NON-EXISTENCE OF AXISYMMETRIC OPTIMAL DOMAINS WITH SMOOTH BOUNDARY FOR THE FIRST CURL EIGENVALUE

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Abstract. We say that a bounded domain \( \Omega \) is optimal for the first positive curl eigenvalue \( \mu_1(\Omega) \) if \( \mu_1(\Omega) \leq \mu_1(\Omega') \) for any domain \( \Omega' \) with the same volume. In spite of the fact that \( \mu_1(\Omega) \) is uniformly lower bounded in terms of the volume, in this paper we prove that there are no axisymmetric optimal (and even locally minimizing) domains with \( C^{2,\alpha} \) boundary that satisfies a mild technical assumption. As a particular case, this rules out the existence of \( C^{2,\alpha} \) optimal axisymmetric domains with a convex section. An analogous result holds in the case of the first negative curl eigenvalue.

1. Introduction

Given a bounded domain \( \Omega \subset \mathbb{R}^3 \), a classical result of Giga and Yoshida [11] (see also [8]) states that curl defines a self-adjoint operator on \( \Omega \) with compact resolvent whose domain \( D_\Omega \) is dense in the space

\[
K(\Omega) = \left\{ v \in L^2(\Omega) : \text{div } v = 0, \ v|_{\partial\Omega} \cdot N = 0, \ \int_{\Omega} v \cdot h \, dx = 0 \quad \text{for all } h \in H_\Omega \right\}.
\]

Here \( H_\Omega \) denotes the space of harmonic fields on \( \Omega \) that are tangent to the boundary, and \( N \) is the outward-pointing normal to the boundary (of course, \( v|_{\partial\Omega} \cdot N = 0 \) has to be understood in the sense of traces). In this work we will only consider domains which are smooth enough, e.g., with a \( C^{2,\alpha} \) boundary \( \partial\Omega := \overline{\Omega} \setminus \Omega \).

The eigenfunctions of curl are then vector fields on \( \Omega \) that satisfy

\[
(1.1) \quad \text{curl } u_k = \mu_k(\Omega) u_k \quad \text{in } \Omega,
\]

and belong to \( D_\Omega \). It is well known that there are infinitely many positive and negative eigenvalues \( \{\mu_k(\Omega)\}_{k=-\infty}^{\infty} \) of curl, which tend to \( \pm \infty \) as \( k \to \pm \infty \) and which one can label so that

\[
\cdots \leq \mu_{-3}(\Omega) \leq \mu_{-2}(\Omega) \leq \mu_{-1}(\Omega) < 0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \mu_3(\Omega) \leq \cdots
\]

We will refer to \( \mu_1(\Omega) \) and \( \mu_{-1}(\Omega) \) as the first positive eigenvalue and first negative eigenvalue of the curl operator, respectively. Notice that their multiplicities can be higher than 1. One should recall that, when considered in absolute value, the first curl eigenvalue admits a variational formulation: indeed,

\[
\min \{\mu_{-1}^2(\Omega), \mu_1^2(\Omega)\} = \inf_{v \in D_{\Omega} \setminus \{0\}} \frac{\int_{\Omega} |\text{curl } v|^2 \, dx}{\int_{\Omega} |v|^2 \, dx}.
\]
In this work we are interested in domains that minimize the first (positive or negative) curl eigenvalue among any other domain with the same volume. More precisely, we introduce the following:

**Definition 1.1.** A $C^{2,\alpha}$ bounded domain $\Omega$ is **optimal** (respectively, **locally optimal**) for the first positive curl eigenvalue if

$$\mu_1(\Omega) \leq \mu_1(\Omega')$$

for any $C^{2,\alpha}$ domain $\Omega'$ of the same volume (respectively, for any $C^{2,\alpha}$-small perturbation $\Omega'$ of $\Omega$ with the same volume). Optimal and locally optimal domains for the first negative curl eigenvalue are defined analogously: $|\mu_{-1}(\Omega)| \leq |\mu_{-1}(\Omega')|$.

The analysis of optimal domains, including questions of existence, uniqueness and regularity, is a classical subject in spectral theory. For the Dirichlet Laplacian, the Faber–Krahn inequality implies that the ball is the only optimal domain for the first eigenvalue. Even in the case of higher eigenvalues of the Dirichlet Laplacian, the situation is much less clear-cut, and in general optimal domains are only known to exist in the class of quasi-open sets \cite{2,7}; in fact, the proof that the corresponding eigenfunctions are Lipschitz continuous is very recent \cite{1}. See \cite{7} for a general account on the subject.

Even though the curl operator plays a preponderant role in different physical contexts such as fluid mechanics and electromagnetic theory, the literature about the corresponding optimal domains is surprisingly scarce. For example, using numerical computations it is easy to show that the ball is not a (locally) optimal domain. A different but somehow related optimization problem was considered in \cite{3}; there, the stress is on the Biot–Savart operator, which is an inverse of sorts for the curl operator that appears in the definition of the helicity. In this paper, the authors obtain necessary conditions for the existence of optimal domains for this problem and conjecture that there should not exist any smooth axisymmetric optimal domains. As we will see later, there are interesting similarities between this problem and the question of optimal domains for the curl operator. For related minimization problems in the context of helicity of compactly supported vector field in $\mathbb{R}^3$ see e.g. \cite{6,9}.

Our goal in this paper is to show that there are no $C^{2,\alpha}$-smooth axisymmetric optimal domains for the first positive (or negative) eigenvalue of curl operator, modulo an additional technical assumption. Of course, given the symmetries of the problem, axisymmetric domains are a particularly relevant class of sets to analyze. We stress that $\mu_1(\Omega)$ and $-\mu_{-1}(\Omega)$ are lower bounded by a constant that only depends on $|\Omega|$ as we shall prove in Appendix A but probably this bound cannot be achieved (at least within the class of smooth enough domains).

