CHERN CLASS INEQUALITIES FOR NON-UNIRULED PROJECTIVE VARIETIES

ERWAN ROUSSEAU AND BEHROUZ TAJI

Abstract. It is known that projective minimal models satisfy the celebrated Miyaoka–Yau inequalities. In this article, we extend these inequalities to the set of all smooth, projective and non-uniruled varieties.

1. Introduction

Interconnections between topological, analytic and algebraic structures of compact complex varieties is a central theme in various branches of geometry and topology. Many of the classical results in this area use characteristic classes, in particular Chern classes to describe such connections. Apart from the celebrated Hironaka-Riemann-Roch theorems, prominent examples were found by Bogomolov [Bog79] and—in a different direction—Yau [Yau77], as a consequence of his (and Aubin’s) solution to Calabi’s conjecture. More precisely, he established that an \( n \)-dimensional compact Kähler manifold \((X, w)\) with \( c_1(X) < 0 \) satisfies the inequality

\[
\int_X \left( 2(n+1)c_2(X) - nc_1^2(X) \right) \wedge w^{n-2} \geq 0.
\]

In a more general setting, using his generic semipositivity result, Miyaoka [Miy87] showed that any minimal variety\(^1\) \( X \) satisfies the inequality

\[
(3c_2(X) - c_1^2(X)) \cdot H^{n-2} \geq 0,
\]

for every ample divisor \( H \subset X \). The combination of the two inequalities (1.0.1), (1.0.2) and their analogues are nowadays referred to as the \textit{Miyaoka–Yau inequalities}.

The purpose of this article is to establish that, as long as \( X \) is not covered by rational curves, it satisfies a Chern class inequality generalizing (1.0.2). Throughout this paper all varieties will be over \( \mathbb{C} \).

Theorem 1.1. Let \( X \) be a smooth, projective and non-uniruled variety of dimension \( n \), and \( H \) any ample divisor. There is a decomposition \( K_X = P + N \) into \( \mathbb{Q} \)-divisors, with \( P \cdot N \cdot H^{n-2} = 0 \), such that

\[
(3c_2(X) - c_1^2(X)) \cdot H^{n-2} \geq N^2 \cdot H^{n-2}.
\]

Moreover, we have \( N = 0 \), when \( K_X \) is nef.

\(^1\)Here minimal is in the sense of the minimal model program, that is \( K_X \) is assumed to be nef, with \( X \) having only terminal singularities.
A few remarks about the statement of Theorem 1.1. First, we note that it is well-known that, for a non-uniruled variety, Chern class inequalities of the form (1.1.1) cannot be gleaned from the ones for its minimal models, when they exist. Second, as is evident from the statement of the theorem, the quantity on the right-hand-side of (1.1.1) (which we may think of as an error term) is forced on us by the negative part of Zariski decomposition. More precisely, given a compete intersection surface $S \subset X$ defined by very general members of very ample linear systems, the divisor $N$ is an extension of the negative part of the Zariski decomposition for $K_X|_S$ (see Section 3 for the details). This extension is in the sense of the Noether-Lefschetz-type theorems (Proposition 3.1).

Theorem 1.1 is a special case of the following more general result that we obtain in this article, which is in fact needed for the proof of Theorem 1.1.

**Theorem 1.2.** Let $(X,D)$ be a log-smooth pair of dimension $n$, with $D$ being a rational divisor. Assume that $H$ is an ample divisor. If $K_X+D$ is pseudo-effective, then, there is a decomposition $K_X+D = P + N$, that is $H$-orthogonal in the sense that $P \cdot N \cdot H^{n-2} = 0$, and for which the inequality
\[
(3\widehat{c}_2(X,D) - \overline{c}_1^2(X,D)) \cdot H^{n-2} \geq N^2 \cdot H^{n-2}
\]
holds. Furthermore, when $K_X + D$ is nef, we have $N = 0$.

The Chern classes $\widehat{c}_i(\cdot)$ in Theorem 1.2 are in the sense of orbifolds (see (2.10.1)), and when $D$ is reduced, they coincide with the usual notion of Chern classes.

1.1. **General strategy of the proof.** For simplicity we will focus mostly on Theorem 1.1; the case where $D = 0$.

As was observed by Miyaoka [Miy87] and later on Simpson [Sim88], it is sometimes possible to use the Bogomolov inequality [Bog79] to establish Miyaoka–Yau inequalities. But the Bogomolov inequality is generally valid when the polarization is defined by ample or nef divisors, which is applicable—for the purpose of Miyaoka–Yau inequalities—when the variety is minimal. But for a general non-uniruled variety $X$ no such polarization exists. On the other hand, we show in the current article that, thanks to the result of Boucksom–Demailly–Păun–Peternell [BDPP13], after cutting down by hyperplanes, the above divisor $P$ defines a so-called movable cycle; a potentially natural choice for a polarization. But in general there is no topological Bogomolov-type inequality for sheaves that are semistable with respect to a movable class $\gamma$. At best, assuming that $\mathcal{E}$ is locally free, one can use a Gauduchon metric $w_G$ constructed in [CP11, Append.], with $w_G^{n-1} \equiv \gamma$, and Li–Yau’s result on the existence of Hermitian-Einstein metrics [LY87] to establish the Bogomolov inequality with respect to $w_G^{n-2}$. But since $w_G^{n-2}$ is not closed, this would not yield a topological inequality. However, thanks to a fundamental result of Langer [Lan04, Thm. 3.4], semistability with respect to a certain subset of movable classes does lead to the classical Bogomolov inequality. Having this important fact in mind, we use the definition of the Zariski decomposition to show that the intersection of $P$ with $H^{n-2}$ belongs to this smaller subset, as long as $X$ is of general type. With this observation, and using further properties of $P$, we then show that, thanks to Campana-Păun’s result on positivity properties of the (log-)cotangent sheaf with respect to movable cycles [CP19], much of Miyaoka’s original approach can then be adapted to establish (1.1.1).

In the more general setting of non-uniruled varieties, that is when $K_X$ is pseudo-effective [BDPP13], given an ample divisor $A$ and any $m \in \mathbb{N}$, we consider the pair $(X, \frac{1}{m}A)$, which is now of log-general type. Here, the log-version of Theorem 1.1 is needed, forcing us to
resort to orbifolds (in the sense of Campana) and their Chern classes as was defined in [GT16, Sect. 2], following [Mum83]. With the inequality (1.2.1) at hand, one can then extract the inequality (1.1.1) through a limiting process, which is reminiscent of [GT16], but employed for somewhat different reasons.

1.2. Related results. Chern class inequalities for surfaces and their connection to the Zariski decomposition was first studied by Miyaoka in [Miy84] and later on by Wahl [Wah94], Megyesi [Meg99], Langer [Lan01] and others. In higher dimensions, when \( K_X + D \) is movable and \( \dim X = 3 \), the inequality (1.2.1) is established in [RT16] for (mildly) singular pairs. Under the assumption that \( K_X + D \) is nef and big, such Chern class inequalities have a rich history and were discovered by Kobayashi [Kob84], Tsuji [Tsu88], Tian [Tian94], to name a few. More recently, and in a more general setting, they have been studied in joint papers with Greb–Kebekus–Peternell [GKPT15] and with Guenancia [GT16]. Further results have been established by Deng [Den21] and Hai–Schreieder [HS20]. Finally, we note that the methods that we use in this article show that coefficient of \( N \) in (1.2.1) can be sharpened. In Section 4 we make some predictions about possibly optimal versions of Theorem 1.2.

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2. Preliminaries

In this section we review certain cones of divisors and curves that are needed for the rest of the article. Relevant notions of (slope) stability, orbifolds and Chern classes will also be introduced.

