Electric Charge as a Vector Quantity

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Abstract
Starting with the premise that the electric charge associated with fundamental fermions (quarks and leptons) can, under certain circumstances, be appropriately represented as a real internal 2-vector, the mathematical “machinery” implicit in the associated internal 2-space is shown to apply to all fundamental fermions. In particular, it is shown that flavor eigenstates, flavor doublets and families of fundamental fermions can all be represented in the 2-space, and that such things as internal colors, family replication, and the observed number (three) of families, are more-or-less implicit in the new 2-space description. Moreover, the model predicts that, unlike the case in the standard model, particles such as the $u$, $c$, and $t$ quarks are characterized by significant internal (topological and other) differences. Similar differences may help explain recent observations of (nearly) maximal $\nu_\mu - \nu_\tau$ mixing.

1 Introduction
If quantum effects are ignored it is completely appropriate to treat electric charge in external spacetime as a scalar quantity. In particular, in 4-dimensional spacetime, the electric charge carried by an isolated particle is appropriately treated as a Lorentz invariant 4-scalar [1]. However, when
quantum effects are taken into account it is not at all clear that the internal description of electric charge should be limited to scalar quantities. The purpose of this paper will be to argue that in the case of fundamental fermions (quarks and leptons) there are good reasons for treating electric charge as a vector quantity associated with a new (abstract) internal 2-space \[2\]. The motivation for taking such an unusual step is that it will be shown to lead to an extension of the standard model description of quarks and leptons.

Before embarking on this exploration, it is appropriate to briefly review the conventional description of flavor doublets of fundamental fermions. Certain aspects of the conventional description via the \(SU(2)\) isospin formalism will suggest the new description.

1.1 The conventional description of flavor doublets

According to the standard model of particle physics, all left-handed quarks and leptons (right-handed antiquarks and antileptons) are members of \(SU(2)\) weak isospin doublets \[3\]. To be specific let us limit the present discussion to the first-family quark states \(|u\rangle\) and \(|d\rangle\), which constitute an \(SU(2)\) weak isospin doublet (also called a flavor doublet). These states are properly to be thought of as being two different (isospin) states of a single quark field \[4\].

Using the conventional isospin language, there exists an isospin (vector) operator \(\tau = \frac{1}{2} \sigma\), where \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) is a vector form whose components are the familiar Pauli matrices \(\sigma_1, \sigma_2\) and \(\sigma_3\). All physically observable states carrying (weak) isospin (e.g., \(|u\rangle\) and \(|d\rangle\)) must be simultaneous eigenstates of the square of the total isospin vector \(\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2\), and the third-component of isospin \(\tau_3\), i.e., they take the form \(|\tau^2, \tau_3\rangle\). The states \(|u\rangle\) and \(|d\rangle\) are said to span a two-dimensional Hilbert space, and to constitute a 2-dimensional representation \(D^{(1/2)}\) of \(SU(2)\). In particular, given that the states \(|u\rangle\) and \(|d\rangle\) are eigenstates of

\[
\tau_3 = \frac{1}{2} \sigma_3, \tag{1}
\]

where

\[
\sigma_3 = 
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \tag{2}
\]

we have the eigenvalue equations (use \(\tau_3|\tau^2, \tau_3\rangle = \tau_3|\tau^2, \tau_3\rangle\) or \(\tau_3|\frac{3}{2}, \pm \frac{1}{2}\rangle =\)
\[ \pm \frac{1}{2} \pm \frac{1}{2} \] and/or the column-vector forms \(|u\rangle = \{1, 0\} \) and \(|d\rangle = \{0, 1\} \)

\[
\begin{align*}
\tau_3 |u\rangle &= +\frac{1}{2} |u\rangle \\
\tau_3 |d\rangle &= -\frac{1}{2} |d\rangle
\end{align*}
\] (3)

Now the electric charge of the quark field in either of the states \(|u\rangle\) or \(|d\rangle\) is given by operating on these states with the operator for electric charge, which may be expressed in this particular case as

\[ Q = \tau_3 + \frac{1}{6} \] (4)

Then we have for the electric charges of these two possible states of the quark field

\[ Q |u\rangle = q_1 |u\rangle \] (5)

and

\[ Q |d\rangle = q_2 |d\rangle, \] (6)

where

\[ q_1 = +\frac{2}{3} \text{ and } q_2 = -\frac{1}{3} \] (7)

are the specific electric charges in question.

While the foregoing description is certainly correct as far as it goes it is, nevertheless, unnecessarily restrictive. In particular, we will show in the next section that the foregoing properties of flavor doublets are suggestive of a (complementary) extension of this conventional description, which treats electric charge as a real vector quantity in a new internal 2-dimensional linear vector space.

### 1.2 Electric charge as a vector quantity

It is implicit in the description of \(SU(2)\) isospin doublets that if one knows the state \(|u\rangle\) one can (must) infer the existence of a second state \(|d\rangle\), and vice versa. The \(2 \times 2\) matrix form of the charge operator \(Q\), and the isospin operator \(\tau_3\) makes this abundantly clear.
If one has a state with $\tau_3 = +\frac{1}{2}$, and one knows that one is dealing with an $SU(2)$ isospin doublet field, then one must also have a state with $\tau_3 = -\frac{1}{2}$. As a consequence of these simple facts, one can have simultaneous knowledge of the two electric charges $q_1$ and $q_2$ associated with the two states $|u\rangle$ and $|d\rangle$, respectively, given either one of the states $|u\rangle$ or $|d\rangle$.

