A FOURIER-MUKAI TRANSFORM FOR
STABLE BUNDLES ON K3 SURFACES

C. Bartocci, U. Bruzzo, and D. Hernández Ruipérez

Dipartimento di Matematica, Università di Genova, Italia

Scuola Internazionale Superiore di Studi Avanzati (SISSA — ISAS), Trieste, Italia

Departamento de Matemática Pura y Aplicada, Universidad de Salamanca, España

Revised — 6 August 1996

Abstract. We define a Fourier-Mukai transform for sheaves on K3 surfaces over \( \mathbb{C} \), and show that it maps polystable bundles to polystable ones. The rôle of “dual” variety to the given K3 surface \( X \) is here played by a suitable component \( \hat{X} \) of the moduli space of stable sheaves on \( X \). For a wide class of K3 surfaces \( \hat{X} \) can be chosen to be isomorphic to \( X \); then the Fourier-Mukai transform is invertible, and the image of a zero-degree stable bundle \( F \) is stable and has the same Euler characteristic as \( F \).

1. Introduction and preliminaries

Mukai’s functor may be defined within a fairly general setting: given two schemes \( X, Y \) (of finite type over an algebraically closed field \( k \)), and an element \( Q \) in the derived category \( D(X \times Y) \) of \( \mathcal{O}_{X \times Y} \)-modules, Mukai [18] defined a functor \( S_{X \to Y} \) from the derived category \( D^{-}(X) \) to \( D^{-}(Y) \),

\[
S_{X \to Y}(\mathcal{E}) = R\hat{\pi}_{*}(Q \otimes \pi^{*}\mathcal{E})
\]

(here \( \pi : X \times Y \to X \) and \( \hat{\pi} : X \times Y \to Y \) are the natural projections).

Mukai has proved that, when \( X \) is an abelian variety, \( Y = \hat{X} \) its dual variety, and \( Q \) the Poincaré bundle on \( X \times \hat{X} \), the functor \( S_{X \to \hat{X}} \) gives rise to an equivalence of categories. If one is interested in transforming sheaves rather than complexes one can introduce (following Mukai) the notion of \( \text{WIT}_{i} \) sheaves: an \( \mathcal{O}_{X} \)-module \( \mathcal{E} \)

1991 Mathematics Subject Classification. 14F05, 32L07, 32L25, 58D27.

Key words and phrases. Fourier-Mukai transform, stable bundles, K3 surfaces, instantons.

Research partly supported by the Italian Ministry for University and Research through the research projects ‘Metodi geometrici e probabilistici in fisica matematica’ and ‘Geometria delle varietà differenziabili,’ by the Spanish DGICYT through the research projects PB91-0188 and PB92-0308, by EUROPROJ, and by the National Group GNSAGA of the Italian Research Council.
is said to be WIT$_i$ if its Fourier-Mukai transform is concentrated in degree $i$, i.e.

$$H^k \pi_!^* (Q \otimes \pi^* \mathcal{E}) = 0 \text{ for all } k \neq i.$$

Let $X$ be a polarized abelian surface and $\widehat{X}$ is its dual. If $\mathcal{F}$ is a $\mu$-stable vector bundle of rank $\geq 2$ and zero degree, then it is WIT$_1$, and its Mukai transform $\widehat{\mathcal{F}}$ is again a $\mu$-stable bundle (with respect to a suitable polarization on $\widehat{X}$) of degree zero. This result was proved by Fahlaoui and Laszlo [10] and Maciocia [14], albeit Schenk [21] and Braam and van Baal [7] had previously obtained a completely equivalent result in the differential-geometric setting: the Nahm-Fourier transform of an instanton on a flat four-torus (with no flat factors) is an instanton over the dual torus (cf. also [9], and, for a detailed proof of the equivalence of the two approaches, [3]). The key remark which makes it possible to relate the algebraic-geometric treatment to the differential-geometric one is that the Fourier-Mukai transform is interpretable as the index of a suitable family of elliptic operators parametrized by the target space $\widehat{X}$, very much in the spirit of Grothendieck-Illusie’s approach to the definition of index of a relative elliptic complex.

The Fourier-Mukai transform can be studied also in the case of nonabelian varieties. The general idea is to consider a variety $X$, some moduli space $Y$ of ‘geometric objects’ over $X$, and (if possible) a ‘universal sheaf’ $Q$ over the product $X \times Y$. In the present paper we consider the case of a smooth algebraic K3 surface $X$ over $\mathbb{C}$ with a fixed ample line bundle $H$ on it; the rôle of dual variety is here played by the moduli space $Y = M_H(v)$ of Gieseker-stable sheaves (with respect to $H$) $\mathcal{E}$ on $X$ having a fixed $v$, where

$$v(\mathcal{E}) = \text{ch}(\mathcal{E}) \sqrt{\text{td}(X)} = (r, \alpha, s) \in H^0(X, \mathbb{Z}) \oplus H^{1,1}(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}).$$

This space, which is a quasi-projective subscheme of the moduli space of simple sheaves over $X$, has been extensively studied by Mukai [19-20]. In particular, he proved the following crucial result: if $M_H(v) = Y$ is nonempty and compact of dimension two, then it is a K3 surface, isogenous to the original K3 surface $X$.

If $X$ is a K3 surface satisfying these conditions and a few additional assumptions, we can prove our first result (already announced in [2]), by exploiting the relationship between stable bundles and instanton bundles (the so-called Hitchin-Kobayashi correspondence).

**Theorem 1.** Let $\mathcal{F}$ be a $\mu$-polystable IT$_1$ locally free sheaf of zero degree on $X$. The Fourier-Mukai transform of $\mathcal{F}$ is a $\mu$-polystable locally free sheaf on $Y$. 

For a wide family of K3 surfaces $X$, we can choose $v$ so that $\widehat{X} = M_H(v)$ is a K3 surface isomorphic to $X$. The nonemptiness of $M_H(v)$ in this case has been proved in [4] by direct methods. Then we prove that the Fourier-Mukai transform is invertible, just as in the case of abelian surfaces:

**Theorem 2.** Let $\mathcal{F}$ be an WIT$_i$ sheaf on $X$. Then its Fourier-Mukai transform $\widehat{\mathcal{F}} = R^i \pi_* (\pi^* \mathcal{F} \otimes Q)$ is a WIT$_{2-i}$ sheaf on $\widehat{X}$, whose Fourier-Mukai transform $R^{2-i}_* \pi_* (\pi^* \widehat{\mathcal{F}} \otimes Q^*)$ is isomorphic to $\mathcal{F}$. 

Furthermore, the Fourier-Mukai transform of a stable bundle is stable:

**Theorem 3.** Let $\mathcal{F}$ be a zero-degree $\mu$-stable bundle on $X$, with $v(\mathcal{F}^*) \neq v$. Then $\mathcal{F}$ is IT$_1$, and its Fourier-Mukai transform $\widehat{\mathcal{F}}$ is $\mu$-stable.
We end this section by fixing some terminology. Let $X$ and $Y$ be compact complex manifolds, and let $Q$ be a fixed coherent sheaf on $X \times Y$, flat over $O_Y$. Let $\pi$, $\hat{\pi}$ be the projections onto the two factors of $X \times Y$.

**Definition 1.** A sheaf $E$ on $X$ satisfies the $i$th Weak Index Theorem condition (i.e. it is WIT$_i$) if $R^j\hat{\pi}_*(\pi^*E \otimes Q) = 0$ for $j \neq i$; similarly, the sheaf $E$ is said to satisfy the $i$th Index Theorem condition (i.e. it is IT$_i$) if $H^j(\hat{\pi}^{-1}(y), E \otimes Q|_{\hat{\pi}^{-1}(y)}) = 0$ for $j \neq i$ and all $y \in Y$.

