Global well-posedness for the defocusing, quintic nonlinear Schrödinger equation in one dimension

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Abstract: In this paper, we prove global well-posedness for low regularity data for the one dimensional quintic defocusing nonlinear Schrödinger equation. We show that a unique solution exists for $u_0 \in H^s(\mathbb{R})$, $s > \frac{8}{29}$. This improves the result in [3], which proved global well-posedness for $s > \frac{1}{3}$. The main new argument is that we obtain almost Morawetz estimates with improved error.

1 Introduction

In this paper we study the initial value problem for the quintic, defocusing, nonlinear Schrödinger equation in one dimension,

\begin{align}
\begin{aligned}
&iu_t + \Delta u = |u|^4 u, \\
&u(0, x) = u_0 \in H^s(\mathbb{R}).
\end{aligned}
\end{align}

This is an $L^2$-critical equation. By the results of [1], this equation has a local solution on some $[0, T]$, $T(\|u_0\|_{H^s(\mathbb{R})}) > 0$, when $s > 0$. If a solution to (1.1) fails to be global and only exists on $[0, T_*)$, $T_* < \infty$, then

\begin{equation}
\lim_{t \to T_*} \|u(t)\|_{H^s(\mathbb{R})} = \infty.
\end{equation}

[3] proved

\begin{equation}
M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)),
\end{equation}

\begin{equation}
E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{6} \int |u(t, x)|^6 dx = E(u(0)),
\end{equation}

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are conserved, giving global well-posedness for \( u_0 \in H^1(\mathbb{R}) \). The regularity necessary for global well-posedness has since been lowered to \( s > \frac{1}{3} \), (see [13]). In this paper we will prove

**Theorem 1.1** ([13]) is globally well-posed for all \( u_0 \in H^s(\mathbb{R}) \), \( s > \frac{8}{29} \). Moreover,

\[
\sup_{t \in [0,T]} \| u(t) \|_{H^s(\mathbb{R})} \lesssim (1 + T)^{\frac{4(1-s)}{29s-8}}. \tag{1.5}
\]

[13] used the I-method, a method that we will utilize in this paper as well. The I-method was first introduced for the defocusing, cubic initial value problem (see [7]).

In §2, we will start with some preliminary information, including the Strichartz estimates, Littlewood-Paley theory, a description of the I-method, and a local well-posedness result. In §3, an energy increment will be obtained. In §4, the almost Morawetz estimates will be proved. In §5, we will prove the theorem.

## 2 Local Well-posedness

The proof of local well-posedness makes use of the Strichartz estimates.

**Theorem 2.1** A pair \((p, q)\) is called an admissible pair if \( \frac{2}{p} + \frac{1}{q} = \frac{1}{2} \). If \((p, q)\) and \((\tilde{p}, \tilde{q})\) are admissible pairs and and \( u(t, x) \) solves

\[
iu_t + \Delta u = F(t), \quad u(0, x) = u_0, \tag{2.1}
\]

then

\[
\| u(t, x) \|_{L^p_t L^q_x(J \times \mathbb{R})} \lesssim \| u_0 \|_{L^2(\mathbb{R})} + \| F(t) \|_{L^p_t L^q_x(J \times \mathbb{R})}. \tag{2.2}
\]

**Proof:** See [23]. \( p' \) denotes the Lebesgue exponent \( \frac{p}{p-1} \).

The Strichartz space will be defined by the norm

\[
\| u \|_{S^0(J \times \mathbb{R})} = \sup_{(p, q) \text{ admissible}} \| u \|_{L^p_t L^q_x(J \times \mathbb{R})}. \tag{2.3}
\]
The space $N^0(J \times \mathbb{R})$ is the dual space to $S^0(J \times \mathbb{R})$. See [23] for more details.

We will also make use of the Littlewood-Paley decomposition. Suppose $\phi(x)$ is a smooth function,

$$\phi(x) = \begin{cases} 1, & |x| \leq 1/2; \\ 0, & |x| > 1. \end{cases}$$

(2.4)

$$\mathcal{F}(P_{\leq Nu}) = \phi\left(\frac{\xi}{N}\right)\hat{u}(\xi),$$

$$\mathcal{F}(P_{>Nu}) = (1 - \phi\left(\frac{\xi}{N}\right))\hat{u}(\xi),$$

$$\mathcal{F}(P_{N}u) = P_{\leq Nu} - P_{< \frac{N}{2}}u.$$  

(2.5)

For convenience, let $u_N = P_{N}u$, similarly for $u_{\leq N}, u_{> N}$.

The $I$-operator is a Fourier multiplier,

$$I_N : H^s(\mathbb{R}) \to H^1(\mathbb{R}),$$

(2.6)

$$\hat{I_Nf}(\xi) = m_N(\xi)\hat{f}(\xi),$$

(2.7)

$$m_N(\xi) = \begin{cases} 1, & |\xi| \leq N; \\ \left(\frac{N}{|\xi|}\right)^{s-1}, & |\xi| > 2N. \end{cases}$$

(2.8)

$$\|Iu\|_{H^1(\mathbb{R})} \lesssim N^{1-s}\|u\|_{H^s(\mathbb{R})},$$

$$\|u\|_{H^s(\mathbb{R})} \lesssim \|Iu\|_{H^1(\mathbb{R})},$$

(2.9)

therefore, controlling $E(Iu(t))$ gives control of $\|u(t)\|_{H^s(\mathbb{R})}$. For the rest of the paper, $I f$ denotes $I_N f$, and the presence of an $N$ is implied.

**Lemma 2.2** Let $I$ be a compact time interval, $t_0 \in I$, $N > 0$, and suppose $u_1, u_2$ are two solutions to (2.2) such that $u_j(t)$ has Fourier support in the region $\{ |\xi| \leq N \}$ for $j = 1, 2$. Suppose also that the Fourier supports of $u_1, u_2$ are separated by at least $\geq cN$. Then for any $q > 2$,

$$\|u_1u_2\|_{L^{q}_{t,x}(I \times \mathbb{R})} \lesssim N^{1-3/q}\|u_1\|_{S^0(I \times \mathbb{R})}\|u_2\|_{S^0(I \times \mathbb{R})},$$

(2.10)

where
\[ \|u\|_{S_0(I \times \mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})} + \|(i\partial_t + \Delta)u\|_{L^{6/5}_{t,x}(I \times \mathbb{R})}. \]  

(2.11)

Proof: See [22].

To this end, let \( q = 2 + \delta, \frac{1}{p} = \frac{1}{2} - \frac{1}{2+\delta} \).

In proving theorem [11] we will make use of a linear-nonlinear decomposition. See [19] for the linear-nonlinear decomposition for the defocusing, semilinear wave equation, [14] for the linear-nonlinear decomposition used for the three dimensional cubic defocusing nonlinear Schrödinger equation.

