A $p$-adic monodromy theorem for de Rham local systems

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Abstract

We study horizontal semistable and horizontal de Rham representations of the absolute Galois group of a certain smooth affinoid over a $p$-adic field. In particular, we prove that a horizontal de Rham representation becomes horizontal semistable after a finite extension of the base field. As an application, we show that every de Rham local system on a smooth rigid analytic variety becomes horizontal semistable étale locally around every classical point. We also discuss potentially crystalline loci of de Rham local systems and cohomologically potentially good reduction loci of smooth proper morphisms.

1. Introduction

Let $p$ be a prime and let $K$ be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic $p$.

Fontaine introduced the notions of crystalline, semistable, and de Rham representations of the absolute Galois group of $K$ to study $p$-adic étale cohomology of algebraic varieties over $K$. It is now known that Galois representations arising from geometry in this way are de Rham. He also conjectured that every de Rham representation is potentially semistable. This reflects the semistable reduction conjecture stating that every smooth proper algebraic variety over $K$ acquires semistable reduction after a finite field extension. Fontaine’s conjecture was proved by Berger [Ber02] and is often referred to as the $p$-adic monodromy theorem.

The present paper discusses a similar question for families of $p$-adic Galois representations parametrized by a variety. More precisely, let $X$ be a smooth rigid analytic variety over $K$ (regarded as an adic space) and let $L$ be an étale $\mathbb{Z}_p$-local system on $X$. For every classical point $x \in X$ (i.e. a point whose residue class field $k(x)$ is finite over $K$) and a geometric point $\overline{x}$ above $x$, the stalk $L_{\overline{x}}$ is a Galois representation of $k(x)$. In this way, we regard $L$ as a family of Galois representations. In particular, we can ask whether $L$ is de Rham, semistable, etc., at every classical point. Note that building on earlier works [Fal88, Hyo89, Fal02, Bri08, AI13], Scholze [Sch13] defines the notion of de Rham local systems, and Liu and Zhu [LZ17] prove that $L$ is a de Rham local system if and only if $L$ is de Rham at every classical point of $X$ (or even at a single classical point on each connected component). In this context, we prove the following $p$-adic monodromy theorem.

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Theorem 1.1 (cf. Theorem 9.2). Let $L$ be a de Rham $\mathbb{Z}_p$-local system on a smooth rigid analytic variety $X$ over $K$. For every $K$-rational point $x \in X$, there exist an open neighborhood $U \subset X$ of $x$ and a finite extension $L$ of $K$ such that $L|_{U \otimes_K L}$ is semistable at every classical point. Moreover, if $L_x$ is potentially crystalline as a Galois representation of $k(x)$, then we can choose $U$ and $L$ so that $L|_{U \otimes_K L}$ is crystalline at every classical point.

We obtain a similar result for every classical point. Note that $X$ can have a proper open subset containing all the $K$-rational points or even all the classical points in general. Hence, the union of open sets $U$ in the theorem may not be the entire $X$.

Remark 1.2. Currently, crystalline or semistable local systems on $X$ are defined only when we fix an integral or formal model of $X$ (cf. [AI13, AI12, Fal89, Fal90, Fal02, TT19, Tsu]). This is the reason why our $p$-adic monodromy theorem concerns pointwise semistability.

Remark 1.3. Kisin [Kis99] studies torsion and $\ell$-adic étale local systems on $X$ for a prime $\ell \neq p$, and shows that around each $K$-rational point, such a local system is locally constant in the analytic topology of $X$. In contrast, $p$-adic étale local systems on $X$ are not necessarily locally constant in the analytic topology.

Remark 1.4. Theorem 1.1 is of local nature. We may also pose a global version of the $p$-adic monodromy theorem concerning semistability. Let $X$ be a smooth rigid analytic variety over $K$ and $L$ a de Rham $\mathbb{Z}_p$-local system on $X$. Does there exist a rigid analytic variety $X'$ finite étale over $X$ such that $L|_{X'}$ is semistable at every classical point? We have an affirmative answer in the following two cases:

(i) the case where $L$ is the $p$-adic Tate module of an abelian variety over $X$ (use Raynaud’s criterion [SGA7, Exposé IX, Proposition 4.7]);
(ii) the case where $L$ has a single Hodge–Tate weight [Shi18, Theorem 1.6].

On the other hand, Lawrence and Li [LL21] show that there exist an algebraic variety $X$ over $K$ and a de Rham $\mathbb{Z}_p$-local system $L$ on $X$ such that for every finite extension $L$ of $K$, $L|_{X \otimes_K L}$ is not semistable at some closed point.

Remark 1.5. It is an interesting attempt to formulate a semistable reduction conjecture in families over a field of mixed characteristic; Theorem 1.1 will be $p$-adic Hodge-theoretic evidence for such a conjecture. In the case of fields of characteristic zero, Abramovich and Karu [AK00] make a semistable reduction conjecture in families, and Adiprasito, Liu, and Temkin [ALT18] prove the conjecture of Abramovich and Karu in a larger generality.

Let us turn to applications of Theorem 1.1. Note that Liu and Zhu’s result implies that if an étale $\mathbb{Q}_p$-local system $L$ on $X$ is de Rham at a classical point on each connected component, then $L$ is de Rham at every classical point. Such a strong result does not hold for potentially crystalline representations. However, we prove that if $L$ is potentially crystalline at a classical point $x$, then it is so at every classical point in an open neighborhood of $x$.

Theorem 1.6 (Theorem 10.1). If $L$ is a de Rham $\mathbb{Q}_p$-local system on a smooth rigid analytic variety $X$ over $K$, then there exists an open subset $U \subset X$ such that the following assertions hold for every classical point $x \in X$:

(i) if $x \in U$, then $L$ is potentially crystalline at $x$;
(ii) if $x \not\in U$, then $L$ is not potentially crystalline at $x$.

The second application concerns the special case where $L$ comes from geometry. For example, suppose that we have a family of elliptic curves $f : E \to X$. For a point $x \in X(K)$, the elliptic
curve $E_x$ has good reduction if and only if the $j$-invariant of $E_x$ is a $p$-adic integer. The set of such points may well be called the (potentially) good reduction locus of $E$. In general, if we have a smooth proper family of algebraic varieties, the étale cohomology of the fibers yields both $\ell$-adic and $p$-adic étale local systems on the base. By combining known cases of $\ell$-adic/$p$-adic monodromy-weight conjecture and the $\ell$-independence with our result or Kisin’s result, we obtain the following theorem:

**Theorem 1.7 (Theorem 10.9).** Assume that the residue field of $K$ is finite. Let $f : Y \to X$ be a smooth proper morphism between smooth algebraic varieties over $K$ such that the relative dimension of $f$ is at most two. For each $m \in \mathbb{N}$, there exists an open subset $U$ of the adic space $X^{\text{ad}}$ associated to $X$ such that the following assertions hold for every classical point $x$ of $X^{\text{ad}}$:

1. If $x \in U$, then $R^mf_*\mathbb{Q}_\ell$ is potentially unramified at $x$ for $\ell \neq p$ and $R^mf_*\mathbb{Q}_p$ is potentially crystalline at $x$;
2. If $x \notin U$, then $R^mf_*\mathbb{Q}_\ell$ is not potentially unramified at $x$ for $\ell \neq p$ and $R^mf_*\mathbb{Q}_p$ is not potentially crystalline at $x$.

**Remark 1.8.** Imai and Mieda [IM20] introduce the notion of the potentially good reduction locus of a Shimura variety using automorphic local systems. They prove the existence of the potentially good reduction loci of Shimura varieties of preabelian type, and use these loci to study the étale cohomology of Shimura varieties.

We now explain ideas behind the proof of Theorem 1.1. We start with an example that motivates our proof. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a smooth proper morphism of smooth $p$-adic formal schemes over $\text{Spf} \mathcal{O}_K$, and let $f : Y \to X$ denote its adic generic fiber. Then $L := R^mf_*\mathbb{Q}_p$ is a de Rham $\mathbb{Q}_p$-local system. In fact, we even know that it is crystalline at every classical point (hence there is nothing to prove in this case).

Let $k$ denote the residue field of $K$. Let $z$ be a $k$-rational point in the special fiber of $\mathfrak{X}$ and let $U \subset X$ be the residue open disk of $z$. For $K$-rational points $x_1, x_2 \in U$, there is a canonical identification of $\varphi$-modules$^1$

$$D_{\text{cris}}(L_{\mathfrak{X}}) \cong H_{\text{cris}}^m(\mathfrak{Y}_z/W(k))[p^{-1}] \cong D_{\text{cris}}(L_{\mathfrak{Y}}).$$

One key property of $U$ is that the Gauss–Manin connection on the vector bundle $H^m_{\text{dr}}(Y/X)$ on $X$ has a full set of solutions when restricted to $U$ (cf. [Kat73] and [BO83, Remark 2.9]). Heuristically, this means that we can compare $\varphi$-modules associated to stalks of $L$ at different points on $U$ by parallel transport. Our proof articulates this idea despite the absence of integral models of $X$ or motives for $L$.

Let us return to the general setting of Theorem 1.1. Liu and Zhu [LZ17] introduce a natural functor $D_{\text{dr}}$ from the category of étale $\mathbb{Q}_p$-local systems on $X$ to the category of filtered vector bundles with integrable connections on $X$. When $X = \text{Spa}(K, \mathcal{O}_K)$, this agrees with Fontaine’s $D_{\text{dr}}$-functor for Galois representations of $K$. The functor commutes with arbitrary pullbacks, and an étale $\mathbb{Q}_p$-local system $L$ on $X$ is de Rham if and only if $\text{rank } D_{\text{dr}}(L) = \text{rank } L$. Note that in the above geometric situation, $D_{\text{dr}}^m(R^mf_*\mathbb{Q}_p)$ coincides with $H^m_{\text{dr}}(Y/X)$ equipped with the Gauss–Manin connection and the Hodge filtration.

Suppose that $L$ is a de Rham $\mathbb{Z}_p$-local system and fix a $K$-rational point $x$ of $X$. Since $X$ is smooth, the theory of $p$-adic differential equations over a polydisk (Theorem 9.7) tells us that there is an open neighborhood $U \subset X$ of $x$ such that $D^m_{\text{dr}}(L|_U)$ has a full set of solutions.

$^1$ They need not be isomorphic as filtered $\varphi$-modules, and thus $L_{\mathfrak{X}}$ and $L_{\mathfrak{Y}}$ may not be isomorphic as Galois representations of $K$. 

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By shrinking $U$ if necessary, we may further assume that $U$ is isomorphic to a rigid torus $\Spa(A, A^\circ)$, where $A^\circ := \mathcal{O}_K[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$, the $p$-adic completion of the Laurent polynomial ring $\mathcal{O}_K[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$, and $A := A^\circ[p^{-1}]$.

Let $\mathcal{G}_A$ denote the absolute Galois group of $A$. Following works of Colmez [Col02] and Brinon [Bri08], we define horizontal crystalline, semistable, and de Rham period rings $B^\nabla_{\text{max}}(A^\circ)$, $B^\nabla_{\text{st}}(A^\circ)$, $B^\nabla_{\text{dR}}(A^\circ)$. Using these period rings, we define the notions of horizontal crystalline, semistable, and de Rham representations of $\mathcal{G}_A$ (Definition 4.13). In our setting, $L|_U$ corresponds to a $p$-adic representation $V$ of $\mathcal{G}_A$. Since $L|_U$ is de Rham and $D_{\text{dR}}(L|_U)$ has a full set of solutions, $V$ is horizontal de Rham. Now the first part of Theorem 1.1 follows from a $p$-adic monodromy theorem for horizontal de Rham representations.

**Theorem 1.9 (Theorem 7.3).** If a $p$-adic representation $V$ of $\mathcal{G}_A$ is horizontal de Rham, then there exists a finite extension $L$ of $K$ such that $V|_{\mathcal{G}_A_L}$ is horizontal semistable, where $A_L := A \otimes_K L$.

The second part of Theorem 1.1 follows from the study of the monodromy operator (cf. Theorem 7.4).

To explain the proof of Theorem 1.9, we need some more notation. Let $\mathcal{O}_K$ denote the $p$-adic completion of the localization $A^\circ_{(\pi)}$, where $\pi$ is a uniformizer of $K$. Then $\mathcal{K} := \mathcal{O}_K[p^{-1}]$ is a complete discrete valuation field with imperfect residue field. We similarly have the notions of horizontal semistable and de Rham representations of the absolute Galois group of $\mathcal{K}$. In [Ohk13], Ohkubo shows that every horizontal de Rham representation becomes horizontal semistable after restricting to the absolute Galois group of the composite $L\mathcal{K}$ for some finite extension $L$ of $K$.

Hence, Theorem 1.9 follows from Ohkubo’s result and the following purity theorem for horizontal semistable representations of $\mathcal{G}_A$.

**Theorem 1.10 (Theorem 6.1).** If a horizontal de Rham representation of $\mathcal{G}_A$ is horizontal semistable when restricted to the absolute Galois group of $\mathcal{K}$, then it is also horizontal semistable as a representation of $\mathcal{G}_A$.

This is the key technical result of our paper, and the proof requires detailed analysis of period rings (e.g. Proposition 2.37). Our purity theorem and its proof are inspired by similar results in different contexts: a $p$-adic monodromy theorem for de Rham representations of the absolute Galois group of $\mathcal{K}$ with values in a reduced affinoid algebra by Berger and Colmez [BC08], and a purity theorem for crystalline local systems in the case of good reduction by Tsuji [Tsu, Theorem 5.4.8].

At this point, we emphasize that the main goal of our work is to develop $p$-adic Hodge theory of horizontal crystalline, semistable, and de Rham representations of $\mathcal{G}_A$. These classes of representations are much more restrictive than those of crystalline and de Rham representations studied by Brinon [Bri08]. However, it is still worth studying them. In fact, as we have explained, de Rham local systems come from horizontal de Rham representations (étale) locally around each classical point. For another instance, Faltings [Fal10] studies the admissible loci of certain $p$-adic period domains using horizontal crystalline representations (see also [Har13, Ked10b, Far18] for relevant works).

This paper establishes two results in $p$-adic Hodge theory of $\mathcal{G}_A$. First, we prove the $p$-adic monodromy theorem for horizontal de Rham representations as we have discussed. Second, we define Fontaine’s functors $D_{\text{pst}}$ and $V_{\text{pst}}$ in our setting (§5). We prove that these functors induce an equivalence of categories between the category of horizontal de Rham representations of $\mathcal{G}_A$ and that of admissible discrete filtered $(\varphi, N, \text{Gal}(K/K), A)$-modules (Theorem 5.10).
Finally, we explain the organization of this paper. The first part (§§2–7) develops $p$-adic Hodge theory for rings (i.e. representations of $G_A$) and the second part (§§8–10) studies $p$-adic Hodge theory for rigid analytic varieties.

In the first part, we need to define horizontal period rings such as $B_{dR}(A^o)$. We define them in two steps. We define horizontal period rings such as $B_{dR}(\Lambda, \Lambda^+)$ for every perfectoid Banach pair $(\Lambda, \Lambda^+)$ over the completed algebraic closure of $K$. For a base ring $R$ (e.g. $A^o$), the $p$-adic completion $\hat{R}$ of the integral closure of $R$ in the algebraic closure of $R[1/p]$ gives such a perfectoid Banach pair, and we set $B_{dR}(R) := B_{dR}(\hat{R}[1/p], \hat{R})$.

In §2, we define horizontal period rings for perfectoid Banach pairs, following [Col02, Bri08]. Most of the arguments in this section are standard in $p$-adic Hodge theory. One exception might be Proposition 2.37, which will be used crucially in the proof of the purity theorem in §6. Section 3 studies the class of base rings $R$ over which Brinon [Bri08] develops $p$-adic Hodge theory. In §4, we define horizontal crystalline, semistable, and de Rham representations. Section 5 introduces filtered $(\varphi, N, Gal(L/K), A)$-modules, and the functors $D_{pst}$ and $V_{pst}$. This section is not used in later sections. After these preparations, we state and prove the purity theorem in §6. We prove Theorem 1.9 in §7.

The second part starts with §8, where we review étale local systems on smooth rigid analytic varieties and period sheaves on the pro-étale site. Our $p$-adic monodromy theorem is proved in §9. Finally, we discuss potentially crystalline loci and cohomologically potentially good reduction loci in §10.

**Notation** 1.11. For a ring $S$ of characteristic $p$, the Frobenius map $x \mapsto x^p$ is denoted by $\varphi$. We say that $S$ is perfect if $\varphi$ is an isomorphism. In this case, $W(S)$ denotes the ring of $p$-typical Witt vectors of $S$. By functoriality, $W(S)$ admits the Witt vector Frobenius $W(\varphi)$, which we simply denote by $\varphi$.

A rigid analytic variety over $K$ refers to a quasi-separated adic space locally of finite type over Spa$(K, \mathcal{O}_K)$ (cf. [Hub94, Proposition 4.5(iv)]). For a rigid analytic variety $X$ over $K$, an open subset of $X$ refers to an open subset of $X$ as an adic space. In particular, open subsets of $X$ may not correspond to admissible open subsets in the sense of [BGR84, 9.1.4 Proposition 2(i)]. We write $\mathbb{B}_{dR}^n_K$ for the $n$-dimensional unit polydisk Spa$(K \langle T_1, \ldots, T_n \rangle, \mathcal{O}_K \langle T_1, \ldots, T_n \rangle)$ and write $T_{dR}^n_K$ for the $n$-dimensional rigid torus Spa$(K \langle T_1^{\pm 1}, \ldots, T_n^{\pm 1} \rangle, \mathcal{O}_K \langle T_1^{\pm 1}, \ldots, T_n^{\pm 1} \rangle)$. Definitions of étale $\mathbb{Z}_p$-local systems and étale $\mathbb{Q}_p$-local systems on $X$ are explained in §8.1.

## 2. Period rings

In this section, we introduce horizontal period rings $\mathbb{B}_{dR}, \mathbb{B}_{\text{max}}, \mathbb{B}_{st}$, etc., and study their basic properties. We will use these period rings to define horizontal de Rham, crystalline, and semistable representations in §4. Generalizing [Fon94a], Colmez defines these period rings for sympathetic algebras in [Col02], and we follow his approach. Proposition 2.37 is a key technical result used in the proof of the purity theorem in §6.

### 2.1 Review of Banach algebras and perfectoid pairs

In this subsection, let $L$ be a field that is complete with respect to a non-archimedean $p$-adic norm $\| \|$ satisfying $|p| = p^{-1}$.

**Definition 2.1.** For an $L$-Banach algebra $\Lambda$, write $\| \cdot \|_{\Lambda}$ for the norm. We denote by $\Lambda^o$ the set of power-bounded elements of $\Lambda$. We say that $\Lambda$ is uniform if $\Lambda^o$ is bounded
(cf. [KL15, Definition 2.8.1]). Note that $\Lambda$ is uniform if and only if the norm on $\Lambda$ is equivalent to its spectral (semi)norm.2

An $L$-Banach pair is a pair $(\Lambda, \Lambda^+)$ in which $\Lambda$ is an $L$-Banach algebra and $\Lambda^+$ is a ring of integral elements, that is, a subring of $\Lambda^0$ which is open and integrally closed in $\Lambda$.

**Remark 2.2.** If $(\Lambda, \Lambda^+)$ is an $L$-Banach pair and if $\Lambda$ is uniform, then $\Lambda^+$ is reduced, $p$-torsion-free, and $p$-adically complete and separated.

**Definition 2.3.** Assume further that $L$ is algebraically closed. An $L$-Banach pair $(\Lambda, \Lambda^+)$ is called perfectoid if $\Lambda$ is uniform and if the Frobenius morphism $x \mapsto x^p$ on $\Lambda^+/p\Lambda^+$ is surjective.

**Remark 2.4.** When $L$ is algebraically closed, an $L$-Banach pair $(\Lambda, \Lambda^+)$ is perfectoid if and only if $(\Lambda, \Lambda^0)$ is perfectoid, or equivalently, $\Lambda$ is a perfectoid $L$-algebra in the sense of [Sch12, Definition 5.1(i)]; see [KL15, Proposition 3.6.2]. In this case, we will simply say that $\Lambda$ is perfectoid (over $L$).

### 2.2 Period rings for perfectoid pairs

Let $k$ be a perfect field of characteristic $p$ and set $K_0 := W(k)[p^{-1}]$. Let $K$ be a totally ramified finite extension of $K_0$. We fix a uniformizer $\pi$ of $K$ and an algebraic closure $\overline{K}$ of $K$. Let $C$ denote the $p$-adic completion of $\overline{K}$. We denote by $\mathcal{O}_C$ the ring of integers of $C$.

**Set-up 2.5.** Let $(\Lambda, \Lambda^+)$ be a perfectoid $C$-Banach pair such that $\Lambda^+$ is an $\mathcal{O}_C$-algebra. Note that $\Lambda = \Lambda^+[p^{-1}]$.

**Remark 2.6.** When we discuss a morphism $(\Lambda, \Lambda^+) \to (\Lambda', \Lambda'^+)$ of such objects, we consider a morphism of $K$-Banach pairs, namely, a morphism $f: \Lambda \to \Lambda'$ of $K$-Banach algebras such that $f(\Lambda^+) \subset \Lambda'^+$. For example, we will consider $\sigma: (C, \mathcal{O}_C) \to (C, \mathcal{O}_C)$ for $\sigma \in \text{Gal}(\overline{K}/K)$.

We set $\Lambda^{+\varphi} := \lim_{\varphi} \Lambda^+/p\Lambda^+$, where $\varphi$ is the Frobenius of $\Lambda^+/p\Lambda^+$. By [KL15, Remark 3.4.10], this is a ring of characteristic $p$ (called the tilt of $\Lambda^+$) and there is a natural multiplicative monoid isomorphism $\lim_{\varphi} \varphi^\infty \Lambda^+ \overset{\cong}{\to} \Lambda^{+\varphi}$. We denote by $\Lambda^{+\varphi} \to \Lambda^+; x \mapsto x^\varphi$ the inverse of the above isomorphism followed by the first projection. Note that $\Lambda^{+\varphi}$ has a norm $\| \cdot \|$ defined by $\|x\| := |x^\varphi|$ and it is complete with respect to this norm [KL15, Lemma 3.4.5].

We set

$$\mathcal{A}_{\inf}(\Lambda^+) = \mathcal{A}_{\inf}(\Lambda, \Lambda^+) := W(\Lambda^{+\varphi}),$$

$$\mathcal{A}_{\inf,K}(\Lambda^+) = \mathcal{A}_{\inf,K}(\Lambda, \Lambda^+) := \mathcal{O}_K \otimes_{W(k)} \mathcal{A}_{\inf}(\Lambda^+).$$

By functoriality of $p$-typical Witt vectors, $\mathcal{A}_{\inf}(\Lambda^+)$ carries the Frobenius $\varphi$ lifting $x \mapsto x^p$. There exists a unique surjective $W(k)$-algebra homomorphism

$$\theta_{\mathcal{A}_{\inf}(\Lambda^+)}: \mathcal{A}_{\inf}(\Lambda^+) \to \Lambda^+$$

characterized by $\sum_{i=0}^{\infty} [x_i]_i := \sum_{i=0}^{\infty} x_i^p$, where $[x_i] \in \mathcal{A}_{\inf}(\Lambda^+)$ is the Teichmüller lift of $x_i$ (cf. [KL15, Lemma 3.2.2, Definition 3.4.3]). This map also extends to a surjective $\mathcal{O}_K$-algebra homomorphism $\theta_{\mathcal{A}_{\inf,K}(\Lambda^+)}: \mathcal{A}_{\inf,K}(\Lambda^+) \to \Lambda^+$.

Choose a compatible system $(p_m)_{m \in \mathbb{N}}$ of $p$-power roots of $p$ in $\overline{K}$, that is, $p_m \in \overline{K}$ with $p_0 = p$ and $p_{m+1} = p m$. Set $p_i := (p_0 \mod p, p_1 \mod p, \ldots) \in \mathcal{O}_C \subset \Lambda^{+\varphi}$. Similarly, choose a compatible system $(\pi_m)_{m \in \mathbb{N}}$ of $p$-power roots of $\pi$ in $\overline{K}$ and set $\pi_i := (\pi_0 \mod p, \pi_1 \mod p, \ldots) \in \mathcal{O}_C \subset \Lambda^{+\varphi}$.  

2 If $\Lambda$ is uniform, then $\Lambda$ with the spectral norm is uniform in the sense of [Ber90, p. 16].
Lemma 2.7 (cf. [KL15, Lemma 3.6.3]). The ideal \( \ker \theta_{a,\inf,K}(\Lambda^+)^{\dR} \) is principal and generated by \( [p] - p \). Similarly, \( \ker \theta_{a,\inf,K}(\Lambda^+)^{\dR} \) is principal with a generator \( [\pi^+] - \pi \).

Let us define the horizontal de Rham period rings.

**Definition 2.8.** We set
\[
\mathbb{B}^{+}_{\dR}(\Lambda^+) = \mathbb{B}^{+}_{\dR}(\Lambda, \Lambda^+) := \varprojlim \Lambda^{\inf}(\Lambda^+)/([\ker \theta_{\inf,K}(\Lambda^+)^{\dR}][p^{-1}])^m,
\]
\[
\mathbb{B}^{+}_{\dR,K}(\Lambda^+) = \mathbb{B}^{+}_{\dR,K}(\Lambda, \Lambda^+) := \varprojlim \Lambda^{\inf,K}(\Lambda^+)/([\ker \theta_{\inf,K}(\Lambda^+)^{\dR}][p^{-1}])^m.
\]

Note that there is a natural ring homomorphism \( \mathbb{B}^{+}_{\dR}(\Lambda^+)^{\dR} \to \mathbb{B}^{+}_{\dR,K}(\Lambda^+)^{\dR} \), and we will explain that this is an isomorphism.

The map \( \theta_{\inf,K}(\Lambda^+) \) extends to \( \theta_{\inf,K}(\Lambda^+)^{\dR} : \mathbb{B}^{+}_{\dR}(\Lambda^+) \to \Lambda^+ \). Since \( \ker \theta_{\inf,K}(\Lambda^+)^{\dR} \) is principal, \( \mathbb{B}^{+}_{\dR}(\Lambda^+) \) is complete with respect to \( \ker \theta_{\inf,K}(\Lambda^+)^{\dR} \). Similarly, \( \theta_{\inf,K}(\Lambda^+) \) extends to \( \theta_{\inf,K}(\Lambda^+)^{\dR} : \mathbb{B}^{+}_{\dR,K}(\Lambda^+) \to \Lambda^+ \), and \( \mathbb{B}^{+}_{\dR,K}(\Lambda^+) \) is complete with respect to \( \ker \theta_{\inf,K}(\Lambda^+)^{\dR} \).

Choose a compatible system \( (\varepsilon_m)_{m \in \mathbb{N}} \) of non-trivial \( p \)-power roots of unity in \( \overline{K} \), that is, \( \varepsilon_m \in \overline{K} \) with \( \varepsilon_0 = 1, \varepsilon_1 \neq 1, \) and \( \varepsilon_{m+1} = \varepsilon_m \). Let \( \varepsilon := (\varepsilon_0 \mod p, \varepsilon_1 \mod p, \ldots) \in \mathcal{O}_C \subset \Lambda^+ \). Note that \( \varepsilon - 1 \in \Lambda^{\inf}(\Lambda^+) \) satisfies \( \theta_{\inf,K}(\Lambda^+)(\varepsilon - 1) = 0 \). In particular, we define
\[
t := \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(|\varepsilon| - 1)^m
\]
as an element of \( \mathbb{B}^{+}_{\dR}(\Lambda^+) \).

The ideal \( \ker \theta_{\inf,K}(\Lambda^+) \) is generated by \( t \) (cf. [Bri08, Proposition 5.1.3]). Similarly, \( \ker \theta_{\inf,K}(\Lambda^+) \) is generated by \( t \). Combining this with the completeness of \( \mathbb{B}^{+}_{\dR}(\Lambda^+) \) and \( \mathbb{B}^{+}_{\dR,K}(\Lambda^+) \), we obtain the following lemma.

**Lemma 2.9.** The natural map \( \mathbb{B}^{+}_{\dR}(\Lambda^+) \to \mathbb{B}^{+}_{\dR,K}(\Lambda^+) \) is an isomorphism.

We identify these rings and denote them by \( \mathbb{B}^{+}_{\dR}(\Lambda^+) \) in what follows. In particular, \( \mathbb{B}^{+}_{\dR}(\Lambda^+) \) is a \( K \)-algebra.

