Derivation algebras of toric varieties

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1 Introduction

Normal affine algebraic varieties in characteristic 0 are uniquely determined (up to isomorphism) by the Lie algebra of derivations of their coordinate ring. This was shown by Siebert [S] and, independently, by Hauser and the third author [HM]. In both papers the assumption of normality is essential. There are non-isomorphic non-normal varieties with isomorphic Lie algebras. The third author [M] treated certain non-normal varieties defined in combinatorial terms by showing that closed simplicial complexes can be reconstructed from the Lie algebra of their Stanley-Reisner ring. Here we study this problem for (in general, non-normal) toric varieties defined by simplicial affine semigroups.

We show that such toric varieties are uniquely determined by their Lie algebra if they are supposed to be Cohen-Macaulay of dimension \( \geq 2 \). The corresponding statement is false in dimension 1. For toric curves we need the stronger hypothesis that they are Gorenstein. In fact, we can reconstruct from the Lie algebra the semigroup defining the variety. Our result should be compared with a recent one of Gubeladze [Gu] saying that an affine semigroup is uniquely determined by the toric variety it defines (more precisely, by its coordinate ring as an augmented algebra).

The main tool in our proofs is a root space decomposition of the Lie algebra of derivations of a Buchsbaum semigroup ring. The set of roots is closely related to the underlying semigroup. This structural description will be used to prove two more results. We show, in the Cohen-Macaulay case, that every automorphism of the Lie algebra is induced from a unique automorphism of the variety. And we establish an infinitesimal analogue of the last statement: Every derivation of the Lie algebra is inner, i.e., the first cohomology of the Lie algebra with coefficients in the adjoint representation vanishes.

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2 The root space decomposition

Let $S$ be an affine semigroup, i.e., a finitely generated subsemigroup of some $\mathbb{N}^n$. We stress that, in this paper, semigroup always means semigroup with zero element. Denote by $G = G(S)$ the subgroup of $\mathbb{Z}^n$ generated by $S$ and by $r = \text{rk } S = \text{rk } G(S)$ its rank. Let $C_S$ be the convex polyhedral cone spanned by $S$ in $\mathbb{Q}^n$. We shall suppose throughout that $S$ is simplicial, i.e., that the convex cone $C_S$ can be spanned by $r$ elements of $S$. For an algebraically closed field $k$ of characteristic 0 let $k[\bar{S}] \subseteq k[t] = k[t_1, \ldots, t_n]$ denote the corresponding semigroup ring. We need to recall how the property of $k[\bar{S}]$ being Cohen-Macaulay or Buchsbaum can be described in terms of $S$. For this purpose, let $F_1, \ldots, F_m$ be the $(r - 1)$-dimensional faces of $C_S$. Set

$$S'_i = \{ \lambda \in G, \lambda + s \in S \text{ for some } s \in S \cap F_i \}$$

for $i = 1, \ldots, m$, and $S' = \bigcap S'_i$.

**Proposition 1.** For a simplicial affine semigroup $S$ the semigroup ring $k[S]$ is Cohen-Macaulay (resp. Buchsbaum) if and only if $S' = S$ (resp. $S' + (S \setminus \{0\}) \subseteq S$).

For the proof see [GSW], [St, Theorem 6.4], [TH, section 4], and [SS, section 6].

The semigroup $S'$ is called the Cohen-Macaulayfication of $S$. Let

$$\bar{S} = \{ s \in G, ms \in S \text{ for some } m \in \mathbb{N}, m \neq 0 \}.$$ 

It is known [H, section 1] that $k[\bar{S}]$ is the normalization of $k[S]$. An affine semigroup $S$ is called standard if

(i) $\bar{S} = G(S) \cap \mathbb{N}^n$.

(ii) For all $i$ the image of $S$ under the the projection $\pi_i$ on the $i$-th component is a numerical semigroup, i.e., the complement $\mathbb{N} \setminus \pi_i(S)$ is finite.

(iii) The semigroups $S \cap \ker \pi_i$, $i = 1, \ldots, n$, are distinct of rank equal to $\text{rk } S - 1$.

It was shown by Hochster [H, section 2] that every affine semigroup is isomorphic to a standard one. Hence we shall assume throughout that $S$ is standard. In that case the cone $C_S$ has exactly $n$ faces of dimension $r - 1$, namely the convex cones spanned by the $S \cap \ker \pi_i$. Hence

$$S'_i = \{ \lambda \in \mathbb{N}^n, \lambda + s \in S \text{ for some } s \in S \text{ with } s_i = 0 \}$$

for $i = 1, \ldots, n$. A standard affine semigroup $S$ is simplicial if and only if $S$ has elements on every coordinate axis. In fact, the cone of a simplicial affine semigroup of rank $r$ has only $r$ faces of dimension $r - 1$. Standardness gives $r = n$. Then the edges of $C_S$ are the intersections of $C_S$ with the coordinate axes, see [SS, section 1]. The reversed implication is obvious. Let $a_i \in \mathbb{N}, a_i \neq 0$, be the minimal number such that $\alpha^i = (0, \ldots, 0, a_i, 0, \ldots, 0) \in S$, where the nonzero entry is at the $i$-th place.
Proposition 2. Every \( k \)-linear derivation \( D \) of \( k[S] \) extends uniquely to a derivation of the polynomial ring \( k[t] \).

Proof. As \( S \subseteq \mathbb{N}^n \) is standard and simplicial it has rank \( n \) and \( k[S] \) has dimension \( n \). Hence the rational function field \( k(t) \) is a separable finite extension of the quotient field \( k(S) \) of \( k[S] \). Therefore \( D \) extends uniquely to a derivation \( D \) of \( k(t) \). Write \( D = \sum f_i \partial_i \) with \( f_i \in k(t) \), say \( f_i = g_i/h_i \) with coprime \( g_i, h_i \in k[t] \).

