Semi-analytical theory of emission and transport in a LAFE-based diode

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A large area field emitter (LAFE) typically consists of several thousands of nanoscale emitting tips. These are difficult to simulate using purely numerical methods based on finite/boundary element or finite difference methods. We show here that a semi-analytically obtained electrostatic field allows tracking of field emitted electrons of a LAFE fairly accurately using the knowledge of only the LAFE geometry. Using a single and a 9-emitter configuration, the beam parameters calculated using this method are compared with the results of tracking using fields generated by COMSOL. The net emission current, energy conservation and the transverse trace-emittance are found to be reproduced with reasonable accuracy.

I. INTRODUCTION

Large area field emitter (LAFE) based electron guns continue to be researched due to their potential advantages such as fast switching, moderate to high beam brightness and a narrow energy distribution. They are also being studied as an alternate to photocathode injectors for use in free-electron lasers. Carbon nanotube based emitter tips are particularly promising due to their inertness and stability, and have been used in various devices such as portable x-ray generators. While there has been considerable experimental progress, theoretical studies and simulations have been few due to several challenges. A recent exception has been the case of smooth vertically aligned emitters in a parallel plate diode geometry, where, it is now possible to predict local fields at each of the individual emitter apex and the corresponding emission current to a reasonable accuracy using analytical methods. Note that there are typically several thousands of high-aspect ratio emitting sites in a LAFE, often randomly placed, and simulating these using a finite/boundary element software is a challenge in terms of the sheer amount of resources required since there may be few or no symmetries that can be exploited for reducing the size of the problem. It thus seems reasonable to pursue a semi-analytical approach towards the simulation of a LAFE based diode.

A prerequisite for any such simulation requires an accurate estimate of the net current from each emitter, as well as the distribution of electrons with respect to the position, the normal energy and the total energy. The number of electrons together with their position and momentum constitute the initial condition. This must be supplemented with an accurate knowledge of the electric field in the diode region, including the immediate neighbourhood of the sharp emitting tip where the field changes rapidly, in order to track the electrons as they move from the cathode towards the anode.

For a single or a few-emitter diode, there are numerous commercial software available for tracking. A self-consistent solution requires more stringent particle-in-cell (PIC) computations, which typically use finite-difference algorithms for the ease of implementing charge conservation. In either case, the demand on computational resources mount as the ratio between the height \(h\) and the apex radius of curvature \(R_a\) increases since even a small error in the local field on the emitter-endcap gets exponentially amplified in the computation of the field-emission current. Values of \(h/R_a\) in the range \(100 < h/R_a < 1000\) are not uncommon with \(R_a\) as small as 5nm. An accurate computation of the local fields for such high aspect-ratio emitters, requires a minimum mesh size smaller than \(R_a/100\) and hence a very high mesh density. With a hundred such high-aspect ratio emitters placed close together, the task of finding just the field emission current from each distinct tip becomes an impossible task for an average workstation. When particle data is added, the challenges increase. Further, since the electrostatic field drops sharply away from the apex within a few \(R_a\), the grid size required for accurate tracking adds to the computational resources. When thousands of such emitters are present, the task becomes daunting even for a parallel machine. It is thus necessary to explore analytical methods, both for emission and tracking of the electrons in the diode region to supplement standard software used for simulating macro-scale devices. If such an exercise is reasonably successful, space-charge effects may subsequently be incorporated semi-analytically in the modelling of LAFE-based diodes.

Analytical tracking methods have been used for a spherical diode and for an emitter shape modelled by an equipotential of a ‘sphere mounted on a cone’ with another distant equipotential acting as an anode. The availability of analytical solutions for such geometries makes them particularly attractive and the model has been used to track and evaluate beam parameters.

Our interest in this work lies in a parallel plate diode geometry with identical vertically aligned emitters mounted on the cathode plate, and the anode placed in close proximity to the tip(s) such as in a gated diode. We shall employ semi-analytical approximate methods, based on the non-linear line charge model (NLCM), to estimate the current from each emitter and subsequently track the emitted electrons, for a single emitter initially and eventually for several emitters. The effec-

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tiveness of the semi-analytical method can be gauged from a comparison with COMSOL generated electrostatic field data, both for the computation of the field-emission current and the tracking of electrons in the diode region. The emitted electrons in both the semi-analytical and COMSOL routes follow the same distribution but may differ in number due to the difference in emission current which in turn is related to the apex field. Note that the semi-analytical apex field and the COMSOL generated apex field are bound to differ even if slightly, leading to errors in the emission current that may be considerably amplified. We shall ignore space charge effects in the following and focus merely on the net emission current, energy-conservation and the average transverse emittance. The issue of emitters having non-identical height or apex radius of curvature is yet another complication that will not be addressed in the present communication.

