Thermodynamics of charged and rotating black strings

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Abstract: We study thermodynamics of cylindrically symmetric black holes. Uncharged as well as charged and rotating objects have been discussed. We derive surface gravity and hence the Hawking temperature and entropy for all these cases. We correct some results in the literature and present new ones. It is seen that thermodynamically these black configurations behave differently from spherically symmetric objects.
1. Black strings

The theory of thermal radiation and evaporation of black holes [1, 2] was a break-through in theoretical physics, and it became a subject where three different areas - the quantum theory, general relativity and thermodynamics - converge. The laws of black hole mechanics were established [3, 4] which are analogous to the laws of ordinary thermodynamics. This also made possible to apply physical investigations to these queer objects. Initially only spherically symmetric black holes were investigated. Later, spherical black holes with an axis of symmetry, like the Kerr black hole were also studied. The study of prolate gravitational collapse of cylindrical and other similar objects resulted in the formulation of the hoop conjecture which states that horizons form if and only if a mass gets compressed into a region whose circumference in every direction is less than its Schwarzschild circumference, $4\pi GM$. This excluded the possibility of the formation of a cylindrical black hole. However, the hoop conjecture holds only when the cosmological constant vanishes. Thus in the presence of negative cosmological constant cylindrically symmetric black holes (black strings) have been studied with great interest. These represent asymptotically anti-de Sitter spaces in the transverse and axial directions. After the pioneering work [5, 6, 7, 8, 9] on their fundamental structure and properties, these interesting objects have been investigated in other contexts as well. They have been studied, for example, in the presence of Born-Infeld and Maxwell fields, in higher dimensional and Gauss-Bonnet gravity theories [10, 11, 12, 13], and in the framework of supergravity theories, low energy string theories and topological defects [14, 15, 16]. Their study is significant from the point of view of cosmic strings and toroidal black holes as well which look locally like black strings. In this paper we study thermodynamic properties of uncharged, charged and rotating black strings. Apart from recovering results that already exist, we correct some formulae in the literature and present new ones as well.

We consider the Einstein-Hilbert action in the presence of the cosmological constant and an electromagnetic field. The total action is [7]

$$S + S_{em} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) - \frac{1}{16\pi} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}, \quad (1.1)$$

where $S$ is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda), \quad (1.2)$$
and $R$ is the curvature scalar, $g$ is the determinant of the metric. The Maxwell tensor is given as
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]
where $A_\mu$ being the vector potential is given by
\[
A_\mu = -h(r)\delta^0_\mu,
\]
h $(r)$ is an arbitrary function of the radial coordinate $r$. In this paper we study the solutions of the Einstein-Maxwell equations in cylindrical coordinates $(t, r, \phi, z)$ with
\[
-\infty < t < \infty, 0 \leq r < \infty, 0 \leq \phi \leq 2\pi, -\infty < z < \infty.
\]

We will give explicit form of metrics for the uncharged, charged and rotating cases in subsequent sections.

2. Thermodynamics of uncharged black string

The general form of the static uncharged black string metric with negative cosmological constant, $\alpha^2 = -\frac{1}{3}\Lambda > 0$, in asymptotically anti-de Sitter direction has been constructed as [9]
\[
ds^2 = -\left(\alpha^2 r^2 - \frac{4M}{\alpha r}\right)dt^2 + \left(\alpha^2 r^2 - \frac{4M}{\alpha r}\right)^{-1}dr^2 + r^2d\phi^2 + \alpha^2 r^2dz^2,
\]
where $\alpha = -\Lambda/3$, $\Lambda$ being the cosmological constant, and $M$ is related to the ADM mass density of the black string.

