Vortex Lattice in Planar Bose-Einstein Condensates with Dipolar Interactions

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In this letter we investigate the effects of dipole-dipole interactions on the vortex lattices in fast rotating Bose-Einstein condensates. For single planar condensate, we show that the triangular lattice structure will be unfavorable when the s-wave interaction is attractive and exceeds a critical value. It will first change to a square lattice, and then become more and more flat with the increase of s-wave attraction, until the collapse of the condensate. For an array of coupled planar condensates, we discuss how the dipole-dipole interactions between neighboring condensates compete with the quantum tunneling processes, which affects the relative displacement of two neighboring vortex lattices and leads to the loss of phase coherence between different condensates.

This work contains two parts. First, we consider an isolated planar condensate under fast rotation. When the rotation frequency is very close to the trapping frequency, the size of vortex core becomes comparable to the spacing between vortices, and its effect becomes important. Therefore the repulsive s-wave interaction favors a triangular vortex lattice where the vortex cores are arranged in a most symmetric way so that the modulation of the condensate density is spatially homogenous, while the attractive interaction favors the lattice structure such as the square lattice and the flat lattice, which are less symmetric than the triangular lattice. However, in the absence of the dipolar interactions, such lattices can not exist because the condensate itself collapses when the s-wave interaction becomes attractive. The presence of dipolar interaction prevents the condensate from collapse for small s-wave attraction and thus the effect of attractive interaction on the lattice structure becomes observable. Here we are going to show that a triangular lattice will be unfavorable in the parameter region that the strength...
of s-wave attraction exceeds a critical value but is not enough to induce collapse. With the increase of s-wave attraction, the vortex lattice will experience a structure transition from triangular to square, and then to very flat lattice as shown in Fig. 1b).

Secondly, we study the effects of the coupling between the neighboring condensates. In the situation that the quantum tunneling is dominative, the condensates in different sites are coherent, and the positions of vortices in different layers are coincident with each other (see the left one of Fig. 1c). However, because the inter-layer dipolar interacting energy will be reduced when the high density region of one condensate coincides with the low density region of its neighboring condensate, it favors that the vortex lattices are stagger-arranged among the lattices (see the right one of Fig. 1c). Here we will show that the competition between the quantum tunneling and the inter-layer dipolar interactions will lead to a quantum phase transition from the coincident phase to the staggered phase. Moreover, the tunneling amplitude will be suppressed exponentially when the vortex lattices are staggered, and consequently the fluctuation of relative phases between neighboring condensates will be enhanced. Therefore, this quantum phase transition is characterized by the loss of phase coherence between sites, in analogy with the superfluid-Mott insulator transition.

Model and Method. We consider a two-dimensional BEC, which is trapped by a harmonic potential $m\omega^2 r^2/2$ in the $xy$ plane and fast rotated with fixed frequency $\Omega$, the system is described by following energy functional

$$K = \int d^2 r \Psi^* (r) \hat{H}_0 \Psi (r) + \int d^2 r d^2 r' \rho (r) V (r - r') \rho (r'), \tag{1}$$

where $\Psi (r)$ is the macroscopic wave function of the condensate, $\rho (r)$ is the condensate density, and $V (r - r')$ is the interactions between atoms which include both the s-wave contact interaction and the magnetic dipolar interaction. The single particle Hamiltonian $\hat{H}_0$ is written as $-\hbar^2 \nabla^2 / (2m) + m\omega^2 r^2 / 2 - \Omega L$, whose eigenstates are Landau levels in the fast rotating limit where $\Omega$ is very close to $\omega$, and the interaction energy is much weaker than the spacing between different Landau levels $2\hbar \omega$ when the condensate density is very low, therefore we can restrict ourselves in the lowest Landau level (LLL) $|1\rangle$. When the vortices form a uniform lattice, the wave function $\Psi (r)$, excluding the Gaussian factor, is a double-periodic analytical function, and can be uniquely given by the Jacobi theta function $\theta_0$. Although the Gaussian profile of the global density is unstable with respect to very weak lattice distortion, the LLL wave function serves very well as a trial wave function to determine the vortex lattice structure. For example, this wave function has been successfully used to explain the dynamic formation of vortex stripe observed in a recent experiment and to predict the structure transition of vortex lattice from triangular to square in a two-component condensate, which has been observed later.

