Hermite–Hadamard–Fejér-Type Inequalities and Weighted Three-Point Quadrature Formulae

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Abstract: The goal of this paper is to derive Hermite–Hadamard–Fejér-type inequalities for higher-order convex functions and a general three-point integral formula involving harmonic sequences of polynomials and w-harmonic sequences of functions. In special cases, Hermite–Hadamard–Fejér-type estimates are derived for various classical quadrature formulae such as the Gauss–Legendre three-point quadrature formula and the Gauss–Chebyshev three-point quadrature formula of the first and of the second kind.

Keywords: Hermite–Hadamard–Fejér inequalities; weighted three-point formulae; higher-order convex functions; w-harmonic sequences of functions; harmonic sequences of polynomials

MSC: 26D15; 65D30; 65D32

1. Introduction

The Hermite–Hadamard inequalities and their weighted versions, the so-called Hermite–Hadamard–Fejér inequalities, are the most well-known inequalities related to the integral mean of a convex function (see [1] (p. 138)).

Theorem 1 (The Hermite–Hadamard–Fejér inequalities). Let \( h : [a, b] \to \mathbb{R} \) be a convex function. Then

\[
h \left( \frac{a + b}{2} \right) \int_a^b u(x) \, dx \leq \int_a^b u(x)h(x) \, dx \leq \left[ \frac{1}{2} h(a) + \frac{1}{2} h(b) \right] \int_a^b u(x) \, dx, \tag{1}\]

where \( u : [a, b] \to \mathbb{R} \) is nonnegative, integrable and symmetric about \( \frac{a+b}{2} \). If \( h \) is a concave function, then the inequalities in (1) are reversed.

If \( u \equiv 1 \), then we are talking about the Hermite–Hadamard inequalities.

Hermite–Hadamard and Hermite–Hadamard–Fejér-type inequalities have many applications in mathematical analysis, numerical analysis, probability and related fields. Their generalizations, refinements and improvements have been an important topic of research (see [1–13], and the references listed therein). In the past few years, Hermite–Hadamard–Fejér-type inequalities for superquadratic functions [2], GA-convex functions [7], quasi-convex functions [11] and convex functions [13] have been largely investigated in the literature.

The importance and significance of our paper are reflected in the way in which we prove new Hermite–Hadamard–Fejér-type inequalities for higher-order convex functions and the general weighted three-point quadrature formula by using inequality (1), and a weighted version of the integral identity expressed by w-harmonic sequences of functions.

For this purpose, let us introduce the notations and terminology used in relation to w-harmonic sequences of functions (see [14]).
Let us consider a subdivision \( \sigma = \{ a = x_0 < x_1 < \cdots < x_m = b \} \) of the segment \([a, b]\), \(m \in \mathbb{N}\). Let \(w : [a, b] \to \mathbb{R}\) be an arbitrary integrable function. For each segment \([x_{j-1}, x_j]\), \(j = 1, \ldots, m\), we define \(w\)-harmonic sequences of functions \(\{w_{jk}\}_{k=1}^{n}\) by:

\[
\begin{align*}
    w'_{jk}(t) &= w(t), \ t \in [x_{j-1}, x_j], \\
    w''_{jk}(t) &= w_{j,k-1}(t), \ t \in [x_{j-1}, x_j], \ k = 2, 3, \ldots, n.
\end{align*}
\]  

(2)

Further, the function \(W_{n,w}\) is defined as follows:

\[
W_{n,w}(t, \sigma) = \begin{cases} 
    w_{1n}(t), & t \in [a, x_1], \\
    w_{2n}(t), & t \in (x_1, x_2), \\
    \vdots \\
    w_{mn}(t), & t \in (x_{m-1}, b). 
\end{cases}
\] 

(3)

The following theorem gives a general integral identity (see [14]).

**Theorem 2.** Let \(f : [a, b] \to \mathbb{R}\) be such that \(f^{(n)}\) is piecewise continuous on \([a, b]\). Then, the following holds:

\[
\begin{align*}
    \int_a^b w(t) f(t) \, dt &= \sum_{k=1}^{n} (-1)^{k-1} \left[ w_{mk}(b) f^{(k-1)}(b) \\ + \sum_{j=1}^{m-1} \left[ w_{jk}(x_j) - w_{j+1,k}(x_j) \right] f^{(k-1)}(x_j) - w_{1k}(a) f^{(k-1)}(a) \right] \\
    &+ (-1)^n \int_a^b W_{n,w}(t, \sigma) f^{(n)}(t) \, dt.
\end{align*}
\]  

(4)

In [15], the authors proved the following Fejér-type inequalities by using identity (4).

**Theorem 3.** Let \(f : [a, b] \to \mathbb{R}\) be \((n+2)\)-convex on \([a, b]\) and \(f^{(n)}\) piecewise continuous on \([a, b]\). Further, let us suppose that the function \(W_{n,w}\), defined in (3), is nonnegative and symmetric about \(\frac{a+b}{2}\) (i.e., \(W_{n,w}(t, \sigma) = W_{n,w}(a+b-t, \sigma)\)). Then

\[
\begin{align*}
    U_n(\sigma) \cdot f^{(n)}(\frac{a+b}{2}) &
\leq (-1)^n \left\{ \int_a^b w(t) f(t) \, dt - \sum_{k=1}^{n} (-1)^{k-1} \left[ w_{mk}(b) f^{(k-1)}(b) \\ + \sum_{j=1}^{m-1} \left[ w_{jk}(x_j) - w_{j+1,k}(x_j) \right] f^{(k-1)}(x_j) - w_{1k}(a) f^{(k-1)}(a) \right] \right\} \\
    &\leq U_n(\sigma) \cdot \left[ \frac{1}{2} f^{(n)}(a) + \frac{1}{2} f^{(n)}(b) \right],
\end{align*}
\]  