Putting $g^{11} = 0$ gives
\[
\alpha^2 r^2 - \frac{4M}{\alpha r} = 0,
\]
so that the event horizon of the black string is
\[
r_+ = \left(\frac{4M}{\alpha}\right)^{\frac{1}{3}}.
\]
there is a coordinate singularity at \( r = r_+ \) i.e. \( g(r_+) = 0 \) which can be removed by using the Painlevé-type coordinate transformation

\[
dt \to dt - \sqrt{\frac{1 - g}{fg}} dr, \tag{2.5}
\]

so that the metric becomes

\[
ds^2 = -f dt^2 + dr^2 + 2f \sqrt{\frac{1 - g}{fg}} dr dt + r^2 d\phi^2 + \alpha^2 r^2 dz^2. \tag{2.6}
\]

Near the event horizon we use Taylor’s series to expand the functions, \( f \) and \( g \) as

\[
f(r_+) = f'(r_+)(r - r_+) + O((r - r_+)^2), \tag{2.7}
\]

\[
g(r_+) = g'(r_+)(r - r_+) + O((r - r_+)^2). \tag{2.8}
\]

In this form the surface gravity becomes

\[
\kappa = \left| \Gamma^0_{00} \right|_{r=r_+} = \frac{1}{2} \left| \sqrt{\frac{1 - g}{fg}} g \frac{df}{dr} \right|_{r=r_+}. \tag{2.9}
\]

Using this formula for the black string (2.1) we obtain

\[
\kappa = \frac{\alpha^3 r_+^3 + 4M}{\alpha r_+^2}. \tag{2.10}
\]

Using Eq. (2.3) in Eq. (2.10) the surface gravity becomes

\[
\kappa = 3\alpha \left( \frac{M}{2} \right)^{\frac{1}{3}}. \tag{2.11}
\]

The Hawking temperature, \( T = \kappa/2\pi \), of uncharged black string is thus given by

\[
T = \frac{3\alpha}{2\pi} \left( \frac{M}{2} \right)^{\frac{1}{3}}, \tag{2.12}
\]

The temperature goes as \( M^{\frac{4}{3}} \), which is very different from that of the Schwarzschild black hole [17]. Thus the negative cosmological constant and topological difference change the thermodynamical behaviour of black configurations.

The area of the event horizon [9] of a black string, \( \sigma = 2\pi \alpha r_+^2 \), in this case becomes

\[
\sigma = \frac{2\pi (4M)^{\frac{2}{3}}}{\alpha}. \tag{2.13}
\]

Thus the Bekenstein-Hawking entropy relation in geometrized units, \( S = \sigma/4 \), takes the form

\[
S = \frac{\pi}{2\alpha} (4M)^{\frac{2}{3}}. \tag{2.14}
\]
3. Thermodynamics of charged black string

In this section we discuss thermodynamics of static charged black string. The space-time in asymptotically anti-de-Sitter direction can be written as [9]

\[ ds^2 = - \left( \frac{\alpha^2 r^2 - 4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2} \right) dt^2 + \left( \frac{\alpha^2 r^2 - 4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2} \right)^{-1} dr^2 + r^2 d\phi^2 + \alpha^2 r^2 dz^2, \]

where \( M \) and \( Q \) are constants. Note that for a cylinder of infinite radius and height \( \Delta z \) the total charge will be infinite. However, the line charge density \( Q z / \Delta z \) will be finite. We take this quantity as \( Q \) and it is related to ADM charge. Similarly \( M \) is the ADM mass per unit length in the \( z \) direction.

The event horizon can be found by putting \( g_{11} = 0 \) as

\[ \alpha^2 r^2 - \frac{4M}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2} = 0. \]

Solving this quartic equation and discarding the two imaginary roots, we see that the horizons of charged black string are

\[ r_{\pm} = \frac{(4M)^{\frac{1}{3}}}{2\alpha} \left[ \sqrt{s} \pm \sqrt{\frac{2}{s^2} - Q^2 \left( \frac{2}{M} \right)^{\frac{1}{2}} - s} \right], \]

where \( s \) is given by

\[ s = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{64Q^6}{27M^4}} \right)^{\frac{1}{3}} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{64Q^6}{27M^4}} \right)^{\frac{1}{3}}. \]

