Second Order Optimality Conditions for Optimal Control Problems of Stochastic Evolution Equations

Qi Lü*, Haisen Zhang† and Xu Zhang‡

Abstract

In this paper, we establish some second order necessary/sufficient optimality conditions for optimal control problems of stochastic evolution equations in infinite dimensions. The control acts on both the drift and diffusion terms and the control region is convex. The concepts of relaxed and $V$-transposition solutions (introduced in our previous works) to operator-valued backward stochastic evolution equations are employed to derive these optimality conditions. The correction part of the second order adjoint equation, which does not appear in the (first order) Pontryagin-type stochastic maximum principle, plays a fundamental role in our second order optimality conditions.

2010 Mathematics Subject Classification. Primary 93E20; Secondary, 60H07, 60H15.

Key Words. Stochastic optimal control, relaxed transposition solution, $V$-transposition solution, operator-valued backward stochastic evolution equation, second order optimality condition.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$, on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined. Let $T > 0$, and let $X$ be a Banach space with norm $|\cdot|_X$. For any $t \in [0, T]$ and $r \in [1, \infty)$, denote by $L^r_{\mathcal{F}_t}(\Omega; X)$ the Banach space of all $\mathcal{F}_t$-measurable random variables $\xi : \Omega \to X$ such that

---

*School of Mathematics, Sichuan University, Chengdu 610064, Sichuan Province, China. The research of this author is partially supported by NSF of China under grant 11471231 and the NSFC-CNRS Joint Research Project under grant 11711530142. E-mail: luitongwang@scu.edu.cn.

†School of Mathematical Sciences, Sichuan Normal University, Chengdu 610068, China. The research of this author is partially supported by NSF of China under grants 11401404, 11471231 and 11701470, and the NSF of CQ CSTC under grants 2015jcyjA00017 and the Advance and Basic Research Project of Chongqing under grant cstc2016jcyjA0239. E-mail: haisenzhang@yeah.net.

‡School of Mathematics, Sichuan University, Chengdu 610064, Sichuan Province, China. The research of this author is partially supported by the NSFC-CNRS Joint Research Project under grant 11711530142, the PCSIRT under grant IRT_16R53 and the Chang Jiang Scholars Program from the Chinese Education Ministry. E-mail: zhang_xu@scu.edu.cn.
\[ \mathbb{E}[|\xi|^r_X] < \infty, \] with the canonical norm. Also, denote by \( D_\mathcal{F}([0,T]; L^r(\Omega; X)) \) the vector space of all \( X \)-valued \( \mathbb{F} \)-adapted processes \( \phi(\cdot) \) such that \( \phi(\cdot) : [0,T] \to L^r_{\mathcal{F}_t}(\Omega; X) \) is càdlàg, i.e., right continuous with left limits. Clearly, \( D_\mathcal{F}([0,T]; L^r(\Omega; X)) \) is a Banach space with the following norm

\[ \|\phi(\cdot)\|_{D_\mathcal{F}([0,T]; L^r(\Omega; X))} = \sup_{t \in [0,T]} \left[ \mathbb{E}[|\phi(t)|^r_X] \right]^{1/r}. \]

Denote by \( C_\mathcal{F}([0,T]; L^r(\Omega; X)) \) the Banach space of all \( X \)-valued \( \mathbb{F} \)-adapted processes \( \phi(\cdot) \) such that \( \phi(\cdot) : [0,T] \to L^r_{\mathcal{F}_t}(\Omega; X) \) is continuous, with norm inherited from \( D_\mathcal{F}([0,T]; L^r(\Omega; X)) \). Fix any \( r_1, r_2, r_3, r_4 \in [1, \infty) \). Put

\[ L^0_\mathcal{F}(\Omega; L^{r_2}(0,T; X)) = \left\{ \phi : (0,T) \times \Omega \to X \mid \phi(\cdot) \text{ is } \mathbb{F}\text{-adapted and} \right. \]

\[ \|\phi\|_{L^0_\mathcal{F}(\Omega; L^{r_2}(0,T; X))} \triangleq \left[ \mathbb{E}\left( \int_0^T |\phi(t)|^{r_2}_X dt \right)^{\frac{r_1}{r_2}} \right]^{\frac{1}{r_1}} < \infty, \]

\[ L^2_\mathcal{F}(0,T; L^{r_1}(\Omega; X)) = \left\{ \phi : (0,T) \times \Omega \to X \mid \phi(\cdot) \text{ is } \mathbb{F}\text{-adapted and} \right. \]

\[ \|\phi\|_{L^2_\mathcal{F}(0,T; L^{r_1}(\Omega; X))} \triangleq \left[ \int_0^T \left( \mathbb{E}[|\phi(t)|^{r_1}_X] \right)^{\frac{r_2}{r_1}} dt \right]^{\frac{1}{r_2}} < \infty. \]

Clearly, both \( L^0_\mathcal{F}(\Omega; L^{r_2}(0,T; X)) \) and \( L^2_\mathcal{F}(0,T; L^{r_1}(\Omega; X)) \) are Banach spaces with the canonical norms. If \( r_1 = r_2 \), we simply write the above spaces as \( L^0_\mathcal{F}(0,T; X) \). Let \( Y \) be another Banach space. Denote by \( \mathcal{L}(X; Y) \) the (Banach) space of all bounded linear operators from \( X \) to \( Y \), with the usual operator norm (When \( Y = X \), we simply write \( \mathcal{L}(X) \) instead of \( \mathcal{L}(X; Y) \)). Further, we denote by \( \mathcal{L}_{pd}(L^0(0,T; L^{r_2}(\Omega; X)), L^2(0,T; L^{r_1}(\Omega; Y))) \) (resp. \( \mathcal{L}_{pd}(X, L^2(0,T; L^{r_1}(\Omega; Y))) \)) the vector space of all bounded, pointwisely defined linear operators \( \mathcal{L} \) from \( L^0(0,T; L^{r_2}(\Omega; X)) \) (resp. \( X \)) to \( L^2(0,T; L^{r_1}(\Omega; Y)) \), i.e., for a.e. \( (t,\omega) \in (0,T) \times \Omega \), there exists an \( L(t,\omega) \in \mathcal{L}(X; Y) \) verifying that \( \langle \mathcal{L}(\phi)(t,\omega) \rangle(t,\omega) = L(t,\omega)\phi(t,\omega), \forall \phi(\cdot) \in L^0(0,T; L^{r_2}(\Omega; X)) \) (resp. \( \langle \mathcal{L}(x)(t,\omega) \rangle(t,\omega) = L(t,\omega)x, \forall x \in X \)). Similarly, one can define the spaces \( \mathcal{L}_{pd}(L^2(\Omega; X), L^0(0,T; L^{r_1}(\Omega; Y))) \) and \( \mathcal{L}_{pd}(L^2(\Omega; X), L^0(\Omega; Y), L^{r_1}(\Omega; Y)) \), etc.

Let \( H \) be a separable Hilbert space with the norm \( |\cdot|_H \) and the inner product \( \langle \cdot, \cdot \rangle_H \), and let \( A \) be an unbounded linear operator (with domain \( D(A) \) on \( H \)), which generates a \( C_0 \)-semigroup \( \{e^{At}\}_{t \geq 0} \). Denote by \( A^* \) the adjoint operator of \( A \). Clearly, \( D(A) \) is a Hilbert space with the usual graph norm, and \( A^* \) is the infinitesimal generator of \( \{e^{A^*t}\}_{t \geq 0} \), the adjoint \( C_0 \)-semigroup of \( \{e^{At}\}_{t \geq 0} \). Let \( U \) be a closed convex subset of another separable Hilbert space \( H_1 \) (with norm \( |\cdot|_{H_1} \) and inner product \( \langle \cdot, \cdot \rangle_{H_1} \)). For any \( \beta \geq 2 \), put

\[ U^\beta([0,T]) \triangleq \left\{ u(\cdot) \in L^\beta_\mathcal{F}(0,T; H_1) \mid u(t,\omega) \in U, \text{ a.e. } (t,\omega) \in [0,T] \times \Omega \right\}. \]

Consider the following controlled (forward) stochastic evolution equation (SEE for short):

\[
\begin{aligned}
\begin{cases}
&dx = (Ax + a(t, x, u))dt + b(t, x, u)dW(t) \quad \text{in } (0,T], \\
&x(0) = x_0,
\end{cases}
\end{aligned}
\]

where \( a, b \) are two suitable functions from \([0,T] \times H \times U \) to \( H \), \( u \in U^\beta([0,T]) \) and \( x_0 \in \)
We call \( x(\cdot) = x(\cdot; x_0, u) \in C_T([0, T]; L^\beta(\Omega; H)) \) a mild solution to (1.1) if
\[
x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}a(s, x(s), u(s))ds + \int_0^t e^{A(t-s)}b(s, x(s), u(s))dW(s), \quad \mathbb{P}\text{-a.s.,} \quad \forall \, t \in [0, T].
\]

Define a cost functional \( \mathcal{J}(\cdot) \) (for the control system (1.1)) as follows:
\[
\mathcal{J}(u(\cdot)) \triangleq \mathbb{E}\left[ \int_0^T g(t, x(t), u(t))dt + h(x(T)) \right], \quad u(\cdot) \in U^\beta[0, T], (1.2)
\]
where \( g : [0, T] \times H \times U \to \mathbb{R} \) and \( h : H \to \mathbb{R} \) are suitably given functions, and \( x(\cdot) \) is the corresponding solution to (1.1).

In this paper we are concerned with the following optimal control problem for (1.1):

**Problem (P)** Find a \( \bar{u}(\cdot) \in U^2[0, T] \) such that
\[
\mathcal{J}(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U^2[0, T]} \mathcal{J}(u(\cdot)). (1.3)
\]

Any \( \bar{u}(\cdot) \) satisfying (1.3) is called an *optimal control*. The state \( \bar{x}(\cdot) \) corresponding to \( \bar{u}(\cdot) \) is called the *optimal state*, and \((\bar{x}(\cdot), \bar{u}(\cdot))\) is called an *optimal pair*.

It is one of the most important issues in optimal control theory to establish optimality conditions for optimal controls, which can be employed to distinguish optimal controls from the other admissible controls. Since the landmark work in [38], first-order necessary conditions are studied extensively in the literature for different kinds of control systems, such as systems governed by ordinary differential equations (e.g. [38]), systems governed by ordinary difference equations (e.g. [5]), systems governed by partial differential equations (e.g. [24]), systems governed by stochastic ordinary differential equations (e.g. [37, 41]), systems governed by SEEs (e.g. [27]), etc.

Similar to the Calculus of Variations (or even the elementary calculus), in addition to the first-order necessary conditions, some second order necessary conditions should be established to distinguish optimal controls from the candidates which satisfy the first order necessary conditions, especially when the optimal controls are singular, i.e., optimal controls satisfy the first order necessary conditions trivially. For instance, when the Hamiltonian corresponding to optimal controls is equal to a constant in a subset of the control region or the gradient and the Hessian (with respect to the control variable \( u \)) of the corresponding Hamiltonian vanish/degenerate. In these cases, the first order necessary conditions are not enough to determine the optimal controls. For more details, we refer the reader to the introduction of [45].

The study of second order necessary conditions for controlled (deterministic) ordinary differential equations may date back to the early time of modern control theory (e.g. [3, 16, 17, 23]) and attracts lots of attention until recently (see [6, 12, 22, 23, 36] and the rich references cited therein). However, as far as we know, there are merely a few published papers for second-order necessary conditions for stochastic optimal control problems in finite dimensions:
• In [1, 31, 39], the main concern focused on the case that the diffusion term is independent of the control variable. In [31, 39], pointwise second-order maximum principles for stochastic singular optimal controls in the sense of Pontryagin-type maximum principle were established, while in [1], the control system with time delay was discussed;

• When the diffusion terms of the control systems contain the control variable, in [9], an integral-type second-order necessary condition for stochastic optimal controls was derived under the assumption that the control region is convex;

• Recently, in [43, 44] (see also [45]) and [13], under some assumptions in terms of the Malliavin Calculus, the authors established pointwise second order necessary conditions for stochastic singular optimal controls with both the convex and the general control constrains;

• Very recently, some first and second order integral type necessary optimality conditions for stochastic optimal control problems with state constraints and closed control constraints were obtained in [14, 15].

The research on the second order sufficient condition for optimal controls also has a long history. It is found that the second order sufficient condition has important applications in the sensitivity analysis and the numerical methods for the optimal control problems. The corresponding theory for the deterministic cases has been extensively studied (e.g. [7, 10, 19, 21, 25, 33, 35, 42]). However, as far as we know, [9] is the only one reference which contains a sort subsection on the second order sufficient condition for optimal controls of stochastic control systems in finite dimensions.

To the best of our knowledge, before our work there exists no literature addressed to the second optimality condition for optimal controls of stochastic control systems in infinite dimensions.

The main purpose of this paper is to establish the second order necessary and sufficient conditions for optimal control problems of SEEs. In this work, both drift and diffusion terms, i.e., $a(t, x, u)$ and $b(t, x, u)$, may contain the control variable $u$, and we assume that the control region $U$ is convex. The key difference between [9, 13, 43] and the present work is that we consider here the SEEs in infinite dimensions. For such kind of control systems, the second order adjoint equation, which is an operator-valued backward stochastic evolution equation (BSEE for short), is much more complex than that in finite dimensions. The main difficulty to study the well-posedness of backward stochastic evolution equations is that, there exists no proper definition of the Itô integral for operator-valued stochastic processes (e.g. [40]). This leads to some essential obstacle to obtain the representation of the correction part of the solutions to such sort of BSEEs. However, it can be found in [45] that, the correction part of the second order adjoint equation plays an indispensable role in the second order necessary conditions.

In this paper, we first employ the notion of relaxed transposition solution (introduced in [27, 28]) for the second order adjoint equations to derive an integral-type second order necessary condition for optimal controls. Then, we use the notion of $V$-transposition solution (introduced in [29]) for the second order adjoint equations to obtain a pointwise second order necessary condition. We remark that, quite different from that in the deterministic setting,
there exist some essential difficulties to derive the pointwise second-order necessary condition from an integral-type one when the diffusion term of the control system depends on the control variable, even for the special case of convex control constraint. We overcome these difficulties by some technique developed in [43], which is for stochastic control problems in finite dimensions.

Also, we establish a second order sufficient condition for optimal controls. This type of condition essentially ensures that the cost functional has a quadratic growth property near an admissible control and hence ensures the local optimality and uniqueness of the minimizer. The basic idea comes from the second order sufficient conditions in optimization theory.

The rest of this paper is organized as follows: In Section 2, we prove some useful estimates corresponding to the control system and present some results for (operator-valued) BSEEs. Section 3 is devoted to establishing the integral type second order necessary conditions for stochastic optimal controls. In Section 4, we obtain a pointwise second order necessary optimality condition. Section 5 is addressed to the second order sufficient optimality conditions. Finally, in Section 6, two simple examples are provided to show the applications of the second order optimality conditions established in Sections 4 and 5.

Partial results of this paper have been announced in [26] without detailed proof.

2 Some preliminaries

Throughout this paper, we assume the following condition.

(A1) Suppose that \( a(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to H \) and \( b(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to H \) are two maps satisfying:

i) For any \((x, u) \in H \times U\), both \( a(\cdot, x, u) : [0, T] \to H \) and \( b(\cdot, x, u) : [0, T] \to H \) are Lebesgue measurable;

ii) There is a constant \( C_L > 0 \) such that, for a.e. \( t \in [0, T] \), any \( x, \tilde{x} \in H \) and any \( u, \tilde{u} \in U \),

\[
\begin{align*}
|a(t, x, u) - a(t, \tilde{x}, \tilde{u})|_H &+ |b(t, x, u) - b(t, \tilde{x}, \tilde{u})|_H \leq C_L \left( |x - \tilde{x}|_H + |u - \tilde{u}|_{H_1} \right), \\
|a(t, 0, 0)|_H &+ |b(t, 0, 0)|_H \leq C_L.
\end{align*}
\]

(2.1)

In the sequel, we shall denote by \( C \) a generic constant, depending on \( T, A, \beta \) and \( C_L \) (or \( F, J \) and \( K \) to be introduced later), which may be different from one place to another. Similar to [11, Chapter 7], for any \( u(\cdot) \in U^\beta[0, T] \), it is easy to show that, under the assumption (A1), the equation (1.1) is well-posed in the sense of mild solution and

\[
\|x\|_{C_T([0,T];L^\beta(\Omega,H))} \leq C \left( 1 + \|x_0\|_{L^\beta_T(\Omega,H)} + \|u\|_{L^\beta_T(\Omega,L^2(0,T;H_1))} \right).
\]

Also, we need the following condition:

(A2) Suppose that \( g(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to \mathbb{R} \) and \( h(\cdot) : H \to \mathbb{R} \) are two functions satisfying:

i) For any \((x, u) \in H \times U\), \( g(\cdot, x, u) : [0, T] \to \mathbb{R} \) is Lebesgue measurable;
ii) There is a constant $C_L > 0$ such that, for a.e. $t \in [0, T]$, any $x \in H$ and $u \in U$,

$$|g(t, x, u)| + |h(x)| \leq C_L(1 + |x|^2_H + |u|^2_{H_1}).$$

(2.2)

Under Assumptions (A1) and (A2), the optimal control problem (1.3) (with $\beta \geq 2$) is well-defined.

To establish second order necessary conditions, we need to introduce further assumptions for $a(\cdot, \cdot, \cdot), b(\cdot, \cdot, \cdot), g(\cdot, \cdot, \cdot)$ and $h(\cdot)$. To simplify the notation, for $\varphi = a, b, f$ and $g$, we denote by $\varphi_x(t, x, u)$ and $\varphi_u(t, x, u)$ respectively the first order partial derivatives of $\varphi$ with respect to $x$ and $u$ at $(t, x, u)$, by $\varphi_{xx}(t, x, u)$, $\varphi_{xu}(t, x, u)$ and $\varphi_{uu}(t, x, u)$ the second order partial derivatives of $\varphi$ at $(t, x, u)$.

(A3) The maps $a(t, \cdot, \cdot)$ and $b(t, \cdot, \cdot)$, and the functional $g(t, \cdot, \cdot)$ and $h(\cdot)$ are $C^2$ with respect to $x$ and $u$. Moreover, there exists a constant $C_L > 0$ such that, for a.e. $t \in [0, T]$ and any $(x, u) \in H \times U$,

$$\begin{cases}
\|a_x(t, x, u)\|_{L(H)} + \|b_x(t, x, u)\|_{L(H)} + \|a_u(t, x, u)\|_{L(H_1; H)} + \|b_u(t, x, u)\|_{L(H_1; H)} \leq C_L, \\
\|g_x(t, x, u)\|_{H} + \|g_u(t, x, u)\|_{H_1} + \|h_x(x)\|_{H} \leq C_L(1 + |x| + |u|), \\
\|a_{xx}(t, x, u)\|_{L(H \times H; H)} + \|b_{xx}(t, x, u)\|_{L(H \times H; H)} + \|a_{xu}(t, x, u)\|_{L(H_1 \times H_1; H)} + \|b_{xu}(t, x, u)\|_{L(H_1 \times H_1; H)} \leq C_L, \\
\|g_{xx}(t, x, u)\|_{L(H)} + \|g_{xu}(t, x, u)\|_{L(H_1; H_1)} + \|g_{uu}(t, x, u)\|_{L(H_1)} + \|h_{xx}(x)\|_{L(H)} \leq C_L.
\end{cases}$$

(2.3)

First, using Assumptions (A1) and (A3), we give some estimates for the control system (1.1) and its linearized systems.

