Communication between agents in dynamic epistemic logic *

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Abstract

This manuscript studies actions of communication between epistemic logic agents. It starts by looking into actions through which all/some agents share all their information, defining the model operation that transforms the model, discussing its properties, introducing a modality for describing it and providing an axiom system for the latter. The main part of the manuscript focuses on an action through which some agents share part of their information: they share all that they know about a topic defined by a given formula. Once again, the manuscript defines the model operation that transforms the model, discusses its properties, introduces a modality for describing it and provides an axiom system for the latter.

Keywords: epistemic logic · distributed knowledge · dynamic epistemic logic · full communication · partial communication

1 Introduction

Epistemic logic (EL; Hintikka 1962) is a logical system for reasoning about the knowledge a set of agents might have. On the syntactic side, its language extends propositional logic with a modality $K_i$ for every agent $i$, with formulas of the form $K_i \varphi$ read as “agent $i$ knows that $\varphi$ is the case”. On the semantic side, it typically relies on relational ‘Kripke’ models, assigning to each agent an indistinguishability relation among epistemic possibilities. ¹ The crucial idea is that knowledge is defined in terms of uncertainty: agent $i$ knows that $\varphi$ is the case when $\varphi$ holds in all situations she considers possible.² Despite its simplicity (or maybe because of it), EL has become a widespread tool, contributing to the formal study of complex multi-agent epistemic notions in philosophy (Hendricks 2006), computer science (Fagin et al. 1995, Meyer and van der Hoek 1995) and economics (de Bruin 2010, Perea 2012).

One of the reasons for the success of EL and its variations is that it allows a natural representation of actions that affect the agents’ information (e.g.,

¹ There are other alternatives; see Footnote 5.
² This is the “information as range” discussed in van Benthem and Martinez (2008).
knowledge and beliefs). The two paradigmatic examples are public announcements (Plaza 1989, Gerbrandy and Groeneveld 1997), representing the effect of agents receiving truthful information, and belief revision (van Ditmarsch 2005, van Benthem 2007, Baltag and Smets 2008), representing actions of agents receiving information that is reliable and yet potentially fallible. These two frameworks are part of what is known as dynamic epistemic logic (DEL; van Ditmarsch et al. 2008, van Benthem 2011), a field whose main feature is that actions are semantically represented not as relations (as done, e.g., in propositional dynamic logic, Harel et al. 2000), but rather as operations that transform the underlying semantic model.

The mentioned DEL frameworks have been used for representing communication between agents (e.g., Ågotnes et al. 2010, van Ditmarsch 2014, Baltag and Smets 2013, Galimullin and Alechina 2017). Yet, they were originally designed to represent the effect of external communication, with the information’s source being some entity that is not part of the system. This can be observed by noticing that, in these settings, the incoming information $\chi$ does not need to be known/believed by any of the involved agents.

This manuscript studies epistemic actions in which the information that is being shared is information some of the agents already have. In this sense, the actions studied here are true actions of inter-agent communication. For this, the crucial notion is that of distributed knowledge (Hilpinen 1977, Halpern and Moses 1984, 1985, 1990), representing what a group of agents would know by putting all their information together. Distributed knowledge thus ‘pre-encodes’ the information a group of agents would have if they were to share their individual pieces. Then, the actions studied here can be seen as (variations of) actions that fulfil this promise, doing so by defining the model that is obtained after communication takes place.

In defining these communication actions, it is important to emphasise that, under relational ‘Kripke’ models, epistemic logic defines knowledge in terms of uncertainty. This is because these models only represent the epistemic uncertainty of the agent, without ‘explaining’ why some uncertainty (i.e., epistemic possibility) has been discarded and why some other remains. This has two important consequences.

- First, as discussed in van der Hoek et al. (1999), distributed knowledge does not satisfy the “principle of full communication”: there are situations in which a group knows distributively a formula $\varphi$, and yet $\varphi$ does not follow from the individual knowledge of the groups’ members. Thus, under relational models, distributed knowledge is better understood as what a group of agents would know (in the “information as range” sense) if they indicated to one another which epistemic possibilities they have already discarded.

- Second, recall that an agent’s uncertainty is represented by her indistinguishability relation. Thus, although changes in uncertainty can be represented by changing what each epistemic possibility describes (technically, by changing the model’s atomic valuation), they are more naturally represented by changes in the relation itself.3

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3Note that changing the model’s domain (removing worlds, as when representing public announcements, or adding them, as when representing non-public forms of communication) is an indirect way of changing indistinguishability relations.
This text is organised as follows. Section 2 recalls the basics of EL, including the semantic model representing the agents’ uncertainty, the formal language used for describing them and an axiom system characterising validities. Then, while Section 3 discusses communication actions through which all/some agents share all their information with everybody (comparing them with proposals in the literature), Section 4 discusses a novel action through which some agents share part of their information with everybody. Section 5 is a brief discussion of the issues arising when only some agents receive the shared information. Finally, Section 6 summarises the work, discussing also further research lines. While the proofs of propositions are found within the text, the proofs of theorems can be found in the appendix.

2 Basic system

Throughout this text, let A be a finite non-empty set of agents, and let P be a non-empty enumerable set of atomic propositions.

Definition 2.1 (Multi-agent relational model) A multi-agent relational model (or, simply, a model) is a tuple M = <W, R, V> where W (also denoted as \( \mathcal{T}(M) \)) is a non-empty set of objects called possible worlds, R = \( \{ R_i \subseteq W \times W | i \in A \} \) contains a binary indistinguishability relation on W for each agent in A, and V : P \( \rightarrow \) \( \varphi(W) \) is the atomic valuation indicating the set of possible worlds in which each atom holds. The class of (multi-agent relational) models is denoted by \( M_A \). A pair \((M, w)\) with \( M \in M_A \) and \( w \in \mathcal{T}(M) \) is called a pointed \( M_A \) model (or, simply, a pointed model), with \( w \) being the evaluation point.

Let \( M = <W, R, V> \) be a model. For \( G \subseteq A \), define \( R^D_G := \bigcap_{k \in G} R_k \), with edges in \( R^D_G \) called G-edges. For \( S \subseteq W \times W \) and \( w \in W \), define \( S(w) := \{ u \in W | Swu \} \).

Note: in a model, the indistinguishability relations are arbitrary binary relations. In particular, they need to be neither reflexive nor symmetric nor Euclidean nor transitive, and hence knowledge here is neither truthful nor positively/negatively introspective. The notion of knowledge used here corresponds simply to “what is true in all the agent’s epistemic possibilities”.

Pointed models are described by the following language.

Definition 2.2 (Language \( L_D \)) Formulas \( \varphi, \psi \) of the language \( L_D \) are given by

\[ \varphi, \psi ::= p \mid \neg \varphi \mid \varphi \land \psi \mid D_G \varphi \]

for \( p \in P \) and \( \emptyset \subseteq G \subseteq A \). Boolean constants (\( \top, \bot \)) and other Boolean operators (\( \lor, \rightarrow, \leftrightarrow \)) are defined as usual. Additionally, define \( K_i \varphi ::= D_{\{i\}} \varphi \).

Note how \( L_D \) contains a modality \( D_G \) for each non-empty set of agents \( G \subseteq A \), thanks to which one can build formulas of the form \( D_G \varphi \), read as “the agents in \( G \) have distributed knowledge of \( \varphi \)”. Thus, \( K_i \varphi \) is read as “agent \( i \) has distributed knowledge of \( \varphi \)” or, in other words, “agent \( i \) knows \( \varphi \)”.

Formulas of \( L_D \) are semantically interpreted in pointed models.

Definition 2.3 (Interpreting \( L_D \) on pointed models) Let \((M, w)\) be a pointed model with \( M = <W, R, V> \). The satisfiability relation \( \vdash \) between \((M, w)\) and a formula in \( L_D \) is defined inductively. Boolean cases are as usual; for the rest,
(M, w) \vdash \phi \quad \text{iff}_M w \in V(\phi),
(M, w) \vdash D_{\phi} q \quad \text{iff}_M \text{ for all } u \in W, \text{ if } R^D_{\phi} w \text{ then } (M, u) \not\vdash \phi.

A formula \( \phi \) is valid on \( M_A \) (notation: \( \vdash \phi \)) if and only if \( (M, w) \vdash \phi \) for every \( w \in \mathcal{W}(M) \) of every \( M \) in \( M_A \). By defining the truth-set of a formula as 
\[ \llbracket \phi \rrbracket^M := \{ w \in W \mid (M, w) \vdash \phi \} \] (so \( \llbracket \phi \rrbracket^M \) is the set of \( \phi \)-worlds in \( M \), that is, the worlds in \( M \) where \( \phi \) holds), one can state equivalently that \( \phi \) is valid on \( M_A \) if and only if \( \llbracket \phi \rrbracket^M = \mathcal{W}(M) \) for every \( M \) in \( M_A \).

The semantic interpretation of \( D_{\phi} \psi \) deserves some comments. Recall: \( R^D_{\phi} \) is the intersection of the relations of agents in \( G \). Thus, \( R^D_{\phi} w u \) holds if and only if \( R_i w u \) holds for every \( i \) in \( G \), that is, if and only if every agent in \( G \) considers \( u \) possible when at \( w \) or, equivalently, if and only if no agent in \( G \) can discard \( u \) when at \( w \). Using the notation \( \llbracket \cdot \rrbracket \), the semantic interpretation of \( D_{\phi} \psi \) is equivalently stated as
\[ (M, w) \vdash D_{\phi} \psi \iff R^D_{\phi}(w) \subseteq \llbracket \psi \rrbracket^M. \]

Note also that the abbreviation \( K_i \psi \) behaves as expected:
\[ (M, w) \vdash K_i \psi \iff (M, w) \vdash D_{\{i\}} \psi \iff R^D_{\{i\}}(w) \subseteq \llbracket \psi \rrbracket^M \iff R_{\{i\}}(w) \subseteq \llbracket \psi \rrbracket^M, \]
so agent \( i \) knows \( \psi \) at \( (M, w) \) if and only if every world she cannot distinguish from \( w \) is a \( \psi \)-world.

Example 2.1 Here are some examples of this setting.

(i) Take \( A = \{a, b, c\} \) and \( P = \{p, q, r\} \). Consider \( M_1 = \langle \{w_0, w_1, w_2, w_3\}, R, V \rangle \), a model whose indistinguishability relations and valuation function are as in the diagram below (each world shows exactly the atoms true at it); take \( w_0 \) to be the evaluation point (double-circled in the diagram).

At \( (M_1, w_0) \) all atoms are true; yet, no agent knows this. First, agent \( a \) knows that \( p \) holds, but knows the truth value of neither \( q \) nor \( r \):
\[ (M_1, w_0) \vdash K_a p \land (\neg K_a q \land \neg K_a \neg q) \land (\neg K_a r \land \neg K_a \neg r). \]

Then, \( b \) knows that \( q \) holds, but knows the truth value of neither \( p \) nor \( r \):
\[ (M_1, w_0) \vdash (\neg K_b p \land \neg K_b \neg p) \land K_b q \land (\neg K_b r \land \neg K_b \neg r). \]
Finally, $c$ knows that $r$ holds, but knows the truth value of neither $p$ nor $q$:

$$(M_1, w_0) \vDash (\neg K_c p \land \neg K_c \neg p) \land (\neg K_c q \land \neg K_c \neg q) \land K_c r.$$  

Still, each agent knows that $a$ knows $p$’s truth-value, that $b$ knows $q$’s truth-value, and that $c$ knows $r$’s truth-value:

$$(M_1, w_0) \vDash \bigwedge_{i \in \{a, b, c\}} K_i \left((K_a p \lor K_a \neg p) \land (K_b q \lor K_b \neg q) \land (K_c r \lor K_c \neg r)\right).$$

Finally, agents would benefit from sharing their individual information. In particular, if they all shared, they would know which the real situation is:

$$(M_1, w_0) \vDash D_{\{a, b\}}(p \land q) \land D_{\{a, c\}}(p \land r) \land D_{\{b, c\}}(q \land r) \land D_{\{a, b, c\}}(p \land q \land r).$$

(ii) Let $A$ and $P$ be as in Item (i); consider the pointed model depicted below.

![Diagram](image)

Again, all atoms are true in the real situation; yet, no agent knows this. On the one hand, $a$ knows $p \lor q$ without knowing the truth-value of $p$ or $q$,

$$(M_2, w_0) \vDash K_a (p \lor q) \land (\neg K_a p \land \neg K_a \neg p) \land (\neg K_a q \land \neg K_a \neg q).$$

On the other hand, $b$ knows $q \lor r$ without knowing the truth-value of $q$ or $r$:

$$(M_2, w_0) \vDash K_b (q \lor r) \land (\neg K_b q \land \neg K_b \neg q) \land (\neg K_b r \land \neg K_b \neg r).$$

Agent $c$ has slightly more information, as she knows that $q$ is true but still ignores the truth-value of $p$ and $r$:

$$(M_2, w_0) \vDash (\neg K_c p \land \neg K_c \neg p) \land K_c q \land (\neg K_c r \land \neg K_c \neg r).$$

This time, while communicating would help $a$ and $b$, it would not help $c$. In fact, collectively, the agents do not have enough information to find out which the real situation is:

$$(M_2, w_0) \vDash D_{\{a, b\}} q \land D_{\{a, c\}} q \land D_{\{b, c\}} q \land \neg D_{\{a, b, c\}} (p \land r).$$

(iii) Take $A = \{a, b\}$ and $P = \{p, q\}$; consider $(M_3, w_0)$ depicted below.
In the pointed model, both \( p \) and \( q \) are true. Both agents have partial information about this: while agent \( a \) knows \( p \) but does not know whether \( q \), agent \( b \) does not know whether \( p \) but knows \( q \):

\[
(M_3, w_0) \vdash K_a p \land (\neg K_a q \land \neg K_a \neg q) \land (\neg K_b p \land \neg K_b \neg p) \land K_b q.
\]

However, both agents have misleading information about what the other knows: \( a \) thinks \( b \) knows \( p \) without knowing whether \( q \), and \( b \) thinks \( a \) does not know whether \( p \) but knows \( q \):

\[
(M_3, w_0) \vdash K_a(K_b p \land (\neg K_b q \land \neg K_b \neg q)) \land K_b((\neg K_a p \land \neg K_a \neg p) \land K_a q).
\]

If they were to share their (partially misleading) information, they would believe inconsistencies:

\[
(M_3, w_0) \vdash D_{\{a,b\}} \bot.
\]

**Axiom system** As the reader can imagine, there are other alternatives for defining a logical framework describing the individual and distributed knowledge a set of agents might have. The epistemic logic framework recalled in this section is based on some concrete choices (e.g., relying on a model that represents uncertainty [via relations], and then defining knowledge in terms of it), and these choices define the properties of the notions of knowledge that arise. Still, some properties might not be easy to identify just from the model and the language’s semantic interpretation. In such cases, it is useful to look for an axiom system characterising the formulas in \( L_D \) that are valid in pointed (multi-agent relational) models. By doing so, the axiom system provides a list of the essential laws governing the behaviour and interaction of individual and distributed knowledge in this setting.

It is well-known (e.g., Halpern and Moses 1985, Fagin et al. 1995, Baltag and Smets 2020) that the axiom system \( L_D \), whose axioms and rules are shown on Table 1, characterise the formulas in \( L_D \) that are valid in pointed \( M_k \)-models. While PR and MP characterise the behaviour of Boolean operators, axioms \( K_D \) and rule \( G_D \) characterise distributed knowledge: \( D_k \) contains all validities (rule \( G_D \)), it is closed under modus ponens (axiom \( K_D \)) and it is monotone on the set of agents (axiom \( M_D \): \( \varphi \) is distributively known by the agents in \( G \), then it is also distributively known by any larger group \( G' \) ).

