the form of a differential equation:

\[
Q \frac{d\xi}{d\theta} = -\frac{\dot{\lambda}_f t_{\text{surf}}}{\xi}
\]  

(1)

where \( \xi \) is the seasonal frost depth, \( \theta \) is the frost heaving time, \( \dot{\lambda}_f \) is the thermal conduction coefficient of frozen soil, \( Q \) is the specific heat of phase transitions (heat of crystallization); \( t_{\text{surf}} \) is the temperature of the soil surface.

Solving the equation 1 we obtain a function which allows to determining the frost depth position at a constant temperature of soil surface.

\[
\xi(\theta) = \left( \frac{2\dot{\lambda}_f |t_{\text{surf}}| \theta}{Q} \right)^{1/2}
\]  

(2)

where \( |t_{\text{surf}}| \) is the absolute value of negative soil surface temperature.

Formula 2 is also applied at a variable temperature of soil surface. Using the linear law of temperature varying in depth, the Stefan condition for movable phase boundary, and considering that at the phase boundary \( t_f < 0 \) (\( t_f \) is the temperature of frozen soil), \( t_{\text{traved}} \geq 0 \) (\( t_{\text{traved}} \) is the temperature of thawed soil), we obtain the equation of frost depth position for the design-basis time interval from \( \theta_1 \) to \( \theta_{i+1} \) with corresponding soil temperatures in the depth at constant surface temperature during the design-basis time interval:

\[
\xi_i = \sqrt{\frac{2\dot{\lambda}_f |t_{\text{surf}}| \theta_i}{Q}} - \left( \frac{2\dot{\lambda}_f |t_{\text{surf}}| \theta_i}{Q} \right)^{1/2}\left( \theta_{i+1} - \theta_i \right)
\]  

(3)

Expression 3 for the case of soil surface variable temperature is obtained through temperature time averaging. Design-basis time interval for temperature averaging can be ten days or a month.

The linear depth temperature distribution function presented in this model is due to surface temperature and frost depth position. The surface temperature in its turn depends on free-air temperature which is registered by meteorological stations at 1-2 m above the surface of the earth. Some researchers point to a distinction between the surface air temperature and surface temperature of
freezing soil [26]. However drawing on existing studies of Orlov (1970) [26] we have a reason to believe that the soil surface temperature can be assumed equal to the surface air temperature when calculating the frost heave process. Thus with regard for the discussed above variable temperature of soil surface and frost depth position, the negative ground temperature at the depths of the freezing conditions are formed in the experimental area, a placements corresponds to placements.

Along with this shear displacement arises between the inclusions of ice or at interface between ice and mineral particles due to their lower shear strength. According to the first soil particles cohesion are suggested that the distribution of the shear displacement occurs following the breakdown of frost failure under shear; employing of Coulomb’s law; existing concepts of the tangential stresses development stages; impact of normal stresses on the tangential stresses and specific cohesion of frozen soil. In the light of the foregoing the tangential stresses expression can be written as

\[ t_f(z, \theta, \xi) = t_{wz}(\theta) - \frac{t_{wz}(\theta)}{\xi(\theta)} z \]  

where \( z \) is the selected pile section coordinate.

**Temperature gradient** in borehole during design-basis time interval is relatively constant in depth but it is varies for different soil surface temperatures and there is linear distribution function of soil temperature in depth.

Frost heave simulation is based on the stress state of expanding soil under freezing taking into account the suggested hypothesis of elastic state.

The frost heave normal stresses are defined in the open system. The open system is a soil body which is freezing under natural conditions. These conditions are formed in the experimental area, a construction site or in the soil body next to the embedded structures built earlier. The open system is characterized by water sucking up from adjacent soil layers. In addition this system is exposed to the influence of the stress-strain state of the adjacent layers subject to the shrinkage or increase in volume. In most cases, water sucking up in the unfrozen zone causes soil shrinkage in the adjacent unfrozen layers.

