Effective Lagrangians for \((0 + 1)\) and \((1 + 1)\) dimensionally reduced versions of \(D = 4, \mathcal{N} = 2\) SYM theory.

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Abstract

We consider dimensionally reduced versions of \(\mathcal{N} = 2\) four–dimensional supersymmetric Yang-Mills theory and determine the one-loop effective Lagrangians associated with the motion over the corresponding moduli spaces. In the \((0 + 1)\) case, the effective Lagrangian describes an \(\mathcal{N} = 4\) supersymmetric quantum mechanics of the Diaconescu–Entin type. In \((1 + 1)\) dimensions, the effective Lagrangian represents a twisted \(\mathcal{N} = 4\) supersymmetric \(\sigma\) model due to Gates, Hull, and Roček. We discuss the genetic relationship between these two models and present the explicit results for all gauge groups.

1 Introduction.

Perturbative structure of supersymmetric 4–dimensional gauge theories has been a subject of intense studies since the beginning of the eighties. Supersymmetry brings about restrictions on the perturbative series, and the larger is the supersymmetry, the more stringent the restrictions are. Thus, in \(\mathcal{N} = 4\) SYM theory \(^2\), the effective charge is not renormalized at all. In

\(^1\)On leave of absence from ITEP, Moscow, Russia.

\(^2\)To avoid confusion, note that our \(\mathcal{N}\) measures the number of the irreducible spinor multiplets of supercharges in 4 dimensions and the number of complex supercharges in \((0 + 1)\) and \((1 + 1)\) dimensions.
the $\mathcal{N} = 2$ case, only one-loop contribution to the $\beta$ function survives while all other vanish. In $\mathcal{N} = 1$ theories, all loops contribute, but there is a rigid relationship between the $\beta$ function and the anomalous dimensions [1].

The perturbative structure of the dimensionally reduced versions of the original 4-dimensional theories is equally non-trivial and interesting. In Refs. [2, 3, 4], we calculated the effective Lagrangians for the QM versions of $\mathcal{N} = 1$ supersymmetric electrodynamics and Yang–Mills theories. The one–loop renormalization of the kinetic term in the reduced theories involve a power infrared rather than the logarithmic ultraviolet integral. Still, the corresponding coefficients turn out to be rigidly related to the one–loop contribution in the 4-dimensional $\beta$ function [5].

The effective QM Lagrangian represents a nonstandard $\mathcal{N} = 2$ $\sigma$ model: its bosonic part describes the motion over $3r$–dimensional target space, $r$ being the rank of the group. (There are certain restrictions for the metric. See Ref.[4] for details.) On the other hand, the effective $(1 + 1)$ Lagrangian represents a standard Kählerian $\sigma$-model. In the simplest case of the $\mathcal{N} = 1$ SQED, the target space is two–dimensional with the metric

$$ds^2_{1+1} = \left[1 + \frac{e^2}{4\pi\phi\phi} + \ldots\right] d\phi d\phi, \quad (1.1)$$

where $\phi = L(A_1 + i A_2)/\sqrt{2}$ are the moduli related to the components of the original 4-dim vector potential $A_\mu$ in two compactified directions. ³ This can be compared with the QM metric

$$ds^2_{0+1} = \left[1 + \frac{e^2}{2|A|^3} + \ldots\right] dA^2, \quad (1.2)$$

with 3–dimensional $A$.

In the present paper, we address the dimensionally reduced versions of the $D = 4, \ \mathcal{N} = 2$ theories. As was noticed earlier [6, 5, 7] the latter enjoy non-renormalization theorems that are pretty much similar to their 4–dimensional counterparts. In particular, the renormalization of the kinetic term in $\mathcal{L}_{\text{eff}}$ receives contributions only at the one–loop level, and for $\mathcal{N} = 4$ theories it

³$L = 2\pi R$, where $R$ is the radius of compactification. In what follows we set $L = 1$. Eq. (1.1) is valid when $\phi\phi \gg e^2$ so that higher loop corrections are suppressed. On the other hand, $\phi\phi \ll 1$, otherwise the effects due to a finite compactification radius would be important.
is not renormalized at all (there are nontrivial contributions to the higher-derivative structures, however [8]). The exact reason why nonrenormalization theorems in different dimensions are so similar is not quite clear yet. We think that a better understanding of this question could shed also some more light on the renormalization of supersymmetric theories in 4 dimensions, and this is our own main motivation for these studies.

This particular paper, does not address these questions, however, and has a technical nature. In the next section, we write some old and some new formulae referring to the $\mathcal{N} = 4$ effective QM theories. They represent nonstandard $\mathcal{N} = 4$ $\sigma$ models living on 5$\ell$–dimensional target space and belong to the class studied earlier in Ref.[9]. Sect. 3 is devoted to the $(1+1)$-dimensional effective theories. They are $\mathcal{N} = 4$ $\sigma$ models. Somewhat surprisingly, they are not hyper-Kählerian but belong to the class of so called twisted $\sigma$ models. (The latter are not so widely known, though they were first described back in 1984 [10].)

2 Effective QM Lagrangians

As was mentioned above, the models we are looking for belong to the class of $\mathcal{N} = 4$ SQM models introduced in [9]. We start with recalling the Diaconescu–Entin construction, translating it into more standard notations and correcting some errors in the coefficients.