Recent progress in using an NLCM-based hybrid model\textsuperscript{33} shows that effects such as shielding, anode-proximity and inverse-shielding mediated through the anode, can be captured fairly accurately using purely geometric quantities such as the height ($h$), apex radius of curvature ($R_a$), the coordinates of each emitter ($x_i, y_i$) on the cathode plane ($z = 0$) and the anode-cathode distance ($D$). The estimate is good if the emitters are not too close although even an error as small as 5% in the apex field can translate into 25-50% change in the emitted current. Deviations of this order from the true value are par for the course, keeping in mind that small experimental uncertainties in estimating the height and radius of curvature can result in even larger changes in current. Having estimated the current, the emitted electrons from each emission site needs to be transported across the diode using fields generated using the NLCM based hybrid model. The parameters for judging the efficacy of the semi-analytical transport include energy conservation and transverse emittance at the anode far away, the system is identical to a line charge potential $V_g$. If the field far away from the tip is expressed as $E_0 = V_g / D$, it can be shown that

$$\lambda = -\frac{4\pi \varepsilon_0 E_0}{\ln((h + L)/(h - L)) - 2L/h}. \tag{1}$$

For a sharp emitter where $h/R_a \gg 1$, $\lambda \approx -4\pi \varepsilon_0 E_0 / (\ln(4h/R_a) - 2)$. The line charge is in effect the projection of the induced surface charge on the hemiellipsoid surface on to the axis of symmetry.

The potential at any point $(\rho, z)$ due to a vertical line charge placed on a grounded conducting plane can be expressed as

$$V(\rho, z) = \frac{1}{4\pi \varepsilon_0} \left[ \int_0^L \frac{\Lambda(s)}{[\rho^2 + (z - s)^2]^{1/2}} ds - \int_0^L \frac{\Lambda(s)}{[\rho^2 + (z + s)^2]^{1/2}} ds \right] + E_0 z \tag{2}$$

where $\Lambda(s) = \lambda s$ and $L$ is the extent of the line charge distribution. The equipotential curve $V = 0$ generated by the line charge and its image together with the macroscopic field $E_0$, has the shape of a hemiellipsoid.

As the anode is brought near, this description becomes somewhat inaccurate since a linear line charge density $\lambda(s)$ cannot strictly generate a hemiellipsoidal equipotential surface when the effects of the anode are incorporated. Nevertheless, a reasonably accurate description can be obtained by persisting with a linear line charge density, modified as\textsuperscript{34}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The emitter can be represented as an equipotential generated by a line charge and an applied electric field $-E_0 \hat{z}$.}
\end{figure}

\section*{II. THE NLCM BASED HYBRID MODEL}

Consider an emitter of height $h$ and apex radius of curvature $R_a$ as shown in Fig. 1 mounted on a grounded cathode plate at $z = 0$ with the anode at $z = D$ having a potential $V_g$. If the emitter is a hemi-ellipsoid and the anode far away, the system is identical to a line charge placed vertically on the cathode plane extending from $z = 0$ to $z = L = h - R_a / 2$ having a linear charge density $\lambda$, and its image extending from $z = 0$ to $z = -L$ having a linear charge density $-\lambda$. If the field far away from the tip is expressed as $E_0 = V_g / D$, it can be shown that

$$\lambda = -\frac{4\pi \varepsilon_0 E_0}{\ln((h + L)/(h - L)) - 2L/h}. \tag{1}$$

For a sharp emitter where $h/R_a \gg 1$, $\lambda \approx -4\pi \varepsilon_0 E_0 / (\ln(4h/R_a) - 2)$. The line charge is in effect the projection of the induced surface charge on the hemiellipsoid surface on to the axis of symmetry.\textsuperscript{33}

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\[
\lambda \approx -\frac{4\pi\varepsilon_0 E_0}{\ln[4h/R_a] - 2 - \alpha_A}
\]

where \(\alpha_A\) represents the infinitely many images of the original line-charge pair formed due to reflection by the anode and cathode planes and is expressed as

\[
\alpha_A = 2 \frac{h}{2nD + h} \sum_{i=1}^{\infty} \left[ \frac{L}{2nD - h} - (2nD + h) \tan^{-1} \frac{L}{2nD + h} \right].
\]