With the semigroup elements \( \alpha^i \) introduced above we have

\[
\alpha_i t_i^{a_i - 1} f_i = D(t^{a_i}) \in k[S] \subseteq k[t]
\]

and \( h_i \) divides \( t_i^{a_i - 1} \). As \( \pi_i(S) \) is a numerical semigroup there is \( s \in G \) with the \( i \)-th component \( s_i = 1 \). Using simpliciality we may assume that \( s \in \mathbb{N}^n \), hence \( s \in \tilde{S} \). It was shown by Seidenberg [Se] that \( D \) maps the normalization \( k[\tilde{S}] \) of \( k[S] \) into itself. Then

\[
\sum a_j t^s f_j/t_j = D(t^s) \in k[\tilde{S}] \subseteq k[t]
\]

implies \( \prod_{j \neq i} t_j^{a_j - 1} t^s f_j/t_i \in k[t] \). Hence \( h_i \) divides \( \prod_{j \neq i} t_j^{a_j - 1} t^s/t_i \). But \( t_i \) does not divide this product since \( s_i = 1 \). Thus \( h_i \in k \) and \( f_i \in k[t] \). This means that \( D \) restricts to a derivation of \( k[t] \). \( \square \)

By Proposition 2 the Lie algebra \( \Theta(S) = \text{Der} k[S] \) of \( k \)-linear derivations of the semigroup ring may be viewed as a subalgebra of \( D = \text{Der} k[t] \). Let us first describe the latter Lie algebra. The derivations \( D_i = t_i \partial_i \) span an Abelian subalgebra \( H \). For a linear form \( \lambda \in H^* \) let

\[
D_\lambda = \{ D \in D, [h, D] = \lambda(h) \cdot D \text{ for all } h \in H \}.
\]

Then \( D \) admits a root space decomposition

\[
D = \bigoplus_{\lambda \in H^*} D_\lambda.
\]

Given the basis \( D_1, \ldots, D_n \) of \( H \) one may identify \( H^* \) with \( k^n \) by identifying the form \( \lambda \) with the vector \( (\lambda(D_1), \ldots, \lambda(D_n)) \). Then the set of \( \lambda \in H^* \) with \( D_\lambda \neq 0 \) equals

\[
\mathbb{N}^n \cup \{ \lambda \in \mathbb{Z}^n, \lambda_i = 1 \text{ for exactly one } i \text{ and } \lambda_j \geq 0 \text{ for all } j \neq i \}.
\]

In fact, for \( \lambda \in \mathbb{N}^n \) the root space \( D_\lambda \) is spanned by all \( D_{\lambda j} = t^\lambda t_j \partial_j \), \( j = 1, \ldots, n \).

In particular, \( D_0 = H \). And if \( \lambda \in \mathbb{Z}^n \) with \( \lambda_i = 1 \) and \( \lambda_j \geq 0 \) for \( j \neq i \) then \( D_\lambda \) is spanned by the single element \( D_{\lambda i} = t^\lambda t_i \partial_i \). All these statements follow from the commutator relation

\[
[D_i, D_{\lambda j}] = \lambda_i \cdot D_{\lambda j}.
\]

In order to describe the subalgebra \( \Theta(S) \) we need some more notation. Let

\[
\Lambda_i = \{ \lambda \in \mathbb{Z}^n, \lambda + s \in S \text{ for all } s \in S \text{ with } s_i \neq 0 \}, \quad i = 1, \ldots, n
\]

\[
\Lambda = \Lambda(S) = \bigcup \Lambda_i
\]

\[
\tilde{S} = \{ \lambda \in \mathbb{N}^n, \lambda + (S \setminus \{0\}) \subseteq S \}.
\]
Remarks. (i) Let \( n = 1 \). Then \( k[S] \) is always Cohen-Macaulay, and the cardinality of \( \Lambda \setminus S \) equals the Cohen-Macaulay type of \( k[S] \), see [HK]. For \( S = \mathbb{N} \) one has \( \tilde{S} = \mathbb{N} \) and \( \Lambda = \tilde{S} \cup \{-1\} \). Otherwise \( 1 \not\in S \). Then our assumption that \( \mathbb{N} \setminus S \) is finite implies \( \Lambda \subseteq \mathbb{N} \) and \( \Lambda = \tilde{S} \).

(ii) Let \( n \geq 2 \). From \( \lambda + \alpha^i \in S \) for \( \lambda \in \tilde{S} \) and two indices \( i \) one sees \( \tilde{S} \subseteq S' \). Hence \( \tilde{S} = S' \) in the Buchsbaum case and \( \tilde{S} = S \) in the Cohen-Macaulay case.

**Proposition 3.** (i) The Lie algebra \( \Theta(S) \) admits a root space decomposition

\[
\Theta(S) = \bigoplus_{\lambda \in H^*} \Theta_{\lambda},
\]

with \( \Theta_{\lambda} = \Theta(S) \cap \mathbb{D}_{\lambda} \).

(ii) Suppose that \( k[S] \) is Buchsbaum. Then the set of \( \lambda \in H^* \) with \( \Theta_{\lambda} \neq 0 \) equals \( \Lambda(S) \). If \( \lambda \in \tilde{S} \) then \( \Theta_{\lambda} \) is spanned by \( D_{\lambda_1}, \ldots, D_{\lambda_n} \). And if \( \lambda \in E_i = \Lambda_i \setminus \tilde{S} \) then \( \Theta_{\lambda_i} \) is spanned by the single element \( D_{\lambda_i} \). In particular, \( \Lambda(S) = \tilde{S} \cup \bigcup E_i \) is a disjoint union.

The elements of \( \tilde{S} \) (resp. \( E_i \)) will be called ordinary (resp. \( i \)-exceptional) roots.

**Proof.** (i) For \( D_{\lambda} = \sum b_{\lambda_i} D_{\lambda_i} \in \mathbb{D}_{\lambda} \) one has \( D_{\lambda} t^s = \sum b_{\lambda_i} s_i \cdot t^{\lambda+s} \). Hence \( \sum_{\lambda} D_{\lambda} \in \Theta(S) \) if and only if \( \lambda + s \in S \) for all \( s \in S \) and all occurring \( \lambda \) with \( \sum b_{\lambda_i} s_i \neq 0 \) if and only if \( D_{\lambda} \in \Theta(S) \) for all occurring \( \lambda \).

(ii) Consider \( \lambda \in \tilde{S} \). Then \( D_{\lambda_1}, \ldots, D_{\lambda_n} \) are defined and contained in \( \Theta(S) \). Next consider \( \lambda \in \Lambda_i \). From \( \lambda + \alpha^i \in S \) we see \( \lambda_j \geq 0 \) for all \( j \neq i \). Moreover, \( \lambda_i \in \Lambda(\pi_i(S)) \) and Remark (i) above yields \( \lambda_i \geq -1 \). Hence \( D_{\lambda_i} \) is defined and contained in \( \Theta(S) \). Conversely, if \( D_{\lambda_i} \in \Theta(S) \) then \( \lambda \in \Lambda_i \). The proof is completed by the following claim: If \( \Theta_{\lambda} \) contains a linear combination of the \( D_{\lambda_i} \) with at least two non-vanishing coefficients then \( \lambda \in \tilde{S} \). In fact, if \( \sum_i b_i D_{\lambda_i} \in \Theta(S) \) with \( b_1, b_2 \neq 0 \) then \( \lambda + \alpha^1 \) and \( \lambda + \alpha^2 \) are contained in \( S \). This gives \( \lambda \in S' \subseteq \tilde{S} \) as \( k[S] \) is Buchsbaum.