(5)

where

\[
\begin{align*}
    U_n(\sigma) &= \frac{(-1)^n}{n!} \int_a^b w(t) \cdot t^n \, dt - (-1)^n \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n-k+1)!} \\
    &\quad \cdot \left( w_{mk}(b) a^{n-k+1} + \sum_{j=1}^{m-1} \left( w_{jk}(x_j) - w_{j+1,k}(x_j) \right) a^{n-k+1} - w_{1k}(a) a^{n-k+1} \right).
\end{align*}
\]  

(6)
If \( W_{n+1}(t, \sigma) \leq 0 \) or \( f \) is an \( (n+2) \)-concave function on \([a, b]\), then the inequalities in (5) hold with reversed inequality signs.

Further, let us recall the definition of the divided difference and the definition of an \( n \)-convex function (see [1] (p. 15)).

**Definition 1.** Let \( f \) be a real-valued function defined on the segment \([a, b]\). The divided difference of order \( n \) of the function \( f \) at distinct points \( x_0, \ldots, x_n \in [a, b] \) is defined recursively by

\[
f[x] = f(x_i), \quad (i = 0, \ldots, n)
\]

and

\[
f[x_0, \ldots, x_n] = f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}] \frac{x_n - x_0}{x_n - x_0}.
\]

The value \( f[x_0, \ldots, x_n] \) is independent of the order of points \( x_0, \ldots, x_n \).

**Definition 2.** A function \( f : [a, b] \to \mathbb{R} \) is said to be \( n \)-convex on \([a, b] \), \( n \geq 0 \), if, for all choices of \((n+1)\) distinct points \( x_0, \ldots, x_n \in [a, b] \), the \( n \)-th order divided difference in \( f \) satisfies

\[
f[x_0, \ldots, x_n] \geq 0.
\]

From the previous definitions, the following property holds: if \( f \) is an \( (n+2) \)-convex function, then there exists the \( n \)-th order derivative \( f^{(n)} \), which is a convex function (see, e.g., [1] (pp. 16, 293)).

The paper is organized as follows. After this introduction, in Section 2, we establish Hermite–Hadamard–Fejér-type inequalities for weighted three-point quadrature formulae by using the integral identity with \( w \)-harmonic sequences of functions, the properties of \( w \)-convex functions, and the properties of \( n \)-convex functions. Since we deal with three-point quadrature formulae that contain values of the function in nodes \( x, \frac{a+b}{2} \) and \( a+b-x \) and values of higher-ordered derivatives in inner nodes, the level of exactness of these quadrature formulae is retained. In Section 3, we derive Hermite–Hadamard–Fejér-type estimates for a generalization of the Gauss–Legendre three-point quadrature formula and a generalization of the Gauss–Chebyshev three-point quadrature formula of the first and of the second kind.

Throughout the paper, the symbol \( B \) denotes the beta function defined by

\[
B(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1} \, ds,
\]

\( \Gamma \) denotes the gamma function defined as:

\[
\Gamma(x) = 2 \int_0^\infty s^{2x-1} e^{-s} \, ds,
\]

and

\[
F(\alpha, \beta, \gamma; z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1}(1-zt)^{-\alpha} \, dt
\]

is a hypergeometric function with \( \gamma > \beta > 0, z < 1 \).

In the paper, we assume that all considered integrals exist and that they are finite.

2. Hermite–Hadamard–Fejér-Type Inequalities for Three-Point Quadrature Formulae

In this section, we establish Hermite–Hadamard–Fejér-type inequalities for the weighted three-point formula using a weighted version of the integral identity expressed by \( w \)-
harmonic sequences of functions that are given in Theorem 2 and the method that originated in [15].

In [16] (p. 54), the authors proved the following theorem.

**Theorem 4.** Let \( w : [a, b] \to \mathbb{R} \) be an integrable function, \( x \in \left[ a, \frac{a+b}{2} \right) \), and let \( \{ L_{j,x} \}_{j=0,1,...,n'} \), \( n \in \mathbb{N} \), be a sequence of harmonic polynomials such that \( \deg L_{j,x} \leq j - 1 \) and \( L_{0,x} \equiv 0 \). Further, let us suppose that \( \{ w_{jk} \}_{k=1,..,n} \) are \( w \)-harmonic sequences of functions on \( [x_{j-1}, x_j] \), for \( j = 1, 2, 3, 4 \), defined by the following relations:

\[
\begin{align*}
w_{1k}(t) &= \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} w(s) \, ds, \quad t \in [a, x], \\
w_{2k}(t) &= \frac{1}{(k-1)!} \int_x^t (t-s)^{k-1} w(s) \, ds + L_{k,x}(t), \quad t \in \left( x, \frac{a+b}{2} \right], \\
w_{3k}(t) &= -\frac{1}{(k-1)!} \int_t^a (t-s)^{k-1} w(s) \, ds + (-1)^k L_{k,x}(a + b - t), \quad t \in \left( \frac{a+b}{2}, a + b - x \right], \\
w_{4k}(t) &= -\frac{1}{(k-1)!} \int_t^b (t-s)^{k-1} w(s) \, ds, \quad t \in (a + b - x, b].
\end{align*}
\]