We can also express it in another form as

\[ r_{\pm} = \frac{M^{\frac{1}{3}}}{2^{\frac{2}{3}} \alpha} \left[ \sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}} + \sqrt{-x^{\frac{1}{3}} - y^{\frac{1}{3}}} + 2 \sqrt{-z + \left[ x^{\frac{1}{3}} + y^{\frac{1}{3}} \right]^2} \right], \]

where

\[ x = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \left( \frac{2}{M} \right)^{\frac{2}{3}} Q^2}, \quad y = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \left( \frac{2}{M} \right)^{\frac{2}{3}} Q^2}, \quad z = \left( \frac{2}{M} \right)^{\frac{2}{3}} Q^2. \]

As \( g^{11} \geq 0 \) when \( 0 \leq r \leq r_- \) and \( r \geq r_+ \); \( g^{11} \leq 0 \) when \( r_- \leq r \leq r_+ \), therefore, the two positive roots can be taken as the outer and inner horizons of the black string.
The singularity at \( r = 0 \) is enclosed by event horizons. We note that Eq. (3.2) has four solutions: \( z = \pm r_+ \) give two outer horizons, and \( z = \pm r_- \) two inner horizons. These enclose the singularity at \( z = 0 \).

Now, the surface gravity of a cylindrical metric (2.4) can be evaluated from [18]

\[
\kappa = \frac{1}{2\sqrt{-h}} \frac{\partial}{\partial x^a} \left( \sqrt{-h} h^{ab} \frac{\partial r}{\partial x^b} \right),
\]

(3.7)

where the second order diagonal metric \( h_{ab} \) and its inverse \( h^{ab} \), are from the \( t - r \) sector of the metric. Substituting the values we get the surface gravity as

\[
\kappa = \alpha^2 r_+ + \frac{2M}{\alpha r_+^2} - \frac{4Q^2}{\alpha^2 r_+^3},
\]

(3.8)

or using the value of \( r_+ \) this takes the form

\[
\kappa = \frac{2\alpha \left[ (M/2)^{\frac{1}{3}}D^4 + (4M^4)^{\frac{1}{3}}D - 4Q^2 \right]}{MD^3},
\]

(3.9)

where

\[
D = \sqrt{x^\frac{1}{3} + y^\frac{1}{3} + \sqrt{-x^\frac{1}{3} - y^\frac{1}{3} + 2\sqrt{-z + \left[ x^\frac{1}{3} + y^\frac{1}{3} \right]^2}}}. \quad (3.10)
\]

Thus the Hawking temperature of charged black string becomes

\[
T = \frac{\alpha \left[ (M/2)^{\frac{1}{3}}D^4 + (4M^4)^{\frac{1}{3}}D - 4Q^2 \right]}{\pi MD^3}.
\]

(3.11)

The area of the horizon per unit length in this case takes the form

\[
\sigma = \frac{\pi (2M^2)^{\frac{1}{3}}D^2}{\alpha},
\]

(3.12)

and thus the Bekenstein-Hawking entropy, \( S = \sigma/4 \), becomes

\[
S = \frac{\pi (2M^2)^{\frac{1}{3}}D^2}{4\alpha}.
\]

(3.13)

In these terms the electric potential, \( \Phi = 2Q/\alpha r_+ \), of the charged black string is given by

\[
\Phi = \left( \frac{16}{M} \right)^{\frac{1}{3}} \frac{Q}{D}.
\]