**Examples.** (i) ([MT, Remark 1.3]) Let \( S \subseteq \mathbb{N}^2 \) be generated by \((0,10),(3,7),(7,3),(8,2),(10,0)\) and let \( \lambda = (9,11) \). Then \( \lambda + (3,7) \not\in S \) but \( \lambda + s \in S \) for the remaining generators \( s \). Hence \( \lambda \in S' \setminus \tilde{S} \) and \( k[S] \) is not Buchsbaum. Moreover, \( \lambda \notin \Lambda(S) \) but \( \Theta_{\lambda} \neq 0 \). In fact, \( 7D_{\lambda_1} - 3D_{\lambda_2} \in \Theta_{\lambda} \).

(ii) Let \( S \subseteq \mathbb{N}^2 \) correspond to the affine cone over the \( d \)-uple embedding of \( \mathbb{P}^1 \) in \( \mathbb{P}^d \), \( d \geq 2 \), i. e., \( S \) is generated by \((0,d),(1,d-1),\ldots,(d-1,1),(d,0)\). Then \( k[S] \) is normal and Cohen-Macaulay. The exceptional roots are \((-1,1) + m(0,d)\) and \((1,-1) + m(d,0)\) with \( m \in \mathbb{N} \).

(iii) Let \( S \subseteq \mathbb{N}^2 \) correspond to the product of a cusp with a line, i. e., \( S \) is generated by \((2,0),(3,0)\) and \((0,1)\). Then \( k[S] \) is Cohen-Macaulay. The 1-exceptional roots are \((1,0) + m(0,1)\) with \( m \in \mathbb{N} \). The 2-exceptional roots are \((0,-1) + m(2,0)\) and \((3,-1) + m(2,0)\) with \( m \in \mathbb{N} \).
Examples (ii) and (iii) illustrate the second part of the next result.

**Proposition 4.** (i) $\tilde{S}$ is a finitely generated subsemigroup of $\mathbb{N}^n$.

(ii) Suppose that $k[S]$ is Buchsbaum and $n \geq 2$. For fixed $i$ let $A_i$ be the semigroup generated by all $\alpha_j$ with $j \neq i$. Then the set $E_i$ of $i$-exceptional roots is a finitely generated $A_i$-module.

**Proof.** (i) Clearly $\tilde{S}$ is a subsemigroup of $\mathbb{N}^n$. Let $A$ be the semigroup generated by $\alpha^1, \ldots, \alpha^n$. We show more generally that every subsemigroup $T \subseteq \mathbb{N}^n$ containing $A$ is finitely generated. Let $a_i$ be the nonzero entry of $\alpha^i$. For $\beta \in \mathbb{N}^n$ with $\beta_i < a_i$ for all $i$ let $T_\beta = (\beta + A) \cap T$. By Dickson’s Lemma each $T_\beta$ is a finitely generated $A$-module (or empty). Since $T = \bigcup T_\beta$ is a finite union, $T$ is finitely generated as an $A$-module and hence as a semigroup.

(ii) We may assume $i = 1$. If $\lambda \in E_1 = \Lambda_1 \setminus S$ then clearly $\lambda + \alpha_1^2 \in \Lambda_1$. Moreover, $\lambda + \alpha_1^i \in S$ so that $\lambda \in S'_i$ for $i \geq 2$. If $\lambda + \alpha_2^2 \in S$ then $\lambda + 2\alpha_2^2 \in S$, hence $\lambda \in S'_1$ and $\lambda \in S' = \tilde{S}$, contradiction. Thus $\lambda + \alpha_2^2 \in E_1$. This proves that $E_1$ is an $A_1$-module. It remains to show that it is finitely generated. For $\gamma \in \mathbb{N} \times \{0\} \subseteq \mathbb{N}^n$ and $\beta \in \{0\} \times \mathbb{N}^{-1} \subseteq \mathbb{N}^n$ with $\beta_i < a_i$ for all $i$ let $E_{\gamma\beta} = (\gamma + \beta + A_1) \cap E_1$. As above this is a finitely generated $A_1$-module (or empty). If $E_{\gamma\beta} \neq \emptyset$ and $\gamma' = \gamma + m\alpha_1^i$ for some $m \in \mathbb{N}$, $m \neq 0$ then $E_{\gamma'\beta} = \emptyset$. Otherwise, there is $\lambda \in A_1$ with $\gamma + \beta + \lambda, \gamma' + \beta + \lambda \in E_1$, contradicting $\gamma' + \beta + \lambda = \gamma + \beta + \lambda + m\alpha_1^1 \in S \subseteq S$. Since there are only finitely many congruence classes of $\mathbb{N}$ modulo $\alpha^1$ the Proposition is proven. \qed

### 3 Reconstruction of the semigroup

Before we explain how to reconstruct the semigroup $S$ from its Lie algebra $\Theta(S)$ we make a remark concerning the reconstruction of $S$ from its semigroup ring $k[S]$ discussed by Gubeladze [21]. Consider the augmentation $k[S] \to k$ defined by $t^s \mapsto 0$ for all $s \in S \setminus \{0\}$. Gubeladze [21, Theorem 2.1] proved that affine semigroups $S_1$ and $S_2$ are isomorphic if $k[S_1]$ and $k[S_2]$ are isomorphic as augmented algebras. Moreover [21, Lemma 2.8], if $k[S_1]$ and $k[S_2]$ are normal and isomorphic just as algebras then they are isomorphic as augmented algebras. We shall extend this result (for simplicial semigroups) to the Buchsbaum case.

Let us say that $S$ corresponds to a *product along a line* if, after permutation of coordinates, $S = \mathbb{N} \oplus M$ for some semigroup $M \subseteq \mathbb{N}^{n-1}$. We shall see that this property only depends on the algebra $k[S]$ and even on the Lie algebra $\Theta(S)$. Let $L = [\Theta(S), \Theta(S)]$ be the derived algebra.

**Proposition 5.** Suppose that $k[S]$ is Buchsbaum. Then the following are equivalent:

(a) The semigroup $S$ corresponds to a product along a line.

(b) There is $\lambda \in \Lambda(S)$ with $|\lambda| < 0$. 

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(c) \( L = \Theta(S) \).

Proof. (a) \( \Leftrightarrow \) (b) If \((-1, 0, \ldots, 0)\) is a root then \((1, 0, \ldots, 0) \in S\) and \(S = \mathbb{N} \oplus M\) with \(M = S \cap \ker \pi_1\). The converse is clear.

