Estimates for norms of two-weighted summation operators on a tree under some conditions on weights

A.A. Vasil’eva

1 Introduction

In this paper, estimates for norms of weighted summation operators (discrete Hardy-type operators) on a tree were obtained for some conditions on weights.

The inequalities

$$\left( \sum_{k=0}^{\infty} w_k \left( \sum_{j=0}^{k} u_j f_j \right)^q \right)^{\frac{1}{q}} \leq C \left( \sum_{k=0}^{\infty} |f_k|^p \right)^{\frac{1}{p}}, \quad (f_k)_{k \in \mathbb{Z}^+} \in l_p,$$

were studied in papers of Leindler [17], Bennett [2,4], Braverman and Stepanov [6], Goldman [12]. The order estimates of the minimal constant $C$ in (1) were first obtained in [4] and [6] (for $1 \leq p, q \leq \infty$ the upper estimates were proved by Heinig and Andersen [1,14]). The similar problem for two-weighted integration operators on a semiaxis was solved by Bradley [5], Mazya and Rozin [21]. Later, these results were generalized for matrix operators and integration operators with different kernels (see, e.g., papers of Heinig and Andersen [1,14], Stepanov [30,31], Oinarov [23], Prokhorov and Stepanov [27], Stepanov and Ushakova [32], Rautian [28], Farsani [11], Oinarov, Persson and Temirkhanova [24], Okpoti, Persson and Wedestig [25,26], and the books [13,15,16]). In the case $p = q = 2$ Naimark and Solomyak [22] showed that the problem of estimating the norm of weighted integration operator on a regular tree with weights depending only on distance from the root can be reduced to a problem on estimating the norm of some weighted Hardy-type operator on a half-axis.

The criterion of boundedness of a two-weighted integration operator on a metric tree and order estimates for its norm were obtained by Evans, Harris and Pick [10]. The estimate for the norm of a summation operator on a combinatorial tree can be derived from their result (it will be made in [2] for $p \leq q$). However, this estimate in general case is rather complicated. Here under some conditions on weights we obtain estimates which are more simple and convenient for applications.

The Hardy-type inequalities on trees are used in order to prove embedding theorems for weighted Sobolev classes on a domain (see [7,33,54]) and in estimating
widths of functional classes, $s$-numbers and entropy numbers of embedding operators (see \cite{17 18, 20, 21}).

Let $X$, $Y$ be sets, $f_1$, $f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \lesssim_y f_2(x, y)$ (or $f_2(x, y) \gtrsim_y f_1(x, y)$) if, for any $y \in Y$, there exists $c(y) > 0$ such that $f_1(x, y) \leq c(y)f_2(x, y)$ for each $x \in X$; $f_1(x, y) \approx_y f_2(x, y)$ if $f_1(x, y) \lesssim_y f_2(x, y)$ and $f_2(x, y) \lesssim_y f_1(x, y)$.

Throughout this paper we consider graphs $\mathcal{G}$ with finite or countable vertex set, which will be denoted by $V(\mathcal{G})$. Also we suppose that the graphs have neither multiple edges nor loops. The set of edges we denote by $E(\mathcal{G})$ and identify pairs of adjacent vertices with edges that connect them.

Let $\mathcal{T} = (\mathcal{T}, \xi_0)$ be a tree rooted at $\xi_0$. We introduce a partial order on $V(\mathcal{T})$ as follows: we say that $\xi' \succ \xi$ if there exists a simple path $(\xi_0, \xi_1, \ldots, \xi_n, \xi')$ such that $\xi = \xi_k$ for some $k \in \mathbb{N}$; by the distance between $\xi$ and $\xi'$ we mean the quantity $\rho_\mathcal{T}(\xi, \xi') = \rho_\mathcal{T}(\xi', \xi) = n + 1 - k$. In addition, set $\rho_\mathcal{T}(\xi, \xi) = 0$. For $j \in \mathbb{Z}_+$ and $\xi \in V(\mathcal{T})$ write

$$V_j(\xi) := V_j^\mathcal{T}(\xi) := \{\xi' \succ \xi : \rho_\mathcal{T}(\xi, \xi') = j\}.$$  

For vertices $\xi \in V(\mathcal{T})$, denote by $T_\xi = (T_\xi, \xi)$ the subtree in $\mathcal{T}$ with the vertex set

$$\{\xi' \in V(\mathcal{T}) : \xi' \succ \xi\}.$$  

Let $W \subset V(\mathcal{T})$. We say that $\mathcal{G} \subset \mathcal{T}$ is a maximal subgraph on the set of vertices $W$ if $V(\mathcal{G}) = W$ and if any two vertices $\xi', \xi'' \in W$ that are adjacent in $\mathcal{T}$ are also adjacent in $\mathcal{G}$.

Let $\mathcal{G}$ be a subgraph in $\mathcal{T}$. Denote by $V_{\max}(\mathcal{G})$ and $V_{\min}(\mathcal{G})$ the set of maximal and minimal vertices in $\mathcal{G}$, respectively. Given a function $f : V(\mathcal{G}) \to \mathbb{R}$, we set

$$\|f\|_{l_p(\mathcal{G})} = \left( \sum_{\xi \in V(\mathcal{G})} |f(\xi)|^p \right)^{1/p}.$$  \hspace{1cm} (2)

Denote by $l_p(\mathcal{G})$ the space of functions $f : V(\mathcal{G}) \to \mathbb{R}$ with a finite norm $\|f\|_{l_p(\mathcal{G})}$.

Let $(\mathcal{G}, \xi_0)$ be a disjoint union of trees, $1 \leq p \leq \infty$, and let $u$, $w : V(\mathcal{G}) \to (0, \infty)$ be weight functions. Define the summation operator $S_{u,w,\mathcal{G}}$ by

$$S_{u,w,\mathcal{G}}f(\xi) = w(\xi) \sum_{\xi' \prec \xi} u(\xi')f(\xi'), \quad \xi \in V(\mathcal{G}), \quad f : V(\mathcal{G}) \to \mathbb{R}.$$  

By $\mathcal{G}_{u,w}$ we denote the operator norm of $S_{u,w,\mathcal{G}} : l_p(\mathcal{G}) \to l_q(\mathcal{G})$, i.e., the minimal constant $C$ in the inequality

$$\left( \sum_{\xi \in V(\mathcal{G})} w^q(\xi) \left( \sum_{\xi' \prec \xi} u(\xi')f(\xi') \right)^q \right)^{1/q} \leq C \left( \sum_{\xi \in V(\mathcal{G})} |f(\xi)|^p \right)^{1/p}, \quad f : V(\mathcal{G}) \to \mathbb{R}.$$  

Let us formulate the main results of this paper.
Theorem 1. Let \( \mathcal{A} \) be a tree and let \( 1 < p < q < \infty \). Suppose that there are \( K \geq 1 \), \( l_0 \in \mathbb{N} \) and \( \lambda \in (0, 1) \) such that

\[
\text{card } V^A_1(\xi) \leq K, \quad \xi \in V(\mathcal{A}),
\]

\[
\frac{u(\xi')}{u(\xi)} \leq K, \quad \frac{\|w\|_{l_0(\mathcal{A}_C)}}{\|w\|_{l_0(\mathcal{A}_C)}} \leq \lambda, \quad \xi \in V(\mathcal{A}), \quad \xi' \in V^A_1(\xi), \quad \xi'' \in V^A_{l_0}(\xi).
\]

Then \( \mathcal{S}^{p,q}_{A,u,w} \underset{K,\lambda,l_0,p,q}{\asymp} \sup_{\xi \in V(\mathcal{A})} u(\xi) \|w\|_{l_0(\mathcal{A}_C)}. \)

Let \( N \in \mathbb{N} \cup \{+\infty\} \), and let \( (A, \xi_0) \) be a tree such that \( V_{\max}(A) = V^A_N(\xi_0) \). Suppose that there exist a non-decreasing function \( \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and a constant \( C_0 \geq 1 \) such that \( \psi(0) = 0 \) and for any \( 0 \leq j \leq j' < N + 1 \), \( \xi \in V^A_j(\xi_0) \)

\[
C_0^{-1} \cdot 2^{\psi(j')-\psi(j)} \leq \text{card } V^A_{j'-j}(\xi) \leq C_0 \cdot 2^{\psi(j')-\psi(j)}.
\]

Let \( u, w : V(\mathcal{A}) \rightarrow (0, \infty), u(\xi) = u_j, w(\xi) = w_j \) for \( \xi \in V^A_j(\xi_0), 1 \leq q \leq p \leq \infty \). Estimate the value \( \mathcal{S}_{A,u,w}^{p,q} \).

Denote by \( \mathcal{S}_{A,u,w}^{p,q} \) the minimal constant \( C \) in the inequality

\[
\left( \sum_{\xi \in V(\mathcal{A})} w^q(\xi) \left( \sum_{\xi' \leq \xi} u(\xi') f(\xi') \right)^q \right)^{\frac{1}{q}} \leq C \|f\|_{l_p(\mathcal{A})},
\]

\[
f : V(\mathcal{A}) \rightarrow \mathbb{R}_+, \quad f(\xi) = f_j \quad \text{for any } \quad \xi \in V^A_j(\xi_0), \quad 0 \leq j < N + 1.
\]

For such functions \( f \) we have

\[
\left( \sum_{\xi \in V(\mathcal{A})} w^q(\xi) \left( \sum_{\xi' \leq \xi} u(\xi') f(\xi') \right)^q \right)^{\frac{1}{q}} \textcolor{blue}{\underset{C_0}{\approx}} \left( \sum_{j=0}^{N} w^q_j \cdot 2^{\psi(j)} \left( \sum_{i=0}^{j} u_i f_i \right)^q \right)^{\frac{1}{q}},
\]

\[
\|f\|_{l_p(\mathcal{A})} \textcolor{blue}{\underset{C_0}{\approx}} \left( \sum_{j=0}^{N} f^p_j \cdot 2^{\psi(j)} \right)^{\frac{1}{p}}
\]

(if \( C_0 = 1 \), then we have exact equalities). Set \( \hat{w}_j = w_j \cdot 2^{\frac{\psi(j)}{q}}, \hat{u}_j = u_j \cdot 2^{\frac{\psi(j)}{p}}, 0 \leq j < N + 1. \) Then \( \mathcal{S}_{A,u,w}^{p,q} \underset{C_0}{\approx} \mathcal{S}_{\hat{u},\hat{w}}^{p,q} \), where \( \mathcal{S}_{\hat{u},\hat{w}}^{p,q} \) is the minimal constant in the inequality

\[
\left( \sum_{j=0}^{N} \hat{u}^q_j \left( \sum_{i=0}^{j} \hat{u}_i \varphi_j \right)^q \right)^{\frac{1}{q}} \leq C \left( \sum_{j=0}^{N} \varphi_j^p \right)^{\frac{1}{p}}, \quad \varphi_j \geq 0, \quad 0 \leq j < N + 1.
\]

Moreover, if \( C_0 = 1 \), then we have the exact equality.
Theorem 2. Let \( p \geq q \). Then \( \mathcal{G}^{p,q}_{A,u,w} \asymp \mathcal{G}^{p,q}_{u,w} \) if \( C_1 = 1 \), then \( \mathcal{G}^{p,q}_{A,u,w} = \mathcal{G}^{p,q}_{u,w} \).

