Distribution of Beurling primes
and zeroes of the Beurling $\zeta$ function I.
Distribution of the zeroes of the $\zeta$ function of Beurling

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Dedicated to János Pintz on the occasion of his 70th birthday

Abstract

We consider the oscillation properties of the remainder term $\Delta(x)$ in the prime number formula for Beurling primes, and their relation to the distribution of the nontrivial zeroes of the Beurling zeta function $\zeta$. The two main directions in this study are obtaining correspondingly high oscillation given the existence of a certain zero $\rho$ of $\zeta$ (a direction of study initiated by Littlewood in 1937 for the Riemann zeta function), and to describe some function theoretical connection between the general oscillation behavior of $\Delta$ on the one hand and the general distribution of nontrivial zeroes (a direction initiated by Phragmen and Ingham). As a first part of our study, here we describe results about the behavior, and in particular fine distribution of zeroes, of $\zeta$.

The analysis here brings about some news, sometimes even for the classical case of the Riemann zeta function. Theorem 4 provides a zero density estimate, which is a complement to known results for the Selberg class, relying on use of the Riemann-type functional equation, which we do not assume in the Beurling context. In Theorem 5 we deduce a variant of a well-known theorem of Turán, extending its range of validity even for rectangles of height only $h = 2$. In Theorem 6 we will extend a zero clustering result of Ramachandra from the Riemann zeta case. A weaker result – which, on the other hand, is a strong sharpening of the average result from the classic book [18] of Montgomery – was worked out by Diamond, Montgomery and Vorhauer [8]. Here we show that the obscure technicalities of the Ramachandra paper [26] (like a polynomial with coefficients like $10^8$) can be gotten rid of, providing a more transparent proof of the validity of this clustering phenomenon.

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1 Introduction

Beurling’s theory fits well to the study of several mathematical structures. A vast field of applications of Beurling’s theory is nowadays called arithmetical semigroups, which are described in detail e.g. by Knopfmacher, [15].

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Here $G$ is a unitary, commutative semigroup, with a countable set of indecomposable generators, called the *primes* of $G$ and denoted usually as $p \in \mathcal{P}$, (with $\mathcal{P} \subset G$ the set of all primes within $G$), which freely generate the whole of $G$: i.e., any element $g \in G$ can be (essentially, i.e. up to order of terms) uniquely written in the form $g = p_1^{k_1} \cdots p_m^{k_m}$: two (essentially) different such expressions are necessarily different as elements of $G$, while each element has its (essentially) own unique prime decomposition.

Moreover, there is a norm $|\cdot|: G \to \mathbb{R}_+$ so that the following hold. Firstly, the image of $G$, $|G| \subset \mathbb{R}_+$ is discrete, i.e. any finite interval of $\mathbb{R}_+$ can contain the norm of only a finite number of elements of $G$; thus the function

$$\mathcal{N}(x) := \# \{ g \in G : |g| \leq x \}$$

exists as a finite, nondecreasing, nonnegative integer valued function on $\mathbb{R}_+$.

Second, the norm is multiplicative, i.e. $|gh| = |g| \cdot |h|$; it follows that for the unit element $e$ of $G$ $|e| = 1$, and that all other elements have norms strictly larger than 1 (otherwise the different elements $g^n$ – which are really different by their different unique prime decomposition – would have a sequence of norms converging to 0).

With these defined, also arithmetical functions can be defined on $G$, see e.g. [15]: we will use in this work the identically one function $u$, the Möbius function $\mu$ and the divisor (number of divisors) function $d$ only: for their analogous definition see pages 73-79 in [15].

E.g. let $\mathcal{A}$ be the category of all finite Abelian groups. By the fundamental theorem of finite abelian groups, any abelian group of finite order decomposes as the direct product of cyclic groups of prime power order and these cyclic groups are isomorphically unique and indecomposable. So here the semigroup is the (isomorphism classes of the) category and the multiplication is the direct product. The number of elements of an abelian group can be taken as norm, and the number of elements of a given norm, i.e. isomorphically nonequivalent Abelian groups of order $n$, is a function $G(n)$ which then satisfies $\mathcal{N}(x) = \sum_{n \leq x} G(n)$.

The ideals $I_K$ of an algebraic number field $K$ over the rational numbers form another basic example.

There are many others: algebraic structures like e.g. semisimple rings, graphs of some required properties, finite pseudometrizable topological spaces, symmetric Riemannian manifolds, compact Lie groups etc., see Knopfmacher’s book, pages 11-22.

2 **Beurling zeta functions admitting analytic continuation**

In this work we assume the so-called ”Axiom A” of Knopfmacher, which he explains in great detail, see pages 73-79 of his fundamental book [15] on Beurling-type generalized arithmetical semigroup theory.

**Definition 1.** It is said that $\mathcal{N}$ (or, loosely speaking, $\zeta$) satisfies Axiom A – more precisely, *Axiom $A(\kappa, \delta, \eta)$* with the suitable constants $\kappa > 0$ and $0 < \eta < \delta$ – if we have

$$\mathcal{N}(x) = \kappa x^{\delta} + O(x^\eta).$$

(2)

Note that by a slight change – renorming all elements $g \in G$ by taking the new norm $\|g\| := |g|^{1/\delta}$ – one can assume that $\delta = 1$, while $\eta$ changes to $\theta := \eta/\delta$ (and all the other constants change respectively, so e.g. in the prime number formula $x^\delta / \delta \log x$ changes to $x / \log x$, etc.). For technical convenience, we decided to use this version, although the above formulation is fully equivalent to ours: so henceforth we refer to Axiom A as meaning the following form:

$$\mathcal{R}(x) := \mathcal{N}(x) - \kappa x \quad |\mathcal{R}(x)| \leq Ax^\theta \quad (\kappa, A > 0, 0 < \theta < 1 \text{ constants, } x \geq 1 \text{ arbitrary}).$$

(3)
It is clear that under Axiom A the Beurling zeta function
\[
\zeta(s) := \int_1^\infty x^{-s}d\mathcal{N}(x) = \sum_{g \in \mathcal{G}} \frac{1}{|g|^s}
\]
(4)
admits a meromorphic, essentially analytic continuation \(\kappa \frac{1}{s-1} + \int_1^\infty x^{-s}d\mathcal{R}(x)\) up to \(\Re s > \theta\) with only one, simple pole at 1.

Later in \(\S\) we will also assume some further conditions, explained in due course.

The Beurling zeta function (4) can be used to express the generalized von Mangoldt function
\[
\Lambda(g) := \Lambda_N(g) := \begin{cases} 
\log |p| & \text{if } g = p^k, \; k \in \mathbb{N} \text{ with some prime } p \in \mathcal{G} \\
0 & \text{if } g \in \mathcal{G} \text{ is not a prime power in } \mathcal{G} 
\end{cases}
\]
(5)
as coefficients of the logarithmic derivative of the zeta function
\[-\frac{\zeta'}{\zeta}(s) = \sum_{g \in \mathcal{G}} \frac{\Lambda(g)}{|g|^s}.\]
(6)

The Beurling theory of generalized primes is mainly concerned with the analysis of the summatory function
\[
\psi(x) := \sum_{g \in \mathcal{G}, \; |g| \leq x} \Lambda(g).
\]
(7)

Apart from generality and applicability to e.g. distribution of prime ideals in number fields, the interest in these things were greatly boosted by a recent construction of Diamond, Montgomery and Vorhauer [8].

They basically showed that under Axiom A in the form (3) the Riemann hypothesis may still fail: moreover, nothing better than the most classical zero-free region and error term of
\[
\zeta(s) \neq 0 \quad \text{whenever } s = \sigma + it, \; \sigma > 1 - \frac{c}{\log t},
\]
and
\[
\psi(x) = x + O(x \exp(-c\sqrt{\log x})
\]
follows from (3) at least if \(\theta > 1/2\).

Therefore, Vinogradov mean value theorems on trigonometric sums and many other stuff are certainly irrelevant in this generality, and for Beurling zeta functions a careful revival of the combination of ”ancient-classical” methods, as summarized e.g. in [14], and ”elementary” arguments can only be implemented.

The present paper is the first part of a series. For motivation, we considered the following nice results.

Denote by \(\eta(t) : (0, \infty) \to (0, 1/2)\) a nonincreasing function and consider the domain
\[
\mathcal{D}(\eta) := \{s = \sigma + it \in \mathbb{C} : \sigma > 1 - \eta(t), \; t > 0\}.
\]
(8)
Following Ingham [11] and Pintz [20, 21] we will then use the derived function
\[
\omega(x) := \omega_\eta(x) := \inf_{y \geq 1} (\eta(y) \log x + \log y).
\]
(9)
Also, we denote, as usual,
\[
\Delta(x) := \psi(x) - x.
\]
(10)
Theorem 1 (Révész). Let \( \zeta(\rho) = 0 \) with \( \rho = \beta + i\gamma \) be a zero of the Riemann zeta function. Then for arbitrary \( \epsilon > 0 \) we have for arbitrarily large values of \( x \) the lower estimate \( |\Delta(x)| \geq (\pi/2 - \epsilon)|\rho| \).

Theorem 2 (Pintz). Assume that there is no zero of the Riemann \( \zeta \) function in \( D(\eta) \). Then for arbitrary \( \epsilon > 0 \) we have

\[
\Delta(x) = O(x \exp(-(1 - \epsilon)\omega(x))).
\]

Theorem 3 (Pintz). Conversely, assuming that there are infinitely many zeroes within the domain \( \eta \), we have for any \( \epsilon > 0 \) the oscillation \( \Delta(x) = \Omega(x \exp(-(1 + \epsilon)\omega(x))) \).

These results, in their original proofs and/or sharpest forms relied on particular things generally not available for the Beurling zeta functions. Therefore, it was unclear how much of these relations can as well be stated for the distribution of Beurling primes?

Our aim with the series is to generalize these and related results (e.g. oscillation results for the summatory function of the respective Möbius function) to the Beurling context eventually.

A century old and ineffective result of Phragmen – which established that the supremum of the real parts of Riemann zeta zeroes agrees with the infimum of the exponents \( a \) with which \( \Delta(x) = O(x^a) \) holds – was first extended to zero-free regions asymptotic to the 1-line in the famous book of Ingham [11]. Soon after Littlewood [17] set the goal to derive effective (and essentially optimal, at least in order) lower bounds on the oscillation size of \( \Delta(x) \), "which a hypothetical zero \( \rho_0 = \beta_0 + i\gamma_0 \) of the Riemann zeta function may cause". In 1950 Turán solved the original problem [35] by use of his powersum method [34]. Several papers followed with extensions and further sharpenings [14, 32, 29, 30, 31, 12], in particular Pintz [20, 21] obtaining impressively precise oscillation estimates as mentioned above. Further developments keep appearing, see e.g. [23, 25] and the references therein.

The more general situation in the Beurling context was also well studied ever since Beurling [2], gaining a further impetus in particular by the seminal paper of Diamond, Montgomery and Vorhauer [8]. We plan to give detailed description of these important developments in the forthcoming more number theoretic parts of the series, so here let us only call attention to the recent works [3, 4, 6, 7].

In order to achieve the above goal, first we must work out a number of auxiliary theorems, which, to the best of our knowledge, have not been developed yet. Although technical for our purposes, these may have their own importance for future discussion of Beurling zeta functions. Therefore, we intend to give full, detailed proofs, even explicit constants, wherever possible. More concretely, the content of the present, first item of the series is the description of the distribution of the zeroes of the Beurling zeta function when the error term of the integer counting function allows an analytic continuation to a vertical strip beyond the 1-line. In this respect our paper is a continuation of [8]. To save space, we do not reproduce the otherwise standard proof of the classical zero-free region, which, as remarked earlier, is in general essentially best possible. Basically we describe three results on the distribution of the zeroes of Beurling zeta functions satisfying a few assumptions of the sort of Axiom A.

First in §7 we prove a "global" density theorem for the zeroes of the Beurling zeta function. Here we do not have the functional equation, but do have the positivity of the coefficients of the Dirichlet series of the \( \zeta \) function, and an Euler product. On the other hand, in the Selberg class positivity is lacking, but the functional equation and the Euler product is available. Thus our density theorem is a complement to the similar result for the Selberg class, proved by Kaczorowsky and Perelli [13].
In §8 we prove a Turán type "local" density estimate. Turán’s theorem works with a zero-free domain, which extends to the critical, generally not zero-free domain, and by this assumption gives an estimate for the number of close-by zeroes. In our analysis we also show how this can be done even if the zero-free region is of constant height – in place of being extended to some height of the log log t order, as given originally in Turán’s work.

Finally, in §9 we generalize a result of Ramachandra, proving that zeroes of the Beurling zeta function, close to the 1-line, occur in clusters. This theorem sharpens Theorem 2 of [8], since the latter gives a nonvanishing estimate only for 1 − β0 < 1/√log γ0, while our version works even for 1 − β0 < 1/ log log γ0.

3 Basic properties of the Beurling ζ

The following basic lemmas are slightly elaborated forms of 4.2.6. Proposition, 4.2.8. Proposition and 4.2.10. Corollary of [15]. These form the most basic facts concerning the Beurling zeta function. Still, we give a proof for explicit handling of the arising constants in these estimates.

Lemma 1. Denote the "partial sums” (partial Laplace transforms) of N(x) as ζX for arbitrary X ≥ 1:

$$\zeta_X(s) := \int_1^X x^{-s}dN(x).$$

Then ζX(s) is an entire function and for σ := ℜs > θ it admits

$$\zeta_X(s) = \begin{cases} 
\zeta(s) - \frac{\kappa X^{1-s}}{s-1} - \int_1^X x^{-s}dR(x) & \text{for all } s \neq 1, \\
\frac{\kappa}{s-1} - \frac{\kappa X^{1-s}}{s-1} + \int_1^X x^{-s}dR(x) & \text{for all } s \neq 1, \\
\kappa \log X + \int_1^X \frac{dR(x)}{x} & \text{for } s = 1,
\end{cases}$$

(12)

together with the estimate

$$|\zeta_X(s)| \leq \zeta_X(\sigma) \leq \begin{cases} 
\min \left( \frac{\kappa X^{1-s}}{s-1} + \frac{A}{s-\theta}, \ \kappa + \kappa X^{1-\sigma} \log X + \frac{A}{s-\theta} \right) & \text{if } \theta < \sigma < 1, \\
\kappa \log X + \frac{A}{1-\theta} & \text{if } \sigma = 1, \\
\min \left( \kappa \log X + \frac{\sigma(A+\kappa)}{s-\frac{\sigma}{\theta}}, \ \frac{\sigma(A+\kappa)}{s-\frac{\sigma}{\theta}} \right) & \text{if } \sigma > 1.
\end{cases} \quad (13)$$

Moreover, the above remainder terms can be bounded as follows.

$$\left| \int_X^\infty x^{-s}dR(x) \right| \leq A \left| \frac{s}{\sigma - \theta} X^{\theta - \sigma} \right. \quad (14)$$

and

$$\left| \int_1^X x^{-s}dR(x) \right| \leq A \left( \frac{1}{\sigma - \theta} X^{\theta - \sigma} + X^{\theta - \sigma} \right) \leq A \min \left( \frac{1}{\sigma - \theta}, \ |s| \log X + X^{\theta - \sigma} \right). \quad (15)$$

Proof. Clearly, both ζX and \int_1^X x^{-s}dR(x) are entire functions, ζ is regular for ℜs > θ except at s = 1, X^{1-s}/(s-1) is regular all over C \ {1}, while \int_X^\infty x^{-s}dR(x) is regular for ℜs > θ. Hence it suffices to prove the formulae (12) for ℜs > 1, and then refer to analytic continuation in extending them all over the common domain of regularity of both sides, i.e. \{ℜs > θ\} \ {1}.