6
\[ \vdash \phi \] for \( \phi \) a propositionally valid scheme

**Table 1: Axiom system \( L_D \), for \( L_D \) w.r.t. models in \( M_k \).**

| PR: \( \vdash \phi \) for \( \phi \) a propositionally valid scheme |
|-------------------------|
| MP: \( \vdash \phi \) and \( \vdash \phi \rightarrow \psi \) then \( \vdash \psi \) |
| \( K D: \vdash D_G(\phi \rightarrow \psi) \rightarrow (D_G \phi \rightarrow D_G \psi) \) |
| \( M_D: \vdash D_G \phi \rightarrow D_G' \phi \) for \( G \subseteq G' \) |

**Theorem 1** The axiom system \( L_D \) (Table 1) is sound and strongly complete for formulas in \( L_D \) w.r.t. models in \( M_k \). \( \blacksquare \)

### 3 Sharing everything with everybody

As discussed before, the concept of distributed knowledge relies on the idea of agents communicating their individual information. Indeed, the fact that a set of agents \( G \) has distributed knowledge of \( \phi \) (\( D_G \phi \)) encodes the intuition that, if they all would share their information, then afterwards each one of them would know that \( \phi \) is the case. This ‘encoding’ can be made explicit by following the \( DEL \) approach: define an operation that, when receiving an initial model representing the agents’ individual information, returns the model that results from the agents sharing their information. The current section explores two variations of this idea: an operation representing an action through which **all** agents share all her information with everybody (Subsection 3.1), and an operation representing an action through which **some** agents share all her information with everybody (Subsection 3.2).

#### 3.1 Everybody shares everything with everybody

The simplest form of communication among \( EL \) agents is one through which **every** agent shares everything with everybody. This will be called an act of \( \forall \forall \forall \)-communication.

**Operation and modality** Recall that a multi-agent relational model represents not the knowledge of each agent, but rather her uncertainty: the worlds she considers possible from a given one. Thus, in this setting, an action through which every agent shares everything she knows with everybody corresponds, formally, to a model operation through which every agent discards every possibility that has been already discarded by any agent. Such an operation is straightforward: the indistinguishability relation of each agent in the new model is defined as the intersection of the relations of all agents in the original model.

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\(^{4}\)Recall that \( D_\emptyset \) is not a modality in \( L_D \). If it were, then \( L_D \) would need further axioms and rules. In fact, since \( R^D(w) = \Upsilon(M) \) for any \( w \in \Upsilon(M) \) and any \( M \in M_k \), it follows that \( (M,w) \vdash D_\emptyset \phi \) if and only if \( (M,u) \vdash \phi \) for all \( u \in \Upsilon(M) \). In other words, \( D_\emptyset \) is nothing but the global universal modality (Goranko and Passy 1992). Because of this, an axiom system for \( D_\emptyset \) requires not only extending \( K_D, G_D \), and \( M_D \) to allow \( G = \emptyset \), but also the use of three additional axioms: \( \vdash D_\emptyset \psi \rightarrow \psi \), \( \vdash D_\emptyset D_\emptyset \phi \rightarrow D_\emptyset \phi \) and \( \vdash D_\emptyset \neg D_\emptyset \psi \rightarrow \neg D_\emptyset \psi \).
Definition 3.1 (∀∀∀-communication) Let \( M = \langle W, R, V \rangle \) be in \( M_A \). The relations in the \( M_A \)-model \( M' = \langle W, R', V \rangle \) are given, for each \( i \in A \), as

\[
R'_i := R'_A.
\]

Thus, after the operation an agent \( i \) cannot distinguish \( u \) from \( w \) (that is, \( R'_i wu \)) if and only if, before the operation, no agent in \( A \) could distinguish \( u \) from \( w \) (that is, \( R_j wu \) for all \( j \in A \)).

It should be emphasised that, despite the given intuition, the just defined operation is not one through which the agents share the information that has allowed them to discard certain possibility. As discussed in the introduction, relational models only represent the epistemic uncertainty of the agents (the edges). Thus, an agent’s communication amounts to sharing the epistemic possibilities she has already discarded (by indicating which edges are not in her indistinguishability relation), so others can discard them too.\(^5\)

Here are some small yet useful observations.

- Each new relation \( R'_i \) can be equivalently defined as \( R_i \cap R'_A \{i\} \).
- Since \( R'_i(w) = R'_A(w) \) is a subset of \( R_i(w) \) for any world \( w \) in any model \( M \), it follows that the action of \( ∀∀∀-\)communication can only reduce the uncertainty of each agent.
- The operation preserves universal relational properties: if all relations in \( \{R_j \mid j \in A\} \) are, e.g., reflexive/transitive/symmetric/Euclidean, then so is each resulting \( R'_i \). Thus, in particular, the operation preserves equivalence relations.

Here is a modality for describing the effects of \( ∀∀∀-\)communication.

Definition 3.2 (Modality \([!]\); language \( L_{D,[!]}/\)) The language \( L_{D,[!]} \) extends \( L_D \) with a modality \([!]\), semantically interpreted in a pointed model \((M, w)\) as

\[
(M, w) \vdash [!] \varphi \quad \text{iff} \quad (M', w) \vdash \varphi.
\]

Using \( \llbracket \cdot \rrbracket \), observe how \( w \in \llbracket [!] \varphi \rrbracket^M \) if and only if \( w \in \llbracket \varphi \rrbracket^M \). Thus,

\[
\llbracket [!] \varphi \rrbracket^M = \llbracket \varphi \rrbracket^M.
\]

Example 3.1 Here are examples of this operation at work.

(i) Recall the model \( M_1 \) from Example 2.1.(i) (diagram below on the left).

\(^5\)Other proposals for representing an agent’s knowledge do keep track of the justifications (Artémov 2008), evidence (van Benthem and Pacuit 2011, Baltag et al. 2016, Özgür 2017) or arguments (Shi et al. 2017, 2021) this knowledge is based on. In such settings, the ‘reasons’ each agent has for her knowledge are present, and thus one can define model operations through which this information is shared.
As shown in the diagram above on the right, the $\forall\forall\forall$ operation removes those edges that were not in the indistinguishability relation of every agent (i.e., it preserves only $A$-edges). Thus,

$$(M_1, w_0) \vDash D_{\{a,b,c\}}(p \land q \land r) \land \left[\bigwedge_{i \in \{a,b,c\}} K_i (p \land q \land r)\right].$$

(ii) Analogously, recall $M_2$ from Example 2.1.(ii), shown below on the left.

Hence, $(M_2, w_0) \vDash D_{\{a,b,c\}} q \land \left[\bigwedge_{i \in \{a,b,c\}} K_i q\right].$

(iii) Finally, recall $M_3$ (Example 2.1.(iii)), shown below on the left.

Thus, $(M_3, w_0) \vDash D_{\{a,b\}} \perp \land \left[\bigwedge_{i \in \{a,b\}} K_i \perp\right].$

**Properties** Here are some properties describing the effects of the $\forall\forall\forall$ operation. First, intuitively, the action turns distributive knowledge of the whole group into individual knowledge. The following proposition shows that this is true only up to a certain extent.
Proposition 3.1 Let \( (M, w) \) be a pointed \( \mathbf{M}_A \) model; take \( \vec{i} \in \mathbf{A} \). Then, \((M, w) \vdash D_\vec{i} \phi \rightarrow [!] \mathbf{K}_\vec{i} \phi \) holds when \( M \) and \( \phi \) are such that \( \llbracket \phi \rrbracket^M \subseteq \llbracket \phi \rrbracket^{M'} \) (i.e., when applying "\( \vdash \)" to \( M \) does not reduce \( \phi \)'s truth-set). However, \( \nvdash D_\vec{i} \phi \rightarrow [!] \mathbf{K}_\vec{i} \phi \).

Proof. By semantic interpretation, \((M, w) \vdash D_\vec{i} \phi \) holds for a given \((M, w)\) if and only if \( R^D_\vec{i}(w) \subseteq \llbracket \phi \rrbracket^M \). But, by the definition of \( R^D_\vec{i} \) (Definition 3.1), \( R^D_\vec{i}(w) \subseteq R^D_\vec{i}(w) \) (\( \forall \forall \forall \) can only reduce uncertainty), so \( R^D_\vec{i}(w) \subseteq \llbracket \phi \rrbracket^M \). Hence, by the assumption, \( R^D_\vec{i}(w) \subseteq \llbracket \phi \rrbracket^M \) and therefore \((M, w) \vdash \mathbf{K}_\vec{i} \phi \), that is, \((M, w) \vdash [!] \mathbf{K}_\vec{i} \phi \).

However, \( \nvdash D_\vec{i} \phi \rightarrow [!] \mathbf{K}_\vec{i} \phi \), as shown by taking \( \phi := \neg \mathbf{K}_\vec{p} \) and observing that, on \( M_1 \) in Example 3.1.(ii), \( (M_1, w_1) \not\vdash D_{\vec{a}, \vec{b}, \vec{c}} \neg \mathbf{K}_\vec{p} \land \neg [!] \mathbf{K}_\vec{b} \mathbf{K}_\vec{p} \). In general, there is not guarantee that every world making \( \phi \) true in a given \( M \) (a world in \( \llbracket \phi \rrbracket^M \)) is also a world making \( \phi \) true in \( M' \) (a world in \( \llbracket \phi \rrbracket^{M'} \)). ■

The fact that \( D_\vec{i} \phi \rightarrow [!] \mathbf{K}_\vec{i} \phi \) is not valid should not be taken to mean that \( \forall \forall \forall \) is flawed. First, this formula one intuitively expects to be valid is so in those situations one intuitively considers: those in which \( \phi \) describes ontic (i.e., propositional) facts.\(^6\) The formula is in fact valid for a larger class of formulas in \( \mathcal{L}_D \) including, e.g., those describing knowledge about propositional formulas \( \{ \mathbf{K}_\vec{p} \gamma \} \gamma \) is propositional\(^7\) and more. Second, the formula is not valid for an arbitrary \( \phi \in \mathcal{L}_D \) because \( \mathcal{L}_D \) can express not only an agent’s (distributed) knowledge but also her ignorance, which might be reduced by the operation. This is used in the counterexample in Proposition 3.1, as \( \neg \mathbf{K}_\vec{b} \mathbf{K}_\vec{p} \) states that “agent \( \vec{b} \) does not know that \( p \) holds”.

Here are two further results.

Proposition 3.2 Let \( M = (W, R, V) \) be in \( \mathbf{M}_A \); let \( M' = (W, R', V) \) and \( (M')' = (W, (R')', V) \) as indicated in Definition 3.1. Then,

\[
\begin{align*}
(i) \quad & (R')_G^D = R^D_G \quad \text{for every } G \subseteq \mathbf{A}; \\
(ii) \quad & (R')^{(i)}_\vec{i} = R^{(i)}_\vec{i} \quad \text{for every } \vec{i} \in \mathbf{A}.
\end{align*}
\]

Proof. (i) For any \( G \subseteq \mathbf{A} \), the relation \( (R')_G^D \) is \( \bigcap_{\vec{k} \in G} R^D_{\vec{k}} \) (def. of \( (R')_G^D \)), equal to \( \bigcap_{\vec{k} \in G} R^D_{\vec{k}} \) (def. of \( (R')_\vec{k}^D \)). (ii) For any \( \vec{i} \in \mathbf{A} \), the relation \( (R')^{(i)}_\vec{i} \) is \( (R')^D_{\vec{i}} \) (def. of \( (R')_\vec{i}^D \)), equal to \( R^D_{\vec{i}} \) (Item (ii)), equal to \( R^{(i)}_{\vec{i}} \) for any \( \vec{i} \in \mathbf{A} \) (def. of \( (R')_\vec{i}^D \)). ■

Proposition 3.2 provides two interesting observations. First, the relation for interpreting \( D_\vec{G} \) in the new model, \( (R')_G^D \), is the same as the relation for interpreting \( D_\vec{G} \) in the original model, \( R^D_\vec{G} \). Thus, for talking about distributed knowledge after \( \forall \forall \forall \) communication (what \( R^D_\vec{G} \) encodes), the modality \([!] \) is not needed: it is enough to use the modality \( D_\vec{G} \) (the one for \( R^D_\vec{G} \)) in the appropriate way (see the translation of Definition A.1, used for the completeness argument).

The second observation states that each relation \( (R')_\vec{i}^D \) in the model after two \( \forall \forall \forall \) acts is exactly as \( R^D_{\vec{i}} \), its matching relation in the model after a single \( \forall \forall \forall \) act. Since \( M' \) and \( (M')' \) have the same domain and atomic valuation, this implies that the \( \forall \forall \forall \) operation is idempotent: \( M' = (M')' \) for any \( M \) in \( \mathbf{M}_A \).

Thus,

\(^6\) The \( \forall \forall \forall \)-act does not affect atomic valuations, so \( \llbracket \gamma \rrbracket^M = \llbracket \gamma \rrbracket^{M'} \) for every \( M \) and every propositional \( \gamma \). Thus, by Proposition 3.1, \( \vdash D_\vec{G} \gamma \rightarrow [!] \mathbf{K}_\vec{G} \gamma \).

\(^7\) By definition, \( w \in \llbracket \mathbf{K}_\vec{p} \gamma \rrbracket^M \) if and only if \( R^D_\vec{p}(w) \subseteq \llbracket \gamma \rrbracket^M \). But \( R^D_\vec{p}(w) \subseteq R_\vec{p} \) and \( \llbracket \gamma \rrbracket = \llbracket \gamma \rrbracket^{M'} \), so \( R^D_\vec{p}(w) \subseteq \llbracket \gamma \rrbracket^{M'} \) and thus \( w \in \llbracket \mathbf{K}_\vec{p} \gamma \rrbracket^{M'} \). Hence, \( \llbracket \mathbf{K}_\vec{p} \gamma \rrbracket^{M'} \subseteq \llbracket \mathbf{K}_\vec{p} \gamma \rrbracket^{M'} \) so \( \vdash D_\vec{p} \mathbf{K}_\vec{p} \gamma \rightarrow [!] \mathbf{K}_\vec{p} \mathbf{K}_\vec{p} \gamma \).
**Proposition 3.3** (i) ⊩ [[!] D_φ ⇔ D_A [[!] φ]; (ii) ⊩ [[!] [[!] φ ⇔ [[!] φ.

**Proof.** Let M = ⟨W, R, V⟩ be in M_A. (i) By definition, w ∈ [[!] D_φ]_M holds if and only if w ∈ [[D_φ]_M, which holds if and only if (R^D_A(w) ⊆ [φ]_M). But, from **Proposition 3.2(i)** and semantic interpretation, the latter is equivalent to R^D_A(w) ⊆ [[!] φ]_M, which holds if and only if w ∈ [[D_A [[!] φ]_M. (ii) By definition, w ∈ [[!] [[!] φ]_M if and only if w ∈ [[!] φ]_M, i.e., if and only if w ∈ [[φ]_M. Then, as a consequence of **Proposition 3.2(ii)**, the latter holds if and only if w ∈ [[φ]_M, that is, if and only if w ∈ [[!] φ]_M.

In particular, the first item of **Proposition 3.3** provides a characterisation of the knowledge any agent 1 has after an act of V-V-communication:

\[ \vdash [[!] K_1 \phi \iff D_A [[!] \phi. \]

**Axiom system** To axiomatise a modality whose semantic interpretation relies on a model operation, a common DEL strategy is to provide *recursion axioms*: valid formulas and validity-preserving rules defining a translation that takes a formula with model-changing modalities (a formula in the ‘dynamic’ language) and returns one without them (a formula in the initial ‘static’ language). To prove soundness within this strategy, it is enough to show that the new axioms and rules are valid and preserve validity; this also shows that a formula and its translation are *semantically* equivalent. To prove completeness, notice that the recursion axioms make a formula and its translation provably equivalent, and thus one can rely on the completeness of the axiom system for the ‘static’ language. The reader is referred to van Ditmarsch et al. (2008, Chapter 7) and Wang and Cao (2013) for an extensive explanation of this technique.

For the case at hand, the recursion axioms appear on **Table 2**. Axiom A^P_1 states that [[!] does not affect the truth value of atomic propositions, and axioms A^P_2 and A^P_3 state, respectively, that [[!] commutes with negation and distributes over conjunction. Axiom A^P_4 (**Proposition 3.3**) indicates that what a group G knows distributively after the operation is exactly what the whole group A knew distributively about [[!]’s effects. Finally, RE, states that replacing logical equivalents within the scope of [[!] preserves logical equivalence. As detailed in the proof of **Theorem 2** (Subsection A.1), these axioms and rule define an ‘inside-out’ translation (see Plaza 1989 and Gerbrandy 1999, Section 4.4) that, when dealing with nested [[!], works first with the deepest occurrence before dealing with the rest.

| A^L_1: \vdash [[!] p ⇔ p | A^P_1: \vdash [[!] D_φ ⇔ D_A [[!] φ |
| A^L_2: \vdash [[!] \neg \phi ⇔ \neg [[!] \phi | RE: If \vdash \phi_1 \iff \phi_2 then \vdash [[!] \phi_1 \iff [[!] \phi_2 |
| A^L_3: \vdash [[!] (\phi_1 \land \phi_2) ⇔ ([![] \phi_1 \land [[!] \phi_2 |

**Table 2:** Additional axioms and rules for L_{D,[!]} that characterises the formulas in L_{D,[!]} that are valid on models in M_A.

