Quantum effects of compactified AdS$_5$ geometry on the higgs potential

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Abstract

In this paper we determine the one loop radiative correction to the higgs potential due to quantum fluctuations about the background metric of Randall-Sundrum model. We then examine the effects of the one loop effective potential on the stability of the classical vacuum paying particular attention to the tadpole terms which could dominate over the classical potential for small field configurations. We find that although the one loop potential due to scale fluctuations could develop a tadpole term in certain regions of the parameter space, it is positive for either sign of H. The quadratic and quartic terms in the radiative correction are too small to cause any instability to the classical vacuum for field configurations that do not violate the limits of ordinary perturbation theory.
Introduction

Recently several proposals based on theories in extra dimensions have been put forward to explain the hierarchy problem. Among them the Randall-Sundrum model is particularly interesting since it proposes a five dimensional world based on a non-factorizable metric

\[ ds^2 = e^{-2kr_c|\theta|} \eta_{\mu\nu} dx^\mu dx^\nu - r_c^2 d\theta^2. \]  

(1)

Here \( r_c \) measures the size of the extra dimensions which is an \( \mathbb{S}^1/\mathbb{Z}_2 \) orbifold. \( \theta \) is the coordinate of the extra dimension with \( \theta \) and \(-\theta\) identified. \( k \) is a mass parameter of the order of the fundamental Planck mass \( M \). Two 3 branes are placed at the orbifold fixed points \( \theta = 0 \) (hidden brane) and \( \theta = \pi \) (visible brane). Randall and Sundrum showed that any field on the visible brane with a fundamental mass parameter \( m_0 \) gets an effective mass \( m = m_0 e^{-kr_c\pi} \) due to the exponential warp factor. Therefore for \( kr_c \approx 12 \) the weak scale is generated from the Planck scale by geometry. Note that the compactification scale \( \mu_c \approx \frac{1}{r_c} \approx \frac{k}{12} \) that is required to generate the large hierarchy is only an order of magnitude smaller than \( M \) unlike in theories with large extra dimensions.

In the Randall-Sundrum model \( r_c \) is the vacuum expectation value of a massless scalar field \( T(x) \). The modulus is therefore not stabilized by some dynamics. In order to stabilize the modulus Goldberger and Wise introduced a scalar field \( \chi(x, \theta) \) in the bulk with interaction potentials localised on the branes. This they showed could generate a potential for \( T(x) \) and stabilize the modulus at the right value \( (kr_c \approx 12) \) needed for the hierarchy without any excessive fine tuning of the parameters.

The couplings of the radion (scale fluctuations) and the graviton (tensor fluctuations) to the SM higgs scalar localized on the visible brane is completely determined by general covariance. In this paper we shall determine the couplings of the radion and the graviton to the SM higgs scalar. We then use these couplings to derive the one loop effective potential of the higgs boson arising from fluctuations about the background metric. Finally we examine the effect of the radiative corrections on the stability of the classical vacuum.
by paying particular attention towards the possible occurrence of tadpole terms in the radiative corrections which could dominate over the classical potential for small $H$.

**Radion couplings to the higgs scalar**

We shall determine the effects of scale fluctuations and tensor fluctuations on the higgs potential separately. First consider the effect of scale fluctuations. The couplings of the radion to the higgs scalar can be determined from the following action

$$S = \int d^4x \sqrt{-g_v} [g_v^{\mu\nu} \frac{1}{2} \partial_\mu H \partial_\nu H - V(H)].$$  \hspace{1cm} (2)

where $V(H) = \frac{1}{2} m_H^2 H^2 + \lambda v H^3 + \frac{\lambda}{4} H^4$. $H$ is a small fluctuation of the higgs field from its classical vacuum $v$. The constant vacuum energy $V(0)$ has been subtracted out from $V(H)$. In the absence of scale fluctuations we have $g_v^{\mu\nu} = e^{2k\pi T(x)} \eta^{\mu\nu} = (\frac{\phi}{f})^{-2} \eta^{\mu\nu}$, $\sqrt{-g_v} = (\frac{\phi}{f})^4$ and $\phi = f e^{-k\pi T(x)}$. Here $f$ is a mass parameter of the order of the fundamental Planck mass $M$. Rescaling $H$ and $v$ as $H \rightarrow \frac{f}{\langle \phi \rangle} h$ and $v \rightarrow \frac{f}{\langle \phi \rangle} v$ we get

$$S = \int d^4x [(\frac{\phi}{\langle \phi \rangle})^2 \frac{1}{2} \eta^{\mu\nu} \partial_\mu H \partial_\nu H - (\frac{\phi}{\langle \phi \rangle})^4 V(H)].$$  \hspace{1cm} (3)

