The energy level shifts and the decay rate of an atom in the presence of a conducting wedge

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In the present article explicit expressions for the decay rate and energy level shifts of an atom in the presence of an ideal conducting wedge and a half-sheet are obtained in the framework of the canonical quantization approach. The angular and radial dependence of the decay rate for different atomic polarizations of an excited atom and also of the energy level shifts are depicted and discussed. The consistency of the present approach in some limiting cases is investigated by comparing the relevant results obtained here to the previously reported results.

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I. INTRODUCTION

An immediate consequence of the quantization of the electromagnetic field is the occurrence of the field fluctuation in the vacuum state. The effect of vacuum fluctuations on atomic systems leads for example to observable phenomena like spontaneous emission or atomic energy level shifts [1]. These radiative properties are explained as the reaction of atom against the existence of the zero-point field.

In quantum field theory the existence of any modification in the presence of an environment is an interesting subject, which have been widely studied. In principle, the calculations of these radiation properties in the presence of an environment, therefore, become the search for the quantized electromagnetic field in the presence of material fields in order to have a correct picture of fluctuating induced effects on atomic systems.

Therefore we should quantize the electromagnetic field in the presence of material fields [2-5] in order to have the explicit form of the field operators. A similar situation arise in static or dynamical Casimir effects which is a consequence of constrained vacuum fluctuations imposed by boundary conditions on macroscopic objects [6]. The presence of a boundary surface gives rise to alterations of vacuum field fluctuations and accordingly the energy level shifts and the decay rate of atomic systems change as the atom changes its position with respect to boundary surfaces [7-12]. The formalisms using different methods for the only dipole decay rate was explained in other literatures [13-15]. In this paper, we use another approach that describe both of the decay rate and the energy level shifts of an atom in terms of the imaginary part of the dimensionless vector potential Green’s function. We will see that for the dipole decay rate agreement with results of the other approaches are found at the end.

As expected, the de-excitation process for different wedges can also occur for an atom inside a wedge. This phenomenon suggesting that the work presented here is applicable to the area of quantum information processing and the system might serve as a qubit.

In the present work we investigate the decay rate and energy level shift of an atom in the presence of a conducting wedge in the frame work of canonical quantization based on previous works [3-5]. We note that the formalism using here requires the components of the dyadic Green’s tensor of the electromagnetic field in the presence of material fields.

The paper is organized as follows. We present basic formulation in Sec. II. We use the components of the dyadic Green’s tensor to investigate how the decay rate and the energy level shifts of an atom is a effected by the presence of a perfectly conducting wedge in Sec. III and then focus on case in which the wedge forms a half-sheet in Sec. IV. It is shown that the analytical framework reproduces the well-known results for an atom in front of a conducting plane.
II. BASIC FORMULAE

A. The decay rate of an initially excited atom

To find the radiative properties of an atom in the presence of a boundary surface we need the explicit form of the field operators. Finding the explicit form of these operators in a general geometry, due to the intricate structure of field expressions, is a very difficult task or even impossible if we are not invoking to numerical calculations. But one can find an alternative approach to find the radiative properties of an atom without dealing with the explicit forms of the field operators. In this approach we try to find the electromagnetic dyadic tensor satisfying all boundary conditions imposed on the boundaries. Here the boundary which we are interested in is an ideal conducting wedge which include plane and half-plane as limiting cases. Here a very short introduction to the derivation of the basic formulae is given and the details of the calculations can be found for example in [16].

Up to the dipole approximation, the decay rate of an excited atom is given by Fermi’s golden rule

$$\Gamma = \frac{2\pi}{\hbar} \sum_f |\langle f|\mathbf{\mu} \cdot \hat{\mathbf{E}}(r_0, t)|0\rangle|^2 \delta(\omega_f - \omega_0),$$  

where \(r_0, \omega_0, \) and \(\mathbf{\mu}\) are the position, transition frequency, and dipole moment of the atom, respectively. The kets \(|f\rangle\) and \(|0\rangle\) are the final and vacuum states of the electromagnetic field, respectively. If we decompose the electric field to positive and negative frequency parts and make use of the fluctuation-dissipation theorem and Kubo’s formula [17]

$$\langle 0|\hat{E}_\alpha^+(r, \omega) \hat{E}_\beta^-(r', \omega')|0\rangle = 2\hbar\omega^2 \text{Im}D_{\alpha\beta}(r, r', \omega) \delta(\omega - \omega'),$$  

one can find the decay rate of an initially excited atom as [16]

$$\Gamma = \frac{2\omega^2}{\hbar} \text{Im}[\mathbf{\mu} \cdot \mathbf{D}(r_0, r_0, \omega) \cdot \mathbf{\mu}],$$  

where \(\mathbf{D}(r_0, r_0, \omega)\) is the Green’s tensor of the electromagnetic field in the presence of boundaries with components \(D_{\alpha\beta}\) appearing in [2]. For dimensional considerations, usually Green tensor is written in terms of the dimensionless Green’s tensor \(\mathcal{G}_{\alpha\beta}(r, r', \omega)\), as

