Third order open mapping theorems and applications to the end-point map

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Abstract

This paper is devoted to a third order study of the end-point map in sub-Riemannian geometry. We first prove third order open mapping results for maps from a Banach space into a finite dimensional manifold. In a second step, we compute the third order term in the Taylor expansion of the end-point map and we specialize the abstract theory to the study of length-minimality of sub-Riemannian strictly singular curves. We conclude with the third order analysis of a specific strictly singular extremal that is not length-minimizing.

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1. Introduction

The most challenging open problems in sub-Riemannian geometry, such as Sard’s problem and the regularity of length-minimizing curves, are related to our limited understanding of the end-point map, see \cite{1, 2}. In this work, we extend the analysis of the end-point map from the second to the third order. In a preliminary part of independent interest, we study open mapping theorems of the third order for maps from a Banach space into a manifold.

Let $M$ be a smooth manifold and $\Delta \subset TM$ be a totally non-holonomic (i.e. completely non-integrable) distribution with rank $2 \leq k < \dim(M)$. For every point $q_0 \in M$, there exist a neighborhood $U \subset M$ of $q_0$ and linearly independent smooth vector fields $f_1, \ldots, f_k \in \text{Vec}(U)$.
such that $\Delta = \text{span}\{f_1, \ldots, f_k\}$ on $U$. The distribution $\Delta$ is non-holonomic (i.e. it satisfies the H"{o}rmander condition) if

$$\text{Lie}_q \{f_1, \ldots, f_k\} = T_qM \quad \text{for every } q \in U,$$

(1.1)

where $\text{Lie}_q \{f_1, \ldots, f_k\}$ denotes the evaluation at $q$ of the Lie algebra generated by $f_1, \ldots, f_k$. Given $q \in U$, we say that $\Delta$ has step $s \in \mathbb{N}$ at $q$ if, to recover the equality in (1.1), we need Lie brackets of length $s$ and $s$ is the least integer with this property. We say that $\Delta$ has step $s$ on $U$ if $\Delta$ has step less than or equal to $s$ at every $q \in U$ and $s$ is the least integer with this property.

We fix on $\Delta$ the metric that makes $f_1, \ldots, f_k$ orthonormal. A curve $\gamma \in \text{AC}([0,1]; U)$ is admissible if $\dot{\gamma} \in \Delta$ a.e. on $[0,1]$. In this case, we have

$$\dot{\gamma} = \sum_{i=1}^k u_i f_i(\gamma), \quad \text{a.e. on } [0,1],$$

(1.2)

for some unique vector of functions $u \in L^1([0,1]; \mathbb{R}^k)$, called the control of $\gamma$. The length of $\gamma$ is $\text{length}(\gamma) := \|u\|_{L^1([0,1]; \mathbb{R}^k)}$. Since our considerations are local around a reference curve $\gamma$, in the sequel we will assume $U = M$.

Fix a point $q_0 \in M$ and let $X = L^1([0,1]; \mathbb{R}^k)$. The end-point map is the map $F = F_{q_0} : X \to M$ defined by $F(u) = \gamma(1)$ where $\gamma$ is the unique solution of (1.2) such that $\gamma(0) = q_0$. The curve $\gamma$ is said to be singular (or abnormal) if the corresponding control $u$ is a critical point of the differential $d_u F : X \to T_{F(u)}M$, i.e. if the differential is not surjective. The corank of $u$ is the dimension of $T_{F(u)}M / \text{Im}(d_u F)$. Singular curves do not depend on the metric fixed on $\Delta$ but nonetheless they may be length-minimizers. They do not have a counterpart in Riemannian geometry and do not obey the classical Hamiltonian formalism.

The Sard’s problem investigates the size (dimension, measure, structure) of the set of points of $M$ that are reachable from $q_0$ by singular curves. Even though Sard’s theorem does not hold in infinite-dimensional spaces [3], it is expected that for the end-point map this set is not too big, see [4–6].

Another important problem is the regularity of length-minimizing curves. Montgomery first showed in [7] the existence of smooth strictly singular curves that are in fact length-minimizing. For the notion of strict singularity we defer to definition 4.1. For these curves, however, the first order necessary conditions provided by the Pontryagin maximum principle [8] do not typically give any further regularity beyond the starting one (Lipschitz or AC). Some results on the regularity of singular sub-Riemannian geodesics are in [9–13], see also the surveys [14, 15]. The difficulty of the problem, again, lies in the complicated structure of the end-point map at critical points.

Similar problems are addressed e.g. in [16, 17], where the authors study generic properties of singular trajectories, and in [18–20], where some regularity results are established for the more general class of control systems affine in the control. A different approach towards singular length-minimizing curves can be found in [21–23], where the authors follow the topological viewpoint rather than the differential one, and study singular curves via homotopy theory and results à la Morse. In the case of Carnot groups, singular curves are contained in the zero set of specific polynomials, see [24, 25].

The second order analysis of the end-point map was developed by Agrachev and Sarychev in [26]. This theory provides necessary conditions for strictly singular length-minimizers. These conditions are deduced from second order open mapping theorems that exploit the notion of regular zero together with Morse’s index theory ([27], chapter 20). This is the starting point of our work.
The openness at singular points of a mapping \( F : X \to Y \) between Banach spaces is an active field of research. Some recent results are e.g. in [28, 29] for the second order case, while singularities of higher-order are studied in [30, 31]. These results, however, are not formulated in the Lie-algebraic language that is pursued in this work.

In a first step, in section 2, we prove abstract third order open mapping theorems for functions \( F : X \to M \), where \( X \) is a Banach space and \( M \) a smooth manifold. In definition 2.4, we introduce an intrinsic notion of third differential \( D^3_0F : \text{dom}(D^3_0F) \to \text{coker}(d_0F) \), where \( \text{dom}(D^3_0F) \subset \ker(d_0F) \) is a precise subspace of the kernel of the differential of \( F \) at 0 \( \in X \). Then, we adapt the notion of regular zero to the third differential. For a given isotropic vector of the second differential \( w_0 \in \text{Iso}(D^3_0F) \), in definition 2.7 we introduce the notion of \( w_0 \)-regular zero.

**Theorem 1.1.** Let \( X \) be a Banach space and \( U \subset X \) an open neighborhood of the origin. Let \( F : U \to M \) be a smooth mapping having a critical point at 0 of corank \( h \geq 1 \). Then:

(a) If \( h = 1 \) and there exists \( v \in \text{dom}(D^3_0F) \) such that \( D^3_0F(v) \neq 0 \), then \( F \) is open at the origin.

(b) For any \( h \geq 1 \), if there exist \( w_0 \in \text{Iso}(D^3_0F) \) and \( v_0 \in \text{dom}(D^3_0F) \) such that \( v_0 \) is a \( w_0 \)-regular zero for \( D^3_0F \), then \( F \) is open at the origin.

The first statement is proved in section 2.2, while the latter is shown in section 2.3. The two statements are different in nature: indeed, the first one does not use the notion of regular zero. Also, point (b) can be seen as a more geometric version of the third order open mapping theorem proved by Sussmann in [32]. Its rephrasement in algebraic terms can be found in theorem 2.8. However, this algebraic version is less satisfactory than its second order counterpart, where the notion of index of a quadratic form produces effective conditions on the end-point map. In our case, finding sufficient conditions of the algebraic type ensuring the existence of a regular zero for a vector valued cubic map (polynomials of degree 3) seems a difficult task.

In section 3, we use tools of chronological calculus to compute the third order term in the Taylor expansion of the end-point map, see proposition 3.5. In fact, our procedure is algorithmic and can be used, in principle, to compute also higher order terms. We shall see that, differently from the second order, the representation of the third differential in terms of Lie brackets is not unique. However, the scalarizations onto the cokernel of the first differential are uniquely defined.

Theorem 1.1 and the formula for the third differential of the end-point map yield the following necessary condition satisfied by any adjoint curve of a singular length-minimizing trajectory \( \gamma \) of corank 1. The construction of adjoint curves is recalled in section 4. We denote by \( d_0F \) the differential of the end-point map \( F : L^1([0,1]; \mathbb{R}^3) \to M \) starting from \( \gamma(0) = q_0 \), and computed at the point \( u \in L^1([0,1]; \mathbb{R}^3) \), the control of \( \gamma \).

**Theorem 1.2.** Let \( (M, \Delta, g) \) be a sub-Riemannian manifold with \( \Delta = \text{span}\{f_1, \ldots, f_k\} \) for \( f_1, \ldots, f_k \in \text{Vec}(M) \). Assume that:

(a) \( \gamma : [0,1] \to M \) is a strictly singular length-minimizing curve of corank 1;

(b) the domain \( \text{dom}(D^3_0F) \) is of finite codimension in \( \ker(d_0F) \).

Then any adjoint curve \( \lambda : [0,1] \to T^*M \) satisfies, for every \( t \in [0,1] \) and for every \( i, j, \ell = 1, \ldots, k \),

\[
\langle \lambda(t), [f_i, [f_j, f_{\ell}]](\gamma(t)) + [f_i, [f_j, f_{\ell}]](\gamma(t))\rangle = 0. \tag{1.3}
\]

This result is proved in section 4. Notice the nontrivial assumption (b) on the codimension of the domain of the third differential. Condition (1.3) is the extension to the third order of...
the first and second order necessary conditions for length-minimality. In fact, if \( \gamma \) is a corank-one singular length-minimizing curve with adjoint curve \( \lambda \), then by the Pontryagin maximum principle we have \( \langle \lambda, f_j \rangle = 0 \) identically along the curve, for every \( j = 1, \ldots, k \). If in addition \( \gamma \) is strictly singular, then \( \langle \lambda, [f_i, f_j] \rangle = 0 \) identically along \( \gamma \), for every \( i, j = 1, \ldots, k \). This is known as Goh condition, see [33].

In section 5 we show an application of the general theory to a specific example of singular curves. We recall that a horizontal curve \( \gamma \) is an extremal if it has an adjoint curve \( \lambda \) that satisfies the Pontryagin maximum principle. Any length-minimizing curve is an extremal, but the converse needs not hold. The notion of (strict) singularity applies to extremal curves as well, see definition 4.1.

**Theorem 1.3.** Consider on \( M = \mathbb{R}^3 \) the distribution \( \Delta = \text{span} \{ f_1, f_2 \} \), where

\[
  f_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad f_2 = (1 - x_1) \frac{\partial}{\partial x_2} + x_1^p \frac{\partial}{\partial x_3},
\]

and \( p \in \mathbb{N} \). Fix on \( \Delta \) the metric \( g \) that makes \( f_1 \) and \( f_2 \) orthonormal. Then:

(a) For any \( p \geq 2 \) the curve \( t \mapsto \gamma(t) = (0, t, 0) \) is a strictly singular extremal in \( (\mathbb{R}^3, \Delta) \).

(b) If \( p \) is an even integer then \( \gamma \) is locally length-minimizing in \( (\mathbb{R}^3, \Delta, g) \).

(c) If \( p = 3 \) then \( \gamma \) is not locally length-minimizing in \( (\mathbb{R}^3, \Delta, g) \).

While claims (a) and (b) are implicitly present in the literature, here we focus on claim (c). Using theorem 1.1, or alternatively theorem 1.2, we show that when \( p = 3 \) the end-point map is open at the control of \( \gamma \). For \( p = 5, 7, \ldots \), the curve \( \gamma \) is probably not length-minimizing. To prove this we would need open mapping theorems of order higher than 3.

2. Third order open mapping theorems

2.1. Intrinsic third differential

Let \( (X, \| \cdot \|) \) be a Banach space and let \( U \subset X \) be an open neighborhood of the origin. We consider a smooth mapping \( F : U \to M \), where \( M \) is a smooth manifold of dimension \( m \in \mathbb{N} \). Here and hereafter, by ‘smooth’ we always mean ‘\( C^\infty \)-smooth’.

By fixing a local chart for \( M \) centered at \( F(0) \), we may consider the representative of \( F \) in this chart as a map from \( U \) to \( \mathbb{R}^m \), and accordingly consider its \( k \)th directional derivative \( d^k_0 F : X \to \mathbb{R}^m \)

\[
  d^k_0 F(v) := \left. \frac{d^k}{dt^k} F(tv) \right|_{t=0}, \quad v \in X.
\]

We denote by \( (v_1, \ldots, v_k) \mapsto d^k_0 F(v_1, \ldots, v_k) \) the associated \( k \)-multilinear map. Then we may expand \( F \) as a Taylor series at 0:

\[
  F(v) = \sum_{j=0}^k \frac{1}{j!} d^j_0 F(v) + o(\|v\|^k).
\]

For \( k \geq 2 \), the maps \( d^k_0 F \) do not behave tensorially and depend on the specific choice of the local chart of \( M \).

