ON THE SEMI-CLASSICAL ANALYSIS OF THE GROUNDSTATE ENERGY OF THE DIRICHLET PAULI OPERATOR III: MAGNETIC FIELDS THAT CHANGE SIGN

Bernard Helffer, Hynek Kovařík, Mikael Sundqvist

To cite this version:

Bernard Helffer, Hynek Kovařík, Mikael Sundqvist. ON THE SEMI-CLASSICAL ANALYSIS OF THE GROUNDSTATE ENERGY OF THE DIRICHLET PAULI OPERATOR III: MAGNETIC FIELDS THAT CHANGE SIGN. 2017. hal-01622813

HAL Id: hal-01622813
https://hal.archives-ouvertes.fr/hal-01622813
Submitted on 24 Oct 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
ON THE SEMI-CLASSICAL ANALYSIS OF 
The GroundState Energy Of 
The Dirichlet Pauli Operator III: 
Magnetic Fields That Change Sign

BERNARD HELFFER, HYNEK KOVAŘÍK, AND MIKAEL P. SUNDQVIST

Abstract. We consider the semi-classical Dirichlet Pauli operator in 
bounded connected domains in the plane, and focus on the case when 
the magnetic field changes sign. We show, in particular, that the ground 
state energy of this Pauli operator will be exponentially small as the semi- 
classical parameter tends to zero and estimate this decay rate, extending 
previous results by Ekholm–Kovařík–Portmann and Helffer–Sundqvist.

1. Introduction

1.1. The Pauli operator. Let \( \Omega \) be a bounded, open, and connected domain 
in \( \mathbb{R}^2 \), let \( B : \Omega \to \mathbb{R} \) be a bounded magnetic field and \( h > 0 \) a semi-classical 
parameter. We are interested in the analysis of the ground state energy \( \Lambda^D(h, B, \Omega) \) of the Dirichlet realization of the Pauli operator

\[
P(B, h) = \begin{pmatrix} P_+(B, h) & 0 \\ 0 & P_-(B, h) \end{pmatrix},
\]

in \( L^2(\Omega, \mathbb{C}^2) \). Here, the spin-up component \( P_+(B, h) \) and spin-down compo-

\[
P_\pm(B, h) := (hD_{x_1} - A_1)^2 + (hD_{x_2} - A_2)^2 \pm hB(x),
\]

\( D_{x_j} = -i\partial_{x_j} \) for \( j = 1, 2 \), and the vector potential \( A = (A_1, A_2) \) satisfies

\[
B(x) = \partial_{x_1} A_2 - \partial_{x_2} A_1, \quad \forall x \in \Omega.
\]

The reference to \( A \) is not necessary when \( \Omega \) is simply connected, in which 
case it will be omitted, but it could play an important role if the domain is 
not simply connected. In the sequel we will write

\[
\lambda^D_\pm(h, A, B, \Omega) = \inf \sigma(P_\pm(B, h))
\]

and skip the reference to \( A \) when not needed. The smallest eigenvalue \( \Lambda^D(h, B, \Omega) \) of \( P(B, h) \) is given by

\[
\Lambda^D(B, h, \Omega) = \min\{\lambda^D_-(h, B, \Omega), \lambda^D_+(h, B, \Omega)\}.
\]

The Pauli operator is non-negative (this follows from an integration by parts, 
or from the view-point that the Pauli operator is the square of a Dirac 
operator) and, as a consequence, the bottom of the spectrum is non-negative,

\[
\lambda^D_\pm(h, A, B, \Omega) \geq 0.
\]

2010 Mathematics Subject Classification. 35P15; 81Q05, 81Q20.

Key words and phrases. Pauli operator, Dirichlet, semi-classical, flux effects.
Moreover, if $\Gamma$ is defined by $\Gamma u = \bar{u}$, then
\[ P_+(B, h)\Gamma = \Gamma P_-(-B, h). \]

It immediately follows that
\[ \lambda^D_D(h, B, \Omega) + (h, B, \Omega) = \lambda^D_D(h, -B, \Omega). \] (6)

Hence, to understand the properties of $\Lambda^D_D(B, h, \Omega)$ it suffices to study $\lambda^D_D(h, A, B, \Omega)$, and we will mostly do so.

Let us now specify the amount of regularity we assume about our domain $\Omega$ and magnetic field $B$. To do so, we introduce the notation $C^{p, \alpha}$ to mean the Hölder class $C^{p, \alpha}$, for some unspecified $\alpha > 0$.

**Assumption 1.** The boundary of $\Omega$ is continuous and piecewise in the Hölder class $C^2$. We allow the boundary to have at most a finite number of corners, each with aperture less than $\pi$. The magnetic field $B$ is assumed to be in class $C^0(\Omega)$.