**Theorem 2** The axiom system L_{D,[!]} (L_D [**Table 1**]+**Table 2**) is sound and strongly complete for formulas in L_{D,[!]} valid over models in M_A.  ■
3.2 *Somebody* shares everything with everybody

A simple generalisation of the act of $\forall\forall\forall$ communication is one where only some agents share everything they know with everybody. This will be called an act of $\exists\forall\forall$-communication.

**Operation and modality** The operation representing $\exists\forall\forall$ communication is a simple variation of the $\forall\forall\forall$-case. First, worlds and valuation are preserved, as before. Then, the indistinguishability relation of each agent in the new model is defined as the intersection of her original relation with those of the agents that share their information (the senders).

**Definition 3.3 ($\exists\forall\forall$-communication)** Let $M = \langle W, R, V \rangle$ be in $M_A$; take $S \subseteq A$. The relations in the $M_A$-model $M^S_A = \langle W, R^S, V \rangle$ are given, for each $i \in A$, as

$$R^S_i := R_i \cap R^D_S.$$  

Thus, after the operation, an agent $i$ cannot distinguish $u$ from $w$ (that is, $R^S_i w u$) if and only if, before the operation, neither she nor any agent in $S$ could distinguish $u$ from $w$ (that is, $R_i w u$ and $R^D_S w u$). Here are some small and yet useful observations.

- Obviously, $R^S_i = R^D_{S \cup \{i\}}$ and, moreover, $i \in S$ implies $R^S_i = R^D_S$.

- For any world $w$ in any model $M$, the set $R^S_i(w) = R_i(w) \cap R^D_S(w)$ is a subset of $R_i(w)$. Hence, just as $\forall\forall\forall$, the action of $\exists\forall\forall$-communication can only reduce the uncertainty of each agent.

- As one might expect, an $\exists\forall\forall$-communication with $A$ the communicating agents is exactly an $\forall\forall\forall$-communication, as $R^A_i = R^S_i$ for every $i \in A$ in any model $M = \langle W, R, V \rangle$.

- If $R_i$ and the relations in $\{R_j \mid j \in S\}$ are all reflexive (resp., transitive, symmetric, Euclidean), then so is the resulting $R^S_i$. In particular, the $\exists\forall\forall$ operation preserves equivalence relations.

Here is the associated modality.

**Definition 3.4 (Modality $[S!]$; language $L_{D,[S]}$)** The language $L_{D,[S]}$ extends $L_D$ with a modality $[S!]$ for each set of agents $S \subseteq A$. Its semantic interpretation in a pointed model $(M, w)$ is given by

$$(M, w) \vdash [S!] \varphi \iff \text{def} \ (M^S, w) \vdash \varphi.$$  

Using the alternative notation, $w \in \llbracket [S!] \varphi \rrbracket^M$ if and only if $w \in \llbracket \varphi \rrbracket^{M^S}$. Thus,

$$\llbracket [S!] \varphi \rrbracket^M = \llbracket \varphi \rrbracket^{M^S}.$$  

**Example 3.2** Here are examples of this operation at work.

(i) Recall the model $M_1$ from Example 2.1(i) (diagram below on the left).
The action \( \{a, b\}\) produces the model depicted by the diagram above on the right. The new uncertainty of each sharing agent is given by the old uncertainty of the sharers (thus, e.g., \( R_{\{a, b\}\_a} = R_{\{a, b\}\_a}^D \)), and the new uncertainty of each non-sharing agent is given by the old uncertainty of the sharers and herself (thus, e.g., \( R_{\{a, b\}\_c} = R_{\{a, b\}\_c}^D \)). Thus,

\[
(M_1, w_0) \models \bigwedge \left\{ D_{\{a, b\}}(p \land q) \land \bigwedge_{r} K_c r, \left[ \{a, b]\}\right] \left( \bigwedge_{k \in \{a, b\}} (K_k(p \land q) \land (\neg K_r r \land \neg K_r \neg r)), K_c(p \land q \land r) \right) \right\}.
\]

In words, while \( a \) and \( b \) know \( p \land q \) distributively, \( c \) knows \( r \). Then, after \( a \) and \( b \) share all their information to everyone, they both get to know \( p \land q \) but still do not know whether \( r \). However, \( c \) gets to know what the real situation is. Analogous situations result if the communicating agents are \( \{a, c\} \) or \( \{b, c\} \).

(ii) Now recall \( M_2 \) from Example 2.1.(ii) (diagram below on the left).

The model \( M_{2\{a, b\}} \), which results from \( a \) and \( b \) sharing all their information to everyone, appears above on the right. Note that the same model results if \( c \) is the only communicating agent (i.e., \( M_{2\{a, b\}} = M_{2\{c\}} \)), and also if the communication is \( \forall \forall \forall \) (i.e., \( M_{2\{a, b\}} = M_{2\{c\}}^3 \); see Example 3.1.(iii)). Thus,

\[
(M_2, w_0) \models \left( [\{a, b\}] \varphi (\leftrightarrow [c]) \varphi \right) \land \left( [\{a, b\}] \varphi (\leftrightarrow [1]) \varphi \right).
\]

The reason for this is that what agents in \( \{a, b\} \) know distributively is exactly what \( c \) knows individually (i.e., \( R_{\{a, b\}} = R_c \)).
Properties What is the effect of a \( \exists \forall \) operation? Intuitively, the action turns distributive knowledge of a group into individual knowledge of the group's members. Just as in the \( \forall \forall \) case, this is true only up to a certain extent.

Proposition 3.4 Let \((M, w)\) be a pointed \(M_k\) model; take \(S \subseteq A\) and \(i \in A\). Then, \((M, w) \nvdash D_S \varphi \rightarrow [S!] K_i \varphi\) holds when \(M\) and \(\varphi\) are such that \([\varphi]^M \subseteq [\varphi]^M_k\). However, \(\nvdash D_S \varphi \rightarrow [S!] K_i \varphi\).

Proof. As that of Proposition 3.1, here using \(R^{S_1}_i(w) \subseteq R_1(w) (S_i\) can only reduce uncertainty). For showing \(\vdash D_S \varphi \rightarrow [S!] K_i \varphi\) for \(i \in S\), take \(\varphi := \neg K_p\) and note that, on \(M_1\) in Example 3.2.(i), \((M_1, w_0) \nvdash D_{\{a\}} \neg K_p \land \neg \neg ([a, b!] \neg K_p). Again, there is not guarantee that every world making \(\varphi\) true in a given \(M\) (a world in \([\varphi]^M)\) is also a world making \(\varphi\) true in \(M^{S_i}\) (a world in \([\varphi]^{M^k}\)).

As in the \(\forall \forall \) case, \(\nvdash D_S \varphi \rightarrow [S!] K_i \varphi\) for \(i \in S\) should not be taken as a drawback for the \(\exists \forall \) operation: the formula is valid for the situations one intuitively considers (i.e., when \(\varphi\) is propositional) and also for further fragments of \(L_D\) (see the discussion after Proposition 3.1).

Here are two further results.

Proposition 3.5 Let \(M = \langle W, R, V \rangle\) be in \(M_k\); take \(S, S_1, S_2 \subseteq A\). Let \(M^{S_1} = \langle W, R^{S_1}, V \rangle\) and \((M^{S_2})^{S_1} = \langle W, (R^{S_2})^{S_1}, V \rangle\) be as indicated in Definition 3.3. Then,

\[
(i) \quad (R^{S_1}_i)^D = R^{D_{S_1}}_{S_1} \quad \text{for} \quad G \subseteq A; \quad (ii) \quad (R^{S_1}_{S_1})^{S_1}_i = R^{(S_1 \cup S_1)}_{S_1} \quad \text{for} \quad i \in A.
\]

Proof. (i) For any \(G \subseteq A\), the relation \((R^{S_1}_i)^D\) is \(\bigcap_{k \in G} R^{S_1}_i\) (def. of \((R^{S_1}_i)^D\)), equal to \(\bigcap_{k \in G} R^{D}_{S_1/k}\) (def. of \(R^{D}_{S_1/k}\)), equal to \(R^{D}_{S_1/G}\). (ii) For any \(i \in A\), the relation \((R^{S_1}_{S_1})^{S_1}_i\) is \((R^{S_1}_{S_1})^{S_1}_i\) (def. of \((R^{S_1}_{S_1})^{S_1}_i\)), equal to \(\bigcap_{k \in G} R^{S_1}_{S_1/k}\) (def. of \((R^{S_1}_{S_1})^{S_1}_i\)), equal to \(R^{S_1}_{S_1/G}\) (def. of \(R^{S_1}_{S_1/G}\)), equal to \(R^{D}_{S_1/G}\) (def. of \(R^{D}_{S_1/G}\)), equal to \(R^{D}_{S_1/G}\).

Analogous to Proposition 3.2 for the \(\forall \forall \) case, Proposition 3.5 states two facts. First, it states that the relation for interpreting \(D_g\) in the new model, \((R^{S_1}_i)^D\), is the same as the relation for interpreting \(D_{S_1/G}\) in the original model, \(R^{D}_{S_1/G}\). As in the \(\forall \forall \) case, this provides a validity (Proposition 3.6.(i)) that is crucial for an axiomatisation by translation.

The second fact is relative to how two consecutive \(\exists \forall \) acts can be ‘comprised’ into a single one. More precisely, it indicates how a \(\exists \forall \) construction in which group \(S_1\) shares first and group \(S_2\) shares second can be replaced by a single \(\exists \forall \) act in which the sharing agents are those in \(S_1 \cup S_2\). Given that \((M^{S_1})^{S_1}_i\) and \(M^{S_1 \cup S_2}\) have the same domain and atomic valuation, this implies that \((M^{S_1})^{S_1}_i = M^{S_1 \cup S_2}\) holds for any \(M\) in \(M_k\).

Thus,

Proposition 3.6 (i) \(\nvdash [S!] D_g \varphi \leftrightarrow D_{S_1/G} [S!] \varphi\); (ii) \(\nvdash [S_1!] [S_2!] \varphi \leftrightarrow ([S_1 \cup S_2]!) \varphi\).

Proof. Let \(M = \langle W, R, V \rangle\) be in \(M_k\). (i) By definition, \(w \in [[S!] D_g \varphi]^M\) holds if and only if \(w \in \left[[D_g \varphi]^M\right]_k\), which holds if and only if \((R^{S_1}_i)^D(w) \subseteq [\varphi]^M_k\). But, from Proposition 3.5.(i) and semantic interpretation, the latter is equivalent to \(R^{D}_{S_1/G}(w) \subseteq [\varphi]^M_k\), which holds if and only if \(w \in [[D_g \varphi]^M\) (ii) By
definition, \( w \in [S_1 !] [S_2 !] \varphi \)^M if and only if \( w \in [S_2 !] \varphi \)^M, i.e., if and only if \( w \in [\varphi]^{M^{\neg \neg} [S_2 !]} \). Then, as a consequence of Proposition 3.5.(ii), the latter holds if and only if \( w \in [\varphi]^{M^{\neg \neg} [S_2 !]} \), that is, if and only if \( w \in [[S_1 \cup S_2 !] \varphi]^M \). 

Thanks to these validities and previous observations, one can find further validities describing properties of an act of \( \exists \forall \forall \) communication. First, as it was mentioned, the \( \forall \forall \forall \) operation of the previous subsection is the particular case of the \( \exists \forall \forall \) operation in which all agents share. Thus,

\[
\vdash [!] \varphi \leftrightarrow [A!] \varphi.
\]

From Proposition 3.6.(i), it follows that the knowledge any agent \( i \) has after an act of \( \exists \forall \forall \)-communication is what was distributively known among her and the sharing agents about the effects of the action:

\[
\vdash [S_i ] K_i \varphi \leftrightarrow D_{\{i\} \cup S} [S_i ] \varphi.
\]

From Proposition 3.6.(ii) it follows that an act of \( \exists \forall \forall \)-communication by the same group is idempotent:

\[
\vdash [S_i ] [S_i ] \varphi \leftrightarrow [S_i ] \varphi.
\]

Moreover, it also follows that, when two groups communicate, the order in which they do so is irrelevant:

\[
\vdash [S_1 !] [S_2 !] \varphi \leftrightarrow [S_2 !] [S_1 !] \varphi.
\]

Thus, repeated \( \exists \forall \forall \)-sharing by the same group does not provide anything new, regardless of whether it is immediate or after some other groups have shared:

\[
\vdash [S_1 !] \cdots [S_m !] [S_1 !] \varphi \leftrightarrow [S_1 !] \cdots [S_m !] \varphi.
\]

**Axiom system** The axiom system for the extended language \( \mathcal{L}_{D,[S]} \) relies again on the DEL reduction axioms technique, with the axioms and rule for the case at hand being those on Table 3. Axioms \( \mathbb{A}_{\exists \forall}^\mathcal{D}, \mathbb{A}_{\forall \forall}^\mathcal{D}, \mathbb{A}_{\exists \forall}^\mathcal{D} \) and rule \( \mathbb{R}_{\exists \forall}^\mathcal{D} \) are as in the \( \forall \forall \forall \) case, indicating respectively that \( [S_i ] \) does not affect atomic propositions, commutes with negation, distributes over conjunction and ‘preserves’ logical equivalence. Axiom \( \mathbb{A}_{\exists \forall}^\mathcal{D} \) is the one that distinguishes \( \exists \forall \forall \) from \( \forall \forall \forall \), indicating that what a group \( G \) knows distributively after \( [S_i ] \) is exactly what the group \( S \cup G \) knew distributively about \( [S_i ] \)'s effects. Together, these axioms and rule define an ‘inside-out’ translation from \( \mathcal{L}_{D,[S]} \) to \( \mathcal{L}_{D} \) such that a formula and its translation are both semantically and provably equivalent (see the proof of Theorem 3 on Subsection A.2).

**Theorem 3** The axiom system \( \mathcal{L}_{D,[S]} \) (\( \mathcal{L}_{D} \) [Table 1]+Table 3) is sound and strongly complete for formulas in \( \mathcal{L}_{D,[S]} \) valid over models in \( M_A \).

### 3.3 Other operations for communication in the literature

The operations proposed in this section are not the first ones representing actions of agents sharing their individual information. The action of “tell
us all you know” of Baltag (2010) is one through which a single agent \( a \in A \) shares all her information with every agent. Thus, the new indistinguishability relation of each agent \( i \in A \) is defined as \( R_i \cap R_a \). This action can be seen as the particular instance of the act of \( \exists \forall \forall \)-communication (Subsection 3.2) in which only one agent communicates (thus corresponding to the modality \([i]!])\).

Then there is the action for ‘resolving the distributed knowledge of a group’ studied in Ågotnes and Wáng (2017). Through it, agents in a group \( G \) share all their information only within \( G \) itself. Thus, while the new indistinguishability relation of every agent not in \( G \) remains exactly as before, that of each agent in \( G \) is defined as \( R^D_G \). This action can be also seen as an instance of the act of \( \exists \forall \forall \)-communication (Subsection 3.2) (only agents in \( G \) share, but note that only agents in \( G \) receive the information. In the terminology of this manuscript, this action is better described as an act of ‘\( \exists \forall \exists \)-communication’ (Section 5 will discuss briefly \( \exists \exists \exists \)-communication). Finally, there is the more general action of ‘semi-public reading events’ of Baltag and Smets (2020), where each agent \( i \) gets a set \( a(\alpha(i)) \subseteq A \) satisfying \( i \in a(\alpha(i)) \). Intuitively, \( a(\alpha(i)) \) contains those agents whose information \( i \) will receive when communication occurs. Thus, for each agent \( i \), the operation defines her new indistinguishability relation as \( R^D_{\alpha(i)} \). Note again that this form of communication is semi-public: even though each agent \( i \) only ‘hears’ what agents in \( a(\alpha(i)) \) ‘say’, the definition implies that \( i \) still learns that every agent \( j \) receives the information provided by \( j \)'s sources (agents in \( a(j) \)). This can be seen as a generalisation of a ‘\( \exists \forall \forall \)'-communication: while some agents (those in \( a(\alpha(i)) \) for some agent \( i \)) share all their information, every agent receives information from potentially different agents (each agent \( i \) only ‘listen to’ those agents in \( a(\alpha(i)) \)).