The base change theorem implies that the sheaf $E$ is IT$_i$ if and only if it is both WIT$_i$ and $R^i\hat{\pi}_*(\pi^*E \otimes Q)$ is locally free.

If either $Q$ is flat over $O_{X \times Y}$, or $F$ is flat over $O_X$, the sheaves $R^k\hat{\pi}_*(\pi^*F \otimes Q)$ are the cohomology sheaves of the Fourier-Mukai transform $R\hat{\pi}_*(\pi^*F \otimes Q)$ in the derived category. Then, if $F$ is WIT$_i$ we call the sheaf $\hat{F} = R^i\hat{\pi}_*(\pi^*F \otimes Q)$ on $Y$ its Fourier-Mukai transform.

## 2. Fourier-Mukai transform on K3 surfaces

### 2.1. Hyperkähler manifolds and quaternionic instantons.** A hyperkähler manifold is a $4n$-dimensional Riemannian manifold $X$ which admits three complex structures $I$, $J$ and $K$, compatible with the Riemannian structure, such that $IJ = K$. On a hyperkähler manifold one can introduce a generalized notion of instanton [15]. The three endomorphisms $I$, $J$, $K$ of $TX \otimes \mathbb{C}$ allow one to define an endomorphism $\phi$ of $\Lambda^2T^*X \otimes \mathbb{C}$,

$$\phi = I \otimes I + J \otimes J + K \otimes K.$$  

This satisfies $\phi^2 = 2\phi + 3$, so that at every $x \in X$ one has an eigenspace decomposition

$$\Lambda^2(T^*_xX \otimes \mathbb{C}) = V_1 \oplus V_2$$  

(1)

corresponding to the eigenvalues 3 and $-1$ of $\phi$, respectively. If $E$ is a $C^\infty$ complex vector bundle on $X$, with connection $\nabla$ and curvature $R_\nabla$, we say that the pair $(E, \nabla)$ is a quaternionic instanton if $R_\nabla$, regarded as a section of $\text{End}(E) \otimes \Lambda^2T^*X$, has no component in $V_2$. If $X$ has dimension 4 this agrees with the usual definition of instanton, since in that case the splitting (1) is no more than the decomposition of the space of two-forms into selfdual and anti-selfdual forms. It is quite evident that $(E, \nabla)$ is a quaternionic instanton if and only if the curvature $R_\nabla$ is of type (1,1) with respect to all the complex structures of $X$ compatible with its hyperkähler structure. Moreover, $(E, \nabla)$ is an Einstein-Hermite bundle with respect to all the induced Kähler structures; thus, for any compatible complex structure on $X$, the bundle $E$ admits a holomorphic structure, and the sheaf of its holomorphic section is then $\mu$-polystable with respect to a polarization given by the Kähler form determined by the given complex structure (we recall that a coherent sheaf is said to be $\mu$-polystable if it is a direct sum of stable sheaves having the same slope).

### 2.2. Assumptions on the moduli space.** Let $X$ be a projective K3 surface over $\mathbb{C}$. The cup product defines a $\mathbb{Z}$-valued pairing on the graded ring $H^\bullet(X, \mathbb{Z})$:

$$\langle a, b, c \rangle \langle a', b', c' \rangle = ac' + bc' + cc'$$
For every sheaf $\mathcal{F}$ on $X$ we define the Mukai vector $v(\mathcal{F}) \in H^\bullet(X, \mathbb{Z})$ as the element $\text{ch}(\mathcal{F}) \sqrt{\text{td}(X)} = (\text{rk} \mathcal{F}, c_1(\mathcal{F}), s(\mathcal{F}))$, where $s(\mathcal{F}) = \text{rk} \mathcal{F} + c_2(\mathcal{F}) = \chi(\mathcal{F}) - \text{rk} \mathcal{F}$.

We fix a polarization $H$ on $X$ and denote by $M_H(v)$ the moduli space of sheaves on $X$ which are Gieseker-stable with respect to $H$, having a fixed $v$.

We shall make the following assumption.

**A1.** One can choose a Mukai vector $v = (r, \ell, s)$ which is primitive and isotropic and satisfies $\gcd (r, \deg \ell, s) = 1$.

By results of Maruyama [16,17] and Mukai [19,20] one has:

**Proposition 1.** If $v$ satisfies assumption A1, then $M_H(v)$ is either empty or a projective K3 surface and a fine moduli scheme parametrizing stable sheaves with Mukai vector $v$, so that there is a universal sheaf $Q$ on $X \times M_H(v)$.

Now we fix a Mukai vector $v = (r, \ell, s)$ satisfying the assumption A1 and the following additional assumptions:

**A2.** The divisor $\ell$ has degree zero and $r > 1$;

**A3.** The moduli space $M_H(v)$ is not empty, and parametrizes $\mu$-stable sheaves.

In this situation the universal sheaf $Q$ on $X \times M_H(v)$ is locally free by virtue of [20], Corollary 3.10, that we recall in the following form:

**Proposition 2.** If $v = (r, \ell, s)$ is isotropic and $r > 1$, every $\mu$-stable sheaf $\mathcal{F}$ on $X$ with $v(\mathcal{F}) = v$ is locally free.

### 2.3. Universal sheaf and the Atiyah-Singer bundle.

We consider a K3 surface $X$ and a Mukai vector $v$ satisfying assumptions A1, A2 and A3, so that the fine moduli space $\hat{X} = M_H(v)$ (a K3 surface) is formed by locally free $\mu$-stable sheaves of zero degree. According to the Hitchin-Kobayashi correspondence, these bundles correspond to irreducible $U(n)$-instantons, hence $\hat{X}$ is identified with a moduli space $M$ of instantons.

This identification can be described as follows. The product manifold $X \times M$ carries the universal Atiyah-Singer instanton bundle $P$ [1], which is a hermitian bundle with a universal hermitian connection. The curvature of the latter is of type $(1,1)$ with respect to the natural complex structure of $X \times M$ [12]. Therefore, $P$ can be given a holomorphic structure, and the resulting holomorphic vector bundle $P$ is relatively stable with respect to the projection onto $M$, since the universal connection is a family of instantonic connections. Then there is a morphism $f: M \to \hat{X}$ such that $(\text{id} \times f)^*Q \otimes \hat{\pi}^*L \simeq P$, where $L$ is a holomorphic line bundle on $M$, and $\hat{\pi}: X \times \hat{X} \to \hat{X}$ is the projection. Since $Q$ is defined up to tensoring by the pullback of a line bundle on $\hat{X}$ we may assume that $(\text{id} \times f)^*Q \simeq P$. The map $f$ is exactly the identification given by the Hitchin-Kobayashi correspondence.

To any complex structure on $X$ there corresponds a well-determined complex structure on $\hat{X}$. Then, if $X$ is hyperkähler, we may endow the manifold $X \times \hat{X}$ with a natural hyperkähler structure.

**Proposition 3.** The pair $(Q, \nabla)$, where $Q$ is the smooth bundle underlying $Q$, and $\nabla$ is the universal connection, is a quaternionic instanton.

---

1. We adopt the usual definition of (semi)stability, so that a Gieseker- (or $\mu$-) semistable sheaf is assumed to be torsion-free. Moreover, by “(semi)stable” we shall mean “Gieseker (semi)stable.”
Proof. For any complex structure on $X$ compatible with its hyperkähler structure the curvature of the universal connection is of type $(1,1)$ [12].