If \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n)} \) is piecewise continuous on \( [a, b] \), then we have

\[
\begin{align*}
\int_a^b w(t) f(t) \, dt &= \sum_{k=1}^n A_k(x) \left( f^{(k-1)}(x) + (-1)^k f^{(k-1)}(a + b - x) \right) \\
&\quad + \sum_{k=1}^n B_k(x) f^{(k-1)}(\frac{a+b}{2}) + (-1)^n \int_a^b W_{n,w}(t,x) f^{(n)}(t) \, dt, \quad (7)
\end{align*}
\]

where

\[
\begin{align*}
A_k(x) &= (-1)^{k-1} \left[ \frac{1}{(k-1)!} \int_x^a (x-s)^{k-1} w(s) \, ds - L_{k,x}(x) \right], \quad k \geq 1, \quad (8) \\
B_k(x) &= 2 \left[ \frac{1}{(k-1)!} \int_x^{\frac{a+b}{2}} (x-s)^{k-1} w(s) \, ds + L_{k,x}\left( \frac{a+b}{2} \right) \right], \quad \text{for odd } k \geq 1, \quad (9)
\end{align*}
\]

and

\[
B_k(x) = 0, \quad \text{for even } k \geq 1,
\]

such that

\[
W_{n,w}(t,x) = \begin{cases} 
  w_{1n}(t), & t \in [a, x], \\
  w_{2n}(t), & t \in \left( x, \frac{a+b}{2} \right], \\
  w_{3n}(t), & t \in \left( \frac{a+b}{2}, a + b - x \right], \\
  w_{4n}(t), & t \in (a + b - x, b].
\end{cases} \quad (10)
\]

**Remark 1.** If we assume \( w(t) = w(a + b - t) \), for each \( t \in [a, b] \), then the following symmetry conditions hold for \( k = 1, \ldots, n \):

\[
w_{1k}(t) = (-1)^k w_{4k}(a + b - t), \quad \text{for } t \in [a, x],
\]

and
where $w_{2k}(t) = (-1)^kw_{3k}(a + b - t)$, for $t \in \left[ x, \frac{a + b}{2}\right]$. 

Using Theorems 1 and 4, the properties of both $n$-convex functions and $w$-harmonic sequences of functions, and the method that originated in [15], in the next theorem, we derive new Hermite–Hadamard–Fejér-type inequalities for the weighted three-point quadrature Formula (7).

**Theorem 5.** Let $w : [a, b] \to \mathbb{R}$ be an integrable function such that $w(t) = w(a + b - t)$, for each $t \in [a, b]$ and $x \in [a, \frac{a + b}{2}]$. Let the function $W_{2n,w}$, defined by (10), be nonnegative. If $f : [a, b] \to \mathbb{R}$ is $(2n + 2)$-convex on $[a, b]$ and $f^{(2n)}$ is piecewise continuous on $[a, b]$, then

$$U_{n,w}(x) \cdot f^{(2n)}\left(\frac{a + b}{2}\right)$$

$$\leq \int_a^b w(t)f(t)\,dt - \frac{2n}{(2n)!} \sum_{k=1}^{2n} A_k(x) \left( f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a + b - x) \right)$$

$$- \frac{2n}{(2n)!} \sum_{k=1}^{2n} B_k(x) \left( \frac{a + b}{2} \right) \leq U_{n,w}(x) \cdot \left\{ \frac{1}{2} f^{(2n)}(a) + \frac{1}{2} f^{(2n)}(b) \right\},$$

where

$$U_{n,w}(x) = \frac{1}{(2n)!} \int_a^b w(t) \cdot t^{2n} \, dt$$

and $A_k$ and $B_k$ are defined as in Theorem 4. If $W_{2n,w}(t, x) \leq 0$ or $f$ is a $(2n + 2)$-concave function, then inequalities (11) hold with reversed inequality signs.

**Proof.** Let us observe that the function $f$ is $(2n + 2)$-convex. Hence, $f^{(2n)}$ is a convex function. It follows from Remark 1 that the function $W_{2n,w}$ is symmetric about $\frac{a + b}{2}$, i.e., $W_{2n,w}(t, x) = W_{2n,w}(a + b - t, x)$. Thus, inequalities (11) follow directly from Theorem 1, replacing a nonnegative and symmetric function $n$ by a nonnegative and symmetric function $W_{2n,w}$, and a convex function $h$ by a convex function $f^{(2n)}$, and then using identity (7) in $\int_a^b W_{2n,w}(t, x) f^{(2n)}(t) \, dt$.

Identity (7) yields $U_{n,w}(x)$ by substituting $n$ with $2n$ and putting $f(t) = \frac{t^{2n}}{(2n)!}$. Then, $f^{(2n)}(t) = 1$ and $f^{(k-1)}(t) = \frac{1}{(2n-k+1)!} t^{2n-k+1}$. On the other hand, if $W_{2n,w}(t, x)$ is nonpositive, then $-W_{2n,w}(t, x)$ is nonnegative, from where there follow reversed signs in (11).

Further, let us assume that $f$ is a $(2n + 2)$-concave function. Hence, the function $-f^{(2n)}$ is convex. Reversed signs in (11) are obtained by putting $-f^{(2n)}$ and the nonnegative function $W_{2n,w}(t, x)$ in (1). This completes the proof. \[\square\]

**Remark 2.** The value of $U_{n,w}(x)$ can be obtained from Theorem 3 by taking an appropriate subdivision of the segment $[a, b]$ and applying the properties of functions $w_{1k}, w_{2k}, w_{3k}$ and $w_{4k}$.
To get a maximum degree of exactness of quadrature Formula (7) for fixed \( x \in \left[a, \frac{a+b}{2}\right] \), we consider a sequence of harmonic polynomials \( \{L_{j,x}\}_{j=0,1,\ldots,n} \) defined as follows:

\[
L_{0,x}(t) = 0, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right], \\
L_{1,x}(x) = \int_a^x w(s) \, ds - \frac{2}{(a+b-2x)^2} \int_a^b \left(s^2 - \left(\frac{a+b}{2}\right)^2\right) w(s) \, ds, \quad (13) \\
L_{j,x}(x) = \frac{1}{(j-1)!} \int_a^x (x-s)^{j-1} w(s) \, ds, \quad j = 2, 3, 4, 5, 6, \\
L_{j,x}(t) = \sum_{k=1}^{n} L_{k,x}(x) \frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right], \quad j = 1, \ldots, n.
\]