**Lemma 2.10.** The natural map \( \Lambda^{\inf,K}(\Lambda^+) \to \mathbb{B}^{+}_{\dR}(\Lambda^+) \) is injective.

**Proof.** We can check that \( \Lambda^{\inf,K}(\Lambda^+) \cap \ker \theta_{\inf,K}(\Lambda^+)^{\dR} / (\ker \theta_{\inf,K}(\Lambda^+)^{\dR})^m = (\ker \theta_{\inf,K}(\Lambda^+)^{\dR})^m \). Moreover, since \( \Lambda^{\inf,K}(\Lambda^+)^{\dR} / (\pi) = \Lambda^+ \) has a norm \( | \cdot | \) satisfying \( |\pi| | \Lambda | > 0 \), it follows from Lemma 2.7 that \( \bigcap_m (\ker \theta_{\inf,K}(\Lambda^+)^{\dR})^m = 0 \). These claims imply the assertion.

**Lemma 2.11** (cf. [Bri08, Proposition 5.1.4]). The ring \( \mathbb{B}^{+}_{\dR}(\Lambda^+) \) is \( t \)-torsion-free.

**Definition 2.12.** Set \( \mathbb{B}^{+}_{\dR}(\Lambda^+)^{\dR} = \mathbb{B}^{+}_{\dR}(\Lambda, \Lambda^+) := \mathbb{B}^{+}_{\dR}(\Lambda^+)[t^{-1}] \). We equip \( \mathbb{B}^{+}_{\dR}(\Lambda^+)^{\dR} \) with the decreasing separated and exhaustive filtration defined by \( \text{Fil}^m \mathbb{B}^{+}_{\dR}(\Lambda^+)^{\dR} := t^m \mathbb{B}^{+}_{\dR}(\Lambda^+)^{\dR} \) for \( m \in \mathbb{Z} \).

**Definition 2.13.** Let \( \Lambda^{\inf}(\Lambda^+)[(\ker \theta)/p] \) be the \( \Lambda^{\inf}(\Lambda^+)^{\dR} \)-subalgebra of \( \Lambda^{\inf}(\Lambda^+)^{\dR} \) generated by \( p^{-1} \ker \theta_{\inf}(\Lambda^+) \). We define \( \Lambda_{\max}(\Lambda^+) = \Lambda_{\max}(\Lambda, \Lambda^+) \) to be the \( p \)-adic completion of \( \Lambda^{\inf}(\Lambda^+)[(\ker \theta)/p] \).

Similarly, let \( \Lambda^{\inf,K}(\Lambda^+)[(\ker \theta)/\pi] \) be the \( \Lambda^{\inf,K}(\Lambda^+)^{\dR} \)-subalgebra of \( \Lambda^{\inf,K}(\Lambda^+)^{\dR} \) generated by \( \pi^{-1} \ker \theta_{\inf,K}(\Lambda^+) \). We define \( \Lambda_{\max,K}(\Lambda^+) = \Lambda_{\max,K}(\Lambda, \Lambda^+) \) to be the \( p \)-adic completion of \( \Lambda^{\inf,K}(\Lambda^+)[(\ker \theta)/\pi] \).
Lemma 2.14. The Frobenius $\varphi$ on $A_{\inf}(\Lambda^+)$ extends to an endomorphism $\varphi$ on $A_{\inf}(\Lambda^+)$ $[(\text{Ker } \theta)/p]$ and thus on $A_{\max}(\Lambda^+)$.  

Proof. Since $\text{Ker } \theta_{A_{\inf}(\Lambda^+)}$ is generated by $[p^r] - p$, the assertion follows from the fact that  

$$
\varphi\left(\frac{[p^r] - p}{p}\right) = \frac{[p^r] - p}{p} = p^{r-1}\left(\frac{[p^r] - p}{p} + 1\right) \in A_{\inf}(\Lambda^+)\left[\frac{\text{Ker } \theta}{p}\right].
$$

\hfill \Box

Lemma 2.15. The rings $A_{\max}(\Lambda^+)$ and $A_{\max,K}(\Lambda^+)$ are $p$-torsion-free.  

Proof. Since $A_{\inf}(\Lambda^+)$ and $A_{\inf,K}(\Lambda^+)$ are $p$-torsion-free, so are $A_{\inf}(\Lambda^+)[(\text{Ker } \theta)/p]$ and $A_{\inf,K}(\Lambda^+)[(\text{Ker } \theta)/\pi]$. Hence, the lemma follows by taking $p$-adic completion.  

\hfill \Box

Definition 2.16. Set  

$$
\mathbb{B}^+_{\max}(\Lambda^+) = \mathbb{B}^+(\Lambda, \Lambda^+) := A_{\max}(\Lambda^+)[p^{-1}],
$$

$$
\mathbb{B}^+_{\max,K}(\Lambda^+) = \mathbb{B}^+(\Lambda, \Lambda^+) := A_{\max,K}(\Lambda^+)[p^{-1}].
$$

Lemma 2.17. The series $t = \sum_{m=1}^{\infty}(((-1)^{m-1}/m)(\varepsilon) - 1)m$ converges in $(1/p^r)A_{\max,K}(\Lambda^+)$ for a sufficiently large $r \in \mathbb{N}$. It also converges in $A_{\max}(\Lambda^+)$.  

Proof. Note that $[\varepsilon] - 1 \in \text{Ker } \theta_{A_{\max,K}(\Lambda^+)}$ and  

$$
\frac{(-1)^{m-1}}{m}(\varepsilon) - 1 = \frac{(-1)^{m-1}}{m}\frac{1}{\pi}
$$

Then the first assertion follows from the fact that $(-1)^{m-1}\pi^m/m$ approaches zero $p$-adically when $m \to \infty$. The proof of the second assertion is similar.  

\hfill \Box

Proposition 2.18. We have  

$$
\mathcal{O}_K \otimes_{\mathbb{W}(k)} A_{\inf}(\Lambda^+)\left[\frac{\text{Ker } \theta}{p}\right] \subset A_{\inf,K}(\Lambda^+)\left[\frac{\text{Ker } \theta}{p}\right] \subset \left(\frac{1}{p}\right)\left(\mathcal{O}_K \otimes_{\mathbb{W}(k)} A_{\inf}(\Lambda^+)\left[\frac{\text{Ker } \theta}{p}\right]\right).
$$

In particular, $K \otimes_{K_0} \mathbb{B}^+_{\max}(\Lambda^+) \cong \mathbb{B}^+_{\max,K}(\Lambda^+)$.  

Proof. Set $e = [K : K_0]$ and write $p = a\pi^e$ and $p^r = a^r(\pi^e)^e$ for some $a \in \mathcal{O}_K^\times$ and $a^r \in (\mathcal{O}_C^\times)^r$. For every $m \in \mathbb{N}$, write $m = qe + r$ with $0 \leq r < e$. Then we see that  

$$
\left(\frac{[p^r]}{p}\right)^m = \left(\frac{[\pi^r]}{\pi}\right)^{em}
$$

and  

$$
\left(\frac{[\pi^r]}{\pi}\right)^m = \frac{1}{p}\left(a^{[\pi^r] - 1}\frac{[p^r]}{p}\right)^q\frac{p}{\pi^r} [\pi^r]^r.
$$

The first assertion follows from these equalities and Lemma 2.7. Taking the $p$-adic completion of the first part and inverting $p$ yields the second assertion.  

\hfill \Box

Let us now turn to Colmez’s description of the horizontal period rings $\mathbb{B}^+_{dR}(\Lambda^+)$ and $\mathbb{B}^+_{\max,K}(\Lambda^+)$ as $K$-vector spaces [Col02, §8.4].

Construction 2.19. Fix a family of elements $(e_i)_{i \in I}$ of $\Lambda^+$ such that the images of the $e_i$ in $\Lambda^+/\pi\Lambda^+$ form a $k$-basis of $\Lambda^+/\pi\Lambda^+$. Then every element of $\Lambda$ is uniquely written as $\sum_{i \in I} a_ie_i$ with $a_i \in K$ such that for every $r > 0$ there are only finitely many $i$ with $|a_i|_\Lambda \geq r$. For each $i \in I$, choose $\tilde{e}_i \in A_{\inf,K}(\Lambda^+)$ such that $\theta_{A_{\inf,K}(\Lambda^+)}(\tilde{e}_i) = e_i$. Consider the map  

$$
s : \Lambda \to A_{\inf,K}(\Lambda^+)\left[p^{-1}\right]; \quad \sum_{i \in I} a_ie_i \mapsto \sum_{i \in I} a_i\tilde{e}_i.
$$

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This is a $K$-linear map and satisfies $\theta_{\text{inf},K}(\Lambda^+) \circ s = \text{id}$ and $s(\Lambda^+) \subset \mathcal{A}_{\text{inf},K}(\Lambda^+)$. In particular, $s$ is continuous with respect to $p$-adic topology.

Set $v = \pi^{-1}(\{\pi^3\} - \pi).$ Recall that $\mathbb{B}_d^+(\Lambda^+)$ is $\text{Ker} \theta_{\text{dr}}(\Lambda^+)$-adically complete and that $v$ generates $\text{Ker} \theta_{\text{dr}}(\Lambda^+)$. For $x \in \mathbb{B}_d^+(\Lambda^+)$, we define two sequences $(a_m(x))_{m \in \mathbb{N}}$ and $(b_m(x))_{m \in \mathbb{N}}$ with $a_m(x) \in \mathbb{B}^+_d(\Lambda^+)$ and $b_m(x) \in \Lambda$ by

$$a_0(x) = x,$$

$$b_m(x) = \theta_{\text{dr}}(\Lambda^+)(a_m(x)), \quad a_{m+1}(x) = \frac{1}{v}(a_m(x) - s(b_m(x))).$$

Define $\tilde{\theta}_{s,v} : \mathbb{B}_d^+(\Lambda^+) \to \Lambda[[X]]$ by $\tilde{\theta}_{s,v}(x) := \sum_{m=0}^{\infty} b_m(x)X^m$.

**Lemma 2.20.**

(i) The map $\tilde{\theta}_{s,v}$ is a $K$-linear isomorphism, and the inverse map is given by $\sum_{m=0}^{\infty} c_mX^m \mapsto \sum_{m=0}^{\infty} s(c_m)v^m$.

(ii) For $F(X) \in \mathcal{K}[[X]]$ and $x \in \mathbb{B}_d^+(\Lambda^+)$, we have $\tilde{\theta}_{s,v}(xF(v)) = \tilde{\theta}_{s,v}(x)F(X)$.

(iii) We have an equality $\tilde{\theta}_{s,v}(\mathcal{A}_{\text{inf},K}(\Lambda^+)) = \Lambda^+[[\pi X]]$.

(iv) The map $\tilde{\theta}_{s,v}$ induces natural identifications (as $K$-vector spaces)

$$\mathcal{A}_{\max,K}(\Lambda^+) \cong \Lambda^+\langle X \rangle \quad \text{and} \quad \mathbb{B}^+_d(\Lambda^+) \cong \Lambda(X).$$

Here $\Lambda^+\langle X \rangle$ denotes the $p$-adic completion of the polynomial algebra $\Lambda^+\{X\}$ and $\Lambda(X) := \Lambda^+\langle X \rangle[p^{-1}]$.

**Proof.** Part (i) follows from $\text{Ker} \theta_{\text{dr}}(\Lambda^+) = v\mathbb{B}_d^+(\Lambda^+)$, and part (ii) follows from the definition of $\tilde{\theta}_{s,v}$ and the fact that $\theta_{\text{dr}}(\Lambda^+)$ and $s$ are both $K$-linear.

We show part (iii). Recall from Lemma 2.7 that $\pi v = [\pi^3] - \pi$ generates $\text{Ker} \theta_{\text{inf},K}(\Lambda^+)$. For $x \in \mathcal{A}_{\text{inf},K}(\Lambda^+$, we see $b_0(x) \in \pi^m\Lambda^+$ by induction on $m$. Hence, $\tilde{\theta}_{s,v}(\mathcal{A}_{\text{inf},K}(\Lambda^+)) \subset \Lambda^+[[\pi X]]$. Since $\mathcal{A}_{\text{inf},K}(\Lambda^+)$ is $(\pi, [\pi^3])$-adically complete and thus $\pi v$-adically complete, the other inclusion follows from part (i).

For part (iv), first note that parts (ii) and (iii) imply

$$\tilde{\theta}_{s,v}\left(\mathcal{A}_{\text{inf},K}(\Lambda^+)\left[\frac{\text{Ker} \theta}{\pi}\right]\right) = \Lambda^+[[\pi X]][X].$$

Hence, the identification $\mathcal{A}_{\max,K}(\Lambda^+) \cong \Lambda^+\langle X \rangle$ follows from

$$\lim_{m} \Lambda^+\langle [\pi X] \rangle[X]/p^m\Lambda^+\langle [\pi X] \rangle[X] \cong \Lambda^+\langle X \rangle.$$

By inverting $p$, we obtain $\mathbb{B}^+_d(\Lambda^+) \cong \Lambda(X)$. \qed

We will show that there is an injective $K$-algebra homomorphism $\mathbb{B}^+_d(\Lambda^+) \to \mathbb{B}^+_d(\Lambda^+)$. For each $m$, $\mathbb{B}^+_d(\Lambda^+)_m := \mathcal{A}_{\text{inf},K}(\Lambda^+)\langle [p^{-1}] \rangle/\left(\text{Ker} \theta_{\text{inf},K}(\Lambda^+)\langle [p^{-1}] \rangle\right)$ is a $K$-Banach algebra. Hence, the map $\mathcal{A}_{\text{inf},K}(\Lambda^+)\langle [\text{Ker} \theta]/\pi \rangle \to \mathbb{B}^+_d(\Lambda^+)_m$ extends to $\mathcal{A}_{\max,K}(\Lambda^+) \to \mathbb{B}^+_d(\Lambda^+)_m$. By taking the inverse limit with respect to $m$ and inverting $p$, we obtain $\mathbb{B}^+_d(\Lambda^+) \to \mathbb{B}^+_d(\Lambda^+) = \mathbb{B}^+_d(\Lambda^+)$. We deduce the injectivity of the map from the description of these period rings in terms of $\tilde{\theta}_{s,v}$ as follows.

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Proposition 2.21. The map $\bar{\theta}_{s,v}$ induces identifications (as $K$-vector spaces)

$$
\begin{align*}
\mathbb{A}_{max,K}(\Lambda^+) & \longrightarrow \mathbb{B}_{max,K}^+(\Lambda^+) \longrightarrow \mathbb{B}_{dR}^+(\Lambda^+)
\end{align*}
$$

making the diagram commutative. In particular, the top horizontal maps are injective, and $\mathbb{B}_{max,K}^+(\Lambda)$ is $t$-torsion-free.

Proof. We have already proved that the vertical maps are all isomorphisms. Since the bottom horizontal maps are injective, so are the top horizontal maps. Since $\mathbb{B}_{dR}^+(\Lambda^+)$ is $t$-torsion-free, the last assertion follows. $\square$

Definition 2.22. We set

$$
\begin{align*}
\mathbb{B}_{max}(\Lambda^+) = \mathbb{B}_{max}(\Lambda, \Lambda^+) :&= \mathbb{B}_{max}^+(\Lambda^+)[t^{-1}], \\
\mathbb{B}_{max,K}(\Lambda^+) = \mathbb{B}_{max,K}(\Lambda, \Lambda^+) :&= \mathbb{B}_{max,K}^+(\Lambda^+)[t^{-1}].
\end{align*}
$$

Since $\varphi(t) = pt$, the Frobenius $\varphi$ on $\mathbb{A}_{max}(\Lambda^+)$ extends to an endomorphism on $\mathbb{B}_{max}(\Lambda^+)$, which we still call the Frobenius and denote by $\varphi$.

Let us define the horizontal semistable period rings. Consider the elements

$$
\log \left( \frac{[p^m]}{p^m} \right) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left( \frac{[p^m]}{p^m} - 1 \right), \quad \log \left( \frac{[\pi^m]}{\pi^m} \right) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left( \frac{[\pi^m]}{\pi^m} - 1 \right) \in \mathbb{B}_{dR}^+(\Lambda^+).
$$

Definition 2.23. Define $\mathbb{B}_{st}^+(\Lambda^+) = \mathbb{B}_{st}^+(\Lambda, \Lambda^+)$ to be the $\mathbb{B}_{max}^+(\Lambda^+)$-subalgebra of $\mathbb{B}_{dR}^+(\Lambda^+)$ generated by $\log([p^m]/p)$. Similarly, define $\mathbb{B}_{st,K}^+(\Lambda^+) = \mathbb{B}_{st,K}^+(\Lambda, \Lambda^+)$ to be the $\mathbb{B}_{max,K}^+(\Lambda^+)$-subalgebra of $\mathbb{B}_{dR}^+(\Lambda^+)$ generated by $\log([p^m]/p)$.

Lemma 2.24. The definitions of $\mathbb{B}_{st}^+(\Lambda^+)$ and $\mathbb{B}_{st,K}^+(\Lambda^+)$ are independent of the choice of a compatible system of $p$-power roots $(p_m)$ of $p$.

Proof. If $(p'_m)$ is a different choice, there exists $a \in \mathbb{Z}_p$ such that $p'_m = a m p_m$ for every $m \in \mathbb{N}$. It follows that $\log([p'^m]/p) = \log([p^m]/p) + at$ in $\mathbb{B}_{dR}^+(\Lambda^+)$ with $at \in \mathbb{B}_{max}^+(\Lambda^+)$.

Lemma 2.25. The ring $\mathbb{B}_{st,K}^+(\Lambda^+)$ coincides with the $\mathbb{B}_{max,K}^+(\Lambda^+)$-subalgebra of $\mathbb{B}_{dR}^+(\Lambda^+)$ generated by $\log([\pi^m]/\pi)$.

Proof. Set $e = [K : K_0]$. Write $p = a \pi^e$ and $p^\flat = a^\flat (\pi^e)^e$ for some $a \in \mathcal{O}_K^\times$ and $a^\flat \in (\mathcal{O}_K^\flat)^\times$. Then $[a]a^{-1} \in \mathbb{A}_{inf,K}(\Lambda^+)$ and $\theta_{\mathbb{A}_{inf,K}}(\Lambda^+)([a^\flat]a^{-1} - 1) = 0$. It follows that the series $\sum_{m=1}^{\infty} ((-1)^{m-1}/m) ([a^\flat]/a - 1) m$ converges in $\mathbb{A}_{max,K}(\Lambda^+)$. Let $\log([a^\flat]/a)$ denote the limit. We see that the image of $\log([a^\flat]/a)$ in $\mathbb{B}_{dR}^+(\Lambda^+)$ is given by the same series and $\log([p^m]/p) = \log([a^\flat]/a) + e \log([\pi^m]/\pi)$. Hence, the assertion follows.

We will describe $\mathbb{B}_{st,K}^+(\Lambda^+)$ via $\bar{\theta}_{s,v}$. For this we need the following fact.

Lemma 2.26. Let $A$ be a uniform $\mathbb{Q}_p$-Banach algebra. Then the power series $\log(1 + X) := \sum_{m=0}^{\infty} ((-1)^{m-1}/m)X^m \in A[[X]]$ is transcendental over $A(X)$. Hence, the $A(X)$-subalgebra

[^3]: Strictly speaking, our definitions differ slightly from the ones in [ColmG2, §8.6]. However, our horizontal semistable period rings are isomorphic to Colmez’s period rings by Corollary 2.28.
$A(X)[\log(1 + X)]$ of $A[[X]]$ generated by $\log(1 + X)$ is isomorphic to a polynomial algebra over $A(X)$.

**Proof.** Since $A$ is uniform, equip $A$ with its spectral norm. Then the Gel'fand transform $A \to \prod_{\nu \in \mathcal{M}(A)} \mathcal{H}(x)$ is isometric with each $\mathcal{H}(x)$ a field [Ber90, Corollary 1.3.2(ii)]. We can thus reduce the general case to the case where $A$ is a field. We may further assume that $A$ is complete and algebraically closed. In this case, the assertion is well known. In fact, suppose that there exist $f_0(X), \ldots, f_m(X) \in A(X)$ such that $f_0(X) + f_1(X) \log(1 + X) + \cdots + f_m(X)(\log(1 + X))^m = 0$. If $\zeta$ is a $p$-power root of unity, then $X = \zeta - 1$ is a root of $\log(1 + X)$, and thus it is also a root of $f_0(X)$. Since any non-zero element of $A(X)$ has at most finitely many roots in $\mathcal{O}_A$, we have $f_0(X) = 0$. By repeating the same argument, we conclude that $f_0(X), \ldots, f_m(X)$ are all zero.

**Proposition 2.27.** The map $\tilde{\theta}_{s,v}$ induces identifications (as $K$-vector spaces)

$$
\begin{align*}
\mathbb{B}_{\text{max},K}^+(\Lambda^+) & \xrightarrow{\cong} \mathbb{B}_{st,K}^+(\Lambda^+) & \mathbb{B}_{\text{dR}}^+(\Lambda^+) \\
\Lambda(X) & \xrightarrow{\cong} \Lambda(X)[\log(1 + X)] & \Lambda[[X]]
\end{align*}
$$

sending $\log([\pi^\beta]/\pi)$ to $\log(1 + X)$.

**Proof.** Recall that $v = [\pi^\beta]/\pi - 1$. Hence, we have $\tilde{\theta}_{s,v}(\log([\pi^\beta]/\pi)^m) = (\log(1 + X))^m$ by Lemma 2.20(ii). By Lemma 2.25, $\mathbb{B}_{st,K}^+(\Lambda^+) = \sum_{n=0}^\infty \mathbb{B}_{\text{max},K}^+(\Lambda^+)[\log([\pi^\beta]/\pi)^n]$, for $K$-vector subspaces of $\mathbb{B}_{\text{dR}}^+(\Lambda^+)$. It follows from Lemma 2.20(ii) and (iv) that $\tilde{\theta}_{s,v}(\mathbb{B}_{st,K}^+(\Lambda^+)) = \sum_{n=0}^\infty \Lambda(X)(\log(1 + X))^m \subset \Lambda[[X]]$.

**Corollary 2.28.** The $\mathbb{B}_{\text{max},K}^+(\Lambda^+)$-algebra homomorphism from the polynomial ring $\mathbb{B}_{\text{max},K}^+(\Lambda^+)[u]$ to $\mathbb{B}_{st,K}^+(\Lambda^+)$ sending $u$ to $\log([p^\beta]/p)$ is an isomorphism. Similarly, the $\mathbb{B}_{\text{max},K}^+(\Lambda^+)$-algebra homomorphism $\mathbb{B}_{\text{max},K}^+(\Lambda^+)[u] \to \mathbb{B}_{st,K}^+(\Lambda^+); u \mapsto \log([p^\beta]/p)$ is an isomorphism. In particular, $K \otimes_{K_0} \mathbb{B}_{\text{max},K}^+(\Lambda^+) \cong \mathbb{B}_{st,K}^+(\Lambda^+)$. **Proof.** The first assertion is a special case of the second. The second assertion follows from Lemmas 2.20(ii), 2.25, 2.26, and Proposition 2.27. The last assertion follows from the first two since $K \otimes_{K_0} \mathbb{B}_{\text{max},K}^+(\Lambda^+) \cong \mathbb{B}_{st,K}^+(\Lambda^+)$. 

**Definition 2.29.** We extend the Frobenius on $\mathbb{B}_{\text{max},K}^+(\Lambda^+)$ to the polynomial ring $\mathbb{B}_{\text{max},K}^+(\Lambda^+)[u]$ by setting $\varphi(u) = pu$. Then the $\mathbb{B}_{\text{max},K}^+(\Lambda^+)$-linear derivation $N = -d/du$ on $\mathbb{B}_{\text{max},K}^+(\Lambda^+)[u]$ satisfies $N\varphi = p\varphi N$. Via the $\mathbb{B}_{\text{max},K}^+(\Lambda^+)$-algebra isomorphism

$$
\mathbb{B}_{\text{max},K}^+(\Lambda^+)[u] \xrightarrow{\cong} \mathbb{B}_{st,K}^+(\Lambda^+); \quad u \mapsto \log\left[\frac{[p^\beta]}{p}\right],
$$

we define the Frobenius $\varphi$ and the monodromy operator $N$ on $\mathbb{B}_{st,K}^+(\Lambda^+)$. Note that $N\varphi = p\varphi N$, $\varphi((\log([p^\beta]/p)) = p(\log([p^\beta]/p))$, and $N((\log([p^\beta]/p))^m) = -m((\log([p^\beta]/p)))^m$. As in Lemma 2.24, we can check that the definitions of $\varphi$ and $N$ on $\mathbb{B}_{st,K}^+(\Lambda^+)$ are independent of the choice of a compatible system of $p$-power roots of $p$.

**Definition 2.30.** Set

$$
\begin{align*}
\mathbb{B}_{st}(\Lambda^+) &= \mathbb{B}_{st}(\Lambda,\Lambda^+) := \mathbb{B}_{st,K}^+(\Lambda^+)[t^{-1}], \\
\mathbb{B}_{st,K}(\Lambda^+) &= \mathbb{B}_{st,K}(\Lambda,\Lambda^+) := \mathbb{B}_{st,K}^+(\Lambda^+)[t^{-1}].
\end{align*}
$$
We have isomorphisms $\mathbb{B}_{\text{max}}(\Lambda^+)[u] \xrightarrow{\cong} \mathbb{B}_{\text{st}}(\Lambda^+)$ and $\mathbb{B}_{\text{max},K}(\Lambda^+)[u] \xrightarrow{\cong} \mathbb{B}_{\text{st},K}(\Lambda^+)$ sending $u$ to $\log([p^2]/p)$. Moreover, the Frobenius $\varphi$ (respectively, the monodromy operator $N$) on $\mathbb{B}_{\text{st}}^+(\Lambda^+)$ extends to a ring (respectively, $\mathbb{B}_{\text{max}}^+(\Lambda^+)$-linear) endomorphism on $\mathbb{B}_{\text{st}}(\Lambda^+)$, which we still denote by $\varphi$ (respectively, $N$).

**Lemma 2.31.** The sequence

$$0 \rightarrow \mathbb{B}_{\text{max}}(\Lambda^+) \rightarrow \mathbb{B}_{\text{st}}(\Lambda^+) \xrightarrow{N} \mathbb{B}_{\text{st}}(\Lambda^+) \rightarrow 0$$

is exact.

**Proof.** This follows easily from the isomorphism $\mathbb{B}_{\text{max}}(\Lambda^+)[u] \xrightarrow{\cong} \mathbb{B}_{\text{st}}(\Lambda^+)$ commuting with $N$.

We also introduce the horizontal crystalline period ring for completeness.

**Definition 2.32.** Let $\mathbb{A}_{\text{cris}}'(\Lambda^+)$ denote the PD envelope (enveloppe à puissances divisées) of $\mathbb{A}_{\text{inf}}(\Lambda^+)$ relative to the ideal $\text{Ker} \theta_{\mathbb{A}_{\text{inf}}(\Lambda^+)}$ compatible with the canonical PD structure on $p\mathbb{A}_{\text{inf}}(\Lambda^+)$. Define $\mathbb{A}_{\text{cris}}(\Lambda^+) = \mathbb{A}_{\text{cris}}'(\Lambda^+,\Lambda^+)$ to be the $p$-adic completion of $\mathbb{A}_{\text{cris}}'(\Lambda^+)$.

**Lemma 2.33.** The Frobenius $\varphi$ on $\mathbb{A}_{\text{inf}}(\Lambda^+)$ extends to a ring endomorphism $\varphi$ on $\mathbb{A}_{\text{cris}}(\Lambda^+)$. $\varphi(x) \equiv x^p \mod p\mathbb{A}_{\text{inf}}(\Lambda^+)$ for $x \in \mathbb{A}_{\text{inf}}(\Lambda^+)$, the Frobenius $\varphi$ on $\mathbb{A}_{\text{inf}}(\Lambda^+)$ extends to a ring endomorphism on $\mathbb{A}_{\text{cris}}'(\Lambda^+)$. Passing to the $p$-adic completion, it further extends to a ring endomorphism on $\mathbb{A}_{\text{cris}}(\Lambda^+)$. $\Box$

Note that we have a natural $\varphi$-equivariant inclusion $\mathbb{A}_{\text{cris}}(\Lambda^+) \hookrightarrow \mathbb{A}_{\text{max}}(\Lambda^+)$ (cf. [Bri08, p. 62]).