## 4 Sharing something with everybody

Section 3 discussed model operations for actions through which all/some agents share all the information with everybody; thus, the actions represent acts of full communication (from a subset of agents). As discussed, the operations are small variations of proposals already present in the literature.

This section, which constitutes the core of this contribution, discusses a variation of the \( \exists \forall \forall \) case: one in which the sharing agents communicate only part of their information. This action makes the process of communication between

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Table 3: Additional axioms and rules for \( L_{D,[S]} \), which characterises the formulas in \( L_{D,[S]} \) that are valid on models in \( M_\alpha \).

\[
\begin{align*}
\mathcal{A}_{S_i}^p: \vdash [S_i]p & \iff p & \mathcal{A}_{S_i}^{D_i}: \vdash [S_i]D_{S_i}q & \iff D_{S_i}[S_i]q \\
\mathcal{A}_{\neg}^S: \vdash [S]\neg q & \iff \neg [S]q & \mathcal{R}_{E_S}: \text{if } \vdash q_1 \iff q_2 \text{ then } \vdash [S]q_1 \iff [S]q_2
\end{align*}
\]

---

8With this definition, it is interesting to notice that, while agents not in \( G \) do not receive the information that is being shared, they still get to know that agents in \( G \) shared their information within themselves. Thus, using the terminology of Baltag and Smets (2020), ‘resolving distributed knowledge’ is a semi-public form of communication.
agents a more realistic one. Indeed, there are natural restrictions on the ‘amount’ of information an agent can communicate at once, and operations with such restrictions can represent more realistic ‘conversations’. In defining this operation, probably the most important question is the following: what defines what each agent will communicate? There are indeed several possibilities (see the discussion in Section 6), but a natural one is to assume that the ‘conversation’ is relative to a given subject/topic. This manuscript uses this idea, assuming that this subject/topic is defined by a formula $\chi$. Following the previous notation, this action through which some agents share some of their information (that relative to the given subject $\chi$) with everybody will be called an act of $\exists \forall$-communication.

Operation and modality For the definitions, the following will be useful.

Definition 4.1 (Relations $\prec_{\chi}$ and $\sim_{\chi}$) Let $M$ be in $\mathbf{M}_A$; let $\chi$ be a formula that can be evaluated at worlds in $M$. The relation $\prec_{\chi}^M \subseteq \mathcal{T}(M) \times \mathcal{T}(M)$ is given by

$$\mathcal{R}_{\chi}^M := \left(\mathcal{R}_\chi^M \times \mathcal{R}_{\neg \chi}^M\right) \cup \left(\mathcal{R}_{\neg \chi}^M \times \mathcal{R}_\chi^M\right).$$

Its complement, the (note: equivalence) relation $\sim_{\chi}^M$ given by $(\mathcal{R}_\chi^M \times \mathcal{R}_\chi^M) \cup (\mathcal{R}_{\neg \chi}^M \times \mathcal{R}_{\neg \chi}^M)$, will be denoted rather as $\sim_{\chi}^M$.

Note how $\sim_{\chi}^M$ describes the indistinguishability of an agent that has full uncertainty about $\chi$ (worlds differing on $\chi$’s truth-value cannot be distinguished) while also having full certainty about everything else (all other worlds can be distinguished). Analogously, $\sim_{\chi}^M$ describes the indistinguishability of an agent that has full certainty about $\chi$ (worlds differing on $\chi$’s truth-value can be distinguished) while also having full uncertainty about everything else (worlds agreeing on $\chi$’s truth-value are indistinguishable). In other words, while the relation $\sim_{\chi}^M$ contains the pairs of worlds in $\mathcal{T}(M) \times \mathcal{T}(M)$ that would be indistinguishable if the available information allowed to tell apart any pair of formulas but $\chi$ and $\neg \chi$, the relation $\sim_{\chi}^M$ contains the pairs of worlds in $\mathcal{T}(M) \times \mathcal{T}(M)$ that would be indistinguishable if the available information allowed to tell apart $\chi$ and $\neg \chi$, and nothing else. Thus, while $\sim_{\chi}^M$ can be seen as a relation of full ignorance on $\chi$, $\sim_{\chi}^M$ can be seen as a relation of knowing only $\chi$ fully.

Here is, then, the definition of the operation for $\exists \forall$-communication.

Definition 4.2 ($\exists \forall$-communication) Let $M = (\mathcal{W}, R, V)$ be in $\mathbf{M}_A$; take $S \subseteq A$ and let $\chi$ be a formula. The relations in $M^{S,i} = (\mathcal{W}, R^{S,i}, V)$ (a structure in $\mathbf{M}_A$) are given, for each $i \in A$, as

$$R^{S,i} : = R_i \setminus \bigcup_{j \in S} \left(R_j \cap \mathcal{R}_{\chi}^M\right).$$

In order to understand the intuition behind the definition, note how the set $\overline{R_j} \cap \mathcal{R}_{\chi}^M$ can be seen as the epistemic contribution of agent $j$ about formula $\chi$ in model $M$, as it contains those pairs in the $M$-uncertainty about $\chi$ (the set $\mathcal{R}_{\chi}^M$) that $j$ has already discarded (the set $\overline{R_j}$). With this in mind, the definition states that agent $i$’s uncertainty after agents in $S$ share all their information on
subject $\chi$ (i.e., her uncertainty in the new model, $R^i_{\vee j}$) is given by her previous uncertainty (i.e., $R_i$) minus the sum ($\cup$) of the epistemic contribution (i.e., $R_j \cap \sim M$) of each agent $j$ in $S$.

Note also how, through the operation, agents in $S$ communicate all they know about $\chi$: intuitively, they share all the information that has allowed them to discard any uncertainty between $\chi$- and $\sim \chi$-worlds, and thus only edges in $\sim M$ can be eliminated. This emphasises the fact that $\chi$ is taken to be the subject/topic of the conversation, with agents intuitively sharing what has allowed them to discard edges between worlds disagreeing on $\chi$’s truth-value.

The relations in the model resulting from $\exists \forall$-communication can be described in a simpler way.

**Proposition 4.1** Let $M = (W, R, V)$ be in $M_A$; take $S \subseteq A$ and let $\chi$ be a formula. Then,

$$R^S_{\vee i} = R_i \cap (R^D_S \cup \sim M).$$

**Proof.** By definition, $R^S_{\vee i} := R_i \setminus \bigcup_{j \in S} (R_j \cap \sim M)$. Then, from the definitions of both set subtraction and $\sim M$, the right-hand side becomes $R_i \cap \bigcap_{j \in S} (R_j \cup \sim M)$. On the latter, using $\bigcap_{j \in S} (R_j \cup \sim M) = R^D_S \cup \sim M$ produces the required $R_i \cap (R^D_S \cup \sim M)$.

Before discussing the $\exists \forall$ operation further, here are some examples.

**Example 4.1** Recall the model $M_1$ from **Example 2.1(i)**, where each agent knows the truth-value of one atom ($a$ knows whether $p$, $b$ knows whether $q$, $c$ knows whether $r$) but does not know the truth-value of the others.

(i) The diagrams below (reflexivity assumed) depict $M_1$ three times. They show, respectively, the partition generated by the (recall: equivalence) relations $\sim_p$, $\sim_q$ and $\sim_r$.

```
\begin{center}
\begin{tikzpicture}
\node (p) at (0,0) {$p$};
\node (q) at (1,0) {$q$};
\node (r) at (2,0) {$r$};
\node (pqr) at (1,1) {$p,q,r$};
\draw (p) edge (pqr)
(p) edge (q)
(p) edge (r);
\end{tikzpicture}
\end{center}
```

The diagrams further below (reflexivity assumed) show the result of three communication acts, all with $S = \{a, b, c\}$: the first on topic $p$, the second on topic $q$ and the third on topic $r$. When building the relations of the

---

9The definition might be easier to grasp by taking a ‘knowledge’ perspective. Intuitively, the knowledge of the agent after a conversation in which agents in $S$ share what they know about $\chi$ is her initial knowledge plus the knowledge any agent in $S$ has about $\chi$. The provided definition describes exactly the same idea, stating it in terms of the agent’s uncertainty.

10There is a natural alternative in which $\chi$ is rather taken to be the content of the conversation. In this asymmetric version, agents intuitively share only what has allowed them to discard $\sim \chi$ as a possibility, and thus only edges pointing to $\sim \chi$-worlds can be eliminated.
new model, the operation looks at the original model, focussing only on edges between worlds disagreeing on the topic’s truth-value (edges across partition cells in the diagrams above) and leaving the rest as they are. For example, for topic \( p \), the operation focusses on edges between worlds in \( \{w_0, w_1, w_2\} \) and worlds in \( \{w_3\} \), disregarding (i.e., not affecting) the rest. When all agent share, as in this case, the operation simply removes the edges under consideration (i.e., across partition cells) that are not in \( R_D^p \).

As the diagrams show, the operation behaves as expected. For example, a conversation among all agents about \( p \) (leftmost model) benefits \( b \) and \( c \) (they get to know \( p \)'s truth-value) but does not benefit the only agent who knew \( p \)'s truth-value before, namely \( a \).11

(ii) Again, below are three copies of \( M_1 \) (reflexivity assumed), this time showing (respectively) the partition generated by the relations \( \sim_{p\land q} \), \( \sim_{p\land r} \), and \( \sim_{q\land r} \).

The diagrams further below (reflexivity assumed) show the result of three communication acts: \( \{a, b\}_{p\land q!} \), \( \{a, b\}_{p\land r!} \) and \( \{a, b\}_{q\land r!} \).

11To be more precise, the action does not give \( a \) any factual information. Yet, she gets information, as after the conversation she knows that both \( b \) and \( c \) know \( p \)'s truth-value.
Example 4.1. However, the operation does not preserve transitivity. Still, one can find specific situations (a, b) such that the (reflexive) relation χ satisfies the requirement for every M and every S. There is no χ satisfying χR ⊆ χR for every M and S. For every model M = (W, R, V) and every set of agents S, an ‘S-conversation about χ’ (producing χR ⊆ χR) has the same effects as an ‘S-conversation about everything’ (producing χR ⊆ χR), for every M and every S. If R ⊆ R, then R ∪ R = R. An ‘S-conversation about χ’ has the same effects as an ‘S-conversation about everything’.

After the examples, here are some observations.

- As R ⊆ R, an ‘S-conversation about χ’ produces R = R for every M and every S. If R = R, then R = R. However, the operation does not preserve transitivity and neither ‘Euclideanity’ (see, e.g., Example 4.1(ii), in particular the effects of the action [a, b] on the relation for agent a).

- As R = R, an ‘S-conversation about χ’ produces R = R for every M and every S.

- For reflexivity, take any w ∈ W. By the assumption, R and R are reflexive, moreover, w = wR. Thus, R. For symmetry, if w = w then R and either R or w = wR. But, by the assumptions and R’s symmetry, R and either R or u = wR. Thus, R.
Now, the modality.

Definition 4.3 (Modality $[S,1]$; language $L_D,[S,1]$) The language $L_D,[S,1]$ extends $L_D$ with modalities $[S,1]$ for each set of agents $S \subseteq A$ and each formula $\chi$. For their semantic interpretation,

$$(M,w) \vdash [S,1] \varphi \iff (M^S_1, w) \vdash \varphi.$$  

Using $\models$, note that $w \in \llbracket [S,1] \varphi \rrbracket^M$ if and only if $w \in \llbracket \varphi \rrbracket^{M_1}$. Thus,

$$\llbracket [S,1] \varphi \rrbracket^M = \llbracket \varphi \rrbracket^{M_1}.$$  

Properties Here are some observations about the $\exists \forall$ operation, starting again with a caveat.

Proposition 4.2 Let $(M, w)$ be a pointed $M_A$ model; take $S \subseteq A$ and $i \in A$. Then, $(M, w) \vdash D_S \varphi \rightarrow [S^p,1] K_i \varphi$ holds when $M$ and $\varphi$ are such that $\llbracket \varphi \rrbracket^M \subseteq \llbracket \varphi \rrbracket^{M_1}$ and the relations in $M$ are reflexive. However, $\varphi \not\in D_S \varphi \rightarrow [S^p,1] K_i \varphi$.

Proof. Since $(M, w) \vdash D_S \varphi$, then $R^D_S(w) \subseteq \llbracket \varphi \rrbracket^M$ and thus, by the assumption, $R^D_S(w) \subseteq \llbracket [S,1] \varphi \rrbracket^M$. To obtain $(M, w) \vdash [S^p,1] K_i \varphi$, one requires $(M^S_1, \varphi) \vdash \chi_1$; i.e., for each set of agents $S$.

Both requirements are essential. In Fact 4.1 it will be shown that reflexivity without $\llbracket \varphi \rrbracket^M \subseteq \llbracket [S,1] \varphi \rrbracket^M$ is not enough. For $\llbracket [S,1] \varphi \rrbracket^M \subseteq \llbracket \varphi \rrbracket^{M^S_1}$ without reflexivity, consider a model $M = (W, R, V)$ with $W = \{w_0, w_1, w_2\}$, $R_S = \{w_0, w_1\}$, $R_1 = R_S \cup \{w_0, w_1\}$, and $V(p) = \{w_2\}$. Note how $(M, w_0) \vdash D \sqcap \neg \varphi$; yet, $(M, w_0) \not\models [\square A, \neg \varphi] K_i p$.

The following proposition provides the two useful observations.

Proposition 4.3 Let $M = (W, R, V)$ be in $M_A$; take $S_1, S_2, T_1 \subseteq A$ and let $\chi_1, \chi_2, \chi_3$ be formulas. Let $M^S_1 = (\chi_1, \chi_2, \chi_3)$ and $M^S_2 = (\chi_4, \chi_5, \chi_6)$ be as indicated in Definition 4.2. Then,

(i) $(R^{S_1,1})^D_{G_s} = (R^{S_2,1} \cup \neg \chi_1)$ for every $G_s \subseteq A$;

(ii) $(R^{S_1,1} \cup \neg \chi_1)_{i_1, i_2} = R_1 \cap (R^{S_2,1} \cap \neg \chi_1)_{i_1, i_2}$ for every $i \in A$.

Proof.

(i) For any $G_s \subseteq A$, the relation $(R^{S_1,1})^D_{G_s}$ is the join of $R^{S_1,1}$ (def. of $(R^{S_1,1})^D_{G_s}$, equal to $\bigcap_{k \in G_S} (R_k \cap (R^D \cup \neg \chi_1))$) and thus to $(R_1 \cup \neg \chi_1)$.

(ii) For any $G_s \subseteq A$, the relation $(R^{S_1,1} \cup \neg \chi_1)_{i_1, i_2}$ is the join of $R^{S_1,1}$ (def. of $(R^{S_1,1})^D_{G_s}$) and thus to $(R_1 \cup \neg \chi_1)$.
(ii) By algebraic manipulations starting from
\[(R^{S\chi_1})^1_{\chi_2} = R^{S\chi_1} \cap \left((R^{S\chi_1})^D_{\chi_2} \cup \sim^M_{\chi_2}\right),\]
using the fact that \(\sim^M_{\chi_2} = \sim^M_{\chi_1,\chi_2,1}\).  
\[\]

On the one hand, the second result of the previous proposition shows the indistinguishability relation of an agent \(i\) after two \(\forall\forall\forall\) operations in a row. By comparing it with the same relation after a single \(\forall\forall\forall\) operation, \(R^{S\chi_1} = R_i \cap (R^D \cup \sim^M_i)\), one understands why, different from the two previous cases, two successive \(\forall\forall\forall\) actions cannot be compressed into a single one.

