A new stochastic mode reduction strategy for dissipative systems

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We present a new methodology for studying non-Hamiltonian nonlinear systems based on an information theoretic extension of a renormalization group technique using a modified maximum entropy principle. We obtain a rigorous dimensionally reduced description for such systems. The neglected degrees of freedom by this reduction are replaced by a systematically defined stochastic process under a constraint on the second moment. This then forms the basis of a computationally efficient method. Numerical computations for the generalized Kuramoto-Sivashinsky equation support our method and reveal that the long-time underlying stochastic process of the fast (unresolved) modes obeys a universal distribution which does not depend on the initial conditions and which we rigorously derive by the maximum entropy principle.

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Many nonlinear time dependent problems in science and engineering are too complex to be fully resolved and hence some degrees of freedom need to be neglected. Popular examples of high dimensional problems, which can only be solved or studied by model reduction and approximation, are models for weather and climate prediction, cell biology processes, non-linear networks, or economics. These problems involve many different timescales, e.g. oceans show a characteristic behavior over years whereas atmosphere over days. More generally, any process governed by nonlinear partial differential equations (PDEs) is infinite dimensional and hence requires a reliable finite dimensional representation for numerical purposes. This rises the central question of how can one systematically and reliably reduce the complexity of such high-dimensional systems without neglecting essential information contained in the unresolved/neglected degrees of freedom.

As in the case of weather and climate modelling, timescale separation is a central feature of many dissipative processes in physical and industrial applications. Such a scale separation can be conveniently expressed in Fourier space by differentiating the so-called fast modes which are characterized with large wavenumbers and converge towards equilibrium much faster than the slow (low wavenumber) modes. As a consequence, the long-time behavior of the system is primarily contained in the latter. There are several mode reduction techniques in the literature that take advantage of such scale decomposition, including deterministic methodologies such as adiabatic elimination [1], and the classical center-manifold theory [2], which usually requires the system to be close to criticality and to equilibrium, i.e. close to an invariant solution which is expected to be part of a finite dimensional center manifold [3]. A different line of thought follow the so-called stochastic mode reduction strategies, where the aim is to convert an infinite-dimensional deterministic dynamical system (PDE) into a low-dimensional stochastic one. In the case of Hamiltonian-like systems, there exist well-known powerful techniques such as optimal prediction and the Mori-Zwanzig formalism [4] for the derivation of mode reduced (low-dimensional) stochastic equations. Both approaches make use of the existence of an invariant or canonical probability distribution given by the Hamiltonian structure. Other examples of these include [5] where a Galerkin truncated Burgers-Hopf equation is considered to obtain a Hamiltonian structure and a canonical Gibbs measure.

However, there does not seem to exist a stochastic methodology available for problems which do not have a Hamiltonian structure. It is precisely our purpose to study this open problem. To this end, we properly extend the evolutionary renormalization group (ERG) method [6], which asymptotically defines equations for slow and fast modes, towards a stochastic mode reduction by adapting the principle of maximum information entropy (PMIE) [7] appropriately extended to PDEs, which allows to extract the relevant information from the fast modes to obtain a closed equation for the slow modes only. Separation of scales (slow/fast) can be physically justified by the presence of dissipation. We observe that the ERG method provides a systematic and rigorous (in terms of error estimates) tool to separate a dissipative, nonlinear problem into fast \((w^f)\) and slow \((v^s)\) modes:

\[
\partial_t v^s = f(v^s, w^s), \quad \partial_t w^f = \frac{1}{\epsilon} g(v^s, w^f),
\]

where the parameter \(0 < \epsilon \ll 1\) measures the timescale separation. It is important to note that in the aforementioned deterministic mode reduction strategies such as invariant manifolds, one generally assumes that the model of interest decomposes as \(1\) from the beginning without further specification \(\epsilon \ll 1\). However, such a decomposition and the associated definition of \(\epsilon\) needs to be checked carefully from a practical and theoretical point of view, and in particular for systems exhibiting spatio-temporal
We have also applied formally the notation $N$ to the first and approximating boundary conditions. The ERG method consists in approximating the solutions of the gKS equation by Chow and Hwa. Using the dissipation property defined by the renormalized fast modes, i.e.,

$$v^\epsilon = P_N u^\epsilon = V(t) + \epsilon PF_N R(t/\epsilon, U),$$

$$w^\epsilon = Q_N u^\epsilon = e^{-Q_N t/\epsilon}(W(t) + \epsilon QF_N R(t/\epsilon, U)),$$

where $L$ is the system size, $\tilde{F}_{NR}$ represents the non-resonant part, and $\tau$ is the rescaled time. The slow variable $V$ of the RG equation solves the standard Galerkin approximation of (2) for $2N + 1$ modes, i.e.,

$$\partial_t V = A_* V + P_N F(V).$$

Putting things together finally leads to the renormalized solutions

$$v^\epsilon = P_N u^\epsilon = V(t) + \epsilon PF_N R(t/\epsilon, U),$$

$$w^\epsilon = Q_N u^\epsilon = e^{-Q_N t/\epsilon}(W(t) + \epsilon QF_N R(t/\epsilon, U)),$$