Let us start by introducing some notation. Consider cylindrical coordinates $(z, r, \varphi) \in \mathbb{R} \times (0, \infty) \times \mathbb{T}$ on $\mathbb{R}^3$, where $\mathbb{T} := \mathbb{R}/(2\pi \mathbb{Z})$. We will henceforth assume that the axis of symmetry of the domain $\Omega$ is the $z$-axis, that is, the line $\mathcal{Z} := \{r = 0\}$. Away from the axis, the domain $\Omega$ can be written as

$$\Omega \setminus \mathcal{Z} = \{(z, r, \varphi) : (z, r) \in D, \varphi \in \mathbb{T}\}.$$ 

We will refer to the planar domain $D \subset \mathbb{R} \times (0, \infty)$ as the **section** of $\Omega$. The distance from a point $x$ to the $z$-axis, which is just its coordinate $r$, will be denoted by $r(x)$. 
We denote by
\[ \delta_{\Omega} := \inf \{ r(x) : x \in \Omega \} \]
the distance from the domain \( \Omega \) to the \( z \)-axis. If \( \delta_{\Omega} = 0 \), it means that \( \Omega \) intersects the \( z \)-axis. In contrast, if \( \delta_{\Omega} > 0 \), one can write this domain in terms of its section as \( \Omega = D \times T \), and the closure of \( D \) is contained in the half-space \( \mathbb{R} \times (0, \infty) \). We will use the notation
\[ \mathcal{R}_\Omega := \{ x \in \partial \Omega : r(x) = \delta_{\Omega} \}, \quad \mathcal{R}_D := \{ (z, r) \in \partial D : r = \delta_{\Omega} \} \]
for the set of points on the boundary of the domain \( \Omega \), or of its section \( D \), that are closest to the symmetry axis.

**Theorem 1.2.** Let \( \Omega \) be an axisymmetric bounded domain with a \( C^{2,\alpha} \) boundary. If \( \Omega \) does not intersect the \( z \)-axis, let us further assume that the boundary \( \partial \Omega \) and the set of innermost boundary points \( \mathcal{R}_\Omega \) are connected. Then the domain \( \Omega \) is not locally optimal for the first positive or negative curl eigenvalue.

Note that the condition that \( \mathcal{R}_\Omega \) be connected is generic if \( \delta_{\Omega} > 0 \). In fact, it is easy to check that for any \( C^{2,\alpha} \) axisymmetric domain \( \Omega \) that does not intersect the \( z \)-axis, there is an axisymmetric domain \( \Omega' \) that is \( C^{2,\alpha} \)-close to \( \Omega \) such that \( \mathcal{R}_{\Omega'} \) consists of a single point. An immediate consequence is that there are no optimal domains in the quite natural class of axisymmetric domains whose section \( D \) is convex:

**Corollary 1.3.** There are no \( C^{2,\alpha} \)-smooth locally optimal domains for the first positive or negative curl eigenvalue that are axisymmetric with a convex section. In particular, the ball is not a locally optimal domain.

The paper is organized as follows. In Section 2 we will obtain a necessary condition for a \( C^{2,\alpha} \) bounded domain to be locally optimal, cf. Proposition 2.1. In particular, in Corollary 2.3 we provide a topological obstruction for a domain (not necessarily axisymmetric) to be locally optimal; this result complements Corollary 1.3 above. The proof of Theorem 1.2 is presented in Section 3, where we also show that \( \mu_1(\Omega) \) is simple and the corresponding eigenfield \( u_1 \) is axisymmetric if \( \Omega \) is a locally optimal axisymmetric domain with \( C^{2,\alpha} \) connected boundary (see Corollary 3.1). Finally, we include Appendix A where we prove that the first positive and negative curl eigenvalues are lower bounded by a constant that only depends on the volume of the domain; while this is reminiscent of the Faber–Krahn inequality for the Dirichlet Laplacian, it is quite different from it in the sense that the bound we obtain is not sharp and probably it cannot be achieved.

## 2. A necessary condition for optimal domains

In this section we prove that any curl eigenfield of a (locally) optimal domain associated with the first positive (or negative) eigenvalue must have constant pointwise norm on the boundary (the same constant for all the connected components). In turn this will imply that the boundary of the domain consists of tori and that the (unparametrized) integral curves of the eigenfield are geodesics with respect to the induced metric. These results are analogous to those obtained for the helicity maximization problem considered in [3, Theorem D]. We remark that the domain is not assumed to be axisymmetric in this section.
Let us first recall that the curl eigenfields are smooth \( u_k \in C^\infty(\Omega) \). Indeed, since \( u_k \) also satisfies (component-wise)
\[
\Delta u_k + \mu_k^2 u_k = 0
\]
in \( \Omega \), the result follows by elliptic regularity (in fact, they are real-analytic in \( \Omega \)). Moreover, since \( \partial \Omega \) is \( C^{2,\alpha} \), it is standard that \( u_k \) is \( C^{1,\alpha} \) up to the boundary [5]. The same results hold for harmonic fields. In particular, we conclude that \( u_k \mid_{\partial \Omega} \) and \( h \mid_{\partial \Omega} \) belong to \( C^{1,\alpha}(\partial \Omega) \). We will use this boundary regularity property in what follows without further mention.

**Proposition 2.1.** If \( \Omega \) is a \( C^{2,\alpha} \) locally optimal domain for the first positive curl eigenvalue, then any eigenfield \( u_1 \) with eigenvalue \( \mu_1(\Omega) \) satisfies that its pointwise norm on \( \partial \Omega \) is constant, i.e., \( |u_1|^2 \mid_{\partial \Omega} = c \) for some \( c > 0 \). The analogous statement holds if \( \Omega \) is a locally optimal domain for the first negative eigenvalue.

