APPROXIMATE GRAVITATIONAL FIELD
OF A ROTATING DEFORMED MASS

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Abstract

A new approximate solution of vacuum and stationary Einstein field
equations is obtained. This solution is constructed by means of a power
series expansion of the Ernst potential in terms of two independent and
dimensionless parameters representing the quadrupole and the angular
momentum respectively. The main feature of the solution is a suitable
description of small deviations from spherical symmetry through pertur-
bations of the static configuration and the massive multipole structure
by using those parameters. This quality of the solution might eventually
provide relevant differences with respect to the description provided by
the Kerr solution.

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1 Introduction

Within the context of axisymmetric Einstein vacuum field equations, the good qualities of the Kerr solution \([1]\) are well known and indeed this solution is the essential reference in the stationary domain, as is the Schwarzschild solution in the static one. We are dealing with an exact solution which is very simple and has a very interesting multipole moment structure in order to describe the gravitational field of compact bodies. The Kerr solution is written in terms of only one parameter, namely \(a\), which represents the angular momentum \((J)\) per unit mass. \(M\) being the mass - i.e. the order zero of the multipole moments-, all the other moments are proportional to a power of this parameter \(a\) equal to each multipole order, \([2]\), \([4]\).

\[
M_k = M (ia)^k
\]  

It is worthwhile mentioning that all multipole moments of the Kerr solution equal the corresponding coefficient appearing in the expansion of its Ernst potential on the symmetry axis. This characteristic makes it easier to calculate the multipole moments of the solution by using the FHP algorithm \([4]\).

All these reasons, among others, have led to this solution becoming the one most used for a broad range of purposes concerning the physical behaviour of test particles or properties of the gravitational fields of celestial sources.

By means of the multipole moments of any solution one can obtain relevant information about the physical properties of the gravitational field, and, for instance, the multipole structure \([1]\) allows one to estimate the relative importance of the successive multipole moments in terms of the parameter \(a\), mainly for the sub-extreme case \((J \ll M)\).

Nevertheless, since the Kerr solution is defined with only one arbitrary parameter, it is not possible to distinguish the different contributions from the quadrupole and angular momentum terms, respectively, to any kind of physical event described by this space-time. It would be very useful to question how different a solution is with respect to the Schwarzschild solution in order to know which different contributions (or the new physics involved) are introduced by a new solution when this has no spherical symmetry and this space-time is no longer static. To make such a description, a solution with more than one parameter, themselves themselves related to different multipole moments, allows one to handle them independently.

In this sense, some authors \([6]\), \([7]\) have supplied static and stationary solutions of the Einstein vacuum field equations, with prescribed multipole moments, that attempt to describe slight deviations from the space-time with spherical symmetry. The Monopole-Quadrupole solution \(MQ\) \([6]\) is an exact axisymmetric and static solution constructed in power series of a parameter \(q\) (dimensionless quadrupole moment), whereas the \(MJ\) solution \([7]\) is a stationary approximate solution describing the space-time of a mass with angular momentum and is obtained by an expansion on the symmetry axis of the Ernst potential in power series of a dimensionless parameter \(\mathcal{J}\). This solution represents successive cor-
rections to the spherical symmetry due to the rotation, the sum of the series being the pure Monopole-Dynamic dipole vacuum solution.

One way to provide the physical content to these solutions consists in establishing a link between their multipole moments and quantities measured from well defined and physically reasonable experiments. This was the purpose of paper [8], in which we established such a link by calculating the total precession per revolution of gyroscopes circumventing the symmetry axis.

As a second result, the possibility emerges of comparing different axisymmetric solutions in term of an observable quantity (angle precessed by the gyroscope). In particular, differences in the contribution of the quadrupole moment and the angular momentum for the Erez-Rosen solution [9] and Kerr solution [1] were obtained with respect to the $MQ^{(1)}$ (see [6], [8] or a brief comment below) and $MJ$ solutions respectively.

Some works [10] have addressed the description of non-spherical collapse and the relevance of this circumstance on the fate of the collapse; and other works [24] study the influence of thermal conduction on the evolution of a self-gravitating system out of hydrostatic equilibrium. Most of the life of a star (at any stage of evolution), may be described on the basis of the quasi-static approximation (slowly evolving regime), and this is so, because most relevant processes in star interiors take place on time scales that are usually much longer than the hydrostatic time scale [25], [26]. Instead of following the evolution of the system a long time after its departure from equilibrium, the system is evaluated immediately after such a departure. Here 'immediately’ means on a time scale of the order of the thermal relaxation time, before the establishment of the steady-state resistive flow. In doing so, it is avoided the introduction of numerical procedures which might lead to model-dependent conclusions, and for that evaluation of the system, an analytic solution becomes fundamental to make use of suitable approximations; in particular, it is very important to have a solution constructed in a suitable way to describe the contributions to these scenarios of both the rotation and the quadrupole deformation separately. On the other hand, however, it is only obtained indications about tendency of the object and not a complete description of its evolution. Then, it should be clear that, for sure, the numerical methods used to solve the complicated mathematical problem arising at the study of non-spherical collapse scenarios are much better to obtain a complete description.

