Further validation to the variational method to obtain flow relations for generalized Newtonian fluids

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Abstract

We continue our investigation to the use of the variational method to derive flow relations for generalized Newtonian fluids in confined geometries. While in the previous investigation we used the straight cylindrical tube geometry with six fluid rheological models to demonstrate and establish the variational method, the focus here is on the plane long thin slit geometry using two rheological models. We demonstrate how the variational principle based on minimizing the total stress in the flow conduit can be used to derive analytical expressions that are previously derived by other methods. Although the examples themselves are of little value, the optimization principle which the variational method is based upon has a significant theoretical value as it reveals the tendency of the flow system to assume a configuration that minimizes the total stress. Our proposal also offers a new methodology to tackle common problems in fluid dynamics and rheology.

Keywords: Euler-Lagrange variational principle; fluid mechanics; rheology; generalized Newtonian fluid; slit flow; pressure-flow rate relation; Newtonian; power law.

1 Introduction

The flow of Newtonian and non-Newtonian fluids in various confined geometries, such as tubes and slits, is commonplace in many natural and technological systems. Hence, many methods have been proposed and developed to solve the flow problems in such geometries applying different physical principles and employing a diverse collection of analytical, empirical and numerical techniques. These methods range from employing the first principles of fluid dynamics which are based on the rules of classical mechanics to more specialized techniques such as the use of Weissenberg-Rabinowitsch-Mooney relation or one of the Navier-Stokes adaptations [1, 2].
One of the elegant mathematical branches that is regularly employed in the physical sciences is the calculus of variation which is based on optimizing functionals that describe certain physical phenomena. The variational method is widely used in many disciplines of theoretical and applied sciences, such as quantum mechanics and statistical physics, as well as many fields of engineering. Apart from its mathematical beauty, the method has a big advantage over many other competing methods by giving an insight into the investigated phenomena. The method does not only solve the problem formally and hence provides a mathematical solution but it also reveals the Nature habits and its inclination to economize or lavish on one of the involved physical attributes or the other such as time, speed, entropy and energy. Some of the well known examples that are based on the variational principle or derived from the variational method are the Fermat principle of least time and the curve of fastest descent (brachistochrone). These examples, among many other variational examples, have played a significant role in the development of the modern natural sciences and mathematical methods.

In reference [3] we made an attempt to exploit the variational method to obtain analytic or numeric relations for the flow of generalized Newtonian fluids in confined geometries where we postulated that the flow profile in a flow conduit will adjust itself to minimize the total stress. In the above reference, the flow of six fluid models (Newtonian, power law, Bingham, Herschel-Bulkley, Carreau and Cross) in straight cylindrical tubes was investigated analytically and/or numerically with some of these models confirming the stated variational hypothesis while others, due to mathematical difficulties or limitation of the underlying principle, demonstrated behavioral trends that are consistent with the variational hypothesis.

No mathematically rigorous proof was gives in [3] to establish the proposed variational method that is based on minimizing the total stress in its generality. Furthermore, we do not make any attempt here to present such a proof. However, in the present paper we make an attempt to consolidate our previous proposal and findings by giving more examples, this time from the slit geometry rather than the tube geometry, to validate the use of the variational principle in deriving flow relations in confined geometries for generalized Newtonian fluids.

2 Method

The rheological behavior of generalized Newtonian fluids in one dimensional shear flow is described by the following constitutive relation
\[ \tau = \mu \gamma \]  

where \( \tau \) is the shear stress, \( \gamma \) is the rate of shear strain, and \( \mu \) is the shear viscosity which is normally a function of the rate of shear strain but not of the deformation history although it may also be a function of other physical parameters such as temperature and pressure. The latter parameters are not considered in the present investigation as we assume a static physical setting (i.e. isothermal, isobaric, etc.) apart from the purely kinematical aspects of the deformation process that is necessary to initiate and sustain the flow.

In the following we use the slit geometry, depicted in Figure 1, as our flow apparatus where \( 2B \) is the slit thickness, \( L \) is the length of the slit across which a pressure drop \( P \) is imposed, and \( W \) is the part of the slit width that is under consideration although for the purpose of eliminating lateral edge effects we assume that the total width of the slit is much larger than the considered part \( W \).