Let $\hat{\phi}$ denote a small fluctuation of the radion from its vev $\langle \phi \rangle$. We then get

$$S = \int d^4x [\frac{1}{2} \partial_\mu H \partial^\mu H - V(H)] + \int d^4x [\partial_\mu H \partial^\mu H - 4V(H)] \frac{\hat{\phi}}{\langle \phi \rangle} + ...$$  \hspace{1cm} (4)

where the dots represent terms of order $(\frac{\hat{\phi}}{\langle \phi \rangle})^2$ and higher. It is easy to see that the couplings of cubic and higher order fluctuations to the higgs field do not contribute to the one loop effective potential. Further in this paper to simplify matters we shall ignore the effects of the couplings of quadratic radion fluctuations to the higgs scalar. Under this approximation we need to consider only vertices with one and only one radion field. A more complete treatment that takes into account the effects of the couplings of quadratic radion fluctuations into account will be given elsewhere.
Radion contribution to the one loop effective potential

We would like to note that since in the evaluation of the effective potential the external lines are assumed to have zero momenta the kinetic energy term of the higgs scalar in eqn. (4) does not contribute to the one loop effective potential. To simplify our discussion we shall first consider the radion coupling only to the mass term of the higgs scalar. The generalization to the case where the radion couplings to all the three terms of $V(H)$ are taken into account is straightforward and will be presented at the end of this section. We shall follow the method of Coleman-Weinberg and sum up an infinite series of one loop diagrams with an arbitrary number of higgs lines at zero momenta attached to it. From each $\hat{\phi}hh$ vertex we have to choose one external higgs line leaving one $\hat{\phi}$ and one higgs line to go into the loop. It therefore follows that only loop diagrams with even number of external higgs lines need to be considered. Summing all such one loop diagrams we get

$$\delta V = \sum_{n=1}^{\infty} V_{2n}$$

$$= -\frac{(m_H^2)^2}{16\pi^2} \int xdx \left[ \frac{1}{2} \frac{\beta^2}{y^2} + \frac{1}{4} \frac{\beta^4}{y^4} + \ldots + \frac{1}{2n} \frac{\beta^{2n}}{y^{2n}} \right]$$

$$= -\frac{(m_H^2)^2}{32\pi^2} \int xdx \left[ \ln(y^2 - \beta^2) - \ln y^2 \right].$$  \(\text{(5)}\)

where $\alpha = \frac{m_H^2}{m_H^2}$, $y^2 = (x + \alpha)(x + 1)$ and $\beta = \frac{4H}{\phi}$.

Evaluating the above integral with a physical cut off $M_s$ (the string scale) we get

$$\delta V_r = V_0(M_s) + \frac{(m_H^2)^2}{32\pi^2} \left[ (\frac{M_H^2}{m_H^2})^2 (\ln \frac{M_H^2}{m_H^2} - \frac{1}{2}) + (1 + \alpha)\left(\frac{M_H^2}{m_H^2}\right)^2 \right]$$

$$- \frac{1}{2} (1 + \alpha^2) \left[ \frac{1}{2} \ln \left(\frac{M_H^2}{m_H^2}\right) \right]$$

$$- \frac{(m_H^2)^2}{32\pi^2} \beta^2 \ln \frac{M_H^2}{\mu^2} + \frac{(m_H^2)^2}{32\pi^2} \beta^2 \ln \frac{M_H^2}{\mu^2} \ln \frac{m_H^2}{\mu^2} - \frac{1}{2}$$

$$+ \frac{\gamma^2}{2} \ln \gamma + \frac{\delta^2}{2} \ln \delta.$$  \(\text{(6)}\)

where $V_0(M_s) = -\frac{(m_H^2)^2}{32\pi^2} \int xdx \ln(x + \alpha)(x + 1)dx$, $\gamma = \frac{(1+\alpha)}{2} - \frac{1}{2} \sqrt{(1-\alpha)^2 + 4\beta^2}$ and $\delta = \frac{(1+\alpha)}{2} + \frac{1}{2} \sqrt{(1-\alpha)^2 + 4\beta^2}$. $\mu$ is the renormalization or subtraction scale. The
terms that are independent of $H$ contributes to the vacuum energy and could be subtracted out. The term $\beta^2 \ln \frac{M_s^2}{\mu^2}$ can be absorbed into the mass counter term of the higgs scalar. The remaining $H$ dependent terms are finite (independent of the cut off $M_s$) and they represent the renormalized one loop correction to the higgs potential due to linear radion fluctuations. In deriving the above result we have considered the radion coupling only to the mass term of $V(H)$. If we consider the radion couplings to all the three terms of $V(H)$ then each vertex can be either $\hat{\phi}H^2$, $\hat{\phi}H^3$ or $\hat{\phi}H^4$. It can be shown that under this condition the radiative correction (unrenormalized) to the higgs potential is given by