The estimates of \( \mathcal{G}^{p,q}_{u,w} \) were obtained by Heinig, Andersen and Bennett [14].

Theorem A. Let \( 1 \leq p, q \leq \infty \), and let \( \{ u_n \}_{n \in \mathbb{Z}_+}, \{ w_n \}_{n \in \mathbb{Z}_+} \) be non-negative sequences such that

\[
M_{u,w} := \sup_{m \in \mathbb{Z}_+} \left( \sum_{n=m}^{\infty} w_n^q \right)^{1/q} \left( \sum_{n=0}^{m} u_n^{p'} \right)^{1/p'} < \infty \quad \text{for} \quad 1 < p \leq q < \infty,
\]

\[
M_{u,w} := \left( \sum_{m=0}^{\infty} \left( \sum_{n=m}^{\infty} w_n^q \right)^{1/q} \left( \sum_{n=0}^{m} u_n^{p'} \right)^{1/p'} \right)^{1/p} < \infty \quad \text{for} \quad 1 \leq q < p \leq \infty.
\]

Let \( \mathcal{G}^{p,q}_{u,w} \) be the minimal constant in the inequality

\[
\left( \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} |w_n f_k| \right)^q \right)^{1/q} \leq C \left( \sum_{n \in \mathbb{Z}_+} |f_n|^p \right)^{1/p}, \quad \{ f_n \}_{n \in \mathbb{Z}_+} \in l_p.
\]

Then \( \mathcal{G}^{p,q}_{u,w} = M_{u,w} \).

2 The discrete analogue of Evans – Harris – Pick theorem

Let \( (\mathcal{T}, \xi_*) \) be a tree, let \( \Delta : E(\mathcal{T}) \to 2^\mathbb{R} \) be a mapping such that for any \( \lambda \in E(\mathcal{T}) \) the set \( \Delta(\lambda) = [a_\lambda, b_\lambda] \) is a non-degenerate segment. By a metric tree we mean

\[
\mathcal{T} = (\mathcal{T}, \Delta) = \{(t, \lambda) : t \in [a_\lambda, b_\lambda], \lambda \in E(\mathcal{T})\};
\]

here we suppose that if \( \xi' \in V_1(\xi), \xi'' \in V_1(\xi') \), \( \lambda = (\xi, \xi'), \lambda' = (\xi', \xi'') \), then \( (b_\lambda, \lambda) = (a_{\lambda'}, \lambda') \). The distance between points of \( \mathcal{T} \) is defined as follows: if \( (\xi_0, \xi_1, \ldots, \xi_n) \) is a simple path in the tree \( \mathcal{T} \), \( n \geq 2 \), \( \iota = (\xi_{i-1}, \xi_i), x = (t_1, \lambda_1), y = (t_n, \lambda_n) \), then we set

\[
|y - x|_\mathcal{T} = |b_{\lambda_1} - t_1| + \sum_{i=2}^{n-1} |b_{\lambda_i} - a_{\lambda_i}| + |t_n - a_{\lambda_n}|;
\]

(6) if \( x = (t', \lambda), y = (t'', \lambda), \) then \( |y - x|_\mathcal{T} = |t' - t''| \).

We say that \( \lambda < \lambda' \) if \( \lambda = (\omega, \xi), \lambda' = (\omega', \xi') \) and \( \xi \leq \omega' \); \( (t', \lambda') \leq (t'', \lambda'') \) if \( \lambda' < \lambda'' \) or \( \lambda' = \lambda'', t' \leq t'' \). If \( (t', \lambda') \leq (t'', \lambda'') \) and \( (t', \lambda') \neq (t'', \lambda'') \), then we write \( (t', \lambda') < (t'', \lambda'') \). For \( a, x \in \mathcal{T}, a \leq x, \) we set \( [a, x] = \{ y \in \mathcal{T} : a \leq y \leq x \} \).
Let $A_\lambda \subset \Delta(\lambda)$, $\lambda \in E(T)$. We say that the subset $A = \{(t, \lambda) : \lambda \in E(T), t \in A_\lambda\}$ is measurable if $A_\lambda$ is measurable for any $\lambda \in E(T)$. Its Lebesgue measure is defined by

$$\text{mes} A = \sum_{\lambda \in E(T)} \text{mes} A_\lambda.$$  

Let $f : A \to \mathbb{R}$. For $\lambda \in E(T)$ define the function $f_\lambda : A_\lambda \to \mathbb{R}$ by $f_\lambda(t) = f(t, \lambda)$. We say that the function $f : A \to \mathbb{R}$ is Lebesgue integrable if $f_\lambda$ is Lebesgue integrable for any $\lambda \in E(T)$ and 

$$\int_A f(x) \, dx = \sum_{\lambda \in E(T)} \int_{A_\lambda} f_\lambda(t) \, dt.$$  

Let $D \subset T$ be a connected set. Denote by $T_D$ the maximal subtree in $T$ such that for any $\lambda \in E(T_D)$ the set $\{t \in \Delta(\lambda) : (t, \lambda) \in D\}$ is a non-degenerated segment. Define $\Delta_D(\lambda)$ as follows: $\Delta_D(\lambda) = \{t \in \Delta(\lambda) : (t, \lambda) \in D\}$, $\lambda \in E(T_D)$. Then $(T_D, \Delta_D)$ is a metric tree. We identify it with the set $D$ and call it a metric subtree of $T$.

Let $D$ be a metric subtree in $T$. We say that the point $x \in D$ is maximal (minimal) if $y \in T \backslash D$ for any $y > x$ (for any $y < x$, respectively). Denote by $D_{\text{max}}$ the set of maximal points in $D$.

Let $T = (T, \Delta)$ be a metric tree, $x_0 \in T$, let $u, w : T \to \mathbb{R}_+$ be measurable functions. Set $T_{x_0} = \{x \in T : x \geq x_0\}$,

$$I_{u, w, x_0}f(x) = w(x) \int_{[x_0, x]} u(t) f(t) \, dt, \quad x \in T_{x_0}.$$  

(7)

Suppose that $V(T)$ is finite.

Denote by $\mathcal{J}_{x_0} = \mathcal{J}_{x_0}(T)$ the family of metric subtrees $D \subset T$ with the following properties:

1. $x_0$ is a minimal point in $D$;
2. if $x \in \partial D \backslash \{x_0\}$, then $x$ is maximal in $D$.

**Example.** Let the metric tree $T = (T, \Delta)$ be defined as follows. The vertex $\xi_0$ is a root of $T$, $V_1(\xi_0) = \{\xi_1\}$, $V_1(\xi_1) = \{\xi_2, \xi_3\}$, $\lambda_1 = (\xi_0, \xi_1)$, $\lambda_2 = (\xi_1, \xi_2)$, $\lambda_3 = (\xi_1, \xi_3)$, $\Delta(\lambda_i) = [0, 1]$, $i = 1, 2, 3$. Let $D = \{(t, \lambda_i) : t \in [0, 1], i = 1, 2\}$, $x_0 = (0, \lambda_1)$. Then $D \notin \mathcal{J}_{x_0}(T)$. Indeed, $(1, \lambda_1) = (0, \lambda_2)$ is a boundary point, but it is not maximal.

Given a subtree $D \subset T$, we denote by $L^p_{\text{discr}}(D)$ the set of functions $\phi : D \to \mathbb{R}$ that are constants on each edge of $D$.  

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Let $\mathbb{D} \in \mathcal{J}_{x_0}(\Gamma)$, $\partial \mathbb{D}\setminus\{x_0\} \subset G \subset \mathbb{D}_{\text{max}}$. We write

$$\alpha_{\mathbb{D},G} = \inf \left\{ \|f\|_{L^p(\mathbb{D})} : f \in L^p(\mathbb{D}), \int_{[x_0,t]} |f(x)|u(x)\,dx = 1 \quad \forall t \in G \right\},$$  \hspace{1cm} (8)

$$\alpha^\text{discr}_{\mathbb{D},G} = \inf \left\{ \|f\|_{L^p(\mathbb{D})} : f \in L^p(\mathbb{D}), \int_{[x_0,t]} |f(x)|u(x)\,dx = 1 \quad \forall t \in G \right\},$$  \hspace{1cm} (9)

Remark 1. If the function $u$ is a constant on each edge of $\mathbb{D}$, then $\alpha_{\mathbb{D},G} = \alpha^\text{discr}_{\mathbb{D},G}$.

The following result was proved by Evans, Harris and Pick [10].

**Theorem B.** Let $1 \leq p \leq q \leq \infty$. Then the operator $I_{u,w,x_0} : L^p(\mathbb{I}_{x_0}) \to L^q(\mathbb{I}_{x_0})$ is bounded if and only if

$$C_{u,w} := \sup_{\mathbb{D} \in \mathcal{J}_{x_0}} \frac{\|w\chi_{x_0\setminus\mathbb{D}}\|_{L^q(\mathbb{I}_{x_0})}}{\alpha_{\mathbb{D}}} < \infty.$$  \hspace{1cm} (10)

Moreover, $C_{u,w} \leq \|I_{u,w}\|_{L^p(\mathbb{I}_{x_0}) \to L^q(\mathbb{I}_{x_0})} \leq 4C_{u,w}$.

The quantity $\alpha_{\mathbb{D}}$ can be calculated recursively. The following result is also proved in [10].

**Theorem C.** Let $\mathbb{D} \in \mathcal{J}_{x_0}$, $\mathbb{D} = \bigcup_{j=0}^m \mathbb{D}_j$, $\mathbb{D}_0 = [x_0, y_0]$, $x_0 < y_0$, $\mathbb{D}_j \in \mathcal{J}_{y_0}$, $1 \leq j \leq m$, $\mathbb{D}_i \cap \mathbb{D}_j = \{y_0\}$, $i \neq j$. Then

$$\frac{1}{\alpha_{\mathbb{D}}} = \left\| \left( \alpha_{\mathbb{D}_0}^{-1} \alpha_{\mathbb{D}_1}^{-1} \cdots \alpha_{\mathbb{D}_m}^{-1} \right)^{1/p} \right\|_{p'}.$$

Let $\mathbf{V}(\mathcal{A})$ be finite. We obtain two-sided estimates of $\mathcal{G}^{p,q}_{\mathcal{A},u,w}$ for $p \leq q$.

Let $\hat{\xi} \in \mathbf{V}(\mathcal{A})$, $\mathcal{D} \subset \mathcal{A}_{\hat{\xi}}$. $\mathbf{V}_{\text{max}}(\mathcal{D}) \setminus \mathbf{V}_{\text{max}}(\mathcal{A}) \subset \Gamma \subset \mathbf{V}_{\text{max}}(\mathcal{D})$. We say that $(\mathcal{D}, \Gamma) \in \mathcal{J}^\circ_{\hat{\xi}}$ if the following conditions hold:

1. $\hat{\xi}$ is minimal in $\mathcal{D}$,
2. if $\xi \in \mathbf{V}(\mathcal{D})$ is not maximal in $\mathcal{D}$, then $\mathbf{V}_1(\xi) \subset \mathbf{V}(\mathcal{D})$.

