Lectures on Isomorphism Theorems

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1 Introduction

These notes originated in a series of lectures I gave in Marseille in May, 2013. I was invited to give an introduction to the isomorphism theorems, originating with Dynkin, [11], [12], which connect Markov local times and Gaussian processes. This is an area I had worked on some time ago, and even written a book about, [25], but had then moved on to other things. However, isomorphism theorems have become of interest once again, both because of new applications to the study of cover times of graphs and Gaussian fields, [6, 7, 8, 9, 10], and because of new isomorphism theorems for non-symmetric Markov processes and their connection with loop soups and Poisson processes, [14, 16, 21, 22, 23, 24]. Thus I felt the time was ripe for a new introduction to this topic.

I greatly enjoyed giving these lectures, since I felt free to focus on what I consider to be the basic ideas. Writing my book with Marcus took a lot of time and effort since we wanted to make sure that all the details were carefully explained. In these notes I have tried to preserve the informal atmosphere of the lectures, and often simply refer the reader to the book [25] and other sources for details.

The actual lectures covered the material which appears in sections 2-7. This begins with some introductory material on Gaussian processes and Markov processes and then studies in turn the isomorphism theorems of Dynkin, Eisenbaum and the generalized second Ray-Knight theorem. In each case we give a proof and a sample application. We then introduce loop soups and permanental processes, the ingredients we use to develop an isomorphism theorem for non-symmetric Markov processes. Along the way we gain new insight into the reason that Gaussian processes appear in the isomorphism theorems in the symmetric case. Chapters 8-10 contain the material I would have liked to include in the lectures, but had to skip because of lack of time. Having developed some general material on Poisson processes in Section 7, we make use of it in the next two sections. Section 8 contains an excursion theory proof of the generalized second Ray-Knight theorem, and section 9 explains a similar theorem for random interlacements. Up till this point, the proofs I give use the method of moments, which for me is the simplest and clearest way to prove isomorphism theorems. In section 10 we explain how to prove
these theorems using the method of Laplace transforms. Some may prefer this approach because it is more ‘automatic’ and doesn’t involve the sometimes subtle combinatorics which come up when dealing with moments.

2 Gaussian processes

A real valued random variable $X$ is called a Gaussian random variable if

$$E\left(e^{i\lambda X}\right) = e^{im\lambda - \sigma^2\lambda^2/2}, \quad \forall \lambda \in R^1$$

(2.1)

for some numbers $m, \sigma$. Differentiating in $\lambda$ we see that

$$E(X) = m, \quad V(X) = \sigma^2.$$  

(2.2)

We can always eliminate the $m$ by subtracting it from $X$. From now on we assume that $E(X) = 0$, so that (2.1) becomes

$$E\left(e^{i\lambda X}\right) = e^{-E((\lambda X)^2)/2}, \quad \forall \lambda \in R^1.$$  

(2.3)

A random vector $X = (X_1, \ldots, X_n) \in R^n$ is called a Gaussian random vector if for each $y \in R^n$, $(y, X)$ is a Gaussian random variable. Thus we have

$$E\left(e^{i(y,X)}\right) = e^{-E((y,X)^2)/2}, \quad \forall y \in R^n.$$  

(2.4)

The $n \times n$ matrix $C = \{C_{i,j}, 1 \leq i, j \leq n\}$ with entries

$$C_{i,j} = E(X_iX_j), \quad 1 \leq i, j \leq n$$

(2.5)

is called the covariance matrix of $X$, and we can write

$$(y, X)^2 = \sum_{i,j=1}^{n} C_{i,j}y_iy_j = (y, Cy),$$

(2.6)

so that (2.4) can be written as

$$E\left(e^{i(y,X)}\right) = e^{-(y,Cy)/2}, \quad \forall y \in R^n.$$  

(2.7)

It follows from (2.5) that $C$ is symmetric and from (2.6) that $C$ is positive definite. We now show that conversely, any symmetric positive definite $n \times n$ matrix $B$ is the covariance matrix of some Gaussian random vector in $R^n$.  


To see this, we first note that if $A$ is any $p \times n$ matrix and we write $Z = AX$, then by (2.7) we have

$$E(e^{i(y,Z)}) = E(e^{i(A'y,X)}) = e^{-(A'y,CA'y)/2} = e^{-(y,ACA'y)/2}, \quad \forall y \in \mathbb{R}^p.$$  \hspace{1cm} (2.8)

so that $Z = AX$ is Gaussian random vector with covariance matrix $ACA^t$.

If $B$ is a symmetric positive definite $n \times n$ matrix, then there exists a symmetric matrix $A$ with $B = A^2$. To see this recall that any symmetric matrix is diagonalizable, so we can find an orthonormal system of vectors $u_i, 1 \leq i \leq n$ such that $Bu_i = \lambda_i u_i, 1 \leq i \leq n$, and the fact that $B$ is positive definite implies that all $\lambda_i \geq 0$. We can then define the matrix $A$ by setting $Au_i = \lambda_i^{1/2} u_i, 1 \leq i \leq n$. If we now take $X$ to be a vector whose components are independent standard normals, so that the covariance matrix of $X$ is $I$, it follows from the above that $Z = AX$ is a Gaussian random vector with covariance matrix $B$.

If $S$ is a general set, a stochastic process $G = \{G_x, x \in S\}$ is called a Gaussian process on $S$ if for any $n$ and any $x_1, \ldots, x_n \in S$, $(G_{x_1}, \ldots, G_{x_n})$ is a Gaussian random vector. Then the function

$$C(x, y) = E(G_x G_y), \quad x, y \in S$$ \hspace{1cm} (2.9)

on $S \times S$ is called the covariance function of $G$. Using the above and Kolmogorov’s extension theorem we see that there is a correspondence between Gaussian processes on $S$ and symmetric positive definite functions on $S \times S$.

Example: Let $S = \mathbb{R}_{++}^1$, and let $C(s, t) = s \wedge t = \int 1_{[0,s]}(x)1_{[0,t]}(x) \, dx$. Then $C(s, t)$ is positive definite since

$$\sum_{i,j=1}^n C(s_i, s_j)y_i y_j = \int \left( \sum_{i=1}^n y_i 1_{[0,s_i]}(x) \right)^2 \, dx \geq 0, \hspace{1cm} (2.10)$$

so there exists a Gaussian process $B = \{B_s, s \in \mathbb{R}_{++}^1\}$. Note that is $s < t < t'$ we have

$$E(B_s(B_{t'} - B_t)) = s \wedge t - s \wedge t' = 0$$ \hspace{1cm} (2.11)

so that $B$ has orthogonal increments, which are then independent by (2.4). Hence $B$ is ‘almost’ Brownian motion. What is missing is a continuous version, which can be established in the usual ways.

### 2.1 Gaussian moment formulas

Let $G = \{G_x, x \in S\}$ be a Gaussian process with covariance function $C$. We present several Gaussian moment formulas which will be used to prove our
Isomorphism Theorems. The basic formula is

\[
E \left( \prod_{i=1}^{n} G_{x_i} \right) = \sum_{p \in R_n} \prod_{(i_1, i_2) \in p} C(x_{i_1}, x_{i_2}) \tag{2.12}
\]

where \( R_n \) denotes the set of pairings \( p \) of the indices \([1, n]\), and the product runs over all pairs in \( p \). In particular, this is empty when \( n \) is odd, in which case the left hand side is zero by symmetry.

Proof: We write (2.7) as

\[
E \left( e^{i \sum_{j=1}^{n} z_j G_{x_j}} \right) = e^{-\sum_{j,k=1}^{n} z_j z_k C(x_j, x_k)/2}. \tag{2.13}
\]

We differentiate successively in \( z_1, \ldots, z_n \) and after differentiating in \( z_j \) we set \( z_j = 0 \).

To begin, we differentiate in \( z_1 \) and then set \( z_1 = 0 \), to obtain

\[
iE \left( G_{x_1} e^{i \sum_{j=2}^{n} z_j G_{x_j}} \right) = \left( -\sum_{k=2}^{n} z_k C(x_1, x_k) \right) e^{-\sum_{j=2}^{n} z_j C(x_j, x_k)/2}. \tag{2.14}
\]

We then differentiate in \( z_2 \), using the product rule for the right hand side, and after setting \( z_2 = 0 \) we obtain

\[
-E \left( G_{x_1} G_{x_2} e^{i \sum_{j=3}^{n} z_j G_{x_j}} \right) = -C(x_1, x_2) e^{-\sum_{j=3}^{n} z_j C(x_j, x_k)/2}
+ \left( -\sum_{k=3}^{n} z_k C(x_1, x_k) \right) \left( -\sum_{k=3}^{n} z_k C(x_2, x_k) \right) e^{-\sum_{j=3}^{n} z_j C(x_j, x_k)/2}. \tag{2.15}
\]

By now it should be clear that by continuing this process we obtain (2.12) \( \square \)

Our next formula is:

\[
E \left( \prod_{i=1}^{n} G_{x_i}^2 \right) = \sum_{A_1 \cup \cdots \cup A_j = [1, n]} \prod_{l=1}^{j} 2^{|A_l| - 1} \text{cy}(A_l), \tag{2.16}
\]

where the sum is over all (unordered) partitions \( A_1 \cup \cdots \cup A_j \) of \([1, n]\) and, if we have \( A_l = \{l_1, l_2, \cdots, l_{|A_l|}\} \) then the cycle function \( \text{cy}(A_l) \) is defined as

\[
\text{cy}(A_l) = \sum_{\pi \in \mathcal{P}_l} C(x_{l_{\pi(1)}}, x_{l_{\pi(2)}}) \cdots C(x_{l_{\pi(|A_l|)}}, x_{l_{\pi(1)}}), \tag{2.17}
\]

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where \( \mathcal{P}_{k} \) denotes the set of permutations of \([1, k]\) on the circle. (For example, \((1, 2, 3), (3, 1, 2)\) and \((2, 3, 1)\) are considered to be the same permutation \( \pi \in \mathcal{P}_{3} \).)

Proof: On the left hand side of (2.16) each \( G_{x_{i}} \) appears twice. We can arbitrarily consider one of the two \( G_{x_{i}} \)’s as the ‘red’ \( G_{x_{i}} \) and the other as the ‘green’ \( G_{x_{i}} \). Consider first the ‘red’ \( G_{x_{1}} \). By (2.12) it is paired with some \( G_{x_{i}} \). If it is paired with the ‘green’ \( G_{x_{1}} \), we set \( A_{1} = \{1\} \), in which case \( c_{y}(A_{1}) = C(x_{1}, x_{1}) \). Otherwise, the ‘red’ \( G_{x_{1}} \) is coupled with one of the two \( G_{x_{j}} \)’s for some \( j \neq 1 \), giving a factor of \( 2 C(x_{1}, x_{j}) \) and so we continue until eventually we are paired with the ‘green’ \( G_{x_{1}} \) which gives the factor \( C(x_{i}, x_{1}) \).

\( A_{1} \) consists of those \( j \) such that \( G_{x_{j}} \) has been used. Beginning again with some \( G_{x_{j}} \) not used yet and iterating we are led to (2.16). \( \square \)

For later reference it will be useful to write (2.16) as

\[
E \left( \prod_{i=1}^{n} G_{x_{i}}^{2}/2 \right) = \sum_{A_{1} \cup \cdots \cup A_{j} = [1, n]} \prod_{l=1}^{j} \frac{1}{2} c_{y}(A_{l}). \tag{2.18}
\]

Our last formula for now is:

\[
E \left( G_{a} G_{b} \prod_{i=1}^{n} G_{x_{i}}^{2}/2 \right) = \sum_{A \subseteq [1, n]} \text{ch}(A; a, b) \sum_{A_{1} \cup \cdots \cup A_{j} = [1, n] - A} \prod_{l=1}^{j} \frac{1}{2} c_{y}(A_{l}), \tag{2.19}
\]

where the sum is over all (unordered) partitions \( A_{1} \cup \cdots \cup A_{j} \) of \([1, n] - A\) and, if \( A = \{l_{1}, l_{2}, \ldots, l_{|A|}\} \) then the chain function \( \text{ch}(A; a, b) \) is defined as

\[
\text{ch}(A; a, b) = \sum_{\pi \in \mathcal{P}_{|A|}} C(x_{a}, x_{l_{\pi(1)}}) C(x_{l_{\pi(1)}}, x_{l_{\pi(2)}}) \cdots C(x_{l_{\pi(|A|)}}, x_{b}), \tag{2.20}
\]

where \( \mathcal{P}_{k} \) denotes the set of permutations of \([1, k]\). Using (2.18) we can rewrite (2.19) as

\[
E \left( G_{a} G_{b} \prod_{i=1}^{n} G_{x_{i}}^{2}/2 \right) = \sum_{A \subseteq [1, n]} \text{ch}_{A}(a, b) E \left( \prod_{i \notin A} G_{x_{i}}^{2}/2 \right). \tag{2.21}
\]

To see this we use the previous procedure but start with \( G_{a} \). Rather than obtain a cycle, since \( G_{a} \) appears only once, eventually we are paired with \( G_{b} \). This forms the chain, and the remaining \( G_{x_{i}}^{2} \)’s lead to cycles as before.

For more details on the material covered in this section, see the beginning of Section 5.1 in [23]. (2.12) is Lemma 5.2.6 in that book, and (2.21) is stated there as (8.93) and proven carefully.
3 Markov processes

Let $S$ be a topological space which is locally compact with countable base (LCCB). Let
\[ \{p_t(x,y), (t,x,y) \in \mathbb{R}_+ \times S \times S\} \]
be a semigroup of sub-probability kernels with respect to some measure $m$ on $S$. That is, $p_t(x,y) \geq 0$ and satisfies
\[ \int p_t(x,y) \, dm(y) \leq 1 \quad (3.1) \]
and
\[ \int p_t(x,y)p_s(y,z) \, dm(y) = p_{t+s}(x,z). \quad (3.2) \]
We write $P_t$ for the semigroup of operators induced by $p_t(x,y)$.
\[ P_t f(x) = \int p_t(x,y) f(y) \, dm(y), \quad (3.3) \]
and note that
\[ \|P_t f\|_{\infty} \leq \|f\|_{\infty}. \quad (3.4) \]

It will be useful to introduce the $\Delta$ formalism which turns any semigroup of sub-probability kernels $p_t(x,y)$ into a semigroup of probability kernels $\tilde{p}_t(x,y)$. To do this we introduce a new point $\Delta \notin S$, called the cemetery state and extend $m$ to have unit mass at $\Delta$. Then if we set $\tilde{p}_t(x,y) = p_t(x,y)$ for $y \in S$, $\tilde{p}_t(x,\Delta) = 1 - \int_S p_t(x,y) \, dm(y)$, and $\tilde{p}_t(\Delta,\Delta) = 1$ one can check that the $\tilde{p}_t(x,y)$ form a semigroup of probability kernels. In the following we will denote by $\tilde{p}_t(x,y)$ this extension to a semigroup of probability kernels on $S \cup \Delta$, and use the convention that for any function $f$ on $S$ we set $f(\Delta) = 0$.

Given such a semigroup of kernels $p_t(x,y)$, we say that $X = \{X_t, t \geq 0\}$ is a Markov process with transition densities $p_t(x,y)$ if for any bounded measurable functions $f_i, 1 \leq i \leq k$ on $S$, and times $t_1 < \cdots < t_k$
\[ P^x \left( \prod_{i=1}^k f_i(X_{t_i}) \right) \quad (3.5) \]
\[ = \int p_{t_1}(x,y_1)p_{t_2-t_1}(y_1,y_2) \cdots p_{t_k-t_{k-1}}(y_{k-1},y_k) \prod_{i=1}^k f_i(y_i) \, dm(y_i). \]

Constructing a ‘nice’ Markov process from the kernels $p_t(x,y)$ is another story. For now we simply assume that $X$ has right continuous paths and satisfies the strong Markov property.
For example, for Brownian motion we have $S = \mathbb{R}^1$, $m$ is Lebesgue measure and $p_t(x, y) = p_t(x - y) = e^{-(x-y)^2/2t}/\sqrt{2\pi t}$.

We next introduce the $\alpha$-potential kernels, $\alpha \geq 0$,

$$u^\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y) \, dt. \quad (3.6)$$

We assume that the $u^\alpha(x, y)$ are continuous for some $\alpha \geq 0$.

We note that if $X$ symmetric then $p_t(x, y)$ is positive definite:

$$\sum_{i,j=1}^n a_i a_j p_t(x_i, x_j) = \sum_{i,j=1}^n a_i a_j \int p_{t/2}(x_i, z) p_{t/2}(z, x_j) \, dm(z) \quad (3.7)$$

$$= \int |\sum_{i=1}^n a_i p_{t/2}(x_i, z)|^2 \, dm(z) \geq 0,$$

where the last equality used the symmetry $p_{t/2}(z, x_j) = p_{t/2}(x_j, z)$. This immediately implies that $u^\alpha(x, y)$ is symmetric and positive definite. Hence there exists a Gaussian process $G = \{G_x, x \in S\}$ with covariance

$$E(G_x G_y) = u^\alpha(x, y). \quad (3.8)$$

Of course, $G$ depends on $\alpha$. When $\alpha = 0$ and $u^0$ is finite we refer to $G$ as the Gaussian process associated with $X$. $G$ is one of the key players in the Isomorphism Theorem.

We now introduce the other key player, the local time $L = \{L^y_t, (t, y) \in R^1_+ \times S\}$ defined by

$$L^y_t = \lim_{\epsilon \to 0} \int_0^t f_{\epsilon,y}(X_r) \, dr, \quad (3.9)$$

where $f_{\epsilon,y}$ is an approximate $\delta$-function at $y$. That is, $f_{\epsilon,y}$ is a non-negative function supported in $B(y, \epsilon)$ with $\int f_{\epsilon,y}(x) \, dm(x) = 1$. If $u^\alpha(x, y)$ is continuous for some $\alpha \geq 0$, it can be shown that the limit in (3.9) exists locally uniformly in $t$, $P^x$ a.s. It is then easily seen that $L^y_t$ inherits the following properties from $\int_0^t f_{\epsilon,y}(X_r) \, dr$: $L^y_0 = 0$, $L^y_t$ is continuous and increasing in $t$, and has the additivity property:

$$L^y_{t+s} = L^y_t + L^y_s \circ \theta_t, \quad (3.10)$$

where $\theta_t \omega(r) = \omega(r + t)$.