In a certain sense then, there could be a physical meaning to a geometric object defined by the ordered pair of observable (real numbers) electric charges associated with the states $|u\rangle$ and $|d\rangle$, namely, a real 2-vector or “charge vector”

$$Q = \{q_1, q_2\}.$$ \hfill (8)

However, it should be stressed that if the 2-vector $Q$ were to be “carried” by, or “associated” with, each of the states $|u\rangle$ and $|d\rangle$ there would be definite nontrivial physical consequences.

Clearly, the assignment of $Q$ to $|u\rangle$ and $|d\rangle$ would mean that these states carry additional information (besides isospin) in the quantum sense [5]. In particular, because the 2-vector $Q$ has to “live” in some abstract internal 2-space, all of the mathematical “machinery” associated with this 2-space would have to be taken into account (e.g., the 2-space metric and various internal transformations in the 2-space) when describing fundamental fermions. In short, this new description would promise a far richer internal structure than that implied by the description of flavor doublets using $SU(2)$ alone. This implied richness constitutes nothing less than a possible extension of the standard model [3, 6], and encourages us to seriously consider the idea of representing electric charge (of fundamental fermions) by an internal 2-vector.

In the next section we will show how this idea leads naturally to, among other things, an explanation for flavor doublets in other families, i.e., to an explanation for family replication.

2 Consequences of Treating Electric Charge as a 2-Vector

In this section we derive a number of consequences of applying the 2-space mathematical machinery to fundamental fermions. Many of the results presented here were arrived at in earlier works by a somewhat different route.
In particular, in [7] we began by generalizing the scalar fermion number $f$ to a $2 \times 2$ real, generally non-Hermitian matrix $F$ (i.e., $f \rightarrow F$), and in [8] we arrived at this same matrix $F$ by analytically continuing the fermion number operator $F(\text{op}) \rightarrow F$.

In the present paper, by contrast, we begin by generalizing electric charge (call it $e$) from a scalar to a 2-vector $Q$ (i.e., $e \rightarrow Q$) or “charge vector.” Only later do we arrive at the matrix $F$ described above. The interested reader is encouraged to consult the indicated references for further details regarding these earlier works.

## 2.1 The 2-space metric

If the 2-vector $Q = \{q_1, q_2\}$ is assigned to each of the matter states $|u\rangle$ and $|d\rangle$, then there must exist a vector (call it $Q^c$) that is assigned to each of the corresponding anti-matter states $|\bar{u}\rangle$ and $|\bar{d}\rangle$, respectively.

Recalling the definition of the charge vector $Q$ given in Sec. 1.2 (see Eq. 8), the (anti) charge vector $Q^c$ must be formed in some way from the ordered pair of real numbers $-q_1$ and $-q_2$ corresponding, respectively, to the electric charges of the $|u\rangle$ and $|d\rangle$ ant mater states. Now, assuming that the scalar product of $Q$ and $Q^c$, namely, $Q \cdot Q^c$ should vanish ($Q$ and $Q^c$ should be orthogonal) so as to insure that matter and antimatter states can be distinguished in the 2-space, it follows that no matter what the metric is, as long as it is real and flat, $Q$ and $Q^c$ must also be linearly independent. To see this, notice first that if $Q$ and $Q^c$ are not linearly independent, then they are, by definition, necessarily, linearly dependent, in which case $Q^c = -Q = \{-q_1, -q_2\}$.

Assuming that the 2-space metric $g$ is real and flat, $g$ can be represented by the $2 \times 2$ matrix $g = (g_{11} = 1, g_{22} = s, g_{12} = g_{21} = 0)$ or
\[
g = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix},
\]  

where $|s| = 1$. Then, given $A \cdot B = \sum_{i,j=1}^{2} g_{ij} a_i b_j$, where $A = (a_1, a_2)$ is a row-vector and $B = \{b_1, b_2\}$ is a (conformable) column-vector, the scalar product of $Q$ and $Q^c = -Q$ (Note that $q_1$ is never equal to $q_2$ because these charges correspond to different eigenstates of the electric charge operator $Q$; see Eqs.
5–7) is, from (8) and (9),

\[ Q \cdot Q^c = (q_1, q_2) \begin{pmatrix} \frac{-q_1}{q_2} \\ \frac{-q_2}{q_1} \end{pmatrix} = -q_1^2 - sq_2^2, \]  

(10)

which is generally \textit{nonzero} for any \(|s| = 1\), i.e., \(s = \pm 1\) (also see Footnote 9).

Therefore, if \(Q\) and \(Q^c\) are to be orthogonal \((Q \cdot Q^c = 0)\), the fact that (10) is generally nonzero ensures that \(Q\) and \(Q^c\) must be \textit{linearly independent}. In this case, given \(Q = \{q_1, q_2\}\), it must be true that

\[ Q^c = \{-q_2, -q_1\}, \]  

(11)

which is \textit{not} proportional to \(Q\), as required to ensure \textit{linear independence}.

Finally, given the general form for the metric (9), and the linear independence (and orthogonality) of \(Q\) and \(Q^c\), one has the following result for \(s\)

\[ Q \cdot Q^c = (q_1, q_2) \begin{pmatrix} \frac{-q_2}{q_1} \\ \frac{-q_1}{q_2} \end{pmatrix} = -q_1q_2 - sq_1q_2 = 0, \]  

(12)

if, and only if, \(s = -1\). Therefore, the 2-space \textit{metric} is given by

\[ g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  

(13)

and we see that the requisite 2-space is, necessarily, “Lorentzian” or \textit{non-Euclidean} [10].