Denote by $\mathcal{D}_\Gamma$ the subtree of $\mathcal{D}$ such that $\mathbf{V}(\mathcal{D}_\Gamma) = \mathbf{V}(\mathcal{D}) \setminus \Gamma$. Then

$$\mathbf{V}(\mathcal{A}_{\hat{\xi}} \setminus \mathcal{D}_\Gamma) = \cup_{\xi \in \Gamma} \mathbf{V}(\mathcal{A}_\xi).$$  \hspace{1cm} (11)

If $(\mathcal{D}, \Gamma) \in \mathcal{J}^\circ_{\hat{\xi}}$ and $\Gamma \neq \emptyset$, we write $(\mathcal{D}, \Gamma) \in \mathcal{J}^\circ_{\hat{\xi}}$.  

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For \((\mathcal{D}, \Gamma) \in \mathcal{J}_\xi^\ast\) we set

\[
\beta_{\mathcal{D}, \Gamma} = \inf \left\{ \| \varphi \|_{l_p(A_\xi)} : \sum_{\xi' \leq \xi} |\varphi(\xi')|u(\xi') = 1, \forall \xi \in \Gamma \right\}.
\] (12)

Notice that if \(\Gamma = \emptyset\), then \(\beta_{\mathcal{D}, \Gamma} = 0\).

**Lemma 1.** Let \(p \leq q\). Then

\[
\mathcal{G}_{A_\xi,u,w} \leq \sup_{p,q} \frac{\|w_{X_{A_\xi}} \|_{l_q(A_\xi)} \|I_{p,q}(A_\xi)\|_{l_q(A_\xi)}}{\beta_{\mathcal{D}, \Gamma}}.
\]

**Proof.** Add a vertex \(\xi\), to the set \(V(\mathcal{A})\) and connect it with \(\xi_0\) by an edge. Thus we obtain a tree \((\mathcal{A}^\#, \xi_0)\). Define the mapping \(\Delta\) by \(\Delta(\lambda) = [0, 1], \lambda \in E(\mathcal{A}^\#)\), and set \(A = (\mathcal{A}^\#, \Delta)\). For each \(\xi \in V(\mathcal{A})\) denote by \(\lambda_\xi\) an edge of the tree \(\mathcal{A}^\#\) with the head \(\xi\) (i.e., \(\lambda_\xi = (\xi', \xi), \xi' < \xi\)). Given a function \(\psi : V(\mathcal{A}) \to \mathbb{R}\), we define a function \(\psi^\# : A \to \mathbb{R}\) by \(\psi^\#(t, \lambda_\xi) = \psi(\xi), t \in [0, 1], \xi \in V(\mathcal{A})\).

Let \(x_0 = (0, \lambda_\xi) \in A\). By the Hölder’s inequality and Theorem 13

\[
\mathcal{G}_{A_\xi,u,w} \leq \sup_{p,q} \frac{\|w_{X_{A_\xi}} \|_{l_q(A_\xi)} \|I_{p,q}(A_\xi)\|_{l_q(A_\xi)}}{\alpha_{\mathcal{D}}},
\]

with \(\alpha_{\mathcal{D}}\) defined by (8) and (9).

Let \(D \in J_{x_0}, D \neq A_{x_0}, \mathcal{D}^\# = \mathcal{A}^\#, \mathcal{D} = (\mathcal{D}^\#)_{\xi} \subset \mathcal{A}, \Gamma = \{\xi \in V(D) : \exists t \in (0, 1] : (t, \lambda_\xi) \in \partial D \setminus \{x_0\}\} \). Then \(V_{\max}(D) \setminus V_{\max}(\mathcal{A}) \subset \Gamma \subset V_{\max}(D)\) and \(\Gamma \neq \emptyset\). Prove that \((\mathcal{D}, \Gamma) \in \mathcal{J}_\xi^\ast\). Property 1 holds by construction. Prove Property 2. Indeed, let \(\xi \in V(D)\), and suppose that there are vertices \(\xi' \in V_1(\xi) \setminus V(D)\) and \(\xi'' \in V_1(\xi) \cap V(\mathcal{D})\). Let \(\eta \in V(\mathcal{A}^\#)\) be the direct predecessor of \(\xi\). Then \((1, (\eta, \xi)) = (0, (\xi', \xi'', \xi'))\) is a boundary point in \(D\) and it is not maximal.

Set \(D^+ = (\mathcal{D}^\#, \Delta), D^- = ((\mathcal{D}^\#)_{\Gamma}, \Delta), G = D^+ \setminus \text{int } D\).

We have

\[
\|w_{X_{A_{x_0}}} \|_{l_q(A_{x_0})} \leq \|w_{X_{A_{x_0}}} \|_{l_q(A_{x_0})} = \|w_{X_{A_{x_0}}} \|_{l_q(A_{x_0})};
\]

by Remark 11

\[
\alpha_{D} \geq \alpha_{D_{\max}} = \alpha_{D_{\max}} = \beta_{\mathcal{D}, \Gamma}.
\]

This yields the desired upper estimate for \(\mathcal{G}_{A_\xi,u,w}\).

Prove the lower estimate. Let \((\mathcal{D}, \Gamma) \in \mathcal{J}_\xi^\ast(\mathcal{A})\). Take a function \(f \in l_p(A_\xi)\) such that \(\sum_{\xi \leq \xi' \in \xi} \|f(\xi')\|_{l_q(A_\xi)} = 1\) for any \(\xi \in \Gamma, \|f\|_{l_p(A_\xi)} = \beta_{\mathcal{D}, \Gamma}, \hat{f} = \frac{f}{\beta_{\mathcal{D}, \Gamma}}\). Then \(\hat{f}(\xi') = 0\) for
for any $\xi' > \xi, \xi \in \Gamma, \|\tilde{f}\|_{L_p(A_\xi)} = 1$,

$$\left(\sum_{\xi \in V(A_\xi)} w^q(\xi) \left(\sum_{\xi < \xi' \leq \xi} u(\xi')|\tilde{f}(\xi')|\right)\right)^{1/q} \geq$$

$$\geq \left(\sum_{\xi \in \Gamma} \sum_{\hat{\xi} \geq \xi} w^q(\hat{\xi}) \left(\sum_{\xi < \xi' \leq \xi} u(\xi')|\tilde{f}(\xi')|\right)\right)^{1/q} \beta^{-1}_{D,\Gamma} \|w\|_{L_q(A_\hat{\xi}\setminus D_\Gamma)}.$$  

This completes the proof.

**Proposition 1.** Let $\xi_{\ast} \in V(A), V_1^A(\xi_{\ast}) = \{\xi_1, \ldots, \xi_m\}$, $(\mathcal{D}, \Gamma) \in J_{\xi_{\ast}}'$, $\mathcal{D}_j = \mathcal{D}_{\xi_j}, \Gamma_j = \Gamma \cap V(\mathcal{D}_j)$. Then

$$\beta^{-1}_{D,\Gamma} = \left\|\left(\left\|u(\xi_{\ast})\right\|_{L_q(A_\xi)}, \left\|\left(\beta_{D_j,\Gamma_j}\right)_{j=1}^m\right\|_{L_p}^{-1}\right\|_{D_{p/q}}. \right.$$  

(14)

This proposition follows from Theorem C and Remark 1.

**3 An estimate for the norm of a weighted summation operator on a tree: case $p < q$**

Let $(A, \xi_0)$ be a tree with a finite vertex set, and let $u, w : V(A) \to (0, \infty)$. For $\xi \in V(A)$ and $(\mathcal{D}, \Gamma) \in J_\xi'$ we set $B_{D,\Gamma} = \beta^{-1}_{D,\Gamma}$.

**Lemma 2.** Let $1 < p < q < \infty$. Then there is $\sigma_{\ast} = \sigma_{\ast}(p, q) \in \left(0, \frac{1}{3}\right)$ such that if

$$u(\xi)\|w\|_{L_q(A_\xi)} \leq 1, \quad \xi \in V(A), \quad (15)$$

and

$$\frac{\|w\|_{L_q(A_\xi)}}{\|w\|_{L_q(A_{\xi'})}} \leq \sigma, \quad \xi \in V(A), \quad \xi' \in V_1^A(\xi) \quad (16)$$

with $\sigma \in (0, \sigma_{\ast})$, then for any $(\mathcal{D}, \Gamma) \in J_{\xi_{\ast}}'$

$$B_{D,\Gamma}\|w\|_{L_q(A_{\xi_{\ast}} \setminus D_\Gamma)} \lesssim 1. \quad (17)$$

**Proof.** For each $t \in [1, \infty]$ we set

$$f(t) := \frac{\int_0^t \sigma^{s/3} ds}{\int_0^1 \sigma^{s/3} ds}. \quad (18)$$
Suppose that $\xi_*$ is not minimal in $\mathcal{A}$. Let $\hat{\xi}$ be the direct predecessor of $\xi_*$. Then $\|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)} = \sigma^t \|w\|_{l_q(A_{\hat{\xi}})}$, $t \geq 1$. Prove that there is $c = c(p, q) \geq 1$ such that

$$B_{D, \Gamma} \|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)} \leq cf^\frac{1}{p} (t). \quad (19)$$

If $\xi_*$ is minimal, then we prove that there is $c = c(p, q) \geq 1$ such that

$$B_{D, \Gamma} \|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)} \leq cf^\frac{1}{p} (\infty). \quad (20)$$

Denote by $\nu_D$ the maximal length of a path in $\mathcal{D}$ with beginning at the point $\xi_*$. We shall prove (19) and (20) by induction on $\nu_D$.

If $\nu_D = 0$, then $\mathcal{D} = \{\xi_*\}$, $\Gamma = \{\xi_*\}$, $B_{D, \Gamma} = u(\xi_*)$, $\mathcal{D}_T = \emptyset$, $\mathcal{A}_{\xi_*}\setminus \mathcal{D}_T = \mathcal{A}_{\xi_*}$.

Therefore, (19) and (20) follow from (15) and from the inequality $f(t) \geq 1$, $t \geq 1$.

Suppose that the assertion is proved for any $\mathcal{D}$ such that $\nu_D \leq \nu$. Prove it for $\nu_D = \nu + 1$.

Let $V_1^A(\xi_*) = \{\xi_1, \ldots, \xi_m\}$, $A_i = A_{\xi_i}$, $\mathcal{D}_i = \mathcal{D}_{\xi_i}$, $\Gamma_i = V(\mathcal{D}_i) \cap \Gamma, G_i = A_i \setminus (\mathcal{D}_i)_{\Gamma_i}$. Then

$$A_{\xi_*} = \{\xi_*\} \cup A_1 \cup \cdots \cup A_m, \quad \mathcal{D} = \{\xi_*\} \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_m,$$

$$\|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)} \leq \left( \sum_{i=1}^m \|w\|_{l_q(A_{\xi_i})}^q \right)^{1/q}. \quad (21)$$

Set $\alpha_i = \frac{\|w\|_{l_q(G_i)}}{\|w\|_{l_q(A_{\xi_*})}}$. Then

$$\alpha_i = \sigma^{t_i}, \quad t_i \geq 1, \quad \beta_{D_i, \Gamma_i} \geq c^{-1} f^{-\frac{1}{p}} (t_i) \|w\|_{l_q(G_i)}. \quad (22)$$

Indeed, the first relation follows from (16). If $\Gamma_i \neq \emptyset$, then $(\mathcal{D}_i, \Gamma_i) \in \mathcal{J}^G_\xi$, and the second relation holds by induction hypothesis. If $\Gamma_i = \emptyset$, then $\mathcal{D}_i = A_i$, $(\mathcal{D}_i)_{\Gamma_i} = A_i$ and $\|w\|_{l_q(G_i)} = 0$.