**Proof.** Let \( V \) be a smooth bounded vector field on \( \mathbb{R}^3 \) which is assumed to be divergence-free in a neighborhood of \( \overline{\Omega} \), and let \( \Phi^t \) denote its time-\( t \) flow, which is a diffeomorphism of \( \mathbb{R}^3 \). Let us now define
\[
\Omega^t := \Phi^t(\Omega), \quad v^t := \Phi^t u_1,
\]
where \( \Phi^t u_1 \) denotes the push-forward of the vector field \( u_1 \) along the diffeomorphism \( \Phi^t \) and we normalize the eigenfunction so that its \( L^2(\Omega) \) norm is \( \|u_1\| = 1 \). Obviously, \( |\Omega^t| = |\Omega| \) if \( |t| < \varepsilon_0 \), with \( \varepsilon_0 > 0 \) a small enough constant, because \( V \) is divergence-free in a neighborhood of \( \overline{\Omega} \). Also notice that \( v^t \) depends smoothly on \( t \).

Let \( N^t \) be the outward-point unit normal to the domain \( \Omega^t \) and let \( N^t_{\partial \Omega} \) denote the 1-form dual to \( N^t \) via the Euclidean metric. Since the kernel of \( N^t_{\partial \Omega} \) at a point \( x \in \partial \Omega^t \) is the tangent plane \( T_x(\partial \Omega^t) \), and the diffeomorphism \( \Phi^t \) maps the tangent plane \( T_{\Phi^t(x)}(\partial \Omega^t) \) onto \( T_x(\partial \Omega^t) \), it is immediate that \( N^t_{\partial \Omega} = F_t \Phi^t_{\dagger} N^t \) for some positive function \( F_t \) on \( \partial \Omega^t \). Then considering the coupling of \( N^t_{\partial \Omega} \) with \( v^t \), it is obvious that
\[
(2.1) \quad N^t \cdot v^t \mid_{\partial \Omega} = N^t_{\partial \Omega}(v^t \mid_{\partial \Omega}) = F_t N^t(u_1 \mid_{\partial \Omega}) \circ \Phi^{-t} = F_t(N \cdot u_1 \mid_{\partial \Omega}) \circ \Phi^{-t} = 0;
\]
where we have used that \( \Phi^t_{\dagger} N^t u_1 \mid_{\partial \Omega} = N^t(u_1 \mid_{\partial \Omega}) \circ \Phi^{-1} \) and \( N \cdot u_1 \mid_{\partial \Omega} = 0 \). Moreover, the derivative of \( v^t \) with respect to \( t \) is given by the Lie derivative
\[
\partial_t v^t = (v^t \cdot \nabla) V - (V \cdot \nabla) v^t,
\]
which we can rewrite (if \( |t| < \varepsilon_0 \)) using vector calculus identities and the fact that \( \text{div} V = 0 \) in a neighborhood of \( \partial \Omega \) as
\[
(2.2) \quad \partial_t v^t = \text{curl}(V \times v^t) - (\text{div} v^t) V.
\]
We can take the divergence in this equation to find a linear equation for \( \text{div} v^t \):
\[
\partial_t \text{div} v^t = -(V \cdot \nabla) \text{div} v^t.
\]
Since \( \text{div} v^t = 0 \) at \( t = 0 \), we infer that
\[
(2.3) \quad \text{div} v^t = 0
\]
for all \( |t| < \varepsilon_0 \).
Armed with these facts, we can now prove that $v^t \in K(\Omega^t)$ for all $|t| < \varepsilon_0$, which amounts to showing that
\[
\int_{\Omega^t} v^t \cdot \nabla \psi \, dx = 0 \quad \text{and} \quad \int_{\Omega^t} v^t \cdot h \, dx = 0
\]
for all $\psi \in H^1(\Omega^t)$ and all $h \in H_{\Omega^t}$ by the Hodge decomposition theorem. Let us start with the first integral, where, by a density argument, one can safely assume that $\psi \in C^1(\Omega^t)$. As $\text{div} \, v^t = 0$ by (2.3), we immediately obtain that
\[
(2.4) \quad \int_{\Omega^t} v^t \cdot \nabla \psi \, dx = \int_{\partial \Omega^t} N^t \cdot v^t |_{\partial \Omega^t} \, \psi \, dS = 0
\]
where we have also used (2.1).

To tackle the second integral, let us denote by $\{h_j^t\}_{j=1}^{b_1}$ a basis of the space $H_{\Omega^t}$ depending smoothly on the parameter $t$. We Recall that the dimension of $H_{\Omega^t}$ is independent of $t$ and given by the first Betti number $b_1$ of the domain $\Omega$. Since $\text{curl} \, h_j^t = 0$ on $\Omega^t$ for all $t$, the derivative of $h_j^t$ with respect to $t$ also satisfies the equation
\[
(2.5) \quad \partial_t h_j^t = H_j^t + \nabla \psi_j^t
\]
of a harmonic field
\[
(2.6) \quad H_j^t = \sum_{k=1}^{b_1} c_{jk}(t) \, h_k^t \in H_{\Omega^t}
\]
and the gradient of a scalar function $\psi_j^t \in H^1(\Omega^t)$.

