Entanglement engineering and topological protection by discrete-time quantum walks

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Abstract

Discrete-time quantum walks (QWs) represent robust and versatile platforms for the controlled engineering of single particle quantum dynamics, and have attracted special attention due to their algorithmic applications in quantum information science. Even in their simplest 1D architectures, they display complex topological phenomena, which can be employed in the systematic study of topological quantum phase transitions [1]. Due to the exponential scaling in the number of resources required, most experimental realizations of QWs to date have been limited to single particles, with only a few implementations involving correlated quantum pairs. In this paper we study applications of QWs in the controlled dynamical engineering of entanglement in bipartite bosonic systems. We show that QWs can be employed in the transition from mode entanglement, where indistinguishability of the quantum particles plays a key role, to the standard type of entanglement associated with distinguishable particles. We also show that by carefully tailoring the steps in the QWs, as well as the initial state for the quantum walker, it is possible to preserve the entanglement content by topological protection. The underlying mechanism that allows for the possibility of both entanglement engineering and entanglement protection is the strong ‘spin–orbit’ coupling induced by the QW. We anticipate that the results reported here can be employed for the controlled emulation of quantum correlations in topological phases.

(Some figures may appear in colour only in the online journal)

1. Introduction

Discrete-time quantum walks (QWs), originally inspired by the Feynman path-integral approach for computing observables [2], are the quantum extension of classical random walks, where classical probabilities are replaced by quantum imaginary amplitudes, so that the multiple paths of the quantum walker can interfere, a feature which results in the probability of the quantum walker being quite distinct from their classical counterpart [3]. Within the realm of classical statistical physics, random walks have been extensively employed as fundamental models describing the complex dynamics of several systems of interest, ranging from transport properties in biological and material compounds, to the dynamical evolution of financial markets. On the other hand, the analogous QWs have been used to model a large number of dynamical physical processes involving quantum coherence, such as photosynthesis [4], diffusion [5] and vortex transport [6]. In recent years, QWs have attracted significant attention as they provide a universal platform for the study of quantum information protocols, allowing for speed-up in search algorithms [7, 8], and for universal simulation of quantum circuits [9]. Moreover, it has recently become apparent that QWs can be employed for the efficient dynamical exploration of a wide range of non-trivial topological phases in 1D and 2D [1, 10–14]. In particular, the simplest 1D random walk that presents non-trivial topological features is analogous to the SSH model in polyacetylene [15].
Several experimental realizations of QWs have readily been accomplished, employing trapped ions [16], ultra-cold atoms in optical lattices [17] and photons [18]. The latter are of particular interest, as light particles can be easily produced, manipulated and detected. However, due to the exponential growth in the Hilbert space size and in the number resources, only a few QW implementations to date involve correlated quantum particles [19, 21].

In this paper, we analyse the use of QWs in the controlled engineering of entanglement between bosonic quantum particles. We discuss in detail the different types of entanglement that can be observed between the quantum particles, namely mode entanglement of indistinguishable particles, and standard entanglement between separate distinct particles, the latter being a fundamental resource for quantum communications. The entanglement of elementary states of indistinguishable particles was discussed in detail in the classic book by Peres [20], and has been the subject of continued active research [22, 23]. Here, we show that QWs can be employed to engineer controlled transitions between different types of entanglements. This can be explained in terms of coupling between different internal degrees of freedom in the quantum walker, such as spin and spatial or orbital degrees of freedom, a result which can be exploited for the controlled dynamical engineering of entanglement. This form of ‘spin–orbit’ coupling mechanism induced by the QW is also at the heart of the entanglement engineering and topological protection mechanisms described in this paper. In quasi-momentum space, the effective Hamiltonian up to a constant energy offset, can be written as

\[ \hat{H}_{\text{eff}} = \int_{-\pi}^{\pi} dk \hat{E}(k) \hat{n}(k) \cdot \hat{\sigma} \otimes |k\rangle \langle k| \]

(3)