Thus $L^y_t$ is continuous in $t$, but what about continuity in $y$? The Isomorphism Theorems allow us to give a complete resolution to this question for symmetric Markov processes.
3.1 Local time moment formulas

For ease of notation we assume that \( u^0(x, y) \) is continuous, and write it as \( u(x, y) \). Our first formula is somewhat similar to the chain function (2.20) which appears in the Gaussian moment formula (2.19).

\[
P_x \left( \prod_{i=1}^{k} L_{x_i}^\infty \right) = \sum_{\pi \in P_k} u(x, x_{\pi(1)}) u(x_{\pi(1)}, x_{\pi(2)}) \cdots u(x_{\pi(k-1)}, x_{\pi(k)}). \tag{3.11}
\]

Proof: It follows from (3.5)

\[
P_x \left( \int_{\{0 < t_1 < \cdots < t_k < \infty\}} \prod_{i=1}^{k} f_i(X_{t_i}) \, dt_i \right) \tag{3.12}
\]

\[
= \int u(x, y_1) u(y_1, y_2) \cdots u(y_{k-1}, y_k) \prod_{i=1}^{k} f_i(y_i) \, dm(y_i).
\]

and consequently, since \( R^k_x = \bigcup_{\pi \in P_k} \{0 < t_{\pi(1)} < t_{\pi(2)} < \cdots < t_{\pi(k)} < \infty\} \) (up to sets of Lebesgue measure 0),

\[
P_x \left( \prod_{i=1}^{k} \int_{-\infty}^{\infty} f_i(X_{t_i}) \, dt_i \right) \tag{3.13}
\]

\[
= \sum_{\pi \in P_k} \int u(x, y_1) u(y_1, y_2) \cdots u(y_{k-1}, y_k) \prod_{i=1}^{k} f_{\pi(i)}(y_i) \, dm(y_i).
\]

Taking \( f_i = f_{\epsilon, x_i} \) and then taking the limit as \( \epsilon \to 0 \) gives (3.11).

To prove Dynkin’s Isomorphism Theorem we will need a different sort of measure, known as an \( h \)-transform of our Markov process \( X \). We define a measure \( Q^{x,y} \) by the formula

\[
Q^{x,y}(F_{1_{\tau < \zeta}}) = P^x(F u(X_t, y)), \quad F \in \mathcal{F}_t. \tag{3.14}
\]

That is, if we take some functional \( F \) which depends only on the path up to time \( t \), we first measure \( F \) using \( P^x \), and then, starting at position \( X_t \), the factor \( u(X_t, y) \) measures all possible ways to end up at \( y \). Here is the moment formula we want:

\[
Q^{x,y} \left( \prod_{i=1}^{k} L_{x_i}^\infty \right) \tag{3.15}
\]

\[
= \sum_{\pi \in P_k} u(x, x_{\pi(1)}) u(x_{\pi(1)}, x_{\pi(2)}) \cdots u(x_{\pi(k-1)}, x_{\pi(k)}) u(x_{\pi(k)}, y).
\]

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In other words, comparing with (2.20) for the associated process,

\[ Q^{x,y} \left( \prod_{i=1}^{k} L_{x_i}^{\infty} \right) = \text{ch}([1,k]; x, y). \]  

(3.16)

Proof: If \( t_1 < \cdots < t_k \), it follows from the definition (3.14) that

\[ Q^{x,y} \left( \prod_{i=1}^{k} f_i(X_{t_i}) \right) = P^x \left( \prod_{i=1}^{k} f_i(X_{t_i}) u(X_{t_i}, y) \right) \]

(3.17)

\[ = \int p_{t_1}(x,y_1)p_{t_2-t_1}(y_1,y_2) \cdots p_{t_k-t_{k-1}}(y_{k-1},y_k)u(y_k,y)\prod_{i=1}^{k} f_i(y_i) dm(y_i). \]

Hence

\[ Q^{x,y} \left( \int_{\{0<t_1<\cdots<t_k<\infty\}} \prod_{i=1}^{k} f_i(X_{t_i}) dt_i \right) \]

(3.18)

\[ = \int u(x,y_1)u(y_1,y_2) \cdots u(y_{k-1},y_k)u(y_k,y)\prod_{i=1}^{k} f_i(y_i) dm(y_i). \]

Arguing as before we then see that

\[ Q^{x,y} \left( \prod_{i=1}^{k} \int_{-\infty}^{\infty} f_i(X_{t_i}) dt_i \right) \]

(3.19)

\[ = \sum_{\pi \in \mathcal{P}_k} \int u(x,y_1)u(y_1,y_2) \cdots u(y_{k-1},y_k)u(y_k,y)\prod_{i=1}^{k} f_{\pi(i)}(y_i) dm(y_i). \]

Taking \( f_i = f_{\epsilon,x_i} \) and then taking the limit as \( \epsilon \to 0 \) gives (3.15).

For more details about Markov processes and local times, see [24, Chapter 2]. (3.13) is Theorem 3.3.2 of that book. The moment formula (3.11) is a special case of Theorem 3.10.1, where we take \( T = \infty \), and (3.15) is equivalent to (3.248).

### 4 The Dynkin Isomorphism Theorem

The Dynkin Isomorphism Theorem can be expressed as

\[ E_G Q^{x,y} \left( F \left( L_{x_i}^{\infty} + \frac{1}{2} G_{x_i}^2 \right) \right) = E_G \left( G_x G_y F \left( \frac{1}{2} G_{x_i}^2 \right) \right). \]

(4.1)
Here, for a bounded measurable function $F$ on $R^\infty$ we use the abbreviation

$$F(h_{x_i}) = F(h_{x_1}, h_{x_2}, h_{x_3}, \ldots). \tag{4.2}$$

$E_G$ denotes expectation with respect to the associated Gaussian process $G$. Note that in (4.1) the associated Gaussian process $G$ is independent of the Markov process $X$. (4.1) is not what is usually referred to as an isomorphism: The right hand side contains only the process $G$, but the left hand side is a mixture of the local time process of $X$ and the independent process $G$. Before giving a proof of (4.1), which will be simple since we have already developed most of tools we need, I would like to give an example to illustrate how to ‘decouple’ $L$ and $G$.

Assume that we know that the associated Gaussian process $G$ is a.s. continuous on $S$. We will use the Dynkin Isomorphism Theorem to show that the total local time $L^z_\infty$ is continuous on $S$, $Q^{x,y}$ a.s. Continuity is a local property, so it is sufficient to show that $L^z_\infty$ is continuous on any compact subset $K \subseteq S$, $Q^{x,y}$ a.s. Pick a countable dense subset $D \subseteq K$, and let $F_D$ be the indicator function of the event that a function $h$ is uniformly continuous on $D$. Since by assumption $G$ is a.s. continuous on $S$, we have that $F_D(G^2/2) = 1$, a.s. Hence the right hand side of (4.1) is equal to $E_G(G_2 G_y) = u(x, y)$, which is precisely the total mass of the measure $E_G Q^{x,y}$. Therefore $F_D(L_\infty + G^2/2) = 1$, a.s. That is, $L^z_\infty + G^2/2$ is almost surely uniformly continuous on $D$, and since we know this is true of $G^2/2$ we have established that $L^z_\infty$ is almost surely uniformly continuous on $D$. This is basically what we wanted to show. Standard techniques allow us to extend $L^z_\infty$ by continuity to $K$, and verify that this extension is indeed the total local time $L^z_\infty$, $z \in K$.

By the way, this result is not purely academic. Necessary and sufficient conditions for the a.s. continuity of a Gaussian process in terms of its covariance are known. We describe this in the next section.

Proof of the Dynkin Isomorphism Theorem: We first take $F$ to be a product, and show that

$$E_G Q^{x,y} \left( \prod_{i=1}^k \left( L^x_i + G^2_{x_i} \right) \right) = E_G \left( G_x G_y \prod_{i=1}^k \frac{1}{2} G^2_{x_i} \right). \tag{4.3}$$

Expanding the product on the left hand side, (4.3) is

$$\sum_{A \subseteq \{1, k\}} Q^{x,y} \left( \prod_{i \in A} L^x_i \right) E_G \left( \prod_{i \notin A} \frac{1}{2} G^2_{x_i} \right) = E_G \left( G_x G_y \prod_{i=1}^k \frac{1}{2} G^2_{x_i} \right). \tag{4.4}$$

In view of (3.16) this is just (2.21).
To extend this to general bounded measurable $F$, we need only show that the two sides of (4.1) are determined by their moments and this will follow once we show that both $L_z^\infty$ and $G_z^2$ are exponentially integrable. But it follows from (3.15) that

$$Q^{x,y} \left((L_z^\infty)^n\right) = n!u(x,z)(u(z,z))^{n-1}u(z,y)$$

and from (2.12) that

$$E \left(G_z^{2n}\right) = |\mathcal{R}_n|^u(z,z),$$

and $|\mathcal{R}_n|$, the number of pairings of $2n$ objects, is bounded by $n!e^a$. (In fact, the exponential integrability of the square of a normal random variable is well known and easy to compute explicitly.)

The proof of the Dynkin Isomorphism Theorem given here is found in [25, Section 8.3.1].

5 The Eisenbaum Isomorphism Theorem

One problem with the Dynkin Isomorphism Theorem is the appearance of the measure $Q^{x,y}$. The following Isomorphism Theorem of Eisenbaum deals with the natural measure $P^x$, but at some cost. It says that for any $s > 0$

$$E_G P^x \left(F\left(L_x^\infty + \frac{1}{2}(G_x + s)^2\right)\right) = E_G \left(\left(1 + \frac{G_x}{s}\right) F\left(\frac{1}{2}(G_x + s)^2\right)\right).$$

(5.1)

Proof: Once again it suffices to prove this when $F$ is a product, in which case, after expanding the first factor on the right hand side of (5.1), it takes the form

$$\sum_{A \cup B = [1,k]} P^x \left(\prod_{i \in A} L_x^i\right) E_G \left(\prod_{i \in B} \frac{1}{2}(G_x + s)^2\right)$$

$$= E_G \left(\prod_{i=1}^k \left(\frac{1}{2}(G_x + s)^2\right)\right) + E_G \left(\frac{G_x}{s} \prod_{i=1}^k \left(\frac{1}{2}(G_x + s)^2\right)\right).$$

(5.2)

The first term on the right hand side corresponds to the term on the on the left hand side with $A = \emptyset$. If $A \neq \emptyset$, recall that by (3.111), if $A = \{a_1, a_2, \ldots, a_{|A|}\}$

$$P^x \left(\prod_{i \in A} L_x^i\right) = \sum_{\pi \in \mathcal{P}_{|A|}} u(x, x_{a_{\pi(1)}})u(x_{a_{\pi(1)}}, x_{a_{\pi(2)}}) \cdots u(x_{a_{\pi(|A|)-1}}, x_{a_{\pi(|A|)}}),$$

(5.3)
For the expectation on the right of (5.2), start with \( G_x \) and apply the Gaussian moment formula (2.12). \( G_x \) must be paired with something. It can be paired with one of the two factors of \( G_{xa_π(1)} \), canceling the 1/2 and giving rise to the factor \( u(x, xa_π(1)) \). The other factor \( G_{xa_π(2)} \) might be paired with one of the two factors of \( G_{xa_π(1)} + s \), canceling the 1/2 and giving rise to the factor \( u(xa_π(1), xa_π(2)) \). We proceed in the way until we pair \( G_{xa_π(\mid A\mid - 1)} \) with \( G_{xa_π(|A|)} \) from one of the two factors of \( (G_{xa_π(|A|)} + s)^2 \). From the other factor we take \( s \), canceling the 1/s from \( G_x \). Thus we have obtained (5.3) and what remains from this expectation on the right of (5.2) is precisely

\[
E_G \left( \prod_{i \in B} \left( G_{x_i} + s \right)^2 \right)
\]

This completes the proof of the Eisenbaum Isomorphism Theorem, but it is of interest, and will be useful later on, to figure out explicitly the other terms. We show that

\[
E_G \left( \prod_{i=1}^{k} \left( \frac{1}{2} (G_{x_i} + s)^2 \right) \right) = \sum_{A_1 \cup \ldots \cup A_l = [1,k]} \prod_{i=1}^{l} \frac{1}{2} \text{cy}(A_i) \prod_{j=1}^{m} \frac{s^2}{2} \text{ch}(B_j) \tag{5.4}
\]

where \( \text{cy}(A) \) is defined in (2.17). \( \text{ch}(B) = 1 \) if \( |B| = 1 \) and, if \( |B| > 1 \) with \( B = \{ b_1, b_2, \ldots , b_{|B|} \} \) then the chain function \( \text{ch}(B) \) is defined as

\[
\text{ch}(B) = \sum_{\pi \in \mathcal{P}_{|B|}} \text{u}(x_{b_{\pi(1)}}, x_{b_{\pi(2)}}) \cdots \text{u}(x_{b_{\pi(|B|-1)}}, x_{b_{\pi(|B|)})}). \tag{5.5}
\]

Note that the ‘chains’ in \( \text{ch}(B) \) are oriented. For example if \( B = \{1, 2\} \) then \( \text{ch}(B) = u(x_1, x_2) + u(x_2, x_1) \). For the symmetric case we are dealing with this is \( 2u(x_1, x_2) \).

Proof of (5.4): It will be convenient to rewrite this as

\[
E_G \left( \prod_{i=1}^{k} \left( \frac{1}{2} (G_{x_i} + s)^2 \right) \right) = \sum_{A \cup B = [1,k]} \left( \prod_{A_1 \cup \ldots \cup A_l = A} \prod_{B_1 \cup \ldots \cup B_m = B} \frac{1}{2} \text{cy}(A_i) \frac{s^2}{2} \text{ch}(B_j) \right) \tag{5.6}
\]

There are many terms in the expansion of \( \prod_{i=1}^{k} \left( \frac{1}{2} (G_{x_i} + s)^2 \right) \). If we look at \( \prod_{i \in A} \frac{1}{2} G^2_{x_i} \) and pair together all factors in this product, then using (2.18) we obtain the term on the right hand side of (5.6) containing cycles. To
obtain the term involving chains, if for example $B_j = \{b_1, b_2, \cdots, b_{|B_j|}\}$ we can obtain $u(x_{b_{\pi(1)}}, x_{b_{\pi(2)}}) \cdots u(x_{b_{\pi(|B|-1)}}, x_{b_{\pi(|B|)}})$ by looking at a specific pairing of $H =: sG_{x_{b_{\pi(1)}}} \left( \prod_{i=2}^{|B|-1} \frac{1}{2} G_{x_{b_{\pi(i)}}}^2 \right) sG_{x_{b_{\pi(|B|)}}}$. That is, we pair $G_{x_{b_{\pi(2)}}}$ with one of the two factors $G_{x_{b_{\pi(2)}}}$, pair the other factor $G_{x_{b_{\pi(2)}}}$ with one of the two factors $G_{x_{b_{\pi(3)}}}^1$, until finally we pair the remaining factor $G_{x_{b_{\pi(|B|)-1}}}$ with $G_{x_{b_{\pi(|B|)}}}$. In this way we have cancelled all the factors of $1/2$ in $H$ and obtained a factor of $s^2$. But note that this pairing is unoriented, while as we mentioned the ‘chains’ in $\text{ch}(B_j)$ are oriented. This accounts for the factor $1/2$ multiplying $\text{ch}(B_j)$ in (5.6).

5.1 Bounded discontinuities

We now present an application of the Eisenbaum Isomorphism Theorem. We first recall the fundamental result of Talagrand that a Gaussian process $G = \{G_x, x \in S\}$ is continuous a.s. if and only if there exists a probability measure $\nu$ on $S$ such that

$$\lim_{\delta \to 0} \sup_{s \in S} \int_0^\delta \left( \frac{1}{\nu(B_s(s, u))} \right)^{1/2} du = 0. \quad (5.7)$$

Here, continuity is with respect to the metric $d(x, y) = \left( \mathbb{E} \left( \{G_x - G_y\}^2 \right) \right)^{1/2}$ which can be expressed in terms of the covariance $u(x, y)$ of $G$.

Marcus and I used Isomorphism Theorems to show that for symmetric Markov processes with continuous potential densities, the total local time $L = \{L^z, z \in S\}$ will be $P^x$ almost surely continuous for each $x \in S$ if and only if the associated Gaussian process $G$ is almost surely continuous. By the result of Talagrand we have an explicit condition in terms of the potential densities $u(x, y)$.

We have already indicated how to use Isomorphism Theorems to show that if the associated Gaussian process $G$ is almost surely continuous, then the total local time $L = \{L^z, z \in S\}$ will be almost surely continuous. We now show how to use the Eisenbaum Isomorphism Theorem to go in the other direction, that is, to show that if the associated Gaussian process $G$ is not almost surely continuous, then the total local time $L = \{L^z, z \in S\}$ cannot be $P^x$ almost surely continuous for each $x \in S$. The key to this result is the fact that a Gaussian process can only be discontinuous in very special ways, which we now recall.
Set

\[ M_f(x_0) = \lim_{\epsilon \to 0} \sup_{x \in B_\delta(x_0, \epsilon)} f(x), \quad m_f(x_0) = \lim_{\epsilon \to 0} \inf_{x \in B_\delta(x_0, \epsilon)} f(x). \]  

(5.8)

Let \( G = \{G_x, x \in S\} \) be a Gaussian process with continuous covariance. If \( G \) is not almost surely continuous then there exists \( x_0 \in S \), a \( \beta(x_0) > 0 \) and a countable dense subset \( C \subseteq S \) such that

\[ M_{G|C}(x_0) = G_{x_0} + \beta(x_0) \quad \text{and} \quad m_{G|C}(x_0) = G_{x_0} - \beta(x_0) \quad \text{a.s.} \]  

(5.9)

When \( 0 < \beta(x_0) < \infty \) we say that \( G \) has a bounded discontinuity at \( x_0 \). If \( \beta(x_0) = \infty \) we say that \( G \) has an unbounded discontinuity at \( x_0 \). We will now use the Eisenbaum Isomorphism Theorem to show that if \( G \) has a bounded discontinuity at \( x_0 \) then \( L = \{L^z, z \in S\} \) will be discontinuous at \( x_0, P^{x_0} \) almost surely. The case of an unbounded discontinuity is somewhat more complicated and we refer the interested reader to [25, Chapter 9.2].