If we are dealing with a collection of emitters placed at \((x_i, y_i)\) on the cathode plane, their collective effect can be expressed by modifying the line charge density of the \(i^{th}\) emitter as

\[
\lambda_i \approx -\frac{4\pi\varepsilon_0 E_0}{\ln[4h/R_a] - 2 - \alpha_A + \alpha S_i - \alpha S_{A_i}}
\]

where \(\alpha S_i\) is the shielding contribution on the \(i^{th}\) emitter due to all the other emitters while \(\alpha S_{A_i}\) is the inverse shielding effect mediated through the anode. They are expressed respectively

\[
\alpha S_i \approx \sum_{j \neq i}^{N} \left[ \frac{1}{h} \frac{\rho_{i}^2 + (h - L)^2}{\rho_{i}^2 + (h + L)^2} - \frac{1}{h} \frac{\rho_{i}^2 + (h + L)^2}{\rho_{j}^2 + (h - L)^2} \right] + \ln \left( \frac{\sqrt{\rho_{i}^2 + (h + L)^2} + h + L}{\sqrt{\rho_{i}^2 + (h - L)^2} + h - L} \right)
\]

while

\[
\alpha S_{A_i} \approx \sum_{n=1}^{\infty} \sum_{j \neq i}^{N} \left[ \frac{D_{mm}}{h} - \frac{D_{mp}}{h} - \frac{D_{pm}}{h} + \frac{D_{pp}}{h} \right]
\]

\[
+ \frac{2nD - h}{h} \ln \left( \frac{D_{mm} + 2nD - h + L}{D_{mm} + 2nD - h - L} \right)
\]

\[
- \frac{2nD + h}{h} \ln \left( \frac{D_{pp} + 2nD + h + L}{D_{pp} + 2nD + h - L} \right)
\]

with

\[
D_{mm} = \sqrt{\rho_{i}^2 + (2nD - h - L)^2}
\]

\[
D_{mp} = \sqrt{\rho_{i}^2 + (2nD - h + L)^2}
\]

\[
D_{pm} = \sqrt{\rho_{i}^2 + (2nD + h - L)^2}
\]

\[
D_{pp} = \sqrt{\rho_{i}^2 + (2nD + h + L)^2}
\]

In the above \(\rho_{ij}\) is the distance between the \(i^{th}\) and \(j^{th}\) emitter on the cathode plane (XY).

**A. The field at the apex**

With these inputs, the field at the apex of the \(i^{th}\) emitter can be expressed as

\[
E_a^{(i)} \approx E_0 \frac{2h/R_a}{\ln[4h/R_a] - 2 - \alpha_A + \alpha S_i - \alpha S_{A_i}} = \gamma_a(i) E_0
\]

where \(\gamma_a(i)\) is referred to as the apex field enhancement factor of the \(i^{th}\) emitter. Thus, if the co-ordinates of a collection of \(N\) emitters (\(N\) can be large) are known, the approximate field at the apex of each emitter can be evaluated fairly easily. Coupled with the generalized cosine law of local field variation near the emitter apex[39-40]

\[
E_1(\tilde{\theta}) = E_a \frac{z/h}{\sqrt{(z/h)^2 + (\rho/R_a)^2}} = E_a \cos \tilde{\theta}
\]

the current from each of them can be evaluated simply from a knowledge of the apex field \(E_a^{(i)}\) and the radius of curvature \(R_a\).

As stated earlier, in reality a linear line charge does not have a hemiellipsoidal equipotential when the anode and all the other emitters are thrown in. Besides, if the shape of the emitter itself is different (such as a rounded cone or a cylindrical post), the line charge density is required to be nonlinear to begin with even if the emitter is isolated. The use of a nonlinear line charge leads to expressions of the form[3]

\[
\gamma_a \approx \frac{(1 - C_0)}{(1 - C_1) \ln[4h/R_a] - (1 - C_2)2 - (1 - C_3)\alpha_A}
\]

when an isolated emitter is in close proximity to the anode. Here \((1 - C_i)\), are corrections due to the nonlinear line charge density but are a-priori unknown. The number of such unknowns can be reduced and \(\gamma_a\) simply expressed as

\[
\gamma_a \approx \frac{2h/R_a}{\alpha_1 \ln[4h/R_a] - \alpha_2}
\]

where \(\alpha_1\) and \(\alpha_2\) can be determined by re-writing Eq. [11] and plotting \(2h/(R_a\gamma_a)\) vs \(\ln(4h/R_a)\). The quantities \(h/R_a\) can be varied by keeping \(h\) and \(D\) constant while changing \(R_a\) locally around the desired value. For each such value of \(h/R_a\), the apex field enhancement factor \(\gamma_a = E_a/E_0\) can be determined using a suitable software such as COMSOL. A straight line fit to the data has \(\alpha_1\) as the approximate slope and \(\alpha_2\) as the intercept.
Once $\alpha_1$ and $\alpha_2$ are known, the apex field can be determined using

$$E_a^{(i)} \simeq E_0 \frac{2h/R_a}{\alpha_1 \ln(4h/R_a) - \alpha_2 + \alpha_S - \alpha_{S4}},$$

(13)

Note that the anode correction $\alpha_S$ has been incorporated in $\alpha_2$ and can be dropped altogether. Also, we have ignored the corrections due to the nonlinear line charge while expressing the contributions of the other emitters. Thus, Eq. (13) is not expected to be too accurate when the emitters are closely packed but is a fair approximation so long as the mean spacing is greater than the height of the emitters. Since the approximate apex field is known, the current from an $N$-emitter system can thus be determined semi-analytically.