2.1.1 Scalar products of 2-vectors

Using the metric given in (13), and the general formula for \(A \cdot B\) given in Sec. 2.1, we immediately have the \textit{scalar product} of two different real 2-vectors

\[ (a, b)\{e, f\} = ae - bf. \]  

(14)

Similarly, the \textit{square} of a real 2-vector is given by

\[ (a, b)\{a, b\} = a^2 - b^2. \]  

(15)

Here we remind the reader that \((, )\) is a \textit{row} vector while \{, \} is a (conformable) \textit{column} vector. Clearly, the scalar products in (14) and (15) transform like charge-conjugation-reversing or \(C\)-reversing (2-scalar) \textit{charges}. 
For example, using (8), (11) and (15) we immediately have the square of the 2-vectors $Q$ and $Q^c$, namely,

$$Q^2 = Q \cdot Q = q_1^2 - q_2^2$$

and

$$(Q^c)^2 = Q^c \cdot Q^c = q_2^2 - q_1^2.$$  \hfill (17)

Therefore,

$$Q^2 = -(Q^c)^2,$$  \hfill (18)

which means that $Q^2$ and $(Q^c)^2$ each transform like C-reversing 2-scalar charges [9].

It happens that these particular charges can be identified with the baryon- or lepton-number carried by quarks or leptons, respectively (see Ref. 7, p. 72). We will see in a later section that when charge vectors such as $Q$ (or $Q^c$) are resolved in the 2-space into pairs of linearly independent vectors (e.g., $Q = U + V$), not only are the components of $Q$, $U$ and $V$, C-reversing charges, but also given

$$Q^2 = U^2 + 2U \cdot V + V^2.$$  \hfill (19)

$U^2$, $2U \cdot V$ and $V^2$ are, like $Q^2$, C-reversing charges. The foregoing collection of 2-scalar charges will be used to define and describe flavor eigenstates, flavor doublets, and eventually families of fundamental fermions.

### 2.1.2 A conjectured “duality”

Given the number of flavors of quarks and leptons, and an appropriate (renormalizable) Lagrangian, the so-called “accidental symmetries” of the Lagrangian [11] are known to “explain” the separate conservation of various (global) flavor-defining (Lorentz 4-scalar) “charges” [e.g., lepton number, baryon number, strangeness, charm, the third-component of (strong or global) isospin, truth, beauty, electron-, muon-, and tau-numbers]. Now, as demonstrated in detail in Ref. 7, pp. 67–71, given certain real C-reversing scalars—components of various vectors and matrices defined on the internal non-Euclidean 2-space, and various scalar products of 2-vectors—it is possible to describe the flavor eigenstates of fundamental fermions [12].
In principle, what one does is to identify the mutually-commuting $C$-reversing 2-space “charges” (call them $C_i$) or charge-like quantum numbers associated with a particular flavor, and then write the corresponding simultaneous flavor-eigenstate as

$$|C_1, C_2, C_3, \ldots, C_n\rangle. \quad (20)$$

Here $C_1, C_2, C_3, \ldots, C_n$, are said to be the “good” charge-like quantum numbers (charges) associated with a particular flavor [13]. Now it also happens that these observable real numbers can be identified with quantum numbers such as electric charge, lepton number, baryon number, strangeness, charm, the third-component of (strong or global) isospin, truth and beauty (see Ref. 7, p. 72). In short, these 2-space charges look very much like those associated with the aforementioned “accidental symmetries” of the Lagrangian! The foregoing properties of the non-Euclidean charge-like scalars, leads naturally to the following “duality” conjecture:

*The global (flavor-defining) charges associated with the “accidental symmetries” of the Lagrangian describing strong and electroweak interactions, and the global (flavor-defining) charges associated with the non-Euclidean 2-space, are (essentially) one and the same charges.*

### 2.2 Charge conjugation in the 2-space

Given that there are numerous $C$-reversing scalars in the 2-space (see Sec. 2.1.1), there must exist a $2 \times 2$ matrix, call it $X$, that serves to transform these scalars, various 2-vectors such as $Q$ or $Q^c$, and various $2 \times 2$ matrices, to their corresponding $C$-reversed (2-space) counterparts. In particular, a matrix $X$ should exist such that (use Eqs. 8 and 11)

$$XQ = Q^c \quad (21)$$

and

$$XQ^c = Q. \quad (22)$$

From (21) and (22) it follows that $X$ must equal its multiplicative inverse ($X = X^{-1}$), and thus

$$X^2 = I_2, \quad (23)$$
where $I_2$ is the $2 \times 2$ identity matrix.

Write $X$ in the general form ($X$ is real)

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (24)$$

and consider the situation where one of the two charges (say $q_2$) associated with $Q = \{q_1, q_2\}$ is zero, and the other charge ($q_1$) is nonzero (this is actually the case for leptons).

In this particular case, we have (use Eqs. 8, 11, 21 and 24)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -q_1 \end{pmatrix}, \quad (25)$$

which means that

$$aq_1 = 0 \quad (26)$$

and

$$cq_1 = -q_1. \quad (27)$$

And therefore, since $q_1 \neq 0$, it must be true from (26) and (27) that $a = 0$ and $c = -1$.

Since $XQ^c = Q$ it must also be true that

$$\begin{pmatrix} 0 & b \\ -1 & d \end{pmatrix} \begin{pmatrix} 0 \\ -q_1 \end{pmatrix} = \begin{pmatrix} q_1 \\ 0 \end{pmatrix}, \quad (28)$$

which means that

$$-bq_1 = q_1 \quad (29)$$

and

$$-dq_1 = 0. \quad (30)$$

Finally, since $q_1 \neq 0$ it must be true from (29) and (30) that $b = -1$ and $d = 0$.

