A NEW TRIANGULATED CATEGORY FOR RATIONAL SURFACE SINGULARITIES

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Abstract. In this short paper we introduce a new triangulated category for rational surface singularities which in the non-Gorenstein case acts as a substitute for the stable category of matrix factorizations. The category is formed as a Frobenius quotient of the category of special CM modules, and we classify the relatively projective-injective objects and thus describe the AR quiver of the quotient. Connections to the corresponding reconstruction algebras are also discussed.

1. Introduction

The theory of almost split sequences first entered the world of quotient singularities through the work of Auslander [Aus86]. Rather than interpreting the McKay correspondence for finite subgroups of SL(2, C) in terms of representations of $G$, he instead viewed the representations as CM modules and showed that the AR quiver coincides with the McKay quiver, thus linking with the geometry through the dual graph of the minimal resolution.

There is a benefit to this viewpoint, since considering representations as modules we may sum them together (without multiplicity) and consider their endomorphism ring; this is Morita equivalent to the skew group ring $\mathbb{C}[x, y]/\#G$. Through projectivization ([Aus71], [ARS97]) the theory of almost split sequences can be used to gain homological insight into the structure of the endomorphism ring, and furthermore it can be used to recover the relations on the McKay quiver which yields a presentation of the algebra [RVS9].

Recently [Wem08] it was realized that for quotients by groups not inside SL(2, C) the skew group ring is far too large, and instead we should sum less CM modules together and consider this endomorphism ring instead. The modules that we sum are the special CM modules, and the resulting endomorphism ring is called a reconstruction algebra. These algebras are in fact defined for all rational surface singularities (not just quotients), are always derived equivalent to the minimal resolution and have global dimension 2 or 3. However the main difference between this new situation and the classical case is that the reconstruction algebra is very non-symmetrical and so for example writing down the relations is a much more delicate and difficult task.

We are thus motivated to study SCM($R$) (or dually $\Omega$CM($R$)), the category of special CM modules (respectively first syzygies of CM modules), from the viewpoint of relative AR theory [ASS1] to try and gain an insight into this problem. This short paper is dedicated to its study, and other related issues.

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In this paper we show that $\text{SCM}(R)$ admits a Frobenius structure and prove that the indecomposable relatively projective objects in $\text{SCM}(R)$ are precisely $R$ together with those special CM modules which correspond to non-$(−2)$ curves in the dual graph of the minimal resolution. Geometrically this means that the quotient $\text{SCM}(R)$ only ‘sees’ the crepant divisors. Note that it is certainly possible that all special CM modules are relatively projective, in which case the quotient category is zero. However there is still enough information to prove some results, for example that at all vertices in a reconstruction algebra corresponding to a $(−2)$ curve in the minimal resolution, there is only one relation which is a cycle at that vertex and further it is (locally) a preprojective relation.

We remark that our triangulated category is a more manageable version of the rather large triangulated category of singularities $\mathcal{D}_{sg}(R) = \mathcal{D}b(\text{mod} R)/\mathcal{K}b(\text{proj} R)$ which is well known in the Gorenstein case to coincide with $\text{CM}(R)$ [Buc87]. Note that our category is definitely not equivalent to $\mathcal{D}_{sg}(R)$ since the category $\mathcal{D}_{sg}(R)$ is always non-zero if $R$ is singular, but $\text{SCM}(R)$ is zero if there are no crepant divisors in the minimal resolution. Furthermore our category $\text{SCM}(R)$ is always Krull-Schmidt, a property not enjoyed by $\mathcal{D}_{sg}(R)$ in the case when $R$ is not Gorenstein. It would be interesting to see if there are indeed any connections between the two categories.

2. Syzygies and Chern classes

Throughout this paper we let $R$ be a complete local normal domain of dimension two over an algebraically closed field of characteristic zero which furthermore is a rational singularity. We denote the minimal resolution by $\pi: \tilde{X} \to \text{Spec} R$ and the exceptional curves by $\{E_i\}_{i \in I}$. For any CM module $M$ of $R$ denote by $M := \pi^* M/\text{torsion}$ the corresponding full sheaf on $\tilde{X}$ and also denote by $T(M)$ the torsion submodule of $M$, i.e. the kernel of the natural map $M \to M^{**}$. Recall the following.

Theorem 2.1. For $M \in \text{CM}(R)$, the following conditions are equivalent.