$$D_{\alpha\beta}(r, r', \omega) = \frac{\omega}{4\pi\varepsilon_0 c^3} \mathcal{G}_{\alpha\beta}(r, r', \omega),$$  

where \(\varepsilon_0\) and \(c\) are the permittivity and the velocity of light in free space respectively. Throughout the paper summation convention is assumed i.e., repeated indices are summed over the three Cartesian coordinates \(x, y, z\). In the absence of boundaries or material fields, the decay rate of an excited atom turns out to be

$$\Gamma_0 = \frac{\mu^2 \omega_0^3}{3\pi\varepsilon_0 \hbar c^3}.$$  

By inserting Eqs. (4,5) into (3), we finally find

$$\Gamma_\alpha = \frac{3}{2} \Gamma_0 \text{Im}[\mathcal{G}_{\alpha\alpha}(r_0, r_0, \omega_0)],$$  

where the subscript \(\alpha\) refers to the different orientations of the dipole moment of the atom.

B. Energy level shift

The presence of the quantum vacuum fluctuations is responsible for fluctuations of the position of an atomic electron around a mean value as

$$\hat{R}(r, t) = \hat{R}_0(r, t) + \triangle \hat{R}(r, t),$$  

where \(\hat{R}_0(r, t)\) is the mean value position and \(\triangle \hat{R}(r, t)\) is the fluctuating part. Fluctuations of the position cause fluctuations in the potential, using Taylor expansion

$$V(\hat{R}_0 + \triangle \hat{R}) = V(\hat{R}_0) + (\triangle \hat{R} \cdot \nabla) V(\hat{R}) + \frac{1}{2}(\triangle \hat{R} \cdot \nabla)^2 V(\hat{R}) + \cdots,$$  

(8)
where
\[ V(\hat{R}_0) = -\frac{Ze^2}{4\pi\varepsilon_0 R_0}, \] (9)
is the potential at the mean value position. Now the energy level shift of the atomic energy state \(|n⟩\) can be obtained by evaluating the expectation value of the leading term in (8) as
\[ \Delta E_n = \frac{1}{2} \langle (\Delta \hat{R} \cdot \nabla)^2 V(\hat{R}) \rangle = \frac{Ze^2}{8\pi\varepsilon_0} Q_{\alpha\beta} \langle [\Delta \hat{R}(r_0, t)]_\alpha [\Delta \hat{R}(r_0, t)]_\beta \rangle, \] (10)
where
\[ Q_{\alpha\beta} = -\langle n| \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{\hat{R}} |n⟩. \] (11)

In the dipole approximation, \( \Delta \hat{R}(r, t) \) satisfies the Langevin equation
\[ m\frac{d^2}{dt^2} \Delta \hat{R}(r, t) + m\Gamma \frac{d}{dt} \Delta \hat{R}(r, t) = -e\hat{E}(r, t), \] (12)
where \( m \) is the mass of the electron and the parameter \( \Gamma \) is the damping constant defined by the Bethe’s average excitation energy
\[ \Gamma = c\gamma = |E_n - E_m|_{av}/\hbar = 17.8R_\infty/\hbar, \] (13)
and \( R_\infty \) is the Rydberg unit of energy. Eq.(12) can be solved using Fourier transform, inserting the solution into Eq.(10) and using Eq.(2), we finally find
\[ \Delta E_n = \frac{ze^4\hbar}{32\pi^3\varepsilon_0^2c^3m^2} \int_0^\infty \frac{q}{q^2 + \gamma^2} Q_{\alpha\beta} \text{Im}[\mathcal{G}_{\alpha\beta}(r_0, r_0, \omega)] dq, \] (14)
where \( q = \omega/c \). The cutoff frequency \( mc/\hbar \) in Eq.(14) which is the Compton wavelength of the electron, is needed due to the validity of the dipole approximation applied in Eq.(12). The main ingredient of the basic formulae (5) and (14) is the dimensionless Green tensor. In the next section the decay rate of an initially excited atom and also the energy level shift are obtained for an atom located at an arbitrary point inside a perfectly conducting wedge.

III. THE PERFECTLY CONDUCTING WEDGE

A. The decay rate

Consider an initially excited atom located at an arbitrary point \( P \) inside an infinite wedge with perfect conducting walls and the apex angle \( \alpha \), Fig.1. Due to the symmetry along the z-component, the point \( P \) is determined by \((r, \theta)\) in polar coordinates system. The Green tensor in the presence of a wedge with perfect conducting walls is now a textbook problem and the interested reader can find the details of its derivation for example in [18–20]. According to
Eq. [16], the relevant components are the diagonal components $D_{rr}, D_{\theta \theta},$ and $D_{zz},$ given by \cite{18,20}

$$D_{rr} = -\frac{2ip}{q^3} \int_{-\infty}^{+\infty} dk e^{ik(z-\z')} \sum_{m=0}^{\infty} \left[ \frac{q^2 m^2 \eta^2}{\eta^2 \eta'} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') + k^2 J_{mp}'(\eta \eta') H_{mp}'^{(1)}(\eta \eta') \right] \sin(mp\theta) \sin(mp\theta'),$$

$$= -\frac{2ip}{q^3} \left[ \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \int_{-\infty}^{+\infty} \frac{q^2 dk}{\eta^2} e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \cos(mp\theta) \cos(mp\theta') \right] + \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \int_{-\infty}^{+\infty} \frac{k^2 dk}{\eta^2} e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \sin(mp\theta) \sin(mp\theta'),$$

$$= -\frac{ip}{q^3} \left[ \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \int_{-\infty}^{+\infty} \frac{q^2 dk}{\eta^2} e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \cos(mp(\theta + \theta')) + \cos(mp(\theta - \theta')) \right] + \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \int_{-\infty}^{+\infty} \frac{k^2 dk}{\eta^2} e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \cos(mp(\theta - \theta')) - \cos(mp(\theta + \theta')) \right],$$