**Lemma 2.34.** We have $\varphi(\mathbb{A}_{\text{max}}(\Lambda^+)) \subset \mathbb{A}_{\text{cris}}(\Lambda^+) \subset \mathbb{A}_{\text{max}}(\Lambda^+)$. $\varphi(x) = x^p + py$ for some $y \in \mathbb{A}_{\text{inf}}(\Lambda^+)$. Then we have $\varphi(x/p) = (p-1)!x^{p} + y \in \mathbb{A}_{\text{cris}}'(\Lambda^+)$. Hence, we have $\varphi(\mathbb{A}_{\text{inf}}(\Lambda^+)[(\text{Ker} \theta)/p]) \subset \mathbb{A}_{\text{cris}}'(\Lambda^+)$. Taking $p$-adic completion yields the desired inclusion. $\Box$

**Definition 2.35.** Set

$$\mathbb{B}^+_{\text{cris}}(\Lambda^+) = \mathbb{B}^+_{\text{cris}}(\Lambda,\Lambda^+) := \mathbb{A}_{\text{cris}}(\Lambda^+)[p^{-1}],$$

$$\mathbb{B}_{\text{cris}}(\Lambda^+) = \mathbb{B}_{\text{cris}}(\Lambda,\Lambda^+) := \mathbb{B}_{\text{cris}}(\Lambda^+)[t^{-1}].$$

The Frobenius $\varphi$ on $\mathbb{A}_{\text{cris}}(\Lambda^+)$ naturally extends over these rings, and the previous lemma implies $\varphi(\mathbb{B}_{\text{max}}(\Lambda^+)) \subset \mathbb{B}_{\text{cris}}(\Lambda^+) \subset \mathbb{B}_{\text{max}}(\Lambda^+)$. $\Box$

**Remark 2.36.** The construction of all the period rings is functorial on perfectoid $C$-Banach pairs $(\Lambda,\Lambda^+)$ with morphisms of $K$-Banach pairs.

The following proposition will be heavily used in the proof of the purity theorem for horizontal semistable representations (Theorem 6.1).

**Proposition 2.37.** Let $(\Lambda',\Lambda'^+)$ be another perfectoid $C$-Banach pair such that $\Lambda'^+$ is an $\mathcal{O}_C$-algebra, and let $f^+: \Lambda^+ \rightarrow \Lambda'^+$ be an injective $\mathcal{O}_K$-algebra homomorphism with
p-torsion-free cokernel. Then the natural commutative diagram

$$
\begin{array}{ccc}
\mathcal{B}^+_{st,K}(\Lambda^+) & \longrightarrow & \mathcal{B}^+_{dR,K}(\Lambda^+) \\
\downarrow & & \downarrow \\
\mathcal{B}^+_{st,K}(\Lambda'^+) & \longrightarrow & \mathcal{B}^+_{dR,K}(\Lambda'^+)
\end{array}
$$

is Cartesian with injective vertical maps.

**Proof.** By construction of the period rings, we have a natural commutative diagram as in the statement. We will prove that it is Cartesian.

Let $f: \Lambda \to \Lambda'$ denote the induced $K$-algebra homomorphism from $f^+$ by inverting $p$. We also change the norm on $\Lambda$ (respectively, $\Lambda'$) so that $\Lambda^+$ (respectively, $\Lambda'^+$) becomes the unit ball. Then $f$ is a closed isometric embedding of $K$-Banach algebras; by $p$-torsion-freeness of $\text{Coker } f^+$, the induced map $\Lambda^+/p^n\Lambda^+ \to \Lambda'^+/p^n\Lambda'^+$ is injective for all $n \in \mathbb{N}$, and thus $f$ is closed and isometric.

The commutative diagram in question only depends on $f^+$ as an $\mathcal{O}_K$-algebra homomorphism and not on the entire $C$-Banach structures on $\Lambda$ and $\Lambda'$. Hence, by replacing the $C$-Banach structure $C \to \Lambda'$ by precomposing an isometric $K$-algebra automorphism $C \to C'$, we may further assume $f(\pi_n) = \pi_n$ for every $n \in \mathbb{N}$.

We will explain that Construction 2.19 is made compatibly with $\theta_{\mathcal{A}_{\inf,K}(\Lambda^+)}$ and $\theta_{\mathcal{A}_{\inf,K}(\Lambda'^+)}$. Consider the $K$-linear maps

$$
s: \Lambda \to \mathcal{A}_{\inf,K}(\Lambda^+)[p^{-1}]; \quad \sum_{i \in I} a_i e_i \mapsto \sum_{i \in I} a_i \tilde{e}_i,
$$

$$
s': \Lambda' \to \mathcal{A}_{\inf,K}(\Lambda'^+)[p^{-1}]; \quad \sum_{i \in I} a_i f(e_i) + \sum_{j \in J} b_j e'_j \mapsto \sum_{i \in I} a_i \mathcal{A}_{\inf,K}(f)(\tilde{e}_i) + \sum_{j \in J} b_j \tilde{e}'_j.
$$

Then we have $\mathcal{A}_{\inf,K}(f) \circ s = s' \circ f$.

Since $f(\pi_n) = \pi_n$ for every $n \in \mathbb{N}$, we have $\mathcal{A}_{\inf,K}(f)([\pi^s]) = [\pi^s]$. Hence, we apply Construction 2.19 to the maps $s$ and $s'$ with $v = \pi^{-1}([\pi^s] - \pi)$ and obtain the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{B}^+_{dR,K}(\Lambda^+) \overset{\tilde{\theta}_{s,v}}{\cong} & \Lambda[[X]] \\
\downarrow & & \downarrow \\
\mathcal{B}^+_{dR,K}(\Lambda'^+) \overset{\tilde{\theta}_{s',v}}{\cong} & \Lambda'[[X]]
\end{array}
$$

By these maps and Proposition 2.27, it suffices to show

$$
\Lambda[[X]] \cap (\Lambda'(X)[\log(1+X)]) = \Lambda(X)[\log(1+X)] \quad \text{inside } \Lambda'[[X]].
$$

This follows from the following lemma.

\[ \square \]
Lemma 2.38. Let $A$ be a $\mathbb{Q}_p$-Banach algebra and $B$ a closed subalgebra of $A$. If $g(X) \in \bigoplus_{i=0}^h A(X)(\log(1 + X))^i$ satisfies $g(X) \in B[[X]]$, then $g(X) \in \bigoplus_{i=0}^h B(X)(\log(1 + X))^i$.

Proof. This is stated and proved in the proof of [BC08, Lemma 6.3.1]. For the convenience of the reader, we give a variant of their proof suggested by the referee.

First we claim that if $f(X) \in A(X)$ satisfies $f(\zeta_{p^n} - 1) \in B \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})$ for every $n$ and every $p^n$th root of unity $\zeta_{p^n}$, then $f(X) \in B(X)$. To see this, take any $a \in A \setminus B$. By a consequence of the Hahn–Banach theorem (e.g. [Sch02, Corollary 9.3]), there exists a continuous $\mathbb{Q}_p$-linear map of the form $l: A \to A/B \to \mathbb{Q}_p$ such that $l(a) \neq 0$. Extend $l$ coefficientwise to a continuous $\mathbb{Q}_p$-linear map $l: A(X) \to \mathbb{Q}_p(X)$. By assumption and construction, $l(f(X)) \in \mathbb{Q}_p(X)$ satisfies $l(f)(\zeta_{p^n} - 1) = 0$ for every $n$ and $\zeta_{p^n}$. Hence, $l(f(X)) = 0$ and we conclude that $f(X) \in B(X)$.

Take any

$$g(X) = f^{(0)}(X) + f^{(1)}(X) \log(1 + X) + \cdots + f^{(h)}(X)(\log(1 + X))^h \in \bigoplus_{i=0}^{\overline{h}} A(X)(\log(1 + X))^i$$

with $g(X) \in B[[X]]$. Then we see that $f^{(0)}(\zeta_{p^n} - 1) = g(\zeta_{p^n} - 1) \in B \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})$ for every $n$ and $\zeta_{p^n}$. By the above claim, we obtain $f^{(0)}(X) \in B(X)$. Applying the same argument to $(g(X) - f^{(0)}(X))/\log(1 + X)$ yields $f^{(1)}(X) \in B(X)$. By induction, we conclude $f^{(i)}(X) \in B(X)$ for every $i$. \qed

3. Preliminaries on base rings

In this section, we review the class of base rings which is considered in [Bri08]. As in the previous section, let $k$ be a perfect field of characteristic $p$ and set $K_0 := W(k)[p^{-1}]$. Let $K$ be a totally ramified finite extension of $K_0$ and fix a uniformizer $\pi$ of $K$. We denote by $C$ the $p$-adic completion of an algebraic closure $\overline{K}$ of $K$.

Let $\mathcal{O}_K(T_1^{\pm 1}, \ldots, T_n^{\pm 1})$ denote the $p$-adic completion of $\mathcal{O}_K[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$.

Set-up 3.1 [Bri08, p. 7]. Let $\tilde{R}$ be a ring obtained from $\mathcal{O}_K(T_1^{\pm 1}, \ldots, T_n^{\pm 1})$ by a finite number of iterations of the following operations:

- $p$-adic completion of an étale extension;
- $p$-adic completion of a localization;
- completion with respect to an ideal containing $p$.

We further assume that either $\mathcal{O}_K(T_1^{\pm 1}, \ldots, T_n^{\pm 1}) \to \tilde{R}$ has geometrically regular fibers or $\tilde{R}$ has Krull dimension less than two, and that $k \to \tilde{R}/\pi \tilde{R}$ is geometrically integral. Let $R$ be an $\tilde{R}$-algebra such that:

(i) $R$ is an integrally closed domain that is finite and flat over $\tilde{R}$;
(ii) $R[p^{-1}]$ is étale over $\tilde{R}[p^{-1}]$;
(iii) $K$ is algebraically closed in $R[p^{-1}]$.

Remark 3.2. For the proof of the main theorem (Theorem 9.2), one only needs the case where $R = \mathcal{O}_K(T_1^{\pm 1}, \ldots, T_n^{\pm 1})$. See §9.4.

Definition 3.3. Following [Bri08, p. 9], we say that $R$ satisfies (BR) if $R = \tilde{R}$. This is a condition on good reduction (bonne réduction).

Set $R_K := R[p^{-1}]$. Fix an algebraic closure $\text{Frac} \: \overline{R}$ of $\text{Frac} \: R$ containing $\overline{K}$. Let $\overline{R}$ be the union of all finite $\overline{R}$-subalgebras $\overline{R'}$ of $\text{Frac} \: \overline{R}$ such that $\overline{R'}[p^{-1}]$ is étale over $R_K$. Note that $\overline{R}$ is
a normal domain. Since $\widetilde{R}[p^{-1}]$ is the union of all finite Galois extensions of $R_K$, we set
\[
\mathcal{G}_{R_K} := \text{Gal}(\widetilde{R}[p^{-1}]/R_K).
\]
Let $\widehat{\mathcal{R}}$ denote the $p$-adic completion of $\mathcal{R}$. Then $\widehat{\mathcal{R}}$ is a $p$-torsion-free $O_C$-algebra, the map $\mathcal{R} \rightarrow \widehat{\mathcal{R}}$ is injective, and $\mathcal{R} \cap p\widehat{\mathcal{R}} = p\mathcal{R}$ [Bri08, Proposition 2.0.3]. The action of $\mathcal{G}_{R_K}$ on $\mathcal{R}$ extends to $\widehat{\mathcal{R}}$ continuously.

We will explain that $\widehat{\mathcal{R}}$ is naturally a perfectoid $C$-Banach pair and thus the results in §2 apply. For $x \in \widehat{\mathcal{R}}[p^{-1}]$, set $|x| = \inf\{|a|_C | a \in C, x \in a\mathcal{R}\}$. Here we normalize the norm $|\cdot|_C$ on $C$ by $|p|_C = p^{-1}$. The proof of the following lemma is clear and left to the reader.

**Lemma 3.4.** The function $|\cdot|$ on $\widehat{\mathcal{R}}[p^{-1}]$ is a $C$-algebra norm making $\widehat{\mathcal{R}}[p^{-1}]$ a $C$-Banach algebra. Moreover, $\widehat{\mathcal{R}}$ is an open subring of $\widehat{\mathcal{R}}[p^{-1}]$, and the induced topology on $\widehat{\mathcal{R}}$ agrees with the $p$-adic topology.

**Lemma 3.5.** The $C$-Banach algebra $\widehat{\mathcal{R}}[p^{-1}]$ is uniform.

**Proof.** Since $p^{-1}\mathcal{R}$ is a bounded subset, it suffices to show $(\widehat{\mathcal{R}}[p^{-1}])^\circ \subset p^{-1}\mathcal{R}$.

First we prove $(\widehat{\mathcal{R}}[p^{-1}])^\circ \cap \mathcal{R}[p^{-1}] \subset p^{-1}\mathcal{R}$. Take $x \in (\widehat{\mathcal{R}}[p^{-1}])^\circ \cap \mathcal{R}[p^{-1}]$. Then the sequence $(p^n x^n)_n$ converges to zero as $n \rightarrow \infty$. In particular, $(px)^n \in \mathcal{R}$ for some $n$. Since $\mathcal{R}$ is a normal domain, we have $px \in \mathcal{R}$ and thus $x \in p^{-1}\mathcal{R}$.

For $x \in (\widehat{\mathcal{R}}[p^{-1}])^\circ$, write $x = u + v$ ($u \in \mathcal{R}[p^{-1}]$, $v \in \widehat{\mathcal{R}}$). Then $u = x - v \in (\widehat{\mathcal{R}}[p^{-1}])^\circ$. Hence, $u \in p^{-1}\mathcal{R}$ by what we have proved, and thus $x \in p^{-1}\mathcal{R}$. \hfill $\square$

**Lemma 3.6.** The subring $\widehat{\mathcal{R}}$ of $\widehat{\mathcal{R}}[p^{-1}]$ is a ring of integral elements.

**Proof.** We need to show that $\widehat{\mathcal{R}}$ is integrally closed in $\widehat{\mathcal{R}}[p^{-1}]$.

Let $x \in \widehat{\mathcal{R}}[p^{-1}]$ be an element that is integral over $\widehat{\mathcal{R}}$ and suppose that $x$ satisfies $x^m + a_{m-1}x^{m-1} + \cdots + a_0 = 0$ for some $a_0, \ldots, a_{m-1} \in \widehat{\mathcal{R}}$. For each $l \in \mathbb{N}$, there exist $y_l \in \mathcal{R}[p^{-1}]$ and $z_l \in p^l\mathcal{R}$ such that $x = y_l + z_l$. Similarly, for each $i = 0, \ldots, m-1$, there exists $a_{i,l} \in \mathcal{R}$ such that $a_i - a_{i,l} \in p^l\mathcal{R}$. We compute

\[
|y_l^m + a_{m-1,l}y_l^{m-1} + \cdots + a_{0,l}| \\
\leq \max\{|y_l^m + a_{m-1,l}y_l^{m-1} + \cdots + a_{0,l}| - (x^m + a_{m-1,l}x^{m-1} + \cdots + a_{0,l})|, \\
|\frac{x^m + a_{m-1,l}x^{m-1} + \cdots + a_{0,l}}{x^m + a_{m-1,l}x^{m-1} + \cdots + a_{0,l}}| - (x^m + a_{m-1,l}x^{m-1} + \cdots + a_{0,l})|\} \\
\leq |p^l| \max\{1, |x|^{m-1}\}.
\]

Since $\{y \in \widehat{\mathcal{R}}[p^{-1}] | |y| < 1\} \subset \widehat{\mathcal{R}}$ and $\mathcal{R}[p^{-1}] \cap \widehat{\mathcal{R}} = \mathcal{R}$, we see that $y_l^m + a_{m-1,l}y_l^{m-1} + \cdots + a_{0,l} \in \mathcal{R}$ for $l \gg 0$. For such $l$, we have $y_l \in \mathcal{R}$ by normality of $\mathcal{R}$ and thus $x = y_l + z_l \in \widehat{\mathcal{R}}$. \hfill $\square$

**Lemma 3.7.** The $C$-Banach pair $(\widehat{\mathcal{R}}[p^{-1}], \widehat{\mathcal{R}})$ is perfectoid.

**Proof.** It remains to prove that the Frobenius is surjective on $\widehat{\mathcal{R}}/p\widehat{\mathcal{R}} = \mathcal{R}/p\mathcal{R}$. Let $\varphi \in \mathcal{R}/p\mathcal{R}$ and choose a lift $x \in \mathcal{R}$ of $\varphi$. Set $R' := R[x][T]/(T^p - T^p + pT - x)$. Then $R'$ is finite over $R$ and $R'[p^{-1}]$ is étale over $R_K$. Choose an embedding of $R'$ into $\text{Frac} \mathcal{R}$ and let $y$ denote the image of $T$. Then $y \in \mathcal{R}$ and it satisfies $(y^p)^p \equiv x \bmod p\mathcal{R}$. Hence, the Frobenius on $\mathcal{R}/p\mathcal{R}$ is surjective. \hfill $\square$

\footnote{Our $\mathcal{G}_{R_K}$ is denoted by $\mathcal{G}_R$ in [Bri08].}
4. Horizontal crystalline, semistable, and de Rham representations

In this section, we define horizontal crystalline, semistable, and de Rham representations of $G_{R_K}$, and study their basic properties. We mainly follow [Bri08, §8], where horizontal crystalline and de Rham representations are studied.

4.1 Admissible representations

We review the notion of admissible representations. Let $E$ be a finite extension of $\mathbb{Q}_p$ and $G$ be a topological group.

**Definition 4.1.** An $E$-representation of $G$ is a finite-dimensional $E$-vector space equipped with a continuous action of $G$. A morphism of $E$-representations of $G$ is an $E$-linear map commuting with actions of $G$. We denote the category of $E$-representations of $G$ by $\text{Rep}_E(G)$. It is naturally a Tannakian category.

**Remark 4.2.** Note that [Fon94b, §1] deals with an abstract field $E$ and an abstract group $G$ but that properties discussed there continue to hold in our setting.

**Definition 4.3** (cf. [Fon94b, 1.3]).

(i) An $(E,G)$-ring is an $E$-algebra $B$ equipped with an $E$-linear action of $G$.

(ii) Let $B$ be an $(E,G)$-ring and let $V \in \text{Rep}_E(G)$. We set $D_B(V) := (B \otimes_E V)^G$, and we let

$$\alpha_B(V) : B \otimes_{B^G} D_B(V) \to B \otimes_E V$$

denote the $B$-linear map induced from the inclusion $D_B(V) \subset B \otimes_E V$. We say that $V$ is $B$-admissible if $\alpha_B(V)$ is an isomorphism of $B$-modules.

**Lemma 4.4.** Let $\alpha : G' \to G$ be a continuous group homomorphism between topological groups. Let $B$ be an $(E,G)$-ring and $B'$ an $(E,G')$-ring equipped with an $E$-algebra homomorphism $\beta : B \to B'$ satisfying

$$\beta(\alpha(g') \cdot b) = g' \cdot \beta(b) \quad \text{for all } g' \in G' \text{ and } b \in B.$$

Let $V \in \text{Rep}_E(G)$ and let $V|_{G'}$ denote the corresponding $E$-representation of $G'$. Assume that $V$ is $B$-admissible and $\alpha_B'(V|_{G'})$ is injective. Then $V|_{G'}$ is $B'$-admissible. Moreover, if $(B')^{G'}$ is a field, then the natural map $(B')^{G'} \otimes_{B^G} D_B(V) \to D_{B'}(V|_{G'})$ is an isomorphism.

**Proof.** Consider the following commutative diagram.

$$\begin{array}{ccc}
B \otimes_{B^G} D_B(V) & \xrightarrow{\alpha_B(V)} & B \otimes_E V \\
\downarrow & & \downarrow \\
B' \otimes_{(B')^{G'}} D_{B'}(V|_{G'}) & \xrightarrow{\alpha_{B'}(V|_{G'})} & B' \otimes_{E} V
\end{array}$$

It follows that $V \subset B' \otimes_E V$ is contained in the image of $\alpha_{B'}(V|_{G'})$. Since $\alpha_{B'}(V|_{G'})$ is injective and $B'$-linear, it is an isomorphism. Hence, $V|_{G'}$ is $B'$-admissible. It also follows that the left
vertical map induces a $G'$-equivariant isomorphism

$$B' \otimes_{(B')^{G'}} \left( (B')^{G'} \otimes_{B} D_{B}(V) \right) = B' \otimes_{B} \left( B \otimes_{B} D_{B}(V) \right) \xrightarrow{\cong} B' \otimes_{(B')^{G'}} D_{B'}(V|_{G'})\).$$

Now assume that $(B')^{G'}$ is a field. Then taking $G'$-invariants of the above isomorphism yields an isomorphism $(B')^{G'} \otimes_{B} D_{B}(V) \xrightarrow{\cong} D_{B'}(V|_{G'})$.

\[\square\]

4.2 Horizontal crystalline, semistable, and de Rham representations

Let $R$ be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1. Recall that we set $\mathcal{G}_{R_{K}} := \text{Gal}(\overline{K}/K_{R})$ and that $(\overline{K}[p^{-1}], \overline{R})$ is a perfectoid $C$-Banach pair.

**Definition 4.5.** Set

- $B_{\text{max}}^{\nabla}(R) := \mathbb{B}_{\text{max}}(\overline{R})$,
- $B_{\text{max},K}^{\nabla}(R) := \mathbb{B}_{\text{max},K}(\overline{R})$,
- $B_{\text{cris}}^{\nabla}(R) := \mathbb{B}_{\text{cris}}(\overline{R})$,
- $B_{\text{st}}^{\nabla}(R) := \mathbb{B}_{\text{st}}(\overline{R})$,
- $B_{\text{st},K}^{\nabla}(R) := \mathbb{B}_{\text{st},K}(\overline{R})$,
- $B_{\text{dr}}^{\nabla}(R) := \mathbb{B}_{\text{dr}}(\overline{R})$, and
- $B_{\text{dr}}^{\nabla,+}(R) := \mathbb{B}_{\text{dr}}^{+}(\overline{R})$.

By functoriality of the construction, all these period rings have an action of $\mathcal{G}_{R_{K}}$. In particular, they are $(\mathbb{Q}_p, \mathcal{G}_{R_{K}})$-rings. To simplify the notation, for $V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_{K}})$, we write $D_{\text{max}}^{\nabla}(V)$ for $D_{\text{max}}^{\nabla}(r)(V)$ and $\alpha_{\text{max}}^{\nabla}(V)$ for $\alpha_{\text{max}}^{\nabla}(r)(V)$. We use similar conventions for other period rings.

**Remark 4.6.**

(i) Our period rings $B_{\text{max}}^{\nabla}(R)$, $B_{\text{cris}}^{\nabla}(R)$, and $B_{\text{dr}}^{\nabla}(R)$ coincide with the ones defined in [Bri08, Definitions 5.1.2, 6.1.2] and denoted by the same symbols.

(ii) The period rings $B_{\text{max}}^{\nabla}(R)$ and $B_{\text{cris}}^{\nabla}(R)$ admit the Frobenius $\varphi$. Similarly, $B_{\text{st}}^{\nabla}(R)$ admits the Frobenius $\varphi$ and the monodromy operator $N$ satisfying $N\varphi = p_{\varphi}N$. The action of $\mathcal{G}_{R_{K}}$ on $B_{\text{max}}^{\nabla}(R)$, $B_{\text{cris}}^{\nabla}(R)$, and $B_{\text{st}}^{\nabla}(R)$ commutes with these endomorphisms. In particular, for $V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_{K}})$, the modules $D_{\text{max}}^{\nabla}(V)$ and $D_{\text{cris}}^{\nabla}(V)$ (respectively, $D_{\text{st}}^{\nabla}(V)$) are equipped with the Frobenius $\varphi$ (respectively, the Frobenius $\varphi$ and the monodromy operator $N$). We have $\varphi(D_{\text{max}}^{\nabla}(V)) \subset D_{\text{cris}}^{\nabla}(V) \subset D_{\text{st}}^{\nabla}(V)$ and $(D_{\text{st}}^{\nabla}(V))^{N=0} = D_{\text{max}}^{\nabla}(V)$.

**Proposition 4.7** [Bri08, Corollary 5.3.7]. We have $(B_{\text{dr}}^{\nabla}(R))^\mathcal{G}_{R_{K}} = K$.

**Corollary 4.8.** We have

$$\begin{align*}
(B_{\text{max},K}^{\nabla}(R))^\mathcal{G}_{R_{K}} &= (B_{\text{st},K}^{\nabla}(R))^\mathcal{G}_{R_{K}} = K, \\
(B_{\text{cris}}^{\nabla}(R))^\mathcal{G}_{R_{K}} &= (B_{\text{max}}^{\nabla}(R))^\mathcal{G}_{R_{K}} = (B_{\text{st}}^{\nabla}(R))^\mathcal{G}_{R_{K}} = K_0.
\end{align*}$$

**Proof.** Since we have $\mathcal{G}_{R_{K}}$-equivariant $K$-algebra homomorphisms

$$K \otimes_{K_0} B_{\text{max}}^{\nabla}(R) \xrightarrow{\cong} B_{\text{max},K}^{\nabla}(R) \hookrightarrow B_{\text{dr}}^{\nabla}(R),$$

we obtain $(B_{\text{max},K}^{\nabla}(R))^\mathcal{G}_{R_{K}} = K$, and $(B_{\text{max}}^{\nabla}(R))^\mathcal{G}_{R_{K}} = K_0$. By the inclusions $\varphi(B_{\text{max}}^{\nabla}(R)) \subset B_{\text{cris}}^{\nabla}(R) \subset B_{\text{max}}^{\nabla}(R)$, we have $(B_{\text{cris}}^{\nabla}(R))^\mathcal{G}_{R_{K}} = K_0$. We can prove the remaining assertions in the same way. $\square$

The following result is clear from what we have discussed.
**Lemma 4.9.** Let $V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$.

(i) $D_{\text{max}}^\nabla(V)$ and $D_{\text{st}}^\nabla(V)$ are $K_0$-vector spaces, and $D_{\text{max},K}^\nabla(V)$, $D_{\text{st},K}^\nabla(V)$, and $D_{\text{dr}}^\nabla(V)$ are $K$-vector spaces.

(ii) We have $K \otimes_{K_0} D_{\text{max}}^\nabla(V) \cong D_{\text{max},K}^\nabla(V)$ and $K \otimes_{K_0} D_{\text{st}}^\nabla(V) \cong D_{\text{st},K}^\nabla(V)$. Moreover, $\alpha_{\text{max}}^\nabla(V)$ (respectively, $\alpha_{\text{st}}^\nabla(V)$) is an isomorphism if and only if $\alpha_{\text{max},K}^\nabla(V)$ (respectively, $\alpha_{\text{st},K}^\nabla(V)$) is an isomorphism.

(iii) We have injective $K$-linear maps $D_{\text{max},K}^\nabla(V) \hookrightarrow D_{\text{st},K}^\nabla(V) \hookrightarrow D_{\text{dr}}^\nabla(V)$.

**Lemma 4.10.** For every $V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, the maps $\alpha_{\text{cris}}^\nabla(V)$, $\alpha_{\text{max}}^\nabla(V)$, $\alpha_{\text{max},K}^\nabla(V)$, $\alpha_{\text{st},K}^\nabla(V)$, and $\alpha_{\text{dr}}^\nabla(V)$ are all injective. Moreover, we have

$$\dim_{K_0} D_{\text{max}}^\nabla(V) = \dim_{K_0} D_{\text{max},K}^\nabla(V) \leq \dim_{K_0} D_{\text{st}}^\nabla(V) = \dim_{K_0} D_{\text{st},K}^\nabla(V) \leq \dim_K D_{\text{dr}}^\nabla(V) \leq \dim_{\mathbb{Q}_p} V.$$

**Proof.** The map $\alpha_{\text{dr}}^\nabla(V)$ is injective by [Bri08, Proposition 8.2.9]. Consider the following commutative diagram.

$$
\begin{array}{ccc}
K \otimes_{K_0} (B_{\text{max}}^\nabla(R) \otimes_{K_0} D_{\text{max}}^\nabla(V)) & \xrightarrow{K \otimes_{K_0} \alpha_{\text{max}}^\nabla(V)} & K \otimes_{K_0} (B_{\text{max}}^\nabla(R) \otimes_{\mathbb{Q}_p} V) \\
\downarrow & & \downarrow \\
B_{\text{max},K}^\nabla(R) \otimes_K D_{\text{max},K}^\nabla(V) & \xrightarrow{\alpha_{\text{max},K}^\nabla(V)} & B_{\text{max},K}^\nabla(R) \otimes_{\mathbb{Q}_p} V \\
\downarrow & & \downarrow \\
B_{\text{dr}}^\nabla(R) \otimes_K D_{\text{dr}}^\nabla(V) & \xrightarrow{\alpha_{\text{dr}}^\nabla(V)} & B_{\text{dr}}^\nabla(R) \otimes_{\mathbb{Q}_p} V
\end{array}
$$

The vertical equalities follow from Lemma 4.9(ii). Hence, the injectivity of $\alpha_{\text{dr}}^\nabla(V)$ implies that of $\alpha_{\text{max}}^\nabla(V)$ and $\alpha_{\text{max},K}^\nabla(V)$. The same argument works for the injectivity of $\alpha_{\text{st}}^\nabla(V)$ and $\alpha_{\text{st},K}^\nabla(V)$. The injectivity of $\alpha_{\text{cris}}^\nabla(V)$ follows from that of $\alpha_{\text{max}}^\nabla(V)$ and the inclusion $B_{\text{cris}}^\nabla(R) \subseteq B_{\text{max}}^\nabla(R)$ by a similar argument. The second assertion follows from Lemma 4.9(iii).