In ([27], chapter 20), the authors study a chart-independent (or ‘intrinsic’) notion of Hessian, by quotienting out the action of the differential. Recall that 0 is a critical point of
F if the differential $d_0 F : X \to T_{F(0)} M$ is not surjective. The cokernel of $d_0 F$ is the quotient space

$$\text{coker}(d_0 F) = T_{F(0)} M / \text{Im}(d_0 F),$$

and the corank of this critical point is its dimension: $\dim \left( T_{F(0)} M / \text{Im}(d_0 F) \right) = \dim(M) - \dim(\text{Im}(d_0 F))$. The central definition for the theory is the following.

**Definition 2.1.** The intrinsic Hessian of $F$ at $u = 0$ is the quadratic map $D_{0}^2 F : \ker(d_0 F) \to \text{coker}(d_0 F)$ defined by

$$D_{0}^2 F(v) := \pi_{\text{coker}(d_0 F)}(d_{0}^2 F(v)),$$

where $d_{0}^2 F$ is computed with respect to any chart centered at $F(0)$ and $\pi_{\text{coker}(d_0 F)}$ is the projection onto $\text{coker}(d_0 F)$.

This definition is independent of the chosen chart and for any linear form $\lambda \in \text{Im}(d_0 F) \perp = \{ \lambda \in T_{F(0)}^* M | \lambda(d_0 F(x)) = 0 \text{ for all } x \in X \}$, and any vector $v \in \ker(d_0 F)$ there holds

$$\lambda D_{0}^2 F(v) = \mathcal{L}_V \circ \mathcal{L}_V(a \circ F)|_{0},$$

where $a \in C^\infty(M)$ is any function such that $d_0 a = \lambda$, $V \in \text{Vec}(U)$ is any smooth vector field such that $V(0) = v$, and $\mathcal{L}_V$ denotes the Lie derivative along $V$.

We denote by $(v, w) \mapsto D_{0}^2 F(v, w)$ the bilinear form associated with the quadratic map $D_{0}^2 F(v)$.

**Definition 2.2.** A regular zero for the intrinsic Hessian $D_{0}^2 F$ is an element $v \in \ker(d_0 F)$ such that:

(a) $D_{0}^2 F(v) = 0$;

(b) the linear map $w \mapsto D_{0}^2 F(v, w)$ is surjective from $\ker(d_0 F)$ onto $\text{coker}(d_0 F)$.

With these notions, the following theorem holds, see ([27], theorem 20.3).

**Theorem 2.3** (Agrachev–Sarychev). *If the intrinsic Hessian $D_{0}^2 F$ has a regular zero then $F$ is open at the origin.*

Necessary conditions for the existence of a regular zero can be found in [26, 34]. Sufficient conditions are given by the Morse-index theory, see [27]. The existence of a regular zero is only a sufficient condition for the openness of a quadratic form. For example, the map $Q : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $Q(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$ does not have nontrivial zeros and, in particular, it has no regular zeros, but nevertheless it is open.

Our objective is to carry this program over to third-order derivatives and to deduce third-order sufficient conditions for the map $F$ to be open at the origin. We first need to define an ‘intrinsic’ third differential. Let $P : M \to M$ be any diffeomorphism leaving the point $F(0)$ fixed and let $\phi : (-\varepsilon, \varepsilon) \to U$ be a smooth curve such that $\phi(0) = 0$. Let us fix a local chart for $M$
centered at $F(0)$. Here and hereafter, we assume that $F(0) = 0$. Then, locally in this chart, we have

$$\frac{d^3}{d\varepsilon^3}P(F(\varepsilon))) \bigg|_{\varepsilon=0} = d_0 P \left( d_0^3 F(\dot{\phi}) + 3d_0^2 F(\ddot{\phi}, \dot{\phi}) + d_0 F(\dddot{\phi}) \right) + 3d_0^2 P \left( d_0^2 F(\dddot{\phi}) + d_0 F(\ddot{\phi}), d_0 F(\dot{\phi}) \right) + d_0^3 P \left( d_0 F(\dot{\phi}), d_0 F(\dot{\phi}), d_0 F(\dot{\phi}) \right).$$

(2.2)

The third derivative in the left-hand side of (2.2) transforms on $T_0M$ as a tangent vector (i.e. according to the first differential $d_0 F$ only) as soon as $\dot{\phi} \in \ker(d_0 F)$. Moreover, a good definition of the third differential should only depend tensorially on tangent vectors. This means that the third derivative

$$\frac{d^3}{d\varepsilon^3} F(\varepsilon) \bigg|_{\varepsilon=0} = d_0^3 F(\dot{\phi}) + 3d_0^2 F(\ddot{\phi}, \dot{\phi}) + d_0 F(\dddot{\phi})$$

should only depend on $\dot{\phi}$. This happens when $d_0^3 F(\dddot{\phi}, \dot{\phi}) = 0$ modulo the image of $d_0 F$. These considerations motivate the following definition.

**Definition 2.4** (Intrinsic third differential). Let $F : U \to M$ be a smooth map. The domain of the third differential of $F$ at $u = 0$ is:

$$\text{dom}(D^3_0 F) := \{ v \in \ker(d_0 F) | \pi_{\text{coker}(d_0 F)}(d_0^3 F(v, x)) = 0 \quad \text{for all } x \in X \}. \quad (2.3)$$

where $d_0^3 F$ is computed with respect to any chart centered at $F(0)$. The third differential of $F$ at $u = 0$ is the cubic map $D^3_0 F : \text{dom}(D^3_0 F) \to \text{coker}(d_0 F)$ defined by

$$D^3_0 F(v) := \pi_{\text{coker}(d_0 F)} \left( d_0^3 F(v) \right),$$

where $d_0^3 F$ is computed with respect to any chart centered at $F(0)$ and $\pi_{\text{coker}(d_0 F)}$ is the projection onto $\text{coker}(d_0 F)$.

**Remark 2.5.** Similarly to the Hessian, these definitions do not depend on the chosen chart. In particular, while $d_0^3 F$ depends on the chart, the condition $\pi_{\text{coker}(d_0 F)}(d_0^3 F(v, x)) = 0$ for all $x \in X$ is independent of this choice.

To see this, we proceed similarly as in (2.2), and we consider smooth curves $\phi, \psi : (-\varepsilon, \varepsilon) \to U$ such that $\phi(0) = \psi(0) = 0$, $\psi = v \in \ker(d_0 F)$ and $\phi = x \in X$. Also, we consider $P : M \to M$ to be any diffeomorphism fixing $F(0) = 0$ and we fix a local chart around 0. Then, by polarization, it is not difficult to see that

$$\frac{d^2}{d\varepsilon^2} P \left( \frac{F(\phi(\varepsilon) + \psi(\varepsilon)) - F(\phi(\varepsilon) - \psi(\varepsilon))}{4} \right) \bigg|_{\varepsilon=0} = d_0 P \left( d_0^2 F(v, x) - d_0 F \left( \frac{\psi}{2} \right) \right),$$

and our assertion follows.

As for $D^3_0 F$, see (2.1), for every non-zero linear form $\lambda \in \text{Im}(d_0 F)^\ast$ and every vector $v \in \text{dom}(D_0 F)$, there holds

$$\lambda D^3_0 F(v) = L_v \circ L_{\psi} \circ L_{\phi} (\alpha \circ F)\big|_0^\ast.$$
where \( a \in C^\infty(M) \) is any function such that \( d_\partial a = \lambda, V \in \text{Vec}(U) \) is any smooth vector field such that \( V(0) = v \), and \( \mathcal{L}_V \) denotes the Lie derivative along \( V \). Indeed, since by assumption we have \( d_\partial F(v) = 0 \), the following identity holds

\[
\mathcal{L}_V \circ \mathcal{L}_V \circ \mathcal{L}_V(a \circ F)|_0 = d_\partial^3 a \left( d_\partial F(v) \right) + 3 d_\partial^2 a \left( \mathcal{L}^2 \mathcal{L}(d_\partial F(v), d_\partial F(v)) + d_\partial a \left( \mathcal{L}^2 \mathcal{L}(d_\partial F(v)) \right) \right)
= \lambda d_\partial^3 F(v).
\]

In particular, \( \mathcal{L}_V \circ \mathcal{L}_V \circ \mathcal{L}_V(a \circ F)|_0 \) does not depend upon higher order differentials of \( a \) at zero.

### 2.2. Open mapping at corank-one critical points

Assume that \( a = 0 \) is a critical point of \( F \) with corank one, i.e. \( \text{Im}(d_\partial F)^+ \) is one-dimensional and for some non-zero linear form \( \lambda \) we have \( \text{Im}(d_\partial F)^+ = \text{span} \{ \lambda \} \). To prove point (i) in theorem 1.1, we adapt an idea used in ([27], theorem 20.3), which consists in finding a suitable perturbation \( \phi : X \to X \) with \( \phi(0) = 0 \), so that \( F \circ \phi \) is open at 0, thus implying that \( F \) is itself open at 0.

**Proof of Theorem 1.1.** - (i) Since 0 is a corank-one critical point, there exists an \((m - 1)\)-dimensional subspace \( E \subset X \), with \( m = \text{dim}(M) \), such that \( X = E \oplus \ker(d_\partial F) \). Since \( E \) is isomorphic to \( \text{Im}(d_\partial F) \) via \( d_\partial F \), we identify it with \( \mathbb{R}^{m-1} \). Namely, we choose a local chart for \( M \) centered at \( F(0) \) and we endow \( T_{F(0)}M \) with a scalar product so that we may identify \( T_{F(0)}M \) with \( \mathbb{R}^m \) and \( \text{Im}(d_\partial F) \) with \( \mathbb{R}^{m-1} \). We then fix a basis \( (e_i)_{1 \leq i \leq m-1} \) of \( E \).

Let \( v \in \text{dom}(d^3_\partial F) \) be such that \( d^3_\partial F(v) \neq 0 \), and let \( z_0, z_1 \in E \) to be fixed later. We define the map \( \phi : \mathbb{R}^{m-1} \times \mathbb{R} \to X \),

\[
\phi(x, y) := \frac{y^3}{3!} v + \frac{y^6}{6!} z_0 + \frac{y^9}{9!} z_1 + x,
\]

where \( x = (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1} \) is identified with \( (x, 0) \in \mathbb{R}^m \). Notice that \( \phi(0) = 0 \). If the composition \( \Phi := F \circ \phi : \mathbb{R}^{m-1} \times \mathbb{R} \to M \) is open at the origin, then \( F \) is a fortiori open at the origin.

We compute the Taylor expansion at zero of \( \Phi \). The only non-trivially zero partial derivative in \( y \) in this expansion are:

\[
\Phi^{(5)}_y(x, y)|_{y=0} = d_\partial F(v),
\]

\[
\Phi^{(6)}_y(x, y)|_{y=0} = 10 d^2_\partial F(v) + d_\partial F(z_0),
\]

\[
\Phi^{(9)}_y(x, y)|_{y=0} = 280 d^3_\partial F(v) + 84 d^2_\partial F(v, z_0) + d_\partial F(z_1),
\]

where \( \Phi^{(k)}_y \) denotes the \( k \)th partial derivative of \( \Phi \) in \( y \).

The term in the first line is zero since \( v \in \ker(d_\partial F) \). The term in line (2.4) is also zero as soon as we choose \( z_0 \in E \) such that \( d_\partial F(z_0) = -10 d^2_\partial F(v, v) \). This \( z_0 \) does exist because \( v \in \text{dom}(d^3_\partial F) \) implies that \( d^2_\partial F(v, v) \in \text{Im}(d_\partial F) \). Finally, in line (2.5) we can choose \( z_1 \in E \) such that

\[
d_\partial F(z_1) = -84 d^2_\partial F(v, z_0).
\]

This \( z_1 \) does exist because, again, \( v \in \text{dom}(d^3_\partial F) \) implies that \( d^2_\partial F(v, z_0) \in \text{Im}(d_\partial F) \).
Eventually, $\Phi$ has the expansion

$$
\Phi(x, y) = d_0 F(x) + \frac{280}{9!} y^9 d_9 F(v) + R(x, y),
$$

where the remainder $R(x, y)$ satisfies

$$
\lim_{(x, y) \to 0} \frac{R(x, y)}{|x| + |y|^9} = 0. \tag{2.6}
$$

The function $\Psi : \mathbb{R}^{n-1} \times \mathbb{R} \to X$, $\Psi(x, y) = \Phi(x, y^{4/3})$, is the composition of $\Phi$ with a homeomorphism, and thus $\Phi$ is open at the origin if so is $\Psi$. After a linear change of coordinates the openness at the origin of $\Psi$ reduces to the openness of $\hat{\Psi}(x, y) = (x, y) + R(x, y^{4/3})$.

