It is well known that in a generally covariant gravitational theory the choice of spacetime scalars as coordinates yields phase-space observables (or "invariants"). However their relation to the symmetry group of diffeomorphism transformations has remained obscure. In a symmetry-inspired approach we construct invariants out of canonically induced active gauge transformations. These invariants may be interpreted as the full set of dynamical variables evaluated in the intrinsic coordinate system. The functional invariants can explicitly be written as a Taylor expansion in the coordinates of any observer, and the coefficients have a physical and geometrical interpretation. Surprisingly, all invariants can be obtained as limits of a family of canonical transformations. This permits a short (again geometric) proof that all invariants, including the lapse and shift, satisfy Poisson brackets that are equal to the invariants of their corresponding Dirac brackets.

Precisely as a consequence of the general covariance of Einstein’s theory of gravity, spatial and temporal coordinates possess no \textit{a priori} physical significance. Consequently it has long been recognized that physically observable quantities in a Hamiltonian formulation of general relativity must be insensitive to transformations from any one set of coordinates to any other set. However, procedures that have been proposed in the literature to date for constructing observable quantities in generally covariant theories suffer a major flaw. The procedures do not establish a link to the full group of underlying diffeomorphism symmetries, including transformations that alter the time coordinate. Consequently, although the resulting phase space functionals that have been proposed are indeed invariant under transformations generated by the first class constraints of the theory, it has not been clear what relationship these constraints bear to the original symmetry group.

Initial steps in elucidating this relationship were undertaken by Pons, Salisbury, and Shepley\textsuperscript{[1]}. It was pointed out that there is a realizable symmetry group in the Hamiltonian formulation of general relativity that is induced by the original diffeomorphism group. Yet every transformation of the metric field and its time derivatives that results from the original group is faithfully represented by the induced symmetry group. Pons and Salisbury\textsuperscript{[2]} suggested that this group could be employed to perform a transformation to intrinsic coordinates, defined through gauge conditions in terms of appropriately selected spacetime scalar fields. The resulting invariants are then simply correlations between the values of these scalars and the remaining phase space variables of the theory\textsuperscript{[2]}. Furthermore, invariants are nothing other than the full set of dynamical variables evaluated in the intrinsic coordinate system, including the conventionally discarded lapse and shift fields.

We follow up here with an explicit general symmetry-based construction of invariants, based on the canonical implementation of intrinsic gauge conditions. We are able to display the invariants as power series in the coordinates. Since time and spatial position are intrinsically defined, they are themselves invariant. The coefficients of the series terms are also all invariants and \textit{ipso facto} constants of the motion. We prove in addition that the invariants so obtained satisfy the Dirac algebra that follows from the intrinsic coordinate-dependent gauge fixing. Finally, we show that all invariants may be obtained as a limit of a family of canonical transformations, thus affording a potentially useful approximation scheme.

The diffeomorphism-induced canonical symmetry group is the group of transformations of metric components and their canonical conjugates that is projectable under the Legendre map from configuration/velocity space to phase space. The corresponding infinitesimal coordinate transformations have a compulsory dependence on the lapse and shift metric components, and due to the group property, a spatially non-local dependence on the spatial components of the metric must also be admitted. Explicitly the coordinate transformations that produce

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projectable variations in the metric are of the form

$$x^\mu = x^\mu - n^\mu(x) 0(g_{ab}; x) - \delta^\mu_\nu \xi^a(g_{bc}; x),$$  \(1\)

where Greek indices are in the range 0, \cdots, 3, and Latin indices have the spatial range 1, \cdots, 3. The “descriptors” \(\xi^a\) are arbitrary functionals. The normal to the fixed time hypersurface is expressed in terms of the lapse field \(N\) and shift fields \(N^a\) as \(n^\mu = (N^{-1}, -N^{-1}N^a)\). The phase space generator of these active transformations is

$$G_\xi(t) = P_\mu \dot{\xi}^\mu + (H_\mu + N^\rho C^\nu_{\rho\sigma} P_\nu) \xi^\sigma.$$