The formulae then follow as for ℜs > 1 the defining integrals are absolutely convergent, and to get the first form we only need to write

$$\zeta_X(s) = \zeta(s) - \int_X^\infty x^{-s}dN(x) = \zeta(s) - \kappa \int_X^\infty x^{-s}dx - \int_X^\infty x^{-s}dR(x);$$
while to obtain the second and also the third ones it suffices to separate \( N(x) \) as \( \kappa x + R(x) \) and execute the integration for the main term.

To prove the estimate (13) we utilize that by definition \( N(x) \) is increasing, hence \( dN(x) \) is positive, and thus \( |\zeta_X(s)| \leq \zeta_X(\sigma) \), which in turn can be calculated by partial integration and substituting \( s = s \). Namely, for \( \sigma \neq 1 \) and \( \sigma > \theta \)

\[
\zeta_X(\sigma) = \left[ x^{-\sigma} N(x) \right]_1^X = \int_1^X x^{-\sigma-1} N(x) \, dx
\]

\[
\leq \frac{\kappa X + AX^\theta}{X^\sigma} + \sigma \int_1^X \frac{\kappa}{x^\sigma} \, dx + \sigma \int_1^X \frac{A}{x^{\sigma-\theta+1}} \, dx
\]

\[
= \kappa X^{1-\sigma} + AX^{\theta-\sigma} + \frac{\sigma \kappa}{1-\sigma} (X^{1-\sigma} - 1) + \frac{\sigma A}{\sigma - \theta} (1 - X^{\theta-\sigma})
\]

\[
\leq \kappa + \frac{\kappa}{1-\sigma} (X^{1-\sigma} - 1) + \frac{\sigma A}{\sigma - \theta}
\]

\[
\leq \begin{cases} 
\min \left( \frac{\kappa X^{1-\sigma}}{1-\sigma} + \frac{A}{\sigma - \theta}, \kappa + \frac{\sigma \kappa}{1-\sigma} \right) & \text{if } \sigma < 1 \\
\min \left( \frac{\sigma (A+\kappa)}{\sigma - 1}, \kappa \log X + \frac{\sigma (A+\kappa)}{\sigma - \theta} \right) & \text{if } \sigma > 1
\end{cases}
\]

with an application of the Lagrange mean value theorem in the very last estimations involving \( \log X \). The \( \sigma = 1 \) case follows e.g. by taking the limit of \( \sigma \to 1+ \), as \( \zeta_X(\sigma) \) is regular.

It remains to estimate the error terms arising from \( R(x) \). After partial integration

\[
\int x^{-\sigma} dR(x) = \left[ R(x)x^{-\sigma} \right] + s \int \frac{R(x)}{x^{\sigma+1}} \, dx
\]

and computing the actual values of the integrated part and applying \( |R(x)| \leq \kappa x^\theta \) both there and in the integrals leads to the estimates in (15) and (14), resp. The last estimate in (15) follows again by Lagrange’s theorem.

**Lemma 2.** We have

\[
|\zeta(s) - \frac{\kappa}{s-1}| \leq \frac{A|s|}{\sigma - \theta} \quad (\theta < \sigma, \ t \in \mathbb{R}, \ s \neq 1),
\]

(16)

and in particular

\[
|\zeta(s)| \leq \frac{\sqrt{2} (A + \kappa)|t|}{\sigma - \theta} \quad (\theta < \sigma \leq 4, \ |t| \geq 4).
\]

(17)

On the other hand for small values of \( t \)

\[
|\zeta(s)(s-1) - \kappa| \leq \frac{A|s|(s-1)}{\sigma - \theta} \leq \frac{100A}{\sigma - \theta} \quad (\theta < \sigma \leq 4, \ |t| \leq 9)
\]

(18)

holds true, and thus we also have

\[
\zeta(s) \neq 0 \quad \text{for} \quad |s-1| \leq \frac{\kappa (1-\theta)}{A + 2\kappa}
\]

(19)

**Proof.** Actually, by \( N(x) = \kappa x + R(x) \) and the trivial integral \( \int_1^\infty x^{-\sigma} \, dx = 1/(s-1) \), it remains only to see that \( |\int_1^\infty x^{-\sigma} dR(x)| \leq A|s|/(\sigma - \theta) \). That is straightforward (and is contained, after \( X \to \infty \), in (15), too). Whence we have (16).

To deduce (17) one has to take into account that in the given domain \( t^2 + \sigma^2 \leq 2t^2 \), hence \( |s| \leq 2|t| \), and that \( t^2 |s-1|^2/(\sigma - \theta)^2 \geq t^4/\sigma^2 > 1/2 \), hence \( 1/|s-1| < \sqrt{2}|t|/(\sigma - \theta) \), too.
Estimating the arising factor $|s||s-1|$ trivially, we obtain (18). Moreover, to have that the two terms in $\zeta(s)(s-1) = \kappa + (s-1)\int_1^\infty x^{-s}dR(x)$ do not give zero it suffices that $\kappa > |s(s-1)|\frac{A}{\sigma-\theta}$, that is $(\kappa/A)(1-\theta) > |s(s-1)| + (\kappa/A)(1-\sigma)$.

Taking $r := |s-1| < \kappa(1-\theta)/(A+2\kappa)$ and estimating trivially we get $|s(s-1)| + (\kappa/A)(1-\sigma) \leq [1 + r + (\kappa/A)r \leq [A/A + \kappa/A + \kappa/(A+2\kappa) = (1-\theta)\kappa/A$, as needed.

\[\text{Lemma 3. We have} \quad |\zeta(s)| \leq \frac{(A+\kappa)\sigma}{\sigma-1} \quad (\sigma > 1). \quad (20)\]

and also

\[|\zeta(s)| \geq \frac{1}{\zeta(\sigma)} > \frac{\sigma-1}{(A+\kappa)\sigma} \quad (\sigma > 1). \quad (21)\]

\[\text{Proof. The first estimate follows from (13), last line after Lemma 5. In the critical strip we have (14), we have } \zeta(s) = \zeta(s) + \frac{\kappa X^{1-s}}{s-1} - \int_X^\infty \frac{dR(x)}{x^s}. \quad (23)\]

From here by termwise estimation, using the first part of the third line in (13) and (14), we arrive at

\[|\zeta(s)| \leq \kappa \log X + \frac{\sigma(A+\kappa)}{\sigma-\theta} + \frac{\kappa}{|t|}X^{1-\sigma} + A(|s| + |s|^\sigma)X^{\theta-\sigma}. \]

Since $|t| \geq 4$ and $1 < \sigma$, we have $|s|/(|t|\sigma) = \sqrt{|t|^2 + \sigma^2} \leq (33/32)$, and $1/|s-1| \leq 1/4 \leq \sigma/(\sigma-\theta)$, so for $X := |t|^{1/(\sigma-\theta)} \geq 1$ we also have

\[|\zeta(s)| \leq \frac{\kappa}{\sigma-\theta} \log |t| + \frac{\sigma(A+\kappa)}{\sigma-\theta} + \frac{\kappa}{4} + \frac{A\left(\frac{33}{32}|t|\sigma + |s|/|t|^\sigma\right)}{\sigma-\theta} \cdot \frac{1}{|t|} \leq \frac{\kappa}{\sigma-\theta} \log |t| + \frac{7\sigma(A+\kappa)}{3(\sigma-\theta)}. \]

\[\text{Lemma 5. In the critical strip we have} \quad |\zeta(s)| \leq \frac{5}{2}(A+\kappa) \max \left(\frac{1}{\theta-\sigma}, \frac{1}{1-\sigma}\right) \cdot |t|^{\frac{1-\sigma}{\sigma-\theta}} \quad (\theta < \sigma < 1, \ |t| \geq 4), \quad (24)\]

and also

\[|\zeta(s)| \leq 2\frac{A+\kappa}{\sigma-\theta} |t|^{\frac{1-\sigma}{\sigma-\theta}} \cdot \log |t| \quad (\theta < \sigma < 1, \ |t| \geq 4). \quad (25)\]

\[\text{Proof. Again we start with (23), coming from the first formula in (12). Since now } |t| \geq 4 \quad \text{and } \theta < \sigma < 1, \text{ we have } |s| \leq (5/4)|t|, \text{ and } |s| + \sigma - \theta < 3/2|t|, \text{ and } 1/|s-1| \leq 1/4, \text{ so the estimates of the first part of the first line of (13) and (14) yield whenever } X \geq 1 \text{ that}

\[|\zeta(s)| \leq \frac{5\kappa X^{1-\sigma}}{4} + \frac{A}{\sigma-\theta} \left(1 + \frac{3}{2} |t| X^{\theta-\sigma}\right). \]
Here choosing \( X := |t|^{1/(1-\theta)} \geq 1 \) implies the first estimate in (23).  

To get also the second estimate in (23), we use the second part of the first line of (14). Similarly as above we again choose \( X := |t|^{1/(1-\theta)} \geq 1 \) and obtain

\[
|\zeta(s)| \leq \left( \kappa + \kappa X^{1-\sigma} \log X + \frac{A}{\sigma - \theta} \right) + \frac{\kappa}{4} X^{1-\sigma} + \frac{3}{2} \frac{A}{\sigma - \theta} |t| X^{\theta - \sigma} \\
\leq \frac{5}{4} \kappa |t|^{\frac{1-\sigma}{1-\theta}} + \frac{\kappa}{1-\theta} |t|^{\frac{1-\sigma}{1-\theta}} \log |t| + \frac{5}{2} \frac{A}{\sigma - \theta} |t|^{\frac{1-\sigma}{1-\theta}}.
\]

As \( |t| \geq 4, \log |t| \geq \log 4 = 1.386... \) and the result follows.

\[\square\]

4 Estimates for the number of zeros of \( \zeta \)

Lemma 6. Let \( \theta < b < 1 \) and consider the rectangle \( H := \{ z \in \mathbb{C} : \Re z \in [b, 1], \Im z \in (t-h, t+h) \} \), where \( h := \frac{\sqrt{7}}{6} \sqrt{(b-\theta)(1-\theta)} \) and \( |t| \geq 6 \) is arbitrary. Then the number of zero-zeros \( n(H) \) in the rectangle \( H \) satisfy

\[
n(H) \leq \frac{1 - \theta}{b - \theta} \left( A_1 + \frac{3}{2} \log |t| \right) \quad A_1 := c + 6 \log(A + \kappa) + 6 \log \frac{1}{1 - \theta} \quad (26)
\]

Remark 1. Note that the very same estimate includes also the total number \( N \) of zeroes in the disc \( |s - (p + it)| \leq r := p - q \), where \( p = 1 + (1 - \theta) \) and \( q := \theta + \frac{2}{3}(b - \theta) \) are parameters introduced in the proof.

Proof. Let \( \theta < a < q < b < 1 < p \leq 2 \). Let us draw the circles \( C_R \) of radius \( R := p - a \) and \( C_r \) of radius \( r := p - q \) around \( z_0 := p + it \). Then the disk \( D_r \) bounded by \( C_r \) will cover the rectangle \( H_1 := [b, 1] \times [t - ih_1, t + ih_1] \) with \( h_1 := \sqrt{(b-q)(2p-b-q)} \). Actually, in the following we estimate the number of roots in \( D_r \); since \( H_1 \subset D_r \), we will be able to conclude the proof.

We use Jensen's inequality (see [10, p. 43]) in the form that for \( f \) regular in \( D(z_0, R) \) and \( r < R \)

\[
\log |f(z_0)| + n \log \frac{R}{r} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(z_0 + R e^{i\varphi})| d\varphi,
\]

where \( n \) is the number of zeroes of \( f \) in the disk \( D(z_0, r) \). Let now for an application of this estimate \( z_0 := p + it \) with \( |t| \geq 6, R = p - a < 2 - \theta \leq 2 \), \( r := p - q \), and \( f := \zeta \). Then the zeta function is indeed regular in the disc, and Jensen's inequality yields

\[
\log |\zeta(p + it)| + n \log \frac{R}{r} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\zeta(z_0 + R e^{i\varphi})| d\varphi.
\]

The inner disk of radius \( r \) covers the rectangle \( H_1 \), hence with the number of zeroes in this rectangle denoted by \( n_1 := n(H_1) \), obviously \( n_1 \leq n \). Clearly \( \log |\zeta(p + it)| \geq \log \left( \frac{R}{r} \right)^{\frac{1}{1-\kappa}\frac{1}{\kappa}} \) by Lemma 3 thus using also \( \log R/r = - \log r/R = - \log \left( 1 - \frac{R-r}{R} \right) > \frac{R-r}{R} \) we are led to

\[
\log \left( \frac{p-1}{A+\kappa} \right) + n_1 \log \frac{R}{r} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\zeta(z_0 + R e^{i\varphi})| d\varphi.
\]

It remains to estimate the integral. Combining (20) and (24) we are led to

\[
\log |\zeta(s)| \leq \log(4(A + \kappa)) + \max \left( \log \frac{1}{\sigma - \theta}, \log \frac{1}{1 - \sigma} \right) + (1 - \sigma) \log |t| \quad (s = \sigma + it),
\]

\[\text{In fact, the constant } 5/2 \text{ can even be replaced by } 2. \text{ Indeed, if } \theta < \sigma \leq \frac{1 + \theta}{2}, \text{ then } |t|^{\frac{1-\sigma}{1-\theta}} \geq 4^{1/2} \text{ allows to estimate } 1 \text{ by } \frac{1}{2} |t|^{\frac{1-\sigma}{1-\theta}}. \text{ If } \frac{1 + \theta}{2} \leq \sigma < 1, \text{ then } \frac{1}{\sigma - \theta} \leq \frac{1}{1-\sigma} |t|^{\frac{1-\sigma}{1-\theta}} \text{ for } |t| \geq 4, \text{ since } \frac{1-\sigma}{1-\sigma} = \left( \frac{1-\theta}{1-\sigma} - 1 \right) \frac{1}{\sigma - \theta} \text{ and the function } (1/u - 1)^{4u} \text{ is decreasing in } (0, 1/2).\]
all over \( \theta < \sigma \leq 4 \) and \( 4 \leq |\tau| \). So now
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\zeta(z_0 + Re^{i\varphi})| d\varphi \leq c + \log(A + \kappa) + \frac{1}{2} \log |t| + \log \frac{1}{R} + \frac{1}{\pi} \int_{\pi/2}^{\pi} \log \frac{R}{p + R \cos \varphi - \theta} d\varphi + \frac{1}{\pi} \int_{0}^{\pi} \log \frac{R}{|p + R \cos \varphi - 1|} d\varphi.
\]
With some calculation it can be seen that both integrals are bounded by absolute constants, hence we obtain
\[
n_1 \leq \frac{p-a}{q-a} \left( c + 2 \log(A + \kappa) + \log \frac{p}{R(p-1)} + \frac{1}{2} \log |t| \right)
\]
Observe that the uniform estimate in the brackets does not depend on the choice of \( \theta < a < q \), hence we can now take \( a \to \theta^+ \).