As is well known, the Weyl metrics [13] are static axisymmetric solutions to vacuum Einstein equations which are given by the line element,

$$ds^2 = -e^{2\Psi} dt^2 + e^{-2\Psi} (e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2)$$

(2)

where metric functions have to satisfy

$$\Psi,\rho + \rho^{-1}\Psi,\rho + \Psi,zz = 0$$

(3)

and

$$\gamma,\rho = \rho(\Psi,\rho - \Psi,\rho) \quad , \quad \gamma,zz = 2\rho\Psi,\rho\Psi,\rho$$

(4)

3
Observe that (3) is just the Laplace equation for $\Psi$ (in the Euclidean space), and furthermore it represents the integrability condition for (4), implying that for any "Newtonian" potential we have a specific Weyl metric. Another interesting way of writing the general solution of Laplace equation, representing an asymptotically flat behaviour, was obtained by Erez-Rosen [9] and Quevedo [12], integrating equations (3), (4) in prolate spheroidal coordinates. A sub-family of Weyl solutions has been obtained by Gutsunayev and Manko [11], starting from the Schwarzschild solution as a seed solution.

In [6], [21] the $MQ$ solution was presented in the following way

$$\Psi_{M-Q} = \Psi_{q^0} + q\Psi_{q^1} + q^2\Psi_{q^2} + \ldots = \sum_{\alpha=0}^{\infty} q^\alpha \Psi_{q^\alpha},$$

(5)

where the zeroth order $\Psi_{q^0}$ corresponds to the Schwarzschild solution, and $\Psi_{q^\alpha}$ are series that can be summed to obtain exact solutions of the Gutsunayev-Manko family [11] and the Erez-Rosen family as well [12]. It appears that each power in $q$ adds a quadrupole correction to the spherically symmetric solution, the first one being referred to as $MQ^{(1)}$ [6]. It should be observed that due to the linearity of Laplace equation, these corrections give rise to a series of exact solutions, and so, the power series of $q$ may be cut at any order, and the partial summatory, up to that order, gives an exact solution representing a quadrupolar correction to the Schwarzschild solution.

The existence of so many different (physically distinguishable [27]) Weyl solutions gives rise to the question: which among Weyl solutions is better entitled to describe small deviations from spherical symmetry?. Although it should be obvious that such a question does not have a unique answer (there is an infinite number of ways of being non-spherical, so to speak), we shall invoke a very simple criterion, emerging from Newtonian gravity, in order to choose that solution: for example, massive multipole moments of an ellipsoid of rotation, with homogeneous density, mass $M$ and axes $(a, a, b)$ read:

$$M_{2n} = \frac{(-2)^n 3Ma^{2n}\epsilon^n(1-\epsilon/2)^n}{(2n+1)(2n+3)}, \quad \epsilon \equiv (a-b)/a,$$

(6)

$$M_{2n+1} = 0$$

(7)

because of the factor $\epsilon^n$, this equation clearly exhibits the progressive decreasing of the relevance of multipole moments as $n$ increases. Same conclusion can be obtained for the multipole moments of rotation associated to a homogeneous distribution of charge, with total charge $Q$ and ellipsoidal geometry with axes $(a, a, b)$ [16]:

$$J_{2n} = 0$$

(8)

$$J_{2n+1} = \frac{(-2)^n 3Qa^{2n+2}\epsilon^n(1-\epsilon/2)^n\Omega}{(2n+1)(2n+3)(2n+5)}$$

(9)

$\Omega$ being the angular velocity of the rigid rotating distribution.
Thus, in order to describe small departures from sphericity, by means of a solution of Einstein equations, it would be required a solution whose multipole structure shares the property mentioned above. For that reason, the \textit{MQ} solution is particularly suitable for the study of perturbations of spherical symmetry. The main argument to support this statement is based on the fact that the previously known Weyl \cite{12} metrics (e.g. Gutsunayev-Manko, Manko, \textit{γ}-metric \cite{13}, etc) present a drawback when describing quasi-spherical space-times. It consists of the fact that its multipole structure is such that all the moments higher than the quadrupole are of the same order as the quadrupole moment. Instead, as it is intuitively clear and as it is shown above for the classical case, the relevance of such multipole moments should decrease as we move from lower to higher moments, the quadrupole moment being the most relevant for a small departure from sphericity.