Let $\bar{u}(\cdot) \in U^\beta[0, T]$ and $\bar{x}(\cdot)$ be the corresponding state of control system (1.1). For $\varphi = a, b$ and $g$, put

$$\varphi_1(t) = \varphi_x(t, \bar{x}(t), \bar{u}(t)), \quad \varphi_2(t) = \varphi_u(t, \bar{x}(t), \bar{u}(t))$$

and

$$\varphi_{11}(t) = \varphi_{xx}(t, \bar{x}(t), \bar{u}(t)), \quad \varphi_{22}(t) = \varphi_{uu}(t, \bar{x}(t), \bar{u}(t)), \quad \varphi_{12}(t) = \varphi_{xu}(t, \bar{x}(t), \bar{u}(t)).$$

Let $u(\cdot) \in U^\beta[0, T]$ be another admissible control with related state $x(\cdot)$. Set $\delta u(\cdot) = u(\cdot) - \bar{u}(\cdot)$ and $\delta x(\cdot) = x(\cdot) - \bar{x}(\cdot)$. Consider the following first and second order linearized evolution equations:

$$\begin{cases}
\frac{dy}{dt} = [Ay + a_1(t)y + a_2(t)\delta u] dt + [b_1(t)y + b_2(t)\delta u] dW(t) \quad \text{in } (0, T], \\
y(0) = 0;
\end{cases}$$

(2.4)

$$\begin{cases}
\frac{dz}{dt} = [Az + a_1(t)z + a_{11}(t)y, y) + 2a_{12}(t)y, \delta u) + a_{22}(t)(\delta u, \delta u)] dt \\
\quad + [b_1(t)z + b_{11}(t)y, y) + 2b_{12}(t)y, \delta u) + b_{22}(t)(\delta u, \delta u)] dW(t) \quad \text{in } (0, T],
\end{cases}$$

(2.5)

We have the following estimates.
Lemma 2.1 Let (A1) and (A3) hold. Then, for any $\beta \geq 2$,
\[
\begin{aligned}
&\left\| \delta x \right\|_{C^\beta([0,T]};L^2(\Omega;H)) \leq C\left\| \delta u \right\|_{L^2(\Omega;L^2(0,T;H_1))}, \\
&\left\| y \right\|_{C^\beta([0,T]};L^2(\Omega;H)) \leq C\left\| \delta u \right\|_{L^2(\Omega;L^2(0,T;H_1))}, \\
&\left\| z \right\|_{C^\beta([0,T]};L^2(\Omega;H)) \leq C\left\| \delta u \right\|_{L^2(\Omega;L^2(0,T;H_1))}^2, \\
&\left\| \delta x - y \right\|_{C^\beta([0,T]};L^2(\Omega;H)) \leq C\left\| \delta u \right\|_{L^2(\Omega;L^2(0,T;H_1))}^2.
\end{aligned}
\]

Proof: We divide the proof into two steps.

Step 1. In this step, we prove the estimates for $\delta x$, $y$ and $z.$

Put
\[
\begin{aligned}
\tilde{a}_1(t) &\triangleq \int_0^1 a_x(t, \tilde{x}(t) + \theta \delta x(t), u(t))d\theta, \\
\tilde{a}_2(t) &\triangleq \int_0^1 a_u(t, \tilde{x}(t), \tilde{u}(t) + \theta \delta u(t))d\theta, \\
\tilde{b}_1(t) &\triangleq \int_0^1 b_x(t, \tilde{x}(t) + \theta \delta x(t), u(t))d\theta, \\
\tilde{b}_2(t) &\triangleq \int_0^1 b_u(t, \tilde{x}(t), \tilde{u}(t) + \theta \delta u(t))d\theta.
\end{aligned}
\]

It is easy to see that $\delta x(\cdot)$ satisfies the following SEE:
\[
\begin{aligned}
d\delta x &= (A\delta x + \tilde{a}_1(t)\delta x + \tilde{a}_2(t)\delta u)dt + (\tilde{b}_1(t)\delta x + \tilde{b}_2(t)\delta u)dW(t) &\quad\text{in } (0,T), \\
\delta x(0) &= 0.
\end{aligned}
\]

Then, by Assumption (A3), we find that
\[
\begin{aligned}
\mathbb{E}\left| \delta x(t) \right|^\beta_H &= \mathbb{E}\left[ \int_0^t e^{A(t-s)}\tilde{a}_1(s)\delta x(s)ds + \int_0^t e^{A(t-s)}\tilde{a}_2(s)\delta u(s)ds \\
&\quad + \int_0^t e^{A(t-s)}\tilde{b}_1(s)\delta x(s)dW(s) + \int_0^t e^{A(t-s)}\tilde{b}_2(s)\delta u(s)dW(s) \right|^\beta_H \\
&\leq C\mathbb{E}\left[ \left( \int_0^t e^{A(t-s)}\tilde{a}_1(s)\delta x(s)ds \right)^\beta_H + \left( \int_0^t e^{A(t-s)}\tilde{b}_1(s)\delta x(s)dW(s) \right)^\beta_H \\
&\quad + \left( \int_0^t e^{A(t-s)}\tilde{a}_2(s)\delta u(s)ds \right)^\beta_H + \left( \int_0^t e^{A(t-s)}\tilde{b}_2(s)\delta u(s)dW(s) \right)^\beta_H \right] \\
&\leq C\left[ \int_0^t \mathbb{E}\left| \delta x(s) \right|^\beta_H ds + \mathbb{E}\left( \int_0^T \left| \delta u(s) \right|^\beta_{H_1} ds \right)^\beta_T \right].
\end{aligned}
\]

It follows from (2.7) and Gronwall’s inequality that
\[
\sup_{t \in [0,T]} \mathbb{E}\left| \delta x(t) \right|^\beta_H \leq C\left\| \delta u \right\|_{L^2(\Omega;L^2(0,T;H_1))}^\beta.
\]

In the same way, by (2.4) and Gronwall’s inequality we get that
\[
\sup_{t \in [0,T]} \mathbb{E}\left| y(t) \right|^\beta_H \leq C\left\| \delta u \right\|_{L^2(\Omega;L^2(0,T;H_1))}^\beta.
\]
Then, by Assumption (A3) and (2.9),
\[
\mathbb{E}|z(t)|^\beta_H = \mathbb{E}\left[\int_0^t e^{A(t-s)}[a_1(s)z(s) + a_{11}(s)(y(s), y(s)) + 2a_{12}(s)(y(s), \delta u(s)) + a_{22}(s)(\delta u(s), \delta u(s))]ds + \int_0^t e^{A(t-s)}[b_1(s)z + b_{11}(s)(y(s), y(s)) + 2b_{12}(s)(y(s), \delta u(s)) + b_{22}(s)(\delta u(s), \delta u(s))]dW(s)\right]^{\beta_H}
\leq C\left[\int_0^t \mathbb{E}|z(s)|^\beta_H ds + \mathbb{E}\left(\int_0^T |y(s)|^4_{H^1} ds\right)^{\frac{2}{7}} + \mathbb{E}\left(\int_0^T |\delta u(s)|^4_{H_1} ds\right)^{\frac{2}{7}}\right]
\leq C\left[\int_0^t \mathbb{E}|z(s)|^\beta_H ds + \mathbb{E}\left(\int_0^T |y(s)|^4_{H^1} ds\right)^{\frac{2}{7}} + \mathbb{E}\left(\int_0^T |\delta u(s)|^4_{H_1} ds\right)^{\frac{2}{7}}\right].
\]

Therefore,
\[
\sup_{t \in [0,T]} \mathbb{E}|z(t)|^\beta_H \leq C\|\delta u\|^{2\beta}_{L_2^2(\Omega; L^4(0,T; H_1))}.
\]

**Step 2.** In this step, we show that
\[
\|\delta x - \delta y\|_{C^2_p([0,T]; L^2(H^1; H))} \leq C\|\delta u\|^{2\beta}_{L_2^2(\Omega; L^4(0,T; H_1))}.
\]

Let \( r_1(\cdot) = \delta x(\cdot) - \delta y(\cdot) \). Then \( r_1 \) solves
\[
\begin{cases}
    dr_1 = \left[Ar_1 + \tilde{a}_1(t)r_1 + (\tilde{a}_1(t) - a_1(t))y + (\tilde{a}_2(t) - a_2(t))\delta u\right]dt + \left[\tilde{b}_1(t)r_1 + (\tilde{b}_1(t) - b_1(t))y + (\tilde{b}_2(t) - b_2(t))\delta u\right]dW(t) & \text{in } (0, T],
    
r_1(0) = 0.
\end{cases}
\]

By (A3), \( a_1(t, \cdot, \cdot), a_2(t, \cdot, \cdot), b_1(t, \cdot, \cdot) \) and \( b_2(t, \cdot, \cdot) \) are Lipschitz on \( H \times U \) with respect to \( t \) uniformly. Then, it follows from (2.8), (2.9) and (2.11) that
\[
\mathbb{E}|r_1(t)|^\beta_H
= \mathbb{E}\left[\int_0^t e^{A(t-s)}\tilde{a}_1(s)r_1(s)ds + \int_0^t e^{A(t-s)}\tilde{b}_1(s)r_1(s)dW(s)
+ \int_0^t e^{A(t-s)}(\tilde{a}_1(s) - a_1(s))y(s)ds + \int_0^t e^{A(t-s)}(\tilde{b}_1(s) - b_1(s))y(s)dW(s)
+ \int_0^t e^{A(t-s)}(\tilde{a}_2(s) - a_2(s))\delta u(s)ds + \int_0^t e^{A(t-s)}(\tilde{b}_2(s) - b_2(s))\delta u(s)dW(s)\right]^{\beta_H}
\leq C\left\{\mathbb{E}\left[\int_0^t |r_1(s)|^{\beta_H} ds + \mathbb{E}\left[\int_0^T (|\tilde{a}_1(s) - a_1(s)|^2_{L(H)} + |\tilde{b}_1(s) - b_1(s)|^2_{L(H^1)}) \cdot |y(s)|^2_{H^1} ds\right]^{\frac{2}{7}}
+ \mathbb{E}\left[\int_0^T (|\tilde{a}_2(s) - a_2(s)|^2_{L(H^1; H)} + |\tilde{b}_2(s) - b_2(s)|^2_{L(H^1; H, H)}) \cdot |\delta u(s)|^2_{H_1} ds\right]^{\frac{2}{7}}\right\}
\leq C\left[\mathbb{E}\left[\int_0^t |r_1(s)|^{\beta_H} ds + \mathbb{E}\left(\int_0^T (|\delta x|^2_{H^1} + |\delta u|^2_{H_1}) \cdot |y(s)|^2_{H^1} ds\right)^{\frac{2}{7}}
+ \mathbb{E}\left(\int_0^T |\delta u(s)|^4_{H_1} ds\right)^{\frac{2}{7}}\right]
\leq C\left[\mathbb{E}\left[\int_0^t |r_1(s)|^{\beta_H} ds + \mathbb{E}\left(\int_0^T |\delta u(s)|^4_{H_1} ds\right)^{\frac{2}{7}}\right]\right],
\]
which, together with Gronwall’s inequality, implies that

$$
\sup_{t \in [0,T]} \mathbb{E}|r_1(t)|^3_H \leq C\|\delta u\|^2_2\|E_{L^2(\Omega; L^4(0,T; H_1))}^3.
$$

This completes the proof of Lemma 2.1.

Next, we give a well-posedness result for the $H$-valued BSEE:

$$
\begin{aligned}
& dp(t) = -A^* p(t) dt + f(t, p(t), q(t)) dt + q(t) dW(t) \quad \text{in } [0, T), \\
& p(T) = p_T.
\end{aligned}
(2.12)
$$

Here $p_T \in L^2_F(\Omega; H)$, $f : [0, T] \times H \times H \times \Omega \to H$ satisfies

$$
\begin{aligned}
& f(\cdot, 0, 0) \in L^1_F(0, T; L^2(\Omega; H)), \\
& \left| f(t, k_1, k_2) - f(t, \tilde{k}_1, \tilde{k}_2) \right|_H \leq C_L \left( |k_1 - \tilde{k}_1|_H + |k_2 - \tilde{k}_2|_H \right),
\end{aligned}
(2.13)
$$
a.e. $(t, \omega) \in [0, T] \times \Omega$, $\forall k_1, k_2, \tilde{k}_1, \tilde{k}_2 \in H$.

Since neither the usual natural filtration condition nor the quasi-left continuity is assumed for the filtration $\mathbb{F}$, we cannot apply the existing results on infinite dimensional BSEEs (e.g. [2, 20, 30, 32]) to obtain the well-posedness of the equation (2.12). In what follows we introduce the concept of the transposition solution to (2.12) and give the well-posedness result. To this end, we consider the following (forward) SEE:

$$
\begin{aligned}
& d\varphi = (A\varphi + v_1) ds + v_2 dW(s) \quad \text{in } (t, T], \\
& \varphi(t) = \eta,
\end{aligned}
(2.14)
$$

where $t \in [0, T]$, $v_1 \in L^1_F(t, T; L^2(\Omega; H))$, $v_2 \in L^2_F(t, T; H)$ and $\eta \in L^2_F(\Omega; H)$ (see [11, Chapter 6] for the well-posedness of (2.14) in the sense of the mild solution).

**Definition 2.1** We call $(p(\cdot), q(\cdot)) \in D_\mathbb{F}(\varnothing; L^2(\Omega; H)) \times L^2_F(0, T; H)$ a transposition solution to (2.12) if for any $t \in [0, T]$, $v_1(\cdot) \in L^1_F(t, T; L^2(\Omega; H))$, $v_2(\cdot) \in L^2_F(t, T; H)$, $\eta \in L^2_F(\Omega; H)$ and the corresponding solution $\varphi \in C_\mathbb{F}([t, T]; L^2(\Omega; H))$ to the equation (2.14), it holds that

$$
\begin{aligned}
& \mathbb{E}\langle \varphi(T), p_T \rangle_H = \mathbb{E}\int_t^T \langle \varphi(s), f(s, p(s), q(s)) \rangle_H ds \\
& = \mathbb{E}\langle \eta, p(t) \rangle_H + \mathbb{E}\int_t^T \langle v_1(s), p(s) \rangle_H ds + \mathbb{E}\int_t^T \langle v_2(s), q(s) \rangle_H ds.
\end{aligned}
$$

**Theorem 2.1** [28, Theorem 2.2] Let $p_T \in L^2_F(\Omega; H)$ and $f(\cdot, \cdot, \cdot)$ satisfy (2.13). Then the equation (2.12) admits a unique transposition solution $(p(\cdot), q(\cdot)) \in D_\mathbb{F}([0, T]; L^2(\Omega; H)) \times L^2_F(0, T; H)$. Furthermore,

$$
\|(p(\cdot), q(\cdot))\|_{D_\mathbb{F}([0, T]; L^2(\Omega; H)) \times L^2_F(0, T; H)} \leq C\left( \|f(\cdot, 0, 0)\|_{L^2_F([0, T]; L^2(\Omega; H))} + \|p_T\|_{L^2_F(\Omega; H)} \right).
$$
We also need the following \( \mathcal{L}(H) \)-valued BSEEs:

\[
\begin{aligned}
dP &= -(A^* + J^*)P dt - P(A + J)dt - K^*PKdt - (K^*Q + QK)dt \\
P(T) &= P_T, \\
f dt = -dF dt + QdW(t)
\end{aligned}
\]  

in \([0, T)\), \hspace{1cm} (2.15)

where \( F \in L^2_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H))) \), \( P_T \in L^2_{\mathbb{F}}(\Omega; \mathcal{L}(H)) \), and \( J, K \in L^2_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H))) \).

To define the solution to (2.15), let us introduce two SEEs:

\[
\begin{aligned}
d\phi_1 &= (A + J)\phi_1 dt + u_1 ds + K\phi_1 dW(s) + v_1 dW(s) \quad \text{in } (t, T], \\
\phi_1(t) &= \xi_1
\end{aligned}
\]  

and

\[
\begin{aligned}
d\phi_2 &= (A + J)\phi_2 dt + u_2 ds + K\phi_2 dW(s) + v_2 dW(s) \quad \text{in } (t, T], \\
\phi_2(t) &= \xi_2.
\end{aligned}
\]

Here \( \xi_1, \xi_2 \in L^2_{\mathbb{F}}(\Omega; H) \) and \( u_1, u_2, v_1, v_2 \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)) \).

Write

\[
D_{\mathbb{F}, w}([0, T]; L^2(\Omega; \mathcal{L}(H)) \triangleq \left\{ P(\cdot) \mid P(\cdot) \in \mathcal{L}_{pd}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))) \right\}
\]

and for every \( t \in [0, T] \) and \( \xi \in L^2_{\mathbb{F}}(\Omega; H) \),

\[
P(\cdot, \cdot)\xi \in D_{\mathbb{F}}([t, T]; L^4(\Omega; H)) \quad \text{and} \quad \|P(\cdot, \cdot)\xi\|_{D_{\mathbb{F}}([t, T]; L^4(\Omega; H))} \leq C\|\xi\|_{L^2_{\mathbb{F}}(\Omega; H)}
\]

and

\[
Q[0, T] \triangleq \left\{ (Q^{(i)}, \tilde{Q}^{(i)}) \mid \text{For any } t \in [0, T], \text{ both } Q^{(i)} \text{ and } \tilde{Q}^{(i)} \text{ are bounded linear operators from } L^4_{\mathbb{F}}(\Omega; H) \times L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)) \times L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)) \text{ to } L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)) \text{ and}\right.
\]

\[
\text{and } Q^{(i)}(0, 0, \cdot)^* = \tilde{Q}^{(i)}(0, 0, \cdot).
\]

**Definition 2.2** We call \( (P(\cdot), (Q^{(i)}, \tilde{Q}^{(i)})) \in D_{\mathbb{F}, w}([0, T]; L^2(\Omega; \mathcal{L}(H))) \times Q[0, T] \) a relaxed transposition solution to (2.15) if for any \( t \in [0, T] \), \( \xi_1, \xi_2 \in L^4_{\mathbb{F}}(\Omega; H) \), \( u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)) \) and \( v_1(\cdot), v_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)) \), it holds that

\[
\begin{aligned}
\mathbb{E}\left\langle P_T\phi_1(T), \phi_2(T) \right\rangle_H &= -\mathbb{E} \int_t^T \left\langle F(s)\phi_1(s), \phi_2(s) \right\rangle_H ds \\
&= \mathbb{E}\left\langle P(t)\xi_1, \xi_2 \right\rangle_H + \mathbb{E} \int_t^T \left\langle P(s)u_1(s), \phi_2(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle P(s)\phi_1(s), u_2(s) \right\rangle_H ds \\
&\quad + \mathbb{E} \int_t^T \left\langle P(s)K(s)\phi_1(s), v_2(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle P(s)v_1(s), K(s)\phi_2(s) + v_2(s) \right\rangle_H ds \\
&\quad + \mathbb{E} \int_t^T \left\langle v_1(s), \tilde{Q}^{(i)}(\xi_2, u_2, v_2) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle Q^{(i)}(\xi_1, u_1, v_1)(s), v_2(s) \right\rangle_H ds.
\end{aligned}
\]  

\hspace{1cm} (2.18)

Here, \( \phi_1(\cdot) \) and \( \phi_2(\cdot) \) solve (2.16) and (2.17), respectively.

\footnote{Throughout this paper, for any operator-valued process (resp. random variable) \( R \), we denote by \( R^* \) its pointwisely dual operator-valued process (resp. random variable), e.g., if \( R \in L^2_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H_1; H_1))) \), then \( R^* \in L^2_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H; H_1))) \), and \( \|R\|_{L^2_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H; H_1)))} = \|R^*\|_{L^2_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H_1; H)))}. \)
**Theorem 2.2** Suppose that $L^2_{T^2}(\Omega)$ is a separable Banach space. Then, the equation (2.15) admits a unique relaxed transposition solution $(P(\cdot), (Q^{(i)}, \hat{Q}^{(i)}) \in D_{F,w}([0, T]; L^2(\Omega; \mathcal{L}(H)))) \times Q[0, T]$. Furthermore,
\[
\|P\|_{D_{F,w}([0, T]; L^2(\Omega; \mathcal{L}(H)))} + \|(Q^{(i)}, \hat{Q}^{(i)})\|_{Q[0, T]}
\leq C(\|F\|_{L^2_{\mathcal{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))} + \|P_T\|_{L^2_{\mathcal{F}}(\Omega; \mathcal{L}(H))}).
\]

**Proof:** See [27, Chapter 6] or [28, Section 3].

Finally, we introduce the concept of the $V$-transposition solution to the equation (2.15). Let $V$ be a Hilbert space such that $H \subset V$ and the embedding operator from $H$ to $V$ is Hilbert-Schmidt. Denote by $V'$ the dual space of $V$ with respect to the pivot space $H$. Then we know that the embedding operator from $V'$ to $H$ is also Hilbert-Schmidt. Let $X$ and $Y$ be two Hilbert spaces. Denote by $\mathcal{L}_2(X; Y)$ ($\mathcal{L}_2(X)$ for $X = Y$) the Hilbert space of all Hilbert-Schmidt operators from $X$ to $Y$.