$$\delta V = \frac{(m_H^2)^2}{32\pi^2} \int x[\ln(y^2 - (\beta + \beta' + \beta'')^2) - \ln(x + \alpha)(x + 1)]dx. \quad (7)$$

where $\beta' = \frac{6H^2}{v(\phi)}$ and $\beta'' = \frac{2H^3}{v^2(\phi)}$. We find that for small fluctuations of the higgs field from its classical vacuum (i.e. for $H < v$) the contributions of $\hat{\phi}H^3$ and $\hat{\phi}H^4$ vertices to the higgs potential are suppressed compared to that of $\hat{\phi}H^2$.

**Graviton couplings to the higgs scalar**

We shall now consider the effects of graviton fluctuations on the higgs potential. In the following discussion we shall ignore the radion fluctuations and take the metric to be

$$ds^2 = e^{-2kr_x|\theta|}g_{\mu\nu}d\xi^\mu d\xi^\nu - r_c^2 d\theta^2. \quad (8)$$

where $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$.

The graviton couplings to the higgs scalar are completely determined by general covariance and can be derived from the action given in eqn (2).

As in the case of the radion we shall be interested only in the radiative corrections that arise from the couplings of linear graviton fluctuations to the higgs field. Under this approximation we have

$$S = \int d^4x \left(\frac{\phi}{f}\right)^4 \left(1 + \frac{\kappa}{2} h^{\alpha}_{\alpha}\right) \frac{1}{2} \left(\frac{\phi}{f}\right)^2 (\eta^{\mu\nu} - \kappa h^{\mu\nu}) \partial_\mu H \partial_\nu H - V(H)]. \quad (9)$$
Rescaling the higgs field and its vev properly we get

\[ S = \int d^4x \left[ \frac{1}{2} \eta^{\mu\nu} \partial_\mu H \partial_\nu H - V(H) \right] + \int d^4x \left[ \frac{1}{4} h_\alpha^{\alpha} \eta^{\mu\nu} \partial_\mu H \partial_\nu H - \frac{1}{2} h_\alpha^{\alpha} V(H) - \frac{1}{2} h^{\mu\nu} \partial_\mu H \partial_\nu H \right]. \] (10)

The graviton couplings to the higgs scalar is therefore given by

\[ S = -\frac{\kappa}{2} \int d^4x h^{\mu\nu}(x, \pi) T_{\mu\nu}(H) \]
\[ = -\frac{\kappa}{2} \sum_{n=0}^{\infty} \int d^4x h_n^{\mu\nu}(x) \frac{\chi_n(\pi)}{\sqrt{r_c}} T_{\mu\nu}(H). \] (11)

where \( T_{\mu\nu}(H) = \partial_\mu H \partial_\nu H - \eta_{\mu\nu} [\frac{1}{2} \partial_\alpha H \partial^\alpha H - V(H)] \). It can be shown that as long as the mass \( m_n \) of the Kaluza-Klein mode is much small compared to the fundamental Planck mass \( M \), \( \chi_n(\pi) \approx e^{kr_c \pi} \sqrt{r_c} \) and \( \chi_0(\pi) = \sqrt{kr_c} \). Choosing \( \kappa = \frac{2}{M_p^2} \) we finally get

\[ S = -\frac{1}{M_p} \int d^4x h_0^{\mu\nu}(x) T_{\mu\nu}(H) - \frac{1}{\Lambda} \sum_{n=1}^{\infty} \int d^4x h_n^{\mu\nu}(x) T_{\mu\nu}(H) \] (12)

where \( \Lambda \approx M_p e^{-kr_c \pi} \approx O(Tev) \).

So the coupling of the zero mode to the higgs scalar (or any other SM field) is inversely proportional to the Planck scale but the coupling of the higher Kaluza-Klein modes to the higgs scalar is inversely proportional to the Tev scale. It is this feature that makes the Kaluza-Klein modes accessible at Tev energies. As in the case of the radion the K.E. term of the higgs field does not contribute to the one loop effective potential which is computed with the external lines at zero momentum. Also the contributions of the cubic and quartic terms in \( V(H) \) can be neglected compared to that of the mass term of the higgs scalar. So the relevant interaction Lagrangian for us is given by \( L_I = -\frac{m_H^2}{2\Lambda} \sum_{n=1}^{\infty} h_n^{\mu\nu} \eta_{\mu\nu} H^2 \).