**Theorem 2.3** If

\[ \|\nabla Iu_0\|_{L^2(\mathbb{R})} \leq 1 \]  

(2.12)

and for some \( \epsilon > 0 \) sufficiently small,

\[ \|Iu\|_{L^{16/3}_{t}L^8_{x}(J \times \mathbb{R})} \leq \epsilon, \]  

(2.13)

then

\[ \|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R})} \lesssim 1. \]  

(2.14)

Moreover, the solution has the form

\[ e^{it\Delta}u_0 + u^{nl}(t), \]  

(2.15)

\[ \|P_{\geq cN} (\langle \nabla \rangle Iu^{nl})\|_{S^0(J \times \mathbb{R})} \lesssim \frac{1}{N^{1/2}}. \]  

(2.16)

Proof: The solution obeys the Duhamel formula,

\[ Iu(t, x) = e^{it\Delta}u_0 + \int_0^t e^{i(t-\tau)\Delta}I(|u(\tau)|^4u(\tau))d\tau. \]  

(2.17)

By the Strichartz estimates,

\[ \|(1-I)u\|_{L^{16/3}_{t}L^8_{x}(J \times \mathbb{R})} \lesssim \frac{1}{N}\|\langle \nabla \rangle Iu\|_{S^0(J \times \mathbb{R})}, \]  

(2.18)
\[ \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R})} \lesssim \| \langle \nabla \rangle Iu_0 \|_{L^2(\mathbb{R})} + \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R})} \| u \|^4_{L_t^{16/3}L_x^2(J \times \mathbb{R})} \]

(2.19)

\[ \lesssim \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R})} \lesssim \| \langle \nabla \rangle Iu_0 \|_{L^2(\mathbb{R})} + \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R})} (\epsilon^4 + \frac{|\langle \nabla \rangle Iu|^4}{N^4}). \]

(2.20)

Therefore, by the continuity method,

\[ \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R})} \lesssim 1. \]

(2.21)

This takes care of (2.14). Next, we remark that this also proves

\[ \| \langle \nabla \rangle I(|u|^4u) \|_{L_t^{6/5}L_x^2(J \times \mathbb{R})} \lesssim \| \langle \nabla \rangle Iu \|_{S^0(J \times \mathbb{R})}^5 \lesssim 1. \]

(2.22)

To estimate the nonlinearity,

\[ \nabla Iu^{nl}(t) = \int_0^t \nabla e^{i(t-\tau)\Delta} I(|u(\tau)|^4u(\tau))d\tau. \]

(2.23)

\[ I(|u(\tau)|^4u(\tau)) = I(|u|_{\leq \frac{\epsilon N}{10}})I(|u|_{\leq \frac{\epsilon N}{10}}) + I(O((u|_{\leq \frac{\epsilon N}{100}})^4(u|_{> \frac{\epsilon N}{100}}))) + I(O((u|_{< \frac{\epsilon N}{10}})(u|_{> \frac{\epsilon N}{100}})u(\tau)^3)). \]

(2.24)

The first term, \( I(|u|_{\leq \frac{\epsilon N}{10}})^4u(\tau) \) is supported on \(|\xi| \leq \frac{\epsilon N}{20}\). To estimate the second term, use the bilinear estimates,

\[ \| \nabla I(O((u|_{> \frac{\epsilon N}{100}})^4)) \|_{L_t^1L_x^2(J \times \mathbb{R})} \]

\[ \lesssim \| \nabla Iu \|_{L_t^1L_x^2(J \times \mathbb{R})} \| u \|_{L_t^4L_x^\infty}^2 \lesssim \frac{1}{N^{1/2}}. \]

Finally,

\[ \| \nabla I(O((u|_{> \frac{\epsilon N}{100}})(u|_{< \frac{\epsilon N}{100}})u(\tau)^3)) \|_{L_t^1L_x^2(J \times \mathbb{R})} \]

\[ \lesssim \| \nabla Iu \|_{L_t^1L_x^2(J \times \mathbb{R})} \| u \|_{L_t^{16/3}L_x^2(J \times \mathbb{R})} \| u \|_{L_t^{16/3}L_x^2(J \times \mathbb{R})}^3 \]

\[ \lesssim \frac{\epsilon^3}{N} \| \langle \nabla \rangle Iu \|^2_{S^0(J \times \mathbb{R})} \lesssim \frac{1}{N}. \]
The last estimate follows from

$$\|u_\geq cN\|_{L_t^{16/3}L_x^8(J\times \mathbb{R})} \lesssim \sum_{cN \leq N_J} \|P_{N_j}u\|_{L_t^{16/3}L_x^8(J\times \mathbb{R})}$$

$$\lesssim \|\langle \nabla \rangle Iu\|_{S^0(J\times \mathbb{R})} \sum_{cN \leq N_J} \frac{1}{N_j^3 N^1 - s} \lesssim \frac{1}{N} \|\langle \nabla \rangle Iu\|_{S^0(J\times \mathbb{R})}.$$  \hfill (2.25)

3 Energy Increment

In this section we prove almost conservation of the modified energy \(E(Iu(t))\).

Theorem 3.1 Let

$$E(Iu(t)) = \frac{1}{2} \int |\nabla Iu(t, x)|^2 \, dx + \frac{1}{6} \int |Iu(t, x)|^6 \, dx.$$  \hfill (3.1)

If \(J\) is an interval where a solution \(u(t, x)\) of (1.1) exists, \(\|u\|_{L_t^{16/3}L_x^8(J\times \mathbb{R})} \leq \epsilon, \ E(Iu_0) \leq 1\), then

$$\sup_{t_1, t_2 \in J} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{1}{N^3/2} \|P_{>cN} \nabla Iu\|_{L_t^4L_x^\infty(J\times \mathbb{R})}^2 + \frac{1}{N^2},$$  \hfill (3.2)

where \(c > 0\) is some small constant.

Proof:

$$\frac{d}{dt} E(Iu(t)) = Re \int (\overline{Iu(t, x)})(I(|u(t, x)|^4u(t, x)) - |Iu(t, x)|^4Iu(t, x)) \, dx.$$  \hfill (3.3)

Taking the Fourier transform, let \(\Sigma = \{\xi_1 + \ldots + \xi_6 = 0\}\), \(d\xi\) is the Lebesgue measure on the hyperplane, using the fact that

$$Iu_t = iI\Delta u - iI(|u|^4u),$$  \hfill (3.4)

$$\frac{d}{dt} E(Iu(t)) = -Re \int_{\Sigma} (i|\xi_1|\overline{\widehat{Iu}(t, \xi_1)})(1 - \frac{m(\xi_2 + \ldots + \xi_6)}{m(\xi_2)m(\xi_3) \ldots m(\xi_6)})$$

$$\times \widehat{Iu}(t, \xi_2)\widehat{Iu}(t, \xi_3)\widehat{Iu}(t, \xi_4)\widehat{Iu}(t, \xi_5)\widehat{Iu}(t, \xi_6) \, d\xi$$  \hfill (3.5)
We will estimate (3.5) and (3.6) separately by making a Littlewood-Paley decomposition and consider several cases separately. Without loss of generality let \( N_2 \geq N_3 \geq N_4 \geq N_5 \geq N_6 \).

### The term (3.5):

When estimating this term, we will frequently use the bilinear estimate (2.11).