Now \( \hat{n}(k) \) defines the quantization axis for the spinor eigenstates at each quasi-momentum \( k \). For a coin operator different from the identity operator, the quasi-energy is given by \( \cos(\hat{E}(k)) = \cos(\theta/2) \cos(k) \), typically corresponding to a two-band structure, for a two-level system. The quasi-energy spectrum is plotted in figure 1 for different values of the coin parameter \( \theta \). For \( |\theta| > 0 \) the spectrum has a gap, and for \( \theta \approx 0, 2\pi \) the spectrum becomes gapless, and presents a Dirac-like dispersion relation, where the quasi-energy versus quasi-momentum are linearly related. This already explains why QWs in their simplest architectures can be used to efficiently simulate a gapped system, or a gapless system with Dirac points.}

**Figure 1.** Band structures for the simplest 1D QW in a two-level system. By tuning the quantum coin parameter \( \theta \) it is possible to simulate a gapped system, or a gapless system with Dirac points. \( \theta = 0 \) (black), \( \theta = \pi/2 \) (red), \( \theta = \pi \) (green), \( \theta = 3\pi/2 \) (orange), \( \theta = 2\pi \) (blue).

**2. Formalism**

**2.1. Basic formalism: QW for single two-level systems in 1D**

The simplest QW consists of a protocol for a two-level particle (\( |\uparrow\rangle, |\downarrow\rangle \)), moving in dimension. The unitary evolution operator \( \hat{U} \) for a single step in the QW is given by

\[ \hat{U} = \hat{T} \hat{R}_v(\theta), \]

(1)

where \( \hat{T} = \sum_x |\uparrow\rangle \langle\uparrow| \otimes |x+1\rangle \langle x+1| + |\downarrow\rangle \langle\downarrow| \otimes |x-1\rangle \langle x-1| \) is a spin-dependent translation by one site \( x \), \( \hat{R}_v(\theta) = \exp(-i\theta \hat{\sigma} \cdot \hat{v}/2) \) is the coin operator given by an arbitrary SU(2) rotation around a unitary vector \( \hat{v} \) in the Bloch sphere, and \( \hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \) are the Pauli matrices. It is the quantum equivalent of randomly tossing a coin and choosing which way the particle will move. Conventionally, the rotation is set to \( \hat{R}_v = e^{-i\hat{\sigma}_z/2} \). The unitary evolution operator \( \hat{U} \) can be considered as a stroboscopic regularization of a static effective Hamiltonian \( \hat{H}_{\text{eff}} \) given by

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(2)

We note that the spin-dependent translation can be written in quasi-momentum space as \( \hat{T} = \int_{-\pi}^{\pi} dk e^{i\hat{E}(k) \cdot \hat{\sigma}} \otimes |k\rangle \langle k| \), which readily reveals the spin–orbit coupling mechanism which is at the heart of the entanglement engineering and topological protection mechanisms described in this paper. In quasi-momentum space, the effective Hamiltonian up to a constant energy offset, can be written as

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simulate dynamical properties of complex condensed matter system with topological properties.

In general, non-trivial topological character requires at least two independent parameters, so that an adiabatic variation of these parameters leads to the accumulation of a Berry phase [24], or the analogous Zak phase for one-dimensional systems [25]. This can be done by introducing a second rotation $R_2(\theta_2)$, which is called in the literature the ‘split-step’ QW [1].

### 2.2. Split-step QW

The split-step QW protocol reads

$$\hat{U}(\theta_1, \theta_2) = \hat{T}_1 \hat{R}_2(\theta_2) \hat{T}_1 \hat{R}_2(\theta_1),$$

where

$$\hat{T}_1 = \sum_x |\uparrow\rangle \langle \uparrow| \otimes |x + 1\rangle \langle x| + |\downarrow\rangle \langle \downarrow| \otimes |x\rangle \langle x|,$$

$$\hat{T}_2 = \sum_x |\downarrow\rangle \langle \downarrow| \otimes |x - 1\rangle \langle x| + |\uparrow\rangle \langle \uparrow| \otimes |x\rangle \langle x|.$$  