By (11),

$$B_{D, \Gamma} \|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)}^{p'} = u^{p'}(\xi_*) \|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)}^{p'} + \left( \sum_{i=1}^m B_{D, \Gamma_i}^{-p'} \|w\|_{l_q(A_{\xi_i})}^{p'} \right)^{-\frac{p'}{p}} \|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)} \leq \quad (22)$$

$$\leq u^{p'}(\xi_*) \|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)}^{p'} + c^{p'} \left( \sum_{i=1}^m \|w\|_{l_q(G_i)}^{p'} f^{-\frac{1}{p}} (t_i) \right)^{-\frac{p'}{p}} \|w\|_{l_q(A_{\xi_*}\setminus \mathcal{D}_T)} \quad (21)$$

$$= u^{p'}(\xi_*) \|w\|_{l_q(A_{\xi_*})}^{p'} \frac{\left( \sum_{i=1}^m \|w\|_{l_q(G_i)}^{p'} \right)^{\frac{p'}{p}}}{\|w\|_{l_q(A_{\xi_*})}^{p'}} +$$
\[ \begin{align*}
&\leq \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{q}{q}} + c^p \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{q}{q}} \left( \sum_{i=1}^{m} \alpha_i^p f^{-\frac{p}{p}}(t_i) \right)^{-\frac{q}{p}} ,
\end{align*} \]

i.e.,

\[ B_{D,1}^p \|w\|_{l_q(A_{t_q})}^p \leq \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{q}{q}} + c^p \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{q}{q}} \left( \sum_{i=1}^{m} \alpha_i^p f^{-\frac{p}{p}}(t_i) \right)^{-\frac{q}{p}} =: S. \tag{23} \]

Let \( t_0 = \min_{1 \leq i \leq m} t_i, I_1 = \{ i \in 1, m : t_i = t_0 \}, I_2 = \{ 1, \ldots, m \} \setminus I_1. \)

Since

\[ \int_0^s \sigma^{t/3} dt = \frac{3}{|\log \sigma|} (1 - \sigma^{-s/3}), \tag{24} \]

it follows that for \( 1 \leq i \leq m \)

\[ \frac{f(t_i)}{f(t_0)} = \frac{1 - \sigma^{t_i/3}}{1 - \sigma^{t_0/3}} \leq 1 + \frac{\sigma^{t/3}}{1 - \sigma^{t_0/3}} \leq 1 + 2\sigma^{t_0/3} \]

for \( \sigma \leq \frac{1}{3}. \)

Hence,

\[ \left( \frac{f(t_i)}{f(t_0)} \right)^{-\frac{p}{p}} \geq \left( 1 + 2\sigma^{t_0/3} \right)^{-\frac{p}{p}} \geq 1 - \frac{2p}{p} \sigma^{t_0/3} \]

Thus,

\[ S \leq \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{q}{q}} + c^p \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{q}{q}} \left( \sum_{i \in I_1} \alpha_i^p + \sum_{i \in I_2} \alpha_i^p \left( 1 - \frac{2p}{p} \sigma^{t_0/3} \right) \right)^{-\frac{q}{p}} f(t_0) =: \tilde{S}. \tag{25} \]

Estimate \( \tilde{S} \) for small \( \sigma. \)

Since \( p < q, \) there is \( \varepsilon_0 = \varepsilon_0(p, q) \in (0, \frac{1}{3}) \) such that for any \( \varepsilon \in [0, \varepsilon_0] \)

\[ (1 - \varepsilon)^{\frac{q}{q}} + \frac{\varepsilon^{\frac{q}{p}}}{2} \geq 1. \tag{26} \]

Let \( \sigma \leq \min \left\{ \frac{1}{3}, \left( \frac{1}{2p} \right)^{\frac{3}{q}} \right\} =: \sigma_1. \)

Set \( \beta = \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{1}{q}}. \)

Then

\[ \beta^q = \sum_{i=1}^{m} \alpha_i^q \leq \sum_{i=1}^{m} \frac{\|w\|_{l_q(A_{t_q})}^q}{\|w\|_{l_q(A_{t_q})}^q} \leq 1. \tag{27} \]
First we consider \( \{\alpha_i\}_{i=1}^m \) satisfying the following property: there is \( i_* \in [1, m] \) such that \( \alpha_{i_*} = \beta(1 - \varepsilon)^{1/q} \), \( \sum_{i \neq i_*} \alpha_i^q = \beta^q \varepsilon, \ 0 \leq \varepsilon < \varepsilon_0. \) Recall that \( \alpha_i = \sigma^{i_*} \) and for \( i \in I_1, j \in I_2 \) the inequality \( \alpha_i > \alpha_j \) holds. Hence, \( I_1 = \{i_*\} \) and
\[
\sum_{i \in I_1} \alpha_i^p + \sum_{i \in I_2} \alpha_i^p \left( 1 - \frac{2p}{p'} \sigma^\frac{4}{3} \right) = \alpha_{i_*}^p + \sum_{i \neq i_*} \alpha_i^p \left( 1 - \frac{2p}{p'} \sigma^\frac{4}{3} \right) \geq \alpha_{i_*}^p + \left( \sum_{i \neq i_*} \alpha_i^q \right)^{\frac{p}{q}} \left( 1 - \frac{2p}{p'} \sigma^\frac{4}{3} \right) =
\]
\[
= \beta^p (1 - \varepsilon)^\frac{2}{q} \beta^p \varepsilon^\frac{2}{q} \left( 1 - \frac{2p}{p'} \sigma^\frac{4}{3} \right) \geq \beta^p \left( (1 - \varepsilon)^\frac{2}{q} + \varepsilon^\frac{2}{q} \right) \geq \beta^p. \tag{26}
\]
Thus,
\[
\bar{S} \leq \beta^p + c' f(t_0). \tag{28}
\]

From (27) it follows that
\[
\beta = \sigma^{i_*}, \quad t_* \geq 0. \tag{29}
\]
Show that there is \( \sigma_2 = \sigma_2(p) \in (0, \sigma_1) \) such that for \( 0 < \sigma \leq \sigma_2, c \geq 2 \)
\[
\beta^p + c' f(t_0) \leq c' f(t_* + 1), \tag{30}
\]
i.e.,
\[
\sigma^{i_*} t_* \leq c'(f(t_* + 1) - f(t_0)). \tag{31}
\]
Indeed, \( \sigma^{i_*} = \alpha_{i_*} = \beta(1 - \varepsilon)^\frac{1}{q} = \sigma^{i_*} (1 - \varepsilon)^\frac{1}{q}. \) Let \( t_0 = t_* + \kappa. \) Then \( (1 - \varepsilon)^\frac{1}{q} = \sigma^\kappa. \) Since \( (1 - \varepsilon)^\frac{1}{q} \geq \frac{2}{3}, \) for small \( \sigma \) we get \( \kappa \leq \frac{1}{4}. \) Hence, \( t_0 \leq t_* + \frac{1}{4} \) and \( t_* \geq t_0 - \frac{1}{4} \geq \frac{3}{4}. \) Therefore,
\[
f(t_* + 1) - f(t_0) \geq f(t_* + 1) - f \left( t_* + \frac{1}{4} \right) \geq \frac{1}{4} \frac{1 - \sigma^\frac{1}{3} \kappa}{1 - \sigma} \geq \frac{1}{4} \frac{\frac{1}{3} + \frac{1}{12}}{1 - \sigma}. \tag{18}
\]
If \( \sigma \) is sufficiently small, then \( c' \frac{1 - \sigma^\frac{1}{3} \kappa}{1 - \sigma} \geq 1. \) Thus, in order to prove (31) it is sufficient to check that \( p' t_* \geq \frac{1}{3} + \frac{1}{12} \). Indeed, it follows from the inequalities \( t_* \geq \frac{3}{4} \) and \( \frac{3}{4} > \frac{1}{3} + \frac{1}{12}. \)

This completes the proof of (30). If \( \xi_0 \) is a minimal vertex, then (23), (25), (28) and (30) yield that \( B_{D,1}^p \|w\|_{p'} \|p' \|_{L^p(A_0, \mathbb{R}^+)} \leq c' f(t_* + 1) \leq c' f(\infty), \) which implies (20).
Suppose that the vertex \( \xi_* \) in not minimal. Let \( \hat{\xi} \) be the direct predecessor of \( \xi_* \). Then 
\[
\|w\|_{l_i(A_{\xi_*})} \leq \sigma \text{ by (16). Therefore, }
\]
\[
\frac{\|w\|_{l_i(A_{\xi_*})}}{\|w\|_{l_i(A_{\hat{\xi}})}} = (\sum_{i=1}^{m} \frac{\|w\|_{l_i(G_i)}}{\|w\|_{l_i(A_{\xi_*})}})^{\frac{1}{q}} \leq \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{1}{q}} \leq \left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{1}{q}} \sigma = \sigma^{t_*+1},
\]
i.e., \( \|w\|_{l_i(A_{\xi_*}) \setminus D_T} = \|w\|_{l_i(A_{\xi_*})}\sigma^t, t \geq t_* + 1 \). This together with (23), (25), (28) and (30) yields that \( B_D \|w\|_{l_i(A_{\xi_*})}^p \leq c^p f(t_* + 1) \leq c^p f(t), \) which implies (19).

Let, now, for any \( i \in I, m \) the inequality \( \alpha_i \leq \beta(1 - \varepsilon_0)^{\frac{1}{q}} \) holds. Prove that there is \( a = a(p, q) < 1 \) such that
\[
\left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^{m} \alpha_i^p \right)^{-\frac{1}{p}} \leq a.
\]
(32)

Indeed, consider the problem
\[
\sum_{i=1}^{m} |\alpha_i|^p \to \min, \sum_{i=1}^{m} |\alpha_i|^q = \beta^q, \ |\alpha_i|^q \leq \beta^q(1 - \varepsilon_0), \quad 1 \leq i \leq m.
\]
The compactness argument yields the existence of the point of minimum, which will be denoted by \((\hat{\alpha}_1, \ldots, \hat{\alpha}_m)\). If \( |\hat{\alpha}_i|^q < \beta^q(1 - \varepsilon_0) \) for any \( i = 1, m \), then by Lagrange’s principle we get \( |\hat{\alpha}_i| = \beta k^{-\frac{1}{q}}, i \in I, \hat{\alpha}_i = 0, i \notin I, \) for some \( I \subset \{1, \ldots, m\} \), card \( I = k, k \geq 2 \). Thus,
\[
\left( \sum_{i=1}^{m} |\hat{\alpha}_i|^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^{m} |\hat{\alpha}_i|^p \right)^{-\frac{1}{p}} = k^{\frac{1}{q} + \frac{1}{p}} \leq 2^{\frac{1}{q} + \frac{1}{p}}.
\]
Let \( |\hat{\alpha}_i|^q = \beta^q(1 - \varepsilon_0) \) for some \( i_* \in \{1, \ldots, m\} \). Then
\[
\sum_{i \neq i_*} |\hat{\alpha}_i|^p + |\hat{\alpha}_{i_*}|^p \geq \left( \sum_{i \neq i_*} |\hat{\alpha}_i|^q \right)^{\frac{p}{q}} + |\hat{\alpha}_{i_*}|^p = \beta^p \varepsilon_0^{\frac{p}{q}} + \beta^p (1 - \varepsilon_0)^{\frac{p}{q}},
\]
\[
\left( \sum_{i=1}^{m} |\hat{\alpha}_i|^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^{m} |\hat{\alpha}_i|^p \right)^{-\frac{1}{p}} \leq \left( \varepsilon_0^{\frac{p}{q}} + (1 - \varepsilon_0)^{\frac{p}{q}} \right)^{-\frac{1}{p}} \leq 1.
\]
The inequality (32) is proved. There exist \( \sigma_3 = \sigma_3(p, q) \in (0, \sigma_2) \) and \( \tilde{a} = \tilde{a}(p, q) < 1 \) such that for any \( \sigma \in (0, \sigma_3) \)
\[
\left( \sum_{i=1}^{m} \alpha_i^q \right)^{\frac{p}{q}} \left( \sum_{i \in I_1} \alpha_i^p + \sum_{i \in I_2} \alpha_i^p \left( 1 - \frac{2p}{p'} \sigma^{\frac{p}{q}} \right) \right)^{-\frac{1}{p'}} \leq \tilde{a}^p.
\]
Therefore,
\[ \tilde{S} \leq \beta^{p'} + c^p \tilde{a}^{p'} f(t_0) \leq 1 + c^p \tilde{a}^{p'} \frac{1}{1 - \sigma^{p' \ast}}. \]

