The effect of large decoherence on mixing time in continuous-time quantum walks on long-range interacting cycles

S Salimi and R Radgohar

Faculty of Science, Department of Physics, University of Kurdistan, Pasdaran Ave, Sanandaj, Iran
E-mail: shsalimi@uok.ac.ir and r.radgohar@uok.ac.ir

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Abstract
In this paper, we consider decoherence in continuous-time quantum walks on long-range interacting cycles (LRICs), which are the extensions of the cycle graphs. For this purpose, we use Gurvitz’s model and assume that every node is monitored by the corresponding point-contact induced by the decoherence process. Then, we focus on large rates of decoherence and calculate the probability distribution analytically and obtain the lower and upper bounds of the mixing time. Our results prove that the mixing time is proportional to the rate of decoherence and the inverse of the square of the distance parameter ($m$). This shows that the mixing time decreases with increasing range of interaction. Also, what we obtain for $m = 0$ is in agreement with Fedichkin, Solenov and Tamon’s results [47] for cycle, and we see that the mixing time of CTQWs on cycle improves with adding interacting edges.

1. Introduction
Random walks or Markov chains on graphs have broad applications in various fields of mathematics [1], computer sciences [2] and the natural sciences, such as modelling of crystals in solid state physics [3]. The quantum walk (QW) is a generalization of the classical random walk (CRW) developed by exploiting the aspects of quantum mechanics, such as superposition and interference [4, 5]. The QW is widely studied in many distinct fields, such as polymer physics, solid state physics, biological physics and quantum computation [6–8]. There are two distinct variants of QWs: the discrete-time QWs (DTQWs) [9, 10] and the continuous-time QWs (CTQWs) [11]. In the CTQW, one can directly define the walk on the position space, whereas in the DTQW, it is necessary to introduce a quantum coin operation to define the direction in which the particle has to move. Experimental implementations of both QW variants have been reported, e.g., on microwave cavities [12], ground state atoms [13], the orbital angular momentum of photons [14], waveguide arrays [15] or Rydberg atoms [16, 17] in optical lattices. Recently, quantum walks have been studied in numerous publications. For instance, the DTQWs have been considered on random environments [18], on quotient graphs [19], in phase space [20] and for single and entangled particles [21]. Also, the CTQWs have been investigated on the $n$-cube [22], on star graphs [23, 24], on quotient graphs [25], on circulant bunkbeds [26], on trees [27], on ultrametric spaces [28], on odd graphs [29], on simple one-dimensional lattices [30], via modules of Bose–Mesner and Terwilliger algebras [31], via spectral distribution associated with adjacency matrix [32] and by using the Krylov subspace-Lanczos algorithm [33].

In this paper, we focus on CTQWs on one-dimensional networks. These networks provide the better understanding of the dynamics of physical systems. For example, they have been used to describe the behaviour of metals in solid state physics [3], to explain the dynamics of atoms in optical lattices [13, 16], to demonstrate Anderson localization in the systems with energetic disorder [34] and to address various aspects of normal and anomalous diffusion [35]. The simplest structure of the one-dimensional networks generated by connecting the nearest neighbours can be modelled by physical systems with short-range interactions. Gravity and electromagnetism are the only known fundamental interactions extending to a macroscopic distance. Due to its basic importance, it has long been a tradition to search for extra long-range interactions [36]. In some of the above-mentioned experiments, e.g. the clouds of ultra-cold Rydberg atoms assembled in a chain over which an exciton migrates, the trapping of the exciton occurs at the ends of the chain;
one finds that long-range interactions have to be considered [17, 37]. Recently, it has been shown, from the view point of quantum electrodynamics, that the CTQW for all long-range interactions in a linear system decaying as $R^{-\nu}$ (where $R$ is the distance between two nodes of the network) belongs to the same universality class for $\nu > 2$, while for classical continuous-time random walk (CTRW) universality only holds for $\nu > 3$ [38]. It has been shown that the long-range interaction leads to a slowing-down of the decay of the average survival probability, which is counter intuitive since for the corresponding classical process one observes a speed-up of the decay [39]. In [40], the authors studied coherent exciton transport on a new network model, namely long-range interacting cycles. LRICs are constructed by considering all the two nodes of distance $m$ on the cycle graphs. Since all the LRICs have the same value of connectivity $k = 4$ (the number of edges which exit from every node), LRICs provide a good facility to study the influence of long-range interaction on the transport dynamics on various coupled dynamical systems, including Josephson junction arrays [41], synchronization [42], small-world networks [43] and many other self-organizing systems. All of the mentioned articles have focused on a closed quantum system without any interaction with its environment. Recently, more realistic analysis of quantum walks by using the decoherence concept has been provided on line [44], on circulant [45] and on hypercube [46]. In [47], the effect of decoherence in the CTQWs on cycles has been considered analytically. The results showed that, for small rates of decoherence, the mixing time improves linearly with decoherence, whereas for large rates of decoherence, the mixing time deteriorates linearly towards the classical limit. In [48], the present authors have studied the influence of small decoherence in the CTQWs on long-range interacting cycles (LRICs). They proved that the mixing time is inversely proportional to the decoherence rate and also independent of the distance parameter $m$. Moreover, they showed that the mixing time upper bound for CTQWs on LRIC remains close to its value in the absence of shortcut links (cycle). Now, we want to investigate the effect of large decoherence on the mixing time in the CTQWs on LRICs. For this end, we use an analytical model developed by Gurvitz [49]. In this model, every vertex is monitored by an individual point-contact induced by the decoherence process.