(3.14)
4. Thermodynamics of charged rotating black string

In this section, we extend our study of thermodynamics to charged and rotating black string. This is the most general case and other cases can be derived from this one. The most general form of the cylindrical symmetric spacetime asymptotically anti-de Sitter is given as [7]

\[
ds^2 = - \left( \alpha^2 r^2 - \frac{4M}{\alpha r} \left( 1 - \frac{\alpha^2 a^2}{2} \right) + \frac{4Q^2}{\alpha^2 r^2} \right) \, dt^2 \\
- \frac{4aM}{\alpha r} \sqrt{1 - \frac{\alpha^2 a^2}{2}} \left( 1 - \frac{Q^2}{M \left( 1 - \frac{\alpha^2 a^2}{2} \right)} \right) \, 2dt \, d\phi \\
+ \left( \alpha^2 r^2 - \frac{4M}{\alpha r} \left( 1 - \frac{3\alpha^2 a^2}{2} \right) + \frac{4Q^2}{\alpha^2 r^2} \left( 1 - \frac{3\alpha^2 a^2}{2} \right) \right)^{-1} \, dr^2 \\
+ \left[ r^2 + \frac{4M a^2}{\alpha r} \left( 1 - \frac{Q^2}{(1 - \frac{\alpha^2 a^2}{2}) M \alpha r} \right) \right] \, d\phi^2 + \alpha^2 r^2 \, dz^2. \tag{4.1} \]

As mentioned earlier the two constants \( M \) and \( Q \) are ADM mass and charge per unit length in the \( z \) direction. This corresponds to the Kerr-Newman solution in spherical symmetry. The parameter \( a \) is defined in the terms of units of angular momentum per unit mass:

\[
\alpha^2 a^2 = 1 - \frac{\Omega'}{M}. \tag{4.2} \]

The relation between \( J \) and \( a \) is given by

\[
J = \frac{3}{2} aM \sqrt{1 - \frac{\alpha^2 a^2}{2}}, \tag{4.3} \]

and the range of \( a \) is \( 0 \leq a \alpha \leq 1 \). The event horizons can be found by putting \( g^{11} \) equal to zero, giving

\[
\alpha^2 r^2 - \frac{4M}{\alpha r} \left( 1 - \frac{3\alpha^2 a^2}{2} \right) + \frac{4Q^2}{\alpha^2 r^2} \left( 1 - \frac{3\alpha^2 a^2}{2} \right) = 0, \tag{4.4} \]
which is a quartic equation in \( r \). Discarding the two imaginary roots, the two real roots give the horizons of rotating and charged black string at

\[
    r_{\pm} = \frac{b^{\frac{1}{3}}}{2\alpha} \left[ \sqrt{s} \pm \sqrt{2 \sqrt{s^2 - Q^2 \left( \frac{2}{M} \right)^{\frac{3}{2}}} - s} \right],
\]

where

\[
    s = x^{\frac{1}{3}} + y^{\frac{1}{3}},\ b = 4M \left( 1 - \frac{3\alpha^2 a^2}{2} \right).
\]  

We can also write it in a more convenient form as

\[
    r_{\pm} = \left[ M \left( 1 - \frac{3\alpha^2 a^2}{2} \right) \right]^{\frac{1}{2}} \left[ \sqrt{x^{\frac{1}{3}} + y^{\frac{1}{3}}} + \sqrt{-x^{\frac{1}{3}} - y^{\frac{1}{3}}} + 2 \sqrt{-z + \left[ x^{\frac{1}{3}} + y^{\frac{1}{3}} \right]^2} \right],
\]

with

\[
    x = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 \left( \frac{4^{\frac{2}{3}} Q^2}{3 \left( 1 - \frac{\alpha^2 a^2}{2} \right) \left( 1 - \frac{\alpha^2 a^2}{2} \right)^{\frac{1}{4}}} \right)^3},
\]

\[
    y = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \left( \frac{4^{\frac{2}{3}} Q^2}{3 \left( 1 - \frac{\alpha^2 a^2}{2} \right) \left( 1 - \frac{\alpha^2 a^2}{2} \right)^{\frac{1}{4}}} \right)^3},
\]

\[
    z = \frac{4^{\frac{2}{3}} Q^2}{\left( 1 - \frac{\alpha^2 a^2}{2} \right) \left( 1 - \frac{\alpha^2 a^2}{2} \right)^{\frac{1}{4}}},
\]

As before we take \( r_{\pm} \) as the outer and inner horizons.