Therefore, we have

\[
A_1(x) = \frac{2}{(a+b-2x)^2} \int_a^b \left(s^2 - \left(\frac{a+b}{2}\right)^2\right) w(s) \, ds, \quad (14) \\
B_1(x) = \int_a^b w(s) \, ds - 2A_1(x), \quad A_k(x) = 0, \quad \text{for } k = 2, 3, 4, 5, 6 \text{ and } B_k(x) = 0, \quad \text{for } k = 2, 3, 4.
\]

Finally, from identity (7), for \( x \in \left[a, \frac{a+b}{2}\right] \), we obtain the following three-point weighted integral formula:

\[
\int_a^b w(t) f(t) \, dt = A_1(x) \left[f(x) + f(a+b-x)\right] + \left(\int_a^b w(s) \, ds - 2A_1(x)\right) f\left(\frac{a+b}{2}\right) + T_{n,w}(x) + (-1)^n \int_a^b W_{n,w}(t,x) f^{(n)}(t) \, dt, \quad (15)
\]

where

\[
T_{n,w}(x) = \sum_{k=2}^{n} A_k(x) \left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) + \sum_{k=5, \text{odd}}^{n} B_k(x) f^{(k-1)}\left(\frac{a+b}{2}\right). \quad (16)
\]

Now, applying results from Theorem 5 to identity (15), we get the following results.

**Corollary 1.** Let \( w : [a, b] \to \mathbb{R} \) be an integrable function such that \( w(t) = w(a+b-t) \), for each \( t \in [a, b] \) and let \( x \in \left[a, \frac{a+b}{2}\right] \). Let the function \( W_{2n,w} \), defined by (10), be nonnegative and let \( L_{j,x} \) be defined by (13). If \( f : [a, b] \to \mathbb{R} \) is \((2n+2)\)-convex on \([a, b]\) and \( f^{(2n)} \) is piecewise continuous on \([a, b]\), then
\[ U_{n,w}(x) \cdot f^{(2n)} \left( \frac{a + b}{2} \right) \]  
\[ \leq \int_a^b w(t) f(t) \; dt - A_1(x) [f(x) + f(a + b - x)] \]  
\[ - \left( \int_a^b w(s) \; ds - 2A_1(x) \right) f \left( \frac{a + b}{2} \right) - T_{2n,w}(x) \]  
\[ \leq U_{n,w}(x) \cdot \left[ \frac{1}{2} f^{(2n)}(a) + \frac{1}{2} f^{(2n)}(b) \right], \]

where

\[ U_{n,w}(x) = \frac{1}{(2n)!} \int_a^b w(t) \cdot t^{2n} \; dt - A_1(x) \frac{x^{2n} + (a + b - x)^{2n}}{(2n)!} \]  
\[ - \left( \int_a^b w(s) \; ds - 2A_1(x) \right) (a + b)^{2n} \frac{2n}{2^{2n}(2n)!} \]  
\[ - \sum_{k=7}^{2n} A_k(x) \frac{x^{2n-k+1} + (-1)^{k-1}(a + b - x)^{2n-k+1}}{(2n - k + 1)!} \]  
\[ - \sum_{k=5, k \text{ odd}}^{2n} B_k(x) \frac{(a + b)^{2n-k+1}}{2^{2n-k+1}(2n - k + 1)!}. \]

If \( W_{2n,w}(t, x) \leq 0 \) or \( f \) is a \((2n + 2)\)-concave function, then inequalities (17) hold with reversed inequality signs.

**Proof.** The proof follows from Theorem 5 for the special choice of the polynomials \( L_{j,x} \). \( \Box \)

**Remark 3.** If we assume \( B_5(x) = 0 \), then we get

\[ x = \frac{a + b}{2} - \frac{1}{\sqrt{\int_a^b \left( s - \frac{a + b}{2} \right)^4 w(s) \; ds}} \cdot \frac{1}{\sqrt{\int_a^b \left( s^2 - \left( \frac{a + b}{2} \right)^2 \right)^2 w(s) \; ds}}. \]

Therefore, for such a choice of \( x \), we obtain the quadrature formula with three nodes, which is accurate for the polynomials of degree at most 5, and the approximation formula includes derivatives of order 6 and more.

### 3. Special Cases

Considering some special cases of the weight function \( w \), in our results given in the previous section, we obtain estimates for the Gauss–Legendre three-point quadrature formula and for the Gauss–Chebyshev three-point quadrature formula of the first and of the second kind.

#### 3.1. Gauss–Legendre Three-Point Quadrature Formula

Let us assume that \( w(t) = 1 \), \( t \in [a, b] \) and \( x \in \left[ a, \frac{a+b}{2} \right] \).

Now, from Theorem 4, we calculate
Proof. A special case of Theorem 5 for \( w(t) = 1, \ t \in [a, b] \), and a nonnegative function \( W_{2n}^{GL} \) defined by (19).