**Definition 2.3** We call
\[(P(\cdot), (Q^{(i)})) \in D_{w,F}([0, T]; L^2(\Omega; \mathcal{L}(H))) \times L^2_{\mathcal{F}}(0, T; \mathcal{L}_2(H; V))\]
a $V$-transposition solution to (2.15) if for any $t \in [0, T]$, $\xi_1, \xi_2 \in L^2_{\mathcal{F}}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathcal{F}}(t, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^2_{\mathcal{F}}(t, T; L^4(\Omega; V'))$, it holds that
\[
\begin{align*}
\mathbb{E}\langle P_T \phi_1(T), \phi_2(T) \rangle_H - \mathbb{E} & \int_t^T \langle F(s) \phi_1(s), \phi_2(s) \rangle_H ds \\
= \mathbb{E}\langle P(t) \xi_1, \xi_2 \rangle_H + \mathbb{E} & \int_t^T \langle P(s) u_1(s), \phi_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) \phi_1(s), u_2(s) \rangle_H ds \\
& + \mathbb{E} \int_t^T \langle P(s) K(s) \phi_1(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) v_1(s), K(s) \phi_2(s) + v_2(s) \rangle_H ds \\
& + \mathbb{E} \int_t^T \langle v_1(s), Q^*(s) \phi_2(s) \rangle_{V', V} ds + \mathbb{E} \int_t^T \langle Q(s) \phi_1(s), v_2(s) \rangle_{V', V} ds.
\end{align*}
\]

(2.19)

Here, $\phi_1(\cdot)$ and $\phi_2(\cdot)$ solve (2.16) and (2.17), respectively.

Set
\[
\mathcal{L}_{HV} : \Delta \{ B \in \mathcal{L}(H) | \text{The restriction of } B \text{ on } V' \text{ belongs to } \mathcal{L}(V') \}
\]
with the norm
\[
|B|_{\mathcal{L}_{HV}} = |B|_{\mathcal{L}(H)} + |B|_{\mathcal{L}(V')}.
\]

Let us introduce the following condition:

(A4) A generates a $C_0$-semigroup on $V'$ and $J, K \in L^\infty(0, T; \mathcal{L}_{HV})$.

**Lemma 2.2 (29, Theorem 3.3)** Suppose that (A4) hold. Then the equation (2.15) admits a unique $V$-transposition solution $(P, Q)$. Furthermore,
\[
\|(P, Q)\|_{D_{w,F}([0, T]; L^2(\Omega; \mathcal{L}(H)))) \times L^2_{\mathcal{F}}(0, T; \mathcal{L}_2(H; V)) \\
\leq C(\|F\|_{L^2_{\mathcal{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))} + \|P_T\|_{L^2_{\mathcal{F}}(\Omega; \mathcal{L}(H))}).
\]

(2.20)
3 Integral-type second order necessary conditions

In this section, we give some integral-type second order necessary conditions for optimal controls.

Define

$$
\mathbb{H}(t, x, u, k_1, k_2) \triangleq \langle k_1, a(t, x, u) \rangle_H + \langle k_2, b(t, x, u) \rangle_H - g(t, x, u),
$$

\begin{equation}
(t, x, u, k_1, k_2) \in [0, T] \times H \times U \times H \times H.
\end{equation}

Let \((\bar{x}(\cdot), \bar{u}(\cdot))\) be an optimal pair, \((p(\cdot), q(\cdot))\) be the transposition solution of the equation \((2.12)\), where \(p_T\) and \(f(\cdot, \cdot, \cdot)\) are given by

$$
\begin{align*}
\left\{ 
\begin{array}{l}
  p_T = -h_x(\bar{x}(T)), \\
  f(t, k_1, k_2) = -a_x(t, \bar{x}(t), \bar{u}(t))^* k_1 - b_x(t, \bar{x}(t), \bar{u}(t))^* k_2 + g_x(t, \bar{x}(t), \bar{u}(t)).
\end{array}
\right.
\end{align*}
$$

Put

$$
\begin{align*}
&\mathbb{H}_p(t) = \mathbb{H}_p(t, \bar{x}(t), \bar{u}(t), p(t), q(t)), \\
&\mathbb{H}_u(t) = \mathbb{H}_u(t, \bar{x}(t), \bar{u}(t), p(t), q(t)), \\
&\mathbb{H}_{xx}(t) = \mathbb{H}_{xx}(t, \bar{x}(t), \bar{u}(t), p(t), q(t)), \\
&\mathbb{H}_{ux}(t) = \mathbb{H}_{ux}(t, \bar{x}(t), \bar{u}(t), p(t), q(t)), \\
&\mathbb{H}_{uu}(t) = \mathbb{H}_{uu}(t, \bar{x}(t), \bar{u}(t), p(t), q(t)).
\end{align*}
$$

Let \((P(\cdot), (Q^c(\cdot), \hat{Q}^c(\cdot)))\) be the relaxed transposition solution to the equation \((2.15)\) in which \(P_T, J(\cdot), K(\cdot)\) and \(F(\cdot)\) are given by

$$
\begin{align*}
\left\{ 
\begin{array}{l}
  P_T = -h_{xx}(\bar{x}(T)), \\
  J(t) = a_x(t, \bar{x}(t), \bar{u}(t)), \\
  K(t) = b_x(t, \bar{x}(t), \bar{u}(t)), \\
  F(t) = -\mathbb{H}_{xx}(t).
\end{array}
\right.
\end{align*}
$$

Our main result in this section is as follows.

**Theorem 3.1** Assume that \(x_0 \in L^2_\mathcal{F}_0(\Omega; H)\) and \(L^2_\mathcal{F}_T(\Omega)\) is separable. Let \((A1)-(A3)\) hold, and let \(u(\cdot) \in U^4[0, T]\) be an optimal control and \(\bar{x}(\cdot)\) be the corresponding optimal state. Then, for any \(u(\cdot) \in U^4[0, T]\) such that

$$
\mathbb{E} \int_0^T \langle \mathbb{H}_u(t), u(t) - \bar{u}(t) \rangle_{H_1} dt = 0,
$$

the following second order necessary condition holds:

$$
\begin{align*}
\mathbb{E} & \int_0^T \left[ \langle \mathbb{H}_{uu}(t)(u(t) - \bar{u}(t)), u(t) - \bar{u}(t) \rangle_{H_1} + \langle b_2(t)^* P(t) b_2(t)(u(t) - \bar{u}(t)), u(t) - \bar{u}(t) \rangle_{H_1} \right] dt \\
&+ 2\mathbb{E} \int_0^T \langle \mathbb{H}_{xx}(t) + a_2(t)^* P(t) + b_2(t)^* P(t) b_1(t), y(t), u(t) - \bar{u}(t) \rangle_{H_1} dt \\
&+ \mathbb{E} \int_0^T \langle (\hat{Q}^c(0) + Q^c(0))^2, a_2(t)(u(t) - \bar{u}(t)) + b_2(t)(u(t) - \bar{u}(t)) \rangle_{H_1} dt \leq 0,
\end{align*}
$$

where \(y(\cdot)\) is the solution to the equation \((2.4)\) corresponding to \(\delta u(\cdot) = u(\cdot) - \bar{u}(\cdot)\) and \((P(\cdot), (Q^c(\cdot), \hat{Q}^c(\cdot)))\) is the relaxed transposition solution to the equation \((2.15)\) with the coefficients given by \((3.5)\).
Proof: Let us divide the proof into four steps.

**Step 1.** In this step, we introduce some notations. Obviously, \( \delta u(\cdot) = u(\cdot) - \bar{u}(\cdot) \in L^4_T(0, T; H_1) \). Since \( U \) is convex, we see that

\[
u^\varepsilon(\cdot) = \bar{u}(\cdot) + \varepsilon \delta u(\cdot) = (1 - \varepsilon) \bar{u}(\cdot) + \varepsilon u(\cdot) \in U^\varepsilon[0, T] \subset U^2[0, T], \quad \forall \varepsilon \in [0, 1].
\]

Denote by \( x^\varepsilon(\cdot) \) the state process of (1.1) corresponding to the control \( u^\varepsilon(\cdot) \). Let \( \delta x^\varepsilon(\cdot) = x^\varepsilon(\cdot) - \bar{x}(\cdot) \) and for \( \psi = a, b, g, \) put

\[
\begin{align*}
\tilde{\psi}^\varepsilon_{11}(t) &\triangleq \int_0^1 (1 - \theta)\psi_{xx}(t, \bar{x}(t) + \theta\delta x^\varepsilon(t), \bar{u}(t) + \theta\varepsilon\delta u(t))d\theta, \\
\tilde{\psi}^\varepsilon_{12}(t) &\triangleq \int_0^1 (1 - \theta)\psi_{xu}(t, \bar{x}(t) + \theta\delta x^\varepsilon(t), \bar{u}(t) + \theta\varepsilon\delta u(t))d\theta, \\
\tilde{\psi}^\varepsilon_{22}(t) &\triangleq \int_0^1 (1 - \theta)\psi_{uu}(t, \bar{x}(t) + \theta\delta x^\varepsilon(t), \bar{u}(t) + \theta\varepsilon\delta u(t))d\theta.
\end{align*}
\]

Also, we define

\[
\tilde{h}^\varepsilon_{xx}(T) \triangleq \int_0^1 (1 - \theta)h_{xx}(\bar{x}(T) + \theta\delta x^\varepsilon(T))d\theta.
\]

**Step 2.** It follows from Lemma 2.1 that for any \( \beta \geq 2 \),

\[
\begin{align*}
&\left\{\begin{array}{l}
\|\delta x^\varepsilon\|_{C^\beta([0, T]; L^2(\Omega; H))} \leq C\varepsilon\|\delta u\|_{L^2_T(\Omega; L^2(0, T; H_1))}, \\
\|\delta x^\varepsilon - \varepsilon y\|_{C^\beta([0, T]; L^2(\Omega; H))} \leq C\varepsilon^2\|\delta u\|_{L^2_T(\Omega; L^2(0, T; H_1))}^2.
\end{array}\right.
\end{align*}
\]

We claim that there exists a subsequence \( \{\varepsilon_n\}_n^\infty \) such that

\[
\left\|\delta x^{\varepsilon_n} - \varepsilon_n y - \frac{\varepsilon_n^2}{2}\right\|_{C^\beta([0, T]; L^2(\Omega; H))} = o(\varepsilon_n^2).
\]

Obviously, \( \delta x^\varepsilon \) solves the following SEE:

\[
\begin{align*}
d\delta x^\varepsilon &\left[ A\delta x^\varepsilon + a_1(t)\delta x^\varepsilon + \varepsilon a_2(t)\delta u + \bar{a}_{11}(t)(\delta x^\varepsilon, \delta x^\varepsilon) \\
&\quad + 2\varepsilon\bar{a}_{12}(t)(\delta x^\varepsilon, \delta u) + \varepsilon^2\bar{a}_{22}(t)(\delta u, \delta u) \right] dt \\
&\quad + \left[ b_1(t)\delta x^\varepsilon + \varepsilon b_2(t)\delta u + \bar{b}_{11}(t)(\delta x^\varepsilon, \delta x^\varepsilon) \\
&\quad + 2\varepsilon\bar{b}_{12}(t)(\delta x^\varepsilon, \delta u) + \varepsilon^2\bar{b}_{22}(t)(\delta u, \delta u) \right] dW(t) \quad \text{in } (0, T], \\
\delta x^\varepsilon(0) &\equiv 0.
\end{align*}
\]
Let \( r_2^\varepsilon(\cdot) = \varepsilon^{-2} \left( \delta x^\varepsilon(\cdot) - \varepsilon y(\cdot) - \frac{\varepsilon^2}{2} z(\cdot) \right) \). Then \( r_2^\varepsilon(\cdot) \) fulfills

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dr_2^\varepsilon}{dt} = \left\{ A r_2^\varepsilon + a_1(t) r_2^\varepsilon + \left[ a_{11}(t) \left( \frac{\delta x^\varepsilon}{\varepsilon}, \frac{\delta x^\varepsilon}{\varepsilon} \right) - \frac{1}{2} a_{11}(t) (y, y) \right] \\
\quad + \left[ 2 \bar{a}_{12}(t) \left( \frac{\delta x^\varepsilon}{\varepsilon}, \delta u(t) - a_{12}(t)(y, \delta u(t)) \right] + \left( \bar{a}_{22}(t) - \frac{1}{2} a_{22}(t) \right) (\delta u, \delta u) \right) \bigg\} dt \\
\quad + \left\{ b_1(t) r_2^\varepsilon + \left[ \bar{b}_{11}(t) \left( \frac{\delta x^\varepsilon}{\varepsilon}, \frac{\delta x^\varepsilon}{\varepsilon} \right) - \frac{1}{2} b_{11}(t)(y, y) \right] \\
\quad + \left[ 2 \bar{b}_{12}(t) \left( \frac{\delta x^\varepsilon}{\varepsilon}, \delta u(t) - b_{12}(t)(y, \delta u(t)) \right] + \left( \bar{b}_{22}(t) - \frac{1}{2} b_{22}(t) \right) (\delta u, \delta u) \right) \bigg\} dW(t) \quad \text{in} \, (0, T], \end{array} \right.
\end{align*}
\]

\( r_2^\varepsilon(0) = 0. \) (3.9)

Put

\[
\Psi_{1, \varepsilon}(t) = \left[ \bar{a}_{11}(t) \left( \frac{\delta x^\varepsilon(t)}{\varepsilon}, \frac{\delta x^\varepsilon(t)}{\varepsilon} \right) - \frac{1}{2} a_{11}(t)(y(t), y(t)) \right] \\
\quad + \left[ 2 \bar{a}_{12}(t) \left( \frac{\delta x^\varepsilon(t)}{\varepsilon}, \delta u(t) \right) - a_{12}(t)(y(t), \delta u(t)) \right] + \left( \bar{a}_{22}(t) - \frac{1}{2} a_{22}(t) \right) (\delta u(t), \delta u(t))
\]

and

\[
\Psi_{2, \varepsilon}(t) = \left[ \bar{b}_{11}(t) \left( \frac{\delta x^\varepsilon(t)}{\varepsilon}, \frac{\delta x^\varepsilon(t)}{\varepsilon} \right) - \frac{1}{2} b_{11}(t)(y(t), y(t)) \right] \\
\quad + \left[ 2 \bar{b}_{12}(t) \left( \frac{\delta x^\varepsilon(t)}{\varepsilon}, \delta u(t) \right) - b_{12}(t)(y(t), \delta u(t)) \right] + \left( \bar{b}_{22}(t) - \frac{1}{2} b_{22}(t) \right) (\delta u(t), \delta u(t)).
\]

We have that

\[
\mathbb{E} \left| r_2^\varepsilon(t) \right|^2_H = \mathbb{E} \left[ \int_0^t e^{A(t-s)} a_1(s) r_2^\varepsilon(s) ds + \int_0^t e^{A(t-s)} b_1(s) r_2^\varepsilon(s) dW(s) \right. \\
\left. + \int_0^t e^{A(t-s)} \Psi_{1, \varepsilon}(s) ds + \int_0^t e^{A(t-s)} \Psi_{2, \varepsilon}(s) dW(s) \right]^2_H 
\leq C \left( \mathbb{E} \int_0^t \left| r_2^\varepsilon(s) \right|^2_H ds + \mathbb{E} \int_0^t \left| \Psi_{1, \varepsilon}(s) \right|^2_H ds + \mathbb{E} \int_0^t \left| \Psi_{2, \varepsilon}(s) \right|^2_H ds \right). \quad (3.10)
\]

By (3.6), there exists a subsequence \( \{ \varepsilon_n \}_{n=1}^{\infty} \) such that \( x^{\varepsilon_n}(\cdot) \to \bar{x}(\cdot) \) (in \( H \)) a.e. in \( \Omega \times [0, T] \), as \( n \to \infty \). Then, by (3.4), Assumption (A3) and Lebesgue’s dominated convergence theorem, we deduce that

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T |\Psi_{1, \varepsilon_n}(t)|^2_H dt \\
\leq \lim_{n \to \infty} \mathbb{E} \int_0^T \left[ \left| \bar{a}_{11}(t) \left( \frac{\delta x^{\varepsilon_n}(t)}{\varepsilon_n}, \frac{\delta x^{\varepsilon_n}(t)}{\varepsilon_n} \right) - \frac{1}{2} a_{11}(t)(y(t), y(t)) \right] \\
\quad + \left[ 2 \bar{a}_{12}(t) \left( \frac{\delta x^{\varepsilon_n}(t)}{\varepsilon_n}, \delta u(t) \right) - a_{12}(t)(y(t), \delta u(t)) \right] \\
+ \left( \bar{a}_{22}(t) - \frac{1}{2} a_{22}(t) \right) (\delta u(t), \delta u(t)) \right|^2_H dt
\]
Similarly, combining (3.10), (3.11) with (3.12) and using Gronwall’s inequality, we obtain (3.7).

\[ \lim_{n \to \infty} \mathbb{E} \int_0^T \left| \tilde{a}_{11}^{\varepsilon_n}(t) \left( \frac{\delta x_{1}^{\varepsilon_n}(t)}{\varepsilon_n}, \frac{\delta x_{2}^{\varepsilon_n}(t)}{\varepsilon_n} \right) - \tilde{a}_{11}(t)(y(t), y(t)) \right|^2_H \\
+ \left\| \tilde{a}_{11}^{\varepsilon_n}(t) - \frac{1}{2} a_{11}(t) \right\|^2_{L(H \times H, H)} \cdot |y(t)|^2_H \\
+ 2 \left\| \tilde{a}_{12}^{\varepsilon_n}(t) \left( \frac{\delta x_{1}^{n}(t)}{\varepsilon_n}, \delta u(t) \right) - \tilde{a}_{12}(t)(y(t), \delta u(t)) \right\|^2_H \\
+ \left\| 2 \tilde{a}_{12}^{\varepsilon_n}(t) - a_{12}(t) \right\|^2_{L(H \times H_1, H)} \cdot |y(t)|^2_H \cdot |\delta u(t)|^2_{H_1} \\
+ \left\| \tilde{a}_{22}^{\varepsilon_n}(t) - \frac{1}{2} a_{22}(t) \right\|^2_{L(H_1 \times H_1, H)} \cdot |\delta u(t)|^4_{H_1} dt = 0. \]

\[ \lim_{n \to \infty} \mathbb{E} \int_0^t |\Psi_{2, \varepsilon_n}(t)|^2_{H_2} dt = 0. \] (3.12)

Combining (3.10), (3.11) with (3.12) and using Gronwall’s inequality, we obtain (3.7).