As a result we obtain the formula of normal stress as the function of moisture that resulted in the formation of “excess ice” (Tretiakova, 2017) [27].

\[ \sigma_f = E_f \frac{1.095SP \cdot \theta \cdot \text{grad } t}{z} \left[ 1 - e \left( 1 - w_e \frac{\rho_d}{\rho_w} - 1.09w \frac{\rho_d}{\rho_w} \right) \right] k_{an} \]  

where \( E_f \) is the deformation modulus of frozen soil, \( SP \) is the segregation soil potential, \( \text{grad } t \) is the temperature gradient, \( e \) is the porosity ratio, \( w_e \) is the water content due to unfrozen water, \( w \) is the natural moisture of soil, \( \rho_d / \rho_w \) is the gravity water content to volumetric water content conversion factor and \( k_{an} \) is the anisotropy factor taking into account the direction of heaving forces.

Theoretical prerequisites for frost heave tangential stress calculation include heterogeneity of interacting continuous media i.e. the foundation material and freezing soil; ability of frozen soil for failure under shear; employing of Coulomb’s law; existing concepts of the tangential stresses development stages; impact of normal stresses on the tangential stresses development. This permitted to suggest that shear stresses resulting from heaving soil occur not in the boundary between the pile and freezing soil but in the boundary freezing layer of soil which flank to the pile surface.

This paper presents two approaches to the tangential stress calculation (Tretiakova, 2017) [28-29] The swell out freezing soil constrained by the pile lateral surface creates compressive stresses near the pile surface. This gives rise to tangential stresses and shear forces of soil relative to the pile surface. The shear forces decompose on numerous shear planes which arise due to different shear strengths of soil components in the boundary layer. Shear displacement occurs following the breakdown of frost soil particles cohesion. Along with this shear displacement arises between the inclusions of ice or at interface between ice and mineral particles due to their lower shear strength. According to the first approach one can reasonably suggest that the distribution of the shear displacements corresponds to quantitative soil-ice particle relation. This makes it possible to trace the connection of tangential stress with frozen soil moisture which defines ice quantity in the soil, and allows to finding the relationship between these stresses and specific cohesion of frozen soil. In the light of the foregoing the tangential stresses expression can be written as (Tretiakova, 2017) [28-29].
\[ \tau_f = \left[ c_f - 1.09(w + (SP \cdot \theta \cdot (grad t) \cdot k_{an} / z))(c_f - c_{sf}) \right] 
+ \eta_f \gamma_{ic}(\tan \phi_f)(1 - 1.09(w + (SP \cdot \theta \cdot (grad t) \cdot k_{an} / z))) 
+ \eta_{ic} \gamma_{ic}(\tan \phi_{ic})1.09(w + (SP \cdot \theta \cdot (grad t) \cdot k_{an} / z)), \]

where \( c_f \) is the soil specific cohesion; \( c_{sf} \) is the ice shear resistance; \( \eta_f \) and \( \eta_{ic} \) are the coefficients of the lateral pressure of soil and ice, respectively; \( \gamma_s \) and \( \gamma_{ic} \) are the volume density of soil and ice, respectively; \( \phi_f \) and \( \phi_{ic} \) are the angles of internal friction of soil and ice, respectively.

As discussed earlier, the frost heave tangential stresses are caused by the compression from normal pressure that heaving soil creates on the lateral surface of the pile. According to the second approach the expression for frost heave tangential stresses is written applying Coulomb’s law as a function of frost heave normal stresses (Tretiakova, 2017) [28-29].

\[ \tau_f = c_f + \sigma_f \tan \phi_f \]

where \( \phi_f \) is the angle of internal friction of frozen soil; \( c_f \) is the frozen soil specific cohesion.

Expression 7 for tangential stresses with regard to the formula 5 is as follows:

\[ \tau_f = c_f + \left\{ \frac{1.09SP \cdot \theta \cdot (grad t)}{z} \left[ 1 - e \left( 1 - w_w \frac{\rho_d}{\rho_w} - 1.09w_w \frac{\rho_d}{\rho_w} \right) \right] k_{an} \right\} \tan \phi_f \]

4.3. Relationship between the pile geometry and stress strain state of heaving soil around the pile

Behavior of reverse taper pile in frozen soil shall be deemed rational provided there is absence of its uplift by frost heave forces. This can be achieved by creating equilibrium of all forces acting on the pile. The equilibrium equations of forces acting on the pile were generated for the aims of this research. These equations describe the interaction between the pile and heaving soil with regard to the combined action of normal and tangential forces on the pile under frost heave; they take into account the pile geometry. The equilibrium equations were worked out for two cases as shown in figure 1. If the frost depth position is located in the pile reverse taper zone (figure 1 (a)); if the frost depth position is located below the part of pile with the reverse taper i.e. in the cylindrical part zone of the pile (figure 1 (b)).