(2)

The \(P_\mu\) are the momenta conjugate to \(N^\mu\), and are primary constraints. The \(H_\mu\) are secondary constraints satisfying the first class closed Poisson bracket condition \(\{H_\mu, H_\nu\} = C^\rho_{\mu\nu} H_\rho\). We employ the convention that repeated indices represent both a sum and an integration over spatial coordinates. As opposed to (3), the Hamiltonian generator of time evolution is \(H = H_\mu N^\rho + P_\mu \Lambda^\rho\), where the \(\Lambda^\rho\) are, excepting for the condition that \(\Lambda^0 > 0\), arbitrary spacetime functions.

As a first step towards the explicit form of the functional invariants we impose an intrinsic coordinate-dependent gauge condition of the form \(\chi^{(1)\mu} := x^\mu - X^\mu(x) = 0\), where the \(X^\mu\) are spacetime scalar functions of the canonical fields. Our task is then to find the canonical transformation that moves the field variables to that location on the gauge orbit where the gauge conditions are satisfied. Once we are in possession of this finite transformation we may employ it to transform all the remaining fields. These actively transformed fields are the invariants that we seek.

We note that preservation of the gauge conditions under temporal evolution leads to additional constraints \(\chi^{(2)\mu} := \delta^\mu_\nu - A^\mu_\nu N^\nu \approx 0\). where we have introduced the matrix \(A^\mu_\nu := \{X^\mu, H_\nu\}\). Following the lead of Henneaux and Teitelboim[9], further exploited by Dittrich[8] and Thiemann[10], we find it convenient to work with linear combinations \(\xi^{(1)\mu}(x)\) of the first class constraints \(\xi^{(1)\mu} := H_\mu, \xi^{(2)\mu} := P_\mu\) having the property that \(\{\xi^{(1)\mu}, \xi^{(2)\nu}\} \approx -\delta^{(1)}_{\mu\nu}\). For this purpose we need the inverse of the matrix \(\{\chi^{(1)\mu}, \chi^{(2)\mu}\}\). The appropriate linear combinations are therefore \(\xi^{(1)}_{\mu} = B^\nu_{\mu} P_\nu\), where \(B^\nu_{\mu}\) is the inverse of \(A^\mu_\nu\), and \(\xi^{(2)}_\mu = B^\nu_{\mu}(H_\mu - B^\nu_{\alpha} N^\alpha(A_\lambda^\beta, H_\rho) P_\rho)\). This new basis exists in general only locally. The gauge generator expressed in terms of this new basis is

$$G_\xi(t) = A^\mu_\nu \xi_\nu + (A^\mu_\nu H_\nu + B^\mu_\rho N^\rho A^\nu_{\sigma}, H_\mu) P_\lambda + N^\rho C^\lambda_{\rho\sigma}(P_\lambda) \xi^\mu.$$  \(3\)

Letting \(\xi = B^\mu_\sigma \xi^\sigma\) we find

$$\dot{\xi}^\mu = \frac{d}{dt}(B^\mu_\sigma \xi^\sigma) = \{B^\mu_\sigma, N^\lambda H_\lambda\} \xi^\sigma + B^\mu_\sigma \xi^\sigma,$$

and substituting into (3) we obtain

$$G_\xi(t) = \xi^\mu + \xi^\mu + P_\mu N^\rho S^\mu_{\rho\sigma} \xi^\sigma.$$  \(4\)

where

$$S^\mu_{\rho\sigma} = \{B_\rho^\nu, H_\sigma\} + B_\rho^\nu B_\gamma^\rho \{A^\gamma_{\sigma}, H_\nu\} + B_\rho^\nu C^\nu_{\rho\sigma}.$$  \(5\)