Let us first fix our notations. For 2-component complex spinors, the indices are raised and lowered with the $\epsilon$ symbol (which is nothing but a two- or three-dimensional charge conjugation matrix),

$$ \psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta, \quad \epsilon^{12} = -\epsilon_{12} = 1. \quad (2.3) $$

Now, $\psi_\alpha$ goes over to $\bar{\psi}^\alpha$ after Hermitian conjugation. Also, $\psi\chi = \epsilon^{\alpha\beta} \psi_\alpha \chi_\beta$, $\bar{\psi}\bar{\chi} = \epsilon_{\alpha\beta} \bar{\psi}^\alpha \bar{\chi}_\beta$.

We will use also the 4-component spinors $\eta_A$ lying in the fundamental representation of $Sp(4) \equiv SO(5)$. In this case, we prefer not to distinguish between lower and upper indices and, whenever necessary, write the charge conjugation matrix $C$ [the symplectic matrix of $Sp(4)$] explicitly. The latter satisfies the property $C \gamma_J^T = \gamma_J C$, where $\gamma_J$, $J = 1, \ldots, 5$ are Euclidean
5-dimensional $\gamma$ matrices. One of the possible choices for the latter is
\[
\gamma_{1,2,3} = \sigma_{1,2,3} \otimes \sigma_3, \quad \gamma_4 = 1 \otimes \sigma_2, \quad \gamma_5 = 1 \otimes \sigma_1.
\]
Let us take a standard chiral supervariable
\[
\Phi = \phi + \sqrt{2} \theta \chi + \theta^2 F + i \bar{\phi} \bar{\theta} \theta + 2 \theta \chi \bar{\theta} \theta - \frac{1}{2} \bar{\phi} (\bar{\theta} \theta)^2 \tag{2.5}
\]
It satisfies the constraint $D_\alpha \Phi = [\partial / \partial \bar{\theta} \theta + i \theta_\alpha \partial / \partial t] \Phi = 0$. Consider also a real 3-component supervariable [3]
\[
V_k = -\frac{1}{4} \epsilon^{\beta \gamma} (\sigma_k)_\alpha^\gamma V_{\alpha \beta}, \quad V_{\alpha \beta} = V_{\beta \alpha}
\]
satisfying the constraint
\[
D_\alpha V_{\beta \gamma} + D_\beta V_{\alpha \gamma} + D_\gamma V_{\alpha \beta} = 0.
\]
$V_{\alpha \gamma}$ can also be obtained from a generic real $\mathcal{N} = 2$ supervariable $C$ as
\[
V_{\alpha \beta} = (D_\alpha \bar{D}_\beta + D_\beta \bar{D}_\alpha) C.
\tag{2.6}
\]
$V_k$ represents a supersymmetric generalization of the vector potential. \[4\] It is expressed into components as follows:
\[
V_k = A_k + \bar{\psi} \sigma_k \theta + \bar{\theta} \sigma_k \psi + \epsilon_{kj} A_j \bar{\psi} \sigma_j \theta + D \bar{\theta} \sigma_k \theta + i(\bar{\theta} \sigma_k \bar{\psi} - \bar{\psi} \sigma_k \theta) \bar{\theta} \theta + \frac{\bar{A}_k}{4} \theta^2 \bar{\theta}^2,
\tag{2.7}
\]
where $\psi_\alpha$ is a 2-component spinor and $D$ is the auxiliary field.

Consider the Lagrangian
\[
\int d^2 \theta d^2 \bar{\theta} \mathcal{K}(V_k, \bar{\Phi}, \Phi) \tag{2.8}
\]
\[4\] A 4-dimensional counterpart of (2.6) is not invariant under gauge transformations $\delta C = i(\Lambda - \Lambda)$. But the QM supervariable (2.6) is.
with real $\mathcal{K}$. The Lagrangian (2.8) has manifest $\mathcal{N} = 2$ supersymmetry. As was shown in [9], for a restricted class of functions satisfying the 5-dimensional harmonicity condition

$$
\frac{\partial^2 \mathcal{K}}{\partial V_k^2} + 2 \frac{\partial^2 \mathcal{K}}{\partial \Phi \partial \Phi} = 0 \ ,
$$

the Lagrangian (2.8) enjoys the full $\mathcal{N} = 4$ supersymmetry. To understand this, express (2.8) into components bearing in mind the condition (2.9)