It is time to make a few choices, so let
\[
p := 1 + (1-\theta)(< 2), \quad q := \theta + \frac{2}{3}(b-\theta) = b - \frac{b-\theta}{3}.
\]
Then we are led to
\[
n_1 \leq \frac{3(1-\theta)}{b-\theta} \left( c + 2 \log(A + \kappa) + 2 \log \frac{1}{1-\theta} + \frac{1}{2} \log |t| \right).
\]
Clearly, the height of the rectangle \( H_1 \) is now
\[
h_1 := \sqrt{(b-q)(2p-b-q)} > \sqrt{\frac{1}{3}(b-\theta)(2(1-\theta) + \frac{1}{3}(1-\theta))} = \frac{\sqrt{7}}{3} \sqrt{(b-\theta)(1-\theta)} = h,
\]
therefore \( H_1 \supset H \) and the assertion is proved.

Joining the necessary number of such rectangles we obviously get.

**Lemma 7.** Let \( \theta < b < 1 \) and consider the rectangle \( H := \{ z \in \mathbb{C} : \Re z \in [b, 1], \Im z \in (t-1, t+1) \} \). Then the number of zeta-zeros \( n(H) \) in the rectangle \( H \) satisfy
\[
n(H) \leq \frac{\sqrt{1-\theta}}{\sqrt{(b-\theta)^3}} \left( A_2 + 3 \log |t| \right) \quad A_2 := c + 12 \log(A + \kappa) + 12 \log \frac{1}{1-\theta}
\]
and consequently the number of zeroes in \([b, 1] \times [-iT, iT] \) is
\[
N(b, T) := \# \{ \rho = \beta + i\gamma : \zeta(\rho) = 0, \beta \geq b, |\gamma| \leq T \} \leq \frac{\sqrt{1-\theta}}{\sqrt{(b-\theta)^3}} \left( 6T \log T + A_3 T \right) \quad \text{with} \quad A_3 := 2A_2.
\]

This can slightly be improved, in particular for the case when \( b \) is closer to \( \theta \) than to 1, as follows.

**Lemma 8.** Let \( \theta < b < 1 \) and consider any height \( T \geq 6 \) together with the rectangle \( Q := Q(b, T) := \{ z \in \mathbb{C} : \Re z \in [b, 1], \Im z \in (-T, T) \} \). Then the number of zeta-zeros \( N(b, T) \) in the rectangle \( Q \) satisfy
\[
N(b, T) \leq \frac{1}{b-\theta} \left\{ \frac{2}{\pi} T \log T + \left( \frac{8}{3} \log(A + \kappa) + \frac{4}{3} \log \frac{1}{b-\theta} + 3.6 \right) T \right\}.
\]
Proof. Instead of Jensen’s theorem, we now apply Littlewood’s theorem (see \cite{5} p. 166) to the rectangle \(Q(a, T) := [a, 2] \times [-iT, iT]\), with \(\theta < a \leq (1 + \theta)/2\) and also \(a < b\), assuming momentarily that on the boundary \(\partial Q(a, T)\) there is no zero of \(\zeta\) (and then get it even for other values of \(a\) and \(T\) by taking limits). This theorem provides the average formula

\[
2\pi \int_a^2 N(q, T) dq = \int_{-T}^T [\log|\zeta(a + it)| - \log|\zeta(2 + it)|] \, dt + \int_a^2 [\arg\zeta(\sigma + iT) - \arg\zeta(\sigma - iT)] \, d\sigma,
\]

where \(\arg\zeta\) is defined by continuous variation from \(\zeta(2 + it)\). As in \cite{5}, the estimation of the second integral is executed by an appeal to Backlund’s observation: \(\arg\zeta(s) = 3 \log\zeta(s) = \frac{1}{2\pi} \left[ \log\zeta(s) - \log\zeta(s) \right]\), hence the second integral is

\[
\int_a^2 [\arg\zeta(\sigma + iT) - \arg\zeta(\sigma - iT)] \, d\sigma = -i \int_a^2 [\log\zeta(\sigma + iT) - \log\zeta(\sigma - iT)] \, d\sigma
\]

At \(\sigma = 2\), Lemma 3 \cite{21} provides that either \(|\Re\zeta(2 \pm iT)| \geq 1/(2\sqrt{2}(A + \kappa))\), or \(|3\zeta(2 \pm iT)| \geq 1/(2\sqrt{2}(A + \kappa))\); e.g. in the first case let

\[
g(s) := g_T(s) := \frac{1}{2} [\zeta(s + iT) + \zeta(s - iT)].
\]

The number of zeroes \(\nu(\sigma)\) of \(g\) along the segment \([\sigma, 2]\) is the number of cases when \(\Re\zeta(\alpha \pm iT)\) vanishes along this segment, hence by the definition using continuous variation, we must have \(|\arg\zeta(\sigma \pm iT)| \leq (\nu(\sigma) + 1)\pi\). Therefore,

\[
\frac{1}{\pi} \int_a^2 \arg\zeta(\sigma \pm iT) \, d\sigma \leq \int_a^2 \nu(\sigma) \, d\sigma + (2-a) \leq \int_a^2 n_g(2 - \sigma) \, d\sigma + 2,
\]

where now \(n(2-g) := n_g(2-\sigma) := \#\{s : |s-2| = 2-\sigma, g(s) = 0\}\), which is clearly at least as large as \(\nu(\sigma)\). Now we use Jensen’s formula to estimate the last integral as

\[
\int_a^2 n_g(2 - \sigma) \, d\sigma = \int_0^{2-a} n(r) \, dr < 2 \int_0^{2-a} n(r) \, dr \leq \frac{1}{\pi} \int_0^{2\pi} \log|g(2+(2-a)e^{i\varphi})|\,d\varphi - 2 \log|g(2)|.
\]

By the above, \(-\log|g(2)| = -\log|\Re\zeta(2 + iT)| \leq \log(2\sqrt{2}(A + \kappa))\), while for the integral we can apply \(|g(2 + (2-a)e^{i\varphi})| \leq \max(|\zeta(2 + iT + (2-a)e^{i\varphi})|, |\zeta(2 - iT + (2-a)e^{i\varphi})|)\).

This last expression can be estimated using the uniform estimates of Lemma 2 \cite{17} (note that \(T \geq 6\) so the variables considered are all in the range \(|t| \geq 4\)). We are led to

\[
\int_a^2 n_g(2 - \sigma) \, d\sigma \leq 2 \left( \log\frac{\sqrt{2}(A + \kappa)}{a - \theta} + \log(T + 2) + \log(2\sqrt{2}(A + \kappa)) \right)
\leq 2 \log T + 4 \log(A + \kappa) + 2 \log\frac{1}{a - \theta} + 3.4.
\]

In all for the argumentum integrals we obtain

\[
\int_a^2 [\arg\zeta(\sigma + iT) - \arg\zeta(\sigma - iT)] \, d\sigma \leq 2\pi \left( \int_a^2 n_g(2 - \sigma) \, d\sigma + 2 \right)
\leq 4\pi \log T + 8\pi \log(A + \kappa) + 4\pi \log\frac{1}{a - \theta} + 34.
\]
Now let us go to the evaluation of the other integral. Using once again the estimate of Lemma \(3\) \(24\), the term along the \(\sigma = 2\) line is of no problem, contributing
\[
\int_{-T}^{T} -\log |\zeta(2 + it)| dt \leq 2T (\log 2 + \log(A + \kappa)).
\]
For the most essential part, the integral along the \(\sigma = a\) line, in the finite range between \([-4, 4]\) an application of \(16\) of Lemma \(2\) yields, using also \(\theta < a \leq (1 + \theta)/2\),
\[
\int_{0}^{4} \log |\zeta(a + it)| dt \leq \int_{0}^{4} \log \left( \frac{A|a + it| + \kappa}{\sqrt{1 + t^2}} \right) dt \leq \int_{0}^{4} \log \left( \frac{(A + \kappa)\sqrt{1 + t^2}}{a - \theta} \right) dt
\]
\[= 4 \log \left( \frac{A + \kappa}{a - \theta} \right) + \frac{1}{2} \int_{0}^{4} \log(1 + t^2) dt = 4 \log \left( \frac{A + \kappa}{a - \theta} \right) + 2.992244.\]
For the range with \(4 \leq |t| \leq T\) we can refer to Lemma \(2\) \(17\), thus we obtain
\[
\int_{-T}^{T} \log |\zeta(a + it)| dt < 8 \log \left( \frac{A + \kappa}{a - \theta} \right) + 6 + 2 \int_{4}^{T} \left( \log \frac{\sqrt{2}(A + \kappa)}{a - \theta} + \log t \right) dt
\]
\[\leq 2T \log \left( \frac{A + \kappa}{a - \theta} \right) + 6 + (T - 4) \log 2 + 2 \{ (T \log T - T) - (4 \log 4 - 4) \}
\]
\[= 2T \log T + \log \frac{\sqrt{2}(A + \kappa)}{e(a - \theta)} \cdot 2T + 14 - 20 \log 2
\]
\[< 2T \log T + \log \frac{\sqrt{2}(A + \kappa)}{e(a - \theta)} \cdot 2T + 1.4.\]
Collecting the last three estimates we obtain from Littlewood’s theorem that
\[
2\pi \int_{a}^{2} N(q, T) dq \leq 2T \log T + 2 \log \left( \frac{2\sqrt{2}(A + \kappa)^2}{e(a - \theta)} \right) T + 4\pi \log T
\]
\[+ 8\pi \log(A + \kappa) + 4\pi \log \frac{1}{a - \theta} + 35.4.
\]
Here we choose \(a := (b + \theta)/2\) and apply
\[
\frac{b - \theta}{2} N(b, T) \leq \int_{a}^{2} N(q, T) dq
\]
to obtain
\[
N(b, T) \leq \frac{1}{b - \theta} \left\{ \frac{2}{\pi} T \log T + \left( \frac{4}{\pi} \log(A + \kappa) + \frac{5 \log 2 - 2}{\pi} + \frac{2}{\pi} \log \frac{1}{b - \theta} \right) T
\]
\[+ 4 \log T + 8 \log(A + \kappa) + 4 \log \frac{1}{b - \theta} + 11.3 \right\}
\]
\[\leq \frac{1}{b - \theta} \left\{ \frac{2}{\pi} T \log T + \left( \frac{8}{3} \log(A + \kappa) + \frac{4}{3} \log \frac{1}{b - \theta} + 3.6 \right) T \right\}.
\]

\[\square\]

**Corollary 1.** Let \(\theta < b < 1\) and consider any heights \(T > R \geq 6\) together with the rectangle \(Q := Q(b, R, T) := \{ z \in \mathbb{C} : \Re z \in [b, 1], \forall z \in (R, T) \}\). Then the number of zeta-zeros \(N(b, R, T)\) in the rectangle \(Q\) satisfy
\[
N(b, R, T) \leq \frac{2}{b - \theta} \int_{(b + \theta)/2}^{2} N(q, R, T) dq \leq \frac{2(T - R) + 8\pi}{\pi(b - \theta)} \left( \log T + \log \left( \frac{30(A + \kappa)^2}{b - \theta} \right) \right).
\]
\[(30)\]
In particular, for the zeroes between \( T - 1 \) and \( T + 1 \) we have for \( T \geq 7 \)

\[
N(b, T - 1, T + 1) \leq \frac{2}{b - \theta} \int_{(b+\theta)/2}^{2} N(q, T - 1, T + 1) dq \\
\leq \frac{30}{\pi(b - \theta)} \left( \log T + \log \left( \frac{35(A + \kappa)^2}{b - \theta} \right) \right).
\]

Proof. We can apply the Littlewood theorem not only to \( Q(b, T) \), but also to the rectangle \( Q(b, R) := [b, 2] \times [-iR, iR] \): moreover, we can simply subtract the contributions of the two formulae to estimate the contribution of the zeroes in between. Thus we obtain (again with \( a := (b + \theta)/2 \)) the estimations

\[
2\pi \frac{b - \theta}{2} [N(b, T) - N(b, R)] \leq 2\pi \int_{a}^{2} [N(q, T) - N(q, R)] dq \\
= \int_{[-T,-R] \cup [R,T]} \left[ \log |\zeta(a + it)| - \log |\zeta(2 + it)| \right] dt \\
+ \int_{a}^{2} [\arg \zeta(\sigma + iT) - \arg \zeta(\sigma - iT)] d\sigma \\
- \int_{a}^{2} [\arg \zeta(\sigma + iR) - \arg \zeta(\sigma - iR)] d\sigma.
\]

As before, for the difference of the two argumentum integrals (and writing in the chosen value of \( a = (b + \theta)/2 \)) we obtain the estimation

\[
\int_{a}^{2} (\ldots T \ldots) d\sigma - \int_{a}^{2} (\ldots R \ldots) d\sigma \leq 8\pi \log T + 16\pi \log(A + \kappa) + 8\pi \log \frac{2}{b - \theta} + 68.
\]

On the other hand for the contribution of the main term integrals of \( \log |\zeta| \) we have a better estimation, due to the fact that here the part over \([-R, R]\) of the integral is canceled. For the part over \( \sigma = 2 \) by the same uniform estimation we obtain

\[
\int_{[-T,-R] \cup [R,T]} -\log |\zeta(2 + it)| dt \leq 2(T - R) \log (2(A + \kappa)) .
\]

Lastly, the contribution over the \( \sigma = a \) line can be estimated as

\[
\int_{[-T,-R] \cup [R,T]} \log |\zeta(a + it)| dt \leq 2 \int_{[-T,-R] \cup [R,T]} \left( \log \left( \frac{\sqrt{2}(A + \kappa)}{a - \theta} + \log t \right) \right) dt. \\
\leq 2(T - R) \left( \log \frac{2\sqrt{2}(A + \kappa)}{b - \theta} + \log T \right).
\]

So collecting our estimates we are led to

\[
\pi(b - \theta) N(b, R, T) \leq 2\pi \int_{a}^{2} [N(q, T) - N(q, R)] dq \\
\leq 2(T - R) \left( \log \left( \frac{4\sqrt{2}(A + \kappa)^2}{b - \theta} \right) + \log T \right) \\
+ 8\pi \log T + 16\pi \log(A + \kappa) + 8\pi \log \frac{1}{b - \theta} + 68 + 8\pi \log 2 \\
\leq 2(T - R + 4\pi) \left( \log T + \log \left( \frac{30(A + \kappa)^2}{b - \theta} \right) \right).
\]

This formula applies also for \( N(b, T + 1, T - 1) \) on noting that \( \log(T + 1) \leq \log T + \log \frac{8}{T} \). \( \square \)
5 The logarithmic derivative of the Beurling $\zeta$

Lemma 9. Let $z = a + it_0$ with $t_0 \geq 8$ and $\theta < a \leq 1$. With $\delta := (a - \theta)/4$ denote by $S$ the (multi)set of the $\zeta$-zeroes (listed according to multiplicity) not farther from $z$ than $\delta$. Then we have with an absolute constant $C$

$$\left| \frac{\zeta'}{\zeta}(z) - \sum_{\rho \in S} \frac{1}{z - \rho} \right| < C \frac{1 - \theta}{(a - \theta)^2} \left( A_3 + \log t_0 + \log \frac{1}{a - \theta} \right). \quad (32)$$

Proof. Let now $z_0 := p + it_0$, $p := 1 + (1 - \theta)$, and denote by $D_j$ the disk around it with the radii $R_j := p - (\theta + j\delta)$ for $j = 1, 2, 3$: we chose $\delta := (a - \theta)/4$. By Lemma 6 and Remark 11 (applied with $b := \theta + \frac{3}{2}\delta$) we know that the number $N$ of $\zeta$-zeroes in the disk $D_1$ satisfies

$$N \leq \frac{1 - \theta}{\delta} \left( \frac{2}{3} A_1 + \log t_0 \right). \quad (33)$$

Now denote the (multi)set of all these zeroes (again taking them according to multiplicities) as $S'$ and define

$$P(s) := \prod_{\rho \in S'} \alpha_\rho(s), \quad \text{where} \quad \alpha(s) := \alpha_\rho(s) := \frac{(s - z_0)R_1 - (\rho - z_0)R_1}{R_1^2 - (s - z_0)(\rho - z_0)}.$$ 

Let us consider now

$$f(s) := \frac{\zeta(s)}{P(s)}$$

which is a regular function in $D_1$, moreover, by construction, it is non-vanishing in $D_1$. Hence also $g(s) := \log f(s)$ is regular at least in $D_1$; by appropriate choice of the logarithm we may assume that $3g(z_0) = \arg f(z_0) \in [-\pi, \pi]$.