By denoting $b_1$ and $b_2$ as the basis vectors of the lattice, a uniform vortex lattice in the LLL is characterized by two parameters, which are $b_2/b_1 = u + iv$ describing the lattice type and $v_\perp = b_2^2/2$ denoting the area of a unit cell. The condensate density takes the form as $\rho (r) = 1/(\pi \sigma^2) \sum_{\bf k} g_{\bf k} \exp (i\bf{K} \cdot \bf{r}) \exp (-r^2/\sigma^2)^2 |\bf{k}| \exp (-r^2/2\sigma^2)$, where the summation is taken over all reciprocal lattice vectors $m_1 \bf{K}_1 + m_2 \bf{K}_2$, with $\bf{K}_1$ and $\bf{K}_2$ being the basis of the reciprocal lattice. Here $\sigma$ is the condensate radius which is given by $\sqrt{(a_\perp^2 - \pi v_\perp)^{-1}}$ where $a_\perp = \sqrt{\hbar/m \omega}$. The coefficients $g_{\bf k}$ denotes $g_{\bf k} g_{\bf k}/g_0$, and $g_{\bf k}$ is explicitly written as $g_{\bf k} = (-1)^{m_1 + m_2 + m_3} e^{-v_\perp |\bf{k}|^2/8\pi \sigma^2}$. With $v_\perp = (2\pi)^2 v_\perp^{-1} [(v_{m_1} + (m_2 + v_{m_1})^2]$. In the LLL region the mean value of the single particle Hamiltonian $E_0$ turns out to be $h(\omega_\perp - \Omega) \sigma^2 / 14$, which is independent of the vortex lattice structure. The energy of s-wave interaction, $V_s (r - r') = g_s (r - r')$, can also be evaluated as $E_s = g \int d^2 r \rho^2 (r) \approx g \int (\pi \sigma^2)^2$, where $I$ denotes the summation of $\sum_{\bf k} |g_{\bf k}| g_{\bf k}/g_0^2$. When the s-wave interaction is repulsive, i.e. $g > 0$, we can find that the triangular lattice is stable by minimizing $E_0$, and the minimization of the total energy, including $E_0$ and $E_s$, yields a finite value of condensate radius $\sigma$. However, when $g$ is negative, the minimization of the total energy results in $\sigma$ vanishing, which implies the collapse of the condensate.

Single Planar Condensate. Within the LLL mean field ansatz, we are going to investigate how the dipolar interactions affect the structure of vortex lattice of a planar condensate. Considering a very deep lattice, at each lattice site the motion of atoms along $z$ direction is strongly confined, and the magnetic dipole moment $\mu$ of atoms are also polarized along $z$ axis, the magnetic dipolar interaction in $xy$ plane reads

$$V_d (r_1, r_2) = \frac{\mu_0 \mu^2}{4\pi} \frac{1}{|r_1 - r_2|^3}, \tag{2}$$

and the dipolar interaction energy $E_d$ is given by $\int d^2 r_1 d^2 r_2 \rho (r_1) V_d (r_1 - r_2) \rho (r_2)$. Denoting the relative displacement $r = r_1 - r_2$, and the center of mass displacement $R = (r_1 + r_2)/2$, we can first integrate $R$ out and $E_d$ turns out to be

$$E_d = \frac{\mu_0 \mu^2}{8\pi \sigma^2 \sigma'} \sum_{\bf K} \left( \left| g_{\bf k} / g_0 \right|^2 \int d^2 r \frac{1}{r^3} e^{-\left(2\sigma^2 \right)} \right). \tag{3}$$