2.4. Polystability of the Fourier-Mukai transform. We turn now to the main purpose of this paper, namely, the analysis of the Fourier-Mukai functor for vector bundles on a K3 surface, and prove that the Fourier-Mukai transform of a polystable bundle is polystable.

Let $X$ and the Mukai vector $v$ satisfy the assumptions A1, A2 and A3, so that $X$ is a projective K3 surface and $\hat{X} = M_H(v)$, a moduli space of zero-degree $\mu$-stable bundles on $X$, is a projective K3 surface. Let $\pi, \hat{\pi}$ be the projections onto the two factors of $X \times \hat{X}$.

We consider a $\mu$-polystable $\text{IT}_1$ locally free sheaf $\mathcal{F}$ of zero degree on $X$ and its Fourier-Mukai transform $\hat{\mathcal{F}} = R^1 \hat{\pi}_*(\pi^* \mathcal{F} \otimes \mathcal{Q})$ on $\hat{X}$. According to the Hitchin-Kobayashi correspondence, $\mathcal{F}$ is the sheaf of holomorphic sections of a vector bundle $\mathcal{F}$ which carries an hermitian metric, such that if $\nabla$ is the connection determined by the holomorphic and hermitian structures of $\mathcal{F}$, the pair $(\mathcal{F}, \nabla)$ is an instanton.

The bundle $\pi^* \mathcal{F} \otimes \mathcal{Q}$ on $X \times \hat{X}$ may be endowed with a connection $\hat{\nabla}$ obtained by coupling the universal connection of $\mathcal{Q}$ with the pullback connection $\pi^* \nabla$, and the pair $(\pi^* \mathcal{F} \otimes \mathcal{Q}, \hat{\nabla})$ is a quaternionic instanton.

We can now state the main result of this section. By regarding $\hat{X}$ as an instanton moduli space, it carries a natural Kähler metric $\Phi_{\hat{X}}$, usually called the Weil-Petersson metric.

**Theorem 1.** Let $\mathcal{F}$ be a $\mu$-stable $\text{IT}_1$ locally free sheaf of zero degree on $X$. The Fourier-Mukai transform $\hat{\mathcal{F}}$ is a $\mu$-polystable locally free sheaf on $\hat{X}$ (with respect to the Kähler form of $\hat{X}$).

Proof. Let us denote by $\mathcal{C}$ the set of complex structures in $X$ which are compatible with its hyperkähler structure, and shall write $X_I$ for $X$ endowed with the complex structure $I$. For every choice of $I$ the moduli space $X$ has an induced complex structure, so that $\mathcal{C}$ parametrizes also the complex structures in $\hat{X}$ compatible with its hyperkähler structure. For every choice of $I$ we consider on the product manifold the complex structure $X_I \otimes \hat{X}_I$, so that $X \times \hat{X}$ acquires a hyperkähler structure, and $\mathcal{C}$ parametrizes the compatible complex structures.

Let $G$ denote the so-called isotropy group of $X$, i.e. the group which permutes the elements in $\mathcal{C}$. The action of $G$ on $\mathcal{C}$ is transitive; we denote by $g_{II'}$ an element in $G$ mapping $I$ to $I'$.

For any choice of $I \in \mathcal{C}$ we have a relative Dolbeault complex $\Omega^{0,*}_I$ on $X \times \hat{X} \to \hat{X}$, and a relative Dirac complex $\mathcal{S}^+_I$, where

$$S^{-}_I = \Omega^{0,1}_I, \quad S^{+}_I = \Omega^{0,0}_I \oplus \Omega^{0,2}_I.$$ 

The relative Dirac operator $D_I$ acts as a $C^\infty$-linear morphism $\mathcal{S}^+_I \to \mathcal{S}^{-}_I$. The element $g_{II'}$ of the isotropy group intertwines the Dirac operators associated with...
the two complex structures, so that one has a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & F_I^\infty & \longrightarrow & \hat{\pi}_*(\tilde{F}\otimes S_I^-) & \longrightarrow & \hat{\pi}_*(\tilde{F}\otimes S_I^+) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow g_{II'} & & \downarrow g_{II'} & & \\
& & 0 & \longrightarrow & \hat{\pi}_*(\tilde{F}\otimes S_I'^-) & \longrightarrow & \hat{\pi}_*(\tilde{F}\otimes S_I'^+) & \longrightarrow & 0
\end{array}
\]

(2)

where \( D_I^* \) is actually the adjoint of the Dirac operator associated with the complex structure \( I \) coupled with the connection of \( \pi^*F\otimes Q \); moreover, \( \infty \) means that we are considering the sheaf of smooth sections of a holomorphic bundle. The operators \( D^* \) are surjective due to the \( IT_1 \) condition, and every sheaf \( \hat{\pi}_*F \) is the sheaf of smooth sections of the Fourier-Mukai transforms \( \hat{F}_I \) (cf. [3,6]).

The diagram (2) induces an isomorphism \( \hat{g}_{II'}: \hat{F}_I \rightarrow \hat{F}_I' \) of \( C^\infty \) vector bundles; moreover, since the bundle \( \pi^*F\otimes Q \) has a natural hermitian metric, the horizontal arrows in this diagram allow one to introduce an hermitian metric \( \hat{h}_I \) on every bundle \( \hat{F}_I \), and \( \hat{g}_{II'} \) is then an isometry. By coupling the relative connection induced by the connection on \( \pi^*F\otimes Q \) with the relative Dolbeault operator associated with a complex structure \( I \) and by taking direct images one defines a connection \( \hat{\nabla}_I \) on \( \hat{F}_I \). Since this connection is also induced by the direct images of the coupled connections on \( \hat{\pi}_*(\hat{F}_I^\infty \otimes S_I^\pm) \), diagram (2) proves that \( \hat{g}_{II'} \) transforms \( \hat{\nabla}_I \) into \( \hat{\nabla}_I' \).

So actually one has a single hermitian bundle \( (\hat{F}_I, \hat{h}_I) \) with a single connection \( \hat{\nabla}_I \) whose curvature is of type \( (1,1) \) with respect to all compatible complex structures.

Then, the pair \( (\hat{F}_I, \hat{\nabla}_I) \) is an instanton. As a consequence, the Fourier-Mukai transform \( \hat{F}_I \) of \( F_I \) is \( \mu \)-polystable with respect to the Kähler form over \( \hat{X} \) determined by \( I \).

3. Nonemptiness of moduli spaces.

Moduli spaces isomorphic to the surface.

In this section we consider a wide class of K3 surfaces for which the hypotheses on \( X \) and on the moduli space \( \hat{X} \) stated in the previous section may be satisfied. When \( X \) is such a surface, \( \hat{X} \) may be chosen so as to be isomorphic to \( X \) itself, and \( X \) may be identified with a moduli space of stable bundles on \( \hat{X} \), in such a way that the relevant universal sheaf is \( Q^* \).

3.1. Nonemptiness of the moduli space. Let \( X \) be a K3 surface endowed with a polarization \( H \) and a divisor \( \ell \) such that \( H^2 = 2 \), \( H \cdot \ell = 0 \) and \( \ell^2 = -12 \), so that \( v^2 = 0 \) with \( v = (2, \ell, -3) \). We shall call the K3 surfaces satisfying these assumptions reflexive.

Throughout the remaining part of this paper we shall also make the further technical assumption that on \( X \) there are no nodal curves of degree 1 or 2 with respect to \( H \). This will hold generically. Indeed, the ample divisor \( H \) defines a double cover of \( \mathbb{P}^2 \) branched over a sextic; the image of a nodal curve of degree 1 is a line tritangent to the sextic, while the image of a nodal curve of degree 2 is a conic, tangent to the sextic at six points. Neither situation can arise in the generic case [5].
Lemma 1. If \( X \) is a reflexive K3 surface and \( D \cdot H > 2 \) for every nodal curve \( D \), the divisor \( E = \ell + 2H \) is not effective. Then, \( H^i(X, \mathcal{O}_X(\ell + 2H)) = 0 \) for \( i \geq 0 \).