It then follows that the time derivative of
\[
f_j(t) := \int_{\Omega^t} v^t \cdot h_j^t \, dx
\]
is of the form
\[
f_j'(t) = \int_{\Omega^t} v^t \cdot (\partial_t h_j^t) \, dx + \int_{\Omega^t} (\partial_t v^t) \cdot h_j^t \, dx + \int_{\Omega^t} V \cdot \nabla (v^t \cdot h_j^t) \, dx,
\]
where the last term (which corresponds to the so-called material derivative) arises from the fact that the domain $\Omega^t$ moves along the flow of $V$. The first term can be readily computed using (2.4), (2.6):
\[
\int_{\Omega^t} v^t \cdot (\partial_t h_j^t) \, dx = \sum_{k=1}^{b_1} c_{jk}(t) \int_{\Omega^t} v^t \cdot h_k^t \, dx + \int_{\Omega^t} v^t \cdot \nabla \psi_j^t \, dx
\]
\[
= \sum_{k=1}^{b_1} c_{jk}(t) \, f_k(t).
\]
Here we have used that $v^t$ is orthogonal to all gradients by Equation (2.4). Using (2.2) and the fact that $\text{div} \, v^t = \text{div} \, V = 0$, the second and third terms yield
\[
I := \int_{\Omega^t} [(\partial_t v^t) \cdot h_j^t + V \cdot \nabla (v^t \cdot h_j^t)] \, dx
\]
\[
= \int_{\Omega^t} \text{curl}(V \times v^t) \cdot h_j^t \, dx + \int_{\partial \Omega^t} (V \cdot N^t) (v^t |_{\partial \Omega^t} \cdot h_j^t |_{\partial \Omega^t}) \, dS,
\]
where we know that the boundary term is well defined because

$$v^t |_{\partial \Omega} = \Phi^t_\ast (u_1 |_{\partial \Omega})$$

with $u_1 |_{\partial \Omega} \in C^{1,\alpha}(\partial \Omega)$. Integrating by parts in the first integral and using Equation (2.1) and that $h_j^t$ is curl-free, we obtain

$$\int_{\Omega^t} \text{curl}(V \times v^t) \cdot h_j^t \, dx = - \int_{\partial \Omega^t} (V \cdot N^t) (v^t |_{\partial \Omega^t} \cdot h_j^t |_{\partial \Omega^t}) \, dS,$$

which implies that $I = 0$. Hence, putting everything together, for all $1 \leq j \leq b_1$ we have

$$f_j'(t) = \sum_{k=1}^{b_1} c_{jk}(t) f_k(t).$$

Since $f_k(0) = 0$, we infer that $f_k(t) = 0$ for all $t$ and all $k$. This completes the proof that $v^t \in K(\Omega^t)$ for all $|t| < \varepsilon_0$.

Since curl defines a self-adjoint operator on $K(\Omega^t)$ with domain $D_{\Omega^t}$, let us denote by $T^t$ its inverse, which is a compact operator on $K(\Omega^t)$. Let us denote by $u_k^t$ and $\mu_k^t$ ($k \in \mathbb{Z} \setminus \{0\}$) an orthonormal basis of eigenfunctions of the curl operator on $D_{\Omega^t}$ and the corresponding eigenvalues. Expanding an arbitrary vector field $w \in K(\Omega^t)$ as

$$w = \sum_{k \in \mathbb{Z} \setminus \{0\}} w_k u_k^t,$$

with $w_k \in \mathbb{R}$, it follows that

$$\langle T^t w, w \rangle = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{w_k^2}{\mu_k^t}, \quad \|w\|^2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} w_k^2,$$

where the inner product and the norm are obviously those of $L^2(\Omega^t)$. Hence for all $w \in K(\Omega^t)$,

$$\frac{1}{\mu_{-1}^t} \leq \frac{\langle T^t w, w \rangle}{\|w\|^2} \leq \frac{1}{\mu_1^t},$$

and these inequalities are sharp because they are saturated for $w = u_1^t$ or $w = u_{-1}^t$.

It follows from the above argument that if $\Omega$ is a locally optimal domain for the first positive curl eigenvalue, then the time derivative of the function

$$R(t) := \frac{\langle T^t v^t, v^t \rangle}{\|v^t\|^2}$$

must satisfy $R'(0) = 0$. Notice that $R(0) = 1/\mu_1$. Although it is not obvious a priori, in the following computations we shall prove that $R(t)$ is smooth in $t$, which justifies to take the time derivative at $t = 0$. 
By the Hodge decomposition theorem, Equation (2.8) implies that the re is a 1-form 
\( \beta \) where 
\[ t \in K \]

1-forms one has 
\[ \frac{\partial}{\partial t}(\nabla \times u_1) = \nabla \times (\nabla \times u_1) \]

The second term is very easy to compute, since by the definition of \( t \), 
\[ \text{curl}(V \times u_1) = (V \times u_1) \cdot dx_1 \wedge dx_2 \wedge dx_3 \]

The computation of the time derivative of the numerator is more involved. We 
\[ d(\alpha^t - \Phi^t \alpha^0) = d\alpha^t - \Phi^t d\alpha^0 \]

where \( d \) respectively denote the exterior product, the differential and the 
contraction with the vector field \( W \). We have also employed that the push-forward 
commutes with the exterior derivative and the volume 3-form \( \Omega^t \) is 
invariant under the flow \( \Phi^t \) (that is, \( \Phi^t_i(dx_1 \wedge dx_2 \wedge dx_3) = dx_1 \wedge dx_2 \wedge dx_3 \) for 
\[ (t) < \varepsilon_0 \]) because \( V \) is divergence-free in a neighborhood of \( \mathcal{O} \). Furthermore, we 
have applied to \( Y := T^t v^t \) the well-known formula 
\[ i_{\text{curl} Y}(dx_1 \wedge dx_2 \wedge dx_3) = d\beta, \]

where \( \beta \) is the 1-form dual to the vector field \( Y \), and used that \( \text{curl} T^t v^t = v^t \).