#### Case 1, \( N_2 \ll N \): In this case, \( m(\xi_i) \equiv 1 \), so

\[
1 - \frac{m(\xi_2 + \ldots + \xi_6)}{m(\xi_2) \cdots m(\xi_6)} \equiv 0. \tag{3.7}
\]

#### Case 2, \( N_2 \gg N \gg N_3 \): By the fundamental theorem of calculus,

\[
|1 - \frac{m(\xi_2 + \ldots + \xi_6)}{m(\xi_2) \cdots m(\xi_6)}| \lesssim \frac{N_3}{N_2}. \tag{3.8}
\]

Recall that \( q = 2 + \delta \), \( \frac{1}{p} = \frac{1}{2} - \frac{1}{q} \), and \( N_1 \sim N_2 \), so

\[
\sum_{N_1 \sim N_2} \sum_{N_6 \leq N_5 \leq N_4 \leq N_3 \ll N} \frac{N_1}{N_2^2} \left\| (P_{N_1} \nabla Iu)(P_{N_3} Iu) \right\|_{L^q_{t,x}} \times \left\| (P_{N_2} \nabla Iu)(P_{N_4} Iu) \right\|_{L^p_{t,x}} \left\| P_{N_5} Iu \right\|_{L^p_{t,x}} \left\| P_{N_6} Iu \right\|_{L^p_{t,x}} \lesssim \frac{1}{N_2^{2-}}.
\]

#### Case 3, \( N_2 \geq N_3 \gg N \gg N_4 \): In this case, estimate the multiplier by

\[
|1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)}| \lesssim \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)}. \tag{3.9}
\]

Consider three subcases separately.
Case 3(a), \( N_3 << N_1 \sim N_2 \):

\[
\sum_{N \lesssim N_3 N_1 \sim N_2 N_0 \lesssim N_3 N_4 \ll N} \sum_{m(N_2) m(N_3) N_2} \frac{1}{m(N_3) N_2} \|(P_{N_1} \nabla Iu)(P_{N_3} Iu)\|_{L^q_{t,x}}^q
\]

\[
\|(P_{N_2} \nabla Iu)(P_{N_4} Iu)\|_{L^q_{t,x}} \|P_{N_5} Iu\|_{L^p_{t,x}} \|P_{N_6} Iu\|_{L^p_{t,x}}
\]

\[
\lesssim \sum_{N \lesssim N_3 N_1 \sim N_2 N_0 \lesssim N_3 N_4 \ll N} \sum_{m(N_2) m(N_3) N_2} \frac{N_1^{1/2}}{N_2^{3/2}} \frac{1}{N_3 N_1^{1-\varepsilon}} \frac{1}{(N_4) (N_5)^{1/2} (N_6)^{1/2}} \lesssim \frac{1}{N^2}.
\]

Case 3(b), \( N_1 << N_2 \sim N_3 \):

\[
\sum_{N \lesssim N_3 N_0 \lesssim N_1 \sim N_2 N_0 \lesssim N_3 N_4 \ll N} \frac{m(N_1)}{m(N_2) m(N_3) N_2} \|(P_{N_1} \nabla Iu)(P_{N_4} Iu)\|_{L^q_{t,x}}^q
\]

\[
\|(P_{N_3} Iu)(P_{N_4} Iu)\|_{L^q_{t,x}} \|P_{N_5} Iu\|_{L^p_{t,x}} \|P_{N_6} Iu\|_{L^p_{t,x}}
\]

\[
\lesssim \sum_{N \lesssim N_3 N_1 \sim N_2 N_0 \lesssim N_3 N_4 \ll N} \sum_{m(N_2) m(N_3) N_2} \frac{N_1^{1/2}}{N_2^{3/2}} \frac{1}{N_3 N_1^{1-\varepsilon}} \frac{1}{(N_4) (N_5)^{1/2} (N_6)^{1/2}} \times \frac{1}{(N_4) (N_5)^{1/2} (N_6)^{1/2}} \lesssim \frac{1}{N^2}.
\]

Case 3(c), \( N_1 \sim N_2 \sim N_3 \): In this case,

\[
\frac{m(N_1)}{m(N_2) m(N_3)} \sim \frac{1}{m(N_3)}.
\]

\[
\sum_{N \lesssim N_1 \sim N_3 N_0 \lesssim N_1 \sim N_2 N_0 \lesssim N_3 N_4 \ll N} \frac{N_1}{N_2} \|(P_{N_1} \nabla Iu)(P_{N_4} Iu)\|_{L^q_{t,x}}^q
\]

\[
\|P_{N_2} \nabla Iu\|_{L^q_{t,x}} \|P_{N_3} Iu\|_{L^p_{t,x}} \|P_{N_5} Iu\|_{L^p_{t,x}} \|P_{N_6} Iu\|_{L^p_{t,x}}
\]

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\[
\lesssim \sum_{N \leq N_2 \sim N_3} \frac{1}{N_2^{1/2} \cdot N_1^{1-s} N_3^s} \|P_{N_2} \nabla Iu\|_{L_t^4 L_x^\infty} \|P_{N_3} \nabla Iu\|_{L_t^4 L_x^\infty} \\
\times \sum_{N_6 \leq N_2 \leq N_4 \ll N} \frac{1}{(N_4)(N_5)(N_6)^{1/2}} \lesssim \frac{1}{N_3^{s/2}} \|P_{>N} Iu\|_{L_t^4 L_x^\infty}^2 (J \times R).
\]

**Case 4,** \(N_2 \geq N_3 \geq N_4 \gtrsim N\):

**Case 4(a),** \(N_1 \sim N_2\): In this case

\[
|1 - \frac{m(\xi_1)}{m(\xi_2) \cdots m(\xi_6)}| \lesssim \frac{1}{m(\xi_3) \cdots m(\xi_6)}.
\]

\[
\sum_{N \leq N_1 \sim N_2} \sum_{N_6 \leq N_5 \leq N_4 \leq N_3} \frac{N_1}{N_2^2} \|P_{N_1} \nabla Iu\|_{L_t^6 L_x^\infty} \|P_{N_2} \nabla Iu\|_{L_t^6 L_x^\infty} \\
\times \|P_{N_3} u\|_{L_t^6 L_x^\infty} \|P_{N_4} u\|_{L_t^6 L_x^\infty} \|P_{N_5} u\|_{L_t^6 L_x^\infty} \|P_{N_6} u\|_{L_t^6 L_x^\infty} \lesssim \frac{1}{N^2}.
\]

**Case 2,** \(N_1 \ll N_2\): In this case \(N_2 \sim N_3\).

\[
\sum_{N \leq N_2 \sim N_3} \sum_{N_6 \leq N_4 \leq N_5 \leq N_2} \frac{N_1 m(N_1)}{N_2 m(N_2)} \|P_{N_1} \nabla Iu\|_{L_t^6 L_x^\infty} \|P_{N_2} \nabla Iu\|_{L_t^6 L_x^\infty} \\
\|P_{N_3} u\|_{L_t^6 L_x^\infty} \|P_{N_4} u\|_{L_t^6 L_x^\infty} \|P_{N_5} u\|_{L_t^6 L_x^\infty} \|P_{N_6} u\|_{L_t^6 L_x^\infty} \lesssim \frac{1}{N^2}.
\]
This takes care of (3.5).