Using this interaction it can be shown that
\[ V_2 = \frac{1}{2} \sum_{n} \left( \frac{m_H^2}{\Lambda} \right)^2 \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu\alpha\beta}^n(k) \frac{1}{(k^2 - m_H^2)^2} \eta_{\mu\nu} \eta_{\alpha\beta} \]

\[ = - \sum_{n} \left( \frac{m_H^2}{\Lambda} \right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{2}{3} \left( \frac{k^2}{m_n^2} - 2 \right) \left( \frac{k^2}{m_n^2} + 1 \right) (k^2 + m^2_n) (k^2 + m_H^2) \]

\[ = - \frac{(m_H^2)^2}{16\pi^2} \sum_{n} \frac{1}{2} \left( \frac{a_n H}{\Lambda} \right)^2 \int_{m_n^2}^{M_s^2} \frac{d^4k}{4 \pi^2} \left( \frac{m_n^2}{3} \left( \frac{2}{3} f(x) \right)^2 \right). \]

\[ V_4 = - \frac{(m_H^2)^2}{16\pi^2} \sum_{n} \frac{1}{4} \left( \frac{a_n H}{\Lambda} \right)^4 \int_{m_n^2}^{M_s^2} \frac{d^4k}{4 \pi^2} \left( \frac{m_n^2}{3} \left( \frac{2}{3} f(x) \right)^4 \right) \]

and

\[ V_{2m} = - \frac{(m_H^2)^2}{16\pi^2} \sum_{n} \frac{1}{2m} \left( \frac{a_n H}{\Lambda} \right)^{2m} \int_{m_n^2}^{M_s^2} \frac{d^4k}{4 \pi^2} \left( \frac{m_n^2}{3} \left( \frac{2}{3} f(x) \right)^{2m} \right) \]

where \( a_n = \frac{m_n^2}{m_H^2} \) and \( f(x) = \sqrt{\frac{x - a_n}{x + 1}} \). The mass \( m_n \) of the Kaluza-Klein modes are given by \( m_n = e^{-k\pi r_k} x_n \) where \( x_n \) are the roots of the Bessel function \( J_1(x_n) \) [6]. The sum runs over all the massive Kaluza-Klein graviton modes. Adding all such contributions arising from diagrams with an even number of external higgs lines we get

\[ \delta V_g = \frac{(m_H^2)^2}{32\pi^2} \sum_{n} \int dx \ln(1 - \frac{a_n^2}{\Lambda^2} H^2 f^2(x)) \]

\[ = [V_0(M_s) + \frac{M_s^4}{64\pi^2} (\ln \frac{M_s^2}{m_H^2} - 1)] + \frac{(m_H^2)^2}{32\pi^2} \frac{1}{2 m_H^2} \ln(1 - \alpha_{n}) \]

\[ + \frac{(1 + 2\alpha_{n})}{(1 - \alpha_{n})} \frac{M_s^2}{m_H^2} \frac{1}{2(1 - \alpha_{n})^n} \ln(\frac{M_s^2}{\mu^2}) + \frac{(m_H^2)^2}{64\pi^2} \frac{[1 + 2\alpha_{n}]^2}{(1 - \alpha_{n})^2} \ln(\frac{M_s^2}{\mu^2} - \frac{1}{2}) \]

\[ - \frac{(1 + 2\alpha_{n})}{(1 - \alpha_{n})^2} \ln(1 - \alpha_{n}) + \frac{(1 + 2\alpha_{n})}{(1 - \alpha_{n})^2} \ln(1 + 2\alpha_{n})]. \]

(14)