Given $r > 0$, we denote by $B_r \subset \mathbb{R}^m$ the ball of radius $r$ centered at the origin. We show that there exists $\delta_0 > 0$ such that $B_{\delta} \subset B_r$ for all $\delta \in (0, \delta_0)$. In fact, (2.6) implies that there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and for all $(x, y) \in B_\delta$ we have

$$
|R(x, y^{4/3})| \leq \frac{1}{4}(|x| + |y|) \leq \frac{\delta}{2}.
$$

Then, given $\xi \in B_{\delta}/2$ and letting $\chi_\xi(x, y) := \xi + (x, y) - \hat{\Psi}(x, y)$, the triangle inequality implies that $\chi_\xi$ maps $B_\delta$ into itself. It follows by the Brouwer’s fixed point theorem that $\chi_\xi$ has a fixed point in $B_\delta$ for every $\xi \in B_{\delta/2}$, and the openness of $\Psi$ follows. \hfill \Box

### 2.3. Open mapping at critical points of arbitrary corank

We turn to the case of critical points of corank higher than one, and to the proof of point (ii) in theorem 1.1. We begin with adapting to the third order setting the notion of regular zero.

**Definition 2.6.** Let $F : U \subset X \to M$ be a smooth map. The *isotropy space* of $D_0^F$ is

$$
\text{Iso}(D_0^F) := \{ u \in \text{ker}(d_0 F) \mid D_0^F F(u) = 0 \}.
$$

Given an isotropic vector $w_0 \in \text{Iso}(D_0^F)$, we define the *second-order image* of $F$ at $w_0$ as the subspace of $\text{coker}(d_0 F)$

$$
\text{Im}(F, 2, w_0) = \text{Im} \left( D_0^F F(w_0, \cdot) \right).
$$

Finally, we define the *second-order cokernel of $F$ at $w_0$* as the quotient

$$
\text{coker}(F, 2, w_0) = \text{coker}(d_0 F)/\text{Im}(F, 2, w_0).
$$

Note that we have $\text{Im}(F, 2, w_0) = 0$ if and only if $w_0 \in \text{ker}(D_0^F) = \{ w_0 \in \text{ker}(d_0 F) \mid D_0^F F(w_0, \cdot) = 0 \}$.

**Definition 2.7.** Let $w_0 \in \text{Iso}(D_0^F)$. A $w_0$-regular zero for $D_0^F$ is an element $v \in \text{dom}(D_0^F)$ such that:

- (a) $D_0^F F(v) = 0$;
- (b) The linear map $\pi_{\text{coker}(F, 2, w_0)}(D_0^F F(v, \cdot)) : \text{dom}(D_0^F) \to \text{coker}(F, 2, w_0)$ is surjective.
Above, $\pi_{\text{coker}(F,2,w_0)}$ is the projection onto $\text{coker}(F,2,w_0)$.

Proof of theorem 1.1-(ii). We fix on $T_{F|0}M$ a scalar product so that we can regard all the spaces $\text{coker}(d_0F)$, $\text{Im}(D_0^2F(w_0,\cdot))$ and $\text{coker}(F,2,w_0)$ as subspaces of $T_{F|0}M$ with direct sums:

$$T_{F|0}M = \text{Im}(d_0F) \oplus \text{coker}(d_0F),$$

$$\text{coker}(d_0F) = \text{Im}(D_0^2F(w_0,\cdot)) \oplus \text{coker}(F,2,w_0).$$

Let $E_1 \subset X$, $E_2 \subset \text{dom}(D_0^2F) = \ker(d_0F)$ and $E_3 \subset \text{dom}(D_0^3F)$ be linear subspaces such that the following mappings are linear isomorphisms:

$$dF_0 : E_1 \rightarrow \text{Im}(d_0F),$$

$$D_0^2F(w_0,\cdot) : E_2 \rightarrow \text{Im}(D_0^2F(w_0,\cdot)), $$

$$D_0^3F(w_0,\cdot) : E_3 \rightarrow \text{coker}(F,2,w_0).$$

We identify $E_1 = \mathbb{R}^{m_1}$, $E_2 = \mathbb{R}^{m_2}$, and $E_3 = \mathbb{R}^{m_3}$ with $m_1 + m_2 + m_3 = m$ and with coordinates $r \in \mathbb{R}^{m_1}, s \in \mathbb{R}^{m_2}$ and $t \in \mathbb{R}^{m_3}$, $E_1$ and $E_3$ with direct sums such that for $s$ and $t$. We denote by $e_1, \ldots, e_{m_3}$ a basis for $E_1$, and by $e_1, \ldots, e_{m_3}$ a basis for $E_2$.

Let $\nu, \xi, \mu, \eta, \xi, \mu, \xi, \eta, \xi$ and $\xi, \eta$ be points in $E_1$ to be fixed later. For $\varepsilon > 0$ we define the map $\phi_\varepsilon : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \rightarrow X$ by:

$$\phi_\varepsilon(r, s, t) = \frac{\varepsilon^6}{6!} r + \frac{\varepsilon^7}{7!} s + \frac{\varepsilon^8}{8!} w_0 + \frac{\varepsilon^9}{9!} s + \frac{\varepsilon^{10}}{10!} s + \frac{\varepsilon^{11}}{11!} s + \frac{\varepsilon^{12}}{12!} t + \frac{\varepsilon^{13}}{13!} \eta + \frac{\varepsilon^{14}}{14!} \xi + \frac{\varepsilon^{15}}{15!} \eta + \frac{\varepsilon^{16}}{16!} \mu + \frac{\varepsilon^{17}}{17!} \xi + \frac{\varepsilon^{18}}{18!} \eta + \frac{\varepsilon^{19}}{19!} \xi.$$

Then we consider the composition $\Phi_\varepsilon := F \circ \phi_\varepsilon : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \rightarrow M$. To prove that $F$ is open at the origin it is sufficient to show that, for small $\varepsilon > 0$, $\Phi_\varepsilon$ is open at the origin.

We compute the derivatives of $\varepsilon \mapsto \Phi_\varepsilon$ and we evaluate them at $\varepsilon = 0$. We use the shorthand notation $\Phi = \Phi_\varepsilon$ and $\phi = \phi_\varepsilon$. The first non-trivially zero derivative at $\varepsilon = 0$ is the sixth one:

$$\Phi^{(6)} = F[\phi^{(6)}] + O(\varepsilon),$$

that for $\varepsilon = 0$ gives $\Phi^{(6)}(0) = d_0F(v_0) = 0$ because $v_0 \in \text{dom}(D_0^2F) \subset \ker(d_0F)$. For $k = 7, \ldots, 19$ we have

$$\Phi^{(k)} = F[\phi^{(k)}] + \sum_{h=1}^{[k/2]} c_{hk} F'[\phi^{(h)}] \phi^{(k-h)}] + \sum_{1 \leq h,k \leq p \leq k} c_{hlp} F'''[\phi^{(h)}] \phi^{(l)} \phi^{(p)} + O(\varepsilon),$$

where $c_{hk}$ and $c_{hlp}$ are positive integers. For $k = 7, \ldots, 11$ we have $\Phi^{(k)} = F[\phi^{(k)}] + O(\varepsilon)$. The only non-trivially zero cases are $k = 7, 8, 11$, for which we have $\Phi^{(k)}(0) = 0$. Indeed, for $k = 7$ we have $d_0F(t) = 0$ because $s \in E_2 \subset \ker(d_0F)$; for $k = 8$ we have $d_0F(w_0) = 0$ because $w_0 \in \text{Iso}(D_0^2F) \subset \ker(d_0F)$; for $k = 11$ we have $d_0F(x) = 0$ because $t \in E_3 \subset \ker(d_0F).$
For $k = 12, \ldots, 19$ we have the following expansions:

$$
\Phi^{(12)} = F[\phi^{(12)}] + c_{66}F'[\phi^{(6)}, \phi^{(6)}] + O(\varepsilon)
$$

$$
\Phi^{(13)} = F[\phi^{(13)}] + c_{66}F''[\phi^{(6)}, \phi^{(7)}] + O(\varepsilon)
$$

$$
\Phi^{(14)} = F[\phi^{(14)}] + c_{66}F''[\phi^{(6)}, \phi^{(8)}] + c_{77}F'[\phi^{(7)}, \phi^{(7)}] + O(\varepsilon).
$$

$$
\Phi^{(15)} = F[\phi^{(15)}] + c_{66}F''[\phi^{(6)}, \phi^{(9)}] + c_{55}F'[\phi^{(7)}, \phi^{(8)}] + O(\varepsilon)
$$

$$
\Phi^{(16)} = F[\phi^{(16)}] + c_{66}F''[\phi^{(6)}, \phi^{(10)}] + c_{78}F''[\phi^{(7)}, \phi^{(8)}] + O(\varepsilon)
$$

$$
\Phi^{(17)} = F[\phi^{(17)}] + c_{66}F''[\phi^{(6)}, \phi^{(11)}] + c_{78}F''[\phi^{(7)}, \phi^{(10)}] + O(\varepsilon)
$$

$$
\Phi^{(18)} = F[\phi^{(18)}] + \sum_{k=6}^{9} c_{k,18-k}F''[\phi^{(k)}, \phi^{(18-k)}] + c_{66}F''[\phi^{(6)}, \phi^{(6)}, \phi^{(6)}] + O(\varepsilon),
$$

$$
\Phi^{(19)} = F[\phi^{(19)}] + \sum_{k=6}^{9} c_{k,19-k}F''[\phi^{(k)}, \phi^{(19-k)}] + c_{66}F''[\phi^{(6)}, \phi^{(6)}, \phi^{(7)}] + O(\varepsilon).
$$

The third differential $F'''$ becomes relevant for $k = 18, 19$.

The equations $\Phi^{(i)}(0) = 0$ lead to the following list of conditions:

$$
d_{0}F(\nu) + c_{66}d_{0}^{2}F(v_{0}, v_{0}) = 0,
$$

(2.7)

$$
d_{0}F(\eta_{i}) + c_{66}d_{0}^{2}F(v_{0}, e_{i}) = 0, \quad i = 1, \ldots, m_{3},
$$

(2.8)

$$
d_{0}F(\xi_{i}) + c_{55}d_{0}^{2}F(v_{0}, v_{0}) = 0,
$$

(2.9)

$$
d_{0}F(\eta_{i}) + c_{78}d_{0}^{2}F(e_{i}, e_{i}) = 0, \quad i, j = 1, \ldots, m_{3},
$$

(2.10)

$$
d_{0}F(\xi_{i}) + c_{78}d_{0}^{2}F(e_{i}, v_{0}) = 0, \quad i = 1, \ldots, m_{3},
$$

(2.11)

$$
d_{0}F(\nu) + c_{55}d_{0}^{2}F(w_{0}, v_{0}) = 0,
$$

(2.12)

$$
d_{0}F(\ell) + c_{66}d_{0}^{2}F(v_{0}, e_{\ell}) = 0, \quad \ell = 1, \ldots, m_{2},
$$

(2.13)

$$
d_{0}F(\zeta_{\ell}) + c_{66}d_{0}^{2}F(v_{0}, v_{0}, v_{0}) = 0,
$$

(2.14)

$$
d_{0}F(\zeta_{\ell}) + c_{55}d_{0}^{2}F(e_{\ell}, e_{\ell}) = 0.
$$

(2.15)

Both (2.9) and (2.10) origin from $\Phi^{(14)}(0) = 0$. Both (2.14) and (2.15) origin from $\Phi^{(18)}(0) = 0$.

Equation (2.7) has a solution $\nu \in E_{1}$ because the vector $v_{0} \in \text{dom}(D_{0}^{3}F)$ satisfies $D_{0}^{3}F(v_{0}) = 0$. Equation (2.8) has a solution $\eta_{i} \in E_{1}$ because, again, the points $d_{0}^{2}F(v_{0}, e_{i})$ are in the image of the differential. For the same reason, there exist solutions $\zeta\ell, \eta_{i}, \zeta\ell, \mu \in E_{1}$ of (2.9)–(2.12).

We study (2.13). Since $v_{0} \in \text{dom}(D_{0}^{3}F)$ we have $\pi_{\text{coker}(d_{0}F)}(d_{0}^{3}F(v_{0}, x)) = 0$ for all $x \in X$. Then $d_{0}F(\nu, e_{\ell})$ also belongs to the image of the differential and so there exists a solution $\zeta_{\ell} \in E_{1}$ to (2.13).

Combining the facts that $v_{0} \in \text{dom}(D_{0}^{3}F)$, and $D_{0}^{3}F(v_{0}) = 0$ by assumption, we can fix $\zeta \in E_{1}$ solving (2.14), since then

$$
c_{6,12}d_{0}^{3}F(v_{0}, \nu) + c_{66}d_{0}^{3}F(v_{0}, v_{0}, v_{0}) \in \text{Im}(d_{0}F).\]
solving (2.15).