We calculate the probability distribution analytically and then, for large rates of decoherence, obtain the lower and upper bounds of the mixing time. Our analytical results prove that the bounds of the mixing time are proportional to the decoherence rate that is in agreement with the results of [47]. Moreover, we prove that these bounds are inversely proportional to the square of the distance parameter $m$, i.e. the mixing time decreases with increasing $m$.

This paper is organized as follows: in section 2, we briefly describe the network structure LRIC. The CTQWs on LRICs are considered in section 3. In section 4, we study the effect of decoherence in CTQWs on LRICs, then we focus on the large rates of decoherence and calculate the probability distribution in section 5. In section 6, we define the mixing time and obtain its lower and upper bounds. We conclude with a summary of results in section 7.

2. Structure of LRIC

A long-range interacting cycle (LRIC) can be generated as follows [40]: first, the network be composed of a cycle graph. Second, two nodes of distance $m$ on the cycle graph are linked by an additional bond. We continue the second step until all the two nodes of distance $m$ have been connected. A LRIC denoted by $G(N, m)$ is characterized by the network size $N$ and the long-range interaction parameter $m$. Figure 1 shows the sketches of $G(8, 3)$ and $G(10, 4)$.

3. CTQWs on LRICs

In general, every network is characterized by a graph whose bonds connect nodes with a wide distribution of mutual distances. Algebraically, every graph corresponds to a discrete Laplace operator $A$. We introduce the states $|j\rangle$ which are localized at the nodes $j$ of the graph and take the set $\{|jj\rangle\}$ to be orthonormal. We assume that transition rates $\gamma$ between all nodes are equal and $\gamma = 1$. The non-diagonal element of matrix $A (A_{ij})$ equals 1 if nodes $i$ and $j$ are connected by a bond and zero otherwise. The diagonal element $A_{ii}$ equals $-\gamma$, where $\gamma$ is the number of bonds which exit from node $i$ [11, 50]. Since the states $|j\rangle$ span the whole Hilbert space, the time evolution of a state $|j\rangle$ starting at time $0$ is determined by the system Hamiltonian $H = A |j, t\rangle = U(t)|j\rangle$, where $U(t) = \exp[-i H t]$ is the quantum mechanical time evolution operator [11, 50]. Thus, the Hamiltonian matrix $H$ of $G(N, m)$ ($m \geq 2$) can be written as

$$H_{ij} = (i|H|j) = \begin{cases} -4, & \text{if } i = j; \\ 1, & \text{if } i = j \pm 1; \\ 1, & \text{if } i = j \pm m; \\ 0, & \text{otherwise}. \end{cases}$$

The Hamiltonian acting on the state $|j\rangle$ is given by

$$H|j\rangle = -4|j\rangle + |j - 1\rangle + |j + 1\rangle + |j - m\rangle + |j + m\rangle.$$  (2)

