Correspondence Between 3D Shapes Based on Improved Functional Map

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Abstract. We propose a shape matching algorithm based on an improved functional map in order to calculate correspondence between two given 3D non-rigid shapes, in which shape correspondence can be represented as a mapping function of mixed transformation matrix $B$ and calibration matrix $P$ of basis functions. First, Laplace matrix is calculated and a new matrix is constructed as basis matrix of function space by using eigenvectors of Laplace matrix after eigen-decomposition. Then, a calibration algorithm based on statistical covariance is proposed to calculate the matrix $P$ that is used to calibrate basis matrices of function space of two shapes. Finally, the matrix $B$ is optimized by an improved ICP(Iterative Closest Point) algorithm in order to calculate shape correspondence with the matrix $P$. Experimental results show that the proposed algorithm avoids excessive initial conditions, obtains accurate shape correspondence and significantly solves symmetry ambiguities of 3D shapes during matching process.

1. Introduction

It is a very important basic research that it tries to find a meaningful correspondence between two given 3D non-rigid shapes [1]. In the area of computer graphics and computer vision, shape correspondence have wide range of applications, such as shape morphing and interpolation [2, 3], shape segmentation [4], mesh parameterization [5], shape recognition and retrieval [6] and so on.

Correspondence can be generally stated as: based on local or global features and content, find a meaningful mapping (or relation) between elements of given 3D non-rigid shapes, which have same or similar features, or the same semantic content. Shape matching is an essentially pivotal step of computing shape correspondence. The majority of existing methods try to tackle these challenges by limiting their search space for rapid construction of correspondence between a small set of landmark points and extending those to a dense correspondence on entire shapes using a coarse-to-fine strategy. Nevertheless, although these approaches based on point-to-point correspondence reduce the complexity of the solution space, due to the presence of coarse similarities in the shape features or symmetry ambiguities, it is difficult to incorporate with global constraints or return meaningful correspondence. The goal for the functional map representation is to find the underlying mappings represented as a parameter matrix $C$ without establishing point-to-point correspondence. However, the symmetry ambiguity that affects the computation results of correspondence cannot be effectively addressed. We present an Improved Functional Map (IFM) for computation of maps between shapes as linear transformations between the corresponding function spaces, i.e., a mixed transformation matrix $B$ and a calibration matrix $P$ of basis functions. By computing matrix $P$, it does not only calibrate spatial basis to solve the symmetry ambiguities, but also provide a good initial value that can
optimize calculation and obtain an accurate shape correspondence.

2. Related work
Rigid matching method is used to calculate shape correspondence, because 3D shapes in Euclidean space have been embedded into a target space including Euclidean space, spherical domain, and topological graph, to name just a few, in which 3D non-rigid shapes are approximately represented as rigid shapes [7-10]. Other embedding domains spanned by eigenfunctions of a graph Laplacian operator or Laplace-Beltrami operator defined on triangulated shapes were suggested by Knossow et al. [11], Rustamov [12]. Euclidean embedding can simplify the calculation of shape correspondence by reducing its dimensions, however important shape information is lost during this process. The other approach computes dense correspondence using strategy of greedy optimization and coarse-to-fine combinatorial matching based on embedding [13], but the approach needs a good initial result in advance, and can’t effectively address symmetry ambiguities. Vladimir G. Kim et al. [14] developed a method for finding a map between surfaces by blending a collection of low dimensional maps. However, a limitation of their approach is that it is able to find only global mappings.

Some new shape descriptors, such as Heat Kernel Signature (HKS) [15, 16], Wave Kernel Signature (WKS) [17], and Learning Spectral Descriptors (LSD) [18], were proposed for characterizing points on non-rigid three-dimensional shapes. All the Laplacian-based descriptors achieve state-of-the-art performance in numerous shape analysis tasks. They are computationally efficient, isometry-invariant by construction, and can gracefully cope with a variety of transformations. However, they also can’t solve symmetry ambiguities.