By a variety we mean a reduced, irreducible complex scheme of finite type. Given a variety \( X \) of dimension \( n \), by \( \text{Div}(X) \) and \( \text{Pic}(X) \) we denote the group of Cartier divisors and isomorphism classes of line bundles, respectively. \( N^1(X) \) denotes the Néron-Severi group consisting of numerical classes of elements of \( \text{Div}(X) \), i.e. \( N^1(X) = \text{Div}(X)/\equiv \). We use \( \text{Div}(X)_\mathbb{Q} \), \( N^1(X)_\mathbb{Q} \) to denote \( \text{Div}(X) \otimes \mathbb{Q} \) and \( N^1(X) \otimes \mathbb{Q} \), respectively. The spaces \( \text{Div}(X)_\mathbb{R} \) and \( N^1(X)_\mathbb{R} \) are similarly defined. We recall that via intersection products \( N^1(X) \) is dual to the Abelian group of classes of curves \( N_1(X) \), which extends to cycle classes with rational or real coefficients (see [Laz04, Sects. 1.1.1.3] for more details). The notation \( A^i(X) \) (and \( A_i(X) \)) refers to the Chow group of \( i \)-cocycles (resp. \( i \)-cycles) in \( X \).

Notation 2.1. Given \( D_i \in \text{Div}(X)_\mathbb{Q} \), for \( 1 \leq i \leq n-1 \), by \([D_1 \cdot \ldots \cdot D_{n-1}]\) we denote the \( n \)-cycle in \( N_1(X)_\mathbb{Q} \) canonically defined by \( D_i \)'s, i.e. the image of \((c_1(\mathcal{O}_X(D_1)) \cdot \ldots \cdot c_1(\mathcal{O}_X(D_n)) \cdot [X]) \in A_1(X) \otimes \mathbb{Q} \) under the cycle map, with \([X]\) denoting the fundamental cycle.

2.1. Cones of curves and divisors, and stability notions. Assuming that \( X \) is projective, let \( NE^1(X)_\mathbb{Q} \subset N^1(X)_\mathbb{Q} \) denote the convex cone of classes of effective \( \mathbb{Q} \)-divisors and set \( NE^1(X)_\mathbb{Q} \) to be its closure, called the pseudo-effective cone.

We define the notion of slope stability in the following general setting.

Definition 2.2 (Slope stability). Given a torsion free sheaf \( \mathcal{F} \) on a smooth projective variety \( X \) and \( \gamma \in N_1(X)_\mathbb{Q} \), we define the slope \( \mu_\gamma(\mathcal{F}) \) of \( \mathcal{F} \) with respect to \( \gamma \) by \( \frac{\sup \{ \langle \mathcal{F}, c_1(\mathcal{F}) \rangle \} \cdot \gamma}{\dim(X)} \). A torsion free sheaf \( \mathcal{E} \) is said to be stable (or semistable) with respect to \( 0 \neq \gamma \), if \( \mu_\gamma(\mathcal{F}) < \mu_\gamma(\mathcal{E}) \) (resp. \( \mu_\gamma(\mathcal{F}) \leq \mu_\gamma(\mathcal{E}) \)), for every nontrivial torsion free subsheaf \( \mathcal{F} \subset \mathcal{E} \).
Through the Harder-Narasimhan filtration, semistable sheaves form the building blocks of coherent, torsion free sheaves. But to ensure the existence of such (unique) filtrations, we generally need more assumptions on $\gamma$ in Definition 2.2. In this article we require the existence of Harder-Narasimhan filtration under the assumption that $\gamma \in N_1(X)_{\mathbb{Q}}$ is movable. Thankfully, such filtrations are known to exist for such classes [GKP16, Sect. 2].

**Definition 2.3** (Movable classes). We say $\gamma \in N_1(X)_{\mathbb{Q}}$ is strongly movable, if there are a projective birational morphism $\pi : \tilde{X} \to X$ and a set of ample divisors $H_1, \ldots, H_{n-1}$ on $\tilde{X}$ such that $\gamma = \pi_* [H_1 \cdot \ldots \cdot H_{n-1}]$. The convex cone in $N_1(X)_{\mathbb{Q}}$ generated by such classes is denoted by $\text{Mov}_1(X)$. We call its closure $\overline{\text{Mov}}_1(X)$ the movable cone. Nontrivial members of $\overline{\text{Mov}}_1(X)$ are referred to as movable or mobile classes.

As discussed in the introduction, for semistability to lead to a suitable Bogomolov inequality we need to work with a smaller set of 1-cycles than those in $\overline{\text{Mov}}_1(X)$. To do so we use the following definition.

**Definition 2.4.** Given an ample class $H \in N^1(X)_{\mathbb{Q}}$, we define

$$K^+_H(X) := \{ D \in N^1(X)_{\mathbb{Q}} \mid D^2 \cdot H^{n-2} > 0 \text{ and } D \cdot H^{n-1} > 0 \} \subset N^1(X)_{\mathbb{Q}}.$$ 

Furthermore, we set

$$B^+_H(X) := \{(D \cdot H^{n-2}) \in N^1(X)_{\mathbb{Q}} \mid D \in K^+_H(X)\} \subset N^1(X)_{\mathbb{Q}}.$$ 

A key property of $K^+_H$ is its “self-duality” in the sense that

$$K^+_H(X) = \{ D \in N^1(X)_{\mathbb{Q}} \mid D \cdot B \cdot H^{n-2} > 0, \forall 0 \neq B \in \overline{K}^+_H(X) \},$$

cf. [Lan04, p. 261] and [HL10, 7.4]. In particular $B^+_H(X) \cup \{0\}$ forms a convex cone. The next theorem of Langer explains our interest in $B^+_H(X)$.

**Theorem 2.5** ([Lan04, Thm. 3.4]). For any torsion free sheaf $\mathcal{F}$ of rank $r$ on $X$, satisfying the inequality

$$\left(2rc_2(\mathcal{F}) - (r - 1)c_1^2(\mathcal{F})\right)H^{n-2} < 0$$

there is a saturated subsheaf $0 \neq \mathcal{F}' \subset \mathcal{F}$ of rank $r'$ such that

$$\left(\frac{1}{r}c_1(\mathcal{F}') - \frac{1}{r}c_1(\mathcal{F})\right) \in K^+_H(X)^2.$$ 

**Remark 2.6.** We note that by the self-duality property (2.4.1), the subsheaf $\mathcal{F}'$ in Theorem 2.5 is a properly destabilizing subsheaf with respect to any $\gamma \in B^+_H(X)$. That is, if $\mathcal{F}$ is semistable with respect to some $\gamma \in B^+_H(X)$, then it verifies the Bogomolov inequality $\Delta_B(\mathcal{F}) \cdot H^{n-2} \geq 0$.

2.2. **Orbifold sheaves and Chern classes.** We follow the definitions and constructions of [GT16, Sects. 2,3] in the generally simpler context of log-smooth pairs. We refer to [Cam04], [JK11], [Taj16], and [CKT16] for more examples and details on pairs and associated notions of adapted morphisms.
A pair \((X, D)\) consists of a variety \(X\) and \(D = \sum d_i \cdot D_i \in \text{Div}(X)_\mathbb{Q}\), with \(d_i = 1 - \frac{b_i}{a_i} \in [0, 1] \cap \mathbb{Q}\), for some \(a_i, b_i \in \mathbb{N}\). A pair \((X, D)\) is said to be log-smooth, if \(X\) is smooth and \(D\) has simple normal crossing support. We say \((X, D)\) is (quasi-)projective, if \(X\) is so.

We now recall a few basic notions regarding morphisms, sheaves and Chern classes encoding the fractional part of \(D\) in \((X, D)\).

**Definition 2.7** (Adapted morphisms). Given a quasi-projective pair \((X, D)\), a finite, Galois and surjective morphism \(f : Y \to X\) of schemes is called adapted (to \((X, D)\)), if the following conditions are satisfied.