**Lemma 4.11.** For $V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, $\alpha_{\text{max}}^\nabla(V)$ is an isomorphism if and only if $\alpha_{\text{cris}}^\nabla(V)$ is an isomorphism. Similarly, $\alpha_{\text{st}}^\nabla(V)$ is an isomorphism if and only if $\alpha_{B_{\text{cris}}(R)[\log(|p'|/p)]}^\nabla(V)$ is an isomorphism.

**Proof.** The sufficiency follows from Lemma 4.4. Since the proof of the necessity of the two assertions is similar, we deal with the necessity for the first. Assume that $\alpha_{\text{max}}^\nabla(V)$ is an isomorphism. Then the Frobenius $\varphi : B_{\text{max}}^\nabla(R) \rightarrow B_{\text{max}}^\nabla(R)$ induces the following commutative diagram.
Here we also use the fact that \( \varphi : B_{\text{max}}^\nabla (R) \to B_{\text{max}}^\nabla (R) \) factors through \( B_{\text{cris}}^\nabla (R) \subset B_{\text{max}}^\nabla (R) \). In particular, the image of \( \alpha_{\text{cris}}^\nabla (V) \) contains \( V \). Since \( \alpha_{\text{cris}}^\nabla (V) \) is injective and \( B_{\text{cris}}^\nabla (R) \)-linear, it is an isomorphism.

**Lemma 4.12.** For \( V \in \text{Rep}_{Q_p} (G_{\mathcal{R}K}) \), \( \alpha_{\text{max}}^\nabla (V) \) is an isomorphism if and only if \( \alpha_{\text{st}}^\nabla (V) \) is an isomorphism and the monodromy operator \( N \) on \( D_{\text{st}}^\nabla (V) \) is zero.

**Proof.** The necessity follows from Lemma 4.4 and \( (D_{\text{st}}^\nabla (V))^N = 0 = D_{\text{max}}^\nabla (V) \). We prove the sufficiency. Note that the multiplication map \( B_{\text{st}}^\nabla (R) \otimes_K B_{\text{st}}^\nabla (R) \to B_{\text{st}}^\nabla (R) \) is compatible with \( N \otimes \text{id} + \text{id} \otimes N \) on the source and \( N \) on the target. Since \( \alpha_{\text{st}}^\nabla (V) \) is an isomorphism, this yields the following diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & B_{\text{max}}^\nabla \otimes_{Q_p} V & \longrightarrow & B_{\text{st}}^\nabla \otimes_{Q_p} V & \longrightarrow & 0 \\
& & \downarrow \cong & \downarrow \alpha_{\text{st}}^\nabla (V) & \downarrow \cong & \alpha_{\text{st}}^\nabla (V) & \cong \\
0 & \longrightarrow & \text{Ker}(f) & \longrightarrow & B_{\text{st}}^\nabla \otimes_K D_{\text{st}}^\nabla (V) & \longrightarrow & B_{\text{st}}^\nabla \otimes_K D_{\text{st}}^\nabla (V) & \longrightarrow & 0
\end{array}
\]

where \( B_{\text{max}}^\nabla = B_{\text{max}}^\nabla (R) \), \( B_{\text{st}}^\nabla = B_{\text{st}}^\nabla (R) \), and \( f \) denotes \( N \otimes \text{id} + \text{id} \otimes N \). Since the monodromy operator \( N \) on \( D_{\text{st}}^\nabla (V) \) is assumed to be zero, we conclude that \( \text{Ker} f = B_{\text{max}}^\nabla (R) \otimes_K D_{\text{max}}^\nabla (V) \) and the left vertical isomorphism is \( \alpha_{\text{max}}^\nabla (V) \).

**Definition 4.13.** Let \( V \in \text{Rep}_{Q_p} (G_{\mathcal{R}K}) \). We say that \( V \) is **horizontal crystalline** if \( \alpha_{\text{max}}^\nabla (V) \) is an isomorphism. Similarly, we say that \( V \) is **horizontal semistable** (respectively, **horizontal de Rham**) if \( \alpha_{\text{st}}^\nabla (V) \) (respectively, \( \alpha_{\text{dr}}^\nabla (V) \)) is an isomorphism.

See Lemmas 4.9(ii), 4.11, and 4.12 for equivalent definitions. By Lemma 4.4, a horizontal crystalline representation is horizontal semistable, and a horizontal semistable representation is horizontal de Rham.

**Remark 4.14.** Our definition of horizontal de Rham representations coincides with that of [Bri08, p. 117]. By Lemma 4.11, our definition of horizontal crystalline representations coincides with the definition of horizontal \( R_0 \)-crystalline representations in [Bri08, p. 117].

**Remark 4.15.** The full subcategory of horizontal crystalline (respectively, semistable, de Rham) representations of \( G_{\mathcal{R}K} \) is stable under taking tensor products, subquotients, and exterior products. This follows from [Bri06, Proposition 3.5] since the period rings \( B_{\text{max}}^\nabla (R) \), \( B_{\text{st}}^\nabla (R) \), and \( B_{\text{dr}}^\nabla (R) \) all satisfy the assumptions of the proposition.

**Lemma 4.16.** The full subcategory of horizontal de Rham representations of \( G_{\mathcal{R}K} \) is stable under taking duals. If \( R \) satisfies condition (BR), then the full subcategory of horizontal crystalline (respectively, semistable) representations of \( G_{\mathcal{R}K} \) is stable under taking duals.

**Proof.** By [Bri06, Proposition 3.7], it suffices to show the assertion for one-dimensional representations. For horizontal de Rham representations (respectively, horizontal crystalline representations under condition (BR)), it is [Bri08, Theorem 8.4.2]. In general, if \( D_{\text{st}}^\nabla (V) \) is one-dimensional over \( K_0 \) for \( V \in \text{Rep}_{Q_p} (G_{\mathcal{R}K}) \), then \( N = 0 \) by the relation \( N \varphi = p \varphi N \). In particular, every one-dimensional horizontal semistable representation is horizontal crystalline, and thus so is its dual.

**Proposition 4.17.** The sequence

\[
0 \to \mathbb{Q}_p \to B_{\text{max}}^\nabla (R)^{\varphi = 1} \to B_{\text{dr}}^\nabla (R)/B_{\text{dr}}^\nabla (R) \to 0
\]

is exact.
Proof. Since $\varphi(B_{\text{max}}^\nabla(R)) \subset B_{\text{cris}}^\nabla(R) \subset B_{\text{max}}^\nabla(R)$, we get $B_{\text{cris}}^\nabla(R)^{\varphi=1} = B_{\text{max}}^\nabla(R)^{\varphi=1}$. Hence, the assertion is [Bri08, Proposition 6.2.24]. □

The exact sequence in the proposition is called the fundamental exact sequence.

4.3 Functoriality

Let $R$ be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1 and fix an algebraic closure $\overline{\text{Frac } R}$ of $\text{Frac } R$ as before. Let us discuss functorial aspects of our formalism in two special cases.

Let $R'$ be another $\mathcal{O}_R$-algebra satisfying the conditions in Set-up 3.1 and let $f: R \to R'$ be an $\mathcal{O}_K$-algebra homomorphism. Recall that

$$G_{R_K} = \text{Gal}(\overline{R}/[p^{-1}]/R_K) = \pi_1^{\text{et}}(\text{Spec } R_K, \overline{\text{Frac } R}).$$

Note that $R_K \to R'_K \to \overline{\text{Frac } R'}$ defines another geometric point of $\text{Spec } R_K$. Choose a path on $\text{Spec } R_K$ from $\overline{\text{Frac } R'}$ to $\overline{\text{Frac } R}$ (cf. [SGA1, Exposé V, 7]). Concretely, this is equivalent to choosing an $\mathcal{O}_K$-algebra homomorphism $\overline{f}: \overline{R} \to \overline{R'}$ extending $f$. The path defines a group homomorphism

$$G_{R'_K} = \pi_1^{\text{et}}(\text{Spec } R'_K, \overline{\text{Frac } R'}) \to \pi_1^{\text{et}}(\text{Spec } R_K, \overline{\text{Frac } R}) = G_{R_K}.$$ 

In what follows (e.g. Proposition 4.20), we always choose a path so that $\overline{f}$ is an $\mathcal{O}_K$-algebra homomorphism. Then $\overline{f}$ gives rise to a homomorphism $(\widehat{\overline{R}}/\overline{R}, \overline{R}) \to (\widehat{\overline{R'}}/\overline{R'}, \overline{R'})$ of $C$-Banach pairs and induces homomorphisms

$$B_{\text{max}}^\nabla(R) \to B_{\text{max}}^\nabla(R'), \quad B_{\text{st}}^\nabla(R) \to B_{\text{st}}^\nabla(R'), \quad \text{and } B_{\text{dR}}^\nabla(R) \to B_{\text{dR}}^\nabla(R').$$

Moreover, they are compatible with $G_{R'_K} \to G_{R_K}$.

**Lemma 4.18.** If a $\mathbb{Q}_p$-representation $V$ of $G_{R_K}$ is horizontal crystalline (respectively, semistable, de Rham), then so is the representation $V|_{G_{R'_K}}$ of $G_{R'_K}$.

**Proof.** This follows from Lemma 4.4. □

We also consider the case of changing the base field. Let $L$ be a finite extension of $K$, and set $R_{OL} := R \otimes_{\mathcal{O}_K} \mathcal{O}_L$ and $R_L := R_{OL}/[p^{-1}] = R_K \otimes_K L$. Observe that $R_{OL}$ satisfies the conditions in Set-up 3.1 with respect to $L$. Since $\overline{\text{Frac } R}$ is an algebraic closure of $\text{Frac } R_{OL}$, we have $\overline{R_{OL}} = \overline{R}$. It follows that $G_{R_L}$ is an open subgroup of $G_{R_K}$ of finite index.

**Lemma 4.19.** Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_{R_K})$.

(i) If $V$ is horizontal crystalline, then the representation $V|_{G_{R_L}}$ of $G_{R_L}$ is also horizontal crystalline. Similarly, if $V$ is horizontal semistable, then so is $V|_{G_{R_L}}$.

(ii) $V$ is horizontal de Rham if and only if $V|_{G_{R_L}}$ is horizontal de Rham.

**Proof.** Part (i) and the necessity of part (ii) follow from Lemma 4.4. Assume that $V|_{G_{R_L}}$ is horizontal de Rham. Using the necessity, we may reduce to the case where $L$ is Galois over $K$. Then $G_{R_L}$ is a normal subgroup of $G_{R_K}$ with quotient $\text{Gal}(L/K)$. Since $B_{\text{dR}}^\nabla(R) = B_{\text{dR}}^\nabla(R_{OL})$, the finite-dimensional $L$-vector space $D_{\text{dR}}^\nabla(V|_{G_{R_L}})$ has a semilinear $\text{Gal}(L/K)$-action and $D_{\text{dR}}^\nabla(V) = D_{\text{dR}}^\nabla(V|_{G_{R_L}})^{\text{Gal}(L/K)}$.

By Galois descent, we have $L \otimes_K D_{\text{dR}}^\nabla(V) \xrightarrow{\text{iso}} D_{\text{dR}}^\nabla(V|_{G_{R_L}})$. It follows that $\alpha_{\text{dR}}^\nabla(V) = \alpha_{\text{dR}}^\nabla(V|_{G_{R_L}})$ and it is an isomorphism by assumption. □
By combining these two cases, we can consider the following setting.

**Proposition 4.20.** Let \( L \) be a finite extension of \( K \) and \( R' \) an \( \mathcal{O}_L \)-algebra satisfying the conditions in Set-up 3.1 with respect to \( L \). Let \( f : R \to R' \) be an \( \mathcal{O}_K \)-algebra homomorphism. Fix a path on \( \text{Spec} \, R_K \) from \( \text{Frac} \, R \to \text{Frac} \, R \) and consider the associated homomorphism \( \mathcal{G}_{R'_L} \to \mathcal{G}_{R_K} \). Let \( V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K}) \) and set \( f^*V := V|_{\mathcal{G}_{R'_L}} \). If \( V \) is horizontal semistable and \( f^*V \) is horizontal crystalline, then \( V \) is horizontal crystalline.

**Proof.** Denote the residue field of \( L \) by \( k_L \) and set \( L_0 = W(k_L)[p^{-1}] \). By Lemma 4.4, we have \( D^N_{\text{st}}(f^*V) \cong L_0 \otimes_{k_0} D^N_{\text{st}}(V) \). Moreover, it follows from the construction that this isomorphism is compatible with the monodromy operator \( \eta \) on the left-hand side and \( \id \otimes \eta \) on the right-hand side. Since \( f^*V \) is horizontal crystalline, we have \( N = 0 \) on \( D^N_{\text{st}}(f^*V) \). By faithful flatness of \( K_0 \to L_0 \), we also have \( N = 0 \) on \( D^V_{\text{st}}(V) \). Hence, \( V \) is horizontal crystalline by Lemma 4.12. \( \square \)

**Remark 4.21.** We will use the proposition when \( R' = \mathcal{O}_L \). In this case, it is informally phrased as follows: if a horizontal semistable representation is crystalline at one classical point, then it is horizontal crystalline.

### 4.4 Horizontal de Rham representations and de Rham representations

In this subsection, we review the de Rham period ring studied in [Bri08]. This ring will be used in the next section to define the \( V_{\text{st}} \)-functor. Let \( R \) be an \( \mathcal{O}_K \)-algebra satisfying the conditions in Set-up 3.1. Set \( \hat{\Omega}_R := \lim_{\leftarrow m} \Omega^1_{R/\mathbb{Z}}/p^m\Omega^1_{R/\mathbb{Z}} \).

**Definition 4.22.** Let \( B_{\text{dR}}(R) \) denote the de Rham period ring of \( R \) defined in [Bri08, Définition 5.1.5]. We briefly recall the definition below.

The map \( \theta_{A_{\text{inf}}(R)} : A_{\text{inf}}(\hat{R}) \to \hat{R} \) extends \( R \)-linearly to \( \theta_R : R \otimes_{\mathbb{Z}} A_{\text{inf}}(\hat{R}) \to \hat{R} \). Let \( A_{\text{inf}}(\hat{R}/R) \) denote the \( \theta_R^{-1}(p\hat{R}) \)-adic completion of \( R \otimes_{\mathbb{Z}} A_{\text{inf}}(\hat{R}) \) [Bri08, Définition 5.1.3]. It is equipped with a map \( \theta_R : A_{\text{inf}}(\hat{R}/R)[p^{-1}] \to \hat{R}/p^1 \). We set

\[
B_{\text{dR}}^+(R) := \lim_{\leftarrow m} A_{\text{inf}}(\hat{R}/R)[p^{-1}]/(\text{Ker} \, \theta_R)^m,
\]

and denote the natural map \( B_{\text{dR}}^+(R) \to \hat{R}/p^1 \) by \( \theta_R \). For \( m \in \mathbb{N} \), we set \( \text{Fil}^m B_{\text{dR}}^+(R) := (\text{Ker} \, \theta_R)^m \). Define \( B_{\text{dR}}(R) := B_{\text{dR}}^+(R)[t^{-1}] \). The ring has an action of \( \mathcal{G}_{R_K} \), a decreasing separated and exhaustive filtration \( \text{Fil}^\bullet \) given by

\[
\text{Fil}^0 B_{\text{dR}}(R) = \sum_{i=0}^{\infty} t^{-i} \text{Fil}^i B_{\text{dR}}^+(R) \quad \text{and} \quad \text{Fil}^m B_{\text{dR}}(R) = t^m \text{Fil}^0 B_{\text{dR}}(R) \quad (m \in \mathbb{Z}),
\]

and a \( B_{\text{dR}}^+(R) \)-linear integrable connection \( \nabla : B_{\text{dR}}(R) \to B_{\text{dR}}(R) \otimes_R \hat{\Omega}_R \) satisfying the Griffiths transversality \( \nabla(\text{Fil}^m B_{\text{dR}}(R)) \subset \text{Fil}^{m-1} B_{\text{dR}}(R) \otimes_R \hat{\Omega}_R \). We also have \( (B_{\text{dR}}(R))^\mathcal{G}_{R_K} = R_K \) and \( B_{\text{dR}}^+(R) = B_{\text{dR}}(R)^{\nabla = 0} \) [Bri08, Propositions 5.2.12, 5.3.3]. The ring \( B_{\text{dR}}(R) \) is a \( (\mathbb{Q}_p, \mathcal{G}_{R_K}) \)-ring.

**Definition 4.23.** For \( V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K}) \), set \( D_{\text{dR}}(V) := D_{B_{\text{dR}}(R)}(V) \) and \( \alpha_{\text{dR}}(V) := \alpha_{B_{\text{dR}}(R)}(V) \). We say that \( V \) is de Rham if it is \( B_{\text{dR}}(R) \)-admissible, that is, \( \alpha_{\text{dR}}(V) \) is an isomorphism. Note that \( \alpha_{\text{dR}}(V) \) is always injective by [Bri08, Proposition 8.2.4].

The \( R_K \)-module \( D_{\text{dR}}(V) \) is projective of rank at most \( \dim_{\mathbb{Q}_p} V \) by [Bri08, Proposition 8.3.1], and it is equipped with induced connection \( \nabla : D_{\text{dR}}(V) \to D_{\text{dR}}(V) \otimes_R \hat{\Omega}_R \).
and filtration $\text{Fil}^\bullet D_{\text{dr}}(V)$. We also have a natural $R_K$-linear map

$$\beta_{\text{dr}}(V) : R_K \otimes_K D^\nabla_{\text{dr}}(V) \to D_{\text{dr}}(V).$$

**Lemma 4.24** [Bri08, Proposition 8.2.10]. For $V \in \text{Rep}_{\mathbb{Q}_p}(G_{R_K})$, it is horizontal de Rham if and only if it is de Rham and $\beta_{\text{dr}}(V)$ is an $R_K$-linear isomorphism.

**Lemma 4.25** [Bri08, Proposition 8.4.3]. Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_{R_K})$. If $V$ is de Rham, then the isomorphism $\alpha_{\text{dr}}(V) : B_{\text{dr}}(R) \otimes_K D^\nabla_{\text{dr}}(V) \xrightarrow{\sim} B_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V$ is compatible with $G_{R_K}$-actions and connections, and strictly compatible with filtrations.

**Lemma 4.26** [Bri08, Corollary 5.2.11]. For $m \in \mathbb{Z}$, we have

$$B^\nabla_{\text{dr}}(R) \cap \text{Fil}^m B_{\text{dr}}(R) = t^m B^\nabla_{\text{dr}}(R).$$

**Remark 4.27.** Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_{R_K})$ be a horizontal de Rham representation. The filtration $\text{Fil}^m B^\nabla_{\text{dr}}(R) := t^m B^\nabla_{\text{dr}}(R)$ on $B^\nabla_{\text{dr}}(R)$ induces a filtration on $D^\nabla_{\text{dr}}(V)$. The maps $\alpha^\nabla_{\text{dr}}(V)$ and $\beta_{\text{dr}}(V)$ preserve filtrations, that is,

$$\alpha_{\text{dr}}(V)(\text{Fil}^m(B_{\text{dr}}(R) \otimes_K D^\nabla_{\text{dr}}(V))) \subset \text{Fil}^m B^\nabla_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V,$$

$$\beta_{\text{dr}}(V)(\text{Fil}^m(R_K \otimes_K D^\nabla_{\text{dr}}(V))) \subset \text{Fil}^m D_{\text{dr}}(V).$$

However, neither $\alpha_{\text{dr}}(V)$ nor $\beta_{\text{dr}}(V)$ is strictly compatible with filtrations in general, and the claim on $\alpha_{\text{dr}}(V)$ in [Bri08, Proposition 8.4.3] is incorrect. See the following example.

**Example 4.28.** Assume $K = K_0$ for simplicity. Let $R = O_K(T^\pm)$ and let $h(T) = 1 + pT \in R$. Fix a compatible system $(h, h^{1/p}, \ldots)$ of $p$-power roots of $h$ and set $h^b := (h \mod p, h^{1/p} \mod p, \ldots) \in \hat{T}^b$.

Let $V = \mathbb{Q}_p e_1 + \mathbb{Q}_p e_2$ be a representation of $G_{R_K}$ corresponding to $(\varepsilon, h^b)$; $g \in G_{R_K}$ acts on $V$ by $ge_1 = \chi(g)e_1$ and $ge_2 = \eta(g)e_1 + e_2$, where $\chi : G_{R_K} \to \text{Gal}(K/K) \to \mathbb{Z}_p^\times$ is the $p$-adic cyclotomic character defined by $g(\varepsilon_m) = \varepsilon^{(g)}_m$, and $\eta : G_{R_K} \to \mathbb{Z}_p$ is a map defined by $g(h^{1/p^n}) = \varepsilon^{n}_{m} h^{1/p^{n}}$.

We show that $V$ is horizontal crystalline. Note that the series $\log[h^b] := \sum_{m=1}^{\infty} (-1)^{m-1}((h^b)^m/m)$ converges in $\Lambda_{\text{max}}^\nabla(R) := \Lambda_{\text{max}}(R)$. Moreover, $\log[h^b] \in \text{Fil}^0 B^\nabla_{\text{dr}}(R) \setminus \text{Fil}^1 B^\nabla_{\text{dr}}(R)$. Set

$$f_1 = \frac{1}{t} e_1, \quad f_2 = e_2 - \frac{\log[h^b]}{t} e_1 \in B^\nabla_{\text{max}}(R) \otimes_{\mathbb{Q}_p} V.$$

Then we see that $V$ is horizontal crystalline with $D^\nabla_{\text{max}}(V) = K f_1 + K f_2$. Note that $D^\nabla_{\text{dr}}(V) = D^\nabla_{\text{max}}(V)$, and if we equip $D^\nabla_{\text{dr}}(V)$ with a decreasing filtration induced from $\text{Fil}^m(B^\nabla_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V) := t^m B^\nabla_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V$ we have

$$\text{Fil}^{-1} D^\nabla_{\text{dr}}(V) = D^\nabla_{\text{dr}}(V), \quad \text{Fil}^0 D^\nabla_{\text{dr}}(V) = 0.$$

In particular, the $B^\nabla_{\text{dr}}(R)$-linear isomorphism

$$\alpha_{\text{dr}}(V) : B^\nabla_{\text{dr}}(R) \otimes_K D^\nabla_{\text{dr}}(V) \xrightarrow{\sim} B^\nabla_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V$$

is not strictly compatible with filtrations. Indeed, we have $e_2 = 1 \otimes f_2 + \log[h^b] \otimes f_1 \notin \text{Fil}^0(B^\nabla_{\text{dr}}(R) \otimes_K D^\nabla_{\text{dr}}(V))$ while $e_2 \in \text{Fil}^0(B^\nabla_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V)$.

Let us turn to $D_{\text{dr}}(V) = R_K f_1 + R_K f_2$. We will give another $R_K$-basis. Since $h$ is invertible in $R$ and $[h^b]/h - 1 \in \ker \theta_R$, the series $\log([h^b]/h) := \sum_{m=1}^{\infty} (-1)^{m-1}((h^b)^m/m)$ converges in $B^+_{\text{dr}}(R)$. Let $h := \sum_{m=1}^{\infty} (-1)^{m-1}((h - 1)^m/m) = \sum_{m=1}^{\infty} (-1)^{m-1}(p^m/m)T^m \in R$. 2178
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We can check \( \log[h^p] = \log([h^p]/h) + \log h \) in \( B_{\text{dr}}^+(R) \). Set
\[
f_3 = e_2 - \frac{\log[h^p]/h}{t} e_1 = f_2 + \frac{\log h}{t} e_1 \in B_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V.
\]
Then \( f_3 \) is fixed by \( G_{R_K} \), and \( D_{\text{dr}}(V) = R_K f_1 + R_K f_3 \). Consider the filtration on \( D_{\text{dr}}(V) \) induced from \( \text{Fil}^m(B_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V) := (\text{Fil}^m B_{\text{dr}}(R)) \otimes_{\mathbb{Q}_p} V \). Since \( \log([h^p]/h) \in \text{Fil}^1 B_{\text{dr}}(R) \), we see that
\[
\text{Fil}^{-1} D_{\text{dr}}(V) = D_{\text{dr}}(V), \quad \text{Fil}^0 D_{\text{dr}}(V) = R_K f_3, \quad \text{Fil}^1 D_{\text{dr}}(V) = 0.
\]
It follows that the natural \( R_K \)-linear isomorphism
\[
\beta_{\text{dr}}(V): R_K \otimes_K D_{\text{dr}}^\Sigma(V) \overset{\sim}{\longrightarrow} D_{\text{dr}}(V)
\]
is not strictly compatible with filtrations. Note that the \( B_{\text{dr}}(R) \)-linear isomorphism
\[
\alpha_{\text{dr}}(V): B_{\text{dr}}(R) \otimes_{R_K} D_{\text{dr}}(V) \overset{\sim}{\longrightarrow} B_{\text{dr}}(R) \otimes_{\mathbb{Q}_p} V
\]
is strictly compatible with filtrations.

5. Filtered \((\varphi, N, \text{Gal}(L/K), R_K)\)-modules

In this section, we follow [Fon94b] and define filtered \((\varphi, N, \text{Gal}(L/K), R_K)\)-modules and the functors \( D_{\text{pst}} \) and \( V_{\text{pst}} \).

As before, let \( k \) be a perfect field of characteristic \( p \) and set \( K_0 := W(k)[p^{-1}] \). Let \( K \) be a totally ramified finite extension of \( K_0 \). We denote the maximal unramified extension of \( K_0 \) inside \( \overline{K} \) by \( K_0^\text{ur} \). Let \( L \) be a Galois extension of \( K \) inside \( \overline{K} \). Let \( L_0 \) be the maximal unramified extension of \( K_0 \) in \( L \) and \( \sigma \) the absolute Frobenius on \( L_0 \). Note that \( (\overline{K})_0 = K_0^\text{ur} \).

Let us first recall the definition of \((\varphi, N, \text{Gal}(L/K))\)-modules.

**Definition 5.1** [Fon94b, 4.2.1]. A \((\varphi, N, \text{Gal}(L/K))\)-module is a finite-dimensional \( L_0 \)-vector space \( D \) equipped with
\begin{itemize}
  \item[(i)] an injective \( \sigma \)-semilinear map \( \varphi: D \to D \) (Frobenius),
  \item[(ii)] an \( L_0 \)-linear endomorphism \( N: D \to D \) (monodromy operator), and
  \item[(iii)] a semilinear action of \( \text{Gal}(L/K) \),
\end{itemize}
such that they satisfy the compatibilities
\begin{itemize}
  \item \( N \varphi = p \varphi N \),
  \item for every \( g \in \text{Gal}(L/K) \), \( g \varphi = \varphi g \) and \( gN = Ng \).
\end{itemize}
When \( L = K \), we simply call it a \((\varphi, N)\)-module.

A morphism of \((\varphi, N, \text{Gal}(L/K))\)-modules is an \( L_0 \)-linear map that commutes with \( \varphi, N \), and \( \text{Gal}(L/K) \)-action.

We say that a \((\varphi, N, \text{Gal}(L/K))\)-module \( D \) is discrete if the action of \( \text{Gal}(L/K) \) on \( D \) is discrete, that is, for every \( d \in D \), the stabilizer subgroup of \( d \) is open in \( \text{Gal}(L/K) \).

Let \( R \) be an \( \mathcal{O}_K \)-algebra satisfying the conditions in Set-up 3.1. We set \( R_{\mathcal{O}_L} := \mathcal{O}_L \otimes_{\mathcal{O}_K} R \) and \( R_L := L \otimes_K R_K \). Note that \( R_{\mathcal{O}_L} \) is \( p \)-adically complete if \( L \) is finite over \( K \) but that this is not the case in general. For an \( L_0 \)-vector space \( D \), we write \( D_{R_L} \) for \( R_L \otimes_{L_0} D \).