For the same reason, i.e. since \( e_\ell \in \text{dom}(D_0^3F) \) for \( \ell = 1, \ldots, m_3 \), we can also pick \( \zeta_{\ell} \in E_1 \) solving (2.15).

Finally, we require that \( u_0 \in E_1 \) solves the equation

\[
d_0 F(u_0) + c_{0,13} d_{0}^2 F(v_0, \eta_1) + c_{7,12} d_{0}^2 F(e, \nu) = 0.
\]

In this way we have \( \Phi^{(9)}(0) = d_0 F(r) + c_{8,11} d_{0}^2 F(w_0, s) + c_{667} d_{0}^3 F(v_0, v_0, t) \), so that the map \( \Phi \) has the following expansion

\[
\Phi_x(r, s, t) = \varepsilon^{19} \left( d_0 F(r) + c_{8,11} d_{0}^2 F(w_0, s) + c_{667} d_{0}^3 F(v_0, v_0, t) \right) + O(\varepsilon^{20}),
\]

with \( c_{8,11} \neq 0 \) and \( c_{667} \neq 0 \). It follows that the map \( \Psi : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \times \mathbb{R} \rightarrow M \).

\[
\Psi(r, s, t, \varepsilon) = \varepsilon^{-19} \Phi_x(r, s, t)
\]
is of class \( C^1 \), with \( \Psi(0) = 0 \) and such that the Jacobian \( J_{(r,s,t)} \Psi(0) \) is surjective onto \( T_0 M \).

By the implicit function theorem, there exists \( \varepsilon_0 > 0 \) and \( C^1 \)-functions \( (r, s, t) : (\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \) such that, for every \( \varepsilon \in (\varepsilon_0, \varepsilon_0) \)

(a) \( \Psi(r(\varepsilon), s(\varepsilon), t(\varepsilon); \varepsilon) = 0 \), and

(b) \( J_{(r,s,t)} \Psi(r(\varepsilon), s(\varepsilon), t(\varepsilon); \varepsilon) \) is surjective onto \( T_0 M \).

This proves that \( \Psi \) is open at the origin for small \( \varepsilon > 0 \), and eventually that \( F \) is open at the origin.

Theorem 1.1-(ii) reduces the open mapping property for \( F \) at 0 to the existence of \( u_0 \in \text{Iso}(D_0^3F) \) such that the third differential

\[
D_0^3F : \text{dom}(D_0^3F) \rightarrow \text{coker}(F, 2, u_0)
\]

admits a \( u_0 \)-regular zero, and since the manifold \( M \) is finite-dimensional, it is enough to consider the case when the source space is finite-dimensional.

Let us recall some facts about cubic maps. Given a cubic map \( P : \mathbb{R}^n \rightarrow \mathbb{R}^n \), for integers \( n \) and \( N \), we denote by \( T : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) the trilinear map associated with \( P \). Then there hold the following facts:

(a) For \( v \in \mathbb{R}^N \), the differential \( d_v P : \mathbb{R}^N \rightarrow \mathbb{R}^n \) is the linear mapping given by \( d_v P(x) = 3T(v, v, x) \), for \( x \in \mathbb{R}^N \).

(b) For \( v \in \mathbb{R}^N \), the second differential \( d^2_v P : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^n \) is the vector-valued symmetric bilinear form given by \( d^2_v P(x, y) = 6T(v, x, y) \), for \( x, y \in \mathbb{R}^N \).

(c) The third differential \( d^3 P : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^n \) is the vector-valued symmetric trilinear form given by \( d^3 P(x, y, z) = 6T(x, y, z) \), for \( x, y, z \in \mathbb{R}^N \).

(d) The third differential defines the linear map \( L : \mathbb{R}^N \rightarrow \text{Sym}(\mathbb{R}, N)^n \) into the space of \( n \)-tuples of \( N \times N \) symmetric matrices given by

\[
L(x) = d^3 P(x, \cdot, \cdot) = 6T(x, \cdot, \cdot), \quad \text{for } x \in \mathbb{R}^N.
\]

We clearly have the identity \( L(x) = d^3 P \) as vector-valued symmetric bilinear maps, for every \( x \in \mathbb{R}^N \).
Theorem 2.8. Let $P : \mathbb{R}^N \to \mathbb{R}^n$ be a cubic map and assume that:

(a) $N \geq n + 1$;

(b) if $e_1, \ldots, e_N$ denotes the canonical basis of $\mathbb{R}^N$, for every non-zero $\lambda \in (\mathbb{R}^n)^*$ the quadratic forms $Q^\lambda_i : \mathbb{R}^N \to \mathbb{R}$, for $i = 1, \ldots, N$,

$$Q^\lambda_i(x) = \lambda d^3 P(e_i, x, x)$$

do not have common isotropic vectors $x \neq 0$.

Then $F$ is open at the origin.

Proof. Since $N \geq n + 1$ and $P$ is a cubic map, $P$ has a non-trivial zero $v \in \mathbb{R}^N$ by the Bézout theorem [see, e.g. ([35], theorem 1, chapter IV §2)]. We claim that this zero is regular.

Suppose by contradiction that $v$ is not regular, i.e. there exists a non-zero $\lambda \in (\mathbb{R}^n)^*$ such that

$$\lambda d^3 P(x) = 3\lambda T(v, v, x) = 0, \quad \text{for } x \in \mathbb{R}^N.$$ 

Denoting by $\langle \cdot, \cdot \rangle$ the scalar product on $\mathbb{R}^N$, we recall the identity (compare with (2.16))

$$\lambda T(u, v, w) = \langle u, \lambda L(v)w \rangle, \quad \text{for } u, v, w \in \mathbb{R}^N. \quad (2.17)$$

Cycling the variables in (2.17), we deduce that $\lambda T(v, x, v) = \langle v, \lambda L(x)v \rangle = 0$ for every $x \in \mathbb{R}^N$, i.e. $v$ is a common isotropic vector for the quadratic forms $L(x)$ as $x$ varies in $\mathbb{R}^N$, which contradicts (ii).

Remark 2.9. In the case of scalar cubic maps, that is $P : \mathbb{R}^N \to \mathbb{R}$, theorem 2.8 can be made more precise. Indeed, if $N \geq 2$, one can prove that the following are equivalent:

(a) $P$ has a regular zero.

(b) $P$ is not a perfect cube.

(c) The linear map $L : \mathbb{R}^N \to \text{Sym}(\mathbb{R}, N)$ is of rank strictly greater than one.

We go back to the case of a smooth map $F : X \to M$.

Corollary 2.10. Let $X$ be a Banach space, $U \subset X$ a neighborhood of $0 \in X$, $M$ a smooth manifold, and $F : U \to M$ a smooth mapping. Assume that there exists $w_0 \in \text{Is}(D^1_0 F)$ such that:

(a) $\dim(\text{dom}(D^3_0 F)) + \dim(\text{Im}(d_0 F)) + \dim(\text{Im}(F, 2, w_0)) > \dim(M)$.

(b) For every non-zero $\lambda \in \text{Im}(F, 2, w_0)^*$ and $x \in \text{dom}(D^3_0 F)$ there exists $v \in \text{dom}(D^3_0 F)$ such that $\lambda D^3_0 F(v, x, x) \neq 0$.

Then $F$ is open at the origin.

Proof. We assume without loss of generality that $\dim(\text{Im}(F, 2, w_0)) + \dim(\text{Im}(d_0 F)) < \dim(M)$ for every $w_0 \in \text{Is}(D^1_0 F)$. Indeed, if there exists $w_0 \in \text{Is}(D^1_0 F)$ such that

$$\dim(\text{Im}(F, 2, w_0)) + \dim(\text{Im}(d_0 F)) = \dim(M)$$

then $F$ is open at the origin by theorem 2.3.

We apply theorem 2.8 to the cubic map $P := D^3_0 F : \text{dom}(D^3_0 F) \to \text{coker}(\text{Im}(F, 2, w_0))$. Letting $N := \dim(\text{dom}(D^3_0 F))$ and
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\[ n := \dim(\text{coker}(\text{Im}(F, 2, w_0))) = \dim(M) - \dim(\text{Im}(d_0 F)) - \dim(\text{Im}(F, 2, w_0)), \]

assumption (a) reads \( N \geq n + 1 \), that is (i) in theorem 2.8.

For every nonzero \( \lambda \in \text{coker}(F, 2, w_0)^* = \text{Im}(F, 2, w_0)^\perp \), the quadratic forms \( Q_\lambda(x) = \lambda D_3^0 F(v, x, x) \) do not have, for varying \( v \in \text{dom}(D_3^0 F) \), a common isotropic vector \( x \in \text{dom}(D_3^0 F) \), by (b). So also assumption (ii) in theorem 2.8 is satisfied.

It follows that \( P = D_3^0 F \) has a \( w_0 \)-regular zero, in the sense of definition 2.7. By theorem 1.1, the map \( F \) is open at the origin. \( \square \)

**Remark 2.11.** The conclusions of corollary 2.10 are unsatisfactory because they are not easily exploitable in the study of the end-point map, in particular at critical points of corank higher than one.

While in the second order analysis the Morse theory provides, via the algebraic notion of index, effective sufficient conditions ensuring the open mapping property, in the third order case we lack a solid algebraic theory describing the invariants of symmetric tensors of order 3, where not even the concepts of rank and symmetric rank necessarily coincide [36], and the diagonalization process is not clear. This makes it difficult to find effective conditions ensuring the existence of regular zeros for cubic maps.

### 3. Third order analysis of the end-point map

In this section we expand the end-point map and we compute the precise structure of its third order term. The computations use the language of chronological calculus for non-autonomous vector fields. The reader can find a detailed introduction to this formalism in ([27], chapter 2). Here, we recall only some basic notation.

Let \( M \) be a smooth manifold and let \( V = (V_t)_{t \in [0, 1]} \) be a time-dependent vector field, that is a map \( M \times [0, 1] \to TM \) so that \( V(q, t) = V_t(q) \in T_q M \) for every \( t \in [0, 1] \). The flow of \( V \) is the map \( P : M \times [0, 1] \times [0, 1] \to M \), \( P(q_0, t_0, t) = P_{t_0}^t(q_0) \), given by evaluating at time \( t \) the solution to the Cauchy problem \( \dot{q}(\tau) = V_t(q(\tau)) \) and \( q(t_0) = q_0 \). The flow \( P_{t_0}^t \) is also called the right chronological exponential of \( V \) and denoted by

\[ \exp \int_{t_0}^t V_\tau \, d\tau := P_{t_0}^t. \] (3.1)

The flow can be formally expanded in the following Volterra series:

\[ P_{t_0}^t = \text{Id} + \sum_{k=1}^{\infty} \int_{\Sigma_3(t_0, t)} V_{\tau_3} \circ \cdots \circ V_{\tau_1} \, d\tau_3 \cdots d\tau_1, \quad t \geq t_0, \]

where \( \Sigma_3(t_0, t) := \{ (\tau_1, \ldots, \tau_3) \in \mathbb{R}^3 | t_0 \leq \tau_2 \leq \cdots \leq \tau_3 \leq t \} \). We agree that \( \Sigma_3(0, t) = \Sigma_3(t) \).

Here and hereafter, we denote by \( \circ \) the composition of operators on \( C^\infty(M) \). When \( t \leq t_0 \) there are analogous formulas. The inverse of \( P_{t_0}^t \), that is the mapping such that \( P_{t_0}^t \circ Q_{t_0}^t = \text{Id} \), is called left chronological exponential and denoted by

\[ \exp \int_{t_0}^t (-V_\tau) \, d\tau := Q_{t_0}^t. \]
3.1. Expansion of the end-point map

Let $M$ be a smooth manifold and let $f_1, \ldots, f_k \in \text{Vec}(M)$ be smooth vector fields on $M$. Given $u \in L^1([0,1]; \mathbb{R}^k)$ we will use the short-hand notation $f_{a(t)} := \sum_{i=1}^k u_i(t) f_i$. Note that $f_a$ is a time-dependent vector field as in the previous subsection.

