On computing special functions in marine engineering

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Abstract. Important modeling applications in marine engineering conduct us to a special class of solutions for difficult differential equations with variable coefficients. In order to be able to solve and implement such models (in wave theory, in acoustics, in hydrodynamics, in electromagnetic waves, but also in many other engineering fields), it is necessary to compute so called special functions: Bessel functions, modified Bessel functions, spherical Bessel functions, Hankel functions. The aim of this paper is to develop numerical solutions in Matlab for the above mentioned special functions. Taking into account the main properties for Bessel and modified Bessel functions, we shortly present analytically solutions (where possible) in the form of series. Especially it is studied the behavior of these special functions using Matlab facilities: numerical solutions and plotting. Finally, it will be compared the behavior of the special functions and point out other directions for investigating properties of Bessel and spherical Bessel functions. The asymptotic forms of Bessel functions and modified Bessel functions allow determination of important properties of these functions. The modified Bessel functions tend to look more like decaying and growing exponentials.

1. Introduction
Solutions of many engineering and mathematical physics problems are connected with so called special functions. One of these functions, Bessel and modified Bessel functions, are especially important for a lot of models from acoustics, wave propagation, heat conduction theory, aeronautics, hydrodynamics, mechanics of solids, elasticity, atomic and nuclear physics. Some complicated models in marine engineering related with wave propagation theory, vibration, hydrodynamics, heat conduction or elasticity could be solved using Bessel functions.

Differential equations with variable coefficients cannot be solved using the same techniques like in differential equations with constant coefficients (by algebraic methods obtaining solutions as elementary functions). In such cases, it is searched for solutions in form of infinite series (method of Frobenius, method of Picard). The Bessel’s equations are of this type. In engineering, the Bessel’s equations are always to be expected when partial differential equations are used in the study of configurations possessing cylindrical symmetry. We also encountered Bessel functions when we solve Laplace equation in cylindrical coordinates.

In mathematics, Bessel functions, denoted as $J_n(x)$, are canonical solutions Bessel’s differential equation [1]:
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left( x^2 - n^2 \right) y = 0 \]  

(1)

where \( n \) is called the order of the Bessel function and represents a real or complex number. The most used case is for integer \( n \). For example, the solutions of Laplace’s equation in cylindrical coordinates are Bessel functions of integer order, often called cylinder functions. Other very important case is for \( n \) half-integer for spherical Bessel functions as solutions of Helmholtz equation transformed in spherical coordinates. It’s important to observe that simply replace \( x \) with \( \lambda x \) will obtain a more general parameter form of Bessel’s equations [1]:

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left( \lambda^2 x^2 - n^2 \right) y = 0 \]  

(2)

In practical situations, when modeling real systems, it’s difficult to recognize the form of the equation and then develop Bessel functions as solutions.

2. Bessel functions

Using the method of Frobenius, it could be searched Bessel’s differential equation solution in the form

\[ y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \]; after substitution in equation (1), by identification, the indicial equation becomes

\[(r+n)(r-n) = 0, \text{ so } r_1 = n, \ r_2 = -n. \]

In the first case, it can be obtained the solution [2]

\[ J_n(x) = \sum_{k=0}^{\infty} (-1)^k \left( \frac{x}{2} \right)^{n+2k} \frac{1}{k! \Gamma(n+k+1)} \quad n \geq 0, \]

(3)
called Bessel function of the first kind, order \( n \) (this series converges for all values of \( x \)). Let’s mention that \( \Gamma(n) \) is the Gamma function (the generalized factorial function).

Taking into account Gamma function properties, if \( n \) are positive integer or zero, then \( \Gamma(n+k+1) = (n+k)! \) and the formula for Bessel function of order \( n \) becomes [3]:

\[ J_n(x) = \sum_{k=0}^{\infty} (-1)^k \left( \frac{x}{2} \right)^{n+2k} \frac{1}{k!(n+k)!} \quad n = 0, 1, 2, \ldots \]

(4)

The behavior of Bessel functions could be seen in the figure 1 plotted using Matlab:

![Figure 1. Bessel functions \( J_n(x) \), \( n = 1,2,3 \).](image-url)
It can be observed that the functions are bounded at 0. These graphs look like the graphs of the functions sine or cosine that decompose proportionally to \( \frac{1}{\sqrt{x}} \), which could be explained if it is taken into account the asymptotic forms for positive \( n \) of Bessel functions \([5]\). \( J_0(x) \), \( J_1(x) \) and \( J_2(x) \) have no complex roots, but each of them has an infinite number of distinct roots which separate each other.

