ON THE CALCULATION OF TWO ESSENTIAL HARMONIC SERIES WITH A WEIGHT 5 STRUCTURE, INVOLVING HARMONIC NUMBERS OF THE TYPE $H_{2n}$

CORNEL IOAN VĂLEAN

Abstract. The core of the present paper is represented by the calculation of two essential harmonic series with a weight 5 structure, involving harmonic numbers of the type $H_{2n}$. The two main series are evaluated by also exploiting results and strategies presented in the book, (Almost) Impossible Integrals, Sums, and Series, 2019.

1. Introduction

The central results of the paper are represented by two harmonic series, involving harmonic numbers of the type $H_{2n}$.

The $n\text{th}$ generalized harmonic number of order $m$ is defined by

$$H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},$$

where $m$ is a positive integer.

We call the two main series presented in the paper essential since they are usually a critical part in the derivation process of other such harmonic series.

In the classical sense, by the weight of a harmonic series we understand the value $W = a_1 + a_2 + \cdots + a_k + a$ we obtain from the summand of the harmonic series

$$\sum_{n=1}^{\infty} \frac{H_n^{(a_1)} H_n^{(a_2)} \cdots H_n^{(a_k)}}{n^a},$$

where $a_1, a_2, \ldots, a_k$, and $a$ are positive integers.

By analogy with the series presented above, we may consider that the two main series have a weight 5 structure.

During the calculations we will also make use of results and strategies presented in the book, (Almost) Impossible Integrals, Sums, and Series.

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2. The lemmas and their proofs

**LEMMA 1.** *(Two useful series representations)* The following equalities hold:

i)

\[-\log(1+x)\log(1-x) = \sum_{n=1}^{\infty} x^{2n} \left( \frac{H_{2n} - H_n}{n} + \frac{1}{2n^2} \right), \ |x| < 1;\]

ii)

\[\log(1-x)Li_2(x) = 3 \sum_{n=1}^{\infty} \frac{x^n}{n^3} - 2 \sum_{n=1}^{\infty} \frac{x^n H_n}{n^2} - \sum_{n=1}^{\infty} x^n \frac{H_n^{(2)}}{n}, \ |x| \leq 1 \wedge x \neq 1,\]

where \(Li_n\) denotes the Polylogarithm function.

**Proof.** The proofs of both results are straightforward if we apply the Cauchy product of two series (e.g. check the proof presented in [2, Chapter 6, p. 344]).

**LEMMA 2.** *(Special logarithmic integrals)* Let \(n\) be a positive integer. The following equalities hold:

i)

\[\int_0^1 x^{n-1} \log(1-x) dx = -\frac{H_n}{n};\]

ii)

\[\int_0^1 x^{2n-1} \log(1+x) dx = \frac{H_{2n} - H_n}{2n}.\]

**Proof.** For a straightforward proof of \(i\), make use of the series representation, \(\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\). Then, for the point \(ii\) we may use that \(\log(1+x) = \log(1-x^2) - \log(1-x)\), and we see immediately the resulting integrals may be calculated by using \(i\). For an alternative way to the integral \(i\), see [2, Chapter 3, p. 59].

**LEMMA 3.** *(An identity with harmonic numbers)* The following equality holds:

\[\sum_{k=1}^{\infty} \frac{H_k}{(k+1)(k+n+1)} = \frac{H_n^2 + H_n^{(2)}}{2n}.\]

**Proof.** The present result may be viewed as a particular case of a more general result which may be found in [2, Chapter 4, p. 289] and proved in [2, Chapter 6, p. 372–374] by using The Master Theorem of Series defined in [3] and [2, Chapter 4, p. 288–289].
**Lemma 4.** (A powerful identity involving harmonic numbers) The following equality holds:

\[
\sum_{n=1}^{\infty} \frac{1}{(2k+2n+1)2n} = \frac{1}{(2k+1)^2} + \frac{H_{2k}}{2(2k+1)} - \frac{H_k}{2k+1} - \frac{\log(2)}{2k+1}.
\]

**Proof.** The calculations are straightforward if we use the partial fraction expansion, add and subtract \(1/(2n+1)\) inside the summand, and then split the series. The details of such an approach may be found in [2, Chapter 6, p. 531].

