Escape from black holes in materials:
Type II Weyl semimetals and generic edge states

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Type II Weyl semimetals are dictated by bulk excitations with tilted light cones, resembling the inside of black holes. We obtain generic boundary conditions for surface boundaries of the type II Weyl semimetals near Weyl nodes, and show that for a certain boundary condition edge states can escape out of the “black hole” event horizon. This means that for realization of the material “black hole” by the Type II Weyl semimetals a careful choice of the boundary condition is necessary.

I. INTRODUCTION

Among various interplay between condensed matter physics and particle physics, recent advances in physics on Weyl semimetals (see [1] for a recent review) is of particular interest, because of its uniqueness about relativistic nature of quasiparticle excitations. Study of Weyl fermions in the Weyl semimetals enlarges the common grounds of the two subjects, not only through the anomaly and topological nature of Weyl fermions leading to the bulk-edge correspondence [2,4], but also with relativistic properties of Weyl fermions.

An intriguing picture of the latter was proposed by Volovik and Zhang [5], concerning in particular Type II Weyl semimetals [6]. Type II Weyl semimetals are defined by Weyl points associated with overtilted Weyl cones, and Ref. [5] clarified that they correspond to light cones allowing propagation only in a certain direction, which in particle physics typically appears behind event horizons of black holes.

In this paper we combine theoretically the idea [5] of equivalence between the Type II Weyl semimetals and black holes, and the bulk-edge correspondence. We analyze most generic edge dispersion of continuum Type II Weyl semimetals. The aim is to study whether the idea of identifying the Type II Weyl semimetals with the inside of the black holes is valid even with the presence of the edge modes. We follow the strategy developed in Ref. [7] on all possible allowed boundary conditions in the continuum limit to seek for a possibility of escaping out of the “black hole.” We find that for a certain class of the boundary conditions of the surface of the semimetal, edge modes can escape from the black hole. This means that the identification needs a proper choice of the boundary condition.

The organization of this paper is as follows. First, in Section II we briefly review continuum Type II Weyl semimetals and their relation to black holes. Then in Section III we introduce generic boundary condition analysis for Type II Weyl semimetals with surfaces. In Section IV we explicitly calculate the generic edge dispersion of Type II Weyl semimetals. In Section V we provide a useful theorem that any edge dispersion is tangential to and ending at bulk dispersion, for generic Weyl semimetals. Then finally in Section VI we calculate spacetime light cone structure for the edge modes and find that they can escape from the black hole for a choice of the surface boundary conditions. The final section is for a summary and discussions.

II. TYPE II WEYL SEMIMETALS AND BLACK HOLES

Let us briefly review the relation between the Type II Weyl semimetals and light cone structure [5,8]. We consider a 3-dimensional Weyl semimetal in the continuum limit, whose Hamiltonian is given by

\[ H = p_i \sigma_i + \alpha_i p_i \mathbf{1} \]  

(1)

where the summation is made for \( i = 1, 2, 3 \) and \( \sigma_i \) is the Pauli matrices. This Hamiltonian is general enough to capture the topological charge of the Weyl semimetal, chirality = +1, after a proper redefinition of the momentum axis and its normalization. The parameters \( \alpha_i (i = 1, 2, 3) \) are real constants. [9]

The bulk dispersion which follows from \( \mathbf{H} \) is

\[ E_{\text{bulk}} = \alpha_i p_i \pm |p|. \]  

(2)

For \( (\alpha_i)^2 > 1 \), the bulk dispersion at \( E = 0 \) is not a single point, but forms a set of flat surfaces in the momentum space, which defines the Type II Weyl semimetals. See fig. [1]

Let us derive the light cone structure of the propagation from the dispersion relation \( \mathbf{H} \). It can be recast to the form \( g^\mu_\nu p_\mu p_\nu = 0 \) with the effective metric

\[ g_{\mu\nu} = \begin{pmatrix} 1 - \alpha_i^2 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & -1 & 0 & 0 \\ \alpha_2 & 0 & -1 & 0 \\ \alpha_3 & 0 & 0 & -1 \end{pmatrix} \]  

(3)

with the standard identification \( p_0 = -E \) (to make sure that the wave function is written as \( \exp[-iEt + ip_i x^i] \)).