**The term (3.6):** Recall that this is the 10-linear term

\[ \text{Re} \int_0^t \int \overline{I(|u|^4 u)} I(|u|^4 u) - |Iu|^4 (Iu)] dx dt. \quad (3.10) \]

The term \(I(|u|^4 u)\) poses a slight technical problem. Ideally, this term would be placed in \(L^6_{t,x}\), and we would then repeat the analysis used in (3.5). However, in general this is not possible, so instead let \(u_l = P_{\leq \frac{N}{30}} u, u_h = P_{\geq \frac{N}{30}} u,\)

\[ F(t, x) = \overline{I(|u|^4 u)} - I(O(u_h^4 u_l)) - I(|u_h|^4 u_l), \quad (3.11) \]

where \(O(u_h^4 u_l)\) consists of those terms in \(|u_l + u_h|^4 (u_l + u_h)\) consisting of four \(u_h\) terms and one \(u_l\) term.

\[
\|F(t, x)\|_{L^1_t L^\infty_x(J \times \mathbb{R})} \lesssim \|\langle \nabla \rangle F(t, x)\|_{L^1_t L^5_x(J \times \mathbb{R})} \\
\lesssim \|\langle \nabla \rangle Iu\|_{L^2_t L^\infty_x(J \times \mathbb{R})} \|u\|_{L^\infty_t L^2_x(J \times \mathbb{R})} \lesssim \|\langle \nabla \rangle Iu\|_{S_0^5(J \times \mathbb{R})}. \quad (3.12)
\]

Now evaluate

\[
\int_0^t \int_{\mathbb{R}} \hat{F}(t, \xi_1) \left[ 1 - \frac{m(\xi_2 + ... + \xi_6)}{m(\xi_2) \cdots m(\xi_6)} \right] \hat{Iu}(t, \xi_2) \hat{Iu}(t, \xi_3) \hat{Iu}(t, \xi_4) \hat{Iu}(t, \xi_5) \hat{Iu}(t, \xi_6), \quad (3.13)
\]

via a Littlewood - Paley partition of unity and considering several subcases separately. Without loss of generality, let \(N_2 \geq N_3 \geq \ldots \geq N_6\).

**Case 1, \(N_2 \ll N\):** Once again, \(m(\xi_i) \equiv 1,\) so

\[ 1 - \frac{m(\xi_2 + ... + \xi_6)}{m(\xi_2) \cdots m(\xi_6)} \equiv 0. \quad (3.14) \]

**Case 2, \(N_2 \gg N \gg N_3\):** In this case, apply the fundamental theorem of calculus,

\[ |1 - \frac{m(\xi_2 + ... + \xi_6)}{m(\xi_2) \cdots m(\xi_6)}| \lesssim \frac{N_3}{N_2}. \quad (3.15) \]

\[
\sum_{N < N_2} \sum_{N_0 \leq N_0 \leq N_3 \leq N_3 < \ll N} \frac{N_3}{N_2} \|P_{N_1} F\|_{L^1_t L^\infty_x(J \times \mathbb{R})} \|P_{N_2} Iu\|_{L^1_t L^\infty_x(J \times \mathbb{R})}
\]

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To evaluate it is necessary to take advantage of some cancellations. 

Case 3, \( N_2 \geq N_3 \gtrsim N \) In this case make the crude estimate

\[
|1 - \frac{m(\xi_2 + \cdots + \xi_6)}{m(\xi_2) \cdots m(\xi_6)}| \lesssim \frac{1}{m(\xi_2) \cdots m(\xi_6)}.
\] (3.16)

\[
\sum_{N_1 \leq N_2, N_3 \leq N_2} \sum_{N_3 \leq N_3} \|P_{N_1} F\|_{L^4_t L^\infty_x(J \times \mathbb{R})} \|P_{N_2} u\|_{L^4_t L^\infty_x(J \times \mathbb{R})} \|P_{N_3} u\|_{L^4_t L^\infty_x(J \times \mathbb{R})}
\times \|P_{N_4} u\|_{L^4_t L^\infty_x(J \times \mathbb{R})} \|P_{N_5} u\|_{L^4_t L^\infty_x(J \times \mathbb{R})} \|P_{N_6} u\|_{L^\infty_t L^2_x(J \times \mathbb{R})}
\lesssim \sum_{N_1 \leq N_2, N_3 \leq N_2} \sum_{N_3 \leq N_3} \frac{1}{N_3^N N^1-s} \frac{1}{N_3^N N^1-s} \frac{1}{(N_4) m(N_4) \langle N_5 \rangle m(N_5) \langle N_6 \rangle m(N_6)} \lesssim \frac{1}{N^2}.
\]

It only remains to consider

\[
\text{Re} \int_0^t \int (i I(|u_h|^4 u_h)) |I(|u|\,|u|) - |I u|^4(I u)| \, dx \, dt
\]
\[
+ \text{Re} \int_0^t \int i I(3 |u_h|^4 u_l + 2 |u_h|^2 u_h^2 u_l) |I(|u|^4 u) - |I u|^4(I u)| \, dx \, dt.
\] (3.17)

To evaluate

\[
\text{Re} \int_0^t \int (i I(|u_h|^4 u_h)) I(|u|^4 u) \, dx \, dt
\]
\[
+ \text{Re} \int_0^t \int i I(3 |u_h|^4 u_l + 2 |u_h|^2 u_h^2 u_l) I(|u|^4 u) \, dx \, dt,
\] (3.18)

it is necessary to take advantage of some cancellations.

\[
\text{Re} \int_0^t \int (i I(|u_h|^4 u_h)) I(|u_h|^4 u_h) \, dx \, dt \equiv 0.
\]

\[
\text{Re} \int_0^t \int i I(3 |u_h|^4 u_l + 2 |u_h|^2 u_h^2 u_l) I(|u_h|^4 u_h) \, dx \, dt
\]
+ \text{Re} \int_0^t \int i(I(|u_h|^4 u_h)(3|u_h|^4 u_l + 2|u_h|^2 u_h^2 u_l^2))dxdt \equiv 0.

It remains to evaluate

\begin{align*}
\int_0^t \int I(|u_h|^4 u_h)I(O(u_l^2 u^3))dxdt + \int_0^t \int I(O(u_h^4 u_l))I(O(u_l u^4))dxdt. \tag{3.19}
\end{align*}

By (3.12),

\begin{align*}
\|I(u_l^2 u^3)\|_{L^4_t L^\infty_x(J \times \mathbb{R})} \lesssim \|(\nabla) I u\|_{S^0(J \times \mathbb{R})}^{5},
\end{align*}

so

\begin{align*}
\int_0^t \int I(|u_h|^4 u_h)I(O(u_l^2 u^3))dxdt \lesssim \|I(u_l^2 u^3)\|_{L^4_t L^\infty_x(J \times \mathbb{R})} \|u_h\|_{L^4_t L^\infty_x(J \times \mathbb{R})}^2 \|u_l\|_{L^\infty_x(J \times \mathbb{R})}^2 \lesssim \frac{1}{N^5}. 
\end{align*}

\begin{align*}
\int_0^t \int I(O(u_h^4 u_l))I(O(u_l u^4))dxdt \lesssim \|I(u_h^4 u_l)\|_{L^2_t L^1_x(J \times \mathbb{R})} \|I(u_l u^4)\|_{L^2_t L^1_x(J \times \mathbb{R})}
\end{align*}

\begin{align*}
\lesssim \|\langle \nabla \rangle I u\|_{L^\infty_x L^2_t(J \times \mathbb{R})} \|u_l\|_{L^\infty_x L^6_t(J \times \mathbb{R})} \|u_h\|_{L^6_x L^\infty_t(J \times \mathbb{R})}\|u_l\|_{L^6_x L^\infty_t(J \times \mathbb{R})} \|u\|_{L^6_x L^\infty_t(J \times \mathbb{R})} \lesssim \frac{1}{N^3}.
\end{align*}

