On general Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities

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To our dear friend Vladimir Maz’ya

Abstract

These classical inequalities allow one to estimate the number of negative eigenvalues and the sums $S_\gamma = \sum |\lambda_i|^\gamma$ for a wide class of Schrödinger operators. We provide a detailed proof of these inequalities for operators on functions in metric spaces using the classical Lieb approach based on the Kac-Feynman formula. The main goal of the paper is a new set of examples which include perturbations of the Anderson operator, operators on free, nilpotent and solvable groups, operators on quantum graphs, Markov processes with independent increments. The study of the examples requires an exact estimate of the kernel of the corresponding parabolic semigroup on the diagonal. In some cases the kernel decays exponentially as $t \to \infty$. This allows us to consider very slow decaying potentials and obtain some results that are precise in the logarithmical scale.

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1 Introduction

Let us recall the classical estimates concerning the negative eigenvalues of the operator $H = -\Delta + V(x)$ on $L^2(R^d)$, $d \geq 3$. Let $N_E(V)$ be the number of eigenvalues $E_i$ of the operator $H$ that are below or equal to $E \leq 0$. In particular, $N_0(V)$ is the number of non-positive eigenvalues. Let

$$N(V) = \# \{ E_i < 0 \}$$

be the number of strictly negative eigenvalues of the operator $H$. Then the Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities have the following form, respectively, (see [4],

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They become particularly transparent (see [16]) if in this case by substitution [15]-[18], [22], [21])

\[ N(V) \leq C_d \int_{R^d} W^\frac{d}{2}(x) dx, \]  
\[ \sum_{i: E_i < 0} |E_i|^\gamma \leq C_{d, \gamma} \int_{R^d} W^{\frac{d}{2} + \gamma}(x) dx. \]

Here \( W = |V_\gamma| \), \( V_\gamma(x) = \min(V(x), 0) \), \( d \geq 3 \), \( \gamma \geq 0 \). The inequality (1) can be considered as a particular case of (2) with \( \gamma = 0 \). Conversely, the inequality (2) can be easily derived from (1) (see [21]). So, below we will mostly discuss the Cwikel-Lieb-Rozenblum inequality and its extensions, although some new results concerning the Lieb-Thirring inequality will also be stated.

A review of different approaches to the proof of (1) can be found in [24]. We will remind only several results. E. Lieb [15], [16] and I. Daubechies [5] offered the following general form of (1) and (2). Let \( H = H_0 + V(x) \), and \( V(x) = V_+(x) - V_-(x) \), \( V_\pm \geq 0 \). Then

\[ N(V) \leq \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} dt \int_X G(tW(x)) \mu(dx). \]  
\[ \sum_{i: E_i < 0} |E_i|^\gamma \leq \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} dt \int_X G(tW(x)) W^\gamma(x) \mu(dx). \]

Here \( W = V_\gamma = \max(0, -V(x)) \), \( G \) is a continuous, convex, non-negative function which grows at infinity not faster than a polynomial, and is such that \( z^{-1}G(z) \) is integrable at zero (hence, \( G(0) = 0 \)), and the integral (3) is finite. The function \( g(\lambda), \lambda \geq 0 \), is defined by

\[ g(\lambda) = \int_0^\infty z^{-1}G(z) e^{-\lambda z} dz, \text{ i.e. } g(1) = \int_0^\infty z^{-1}G(z) e^{-z} dz. \]  

(5)

Note that \( \pi(t) = (2\pi t)^{\frac{d}{2}} \) in the classical case of \( H_0 = -\Delta \) on \( L^2(R^d) \), and (1) follows from (3) in this case by substitution \( t \to \tau = tW(x) \) if \( G \) is such that \( \int_0^\infty z^{-1-\frac{d}{2}}G(z) dz < \infty \).

The inequalities above are meaningful only for those \( W \) for which integrals converge. They become particularly transparent (see [16]) if \( G(z) = 0 \) for \( z \leq \sigma \), \( G(z) = z - \sigma \) for \( z > \sigma, \sigma \geq 0 \). Then (3), (4) take the form

\[ N(V) \leq \frac{1}{c(\sigma)} \int_X W(x) \int_{W(x)}^\infty \pi(t) dt \mu(dx), \]  
\[ \sum_{i: E_i < 0} |E_i|^\gamma \leq \frac{1}{c(\sigma)} \int_X W^{\gamma+1}(x) \int_{W(x)}^\infty \pi(t) dt \mu(dx), \]

where \( c(\sigma) = e^{-\sigma} \int_0^\infty \frac{e^{-z} dz}{z+\sigma} \).