(1) $\text{Ext}^1_\tilde{X}(M, \mathcal{O}) = 0$,
(2) $(M \otimes_R \omega)/T(M \otimes_R \omega) \in \text{CM}(R)$,
(3) $\text{Ext}^2_R(\text{Tr} M, \omega) = 0$,
(4) $\Omega \text{Tr} M \in \text{CM}(R)$,
(5) $\text{Ext}^1_R(M, R) = 0$,
(6) $M^* \in \Omega \text{CM}(R)$,
(7) $\Omega M \cong M^*$ up to free summands.

We call such a module $M$ a special CM module.

Proof. (1) $\iff$ (2) is due to Wunram [Wun88], the remainder can be found in [IW08, 2.7, 3.5].

For the remainder of this paper we denote the non-free indecomposable special CM modules by $\{M_i\}_{i \in I}$, where $M_i$ corresponds to the curve $E_i$. The corresponding special full sheaves will be denoted by $\{\mathcal{M}_i\}_{i \in I}$. For any CM $R$-module $M$, we denote the chern class of the corresponding full sheaf by $c_1(M)$.
We denote by $\text{CM}(R)$ the category of $CM$-$R$-modules, by $\text{SCM}(R)$ the category of special $CM$-$R$-modules, and by $\Omega \text{CM}(R)$ the category of first syzygies of $CM$-$R$-modules. By Theorem 2.1 above, we have a duality

\[ (-)^* : \text{SCM}(R) \to \Omega \text{CM}(R). \]

Note that $\text{SCM}(R)$ (and dually $\Omega \text{CM}(R)$) have finite type since the indecomposable special $CM$ modules are in one-to-one correspondence with the exceptional curves in the minimal resolution of $\text{Spec}R$. Moreover $\text{CM}(R)$ has finite type if and only if $R$ is a quotient singularity [Aus86], so by passing to the specials we can use finite-type algebra even in the case when there are infinitely many $CM$ modules. Also, we remark that Gorenstein rational surfaces (i.e. ADE surface singularities) are very rare, and passing to this more general setting vastly increases the number of examples at our disposal.

Now when $R$ is Gorenstein every $CM$ module is special and so the categories $\Omega \text{CM}(R)$ and $\text{SCM}(R)$ coincide; they both equal $\text{CM}(R)$. It is therefore natural to ask about the intersection of the categories $\Omega \text{CM}(R)$ and $\text{SCM}(R)$ when $R$ is not Gorenstein.

Proposition 2.2. If $R$ is not Gorenstein, then

1. If $X \in \text{CM}(R)$ such that $\text{Ext}^1_R(X, R) = 0$ for $i = 1, 2$ then $X$ is free.
2. $\text{SCM}(R) \cap \Omega \text{CM}(R) = \text{add}R$.

Proof. (1) Since $\text{Ext}^1(X, R) = 0$, by Theorem 2.1 we know that $X$ is special and so $\Omega X \cong X^*$. Further $0 = \text{Ext}^1_R(\Omega X, R) = \text{Ext}^1_R(X^*, R)$ and so $X^*$ is also special. Now applying Theorem 2.1(7) to both $X$ and $X^*$ there exist short exact sequences

\[ (2.1) \quad 0 \to X^* \to P \to X \to 0 \]
\[ (2.2) \quad 0 \to X \to Q \to X^* \to 0 \]

with $P, Q \in \text{add}R$. We want to prove that $\text{Ext}^t_R(X, R) = 0$ for all $t \geq 1$ so since we know this holds for $t = 1$, inductively suppose that it holds for $t - 1$. Then by (2.1) we know

$\text{Ext}^t_R(X, R) \cong \text{Ext}^{t-1}_R(X^*, R) = 0$

and by (2.2) we know

$\text{Ext}^t_R(X^*, R) \cong \text{Ext}^{t-1}_R(X, R) = 0$.

Thus by induction it follows that $\text{Ext}^t_R(X \oplus X^*, R) = 0$ for all $t \geq 1$ and hence by definition $X$ is a totally reflexive module (see [Tak04, CPST08]). But since $R$ has only finitely many special $CM$ modules it has in particular only finitely many totally reflexive modules. If $X$ is non-free then by [Tak04] (see also [CPST08, 4.3]) it follows that $R$ is Gorenstein. Hence $X$ must be free.

(2) It follows from (1) that if $X$ and $X^*$ are special, then $X^*$ is free.

Remark 2.3. An easy conclusion of Proposition 2.2 and Theorem 2.1(7) is that if $M$ is a non-free special $CM$ module, then $\Omega^2M \cong \Omega M^*$ is never isomorphic to $M$. This is in contrast to the Gorenstein case in which $\Omega^2M$ is always isomorphic to $M$ [Eis80].