Proof. Since \( E^2 = -4 \), if \( E \) is effective, it is not irreducible and \( E = D + F \) for some nodal curve \( D \). Then \( D \cdot H = 3 \) so that \( F \cdot H = 1 \) and \( F \) is also irreducible. It follows that \( F^2 \geq -2 \). If \( F^2 \geq 0 \), then \( D \cdot F \leq -1 \), so that \( D = F \) which is absurd. Thus, \( F^2 = -2 \) and \( F \) is a nodal curve of degree 1, a situation we are excluding. \( \blacksquare \)

The following fundamental result is proved in [4].

Proposition 4. Let \( X \) be a reflexive K3 surface such that \( \ell + 2H \) is not effective. The moduli space \( \hat{X} \) of stable sheaves \( \mathcal{E} \) with \( v(\mathcal{E}) = (2, \ell, -3) \) is not empty. Moreover, every element in \( \hat{X} \) is \( \mu \)-stable, so that a reflexive K3 surface meets assumptions \( A1, A2 \) and \( A3 \). \( \blacksquare \)

One should notice that the elements in \( \hat{X} \) do not satisfy the lower bounds on the discriminant established by Sorger\(^3\) and Hirschowitz and Laszlo [11].

3.2. The isomorphism \( \hat{X} \cong X \). We consider now a reflexive K3 surface \( X \) satisfying the assumption on nodal curves described in the previous section, and show that there is a natural isomorphism between \( X \) and \( \hat{X} = M_H(v) \).

The following result is a direct consequence of Lemma 1.

Lemma 2. \( \dim \text{Ext}^1(\mathcal{I}_p(\ell + 2H), \mathcal{O}_X) = \dim H^1(X, \mathcal{I}_p(\ell + 2H)) = 1 \) for every point \( p \in X \), where \( \mathcal{I}_p \) is the ideal sheaf of \( p \). \( \blacksquare \)

Lemma 3. Let \( [\mathcal{E}] \in \hat{X} \). For every section of \( \mathcal{E}(H) \) there is an exact sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}(H) \rightarrow \mathcal{I}_p(\ell + 2H) \rightarrow 0,
\]

where \( \mathcal{I}_p \) is the ideal sheaf of a point \( p \in X \). Moreover, \( \dim H^0(X, \mathcal{E}(H)) = 1 \), so that the point \( p \) depends only on the sheaf \( \mathcal{E} \).

Proof. Since \( H^2(X, \mathcal{E}(H)) = 0 \) and \( \chi(\mathcal{E}(H)) = 1 \), the sheaf \( \mathcal{E}(H) \) has at least one section, and we have an exact sequence \( 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}(H) \rightarrow \mathcal{K} \rightarrow 0 \). By taking double duals we obtain the exact sequence

\[
0 \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{E}(H) \rightarrow \mathcal{I}_Z \otimes \mathcal{O}_X(\ell + 2H - D) \rightarrow 0,
\]

where \( Z \) is a zero-dimensional subscheme and \( D \) is an effective divisor with \( 0 \leq D \cdot H \leq 1 \).

If \( D \cdot H = 1 \), \( D \) is an irreducible curve, so that \( D^2 \geq -2 \). Moreover, \( H - 2D \neq 0 \), thus Hodge index theorem implies that \( 0 > (H - 2D)^2 = 4D^2 - 2 \) and we have two cases, \( D^2 = 0 \) and \( D^2 = -2 \). If \( D^2 = 0 \), then \( (H - 2D)^2 = -2 \), so that either \( H - 2D \) or \( 2D - H \) is effective, which is absurd. Thus, \( D^2 = -2 \), and \( D \) is a nodal curve of degree 1, a situation we are excluding. Hence \( D = 0 \) and \( \text{length}(Z) = 1 \).

The second statement follows from the previous Lemma. \( \blacksquare \)

These results imply that there exists a one-to-one map of sets \( \Psi: \hat{X} \leftrightarrow X \), given by \( \Psi([\mathcal{E}]) = p \), where \( p \) is the point determined by Lemma 3. Our next aim is to prove that this map is actually an isomorphism of schemes; to this end we need a result that follows from Grauert’s base change theorem and Lemma 2.

---

\(^3\)Talk given at the Europroj Workshop “Vector bundles and structure of moduli”, Lambrecht 1994.
Corollary 1. The sheaf $\mathcal{O}_X(H)$ is IT$_0$, and its Fourier-Mukai transform $\mathcal{N} = \hat{\pi}_*(\mathcal{Q} \otimes \pi^*\mathcal{O}_X(H))$ is a line bundle on $\hat{X}$. 

Now, the natural morphism $\hat{\pi}^*\mathcal{N} \to \mathcal{Q} \otimes \pi^*\mathcal{O}_X(H)$ provides a section

$$0 \to \mathcal{O}_{X \times \hat{X}} \xrightarrow{\sigma} \mathcal{Q} \otimes \pi^*\mathcal{O}_X(H) \otimes \hat{\pi}^*\mathcal{N}^{-1} \to K \to 0.$$ 

Let $j: Z \hookrightarrow X \times \hat{X}$ be the closed subscheme of zeroes of $\sigma$, and let $p = \pi \circ j: Z \to X$, $\hat{p} = \hat{\pi} \circ j: Z \to \hat{X}$ be the proper morphisms induced by the projections $\pi$ and $\hat{\pi}$.

Proposition 5. The morphism $\hat{p}: Z \cong \hat{X}$ is an isomorphism of schemes and the map $\Psi$ is the composite morphism $p \circ \hat{p}^{-1}: \hat{X} \to X$. Moreover, $\Psi$ is an isomorphism of schemes.

Proof. One easily sees that for every (closed) point $\xi \in \hat{X}$, $\sigma$ induces a section

$$0 \to \mathcal{O}_X \xrightarrow{\sigma_{\xi}} \mathcal{E}(H) \to K_{\xi} \to 0$$

of $\mathcal{E}(H)$. By Lemma 3, $K_{\xi} \simeq \mathcal{I}_{p(\xi)}(\ell + 2H)$ for a well-defined point $p(\xi) \in X$. Then, every closed fibre of $\hat{p}: Z \to \hat{X}$ consists of a single point and $\hat{p}$ is a proper finite epimorphism of degree 1 by Zariski's Main Theorem. Since $\hat{X}$ is smooth, $\hat{p}$ is an isomorphism. Moreover one has $\Psi = p \circ \hat{p}^{-1}$.

By Lemma 2 for every (closed) point $\xi \in \hat{X}$ the fibre $\Psi^{-1}(\Psi(\xi))$ is a single point. If $\Psi(\hat{X})$ is the scheme-theoretic image of $\Psi$, $\hat{X} \to \Psi(\hat{X})$ is a finite epimorphism of degree 1 as above, so that $\dim \Psi(\hat{X}) = 2$ and $\Psi(\hat{X}) = X$. The smoothness of $X$ yields once more the result. 

Corollary 2. Let $\mathcal{E}$ be a sheaf which fits into an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E}(H) \to \mathcal{I}_p(\ell + 2H) \to 0,$$

where $\mathcal{I}_p$ is the ideal sheaf of a point $p \in X$. Then $\mathcal{E}$ is $\mu$-stable and locally free with $v(\mathcal{E}) = v = (2, \ell, -3)$ and $\Psi([\mathcal{E}]) = p$. 