$$
\mathcal{L} = h \left[ \frac{1}{2} A_j^2 + \frac{i}{2} \dot{\psi} \dot{\psi} - \frac{i}{2} \ddot{\psi} \dot{\psi} + \ddot{\chi} \dot{\chi} - \dot{\chi} \dot{\chi} \right] + \frac{1}{2} (\partial h)_{kjp} \dot{A}_j [\bar{\psi} \sigma_p \psi + \bar{\chi} \sigma_p \chi] + \frac{i}{\sqrt{2}} \dot{A}_j \left[ \frac{\partial h}{\partial \Phi} \bar{\psi} \sigma_j \chi - \frac{\partial h}{\partial \Phi} \bar{\psi} \sigma_j \chi \right] + \frac{i}{2} \left( \ddot{\psi} - \dot{\Phi} \partial \frac{\partial h}{\partial \Phi} \right) (\ddot{\psi} \dot{\psi} + \ddot{\chi} \dot{\chi}) + \frac{i}{\sqrt{2}} \partial_k h \left( \dot{\psi} \sigma_k \chi - \dot{\chi} \sigma_k \chi \right) - \frac{1}{8 h} \left[ \partial_j h (\ddot{\psi} \sigma_j \psi - \ddot{\chi} \sigma_j \chi) + \sqrt{2} \left( \frac{\partial h}{\partial \Phi} \psi \chi + \frac{\partial h}{\partial \Phi} \bar{\psi} \bar{\chi} \right) \right]^2 - \frac{1}{4 h} \left| \psi^2 \frac{\partial h}{\partial \Phi} - \chi^2 \frac{\partial h}{\partial \Phi} - \sqrt{2} \partial_j h \ddot{\psi} \sigma_j \psi \right|^2 - \frac{1}{8} \partial^2 h \left( \ddot{\psi} \dot{\psi}^2 + \ddot{\chi} \dot{\chi}^2 \right) - \frac{1}{4} \left( \frac{\partial^2 h}{\partial \Phi^2} \psi^2 \chi^2 + \frac{\partial^2 h}{\partial \Phi^2} \ddot{\psi} \ddot{\chi} \right) + \frac{1}{2} \partial_j \partial_k h (\ddot{\psi} \sigma_j \chi) (\bar{\chi} \sigma_k \psi) + \frac{1}{2 \sqrt{2}} \left[ \frac{\partial^2 h}{\partial A_j \partial \Phi} \left( \chi^2 \ddot{\chi} \sigma_j \psi - \ddot{\psi} \ddot{\psi} \sigma_j \chi \right) + \frac{\partial^2 h}{\partial A_j \partial \Phi} \left( \ddot{\chi} \ddot{\chi} \sigma_j \psi - \ddot{\psi} \ddot{\psi} \sigma_j \chi \right) \right],
$$

where $h = 2 \partial^2 \mathcal{K} / \partial A_j^2$.

The bosonic part of Eq.(2.10) describes the motion over a 5-dimensional target space with conformally flat metric

$$
d s^2 = h dA_J^2 = h \left( dA^2 + 2 d\phi d\phi \right).
$$

($J = 1, \ldots, 5; \phi = (A_4 + i A_5) / \sqrt{2}$. Harmonicity of $\mathcal{K}$ implies the harmonicity of $h$: $\partial_J \partial_J h = 0$. The fermion variables $\psi_\alpha$ and $\chi_\alpha$ enter the expression (2.10) symmetrically. More exactly, (2.10) is invariant under the transformation

$$
\psi \rightarrow i \chi, \quad \chi \rightarrow i \psi.
$$

(2.12)
And this implies that besides manifest supersymmetry transformations mixing $A$ with $\psi$ and $\phi$ with $\chi$, (2.10) is also invariant under two other transformations mixing $A$ with $\chi$ and $\phi$ with $\psi$. One can exploit the symmetry (2.12) and define a bispinor

$$\eta_A = \begin{pmatrix} \psi_\alpha \\ i\bar{\chi}_\alpha \end{pmatrix} .$$

(2.13)

$\eta_A$ belongs to the fundamental representation of $Sp(4) \equiv$ spinor representation of $SO(5)$. The Lagrangian (2.10) can then be rewritten in $O(5)$ notations as

$$\mathcal{L} = \frac{1}{2} \left[ \frac{1}{2} \dot{A}_J^2 + \frac{i}{2} (\bar{\eta} \eta - \dot{\bar{\eta}} \dot{\eta}) \right] + \frac{i}{2} \partial_J h \dot{A}_K \, \bar{\eta} \sigma_{JK} \eta + \frac{1}{24} \left( 2 \partial_J \partial_K h - \frac{3}{h} \partial_J h \partial_K h \right) \left( \bar{\eta} \gamma_{JK} \eta - \eta \gamma_{JL} \bar{\eta} \gamma_{KL} \eta - \eta C \gamma_{JL} \bar{\eta} \gamma_{KL} C \eta \right) ,$$

(2.14)

where $\sigma_{JK} = (1/2)(\gamma_J \gamma_K - \gamma_K \gamma_J)$. The Lagrangian (2.14) is symmetric under charge conjugation $\eta \rightarrow C\bar{\eta}$ [that just coincides with the symmetry (2.12) !]. It enjoys $\mathcal{N} = 4$ supersymmetry for any harmonic function $h(A_J)$. If we also require the Lagrangian to be $O(5)$ invariant, the function $h$ must have the form

$$h = a + \frac{c}{(A_J^2)^{3/2}}$$

(2.15)

(We will assume that $a = 1$, which can always be achieved by a proper rescaling. The second term is proportional to the Green’s function of the Laplacian in 5 dimensions). The corresponding prepotential can be chosen as

$$\mathcal{K} = \frac{R^2}{12} - \frac{\rho^2}{8} - \frac{c}{2R} \ln \left( R + \sqrt{R^2 + \rho^2} \right) ,$$

(2.16)

where $R^2 = V_k$ and $\rho^2 = 2\bar{\Phi}\Phi$. Note that $\mathcal{K}$ need not be and is not $O(5)$ invariant.