We can now write the numerator of the function \( R(t) \) as 
\[ g(t) := \langle T^t v^t, v^t \rangle = \int_{\Omega^t} \alpha^t \wedge d\alpha^t. \]

By the Hodge decomposition theorem, Equation (2.8) implies that there is a 1-form 
\( \beta^t \), dual to a vector field \( h^t \in \mathcal{H}_{\Omega^t} \), and a function \( \psi^t \in L^1(\Omega^t) \) such that 
\[ \alpha^t = \Phi^t \alpha^0 + \beta^t + d\psi^t. \]

One then has 
\[ g(t) = \int_{\Omega^t} \Phi^t_i \alpha^0 \wedge d\alpha^t + \int_{\Omega^t} \beta^t \wedge d\alpha^t + + \int_{\Omega^t} d\psi^t \wedge d\alpha^t. \]

The second term is very easy to compute, since by the definition of the various 
1-forms one has 
\[ \int_{\Omega^t} \beta^t \wedge d\alpha^t = \int_{\Omega^t} h^t \cdot \text{curl}(T^t v^t) dx = \int_{\Omega^t} h^t \cdot v^t dx = 0 \]
because \( v^t \in K(\Omega^t) \), and the third term is similar: 
\[ \int_{\Omega^t} d\psi^t \wedge d\alpha^t = \int_{\Omega^t} \nabla \psi^t \cdot \text{curl}(T^t v^t) dx = \int_{\Omega^t} \nabla \psi^t \cdot v^t dx = 0. \]
Hence $g(t)$ coincides with the first summand, which can be rewritten using (2.8) as

$$
g(t) = \int_{\Omega_t} \Phi_t^* \alpha_0 \wedge d(\Phi_t^* \alpha_0)$$

$$= \int_{\Omega_t} \Phi_t^* (\alpha_0 \wedge d\alpha_0)$$

$$= \int_{\Omega} \alpha_0 \wedge d\alpha_0$$

$$= \int_{\Omega} u_1 \cdot T^0 u_1 \, dx$$

$$= \frac{1}{\mu_1}.$$  

This shows that

$$R(t) = \frac{1}{\mu_1 \|v^t\|^2},$$

and in particular, $R(t)$ is smooth in $t$. The identity (2.7) and the fact that $\|u_1\| = 1$ readily yields

$$R'(0) = \frac{1}{\mu_1} \int_{\partial\Omega} (V \cdot N) |u_1|^2 \, dS.$$  

It is standard that this integral vanishes for any divergence-free vector field $V \in C^\infty(\Omega)$ if and only if $|u_1|^2 \, |_{\partial\Omega}$ is a constant $c$, the same on each connected component of $\partial\Omega$. 

Finally, assume that $c = 0$. It is then easy to check that the vector field

$$u(x) := \begin{cases} u_1(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

is in $H^1(\mathbb{R}^3)$ and satisfies the equation \text{curl} \, u = \mu_1(\Omega)u$ in $\mathbb{R}^3$ in the sense of distributions. The Liouville theorem for Beltrami flows [10, 3] then implies that $u = 0$ in $\mathbb{R}^3$, which is a contradiction, so we conclude that $c > 0$. Analogous arguments work for the first negative curl eigenvalue, and the proposition then follows. $\square$

Remark 2.2. Although we will only consider $C^2$ domains in this article, Proposition 2.1 also holds for Lipschitz optimal domains, where all the boundary restrictions have to be understood in the sense of traces.

Corollary 2.3. If $\Omega$ is a $C^{2,\alpha}$ locally optimal domain for the first positive curl eigenvalue then each connected component of $\partial\Omega$ is diffeomorphic to $\mathbb{T}^2$ and the (unparameterized) integral curves of $u_1 \, |_{\partial\Omega}$ are geodesics with respect to the induced metric. The analogous statement holds for the first negative eigenvalue. In particular, there are no $C^{2,\alpha}$-smooth locally optimal domains that are convex.

Proof. Let $\Sigma_k$ be a connected component of $\partial\Omega$. If the Euler characteristic $\chi(\Sigma_k) \neq 0$, then $u_1|_{\Sigma_k}$ vanishes at some point by the Poincaré–Hopf index theorem. In view of Proposition 2.1 we conclude that $c = |u_1|^2 \, |_{\partial\Omega} = 0$, which is a contradiction, so we deduce that $\chi(\Sigma_k) = 0$ for all the connected components of $\partial\Omega$, which means that they are diffeomorphic to $\mathbb{T}^2$. Now, the well-known identity

$$\nabla_u u = \frac{1}{2} \nabla |u|^2 - u \times \text{curl} \, u$$
implies that $\nabla u_1 = \frac{1}{2} \nabla |u_1|^2$. Since $u_1$ is $C^{1,\alpha}$ up to the boundary and $|u_1|^2 |_{\partial \Omega} = c$, the restriction $v := u_1 |_{\partial \Omega}$ on $\partial \Omega$ satisfies $\nabla v = 0$, where $\nabla v$ is the covariant derivative along $v$ with respect to the induced metric. This is equivalent to saying that the (unparametrized) integral curves of $v$ are geodesics. The last claim in the statement follows from the fact that the boundary of any $C^{2,\alpha}$ convex domain is diffeomorphic to a sphere.

3. Proof of Theorem 1.2

Let $\Omega$ be an axisymmetric $C^{2,\alpha}$ locally optimal domain for the first positive curl eigenvalue and consider an eigenfield $u_1$ as in Proposition 2.1. Corollary 2.3 implies that all the connected components of $\partial \Omega$ are diffeomorphic to $T^2$, and hence $\delta \Omega > 0$, i.e., $\Omega$ does not intersect the $z$-axis. The domain $\Omega$ is then of the form $D \times T$ where $D \subset \mathbb{R} \times (0, \infty)$ is the section of $\Omega$. We assume in what follows that $\partial D$ (and so $\partial D$) is connected.