4. Fourier-Mukai transform on reflexive K3 surfaces

In this section we investigate the main properties of the Mukai transform in the case of reflexive K3 surfaces satisfying the assumption on nodal curves described in the previous section. For these K3 surfaces the Fourier-Mukai functor is invertible, and Theorem 1 holds in a stronger form, in that the Fourier-Mukai transform $\hat{F}$ of a stable bundle $\mathcal{F}$ is itself stable. We also prove the nice formula $\chi(\mathcal{F}) = \chi(\hat{\mathcal{F}})$.

4.1. A natural polarization for $\hat{X}$. If $[\mathcal{E}] \in \hat{X}$, then $H^0(X, \mathcal{E}) = H^2(X, \mathcal{E}) = 0$ and $\dim H^1(X, \mathcal{E}) = 1$. It follows that $\mathcal{O}_X$ is IT$_1$ and that $\mathcal{O}_X$ is a line bundle. We can then normalize $\mathcal{Q}$ by twisting it by $\hat{\pi}^*(R^1\hat{\pi}_*\mathcal{Q})^{-1}$, so that $R^1\hat{\pi}_*\mathcal{Q} \simeq \mathcal{O}_{\hat{X}}$. We shall henceforth assume that $\mathcal{Q}$ is normalized in this way.

Let us denote $\gamma = ch(\mathcal{Q})$. If $\gamma^{i,j}$ is the $(i, j)$ Künneth component of $\gamma$, one has $\gamma^{2,0} = \ell$; we set $\gamma^{0,2} = -\ell$. Then, Riemann-Roch theorem gives $(-1)^ic_1(\hat{\mathcal{L}}) = -ch_2(\mathcal{L}) \hat{\ell} + \hat{\pi}_*(\gamma^{2,2} c_1(\mathcal{L}))$ for every WIT, line bundle $\mathcal{L}$. In particular, since $\mathcal{O}_X(H)$ is IT$_0$ and $\mathcal{O}_X(H)$ is a line bundle (Corollary 1), one has

$$(-1)^x c_1(\mathcal{O}_X(H)) = \hat{\ell} \hat{\mathcal{H}} . \mathcal{O}_X(H) \simeq \mathcal{O}_X(\hat{\mathcal{H}}) \bigotimes \mathcal{O}_X(0) \simeq \mathcal{O}_X(\hat{\mathcal{H}}).$$
where \( \hat{H} = -\hat{\pi}_*(\gamma^{2,2}H) \).

We will show that the divisor \( \hat{H} \) is a natural polarization on \( \hat{X} \). Indeed, the space \( \hat{X} \), regarded as a moduli space of instantons on \( X \), carries the Weil-Petersson metric \( \Phi_X \), already considered in Theorem 1, and we can prove, in the spirit of [13], that the class of this metric may be identified with the class \( \hat{H} \).

**Proposition 6.** \( \hat{H} = \left[ \frac{1}{8\pi^2} \Phi_X \right] \).

**Proof.** Let \( \mathfrak{R} \) denote the curvature of the universal connection on \( Q \) (cf. Proposition 3). Let \( \Phi_X \) be the Kähler form of \( X \), so that \( [\Phi_X] = H \). We note the identities

\[
\int_X \Phi_X \wedge (\text{tr} \mathfrak{R})^2 = \left( \frac{2\pi}{i} \right)^2 \int_X \Phi_X \wedge (c_1(Q))^2 = 2(4\pi)^2(\ell \cdot H)\ell = 0
\]

\[
\int_X \Phi_X \wedge \text{tr} \mathfrak{R}^{2,0} \wedge \mathfrak{R}^{0,2} = 0.
\]

By representing the Chern character \( \gamma \) in terms of the curvature form \( \tilde{\mathfrak{R}} \) we may compute

\[
\hat{H} = \frac{1}{8\pi^2} \int_X \Phi_X \wedge \text{tr} \mathfrak{R}^2
\]

\[
= \frac{1}{8\pi^2} \int_X \Phi_X \wedge \text{tr}(2\mathfrak{R}^{2,0} \wedge \mathfrak{R}^{0,2} + \mathfrak{R}^{1,1} \wedge \mathfrak{R}^{1,1}) = \frac{1}{8\pi^2} \Phi_X.
\]

Thus \( \hat{H} \) is ample, and can be taken as a polarization on \( \hat{X} \); moreover, \( \hat{\ell} \cdot \hat{H} = 0 \), and \( \ell^2 = -12 \), so that \( \hat{\nu} = (2, \hat{\ell}, -3) \) is an isotropic Mukai vector and \( \hat{X} \) is a reflexive K3 surface with respect to \( (\hat{H}, \hat{\ell}) \).

4.2. \( X \) as a moduli space of bundles on \( \hat{X} \). In this section we prove that we can regard \( \hat{X} \) as a moduli space of bundles with topological invariants \( (2, \ell, -3) \) that are \( \mu \)-stable with respect to the natural polarization \( \hat{H} \); it turns out that the relevant universal bundle in this case is simply \( Q^* \).

Lemma 3 suggests that the universal sheaf \( Q \) can be obtained as an extension of suitable torsion-free rank-one sheaves on \( X \times \hat{X} \). Let \( \mathcal{I}_\Psi \) be the ideal sheaf of the graph \( \Gamma_\Psi: \hat{X} \hookrightarrow X \times \hat{X} \) of \( \Psi \).

**Lemma 4.** The direct image \( \hat{\pi}_*[\mathcal{E}xt^1(\mathcal{I}_\Psi \otimes \pi^*\mathcal{O}_X(\ell + 2H), \mathcal{O}_{X \times \hat{X}})] \) is a line bundle \( \mathcal{L} \) on \( \hat{X} \).

**Proof.** Write \( E = \ell + 2H \) and \( \mathcal{O}_\Psi = (\Gamma_\Psi)_*\mathcal{O}_{\hat{X}} \). Then,

\[
\mathcal{E}xt^1(\mathcal{I}_\Psi \otimes \pi^*\mathcal{O}_X(E), \mathcal{O}_{X \times \hat{X}}) \cong \mathcal{O}_\Psi \otimes \pi^*\mathcal{O}_X(-E).
\]

By Lemma 1, \( R^i\hat{\pi}_*\pi^*\mathcal{O}_X(-E) = 0 \) for \( i \geq 0 \), hence, from the exact sequence

\[
0 \to \mathcal{I}_\Psi \otimes \pi^*\mathcal{O}_X(-E) \to \pi^*\mathcal{O}_X(-E) \to \mathcal{O}_\Psi \otimes \pi^*\mathcal{O}_X(-E) \to 0,
\]

we obtain \( \hat{\pi}_*(\mathcal{O}_\Psi \otimes \pi^*\mathcal{O}_X(-E)) \cong R^1\hat{\pi}_*(\mathcal{I}_\Psi \otimes \pi^*\mathcal{O}_X(-E)) \). But for every \( \xi \in \hat{X} \) one has \( H^1(X, \mathcal{I}_\Psi \otimes \pi^*\mathcal{O}_X(-E) \otimes \kappa(\xi)) = H^1(X, \mathcal{I}_p(-E)) \), where \( p = \Psi(\xi) \), and one concludes by Lemma 1 and by Grauert’s base change theorem. ■
It follows that the sheaf $\mathcal{E}xt^1(I_\Psi \otimes \pi^*\mathcal{O}_X(\ell + 2H), \hat{\pi}^*(\mathcal{L}^{-1}))$ has a section, so that there is an extension

$$ (4) \quad 0 \to \hat{\pi}^*(\mathcal{L}^{-1}) \to \mathcal{P} \to I_\Psi \otimes \pi^*\mathcal{O}_X(\ell + 2H) \to 0. $$