This takes care of (3.18). To finish estimating (3.17),

\begin{align*}
\text{Re} \int_0^t \int (iI(|u_h|^4 u_h))(I u^4)(I u)dxdt \\
+ \text{Re} \int_0^t \int iI(3|u_h|^4 u_l + 2|u_h|^2 u_h^2 u_l^2))(I u^4)(I u)dxdt, \tag{3.20}
\end{align*}

\begin{align*}
\lesssim \|u_h\|_{L^4_{t,x}(J \times \mathbb{R})}^4 \|u\|_{L^6_{t,x}(J \times \mathbb{R})} \|I u\|_{L^6_{t,x}(J \times \mathbb{R})} \|I u\|_{L^6_{t,x}(J \times \mathbb{R})} \|u\|_{L^6_{t,x}(J \times \mathbb{R})} \|u^4\|_{L^6_{t,x}(J \times \mathbb{R})} \lesssim \frac{1}{N^4},
\end{align*}

and the proof of theorem 3.1 is complete. □
4 Morawetz estimates

**Theorem 4.1** Let $u$ be the solution to the nonlinear Schrödinger equation in one dimension,

$$iu_t + \Delta u = |u|^4 u. \quad (4.1)$$

Then

$$\|Iu\|^{8}_{L^{8}_{t,x}([0,T] \times \mathbb{R})} \lesssim \|Iu\|^{7}_{L^{\infty}_{t}H^{\frac{1}{2}}_{x}([0,T] \times \mathbb{R})} + \sum_{J_k} \frac{1}{N^2} \|\langle \nabla \rangle Iu\|^{12}_{S^{0}_N(J_k \times \mathbb{R})}, \quad (4.2)$$

where $[0, T] = \cup_k J_k$.

**Proof:** We start with the case $I = 1$. We will use the method found in [13] and [6]. Let $\omega(t, z) : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}$,

$$\omega(t, z) = u(t, x_1)u(t, x_2)u(t, x_3)u(t, x_4), \quad (4.3)$$

where $u(t, x)$ is a solution to (1.1). Then $\omega(t, z)$ obeys the equation

$$i\omega_t + \Delta z \omega = (\sum_{i=1}^{4} |u(t, x_i)|^4)\omega(t, z) = N. \quad (4.4)$$

Next, define the interaction Morawetz quantity,

$$M_a(t) = 2 \int_{\mathbb{R}^4} \partial_j a(z) Im(\overline{\omega(t, z)} \partial_j \omega(t, z))dz, \quad (4.5)$$

following the convention that repeated indices are summed. Let

$$T_{0j}(t, z) = 2 Im(\overline{\omega(t, z)} \partial_j \omega(t, z)), \quad (4.6)$$

$$L_{jk}(t, z) = -\partial_{jk}(|\omega(t, z)|^2) + 4 Re(\overline{\partial_j \omega} \partial_k \omega)(t, z), \quad (4.7)$$

$$\partial_t T_{0j} + \partial_k L_{jk} = 2\{N, \omega\}_p. \quad (4.8)$$

$$\int_0^T \int_{\mathbb{R}^4} \partial_j \partial_j a(z) Im(\overline{\omega(t, z)} \partial_j \omega(t, z))dzdt \quad (4.9)$$

$$= \int_0^T \int_{\mathbb{R}^4} \partial_j a(z) \partial_{jk}(|\omega(t, z)|^2)dzdt \quad (4.10)$$

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\[ -4 \int_0^T \int_{\mathbb{R}^4} \partial_j a(z) \partial_k \text{Re}(\partial_j \omega \partial_k \omega)(t, z) dz dt \]  
(4.11)

\[ + 2 \int_0^T \int_{\mathbb{R}^4} a_j(z) \{ \mathbf{N}, \omega \}_p^j dz dt. \]  
(4.12)

Now, evaluate each term separately. Make a change of variables, \( y =Az \), where

\[
A = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1
\end{pmatrix}
\]  
(4.13)

is an orthonormal matrix with inverse

\[
A^{-1} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1
\end{pmatrix}
\]  
(4.14)

In the new variables, let \( a(y) = (y_2^2 + y_3^2 + y_4^2)^{1/2} \),

\[-\Delta \Delta a(y) = 4\pi \delta (y_2, y_3, y_4).\]

\[
A^{-1} \begin{pmatrix}
y_1 \\
0 \\
0 \\
0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
y_1 \\
y_1 \\
y_1 \\
y_1
\end{pmatrix}
\]  
(4.15)

Therefore, integrating (4.10) by parts,

\[
(4.10) = \int_0^T \int (\Delta \Delta a(y)) |\omega(t, z)|^2 dz dt = 8\pi \int_0^T \int_{\mathbb{R}} |u(t, x)|^2 dx dt. \tag{4.16}
\]

\( \partial_j a(z) \) is a positive semidefinite matrix, so integrating (4.11) by parts,

\[
4 \int_0^T \int_{\mathbb{R}^4} (\partial_j a(z)) \text{Re}(\partial_j \omega \partial_k \omega)(t, z) \geq 0. \tag{4.17}
\]

Finally, for (4.12),
\( \{N, \omega\}_p^j(t,z) = -2|\omega(t,z)|^2 \partial_j(\sum_{i=1}^{4} |u(t,x_i)|^4). \)  

(4.18)

\[ |\omega(t,z)|^2 \partial_j(\sum_{i=1}^{4} |u(t,x_i)|^4) = \frac{2}{3} \sum_{j=1}^{4} \partial_j(|u(t,x_j)|^6(|\prod_{i \neq j} u(t,x_i)|^2)), \]

so integrating (4.12) by parts,

\[
-2 \int_0^T \int_R a_j(z) \{N, \omega\}_p^j(t,z) dz dt = \frac{4}{3} \int_0^T \int_R a_{jj}(z)|\omega(t,z)|^2|u(t,x_j)|^4 dz dt \geq 0.
\]

(4.19)

Therefore,

\[
\int_0^T \int_R |u(t,x)|^8 dx dt \lesssim \int_0^T \int R |\partial_t(a_j(z)Im[\overline{\omega(t,z)}\partial_j \omega(t,z)]| dz dt \lesssim \|u\|_{L^\infty_t H^{1/2}}(0,T) \sum_{j=1}^6 |u_0|_{L^2(R)}^6.
\]

(4.20)

Next, we prove an almost Morawetz estimate (see [5], [11], [15] for the two dimensional case; [5], [12], [6] for discussion of the one dimensional case).

If \( u(t,x) \) solves (4.11), then \( Iu(t,x) \) solves

\[ iIu_t(t,x) + \Delta Iu(t,x) = I(|u(t,x)|^4u(t,x)) = N. \]  

(4.21)

Split the nonlinearity into "good" and "bad" pieces, \( N = N_g + N_b. \)

\[ N_g = \sum_{i=1}^{4} |Iu(t,x_i)|^4 Iu(t,x_i) \prod_{j \neq i} Iu(t,x_j), \]  

(4.22)

\[ N_b = \sum_{i=1}^{4} [I(|u(t,x_i)|^4u(t,x_i)) - |Iu(t,x_i)|^4(Iu(t,x_i))] \prod_{j \neq i} Iu(t,x_j). \]  

(4.23)