On the other hand, the first result indicates that the relation for interpreting \(D\) in the new model, \((R^D)^D\) can be described as \(R^{D \cap \sim M_i}\). This, together with the abbreviation
\[D^\chi_i \varphi := (\chi \rightarrow D_\varphi (\chi \rightarrow \varphi)) \land (\sim \chi \rightarrow D_\varphi (\sim \chi \rightarrow \varphi))\]
so \((M, w) \models D^\chi_i \varphi\) if and only if \((R^D \cap \sim M)(w) \subseteq \llbracket \varphi \rrbracket^M\) provides the following validity, which will be useful for the axiomatisation.

**Proposition 4.4** \(\models [S_i \chi] \varphi \iff \left(D_{S,\chi} [S_i \chi] \varphi \land D_{\chi}^D [S_i \chi] \varphi\right)\).

**Proof.** Let \(M = \langle W, R, V \rangle\) be in \(M_{\chi}\). By definition, \(w \in \llbracket [S_i \chi] \varphi \rrbracket^M\) if and only if \(w \in \llbracket D_\varphi \rrbracket^M\), which holds if and only if \((R_{S,\chi})^D(w) \subseteq \llbracket \varphi \rrbracket^M\). But, from **Proposition 4.3(i)** and semantic interpretation, the latter is equivalent to \((R^D_{S,\chi} \cup (R^D \cap \sim M_i)(w)) \subseteq \llbracket \varphi \rrbracket^M\), that is, to the conjunction \(R^D_{S,\chi} \subseteq \llbracket \varphi \rrbracket^M\) and \((R^D \cap \sim M_i) \subseteq \llbracket \varphi \rrbracket^M\), whose conjuncts are equivalent, respectively, to \(w \in \llbracket D_{S,\chi} [S_i \chi] \varphi \rrbracket^M\) and \(w \in \llbracket D_{\chi}^D [S_i \chi] \varphi \rrbracket^M\).

This validity yields immediately the following one, characterising the knowledge of an agent after \(\forall\forall\forall\) communication:
\[\models [S_i \chi] K_\chi \varphi \iff \left(D_{S,\chi} [S_i \chi] \varphi \land D_{\chi}^D [S_i \chi] \varphi\right)\]

Here is another useful validity, a consequence of an earlier observation:
\[\models [S_i \chi] \varphi \iff [S_{\sim \chi}] \varphi.\]

In the actions of **Section 3**, each relation in the new model is defined in terms of relations in the initial one. However, the \(\forall\forall\forall\) operation defines each 'new' relation in terms of 'old' ones and the relation of only knowing \(\chi\) fully, \(\sim \chi\) (with \(\chi\) the communication's topic). This fact and that model operations can change the truth-set of a formula is what makes \(\forall\forall\forall\) behaves less similar to the \(\forall\forall\forall\)- and \(\forall\forall\forall\)-actions, and more similar to public announcements.

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14Indeed, by definition, \(\sim^M_{\chi_2} = (\llbracket \chi_2 \rrbracket^M \times \llbracket \chi_2 \rrbracket^M) \cup (\llbracket \sim \chi_2 \rrbracket^M \times \llbracket \sim \chi_2 \rrbracket^M)\). But recall: \(\llbracket \chi \rrbracket^M = \llbracket [S_i \chi] \varphi \rrbracket^M\), moreover, \(\llbracket \sim \rrbracket^M = W \setminus \llbracket [S_i \chi] \varphi \rrbracket^M\). Then, \(\sim^M_{\chi_2} = (\llbracket [S_i \chi_2] \chi_2 \rrbracket^M \times \llbracket [S_i \chi_2] \chi_2 \rrbracket^M) \cup (\llbracket [S_i \chi_2] \chi_2 \rrbracket^M \times \llbracket [S_i \chi_2] \chi_2 \rrbracket^M)\).
Fact 4.1

(i) $\not\in [S_1][S_1] \varphi \leftrightarrow [S_1] \varphi$: two successive $\exists \forall$-acts by the same group and on the same topic cannot be collapsed into a single one on the same topic.

(ii) $\not\in [S_1,2][S_2] \varphi \leftrightarrow [S_2] \varphi$: $\exists \forall$-acts do not commute.

(iii) $\not\in [S_1] [S_2] \varphi \leftrightarrow ([S_1 \cup S_2]) \varphi$: successive $\exists \forall$-acts by different groups on the same topic cannot be 'compressed' by using the union of the groups.

(iv) $\not\in [S_1][S_2] \varphi \leftrightarrow [S_1 \times S_2] \varphi$: successive $\exists \forall$-acts by the same group on different topic cannot be 'compressed' by using the conjunction of the topics.

Proof. In the model below (with equivalence relations), each world indicates the truth-value of atoms $m_a, m_b, m_c$ (in that order) by using "•" (the atom holds) or "◦" (the atom fails).

\[
M_1:
\]

$\chi_i := K_i m_i \vee K_i \neg m_i$ for each $i \in A = \{a, b, c\}$, stating that agent $i$ knows $m_i$'s truth-value. Define their disjunction $\chi := \chi_a \vee \chi_b \vee \chi_c$. Note how, at worlds in $\{w_0, w_2, w_3, w_6\}$, every agent $i$ has uncertainty about whether $m_i$ holds (i.e., $\chi_i$ fails for every $i$, and thus so does $\chi$). However at worlds in $\{w_1, w_4, w_5, w_7\}$, at least one agent $i$ knows whether $m_i$ (so $\chi_i$ is the case).

(i) Since $\llbracket \chi \rrbracket^{M_1} = \{w_0, w_2, w_3, w_6\}$, applying $A_{\chi}!$ to $M_1$ (note the generated partition) yields $M_2$ below. Then, since $\llbracket \chi \rrbracket^{M_2} = \{w_2\}$, a further application of $A_{\chi}!$ yields $M_3$.

\[
M_1 \xrightarrow{A_{\chi}!} M_2 \xrightarrow{A_{\chi}!} M_3
\]

Thus, $(M_1, w_2) \not\in [A_{\chi}!] [A_{\chi}!] \chi_a$ and yet $(M_1, w_2) \not\in [A_{\chi}!] \chi_a$. Note also how $(M_1, w_2) \vdash D_a \supset \chi_a \wedge \neg [A_{\chi}!] K_a \supset \chi_a$, thus showing that $\not\in D_a \varphi \rightarrow [S_1] \varphi$ for $i \in S$, even when the model is reflexive (cf. the discussion in the proof of Proposition 4.2).

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[15] Thus, e.g., at the atom $m_a$ is true, $m_b$ is false and $m_c$ is true.

[16] As a visual aid, in this model each atom $\chi_i$ holds in those worlds without outgoing 1-edges (other than the implicitly present reflexive ones).
(ii) On the one hand, \( [\chi a]_{M^1} = \{w_5, w_7\} \), so applying \([a]_{\chi c} \) to \( M_1 \) yields \( M'_2 \) below. Then, \( [\chi c]_{M'_2} = \{w_1, w_3, w_5, w_7\} \) so a further \([b, c]_{\chi c} \) yields \( M'_3 \).

On the other hand, \( [\chi c]_{M^1} = \{w_1, w_5\} \), so applying \([b, c]_{\chi c} \) to \( M_1 \) yields \( M'_2 \) below. Then, \( [\chi a]_{M'_2} = \{w_1, w_3, w_5, w_7\} \) so a further \([a]_{\chi c} \) yields \( M'_3 \).

Thus, \((M_1, w_3) \models [a]_{\chi a} \land [b, c]_{\chi c} \) and yet \((M_1, w_3) \not\models [b, c]_{\chi c} \lor [a]_{\chi a} \).

(iii) As it can be seen from Item (i), \((M_1, w_2) \models [a, b]_{\chi a} \land [c]_{\chi c} \) (the model that results from \([a, b]_{\chi a} \) is also \( M_2 \), and a further application of \([c]_{\chi c} \) yields a model \( M''_2 \) differing from \( M_2 \) in having an additional \( c \)-edge between \( w_2 \) and \( w_6 \), and yet \((M_1, w_2) \not\models [a, b, c]_{\chi a} \).

(iv) As it can be seen from Item (iii), \((M_1, w_2) \models [A_{\chi a} \land [a, b]_{\chi a}]_{\chi a} \) (the model that results from \( A_{\chi a} \) is also \( M'_2 \), and a further \( A_{\chi a} \) yields \( M'_3 \) too). However, \( [\chi a \land [\chi c]]_{M^1} = \{w_5\} \), so applying \( A_{[\chi a \land [\chi c]]} \) on \( M_1 \) has no effect, and thus \((M_1, w_3) \not\models [A_{[\chi a \land [\chi c]]}]_{\chi a} \).

The provided counterexamples show that the given formulas are not valid, even under equivalence relations.

Axiom system Once again, the axiom system for \( \mathcal{L}_{D, [S], !} \) relies on reduction axioms (Table 4). Axioms \( A^P_{S, !}, A^-_{S, !}, A^\circ_{S, !} \), and rule \( RE_{S, !} \) are similar to their previous matching cases. Axiom \( A^D_{S, !} \) is the important one, indicating that a group \( G \) knows \( \varphi \) distributively after the action \(([S, !]D_6 \varphi) \) if and only if the group \( S \cup G \) knew, distributively, that \( \varphi \) would hold after the action \((D_{S \cup G} [S, !] \varphi) \) and the group \( G \) knew, distributively and relative to similarity on \( \chi \), that \( \varphi \) would hold after the action \((D^G_{\chi} [S, !] \varphi) \).
\[ A_{\chi}^{C} : \vdash [S_{\chi}] p \leftrightarrow p \]
\[ A_{\chi}^{D} : \vdash [S_{\chi}] D_{\phi} \leftrightarrow (D_{S_{\chi}}[S_{\chi}] \phi \land D_{\chi}[S_{\chi}] \phi) \]
\[ A_{\chi}^{E} : \vdash [S_{\chi}] \neg \phi \leftrightarrow \neg [S_{\chi}] \phi \]
\[ \text{RE}_{\chi} : \text{If } \vdash \phi_1 \leftrightarrow \phi_2 \text{ then } \vdash [S_{\chi}] \phi_1 \leftrightarrow [S_{\chi}] \phi_2 \]
\[ A_{\chi}^{F} : \vdash [S_{\chi}](\phi \land \psi) \leftrightarrow ([S_{\chi}] \phi \land [S_{\chi}] \psi) \]

Table 4: Additional axioms and rules for \( L_{D,[S_{\chi}]} \), which characterises the formulas in \( L_{D,[S_{\chi}]} \) that are valid on models in \( M_{\chi} \).

**Theorem 4** The axiom system \( L_{D,[S_{\chi}]} (L_{D} + \text{Table 1} + \text{Table 4}) \) is sound and strongly complete for formulas in \( L_{D,[S_{\chi}]} \) valid over models in \( M_{\chi} \).■

**5 Coda: sharing with somebody**

Section 3 introduced two forms of inter-agent communication: \( \forall \forall \forall \), through which all agents share all their information with everybody, and \( \exists \forall \forall \), through which some agents share all their information with everybody. Then, Section 5 introduced \( \exists \forall \forall \), a form of communication through which some agents share some of their information with everybody. The next natural step is defining a form of communication through which some agents share some of their information with somebody. This operation, which could be denoted by \( \exists \exists \exists \), will not be studied in this manuscript. Instead, the text will only discuss some of the modelling options that arise.

First, when defining the \( \exists \exists \exists \) operation, one might decide that the shared information is received only by agents in a given group \( R \subseteq A \). This would represent a scenario similar to a round table with a potentially distracted audience: the people on ‘the table’ get to talk about a given topic, with some members of the audience listening and the distracted ones missing the information. However, one could also think of a different scenario, one in which every sharing agent \( i \) has a specific set of listeners \( R_i \). This is closer to what happens in some online social networks, where only ‘friends’ or ‘followers’ can receive what a given agent sends (cf. with Baltag and Smets 2020, discussed in Subsection 3.3).

In fact, the scenario with a single set \( R \) can be seen as the special case of the scenario with \( \{R_i\}_{i \in A} \) in which all sets \( R_i \) are the same.

A further possibility is to take the social networks idea seriously and work with models that represent explicitly the social connections each agent has. These structures have been used by the logic community (Seligman et al. 2011 is one of the earliest proposals), either for studying information flow in social networks (peer pressure: Zhen and Seligman 2011; reflexive influence: Christoff et al. 2016; diffusion and prediction: Baltag et al. 2019) or for studying social network formation (as a side effect of peer pressure: Pedersen and Slavkovik 2017; by similarity: Smets and Velázquez-Quesada 2018, 2017, 2019).

A second decision to make when defining an \( \exists \exists \exists \) operation is the following: what information, if any, obtain the agents that do not ‘hear’ the communication? On one extreme, these agents might be oblivious, not only to the content of the communication, but also to the fact that a communication took place. On the technical side, representing this form of ‘private’ communication requires an operation that changes not only the model’s indistinguishability relations,
but also its domain. Indeed, while the ‘real’ part of the model would change as the communication takes place (the receiving agents might learn something), for non-receiving agents one should keep ‘a copy’ of the original model, to indicate that they see no change at all.\footnote{This is how the action models of Baltag et al. (1998, 1999) deal with private announcements.} On the other extreme, non-receiving agents might not hear the content of the communication, but they might notice that the communication takes place (the semi-public form of communication from Ågotnes and Wang 2017 and Baltag and Smets 2020 discussed in Subsection 3.3). Even more, they might know the topic of the conversation. Combining this with some previous knowledge about what the communicating agents know, the non-receivers might get to know part of what is being shared, and thus they might get to know (part of) the new epistemic state of the communicating agents.

### 6 Summary and further research lines

This paper has discussed three communication actions. Different from most epistemic acts present in the literature, they truly describe processes of inter-agent communication: those in which the information that is being shared is information some of the agents already have. Through the first two actions, $\forall \forall \forall$ and $\exists \forall \forall$, all/some agents share all their information with everybody. Through the third, $\exists \forall \forall$, the core of this contribution, some agents share some of their information with everybody. The text has presented examples of these actions at work, discussing some of their basic properties and providing, in all cases, a sound and complete axiom system for modalities describing their effects.

There are several research lines that arise from the current proposal. Here are some of them.

- Distributed knowledge has played a crucial role in this manuscript. Yet, the equally important notion of common knowledge (referring to the infinite iteration including everybody knows, everybody knows that everybody knows, and so on; Lewis 1969, Vanderschraaf and Sillari 2014) has been absent. Common knowledge is an important piece when discussing communication between agents, as one would like to know not only what each individual agent gets from the action (their individual knowledge), but also what a group as a whole learns from it (the group’s common knowledge). Adding the common knowledge operator to the studied languages involves the use of additional tools, in particular for the axiomatisation.

- In a $\exists \forall \forall$ action, what each sharing agent communicates is given by a formula $\chi$, understood as the subject/topic of the conversation. This is the reason why the operation only considers for elimination edges between worlds differing in $\chi$’s truth-value. However, as mentioned, there are other alternatives. One can understand $\chi$ as the content of the conversation instead; in such case, the operation should consider for elimination only edges pointing to $\neg \chi$-worlds. One could also take a set of formulas. If this represents the set of topics then, under some reasonable restrictions on the class of models, such variation could turn out to be a generalisation of one of the other actions: an $\exists \forall \forall$-communication on all topics could be equivalent to an $\forall \forall \forall$ communication.


In any of these alternatives, one can also look for a less uniform treatment, assigning to each sharing agent (and even to sets of them) a potentially different (set of) formula(s) (cf. van Benthem and Minica 2012, Baltag et al. 2018). All these options deserve a proper exploration.

- In the communication actions studied in this text, each agent communicates by sharing (some of) the possibilities she has already discarded (those in $R_1$). Thus, it has been implicitly assumed that, while agents might not share ‘everything that they know’, what they share is ‘something that they do know’ (in other words, they ‘communicate’ a set $S \subseteq R_1$). But one can also find agents that share ‘more than what they know’ (they communicate a set $S \supset R_1$), and even only ‘things they do not know’ (they communicate a set $S$ with $S \cap R_1 = \emptyset$). These alternatives can be used for studying acts of lying, which so far have been modelled only by adapting tools from public announcement logic (van Ditmarsch 2014).

