Extensions in the category of divisible, locally compact abelian groups

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Abstract

In this paper, we prove that an extension of a divisible, torsion-free group by a compact torsion group split. Also, we show that an extension of a torsion-free, locally compact abelian group by a compact torsion group need not to be split.

1 Introduction

Throughout, all groups are Hausdorff abelian topological groups and will be written additively. Let $\mathcal{L}$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. The Pontryagin dual of a group $G$ is denoted by $\hat{G}$. A morphism is called proper if it is open onto its image and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\mathcal{L}$ is said to be proper exact if $\phi$ and $\psi$ are proper morphisms. In this case the sequence is called an extension of $A$ by $C$ (in $\mathcal{L}$). Following [5], we let $Ext(C, A)$ denote the (discrete) group of extensions of $A$ by $C$. In [4, Theorem 1], it is proved that if $C$ is a compact torsion group and $G$ a divisible LCA group, then $Ext(C, G) = 0$. However, the suggested proof in
[4] appears to be incomplete as it uses the incorrect Proposition 8 of [2]. In [7], we proved that if $G$ is $\sigma$-compact, then $\text{Ext}(C, G) = 0$. In this paper, we show that if $G$ is a divisible, torsion-free LCA group, then $\text{Ext}(C, G) = 0$.

The additive topological group of real numbers is denoted by $R$, $Q$ is the group of rationales with discrete topology and $Z$ is the group of integers. For a prime number $p$, $F_p$ is the $p$-adic number groups which is the minimal divisible extension of $J_p$. The topological isomorphism will be denote by $\cong$. For more on locally compact abelian groups see [6].

2 Main Results

Lemma 2.1. Let $X \in \mathcal{L}$. Then $n\text{Ext}(X, F_p) = \text{Ext}(X, F_p)$ for every positive integer $n$.

Proof. It is sufficient to consider that by [1, Lemma 2], $f : F_p \to F_p$ defined by $f(x) = nx$ is a proper morphism.

Lemma 2.2. Let $X$ be a compact torsion group. Then $\text{Ext}(X, F_p) = 0$.

Proof. Since $F_p$ is totally disconnected, so it contains a compact open subgroup $K$. Consider the exact sequence $0 \to K \to F_p \to F_p/K \to 0$. By [5, Corollary 2.10], there exists a short exact sequence

$$
\text{Hom}(X, F_p) \to \text{Hom}(X, F_p/K) \to \text{Ext}(X, K) \to \text{Ext}(X, F_p) \to \text{Ext}(X, F_p/K) \to 0
$$

Since $F_p$ is divisible, so by[5, Theorem 3.4] $\text{Ext}(X, F_p/K) = 0$. On the other hand, $X$ is torsion and $F_p$ torsion-free. Hence, $\text{Hom}(X, F_p) = 0$. So, we have the following exact sequence

$$
0 \to \text{Hom}(X, F_p/K) \to \text{Ext}(X, K) \to \text{Ext}(X, F_p) \to 0(*)
$$

Since $X$ is compact torsion, so $nX = 0$ for some $n$. Hence, $n\text{Ext}(X, K) = 0$. Since (*) is exact, so $n\text{Ext}(X, F_p) = 0$. Hence by Lemma1, $\text{Ext}(X, F_p) = 0$.

Remark 2.3. Let $X$ be a group. If $f : X \to X, f(x) = nx$ is topological isomorphism for each positive integer $n$, then $X$ is a divisible, torsion-free group.
Lemma 2.4. Let $X$ be a compact group. Then $\text{Ext}(X, F_p)$ is a divisible, torsion-free group.

Proof. Let $n$ be an arbitrary positive integer. Then the exact sequence $0 \to X \xrightarrow{\times n} X \to X/n \to 0$ induces the following exact sequence

$$\text{Ext}(X/n, F_p) \to \text{Ext}(X, F_p) \xrightarrow{\times n} \text{Ext}(X, F_p) \to 0$$

By Lemma 2.2, $\text{Ext}(X/n, F_p) = 0$. So $\text{Ext}(X, F_p) \xrightarrow{\times n} \text{Ext}(X, F_p)$ is a topological isomorphism. Hence by Remark 2.3, $\text{Ext}(X, F_p)$ is a divisible, torsion-free group.