It is not so difficult to write a generalized DE model including an arbitrary large number of supervariables. Consider a set of chiral and real 3–vector supervariables $(V^a_k, \bar{\Phi}^a, \Phi^a) \equiv V^a_j$, $a = 1, \ldots , r$. The Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \, \mathcal{K}(V^a_j)$$

(2.17)

$^5$Note the difference in coefficients with Eq.(3.10) of Ref. [9]!
is $\mathcal{N} = 4$ supersymmetric if the function $\mathcal{K}$ satisfies the generalized harmonicity condition

$$\frac{\partial^2 \mathcal{K}}{\partial V_a^i \partial V_b^j} = 0 \quad (2.18)$$

for all $a, b$.

Our goal is to find the QM effective Lagrangians for $\mathcal{N} = 2, D = 4$ theories. Consider first Abelian theory. The moduli space of the $\mathcal{N} = 2$ SQED includes three components of the vector potential $A_k$ and also a neutral complex scalar $\phi$. This gives us 5 bosonic degrees of freedom in $\mathcal{L}_{\text{eff}}$. The requirements of $O(5)$ invariance [the original Lagrangian reduced to (0+1) dimensions had it and the effective Lagrangian should also have it] and of $\mathcal{N} = 4$ supersymmetry rigidly determines the form of $\mathcal{L}_{\text{eff}}$. It is given by Eqs.(2.14), (2.15) and the only question is the numerical value of the coefficient $c$. It is fixed by the explicit one–loop calculation $[5, 7]$,

$$c_{\mathcal{N}=2 \text{ SQED}} = c_{\mathcal{N}=1 \text{ SQED}} = \frac{e^2}{2}. \quad (2.19)$$

Now let the original theory be non–Abelian. The moduli space where the effective Lagrangian lives is now $5r$–dimensional. This is best seen if treating the QM theory as a result of the dimensional reduction of (5+1)–dimensional SYM theory. Classical vacua correspond to the vanishing field strength $F_{JK} = 0$. In the QM limit, this implies $[A_J, A_K] = 0$ and hence $A_J$ belongs to the ($r$–dimensional) Cartan subalgebra of the original Lie algebra. In the simplest $SU(2)$ case, the rank $r$ is 1, moduli space is 5–dimensional, and theory has the same form (2.14), (2.15) as in the Abelian case. Again, the coefficient $c$ is fixed from the one–loop calculation,

$$c_{\mathcal{N}=2 \text{ SYM}}^{SU(2)} = -g^2. \quad (2.20)$$

Consider now an arbitrary simple gauge group. The prepotential $\mathcal{K}(V^a_k, \Phi^a, \Phi^a)$ can be fixed by the same method as in the $\mathcal{N} = 1$ case $[4]$. The effective Lagrangian is singular when the (positive) root forms $^6 V^{(j)}_k = \alpha_j(V^a_k)$ or $\Phi^{(j)} = \alpha_j(\Phi^a)$ vanish. Indeed, an accurate analysis shows that these are the

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^6In the familiar case of $SU(3)$, there are 3 such root forms corresponding to $T$-spin, $V$-spin, and $U$-spin.
regions in the moduli space where Born–Oppenheimer approximation breaks down. When, for some particular root \( j_0 \), \(|V^{(j_0)}_k|\) and \( |\Phi^{(j_0)}|\) are small compared to the values of other root forms, we can forget about the other roots, and the complicated original Lie algebra is effectively reduced to \( SU(2) \). The effective Lagrangian for the latter was, however, calculated above.

The result is obtained from the requirement that \( L_{\text{eff}} \) has a correct limit at small \(|V^{(j_0)}_k|\) and \( |\Phi^{(j_0)}|\) and from the harmonicity condition (2.18) with the use of group theory relations

\[
\sum_j \alpha_j(X)\alpha_j(Y) = \frac{c_V}{2} \sum_{a=1}^r X^a Y^a ,
\]

\[
\sum_j c_{jj'}^2 = d_j \frac{c_V}{2} ,
\]

(2.21)

where \( c_V \) is the adjoint Kasimir eigenvalue, \( c_{jj'} = \langle \alpha_j | \alpha_{j'} \rangle \), and \( d_j = c_{jj} \) is normalized to 1 for the long roots and to 1/2 or 1/3 (in the case of \( G_2 \)) for the short roots. The sum in Eq.(2.21) runs over all positive roots. We obtain

\[
F = \sum_j \left\{ \frac{1}{6c_V} \left[ (R^{(j)})^2 - \frac{3}{2} (\rho^{(j)})^2 \right] \right. + \left. \frac{g^2}{2R^{(j)}} \ln \left[ R^{(j)} + \sqrt{(R^{(j)})^2 + (\rho^{(j)})^2} \right] \right\} ,
\]

(2.22)

where \((R^{(j)})^2 = (V^{(j)}_k)^2\) and \((\rho^{(j)})^2 = 2\bar{\Phi}^{(j)}\Phi^{(j)}\).