**Step 3.** By Taylor’s formula, we see that

\[ g(t, x^\varepsilon(t), u^\varepsilon(t)) - g(t, \bar{x}(t), \bar{u}(t)) = \langle g_1(t), \delta x^\varepsilon(t) \rangle_H + \varepsilon \langle g_2(t), \delta u(t) \rangle_{H_1} + \langle g_3^\varepsilon(t), \delta x^\varepsilon(t), \delta u(t) \rangle_H \\
+ 2\varepsilon \langle \bar{g}_1(t) \delta x^\varepsilon(t), \delta u(t) \rangle_{H_1} + \varepsilon^2 \langle \bar{g}_2^\varepsilon(t) \delta u(t), \delta u(t) \rangle_{H_1} \] (3.13)

and

\[ h(x^\varepsilon(T)) - h(\bar{x}(T)) = \langle h_x(\bar{x}(T)), \delta x^\varepsilon(T) \rangle_H + \langle \bar{h}_x^\varepsilon(T) \delta x^\varepsilon(T), \delta x^\varepsilon(T) \rangle_H. \] (3.14)

Using a similar argument in the proof of (3.7), we can obtain that for the subsequence \( \{\varepsilon_n\}_{n=1}^\infty \) such that \( x^{\varepsilon_n}(\cdot) \to \bar{x}(\cdot) \) (in \( H \)) a.e. in \( [0, T] \times \Omega \), as \( n \to \infty \),

\[ \lim_{n \to \infty} \frac{1}{\varepsilon_n^2} \mathbb{E} \int_0^T \left( \langle \tilde{g}_{11}^{\varepsilon_n}(t) \delta x^{\varepsilon_n}(t), \delta x^{\varepsilon_n}(t) \rangle_H - \frac{\varepsilon_n^2}{2} \langle g_{11}(t) y(t), y(t) \rangle_H \right) dt = 0, \]

\[ \lim_{n \to \infty} \frac{1}{\varepsilon_n^2} \mathbb{E} \int_0^T \left( 2 \langle \tilde{g}_{12}^{\varepsilon_n}(t) \delta x^{\varepsilon_n}(t), \varepsilon_n \delta u(t) \rangle_{H_1} - \varepsilon_n^2 \langle g_{12}(t) y(t), \delta u(t) \rangle_{H_1} \right) dt = 0, \]

\[ \lim_{n \to \infty} \mathbb{E} \int_0^T \left( \langle \tilde{g}_{22}^{\varepsilon_n}(t) \delta u(t), \delta u(t) \rangle_{H_1} - \frac{1}{2} \langle g_{22}(t) \delta u(t), \delta u(t) \rangle_{H_1} \right) dt = 0 \]

and

\[ \lim_{n \to \infty} \frac{1}{\varepsilon_n^2} \mathbb{E} \left( \langle \tilde{h}_{xx}^{\varepsilon_n}(\bar{x}(T)) \delta x^{\varepsilon_n}(T), \delta x^{\varepsilon_n}(T) \rangle_H - \frac{\varepsilon_n^2}{2} \langle h_{xx}(\bar{x}(T)) y(T), y(T) \rangle_H \right) = 0. \]
These, together with (3.7), imply that

\[ \mathcal{J}(u^n) - \mathcal{J}(\bar{u}) \]

\[ = \mathbb{E} \int_0^T \left[ \varepsilon_n \langle g_1(t), y(t) \rangle_H + \frac{\varepsilon_n^2}{2} \langle g_1(t), z(t) \rangle_H + \varepsilon_n \langle g_2(t), \delta u(t) \rangle_{H_1} ight. 
+ \frac{\varepsilon_n^2}{2} \left( \langle g_{11}(t)y(t), y(t) \rangle_H + 2 \langle g_{12}(t)y(t), \delta u(t) \rangle_{H_1} + \langle g_{22}(t)\delta u(t), \delta u(t) \rangle_{H_1} \right) \right] dt 
+ \mathbb{E} \left( \varepsilon_n \langle h_x(\bar{x}(T)), y(T) \rangle_H + \frac{\varepsilon_n^2}{2} \langle h_x(\bar{x}(T)), z(T) \rangle_H + \frac{\varepsilon_n^2}{2} \langle h_{xx}(\bar{x}(T))y(T), y(T) \rangle_H \right) + o(\varepsilon_n^2). \]  

(3.15)

**Step 4.** By the definition of the transposition solution to (2.12), we have that

\[ \mathbb{E} \langle h_x(\bar{x}(T)), y(T) \rangle_H \]

\[ = -\mathbb{E} \int_0^T \left( \langle p(t), a_2(t)\delta u(t) \rangle_H + \langle q(t), b_2(t)\delta u(t) \rangle_H + \langle g_1(t), y(t) \rangle_H \right) dt \]  

(3.16)

and

\[ \mathbb{E} \langle h_x(\bar{x}(T)), z(T) \rangle_H \]

\[ = -\mathbb{E} \int_0^T \left( \langle p(t), a_{11}(t)(y(t), y(t)) \rangle_H + 2 \langle p(t), a_{12}(t)(y(t), \delta u(t)) \rangle_H 
+ \langle p(t), a_{22}(t)(\delta u(t), \delta u(t)) \rangle_H + \langle q(t), b_{11}(t)(y(t), y(t)) \rangle_H 
+ 2 \langle q(t), b_{12}(t)(y(t), \delta u(t)) \rangle_H + \langle q(t), b_{22}(t)(\delta u(t), \delta u(t)) \rangle_H + \langle g_1(t), z(t) \rangle_H \right) dt. \]  

(3.17)

In addition, by the definition of the relaxed transposition solution to (2.15), we get that

\[ \mathbb{E} \langle h_{xx}(\bar{x}(T))y(T), y(T) \rangle_H \]

\[ = -\mathbb{E} \int_0^T \left( \langle P(t)y(t), a_2(t)\delta u(t) \rangle_H + \langle y(t), P(t)a_2(t)\delta u(t) \rangle_H 
+ \langle P(t)b_1(t)y(t), b_2(t)\delta u(t) \rangle_H + \langle b_1(t)y(t), P(t)b_2(t)\delta u(t) \rangle_H 
+ \langle P(t)b_2(t)\delta u(t), b_2(t)\delta u(t) \rangle_H + \langle \widehat{Q}^{(0)}(0, a_2(t)\delta u, b_2(t)\delta u)(t), b_2(t)\delta u(t) \rangle_H 
+ \langle Q^{(0)}(0, a_2(t)\delta u, b_2(t)\delta u)(t), b_2(t)\delta u(t) \rangle_H \right) dt. \]  

(3.18)

Combining (3.15) - (3.18) with (3.4), we obtain that

\[ 0 \leq \frac{\mathcal{J}(u^n) - \mathcal{J}(\bar{u})}{\varepsilon_n^2} \]

\[ = -\mathbb{E} \int_0^T \left[ \frac{1}{\varepsilon_n} \left( \langle p(t), a_2(t)\delta u(t) \rangle_H + \langle q(t), b_2(t)\delta u(t) \rangle_H - \langle g_2(t), \delta u(t) \rangle_{H_1} \right) 
+ \frac{1}{2} \left( \langle p(t), a_{22}(t)(\delta u(t), \delta u(t)) \rangle_H + \langle q(t), b_{22}(t)(\delta u(t), \delta u(t)) \rangle_H - \langle g_{22}(t)\delta u(t), \delta u(t) \rangle_{H_1} \right) 
+ \frac{1}{2} \langle P(t)b_2(t)\delta u(t), b_2(t)\delta u(t) \rangle_H + \left( -\langle g_{12}(t)y(t), \delta u(t) \rangle_{H_1} + \langle p(t), a_{12}(t)(y, \delta u) \rangle_H \right) \right] dt. \]
Then, letting $n \to \infty$, we finally get (3.5). □

According to Lemma 2.2, to obtain the well-posedness of (2.15) in the sense of $V$-transposition solution, we only need the following assumption:

(A5) A generates a $C_0$-semigroup on $V'$, $a_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)), b_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)), b_u(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)) \in L^\infty_\mathcal{F}(0, T; \mathcal{L}_{HV'})$ and $a_u(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)), b_u(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)) \in L^\infty_\mathcal{F}(0, T; \mathcal{L}(H_1; V'))$.

Let $(P, Q)$ be the $V$-transposition solution to BSEE (2.15) in which $P_T, J(\cdot), K(\cdot)$ and $F(\cdot)$ are given by (3.3). Put

\[ S(t) = \mathbb{H}_{xu}(t) + a_2(t)^*P(t) + b_2(t)^*Q(t) + b_2(t)^*P(t)b_1(t). \]  

(3.19)

The following result holds immediately from Theorem 3.1.

**Corollary 3.1** Let the assumptions in Theorem 3.1 and (A5) hold. If $\bar{u}(\cdot) \in \mathcal{U}^4[0, T]$, then, for any $u(\cdot) \in \mathcal{U}^4[0, T]$ such that

\[ \mathbb{E} \int_0^T \left\langle \mathbb{H}_u(t), u(t) - \bar{u}(t) \right\rangle_{H_1} dt = 0, \]

the following second order condition holds:

\[ \mathbb{E} \int_0^T \left[ \left\langle \left( \mathbb{H}_{uu}(t) + b_2(t)^*P(t)b_2(t) \right) \left( u(t) - \bar{u}(t) \right), u(t) - \bar{u}(t) \right\rangle_{H_1} + 2 \left\langle S(t)y(t), u(t) - \bar{u}(t) \right\rangle_{H_1} \right] dt \leq 0. \]  

(3.20)

### 4 Pointwise second order necessary conditions

In this section, we derive the pointwise second order necessary condition for optimal controls by the integral-type condition (3.20). We assume that $\mathbb{F}$ is the natural filtration generated
by $W(\cdot)$. To begin with, let us introduce some concepts and technical results which will be used in the rest of this section.

First, we give the concept of the singular optimal control as follow:

**Definition 4.1** We call $\tilde{u}(\cdot) \in \mathcal{U}^2[0,T]$ a singular optimal control in the classical sense if it is an optimal control and satisfies

\[
\begin{cases}
\mathcal{H}_u(t, \tilde{x}(t), \tilde{u}(t), p(t), q(t)) = 0, \quad \text{a.e. } (t, \omega) \in [0,T] \times \Omega,

( (\mathcal{H}_u(t, \tilde{x}(t), \tilde{u}(t), p(t), q(t)) + b_u(t, \tilde{x}(t), \tilde{u}(t))^* P(t) b_u(t, \tilde{x}(t), \tilde{u}(t))) (v - \tilde{u}(t)),

v - \tilde{u}(t) \rangle_{H_1} = 0, \quad \forall \ v \in U, \ a.e. \ (t, \omega) \in [0,T] \times \Omega.
\end{cases}
\tag{4.1}
\]

Next, we recall some concepts and results from Malliavin calculus (see [34] for a detailed introduction on this topic).

Let $\tilde{H}$ be a separable Hilbert space. We introduce the Sobolev space $\mathbb{D}^{1,2}(\tilde{H})$ of $\tilde{H}$-valued random variables in the following way.

Denote by $C_b^\infty(\mathbb{R}^m)$ the set of $C^\infty$-smooth functions with bounded partial derivatives. For any $h \in L^2(0,T)$, write $W(h) = \int_0^T h(t) dW(t)$. If $F$ is a smooth $\tilde{H}$-valued random variable of the form

\[ F = \sum_{j=1}^n f_j(W(h_{j_1}), \ldots, W(h_{j_m})) \kappa_j \tag{4.2} \]

where $h_{j_k} \in L^2(0,T)$, $\kappa_j \in \tilde{H}$ and $f_j \in C_b^\infty(\mathbb{R}^{j_m})$, $n, j_m \in \mathbb{N}$, then the derivative of $F$ is defined as

\[ \mathcal{D}F = \sum_{j=1}^n \sum_{k=1}^{j_m} h_{j_k} \frac{\partial f_j}{\partial x_{j_k}}(W(h_{j_1}), \ldots, W(h_{j_m})) \kappa_j. \]

Clearly, $\mathcal{D}F$ is a smooth random variable with values in $L^2(0,T; \tilde{H})$. Denote by $\mathbb{D}^{1,2}(\tilde{H})$ the completion of the class of smooth $\tilde{H}$-valued random variables with respect to the norm

\[ \|F\|_{\mathbb{D}^{1,2}} = \left( \mathbb{E}\|F\|^2_{\tilde{H}} + \mathbb{E} \int_0^T |\mathcal{D}_t F|^2_{\tilde{H}} dt \right)^{\frac{1}{2}}. \]

In particular, given two separable Hilbert spaces $H_1$ and $H_2$ we can consider $\tilde{H} = \mathcal{L}_2(H_1; H_2)$, and in this case, for any $F$ in the space $\mathbb{D}^{1,2}(\mathcal{L}_2(H_1; H_2))$, we have that $\mathcal{D}F \in L^2(\Omega; L^2(0,T; \mathcal{L}_2(H_1; H_2)))$.

When $\zeta \in \mathbb{D}^{1,2}(\tilde{H})$, the following Clark-Ocone representation formula holds:

\[ \zeta = \mathbb{E} \zeta + \int_0^T \mathbb{E}(\mathcal{D}_s \zeta | \mathcal{F}_s) dW(s). \tag{4.3} \]

Furthermore, if $\zeta$ is $\mathcal{F}_t$-measurable, then $\mathcal{D}_s \zeta = 0$ for any $s \in (t,T]$.

Write $\mathbb{L}^{1,2}(\tilde{H})$ for the space of processes $\varphi \in L^2([0,T] \times \Omega; \tilde{H})$ such that

(i) For a.e. $t \in [0,T]$, $\varphi(t, \cdot) \in \mathbb{D}^{1,2}(\tilde{H})$;

(ii) The function $(s,t,\omega) \mapsto \mathcal{D}_s \varphi(t,\omega)$ ($(s,t,\omega) \in [0,T] \times [0,T] \times \Omega$) admits a measurable version; and

18
(iii) \[ |||\varphi|||_{1,2} \Delta = \left( \mathbb{E} \int_0^T |\varphi(t)|^2_H dt + \mathbb{E} \int_0^T \int_0^T |D_s\varphi(t)|^2_H dsdt \right)^{\frac{1}{2}} < +\infty. \]

Denote by \( \mathbb{L}^{1,2}_g(\widetilde{H}) \) the set of all adapted processes in \( \mathbb{L}^{1,2}(\widetilde{H}) \). In addition, put

\[ \mathbb{L}^{1,2}_{2+}(\widetilde{H}) \triangleq \left\{ \varphi(\cdot) \in \mathbb{L}^{1,2}(\widetilde{H}) \mid \exists D^+\varphi(\cdot) \in L^2([0,T] \times \Omega; \widetilde{H}) \text{ such that} \right. \]

\[ f_\varepsilon(s) \triangleq \sup_{s < t < (s+\varepsilon) \land T} \mathbb{E}|D_s\varphi(t) - D^+\varphi(s)|^2_H < \infty, \ a.e. \ s \in [0,T], \]

\[ f_\varepsilon(\cdot) \text{ is measurable on } [0,T] \text{ for any } \varepsilon > 0, \text{ and } \lim_{\varepsilon \to 0^+} \int_0^T f_\varepsilon(s) ds = 0 \}

and

\[ \mathbb{L}^{1,2}_{2-}(\widetilde{H}) \triangleq \left\{ \varphi(\cdot) \in \mathbb{L}^{1,2}(\widetilde{H}) \mid \exists D^-\varphi(\cdot) \in L^2([0,T] \times \Omega; \widetilde{H}) \text{ such that} \right. \]

\[ g_\varepsilon(s) \triangleq \sup_{(s-\varepsilon) \lor 0 < t < s} \mathbb{E}|D_s\varphi(t) - D^-\varphi(s)|^2_H < \infty, \ a.e. \ s \in [0,T], \]

\[ g_\varepsilon(\cdot) \text{ is measurable on } [0,T] \text{ for any } \varepsilon > 0, \text{ and } \lim_{\varepsilon \to 0^+} \int_0^T g_\varepsilon(s) ds = 0 \}

Set

\[ \mathbb{L}^{1,2}_2(\widetilde{H}) = \mathbb{L}^{1,2}_{2+}(\widetilde{H}) \cap \mathbb{L}^{1,2}_{2-}(\widetilde{H}). \]

For any \( \varphi(\cdot) \in \mathbb{L}^{1,2}_2(\widetilde{H}) \), denote \( \nabla \varphi(\cdot) = D^+\varphi(\cdot) + D^-\varphi(\cdot) \).

When \( \varphi \) is adapted, \( D_\varphi(t) = 0 \) for any \( t < s \). In this case, \( D^-\varphi(\cdot) = 0 \), and \( \nabla \varphi(\cdot) = D^+\varphi(\cdot) \). Denote by \( \mathbb{L}^{1,2}_{2,\mathbb{F}}(\widetilde{H}) \) the set of all adapted processes in \( \mathbb{L}^{1,2}_2(\widetilde{H}) \).

Roughly speaking, an element \( \varphi \in \mathbb{L}^{1,2}_2(\widetilde{H}) \) is a stochastic process whose Malliavin derivative has suitable continuity on some neighbourhood of \{ \( (t,t) \mid t \in [0,T] \} \}. Examples of such process can be found in [34]. Especially, if \( (s,t) \mapsto D_\varphi(t) \) is continuous from \{ \( (s,t) \mid |s-t| < \delta, \ s,t \in [0,T] \} \) (for some \( \delta > 0 \)) to \( L^2_{\mathbb{F}}(\Omega; \widetilde{H}) \), then \( \varphi \in \mathbb{L}^{1,2}_2(\widetilde{H}) \) and, \( D^+\varphi(t) = D^-\varphi(t) = D_\varphi(t) \).

We have the following result.

**Lemma 4.1** Let \( \varphi(\cdot) \in \mathbb{L}^{1,2}_{2,\mathbb{F}}(\widetilde{H}) \). Then, there exists a sequence \{ \( \theta_n \}_{n=1}^{\infty} \) of positive numbers such that \( \theta_n \to 0^+ \) as \( n \to \infty \) and

\[ \lim_{n \to \infty} \frac{1}{\theta_n^2} \int_0^{T + \theta_n} \int_0^t \mathbb{E}|D_s\varphi(t) - \nabla \varphi(s)|^2_H dsdt = 0, \ a.e. \ \tau \in [0,T]. \quad (4.4) \]

**Proof:** For any \( \tau, \theta \in [0,\infty) \), we take the convention that

\[ \sup_{t \in [\tau, \tau + \theta] \cap [0,T]} \mathbb{E}|D_\varphi(t) - \nabla \varphi(\tau)|^2_H = 0 \]

whenever \[ [\tau, \tau + \theta] \cap [0,T] = \emptyset \]. It follows from the definition of \( \mathbb{L}^{1,2}_{2,\mathbb{F}}(\widetilde{H}) \) that

\[ \lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_0^T \int_\tau^{\tau + \theta} \int_\tau^t \mathbb{E}|D_s\varphi(t) - \nabla \varphi(s)|^2_H dsdt\, d\tau = 0 \]
which implies (4.4).

The following results will be frequently used in the proof of the main results in this section.

**Lemma 4.2** Let $\phi(\cdot), \psi(\cdot) \in L^2_F(0, T; H)$. Then, for a.e. $\tau \in [0, T)$,

\[
\lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_{\tau}^{T} \int_{s}^{\tau + \theta} \mathbb{E} \left| D_s \phi(t) - \nabla \phi(s) \right|^2_H \, ds \, dt \delta = \frac{1}{2} \mathbb{E} \langle \phi(\tau), \psi(\tau) \rangle_H,
\]

(4.5)

\[
\lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_{\tau}^{T} \int_{s}^{\tau + \theta} \mathbb{E} \left| D_s \psi(t) - \nabla \psi(s) \right|^2_H \, ds \, dt \delta = \frac{1}{2} \mathbb{E} \langle \phi(\tau), \psi(\tau) \rangle_H.
\]

(4.6)

**Proof:** The equality (4.5) is a corollary of the Lebesgue differentiation theorem. Now, we prove (4.6). For any $\tau \in [0, T)$, let $\theta > 0$ and $\tau + \theta < T$. It follows from the Lebesgue differentiation theorem that

\[
\lim_{\theta \to 0^+} \frac{1}{\theta} \int_{\tau}^{\tau + \theta} \mathbb{E} \left| \phi(t) - \phi(\tau) \right|^2_H \, dt = 0, \quad \text{a.e. } \tau \in [0, T),
\]

and

\[
\lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_{\tau}^{T} \int_{s}^{t} \mathbb{E} \left| e^{A(t-s)} \psi(s) \right|^2_H \, ds \, dt = \frac{1}{2} \mathbb{E} \left| \psi(\tau) \right|^2_H, \quad \text{a.e. } \tau \in [0, T).
\]
Therefore,
\[
\begin{align*}
\lim_{\theta \to 0^+} \frac{1}{\theta^2} & \mathbb{E} \int_\tau^{\tau + \theta} \left( \phi(t) - \phi(\tau), \int_\tau^t e^{A(t-s)} \psi(s) ds \right)_H dt \\
& \leq \lim_{\theta \to 0^+} \frac{1}{\theta^2} \left[ \int_\tau^{\tau + \theta} \mathbb{E} \left| \phi(t) - \phi(\tau) \right|^2_H dt \right]^{\frac{1}{2}} \left[ \int_\tau^{\tau + \theta} \mathbb{E} \left| e^{A(t-s)} \psi(s) \right|^2_H ds dt \right]^{\frac{1}{2}} \\
& \leq \lim_{\theta \to 0^+} \frac{1}{\theta^2} \left[ \int_\tau^{\tau + \theta} \mathbb{E} \left| \phi(t) - \phi(\tau) \right|^2_H dt \right]^{\frac{1}{2}} \left[ \int_\tau^{\tau + \theta} \mathbb{E} \left| e^{A(t-s)} \psi(s) \right|^2_H ds dt \right]^{\frac{1}{2}} \\
& = 0, \quad \text{a.e. } \tau \in [0, T).
\end{align*}
\]

From (4.5) and (4.7), we obtain (4.6). This completes the proof of Lemma 4.2.