This integral equals to $2\pi \int d\sigma J_0 (K r) r^2 / (2\sigma^2)^2 r^2$, which has shot-range divergence and therefore requires a short-range cut-off $\Lambda$. Here $\Lambda$ should be choose as $4a_z$ where $a_z$ is the longitudinal harmonic length of each lattice site. It is important to notice that a dimension-
less value $W$ defined as below is regular, i.e.

$$W = \sum_k \frac{2\mathbf{K}}{g_0} \int d\vec{r} \frac{1}{\pi^2} \left[ J_0(K\tilde{r}_a) e^{-\tilde{r}^2 a_\perp^2/(2\sigma^2)} - 1 \right].$$

(4)

with $\tilde{r} = r/a_\perp$, and the remainder term depending on $\Lambda$ is proportional to $I$, thus the total dipolar energy of a rotating condensate can be viewed as an isotropic and locally repulsive part plus an attractive part, because $W$ is always negative. The total interaction energy including the dipolar and $s$-wave interaction turns out to be

$$E_{\text{int}} = \frac{1}{\pi\sigma^2} \left[ \left( \frac{\mu_0\mu^2}{4\Lambda} + g \right) I + \frac{\mu_0\mu^2}{4a_\perp}W \right].$$

(5)

where $\alpha$ are $a_\perp/\Lambda + 4ga_\perp/(\mu_0\mu^2)$ which can be gradually changed in a wide range because the $s$-wave interaction constant $g$ can be tuned via Feshbach resonance technique. The structure of vortex lattice in equilibrium can be obtained by minimizing $E_{\text{int}}$, and the collapse of condensate occurs when $E_{\text{int}}$ becomes negative.

The results of the minimization of the mean field energy are shown in Fig. 2. In the regime that the repulsive interaction is dominant, the ground state is a triangular lattice with $\theta_{\text{min}} = \pi/3$ and $|\mathbf{b}_2/\mathbf{b}_1| = 1$. When $\alpha$ decreases below 2.0, the triangular lattice becomes unstable and will be replaced by the square lattice, which can be seen from Fig. 2 where the energy at $\theta = \pi/2$ is lower than $\theta = \pi/3$. Furthermore, the energy takes its minimum at the place where $|\mathbf{b}_2/\mathbf{b}_1|$ is smaller than unity when $\alpha$ is smaller than 1.9, as shown in Fig. 2. From Fig. 2(a) one can see that the vortex lattice changes from a triangular one to a square one, and then becomes more and more flat until the collapse of the whole condensate, as the $s$-wave attraction increases. In this calculation, we take $\Lambda/a_\perp = 0.1$ which is a typical values of current experiments, for chromium the square lattice occurs when $a_o$ is between $-2.32a_o$ and $-2.28a_o$, and the flat lattice occurs when $a_o$ is between $-2.55a_o$ and $-2.32a_o$, where $a_o = 0.0529$nm.

**Coupled Planar Condensates.** Now we are going to study the effects of coupling between nest-neighboring condensates. Provided that the energy scale of the inter-layer coupling processes is much smaller than the intra-layer energy scale, the structure of vortex lattice will not be affected by the inter-layer coupling. Here we are interested in how the inter-layer dipolar interactions affect the relative displacement of vortex lattices in different layers, and therefore we focus on a typical case of triangular vortex lattice. Neglecting the $s$-wave interaction between layers, the inter-layer interaction is

$$V_d'(r_1, r_2) = \frac{\mu_0\mu^2}{4\pi} \frac{2d^2 - r^2}{(d^2 + r^2)^{3/2}}.$$  

(6)

where $d$ is the distance between two layers, and $r = r_1 - r_2$ denotes the relative displacement in the $xy$ plane. In the large vortex number limit the inter-layer interaction energy turns out to be

$$E_d' = \frac{\mu_0\mu^2}{4\pi\sigma^2 d} \sum K \left( \frac{g_K}{g_0} \right)^2 e^{iK \cdot r_0} F(|K|, d)$$