Let

\[ \omega(t,z) = Iu(t,x_1)Iu(t,x_2)Iu(t,x_3)Iu(t,x_4), \]

performing the same analysis will split

\[ \int_0^T \int_{R^4} \partial_t a_j(z) T_{ij}(t,z) dz dt, \]
into a sum of terms of the form (4.10), (4.11), and (4.12). If \( N = N_g \), then the previous analysis would carry over identically. Indeed,

\[
\int_0^T \int_{\mathbb{R}^4} (\Delta \Delta a(z))|\omega(t, z)|^2 \, dz \, dt = 8\pi \int_0^T \int |Iu(t, x)|^8 \, dx \, dt. \tag{4.24}
\]

\[
4 \int_0^T \int_{\mathbb{R}^4} (\partial_{jk} a(z))Re(\overline{\partial_j \omega} \partial_k \omega)(t, z) \, dz \, dt \geq 0. \tag{4.25}
\]

\[
2 \int_0^T \int a_j(z)\{N_g, \omega\}_p(t, z) \, dz \, dt = \frac{4}{3} \int_0^T \int a_{jj}(z)|\omega(t, z)|^2 |Iu(t, x_j)|^4 \, dz \geq 0. \tag{4.26}
\]

Therefore,

\[
\int_0^T \int |u(t, x)|^8 \, dx \, dt \lesssim \|Iu\|_{L^\infty_t H^\frac{1}{2}_x([0, T] \times \mathbb{R})} \|u_0\|_{L^2_x(\mathbb{R})}^2 + \|a_j(z)\{N_b, \omega\}_p(t, z) \, dz \, dt|. \tag{4.27}
\]

To analyze the remainder \( N_b \), first consider a term of the form

\[
\int J_k \int a_j(z)\overline{N_b(t, z)} \partial_j \omega(t, z) \, dz \, dt. \tag{4.28}
\]

Recall (4.23), without loss of generality let \( i = 1 \) and estimate

\[
\int J_k \int a_j(z)[I(|u(t, x_1)|^4 u(t, x_1)) - |Iu(t, x_1)|^4 |Iu(t, x_2)Iu(t, x_3)Iu(t, x_4)|] \\
\times \partial_j(Iu(t, x_1)Iu(t, x_2)Iu(t, x_3)Iu(t, x_4)) \, dx \, dt. \tag{4.29}
\]

Because \( a_j(z) \in L^\infty \),

\[
4.28 \lesssim \|I(|u(t, x)|^4 u(t, x)) - |Iu(t, x)|^4 |Iu(t, x)|\|_{L^\infty_t L^2_x(\mathbb{R}) \times J_k} \|(\nabla) Iu\|_{L^\infty_t L^2_x(\mathbb{R})}^7.
\]

To evaluate

\[
I(|u(t, x)|^4 u(t, x)) - |Iu(t, x)|^4 |Iu(t, x)|,
\]

make a Littlewood - Paley partition of unity. Let
\[
F(t, \xi) = \int_{\xi = \xi_1 + \ldots + \xi_5} \left[ 1 - \frac{m(\xi_1 + \ldots + \xi_5)}{m(\xi_1) \ldots m(\xi_5)} \right] \hat{u}(t, \xi_1) \hat{u}(t, \xi_2) \hat{u}(t, \xi_3) \hat{u}(t, \xi_4) \hat{u}(t, \xi_5).
\]

(4.31)

Without loss of generality, let \(N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5\). Consider several cases separately.

**Case 1:** \(N_1 \ll N\) In this case,

\[
|1 - \frac{m(\xi_1 + \ldots + \xi_5)}{m(\xi_1) \ldots m(\xi_5)}| = 0.
\]

**Case 2:** \(N_1 \gtrsim N >> N_2\) In this case, by the fundamental theorem of calculus,

\[
|1 - \frac{m(\xi_1 + \ldots + \xi_5)}{m(\xi_1) \ldots m(\xi_5)}| \lesssim \frac{N_2}{N_1}.
\]

(4.28)

\[
\leq \sum_{N_1} \sum_{N_2} \sum_{N_3} \sum_{N_4} \sum_{N_5} \frac{1}{N_1^2} \|P_{N_1} (\nabla) Iu\|_{L_t^\infty L_x^2(J_k \times R)} \|P_{N_2} (\nabla) Iu\|_{L_t^4 L_x^\infty(J_k \times R)}
\]

\[
\times \|P_{N_3} (\nabla) Iu\|_{L_t^4 L_x^\infty(J_k \times R)} \|P_{N_4} (\nabla) Iu\|_{L_t^4 L_x^\infty(J_k \times R)} \|P_{N_5} (\nabla) Iu\|_{L_t^4 L_x^\infty(J_k \times R)} 
\]

\[
\lesssim \frac{1}{N^{2-s}} \|\langle \nabla \rangle Iu\|_{S^0(J \times R)}^{50}.
\]

**Case 3:** \(N_1 \geq N_2 \gtrsim N\) In this case, crudely estimate

\[
|1 - \frac{m(\xi_1 + \ldots + \xi_5)}{m(\xi_1) \ldots m(\xi_5)}| \lesssim \frac{1}{m(N_1)m(N_2)m(N_3)m(N_4)m(N_5)}.
\]

(4.28)

\[
\lesssim \sum_{N_1 \leq N_2 \leq N_1} \|P_{N_1} u\|_{L_t^\infty L_x^2(J_k \times R)} \|P_{N_2} u\|_{L_t^4 L_x^\infty(J_k \times R)}
\]

\[
\times \sum_{N_3 \leq N_4 \leq N_3 \leq N_2} \|P_{N_3} u\|_{L_t^4 L_x^\infty(J_k \times R)} \|P_{N_4} u\|_{L_t^4 L_x^\infty(J_k \times R)} \|P_{N_5} u\|_{L_t^4 L_x^\infty(J_k \times R)}
\]

\[
\lesssim \sum_{N_1 \leq N_2 \leq N_1} \frac{1}{N_1^{1-s} N_2^{1-s} N_2^{1-s}}.
\]
\[
\sum_{N_5 \leq N_4 \leq N_3} \frac{1}{m(N_3)N_3} \frac{1}{m(N_4)N_4} \frac{1}{m(N_5)N_5} \| \langle \nabla \rangle Iu \|_{S^0(J_h \times \mathbb{R})}^5 \lesssim \frac{1}{N^2} \| \langle \nabla \rangle Iu \|_{S^0(J_h \times \mathbb{R})}^5.
\]

For a term of the form
\[
\int_{J_h} \int_{\mathbb{R}^4} a_j(z) (\partial_j N_6(t, z)) \omega(t, z) dz dt,
\]
integrating by parts rewrites this term as a term of the form \(4.28\) plus a term of the form
\[
\int_{J_h} \int_{\mathbb{R}^4} a_{jj}(z) N_6(t, z) \omega(t, z) dz dt.
\]
By \(4.23\) and the estimates on \(4.31\),
\[
\| N_6(t, z) \|_{L^1_t L^2_x(J_h \times \mathbb{R}^4)} \lesssim \frac{1}{N^2} \| \langle \nabla \rangle Iu \|_{S^0(J_h \times \mathbb{R})}^8.
\]