It has been shown analytically [1] that this system still presents chiral symmetry, meaning that the positions $\{\vec{r}(k)\}$ of the eigenstates of $\hat{U}(\theta_1, \theta_2)$ lie on a circle of the Bloch sphere. Nevertheless, depending on the values of $\theta_1$ and $\theta_2$, the set $\{\vec{r}(k)\}$ can describe the full circle ($W = 1$), or not ($W = 0$), where $W$ is the winding number associated with the topological structure of the spectrum. The spectrum presents a gap in both cases $|\tan(\theta_2)/\tan(\theta_1)| < 1$ ($W = 1$) and $|\tan(\theta_2)/\tan(\theta_1)| > 1$ ($W = 0$). When $|\tan(\theta_2)/\tan(\theta_1)| = 1$, the gap closes (at quasi-energy 0 or π), and one cannot properly define a winding number. In this case, the corresponding 0 (respectively π) quasi-energy state corresponds to a so-called topological bound state. The topological phase diagram with two coin parameters becomes a mesh-grid of alternating squares, characterized by alternating topological numbers $W = 0, 1$ (see figure 2). We note that when $\theta_2 = 0$, we recover the standard QW protocol operator.

In order to experimentally observe the presence of the topological nature of the split-step QW, one can decide to make one of the rotations inhomogeneous; for instance, make $\theta_2$ vary as a function of $x$, going from $\theta_2^{-}$ (when $x \to -\infty$) to $\theta_2^{+}$ (when $x \to +\infty$). If one line where the gap closes is crossed by the trajectory of $(\theta_1, \theta_2)$, we know that the local spectra are gapped far from the crossing point, and gapless in the vicinity of this point (see figure 3). The associated bound state (say of 0 quasi-energy) will be localized around the zone where the $(\theta_1, \theta_2)$ trajectory crosses the line (say $x = 0$). Already at a few sites from $x = 0$, there is no arbitrary small quasi-energy states that could permit the propagation of the 0 quasi-energy component of the wavepacket, and consequently the overlap between the initial wavepacket and the localized bound state is topologically ‘trapped’ around $x = 0$. Then, after propagation, the probability to detect the particle localized around $x = 0$ remains significant compared to the case in which bound states are not present in the spectrum. Indeed, the basic feature of the QW evolution of an initially localized (around $x = 0$) particle is a ballistic expansion which is in general anisotropic (not symmetric in 1D) and where the probability for the particle to stay at its initial position decreases very quickly. This difference of behaviours has been indeed experimentally observed [27] and clearly shows the impact of the topological structure on the system dynamics. Note that the trajectory $(\theta_1, \theta_2)$ has to keep $\theta_1$ uniform in order to preserve the chiral symmetry of the system, but the variations of $(\theta_2)$ can be arbitrary.

### 2.3. Basic formalism: QW for two identical two-level systems in 1D

We consider now the dynamics of two identical bosonic particles, such as linearly polarized photons, propagating in the same QW protocol, represented by identical unitary evolution operators $\hat{U}(\theta_1, \theta_2)$, as defined in the previous subsection.
Since photons do not interact, we focus on the evolution of entanglement on the two-particle dynamics in order to analyse the impact of quantum correlations in the system.

In order to maintain the global symmetry of exchange of bosonic particles, the states describing the case in which one particle is at position $x_1$ and the other at $x_2$ are as follows: on the one hand, any possible tensorial product of the symmetric state in positions ($\frac{1}{\sqrt{2}} (|x_1, x_2\rangle + |x_2, x_1\rangle)$) and one of the symmetric states in spins ($|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$) or ($\frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$) or, on the other hand, the tensorial product of the antisymmetric state in positions ($\frac{1}{\sqrt{2}} (|x_1, x_2\rangle - |x_2, x_1\rangle)$) and the antisymmetric state in spins ($\frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$). Since each photon evolves with the same unitary evolution operator, an initially symmetric state with respect to the exchange of particles will remain symmetric after an arbitrary evolution time.