The above equation is the discrete version of the Hamiltonian for a free particle moving on the cycle. Using the Bloch function [3, 51] in solid state physics, the time-independent Schrödinger equation can be written as

$$H|\psi_n\rangle = E_n|\psi_n\rangle.$$  (3)
The Bloch states $|\psi_n\rangle$ can be expanded as a linear combination of states $|j\rangle$:

$$|\psi_n\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i\theta_n j} |j\rangle.$$  (4)

Substituting equations (2) and (4) into equation (3), we obtain the eigenvalues of the system as

$$E_n = -4 + 2 \cos(\theta_n) + 2 \cos(m\theta_n).$$  (5)

The Bloch relation can be obtained by projecting $|\psi_n\rangle$ on the state $|j\rangle$ such that $\psi_n(j) = (j|\psi_n\rangle = e^{-i\theta_n j}/\sqrt{N}$. It follows that $\theta_n = 2n\pi/N$ with $n$ integer and $n \in [0, N)$. From the Schrödinger equation, we have

$$\frac{i}{\hbar} \frac{d}{dt} |\psi_n(t)\rangle = H |\psi_n(t)\rangle.$$  (6)

By assuming $\hbar = 1$ and $|\psi_n(0)\rangle = |0\rangle$, the solution of the above equation is $|\psi_n(t)\rangle = e^{-iHt}|0\rangle$.

The probability of finding the walker at node $j$ at time $t$ is given by

$$P_j(t) = \langle j|\psi(t)\rangle^2.$$  (7)

4. The decoherent CTQWs on LRICs

Here, we want to study the appearance of decoherence in the CTQWs and obtain the probability distribution $P(t)$. For this end, we make use of an analytical model developed by Gurvitz [49, 52]. In this mode, a ballistic point-contact is placed near each node of network that is taken as noninvasive detector. Gurvitz demonstrated that the measurement process is fully described by the Bloch-type equations applied to the whole system. These equations led to the collapse of the density matrix into the statistical mixture in the course of the measurement process.

According to the Gurvitz model, the time evolution of a density matrix for our network (LRIC) has the following form:

$$\frac{d}{dt} \rho_{j,k}(t) = \frac{1}{4} [-\rho_{j-1,k} - \rho_{j+1,k} - \rho_{j-m,k} - \rho_{j+m,k} + \rho_{j-1,k} + \rho_{j+1,k} + \rho_{j,m,k} + \rho_{j,k,m}] - \Gamma (1 - \delta_{j,k}) \rho_{j,k},$$  (8)

that density matrix $\rho(t)$ is as $\rho(t) = \langle \psi(t)|\psi(t)\rangle$ and $\Gamma$ is the decoherence parameter. Also, we observe that $P_j(t) = \rho_{j,j}(t)$.

We define the variable $S_{j,k}$ as [47]

$$S_{j,k} = i^{k-j} \rho_{j,k},$$  (9)

and by substituting it into equation (7), we obtain

$$\frac{d}{dt} S_{j,k} = \frac{1}{4} [-S_{j-1,k} + S_{j+1,k} - i^{m+1} S_{j-m,k} - i^{m+1} S_{j+m,k} - S_{j,k-1} + S_{j,k+1} + i^{m+1} S_{j,k+m} + i^{m+1} S_{j,k,m}] - \Gamma (1 - \delta_{j,k}) S_{j,k}.$$  (10)

Note that by assuming $m = 0$, we achieve the relations mentioned in [47].

5. Large decoherence

Firstly, we review the results obtained in [48] for the decoherent CTQWs on LRICs with small rate of decoherence. The mixing time upper bound for the odd values of $m$ is as

$$T_{mix}(\epsilon) < \frac{\Gamma}{\Gamma - (m/2)} \ln \left( \frac{N}{\epsilon} \right)$$

and for the even $m$s is as

$$T_{mix}(\epsilon) < \frac{\Gamma}{\Gamma - (m/2)} \ln \left( \frac{N}{\epsilon} \right).$$

Thus, the mixing time upper bound for odd $m$ is larger than the mixing time upper bound for even $m$. In addition, these relations show that the upper bound of the mixing time is inversely proportional to the decoherence rate $\Gamma$ and independent of the distance parameter $m$.