**Definition 3.1.** The **end-point map** relative to the vector fields $f_1, \ldots, f_k$ is the map $F : M \times L^1([0,1]; \mathbb{R}^k) \to M$ given by

$$F(q_0, u) := F_{q_0}(u) := q_0 \circ \exp \int_0^1 f_{a(t)} \, dt.$$  

We are assuming that the Cauchy problem for $f_{a(t)}$ has a solution defined on the whole interval $[0,1]$. We perform a perturbation analysis of the end-point map with respect to the control variable. To this aim, by the variation formula in (\cite{27}, §2.7, (2.28)) we have:

$$F_{q_0}(u + v) = q_0 \circ \exp \int_0^1 (f_{a(t)} + f_{a(t)}) \, dt = F_{q_0}(u) \circ \exp \int_0^1 \text{Ad} \left( \exp \int t^1 f_{a(t)} \, dt \right) f_{a(t)} \, dt.$$  

Recall that for a vector field $V \in \text{Vec}(M)$ and for a diffeomorphism $B$ of $M$, the adjoint operator $(\text{Ad}B)V$ is defined by the formula $(\text{Ad}B)V := B \circ V \circ B^{-1}$.

This motivates the following definition.

**Definition 3.2.** The **perturbation map** relative to the vector fields $f_1, \ldots, f_k$ is the map $G : L^1([0,1]; \mathbb{R}^k) \times L^1([0,1]; \mathbb{R}^k) \times M \to M$ given by

$$G(u, v, q_1) := G_{q_1}^u(v) = q_1 \circ \exp \int_0^1 \text{Ad} \left( \exp \int t^1 f_{a(t)} \, dt \right) f_{a(t)} \, dt.$$  

The term ‘perturbation’ is of course motivated by the fact that, by the variation formula, there holds:

$$G(u, v, F_{q_0}(u)) = F_{q_0}(u + v),$$

so when $q_1 = F_{q_0}(u)$ and $v$ is small, $G_{F_{q_0}(u)}(v)$ is a small perturbation of $F_{q_0}(u)$. For $t \in [0,1]$, we define the time-dependent vector field

$$g_{a(t)} := \exp \left( \exp \int t^1 f_{a(t)} \, dt \right) \, f_{a(t)}.$$  

As an operator on $C^\infty(M)$, $G_{q_1}^u(v)$ admits the formal expansion:

$$G_{q_1}^u(v) := q_1 \circ \exp \int_0^1 g_{a(t)} \, dt = q_1 \circ \left( \text{Id} + \sum_{k=1}^\infty \int_{\Sigma k} g_{a(t_2)} \circ \cdots \circ g_{a(t_1)} \, d\tau_k \cdots d\tau_1 \right).$$  

Replacing $v$ by $\varepsilon v$ in (3.4) and dropping the dependence on $q_1$, we introduce the family of diffeomorphisms depending on the parameter $\varepsilon > 0$:

$$G^\varepsilon(v) := \text{Id} + \sum_{k=1}^\infty \varepsilon^k \int_{\Sigma k} g_{a(t_2)} \circ \cdots \circ g_{a(t_1)} \, d\tau_k \cdots d\tau_1.$$  

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Now we compute a different expansion for \( G^\varepsilon(v; \varepsilon) \), where the role of the Lie-brackets of \( f_1, \ldots, f_k \) is more transparent. We can compute the derivative in \( \varepsilon \) of \( G(v; \varepsilon) \) using ([27], §2.8, (2.31)):

\[
\frac{\partial}{\partial \varepsilon} G^\varepsilon(v; \varepsilon) = W(v; \varepsilon) \circ G^\varepsilon(v; \varepsilon),
\]

where the vector field \( W(v; \varepsilon) \) is given by the formula

\[
W(v; \varepsilon) = \int_0^1 \text{Ad} \left( \exp \int_0^\tau \varepsilon g_{\varepsilon(t)}^\tau \, d\tau \right) g_{\varepsilon(t)}^\tau \, d\tau = \int_0^1 \left( \exp \int_0^\tau \varepsilon g_{\varepsilon(t)}^\tau \, d\tau \right) g_{\varepsilon(t)}^\tau \, d\tau.
\]

For the definition of the integral \( \int_0^1 V, d\tau \) of a non-autonomous vector field \( t \mapsto V_t \), we refer to ([27], §2.3).

Comparing (3.6) with the Volterra expansion of the left chronological exponential we deduce that:

\[
G^\varepsilon(v; \varepsilon) = \exp \int_0^\varepsilon W(v; t) \, dt = \text{Id} + \sum_{n=1}^{\infty} \int_{\Sigma_n(t)} W(v; \eta_1) \circ \cdots \circ W(v; \eta_{n-1}) \, d\eta_n \cdots d\eta_1.
\]

Thus the formal series in (3.5) and (3.8) coincide for every \( \varepsilon > 0 \). From formula (3.8) we deduce the following expansion for \( G^\varepsilon(v) \) as an operator on \( C^\infty(M) \).

**Lemma 3.3.** For every \( v \in L^1([0, 1]; \mathbb{R}^3) \) we have:

\[
G^\varepsilon(v) = \text{Id} + d_0 G^\varepsilon(v) + \frac{1}{2} d_0^2 G^\varepsilon(v) + \frac{1}{6} d_0^3 G^\varepsilon(v) + O(\|v\|_{L^1([0, 1]; \mathbb{R}^3)}^3),
\]

where

\[
d_0 G^\varepsilon(v) = \int_0^1 g_{\varepsilon(t0)}^\tau \, d\tau,
\]

\[
d_0^2 G^\varepsilon(v) = \int \sum_\Sigma [g_{\varepsilon(t0)}^{\tau_2}, g_{\varepsilon(t1)}^{\tau_1}] \, d\tau_2 d\tau_1 + \left( \int_0^1 g_{\varepsilon(t0)}^\tau \, d\tau \right) \circ \left( \int_0^1 g_{\varepsilon(t0)}^\tau \, d\tau \right),
\]

\[
d_0^3 G^\varepsilon(v) = 2 \int \sum_\Sigma [g_{\varepsilon(t0)}^{\tau_3}, [g_{\varepsilon(t2)}^{\tau_2}, g_{\varepsilon(t1)}^{\tau_1}]] \, d\tau_2 d\tau_1
\]

\[
+ 2 \left( \int \sum_\Sigma [g_{\varepsilon(t0)}^{\tau_3}, g_{\varepsilon(t1)}^{\tau_1}] \, d\tau_2 d\tau_1 \right) \circ \left( \int_0^1 g_{\varepsilon(t0)}^\tau \, d\tau \right)
\]

\[
+ \left( \int_0^1 g_{\varepsilon(t0)}^\tau \, d\tau \right) \circ \left( \int \sum_\Sigma [g_{\varepsilon(t2)}^{\tau_2}, g_{\varepsilon(t1)}^{\tau_1}] \, d\tau_2 d\tau_1 \right)
\]

\[
+ \left( \int_0^1 g_{\varepsilon(t0)}^\tau \, d\tau \right) \circ \left( \int_0^1 g_{\varepsilon(t0)}^\tau \, d\tau \right) \circ \left( \int_0^1 g_{\varepsilon(t0)}^\tau \, d\tau \right).
\]
Proposition 3.5. For any nonlinearity $f$, then using the formulas in (3.8), we get

$$W_1(v) = \int_0^1 g_0^\mu dt \quad \text{and} \quad W_k(v) = \int \sum_{h=1}^\infty \frac{\varepsilon}{h} W_h(v), \quad k \geq 2.$$  

We then compute the first three terms of the sum in (3.8):

$$\int_{\Sigma_1(v)} W(v; \eta_1) d\eta_1 = \sum_{h=1}^{\infty} \frac{e^h}{h} W_h(v),$$

$$\int_{\Sigma_2(v)} W(v; \eta_1) \circ W(v; \eta_2) d\eta_1 = \sum_{h,k=1}^{\infty} \frac{e^{h+k}}{(h+k)k} W_h(v) \circ W_k(v),$$

$$\int_{\Sigma_3(v)} W(v; \eta_1) \circ W(v; \eta_2) \circ W(v; \eta_3) d\eta_1 = \sum_{h,k,l=1}^{\infty} \frac{e^{h+k+l}}{(h+k+l)(k+l)\ell}.$$  

Then using these formulas in (3.8), we get

$$G^\mu(v) = \text{Id} + \varepsilon W_1(v) + \frac{\varepsilon^2}{2} (W_2(v) + W_1(v) \circ W_1(v)) + \frac{\varepsilon^3}{3} (W_3(v) + W_2(v) \circ W_1(v))$$

$$+ \frac{\varepsilon^3}{6} (W_1(v) \circ W_2(v) + W_1(v) \circ W_1(v) \circ W_1(v)) + O(\varepsilon^4 \|v\|_{L^1(0,1;\mathbb{R}^3)}^4),$$

where the estimate on the remainder follows from remark 3.4 below. From this formula, we can compute the directional derivatives

$$d_0 G^\mu(v) = \left. \frac{d}{d \varepsilon} G^\mu(v) \right|_{\varepsilon=0}, \quad d_0^2 G^\mu(v) = \left. \frac{d^2}{d \varepsilon^2} G^\mu(v) \right|_{\varepsilon=0}, \quad d_0^3 G^\mu(v) = \left. \frac{d^3}{d \varepsilon^3} G^\mu(v) \right|_{\varepsilon=0},$$

obtaining formulas (3.10)–(3.12). \qed

Remark 3.4. Even if lemma 3.3 is enough for our purposes, the computation’s method in the proof is algorithmic and permits to determine the terms of any order in the expansion of $G^\mu(v)$. Indeed, for $k \geq 1$ we have the formal identity

$$\int_{\Sigma_k(v)} W(v; \eta_1) \circ \cdots \circ W(v; \eta_k) d\eta_1 \cdots d\eta_k = \sum_{h_1,\ldots,h_k=1}^{\infty} \frac{\varepsilon^{h_1+\cdots+h_k}}{(h_1 + \cdots + h_k) \cdots (h_{k-1} + h_k) h_k} W_{h_1}(v) \circ \cdots \circ W_{h_k}(v).$$

As consequence of lemma 3.3, we obtain an explicit formula for the intrinsic third differential of $G^\mu_{q_1}$ (recall definition 2.4).

Proposition 3.5. For any $v \in \text{dom}(d_0^3 G^\mu_{q_1})$ and $\lambda \in \text{Im}(d_0 G^\mu_{q_1})$ we have:

$$\lambda d_0^3 G^\mu_{q_1}(v) = 2 \int_{\Sigma_3} \left[ \lambda, [g_{\theta(\eta_1)}, [g_{\theta(\eta_2)}, g_{\theta(\eta_3)}]](q_1) \right] d\tau_3 d\tau_2 d\tau_1.$$  

(3.14)
Proof. Let \( v \in \text{dom}(D_0^3 G_{q_1}^{s}) \) and \( a \in C^\infty(M) \) be such that \( a(q_1) = 0 \) and \( d_{q_1} a = \lambda \). Since \( \text{dom}(D_0^3 G_{q_1}^{s}) \subset \ker(d_0 G_{q_1}^{s}) \), we deduce that

\[
d_0 G_{q_1}^{s}(v) = q_1 \circ \int_0^1 s_{v(q_1)}^{\mu_3} \, d\tau_1 = 0. \tag{3.15}
\]

By the definition of the third differential and by a computation similar to (2.2) we have

\[
\frac{d^3}{d\varepsilon^3} a(G_{q_1}^s(v\varepsilon)) \bigg|_{\varepsilon=0} = \lambda D_0^3 G_{q_1}^{s}(v)
\]

where we used (3.15) to prove that the terms involving second and third order derivatives of \( a \) are zero. Moreover, as \( v \in \text{dom}(D_0^3 G_{q_1}^{s}) \) we also have

\[
\frac{d^2}{d\varepsilon^2} a(G_{q_1}^s(v\varepsilon)) \bigg|_{\varepsilon=0} = \lambda D_0^2 G_{q_1}^{s}(v) = 0.
\]

Returning to the chronological notation, we have to expand to the third order the expression

\[
(G_{q_1}^s(v))a = \left( q_1 \circ \exp \int_0^1 s_{v(t)}^{\mu_3} \, dt \right) a.
\]

Comparing (3.9) with the expansion provided in (3.4), we have to calculate:

\[
2 \left( q_1 \circ \int_\Sigma_2 s_{v(q_1)}^{\mu_2} \circ g_{v(q_1)}^{\mu_1} \, d\tau_2 \, d\tau_1 \right) a, \quad \text{and}
\]

\[
6 \left( q_1 \circ \int_\Sigma_3 s_{v(q_1)}^{\mu_2} \circ g_{v(q_1)}^{\mu_1} \circ g_{v(q_1)}^{\mu_1} \, d\tau_3 \, d\tau_2 \, d\tau_1 \right) a. \tag{3.16}
\]

From formula (3.11) in lemma 3.3 we obtain

\[
(d_0^2 G_{q_1}^{s}(v)) a = 2 \left( q_1 \circ \int_\Sigma_2 s_{v(q_1)}^{\mu_2} \circ g_{v(q_1)}^{\mu_1} \, d\tau_2 \, d\tau_1 \right) a
\]

\[
+ 6 \left( q_1 \circ \int_\Sigma_3 s_{v(q_1)}^{\mu_2} \circ g_{v(q_1)}^{\mu_1} \circ g_{v(q_1)}^{\mu_1} \, d\tau_3 \, d\tau_2 \, d\tau_1 \right) a
\]

\[
= \int_\Sigma_2 \left< \lambda, [s_{v(q_1)}^{\mu_2}, g_{v(q_1)}^{\mu_1}] (q_1) \right> \, d\tau_2 \, d\tau_1 + d_{q_1} a \left( \int_0^1 s_{v(t)}^{\mu_3}(q_1) \, dt, \int_0^1 g_{v(t)}^{\mu_3}(q_1) \, dt \right) = 0.
\]

Indeed, since \( v \in \text{dom}(D_0^3 G_{q_1}^{s}) \), the second term is zero by (3.15), i.e. because \( \int_0^1 g_{v(t)}^{\mu_3}(q_1) \, dt = 0 \), while we deduce from \( d_0^2 G_{q_1}^{s}(v) \in \text{Im}(d_0 G_{q_1}^{s}) \) that

\[
\frac{1}{2} \int_\Sigma_2 [s_{v(q_1)}^{\mu_2}, g_{v(q_1)}^{\mu_1}] (q_1) \, d\tau_2 \, d\tau_1 \in \text{Im}(d_0 G_{q_1}^{s}),
\]

so that the dual product with \( \lambda \in \text{Im}(d_0 G_{q_1}^{s}) \) cancels also the first one.