In order to calculate the angular velocity of charged and rotating black string, \( \Omega = g_{t\phi}/g_{\phi\phi} \), we note that

\[
    g_{t\phi} = -\alpha \sqrt{1 - \frac{\alpha^2 a^2}{2}} \frac{\sqrt{\alpha^2 r^2}}{(1 - \frac{3\alpha^2 a^2}{2})^{\frac{3}{4}}},
\]

and
\[ \phi = r^2 \left( 1 - \frac{a^2r^2}{2} \right). \]  

(4.9)

Thus

\[ \Omega = \frac{ae^2}{1 - \frac{a^2r^2}{2}}. \]  

(4.10)

To work out the surface gravity of this black string, let us consider the general form of rotating cylindrical metric as

\[ ds^2 = -g_{tt} dt^2 + g_{rr} dr^2 + g_{\phi \phi} d\phi^2 + g_{zz} dz^2 - 2g_{t\phi} dt d\phi. \]  

(4.11)

To get rid of the motion of \( \phi \) we employ the following rotating coordinate system that is co-rotating with the horizon

\[ \phi = \phi' + \Omega t, \quad \phi = \phi' - \Omega t. \]  

(4.12)

Using this transformation the above metric becomes

\[ ds^2 = -G_{tt} dt^2 + g_{rr} dr^2 + g_{zz} dz^2 + g_{\phi \phi} d\phi'^2 - 2g_{t\phi} dt d\phi'. \]  

(4.13)

Here

\[ G_{tt} = g_{tt} + 2g_{t\phi} \Omega - g_{\phi \phi} \Omega^2. \]  

(4.14)

This metric has a Killing field

\[ \xi^\mu = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}, \]  

(4.15)

which satisfies

\[ \xi^\mu \xi_\mu = g_{tt} + 2g_{t\phi} \Omega - g_{\phi \phi} \Omega^2. \]  

(4.16)

Comparing Eqs. (4.14) and (4.16) we see that \( \xi^\mu \xi_\mu = G_{tt} \). Writing [19, 20] \( \xi^\mu \xi_\mu = -\lambda^2 \), we note that \( G_{tt} = -\lambda^2 \). Therefore we get

\[ \nabla_\mu (-\lambda^2) = -2\kappa \xi_\mu, \]

\[ g^{\alpha\mu} \nabla_\mu (-\lambda^2) \nabla_\mu (-\lambda^2) = 4\kappa^2 G_{tt}, \]

\[ g^{rr} (\partial_r G_{tt})^2 = 4\kappa^2 G_{tt}. \]  

(4.17)

Thus

\[ \kappa = \frac{1}{2} \frac{\sqrt{g^{rr}} \partial_r (G_{tt})}{\sqrt{G_{tt}}}. \]  

(4.18)
Now using Taylor’s series expansions

\[ G_{tt} = G'_{tt} (r - r_o) + ..., \]  \hspace{1cm} (4.19)  
\[ g^{rr} = g^{rr'} (r - r_o) + ..., \]  \hspace{1cm} (4.20)  

in the above formula, the surface gravity of rotating black string becomes

\[ \kappa = \sqrt{\frac{G'_{tt} g^{rr'}}{2}}. \]  \hspace{1cm} (4.21)  