Moreover, Lemma 3 implies that $\mathcal{P} \otimes \pi^*\mathcal{O}_X(-H)$ is a universal sheaf on $X \times \hat{X}$; thus $\mathcal{P} \cong Q \otimes \hat{\pi}^*\mathcal{N} \otimes \pi^*\mathcal{O}_X(H)$ for a line bundle $\mathcal{N}$ on $\hat{X}$. The sheaves $\mathcal{L}$ and $\mathcal{N}$ are readily determined; by applying $\hat{\pi}_*$ to the sequence above one obtains

$$ \mathcal{L}^{-1} = \mathcal{O}_X(H) \otimes \mathcal{N} \cong \mathcal{O}_{\hat{X}}(-\hat{\ell} - \hat{H}) \otimes \mathcal{N}, $$

where the second equality is due to equation (3). Now, by restricting the exact sequence (4) to a fibre $\pi^{-1}(p)$, we obtain $c_1(\mathcal{N}) = -\hat{\ell} - \hat{H} - c_1(\mathcal{O}_{\pi^{-1}(p)}) = -\hat{H}$. Then we have

**Proposition 7.** The sequence of coherent sheaves on $X \times \hat{X}$

$$ (5) \quad 0 \to \hat{\pi}^*\mathcal{O}_{\hat{X}}(-\hat{\ell} - 2\hat{H}) \to Q \otimes \hat{\pi}^*\mathcal{O}_{\hat{X}}(-\hat{H}) \otimes \pi^*\mathcal{O}_X(H) \to I_\Psi \otimes \pi^*\mathcal{O}_X(\ell + 2H) \to 0 $$

is exact.

This sequence allows us to compute the Chern character of $Q$. In particular, we obtain

$$ (6) \quad \gamma^{2,2} = (\ell + 2H) \cup \hat{H} + H \cup \hat{\ell} - \iota, $$

where $\iota \in H^2(X, \mathbb{Z}) \otimes H^2(\hat{X}, \mathbb{Z})$ is the element corresponding to the isomorphism $\Psi^*: H^2(X, \mathbb{Z}) \to H^2(\hat{X}, \mathbb{Z})$. From this we get

$$ \hat{H} = \Psi^*(2\ell + 5H), \quad \hat{\ell} = \Psi^*(-5\ell - 12H). $$

By taking duals in the sequence (5) and restricting to the fibres of $\hat{\pi}$ we obtain

$$ (7) \quad 0 \to \mathcal{O}_{\hat{X}} \to Q_p^*(\hat{H}) \to I_\xi(\ell + 2\hat{H}) \to 0, $$

where $p = \Psi(\xi)$.

We need to show that the sheaves $Q_p^*$ are $\mu$-stable with respect to $\hat{H}$. We are not in a position to apply Corollary 4 to the reflexive K3 surface $(\hat{X}, \hat{H}, \hat{\ell})$, since we cannot a priori exclude that $\hat{X}$ contains nodal curves of degree 1 with respect to $\hat{H}$. This problem is circumvented as follows. Since $\ell + 2\hat{H} = -\Psi^*(\ell + 2H)$, it has negative degree with respect to $\Psi^*H$, so that it is not effective. Thus, $\dim H^0(X, Q_p^*(\hat{H})) = 1$ and $\dim \mathrm{Ext}^1(I_\xi(\ell + 2\hat{H}), \mathcal{O}_{\hat{X}}) = 1$. An easy calculation shows that the sheaves $Q_p^*$ are simple, so that $Q^*$ defines a morphism

$$ \alpha: X \to \text{Spl}(\hat{\nu}, \hat{X}) $$

into the moduli scheme of simple sheaves on $\hat{X}$ with Mukai vector $\hat{\nu}$. Proceeding as in the proof of Proposition 5, we obtain that $\alpha$ is an isomorphism with a connected component of $\text{Spl}(\hat{\nu}, \hat{X})$. Moreover, since $\ell + 2\hat{H}$ is not effective, Proposition 4 for
Proposition 8. \( (\hat{X}, \hat{H}, \hat{\ell}) \) implies that the moduli space \( M_{\hat{H}}(\hat{v}) \) of stable sheaves on \( \hat{X} \) (with respect to \( \hat{H} \)) with Mukai vector \( \hat{v} \) is a non-empty connected component of \( \text{Spl}(\hat{v}, \hat{X}) \), consisting of locally free \( \mu \)-stable sheaves.

According to the proof of Lemma 3, if \( [\mathcal{F}] \in M_{\hat{H}}(\hat{v}) \) the sheaf \( \mathcal{F} \) fits into an exact sequence like (7) for a well-defined point \( \xi \in \hat{X} \) unless \( \mathcal{F} \) is given by an extension

\[
0 \to \mathcal{O}_{\hat{X}}(D) \to \mathcal{F}(\hat{H}) \to \mathcal{I}_Z(\hat{\ell} + 2\hat{H} - D) \to 0
\]

where \( D \) is a closed subscheme of \( \hat{X} \). In the latter case, \( H^i(\hat{X}, \mathcal{I}_Z(\hat{\ell} + 2\hat{H} - D)) = 0 \) for \( i \geq 0 \) so that \( \text{length}(Z) = 0 \) and \( -4 = (\hat{\ell} + 2\hat{H} - D)^2 \). Then, \( D \cdot \hat{\ell} = -3 \) and \( D \cdot \Psi^*H = D \cdot (5\hat{H} + 2\hat{\ell}) = -1 \), which is absurd since \( \Psi^*H \) is ample. Then, one has

\[
0 \to \mathcal{O}_{\hat{X}} \to \mathcal{F}(\hat{H}) \to \mathcal{I}_\xi(\hat{\ell} + 2\hat{H}) \to 0,
\]

for a point \( \xi \in \hat{X} \), and \( \mathcal{F} \cong \mathcal{Q}_p^* \) with \( p = \Psi(\xi) \), since \( \dim \text{Ext}^1(\mathcal{I}_\xi(\hat{\ell} + 2\hat{H}), \mathcal{O}_{\hat{X}}) = 1 \). Thus, \( M_{\hat{H}}(\hat{v}) \) is contained in \( \alpha(X) \) and the two spaces must coincide. This means that the bundles \( \mathcal{Q}_p^* \) are \( \mu \)-stable with respect to \( \hat{H} \); therefore the sequence (5) exhibits explicitly the parametrization of vector bundles on \( \hat{X} \) with invariants \((2, \hat{\ell}, -3)\) that are \( \mu \)-stable with respect to \( \hat{H} \) by the points of \( X \). As a consequence:

**Proposition 8.** \( X \) is a fine moduli space of \( \mu \)-stable bundles on \( \hat{X} \) (polarized by \( \hat{H} \)) with invariants \((2, \hat{\ell}, -3)\), and the relevant universal sheaf is \( \mathcal{Q}^* \).