- (2.7.1) \(Y\) is normal and quasi-projective.
- (2.7.2) For every \(D_i\), with \(d_i \neq 1\), there are \(m_i \in \mathbb{N}\) and a reduced divisor \(D'_i \subset Y\) such that \(f^*D_i = (m_i \cdot a_i) \cdot D'_i\).
- (2.7.3) The morphism \(f\) is étale at the generic point of \([D]\).

Furthermore, if \(m_i = 1\), for all \(i\), we say \(f\) is strictly adapted.

**Example 2.8.** Constructions of Bloch–Gieseker [BG71] and Kawamata [Laz04, Prop.4.12] provide prime examples of strictly adapted morphisms with the following additional property: the ramification locus of \(f\) is equal to \(\text{supp}([D] + A)\), for some general member \(A\) of a very ample linear system. Moreover, when \((X, D)\) is log-smooth, from their construction it follows that so is \((Y, (f^*D)_{\text{red}})\).

**Notation 2.9.** Given a log-smooth pair \((X, D)\), let \(f : Y \to (X, D = \sum d_i \cdot D_i)\) be strictly adapted. Assume that \(Y\) is smooth. With \(d_i = 1 - (a_i/b_i)\), let \(D'^{\text{ij}}_Y\) be the collection of prime divisors in \(\text{supp}(f^*D_i)\) and define

\[
\hat{D}^{\text{ij}}_Y := b_i \cdot D'^{\text{ij}}_Y.
\]

Let \(G := \text{Gal}(Y/X)\).

**Definition 2.10** (Orbifold cotangent sheaf). In the setting of **Notation 2.9** we define the orbifold cotangent sheaf \(\Omega^{\text{1}}_{(Y, f, D)}\) of \((X, D)\) with respect to \(f\) by the kernel of the morphism

\[
f^*\Omega^1_X(\log[D]) \longrightarrow \bigoplus_{i,j(i)} \mathcal{O}_{\hat{D}^{\text{ij}}_Y},
\]

which is naturally defined using the residue map.

We note that \(\Omega^{\text{1}}_{(Y, f, D)}\) naturally has a structure of a \(G\)-sheaf [CKT16] (see [HL10, Def. 4.2.5] for the definition). Such objects are studied in a much more general setting (called orbifold sheaves) in [GT16, Subsect. 2.6].

**2.3. Orbifold Chern classes.** Let \(f : Y \to (X, D)\) be a strictly adapted morphism for a log-smooth pair \((X, D)\). Assume that \(Y\) is smooth and set \(G := \text{Gal}(Y/X)\). Given a coherent \(G\)-sheaf \(\mathcal{E}\) on \(Y\), we have \(c_i(\mathcal{E}) \in A^i(Y)^G\). Here \(c_i(\cdot)\) denotes the \(i\)th Chern class and \(A^i(Y)^G\) the group of \(G\)-invariant, \(i\)-cocycles in \(Y\). We define the \(i\)th orbifold Chern class of \(\mathcal{E}\) by

\[
\hat{c}_i(\mathcal{E}) := \frac{1}{|G|} \cdot \psi_i(c_i(\mathcal{E})) \in A_{n-i}(X) \otimes \mathbb{Q},
\]

where \(\psi_i\) is the natural map \(\psi_i : A^i(Y)^G \otimes \mathbb{Q} \to A_{n-i}(X) \otimes \mathbb{Q}\) defined by the composition of cap product with \([Y]\) and pushforward.

With the above definition, when \(X\) is projective, \(\hat{c}_i(\mathcal{E})\) defines a multilinear form on \(N^1(X)_{\mathbb{Q}}^{n-1}\). Furthermore, with \(f\) being flat, from [Mum83, Thm. 3.1] it follows that \(\psi\)
is in fact a group isomorphism. Thus, similar to [Mum83] (or [GT16, Append.]) we can use this isomorphism to equip \( A_*(X) \otimes \mathbb{Q} \) with a ring structure compatible with that of \( A^*(Y)^G \otimes \mathbb{Q} \). In this way, products of orbifold Chern classes can also be consistently defined in \( A_*(X) \otimes \mathbb{Q} \).

One can check that for \( G \)-sheaves on \( Y \), defined by pullback of sheaves on \( X \), the above notion of orbifold Chern classes is consistent with the projection formula, when applicable.

**Remark 2.11.** Let \( f : Y \to (X, D) \) be strictly adapted to the log-smooth projective pair as in Example 2.8 and \( \mathcal{F} \) an ample \( \mathbb{Q} \)-divisor. Let \( \gamma \in B^+_H(X) \) and \( \mathcal{F} \) a torsion free, \( G \)-sheaf of rank \( r \) on \( Y \) that is \( (f^* \gamma) \)-semistable. Then, by Theorem 2.5 and Remark 2.6 we know that \( \Delta_B(\mathcal{F}) \cdot f^*H^{n-2} \geq 0 \). Using (2.10.1) we can then deduce that

\[
\Delta_B(\mathcal{F}) = \frac{2rC_2(\mathcal{F}) - (r-1)C_1(\mathcal{F})}{H^{n-2}} \geq 0.
\]

**Notation 2.12.** Given a strictly adapted morphism \( f : Y \to (X, D) \), with log-smooth \((Y, (f^*D)_{\text{red}})\), we define

\[
\bar{c}_i(X, D) := \bar{c}_i(\Omega^1_{(Y,f,D)}) \in A_{n-i}(X) \otimes \mathbb{Q}.
\]

We note that according to [GT16, Prop. 3.5 and Ex. 3.3] the cycle defined by \( \bar{c}_i(\Omega^1_{(Y,f,D)}) \) is independent of the choice of \( f \). In this light, the choice of notation in (2.12.1) is unambiguous within the set of such morphisms.

### 3. Constructing movable cycles in \( B^+_H \) via Zariski decomposition

Our main goal in this section is to use Zariski decomposition on certain complete-intersection surfaces to construct global moving cycles in \( B^+_H \), which is the content of Proposition 3.8.

**Proposition 3.1.** Let \( X \) be a smooth projective variety of dimension \( n \geq 3 \) and \( H \) a very ample divisor. For a sufficiently large \( m \), there are a (Zariski) dense subset \( V_{NL} \subseteq |mH| \) and a smooth complete intersection surface

\[
S_{NL} = H_1 \cap \ldots \cap H_{n-3} \cap A,
\]

where each \( H_i \) is a general member of \(|H|\) and \( A \in V_{NL} \), satisfying the following properties.

1. **(3.1.1)** The restriction map \( \text{Pic}(X) \to \text{Pic}(S_{NL}) \) is an isomorphism.
2. **(3.1.2)** The isomorphism in (3.1.1) extends to an isomorphism \( \mathbb{N}^1(X) \to \mathbb{N}^1(S_{NL}) \).

**Proof.** By a repeated application of the Grothendieck-Lefschetz theorem [Gro68, Exp. XII, Cor. 3.6] (see also [Laz04, Ex. 3.1.25] and further references therein) there are general members \( H_i \) of \(|H|\) such that \( Y := H_1 \cap \ldots \cap H_{n-3} \) is a smooth projective threefold for which the natural map \( \text{Pic}(X) \to \text{Pic}(Y) \) is an isomorphism. Furthermore, for any \( m \in \mathbb{N} \), after shrinking \(|mH|\), if necessary, the complete-intersection surface \( S = Y \cap A \) is smooth, for every \( A \in |mH| \).