By (3.12) and (3.15), for the last term in (3.16) we similarly obtain the identity

\[
(d_0^3 G_{q_1}^{s}(v)) a = 6 \left( q_1 \circ \int_\Sigma_3 s_{v(q_1)}^{\mu_2} \circ g_{v(q_1)}^{\mu_1} \circ g_{v(q_1)}^{\mu_1} \, d\tau_3 \, d\tau_2 \, d\tau_1 \right) a
\]

\[
= 2 \int_\Sigma_3 \left< \lambda, [s_{v(q_1)}^{\mu_3}, g_{v(q_1)}^{\mu_2}, g_{v(q_1)}^{\mu_1}] (q_1) \right> \, d\tau_3 \, d\tau_2 \, d\tau_1;
\]

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indeed in all the other terms of this expansion, at least one of the arguments of either the second or the third differential of \(a\) at 0 is \(\int_0^1 g_{\varepsilon(t)}^{\mu}(q_1)\,dt = 0\). The thesis follows.

**Remark 3.6.** The representation formula (3.14) for \(\mathcal{D}_\varepsilon G^\mu_q(v)\) in terms of Lie brackets is not unique, and a different representation can be obtained in the following way. If we compute the derivative of \(\varepsilon \mapsto G^\mu(v\varepsilon)\) according to ([27], §2.8, (2.32)), we find

\[
\frac{\partial}{\partial \varepsilon} G^\mu(v\varepsilon) = G^\mu(v\varepsilon) \circ \tilde{\mathcal{W}}(v; \varepsilon),
\]

where

\[
\tilde{\mathcal{W}}(v; \varepsilon) := \int_0^1 \left( \exp \int_1^\varepsilon \text{ad} \, g_{\varepsilon(t)}^{\mu} \, dt \right) g_{\varepsilon(t)}^{\mu} \, dt.
\]

Note that the composition order of \(G^\mu(v\varepsilon)\) and \(\tilde{\mathcal{W}}(v; \varepsilon)\) in (3.17) is reversed compared to (3.6). Since we have

\[
\exp \int_1^\varepsilon \text{ad} \, g_{\varepsilon(t)}^{\mu} \, dt = \text{Ad} \left( \exp \int_1^\varepsilon g_{\varepsilon(t)}^{\mu} \, dt \right) = \text{Ad} \left( -\exp \int_1^\varepsilon g_{\varepsilon(t)}^{\mu} \, dt \right) = \exp \int_1^\varepsilon -\text{ad} \, g_{\varepsilon(t)}^{\mu} \, dt,
\]

the expansion in Volterra series of \(\tilde{\mathcal{W}}(v; \varepsilon)\) is

\[
\tilde{\mathcal{W}}(v; \varepsilon) = \int_0^1 g_{\varepsilon(t)}^{\mu} \, dt + \sum_{k=2}^{\infty} (-\varepsilon)^{k-1} \int_{\Sigma_k} \text{ad} g_{\varepsilon(t_1)}^{\mu_1} \circ \cdots \circ \text{ad} g_{\varepsilon(t_{k-1})}^{\mu_{k-1}} (g_{\varepsilon(t_k)}^{\mu_k}) \, dt_k \cdots \, dt_1.
\]

In (3.18), the order of the vector fields in the commutator is reversed with respect to (3.13). Our computation also yields the identity

\[
\int_{\Sigma_1} [g_{\varepsilon(t_1)}^{\mu_1}, [g_{\varepsilon(t_2)}^{\mu_2}, g_{\varepsilon(t_1)}^{\mu_1}]] \, dt_1 \, dt_2 \, dt_1 = \int_{\Sigma_3} [g_{\varepsilon(t_1)}^{\mu_1}, [g_{\varepsilon(t_2)}^{\mu_2}, g_{\varepsilon(t_1)}^{\mu_1}]] \, dt_1 \, dt_2 \, dt_1 + \frac{1}{2} \int_{\Sigma_2} \int_0^1 g_{\varepsilon(t)}^{\mu} \, dt \left[ \int_{\Sigma_2} [g_{\varepsilon(t_2)}^{\mu_2}, g_{\varepsilon(t_1)}^{\mu_1}] \, dt_2 \, dt_1 \right],
\]

thanks to which we may obtain another expression for \(\lambda \mathcal{D}_\varepsilon G^\mu_q(v)\).

Even though the representation for the third differential is not unique, for any \(v \in \text{dom}(\mathcal{D}_\varepsilon G^\mu_q)\) and \(\lambda \in \text{Im}(\mathcal{D}_\varepsilon G^\mu_q)\), we have the identity:

\[
\int_{\Sigma_1} (\lambda, [g_{\varepsilon(t_1)}^{\mu_1}, [g_{\varepsilon(t_2)}^{\mu_2}, g_{\varepsilon(t_1)}^{\mu_1}]](q_1)) \, dt_1 \, dt_2 \, dt_1 = \int_{\Sigma_3} (\lambda, [g_{\varepsilon(t_1)}^{\mu_1}, [g_{\varepsilon(t_2)}^{\mu_2}, g_{\varepsilon(t_1)}^{\mu_1}]](q_1)) \, dt_1 \, dt_2 \, dt_1.
\]

For the second differential, the two series in (3.13) and (3.18) produce the same formula, that was already established e.g. in ([27], §20.3). For further discussions concerning the algebra of all representations for the kth differential we refer to [37].
4. Third order necessary conditions for singular length-minimizers

We use the Taylor formula for the end-point map obtained in section 3, in connection with
our open mapping results, to get third-order necessary conditions satisfied by strictly singular
length-minimizers. It is well-known that length-minimizing curves can be equivalently char-
terized as minimizers of the energy. We will take this perspective in order to use the Hilbert
spaces techniques.

Let $f_1, \ldots, f_k \in \text{Vec}(M)$ be smooth vector fields on the manifold $M$ spanning the distribution
$\Delta$ and satisfying the Hörmander condition (1.1). We denote by $X = L^2([0, 1]; \mathbb{R}^k)$ the space of
controls and by $J : X \to [0, \infty)$

$$J(u) = \frac{1}{2} \|u\|^2_{L^2([0, 1]; \mathbb{R}^k)}$$

the energy-functional. For a fixed $q_0 \in M$, we consider the end-point map $F = F_{q_0} : X \to M$. The extended end-point map is the map $\mathcal{F} : X \to M \times \mathbb{R}$ given by $\mathcal{F}(u) = (F(u), J(u))$.

**Definition 4.1.** A critical point $u \in X$ of $\mathcal{F}$ is regular (singular) if there exists $(\lambda, \lambda_0) \in \text{Im}(duF)^{\perp} \subset T_{F(u)}M \times \mathbb{R}$ such that $\lambda_0 \neq 0$ ($\lambda_0 = 0$, respectively).

A critical point $u \in X$ is strictly singular if for every $(\lambda, \lambda_0) \in \text{Im}(duF)^{\perp}$ we have $\lambda_0 = 0$.

An extremal curve $\gamma$ is regular (singular or strictly singular) if its control is regular (singular or strictly singular, respectively).

In this section, we prove theorem 1.2. During the proof, we assume that condition (1.3) does
not hold and our goal will be to show that the extended end-point map $\mathcal{F}$ is not open at $u$, thus
contradicting the length-minimality (or the energy-minimality) of the strictly singular curve $\gamma$ with control $u$.

The differential analysis of $\mathcal{F}$ will be reduced to the analysis of the end-point map $F$ in the
following way. If $u$ is strictly singular then $\text{coker}(duF)$ and $\text{coker}(duJ)$ are isomorphic and can
be identified, since the energy-coordinate is covered by $\text{Im}(duF)$. Then the second and third
differentials satisfy

$$D_2^3uF = D_2^3uF\bigg|_{\ker(duF)} \quad \text{and} \quad D_3^3uF = D_3^3uF\bigg|_{\ker(duF)},$$

where the kernel of $duF$ is $\ker(duF) = \ker(duF) \cap \ker(duJ)$. In particular, we have

$$\text{dom}(D_3^3uF) = \text{dom}(D_3^3uF) \cap \ker(duJ),$$

and $\text{dom}(D_3^3uF)$ is finitely complemented in $\ker(duF)$ as soon as $\text{dom}(D_3^3uF)$ is finitely com-
plemented in $\ker(duF)$ (this is assumption (ii) of theorem 1.2). In fact, $\ker(duJ)$ has codimension 1.

At this point, we will show that there exists $v \in \text{dom}(D_3^3uF)$ such that $D_3^3uF(v) = D_3^3uF(v) \neq 0$. Here, we use the fact that $\text{dom}(D_3^3uF)$ is finitely complemented. The mapping $\mathcal{F}$ will be thus
open at $u$, by theorem 1.1 part (i). Here, we use the fact that $\gamma$ is of corank 1. This will be the
end of the proof, obtained with lemma 4.5 below.

**Remark 4.2.** The end-point map $F$ is trivially open but this does not imply that the extended
map $\mathcal{F}$ is open. Consider the example $\hat{\mathcal{F}} := (F, J) : \mathbb{R}^2 \to \mathbb{R}^2$.
\[ \tilde{F}(x, y) = (y^2 - x^2, y), \]

where \( \tilde{F}(x, y) = y^2 - x^2 \) is open at the origin. The critical point 0 of \( \tilde{F} \) is strictly singular. However, the intrinsic Hessian \( D^2_u \tilde{F} \) restricted to \( \text{ker}(d_0 \tilde{F}) \) has index 0. And in fact \( \tilde{F} \) is not open at 0.

The analysis of \( F \) at \( u \) is further reduced to the analysis of the perturbed map \( G \) at 0. Let \( q_1 = F(u) \) be the final point and, similarly to formula (3.2), define \( G_{q_1}^u : \text{ker}(d_0 \tilde{F}) \to M \) letting \( G_{q_1}^u(v) = F(u + v) \). In the rest of the section we omit in \( G u \) the subscripts \( q_1 \) and the superscript \( u \). We have the following identities

\[ D^2_0 G = D^2_u \tilde{F}, \quad D^3_0 G = D^3_u \tilde{F}. \]

Thanks to proposition 3.5, given \( \lambda \in \text{Im}(d_0 G)^\perp \) the trilinear map \( \lambda T : \text{dom}(D^3_0 G)^3 \to \mathbb{R} \) associated with \( \lambda D^3_0 G \) is given by:

\[ \lambda T(v_1, v_2, v_3) = \frac{1}{3} \sum_{\sigma \in S_3} \int_{\Sigma_3} \left( \lambda, [g^{u, \sigma_3}_{r_3(f_3)}], [g^{u, \sigma_2}_{r_2(f_2)}], [g^{u, \sigma_1}_{r_1(f_1)}] \right)(q_1) \, d\tau_3 d\tau_2 d\tau_1, \]

where the summation runs over all the permutations \( \sigma \) in the symmetric group \( S_3 \). Corollary 2.10 specializes as follows.