**Lemma 5.** (Two classical Euler sums) The following equalities hold:

i) \(2 \sum_{k=1}^{\infty} \frac{H_k}{k^m} = (m+2)\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1), m \geq 2;\)

ii) \(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{H_k}{k^{2m}} = \left( m + \frac{1}{2} \right) \eta(2m+1) - \frac{1}{2} \zeta(2m+1) - \sum_{i=1}^{m-1} \eta(2i)\zeta(2m-2i+1), m \geq 1,\)

where \(\zeta\) represents the Riemann zeta function and \(\eta\) denotes the Dirichlet eta function.

**Proof.** The result from the point i) is known in the mathematical literature from old times. For example, an elementary solution may be found in [1, Chapter 2, pp. 103–105]. Another way to prove it is based on the identity in Lemma 3, by multiplying its both sides by \(n\) and then considering the differentiation with respect to \(n\). A solution to the series result from the point ii) may be found in [4].

**Lemma 6.** (A bunch of key harmonic series). The following equalities hold:

i) \(\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} = 3\zeta(2)\zeta(3) - \frac{9}{2} \zeta(5);\)

ii) \(\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2} \zeta(5) - \zeta(2)\zeta(3);\)

iii) \(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3} = \frac{11}{4} \zeta(4) - \frac{7}{4} \log(2)\zeta(3) + \frac{1}{2} \log^2(2)\zeta(2) - \frac{1}{12} \log^4(2) - 2 \text{Li}_4 \left( \frac{1}{2} \right);\)
iv)
\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(2)}}{n^3} = \frac{5}{8} \zeta(2) \zeta(3) - \frac{11}{32} \zeta(5);
\]

v)
\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^2}{n^3} = \frac{2}{15} \log^5(2) - \frac{11}{8} \zeta(2) \zeta(3) - \frac{19}{32} \zeta(5) + \frac{7}{4} \log^2(2) \zeta(3)
- \frac{2}{3} \log^3(2) \zeta(2) + 4 \log(2) \text{Li}_4 \left(\frac{1}{2}\right) + 4 \text{Li}_5 \left(\frac{1}{2}\right);
\]

vi)
\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n H_n^{(2)}}{n^2} = \frac{23}{8} \zeta(5) - \frac{7}{4} \log^2(2) \zeta(3) + \frac{3}{2} \log^3(2) \zeta(2) + \frac{15}{16} \zeta(2) \zeta(3)
- \frac{2}{15} \log^5(2) - 4 \log(2) \text{Li}_4 \left(\frac{1}{2}\right) - 4 \text{Li}_5 \left(\frac{1}{2}\right),
\]

where \( \zeta \) represents the Riemann zeta function and \( \text{Li}_n \) denotes the Polylogarithm function.

Proof. All the series results are found in [2, Chapter 4, pp. 292–293, pp. 309–312].

3. The main theorems and their proofs

Theorem 1. (Main results - the first part) The following equality holds:
\[
\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n)^3} = \frac{307}{128} \zeta(5) - \frac{1}{16} \zeta(2) \zeta(3) + \frac{1}{3} \log^3(2) \zeta(2) - \frac{7}{8} \log^2(2) \zeta(3) - \frac{1}{15} \log^5(2)
- 2 \log(2) \text{Li}_4 \left(\frac{1}{2}\right) - 2 \text{Li}_5 \left(\frac{1}{2}\right),
\]

where \( \zeta \) represents the Riemann zeta function and \( \text{Li}_n \) denotes the Polylogarithm function.