Assume that the eigenfield $u_1$ is axisymmetric. It can then be written in cylindrical coordinates (in terms of the orthonormal basis $\{e_z, e_r, e_\phi\}$) as

$$u_1 = \frac{1}{r} [\partial_r \psi e_z - \partial_z \psi e_r + \mu_1 \psi e_\phi],$$

where $\mu_1 \equiv \mu_1(\Omega)$ is the first positive curl eigenvalue, and the function $\psi(z, r)$ satisfies the Grad-Shafranov equation

$$L \psi = -\mu_1^2 \psi$$

in the section $D$. In particular, $\psi$ belongs to $C^{2,\alpha}(\overline{D})$. Here we have set

$$L \psi := \partial_{zz} \psi + \partial_{rr} \psi - \frac{1}{r} \partial_r \psi.$$

Since $u_1$ is tangent to $\partial \Omega$ and $|u_1|^2 |_{\partial \Omega}$ is constant, it follows that $\psi$ satisfies the following boundary conditions

$$\psi |_{\partial D} = c_1,$$

$$\frac{(\nabla \psi)^2 + \mu_1^2 \psi^2}{r^2} |_{\partial D} = c_2,$$

for some constants $c_1$ and $c_2 > 0$.

Since $\partial D$ is connected, $D$ is a simply connected planar domain, so $\Omega$ is diffeomorphic to a solid torus. The space of harmonic fields $\mathcal{H}_\Omega$ then has dimension one, and it is trivial to check that the only harmonic field (up to a constant factor) is given by

$$h = \frac{1}{r} e_\phi.$$

The fact that $u_1 \in K(\Omega)$ implies that

$$0 = \int_{\Omega} u_1 \cdot h dx = 2 \pi \mu_1 \int_D \frac{\psi(z, r)}{r}dzdr.$$

Using Equation (3.2), this yields

$$0 = \int_D \frac{L \psi}{r} dzdr = \int_D \left[ \partial_z \left( \frac{\partial_z \psi}{r} \right) + \partial_r \left( \frac{\partial_r \psi}{r} \right) \right] dzdr = \int_{\partial D} \frac{\nabla \psi \cdot N}{r} dS.$$
Here $\nabla \psi := \partial_z \psi e_z + \partial_r \psi e_r$, and $N$ is the outward-point unit normal to $\partial D$. To pass to the last equality we have simply integrated by parts.

If $c_1 = 0$ in Equation (3.4), then (3.5) and (3.7) imply that $c_2 = 0$, which means that $|u_1|^2 \big|_{\partial D} = 0$, which contradicts Proposition 2.1. Let us now consider the case $c_1 \neq 0$. The connectedness of $\partial D$ and the fact that $\nabla \psi \big|_{\partial D}$ cannot be identically zero, imply that Equation (3.7) can be fulfilled only if the zero set of $\nabla \psi$ on $\partial D$ is nonempty and consists of at least two connected components. Let us characterize the zero set $Z$ of $\nabla \psi \big|_{\partial D}$. Take a point $(r_0, z_0) \in Z$ and assume that there is a point $(r_0, z_0) \in \partial D$ with $r_0 < r_*$; then Equation (3.7) implies that

$$
\frac{\mu_1^2 c_1^2}{r_*^2} = c_2,
$$

and

$$
\frac{\mu_1^2 c_1^2}{r_0^2} = \frac{(\nabla \psi)^2_{(r_0, z_0)}}{r_0^2} > \frac{(\nabla \psi)^2_{(r_0, z_0)}}{r_*^2} = \frac{(\nabla \psi)^2_{(r_0, z_0)}}{r_*^2} + c_2,
$$

which is a contradiction. We hence conclude that $Z \subset \mathcal{R}_D$, i.e., the set of points on $\partial D$ that are closest to the $z$-axis, and being $Z$ nonempty; the inclusion $\mathcal{R}_D \subset Z$ also follows from Equation (3.5). This shows that $\mathcal{R}_D$ and so $\mathcal{R}_\Omega$ consists of at least two connected components if $\Omega$ is a locally optimal domain, thus proving the second claim in Theorem 1.2 provided that $u_1$ is axisymmetric.

In general, an eigenfield $u_1$ associated to $\mu_1$ does not need to be axisymmetric and reads in cylindrical coordinates as

$$
u = u_1^z e_z + u_1^r e_r + u_1^\phi e_\phi,
$$

where $u_1^z, u_1^r, u_1^\phi$ are functions of $(z, r, \phi) \in \Omega$. Let us now define the axisymmetric vector field

$$u_1^S := Su_1^z e_z + Su_1^r e_r + Su_1^\phi e_\phi,
$$

where the axisymmetrization operator $S : C^{k,\alpha}(\Omega) \to C^{k,\alpha}(\Omega)$ is given by

$$Sf(z, r, \phi) := \frac{1}{2\pi} \int_0^{2\pi} f(z, r, \phi) d\phi.
$$

Since $\Omega$ is axisymmetric, it is easy to check that $\text{curl } u_1^S = \mu_1 u_1^S$ and $u_1^S \in K(\Omega)$, so $u_1^S$ is also an eigenfield of curl with eigenvalue $\mu_1$. We claim that $u_1^S$ is not identically zero on $\Omega$.