I. Daubechies [5] used Lieb method to justify the estimates above for some pseudo-differential operators in \( R^d \). She also mentioned there that the Lieb method works in a wider setting. A slightly different approach based on the Trotter formula was used by
G. Rozenblum and M. Solomyak [23], [24]. They proved (3) for a wide class of operators in $L^2(X, \mu)$ where $X$ is a measure space with a $\sigma$-finite measure $\mu = \mu(dx)$. They also suggested the following form of (3). Assume that the function $\pi(t)$ has different power asymptotics as $t \to 0$ and $t \to \infty$. Let

$$p_0(t, x, x) \leq c/t^{\alpha/2}, \quad t \leq h, \quad p_0(t, x, x) \leq c/t^{\beta/2}, \quad t > h,$$

where $h > 0$ is arbitrary. The parameters $\alpha$ and $\beta$ characterize the “local dimension” and the “global dimension” of $X$, respectively. For example $\alpha = \beta = d$ in the classical case of the Laplacian $H_0 = -\Delta$ in the Euclidean space $X = \mathbb{R}^d$. If $H_0 = -\Delta$ is the difference Laplacian on the lattice $X = \mathbb{Z}^d$, then $\alpha = 0, \beta = d$. If $X = S^n \times \mathbb{R}^d$ is the product of $n$-dimensional sphere and $\mathbb{R}^d$, then $\alpha = n + d, \beta = d$.

If $\alpha, \beta > 2$, inequality (3) implies (see [24]) that

$$N(V) \leq C(h)[\int_{\{W(x) \leq h^{-1}\}} W^{\beta/2}(x) \mu(dx) + \int_{\{W(x) > h^{-1}\}} W^{\alpha/2}(x) \mu(dx)],$$

Note that the restriction $\beta > 2$ is essential here in the same way as the condition $d > 2$ in (1). We will show that the assumption on $\alpha$ can be omitted, but the form of the estimate in (9) changes in this case.

The paper consists of two parts. In a shorter first part we will give a detail proof of the general form of Cwikel-Lieb-Rozenblum (3) and Lieb-Thirring (4) inequalities for Schrödinger operator in $L^2(X, \mu)$ where $X$ is a metric space with a $\sigma$-finite measure $\mu$. We shall use the Lieb method which is based on trace inequalities and the Kac-Feynman representation of the Schrödinger parabolic semigroup. This approach could be particularly preferable for readers with a background in probability theory. We do not go there beyond results obtained in [23], [24]. This part has mostly a methodological character. We also will show that inequality (3) is valid for $N_0(V)$, not only for $N(V)$.

The main goal of the paper is a new set of examples. We will consider operators which may have different power asymptotics of $\pi(t)$ as $t \to 0$ or $t \to \infty$ or exponential asymptotics as $t \to \infty$. The latter case will allow us to consider the potentials which decay very slowly at infinity. This is particularly important in some applications, such as Anderson model, where the borderline between operators with a finite and infinite number of eigenvalues is defined by the decay of the perturbation in the logarithmic scale.

The paper is organized as follows. The general statement will be proved in Theorem 2.1 in the next section. Theorems 2.5, 2.6 at the end of that section are consequences of Theorem 2.1. They provide more transparent results under additional assumptions on the asymptotic (power or exponential) behavior of $\pi(t)$. Note that we consider all $\alpha \geq 0$ in (8). Sections 3-6 are devoted to examples. Some cases of a low local dimension $\alpha$ are studied in Section 3. Operators on lattices (see also ([24])) and graphs are considered there. Section 4 deals with perturbations of Anderson operator. Lobachevsky plane (see also ([24])) and pseudo differential operators related to processes with independent increments are considered in Section 5. Section 6 is devoted to operators on free groups, continuous and
discrete Heisenberg group (see also ([9]),([11])), continuous and discrete groups of affine transformations of the line. The Appendix contains the justification of the asymptotics of \(\pi(t)\) for the quantum graph operator.

Note that in order to apply any of estimates (3),(4) or (6)-(9) one needs an exact bound for \(\pi(t)\) which can be a challenging problem in some cases.

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2 General Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities.