Proof: Simple algebra shows that

\[ (G_x + s)^2 - (G_{x_0} + s)^2 = (G_x - G_{x_0})^2 + 2(G_{x_0} + s)(G_x - G_{x_0}). \]  

(5.10)

Using this we claim that almost surely

\[ \lim_{\epsilon \to 0} \sup_{x \in B_\delta(x_0, \epsilon) \cap C} (G_x + s)^2 - (G_{x_0} + s)^2 = \beta^2(x_0) + 2\beta(x_0)|G_{x_0} + s|. \]  

(5.11)

To see this, look at the right hand side of (5.11). If \( G_{x_0} + s > 0 \) we obtain (5.11) by taking a sequence of points \( x_n \to x_0 \) such that, by (5.9), \( G_{x_n} - G_{x_0} \to \beta(x_0) \), while if \( G_{x_0} + s < 0 \) we obtain (5.11) by taking a sequence of points \( x_n \to x_0 \) such that, by (5.9), \( G_{x_n} - G_{x_0} \to -\beta(x_0) \).

We can rewrite (5.11) in form more appropriate to the Eisenbaum Isomorphism Theorem:

\[ \lim_{\epsilon \to 0} \sup_{x \in B_\delta(x_0, \epsilon) \cap C} \frac{1}{2}(G_x + s)^2 - \frac{1}{2}(G_{x_0} + s)^2 = \frac{\beta^2(x_0)}{2} + 2^{1/2}\beta(x_0)\sqrt{\frac{1}{2}(G_{x_0} + s)^2}, \]  

(5.12)

almost surely. Let \( x_i \) be an enumeration of the points in \( C \). We now apply the Eisenbaum Isomorphism Theorem with \( F(f_{x_i}) \) the indicator function of the event

\[ \lim_{\epsilon \to 0} \sup_{x \in B_\delta(x_0, \epsilon)} f_{x_i} - f_{x_0} = \frac{\beta^2(x_0)}{2} + 2^{1/2}\beta(x_0)\sqrt{f_{x_0}}. \]  

(5.13)
By (5.12), $F(\frac{1}{2}(G_{x_i} + s)^2) = 1$ a.s. The Eisenbaum Isomorphism Theorem then implies that for any $s > 0$, $F(L_\infty^{x_i} + \frac{1}{2}(G_{x_i} + s)^2) = 1$ a.s. That is,

$$\lim_{\epsilon \to 0} \sup_{x_i \in B_d(x_0, \epsilon)} \left( L_\infty^{x_i} - L_\infty^{x_0} + \frac{1}{2}(G_{x_i} + s)^2 - \frac{1}{2}(G_{x_0} + s)^2 \right) \quad (5.14)$$

$$= \frac{\beta^2(x_0)}{2} + 2^{1/2} \beta(x_0) \sqrt{L_\infty^{x_0} + \frac{1}{2}(G_{x_0} + s)^2}.$$  

$$\geq \frac{\beta^2(x_0)}{2} + 2^{1/2} \beta(x_0) \sqrt{L_\infty^{x_0}}.$$  

Using the fact that $\limsup_i A_i + \limsup_i B_i \geq \limsup_i (A_i + B_i)$ and then (5.12) we see that almost surely

$$\lim_{\epsilon \to 0} \sup_{x_i \in B_d(x_0, \epsilon)} L_\infty^{x_i} - L_\infty^{x_0} \geq 2^{1/2} \beta(x_0) \sqrt{L_\infty^{x_0}} - 2^{1/2} \beta(x_0)|G_{x_0} + s|. \quad (5.15)$$

At first glance this doesn’t seem very useful. We want to show that $L$ has a discontinuity at $x_0$, that is, that the left hand side is strictly positive, but because we are subtracting $2^{1/2} \beta(x_0)|G_{x_0} + s|$ on the right hand side, the right hand side might be negative!

I will now perform a magic trick. I will make the $\beta(x_0)|G_{x_0} + s|$ disappear before your very eyes! For this purpose recall that $L$ and $G$ are independent and in fact live on different spaces. To emphasize this we write (5.16) as the statement that

$$\lim_{\epsilon \to 0} \sup_{x_i \in B_d(x_0, \epsilon)} L_\infty^{x_i}(\omega) - L_\infty^{x_0}(\omega) \geq 2^{1/2} \beta(x_0) \sqrt{L_\infty^{x_0}(\omega)} - 2^{1/2} \beta(x_0)|G_{x_0}(\omega') + s| \quad (5.16)$$

almost surely with respect to $P^x \times E_G$. By Fubini’s theorem then, this holds $P^x$ almost surely for $P_G$ almost every $\omega'$. That is, (5.16) holds $P^x$ almost surely for all $\omega' \in \Omega'$ where $P_G(\Omega') = 1$. If we could only find an $\omega' \in \Omega'$ with $|G_{x_0}(\omega') + s| = 0$ we would be done. We do something similar. Fix $\delta > 0$ and set $s = \delta$. Since $G_{x_0}$ is a normal random variable, we have

$$P_G(|G_{x_0}| \leq \delta) > 0. \quad (5.17)$$

Since $P_G(\Omega') = 1$ we can find an $\omega' \in \Omega'$ with $|G_{x_0}(\omega')| \leq \delta$. By the above we now see that $P^x$ almost surely

$$\lim_{\epsilon \to 0} \sup_{x_i \in B_d(x_0, \epsilon)} L_\infty^{x_i}(\omega) - L_\infty^{x_0}(\omega) \geq 2^{1/2} \beta(x_0) \sqrt{L_\infty^{x_0}(\omega)} - 2^{1/2} \beta(x_0)2\delta. \quad (5.18)$$
Since this is true for any $\delta > 0$ we have in fact shown that $P^x$ almost surely
\[
\lim_{\epsilon \to 0} \sup_{x_i \in B_\epsilon(x_0, \epsilon)} L^x_\epsilon(\omega) - L^x_\infty(\omega) \geq 2^{1/2}\beta(x_0)\sqrt{L^x_\infty(\omega)}.
\] (5.19)

Is $L^x_\infty(\omega) > 0$ almost surely? This depends on whether or not the path has visited $x_0$. But we can give a simple proof that $L^x_\infty > 0$, $P^{x_0}$ almost surely, so by the above we can conclude that the local time $L$ is discontinuous $P^{x_0}$ almost surely. This is all that we wanted to establish.

The fact that $L^x_\infty > 0$, $P^{x_0}$ almost surely follows from (3.11) which implies that for all $n$
\[
P^{x_0} ((L^x_\infty)^n) = n!(u(x_0, x_0))^n,
\] which implies that $L^x_\infty$ is distributed under $P^{x_0}$ as an exponential random variable (with mean $u(x_0, x_0)$). Since exponential random variables are strictly positive almost surely, we are done. \(\square\)

The proof of the Eisenbaum Isomorphism Theorem given here is similar to that in [25, Section 8.3.2]. For Talagrand’s theorem, see Chapter 6 of that book. The property (5.9) concerning discontinuities of Gaussian processes is Theorem 5.3.7 and our proof that if the associated Gaussian process has a bounded discontinuity, the local time will be discontinuous is given in detail in Chapter 9.1 of the book.

### 6 The generalized second Ray-Knight theorem

We fix some point in $S$ which we denote by 0. Set
\[
\tau(t) = \inf \{s \mid L^0_s > t\},
\] (6.1)
the right continuous inverse local time at 0, and
\[
T_y = \inf \{s \mid X_s = y\},
\] (6.2)
the first hitting time of $y$. For this section we assume that $u^\alpha(x, y)$ is continuous for any $\alpha > 0$ and $u(0, 0) = \infty$. In addition, we assume that $P^0 (T_x < \infty) = P^x (T_0 < \infty) = 1$ for all $x \in S$. The generalized second Ray-Knight theorem states that for all $t > 0$
\[
E_G P^0 \left( F \left( \frac{L^x_{\tau(t)}}{\frac{1}{2} \eta_{x_1}^2} \right) \right) = E_G \left( F \left( \frac{1}{2} \eta_{x_1} + \sqrt{2t} \right)^2 \right),
\] (6.3)
where \( \{ \eta_x, x \in S \} \) is the Gaussian process with covariance

\[
    u_{T_0}(x, y) = E_x^y \left( L_{T_0}^y \right). \tag{6.4}
\]

We will see below that indeed \( u_{T_0}(x, y) \) is symmetric and positive definite. The notation \( u_{T_0}(x, y) \) is meant to suggest that \( u_{T_0}(x, y) \) is the potential density of the Markov process obtained by killing \( X \) at \( T_0 \). This is true, \[25\], Chapter 4.5, but we will not use this fact.

Recall that by (5.4), after replacing \( s \) by \( \sqrt{2t} \).

\[
    E_G \left( \prod_{i=1}^{k} \left( \frac{1}{2}(\eta_{x_i} + \sqrt{2t})^2 \right) \right) = \sum_{\cup B_1 \cup \cdots \cup B_m = [1, k]} \prod_{i=1}^{l} \frac{1}{2} c_{y_0}(A_i) \prod_{j=1}^{m} t_{ch_0}(B_j). \tag{6.5}
\]

Here \( c_{y_0} \) and \( ch_0 \) refer to the covariance \( u_{T_0}(x, y) \) of \( \eta \). We emphasize that the sum is over unordered partitions of \([1, k]\). That is, \( B_1 = \{1, 2, 3\}, B_2 = \{4, 5\} \) and \( B_1 = \{1, 2, 3\}, B_1 = \{4, 5\} \) are not counted separately in the sum. Using (2.18) as before, the generalized second Ray-Knight theorem will be proven once we show that for all \( t > 0 \)

\[
    P^0_\lambda \left( \prod_{i=1}^{k} L_{\tau(t)}^{x_i} \right) = \sum_{\pi \in P_k} \sum_{\text{unordered}} t^m \prod_{j=1}^{m} ch_0(B_j). \tag{6.6}
\]

This, however, is not so simple. Our local time moment formulas are for the total local time of a Markov process, but \( X \) killed at \( \tau(t) \) is not a Markov process. To prove (6.6) we let \( \lambda \) be an independent exponential random variable with mean \( \alpha \) and show that

\[
    P^x_\lambda \left( \prod_{i=1}^{k} L_{\tau(\lambda)}^{x_i} \right) = \sum_{\pi \in P_k} u_{\tau(\lambda)}(x, x_{\pi(1)}) u_{\tau(\lambda)}(x_{\pi(1)}, x_{\pi(2)}) \cdots u_{\tau(\lambda)}(x_{\pi(k-1)}, x_{\pi(k)}) \tag{6.7}
\]

where \( P^x_\lambda = P^x \times P_\lambda \) and

\[
    u_{\tau(\lambda)}(x, y) := u_{T_0}(x, y) + \alpha. \tag{6.8}
\]

Once again, the notation \( u_{\tau(\lambda)}(x, y) \) is meant to suggest that \( u_{\tau(\lambda)}(x, y) \) is the potential density of a symmetric Markov process with probabilities \( P^x_\lambda \) obtained by killing \( X \) at \( \tau(\lambda) \). And once again this is true, \[13\], but we will give a proof of (6.7)-(6.8) which does not use this fact.
Combining (6.7) and (6.8) and expanding the product we see that

\[ P^0_\lambda \left( \prod_{i=1}^k I_{\tau(\lambda)}^{x_i} \right) = \sum_{\pi \in P_k} \alpha \left( u_{\pi(1)}(x_{\pi(1)}), x_{\pi(2)}(2) \right) \cdots \left( u_{\pi(k-1), x_{\pi(k)}}(k) \right) + \alpha \]  

(6.9)

\[ = \sum_{m=1}^k \sum_{\text{ordered}} \prod_{j=1}^m \alpha \ ch_0(B_j), \]

where now the sum is over ordered partitions, since it comes from a sum over permutations, where order counts. Thus we can write

\[ P^0_\lambda \left( \prod_{i=1}^k L_{\tau(\lambda)}^{x_i} \right) = \sum_{m=1}^k \sum_{\text{unordered}} \prod_{j=1}^m \alpha \ ch_0(B_j). \]  

(6.10)

Since \( \int_0^\infty e^{-t/\alpha} t^m dt/\alpha = \alpha^m m! \), we have shown that for any \( \alpha > 0 \)

\[ \int_0^\infty e^{-t/\alpha} P^0_\lambda \left( \prod_{i=1}^k L_{\tau(t)}^{x_i} \right) dt/\alpha = \int_0^\infty e^{-t/\alpha} \sum_{m=1}^k \sum_{\text{unordered}} t^m \prod_{j=1}^m \ ch_0(B_j) dt/\alpha. \]  

(6.11)

Since \( \tau(t) \) is right continuous, we have established (6.6) and hence the generalized second Ray-Knight theorem, (6.3).

We still have to fill in some missing pieces. We first show that \( u_{T_0}(x, y) \) is symmetric and positive definite. In fact, we show that

\[ u_{T_0}(x, y) = \lim_{\alpha \to 0} \left( u^{\alpha}(x, y) - \frac{u^{\alpha}(x, 0)u^{\alpha}(0, y)}{u^{\alpha}(0, 0)} \right). \]  

(6.12)

This will show that \( u_{T_0}(x, y) \) is symmetric, and since if \( G_x \) is the Gaussian process with covariance \( u^{\alpha}(x, y) \) then

\[ E \left( \left( G_x - \frac{u^{\alpha}(x, 0)}{u^{\alpha}(0, 0)} G_0 \right) \left( G_y - \frac{u^{\alpha}(y, 0)}{u^{\alpha}(0, 0)} G_0 \right) \right) = u^{\alpha}(x, y) - \frac{u^{\alpha}(x, 0)u^{\alpha}(0, y)}{u^{\alpha}(0, 0)}. \]  

(6.13)
will also show that $u_{T_0}(x, y)$ is positive definite.

Before showing (6.12), let us illustrate it for Brownian motion. We first show that

$$u^\alpha(x, y) = \frac{e^{-\sqrt{2\alpha}|x-y|}}{\sqrt{2\alpha}}. \quad (6.14)$$

To see this we use the Fourier representation

$$u^\alpha(x, y) = \frac{1}{2\pi} \int_0^\infty e^{-\alpha t} p_t(x, y) \, dt = \frac{1}{2\pi} \int_{-\infty}^\infty e^{iz(x-y)-tz^2/2} \, dz. \quad (6.15)$$

$\alpha + z^2/2$ has two roots in the complex plane $z_{\pm} = \pm i\sqrt{2\alpha}$. When $(x-y) > 0$ we can evaluate the right hand term in (6.15) by using the residue at $z_+$, while if $(x-y) < 0$ we use the residue at $z_-$. This proves (6.14). Applying this to (6.12) we see that

$$u_{T_0}(x, y) = \lim_{\alpha \to 0} \left( \frac{e^{-\sqrt{2\alpha}|x-y|} - e^{-\sqrt{2\alpha}(|x|+|y|)}}{\sqrt{2\alpha}} \right) \quad (6.16)$$

Thus the process $\{\eta_x, x \in \mathbb{R}^1\}$ corresponding to Brownian motion is just $\sqrt{2}$ times two-sided Brownian motion.

We now return to the proof of (6.12). Let $Z$ be the process obtained by killing $X$ at an independent exponential time $\rho$ of mean $1/\beta$. That is, $Z_t(\omega) = X_t(\omega)$ if $t < \lambda$ and $Z_\lambda(\omega) = \Delta$ if $t \geq \rho$. Then, recalling our convention that for a function on $S$ we take $f(\Delta) = 0$,

$$E^x_\rho (f(Z_t)) = E^x_\rho (1_{\{\rho > t\}} f(X_t)) = e^{-\beta t} E^x (f(X_t)) = e^{-\beta t} \int p_t(x, y) f(y) \, dm(y). \quad (6.17)$$

It follows that $Z$ is a symmetric Markov process whose 0-potential density is $u^\beta(x, y)$. And since the total local time of $Z$ at $y$ is $L^y_\rho$, see (3.9), we have

$$u^\beta(x, y) = E^x_\rho (L^y_\rho) = \int_0^\infty e^{-\beta t} E^x (L^y_t) \, \beta dt = E^x \left( \int_0^\infty e^{-\beta t} L^y_t \, \beta dt \right). \quad (6.18)$$

Since $L^y_t$ is continuous and increasing in $t$, integration by parts then shows that

$$u^\beta(x, y) = E^x \left( \int_0^\infty e^{-\beta t} \, dL^y_t \right). \quad (6.19)$$
Using again the additivity of local time and then the strong Markov property we see that for any $\alpha > 0$

$$E^x \left( \int_{T_0}^{\infty} e^{-\alpha t} dL_t^y \right) = E^x \left( e^{-\alpha T_0} \left( \int_{0}^{\infty} e^{-\alpha t} dL_t^y \right) \circ \theta_{T_0} \right)$$  \hspace{1cm} (6.20)

$$= E^x \left( e^{-\alpha T_0} \right) u^\alpha(0, y),$$

where we use the fact that $P^x(T_0 < \infty) = 1$ and that $X_{T_0} = 0$. In particular, for $y = 0$, recalling that $L_t^0$ cannot grow until time $T_0$, this gives

$$u^\alpha(x, 0) = E^x \left( \int_{T_0}^{\infty} e^{-\alpha t} dL_t^0 \right) = E^x \left( e^{-\alpha T_0} \right) u^\alpha(0, 0),$$  \hspace{1cm} (6.21)

showing that $E^x \left( e^{-\alpha T_0} \right) = u^\alpha(x, 0)/u^\alpha(0, 0)$. Putting this back into (6.20) we obtain

$$E^x \left( \int_{T_0}^{\infty} e^{-\alpha t} dL_t^y \right) = \frac{u^\alpha(x, 0) u^\alpha(0, y)}{u^\alpha(0, 0)}. \hspace{1cm} (6.22)$$

Together with (6.19) this shows that

$$E^x \left( \int_{0}^{T_0} e^{-\alpha t} dL_t^y \right) = u^\alpha(x, y) - \frac{u^\alpha(x, 0) u^\alpha(0, y)}{u^\alpha(0, 0)},$$  \hspace{1cm} (6.23)

and letting $\alpha \to 0$ completes the proof of (6.12).