3 Effective (1+1) Lagrangians.

Consider first the Abelian case. For the \( D = 4, N = 2 \) SQED compactified on \( T^2 \) [or if you will the \( D = 6, N = 1 \) SQED compactified on \( T^4 \)], moduli space is 4–dimensional and can be described by two complex variables

\[
\sigma = (A_1 + iA_2)/\sqrt{2} , \quad \phi = (A_4 + iA_5)/\sqrt{2} .
\]

(3.23)

One loop calculation brings about a nontrivial metric in the target space \((\sigma, \bar{\sigma}, \phi, \bar{\phi})\). This metric can be related to the SQM 5–dimensional metric by integrating the latter over \( A_3 \) by the same token as the Kählerian metric.
(1.1) is obtained from the metric (1.2) of the SQM model in the $\mathcal{N} = 1$ case [5] (some further comments about it can be found below),

$$ds_{1+1}^2|_{\mathcal{N}=2} = (1 + \delta h_{1+1})[d\sigma d\bar{\sigma} + d\phi d\bar{\phi}] ,$$

(3.24)

where

$$\delta h_{1+1} = \int_{-\infty}^{\infty} \frac{dA_3}{2\pi} \delta h_{0+1} =$$

$$c \int_{-\infty}^{\infty} \frac{dA_3}{2\pi} \frac{1}{[2\phi \phi + 2\bar{\sigma} \sigma + A_3^2]^{3/2}} = \frac{c}{2\pi(\phi \phi + \bar{\sigma} \sigma)} .$$

(3.25)

We expect the effective action to have the $\sigma$ model form. One could worry at this point because the metric (3.24), (3.25) is not hyper-Kählerian (the Ricci tensor and the scalar curvature do not vanish) while hyper-Kählerian nature of the metric was shown to be necessary for the standard (1+1) $\sigma$ model to enjoy $\mathcal{N} = 4$ supersymmetry [11]. In our case, $\mathcal{N} = 4$ supersymmetry is there but the metric is not hyper-Kählerian, and this seems to present a paradox. The resolution is that the $\sigma$ model in hand is not standard.

Indeed, the bosonic part of the Lagrangian involves besides the standard kinetic term $h(\partial_{\alpha} \bar{\sigma} \partial_{\alpha} \sigma + \partial_{\alpha} \partial_{\alpha} \phi)$ also the "twisted" term $\propto \epsilon_{\alpha\beta} \partial_{\alpha} \sigma \partial_{\beta} \phi$ and $\propto \epsilon_{\alpha\beta} \partial_{\alpha} \bar{\sigma} \partial_{\beta} \bar{\phi}$. To understand where the twisted term comes from, consider a charged fermion loop in the background

$$\sigma = \sigma_0 + \sigma_{\tau} \tau + \sigma_z z, \quad \phi = \phi_0 + \phi_{\tau} \tau + \phi_z z$$

(3.26)

($\tau$ is the Euclidean time). The contribution to the effective action is $\propto \ln \det \| \mathcal{D} \|$, where $\mathcal{D}$ is the 6-dimensional Euclidean Dirac operator, which can be written in the form

$$\mathcal{D} = i \frac{\partial}{\partial \tau} + \gamma_{\beta} \frac{\partial}{\partial z} - i(\gamma_1 A_1 + \gamma_2 A_2 + \gamma_4 A_4 + \gamma_5 A_5)$$

(3.27)

[\gamma matrices are defined in Eq.(2.4) and $A_{1,2,4,5}$ in Eq.(3.23)].

Now, if $A_4$ and $A_5$ were absent, we could write $\mathcal{D} = \gamma_4(i\tilde{\gamma}_\mu \mathcal{D}_\mu)$, with

$\mu = 1, 2, 3, 4; \mathcal{D}_4 = \frac{\partial}{\partial \tau}, \mathcal{D}_3 = \frac{\partial}{\partial z}, \mathcal{D}_{1,2} = -iA_{1,2}; \tilde{\gamma}_4 = \gamma_4, \tilde{\gamma}_{1,2,3} = -i\gamma_4 \gamma_{1,2,3}$

and then use the squaring trick

$$\det \| \mathcal{D} \| = \det \| i\tilde{\gamma}_\mu \mathcal{D}_\mu \| = \det^{1/2} \left\| -\mathcal{D}^2 + i\tilde{\sigma}_{\mu\nu} F_{\mu\nu} \right\| .$$

(3.28)
with $F_{14} = \partial A_1 / \partial \tau$, etc. The effective action would be proportional to

$$\Tr\{\sigma_{\mu\nu}\sigma_{\alpha\beta}\} F_{\mu\nu} F_{\alpha\beta} \int \frac{d^2 p}{4\pi^2} \frac{1}{(p^2 + 2\bar{\sigma}\sigma)^2}, \quad (3.29)$$

which gives the renormalization of the kinetic term while the twisted term does not appear. The squaring trick works also in the case where $A_{4,5}$ are nonzero, but do not depend on $\tau, z$. Then $2\bar{\phi}\phi$ is just added to $-D^2$ in Eq.(3.28) and to $2\bar{\sigma}\sigma$ in Eq.(3.29) leading to Eq.(3.25). But in the generic case the fermion determinant cannot be reduced to $\det^{1/2} \| -D^2 + i\frac{1}{2}\sigma_{\mu\nu} F_{\mu\nu} \|$. The basic reason for this impasse is that one cannot adequately "serve" six components of the gradient with only five $\gamma$ matrices. As a result, the extra twisted term in the determinant appears.

We need not perform an explicit calculation here as the twisted and all other terms in the Lagrangian are fixed by supersymmetry. The twisted $\mathcal{N} = 4$ supersymmetric $\sigma$ model was constructed almost 20 years ago[10]. At that time it did not attract much attention. Recently, there is some revival of interest in the GHR model: it happened to pop up in some string-related problems [12, 13]. It also pops up as the effective $(1 + 1)$ Lagrangian in the case under study.