We already know that the dual of the first syzygy of any $CM$ module is special; the following result gives us precise information about the decomposition into irreducibles.
Theorem 2.4. Let $M$ be a CM $R$-module. Then $\Omega M \cong \bigoplus_{i \in I}(\Omega M^i)^{\oplus c_1(M) \cdot E_i}$.

Proof. Since $M$ is CM, by Artin-Verdier [AV85, 1.2] we have the following exact sequence

$$0 \to \mathcal{O}^{\oplus r} \to M \to \mathcal{O}_D \to 0$$

where $r$ is the rank of $M$ and $D$ represents the chern class of $\mathcal{M}$, i.e. $c_1(M) \cdot E_i = D \cdot E_i$ for a exceptional curves $E_i$. Now the minimal number of generators of $H^0(\mathcal{O}_D)$ is $Z \cdot f \cdot D$ and so choosing such a set of generators yields an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus Z \cdot f \cdot D} \to \mathcal{O}_D \to 0.$$

Exactly as in the proof of [Wun88, 1.2(a)] (Wunram considered the case when $M$ is indecomposable, but his proof works in this more general setting) $\mathcal{K}^*$ is a full sheaf of rank $Z \cdot f \cdot D$ satisfying $H^1(\mathcal{K}) = 0$. Thus there exists some special CM module $N$ such that $\mathcal{K}^* = N$, with $c_1(N) \cdot E_i = D \cdot E_i = c_1(M) \cdot E_i$ for all $i$.

Now $N$ decomposes into a sum of indecomposable special bundles, and the chern class forces $N = \bigoplus_{i \in I} M^i_{\oplus c_1(M) \cdot E_i}$ for some $s \in \mathbb{N}$. The fact that $s = 0$ follows by running the argument in [Wun88, 1.2(a)], or alternatively by using [VdB04, 3.5.3]. Hence we have a short exact sequence

$$0 \to \bigoplus_{i \in I} M^i_{\oplus c_1(M) \cdot E_i} \to \mathcal{O}^{\oplus Z \cdot f \cdot D} \to \mathcal{O}_D \to 0$$

from which taking the appropriate pullback gives us a diagram

\[
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}^{\oplus r} & \rightarrow & \mathcal{K} & \rightarrow & \mathcal{O}^{\oplus Z \cdot f \cdot D} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}^{\oplus r} & \rightarrow & \mathcal{M} & \rightarrow & \mathcal{O}_D & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]

But the singularity is rational and so the middle horizontal sequence splits, giving $\mathcal{K} = \mathcal{O}^{\oplus r + Z \cdot f \cdot D}$. Now since $H^1(\bigoplus_{i \in I} M^i_{\oplus c_1(M) \cdot E_i}) = 0$ we may push down the middle vertical sequence to obtain the short exact sequence

$$0 \to \bigoplus_{i \in I} M^i_{\oplus c_1(M) \cdot E_i} \to R^{\oplus r + Z \cdot f \cdot c_1(M)} \to M \to 0.$$ 

Since by Theorem 2.1 $\Omega M_i \cong M^i_*$, the result follows. \qed

Remark 2.5. The above theorem gives us a global combinatorial method for computing chern classes of full sheaves in the cases of quotient singularities that doesn’t resort to calculating with local co-ordinates on the minimal resolution, since the syzygy of any CM module can be easily calculated by using a counting argument on the AR(=McKay) quiver. For details see [IW08, 4.9, 4.10].
Remark 2.6. Since the above first syzygy contains no free summands it follows that any CM module $M$ is minimally generated by $\text{rk} M + Z_f \cdot c_1(M)$ elements. This gives a new proof of [Wun88, 2.1].

The following observation will be used in the next section.

Corollary 2.7. (1) We have $\Omega \omega \cong \bigoplus_{i \in I} (\Omega M_i) \oplus -E_i^2 - 2$.

(2) If $R$ is not Gorenstein and $\omega$ is a special CM $R$-module, then the exceptional curve corresponding to $\omega$ is a $(-3)$-curve and all other exceptional curves are $(-2)$-curves.

Proof. (1) By Theorem 2.4 $\Omega \omega \cong \bigoplus_{i \in I} (\Omega M_i) \oplus -E_i^2 - 2$. Further the adjunction formula states that $-2 = (K \cdot E_i)$ and so $K \cdot E_i = -E_i^2 - 2$.