### 4.3. Inversion of the Fourier-Mukai transform.

Let \( X \) be a K3 surface and \( \nu \) a Mukai vector satisfying assumptions A1, A2 and A3. We consider the Fourier-Mukai transform as a functor

\[
S_X(F) = R\pi_*(\pi^*F \otimes L)
\]

between the derived categories \( D(X) \) and \( D(\hat{X}) \) (we may use the full derived categories instead of the \( D^- \) categories because \( Q \) is locally free). In view of Proposition 8 a natural candidate for the inverse of \( S_X \) is the functor \( S_{\hat{X}}: D(\hat{X}) \to D(X) \)

\[
S_{\hat{X}}(G) = R\pi_*(\hat{\pi}^*G \otimes Q^*)
\]

Since \( X \) and \( \hat{X} \) are K3 surfaces, the relative dualizing complexes \( \omega_\pi \) and \( \omega_{\hat{\pi}} \) are both isomorphic to \( \mathcal{O}_{X \times \hat{X}}[2] \). Then, a straightforward application of relative duality gives:

**Proposition 9.** For every objects \( F \) in \( D(X) \) and \( G \) in \( D(\hat{X}) \), we have functorial isomorphisms

\[
\text{Hom}_{D(\hat{X})}(G, S_X(F)) \cong \text{Hom}_{D(X)}(S_{\hat{X}}(G), F[-2])
\]

\[
\text{Hom}_{D(X)}(F, S_{\hat{X}}(G)) \cong \text{Hom}_{D(\hat{X})}(S_X(F), G[-2]).
\]
Proposition 10. For every $G \in D(\tilde{X})$ there is a functorial isomorphism

$$S_X(S_X(G)) \cong G[-2]$$

in the derived category $D(\tilde{X})$. Moreover, if $X$ is reflexive and $D \cdot H > 2$ for every nodal curve $D$ in $X$, then for every $F \in D(X)$ there is also a functorial isomorphism

$$S_X(S_X(F)) \cong F[-2]$$

in the derived category $D(X)$.

**Proof.** Let $q_1$ and $q_2$ be the projections onto the two factors of $X \times \tilde{X}$, and $\pi_{ij}$ the projection of $X \times \tilde{X} \times \tilde{X}$ onto the product of the $i$th and $j$th factors. Then, the composite functor is given by $S_X(S_X(G)) = R\pi_{23,*}(\pi_{12}^*Q \otimes \pi_{13}^*Q)$ (see [18]), with

$$\tilde{Q} = R\pi_{23,*}(\pi_{12}^*Q^L \otimes \pi_{13}^*Q) \cong R\pi_{23,*}R\text{Hom}^*(\pi_{12}^*Q, \pi_{13}^*Q),$$

where $R\text{Hom}^*(\ , \ )$ denotes the total derived functor of the complex of sheaf homomorphisms. By [20], Proposition 4.10, the right-hand side is isomorphic in the derived category to $\delta_*\mathcal{M}[-2]$, where $\delta: \tilde{X} \hookrightarrow \tilde{X} \times \tilde{X}$ is the diagonal embedding and $\mathcal{M}$ is an invertible sheaf on $\tilde{X}$. It follows that $S_X(S_X(G)) \cong GL^{\mathcal{M}}[-2]$. Let $\rho: \tilde{X} \times \tilde{X} \to \tilde{X} \times \tilde{X}$ be the permutation morphism; by base-change theory we have

$$\rho^*\tilde{Q} \cong R\pi_{23,*}(\pi_{13}^*Q^L \otimes \pi_{12}^*Q) \cong R\pi_{23,*}[(\pi_{13}^*Q \otimes \pi_{12}^*Q^*)^*] \cong \tilde{Q}^*[-2],$$

by relative duality for $\pi_{23}$. From $\rho \circ \delta = \rho$ one finds that $\mathcal{M}[-2] \cong \mathcal{M}^*[2]$. Then, there is an isomorphism of invertible sheaves $\mathcal{M} \cong \mathcal{M}^*$, and $\mathcal{M} \cong \mathcal{O}_{\tilde{X}}$.

The second statement follows from the first by Proposition 8. \hfill \blacksquare

**Corollary 3.** If $X$ is reflexive and $D \cdot H > 2$ for every nodal curve $D$ in $X$, there are functorial isomorphisms

$$\text{Hom}_{D(X)}(G, G) \cong \text{Hom}_{D(X)}(S_X(G), S_X(G))$$

$$\text{Hom}_{D(X)}(F, \tilde{F}) \cong \text{Hom}_{D(\tilde{X})}(S_X(F), S_X(\tilde{F})),$$

for $F, \tilde{F}$ in $D(X)$ and $G, \tilde{G}$ in $D(\tilde{X})$. \hfill \blacksquare

Thus, there is a duality between the varieties $X$ and $\tilde{X}$, in a complete analogy with the case of the Fourier-Mukai transform on abelian surfaces.

In particular we have the following result.

**Theorem 2.** Let $\mathcal{F}$ be a WIT$_1$ sheaf on $X$. Then its Fourier-Mukai transform $\tilde{\mathcal{F}} = R^i\pi_*\mathcal{F}$ is a WIT$_2$ sheaf on $\tilde{X}$, whose Fourier-Mukai transform $R^{2-i}\pi_*\tilde{\mathcal{F}}$ is isomorphic to $\mathcal{F}$. \hfill \blacksquare

### 4.4. Stability of the Fourier-Mukai transform.

We may now prove a stronger form of Theorem 1. Let $X$ be a reflexive K3 surface such that $D \cdot H > 2$ for every nodal curve $D$, and choose $\tilde{X}$ as before. By proceeding as in Corollary 2.5 of [18] and taking into account Corollary 3, we obtain...
Lemma 5. Let $\mathcal{F}, \mathcal{F}'$ be coherent sheaves on $X$. If $\mathcal{F}$ is WIT$_i$ and $\mathcal{F}'$ is WIT$_j$, we have
\[ \text{Ext}^h(\mathcal{F}, \mathcal{F}') \simeq \text{Ext}^{h+i-j}(\hat{\mathcal{F}}, \hat{\mathcal{F}}). \]
for $h = 0, 1, 2$. In particular, there is an isomorphism $\text{Ext}^h(\mathcal{F}, \mathcal{F}) \simeq \text{Ext}^h(\hat{\mathcal{F}}, \hat{\mathcal{F}})$ for every $h$, so that $\hat{\mathcal{F}}$ is simple for every simple WIT$_i$ sheaf $\mathcal{F}$.

Lemma 5. If $\mathcal{F}$ is a $\mu$-stable vector bundle of degree zero, and $\nu(\mathcal{F}^*) \neq (2, \ell, -3)$, then $\mathcal{F}$ is IT$_1$.

Proof. For every $\xi \in \hat{X}$ we have $H^2(X, \mathcal{F} \otimes Q_\xi)^* \simeq \text{Hom}(Q_\xi, \mathcal{F}^*)$. Since $\mathcal{F}$ and $Q_\xi$ are $\mu$-stable, if there exists a nonzero morphism $Q_\xi \to \mathcal{F}^*$, then it is an isomorphism, which is incompatible with the condition in the statement. The same argument also shows that $H^0(X, \mathcal{F} \otimes Q_\xi) \simeq \text{Hom}(\mathcal{F}^*, Q_\xi) = 0$, thus concluding the proof.

Theorem 3. Let $\mathcal{F}$ be a zero-degree $\mu$-stable bundle on $X$, with $\nu(\mathcal{F}^*) \neq (2, \ell, -3)$. Then its Fourier-Mukai transform $\hat{\mathcal{F}}$ is $\mu$-stable.

Proof. $\hat{\mathcal{F}}$ is $\mu$-polystable by Theorem 1, and simple by Lemma 5, so that it is $\mu$-stable.

4.4. Topological invariants.

We wish to compute the topological invariants of the Fourier-Mukai transform $S_X(\mathcal{F}) = R\pi_*(\pi^*\mathcal{F} \otimes \mathcal{Q}) \in D(\hat{X})$ of a sheaf $\mathcal{F}$ on $X$ in terms of those of $\mathcal{F}$. The formula is obtained as usual by the Riemann-Roch theorem, taking into account that we can compute the Chern character $\gamma$ of $\mathcal{Q}$ from the sequence (5).