In the following we show in two ways that this is the metric inside of a black hole. For simplicity we consider...
\(\alpha_2 = \alpha_3 = 0\). First, consider a Schwarzschild black hole metric in Painlevé-Gullstrand coordinates,

\[
ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - 2\sqrt{\frac{2M}{r}}dtdr - dr^2 - r^2d\Omega_2^2. \tag{4}
\]

Expand the metric around a spatial point near the horizon \((x, y, z) = (2M + \delta x, 0, 0)\) and denote a coordinate \(\alpha^2 \equiv 2M/(2M + \delta x)\), then

\[
ds^2 = (1 - \alpha^2)dt^2 + 2\alpha dt dx - dx^2 - dy^2 - dz^2. \tag{5}
\]

This reproduces the effective metric of the Weyl semimetal. \(\left[3\right]\). If the expansion point is inside of the black hole, \(\delta x < 0\), then \(\alpha^2 > 1\), so it corresponds to the dispersion of the Type II Weyl semimetal.

Another way to see a relation to the black hole is an explicit construction of light cones. A null vector \(n^\mu\) satisfies \(n^\mu n_\mu g_{\mu\nu} = 0\), which means

\[
\frac{1}{\alpha^2 - 1}(n^x)^2 = \frac{1}{\alpha^2 - 1}(\tilde{n}^t)^2 + (n^y)^2 + (n^z)^2 \tag{6}
\]

with \(\tilde{n}^t \equiv n^t + \frac{\alpha}{1 - \alpha^2}n^x\). The section at \(n^y = n^z = 0\) is given by

\[
\frac{n^t}{n^x} = \frac{\alpha \pm 1}{\alpha^2 - 1} \tag{7}
\]

which is always negative (positive) for \(\alpha < -1\) (\(\alpha > 1\)). This means that the light propagation is always in a certain direction, it never goes back, which happens also inside a “black hole.” See Fig. 1 for a pictorial view of the light cone structure.

### III. GENERIC BOUNDARY CONDITIONS FOR TYPE II WEYL SEMIMETALS

For the Type II Weyl material, we introduce a single flat boundary surface at \(x^3 = 0\), with a generic boundary condition

\[
N\psi(x^3 = 0) = 0 \tag{8}
\]

where \(N\) is a constant complex \(2 \times 2\) matrix. With the Hamiltonian \(\left[1\right]\), the hermiticity condition for the system requires

\[
\psi^\dagger (\sigma_3 + \alpha_3 \mathbf{l}) \psi_2 = 0 \tag{9}
\]

for arbitrary wave functions \(\psi_1\) and \(\psi_2\) at the boundary. We like to find the most generic \(N\) which leads to \(\psi\). First, noting \(\det N = 0\) from \(\left[8\right]\), we can write \(N\) as

\[
N = \begin{pmatrix} 1 & \beta \\ \gamma & \gamma\beta \end{pmatrix}, \tag{10}
\]

up to the overall normalization of \(N\) (which is irrelevant to the boundary condition \(\left[8\right]\)), so the solution of \(\left[8\right]\) is written as \(\psi_i = (-\beta, 1)^T f_i\) with a scalar function \(f_i\). Then the condition \(\left[9\right]\) is recast to

\[
|\beta|^2 - 1 + \alpha_3 (|\beta|^2 + 1) = 0. \tag{11}
\]

So we find that a consistent boundary condition exists only when \(|\alpha_3| < 1\) and \(|\beta| = \sqrt{1 - \alpha_3^2}/\alpha_3\). In other words, the most general boundary condition for Weyl semimetals with the Hamiltonian \(\left[1\right]\) is

\[
\left(1, \sqrt{\frac{1 - \alpha_3}{1 + \alpha_3}} e^{i\theta}\right) \psi(x^3 = 0) = 0 \tag{12}
\]

with a boundary condition parameter \(\theta\) \((0 \leq \theta < 2\pi)\).