Collecting the foregoing matrix elements, we have

$$X = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (31)$$
or

\[ X = -\sigma_X, \tag{32} \]

where \( \sigma_X \) or \( \sigma_1 \) is one of the familiar Pauli matrices. In general, the matrix \( X = -\sigma_X \) should apply (in the 2-space) to 2-scalars, 2-vectors, 2 \times 2 matrices, and to both quarks and leptons.

### 2.2.1 Transformation of the metric and other matrices under X

Any 2 \times 2 matrix \( M \), appropriate to the 2-space description of fundamental fermions, should transform under \( X = -\sigma_X \) to its C-reversed counterpart \( M^c \) according to the similarity transformation

\[ X M X^{-1} = M^c, \tag{33} \]

or because \( X = X^{-1} = -\sigma_X \), equivalently as

\[ (-\sigma_X)M(-\sigma_X) = M^c. \tag{34} \]

For example, the metric \( g \) (see Eq. 13) is found to be C-reversing since

\[ (-\sigma_X)g(-\sigma_X) = -g. \tag{35} \]

A matrix that is C-invariant (e.g., the matrix \( X \)) would, necessarily, have the form

\[ N = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \tag{36} \]

where it is clear that

\[ (-\sigma_X)N(-\sigma_X) = N. \tag{37} \]

Now let us apply the foregoing similarity transformation to the matrix \( F \), which represents the generalized fermion number in this 2-space [see Ref. 7, pp. 4–12].
2.3 The generalized fermion number $F(v)$

There are other $2 \times 2$ matrices besides $X$ that act on the 2-vectors $Q$ and $Q^c$. Let us define a matrix $F$ whose eigenvalues $f$ are the fermion numbers $f_m = +1$ for matter, and $f_a = -1$ for antimatter, and whose eigenvectors are the electric-charge vectors $Q$ and $Q^c$, respectively. Then

$$FQ = f_m Q \quad (38)$$

and

$$FQ^c = f_a Q^c. \quad (39)$$

Clearly, (38) and (39) require $F$ to be equal to its multiplicative inverse, i.e., $F = F^{-1}$ or $F^2 = I_2$.

Given that the eigenvalues of $F$ are $f_m$ and $f_a$, the diagonal form for $F$ is simply (see Ref. 7, p. 4)

$$F_{\text{diag}} = \begin{pmatrix} f_m & 0 \\ 0 & f_a \end{pmatrix}. \quad (40)$$

And, from (31) and (33) the $C$-reversed counterpart of $F_{\text{diag}}$ is, necessarily, given by the similarity transformation

$$(-\sigma_X)F_{\text{diag}}(-\sigma_X) = -F_{\text{diag}}. \quad (41)$$

From the minus sign on the right hand side of (41) we see that the scalar fermion numbers $f_m$ and $f_a$ properly change signs (are $C$-reversing “charges”) under $-\sigma_X$.

Consider next a more general (nondiagonal) matrix $F$. Because the trace, determinant and square of $F$ are invariants, one has

$$trF = tr F_{\text{diag}} = f_m + f_a = 0 \quad (42)$$

$$detF = det F_{\text{diag}} = f_m \cdot f_a = -1 \quad (43)$$

and

$$F^2 = F_{\text{diag}}^2 = I_2. \quad (44)$$

And, because $F$ is traceless it must have the general form

$$F = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \quad (45)$$
Finally, because $F$ should transform in the same way as $F_{\text{diag}}$ under $X = -\sigma_X$ we have the similarity transform

$$(-\sigma_X)F(-\sigma_X) = -F. \quad (46)$$

Using $F$ as expressed by (45) and employing (46), one finds that $c = -b$. Hence, the most general form of $F$ is, necessarily,

$$F = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}, \quad (47)$$

where $a^2 - b^2 = 1$ since $\det F = -1$.

Making the following substitutions in (47), namely,

$$a = \cosh v \quad (48)$$

and

$$b = \pm \sinh v, \quad (49)$$

where $-\infty \leq v \leq +\infty$ is a real (dimensionless) parameter in the range indicated, $F$ or $F(v)$ finally assumes the general form

$$F(v) = \begin{pmatrix} \cosh v, & \pm \sinh v \\ \mp \sinh v, & -\cosh v \end{pmatrix}. \quad (50)$$

Here, $F^2(v) = I_2$ for any $v$, $F(v)$ satisfies the boundary condition $F(0) = F_{\text{diag}}$, and the non-Euclidean 2-scalars $Q^2$ and $(Q^c)^2$ are left invariant by the transformation $F(v)$.

### 2.4 Distinguishing quarks and leptons

Choosing the upper signs in (50), the matrix $F(v)$ becomes

$$F(v) = \begin{pmatrix} \cosh v & \sinh v \\ -\sinh v & -\cosh v \end{pmatrix}, \quad (51)$$

where $v$ is also chosen to be a positive real number (see Ref. 7, p. 50 and 54).

As described in Ref. 7, pp. 52–55, the parameter $v$ distinguishes between *quarks* and *leptons*. In particular, the parameter $v$ is found to be quantized and obeys the “quantum condition”:

$$v = \ln M_c, \quad (52)$$
where \( M_c \) counts both the number of fundamental fermions in a strongly-bound composite fermion, and the strong-color multiplicity. That is, \( M_c = 3 \) for quarks (strong-color triplets) and \( M_c = 1 \) for leptons (strong-color singlets). Thus we have found that a connection exists between the 2-space description of quarks and leptons, and their associated strong colors!