**Proposition 4.3.** Assume that there exists \( w_0 \in \text{Iso}(D^3_0 G) \) such that:

(a) \( \text{dim}(\text{dom}(D^3_0 G)) + \text{dim}(\text{Im}(G, 2, w_0)) + \text{dim}(\text{Im}(d_0 G)) > \text{dim}(M) \);

(b) For every non-zero \( \lambda \in \text{Im}(G, 2, w_0)^\perp \) and \( v \in \text{dom}(D^3_0 G) \) the real-valued map

\[ \text{dom}(D^3_0 G) \ni x \mapsto \lambda T(v, v, x) \]

is not the zero mapping.

Then \( G \) is open at zero.

As a consequence we have the following corollary, that is of interest when coker \((G, 2, w_0) \neq 0 \):

**Corollary 4.4.** Let \( u \) be the control of a strictly singular length-minimizing curve. Then, for every \( w_0 \in \text{Iso}(D^3_0 G) \) one of the following holds:

(a) \( \text{dim}(\text{dom}(D^3_0 G)) + \text{dim}(\text{Im}(G, 2, w_0)) + \text{dim}(\text{Im}(d_0 G)) \leq \text{dim}(M) \), or

(b) there exist a non-zero covector \( \lambda \in \text{Im}(G, 2, w_0)^\perp \) and \( v \in \text{dom}(D^3_0 G) \) such that

\[ \lambda T(v, v, x) = 0 \quad \text{for every } x \in \text{dom}(D^3_0 G). \]

For strictly singular length-minimizers of corank one, the negation of theorem 1.1 provides a more refined criterion. Indeed, its contrapositive translates into a point-wise condition as soon as the subspace \( \text{dom}(D^3_0 G) \) is sufficiently large.

Let us first recall the construction of adjoint curves. Let \( \gamma : [0, 1] \to M \) be an admissible curve with control \( u \), with \( \gamma(0) = q_0 \) and \( \gamma(1) = q_1 \). We denote by \( P_{q_0}^u \) the flow of the non-autonomous vector field \( V_u = f_0 q_0^u \) as in (3.1). Then we have \( \gamma(t) = P_{q_0}^u(q_0) \) for \( t \in [0, 1] \). That the differential \( (P_{q_0}^u)_*: T_{q_0} M \to T_{q_1} M \) is given by
Similarly, for any nonlinearity normalisation of the identity:

\[ (P_1^s)_* = \text{Ad}((P_1^s)^{-1}) = \text{Ad}(P_1^s) = \exp \int_0^t f_{\mu(\tau)} d\tau \]

The adjoint map \((P_1^s)^\dagger\) sends \(T_{\gamma(t)}^* M\) to \(T_{\gamma(t)}^* M\). For every \(\lambda \in \text{Im}(d_0 G)^\perp\), the curve of covectors defined by

\[ \lambda(t) := (P_1^s)^\dagger \lambda \in T_{\gamma(t)}^* M, \quad t \in [0, 1], \]

is called the adjoint curve to \(\gamma\) relative to \(\lambda\). In the corank 1 case, this curve is unique up to normalization of \(\lambda \neq 0\).

**Proof of Theorem 1.2.** Proving (1.3) is equivalent to showing that for every choice of \(e_i, e_j, e_l\) in the canonical basis of \(\mathbb{R}^k\), and for every \(\lambda \in \text{Im}(d_0 G)^\perp\), we have

\[ \langle \lambda, [g^i, [g^j, g^l]](q_1) + [g^i, [g^j, g^l]](q_1) \rangle = 0, \quad t \in [0, 1], \]

where, as in (3.3), we set \(g^i := (P_1^s)_* f_i\) for \(t \in [0, 1]\) and \(v \in \mathbb{R}^k\), and \(g^i := g^i_{\lambda}\) for every \(i = 1, \ldots, k\). Indeed, for all \(i, j, l, t \in [0, 1]\), and \(k\) we have

\[ \langle \lambda(t), [f_i, [f_j, f_l]](\gamma(t)) \rangle = \langle (P_1^s)^\dagger \lambda, [f_i, [f_j, f_l]](\gamma(t)) \rangle = \langle \lambda, [g^i, [g^j, g^l]](q_1) \rangle. \]

Let us fix \(\bar{t} \in [0, 1]\). Given \(s > 0\) such that \(\bar{t} + s \leq 1\), for every \(v \in L^2([0, 1]; \mathbb{R}^k)\) we define

\[ v_\gamma(t) := v \left( \frac{t - \bar{t}}{s} \right), \quad \text{for } t \in [\bar{t}, \bar{t} + s], \quad (4.2) \]

and zero elsewhere. We consider the subspace of \(\text{dom}(D_0^3 G)\):

\[ E_s := \left\{ u \in \text{dom}(D_0^3 G) \right| u = v_\gamma \quad \text{for some } v \in L^2([0, 1]; \mathbb{R}^k) \right\} \quad \text{with} \quad \int_0^1 v(t) dt = 0 \right\}. \]

We observe that \(E_s\) has finite codimension in \(L^2([\bar{t}, \bar{t} + s]; \mathbb{R}^k)\) for every \(s\). Indeed, the mapping \(v \mapsto v_\gamma\) is a linear isomorphism of \(L^2([0, 1]; \mathbb{R}^k)\) onto \(L^2([\bar{t}, \bar{t} + s]; \mathbb{R}^k)\); since \(\text{dom}(D_0^3 G)\) is finitely complemented in \(L^2([0, 1]; \mathbb{R}^k)\) by assumption, then

\[ E_s = \text{dom}(D_0^3 G) \cap L^2([\bar{t}, \bar{t} + s]; \mathbb{R}^k) \]

is finitely complemented in \(L^2([\bar{t}, \bar{t} + s]; \mathbb{R}^k)\).

Given \(v \in L^2([0, 1]; \mathbb{R}^k)\), its primitive \(z \in H^1([0, 1]; \mathbb{R}^k)\) with \(z(0) = 0\) is

\[ z(t) = \int_0^t v(\tau) d\tau, \quad t \in [0, 1]. \quad (4.3) \]

Similarly, for any \(v_\gamma\) as in (4.2) let \(z_\gamma\) be its primitive with \(z_\gamma(0) = 0\). It is immediate to establish the identity:

\[ z_\gamma(t) = sz \left( \frac{t - \bar{t}}{s} \right). \quad (4.4) \]

Moreover, if \(v_\gamma \in E_s\) the zero-mean property of \(v\) translates into:

\[ z_\gamma(\bar{t}) = z(\bar{t} + s) = 0. \quad (4.5) \]
In the next lines, we shall use several times the following integration by parts formula. For every \( 0 \leq \alpha < \beta \leq 1 \) and \( v \in L^1([0,1];\mathbb{R}^d) \), denoting by \( z \in H^1([0,1];\mathbb{R}^d) \) the primitive of \( v \), we have:

\[
\int_{\alpha}^{\beta} \frac{d}{dt} g^i_j(t) dt = \int_{\alpha}^{\beta} \sum_{i=1}^{k} g^i_j \cdot \dot{z_i}(t) dt \\
= \sum_{i=1}^{k} \int_{\alpha}^{\beta} g^i_j \dot{z_i}(\beta) dt - \sum_{i=1}^{k} \int_{\alpha}^{\beta} g^i_j \dot{z_i}(\alpha) dt - \int_{\alpha}^{\beta} \sum_{i=1}^{k} (\partial g^i_j) z_i(t) dt \\
= g^i_j(\beta) - g^i_j(\alpha) - \int_{\alpha}^{\beta} (\partial g^i_j) z_i(t) dt.
\]

Starting from proposition 3.5, applying this formula to \( v_A \in E_A \) and using (4.5) we obtain:

\[
\frac{1}{2} \lambda D^3_{\mid} G(v_A) = - \sum_{t_j \leq t_2 \leq t_1} \sum_{t_j \leq t_2 \leq t_3} \left( \lambda, [g^1_{\mid t_2}, [g^2_{\mid t_2}, [g^3_{\mid t_1}]](q(t))] \right) dt_1 dt_2 dt_3 \\
= - \int \int \left( \lambda, [g^1_{\mid t_2}, [g^2_{\mid t_2}, [g^3_{\mid t_1}]](q)] \right) dt_1 dt_2 dt_3 \\
= \int \int \left( \lambda, [g^1_{\mid t_2}, [g^2_{\mid t_2}, [g^3_{\mid t_1}] q] \right) dt_1 dt_2 dt_3 \\
+ \int \int \left( \lambda, [\partial g^3_{\mid t_3}, [g^2_{\mid t_2}, [g^3_{\mid t_1}]](q)] \right) dt_1 dt_2 dt_3 \\
= A(s) + B(s) - C(s),
\]

(4.6)

where \( A, B, \) and \( C \) are defined through the last identity.

From their very definition, we see that the maps

\[ \tau \mapsto g^i_j = \text{Ad} \left( \exp \int_1^\tau f_{\mid t} dt \right) f_i \]

are Lipschitz continuous for every \( i = 1, \ldots, k \) because their derivatives depend on time through a locally bounded vector field. Then we have the expansion

\[ g^{i+\theta} = g^i + O(s), \]

(4.7)

where the error \( O(s) \) is uniform for \( 0 \leq \theta \leq 1 \).

Now we estimate the terms \( A(s), B(s), \) and \( C(s) \) appearing in (4.6). We claim that
\[ A(s) = s^3 \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt + O(s^4). \]

To prove this identity we perform in \( A(s) \) the change of variable \( \tau_2 = t + st \) with \( t \in [0, 1] \), and we use (4.4) and (4.7). With a similar argument, we show that

\[ B(s) = O(s^4) \quad \text{and} \quad C(s) = O(s^4). \]

We conclude that

\[ \frac{1}{2} \lambda D^3 G(v_t) = s^3 \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt + O(s^4). \quad (4.8) \]

Let us introduce the set:

\[ \mathfrak{Z}_x := \{ z \in H_0^1([0, 1]; \mathbb{R}^k) | z_x \in \text{dom}(D^3_0 G) \} \subset H_0^1([0, 1]; \mathbb{R}^k). \]

As in (4.3), in the next lines given \( z \in \mathfrak{Z}_x \) we set \( v := z \) and \( v_x := z_x \). By point (i) of theorem 1.1, the map \( \mathfrak{Z}_x \ni z \mapsto \lambda D^3_0 G(z_x) \) is the zero map, for otherwise the curve \( \gamma \) would not be length-minimizing. This implies that the principal term in (4.8), i.e. the cubic map \( T : \mathfrak{Z}_x \to \mathbb{R} \),

\[ T(z) = \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt, \]

is identically zero. By polarization, we conclude that the trilinear map \( \mathfrak{T} : \mathfrak{Z}_x \times \mathfrak{Z}_x \times \mathfrak{Z}_x \to \mathbb{R} \) associated with \( T \),

\[ \mathfrak{T}(z_1, z_2, z_3) = \frac{1}{6} \sum_{\sigma \in S_3} \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt, \quad (4.9) \]

is zero as well. Integrating by parts and using the Jacobi identity, we obtain

\[ \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt \]

\[ = - \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt - \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt \]

\[ = \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt - \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt \]

\[ + \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt, \]

and a similar expansion holds for \( \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt \), switching the role of \( z_1 \) and \( z_2 \). Plugging these expressions in (4.9), we find:

\[ 2 \mathfrak{T}(z_1, z_2, z_3) = \int_0^1 \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) + \langle \lambda, [g^{l}_{(2)}], [g^{l}_{(2)}], S^{l}_{(0)}] \rangle(q_1) \, dt. \]

To conclude the proof it suffices to show that \( \mathfrak{T} = 0 \) implies that, for every choice of vectors \( e_t, e_y, e_z \) in the standard basis of \( \mathbb{R}^k \), we have
where $W$ in the canonical basis of $\mathbb{R}^4$ such that (4.10) does not hold. We claim that it is possible to find two functions $\alpha, \beta \in H^2_0([0, 1]; \mathbb{R})$ such that:

(a) $\alpha e_i, \alpha e_j, \beta e_j$ belong to $\mathcal{F}$;
(b) $\int_0^1 \alpha(t)\dot{\alpha}(t)\beta(t)dt \neq 0$.

Then, if the claim is true we reach a contradiction because we would have

\[ 2\mathcal{E}(\alpha e_i, \alpha e_j, \beta e_j) = \left( \int_0^1 \alpha(t)\dot{\alpha}(t)\beta(t)dt \right) \cdot \left( \langle \lambda, [g^j, g^j](q_i) \rangle + \langle \lambda, [g^j, g^j](q_1) \rangle \right) \neq 0. \]

To complete the argument it only remains to prove the claim.

**Lemma 4.5.** For every choice of vectors $e_i, e_j, e_t$ in the standard basis of $\mathbb{R}^4$, there exist functions $\alpha, \beta \in H^2_0([0, 1]; \mathbb{R})$ such that (i) and (ii) above hold.