In the second case, by replacing \( n \) with \( -n \), can be found the relation

\[
J_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \cdot \left( \frac{x}{2} \right)^{-n+2k} \cdot \frac{1}{k! \Gamma(-n+k+1)}.
\]

The last function is called Bessel function of negative order \( n \) and let observe that also this series converges for all values of \( x \).

The most used series are for \( n = 0, 1, 2 \). According to \([2]\), it can be mentioned interesting properties of these functions:

\[
\frac{d}{dx} J_0(x) = -J_1(x), \quad \frac{d}{dx} xJ_1(x) = xJ_0(x)
\]

(5)

For \( n \) not an integer, Bessel functions \( J_n \) and \( J_{-n} \) are linearly independent; therefore, \( y(x) = c_0 J_n(x) + c_1 J_{-n}(x) \) will represent the general solution of Bessel’s differential equation.

Bessel functions \( J_n \) and \( J_{-n} \) are important for engineering applications for \( n = \frac{1}{2} \) and \( n = -\frac{1}{2} \). It is not difficult to obtain \( J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \) and \( J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x \). Using the plots (figure 2) of these two functions, it is easy to observe that one is bounded at 0 and the other not. These means that the functions \( J_{\frac{1}{2}}(x) \) and \( J_{-\frac{1}{2}}(x) \) are linear independent.

Figure 2. Bessel functions \( J_n(x) \), \( n = -1/2, 1/2 \).

In the case \( n \in \mathbb{Z} \), using the infinite series representations, it can be obtained the relation: \( J_{-n}(x) = (-1)^n J_n(x) \) \([3]\). It is necessary to mention that Gamma function exist only for positive arguments. Because the functions \( J_n \) and \( J_{-n} \) are not linearly independent, it’s necessary to find another second linearly independent solution for Bessel’s differential equation.

The following recurrence formulas can be proven:

\[
xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x), \quad xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x), \quad 2xJ_n(x) = J_{n+1}(x) + J_{n-1}(x).
\]
The function [1]: 

\[ Y_n(x) = \frac{\cos(n\pi)J_n(x) - J_{-n}(x)}{\sin n\pi}, \]  

(6)

it is known as Bessel functions of the second kind. It’s easy to observe that in origin the functions are infinite. Obviously, \( J_n(x) \) and \( Y_n(x) \) are linearly independent if \( n \) is not an integer. The general solution of Bessel’s differential equation is written as: \( y(x) = AJ_n(x) + BY_n(x) \), where usually the constants are determined from boundary conditions of the equation.

**Figure 3.** Bessel functions \( Y_n(x) \), \( n = 0,1,2,3 \).

It is easy to verify that the Bessel function of second kind satisfy the relation: 

\[ Y_{-m}(x) = (-1)^m Y_m(x). \]

The examination of the surface \( z = J_n(x) \) show the behavior of these function when \( x \) and \( n \) varies continuously. (figure 4, for \( 0 \leq n \leq 10 \), \( 0 \leq x \leq 10 \)).

**Figure 4.** Surface of \( z = J_n(x) \), \( 0 \leq n \leq 10 \), \( 0 \leq x \leq 10 \).

For \( n = 0 \), the function \( J_0(x) \) equals 1 when \( x \) equals 0.

If \( n > 0 \), \( J_n'(0) = 0 \); the tangent line at origin is horizontally.

If \( n < 1 \), \( J_n'(0) = \infty \); the tangent line at the origin is vertically.