(2) Immediate from (1). \qed

3. A Frobenius Structure on SCM(R)

In this section we endow the category $\text{SCM}(R)$ with a Frobenius structure and thus produce a triangulated category $\text{SCM}(R)$. We say that an extension closed subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is an exact category. (This is slightly stronger than the formal definition by Quillen [Qui73]. See also [Kel90, Appendix A].) For example $\text{CM}(R)$ is an exact category.

We start with the following easy observation.

Lemma 3.1. (1) $\text{SCM}(R)$ is an extension closed subcategory of $\text{CM}(R)$.

(2) $\text{SCM}(R)$ forms an exact category.

Proof. (1) is an immediate consequence of Theorem 2.1(5), and (2) is a consequence of (1). \qed

Let us recall the definition of Frobenius categories introduced by Happel [Hap88]. We say that an object $X \in \mathcal{B}$ is relatively projective (respectively, relatively injective) if

$$\text{Ext}^1_{\mathcal{A}}(X, B) = 0 \quad (\text{respectively, } \text{Ext}^1_{\mathcal{A}}(B, X) = 0).$$

We say that $C$ has enough relatively projective objects (respectively, injective) if for any $X \in \mathcal{B}$, there exists an exact sequence

$$0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \quad (\text{respectively, } 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0)$$

in $\mathcal{A}$ such that $Y \in \mathcal{B}$ is relatively projective (respectively, injective) and $Z \in \mathcal{B}$. We say that $\mathcal{B}$ is Frobenius if it has enough relatively projective and enough relatively injective objects, and further the relatively projective and the relatively injective objects coincide. When $\mathcal{B}$ is a Frobenius category with the subcategory $\mathcal{P}$ of relatively projective objects, the factor subcategory

$$\mathcal{B} := \mathcal{B}/[\mathcal{P}]$$

is called the stable category of $\mathcal{B}$. The reason why we use the notation $\mathcal{B}$ is to distinguish the stable category $\text{SCM}(R)$ (which we will study) from the full subcategory $\text{SCM}(R)$ of $\text{CM}(R)$.

Our main result in this section is the following.

Theorem 3.2. (1) $\text{SCM}(R)$ is a Frobenius category.

(2) The stable category $\text{SCM}(R)$ is a triangulated category.
Let us recall the definition of functorially finite subcategories introduced by Auslander-Smalø [AS81]. Let $\mathcal{B}$ be an additive category and $\mathcal{C}$ a full subcategory of $\mathcal{B}$. We say that a subcategory $\mathcal{C}$ of an additive category $\mathcal{B}$ is contravariantly finite (respectively, covariantly finite) if for any $X \in \mathcal{B}$, there exists a morphism $f : Y \to X$ (respectively, $f : X \to Y$) with $Y \in \mathcal{C}$ such that

\[ \text{Hom}_{\mathcal{B}}(\mathcal{C}, Y) \to \text{Hom}_{\mathcal{B}}(\mathcal{C}, X) \quad (\text{respectively, } \text{Hom}_{\mathcal{B}}(Y, \mathcal{C}) \to \text{Hom}_{\mathcal{B}}(X, \mathcal{C})) \]

is surjective. We say that $\mathcal{C}$ is a functorially finite subcategory of $\mathcal{B}$ if it is both contravariantly and covariantly finite.

We need the following rather general observation.

**Proposition 3.3.** Let $\mathcal{B}$ be a Krull-Schmidt exact category with enough relatively injective (respectively, projective) objects, and $\mathcal{C}$ a contravariantly (respectively, covariantly) finite extension closed subcategory of $\mathcal{B}$. Then $\mathcal{C}$ is an exact category with enough relatively injective (respectively, projective) objects.

**Proof.** We prove the statement regarding relatively injective objects; the proof for relatively projective objects is similar. It is clear that $\mathcal{C}$ is also an exact category. Let $X$ be in $\mathcal{C}$ and take an exact sequence $0 \to X \to I \to X' \to 0$ with $I$ relatively injective in $\mathcal{B}$. Then we have an exact sequence of functors $\text{Hom}_{\mathcal{B}}(-, X') \to \text{Ext}^1_A(-, X) \to 0$. Since $\mathcal{C}$ is Krull-Schmidt and contravariantly finite in $\mathcal{B}$, we can take a projective cover $\phi : \text{Hom}_{\mathcal{C}}(-, Y) \to \text{Ext}^1_A(-, X)|_{\mathcal{C}} \to 0$ of $\mathcal{C}$-modules (for the definition of $\mathcal{C}$-modules see for example [Yos90]). This is induced by an exact sequence $0 \to X \to Z \to Y \to 0$ with terms in $\mathcal{C}$.