(15)

$$D_{\theta \theta} = -\frac{2ip}{q^3} \int_{-\infty}^{+\infty} dk e^{ik(z-\z')} \sum_{m=0}^{\infty} \left[ \frac{k^2 m^2 \eta^2}{\eta^2 \eta^2} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') + q^2 J_{mp}'(\eta \eta') H_{mp}'^{(1)}(\eta \eta') \right] \cos(mp\theta) \cos(mp\theta'),$$

$$= -\frac{2ip}{q^3} \left[ \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \int_{-\infty}^{+\infty} \frac{k^2 dk}{\eta^2} e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \sin(mp\theta) \sin(mp\theta') \right] + \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \int_{-\infty}^{+\infty} \frac{q^2 dk}{\eta^2} e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \cos(mp\theta) \cos(mp\theta'),$$

$$= -\frac{ip}{q^3} \left[ \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \int_{-\infty}^{+\infty} \frac{k^2 dk}{\eta^2} e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \cos(mp(\theta + \theta')) - \cos(mp(\theta - \theta')) \right] + \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \int_{-\infty}^{+\infty} \frac{q^2 dk}{\eta^2} e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \cos(mp(\theta + \theta')) + \cos(mp(\theta - \theta')) \right],$$

(16)

$$D_{zz} = -\frac{2ip}{q^3} \int_{-\infty}^{+\infty} dk \eta^2 e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \sin(mp\theta) \sin(mp\theta'),$$

$$= -\frac{ip}{q^3} \int_{-\infty}^{+\infty} dk \eta^2 e^{ik(z-\z')} \sum_{m=0}^{\infty} J_{mp}(\eta \eta') H_{mp}^{(1)}(\eta \eta') \sin(mp(\theta + \theta')) \cos(mp(\theta + \theta')) \right],$$

(17)

where $p = \pi/\alpha, \eta = \sqrt{q^2 - k^2},$ and since there are not derivatives with respect to $z$ or $z'$, so we can simply set $z = z'$. For the special case $\alpha = \pi/n,$ where $n$ is a natural number, the parameter $p$ is an integer ($p = n$), and the summation over $m$ can be done using Graf’s addition theorem \cite{21}

$$\sum_{n=0}^{p-1} K_0(\zeta R_n) = 2p \sum_{m=0}^{\infty} I_{mp}(\zeta r_1) K_{mp}(\zeta r_2) \cos(mp\phi),$$

(18)

where

$$R_n = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\phi + 2n\pi/p)].}$$

(19)
FIG. 1: (Color online) Atom is located at point P inside a wedge with perfectly conducting walls.

Therefore, using Eq. (18) and changing the integration variable \( u = k/q \), the diagonal components can be written as

\[
\mathcal{D}_{rr} = \frac{2}{\pi q^2} \left[ \frac{1}{rr'} \sum_{n=0}^{p-1} \int_0^{+\infty} \frac{du}{(u^2-1)} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \left( K_0(\sqrt{u^2-1}qR_1) + K_0(\sqrt{u^2-1}qR_2) \right) \right] \\
+ \sum_{n=0}^{p-1} \int_0^{+\infty} \frac{u^2}{(u^2-1)} \frac{\partial}{\partial r} \frac{\partial}{\partial r'} \left( K_0(\sqrt{u^2-1}qR_2) - K_0(\sqrt{u^2-1}qR_1) \right),
\]

(20)

\[
\mathcal{D}_{\theta\theta} = \frac{2}{\pi q^2} \left[ \frac{1}{rr'} \sum_{n=0}^{p-1} \int_0^{+\infty} \frac{du}{(u^2-1)} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta'} \left( K_0(\sqrt{u^2-1}qR_2) - K_0(\sqrt{u^2-1}qR_1) \right) \right] \\
+ \sum_{n=0}^{p-1} \int_0^{+\infty} \frac{du}{(u^2-1)} \frac{\partial}{\partial r} \frac{\partial}{\partial r'} \left( K_0(\sqrt{u^2-1}qR_1) + K_0(\sqrt{u^2-1}qR_2) \right),
\]

(21)

\[
\mathcal{D}_{zz} = \frac{2}{\pi} \sum_{n=0}^{p-1} \int_0^{+\infty} (u^2-1) \left( K_0(\sqrt{u^2-1}qR_2) - K_0(\sqrt{u^2-1}qR_1) \right),
\]