A novel representation, Functional Map Representation [19, 20], is introduced for maps between pairs of shapes that generalizes the standard notion of a map as a pairing of points by concentrating on finding correspondences between generic functions defined on shapes. This approach can obtain accurate results, but it need cut up the shapes and compute the segment correspondence constrains in advance, and can’t effectively address symmetry ambiguities. Maks Ovsjanikov et al. [21] propose a novel method for non-rigid shape matching, designed to address the symmetric ambiguity problem directly by performing shape matching in an appropriate quotient space, where the symmetry has been identified and factored out. However, they still need cut up the function space and calculate a symmetry-self correspondence in advance.

3. Theory of functional map

3.1. Functional map
Correspondence between shapes can be represented as a mapping problem \( T : M \rightarrow N \). \( T \) induces a natural transformation of derived quantities, such as functions on \( M \). To be precise, if we are given a scalar function \( f : M \rightarrow \mathbb{R} \), then we obtain a corresponding function \( g : N \rightarrow \mathbb{R} \) by composition, as in \( g = f \circ T^{-1} \). Therefore, it can denote this induced transformation by \( T_{f} : F(M, \mathbb{R}) \rightarrow F(N, \mathbb{R}) \), where \( F(\ast, \mathbb{R}) \) denotes a generic space of real-valued functions. \( T_{f} \) can be called the functional representation of the mapping \( T \). Maks Ovsjanikov et al. [19] make the following three simple remarks:

Remark 1 The original mapping \( T \) can be recovered from \( T_{f} \).

Remark 2 For any fixed bijective map \( T : M \rightarrow N \), \( T_{f} \) is a linear map between function spaces.

Let \( \varphi_{a} \) be a basis matrix which consists of function spatial base of \( M \), the scalar function \( f \) of \( M \) can be represented as \( f = \varphi_{a} a \), in which \( a \) is a vector of coefficients of function \( f \) based on basis matrix \( \varphi_{M} \). Same can \( g \) be said of the scalar function \( g \) of \( N \) in \( g = \varphi_{b} b \) according to formula (1). Given a matrix \( C \) to let \( T_{f}(\varphi_{a}) = \varphi_{b} C \), conclusion can be drawn from the follow key observation:

Remark 3 The map \( T_{f} \) can be represented as a (possibly infinite) matrix \( C \) s.t. for any function \( f \) represented with the vector \( a \) then \( T_{f}(a) = Ca \).

\[ \varphi_{a} b = g = T_{f}(f) = T_{f}(\varphi_{a} a) \] (1)

\[ \varphi_{b} a = T_{f}(\varphi_{a}) a \] (2)
\[ \varphi_n b = \varphi_n C a \]  
\[ b = C a \]  

Several approaches used the first \( n \) eigenfunctions of Laplace operator as the bases of function space [19-21], so function map can be represented as sparse matrix \( C_{n \times n} \).

3.2. Calibration matrix of basis matrix
Eigen-decomposition of shape Laplace-Beltrami operator also known as eigenfunctions can obtain eigenvalues and eigenvectors. Given two shapes \( M \) and \( N \), matrices \( \varphi_M \) and \( \varphi_N \) of eigenfunctions of these shapes can be said to be different basis matrixes in same embedding space. According to space theory of liner algebra, there is a transition matrix \( J \) that makes transition between these basis matrixes, as indicated in formula (5).

\[ \varphi_M = \varphi_N J \]  

In fact, there are some special circumstances in the process of eigen-decomposition, such as repeated root and inconsistent distribution of eigenfunctions, in which errors occur when computing shape correspondence.

On the one hand, because an elementary matrix \( P \) can calibrate basis matrix, matrix \( P \) is a part of the transition matrix \( J \). On the other hand, matrix \( P \) isn’t a part of the transition matrix \( J \), but it can be converted by a matrix \( E: JE=P (J=PE^{-1}) \). So formula (5) can be written as:

\[ \varphi_M = \varphi_N PD \]  

3.3. Theory of improved functional map
To avoid overmuch initial steps and solve the symmetry ambiguities, we propose an improved functional map (IFM) algorithm. Given two shapes \( M \) and \( N \) with scalar functions \( f: M \rightarrow \mathbb{R} \) and \( g: N \rightarrow \mathbb{R} \) respectively, there exists a map \( T: M \rightarrow N \), and \( f \) and \( g \) respectively can be represented as \( f=\varphi_M a \) and \( g=\varphi_N b \).