B. The field in the diode region

The field at the apex of the sharp emitters is much larger than $E_0 = V_a/D$. In its immediate vicinity outside the emitter surface, the field remains high. Determining this variation accurately as the point moves away from the apex requires enormous computational resources. The line charge model provides an alternate approximate economical means to determine the field components in the diode region since $\lambda$ in Eq. (3) does approximately take into account the anode and other emitters in the neighbourhood. We shall continue with the use of $\lambda$ even when the line charge density is expected to be nonlinear with the caveat that $\alpha_1$ and $\alpha_2$ are determined numerically for a single emitter as described in section II A (in particular Eq. 12). This implies that the effect of nonlinearity at the single-emitter level is incorporated but it is ignored while expressing the contribution of other emitters.

Within this approximate model, the $\dot{\rho}$ component of the electric field, $-\partial V/\partial \rho$ at the point $(\rho, z)$ can be evaluated to yield

$$E_\rho = \frac{\lambda}{4\pi\epsilon_0} \frac{1}{\rho} \left[ \frac{\rho^2 + z(z + L)}{\sqrt{\rho^2 + (z + L)^2}} - \frac{\rho^2 + z(z - L)}{\sqrt{\rho^2 + (z - L)^2}} \right],$$

(14)

while

$$E_z = \frac{\lambda}{4\pi\epsilon_0} \left[ \frac{L}{\sqrt{\rho^2 + (z + L)^2}} + \frac{L}{\sqrt{\rho^2 + (z - L)^2}} + \ln \left\{ \frac{\sqrt{\rho^2 + (z + L)^2} - (z + L)}{\sqrt{\rho^2 + (z - L)^2} - (z - L)} \right\} \right] - E_0. $$

(15)

For small values of $\rho$ (i.e. close to the axis of symmetry),

$$E_\rho \simeq \frac{\lambda \rho}{4\pi\epsilon_0} \left[ \frac{2z^2L}{(z^2 - L^2)^2} - \frac{2L}{(z^2 - L^2)} \right],$$

(16)

while

$$E_z = \frac{\lambda}{4\pi\epsilon_0} \left[ \frac{L}{\sqrt{\rho^2 + (z + L)^2}} + \frac{L}{\sqrt{\rho^2 + (z - L)^2}} \right] - E_0. $$

(17)

We shall first test how these fields fare for a single emitter before applying them to a collection of emitters.

III. NUMERICAL TESTS

A. Single emitters

We shall first consider two examples of single field emitters. The first is a hemiellipsoid (HE) with $h/R_a = 300$ and $R_a = 500\mu$m. The other is a hemiellipsoid on a cylindrical post (HECP), again with $h/R_a = 300$ with $R_a = 500\mu$m. The hemiellipsoid endcap has a height $h_c = 5R_a$ while the cylindrical post has height $h = 5R_a$ and radius $\sqrt{h_c}R_a/2$. The anode in both cases is placed at a distance $20R_a$ from the tip so that the anode-cathode plate distance is $D = 160\mu$m and $h = 150\mu$m. The hemiellipsoid with the anode far away (e.g. $D > 5h$) is analytically solvable with $\alpha_1 = 1$ and $\alpha_2 = 2$ for the values of $h$ and $R_a$ chosen. It is expected that tracking would be accurate in such a case. With the anode placed close to the tip, a simple closed form analytical solution does not exist and this scenario presents the first non-trivial test of the method outlined in section III. The HECP emitter on the other hand has no analytical solution even when the anode is far away and is a typical case where the line charge density is nonlinear. The two examples thus present non-trivial test cases for the methodology described in section III.

In order to use the semi-analytical model outlined in section III and test its accuracy, we shall first determine the electrostatic field using the finite-element based multiphysics simulation software, COMSOL. Since the problem eventually involves multiple emitters, a 3-dimensional depiction of the hemiellipsoid and HECP emitters is used. These are placed in a parallel plate geometry with the bottom (cathode)-plate having $V = 0$ while the anode is another equipotential with $V = 4800V$. The computational domain is a cuboid with the other 4 faces placed at a distance $10h$ from the centre of the box, each having Neumann boundary condition. With $h/R_a = 300$ and $R_a = 500\mu$m, the minimum domain size is chosen to be less than or equal to $R_a/100$. Convergence is ensured in each case by adjusting the mesh parameters.