First we estimate the order of magnitude of $f$ on the perimeter of $D_1$. If $s \in \partial D_1$, then any factor of $P(s)$ is exactly 1 in modulus, hence $|f(s)| = |\zeta(s)|$. Note that $\sigma - \theta$ is at least $\delta$, $\sigma < p + R_1 < 2p - \theta < 4$, $R_1 < p < 2$ and $t > t_0 - R_1 > 6$. Here we can consider the four cases according to $\sigma \geq p$, $1 < \sigma < p$, $\frac{1 + \theta}{2} \leq \sigma \leq 1$ and $\theta < \sigma < \frac{1 + \theta}{2}$, applying (20), (22), (23) and (24), respectively. With some calculation we obtain for all cases

$$|f(s)| = |\zeta(s)| \leq \frac{5}{2} \frac{(A + \kappa)t}{\delta} \quad (s \in \partial D_1).$$

As $g(s) := \log f(s)$ is analytic on $D_1$, we can apply the Borel-Carathéodory lemma:

$$\max_{|s-z_0| \leq R_2} |g(s) - g(z_0)| \leq \frac{2R_2}{R_1 - R_2} \max_{|s-z_0| \leq R_1} \Re(g(s) - g(z_0)).$$

That is, with a slight reformulation we obtain for $s \in D_2$

$$|g(s)| \leq |g(s) - g(z_0)| + |\Im g(z_0)| + |\Re g(z_0)| \leq \pi + \frac{2R_2}{\delta} \max_{|s-z_0| \leq R_1} \Re g(s) + \left( \frac{2R_2}{\delta} \pm 1 \right) \Re(-g(z_0)),$$

where here by the choice of our radii and parameters, it is clear that the sign of the coefficient in front of $\Re(-g(z_0))$ is positive. Firstly,

$$\Re g(s) = \log |f(s)| \leq \log \left( \frac{5}{2} \frac{(A + \kappa)|t|}{\delta} \right) \quad (s \in \partial D_1),$$

---

2For the four cases the quoted inequalities yield in respective order: 1) $\frac{\pi - \theta}{\alpha + 1} \leq \frac{\pi - \theta}{\alpha - 1} = 2$, hence $(A + \kappa - \overline{t})/\alpha \leq (A + \kappa)(1 + \frac{1}{\alpha + 1}) \leq (A + \kappa)(\frac{\pi - \theta}{\alpha - 1}) < (A + \kappa)\frac{\pi}{\alpha} < \frac{\pi}{\alpha}(t); 2) \frac{2R_2}{\delta} \log t + \frac{\pi}{\alpha}(t) \leq \frac{2R_2}{\delta} \log t + \frac{\pi}{\alpha}(t); 3) 2 \frac{2R_2}{\delta} \log t \leq \frac{2R_2}{\delta} \log t < \frac{2R_2}{\delta} \log t$ and finally 4) $\delta \frac{2R_2}{\delta} t$. 13
that is, for the maximum of $\Re g(s)$ on $\partial D_1$ we get

$$\max_{|s-z_0| \leq R_1} \Re g(s) \leq \log \left( \frac{25(A + \kappa)|t_0|}{8 \delta} \right).$$

Application of the Borel-Caratheodory lemma on $D_2$ thus yields

$$\max_{|s-z_0| \leq R_2} |g(s)| \ll \frac{1 - \theta}{\delta} \left( A_1 + \log t_0 + \log \frac{1}{\delta} + \Re(-g(z_0)) \right).$$

Since the factors $\alpha(s)$ of $P(s)$ are all less than 1 at $z_0$, by (21) we obtain

$$-\Re g(z_0) = \log \left| \frac{1}{\zeta(z_0)} \right| + \log |P(z_0)| \leq \log \frac{2(A + \kappa)}{1 - \theta} < \frac{1}{6} A_1.$$

Hence we are led to

$$\max_{|s-z_0| \leq R_2} |g(s)| \ll \frac{1 - \theta}{\delta} \left( A_1 + \log t_0 + \log \frac{1}{\delta} \right).$$

Finally we apply the usual Cauchy estimate to the regular function $g = \log f$ in $D_2$ to obtain a generally valid estimate of the derivative in $D_3$. Thus the radius of the estimation is $\delta$, which will divide on the right hand side to give

$$\max_{|s-z_0| \leq R_3} |g'(s)| \ll \frac{1 - \theta}{\delta^2} \left( A_1 + \log t_0 + \log \frac{1}{\delta} \right). \quad (34)$$

In view of the definition of $f(s)$, $P(s)$ and $S'$ we have $f'/f = \zeta'/\zeta - P'/P$, hence

$$g'(s) = \frac{\zeta'}{\zeta}(s) - \sum_{\rho \in S'} \frac{1}{s - \rho} - \sum_{\rho \in S'} \frac{(\rho - z_0)}{R_1^2 - (s - z_0)(\rho - z_0)},$$

and the last sum can be estimated by $N/\delta$, for any $s \in D_3$ the difference in the denominator is at least $2\delta R_1$ in modulus, while the numerator is at most $R_1$. However, applying (34) to $N$, the so resulting estimation can be melted in the main term.

Finally we need to take care of the difference between $S$ and $S'$. Clearly, $S \subset S'$, and for any $\rho \in S \setminus S \ |z - \rho| > \delta$. Therefore, $\sum_{\rho \in S \setminus S} \frac{1}{z - \rho}$ does not exceed $N/\delta$, which is already known to be feasible. This yields the result. $\square$

**Lemma 10.** For any given two parameters $\theta < a < b < 1$ with $b \geq \theta + 2(a - \theta)$ i.e. $a \leq (b + \theta)/2$, there exists a broken line $\Gamma = \Gamma(a, b)$, symmetric to the real axis and consisting of horizontal and vertical line segments only, so that its upper half is

$$\Gamma_+ = \bigcup_{k=1}^{\infty} \{[\sigma_{k-1} + it_{k-1}, \sigma_{k-1} + it_k] \cup [\sigma_{k-1} + it_k, \sigma_k + it_k]\}$$

with $\sigma_j \in [a, b]$, $t_0 = 0$, $t_1 \in [5, 6]$ and $t_j \in [t_{j-1} + 1, t_{j-1} + 2] \ (j \geq 2)$ and satisfying that the distance of any $\zeta$-zero $\rho = \beta + i\gamma$ from $\Gamma$ is at least

$$d := d(\rho, \Gamma) > \frac{(b - \theta)^2}{7} \frac{1}{\log |\gamma| + B(b)} \quad \text{with} \quad B(b) := c \log \left( \frac{A + \kappa}{b - \theta} \right). \quad (35)$$

Moreover, the same separation from $\zeta$-zeroes holds also for the whole horizontal line segments $[a + it_k, 2 + it_k]$, $k = 1, \ldots, \infty$. 

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Proof. The proof is a straightforward inductive construction, using each step the above Corollary, to ensure two-sided avoidance of all the zeroes. For a very similar construction see [27, 28].

With a very slight modification of the same construction, one can obtain the following, technical strengthening of the lemma.

Lemma 11. For any given two parameters \( \theta < a < b < 1 \) with \( b \geq \theta + 2(a - \theta) \) i.e. \( a \leq (b + \theta)/2 \), and for any finite set \( \mathcal{A} := \{0 =: \alpha_0 < \alpha_1 < \cdots < \alpha_N\} \), there exists a broken line \( \Gamma = \Gamma^4(a, b) \), symmetric to the real axis and consisting of horizontal and vertical line segments only, so that its upper half is

\[
\Gamma_+ = \bigcup_{k=1}^{\infty} \{[\sigma_{k-1} + it_{k-1}, \sigma_{k-1} + it_k] \cup [\sigma_{k-1} + it_k, \sigma_k + it_k]\}
\]

with \( \sigma_j \in [a, b] \), \( t_0 = 0 \), \( t_1 \in [5, 6] \) and \( t_j \in [t_{j-1} + 1, t_{j-1} + 2] \) \((j \geq 2)\) and satisfying that the distance of all translates \( \rho \pm i\alpha_n \) \((n = 0, \ldots, N)\) of any \( \zeta \)-zero \( \rho = \beta + i\gamma \) from \( \Gamma \) is at least

\[
d := d(\rho, \Gamma(\mathcal{A})) := \min_{n=0,\ldots,\infty} (d(\rho + i\alpha_n, \Gamma), d(\rho - i\alpha_n, \Gamma)) > \frac{(b - \theta)^2}{14(N + 1) \log(\alpha_N + |\gamma|) + B(b)},
\]

with the same \( B := B(b) \) as in (35). Moreover, the same separation from \( \zeta \)-zeros and their \( \mathcal{A} \)-translates holds also for the whole horizontal line segments \([a + it_k, 2 + it_k], k = 1, \ldots, \infty\) as well.

Lemma 12. For any \( \theta < a < b < 1 \), on the line \( \Gamma = \Gamma(a, b) \), as well as on the horizontal line segments \([a + it_k, 2 + it_k], k = 1, \ldots, \infty\) constructed in Lemma 10, we have

\[
\left| \frac{\zeta'(s)}{\zeta(s)} \right| \ll \frac{1}{(a - \theta)^3} (A_3 + \log t) \left( A_3 + \log t + \log \frac{1}{a - \theta} \right).
\]

A similar inequality holds (with the \( O \)-constant multiplied by \( N := \# \mathcal{A} \)) for any finite set \( \mathcal{A} := \{0 =: \alpha_0 < \alpha_1 < \cdots < \alpha_N\} \), and the corresponding broken line \( \Gamma = \Gamma^4(a, b) \), constructed in Lemma 11.

Proof. The combination of Lemmas 9 and 10 give that for any \( s \in \Gamma \cup (\cup_{k=1}^{\infty} [a + it_k, 2 + it_k]) \), it suffices to estimate the contribution of the zeroes in the formula (32). The total number of zeroes in this sum can at most be \( N(a', t - 1, t + 1) \), with \( a' := (a + \theta)/2 \), so according to estimate (31) of Corollary 1 at most \( \ll (a' - \theta)^{-1}(\log t + A_3 - \log (a' - \theta)) \). Denote \( h' := \sqrt{7}/3 \sqrt{(a' - \theta)(1 - \theta)} \) and \( H' := H(t; a', h') := [a', 1] \times [\sqrt{t - h'}, \sqrt{t + h'}] \). We separate those zeros which are closer to \( s \) than \( h' \): the number of these zeros is, by Lemma 6, at most \( n(H') \leq \frac{1 - \theta}{a' - \theta} (A_1 + \frac{3}{2} \log |t|) \). However, even these zeroes, by construction of \( \Gamma \), stay away from \( s \) at least as far as \( d \) in Lemma 10. Hence the contribution of zeros in formula (32), when applied to \( s \), can be estimated as

\[
\sum_{\rho \in \mathcal{S}} \frac{1}{|\rho - s|} \ll n(H'(t; a', h')) \frac{1}{d} + N(a', t - 1, t + 1) \frac{1}{h'}
\]

\[
\ll \frac{1 - \theta}{a' - \theta} (A_1 + \log |t|) \frac{\log t + A_3 - \log (a' - \theta)}{(a' - \theta)^2} + \frac{\log t + A_3 - \log (a' - \theta)}{a' - \theta} \left( \frac{1}{\sqrt{(a' - \theta)(1 - \theta)} - 1} \right)
\]

Since \( a - \theta = 2(a' - \theta) \), this gives the assertion. The proof for \( \Gamma^4(a, b) \) is analogous. \( \square \)
6 Riemann-von Mangoldt type formulae of prime distribution

with zeroes of the Beurling $\zeta$

In the following we will denote the set of $\zeta$-zeroes, lying to the right of $\Gamma$, by $\mathcal{Z}(\Gamma)$, and denote $\mathcal{Z}(\Gamma,T)$ the set of those zeroes $\rho = \beta + i\gamma \in \mathcal{Z}(\Gamma)$ which satisfy $|\gamma| \leq T$.