For later reference, we introduce the notation $C^{p, \alpha}_{pw}$ for the piecewise $C^{p, \alpha}$ condition from the assumption. From now we will always work under Assumption 1.

1.2. **The state of art.** Given a magnetic field $B$, we introduce the sets $\Omega^+_B$ and $\Omega^-_B$ as the subsets of $\Omega$ where the magnetic field $B$ is positive and negative, respectively,
\[ \Omega^+_B = \{ x \in \Omega : B(x) > 0 \} \quad \text{and} \quad \Omega^-_B = \{ x \in \Omega : B(x) < 0 \}. \] (7)

Assuming that $\Omega^+_B$ is non-empty, we know from [3, 8] that $\lambda^D_D(h, A, B, \Omega)$ is exponentially small as the semi-classical parameter $h > 0$ tends to zero. More precisely, we have

**Theorem 1.1 ([3, 8]).** Let $\Omega$ be a connected domain in $\mathbb{R}^2$. If $B$ does not vanish identically in $\Omega$ there exists $\epsilon > 0$ such that, for all $h > 0$ and for all $A$ such that $\text{curl} A = B$,
\[ \lambda^D_D(h, A, B, \Omega) \geq \lambda^D_D(\Omega) h^2 \exp(-\epsilon/h). \] (8)

Here, $\lambda^D_D(\Omega)$ denotes the ground state energy of the Dirichlet Laplacian in $\Omega$.

In [3], the statement in the theorem is proved under the assumption that $\Omega$ is simply connected. The generalization to non-simply connected domains, given in [8], was relatively straightforward using domain monotonicity of the ground state energy in the case of the Dirichlet problem,
\[ \lambda^-_D(h, A, B, \Omega) \geq \lambda^-_D(h, \tilde{B}, \tilde{\Omega}), \]
as soon as extensions of $A$ and $B$ to the simply connected envelope $\tilde{\Omega}$ of $\Omega$ were constructed.

The proof in [3] gives a way of computing a lower bound for $\epsilon$, by considering the oscillation of the scalar potential $\psi$, i.e. of any solution of $\Delta \psi = B$, and optimizing over $\psi$. For future reference, we introduce a specific choice of the scalar potential by letting the function $\psi_0$ be a solution of
\[ \Delta \psi_0 = B(x) \text{ in } \Omega, \quad \psi_0 = 0 \text{ on } \partial \Omega. \] (9)
The regularity conditions in Assumption 1 guarantee that $\psi_0$ belongs to $C^0(\overline{\Omega}) \cap C^2(\Omega)$ and that it has $C^1$ regularity up to the boundary, away from the corners. With this particular choice, proposed for simply connected domains in [7], we have

**Theorem 1.2** ([3]). Assume that $\Omega$ is a simply connected domain in $\mathbb{R}^2$ and that $\psi_0$ satisfies (9). Then

$$\lambda^D(h, A, B, \Omega) \geq \lambda^D(\Omega) h^2 \exp(-2 \operatorname{Osc}_\Omega \psi_0/h).$$

(10)

For positive magnetic fields the following theorem was proved in [7]:

**Theorem 1.3** ([7]). Let $\Omega$ be a simply connected domain in $\mathbb{R}^2$. If $B > 0$, and if $\psi_0$ satisfies (9), then, for any $h > 0$,

$$\lambda_-^D(h, B, \Omega) \geq \lambda^D(\Omega) h^2 \exp(2 \inf_\Omega \psi_0/h).$$

In the semi-classical limit,

$$\lim_{h \to 0} h \log \lambda^D(h, B, \Omega) = 2 \inf_\Omega \psi_0.$$

In the non-simply connected case, effects from the the circulations of the magnetic potential along different components of the boundary could in principle introduce a different constant than $\inf_\Omega \psi_0$ inside the exponential function. Thus, a modified scalar potential, taking the circulation along the holes into account, was used in [8]. It turns out, however, that in the semi-classical limit, such effects disappear, and it is again $\psi_0$ that gives the correct asymptotics.

**Theorem 1.4** ([8]). Assume that $\Omega$ is a connected domain in $\mathbb{R}^2$, and that $B > 0$. If $\psi_0$ is the solution of (9), then, for any $A$ such that $\operatorname{curl} A = B$,

$$\lim_{h \to 0} h \log \lambda^D(h, A, B, \Omega) = 2 \inf_\Omega \psi_0.$$

2. Main results

The aim of this paper is to extend the above mentioned results to Pauli operators with sign changing magnetic fields.

2.1. The ground state energy of $P_\pm(B, h)$. It turns out that the scalar potential $\psi_0$, the solution of (9), still plays a main role for the asymptotic of the bottom of the spectrum of the Pauli operator. If $B > 0$ then, by the maximum principle $\psi_0 < 0$ in $\Omega$, and similarly, if $B < 0$ then $\psi_0 > 0$ in $\Omega$. For $B$ with varying sign, it might still be the case that $\psi_0$ is of constant sign in $\Omega$, but that will depend on $B$, and the situation is delicate (we study some examples in the sections 5 and 6).