\(\Delta a\) does not lie in \(L^\infty(\mathbb{R}^4)\), rather,
\[
a_{jj}(y) \lesssim \frac{1}{(y_2^2 + y_3^2 + y_4^2)^{1/2}}.
\]
Integrating
\[
\int_{-\infty}^{\infty} \int_{y_2^2 + y_3^2 + y_4^2 \leq 1} \frac{1}{y_2^2 + y_3^2 + y_4^2} |Iu(t, x_1) Iu(t, x_2) Iu(t, x_3) Iu(t, x_4)|^2 dy_1 dy_2 dy_3 dy_4
\]
along the tube \(y_2^2 + y_3^2 + y_4^2 \leq 1\), using \(4.15\),
\[
\lesssim \left( \int_0^{1} \frac{r^2}{r^2} dr \right) \| Iu \|_{L^\infty_t L^5_x(J_h \times \mathbb{R})}^8 \lesssim \| \langle \nabla \rangle Iu \|_{S^0(J_h \times \mathbb{R})}^8,
\]

so
\[
\| a_{jj}(y) \omega(t, y) \|_{L^\infty_t L^5_x(J_h \times \mathbb{R}^4)} \lesssim \| \langle \nabla \rangle Iu \|_{S^0(J_h \times \mathbb{R})}^4.
\]
On the other hand, when $y_2^2 + y_3^2 + y_4^2 \geq 1$, $\partial_{jj} a(z)$ is bounded and

$$\|Iu(t, x_1)Iu(t, x_2)Iu(t, x_3)Iu(t, x_4)\|_{L^1_tL^2_x(\mathbb{R}^4)} \lesssim \|\langle \nabla \rangle Iu\|_{S_0(\mathbb{R}^4)}^{1/2}.$$  \hspace{1cm} (4.36)

Since

$$\|I(|u(t, x_1)|^4u(t, x_1)) - |Iu(t, x_1)|^4Iu(t, x_1)\|_{L^1_tL^2_x(\mathbb{R}^4)} \lesssim \frac{1}{N^2} \|\langle \nabla \rangle Iu\|_{S_0(\mathbb{R}^4)}^{3/2},$$

theorem 4.1 is proved.

\section{Proof of Theorem 1.1}

\textbf{Theorem 5.1} \hspace{0.1cm} $Iu_0(t, x) \in H^s(\mathbb{R})$, $s > \frac{8}{27}$.

\textbf{Proof:}

$$\int |\nabla Iu_0(x)|^2 dx \lesssim N^{2(1-s)}\|u_0\|_{H^s(\mathbb{R})}. \hspace{1cm} (5.1)$$

$$\int |Iu_0(x)|^6 dx \lesssim N^{2-6s}\|u_0\|_{H^s(\mathbb{R})}^6.$$  

If $u(t, x)$ solves (1.1) on $[0, T_0]$, then rescaling,

$$\frac{1}{\lambda^{1/2}}u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

solves (1.1) on $[0, \lambda^2T_0]$, we will call the rescaled solution $u_\lambda(t, x)$. Choose $\lambda \sim N^{(1-s)/s}$ so that $E(Iu_\lambda(0)) = 1/2$. Let

$$W = \{t : E(Iu_\lambda(t)) \leq \frac{9}{10}\}. \hspace{1cm} (5.3)$$

$W$ is closed by the dominated convergence theorem and nonempty since $0 \in W$. To prove $W = [0, \lambda^2T_0]$, it suffices to prove $W$ is open in $[0, \lambda^2T_0]$. Suppose $W = [0, T]$, then by continuity there exists $\delta > 0$ such that $E(Iu_\lambda(t)) \leq 1$ on $[0, T + \delta]$.

\textbf{Lemma 5.2} When $s > 1/4$, 

$$\|Iu_\lambda\|_{L^8_t([0, T + \delta] \times \mathbb{R})} \leq \frac{3}{2}C_0.$$  \hspace{1cm} (5.4)

for some $C_0(T_0)$. 

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Proof: Let $\tau = \sup \{ T_\epsilon \in [0, T + \delta] : \| Iu_\lambda \|_{L_{t,x}^8([0,T_\epsilon] \times \mathbb{R})} \leq \frac{3}{2} C_0 \}$. If $\tau < T + \delta$, there exists $\delta' > 0$ such that

$$\| Iu_\lambda \|_{L_{t,x}^8([0,\tau+\delta'] \times \mathbb{R})} \leq 2 C_0. \quad (5.5)$$

Recall $E(Iu_\lambda(t)) \leq 1$ on $[0, \tau + \delta']$,

$$\| Iu_\lambda \|_{L_t^{16/3}L_x^8([0,\tau+\delta'] \times \mathbb{R})} \leq (\lambda^2 T_0)^{1/16} \| Iu_\lambda \|_{L_{t,x}^8([0,\tau+\delta'] \times \mathbb{R})} \leq (\lambda^2 T_0)^{1/16} (2C_0)^{1/8}. \quad (5.6)$$

Partition $[0, \tau + \delta']$ into

$$\frac{(\lambda^2 T_0)^{1/3} (2C_0)^{2/3}}{\epsilon^{16/3}}$$

subintervals such that

$$\| Iu_\lambda \|_{L_t^{16/3}L_x^8([0,\tau+\delta'] \times \mathbb{R})} \leq \epsilon \quad (5.7)$$
on each subinterval. Then apply the almost Morawetz estimate,

$$\| Iu_\lambda \|_{L_t^8L_x^8([0,\tau+\delta'] \times \mathbb{R})} \leq C \| u_0 \|_{L_t^2L_x^6(\mathbb{R})} \| Iu_\lambda \|_{L_t^\infty H^1([0,\tau+\delta'] \times \mathbb{R})} + \sum_k \| \langle \nabla \rangle Iu \|_{H^0(J_k \times \mathbb{R})} \leq C_0 + C \frac{(\lambda^2 T_0)^{1/3} (2C_0)^{2/3}}{\epsilon^{16/3} N^{2-}} \leq C_0 + C \frac{N^{2(1-s)}}{\epsilon^{16/3} N^{2-}} \leq \frac{3}{2} C_0, \quad (5.8)$$

when $N$ is sufficiently large, as long as $\frac{2}{3} \left( \frac{1-s}{s} \right) < 2$, or $s > 1/4$. This proves the lemma. $\square$

Returning to the theorem,

$$\| Iu_\lambda \|_{L_{t,x}^8([0,T+\delta] \times \mathbb{R})} \leq \frac{3}{2} C_0, \quad (5.9)$$

$$\| Iu_\lambda \|_{L_t^{16/3}L_x^8([0,T+\delta] \times \mathbb{R})} \lesssim \lambda^{1/8} T_0^{1/16}. \quad (5.9)$$

Partition $[0, T + \delta]$ into $\lesssim (\lambda^2 T_0)^{1/3}$ subintervals. We will call these little intervals. Take the union of the first $N^{1/2-}$ little subintervals, and call this big interval

$$J_1 = \bigcup_{l=1}^{N^{1/2-}} J_{1,l}. \quad (5.9)$$

Take the union of the next $N^{1/2-}$ subintervals and call this big interval $J_2$, and so on.
\[ [0, T + \delta] = \bigcup_{k=1}^{N^{1/2}} J_k = \bigcup_{k=1}^{N^{1/2}} \bigcup_{l=1}^{N^{1/2}} J_{k,l}. \quad (5.10) \]

**Lemma 5.3** Suppose \( E(Iu(t)) \leq 1 \) on \( J_k \).

\[
\sup_{t_1, t_2 \in J_k} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{1}{N^{5/4}}. \quad (5.11)
\]

**Proof:** By theorem (3.1),

\[
\sup_{t_1, t_2 \in J_{k,l}} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{1}{N^{3/2}} \|P_{cN^2} \nabla Iu\|^2_{L^4_t L^\infty_x(J_{k,l} \times \mathbb{R})} + \frac{1}{N^{2-}}. \quad (5.12)
\]

\[
\sum_{l=1}^{N^{1/2}} \|P_{cN^2} \nabla Iu\|^2_{L^4_t L^\infty_x(J_{k,l} \times \mathbb{R})} \lesssim N^{1/4} \|P_{cN^2} \nabla Iu\|^2_{L^4_t L^\infty_x(J_k \times \mathbb{R})}.
\]

Let \( J_k = [a, b] \), \( J_{k,m} = [a_m, b_m] \), \( a_{m+1} = b_m \), \( a_1 = a \), \( b_{N^{1/2}-} = b \). On each little subinterval, perform the linear-nonlinear decomposition in theorem 2.3.