Indeed, assume that there is a point $p_0 = (z_0, r_0, \phi_0) \in \partial \Omega$ such that $u_1^\phi(p_0) = 0$. Then $u_1 \big|_{\partial \Omega} (p_0)$ is tangent to the meridian of $\partial \Omega$ passing by the point $p_0$. Noticing that the (unparametrized) integral curves of $u_1 \big|_{\partial \Omega}$ are geodesics on $\partial \Omega$ by Corollary 2.3 and the fact that the meridian circles of an axisymmetric surface are geodesics, it immediately follows that the meridian $\gamma_0$ on $\partial \Omega$ corresponding to the point $p_0$ is an (unparametrized) integral curve of $u_1 \big|_{\partial \Omega}$. Since $|u_1|^2 \big|_{\partial \Omega} = c > 0$, calling $D_0$ the disk $\{ \phi = \phi_0 \}$ in $\Omega$ bounded by $\gamma_0$, we can write

$$0 \neq \int_{\gamma_0} u_1 \, dl = \int_{D_0} \text{curl } u_1 \cdot N \, dS = \mu_1 \int_{D_0} u_1 \cdot N \, dz \, dr = \mu_1 \int_\Omega u_1 \cdot N \, dz \, dr \, d\phi = \frac{\mu_1}{2\pi} \int_{\Omega} u_1 \cdot h \, dz \, dr = 0.
$$

Here $N = e_\phi$ is a unit vector normal to $D_0$ and $h$ is the unique harmonic field, cf. Equation (3.3), in $\Omega$. In the first equality we have used Stokes theorem and to
pass from the integral on $D_0$ to an integral on $\Omega$ we have used that $u_1$ is divergence-free and hence its flux through any disk $\{ \varphi = \varphi_0 \}$ does not depend on the angle. Since this equation yields a contradiction, we conclude that the component $u_1\varphi$ does not vanish at any point of $\partial \Omega$. Accordingly, the axisymmetric vector field $u_1^S$ cannot be identically zero, as claimed.

Summarizing, we have proved that for any optimal axisymmetric domain there exists a nontrivial axisymmetric curl eigenfield $u_1^S$ associated with the first positive curl eigenvalue $\mu_1$. The theorem then follows applying the previous discussion to the field $u_1^S$. The case of the first negative curl eigenvalue is completely analogous.

We conclude this section with a corollary that establishes that the first positive (or negative) curl eigenvalue of a locally optimal axisymmetric domain is simple. Notice that this is a very special property of optimal domains, because the first curl eigenvalue of a general bounded domain does not need to be simple (in contrast with the case of the Dirichlet Laplacian).

**Corollary 3.1.** Let $\Omega$ be an axisymmetric bounded domain with $C^{2,\alpha}$ connected boundary. If it is locally optimal for the first positive curl eigenvalue $\mu_1(\Omega)$, then this eigenvalue is simple and the corresponding curl eigenfield $u_1$ is axisymmetric. The same result holds if the domain is locally optimal for the first negative curl eigenvalue.

**Proof.** Since $\Omega$ is locally optimal and $C^{2,\alpha}$, Theorem 1.2 implies that $\Omega = D \times \mathbb{T}$ for some section $D$ whose closure is contained in $\mathbb{R} \times (0, \infty)$. Let $u_1$ and $v_1$ be two linearly independent eigenfields associated to $\mu_1$ and consider their axisymmetrizations $Su_1$ and $Sv_1$ as defined above. Recall that we have proved that the axisymmetrization of any curl eigenfield of $\mu_1$ is another nontrivial curl eigenfield. It follows from Proposition 2.1 that any linear combination $aSu_1 + bSv_1$, $a, b \in \mathbb{R}$, has constant pointwise norm on $\partial \Omega$, i.e.

$$|aSu_1 + bSv_1|^2|_{\partial \Omega} = c(a, b),$$

a constant that may depend on $a$ and $b$. It is then easy to check that the angle $\Theta$ formed by the vectors $Su_1(x)$ and $Sv_1(x)$ does not depend on the point $x \in \partial \Omega$.

The fields $Su_1$ and $Sv_1$ can be written as in Equation (3.1) for some functions $\psi$ and $\hat{\psi}$ on $D$, respectively. Since $\partial \Omega$ is connected, the proof of Theorem 1.2 also shows that $\nabla \psi|_{\partial D}$ and $\nabla \hat{\psi}|_{\partial D}$ vanish exactly on the same set $\mathcal{R}_D$. In view of Equation (3.1), the fields $Su_1|_{\partial \Omega}$ and $Sv_1|_{\partial \Omega}$ are then collinear at any point $p \in \mathcal{R}_\Omega$ (the are tangent to the rotation field $e_\varphi$), so from the argument above we conclude that they are collinear everywhere on $\partial \Omega$. The fact that they have constant pointwise norm on $\partial \Omega$ hence implies that there are constants $a_0, b_0$ such that the curl eigenfield $a_0Su_1 + b_0Sv_1$ satisfies

$$(a_0Su_1 + b_0Sv_1)|_{\partial \Omega} = 0.$$  

In view of Proposition 2.1 it follows that $a_0Su_1 + b_0Sv_1 = 0$ in $\Omega$, which means that the axisymmetrization of the curl eigenfield $a_0u_1 + b_0v_1$ is identically zero. Since we proved above that this cannot happen unless the linear combination $a_0u_1 + b_0v_1$ is zero itself, we finally conclude that the eigenvalue $\mu_1$ is simple. The same axisymmetrization argument also shows that the corresponding eigenfield $u_1$ must be axisymmetric. \qed
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Appendix A. A Uniform Lower Bound for the First Curl Eigenvalue

In this section we show that both the first positive and negative curl eigenvalues are lower bounded by a constant that only depends on the volume of the domain. However, this is rather different from the Faber–Krahn inequality for the Dirichlet Laplacian because here the bound is far from sharp, and probably it cannot be achieved. A similar bound (but upper instead of lower) holds for the helicity maximization problem considered in [3, Theorem E].

Theorem A.1. For any bounded $C^2$ domain $\Omega \subset \mathbb{R}^3$,

$$\min\{\mu_1(\Omega), -\mu_{-1}(\Omega)\} \geq \left(\frac{4\pi}{3|\Omega|}\right)^{1/3}.$$  

Proof. Since the curl operator is self-adjoint on the domain $D_\Omega$, let us denote by $\text{curl}^{-1}$ the compact self-adjoint operator on $\mathcal{K}(\Omega)$ defined by its inverse. It is then clear that

$$\frac{\|v\|^2}{\mu_1} \leq \langle \text{curl}^{-1} v, v \rangle \leq \frac{\|v\|^2}{\mu_{-1}}$$  

for all $v \in \mathcal{K}(\Omega)$, and that these inequalities are in fact equalities when $v$ is $u_{-1}$ or $u_1$, respectively.