If we assume that the polynomials \( L_{j,a}(t) \) are such that

\[
L_{0,a}(t) = 0, \quad \text{for } t \in \left[ x, \frac{a+b}{2} \right],
\]

\[
L_{1,a}(x) = x - a - \frac{(b-a)^3}{6(a+b-2x)^2},
\]

\[
L_{j,a}(x) = \frac{(x-a)^j}{j!}, \quad j = 2, 3, 4, 5, 6,
\]

\[
L_{j,a}(t) = \sum_{k=1}^{6} L_{k,a}(x) \frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t \in \left[ x, \frac{a+b}{2} \right], \ j = 1, \ldots, n,
\]

and

\[
W_{n}^{GL}(t, x) = \begin{cases} 
  w_{1n}(t) = \frac{(t-a)^n}{n!}, & t \in [a, x], \\
  w_{2n}(t) = \frac{(t-b)^n}{n!} + L_{n,a}(t), & t \in \left( x, \frac{a+b}{2} \right], \\
  w_{3n}(t) = \frac{(t-a-b+x)^n}{n!} + (-1)^n L_{n,a}(a+b-t), & t \in \left( \frac{a+b}{2}, a+b-x \right], \\
  w_{4n}(t) = \frac{(t-b)^n}{n!}, & t \in (a+b-x, b], 
\end{cases}
\]  

(19)

and

\[
A_{k}^{GL}(x) = (-1)^{k-1} \left[ \frac{(x-a)^k}{k!} - L_{k,a}(x) \right], \quad \text{for } k \geq 1,
\]

\[
B_{k}^{GL}(x) = 2 \left( \frac{\frac{a+b}{2} - x}{k!} + L_{k,a} \left( \frac{a+b}{2} \right) \right), \quad \text{for odd } k \geq 1,
\]

and

\[
B_{k}^{GL}(x) = 0, \quad \text{for even } k > 1.
\]

Corollary 2. Let \( w_{2,2n}(t) \geq 0, \) for all \( t \in \left( x, \frac{a+b}{2} \right] \) and for \( n \in \mathbb{N} \). If \( f : [a, b] \to \mathbb{R} \) is a \((2n+2)\)-concave function and \( f^{(2n)} \) is piecewise continuous on \([a, b]\), then

\[
U_{n}^{GL}(x) \cdot f^{(2n)} \left( \frac{a+b}{2} \right) \leq \int_{a}^{b} \frac{f(t) dt}{\sum_{k=1}^{2n} A_{k}^{GL}(x) \left( f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x) \right) - \sum_{k=1,k \text{ odd}}^{2n} B_{k}^{GL}(x) f^{(k-1)} \left( \frac{a+b}{2} \right) \leq U_{n}^{GL}(x) \cdot \left[ \frac{1}{2} f^{(2n)}(a) + \frac{1}{2} f^{(2n)}(b) \right],
\]

where

\[
U_{n}^{GL}(x) = \frac{b^{2n+1} - a^{2n+1}}{(2n+1)!} - \sum_{k=1}^{2n} A_{k}^{GL}(x) \frac{x^{2n-k+1} + (-1)^{k-1} (a+b-x)^{2n-k+1}}{(2n-k+1)!} - \sum_{k=1,k \text{ odd}}^{2n} B_{k}^{GL}(x) \frac{(a+b)^{2n-k+1}}{2^{2n-k+1}(2n-k+1)!}.
\]  

(21)

If \( f \) is a \((2n+2)\)-concave function, then inequalities (20) hold with reversed inequality signs.

Proof. A special case of Theorem 5 for \( w(t) = 1, \ t \in [a, b] \), and a nonnegative function \( W_{2n}^{GL} \) defined by (19).
we get $A_{1}^{GL}(x) = \frac{(b-a)^3}{6(a+b-2x)^2}$, $A_{k}^{GL}(x) = 0$, for $k = 2, 3, 4, 5, 6$, $B_{1}^{GL}(x) = b - a - 2A_{1}^{GL}(x)$ and $B_{3}^{GL}(x) = 0$. Thus, we obtain the following non-weighted three-point quadrature formulae:

$$\int_{a}^{b} f(t) \, dt = \frac{(b-a)^3}{6(a+b-2x)^2}[f(x) + f(a + b - x)]$$

$$+ \left( b - a - \frac{(b-a)^3}{3(a+b-2x)^2} \right) f\left( \frac{a+b}{2} \right)$$

$$+ T_{n}^{GL}(x) + (-1)^{n} \int_{a}^{b} W_{n}^{GL}(t, x) f^{(n)}(t) \, dt,$$

where

$$T_{n}^{GL}(x) = \sum_{k=7}^{n} A_{k}^{GL}(x) \left( f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a + b - x) \right)$$

$$+ \sum_{k=5, odd}^{n} B_{k}^{GL}(x) f^{(k-1)}\left( \frac{a+b}{2} \right).$$

(24)

In particular, according to Remark 3, for $[a, b] = [-1, 1]$ and $x = \frac{-\sqrt{15}}{5}$, we get $B_{5}^{GL}(x) = 0$, and there follows a generalization of the Gauss–Legendre three-point formula. Now, we derive Hermite–Hadamard–Fejér-type estimates for this generalization of the Gauss–Legendre three-point formula.