**Proof.** We endow $H^2_0([0, 1]; \mathbb{R}^4)$ with the inner product given by

\[ (v, w) := \int_0^1 \langle \bar{\nu}(t), \bar{\nu}(t) \rangle dt \]

for functions $v, w \in H^2_0([0, 1]; \mathbb{R}^4)$. Since the map $v \mapsto \bar{v}$ in (4.2) is a linear isomorphism of $L^2([0, 1]; \mathbb{R}^4)$ onto $L^2([t, t+1]; \mathbb{R}^4)$, and $E_t$ is finitely complemented in $L^2([t, t+1]; \mathbb{R}^4)$, it follows that we have the orthogonal decomposition

\[ H^2_0([0, 1]; \mathbb{R}^4) = \mathcal{F} \oplus W, \]

where $W = \text{span}\{w_1, \ldots, w_q\}$ has dimension $q < \infty$. Given $\phi \in H^2_0([0, 1]; \mathbb{R})$ and a vector $e_t$ in the canonical basis of $\mathbb{R}^4$, we have that $\phi \cdot e_t \in \mathcal{F}$ if and only if

\[ \int_0^1 \dot{\phi}(t)\dot{w}_t dt = 0, \quad \text{for every } p = 1, \ldots, q, \]

where we denoted by $w^j_p$ the $j$th component of $w_p$. The Fourier expansion of the function $\dot{w}^j_p$ is

\[ \dot{w}^j_p(t) = \Re \left( \sum_{n=1}^{\infty} \dot{w}^j_{p,n} e^{2\pi in} \right), \]

for suitable complex coefficients $\dot{w}^j_{p,n} \in \mathbb{C}$. We look for functions $\alpha \in H^2_0([0, 1]; \mathbb{R})$ and $\beta \in H^2_0([0, 1]; \mathbb{R})$ satisfying (i) and (ii) of the form:

\[ \alpha(t) = \Re \left( \sum_{n=1}^{\infty} a_n \frac{e^{2\pi in} - 1}{2\pi in} \right), \quad \beta(t) = \Re \left( \sum_{n=1}^{\infty} b_n \frac{e^{2\pi in} - 1}{2\pi in} \right), \]

with real-valued coefficients $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$, i.e. $a_n, b_n \in \mathbb{R}$. A routine computation shows that

\[ \int_0^1 \alpha(t)\dot{\alpha}(t)\beta(t)dt = \frac{1}{16\pi^2} \sum_{n,m=1}^{\infty} \frac{a_n}{n+m} \frac{m-n}{n} \int_0^1 \frac{a_{n+m}b_m}{m} - \frac{a_m b_{n+m}}{n} \right) = \sum_{n=1}^{\infty} e_n(a) b_n, \]
where the coefficients $c_n(a)$ are defined through the last identity. If we choose $a$ such that $a_n = 0$ for every $n > k$ for some integer $k$, then $c_n(a) = 0$ for all $n > 2k$. Moreover, the last nontrivial coefficient $c_n(a)$ is

$$c_{2k}(a) = -\frac{a_k^2}{32\pi^2 k^2}.$$  (4.11)

The functions $\alpha$ and $\beta$ satisfy (i) and (ii) if and only if the coefficients $a$ and $b$ satisfy

$$\sum_{n=1}^{\infty} a_n \Re(\hat{w}_{n,p}^i) = \sum_{n=1}^{\infty} a_n \Im(\hat{w}_{n,p}^i) = \sum_{n=1}^{\infty} b_n \Re(\hat{w}_{n,p}^i) = 0, \quad p = 1, \ldots, q,$$

$$\sum_{n=1}^{\infty} c_n(a) b_n \neq 0.$$  (4.12)

Given $r \in \mathbb{N}$, we introduce the matrices

$$\mathcal{M}_r := \begin{pmatrix} \Re(\hat{w}_{n,p}^i) \\ \Im(\hat{w}_{n,p}^i) \end{pmatrix} \quad \text{for } p = 1, \ldots, q, n = 1, \ldots, r \in M_{2q \times r}(\mathbb{R}),$$

$$\mathcal{N}_r := \begin{pmatrix} \Re(\hat{w}_{n,p}^i) \end{pmatrix} \quad \text{for } n = 1, \ldots, r \in M_{q \times r}(\mathbb{R}).$$

We claim that there exist integers $k_1, \ldots, k_{q+1}$ such that $2q + 1 \leq k_1 < \cdots < k_{q+1} \leq q_0 := q^2 + 4q + 1$ and such that for every $i = 1, \ldots, q+1$, there exists a vector $a^i := (a_1^i, \ldots, a_{q_0}^i) \in \text{ker}(\mathcal{M}_k)$ with the property that $a_k^i \neq 0$.

Indeed, for every $1 \leq i \leq q+1$ there exists $(i+1)q+i \leq k \leq (i+2)q+i-1$ such that the matrices $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$ have the same image in $\mathbb{R}^{2q}$. Thus there exists a vector $a \in \text{ker}(\mathcal{M}_k)$ such that $a_{k+1} \neq 0$, otherwise $\text{ker}(\mathcal{M}_{k+1})$ would be isomorphic to $\text{ker}(\mathcal{M}_k)$, contradicting the rank-nullity theorem. Then we define $k_i := k+1$ and $a^i := a$, proving the claim.

We embed the vectors $a^1, \ldots, a^{q+1}$ into $\mathbb{R}^{2q_0}$ by adding zero coordinates if necessary. By construction, we have that $a^i \in \text{ker}(\mathcal{M}_{q_0})$ for every $i = 1, \ldots, q+1$. The vectors $c^i := c(a^i) \in \mathbb{R}^{2q_0}$ are linearly independent. In fact, for every $i = 1, \ldots, q+1$ we have $c_{2k_i}^i \neq 0$ by (4.11), while $c_j^i = 0$ for every $j = 2k_i+1, \ldots, 2q_0$.

The image of the transposed mapping $\mathcal{N}_{q_0}^T : \mathbb{R}^q \to \mathbb{R}^{2q_0}$ is a vector subspace of $\mathbb{R}^{2q_0}$ of dimension at most $q$. Then there exists $i \in \{1, \ldots, q+1\}$ such that $c^i \notin \text{Im}(\mathcal{N}_{q_0}^T)$, and thus it is possible to find a vector $b = (b_1, \ldots, b_{2q_0}) \in \mathbb{R}^{2q_0}$ such that $b \perp \text{Im}(\mathcal{N}_{q_0}^T)$, i.e. $b \notin \text{ker}(\mathcal{N}_{q_0})$, with $\sum_{k=1}^{2q_0} c_k^i b_k \neq 0$.

Eventually, choosing $a = a^i$ and $b$ as above we obtain a solution to (4.12), as desired. \(\square\)

**Remark 4.6.** In accordance with the two possible expressions of $\lambda \mathcal{D}_G^T G$ given in (3.19), we observe that (1.3) is symmetric with respect to $i$ and $\ell$, being therefore independent of the choice of the representation.
5. Third order analysis of a singular extremal

We prove in this section part (iii) of theorem 1.3. The proof of part (i) is elementary and can be omitted. The proof of part (ii) is identical to the case $p = 2$ that is discussed in ([38], section 7.1).

Proof of theorem 1.3-(iii). Let $\mathcal{F} = (F, J) : L^2([0, 1]; \mathbb{R}^2) \to \mathbb{R}^3 \times \mathbb{R}$ be the extended end-point map with initial point $q_0 = 0$ introduced below (4.1), where $J$ is the energy functional.

We claim that $\mathcal{F}$ is open at the point $u = (0, 1) \in L^2([0, 1]; \mathbb{R}^2)$, the control of the singular trajectory $\gamma$. As in (3.2) and (3.4), we let $G(v) = G_{q_0}(v) = F_{q_0}(u + v)$, where $q_1 = (0, 1, 0)$. The infinitesimal analysis of $F$ at $u$ is reduced to the infinitesimal analysis of $G$ at 0. By lemma 3.3 the differential of $G$ at 0 is given by $d_0 G(v) = \int_0^1 g^i v_i(t) dt$, where

$$
g^i = \sum_{i=1}^2 v_i(t) \text{Ad} \left( \exp \int_0^t f_i(\tau) d\tau \right) f_i,
$$

and $\text{Ad} \left( \exp \int_0^t f_i(\tau) d\tau \right)$ is the differential of the inverse of the flow $(x, t) \mapsto P^i_t(x)$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \in [0, 1]$, and $P^i_t(x) = \gamma(t)$, where $t \mapsto \gamma(t)$ is the horizontal trajectory with control $u$ such that $\gamma(1) = x \in \mathbb{R}^3$. Using the formulas for $f_1$ and $f_2$ in (1.4), we find

$$
\gamma_1(t) = x_1, \quad \gamma_2(t) = (t - 1)(1 - x_1) + x_2, \quad \gamma_3(t) = (t - 1)x_3 + x_3.
$$

The inverse of the differential $(P^1_i)_{=1}^{-1} = (P^1_i)_{=1} : T_{(0,1,0)} \mathbb{R}^3 \to T_{(0,1,0)} \mathbb{R}^3$ is given by

$$(P^1_i)_{=1} = \begin{pmatrix}
1 & 0 & 0 \\
(1 - t) & 1 & 0 \\
-3(t - 1)x_3^2 & 0 & 1
\end{pmatrix}.
$$

Accordingly, the vector fields $g_1^i$ and $g_2^i$ are

$$
g_1^i := (P^1_i)_{=1} f_1 = \frac{\partial}{\partial x_1} + (t - 1) \frac{\partial}{\partial x_2} - 3(t - 1)x_3^2 \frac{\partial}{\partial x_3},
$$

$$
g_2^i := (P^1_i)_{=1} f_2 = f_2,
$$

and we obtain the following formula for the differential of $G$:

$$
d_0 G(v) = \begin{pmatrix}
\int_0^1 v_1(t) dt \\
\int_0^1 \{ (t - 1)v_1(t) + v_2(t) \} dt \\
0
\end{pmatrix}.
$$

We then see that a generator of $\text{Im}(d_0 G)^\perp$ is the covector $\lambda = (0, 0, 1)$.

We compute the intrinsic Hessian $D^2_0 G$, again using lemma 3.3. By (5.1), for every $0 \leq t_1, t_2 \leq 1$ and every $v, w \in \mathbb{R}^3$ at the point $q_1 = (0, 1, 0)$ we have $\langle \lambda, [g_1^i, g_2^i](q_1) \rangle = 0$. Then $\lambda d_0^2 G(v) = 0$ for every $v \in \ker(d_0 G)$, hence $D^2_0 G = 0$.

Finally, we compute the intrinsic third differential $D^3_0 G$. Note first that since the intrinsic Hessian vanishes, by our definition in (2.3) we also have $\text{dom}(D^3_0 G) = \ker(d_0 G)$. The only
commutator of length three which has non-zero third component is \([g^{[3]}_1, [g^{[2]}_1, g^{[1]}_1]](q_1)\), and namely we have
\[
\langle \lambda, [g^{[3]}_1, [g^{[2]}_1, g^{[1]}_1]](q_1) \rangle = 6(t_2 - t_3).
\]

Then, by lemma 3.3 we obtain the formula
\[
\lambda \mathcal{D}^3_0 G(v) = 12 \int_0^1 \int_0^t v_1(t_1) v_1(t_2) v_1(t_3) (t_2 - t_3) \, dt_3 \, dt_2 \, dt_1.
\]

We claim that there exists \(v \in L^2([0, 1]; \mathbb{R}^2)\) such that \(v \in \ker(d_u \mathcal{F})\) and \(\lambda \mathcal{D}^3_0 G(v) \neq 0\). By theorem 1.1 and by our discussion at the beginning of section 4, the mapping \(\mathcal{F}\) is then open at 0 and thus the singular trajectory \(\gamma\) is not optimal (i.e. of minimal length).

The vector \(v = (v_1, v_2)\) belongs to \(\ker(d_u \mathcal{F})\) precisely when the function \(v_2\) has zero mean. Then, by (5.2) the condition \(v \in \ker(d_u \mathcal{F})\) is equivalent to
\[
\int_0^1 v_1(t) \, dt = \int_0^1 t v_1(t) \, dt = \int_0^1 v_2(t) \, dt = 0.
\]

Let \(v_2\) be any function with zero mean and choose \(v_1(t) = \sin(2\pi t) - 2\sin(4\pi t)\). This function is orthogonal to 1 and \(t\), and moreover
\[
\int_0^1 \int_0^t \int_0^{t_2} v_1(t_1) v_1(t_2) v_1(t_3) (t_2 - t_3) \, dt_3 \, dt_2 \, dt_1 = \frac{3}{64\pi^3} \neq 0.
\]

This proves the claim and finishes the proof of the theorem.

\(\square\)

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