Let us define the Mukai vector and the Euler characteristic of the Fourier-Mukai transform $S_X(\mathcal{F}) \in D(\hat{X})$ by $v(S_X(\mathcal{F})) = \sum_{i=0}^{2} (-1)^{i} v(R^{i}\pi_*(\pi^*\mathcal{F} \otimes \mathcal{Q}))$ and $\chi(S_X(\mathcal{F})) = \sum_{i=0}^{2} (-1)^{i+1} \chi(R^{i}\pi_*(\pi^*\mathcal{F} \otimes \mathcal{Q}))$.

Proposition 11. Given a coherent sheaf $\mathcal{F}$ on $X$, let $u = (\rho, c_1, \sigma) = (\text{rk}(\mathcal{F}), c_1(\mathcal{F}), \text{rk}\mathcal{F} + \text{ch}_2(\mathcal{F}))$ be the Mukai vector of $\mathcal{F}$, and $d = c_1 \cdot H$. If $v = v(S_X(\mathcal{F})) = (\hat{\rho}, \hat{c}_1, \hat{\sigma})$, one has
\[ \hat{\rho} = -3\rho + 2\sigma + \ell \cdot c_1, \]
\[ \hat{c}_1 = (\ell \cdot c_1 + 2d)\hat{H} + (\rho + d - s)\ell - \Psi^*(c_1), \]
\[ \hat{\sigma} = 2\rho - 3\sigma - \ell \cdot c_1. \]

Then $\chi(S_X(\mathcal{F})) = -\chi(\mathcal{F})$ and $\hat{u}^2 = u^2$.

Corollary 8. The Fourier-Mukai transform preserves the Euler characteristic and the degree of WIT$_1$-sheaves, that is, if $\mathcal{F}$ is WIT$_1$ then $\chi(\hat{\mathcal{F}}) = \chi(\mathcal{F})$ and $c_1(\mathcal{F}) \cdot H = c_1(\hat{\mathcal{F}}) \cdot \hat{H}$.

Final remarks. Theorem 3 can be exploited to investigate the structure of the entire moduli space of stable sheaves on a reflexive K3 surface $X$. Let $u$ be a Mukai vector, $u \neq (2, -\ell, -3)$. The locally free $\mu$-stable sheaves $\mathcal{F}$ on $X$ with $\nu(\mathcal{F}) = u$ are a Zariski open set $M_H^\mu(u) \subset M_H(u)$; they are mapped by the Fourier-Mukai transform onto the Zariski open set of locally free $\mu$-stable sheaves in $M_H^\mu(\hat{u})$, where $\hat{u}$ is given in terms of $u$ according to Proposition 11. This map preserves the number of $U$-automorphisms and the birational type of these schemes.
holomorphic symplectic structures of these spaces. Moreover, one has \( \dim M^H(u) = \dim M^\hat{H}(\hat{u}) \) (since \( u^2 = \hat{u}^2 \)), and so — provided \( M^\mu_H(u) \) is not empty — there is a birational correspondence \( M^H(u) \to M^\hat{H}(\hat{u}) \).

In some cases stronger results can be obtained; for instance it can be shown that for any \( n \geq 1 \) the moduli space \( M^H(1 + 2n, -n\ell, 1 - 3n) \) is biholomorphic to the punctual Hilbert scheme \( \text{Hilb}^n(X) \) [8].

In [4] we give a completely algebraic proof of Theorem 3. Also in [4] we prove algebraically that \( \mathcal{O}_X(2\hat{H}) \) is the determinant line bundle, which is an alternative proof of the ampleness of \( \hat{H} \).

The transcendental proof of Theorem 1 extends directly to higher dimensions, providing a proof of the the fact that the Fourier-Mukai transform on hyperkähler manifolds maps quaternionic instantons to quaternionic instantons [6].

Acknowledgments. We thank P. Francia, J. M. Muñoz Porras, K. O’Grady and especially A. Maciocia for useful discussions and suggestions. The first author also thanks S. Donaldson for advice at a preliminary stage of this work. This work was partly done while the first author was visiting the State University of New York at Stony Brook, and the second author was visiting Victoria University of Wellington, New Zealand. They thank the respective Departments of Mathematics for their warm hospitality and for providing excellent working conditions.

References

[1] Atiyah M.F., Singer I.M., Dirac operators coupled to vector potentials, Proc. Natl. Acad. Sci. U.S.A. 81 (1984), 2597–2600.

[2] Bartocci C., Instantons over K3 surfaces, Group theoretical methods in physics, Vol. II, M.A. del Olmo, M. Santander and J. Mateos Guilarte (eds.), Anales de Física. Monografias, 1. CIEMAT, Madrid, 1993, pp. 64–67.

[3] Bartocci C., Bruzzo U., Hernández Ruipérez D., Fourier-Mukai transform and index theory, Manuscripta Math. 85 (1994), 141–163.

[4] ______, Existence of \( \mu \)-stable vector bundles on K3 surfaces and the Fourier-Mukai transform, in “Proceedings of EUROPROJ 94”, P. Newstead ed., M. Dekker (to appear).

[5] ______, Moduli of reflexive K3 surfaces, in “Complex analysis and geometry”, E. Ballico et al. eds., M. Dekker (to appear).

[6] ______, A hyperkähler Fourier transform, Preprint.

[7] Braam P.J., Van Baal P., Nahm’s transformation for instantons, Commun. Math. Phys. 122 (1989), 267–280.

[8] Bruzzo U., Maciocia A., Hilbert schemes of points on some K3 surfaces and Gieseker stable bundles, to appear, Math. Trans. Cambridge Phil. Soc. (1996).

[9] Donaldson S.K., Kronheimer P.B., The geometry of four-manifolds, Clarendon Press, Oxford, 1990.

[10] Fahlaoui R., Laszlo Y., Transformée de Fourier et stabilité sur les surfaces abéliennes, Comp. Math. 79 (1991), 271–278.

[11] Hirschowitz A., Laszlo Y., A propos de l’existence de fibrés stables sur les surfaces, Preprint [alg-geom/9310008].

[12] Itoh M., Yang-Mills connections and the index bundles, Tsukuba Math. J. 13 (1989), 423–441.

[13] ______, Poincaré bundle and Chern classes, Adv. Studies Pure Math. 18-1 (1990), 271–281.

[14] Maciocia A., Gieseker stability and the Fourier-Mukai transform for abelian surfaces, to appear, Quart. J. Math. (1996).
[15] Mamone Capria M., Salamon S.M., Yang-Mills fields on quaternionic spaces, Nonlinearity 1 (1988), 517–530.
[16] Maruyama M., Moduli of stable sheaves I, J. Math. Kyoto Univ. 17 (1977), 91–126.
[17] ———, Moduli of stable sheaves II, J. Math. Kyoto Univ. 18 (1981), 557–614.
[18] Mukai S., Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J. 81 (1981), 153–175.
[19] ———, Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, Invent. Math. 77 (1984), 101–116.
[20] ———, On the moduli space of bundles on a K3 surface I, Vector bundles on algebraic varieties, Tata Institute of Fundamental Research, Oxford University Press, Bombay and London, 1987.
[21] Schenk H., On a generalized Fourier transform of instantons over flat tori, Commun. Math. Phys. 116 (1988), 177–183.

⋆ Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy.
E-mail address: bartocci@dima.unige.it

♯ S.I.S.S.A., Via Beirut 2-4, 34014 Miramare, Trieste, Italy.
E-mail address: bruzzo@sissa.it

¶ Departamento de Matemática Pura y Aplicada, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain.
E-mail address: ruiperez@gugu.usal.es