We will show that $Z$ is relatively injective. Take any exact sequence $0 \to Z \to Z' \to Z'' \to 0$ with terms in $\mathcal{C}$. We will show that this splits. Consider the following exact commutative diagram:

\[
\begin{array}{cccccccccc}
& & & & & & 0 & & 0 \\
& & & & & \downarrow & & \downarrow \\
0 & \to & X & \to & Z & \to & Y & \to & 0 \\
\| & & & & & & \| & & & & & & & & & & & \| \\
0 & \to & X & \to & Z' & \to & Y' & \to & 0 \\
\| & & & & & & \downarrow & & & & & & & & & & & \| \\
Z'' & \to & Z'' & \to & 0 & & 0 \\
\end{array}
\]

(3.1)

Then $Y' \in \mathcal{C}$, and we have the commutative diagram

\[
\begin{array}{cccccccccc}
0 & \to & \text{Hom}_{\mathcal{C}}(-, X) & \to & \text{Hom}_{\mathcal{C}}(-, Z) & \to & \text{Hom}_{\mathcal{C}}(-, Y) & \to & \text{Ext}^1_A(-, X)|_{\mathcal{C}} & \to & 0 \\
\| & & & & & & \| & & & & & & & & & & & \| \\
0 & \to & \text{Hom}_{\mathcal{C}}(-, X) & \to & \text{Hom}_{\mathcal{C}}(-, Z') & \to & \text{Hom}_{\mathcal{C}}(-, Y') & \to & \text{Ext}^1_A(-, X)|_{\mathcal{C}} & \to & 0 \\
\end{array}
\]

(3.2)

of exact sequences of $\mathcal{C}$-modules. Since $\phi$ is a projective cover, we have that $(-a)$ is a split monomorphism. Thus $a$ is a split monomorphism. We see that the sequence $0 \to \text{Ext}^1(Z'', Z) \to \text{Ext}^1(Z'', Y)$ is exact by evaluating the upper sequence in (3.2) at $Z''$. Under this map the middle vertical exact sequence in (3.1) gets sent to the right vertical exact sequence in (3.1), so since this splits it follows that the middle vertical sequence in (3.1) splits. Hence $Z$ is relatively injective, and consequently $\mathcal{C}$ has enough relatively injective objects. \qed
In particular, since CM($R$) has enough relatively projective and injective objects and further SCM($R$) is a functorially finite subcategory of CM($R$), we conclude that SCM($R$) has enough relatively projective and injective objects.

We need the following observation:

**Lemma 3.4.** For any $X, Y \in SCM(R)$ we have $\Ext^1_R(X, Y) \cong \Ext^1_R(Y, X)$.

**Proof.** Since $X$ is special we have an exact sequence $0 \to X^* \to P \to X \to 0$ with $P \in \text{add } R$. Applying $\Hom_R(\_, Y)$ and $\Hom_R(Y^*, \_)$ respectively and using the fact that $\Ext^1_R(Y, P) = 0$ since $Y$ is special we obtain a commutative diagram

$$
\begin{array}{c}
\Hom_R(P, Y) \longrightarrow \Hom_R(X^*, Y) \longrightarrow \Ext^1_R(X, Y) \longrightarrow 0 \\
\downarrow \quad \quad \quad \quad \downarrow \\
\Hom_R(Y^*, P^*) \longrightarrow \Hom_R(Y^*, X)
\end{array}
$$

in which the vertical maps are isomorphisms. But it is easy to see that the cokernel of the bottom map is simply $\Hom_R(Y^*, X)$ and so we obtain an isomorphism $\Ext^1_R(X, Y) \cong \Hom_R(Y^*, X)$. By repeating this argument with the role of $X$ and $Y$ swapped we obtain

$$
\Ext^1_R(X, Y) \cong \Hom_R(Y^*, X) \cong \Hom_R(X, Y^*) \cong \Ext^1_R(Y, X).
$$

Thus in SCM($R$) the relatively projective objects and the relatively injective objects coincide and so consequently SCM($R$) is a Frobenius category and thus SCM($R$) is triangulated [Hap88]. This completes the proof of Theorem 3.2.

The next result gives a precise description of the relatively projective objects:

**Theorem 3.5.** Let $i \in I$. Then $M_i$ is relatively projective in SCM($R$) if and only if $E_i$ is not a ($-2$)-curve.

We divide the proof into two lemmas.

**Lemma 3.6.** If $E_i$ is a ($-2$)-curve, then $M_i$ is not relatively projective and further $\Ext^1_R(M_i, M_i) \neq 0$.