We will assume that \(X\) is a complete \(\sigma\)-compact metric space with Borel \(\sigma\)-algebra \(\mathcal{B}(X)\) and a \(\sigma\)-finite measure \(\mu(dx)\). Let \(H_0\) be a self-adjoint non-negative operator on \(L^2(X,\mathcal{B},\mu)\) with the following two properties:

(a) Operator \(-H_0\) is the generator of a semigroup \(P_t\) acting on \(C(X)\). The kernel \(p_0(t,x,y)\) of \(P_t\) is continuous with respect to all the variables when \(t>0\) and satisfies the relations

\[
\frac{\partial p_0}{\partial t} = -H_0p_0, \quad t > 0, \quad p_0(0,x,y) = \delta_y(x), \quad \int_X p_0(t,x,y)\mu(dy) = 1,
\]

i.e. \(p_0\) is a fundamental solution of the corresponding parabolic problem. We assume that \(p_0(t,x,y)\) is symmetric, non-negative, and it defines a Markov process \(x_s, s \geq 0,\) on \(X\) with the transition density \(p_0(t,x,y)\) with respect to the measure \(\mu\).

Note that this assumption implies that \(p_0(t,x,x)\) is strictly positive for all \(x \in X, t > 0,\) since

\[
p_0(t,x,x) = \int_X p_0^2\left(\frac{t}{2},x,z\right)p_0\left(\frac{t}{2},z,y\right)\mu(dz) > 0.
\]

(b) There exists a function \(\pi(t)\) such that \(p_0(t,x,x) \leq \pi(t)\) for \(t \geq 0\) and all \(x \in X\). We also assume that \(\pi(t)\) has at most power singularity at \(t \to 0\) and is integrable at infinity, i.e. there exists \(m\) such that

\[
\int_0^\infty \frac{t^m}{1 + t^m}\pi(t)dt < \infty.
\]

Note that condition (b) implies that

\[
p_0(t,x,y) \leq \pi(t), \quad x,y \in X.
\]

In fact,

\[
p_0(t,x,y) = \int_X p_0\left(\frac{t}{2},x,z\right)p_0\left(\frac{t}{2},z,y\right)\mu(dz) \leq \left( \int_X p_0^2\left(\frac{t}{2},x,z\right)\mu(dz) \right)^{1/2} \left( \int_X p_0^2\left(\frac{t}{2},z,y\right)\mu(dz) \right)^{1/2},
\]
which implies (13) due to (11). Let us note that (12), (13) imply that the process \( x_s \) is transient.

We decided to put an extra requirement on \( X \) to be a metric space in order to be able to assume that \( p_0 \) is continuous and use a standard version of the Kac-Feynman formula. This makes all the arguments more transparent. In fact, \( X \) is a metric space in all examples below. However, all the arguments can be modified to be applicable to the case when \( X \) is a measure space by using \( L^2 \)-theory of Markov processes based on the Dirichlet forms.

Many examples of operators which satisfy conditions (a) and (b) will be given later. At this point we would like to mention only a couple of examples. First, note that self-adjoint uniformly elliptic operators of second order satisfy conditions (a) and (b). Condition (b) holds with \( \pi(t) = C t^{-d/2} \) due to Aronson inequality.

Another wide class of operators with conditions (a) and (b) consists of operators which satisfy condition (a) and are invariant with respect to transformations from a rich enough subgroup \( \Gamma \) of the group of isometries of \( X \). The subgroup \( \Gamma \) has to be transitive, i.e., for some reference point \( x_0 \in X \) and each \( x \in X \) there exists an element \( g_x \in \Gamma \) for which \( g_x(x_0) = x \). Then \( p_0(t, x, x) = p_0(t, x_0, x_0) = \pi(t) \). The simplest example of such an operator is given by \( H_0 = -\Delta \) on \( L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx) \). The group \( \Gamma \) in this case is the group of translations or the group of all Euclidean transformations (translations and rotations). Another example is given by \( X = \mathbb{Z}^d \) being a lattice and \( -H_0 \) a difference Laplacian. Other examples will be given later.

(c) Our next assumption mostly concerns the potential. We need to know that the perturbed operator \( H = H_0 + V(x) \) is well defined and has pure discrete spectrum on the negative semiaxis. For this purpose it is enough to assume that the operator \( V(x)(H_0 - E)^{-1} \) is compact for some \( E > 0 \). This assumption can be weakened. If the domain of \( H_0 \) contains a dense in \( L^2(X, \mathcal{B}, \mu) \) set of bounded compactly supported functions, then it is enough to assume that \( V_-(x)(H_0 - E)^{-1} \) is compact for some \( E > 0 \) and the positive part of the potential is locally integrable (see [1]).