The proof of (6.7)-(6.8) is more complicated and we defer the proof until after we present an application of the generalized second Ray-Knight theorem. However, we point out that if we knew that the process obtained by killing $X$ at $\tau(\lambda)$ is a symmetric Markov process with continuous potential densities $u_{\tau(\lambda)}(x, y)$, then (6.7) would simply be our moment formula (3.11), and in particular we would have

$$E^x \left( L_{\tau(\lambda)}^y \right) = u_{\tau(\lambda)}(x, y).$$  \hspace{1cm} (6.24)

Since $\tau(t)$ cannot grow until the process first reaches 0 we have $\tau(\lambda) = T_0 + \tau(\lambda) \circ \theta_{T_0}$. Hence, using (3.10), the additivity of local times,

$$E^x \left( L_{\tau(\lambda)}^y \right) = E^x \left( L_{T_0+\tau(\lambda)\circ\theta T_0}^y \right)$$

$$= E^x \left( L_{T_0}^y \right) + E^x \left( L_{\tau(\lambda)\circ\theta_{T_0}}^y \circ \theta_{T_0} \right)$$

$$= E^x \left( L_{T_0}^y \right) + E^0 \left( L_{\tau(\lambda)}^y \right),$$

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where the last step used the strong Markov property at the stopping time $T_0$. Thus
\[ u_{\tau(\lambda)}(x, y) = u_{T_0}(x, y) + u_{\tau(\lambda)}(0, y). \]

By symmetry we see that
\[ u_{\tau(\lambda)}(0, y) = u_{\tau(\lambda)}(y, 0) = E_{\lambda} \left( L^0_{\tau(\lambda)} \right) = E_{\lambda}(\lambda) = \alpha. \quad (6.25) \]

which combined with the previous display gives (6.8).

### 6.1 Favorite points

We now illustrate how one can apply the generalized second Ray-Knight theorem. Let $X$ be a Markov process in $\mathbb{R}^1$ with continuous potential densities $u^\alpha(x, y)$ and jointly continuous local times $L^x_t$. Let
\[ V_t = \{ x \in \mathbb{R}^1 \mid L^x_t = \sup_y L^y_t \}, \quad (6.26) \]
which we call the set of favorite points at time $t$. At any time $t$ there may be more than one favorite point. Let
\[ V_t = \inf \{ |x| \mid x \in V_t \}. \quad (6.27) \]

$V_t$ is a stochastic process in $t$. Does $\lim_{t \to \infty} V_t = \infty$? If so, how fast does $V_t$ grow? The generalized second Ray-Knight theorem has been used to give information about the rate of growth of $V_t$ for the symmetric stable processes, see Bass, Eisenbaum and Shi, [1] and the notes at the end of this chapter. We will illustrate this for the case of Brownian motion, although for this case one can avoid use of the Ray-Knight theorem. Furthermore, in order not to get bogged down in details we consider only the following result:
\[ \lim_{t \to \infty} \inf \frac{\log \gamma t}{\sqrt{t}} V_t = \infty, \quad P^0 \ \text{a.s.} \quad (6.28) \]

for any $\gamma > 6$. Note that the law of the iterated logarithm says that
\[ \lim_{t \to \infty} \sup \frac{B_t}{\sqrt{2t \log \log t}} = 1, \quad P^0 \ \text{a.s.} \]

so that in some sense the favorite points are near the boundary of the Brownian motion, $\approx \sqrt{t}$. Our techniques actually allow us to conclude that this holds for any $\gamma > 3$. It has been conjectured that $\gamma = 1$ is the critical value.
Furthermore, since our goal is only to illustrate how one can apply the generalized second Ray-Knight theorem, we only discuss the proof for one direction of (6.28).

We use the generalized second Ray-Knight theorem to prove the following. Let \( h(t) = t/\log^5 t \). Then

\[
\lim_{t \to \infty} \sup_{\{x : |x| \leq h(t)\}} \frac{\log^2 t} t \left( \frac{L_x(t)} t - t \right) = 0, \quad P^0 \text{ a.s.} \tag{6.29}
\]

For this, we first fix \( \lambda \) large and bound

\[
P^0 \left( \sup_{\{x : |x| \leq h(t)\}} L_x(t) - t \geq 2\lambda \right). \tag{6.30}
\]

This is certainly bounded by the following, which allows us the opportunity to use the generalized second Ray-Knight theorem in the second line:

\[
\leq P^0 \left( \sup_{\{x : |x| \leq h(t)\}} \left( \frac{L_x(t)} t + \frac{\eta_x^2} 2 - t \right) \geq 2\lambda \right) \tag{6.31}
\]

\[
= P_{\eta} \left( \sup_{\{x : |x| \leq h(t)\}} \left( \frac{\eta_x^2} 2 + \sqrt{2t} \eta_x \right) \geq 2\lambda \right)
\]

\[
= P_{\eta} \left( \sup_{\{x : |x| \leq h(t)\}} \left( \frac{\eta_x^2} 2 + \sqrt{2t} \eta_x \right) \geq 2\lambda \right)
\]

Using the fact that \( \eta_x \) is just \( \sqrt{2} \) times two-sided Brownian motion together with the reflection principle allows us to bound the last line by

\[
Ce^{-\lambda h(t)} + Ce^{-\lambda^2/4t h(t)}, \tag{6.32}
\]

Taking \( \lambda = \epsilon t / \log^2 t \) we have shown that

\[
P^0 \left( \sup_{\{x : |x| \leq h(t)\}} \frac{\log^2 t} t \left( \frac{L_x(t)} t - t \right) \geq 2\epsilon \right) \leq C/t^2/4. \tag{6.33}
\]

Using Borel-Cantelli on the sequence \( t_n = n^{8/\epsilon^2} \) and then interpolating we find that the left hand side of (6.29) is less than \( 2\epsilon \) for any \( \epsilon > 0 \), which establishes (6.29).
Now assume that we can show that for $t \in [\tau(r)^-, \tau(r)]$ we have the following lower bound on the absolute maximum local time
\[
\sup_x L^x_t > r + r/\log^2 r. \tag{6.34}
\]
By (6.29), using the fact that for $t \in [\tau(r)^-, \tau(r)]$, $L^0_t \leq L^0_{\tau(r)} = r$ we have that for large $t$
\[
\sup_{\{x \mid |x| \leq h(L^0_t)\}} L^x_t \leq \sup_{\{x \mid |x| \leq h(r)\}} L^x_{\tau(r)} \leq r + r/\log^2 r. \tag{6.35}
\]
Comparison of this with (6.34) shows that
\[
V_t \geq h(L^0_t). \tag{6.36}
\]
We show below that for any $\epsilon > 0$
\[
\lim_{t \to \infty} \frac{(\log t)^{1+\epsilon} L^0_t}{\sqrt{t}} = \infty, \quad P^0 \text{ a.s.} \tag{6.37}
\]
and together with (6.36) this gives the lower bound in (6.28).

Before proving (6.37), we observe that (6.34) can be obtained from
\[
\lim_{t \to \infty} \sup_x \frac{\log^2 t (L^x_t - t)}{t} = \infty, \quad P^0 \text{ a.s.} \tag{6.38}
\]
which can be shown by another application of the generalized second Ray-Knight theorem. However, for these lecture notes, one illustration is enough!

In order to prove (6.37) we need some basic facts about the inverse local time $\tau(t)$. Since
\[
\tau(t+s) = \tau(s) + \tau(t) \circ \theta_{\tau(s)}, \tag{6.39}
\]
it follows using the strong Markov property and the fact that $X_{\tau(s)} = 0$ that
\[
f(t+s) = \mathbb{P}^0 \left( e^{-\beta \tau(t+s)} \right) = \mathbb{P}^0 \left( e^{-\beta \tau(s)} \left( e^{-\beta \tau(t)} \circ \theta_{\tau(s)} \right) \right) = \mathbb{P}^0 \left( e^{-\beta \tau(s)} \right) \mathbb{P}^0 \left( e^{-\beta \tau(t)} \right) = f(s)f(t). \tag{6.40}
\]
Since for $\beta > 0$, $f(t)$ is decreasing, bounded by 1 and right continuous, we must have $f(t) = e^{-t \nu(\beta)}$ for some function $\nu(\lambda)$ which we now evaluate.

Note first that for any function $g$
\[
\int_0^\infty g(t) dL^0_t = \int_0^\infty g(\tau(s)) ds. \tag{6.41}
\]
To see this it suffices to verify it for functions of the form $g(t) = 1_{[0,r]}(t)$, for which (6.41) is the statement that $L_0^r = \{ s \mid \tau(s) \leq r \}$ which is easily verified. Then, using (6.41)
\[
\frac{1}{v(\beta)} = \int_0^\infty f(s) \, ds = P^0 \left( \int_0^\infty e^{-\beta \tau(s)} \, ds \right) = P^0 \left( \int_0^\infty e^{-\beta t} \, dL_0^t \right) = u^2(0,0)
\]
by (6.19). Thus we have
\[
P^0 \left( e^{-\beta \tau(t)} \right) = e^{-t/u^2(0,0)}. \tag{6.43}
\]
In particular for Brownian motion, by (6.14)
\[
P^0 \left( e^{-\beta \tau(t)} \right) = e^{-t \sqrt{2\beta}}. \tag{6.44}
\]
We next use this to show that for Brownian motion
\[
\limsup_{r \to \infty} \frac{r}{r^2 \log 2 + \epsilon} \log \frac{1}{r} = 0, \quad P^0 \text{ a.s.} \tag{6.45}
\]
from which we will then derive (6.37). To prove (6.45) note that
\[
P^0 \left( \tau(r) \geq x \right) = P^0 \left( 1 - e^{-\tau(r)/x} \geq 1 - e^{-1} \right) \leq \frac{1}{1 - e^{-1}} P^0 \left( 1 - e^{-\tau(r)/x} \right) = c \left( 1 - e^{-r \sqrt{2/x}} \right) \leq c r / \sqrt{x}.
\]
Thus
\[
P^0 \left( \tau(r) \geq r^2 \log 2 + \epsilon \right) \leq c / \log 1 + t / 2,
\]
so that taking $r_n = e^n$ by Borel-Cantelli
\[
\limsup_{n \to \infty} \frac{\tau(r_n)}{r_n^2 \log 2 + \epsilon} = 0, \quad P^0 \text{ a.s.} \tag{6.48}
\]
and (6.45) follows by interpolation.

Finally, (6.45) says that it takes less than $r^2 \log 2 + \epsilon r$ for the local time at 0 to reach the level $r$, so that for large $r$
\[
L_{r^2 \log 2 + \epsilon}^0 \geq r. \tag{6.49}
\]
Taking $r = \left( \frac{t}{\log 1 + t} \right)^{1/2}$ leads to (6.37).
6.2 Proof of the moment formula for \( L^y_{\tau(\lambda)} \)

We first prove that

\[
E^x_\lambda \left( L^y_{\tau(\lambda)} \right) = u_{T_0} (x, y) + \alpha. \tag{6.50}
\]

Since \( \tau(t) \) cannot grow until the process first reaches 0 we have \( \tau(\lambda) = T_0 + \tau(\lambda) \circ \theta_{T_0} \). Hence for any \( \beta > 0 \),

\[
E^x_\lambda \left( \int_{\tau(\lambda)}^{\infty} e^{-\beta t} \, dL^y_t \right) = E^x_\lambda \left( \int_{T_0 + \tau(\lambda) \circ \theta_{T_0}}^{\infty} e^{-\beta t} \, dL^y_t \right) \tag{6.51}
\]

\[
= E^x_\lambda \left( e^{-\beta T_0} \int_{\tau(\lambda) \circ \theta_{T_0}}^{\infty} e^{-\beta t} \, dL^y_t \right)
\]

\[
= E^x_\lambda \left( e^{-\beta T_0} \int_{\tau(\lambda)}^{\infty} e^{-\beta t} \, dL^y_t \circ \theta_{T_0} \right)
\]

\[
= E^x \left( e^{-\beta T_0} E^0_{\lambda, T_0} \left( \int_{\tau(\lambda)}^{\infty} e^{-\beta t} \, dL^y_t \right) \right)
\]

\[
= E^x \left( e^{-\beta T_0} E^0_{\lambda, T_0} \left( \int_{\tau(\lambda)}^{\infty} e^{-\beta t} \, dL^y_t \right) \right)
\]

using the strong Markov property at \( T_0 \), the fact that \( P^x (T_0 < \infty) = 1 \) and that \( X_{T_0} = 0 \). Similarly,

\[
E^0_{\lambda} \left( \int_{\tau(\lambda)}^{\infty} e^{-\beta t} \, dL^y_t \right) \tag{6.52}
\]

\[
= E^0_{\lambda} \left( 1_{\{\tau(\lambda) < \infty\}} \int_{\tau(\lambda)}^{\infty} e^{-\beta t} \, dL^y_t \right)
\]

\[
= E^0_{\lambda} \left( e^{-\beta \tau(\lambda)} 1_{\{\tau(\lambda) < \infty\}} \left( \int_0^{\infty} e^{-\beta t} \, dL^y_t \right) \circ \theta_{\tau(\lambda)} \right)
\]

\[
= E^0_{\lambda} \left( e^{-\beta \tau(\lambda)} 1_{\{\tau(\lambda) < \infty\}} E^{X_{\tau(\lambda)}} \left( \int_0^{\infty} e^{-\beta t} \, dL^y_t \right) \right)
\]

\[
= E^0_{\lambda} \left( e^{-\beta \tau(\lambda)} \right) E^0 \left( \int_0^{\infty} e^{-\beta t} \, dL^y_t \right)
\]

using the strong Markov property at \( \tau(\lambda) \), and the fact that \( X_{\tau(\lambda)} = 0 \) on \( \tau(\lambda) < \infty \). Combining \( (6.51) \) and \( (6.52) \), we see that

\[
E^x_\lambda \left( \int_{\tau(\lambda)}^{\infty} e^{-\beta t} \, dL^y_t \right) = E^x \left( e^{-\beta T_0} \right) E^0_{\lambda, T_0} \left( e^{-\beta \tau(\lambda)} \right) u^\beta (0, y). \tag{6.53}
\]

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Using this and proceeding exactly as in (6.20)–(6.23), we see that

\[ u_{\tau(\lambda)}(x, y) = \lim_{\beta \to 0} \left\{ u^\beta(x, y) - \frac{u^\beta(x, 0)u^\beta(0, y)}{u^\beta(0, 0)} \right\} \]  

(6.54)

\[ + \lim_{\beta \to 0} \frac{u^\beta(x, 0)u^\beta(0, y)}{u^\beta(0, 0)} (1 - E_\lambda^0(e^{-\beta\tau(\lambda)})). \]

By (6.12),

\[ \lim_{\beta \to 0} \left\{ u^\beta(x, y) - \frac{u^\beta(x, 0)u^\beta(0, y)}{u^\beta(0, 0)} \right\} = u_{\tau_0}(x, y). \]  

(6.55)

Also, by (6.21),

\[ \lim_{\beta \to 0} \frac{u^\beta(x, 0)u^\beta(0, y)}{u^\beta(0, 0)} (1 - E_\lambda^0(e^{-\beta\tau(\lambda)}) = \lim_{\beta \to 0} E^x \left( e^{-\beta T_0} \right) u^\beta(0, 0)(1 - E_\lambda^0(e^{-\beta\tau(\lambda)}) \]

(6.56)

and, by (6.43),

\[ u^\beta(0, 0)(1 - E_\lambda^0(e^{-\beta\tau(\lambda)}) \]

(6.57)

\[ = \frac{1}{1/\alpha + 1/u^\beta(0, 0)} = \frac{1}{1/\alpha + 1/u^\beta(0, 0)}. \]

Since \( \lim_{\beta \to 0} u^\beta(0, 0) = \infty \) we get (6.50).