It turns out that \(S^\mu_{\rho\sigma} = -B^\mu_\rho B_\sigma^\nu \{X^\beta, C^\gamma_{\alpha\beta}\} H_\lambda \overline{\eta} \). Thus since the last term in (4) is quadratic in the constraints, it can be discarded “on-shell” (on the first class constraint hypersurface). We obtain the following simple form for the gauge generator,

$$G_\xi(t) = \xi^\sigma + \xi^\nu + P_\mu N^\rho S^\nu_{\rho\sigma} \xi^\sigma.$$  \(6\)

The finite active gauge transformation of any dynamical field \(\Phi\) takes the form

$$\exp\left\{-\frac{1}{2} G_\xi(t)\right\} \Phi = \Phi + \{\Phi, \xi\} + \frac{1}{2} \{\Phi, \xi\} + \cdots.$$  \(7\)

Since by assumption the scalars \(X^\mu\) do not depend on the lapse and shift the descriptors for the finite gauge transformation that transforms \(X^\mu\) to \(x^\mu\) is easily seen to satisfy, on shell,

$$x^\mu = X^\mu + \xi^\mu.$$  \(8\)

where \(\xi^\mu := A^\mu_\sigma \xi^\sigma\) is considered a function of the space-time coordinates and we have made use of the fact that the Poisson brackets \(\overline{\eta}_\mu\) with themselves vanish “strongly”, i.e., they are proportional to terms at least quadratic in the \(\overline{\eta}_\mu\). The descriptors \(\xi = \chi^{(1)\mu}\) and \(\tilde{\xi} = \chi^{(2)\mu}\) may therefore be substituted into (8), after the Poisson brackets have been computed, to obtain all invariant functionals \(I_\Phi\) associated with the fields \(\Phi\).

Although the following results hold for all fields \(h\) we focus here on fields other than the lapse and shift. Then the explicit expressions for the invariants are

$$I_\Phi = \Phi + \chi^{(1)n} \{\Phi, \xi\} + \frac{1}{2!} \chi^{(2)n} \chi^{(1)n} \{\Phi, \xi\} + \cdots$$  \(9\)

where \(\approx\) is the symbol for Dirac’s weak equality, that is, an equality which holds “on-shell”. The time rate of change of these invariant functionals satisfies

$$\frac{d}{dt} I_\Phi = \frac{\partial}{\partial t} I_\Phi + \{I_\Phi, N^\mu H_\mu\} \approx \frac{\partial}{\partial t} I_\Phi \approx \{\Phi, \overline{\eta}_0\}$$

and substituting into (9) we obtain

$$I_\Phi = \sum_{n=0}^{\infty} \frac{1}{n!} (\chi^{(1)n}) \{\Phi, \overline{\eta}_n\} \{\Phi, \overline{\eta}_n\} \< \cdots$$

where in the third line we used the strong vanishing of the Poisson brackets of the \(\overline{\eta}_n\). But recognizing that the invariant fields are simply the fields at the gauge-fixed point \(p_0\) on the gauge orbit, we note that
\[ I_{\Phi, \overline{\sigma}} = \{ \Phi, \overline{H}_0 \}_{\rho_0} = \{ \Phi, N^\mu \mathcal{H}_\mu \}_{\rho_0}. \]

Therefore the invariants satisfy the equations of motion of the gauge fixed fields. We note also that repeated time derivatives of the invariant fields yield the simple expression \( \frac{\partial^n}{\partial t^n} I_{\Phi} \approx I_{\Phi, \overline{\sigma}} \), and that we may therefore express the invariants as follows as a Taylor series in \( t \):

\[ I_{\Phi} \approx \sum_{n=0}^{\infty} \frac{t^n}{n!} I_{\Phi, \overline{\sigma}}(\sigma) \big|_{t=0}. \tag{10} \]

This expression displays in a striking manner the notion of “evolving constant of the motion” introduced originally by Rovelli\cite{11}. We stress however, that the evolution we obtain here is nothing other than the evolution determined by Einstein’s equations in the gauge-fixed coordinate system. We point out in addition that according to Batte et al\cite{10} the potentially infinite set of invariants \( I_{\Phi, \overline{\sigma}}(\sigma) \) will correspond to Noether symmetries. In generic general relativity the conserved quantities must necessarily be spatially non-local, as pointed out by Torre\cite{11}.