According to formula (6), the scalar function \( f \) of shape \( M \) can be converted into \( f=\varphi_N PDa \) where \( PDa \) is a coefficient vector of function \( f \) in basis matrix \( \varphi_N \). The original functional map \( T_p: F(M, \mathbb{R}) \rightarrow F(N, \mathbb{R}) \) becomes a linear transformation \( T_p: F(N, \mathbb{R}) \rightarrow F(N, \mathbb{R}) \) with the same base. According to space theory of linear algebra, a linear transformation can be represented as a matrix \( G \), as shown in formula (7).

\[ T_p (\varphi_N) = \varphi_N G \]  

On the basis of preceding analysis, substituting formula (6) and formula (7) into formula (2), will give formula (8) and formula (9). Because the calibration matrix \( P \) is an elementary matrix that consists of -1, 0, 1, it can be computed easily. \( D \) and \( G \) are unknown matrices, there is \( B=DG \) which is called as mixed transformation matrix, so formula (9) can be converted into formula (10).

\[ \varphi_N b = T_p (\varphi_N PD)a \]  
\[ b = PDGa \]  
\[ b = PBa \]  

According to the improved functional map, now computing correspondence between shape \( M \) and \( N \) is translated into calculations of \( P \) and \( B \).

4. Calculation of shape correspondence
Our algorithm avoids the excessive initial conditions during calculation of shape correspondence, and only need input two shapes \( M \), \( N \) and the number of eigenfunctions.

4.1. Laplacian matrix of shapes
The restriction of Laplace-Beltrami operator to a point is expected to be close to combinatorial mesh...
Laplacian when shapes become sufficiently dense. For convenient calculation and dealing with dense triangulation in mesh, the bases of functional space can be calculated by eigen-decomposition of combinatorial mesh Laplacian based on Gaussian kernel. A triangular-mesh shape \( M \) can be regarded as a graph \( I = \{ V, E \} \), where \( V \) is vertex set and \( E \) is edge set. Matrix \( W \) denotes connection matrix of graph \( I \), where \( W_{ij} \) is the element of the \( i \)th row and the \( j \)th column in \( W \). An edge \( e_{ij} \) is defined by vertex \( v_i \) and \( v_j \) (\( v_i, v_j \in V \)), if the edge \( e_{ij} \) is part of \( E \), \( W_{ij} = \gamma_{ij} \); otherwise, then \( W_{ij} = 0 \), where \( w_{ij} \) is weight of vertex \( v_i \) and \( v_j \) (it can be defined by length of \( e_{ij} \)) as shown in formula (11), where \( l(e_{ij}) \) denotes length of \( e_{ij} \) and \( \sigma \) is parameter of the width.

\[
w_{ij} = \exp \left( \frac{-l^2(e_{ij})}{2\sigma^2} \right)
\] (11)

We define a diagonal matrix \( D \) that expresses the degree of matrix \( W \). Let \( d_i \) be the element of the \( i \)th row and the \( i \)th column in \( D \), then \( d_i = \sum w_{ij} \). Laplacian matrix \( L \) of shape can be as indicated in formula (12).

\[
L = D - W
\] (12)

The eigen-decomposition of Laplacian matrix \( L \) can obtain the bases of functional space of 3D shapes.

4.2. Calibration of basis matrix

Given shapes \( M \) and \( N \), \( L_M \) and \( L_N \) denote Laplacian matrix respectively. For eigen-decomposition and standardization of \( L_M \) and \( L_N \), we use eigenfunctions corresponding to the first \( n \) minimum eigenvalues to obtain the basis matrix \( \phi_M \) and \( \phi_N \). It shows that there is a matrix \( P \) to calibrate these basis matrices, and the calibration matrix \( P \) is an elementary transformation matrix consisting of -1, 0, 1. In order to calculate the calibration matrix \( P \), we propose a new calibration algorithm based on statistics covariance, in which each eigenfunction can be regarded as a dataset, subsequently, the covariance matrix can be computed by these datasets. Because covariance matrix shows similarity between these datasets, matrix \( P \) can be calculated accurately. Let \( \phi_M^i \) denote the \( i \)th column of \( \phi_M \) which is also the \( i \)th eigenfunction.