We note that in the case of photon pairs produced by type-II spontaneous parametric down conversion (SPDC), the two-photon state has the general form $|\psi\rangle = \int dq_1 \int dq_2 \int dw_1 \int dw_2 [\phi_{HV}(q_1, w_1; q_2, w_2) + \phi_{HV}(q_1, w_1; q_2, w_2)]|V_1, q_1, w_1; H_2, q_2, w_2\rangle$, where $(H, V)$ stand for orthogonal linear polarizations ($\uparrow$, $\downarrow$), and $(q, w)$ represent the spatial and frequency degrees of freedom, respectively. For the case of indistinguishable particles (i.e. no temporal or spatial walk-off), the two-photon amplitude wavefunctions should be identical $\phi_{HV}(q_1, w_1; q_2, w_2) = \phi_{HV}(q_1, w_1; q_2, w_2)$, which forces the spatial and frequency part of the wavefunction to be symmetric. In this case, the state is maximally entangled in polarization (i.e. Bell state). Any uncontrolled spatial or temporal walk-off reduces the indistinguishability of the photon pairs, and therefore the entanglement content.

In the following, in order to quantify the entanglement of a given two-particle state, we will compute the negativity either in mode representation, when both particles are localized at the same position, or in polarization representation when the particles are physically separated (i.e. the overlap between the two wavepackets can be neglected), and can then be treated as distinguishable particles. The difference between polarization (for distinguishable particles) and mode (for indistinguishable particles) entanglements is directly induced by the difference between first and second quantizations. Appendix A is fully dedicated to the concept of entanglement in an indistinguishable particle system [23, 22].

**Figure 4.** Spatial probability distributions in logarithmic scale of $|\Psi_A\rangle$ (top) and $|\Psi_B\rangle$ (bottom) after 60 iterations.

**Figure 5.** The red solid (respectively dashed blue) line represents the computed polarization negativity of the measured reduced density matrix for initial state $|\Psi_A\rangle$ (respectively $|\Psi_B\rangle$), versus time $n$. In both cases, it is maximum ($\Lambda(\rho) = 1$) at the initial time.

### 3. Entanglement engineering by discrete-time quantum walks

In order to illustrate the transversality of the concept of entanglement between identical particles, we engineer a split-step QW protocol for which two indistinguishable photons initially localized around $x = 0$ can be measured with a reasonably high probability at different positions after the propagation, and can then be distinguished. With a well-chosen set of coin parameters $(\theta_1, \theta_2)$, we will show that an initially mode-entangled state leads to non-vanishing entanglement in polarization between the two photons, whereas an initial state which is separable in terms of modes does not exhibit this behaviour. We will refer to this phenomenon as ‘entanglement conversion’, from mode entanglement to polarization entanglement, during a temporal transition from indistinguishable to distinguishable particles.

#### 3.1. Numerical results

We numerically compute the split-step QW dynamics of the two following initial states: $|\Psi_A\rangle = |x_1 = 0, x_2 = 0\rangle \otimes \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$ and $|\Psi_B\rangle = |x_1 = 0, x_2 = 0\rangle \otimes |\uparrow\downarrow\rangle$. The chosen coin parameters are $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \pm \frac{\pi}{4}$, and then the topological aspects of the split-step QW are not involved here since the whole system belongs to the same topological zone (see figure 2).

We clearly see in figure 4 that there is almost no difference in the probability distributions between the two
cases. Indeed, both present important symmetric peaks of probability on the off-diagonal line: \( x_1 + x_2 = 0 \), which represents one photon on each side. For each initial state and at each time step \( n \), we first project the two-particle state \( \langle U(\theta_1, \theta_2^\pm) \otimes \bar{U}(\theta_1, \theta_2^\mp) \rangle \) on the two-position state \( |\psi_{x_{\text{max}}, -x_{\text{max}}}| \) corresponding to the maximum probability lying on the off-diagonal line, and then normalize the result. Hence, one can build the 4 x 4 ‘reduced’ density matrix \( \hat{\rho} \) and evaluate the entanglement between polarizations by computing the polarization negativity: \( N(\hat{\rho}) = \text{Tr}(|\hat{\rho}|) - 1 \), where \( |\hat{\rho}| \) is the partially transposed density operator (see appendix B).

A radical difference is observed in terms of polarization entanglement during time evolution. In the case of a non mode-entangled initial state (|\( \psi_1 \rangle \)), some fluctuations of the polarization negativity are observed, close to the minimum value 0. On the other hand, if the state is initially entangled in modes (|\( \psi_0 \rangle \)), part of this entanglement is transferred (or converted) to a traditional polarization entanglement during time evolution (see figure 5).