### 2.4.1 Quark and lepton electric charges and \( B \) and \( L \)

It has been shown (see Sec. 2.3, Eqs. 38 and 39; Ref. 7, pp. 52–55, and Ref. 8) that the quark and lepton electric charges are the “up”-“down” components of the eigenvectors of the matrix \( F(v) \) specified by (51) and (52). In particular, the quark charges are given by (\( M_c = 3 \))

\[
q_1(f) = \frac{(M_c^2 - 1)}{2M_c(M_c - f)} = +\frac{2}{3} \text{ for } f = +1 \quad \text{and} \quad +\frac{1}{3} \text{ for } f = -1, \quad (53)
\]

\[
q_2(f) = q_1(f) - 1, \quad (54)
\]

where the baryon number for quarks is \( B = q_1^2(f) - q_2^2(f) = \pm \frac{1}{3} \) for \( f = \pm 1 \). Similarly, the lepton electric charges are given by (\( M_c = 1 \))

\[
q'_1(f) = \frac{-(M_c^2 - 1)}{2M_c(M_c - f)} = -1 \text{ for } f = +1 \quad \text{and} \quad 0 \text{ for } f = -1, \quad (55)
\]

\[
q'_2(f) = q'_1(f) + 1, \quad (56)
\]

where the lepton number for leptons is \( L = [q'_1(f)]^2 - [q'_2(f)]^2 = \pm 1 \) for \( f = \pm 1 \).

In summary, the new 2-space description, and \( F(v) \), is found to provide an explanation for the quark-lepton “dichotomy” of fundamental fermions in addition to the matter-antimatter, and “up”-“down” type flavor-dichotomies.

### 2.5 Representing flavor doublets in the 2-space

Consider again the eigenvectors \( Q \) of \( F(v) \) for fundamental fermions. Since the space on which \( F(v) \) “acts” is two-dimensional, an observable vector \( Q \) can be “resolved” into two (no more or less) observable, linearly-independent vectors, call them \( U \) and \( V \), as \( Q = U + V \) [14]. Now, because these three vectors (\( Q, U, \) and \( V \)) are simultaneous observables, it makes sense to speak of this “triad” of vectors as being a well defined geometric object, namely, a “vector triad.”
Recognizing that the components of $Q$, $U$ and $V$ are $C$-reversing charge-like observables [13] we can write these observable “charge” vectors as
\[ Q = \{q_1, q_2\}, \quad (57) \]
\[ U = \{u_1, u_2\}, \quad (58) \]
\[ V = \{v_1, v_2\}, \quad (59) \]
where $q_1$, $q_2$, $u_1$, $u_2$, $v_1$ and $v_2$ are the various observable “charges” (e.g., $q_1$ and $q_2$ are the electric charges). Given $Q = U + V$, the non-Euclidean metric (13), and Eqs. (57) through (59), we find the associated observable quadratic-“charges” [13]
\[ Q^2 = U^2 + 2U \cdot V + V^2 \quad (60) \]
\[ 2U \cdot V = 2(u_1 v_1 - u_2 v_2) \quad (61) \]
\[ U^2 = u_1^2 - u_2^2 \quad (62) \]
\[ V^2 = v_1^2 - v_2^2. \quad (63) \]
Finally, using the foregoing collection of (global) flavor-defining charges, we can express the two quantum states (simultaneous flavor-eigenstates) associated with a single vector-triad in the form of “ket” vectors as follows (Ref. 7, pp. 16–18)
\[ |q_1, u_1, v_1, Q^2, U^2, 2U \cdot V, V^2\rangle \]
\[ |q_2, u_2, v_2, Q^2, U^2, 2U \cdot V, V^2\rangle . \quad (64) \]
Here, the state $|q_1, u_1, v_1, Q^2, U^2, 2U \cdot V, V^2\rangle$ represents the “up”-type flavor-eigenstate, and $|q_2, u_2, v_2, Q^2, U^2, 2U \cdot V, V^2\rangle$ represents the corresponding “down”-type flavor-eigenstate in a flavor doublet of fundamental fermions [12, 15].

### 2.6 Family replication and the number of families

In Ref. 7, pp. 39–49 and pp. 59–65, it is shown that flavor doublets (hence families) are replicated and that there are only three families of quarks and leptons. We refer the reader to [7] for a full and detailed account. Here we simply outline how this situation comes about.

Once again, by the definition of a linear-vector 2-space, a 2-vector such as $Q$ can always be resolved into a pair (no more, or less) of linearly-independent
vectors \( \mathbf{U} \) and \( \mathbf{V} \) as \( \mathbf{Q} = \mathbf{U} + \mathbf{V} \) (see Sec. 2.5). And, since \( \mathbf{Q} \) represents a flavor doublet, so should \( \mathbf{U} \) and \( \mathbf{V} \) represent this same flavor doublet. But, if this is so, *different vector-resolutions of \( \mathbf{Q} \) (i.e., different vector-triads) should correspond to different flavor-doublets having the same \( \mathbf{Q} \).* In other words, flavor doublets should be replicated.

Since \( \mathbf{Q} \) can be resolved (mathematically) in an infinite number of ways, we might suppose that there are an infinite number of flavor doublets, and hence, families. But, because of various “quantum constraints,” it is possible to show that \( \mathbf{Q} \) can be resolved in only three physically acceptable ways for \( \mathbf{Q} \)-vectors associated with either quarks or leptons. In other words, there can be only six quark flavors and six lepton flavors, which leads to the (*ex post facto*) “prediction” of three quark-lepton families.