In the following, we assume that the rate decoherence $\Gamma$ is larger than the mixing time upper bound for even $m$. Firstly, as mentioned in [47], we define the diagonal sum

$$D_k = \sum_{j=0}^{N-1} S_{j,j+k(mod N)},$$  (11)

where the indices are treated as integers modulo $N$.

From equation (9), one can achieve the following form:

$$\frac{d}{dt} D_k = -\Gamma (1 - \delta_{k,0}) D_k.$$  (12)

Thus, the minor diagonal sum ($D_k$ for $k \neq 0$) decay with characteristic time of order $1/\Gamma$.

Also, by applying equation (9) for the elements on the two minor diagonals nearest to the major diagonal, we observe that these elements make limit to small values of order $1/\Gamma^2$, etc.

Now, we consider only matrix elements of the order of $1/\Gamma$ and get

$$S'_{j,j} = \frac{1}{4} [-S_{j-1,j} + S_{j+1,j} - i^{m+1} S_{j-m,j} - i^{m+1} S_{j+m,j} - S_{j-1,j} + S_{j+1,j} + i^{m+1} S_{j-m,j} + i^{m+1} S_{j+m,j}],$$  (13)

$$S'_{j,j+1} = \frac{1}{4} [S_{j+1,j+1} - S_{j+1,j}] - \Gamma S_{j,j+1},$$  (14)

$$S'_{j,j-1} = \frac{1}{4} [-S_{j-1,j-1} + S_{j,j}] - \Gamma S_{j,j-1},$$  (15)

$$S'_{j,j,m} = \frac{1}{4} [-i^{m+1} S_{j+m,j,m} + i^{m+1} S_{j-j,m} - \Gamma S_{j,j,m}].$$  (16)

For the sake of simplicity, we define

$$a_j = S_{j,j}, \quad d_j = S_{j,j+1} + S_{j+1,j},$$  (17)

$$f_j = i^{m+1} S_{j,j,m} - i^{m+1} S_{j-j,m}.$$

After some algebra, one can get

$$a'_j = \frac{1}{4} [-d_{j-1} + d_j + f_j - f_{j-m}],$$  (18)

$$d'_j = \frac{1}{4} [a_{j+1} - a_j] - \Gamma d_j,$$  (19)

$$f'_j = \frac{1}{4} [a_{j+m} - a_j] - \Gamma f_j.$$  (20)

Differentiation of the above equations gives

$$a''_j = \frac{1}{4} [-d'_{j-1} + d'_j + f'_j - f'_{j-m}],$$  (21)

$$d''_j = \frac{1}{4} [a'_{j+1} - a'_j] - \Gamma d'_j.$$  (22)
where \( \gamma_k, D_k, A_k, \) and \( F_k \) are unknown.

To obtain \( \gamma_k \), we substitute these solutions into the abovementioned equations.

Let us now assume that we have the solutions of equations (21), (22) and (23)

\[
f_j = \sum_{k=0}^{N-1} F_k e^{\frac{i2\pi}{N} k j}.
\]

where \( \gamma_k, A_k, D_k, \) and \( F_k \) are unknown.

The general solutions of equations (21), (22) and (23) are

\[
a_j = \frac{1}{N} \sum_{k=0}^{N-1} \left[ A_{k,0} e^{-\gamma_k t} + A_{k,1} e^{-\gamma_k t} + A_{k,2} e^{-\gamma_k t} + A_{k,3} e^{-\gamma_k t} \right] \omega^{jk},
\]

\[
d_j = \frac{1}{N} \sum_{k=0}^{N-1} \left[ D_{k,0} e^{-\gamma_k t} + D_{k,1} e^{-\gamma_k t} + D_{k,2} e^{-\gamma_k t} + D_{k,3} e^{-\gamma_k t} \right] \omega^{jk},
\]

\[
f_j = \frac{1}{N} \sum_{k=0}^{N-1} \left[ F_{k,0} e^{-\gamma_k t} + F_{k,1} e^{-\gamma_k t} + F_{k,2} e^{-\gamma_k t} + F_{k,3} e^{-\gamma_k t} \right] \omega^{jk},
\]