It was shown that, for $\mathcal{N} = 4$ supersymmetric generalization to be possible, the conformal factor in the metric $h(\bar{\sigma}, \sigma, \bar{\phi}, \phi)$ should satisfy the harmonicity condition

$$\frac{\partial^2 h}{\partial \bar{\sigma} \partial \sigma} + \frac{\partial^2 h}{\partial \bar{\phi} \partial \phi} = 0. \quad (3.30)$$

Obviously, (3.25) satisfies it everywhere besides the origin. The relationship of (3.30) to the 5-dimensional harmonicity condition for the metric in the effective SQM model (2.14) is also obvious. Indeed, integrating a $D$-dimensional harmonic function over one of the coordinates like in (3.25), we always arrive at a $(D - 1)$-dimensional harmonic function.

To construct the full action, consider along with the standard chiral multiplet $\Phi$ satisfying the conditions $\mathcal{D}_a \Phi = 0$ also a twisted chiral multiplet $\Sigma$ which satisfies the constraints

$$\mathcal{D}_1 \Sigma \equiv \bar{\mathcal{D}}_+ \Sigma = 0, \quad \mathcal{D}_2 \Sigma \equiv \mathcal{D}_- \Sigma = 0. \quad (3.30)$$

\footnote{By the same reason, the squaring trick does not work for Weyl 2-component fermions in 4 dimensions: three Pauli matrices that are available in that case are not enough to do the job.}
Now, $\Phi$ and $\Sigma$ are expressed into components as follows

$$
\Phi = \phi + \sqrt{2}(\theta+\chi- - \theta_+\chi_+) + i(\partial_+\phi)\bar{\theta}_+\theta_+ + i(\partial_-\phi)\bar{\theta}_-\theta_-
+ i\sqrt{2}[\bar{\theta}_+(\partial_+\chi_+) + \bar{\theta}_-(\partial_-\chi_-)] \theta_+\theta_- - (\partial_+\partial_-\phi)\bar{\theta}_+\theta_+\bar{\theta}_-\theta_- + 2\theta_+\theta_- F ,
$$

(3.31)

and

$$
\Sigma = \sigma + \sqrt{2}(\bar{\theta}_+\psi_+ - \theta_-\bar{\psi}_+) - i(\partial_+\sigma)\bar{\theta}_+\theta_+ + i(\partial_-\sigma)\bar{\theta}_-\theta_- + i\sqrt{2}[\theta_+(\partial_+\bar{\psi}_+) + \bar{\theta}_-\partial_-\psi_-] \theta_+\bar{\theta}_+ + (\partial_+\partial_-\sigma)\bar{\theta}_+\theta_+\bar{\theta}_-\theta_- + 2\bar{\theta}_+\theta_- G ,
$$

(3.32)

where $\partial_\pm = \partial_t \pm \partial_z$. The twisted multiplet (3.32) is closely related to the multiplet (2.7). Actually, the QM version of Eq.(3.32) (obtained when the spatial derivatives are suppressed, $\partial_\pm \to \partial_t$) just coincides with $\left(V_1 + iV_2\right)/\sqrt{2}$ (and $G = (D + i\bar{A}_3)/\sqrt{2}$).

We see that the twisted multiplet differs from the standard one by a pure convention: $\Sigma$ is obtained from $\Phi$ by interchanging $\theta_+\bar{\theta}_+$ and $\theta_-\bar{\theta}_-$. This means that the change $\Phi \to \Sigma$ in any standard action involving $\Phi$ would change nothing. However, one can write nontrivial Lagrangians involving both $\Phi$ and $\Sigma$. The twisted $\sigma$ model is determined by the expression

$$
L = \int d^2\theta d^2\bar{\theta} \mathcal{K}(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma}) ,
$$

(3.33)

where the prepotential $\mathcal{K}$ satisfies the harmonicity condition,

$$
\frac{\partial^2 \mathcal{K}}{\partial \Sigma \partial \bar{\Sigma}} + \frac{\partial^2 \mathcal{K}}{\partial \Phi \partial \bar{\Phi}} = 0 .
$$

(3.34)

The condition (3.34) is required if we want the theory to be $\mathcal{N} = 4$ supersymmetric. Only for a harmonic $\mathcal{K}$, the fermion interchange symmetry (2.12) holds for the fermion kinetic term

$$
L_{\text{kin}}^{\text{form}} = ih \left[ \bar{\chi}_+\partial_+\chi_+ + \bar{\chi}_-\partial_-\chi_- + \bar{\psi}_+\partial_+\psi_+ + \bar{\psi}_-\partial_-\psi_- \right] ,
$$

(3.35)

(h = $4\partial^2 \mathcal{K}/\partial \bar{\sigma} \partial \sigma = -4\partial^2 \mathcal{K}/\partial \bar{\phi} \partial \phi$) and for the full Lagrangian (cf. Eq.(6.11) of Ref. [14])

$$
L = L_{\text{kin}}^{\text{bos}} + L_{\text{kin}}^{\text{form}} + L_{\text{mixed}} + L_{4f} ,
$$