Proof. Based on Lemma 1, the point i), we obtain by integration that
\[
-\int_0^x \log(1+y) \log(1-y) \frac{dy}{y} = \sum_{n=1}^{\infty} x^{2n} \left( \frac{H_{2n} - H_n}{2n^2} + \frac{1}{4n^3} \right),
\]

and if we multiply both sides by \( \log(1+x)/x \) and integrate from \( x = 0 \) to \( x = 1 \), using Lemma 2, the point ii), we have
\[
-\int_0^1 \frac{\log(1+x)}{x} \left( \int_0^x \log(1+y) \log(1-y) \frac{dy}{y} \right) \frac{dx}{y} = \sum_{n=1}^{\infty} \frac{H_{2n} - H_n}{2n^2} \left( \frac{H_{2n} - H_n}{2n^2} + \frac{1}{4n^3} \right).
\]
If in (1) we integrate by parts, then use \( \sum_{n=1}^{\infty} a_{2n} = \frac{1}{2} \left( \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} (-1)^{n-1} a_n \right) \) for the series in the right-hand side, and afterwards rearrange, we get

\[
\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n)^3} = \frac{7}{32} \sum_{n=1}^{\infty} \frac{H_n}{n^4} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^4} + \frac{5}{16} \sum_{n=1}^{\infty} \frac{H_n^2}{n^3} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{n^3} - \frac{5}{64} \zeta(2) \zeta(3) \\
+ \frac{1}{4} \int_0^1 \frac{\log(1+x) \log(1-x) \text{Li}_2(-x)}{x} \, dx
\]

where in the calculations we also used Lemma 5, the point \( i) \) with \( m = 4 \), the point \( ii) \) with \( m = 2 \), and Lemma 6, the points \( ii) \) and \( v) \).

On the other hand, using Lemma 1, the point \( ii) \), Lemma 2, the point \( i) \), and then the series results from Lemma 5, the point \( ii) \) with \( m = 2 \), and Lemma 6, the points \( v) \) and \( vi) \), the integral in (2) may be written as

\[
\int_0^1 \frac{\log(1+x) \log(1-x) \text{Li}_2(-x)}{x} \, dx \\
= \int_0^1 \left( 2 \sum_{n=1}^{\infty} (-1)^n x^{n-1} \frac{H_n}{n^2} + \sum_{n=1}^{\infty} (-1)^n x^{n-1} \frac{H_n^{(2)}}{n} - 3 \sum_{n=1}^{\infty} (-1)^n x^{n-1} \frac{H_n^2}{n^3} \right) \log(1-x) \, dx \\
= 3 \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^4} - 2 \sum_{n=1}^{\infty} (-1)^n \frac{H_n^2}{n^3} - \sum_{n=1}^{\infty} (-1)^n \frac{H_n^2}{n^2} \\
= \frac{123}{32} \zeta(5) + \frac{5}{16} \zeta(2) \zeta(3) + \frac{2}{3} \log^3(2) \zeta(2) - \frac{7}{4} \log^2(2) \zeta(3) - \frac{2}{15} \log^5(2) \\
- 4 \log(2) \text{Li}_4 \left( \frac{1}{2} \right) - 4 \text{Li}_5 \left( \frac{1}{2} \right).
\]

By combining (2) and (3) the desired result follows.

**Theorem 2.** (Main results - the second part) The following equality holds:

\[
\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n+1)^3} = \frac{1}{12} \log^5(2) + \frac{31}{128} \zeta(5) - \frac{1}{2} \log^3(2) \zeta(2) + \frac{7}{4} \log^2(2) \zeta(3) \\
- \frac{17}{8} \log(2) \zeta(4) + 2 \log(2) \text{Li}_4 \left( \frac{1}{2} \right),
\]

where \( \zeta \) represents the Riemann zeta function and \( \text{Li}_n \) denotes the Polylogarithm function.
Proof. Using the identity with harmonic numbers in Lemma 3 where we replace $n$ by $2n$, then multiply both sides by $1/n^2$, and consider the summation from $n = 1$ to $\infty$, we have