Combining (6.50) with the definition (6.7) we have

\[ u_{\tau(\lambda)}(x, y) := u_{T_0}(x, y) + \alpha = E_\lambda^x \left( L_{\tau(\lambda)}^y \right) \]  

(6.58)

\[ = E_\lambda^x \left( \int_0^{\infty} 1_{\{\tau(\lambda) > t\}} \ dL_t^y \right) \]

\[ = E_\lambda^x \left( \int_0^{\infty} 1_{\{\lambda > L_0^t\}} \ dL_t^y \right) \]

\[ = E^x \left( \int_0^{\infty} P_\lambda(\lambda > L_0^t) \ dL_t^y \right) \]

\[ = E^x \left( \int_0^{\infty} e^{-L_0^t/\alpha} \ dL_t^y \right). \]
We now prove (6.7). We have
\[ E^x(\prod_{i=1}^n L^y_{\tau(\lambda)}) = E^x \left( \prod_{i=1}^n \int_0^{\tau(\lambda)} dL^y_{t_i} \right) \] (6.59)
\[ = \sum_{\pi \in P_n} E^x \left( \int_{\{0<t_1<...<t_n<\tau(\lambda)\}} \prod_{i=1}^n dL^y_{t_i}^{\pi_i} \right). \]

Let \( y_{\pi_i} = z_i, i = 1, \ldots, n \). Note that
\[ \int_{\{0<t_1<...<t_n<\tau(\lambda)\}} \prod_{i=1}^n dL^z_{t_i} = \int_0^{\tau(\lambda)} \int_{t_1}^{\tau(\lambda)} \cdots \int_{t_{n-1}}^{\tau(\lambda)} \prod_{i=1}^n dL^z_{t_i}. \] (6.60)

Therefore, setting
\[ F(t, \tau(\lambda)) = \int_{\{t<t_2<...<t_n<\tau(\lambda)\}} \prod_{i=2}^n dL^z_{t_i}, \]
we have
\[ E^x \left( \int_{\{0<t_1<...<t_n<\tau(\lambda)\}} \prod_{i=1}^n dL^z_{t_i} \right) = E^x \left( \int_0^{\tau(\lambda)} F(t, \tau(\lambda)) dL^z_{t_i} \right) \]
\[ = E^x \left( \int_0^{\infty} 1_{\{\lambda>L_0^t\}} F(t, \tau(\lambda)) dL^z_{t} \right). \] (6.62)

Since \( \lambda > L_0^t \) implies that \( \tau(\lambda) = t + \tau(\lambda - L_0^t) \circ \theta_t \),
\[ E^\lambda \left( 1_{\{\lambda>L_0^t\}} F(t, \tau(\lambda)) \right) \]
\[ = E^\lambda \left( 1_{\{\lambda>L_0^t\}} F(t, t + \tau(\lambda - L_0^t) \circ \theta_t) \right) \]
\[ = \int_{L_0^t}^{\infty} F(t, t + \tau(y - L_0^t) \circ \theta_t) e^{-y/\alpha} dy/\alpha \]
\[ = e^{-L_0^t/\alpha} \int_0^{\infty} F(t, t + \tau(y) \circ \theta_t) e^{-y/\alpha} dy/\alpha \]
\[ = e^{-L_0^t/\alpha} E^\lambda (F(t, t + \tau(\lambda) \circ \theta_t)). \] (6.63)

Using this in (6.62)
\[ E^x \left( \int_0^{\infty} 1_{\{\lambda>L_0^t\}} F(t, \tau(\lambda)) dL^z_{t} \right) \]
\[ = E^x \left( \int_0^{\infty} e^{-L_0^t/\alpha} F(t, t + \tau(\lambda) \circ \theta_t) dL^z_{t} \right) \]
\[ = E^x \left( \int_0^{\infty} e^{-L_0^t/\alpha} F(0, \tau(\lambda) \circ \theta_t) dL^z_{t} \right), \] (6.64)
where the last equality uses the additivity of local times.

Let $\tau_{z_1}(s)$ denote the right continuous local time for $z_1$. Using the analogue of (6.41) for $\tau_{z_1}(s)$ and then the strong Markov property at $\tau_{z_1}(s)$ we have

\[
E_x^\lambda \left( \int_0^\infty e^{-L_0^0 t/\alpha} F(0, \tau(\lambda)) \circ \theta_t \, dL_{t}^z \right) = E_x^\lambda \left( \int_0^\infty e^{-L_0^0 \tau_{z_1}(s)/\alpha} F(0, \tau(\lambda)) \circ \theta_{\tau_{z_1}(s)} \, ds \right)
\]

\[
= \int_0^\infty E^x \left( e^{-L_0^0 \tau_{z_1}(s)/\alpha} E^z_{\tau_{z_1}(s)} (F(0, \tau(\lambda))) \right) \, ds
\]

\[
= E^z_{\tau_{z_1}(s)} (F(0, \tau(\lambda))) E^{x} \left( \int_0^\infty e^{-L_0^0 t/\alpha} \, dL_{t}^z \right) = E^z_{\tau_{z_1}(s)} (F(0, \tau(\lambda))) u_{\tau(\lambda)}(x, z_1),
\]

where, for the next to last equation, we use (6.41) and for the last equation, we use (6.58).

Combining (6.59)–(6.65) we see that

\[
E^x_{\tau_{z_1}(s)} \left( \int_{\{0<t_1<...<t_n<\tau(\lambda)\}} \prod_{i=1}^n dL_{t_i}^{z_i} \right) = u_{\tau(\lambda)}(x, z_1) E^z_{\tau_{z_1}(s)} \left( \int_{\{0<t_2<...<t_n<\tau(\lambda)\}} \prod_{i=2}^n dL_{t_i}^{z_i} \right).
\]

Iterating this argument we get

\[
E^x_{\tau_{z_1}(s)} \left( \int_{\{0<t_1<...<t_n<\tau(\lambda)\}} \prod_{i=1}^n dL_{t_i}^{z_i} \right) = u_{\tau(\lambda)}(x, z_1) u_{\tau(\lambda)}(z_1, z_2) \cdots u_{\tau(\lambda)}(z_{n-1}, z_n).
\]

Summing over $\pi$ we get (6.7).

The proof of the generalized second Ray-Knight theorem given here is similar to that in [25, Section 8.3.3]. Chapter 11 gives a full treatment of favorite points for Brownian motion and stable processes.

We have assumed that our Markov process is recurrent. For the transient case see [25, Theorem 8.2.3].
7 Loop soups

Until now we have considered only symmetric Markov processes. This was natural since our Isomorphism theorems related Markov local times to the squares of an associated Gaussian process whose covariance was the potential density of our Markov process, and covariance functions are always symmetric. If we want to have an Isomorphism theorem for the local times of a not necessarily symmetric Markov processes we will need to find a substitute for Gaussian squares. The route we take is long, but quite interesting. In the end it should help remove some of the mystery of Isomorphism theorems, even in the symmetric case. The mystery I refer to is this: Even after all the proofs we have seen, why, intuitively, should Gaussian squares be related to Markov local times?

7.1 The loop measure

Once again, we assume a Markov process $X \in S$ with transition densities $p_t(x, y)$ with respect to a measure $m$ on $S$. But we do not assume that $p_t(x, y)$ is symmetric. Our first step is to define bridge measures $Q^{x,y}_t$ for $X$. Consider

$$M_s = p_{t-s}(X_s, x). \quad (7.1)$$

Let us show that $M_s$ is a martingale, $0 \leq s \leq t$. Note that

$$M_s = p_{t-s}(X_s, x) = p_{t-s}(X_{s-r}, x) \circ \theta_r. \quad (7.2)$$

Hence, using the Markov property

$$E^x(M_s | \mathcal{F}_r) = E^x_{X_r}(p_{t-s}(X_{s-r}, x)) \quad (7.3)$$

$$= \int p_{s-r}(X_r, z)p_{t-s}(z, x) \, dm(z) = M_r.$$

It follows that if we set

$$Q^{x,y}_t(G) = P^x(G \, M_s) = P^x(G \, p_{t-s}(X_s, y)), \quad (7.4)$$

for all $G \in \mathcal{F}_s$ with $s < t$, then $Q^{x,y}_t$ is well-defined. That is, if in fact $G \in \mathcal{F}_r$ with $r < s < t$, then we obtain the same value for $Q^{x,y}_t(G)$ if we use

$$Q^{x,y}_t(G) = P^x(G \, M_r), \quad (7.5)$$

which follows from the fact that $M_s$ is a martingale, $0 \leq s \leq t$. We note that $Q^{x,y}_t$ extends naturally to $\mathcal{F}^-_t$. 

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Given bridge measures we can now define the loop measure

\[ \mu(F) = \int_0^\infty \frac{1}{t} \int Q_t^{x,x}(F \circ k_t) \, dm(x) \, dt \]  

(7.6)

for all \( F \in \mathcal{F} \), where \( k_t \) is the killing operator: \( k_t \omega(s) = \omega(s) \) if \( s < t \), and \( k_t \omega(s) = \Delta \) if \( s \geq t \).

We need to see how to compute with \( \mu \). Our goal is to obtain the following moment formula for local times under \( \mu \).

\[ \mu \left( \prod_{j=1}^k L_{x_j}^{\infty} \right) = cy([1,k]) \]  

(7.7)

\[ = \sum_{\pi \in \mathcal{P}_k} u(x_{\pi(1)}, x_{\pi(2)}) \cdots u(x_{\pi(k-1)}, x_{\pi(k)}) u(x_{\pi(k)}, x_{\pi(1)}). \]

Proof: If \( 0 < t_1 < t_2 < \cdots < t_k \) it follows from the definition of the killing operator \( k_t \) that

\[ \prod_{j=1}^k f_j(X_{t_j}) \circ k_t = 1_{\{t > t_k\}} \prod_{j=1}^k f_j(X_{t_j}). \]  

(7.8)

Hence from the definition of the bridge measure \( Q_t^{x,x} \)

\[ Q_t^{x,x} \left( \prod_{j=1}^k f_j(X_{t_j}) \circ k_t \right) = 1_{\{t > t_k\}} \int p_{t_1}(x, y_1) f_1(y_1) p_{t_2-t_1}(y_1, y_2) f_2(y_2) \cdots \]

\[ \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) f_k(y_k) p_{t_k-y_{k}}(y_k, x) \, dm(y_1) \cdots dm(y_k) \]

Integrating with respect to \( dm(x) \) and using \( \int p_{t_k}(y, x) p_{t_1}(x, y_1) \, dm(x) = p_{t_1+t-k}(y, y_1) \) we obtain

\[ \mu \left( \prod_{j=1}^k f_j(X_{t_j}) \right) = \int_{t_k}^\infty \frac{1}{t} \int f_1(y_1) p_{t_2-t_1}(y_1, y_2) f_2(y_2) \cdots \]

\[ \cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k) f_k(y_k) p_{t_1+t-k}(y_k, y_1) \, dm(y_1) \cdots dm(y_k). \]  

(7.9)

This does not look very enlightening, but we plough ahead and integrate over
time to obtain

\[
\mu \left( \int_{\{0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t < \infty\}} \prod_{j=1}^{k} f_j(X_{t_j}) \, dt_j \right)
\]

(7.10)

\[
= \int_{\{0 \leq t_1 \leq \cdots \leq t_k \leq t < \infty\}} 1 \frac{1}{t} \int f_1(y_1)p_{t_2-t_1}(y_1, y_2)f_2(y_2) \cdots \\
\cdots p_{t_k-t_{k-1}}(y_{k-1}, y_k)f_k(y_k)p_{t_{k+1}-t_k}(y_k, y_1) \prod_{j=1}^{k} dm(y_j) \, dt_j \, dt.
\]

Note that we are integrating over \(k + 1\) time variables, \(t_1 \leq \cdots \leq t_k \leq t\). We make the following change of variables: \(r_1 = t_1 + t - t_k, r_2 = t_2 - t_1, \ldots r_k = t_k - t_{k-1}\), and retain \(t_1\) as our \(k + 1\)st time variable. The range of integration is \([0, \infty)\) for all \(r_j\) and \(t_1 \leq r_1\). We also note that \(r_1 + r_2 + \cdots + r_k = t\) so that (7.10) becomes

\[
\mu \left( \int_{\{0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t < \infty\}} \prod_{j=1}^{k} f_j(X_{t_j}) \, dt_j \right)
\]

(7.11)

\[
= \int_{R^k_+} \frac{1}{r_1 + \cdots + r_k} \left( \int f_1(y_1)p_{r_2}(y_1, y_2)f_2(y_2) \cdots \\
\cdots p_{r_k}(y_{k-1}, y_k)f_k(y_k)p_{r_1}(y_k, y_1) \prod_{j=1}^{k} dm(y_j) \right) \left( \int_{0}^{r_1} 1 \, dt_1 \right) \prod_{j=1}^{k} dr_j
\]

\[
= \int_{R^k_+} \frac{r_1}{r_1 + \cdots + r_k} \int f_1(y_1)p_{r_2}(y_1, y_2)f_2(y_2) \cdots \\
\cdots p_{r_k}(y_{k-1}, y_k)f_k(y_k)p_{r_1}(y_k, y_1) \prod_{j=1}^{k} dm(y_j) \, dr_j
\]

Using the fact that \(R^k_+ = \cup_{\pi \in \mathcal{P}_k} \{0 \leq t_{\pi(1)} \leq \cdots \leq t_{\pi(k)} \leq t_{\pi(k-1)} < \infty\} \)

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we then obtain
\[
\mu \left( \prod_{j=1}^{k} \left( \int_{0}^{\infty} f_j(X_t) \, dt \right) \right) = \sum_{\pi \in \mathcal{P}_k} \mu \left( \int_{\{0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} \prod_{j=1}^{k} f_{\pi(j)}(X_t) \, dt \right)
\]
\[
= \int \frac{r_1}{r_1 + \cdots + r_k} \sum_{\pi \in \mathcal{P}_k} \int f_{\pi(1)}(y_1)p_{\pi(2)}(y_2)f_{\pi(3)}(y_3) \cdots \cdots \prod_{j=1}^{k} dm(y_j) \, dr_j
\]
\[
= \int \frac{r_1}{r_1 + \cdots + r_k} h(r_1, r_2, \ldots, r_k) \prod_{j=1}^{k} dr_j,
\]
where
\[
h(r_1, r_2, \ldots, r_k) = \sum_{\pi \in \mathcal{P}_k} \int p_{\pi(2)}(y_2) \cdots p_{\pi(1)}(y_1) \prod_{j=1}^{k} f_{\pi(j)}(y_k) \, dy.
\]

The basic idea we now use is that since \( h \) involves a sum over permutations, there is no longer anything special about \( r_1 \), which will allow us to eliminate the factor \( \frac{r_1}{r_1 + \cdots + r_k} \) and end up with a nice formula. In more detail, observe that
\[
h(r_1, r_2, \ldots, r_k) = h(r_2, r_3, \ldots, r_1).
\] (7.14)

Hence, first changing variables in (7.12), and then using (7.14) we obtain
\[
\mu \left( \prod_{j=1}^{k} \left( \int_{0}^{\infty} f_j(X_t) \, dt \right) \right) = \int \frac{r_2}{r_1 + \cdots + r_k} h(r_2, r_3, \ldots, r_1) \prod_{j=1}^{k} dr_j
\]
\[
= \int \frac{r_2}{r_1 + \cdots + r_k} h(r_1, r_2, \ldots, r_k) \prod_{j=1}^{k} dr_j. \tag{7.15}
\]
turn by \(r_3, \cdots, r_k\) we have shown that

\[
\mu \left( \prod_{j=1}^{k} \left( \int_{0}^{\infty} f_j(X_t) \, dt \right) \right) = \frac{1}{k} \int h(r_1, r_2, \ldots, r_k) \prod_{j=1}^{k} \, dr_j
\]  

\[
= \frac{1}{k} \sum_{\pi \in P_k} \int f_{\pi(1)}(y_1)u(y_1, y_2)f_{\pi(2)}(y_2) \cdots u(y_{k-1}, y_k)f_{\pi(k)}(y_k)u(y_k, y_1) \prod_{j=1}^{k} \, dm(y_j),
\]

Letting \(f_j = f_{x_j, \delta}\) and taking the limit \(\delta \to 0\) we obtain a simple moment formula:

\[
\mu \left( \prod_{j=1}^{k} L_{\infty}^{x_j} \right) = \frac{1}{k} \sum_{\pi \in P_k} u(x_{\pi(1)}, x_{\pi(2)}) \cdots u(x_{\pi(k-1)}, x_{\pi(k)})u(x_{\pi(k)}, x_{\pi(1)}).
\]  

\[
(7.16)
\]

The product of \(u\)'s on the right is invariant under the \(k\) rotations \((1, 2, \ldots, k) \to (i, i+1, \ldots, i+k)\) mod \(k\). Thus we have

\[
\mu \left( \prod_{j=1}^{k} L_{\infty}^{x_j} \right) = \sum_{\pi \in P_k^0} u(x_{\pi(1)}, x_{\pi(2)}) \cdots u(x_{\pi(k-1)}, x_{\pi(k)})u(x_{\pi(k)}, x_{\pi(1)}),
\]  

\[
(7.17)
\]

which is (7.7).

Fixing a point \(x_0\) then gives us

\[
\mu \left( L_{\infty}^{x_0} \prod_{j=1}^{k} L_{\infty}^{x_j} \right) = \sum_{\pi \in P_k} u(x_0, x_{\pi(1)})u(x_{\pi(1)}, x_{\pi(2)}) \cdots u(x_{\pi(k-1)}, x_{\pi(k)})u(x_{\pi(k)}, x_0),
\]  

\[
(7.19)
\]

or, using (3.15),

\[
\mu \left( L_{\infty}^{x_0} \prod_{j=1}^{k} L_{\infty}^{x_j} \right) = cy([0, k]) = Q^{x_0, x_0} \left( \prod_{j=1}^{k} L_{\infty}^{x_j} \right).
\]  

\[
(7.20)
\]

Recalling the role that \(Q^{x,x}\) played in Dynkin’s isomorphism theorem in the symmetric case, we can feel we are getting closer to an isomorphism theorem in the non-symmetric case. We need to recall some basic facts about Poisson processes.

But first we make a slight improvement on (7.20). Since we have already seen that total local times are exponentially integrable under \(Q^{x,x}\), and this does not depend on symmetry, (7.20) implies that

\[
\mu (L_{\infty}^{x} F (L_{\infty}^{x})) = Q^{x,x} (F (L_{\infty}^{x})).
\]  

\[
(7.21)
\]
7.2 Poisson processes

Let $\mathcal{L}_\alpha$ be a Poisson process on $\Omega_\Delta$ with intensity measure $\alpha \mu$. Thus, each realization of $\mathcal{L}_\alpha$ is a countable collection of points in $\Omega_\Delta$, and if

$$N(A) := \#\{\mathcal{L}_\alpha \cap A\}, \quad A \subseteq \Omega_\Delta$$

(7.22) then

$$P_{\mathcal{L}_\alpha} (N(A) = k) = \frac{(\alpha \mu(A))^k}{k!} e^{-\alpha \mu(A)}, \quad (7.23)$$

and $N(A_1), \ldots, N(A_k)$ are independent for disjoint $A_1, \ldots, A_k$. For any bounded measurable functional $f$ on $\Omega_\Delta$ let

$$N(f) = \sum_{\omega \in \mathcal{L}_\alpha} f(\omega), \quad (7.24)$$

so that $N(A) = N(1_{\{A\}})$.