### 2.7 New internal differences and neutrino mixing

It is important to understand that the new 2-space description of fundamental fermions (quarks and leptons) provides a distinction between these particles that goes beyond differences that can be explained by mass differences alone. For example, in the standard model the only difference between the \( u \), \( c \) and \( t \) quarks is that they have different masses. Otherwise, these particles experience identical strong and electroweak interactions. Moreover, as described in Section 2.1.2, the separate conservation of quantum numbers such as “charm” and “truth” can be attributed to certain unavoidable “accidental symmetries” associated with the (renormalizable) Lagrangian describing the (strong) interactions of these particles [11].

Taken at face value, these accidental symmetries would seem to imply that there are no internal “wheels and gears” that would distinguish a \( u \) quark from a \( c \) quark, for example. But, if the string theories are correct, these particles would be associated with different “handles” on the compactified space (see Ref. 16, Vol. 2, p. 408), and so would be different in this *additional* sense. Likewise, in the present non-Euclidean 2-space description, topological differences in addition to a variety of (global) 2-scalars, which are only *indirectly* related to the accidental symmetries of the Lagrangian, serve to provide further distinctions between matter particles.

A possible experimental signal of such “internal” differences is to be found in the recent observations at the Super Kamiokande of bi-maximal neutrino mixing [17]. Models which begin by positing a neutrino mass-matrix and associated mixing-parameters, such as the three-generation model proposed
by Georgi and Glashow [18], do an acceptable job of describing the observations. However, bi-maximal mixing may have a deeper explanation in terms of internal topological differences (in the non-Euclidean 2-space) between $\nu_e$, and $\nu_\mu$ or $\nu_\tau$ neutrinos.

With respect to the internal transformation $F(v)$, the topology of the non-Euclidean “vector triad” (see Sec. 2.6, Ref. 7, p. 57, Ref. 20, and the qualifying remarks in Footnote 19) representing the $\nu_e$ ($\nu_\mu$ or $\nu_\tau$), is found to be that of a cylinder (Möbius strip). And, assuming that a change in topology during neutrino mixing is suppressed by energy “barriers,” or other topological “barriers” (e.g., one cannot continuously deform a doughnut into a sphere), while neutrino mixing without topology-change is (relatively) enhanced, one can readily explain the experimental observation of (nearly) maximal $\nu_\mu - \nu_\tau$ neutrino mixing—at least maximal $\nu_\mu - \nu_\tau$ mixing over long distances, where the proposed topological influences are expected to be cumulative [19, 20]. If this qualitative explanation is basically correct, then it follows that the neutrino mass-matrix and associated mixing-parameters needed to explain bi-maximal neutrino mixing, would be the result, at least in part, of these deeper (internal) topological differences between neutrinos, and not their cause.

3 Summary and Conclusions

By insisting that the electric charge associated with $SU(2)$ flavor doublets, (i.e., weak isospin doublets) of fundamental fermions (quarks and leptons) can be treated as a real, internal 2-vector, considerations of self consistency dictate that all of the mathematical “machinery” associated with the linear vector 2-space (e.g., the metric, and various transformations) must be brought to bear when describing fundamental fermions. While this mathematical machinery is very simple, its application to fundamental fermions immediately leads to a (modest) extension of the standard model description of quarks and leptons, and to a number of predictions.

The model predicts, among other things, that unlike the situation in the standard model [3, 6], particles such as the $u$, $c$ and $t$ quarks are characterized by significant internal (topological and other) differences. Similar differences may help explain recent observations of (nearly) maximal $\nu_\mu - \nu_\tau$ mixing [17–20]. Moreover, while we began this paper with the introduction of an $SU(2)$ flavor doublet of “quarks” $|u\rangle, |d\rangle$ and their antiparticle coun-
terparts (see Sec. 1.0), very little else was assumed about quarks, leptons, internal (strong) colors or family replication. And yet, the subsequent 2-space description correctly predicts that all of these things, and more, are properties of fundamental fermions. For example, to maintain compatibility with the standard model, we were forced to (tentatively) conclude that certain global “charges” associated with the “accidental symmetries” of the Lagrangian describing strong and electroweak interactions, and various 2-space charges, are (essentially) one and the same charges (see the “duality” conjecture in Sec. 2.1.2). Finally, family replication emerged here as little more than the number of different physically acceptable ways (three physically acceptable ways are predicted) the vector $Q$ for quarks and leptons can be resolved in the 2-space (see Sec. 2.6 and Ref. 7, pp. 39–49) into pairs of linearly independent “basis” vectors $U$ and $V$ (i.e., $Q = U + V$).

In closing it should be pointed out that popular extensions of the standard model such as the so-called realistic (free fermionic) three-generation string models [21, 22], also provide an “explanation” for family replication and the number of families. Does this mean that in spite of very significant and obvious differences there are, nevertheless, “deep” connections between the proposed 2-space description and string theories?

I thank R. Zannelli for pointing out a problem with the existing 2-space description of muons (see Tables II and IV, and p. 57 in Ref. 7). This problem, together with my proposed “remedy,” are briefly described in [19].

References and Footnotes

[1] J. D. Jackson, Classical Electrodynamics, John Wiley & Sons, Inc., New York, 1962, p. 377; Paul Lorrain and Dale Corson, Electromagnetic Fields and Waves, (Second Edition), W. H. Freeman and Co., San Francisco, 1970, pp. 228–229.

[2] Of course, in external 4-D spacetime, observed scalars such as the electric charge of a quark or a lepton are taken to correspond to a projection of a real 2-vector on one of the two internal and orthogonal coordinate “axes” of such a 2-space.