In \cite{15}, the behaviour of geodesics is compared with the spherically symmetric situation, throwing light on the sensitivity of the trajectories to deviations from spherical symmetry. The change of sign in the proper radial acceleration of test particles moving radially along the symmetry axis, close to the $r = 2M$ surface, and related to the quadrupole moment of the source deserves particular attention.

The ultimate aim of the above works \cite{6}, \cite{7} (see \cite{16} for general expressions), and the main contribution of the solution presented here, is to obtain a hierarchy of solutions describing pure multipole moment space-times. Since 1918, when Schwarzschild published its static solution of the Einstein vacuum field equations, one of the more active topics of relativistic scientists is devoted to obtain exact solutions to those equations. So, up to now a lot of works have contributed not only with new solutions but developing several techniques to construct new solutions, specifically in the axial symmetry case, nevertheless with arbitrary, in general, physical content. Afterwards, main aim aroused was to get the correct physical interpretation of the solutions obtained. Some of these solutions are already known and studied but in general there is no specific method which allows us both to understand which is the physical relevance of a new solution as well as to construct solutions with prescribed physical behaviour beyond of the symmetry properties of the problem. In \cite{6}, \cite{7}, a generalization of the classical gravitational field to General Relativity theory was planned, by describing the solution defined as the sum of different pure multipole contributions, (following an analogy of the solutions for the classical potential), constructed by means of suitable harmonic solutions of the Laplace equation. A recent work, \cite{18}, establishes the reciprocal relationship between static solutions with axial symmetry and any prescribed multipole structure. The authors in \cite{17}, \cite{18} show how to obtain the Weyl moments required for constructing the pure multipole solutions sought. Although these works are highly relevant, we have not yet a generalization of those results to the stationary general case, and this is the motivation of the solution presented here.

The following step to the \textit{MQ} solution is the pure multipole solution, which has only mass, angular momentum and quadrupole moment, referred to hereafter as the \textit{MJQ} solution. Here we present an approximation to this solution
by means of an expansion of the corresponding Ernst potential. It will be shown that the resulting solution is a very good candidate to describe corrections to the spherical and static configuration due to the rotation and the quadrupole deformation independently.

For this exterior metric that is presented here (MJQ-solution) it would be very interesting to construct a source. The motivation for this is twofold: on the one hand, it is always interesting to propose bounded and physically reasonable sources of gravitational fields, which may serve as models of compact objects. On the other hand, spherical symmetry is a common assumption in the study of compact self-gravitating objects (white dwarfs, neutron stars, black holes). Therefore it is pertinent to ask, how do small deviations from this assumption, related to any kind of perturbation (e.g. fluctuations of the stellar matter, external perturbations, etc.), affect the properties of the system? However, for sufficiently strong fields, in order to answer to this question it is necessary to deal with non-spherically symmetric solution of Einstein equations. Work towards this way is in progress, and, for example, in [21] an interior solution for the $MQ^{(1)}$ solution was already obtained.

The paper is organized as follows: In the next section, all relevant equations are given; we develop the construction of the approximate MJQ solution and analyze the behaviour of the solution for limiting cases. Reasons for appreciating the physical relevance of this solution and its meaningful contribution to the description of small deviations from spherical symmetry are given. Finally, a discussion of the results is presented in the last section, as well as some comments on forthcoming works and the proposals therein.

2 The MJQ Solution

2.1 The approximation of the solution

In 1968 Ernst [23] showed a simplification of the field equations derived from the vacuum stationary and axisymmetric Einstein equations. In order to do that, Ernst made use of a variational principle on the Lagrange function constructed with the metric functions $f$ and $\omega$ of the line element describing those metrics, which in Weyl canonical coordinates [13] reads as follows:

$$ds^2 = -f(dt - \omega d\phi)^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2d\phi^2]$$

being $f$, $\omega$ and $\gamma$ functions only depending on $(\rho, z)$.

A very simple way to obtain the Ernst equations, is by means of a suitable rewrite of the field equations; a redefinition of the metric function $\omega$ is made by introducing a scalar function $W$ whose existence is guaranteed by one of the Einstein field equations\(^1\). Then, the Ernst potential $E$ is defined as a complex function whose real and imaginary part are the metric functions $f$ and $W$ respectively, the real part representing the norm of the Killing vector.