**Lemma 13** (von Mangoldt formula). Let $\theta < a < 1$ and $b := 2a - \theta < 1$, and let $\Gamma = \Gamma(a,b)$ be the curve defined in Lemma 10 with $t_k$ the corresponding set of abscissae in the construction. Then for any $5 \leq t_k < x$ we have

$$
\psi(x) = x - \sum_{\rho \in \mathcal{Z}(\Gamma,t_k)} \frac{x^\rho}{\rho} + O \left( \frac{1}{a-\theta} \left( A_3 + \log \frac{x}{a-\theta} \right)^3 \left( \frac{x}{t_k} + x^b \right) \right).
$$

**Proof.** Analogously to the effective version of the classical Perron formula, as given in e.g. [33, Théorème 2, Chapitre II.2, p. 135], for any fixed $T$, $x > 1$, $1 < p < 2$ one can prove the effective representation

$$
\psi(x) = \frac{1}{2\pi i} \int_{p-iT}^{p+iT} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + O \left( \int_{1}^{\infty} \frac{x^p}{u^p(1+T|\log(x/u)|)} du \right).
$$

Here we move the path of integration to the curve $\Gamma$ of Lemma 10, more precisely, to the curve $\Gamma_T$ consisting of the part of $\Gamma$ in the strip $-T \leq s \leq T$ joined by the two segments $[p-iT, q-iT]$ and $[q+iT, p+iT]$, with $q \pm iT \in \Gamma$. Obviously, the set of zeroes of $\zeta$ in the domain encircled by $[p-iT, p+iT]$ and $\Gamma_T$ is $\mathcal{Z}(\Gamma,T)$. Assuming that no zero lies on $\Gamma_T$ it follows by an application of the residue theorem

$$
\psi(x) = x - \sum_{\rho \in \mathcal{Z}(\Gamma,T)} \frac{x^\rho}{\rho} + \int_{\Gamma_T} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + O \left( \int_{1}^{\infty} \frac{x^p}{u^p(1+T|\log(x/u)|)} du \right). \tag{38}
$$

Now let us put $T = t_k \geq 5$. Since the arc length of $\Gamma_{t_k}$ is at most $5t_k$, by an application of Lemma 12 we find

$$
\int_{\Gamma_T} \left| \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \right| ds \ll \frac{1}{(a-\theta)^3} \left( A_3 + \log t_k + \log \frac{1}{a-\theta} \right)^2 \left( \frac{x^p}{t_k} + x^b \log \frac{t_k}{a} \right). \tag{39}
$$

For the $O$-term we execute a detailed calculus here. First, observe that by the definition of $\Lambda(g)$, we have $d\psi(u) \leq \log u \ dN(u)$, (the inequality interpreted as between positive measures), whence it suffices to treat

$$
J := J(x,p,T) := x^p \int_{1}^{\infty} \frac{\log u}{u^p(1+T|\log(x/u)|)} dN(u). \tag{40}
$$

First we substitute here in the integral the log by $\log x$, with an error

$$
\int_{1}^{\infty} \frac{|\log(x/u)|}{u^p(1+T|\log(x/u)|)} dN(u) \leq \int_{1}^{\infty} \frac{1}{Tu^p} dN(u) = \frac{1}{T^2} \zeta(p) \leq \frac{A + k}{T^p(1-p)},
$$

invoking (20) in the last step. Writing in $N(u) = \kappa u + \mathcal{R}(u)$, we thus find

$$
|J - \kappa \log xI - x^p \log xL| \leq \frac{2(A + k)}{(p-1)T} x^p, \tag{41}
$$

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where
\[ I := \int_1^\infty \frac{(x/u)^p}{1 + T|\log(x/u)|} \, du \] (42)
and
\[ L := \int_1^\infty \frac{dR(u)}{u^p 1 + T|\log(x/u)|}. \] (43)

Cutting the interval of integration at \( x \), we denote \( I' \) and \( L' \) the integrals on \([1,x]\), and \( I', L' \) the integrals on \([x,\infty]\). On the interval \([1,x]\) the absolute value is \( \log(x/u) \), and conversely, on \([x,\infty]\) it is \( \log(u/x) \). On both parts, the integrand is a nice, continuously differentiable function. Substituting \( v := x/u \) and then \( w := T \log v \) we obtain
\[ I' = \int_1^x \frac{x v^{p-2} dv}{1 + T \log v} = \frac{x}{T} \int_0^\infty \frac{e^{\alpha w} dw}{1 + w} \quad \text{with} \quad \alpha := \frac{p - 1}{T} \]
Estimating the exponential by \( e \) up to \( 1/\alpha - 1 \), the part until \( 1/\alpha - 1 \) is at most \( e \log(1/\alpha) \). The rest is at most
\[ \int_{1/\alpha - 1}^{\infty} e^{\alpha w} dw < e^{\alpha T} x^\alpha, \]
whence
\[ I' \leq \frac{e x (\log T + \log \frac{1}{p-1})}{T} + \frac{x^p}{T} \] (44)

To estimate \( I'' \) we substitute first \( v := u/x \), and second \( w := T \log v \) to find
\[ I'' = x \int_1^\infty \frac{dv}{v^p (1 + T \log v)} = \frac{x}{T} \int_0^\infty \frac{e^{-\alpha w} dw}{1 + w}. \]
with the very same \( \alpha \). Here we can calculate further as
\[ \int_0^\infty \frac{e^{-\alpha w} dw}{1 + w} = \left[ -\frac{e^{-\alpha w}}{\alpha(1+w)} \right]_0^\infty - \frac{1}{\alpha} \int_0^\infty \frac{e^{-\alpha w} dw}{(1+w)^2} = \frac{1}{\alpha} \int_0^\infty \frac{1 - e^{-\alpha w}}{(1+w)^2} dw \]
\[ \leq \frac{1}{\alpha} \int_0^\infty \frac{\min(1,\alpha w)}{(1+w)^2} dw = \int_0^{1/\alpha} \frac{w \, dw}{(1+w)^2} + \frac{1}{\alpha} \int_{1/\alpha}^\infty \frac{dw}{(1+w)^2} \]
\[ \leq \log(1 + 1/\alpha) + \frac{1}{1 + \alpha} \leq \log(1/\alpha) + 2 \]
\[ = \log T + \log \frac{1}{p - 1} + 2. \] (45)

So in all we obtain
\[ I \ll \frac{x^p + x \log T + x \log \frac{1}{p-1}}{T}. \] (46)

In case of \( L \) we must pay some attention to the possible sign changes of \( R \), although the order of this term is in general smaller. Now
\[ L' = \left[ \frac{R(u)}{u^p (1 + T \log(x/u))} \right]_1^x + \int_1^x \frac{R(u) \left( p - T (1 + T \log(x/u))^{-1} \right)}{u^{p+1} (1 + T \log(x/u))} \, du \]
so
\[ \left| L' - \frac{R(x)}{x^p} \right| \leq \int_1^x \frac{|R(u)| \left( p (1 + T \log(x/u)) + T \right)}{u^{p+1} (1 + T \log(x/u))^2} \, du. \] (47)
Writing in $|R(u)| \leq Au^\theta$ and substituting $v := x/u$ we obtain\(^3\)

\[
\left| L - \frac{R(x)}{x^p} \right| \leq x^{\theta-p-1} \int_1^x \frac{Ax^{1+p-\theta}(2(1 + T \log v) + T)x dv}{(1 + T \log v)^2 v^2} \leq 2ATx^{\theta-p} \int_1^x \frac{v^{p-\theta}(1 + \log v) dv}{(1 + T \log v)^2 v} = 2ATx^{\theta-p} \int_0^{\log x} e^{(p-\theta)w} \frac{1 + w}{(1 + tw)^2} dw = 2ATx^{\theta-p} \left( \int_0^1 + \int_1^{\log x} \right) \leq ATx^{\theta-p} \int_0^1 \frac{dw}{(1 + Tw)^2} + Ax^{\theta-p} \int_1^{\log x} \frac{e^{(p-\theta)w}}{1 + Tw} dw = Ax^{\theta-p} \left( 1 - \frac{1}{T + 1} + \frac{1}{T} \int_T^{\log x} \frac{e^{\beta \xi}}{1 + \xi} d\xi \right) \quad (with \ \beta := \frac{p - \theta}{T}) \leq Ax^{\theta-p} \left( 1 + \frac{1}{(p-1)T} + \frac{x^{p-\theta}}{T} \right) = Ax^{\theta-p} \left( 1 + \frac{1}{(p-1)T} \right) + A \frac{x}{T}.
\]

Similarly, integration by parts gives

\[
L' = \left[ \frac{R(u)}{u^p(1 + T \log(u/x))} \right]_x^\infty + \int_x^\infty \frac{R(u)}{u^{p+1}(1 + T \log(u/x))} \frac{du}{u^p}
\]

whence with the very parameter $\beta$

\[
\left| L' + \frac{R(x)}{x^p} \right| \leq A \int_x^\infty \frac{u^{\theta-p-1}[T + p(1 + T \log(u/x))]}{(1 + T \log(u/x))^2} du \leq Ax^{\theta-p} \int_1^\infty \frac{v^{\theta-p-1}[T + p(1 + \log v)]}{(1 + T \log v)^2} dv \leq Ax^{\theta-p} \int_0^\infty \frac{e^{-\beta w} v^{p-\theta}(1 + w)}{(1 + w)^2} \frac{dw}{T} \leq 2Ax^{\theta-p} \int_0^\infty \frac{e^{-\beta w} \left( \frac{1}{T(1 + w)} + \frac{1}{(1 + w)^2} \right)}{T(p-\theta) + 2} dw \leq 2Ax^{\theta-p} \left( \frac{1}{T(p-\theta)} + \frac{2}{T} \right) \quad (49)
\]

estimating the integral as in \((45)\).

Adding \((48)\) and \((49)\) we thus arrive at

\[
|L| \ll \frac{A}{T} + Ax^{\theta-p} \left( \frac{1}{(p-1)T} + \frac{\log \frac{T}{T}}{T} \right) \quad (50)
\]

\(^3\)Deriving the last line from the last but one, we calculate the integral as follows. $\int_T^{\log x} e^{\beta \xi} d\xi = \left[ \frac{e^{\beta \xi}}{\beta} \right]_T^{\log x} + \beta = \frac{e^{\beta \log x} - 1}{\beta} < \frac{e^{(p-\theta) \log x} - 1}{\frac{p-\theta}{\log x}}$. If $p - \theta < 1$, the part of the last integral below $v = 1$ is below $e^{p-\theta}/(p-\theta) < e^{p}/p - 1$. Here we distinguish two cases, according to $(p-\theta) \log x \geq 1$ or not. In the first case the integral for $1 \leq v \leq (p-\theta) \log x$ is at most $e^{(p-\theta) \log x} \int_1^{(p-\theta) \log x} \frac{d\xi}{\frac{p-\theta}{\log x}} < x^{p-\theta}$, and also $e^{(p-\theta) \log x} / (p-\theta) \log x < x^{p-\theta}$, which yields the estimate $\ll 1/(p-1) + x^{p-\theta}$. On the other hand for $(p-\theta) \log x < 1$ we have $\frac{e^{(p-\theta) \log x}}{(p-\theta) \log x} < \frac{1}{(p-1) \log x} \ll 1/(p-1)$ and the overall estimate is of the same order.
Taking into account (46) and (50) in (41), we are led to
\[
J \ll \left( \frac{(A + \kappa)x^p}{(p - 1)T} + \kappa \log x \frac{x^p + x \log x \frac{T - 1}{T}}{T} + Ax^p \log x \left( \frac{1}{T} + x^{\theta - p} + \frac{x^{\theta - p}}{(p - 1)T} + \frac{x^{\theta - p} \log \frac{T - 1}{T}}{T} \right) \right)
\]
\[
\ll (A + \kappa) \left( \frac{x^p \log x}{T} + \frac{x^p + x \log x}{(p - 1)T} + x \log x \frac{\log \frac{T - 1}{T}}{T} + x^\theta \log x \right). \tag{51}
\]

Let us take now \( p := 1 + 1/\log x \). Finally, if we substitute the estimates of (39) and (51) (applied for \( T := t_k \)) in (38), the assertion of the Lemma follows after an easy calculation. \( \square \)

7 A density theorem for \( \zeta \)-zeros close to the 1-line

In this section we will make two additional assumptions, both quite frequent and general, but still forming some restrictions to our general treatment.

One very generally used condition is that the norm would actually map to the natural integers. In cases of counting type problems, as well as e.g. for algebraic number fields where certain indexes are used as norms (equivalence classes modulo an ideal, e.g.), this is all self-evident.

**Definition 2** (Condition B). We say that Condition B is satisfied, if \(| \cdot | : \mathcal{G} \to \mathbb{N} \), that is, the norm \( |g| \) of any element \( g \in \mathcal{G} \) is a natural number.

As is natural, we will write \( \nu \in \mathcal{G} \) if there exists \( g \in \mathcal{G} \) with \( |g| = \nu \). Under Condition B we can introduce the arithmetical function \( G(\nu) := \sum_{g \in \mathcal{G}, |g| = \nu} 1 \), which is then a multiplicative function on \( \mathbb{N} \). The next condition is a kind of “average Ramanujan condition” for the Beurling zeta function.

**Definition 3** (Condition G). We say that Condition G is satisfied, if with a certain \( p > 1 \) we have for the function
\[
F_p(X) := \frac{1}{X} \sum_{g \in \mathcal{G}, |g| \leq X} G(|g|)^p = \frac{1}{X} \sum_{\nu \in \mathcal{G}, \nu \leq X} G(\nu)^{1+p} = \frac{1}{X} \int_1^X G^p(x) dN(x) \tag{52}
\]
the property that
\[
\log F_p(X) = o(\log X) \quad (X \to \infty), \tag{53}
\]
that is, for any fixed \( \varepsilon > 0 \) \( F_p(X) = O(X^\varepsilon) \).

Note that in case \( \log G(\nu) = o(\log \nu) \), i.e. when for all \( \varepsilon > 0 \) we have \( G(\nu) = O(\nu^\varepsilon) \), then Condition G is automatically satisfied. Even this stronger \( O(\nu^\varepsilon) \) order estimate is proved for many important cases, see e.g. 2.4. Theorem and 2.5. Corollary of [15].

There are many natural examples of the above condition. E.g. if \( \mathcal{G} \) is the ideal ring of an algebraic number field \( \mathbb{K} \), then a well-known result, see e.g. Lemma 4.9. on p. 143 of [19], provides the estimate \( G(m) \leq d_n(m) = O(m^\varepsilon) \) for all \( \varepsilon > 0 \) (where \( d_n(m) \) is the classical \( n \)-term divisor function and \( n \) is the degree of the algebraic number field \( \mathbb{K} \) in question). It is clear that in case \( G(\nu) = O(\nu^\varepsilon) \) also Condition G must hold. Actually, \( d_n(m) \leq d^{n-1}(m) \) and by well-known number theory we also have
\[
\sum_{m \leq X} d^n(m) \sim C_q X \log^{2^{n-1}} X \quad (X \to \infty),
\]
so our Condition G holds for all exponent \( p \).
More generally, let $\mathfrak{A}$ denote the category of all finite abelian groups, $\mathfrak{S}$ be the category of all semisimple associative rings of finite cardinal, and $\mathfrak{F}$ be the category of all finitely generated torsion modules over the ring $D$ of all algebraic integers in some given algebraic number field $K$. One can also consider $\mathfrak{S}^{(k)}$, for any given finite or infinite sequence $(k) = k_1, \ldots, k_n, \ldots$, the category of all modules $M \in \mathfrak{S}$ such that every indecomposable direct summand of $M$ is isomorphic to a cyclic module of the form $D/P^{r_k}$, where $P$ is a prime ideal in $D$ and $r = k_i$ for some $i$, see [15] p. 117. These structures contain, as sub-semigroups, many other important arithmetical categories like semisimple finite dimensional associative algebras over a given field, certain Galois fields, etc.: see [15] Ch. 1 for details, in particular page 16-21 for more detailed description of these and related structures.

For abelian groups of finite order, the counting function $G_\mathfrak{A}(m)$ has, by [15] V.1.10 Corollary], an asymptotical $k^{th}$ moment for every $k \in \mathbb{N}$, which is a strong form of the above Condition G: the $o(\log X)$ function is just $C + o(1)$. In $\mathfrak{S}$ the counting function $G_\mathfrak{S}(n)$ also has, by [15] V.1.13 Theorem], an asymptotical $k^{th}$ moment for every $k \in \mathbb{N}$, implying again the strong form of Condition G. For $\mathfrak{F}$ this is found in [15] V.1.9. Theorem], too. (The reason of this is the intimate relationship of the value of $G$ on prime powers $p^r$ with the partition function $p(r)$, see p. 124 of [15].)

These categories (and many others) all satisfy Axiom A, see [15], p. 16-20 and 120-121, e.g.

Let us recall the following facts from the theory of arithmetical semigroups (see [15], IV.4.1. Proposition and V.2.9. Theorem.)

**Lemma 14** (Knopfmacher). Let $\mathcal{G}$ be an arithmetical semigroup satisfying Axiom A. Then for the divisor function $d(g)$ on $\mathcal{G}$ we have with any $k \in \mathbb{N}$ the asymptotic equivalence formula

$$
\sum_{g \in \mathcal{G}: |g| < X} d^k(g) \sim \frac{K^{2k}}{(2k - 1)!} A_0 X \log^{2k - 1} X \quad (X \to \infty). 
$$

(54) We also have

$$
\limsup_{|g| \to \infty} \frac{\log d(g) \log \log |g|}{\log |g|} = 2, 
$$

(55) so $\log d(g) = o(\log |g|)$, too.