With (6) in mind we focus again on the eigenvalue $\lambda_-^D(h, B, \Omega)$. Our first result concerns the case when $\psi_0$ attains negative values in $\Omega$.

**Theorem 2.1.** Assume that $\Omega$ is a simply connected domain in $\mathbb{R}^2$, and that $\inf_\Omega \psi_0 < 0$, where $\psi_0$ is satisfying (9). Then

$$\limsup_{h \to 0} h \log \lambda_-^D(h, B, \Omega) \leq 2 \inf_\Omega \psi_0.$$
We recall from the definitions of $\Omega^+_B$ and $\Omega^-_B$, and will now turn to the case when both of them are non-void.

We assume in addition that $\Gamma := B^{-1}(0) \subset \Omega$ is of class $C^{2,+}$ and that $\Gamma \cap \partial \Omega$ is either empty or, if non empty, that the intersection is a finite set, avoiding the corner points, with transversal intersection.

Under this assumption $\Omega^+_B$ satisfies the same condition as $\Omega$ from Assumption 1, and we will denote by $\hat{\psi}_0$ the solution of

$$\Delta \hat{\psi}_0 = B(x) \text{ in } \Omega^+_B, \quad \hat{\psi}_0 = 0 \text{ on } \partial \Omega^+_B.$$  

(11)

By domain monotonicity, with $\Omega^+_B \subset \Omega$, we can apply Theorem 2.1 with $\Omega$ replaced by $\Omega^+_B$ and get

Corollary 2.2. Assume that $\Omega$ is a connected domain in $\mathbb{R}^2$. Assume further that $\hat{\psi}_0$ satisfies (11). Then

$$\limsup_{h \to 0} h \log \lambda^D_{\Omega^+_B}(h, B, \Omega) \leq 2 \inf_{\Omega^+_B} \hat{\psi}_0.$$  

(12)

Now, the main problem is to determine if one of the bounds above, i.e. (10) and (12), is optimal. We have two possibly enlightening statements on this question. The first one gives a simple criterion under which the upper bound given in Corollary 2.2 is not optimal.

Theorem 2.3. Assume that $\Omega$ is a simply connected domain in $\mathbb{R}^2$, $\Omega^+_B \neq \emptyset$, and that $\hat{\psi}_0$ satisfies (11). If $B^{-1}(0)$ either is a compact $C^{2,+}$ closed curve in $\Omega$ or a $C^{2,+}$ line crossing $\partial \Omega$ transversally away from the corners, then

$$\limsup_{h \to 0} h \log \lambda^D_{\Omega^+_B}(h, B, \Omega) < 2 \inf_{\Omega^+_B} \hat{\psi}_0.$$  

Two examples where this condition is satisfied is when $\Omega$ is a disk, and the magnetic field is either radial, vanishing on a circle, or affine, vanishing on a line. We will return to them later.

We mentioned earlier that even though $B$ changes sign, it might happen that the scalar potential $\psi_0$ does not. Our second statement says that in this case we actually have the optimal result.

Theorem 2.4. Suppose that $\Omega$ is a simply connected domain in $\mathbb{R}^2$. If $\psi_0 < 0$ in $\Omega$, where $\psi_0$ is the solution of (9), then

$$\lim_{h \to 0} h \log \lambda^D_{\Omega^+_B}(h, B, \Omega) = 2 \inf_{\Omega} \psi_0.$$  

(13)

2.2. The ground state energy of the Pauli operator. We have already mentioned, that from (6) it follows that to understand the lowest eigenvalue of each of the components of $P(B, h)$, it suffices to study one of them, with the extra price that we must do it both for $B$ and $-B$. To discuss the lowest eigenvalue $\Lambda^D_{\Omega^+_B}(B, h, \Omega)$ of the Pauli operator $P(B, h)$, we will compare the eigenvalues for the spin-up and spin-down components.

If the scalar potential $\psi_0$ does not changes sign in $\Omega$, we can transfer the earlier results to $\Lambda^D_{B, h, \Omega}$.

Theorem 2.5. Let $\Omega$ be a simply connected domain in $\mathbb{R}^2$, and let $\psi_0$ be given by (9). If $\psi_0$ does not change sign in $\Omega$, then

$$\lim_{h \to 0} h \log \Lambda^D_{B, h, \Omega} = -2 \text{ Osc } \psi_0.$$  

(13)
Our final result concerns a case when $\psi_0$ changes sign.