Note that introduction of the boundary at \(x^3 = 0\) does not allow \(|\alpha_3| > 1\). This also implies that the vector \(\alpha\) of Type II Weyl semimetals cannot be normal to the boundary. \(\left[11\right]\) Of course, putting \(\alpha_3 = 0\) brings us back to the generic boundary condition studied in Ref. \(\left[7\right]\).
we show only the result here. The energy eigenvalue is

\[ E = \alpha_1 p_1 + \alpha_2 p_2 - \sqrt{1 - \alpha_3^2} (p_1 \cos \theta - p_2 \sin \theta). \]  

(13)

The edge state wave function is

\[ \psi = \left( -\sqrt{\frac{1-\alpha_3^2}{1+\alpha_3^2}} e^{i\theta} \right) \exp[i k_3 x^3] \]  

(14)

with the complex momentum \( k_3 \),

\[ k_3 \equiv \frac{\alpha_3(p_1 \cos \theta - p_2 \sin \theta) - i(p_1 \sin \theta + p_2 \cos \theta)}{\sqrt{1-\alpha_3^2}}. \]  

(15)

The imaginary part of \( k_3 \) shows the localization of the edge state at the boundary. When the material exits in the region \( x^3 \geq 0 \), the normalizability condition for the wave function is \( \beta \equiv \text{Im}[k_3] > 0 \), which is equivalent to

\[ p_1 \sin \theta + p_2 \cos \theta < 0. \]  

(16)

IV. EDGE DISPERSION OF TYPE II WEEY SEMIMETALS

The edge state should exist as a result of the topological protection, since the bulk-edge correspondence \[2, 4\] works also for the Type II Weyl semimetals \[5, 12\]. The edge state is localized at the boundary because of the imaginary part of the momentum normal to the boundary. Although the bulk mode satisfies the boundary condition by taking an appropriate linear combination of the incoming and outgoing modes at the boundary, such a linear combination cannot be taken for the edge mode since only one of these two modes corresponds to the edge mode and the other is an unphysical non-normalizable mode. Thus, the boundary condition gives an additional condition to the momenta of the edge mode.

Let us solve the Hamiltonian eigen equation \( H \psi = E_{\text{edge}} \psi \) for the edge states, by imposing the most generic boundary condition \[12\]. It is quite straightforward and we show only the result here. The energy eigenvalue is

\[ E_{\text{edge}} = \alpha_1 p_1 + \alpha_2 p_2 - \sqrt{1-\alpha_3^2} (p_1 \cos \theta - p_2 \sin \theta). \]  

The edge dispersion is a straight line in the \((p_1, p_2)\) plane at the constant energy slice. We show some of the examples of the edge and bulk dispersions in Fig. 2.

![Fig. 2](image)

FIG. 2. Upper row: Energy dispersion \( E_{\text{bulk}} \) as a function of \( p_1 \) and \( p_2 \) at the slice \( p_3 = \text{Re}[k_3] \), and the edge dispersion \( E_{\text{edge}} \) given in \[13\] with \[16\]. We chose \( \alpha_2 = \alpha_3 = 0 \) with \( \alpha_1 = -1.2 \) (Type II), and the boundary condition parameter \( \theta = 0, \theta = \pi/2, \theta = \pi \) and \( \theta = (3\pi/2) \) (From left to right). The edge dispersion is always flat, and tangential to the bulk edge dispersion. Lower row: Corresponding slices at \( E = 0.3 \), shown in the \((p_1, p_2)\)-plane. Blue curved lines are for the bulk dispersion, and red half lines are for the edge dispersion.

The statement was explicitly made by Haldane \[13\] for generic Weyl semimetals and here we provide a proof of it. This theorem is not only for the Type II Weyl semimetals but applicable to any bulk and edge state which satisfies the definitions that we will provide below.