![Figure 1: Schematic drawing of the slit geometry which is used in the present investigation.](image)

For the slit geometry of Figure 1 the total stress is given by
\[
\tau_t = \int_{-B}^{+B} \! d\tau = \int_{-B}^{+B} \frac{d\tau}{dz} \! dz = \int_{-B}^{+B} \left( \mu \gamma \right) \! dz = \int_{-B}^{+B} \left( \gamma \frac{d\mu}{dz} + \mu \frac{d\gamma}{dz} \right) \! dz \quad (2)
\]

where \( \tau_t \) is the total stress, and \( \tau_{\pm B} \) is the shear stress at the slit walls corresponding to \( z = \pm B \).

The total stress, as given by Equation 2, can be minimized by applying the Euler-Lagrange variational principle which, in one of its forms, is given by

\[
\frac{d}{dx} \left( f - y' \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial x} = 0 \quad (3)
\]

where the symbols corresponding to our problem statement are defined as

\[
x \equiv z, \quad y \equiv \gamma, \quad f \equiv \gamma \frac{d\mu}{dz} + \mu \frac{d\gamma}{dz}, \quad \text{and} \quad \frac{\partial f}{\partial y'} \equiv \frac{\partial}{\partial \gamma'} \left( \gamma \frac{d\mu}{dz} + \mu \frac{d\gamma}{dz} \right) = \mu \quad (4)
\]

On substituting these symbols into Equation 3 the following equation is obtained

\[
\frac{d}{dz} \left( \gamma \frac{d\mu}{dz} + \mu \frac{d\gamma}{dz} - \gamma \frac{d\mu}{dz} - \mu \frac{d\gamma}{dz} \right) - \frac{\partial}{\partial z} \left( \gamma \frac{d\mu}{dz} + \mu \frac{d\gamma}{dz} \right) = 0 \quad (5)
\]

i.e.

\[
\frac{d}{dz} \left( \gamma \frac{d\mu}{dz} - \mu \frac{d\gamma}{dz} \right) - \frac{\partial}{\partial z} \left( \gamma \frac{d\mu}{dz} + \mu \frac{d\gamma}{dz} \right) = 0 \quad (6)
\]

Since for the single variable dependency the ordinary derivative is equivalent to the partial derivative, we can write this equation as

\[
\frac{\partial}{\partial z} \left( \gamma \frac{d\mu}{dz} - \gamma \frac{d\mu}{dz} - \mu \frac{d\gamma}{dz} \right) = 0 \quad (7)
\]

that is

\[
\frac{\partial}{\partial z} \left( \mu \frac{d\gamma}{dz} \right) = 0 \quad (8)
\]

3 Validation

In this section we use Equation 8 to derive flow relations correlating the volumetric flow rate \( Q \) through the slit to the pressure drop \( P \) across the slit length \( L \) for two
generalized Newtonian fluids, Newtonian and power law, in the above stated slit geometry. We assume a laminar, incompressible, time-independent, fully-developed, isothermal flow where entry and exit edge effects are negligible. We also assume negligible body forces and a blunt flow velocity profile with a no-shear stationary region at the profile center line which is consistent with the considered type of fluids and flow conditions, i.e. viscous generalized Newtonian fluids in a pressure-driven laminar flow. As for the plane slit geometry, we assume, following what is stated in the literature [2], a long thin slit with \( B \ll W \ll L \) although we believe that some of these conditions are redundant according to our own statement and problem settings.

3.1 Newtonian

The viscosity of Newtonian fluids is constant, that is

\[ \mu = \mu_o \] (9)

and therefore Equation 8 becomes

\[ \frac{\partial}{\partial z} \left( \mu_o \frac{d\gamma}{dz} \right) = 0 \] (10)

On performing the outer integration we obtain

\[ \mu_o \frac{d\gamma}{dz} = A \] (11)

where \( A \) is the constant of integration. On performing the inner integration we obtain

\[ \gamma = \frac{A}{\mu_o} z + C \] (12)

where \( C \) is a second constant of integration.