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} + 2 \sum_{n=1}^{\infty} \frac{H_{2n}^2}{(2n)^3} = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{H_k}{(k+1)(k+2n+1)n^2} \right) \sum_{n=1}^{\infty} \frac{H_{2k}}{(2k+1)(2k+2n+1)n^2}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{H_{2k-1}}{k(k+n)n^2} \right) + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{H_{2k}}{2k+1} \right) \left( \sum_{n=1}^{\infty} \frac{1}{(2k+1)(2k+2n+1)n^2} \right)$$

$$= \frac{1}{4} \sum_{k=1}^{\infty} \frac{H_{2k}}{(2k+1)^2} \left( \sum_{n=1}^{\infty} \frac{1}{(2k+2n+1)n^2} \right)$$

$$+ \frac{1}{4} \zeta(2) \sum_{k=1}^{\infty} \frac{H_{2k} - 1/(2k)}{k^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{H_k}{k^2} \left( \sum_{n=1}^{\infty} \frac{1}{n(k+n)} \right) + \frac{1}{4} \sum_{k=1}^{\infty} \frac{H_{2k+1} - 1/(2k+1)}{(2k+1)^2}$$

$$- \frac{1}{4} \sum_{k=1}^{\infty} \frac{H_{2k}}{(2k+1)^2} \left( \frac{1}{(2k+1)^2} + \frac{H_{2k}}{2k+1} - \frac{H_k}{2(2k+1)} - \log(2) \right) \frac{2(2k+1)}{2k+1}$$

$$= \frac{1}{8} \sum_{n=1}^{\infty} \frac{H_n}{n^3} + \zeta(2) \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^2} + \zeta(2) \sum_{n=1}^{\infty} \frac{H_{2n+1}}{(2n+1)^2} + 4 \log(2) \sum_{n=1}^{\infty} \frac{H_{2n+1}}{(2n+1)^3}$$

$$+ 4 \sum_{n=1}^{\infty} \frac{H_{2n+1}}{(2n+1)^3} - 4 \sum_{n=1}^{\infty} \frac{H_{2n+1}^2}{(2n+1)^3} + 2 \sum_{n=1}^{\infty} \frac{H_n H_{2n}}{n^3} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{H_n H_{2n}}{n^3}$$

$$+ \zeta(2) - \zeta(2) \zeta(3) - \frac{15}{4} \log(2) \zeta(4) + 4 \log(2),$$

and using that $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=1}^{\infty} a_{2n} + \sum_{n=1}^{\infty} a_{2n+1}$ and $\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2} \left( \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} (-1)^{n-1} a_n \right)$, we arrive at

$$\sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n+1)^3}$$

$$= \frac{3}{2} \sum_{n=1}^{\infty} \frac{H_n^2}{n^3} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^3} - \frac{1}{2} \zeta(2) \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \log(2) \sum_{n=1}^{\infty} \frac{H_n}{n^3} - \frac{17}{16} \sum_{n=1}^{\infty} \frac{H_n}{n^4} + \sum_{n=1}^{\infty} \frac{H_n H_{2n}}{(2n)^3}$$

$$+ \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^2}{n^3} - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3} - \log(2) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^4}$$

$$+ \frac{15}{8} \log(2) \zeta(4) + \frac{1}{2} \zeta(2) \zeta(3)$$

$$= \frac{1}{12} \log^5(2) + \frac{31}{128} \zeta(5) - \frac{1}{2} \log^3(2) \zeta(2) + \frac{7}{4} \log^2(2) \zeta(3) - \frac{17}{8} \log(2) \zeta(4)$$
\[ + 2 \log(2) \text{Li}_4 \left( \frac{1}{2} \right). \]

In the calculations we have used results from Lemma 4, Lemma 5, the point \( i \) with \( m = 2, 3, 4 \), the point \( ii \) with \( m = 2 \), Lemma 6, the points \( i), ii), iii), iv), v) \), and Theorem 1.

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REFERENCES

[1] H.M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Springer (originally published by Kluwer), Dordrecht, 2001.
[2] C.I. Vâlean, (Almost) Impossible Integrals, Sums, and Series, Springer, New York, 2019.
[3] C.I. Vâlean, A master theorem of series and an evaluation of a cubic harmonic series, JCA 10, no.2, 97–107, 2017.
[4] C.I. Vâlean, A new powerful strategy of calculating a class of alternating Euler sums, https://www.researchgate.net/publication/333999069, 2019.