One interesting observation is that the edge dispersion always tangential to the bulk dispersion. The next section is devoted for a proof that the edge dispersion is always tangential to the bulk dispersion at the merging point.

V. TANGENTIALITY THEOREM OF EDGE AND BULK DISPERSIONS

In this section, we show that any edge dispersion is tangential to the bulk dispersion at the merging point. The statement was explicitly made by Haldane \[13\] for generic Weyl semimetals and here we provide a proof of it. This theorem is not only for the Type II Weyl semimetals but applicable to any bulk and edge state.
where \( p_i \) are real momenta. The edge dispersion is given in terms of the same function \( F \) as

\[
E = F(p_i(x^\mu), p_z + i\beta) ,
\]

but the momenta satisfy additional constraints which come from the boundary condition. If the edge dispersion continues to \( \beta = 0 \), it is merged into the bulk dispersion there.

Now, it is straightforward to show that the edge dispersion is tangential to the bulk dispersion. The tangent space of the bulk dispersion is given by

\[
0 = dE = \sum_i \frac{\partial F}{\partial p_i} dp_i .
\]

On the other hand, the tangent space of the edge dispersion is expressed as

\[
0 = dE = \sum_{i(z)} \frac{\partial F}{\partial p_i} dp_i + \frac{\partial F}{\partial p_z} (dp_z + id\beta) .
\]

The Hermiticity condition for the bulk mode implies that all \( \frac{\partial F}{\partial p_i} \) must be real since all real momenta \( p_i \) are independent for the bulk mode. Then, the real and imaginary parts of \((23)\) give

\[
0 = dE = \sum_i \frac{\partial F}{\partial p_i} dp_i ,
\]

\[
0 = \frac{\partial F}{\partial p_z} ,
\]

respectively. At the merging point \( \beta = 0 \), the first equation agrees with the tangent space of the bulk dispersion there. Therefore, the edge dispersion is tangent to the bulk dispersion at the merging point. The imaginary part \((23)\) must be satisfied on the merging point, for the energy of the edge mode to be real.

Finally we emphasize again that the above proof is valid for any system, for example, a system on a discrete lattice, as long as it satisfies the conditions (i)-(iv) above, though in this paper we focus on the continuum limit in the Type II Weyl semimetals. For the case of Type II Weyl semimetals, it can be seen in Fig. 2 that the edge dispersion is tangential to the bulk dispersion.

**VI. ESCAPE FROM BLACK HOLES**

Let us study the propagation direction of the edge state to see whether it can escape from the “black hole.” The relation between the propagation direction \( n^\mu \) and the four-momentum \( p_\mu \) is \( n^\mu = g^\mu_\nu p_\nu \). Substituting the edge direction
dispersion (13) and $p_3 = \text{Re}[k_3]$, we find

\begin{align*}
n^0 &= \frac{1}{\sqrt{1-\alpha_3^2}} (p_1 \cos \theta - p_2 \sin \theta), \\
n^1 &= -p_1 + \frac{\alpha_1}{\sqrt{1-\alpha_3^2}} (p_1 \cos \theta - p_2 \sin \theta), \\
n^2 &= -p_2 + \frac{\alpha_2}{\sqrt{1-\alpha_3^2}} (p_1 \cos \theta - p_2 \sin \theta), \\
n^3 &= 0.
\end{align*}

(26) (27) (28) (29)

Note that automatically we obtained $n^3 = 0$, which is consistent with the fact that the edge mode propagates along the boundary $x^3 = 0$.

The expression above applies to any $\alpha_1$ and $\alpha_2$. The Type II Weyl semimetal has $\alpha_1^2 + \alpha_2^2 > 1 - \alpha_3^2$. Without loss of generality, we can take $\alpha_1 < -\sqrt{1-\alpha_3^2}$ and $\alpha_2 = 0$, by using the rotation in $(x^1, x^2)$-plane. So let us concentrate on this case. All the bulk modes propagate in the negative direction of $x^1$. So, if we can find an edge mode which propagates in the positive direction of $x^1$, that is $n^1/n^0 > 0$, we conclude that the edge mode can escape from the black hole. In other words, for $(p_1, p_2)$ satisfying (13) and $E_{\text{edge}} > 0$ with (13), if there exists $(p_1, p_2)$ giving $n^1/n^0 > 0$, the edge mode can escape from the black hole. As we will see below, the answer depends on the parameter $\theta$ of the boundary condition.