(7)

where $F(|K|, d)$ denotes a dimensionless integral

$$F(|K|, d) = \int \frac{d\tilde{r}}{\pi^2} J_0(|K|\tilde{r}) e^{-\tilde{r}^2 a^2/(2\sigma^2)} \frac{\tilde{r}(2 - \tilde{r}^2)}{(1 + \tilde{r}^2)^{3/2}}.$$  

(8)

with $\tilde{r} = r/d$, and $r_0$ is the relative displacement between two neighboring vortex lattices. Because $|g_K|$ exponentially decreases as the increase of $|K|^2$, it is sufficient to keep the terms with small value of $|K|$ in the summation, namely, $K_0 = 0$ and $K_1 = |K_{\perp \pm 1}| = |K_{\pm 1, \pm 1}| = |K_{\pm 1, \mp 1}|$ for the triangular lattice. Thus

$$E_d' = \frac{\mu_0\mu^2}{4\pi\sigma^2 d} \left( F(K_0, d) + 2CF(K_1, d) \sum' \cos(K \cdot r_0) \right).$$

(9)

Here both $F(K_0, d)$ and $F(K_1, d)$ are positive, the constant $C$ denotes the value of $|g_{K_{1,0}}/g_0|^2 (= |g_{K_{1,1}}/g_0|^2 = |g_{K_{1,-1}}/g_0|^2)$ which equals to 0.0532 for the triangular lattice, and $\sum'$ denotes the summation over $K_{1,0}, K_{0,1}$.
and \(K_{1,-1}\). Taking the symmetry of the triangular lattice into account, we set \(\mathbf{r}_0 = x(\mathbf{b}_1 + \mathbf{b}_2)\) with \(x \in [0, 1/2]\), \(\mathbf{E}_d\) takes its minimum at \(x = 1/3\), which means that the vortex cores of one lattice plate to the center of unity cells of its neighboring lattice.

Another inter-layer process is the quantum tunneling between neighboring condensates. Within the LLL mean field theory, the tunnelling energy can be expressed as \(E_t = -t \cos \phi \exp(-N_v \pi r_0^2/v_c)/2\) where \(N_v\) is the number of vortices, \(t\) is the hopping amplitude between two neighboring sites, and \(\phi\) is the relative phase between two neighboring condensates. This term favors \(\mathbf{r}_0 = \mathbf{0}\), which corresponds to binding the vortices in different layers at the same positions in the \(xy\) plane. Therefore the total energy of the processes coupling two neighboring condensates, \(E_{\text{cou}}\), includes \(E_d\) and \(E_t\). By minimizing \(E_{\text{cou}}\), it can be found that the competition between these two processes will result in a transition of the ground state. Two typical curves of \(E_{\text{cou}}\) are shown in Fig. \(\text{E}\) (a) and (b), in the case (a) the vortex lattices are coincident and in the case (b) they are staggered. As illustrated in Fig. \(\text{F}\) (c), this transition is driven by the parameter \(\beta\) defined as 
\[
\beta = \frac{\mu_0 \mu/(10\pi \sigma^2 a_{\perp})}{\mu_0 \mu/(10\pi \sigma^2 a_{\perp})}.
\]

It is worthwhile to consider the relative phase fluctuations between two neighboring condensates. Because \(\delta \phi\) is proportional to \(1/\sqrt{t \exp(-N_v \pi r_0^2/v_c)}\), the denominator of which will be suppressed to a exponentially small value when \(r_0^2\) becomes comparable to \(v_c\), in the staggered phase, the transition of \(\mathbf{r}_0\) is accompanied by the loss of phase coherence between different layers and the suppression of particle number fluctuations in each site. In this sense, this transition is similar to the superfluid-Mott insulator transition. However, a peculiar point is that this transition is driven by nearest-neighbor interactions instead of the on-site interactions.

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**Note added.** After finishing an initial version of this paper, we became aware of the work by Cooper, Rezayi and Simon in which the vortex lattice structure of a planar dipolar condensate was also discussed.

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