Typically (in particular, in all the examples below) \( H_0 \) is an elliptic operator, the kernel of the resolvent \( (H_0 - E)^{-1} \) has singularity only at \( x = y \), this singularity is weak, and the assumptions (c) holds if the potential has an appropriate behavior at infinity. Therefore we do not need to discuss the validity of this assumption in the examples below.

**Theorem 2.1.** Let \((X, \mathcal{B}, \mu)\) be a complete \( \sigma \)-compact metric space with the Borel \( \sigma \)-algebra \( \mathcal{B} \) and a \( \sigma \)-finite measure \( \mu \) on \( \mathcal{B} \).

Let \( H = H_0 + V(x) \), where \( H_0 \) is a self-adjoint, non-negative operator on \( L^2(X, \mathcal{B}, \mu) \), the potential \( V = V(x) = V_+ - V_- \), \( V_\pm \geq 0 \), is real valued, and the assumptions (a)-(c) hold.

Then

\[
N_0(V) \leq \frac{1}{\tilde{g}(1)} \int_0^\infty \frac{\pi(t)}{t} \int_X G(tW(x)) \mu(dx)dt,
\]

(14)
and
\[ \sum_{i : E_i < 0} |E_i|^7 \leq \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} \int_X G(tW(x))W(x)^7 \mu(dx)dt, \] (15)
where \( W(x) = V_-(x) \), and functions \( G \) and \( g \) are introduced above in (3) and (5).

**Remark 2.2.** Note that (14) differs from (3) only by inclusion of the dimension of the null space of the operator \( H \) into the left-hand side of (14). This difference is not very essential, and the first goal of this part of the paper is to give an alternative proof of (3) suitable for readers with a background in probability theory.

**Remark 2.3.** If \( G(z) = 0 \) for \( z \leq \sigma \), \( G(z) = z - \sigma \) for \( z > \sigma \), \( \sigma \geq 0 \), then (14), (15) take the form
\[ N_0(V) \leq \frac{1}{c(\sigma)} \int_X W(x) \int_{\frac{\pi(t)}{W(x)}}^\infty \pi(t) dt \mu(dx), \] (16)
\[ \sum_{i : E_i < 0} |E_i|^7 \leq \frac{1}{c(\sigma)} \int_X W^{\gamma + 1}(x) \int_{\frac{\pi(t)}{W(x)}}^\infty \pi(t) dt \mu(dx), \] (17)
where \( c(\sigma) = e^{-\sigma} \int_0^\infty \frac{e^{-x} dx}{x+\sigma} \). Some applications of these inequalities will be given below.

**Remark 2.4.** Inequalities (14), (15) are valid with \( \pi(t) \) moved under sign of the interior integrals and replaced by \( p_0(t, x, x) \). For example, (14) holds in the following form
\[ N_0(V) \leq \frac{1}{g(1)} \int_0^\infty \frac{1}{t} \int_X p_0(t, x, x)G(tW(x)) \mu(dx) dt. \]

The same change can be made in (16), (17). A very minor change in the proof of the theorem is needed in order to justify this remark. Namely, one needs only to omit the last line in (32).

**Proof of Theorem 2.1.** *Step 1.* Since the eigenvalues \( E_i \) depend monotonically on the potential \( V(x) \), without loss of generality one can assume that \( V(x) = -W(x) \leq 0 \).

First (steps 1-6), we’ll prove inequality (14) for \( N(V) \) instead of \( N_0(V) \). Here we can assume that \( V(x) \in C_{\text{com}}(X) \). Indeed, when \( N(V) \) is considered, inequality (14) with \( V(x) \in C_{\text{com}}(X) \) implies the same inequality with any \( V \) such that the integral in (14) converges (see [21]). Then (step 7), we’ll show that inequality (14) for \( N(V) \) leads to the same inequality for \( N_0(V) \). Finally (step 8), we will remind the reader of standard arguments which allow us to derive (15) from (14).