The invariant functionals \( I_{\Phi, \overline{\sigma}}(\sigma) \big|_{t=0} \) have an immediate physical interpretation: They are the \( n \)’th time derivative of \( \Phi \) evaluated at time zero. A similar result holds for spatial derivatives, as we now show. But first we point out that the \( X^\mu \) are to be spacetime scalars, i.e., under the projectable infinitesimal coordinate transformations \( \{ \chi \} \) we require that \( \{ X^\mu, \xi^\nu \mathcal{H}_\nu \} = X^\mu \xi^\nu e^\nu. \) It follows that \( A^\mu_{\alpha} = X^\mu_{\alpha} \), but there are no restrictions on \( A^\mu_{\nu} \). At the gauge fixed point on the gauge orbit we find therefore that \( A^\mu_{\nu} = \delta^\mu_{\nu} \) and \( B^\mu_{\nu} = N^\mu \delta^\nu_{\nu} + \delta^\mu_{\nu} \delta^\nu_{\nu} \). Parallelising for the spatial coordinates the computation in \( \tau \) we find that \( \frac{\partial^n}{\partial t^n} \mathcal{I}_{\Phi} \approx I_{\Phi, \overline{\sigma}} \). But again we recognize that the invariant associated with \( \{ \Phi, \overline{\sigma}_a \} \) is this quantity evaluated where the gauge condition is satisfied. At this point \( \overline{\sigma}_a = B^\mu_{\alpha} \mathcal{H}_\mu = \mathcal{H}_a \), confirming that this invariant is indeed nothing other than the spatial derivative. In addition we may expand in powers of the spatial coordinate \( x^a \), resulting in the interpretation of the associated invariant coefficients of \( n \)’th power of \( x^a \) as the \( n \)’th spatial derivative evaluated at \( x^a = 0 \). All observers end up with the same explicit functions of their own coordinates as does the observer at the gauge fixed location. This essentially is the physical content of the invariants \( \mathcal{I}_{\Phi} \).

The map \( \Phi \rightarrow \mathcal{I}_{\Phi} \) cannot be canonical because it sends all the gauge equivalent configurations to the same configuration, the one satisfying the gauge fixing conditions. But surprisingly, it can be understood as a limit of a one-parameter family of canonical maps. Let us consider the functionals, obtained from canonical maps, \( \mathcal{K}^{(A)} = \exp \left( \{ -, \Lambda x^{(1)} \mathcal{H}_\nu \} \right) \Phi. \) One can prove that, on shell, \( \mathcal{K}^{(A)} \approx \sum_{n=0}^{\infty} \frac{1}{n!} (1 - e^{-\Lambda})^n (x^{(1)})^n \{ \Phi, \mathcal{H}_\nu \}_{(n)} \approx \mathcal{K}^{(A)}, \tag{11} \)

It is remarkable that the same functional \( \mathcal{K}^{(A)} \) exhibits two different power series expansions: the first is just its definition and is in terms of the parameter \( \Lambda \), the second, as in \([11]\), is in terms of \((1 - e^{-\Lambda})\) and is only valid on shell. The important result is that we recover the invariants,

\[ \lim_{\Lambda \rightarrow \infty} \mathcal{K}^{(A)} \approx \mathcal{I}_{\Phi}. \]

Another result of interest is

\[ \{ \mathcal{K}^{(A)} \Phi, \overline{\sigma}_\nu \} \approx (1 - e^{-\Lambda}) \{ \Phi, \mathcal{H}_\nu \} \{ \chi^{(1)}, \chi^{(1)} \}. \tag{12} \]