Using the values

\[ G_{tt} = \frac{1 - \frac{1}{2} \alpha^2 a^2}{1 - \frac{3}{2} \alpha^2 a^2} \left[ \alpha^2 r^2 - \frac{4M (1 - \frac{3}{2} \alpha^2 a^2)}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2} \left( 1 - \frac{3}{2} \alpha^2 a^2 \right) \right], \]  \hspace{1cm} (4.22)  
and

\[ g^{rr} = \alpha^2 r^2 - \frac{4M (1 - \frac{3}{2} \alpha^2 a^2)}{\alpha^2 r} + \frac{4Q^2}{\alpha^2 r^2} \left( 1 - \frac{3}{2} \alpha^2 a^2 \right), \]  \hspace{1cm} (4.23)  

this takes the form

\[ \kappa = \sqrt{\frac{1 - \frac{1}{2} \alpha^2 a^2}{1 - \frac{3}{2} \alpha^2 a^2} \left[ \alpha^2 r_+ + \frac{2M (1 - \frac{3}{2} \alpha^2 a^2)}{\alpha^2 r_+^2} - \frac{4Q^2}{\alpha^2 r_+^3} \left( 1 - \frac{3}{2} \alpha^2 a^2 \right) \right]}. \]  \hspace{1cm} (4.24)  

On using Eq. (4.7) in this expression we obtain after simplification

\[ \kappa = \frac{\alpha M^\frac{1}{2} \sqrt{1 - \frac{a^2 \alpha^2}{2} D}}{2^\frac{3}{4} (1 - \frac{3a^2 \alpha^2}{2})^{\frac{3}{4}}} + \frac{2^\frac{5}{4} M^{\frac{1}{2}} \sqrt{(1 - \frac{1}{2} \alpha^2 a^2)}}{1 - \frac{3}{2} \alpha^2 a^2} D^2 - \frac{8\alpha Q^2}{MD^3 \sqrt{(1 - \frac{a^2 \alpha^2}{2}) (1 - \frac{3a^2 \alpha^2}{2})}}. \]  \hspace{1cm} (4.25)  

Here

\[ D = \sqrt{x^\frac{1}{3} + y^\frac{1}{3}} + \sqrt{-x^\frac{1}{3} - y^\frac{1}{3} + 2\sqrt{-z + (x^\frac{1}{3} + y^\frac{1}{3})^2}}. \]  \hspace{1cm} (4.26)  

Thus the Hawking temperature for the charged rotating black string is given by

\[ T = \frac{1}{2\pi} \sqrt{\frac{1 - \frac{1}{2} \alpha^2 a^2}{1 - \frac{3}{2} \alpha^2 a^2} \left[ \alpha^2 r_+ + \frac{2M (1 - \frac{3}{2} \alpha^2 a^2)}{\alpha^2 r_+^2} - \frac{4Q^2}{\alpha^2 r_+^3} \left( 1 - \frac{3}{2} \alpha^2 a^2 \right) \right]} \].  \hspace{1cm} (4.27)  

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In terms of $D$ this can be written as

$$T = \frac{1}{2\pi} \left[ \frac{\alpha M^{\frac{1}{3}} \sqrt{1 - \frac{\alpha^2 a^2}{2}} D}{2^{\frac{5}{2}} (1 - \frac{3\alpha^2 a^2}{2})^{\frac{5}{3}}} + \frac{2^{\frac{1}{2}} M^{\frac{1}{3}} \sqrt{1 - \frac{1}{2}\alpha^2 a^2}}{(1 - \frac{3}{2}\alpha^2 a^2)^{\frac{1}{3}} D^2} - \frac{8\alpha Q^2}{MD^3 \sqrt{1 - \frac{\alpha^2 a^2}{2}} (1 - \frac{3\alpha^2 a^2}{2})} \right].$$

(4.28)

The area of the horizon per unit length of charged rotating black string takes the form

$$\sigma = \frac{2^{\frac{1}{2}} \pi \left[ M \left( 1 - \frac{3\alpha^2 a^2}{2} \right) \right]^{\frac{2}{3}} D^2}{\alpha},$$

(4.29)

and the entropy in term of the horizon area becomes

$$S = \frac{\pi \left[ M \left( 1 - \frac{3\alpha^2 a^2}{2} \right) \right]^{\frac{1}{3}} D^2}{2^{\frac{1}{2}} \alpha}.$$

(4.30)

The electric potential in this case takes the form

$$\Phi = \frac{2^{\frac{1}{2}} Q}{\left[ M \left( 1 - \frac{3\alpha^2 a^2}{2} \right) \right]^{\frac{1}{3}} D}.$$

(4.31)

5. Discussion

We have studied thermodynamics of uncharged, charged and rotating black strings having a negative cosmological constant. These black configurations are asymptotically anti-de Sitter. We note that thermodynamic behaviour of these objects is quite different from their corresponding spherical configurations like the Schwarzschild, Kerr and Kerr-Newman black holes. This shows that thermodynamic properties change with the change in topological structure. The negative cosmological constant also has a significant role here.