If the assumptions of Corollary 1 hold for $w(t) = 1$, $t \in [-1, 1]$, and if $f : [-1, 1] \to \mathbb{R}$ is a $(2n + 2)$-convex function, we derive:

$$U_{n}^{GL}\left( \frac{-\sqrt{15}}{5} \right) \cdot f^{(2n)}(0)$$

$$\leq \frac{1}{9} \left[ 5f\left( \frac{-\sqrt{15}}{5} \right) + 8f(0) + 5f\left( \frac{\sqrt{15}}{5} \right) \right] - T_{2n}^{GL}\left( \frac{-\sqrt{15}}{5} \right)$$

$$\leq U_{n}^{GL}\left( \frac{-\sqrt{15}}{5} \right) \cdot \left[ \frac{1}{2} f^{(2n)}(-1) + \frac{1}{2} f^{(2n)}(1) \right],$$

where

$$U_{n}^{GL}\left( \frac{-\sqrt{15}}{5} \right) = \frac{2 \cdot 5^{n-1} - 2(2n + 1) \cdot 3^{n-2}}{5^{n-1}(2n+1)!}$$

$$- \sum_{k=7}^{2n} A_{k}^{GL}\left( \frac{-\sqrt{15}}{5} \right) \left( \frac{-\sqrt{15}}{5} \right)^{2n-k+1} + (-1)^{k-1}(\frac{\sqrt{15}}{5})^{2n-k+1} \frac{1}{5^{2n-k+1}(2n-k+1)!}.$$

(25)

In a special case, for $n = 3$, we get

$$\frac{1}{15,750} \cdot f^{(6)}(0)$$

$$\leq \frac{1}{9} \left[ 5f\left( \frac{-\sqrt{15}}{5} \right) + 8f(0) + 5f\left( \frac{\sqrt{15}}{5} \right) \right]$$

$$\leq \frac{1}{15,750} \cdot \left[ \frac{1}{2} f^{(6)}(-1) + \frac{1}{2} f^{(6)}(1) \right].$$

(26)
3.2. Gauss–Chebyshev Three-Point Quadrature Formula of the First Kind

Let us assume that \( w(t) = \frac{1}{\sqrt{1-t^2}} \), \( t \in (-1, 1) \) and \( x \in [-1, 0) \).

From Theorem 4, there follow:

\[
W_{n,w}^{GC_1}(t, x) = \begin{cases} 
 w_{1n}(t) = \frac{1}{(n-1)!} \int_{-1}^{t} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}} \, ds, & t \in [-1, x], \\
 w_{2n}(t) = \frac{1}{(n-1)!} \int_{x}^{t} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}} \, ds + L_{n,x}(t), & t \in (x, 0), \\
 w_{3n}(t) = -\frac{1}{n-1} \int_{x}^{t} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}} \, ds + (-1)^n L_{n,x}(-t), & t \in (0, -x), \\
 w_{4n}(t) = -\frac{1}{n-1} \int_{t}^{x} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}} \, ds, & t \in (-x, 1),
\end{cases}
\]

(27)

\[
A_k^{GC_1}(x) = (-1)^{k-1} \left[ \frac{(x + 1)^{k-1/2}}{\sqrt{21}} \frac{\sqrt{\pi}}{2} \Gamma \left( \frac{1}{2}, 0, 2 \cdot \frac{x}{2} + 1 + \frac{1}{2} \right) \right], \quad k \geq 1,
\]

and

\[
B_k^{GC_1}(x) = 2 \left[ \frac{(-1)^{k-1}}{(k-1)!} \int_{x}^{0} \frac{s^{k-1}}{\sqrt{1-s^2}} \, ds + L_{k,x}(0) \right], \quad \text{for odd } k \geq 1,
\]

and

\[
B_k^{GC_1}(x) = 0, \quad \text{for even } k > 1.
\]

Corollary 3. Let \( w_{2n}(t) \geq 0 \), for all \( t \in [x, 0] \) and for \( n \in \mathbb{N} \). If \( f : [-1, 1] \rightarrow \mathbb{R} \) is a \((2n + 2)\)-convex function and \( f^{(2n)} \) is piecewise continuous on \([-1, 1] \), then

\[
U_n^{GC_1}(x) \cdot f^{(2n)}(0) \leq \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} \, dt - \sum_{k=1}^{2n} A_k^{GC_1}(x) \left( f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(-x) \right)
\]

\[
- \sum_{k=1, k \text{ odd}}^{2n} B_k^{GC_1}(x) f^{(k-1)}(0) \leq U_n^{GC_1}(x) \cdot \left[ \frac{1}{2} f^{(2n)}(-1) + \frac{1}{2} f^{(2n)}(1) \right],
\]

where

\[
U_n^{GC_1}(x) = \frac{1}{(2n)!} B \left( \frac{1}{2} \cdot \frac{1}{2} + n \right)
\]

\[
- \sum_{k=1}^{2n} A_k^{GC_1}(x) \sum_{j=0}^{2n-k+1} (-1)^{k-1} (-x)^{2n-k+1} \frac{(2n-k+1)!}{(2n-k+1)!}.
\]

If \( f \) is a \((2n + 2)\)-concave function, then inequalities (28) hold with reversed inequality signs.

Proof. A special case of Theorem 5 for \( w(t) = \frac{1}{\sqrt{1-t^2}} \), \( t \in (-1, 1) \), and a nonnegative function \( W_{2n,w}^{GC_1} \) defined by (27). \( \Box \)

If we assume that the polynomials \( L_{j,x}(t) \) are such that
where

\[ L_{0,x}(t) = 0, \text{ for } t \in [x,0), \]
\[ L_{1,x}(x) = \arcsin x + \pi \frac{1}{2} - \pi \frac{1}{4} x^2, \]
\[ L_{j,x}(x) = \frac{(x+1)^{j-1/2}}{\sqrt{2^j F\left(\frac{1}{2}, \frac{1}{2} + j, \frac{x+1}{2}\right)}}, j = 2,3,4,5,6, \]
\[ L_{j,x}(t) = \frac{6^n j}{k!} \left(\frac{t-x}{(j-k)!}\right)^{j-k}, \text{ for } t \in [x,0], j = 1, \ldots, n, \]

we calculate \( A_{G1}^{G1}(x) = \frac{\pi}{4x^2}, A_{G1}^{G1}(x) = 0, \text{ for } k = 2,3,4,5,6, B_{G1}^{G1}(x) = \pi - \frac{\pi}{4x^2} \text{ and } B_{G3}^{G1}(x) = 0. \)