We shall consider a single emitter placed at the centre of the cathode plate. Fig. 2 shows the distribution of $(2h/R_a)/\gamma_a$ values plotted against $\ln(4h/R_a)$. The apex field enhancement factor $\gamma_a$ is obtained using COMSOL for both the Hemiellipsoid (HE) and hemiellipsoid on a cylindrical post (HECP). Notice that both sets of points lie approximately on a straight line. The best fits are also
shall use the best fit values of $\alpha$ linear line charge and (HECP), the best fit value of the pair is $\alpha = 1$ and $\alpha_2 = 2$. The increase in $\alpha_2$ is due to the anode proximity effect.

In case of the hemiellipsoid on a cylindrical post (HECP), the best fit value of the pair is $\alpha_1 = 0.632927$ and $\alpha_2 = 1.19338$. Since, the case corresponds to a non-linear line charge and $C_0, C_1, C_3$ and $C_4$ are unknown, we shall use the best fit values $\alpha_1$ and $\alpha_2$ to represent an HECP with the anode placed at $D = h + 20R_a$.

shown with the slope representing $\alpha_1$ and the intercept $\alpha_2$. Thus, for the hemiellipsoid emitter with the anode at $D = h + 20R_a$, $\alpha_1 = 1.01724$ while $\alpha_2 = 3.36553$. Notice that the anode-at-infinity should have $\alpha_1 = 0$ and $\alpha_2 = 2$. The increase in $\alpha_2$ is due to the anode proximity effect.

Having determined $\alpha_1$ and $\alpha_2$, the apex field $E_a$ and the line charge density $\lambda$ can be obtained using the value of the macroscopic field $E_0 = 3 \times 10^7$ V/m. These can in turn be used to generate the $N$ particle positions on the endcap using the distribution function

$$f(\theta) \approx 2\pi R_a^2 \frac{\sin \theta}{\cos^2 \theta} J_{MG}(\theta)$$

(18)

where $\theta$ is given by Eq. (8) and $J_{MG}$ is the Murphy-Good current density

$$J_{MG}(\theta) = \frac{1}{t_F^2} \frac{A_{FN}}{\phi} (E_l(\theta))^2 \exp \left(-B_{FN} v_F \theta^{3/2}/E_l(\theta)\right)$$

(19)

Here $A_{FN} \approx 1.541443 \mu A$ eV V$^{-2}$, $B_{FN} \approx 6.830890$ eV$^{-3/2}$ V nm$^{-1}$ are the conventional Fowler-Nordheim constants while $t_F \approx 1 - f + (f/6) \ln f$ and $t_F \approx 1 + f/9 - (f/18) \ln f$ are corrections due to the image charge potential with $f \equiv c_S^2 E_l(\theta)/\phi^2$ where $c_S$ is the Schottky constant and $c_S^2 = 1.439965$ eV$^2$V$^{-1}$nm. $E_l(\theta)$ refers to the local electric field on the emitter surface, while $\phi$ is the work function of the emitter material which we shall consider to be 4.5eV. In all the studies presented here, $N = 20000$.

As mentioned in Ref. [18][23][24], first $\theta$ is sampled from the distribution described above. The value of the normal energy $E_N$ and total energy $E_T$ are thereafter obtained using the conditional distributions $f(E_N|\theta)$ and $f(E_T|E_N, \theta)$ respectively. These distributions can be arrived at using the joint distribution $f(\theta, E_N, E_T)$[18]. Once the angle $\theta$, normal energy and total energy are calculated, velocity components in local co-ordinate system are obtained using suitable transformations.

There are three kinds of tracking that we shall focus on. The first uses the analytical fields directly to integrate the trajectory from the emitter surface to the anode. The second makes use of analytical field computed on a grid while the third uses the COMSOL field on a grid [18][19]. As the data typically available in a simulation is on a grid (with appropriate weighting schemes to find the point data), the analytical and COMSOL fields are best compared by considering identical grids. The grid chosen here consists of $64 \times 64 \times 256$ points with 64 each in the X and Y directions extending from $[-3.25 \mu m, 3.25 \mu m]$ and 256 points along the Z-direction from $149 \mu m$ to $161 \mu m$. The anode is placed at $160 \mu m$ and is assumed to have a thickness of $1 \mu m$. The position and momentum of the electrons are recorded when they reach the anode and these are subsequently used for further analysis.

A phase space $x-p_x$ plot for the hemiellipsoid emitter is shown in Fig. 3. Clearly, the grid data for the analytical and COMSOL fields are close to each other while non-grid analytical result shows some deviation at the extremities. It is common however to study the trace-space plots where $x' = p_x/p_z$ and $y' = p_y/p_z$ are used instead. A plot of $x-x'$ is shown in Fig. 4 for the hemiellipsoid emitter. The difference between the analytical and

![FIG. 2. The values of $(2h/R_a)/\gamma$ are shown plotted against $\ln(4h/R_a)$. They appear to be on a straight line. The best fit is also shown. The slope and the intercept are estimates of $\alpha_1$ and $\alpha_2$ respectively.]