**Proof.** We show that $\Ext^1_R(M_i, M_i) \neq 0$. Denote $M := R \oplus \bigoplus_{j \in I} M_j$ and let $A := \text{End}_R(M)$. For all $j \in I$ denote $S_j$ to be the top of $\Hom_R(M, M_j)$ and $S_i$ to be the top of $\Hom_R(M, R)$. Then by inspecting the proof of [Wen08 3.3] we see that $\Ext^3_A(S_i, S_j) = 0$ for all $j \in I$ and further $\Ext^3_A(S_i, S_i) = -E_i^2 - 2$. Thus since $M_i$ corresponds to a ($-2$) curve, $\Ext^3_A(S_i, -) = 0$ against all simple $A$ modules and so $\text{pd}_A S_i = 2$. Consider a minimal projective resolution of $A$-modules

$$
0 \to \Hom_R(M, T) \to \Hom_R(M, Y) \to \Hom_R(M, M_i) \to S_i \to 0.
$$

We know that it comes from a non-split exact sequence

$$
0 \to T \to Y \to M_i \to 0
$$

with $T, Y \in SCM(R)$. Now by [Wen08 3.3] we know that $\Ext^2_A(S_i, S_*) = ((Z_K - Z_f) \cdot E_i)_- = 0$ (since $Z_K \cdot E_i = 0$ and so $(Z_K - Z_f) \cdot E_i \geq 0$), $\Ext^2_A(S_i, S_j) = 0$ if $i \neq j$ and further $\Ext^2_A(S_i, S_i) = -E_i^2 - 1 = 1$ since $M_i$ corresponds to a ($-2$)-curve. Hence $\Hom_R(M, T) = \Hom_R(M, M_i)$ and so $T = M_i$, forcing $\Ext^1_R(M_i, M_i) \neq 0$. 

□
Lemma 3.7. If $E_i$ is not a $(-2)$-curve, then $M_i$ is relatively projective in $SCM(R)$ and $\text{Ext}^1_R(M_i, M_i) = 0$.

Proof. Firstly note that for all $X, Y \in CM(R)$, if $\text{Ext}^1_R(X, Y) = 0$ then necessarily $\text{Ext}^1_R(\tau^{-1}\Omega^{-1}Y, X) = 0$. To see this, just take the short exact sequence $0 \to Y \to I \to \Omega^{-1}Y \to 0$ with $I \in \text{add } \omega$ and apply $\text{Hom}_R(-, -)$ to get

$$0 \to \text{Hom}_R(Y, X) \to \text{Hom}_R(I, X) \to \text{Hom}_R(\Omega^{-1}Y, X) \to \text{Ext}^1_R(X, Y) = 0.$$

Consequently every map from $X$ to $\Omega^{-1}Y$ factors through an injective object and hence by AR duality $0 = D\text{Hom}_R(X, \Omega^{-1}Y) = \text{Ext}^1_R(\tau^{-1}\Omega^{-1}Y, X)$. 

Now if $X \in SCM(R)$ then $\text{Ext}^1_R(X, R) = 0$ and so applying the above with $Y = R$ we get $\text{Ext}^1_R(\tau^{-1}\Omega^{-1}R, X) = 0$. But $\tau^{-1}\Omega^{-1}R = \text{Hom}_R(\Omega^{-1}R, \omega)^* = (\Omega\omega)^*$ and so this shows that $\text{Ext}^1_R((\Omega\omega)^*, -) = 0$ on $SCM(R)$, hence $(\Omega\omega)^*$ is relatively projective. But now by Corollary 2.7 we know that $(\Omega\omega)^*$ has as summands all the indecomposable special CM modules corresponding to non-$(-2)$ curves and thus all of them are relatively projective. □

This completes the proof of Theorem 3.5. □

We now show the following existence theorem of almost split sequences in $SCM(R)$. The theory of almost split sequences in subcategories was first developed by Auslander and Smalø [AS81] for finite dimensional algebras; here our algebras are not finite dimensional, but the proofs are rather similar:

Proposition 3.8. Let $i \in I$. Then $M_i$ is not relatively projective if and only if there exists an exact sequence

$$0 \to M_i \xrightarrow{g} Y \xrightarrow{f} M_i \to 0$$

such that the sequences

$$0 \to \text{Hom}_R(-, M_i) \xrightarrow{g} \text{Hom}_R(-, Y) \xrightarrow{f} J_{CM(R)}(-, M_i) \to 0,$$

$$0 \to \text{Hom}_R(M_i, -) \xleftarrow{f} \text{Hom}_R(Y, -) \xrightarrow{g} J_{CM(R)}(M_i, -) \to 0.$$

are exact on $SCM(R)$.