(b) \( \Rightarrow \) (c) Here and later we use the commutator relation

\[
[D_{\lambda t}, D_{\mu j}] = \mu_i D_{\lambda t^{\mu}, \mu} - \lambda_j D_{\lambda t^{\mu}, \mu}.
\]

It shows \( \bigoplus_{\lambda \neq 0} \Theta_{\lambda} \subseteq L \). Let \( \lambda = (-1, 0, \ldots, 0) \in \Lambda \) so that \( \mu = (1, 0, \ldots, 0) \in S \subseteq \tilde{S} \). Then \( L \) contains \( 2D_1 = [D_{\lambda 1}, D_{\mu 1}] \) and \( D_j = [D_{\lambda j}, D_{\mu j}] \) for \( j \geq 2 \). Thus \( \Theta_0 = H \subseteq L \).

(c) \( \Rightarrow \) (b) Assume that \( |\lambda| \geq 0 \) for all roots \( \lambda \). Then \( \eta^1 + \eta^2 = 0 \) for roots \( \eta^1, \eta^2 \neq 0 \) is possible only if (after permutation of coordinates) \( \eta^1 = (-1, 1, 0, \ldots, 0) \), \( \eta^2 = (1, -1, 0, \ldots, 0) \). In this case \( [D_{\eta^1,1}, D_{\eta^2,2}] = D_2 - D_1 \). Since \( \Theta_0 \) is Abelian we obtain

\[
L \subseteq \bigoplus_{\lambda \neq 0} \Theta_{\lambda} \oplus <D_n - D_1, \ldots, D_2 - D_1>.
\]

and \( \Theta_0 \not\subseteq L \). \( \square \)

Proposition 6. Suppose that \( k[S_1] \) and \( k[S_2] \) are Buchsbaum.

(i) If \( k[S_1] \) and \( k[S_2] \) are isomorphic as algebras then they are isomorphic as augmented algebras.

(ii) If \( S_1 \) and \( S_2 \) do not correspond to products along a line then every algebra isomorphism \( \phi : k[S_1] \to k[S_2] \) is augmented.

Proof. Let \( I \subseteq k[S_2] \) be a proper differential ideal, i. e., \( D(I) \subseteq I \) for every \( D \in \Theta(S_2) \). We claim that \( I \) is generated by some monomials \( t^s \), \( s \in S_2 \). In particular, \( I \) is contained in the augmentation ideal generated by all \( t^s \), \( s \in S_2 \setminus \{0\} \). Given \( f = \sum b_t t^s \in I \) fix any \( s \) with \( b_s \neq 0 \). Take any of the remaining \( \lambda \in S_2 \) with \( b_\lambda \neq 0 \) and choose \( j \) with \( \lambda_j \neq s_j \). Then \( \sum_i (\lambda_j - s_j) b_{i,s} t^{\mu} = \lambda_j f - D_j(f) \in I \) contains less monomials than \( f \) but still the monomial \( t^s \). Repeated application yields \( t^s \in I \), proving the claim.

Now assume \( S_1 = \mathbb{N}^m \oplus M \) for some \( M \subseteq \mathbb{N}^{n-m} \) which does not correspond to a product along a line. Let \( J \) be the ideal of \( k[S_1] \) generated by all \( t^\mu \), \( \mu \in M \setminus \{0\} \).

We claim that \( J \) is differential. Consider any \( \lambda \in \Lambda_i \), \( i = 1, \ldots, n \). In order to show \( D_{\lambda i}(t^\mu) = \mu_i t^{\mu, i} \in J \) we may assume \( \mu_i \neq 0 \). Then \( \lambda + \mu \in S_1 \). From \( |\mu| \geq 2 \) we conclude \( \lambda + \mu = \nu + \mu' \) with \( \nu \in \mathbb{N}^m \) and \( \mu' \in M \setminus \{0\} \). Hence \( t^{\nu, \mu'} \in J \).

Let \( \phi : k[S_1] \to k[S_2] \) be an algebra isomorphism. It induces a Lie algebra isomorphism \( \phi^\#: \Theta(S_1) \to \Theta(S_2) \) by \( D \mapsto \phi \circ D \circ \phi^{-1} \). Since \( J \) is differential its image in \( k[S_2] \) is differential and hence contained in the augmentation ideal of \( k[S_2] \). We have \( k[S_1] = k[M][t_1, \ldots, t_m] \). For \( i = 1, \ldots, m \) let \( c_i \) be the constant term of \( \phi(t_i) \). Define the \( k[M] \)-automorphism \( \psi \) of \( k[S_1] \) by \( \psi(t_i) = t_i - c_i \),
Assertion (ii) now also is clear because in that case $J$ equals the augmentation ideal.

\[ \text{Theorem 1.} \] Let $S_1, S_2$ be simplicial affine semigroups such that $k[S_1], k[S_2]$ are Buchsbaum. Suppose that the Lie algebras $\Theta(S_1), \Theta(S_2)$ are isomorphic. Then $S_1, S_2$ have the same rank and the semigroups $S_1, S_2$ are isomorphic.

Proof. If $\Theta(S_1)$ equals its derived algebra then $S_1$ and $S_2$ correspond to products along a line. By a result of Skryabin [3], Theorem 2] the semigroup rings $k[S_1], k[S_2]$ are isomorphic. Then [Gu, Theorem 2.1] and Proposition 6 imply that the semigroups $S_1, S_2$ themselves are isomorphic. Now suppose that the derived algebra is strictly smaller than $\Theta(S_1)$. Then $|\lambda| \geq 0$ for all $\lambda \in \Lambda(S_1)$. As $[\Theta_\lambda, \Theta_\mu] \subseteq \Theta_{\lambda+\mu}$ for all roots $\lambda, \mu$, the subspaces $I_d = \bigoplus_{|\lambda| \geq d} \Theta_\lambda$ are ideals of $\Theta(S_1)$ with finite dimensional quotients $\Theta(S_1)/I_d$ and $\bigcap_{d \in N} I_d = 0$. Given an isomorphism $\Theta(S_1) \simeq \Theta(S_2)$ we obtain an Abelian subalgebra $H_2$ of $\Theta(S_1)$ and another root space decomposition $\Theta(S_1) = \bigoplus_{\mu \in H_2^*} \Theta_\mu$. Every finite dimensional subspace of $\Theta(S_1)$ is mapped isomorphically onto its image in $\Theta(S_1)/I_d$ if $d$ is sufficiently large. Thus, for $d \gg 0$, $H_2$ embeds into $Q = \Theta(S_1)/I_d$. For $\mu \in H_2^*$ consider the root spaces

$$Q_\mu^\prime = \{ D \in Q, [h, D] = \mu(h) \cdot D \text{ for all } h \in H_2 \}.$$  