(22)

where

\[
R_1 = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta + \theta' + 2n\pi/p)}, \\
R_2 = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta' + 2n\pi/p)}.
\]

(23)

Now using the formula [21]

\[
\int_0^{+\infty} dx \frac{K_\nu(\beta \sqrt{x^2 + z^2})}{\sqrt{(x^2 + z^2)^{\nu}}} x^{2\mu+1} = \frac{2\mu \Gamma(\mu + 1)}{\beta^{\mu+1} z^{\nu-\mu-1}} K_{\nu-\mu-1}(\beta z),
\]

(24)
and doing some straightforward calculations, we finally find the imaginary part of the diagonal components of Green tensor as

\[
\text{Im}[\mathcal{D}_{rr}(r_0, r_0, \omega)] = \sum_{n=0}^{p-1} \left[ \frac{\cos x}{x^2} + \sin \frac{x}{x^3} - \sin \frac{x}{x^3} \right] + \sin^2 \left( \frac{n\pi}{p} \right) \left( \frac{\cos x}{x^2} - \sin \frac{x}{x^3} \right)
- \left( \frac{\cos \frac{x}{x^3} + \sin \frac{x}{x^3} - \sin \frac{x}{x^3}}{x^3} \right) - \sin^2 \left( \frac{\theta + \frac{n\pi}{p}}{x^3} \right) \left( \frac{\cos \frac{x}{x^3} - \sin \frac{x}{x^3}}{x^3} \right),
\]

\[
\text{Im}[\mathcal{D}_{\theta\theta}(r_0, r_0, \omega)] = -\sum_{n=0}^{p-1} 2 \left( \frac{\cos x}{x^2} - \sin \frac{x}{x^3} \right) - \sin^2 \left( \frac{n\pi}{p} \right) \left( \frac{\cos x}{x^2} - \sin \frac{x}{x^3} \right)
+ 2 \left( \frac{\cos \frac{x}{x^3} + \sin \frac{x}{x^3} - \sin \frac{x}{x^3}}{x^3} \right) - \sin^2 \left( \frac{\theta + \frac{n\pi}{p}}{x^3} \right) \left( \frac{\cos \frac{x}{x^3} - \sin \frac{x}{x^3}}{x^3} \right),
\]

\[
\text{Im}[\mathcal{D}_{zz}(r_0, r_0, \omega)] = \sum_{n=0}^{p-1} \left[ \frac{\sin x}{x} + \cos x \frac{x^3}{x^3} - \sin x \frac{x^3}{x^3} \right] - \left( \frac{\sin \frac{x}{x^3} + \cos \frac{x}{x^3} - \sin \frac{x}{x^3}}{x^3} \right).
\]

By inserting Eqs. (25, 26, 27) into (6), we find

\[
\frac{\Gamma_r}{\Gamma_0} = 3 \frac{p-1}{2} \sum_{n=0}^{p-1} \left[ \frac{\cos x}{x^2} + \sin \frac{x}{x^3} - \sin \frac{x}{x^3} \right] + \sin^2 \left( \frac{n\pi}{p} \right) \left( \frac{\cos x}{x^2} - \sin \frac{x}{x^3} \right)
- \left( \frac{\cos \frac{x}{x^3} + \sin \frac{x}{x^3} - \sin \frac{x}{x^3}}{x^3} \right) - \sin^2 \left( \frac{\theta + \frac{n\pi}{p}}{x^3} \right) \left( \frac{\cos \frac{x}{x^3} - \sin \frac{x}{x^3}}{x^3} \right),
\]

FIG. 2: (Color online) The dimensionless decay rate of an excited atom for three orientations (a) $\frac{\Gamma_r}{\Gamma_0}$, (b) $\frac{\Gamma_{\theta\theta}}{\Gamma_0}$, (c) $\frac{\Gamma_{zz}}{\Gamma_0}$ in terms of dimensionless distance $\frac{r_0}{\lambda}$ from the origin along the symmetry line for the wedges with different $\alpha$. 

$\alpha = \frac{\pi}{2}$

$\alpha = \frac{\pi}{3}$

$\alpha = \pi$
FIG. 3: (Color online) The dimensionless decay rate for three orientations of the polarization of the excited atom (a) $\Gamma_r/\Gamma_0$, (b) $\Gamma_\theta/\Gamma_0$, (c) $\Gamma_z/\Gamma_0$ in terms of the angle $\theta$, and fixed distance $\frac{x_0}{\lambda} = 20$, for $\alpha = \pi/3, \pi/2, \pi$.

\[
\frac{\Gamma_\theta}{\Gamma_0} = -\frac{3}{2} \sum_{n=0}^{p-1} \left[ 2 \left( \frac{\cos x}{x^2} - \sin x \right) - \sin^2 \left( \frac{n\pi}{p} \right) \left( \frac{\cos x}{x^3} - \frac{\sin x}{x^3} \right) \right] \\
+ 2 \left( \frac{\cos x_\theta}{x_\theta^2} - \frac{\sin x_\theta}{x_\theta^3} \right) - \sin^2 \left( \theta + \frac{n\pi}{p} \right) \left( \frac{\cos x_\theta}{x_\theta^3} - \frac{\sin x_\theta}{x_\theta^3} \right),
\]