3.2. Discussion

The split-step QW protocol allowed us to propose a simple realization exhibiting a dynamical transition from an indistinguishable to distinguishable bipartite system. Additionally, it can offer a nearly direct observation of mode entanglement between identical particles. Let us insist on the fact that at short (i.e. experimentally meaningful) times, the measurability of this effect is enhanced because the probability to measure the two photons on the anti-diagonal line is decreasing with time.

Note that the entanglement increases and decreases in the course of the evolution which is described by a local Hamiltonian. This is indeed possible, due to the transfer of the initial mode entanglement into the polarization entanglement, when the two identical photons are far apart, and can thus be distinguished (see appendix A for more details).

In this section, we have avoided the presence of a non-trivial topological structure of the split-step QW in order to focus on the nature of entanglement between identical particles. We will now include this important feature, and focus on the interplay between entanglement and the eventual presence of topological bound states in the spectrum of the single-particle evolution operator.

4. Entanglement protection by discrete-time quantum walks

Let us consider a configuration of \((\theta_1, \theta_2^\pm)\) that crosses one (figure 6, left panel) or two (figure 6, right panel) gap closing line(s).

Here, we show that when an entangled two-particle state (|\( \psi_0 \rangle \)) follows a QW protocol for which there is one bound state \( |\psi_{\text{BS}} \rangle \) localized in \( x = 0 \), the two-particle wavepacket presents an important localized part around the origin \( x_1 = x_2 = 0 \), which can be locally identified as \( |\psi_{\text{BS}} \rangle \otimes |\psi_{\text{BS}} \rangle \) (see figure 7, top panel). When two gap closing lines are crossed, the localization phenomenon around the origin is even stronger (see figure 7, bottom panel). In this case, two different bound states exist in the spectrum of the single-particle evolution operator, one \( |\psi_0 \rangle \) of 0 quasi-energy, and the other \( |\psi_\pi \rangle \) of \( \pi \) quasi-energy.

We suppose that both photons are now detected at the origin and compare the mode entanglement during time evolution. This mathematically corresponds to projecting the two-particle state on \( |x_1 = 0, x_2 = 0 \rangle \), building up the density matrix represented in modes \( \hat{\rho}_m \) and computing the mode negativity \( N(\hat{\rho}_m) \) (see appendix B).

We can easily observe that when two bound states are present in the system, the phenomenon of localization around the origin is clearly enhanced, and thus the probability of detecting both particles in \( x = 0 \) is increased. We can also note that the entanglement vanishes quickly during time evolution in the first case (one bound state present), whereas in the other scenario (two bound states present), the entanglement oscillates between 0.2 and 0.4 without decreasing at all.

4.1. Discussion

Similarly to the single particle case, the main feature of the QW is a ballistic expansion of an initially localized wavepacket. Here, after a certain time, a significant part of the wavefunction leaves the origin via ballistic expansion, but what remains in the centre is the overlap of the initial state with the bound state(s). In the first case, we can approximate this fraction of the state by a product state: \( |\Psi\rangle \approx_{t_\text{final}} |\psi_{\text{BS}} \rangle \otimes |\psi_{\text{BS}} \rangle \), where \( |\psi_{\text{BS}} \rangle \) represents the zero energy bound state present in the first case. We can then easily understand the fact that the entanglement rapidly decreases because of the presence of one single bound state.
Figure 7. Spatial probability distributions in logarithmic scale of $|\Psi_1^B\rangle$ after 60 iterations. The spectrum of the single-particle evolution operator contains one single (top) or two (bottom) bound states.

In the second column case, two bound states are present ($|\Psi_{10}^B\rangle$ and $|\Psi_{1\pi}^B\rangle$), and as previously explained, after a certain time of propagation, the central part of the wavefunction is mainly due to the presence of bound states. When two bound states (or more) are present, we cannot conclude that the form of the state is a product: in general, it would be a coherent superposition of the different bound states. This qualitatively explains the important difference observed in the evolutions of entanglement. In the latter case, the non-vanishing negativity is directly related to the topological structure of the system and thus, this entanglement should be robust against impurities, or disorder.