The functionals \( \mathcal{K}^{(A)} \) can be used to yield a simple proof that the Poisson brackets of the invariants are simply the invariants associated with the Dirac brackets of the fields. The proof proceeds as follows for variables other than the lapse and shift, but can be generalized to include them. The computation of the first order off shell terms of \( \mathcal{K}^{(A)} \) gives

\[ \mathcal{K}^{(A)} = \mathcal{K}^{(A)} + e^\Lambda \gamma^\mu \mathcal{H}_\mu + \mathcal{O}(2), \tag{13} \]

with

\[ \gamma^\mu := (1 - e^{-\Lambda}) \{ \Phi, \chi^{(1)} \} + \frac{(1 - e^{-\Lambda})^2}{2} \{ \Phi, \mathcal{H}_\nu \} \{ \chi^{(1)}, \chi^{(1)} \}. \]

Now we can compute

\[ \mathcal{K}^{(A)} = \mathcal{K}^{(A)} - e^\Lambda \{ \mathcal{K}^{(A)} \mathcal{H}_\mu \} \gamma^\mu_{\Phi \beta} - e^\Lambda \gamma^\mu_{\Phi \beta} \mathcal{K}^{(A)} + \mathcal{O}(\mathcal{H}, \chi) \]

and when we go on shell \( \mathcal{H} = 0 \), using \( \{12\} \),

\[ \mathcal{K}^{(A)} = \mathcal{K}^{(A)} \approx \mathcal{K}^{(A)} - e^\Lambda \mathcal{K}^{(A)} \mathcal{H}_\mu \gamma^\mu_{\Phi \beta} - e^\Lambda \gamma^\mu_{\Phi \beta} \mathcal{K}^{(A)} + \mathcal{O}(\chi) \]

Next we can take the limit \( \Lambda \rightarrow \infty \) and obtain

\[ \mathcal{I}_{\Phi, \mathcal{H}} = \mathcal{I}_{\Phi, \mathcal{H}} - \mathcal{I}_{\Phi, \mathcal{H}} + \mathcal{O}(\chi) \]

which explicitly shows the role played by the off shell terms of \( \mathcal{K}^{(A)} \) in the computation of \( \{ \mathcal{I}_{\Phi, \mathcal{H}} \} \). These terms will gently conspire to bring the Dirac bracket on the stage. Indeed, sending \( \chi \rightarrow 0 \), that is, examining the configurations at the location \( \rho_0 \) (on the gauge orbit)
where the gauge fixing constraints are satisfied, we obtain

\[
\{q^A, I_{q^B}\}_\gamma = \{\Phi^A, \Phi^B\}
\]

\[
- \{\Phi^A, \mathcal{H}_\gamma\} = \left(\Phi^A, \chi^{\mu}\right) + \frac{1}{2} \{\Phi^A, \chi^{\mu}\} \mathcal{H}_\gamma, \Phi^B
\]

\[
- \left(\Phi^A, \chi^{\mu}\right) + \frac{1}{2} \{\Phi^A, \chi^{\mu}\} \mathcal{H}_\gamma, \Phi^B
\]

\[
\{\Phi^A, \Phi^B\}\gamma = I_{\{\Phi^A, \Phi^B\}}\gamma.
\]

(14)

Now, gauge transformations will move this result at \(p\gamma\) to any other location on the gauge orbit. Since these gauge transformations are canonical and both the Poisson bracket structure and the invariants are preserved by them, we can drop the restriction \(p\gamma\) in the above result and conclude that

\[
\{I_{q^A}, I_{q^B}\} \approx I_{\{\Phi^A, \Phi^B\}}.
\]

(15)

This result has been obtained previously by Thiemann[8] in a much more laborious manner, and not including lapse and shift.