(29)

\[
\frac{\Gamma_z}{\Gamma_0} = \frac{3}{2} \sum_{n=0}^{p-1} \left[ \left( \frac{\sin x}{x} + \frac{\cos x}{x^2} - \frac{\sin x}{x^3} \right) - \left( \frac{\sin x_\theta}{x_\theta} + \frac{\cos x_\theta}{x_\theta^2} - \frac{\sin x_\theta}{x_\theta^3} \right) \right],
\]

(30)

where $x_\theta = 2r_0q\sin(\theta + n\pi/p)$ and $x = 2r_0q\sin(n\pi/p)$. In Fig.2, the decay rate of an initially excited atom for different polarizations and apex angles is depicted in terms of the distance from the $z$-axis for $\theta = \alpha/2$. The decay rates are normalized to the decay rates in free space $\Gamma_0$. Distances are also normalized to wavelength $\lambda$. In all of these diagrams when $r_0 \gg \lambda$, i.e the atom is far away from the axis, the decay rate tends to the free-space decay rate as expected.

In Fig.3, the typical behavior of the decay rate in terms of the angle $\theta$ and fixed distance from the $z$-axis is depicted for different apex angles $\alpha$. For the special case $\alpha = \pi$, the wedge degenerates into a plane sheet. In this case we find for the decay rates

\[
\frac{\Gamma_\parallel}{\Gamma_0} = \frac{\Gamma_\theta + \Gamma_z}{\Gamma_0},
\]

\[
= 1 - \frac{3}{2} \left[ \frac{\sin (2r_0q)}{(2r_0q)^2} + \frac{\cos (2r_0q)}{(2r_0q)^3} - \frac{\sin (2r_0q)}{(2r_0q)^3} \right],
\]

(31)

and

\[
\frac{\Gamma_\perp}{\Gamma_0} = \frac{\Gamma_r}{\Gamma_0},
\]

\[
= 1 - 3 \left[ \frac{\cos (2r_0q)}{(2r_0q)^2} - \frac{\sin (2r_0q)}{(2r_0q)^3} \right],
\]

(32)

which are in agreement with the results reported in [16].
B. The energy level shift

In this section we find the energy level shift of an atom placed in an arbitrary point inside an ideal conducting wedge defined by \((r, \theta)\) in Fig.1. In Eq.\((14)\), the repeated indices are summed over the three Cartesian coordinates \(\alpha, \beta = x, y, z\). The Green’s tensor in Cartesian coordinates can be obtained from the cylindrical one using simple coordinate transformations. Due to the symmetry of the problem, we find

\[
\text{Im}[D_{\alpha z}(r_0, r_0, \omega)] = \text{Im}[D_{z\alpha}(r_0, r_0, \omega)] = 0,
\]

where \(\alpha = x, y\), and for the other off-diagonal components, it is easy to show that

\[
\text{Im}[D_{xy}(r_0, r_0, \omega) + D_{yx}(r_0, r_0, \omega)] \ll \text{Im}[\text{tr}[D(r_0, r_0, \omega)]],
\]

where

\[
\text{tr}[D] = D_{xx} + D_{yy} + D_{zz},
\]

\[
= D_{rr} + D_{\theta\theta} + D_{zz}.
\]

Therefore, we can rewrite Eq.\((14)\) as \([16]\)

\[
\Delta E_n = \frac{z e^4 \hbar}{24 \pi^2 \varepsilon_0^2 c^4 m^2} \left| \psi(0) \right|^2 \int_0^{\frac{m^2}{2 \hbar q}} \frac{q}{q^2 + \gamma^2} \sum_\alpha \text{Im}[D_{\alpha\alpha}(r_0, r_0, \omega)] dq. \tag{36}
\]

Now using Eqs.\((25, 26, 27)\), we will find

\[
\text{Im}\left[\sum_\alpha D_{\alpha\alpha}(r_0, r_0, \omega)\right] = \sum_{n=0}^{p-1} 2 \left[ \frac{\sin x}{x} - \sin^2\left(\frac{n\pi}{p}\right) \left( \frac{\sin x}{x} - \frac{\cos x}{x^2} + \frac{\sin x}{x^3} \right) - \left( \frac{\sin x}{x} + 2 \frac{\cos x}{x^2} - 2 \frac{\sin x}{x^3} \right) \right]. \tag{37}
\]

By inserting Eq.\((37)\) into Eq.\((14)\), we find an expression for the energy level shifts inside the wedge as

\[
\frac{\Delta E_n}{\Delta E_0} = (\ln \frac{mc}{\hbar \gamma})^{-1} \sum_{n=0}^{p-1} \int_0^{\frac{m^2}{2 \hbar q}} \frac{q dq}{q^2 + \gamma^2} \left[ \frac{\sin x}{x} - \sin^2\left(\frac{n\pi}{p}\right) \left( \frac{\sin x}{x} - \frac{\cos x}{x^2} + \frac{\sin x}{x^3} \right) - \left( \frac{\sin x}{x} + 2 \frac{\cos x}{x^2} - 2 \frac{\sin x}{x^3} \right) \right]. \tag{38}
\]
which for the special choice $p = 1$ tends to the familiar expression for an ideal conducting plane [16]. In Fig. 4, the relative energy level shifts of an atom inside the wedge are depicted in terms of the distance from the z axis along the symmetry line for different apex angles $\alpha$. We see that when $\alpha < \frac{\pi}{2}$, at the region near to the narrow end of the wedge, the energy level shifts are much smaller compared to the vacuum case which means that in this region the atom is more stable in its excited state. In Fig. 5, the relative energy level shifts of an excited atom are depicted in terms of the angle $\theta$ for wedges with different apex angles $\alpha$.