Proof. Suppose $M_i$ not relatively projective. We firstly show that there exists an almost split sequence $0 \to Z \to Y \to M_i \to 0$ in $SCM(R)$. Since there are only finitely many indecomposable objects in $SCM(R)$, certainly there exists an exact sequence

$$0 \to Z \xrightarrow{g} Y \xrightarrow{f} M_i \to 0$$

with $Y \in SCM(R)$ and $f$ a minimal right almost split map in $SCM(R)$. We claim that $Z \in SCM(R)$ and further $g$ is a minimal left almost split map.

Since $M_i$ is not relatively projective, there exists an exact sequence

$$0 \to Z' \to Y' \to M_i \to 0$$

with $Z' \in SCM(R)$. Since $f$ is right almost split we have a commutative diagram

$$
\begin{array}{c}
0 \\ \downarrow \quad \downarrow \\
Z' \to Y' \to M_i \\
\downarrow \\
Z \to Y \to M_i \\
\end{array}
$$
and so taking the mapping cone gives the short exact sequence

\[ 0 \to Z' \to Z \oplus Y' \to Y \to 0. \]

Since SCM(R) is closed under extensions we conclude that \( Z \in SCM(R) \). The fact \( g \) is a minimal left almost split map is now routine (see e.g. [Yos90] 2.14).

To finish the proof we must show that \( Z \cong M_i \). But as in the proof of Lemma 3.8 the above gives a minimal projective resolution

\[ 0 \to \text{Hom}_R(M, Z) \to \text{Hom}_R(M, Y) \to \text{Hom}_R(M, M_i) \to S_i \to 0 \]

of the simple \( A \)-module \( S_i \). By [Wem08, 3.3] we know that \( \text{Ext}^2_A(S_i, S) = 0 \) for any simple \( A \)-module \( S \neq S_i \). This implies \( Z \cong M_i \) and so we are done. \( \square \)

The following property of ‘Auslander algebras’ of triangulated categories is useful.

**Proposition 3.9.** Let \( T \) be a Hom-finite \( k \)-linear triangulated category with an additive generator \( M \). Then \( B := \text{End}_T(M) \) is a self-injective \( k \)-algebra.

**Proof.** For any \( X \in \text{mod} B \), we can take a projective resolution

\[ \text{Hom}_T(M, M_1) \xrightarrow{f} \text{Hom}_T(M, M_0) \to X \to 0. \]

Take a triangle \( M_2 \xrightarrow{g} M_1 \xrightarrow{f} M_0 \to M_2[1], \) then we continue a projective resolution

\[ \text{Hom}_T(M, M_2) \xrightarrow{g} \text{Hom}_T(M, M_1) \xrightarrow{f} \text{Hom}_T(M, M_0) \to X \to 0. \]

Applying \( \text{Hom}_B(\cdot, B) \) gives the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_T(M_0, M) & \xrightarrow{f} & \text{Hom}_T(M_1, M) \\
\downarrow & & \downarrow \text{Hom}_T(M, M_0) \\
\text{Hom}_B(\text{Hom}_T(M, M_0), B) & \xrightarrow{g} & \text{Hom}_B(\text{Hom}_T(M, M_1), B) \\
\end{array}
\]

where all vertical maps are isomorphisms. Since the lower sequence is exact (by properties of triangles), so is the top. Hence \( \text{Ext}_B^1(X, B) = 0 \) and so \( B \) is self-injective. \( \square \)

We deduce the following results on our triangulated category \( SCM(R) \) and the stable reconstruction algebra \( End_{SCM(R)}(\bigoplus_{i \in I} M_i) \).

**Corollary 3.10.** (1) The AR quiver of the category \( SCM(R) \) is a disjoint union of the double of Dynkin diagrams, corresponding to the subconfigurations of \((-2)\)-curves in the minimal resolution.

(2) The algebra \( \text{End}_{SCM(R)}(\bigoplus_{i \in I} M_i) \) is a factor algebra of the reconstruction algebra \( \text{End}_R(R \oplus \bigoplus_{i \in I} M_i) \) by the ideal generated by idempotents corresponding to \( R \) and the non-\((-2)\)-curves.

(3) The algebra \( \text{End}_{SCM(R)}(\bigoplus_{i \in I} M_i) \) is self-injective, and the quiver is a disjoint union of the double of Dynkin diagrams.