We will need three basic facts about our Poisson process. The master formula for Poisson processes says that for any bounded measurable functional $f$ on $\Omega_\Delta$

$$E_{\mathcal{L}_\alpha} \left( e^{N(f)} \right) = \exp \left( \alpha \left( \int_{\Omega_\Delta} \left( e^{f(\omega)} - 1 \right) d\mu(\omega) \right) \right). \quad (7.25)$$

Proof: A simple calculation using (7.23) shows that

$$E_{\mathcal{L}_\alpha} \left( e^{zN(A)} \right) = e^{z \alpha \mu(A)} \sum_{k=0}^{\infty} \frac{(\alpha \mu(A))^k}{k!} e^{-\alpha \mu(A)} = \exp \left( \alpha \left( e^z - 1 \right) \mu(A) \right). \quad (7.26)$$

If $A_1 \cup \cdots \cup A_n = \Omega_\Delta$ is a partition of $\Omega_\Delta$, and $f = \sum_{j=1}^{n} z_j 1_{\{A_j\}}$ then

$$N(f) = \sum_{\omega \in \mathcal{L}_\alpha} f(\omega) = \sum_{j=1}^{n} z_j N(A_j), \quad (7.27)$$

so that by independence we have

$$E_{\mathcal{L}_\alpha} \left( e^{N(f)} \right) = \prod_{j=1}^{n} E_{\mathcal{L}_\alpha} \left( e^{z_j N(A_j)} \right) \quad (7.28)$$

$$= \prod_{j=1}^{n} \exp \left( \alpha \left( e^{z_j} - 1 \right) \mu(A_j) \right) = \exp \left( \alpha \left( \int_{\Omega_\Delta} \left( e^{f(\omega)} - 1 \right) d\mu(\omega) \right) \right),$$

and (7.25) for general $f$ follows on taking limits. 

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The second fact is the moment formula

\[ E_{\mathcal{L}_\alpha} \left( \prod_{j=1}^{n} N(f_j) \right) = \sum_{U_{i_1}=1}^{[1,n]} \prod_{i=1}^{\ell} \alpha \mu \left( \prod_{j \in B_i} f_j \right). \]  

(7.29)

Proof: Since \( N(\sum_{j=1}^{n} z_j f_j) = \sum_{j=1}^{n} z_j N(f_j) \), by the master formula

\[ E_{\mathcal{L}_\alpha} \left( e^{\sum_{j=1}^{n} z_j N(f_j)} \right) = \exp \left( \alpha \left( \int_{\Omega} \left( e^{\sum_{j=1}^{n} z_j f_j(\omega)} - 1 \right) d\mu(\omega) \right) \right). \]  

(7.30)

Differentiating with respect to \( z_1 \) and then setting \( z_1 = 0 \) we obtain

\[ E_{\mathcal{L}_\alpha} \left( N(f_1) e^{\sum_{j=1}^{n} z_j N(f_j)} \right) \]

\[ = \alpha \left( \int_{\Omega} f_1 e^{\sum_{j=1}^{n} z_j f_j(\omega)} d\mu(\omega) \right) \exp \left( \alpha \left( \int_{\Omega} \left( e^{\sum_{j=1}^{n} z_j f_j(\omega)} - 1 \right) d\mu(\omega) \right) \right). \]  

(7.31)

Differentiate now with respect to \( z_2 \), using the product rule for the right hand side and then setting \( z_2 = 0 \) we obtain

\[ E_{\mathcal{L}_\alpha} \left( N(f_1) N(f_2) e^{\sum_{j=1}^{n} z_j N(f_j)} \right) \]

\[ = \alpha \left( \int_{\Omega} f_1 f_2 e^{\sum_{j=1}^{n} z_j f_j(\omega)} d\mu(\omega) \right) \exp \left( \alpha \left( \int_{\Omega} \left( e^{\sum_{j=1}^{n} z_j f_j(\omega)} - 1 \right) d\mu(\omega) \right) \right) \]

\[ + \alpha \left( \int_{\Omega} f_1 e^{\sum_{j=1}^{n} z_j f_j(\omega)} d\mu(\omega) \right) \alpha \left( \int_{\Omega} f_2 e^{\sum_{j=1}^{n} z_j f_j(\omega)} d\mu(\omega) \right) \exp \left( \alpha \left( \int_{\Omega} \left( e^{\sum_{j=1}^{n} z_j f_j(\omega)} - 1 \right) d\mu(\omega) \right) \right). \]  

(7.32)

By now it should be clear that iterating this leads to (7.29).

Our last basic fact is the Palm formula which says that for \( f \) as above and \( G \) a symmetric measurable function on \( \Omega^\infty_\Delta \)

\[ E_{\mathcal{L}_\alpha} \left( N(f) G(\mathcal{L}_\alpha) \right) = \alpha \int E_{\mathcal{L}_\alpha} \left( G(\omega' \cup \mathcal{L}_\alpha) \right) f(\omega') d\mu(\omega'). \]  

(7.33)

Proof: By the master formula

\[ E_{\mathcal{L}_\alpha} \left( e^{z N(f) + N(g)} \right) = \exp \left( \alpha \left( \int_{\Omega} \left( e^{z f(\omega) + g(\omega)} - 1 \right) d\mu(\omega) \right) \right). \]  

(7.34)
Differentiation with respect to $z$ and then setting $z = 0$ we obtain
\begin{align*}
E_{\mathcal{L}_\alpha} \left( N(f) \ e^{N(g)} \right) \\
= \alpha \left( \int_{\Omega_\Delta} f(\omega) e^{\theta(\omega)} \ d\mu(\omega) \right) \exp \left( \alpha \left( \int_{\Omega_\Delta} \left( e^{\theta(\omega)} - 1 \right) d\mu(\omega) \right) \right).
\end{align*}

Using the master formula on the right most term we can write the right hand side as
\begin{align*}
\alpha \left( \int_{\Omega_\Delta} f(\omega') e^{\theta(\omega')} \ d\mu(\omega') \right) E_{\mathcal{L}_\alpha} \left( e^{\sum_{\omega \in \mathcal{L}_\alpha} g(\omega)} \right) \\
= \alpha \left( \int_{\Omega_\Delta} f(\omega') E_{\mathcal{L}_\alpha} \left( e^{\sum_{\omega' \cup \mathcal{L}_\alpha} g(\omega)} \right) d\mu(\omega') \right) .
\end{align*}

This proves (7.33) for the special case when $G(\omega_i) = e^{\sum_{i=1}^{\infty} g(\omega_i)}$, and this would actually be sufficient for our purposes, but in fact the general case follows from this.

\subsection*{7.3 The isomorphism theorem}

We now use loop soups to prove a general isomorphism theorem for not necessarily symmetric Markov processes, which in the symmetric case, with $\alpha = 1/2$, is the Dynkin isomorphism theorem. Let
\begin{equation}
\hat{L}_x^\alpha = N(L_\infty^x) = \sum_{\omega \in \mathcal{L}_\alpha} L_\infty^x(\omega). \tag{7.37}
\end{equation}

Using (7.29) with $f_j = L_\infty^x j$ and then (7.3) we obtain
\begin{align*}
E_{\mathcal{L}_\alpha} \left( \prod_{j=1}^{n} \hat{L}_x^j \right) &= \sum_{\cup_{i=1}^{\ell} B_i = [1,n]} \prod_{i=1}^{\ell} \alpha \mu \left( \prod_{j \in B_i} L_\infty^x \right) \tag{7.38} \\
&= \sum_{\cup_{i=1}^{\ell} B_i = [1,n]} \prod_{i=1}^{\ell} \alpha \mu(B_i) .
\end{align*}

We prove the following general isomorphism theorem
\begin{equation}
E_{\mathcal{L}_\alpha} \left( \hat{L}_x^0 \ F \left( \hat{L}_x^j \right) \right) = \alpha E_{\mathcal{L}_\alpha} Q^{x_0,x_0} \left( F \left( \hat{L}_x^j + L_\infty^x \right) \right) . \tag{7.39}
\end{equation}

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Proof: As before, it suffices to prove that
\[ E_\alpha \left( \prod_{j=1}^{k} \hat{L}_\alpha^{x_j} \right) = \alpha E_\alpha Q^{x_0,x_0} \left( \prod_{j=1}^{k} \left( \hat{L}_\alpha^{x_j} + L^{x_j}_\infty \right) \right), \] (7.40)
which we can write as
\[ E_\alpha \left( \prod_{j=0}^{k} \hat{L}_\alpha^{x_j} \right) = \alpha \sum_{A \subseteq [1,k]} Q^{x_0,x_0} \left( \prod_{i \in A} L^{x_i}_\infty \right) E_\alpha \left( \prod_{i \notin A} \hat{L}_\alpha^{x_i} \right). \] (7.41)

Using (7.38), our theorem is the claim that
\[ \sum_{\cup_{i=0}^{\ell} B_i = \{0, \ldots, n\}} \prod_{i=1}^{\ell} \alpha \text{cy}(B_i) \] (7.42)
\[ = \alpha \sum_{A \subseteq [1,k]} Q^{x_0,x_0} \left( \prod_{i \in A} L^{x_i}_\infty \right) \sum_{\cup_{i=1}^{m} C_i = A^c} \prod_{i=1}^{m} \alpha \text{cy}(C_i). \]

This follows from (7.20), which says the \( Q^{x_0,x_0} \left( \prod_{i \in A} L^{x_i}_\infty \right) = \text{cy}(A \cup \{0\}) \).

Comparing (7.38) with (2.18) proves that in symmetric case, with \( \alpha = 1/2 \), we have
\[ \{ \hat{L}_\alpha^x, x \in S \} \text{law} = \{ \frac{1}{2} G^2_x, x \in S \} \] (7.43)
In this case, (7.39) is Dynkin’s isomorphism theorem (4.1) which we write as
\[ E_G \left( G^2_x F \left( \frac{1}{2} G^2_{x_i} \right) \right) = E_G Q^{x,x} \left( F \left( \frac{1}{2} G^2_{x_i} + L^{x_i}_\infty \right) \right). \] (7.44)

(7.43) explains why, intuitively, Gaussian squares should be related to Markov local times. The reason is that Gaussian squares are themselves sums of local times.

### 7.4 A Palm formula proof of the isomorphism theorem

In this section we show that the isomorphism theorem (7.39) is just a simple application of the Palm formula. We apply the Palm formula (7.33) with \( f(\omega) = L^{x}_\infty(\omega) \) and \( G(\mathcal{L}) = F \left( \hat{L}_\alpha^{x} \right) \) where as before
\[ \hat{L}_\alpha^{x} = N(L^{x}_\infty) = \sum_{\omega \in \mathcal{L}_\alpha} L^{x}_\infty(\omega). \] (7.45)
Using the fact that $\mu$ is non-atomic, we see that for any fixed $\omega' \in \Omega_\Delta$, almost surely $\omega' \notin \mathcal{L}_\alpha$, and consequently

$$\tilde{L}_\alpha^{x_j}(\omega' \cup \mathcal{L}_\alpha) = \sum_{\omega \in \omega' \cup \mathcal{L}_\alpha} L^{x_j}_\infty(\omega) = \tilde{L}_\alpha^{x_j}(\mathcal{L}_\alpha) + L^{x_j}_\infty(\omega').$$  \hspace{1em} (7.46)

Thus

$$G(\omega' \cup \mathcal{L}_\alpha) = F\left(\tilde{L}_\alpha^{x_j}(\mathcal{L}_\alpha) + L^{x_j}_\infty(\omega')\right).$$  \hspace{1em} (7.47)

Then by the Palm formula (7.33)

$$E_{\mathcal{L}_\alpha}\left(L^{x_j}_\alpha F\left(\tilde{L}_\alpha^{x_j}\right)\right) = \alpha E_{\mathcal{L}_\alpha} \int \left(L^{x_j}_\infty(\omega') F\left(\tilde{L}_\alpha^{x_j} + L^{x_j}_\infty(\omega')\right)\right) d\mu(\omega').$$  \hspace{1em} (7.48)

It follows from (7.21) that we can rewrite this as

$$E_{\mathcal{L}_\alpha}\left(L^{x_j}_\alpha F\left(\tilde{L}_\alpha^{x_j}\right)\right) = \alpha E_{\mathcal{L}_\alpha} Q^{x,x}\left(F\left(\tilde{L}_\alpha^{x_j} + L^{x_j}_\infty\right)\right),$$  \hspace{1em} (7.49)

which is (7.39).

### 7.5 Permanental processes

Our goal in this sub-section is to better understand the stochastic process $\{\tilde{L}_\alpha^x, x \in S\}$ which appears in our isomorphism theorem (7.49).

Using (7.38), (7.39) and writing $B_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,|B_i|}\}$ we have

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^n \tilde{L}_\alpha^{x_j}\right) = \sum_{\ell=1}^n \sum_{\cup_{i=1}^\ell B_i = [1,n]} \alpha^{\ell} \prod_{i=1}^\ell u(x_{i,\pi_1}, x_{i,\pi_2}) \cdots u(x_{i,\pi_{|B_i|}}, x_{i,\pi_1}).$$  \hspace{1em} (7.50)

(7.50) can also be written as

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^n \tilde{L}_\alpha^{x_j}\right) = \sum_{\pi \in \mathcal{P}_n} \alpha^{c(\pi)} u(x_1, x_{\pi_1}) u(x_2, x_{\pi_2}) \cdots u(x_n, x_{\pi_n}).$$  \hspace{1em} (7.51)

In particular

$$E_{\mathcal{L}_\alpha}\left(\tilde{L}_\alpha^x\right) = \alpha u(x, x), \quad \text{Cov}\left(\tilde{L}_\alpha^x, \tilde{L}_\alpha^y\right) = \alpha u(x, y) u(y, x).$$  \hspace{1em} (7.52)

When $\alpha = 1$ the right hand side of (7.51) is the permanent of the matrix $\{u(x_i, x_j)\}$, while if $\alpha = -1$ we obtain the determinant. In general, this is
referred to as the $\alpha$-permanent, see [33], and a process satisfying (7.51) is called an $\alpha$-permanent process.

By (7.52), $u(x,y)u(y,x)$ is positive definite, hence so is $\sqrt{u(x,y)u(y,x)}$. Let $(G_x, G_y)$ be the Gaussian random vector with covariance $\sqrt{u(x,y)u(y,x)}$. An important property of the $1/2$-permanental process $\tilde{L}_\alpha^x$, $x \in S$ is that the bivariate distributions $(\tilde{L}_\alpha^x, \tilde{L}_\alpha^y)$ are the same as $(G_x^2/2, G_y^2/2)$. To see this, it suffices to show that

$$E_{\mathcal{L}_\alpha} \left( \left( \tilde{L}_\alpha^x \right)^j \left( \tilde{L}_\alpha^y \right)^k \right) = E \left( (G_x^2/2)^j (G_y^2/2)^k \right)$$

for all $j, k$. Comparing (7.38) with (2.18) with $\alpha = 1/2$, shows that both involve cycles, the only difference being that the left hand side involves cycles with respect to $\sqrt{u(x,y)u(y,x)}$ while the right hand side uses $u(x,y)$. $u(x,y)$ is the same as $\sqrt{u(x,y)u(y,x)}$ when $x = y$, but note that in the left hand side of (7.33), whenever we have $u(x,y)$ with $x \neq y$, we must also have a corresponding $u(y,x)$. If we replace both elements of this pair by $\sqrt{u(x,y)u(y,x)}$, we will not change the value of the left hand side. Implementing this change for all $u(x,y)$ with $x \neq y$ establishes (7.53).

The importance of the fact that $(\tilde{L}_\alpha^x, \tilde{L}_\alpha^y) \overset{\text{dist}}{=} (G_x^2/2, G_y^2/2)$ comes from the fact that in proving the sufficiency of the condition (5.7) for the continuity of Gaussian processes, all that is used is the bivariate distributions. This allows us to obtain a similar result for permanental processes, see [26].

See [22] and the earlier Arxiv version of [26]. See [15] for Markovian bridges. For Poisson processes see [17]. The Palm formula is given in [2 Lemma 2.3]. This reference assumes that $S$ is Polish, but that assumption is not necessary. Permanental processes were introduced in [33], and their relevance to isomorphism theorems was established in [14]. For later developments see [16, 23, 24]. For other work on loop soups see [18, 19, 20].

8 A Poisson process approach to the generalized second Ray-Knight theorem

Using excursion theory, we can give a simple proof of the generalized second Ray–Knight Theorem which does not make require us to work with $\tau(\lambda)$ for an independent exponential $\lambda$.

As before we assume that $X$ is symmetric, recurrent, with $P^x (T_0 < \infty) = P^x (T_0 < \infty) = 1$ for all $x \in S$ and $u(0,0) = \infty$. We let $n$ denote the excursion measure for $X$ with respect to the point $0$. $n$ is a $\sigma$-finite measure on $\Omega_\Delta$. Let
\(E_t\) be a Poisson process on \(\Omega_\Delta\) with intensity measure \(tn\). It is a fundamental result of excursion theory, [3, 4] that
\[
\{L^x_{\tau(t)}, x \neq 0, P^0\} = \{N(L^x_{\infty}) = \sum_{\omega \in \mathcal{E}_t} L^x_{\infty}(\omega), x \neq 0, P^0\}. \tag{8.1}
\]
Hence by the moment formula (7.29)
\[
P^0\left(\prod_{j=1}^{n} L^x_{\tau(t)}\right) = E_{\mathcal{E}_t} \left(\prod_{j=1}^{n} N(L^x_{\infty})\right) = \sum_{\cup_{i=1}^{l} B_i = [1, n]} \prod_{i=1}^{l} t_n \left(\prod_{j \in B_i} L^x_{\infty}\right). \tag{8.2}
\]
In view of (6.6) we need only show that
\[
\prod_{j \in B_i} L^x_{\infty} = \text{ch}_0(B_j). \tag{8.3}
\]
This is the content of the next Lemma.