[3] K. Huang, Quarks, Leptons and Gauge Fields, World Scientific Publishing Co., Singapore (1982), pp. 6–8.
[4] W. Greiner and B. Müller, *Quantum Mechanics (Symmetries)*, Springer-Verlag, Berlin, 1989, pp. 95–96; W. Heisenberg, Zeitschrift für Physik 77, 1 (1932); K. Huang, *Quarks, Leptons and Gauge Fields*, World Scientific, Singapore, 1982, pp. 12–14; A. W. Joshi, *Elements of Group Theory for Physics* (Third Edition), Wiley Eastern Limited, New Delhi, 1982, pp. 147–148; O. Nachtmann, *Elementary Particle Physics (Concepts and Phenomena)*, Springer-Verlag, Berlin, 1989, p. 185.

[5] If the 2-vector $Q = \{q_1, q_2\}$ characterizes the SU(2)-doublet of states $|u\rangle$ and $|d\rangle$, this same 2-vector must also characterize the states $|u\rangle$ and $|d\rangle$, individually. That is, each of the states $|u\rangle$ and $|d\rangle$ may be said to “carry” the vector $Q$. It is important to understand that while the assignment of an electric-charge vector $Q$ to each of the two quark states $|u\rangle$ and $|d\rangle$ adds information in the quantum sense, it does so in a way that does not violate quantum mechanics or special relativity. An analogous situation involves the isospin vector $\tau$ (see Sec. 1.1 in the main text). Note that the vector $Q$ is analogous to the vector $\tau$ in the sense that a single vector $\tau$, like $Q$, may be assigned to each of the quark states $|u\rangle$ and $|d\rangle$. Moreover, $\tau^2$ and $\tau_3$ are Lorentz 4-scalars that serve to (partially) define the states $|u\rangle$ and $|d\rangle$ via the states $|\tau^2, \tau_3 = \frac{1}{2}\rangle$ and $|\tau^2, \tau_3 = -\frac{1}{2}\rangle$, respectively. Similarly, the states $|u\rangle$ and $|d\rangle$ are (partially) defined by the Lorentz 4-scalars $Q^2$, $q_1$ and $q_2$ via the states $|Q^2, q_1\rangle$ and $|Q^2, q_2\rangle$, respectively. Here, it happens that $Q^2$ provides additional information (i.e., the baryon number $B = Q^2$) on the states $|u\rangle$ and $|d\rangle$.

[6] C. Quigg, *Gauge Theories of the Strong, Weak and Electromagnetic Interactions*, The Benjamin/Cummings Publishing Co., Reading, Mass., 1983. It is well known that the standard model adds “richness” to the description of SU(2) flavor-doublet states such as $|u\rangle$ and $|d\rangle$ by incorporating additional symmetries such as SU(3) color. We assume here that the added “richness” associated with the new 2-space description, not only incorporates many features of the standard model, but also leads to features that lie outside the domain of the standard model, i.e., to features that effectively extend the standard model.

[7] Gerald L. Fitzpatrick, *The Family Problem-New Internal Algebraic and Geometric Regularities*, Nova Scientific Press, Issaquah, WA (1997). Additional information: [http://physicsweb.org/TIPTOP/](http://physicsweb.org/TIPTOP/) or [http://www.amazon.com/exec/obidos/ISBN=0965569500](http://www.amazon.com/exec/obidos/ISBN=0965569500). In spite of
the many successes of the standard model of particle physics, the observed proliferation of matter-fields, in the form of “replicated” generations or families, is a major unsolved problem. This book proposes a new organizing principle for fundamental fermions, i.e., a minimalistic “extension” of the standard model based, in part, on the Cayley-Hamilton theorem for matrices. In particular, to introduce (internal) global degrees of freedom that are capable of distinguishing all observed flavors, the Cayley-Hamilton theorem is used to generalize the familiar standard-model concept of scalar fermion-numbers \( f \) (i.e., \( f_m = +1 \) for all fermions and \( f_a = -1 \) for all antifermions). This theorem states that every (square) matrix satisfies its characteristic equation. Hence, if \( f_m \) and \( f_a \) are taken to be the eigenvalues of some real matrix \( F(v) \)—a “generalized fermion number”—it follows from this theorem that both \( f \) and \( F(v) \) are square-roots of unity. Assuming further that the components of both \( F(v) \) and its eigenvectors are global charge-like quantum observables, and that \( F(v) \) “acts” on a (real) vector 2-space, both the form of \( F(v) \) and the 2-space metric are determined. The 2-space is found to have a “Lorentzian” or non-Euclidean metric, and various associated 2-scalars are found to serve as global flavor-defining “charges,” which can be identified with charges such as strangeness, charm, baryon and lepton numbers etc.. Hence, these global charges can be used to describe individual flavors (i.e., flavor eigenstates), flavor doublets and families. Moreover, because of the aforementioned non-Euclidean constraints, and certain standard-model constraints, these global charges are effectively-“quantized” in such a way that families are replicated. Finally, because these same constraints dictate that there are only a limited number of values these charges can assume, it is found that families always come in “threes.”

[8] G. L. Fitzpatrick, “Continuation of the Fermion Number Operator and the Puzzle of Families,” in the LANL physics e-Print archive [physics/0007038].

[9] Note that if \( Q_c = -Q \), then \( (Q_c)^2 = (-Q)^2 \) no matter what the metric is (i.e., \( s = \pm 1 \)). That is, if \( Q_c = -Q \), \( (Q_c)^2 \) and \( Q^2 \) are the same for matter and antimatter, so they do not transform like charges, and consequently do not distinguish between matter and antimatter.
The term “non-Euclidean geometry” is usually reserved for the geometry of curved spaces. However, to avoid confusing the flat “Lorentzian” 2-D geometry (and discrete transformations therein) with the flat Lorentzian 4-D geometry of spacetime (and continuous transformations therein), we prefer to use a term other than “Lorentzian” to describe the 2-space. In particular, because this 2-space is not Euclidean, we choose to break with tradition, and refer to this flat space as being “not-Euclidean,” or more correctly, non-Euclidean.