Their sum is direct. Since each $\Theta_\mu^\prime$ is mapped into $Q_\mu^\prime$ and the images of the $\Theta_\mu^\prime$ span $Q$ we see $Q = \bigoplus_{\mu \in H_2^*} Q_\mu^\prime$ and that each $\Theta_\mu^\prime$ is mapped onto $Q_\mu^\prime$. In particular, $Q_0^\prime = H_2$. It follows that $H_2$ equals its normalizer in $Q$ and hence is a Cartan subalgebra of $Q$. Using Proposition 3, Remark (i) preceding it, and Proposition 4 we may assume that the subsemigroup of $H_2^*$ generated by all $\mu$ with $\dim Q_\mu^\prime = \dim H_2 = \text{rk} \ S_2$ equals $\tilde{S}_2$. Analogous statements hold true for $H_1$ and $d \gg 0$. Since $Q$ is finite dimensional there is an automorphism of $Q$ mapping the Cartan subalgebra $H_1$ onto the second Cartan subalgebra $H_2$, [Hu, section 16]. Its dual induces an isomorphism between the semigroups $S_1$ and $S_2$.  \[ \square \]

Using Remark (ii) preceding Proposition 3 we conclude

\[ \text{Corollary 1.} \] Simplicial affine semigroups $S$ of rank $\geq 2$ with $k[S]$ Cohen-Macaulay are uniquely determined by their Lie algebra $\Theta(S)$.

Look again at Gubeladze’s Theorem that $S$ is uniquely determined by the augmented algebra $k[S]$. In the above proof we applied this only in case $S$ does correspond to a product along a line. Therefore, using the Lie algebra $\Theta(S)$ as an intermediate step, we have reproven Gubeladze’s Theorem in the special case that $S$ is simplicial, does not correspond to a product along a line, and $k[S]$ is Cohen-Macaulay of dimension $\geq 2$. But $\Theta(S)$ cannot distinguish between semigroups with the same Cohen-Macaulayfication:
Examples. (i) Fix $d, l \in \mathbb{N}$, both $\geq 2$. Let $S$ consist of all $s \in \mathbb{N}^2$ with $|s| = md$, $m \geq l$. Then $k[S]$ is Buchsbaum and the Cohen-Macaulayfication $S'$ is generated by $(0, d), (1, d - 1), \ldots, (d - 1, 1), (d, 0)$. Both $S$ and $S'$ have the same exceptional roots, see Example (ii) after Proposition 3. Hence $\Theta(S) = \Theta(S')$, independently of $l$.

(ii) Let $S_1$ (resp. $S_2$) be generated by all $\lambda \in \mathbb{N}^2$ with $|\lambda| = 6$ except $\lambda = (3, 3)$ (resp. $\lambda = (2, 4)$). They have a Buchsbaum semigroup ring and the same Cohen-Macaulayfication generated by all $\lambda \in \mathbb{N}^2$ with $|\lambda| = 6$. In both cases the exceptional roots are $(-1, 7) + m(0, 6)$ and $(7, -1) + m(6, 0)$ with $m \in \mathbb{N}$. Hence $\Theta(S_1) = \Theta(S_2)$. But $S_1, S_2$ are not isomorphic. In fact, any isomorphism would map the set of extremal elements $\{(6, 0), (0, 6)\}$ onto itself, hence $(6, 6)$ onto $(6, 6)$. This contradicts $(6, 6) = 2(3, 3)$ in $S_2$ but $(6, 6) \neq 2s$ for all $s \in S_1$. Observe that both semigroups correspond to affine cones over smooth projective curves in $\mathbb{P}^5$.

In the rank 1 case the situation is different. Although the semigroup ring always is Cohen-Macaulay the semigroup is, in general, not determined by the Lie algebra:

Examples. (i) The numerical semigroups generated by 2 and 3 (resp. 3, 4 and 5) have the same $\tilde{S} = \mathbb{N}$, hence the same Lie algebra. Observe that the semigroup ring is Gorenstein in the first case whereas it has Cohen-Macaulay type 2 in the second, see Remark (i) preceding Proposition 3.

(ii) The numerical semigroups generated by 3, 7 and 8 (resp. 4, 5 and 7) have the same $\tilde{S}$ generated by 3, 4 and 5, hence the same Lie algebra. Observe that the Cohen-Macaulay type is 2 in both cases.

Corollary 2. Numerical semigroups $S$ with $k[S]$ Gorenstein are uniquely determined by $\Theta(S)$ and even by the finite dimensional Lie algebra $\Theta(S)/[L, L]$.

Proof. If $L = \Theta(S)$ then $S = \mathbb{N}$. So suppose $L \neq \Theta(S)$. Then $\tilde{S}$ is the set of roots and $L = \bigoplus_{\lambda \neq 0} \Theta_\lambda$. This implies $\Theta_\lambda \cap [L, L] = 0$ for $\lambda$ in the minimal generator system of $\tilde{S}$ and $\Theta_\lambda \subseteq [L, L]$ for every $\lambda$ which can be decomposed as $\lambda = \mu + \nu$ with two different $\mu, \nu \in \tilde{S}$. We see that $\Theta(S)/[L, L]$ is finite dimensional and that we can use the intrinsically defined ideal $[L, L]$ instead of $I_d$ in the proof of Theorem 1. It remains to show that $S$ is uniquely determined by $\tilde{S}$ in the Gorenstein case. By [HK] Satz 1.9, Proposition 2.21] we know $\tilde{S} = S \cup \{c - 1\}$ with the conductor $c$ of $S$. Consider first the case $\tilde{S} = \mathbb{N}$. Then $S$ must be the semigroup $\mathbb{N} \setminus \{1\}$, generated by 2 and 3. Now let $\tilde{S} \neq \mathbb{N}$. Let $a$ be the smallest element of $S$ different from 0. As $S$ is a symmetric semigroup we see $c - 2, \ldots, c - a \in S$ but $c - a - 1 \notin S$. Thus $\tilde{S}$ has conductor $c - a$. Then $c - a \in S \setminus \{0\}$ implies $c - 1 > c - a \geq a$. Hence $a$ is the smallest element of $\tilde{S}$ different from 0. Therefore, $S = \tilde{S} \setminus \{c - 1\}$ is determined via $c - a$ and $a$ by $\tilde{S}$. □
4 Automorphisms of the Lie algebra

Every automorphism $\phi$ of $k[S]$ induces a Lie algebra automorphism

$$\phi^\#: \Theta(S) \to \Theta(S) : D \mapsto \phi \circ D \circ \phi^{-1}.$$ 

The purpose of this section is to show

**Theorem 2.** Let $S$ be a simplicial affine semigroup such that $k[S]$ is Cohen-Macaulay. For every automorphism $\Phi$ of $\Theta(S)$ there is a unique automorphism $\phi$ of $k[S]$ such that $\Phi = \phi^\#$.