Remark 4.2
We can replace (A6) by the following assumption:
\[
\tilde{u}(\cdot) \in L^\infty([0, T]) \cap L^\infty([0, T] \times \Omega; \mathcal{L}(H_1; H)),
\]
and
\[
\mathcal{D} \mathcal{S}(\cdot) \in L^2(0, T; L^\infty([0, T] \times \Omega; \mathcal{L}(H_1; H))).
\]

Remark 4.1 (A6) is a restriction on the regularity of optimal controls. We believe that it is a technical condition. However, we do not know how to get rid of it now.

Remark 4.2 We can replace (A6) by the following assumption:
\[
\tilde{u}(\cdot) \in L^\infty([0, T]) \cap L^\infty([0, T] \times \Omega; \mathcal{L}(H_1; V'))
\]
and
\[
\mathcal{D} \mathcal{S}(\cdot) \in L^2(0, T; L^\infty([0, T] \times \Omega; \mathcal{L}(H_1; V'))).
\]

In (A6'), we relax the restriction of the regularity on H by assuming that \(a_u, b_u\) can map \(H_1\) into a more regular space \(V'\).

By Assumption (A6), for any \(v \in U\), \(\mathcal{S}(t)^*(v - \tilde{u}(t)) \in L^\infty_{\mathcal{F}_s}(H)\) and
\[
\mathcal{S}(t)^*(v - \tilde{u}(t)) = \mathbb{E} \left[ \mathcal{S}(t)^*(v - \tilde{u}(t)) \right] + \int_0^t \mathbb{E} \left[ \mathcal{D}_s \left( \mathcal{S}(t)^*(v - \tilde{u}(t)) \right) \right] dW(s), \quad \mathbb{P}\text{-a.s.} \quad (4.8)
\]

Now we are about to give our main result, the pointwise second order necessary condition for singular optimal controls.

When the optimal control \(\tilde{u}\) is singular in the sense of Definition 4.1, the following result is an immediate consequence of Corollary 3.1.

Corollary 4.1 Assume that \(x_0 \in L^2_{\mathcal{F}_0}(\Omega; H)\). Let Assumptions (A1)–(A3), (A5) hold, and let \(\tilde{u}(\cdot) \in U^4[0, T]\) be a singular optimal control and \(\tilde{x}(\cdot)\) be the corresponding optimal state. Then, for any \(u(\cdot) \in U^4[0, T]\),
\[
\mathbb{E} \int_0^T \left( y(t), \mathcal{S}(t)^*(u(t) - \tilde{u}(t)) \right)_H dt \leq 0. \quad (4.9)
\]
Using (4.8) and (4.9), we have the following pointwise second-order necessary condition for singular optimal controls.

**Theorem 4.1**: Let Assumptions (A1)-(A3) and (A5)-(A6) hold. If \( \bar{u}(\cdot) \in \mathcal{U}^4[0, T] \) is a singular optimal control in the classical sense, then for a.e. \( \tau \in [0, T] \), it holds that

\[
\langle a_2(\tau)(v - \bar{u}(\tau)), S(\tau)^*(v - \bar{u}(\tau)) \rangle_H + \langle b_2(\tau)(v - \bar{u}(\tau)), \nabla S(\tau)^*(v - \bar{u}(\tau)) \rangle_H \\
- \langle b_2(\tau)(v - \bar{u}(\tau)), S(\tau)^* \nabla \bar{u}(\tau) \rangle_H \leq 0, \quad \forall v \in U, \ \mathbb{P}\text{-a.s.}
\]

**Proof**: Since \( W(\cdot) \) is a continuous stochastic process, \( \mathcal{F}_i \) is countably generated for any \( t \in [0, T] \). Hence, one can find a sequence \( \{F_i\}_{i=1}^\infty \subset \mathcal{F}_t \) such that for any \( F \in \mathcal{F}_t \), there exists a subsequence \( \{F_{i_n}\}_{n=1}^\infty \subset \{F_i\}_{i=1}^\infty \) such that \( \lim_{n \to \infty} \mathbb{P}(F \setminus F_{i_n} \cup (F_{i_n} \setminus F)) = 0 \). \( \mathcal{F}_i \) is also said to be generated by the sequence \( \{F_i\}_{i=1}^\infty \).

Denote by \( \{t_i\}_{i=1}^\infty \) the sequence constituted by all rational numbers in \( [0, T] \), by \( \{v^k\}_{k=1}^\infty \) a dense subset of \( U \). As in [18], we choose \( \{F_{ij}\}_{j=1}^\infty \subset \{F_i\}_{i=1}^\infty \) to be a sequence generating \( \mathcal{F}_{t_i} \) (for each \( i \in \mathbb{N} \)). Fix \( i, j, k \in \mathbb{N} \) arbitrarily. For any \( \tau \in [t_i, T) \) and \( \theta \in (0, T - \tau) \), write \( E^i_\theta = [\tau, \tau + \theta) \), and define

\[
u^{k,\theta}_{ij}(t, \omega) = \begin{cases} v^k, & (t, \omega) \in E^i_\theta \times F_{ij}, \\ \bar{u}(t, \omega), & (t, \omega) \in ([0, T] \times \Omega \setminus (E^i_\theta \times F_{ij})). \end{cases}
\]

Clearly, \( u^{k,\theta}_{ij}(\cdot) \in \mathcal{U}^4[0, T] \) and

\[
\nu^{k,\theta}_{ij}(t, \omega) - \bar{u}(t, \omega) = (v^k - \bar{u}(t, \omega))\chi_{F_{ij}}(\omega)\chi_{E^i_\theta}(t), \quad (t, \omega) \in [0, T] \times \Omega.
\]

Then, substituting \( u(\cdot) \) by \( u^{k,\theta}_{ij}(\cdot) \) in (4.9), we obtain that

\[
\mathbb{E} \int_\tau^{\tau+\theta} \langle y^{k,\theta}_{ij}(t), S(t)^*(v^k - \bar{u}(t)) \rangle_H \chi_{F_{ij}}(\omega) dt \leq 0,
\]

where \( y^{k,\theta}_{ij}(\cdot) \) is the solution to the equation (2.4) with \( u(\cdot) \) replaced by \( u^{k,\theta}_{ij}(\cdot) \). Note that \( y^{k,\theta}_{ij}(\cdot) \) is the mild solution to the linear evolution equation (2.4), i.e.,

\[
y^{k,\theta}_{ij}(t) = \int_0^t e^{A(t-s)} \left[ a_1(s)y^{k,\theta}_{ij}(s) + a_2(s)(v^k - \bar{u}(s))\chi_{E^i_\theta}(s)\chi_{F_{ij}}(\omega) \right] ds \\
+ \int_0^t e^{A(t-s)} \left[ b_1(s)y^{k,\theta}_{ij}(s) + b_2(s)(v^k - \bar{u}(s))\chi_{E^i_\theta}(s)\chi_{F_{ij}}(\omega) \right] dW(s),
\]

\( \mathbb{P}\text{-a.s.}, \forall t \in [0, T] \).

Substituting (4.12) into (4.11) and recalling that \( y^{k,\theta}_{ij}(t) = 0 \) for any \( t \in [0, \tau) \), we have

\[
0 \geq \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \left( \int_\tau^t e^{A(t-s)} \left[ a_1(s)y^{k,\theta}_{ij}(s) \\
+ a_2(s)(v^k - \bar{u}(s))\chi_{F_{ij}}(\omega) \right] ds, S(t)^*(v^k - \bar{u}(t)) \right)_H \chi_{F_{ij}}(\omega) dt \\
+ \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau+\theta} \left( \int_\tau^t e^{A(t-s)} \left[ b_1(s)y^{k,\theta}_{ij}(s) \\
+ b_2(s)(v^k - \bar{u}(s))\chi_{F_{ij}}(\omega) \right] dW(s), S(t)^*(v^k - \bar{u}(t)) \right)_H \chi_{F_{ij}}(\omega) dt.
\]
By (A3), using Gronwall’s inequality and Burkholder-Davis-Gundy’s inequality, it is easy to prove that

\[
\sup_{t \in [0, T]} \mathbb{E}|y_{ij}^{k, \theta}(t)|_H^2 \leq C \mathbb{E} \int_0^T |v^k - \bar{u}(s)|_{H, \chi_{F_i}}^2(s) \chi_{F_i}(\omega) ds. \tag{4.14}
\]

Consequently, for a.e. \( \tau \in [t_i, T) \),

\[
\frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau + \theta} \left( \int_\tau^t e^{A(t-s)} a_1(s) y_{ij}^{k, \theta}(s) ds, \mathcal{S}(t)^* (v^k - \bar{u}(t)) \right)_H \chi_{F_i}(\omega) dt \leq \frac{1}{\theta^2} \left( \mathbb{E} \int_\tau^{\tau + \theta} \left| \int_\tau^t e^{A(t-s)} a_1(s) y_{ij}^{k, \theta}(s) ds \right|_H^2 dt \right)^{\frac{1}{2}} \left( \mathbb{E} \int_\tau^{\tau + \theta} \left| \mathcal{S}(t)^* (v^k - \bar{u}(t)) \right|_H^2 dt \right)^{\frac{1}{2}} \leq \frac{C}{\theta^2} \left( \sup_{t \in [0, T]} \mathbb{E} |y_{ij}^{k, \theta}(t)|_H^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \int_\tau^{\tau + \theta} \left| \mathcal{S}(t)^* (v^k - \bar{u}(t)) \right|_H^2 dt \right)^{\frac{1}{2}} \to 0, \quad \theta \to 0^+. \tag{4.15}
\]

Next, by Lemma 4.2 for a.e. \( \tau \in [t_i, T) \),

\[
\lim_{\theta \to 0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau + \theta} \left( \int_\tau^t e^{A(t-s)} a_2(s) \right. \\
\left. \left( (v^k - \bar{u}(s)) \chi_{F_i}(\omega) \right) ds, \mathcal{S}(t)^* (v^k - \bar{u}(t)) \right)_H \chi_{F_i}(\omega) dt = \frac{1}{2} \mathbb{E} \left( (a_2(\tau)(v^k - \bar{u}(\tau)), \mathcal{S}(\tau)^* (v^k - \bar{u}(\tau))) \right)_H \chi_{F_i}(\omega). \tag{4.16}
\]

Therefore, by (4.15) and (4.16), we have already proved that

\[
\lim_{\theta \to 0^+} \frac{1}{\theta^2} \mathbb{E} \int_\tau^{\tau + \theta} \left( \int_\tau^t e^{A(t-s)} \left[ a_1(s) y_{ij}^{k, \theta}(s) \right. \\
\left. + a_2(s) \left( (v^k - \bar{u}(s)) \chi_{F_i}(\omega) \right) ds, \mathcal{S}(t)^* (v^k - \bar{u}(t)) \right)_H \chi_{F_i}(\omega) dt = \frac{1}{2} \mathbb{E} \left( (a_2(\tau)(v^k - \bar{u}(\tau)), \mathcal{S}(\tau)^* (v^k - \bar{u}(\tau))) \right)_H \chi_{F_i}(\omega), \quad \text{a.e. } \tau \in [t_i, T). \tag{4.17}
\]

On the other hand, by Assumption (A6) and (4.8),

\[
\mathbb{E} \int_\tau^{\tau + \theta} \left( \int_\tau^t e^{A(t-s)} \left[ b_1(s) y_{ij}^{k, \theta}(s) \right. \\
\left. + b_2(s) \left( (v^k - \bar{u}(s)) \chi_{F_i}(\omega) \right) \right] dW(s), \mathcal{S}(t)^* (v^k - \bar{u}(t)) \right)_H \chi_{F_i}(\omega) dt = \int_\tau^{\tau + \theta} \mathbb{E} \left\{ \left( \int_\tau^t e^{A(t-s)} \left[ b_1(s) y_{ij}^{k, \theta}(s) + b_2(s) \left( (v^k - \bar{u}(s)) \chi_{F_i}(\omega) \right) \right] dW(s), \\
\mathbb{E} \left[ \mathcal{S}(t)^* (v^k - \bar{u}(t)) \right] \right\}_H \chi_{F_i}(\omega) \right\} dt \tag{4.18}
\]

\[
+ \int_\tau^{\tau + \theta} \mathbb{E} \left\{ \left( \int_\tau^t e^{A(t-s)} \left[ b_1(s) y_{ij}^{k, \theta}(s) + b_2(s) \left( (v^k - \bar{u}(s)) \chi_{F_i}(\omega) \right) \right] dW(s), \\
\mathbb{E} \left[ \mathcal{S}(t)^* (v^k - \bar{u}(t)) \right] \right\}_H \chi_{F_i}(\omega) \right\} dt.
\]
Then, we have

\[ \int_0^t \mathbb{E} \left[ \mathcal{D}_s \langle S(t)^*(v^k - \bar{u}(t)) \rangle \mid \mathcal{F}_s \right] dW(s) \right\}_H \chi_{F_{ij}}(\omega) \}

\[ d\theta(t) \]

By Lemma 4.1, there exists a sequence \( \{\theta_n\}_{n=1}^\infty \) such that \( \theta_n \to 0^+ \) as \( n \to \infty \) and

\[ \frac{1}{\theta_n^2} \int_\tau^{\tau+\theta_n} \int_\tau^{\tau+\theta_n} \mathbb{E} \left[ \langle e^{A(t-s)}b_1(s)y_{ij}^k, \mathcal{D}_s \langle S(t)^*(v^k - \bar{u}(t)) \rangle \rangle_H \chi_{F_{ij}}(\omega) \right] d\theta_n d\theta_n \]

\[ \leq \frac{1}{\theta_n^2} \int_\tau^{\tau+\theta_n} \int_\tau^{\tau+\theta_n} \mathbb{E} \left[ \langle e^{A(t-s)}b_1(s)y_{ij}^k, \mathcal{D}_s \langle S(t)^*(v^k - \bar{u}(t)) \rangle \rangle_H \right] d\theta_n d\theta_n \]

\[ \leq C \left( \sup_{t \in [0,T]} \mathbb{E} |y_{ij}^k| \right)^\frac{1}{2} \cdot \left( \mathbb{E} \int_\tau^{\tau+\theta_n} \int_\tau^{\tau+\theta_n} \mathbb{E} \left[ \mathcal{D}_s \langle S(t)^*(v^k - \bar{u}(t)) \rangle \right] \right) \to 0, \quad n \to \infty, \quad a.e. \, \tau \in [t_i, T). \)

We next prove that there exists a subsequence \( \{\theta_{n_l}\}_{l=1}^\infty \) of \( \{\theta_n\}_{n=1}^\infty \), such that \( \theta_{n_l} \to 0^+ \) as \( l \to \infty \) and

\[ \lim_{l \to \infty} \frac{1}{\theta_{n_l}^2} \int_\tau^{\tau+\theta_{n_l}} \int_\tau^{\tau+\theta_{n_l}} \mathbb{E} \left[ \langle e^{A(t-s)}b_2(s)(v^k - \bar{u}(s)) \rangle_H \chi_{F_{ij}}(\omega) \right] d\theta_n d\theta_n \]

\[ \mathcal{D}_s \langle S(t)^*(v^k - \bar{u}(t)) \rangle \right\}_H \chi_{F_{ij}}(\omega) \}

\[ \frac{1}{2} \mathbb{E} \left[ \langle b_2(\tau)(v^k - \bar{u}(\tau)), \nabla S(\tau)^*(v^k - \bar{u}(\tau)) \rangle_H \chi_{F_{ij}}(\omega) \right) \]

\[ - \frac{1}{2} \mathbb{E} \left[ \langle b_2(\tau)(v^k - \bar{u}(\tau)), S(\tau)^*\nabla \bar{u}(\tau) \rangle \right] H \chi_{F_{ij}}(\omega), \quad a.e. \, \tau \in [t_i, T). \]

By (A6),

\[ \mathcal{D}_s \langle S(t)^*(v^k - \bar{u}(t)) \rangle = \mathcal{D}_s \langle S(t)^*(v^k - \bar{u}(t)) \rangle - S(t)^* \mathcal{D}_s \bar{u}(t). \]

Then, we have

\[ \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^{\tau+\theta} \mathbb{E} \left[ \langle e^{A(t-s)}b_2(s)(v^k - \bar{u}(s)) \rangle_H \chi_{F_{ij}}(\omega) \right] d\theta_n d\theta_n \]

\[ = \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^{\tau+\theta} \mathbb{E} \left[ \langle e^{A(t-s)}b_2(s)(v^k - \bar{u}(s)) \rangle \right] d\theta_n d\theta_n \]

\[ - \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^{\tau+\theta} \mathbb{E} \left[ \langle e^{A(t-s)}b_2(s)(v^k - \bar{u}(s)) \rangle \right] d\theta_n d\theta_n \]
For the first part in the right hand side of (4.21),
\[
\frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E} \left( \langle e^{A(t-s)} b_2(s)(v^k - \bar{u}(s)), D_s S(t)^* (v^k - \bar{u}(t)) \rangle_{H \theta \chi_F}(\omega) \right) ds dt
\]
\[
eq \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E} \left[ \langle e^{A(t-s)} b_2(s)(v^k - \bar{u}(s)), (D_s S(t) - \nabla S(s))^* (v^k - \bar{u}(t)) \rangle_{H \theta \chi_F}(\omega) \right] ds dt
\]
\[
+ \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E} \left( \langle e^{A(t-s)} b_2(s)(v^k - \bar{u}(s)), \nabla S(s)^* (v^k - \bar{u}(t)) \rangle_{H \theta \chi_F}(\omega) \right) ds dt.
\]
(4.22)

Since
\[
\left| \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E} \left[ \langle e^{A(t-s)} b_2(s)(v^k - \bar{u}(s)), (D_s S(t) - \nabla S(s))^* (v^k - \bar{u}(t)) \rangle_{H \theta \chi_F}(\omega) \right] ds dt \right|
\]
\[
\leq C \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E} \left[ |e^{A(t-s)} b_2(s)(v^k - \bar{u}(s))|_{H^r} \right] ds dt
\]
\[
\leq C \left( \mathbb{E} \int_\tau^{\tau+\theta} \int_\tau^t |(v^k - \bar{u}(s))|_{H^r}^2 \cdot |(v^k - \bar{u}(t))|_{H^r}^2 ds dt \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \mathbb{E} \int_\tau^{\tau+\theta} \int_\tau^t |D_s S(t) - \nabla S(s)|_{L_2(H^r:H^r)}^2 ds dt \right)^{\frac{1}{2}}.
\]

By Lemma 4.1, there exists a subsequence of \(\theta_{n_i} \to 0^+\) as \(l \to \infty\) and
\[
\lim_{l \to \infty} \frac{1}{\theta_{n_i}^2} \int_\tau^{\tau+\theta_{n_i}} \int_\tau^t \mathbb{E} \left[ \langle e^{A(t-s)} b_2(s)(v^k - \bar{u}(s)), (D_s S(t) - \nabla S(s))^* (v^k - \bar{u}(t)) \rangle_{H \theta \chi_F}(\omega) \right] ds dt
\]
\[
= 0, \quad \text{a.e. } \tau \in [0, T).
\]
(4.23)

For the second part in the right hand side of (4.22), by Lemma 4.2 it follows that
\[
\lim_{\theta \to 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \mathbb{E} \left( \langle e^{A(t-s)} b_2(s)(v^k - \bar{u}(s)), \nabla S(s)^* (v^k - \bar{u}(t)) \rangle_{H \theta \chi_F}(\omega) \right) ds dt
\]
\[
= \frac{1}{2} \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), \nabla S(\tau)^* (v^k - \bar{u}(\tau)) \rangle_{H \theta \chi_F}(\omega) \right), \quad \text{a.e. } \tau \in [t_i, T).
\]
(4.24)