![FIG. 3. An $x-p_x$ plot at the anode located at $z = 160 \mu m$. The $y-p_y$ plot looks similar. The emitter is a hemiellipsoid with apex radius $R_a = 500 \mu m$ and height $h = 150 \mu m$.]
COMSOL grid-data is more evident now. A similar plot for the HECP emitter is shown in Fig. 5.

A quantitative estimate of the deviation can be obtained from the rms trace-emittance defined as

$$\epsilon_{\text{Tr,rms}} = \sqrt{\langle x^2 \rangle < x'^2 > - < xx' >^2} = \epsilon_x$$  \quad (20)

where

$$< x^2 > = \frac{\sum x_i^2}{N} - \left( \frac{\sum x_i}{N} \right)^2$$  \quad (21)

$$< x'^2 > = \frac{\sum x'_i^2}{N} - \left( \frac{\sum x'_i}{N} \right)^2$$  \quad (22)

$$< xx' >^2 = \frac{\sum x_i x'_i}{N} - \frac{\sum x'_i \sum x_i}{N^2}$$  \quad (23)

Table I which summarizes the results for the single-emitters considered. The quantities of interest are (i) the emitter current (ii) the apex field (iii) $\int E.dl = \int_h^D E_z dz$ along the axis of symmetry and finally $\epsilon_x$.

It is clear from Table I that the values of $\alpha_1$ and $\alpha_2$, obtained through a fit, allow accurate determination of the apex field and hence the emission current. The values of $\int E.dl$ clearly show that while COMSOL evaluates the diode field accurately, the analytical field has errors which reflects both in the grid and non-grid evaluations.

For the HE emitter, the analytical field under-predicts $\int E.dl$ while for the HECP emitter, it over-predicts. The error ranges from 2.6% for HE to 12.6% for HECP for non-grid data, while the grid-data has errors of 8.9% for HE and 6.3% for HECP. The transverse emittance values are reasonably close for the COMSOL and analytical grid data, supportive of Figs. 4 and 5.

In summary, single emitter predictions for the current and transverse-emittance are reasonably accurate for single emitters.

### B. A 9-emitter array

Multiple emitters provide a comprehensive testing ground for the effectiveness of the hybrid model. Even a $3 \times 3$ emitter array can highlight the role of shielding and inverse shielding mediated through the anode. Consider thus, 9 identical hemiellipsoid emitters arranged on a square grid with the nearest neighbour spacing denoted by $c$. Their height $h = 150\mu m$ while $R_a = 500nm$ and $D = 160\mu m$ as in the previous section. The values of $\alpha_1$ and $\alpha_2$ are thus unchanged. The presence of other

![Fig. 4](image1.png)  
Fig. 4. A trace-space $x - x'$ plot at $z = 160\mu m$ where $x' = p_x/p_z$. The emitter is a hemiellipsoid of apex radius $R_a = 500nm$.

![Fig. 5](image2.png)  
Fig. 5. A trace-space plot at $z = 160\mu m$ for the HECP emitter.

| Emitter | Method | Grid | $E_a$ (V/nm) | I (mA) | $\int E.dl$ (V) | $\epsilon_x$ (nm) |
|---------|--------|------|-------------|-------|----------------|------------------|
| HE      | ANLYTL | N    | 4.7081      | 0.18961 | 4674           | 3.748           |
| HE      | ANLYTL | Y    | 4.7081      | 0.18961 | 4371           | 9.958           |
| HE      | COMSOL | Y    | 4.6821      | 0.17347 | 4799           | 5.985           |
| HECP    | ANLYTL| N    | 5.4899      | 1.91510 | 5405           | 5.203           |
| HECP    | ANLYTL| Y    | 5.4899      | 1.92510 | 5101           | 13.921          |
| HECP    | COMSOL | Y    | 5.4648      | 1.79525 | 4796           | 12.436          |

In summary, single emitter predictions for the current and transverse-emittance are reasonably accurate for single emitters.
emitters leads to non-zero values of $\alpha_S$ and $\alpha_{SA}$, and this leads to a change in apex electric field. We summarize the results for $c = 2h/3$ in Table [II]. Note that due to the symmetry, there are only 3 kinds of emitters. The one at the centre, the 4 nearest neighbours and the 4 next-nearest neighbours located at the corners. Thus, analytical results show identical values for the groups of 4 emitters. Note that, $\int E.dl$ computation uses grid data for both COMSOL and analytical fields. A typical potential profile is shown in Fig. [6].