**Proof.** If $\Phi = \phi^\#$ then $\Phi(f \cdot \Phi^{-1}(D)) = \phi(f) \cdot D$ for all $f \in k[S]$ and $D \in \Theta(S)$.

This shows uniqueness. Now take an arbitrary automorphism $\Phi$ of $\Theta(S)$. If $S$ corresponds to a product along a line the assertion follows from [Sk, Theorem 2].

By Proposition 5 we may assume $|\lambda| \geq 0$ for all $\lambda \in \Lambda$. The Lie algebra $\Theta(S)$ is graded by $\Theta^d = \bigoplus_{|\lambda|=d} \Theta_\lambda$. Note that $\Theta^0$ consists of the linear vector fields in $\Theta(S)$, i. e., those $\sum f_i \partial_i \in \Theta(S)$ where the $f_i$ are linear forms in the variables $t_i$.

The homogeneous component of smallest degree of $D \in \Theta(S)$, $D \neq 0$, will be called the leading form of $D$. We claim that every $h' = \Phi(h) \in \Phi(H)$, $h' \neq 0$, has leading form of degree zero. In fact, choose $\lambda \in \Lambda$ with $\lambda(h) \neq 0$ and $Y \in \Phi(\Theta_\lambda)$, $Y \neq 0$. Comparison of leading forms in $[h', Y] = \lambda(h) \cdot Y$ yields the claim. Hence the leading forms of the vector fields $Y_i = \Phi(D_i)$, $i = 1, \ldots, n$, are linear vector fields and linearly independent. We can thus find a point $p$ in affine space $k^n$ such that the tangent vectors $Y_1(p), \ldots, Y_n(p)$ are linearly independent.

Now consider the polynomial ring $k[t]$ as a subring of the ring $F = k[[t - p]]$ of formal power series centered at $p$ and $D_k[t]$ as a subalgebra of the Lie algebra Der $F$. By Proposition 7 below there are formal coordinates $s_1, \ldots, s_n$ at $p$, i. e., elements of $F$ vanishing at $p$ with $k[[s]] = F$, such that $Y_i = \partial_{s_i}$ in Der $F$. Let $x_i = \exp s_i$ for $i = 1, \ldots, n$. If $\lambda \in \mathbb{Z}^n$ then

$$Y_i(x^\lambda) = \lambda_i \cdot x^\lambda \quad \text{for} \quad i = 1, \ldots, n$$

and, up to multiplication with a constant, $x^\lambda$ is the unique element of $F$ with this property. This implies that for $\lambda \in \mathbb{Z}^n$ the root space

$$\Theta_\lambda = \{ D \in \text{Der} F, [Y_i, D] = \lambda_i \cdot D \text{ for all } i \}$$

is spanned by the $Y_\lambda i = x^\lambda Y_i$, $i = 1, \ldots, n$. We conclude that $\Phi(\Theta_\lambda) = \Theta'_\lambda$ for ordinary roots $\lambda \in \hat{S}$ and $\Phi(\Theta_\lambda) \subseteq \Theta'_\lambda$ for exceptional roots $\lambda \in \bigcup E_i$. Next we claim

$$\Phi(D_{\lambda i}) = b_{\lambda i} Y_{\lambda i} \quad \text{for all } \lambda \text{ and } i$$

with suitable constants $b_{\lambda i} \neq 0$. To prove this, note that $[D_{\lambda i}, D_{\mu j}] = -\lambda_i D_{\lambda + \mu, j}$ if $\mu_i = 0$ and thus $Y = \Phi(D_{\lambda i})$ has the following property: For all $\mu \in \hat{S}$ with $\mu_i = 0$ the image of $\text{ad} Y : \Theta'_\mu \to \Theta'_{\lambda + \mu}$ has dimension $\leq 1$. Hence it is enough to show that, up to multiplication with a constant, $Y_{\lambda i}$ is the unique element of
$\Theta'_S$ with this property. In fact, for $Y = \sum_k c_k Y^k$ the matrix of coefficients of $(Y, Y^j_\mu)_j$ with respect to the basis $(Y^\lambda_\mu,k)_k$ has determinant equal to the value at $\sum c_k \mu_k$ of the characteristic polynomial of the matrix $(\lambda^k c_k)_{j,k}$. This value does not vanish for a suitable choice of $\mu \in \hat{S}$ with $\mu_i = 0$ if $c_k \neq 0$ for some $k \neq i$, and the claim is proven.

For fixed $\lambda$ choose $\mu \in \hat{S}$ with $\mu_1 \neq \lambda_1$ and $\mu_i \neq 0$ for $i \neq 1$. Then the usual commutator relation implies $b_{\lambda i} b_{\mu 1} = b_{\lambda + \mu, 1}$ for all $i$. Hence the $b_{\lambda i}$ are independent of $i$, say $b_{\lambda i} = b_{\lambda}$. We have

$$\Phi(D_{\lambda i}) = b_{\lambda} Y_{\lambda i} \quad \text{for all } \lambda \text{ and } i.$$ 

Denote by $\Gamma$ the subgroup of $\mathbb{Z}^n$ generated by $\Lambda$. As $b_{\lambda} b_{\mu} = b_{\lambda + \mu}$ for all $\lambda, \mu \in \Lambda$ with $\lambda + \mu \in \Lambda$ the map $\lambda \mapsto b_{\lambda}$ can be extended to a homomorphism $\Gamma \to k^\ast$. The group $\Gamma$ is free of rank $n$, say generated by $\gamma_1, \ldots, \gamma_n$. There is a rational matrix $Q = (q_{ij})_{i,j}$ such that $l = Q \cdot \lambda$ if $\lambda = \sum l_i \gamma_i \in \Gamma \subseteq \mathbb{Z}^n$ and $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$.