S. Weinberg, *The Quantum Theory of Fields, Vol. I, Foundations*, Cambridge University Press, New York, NY (1995), pp. 529–531; *The Quantum Theory of Fields, Vol. II, Modern Applications*, Cambridge University Press, New York, NY (1996), p. 155.

When weak interactions are “turned off” flavor eigenstates and mass eigenstates are one and the same. For the most part, when we speak here of flavor eigenstates, we are referring to the situation where flavor eigenstates and mass eigenstates are the same.

It is important to point out that the various C-reversing 2-scalars associated with the internal 2-space description are taken to be Lorentz 4-scalars in an external spacetime setting. That is, because $q_1$ and $q_2$ are Lorentz 4-scalars in 4-D spacetime, the components of the 2-vectors $\mathbf{U}$ and $\mathbf{V}$, and the scalar products $Q^2$, $U^2$, $V^2$, $\mathbf{U} \cdot \mathbf{V}$ are also Lorentz 4-scalars. This connection strengthens the idea that these numbers define flavors.

When we say that the vectors $\mathbf{Q}$, $\mathbf{U}$ and $\mathbf{V}$ are observables, we mean that their associated component-“charges” are mutually-commuting simultaneous observables. Hence, all of these charge-like components can be known in principle, at the same time, meaning that the vectors $\mathbf{Q}$, $\mathbf{U}$ and $\mathbf{V}$ can be known simultaneously. Thus the vector “triad” ($\mathbf{Q}, \mathbf{U}, \mathbf{V}$) is a well defined geometric object.

In Section 2.4 in the main text, it is shown that strong colors are more-or-less implicit in the 2-space description of quarks and leptons. However, the detailed connections between the 2-space and $SU(3)$ color, have yet to be worked out. And, because each flavor doublet of quarks or leptons is also a weak isospin doublet, there are other weak colors (i.e.,
weak isospin and weak hypercharge) that must be taken into account. Strictly speaking, then, besides the specification of global charges, the overall quantum state of a fundamental fermion would, necessarily, involve a specification of the spin state, the energy-momentum state and so on, together with a specification of the particular mix of local color (gauge)-charges $R, W, B, G$ and $Y$ carried by each fundamental fermion. This color-mix would be determined, in turn, by something like a complementary, local $SU(5)$ color-dependent gauge description.

[16] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory, Vols. 1 and 2*, Cambridge University Press, 1987.

[17] T. Kajita, for the Super-Kamiokande, Kamiokande Collaboration, in the LANL physics e-Print archive [hep–ex/9810001].

[18] H. Georgi and S. L. Glashow, “Neutrinos on Earth and in the Heavens,” in the LANL physics e-Print archive [hep–ph/9808293].

[19] With the possible exception of one family, all quarks and leptons within a given family are found to exhibit the same topology with respect to $F$. This is certainly the case for the first and third families. And, because of these facts, together with the requirement of quark-lepton “universality,” at least within any given family, we naturally expect this to be the case for the second family as well. It happens that the second-family $c$ and $s$ quarks exhibit Möbius topology with respect to $F$. However, strictly speaking, the second-family leptons, namely, the muon and its associated neutrino (and associated antiparticles) exhibit the requisite Möbius topology only if the components of the associated $V$-vectors are not exactly zero (see p. 57, and Tables II and IV in Ref. 7). Or, to put it another way, the associated $U$-vectors must fall slightly inside the physical lepton quadrant (quadrant II) of the 2-space (see Ref. 7, Chapter 3 and Sec. 4.2.2). Therefore, if we wish to maintain a kind of quark-lepton “universality” within families, we must assume that some physical mechanism exists, which ensures that the $V$-vector components associated with muons, though predicted to be very small, rarely if ever actually equal zero. This assumption might be justified in a more fundamental treatment where (hypothetical) quantum fluctuations (of the $V$-vector components) are taken into account. Such a proposal is encouraged by the fact that (small) fluctuations of this kind would have
no effect on the topology of other quarks and leptons, simply because their associated V-vector components are significantly different from zero to begin with (see Tables II and IV in Ref. 7).

[20] G. L. Fitzpatrick, “Topological Constraints on Long-Distance Neutrino Mixtures,” in the LANL physics e-Print archive [physics/0007039]. Equation (11) in this paper describes a hypothetical situation in which topology-maintaining (topology-changing) influences in three-flavor neutrino mixtures are dominant (nonexistent). Because the matrix element $a[b = (1 - a)/2]$ in equation (11) assumes its largest [smallest] possible value, namely unity [zero], the matrix element $a[b = (1 - a)/2]$ is taken to be a pure measure of topology-maintaining [topology-changing] influences in general. Hence, the product $f(a) = ab = a(1 - a)/2$ is taken to be a pure measure of the balance between topology-maintaining and topology-changing influences. This balance should occur at the maximum of $f(a)$, namely, when $a = \frac{1}{2}$. Given this “equilibrium” value of the matrix element $a$, one easily establishes that the (predicted) matrix describing long-distance three-flavor neutrino (equilibrium) mixtures is given by

$$M = \frac{1}{8} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}.$$

[21] A. E. Faraggi, “Towards the Classification of the Realistic Free Fermionic Models,” in the LANL physics e-Print archive [hep-th/9708112].

[22] G. B. Cleaver, A. E. Faraggi, D. V. Nanopoulos and T. ter Veldhuis, “Towards String Predictions,” in the LANL physics e-Print archive [hep-ph/0002292].