**Theorem 2.6.** Let $\Omega$ be a simply connected domain in $\mathbb{R}^2$, and let $\psi_0$ be given by (9). Assume that

$$\psi_{\text{min}} = \inf_{\Omega} \psi_0 < 0 < \sup_{\Omega} \psi_0 = \psi_{\text{max}}.$$ 

Assume further that $\psi_0^{-1}(\psi_{\text{max}})$ contains a $C^2_+ \cap \Omega$ closed curve enclosing a non-empty part of $\psi_0^{-1}(\psi_{\text{min}})$ or that $\psi_0^{-1}(\psi_{\text{min}})$ contains a $C^2_+ \cap \Omega$ closed curve enclosing a non-empty part of $\psi_0^{-1}(\psi_{\text{max}})$. Then

$$\lim_{h \to 0} h \log \Lambda^D(h, B, \Omega) = -2 \text{ Osc } \psi_0.$$ 

(14)

**Remark 2.7.** Since $\psi_0$ can attain its minimum only in $\Omega + B$ and its maximum only in $\Omega - B$, see equation (7), a necessary condition for the hypothesis of Theorem 2.6 is that $B^{-1}(0)$ contain a closed curve. A radial magnetic field which changes sign on a disc is a typical example in which this condition is satisfied, see Section 5.

3. **Proof of Theorems 2.1, 2.4, 2.5, and 2.6**

We assume in this section that $\Omega$ is simply connected. As it is standard, we can assume up to a gauge transform that

$$\text{div } A = 0 \text{ in } \Omega, \quad A \cdot \nu \text{ on } \partial \Omega.$$ 

In this case, the solution $\psi_0$ of (9) satisfies:

$$A = \nabla \psi_0 = (-\partial_{x_2} \psi_0, \partial_{x_1} \psi_0).$$

We let $\psi_{\text{min}} = \inf_{\Omega} \psi_0$ and $\psi_{\text{max}} = \sup_{\Omega} \psi_0$, and assume that

$$\psi_{\text{min}} < \psi_{\text{max}} = 0.$$ 

(15)

We note that this is the case when $B > 0$ in $\Omega$ and that this condition implies that $\int_{\partial \Omega} B(x) \, dx > 0$. An example of such a magnetic field will be given in Section 6.

**Proof of Theorem 2.4.** Under Assumption (15), the proofs of Section 4 in [8] go through. One can first consider trial states in the form $\psi \eta \exp(-\psi_0/h)$ with $\psi \eta$ compactly supported in $\Omega$ and $\psi \eta$ being equal to 1 outside a sufficiently small neighborhood of the boundary, or in the form (this equality defines $\psi$)

$$\exp(-\psi_0/h) - \exp(\psi_0/h) := \exp(-\psi_0/h)v.$$ 

The only change consists in replacing [8, equation (4.4)] by

$$h^2 \int_{\Omega} \exp\left(-\frac{2 \psi}{h}\right) |(\partial_{x_1} + i \partial_{x_2}) v|^2 \, dx \leq 4h \int_{\Omega} |B(x)| \, dx.$$ 

(16)

Hence all the statements are unchanged under the condition of replacing $\Phi = \int B(x) \, dx$ by $2 \int |B(x)| \, dx$. In particular, the statement in Theorem 2.4 follows.

**Proof of Theorem 2.1.** If we only assume $\psi_{\text{min}} < 0$, we get by the same calculation as above that

$$2(\psi_{\text{min}} - \psi_{\text{max}}) \leq \lim_{h \to 0} h \log \lambda^D(h, A, B, \Omega)$$

$$\leq \lim_{h \to 0} \sup h \log \lambda^D(h, A, B, \Omega) \leq 2 \psi_{\text{min}}.$$ 

(17)

This proves Theorem 2.1.

□
Proof of Theorem 2.3. We first note that, since \( \text{Osc}_\Omega \psi_0 = \text{Osc}_\Omega (-\psi_0) \), Theorem 1.2 in combination with (9) gives

\[
\lim_{h \to 0} h \log \lambda_D^+(h, B, \Omega) \geq -2 \text{Osc}_\Omega \psi_0. \tag{18}
\]

If \( \psi_0 < 0 \) in \( \Omega \), the claim follows from Theorem 2.3 and (18). If \( \psi_0 > 0 \) in \( \Omega \), then Theorem 2.4 together with equation (6) implies that

\[
\lim_{h \to 0} h \log \lambda_D^+(h, B, \Omega) = 2 \inf_{\Omega} (-\psi_0) = -2 \text{Osc}_\Omega \psi_0. \tag{19}
\]

The lower bound (18) then again completes the proof.