\(^1\)It can be proved that the gradient of this scalar $W$ is exactly the projection of the twist of the time Killing vector, describing stationarity, over the 3-dimensional quotient manifold.
On symmetry axis, the potential \( \xi \), which is the transformation \( \xi \equiv \frac{1 - E}{1 + E} \) from the original Ernst potential, can be expanded by means of a power series of the inverse Weyl’s canonical coordinate \( z \) as follows:

\[
\xi(\rho = 0, z) = \sum_{n=0}^{\infty} m_n z^{-(n+1)}
\] (11)

where \( \rho \) represents the Weyl radial coordinate.

In [7] a method was proposed for obtaining an approximate solution of the stationary vacuum Einstein field equations (\( MJ \)-solution or pure monopole-dynamic dipole solution) by means of an expansion of the Ernst potential \( \xi \) in power series of a dimensionless parameter, \( J \), directly related to the angular momentum of the solution. The expansion to describe that solution (\( MJ \)) is quite appropriate, since it is known that its Ernst potential on the symmetry axis can be written in terms of successive powers of that parameter. This conclusion is obtained from the fact that Fodor, Hoenselaers and Perjés [4] have developed an algorithm (FHP) which allows us to calculate the Geroch [3] and Hansen [2] relativistic multipole moments, related to a vacuum stationary axisymmetric solution, in terms of the coefficients \( (m_n) \) of the expansion on the symmetry axis of the Ernst potential \( \xi \). Both the result obtained up to multipole order 10 by these authors, as well as the calculations we have carried out up to order 20 [7], show that the relation between multipole moments and coefficients \( m_n \) is triangular: that is to say, the multipole moment and the corresponding coefficient \( m_n \) at every order differ in certain combination of lesser order \( m_k \) coefficients. Therefore, these relations can be suitably inverted, and it enables us to determine unequivocally the coefficients \( m_n \) which define the Ernst potential \( \xi \) in terms of the multipole moments for any given solution. For example in [7] was obtained the coefficients which characterize the Ernst potential of the solution having only massive monopole (the mass) and dynamic dipole (angular momentum).

We are now looking for the stationary axisymmetric space-time whose multipole moment structure only has angular momentum and quadrupole moment in addition to the mass. By imposing this condition on the coefficients of the series expansion of the Ernst potential along the symmetry axis [11] we obtain the following expressions (only the first ones are shown, but they have been calculated up to order 20):

\[
\begin{align*}
    m_0 &= M \\
    m_1 &= iJ \\
    m_2 &= Q \\
    m_3 &= 0 \\
    m_4 &= \frac{1}{7} M J^2 + \frac{1}{7} M^2 Q \\
    m_5 &= -\frac{i}{21} J^3 - \frac{8}{21} M J Q
\end{align*}
\]
\[ m_6 = \frac{1}{21} M^4 Q + \frac{1}{21} M^3 J^2 - \frac{51}{231} M Q^2 - \frac{23}{231} Q J^2 \]  

(12)

\( J, Q \) and \( M \) being the angular momentum, the quadrupole moment and the mass of the solution respectively. This result leads to the Ernst potential \( \xi \) on the symmetry axis of the following type

\[ \xi(\rho = 0, z) = \sum_{n=0}^{\infty} J^n \sum_{k=0}^{\infty} q^k F(n, k) , \]  

(13)

where \( \mathcal{J} = i \frac{J}{M^2} \) and \( q = \frac{Q}{M^3} \), and the double index function \( F(n, k) \) is a series that contains information about the dependence on the Weyl coordinate, \( z \), along the axis with the corresponding numerical factor derived from the coefficients \( m_n \). For instance, it can be seen that \( F(0, 0) = \frac{M}{z} \) and \( F(1, 0) = \frac{M^2}{z^2} \). Furthermore, by considering only \( k = 0 \) in the double sum of (13), then the expression for the \( MJ \) solution can be obtained.

Since we are interested in the description of slight deviations from spherical symmetry, we shall take into account the leading terms in the expansion (13) assuming that \( q, \mathcal{J} << 1 \), and neglecting terms involving powers or products of both parameters.