**Step 2.** We denote by \( B \) and \( B_n \) the operators
\[ B = W^{1/2}(H_0 + \kappa^2)^{-1}W^{1/2}, \quad B_n = W^{1/2}(H_0 + \kappa^2 + nW)^{-1}W^{1/2}, \quad W = W(x). \]
If \( N_{-\kappa^2}(V) = \#\{E_i \leq -\kappa^2 < 0\} \), \( \lambda_k \) are eigenvalues of the operator \( B \) and \( n(\lambda, B) = \#\{k : \lambda_k \geq \lambda\} \), then the Birman-Schwinger principle implies
\[ N_{-\kappa^2}(V) = n(1, B), \] (18)
Thus, if \( F = F(\lambda), \ \lambda \geq 0 \), is a non-negative strictly monotonically growing function, and \( \{\mu_k\} \) is the set of eigenvalues of the operator \( F(B) \), then

\[
N_{-\kappa^2}(V) \leq \sum_{k: \mu_k \geq F(1)} 1 \leq \frac{1}{F(1)} \sum_{k: \mu_k \geq F(1)} \mu_k \leq \frac{1}{F(1)} \text{Tr} F(B). \tag{19}
\]

This inequality will be used with the function \( F \) of the form

\[
F(\lambda) = \int_0^\infty P(e^{-z}) e^{-\lambda t} \, dz, \quad P(t) = \sum c_n t^n, \tag{20}
\]

The exponential polynomial \( P(e^{-z}), \ z > 0 \), will be chosen later, but it will be a non-negative function with zero of order \( m \) at \( z = 0 \), i.e.

\[
P(e^{-z}) \leq C \frac{z^m}{1 + z^m}, \quad z \geq 0, \tag{21}
\]

where \( m \) is defined in the condition (b). Since \( P(e^{-z}) \geq 0 \), (20) implies that \( F \) is non-negative and monotonic, and therefore (19) holds.

From (20) it follows that

\[
F(\lambda) = \sum_{n=0}^N c_n \frac{\lambda}{1 + n\lambda},
\]

and the obvious relation \( B_n = B(1 + nB)^{-1} \) implies that

\[
F(B) = \sum_{n=0}^N c_n B_n = W^\frac{1}{2} \sum_{n=0}^N c_n (H_0 + \kappa^2 + nW)^{-1} W^\frac{1}{2}.
\]

For an arbitrary operator \( K \), we denote its kernel by \( K(x, y) \). The kernel of the operator \( F(B) \) can be expressed through the fundamental solutions \( p = p_n(t, x, y) \) of the parabolic problem

\[
p_t = (H_0 + nW(x)) p, \ t > 0, \quad p(0, x, y) = \delta_y(x).
\]

Namely,

\[
F(B)(x, y) = W^\frac{1}{2}(x) \int_0^\infty e^{-\kappa^2 t} \sum_{n=0}^N c_n p_n(t, x, y) dt W^\frac{1}{2}(y). \tag{22}
\]

It will be shown below that the integral above converges uniformly in \( x \) and \( y \) when \( \kappa = 0 \). Hence, the kernel \( F(B)(x, y) \) is continuous. Since the operator \( F(B) \) is non-negative, from the last relation and (19), after passing to the limit as \( \kappa \to 0 \), it follows that

\[
N(V) \leq \frac{1}{F(1)} \int_0^\infty \int_X W(x) \sum_{n=0}^N c_n p_n(t, x, x) dt \mu(dx). \tag{23}
\]
Step 3. The Kac-Feynman formula allows us to write an "explicit" representation for the Schrödinger semigroup $e^{t(-H_0-nW(x))}$ using the Markov process $x_s$ associated to the unperturbed operator $H_0$. Namely, the solution of the parabolic problem

$$\frac{\partial u}{\partial t} = -H_0u - nW(x)u, \quad t > 0, \quad u(0, x) = \varphi(x) \in C(X),$$

(24)
can be written in the form

$$u(t, x) = E_x e^{-\int_0^t W(x_s) ds} \varphi(x_t).$$

Note that the finite-dimensional distributions of $x_s$ (for $0 < t_1 < \ldots < t_n, \Gamma_1, \ldots \Gamma_n \in B(X)$) are given by the formula

$$P_{x \rightarrow y}(\hat{b}_{t_1} \in \Gamma_1, \ldots, \hat{b}_{t_n} \in \Gamma_n) = \int_{\Gamma_1} \ldots \int_{\Gamma_n} p_0(t_1, x, x_1)p_0(t_2 - t_1, x_1, x_2)\ldots p_0(t_n - t_{n-1}, x_{n-1}, x_n) \mu(dx_1)\ldots \mu(dx_n).$$

If $p_0(t, x, y) > 0$, then one can define the conditional process (bridge) $\hat{b}_s = \hat{b}_s^{x \rightarrow y, t}, s \in [0, t]$, which starts at $x$ and ends at $y$. Its finite-dimensional distributions are

$$P_{x \rightarrow y}(\hat{b}_{t_1} \in \Gamma_1, \ldots, \hat{b}_{t_n} \in \Gamma_n) = \int_{\Gamma_1} \ldots \int_{\Gamma_n} p_0(t_1, x, x_1)\ldots p_0(t_n - t_{n-1}, x_{n-1}, x_n)p_0(t - t_n, x_n, y) \mu(dx_1)\ldots \mu(dx_n)\mu(dx_n) p_0(t, x, y).$$

In particular, the bridge $\hat{b}_s^{x \rightarrow y, t}, s \in [0, t]$, is defined, since $p_0(t, x, x) > 0$ (see condition (a)).