Therefore, by (4.22)–(4.24), we conclude that
\[
\lim_{l \to \infty} \frac{1}{\theta_{n_i}^2} \int_\tau^{\tau+\theta_{n_i}} \int_\tau^t \mathbb{E} \left( \langle e^{A(t-s)} b_2(s)(v^k - \bar{u}(s)), D_s S(t)^* (v^k - \bar{u}(t)) \rangle_{H \theta \chi_F}(\omega) \right) ds dt
\]
\[
= \frac{1}{2} \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), \nabla S(\tau)^* (v^k - \bar{u}(\tau)) \rangle_{H \theta \chi_F}(\omega) \right), \quad \text{a.e. } \tau \in [t_i, T).
\]
(4.25)
In a similar way, we can prove that there exists a subsequence \( \{ \theta_{n_l} \}_{l=1}^{\infty} \) of \( \{ \theta_n \}_{n=1}^{\infty} \) such that

\[
\lim_{l \to \infty} \frac{1}{\theta_{n_l}^2} \int_{\tau}^{\tau + \theta_{n_l}} \int_{\tau}^{t} \mathbb{E} \left( \langle e^{A(t-s)} b_2(s)(v^k - \bar{u}(s)), S(t)^* D_s \bar{u}(t) \rangle_{H \chi_{F_{ij}}(\omega)} \right) d\sigma dt
= \frac{1}{2} \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), S(\tau)^* \nabla \bar{u}(\tau) \rangle_{H \chi_{F_{ij}}(\omega)} \right), \quad \text{a.e. } \tau \in [t_i, T].
\] (4.26)

Combining (4.21), (4.24) and (4.20), we obtain (4.22). Then, by (4.18), (4.19) and (4.20), we obtain that there exists a subsequence \( \{ \theta_{n_l} \}_{l=1}^{\infty} \), \( \theta_{n_l} \to 0^+ \) as \( l \to \infty \) and

\[
\lim_{l \to \infty} \mathbb{E} \int_{\tau}^{\tau + \theta_{n_l}} \left\langle \int_{\tau}^{t} e^{A(t-s)} \left[ b_1(s) y_{ij} \theta_{n_l}(s)ight.ight.
\left. + b_2(s)(v^k - \bar{u}(s)) \chi_{F_{ij}}(\omega) \right] dW(s), S(t)(v^k - \bar{u}(t)) \right\rangle_{H \chi_{F_{ij}}(\omega)} dt
= \frac{1}{2} \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), \nabla S(\tau)^* (v^k - \bar{u}(\tau)) \rangle_{H \chi_{F_{ij}}(\omega)} \right)
- \frac{1}{2} \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), S(\tau)^* \nabla \bar{u}(\tau) \rangle_{H \chi_{F_{ij}}(\omega)} \right), \quad \text{a.e. } \tau \in [t_i, T].
\] (4.27)

Finally, by (4.13), (4.17) and (4.27) we conclude that, for any \( i, j, k \in \mathbb{N} \), there exists a Lebesgue measurable set \( E_{i,j}^k \subset [t_i, T] \) with \( |E_{i,j}^k| = 0 \) such that

\[
0 \geq \frac{1}{2} \mathbb{E} \left( \langle a_2(\tau)(v^k - \bar{u}(\tau)), S(\tau)^* (v^k - \bar{u}(\tau)) \rangle_{H \chi_{F_{ij}}(\omega)} \right)
+ \frac{1}{2} \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), \nabla S(\tau)^* (v^k - \bar{u}(\tau)) \rangle_{H \chi_{F_{ij}}(\omega)} \right)
- \frac{1}{2} \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), S(\tau)^* \nabla \bar{u}(\tau) \rangle_{H \chi_{F_{ij}}(\omega)} \right), \quad \forall \, \tau \in [t_i, T] \setminus E_{i,j}^k.
\]

Let \( E_0 = \bigcup_{i,j,k \in \mathbb{N}} E_{i,j}^k \). Then \( |E_0| = 0 \), and for any \( i, j, k \in \mathbb{N} \),

\[
\mathbb{E} \left( \langle a_2(\tau)(v^k - \bar{u}(\tau)), S(\tau)^* (v^k - \bar{u}(\tau)) \rangle_{H \chi_{F_{ij}}(\omega)} \right)
+ \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), \nabla S(\tau)^* (v^k - \bar{u}(\tau)) \rangle_{H \chi_{F_{ij}}(\omega)} \right)
- \mathbb{E} \left( \langle b_2(\tau)(v^k - \bar{u}(\tau)), S(\tau)^* \nabla \bar{u}(\tau) \rangle_{H \chi_{F_{ij}}(\omega)} \right)
\leq 0, \quad \forall \, \tau \in [t_i, T] \setminus E_0.
\]

By the construction of \( \{ F_{ij} \}_{i=1}^{\infty} \), the continuity of the filter \( \mathbb{F} \) and the density of \( \{ v^k \}_{k=1}^{\infty} \), we conclude that

\[
\langle a_2(\tau)(v - \bar{u}(\tau)), S(\tau)^* (v - \bar{u}(\tau)) \rangle_H + \langle b_2(\tau)(v - \bar{u}(\tau)), \nabla S(\tau)^* (v - \bar{u}(\tau)) \rangle_H
- \langle b_2(\tau)(v - \bar{u}(\tau)), S(\tau)^* \nabla \bar{u}(\tau) \rangle_H \leq 0, \quad \text{a.s.}, \quad \forall \, (\tau, v) \in ([0, T] \setminus E_0) \times U.
\]

This completes the proof of Theorem 4.1. \( \square \)
5 Second order sufficient conditions

In this section, we discuss the second order sufficient condition for the optimal control problem (3.3). We first give a simple and direct result, and then we generalize it under some proper assumptions and obtain a second order sufficient condition which has minimal gap with the second order necessary condition. The basic idea comes from optimization theory.

In addition to Assumption (A1)–(A3), we assume that

(A7) $U$ is a bounded closed convex set.

(A8) There exists a constant $C_L > 0$ such that, for any $t \in [0,T]$ and $(x,u),(\tilde{x},\tilde{u}) \in H \times U$,

$$
\begin{align*}
\|a_{xx}(t,x,u) - a_{xx}(t,\tilde{x},\tilde{u})\|_{L(H \times H;H)} + \|b_{xx}(t,x,u) - b_{xx}(t,\tilde{x},\tilde{u})\|_{L(H \times H;H)} \\
+ \|a_{xu}(t,x,u) - a_{xu}(t,\tilde{x},\tilde{u})\|_{L(H \times H;H)} + \|b_{xu}(t,x,u) - b_{xu}(t,\tilde{x},\tilde{u})\|_{L(H \times H;H)} \\
+ \|a_{uu}(t,x,u) - a_{uu}(t,\tilde{x},\tilde{u})\|_{L(H \times H;H)} + \|b_{uu}(t,x,u) - b_{uu}(t,\tilde{x},\tilde{u})\|_{L(H \times H;H)} \\
\leq C_L (|x - \tilde{x}|_H + |u - \tilde{u}|_H),
\end{align*}
$$

(5.1)

Under Assumption (A7), any $U$-valued measurable adapted process $u(\cdot)$ belongs to $U^\beta[0,T] \subset U^{\beta}[0,T]$ ($\beta \geq 2$). Let $u(\cdot), \bar{u}(\cdot) \in U^\beta[0,T]$, $x(\cdot)$ and $\bar{x}(\cdot)$ be solutions to the control system (1.1) with respect to $u(\cdot)$ and $\bar{u}(\cdot)$, respectively. Let $\delta u$, $\delta x$, $y$ and $z$ be defined as in Section 2. We first give the following estimate:

Lemma 5.1 Let (A1), (A3) and (A7)–(A8) hold. Then, for any $\beta \geq 2$,

$$
\|\delta x - y - \frac{1}{2}z\|_{C^\beta_y([0,T];L^\beta(0,T;H_1))} \leq C (\|\delta u\|_{L^\beta_y(0,T;H_1)} \cdot \|\delta u\|^2_{L^\beta_y(0,T;H_1)}).
$$

(5.2)

Proof: For $\psi = a, b$, put

$$
\begin{align*}
\hat{\psi}_{11}(t) &\triangleq \int_0^1 (1 - \theta)\psi_{xx}(t,\tilde{x}(\theta),\bar{u}(\theta) + \theta\delta u(\theta))d\theta, \\
\hat{\psi}_{12}(t) &\triangleq \int_0^1 (1 - \theta)\psi_{xu}(t,\tilde{x}(\theta),\bar{u}(\theta) + \theta\delta u(\theta))d\theta, \\
\hat{\psi}_{22}(t) &\triangleq \int_0^1 (1 - \theta)\psi_{uu}(t,\tilde{x}(\theta),\bar{u}(\theta) + \theta\delta u(\theta))d\theta.
\end{align*}
$$

Similar to the proof of Step 1 in Theorem 3.1, $\delta x$ solves the following SEE:

$$
\begin{align*}
d\delta x &= \left[ A\delta x + a_1(t)\delta x + a_2(t)\delta u + \hat{a}_{11}(t)(\delta x, \delta x) \\
&\quad + 2\hat{a}_{12}(t)(\delta x, \delta u) + \hat{a}_{22}(t)(\delta u, \delta u) \right] dt \\
&\quad + \left[ b_1(t)\delta x + b_2(t)\delta u + \hat{b}_{11}(t)(\delta x, \delta x) \\
&\quad + 2\hat{b}_{12}(t)(\delta x, \delta u) + \hat{b}_{22}(t)(\delta u, \delta u) \right] dW(t)
\end{align*}
$$

(5.3)

in $(0,T]$.
Let \( r_2(\cdot) = \delta x(\cdot) - y(\cdot) - \frac{1}{2} z(\cdot) \). Then \( r_2(\cdot) \) fulfills
\[
\begin{cases}
    dr_2 = (Ar_2 + a_1(t)r_2 + \Upsilon_1(t))\,dt + (b_1(t)r_2 + \Upsilon_2(t))\,dW(t) & \text{in } (0, T], \\
r_2(0) = 0,
\end{cases}
\] (5.4)
where
\[
\Upsilon_1(t) = \left( \hat{a}_{11}(t)(\delta x(t), \delta x(t)) - \frac{1}{2} a_{11}(t)(y(t), y(t)) \right)
+ (2\hat{a}_{12}(t)(\delta x(t), \delta u(t)) - a_{12}(t)(y(t), \delta u(t))) + \left( \hat{a}_{22}(t) - \frac{1}{2} a_{22}(t) \right)(\delta u(t), \delta u(t))
\]
and
\[
\Upsilon_2(t) = \left( \hat{b}_{11}(t)(\delta x(t), \delta x(t)) - \frac{1}{2} b_{11}(t)(y(t), y(t)) \right)
+ (2\hat{b}_{12}(t)(\delta x(t), \delta u(t)) - b_{12}(t)(y(t), \delta u(t))) + \left( \hat{b}_{22}(t) - \frac{1}{2} b_{22}(t) \right)(\delta u(t), \delta u(t)).
\]
We have that
\[
\mathbb{E}\left[ \beta |r_2(t)|_H^\beta \right] = \mathbb{E}\left[ \left| \int_0^t e^{A(t-s)}a_1(s)r_2(s)\,ds + \int_0^t e^{A(t-s)}b_1(s)r_2(s)\,dW(s) \right|_H^\beta \right]
+ \mathbb{E}\left[ \left| \int_0^t e^{A(t-s)}\Upsilon_1(s)\,ds + \int_0^t e^{A(t-s)}\Upsilon_2(s)\,dW(s) \right|_H^\beta \right]
\leq C \mathbb{E}\left[ \left| \int_0^t |r_2(s)|_H^\beta \,ds + \left( \int_0^t |\Upsilon_1(s)|_H \,ds \right)^\beta + \left( \int_0^t |\Upsilon_2(s)|_H^2 \,ds \right)^{\frac{\beta}{2}} \right].
\] (5.5)
By (A3), (A7)–(A8) and Lemma 2.1 we deduce that
\[
\mathbb{E}\left( \int_0^t |\Upsilon_1(s)|_H \,ds \right)^\beta \leq C \mathbb{E}\left( \int_0^T \left( \hat{a}_{11}(t)(\delta x(t), \delta x(t)) - \frac{1}{2} a_{11}(t)(y(t), y(t)) \right)
+ (2\hat{a}_{12}(t)(\delta x(t), \delta u(t)) - a_{12}(t)(y(t), \delta u(t)))
+ \left( \hat{a}_{22}(t) - \frac{1}{2} a_{22}(t) \right)(\delta u(t), \delta u(t)) \right)_H^\beta \right.
\leq C \mathbb{E}\left[ \int_0^T \left( \hat{a}_{11}(t)(\delta x(t), \delta x(t)) - \hat{a}_{11}(t)(y(t), y(t)) \right)_H + \left| \hat{a}_{11}(t) - \frac{1}{2} a_{11}(t) \right| \left| L(H \times H, H) \cdot |y(t)|_H^2 \right.
+ 2\hat{a}_{12}(t)(\delta x(t), \delta u(t)) - \hat{a}_{12}(t)(y(t), \delta u(t)) \right)_H
\left. \left. + \left| 2\hat{a}_{12}(t) - a_{12}(t) \right| \left| L(H \times H_1, H) \cdot |y(t)|_H \cdot |\delta u(t)|_{H_1} \right|
\left. + \left| \hat{a}_{22}(t) - \frac{1}{2} a_{22}(t) \right| \left| L(H_1 \times H_1, H) \cdot |\delta u(t)|_{H_1}^2 \right| \right) \,dt \right]^{\beta}
\leq C \mathbb{E}\left[ \int_0^T \left( |\delta x(t) + y(t)|_H \cdot |\delta x(t) - y(t)|_H + \left( |\delta x(t)|_H + |\delta u(t)|_{H_1} \right) \cdot |y(t)|_H \right.
\left. + 2|\delta x(t) - y(t)|_H \cdot |\delta u(t)|_{H_1} + \left( |\delta x(t)|_H + |\delta u(t)|_{H_1} \right) \cdot |y(t)|_H \cdot |\delta u(t)|_{H_1} \right]^{\beta}
\]
Suppose that \( y > 0 \) and be an admissible control and \( \bar{u}(\cdot) \) for any \( \bar{\beta} \), such that for any \( \bar{\beta} \), \( \| \delta u \|_{L^\beta_{F_0}(0,T;H_1)} \cdot \| \delta u \|_{L^\beta_{F_0}(0,T;H_1)} \).

Similarly,

\[
\mathbb{E}\left( \int_0^t |\gamma_2(s)|^2_H ds \right)^{\frac{2}{\beta}} \leq C \| \delta u \|_{L^\beta_{F_0}(0,T;H_1)} \cdot \| \delta u \|_{L^\beta_{F_0}(0,T;H_1)}.
\]

Combining (5.6), (5.7) with (5.7), we obtain (5.2).

Now, we put

\[
\Lambda(v(\cdot)) \triangleq \mathbb{E} \int_0^T \left[ \langle \mathbb{H}_{uu}(t)v(t), v(t) \rangle_{H_1} + \langle b_2(t)^* P(t)b_2(t)v(t), v(t) \rangle_{H_1} \right] dt
\]

\[
+ 2\mathbb{E} \int_0^T \left[ \langle (\mathbb{H}_{uu}(t)+a_2(t)^* P(t)+b_2(t)^* P(t)b_1(t))v(t), v(t) \rangle_H \right] dt
\]

\[
+ \mathbb{E} \int_0^T \left[ \langle (\hat{Q}^{(0)} + Q^{(0)})(0, a_2(t)v(t), b_2(t)v(t)), b_2(t)v(t) \rangle_H \right] dt,
\]

and

\[
\tilde{\Lambda}(v(\cdot)) \triangleq \mathbb{E} \int_0^T \left[ \langle \mathbb{H}_{uu}(t)v(t), v(t) \rangle_{H_1} + \langle b_2(t)^* P(t)b_2(t)v(t), v(t) \rangle_{H_1} \right] dt
\]

\[
+ 2\mathbb{E} \int_0^T \langle (S(t)v(t), v(t)) \rangle_{H_1} dt.
\]

where \( y^v(\cdot) \) is the solution to the equation (2.14) with \( \delta u \) replaced by \( v \) and \( (P(\cdot), (Q^{(\cdot)}, \hat{Q}^{(\cdot)})) \) (resp. \( (P, Q) \)) is the relaxed transposition solution (resp. the V-transposition solution) of BSEE (2.15) with \( P_7, J(\cdot), K(\cdot) \) and \( F(\cdot) \) given by (3.3). Note that the mapping

\[
v(\cdot) \mapsto y^v(\cdot)
\]

from \( L^2_{F_0}(0,T;H_1) \) to \( C_{F}([0,T]; L^2(\Omega;H)) \) is a linear for any \( \beta \geq 2 \). The mapping \( \Lambda \) and \( \tilde{\Lambda} \) are actually two quadratic-like forms defined on the Banach space \( L^2_{F_0}(0,T;H_1) \) for any \( \beta \geq 4 \).

By Lemmas 2.3 and 5.1 we obtain the following second order sufficient condition.

**Theorem 5.1** Suppose that \( x_0 \in L^2_{F_0}(\Omega;H) \). Let (A1)–(A3) and (A7)–(A8) hold, and let \( \bar{u}(\cdot) \) be an admissible control and \( \bar{x}(\cdot) \) be the corresponding state. If there exists a constant \( \varrho > 0 \) such that for any \( u(\cdot) \in \mathcal{U}^\infty[0,T] \),

\[
\mathbb{E} \int_0^T \langle \mathbb{H}_u(t), u(t) - \bar{u}(t) \rangle_{H_1} dt \leq 0
\]

and

\[
\Lambda(u - \bar{u}) \leq -2\varrho \| u - \bar{u} \|^2_{L^2_{F_0}(0,T;H_1)},
\]

29
then there exists a constant $\sigma > 0$ such that for any $u(\cdot) \in U^{\infty}[0, T]$ with $\|u - \bar{u}\|_{L^2_{\tilde{c}}(0, T; H_1)} \leq \sigma$,

$$J(u) \geq J(\bar{u}) + \frac{\sigma}{2} \|u - \bar{u}\|^2_{L^2_{\tilde{c}}(0, T; H_1)}. \quad (5.12)$$

Especially, $\bar{u}$ is a local minima of the optimal control problem (1.2).