For the special case $\alpha = 2\pi$, we have $p = \frac{1}{2}$, that is the wedge degenerates into a half-sheet. In this case $p$ is not an integer, so we can not use Eq. (18). In the next section using the dyadic Green tensor [22] we find the decay rate and energy level shift of an atom in the vicinity of an ideal conducting half-sheet.

IV. THE CONDUCTING HALF-SHEET

A. Green tensor

To find the decay rate and energy level shifts of an atom near a conducting half-sheet, let us consider the geometry depicted in Fig. 6. The conducting half-sheet is defined by the $xz$ plane for $x \geq 0$. For this geometry, the electric type dyadic Green’s function or Green’s tensor $D_{e1}$, satisfying boundary conditions on the walls, is given by [19, 20, 22]

$$D_{e1} = \frac{1}{2q} \left\{ \left[ (I + \frac{\nabla \cdot \nabla'}{q^2}) \frac{e^{-iqr_i}}{r_i} - iqI(\zeta_i, r_i) \right] - [I_r, (I + \frac{\nabla \cdot \nabla'}{q^2})] \frac{e^{-iqr_r}}{r_r} - iqI(\zeta_r, r_r) \right\}$$

$$- i \frac{q}{q}(\bar{x} \sin(\theta) \cos(\frac{\theta'}{2}) - \bar{y} \cos(\frac{\theta}{2})) (\bar{x} \sin(\frac{\theta'}{2}) - \bar{y} \cos(\frac{\theta'}{2})) H_0^{(2)}(q p) + \frac{1}{q} \nabla' \sin(\frac{\theta'}{2}) \frac{r + r'}{\sqrt{rr'}} H_1^{(2)}(q p) \right\},$$

(39)
where

\[ I = \hat{x}x + \hat{y}y + \hat{z}z, \]
\[ I_r = \hat{x}x - \hat{y}y + \hat{z}z, \]
\[ p = \sqrt{r + r'}^2 + (z - z')^2, \]
\[ r = \sqrt{(x-x')^2 + (y-y')^2 - (z-z')^2} = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} - (z-z')^2, \]
\[ r_r = \sqrt{(x-x')^2 + (y+y')^2 - (z-z')^2} = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta + \theta')} - (z-z')^2, \]
\[ I(\zeta, \eta) = \int_0^\zeta dt \frac{H(2)(\omega \sqrt{t^2 + \eta^2})}{\sqrt{t^2 + \eta^2}}, \]

and \( \zeta_\pm = 2 \sqrt{rr'} \cos(\frac{q\pm q'}{2}) \).

By making use of Eq. (39), we find the diagonal components of Green’s tensor as

\[ \mathcal{D}_{yy} = \frac{1}{2q} \left\{ (1 + \frac{1}{q^2} \frac{\partial}{\partial y} \frac{\partial}{\partial y'})(e^{-iqr_i} \frac{e^{-iqr_r}}{r_r} - iqI(\zeta_-, r_i)) + (1 + \frac{1}{q^2} \frac{\partial}{\partial y} \frac{\partial}{\partial y'})(e^{-iqr_i} \frac{e^{-iqr_r}}{r_r} - iqI(\zeta_+, r_r)) \right\} 
- \frac{i}{q} \left\{ \cos(\frac{\theta}{2}) \cos(\frac{\theta'}{2}) J_0(qp) \frac{(r + r')}{\sqrt{rr'}} \left( \frac{r + r'}{\sqrt{rr'}} \right) \right\}, \]

\[ \mathcal{D}_{xx} = \frac{1}{2q} \left\{ (1 + \frac{1}{q^2} \frac{\partial}{\partial x} \frac{\partial}{\partial x'})(e^{-iqr_i} \frac{e^{-iqr_r}}{r_r} - iqI(\zeta_-, r_i)) - (1 + \frac{1}{q^2} \frac{\partial}{\partial x} \frac{\partial}{\partial x'})(e^{-iqr_i} \frac{e^{-iqr_r}}{r_r} - iqI(\zeta_+, r_r)) \right\} 
- \frac{i}{q} \left\{ \sin(\frac{\theta}{2}) \sin(\frac{\theta'}{2}) J_0(qp) \frac{(r + r')}{\sqrt{rr'}} \left( \frac{r + r'}{\sqrt{rr'}} \right) \right\}, \]

\[ \mathcal{D}_{zz} = \frac{1}{2q} \left\{ (1 + \frac{1}{q^2} \frac{\partial}{\partial z} \frac{\partial}{\partial z'})(e^{-iqr_i} \frac{e^{-iqr_r}}{r_r} - iqI(\zeta_-, r_i)) - (1 + \frac{1}{q^2} \frac{\partial}{\partial z} \frac{\partial}{\partial z'})(e^{-iqr_i} \frac{e^{-iqr_r}}{r_r} - iqI(\zeta_+, r_r)) \right\}. \]
FIG. 7: (Color online) The dimensionless decay rate $\Gamma_y/\Gamma_0$ of an excited atom in the vicinity of a half-sheet along the lines (a) $\theta = \pi/6$, (b) $\theta = \pi/2$ and (c) $\theta = 5\pi/6$. 