Proof of Theorem 2.6. Suppose first that \( \psi_0^{-1}(\psi_{\text{max}}) \) contains a \( C^{2,+} \) closed curve \( \gamma_1 \) enclosing a non-empty part of \( \psi_0^{-1}(\psi_{\text{min}}) \). Let \( \Omega_1 \subset \Omega \) be the region enclosed by \( \gamma_1 \) and define on \( \Omega_1 \) the function \( \psi_1 = \psi_0 - \psi_{\text{max}} \). Then \( \Delta \psi_1 = B, \ psi_1 < 0 \) and \( \psi_1 = 0 \) on \( \partial \Omega_1 = \gamma_1 \). The domain monotonicity and Theorem 2.1 thus imply that

\[
\lim_{h \to 0} h \log \lambda_D^+(h, B, \Omega) \leq \limsup_{h \to 0} h \log \lambda_D^+(h, B, \Omega_1)
\]

\[
\leq -2 \inf_{\Omega_1} \psi_1 = -2 \text{Osc}_\Omega \psi_0.
\]

On the other hand, if \( \psi_0^{-1}(\psi_{\text{min}}) \) contains a \( C^{2,+} \) closed curve \( \gamma_2 \) enclosing a non-empty part of \( \psi_0^{-1}(\psi_{\text{max}}) \), then we denote by \( \Omega_2 \subset \Omega \) the region enclosed by \( \gamma_2 \) and define \( \psi_2 = -\psi_0 + \psi_{\text{min}} \). Hence \( \Delta \psi_2 = B, \ psi_2 < 0 \) and \( \psi_2 = 0 \) on \( \partial \Omega_2 = \gamma_2 \). In view of equation (6), Theorem 2.4 and the domain monotonicity we then get

\[
\lim_{h \to 0} h \log \lambda_D^+(h, B, \Omega) \leq \limsup_{h \to 0} h \log \lambda_D^+(h, B, \Omega_2)
\]

\[
= \limsup_{h \to 0} h \log \lambda_D^+(h, -B, \Omega_2)
\]

\[
\leq -2 \inf_{\Omega_2} \psi_2 = -2 \text{Osc}_\Omega \psi_0.
\]

In either case, an application of inequality (18) completes the proof.

4. Proof of Theorem 2.3

4.1. A deformation argument. We have seen that we can have \( \psi < 0 \) in \( \Omega \) without to assume \( B > 0 \) and that once this property is satisfied we can obtain an upper bound by restricting to the subset of \( \Omega \) where \( \psi \) is negative instead. Hence a natural idea is to consider the family of subdomains of \( \Omega \) defined by

\[
\mathcal{F} = \{ \omega \subset \Omega, \partial \omega \subset C^{2,+}_\text{pw} : \Delta \psi = B \text{ in } \omega \text{ and } \psi|_{\partial \omega} = 0 \Rightarrow \psi < 0 \text{ in } \omega \}.
\]

We know that \( \Omega_B^\omega \in \mathcal{F} \). The idea behind the proof of Theorem 2.3 is to show that there exists \( \omega \in \mathcal{F} \) such that \( \Omega_B^\omega \subset \omega \) with strict inclusion. More precisely, we have

**Proposition 4.1.** Let \( \omega \in \mathcal{F} \) and let \( \psi_\omega \) be the solution of \( \Delta \psi = B \) in \( \omega \) such that \( \psi_\omega = 0 \) on \( \partial \omega \). If \( \partial_\nu \psi_\omega > 0 \) at some point \( M_\omega \) of \( \partial \omega \cap \Omega \), there exists \( \psi_\bar{\omega} \) attached to \( \bar{\omega} \), with \( \omega \subset \bar{\omega} \) such that

\[
\inf_{\omega} \psi_\bar{\omega} < \inf_{\omega} \psi_\omega. \tag{20}
\]
Proof. First we deform $\omega$ into $\tilde{\omega}$ (smooth and small perturbation). We refer to [10, Chapter 5] for different ways to do this. We can for example extend the outward normal vector field to a vector field defined in a tubular neighborhood of $\partial \omega$. We call this vector field $X_0$, and take a function $\theta$ in $C^\infty_0(\mathbb{R}^2)$ with compact support near $M_\omega$ and equal to 1 near $M_\omega$. We then consider the vector field $X := \theta X_0$, which is naturally defined in $\mathbb{R}^2$. If we consider the associated flow $\Phi_t$ of the vector field $X$, $t \mapsto \Phi_t(\omega)$ defines the desired deformation for $t$ small. We then define $\tilde{\omega} = \Phi_t(\omega)$ for some $t_0 > 0$, and construct the corresponding scalar potential $\psi_{\tilde{\omega}}$. We claim that comparing $\psi_\omega$ and $\psi_{\tilde{\omega}}$, which can be done by comparing $\psi_\omega$ and $\psi_{\tilde{\omega}} \circ \Phi^{-1}_{t_0}$ which are defined on $\omega$ and arbitrarily close (for $t_0$ small enough) in $H^2(\omega)$, hence in $C^0(\bar{\omega})$, we get, noting that $\psi_{\tilde{\omega}} - \psi_\omega$ is harmonic in $\omega$, non positive on $\partial \omega$, strictly negative near $M_\omega$ and using the maximum principle, that $\psi_{\tilde{\omega}} < \psi_\omega$ in $\omega$. In particular, we get (20). □