(3.36)
where

$$\mathcal{L}_{\text{kin}}^{\text{bos}} = h \left[ |\partial_\sigma \phi|^2 + |\partial_\sigma \sigma|^2 \right] + 4 \left[ \frac{\partial^2 \mathcal{K}}{\partial \sigma \partial \phi} \epsilon_{\alpha \beta} (\partial_\alpha \sigma)(\partial_\beta \phi) + \frac{\partial^2 \mathcal{K}}{\partial \sigma \partial \phi} \epsilon_{\alpha \beta} (\partial_\alpha \bar{\sigma})(\partial_\beta \bar{\phi}) \right] , \quad (3.37)$$

$$\mathcal{L}_{\text{mixed}} = \frac{i}{2} (\bar{\psi}_+ \psi_+ + \bar{\chi}_+ \chi_+) \left( \frac{\partial h}{\partial \sigma} - \frac{\partial h}{\partial \phi} \right) + \frac{i}{2} (\bar{\psi}_- \psi_- + \bar{\chi}_- \chi_-) \left( \frac{\partial h}{\partial \sigma} - \frac{\partial h}{\partial \phi} \right) - i \bar{\psi}_+ \chi_+ \left( \frac{\partial h}{\partial \sigma} - \frac{\partial h}{\partial \phi} \right) +$$

$$-i \bar{\psi}_- \chi_- \left( \frac{\partial h}{\partial \sigma} - \frac{\partial h}{\partial \phi} \right) - i \bar{\psi}_- \chi_- \left( \frac{\partial h}{\partial \sigma} - \frac{\partial h}{\partial \phi} \right) , \quad (3.38)$$

$$\mathcal{L}_4 = -\frac{\partial^2 h}{\partial \sigma \partial \sigma} (\bar{\psi}_+ \psi_+ + \bar{\chi}_+ \chi_+) (\bar{\psi}_- \psi_- + \bar{\chi}_- \chi_-) + \frac{\partial^2 h}{\partial \sigma^2} \bar{\psi}_- \psi_+ \chi_+ +$$

$$-\frac{\partial^2 h}{\partial \phi^2} \bar{\psi}_- \psi_+ \chi_+ + \frac{\partial^2 h}{\partial \sigma^2} \bar{\psi}_+ \chi_+ \psi_- + \frac{\partial^2 h}{\partial \phi^2} \bar{\psi}_+ \chi_- \psi_- +$$

$$+(\bar{\psi}_+ \psi_+ + \bar{\chi}_+ \chi_+) \left( \frac{\partial^2 h}{\partial \sigma \partial \phi} \bar{\psi}_- \chi_- - \frac{\partial^2 h}{\partial \sigma \partial \phi} \psi_- \chi_- \right) +$$

$$+(\bar{\psi}_- \psi_- + \bar{\chi}_- \chi_-) \left( \frac{\partial^2 h}{\partial \sigma \partial \phi} \psi_+ \chi_+ - \frac{\partial^2 h}{\partial \sigma \partial \phi} \bar{\psi}_+ \bar{\chi}_+ \right) -$$

$$-\frac{1}{h} \left| \frac{\partial h}{\partial \sigma} \bar{\chi}_- \psi_+ - \frac{\partial h}{\partial \phi} \bar{\psi}_+ \psi_- + \frac{\partial h}{\partial \sigma} \bar{\chi}_+ \psi_- - \frac{\partial h}{\partial \phi} \bar{\psi}_+ \bar{\chi}_- \right|^2$$

$$-\frac{1}{h} \left| \frac{\partial h}{\partial \sigma} \bar{\chi}_- \psi_+ + \frac{\partial h}{\partial \phi} \psi_+ \psi_- - \frac{\partial h}{\partial \sigma} \bar{\psi}_+ \bar{\chi}_+ - \frac{\partial h}{\partial \phi} \bar{\psi}_+ \bar{\chi}_+ \right|^2 . \quad (3.39)$$

By the same token as in the DE model discussed in the previous section, the symmetry (2.12) brings about two extra supersymmetries mixing $\sigma$ with $\chi$ and $\phi$ with $\psi$ on top of two manifest supersymmetries mixing $\sigma$ with $\psi$ and $\phi$ with $\chi$. 

12
The metric \( h \) was fixed in Eqs. (3.25), (2.19), (2.20). There is a freedom in the choice of the prepotential: two functions \( K \) and \( K' \) related as

\[
K' = K + f(\bar{\sigma}, \phi) + \bar{f}(\sigma, \bar{\phi}) + g(\sigma, \phi) + \bar{g}(\bar{\sigma}, \bar{\phi}) \tag{3.40}
\]
define one and the same theory (adding \( f + \bar{f} \) leaves \( \mathcal{L} \) invariant while adding \( g + \bar{g} \) changes it by a total derivative). One of the possible choices for \( K \) is \([15, 13]\)

\[
K = \frac{\Sigma \Sigma - \bar{\Phi} \Phi}{4} + \frac{c}{8\pi} \left[ F \left( \frac{\Sigma \Sigma}{\Phi \Phi} \right) - \ln \Phi \ln \bar{\Phi} \right], \tag{3.41}
\]

where

\[
F(\eta) = \int_{\eta}^{1} \frac{\ln(1 + \xi)}{\xi} \, d\xi \tag{3.42}
\]
is the Spence function. Substituting it in \( \mathcal{L}^{\text{bos}}_{\text{kin}} \), we obtain