B. The decay rate

To calculate $\Gamma_{\perp}(\Gamma_y$ in our notation), we need the $yy$-component of the Green’s tensor, by inserting Eq.(46) into Eq.(40), we find

$$
\Gamma_y = -\frac{3}{4q} \text{Im} \left\{ [(1 + \frac{1}{q^2} \frac{\partial}{\partial y'} \frac{\partial}{\partial y}) \left( e^{-iqr'/r_r} - i q I(\zeta_+, r_r) \right) + (1 + \frac{1}{q^2} \frac{\partial}{\partial y'} \frac{\partial}{\partial y}) \left( e^{-iqr''/r_i} - i q I(\zeta_-, r_i) \right)] 
- 2i \left[ \cos(\theta/2) \cos(\theta'/2) \frac{J_0(q p)}{\sqrt{rr'}} - \frac{1}{q} \cos(\theta) \frac{\partial}{\partial y'} (\sin(\theta'/2) \frac{r + r'}{\sqrt{rr'}} \frac{J_1(q p)}{p}) \right] \right\}_{x \to x', y \to y', z \to z'}
$$

and for the in-plane polarizations $\Gamma_x$ or $\Gamma_z$ we will find

$$
\Gamma_x = \frac{3}{4q} \text{Im} \left\{ [(1 + \frac{1}{q^2} \frac{\partial}{\partial x'} \frac{\partial}{\partial x}) \left( e^{-iqr'/r_r} - i q I(\zeta_+, r_r) \right) - (1 + \frac{1}{q^2} \frac{\partial}{\partial x'} \frac{\partial}{\partial x}) \left( e^{-iqr''/r_i} - i q I(\zeta_-, r_i) \right)] 
+ 2i \left[ \sin(\theta/2) \sin(\theta'/2) \frac{J_0(q p)}{\sqrt{rr'}} + \frac{1}{q} \sin(\theta) \frac{\partial}{\partial x'} (\sin(\theta'/2) \frac{r + r'}{\sqrt{rr'}} \frac{J_1(q p)}{p}) \right] \right\}_{x \to x', y \to y', z \to z'}
$$

$$
\Gamma_z = \frac{3}{4q} \text{Im} \left\{ (1 + \frac{1}{q^2} \frac{\partial}{\partial z'} \frac{\partial}{\partial z}) \left( e^{-iqr'/r_r} - i q I(\zeta_+, r_r) \right) - (1 + \frac{1}{q^2} \frac{\partial}{\partial z'} \frac{\partial}{\partial z}) \left( e^{-iqr''/r_i} - i q I(\zeta_-, r_i) \right) \right\}_{x \to x', y \to y', z \to z'}.
$$

The decay rate for the $z$-polarization of the excited atom in terms of the dimensionless distance $x/A$ along the lines (a) $\theta = \pi/6$, (b) $\theta = \pi/2$ and (c) $\theta = 5\pi/6$ is depicted in Fig.7. The decay rate for the $z$-polarization in terms of the $\theta$ and fixed distance is also depicted in Fig.8, showing a symmetry behaviour around $\theta = \pi$, as expected. It is interesting to note that in the Fig.8, the presence of the half-sheet affect the decay rate only for angles $\theta < \pi/2$ or $\theta > 3\pi/2$, this is because the emitted photon from the excited atom when emitted inside the angle $\pi/2 < \theta < 3\pi/2$ can not be reflected back to the atom.

As a consistency check, let us find the limiting case where in Fig.6, the atom is placed at a distance far from the $z$-axis or the edge of the half-sheet, that is $d$ is fixed and $h \to \infty$. In this limiting case using Eq.(24) we will find

$$
\Gamma_y = -\frac{3}{4q} \left\{ \left[ (1 + \frac{1}{q^2} \frac{\partial}{\partial y'} \frac{\partial}{\partial y}) \left( -\frac{2 \sin(q r_r)}{r_r} \right) + (1 + \frac{1}{q^2} \frac{\partial}{\partial y'} \frac{\partial}{\partial y}) \left( -\frac{2 \sin(q r_i)}{r_i} \right) \right] \right\}_{x \to x', y \to y', z \to z'},
$$

$$
\Gamma_x = \frac{3}{4q} \left\{ (1 + \frac{1}{q^2} \frac{\partial}{\partial x'} \frac{\partial}{\partial x}) \left( -\frac{2 \sin(q r_r)}{r_r} \right) - (1 + \frac{1}{q^2} \frac{\partial}{\partial x'} \frac{\partial}{\partial x}) \left( -\frac{2 \sin(q r_i)}{r_i} \right) \right\}_{x \to x', y \to y', z \to z'},
$$