Now, we derive

\[
\frac{1}{\sqrt{1-t^2}} f(t) = \frac{\pi}{4x^2} f(x) + \left(\pi - \frac{\pi}{2x^2}\right) f(0) + \frac{\pi}{4x^2} f(-x)
\]
\[
+ T_{n,0}^{G1}(x) + (-1)^n \int_{-1}^{1} W_{n,0}^{G1}(t) f^{(n)}(t) dt,
\]

where

\[
T_{n,0}^{G1}(x) = \sum_{k=2}^{n} A_{k}^{G1}(x) \left(f^{(k-1)}(x) + (-1)^{k} f^{(k-1)}(-x)\right)
\]
\[
+ \sum_{k=3,odd}^{n} B_{k}^{G1}(x) f^{(k-1)}(0). \tag{31}
\]

In particular, there follows a generalization of the Gauss–Chebyshev three-point quadrature formula of the first kind for \( x = -\frac{\sqrt{3}}{2} \). Now, we derive Hermite–Hadamard-type estimates for the Gauss–Chebyshev three-point quadrature formula of the first kind.

If the assumptions of Corollary 1 hold for \( w(t) = \frac{1}{\sqrt{1-t^2}}, t \in (-1,1), \) and if \( f : [-1,1] \rightarrow \mathbb{R} \) is a \((2n+2)\)-convex function, we get

\[
U_{n}^{G1}\left(-\frac{\sqrt{3}}{2}\right) \cdot f^{(2n)}(0) \tag{32}
\]
\[
\leq \frac{1}{\sqrt{1-t^2}} f(t) dt - \pi \left[ f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right)\right] - T_{2n,0}^{G1}\left(-\frac{\sqrt{3}}{2}\right)
\]
\[
\leq U_{n}^{G1}\left(-\frac{\sqrt{3}}{2}\right) \cdot \left[\frac{1}{2} f^{(2n)}(-1) + \frac{1}{2} f^{(2n)}(1)\right],
\]

where

\[
U_{n}^{G1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{(2n)!} \left[ B\left(\frac{1}{2}, \frac{1}{2} + n\right) - \pi \frac{3^{n-1}}{2^{2n-1}(2n)!}\right]
\]
\[ - \sum_{k=3}^{2n} A_{k}^{G1}\left(-\frac{\sqrt{3}}{2}\right) (-1)^{k-1} \frac{3^{2n-k+1} + (-1)^{k-1} 3^{2n-k+1}}{2^{2n-k+1}(2n-k+1)!}. \]

In a special case, for \( n = 3 \), we obtain
\[
\frac{\pi}{23,040} \cdot f^{(6)}(0)
\]
(33)

\[
\leq \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt - \frac{\pi}{3} \left[ f\left(\frac{-\sqrt{3}}{2}\right) + f\left(\frac{\sqrt{3}}{2}\right) \right]
\]
(34)

3.3. Gauss–Chebyshev Three-Point Quadrature Formula of the Second Kind

Assuming \( w(t) = \sqrt{1-t^2}, t \in [-1, 1] \) and \( x \in [-1, 0) \) and using Theorem 4, we obtain

\[
W_{n, w}^{GC2}(t, x) = \begin{cases} 
  w_{1n}(t) = \frac{1}{(n-1)!} \int_{-1}^{1} (t-s)^{n-1} \sqrt{1-s^2} \ ds, & t \in [-1, x], \\
  w_{2n}(t) = \frac{1}{(n-1)!} \int_{-1}^{1} (t-s)^{n-1} \sqrt{1-s^2} \ ds + L_n(x), & t \in (x, 0], \\
  w_{3n}(t) = -\frac{1}{(n-1)!} \int_{-1}^{t} (s-x)^{n-1} \sqrt{1-s^2} \ ds + (-1)^n L_n(x), & t \in (0, -x], \\
  w_{4n}(t) = -\frac{1}{(n-1)!} \int_{-1}^{t} (s-x)^{n-1} \sqrt{1-s^2} \ ds, & t \in (-x, 1], 
\end{cases}
\]

(34)

\[
A_k^{GC2}(x) = (-1)^{k-1} \left[ \frac{(x+1)^{k+1/2}}{2\pi} \right] F\left( -\frac{1}{2}, \frac{3}{2}, \frac{3}{2} + k, -\frac{x+1}{2} \right) - L_k(x), \quad k \geq 1,
\]

\[
B_k^{GC2}(x) = 2 \left[ \frac{(-1)^{k-1}}{(k-1)!} \int_{x}^{0} s^{k-1} \sqrt{1-s^2} \ ds + L_k(x) \right], \quad \text{for odd } k \geq 1,
\]

and

\[
B_k^{GC2}(x) = 0, \quad \text{for even } k > 1.
\]

**Corollary 4.** Let \( w_{2,2n}(t) \geq 0, \) for all \( t \in (x, 0] \) and for \( n \in \mathbb{N} \). If \( f : [-1, 1] \to \mathbb{R} \) is a \((2n+2)\)-concave function and \( f^{(2n)} \) is piecewise continuous on \([-1, 1]\), then

\[
U_n^{GC2}(x) \cdot f^{(2n)}(0) \leq \int_{-1}^{1} f(t) \sqrt{1-t^2} dt - 2n \sum_{k=1}^{2n} A_k^{GC2}(x) \left( f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(-x) \right) - \sum_{k=1, k \text{ odd}}^{2n} B_k^{GC2}(x) f^{(k-1)}(0) \leq U_n^{GC2}(x) \cdot \left[ \frac{1}{2} f^{(2n)}(-1) + \frac{1}{2} f^{(2n)}(1) \right],
\]

(35)

where

\[
U_n^{GC2}(x) = \frac{1}{(2n)!} B \left( 3 \frac{1}{2}, \frac{1}{2} + n \right)
\]
(36)

\[
- \sum_{k=1}^{2n} A_k^{GC2}(x) x^{2n-k+1} + (-1)^{k-1} (-x)^{2n-k+1} \frac{2n-k+1}{(2n-k+1)!}.
\]

If \( f \) is a \((2n+2)\)-concave function, then inequalities (35) hold with reversed inequality signs.