\[
\mathcal{L}^{\text{bos}}_{\text{kin}} = \left[ 1 + \frac{c}{2\pi(\bar{\sigma} \sigma + \bar{\phi} \phi)} \right] \left[ |\partial_\alpha \phi|^2 + |\partial_\alpha \sigma|^2 \right] - \frac{c}{2\pi(\bar{\sigma} \sigma + \bar{\phi} \phi)} \left[ \frac{\sigma}{\phi} \epsilon_{\alpha \beta} (\partial_\alpha \bar{\sigma})(\partial_\beta \bar{\phi}) + \frac{\bar{\sigma}}{\phi} \epsilon_{\alpha \beta} (\partial_\alpha \sigma)(\partial_\beta \phi) \right]. \tag{3.43}
\]

The twisted term is a 2–form \( F \). Its external derivative \( dF \) can be associated with the torsion (the freedom (3.40) of choice of \( K \) corresponds to adding to \( F \) the external derivative of the 1-form “organized” from the functions \( f, \bar{f}, g, \bar{g} \). The torsion is invariant under such a change.) Now, \( F \) is self-dual, \( F = F^* \) (with the convention \( \sigma = (x + iy)/\sqrt{2} \), \( \phi = (z + it)/\sqrt{2} \)). One can observe that the “action” \( \int F \wedge F^* \) diverges logarithmically.  

It is clear that the Lagrangians (3.33), (3.36) on one hand and (2.8), (2.14) on the other hand are closely related, like the superfields (3.32) and (2.7) are. Of course, (2.14) is not obtained from (3.36) by a trivial dimensional reduction: the degrees of freedom counting is different, etc. The relationship is established in the same way as in the \( \mathcal{N} = 1 \) case [5]: one should take the functional integral with the Lagrangian (2.14) and perform the integration

\[\]
over $\prod_t dA_3(t)$. After that the metric is transformed as in Eq. (3.25), the terms involving the derivatives with respect to $A_3$ disappear and the 4–fermion term is transformed as

$$\mathcal{L}_{4f} \rightarrow \mathcal{L}_{4f} + \frac{(\partial_j h)(\partial_K h)}{8h} \bar{\eta} \sigma_{j3} \eta \bar{\eta} \sigma_{K3} \eta . \quad (3.44)$$

The QM Lagrangian thus obtained coincides with the Lagrangian (3.36) where all spatial derivatives (and thereby the twisted bosonic term) are suppressed, $\partial_\pm \to \partial_t$.

Consider now a generic non–Abelian case. For a simple Lie group of rank $r$, the effective Lagrangian is

$$\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \mathcal{K}(\bar{\Phi}^a, \Phi^a; \bar{\Sigma}^a, \Sigma^a) , \quad (3.45)$$

where $a = 1, \ldots, r$ and the expression for $\mathcal{K}$ is derived exactly in the same way as for the SQM model of the previous section [see Eq.(2.22)]. We have

$$\mathcal{K} = \sum_j \left\{ \frac{1}{2c_V} \left[ \bar{\Sigma}^{(j)} \Sigma^{(j)} - \bar{\Phi}^{(j)} \Phi^{(j)} \right] 
- \frac{g^2}{8\pi} \left[ F \left( \frac{\Sigma^{(j)}}{\bar{\Phi}^{(j)}} \frac{\Sigma^{(j)}}{\Phi^{(j)}} \right) - \ln \Phi^{(j)} \ln \bar{\Phi}^{(j)} \right] \right\} , \quad (3.46)$$

where $\Sigma^{(j)} = \alpha_j(\Sigma^a)$, etc. The prepotential (3.46) satisfies a generalized harmonicity condition

$$\frac{\partial^2 \mathcal{K}}{\partial \Sigma^a \partial \Sigma^b} + \frac{\partial^2 \mathcal{K}}{\partial \Phi^a \partial \bar{\Phi}^b} = 0 \quad (3.47)$$

for all $a, b$.

4 Discussion

The results obtained in this paper are closely parallel to the well-known results derived earlier in Refs.[16, 17]. In [16] the effective Lagrangian for the 4D $\mathcal{N} = 2$ SYM theory was constructed. It involved $r$ different Abelian gauge fields $V^a$, the moduli complex variables $\phi^a$, and their superpartners. The Lagrangian is exact as far as the quadratic in derivatives terms are concerned.
The same program was partly carried out for the $\mathcal{N} = 2$ SYM theory with one spatial dimension compactified [17]. The Lagrangian represents a complicated hyper–Kählerian $\sigma$ model. Again, it is exact when higher derivative terms are disregarded.

We have solved here the same problem, but for the theories with two and three spatial dimensions compactified. Our results are exact in the same sense as above. As we have shown, the effective $(0 + 1)$ and $(1 + 1)$ models are closely related. It would be interesting to explore their relationship to $(3 + 1)$ and $(2 + 1)$ models in more details. But one difference is already seen. The effective Seiberg–Witten Lagrangian not only takes into account the one loop renormalization of the effective charge (at this level, the relationship was explored back in [5]), but also sums up nontrivial multi-instanton effects. No trace of these nonperturbative effects is left in $(1+1)$ and in $(0+1)$ dimensions.

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