The full solution for \( S(t) \) has the form as

\[
S_{j,k}(t) = \begin{cases} a_j, & \text{if } j = k; \\ d_j/2, & \text{if } j = k \pm 1; \\ f_j/2, & \text{if } j = k \pm m; \\ 0, & \text{otherwise} \end{cases}
\]

Thus, the probability distribution \( P_j(t) \) is the same \( a_j \). At large rates of decoherence \( \Gamma \), equation (28) reduces to

\[
a_j(t) = \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \exp \left[ -\frac{1}{2\Gamma} \left( \sin^2 \left( \frac{\pi k}{N} \right) + \sin^2 \left( \frac{\pi km}{N} \right) \right) \right] \right\} \omega^{jk}.
\]

One can see that the above distribution does not converge to any stationary distribution. The reason being that the evolution of the QW, as mentioned in section 3, is given by the unitary operator \( U = e^{-i\mathcal{H}t} \). This is the fact that unitary operators preserve the norm of states, and hence the distance between the states describing the system at subsequent times does not converge to zero [9]. This implies that the probability distribution of CTQW does not converge to CRW.

6. The bounds of mixing time

The mixing time is the time it takes for the walk to approximate the uniform distribution [9, 47], i.e.

\[
T_{\text{max}} = \min \left\{ T : \sum_{j=0}^{N-1} \left| P_j(t) - \frac{1}{N} \right| \leq \epsilon \right\}.
\]

From equation (35), we obtain

\[
\sum_{j=0}^{N-1} \left| a_j(t) - \frac{1}{N} \right| = \sum_{j=0}^{N-1} \left| \frac{1}{N} \sum_{k=0}^{N-1} \exp \left[ -\frac{1}{2\Gamma} \left( \sin^2 \left( \frac{\pi k}{N} \right) + \sin^2 \left( \frac{\pi km}{N} \right) \right) \right] e^{\frac{2\pi i j t}{N}} - \frac{1}{N} \right|.
\]

that simplifies to

\[
\sum_{j=0}^{N-1} \left| a_j(t) - \frac{1}{N} \right| = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \exp \left[ -\frac{1}{2\Gamma} \left( \sin^2 \left( \frac{\pi k}{N} \right) + \sin^2 \left( \frac{\pi km}{N} \right) \right) \right] \cos \left( \frac{2\pi k j}{N} \right).
\]
**Lower bound**

To obtain a lower bound, we apply the following method.

We retain only the terms \( j = 0 \) and \( k = 1, N - 1 \).

\[
\sum_{j=0}^{N-1} a_j(t) - \frac{1}{N} \geq a_0(t) - \frac{1}{N} \leq \frac{1}{N} \sum_{k=1}^{N-1} \exp \left( - \frac{(\sin^2 \left( \frac{\pi}{N} k \right) + \sin^2 \left( \frac{\pi}{N} m \right))}{2\Gamma} t \right) \geq \frac{2}{N} \exp \left( - \frac{(\sin^2 \left( \frac{\pi}{N} j \right) + \sin^2 \left( \frac{\pi}{N} m \right))}{2\Gamma} t \right).
\]

According to the mixing time definition, we have

\[
T_{lower} = \frac{2\Gamma}{\sin^2 \left( \frac{\pi}{N} \right) + \sin^2 \left( \frac{\pi}{N} m \right)} \ln \left( \frac{2}{N\epsilon} \right),
\]

and for \( N \gg 1 \)

\[
T_{lower} \approx \frac{2\Gamma N^2}{\pi^2 (1 + m^2)} \ln \left( \frac{2}{N\epsilon} \right).
\]

Note that for \( m = 0 \),

\[
T_{lower} \approx \frac{2\Gamma N^2}{\pi^2} \ln \left( \frac{2}{N\epsilon} \right),
\]

which is the same result as in [47]. One observe the mixing time lower bound in CTQWs on cycle decreases with adding interacting links.