For \( n = 1 \), \( J_1'(0) = \frac{1}{2} \); tangent line is oblique.
3. Spherical Bessel functions

Let consider the equation \( x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + \left[ k^2 x^2 - l(l+1) \right] y = 0 \). The general solution of this equation is

\[
y = A \sqrt{-x} J_{l+\frac{1}{2}}(kx) + B \sqrt{-x} Y_{l+\frac{1}{2}}(kx),
\]

where \( J_{l+\frac{1}{2}}(x) \) and \( Y_{l+\frac{1}{2}}(x) \) are the spherical Bessel functions. The spherical Bessel function of most interest are those of integer and half integer order. Important engineering applications of these functions are diffusion problems (especially Helmholtz equation) and asymptotic analysis. According to [3], these functions can be written in terms of elementary functions, for \( n \) a non-negative integer:

\[
j_n(x) = (-1)^n x^n \left( \frac{1}{x} \frac{d}{dx} \right)^n \sin x,
\]

\[
y_n(x) = (-1)^n x^n \left( \frac{1}{x} \frac{d}{dx} \right)^n \cos x,
\]

\[
h^{(1)}_n(x) = -i(-1)^n x^n \left( \frac{1}{x} \frac{d}{dx} \right)^n e^{ix},
\]

\[
h^{(2)}_n(x) = i(-1)^n x^n \left( \frac{1}{x} \frac{d}{dx} \right)^n e^{-ix}.
\]

We shall plot the graph of spherical Bessel functions for \( n \in \{1, 2, 3\} \).

4. Application

Bessel functions are connected with other special functions such as: Gamma, Hypergeometric, Struve function. Bessel-real and Bessel-imaginary have applications in physics, such as the resistance of conductors of alternating current.

Bessel functions are important in problems of wave propagation, static potentials, studying electromagnetic waves in a cylindrical waveguide, heat conduction in a cylindrical object.

Solving problems, hardly find Bessel equation in canonical form (1). When it is possible, it is useful to know the reduction of the equation to the form (1).

To simplify writing the solution of presented examples, it could be used \( AJ_n(z) + BY_n(x) = Z_n(z) \)
### Table 1. Examples of equations which can be reduced to Bessel equation.

| Name | Equation | Solution |
|------|----------|----------|
| 01 | \( \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(k^2 - \frac{n^2}{x^2}\right)y = 0 \) | \( Z_n(kx) \) |
| Oscillations of a heavy wire suspended from one end | \( \frac{d^2 \phi}{dx^2} + \frac{1}{x} \frac{d\phi}{dx} + \frac{\omega^2 \phi}{g} = 0 \) | \( Z_n\left(2\omega \sqrt{\frac{x}{g}}\right) \) |
| Perfect circular motion of a membrane stretched | \( \frac{d^2 z}{dp^2} + \frac{1}{\rho} \frac{dz}{dp} + \frac{1}{\rho^2} \frac{\partial^2 z}{\partial p^2} = k^2 \frac{\partial^2 z}{\partial t^2} \) | \( e^{i\omega t} \cos nqJ_n(k\omega \rho) \) |
| Electromagnetic oscillation of a cavity in the form of rotating drum wave modulated frequency spectrum | \( \frac{d^2 U}{dz^2} + \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \rho^2} + k^2 U = 0 \) | \( U = J_n(\alpha) \cos n\phi \sin qz e^{-\beta r} \) |
| \( u(t) = U \sin \left(\Omega_0 t + \beta \int_0^t \sigma(t) dt\right) \) | \( u(t) = \sum_{n=-\infty}^{\infty} J_n(\xi) \sin(\Omega_0 + n \alpha) t \) |

#### 5. Conclusions

It have been briefly analyzed, analytical and numerical, the most known Bessel functions and also other modified Bessel functions. Using Matlab facilities seems to be much easier to investigate the functional behavior of the Bessel functions. Since the low order integer Bessel functions are so much used in various applications, it was mainly numerically treated the Bessel functions, spherical Bessel functions for order \( n = 0, 1, 2, 3 \). Obviously, the ordinary Bessel functions are oscillating as functions of a real argument. On the other hand, the modified Bessel functions tend to look more like decaying and growing exponentials.

It’s also interesting to compare the behavior of low order integer Bessel functions and modified Bessel functions for \( x \to 0 \) and for \( x \to \infty \). These results are used in evaluating boundary conditions for boundary value problems. Using Matlab, the asymptotic forms of Bessel functions and modified Bessel functions allow determination of important properties of these functions.

Taking into account the large number of engineering problems and also mathematical physics problems connected with Bessel functions, it is sure that using Matlab (as numerical computing environment and programming language) could improve not only computing Bessel functions and modified Bessel functions, but also studying properties of these special functions.

#### References

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