Write $q_{ij} = r_{ij}/s$ with integers $r_{ij}, s$, choose $\zeta_i \in k$ such that $b_{\gamma_i} = \zeta_i^s$, and let $c_j = \prod_i c_i^{r_{ij}}$. Then $b_{\lambda} = c_{\lambda}$ for all $\lambda \in \Gamma$. Thus, if we replace the $x_j$ by $c_j x_j$ we obtain for the new $Y_{\lambda i} = x^\lambda Y_i$ the equations

$$\Phi(D_{\lambda i}) = Y_{\lambda i} \quad \text{for all } \lambda \text{ and } i.$$ 

We have seen above that $\Theta(S)$ is spanned by all $x^\lambda Y_i$ with $\lambda \in \Lambda_i$ and $i = 1, \ldots, n$. Using $|\lambda| \geq 0$ for all $\lambda$ it is easy to show that each $\Lambda_i$ is an $\hat{S}$-module. Hence $\Theta(S)$ is a module over the subalgebra of $\mathcal{F}$ generated by all $x^s$, $s \in \hat{S}$. Fix $s \in \hat{S}$. From $x^st_i \partial_i = x^s D_i \in \Theta(S) \subseteq \text{Der} k[t]$ we conclude that the element $x^s t_i$ of $\mathcal{F}$ actually is contained in the subalgebra $k[t]$. Since the same is true for $x^{2s} t_i$ we obtain $x^s \in k[t]$. Even more: $x^s D_i \in \Theta(S)$ for $i = 1, \ldots, n$ shows $x^s \in k[\hat{S}]$. The $x_i$ are algebraically independent. Therefore, an algebraic relation between finitely many $t_1^s, \ldots, t_n^s$ holds if and only if the same relation holds between $x_1^s, \ldots, x_n^s$. This means that we can define an injective homomorphism $\phi : k[\hat{S}] \to k[\hat{S}]$ by $\phi(t^s) = x^s$. The equations $\Phi(t^s D_i) = x^\lambda Y_i$ established above translate into $\Phi(D) \circ \phi = \phi \circ D$ for all $D \in \Theta(S)$. Using $\Phi^{-1}$ instead of $\Phi$ we get an injective endomorphism $\psi$ of $k[\hat{S}]$ with $D \circ \psi = \psi \circ \Phi(D)$ for all $D$.

Then $D_i \circ \psi \circ \phi = \psi \circ \phi \circ D_i$ for all $i$. Using this information one shows that $\psi \circ \phi$ maps each one-dimensional subspace of $k[\hat{S}]$ spanned by some $t^s$ into itself. Hence injectivity of $\phi$ and $\psi$ implies surjectivity of both. In case $n \geq 2$ we are done because then $S = \hat{S}$ by our hypothesis on $k[S]$ and $\phi$ is an automorphism of $k[S] = k[\hat{S}]$ with $\Phi(D) = \phi \circ D \circ \phi^{-1}$ for all $D \in \Theta(S)$, i. e., $\Phi = \phi^\ast$.

Finally, consider the case $n = 1$. Then $x$ is a single element of $\mathcal{F}$ with $x^s \in k[t]$ for all $s \in \hat{S}$. Since $\hat{S}$ is a numerical semigroup $x$ must be contained in $k(t)$ and, being integral over $k[t]$, even in $k[t]$. As $\phi : t^s \mapsto x^s$ defines an automorphism of $k[\hat{S}]$ the polynomial $x$ has degree 1, say $x = a + bt$. We had $Y(x) = x$ for $Y = \Phi(t \partial_t) \in \Theta(S)$. This implies $Y = (a/b + t) \partial_t$ and $a = 0$ because $S \neq \mathbb{N}$, i. e., $-1$ is not a root. Therefore, $\phi$ restricts to an automorphism of $k[S]$ with $\Phi = \phi^\ast$. 

$\square$
It remains to show

**Proposition 7.** Let \( F = k[t_1, \ldots, t_n] \). Suppose that \( Y_1, \ldots, Y_n \in \text{Der} F \) satisfy \([Y_i, Y_j] = 0\) for all \( i, j \) and that \( Y_1(0), \ldots, Y_n(0) \) are linearly independent. Then there are formal coordinates \( s_1, \ldots, s_n \) such that \( Y_i = \partial_{s_i} \) for all \( i \).

**Proof.** Write \( Y_i = \sum_j f_{ij} \partial_{t_j} \). By hypothesis the matrix \( F = (f_{ij})_{i,j} \) is invertible over \( F \), say with inverse \( G = (g_{jk})_{j,k} \). Application of \( Y_m \) to

\[
\sum_j f_{ij} g_{jk} = \delta_k^i
\]

yields

\[
\sum_{i,j} f_{im}^j f_{ij}^l (\partial_{t_l} g_{jk}) = -\sum_{i,j} f_{im}^j (\partial_{t_l} f_{ij}^l) g_{jk}^i
\]

for all \( j \). Hence we may interchange \( i \) and \( m \) in the right hand side and, therefore, in the left hand side of (*)

After renaming the summation indices we obtain

\[
\sum_{i,j} f_{im}^j f_{ij}^l (\partial_{t_l} g_{jk} - \partial_{t_j} g_{lk}^i) = 0.
\]

Invertibility of \( F \) implies

\[
\partial_{t_l} g_{jk} = \partial_{t_j} g_{lk}^i
\]

for all \( l, j \) and \( k \). This condition is equivalent (over a field of characteristic 0) to the existence of \( s_1, \ldots, s_n \in F \) vanishing at 0 with

\[
g_{jk}^i = \partial_{t_l} s_k
\]

for all \( j \) and \( k \). These \( s_k \) form a system of coordinates because \( G \) is invertible.

And clearly \( Y_i s_k = \delta_k^i \) for all \( i, k \). \( \square \)

**Remark.** Proposition 7 is the special case \( r = n \) of a more general statement involving an arbitrary number \( r \leq n \) of vector fields. The latter usually is stated for differentiable or analytic vector fields over the fields of real or complex numbers and appears in the literature in connection with Frobenius’ Theorem. It is surely known to hold for formal power series vector fields over arbitrary fields of characteristic 0. But lacking an explicit reference we have chosen to provide the very simple proof above.
5 Derivations of the Lie algebra

In this section we show

**Theorem 3.** Let $S \subseteq \mathbb{N}^n$ be a simplicial affine semigroup such that $k[S]$ is Buchsbaum. Then every derivation $\Delta$ of $\Theta(S)$ is inner: $\Delta = \text{ad} D$ for some $D \in \Theta(S)$.