In order to see whether the edge mode can escape from the black hole, it is convenient to rewrite $n^0$ and $n^1$ in terms of energy $E_{\text{edge}}$:

\begin{align*}
n^0 &= \frac{E_{\text{edge}} - \alpha_1 p_1}{1 - \alpha_3^2}, \\
n^1 &= \frac{\alpha_1}{1 - \alpha_3^2} E_{\text{edge}} + \frac{1 - \alpha_3^2 \alpha_2^2}{1 - \alpha_3^2} p_1.
\end{align*}

(30) (31)

Here, we consider only the edge modes with positive energy $E_{\text{edge}} > 0$, and the other parameters satisfy $\alpha_1 < -\sqrt{1-\alpha_3^2}$ and $-1 < \alpha_3 < 1$. The sign of $n^0$ depend on given energy $E_{\text{edge}}$ and momentum $p_1$ as

\begin{align*}
n^0 > 0 \quad &\text{for} \quad p_1 > \alpha_1^{-1} E_{\text{edge}}, \\
n^0 < 0 \quad &\text{for} \quad p_1 < \alpha_1^{-1} E_{\text{edge}},
\end{align*}

(32) (33)

while the sign of $n^1$ flips as

\begin{align*}
n^1 > 0 \quad &\text{for} \quad p_1 < -\frac{\alpha_1}{1 - \alpha_3^2 - \alpha_3^2} E_{\text{edge}}, \\
n^1 < 0 \quad &\text{for} \quad p_1 > -\frac{\alpha_1}{1 - \alpha_3^2 - \alpha_3^2} E_{\text{edge}},
\end{align*}

(34) (35)

where both $\alpha_1^{-1} E_{\text{edge}}$ and $-\frac{\alpha_1}{1 - \alpha_3^2 - \alpha_3^2} E_{\text{edge}}$ are negative. From the conditions $E_{\text{edge}} > 0$, $\alpha_1 < -\sqrt{1-\alpha_3^2}$ and $-1 < \alpha_3 < 1$, it is straightforward to obtain the following relation:

\begin{align*}
-\frac{\alpha_1}{1 - \alpha_3^2 - \alpha_3^2} E_{\text{edge}} < p_1 < \alpha_1^{-1} E_{\text{edge}}.
\end{align*}

(36)

Thus there is always a range of momentum $p_1$;

\begin{align*}
-\frac{\alpha_1}{1 - \alpha_3^2 - \alpha_3^2} E_{\text{edge}} < p_1 < \alpha_1^{-1} E_{\text{edge}},
\end{align*}

(37)

which shows $n^1/n^0 > 0$, or equivalently, a possible edge mode escaping away from the black hole.

However, note that it does not immediately mean that there exists such an edge mode which can escape from the black hole. This edge mode needs a value of $p_1$ which is in the range (37), that is, the edge dispersion needs to allow $p_1$ to overlap with (37). This can be seen in Fig. 4: the edge dispersions for various $\theta$ are shown pictorially in Fig. 4 for the case of $\alpha_1 = -1.2$ and $\alpha_2 = \alpha_3 = 0$. If the edge dispersion (colored in red) intersects with the range (37) (the grey region), then that is the edge mode escaping away from the black hole.
For $0 < \theta < \cos^{-1}\left(\alpha_1^{-1}\sqrt{1-\alpha_3^2}\right) < \pi$, the edge dispersion has $p_2 > 0$ at the merging point $\beta = \text{Im}k_3 = 0$, and the condition $\beta > 0$ gives the upper bound of $p_1$ but no lower bound. Since the edge dispersion is a straight line with $p_1 > \alpha_1^{-1}E_{\text{edge}}$ at the merging point, the edge dispersion extends to the region (37). Thus the edge mode can escape from the black hole.