![Fig. 6. A typical potential profile for a 9-emitter arrangement.](image)

**TABLE II. Characteristics for a 9-emitter system with $c = 2h/3$ with $h = 150\mu m$.** The symbols are CMSL (COMSOL), ANLY (Analytical), I (Current) and G (Grid). The correct value of $\int E.dl$ is 4800V.

| Location $x_i, y_i$ ($\mu m$) | $E_a$ CMSL (V/µm) | $E_a$ ANLY (V/µm) | $I_i$ CMSL (mA) | $I_i$ ANLY (mA) | $\int E.dl$ CMSL (V) | $\int E.dl$ ANLY (G) |
|-----------------------------|-------------------|-------------------|----------------|----------------|-----------------|----------------|
| 0, 0                        | 4.595             | 4.498             | 0.128          | 0.0898         | 4792            | 4237            |
| 100, 0                      | 4.621             | 4.554             | 0.140          | 0.110          | 4792            | 4267            |
| 0, 100                      | 4.623             | 4.554             | 0.141          | 0.110          | 4794            | 4267            |
| -100, 0                     | 4.632             | 4.554             | 0.146          | 0.110          | 4794            | 4267            |
| 0, -100                     | 4.623             | 4.554             | 0.141          | 0.110          | 4793            | 4267            |
| 100, 100                    | 4.635             | 4.601             | 0.147          | 0.131          | 4795            | 4293            |
| 100, -100                   | 4.632             | 4.601             | 0.150          | 0.131          | 4794            | 4293            |
| -100, 100                   | 4.627             | 4.601             | 0.143          | 0.131          | 4796            | 4293            |
| -100, -100                  | 4.637             | 4.601             | 0.148          | 0.131          | 4793            | 4293            |

The average error in net emission current from the 9-pin array is about 18% at $c = 2h/3$. As explained earlier, this is a small error given the large deviations that small uncertainties in geometries can cause. The average deviation of $\int E.dl$ using non-grid fields is 4.6% while the error on using grid-data is higher at 10.9%. The COMSOL grid data is close to the true value of 4800V. The average transverse emittance $\sigma_x$ at the anode plate ($D = 160\mu m$) using COMSOL data is 6.56nm while the average $\sigma_x$ using analytical grid data is 9.92nm. The average $\sigma_x$ using non-grid analytical fields is expectedly lesser.

A similar study for $c = h/2$ leads to larger errors since the method under-predicts the apex field. Thus, the net current is smaller by about 48%, the error in non-grid $\int E.dl$ is 6.2% while the error on using grid-data is 12.5%. The average $\sigma_x$ using COMSOL grid-data is 6.4nm while the average $\sigma_x$ using analytical fields on the grid is 8.6nm.

It is expected that at a fixed anode-cathode distance $D$, the errors will further increase as $c$ decreases. If, on the other hand, the emitters are randomly distributed having a mean separation $c$, the errors are expected to be smaller as compared to the regular arrangement. This expectation stems from the observation that pins that are closer to each other will have larger individual errors but their weighted contribution to the net current is smaller due to the larger shielding effects. On the other hand, the sparsely populated regions are expected to have smaller individual errors and larger weights.

As the anode moves further away from the emitter tips, the effect of the term $\alpha_{SA}$ will be smaller and shielding will dominate. Thus, the optimum emitter separation $c$ is expected to be larger than the height $c$ in order to achieve maximum current density. In such a scenario, the errors on using the analytical fields will be small.

**IV. CONCLUDING REMARKS**

The semi-analytical fields provide a useful, reasonably accurate and fast alternative for a LAFE-based diode simulation. It has been tested here using a $3 \times 3$ grid and the results are encouraging when the spacing is not too small compared to the height of the emitters. The results can thus be used to simulate larger $N \times N$ grids (e.g. $N > 100$) or emitters placed randomly. As mentioned earlier, this is still an idealization of a typical realistic situation where emitter heights need not be identical or their apex radius of curvature may differ. These aspects are of immediate interest and under investigation. Other issues relate to space-charge effects and emitter degradation (in a statistical sense), both of which can be approximately incorporated within the model proposed here.

Finally, the anode at close proximity is of relevance for gated diodes. The results can be used to transmit electrons through the anode plate with a pre-defined energy dependent transparency. Alternately, the gate can be simulated using a circular apertures centred at each emitter. A COMSOL study using a 9-pin array shows that if the gate aperture radius $R_g = h/6$ and $c = h/2$,
the apex fields reduce as compared to a plate anode and are somewhat smaller than the analytical values. Thus, the analytical modelling discussed in this paper approximates gated diodes with circular apertures reasonably well and can be used to obtain estimates of the current and transverse emittance.

V. AUTHOR DECLARATIONS

A. Conflict of interest

There is no conflict of interest to disclose.

B. Data Availability

The data that supports the findings of this study are available within the article.