4.2. Proof of Theorem 2.3. With the deformation argument at hand, we are now ready to give a

Proof of Theorem 2.3. This is now an immediate application of Proposition 4.1 if we can show that at some point of $B^{-1}(0)$, $\partial_r \psi_0 \neq 0$. However, from the Hopf boundary lemma it follows that at every regular point of $\partial(\Omega_B \cap \Omega)$ we have $\partial_r \psi_0 > 0$. Hence the claim. □

5. Radial magnetic fields

A typical application of the general theorems concerns radial magnetic fields in $\Omega = D(0, R)$, the disk of radius $R$ centered at 0. The following result is a combination of the theorems appearing in Section 2.

Theorem 5.1. Assume that $\Omega = D(0, R)$, and that the magnetic field $B$ is radial and continuous. Then,

$$\lim_{h \to 0} h \log \Lambda^D(h, B, \Omega) = -2 \text{Osc } \psi_0.$$ 

Proof. We observe that the solution $\psi_0$ of (9) is radial, and write $r = (x_1^2 + x_2^2)^{1/2}$ and $\phi_0(r) = \psi_0(x_1, x_2)$. In view of Theorem 2.5 we may assume without loss of generality that $\phi_0$ changes sign in $(0, R)$. The claim then follows from Theorem 2.6. □

Example. As an example of a radial field we consider the function

$$B_\beta(x_1, x_2) = \beta^2 - r^2$$

on the unit disc $D(0, 1)$. An explicit solution of

$$\Delta \psi_0 = B_\beta \text{ in } D(0, 1), \quad \psi_0 = 0 \text{ on } \partial D(0, 1),$$

is radial, and given by

$$\psi_0(x_1, x_2) = \frac{1}{16} (r^2 + 1 - 4\beta^2)(1 - r^2).$$
Hence for $\beta \in (0, \frac{1}{2}) \cup (\frac{1}{\sqrt{2}}, 1)$ we can apply Theorem 2.5. while the case $\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$ is covered by Theorem 5.1. A straightforward calculation then shows that

$$\lim_{h \to 0} h \log \Lambda^D(h, B_\beta, D(0, 1)) = -2 \text{Osc } \psi_0 = -\begin{cases} \frac{1}{8}(2\beta^2 - 1)^2 & \beta \in (0, \frac{1}{2}) \\ \frac{1}{2}\beta^4 & \beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}] \\ \frac{1}{8}(4\beta^2 - 1) & \beta \in (\frac{1}{\sqrt{2}}, 1). \end{cases}$$

**Remark 5.2.** Since $\Lambda^D(h, B_\beta, D(0, 1)) = \Lambda^D(h, -B_\beta, D(0, 1))$ the example above also covers the magnetic field $-B_\beta$, i.e. the case when the magnetic field is negative inside the domain delimited by the circle $B_\beta^{-1}(0)$.

### 6. A Magnetic Field Vanishing on a Line Joining Two Points of the Boundary

#### 6.1. Preliminaries.

We present a numerical study for the case when the zero-set of the magnetic field, $B_\beta^{-1}(0)$, is a line joining two points of the boundary. We consider again $\Omega = D(0, 1)$, the disk of radius 1, and assume that

$$B_\beta(x_1, x_2) = \beta - x_1, \quad -1 < \beta < 1.$$ 

This means that $B_\beta^{-1}(0)$ is given by the line $x_1 = \beta$. The solution of

$$\Delta \psi_\beta = (\beta - x_1) \text{ in } D(0, 1), \quad \psi_\beta = 0 \text{ on } \partial D(0, 1),$$

is given by

$$\psi_\beta(x_1, x_2) = \frac{1}{8}(x_1 - 2\beta)(1 - x_1^2 - x_2^2). \quad (21)$$

A straightforward calculation shows that

$$\max \psi_\beta = \begin{cases} \psi_\beta \left(\frac{2\beta + \sqrt{3 + 4\beta^2}}{3}, 0\right) & \text{for } -1 < \beta < 1/2, \\ 0 & \text{for } 1/2 \leq \beta < 1, \end{cases} \quad (22)$$

and

$$\min \psi_\beta = \begin{cases} 0 & \text{for } -1 < \beta \leq -1/2, \\ \psi_\beta \left(\frac{2\beta - \sqrt{3 + 4\beta^2}}{3}, 0\right) & \text{for } -1/2 < \beta < 1. \end{cases} \quad (23)$$