where $F_N R(s, U) := \int_0^t \tilde{F}_{NR} (\tau, U) \, d\tau$. Note that the fast modes $W$ (required in (4)) are still infinite dimensional. To obtain a finite-dimensional representation, we replace $w^\epsilon$ with a random process which is defined by the original (i.e., not renormalized) fast variable $w^\epsilon$ which contains more information about the dynamics. We obtain the probability distribution for $w^\epsilon$ via PMIE by maximizing the information entropy

$$S(f(w_j^\epsilon)) := -\int_\Omega f(w_j^\epsilon) \log \left( \frac{f(w_j^\epsilon)}{\nu(\omega)} \right) \, d\omega,$$

under the constraint

$$\int_\Omega f(w_j^\epsilon) \frac{d}{dt} C_N(w_j^\epsilon) \, d\omega = \delta_j(t),$$

where $\Omega$ is the space of events and $f(w_j^\epsilon)$ is the probability density of the $j$-th fast mode $w_j(t, \omega)$; $\delta_j(t)$ is a characteristic dissipation rate, which for simplicity we approximate by the $j$-th Fourier mode $\tilde{W}_j(t) := e^{-\epsilon \beta j^2/2} w_j(0)$ of the leading order term in (1) by setting $\delta_j(t) := \frac{d}{dt} C_N(\tilde{W}_j) := -\frac{1}{\beta} \tilde{W}_j^2$. The measure $\nu$ is defined by prior or background knowledge on the system, such as uncertainties associated with the model (which turns out to be a uniform distribution after applying the PMIE), and the stochastic process $W$ defined via (8) and (9) finally leads to a random process for the solution $v^\epsilon$ in (7).
The gKS equation.— We exemplify the above procedure with the gKS equation, i.e.,
\[ A := -(\partial_x^2 + \kappa \partial_t^2 + \partial_z^2), \quad F(u) := -u \partial_x u, \]
defined on the periodic domain \( \mathcal{D}_L := [-L/2, L/2] \). The ERG provides the deterministic approximation \( \tilde{A} \) via (7) and
\[
\mathcal{P}F_{\tilde{A}}(s, U) = 2\lambda \sum_{|j| \leq N} e^{i j x} \sum_{k, l, m} \frac{e^{-\rho_k^{w} s}}{\rho_k^{w} t} V_k^j W_l^j \]
\[ + i\lambda \sum_{|j| \leq N} e^{i j x} \sum_{k+l=j} \frac{e^{-\rho_k^{w} s}}{\rho_k^{w} + \rho_l^{w}} V_k W_l, \]
where \( \rho_k^{w} := -\frac{1}{\lambda} \left( (l/\alpha)^2 + i\kappa (l/\alpha)^3 + (l/\alpha)^4 \right) \) are the eigenvalues of \( \mathcal{A}_w \) with eigenvectors \( e^{i j x} \), and \( \lambda := L/2\pi \).

We choose \( C_N(w_j^f(t, \omega)) := -\frac{1}{2} (w_j^f)^2(t, \omega) \) and hence from the solutions of the equation for \( w_j^f \), we obtain
\[
\frac{d}{dt} C_N(w_j^f) = \rho_j^{w} (w_j^f)^2 + i\omega_j \sum_{|k| \leq N} v_{jk} \frac{j-k}{\alpha} w_j^f - k, \]
Maximizing now the information entropy \( \mathcal{S} \) under the constraint \( \mathcal{I} \), see [2, Chapt. 9], leads to the following probability density \( f(w_j^f) := \frac{1}{Z_j^f} \mu_j \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-(w_j^f - \mu_j)^2}{2\sigma_j^2} \right) \), where \( \mu_j := c_j^{-1} \sigma_j \sqrt{2\pi} \exp \left( -\mu_j^2 / (2\sigma_j^2) \right) \), and
\[
\sigma_j^2 := \frac{1}{\lambda \lambda_j^{w}}, \quad \lambda := \frac{\nu_j}{2\lambda_j^{w}}. \quad (10) \]

Note that if one replaces \( \mathcal{S} \) by \( \int f(w_j^f) dw_j^f \) we get the classical result \( w_j^f \sim N(0, \sigma_j^2) \). We apply the PMIE with respect to the original dynamics \( w^f \) in \( \mathcal{S} \) and hence added complete dynamical information by to the Fourier modes \( W_j \) in \( \mathcal{I} \) leading to \( W_j \sim N(\mu_j, \sigma_j^2) \). It follows from (10) and \( \mu_j \) denotes the complex conjugate of \( \mu_j \), that \( \mu_j = 0 \) for \( j > N \) and the constraint \( \mathcal{I} \) finally defines \( \delta_j(t) = \frac{d C_N(W_j)}{dt} = \rho_j^{w} e^{-2\nu_j^t} w_j^f(0) \) for the dissipation rate in (9). For random ICs, the above methodology carries over but \( w_j^f(0) \) is replaced with \( \langle w_j^f(0) \rangle \), where brackets denote average over different ICs.