For $\theta = \cos^{-1}\left(\alpha_1^{-1}\sqrt{1-\alpha_3^2}\right) < \pi$, the edge mode is on the asymptote of the hyperboloid of the bulk mode. There is no upper or lower bound on $p_1$, and the edge dispersion extends to the region (37). The edge mode can escape from the black hole.

For $\cos^{-1}\left(\alpha_1^{-1}\sqrt{1-\alpha_3^2}\right) < \theta < \pi$, the edge dispersion has $p_2 < 0$ at the merging point $\beta = \text{Im}k_3 = 0$, and the condition $\beta > 0$ gives the lower bound of $p_1$ but no upper bound. Since the edge dispersion is a straight line with $p_1 < -\frac{\alpha_1}{1-\alpha_1^{-1}\alpha_3^2}E_{\text{edge}}$ at the merging point, the edge dispersion extends to the region (37). Thus the edge mode can escape from the black hole.

For $\theta = \pi$, the edge dispersion is given by

$$E_{\text{edge}} = \left(\alpha_1 + \sqrt{1-\alpha_3^2}\right)p_1.$$  

(39)

Since the momentum is fixed for given $E_{\text{edge}}$ and satisfies $p_1 < -\frac{\alpha_1}{1-\alpha_1^{-1}\alpha_3^2}E_{\text{edge}}$, the edge mode cannot escape from the black hole.

For $\pi < \theta < \cos^{-1}\left(\alpha_1^{-1}\sqrt{1-\alpha_3^2}\right)$, the edge dispersion has $p_2 > 0$ at the merging point $\beta = \text{Im}k_3 = 0$, and the condition $\beta > 0$ gives the upper bound of $p_1$. Since the edge dispersion is a straight line with $p_1 < -\frac{\alpha_1}{1-\alpha_1^{-1}\alpha_3^2}E_{\text{edge}}$ at the merging point, which has maximum of $p_1$, the edge mode cannot escape from the black hole.

For $\theta = \cos^{-1}\left(\alpha_1^{-1}\sqrt{1-\alpha_3^2}\right) > \pi$, no edge mode is allowed near the Weyl point. The bulk dispersion is approximately given by a hyperboloid. The merging point of edge and bulk mode are in $p_1 \rightarrow \pm \infty$, and the edge dispersion extends outward from the merging point.

For $\cos^{-1}\left(\alpha_1^{-1}\sqrt{1-\alpha_3^2}\right) < \theta < 2\pi$, the edge dispersion has $p_2 < 0$ at the merging point $\beta = \text{Im}k_3 = 0$, and the condition $\beta > 0$ gives the lower bound of $p_1$. Since the edge dispersion is a straight line with $p_1 > \alpha_1^{-1}E_{\text{edge}}$ at the merging point, which has minimum of $p_1$, the edge mode cannot escape from the black hole.

In summary, in the convention $\alpha_1 < \sqrt{-1-\alpha_3^2}$ and $\alpha_2 = 0$, the edge mode can escape away from the black hole, when the boundary condition parameter $\theta$ satisfies

$$0 < \theta < \pi.$$  

(40)

This means that for a randomly chosen consistent boundary condition $\theta$, it may allow the edge modes propagating out of the black hole defined by the bulk mode of the Type II Weyl semimetals. Therefore, in building a black hole analogue by the Type II Weyl semimetals, one needs to carefully choose the surface boundary conditions of the material, such that the edge modes do not violate the causality produced by the “black hole.”