Equilibrium equations of the reverse taper pile in soil affected by frost heave forces are written as follows:

Case (a) if \( \xi < z_i \):

\[ -P - S_f \sin \alpha + (T_{t1} - F_z) \cos \alpha - F_z = 0 \]

Case (b) if \( \xi > z_i \):

\[ -P - S_f \sin \alpha + T_{t1} \cos \alpha + T_{t1} - F_z = 0 \]

Case (a) if \( \xi < z_i \): \( S_f = \int \sigma_f 2 \pi R_{t1}(z) dz \), \( T_{t1} = \int \tau_{t1} 2 \pi R_{t1}(z) dz \), \( F_z = \int f_z 2 \pi R_{t1}(z) dz \), \( F_z = \int f_z 2 \pi R_{t1}(z) dz \),

Case (b) if \( \xi > z_i \): \( S_f = \int \sigma_f 2 \pi R_{t1}(z) dz \), \( T_{t1} = \int \tau_{t1} 2 \pi R_{t1}(z) dz \), \( F_z = \int f_z 2 \pi R_{t1}(z) dz \), \( F_z = \int f_z 2 \pi R_{t1}(z) dz \).
Where $S_f$ is integral normal force of frost heave at the part of the pile with reverse taper,
$T_{f1}$ is integral tangential force of frost heave at the part of the pile with a reverse taper,
$T_{f2}$ is integral tangential force of frost heave at the part of the pile with a constant cross-section,
$F_f$ is integral friction resistance force at the lateral surface of the pile part with a reverse taper
located in thawed soil,
$F_2$ is integral friction resistance force at the lateral surface of the pile part with a constant cross-
section located in thawed soil.

$\sigma_f$ is the frost heave normal force acting perpendicular to the lateral surface of the pile;
$\tau_f$ is the frost heave tangential force acting along the lateral surface of the pile; $f$ is the friction resistance force
at the part of the pile located in thawed soil; $P$ is the external permanent load and the proper weight of
the pile; $\alpha$ is the taper angle of the pile; $R_t(z)$ is the variable radius of the taper; $R_c$ is the radius of the pile
cylindrical part; $z_t$ and $z_c$ – are the coordinate of base taper part and cylindrical part of the pile
respectively (see figure 1).

The area of elementary annular strip along the taper perimeter ($dF_t$) is given in figure 2.

The following simplifications are adopted in figure 2.

The selected pile section coordinate ($z$); the variable radius of the taper ($R_t(z)$)

The area of an elementary annular strip along the perimeter of cylindrical part of the pile is as
follows:

$$dF = 2\pi R_c \sin \alpha (z_c - z_c) dz$$

(11)

The following assumptions are taken to simplify the solution of the equation:

$$\tan \alpha \approx \sin \alpha$$

(13)

$$\cos \alpha \approx 1$$

(14)

Scheme of the pile taper geometry is shown in figure 2. According to the scheme we have:

$$\tan \alpha = \frac{R_c - R_t}{L_t} \Rightarrow R_c - R_t = L_t \tan \alpha$$

(15)

$$\sin \alpha = \frac{R_c - R_t}{L_c} \Rightarrow L_c = \frac{R_c - R_t}{\sin \alpha}$$

(16)

With regard to expressions 16, 15 and 13 we get:
cos \alpha = \frac{L_1}{L} = \frac{L_1 \sin \alpha}{R - R_1} = \frac{L_1 \sin \alpha}{L_1 \tan \alpha} \approx 1 \quad (17)

Let us consider the case (a) for the equilibrium of the forces acting on the pile in heaving soil when the frost depth position is located in the inverted taper zone i.e. \( \xi < z_t \). If \( \cos \alpha \approx 1 \) (expression 13), then equilibrium equation 9 acquires the form:

\[-P - S_j \sin \alpha + T_{j_{\perp}} - F_i - F_z = 0 \quad (18)\]