$$
\Gamma_z = \frac{3}{4q} \left\{ (1 + \frac{1}{q^2} \frac{\partial}{\partial z'} \frac{\partial}{\partial z}) \left( -\frac{2 \sin(q r_r)}{r_r} \right) - (1 + \frac{1}{q^2} \frac{\partial}{\partial z'} \frac{\partial}{\partial z}) \left( -\frac{2 \sin(q r_i)}{r_i} \right) \right\}_{x \to x', y \to y', z \to z'}.
$$
FIG. 8: (Color online) The dimensionless decay rate $\frac{\Gamma_z}{\Gamma_0}$ of an excited atom in the vicinity of a half-sheet in terms of the angle $\theta \in (0, 2\pi)$, for a fixed distance $\frac{r_0}{\lambda} = 5$.

\[
\frac{\Gamma_z}{\Gamma_0} = \frac{3}{4q} \left[ (1 + \frac{1}{q^2} \frac{\partial}{\partial z'} \frac{\partial}{\partial z}) \left( -2 \sin(q r) \right) - (1 + \frac{1}{q^2} \frac{\partial}{\partial z^*} \frac{\partial}{\partial z^*}) \left( -2 \sin(q r^*) \right) \right] \bigg|_{x \to x', y \to y', z \to z'}. \tag{54}
\]

By taking the derivatives and evaluating the expressions at $r' = r$ we finally find

\[
\frac{\Gamma_y}{\Gamma_0} = 1 - 3 \frac{\cos(2qd)}{(2qd)^2}, \tag{55}
\]

and

\[
\frac{\Gamma_x}{\Gamma_0} = \frac{\Gamma_z}{\Gamma_0} = 1 - 3 \frac{\sin(2qd)}{(2qd)^2} \frac{\cos(2qd)}{(2qd)^2} - \frac{\sin(2qd)}{(2qd)^3}, \tag{56}
\]

which are the same results reported in [16] for an ideal conducting plane, as expected.

C. The energy level shifts

By inserting Eqs.(46,47,48) into Eq.(14), we can find the energy level shift of an atom in the presence of an ideal conducting half-sheet as

\[
\frac{\Delta E_n}{\Delta E_0} = \frac{1}{4 \ln \frac{mc}{q^2}} \int_0^{\frac{mc}{q^2}} \frac{dq}{q^2 + \gamma^2} A, \tag{57}
\]

where

\[
A = \text{Im} \left\{ [1 + \frac{1}{q^2} \left( \frac{\partial}{\partial x'} \frac{\partial}{\partial x'} - \frac{\partial}{\partial y'} \frac{\partial}{\partial y'} + \frac{\partial}{\partial z'} \frac{\partial}{\partial z'} \right)] \left[ \frac{e^{-iqr}}{r} - i q I(\zeta_+, r) \right] 
- [3 + \frac{1}{q^2} \left( \frac{\partial}{\partial x'} \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'} \frac{\partial}{\partial y'} + \frac{\partial}{\partial z'} \frac{\partial}{\partial z'} \right)] \left[ \frac{e^{-iqr_i}}{r_i} - i q I(\zeta_-, r_i) \right] 
+ 2i \left[ \cos\left( \frac{\theta - \theta'}{2} \right) J_0(q p) \sqrt{rr'} + \frac{1}{q} \sin\left( \frac{\theta}{2} \right) \frac{\partial}{\partial x} \left( \sin\left( \frac{\theta}{2} \right) \frac{r + r'}{\sqrt{rr'}} \frac{J_1(q p)}{p} \right) 
- \frac{1}{q} \cos\left( \frac{\theta}{2} \right) \frac{\partial}{\partial y} \left( \sin\left( \frac{\theta}{2} \right) \frac{r + r'}{\sqrt{rr'}} \frac{J_1(q p)}{p} \right) \right] \bigg|_{x \to x', y \to y', z \to z'}. \tag{58}
\]
The above expression for the energy level shift cannot be reduced any further to a simple analytical form and one should invoke numerical calculations.

As before, when $d$ is fixed and $h \to \infty$ (see Fig.6), the three last terms in Eq.(58) tend to zero and we find

$$\frac{\Delta E_n}{\Delta E_0} = 1 - \left(\ln \frac{mc}{\hbar \gamma}\right)^{-1} \int_0^{\frac{mc}{\hbar \gamma}} \frac{q dq}{q^2 + \gamma^2} \left[\sin(2qd) \frac{\cos(2qd)}{(2qd)^2} - 2\frac{\sin(2qd)}{(2qd)^3}\right]$$

again in agreement with the result reported in [16], as expected.

V. CONCLUSIONS

Explicit expressions for the decay rate and energy level shifts of an atom in the presence of an ideal conducting wedge and a half-sheet are obtained in the frame work of the canonical quantization approach. The angular and radial dependence of the decay rate for different atomic polarizations of an excited atom and also of the energy level shifts are depicted and discussed. The consistency of the present approach in some limiting cases is investigated by comparing the relevant results obtained here to the previously reported results.

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