Thus, with these considerations the approximate solution up to order \( q \) and \( \mathcal{J} \), describing the space-time with mass, angular momentum and quadrupole moment, is given by the following Ernst potential on the symmetry axis:

\[ \xi(\rho = 0, z) = \xi_0(\rho = 0, z) + \xi_1(\rho = 0, z) \mathcal{J} + \xi_2(\rho = 0, z) q , \]  

(14)

where

\[ \xi_0(\rho = 0, z) = \frac{M}{z} \]  

(15)

\[ \xi_1(\rho = 0, z) = \frac{M^2}{z^2} \]  

(16)

\[ F(0, 1) = \xi_2(\rho = 0, z) = \sum_{j=1}^{\infty} m_{2j} z^{-(2j+1)} = \sum_{j=1}^{\infty} \hat{\lambda}^{2j+1} F_{2j}^q , \]  

(17)

with \( \hat{\lambda} = \frac{M}{z} \) and \( F_{2j}^q = \frac{15}{(2j + 3)(2j + 1)(2j - 1)} \) because we know from the FHP algorithm (see coefficients \( m_n \) of (13)) that the \( MJQ \) solution, up to order \( q \) and \( \mathcal{J} \), is defined by the following coefficients:

\[ m_{2j+1} = 0 , \quad m_{2j} = F_{2j}^q M^{2j+1} q , \quad \forall j \geq 1 . \]  

(18)

Now, in what follows, and according to the scheme developed in [7], we proceed to solve the Ernst equation

\[ (\xi^* - 1) \Delta \xi = 2 \xi^*(\nabla \xi)^2 \]  

(19)
to each order in $J$ and $q$ for the full Ernst potential $\xi$, with the following form

$$\xi = \xi_0 + \xi_1 J + \xi_2 q$$  \hspace{1cm} (20)

The set of constants appearing in the general solution of (19) to each order, will be determined by restricting the solution on the symmetry axis and comparison with expression (14). The zero order in $J$ and $q$ gives

$$\left(\xi_0^2 - 1\right) \triangle \xi_0 = 2\xi_0(\nabla \xi_0)^2 ,$$  \hspace{1cm} (21)

and we take $\xi_0$ to be the Ernst potential corresponding to the Schwarzschild solution.

The first order in parameter $J$, i.e., $\xi_1$ is already obtained in [7] by solving the equation

$$\left(\xi_0^2 - 1\right) \triangle \xi_1 - 4\xi_0 \nabla \xi_0 \nabla \xi_1 + 2\xi_1(\nabla \xi_0)^2 = 0 ,$$  \hspace{1cm} (22)

and by imposing as a neighbourhood condition the known behaviour of the solution on the symmetry axis (16). It should be noted that the approximate solution to this order, which in prolate coordinates is written

$$\xi = \xi_0 + \xi_1 J = \frac{1}{x} + \frac{y}{x^2} J ,$$  \hspace{1cm} (23)

is the same as that arising from the expansion of the Kerr solution on parameter $J = \frac{a}{M}$ up to first order.

The first order in parameter $q$ ($\xi_2$) must fulfills the following equation:

$$\left(\xi_0^2 - 1\right) \triangle \xi_2 - 4\xi_0 \nabla \xi_0 \nabla \xi_2 + 2\xi_2(\nabla \xi_0)^2 \frac{\xi_2^2 + 1}{\xi_0^2 - 1} = 0 .$$  \hspace{1cm} (24)

This homogeneous equation can be simplified by means of the following redefinition of the function $\xi_2$:

$$\zeta_2 \equiv \frac{\xi_2}{\xi_0^2 - 1} ,$$  \hspace{1cm} (25)

which leads to the Laplace equation for the function $\zeta_2$. It can be easily checked that, as is known, the equation $\triangle \zeta_2 = 0$ is separable and the general solution with regular behaviour on the symmetry axis ($y = \pm 1$) as well as in the neighbourhood of infinity, affords:

$$\zeta_2 = \sum_{n=0}^{\infty} h_n^q Q_n(x) P_n(y) ,$$  \hspace{1cm} (26)

and therefore:

$$\xi_2 = \frac{1 - x^2}{x^2} \sum_{n=0}^{\infty} h_n^q Q_n(x) P_n(y) ,$$  \hspace{1cm} (27)

where $\{x, y\}$ are prolate coordinates; $h_n^q$ are constants of integration, and $Q_n(x)$, $P_n(y)$ are associated Legendre functions of the second kind and Legendre polynomials respectively.
In order to compare the restriction on the symmetry axis of this result (27) with expression (17), we rewrite (17) as follows:

\[ \xi_2(\rho = 0, z) = \frac{M^2 - z^2}{z^2} \left[ \frac{1}{\lambda^2 - 1} \xi_2(\rho = 0, z) \right] , \tag{28} \]

and by carrying out an expansion on the parameter \( \lambda \) we have:

\[ \xi_2(\rho = 0, z) = \frac{z^2 - M^2}{z^2} \left[ \sum_{i=0}^{\infty} \lambda^{2i} \sum_{j=1}^{\infty} \lambda^{2j+1} F_{2j}^q \right] = \frac{z^2 - M^2}{z^2} \left[ \sum_{j=1}^{\infty} \lambda^{2j+1} I_j^q \right] , \tag{29} \]

with the definition \( I_j^q \equiv \sum_{i=1}^{j} F_{2i}^q \). By using Lemma 4 of the Appendix in [7] we rewrite this expression in terms of associated Legendre functions of the second kind as follows:

\[ \xi_2(\rho = 0, z) = \frac{z^2 - M^2}{z^2} \sum_{i=1}^{\infty} Q_{2i}(1/\lambda) (4i+1) I_j^q \sum_{j=1}^{\infty} I_j^q L_{2i,2j} . \tag{30} \]