Now from the no-shear condition at the slit center line \( z = 0 \), \( C \) can be determined, that is

\[ \gamma (z = 0) = 0 \quad \Rightarrow \quad C = 0 \] (13)

Similarly, from the no-slip boundary condition [4] at \( z = \pm B \) which controls the wall shear stress we determine \( A \), i.e.
\[ \tau_{\pm B} = \frac{F_{\perp}}{\sigma_{\parallel}} = \frac{2BWp}{2WL} = \frac{BP}{L} \]  

(14)

where \( \tau_{\pm B} \) is the shear stress at the slit walls, \( F_{\perp} \) is the flow driving force which is normal to the slit cross section in the flow direction, and \( \sigma_{\parallel} \) is the slit wall area which is tangential to the flow direction. Hence

\[ \gamma(z = \pm B) = \frac{\tau_{\pm B}}{\mu_o} = \frac{BP}{\mu_o L} = \frac{AB}{\mu_o} \Rightarrow A = \frac{P}{L} \]  

(15)

Therefore

\[ \gamma(z) = \frac{P}{\mu_o L} z \]  

(16)

On integrating the rate of shear strain with respect to \( z \), the standard parabolic speed profile is obtained, that is

\[
\begin{align*}
  v(z) &= \int dv = \int \frac{dv}{dz} dz = \int \gamma dz = \int \frac{P}{\mu_o L} z^2 + D
\end{align*}
\]

(17)

where \( v(z) \) is the fluid speed at \( z \) in the \( x \) direction and \( D \) is another constant of integration which can be determined from the no-slip at the wall boundary condition, that is

\[ v(z = \pm B) = 0 \Rightarrow D = -\frac{P}{2\mu_o L} B^2 \]  

(18)

that is

\[ v(z) = \frac{-P}{2\mu_o L} (B^2 - z^2) \]  

(19)

where the leading negative sign is justified by the fact that the pressure gradient is opposite in direction to the flow velocity vector. The volumetric flow rate is then obtained by integrating the flow speed profile over the slit cross sectional area in the \( z \) direction, that is

\[
Q = \int_{-B}^{+B} |v| W dz = \frac{2WP}{2\mu_o L} \int_{0}^{+B} (B^2 - z^2) dz = \frac{WP}{\mu_o L} \left[ B^2 z - \frac{z^3}{3} \right]_{0}^{+B}
\]

(20)

that is

\[ Q = \frac{2WB^3P}{3\mu_o L} \]  

(21)
which is the well known volumetric flow rate formula for the flow of Newtonian fluids in a plane long thin slit as obtained by other methods which are not based on the variational principle (refer for instance to Bird et al. [2] Table 4.5-14 where \( \mu_o \equiv \mu \) and \( P \equiv P_0 - P_L \).

### 3.2 Power Law

The shear dependent viscosity of power law fluids is given by

\[
\mu = k\gamma^{n-1}
\]

where \( k \) is the power law consistency coefficient and \( n \) is the flow behavior index. On applying the Euler-Lagrange variational principle (Equation 8) we obtain

\[
\frac{\partial}{\partial z} \left( k\gamma^{n-1} \frac{d\gamma}{dz} \right) = 0
\]

On performing the outer integral we obtain

\[
k\gamma^{n-1} \frac{d\gamma}{dz} = A
\]

On separating the two variables in the last equation and integrating both sides we obtain

\[
\gamma = \sqrt[n]{\frac{n}{k} (Az + C)}
\]

where \( A \) and \( C \) are the constants of integration which can be determined from the two limiting conditions, that is

\[
\gamma (z = 0) = 0 \quad \Rightarrow \quad C = 0
\]

and

\[
\gamma (z = B) = \sqrt{\frac{\tau_B}{k}} = \sqrt{\frac{BP}{Lk}} = \sqrt{\frac{n}{k} AB} \quad \Rightarrow \quad A = \frac{P}{nL}
\]

where the first step in the last equation is obtained from the constitutive relation of power law fluids, i.e.

\[
\tau = k\gamma^n
\]

with the substitution \( z = B \) in Equation 25. Hence, from Equation 25 we obtain
\[ \gamma = \sqrt[n]{\frac{P}{kL} z^{1/n}} \quad (29) \]