We have used two different methods to work out the surface gravity of non-rotating and rotating strings, and the results are found to be consistent. For rotating and charged black string we have used the Killing field $\xi^\mu = \partial/\partial t + \Omega \partial/\partial \phi$, where $\partial/\partial t$ is the time like Killing vector, $\partial/\partial \phi$ is the rotational Killing vector and $\Omega$ is the angular velocity. In the most general case of charged and rotating black string the Hawking temperature is found to be

$$T = \frac{1}{2\pi} \sqrt{\frac{1 - \frac{1}{2}\alpha^2 a^2}{1 - \frac{3}{2}\alpha^2 a^2}} \left[ \alpha^2 r_+ + \frac{2M \left( 1 - \frac{3}{2}\alpha^2 a^2 \right)}{\alpha^2 r_+^2} - \frac{4Q^2}{\alpha^2 r_+^3} \left( 1 - \frac{3}{2}\alpha^2 a^2 \right) \right].$$

(5.1)
For uncharged rotating black string, i.e. \( Q = 0 \), it reduces to
\[
T = \frac{1}{2\pi} \sqrt{\frac{1 - \frac{1}{2}\alpha^2 a^2}{1 - \frac{3}{2}\alpha^2 a^2}} \left[ \alpha^2 r_+ + \frac{2M(1 - \frac{3}{2}\alpha^2 a^2)}{\alpha^2 r_+^2} \right].
\] (5.2)

For charged black string without rotation, i.e. we put \( a = 0 \), the Hawking temperature becomes
\[
T = \frac{1}{2\pi} \left( \alpha^2 r_+ + \frac{2M}{\alpha^2 r_+^2} - \frac{4Q^2}{\alpha^2 r_+^2} \right),
\] (5.3)
which is the same as in [9] and has been confirmed by the Hamilton-Jacobi method in the quantum tunneling approach [21]. For the simplest case when \( Q = 0 = a \) the temperature takes the form
\[
T = \frac{1}{2\pi} \left( \alpha^2 r_+ + \frac{2M}{\alpha^2 r_+^2} \right).
\] (5.4)

It must be mentioned here that in the case of uncharged rotating black string the horizon can be written from Eq. (4.5) by taking \( Q = 0 \) as
\[
r_+^3 = \frac{2}{\alpha^3} \left[ -M + 3\sqrt{M^2 - \frac{8}{9}\alpha^2 J^2} \right],
\] (5.5)
which is different from the value given in Ref. [6] by a factor of 8. From the horizon given in Eq. (5.5) we can correctly obtain the value for the case of uncharged non-rotating black string. However, the horizons given in Ref. [7] are correct and reduce to the uncharged case by putting \( Q = 0 \). The Hawking temperature is not given in Ref. [7], and is calculated in Ref. [6] for the uncharged rotating black string, but it is not correct. If we use the notation of Ref. [6] i.e. we write Eq. (5.2) in terms of \( J \) by using Eq. (4.3) then the formula comes out to be
\[
T = \frac{1}{2\pi \alpha^3} \left( 3\sqrt{M^2 - \frac{8}{9}\alpha^2 J^2} - M \right)^{\frac{1}{3}} \left( \sqrt{1 - \sqrt{1 - \frac{8}{9}\alpha^2 J^2}} + 1 \right),
\] (5.6)
which is different from the one given in Ref. [6].

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