In this example the entangled particles are trapped at the origin because the topological boundary is located at $x = 0$ (see figure 3). Nevertheless, the spatial location of this boundary is arbitrary. Delocalized entangled states of distinguishable particles involving two spatial locations $x_{1,2}$ could in principle also be topologically protected by introducing additional topological boundaries via distinguishable inhomogeneous coin operators. For the case of two-dimensional systems, the analogous edge modes [1] could also be employed for entanglement topological protection of delocalized states. Detection of topologically protected states can be accomplished via quantum state tomography and measurements of joint probability distributions [26, 27].

The co-existence of pairs of bound states with quasi-energy difference $\pi$ is a robust phenomenon with topological origin, characteristic of periodically driven systems [28]. In our case we find a similar effect for a static effective Hamiltonian due to the presence of two distinct bound states with quasi-energy $E = 0$ and $E = \pi$, when crossing two distinct topological boundaries for our specific choice of coin operators. The co-existence of these bound states, in turn, explains the periodic oscillations in the negativity displayed in figure 8, and represents a novel feature that can be exploited for entanglement protection. Other similar localization mechanisms could also be considered for entanglement protection. In particular, it has recently been shown that adding local impurities in a standard QW process gives rise to the existence of localized bound states [33]. The insight into topological protection gained by our approach can help to explore topological properties in novel correlated materials, where current technology readily enables experimental realizations.

Figure 8. Computed negativity of mode entanglement when the two photons are detected at the origin, for the split-step QW evolution of the initial state $|\Psi_{1}^B\rangle$. The red full line corresponds to the presence of a single bound state, and the oscillatory behaviour displayed in the blue dashed line is a clear signature of the presence of two bound states with quasi-energy difference $\pi$.

5. Outlook

To conclude, in this paper we proposed the use of quantum walks (QWs) for the controlled engineering and protection of entanglement in bosonic bipartite systems. The underlying mechanism that allows for the manipulation and control of entanglement, as well as for its topological protection, is the strong coupling between internal degrees of freedom in the quantum walker, which plays an analogous role to ‘spin–orbit’ coupling in topological systems. In particular, we showed that QWs can be employed in the transition from bound entanglement to mode entanglement and that such entanglement content can be topologically protected by tailoring the quantum coin, and the initial state for the quantum walker. The extension of QWs to correlated particle pairs opens the door to simulations of quantum correlations in condensed matter systems with topological order.

The results presented here have direct experimental applications in photonic quantum systems, and could improve the performance of photonic quantum technologies [29]. In this regard, we note that the phase stability of multi-path QW interferometers is not expected to represent a significant experimental limitation, since topologically protected states are expected to be resilient against local phase noise and perturbations. However, temporal or spatial walk-off in the
system with increased number of iterations can play a deleterious dephasing effect which, in combination with losses, can limit the number of iterations. Current bulk optics implementations are limited to \( n = 7 \) iterations at the most [27].

Extensions of the proposed scheme to multi-photon QWs could also be considered [30], allowing for the simulation of novel many-body effects in topological phenomena. In this case, the statistical properties of the photon emission process can limit the visibility in multi-photon quantum interferometers. In this scenario, phase-matching engineering techniques can be a fundamental requirement alleviating, in turn, the need for passive filtering [31]. Trapped ion QW realizations [16], as well as ultra-cold atom implementations [32] can also be envisioned, naturally allowing for the exploration of interaction-driven topological states.

Acknowledgments

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Appendix A. Entanglement of identical particles

The notion of quantum entanglement is associated with states (i.e. density operators) belonging to a Hilbert space which possesses a tensorial product structure. The canonical example is the case of two distinguishable particles (parties) \( A \) and \( B \), for which the Hilbert space of the bipartite system is given by \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \). In the case of indistinguishable particles, as considered in this work, the notion of entanglement becomes ambiguous because one cannot create a partition of the Hilbert as previously: labelling the particles is not possible because of their indistinguishability. A new tensorial product structure of the Hilbert space is needed because both particles ‘live’ in the same Fock space. Consequently, the notion of entanglement between the particles does not have any physical meaning. The adapted formalism in this context is the second quantization, that automatically symmetrizes (respectively antisymmetrizes) the bosonic (respectively fermionic) wavefunctions.