8.1 Excursion local time

**Lemma 8.1** For any \(y \neq 0\)
\[
n(L^y_{\infty}) = 1, \tag{8.4}
\]
and for \(y_1, \cdots, y_k \neq 0, k \geq 2,\)
\[
n\left(\prod_{j=1}^{k} L^{y_j}_{\infty}\right) = \sum_{\pi \in P_k} \prod_{j=1}^{k-1} u_{T_0}(y_{\pi(j)}, y_{\pi(j+1)}). \tag{8.5}
\]

**Proof:** Under \(n\), the coordinate process is Markovian with an entrance law which we denote by \(\eta_t, t > 0\), and transition probabilities given by the stopped process \(X^0_t = X_{t \wedge T_0}\). \(X^0\) has potential densities \(u_{T_0}(x, y)\) for \(x, y \neq 0\), where \(u_{T_0}(x, y)\) are the potential densities for the process obtained by killing \(X\) at \(T_0\).

Let \(\eta_t, t > 0\), denote the entrance law for \(n\). Then
\[
n \left( f(\eta_t) \right) = \eta_t(f). \tag{8.6}
\]
It follows from [1 XV, (78.3)] or [28 VI, (50.3)] that for any \(f\) which is zero at 0
\[
\int_0^{\infty} e^{-\alpha t} \eta_t(f) dt = \frac{1}{u^\alpha(0, 0)} \int u^\alpha(0, x) f(x) dm(x). \tag{8.7}
\]
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Hence, if \( f_{y,\epsilon} \) is an approximate \( \delta \)-function for \( y \) supported in the ball of radius \( \epsilon \) centered at \( y \), then for \( \epsilon \) sufficiently small
\[
 n \left( \int_0^\infty e^{-\alpha t} f_{y,\epsilon} (X_t) \, dt \right) = \frac{1}{u^\alpha(0,0)} \int u^\alpha(0, x) f_{y,\epsilon} (x) \, dm(x). \tag{8.8}
\]

We claim that for any bounded measurable function \( g \)
\[
 \lim_{\epsilon \to 0} \int g(t) f_{y,\epsilon} (X_t) \, dt = \int g(t) \, dL^y_t. \tag{8.9}
\]

It suffices to prove this for \( g \) of the form \( g(t) = 1_{(0,r]}(t) \), in which case it follows from \( (3.9) \). Hence, letting \( \epsilon \to 0 \) in \( (8.8) \) and then using \( (6.21) \) gives
\[
 n \left( \int_0^\infty e^{-\alpha t} \, dL^y_t \right) = u^\alpha(0, y) = E_y(e^{-\alpha T_0}). \tag{8.10}
\]

Letting \( \alpha \to 0 \) gives
\[
 n \left( \int_0^\infty \, dL^y_t \right) = P_y(T_0 < \infty) = 1 \tag{8.11}
\]
by our assumption. This proves \( (8.4) \).

Now let \( 0 < t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty \). Then, if \( P^0_t \) denotes the transition operator for the stopped process \( X^0_t \),
\[
 n \left( \prod_{j=1}^k f_{y_j,\epsilon} (X_{t_j}) \right) \tag{8.12}
\]
\[
 = \int f_{y_1,\epsilon}(x_1) \prod_{j=2}^k P^0_{t_{j-1} - t_{j-2}}(x_{j-1}, dx_j) f_{y_j,\epsilon}(x_j) t_{t_1}(dx_1),
\]

Hence, using \( (8.7) \) with \( \epsilon \) sufficiently small
\[
 n \left( \int_{\{0 < t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} e^{-\alpha t_k} \prod_{j=1}^k f_{y_j,\epsilon} (X_{t_j}) \, dt \right) \tag{8.13}
\]
\[
 = \frac{1}{u^\alpha(0,0)} \int u^\alpha(0, x_1) f_{y_1,\epsilon}(x_1) \prod_{j=2}^k u^\alpha_{T_0}(x_{j-1}, x_j) f_{y_j,\epsilon}(x_j) \prod_{j=1}^k dm(x_j).
\]

Letting \( \epsilon \to 0 \) gives
\[
 n \left( \int_{\{0 < t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} e^{-\alpha t_k} \prod_{j=1}^k dL^y_t \right) \tag{8.14}
\]
\[
 = \frac{u^\alpha(0, y_1)}{u^\alpha(0,0)} \prod_{j=2}^k u^\alpha_{T_0}(y_{j-1}, y_j),
\]

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and (8.3) follows as before on letting $\alpha \to 0$.

The passage to the limit $\epsilon \to 0$ under the measure $n$ in both (8.10) and (8.14) requires more justification. However, $n(1_{\{\zeta \geq \delta\}} \cdot)$ is a finite measure for any $\delta > 0$. Using the material we have presented, it is easy to check that the integrands are uniformly bounded in $L^2(dn)$ and hence uniformly integrable in $L^2(1_{\{\zeta \geq \delta\}} dn)$. We can thus take the $\epsilon \to 0$ in $L^2(1_{\{\zeta \geq \delta\}} dn)$, and then the $\delta \to 0$ limit using the monotone convergence theorem.

9 Another Poisson process isomorphism theorem: random interlacements

Sznitman has recently developed an isomorphism theorem related to a Poisson process for quasi-processes which he refers to as random interlacements, [32]. This isomorphism theorem has the structure of the generalized second Ray-Knight theorem. Typically, the underlying Markov processes (think of Brownian motion in two or more dimensions) do not have finite potential densities, hence there are no local times nor associated Gaussian processes $G_x$ indexed by points in the state space $S$. We first develop material for the associated Gaussian process which is now indexed by measures on $S$. We then introduce quasi-processes and random interlacements. At that stage the isomorphism theorem will be straightforward.

9.1 Gaussian fields

We assume that $X$ is a symmetric Markov process in $S$ with transition densities $p_t(x, y)$. As before, these are positive definite and consequently the potential densities $u(x, y)$ will be positive definite in the wide sense, that is

$$\int \int u(x, y) \, d\nu(x) \, d\nu(y) \geq 0$$

(9.1)

for any positive measure $\nu$ on $S$. Let $\mathcal{M}$ denote the set of finite positive measures on $S$ and

$$\mathcal{G}^1 = \{ \nu \in \mathcal{M} \mid \int \int u(x, y) \, d\nu(x) \, d\nu(y) < \infty \}. \quad (9.2)$$

Let $G_{\nu}$ denote the mean zero Gaussian process on $\mathcal{G}^1$ with covariance

$$E(G_{\nu}G_{\nu'}) = \int \int u(x, y) \, d\nu(x) \, d\nu'(y).$$

(9.3)
We would like to find an analogue of \( \int G^2 \, d\nu(x) \) to obtain some version of (2.18), but if any of the sets \( A_l \) in (2.18) are singletons, then the cycle term would be \( \int u(x, x) \, d\nu(x) \) and for the processes we would like to consider, \( u(x, x) = \infty \) for all \( x \). It is the need to eliminate such singletons that leads us to define the Wick square. In the following we assume that \( u_\delta(x, y) =: \int_\delta^\infty p_t(x, y) \, dt < \infty \) for all \( x, y \in S \), and \( \delta > 0 \).

It is then easy to check that \( p_\epsilon(x, y) \, dm(y) \in G^1 \) for any \( \epsilon > 0 \) and \( x \in S \). Set

\[
G_{x, \epsilon} = G_{p_\epsilon(x, y) \, dm(y)}.
\]  

(9.4)

Then \( G_{x, \epsilon} \) is a Gaussian process on \( S \times (0, \infty) \) with covariance

\[
E(G_{x, \epsilon} G_{x', \epsilon'}) = u_\epsilon + \epsilon'(x, y).
\]  

(9.5)

If we set \( G^2_{x, \epsilon} := G_{x, \epsilon} - E(G^2_{x, \epsilon}) \) then it is easy to see that

\[
E\left( \prod_{i=1}^n : G^2_{x_i, \epsilon} : / 2 \right) = \sum_{A_1 \cup \cdots \cup A_j = [1, n]} \prod_{l=1}^j \frac{1}{2} c_\epsilon(A_l),
\]  

(9.6)

where

\[
c_\epsilon(A_l) = \sum_{\pi \in \mathcal{P}_{|A_l|}} u_2(x_{l_\pi(1)}, x_{l_\pi(2)}) \cdots u_2(x_{l_\pi(|A_l|)}, x_{l_\pi(1)}).
\]  

(9.7)

Define the the Wick square

\[
: G^2 : (\nu) = \lim_{\epsilon \to 0} \int : G^2_{x, \epsilon} : \, d\nu(x).
\]  

(9.8)

Using (9.5) we can show that if

\[
\int \int u^2(x, y) \, d\nu(x) \, d\nu(y) < \infty,
\]  

(9.9)

then the limit in (9.8) exists in all \( L^p \), and we have

\[
E\left( \prod_{i=1}^n : G^2 : (\nu_i) / 2 \right) = \sum_{A_1 \cup \cdots \cup A_j = [1, n]} \prod_{l=1}^j \frac{1}{2} c(A_l, \nu),
\]  

(9.10)

where

\[
c(A_l, \nu) = \sum_{\pi \in \mathcal{P}_{|A_l|}} \int u(x_{l_\pi(1)}, x_{l_\pi(2)}) \cdots u(x_{l_\pi(|A_l|)}, x_{l_\pi(1)}) \prod_{j \in A_l} d\nu_j(x_j).
\]  

(9.11)
(See [27, Lemma 3.3] for an important ingredient in the proof). Set
\[
\mathcal{G}^2 = \{ \nu \mid \int \int u^2(x, y) \, d\nu(x) \, d\nu(y) < \infty \}, \tag{9.12}
\]
and let \( \mathcal{G}^2_K \) denote the subset of measures \( \nu \in \mathcal{G}^2 \) with support in the compact set \( K \subseteq S \).

Let \( |\nu| \) denote the mass of \( \mu \). Exactly as in (5.4) we can then show that for \( \nu_i \in \mathcal{G}^2 \)
\[
E_G \left( \prod_{i=1}^k : G^2 : (\nu_i)/2 + \sqrt{2t} G_{\nu_i} + t|\nu_i| \right) \tag{9.13}
\]
\[
= \sum_{\cup_{i=1}^k \cup_{j=1}^m (B_j \subseteq [1,k], |A_i| \geq 2)} \frac{1}{2} \sum_{i=1}^l \text{cy}(A_i, \nu) \prod_{j=1}^m t \, \text{ch}(B_j, \nu)
\]
where \( \text{ch}(B) = |\nu_i| \) if \( B = \{i\} \) and, if \( |B| > 1 \) with \( B = \{b_1, b_2, \cdots, b_{|B|}\} \) then the chain function \( \text{ch}(B, \nu) \) is defined as
\[
\text{ch}(B, \nu) = \sum_{\pi \in \mathcal{P}[B]} \int u(x_{b_{(1)}}, x_{b_{(2)}}) \cdots u(x_{b_{(|B|-1)}}, x_{b_{(|B|)}}) \prod_{j \in B} d\nu_j(x_j). \tag{9.14}
\]

It follows as in proof of the generalized second Ray-Knight theorem that if we can find a family of random variables \( \{S_{\nu,t}, \nu \in \mathcal{G}^2\} \) such that
\[
P \left( \prod_{i=1}^k S_{\nu_i,t} \right) = \sum_{m=1}^k \sum_{\text{unordered } B_1 \cup \cdots \cup B_m = [1,k]} t^m \prod_{j=1}^m \text{ch}(B_j, \nu), \tag{9.15}
\]
then we will have established the isomorphism theorem
\[
E_G P \left( F \left( S_{\nu_i} + : G^2 : (\nu_i)/2 \right) \right) = E_G \left( F \left( : G^2 : (\nu_i)/2 + \sqrt{2t} G_{\nu_i} + t|\nu_i| \right) \right), \tag{9.16}
\]
Such random variables \( S_{\nu} \) will come from additive functionals of random interlacements.

### 9.2 Quasi-processes and additive functionals

Let \( X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x) \) be a ‘nice’ symmetric transient Markov process
as before with LCCB state space \( S \) and transition densities \( p_t(x, y) \) with respect to a \( \sigma \)-finite measure \( m \). We assume that \( m \) is dissipative, that is, that
\( \int u(x, y) f(y) \, m(dy) < \infty \) \( m \)-a.e for each non-negative \( f \in L^1(m) \). This will hold for example for Brownian motion in \( R^3 \) or exponentially killed Brownian motion in \( R^2 \), with \( m \) being Lebesgue measure.

Let \( W \) denote the set of paths \( \omega : R^1 \to S \cup \Delta \) which are \( S \) valued and right continuous on some open interval \( (\alpha(\omega), \beta(\omega)) \) and \( \omega(t) = \Delta \) otherwise. Let \( Y_t = \omega(t) \), and define the shift operators

\[
(\sigma_t \omega)(s) = \omega(t+s), \quad s, t \in R^1.
\]  

(9.17)

Set \( \mathcal{H} = \sigma(Y_s, s \in R^1) \) and \( \mathcal{H}_t = \sigma(Y_s, s \leq t) \). Let \( A \) denote the \( \sigma \)-algebra of shift invariant events in \( \mathcal{H} \). The quasi-process associated with \( X \) is the measure \( P_m \) on \( (W, A) \) which satisfies the following two conditions:

\[
(i) \quad P_m \left( \int_{R^1} f(Y_t) \, dt \right) = m(f),
\]  

(9.18)

for all measurable \( f \) on \( S \), and (ii): for any intrinsic stopping time \( T, Y_{T+t}, t > 0 \) is Markovian with semigroup \( P_t \), recall (3.3), under \( P_m \mid\{T < \infty\} \). An \( \mathcal{H}_t \)-stopping time \( T \) is called intrinsic if \( \alpha \leq T \leq \beta \) on \( \{T < \infty\} \) and \( T = t + T \circ \sigma_t \) for all \( t \in R^1 \). A first hitting time is an example of an intrinsic stopping time.

If \( L_{\nu}^t, t \geq 0 \) denotes the continuous additive functional, (recall (3.10)), on \( \Omega \) with \( E_x(L_{\nu}^\infty) = \sup_x \int u(x, y) \, d\nu(y) < \infty \) then there is an extension to \( W \), which we also denote by \( L_{\nu}^t, t \in R^1 \) with the property that

\[
P_m \left( \int_{R^1} g(Y_t) \, dL_{\nu}^t \right) = \nu(g),
\]  

(9.19)

for all measurable \( g \), see [5, XIX, (26.5)]. For example, if \( \nu = f \, dm \) then

\[
L_{L_{\nu}^t}^{\nu dm} = \int_{-\infty}^{t} f(Y_s) \, ds
\]  

(9.20)

and (9.19) follows easily from (9.18). In general one can think of \( L_{\nu}^t \) as

\[
L_{\nu}^t = \lim_{\epsilon \to 0} \int_{S} f(x, \epsilon(Y_s)) \, ds \, d\nu(x).
\]  

(9.21)

**Lemma 9.1** For any \( \nu_1, \ldots, \nu_k \), with support in some compact \( K \subset S \)

\[
P_m \left( \prod_{j=1}^{k} L_{\nu_j}^\infty \right) = \sum_{\pi \in \mathcal{P}_k} \prod_{j=1}^{k-1} u(y_j, y_{j+1}) \prod_{j=1}^{k} d\nu_{\pi(j)}(y) = ch([1, k], \nu).
\]  

(9.22)
Compare Lemma 8.1.

**Proof:** Let $T_K$ denote the first hitting time of $K$. Since the measures $\nu_i$ are supported in $K$, it follows that the functionals $L_t^{\nu_i}$ do not grow until time $T_K$. Hence

$$\mathbb{P}_m\left(\int_{\{\infty < t_1 \leq \cdots \leq t_k < \infty\}} \prod_{j=1}^k dL_{t_j}^{\nu_j}\right)$$

(9.23)

$$= \mathbb{P}_m\left(\int_{\{0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} \prod_{j=1}^k dL_{T_K+t_j}^{\nu_j}\right).$$

Hence by the second property of $\mathbb{P}_m$ this equals

$$\mathbb{P}_m\left(\int_0^\infty h\left(Y_{T_K+t_1}\right) dL_{T_K+t_1}^{\nu_1}\right),$$

(9.24)

where

$$h(x) = E^x\left(\int_{\{0 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k < \infty\}} \prod_{j=2}^k dL_{t_j}^{\nu_j}\right)$$

(9.25)

$$= \int u(x, y_2) \prod_{j=2}^{k-1} u(y_j, y_{j+1}) \prod_{j=2}^k d\nu_j(y).$$

(For those unfamiliar with such calculations, think of (9.20) or more generally (9.21)). Using once again the fact that $L_t^{\nu_i}$ doesn’t grow until time $T_K$ and then (9.19) shows that

$$\mathbb{P}_m\left(\int_{\{\infty < t_1 \leq \cdots \leq t_k < \infty\}} \prod_{j=1}^k dL_{t_j}^{\nu_j}\right)$$

(9.26)

$$= \mathbb{P}_m\left(\int_{R^1} h\left(Y_{t_1}\right) dL_{t_1}^{\nu_1}\right)$$

$$= \int \prod_{j=1}^{k-1} u(y_j, y_{j+1}) \prod_{j=1}^k d\nu_j(y),$$

and (9.22) follows. \qed
9.3 Interlacements

Interlacements are the soup of a quasi-process. More precisely, the interlacement $I_t$ is the Poisson process with intensity measure $tP_m$. We let $P_{I_t}$ denote probabilities for the process $I_t$. Let

$$L_{\infty}^\nu = \sum_{\omega \in I_t} L_{\infty}^\nu(\omega).$$

(9.27)

Using (9.22) and the moment formula (7.42), we see that the functionals $\tilde{L}_{\infty}^\nu$, under the measure $P_{I_t}$, satisfy (9.15). In view of (9.16) we have the following interlacement Isomorphism theorem which is essentially due to Sznitman, [32].

**Theorem 9.1** For any $t > 0$, compact $K \subset S$ and countable $D \subseteq \mathcal{G}_K^2$,

$$\left\{ \tilde{L}_{\infty}^{\nu} \right\} \text{law} = \left\{ \frac{1}{2} : G^2 : (\nu), \nu \in D, P_{I_t} \times P_G \right\}.$$

(9.28)

10 Isomorphism theorems via Laplace transforms

In this section we give alternate proofs for our Isomorphism theorems. The innovation here is that we use the moment generating function of Gaussian squares, described in the next subsection, instead of the Gaussian moment formulas of Section 2.1. On the other hand, we still need the local time moment formulas, and in particular for the generalized second Ray-Knight theorem we have seen that the derivation is not trivial.