**Definition 5.2** (cf. [Fon94b, 4.3.2]). A \((\varphi, N, \text{Gal}(L/K), R_K)\)-module is a pair consisting of
\begin{itemize}
  \item[(i)] a \((\varphi, N, \text{Gal}(L/K))\)-module \( D \) and
  \item[(ii)] a \( \text{Gal}(L/K) \)-stable separated and exhaustive decreasing filtration \((\text{Fil}^i D_{R_L})_{i \in \mathbb{Z}} \) of \( D_{R_L} \).
\end{itemize}
such that each $\text{gr}^i D_{R_L}$ is a projective $R_L$-submodule of $D_{R_L}$. When $L = K$, we simply call it a $(\varphi, N, R_K)$-module.

A morphism of filtered $(\varphi, N, \text{Gal}(L/K), R_K)$-modules is a morphism of $(\varphi, N, \text{Gal}(L/K))$-modules whose scalar extension from $L_0$ to $R_L$ preserves filtrations. A filtered $(\varphi, N, \text{Gal}(L/K), R_K)$-module is called discrete if it is discrete as a $(\varphi, N, \text{Gal}(L/K))$-module.

We write $\text{MF}_{L/K}(\varphi, N, R_K)$ for the full subcategory of discrete filtered $(\varphi, N, \text{Gal}(L/K), R_K)$-modules.

**Proposition 5.3.** Let $L$ be a finite Galois extension of $K$ and let $V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$. If $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable, then $D_{\text{st}}^\nabla(V|_{\mathcal{G}_{R_L}})$ is a $(\varphi, N, \text{Gal}(L/K))$-module. Moreover, if we equip $D_{\text{st}}^\nabla(V|_{\mathcal{G}_{R_L}})$ with the filtration induced from $D_{\text{dR}}(V|_{\mathcal{G}_{R_L}})$ via the isomorphism

$$R_L \otimes_{L_0} D_{\text{st}}^\nabla(V|_{\mathcal{G}_{R_L}}) \cong D_{\text{dR}}(V|_{\mathcal{G}_{R_L}}),$$

then $D_{\text{st}}^\nabla(V|_{\mathcal{G}_{R_L}})$ is a filtered $(\varphi, N, \text{Gal}(L/K), R_K)$-module.

**Proof.** Recall that $B_{\text{st}}^\nabla(R_{O_L}) \otimes_{\mathbb{Q}_p} V$ admits an injective $\sigma$-semilinear map $\varphi \otimes \text{id}$ and an $L_0$-linear endomorphism $N \otimes \text{id}$ satisfying $(N \otimes \text{id})(\varphi \otimes \text{id}) = p(\varphi \otimes \text{id})(N \otimes \text{id})$. Moreover, by the identification $B_{\text{st}}^\nabla(R_{O_L}) \otimes_{\mathbb{Q}_p} V = B_{\text{st}}^\nabla(R) \otimes_{\mathbb{Q}_p} V$, it is equipped with diagonal semilinear $\mathcal{G}_{R_K}$-action commuting with $\varphi \otimes \text{id}$ and $N \otimes \text{id}$. The first assertion follows from these remarks and $\mathcal{G}_{R_K}/\mathcal{G}_{R_L} \cong \text{Gal}(L/K)$. The second assertion follows from the first and [Bri08, Proposition 8.3.4(a)].

**Definition 5.4.** Assume that $L$ is finite over $K$. Recall the natural identifications $B_{\text{st}}^\nabla(R_{O_L}) = B_{\text{st}}^\nabla(R)$ and $B_{\text{dR}}^\nabla(R_{O_L}) = B_{\text{dR}}^\nabla(R)$. For a filtered $(\varphi, N, \text{Gal}(L/K), R_K)$-module $D$, set

$$V_{\text{st},L}(D) := (B_{\text{st}}^\nabla(R) \otimes_{L_0} D)^{\varphi = 1,N = 0} \cap \text{Fil}^0 (B_{\text{dR}}^\nabla(R) \otimes_{R_L} D_{R_L}),$$

where $\varphi$ (respectively, $N$) on $B_{\text{st}}^\nabla(R) \otimes_{L_0} D$ denotes $\varphi \otimes \varphi$ (respectively, $N \otimes \text{id} + \text{id} \otimes N$) and $\text{Fil}^m$ on $B_{\text{dR}}^\nabla(R) \otimes_{R_L} D_{R_L}$ denotes $\sum_{m \in \mathbb{Z}} \text{Fil}^m B_{\text{dR}}^\nabla(R) \otimes_{R_L} \text{Fil}^{*-m} D_{R_L}$. It is a $\mathbb{Q}_p$-vector space equipped with a $\mathbb{Q}_p$-linear action of $\mathcal{G}_{R_K}$.

**Proposition 5.5.** Let $L$ be a finite Galois extension of $K$ and let $V \in \text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$. If $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable, then the natural map $V \to V_{\text{st},L}(D_{\text{st}}^\nabla(V|_{\mathcal{G}_{R_L}}))$ is a $\mathcal{G}_{R_K}$-equivariant isomorphism.

**Proof.** We use the identification $B_{\text{st}}^\nabla(R_{O_L}) = B_{\text{st}}^\nabla(R)$. For simplicity, we write $V_L$ for $V|_{\mathcal{G}_{R_L}}$. Since $V_L$ is horizontal semistable, we have the following commutative diagram.

$$
\begin{array}{c}
B_{\text{st}}^\nabla(R) \otimes_{L_0} D_{\text{st}}^\nabla(V_L) \\
\downarrow \\
B_{\text{dR}}^\nabla(R) \otimes_{R_L} D_{\text{dR}}(V_L)
\end{array}
\xrightarrow{\alpha_{\text{st}}^\nabla(V_L)}
\begin{array}{c}
B_{\text{st}}^\nabla(R) \otimes_{\mathbb{Q}_p} V \\
\downarrow \\
B_{\text{dR}}^\nabla(R) \otimes_{\mathbb{Q}_p} V
\end{array}
\cong
\begin{array}{c}
B_{\text{st}}^\nabla(R) \otimes_{\mathbb{Q}_p} V \\
\downarrow \\
B_{\text{dR}}(R) \otimes_{\mathbb{Q}_p} V
\end{array}
\cong
\begin{array}{c}
B_{\text{dR}}(R) \otimes_{\mathbb{Q}_p} V
\end{array}
$$

Furthermore, the isomorphism $\alpha_{\text{st}}^\nabla(V_L)$ is compatible with $\mathcal{G}_{R_K}$-actions, Frobenius, and monodromy operators, and the isomorphism $\alpha_{\text{dR}}(V_L)$ is compatible with $\mathcal{G}_{R_K}$-actions and strictly compatible with filtrations. Note also that $D_{\text{st}}^\nabla(V_L)_{R_L} \cong D_{\text{dR}}(V_L)$. Hence, we have
$\mathcal{G}_{R_\K}$-equivariant isomorphisms

$$V_{\text{st}}(D^\nabla_{\text{st}}(V_{L})) = (B^\nabla_{\text{st}}(R) \otimes_{L_0} D^\nabla_{\text{st}}(V_{L}))^{\varphi=1,N=0} \cap \text{Fil}^0(B_{\text{dR}}(R) \otimes_{RL} D^{\text{st}}(V_{L})_{RL})$$

$$\cong (B^\nabla_{\text{st}}(R) \otimes_{L_0} D^\nabla_{\text{st}}(V_{L}))^{\varphi=1,N=0} \cap \text{Fil}^0(B_{\text{dR}}(R) \otimes_{RL} D_{\text{dR}}(V_{L}))$$

$$\cong (B^\nabla_{\text{st}}(R) \otimes_{Q_p} V)^{\varphi=1,N=0} \cap \text{Fil}^0(B_{\text{dR}}(R) \otimes_{Q_p} V)$$

$$\cong ((B^\nabla_{\text{st}}(R))^{\varphi=1,N=0} \cap \text{Fil}^0(B_{\text{dR}}(R))) \otimes_{Q_p} V.$$

By Lemma 4.26 and Proposition 4.17, we have

$$(B^\nabla_{\text{st}}(R))^{\varphi=1,N=0} \cap \text{Fil}^0(B_{\text{dR}}(R)) = (B^\nabla_{\text{st}}(R))^{\varphi=1,N=0} \cap (B^\nabla_{\text{dR}}(R) \cap \text{Fil}^0(B_{\text{dR}}(R)))$$

$$= (B^\nabla_{\text{max}}(R))^{\varphi=1} \cap B^{\text{st}}_{\text{dR}}(R) = Q_p.$$

Hence, $((B^\nabla_{\text{st}}(R))^{\varphi=1,N=0} \cap \text{Fil}^0(B_{\text{dR}}(R))) \otimes_{Q_p} V$ is $\mathcal{G}_{R_\K}$-equivariantly isomorphic to $V$.  

**Remark 5.7.** Every object of $\text{Rep}_{\text{Q}_p}(\mathcal{G}_{R_\K})$ is horizontal de Rham by Lemma 4.19(ii). We will prove that $\text{Rep}_{\text{Q}_p}(\mathcal{G}_{R_\K})$ is exactly the full subcategory of horizontal de Rham representations if $R$ satisfies condition (BR) (Theorem 7.3). In this case, $\text{Rep}_{\text{Q}_p}(\mathcal{G}_{R_\K})$ is a Tannakian subcategory of $\text{Rep}_{\text{Q}_p}(\mathcal{G}_{R_\K})$ by Remark 4.15 and Lemma 4.16.

**Lemma 5.8.** Let $V \in \text{Rep}_{\text{Q}_p}(\mathcal{G}_{R_\K})$. Let $L$ be a finite Galois extension of $K$ inside $\K$ such that $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable. Then there is a natural identification $D^\nabla_{\text{pst}}(V)_{R_\K} \cong R_\K \otimes_L D_{\text{dR}}(V|_{\mathcal{G}_{R_L}})$. Moreover, if we set

$$\text{Fil}^\bullet D^\nabla_{\text{pst}}(V)_{R_\K} := R_\K \otimes_{R_L} \text{Fil}^\bullet D_{\text{dR}}(V|_{\mathcal{G}_{R_L}}),$$

then $D^\nabla_{\text{pst}}(V)$ is a discrete filtered $(\varphi, N, \text{Gal}(\K/K), R_K)$-module and the filtration is independent of the choice of $L$.

**Proof.** For every finite Galois extension $L'$ of $K$ containing $L$, $V|_{\mathcal{G}_{R_{L'}}}$ is horizontal semistable and we have

$$D^\nabla_{\text{st}}(V|_{\mathcal{G}_{R_{L'}}}) \cong L'_0 \otimes_{L_0} D^\nabla_{\text{st}}(V|_{\mathcal{G}_{R_L}}),$$

$$D_{\text{dR}}(V|_{\mathcal{G}_{R_{L'}}}) \cong R_{L'} \otimes_{R_L} D_{\text{dR}}(V|_{\mathcal{G}_{R_L}}) \cong R_{L'} \otimes_{L_0} D^\nabla_{\text{st}}(V|_{\mathcal{G}_{R_L}}).$$

Moreover, the first isomorphism is compatible with Frobenii, monodromy operators and $\text{Gal}(L'/K)$-actions, and the second isomorphism is compatible with $\text{Gal}(L'/K)$-actions and strictly compatible with filtrations. The assertions follow from these remarks and Proposition 5.3.  

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Definition 5.9. Let $D$ be a filtered $(\varphi, N, \text{Gal}(\overline{K}/K), R_K)$-module. Set
\[
V_{\text{pst}}(D) := (\mathbb{B}^\varphi_{st}(R) \otimes_{K_0^{ur}} D)^{\varphi=1,N=0} \cap \text{Fil}^0(\mathbb{B}_{dR}(R) \otimes_{R_K} D_{R_K}),
\]
where $\varphi$ (respectively, $N$) on $\mathbb{B}^\varphi_{st}(R) \otimes_{K_0^{ur}} D$ denotes $\varphi \otimes \varphi$ (respectively, $N \otimes \text{id} + \text{id} \otimes N$) and $\text{Fil}^\bullet$ on $\mathbb{B}_{dR}(R) \otimes_{R_K} D_{R_K}$ denotes $\sum_{m \in \mathbb{Z}} \text{Fil}^m \mathbb{B}_{dR}(R) \otimes_{R_K} \text{Fil}^{*-m} D_{R_K}$. It is a $\mathbb{Q}_p$-vector space equipped with a $\mathbb{Q}_p$-linear action of $G_{R_K}$.

We say that $D$ is admissible if it is isomorphic to $\mathbb{D}^\varphi_{\text{pst}}(V)$ for some $V \in \text{Rep}^\varphi_{\text{pst}}(G_{R_K})$. Let $\text{MF}^{\text{ad}}_{K/K}(\varphi, N, R_K)$ be the full subcategory of $\text{MF}_{K/K}^\varphi(\varphi, N, R_K)$ consisting of admissible objects.

Theorem 5.10. Assume that $R$ satisfies condition (BR). Then the functor
\[
D^\varphi_{\text{pst}} : \text{Rep}^\varphi_{\text{pst}}(G_{R_K}) \rightarrow \text{MF}^{\text{ad}}_{K/K}(\varphi, N, R_K)
\]
is an equivalence of categories with quasi-inverse given by $V_{\text{pst}}$.

Proof. By Proposition 5.5, we see that the $\mathbb{Q}_p$-linear map $V \rightarrow V_{\text{pst}}(D_{\text{pst}}(V))$ is a $G_{R_K}$-equivariant isomorphism. For $V_1, V_2 \in \text{Rep}^\varphi_{\text{pst}}(G_{R_K})$, this result for $V = \text{Hom}(V_1, V_2) \in \text{Rep}^\varphi_{\text{pst}}(G_{R_K})$ implies that $D_{\text{pst}}$ is fully faithful. Hence, the assertion follows. \hfill \Box

Remark 5.11. It is desirable to characterize the admissible filtered $(\varphi, N, \text{Gal}(\overline{K}/K), R_K)$-module intrinsically. When $R = K$, Colmez and Fontaine [CF00] prove that admissibility is equivalent to weak admissibility. Results on $p$-adic period domains suggest that admissibility will not be characterized by weak admissibility after every specialization of $R$ to a finite extension of $K$ (see [Har13, Example 6.7]).

6. Purity for horizontal semistable representations

6.1 Statement

We first recall our set-up. Let $k$ be a perfect field of characteristic $p$ and set $K_0 := W(k)[p^{-1}]$. Let $K$ be a totally ramified finite extension of $K_0$ and fix a uniformizer $\pi$ of $K$. We denote by $C$ the $p$-adic completion of an algebraic closure $\overline{K}$ of $K$. Let $R$ be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1. We further assume that $R$ satisfies condition (BR) of Definition 3.3. In particular, $(\pi) \subset R$ is a prime ideal.

Consider the localization $R_{(\pi)}$ of $R$ at the prime ideal $\mathfrak{p}$ of $R$. Let $\mathcal{O}_K$ denote the $p$-adic completion of $R_{(\pi)}$. It is a complete discrete valuation ring with residue field admitting a finite $p$-basis. Set $\mathcal{K} := \mathcal{O}_K[p^{-1}]$ and fix an algebraic closure $\overline{K}$ of $\mathcal{K}$. Let $\mathcal{O}_{\overline{K}}$ denote the ring of integers of $\overline{K}$. We denote the $p$-adic completion of $\mathcal{O}_K$ by $\widehat{\mathcal{O}}_{\overline{K}}$.

Representations of $\text{Gal}(\overline{K}/\mathcal{K})$ are studied in [Bri06, Mor14, Oik13]. In particular, we can discuss whether such a representation is horizontal crystalline, semistable, or de Rham (cf. [Oik13]). Note that these classes of representations are defined by admissibility of the period rings $\mathbb{B}_{\text{cris}}(\Lambda^+)$ (or equivalently $\mathbb{B}_{\text{max}}(\Lambda^+)$, $\mathbb{B}_{\text{st}}(\Lambda^+)$, and $\mathbb{B}_{dR}(\Lambda^+)$ for $\Lambda^+ = \widehat{\mathcal{O}}_{\overline{K}}$, respectively (see Lemma 4.11 and [Oik13, §4] for horizontal semistable representations).

Let $T$ denote the set of prime ideals of $\overline{R}$ above $(\pi) \subset R$. Note that $G_{R_K}$ acts transitively on $T$. For $\mathfrak{p} \in T$, consider the decomposition subgroup
\[
G_{R_K}(\mathfrak{p}) := \{ g \in G_{R_K} \mid g(\mathfrak{p}) = \mathfrak{p} \}.
\]
We now fix an $R$-algebra embedding $\overline{R} \hookrightarrow \mathcal{O}_{\overline{K}}$. This determines a prime ideal of $\overline{R}$ above $(\pi) \subset R$, which we denote by $p_0$, and it gives rise to a homomorphism
\[
\widehat{G}_{R_K}(p_0) := \text{Gal}(\overline{K}/\mathcal{K}) \rightarrow G_{R_K}.
\]
Note that the map as \( \hat{G}_{R,K}(p_0) \to G_{R,K}(p_0) \hookrightarrow G_{R,K} \) by [Bri08, Lemma 3.3.1]. We also remark that a different choice of the embedding \( \mathcal{T} \hookrightarrow \mathcal{O}_K \) defines another ideal \( p'_0 \in T \) and it changes the homomorphism \( \text{Gal}(\overline{K}/K) \to G_{R,K} \) with its conjugate by an element \( g \) of \( G_{R,K} \) such that \( g(p_0) = p'_0 \).

**Theorem 6.1.** Let \( V \in \text{Rep}_{\mathbb{Q}_p}(G_{R,K}) \). If \( V \) is horizontal de Rham and \( V|_{\hat{G}_{R,K}(p_0)} \) is horizontal semistable, then \( V \) is horizontal semistable.

**Remark 6.2.** The converse of Theorem 6.1 also holds by Lemma 4.4.

The rest of this section is devoted to the proof of Theorem 6.1. Our proof is modeled after Tsuji’s work on a purity theorem on crystalline local systems [Tsu, Theorem 5.4.8].

### 6.2 Preliminaries on relevant period rings

**Lemma 6.3.** Consider \( d: K \to K \otimes \mathcal{O}_K \hat{\Omega}_{\mathcal{O}_K} \), where \( \hat{\Omega}_{\mathcal{O}_K} := \lim_{\longrightarrow} \Omega_{\mathcal{O}_K/Z}/p^m \Omega_{\mathcal{O}_K/Z} \). We have \( \text{Ker}(d: K \to K \otimes \mathcal{O}_K \hat{\Omega}_{\mathcal{O}_K}) = K \).

**Proof.** Since \( K \) is algebraically closed in \( R \), it is also algebraically closed in \( K \). Hence, the lemma follows from [Bri06, Proposition 2.28]. \( \square \)

**Remark 6.4.** In [Ohk13], this kernel is denoted by \( K_{\text{can}} \) (cf. [Ohk13, Definition 1.3(i), Remark 1.4(i)]).

Note that \( \hat{\Omega}_{\mathcal{O}_{[p^{-1}]}} \) is a complete algebraically closed field containing \( C \). Hence, the pair \( (\hat{\Omega}_{\mathcal{O}[p^{-1}]}, \mathcal{O}_{\mathcal{O}_K}) \) is a perfectoid \( C \)-Banach pair.

For \( p \in T \), let \( \hat{R}_p \) denote the \( p \)-adic completion of the localization \( \hat{R}_p \) of \( \hat{R} \) at \( p \). It is naturally an \( \mathcal{O}_{C} \)-algebra. Since \( \hat{R}_p \) can be written as the union of one-dimensional Noetherian normal local domains with \( p \) non-unit, \( \hat{R}_p [p^{-1}] \) is a complete valuation field with respect to \( p \)-adic valuation. In particular, \( \hat{R}_p[p^{-1}] \) is a uniform \( C \)-Banach algebra and \( \hat{R}_p \) is a ring of integral elements.

**Lemma 6.5.** The \( C \)-Banach pair \( (\hat{R}_p[p^{-1}], \hat{R}_p) \) is perfectoid.

**Proof.** Since \( \hat{R}_p / p\hat{R}_p = \hat{R}_p/p\hat{R}_p \) is the localization of \( \hat{R}/p\hat{R} \) at the ideal \( p \) mod \( p\hat{R} \), the Frobenius is surjective on \( \hat{R}_p / p\hat{R}_p \) as it is so on \( \hat{R}/p\hat{R} \). \( \square \)

To simplify our notation, set

\[
\Lambda_{\hat{R}} := \hat{R}, \quad \Lambda_{\hat{R}_p} := \hat{R}_p \quad (p \in T), \quad \text{and} \quad \Lambda_{\mathcal{O}_K} := \hat{\Omega}_{\mathcal{O}_K}.
\]

We also set \( \Lambda_{*,\hat{R}} := \Lambda_{\hat{R}}[p^{-1}] \) for \( * \in \{ R, p, \mathcal{O}_K \} \).

Consider the \( \hat{G}_{R,K}(p_0) \)-equivariant inclusion \( \hat{R}_{p_0} \hookrightarrow \mathcal{O}_{K} \), where \( \hat{G}_{R,K}(p_0) \) acts on \( \hat{R}_{p_0} \) via \( \hat{G}_{R,K}(p_0) \to G_{R,K}(p_0) \). This extends to \( \hat{G}_{R,K}(p_0) \)-equivariant injective maps \( \Lambda_{\hat{R}_0} \hookrightarrow \Lambda_{\mathcal{O}_K} \) and \( \Lambda_{p_0} \hookrightarrow \Lambda_{\mathcal{O}_K} \). Note that the former is a map of complete valuation rings of rank one and thus it has \( p \)-torsion-free cokernel.

**Lemma 6.6.** The natural \( \hat{G}_{R,K}(p_0) \)-equivariant commutative diagram

\[
\begin{array}{ccc}
\mathbb{B}^+_\text{st,K}(\Lambda_{\hat{R}_0}) & \longrightarrow & \mathbb{B}^+_\text{dR}(\Lambda_{\hat{R}_0}) \\
\downarrow & & \downarrow \\
\mathbb{B}^+_\text{st,K}(\Lambda_{\mathcal{O}_K}) & \longrightarrow & \mathbb{B}^+_\text{dR}(\Lambda_{\mathcal{O}_K})
\end{array}
\]

is Cartesian.
Lemma 6.9. We have the maximal unramified extension of $K$ following that $L$ follows from $[\text{Ohk13}, \text{Corollary 4.3, third}]$, Lemma 6.3, and the inclusion $\Lambda \subset L$. Thus it is an integral domain. Hence, we conclude that $b$ is invertible in $\Lambda$. Since $L \cong K \otimes_{K_0} B$ and $B_{st}(\Lambda^+_K)$ is a $P_0$-algebra. Note that $L \cong K \otimes_{K_0} B$ and $B_{st}(\Lambda^+_K) = L \otimes_{P_0} B_{st}(\Lambda^+_K)$. Let $\overline{\mathcal{L}}$ denote the algebraic closure of $L$ in $\Lambda_K$. So $\Lambda_K$ is the $p$-adic completion of $\overline{\mathcal{L}}$. We will show that every one-dimensional $\mathcal{L}$-vector subspace $\Delta$ of $B_{st}(\Lambda^+_K)$ that is stable by $\text{Gal}(\overline{\mathcal{L}}/\mathcal{L})$ is written as $\mathcal{L}^i$ for some $i \in \mathbb{Z}$. Since $t$ is invertible in $B_{st}(\Lambda^+_K)$ and $\text{Gal}(\overline{\mathcal{L}}/\mathcal{L})$ can be identified with a subgroup of $\hat{G}_{K_0}(p_0)$, this will complete the proof.

Let $b \in \Delta$ be a generator. By replacing $b$ by $bt^{-i}$ for some $i \in \mathbb{Z}$, we may assume that $b \in B_{dr}(\Lambda^+_K)$. It is enough to show that $b \in \mathcal{L}$. Consider $\theta = \theta_{B_{dr}(\Lambda^+_K)} : B_{dr}(\Lambda^+_K) \to \Lambda_K$. The image $\theta(\Delta)$ is a one-dimensional $\mathcal{L}$-vector subspace generated by $\theta(b)$ and stable by $\text{Gal}(\overline{\mathcal{L}}/\mathcal{L})$. By a result of Sen (cf. [Fon94b, Remark 3.3(i)]), we have $\theta(b) \in \overline{\mathcal{L}} \subset \Lambda_K$. In particular, the action of $\text{Gal}(\overline{\mathcal{L}}/\mathcal{L})$ on $\theta(\Delta)$ factors through $\text{Gal}(\mathcal{L}'/\mathcal{L})$ for some finite Galois extension $\mathcal{L}'$ of $\mathcal{L}$ inside $\overline{\mathcal{L}}$, and so does its action on $\Delta$.

Let $\mathcal{L}(b)$ denote the $\mathcal{L}$-subalgebra of $B_{st,K}(\Lambda^+_K)$ generated by $b$. Since $b \in B_{dr}(\Lambda^+_K) \setminus tB_{dr}(\Lambda^+_K)$, we see that $\mathcal{L}(b)$ is $\text{Gal}(\overline{\mathcal{L}}/\mathcal{L})$-equivariantly isomorphic to its image $\theta(\mathcal{L}(b))$. It follows that $\mathcal{L}(b)$ is isomorphic to a subextension of $\mathcal{L}$ inside $\mathcal{L}'$. This implies that $\mathcal{L}' \otimes_{\mathcal{L}} \mathcal{L}(b)$ is not an integral domain unless $\mathcal{L}(b) = \mathcal{L}$. On the other hand, we know that $\mathcal{L}' \otimes_{\mathcal{L}} B_{st,K}(\Lambda^+_K) = \mathcal{L}' \otimes_{P_0} B_{st}(\Lambda^+_K)$ embeds into $B_{dr}(\Lambda^+_K)$ by Corollary 2.28 (with respect to $\mathcal{L}'$ instead of $K$) and thus it is an integral domain. Hence, we conclude that $b \in \mathcal{L}$.

Lemma 6.10. We have $(B_{st}(\Lambda^+_K))^G_{K_0}(p) = K_0$ and $(B_{dr}(\Lambda^+_p))^G_{K_0}(p) = K$. Moreover, the $(\mathbb{Q}_p, \hat{G}_{K_0}(p))$-rings $B_{st,K}(\Lambda^+_p)$ and $B_{dr}(\Lambda^+_p)$ are $\hat{G}_{K_0}(p)$-regular.
Proof. Without loss of generality, we may assume that \( p = p_0 \). The first assertion follows from Lemmas 6.6 and 6.9. Since \( \Lambda_{p_0} \) is a field, so is \( \mathbb{B}_{dR}(\Lambda_{p_0}^+) \). Hence, \( \mathbb{B}_{dR}(\Lambda_{p_0}) \) is \( G_K(p_0) \)-regular. It remains to verify that \( \mathbb{B}_{st}(\Lambda_{p_0}^+) \) satisfies condition \((G \cdot R_3)\) by [Fon94b, Proposition 1.6.5]. This follows from the \( G_K(p_0) \)-equivariant Cartesian diagram in Lemma 6.6 and the fact that \( \mathbb{B}_{dR}(\Lambda_{p_0}), \mathbb{B}_{st}(\Lambda_{p_0}^+), \) and \( \mathbb{B}_{dR}(\Lambda_{p_0}^+) \) satisfy condition \((G \cdot R_3)\).

6.3 Proof of Theorem 6.1

Let us return to the setting of Theorem 6.1. Since the proof is long and technical, we start by explaining its main idea. We need to show that \( \alpha_{st,K}(V) : \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{K_0} D_{st,K}(V) \to \mathbb{B}_{st,K}(\Lambda_{R}^+) \otimes_{\mathbb{Q}_p} V \) is surjective. Heuristically, this means that \( \mathbb{B}_{st,K}(\Lambda_{R}^+) \otimes_{\mathbb{Q}_p} V \) has enough Galois invariant elements. By assumption, we know that \( \mathbb{B}_{dR}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V \) has enough Galois invariant elements and that \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V \) has enough \( G_K(p_0) \)-invariants. Hence, one might hope for an equality \( \mathbb{B}_{dR}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} \mathbb{B}_{st,K}(\Lambda_{p_0}^+) = \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V \) and deduce that \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V \) has enough Galois invariant elements. Although such an argument does not work literally, our proof pursues this naive idea by using Proposition 2.37 and its consequences (Lemmas 6.6 and 6.14).

Concretely, we first show that \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V \) has enough \( G_K(p) \)-invariants for every \( p \in T \) (Lemma 6.12). Then we study the period ring \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \) where \( \Lambda_{p_0}^+ := \prod_{p \in T} \Lambda_{p_0}^+ \); we analyze the map \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \to \prod_{p \in T} \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \) and deduce that \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V \) has enough Galois invariants (Lemma 6.15). Combining this with the Cartesian diagram in Lemma 6.14, we prove that \( \mathbb{B}_{st,K}(\Lambda_{R}^+) \otimes_{\mathbb{Q}_p} V \) has enough Galois invariants and that \( \alpha_{st,K}(V) \) is surjective.