Given $v \in D_\Omega$, let us consider the vector field defined by $v$ via the Biot–Savart integral

$$\text{BS} v(x) := \int_\Omega \frac{v(y) \times (x - y)}{4\pi|x - y|^3} \, dy.$$  

It is standard (see e.g. [2]) that, as $N \cdot v|_{\partial \Omega} = 0$,

$$\text{div BS} v = 0, \quad \text{curl BS} v = v$$  

in $\Omega$. Since $\text{BS} v - \text{curl}^{-1} v$ is curl-free, the Hodge decomposition theorem then implies

$$\text{curl}^{-1} v = \text{BS} v + h_v + \nabla \varphi_v,$$

where $h_v \in \mathcal{H}_\Omega$ and $\varphi_v$ is a scalar function in $H^1(\Omega)$.

Using this formula, we obtain that

$$\langle \text{curl}^{-1} v, v \rangle = \langle \text{BS} v, v \rangle + \langle h_v, v \rangle + \langle \nabla \varphi_v, v \rangle = \langle \text{BS} v, v \rangle,$$  



where we have used that the other two terms vanish because a vector field \(v \in D_\Omega\) is orthogonal to the kernel of curl. Notice now that

\[
|BSv(x)| = \left| \int_\Omega \frac{v(y) \times (x-y)}{4\pi|x-y|^3} \, dy \right|
\leq \frac{1}{4\pi} \int_\Omega \frac{|v(y)|}{|x-y|^2} \, dy
\leq \frac{1}{4\pi} \left( \int_\Omega \frac{|v(y)|^2}{|x-y|^2} \, dy \right)^{1/2} \left( \int_\Omega \frac{dy}{|x-y|^2} \right)^{1/2}.
\]

(A.3)

Denoting by \(\Omega^*\) the ball centered at the origin whose volume equals that of \(\Omega\), the rearrangement inequality ensures that

\[
\sup_{x \in \Omega} \int_\Omega \frac{dy}{|x-y|^2} \leq \sup_{z \in \Omega^*} \int_{\Omega^*} \frac{dy}{|z-y|^2} = \int_{\Omega^*} \frac{dy}{|y|^2} = (48\pi^2|\Omega|)^{1/3}.
\]

This estimate implies that

\[
\int_\Omega \int_\Omega \frac{|v(y)|^2}{|x-y|^2} \, dx \, dy \leq \left( \sup_{Y \in \Omega} \int_\Omega \frac{dx}{|x-Y|^2} \right) \left( \int_\Omega |v(y)|^2 \, dy \right)
\leq (48\pi^2|\Omega|)^{1/3} \|v\|^2,
\]

so now we can go back to the inequality (A.3), square it and integrate in \(\Omega\) to estimate the \(L^2\) norm of \(BSv\) as

\[
\|BSv\| \leq \left( \frac{3|\Omega|}{4\pi} \right)^{1/3} \|v\|.
\]

By (A.2), this yields

\[
|\langle \text{curl}^{-1} v, v \rangle| \leq \left( \frac{3|\Omega|}{4\pi} \right)^{1/3} \|v\|^2
\]

for all \(v \in D_\Omega\). Since \(D_\Omega\) is dense in \(K(\Omega)\) and \(\text{curl}^{-1}\) is a bounded linear operator, it then follows that the estimate holds for all \(v \in K(\Omega)\). In turn, this implies the eigenvalue estimate presented in the statement of the theorem because the inequalities (A.1) are saturated when \(v = u_1\) or \(v = u_{-1}\), in each case. \(\Box\)

**References**

1. D. Bucur, D. Mazzoleni, A. Pratelli, B. Velichkov, Lipschitz regularity of the eigenfunctions on optimal domains, Arch. Rat. Mech. Anal. 216 (2015) 117–151.
2. G. Buttazzo, G. Dal Maso, An existence result for a class of shape optimization problems, Arch. Rat. Mech. Anal. 122 (1993) 183–195.
3. J. Cantarella, D. DeTurck, H. Gluck, M. Teytel, Isoperimetric problems for the helicity of vector fields and the Biot–Savart and curl operators, J. Math. Phys. 41 (2000) 5615–5641.
4. D. Chae, P. Constantin, Remarks on a Liouville-type theorem for Beltrami flows, Int. Math. Res. Not. 2015, 10012–10016.
5. A. Enciso, M.A. García-Ferrero, D. Peralta-Salas, The Biot–Savart operator of a bounded domain, J. Math. Pures Appl. 119 (2018) 85–113.
6. M.H. Freedman, Z.X. He, Divergence-free fields: energy and asymptotic crossing number, Ann. of Math. 134 (1991) 189–229.
7. A. Henrot, Shape Optimization and Spectral Theory. De Gruyter, Warsaw/Berlin, 2017.
8. R. Hiptmair, P.R. Kotiuga, S. Tordeux, Self-adjoint curl operators, Ann. Mat. Pura Appl. 191 (2012) 431–457.
9. P. Laurence, E. Stredulinsky, A lower bound for the energy of magnetic fields supported in linked tori, C. R. Acad. Sci. Paris 331 (2000) 201–206.
10. N. Nadirashvili, Liouville theorem for Beltrami flow, Geom. Funct. Anal. 24 (2014) 916–921.
11. Z. Yoshida, Y. Giga, Remarks on spectra of operator rot, Math. Z. 204 (1990) 235–245.

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