Further, there are \( \sigma_4 = \sigma_4(p, q) \in (0, \sigma_3) \) and \( a_4 = a_4(p, q) < 1 \) such that for \( \sigma \in (0, \sigma_4) \) the inequality \( \tilde{a}^{p'} \frac{1}{1 - \sigma^{p' \ast}} \leq a_4^{p'} \) holds. Hence, there exists \( c_0(p, q) \geq 2 \) such that \( \tilde{S} \leq 1 + c^p a_4^{p'} \leq c^p \) for \( c \geq c_0(p, q) \). \( \square \)

**Corollary 1.** Let \( u, w : V(A) \to [0, \infty) \). Suppose that \( 1 < p < q < \infty \) and (16) holds with \( 0 < \sigma < \sigma_4(p, q) \). Then
\[ \mathcal{S}_{A, u, w}^{p,q} \lesssim \sup_{p,q \xi \in V(A)} u(\xi) \|w\|_{l_p(A_\xi)}. \]

**Proof.** It suffices to consider \( u, w : V(A) \to (0, \infty) \). In this case, the assertion follows from Lemmas [1] and [2]. \( \square \)

The following lemma gives a lower estimate.

**Lemma 3.** Let \( 1 < p \leq q < \infty, \xi_* \in V(A) \). Then
\[ \mathcal{S}_{A, u, w}^{p,q} \geq \sup_{p,q \xi \in V(A_\xi)} \left( \sum_{\xi_* \leq \xi' \leq \xi} u^{p'}(\xi') \right)^{1/p} \|w\|_{l_p(A_\xi)}. \]

**Proof.** Let \( \xi \in V(A_{\xi_*}) \). Define the tree \( \mathcal{D} \) by \( V(\mathcal{D}) = (V(A_{\xi_*}) \cup V(A_\xi)) \cup \{\xi\} \) and set \( \Gamma = \{\xi\} \). Then \( (\mathcal{D}, \Gamma) \in \mathcal{J}_{\xi_*}', V(\mathcal{D}_\Gamma) = V(A_{\xi_*}) \cup V(A_\xi), A_\xi \setminus D_\Gamma = A_\xi \). By Lemma [1]
\[ \mathcal{S}_{A, u, w}^{p,q} \geq \beta^{-1}_{D, \Gamma} \|w\|_{l_p(A_\xi)}. \] (33)

We have
\[ \beta_{D, \Gamma} = \inf \left\{ \|f\|_{l_p(D)} : \sum_{\xi_* \leq \xi' \leq \xi} u(\xi') |f(\xi')| = 1 \right\} = \inf \left\{ \left( \sum_{\xi_* \leq \xi' \leq \xi} |f(\xi')|^p \right)^{1/p} : \sum_{\xi_* \leq \xi' \leq \xi} u(\xi') |f(\xi')| = 1 \right\} = \left( \sum_{\xi_* \leq \xi' \leq \xi} u^{p'}(\xi') \right)^{-1/p'}. \]

This completes the proof. \( \square \)

Let \( (A, \xi_0) \) be a tree, \( u, w : V(A) \to [0, \infty) \), \( \xi_* \in V^A_{j_0}(\xi_0), m \in \mathbb{Z}_+ \cup \{+\infty\}, j_0 < j_1 < j_2 < \ldots < j_k < \ldots, J = \{j_k\}_{0 \leq k < m+1} \). For \( 0 \leq k < m+1 \) denote by \( \mathcal{G}_k \) the maximal subgraph of \( A \) on the set of vertices \( \cup_{j_k \leq j < j_{k+1}} V^A_{j_k} \xi_0 \), and by \( \{A_{k,i}\}_{i \in I_k} \), the set of its connected components. Let \( \xi_{k,i} \) be the minimal vertex of the tree \( A_{k,i} \).

Define the tree \( A_J \) by
\[ V(A_J) = \{\xi_{k,i}\}_{0 \leq k < m+1, i \in I_k}, \quad V^A_{I,J}(\xi_{k,i}) = V^A_{J_{k+1}-j_k}(\xi_{k,i}), \quad 0 \leq k < m. \] (34)
For $0 \leq k < m + 1, i \in I_k$ we set

$$u_f(\xi_{k,i}) = \|u\|_{l_p(A_{k,i})}, \quad w_f(\xi_{k,i}) = \|w\|_{l_q(A_{k,i})}.$$  \hfill (35)

**Lemma 4.** The inequality $\mathcal{S}_{A_{k,i},u,w}^{p,q} \leq \mathcal{S}_{A_{k,i},u,w}^{p,q}$ holds.

**Proof.** Let $f : V(A_{k,i}) \to \mathbb{R}_+$, $\|f\|_{l_p(A_{k,i})} = 1$. Denote $f_J(\xi_{k,i}) = \|f\|_{l_p(A_{k,i})}$, $0 \leq k \leq m, i \in I_k$. Then $\|f_J\|_{l_p(A_{k,i})} = 1$.

Let $\xi \in V(A_{k,i})$. Then for any $0 \leq l \leq k$ there exists $i_l \in I_l$ such that $\xi_{l,i_l} \leq \xi$.

This together the Hölder’s inequality yields

$$\sum_{\xi \leq \xi' \leq \xi} u(\xi')f(\xi') \leq \sum_{l=0}^{k} \sum_{\xi' \in V(A_{l,i_l})} u(\xi')f(\xi') \leq \sum_{l=0}^{k} u_J(\xi_{l,i_l})f_J(\xi_{l,i_l}) = \sum_{\xi' \in V(A), \xi' \leq \xi, i} u_f(\xi_{l,i})f_f(\xi_{l,i}).$$

Hence,

$$\sum_{\xi \in V(A_{k,i})} w^q(\xi)\left(\sum_{\xi \leq \xi' \leq \xi} u(\xi')f(\xi')\right)^q = \sum_{k=0}^{m} \sum_{i \in I_k} \sum_{\xi \in V(A_{k,i})} w^q(\xi)\left(\sum_{\xi' \in V(A_{k,i}), \xi' \leq \xi, i} u(\xi')f(\xi')\right)^q \leq \sum_{k=0}^{m} \sum_{i \in I_k} w^q_J(\xi_{k,i})\left(\sum_{\xi' \in V(A_{k,i}), \xi' \leq \xi, i} u(\xi')f(\xi')\right)^q = \sum_{\xi \in V(A_{k,i})} w^q_J(\xi)\left(\sum_{\xi' \in V(A_{k,i}), \xi' \leq \xi, i} u_f(\xi_{l,i})f_f(\xi_{l,i})\right)^q \leq \left[\mathcal{S}_{A_{k,i},u,w}^{p,q}\right]^q.$$ (35)

This completes the proof. \[ \square \]

**Proof of Theorem 1.**

Denote by $\xi_0$ the minimal vertex of $A$ and set $\hat{\delta} = (K, \lambda, l_0, p, q)$. \n
Let $\sigma_\ast = \sigma_\ast(p, q) \in (0, 1)$ be such as in Lemma 4 and let $t_\ast = t_\ast(\hat{\delta}) \in \mathbb{N}$ be such that $\lambda^{t_\ast} \leq \frac{\alpha}{2}$. Set $l_\ast = l_0t_\ast$. For $m \in \mathbb{N}$ we define the function $u_m : V(A) \to \mathbb{R}_+$ by

$$u_m(\xi) = \begin{cases} u(\xi), & \xi \in V(A(\xi_0)), \quad j \leq l_\ast m, \\ 0, & \xi \in V(A(\xi_0)), \quad j > l_\ast m. \end{cases}$$

Prove that

$$\mathcal{S}_{A_{k,i},u,w}^{p,q} \leq \sup_{\hat{\delta} \in V(A)} u(\xi)\|w\|_{l_q(A_{k,i})}.$$ (36)
This together with B. Levi’s theorem gives the desired estimate.

For \( k \in \mathbb{Z}_+ \) we set \( j_k = l_k k \). Denote \( J = \{ j_k \}_{0 \leq k \leq m} \) and define the tree \( A_J \) by (34). Then Lemma 4 yields

\[
\mathcal{G}^{p,q}_{A_J,(u_m),w,J} \leq \mathcal{G}^{p,q}_{A_J,(u_m),J,J} \; (37)
\]

here \((u_m)_J, w_J\) are defined by (35).

Let \( 0 \leq k \leq m - 1, \xi_{k,i} \in V(A_J), \xi_{k+1,i} \in V^{A_J}(\xi_{k,i}), \xi_{k,i} = \eta_0 < \eta_1 < \cdots < \eta_s = \xi_{k+1,i}, \eta_j \in V^{A_J}(\eta_{j-1}), 1 \leq j \leq l_s \). Then

\[
\frac{\|w_j\|_q(A_{\xi_{k+1,i}})}{\|w_j\|_q(A_{\xi_{k,i}})} = \frac{\|w\|_q(A_{\xi_{k+1,i}'})}{\|w\|_q(A_{\xi_{k,i}'})} = \prod_{j=1}^{l_s} \frac{\|w\|_q(A_{\eta_j})}{\|w\|_q(A_{\eta_{j-1}})} \; (38)
\]

By Corollary 1

\[
\mathcal{G}^{p,q}_{A_J,(u_m),J,J} \leq \sup_{p,q,\xi \in V(A_J)} (u_m)_{J}(\xi) \|w\|_q(A_{\xi}) = \sup_{0 \leq k \leq m, \ i \in I_k} \|u_m\|_{p,\iota(A_{\xi_{k,i}})} \|w\|_q(A_{\xi_{k,i}}).
\]

If \( k < m \), then by (3) we have card \( V(A_{\xi_{k,i}}) \leq 1 \); this together the first relation in (4) yield that \( \|u_m\|_{p,\iota(A_{\xi_{k,i}})} \leq u(\xi_{k,i}). \) If \( k = m \), then \( \|u_m\|_{p,\iota(A_{\xi_{j,i}})} = u(\xi_{j,i}). \) This together with (37) and (38) implies (39).