Taking into account the given above Expressions we write the equation 18 in integral form:

\[-P - \left( \int_0^\xi \sigma_f \, df \right) \sin \alpha + \int_0^\xi \tau_{f_{\perp}} \, df_{\perp} - \int_0^\xi f_1 \, df_1 - \int_0^\xi f_2 \, df_2 = 0 \quad (19)\]

Equation 19 was transformed to the quadratic equation with the consideration of the variable radius of the taper, the area of elementary annular strip along the taper perimeter (expression 11) and the area of an elementary annular strip along the perimeter of cylindrical part of the pile (expression 12). Thus the quadratic equation to calculation the inverted taper angle inclination has the following form:

\[
\sigma_j (z_{\xi} - 0.5\xi^2) (\sin \alpha)^2 + \left[ -R_c \sigma_j \xi - r_j (z_{\xi} - 0.5\xi^2) + 0.5f_j (z_{\xi} - \xi) \right] (\sin \alpha)
+ R_c [r_{j_{\perp}} - f_j (z_{\xi} - \xi) - f_1 (z_{\xi} - z_t)] - 0.5\pi^{-1} P = 0
\quad (20)\]

The sine of the taper angle is the solution of expression 20.

Let us consider case “b” for the equilibrium forces acting on the pile in heaving soil when the frost depth position is located below the inverted taper zone i.e. \( \xi > z_t \). If \( \cos \alpha \approx 1 \) (expression 13), then equilibrium equation 10 acquires the form:

\[-P - S_j \sin \alpha + T_{j_{\perp}} + T_{n_{\perp}} - F_z = 0 \quad (21)\]

Taking into account the given above expressions we write equation 21 in integral form:

\[-P - \left( \int_0^{z_T} \sigma_f \, df \right) \sin \alpha + \int_0^{z_T} \tau_{f_{\perp}} \, df_{\perp} + \int_0^{z_T} \tau_{n_{\perp}} \, df_{n_{\perp}} - \int_0^{z_T} f_2 \, df_2 = 0 \quad (22)\]

Equation 22 was transformed to the quadratic equation with consideration the variable radius of the taper, the area of elementary annular strip along the taper perimeter (Expression 11), the area of an elementary annular strip along the perimeter of cylindrical part of the pile (Expression 12) and is written to the following form:

\[0.5\sigma_f z_T^2 (\sin \alpha)^2 + \left[ -R_c \sigma_f z_T - 0.5\tau f_1 z_T^2 \right] (\sin \alpha) +
+ R_c [r_{j_{\perp}} z_T + r_{f_2} (\xi - z_T) - f_2 (z_{c} - \xi)] - 0.5\pi^{-1} P = 0\]

Expression 23 allowed to determining the sine of taper angle.

4.4. Mathematical soil-pile interaction models for reverse taper piles

The mathematical soil-pile interaction model for reverse taper piles in frost heaving soils represents a system of equations:

\[\xi_{\perp,\alpha} = \left[ \xi_{\perp,\alpha}^2 - \frac{2L_{surf(t)} Q}{Q} (\theta_{\perp,\alpha} - \theta_{\perp}) \right]^{\frac{1}{2}} \quad (24)\]
\[ t_r(z, \theta, \xi) = \frac{t_{aw}(\theta)}{\xi(\theta)} \cdot z \]  

\[ \sigma_r = \frac{1.09SP \cdot \theta \cdot \text{grad} t}{z} \left[ 1 - q \left( 1 - w_u \frac{P_c}{\rho_c} - 1.09w \frac{P_f}{\rho_f} \right) \right] k_{aw} \]  

\[ \tau_r = \left[ c_r - 1.09(w + (SP \cdot \theta \cdot \text{grad} t \cdot k_{aw}/z))(c_r - c_{aw}) \right] \]  

\[ + \eta \gamma_s(z(tg \phi_s)(1 - 1.09(w + (SP \cdot \theta \cdot \text{grad} t \cdot k_{aw}/z))) \]  

\[ + \eta \gamma_w(z(tg \phi_w)1.09(w + (SP \cdot \theta \cdot \text{grad} t \cdot k_{aw}/z)), \]  

\[ \tau_r = c_r + \sigma_r \tan \phi_r \]  

\[ dF_i = 2\pi[R - \sin \alpha(z_i - z)]dz \]  

\[ dF_i = 2\pi R_idz \]  