We illustrate some of these classical ideas with a spatially homogeneous isotropic cosmological model that has proven useful in loop quantum gravitational approaches to quantum cosmology[8]. The spacetime increment squared in this model is

\[
ds^2 = -N^2 dt^2 + a^2 (d\chi^2 + dy^2 + dz^2)
\]

where the lapse \(N\) and expansion factor \(a\) depend on the coordinate time \(t\). The material source is taken to be a massless spacetime scalar field \(\phi(t)\). It is convenient to make a change of variables in this model to new variables \(n, u, v\) satisfying \(N = 2n/9\), \(a = (2\pi/3)^{1/3} \exp\left((u + v)/3\right)\), and \(\phi = (27\pi)^{-1/2} \ln(u/v)\). Then the Lagrangian becomes simply

\[
L = -\dot{w}/n
\]

resulting in the primary constraints

\[
H := -p_nv_n \approx 0
\]

and \(p_n \approx 0\). The generator of time evolution is \(nH + \lambda p_n\).

Choosing \(p_n \approx 0\) we find that if we fix the gauge by \(\chi^{(1)} = t - v = 0\), this model possesses the curious property that the only physical variable that changes in time is time itself. This property must be distinguished from the conventional notion of “frozen time” in which it was understood that time did not exist. Of course, by other gauge choices the remaining variables actually undergo non-trivial time evolution. One can simply perform a passive coordinate transformation on the invariant variables that we obtain presently.

With the choice \(X^0 = v\) we find \(A := \{v, H\} = -p_n\), so \(B = -1/p_n\), resulting in the constraints \(H = p_v\) and \(p_n = -p_n/p_v\). The gauge generator is therefore

\[
G_r = -\xi p_n + \xi p_v.
\]

All canonical variables except \(n\) commute with \(G_r\), so \(I_u = u, I_{p_n} = p_u\). Furthermore

\[
\{n, G_r\} = -\xi /p_n
\]

has a vanishing Poisson bracket with \(G_r\), and when evaluated at \(\xi = 1 + p_n n = \chi^{(2)}\) results in \(I_n = n + \{n, G_r\} = -1/p_n\). Note that, as is generally the case, this last invariant can be obtained through the equivalent passive coordinate transformation from an arbitrary time \(t\) to \(t = v\). Representing the functional form in the intrinsic coordinate system by \(\bar{n}\), we have

\[
\bar{n}(v(t)) = n(t) \frac{dv}{dt} = n(t)/\frac{dv}{dt} = -1/p_n.
\]

Illustrating the Noether symmetry generated by the constant of motion \(u\), we find \(\{\bar{t}, H\} = H/p_n\), and therefore the infinitesimal rigid symmetry generator is \(\epsilon t = \epsilon u + n p_n/p_n\). The sole resulting variation is \(\delta n = cn/p_n = -\epsilon /u\). We confirm that the corresponding variation of the Lagrangian is a total time derivative: \(\delta L = -\epsilon\dot{v}\).

Let us next consider the limit of the canonical transformation engendered by \(\Lambda(\chi^{(2)}p_n/p_n + \chi^{(1)}p_n)\). \(\Lambda(-p_n/p_n + p_n(t - t)p_v)\). We find, for example,

\[
K^{(A)}_\gamma = -1/p_n + \exp(-\Lambda)(1/p_n + n),
\]

confirming that \(K^{(A)}_\gamma \rightarrow I_n\) as \(\Lambda \rightarrow \infty\). Finally we note that since the only non-vanishing elements of the inverse of the matrix of Poisson brackets of constraints and gauge conditions

\[
C_1 := H, C_2 := \chi^{(1)}, C_3 := p_n, \text{ and } C_4 := \chi^{(2)}
\]

\(M_{12} = M_{34} = 1/p_n\) and \(M_{23} = 1/p_n^2\), the Dirac bracket

\[
\{n, p_n\}^* = \{n, p_n\} M_{34} \{n, p_n, u\} = n/p_n,
\]

and therefore \(I\{n, u\}^* = -1/p_n^2\). As expected, this is equal to the ordinary Poisson bracket \(I_n, I_u\).

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