The lower estimate follows from Lemma 3.

Consider two examples.

**Example 1.** Suppose that there is \( C_s \geq 1 \) such that for any \( j \in \mathbb{Z}_+, j' \geq j, \xi \in V^A_j(\xi) \)

\[
C_s^{-1} \cdot 2^{\psi(j')-\psi(j)} \leq \text{card } V^A_{j'-j}(\xi) \leq C_s \cdot 2^{\psi(j')-\psi(j)}, \quad 2^{\psi(t)} = 2^{\theta s t} \Lambda_s(2^{st});
\]

here \( \theta > 0, s \in \mathbb{N}, \Lambda_s : (0, \infty) \rightarrow (0, \infty) \) is an absolutely continuous function such that \( \lim_{y \rightarrow \infty} \frac{\Lambda_s(y)}{\Lambda_s(2^y)} = 0. \) Suppose that for \( \xi \in V^A_j(\xi_0) \)

\[
u(\xi) = u_j = 2^{\frac{\theta s j}{\nu}} \Psi_j(2^j), \quad w(\xi) = w_j = 2^{\frac{\theta s j}{\nu}} \Psi_j(2^j).
\]

Here \( \Psi_j, \Psi_j : (0, \infty) \rightarrow (0, \infty) \) are absolutely continuous functions such that \( \lim_{y \rightarrow \infty} \frac{\psi_j(y)}{\psi_j(2^j)} = \lim_{y \rightarrow \infty} \frac{\nu \psi_j(y)}{\nu \psi_j(2^j)} = 0. \)

Set \( 3 = (u, w, \psi, C_s, p, q) \).

For \( j_0 \in \mathbb{Z}_+ \) we write

\[
M_{j_0} = \sup_{j \in \mathbb{Z}_+, j \geq j_0} \Psi_j(2^j) \left( \sum_{i \geq j} \Psi_i(2^i) \frac{\Lambda_s(2^i)}{\Lambda_s(2^j)} \right)^{\frac{1}{2}}.
\]

The proof of the following lemma is straightforward and will be omitted.
Lemma 5. Let $\Lambda_\ast : (0, +\infty) \to (0, +\infty)$ be an absolutely continuous function such that $\lim_{y \to +\infty} \frac{\Lambda_\ast(y)}{\Lambda_\ast(0)} = 0$. Then for any $\varepsilon > 0$

$$t^{-\varepsilon} \leq \frac{\Lambda_\ast(\varepsilon t)}{\Lambda_\ast(\varepsilon)} \leq t^\varepsilon, \quad 1 \leq \varepsilon < \infty, \quad 1 \leq t < \infty.$$  

(41)

Theorem 3. Let $M_{j_0} < \infty$, $\xi_\ast \in V^A_{j_0}(\xi_0)$. Then $\mathcal{G}^{\rho, q}_{A_{\xi_\ast}, w, u} \preceq \frac{1}{3} M_{j_0}$.

Proof. Let $\xi \in V^A_{j_{\ast - j_0}}(\xi_\ast)$. Then

$$\|w\|_{l_q(A_{\xi})} \leq \left( \sum_{j' \geq j} w^q(j') 2^{\psi(j') - \psi(j)} \right)^{\frac{1}{q}} \leq \sum_{j' \geq j} 2^{-\theta sj' \Psi_w^q(2sj')} \cdot \frac{\Lambda_\ast(2sj')}{\Lambda_\ast(2sj)} \left( \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj')} \right)^{\frac{1}{q}} = 2^{-\frac{\theta sj}{q}} \left( \sum_{j' \geq j} \Psi_w^q(2sj') \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj')} \right)^{\frac{1}{q}},$$

i.e.,

$$\|w\|_{l_q(A_{\xi})} \leq \frac{2^{\frac{\theta sj}{q}} \left( \sum_{j' \geq j} \Psi_w^q(2sj') \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj')} \right)^{\frac{1}{q}}}{3}.$$  

(42)

For any $l_0 \in \mathbb{N}$

$$\sum_{j' \geq j + l_0} \frac{\Lambda_\ast(2sj')}{\Lambda_\ast(2sj)} \Psi_w^q(2sj') \leq \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj + l_0)} \sum_{j' \geq j} \frac{\Lambda_\ast(2sj')}{\Lambda_\ast(2sj)} \Psi_w^q(2sj').$$

Therefore, for any $\xi \in V^A_{j_{\ast - j_0}}(\xi_\ast), \xi' \in V^A_{l_0}(\xi)$

$$\frac{\|w\|_{l_q(A_{\xi'})}}{\|w\|_{l_q(A_{\xi})}} \leq \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj + l_0)} \left( \sum_{j' \geq j} \Psi_w^q(2sj') \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj')} \right)^{\frac{1}{q}} \leq 2^{-\frac{\theta sj}{2q}}.$$

Hence, for sufficiently large $l_0$ there is $\lambda \in (0, 1)$ such that $\frac{\|w\|_{l_q(A_{\lambda \xi'})}}{\|w\|_{l_q(A_{\xi})}} \leq \lambda, \xi' \in V^A_{l_0}(\xi)$.

For any $\xi \in V^A_{j_{\ast - j_0}}(\xi_\ast)$

$$\|w\|_{l_q(A_{\xi})} \Lambda_{\ast}(\xi) \leq \frac{2^{\frac{\theta sj}{q}} \left( \sum_{j' \geq j} \Psi_w^q(2sj') \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj')} \right)^{\frac{1}{q}}} \leq \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj + l_0)} \left( \sum_{j' \geq j} \Psi_w^q(2sj') \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj')} \right)^{\frac{1}{q}} \cdot 2^{-\frac{\theta sj}{2q}} \Psi_u(2sj) \leq$$

$$= \left( \sum_{j' \geq j} \Psi_w^q(2sj') \frac{\Lambda_\ast(2sj)}{\Lambda_\ast(2sj')} \right)^{\frac{1}{q}} \Psi_u(2sj).$$

(43)

It remains to take the supremum over $j \geq j_0$ and apply Theorem 3. □

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Example 2. Suppose that there exists $C_* \geq 1$ such that for any $j \in \mathbb{Z}_+, j' \geq j$, $\xi \in V^A_j(\xi)$

$$C_*^{-1} \cdot 2^{\psi(j')-\psi(j)} \leq \text{card } V^A_{j'-j}(\xi) \leq C_* \cdot 2^{\psi(j')-\psi(j)}, \quad 2^{\psi(j)} = j^{\gamma_\tau}(j); \quad (43)$$

here $\gamma_\tau > 0$, $\tau_\xi : (0, \infty) \to (0, \infty)$ is an absolutely continuous function such that $\lim_{y \to \infty} \frac{\psi_\tau(y)}{\tau_\xi(y)} = 0$. Suppose that for any $\xi \in V^A_j(\xi)$

$$u(\xi) = u_j = j^{-\alpha_\rho} \rho_u(j), \quad w(\xi) = w_j = j^{-\alpha_\rho} \rho_w(j), \quad (44)$$

where $\rho_u, \rho_w : (0, \infty) \to (0, \infty)$ are absolutely continuous functions such that $\lim_{y \to \infty} \frac{\psi_\rho(y)}{\rho_u(y)} = \lim_{y \to \infty} \frac{\psi_y(y)}{\rho_w(y)} = 0$.

As in Example 1, we set $\mathcal{J} = (u, w, \rho, C, \alpha, q)$.

Theorem 4. Suppose that $j_0 = 2^k_0$, $k_0 \in \mathbb{Z}_+, \xi_* \in V^A_{j_0}(\xi_*)$.

1. Let $-\alpha_u + \frac{1}{q} + \frac{\tau_\xi}{q} < 0$. Set $\alpha = \alpha_u + \alpha_w$, $\rho(t) = \rho_u(t) \rho_w(t)$. If $M_{j_0} := \sup_{j \geq j_0} j^{-\alpha_\rho + \frac{1}{p} + \frac{\tau_\xi}{q}} \rho(j) < \infty$, then $\mathcal{G}^{p,q}_{A_{\xi_*}, u,w} \simeq M_{j_0}$. In particular, if $-\alpha + \frac{1}{q} + \frac{\tau_\xi}{q} < 0$, then $\mathcal{G}^{p,q}_{A_{\xi_*}, u,w} \simeq j_0^{-\alpha_\rho + \frac{1}{p} + \frac{\tau_\xi}{q}} \rho(j_0)$.

2. Let $-\alpha_u + \frac{1}{q} + \frac{\tau_\xi}{q} = 0$, $-\alpha_u + \frac{1}{p} - \frac{\tau_\xi}{q} = 0$,

$$\hat{M}_k := \sup_{k \in \mathbb{Z}_+} \rho_u(2^{k_0 + k}) \left( \sum_{l \geq k} \rho_y(l, 2^{k_0 + l}) \frac{\tau_\xi(2^{k_0 + k})}{\tau_\xi(2^{k_0 + l})} \right)^{\frac{1}{q}} < \infty.$$ 

Then $\mathcal{G}^{p,q}_{A_{\xi_*}, u,w} \simeq \hat{M}_k$.

Proof. Prove the upper estimate. Let $j_k = 2^{k_0 + k}$, $k \in \mathbb{Z}_+$, $J = \{ j_k \}_{k \in \mathbb{Z}_+}$. Define the tree $A_{\xi}$ and weights $w_{j_k}$, $u_j$ by (34), (35). Since

$$\text{card } \{ \xi \in V^A_{j-j_k}(\xi_{j_k}) \} \lesssim \frac{2^{k_0 + k}}{3}, \quad j_k \leq j < j_{k+1}, \quad (45)$$

we have $\text{card } V(\mathcal{A}_{\xi,j_{k+1}}) \simeq \frac{2^{k_0 + k}}{3}$. Hence,

$$(u_{j_j})_{(\xi_{j_k})} \lesssim \frac{2^{-(\alpha_u + \frac{1}{p})}}{3} \rho_u(2^{k_0 + k}), \quad (w_{j_j})_{(\xi_{j_k})} \lesssim \frac{2^{-(\alpha_u + \frac{1}{p})}}{3} \rho_w(2^{k_0 + k}),$$

$$\text{card } V^A_{k'-k}(\xi_{j_k}) = \text{card } V^A_{j_{k'}-j_k}(\xi_{j_k}) \lesssim \frac{2^{\psi(j')-\psi(j)}}{3} 2^{\psi(j')-\psi(j)}, \quad k' \geq k, \quad 2^{\psi(j')} = 2^{\gamma_\tau(k' + 1)} \tau_\xi(2^{k_0 + l}).$$

In the case 1 we get

$$\| w_j \|_{L_p(\mathcal{A}_{\xi,j_{k+1}})} \lesssim \frac{2^{-(\alpha_u + \frac{1}{q})}}{3} \rho_w(2^{k_0 + k}).$$
Suppose that conditions of Theorem 2 hold.

The tree obtained from $T \rightarrow (\xi,T)$ if each element of $k \in \mathbb{N}$, $T$, $T_1$, ..., $T_k$ be trees that have no common vertices, let $v_1, \ldots, v_k \in V(T)$, $w_j \in V(T_j)$, $j = 1, \ldots, k$. Denote by

$$J(T, T_1, \ldots, T_k; v_1, w_1, \ldots, v_k, w_k)$$

the tree obtained from $T$, $T_1$, ..., $T_k$ by connecting the vertices $v_j$ and $w_j$ by an edge for each $j = 1, \ldots, k$.