Now, consider the short exact sequence

\[
0 \longrightarrow \mathcal{O}_Y(K_Y) \longrightarrow \mathcal{O}_Y(K_Y + S) \longrightarrow \mathcal{O}_S(K_S) \longrightarrow 0,
\]

which is naturally defined by using adjunction. By the Kodaira vanishing \( H^1(K_Y + S) = 0 \), the induced exact cohomology sequence partially reads

\[
0 \longrightarrow H^0(K_Y) \xrightarrow{i} H^0(K_Y + S) \longrightarrow H^0(K_S) \xrightarrow{\alpha} H^1(K_Y) \longrightarrow 0.
\]
Claim 3.2. We have \( h^{2,0}(Y) < h^{2,0}(S) \), if and only if \( i \) is a strict inclusion.

Proof of Claim 3.2. Noting that

\[
\begin{align*}
    h^{2,0}(Y) &= h^{0,2}(Y) = h^2(\mathcal{O}_Y) = h^1(K_Y), \\
    h^{2,0}(S) &= h^{0,2}(S) = h^2(\mathcal{O}_S) = h^0(K_S)
\end{align*}
\]

and the surjectivity of \( \alpha \), we find that \( h^{2,0}(Y) < h^{2,0}(S) \), if and only if \( \ker(\alpha) \neq 0 \). The rest now follows from a straightforward diagram chasing. \( \square \)

Now, let \( m \) be sufficiently large so that \( i : H^0(K_Y) \to H^0(K_Y + S) \) is a strict injection. Thanks to a theorem of Moishezon [Moi67, Thm. 7.5], after removing a countable number of closed subschemes from \( |mH| \), we find a subset \( V_{NL} \subseteq |mH| \) such that, for every \( A \in V_{NL} \) and \( S_{NL} := Y \cap A \), the natural map \( \text{Pic}(Y) \to \text{Pic}(S_{NL}) \) is an isomorphism.

For Item (3.1.2), we will keep the notations for the proof of Item (3.1.1). Again, since \( N^1(X) \cong N^1(Y) \) (see for example [Laz04, Ex. 3.1.29]), it suffices to prove \( N^1(Y) \cong N^1(S_{NL}) \). As \( S_{NL} \) is reduced, we have a commutative diagram of long exact cohomology sequences arising from the two exponential sequences on \( Y \) and \( S_{NL} \). In particular we have

\[
\begin{array}{ccc}
H^1(Y, \mathcal{O}_Y^*) & \longrightarrow & H^2(Y; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(S_{NL}, \mathcal{O}_{S_{NL}}^*) & \longrightarrow & H^2(S_{NL}; \mathbb{Z}).
\end{array}
\]

Now, with the vertical arrow on the left being an isomorphism by Item (3.1.1) and the one on the right being an injection by the Lefschetz hyperplane theorem ([Laz04, Thm. 3.1.17] or [Voi07, 2.3.2]), the isomorphism \( \text{Pic}(Y) \to \text{Pic}(S_{NL}) \) descends to an isomorphism \( (\text{Pic}(Y)/\equiv) \to (\text{Pic}(S_{NL})/\equiv) \), as required. \( \square \)

Before stating the application of Proposition 3.1 that we need, we briefly review Nakayama’s \( \sigma \)-decomposition.

3.1. \( \sigma \)-decomposition. According to [Nak04, Chapt. III], given a smooth projective variety and a big divisor \( B \in \text{Div}(X)_{\mathbb{R}} \), for every prime divisor \( \Gamma \subset X \), we define

\[
\sigma_{\Gamma}(B) := \inf \{ \text{mult}_B(B') \mid B' \equiv_{\mathbb{R}} B, B' \geq 0 \}
\]

and set \( N_{\sigma}(B) := \sum_{\text{prime } \Gamma} \sigma_{\Gamma}(B) \cdot \Gamma \). We further define \( P_{\sigma}(B) := B - N_{\sigma}(B) \). Now, let \( D \) be a pseudo-effective divisor. For any ample divisor \( A \), according to [Nak04, Lem. III. 1.5] and [Nak04, Lem. III. 1.7], \( \lim_{\epsilon \to 0^+} \sigma_{\Gamma}(D + \epsilon A) \) exists and is independent of the choice of \( A \). We now set

\[
\sigma_{\Gamma}(D) := \lim_{\epsilon \to 0^+} \sigma_{\Gamma}(D + \epsilon A)^3
\]

and define the negative part of \( D \) by

\[
N_{\sigma}(D) := \sum_{\text{prime } \Gamma} \sigma_{\Gamma}(D) \cdot \Gamma,
\]

which by [Nak04, Cor. III.1.11] is a finite sum. We set \( P_{\sigma}(D) := D - N_{\sigma}(D) \) and call the decomposition \( D = N_{\sigma}(D) + P_{\sigma}(D) \) the \( \sigma \)-decomposition of \( D \). We sometimes refer to \( P_{\sigma} \) as the positive part of this decomposition.

Next, we establish the fact that negative parts behave well under taking limits.

\( ^3 \)When \( D \) is big, this is consistent with (3.2.1), cf. [Nak04, Lem. III. 1.7].
Proposition 3.3.

\((3.3.1)\) \[ \lim_{\epsilon \to 0^+} N_\sigma(D + \epsilon A) = N_\sigma(D). \]

Proof. First, let us observe that for any two pseudo-effective divisors \(D_1\) and \(D_2\) we have
\[
\sigma_\Gamma(D_1 + D_2) \leq \sigma_\Gamma(D_1) + \sigma_\Gamma(D_2).
\]
Indeed, by definition, we have
\[(3.3.2) \quad \sigma_\Gamma(D_1 + D_2) = \lim_{\epsilon \to 0^+} \sigma_\Gamma(D_1 + \frac{\epsilon}{2} A + D_2 + \frac{\epsilon}{2} A).\]

By [Nak04, Chapt. III, p. 79] we have
\[\sigma_\Gamma(D_1 + \frac{\epsilon}{2} A + D_2 + \frac{\epsilon}{2} A) \leq \sigma_\Gamma(D_1 + \frac{\epsilon}{2} A) + \sigma_\Gamma(D_2 + \frac{\epsilon}{2} A).\]

Therefore, after taking the limit and using (3.3.2), we find
\[\sigma_\Gamma(D_1 + D_2) \leq \lim_{\epsilon \to 0^+} (\sigma_\Gamma(D_1 + \frac{\epsilon}{2} A) + \sigma_\Gamma(D_2 + \frac{\epsilon}{2} A)) = \sigma_\Gamma(D_1) + \sigma_\Gamma(D_2).\]

It now follows that \(N_\sigma(D_1 + D_2) \leq N_\sigma(D_1) + N_\sigma(D_2)\). In particular for every real \(\epsilon > 0\), we have: \(N_\sigma(D + \epsilon A) \leq N_\sigma(D)\).

Next, define the set \(S := \{\Gamma \text{ prime } | \sigma_\Gamma(D) \neq 0\}\), which is finite by [Nak04, Cor. III.1.11]. By definition we now have
\[N_\sigma(D) = \sum_{\Gamma \in S} \sigma_\Gamma(D) \cdot \Gamma\]
\[= \lim_{\epsilon \to 0^+} \left( \sum_{\Gamma \in S} \sigma_\Gamma(D + \epsilon A) \cdot \Gamma \right).\]

On the other hand, for every \(\epsilon > 0\), we have
\[\sum_{\Gamma \in S} \sigma_\Gamma(D + \epsilon A) \cdot \Gamma \leq \sum_{\Gamma} \sigma_\Gamma(D + \epsilon A) \cdot \Gamma \leq N_\sigma(D).
\]
Now, taking the limit \(\epsilon \to 0^+\) finishes the proof. \(\square\)

Remark 3.4. For a pseudo-effective integral divisor \(D\) on a smooth projective surface, the \(\sigma\)-decomposition of \(D\) coincides with the usual Zariski decomposition, cf. [Nak04, Rem. 1.17.(1)]. In particular, in this case \(P_\sigma\) and \(N_\sigma\) are \(\mathbb{Q}\)-divisors.