where \( \alpha_{n} = \frac{2}{\pi} a_n^2 \frac{H^2}{\Lambda^2} \) and \( V_0(M_s) = -\frac{(m_H^2)^2}{32\pi^2} \sum_n \int x \ln(1 + x) dx \). The mass \( m_n \) of the first few Kaluza-Klein modes usually lie in the several Tev range. Thus if the higgs boson
is light i.e. $m_H < 1$ Tev the coefficients $a_n$ are much small compared to unity. Hence for small field configurations the radiative correction due to radion dominates over that of gravitons. The terms inside the first pair of brackets are independent of $H$ and can be absorbed in the vacuum energy. The terms within the second pair of brackets depend both on the cut off and $H$. These terms can be absorbed in the counter terms that has to be added to the classical higgs potential to make the one loop effective potential insensitive to the cut off. The terms inside the third pair of brackets depend on $H$ but not on the cut off. They represent the renormalized radiative correction to the higgs potential due to graviton fluctuations. We would like to note that if $\alpha_n$ and $\frac{\alpha_n}{a_n}$ are small then $\ln(1 - \alpha_n)$, $\frac{(1+2\alpha_n)}{(1-\alpha_n)}$ and $\frac{(1+2\alpha_n)^2}{(1-\alpha_n)^2}$ can be expanded into an infinite power series in $H$. Since the interaction of the KK graviton modes to the higgs scalar is non-renormalizable we need an infinite number of local counter terms to absorb the cut off dependent terms of the one loop effective potential.

**Effect of the radiative correction on the vacuum stability**

Examining the one loop radiative correction due to graviton fluctuations we find that there is no potentially dangerous tadpole term that could destabilize the classical vacuum. The lowest order term is quadratic in the higgs field. Further it is suppressed by a loop factor of $\frac{1}{64\pi^2}$ and another factor of $\frac{m_H^4}{m_n^4 \Lambda^2}$ arising from KK graviton couplings to higgs scalar. The quadratic term arising from radiative corrections due to gravitons is therefore smaller than the quadratic term present in the classical potential. Further although there are higher order terms in the graviton induced radiative correction these terms are small within the limits of validity of perturbation theory. The higgs field configurations that are needed for these terms to dominate and cause any instability occur outside the limits of ordinary perturbation theory.

The form of the lowest order term in the radiative correction due to the radion is difficult to determine in general since $\gamma$ and $\delta$ have a complicated dependence on $H$. We shall therefore study it under two special cases.
Case I: In this case we shall assume that $1 - \alpha \ll 2\beta$. It then follows that $\gamma \approx \frac{1+\alpha}{2} - \beta$ and $\delta \approx \frac{1+\alpha}{2} + \beta$. Substituting the values of $\gamma$ and $\delta$ we get

\[
\delta V_r \approx \frac{(m_H^2)^2}{32\pi^2} \left[ \beta^2 \left( \frac{m_H^2}{\mu^2} - \frac{1}{2} + \frac{(1 + \alpha)\beta}{2} \right) \ln \frac{1+\alpha + \beta}{1+\alpha - \beta} + \frac{(1 + \alpha)^2}{8} (1 + \frac{4\beta^2}{(1 + \alpha)^2}) \ln \left( \frac{(1 + \alpha)^2}{4} - \beta^2 \right) \right]. \tag{15}
\]

Therefore in this case the radion induced radiative correction give a tadpole term which can dominate over the quadratic and quartic terms in the classical potential for small $H$. However the tadpole term is positive for both signs of $h$ and therefore it does not destabilize the classical vacuum.

Case II: We shall now consider the case when $1 - \alpha \gg 2\beta$. It then follows that $\gamma \approx \alpha - \frac{\beta^2}{(1-\alpha)}$ and $\delta \approx 1 + \frac{\beta^2}{(1-\alpha)}$. Substituting the values of $\gamma$ and $\delta$ in $\delta V_r$ we get

\[
\delta V_r \approx \frac{(m_H^2)^2}{32\pi^2} \left[ \beta^2 \left( \frac{m_H^2}{\mu^2} - \frac{1}{2} + \frac{(1 + \alpha)^2}{2} \ln \frac{1+\alpha + \beta}{1+\alpha - \beta} \right) + \frac{1}{2} \frac{\beta^2}{(1 - \alpha)} (1 + \frac{\beta^2}{1 - \alpha})^2 \right]. \tag{16}
\]

The lowest order term in this case is quadratic in the higgs field. However it is suppressed by a loop factor of $\frac{1}{32\pi^2}$ and an additional factor of $\frac{m_H^2}{(\phi)^2}$ coming from radion couplings. Hence in this case we do not expect any instability to the classical vacuum to develop within the limits of perturbation theory.

Conclusion

In this paper we have determined the one loop radiative correction to the higgs potential due to linear radion and graviton fluctuations in the Randall-Sundrum model. We find that the radiative correction due to graviton fluctuations do not give rise to tadpole term. On the other hand although the radiative correction due to radion fluctuations does give rise to a tadpole term under some cases it is positive for both sign of $H$. The quadratic
and higher order terms in the radiative correction for both kinds of fluctuations are too small compared to the classical potential for any instability to develop within the limits of ordinary perturbation theory.

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