**Proof:** Let $u(\cdot) \in U^{\infty}[0, T]$ and $x(\cdot)$ (resp. $\bar{x}(\cdot)$) be the solutions to the control system (1.1) with respect to $u(\cdot)$ (resp. $\bar{u}(\cdot)$). Let $\delta u$, $\delta x$, $y$ and $z$ be defined as in Section 2. Put

$$
\begin{align*}
\hat{g}_{11}(t) &\triangleq \int_0^1 (1 - \theta)g_{xx}(t, \bar{x}(t) + \theta \delta x(t), \bar{u}(t) + \theta \delta u(t))d\theta, \\
\hat{g}_{12}(t) &\triangleq \int_0^1 (1 - \theta)g_{xu}(t, \bar{x}(t) + \theta \delta x(t), \bar{u}(t) + \theta \delta u(t))d\theta, \\
\hat{g}_{22}(t) &\triangleq \int_0^1 (1 - \theta)g_{uu}(t, \bar{x}(t) + \theta \delta x(t), \bar{u}(t) + \theta \delta u(t))d\theta, \\
\hat{h}_{xx}(T) &\triangleq \int_0^1 (1 - \theta)h_{xx}(\bar{x}(T) + \theta \delta x(T))d\theta.
\end{align*}
$$

By Taylor’s formula, we see that

$$
g(t, x(t), u(t)) - g(t, \bar{x}(t), \bar{u}(t)) = \langle \hat{g}_{11}(t)\delta x(t), \delta x(t) \rangle_{H_1} + \langle \hat{g}_{12}(t)\delta u(t), \delta x(t) \rangle_{H_1}
+ \langle \hat{g}_{22}(t)\delta u(t), \delta u(t) \rangle_{H_1},
$$

and

$$h(x(T)) - h(\bar{x}(T)) = \langle \hat{h}_{xx}(T)\delta x(T), \delta x(T) \rangle_{H_1}.
$$

Using a similar method in the proof of (5.6), we obtain that

$$
\begin{align*}
\left| \mathbb{E} \int_0^T \left( \langle \bar{g}_{11}(t)\delta x(t), \delta x(t) \rangle_{H_1} - \frac{1}{2} \langle \bar{g}_{11}(t)y(t), y(t) \rangle_{H_1} \right) dt \right| &\leq C\left( \|\delta u\|_{L^2_{\tilde{c}}(0, T; H_1)} \cdot \|\delta u\|^2_{L^2_{\tilde{c}}(0, T; H_1)} \right), \\
\left| \mathbb{E} \int_0^T \left( 2\langle \bar{g}_{12}(t)\delta x(t), \delta u(t) \rangle_{H_1} - \langle \bar{g}_{12}(t)y(t), \delta u(t) \rangle_{H_1} \right) dt \right| &\leq C\left( \|\delta u\|_{L^2_{\tilde{c}}(0, T; H_1)} \cdot \|\delta u\|^2_{L^2_{\tilde{c}}(0, T; H_1)} \right),
\end{align*}
$$

and

$$
\begin{align*}
\left| \mathbb{E} \left( \langle \bar{h}_{xx}(\bar{x}(T))\delta x(T), \delta x(T) \rangle_{H_1} - \frac{1}{2} \langle \bar{h}_{xx}(\bar{x}(T))y(T), y(T) \rangle_{H_1} \right) \right| &\leq C\left( \|\delta u\|_{L^2_{\tilde{c}}(0, T; H_1)} \cdot \|\delta u\|^2_{L^2_{\tilde{c}}(0, T; H_1)} \right).
\end{align*}
$$

30
Also, by (5.2) and (A3),
\[
\| \mathbb{E} \int_0^T \left\langle g_1(t), \delta x(t) - y(t) - \frac{1}{2} \delta z(t) \right\rangle_H dt + \mathbb{E} \left\langle h_x(\bar{x}(T)), \delta x(T) - y(T) - \frac{1}{2} \delta z(T) \right\rangle_H \| \\
\leq \left\| \delta x - y - \frac{1}{2} \delta z \right\|_{C^0([0,T];L^2(\Omega;H))} \leq C\left(\|\delta u\|_{L^\infty(0,T;H_1)} \cdot \|\delta u\|^2_{L^2(0,T;H)} \right).
\] (5.19)

Combining (5.13)–(5.14) with (5.15)–(5.19), we have that
\[
\mathcal{J}(u) - \mathcal{J}(\bar{u}) \geq \mathbb{E} \int_0^T \left[ \left\langle g_1(t), y(t) \right\rangle_H + \frac{1}{2} \left\langle g_2(t), \delta u(t) \right\rangle_H \\
+ \frac{1}{2} \left( \left\langle g_{11}(t)y(t), y(t) \right\rangle_H + 2\left\langle g_{12}(t)y(t), \delta u(t) \right\rangle_H + \left\langle g_{22}(t)\delta u(t), \delta u(t) \right\rangle_H \right) \right] dt \\
+ \mathbb{E} \left[ \left\langle h_x(\bar{x}(T)), y(T) \right\rangle_H + \frac{1}{2} \left\langle h_x(\bar{x}(T)), z(T) \right\rangle_H + \frac{1}{2} \left\langle h_{xx}(\bar{x}(T))y(T), y(T) \right\rangle_H \\
- C\|\delta u\|_{L^\infty(0,T;H_1)} \cdot \|\delta u\|^2_{L^2(0,T;H_1)} \right].
\] (5.20)

Substituting (3.16)–(3.18) into (5.20) and combining with (5.10), we get that
\[
\mathcal{J}(u) - \mathcal{J}(\bar{u}) \geq -\mathbb{E} \int_0^T \left\langle \mathbb{H}_u(t), \delta u(t) \right\rangle_{H_1} dt \\
- \mathbb{E} \int_0^T \left( \frac{1}{2} \left\langle \mathbb{H}_{uu}(t) \delta u(t), \delta u(t) \right\rangle_{H_1} + \frac{1}{2} \left\langle b_2(t)^* P(t) b_2(t) \delta u(t), \delta u(t) \right\rangle_{H_1} \right) dt \\
- \mathbb{E} \int_0^T \left( \left\langle \mathbb{H}_{xx} + a_2(t)^* P(t) + b_2(t)^* P(t) b_1(t) \right\rangle y(t), \delta u(t) \right)_{H_1} dt \\
- \frac{1}{2} \mathbb{E} \int_0^T \left\langle (\hat{Q}^{(0)} + Q^{(0)})(0, a_2(t) \delta u(t), b_2(t) \delta u(t)), b_2(t) \delta u(t) \right\rangle_{H_1} dt \\
- C\left(\|\delta u\|_{L^\infty(0,T;H_1)} \cdot \|\delta u\|^2_{L^2(0,T;H_1)} \right) \\
= -\mathbb{E} \int_0^T \left\langle \mathbb{H}_u(t), \delta u(t) \right\rangle_{H_1} dt - \frac{1}{2} \Lambda(\delta u) - C\left(\|\delta u\|_{L^\infty(0,T;H_1)} \cdot \|\delta u\|^2_{L^2(0,T;H_1)} \right) \\
\geq \varrho\|\delta u\|^2_{L^2(0,T;H_1)} - C\left(\|\delta u\|_{L^\infty(0,T;H_1)} \cdot \|\delta u\|^2_{L^2(0,T;H_1)} \right).
\] (5.21)

Then, choosing $\sigma$ small enough such that
\[
C\|\delta u\|_{L^\infty(0,T;H_1)} \leq C\sigma \leq \frac{\varrho}{2},
\]
we finally obtain (5.12).

When the BSEE (2.15) has a unique $V$-transposition solution $(P, Q)$ with which $P_T, J(\cdot), K(\cdot)$ and $F(\cdot)$ are given by (5.3), the following result immediately follows from Theorem 5.1.
Corollary 5.1 In addition to the assumptions in Theorem 5.1, assume that (A5) holds. If there exists a constant $\varrho > 0$ such that for any $u(\cdot) \in \mathcal{U}[0,T]$, 
\[
\mathbb{E} \int_0^T \langle \mathcal{H}_u(t), u(t) - \bar{u}(t) \rangle_{H_1} dt \leq 0
\]
and
\[
\Lambda(u - \bar{u}) \leq -2\varrho \| u - \bar{u} \|^2_{L^2_\mathcal{F}(0,T;H_1)},
\]
then there exists a constant $\sigma > 0$ such that for any $u(\cdot) \in \mathcal{U}[0,T]$ with $\| u - \bar{u} \|_{L^2_\mathcal{F}(0,T;H_1)} \leq \sigma$, 
\[
\mathcal{J}(u) \geq \mathcal{J}(\bar{u}) + \frac{\varrho}{2} \| u - \bar{u} \|^2_{L^2_\mathcal{F}(0,T;H_1)},
\]
and, $\bar{u}$ is a local minima of the optimal control problem (1.2).

In what follows we refine the second order sufficient conditions in Theorem 5.1 and Corollary 5.1 by the general Legendre form.

Definition 5.1 Let $X$ be a reflexive Banach space. A functional $\Psi : X \to \mathbb{R}$ is called a general Legendre form if $\Psi$ is weakly lower semicontinuous, positively homogeneous of degree 2, i.e., for any $x \in X$, $t > 0$, $\Psi(tx) = t^2\Psi(x)$ and if $x_k \xrightarrow{w} x$ and $\Psi(x_k) \to \Psi(x)$, it holds that $x_k \to x$ strongly.

Some sufficient and necessary conditions to ensure a functional to be a Legendre form can be found in [8].

Define 
\[
T_{\mathcal{U}[0,T]}(\bar{u}) \triangleq \text{cl}_s \left\{ v = \alpha(u - \bar{u}) \mid u \in \mathcal{U}[0,T], \alpha \geq 0 \right\},
\]
where $\text{cl}_s(\mathcal{A})$ is the closure of a set $\mathcal{A}$ under the norm topology of the Banach space $L^2_\mathcal{F}(0,T;H_1)$. If $-\Lambda$ is a general Legendre form defined on $L^2_\mathcal{F}(0,T;H_1)$, the negative definite condition (5.11) can be weaken to the following directional negative definite condition:

(A9) 
\[
\Lambda(v) < 0, \quad \forall v \in C_{\mathcal{U}[0,T]}(\bar{u}),
\]
where
\[
C_{\mathcal{U}[0,T]}(\bar{u}) \triangleq \left[ T_{\mathcal{U}[0,T]}(\bar{u}) \right] \bigcap \left\{ v \in \mathcal{U}[0,T] \mid \mathbb{E} \int_0^T \langle \mathcal{H}_u(t), v(t) \rangle_{H_1} dt = 0 \right\} \setminus \{ 0 \}.
\]

When $-\Lambda$ is a general Legendre form, we have the following second order sufficient condition:

Theorem 5.2 Assume that $x_0 \in L^2_\mathcal{F}_0(\Omega;H)$ and $-\Lambda$ is a general Legendre form on $L^2_\mathcal{F}(0,T;H_1)$. Let (A1) - (A3) and (A7) - (A9) hold, and let $u(\cdot)$ be an admissible control and $\bar{u}(\cdot)$ be the corresponding state. If for any $u(\cdot) \in \mathcal{U}[0,T]$, 
\[
\mathbb{E} \int_0^T \langle \mathcal{H}_u(t), u(t) - \bar{u}(t) \rangle_{H_1} dt \leq 0,
\]
then there exist constants $\sigma > 0$ and $\varrho > 0$ such that for any $u(\cdot) \in \mathcal{U}[0,T]$ with $\| u - \bar{u} \|_{L^2_\mathcal{F}(0,T;H_1)} \leq \sigma$, the quadratic growth condition (5.12) holds.
Proof: We prove this conclusion through a contradiction argument. If one could not find \( \sigma > 0 \) and \( \rho > 0 \) such that (5.12) holds, then there must exist sequences \( \{\varrho_n\}_n^\infty \) and \( \{u_n\}_n^\infty \) such that for any \( n, \varrho_n > 0, \varrho_n \to 0, u_n \in U^\infty[0, T], \|u_n - \bar{u}\|_{L^8_F(0, T; H_1)} \to 0 \) (as \( n \to \infty \)) and

\[
J(u_n) < J(\bar{u}) + \frac{\varrho_n}{2}\|u_n - \bar{u}\|^2_{L^8_F(0, T; H_1)}.
\]

Let

\[ v_n = \frac{u_n - \bar{u}}{\|u_n - \bar{u}\|_{L^8_F(0, T; H_1)}}. \]

It is clear that \( v_n \) is a unit vector of \( L^8_F(0, T; H_1) \) for any \( n \in \mathbb{N} \), and there exists a subsequence \( \{u_{n_k}\}_{k=1}^\infty \) which converges weakly to a vector \( v \in L^8_F(0, T; H_1) \). Without loss of generality, we assume \( v_n \xrightarrow{w} v \). Since \( U \) is convex, \( T_{ud^8[0, T]}(\bar{u}) \) is a closed convex set. Noting that \( L^8_F(0, T; H_1) \) is a reflexive Banach space, we have \( v \in T_{ud^8[0, T]}(\bar{u}) \).

Let us divide the rest of the proof into three steps.

**Step 1:** In this step, we prove that \( \mathbb{E} \int_0^T \langle \mathbb{H}_u(t), v(t) \rangle dt = 0 \). By (5.23),

\[
\mathbb{E} \int_0^T \langle \mathbb{H}_u(t), v_n(t) \rangle_{H_1} dt \leq 0,
\]

and hence

\[
\mathbb{E} \int_0^T \langle \mathbb{H}_u(t), v(t) \rangle_{H_1} dt \leq 0.
\]

If for some \( \varepsilon > 0 \),

\[
\mathbb{E} \int_0^T \langle \mathbb{H}_u(t), v \rangle_{H_1} dt < -\varepsilon < 0,
\]

by (5.21), it is easy to find that

\[
J(u_n) \geq J(\bar{u}) - \mathbb{E} \int_0^T \langle \mathbb{H}_u(t), u_n(t) - \bar{u}(t) \rangle_{H_1} dt - o(\|u_n(t) - \bar{u}(t)\|_{L^8_F(0, T; H_1)})
\]

\[
= J(\bar{u}) - \|u_n - \bar{u}\|_{L^8_F(0, T; H_1)}^2 \mathbb{E} \int_0^T \langle \mathbb{H}_u(t), v_n(t) \rangle_{H_1} dt - o(\|u_n(t) - \bar{u}(t)\|_{L^8_F(0, T; H_1)}).
\]

Then, by assumption (5.24), we have that

\[ -\mathbb{E} \int_0^T \langle \mathbb{H}_u(t), v_n(t) \rangle_{H_1} dt - o(1) < \frac{\varrho_n}{2}\|u_n - \bar{u}\|_{L^8_F(0, T; H_1)}. \]

Letting \( n \to \infty \), we get that

\[ 0 < \varepsilon < -\mathbb{E} \int_0^T \langle \mathbb{H}_u(t), v(t) \rangle_{H_1} dt \leq 0, \]

a contradiction. Therefore,

\[ \mathbb{E} \int_0^T \langle \mathbb{H}_u(t), v(t) \rangle dt = 0. \]
Step 2: In this step, we prove that \( v \neq 0 \).

If not, \( \Lambda(v) = \Lambda(0) = 0 \). Using (5.21) again, we obtain that

\[
J(u_n) \geq J(\tilde{u}) - \mathbb{E} \int_0^T \langle \mathbb{H}_n(t), u_n(t) - \tilde{u}(t) \rangle_{H_1} dt - \frac{1}{2} \Lambda(u_n(t) - \tilde{u}(t)) - C \left( \| u_n(t) - \tilde{u}(t) \|_{L^{\infty}_F(0,T;H_1)} \right) \| u_n(t) - \tilde{u}(t) \|_{L^2_F(0,T;H_1)}^2 \geq J(\tilde{u}) - \frac{1}{2} \Lambda(u_n(t) - \tilde{u}(t)) - C \left( \| u_n(t) - \tilde{u}(t) \|_{L^{\infty}_F(0,T;H_1)} \right) \| u_n(t) - \tilde{u}(t) \|_{L^2_F(0,T;H_1)}^2.
\]

By (5.24),

\[
- \frac{1}{2} \Lambda(u_n(t) - \tilde{u}(t)) - C \left( \| u_n(t) - \tilde{u}(t) \|_{L^{\infty}_F(0,T;H_1)} \right) \| u_n(t) - \tilde{u}(t) \|_{L^2_F(0,T;H_1)}^2 < \frac{\varrho_n}{2} \| u_n - \tilde{u} \|_{L^2_F(0,T;H_1)}^2,
\]

which implies

\[
- \frac{1}{2} \Lambda(v_n) - C \| u_n(t) - \tilde{u}(t) \|_{L^{\infty}_F(0,T;H_1)} < \frac{\varrho_n}{2}.
\]

Since \(-\Lambda\) is weakly lower semicontinuous,

\[
\lim_{n \to \infty} -\Lambda(v_n) \geq -\Lambda(v).
\]

Then, by (5.25),

\[
0 = -\frac{1}{2} \Lambda(v) \leq \lim_{n \to \infty} \left( -\frac{1}{2} \Lambda(v_n) - C \| u_n(t) - \tilde{u}(t) \|_{L^{\infty}_F(0,T;H_1)} \right) \leq \lim_{n \to \infty} \frac{\varrho_n}{2} = 0,
\]

which implies that there exists a subsequence \( \{v_{n_k}\}_{k=1}^{\infty} \) (of \( \{v_n\}_{n=1}^{\infty} \)) such that \(-\Lambda(v_{n_k}) \to -\Lambda(v) = 0\) as \( k \to \infty \). Since \(-\Lambda\) is a Legendre form and \( v_{n_k} \xrightarrow{w} v \), we have \( v_{n_k} \to v \) strongly. But \( \| v_{n_k} \|_{L^2_F(0,T;H_1)} = 1 \) and therefore \( \| v \|_{L^2_F(0,T;H_1)} = 1 \), contradicting to the assumption that \( v = 0 \).

Step 3: By Steps 1 and 2, we have proved that \( v \in C_{\mathcal{U}^S[0,T]}(\tilde{u}) \). Then by Assumption (A8), there exists a constant \( \varepsilon > 0 \) such that

\[
-\Lambda(v) \geq \varepsilon > 0,
\]

which gives

\[
0 < \frac{\varepsilon}{2} \leq -\frac{1}{2} \Lambda(v) \leq \lim_{n \to \infty} \left( -\frac{1}{2} \Lambda(v_n) - C \| u_n(t) - \tilde{u}(t) \|_{L^{\infty}_F(0,T;H_1)} \right) \leq \lim_{n \to \infty} \frac{\varrho_n}{2} = 0,
\]

a contradiction. This completes the proof of Theorem 5.2. \(\square\)

**Corollary 5.2** Assume that \( x_0 \in L^2_F(\Omega; H) \), (A1)–(A3), (A5) and (A7)–(A8) hold. Let \((P,Q)\) be the unique \( V \)-transposition solution to BSEE (2.15) with \( P_T, J(\cdot), K(\cdot) \) and \( F(\cdot) \) given by (5.23) and let \( \bar{u}(\cdot) \) be an admissible control with \( \bar{x}(\cdot) \) the corresponding state. If \(-\bar{\Lambda}\) is a Legendre form on \( L^2_F(0,T;H_1)\),

\[
\bar{\Lambda}(v) < 0, \quad \forall \ v \in C_{\mathcal{U}^S[0,T]}(\tilde{u})
\]

34
and for any \( u(\cdot) \in U^\infty[0,T] \),

\[
E \int_0^T \langle \mathcal{H}_u(t), u(t) - \bar{u}(t) \rangle_{H_1} \, dt \leq 0,
\]

then there exist constants \( \sigma > 0 \) and \( \rho > 0 \) such that for any \( u(\cdot) \in U^\infty[0,T] \) with \( \| u - \bar{u} \|_{L^\infty(0,T;H_1)} \leq \sigma \), the quadratic growth condition \( (5.12) \) holds.

**Remark 5.1** The proof of Theorem 5.2 is a modification of the related conclusion in deterministic optimization problem, see [8, Chapter 3]. The corresponding results in the deterministic optimal control problem can be found in [7], and that in optimal control problems of stochastic differential equations can be found in [9]. Note that, in Theorem 5.2, we do not need the assumptions that \( b_{ww}(t,x,u) \equiv 0 \) or the maps \( (x,u) \mapsto a(t,x,u) \) and \( (x,u) \mapsto b(t,x,u) \) are affine for a.e. \( t \in [0,T] \). Therefore, Theorem 5.2 is much more general than [8, Proposition 4.15]. In addition, even though the condition \( (5.22) \) in Theorem 5.2 is weaker than the condition \( (5.11) \) in Theorem 5.1 in the stochastic cases, there exist some essential difficulties to verify if the corresponding quadratic-like forms \(-\Lambda\) or \(-\bar{\Lambda}\) are Legendre form (see [9, Examples 4.16–4.17]). Therefore, sometimes it is much more convenient to use Theorem 5.1 in practice.

### 6 Examples

In this section, we shall give some examples. Firstly, we apply our second order necessary condition for systems of controlled stochastic heat equations. The same thing can be done for lots of other systems, such as stochastic Schrödinger equations, stochastic Korteweg-de Vries equations, stochastic Kuramoto-Sivashinsky equations, stochastic Cahn-Hilliard equations, etc.