Let $(D, \xi_0)$ be a tree, $\xi \in V(D)$, $n \in \mathbb{N}$, let $T = \{A_1, \ldots, A_n\}$ be a partition of $V^p(D)$ into nonempty subsets, $A_j = \{\xi_{j,1}, \ldots, \xi_{j,k_j}\}$. Define the graph $G_{\xi,T}(D)$ as follows.

1. Let $\xi = \xi_0$. Then we denote by $G_{\xi,T}(D)$ the graph that is a disjoint union of trees $D_j := J(\{\eta_j\})$, $D_{\xi_{j,1}}$, ..., $D_{\xi_{j,k_j}}$; $\eta_j$, $\xi_{j,1}$, ..., $\eta_j$, $\xi_{j,k_j}$.

2. Let $\xi > \xi_0$, and let $\eta$ be the direct predecessor of $\xi$. Then we set

$$G_{\xi,T}(D) = J(D \setminus D_\xi, D_1, \ldots, D_n; \eta, \eta_1, \ldots, \eta_n),$$

where the vertices $\eta_j$ and trees $D_j$ are defined above.

Let $\overline{\pi}, \overline{w} : V(D) \to (0, \infty)$, $\xi \in V(D)$. Define weights $\overline{\pi}_\xi, T$ and $\overline{w}_\xi, T$ on the graph $G_{\xi,T}(D)$ as follows. If $\xi \in V(D) \setminus V(D_\xi)$ or $\xi \in \bigcup_{j=1}^n \bigcup_{i=1}^{k_j} V(D_{\xi_{j,i}})$, then we set $\overline{\pi}_\xi, T(\xi) = \overline{\pi}(\xi)$, $\overline{w}_\xi, T(\xi) = \overline{w}(\xi)$; if $\xi = \eta_j$ for some $j \in \{1, \ldots, n\}$, then we set

$$\overline{\pi}_\xi, T(\eta_j) = n^{1/2} \overline{\pi}(\xi), \quad \overline{w}_\xi, T(\eta_j) = n^{-1/2} \overline{w}(\xi).$$

If each element of $T$ is a singlepoint, then we denote

$$G_{\xi,T}(D) = G_{\xi}(D), \quad \overline{\pi}_\xi, T = \overline{\pi}_\xi, \quad \overline{w}_\xi, T = \overline{w}_\xi.$$

4 An estimate for the norm of a weighted summation operator on a tree: case $p \geq q$

Suppose that conditions of Theorem 2 hold.

We shall use the following notation.

Let $(D, \xi_0)$ be a tree, $\xi \in V(D)$, $n \in \mathbb{N}$, let $T = \{A_1, \ldots, A_n\}$ be a partition of $V^p(D)$ into nonempty subsets, $A_j = \{\xi_{j,1}, \ldots, \xi_{j,k_j}\}$. Define the graph $G_{\xi,T}(D)$ as follows.

$$\sup_{J \in \mathbb{Z}_+} \|w_J\|_{l_p(A_{\xi,k,i})} u_J(\xi_{k,i}) \leq \sup_{j \geq k_0} \frac{2^{\frac{\alpha+1}{\gamma} + \frac{1}{\gamma}}}{3} \rho(2^j) = M_{j_0}. \quad (46)$$

In the case 2 we have

$$(u_J)(\xi_{k,i}) \leq 2^{-2^{(k_0+k)}} \rho_u(2^{k_0+k}), \quad (w_J)(\xi_{k,i}) \leq 2^{-2^{(k_0+k)}} \rho_w(2^{k_0+k}). \quad (47)$$

The further arguments are the same as in Example 1.

In order to prove the lower estimate, we notice that $\|w\|_{l_p(A_{\xi,k,i})} = \|w_J\|_{l_p(A_{\xi,k,i})}$,

$$\left( \sum_{\xi \leq \xi \leq \xi_{k,i}} w^\rho(\xi') \right)^{\frac{1}{\rho}} \gtrsim \frac{1}{3} (u_J)(\xi_{k,i}) \text{ for } k \geq 1 \text{ and apply Lemma 3 together with (46) and (47).} \quad \Box$$

$$\sum_{\xi \leq \xi \leq \xi_{k,i}} w^\rho(\xi') \gtrsim \frac{1}{3} (u_J)(\xi_{k,i}) \text{ for } k \geq 1 \text{ and apply Lemma 3 together with (46) and (47).} \quad \Box$$
Lemma 6. For any $1 \leq p, q \leq \infty$

$$\mathcal{E}^{p,q}_{\varpi,\pi,D} \subseteq \mathcal{E}^{p,q}_{\varpi,\pi,T,\varpi,G_{\xi,T}(D)}.$$  \hspace{1cm} (50)

Proof. Let $f : V(D) \to \mathbb{R}_+$, $\|f\|_{p(D)} = 1$. Define the function $f_{\xi,T} : V(G_{\xi}(D)) \to \mathbb{R}_+$ as follows: we set $f_{\xi,T}(\zeta) = f(\zeta)$ for $\zeta \in V(D) \setminus V(D_\xi)$ or for $\zeta \in \bigcup_{j=1}^n \bigcup_{i=1}^{k_j} V(D_{\xi_{ji}})$, and we set $f_{\xi,T}(\eta_j) = n^{-\frac{1}{p}}f(\xi)$, $1 \leq j \leq n$. Then $\|f_{\xi,T}\|_{p(G_{\xi,T}(D))} = \|f\|_{p(D)}$. We have

$$\sum_{\zeta \in V(D)} \overline{w}^q(\xi) \left( \sum_{\zeta' \leq \xi} \overline{u}(\zeta') f(\zeta') \right)^q = \sum_{\zeta \in V(D) \setminus V(D_\xi)} \overline{w}^q(\xi) \left( \sum_{\zeta' \leq \xi} \overline{u}(\zeta') f(\zeta') \right)^q +$$

$$+ \overline{w}^q(\xi) \left( \sum_{\zeta' \leq \xi} \overline{u}(\zeta') f(\zeta') \right)^q + \sum_{j=1}^n \sum_{i=1}^{k_j} \overline{w}^q(\xi) \left( \sum_{\zeta' \leq \xi} \overline{u}(\zeta') f(\zeta') \right)^q =: S.$$  \hspace{1cm} (51)

Since $V(D) \setminus V(D_\xi) \subset V(G_{\xi,T}(D))$, by definitions of $\varpi_{\xi,T}$, $\overline{w}_{\xi,T}$ and $f_{\xi,T}$ we get

$$\sum_{\zeta \in V(D) \setminus V(D_\xi)} \overline{w}^q(\xi) \left( \sum_{\zeta' \leq \xi} \overline{u}(\zeta') f(\zeta') \right)^q = \sum_{\zeta \in V(D) \setminus V(D_\xi)} \overline{w}^q_{\xi,T}(\xi) \left( \sum_{\zeta' \leq \xi} \overline{u}_{\xi,T}(\zeta') f_{\xi,T}(\zeta') \right)^q.$$  \hspace{1cm} (52)

Let $1 \leq j \leq n$. Then

$$\overline{w}^q_{\xi,T}(\eta_j) \left( \sum_{\zeta' \in V(G_{\xi,T}(D)), \zeta' \leq \eta_j} \overline{u}_{\xi,T}(\zeta') f_{\xi,T}(\zeta') \right)^q =$$

$$= n^{-1} \overline{w}^q(\xi) \left( \sum_{\zeta' \in V(D), \zeta' \leq \xi} \overline{u}(\zeta') f(\zeta') + n^{-\frac{1}{p}} \overline{u}(\xi) \cdot n^{-\frac{1}{p}} f(\xi) \right)^q =$$

$$= n^{-1} \overline{w}^q(\xi) \left( \sum_{\zeta' \in V(D), \zeta' \leq \xi} \overline{u}(\zeta') f(\zeta') \right)^q.$$

Hence,

$$\overline{w}^q(\xi) \left( \sum_{\zeta' \in V(D), \zeta' \leq \xi} \overline{u}(\zeta') f(\zeta') \right)^q = \sum_{j=1}^n \overline{w}^q_{\xi,T}(\eta_j) \left( \sum_{\zeta' \in V(G_{\xi,T}(D)), \zeta' \leq \eta_j} \overline{u}_{\xi,T}(\zeta') f_{\xi,T}(\zeta') \right)^q.$$  \hspace{1cm} (52)

Let $\zeta \in V(D_{\xi_{ji}})$, $1 \leq j \leq n$, $1 \leq i \leq k_j$. Then

$$\sum_{\zeta' \in V(G_{\xi,T}(D)), \zeta' \leq \xi} \overline{u}_{\xi,T}(\zeta') f_{\xi,T}(\zeta') = \sum_{\zeta' \in V(G_{\xi,T}(D)), \zeta' \leq \xi, \zeta' \neq \eta_j} \overline{u}_{\xi,T}(\zeta') f_{\xi,T}(\zeta') + \overline{u}_{\xi,T}(\eta_j) f_{\xi,T}(\eta_j) =$$

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Proof of Theorem 2.
It suffices to consider the case
\[ \sum_{\xi \in V(D), \xi' \leq \xi} \pi'(\xi') f(\xi') = \sum_{\xi' \in V(D), \xi' \leq \xi} \pi(\xi') f(\xi'). \]
Therefore,
\[ \overline{w}(\xi) \left( \sum_{\xi' \in V(D), \xi' \leq \xi} \pi(\xi') f(\xi') \right)^q = \overline{w}_{\xi,T}(\xi) \left( \sum_{\xi' \in V(G_{\xi,T}(D)), \xi' \leq \xi} \overline{w}_{\xi,T}(\xi') f_{\xi,T}(\xi') \right)^q. \]

From (51), (52) and (53) it follows that
\[ S = \sum_{\xi \in V(G_{\xi,T}(D))} \overline{w}_{\xi,T}(\xi) \left( \sum_{\xi' \leq \xi} \overline{w}_{\xi,T}(\xi') f_{\xi,T}(\xi') \right)^q \leq \left( \overline{w}_{\xi,T}, \overline{w}_{\xi,T}, G_{\xi,T}(D) \right)^q. \]

This completes the proof of (51). □

Denote by \([A]_{\leq n}\) a subtree in \(A\) such that
\[ V([A]_{\leq n}) = \bigcup_{j=0}^{n} V_{\xi}^A(\xi_0). \]

Proof of Theorem 2. It suffices to consider the case \(p < \infty\) and \(N < \infty\).