It follows that

$$\text{Osc } \psi_\beta = \begin{cases} \psi_\beta \left(\frac{2\beta + \sqrt{3 + 4\beta^2}}{3}, 0\right) & -1 < \beta \leq -1/2, \\ \psi_\beta \left(\frac{2\beta + \sqrt{3 + 4\beta^2}}{3}, 0\right) - \psi_\beta \left(\frac{2\beta - \sqrt{3 + 4\beta^2}}{3}, 0\right) & -1/2 < \beta < 1/2, \\ -\psi_\beta \left(\frac{2\beta - \sqrt{3 + 4\beta^2}}{3}, 0\right) & 1/2 \leq \beta < 1. \end{cases} \quad (24)$$

Next, we introduce $B_\beta^+$ as the subset of $D(0, 1)$ where $B_\beta$ is positive, i.e. $B_\beta^+ = \{(x_1, x_2) \in D(0, 1) \mid x_1 < \beta\}$. The solution $\hat{\psi}_\beta$ of

$$\Delta \hat{\psi}_\beta = (\beta - x_1) \text{ in } \Omega_\beta^+, \quad \hat{\psi}_\beta = 0 \text{ on } \partial \Omega_\beta^+, \quad (25)$$

is not explicit, except in the case $\beta = 0$, where we have $\hat{\psi}_0 = \psi_0$ in $\Omega_0^+$. The oscillation of $\hat{\psi}_\beta$ can be calculated numerically. When $\beta \in (\frac{1}{2}, 1)$, the oscillation of $\hat{\psi}_\beta$ is strictly smaller than the oscillation of $\psi_\beta$. Indeed, the function $\Psi_\beta = \psi_\beta - \hat{\psi}_\beta$ will in this case satisfy $\Delta \Psi_\beta = 0$ in $\Omega_\beta$ and $\Psi_\beta = 0$
on the circular part of the boundary of $\Omega_\beta$ and $\Psi_\beta = \psi_\beta \leq 0$ on the line $x = \beta$. The Maximum principle gives $\Psi_\beta < 0$ in $\Omega_\beta$, and so $\psi_\beta < \hat{\psi}_\beta$ in $\Omega_\beta$. See also Figure 1.

![Figure 1. Here we have plotted Osc $\psi_\beta - \text{Osc} \hat{\psi}_\beta$ as a function of $\beta \in (-1,1)$.](image)

6.2. Application. We assume that

$$-\frac{1}{2} < \beta < \frac{1}{2}.$$  

Using the restriction to $\Omega_{2\beta}^+$, we get by (17) an upper-bound (using (23)) involving $-\psi_\beta((2\beta - \sqrt{3} + 4\beta^2)/3)$. On the other hand we have a lower bound by the general theorem involving the oscillation of $\psi_\beta$ (see (24) for the computation). Here we will discuss the application of Proposition 4.1 when applied with $\omega = \Omega_{2\beta}$. What we need is to compute

$$\partial_{x_1}\psi_\beta(2\beta, x_2) = \frac{1}{8}(1 - 4\beta^2 - x_2^2),$$

with $\psi_\beta$ defined in (21), and to observe that it does not vanish. As a consequence the upper bound of $\limsup_{h \to 0} h \log \lambda_1$ given by Corollary 2.2 when applied with $\omega = \Omega_{2\beta}$ is not optimal. Pushing the boundary will indeed improve the upper bound. The question of the optimality of the lower bound remains open.

6.3. Researching a better upper bound. Given $\beta$ it is interesting to find the largest possible domain $\Omega_{\max} \subset D(0,1)$ such that the solution to

$$\Delta \psi = \beta - x_1, \text{ in } \Omega_{\max}, \quad \psi = 0, \text{ on } \partial \Omega_{\max}$$  

is strictly negative in $\Omega_{\max}$. The oscillation of that solution could contribute to a candidate for the optimal constant in the asymptotics of the Pauli eigenvalue. We are only able to consider this problem numerically.

To find $\Omega_{\max}$ numerically, we follow (a slightly modified version of) an iterative procedure that was kindly suggested by Stephen Luttrell [11], described below. We start by numerically solving the problem on a regular polygon with (many) corners, positioned on the unit disk. Then we look at the sign of the solution close to each corner of the polygon, and move the corresponding points to make the new domain smaller if the calculated
value is positive, and larger if it is negative. We also make sure that no point moves out from the disk. This gives us a new set of points. We build a new polygon, and repeat the procedure until the Euclidean distance between the corners of two iterations becomes as small as we wish (see Figure 2 for an example).

![Image of converging domains](image)

**Figure 2.** The set of domains converge quickly. In this example we start with 200 vertices, \( \beta = 0.2 \), and we exit the loop when the square norm of the difference between two consecutive iterations become less than 0.005 (after five steps). The red dots show the vertices used in the domain of the current step, and the black dots the ones that are calculated for the next step.