\( L_{2i,2j} \) being the coefficient of the Legendre polynomial of order \( 2i \), with the power \( 2j \) of the variable. Reordering the sums, we obtain:

\[ \xi_2(\rho = 0, z) = \frac{z^2 - M^2}{z^2} \sum_{i=1}^{\infty} Q_{2i}(1/\lambda) (4i+1) \sum_{j=1}^{i} I_j^q L_{2i,2j} . \tag{31} \]

We now calculate \( I_j^q \) for our case:

\[ I_j^q = \frac{5j(j^2 + 2)}{(2j + 3)(2j + 1)} = \frac{5}{4} - \frac{15}{8} \frac{1}{2j + 1} + \frac{15}{8} \frac{1}{2j + 3} \tag{32} \]

and by comparing expression (27) on the symmetry axis \( (y = \pm 1, x = \frac{z}{M}) \) with (31), we conclude that:

\[ h_{2n+1}^q = 0 \quad \forall n \geq 0 \tag{33} \]
\[ h_0^q = 0 \]
\[ h_{2n}^q = -(4n + 1) \sum_{j=1}^{n} I_j^q L_{2n,2j} \quad \forall n \geq 1 . \tag{34} \]

Using (32), it can be verified that in the expression for the constants \( h_{2n}^q \), i.e.,

\[ h_{2n}^q = -(4n + 1) \sum_{j=0}^{n} \frac{5}{4} L_{2n,2j} - \sum_{j=0}^{n} \frac{15}{8} \frac{L_{2n,2j}}{2j + 1} + \sum_{j=0}^{n} \frac{15}{8} \frac{L_{2n,2j}}{2j + 3} , \tag{34} \]

for \( n \geq 2 \) the two last terms are equal to zero (see Lemma 2 of the appendix in [7]), and taking into account the orthonormality of Legendre’s polynomials we have:

\[ h_{2n}^q = -\frac{5}{4} (4n + 1) , \quad \forall n \geq 2 , \tag{35} \]
whereas for $n = 1$ we see that in (34) the last term is not zero (by virtue of Lemma 2 in [7]) and therefore:

$$h_2^q = -\frac{15}{2^3}.$$  \hspace{1cm} (36)

Once we have determined the constants $h_2^{2n}$, we can state that the contribution on the parameter $q$ to the solution is given by the potential

$$\zeta_2 = -\frac{15}{2} Q_2(x) P_2(y) - \sum_{n=2}^{\infty} \frac{5}{4} (4n + 1) Q_{2n}(x) P_{2n}(y).$$  \hspace{1cm} (37)

Alternatively, by using the Heine identity [19], [20] (see also equation (72) in [7]) we have:

$$\zeta_2 = \frac{5}{4} Q_0(x) P_0(y) - \frac{5}{4} Q_2(x) P_2(y) - \sum_{n=0}^{\infty} \frac{5}{4} (4n + 1) Q_{2n}(x) P_{2n}(y)$$

$$= \frac{5}{4} Q_0(x) P_0(y) - \frac{5}{4} Q_2(x) P_2(y) - \frac{5}{4} \frac{x}{x^2 - y^2}.$$  \hspace{1cm} (38)

### 2.2 Analysis of the solution

We shall make some comments on the approximate solution obtained $^2$:

$$\xi = \frac{1}{x} + y \frac{1}{x^2} \mathcal{J} + q \frac{1}{x^2} (\frac{5}{4} Q_0(x) P_0(y) - Q_2(x) P_2(y) - \frac{x}{x^2 - y^2}) \mathcal{J}$$  \hspace{1cm} (39)

As is well known, the general stationary axisymmetric gravitational field can be described by the line element [20]

$$ds^2 = -f(dt - \omega d\phi)^2 + \sigma^2 f^{-1} e^{2\gamma(x^2 - y^2)} \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) d\phi^2,$$  \hspace{1cm} (40)

where the coordinate system is $\{t, x, y, \phi\}$, the space-like coordinates $\{x, y\}$ being the prolate spheroidal coordinates, $x$ represents a radial coordinate, whereas the other coordinate $y$ represents the cosine function of the polar angle, with $\sigma$ a real constant that, for our case, can be identified with the mass; the unknown functions $f$, $\gamma$ and $\omega$ only depend on $x$ and $y$.