Let us first consider the case in which the two photons are at the same spatial location \( x \), have identical frequencies (i.e. the monochromatic case) and can only be described by the polarization degree of freedom (modes \( \uparrow \) and \( \downarrow \)). Any bosonic two-particle wavefunction can be built by the superposition of the states of the canonical symmetric basis composed by \( a_{\uparrow}^\dagger a_{\downarrow}^\dagger \ket{\text{vac}} = \ket{\uparrow\downarrow} \), \( a_{\downarrow}^\dagger a_{\uparrow}^\dagger \ket{\text{vac}} = \ket{\downarrow\uparrow} \) and \( a_{\uparrow}^\dagger a_{\uparrow}^\dagger \ket{\text{vac}} = \frac{1}{\sqrt{2}} (\ket{\uparrow\uparrow} + \ket{\downarrow\downarrow}) \), where \( \ket{\text{vac}} \) represents the vacuum state of the Fock space. Writing a state in the first quantization form (for example \( \frac{1}{\sqrt{2}} (\ket{\uparrow\uparrow} + \ket{\downarrow\downarrow}) \)) is not correct, as stressed above, because it supposes that one can unequivocally know which particle corresponds to the first and second positions in writing a ket. We then write these states in a more synthetic form, in terms of mode (\( \uparrow \) or \( \downarrow \)) with an occupation number : \( a_{\uparrow}^\dagger a_{\uparrow}^\dagger \ket{\text{vac}} = \ket{2 \uparrow \downarrow} \), \( a_{\downarrow}^\dagger a_{\downarrow}^\dagger \ket{\text{vac}} = \ket{0 \uparrow 2 \downarrow} \), \( a_{\uparrow}^\dagger a_{\downarrow}^\dagger \ket{\text{vac}} = \ket{1 \uparrow 1 \downarrow} \). The underlying tensorial product structure of the new Hilbert space is \( \mathcal{H} = \mathcal{H}_\uparrow \otimes \mathcal{H}_\downarrow \), where \( \mathcal{H}_{\uparrow/\downarrow} \) stands for the mode \( \uparrow / \downarrow \).

Additionally, another advantage of this formulation is that we clearly see now that the apparent entanglement in polarization of \( |\frac{1}{\sqrt{2}} (\ket{\downarrow\uparrow} + \ket{\uparrow\downarrow}) \rangle \) is just the consequence of the bosonic character of the photon pair because in the second quantization language, it is written as a product state in modes \( |1 \uparrow, 1 \downarrow \rangle \), as well as the two other states mentioned above. In particular, the state \( \frac{1}{\sqrt{2}} (\ket{\uparrow\uparrow} + \ket{\downarrow\downarrow}) \rangle \) which looks maximally entangled in the polarization representation (first quantization) can be rewritten \( \frac{1}{\sqrt{2}} (|2 \uparrow, 0 \downarrow \rangle + |0 \uparrow, 2 \downarrow \rangle) \rangle \) in the mode representation (second quantization), which shows that this state is indeed maximally entangled, but in terms of modes [22, 23].

More precisely, it has been shown that the partial transposition criterion for bipartite entanglement is appropriate also in the context of \( N \) identical particles [22]. For that purpose, one need to express the density operator in the canonical basis of second quantization: \( |k \uparrow, N-k \downarrow \rangle \), with \( k \in [0, N] \), and transpose the occupation numbers. Consider the following density operator, written in the mode representation,

\[
\hat{\rho} = \sum_{k,l=0}^{N} \rho_{k,l} |k, N-k \rangle \langle l, N-l |,
\]

\[
\sum_{k} \rho_{k,k} = 1.
\]

The partially transposed density operator \( \hat{\rho} \) reads

\[
\hat{\rho} = \sum_{k,l=0}^{N} \rho_{k,l} |l, N-k \rangle \langle k, N-l |,
\]

which involves states associated with 0 to 2\( N \) particles. This particular structure leads to a simple form of negativity

\[
N(\rho) = \frac{1}{2} \sum_{k,l=0}^{N} |\rho_{k,l}|.
\]  

(A.1)

In the case in which particles are spatially separated (say one in \( x_1 \) and the other in \( x_2 \)), one can distinguish them, and thus the first quantization is allowed in order to compute entanglement between the particle located in \( x_1 \) and the one located in \( x_2 \).