On integrating the rate of shear strain with respect to \( z \), the flow speed profile is determined, i.e.

\[ v(z) = \int dv = \int \frac{dv}{dz} dz = \int \int \sqrt[n]{\frac{P}{kL} z^{1/n}} dz = \frac{n}{n+1} \sqrt[n]{\frac{P}{kL} z^{1+1/n} + D} \quad (30) \]

where \( D \) is another constant of integration which can be determined from the no-slip at the wall condition, that is

\[ v(z = B) = 0 \quad \Rightarrow \quad D = -\frac{n}{n+1} \sqrt[n]{\frac{P}{kL} B^{1+1/n}} \quad (31) \]

i.e.

\[ v(z) = -\frac{n}{n+1} \sqrt[n]{\frac{P}{kL}} (B^{1+1/n} - z^{1+1/n}) \quad (32) \]

The volumetric flow rate can then be obtained by integrating the flow speed profile with respect to the cross sectional area in the \( z \) direction, that is

\[ Q = \int_{-B}^{+B} |v| W dz = \frac{2Wn}{n+1} \sqrt[n]{\frac{P}{kL}} \int_{0}^{+B} (B^{1+1/n} - z^{1+1/n}) dz \quad (33) \]

\[ = \frac{2Wn}{n+1} \sqrt[n]{\frac{P}{kL}} \left[ B^{1+1/n}z - \frac{z^{2+1/n}}{2 + 1/n} \right]_{0}^{+B} \quad (34) \]

\[ = \frac{2Wn}{n+1} \sqrt[n]{\frac{P}{kL}} \left[ B^{2+1/n} - \frac{B^{2+1/n}}{2 + 1/n} \right] \quad (35) \]

i.e.

\[ Q = \frac{2WB^{2/n}}{2n + 1} \sqrt[n]{\frac{PB}{kL}} \quad (36) \]

which is the well known volumetric flow rate relation for the flow of power law fluids in a long thin slit as obtained by other non-variational methods (refer for instance to Bird et al. [2] Table 4.2-1 where \( k \equiv m \) and \( P \equiv P_0 - P_L \)).
4 Conclusions

In this paper we provided further evidence for the validity of the variational principle which is based on minimizing the total stress in the flow conduit to obtain flow relations for the generalized Newtonian fluids in confined geometries. Our investigation in the present paper to the plane long thin slit geometry confirms our previous findings which were established using the straight cylindrical tube geometry.

Although the derived expressions are not of interest of their own as they have already been obtained from other non-variational methods, the theoretical aspect of our investigation should be of great interest as it reveals a new science that is the tendency of the flow system to minimize the total stress which the variational principle is based upon; hence giving an insight into the underlying physical principles that control the flow of fluids.

The value of our investigation is not limited to the above mentioned theoretical aspect but it has a practical aspect as well since the variational method can be used as an alternative to other methods for other geometries and other rheological fluid models where mathematical difficulties may be overcome in one formulation based on one of these methods but not the others. The variational method is also more general and hence enjoys a wider applicability than some of the other methods which are based on more special or restrictive physical or mathematical principles.

5 Nomenclature

\[\gamma\] rate of shear strain (s\(^{-1}\))
\[\mu\] fluid shear viscosity (Pa.s)
\[\sigma_{||}\] slit wall area tangential to the flow direction (m\(^2\))
\[\tau\] shear stress (Pa)
\[\tau_{\pm B}\] shear stress at slit walls corresponding to \(z = \pm B\) (Pa)
\[\tau_t\] total shear stress (Pa)

\[B\] slit half thickness (m)
\[F_{\perp}\] force normal to the slit cross section in the flow direction (N)
\[k\] consistency coefficient in the power law model (Pa.s\(^n\))
\[L\] slit length (m)
$n$  flow behavior index in the power law model
$P$  pressure drop across the slit length (Pa)
$P_0$  pressure at the slit entrance (Pa)
$P_L$  pressure at the slit exit (Pa)
$Q$  volumetric flow rate (m$^3$.s$^{-1}$)
$v$  fluid speed (m.s$^{-1}$)
$W$  slit width (m)

References

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