VI. REFERENCE

1J.W. Lewellen, and J. Noonan, Phys. Rev. ST Accel. Beams 8, 033502 (2005).
2S.C. Leemann, A. Streun, and A. Wrulich, Phys. Rev. ST Accel. Beams 10, 071302 (2007).
3J.D. Jarvis, B.K. Choi, A.B. Hmelo, B. Ivanov, and C.A. Brau, J. Vac. Sci. Technol. B 30, 042201 (2012).
4Y. Y. Yu and K. C. Park, J. Vac. Sci. Technol. B 40, 022204 (2022).
5J. H. Hong, J. S. Kang and K. C. Park, J. Vac. Sci. Technol. 36, 02C109 (2018).
6K.L. Jensen, P.G. O’Shea, D.W. Feldman, and J.L. Shaw, J. Appl. Phys. 107, 014903 (2010).
7D. Biswas and R. Rudra, Physics of Plasmas 25, 083105 (2018).
8D. Biswas, Phys. Plasmas 25, 043113 (2018).
9D. Biswas, Physics of Plasmas, 26, 073106 (2019).
10R. Rudra and D. Biswas, AIP Advances, 9, 125207 (2019).
11D. Biswas and R. Rudra, J. Vac. Sci. Technol. B, 38, 023207 (2020).
12D. Biswas, J. Vac. Sci. Technol. B38, 063201 (2020).
13T. A. de Assis, F. F. Dall’Agnol and M. Cahay, Appl. Phys. Lett.116, 203101 (2020).
14E. L. Murphy and R. H. Good, Phys. Rev. 102, 1464 (1956).
15R. G. Forbes, App. Phys. Lett. 89, 133122 (2006).
16R. G. Forbes and J. H. B. Deane, Proc. R. Soc. A 463, 2907 (2007).
17K. L. Jensen, Introduction to the physics of electron emission, Chichester, U.K., Wiley, 2018.
18D. Biswas, Physics of Plasmas 25, 043105 (2018).
19D. Biswas and R. Ramachandran, J. Vac. Sci. Technol. B 37, 021801 (2019).
20D. Biswas and R. Ramachandran, J. Appl. Phys. 129, 194301 (2021).
21D. Biswas, J. Appl. Phys. 131, 154301 (2022).
22D. Biswas and R. Kumar, J. Vac. Sci. Technol. B 37, 040603 (2019).
23G. Sarkar, R. Kumar, G. Singh, and D. Biswas, Phys. Plasmas 28, 013111 (2021).
24R. Kumar, G. Singh and D. Biswas, Physics of Plasmas Phys. Plasmas 28, 093110 (2021).
25R. Ramachandran and D. Biswas, J. Appl. Phys. 129, 184301 (2021).
26T. A. de Assis and F. F. Dall’Agnol, J. Vac. Sci. Technol. B 37, 022202 (2019).
27D. Biswas, R. Kumar and G. Singh, J. Appl. Phys. 130, 185302 (2021).
28T. E. Everhart, Journal of Applied Physics 38, 4944 (1967).
29J. C. Wiesner and T. E. Everhart, Journal of Applied Physics 44, 2140 (1973).
30J. C. Wiesner and T. E. Everhart, Journal of Applied Physics 45, 2797 (1974).
31D. Biswas, G. Singh and R. Kumar, J. App. Phys. 120, 124307 (2016).
32E. Mesa, E. Dubado-Fuentes, and J. J. Sanz, J. Appl. Phys. 79, 39 (1996).
33E. G. Pogorelov, A. I. Zhabnov, and Y.-C. Chang, Ultramicroscopy 109, 373 (2009).
34J. R. Harris, K. L. Jensen, D. A. Schiffler, and J. J. Petillo, Appl. Phys. Lett. 106, 201603 (2015).
35J. R. Harris, K. L. Jensen, W. Tang and D. A. Schiffler, J. Vac. Sci. Technol. B 34, 041215 (2016).
36M. Reiser, Theory and Design of Charged Particle Beams (Wiley, New York, 1994).
37K. Floettmann, Phys. Rev. ST Accel. Beams 6, 034202 (2003); ibid 6, 079901 (2003).
38C.A. Brau, What Brightness Means, in Physics and Applications of High Brightness Electron Beams: Proceedings of the ICFA Workshop Chia Laguna, Sardinia, Italy 1-6 July 2002, edited by J. Rosenzweig, G. Travish, and L. Serafini (World Scientific Publishing Company, 2004), p. 20-27.
39D. Biswas, G. Singh, S. G. Sarkar and R. Kumar, Ultramicroscopy 185, 1 (2018).
40D. Biswas, G. Singh and R. Ramachandran, Physica E 109, 179 (2019).
41S. Sarkar and D. Biswas, J. Vac. Sci. Technol. B 37, 062203 (2019).
42The AC/DC module of COMSOL v5.4 is used in the simulation. For further details, see Ref. [19].