Let $p = p_n(t, x, y)$ be the fundamental solution of the problem (24). Then $p_n(t, x, y)$ can be expressed in terms of the bridge $\hat{b}_s = \hat{b}_s^{x \rightarrow y, t}, s \in [0, t]$:

$$p_n(t, x, y) = p_0(t, x, y)E_{x \rightarrow y} e^{-\int_0^t W(\hat{b}_s) ds}.$$

(25)

One of the consequences of (25) is that

$$p_n(t, x, y) \leq p_0(t, x, y).$$

(26)

Another consequence of (25) is the uniform convergence of the integral in (22) (and in (23)). In fact, (21) implies that

$$\sum_{n=0}^N c_n e^{-\int_0^t W(\hat{b}_s) ds} \leq C \frac{t^m}{1 + t^m}.$$
Hence from (25) and (13) it follows that the integrand in (22) can be estimated from above by \( C\pi(t)\frac{t^n}{1 + t^n} \). Then the uniform convergence of the integral in (22) follows from (12).

Now (23) and (25) imply

\[
N(V) \leq \frac{1}{F(1)} \int_0^\infty \int_X W(x)p_0(t, x, x)E_{x \to x} \left[ \sum_{n=0}^N c_ne^{-n\int_0^t W(\hat{b}_u)du} \right] \mu(dx)dt,
\]

where \( \tau \) above is

\[
\hat{b}_s = \hat{b}_s^{x \to x,t}.
\]

**Step 4.** We would like to rewrite the last inequality in the form

\[
N(V) \leq \frac{1}{F(1)} \int_0^\infty \int_X p_0(t, x, x)E_{x \to x} [W(\hat{b}_t) \sum_{n=0}^N c_ne^{-n\int_0^t W(\hat{b}_u)du}] \mu(dx)dt
\]

for an arbitrary \( \tau \in [0, t] \). For that purpose, it is enough to show that

\[
\int_X p_0(t, x, x)E_{x \to x} [W(\hat{b}_t) e^{-\int_0^t mW(\hat{b}_u)du}] \mu(dx) = \int_X p_0(t, x, x)W(x)E_{x \to x} [e^{-\int_0^t mW(\hat{b}_u)du}] \mu(dx).
\]

The validity of (28) can be justified using the Markov property of \( \hat{b}_s \) and its symmetry (reversibility in time). We fix \( \tau \in (0, t) \). Let \( y = \hat{b}_\tau \). We split \( \hat{b}_s \) into two bridges \( \hat{b}_u^{x \to y, \tau}, \)

\( u \in [0, \tau] \), and \( \hat{b}_v^{x \to y, \tau}, \)

\( v \in [\tau, t] \). The first bridge starts at \( x \) and ends at \( y \), the second one starts at \( y \) and goes back to \( x \). Using these bridges, one can represent the left hand side above as

\[
\int_X \int_X W(y) [p_0(\tau, x, y)p_0(t - \tau, y, x) - p_m(\tau, x, y)p_m(t - \tau, y, x)] \mu(dx)\mu(dy)
\]

which coincides with the right hand side of (28). This proves (27).

**Step 5.** We take the average of both sides of (27) with respect to \( \tau \in [0, t] \) and rewrite it in the form

\[
N(V) \leq \frac{1}{F(1)} \int_0^\infty \int_X p_0(t, x, x)E_{x \to x} \left[ \sum_{n=0}^N c_m \int_0^t W(\hat{b}_s)ds e^{-\int_0^s mW(\hat{b}_u)du} \right] \mu(dx)dt
\]

\[
= \frac{1}{F(1)} \int_0^\infty \int_X p_0(t, x, x)E_{x \to x} (uP(e^{-u})) \mu(dx)dt, \quad u = \int_0^t W(\hat{b}_s)ds,
\]

where \( P \) is the polynomial defined in (20) and (23).