Theorem 2.5. Let $X \in \mathcal{L}$. Then $\text{Ext}(X, F_p)$ is a divisible, torsion-free group.

Proof. Let $X \in \mathcal{L}$. By [6, Theorem 24.3], $X = R^n \oplus H$ where $H$ contains a compact open subgroup $K$. Consider the exact sequence

$$\text{Ext}(H/K, F_p) \to \text{Ext}(H, F_p) \to \text{Ext}(K, F_p) \to 0$$

Since $H/K$ is a discrete group and $F_p$ a divisible group, so $\text{Ext}(H/K, F_p) = 0$. Hence $\text{Ext}(H, F_p) \cong \text{Ext}(K, F_p)$. By Lemma 2.4, $\text{Ext}(K, F_p)$ is a divisible, torsion-free group. So $\text{Ext}(X, F_p)$ is a divisible, torsion-free group.

Theorem 2.6 Let $X$ be a compact torsion group and $G$ a divisible, torsion-free group. Then $\text{Ext}(X, G) = 0$.

Proof. By [6, 25.33], $G \cong R^n \oplus A \oplus M \oplus \prod_p F_p^{\times p}$, where $A$ is a discrete, torsion-free, divisible group and $M$ a compact connected torsion-free group. By [5, Theorem 3.4], $\text{Ext}(X, A) = 0$. Also $\text{Ext}(X, M) \cong \text{Ext}(\hat{M}, \hat{X})$. Since $\hat{X}$ is a discrete bounded group and $\hat{M}$ a discrete torsion-free group, so by [2, Theorem 27.5], $\text{Ext}(\hat{M}, \hat{X}) = 0$. By Lemma 2.2, $\text{Ext}(X, F_p) = 0$. Hence $\text{Ext}(X, G) = 0$.

Lemma 2.7. Let $X$ be a compact torsion group. Then $\text{Hom}(X, Q/Z) \cong \hat{X}$.
Proof. The exact sequence $0 \to Z \to Q \to Q/Z$ induces the following exact sequence

$$\text{Hom}(X, Q) \to \text{Hom}(X, Q/Z) \to \text{Ext}(X, Z) \to \text{Ext}(X, Q)$$

Since $X$ is torsion and $Q$ is torsion-free, so $\text{Hom}(X, Q) = 0$. Also by [5, Theorem 3.4], $\text{Ext}(X, Q) = 0$. Hence $\text{Hom}(X, Q/Z) \cong \text{Ext}(X, Z)$. By [5, Theorem 2.12 and Proposition 2.17], $\text{Ext}(X, Z) \cong \text{Ext}(\hat{Z}, \hat{X}) \cong \hat{X}$. So $\text{Hom}(X, Q/Z) \cong \hat{X}$.

**Theorem 2.8.** Let $X$ be a compact torsion group and $G$ a torsion-free group. Then $\text{Ext}(X, G) \neq 0$.

Proof. Let $G^*$ be the minimal divisible extension of $G$. By [6, A.13], $G^*$ is a divisible, torsion-free group. Hence by Theorem 2.6, $\text{Ext}(X, G^*) = 0$. By [5, Corollary 2.10], we have the following exact sequence

$$\text{Hom}(X, G^*) \to \text{Hom}(X, G^*/G) \to \text{Ext}(X, G) \to \text{Ext}(X, G^*) = 0$$

Since $X$ is torsion and $G^*$ torsion-free, so $\text{Hom}(X, G^*) = 0$. Hence, $\text{Hom}(X, G^*/G) \cong \text{Ext}(X, G)$. Since $G^*/G$ is a discrete, torsion divisible group, so $\text{Hom}(X, G^*/G)$ containing a copy of $\text{Hom}(X, Q/Z)$. Hence by Lemma 2.7, $\text{Ext}(X, G) \neq 0$.

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