From the formula for \( f(w_j^f) \) we deduce that the normalization constant, i.e. the partition function is \( Z_j^f := m_j. \) Based on \( f \), the distribution of the fast renormalized variable \( W(x, t) := \sum_{|j| > N} W_j e^{i j / \alpha (x + 2V_0 t)} \) can be determined as \( W \sim N(\mu_W, \sigma_W^2) \) where \( \mu_W := \sum_{|j| > N} e^{i j / \alpha (x + 2V_0 t)} \mu_j \), and \( \sigma_W^2 := \sum_{|j| > N} e^{i j / \alpha (x + 2V_0 t)} \sigma_j^2 \). Using \( W \sim N(\mu_W, \sigma_W^2) \) in equation (7) gives the final result of our stochastic mode reduction method. Not only this result offers a systematic way of accurately representing deterministic equations with low-dimensional stochastic ones but it also allows for efficient computations as we only need to solve the reduced model and add the noise \textit{a posteriori}.

Numerical results and physical interpretations.— We start by looking at the statistics of the fast modes by numerically solving the gKS equation for \( 2\Lambda + 1 \) Fourier modes with \( \Lambda := 2048 \), \( \kappa = 0.1 \), and using different types of random ICs of the form \( u_0(x) = a\xi(x) \), where \( \xi(x) \) corresponds to either spatial white noise \( i.e. \langle \xi(x)\xi(x') \rangle = 2\delta(x-x') \) with zero mean and unit variance or a uniform distribution \( \xi(x) \in [-1, 1] \). We choose also different values for the noise amplitude, namely \( a = 0.1, 1, 3, 6 \), and perform 2,000 noise realizations each. The spatio-temporal solution of \( u(x, t) \) rapidly evolves into a complex dynamics characterised by a chaotic behaviour (see e.g. [8]). Figures (1a, b, c) show that after some time, the distribution of the fast modes relaxes to a universal PDF which is independent of the ICs and corresponds to a Gaussian distribution \( N(0, \sigma_W^2) \). This relaxation can also be seen by computing the evolution of the entropy \( S_1 \), observing that the final state is independent of the ICs [cf. (1b)]. Our results also suggest that the variance of the fast modes has a \( k \)-dependency which is an exponen-
Fourier mode by defining: \( \delta v_k = ((v_k - v_k^\prime)/\max(v_k))^2 \), where \( v_k \) and \( v_k^\prime \) correspond to the absolute value (amplitude) of the slow modes from the full and ERG solution. Figure 3 shows the results for sufficiently long times (i.e. the entropy function has already equilibrated) and for different values of \( N \). It is interesting to note that the error distance for the linearly unstable (slow) modes, which are defined as those with positive eigenvalues \( \rho_k > 0 \) [see Fig. 3(b)] is practically not affected as we change \( N \). On the other hand, the stable modes between 1 < \( k < 3 \) largely depend on the truncation number \( N \) observing as before a rapid convergence to zero as \( N \to \Lambda/2 \). This indicates that (a) the dynamics of the unstable modes is robust and seems not to depend on the number of stable modes used for the reduced model, and (b) less than half of the stable modes, i.e., \( N < \Lambda/2 \), provides already a reliable representation of the full system solution.

Finally, we also look at how time-correlations of a single mode can be well represented by the reduced model. To this end, we compute the frequency power spectrum of the absolute value of a given mode for 0 < \( t < T \), i.e. \( s_f(k, w) = (1/T) \sum_t v_k(t) \exp(iwt) \) (See Supplemental Material at []). As before, all solutions with different \( N \) give similar results for the unstable modes, whereas the difference between both solutions for the stable slow modes grows as \( N \) is decreased (see Fig. 1 in Supplemental Material at []).

To conclude, we have outlined a new stochastic mode reduction methodology for dissipative dynamical systems of the general form (2). It was exemplified with a paradigm for nonlinear evolution and pattern formation, the gKS equation. The cornerstone of our methodology is the information entropy which combined with an ERG formalism gives a rigorous and systematic justification of fast-slow scale separation as well as of randomness in dissipative systems. We demonstrated numerically the validity of the method and its efficiency, i.e. that one only needs to solve the reduced model (which can contain as few as half of the whole number of modes) and then add the particular type of the underlying stochastic process resulting from the maximum entropy principle. We further showed that the methodology allows to uncover new physical insights. These include: a universal PDF for the fast modes that emerges independently of the ICs and a clear distinction between the modes which are relevant to describe the dynamics of the full system based on a reduced model, from those that have a faster decay. Moreover, our method uncovers an appropriate definition of entropy for dissipative non-equilibrium processes which show a universal characteristic such as a Gaussian PDF, thus providing a systematic means for quantifying the evolution of dissipative systems.

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