Let now \( P \) be such that

\[
u P(e^{-u}) \leq G(u),
\]

(30)
where $G$ is defined in the statement of Theorem 2.1. Then one can replace $uP(e^{-u})$ in (29) by $G(u)$. Then the Jensen inequality implies that

$$G(\int_0^t W(\hat{b}_s))ds = G(\frac{1}{t} \int_0^t tW(\hat{b}_s))ds \leq \frac{1}{t} \int_0^t G(tW(\hat{b}_s))ds.$$ 

This allows us to rewrite (29) in the form

$$N(V) \leq \frac{1}{F(1)} \int_0^\infty \int_X \frac{p_0(t, x, x)}{t} \int_0^t E_{x \to x} G(tW(\hat{b}_s))ds \mu(dx)dt. \quad (31)$$

It is essential that one can use the exact formula for the distribution above:

$$E_{x \to x} G(tW(\hat{b}_s)) = \int_X G(tW(z)) \frac{p_0(s, x, z)p_0(t - s, z, x)}{p_0(t, x, x)} \mu(dx).$$

From here and (31) it follows that

$$N(V) \leq \frac{1}{F(1)} \int_0^\infty \frac{1}{t^2} \int_0^t ds \int_X \int_X G(tW(z))p_0(s, x, z)p_0(t - s, z, x)\mu(dx)\mu(dz)dt$$

$$= \frac{1}{F(1)} \int_0^\infty \frac{1}{t^2} \int_0^t ds \int_X \mu(dx)G(tW(z))p_0(t, z, z)\mu(dz)dt$$

$$= \frac{1}{F(1)} \int_0^\infty \frac{1}{t} \int_X G(tW(z))p_0(t, z, z)\mu(dz)dt$$

$$\leq \frac{1}{F(1)} \int_0^\infty \frac{\pi(t)}{t} \int_X G(tW(z))\mu(dz)dt, \quad (32)$$

where $F(1)$ is defined in (20).

**Step 6.** Now we are going to specify the choice of the polynomial $P$ which was used in the previous steps. It must be non-negative and satisfy (12) and (30). Polynomial $P$ will be determined by the choice of the function $G$. Note that it is enough to prove (14) for functions $G$ which are linear at infinity. In fact, for arbitrary $G$, let $G_N \leq G$ be a continuous function which coincides with $G$ when $z \leq N$ and is linear when $z \geq N$. For example, if $G$ is smooth, $G_N$ can be obtained if the graph of $G$ for $z \geq N$ is replaced by the tangent line through the point $(N, G(N))$. Since $G_N \leq G$, the validity of (14) for $G_N$ implies (14) with the function $G$ in the integrand and $g(1)$ being replaced by $g_N(1)$. Passing to the limit as $N \to \infty$ in this inequality, one gets (14), since $g_N(1) \to g(1)$ as $N \to \infty$. Similar arguments allow us to assume that $G = 0$ in a neighborhood of the origin. (The validity of (14) for $G_\varepsilon(z) = G(z - \varepsilon) \leq G(z)$ implies (14)). Now consider $G^\varepsilon(z) = \max(G(z), y(\varepsilon, z))$ where $y(\varepsilon, z) = zm^{m+1}, z \leq \varepsilon, y(\varepsilon, z) = (m + 1)(z - \varepsilon) + \varepsilon m^{m+1}, z > \varepsilon$, with $m$ defined in condition (b). We will show later that the right-hand side of (14) is finite for $G = G^\varepsilon$. Thus if (14) is proved for $G = G^\varepsilon$, then passing to the limit as $\varepsilon \to 0$ one gets (14) for $G$. Hence we can assume that $G = az$ at infinity and $G = zm^{m+1}$ in a neighborhood of the origin. Note that $a \neq 0$, since $G$ is convex.
A special approximation of the function $G$ by exponential polynomials will be used. Consider function $H(z) = \frac{G(z)}{z^{1-\epsilon}}$, $z > 0$. It is continuous, nonnegative and has positive limits as $z \to 0$ and $z \to \infty$. Hence there is an exponential polynomial $p_\epsilon(e^{-z})$ which approximates $H(z)$ from below, i.e.

$$|H(z) - p_\epsilon(e^{-z})| < \epsilon, \quad 0 < p_\epsilon(e^{-z}) \leq H(z) \leq 2p_\epsilon(e^{-z}), \quad z > 0.$$ 