**Upper bound**

Now, we want to obtain the upper bound as follows:

\[
\sum_{j=0}^{N-1} a_j(t) - \frac{1}{N} = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{N-1} \exp \left( - \frac{1}{2\Gamma} \left( \sin^2 \left( \frac{\pi}{N} k \right) + \sin^2 \left( \frac{\pi}{N} m \right) \right) t \right) \leq \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{N-1} \exp \left( - \frac{1}{2\Gamma} \left( \sin^2 \left( \frac{\pi}{N} j \right) + \sin^2 \left( \frac{\pi}{N} m \right) \right) t \right)
\]

Since for \( 0 < x < \frac{\pi}{2} \), there \( \sin x > \frac{2}{\pi} \) [53], we get

\[
\sum_{j=0}^{N-1} a_j(t) - \frac{1}{N} \leq \frac{2}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{N-1} \exp \left( - \frac{1}{2\Gamma} \left( \sin^2 \left( \frac{\pi}{N} k \right) + \sin^2 \left( \frac{\pi}{N} m \right) \right) t \right) \leq \frac{2}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{\left( \frac{N}{2} \right)} \exp \left( - \frac{1}{\Gamma} \left( \frac{2k^2 + k^2 m^2}{N^2} \right) t \right)
\]

\[
\leq \frac{2}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{\left( \frac{N}{2} \right)} \exp \left( - \frac{1}{\Gamma} \left( \frac{2k + km}{N^2} \right) t \right)
\]

\[
\leq \frac{2}{N} \sum_{j=0}^{N-1} \sum_{k=1}^{\infty} \exp \left( - \frac{1}{\Gamma} \left( \frac{2k + km}{N^2} \right) t \right),
\]

where in the third inequality, we used of the relation \( k^2 \geq k \) for \( k \geq 1 \).

\[
\sum_{j=0}^{N-1} \left| a_j(t) - \frac{1}{N} \right| \leq \exp \left[ \frac{1}{\Gamma} \left( \frac{2}{N^2} + \frac{2m^2}{N^2} \right) t \right] - 1.
\]

Thus,

\[
T_{upper} = \frac{\Gamma N^2}{2(1 + m^2)} \ln \left( \frac{2 + \epsilon}{\epsilon} \right).
\]

Note that for \( m = 0 \), we get

\[
T_{upper} = \frac{\Gamma N^2}{2} \ln \left( \frac{2 + \epsilon}{\epsilon} \right),
\]

which is in agreement with the result of [47]. One can see that the mixing time upper bound for LRIC is smaller than the mixing time upper bound for cycle. Based on the above analysis, the mixing time in CTQWs on cycle, decreases with adding interacting links. Moreover, we observe that the mixing time is proportional to the square of the distance parameter \( m \) such that it decreases with increasing interaction range. Now, we want to compare the lower bound with the upper bound. Since equation (41) was obtained by assuming \( N \ll 1 \) and \( \epsilon \ll 1 \), we have \( \ln \left( \frac{2}{N\epsilon} \right) < \ln \left( \frac{2}{\epsilon} \right) \). Thus, we can write

\[
\frac{2\Gamma N^2}{\pi^2 (1 + m^2)} \ln \left( \frac{2}{N\epsilon} \right) < T_{mix} < \frac{\Gamma N^2}{2(1 + m^2)} \ln \left( \frac{2 + \epsilon}{\epsilon} \right).
\]

7. Conclusion

We studied the continuous-time quantum walk on long-range interacting cycles (LRICs) under large decoherence \( \Gamma \gg 1 \). We obtained the probability distribution analytically and found that the mixing time is bounded as

\[
\frac{2\Gamma N^2}{\pi^2 (1 + m^2)} \ln \left( \frac{2}{N\epsilon} \right) < T_{mix} < \frac{\Gamma N^2}{2(1 + m^2)} \ln \left( \frac{2 + \epsilon}{\epsilon} \right).
\]

We proved that \( T_{mix} \) is proportional to the decoherence rate \( \Gamma \) and this result accords with the conclusion obtained in [47]. In [48] it has been proved that \( T_{mix} \), for small rates of decoherence, is independent of the distance parameter \( m \), while in this paper we showed that \( T_{mix} \), for large rates of decoherence, is inversely proportional to the square of \( m \). In other words, the mixing time decreases with increasing range of interaction. Also, since LRIC is the same generalized cycle, by replacing \( m = 0 \) in all the above relations, one achieves the results of [47]. Moreover, we proved that the mixing times improve with adding interacting links.

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