Let us now start the proof of Theorem 6.1. By assumption, \( V |_{G_K(p_0)} \) is \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \)-admissible and thus \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \)-admissible. By replacing \( V \) by its Tate twist \( V(-n) \) for \( n \gg 0 \), we may assume

\[
(\mathbb{B}_{dR}(\Lambda_{R}^+) \otimes_{\mathbb{Q}_p} V)^{G_K} \cong (\mathbb{B}_{dR}(\Lambda_{R}^+) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}}
\]

and

\[
(\mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}} \Rightarrow (\mathbb{B}_{dR}(\Lambda_{R}^+) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}(p_0)} \Rightarrow (\mathbb{B}_{dR}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}(p_0)}.
\]

In particular, the natural map \( (\mathbb{B}_{dR}(\Lambda_{R}^+) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}} \to (\mathbb{B}_{dR}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}(p_0)} \) is an isomorphism.

Lemma 6.11. The representation \( V |_{G_{R_K}(p_0)} \) is \( \mathbb{B}_{st,K}(\Lambda_{p_0}^+) \)-admissible. Moreover, \( (\mathbb{B}_{dR}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}(p_0)} = (\mathbb{B}_{dR}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}(p_0)} \).

Proof. Consider the following commutative diagram.

\[
\begin{array}{ccc}
(\mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{\mathbb{B}_{R_K}(p_0)} & \cong & (\mathbb{B}_{dR}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{\mathbb{B}_{R_K}(p_0)} \\
\downarrow & & \downarrow \\
(\mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{\mathbb{G}_{R_K}(p_0)} & \Rightarrow & (\mathbb{B}_{dR}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{\mathbb{G}_{R_K}(p_0)}
\end{array}
\]

It follows from the remark before the lemma that the right vertical map is an isomorphism of \( K \)-vector spaces of dimension \( \dim_{\mathbb{Q}_p} V \). By Lemma 6.6, we conclude that all four \( K \)-vector spaces in the diagram are equal, and we obtain the second assertion. We also have

\[
\dim_{\mathbb{Q}_p} V = \dim_K(\mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{\mathbb{G}_{R_K}(p_0)} \leq \dim_K(\mathbb{B}_{st,K}(\Lambda_{p_0}^+) \otimes_{\mathbb{Q}_p} V)^{\mathbb{G}_{R_K}(p_0)}.
\]
By Proposition 6.8 and Lemma 6.10, we have \( \dim_K (B_{st,K}(\Lambda^+_{p_0}) \otimes \mathbb{Q}_p) V_{G_{R_K}(p_0)} = \dim_{\mathbb{Q}_p} V \) and thus \( V|_{G_{R_K}(p_0)} \) is \( B_{st,K}(\Lambda^+_{p_0}) \)-admissible.

Observe that each \( \sigma \in G_{R_K} \) induces an \( R \)-algebra isomorphism \( \Lambda^+_p \rightarrow \Lambda^+_p \langle p \rangle \) for \( p \in T \), and \( G_{R_K}(\sigma(p)) = \sigma G_{R_K}(p) \sigma^{-1} \) in \( G_{R_K} \).

**Lemma 6.12.** For each \( p \in T \), the representation \( V|_{G_{R_K}(p)} \) is \( B_{st,K}(\Lambda^+_{p}) \)-admissible. Moreover, \( (B_{st,K}(\Lambda^+_{p}) \otimes \mathbb{Q}_p) V_{G_{R_K}(p)} = (B_{st,K}(\Lambda^+_{p}) \otimes \mathbb{Q}_p) V_{G_{R_K}(p)} \).

**Proof.** Choose \( \sigma \in G_{R_K} \) with \( \sigma(p_0) = p \). Hence, \( G_{R_K}(p) = \sigma G_{R_K}(p_0) \sigma^{-1} \) in \( G_{R_K} \), and \( \sigma \) induces isomorphisms \( \Lambda^+_p \cong \Lambda^+_p \) and \( \sigma: B_{st,K}(\Lambda^+_{p_0}) \cong B_{st,K}(\Lambda^+_p) \). These maps are compatible with the group homomorphism \( G_{R_K}(p_0) \rightarrow G_{R_K}(p) \) given by \( \sigma \)-conjugation.

Hence, for every \( \tau \in G_R(p_0) \), we have a commutative diagram as follows.

\[
\begin{array}{ccc}
B_{st,K}(\Lambda^+_{p_0}) \otimes \mathbb{Q}_p V & \xrightarrow{\sigma \otimes \tau} & B_{st,K}(\Lambda^+_{p}) \otimes \mathbb{Q}_p V \\
\tau \otimes \sigma & & \sigma \tau \sigma^{-1} \otimes \sigma \tau \sigma^{-1} \\
B_{st,K}(\Lambda^+_{p_0}) \otimes \mathbb{Q}_p V & \cong & B_{st,K}(\Lambda^+_{p}) \otimes \mathbb{Q}_p V
\end{array}
\]

In particular, if \( x \in B_{st,K}(\Lambda^+_{p_0}) \otimes \mathbb{Q}_p V \) is \( G_{R_K}(p_0) \)-invariant, then \( (\sigma \otimes \tau)(x) \in B_{st,K}(\Lambda^+_{p}) \otimes \mathbb{Q}_p V \) is \( G_{R_K}(p) \)-invariant. It follows that \( \sigma \otimes \tau \) restricts to a \( K \)-linear isomorphism \( D_{B_{st,K}(\Lambda^+_{p_0})}(V|_{G_{R_K}(p_0)}) \cong D_{B_{st,K}(\Lambda^+_{p})}(V|_{G_{R_K}(p)}) \). Hence, we have \( \dim_K D_{B_{st,K}(\Lambda^+_{p})}(V|_{G_{R_K}(p)}) = \dim_{\mathbb{Q}_p} V \).

By Proposition 6.8 and Lemma 6.10, we conclude that \( V|_{G_{R_K}(p)} \) is \( B_{st,K}(\Lambda^+_{p}) \)-admissible.

Similarly, the second assertion follows from the one for \( p_0 \) via \( \sigma \otimes \sigma \).

Set \( \Lambda_T^+ := \prod_{p \in T} \Lambda^+_p = \prod_{p \in T} \widehat{R}_p \) and \( \Lambda_T := \Lambda_T^+[p^{-1}] \).

Hence, \( \Lambda_T \) is the product of \( \Lambda_p \) over \( p \in T \) as a \( C \)-Banach algebra.

**Lemma 6.13.** The map \( \Lambda_R^+ \rightarrow \Lambda_T^+ \) is injective with \( p \)-torsion-free cokernel.

**Proof.** Recall that \( \widehat{R} \) can be written as the union of Noetherian normal domains finite over \( R \). Since \( R/\pi R \) is an integral domain, we deduce that the natural map \( \widehat{R}/\pi \widehat{R} \rightarrow \prod_{p \in T} \widehat{R}_p/\pi \widehat{R}_p \) is injective. It follows that \( \widehat{R}/\pi^m \widehat{R} \rightarrow \prod_{p \in T} \widehat{R}_p/\pi^m \widehat{R}_p \) is injective for every \( m \in \mathbb{N} \). By taking the inverse limit over \( m \), we see that \( \Lambda_T^+ \rightarrow \prod_{p \in T} \Lambda_T^+ = \Lambda_T^+ \) is injective with \( \pi \)-torsion-free cokernel. Hence, the cokernel is also \( p \)-torsion-free.

Each \( \sigma \in G_{R_K} \) induces an \( R \)-algebra isomorphism \( \Lambda^+_p \rightarrow \Lambda^+_p \langle p \rangle \) for \( p \in T \). This gives rise to a natural \( G_{R_K} \)-action on \( \Lambda_T^+ \) making the injection \( \Lambda_R^+ \rightarrow \Lambda_T^+ \) \( G_{R_K} \)-equivariant (cf. [Bri08, Remark 3.3.2]).

Since products commute with inverse limits and the Witt vector functor, we have \( \Lambda_{\text{inf}}(\Lambda_T^+) = \prod_{p \in T} \Lambda_{\text{inf}}(\Lambda_T^+) \). Since \( O_K \) is finite free over \( W(k) \), we conclude that \( \Lambda_{\text{inf},K}(\Lambda_T^+) = \prod_{p \in T} \Lambda_{\text{inf},K}(\Lambda_T^+) \), and \( \theta_{\Lambda_{\text{inf},K}(\Lambda_T^+)} \) is the product of \( \theta_{\Lambda_{\text{inf},K}(\Lambda_T^+)} \) over \( p \in T \).
Lemma 6.14. The following commutative diagram is Cartesian.

\[
\begin{array}{ccc}
\mathbb{B}^+_{st,K}(\Lambda^+_R) & \longrightarrow & \mathbb{B}^+_{dR}(\Lambda^+_R) \\
\downarrow & & \downarrow \\
\mathbb{B}^+_{st,K}(\Lambda^+_T) & \longrightarrow & \mathbb{B}^+_{dR}(\Lambda^+_T)
\end{array}
\]

Proof. This follows from Proposition 2.37 and Lemma 6.13. \qed

For each \( p \in T \), choose \( \sigma_p \in G_{RK} \) such that \( \sigma_p(p_0) = p \) and \( \sigma_p|_R = \text{id} \). This is possible since \( R/\pi R \) is geometrically integral over \( K \). We also assume that \( \sigma_{p_0} = \text{id} \). We have an isomorphism

\[
\sigma_p \otimes \sigma_p : \mathbb{B}^+_{st,K}(\Lambda^+_p) \otimes_{Q_p} V \xrightarrow{\cong} \mathbb{B}^+_{st,K}(\Lambda^+_p) \otimes_{Q_p} V
\]

compatible with the homomorphism \( G_{RK}(p_0) \to G_{RK}(p) \) given by \( \sigma_p \)-conjugation.

Lemma 6.15.

(i) The map

\[
\mathbb{B}^+_{st,K}(\Lambda^+_T) \to \prod_{p \in T} \mathbb{B}^+_{st,K}(\Lambda^+_p)
\]

is injective.

(ii) If \( x = (x_p)_p \in \prod_{p \in T}(\mathbb{B}^+_{st,K}(\Lambda^+_p) \otimes_{Q_p} V) \) satisfies \( (\sigma_p \otimes \sigma_p)(x_p) = x_p \) for every \( p \in T \), then

\[
x \in \mathbb{B}^+_{st,K}(\Lambda^+_T) \otimes_{Q_T} V.
\]

Proof. Consider \( \theta_{\Lambda_{inf,K},(\Lambda^+_p)} : \Lambda_{inf,K}(\Lambda^+_p) \to \Lambda^+_p \) and choose an \( O_K \)-linear section \( s_{p_0} : \Lambda^+_p \to \Lambda_{inf,K}(\Lambda^+_p) \) of \( \theta_{\Lambda_{inf,K},(\Lambda^+_p)} \) as in Construction 2.19. By the argument there, it defines a \( K \)-linear isomorphism

\[
\tilde{\theta}_{v, p_0} : \mathbb{B}^+_{st,K}(\Lambda^+_p) \xrightarrow{\cong} \Lambda^+_p(X)[\log(1 + X)].
\]

For each \( p \in T \), we set

\[
s_p := \sigma_p \circ s_{p_0} \circ \sigma_p^{-1} : \Lambda^+_p \to \Lambda^+_p \to \Lambda_{inf,K}(\Lambda^+_p) \to \Lambda_{inf,K}(\Lambda^+_p).
\]

Then \( s_p \) is an \( O_K \)-linear section of \( \theta_{\Lambda_{inf,K},(\Lambda^+_p)} \) satisfying \( s_p \circ \sigma_p = \sigma_p \circ s_{p_0} \). It also yields a \( K \)-linear isomorphism

\[
\tilde{\theta}_{v, p} : \mathbb{B}^+_{st,K}(\Lambda^+_p) \xrightarrow{\cong} \Lambda^+_p(X)[\log(1 + X)].
\]

Let \( \sigma_p : \Lambda_{p_0}(X)[\log(1 + X)] \to \Lambda_p(X)[\log(1 + X)] \) denote the continuous \( K \)-linear map extending \( \sigma_p : \Lambda_{p_0} \to \Lambda_p \) by \( \sigma_p(X'(\log(1 + X))^m) = X'(\log(1 + X))^m \). We see that \( \tilde{\theta}_{v, p} \circ \sigma_p = \sigma_p \circ \tilde{\theta}_{v, p_0} \) by Lemma 2.20(ii), Proposition 2.27, and the assumption \( \sigma_p|_R = \text{id} \).

Similarly,

\[
\prod_{p \in T} s_p : \Lambda^+_T = \prod_{p \in T} \Lambda^+_p \to \prod_{p \in T} \Lambda_{inf,K}(\Lambda^+_p) = \Lambda_{inf,K}(\Lambda^+_T)
\]

is an \( O_K \)-linear section of \( \theta_{\Lambda_{inf,K},(\Lambda^+_T)} \), and it yields a \( K \)-linear isomorphism

\[
\tilde{\theta}_{v, T} : \mathbb{B}^+_{st,K}(\Lambda^+_T) \xrightarrow{\cong} \Lambda_T(X)[\log(1 + X)].
\]

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Then we a commutative diagram as follows.

\[
\begin{array}{ccc}
\mathbb{B}^+_\text{st,K}(\Lambda^+_T) & \longrightarrow & \prod_{p \in T} \mathbb{B}^+_\text{st,K}(\Lambda^+_p) \\
\tilde{\theta}_{\text{st},T} & \cong & \prod \tilde{\theta}_{\text{st},p} \\
\Lambda_T\langle X \rangle [\log(1 + X)] & \longrightarrow & \prod_{p \in T} \Lambda_p\langle X \rangle [\log(1 + X)]
\end{array}
\]

For \( h \in \mathbb{N} \) and \( * \in \{ T, p \} \), we denote the \( \Lambda_{\text{max,K}}^+(\Lambda^+_T) \)-submodule \( \bigoplus_{i=0}^h \Lambda_{\text{st,K}}^+(\Lambda^+_T) \) of \( \mathbb{B}^+_\text{st,K}(\Lambda^+_T) \) by \( \Lambda_{\text{st,K}}^{\leq h}(\Lambda^+_T) \). Note that \( \mathbb{B}^+_\text{st,K}(\Lambda^+_T) = \bigcup_h \Lambda_{\text{st,K}}^{\leq h}(\Lambda^+_T)[p^{-1}] \). By Lemma 2.20(ii) and (iv), we have the functorial identification

\[
\tilde{\theta}_{\text{st},*} : \Lambda_{\text{st,K}}^{\leq h}(\Lambda^+_T) \cong \bigoplus_{i=0}^h \Lambda^+_T\langle X \rangle (\log(1 + X))^i =: \Lambda^+_T\langle X \rangle [\log(1 + X)]^{\leq h}
\]

as \( \mathcal{O}_K \)-modules.

Recall that

\[
\Lambda^+_T\langle X \rangle = \left\{ \sum_{i=0}^{\infty} \lambda_i X^i \in \Lambda^+_T[[X]] \mid \forall \ l \in \mathbb{N}, \ \lambda_i \in p^l \Lambda^+_T \text{ for all but finitely many } i \right\}.
\]

In particular, \( \Lambda^+_T\langle X \rangle \to \prod_{p \in T} (\Lambda^+_p\langle X \rangle [\log(1 + X)]^{\leq h}) \) is injective.

We now prove part (i). By the above remark and the transcendence of \( \log(1 + X) \) (Lemma 2.26), we see that the map

\[
\Lambda_T^+\langle X \rangle [\log(1 + X)]^{\leq h} \to \prod_{p \in T} (\Lambda_p^+\langle X \rangle [\log(1 + X)]^{\leq h})
\]

is injective. Hence, the map \( \Lambda_{\text{st,K}}^{\leq h}(\Lambda^+_T) \to \prod_{p \in T} \Lambda_{\text{st,K}}^{\leq h}(\Lambda^+_T) \) is injective and part (i) follows.

Turning to part (ii), consider the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{B}^+_\text{st,K}(\Lambda^+_p) \otimes_{\mathbb{Q}_p} V & \longrightarrow & \Lambda_p^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Q}_p} V \\
\sigma_p \otimes \sigma_p & \cong & \sigma_p \otimes \sigma_p \\
\mathbb{B}^+_\text{st,K}(\Lambda^+_p) \otimes_{\mathbb{Q}_p} V & \longrightarrow & \Lambda_p^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Q}_p} V
\end{array}
\]

Then the assertion is equivalent to saying that if \( x = (x_p)_p \in \prod_{p \in T} (\Lambda_p^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Q}_p} V) \) satisfies \( (\sigma_p \otimes \sigma_p)(x_p) = x_p \) for every \( p \in T \), then \( x \in \Lambda_T^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Q}_p} V \).

Fix a \( \mathcal{G}_K \)-stable \( \mathbb{Z}_p \)-lattice \( V_{\mathbb{Z}_p} \) of \( V \) and choose such \( x = (x_p)_p \). There exist \( m, h \in \mathbb{N} \) such that \( p^m x_p \in \Lambda_p^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p} \). Then for every \( p \in T \), we have

\[
p^m x_p = (\sigma_p \otimes \sigma_p)(p^m x_p) \in \Lambda_p^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}.
\]

Moreover, we see that if there exist \( l \in \mathbb{N} \) and \( f(X) \in \Lambda_p^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p} \) (i.e. a polynomial in \( X \) and \( \log(1 + X) \)) such that

\[
p^m x_p - f(X) \in p^l (\Lambda_p^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}),
\]

then

\[
p^m x_p - (\sigma_p \otimes \sigma_p)(f(X)) \in p^l (\Lambda_p^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}).
\]

This implies that \( p^m x \in \Lambda_T^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p} \), and thus \( x \in \Lambda_T^+\langle X \rangle [\log(1 + X)] \otimes_{\mathbb{Q}_p} V \).
Remark 6.16. With the notation as in the proof, the same argument as in the last part of the proof shows the following assertion. Let $D_{O_K}$ be a finite free $O_K$-module and let $W \subset A_{st, K}^h(\Lambda^+_p) \otimes O_K D_{O_K}$ be a finite $\mathbb{Z}_p$-submodule. Then the submodule $\prod_{p \in T}((\sigma_p \otimes \text{id})(W))$ of $\prod_{p \in T}(A_{st, K}^h(\Lambda^+_p) \otimes O_K D_{O_K})$ lands in $A_{st, K}^h(\Lambda^+_T) \otimes O_K D_{O_K}$. In particular, if $x = (x_p) \in \prod_{p \in T}(A_{st, K}^h(\Lambda) \otimes O_K D_{O_K})$ satisfies $x_p \in (\sigma_p \otimes \text{id})(W) \subset A_{st, K}^h(\Lambda^+_p) \otimes O_K D_{O_K}$ for every $p \in T$, then $x \in A_{st, T}^h(\Lambda^+_T) \otimes O_K D_{O_K}$.

Lemma 6.17. Let $x \in (\mathbb{B}^+_{dR}(\Lambda_R^+) \otimes Q_p)_{\mathcal{G}_{R_K}}$ and let $x_p$ denote the image of $x$ in $(\mathbb{B}^+_{dR}(\Lambda_p^+) \otimes Q_p)_{\mathcal{G}_{R_K}(p)}$. Then for each $p \in T$, we have $(\sigma_p \otimes \sigma_p)(x_{p_0}) = x_p$.

Proof. This results from the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{B}^+_{dR}(\Lambda_R^+) \otimes Q_p V & \xrightarrow{\sigma_p \otimes \sigma_p} & \mathbb{B}^+_{dR}(\Lambda_R^+) \otimes Q_p V \\
| & & | \\
\mathbb{B}^+_{dR}(\Lambda_{p_0}^+) \otimes Q_p V & \xrightarrow{\sigma_p \otimes \sigma_p} & \mathbb{B}^+_{dR}(\Lambda_{p_0}^+) \otimes Q_p V
\end{array}
\]

Namely, we have $(\sigma_p \otimes \sigma_p)(x_{p_0}) = ((\sigma_p \otimes \sigma_p)(x))_p = x_p$. \hfill \Box

Lemma 6.18. The inclusion

$$(\mathbb{B}^+_{st, K}(\Lambda_R) \otimes Q_p V)^{\mathcal{G}_{R_K}} \subset (\mathbb{B}^+_{dR}(\Lambda_R) \otimes Q_p V)^{\mathcal{G}_{R_K}}$$

is an equality.

Proof. Let $x \in (\mathbb{B}^+_{dR}(\Lambda_R^+) \otimes Q_p V)_{\mathcal{G}_{R_K}}$. For each $p \in T$, let $x_p$ denote the image of $x$ in $(\mathbb{B}^+_{dR}(\Lambda_p^+) \otimes Q_p V)_{\mathcal{G}_{R_K}(p)}$. Then we have $(\sigma_p \otimes \sigma_p)(x_{p_0}) = x_p$ by the previous lemma. Since $(\mathbb{B}^+_{st, K}(\Lambda_p^+) \otimes Q_p V)^{\mathcal{G}_{R_K}(p)} = (\mathbb{B}^+_{dR}(\Lambda_p^+) \otimes Q_p V)^{\mathcal{G}_{R_K}(p)}$ by Lemma 6.12, we have $(x_p)_{p \in T} \in \prod_{p \in T}(\mathbb{B}^+_{st, K}(\Lambda_p^+) \otimes Q_p V)$. Hence, we get $x \in \mathbb{B}^+_{st, K}(\Lambda_T^+) \otimes Q_p V$ by Lemma 6.15(ii). Since the commutative diagram

\[
\begin{array}{ccc}
\mathbb{B}^+_{st, K}(\Lambda_R^+) \otimes Q_p V & \longrightarrow & \mathbb{B}^+_{dR}(\Lambda_R^+) \otimes Q_p V \\
| & & | \\
\mathbb{B}^+_{st, K}(\Lambda_T^+) \otimes Q_p V & \longrightarrow & \mathbb{B}^+_{dR}(\Lambda_T^+) \otimes Q_p V
\end{array}
\]

is Cartesian by Lemma 6.14, we conclude that $x \in \mathbb{B}^+_{st, K}(\Lambda_R^+) \otimes Q_p V$. \hfill \Box

Let us complete the proof of Theorem 6.1. Set

$$D := (\mathbb{B}^+_{st, K}(\Lambda_R^+) \otimes Q_p V)^{\mathcal{G}_{R_K}} = (\mathbb{B}^+_{dR}(\Lambda_R^+) \otimes Q_p V)^{\mathcal{G}_{R_K}} = (\mathbb{B}^+_{dR}(\Lambda_R^+) \otimes Q_p V)^{\mathcal{G}_{R_K}}.$$

This is a $K$-vector space and $\dim_K D = \dim_{Q_p} V$. Consider the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{B}^+_{st, K}(\Lambda_R^+) \otimes K D & \xrightarrow{\alpha^+_{st, K}(V)} & \mathbb{B}^+_{st, K}(\Lambda_R^+) \otimes Q_p V \\
\downarrow & & \downarrow \\
\mathbb{B}^+_{dR}(\Lambda_R^+) \otimes K D & \xrightarrow{\alpha^+_{dR}(V)} & \mathbb{B}^+_{dR}(\Lambda_R^+) \otimes Q_p V
\end{array}
\]
We need to show that $\alpha_{\text{st}, K}^\nabla(V)$ is surjective. Choose $m \in \mathbb{N}$ such that the $\mathbb{Q}_p$-vector subspace $\mathbb{Q}_p^{im} \otimes_{\mathbb{Q}_p} V$ of $\mathbb{B}_{\text{dR}}(\Lambda^+_{R}) \otimes_{\mathbb{Q}_p} V$ is contained in the image of $\mathbb{B}_{\text{dR}}^+(\Lambda^+_{R}) \otimes_K D$ under $\alpha_{\text{dR}}(V)$. Fix a $G_{R_K}$-stable $\mathbb{Z}_p$-lattice $V_{Z_p}$ of $V$ and set $V'_{Z_p} := \mathbb{Q}_p^{im} \otimes_{\mathbb{Z}_p} V_{Z_p}$. It suffices to prove that $V'_{Z_p}$, as a submodule of $\mathbb{B}_{\text{st}, K}(\Lambda^+_{R}) \otimes_{\mathbb{Q}_p} V$, is contained in the image of $\alpha_{\text{st}, K}^\nabla(V)$.

For $h \in \mathbb{N}$ and $\ast \in \{R, T, p\}$, consider the $A_{\text{max}, K}(\Lambda^+_{R})$-submodule

$$A_{\text{st}, K}^\leq h(\Lambda^+_{R}) := \bigoplus_{i=0}^{h} A_{\text{max}, K}(\Lambda^+_p) \left( \log \left( \frac{\pi^n}{\pi^i} \right) \right) \subset \mathbb{B}_{\text{st}, K}^+(\Lambda^+_p).$$

Note that $\mathbb{B}_{\text{st}, K}^+(\Lambda^+_p) = \bigcup_{h} A_{\text{st}, K}^\leq h(\Lambda^+_p)[p^{-1}]$. Observe that each $A_{\text{st}, K}^\leq h(\Lambda^+_p)[p^{-1}]$ is stable under the action of $G_{R_K}$ and

$$(A_{\text{st}, K}^\leq h(\Lambda^+_p)[p^{-1}]) G_{R_K} = (A_{\text{st}, K}^\leq h(\Lambda^+_p)[p^{-1}]) G_{R_K}(p) = K.$$ 

Since $D$ is finite-dimensional over $K$, there exists $h \in \mathbb{N}$ such that

$$D = (A_{\text{st}, K}^\leq h(\Lambda^+_p)[p^{-1}] \otimes_{\mathbb{Q}_p} V)^{G_{R_K}}.$$ 

It follows that for each $p \in T$, we have the isomorphism

$$D \xrightarrow{\cong} (A_{\text{st}, K}^\leq h(\Lambda^+_p)[p^{-1}] \otimes_{\mathbb{Q}_p} V)^{G_{R_K}(p)} = (\mathbb{B}_{\text{dR}}(\Lambda^+_p) \otimes_{\mathbb{Q}_p} V)^{G_{R_K}(p)}.$$ 

By increasing $h$ if necessary, we may further assume that $V'_{Z_p}$, as a submodule of $\mathbb{B}_{\text{st}, K}(\Lambda^+_{R_0}) \otimes_{\mathbb{Q}_p} V$, is contained in the image of $A_{\text{st}, K}^\leq h(\Lambda^+_{R_0})[p^{-1}] \otimes_K D$ under the isomorphism

$$\alpha_{\text{st}, K}^\nabla(V|_{G_{R_K}(p_0)}): \mathbb{B}_{\text{st}, K}(\Lambda^+_{R_0}) \otimes_K D \xrightarrow{\cong} \mathbb{B}_{\text{st}, K}(\Lambda^+_{R_0}) \otimes_{\mathbb{Q}_p} V.$$ 

Choose an $O_{K}$-lattice $D_{O_K}$ of $D$ such that $V'_{Z_p}$, as a submodule of $\mathbb{B}_{\text{st}, K}(\Lambda^+_{R_0}) \otimes_{\mathbb{Q}_p} V$, is contained in the image of $A_{\text{st}, K}^\leq h(\Lambda^+_{R_0}) \otimes_{O_K} D_{O_K}$ under $\alpha_{\text{st}, K}^\nabla(V|_{G_{R_K}(p_0)})$.

For each $p \in T$, the functoriality of the construction gives a commutative diagram as follows.

$$\begin{CD}
A_{\text{st}, K}^\leq h(\Lambda^+_{R_0}) \otimes_{O_K} D_{O_K} @>\alpha_{\text{st}, K}^\nabla(V|_{G_{R_K}(p_0)})>> \mathbb{B}_{\text{st}, K}(\Lambda^+_{R_0}) \otimes_{\mathbb{Q}_p} V \\
@V\sigma_p \otimes \text{id} VV
A_{\text{st}, K}^\leq h(\Lambda^+_{R_0}) \otimes_{O_K} D_{O_K} @>\alpha_{\text{st}, K}^\nabla(V|_{G_{R_K}(p)})>> \mathbb{B}_{\text{st}, K}(\Lambda^+_{R_0}) \otimes_{\mathbb{Q}_p} V
\end{CD}$$

Since $V'_{Z_p}$ is $G_{R_K}$-stable, we conclude that $V'_{Z_p}$, as a submodule of $\mathbb{B}_{\text{st}, K}(\Lambda^+_{R}) \otimes_{\mathbb{Q}_p} V$, is contained in the image of $A_{\text{st}, K}^\leq h(\Lambda^+_{R}) \otimes_{O_K} D_{O_K}$ under $\alpha_{\text{st}, K}^\nabla(V|_{G_{R_K}(p)})$.