**Proof.** (1) A subtree of a rational tree is rational, thus the remaining \((-2)\) configurations are all Dynkin diagrams. Alternatively, it is well-known that the AR quiver of a Hom-finite \( k \)-linear triangulated category of finite type is a disjoint union of Dynkin diagrams [XZ05].

(2) This is clear.
Immediate from Lemma 3.9 since $\bigoplus_{i \in I} M_i$ is an additive generator of the triangulated category $\underline{SCM}(R)$. □

The following are examples which illustrate the above results. Note that the quiver of the reconstruction algebra follows easily from combinatorics on the dual graph, see [Wem08] for details.

**Example 3.11.**

| Dual graph | Reconstruction Algebra | AR quiver of $\underline{SCM}(R)$ |
|------------|------------------------|----------------------------------|
| ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) |
| ![Diagram](image4) | ![Diagram](image5) | ![Diagram](image6) |
| ![Diagram](image7) | ![Diagram](image8) | ![Diagram](image9) |

**Remark 3.12.** Since the non $(-2)$-curves (and $R$) die in the quotient, often the AR quiver of $\underline{SCM}(R)$ has components. In fact although the number of components is always finite, the number of possible components is arbitrarily large, as can be seen by constructing the following well-known rational tree: for any graph $\Gamma$ with vertices $E_i$, add self-intersection numbers as $E_i^2 :=\begin{cases} -2 & \text{if the number of neighbours of } E_i \text{ is one} \\ -(\text{number of neighbours of } E_i) & \text{else} \end{cases}$ then this is the dual graph of some rational surface singularity. In particular

![Diagram](image10)

(where in the region . . . we repeat the block on the right hand side) corresponds to some rational surface singularity. On taking the quotient there are many components; increasing the size of the dual graph increases the number of such components.

**Remark 3.13.** Note that the above examples also illustrate that in many cases the category $\underline{SCM}(R)$ is equivalent to $\underline{CM}(R')$ for some Gorenstein ring $R'$.

We end by using our results to characterize those rational surfaces for which the category $CM(R)$ contains an $n$-cluster tilting object. Recall that $M \in CM(R)$
is called \( n \)-cluster tilting (or maximal \( (n - 1) \)-orthogonal) for a positive integer \( n \). If \( \omega \) is not Gorenstein so by Corollary 2.7 the exceptional curve corresponding to \( M \sim \text{indecomposable summand of} \ M \) in \( \text{CM}(R) \), this implies that \( \omega \) is not \( n \)-cluster tilting object for some \( n > 2 \) if and only if \( R \) is regular.

Proof. If \( R \) is regular, then \( \text{CM}(R) = \text{add} \ R \) and so \( R \) is an \( n \)-cluster tilting object in \( \text{CM}(R) \) for any \( n \geq 1 \). Hence we only need to consider the case when \( R \) is not regular.

(1) By the Krull-Schmidt property, \( \text{CM}(R) \) has a 1-cluster tilting object if and only if \( \text{CM}(R) \) has finite type. By [Aus86] this is equivalent to \( R \) being a quotient singularity.

(2) Let \( M \) be a basic 2-cluster tilting object of \( \text{CM}(R) \). Since \( R \in \text{add} \ M \) and \( \text{Ext}^1_R(M, M) = 0 \), we have that \( M \) is special. Now since \( \omega \) is a summand of \( M \), this implies that \( \omega \) is special. Since \( \text{Ext}^1_R(M, M) = 0 \), by Lemma 3.6 any non-free indecomposable summand of \( M \) corresponds to a non-\((–2)\)-curve. In particular \( R \) is not Gorenstein so by Corollary 2.7 the exceptional curve corresponding to \( \omega \) is a \((–3)\)-curve and all other exceptional curves are \((–2)\)-curves. This implies \( M \cong R \oplus \omega \), so by Lemma 3.6 we have that \( M \) is relatively projective in \( \text{SCM}(R) \).

Since \( \text{Ext}^1_R(M, \text{SCM}(R)) = 0 \), we have \( \text{SCM}(R) = \text{add} \ M \). Thus the minimal resolution of \( \text{Spec} R \) consists only of one \((–3)\)-curve, so \( R \cong k[[x, y]]^{\pm(1,1)} \). By inspection, in this case \( R \oplus \omega \) is a 2-cluster tilting object.

(3) \( \text{CM}(R) \) does not have an \( n \)-cluster tilting object for \( n > 2 \) by Proposition 2.2(1) in the non-Gorenstein case, and by Lemma 3.6 in the Gorenstein case. \( \square \)

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