4.3. Mixed transformation matrix based on ICP optimization algorithm

The mixed transformation matrix \( B \) can be optimized by ICP (Iterative Closet Point) algorithm according to descriptor preservation, and the matrix \( B \) is subsequently used to compute shape correspondences. Given shapes \( M \) and \( N \), \( \lambda_M \) and \( \lambda_N \) are the first \( n \) eigenvalue of Laplace-Beltrami operator respectively. \( \phi_M \) and \( \phi_N \) respectively represent matrices of corresponding eigenfunctions, \( \delta_M^i \) and \( \delta_N^i \) are it’s \( i \)th embedding coordinate (i.e. the \( i \)th coefficient vector in \( \phi_M^T \) and \( P^T \phi_N^T \)) of shape \( M \) and \( N \) respectively.

5. Experimental results

In this paper, we compared Blended Intrinsic Maps (BIM) [14] which is classic algorithm, Functional Map (FM) which is the newest theory of correspondence with the result of our algorithm, Improved Functional Map (IFM). The experimental results showed that our method can accurately compute shape correspondence and effectively solve symmetry ambiguities of 3D shapes during matching process.

Firstly, we do our experiments on TOSCA dataset. We select the first 15 eigenfunctions and calculate Geodesic Error by using same correspondence with the BIM and FM as shown in Figure 1. It shows the percent of correspondences which have Geodesic Error smaller than a threshold, for the maps generated by our IFM algorithm denoted by the red line, the maps after BIM algorithm denoted by the green line, and the maps after FM algorithm denoted by blue line. The results show that our method detects over 61.15% correct correspondences for a small threshold of 0.025 and converges to finding almost all correct correspondences within geodesic error 0.125, in comparison that surpasses BIM. Comparing with FM, the correct correspondences of IFM are all greater except when Geodesic
Error is 0. This plot reveals that method based on FM can not solve symmetry ambiguities problem very well, because correspondence rate is still low when Geodesic Error is over 0.25.

2 shows distributions of Geodesic Error of different kind of shapes in TOSCA dataset based on IFM. Correspondence rate of human shapes (Michael, David and Victoria) are greater than animal shapes when Geodesic Error is over 0.05. Compared red line in Figure 1 with green line in Figure 2, distribution of Geodesic Error of Gorilla is similar to the whole TOSCA dataset, so we analyses influence of quantity of eigenfunctions by using the distribution of Geodesic Error of Gorilla. The next experiment compared maps found by selecting 15, 35, 50 and 100 eigenfunctions which are represented in Figure 3 as IFM-15, IFM-35, IFM-50 and IFM-100 respectively. The correspondence rate reaches 95.83% when the threshold of Geodesic Error is 0.05 for IFM-100 (100 eigenfunctions are selected), which is better than IFM-15, IFM-35 and IFM-50. Therefore, this experiment demonstrates that the more quantity of eigenfunctions, the better correspondence of Gorilla shape.

We select the first 35 eigenfunctions to calculate Geodesic Error by using same correspondence with BIM as shown in Figure 4. The red line denotes our IFM algorithm, and the green and blue lines denote BIM and FM respectively. For all threshold of Geodesic Error, correspondence rate of IFM is greater than BIM. Compared to FM, the correspondence rate of IFM is 90.21% and FM algorithm only reaches 83.5% when Geodesic Error is 0.075. This plot also reveals that the correspondence rate of our method reaches 100% within Geodesic Error 0.2. That means our IFM can effectively solve symmetry ambiguities problem.

6. Conclusion and future work
Correspondence between shapes is a basic problem in research field of computer vision and computer graphics. We have developed an algorithm based on Improved Functional Map, such that computing
correspondence is translated into calculations of mixed transformation matrix $B$ and calibration matrix $P$. The experimental results show that our method can avoid excessive initial conditions, obtain accurate shape correspondence and solve symmetry ambiguities of 3D shapes during the matching process.

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