10.1 Moment generating functions of Gaussian squares

Let $G = (G_1, \ldots, G_n) \in R^n$ be a Gaussian random vector with covariance matrix $C$. If $C$ is invertible we first show that for all bounded measurable functions $F$ on $R^d$

$$E(F(G_1, \ldots, G_n)) = \frac{1}{(2\pi)^{n/2} \sqrt{|C|}} \int_{R^n} F(x) e^{-\langle x, C^{-1}x \rangle/2} dx$$

(10.1)

where $|C|$ denotes the determinant of $C$. To see this it suffices to prove it for $F$ of the form $F(x) = e^{i\langle y, x \rangle}$, in which case we need to show that

$$E\left(e^{i\langle y, G \rangle}\right) = \frac{1}{(2\pi)^{n/2} \sqrt{|C|}} \int_{R^n} e^{i\langle y, x \rangle} e^{-\langle x, C^{-1}x \rangle/2} dx.$$

(10.2)
Setting \( x = C^{1/2}z \), (recall the paragraph following (2.8)), so that \( dx = |C|^{1/2} dz \), the right hand side of (10.2) becomes
\[
\frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{i(C^{1/2}y,z)} e^{-(z,z)/2} \, dz = e^{-(y,Cy)/2}
\]
which, by (2.7) equals the left hand side of (10.2).

We now show that for any Gaussian random vector \( G = (G_1, \ldots, G_n) \) with covariance matrix \( C \), any vector \( u = (u_1, \ldots, u_n) \) and \( \lambda_1, \ldots, \lambda_n \) sufficiently small
\[
E \left( e^{\sum_{j=1}^n \lambda_j u_j G_j + \lambda_j G_j^2/2} \right) = \frac{1}{\sqrt{|I - \Lambda C|}} e^{(u, \Lambda \bar{C} \Lambda u)/2}
\]
where \( \Lambda \) is the diagonal matrix with entries \( (\lambda_1, \ldots, \lambda_n) \) and
\[
\bar{C} = (I - C\Lambda)^{-1} C.
\]
(10.4) will be the key to the alternative proofs of the Isomorphism theorems given in this section.

Proof of (10.4): Assume first that \( C \) is invertible. Then from (10.5)
\[
\bar{C}^{-1} = C^{-1}(I - C\Lambda) = C^{-1} - \Lambda.
\]
Hence, using (10.1) we have
\[
E \left( e^{\sum_{j=1}^n \lambda_j u_j G_j + \lambda_j G_j^2/2} \right)
= \frac{1}{(2\pi)^{n/2} \sqrt{|C|}} \int_{R^n} e^{(\Lambda u, x)} e^{(x, \Lambda x)/2} e^{-(x, C^{-1} x)/2} \, dx
\]
\[
= \frac{1}{(2\pi)^{n/2} \sqrt{|C|}} \int_{R^n} e^{(\Lambda u, x)} e^{-(x, \bar{C}^{-1} x)/2} \, dx
\]
It is clear from (10.6) that for \( \lambda_1, \ldots, \lambda_n \) sufficiently small, \( \bar{C}^{-1} \) is invertible, symmetric and positive definite. Hence changing variables \( x = C^{1/2}z \) as before the last display
\[
= \frac{\sqrt{|C|}}{\sqrt{|C|}} \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{(C^{1/2} \Lambda u, z)} e^{-(z, z)/2} \, dz = \sqrt{|C|/|C|} e^{(u, \Lambda C \Lambda u)/2},
\]
and (10.4) for \( C \) invertible follows from (10.5).

For general \( C \), recall that we can find an orthonormal system of vectors \( u_i, 1 \leq i \leq n \) such that \( Cu_i = c_i u_i, 1 \leq i \leq n \), and the fact that \( C \) is positive definite implies that all \( c_i \geq 0 \). We can then define the matrix \( C_{\epsilon} \) by setting
\[ C_\epsilon u_i = (c_i + \epsilon)u_i, 1 \leq i \leq n. \]  
\[ C_\epsilon \] is clearly symmetric, positive definite and invertible. We then obtain (10.4) by first proving it for the Gaussian random vector \( G_\epsilon \) with covariance matrix \( C_\epsilon \) and then taking the limit as \( \epsilon \to 0 \). That \( G_\epsilon \to G \) in distribution follows from (2.7).

We note that (10.4) immediately implies that
\[
E\left( e^{\sum_{j=1}^n \lambda_j (G_j + u_j)^2 / 2} \right) = \frac{1}{\sqrt{|I - \Lambda C|}} e^{(u, \Lambda u)/2} e^{(u, \Lambda C \Lambda u)/2}. \tag{10.9}
\]

We also note the following computation for later use:
\[
\Lambda + \Lambda C \Lambda = \Lambda + \Lambda (I - CA)^{-1} CA
\]
\[
= \Lambda (I + (I - CA)^{-1} CA) = \Lambda \left( \sum_{k=0}^\infty (CA)^k \right). \tag{10.10}
\]

### 10.2 Another proof of the Dynkin Isomorphism theorem

It suffices to show that
\[
E_G Q_{x_1, x_2} \left( e^{\sum_{j=1}^n \lambda_j L_{x_j}^2} \right) = E_G \left( G_{x_1} G_{x_2} e^{\sum_{j=1}^n \lambda_j \frac{1}{2} G_{x_j}^2} \right). \tag{10.11}
\]

for all \( n, x_1, \ldots, x_n \in S \) and \( \lambda_1, \ldots, \lambda_n \) sufficiently small, since we can always take \( \lambda_1 = \lambda_2 = 0 \). By the independence of \( X \) and \( G \) this is equivalent to showing that
\[
Q_{x_1, x_2} \left( e^{\sum_{j=1}^n \lambda_j L_{x_j}^2} \right) = \frac{E_G \left( G_{x_1} G_{x_2} e^{\sum_{j=1}^n \lambda_j \frac{1}{2} G_{x_j}^2} \right)}{E_G \left( e^{\sum_{j=1}^n \lambda_j \frac{1}{2} G_{x_j}^2} \right)}. \tag{10.12}
\]

Differentiating (10.4) with respect to \( u_1, u_2 \) and then setting all \( u_j = 0 \) for the numerator and using (10.4) with all \( u_j = 0 \) for the denominator we see that
\[
\frac{E_G \left( G_{x_1} G_{x_2} e^{\sum_{j=1}^n \lambda_j \frac{1}{2} G_{x_j}^2} \right)}{E_G \left( e^{\sum_{j=1}^n \lambda_j \frac{1}{2} G_{x_j}^2} \right)} = \bar{C}_{1,2}. \tag{10.13}
\]

To evaluate the left hand side of (10.12) it is useful to introduce the atomic measure on \( S \)
\[
\nu = \sum_{j=1}^n \lambda_j \delta_{x_j}. \tag{10.14}
\]
and to write \( \sum_{j=1}^{n} \lambda_j L_{xj}^x = \int L_{xj}^x \, d\nu(x) \). With this notation we obtain from (3.15)

\[
Q^{x_1, x_2} \left( \left( \int L_{xj}^x \, d\nu(x) \right)^k \right) = k! \int u(x_1, y_1) u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, x_2) \prod_{j=1}^{k} d\nu(y_j)
\]

\[
= k! \left( C \Lambda \cdots C \Lambda C \right)_{1,2} = k!((CA)^kC)_{1,2}.
\]

Thus

\[
Q^{x_1, x_2} \left( e^{\sum_{j=1}^{n} \lambda_j L_{xj}^x} \right) = \sum_{k=0}^{\infty} ((CA)^kC)_{1,2} = C_{1,2}.
\]

10.3 Another proof of the Eisenbaum Isomorphism theorem

As in the last subsection, it suffices to prove that

\[
P^{x_1} \left( e^{\sum_{j=1}^{n} \lambda_j L_{xj}^x} \right) = \frac{E_G \left( \left( 1 + \frac{G_{x1}}{s} \right) e^{\sum_{j=1}^{n} \lambda_j \frac{1}{2}(G_{xj}+s)^2} \right)}{E_G \left( e^{\sum_{j=1}^{n} \lambda_j \frac{1}{2}(G_{xj}+s)^2} \right)}.
\]

(10.17)

for all \( n, x_1, \ldots, x_n \in S \) and \( \lambda_1, \ldots, \lambda_n \) sufficiently small. We can write the right hand side as

\[
1 + \frac{E_G \left( G_{x1} e^{\sum_{j=1}^{n} s \lambda_j G_{xj} + \lambda_j \frac{1}{2} G_{xj}^2} \right)}{sE_G \left( e^{\sum_{j=1}^{n} s \lambda_j G_{xj} + \lambda_j \frac{1}{2} G_{xj}^2} \right)} = 1 + \sum_{j=1}^{n} \bar{C}_{1,j} \lambda_j,
\]

(10.18)

where the last equality comes from differentiating (10.4) with respect to \( u_1 \) and then setting all \( u_j = s \) for the numerator, and setting all \( u_j = s \) for the denominator. We can then rewrite

\[
1 + \sum_{j=1}^{n} \bar{C}_{1,j} \lambda_j = 1 + \sum_{j=1}^{n} \left\{ (I - CA)^{-1}CA \right\}_{1,j} = 1 + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (CA)^k_{1,j}.
\]

(10.19)
Using again the notation (10.14), we obtain from (3.11)
\[
\begin{align*}
P^x_1 \left( \left( \int L^x_{\infty} \, d\nu(x) \right)^k \right) &= k! \int u(x_1, y_1) u(y_1, y_2) \cdots u(y_{k-1}, y_k) \prod_{j=1}^k d\nu(y_j) \\
&= k! \sum_{j=1}^n (CA \cdots CA)_{1,j} = k! \sum_{j=1}^n ((CA)^k)_{1,j}.
\end{align*}
\]
Hence
\[
P^x_1 \left( e^{\sum_{j=1}^n \lambda_j L^x_{\tau(t)}} \right) = \sum_{j=1}^n (\sum_{k=0}^{\infty} (CA)^k)_{1,j} = \sum_{j=1}^n I_{1,j} = 1.
\] (10.21)

10.4 Another proof of the generalized second Ray-Knight theorem

As in the last two subsections, it suffices to prove that for all \( t \)
\[
P^0 \left( e^{\sum_{j=1}^n \lambda_j L^x_{\tau(t)}} \right) = \frac{E_\eta \left( e^{\sum_{j=1}^n \lambda_j \frac{1}{2} (\eta_j + \sqrt{2t})^2} \right)}{E_\eta \left( e^{\sum_{j=1}^n \lambda_j \frac{1}{2} \eta_j^2} \right)}.
\] (10.22)

for all \( n, x_1, \ldots, x_n \in S \) and \( \lambda_1, \ldots, \lambda_n \) sufficiently small. Let \( C_0 \) denote the covariance matrix of \( (\eta_1, \ldots, \eta_n) \) and let \( \mathbf{1} = (1, \ldots, 1) \), that is, the vector in \( \mathbb{R}^n \) with all components equal to 1. Using (10.9) with all \( u_j = \sqrt{2t} \) for the numerator, and all \( u_j = 0 \) for the denominator we obtain
\[
E_\eta \left( e^{\sum_{j=1}^n \lambda_j \frac{1}{2} (\eta_j + \sqrt{2t})^2} \right) = e^{t(1, \mathbf{1})} e^{t(1, \mathbf{1} C_0 \mathbf{1})} = e^{t(1, C_0 (C_0^{-1} C_0 \mathbf{1}))}.
\] (10.23)

where the last equality used (10.10).

Using once again the notation (10.14), we obtain from (6.6)
\[
P^0 \left( \left( \int L^x_{\tau(t)} \, d\nu(x) \right)^k \right) = \sum_{m=1}^k \sum_{\text{unordered}} \int_{B_1 \cup \cdots \cup B_m = [1, k]} t^m \prod_{j=1}^m \mathbf{c}_0(B_j) \prod_{l=1}^k d\nu(y_l).
\] (10.24)
Hence if we set
\[ h(k) = \int u_0(y_1, y_2) \cdots u_0(y_{k-1}, y_k) \prod_{t=1}^{k} d\nu(y_t) = (1, \Lambda(C_0 \Lambda)^{k-1}) \] (10.25)
we see that for any partition \( B_1 \cup \cdots \cup B_m = [1, k] \)
\[ \int \prod_{j=1}^{m} c_h(B_j) \prod_{t=1}^{k} d\nu(y_t) = \prod_{j=1}^{m} |B_j|! h(|B_j|). \] (10.26)
Since there are \( \frac{k!}{m!} \) ways to partition \( k \) objects into \( m \) unordered subsets of size \( k_1, \ldots, k_m \) we see that
\[ P^0 \left( \left( \int L^x_{\tau(t)} d\nu(x) \right)^k \right) = k! \sum_{m=1}^{k} \sum_{k_1+\cdots+k_m=k} \left( \frac{t^m}{m!} \sum_{k_1+\cdots+k_m=k} \prod_{j=1}^{k} h(k_j) \right) \] (10.27)
Hence
\[ P^0 \left( e^{\int L^x_{\tau(t)} d\nu(x)} \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \left( \sum_{j=1}^{\infty} h(j) \right)^m = e^{t \sum_{j=1}^{\infty} h(j)}. \] (10.28)
In view of (10.25), this gives (10.23).

10.5 Yet another proof of the generalized second Ray-Knight theorem using excursion theory

It follows from (8.1) and the master formula (7.25) that for \( \delta \) small
\[ P^0 \left( e^{\delta \int L^x_{\tau(t)} d\nu(x)} \right) = \exp \left( t n \left( e^{\delta \int L^x_{\infty} d\nu(x)} - 1 \right) \right), \] (10.29)
and it follows from (8.3) and (10.26) that
\[ n \left( e^{\delta \int L^x_{\infty} d\nu(x)} - 1 \right) = \sum_{n=1}^{\infty} \delta^n h(n). \] (10.30)
This completes the proof of (10.28) which we have seen is sufficient to prove our theorem.
10.6 Another proof of the interlacement Isomorphism theorem

Because everything is additive in $\nu$ we can write (9.28) as

$$P_{\mathcal{L}} \times P_{G} \left( e^{\frac{\delta}{2}G^2(\nu)} + \frac{\delta}{2}G^2(\nu) + \delta \sqrt{2t}G(\nu) + \delta t|\nu| \right) = P_{G} \left( e^{\frac{\delta}{2}G^2(\nu)} \right)$$  \hspace{1cm} (10.31)

for $\delta$ small. Equivalently, we show that

$$P_{\mathcal{L}} \left( e^{\frac{\delta}{2}L^\infty} \right) = \frac{P_{G} \left( e^{\frac{\delta}{2}G^2(\nu)} + \delta \sqrt{2t}G\nu + \delta t|\nu| \right)}{P_{G} \left( e^{\frac{\delta}{2}G^2(\nu)} \right)}.$$  \hspace{1cm} (10.32)

(9.22) shows that

$$P_{m} \left( (L^\infty)^k \right) = k! \prod_{j=1}^{k-1} u(y_j, y_j+1) \prod_{j=1}^{k} \nu(dy_j).$$  \hspace{1cm} (10.33)

Given (10.33) and our use of the master formula in subsection 10.5 it suffices to show that

$$P_{G} \left( e^{\frac{\delta}{2}G^2(\nu)} + \delta \sqrt{2t}G\nu + \delta t|\nu| \right) = \exp \left( t \sum_{n=1}^{\infty} \delta^n \int \prod_{j=1}^{n-1} u(x_j, x_{j+1}) \prod_{j=1}^{n} \nu(dx_j) \right).$$  \hspace{1cm} (10.34)

To see this, we first note that using (9.8), the Gaussian moment formula and the monotone convergence theorem we have

$$P_{G} \left( e^{\frac{\delta}{2}G^2(\nu)} \right) = \lim_{\epsilon \to 0} P_{G_\delta} \left( e^{\frac{\delta}{2}f(G^2) - E(G^2)} \right) \hspace{1cm} (10.35)$$

and

$$P_{G} \left( e^{\frac{\delta}{2}G^2(\nu)} + \delta \sqrt{2t}G\nu + \delta t|\nu| \right) = \lim_{\epsilon \to 0} P_{G_\delta} \left( e^{\frac{\delta}{2}f(G^2 + \sqrt{2t}G\nu + \delta t|\nu|)} \right) \hspace{1cm} (10.36)$$
Therefore

\[
\begin{align*}
P_G \left( e^{\frac{\delta}{2} \int (e^{\delta} G^2(\nu) + \delta \sqrt{2t} G^2 + \delta t |\nu|)} \right) \\
= \lim_{\epsilon \to 0} \frac{P_G \left( e^{\frac{\delta}{2} \int (G^2(x) + \sqrt{2t})^2 - E(G^2(x))} \nu(x) \right)}{P_G \left( e^{\frac{\delta}{2} \int (G^2(x) - E(G^2(x))) \nu(x) \right)} \\
= \lim_{\epsilon \to 0} \frac{P_G \left( e^{\frac{\delta}{2} \int (G^2(x) + \sqrt{2t})^2} \nu(x) \right)}{P_G \left( e^{\frac{\delta}{2} \int G^2(x) \nu(x) \right)} \\
= \lim_{\epsilon \to 0} \exp \left( t \left( \sum_{n=1}^{\infty} \delta^n \int \prod_{j=1}^{n-1} u_i(x_j, x_{j+1}) \prod_{j=1}^{n} \nu(dx_j) \right) \right)
\end{align*}
\]

as in (10.23). Using the monotone convergence theorem this is

\[
\begin{align*}
&= \exp \left( t \left( \sum_{n=1}^{\infty} \delta^n \int \prod_{j=1}^{n-1} u_i(x_j, x_{j+1}) \prod_{j=1}^{n} \nu(dx_j) \right) \right).
\end{align*}
\]

This completes the proof of (10.34) and hence of (9.28).

\[\square\]

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