- The success of public announcement logic comes from the fact that it provides formal tools for studying different epistemic change phenomena. Among them, one can find studies on arbitrary announcements. Indeed, some proposals (e.g., Balbiani et al. 2008, Galimullin and Ågotnes 2021) have worked with modalities of the form $\langle *! \rangle \varphi$, read as “there is a formula that can be truthfully announced, and after doing so $\varphi$ will be the case”. The setting presented here offers a further alternative for quantifying over information change: one can define an operation through which a given agent (and eventually a set of them) shares an arbitrary set $S \subseteq R_1$. Thus, one can study what a set of agents can get to know (i.e., a form of knowability) by sharing their information.

A Appendix

In the proofs, IH abbreviates “inductive hypothesis”.

A.1 Proof of Theorem 2 (System $L_{D, [!]})$

**Soundness** The soundness of axioms and rules in $L_D$ is well-known. For those in Table 2, the case of $A_\varphi^D$ is straightforward (the operation does not affect atomic valuations), the cases for $A_{\neg}^D$ and $A_\land^D$ follow from the inductive hypotheses, and $A_D^D$ has been proved valid already (Proposition 3.3). For $RE$, suppose $\vdash \varphi_1 \leftrightarrow \varphi_2$, that is, suppose $\llbracket \varphi_1 \rrbracket^M = \llbracket \varphi_2 \rrbracket^M$ for every $M$ in $M_A$. Take any pointed model $(M, w)$ with $M = \langle W, R, V \rangle$. Then, $(M, w) \vdash [!] \varphi_1$ if and only if $w \in \llbracket \varphi_1 \rrbracket^M$. But $M'$ is a model in $M_A$, so the latter holds if and only if $w \in \llbracket \varphi_2 \rrbracket^M$, which holds if and only if $(M, w) \vdash [!] \varphi_2$. Thus, $(M, w) \vdash [!] \varphi_1 \leftrightarrow [!] \varphi_2$.

**Completeness** The argument relies on a translation from $L_{D, [!]})$ to $L_D$, which is defined as follows.
Definition A.1 (Translation) The translation $\tau$ is given by

$$
\begin{align*}
\tau(p) &:= p, & \tau([!]p) &:= \tau(p), \\
\tau(\neg \varphi) &:= \neg \tau(\varphi), & \tau([!]\neg \varphi) &:= \tau(\neg [!]\varphi), \\
\tau(\varphi_1 \land \varphi_2) &:= \tau(\varphi_1) \land \tau(\varphi_2), & \tau([!]\varphi_1 \land [!]\varphi_2) &:= \tau([!]\varphi_1 \land [!]\varphi_2), \\
\tau(D_\varphi \varphi) &:= D_\varphi \tau(\varphi), & \tau([!]D_\varphi \varphi) &:= \tau(D_\varphi [!]\varphi), \\
\tau([!]\varphi) &:= \tau([!]\varphi).
\end{align*}
$$

This translation works with formulas of the form $[!] [!] \varphi$ in an “inside-out” fashion, dealing first with the deepest occurrence of $[!]$ (i.e., translating $[!] [!] \varphi$ before dealing with the rest (i.e., before translating $[!]$ $\tau([!] \varphi)$). Because of this, the strategy proving strong completeness is as follows (cf. Plaza 1989 and Gerbrandy 1999, Section 4.4): (i) show that $\tau$ is a proper recursive translation that returns formulas in $L_D$ (Proposition A.1), (ii) show that a formula and its translation are both provably and semantically equivalent (Proposition A.2), and (iii) use $\tau$ and the completeness of $L_D$ to show that, if $\Psi \cup \{\varphi\} \subseteq L_{D,[!]r}$, then from $\Psi \vdash \varphi$ it follows that $\varphi$ is derivable from $\Psi$ in $L_{D,[!]r}$.

It is clear that the domain of $\tau$ is $L_{D,[!]r}$: cases on the leftmost column take care of formulas that do not start with $[!]$, and cases on the rightmost column take care of formulas that start with $[!]$. Still, one needs to show not only that, if $\varphi$ is in $L_{D,[!]r}$, then $\tau(\varphi)$ is in $L_D$, but also that the calculation of $\tau(\varphi)$ actually ends. In doing so, the following notion of complexity will play a crucial role.

Definition A.2 (Complexity for $L_{D,[!]r}$) The functions $\text{nsc} : L_{D,[!]r} \rightarrow \mathbb{N} \setminus \{0\}$ (nested ‘static’ complexity, focussing on operators in $L_D$) and $\text{ndc} : L_{D,[!]r} \rightarrow \mathbb{N}$ (nested ‘dynamic’ complexity, focussing on $[!]$) are defined as follows.

$$
\begin{align*}
\text{nsc}(p) &:= 1, & \text{ndc}(p) &:= 0, \\
\text{nsc}(\neg \varphi) &:= 1 + \text{nsc}(\varphi), & \text{ndc}(\neg \varphi) &:= \text{ndc}(\varphi), \\
\text{nsc}(\varphi_1 \land \varphi_2) &:= 1 + \max \{\text{nsc}(\varphi_1), \text{nsc}(\varphi_2)\}, & \text{ndc}(\varphi_1 \land \varphi_2) &:= \max \{\text{ndc}(\varphi_1), \text{ndc}(\varphi_2)\}, \\
\text{nsc}(D_\varphi \varphi) &:= 1 + \text{nsc}(\varphi), & \text{ndc}(D_\varphi \varphi) &:= \text{ndc}(\varphi), \\
\text{nsc}([!] \varphi) &:= 2 \text{nsc}(\varphi). & \text{ndc}([!] \varphi) &:= 1 + \text{ndc}(\varphi).
\end{align*}
$$

Then, for any $\varphi_1, \varphi_2 \in L_{D,[!]r}$, write $c(\varphi_1) > c(\varphi_2)$ if and only if

$$
\text{ndc}(\varphi_1) > \text{ndc}(\varphi_2) \quad \text{or} \quad \text{ndc}(\varphi_1) = \text{ndc}(\varphi_2) \text{ and } \text{nsc}(\varphi_1) > \text{nsc}(\varphi_2).
$$

Thus, while $\text{nsc}$ counts a formula’s nested Boolean and modal operators, $\text{ndc}$ counts a formula’s nested dynamic operators. The complexity ordering $c$ relies on both $\text{nsc}$ and $\text{ndc}$, with the latter taking precedence: $\varphi_1$ is more complex than $\varphi_2$ (i.e., $c(\varphi_1) > c(\varphi_2)$) if and only if either $\varphi_1$’s ‘dynamic’ complexity is greater than that of $\varphi_2$ (i.e., $\text{ndc}(\varphi_1) > \text{ndc}(\varphi_2)$), or else both have the same ‘dynamic’ complexity but $\varphi_1$’s ‘static’ complexity is greater than that of $\varphi_2$ (i.e., $\text{nsc}(\varphi_1) = \text{nsc}(\varphi_2)$ and $\text{nsc}(\varphi_1) > \text{nsc}(\varphi_2)$).

Relying on the complexity ordering $c$, the following proposition states that $\tau$ is a proper recursive translation from $L_{D,[!]r}$ to $L_D$.

Proposition A.1 For every $\varphi \in L_{D,[!]r}$,
The following lemma will be useful.

**Lemma A.1** Define the strict subformula function $\text{ssub} : \mathcal{L}_{D,[1]} \rightarrow \varphi(\mathcal{L}_{D,[1]})$ as

$$
\text{ssub}(p) := \emptyset,
\text{ssub}(\neg \varphi) := \varphi \cup \text{ssub}(\varphi),
\text{ssub}(\varphi_1 \land \varphi_2) := \{\varphi_1, \varphi_2\} \cup \text{ssub}(\varphi_1) \cup \text{ssub}(\varphi_2),
$$

Let $\varphi$ be an $\mathcal{L}_{D,[1]}$-formula. Then, $c(\varphi) > c(\psi)$ for every $\psi \in \text{ssub}(\varphi)$.

**Proof.** The proof is by structural induction on $\varphi$. Here are the cases.

- **Base case ($p$).** Immediate, as $\text{ssub}(p) = \emptyset$.

- **Inductive case ($\neg \varphi$).** Note that, by definition,
  
  $\text{(i)}$ $\text{ndc}(\neg \varphi) = \text{ndc}(\varphi),$
  
  $\text{(ii)}$ $\text{nsc}(\neg \varphi) > \text{nsc}(\varphi)$.

  Take any $\psi \in \text{ssub}(\neg \varphi) = \{\varphi\} \cup \text{ssub}(\varphi)$, and consider the cases.

  - **Case $\psi = \varphi$.** By **Item (i)** and **Item (ii)**, it follows that $c(\neg \varphi) > c(\psi)$.
  
  - **Case $\psi \in \text{ssub}(\varphi)$.** By IH, $c(\varphi) > c(\psi)$ for any such $\psi$. By definition of $c$, this implies either
    
    $\text{ndc}(\varphi) > \text{ndc}(\psi)$, so $\text{ndc}(\neg \varphi) > \text{ndc}(\psi)$ (**Item (i)**) hence $c(\neg \varphi) > c(\psi)$, or
    
    $\text{ndc}(\varphi) = \text{ndc}(\psi)$ and $\text{nsc}(\varphi) > \text{nsc}(\psi)$, so $\text{ndc}(\neg \varphi) = \text{ndc}(\psi)$ (by **Item (i)**) and $\text{nsc}(\neg \varphi) > \text{nsc}(\psi)$ (by **Item (ii)**); hence, $c(\neg \varphi) > c(\psi)$.

- **Inductive case ($\varphi_1 \land \varphi_2$).** Note that, by definition, for $i \in \{1, 2\},$
  
  $\text{(i)}$ $\text{ndc}(\varphi_1 \land \varphi_2) \geq \text{ndc}(\varphi_i),$
  
  $\text{(ii)}$ $\text{nsc}(\varphi_1 \land \varphi_2) > \text{nsc}(\varphi_i)$.

  Take any $\psi \in \text{ssub}(\varphi_1 \land \varphi_2) = \{\varphi_1, \varphi_2\} \cup \text{ssub}(\varphi_1) \cup \text{ssub}(\varphi_2)$.

  - **Case $\psi = \varphi_i$.** By **Item (i)**, $\text{ndc}(\varphi_1 \land \varphi_2) \geq \text{ndc}(\psi_i)$. If $\text{ndc}(\varphi_1 \land \varphi_2) > \text{ndc}(\psi_i)$, then $c(\varphi_1 \land \varphi_2) > c(\psi_i)$ follows immediately; otherwise, $\text{ndc}(\varphi_1 \land \varphi_2) = \text{ndc}(\psi_i)$ and $\text{nsc}(\varphi_1 \land \varphi_2) > \text{nsc}(\psi_i)$ (**Item (ii)**), so $c(\varphi_1 \land \varphi_2) > c(\psi_i)$ again.

  - **Case $\psi \in \text{ssub}(\varphi_i)$.** By IH, $c(\varphi_i) > c(\psi)$ for any such $\psi$. Thus, either
    
    $\text{ndc}(\varphi_i) > \text{ndc}(\psi)$, so $\text{ndc}(\varphi_1 \land \varphi_2) > \text{ndc}(\psi)$ (by **Item (i)**) and hence $c(\varphi_1 \land \varphi_2) > c(\psi)$, or
    
    $\text{ndc}(\varphi_i) = \text{ndc}(\psi)$ and $\text{nsc}(\varphi_i) > \text{nsc}(\psi)$. By **Item (ii)**, either $\text{ndc}(\varphi_1 \land \varphi_2) > \text{ndc}(\psi)$ or $\text{ndc}(\varphi_1 \land \varphi_2) = \text{ndc}(\psi)$. In the first case, $\text{ndc}(\varphi_1 \land \varphi_2) > \text{ndc}(\psi)$ and thus $c(\varphi_1 \land \varphi_2) > c(\psi)$; in the second case, $\text{ndc}(\varphi_1 \land \varphi_2) = \text{ndc}(\psi)$ and $\text{nsc}(\varphi_1 \land \varphi_2) > \text{nsc}(\psi)$ (using **Item (ii)**), so $c(\varphi_1 \land \varphi_2) > c(\psi)$.

- **Inductive case ($D_\varphi \varphi$).** Note that, by definition,
  
  $\text{(i)}$ $\text{ndc}(D_\varphi \varphi) = \text{ndc}(\varphi),$
  
  $\text{(ii)}$ $\text{nsc}(D_\varphi \varphi) > \text{nsc}(\varphi)$.

  Take any $\psi \in \text{ssub}(D_\varphi \varphi) = \{\varphi\} \cup \text{ssub}(\varphi)$. Thus, there are two cases.

  - **Case $\psi = \varphi$.** By **Item (i)** and **Item (ii)**, it follows that $c(D_\varphi \varphi) > c(\psi)$. 

(\text{\textit{(T2)}}) \text{ if } \tau(\varphi) \text{ is defined in terms of } \tau(\psi), \text{ then } c(\varphi) > c(\psi). \text{ Thus, the calculation of } \tau(\varphi) \text{ will eventually end. }

(\text{\textit{(T2)}}) \tau(\varphi) \in \mathcal{L}_D.

**Proof.** The following lemma will be useful.
- **Case** $\psi \in \text{ssub}(\varphi)$. By IH, $c(\varphi) > c(\psi)$ for any such $\psi$. This implies either
  - $\text{ndc}(\varphi) > \text{ndc}(\psi)$, so $\text{ndc}(D_\varphi \varphi) > \text{ndc}(\psi)$ (**Item (i)**) then $c(D_\varphi \varphi) > c(\psi)$, or
  - $\text{ndc}(\varphi) = \text{ndc}(\psi)$ and $\text{nsc}(\varphi) > \text{nsc}(\psi)$, so $\text{ndc}(D_\varphi \varphi) = \text{ndc}(\psi)$ (by **Item (ii)**) and $\text{nsc}(D_\varphi \varphi) > \text{nsc}(\psi)$ (by **Item (ii)**); hence, $c(D_\varphi \varphi) > c(\psi)$.

- **Inductive case** ($[!][\varphi]$). Note that, by definition,
  
  (i) $\text{ndc}([!] \varphi) > \text{ndc}(\varphi).

  Take any $\psi \in \text{ssub}([!] \varphi) = \{\varphi\} \cup \text{ssub}(\varphi)$. Thus, there are two cases.

  - **Case** $\psi = \varphi$. By **Item (i)**, it follows that $c([!] \varphi) > c(\psi)$.
  
  - **Case** $\psi \in \text{ssub}(\varphi)$. By IH, $c(\varphi) > c(\psi)$ for any such $\psi$. Then, either
    - $\text{ndc}(\varphi) > \text{ndc}(\psi)$, so $\text{ndc}([!] \varphi) > \text{ndc}(\psi)$ (**Item (i)**) then $c([!] \varphi) > c(\psi)$, or
    - $\text{ndc}(\varphi) = \text{ndc}(\psi)$ and $\text{nsc}(\varphi) > \text{nsc}(\psi)$, so $\text{ndc}([!] \varphi) > \text{ndc}(\psi)$ (by **Item (ii)**) and hence $c([!] \varphi) > c(\psi)$.

Now, for the actual proposition, the proof proceeds by induction on $c(\varphi)$, the complexity of $\varphi$, with both **Item (τ1)** and **Item (τ2)** proved simultaneously.

- **Base case** ($\varphi$ such that $c(\varphi)$ is minimum). A $\varphi$ with the minimum $c$ should be minimum at both ndc (i.e., $\text{ndc}(\varphi) = 0$, so any $\varphi$ without observation operators) and nsc (i.e., $\text{nsc}(\varphi) = 1$). The only such formula is $p$. Proving **Item (τ1)** is immediate, as the definition of $τ(p)$ does not use $τ$; proving **Item (τ2)** is also immediate, as $τ(p) = p$ is a formula in $L_D$.

- **Inductive case** ($\varphi$ such that $c(\varphi)$ is not minimum), case $\neg \varphi$. For **Item (τ1)**, the definition of $τ(\neg \varphi)$ uses $τ(\varphi)$. But $\varphi \in \text{ssub}(\neg \varphi)$ so, by **Lemma A.1**, $c(\neg \varphi) > c(\varphi)$. For **Item (τ2)**, the same $c(\neg \varphi) > c(\varphi)$ implies that, by IH, $τ(\varphi) \in L_D$; hence, so is $\neg τ(\varphi) = τ(\neg \varphi)$.