Corollary 3.5. Let \(S_{NL}\) be the smooth projective surface in the setting of Proposition 3.1.

\((3.5.1)\) Let \(\{D_{NL,m}\} \in N^1(S_{NL})_\mathbb{Q}\) be a sequence converging to \(D_{NL} \in N^1(S_{NL})_\mathbb{Q}\). Then, \(\lim_{m \to \infty} D_m \equiv D\), where \(D_m\) and \(D\) are extensions of \(D_{NL,m}\) and \(D_{NL}\) under the isomorphism in Item (3.1.2).

\((3.5.2)\) Assuming that \(D_{NL} \in N^1(S_{NL})_\mathbb{Q}\) is pseudo-effective and \(A \subset X\) is ample, let \(N_{\Delta_m}\) be the extension of \(N_\sigma(D_{NL} + \frac{m}{A} \cdot S_{NL})\). Then, we have \(\lim_{m \to \infty} N_{\Delta_m} \equiv N\), where \(N\) is the extension of \(N_\sigma(D_{NL})\).

Proof. Item (3.5.1) immediately follows from Item (3.1.2). Item (3.5.2) follows from (3.5.1) and (3.3.1). \(\square\)
3.2. From positive parts to movable cycles in $B^+_H$.

**Notation 3.6.** Let $X$ be a quasi-projective variety of dimension $n$ with a very ample divisor $H$. For any subset $W \subseteq \prod_{j=1}^{n-2} |H|$, we use the notation $S \in W$ to say that $S = T_1 \cap \ldots \cap T_{n-2}$ is a complete-intersection surface defined by some element $(T_1, \ldots, T_{n-2}) \in W$.

**Lemma 3.7.** Let $X$ be a smooth projective variety of dimension $n \geq 3$ and $H$ a very ample divisor. Let $D \in N^1(X)_\mathbb{Q}$ be a divisor class such that, for some $S \in \prod_{j=1}^{n-2} |H|$, the restriction $D|_S$ is nef. Then,

$$(3.7.1) \text{ after removing a countable number of closed subsets of } \prod_{j=1}^{n-2} |H|, \text{ there is a (Zariski dense) subset } W_D \subseteq \prod_{j=1}^{n-2} |H| \text{ such that for every } S_n \in W_D, \text{ the restriction } D|_{S_n} \text{ is nef, and }$$

$$(3.7.2) [D \cdot H^{n-2}] \in \text{Mov}_1(X)_{\mathbb{Q}}.$$ 

**Proof.** Let $i_{|H|} : X \hookrightarrow \mathbb{P}^l$ be the embedding defined by $|H|$, so that $i^*\mathcal{O}_{\mathbb{P}^l}(1) \cong \mathcal{O}_X(H)$. We set

$$\chi^{n-2} \subset \mathbb{P}^l \times \prod_{j=1}^{n-2} \mathbb{P}(H^0(\mathbb{P}^l, \mathcal{O}_{\mathbb{P}^l}(1)))$$

$$\downarrow \cong \Gamma_{n-2}$$

$$\text{to be the (universal) family of complete-intersection surfaces cut out by hyperplanes in } |\mathcal{O}_{\mathbb{P}^l}(1)|. \text{ More precisely, with } \{a_{ij}\}_{0 \leq j \leq l} \text{ being the homogenous coordinates for the } i\text{th factor of } \Gamma_{n-2} \text{ and } \{f_j\}_{0 \leq j \leq l} \text{ a basis for } H^0(\mathcal{O}_{\mathbb{P}^l}(1)), \text{ the variety } \chi^{n-2} \text{ is defined by the vanishing locus of }$$

$$\{g_i := \sum_{0 \leq j \leq l} a_{ij} f_j\}_{1 \leq i \leq n-2}.$$ 

Next, set $\chi_X^{n-2}$ to be the pullback of $\chi^{n-2}$ via the natural injection

$$X \times \prod_{j=1}^{n-2} |H| \xrightarrow{i_{|H|} \otimes \text{isom}} \mathbb{P}^l \times \Gamma_{n-2},$$

with the isomorphism arising from the one for the vector spaces $H^0(X, \mathcal{O}_X(H)) \rightarrow H^0(\mathbb{P}^l, \mathcal{O}_{\mathbb{P}^l}(1))$, as defined by $i_{|H|}$. Let

$$\xymatrix{ \chi_X^{n-2} \ar[d]^\sigma \ar[r] & \chi^{n-2} \ar[d] \ar[r] & \mathbb{P}^l \times \Gamma_{n-2} \ar[d] \ar[l] }$$

be the resulting commutative diagram. We define $\mu := \text{pr}_1 \circ \sigma$ and use $f := \text{pr}_2 \circ \sigma : \chi_X^{n-2} \rightarrow \prod_{j=1}^{n-2} |H|$ to denote the induced proper morphism, with $\text{pr}_k$ denoting the natural projection map to the $k$-th factor.

Now, let $F_0$ be the fiber of $f$ corresponding to $S$. By assumption $\mu^*D|_{F_0}$ is nef. Therefore, thanks to openness of amplitude for projective morphisms (not necessarily flat) to Noetherian schemes [KM98, Prop. 1.41], we find that $\mu^*D|_{F_i}$ is also nef, for the very general fiber $F_i$, proving Item (3.7.1).
For Item (3.7.2), let $B \in \overline{\text{NE}}^1(X)_\mathbb{Q}$ be any pseudo-effective class. With the above construction of $W_D^0$, we can find general, inductively constructed $S' \in W_D^0$ such that $B|_{S'}$ is pseudo-effective. Therefore,

$$B \cdot D \cdot H^{n-2} = B|_{S'} \cdot D|_{S'} \geq 0.$$ 

Now, since the inequality $B \cdot D \cdot H^{n-2} \geq 0$ holds for any $B \in \overline{\text{NE}}^1(X)_\mathbb{Q}$, this means that

$$[D \cdot H^{n-2}] \in (\overline{\text{NE}}^1(X)_\mathbb{Q})^*,$$

that is the cycle $[D \cdot H^{n-2}]$ is dual to the movable cone. On the other hand, by [BDPP13, Thm. 0.2] (and standard facts in convex geometry) we know that $(\overline{\text{NE}}^1(X)_\mathbb{Q})^* = \text{Mov}_1(X)$, which finishes the proof.

**Proposition 3.8.** In the setting of Proposition 3.1 let $P_{NL} \in \mathbb{N}^1(S_{NL})_\mathbb{Q}$ be a nef and big class with the extension $P \in \mathbb{N}^1(X)_\mathbb{Q}$. Then, $[P \cdot H^{n-2}] \in \text{Mov}_1(X) \cap B_H^+(X)$.

**Proof.** Since $P_{NL}$ is nef and big, we have $P^2 \cdot H^{n-2} > 0$ and $P \cdot H^{n-1} > 0$, implying that $P \in K_H^+$, i.e. $[P \cdot H^{n-2}] \in B_H^+(X)$. The rest follows from (3.7.2). \qed

4. **Miyaoka–Yau-type inequalities for non-uniruled varieties**

We now proceed to the proof of Theorem 1.2.

4.1. **The general type case.** Assume that $(X, D)$ is a log-smooth pair of log-general type. We may assume that $H$ is very ample and that the integer $m$ in Proposition 3.1 is equal to 1. Let $S_{NL}$ be a smooth complete-intersection surface as constructed in Proposition 3.1. Note that for a suitable choice of $S_{NL}$ we can ensure that $(K_X + D)|_{S_{NL}}$ is big.