We denote, as usual, the number of $\zeta$-zeros in the rectangle $|t| \leq T$, $\sigma > \alpha$ as $N(\alpha, T)$.

**Theorem 4.** Assume that $\mathcal{G}$ satisfies besides Axiom A also Conditions B and G, too. Then for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, \mathcal{G})$ such that for all $\alpha > (1 + \theta)/2$ we have

$$
N(\alpha, T) \leq CT^{\frac{\pi_2(1 - \alpha) + \varepsilon}{2}}. 
$$

(56) Proof. We apply the general modern treatment of zero-detecting sums and large sieve type estimates: the derivation here follows the relatively simple, straightforward argument in [22]. We also aim at a log-free version, which is motivated in particular by [24].

Let $Y$ be a a large parameter, so that $Y > T^3$. We define the arithmetical function $a(g) := a_T(g)$ on $\mathcal{G}$ as

$$
a(g) := \sum_{h \mid g: |h| \leq T} \mu(h). 
$$

Clearly, $a(1) = 1$ and for $1 < |g| \leq T$ we have $a(g) = 0$: for other values of $g$ we have $|a(g)| \leq d(g)$. For any complex number $z \in \mathbb{C}$ put

$$
H := H(z) := H(T, Y, z) := \sum_{|g| < Y} \frac{a(g)}{|g|^2} = 1 + \sum_{T < |g| < Y} \frac{a(g)}{|g|^2},
$$

(57)
By Lemma [7], we already know about the number of zeroes that \( N(\alpha, X) \ll \sqrt{\frac{1 - \theta}{\alpha - \theta}} X \log X \), so using also \( \alpha > \frac{1 + \theta}{2} \) and choosing \( X := \log^2 T Y^{1 - \alpha} \) the number of zeroes below the height \( X \) can be estimated as

\[
N(\alpha, X) \leq \frac{24}{1 - \theta} \left( A_3 + 6 \log 2 + 6 \log \log T + 6(1 - \alpha) \log Y \right) \log^2 T Y^{1 - \alpha}
\]

\[
\ll \left( \frac{1}{1 - \theta} \log T + \log Y \right)^3 Y^{1 - \alpha}.
\]

(58)

On the other hand for the zeroes counted in \( N(\alpha, T) - N(\alpha, X) \), we select a separated "one-covering", i.e. we take (one) zero \( \rho_1 = \beta_1 + i \gamma_1 \) with \( \beta_1 > \alpha \), \( \gamma_1 \geq X \) and of minimal \( \gamma_1 \), and then inductively, after \( \rho_j \) has already been selected, we take the next \( \rho_{j+1} = \beta_{j+1} + i \gamma_{j+1} \) with minimal \( \gamma_{j+1} \geq \gamma_j + 1 \) but remaining \( \leq T \). Clearly \( \gamma_j + 1 \leq \gamma_{j+1} \) entails that the set \( \mathcal{Z} := \{ \rho_j \} \) of all the zeroes selected is finite: let its number of elements be \( Z := \# \mathcal{Z} \), say.

By construction, the imaginary part of any \( \zeta \)-zero in the rectangle \([\alpha, 1] \times [X, T] \) is within 1 to some element \( \rho_j \) of the set \( \mathcal{Z} \), so by Lemma [7] and taking into account \( \alpha > \frac{1 + \theta}{2} \) and the symmetry of the zeroes with respect to the real line, too, we obtain

\[
N(\alpha, T) - N(\alpha, X) \leq Z \frac{5^{5/2}}{1 - \theta} \left( A_2 + 3 \log T \right).
\]

(59)

Finally, we are to estimate \( \# \mathcal{Z} = Z \). Firstly, for any \( \rho = \rho_j \in \mathcal{Z} \), we can write

\[
H := H(T, Y, \rho_j) = \sum_{|g| \leq Y^\rho} \sum_{|h| \leq T} \mu(h) \leq \sum_{|h| \leq T} \left( \frac{1}{|h|^\rho} + 1 \right) \sum_{|h| \leq T} \left( \frac{\mu(h)}{|h|^\rho} \right) \sum_{|h| \leq T} \frac{1}{|h|^\beta} |\zeta_{Y/|h|}(\rho)|.
\]

Here using \( \zeta(\rho) = 0 \), too, the inner expression can be estimated by Lemma [1] [12] (first line) and [14]. We gain

\[
|\zeta_{Y/|h|}(\rho)| \leq \kappa \frac{Y^{1 - \beta}}{|\rho| - 1} + A \left( \frac{Y}{|h|} \right)^{\theta - \beta} \left( \frac{|\rho|}{\beta - \theta} + 1 \right) \leq \kappa \gamma \left( \frac{Y}{|h|} \right)^{1 - \beta} + \left( \frac{4A}{1 - \theta} \right) \gamma \left( \frac{Y}{|h|} \right)^{\theta - \beta}.
\]

Inserting this into the estimation of \( H \) we are led to

\[
H \leq \frac{\kappa Y^{1 - \beta}}{\gamma} \sum_{|h| \leq T} \frac{1}{|h|^\beta} + \left( \frac{4A}{1 - \theta} \right) \gamma Y^{\theta - \beta} \sum_{|h| \leq T} \frac{1}{|h|^\beta} = \frac{\kappa Y^{1 - \beta}}{\gamma} \zeta_T(1) + \left( \frac{4A}{1 - \theta} \right) \gamma Y^{\theta - \beta} \zeta_T(\theta).
\]

According to the second line of [12] in Lemma [1] we find

\[
|\zeta_T(\theta)| \leq \kappa \frac{T^{1 - \theta}}{1 - \theta} + \int_1^T x^{-\theta} d\mathcal{R}(x) \leq \kappa \frac{T^{1 - \theta}}{1 - \theta} + A + A \theta \log T
\]

by partial integration and \( |\mathcal{R}(x)| \leq A x^\beta \); and

\[
\zeta_T(1) \leq \kappa \log T + \frac{A}{1 - \theta}
\]

according to Lemma [1] [13], middle line. For \( T > \exp(1/(1 - \theta)) \) these expressions can be further estimated as \( \zeta_T(1) \leq (A + \kappa) \log T \) and \( \zeta_T(\theta) < \frac{4 + \kappa}{1 - \theta} T^{1 - \theta} \), so inserting into the last estimation of \( H \) leads to

\[
H \leq \frac{\kappa(A + \kappa) \log T Y^{1 - \beta}}{\gamma} + \left( \frac{4A(A + \kappa)}{(1 - \theta)^2} \right) \gamma T^{1 - \theta} Y^{\theta - \beta}
\]

\[
\leq (4A + \kappa)(A + \kappa) \log T \left( \frac{Y^{1 - \beta}}{\gamma} + \frac{\gamma Y^{\theta - \beta} T^{1 - \theta}}{(1 - \theta)^2} \right).
\]
Assuming now that

\[ Y \geq Y_0(T, \theta) := T \left( \frac{1}{1 - \gamma} \right)^{\frac{2}{\gamma}}, \]

the second term is always below the first in view of \(|\gamma| \leq T\). Furthermore, \(X \leq \gamma\), so writing in \(X := \log^2 T Y^{1-\alpha}\) in place of \(\gamma\) we infer

\[ H \leq 8(A + \kappa)^2 \log T \frac{Y^{1-\beta}}{\gamma} \leq \frac{8(A + \kappa)^2}{\log T} Y^{\alpha-\beta} \leq \frac{8(A + \kappa)^2}{\log T} < \frac{1}{2}, \]

if \(T > \exp(16(A + \kappa)^2)\). Therefore, in (63) the first constant 1 is dominating, and via |\(H - 1| > 1/2\) we obtain

\[ \frac{1}{4} Z \leq \sum_{j=1}^{Z} |H(\rho_j) - 1|^2 = \sum_{j=1}^{Z} \left[ \sum_{|g|<Y} a(g) \left| \frac{\log(Y/T)}{g|p_j}\right|^2 \right] \leq \log^2 \left( \frac{Y}{T} \right) \sum_{j=1}^{Z} \sum_{M < m \leq N} \left( \sum_{|g|=m} a(g) \frac{|\sum_{g : |g|=m} a(g)|}{m^{p_j}} \right)^2, \]

with some appropriate \(T \leq M < N < eM, N \leq Y\). First we want to estimate the coefficients of the Dirichlet series, so let us write

\[ F(m) := \sum_{g : |g|=m} a(g) \]

and use Cauchy's inequality and the trivial upper estimate \(|a(g)| \leq d(g)| to obtain

\[ |F(m)|^2 \leq \sum_{g : |g|=m} 1 \cdot \sum_{g : |g|=m} a^2(g) = G(m) \sum_{g : |g|=m} a^2(g) \leq G(m) \sum_{g : |g|=m} d^2(g) = \sum_{g : |g|=m} G(|g|)d^2(g). \]

Recall that the exponents \(\rho_j\) in the inner Dirichlet polynomials of the last double sum in (61) are all counted in \(N(\alpha, T)\), hence \(\beta \geq \alpha\) and \(|\gamma| \leq T\). By the large sieve type inequality of [18, Theorem 7.5],

\[ \sum_{j=1}^{Z} \sum_{M < m \leq N} \left| \frac{F(m)}{m^{p_j}} \right|^2 \ll (T + N) \log N \sum_{m=M}^{N} \frac{F^2(m)}{m^{2p_j}} \log \frac{2N}{\log 2n} \ll N \log N \sum_{m=M}^{N} \frac{F^2(m)}{N^{2p_j}} \ll \log N N^{1-2\alpha} \sum_{n=1}^{N} F^2(m). \]

Here we use (63) and apply Hölder’s inequality while summing over elements of \(G\) with some exponent \(p\) with which \(p\) our Condition G holds. This yields

\[ \sum_{n=1}^{N} F^2(m) \leq \sum_{g : |g| \leq N} G(|g|)d^2(g) \leq \left( \sum_{g : |g| \leq N} G(|g|) \right)^{1/p} \left( \sum_{g : |g| \leq N} d^q(g) \right)^{1/q}. \]
By Lemma 14, the second sum is $O(N \log^{C(q)} N)$, while the first sum is by Condition G $N^{1+\varepsilon}$. Collecting (61), (64) and (65) leads to

$$
Z \ll \log^2 Y \log N \cdot N^{1-2\alpha} \left( N \log^{C(q)} N \right)^{1/q} N^{(1+\varepsilon)/p} \ll Y^{2-2\alpha+\varepsilon}.
$$

At last, we can choose $Y$ the smallest possible, that is $Y := Y_0$, to get

$$
Z \ll T \left( \frac{1}{1-\theta} \right)^{2-2\alpha+\varepsilon} \ll T^{\frac{6-2\theta}{1-\theta} \left( (1-\alpha)+\varepsilon \right)}
$$

To finish the proof we need only to combine this estimate with (58) and (59).

The result thus shows, that e.g. the functional equation, so fundamentally present in several approaches, is not necessary for a density theorem to hold. On the other hand positivity of the coefficients of the Dirichlet series does play a role here. In this respect Theorem 4 is complement to the similar density theorem of Kaczorowsky and Perelli, [13], where the Selberg class of zeta functions are shown to admit such density estimates. In that approach essential use is made of the analytic functional equation, characteristic for the Selberg class but not assumed here for the Beurling situation.

There seems to be a bridge between the two approaches. Our most essential condition is Axiom A, and the above method gives a nontrivial density result when $\alpha > (1 + \theta)/2$. However, from purely analytic assumptions (like the functional equation with certain general analytic order estimates – compare the remarks after formula (1.9) in [9] – it is possible to deduce similar asymptotic formulae with error, see e.g. [9, Theorem 1.2]. In particular, a degree $m$ Selberg class function admits an Axiom A type asymptotic with error exponent $\theta = 2/(m+1)$, c.f. [9, Proposition 1.1], providing $(1 + \theta)/2 = 1 - 1/(m+1)$ (and even this is improved upon for particular lower degrees like $m = 2, 3$). On the other hand direct approaches (à la Montgomery) to derive a density result seem to yield nontrivial results for $\alpha > 1 - 1/m$ – at least for degree $m = 2$ Kaczorowski and Perelli gave one for $\alpha > 1/2$, see [13].

It would be interesting to analyze what essentially minimal set of properties, assumed on the class of zeta functions considered, can still imply the validity of such density estimates, and what possible use can be made of the situation when both positivity and functional equation (alongside with the Euler product representation and other useful properties) can be assumed.

8 A Turán type local density theorem for $\zeta$-zeros close to the boundary of the zero-free region

For arbitrary $\tau > 0$ and $\theta < b < 1$, we denote the rectangle $Q_{b,h}(\tau) := [b, 1] \times [i(\tau-h), i(\tau+h)]$, and the number of zeroes of $\zeta(s)$ in such a rectangle by $M := M_{b,h}(\tau)$.

**Theorem 5.** Let $(1 + \theta)/2 < b \leq 1$, $2 \leq h$ and $\tau > \max(2h, \tau_0)$ where $\tau_0 = \tau_0(\theta, A, \kappa)$ is a large constant depending on the given parameters of $\zeta(s)$.

Assume that $\zeta(s)$ does not vanish in the rectangle $\sigma \geq b$, $|t-\tau| \leq h$, denoted by $Q_{b,h}(\tau)$, i.e. that $M_{b,h}(\tau) = 0$. Then for any $\delta < (b - \theta)/10$ we have

$$
M_{b-\delta,\delta}(\tau) \leq A' \frac{\delta}{1-\theta} \log \tau
$$

with $A'$ being a constant depending only on $A$ and $\kappa$.  

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The original work of Turán, aiming at almost getting the density hypothesis, involves the condition \( h > \log \tau \). The present analysis reveals that this is not necessary: we get the same result with any \( h > 2 \) as well. Thus the result - at least formally - gives something new even for the Riemann zeta function.

In proving the result we follow closely the original work of Turán, cf. the Appendix of [30].

**Lemma 15.** Let the parameters \( b, h, \tau \) be fixed as in the previous theorem. If \( M_{b, h}(\tau) = 0 \) and \( \lambda := 5 \log \log \tau / \log \log \tau \), then in the domain \( \Omega_{b+\lambda, h-1}(\tau) \) the inequality

\[
|\frac{\zeta'}{\zeta}(s)| \leq \frac{\log t}{16(\log \log t)^2}
\]

(67)

holds true.