- **Inductive case** ($\varphi$ such that $c(\varphi)$ is not minimum), case $\varphi_1 \land \varphi_2$. For **Item (τ1)**, the definition of $τ(\varphi_1 \land \varphi_2)$ uses both $τ(\varphi_1)$ and $τ(\varphi_2)$. But $ϕ_i \in \text{ssub}(ϕ_i)$ for $i \in \{1, 2\}$ so, by **Lemma A.1**, $c(ϕ_i) > c(ϕ_i)$. For **Item (τ2)**, the same $c(ϕ_i) > c(ϕ_i)$ implies that, by IH, $τ(ϕ_i) \in L_D$; hence, so is $τ(ϕ_1) \land τ(ϕ_2) = τ(ϕ_1 \land ϕ_2)$.

- **Inductive case** ($\varphi$ such that $c(\varphi)$ is not minimum), case $D_\varphi \varphi$. For **Item (τ1)**, the definition of $τ(D_\varphi \varphi)$ uses $τ(\varphi)$. But $ϕ \in \text{ssub}(D_\varphi \varphi)$ so, by **Lemma A.1**, $c(D_\varphi \varphi) > c(ϕ)$. For **Item (τ2)**, the same $c(D_\varphi \varphi) > c(ϕ)$ implies that, by IH, $τ(ϕ) \in L_D$; hence, so is $D_\varphi τ(ϕ) = τ(D_\varphi \varphi)$.

- **Inductive case** ($\varphi$ such that $c(\varphi)$ is not minimum), case $[!]p$. For **Item (τ1)**, $τ([!]p)$ uses $τ(p)$. But $p \in \text{ssub}([!]p)$, so $c([!]p) > c(ϕ)$ (**Lemma A.1**). For **Item (τ2)**, from the same $c([!]p) > c(ϕ)$ and IH, $τ(p) = τ([!]p) \in L_D$.

- **Inductive case** ($\varphi$ such that $c(\varphi)$ is not minimum), case $[!]\neg \varphi$. For **Item (τ1)**, the definition of $τ([!] \neg \varphi)$ uses $τ(\neg [!] \varphi)$. Now, on the one hand,
  - $\text{ndc}([!] \neg \varphi) = 1 + \text{ndc}(\varphi),
  
  but, on the other hand,
\[
- \text{nsc}([!] \neg \varphi) = 2 + 2 \text{nsc} (\varphi), \quad - \text{nsc}(\neg [!] \varphi) = 1 + 2 \text{nsc} (\varphi).
\]

Thus, \(c([!] \neg \varphi) > c(\neg [!] \varphi)\). For Item (2), take the just obtained \(c([!] \neg \varphi) > c(\neg [!] \varphi)\); then, by IH, \(\tau (\neg [!] \varphi) = \tau ([!] \neg \varphi) \in \mathcal{L}_D\).

- **Inductive case (\(\varphi\) such that \(c(\varphi)\) is not minimum), case \([!] (\varphi_1 \land \varphi_2)\).** For Item (1), the definition of \(\tau ([!] (\varphi_1 \land \varphi_2))\) uses \(\tau ([!] \varphi_1 \land [!] \varphi_2)\). Now note that, on the one hand, by taking max \([\text{nsc}(\varphi_1), \text{nsc}(\varphi_2)]) = \text{nsc}(\varphi_i),\)

\[
- \text{nsc}([!] (\varphi_1 \land \varphi_2)) = 1 + \text{nsc}(\varphi_i), \quad - \text{nsc}([!] \varphi_1 \land [!] \varphi_2) = 1 + \text{nsc}(\varphi_i),
\]

but, on the other hand, by taking max \([\text{nsc}(\varphi_1), \text{nsc}(\varphi_2)]) = \text{nsc}(\varphi_i),\)

\[
- \text{nsc}([!] (\varphi_1 \land \varphi_2)) = 2 + 2 \text{nsc}(\varphi_i), \quad - \text{nsc}([!] \varphi_1 \land [!] \varphi_2) = 1 + 2 \text{nsc}(\varphi_i).
\]

Thus, \(c([!] (\varphi_1 \land \varphi_2)) > c([!] \varphi_1 \land [!] \varphi_2)\). For Item (2), take the just obtained \(c([!] (\varphi_1 \land \varphi_2)) > c([!] \varphi_1 \land [!] \varphi_2))\); then, by IH, \(\tau ([!] \varphi_1 \land [!] \varphi_2) = \tau ([!] (\varphi_1 \land \varphi_2)) \in \mathcal{L}_D\).

- **Inductive case (\(\varphi\) such that \(c(\varphi)\) is not minimum), case \([!] [!] \varphi)\).** For Item (1), the definition of \(\tau ([!] [!] \varphi)\) uses two instances of \(\tau\), namely \(\tau ([!] [!] \varphi)\) and \(\tau ([!] [!] \varphi)\). For the first, \([!] \varphi \in \text{ssub}([!] [!] \varphi)\) so, by Lemma A.1, \(c([!] [!] \varphi) > c([!] [!] \varphi)\). For the second, note that

\[
- \text{nsc}([!] [!] \varphi) = 2 + \text{nsc}(\varphi),
\]

\[
- \text{nsc}([!] [!] \varphi) = 1 + \text{nsc}(\tau ([!] \varphi)). \quad \text{But, as it has been shown,} \quad c([!] [!] \varphi) > c([!] [!] \varphi);
\]

thus, by IH, \(\tau ([!] \varphi) \in \mathcal{L}_D\) and therefore \(\text{nsc}([!] \varphi) = 0\). Hence, \(\text{nsc}([!] [!] \varphi) = 1\).

Thus, \(c([!] [!] \varphi) > c([!] [!] \varphi))\). For Item (2), the just obtained \(c([!] [!] \varphi) > c([!] [!] \varphi))\) and IH imply \(\tau ([!] [!] \varphi) = \tau ([!] [!] \varphi) \in \mathcal{L}_D\).

Using the ordering \(c\), the proposition below shows that a formula \(\varphi \in \mathcal{L}_{D,[!]\varphi}\) and its translation \(\tau (\varphi) \in \mathcal{L}_D\) are both provably and semantically equivalent.

**Proposition A.2** For every \(\varphi \in \mathcal{L}_{D,[!]\varphi}\),

\((\tau 1) \vdash \varphi \iff \tau (\varphi) \) under \(\mathcal{L}_{D,[!]\varphi}\), \quad (\tau 2) \vdash \varphi \iff \tau (\varphi)

**Proof.** By induction on \(c(\varphi)\). The following rule will be useful.

**Lemma A.2** Let \(\varphi_1, \varphi_2\) be formulas in \(\mathcal{L}_{D,[!]\varphi}\). Then, when using the system \(\mathcal{L}_D\),

\[\vdash \varphi_1 \iff \varphi_2 \text{ then } \vdash D_6 \varphi_1 \iff D_6 \varphi_2.\]

**Proof.** Suppose \(\vdash \varphi_1 \iff \varphi_2\). By propositional reasoning and modus ponens, \(\vdash \varphi_1 \rightarrow \varphi_2\) and \(\vdash \varphi_2 \rightarrow \varphi_1\); so \(\vdash D_6 \varphi_1 \rightarrow \varphi_2\) and \(\vdash D_6 \varphi_2 \rightarrow \varphi_1\) (by rule \(G_3\)). Then, from \(K_D\) and modus ponens, \(\vdash D_6 \varphi_1 \rightarrow D_6 \varphi_2\) and \(\vdash D_6 \varphi_2 \rightarrow D_6 \varphi_1\). Hence, by propositional reasoning and modus ponens, \(\vdash D_6 \varphi_1 \iff D_6 \varphi_2\). ■
Here is the proof of the proposition.

(τ1) Here are the cases.

- **Base case (p).** By propositional reasoning, ⊢ p ↔ p. But τ(p) = p, so the required ⊢ p ↔ τ(p) follows.

- **Inductive case (¬φ).** Since c(¬φ) > c(φ) (same case in Proposition A.1), from IH it follows that ⊢ ¬φ ↔ τ(φ). Then ⊢ ¬φ ↔ ¬τ(φ) (propositional reasoning) and thus, by τ’s definition, ⊢ ¬φ ↔ τ(¬φ).

- **Inductive case (φ1 ∧ φ2).** Since c(φ1 ∧ φ2) > c(φ1) for i ∈ {1, 2} (same case in Proposition A.1), from IH it follows that ⊢ φi ↔ τ(φi). Then ⊢ (φ1 ∧ φ2) ↔ (τ(φ1) ∧ τ(φ2)) (propositional reasoning) and thus, by τ’s definition, ⊢ (φ1 ∧ φ2) ↔ τ(φ1 ∧ φ2).

- **Inductive case (Dg φ).** Since c(Dg φ) > c(φ) (same case in Proposition A.1), from IH it follows that ⊢ φ ↔ τ(φ). Then ⊢ Dg φ ↔ Dg τ(φ) (Lemma A.2, since Lg is a subsystem of Lg,[Γ]) and thus, by τ’s definition, ⊢ Dg φ ↔ τ(Dg φ).

- **Inductive case (Dgφ).** Since c(Dgφ) > c(φ) (same case in Proposition A.1), from IH it follows that ⊢ φ ↔ τ(φ). Hence, by τ’s definition, ⊢ [Γ]φ ↔ τ([Γ]φ).

- **Inductive case (Dg φ).** Since c(Dg φ) > c(Dg[Γ] φ) (same case in Proposition A.1), from IH it follows that ⊢ Dg φ ↔ τ(Dg[Γ] φ). But ⊢ Dg φ ↔ Dg[Γ] τ(φ) (axiom AΓ) so, by propositional reasoning, ⊢ Dg φ ↔ τ(Dg[Γ] φ). Hence, by τ’s definition, ⊢ [Γ]τ(φ) ↔ τ([Γ]τ(φ)).

(τ2) By the previous item, ⊢ φ ↔ τ(φ). But, as it has been shown, Lg,[Γ] is sound for pointed MA-models; therefore, ⊢ φ ↔ τ(φ).

Finally, the argument for strong completeness, which has three steps.

(i) Take Ψ ∪ {φ} ⊆ Lg,[Γ] and suppose Ψ ⊢ φ, i.e., suppose that, for every pointed MA-model (M, w), if (M, w) ⊢ Ψ then (M, w) ⊢ φ or, in other words,
for every such \((M, w)\),
\[(M, w) \vdash \psi \text{ for all } \psi \in \Psi \quad \text{implies} \quad (M, w) \not\vdash \varphi.\]

Since \(\vdash \varphi' \leftrightarrow \tau(\varphi')\) for every \(\varphi' \in \mathcal{L}_{D,[]}\) \((\text{Proposition A.2}(\tau_2))\), it follows that, for every \((M, w)\),
- \((M, w) \vdash \tau(\psi)\) for all \(\psi \in \Psi\) if and only if \((M, w) \vdash \psi\) for all \(\psi \in \Psi\), and
- \((M, w) \vdash \varphi\) if and only if \((M, w) \not\vdash \tau(\varphi)\).

Thus, for every \((M, w)\),
\[(M, w) \vdash \tau(\psi) \text{ for all } \psi \in \Psi \quad \text{implies} \quad (M, w) \not\vdash \tau(\varphi).\]

By defining \(\tau(\Psi) := \{\tau(\psi) \mid \psi \in \Psi\}\), it follows that \((M, w) \vdash \tau(\Psi)\) implies \((M, w) \not\vdash \tau(\varphi)\) for every \((M, w)\); in other words, \(\tau(\Psi) \not\vdash \tau(\varphi)\).

\(\text{(ii)}\) Since \(\tau(\varphi') \in \mathcal{L}_{D,[]}\) for every \(\varphi' \in \mathcal{L}_{D,[]}\) \((\text{Proposition A.1}(\tau_2))\), it follows that \(\tau(\Psi) \cup \{\tau(\varphi)\} \subseteq \mathcal{L}_{D,[]}\); therefore, the just obtained \(\tau(\Psi) \not\vdash \tau(\varphi)\) and Theorem 1 imply \(\tau(\Psi) \not\vdash \tau(\varphi)\) under \(L_D\). Since \(L_D\) is a subsystem of \(L_{D,[]}\), it follows that \(\tau(\Psi) \not\vdash \tau(\varphi)\) under \(L_{D,[]}\).

\(\text{(iii)}\) Since \(\tau(\Psi) \not\vdash \tau(\varphi)\) under \(L_{D,[]}\), there are \(\psi'_1, \ldots, \psi'_n \in \tau(\Psi)\) such that
\[\vdash (\psi'_1 \land \cdots \land \psi'_n) \not\vdash \tau(\varphi).\]

under \(L_{D,[]}\). Then, from the definition of \(\tau(\Psi)\), there are \(\psi_1, \ldots, \psi_n \in \Psi\) such that
\[\vdash (\tau(\psi_1) \land \cdots \land \tau(\psi_n)) \not\vdash \tau(\varphi).\]

under \(L_{D,[]}\). But \(\varphi' \leftrightarrow \tau(\varphi')\) for every \(\varphi' \in \mathcal{L}_{D,[]}\) \((\text{Proposition A.2}(\tau_1))\); hence, from \(\Psi \cup \{\varphi\} \subseteq \mathcal{L}_{D,[]}\) (and using propositional reasoning for the first),
\[\vdash (\psi_1 \land \cdots \land \psi_n) \not\vdash (\tau(\psi_1) \land \cdots \land \tau(\psi_n)) \quad \text{and} \quad \vdash (\psi_1 \land \cdots \land \psi_n) \not\vdash \varphi.\]

Therefore,
\[\vdash (\psi_1 \land \cdots \land \psi_n) \not\vdash \varphi\]

and hence \(\Psi \not\vdash \varphi\) under \(L_{D,[]}\), as required.

### A.2 Proof of Theorem 3 (System \(L_{D,[]}\))

**Soundness** Again, the soundness of axioms and rules in \(L_D\) is well-known. For those in \(\text{Table 3}\), the soundness of \(\mathcal{A}_{S_1}, \mathcal{A}_{S_2}, \mathcal{A}_{S_3}\) and \(\text{RE}_S\) is as in the \(\forall \forall\forall\) case (for the latter, recall that \(M_S\) is a model in \(M_\Lambda\)), and \(\mathcal{A}_{S_4}\) has been proved valid already \((\text{Proposition A.6})\).

**Completeness** The argument relies on the following translation.

**Definition A.3 (Translation)** The translation \(\tau\) is given by
\[
\begin{align*}
\tau(p) &:= p, \\
\tau(\neg \varphi) &:= \neg \tau(\varphi), \\
\tau(\varphi_1 \land \varphi_2) &:= \tau(\varphi_1) \land \tau(\varphi_2), \\
\tau(D_{S} \varphi) &:= D_{S} \tau(\varphi), \\
\tau([S_1] \varphi) &:= [S_1] \tau(\varphi), \\
\tau([S_2] \varphi) &:= [S_2] \tau(\varphi), \\
\tau([S_3] \varphi) &:= [S_3] \tau(\varphi). \\
\end{align*}
\]
This translation works again in an “inside-out” fashion, dealing first with the deepest occurrence of \([S]\) before dealing with the rest. The strategy for proving strong completeness is as in the \(∀∀∀\) case: (i) show that \(τ\) is a proper recursive translation that returns formulas in \(L_D\) (Proposition A.3), (ii) show that a formula and its translation are both provably and semantically equivalent (Proposition A.4), and (iii) use \(τ\) and the completeness of \(L_D\) to show that, if \(Ψ \cup \{ϕ\} \subseteq L_D, [S]\), then from \(Ψ \vdash ϕ\) it follows that \(ϕ\) is derivable from \(Ψ\) in \(L_D, [S]\).

Here is the notion of complexity on which the proofs of Proposition A.3 and Proposition A.4 rely.