Appendix B. Entanglement measurement

We give here all the technical details concerning the computation of the negativity of a two-particle state whether the particles are detected at the same position or not. As stressed in the text, when particles are located at the position, we compute the entanglement between the modes.
and $\downarrow$, whereas when particles are spatially separated, the first quantization formalism is allowed and we then compute the entanglement between effectively distinguishable particles.

In any case, the general form for the evolved two-particle symmetrized (with respect to the exchange of particles) state is

$$|\Psi\rangle = \sum_{x_1 \neq x_2} \left[ \frac{1}{\sqrt{2}} (|x_1, x_2\rangle + |x_2, x_1\rangle) \left( \alpha_{x_1, x_2} |\uparrow\rangle, |\uparrow\rangle + \beta_{x_1, x_2} (|\uparrow\rangle, |\downarrow\rangle + |\downarrow\rangle, |\uparrow\rangle) \right) \right] \right.$$ 

$$+ \frac{\alpha_{x_1, x_2}}{\sqrt{2}} (|\uparrow\rangle, |\downarrow\rangle + |\downarrow\rangle, |\uparrow\rangle) \right] \right.$$ 

$$+ \frac{\gamma_{x_1, x_2}}{2} (|x_1, x_2\rangle - |x_2, x_1\rangle) \left( |\uparrow\rangle, |\downarrow\rangle - |\downarrow\rangle, |\uparrow\rangle \right) \right]$$ 

$$+ \sum_x |x, x\rangle \left( \alpha_{x, x} |\uparrow\rangle, |\uparrow\rangle + \beta_{x, x} \left( |\uparrow\rangle, |\downarrow\rangle + |\downarrow\rangle, |\uparrow\rangle \right) + \gamma_{x, x} |\downarrow\rangle, |\downarrow\rangle \right).$$

**Mode entanglement**

Suppose that we projectively measure both particles at position $x = 0$, which corresponds to projecting $|\Psi\rangle$ onto the bra $\langle x = 0 |$ and normalizing the result. The obtained two-spin 1/2 state is in the form

$$|\Psi_m\rangle = \alpha |\uparrow\rangle, |\uparrow\rangle + \beta |\downarrow\rangle, |\uparrow\rangle + \gamma |\downarrow\rangle, |\downarrow\rangle,$$

with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

In order to evaluate the mode entanglement of this state, we write down the corresponding density matrix $\hat{\rho}_m = |\Psi_m\rangle \langle \Psi_m|$ in the basis of modes (i.e. in the formalism of second quantization): $|k, 2-k\rangle$ with $k \in \{0, 1, 2\}$. As we stressed in the previous appendix, we need to write all the matrix elements $\alpha_{i, j}(k, l)$ with $(i, j, k, l) \in \{0, 1, 2\}^4$ in order to be able to partially transpose $\hat{\rho}_m$. Consequently, $\hat{\rho}_m$ is a $9 \times 9$ complex matrix. Then, the negativity is analytically given by equation (A.1).

**Polarization entanglement**

If we projectively measure particles in two different positions, which corresponds to projecting $|\Psi\rangle$ onto the bra $\langle x_1, x_2 |$ and normalizing the result, we obtain a state in the form

$$|\Psi_m\rangle = \alpha |\uparrow\rangle, |\uparrow\rangle + \beta + \delta |\uparrow\rangle, |\downarrow\rangle + \beta - \delta |\downarrow\rangle, |\uparrow\rangle + \gamma |\downarrow\rangle, |\downarrow\rangle,$$

with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.

Note that in our case, since the initial state is a product of a symmetric state in position and a symmetric state in polarization, $\delta$ will be equal to 0 at all times.

We then build the density operator of the measured state: $\hat{\rho}_m = |\Psi_m\rangle \langle \Psi_m|$ and compute its negativity, representing the entanglement between particle 1 measured in $x_1$ and particle 2, in $x_2$.

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