Let $(K_X + D)|_{S_{NL}} = P_{NL} + N_{NL}$ be the $\sigma$-decomposition (which coincides with the Zariski decomposition by Remark 3.4). Let $P$ be the extension of $P_{NL}$ under the isomorphism $\mathbb{N}^1(X)_\mathbb{Q} \to \mathbb{N}^1(S_{NL})_\mathbb{Q}$ and define $N := (K_X + D) - P$. We note that with $P_{NL}$ being nef and big, using Proposition 3.8, we have

$$\gamma := [P \cdot H^{n-2}] \in \text{Mov}_1(X) \cap B_H^+(X).$$

Therefore, thanks to [CP19, Thm. 1.3], for every strictly adapted morphism $f : Y \to X$ as in Example 2.8, with the ramification locus given by supp([D] + A), for some very ample divisor $A$, the orbifold cotangent sheaf $\Omega^1_{(Y,f,D)}$ is semipositive with respect to $f^* \gamma$. This means that for every torsion free quotient $\mathcal{E}$ of $\Omega^1_{(Y,f,D)}$ we have $c_1(\mathcal{E}) : f^* \gamma \geq 0$.

Now, if $\Omega^1_{(Y,f,D)}$ is semistable with respect to $f^* \gamma$, then by Theorem 2.5 and Remark 2.6 we have $\Delta_B(\Omega^1_{(Y,f,D)}) \cdot H^{n-2} \geq 0$ (see Remark 3.11). Straightforward calculations, using the fact that $N^2 \cdot H^{n-2} < 0$, then show that from this inequality we can deduce $(3c_2 - c_1^2)(X, D) \cdot H^{n-2} \geq N^2 \cdot H^{n-2}$. We may thus assume that $\Omega^1_{(Y,f,D)}$ is not semistable with respect to $f^* \gamma$.

Define $G := \text{Gal}(Y/X)$. Let

$$(e_i)_{0 \leq i \leq t} \subseteq \Omega^1_{(Y,f,D)} , \quad \text{with} \quad e_0 = 0 , e_t = \Omega^1_{(Y,f,D)} \text{ and } t > 1,$$

be the increasing Harder-Narasimhan filtration with respect to $f^* \gamma = f^*(P \cdot H^{n-2})$. For $1 \leq i \leq t$, denote the torsion free, semistable quotients of this filtration by $\mathcal{E}_i := e_i/e_{i-1}$.

**Remark 2.11.** We say $\alpha \in N_1(X)_\mathbb{Q}$ is dual to $\overline{\text{NE}}^1(X)_\mathbb{Q}$, if, for every $D \in \overline{\text{NE}}^1(X)_\mathbb{Q}$ we have $D \cdot \alpha \geq 0$. 

4We say $\alpha \in N_1(X)_\mathbb{Q}$ is dual to $\overline{\text{NE}}^1(X)_\mathbb{Q}$, if, for every $D \in \overline{\text{NE}}^1(X)_\mathbb{Q}$ we have $D \cdot \alpha \geq 0$. 

4
and set \( r_i := \text{rank}(\mathcal{F}_i) \). As each \( \mathcal{F}_i \) is unique, it is equipped with a natural structure of a \( G \)-sheaf, and thus so is each \( \mathcal{F}_i \).

According to Theorem 2.5, Remark 2.6 and Remark 2.11, for every \( i \), we have

(4.0.1) \[ \Delta_M(\mathcal{F}_i) \cdot H^{n-2} \geq 0. \]

For the rest of this subsection we will closely follow the arguments of [Miy87, Prop. 7.1], adapting them to our setting by using the results in Section 3.

4.1.1. Step 1: A lower bound for \((3c^2 - \hat{c}_1^2)(\mathcal{F}_i)\) in terms of \((3\hat{c}^2 - \hat{c}_1^2)(\mathcal{F}_1)\). For \( 1 \leq i \leq t \), define \( \alpha_i \in \mathbb{Q} \) by the equality

\[ r_i \alpha_i = \frac{\hat{c}_1(\mathcal{F}_i) \cdot \gamma}{P^2 \cdot H^{n-2}}. \]

Using \( P \cdot N \cdot H^{n-2} = 0 \), this implies that

(4.0.2) \[ \sum_{i=1}^{t} r_i \alpha_i = \frac{(K_X + D) \cdot \gamma}{P^2 \cdot H^{n-2}} = \frac{(P + N) \cdot P \cdot H^{n-2}}{P^2 \cdot H^{n-2}} = 1. \]

Moreover, as \( \mathcal{F}_i \) is semipositive with respect to \( f^*\gamma \), we find \( \alpha_i \geq 0 \). On the other hand, with \( (\mathcal{F}_i)_{0 \leq i \leq t} \subseteq \Omega^1_{(Y, f, D)} \) being the Harder-Narasimhan filtration, by construction we have \( \mu_{f^*\gamma}(\mathcal{F}_i) > \mu_{f^*\gamma}(\mathcal{F}_{i+1}) \), which implies that

(4.0.3) \[ \alpha_1 > \alpha_2 > \ldots > \alpha_t \geq 0. \]

Furthermore, with \( W_p^0 \) as in Lemma 3.7, we can find \( S \in W_p^0 \) such that the restriction of every \( \mathcal{F}_i|_{\tilde{S}} \) is torsion free and that \( \tilde{S} := f^{-1}S \) is smooth. Using Item (3.7.1) we can then apply the Hodge index theorem for surfaces to conclude

\[ c_1^2(\mathcal{F}_i|_{\tilde{S}}) \cdot (f^*P|_{\tilde{S}})^2 \leq (c_1(\mathcal{F}_i|_{\tilde{S}}) \cdot f^*P|_{\tilde{S}})^2. \]

By writing this latter inequality in terms of orbifold Chern classes, we get

\[ (c_1^2(\mathcal{F}_i) \cdot H^{n-2})(P^2 \cdot H^{n-2}) \leq (\hat{c}_1(\mathcal{F}_i) \cdot P \cdot H^{n-2})^2, \]

which implies that

(4.0.4) \[ -\hat{c}_1^2(\mathcal{F}_i) \cdot H^{n-2} \geq -P^2 \cdot H^{n-2}(r_i \alpha_i)^2. \]

We now consider

(4.0.5) \[ (6\hat{c}_2 - 2\hat{c}_1^2)(\mathcal{F}_i) = \sum_{i=1}^{t} 6\hat{c}_2(\mathcal{F}_i) + 6 \sum_{i<j} \hat{c}_1(\mathcal{F}_i) \cdot \hat{c}_1(\mathcal{F}_j) - 2\hat{c}_1^2(\mathcal{F}_i). \]

Using

\[ \hat{c}_1^2(\mathcal{F}_i) = \sum_{i=1}^{t} \hat{c}_1^2(\mathcal{F}_i) + 2 \sum_{i<j} \hat{c}_1(\mathcal{F}_i) \cdot \hat{c}_1(\mathcal{F}_j), \]

we can then rewrite (4.0.5) as

\[ (6\hat{c}_2 - 2\hat{c}_1^2)(\mathcal{F}_i) = \sum_{i=1}^{t} (6\hat{c}_2 - 3\hat{c}_1^2)(\mathcal{F}_i) + \hat{c}_1^2(\mathcal{F}_i) \]

\[ = 3 \sum_{i>1} (2\hat{c}_2 - \hat{c}_1^2)(\mathcal{F}_i) + (6\hat{c}_2 - 3\hat{c}_1^2)(\mathcal{F}_1) + \hat{c}_1^2(\mathcal{F}_i). \]
Consequently, using the Bogomolov inequality (4.0.1) we have
\[(6\tilde{c}_2 - 2\tilde{c}_1^2)(\mathcal{E}_i) \cdot H^{n-2} \geq \left[ -3 \sum_{i>1} \frac{1}{r_i} \tilde{c}_1^2(\mathcal{D}_i) + (6\tilde{c}_2 - 3\tilde{c}_1^2)(\mathcal{E}_1) + P^2 \right] \cdot H^{n-2} + N^2 \cdot H^{n-2}.\]

By (4.0.4) it thus follows that
\[(6\tilde{c}_2 - 2\tilde{c}_1^2)(\mathcal{E}_i) \cdot H^{n-2} \geq \left[ -3 \sum_{i>1} \frac{1}{r_i} P^2(r_i \alpha_i)^2 + (6\tilde{c}_2 - 3\tilde{c}_1^2)(\mathcal{E}_1) + P^2 \right] \cdot H^{n-2} + N^2 \cdot H^{n-2}.