**Proof.** The cochain complex of the Lie algebra $\Theta(S)$ with coefficients in the adjoint representation has a $\mathbb{Z}^n$-grading given by the root space decomposition. By [F, Theorem 1.5.2b] it is acyclic in degrees different from zero. Hence we may assume that the given $\Delta$ has degree 0, i.e. $\Delta(\Theta_\lambda) \subseteq \Theta_\lambda$ for all $\lambda$. For each root $\lambda$ denote by $M(\lambda)$ the set of $i$ such that $D_{\lambda i} \in \Theta(S)$. Thus $M(\lambda) = \{1, \ldots, n\}$ for ordinary roots and $M(\lambda) = \{i\}$ for $i$-exceptional roots. We have

$$\Delta(D_{\lambda i}) = \sum_{m \in M(\lambda)} b_{\lambda m} D_{\lambda m} \quad \text{for} \quad i \in M(\lambda)$$

with suitable constants $b_{\lambda m} \in k$. The brackets of the generators are given by

$$[D_{\lambda i}, D_{\mu j}] = \mu_i D_{\lambda+\mu, j} - \lambda_j D_{\lambda+\mu, i}$$

Inserting (1) and (2) into the cocycle condition

$$\Delta([D_{\lambda i}, D_{\mu j}]) = [\Delta(D_{\lambda i}), D_{\mu j}] + [D_{\lambda i}, \Delta(D_{\mu j})]$$

gives

$$\sum_{m} (\mu_i \cdot b_{\lambda+\mu, j, m} - \lambda_j \cdot b_{\lambda+\mu, i, m}) D_{\lambda+\mu, m}$$

$$= \sum_{m} (\mu_i \cdot b_{\mu j m} - \lambda_j \cdot b_{\lambda m}) D_{\lambda+\mu, m}$$

$$+ (\sum_{m} \mu_m \cdot b_{\lambda m}) D_{\lambda+\mu, j} - (\sum_{m} \lambda_m \cdot b_{\mu j m}) D_{\lambda+\mu, i}.$$ 

By comparing the coefficients one obtains

$$\mu_i \cdot b_{\lambda+\mu, j, m} - \lambda_j \cdot b_{\lambda+\mu, i, m} = \mu_i \cdot b_{\mu j m} - \lambda_j \cdot b_{\lambda m} \quad \text{for} \quad m \neq i, j$$

$$\mu_i \cdot b_{\lambda+\mu, j, j} - \lambda_j \cdot b_{\lambda+\mu, i, j} = \mu_i \cdot b_{\mu j j} - \lambda_j \cdot b_{\lambda i j} + \sum_{m} \mu_m \cdot b_{\lambda m} \quad \text{for} \quad j \neq i$$

$$(\mu_i - \lambda_i) b_{\lambda+\mu, i, i} = \mu_i \cdot b_{\mu i i} - \lambda_i \cdot b_{\lambda i i} + \sum_{m} \mu_m \cdot b_{\lambda m} - \sum_{m} \lambda_m \cdot b_{\mu i m}$$

Equation (4) with $\lambda = \mu = \alpha^i$ yields

$$b_{2\alpha^i, i, j} = 0 \quad \text{for} \quad i \neq j$$

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Let us show that $b_{\lambda_{ij}} = 0$ for all $\lambda \in \tilde{S}$ and all $i, j \in M(\lambda)$ with $i \neq j$. Set $\mu = 2\alpha^i$. In case $\lambda_i = 0$ the claim follows from (5) and (6). If $\lambda_i \neq 0$ use (3) with $j = i$ and $m$ replaced by $j$ to show $b_{\lambda + \mu, i, j} = b_{\lambda_{ij}}$. Then (4) gives the claim.

Now we have

$$\Delta(D_{\lambda i}) = b_{\lambda i}D_{\lambda i} \quad \text{for} \quad i \in M(\lambda)$$

with suitable $b_{\lambda i} \in k$. Equations (4) and (5) reduce to

$$\mu_i \cdot b_{\lambda + \mu, j} = \mu_i \cdot b_{\mu j} + \mu_i \cdot b_{\lambda i} \quad \text{for} \quad j \neq i \tag{7}$$

$$(\mu_j - \lambda_j)b_{\lambda + \mu, j} = (\mu_j - \lambda_j)(b_{\lambda j} + b_{\mu j}) \tag{8}$$

For fixed $\lambda \in \tilde{S}$ the coefficients $b_{\lambda i}$ are independent of $i \in M(\lambda)$. In fact, for $j \neq i$ apply (7) and (8) where $\mu$ is any element of $\tilde{S}$ with $\mu_i \neq 0$ and $\mu_j \neq \lambda_j$. Thus we may write $b_{\lambda}$ instead of $b_{\lambda i}$.

Consider first the case $n \geq 2$. Then (7) implies $b_{\lambda + \mu} = b_{\lambda} + b_{\mu}$ for $\lambda, \mu \in \tilde{S}$. Let $c_i = b_{\alpha^i}/a_i$ where $a_i$ denotes the nonzero entry of $a^i$. Using the fact that $\tilde{S}$ is torsion modulo the semigroup generated by the $\alpha^i$ one shows $b_{\lambda} = \sum c_i \lambda_i$ for all $\lambda \in \tilde{S}$. The same is seen to hold for $\lambda \in \Lambda_i$ by applying (7) with some $\mu \in S$, $\mu_i \neq 0$. We have proven

$$\left[ \sum c_i D_i, D_{\lambda j} \right] = \sum c_i \lambda_i D_{\lambda j} = b_{\lambda}D_{\lambda j} = \Delta(D_{\lambda j})$$

for all $\lambda \in \Lambda$ and $j \in M(\lambda)$. This means $\Delta = \text{ad} D$ for $D = \sum c_i D_i$.

In the case $n = 1$ only equation (8) is available. Then $b_{5\lambda} = b_{3\lambda} + b_{2\lambda} = 2b_{2\lambda} + b_{\lambda}$ and $b_{5\lambda} = b_{4\lambda} + b_{\lambda} = b_{3\lambda} + 2b_{\lambda} = 3b_{\lambda}$, hence $b_{2\lambda} = 2b_{\lambda}$ and then $b_{m\lambda} = mb_{\lambda}$ for all $m \in \mathbb{N}, \lambda \in \tilde{S}$ with $m, \lambda > 0$. This shows that the ratio $b_{\lambda}/\lambda$ is independent of $\lambda$, say $b_{\lambda}/\lambda = c$. Hence $b_{\lambda} = c\lambda$ for all positive roots. Since the same clearly holds for $\lambda = 0$ (and $\lambda = -1$ in the special case $S = \mathbb{N}$) we have again shown that $\Delta$ is inner. □

**Remark.** In the special case $S = \mathbb{N}^n$ Theorem 2 was proven by Heinze [He, Kap. II, Satz 2.8]. More generally, for semigroups corresponding to a product along a line it follows from work of Skryabin [Sk, Theorem 3].

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