The Ernst potential [20] $E = \frac{1}{2\pi} \xi$ is defined by $E = f + iW$, where

$$W_{,x} = \sigma^{-1} (x^2 - 1)^{-1} f^2 \omega_{,y}$$

$$W_{,y} = -\sigma^{-1} (1 - y^2)^{-1} f^2 \omega_{,x}$$  \hspace{1cm} (41)

$^2$Note that for a real variable $x > 1$, we have [22] the following associated Legendre functions of the second kind: $Q_0(x) = -\frac{1}{4} \ln \frac{1}{x+1}$, $Q_2(x) = -\frac{1}{4} P_2(x) \ln \frac{1}{x+1} - \frac{x}{2}$
and the function \( \gamma \) satisfies the following partial differential equations:

\[
\begin{align*}
\gamma_x &= \frac{1 - y^2}{4f^2(x^2 - y^2)} \\
x(x^2 - 1)E_xE_x^* - x(1 - y^2)E_yE_y^* - y(x^2 - 1)(E_xE_y^* + E_yE_x^*) \\
\gamma_y &= \frac{x^2 - 1}{4f^2(x^2 - y^2)} \\
y(x^2 - 1)E_xE_x^* - y(1 - y^2)E_yE_y^* + x(1 - y^2)(E_xE_y^* + E_yE_x^*)
\end{align*}
\]

Hence, from (39) we obtain:

\[
\begin{align*}
f &= \frac{x - 1}{x + 1} \left\{ 1 + \frac{5}{2}q \left[ Q_0(x)P_0(y) - Q_2(x)P_2(y) - \frac{x}{x^2 - y^2} \right] \right\} \\
W &= -\frac{2y}{(1 + x)^2 M^2}
\end{align*}
\]

and, therefore, by solving equations (42) and (43) up to order \( q \) and \( J \) we have:

\[
\begin{align*}
\omega &= 2 \frac{J y^2 - 1}{M x - 1} \\
\gamma &= \frac{1}{2} \ln \left( \frac{x^2 - 1}{x^2 - y^2} \right) - \frac{15}{8} qx(1 - y^2) \ln \left( \frac{x - 1}{x + 1} \right) + \\
&\quad \frac{5}{8} q \left[ 6y^2 - \frac{2}{(x^2 - y^2)^2} (y^2 + x^2(x^2 - 3y^2 + 1)) - 4 \right].
\end{align*}
\]

The solution fulfils asymptotical flatness conditions since all its metric functions have the good asymptotic behaviour \([20]\), and the metric does not possess any singularity on the symmetry axis, i.e., \( \lim_{x \to \infty} f = 1 \), \( \lim_{x \to \infty} \omega = 0 \), \( \lim_{x \to \infty} \gamma = 0 \), \( \lim_{y^2 \to 1} \gamma = 0 \).

It is possible to check the above-mentioned good quality of the solution for describing the approximate gravitational field of a mass with only quadrupole and angular momentum. First, if we calculate the multipole moments of the solution we obtain:

\[
\begin{align*}
M_0 &= M \\
M_1 &= J M^2 \equiv iJ \\
M_2 &= qM^3 \equiv Q \\
M_3 &= 0 \\
M_4 &= J^2 M^5 \\
M_5 &= \frac{M^6}{21}(J q - J^3) \\
M_6 &= \frac{M^7}{21}(J^2 - \frac{51}{11} q^2 + \frac{23}{11} q J^2).
\end{align*}
\]

These expressions show that the solution possess equatorial symmetry since massive multipole moments of odd order are null whereas dynamic multipole moments of even order are equal to zero.
As can be seen, all multipole moments are several quantities of order higher than $J$, $q$, or products of both; i.e., the order of approximation considered. Take into account that both $J$ and $q$ are very small quantities, since they are the dimensionless parameters $iJ/M^2$ and $Q/M^3$ respectively, and we are considering small deviations from sphericity, i.e., quadrupole moment $Q$ and angular momentum $J$ are less than the first multipolar order, the mass $M$. Obviously, this result was expected because the solution is constructed in such a way that the coefficients $m_n$ are taken up to a linear contribution in both parameters.