**Definition A.4 (Complexity for \(L_D, [S]\))** The functions \(nsc : L_D, [S] \rightarrow \mathbb{N} \setminus \{0\}\) and \(ndc : L_D, [S] \rightarrow \mathbb{N}\) are defined, for atoms, Boolean operators and the modality \(D_G\), as in the \(∀∀∀\) case (Definition A.2). For the dynamic operator \([S]\), the cases are as for \([!]\):

\[
nsc([S]\varphi) := 2 \cdot nsc(ϕ), \quad ndc([S]\varphi) := 1 + ndc(ϕ).\]

Then define \(c\) as before: given \(ϕ_1, ϕ_2 \in L_D, [S]\), write \(c(ϕ_1) > c(ϕ_2)\) if and only if

\[
ndc(ϕ_1) > ndc(ϕ_2) \quad \text{or} \quad ndc(ϕ_1) = ndc(ϕ_2) \quad \text{and} \quad nsc(ϕ_1) > nsc(ϕ_2). \quad \blacksquare
\]

First, \(τ\) is a proper recursive translation from \(L_D, [S]\) to \(L_D\).

**Proposition A.3** For every \(ϕ \in L_D, [S]\),

(τ1) if \(τ(ϕ)\) is defined in terms of \(τ(ψ)\), then \(c(ϕ) > c(ψ)\).

(τ2) \(τ(ϕ) \in L_D\).

**Proof.** The following lemma will be useful.

**Lemma A.3** Let \(ssub : L_D, [S] \rightarrow \mathcal{P}(L_D, [S])\) be the strict subformula function for the language \(L_D, [S]\) (defined in the expected way); let \(ϕ\) be an \(L_D, [S]\)-formula. Then, \(c(ϕ) > c(ψ)\) for every \(ψ \in ssub(ϕ)\).

**Proof.** The proof is by structural induction on \(ϕ\). Given that \(nsc, ndc\) and \(c\) are defined as for the \(∀∀∀\) instance, all cases here are exactly as in their \(∀∀∀\) counterpart (Lemma A.1). ■

The proof of the proposition is by induction on \(c(ϕ)\) relying on Lemma A.3, just as in the \(∀∀∀\) case. Again, Item (τ1) and Item (τ2) are proved simultaneously. Given that \(nsc, ndc, c\) and \(τ\) are defined as for the \(∀∀∀\) instance, all cases here are exactly as in their \(∀∀∀\) counterpart (Proposition A.1). ■

Then, a formula \(ϕ \in L_D, [S]\) and its translation \(τ(ϕ) \in L_D\) are both provably and semantically equivalent.

**Proposition A.4** For every \(ϕ \in L_D, [S]\),

(τ1) \(ϕ \leftrightarrow τ(ϕ)\),

(τ2) \(ϕ \equiv τ(ϕ)\)

**Proof.** Here are the arguments.
Proposition A.3 and the fact that LD is a subsystem of LD,SG. The inductive cases [S]p, [S]¬q, [S]|q₁ ∧ q₂, [S]|Dq and [S]|S₁|S₂|q are also as in Proposition A.2.(τ1) (relying on Proposition A.3 and τ’s definition), this time using axioms A_S, A_S̃, A_S̃D and rule RE_S, respectively.

(τ2) Exactly as in Proposition A.2.(τ2).

Finally, the argument for strong completeness is as the ∀∀∀ case (page 32), relying on Proposition A.3 and Proposition A.4 instead.

A.3 Proof of Theorem 4 (System LD,SG)

Soundness The soundness of axioms and rules in LD is well-known. For those in Table 4, soundness of A_Sₘₐ, A_Sₘₐ̃, A_S₁̃, and RE_S₁, is as in the previous cases (for the latter, recall that M₈,₁ is a model in Mₐ), and A₈,₁ has been proved valid already (Proposition 4.4).

Completeness The argument uses the following “inside-out” translation.

Definition A.5 (Translation) The translation τ is given by

\[
\begin{align*}
\tau(p) & := p, \\
\tau(\neg q) & := \neg \tau(q), \\
\tau(q₁ ∧ q₂) & := \tau(q₁) ∧ \tau(q₂), \\
\tau(Dq) & := Dq \tau(q), \\
\tau([S₁]|q₁ ∧ [S₂]|q₂) & := \tau([S₁]|q₁) ∧ \tau([S₂]|q₂), \\
\tau([S₁]|Dq) & := \tau([S₁]|Dq), \\
\tau([S₁]|S₂|q) & := \tau([S₁]|S₂|q).
\end{align*}
\]

Here is the notion of complexity on which the proofs of Proposition A.5 and Proposition A.6 will rely.

Definition A.6 (Complexity for LD,SG) The functions nsc : LD,SG → N \ {0} and ndc : LD,SG → N are defined, for atoms, Boolean operators and the modality Dq, as in the ∀∀∀ case (Definition A.2). The case of dynamic operator [S₁]|, though, is different:

\[
nsc([S₁]|q) := (8 + nsc(χ))nsc(q), \hspace{1cm} ndc([S₁]|q) := 1 + ndc(χ) + ndc(q).
\]

Define c as before: given q₁, q₂ ∈ LD,SG, write c(q₁) > c(q₂) if and only if

\[
\text{nsc(q₁)} > \text{nsc(q₂)} \quad \text{or} \quad \text{ndc(q₁)} = \text{ndc(q₂)} \text{ and nsc(q₁)} > \text{nsc(q₂)}.
\]

First, τ is a proper recursive translation from LD,SG to LD.

Proposition A.5 For every q ∈ LD,SG,

(τ1) if τ(q) is defined in terms of τ(ψ), then c(q) > c(ψ).
(τ2) τ(q) ∈ LD.
Proof. The following lemma will be useful.

Lemma A.4 Let \( \text{ssub} : \mathcal{L}_{D, [S, \psi]} \to \phi(\mathcal{L}_{D, [S, \psi]}) \) be the strict subformula function for the language \( \mathcal{L}_{D, [S, \psi]} \) (defined for atoms, Boolean operators and the \( D_{\theta} \) modality as before, and for \( [S, \psi] \)) as \( \text{ssub}([S, \psi]) := \{ \chi, \phi \} \cup \text{ssub}(\chi) \cup \text{ssub}(\phi) \); let \( \phi \) be an \( \mathcal{L}_{D, [S, \psi]} \)-formula. Then, \( c(\phi) > c(\psi) \) for every \( \psi \in \text{ssub}(\phi) \).

Proof. The proof is by structural induction on \( \phi \). For atoms, Boolean operators and formulas of the form \( D_{\theta} \), the functions \( nsc, ndc \) and \( c \) are defined as before, and for cases.

\[ \text{L} \] Thus, take any \( \psi \in \text{ssub}([S, \psi]) \) and consider the cases.

- **Cases** \( \psi = \varphi \) and \( \psi = \chi \). By Item (i), it follows that \( c([S, \psi]) > c(\psi) \).
- **Case** \( \psi \in \text{ssub}(\chi) \). By IH, \( c(\chi) > c(\psi) \) for any such \( \psi \). Then, by definition of \( c \), either
  - \( ndc(\chi) > ndc(\psi) \), so \( ndc([S, \psi]) > ndc(\psi) \) (by Item (i)) and therefore \( c([S, \psi]) > c(\psi) \), or
  - \( ndc(\chi) = ndc(\psi) \) and \( nsc(\chi) > nsc(\psi) \), so \( ndc([S, \chi]) > ndc(\psi) \) (by Item (i)) and hence \( c([S, \psi]) > c(\psi) \).
- **Case** \( \psi \in \text{ssub}(\phi) \). Exactly as the previous one.

The proof of the proposition is by induction on \( c(\phi) \) relying on Lemma A.4, analogous to the \( \forall \forall \) case. Again, Item (\( \tau \)I) and Item (\( \tau \)2) are proved simultaneously. For atoms, Boolean operators and formulas of the form \( D_{\theta} \), the functions \( nsc, ndc, c \) and the translation \( \tau \) are defined as for the \( \forall \forall \) instance; thus, cases \( p, \neg \varphi, \varphi_1 \land \varphi_2 \) and \( D_{\theta} \) are exactly as in Proposition A.1. Here are the remaining cases.

- **Inductive case** (\( \phi \) such that \( c(\phi) \) is not minimum), case \( [S, \psi]p \). For Item (\( \tau \)I), the definition of \( \tau([S, \psi])p \) uses \( \tau(p) \). For \( p \in \text{ssub}([S, \psi]p) \), so by Lemma A.4, \( c([S, \psi]p) > c(p) \). For Item (\( \tau \)2), take the just obtained \( c([S, \psi]p) > c(p) \); then, by IH, \( \tau(p) = \tau([S, \psi]p) \in \mathcal{L}_D \).
- **Inductive case** (\( \phi \) such that \( c(\phi) \) is not minimum), case \( [S, \psi] \neg \varphi \). For Item (\( \tau \)I), the definition of \( \tau([S, \psi]) \neg \varphi \) uses \( \tau(\neg [S, \psi]) \neg \varphi \). Now note that, on the one hand,
  - \( ndc([S, \psi]) \neg \varphi = 1 + ndc(\chi) + ndc(\varphi) \),
  - \( nsc([S, \psi]) \neg \varphi = 8 + 8 nsc(\varphi) + nsc(\chi) + nsc(\chi) nsc(\varphi) \),
  - \( nsc([S, \psi]) \neg \varphi = 1 + 8 nsc(\varphi) + nsc(\chi) nsc(\varphi). \)
  
  Thus, \( c([S, \psi]) \neg \varphi > c([S, \psi]) \neg \varphi \). For Item (\( \tau \)2), the just obtained \( c([S, \psi]) \neg \varphi > c([S, \psi]) \neg \varphi \) implies, by IH, \( \tau([S, \psi]) \neg \varphi = \tau([S, \psi]) \neg \varphi \in \mathcal{L}_D \).
• Inductive case (ϕ such that c(ϕ) is not minimum), case \([S_1,!] \land \varphi_2\). For Item (τ1), the definition of \(\tau([S_1,!] \land \varphi_2)\) uses \(\tau([S_1,!] \varphi_1 \land [S_1,!] \varphi_2)\). Now, on the one hand, by taking max \([\text{ndc}(\varphi_1), \text{ndc}(\varphi_2)] = \text{ndc}(\varphi_i)\),
- \(\text{ndc}([S_1,!] \land \varphi_2) = 1 + \text{ndc}(\chi) + \text{ndc}(\varphi_i)\),
- \(\text{ndc}([S_1,!] \varphi_1 \land [S_1,!] \varphi_2) = 1 + \text{ndc}(\chi) + \text{ndc}(\varphi_i)\),
but, on the other hand, by taking max \([\text{nsc}(\varphi_1), \text{nsc}(\varphi_2)] = \text{nsc}(\varphi_i)\),
- \(\text{nsc}([S_1,!] \land \varphi_2) = 8 + 8 \text{nsc}(\varphi_i) + \text{nsc}(\chi) + \text{nsc}(\varphi_i)\),
- \(\text{nsc}([S_1,!] \varphi_1 \land [S_1,!] \varphi_2) = 8 + 8 \text{nsc}(\varphi_i) + \text{nsc}(\chi) \text{nsc}(\varphi_i)\).
Thus, \(c([S_1,!] \land \varphi_2) > c([S_1,!] \varphi_1 \land [S_1,!] \varphi_2)\). For Item (τ2), take the just obtained \(c([S_1,!] \land \varphi_2) > c([S_1,!] \varphi_1 \land [S_1,!] \varphi_2)\); then, by IH, \(\tau([S_1,!] \varphi_1 \land [S_1,!] \varphi_2) = \tau([S_1,!] \land \varphi_2) \in \mathcal{L}_D\).

• Inductive case (ϕ such that c(ϕ) is not minimum), case \([S_1,!] \land \varphi_1\). For Item (τ1), the definition of \(\tau([S_1,!] \land \varphi_1 \land \varphi_2)\) uses \(\tau(D_{\varphi_1} \land [S_1,!] \varphi_1 \land [S_1,!] \varphi_2)\). Now note that, on the one hand,
- \(\text{ndc}([S_1,!] \land \varphi_1) = 1 + \text{ndc}(\chi) + \text{ndc}(\varphi_i)\),
- \(\text{ndc}(D_{\varphi_1} \land [S_1,!] \varphi_1) = 1 + \text{ndc}(\chi) + \text{ndc}(\varphi_i)\),
but, on the other hand,
- \(\text{nsc}([S_1,!] \land \varphi_1) = 8 + 8 \text{nsc}(\varphi_i) + \text{nsc}(\chi) + \text{nsc}(\varphi_i)\),
- \(\text{nsc}(D_{\varphi_1} \land [S_1,!] \varphi_1) = 8 + 8 \text{nsc}(\varphi_i) + \text{nsc}(\chi) \text{nsc}(\varphi_i)\).
Thus, \(c([S_1,!] \land \varphi_1) > c(D_{\varphi_1} \land [S_1,!] \varphi_1)\). This also yields Item (τ2), as from it and IH it follows that \(\tau(D_{\varphi_1} \land [S_1,!] \varphi_1) = \tau([S_1,!] \land \varphi_1) \in \mathcal{L}_D\).

• Inductive case (ϕ such that c(ϕ) is not minimum), case \([S_1,!] \land [S_2,!] \varphi\). For Item (τ1), the definition of \(\tau([S_1,!] \land [S_2,!] \varphi)\) uses two instances of τ, namely \(\tau([S_2,!] \varphi)\) and \(\tau([S_1,!] \land [S_2,!] \varphi)\). For the first, \([S_2,!] \varphi \in \text{ssub}([S_1,!] \land [S_2,!] \varphi)\) so, by Lemma A.4, \(c([S_1,!] \land [S_2,!] \varphi) > c([S_2,!] \varphi)\). For the second, note that,
- \(\text{ndc}([S_1,!] \land [S_2,!] \varphi) = 2 + \text{ndc}(\chi_1) + \text{ndc}(\chi_2) + \text{ndc}(\varphi_i)\),
- \(\text{ndc}([S_1,!] \land [S_2,!] \varphi) = 1 + \text{ndc}(\chi_1) + \text{ndc}(\tau([S_2,!] \varphi))\). But, as it has been shown,
\(c([S_1,!] \land [S_2,!] \varphi) > c([S_2,!] \varphi)\); thus, by IH, \(\tau([S_2,!] \varphi) \in \mathcal{L}_D\) and therefore \(\text{ndc}([S_2,!] \varphi) = 0\). Hence, \(\text{ndc}([S_1,!] \land [S_2,!] \varphi) = 1 + \text{ndc}(\chi_1)\).
Thus, \(c([S_1,!] \land [S_2,!] \varphi) > c([S_1,!] \land [S_2,!] \varphi)\). This also yields Item (τ2) as, from it and IH, \(\tau([S_1,!] \land [S_2,!] \varphi) = \tau([S_1,!] \land [S_2,!] \varphi) \in \mathcal{L}_D\).

Then, a formula \(\varphi \in \mathcal{L}_D, [S_1,!]\) and its translation \(\tau(\varphi) \in \mathcal{L}_D\) are both provably and semantically equivalent.

Proposition A.6 For every \(\varphi \in \mathcal{L}_D, [S_1,!]\),

\((\tau_1) \quad \varphi \leftrightarrow \tau(\varphi), \quad \text{(\tau_2) \quad } \varphi \leftrightarrow \tau(\varphi)\)

Proof. Here are the arguments.

(τ1) The proof proceeds by induction on c(ϕ). Given Proposition A.5 and the fact that τ is defined as in the \(\land \land \land\) instance, the base case (ϕ) and inductive cases \(\neg \varphi, \varphi_1 \land \varphi_2\) and \(D_{\varphi} \varphi\) are as in Proposition A.2 (τ1), the latter
using Lemma A.2 and the fact that $L_D$ is a subsystem of $L_{SD\chi}$. The inductive cases $[S,\chi]|p$, $[S,\chi]|\neg\phi$, $[S,\chi]|(\theta_1 \land \phi_2)$, $[S,\chi]|D_{\chi}\phi$ and $[S,\chi]|D_{S,\chi}\phi$ are also as in Proposition A.2 (τ1) (relying on Proposition A.5 and τ’s definition), this time using axioms $A_{S,\chi}$, $A_{S,\chi}$, $A_{S,\chi}$, $A_{S,\chi}$ and rule $\text{RE}_{S,\chi}$ respectively.

(τ2) Exactly as in Proposition A.2 (τ2).

Finally, the argument for strong completeness is as the $\forall\forall\forall$ case (page 32), relying on Proposition A.5 and Proposition A.6 instead.

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