That is, we have
\[(4.0.6) \quad (6\tilde{c}_2 - 2\tilde{c}_1^2)(\mathcal{E}_i) \cdot H^{n-2} \geq \left[ P^2(1 - 3 \sum_{i=1}^t r_i \alpha_i^2) + (6\tilde{c}_2 - 3\tilde{c}_1^2)(\mathcal{E}_1) \right] \cdot H^{n-2} + N^2 \cdot H^{n-2}.\]

4.1.2. Step 2: Analysis of \((3\tilde{c}_2 - \tilde{c}_1^2)(\mathcal{E}_1)\) based on \(\text{rank}(\mathcal{E}_1)\). We now study the inequality (4.0.6) depending on \(\text{rank}(\mathcal{E}_1)\).

**Claim 4.1.** If \(\text{rank}(\mathcal{E}_1) \geq 3\), then \((3\tilde{c}_2 - \tilde{c}_1^2)(\mathcal{E}_1) \cdot H^{n-2} \geq \frac{1}{2}(N^2 \cdot H^{n-2})\).

**Proof of Claim 4.1.** Using (4.0.1) for \(\mathcal{E}_1 = \mathscr{Q}_1\) and (4.0.4) for \(i = 1\), from (4.0.6) it follows that
\[(6\tilde{c}_2 - 2\tilde{c}_1^2)(\mathcal{E}_1) \cdot H^{n-2} \geq \left[ P^2(1 - 3 \sum_{i=1}^t r_i \alpha_i^2) \right] \cdot H^{n-2} + N^2 \cdot H^{n-2}.\]

On the other hand, by (4.0.3) we have \(\alpha_1 > \alpha_i\) for every \(2 \leq i \leq t\). We thus find
\[\left(1 - 3 \sum_{i=1}^t r_i \alpha_i^2\right) P^2 \cdot H^{n-2} \geq \left(1 - 3 \alpha_1 \sum_{i=1}^t r_i \alpha_i\right) P^2 \cdot H^{n-2} = (1 - 3 \alpha_1) P^2 \cdot H^{n-2} \quad \text{by (4.0.2)},\]
so that
\[(6\tilde{c}_2 - 2\tilde{c}_1^2)(\mathcal{E}_1) \geq (1 - 3 \alpha_1) P^2 \cdot H^{n-2} + N^2 \cdot H^{n-2}.\]

Using the assumption \(r_1 \geq 3\), the equality (4.0.2), and \(\alpha_1 \geq 0\), it follows that \(\alpha_1 \leq \frac{1}{3}\), i.e. \(1 - 3 \alpha_1 \geq 0\), proving the claim.

It remains to consider the case where \(\text{rank}(\mathcal{E}_1) \leq 2\). To do so, we consider the short exact sequence
\[(4.1.1) \quad 0 \rightarrow \mathcal{E}_1 \rightarrow \Omega^1_{(Y,f,D)} \rightarrow \mathscr{Q} \rightarrow 0,\]
with \(\mathscr{Q}\) being the torsion free quotient sheaf.

**Claim 4.2.** Let \(W_p^0\) be as in Lemma 3.7. We can find \(S \in W_p^0\) such that
\begin{enumerate}
\item[(4.2.1)] the pair \((S, (D + A)|_S)\) is log-smooth and thus so is \((\widehat{S}, D_S := (f^* D_S)_\text{red})\), with \(\widehat{S} := f^{-1} S\), \(D_S := D|_S\),
\item[(4.2.2)] \(\mathcal{E}_1|_S\) is locally free, and
\item[(4.2.3)] the support of \(\Omega^1_{(\widehat{S}, f,D_S)} \cap \mathscr{Q}|_S\) is a proper subset of \(\widehat{S}\), where \(\Omega^1_{(\widehat{S}, f,D_S)}\) is the orbifold cotangent sheaf associated to \(f|_S : \widehat{S} \rightarrow (S, D_S)\).
\end{enumerate}
Proof of Claim 4.2. As $(\prod_{j=1}^{n-1} |H| \setminus W_0^j)$ consists of a union of only countable number of closed subsets, by a successive application of the Lefschetz hyperplane theorem, for a general member of $|H|$, items (4.2.1) and (4.2.2) are guaranteed to hold (note that $\mathcal{E}_1$ is reflexive and thus locally free in codimension two). Same is true for Item (4.0.3) by the following observation: after removing a closed subscheme of $Y$, the surjection in (4.1.1) defines $\mathcal{Z}^*$, locally analytically, as a sum of rank one foliations (trivially integrable). Therefore, by choosing $\hat{S}$ transversal to the associated leaves, and using Nakayama’s lemma, we can ensure that $\Omega^1_{(S,f,D_s)} \cap \mathcal{Z}^*_S$ has proper support. □

Now, by Claim 4.2, the composition

$$
(4.2.4) \quad \mathcal{E}_1,\hat{S} := \mathcal{E}_1|_S \rightarrow \Omega^1_{(S,f,D_s)} \subseteq \Omega^1_S(\log \mathcal{D}_S),
$$

is generically injective, where $\alpha_N$ is naturally defined by the orbifold conormal bundle sequence. Since $\mathcal{E}_1|_S$ is torsion free, it follows that the map (4.2.4) is injective over $\hat{S}$. We now consider two cases depending on $r_1$.

Case I: $\text{rank}(\mathcal{E}_1) = 2$. Using the injection (4.2.4)

$$
\mathcal{E}_1,\hat{S} \hookrightarrow \Omega^1_{(S,f,D_s)} \subseteq \Omega^1_S(\log \mathcal{D}_S),
$$

according to [Miy84, Rk. 4.18], we either have $(3c_2 - c_1^2)(\mathcal{E}_1,\hat{S}) \geq 0$, or $\kappa(\mathcal{E}_1,\hat{S}) := \kappa(\det \mathcal{E}_1,\hat{S}) < 0$.

If $(3c_2 - c_1^2)(\mathcal{E}_1,\hat{S}) \cdot H^{n-2} \geq 0$, then by (4.0.6) we have

$$
(6c_2 - 2c_1^2) \cdot H^{n-2} \geq [P^2(1 - 3 \sum_{i > 1} r_i \alpha_i^2) - c_1^2(\mathcal{E}_1)] \cdot H^{n-2} + N^2 \cdot H^{n-2}
\geq [P^2(1 - 3 \sum_{i > 1} r_i \alpha_i^2) - P^2(r_1 \alpha_1)^2] \cdot H^{n-2} + N^2 \cdot H^{n-2} \quad \text{by (4.0.4)}
= \left( P^2(1 - 4\alpha^2 - 3 \sum_{i > 1} r_i \alpha_i^2) \right) \cdot H^{n-2} + N^2 \cdot H^{n-2} \quad \text{as } r_1=2
\geq P^2(1 - 4\alpha^2 - 3 \sum_{i > 1} r_i \alpha_i) \cdot H^{n-2} + N^2 \cdot H^{n-2} \quad \text{by (4.0.3)}
= P^2(1 - 2\alpha(1 + 2\alpha - 3\alpha)) \cdot H^{n-2} + N^2 \cdot H^{n-2} \quad \text{by (4.0.2)}
= P^2(1 - 2\alpha(1 + 2\alpha - 3\alpha)) \cdot H^{n-2} + N^2 \cdot H^{n-2}.
$$

(4.2.5)

On the other hand, using (4.0.3) we have

$$
3\alpha_2 \leq 2\alpha_2 + r_2\alpha_2 \leq 2\alpha_1 + r_2\alpha_2
\leq 1 \quad \text{by (4.0.2)},
$$

implying that $1 - 3\alpha_2 \geq 0$. Furthermore, again by (4.0.2), we have $2\alpha_1 \leq 1$, i.e. $1 - 2\alpha_1 \geq 0$.