Furthermore, this solution has good limits when we consider the parameter $q$ or $J$ to be equal to zero. Of course, $q = J = 0$ reproduces the Schwarzschild solution, as we already mentioned, since the solution is constructed by starting from $\xi_0$, the Ernst potential corresponding to spherical symmetry space-time. If we take in (39) $q = 0$, the resulting solution should represent the first correction to the static configuration of the Schwarzschild solution due to the effect of rotation. In fact, that solution, (23), corresponds to the Kerr solution up to order $J$ in the expansion of the Ernst potential in power series of that parameter $J$, i.e.,

$$
\xi_{kerr} = \frac{1}{x\sqrt{1 + J^2 - Jy}},
$$

(48)

whose expansion in power series of parameter $J$ gives:

$$
\xi_{0}^{kerr} = \frac{1}{x},
$$

(49)

$$
\xi_{1}^{kerr} = \frac{x^2 - 1}{x^2} Q_1^{(2)}(x) P_1(y) = \frac{1}{2} \frac{y}{x^2}
$$

Finally, if we take $J = 0$ in (39), the corresponding solution

$$
\xi = \frac{1}{x} + q \frac{1 - x^2}{x^2} \frac{5}{4} [Q_0(x) P_0(y) - Q_2(x) P_2(y) - \frac{x}{x^2 - y^2}]
$$

(50)

should represent the first contribution to the deformation from the spherical symmetry configuration due to the quadrupole moment. Therefore, it should be the $MQ^{(1)}$ solution constructed with this claim. In order to verify this conjecture, it is necessary to check that the metric function $\Psi$ corresponding to the $MQ^{(1)}$ solution generates the Ernst potential up to order $q$. Since the $MQ^{(1)}$ solution is given by $\Psi = \Psi_0 + q\Psi_1$, we have:

$$
E \equiv e^{2\Psi} = A(1 + 2\Psi_1 q) + O(q^2),
$$

(51)

with $A \equiv e^{2\Psi_0} = \frac{x - 1}{x + 1}$, and,

$$
\xi = \frac{1 - E}{1 + E} = \frac{B - D}{B + D},
$$

(52)

with the following notation: $B \equiv 1 - A$, $C \equiv 1 + A$ and $D \equiv 2Aq\Psi_1$. Hence, up to order $q$, we have that

$$
\xi_{MQ^{(1)}} \approx \frac{B}{C} - \frac{D}{C} \frac{B}{C} + 1 = \frac{1}{x} - \frac{x^2 - 1}{x^2} q\Psi_1 + O(q^2).
$$

(53)
By substituting the expression for Ψ₁, it is clear that (53) is actually equivalent to the solution obtained up to order q, i.e., (50).

3 Conclusion

In previous sections an approximate solution of the vacuum Einstein field equations with axial symmetry was obtained. The metric functions of the Weyl line element are given explicitly by means of the corresponding Ernst potential. To construct the solution we have made use of the relationship between the coefficients of the Ernst potential appearing at its series expansion in powers of the Weyl coordinate along the symmetry axis, and the relativistic multipole moments of the solution.

The relevance of this new solution, in contrast to other known solutions, has a very interesting physical meaning since it can be used to describe -in the perturbative sense- the corrections to the spherical symmetry that an angular and quadrupole moment would incorporate to the solution. As already seen, the solution has two independent parameters, q and J, that are directly related to these multipole moments and, by using those parameters, the approximation up to first order of the pure multipole $M\,J\,Q$ solution is constructed. The good limits of the solution for the cases $q = 0$ and $J = 0$ are verified, thereby recovering the $M\,J$ solution up to first order (which is exactly Kerr at this order) and the $M\,Q^{(1)}$ solution respectively.

A significant difference of this solution with respect to the Kerr solution is that it allows us to control the magnitude of the contributions from the quadrupole deformation and the effect of rotation independently by means of its two parameters. The different behaviour of test particles moving along geodesics in the radial direction of the $M\,Q^{(1)}$ solution with respect to that movement in the case of spherical collapse is already known. We expect that this solution may be a relevant reference to describe non-spherical collapse since it can supply more detailed information about the process. Study of all the implications of this claim for the solution is still in progress and will be addressed in future work, where the good behaviour of the event horizon will be shown as well as a detailed study of the test particle geodesics. In any case, it should be stressed that the approximation used to construct the solution is correct in the sense that it allows one to approach the source without any type of discontinuity, and moreso for the strong field case (where the approximation is better since the parameters $q \equiv Q/M^3$ and $J \equiv i\,J/M^2$ become sufficiently less than 1 to be neglected).

Apart from achieving an interior solution for the $M\,Q^{(1)}$, it is expected that a search for an interior solution matching this new solution would provide a very interesting global model for the description of compact bodies slightly different from the spherical, but never realistic, source.

14
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