**Proof.** Let \( g(s) := \log (\zeta(s)/\zeta(p+it)) \), where \( 1 < p \leq 2 \) is to be determined later. This function is analytic in the disk \( |s - (p+it)| \leq R := p - b < 1 \), as \( \zeta(s) \) does not vanish (for \( |t - \tau| + R \leq h - 1 + R \leq h \)), and so its logarithm is analytic. Clearly, in the center of this disk \( g(p+it) = 0 \). For the upper estimation of the real part of \( g(s) \) along the circle, let us use (17) and (21), which leads to

\[
\Re g(s) \leq \log \left( \frac{\sqrt{2}(A + \kappa)}{b - \theta} \right) + \log(t + 1) + \log \left( \frac{p(1 + \kappa)}{p - 1} \right)
\]

\[
< \log \left( \frac{2\sqrt{2}(A + \kappa)}{1 - \theta} \right) + \log \left( \frac{3}{2} t \right) + \log \left( \frac{p(1 + \kappa)}{p - 1} \right)
\]

\[
< \log t + \log \left( \frac{5p(1 + \kappa)^2}{(1 - \theta)(p - 1)} \right).
\]

We will chose

\[
p := \frac{3}{2} - \Delta \log \log \log t
\]

\[
\Delta := k \log \log \log t < 0.2, \quad (k \geq 4, \quad k \in \mathbb{N}).
\]

So let us assume that \( \tau \) is large enough, say \( \tau > \tau_1 := 30(\frac{A + \kappa)^2}{1 - \theta} \): then by \( t > \tau/2 \) in the last estimate the \( \log t \) term dominates, and \( \Re g(s) \leq 2 \log t \). Let \( R' := R - \Delta < R \): by the Borel-Carathéodory theorem,

\[
\max_{|s - (p+it)| \leq R'} |g(s)| \leq \frac{2R'}{\Delta} \log |t| = \frac{4(p - b - \Delta)}{\Delta} \log t < \frac{4}{\Delta} \log t.
\]

Next we apply the three-circle theorem to the circles \( C_1 := \{ s \mid |s - (p+it)| \leq R' \} \), \( C_2 := \{ s \mid |s - (p+it)| \leq R \} \) with \( R'' := R - 2\Delta \), and \( C_3 := \{ s \mid |s - (p+it)| \leq q \} \), where now \( q := (p - 1)/2 \geq 1/4 \), so on \( C_3 \) by the estimates of Lemma [3] we have

\[
\max_{C_3} |g(s)| \leq \log \left( \frac{2(A + \kappa)^2 q^2}{(p - 1)^2} \right) = \log \left( 18(A + \kappa)^2 \right).
\]

From the three-circle theorem we obtain

\[
\max_{|s - (p+it)| \leq R'} \log |g(s)| \leq \frac{\Delta}{R'} \log \log \left( 18(A + \kappa)^2 \right) + \frac{R' - \Delta}{R'} \log \left( \frac{4}{\Delta} \log t \right)
\]

\[
= \frac{\Delta}{R'} \log \left( \frac{\Delta}{4} \log \left( 18(A + \kappa)^2 \right) \right) + \log \left( 1 - \frac{\Delta}{R'} \right) \log \log t.
\]
As now $t > \exp(\exp(4/\log(18(A + \kappa)^2)))$, the first expression is negative. Moreover, the fraction $\Delta/R'$ is at least $\Delta$, since $R' < R < 1$. Also, $\Delta > 4/\log \log t$ whenever $t > \exp(\exp(\exp(4)))$, i.e. $\log \log \log t > 4$. Therefore, we end up with

$$\max_{|s-(p+it)| \leq R'} |g(s)| \leq \log \frac{4}{\Delta} + (1 - \Delta) \log \log t \leq \log \log \log t + \log \log t - k \log \log \log t.$$ 

Choosing $k = 4$ we conclude

$$\max_{|s-(p+it)| \leq R'} |g(s)| \leq \frac{\log t}{(\log \log t)^2}.$$ 

Finally, we choose $r := R - 3\Delta = R^* - \Delta$, and apply the standard Cauchy estimate for the estimation of $|g'(s)|$ in a circle $|s - (p + it)| \leq r$. We thus obtain

$$|g'(s)| \leq \frac{1}{\Delta} \left( \frac{\log t}{\log \log t} \right)^3 = \frac{1}{4 \log \log t} \frac{\log t}{(\log \log t)^2} < \frac{\log t}{16(\log \log t)^2}.$$ 

We have the same all over the area of these circles, and as $\tau$ is large enough, $\lambda > \Delta$ and the domain $Q_{b+\lambda,h-1}$ is contained in the total area covered by these circles for all $t$ with $|t - \tau| \leq h - 1$. But $g'(s) = (\log \zeta(s))' = \zeta'(s)/\zeta(s)$, and so at any point $s = \sigma + it$, $\sigma \geq b + \lambda$, $|t - \tau| \leq h - 1$ we thus have (67).

**Proof of Theorem 2.** Let $s = u + it \in Q := Q_{b+\lambda,h-1}$ (with $\lambda$ as in Lemma 15) and consider the number of zeroes $N$ in the disk $D(s,r)$ with $r := (u - \theta)/2$. As $u > b > (1 + \theta)/2$, the disc $K := \{z : |z - (1 + (1 - \theta) + it)| \leq R := 2 - 2\theta - 2(u - \theta)/3\}$ contains $D(s,r)$, whence by Remark 11 (applied to $u$ in place of $b$) the number of $\zeta$-zeroes $N$ in $D(s,r)$ can at most be as large as

$$N \leq \frac{1 - \theta - \frac{3}{2}}{u - \theta} \left( A_1 + \frac{3}{2} \log t \right).$$

Note that the same disk $D(s,r)$ occurs in Lemma 9, hence we obtain now for the (multi)set $S$ of zeroes within $D(s,r)$ that

$$\Re \left\{ -\frac{\zeta'}{\zeta} \frac{1}{s - \rho} \right\} \leq \left| \frac{\zeta'}{\zeta} (s) - \sum_{\rho \in S} \frac{1}{s - \rho} \right| \ll \frac{1}{1 - \theta} \left( A_1 + \log t \right).$$

The real parts of the expressions with the zeroes in the sum can be rewritten for $\rho = \beta + i\gamma$ as

$$\Re \frac{1}{s - \rho} = -\frac{u - \beta}{(u - \beta)^2 + (t - \gamma)^2},$$

hence from (68) we obtain with a suitably large constant $C$ and for $\tau$ (and hence $t$) large enough

$$\sum_{\rho \in S} \frac{u - \beta}{(u - \beta)^2 + (t - \gamma)^2} < \Re \frac{\zeta'}{\zeta} (s) + C \frac{\log t}{1 - \theta}$$

We choose now $s := b + \delta + i\tau$, i.e. $u = b + \delta$ and $t := \tau$. Assuming that $\tau$ is large enough, we certainly have $\lambda < \delta$, that is, $u + it \in Q_{b+\lambda,h-1}$ and the estimations of the previous Lemma 15 can be applied to bound the arising $\zeta'(u + it)$ in the right hand side. On the other hand any nonnegative member of the sum can be dropped with the inequality still holding true. In view of the fact that now $Q_{b-\delta,\delta}(\tau) \subset D(s,r)$ we obtain

$$\sum_{\rho \in Q_{b-\delta,\delta}} \frac{u - \beta}{(u - \beta)^2 + (t - \gamma)^2} < \frac{\log \tau}{16(\log \log \tau)^2} + C \frac{\log \tau}{1 - \theta}. $$

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Now each term on the left-hand side is at least $2\delta/5\delta^2$, hence we arrive at

$$M_{b-\delta,\delta} \ll \frac{1}{1-\theta} \log \tau.$$ 

The proof of the theorem concludes. \qed

## 9 Clustering of zeroes in the vicinity of the 1-line

In [8], Theorem 2 the authors prove that the zeroes $\rho = \beta + i\gamma$ of the Beurling zeta function, close to the one-line in the sense that $\beta > c/\sqrt{\log \gamma}$, show a phenomenon of clustering: they do not occur in isolation, but instead once a zero $\rho_0$ occurs, there must be further ones in the union of some small discs $D(1 + i\gamma_0, \lambda)$ and $D(1 + i2\gamma_0, \lambda)$ around $1 + i\gamma_0$ and $1 + i2\gamma_0$.

This theorem is itself a sharpening of what was proved in the average by Montgomery in his monograph [18] for the case of the Riemann zeta function. Yet there is a less quoted, nevertheless sharper result, due to Ramachandra [26], which gives clustering with closeness as small as $1/\log \log \gamma$ and still localizes to the small discs $D(1 + i\gamma_0, \lambda)$ and $D(1 + i2\gamma_0, \lambda)$ around $1 + i\gamma_0$ and $1 + i2\gamma_0$.

It is worthy to work out the result here not only for sake of generality but also to give a somewhat more transparent deduction of the result. Indeed, Ramachandra uses, for some reasons seemingly lying in the later, more general application of his method (to clustering around $1 + i\gamma_1$ and $1 + i\gamma_2$ when two close zeroes are known at height $\gamma_1$ and $\gamma_2$, respectively) a positive trigonometric polynomial with obscure coefficients (like $10^8$). Here we analyze the method and show that the most common $3 + 4\cos \theta + \cos 2\theta$ does the job as well.

Note that the further paper of Balasubramanian and Ramachandra [1], claiming to achieve the even nicer localization of clustering right in $D(1 + i\gamma_0, \lambda)$, is erroneous.

**Theorem 6.** Assume $\zeta(\rho_0) = 0$ with $\rho_0 = \beta_0 + i\gamma_0$, $\gamma_0 > 100$, and let the parameter $0 < \lambda < 1/2$ be arbitrary. Then there exists a constant $A_9 := A_9(\kappa, \theta, A)$, depending only on the parameters of the Beurling zeta function, so that with any value of the further parameter $Y$ satisfying $\log \log \gamma_0 < Y$ we have

$$\sum_{\rho \in D(1+i\gamma_0, \lambda) \setminus \{\rho_0\}} e^{-Y(1-\beta)} + \sum_{\rho \in D(1+i2\gamma_0, \lambda)} e^{-Y(1-\beta)} \gg \frac{\lambda}{Y(1-\beta_0)} - A_9, \quad (70)$$

with the implied constant in $\gg$ being an absolute constant.

**Proof.** We will work with the kernel function

$$K(x, w) := x^w \Gamma \left( \frac{w}{2} \right),$$

satisfying the integral formula

$$d(x, \nu) := \frac{1}{2\pi i} \int_{(2)} \frac{1}{\nu^w} x^w \Gamma \left( \frac{w}{2} \right) dw = 2 \exp \left( -\left( \frac{\nu}{x} \right)^2 \right) \quad (\nu \geq 1, \, \nu \in \mathbb{R}),$$

where, as usual, $\int_{(2)} = \int_{2+i\infty}^{2+i\infty \rho \to \rho_0}$.

At the outset we let $s \in \mathbb{C}$ be arbitrary with $\Re s > 1$. We put

$$f(s) := f^B_{\gamma_0}(s) f_{2\gamma_0}^C(s) \zeta^{-2D}(s), \quad \text{with} \quad f_\alpha(s) := \zeta^2(s)\zeta(s - i\alpha)\zeta(s + i\alpha),$$

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so that

\[
\frac{f'}{f}(s) = B \frac{f'_{\gamma_0}}{f_{\gamma_0}}(s) + C \frac{f'_{2\gamma_0}}{f_{2\gamma_0}}(s) - 2D \frac{c'}{\zeta}(s) = 2(B + C - D) \frac{c'}{\zeta}(s) + \frac{B \left( \frac{c'}{\zeta}(s - i\gamma_0) + \frac{c'}{\zeta}(s + i\gamma_0) \right) + C \left( \frac{c'}{\zeta}(s - 2i\gamma_0) + \frac{c'}{\zeta}(s + 2i\gamma_0) \right)}{|g|^s}.
\]

with the choice of the constant parameters \(B, C, D > 0\) such that

\[
P(u) := (B + C - D) + B \cos u + C \cos(2u) \geq 0 \quad (\forall u \in \mathbb{R}).
\]

For any \(s\) in the right halfplane \(\Re s > 1\) we have

\[
F(s, x) := -\sum_{g \in \mathcal{G}} \frac{\Lambda(g)P(\gamma_0 \log |g|)d(x, |g|)}{|g|^s} = \frac{1}{2\pi i} \int_{(2)} \frac{1}{f(s + w)}K(w, x)dw.
\]

Similarly to the proof of the von Mangoldt formula in Lemma \([13]\), we move the line of integration to the left. For that, we fix some \(0 < a < b\) to be determined later, choose \(\mathcal{A} := \{\gamma_0, 2\gamma_0\}\), and consider the curve \(\Gamma := \Gamma^\mathcal{A}(a, b)\), as constructed in Lemma \([11]\). We now move the contour of integration to the left along the horizontal lines \(t = \pm t_k\). However, now it is easy to see that the \"bridges\" along a horizontal segment \(t = \pm t_k\) give \(o(1)\) contribution, hence by letting \(t_k \to \infty\) the whole line of integration can be moved to \(\Gamma - s\).

Here we make only the sole hypotheses, excluding only countably many values of \(s\), that the line \(\Gamma\) does not contain any singularities of the integrand, i.e. \(\Gamma\) does not contain singularities of \(f'/f\). In fact, we will only consider parameter values \(1 < s < 1 + 1/100\) in the proof. Note that for real \(s\) we can also write \(F(s, x) = \Re F(s, x)\) and

\[
\frac{1}{2} \Re \frac{f'}{f}(s) = \frac{1}{2} \frac{f'}{f}(s) = E \frac{c'}{\zeta}(s) + B \Re \frac{c'}{\zeta}(s - i\gamma_0) + C \Re \frac{c'}{\zeta}(s - 2i\gamma_0) \quad (E := B + C - D).
\]

We thus find, using again the notation \(\mathcal{Z}(L)\) for the set of \(\zeta\)-zeros to the right of the curve \(L\), that

\[
F(s, x) = \frac{1}{2\pi i} \int_{\Gamma - s} \frac{1}{f(s + w)}K(w, x)dw + \frac{1}{2} \sum_{r : \sigma_0 - s < r < \rho} \text{Res}[K(w, x) : w = r] \frac{f'}{f}(s + r)
\]

\[
+ \sum_{\rho \in \mathcal{Z}(\Gamma)} K(\rho - s, x)
\]

\[
+ B \Re \left\{ -K(1 + s - i\gamma_0, x) + \sum_{\rho \in \mathcal{Z}(\Gamma - i\gamma_0)} K(\rho - s + i\gamma_0, x) \right\}
\]

\[
+ C \Re \left\{ -K(1 + s + 2\gamma_0, x) + \sum_{\rho \in \mathcal{Z}(\Gamma + 2\gamma_0)} K(\rho - s + 2\gamma_0, x) \right\},
\]

where \(\sigma_0\) is the abscissa of the line segment of \(\Gamma\) crossing the real axis. The only singularity \(r\) of \(K(w, x)\) in \([\sigma_0 - s, 2 - s]\) is at \(r = 0\) (for \(2 > 1.01 > s > 1 > \sigma_0 > 0\), and \(\Gamma(w/2)\) has singularities only at 0 and at the negative even integers), so the first sum reduces to
2(f'/f)(s) in view of the residuum of x^wΓ(w/2) at w = 0 being exactly 2. Multiplying by the constant factor 1/2 in front and subtracting this, we arrive at

\[
0 \leq \sum_{g \in \mathcal{G}} \Lambda(g) P(\gamma_0 \log |g|) \left( 2 - 2 \exp(-(|x/g|^2)) \right) |g|^s = F(s, x) - \frac{f'}{f}(s)
\]

where

\[
I := \frac{1}{2\pi i} \int_{\Gamma-s} \frac{1}{2} f'(s+w)K(w, x)dw
\]

and

\[
S_j := -K(1-s + ij\gamma_0, x) + \sum_{\rho \in \mathbb{Z}(\Gamma-i-j\gamma_0)} K(\rho - s + ij\gamma_0, x) \quad (j = 0, 1, 2).
\]