Going back to (4.2.5) we now find

$$
(6c_2 - 2c_1^2) \cdot H^{n-2} \geq 2\alpha_1 \cdot P^2 \cdot H^{n-2} + N^2 \cdot H^{n-2}
\geq N^2 \cdot H^{n-2},
$$

establishing our desired inequality.
We now assume that \( \kappa(\mathcal{E}_1^\prime, \mathcal{S}) < 0 \). As \( S \in W_0^0 \), the restriction \( P|_S \) is nef and thus so is \( f^* P|_S \). Moreover, as \( a_1 > 0 \), we have

\[
(4.2.6) \quad e_1(\mathcal{E}_1^\prime, \mathcal{S}) \cdot f^* P|_S > 0
\]

Using Riemann-Roch we thus get \( e_1^2(\mathcal{E}_1^\prime, \mathcal{S}) \leq 0 \). Going back to (4.0.6) we get

\[
(6c_2 - 2c_1^2)(\mathcal{E}_1^\prime) \cdot H^{n-2} \geq [P^2(1 - 3 \sum_{i>1} r_i \alpha_i^2) + 3(6c_2 - 2c_1^2)(\mathcal{E}_1^\prime)] \cdot H^{n-2} + N^2 \cdot H^{n-2}
\]

by (4.0.1)

\[
\geq [P^2(1 - 3 \sum_{i>1} r_i \alpha_i^2) + 3(\frac{-1}{r_1} c_1^2)(\mathcal{E}_1^\prime)] \cdot H^{n-2} + N^2 \cdot H^{n-2}
\]

\[
\geq P^2(1 - 3 \sum_{i>1} r_i \alpha_i^2) \cdot H^{n-2} + N^2 \cdot H^{n-2}
\]

\[
\geq P^2(1 - 3 \alpha_1 \sum_{i>1} r_i \alpha_i) \cdot H^{n-2} + N^2 \cdot H^{n-2} \quad \text{by (4.0.2)}
\]

by (4.0.3)

\[
\geq P^2 \left( \frac{(1 - 3 \alpha_1(1 - 2 \alpha_1))}{1 - 3 \alpha_1 + 6 \alpha_1^2 = 6(\alpha_1 - \frac{1}{2})^2 + \frac{1}{4}} \right) \cdot H^{n-2} + N^2 \cdot H^{n-2} > 0
\]

Case II: rank(\( \mathcal{E}_1^\prime \)) = 1. Again, by using the injection (4.2.4), we have \( \mathcal{E}_1^\prime \rightarrow \mathcal{S} \rightarrow \mathcal{E}_1^\prime \). Therefore, thanks to Bogomolov-Sommese vanishing [Bog79], [Miy77], [SSS85] (see also [EV89] and [EV92] for generalizations), we have \( \kappa(\mathcal{S}, \mathcal{E}_1, \mathcal{S}) \leq 1 \). On the other hand, we have the inequality (4.2.6). With \( f^* P|_S \) being nef, using Riemann-Roch, we thus find \( e_1^2(\mathcal{E}_1, \mathcal{S}) \leq 0 \). Moreover, as rank(\( \mathcal{E}_1, \mathcal{S} \)) = 1, we have \( e_1(\mathcal{E}_1, \mathcal{S}) = 0 \). Now, going back to (4.0.6) we get

\[
(6c_2 - 2c_1^2)(\mathcal{E}_1^\prime) \cdot H^{n-2} \geq P^2(1 - 3 \sum_{i>1} r_i \alpha_i^2) \cdot H^{n-2} + N^2 \cdot H^{n-2}
\]

\[
\geq P^2(1 - 3 \alpha_1 \sum_{i>1} r_i \alpha_i) \cdot H^{n-2} + N^2 \cdot H^{n-2} \quad \text{by (4.0.3)}
\]

\[
= P^2(1 - 3 \alpha_1(1 - \alpha_1)) \cdot H^{n-2} + N^2 \cdot H^{n-2} \quad \text{by (4.0.2)}
\]

\[
= P^2 \left( 3((\alpha_1 - \frac{1}{2})^2 - \frac{1}{4}) + 1 \right) \cdot H^{n-2} + N^2 \cdot H^{n-2}
\]

\[
= P^2 \left( 3(\alpha_1 - \frac{1}{2})^2 + \frac{1}{4} \right) \cdot H^{n-2} + N^2 \cdot H^{n-2}
\]

\[
\geq N^2 \cdot H^{n-2},
\]

which finishes the proof of the log-general type case.

### 4.2. The pseudo-effective case

Assuming that \( K_X + D \) is pseudo-effective, for any very ample divisor \( A \) and \( m \in \mathbb{N} \), we consider the pair \( (X, D + \frac{1}{m}A) \). We may assume that \( H \) in the setting of Theorem 1.2 is very ample. For \( 1 \leq i \leq n-3 \), let \( H_i \in |H| \) be general members such that \( S_{n_1} \) is a complete-intersection surface as in Proposition 3.1 and that \( (K_X + D)|_{S_{n_1}} \) is pseudo-effective.
By the general type case we know that
\[(3\hat{c}_2 - \hat{c}_1^3)(X, D + \frac{1}{m}A) \cdot H^{n-2} \geq N^2_{\frac{1}{m}} \cdot H^{n-2},\]
where $N^2_{\frac{1}{m}}$ denotes the extension of $N_a((K_X + D + \frac{1}{m}A)|_{SNL})$. Using the continuity of orbifold Chern numbers [GT16, Prop. 3.11] and Item (3.5.2), it follows that
\[(3\hat{c}_2 - \hat{c}_1^3)(X, D) \cdot H^{n-2} \geq N^2 \cdot H^{n-2},\]
with $N$ being the extension of $N_a((K_X + D)|_{SNL})$.

4.3. **Concluding remarks.** As is evident from the proof of Theorem 1.2, the inequality (1.2.1) can be sharpened to
\[\left(3\hat{c}_2(X, D) - \hat{c}_1^3(X, D)\right) \cdot H^{n-2} \geq \frac{1}{2} N^2 \cdot H^{n-2}.\]
It would be interesting to know, if this can be improved further by the inequality
\[(3\hat{c}_2(X, D) - \hat{c}_1^3(X, D)) \cdot H^{n-2} \geq \frac{1}{2} (1 - \frac{3}{n}) N^2 \cdot H^{n-2}.\]
We note that (4.2.8) coincides with [Miy84, Rk. 4.18], when $\dim = 2$, and the claimed inequality in [LM97, p. 498] in higher dimensions. We refer to [RT16, Rem. 8.2] for a brief discussion of gaps in the proof of the latter inequality.

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Erwan Rousseau, Univ Brest, CNRS UMR 6205, Laboratoire de Mathematiques de Bretagne Atlantique, F-29200 Brest, France

Email address: erwan.rousseau@univ-brest.fr
URL: http://eroussea.perso.math.cnrs.fr/

Behrouz Taji, School of Mathematics and Statistics - Red Centre, The University of New South Wales, NSW 2052 Australia

Email address: b.taji@unsw.edu.au
URL: https://web.maths.unsw.edu.au/~btaji/