For \(0 \leq j \leq N\) we construct the graph \(G_{j,A}\) and the functions \(u^{(j)}, w^{(j)} : V(G_{j,A}) \to (0, \infty)\) with the following properties:
1. \(G_{N,A} = A, u^{(N)} = u, w^{(N)} = w. \)
2. If \(1 \leq j \leq N - 1\), then \(G_{j,A}\) is a tree with the minimal vertex \(\xi_0\); here
\[ \mathcal{G}_{j,A} \leq j-1 = [A]_{\leq j-1}, \quad V^A_{N-j,A}(\xi_0) = V_{max}(G_{j,A}); \]
\[ \text{card} V^A_{1,j-1}(\xi_0) = \text{card} V^A_{N-j+1}(\xi_0), \quad \text{if} \quad \xi \in V^A_{j-1}(\xi_0); \]
\[ \text{card} V^A_{1,j-1}(\xi_0) = 1, \quad \text{if} \quad \xi \in V^A_{i,j-1}(\xi_0), \quad j \leq i \leq N - 1. \]

In addition,
\[ u^{(j)}(\xi) = u(\xi), \quad w^{(j)}(\xi) = w(\xi), \quad \xi \in V([G_{j,A}]_{\leq j-1}); \]
\[ u^{(j)}(\xi) \asymp \frac{2^{\frac{u(N)-u(j)}{p}}}{C_*}, \quad w^{(j)}(\xi) \asymp \frac{2^{\frac{u(N)-w(j)}{p}}}{C_*}, \quad \xi \in V^A_{i,j-1}(\xi_0), \quad j \leq i \leq N; \]

if \(C_* = 1\), then we have exact equalities in (58).

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3. If \( j = 0 \), then \( G_{j,A} \) is a disjoint union of paths \( \zeta_{k,0} < \zeta_{k,1} < \cdots < \zeta_{k,N} \), \( 1 \leq k \leq \text{card } V_{N}^{A}(\xi_{0}) \). In addition,
\[
  u^{(0)}(\zeta_{k,i}) \asymp u_{i} \cdot 2^{\frac{\psi(N) - \psi(i)}{p}}, \quad w^{(0)}(\zeta_{k,i}) \asymp w_{i} \cdot 2^{\frac{\psi(N) - \psi(i)}{q}}, \quad 0 \leq i \leq N. \tag{59}
\]

If \( C_{*} = 1 \), then we have exact equalities in \( (59) \).

4. \( \mathcal{G}_{p,q}^{\rho,q} \leq \mathcal{G}_{j,A,u(w),w(w)}^{p,q} \).

The graphs \( G_{j,A} \) and the functions \( u^{(j)}, w^{(j)} \) will be constructed by induction on \( j \). Suppose that for some \( 0 \leq k \leq N - 1 \) the trees \( G_{k+1,A} \) and the functions \( u^{(k+1)}, w^{(k+1)} \) are constructed, and suppose that assertions 1–4 hold with \( j := k + 1 \).

Set \( V_{k}^{G_{k+1,A}}(\xi_{0}) = \{ \zeta_{1}, \ldots, \zeta_{m} \} \). From \( (54) \) it follows that \( \{ \zeta_{1}, \ldots, \zeta_{m} \} = V_{k}^{A}(\zeta_{0}), \)
\[
u^{(k+1)}(\zeta_{t}) = w(\zeta_{t}), \quad k+1 \leq t \leq m.
\]

We set
\[
\mathcal{G}_{k,A} = \mathcal{G}_{\zeta_{1}}, \ldots, \mathcal{G}_{\zeta_{m}} \left( \mathcal{G}_{k,A}(\mathcal{G}_{k+1,A}) \right),
\]
\[
u_{k} = (((\nu_{k+1}^{1})_{\zeta_{1}}), \ldots, \nu_{k}^{m} = (((\nu_{k+1}^{1})_{\zeta_{2}}), \ldots, \nu_{k}^{m})
\]
(see \( (19) \)). From Lemma \( 6 \) and the induction assumption we obtain assertion 4. Conditions \( (54), (55), (56) \) for \( j := k > 0 \) and the first part of assertion 3 hold by construction and by the induction hypothesis.

Estimate the values \( u^{(k)}(\eta) \) and \( w^{(k)}(\eta) \), \( \eta \in V(G_{k,A}) \). Let \( \eta \in V_{k}(G_{k,A}) \). Then \( V_{1}^{G_{k,A}}(\eta) = \{ \eta' \} \). There exists \( 1 \leq t \leq m \) such that \( \eta' \in V_{1}^{G_{k+1,A}}(\zeta_{t}) \). From definition of \( u^{(k)} \) and \( w^{(k)} \) and from \( (57) \) applied to \( j := k + 1 \) we get
\[
\begin{align*}
u^{(k)}(\eta) & \overset{(48)}{=} u^{(k+1)}(\zeta_{t}) \left( \text{card } V_{N-k}^{A}(\zeta_{t}) \right)^{-\frac{1}{q}} \overset{(55)}{=} u(\zeta_{t}) 2^{\frac{\psi(N) - \psi(k)}{p}}, \\
u^{(k)}(\eta) & \overset{(48)}{=} w^{(k+1)}(\zeta_{t}) \left( \text{card } V_{N-k}^{A}(\zeta_{t}) \right)^{-\frac{1}{q}} \overset{(55)}{=} w(\zeta_{t}) 2^{\frac{\psi(N) - \psi(k)}{q}}.
\end{align*}
\]
If \( C_{*} = 1 \), then we have exact equalities.

Let \( \eta \in V(G_{k,A}) \setminus V_{k}(G_{k,A}) \). Then \( u^{(k)}(\eta) = u^{(k+1)}(\eta), \quad w^{(k)}(\eta) = w^{(k+1)}(\eta) \). This together with the induction assumption yields \( (57) \) and \( (58) \) for \( k > 0 \) and the second part of assertion 3 for \( k = 0 \).

Let us estimate \( \mathcal{G}_{p,q}^{\rho,q} \mathcal{G}_{0,A,u(w),w(w)}^{p,q} \). Set
\[
m_{*} = \text{card } V_{N}^{A}(\xi_{0}) \overset{(53)}{=} 2^{\psi(N)} \tag{60}
\]
(if \( C_{*} = 1 \), then the exact equality holds). By assertion 3, \( \mathcal{G}_{p,q}^{\rho,q} \mathcal{G}_{0,A,u(w),w(w)}^{p,q} \overset{(55)}{=} \mathcal{G}_{p,q}^{\rho,q} \mathcal{G}_{0,A,\tilde{u},\tilde{w}}^{p,q} \), where
\[
\tilde{u}(\zeta_{k,i}) = \tilde{u}_{i} := u_{i} \cdot 2^{\frac{\psi(N) - \psi(i)}{p}}, \quad \tilde{w}(\zeta_{k,i}) = \tilde{w}_{i} := w_{i} \cdot 2^{\frac{\psi(N) - \psi(i)}{q}}. \tag{61}
\]
Let \( f : \mathbf{V}(\mathcal{G}_{0,A}) \to \mathbb{R}_+ \), \( \|f\|_{p(\mathcal{G}_{0,A})} = 1 \). Set \( \varphi(\zeta_{k,i}) = \varphi_{k,i} = f^p(\zeta_{k,i}) \). Then

\[
\sum_{k=1}^{m_s} \sum_{j=0}^{N} \varphi_{k,i} = 1,
\]

\[
\sum_{\xi \in \mathbf{V}(\mathcal{G}_{0,A})} \tilde{w}_q^q(\xi) \left( \sum_{\xi' \leq \xi} \tilde{u}_j(\xi') f(\xi') \right)^q = \sum_{k=1}^{m_s} \sum_{j=0}^{N} \tilde{w}_j^q \left( \sum_{i=0}^{j} \tilde{u}_i \varphi_{1/p}^{1/p} \right)^q =: \mathcal{F}(\varphi).
\]

Since \( p \geq q \), the function \( t \mapsto t^q \) is concave on \( \mathbb{R}_+ \). This together with the inverse Minkowski inequality implies that \( \mathcal{F}(\varphi) \) is concave on the set of nonnegative functions \( \varphi \).

Set \( \tilde{\varphi}(\zeta_{k,i}) = \tilde{\varphi}_i = \frac{1}{m_s} \sum_{\ell=1}^{m_s} \varphi_{\ell,i}, 1 \leq k \leq m_s, \tilde{f}_i = \tilde{\varphi}_i^{1/p} m_s^{1/p} \). Then

\[
\sum_{j=0}^{N} \tilde{\varphi}_i = \frac{1}{m_s} \sum_{k=1}^{m_s} \sum_{j=0}^{N} \varphi_{k,i} = \frac{1}{m_s}, \quad \sum_{j=0}^{N} \tilde{f}_i^p = 1.
\]

Notice that \( \tilde{\varphi}(\zeta_{k,i}) = \frac{1}{\text{card} S_{m_s}} \sum_{\pi \in S_{m_s}} \varphi_{\pi}(\zeta_{k,i}) \) and \( \mathcal{F}(\varphi) = \mathcal{F}(\varphi_{\pi}) \) for any \( \pi \in S_{m_s} \), where \( S_{m_s} \) is the set of all permutations of \( m_s \) elements and \( \varphi_{\pi}(\zeta_{k,i}) = \varphi(\zeta_{\pi(k),i}) \).

Since \( \mathcal{F} \) is concave, the inequality \( \mathcal{F}(\varphi) \leq \mathcal{F}(\tilde{\varphi}) \) holds. Therefore,

\[
\sum_{k=1}^{m_s} \sum_{j=0}^{N} \tilde{w}_j^q \left( \sum_{i=0}^{j} \tilde{u}_i \varphi_{1/p}^{1/p} \right)^q \lesssim \sum_{k=1}^{m_s} \sum_{j=0}^{N} \tilde{w}_j^q \left( \sum_{i=0}^{j} \tilde{u}_i \varphi_{1/p}^{1/p} \right)^q = \mathcal{F}(\varphi_{\pi}) \lesssim \left[ \mathcal{G}_{p,q} \right]^q.
\]

This completes the proof.

The similar assertion can be obtained for the weighted integration operator on a metric tree. Let \( \mathbb{A} = (\mathbb{A}, \Delta) \), where \((\mathbb{A}, \xi_0)\) satisfies (5) and \( \text{card} V^1_\mathbb{A}(\xi_0) = 1 \). Suppose that \( \Delta((\xi', \xi'')) = [a_j, b_j] \) for any \( \xi' \in V^1_\mathbb{A}(\xi_0), \xi'' \in V^1_\mathbb{A}(\xi') \). Let \( x_0 \) be the minimal point in \( \mathbb{A} \). Consider the weight functions \( g, v : \mathbb{A} \to (0, \infty) \) such that \( g(x) = g_0(|x - x_0|_\mathbb{A}), v(x) = v_0(|x - x_0|_\mathbb{A}) \) (see (3)).

Set \( R = \sum_{j \in \mathbb{Z}_+} (b_j - a_j) \),

\[
\hat{v}_0(t) = v_0(t) \cdot 2^{\frac{v(0)}{q}}, \quad \hat{g}_0(t) = g_0(t) \cdot 2^{-\frac{g(0)}{p}}, \quad t = \sum_{i=0}^{j-1} (b_i - a_i) + s, \quad s \in [a_j, b_j].
\]

\[22\]
Let $I_{g,v,x_0} : L_p(A) \to L_q(A)$ be defined by (7), and let $\tilde{I}_{\tilde{g}_0,\tilde{v}_0} f(t) = \int_0^t \tilde{g}_0(x) f(x) \, dx$, $0 \leq t < R$, $f \in L_p(0, R)$.

**Theorem 5.** Let $1 \leq q \leq p \leq \infty$. Then

$$\| I_{g,v,x_0} \|_{L_p(A) \to L_q(A)} \asymp \| \tilde{I}_{\tilde{g}_0,\tilde{v}_0} \|_{L_p(0, R) \to L_q(0, R)}.$$

If $C_* = 1$, then the exact equality holds.

This result is proved similarly as Theorem 2. For $p = q = 2$ it was obtained in [22].

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