Using that s is chosen to be real, and that we only need ReS_j, we can also write \( \Re S_j = \Re \overline{S_j} \), where now

\[
\overline{S_j} = -K(1-s - ij\gamma_0, x) + \sum_{\rho \in \mathbb{Z}(\Gamma+i+j\gamma_0)} K(\rho - s - ij\gamma_0, x) \quad (j = 0, 1, 2).
\]

Stirling’s formula for \( w = u+iv \) in the strip \(-1-1/100 \leq a-s \leq \Re w \leq b-s < b-1 < 0 \) gives \( |\Gamma(w/2)| \ll \frac{1}{\pi^{1/2}} (1+|v|)^{s/2-1/4} e^{-\pi v^2/4} \). Applying the version of Lemma 12 with plugging in Lemma 11 instead of Lemma 10, i.e. bounding also \((\zeta'/\zeta)(s + i\gamma_0)\) the same way, we obtain

\[
I \ll \frac{1 - \theta}{(a-\theta)^3} \int_0^\infty \left( A_3 + \log(2\gamma_0 + v) + \log \frac{1}{a-\theta} \right)^2 x^{b-s} \frac{1}{1-b} \frac{(b-s)^2}{1} e^{-\pi v^2/4} dv \quad (73)
\]

Next, we estimate the contribution of zeroes in \( \overline{S}_j \), having imaginary part further from \( j\gamma_0 \) than \( Q \), with a parameter \( Q > 1 \) to be chosen later. One summand is at most

\[
|K(\beta - s + i(\gamma - j\gamma_0), x)| \ll x^{1-s} |\gamma - j\gamma_0|^{(1-s)/2-1/4} e^{-\pi |\gamma - j\gamma_0|^2/4} \leq x^{1-s} e^{-\pi |\gamma - j\gamma_0|^2/4}
\]

since for \(-1 - 1/100 < \Re w < 1 - s \) and \( |3w| > Q \geq 1 \) we have a uniform Stirling bound. For the corresponding sums we thus have, by an application of Lemma 8

\[
\sum_{\rho \in \mathbb{Z}(\Gamma \cup \Gamma \gamma_0) \mid \gamma - j\gamma_0 \geq Q} K(\rho - s - ij\gamma_0, x) \ll x^{1-s} \int_{|t-j\gamma_0| > Q} e^{-\pi |t-j\gamma_0|^2/4} dN(a, t)
\]

\[
= x^{1-s} \int_Q^\infty e^{-\pi u^2/4} N(a, u + j\gamma_0) + N(a, u + j\gamma_0 - u) \] 

\[
\ll x^{1-s} \left\{ N(a, j\gamma_0 + Q) e^{-Q/4} + \int_Q^\infty e^{-\pi u^2/4} N(a, u + j\gamma_0) du \right\} 
\]

\[
\ll A_5 x^{1-s} (Q + j\gamma_0)^2 e^{-Q/4},
\]

where as before, \( A_5 \) and later on \( A_6 \) etc. are constants depending on the parameters \( \kappa, A \) and \( \theta \). For \( j = 0 \) we have for the small zeroes of \( S_0 \) the estimate

\[
\sum_{\rho \in \mathbb{Z}(\Gamma) \mid |\gamma| < 1} K(\rho - s, x) \leq \frac{A + 2\kappa}{\kappa(1-\theta)} N(a, 1)x^{1-s} \ll A_6 x^{1-s}
\]
according to the separation of zeroes of $\zeta(s)$ from 1, given in Lemma 2 (19) and using Lemma 8 too. Similarly, for the rest of $S_0$ we have, referring to the uniform Stirling bound here,

$$\left| \sum_{\rho \in \mathbb{Z}((\tau)) \setminus [-1,1]} K(\rho - s, x) \right| \leq \sum_{q=1}^{Q} N(a, q, q + 1) x^{1-s} e^{-q\pi/4} \ll A_7 x^{1-s}.$$  

Note that $1 - s - i\gamma_0$ and $1 - s - 2i\gamma_0$ are separated by at least $\gamma_0 > 1$ from the singularity of $\Gamma(w/2)$ at $w = 0$, whence we have $|K(1 - s - ij\gamma_0)| \ll x^{1-s}$.

Choosing e.g. $Q := [3 \log \gamma_0]$ the above estimates yield

$$0 \leq -E K(1 - s, x) + BS_1^s + CS_2^s + A_8 \left( x^{b-s} \log^2 \gamma_0 + x^{1-s} \right),$$

where for $j = 1, 2$ we denote

$$S_j^s := \Re \sum_{\rho \in \mathbb{Z}((\tau) + i\gamma_0) \setminus [-1,1]} K(\rho - s - ij\gamma_0, x).$$

(74)

Recall that $\lambda \in (0, 1 - \theta)$ is a constant not exceeding 1, hence the number of zeroes in the discs $D(1 + i\gamma_0, \lambda)$ does not exceed $A_3 - \log(a - \theta) + \log(2 + \gamma_0)$. Similarly, the number of summands in both sums $S_j^s$ ($j = 1, 2$) are at most

$$N(a, j\gamma_0 - Q, j\gamma_0 + Q) \ll Q \left( A_3 + \log \frac{1}{a - \theta} + \log(Q + \gamma_0) \right) \ll A_3^s \log^2 \gamma_0$$

(75)

according to Corollary 2 in view of the choice of $Q$ as a large constant.

It is appropriate here to fix a few of our parameters. We choose $a := \theta + (1 - \theta)/3$, $b := \theta + 2(1 - \theta)/3$, so that all the constants $a - \theta, 1 - b, b - \theta$ are equivalent to $1 - \theta$. Then $A_3^s$ will only depend on $\theta$ (and $\kappa, A$), but not on $a, b$. Since $b = 1 - (1 - \theta)/3$, we have $x^{b-s} = x^{-(1-\theta)/3+1-s}$. We will now assume that

$$x \geq X_0 \geq \exp \left( \frac{3}{1 - \theta} \log \log(3 + \gamma_0) \right)$$

so that $x^{b-s} \log^2 \gamma_0 \leq x^{1-s}$. Recall that the constant $A_8$ depends only on the parameters of the Beurling zeta function, more precisely

$$A_8 = A_8(\theta, \kappa, A).$$

Summarizing the above, we have

$$0 \leq -E K(1 - s, x) + BK(\rho_0 - s - i\gamma_0, x) + BS_1^s + CS_2^s + 2A_8 x^{1-s},$$

where $S_1^s$ is essentially $S_1^s$, save the contribution of the known zero at $\rho_0$.

Next we separate the contribution of the zeroes inside $D := D(1 + i\gamma_0, \lambda) \cup D(1 + i2\gamma_0, \lambda)$. Here we use the estimate, proved in [18] and also in [26], that for any $z$ subject to $-3/2 < \Re z < 0$ we have uniformly $\Re z \Gamma(z/2) \ll x^{\Re z} \log x$. In view of this, and our choice of $1 < s < 1.01$, the contribution of these close zeroes is at most a constant times

$$S^s := \sum_{\rho \in D \setminus \{\rho_0\}} x^{\beta - s} \log x.$$ 

Collecting the above estimates we obtain

$$EK(1 - s, x) - BK(\beta_0 - s, x) - BS_1 - CS_2^s - 2A_8 x^{1-s} \ll S^s$$

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where now the sums $S_j^*$ are the same as $S_j^*$, but the summands chosen only from

$$\{\rho \in \mathcal{Z}(\Gamma - ij\gamma_0) : |\gamma - j\gamma_0| \leq Q\} \setminus D(1 + ij\gamma_0, \lambda).$$

Note that all terms in the sums $S_j^* (j = 1, 2)$ have the form $K(z)$ with $|z| > \lambda$.

Now it may be clear that any choice of the coefficients $B, C, D$, hence $E$, which satisfy

$$E < B,$$

suffices. So we choose $B := 4, C := 1$ and $D := 4$ to obtain $E = 3$ (and thus

$$P(u) = 3 + 4 \cos u + \cos(2u)).$$

After multiplying by $x^{s-1}$ we are led to

$$3\Gamma \left( \frac{1 - s}{2} \right) - 4x^{\beta_0 - 1}\Gamma \left( \frac{\beta_0 - s}{2} \right) - 4S_1^{**} - S_2^{**} - 2A_8 \ll \sum_{\rho \in D \setminus \{\rho_0\}} x^{\beta - 1}\log x, \quad (76)$$

where

$$S_j^{**} := \Re \sum_{\rho \in \mathcal{Z}(\Gamma + j\gamma_0) \setminus \{\gamma - j\gamma_0 < Q, |\rho - ij\gamma_0| > \lambda}} x^{\rho - ij\gamma_0 - 1}\Gamma \left( \frac{\rho - s}{2} \right). \quad (77)$$

We chose our parameters so that we will have

$$\log \log \gamma_0 < y := \log x \leq \frac{0.1}{1 - \beta_0}, \quad (78)$$

$$(1 <) 1 + 10(1 - \beta_0) < s < 1.01. \quad (79)$$

Note that in order to have a nonempty interval for $y$ in the first condition it suffices to have

$$1 - \beta_0 < 0.1/\log \log \gamma_0,$$

and to have a nonempty interval for $s$ in the second condition it suffices to have $1 - \beta_0 < 0.001$.

The above conditions allow an interval for the choice of $x$. We will now take a subinterval, so that $\log \log \gamma_0 \leq Y_0 := \log X_0 < Y_1 := \log X_1 \leq 0.1/(1 - \beta_0)$, and consider admissible values $y := \log x = Y_0 + u_1 + \cdots + u_m$ with $0 \leq u_j \leq d$, where $d := 2e^2/\lambda$ is a constant, now depending also on $\lambda$, and $m \in \mathbb{N}$ is an integer parameter. We choose $m := [Y_0]$.

Then the above parametrization of $y = \log x$ will run between $Y_0$ and $Y_0 + md$, whence $Y_1 = Y_0 + md \in (dY_0, (1 + d)Y_0]$ must be below the bound $0.1/(1 - \beta_0)$ for $\log X_1$, i.e. below. We thus require $(1 + d)Y_0 \leq 0.1/(1 - \beta_0)$. Whenever such an $Y_0$ is chosen, the corresponding $m$ and $Y_1 = Y_0 + md$ satisfy the necessary bounds, hence the interval $[X_0, X_1]$ is admissible.

To have a nonempty interval for $Y_0$, we need

$$\log \log \gamma_0 \leq Y_0 \leq \frac{1}{1 + d} \frac{0.1}{1 - \beta_0} \quad \text{(80)}$$

Furthermore, for any values of $\beta_0$ and $s$ satisfying (79) we have

$$3\Gamma \left( \frac{1 - s}{2} \right) - 4x^{\beta_0 - 1}\Gamma \left( \frac{\beta_0 - s}{2} \right) = -\frac{6}{s - 1}\Gamma \left( \frac{3 - s}{2} \right) + \frac{8}{s - \beta_0} e^{-(1 - \beta_0) \log x} \Gamma \left( \frac{2 + \beta_0 - s}{2} \right) \geq \left\{ \begin{array}{ll}
\frac{8}{s - 1} & \text{if } s \geq 1 + \beta_0 \\
6 & \text{if } s \leq 1 + \beta_0 
\end{array} \right\} \Gamma \left( \frac{3 - s}{2} \right)$$

because $\Gamma$ is decreasing around 1 and thus $\Gamma \left( \frac{3 + \beta_0 - s}{2} \right) > \Gamma \left( \frac{3 - s}{2} \right)$. So in all we get

$$3\Gamma \left( \frac{1 - s}{2} \right) - 4x^{\beta_0 - 1}\Gamma \left( \frac{\beta_0 - s}{2} \right) \geq \frac{1}{s - 1}. \quad \text{(81)}$$

Using also $Y_0 = \log X_0 \leq \log x \leq Y_1 \leq (1 + d)Y_0$ we obtain

$$\frac{1}{s - 1} - 2A_8 - 4S_1^{**} - S_2^{**} \ll \sum_{\rho \in D \setminus \{\rho_0\}} x^{\beta - 1}\log x \leq (1 + d)Y_0 \sum_{\rho \in D \setminus \{\rho_0\}} X_0^{\beta - 1}.$$

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Since we have this inequality for all \( x \in [X_0, X_1] \), i.e. for all values of \( u := (u_1, \ldots, u_m) \in [0, d]^m \), we can average it on the left hand side with respect to all the \( u_k \). For the general term of the sums \( S^*_j \) we obtain

\[
\int_0^d \int_0^d \cdots \int_0^d \left\{ R X^{\rho - ij\gamma_0 - 1} e^{(u_1 + u_2 + \cdots + u_m)(\rho - ij\gamma_0 - 1)} \right\} du_1 du_2 \cdots du_m
\]

\[
\ll 2^m \frac{1}{\lambda} \frac{1}{|\rho - ij\gamma_0 - 1|^m} \leq \left( \frac{2}{\lambda} \right)^{m+1},
\]

because \( |\rho - ij\gamma_0 - 1| \geq \lambda \) as \( \rho \) lies outside \( D(1 + ij\gamma_0, \lambda) \) and \( |\Gamma(\frac{\rho - ij\gamma_0}{2})| \ll 1/\lambda \).

Taking into account (75) and the total volume \( d^m \) of the cube \([0, d]^m\), we obtain with an absolute constant \( C^* \) the inequality

\[
\frac{1}{s - 1} - 2A_8 - C^* A_3 \log^2(\gamma_0) \frac{(2)}{d\lambda} \leq (1 + d) Y_0 \sum_{\rho \in D \setminus \{\rho_0\}} X_0^{\beta - 1}.
\]

Now in view of \( d := 2e^2/\lambda \), we have \( \left( \frac{2}{d\lambda} \right)^{m+1} = \exp(-2(m + 1)) \leq \exp(-2Y_0) \leq \log^{-2} \gamma_0 \), hence writing in the value of \( d \) and putting \( C'' := 2e^2 C^* \) we arrive at

\[
\frac{1}{s - 1} - 2A_8 - \frac{C'' A_3}{\lambda} \ll \frac{1}{\lambda} Y_0 \sum_{\rho \in D \setminus \{\rho_0\}} X_0^{\beta - 1}.
\]

Clearly our best choice here is to choose \( s \) as small as possible. We must meet the conditions (78), so the smallest admissible value is \( s = 1 + 10(1 - \beta_0) \). This choice yields

\[
\frac{\lambda}{Y_0 (1 - \beta_0)} - \frac{20A_8 \lambda + 10C'' A_3}{Y_0} \ll \sum_{\rho \in D \setminus \{\rho_0\}} e^{Y_0 (\beta - 1)}.
\]

Applying \( \lambda < 1/2 \) now gives a better inequality than the asserted inequality (70) for \( Y_0 \) in place of \( Y \). It remains to see that the additional condition \( Y_0 \leq \frac{1}{1 + d} \frac{\lambda}{1 - \beta_0} \) from the right hand side of (80) can be dropped, since for \( Y_0 > C\lambda/(1 - \beta_0) \) the first term in the difference on the right hand side of (70) is \( < 1/C \), and thus the difference becomes negative if we take \( A_9 > 1/C \).

\[
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\]

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\[
\begin{align*}
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\]
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