When \( \xi < z_i: \) 

\[ -P - \int_0^\xi \sigma_r \cdot dF_i \sin \alpha + \int_0^\xi \tau_i \cdot dF_i - \int_\xi f_i \cdot dF_i = 0 \]  

\[ \sigma_r(z, \xi - 0.5\xi^2)(\sin \alpha)^2 + [-R \sigma_r \xi - \tau_r(z, \xi - 0.5\xi^2) + 0.5f_i(z_i - \xi)](\sin \alpha) \]  

\[ + R[\tau_r(z, \xi - f_i(z_i - \xi) - f_i(z_i - z_i)] - 0.5\pi^2P = 0 \]  

When \( \xi > z_i: \) 

\[ -P - \int_0^\xi \sigma_r \cdot dF_i \sin \alpha + \int_0^\xi \tau_i \cdot dF_i + \int_\xi f_i \cdot dF_i = 0 \]  

\[ 0.5\sigma_r z_i^2(\sin \alpha)^2 + (-R \sigma_r z_i - 0.5\tau_r z_i^2)(\sin \alpha) + \]  

\[ + R[\tau_r(z_i + \tau_r(z_i - z_i)] - f_i(z_i - \xi)] - 0.5\pi^2P = 0 \]  

5. Discussion

It can be stated we have succeeded in obtaining the dependencies between the sine of the pile taper angle and forces which act on the pile under frost heave. The sine of taper angle as required for relieving the tangential forces of frost heave has been obtained as solution of equilibrium equation for the forces acting on the special pile under given conditions. Taking into account the variableness of taper radius, the forces equilibrium equation for the pile was written in the quadratic form. Equations were developed for case when the frost depth position is within the inverted taper of the pile (equations 20) and for case of the frost depth position below the taper (equations 23).

Furthermore, a mathematical soil-pile interaction model for reverse taper piles in frost heaving soils was formulated on the basis of the forces equilibrium equations and the frost heave phenomenon modeling.

The obtained equations 20 and 23 provide a single step calculation method to find were only one step to calculate the taper pile angle. Whereas in earlier occasions of Kulchitskiy and Khamidullin [6], Yushkov and Repetskii [2] the taper pile angle is found by trial and error method. The abovementioned studies do not calculate the required taper angle but this angle is necessary for each calculation of a pile even when its observation is not possible. Yushkov and Repetskii [2] have made a pile calculation without due regard to variable radius of taper or different cases of frost depth position...
as relating to the pile. These factors are taken into account in our model. Our integral mathematical model allows to calculating the pile, its application ranging from frost heave parameters to the taper pile angle calculation.

Directions for future research include the development of a software application based on the obtained models; it would bring the present research results into more widespread use. A software module to calculate the geometry of the reverse taper pile is being worked out. The investigation of the prismatic piles with the reverse taper is also performed by the research paper authors.

6. Conclusion
To bring the paper to a close, we summaries the main findings.

We have devised a novel methodology of modeling a pile that can resist frost heave tangential forces owing to its form. The analytic models of the reverse taper pile were given for two cases: when the frost depth position is in the reverse taper zone of the pile and when the frost depth position is below that zone i.e. in the cylindrical part zone of the pile.

Analytic expressions for frost heave stresses were derived. The normal stress was determined as the function of moisture that resulted in formation of “excess ice” exceeding the soil interstices volume. The tangential stresses were found taking into account the specific cohesion and frozen soil moisture as well as in function of frost heave normal stresses.

We have managed to establish analytical relationships between the geometrical parameters of the pile and the stress-strain state of the soil around the pile. Based on the obtained analytical dependencies, Equations 20 and 23 were written for calculating the angle of inclination of the pile surface. These Equations reflect the relationship between the geometry of the pile and the forces acting on the pile in the heaving soil. The Equations were written in quadratic form.

In general a mathematical soil-pile interaction model has been proposed for reverse taper piles; it allows calculate the geometry of the pile under given climatic and geological conditions.

The importance of our work lies in the increase of piles design cost effectiveness for the conditions of frost heave of soil. These findings add to a growing body of research in the field of structures effectiveness in the conditions of frost heaving of the soil.

The present study has investigated the reverse taper piles which can be applied as the foundation of shallow tunnels that operate under frost heave conditions. The research results might be transferable for the use of piles in bridges pillar construction in cold regions.

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