**Example 6.1** Let \( H = L^2[0,1] \times L^2[0,1], \ H_1 = H^1_0(0,1) \times H^1_0(0,1), \ V = H^{-1}(0,1) \times H^{-1}(0,1) \) and \( U = H^1_0(0,1) \times B_{H^1_0(0,1)} \) where \( B_{H^1_0(0,1)} \) is the closed unit ball in \( H^1_0(0,1) \). Then \( V' = H^1_0(0,1) \times H^1_0(0,1) \). Define an operator \( A \) by

\[
\begin{cases}
D(A) = H^2(0,1) \cap H^1_0(0,1), \\
A f = \partial_{xx} f, \quad \forall f \in D(A).
\end{cases}
\]

It is clear that the embedding from \( H \) to \( V \) is Hilbert-Schmidt and \( A \) generates a \( C_0 \)-semigroup on \( H^1_0(0,1) \). Consider the following control system:

\[
\begin{aligned}
d\varphi_1 &= \partial_{xx} \varphi_1 \, dt + u_1 \, dt + (\varphi_1 + \varphi_2) \, dW(t) \quad \text{in} \ (0,T] \times (0,1), \\
d\varphi_2 &= \partial_{xx} \varphi_2 \, dt + u_2 \, dW(t) \quad \text{in} \ (0,T] \times (0,1), \\
\varphi_1(t,0) &= \varphi_1(t,1) = 0, \quad \text{in} \ (0,T), \\
\varphi_2(t,0) &= \varphi_2(t,1) = 0, \quad \text{in} \ (0,T), \\
\varphi_1(0,x) &= \varphi(x), \quad \text{on} \ (0,1), \\
\varphi_2(0,x) &= 0, \quad \text{on} \ (0,1),
\end{aligned}
\]  

(6.1)
and the cost functional

\[ J(u) = \frac{1}{2} \mathbb{E} \langle \varphi_1(T), \varphi_1(T) \rangle_{L^2(0,1)}. \]

It is easy to see that (A1)–(A3) hold for the above optimal control problem. Furthermore,

\[ a_x = 0 \in L^\infty_\mathcal{F}(0,T; \mathcal{L}_H\mathcal{V}), \quad b_x = \left( \begin{array}{cc} I & I \\ 0 & 0 \end{array} \right) \in L^\infty_\mathcal{F}(0,T; \mathcal{L}_H\mathcal{V}). \]

Then, we see that (A5) holds.

Let \( \phi = \sum_{n=1}^\infty a_n \sqrt{2} \sin n\pi x \in L^2(0,1) \). We claim that \((u_1, u_2) = (f, 0) \in \mathcal{U}^2[0,T] \), where

\[ f(t, x) = \sum_{n=1}^\infty f_n(t) \sqrt{2} \sin n\pi x \quad \text{for} \quad f_n(t) = -\frac{a_n}{T} e^{-(n^2\pi^2+1/2)t} + W(t) \]

is an optimal control. Indeed, if \((u_1, u_2) = (f, 0) \), then the corresponding solution \((\varphi_1, \varphi_2)\) satisfies that \( \varphi_1(T) = 0 \). Next, direct computations show that \( f \in L^2(0,T; H_0^1(0,1)) \). This verifies our claim. Furthermore, one can show that \( f(\cdot) \in L^{1,2}_{\mathcal{F}}(H_0^1(0,1)) \). Hence, we find that the first condition in (A6) holds.

For this optimal control problem, the Hamiltonian is

\[ \mathbb{H}(t, (\varphi_1, \varphi_2), (u_1, u_2), (p_1, p_2), (q_1, q_2)) = p_1 u_1 + q_1 (\varphi_1 + \varphi_2) + q_2 u_2^2, \]

and the corresponding first order adjoint equation is

\[
\begin{aligned}
dp_1 &= -\partial_x p_1 dt - q_1 dt + q_1 dW(t) \quad \text{in} \ [0,T] \times (0,1), \\
dp_2 &= -\partial_x p_2 dt - q_1 dt + q_2 dW(t) \quad \text{in} \ [0,T] \times (0,1), \\
p_1(\cdot, 0) &= p_1(\cdot, 1) = 0, \quad \text{on} \ [0,T], \\
p_2(\cdot, 0) &= p_2(\cdot, 1) = 0, \quad \text{on} \ [0,T], \\
p_1(T, \cdot) &= p_2(T, \cdot) = 0, \quad \text{in} \ (0,1).
\end{aligned}
\]

Obviously, \((p_1, p_2) \equiv 0\), \((q_1, q_2) \equiv 0\) and therefore,

\[ \mathbb{H}(t, (\varphi_1, \varphi_2), (u_1, u_2), (p_1, q_1), (p_2, q_2)) \equiv 0. \]

Then, the second order adjoint equation reads

\[
\begin{aligned}
dP &= - \left( \begin{array}{cc} A^* P_{11} + P_{11} A^* & A^* P_{12} + P_{12} A^* \\ A^* P_{21} + P_{21} A^* & A^* P_{22} + P_{22} A^* \end{array} \right) dt \\
&\quad - \left[ \begin{array}{cc} 2Q_{11} & Q_{11} + Q_{12} \\ Q_{11} + Q_{21} & Q_{12} + Q_{21} \end{array} \right] dt + \left( \begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right) dW(t) \quad \text{in} \ [0,T], \\
P(T) &= \left( \begin{array}{cc} -I & 0 \\ 0 & 0 \end{array} \right).
\end{aligned}
\]

(6.3)
It is clear that \( Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = 0. \) Then \( P \) is the solution to
\[
\begin{aligned}
&dP = - \begin{pmatrix} A^*P_{11} + P_{11}A^* & A^*P_{12} + P_{12}A^* \\ A^*P_{21} + P_{21}A^* & A^*P_{22} + P_{22}A^* \end{pmatrix} dt - \begin{pmatrix} P_{11} & P_{11} \\ P_{11} & P_{11} \end{pmatrix} dt \quad \text{in } [0, T),
\end{aligned}
\]
\[
P(T) = \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix}.
\]

Obviously,
\[
P(\cdot) = -e^{A(T-\cdot)}J e^{A(T-\cdot)} - \int_0^T e^{A(s-\cdot)} P(s) e^{A(s-\cdot)} ds
\]
\[
\in L^1_{2, \mathbb{P}}(L_2([H_0^1(0,1)]^2; (L^2(0, 1))^2]) \cap L^\infty([0, T] \times \Omega; L_2([H_0^1(0,1)]^2; (L^2(0, 1))^2])
\]

Further, by the classical theory of Riccati equations (see [4, Part IV, Section 2.2, Theorem 2.1]), we know that \( P_{11}(t) < 0, \) for any \( t \in [0, T]. \)

Since \( b_u(t, \bar{x}(t), \bar{u}(t)) \equiv 0, \) we have,
\[
\mathbb{H}_u(t, (\varphi^1, \varphi^2), (u_1, u_2), (p^1, q^1), (p^2, q^2)) \equiv 0,
\]
\[
\mathbb{H}_{uu}(t, (\varphi^1, \varphi^2), (u_1, u_2), (p^1, q^1), (p^2, q^2)) + b_u(t, \bar{x}(t), \bar{u}(t))^* P(t) b_u(t, \bar{x}(t), \bar{u}(t)) \equiv 0,
\]
and
\[
S(t) = \begin{pmatrix} P_{11} & P_{12} \\ 0 & 0 \end{pmatrix}, \quad \nabla S(t) = 0.
\]

Then, we see that the second condition in (A6) holds.

In what follows, we consider an application of the second order sufficient condition in the stochastic LQ problems.

**Example 6.2** Let us consider the following linear control system
\[
\begin{aligned}
dx &= (Ax + B_1x + C_1u) dt + (B_2x + C_2u) dW(t) \quad \text{in } (0, T],
\end{aligned}
\]
and the cost functional
\[
J(u(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T \left( \langle Rx(t), x(t) \rangle_H + 2 \langle x(t), Mu(t) \rangle_H + \langle Nu(t), u(t) \rangle_{H_1} \right) dt.
\]

We assume that \( B_1, B_2 \in \mathcal{L}_{H^\prime}, C_1, C_2, M \in \mathcal{L}(H_1; H), R \in \mathcal{L}(H), N \in \mathcal{L}(H_1). \) Moreover, \( R \) and \( N \) are self-adjoint.

It is clearly that, the optimal control problem (1.3) for control system (6.4) and cost functional (6.5) is well-defined on \( \mathcal{U}^2[0, T]. \)

Let \((\bar{x}, \bar{u})\) be an admissible pair. Define
\[
\mathbb{H}(t, x, u, k_1, k_2) \triangleq \langle k_1, B_1x + C_1u \rangle_H + \langle k_2, B_2x + C_2u \rangle_H
\]
\[-\frac{1}{2} \left( \langle Rx(t), x(t) \rangle_H + 2 \langle x(t), Mu(t) \rangle_H + \langle Nu(t), u(t) \rangle_{H_1} \right),
\]
\[(t, x, u, k_1, k_2) \in [0, T] \times H \times U \times H \times H,
\]

37
and define the first and second order adjoint equations:

\[
\begin{cases}
  dp = -A^*pd - (B_1^*p + B_2^*q - R\bar{x}(t) - M\bar{u}(t))dt + qdW(t) & \text{in } [0, T), \\
p(T) = 0
\end{cases}
\]  
\tag{6.6}

and

\[
\begin{cases}
  dP = -(A^* + B_1^*)Pdt - P(A + B_1)dt - B_2^*PB_2dt \\
    - (B_2^*Q + QB_2)dt + Rdt + QdW(t) & \text{in } [0, T), \\
P(T) = 0.
\end{cases}
\]  
\tag{6.7}

Obviously, BSEE\, [6.6] admits a unique transposition solution \((p, q)\), and, BSEE\, [6.7] admits a unique \(V\)-transposition solution \((P, Q)\). In addition, since the operators \(B_1, B_2\) and \(N\) independent of \((t, \omega)\), we have that \((P, Q)\) is actually the solution to the follow deterministic operator-valued evolution equation:

\[
\begin{cases}
  dP = -(A^* + B_1^*)Pdt - P(A + B_1)dt - B_2^*PB_2dt + Rdt & \text{in } [0, T), \\
P(T) = 0.
\end{cases}
\]  
\tag{6.8}

Let \(u(\cdot) \in U^2[0, T]\) be another admissible control with the corresponding state \(x(\cdot)\) and denote \(\delta x = x(\cdot) - \bar{x}(\cdot), \delta u = u(\cdot) - \bar{u}(\cdot)\). We have that

\[
\begin{align*}
  \mathcal{J}(u) - \mathcal{J}(\bar{u}) &= \mathbb{E} \int_0^T \left( \langle R\bar{x}(t), \delta x(t) \rangle_H + \frac{1}{2} \langle R\delta x(t), \delta x(t) \rangle_H \right. \\
  &\quad + \langle N\bar{u}(t), \delta u(t) \rangle_{H_1} + \frac{1}{2} \langle N\delta u(t), \delta u(t) \rangle_{H_1} \\
  &\quad + \langle \bar{x}(t), M\delta u(t) \rangle_H + \langle \delta x(t), M\bar{u}(t) \rangle_H + \langle \delta x(t), M\delta u(t) \rangle_H \bigg) dt \\
  &= -\mathbb{E} \int_0^T \left( \langle \mathbb{H}_{\bar{u}}(t), \delta u(t) \rangle_{H_1} - \frac{1}{2} \langle R\delta x(t), \delta x(t) \rangle_H \right. \\
  &\quad - \frac{1}{2} \langle N\delta u(t), \delta u(t) \rangle_{H_1} - \langle \delta x(t), M\delta u(t) \rangle_H \bigg) dt.
\end{align*}
\]  
\tag{6.9}

By Itô’s formula,

\[
\begin{align*}
  -\mathbb{E} \int_0^T \left( \langle R\delta x(t), \delta x(t) \rangle_H + \langle N\delta u(t), \delta u(t) \rangle_{H_1} + 2\langle \delta x(t), M\delta u(t) \rangle_H \right) dt \\
  &= -\mathbb{E} \int_0^T \left[ 2\langle S\delta x(t), \delta u(t) \rangle_{H_1} + \langle (N + C_1^*P_2C_2)\delta u(t), \delta u(t) \rangle_{H_1} \right] dt \\
  &= -\tilde{\Lambda}(\delta u(\cdot)),
\end{align*}
\]  
\tag{6.10}

where\[
S = C_1^*P + C_2^*PB_2 - M^*.
\]

Noting that in the this special case, the quadratic form \(\tilde{\Lambda}\) can be extended into the Hilbert space \(L^2_F(0, T; H_1)\). Therefore, the second order sufficient condition

\[
\tilde{\Lambda}(\delta u(\cdot)) \leq -\varrho \|\delta u\|^2_{L^2_F(0, T; H_1)}
\]  
\tag{6.11}
holds true if and only if the quadratic functional (defined on $L^2_F(0,T;H_1)$)
\[
F(\delta u) \triangleq -\mathbb{E} \int_0^T \langle (\Gamma^* R \Gamma + \Gamma^* M + N) \delta u(t), \delta u(t) \rangle_{H_1} dt \leq -\varrho \|\delta u\|_{L^2_F(0,T;H_1)}^2,
\]
(6.12)
where $\Gamma \delta u = \delta x$.

Using a similar argumentation as in Section 5, we have, when condition (6.12) is satisfied, any $(\bar{x}, \bar{u})$ is an local optimal pair if
\[
\mathbb{E} \int_0^T \langle H_u(t), u(t) - \bar{u}(t) \rangle_{H_1} dt \leq 0, \quad \forall u(\cdot) \in U^2[0,T].
\]

Furthermore, since the inequality (6.11) holds true for any $u(\cdot) \in U^2[0,T]$, we know that the $(\bar{x}, \bar{u})$ satisfying the above inequality is the unique globally minimizer.

References

[1] Ch. A. Agayeva, Second order necessary conditions of optimality for stochastic systems with variable delay. Theor. Probability and Math. Statist. 83 (2011), 1–12.

[2] A. Al-Hussein, Backward stochastic partial differential equations driven by infinite dimensional martingales and applications. Stochastics. 81 (2009), 601–626.

[3] D. J. Bell and D. H. Jacobson, Singular Optimal Control Problems, Math. Sci. Eng. 117, Academic Press, New York, 1975.

[4] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, Representation and control of infinite dimensional systems. Birkhäuser Boston, Inc., Boston, MA, 2007.

[5] V. G. Boltyanskiĭ, Optimal Control of Discrete Systems. Translated from the Russian edition (Nauka, Moscow, 1973) by R. Hardin. John Wiley & Sons, New York, 1978.

[6] J. F. Bonnans, X. Dupuis and L. Pfeiffer, Second-order necessary conditions in Pontryagin form for optimal control problems. SIAM J. Control Optim. 52 (2014), 3887–3916.

[7] J. F. Bonnans, X. Dupuis and L. Pfeiffer, Second-order sufficient conditions for strong solutions to optimal control problems. ESAIM Control Optim. Calc. Var. 20 (2014), 704–724.

[8] J. F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems. Springer, New York, 2000.

[9] J. F. Bonnans and F. J. Silva, First and second order necessary conditions for stochastic optimal control problems. Appl. Math. Optim. 65 (2012), 403–439.

[10] E. Casas and F. Tröltzsch, Second order analysis for optimal control problems: improving results expected from abstract theory. SIAM J. Optim. 22 (2012), 261–279.

[11] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge, 1992.

[12] H. Frankowska and D. Tonon, Pointwise second-order necessary optimality conditions for the Mayer problem with control constraints. SIAM J. Control Optim. 51 (2013), 3814–3843.
[13] H. Frankowska, H. Zhang and X. Zhang, First and second order necessary conditions for stochastic optimal controls. J. Differential Equations. 262 (2017), 3689–3736.

[14] H. Frankowska, H. Zhang and X. Zhang, Stochastic optimal control problems with control and initial-final states constraints. SIAM J. Control Optim. 56 (2018), 1823–1855.

[15] H. Frankowska, H. Zhang and X. Zhang, Necessary optimality conditions for local minimizers of stochastic optimal control problems with state constraints. Trans. Amer. Math. Soc. In press.

[16] R. Gabasov and F. M. Kirillova, High order necessary conditions for optimality. SIAM J. Control. 10 (1972), 127–168.

[17] B. S. Goh, Necessary conditions for singular extremals involving multiple control variables. SIAM J. Control. 4 (1966), 716–731.

[18] U. G. Haussmann, General necessary conditions for optimal control of stochastic system. Math. Prog. Study. 6 (1976), 34–48.

[19] D. Hoehener, Variational approach to second-order sufficient optimality conditions in optimal control. SIAM J. Control Optim. 52 (2014), 861–892.

[20] Y. Hu and S. Peng, Adapted solution of backward semilinear stochastic evolution equations. Stoch. Anal. & Appl. 9 (1991), 445–459.

[21] D. H. Jacobson, Sufficient conditions for nonnegativity of the second variation in singular and nonsingular control problems. SIAM J. Control. 8 (1970), 403–423.

[22] H.-W. Knobloch, Higher Order Necessary Conditions in Optimal Control Theory. Springer-Verlag, New York, 1981.

[23] A. J. Krener, The high order maximal principle and its application to singular extremals. SIAM J. Control Optim. 15 (1977), 256–293.

[24] X. Li and J. Yong, Optimal Control Theory for Infinite-Dimensional Systems. Birkhäuser Boston, Inc., Boston, MA, 1995.

[25] H. Lou, Second-order necessary/sufficient conditions for optimal control problems in the absence of linear structure. Discrete Contin. Dyn. Syst. Ser. B. 14 (2010), 1445–1464.

[26] Q. Lü, Second order necessary conditions for optimal control problems of stochastic evolution equations. Proceedings of the 35th Chinese Control Conference. Chengdu, China, July 27-29, (2016), 2620–2625.

[27] Q. Lü and X. Zhang, General Pontryagin-Type Stochastic Maximum Principle and Backward Stochastic Evolution Equations in Infinite Dimensions. Springer Briefs in Mathematics. Springer, Cham, 2014.

[28] Q. Lü and X. Zhang, Transposition method for backward stochastic evolution equations revisited, and its application. Math. Control Relat. Fields. 5 (2015), 529–555.

[29] Q. Lü and X. Zhang, Operator-valued backward stochastic Lyapunov equations in infinite dimensions, and its application. Math. Control Relat. Fields. 8 (2018), 337–381.
[30] J. Ma and J. Yong, *Forward-Backward Stochastic Differential Equations and Their Applications*. Lecture Notes in Math. vol. 1702. Springer-Verlag, New York, 1999.

[31] N. I. Mahmudov and A. E. Bashirov, *First order and second order necessary conditions of optimality for stochastic systems*. Statistics and control of stochastic process (Moscow, 1995/1996), 283–295, World Sci. Publ. Re Horm Edge, NJ, 1997.

[32] N. I. Mahmudov and M. A. McKibben, *On backward stochastic evolution equations in Hilbert spaces and optimal control*. Nonlinear Anal. 67 (2007), 1260–1274.

[33] H. G. Moyer, *Sufficient conditions for a strong minimum in singular control problems*. SIAM J. Control. 11 (1973), 620–636.

[34] D. Nualart, *The Malliavin Calculus and Related Topics*. Springer-Verlag, Berlin, 2006.

[35] N. P. Osmolovskii, *Second-order sufficient optimality conditions for control problems with linearly independent gradients of control constraints*. ESAIM Control Optim. Calc. Var. 18 (2012), 452–482.

[36] N. P. Osmolovskii and H. Maurer, *Applications to Regular and Bang-Bang Control. Second-Order Necessary and Sufficient Optimality Conditions in Calculus of Variations and Optimal Control*. SIAM, Philadelphia, PA, 2012.

[37] S. Peng, *A general stochastic maximum principle for optimal control problems*. SIAM J. Control Optim. 28 (1990), 966–979.

[38] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mischenko, *Mathematical Theory of Optimal Processes*. Wiley, New York, 1962.

[39] S. Tang, *A second-order maximum principle for singular optimal stochastic controls*. Discrete Contin. Dyn. Syst. Ser. B. 14 (2010), 1581–1599.

[40] J. M. A. M. van Neerven, M. C. Veraar and L. Weis, *Stochastic integration in UMD Banach spaces*. Ann. Probab. 35 (2007), 1438–1478.

[41] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer-Verlag, New York, 1999.

[42] V. Zeidan, *First and second order sufficient conditions for optimal control and the calculus of variations*. Appl. Math. Optim. 11 (1984), 209–226.

[43] H. Zhang and X. Zhang, *Pointwise second-order necessary conditions for stochastic optimal controls, Part I: The case of convex control constraint*. SIAM J. Control Optim. 53 (2015), 2267–2296.

[44] H. Zhang and X. Zhang, *Pointwise second-order necessary conditions for stochastic optimal controls, Part II: The general cases*. SIAM J. Control Optim. 55 (2017), 2841–2875.

[45] H. Zhang and X. Zhang, *Second-order necessary conditions for stochastic optimal control problems*. SIAM Rev. 60 (2018), 139–178.