The solution on \([a_m, b_m] \) is of the form

\[
e^{i(t-a_m)\Delta} u(a_m) + u_{nl}^m(t).
\]

By induction,

\[
e^{i(t-a_m)\Delta} u(a_m) = e^{it\Delta} u_0 + \sum_{j=1}^{m-1} e^{i(t-a_j)\Delta} u_{nl}^j(b_j). \quad (5.13)
\]

\[
\|\nabla e^{i(t-a)\Delta} Iu(a)\|_{L^4_t L^\infty_x(J_k \times \mathbb{R})} \lesssim 1. \quad (5.14)
\]

\[
\sum_{m=1}^{N^{1/2-}} \|\nabla e^{i(t-a)\Delta} Iu(b_m)\|_{L^4_t L^\infty_x(J_k \times \mathbb{R})} \lesssim 1. \quad (5.15)
\]

\[
\sum_{l=1}^{N^{1/2-}} \|\nabla Iu_{nl}(t)\|_{L^4_t L^\infty_x(J_{k,l} \times \mathbb{R})} \lesssim \frac{1}{N^{3/2}-}. \quad (5.16)
\]
Therefore, 
\[ \| \nabla P_{>cN} Iu \|_{L^4_t L^\infty_x(J_k \times \mathbb{R})} \lesssim 1. \]
Plugging this back into (5.11),
\[ \sum_{l=1}^{N^{1/2}} \sup_{t_1, t_2 \in J_{k,l}} | E(Iu(t_1)) - E(Iu(t_2)) | \lesssim \frac{1}{N^{5/4}} + \frac{1}{N^{3/2}}, \]
which proves the lemma. □

Returning to the theorem once again, if \((\lambda^2 T_0)^{1/3} < N^{7/4}\), for \(t \in [0, T + \delta]\), or if \(s > \frac{8}{29}\),
\[
E(Iu(t)) \leq \frac{1}{2} + C \left( \frac{\lambda^2 T_0}{N^{7/4}} \right)^{1/3} = \frac{1}{2} + C N^{2(1-s)/3} T_0^{1/3} \leq \frac{9}{10},
\] (5.17)
for sufficiently large \(N\), proving the theorem. We have to take \(N \sim T_0^{4s/29s-8}\).
Since \(\lambda \sim N^{1-s}\), and
\[
\| u(t) \|_{H^s(\mathbb{R})} = \lambda^s \| u(\frac{t}{\lambda^2}) \|_{H^s(\mathbb{R})},
\]
\[
\sup_{t \in [0, T]} \| u(t) \|_{H^s(\mathbb{R})} \lesssim (1 + T) \left( \frac{4(1-s)\delta}{29s-8} \right)^{1/2}.
\] (5.18)
□
References

[1] J. Bourgain. Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity. *International Mathematical Research Notices*, 5:253 – 283, 1998.

[2] J. Bourgain. *Global Solutions of Nonlinear Schrödinger Equations*. American Mathematical Society Colloquium Publications, 1999.

[3] T. Cazenave and F. Weissler. The Cauchy problem for the nonlinear Schrödinger Equation in $H^1$. *Manuscripta Mathematica*, 61:477 – 494, 1988.

[4] T. Cazenave and F. Weissler. The Cauchy problem for the nonlinear Schrödinger Equation in $H^s$. *Nonlinear Analysis*, 14:807 – 836, 1990.

[5] J. Colliander, M. Grillakis, and N. Tzirakis. Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. *Int. Math. Res. Not. IMRN*, (23):90 – 119, 2007.

[6] J. Colliander, J. Holmer, M. Visan, and X. Zhang. Global existence and scattering for rough solutions to generalized nonlinear Schrödinger equations on $\mathbb{R}$. *Commun. Pure Appl. Anal.*, 7(3):467–489, 2008.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation. *Mathematical Research Letters*, 9:659 – 682, 2002.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$. *Communications on Pure and Applied Mathematics*, 21:987 – 1014, 2004.

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Resonant decompositions and the I-method for cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. *Discrete and Continuous Dynamical Systems A*, 21:665 – 686, 2007.

[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Ann. of Math. (2)*, 167(3):767–865, 2008.
[11] J. Colliander and T. Roy. Bootstrapped Morawetz estimates and resonant decomposition for low regularity global solutions of cubic NLS on $\mathbb{R}^2$. preprint, arXiv:0811.1803.

[12] D. De Silva, N. Pavlović, G. Staffilani, and N. Tzirakis. Global well-posedness for the $L^2$-critical nonlinear Schrödinger equation in higher dimensions. to appear, Communications on Pure and Applied Analysis.

[13] D. De Silva, N. Pavlović, G. Staffilani, and N. Tzirakis. Global well-posedness and polynomial bounds for the defocusing $L^2$-critical nonlinear Schrödinger equation in $\mathbb{R}$. Comm. Partial Differential Equations, 33(7-9):1395–1429, 2008.

[14] B. Dodson. Global well-posedness for the defocusing, cubic, nonlinear Schrödinger equation when $n = 3$ via a linear-nonlinear decomposition. arXiv:0910.2260.

[15] B. Dodson. Improved almost Morawetz estimates for the cubic nonlinear Schrödinger equation. arXiv:0909.0757.

[16] M. Keel and T. Tao. Endpoint Strichartz estimates. American Journal of Mathematics, 120:955 – 980, 1998.

[17] R. Killip, T. Tao, and M. Visan. The cubic nonlinear Schrödinger equation in two dimensions with radial data. Journal of the European Mathematical Society, to appear.

[18] R. Killip, M. Visan, and X. Zhang. The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher. Anal. PDE, 1(2):229–266, 2008.

[19] T. Roy. Adapted linear-nonlinear decomposition and global well-posedness for solutions to the defocusing cubic wave equation on $\mathbb{R}^3$. Discrete Contin. Dyn. Syst., 24(4):1307–1323, 2009.

[20] C. Sogge. Fourier Integrals in Classical Analysis. Cambridge University Press, 1993.

[21] E. Stein. Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, 1993.

[22] T. Tao. A sharp bilinear restrictions estimate for paraboloids. Geom. Funct. Anal., 13(6):1359–1384, 2003.
[23] T. Tao. *Nonlinear Dispersive Equations: Local and Global Analysis*. American Mathematical Society, 2006.

[24] T. Tao, M. Visan, and X. Zhang. The nonlinear Schrödinger equation with combined power-type nonlinearities. *Comm. Partial Differential Equations*, 32(7-9):1281–1343, 2007.

[25] M. Taylor. *Pseudodifferential Operators and Nonlinear PDE*. Birkhauser, 1991.

[26] M. Taylor. *Partial Differential Equations*. Springer Verlag Inc., 1996.

[27] Y. Tsutsumi. $L^2$ solutions for nonlinear Schrödinger equation and nonlinear groups. *Funkcional Ekvacioj*, 30:115 – 125, 1987.

[28] M. Visan. The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. *Duke Mathematical Journal*, 138:281 – 374, 2007.