Consider the isomorphism

$$\alpha_{\text{dR}}^\nabla(V): \mathbb{B}_{\text{dR}}(\Lambda^+_{R}) \otimes_K D \xrightarrow{\cong} \mathbb{B}_{\text{dR}}(\Lambda^+_{R}) \otimes_{\mathbb{Q}_p} V$$

and let $W \subset \mathbb{B}_{\text{dR}}(\Lambda^+_{R}) \otimes_K D$ denote the inverse image of $V'_{Z_p}$, as a submodule of $\mathbb{B}_{\text{dR}}(\Lambda^+_{R}) \otimes_{\mathbb{Q}_p} V$, under $\alpha_{\text{dR}}^\nabla(V)$. Observe that $W \subset \mathbb{B}_{\text{dR}}(\Lambda^+_{R}) \otimes_K D$. By the above discussion, the image of $W$ under the map

$$\mathbb{B}_{\text{dR}}(\Lambda^+_{R}) \otimes_K D \rightarrow \prod_{p \in T} (\mathbb{B}_{\text{dR}}(\Lambda^+_{R}) \otimes_K D)$$

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lands in the subspace $\prod_{p \in T}(A_{st,K}^h(\Lambda^+_p) \otimes O_K D_{O_K})$. It follows from Remark 6.16 that the image of $W$ under the map

$$\mathbb{B}^+_\text{dr}(\Lambda^+_R) \otimes_K D \to \mathbb{B}^+_\text{dr}(\Lambda^+_T) \otimes_K D$$

lands in the subspace $A_{st,K}^h(\Lambda^+_T) \otimes O_K D_{O_K}$ and thus in $\mathbb{B}^+_\text{st}(\Lambda^+_T) \otimes_K D$. By Lemma 6.14, we conclude that $W$ lands in the subspace

$$\mathbb{B}^+_\text{st}(\Lambda^+_R) \otimes_K D \subset \mathbb{B}^+_\text{dr}(\Lambda^+_R) \otimes_K D.$$ 

This implies that $v'_p$, as a submodule of $\mathbb{B}_{st,K}(\Lambda^+_R) \otimes \mathbb{Q}_p V$, is contained in the image of $\sigma_{st,K}^*(V)$. Hence, $V$ is a horizontal semistable representation of $G_{R_K}$. This completes the proof of Theorem 6.1.

7. $p$-adic monodromy theorem for horizontal de Rham representations

7.1 Horizontal version of $p$-adic monodromy theorem for Galois representations

Let us recall a result of Ohkubo. We use a slightly different notation from [Ohk13]. Let $\mathcal{K}$ be a complete discrete valuation field of mixed characteristic $(0,p)$ and let $k_{\mathcal{K}}$ denote the residue field of $\mathcal{K}$.

Fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. Define the subfield $\mathcal{K}_{\text{can}}$ of $\mathcal{K}$ as the algebraic closure of $W(k_{\mathcal{K}})[p^{-1}]$ in $\mathcal{K}$, where $k_{\mathcal{K}}^\infty := \bigcap_{m \in \mathbb{N}} k_{\mathcal{K}}^m$. We use a similar notation for finite extensions of $\mathcal{K}$.

Ohkubo proves the $p$-adic monodromy theorem for horizontal de Rham representations of $\mathcal{K}$.

**Theorem 7.1** [Ohk13, Theorem 7.4]. Let $V_\mathcal{K}$ be a horizontal de Rham representation of $\text{Gal}(\overline{\mathcal{K}}/\mathcal{K})$. Then there exists a finite extension $L$ of $\mathcal{K}_{\text{can}}$ such that $V_\mathcal{K}|_{\text{Gal}(\overline{\mathcal{K}}/\mathcal{K}L)}$ is horizontal semistable, where $\mathcal{K}L$ is the composite field of $\mathcal{K}$ and $L$ in $\overline{\mathcal{K}}$.

**Remark 7.2.** [Ohk13, Theorem 7.6] claims that every horizontal de Rham representation of $\text{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ comes from a de Rham representation of $\text{Gal}(\overline{\mathcal{K}}_{\text{can}}/\mathcal{K}_{\text{can}})$ via the pullback along $\text{Gal}(\overline{\mathcal{K}}/\mathcal{K}) \to \text{Gal}(\overline{\mathcal{K}}_{\text{can}}/\mathcal{K}_{\text{can}})$, which is not correct. The error comes from an incorrect argument in the proof of [Ohk13, Proposition 7.3] (cf. Remark 4.27 and Example 4.28). However, these errors are irrelevant to the proof of the horizontal $p$-adic monodromy theorem [Ohk13, Theorem 7.4], and the latter theorem is correct.

7.2 $p$-adic monodromy theorem for horizontal de Rham representations

Let $\mathcal{K}$ be a complete discrete valuation field of characteristic zero with perfect residue field $k$ of characteristic $p$. Fix a uniformizer $\pi$ of $\mathcal{K}$. Let $R$ be an $O_\mathcal{K}$-algebra satisfying the conditions in Set-up 3.1. We further assume that $R$ satisfies condition (BR) of Definition 3.3. For each finite extension $L$ of $\mathcal{K}$, we denote $O_L \otimes_{O_\mathcal{K}} R$ by $R_{O_L}$. Then $R_{O_L}$ also satisfies the same properties with respect to $L$. As before, we write $R_{\mathcal{K}}$ (respectively, $R_L$) for $R[p^{-1}]$ (respectively, $R_{O_L}[p^{-1}]$).

Theorem 1.9 is a special case of the following $p$-adic monodromy theorem for horizontal de Rham representations.

**Theorem 7.3.** Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_{R_K})$. If $V$ is horizontal de Rham, then there exists a finite extension $L$ of $\mathcal{K}$ such that $V|_{G_{R_{O_L}}}^*$ is horizontal semistable.

**Proof.** We use the notation in § 6.1. Consider the complete discrete valuation field $\mathcal{K} := \widehat{R}(\pi)[p^{-1}]$. Fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ and an $R$-algebra embedding $\overline{R} \hookrightarrow \overline{\mathcal{K}}$. Note $\mathcal{K}_{\text{can}} = \mathcal{K}$ by Lemma 6.3 and Remark 6.4.

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5 [Ohk13] uses $K$ for our $\mathcal{K}$. In our paper, $K$ is a finite extension of $\mathbb{Q}_p$.  

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Since $V$ is a horizontal de Rham representation of $\mathcal{G}_{R_K}$, $V_K := V|_{\text{Gal}(\mathcal{K}/K)}$ is a horizontal de Rham representation of $\text{Gal}(\mathcal{K}/K)$. Hence, by Theorem 7.1, there exists a finite extension $L$ of $K$ such that $V_K|_{\text{Gal}(\mathcal{K}/L)}$ is horizontal semistable.

We prove that $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable. Let $\pi_L$ denote a uniformizer of $L$. Since $R/\pi R$ is geometrically integral over $k$, $(\pi_L) \subset R_{O_L}$ is a prime ideal, and thus the ring of integers of $KL$ coincides with the $p$-adic completion $(R_{O_L})_{(\pi_L)}$ of the localization of $R_{O_L}$ at $(\pi_L)$. Hence, $KL = (R_{O_L})_{(\pi_L)}[p^{-1}]$.

We know that $V|_{\mathcal{G}_{R_L}}|_{\text{Gal}(\mathcal{K}/L)} = V_K|_{\text{Gal}(\mathcal{K}/L)}$ is a horizontal semistable representation of $\text{Gal}(\mathcal{K}/L)$. The embedding $R_{O_L} \hookrightarrow \mathcal{K}$ determines a prime ideal $p_0$ of $\overline{R_{O_L}}$ above $(\pi_L) \subset R_{O_L}$, and $\text{Gal}(\overline{K}/L)$ coincides with $\widehat{\mathcal{G}_{R_L}}(p_0)$. Therefore $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable by Theorem 6.1.

**Theorem 7.4.** Let $V$ be a horizontal de Rham representation of $\mathcal{G}_{R_K}$. Assume that there exists a finite extension $K'$ of $K$ and a map $f: R \to K'$ such that $f^*V$ is a potentially crystalline representation of $\text{Gal}(\overline{K}/K')$. Then there exists a finite extension $L$ of $K$ such that $V|_{\mathcal{G}_{R_L}}$ is horizontal crystalline. 

**Proof.** By extending $K'$ if necessary, we may assume that the representation $f^*V$ of $\text{Gal}(\overline{K}/K')$ is crystalline (equivalently, horizontal crystalline). By Theorem 7.3, there exists a finite extension $L$ of $K'$ such that $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable. Then the assertion follows from Proposition 4.20. 

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### 8. $p$-adic Hodge theory for rigid analytic varieties

In this section, we review $p$-adic local systems and period sheaves. Let $K$ be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic $p$.

We work on the étale site $X_{\text{ét}}$ (cf. [Hub96, Definition 2.3.1]) and the pro-étale site $X_{\text{pro ét}}$. For the pro-étale site, we use the one defined in [Sch13, Sch16], and thus we have the projection $\nu: X_{\text{pro ét}} \to X_{\text{ét}}$.

Recall that $U \in X_{\text{pro ét}}$ is called **affinoid perfectoid** if $U$ has a pro-étale presentation $U = \lim U_i \to X$ by affinoid $U_i = \text{Spa}(A_i, A_i^+)$ such that, denoting by $A^+$ the $p$-adic completion of $\lim A_i^+$, and $A = A^+[p^{-1}]$, the pair $(A, A^+)$ is a perfectoid affinoid $(L, L^+)$-algebra for a perfectoid field $L$ over $K$ and an open and bounded valuation subring $L^+ \subset L$ (cf. [Sch13, Definition 4.3(i)]). In this case, we write $\hat{U}$ for $\text{Spa}(A, A^+)$, which is independent of the pro-étale presentation $U = \lim U_i$.

#### 8.1 $p$-adic local systems on $X_{\text{ét}}$

Let us clarify our terminology of local systems.

**Definition 8.1** (cf. [Sch13, Definition 8.1], [KL15, §§ 1.4 and 8.4]). A **$\mathbb{Z}_p$-local system** on $X_{\text{ét}}$ (or an étale $\mathbb{Z}_p$-local system on $X$) is an inverse system $\mathbb{L} = (\mathbb{L}_n)$ of sheaves of $\mathbb{Z}/p^n$-modules $\mathbb{L}_n$ on $X_{\text{ét}}$ such that each $\mathbb{L}_n$ is locally on $X_{\text{ét}}$ a constant sheaf associated with a finitely generated flat $\mathbb{Z}/p^n$-module and such that this inverse system is isomorphic in the procategory to an inverse system for which $\mathbb{L}_{n+1}/p^n \cong \mathbb{L}_n$.

**Definition 8.2** (cf. [KL15, §§ 1.4 and 8.4]). An **isogeny $\mathbb{Z}_p$-local system** on $X_{\text{ét}}$ is an object of the isogeny category of $\mathbb{Z}_p$-local systems on $X_{\text{ét}}$. A **$\mathbb{Q}_p$-local system** on $X_{\text{ét}}$ (or an étale $\mathbb{Q}_p$-local system on $X$) is an object of the stack associated to the fibered category of isogeny $\mathbb{Z}_p$-local systems on $X_{\text{ét}}$. 

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Remark 8.3. Assume that \(X = \text{Spa}(A, A^+)\) is an affinoid. In this case, the natural functor from the category of lisse \(\mathbb{Z}_p\)-sheaves on the étale site (\(\text{Spec}\,A\))_ét to the category of \(\mathbb{Z}_p\)-local systems on \(\text{Spa}(A, A^+)\)_ét is an equivalence of categories by [Hub96, Example 1.6.6 (ii)] (cf. [KL15, Remark 8.4.5]).

Moreover, if \(A^+\) satisfies conditions in Set-up 3.1 relative to \(K\), then the category \(\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_A)\) is equivalent to the category of isogeny \(\mathbb{Z}_p\)-local systems on \(\text{Spa}(A, A^+)\_ét\) (cf. [KL15, Remark 8.4.4]).

Set \(\widehat{\mathbb{Z}}_p = \lim_{\to} \mathbb{Z}/p^n\) and \(\widehat{\mathbb{Q}}_p = \mathbb{Q}_p[p^{-1}]\) as sheaves on \(X_{\text{proét}}\). For a \(\mathbb{Z}_p\)-local system \(\mathbb{L} = (L_n)\) on \(X_\text{ét}\), we set \(\widehat{\mathbb{L}} = \lim_{\to} \nu^*L_n\). Then \(\widehat{\mathbb{L}}\) is a lisse \(\widehat{\mathbb{Q}}_p\)-sheaf on \(X_{\text{proét}}\) in the sense of [Sch13, Definition 8.1]. This functor is an equivalence of categories. Similarly, we can naturally associate to every \(\mathbb{Q}_p\)-local system \(\mathbb{L}\) on \(X_\text{ét}\) a lisse \(\widehat{\mathbb{Q}}_p\)-sheaf on \(X_{\text{proét}}\), which we denote by \(\widehat{\mathbb{L}}\).

8.2 Period sheaves and the Riemann–Hilbert functor \(D_{\text{dr}}\)

We will use the following sheaves on \(X_{\text{proét}}\): the de Rham sheaf \(\mathbb{B}_{\text{dr}}\) (cf. [Sch13, Definition 6.1(iii)]) and the structural de Rham sheaf \(\mathcal{O}\mathbb{B}_{\text{dr}}\) (cf. [Sch13, Sch16]).

By [Sch13, Theorem 6.5(i)], for an affinoid perfectoid \(U \subseteq X_{\text{proét}}\) with \(\widehat{U} = \text{Spa}(A, A^+)\), we have

\[
\mathbb{B}_{\text{dr}}(U) = \mathbb{B}_{\text{dr}}(A, A^+),
\]

where the latter is the period ring defined in §2, which agrees with the one defined in [Sch13, p. 49].

The sheaf \(\mathcal{O}\mathbb{B}_{\text{dr}}\) is a \(\mathbb{B}_{\text{dr}}\)-algebra equipped with a filtration and a \(\mathbb{B}_{\text{dr}}\)-linear connection \(\nabla : \mathcal{O}\mathbb{B}_{\text{dr}} \to \mathcal{O}\mathbb{B}_{\text{dr}} \otimes_{\mathcal{O}_X} \Omega^1_X\) satisfying the Griffiths transversality and \((\mathcal{O}\mathbb{B}_{\text{dr}})\nabla = 0 = \mathbb{B}_{\text{dr}}\).

We have the following description of \(\mathcal{O}\mathbb{B}_{\text{dr}}\). Let \(\text{Spa}(A, A^+)\) be an affinoid admitting standard étale maps to both \(X\) and \(T^1_L\) for some finite extension \(L\) over \(K\). If we replace \(L\) with its algebraic closure in \(A\), then \(R = A^+\) satisfies conditions in Set-up 3.1 relative to \(L\). Write \(\widetilde{R} = \bigcap_{i \in I} A_i^+\) where \(A_i^+\) is a finite normal \(R\)-subalgebra of \(\text{Frac}\, R\) such that \(A_i = A_i^+[p^{-1}]\) is étale over \(A = R_K\). Set \(U := \bigcup_{i \in I} \text{Spa}(A_i, A_i^+) \subseteq X_{\text{proét}}\); this is affinoid perfectoid with \(\widehat{U} = \text{Spa}(\widetilde{R}[p^{-1}], \widetilde{R})\). In this setting, \(\mathbb{B}_{\text{dr}}(U) = \mathbb{B}_{\text{dr}}(\widetilde{R}) = \mathbb{B}_{\text{dr}}^\Sigma(R),\) and there is a natural \(\mathbb{B}_{\text{dr}}^\Sigma(R)\)-linear isomorphism

\[
\mathcal{O}\mathbb{B}_{\text{dr}}(U) \cong \mathbb{B}_{\text{dr}}(R),
\]

compatible with filtrations and connections, where the right-hand side is the de Rham period ring recalled in §4.4. To see this, note that \(\mathcal{O}\mathbb{B}_{\text{dr}}\) is defined to be the sheafification of the presheaf defined in [Sch16, (3)]. From the definitions, we have a \(\mathbb{B}_{\text{dr}}^\Sigma(R)\)-algebra homomorphism \(\mathbb{B}_{\text{dr}}(R) \to \mathcal{O}\mathbb{B}_{\text{dr}}(U)\). By comparing [Sch13, Proposition 6.10], [Sch16, (3)], and [Bri08, Proposition 5.2.2], we conclude that this map is an isomorphism compatible with filtrations and connections.

Definition 8.4 (cf. [Sch13, Definition 8.3]). Let \(\mathbb{L}\) be a \(\mathbb{Q}_p\)-local system on \(X_\text{ét}\). We say that \(\mathbb{L}\) is de Rham if there exists a filtered \(\mathcal{O}_X\)-module \(\mathcal{E}\) with an integrable connection such that there is an isomorphism of sheaves on \(X_{\text{proét}}\),

\[
\widehat{\mathbb{L}} \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbb{B}_{\text{dr}} \cong \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\text{dr}},
\]

compatible with filtrations and connections. Similarly, we say that a(n isogeny) \(\mathbb{Z}_p\)-local system on \(X_\text{ét}\) is de Rham if the associated \(\mathbb{Q}_p\)-local system is de Rham.

Let us recall an important theorem of Liu and Zhu.
Theorem 8.5 [LZ17, Theorem 3.9]. For a $\mathbb{Q}_p$-local system $L$ on $X_{\text{ét}}$, set

$$D_{\text{dR}}(L) := \nu_* (\hat{L} \otimes \hat{\mathbb{Q}}_p \mathcal{O}_{\text{dR}}).$$

(i) $D_{\text{dR}}(L)$ is a vector bundle on $X_{\text{ét}}$ equipped with an integrable connection

$$\nabla_L := \nu_* (\text{id} \otimes \nabla) : D_{\text{dR}}(L) \to D_{\text{dR}}(L) \otimes \mathcal{O}_{X_{\text{ét}}} \Omega_{X_{\text{ét}}}$$

and a decreasing filtration $\text{Fil}^\bullet$ satisfying the Griffiths transversality.

(ii) The functor $L \mapsto (D_{\text{dR}}(L), \nabla_L, \text{Fil}^\bullet)$ commutes with pullbacks.

(iii) The following conditions are equivalent for $L$:

(a) $L$ is de Rham;

(b) $\text{rank } D_{\text{dR}}(L) = \text{rank } L$;

(c) for each connected component, there exist a classical point $x$ of $X$ and a geometric point $\overline{x}$ above $x$ such that the stalk $L_{\overline{x}}$ is a de Rham representation of the absolute Galois group of $k(x)$.

In this case, the natural map $\nu^* D_{\text{dR}}(L) \otimes_{\mathcal{O}_X} \mathcal{O}_{\text{dR}} \to \hat{L} \otimes \hat{\mathbb{Q}}_p \mathcal{O}_{\text{dR}}$ is an isomorphism of sheaves on $X_{\text{proét}}$ compatible with filtrations and connections.

Remark 8.6. The category of vector bundles on $X$ (with the analytic topology) is equivalent to the category of vector bundles on $X_{\text{ét}}$ by [Sch13, Lemma 7.3]. Hence, we also regard $D_{\text{dR}}(L)$ as a vector bundle on $X$.

Lemma 8.7. Assume that $X = \text{Spa}(A, A^+)$ is affinoid such that $R = A^+$ satisfies conditions in Set-up 3.1 relative to $K$. Let $L$ be an isogeny $\mathbb{Z}_p$-local system on $X_{\text{ét}}$ and let $V \in \text{Rep}_{\mathbb{Q}_p}(G_A)$ be the corresponding representation. Then the vector bundle $D_{\text{dR}}(L)$ is associated to the $A$-module $D_{\text{dR}}(V)$ and this identification is compatible with connections and filtrations. Moreover, $L$ is de Rham if and only if $V$ is a de Rham representation.

Proof. Since $D_{\text{dR}}(L)$ is a coherent sheaf on $X$, it suffices to show that $\Gamma(X, D_{\text{dR}}(L))$ is naturally identified with $D_{\text{dR}}(V)$. By definition, we have

$$\Gamma(X, D_{\text{dR}}(L)) = \Gamma(X_{\text{proét}}, \hat{L} \otimes \hat{\mathbb{Q}}_p \mathcal{O}_{\text{dR}}).$$

Write $\overline{R} = \varprojlim_{i \in I} A_i^+$, where $A_i^+$ is a finite normal $R$-subalgebra of $\overline{\text{Frac } R}$ such that $A_i = A_i^+[p^{-1}]$ is étale over $A = R_K$. Set $U := \varprojlim_i \text{Spa}(A_i, A_i^+) \in X_{\text{proét}}$; this is affinoid perfectoid with $\hat{U} = \text{Spa}(\overline{R}[p^{-1}], \overline{R})$. Since every finite étale $A$-algebra splits over $\overline{R}[p^{-1}]$, the lisse $\hat{\mathbb{Q}}_p$-sheaf $\hat{L}$ is trivial over $U$ and identified with $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Q}}_p} V$. Hence, $\Gamma(U, \hat{L}) = V$ and $\Gamma(U, \hat{L} \otimes \hat{\mathbb{Q}}_p \mathcal{O}_{\text{dR}}) = V \otimes_{\hat{\mathbb{Q}}_p} B_{\text{dR}}(R)$.

Note that the map $U \to X$ is a profinite étale Galois cover with Galois group $G_A$. Regard the profinite group $G_A$ as an object of $\text{Spa}(K, \mathcal{O}_K)_{\text{proét}}$ (and thus an object of $X_{\text{proét}}$ by the pullback) and consider $U \times G_A \in X_{\text{proét}}$. Then we can check that there is a natural isomorphism $U \times_X U \cong U \times G_A$ compatible with the first projection to $U$. Under this identification, we have $\Gamma(U \times_X U, \hat{L} \otimes \hat{\mathbb{Q}}_p \mathcal{O}_{\text{dR}}) = \prod_{G_A} (V \otimes_{\hat{\mathbb{Q}}_p} B_{\text{dR}}(R))$. Moreover, the two maps $p_1^*, p_2^*: V \otimes_{\hat{\mathbb{Q}}_p} B_{\text{dR}}(R) \to \prod_{G_A} (V \otimes_{\hat{\mathbb{Q}}_p} B_{\text{dR}}(R))$ induced by $p_1, p_2: U \times_X U \to U$ send $v \in V \otimes_{\hat{\mathbb{Q}}_p} B_{\text{dR}}(R)$ to
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Let $K$ be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic $p$. For a rigid analytic variety $X$ over $K$ (i.e. a quasi-separated adic space locally of finite type over $\text{Spa}(K,\mathcal{O}_K)$), we call a point $x$ of $X$ classical if the residue class field $k(x)$ of $x$ is finite over $K$. For a complete extension $L$ of $K$, we write $X_L$ for $X \times_{\text{Spa}(K,\mathcal{O}_K)} \text{Spa}(L,\mathcal{O}_L)$.

9. A $p$-adic monodromy theorem for de Rham local systems

Let $K$ be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic $p$. For a rigid analytic variety $X$ over $K$ (i.e. a quasi-separated adic space locally of finite type over $\text{Spa}(K,\mathcal{O}_K)$), we call a point $x$ of $X$ classical if the residue class field $k(x)$ of $x$ is finite over $K$. For a complete extension $L$ of $K$, we write $X_L$ for $X \times_{\text{Spa}(K,\mathcal{O}_K)} \text{Spa}(L,\mathcal{O}_L)$.

9.1 Statement of the theorem

**Definition 9.1.** Let $X$ be a smooth rigid analytic variety over $K$, and let $L$ be an isogeny $\mathbb{Z}_p$-local system on $X_{\text{ét}}$.

(i) We say that $L$ is **horizontal de Rham** if $L$ is de Rham and $(D_{\text{DR}}(L),\nabla)$ has a full set of solutions, that is, $(D_{\text{DR}}(L),\nabla) \cong (\mathcal{O}_X,d)^{\text{rank} L}$.

(ii) Let $x$ be a classical point of $X$. We say that $L$ is **semistable** (respectively, **(potentially) crystalline**) at $x$ if for some (equivalently, any) geometric point $\overline{x}$ above $x$, the stalk $L_{\overline{x}}$ is a semistable (respectively, (potentially) crystalline) representation of the absolute Galois group of $k(x)$.

Theorem 1.1 is a special case of the following $p$-adic monodromy theorem for de Rham local systems.

**Theorem 9.2.** Let $X$ be a smooth rigid analytic variety over $K$, and let $L$ be a de Rham isogeny $\mathbb{Z}_p$-local system on $X_{\text{ét}}$. Then for every classical point $x \in X$, there exist an open neighborhood $U \subset X_{k(x)}$ of $x$ and a finite extension $L$ of $k(x)$ such that $L|_{U_L}$ is horizontal de Rham and semistable at every classical point. Moreover, if $L_{\overline{x}}$ is potentially crystalline, then $U$ and $L$ can be chosen in such a way that $L|_{U_L}$ is horizontal de Rham and crystalline at every classical point.

Let us first make a remark and discuss its consequence.

**Remark 9.3.** If $x$ runs over all the classical points of $X$, the images in $X$ of the $U$ in the theorem form a collection of open subsets of $X$. In general, the union of such open subsets may not be the entire $X$. 

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Corollary 9.4. Assume that the residue field of $K$ is finite. Let $X$ be a quasi-compact smooth rigid analytic variety over $K$, and let $L$ be a de Rham $\mathbb{Q}_p$-local system on $X_{\text{ét}}$. Then there exists a finite extension $L$ of $K$ such that for every $K$-rational point $x \in X$ and every geometric point $\overline{x}$ above $x$, the representation $L_{\overline{x}}$ of the absolute Galois group of $k(x)$ becomes semistable over $L$.

Proof of Corollary 9.4. Note that the set of $K$-valued points of $X$ is quasi-compact with respect to the analytic topology; the general case is reduced to the case where $X$ is a closed polydisk, and the assertion is obvious in this case.

If $L$ is an isogeny $\mathbb{Z}_p$-local system, then the assertion follows from Theorem 9.2 and the quasi-compactness of $X(K)$. In general, take an étale covering $\{U_i \to X\}_{i \in I}$ such that $L|_{U_i}$ is an isogeny $\mathbb{Z}_p$-local system on $U_{i,\text{ét}}$ for each $i \in I$. Since $X$ is quasi-compact, we may assume that each $U_i$ is quasi-compact and that the index set $I$ is finite. Set $U := \bigsqcup_{i \in I} U_i$ and let $f : U \to X$ denote the structure morphism. Note that $f$ is étale and quasi-compact. Hence, there exists $n \in \mathbb{N}$ such that every geometric fiber of $f$ has cardinality at most $n$. Now let $K'$ be the composite of all the finite extensions of $K$ of degree at most $n$. Since $K$ is assumed to have finite residue field, $K'$ is finite over $K$. By applying the above discussion to $(U, L|_U, K')$, we obtain a finite extension $L$ of $K'$ such that for every $x \in U(K')$ and every geometric point $\overline{x}$ above $x$, $L_{\overline{x}}$ becomes semistable over $L$. By construction, $X(K)$ is contained in the image of $U(K')$ under $f$. Hence, this $L$ satisfies the requirement.

We prove Theorem 9.2 in the rest of this section. To this end, we show that $(D_{\text{dR}}(L), \nabla)$ is analytic locally horizontal de Rham. Then the assertion follows from Theorems 7.3 and 7.4.

9.2 Preliminaries on smooth rigid analytic varieties

The following lemma is easy to prove and we omit the proof. Recall that $\mathbb{B}^n_K$ denotes the $n$-dimensional closed unit polydisk $\text{Spa}(K(T_1, \ldots, T_n), \mathcal{O}_K(T_1, \ldots, T_n))$.

Lemma 9.5. Let $x \in \mathbb{B}^n_K$ be a point with residue class field $K$. For every open neighborhood $V$ of $x$ in $\mathbb{B}^n_K$, there exists an open immersion of the form $\mathbb{B}^n_L \to V$ sending $0$ to $x$.

Proposition 9.6. Let $X$ be an $n$-dimensional smooth rigid analytic variety over $K$. Let $L$ be a finite extension of $K$ and $x \in X$ a classical point with residue class field $L$. Then there exists an open immersion $\mathbb{B}^n_L \to X_L$ sending $0 \in \mathbb{B}^n_L$ to $x \in X_L$.

Proof. By replacing $X$ by $X_L$, we may assume $L = K$. By shrinking $X$ if necessary, we may assume that the structure morphism $X \to \text{Spa}(K, \mathcal{O}_K)$ factors as $X \xrightarrow{f} \mathbb{B}^n_K \to \text{Spa}(K, \mathcal{O}_K)$, where $f$ is étale and the second is the structure morphism [Hub96, Corollary 1.6.10]. We may further assume that $f(x) = 0$ by translation.