Let us elaborate more on the reason for this conclusion. The effective metric (3) is determined by the bulk excitations, so the light cone structure is fixed by it. The edge modes generically propagate outside of the light cone, so edge modes are tachyonic. With a proper choice of the boundary condition, they can even propagate in the direction opposite to the bulk tilted light cone. Therefore the edge modes can eventually go outside the black hole horizon.
VII. SUMMARY

In this paper, we have studied generic boundary conditions and generic edge dispersions in Type II Weyl semimetals in the continuum and the low energy limits. Based on the bulk dispersion argument \[5\] that Type II Weyl semimetals can be regarded as the inside of a black hole, we have explored possibility of having an edge mode which can escape away from the black hole horizon. We have found that the generic boundary condition is parameterized by a single rotation parameter $\theta$ ($0 \leq \theta < 2\pi$) as \[12\] and for a part of the range of the parameter ($0 < \theta < \pi$ for $\alpha_2 = 0$) there exists an edge mode escaping away from the black hole.

For a realization of the black hole by the Type II Weyl semimetals, since any material has its surface, we need a special care about the choice of the boundary condition. Our analysis shows that $\theta$ needs to be in the range $\pi \leq \theta \leq 2\pi$ not to violate the black hole causal structure. A safe way is to choose, for example, $\theta = 3\pi/2$ which amounts to the boundary condition

$$\left(1, -i \sqrt{\frac{1 - \alpha_3}{1 + \alpha_3}}\right) \psi(x^3 = 0) = 0 \quad (41)$$

for the Hamiltonian \[1\] and the spatial coordinate $x^3 \geq 0$ for the material with the surface at $x^3 = 0$.

In this paper we have dealt only with the continuum limit of the Type II Weyl semimetals, because it has enabled us to study the most generic boundary conditions, which are necessary for checking the possibility of escaping from the black hole. The physical realization of the specific value of $\theta$ depends on discrete lattice models of the Type II Weyl semimetal. Once the bulk discrete model is obtained, one takes the continuum limit and extract the value of $\theta$ from the numerically observed edge mode dispersion \[13\], then one can check whether the edge mode is escaping out of the black hole or not.

The identification of the Weyl semimetals with the black hole can be extended to topological “insulators.” It is known that regarding one of the momenta of Weyl semimetals to be a nonzero constant reduces the system to a topological insulator. The Type I Weyl semimetal with $p_1 = m$ is a 2-dimensional topological insulator of class A, and we can consider the same dimensional reduction from the Type II Weyl semimetal to a topological “insulator” — which is not insulating due to the tilted light cone. Our analysis is valid even with putting $p_2 = m$. So, black hole validity can be checked in the same manner, with the boundary condition parameter $\theta$.

The important part of the analyses in this paper is the most generic boundary conditions in the continuum limit. The idea of the method was used \[14\] to find a topological charge of the edge state, which results in the discovery of states localized at corners \[15, 16\] (which were recently called corner states or hinge states in higher-order topological insulators \[17, 18\]). It would be interesting to explore the edge mode contributions to the black hole interpretation of various deformed topological insulators, as well as Type III and Type IV Weyl semimetals \[19\]. With these deformations of the Weyl semimetal Hamiltonians, D-brane interpretation of the bands \[20\] may not persist, that is also an interesting issue.

Although the propagation of the bulk modes mimics that in a black hole geometry, whether the Hawking radiation emanating from the event horizon (which is the boundary between Type I and Type II semimetals \[21\]) exists or not is rather a subtle question, as the Hawking radiation originates in the change of the quantum vacuum in black hole formation. It is challenging to construct a theoretical framework of Weyl semimetals accompanying a Hawking temperature and possible experimental setups \[22\].

Introducing a surface boundary to the Type II Weyl semimetals in turn means slicing a black hole, which sounds impossible in general relativity. Black holes in brane world scenario would be the closest example in particle physics, and we hope our condensed matter analyses may inspire also particle physics in the future.

ACKNOWLEDGMENTS

We would like to thank D. R. Candido, H. Katsura, M. Koshino, M. Kurkov, M. Ochi, R. Okugawa and A. Zyuzin for valuable comments. This work is supported in part by JSPS KAKENHI Grant No. JP17H06462.
[10] See also Refs. [24][26] for 1d and 2d generic boundary conditions.

[11] This bound was independently studied in Ref. [23]. The authors would like to thank A. Zyuzin for bringing Ref. [23] to our attention.

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