**Proof.** A special case of Theorem 5 for \( w(t) = \sqrt{1-t^2}, t \in [-1, 1] \), and a nonnegative function \( W_{2n,w}^{GC2} \) defined by (34).
If the polynomials \( L_{j,x}(t) \) are such that
\[
\begin{align*}
L_{0,x}(t) &= 0, \text{ for } t \in [x,0], \\
L_{1,x}(t) &= \frac{1}{2} \left( \arcsin x + \frac{\pi}{2} - \frac{\pi}{8x^2} + \frac{x\sqrt{1-x^2}}{2} \right), \\
L_{j,x}(t) &= \frac{(x+1)^{j+1/2}\sqrt{2\pi}}{\Gamma(\frac{3}{2}+j)} F \left( -\frac{3}{2}, \frac{3}{2} + j, \frac{x+1}{2} \right), \quad j = 2, 3, 4, 5, 6, \\
L_{i,j}(t) &= \sum_{k=1}^{n/j} L_{k,x}(t) \frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t \in [x,0], \quad j = 1, \ldots, n,
\end{align*}
\]
we have \( A_{1}^{GC2}(x) = \frac{\sqrt{1-x^2}}{4} - \frac{\pi}{16\pi}, \quad A_{k}^{GC2}(x) = 0, \quad \text{for } k = 2, 3, 4, 5, 6, \quad B_{1}^{GC2}(x) = \frac{x}{2} - \frac{\sqrt{1-x^2}}{2} + \frac{\pi}{8\pi} \) and \( B_{3}^{GC2}(x) = 0, \) so we obtain
\[
\begin{align*}
\frac{1}{1-f(t)} \frac{\sqrt{1-t^2}}{dt} &= A_{1}^{GC2}(x)[f(x) + f(-x)] + B_{1}^{GC2}(x)f(0) \\
&\quad + T_{n,\text{inv}}^{GC2}(x) + (-1)^n \int_{-1}^{1} W_{n,\text{inv}}^{GC2}(t,x) f^{(n)}(t) dt, \quad (37)
\end{align*}
\]
where
\[
T_{n,\text{inv}}^{GC2}(x) = \sum_{k=1}^{n} A_{k}^{GC2}(x) \left( f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(-x) \right) \\
&\quad + \sum_{k=5\text{odd}}^{n} B_{k}^{GC2}(x) f^{(k-1)}(0). \quad (38)
\]
In particular, a generalization of the Gauss–Chebyshev three-point quadrature formula of the second kind follows for \( x = -\sqrt{\frac{n}{2}}. \) Now, we derive Hermite–Hadamard-type estimates for the Gauss–Chebyshev three-point quadrature formula of the second kind.

Applying Corollary 1 to \( w(t) = \sqrt{1-t^2}, \quad t \in [-1,1], \quad x = -\sqrt{\frac{n}{2}}, \) and a \((2n+2)\)-convex function \( f, \) we obtain
\[
\begin{align*}
U_{n}^{GC2} \left( -\frac{\sqrt{n}}{2} \right) \cdot f^{(2n)}(0) \\
&\leq \frac{1}{\sqrt{1-t^2}} \int_{-1}^{1} f(t) dt - \frac{\pi}{8} \left[ f \left( -\frac{\sqrt{n}}{2} \right) + 2f(0) + f \left( \frac{\sqrt{n}}{2} \right) \right] - T_{2n,\text{inv}}^{GC2} \left( -\frac{\sqrt{n}}{2} \right) \\
&\leq U_{n}^{GC2} \left( -\frac{\sqrt{n}}{2} \right) \cdot \left[ \frac{1}{2} f^{(2n)}(-1) + \frac{1}{2} f^{(2n)}(1) \right],
\end{align*}
\]
where
\[
\begin{align*}
U_{n}^{GC2} \left( -\frac{\sqrt{n}}{2} \right) &= \frac{1}{(2n)!} B \left( \frac{3}{2}, \frac{1}{2} + n \right) - \frac{\pi}{2n+2(2n)!} \\
&\quad - \sum_{k=7}^{2n} A_{k}^{GC2} \left( -\frac{\sqrt{n}}{2} \right) (-\sqrt{n})^{2n-k+1} + (-1)^{k-1} \left( \sqrt{n} \right)^{2n-k+1} \\
&\quad \times \frac{(-1)^{k+1}}{2n-k+1(2n-k+1)!}.
\end{align*}
\]
As a special case, for \( n = 3, \) we obtain
\[
\frac{\pi}{92,160} \cdot f^{(6)}(0) \\
\leq \int_{-1}^{1} f(t) \sqrt{1 - t^2} \, dt - \frac{\pi}{8} \left[ f\left( -\frac{\sqrt{2}}{2} \right) + 2f(0) + f\left( \frac{\sqrt{2}}{2} \right) \right]
\leq \frac{\pi}{92,160} \left[ \frac{1}{2} f^{(6)}(-1) + \frac{1}{2} f^{(6)}(1) \right].
\]

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