By [Hub96, Corollary 1.7.4, Lemma 1.6.4], the map of local rings $f^* : \mathcal{O}_{\mathbb{B}^n_K, 0} \to \mathcal{O}_{X,x}$ is finite with the same residue class field $K$. Recall that $\mathcal{O}_{\mathbb{B}^n_K, 0}$ is Henselian (cf. [Hou60, p. 9, Corollary 2], [Ber94, Theorem 2.1.5]). Hence, we conclude that $\mathcal{O}_{\mathbb{B}^n_K, 0} \cong \mathcal{O}_{X,x}$.

It follows that there exist an affine open neighborhood $U$ of $x \in X$, an affine open neighborhood $V$ of $0 \in \mathbb{B}^n_K$, and a morphism $g : V \to U$ such that $f \circ g$ is the inclusion $V \subset \mathbb{B}^n_K$. Note that $g$ is injective and that $g^* : \mathcal{O}_{U,x} \to \mathcal{O}_{V,0}$ is an isomorphism. After shrinking $V$ and $U$ if necessary, we may further assume that $g$ is an isomorphism by [BGR84, 7.3.3 Proposition 5]. By Lemma 9.5, we get an open immersion $\mathbb{B}^n_K \to V \xrightarrow{g} U \to X$ sending $0$ to $x$. □

9.3 Vector bundles with integrable connections on the polydisk $\mathbb{B}^n_K$.

Let $A := K(T_1, \ldots, T_n)$ and let $d : A \to \Omega_A$ denote the universal continuous $K$-derivation of $A$ [Hub96, Definition 1.6.1]. Concretely, $\Omega_A$ is a finite free $A$-module of rank $n$ with generators
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d$T_1, \ldots, dT_n$, and $d: A \to \Omega_A$ sends $f$ to $df = \sum_{i=1}^n (\partial f/\partial T_i) dT_i$ for $f \in A$ [Hub96, 1.6.2]. In other words, $\Omega_A = A \otimes_A^L \Omega_A$ (cf. § 4.4).

The following result will be well known to experts. For example, see [DGS94, Appendix III], [Ked10a, Proposition 9.3.3] for the case $n = 1$.

**Theorem 9.7.** Let $(M, \nabla)$ be a finite free $A$-module equipped with an integrable connection. Then there exists $l \in \mathbb{N}$ such that the induced connection on

$$M \otimes_A K\langle T_1/p^1, \ldots, T_n/p^l \rangle$$

has a full set of solutions, that is, it is isomorphic to $(K\langle T_1/p^1, \ldots, T_n/p^l \rangle, d)_{\text{rank}_A M}$.

**Proof.** Consider the natural inclusion $M \subset M \otimes_A K[[T_1, \ldots, T_n]]$ and extend the integrable connection to the latter. By a classical result (cf. [Kat70, Proposition 8.9]), we know that $M \otimes_A K[[T_1, \ldots, T_n]]$ has a full set of solutions. In fact, we can construct an explicit $K$-linear surjection

$$P: M \to (M \otimes_A K[[T_1, \ldots, T_n]])^{\nabla=0}$$

using the Taylor expansion, which we now explain.

For each $i = 1, \ldots, n$, we set $D_i := \nabla_{\partial/\partial T_i}: M \to M$. Then the integrability of $\nabla$ is equivalent to $D_i \circ D_i = D_i \circ D_i$ for every $i, j = 1, \ldots, n$. For each $\bar{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$, we set $\bar{j}' = j_1 + \cdots + j_n$, $\bar{j}! := (j_1)! \cdots (j_n)!$, $T_{\bar{j}} := T_1^{j_1} \cdots T_n^{j_n}$, and $D_{\bar{j}} := \prod_{i=1}^n (D_i)^{j_i}: M \to M$.

We define $P: M \to M \otimes_A K[[T_1, \ldots, T_n]]$ by

$$P(f) := \sum_{\bar{j} \in \mathbb{N}^n} \frac{(-1)^{|\bar{j}|}}{|\bar{j}|!} T_{\bar{j}} D_{\bar{j}}(f).$$

Then we can check that $P(f)(0, \ldots, 0) = f(0, \ldots, 0)$, $\text{Im} P \subset (M \otimes_A K[[T_1, \ldots, T_n]])^{\nabla=0}$, $\ker P = (T_1, \ldots, T_n)M$, and that $P$ induces an isomorphism

$$P: M/(T_1, \ldots, T_n) \xrightarrow{\cong} (M \otimes_A K[[T_1, \ldots, T_n]])^{\nabla=0}.$$ 

Let $r = \text{rank}_A M$ and choose an $A$-basis of $M$ so that we identify $M = A^r$. For each $f \in A^r$, we can uniquely write $P(f)$ as

$$P(f) = \sum_{\bar{j} \in \mathbb{N}^n} A_{\bar{j}}(f) T_{\bar{j}}, \quad A_{\bar{j}}(f) \in K^r.$$ 

More concretely, we see that

$$A_{\bar{j}}(f) = \left. \left( \left( \frac{\partial}{\partial T_i} \right)^{\bar{j}} (P(f)) \right) \right|_{(0, \ldots, 0)} = \left. \left( \left( \frac{\partial}{\partial T_i} \right)^{\bar{j}} \left( \sum_{\bar{j}' \leq \bar{j}} \frac{(-1)^{|\bar{j}'|}}{|\bar{j}'|!} T_{\bar{j}'} D_{\bar{j}'}(f) \right) \right) \right|_{(0, \ldots, 0)},$$

where $(\partial/\partial T_i)^{\bar{j}} := (\partial/\partial T_i_1)^{j_1} \cdots (\partial/\partial T_i_n)^{j_n}$ for $\bar{j} = (j_1, \ldots, j_n)$, and $j' \leq j$ refers to the inequality entrywise. In this way, we get a $K$-linear map $A_{\bar{j}}: A^r \to K^r$.

Consider the Gauss norm on $A$, and equip $M = A^r$ and $M_r(A)$ with the norm defined by the maximum of the Gauss norm of each entry. Similarly, equip $K^r$ with the $p$-adic norm. We normalize these norms so that the norm of $p$ is $p^{-1}$. We will estimate the operator norm $\|A_{\bar{j}}\|$. For each $i = 1, \ldots, n$, write

$$D_i = \frac{\partial}{\partial T_i} + B_i, \quad B_i \in M_r(A).$$

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Choose $C' > 0$ such that $\|B_i\| \leq C'$ for $i = 1, \ldots, n$. Set $C = \max\{1, C'\}$. Note that both $\partial/\partial T_i$: $A^r \to A^r$ and the multiplication by $T_i$ on $A^r$ have operator norm one and that the $p$-adic norm of $j!$ is greater than or equal to $(1/p^{l/(p-1)})^{j!}$. It follows that
\[
\left\| \frac{(-1)^j}{j!} T^{j} D_{T}^{j} \right\| \leq C^{\frac{1}{l}} p^{\frac{j}{l}} \left( \frac{1}{p-1} \right),
\]
and thus
\[
\left\| A_{j} \right\| = \left\| \left( \frac{\partial}{\partial T} \right)^{j} \left( \sum_{j' \leq j} \frac{(-1)^{j'}}{j'!} T^{j'} D_{T}^{j'} \right) \right\|_{(0, \ldots, 0)} \leq C^{\frac{1}{l}} p^{\frac{j}{l}} \left( \frac{1}{p-1} \right).
\]
Choose $l \in \mathbb{N}$ such that $C p^{1/(p-1)-l} < 1$. By the estimate on $\|A_{j}\|$, we have $\Im P \subset M \otimes_{A} K(T_1/p^l, \ldots, T_n/p^l)$. In other words, we have
\[
(M \otimes_{A} K[[T_1, \ldots, T_n]])^{\nabla=0} \subset M \otimes_{A} K(T_1/p^l, \ldots, T_n/p^l).
\]
We again identify $M = A^r$ and $M \otimes_{A} K(T_1/p^l, \ldots, T_n/p^l) = K(T_1/p^l, \ldots, T_n/p^l)^r$. Choose $s_1 = (s_{11}, \ldots, s_{1r}), \ldots, s_r = (s_{r1}, \ldots, s_{rr}) \in M \otimes_{A} K(T_1/p^l, \ldots, T_n/p^l)$ such that $s_1, \ldots, s_r$ form a $K$-basis of $(M \otimes_{A} K[[T_1, \ldots, T_n]])^{\nabla=0}$ and such that the values at $(T_1, \ldots, T_n) = (0, \ldots, 0)$ satisfy
\[
s_1(0, \ldots, 0) = (1, 0, \ldots, 0), \ldots, s_r(0, \ldots, 0) = (0, \ldots, 0, 1).
\]
Set $s := \det(s_{ij})_{i,j=1,\ldots,r} \in K(T_1/p^l, \ldots, T_n/p^l)$. Then $s$ satisfies $s(0, \ldots, 0) = 1 \neq 0$. In particular, there exists $l' \geq l$ such that $s$ is invertible in $K(T_1/p^{l'}, \ldots, T_n/p^{l'})$.

Let us consider the induced connection $\nabla$ on $M \otimes_{A} K(T_1/p^{l'}, \ldots, T_n/p^{l'})$. Since $s_1, \ldots, s_r \in M \otimes_{A} K(T_1/p^{l'}, \ldots, T_n/p^{l'})$, we have
\[
(M \otimes_{A} K(T_1/p^{l'}, \ldots, T_n/p^{l'}))^{\nabla=0} \subset M \otimes_{A} K(T_1/p^{l'}, \ldots, T_n/p^{l'}).
\]
Moreover, since $s$ is invertible in $K(T_1/p^{l'}, \ldots, T_n/p^{l'})$, we conclude that the map
\[
(M \otimes_{A} K(T_1/p^{l'}, \ldots, T_n/p^{l'}))^{\nabla=0} \otimes_{K} K(T_1/p^{l'}, \ldots, T_n/p^{l'}) \to M \otimes_{A} K(T_1/p^{l'}, \ldots, T_n/p^{l'})
\]
is an isomorphism. This means that $(M \otimes_{A} K(T_1/p^{l'}, \ldots, T_n/p^{l'}), \nabla)$ has a full set of solutions. \hfill \square

9.4 Proof of Theorem 9.2

We prove Theorem 9.2. By Proposition 9.6 and Theorem 9.7, there exists an open neighborhood $U \subset X_{k(x)}$ of $x$ such that $U \cong \mathbb{B}^n_{k(x)}$ and $(D_{\text{dR}}(L)|_U, \nabla)$ has a full set of solutions. Hence, $\mathbb{L}|_U$ is horizontal de Rham because $D_{\text{dR}}(L)|_U = D_{\text{dR}}(L|_U)$. By shrinking $U$ again, we may assume that $U$ is isomorphic to $T^n_{k(x)}$. Write $T^n_{k(x)} = \text{Spa}(A, A^\circ)$. Then $R = A^\circ$ satisfies the conditions in Set-up 3.1 relative to $k(x)$. Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_A)$ be the representation corresponding to $\mathbb{L}|_U$. Then $V$ is horizontal de Rham by Lemma 8.7. By Theorem 7.3, there exists a finite extension $L$ of $k(x)$ such that $V|_{\mathbb{Q}_A_L}$ is horizontal semistable. In particular, $\mathbb{L}|_{U_L}$ is horizontal de Rham and semistable at every classical point. The second assertion follows from Theorem 7.4.

10. Potentially crystalline loci and potentially good reduction loci

As an application of our $p$-adic monodromy theorem, we discuss potentially crystalline loci of de Rham local systems and cohomologically potentially good reduction loci of smooth proper families of relative dimension at most two.

Let $K$ be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic $p$. In §§10.2 and 10.3, we will further assume that the residue field of $K$ is finite.
10.1 Potentially crystalline loci

Theorem 10.1. For every smooth rigid analytic variety $X$ over $K$ and every de Rham $\mathbb{Q}_p$-local system $L$ on $X_{\rig}$, there exists an open subset $U \subset X$ such that the following assertions hold for every classical point $x \in X$:

(i) if $x \in U$, then $L_x$ is potentially crystalline at $x$;
(ii) if $x \not\in U$, then $L_x$ is not potentially crystalline at $x$.

We call the largest open subset satisfying these properties the potentially crystalline locus of $L$.

Proof. Observe that if there exists an étale covering $\{X_i\}_i$ of $X$ such that each $L|_{X_i}$ admits an open $U_i \subset X_i$ satisfying (i) and (ii) for $X_i$, then the union of the images of the $U_i$ in $X$ satisfies (i) and (ii) for $X$. By passing to an étale cover of $X$, we may assume that $L$ is a de Rham isogeny $\mathbb{Z}_p$-local system on $X_{\rig}$.

Let $x$ be a classical point of $X$ such that $L_x$ is potentially crystalline at $x$. By Theorem 9.2, there exists an open neighborhood $U'_x \subset X_{k(x)}$ of $x$ such that $L|_{U'_x}$ is potentially crystalline at every classical point. Let $U_x$ denote the image of $U'_x$ in $X$; it is an open subset of $X$. Then the union of such $U_x$ satisfies the desired properties. \hfill $\square$

10.2 The monodromy-weight conjecture and $\ell$-independence

We will discuss cohomologically potentially good reduction loci of smooth proper maps in the next subsection. For this purpose, let us briefly review basic results and conjectures on cohomologies of smooth proper varieties over $p$-adic fields.

Let $k$ denote the residue field of $K$ and assume that $k$ is finite of cardinality $q = p^h$. Set $P_0 := W(K)[p^{-1}]$ and write $\sigma: P_0 \rightarrow P_0$ for the Witt vector Frobenius lifting $x \mapsto x^p$.

Set $\text{Gal}_K := \text{Gal}(\overline{K}/K)$. Let $I_K$ denote the inertia subgroup of $\text{Gal}_K$ and let $W_K$ denote the Weil group of $K$; they fit in the exact sequence $0 \rightarrow I_K \rightarrow W_K \rightarrow h\mathbb{Z} \rightarrow 0$, where $v$ is normalized so that $h \in h\mathbb{Z} \subset \mathbb{Z} = \text{Gal}_k$ is the geometric Frobenius $a \mapsto a^{-q}$ of $k$. Set $W^+_K := \{g \in W_K \mid v(g) \geq 0\}$.

Let $X$ be a smooth proper scheme purely of dimension $n$ over $K$. For each prime $\ell$, the $\ell$-adic étale cohomology $H^m_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is an $\ell$-adic representation of $\text{Gal}_K$.

First, assume $\ell \neq p$. Let $t_\ell: I_K \rightarrow \mathbb{Z}_\ell(1)$ be the map sending $g \in I_K$ to $(g^{\ell-m})/\pi^{\ell-m})_m \in \mathbb{Z}_\ell(1)$, where $\pi$ is a uniformizer of $K$. By Grothendieck’s monodromy theorem, there exist a nilpotent endomorphism $N \in \text{End}(H^m_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell))(-1)$ and an open subgroup $I \subset I_K$ such that for $g \in I$, the action of $g$ on $H^m_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is given by $\exp(t_\ell(g)N)$. The endomorphism $N$ defines the monodromy filtration $\text{Fil}^N$ on $H^m_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell)$; it is the unique separated and exhaustive increasing filtration such that $N(\text{Fil}^N_s \subset \text{Fil}^N_{s+1}(-1)$ for $s \in \mathbb{Z}$ and such that the induced map $N^*: \text{gr}^N_s H^m_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell) \rightarrow \text{gr}^N_s H^m_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell)\left(-s \right)$ is an isomorphism for $s \geq 0$.

Conjecture 10.2 ($\ell$-adic monodromy-weight conjecture for $\ell \neq p$). For $s \in \mathbb{Z}$ and $g \in W_K$, the eigenvalues of $g$ on $\text{gr}^N_s H^m_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ are algebraic numbers and the complex absolute values of their conjugates are $p^{(m+s)v(g)/2}$.

Remark 10.3. When $X$ has good reduction, the conjecture follows from Deligne’s purity theorem on weights in [Del80]. The conjecture is also known for $m \leq 2$; When $X$ has a strictly semistable model and $\text{dim} X = 2$, it is [RZ82, Satz 2.13]. Using [dJ96], we can reduce the general case to this case (see [Sai03, Lemma 3.9]).

Next, assume $\ell = p$. Let us recall the general recipe of Fontaine [Fon94c].
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Let $V$ be a de Rham $\mathbb{Q}_p$-representation of $\text{Gal}_K$. By the $p$-adic monodromy theorem [Ber02, Theorem 0.7], $V$ is potentially semistable. We set

$$\hat{D}_{\text{pst}}(V) := \lim_{H \subset I_K} (B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^H,$$

where $H$ runs over open subgroups of $I_K$. It is a $P_0$-vector space of dimension $\dim_{\mathbb{Q}_p} V$ equipped with a semilinear action of $\text{Gal}_K$, a Frobenius $\varphi$, and a monodromy operator $N$ satisfying $N \varphi = p \varphi N$. Such an object is called a $p$-module of Deligne in [Fon94c, 1.1]. It follows that $\varphi$ is bijective and that $N$ is nilpotent.

We associate to $\hat{D}_{\text{pst}}(V)$ a Weil–Deligne representation $W \hat{D}_{\text{pst}}(V)$ over $P_0$ as follows. Set $W \hat{D}_{\text{pst}}(V) := \hat{D}_{\text{pst}}(V)$ as a $P_0$-vector space equipped with the monodromy operator $N$. Define a $P_0$-linear action of $W_K$ on $W \hat{D}_{\text{pst}}(V)$ by letting $g \in W_K$ act as $g \cdot \varphi^{g(\varphi)}$. Since $N$ acts nilpotently, there exists a unique separated and exhaustive increasing filtration $\text{Fil}^N_{\text{pst}}$ on $W \hat{D}_{\text{pst}}(V)$ such that $N(\text{Fil}^s_{\text{pst}}) \subset \text{Fil}^{s+1}_{\text{pst}}$ for $s \in \mathbb{Z}$ and such that $N^s \colon \text{gr}^s_{\text{pst}} \rightarrow \text{gr}^{s+1}_{\text{pst}}$ is an isomorphism for every $s \geq 0$.

Let us return to the $p$-adic representation $V = H^m_{\text{pst}}(X_{\overline{K}}, \mathbb{Q}_p)$. It is de Rham by [Tsu99] and [dJ96] (cf. [Tsu02, Theorem A1]). We denote by $H^m_{\text{pst}, p}(X)$ the associated Weil–Deligne representation $W \hat{D}_{\text{pst}}(H^m_{\text{pst}}(X_{\overline{K}}, \mathbb{Q}_p))$ of $K$.

The following statement is part of the conjecture of [Jan88, p. 347] and [Jan10, Conjecture 5.1].

**Conjecture 10.4** ($p$-adic monodromy-weight conjecture). For $s \in \mathbb{Z}$ and $g \in W_K$, the eigenvalues of $g$ on $\text{gr}^s_{\text{pst}} H^m_{\text{pst}, p}(X)$ are algebraic numbers and the complex absolute values of their conjugates are $p^{(m+s) \varphi(g)/2}$.

The next result is well known to experts, but we give a proof for completeness.

**Theorem 10.5.** Conjecture 10.4 holds for $m \leq 2$.

**Proof.** We follow the proof of [Sai03, Lemma 3.9]. Note that for a finite extension $L$ of $K$, $H^m_{\text{pst}, p}(X)$ and $H^m_{\text{pst}, p}(X_L)$ are isomorphic as Weil–Deligne representations of $L$ (after rescaling the monodromy operator; see [Fon94c, 2.2.6]). It follows that the assertion for $H^m_{\text{pst}, p}(X)$ is equivalent to the one for $H^m_{\text{pst}, p}(X_L)$.

We may assume that $X$ is geometrically connected over $K$. By [dJ96, Theorem 6.5], there exist a finite extension $L$ of $K$ inside $\overline{K}$, a smooth projective scheme $X'$ over $L$, and a proper surjective and generically finite morphism $f : X' \rightarrow X$. The morphism $f$ induces a $\text{Gal}_L$-equivariant injection $f^* : H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p) \rightarrow H^m_{\text{et}}(X'_{\overline{K}}, \mathbb{Q}_p)$ and thus an injective map $f^* : H^m_{\text{pst}, p}(X_L) \rightarrow H^m_{\text{pst}, p}(X')$ of Weil–Deligne representations of $L$. In this case, it suffices to prove the assertion for the latter, and thus we may assume that $X$ is projective by replacing $X$ by $X'$ and $K$ by $L$. By replacing $X$ by a general hyperplane section, we may also assume $\dim X \leq 2$.

By applying [dJ96, Theorem 6.5] again, we may assume that $X$ admits a projective and strictly semistable model $\mathcal{X}$ over $\mathcal{O}_K$. Denote the special fiber $\mathcal{X} \otimes_{\mathcal{O}_K} k$ by $Y$. Let $M_Y$ be the fine log structure on $\mathcal{X}$ associated to the simple normal crossing divisor $Y \subset \mathcal{X}$ and let $M_Y$ be the inverse image of $M_Y$ on $Y$. Write $H^m_{\log, \text{cris}}(Y/W)$ for the log-crystalline cohomology of $(Y, M_Y)/(k, N \oplus k^\times)$ over $W := W(k)$ and set $H^m_{\log, \text{cris}}(\mathcal{X}) := H^m_{\log, \text{cris}}(Y/W) \otimes_W K_0$, where $K_0 := W[p^{-1}]$. Then $H^m_{\log, \text{cris}}(\mathcal{X})$ has a natural $(\varphi, N)$-module structure relative to $K$.

By [Tsu99, Theorem 0.2], we have an isomorphism $\Delta_{\text{st}}(H^m_{\log, \text{cris}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^m_{\log, \text{cris}}(\mathcal{X})$ of

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6 A choice of a uniformizer of $K$ equips $H^m_{\log, \text{cris}}(\mathcal{X})$ with a filtration over $K$ by the comparison isomorphism to the log-de Rham cohomology, but we do not need this structure.
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\((\varphi, N)\)-modules relative to \( K \).\(^7\) It follows that \( \bar{D}_{\text{pst}}(H^m_{\text{ét}}(X_K, \mathbb{Q}_p)) \cong H^m_{\text{log cris}}(\mathcal{X}) \otimes K_0 P_0 \) as \( p \)-modules of Deligne.

Note that \( \varphi^h \) acts \( K_0 \)-linearly on \( H^m_{\text{log cris}}(\mathcal{X}) \) and the monodromy operator defines the monodromy filtration \( \text{Fil}^N \) on \( H^m_{\text{log cris}}(\mathcal{X}) \). By the definition of the Weil–Deligne representation \( H^m_{\text{pst},\mathbb{Q}}(X) \) and the above isomorphism of \( p \)-modules of Deligne, it suffices to prove that the eigenvalues of \( \varphi^h \) on \( \text{gr}_s^N H^m_{\text{log cris}}(\mathcal{X}) \) are algebraic numbers and the complex absolute values of their conjugates are \( p^{(m+s)/2} \).

For \( r \geq 1 \), let \( Y^{(r)} \) denote the disjoint union of all \( r \)-fold intersections of the different irreducible components of \( Y \). Then we have the \( p \)-adic weight spectral sequence

\[
E_1^{-l,m+l} = \bigoplus_{j \geq \max\{-l,0\}} H^m_{\text{cris}}(Y^{(j)}(W)/W)(-j-l) \otimes W K_0 \Rightarrow H^m_{\text{log cris}}(\mathcal{X}).
\]

This spectral sequence is \( \varphi \)-equivariant (cf. [Nak05, 9.11.1]) and [Nak06, §2 (B)]). By the purity of weights of the \( E_1 \)-term [KM74], the spectral sequence degenerates at \( E_2 \), inducing the filtration \( \text{Fil}^W \) on \( H^m_{\text{log cris}}(\mathcal{X}) \) such that the eigenvalues of \( \varphi^h \) on \( \text{gr}_s^N H^m_{\text{log cris}}(\mathcal{X}) \) are algebraic numbers whose complex conjugates have absolute value \( q^{(m+s)/2} \). By [Mok93, Theorem 5.3, Corollary 6.2.3] and [Nak06, Theorem 8.3, Corollary 8.4], two filtrations \( \text{Fil}^N \) and \( \text{Fil}^W \) on \( H^m_{\text{log cris}}(\mathcal{X}) \) agree. This completes the proof. \( \square \)

To compare the results for \( \ell \) and \( p \), let us recall a theorem on \( \ell \)-independence.

**Theorem 10.6** [Och99, Theorems B and 3.1]. Let \( X \) be a smooth proper scheme over \( K \) of dimension \( n \). For every \( g \in W^+_K \), \( \sum_{m=0}^{2n} \text{tr}(g^*; H^m_{\text{ét}}(X_K, \mathbb{Q}_\ell)) \) is a rational integer that is independent of \( \ell \) with \( \ell \neq p \). Moreover, it is equal to \( \sum_{m=0}^{2n} \text{tr}(g^*; H^m_{\text{pst},\mathbb{Q}}(X)) \).

**Corollary 10.7.** Let \( X \) be a smooth proper scheme over \( K \) of dimension at most two. For every \( m \in \mathbb{N} \) and \( g \in W^+_K \), the trace \( \text{tr}(g^*; H^m_{\text{ét}}(X_K, \mathbb{Q}_\ell)) \) is a rational integer that is independent of \( \ell \) with \( \ell \neq p \). Moreover, it is equal to \( \text{tr}(g^*; H^m_{\text{pst},\mathbb{Q}}(X)) \).

**Proof.** The cases \( m = 0, \dim X = 2 \) are obvious. The same assertion for an abelian variety with \( m = 1 \) is also known (cf. [Fon94c, Remark 2.4.6(iii)] and [Noo17, Corollary 2.2]). From these results, it remains to deal with the case where \( \dim X = 2 \) and \( m = 2 \); this follows from Theorem 10.6. \( \square \)

**Corollary 10.8.** Let \( X \) be a smooth proper scheme of dimension at most two over \( K \) and let \( m \in \mathbb{N} \). For a rational prime \( \ell \neq p \), the \( \ell \)-adic representation \( H^m_{\text{ét}}(X_K, \mathbb{Q}_\ell) \) is potentially unramified if and only if the \( p \)-adic representation \( H^m_{\text{ét}}(X_K, \mathbb{Q}_p) \) is potentially crystalline.

**Proof.** The former condition in the assertion is equivalent to the condition that \( N = 0 \) on \( H^m_{\text{ét}}(X_K, \mathbb{Q}_\ell) \). Since the \( \ell \)-adic monodromy-weight conjecture is known for \( X \) (Remark 10.3), it is also equivalent to the condition that, for every \( g \in W^+_K \), the eigenvalues of \( g \) on \( H^m_{\text{ét}}(X_K, \mathbb{Q}_\ell) \), which are algebraic numbers, satisfy the property that the complex absolute values of their conjugates are \( \ell^{m(\mu(g)/2)} \). Similarly, by Theorem 10.5, the latter condition in the assertion is equivalent to the condition that, for every \( g \in W^+_K \), the eigenvalues of \( g \) on \( H^m_{\text{pst},\mathbb{Q}}(X) \), which are algebraic numbers, satisfy the property that the complex absolute values of their conjugates are \( \ell^{m(\mu(g)/2)} \). Hence, the assertion easily follows from Corollary 10.7. \( \square \)

\(^7\) Depending on the convention of the monodromy operators, we may need to change the sign of one of the monodromy operators, but this does not affect our argument.
10.3 Cohomologically potentially good reduction locus

Let $K$ be a complete discrete valuation field of characteristic zero with finite residue field of characteristic $p$.

Theorem 10.9. Let $f : Y \to X$ be a smooth proper morphism between smooth algebraic varieties over $K$ and let $X^{\text{ad}}$ denote the adic space associated to $X$. Assume that the relative dimension of $f$ is at most two. Then for each $m \in \mathbb{N}$, there exists an open subset $U$ of $X^{\text{ad}}$ such that the following assertions hold for every classical point $x$ of $X^{\text{ad}}$ and every geometric point $\overline{x}$ above $x$:

(i) if $x \in U$, then $(R^m f_\ast Q_\ell)_x$ is potentially unramified for $\ell \neq p$ and $(R^m f_\ast Q_p)_x$ is potentially crystalline as Galois representations of $k(x)$;
(ii) if $x \notin U$, then $(R^m f_\ast Q_\ell)_x$ is not potentially unramified for $\ell \neq p$ and $(R^m f_\ast Q_p)_x$ is not potentially crystalline as Galois representations of $k(x)$.

We call the largest open subset satisfying these properties the cohomologically potentially good reduction locus of $f$ of degree $m$.

Proof. By Corollary 10.8, the theorem follows from Theorem 10.1 or [Kis99, Proposition 5.2 (2)].

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