We denote by \( \psi_{\beta, \text{opt}} \) the function we end up with after the iterative procedure (ideally a solution of (26)). In Figure 3 we have made a comparison of the oscillation of \( \psi_{\beta, \text{opt}} \) and \( \psi_{\beta} \), and we find that the oscillation of \( \psi_{\beta, \text{opt}} \) is slightly larger than that of \( \psi_{\beta} \).

![Graph of oscillations](image)

**Figure 3.** The oscillation of \( \psi_{\beta, \text{opt}} \) is slightly larger than that of \( \psi_{\beta} \). Here we show \(-2 \text{Osc} \psi_{\beta, \text{opt}}\) for \(-0.5 \leq \beta \leq 1\) (orange) and \(-2 \text{Osc} \psi_{\beta}\) for \(-1 \leq \beta \leq 1\) (black).
7. Conclusion

We have initially obtained from the previous papers [3, 7, 8] two natural upper bounds and a natural lower bound. We have shown that in general these two initial upper bounds cannot be true. We have also presented particular cases where the results are optimal. In all these cases, the oscillation of $\psi_0$ is shown to be optimal. If $\psi_0^{-1}(0, +\infty) \cap \Omega \neq \emptyset$, we have obtained the optimality by constructing an open set $\omega$ in $\Omega$ for which the solution $\psi_{\omega}$ of $\Delta \psi = B$ in $\omega$ and $\psi = 0$ on $\partial \omega$ satisfies in addition $\partial_{\nu} \psi = 0$ on $\partial \omega \cap \Omega$.

Numerically it could be interesting to see how to “push the boundary” in Proposition 4.1 in order to get a maximal domain.

Finally, as already observed in [7], one can also expect to get upperbounds by using previous results devoted to the asymptotics of the ground state energy of the Witten Laplacian (see [1, 2, 4, 5, 6, 13] and the quite recent note of B. Nectoux [12]). This unfortunately does not lead to any improvement in the cases we have considered.

Acknowledgements

B. H. would like to thank D. Le Peutrec for discussions around the work of B. Nectoux. H. K. has been partially supported by Gruppo Nazionale per Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The support of MIUR-PRIN2010-11 grant for the project “Calcolo delle variazioni” (H. K.), is also gratefully acknowledged.

References

[1] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein: Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times. JEMS 6(4), 399–424 (2004).
[2] A. Bovier, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes II: Precise asymptotics for small eigenvalues. JEMS 7(1), 69–99 (2004).
[3] T. Ekholm, H. Kovařík, and F. Portmann. Estimates for the lowest eigenvalue of magnetic Laplacians. J. Math. Anal. Appl. 439 (1), 330–346 (2016).
[4] M.I. Freidlin and A.D. Wentzell. Random perturbations of dynamical systems. Transl. from the Russian by Joseph Szuecs. 2nd ed. Grundlehren der Mathematischen Wissenschaften. 260. New York (1998).
[5] B. Helffer, M. Klein and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. Matematica Contemporanea, 26, 41–85 (2004).
[6] B. Helffer and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary. Mém. Soc. Math. Fr. (N.S.) No. 105 (2006).
[7] B. Helffer and M. Persson Sundqvist. On the semi-classical analysis of the Dirichlet Pauli operator. J. Math. Anal. Appl. 449 (1), 138–153 (2017).
[8] B. Helffer and M. Persson Sundqvist. On the semi-classical analysis of the groundstate energy of the Dirichlet Pauli operator in non-simply connected domains. Journal of Mathematical Sciences 226 (4), 531–544 (2017).
[9] B. Helffer and J. Sjöstrand. A proof of the Bott inequalities. Algebraic Analysis, Vol.1, Academic Press, 171-183 (1988).
[10] A. Henrot and M. Pierre. Variation et optimisation de formes – une analyse géométrique– Mathématiques et Applications. 48. Springer (2005).
[11] S. Luttrell. https://mathematica.stackexchange.com/a/154435/21414 (fetched October 19, 2017).
[12] B. Nectoux. Sharp estimate of the mean exit time of a bounded domain in the zero white noise limit. Personal communication.

[13] E. Witten. Supersymmetry and Morse inequalities. J. Diff. Geom. 17, 661–692 (1982).

(Bernard Helffer) Laboratoire de Mathématiques Jean Leray, Université de Nantes, 2 rue de la Houssinière, 44322 Nantes, France and Laboratoire de Mathématiques, Université Paris-Sud, France.

E-mail address: bernard.helffer@univ-nantes.fr

(Hynek Kovařík) DICATAM, Sezione di Matematica, Università degli studi di Brescia, Italy.

E-mail address: hynek.kovarik@unibs.it

(Mikael P. Sundqvist) Lund University, Department of Mathematical Sciences, Box 118, 221 00 Lund, Sweden.

E-mail address: mikael.persson_sundqvist@math.lth.se