In order to find $p_\epsilon$, one can change the variable $t = e^{-z}$ and reduce the problem to the standard Weierstrass theorem on the interval $(0,1)$. If $P_\epsilon(e^{-z}) = (1 - e^{-z})^m p_\epsilon(e^{-z})$, then

$$|z^{-1}G(z) - p_\epsilon(e^{-z})| < \epsilon, \quad 0 < p_\epsilon(e^{-z}) \leq z^{-1}G(z), \quad z > 0; \quad P_\epsilon(e^{-z}) < Cz^m, \quad z \to 0. \quad (33)$$

We will choose polynomial $P$ in (20) and (23) to be equal to $P_\epsilon$. The last two of relations (33) show that $P = P_\epsilon$ satisfies all the properties used to obtain (32). Function $F$ in (32) is defined by (20) with $P = P_\epsilon$, and therefore $F(1) = F_\epsilon(1)$ depends on $\epsilon$. From the first relation of (33) it follows that $F_\epsilon(1) \to g(1)$ as $\epsilon \to 0$. Thus passing to the limit in (32) as $\epsilon \to 0$ we complete the proof of inequality (14) for $N(V)$.

Step 7. Now we are going to show that inequality (14) for $N(V)$ implies the validity of this inequality for $N_0(V)$ under the assumption that integral (14) converges. We can assume that $G$ is linear at infinity and $G(z) = z^{m+1}$ in a neighborhood of the origin (see step 6). (Note that the above properties of $G$ imply easily the convergence of the integral (14) if $W \in C_{com}(X)$, but (14) must be proved without this additional restriction on $W$).

Let $\{\psi_i\}, \ 1 \leq i \leq n$, be a basis in the null space of the operator $H$. We need to show that $n$ is finite and $N(V) + n$ does not exceed the right-hand side of (14).

Let $V_k = V_k(x), \ k = 1, 2, ..., $ be arbitrary functions such that $V_k(x) = 1$ when $x \in X_k \in X$, $V_k(x) = 0$ when $x \notin X_k$, where the sets $X_k$ have finite measures, $\mu(X_k) = c_k < \infty$, and

$$\int_X V_k(x)|\psi_j|^2(x)\mu(dx) = \int_{X_k} |\psi_j|^2(x)\mu(dx) > 0, \quad 1 \leq j \leq k.$$

Consider the operator

$$H_{\epsilon,k} = -H_0 + V(x) - \epsilon V_k(x) = -H_0 - W(x) - \epsilon V_k(x).$$

This operators has at least $N(V) + k$ strictly negative eigenvalues. In fact, the Dirichlet form

$$\int_X [-(H_0\phi)\phi + (V - \varepsilon V_k)|\phi|^2]\mu(dx)$$

with $\varepsilon > 0$ is negative if $\phi$ is an eigenfunction of $H$ with a negative eigenvalue or $\phi = \psi_j, \ j \leq k$. Therefore there exist at least $N(V) + k$ linearly independent functions $\phi$ for which the Dirichlet form is negative. Now from inequality (14) for strictly negative eigenvalues of the operator $H_{\epsilon,k}$ it follows that

$$N(V) + k \leq \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} \int_X G(tW_\epsilon(x))\mu(dx)dt, \quad W_\epsilon = W + \varepsilon V_k. \quad (34)$$
Assume that the double integral in (34) converges when \( \varepsilon = 1 \). Then one can pass to the limit as \( \varepsilon \to 0 \) in (34) and get

\[
N(V) + k \leq \frac{1}{g(1)} \int_0^\infty \pi(t) G(tW(x)) \mu(dx) dt.
\]

Here \( k \) is arbitrary if \( n = \infty \), and one can take \( k = n \) if \( n < \infty \). This proves (14). Hence it remains only to justify the convergence of the double integral in (34) with \( \varepsilon = 1 \) under the condition that the double integral in (14) converges.

The integrands in (34) and (14) coincide when \( x \notin X_k \). Hence we only need to prove the convergence of the integral (34) with \( X \) replaced by \( X_k' = X_k \cap \{ W(x) > 1 \} \) and \( X_k'' = X_k \cap \{ W(x) < 1 \} \), respectively. Since \( W + V_k \leq 2W \), from the properties of \( G \) mentioned above it follows that

\[
G(t(W + V_k)) \leq CG(tW), \quad x \in X_k'.
\]

This implies that \( I_k < \infty \). Since \( \mu(X_k'') < \infty \) and

\[
G(t(W + V_k)) \leq C \frac{t^{m+1}}{1 + t^{m+1}}, \quad x \in X_k'',
\]

from (12) it follows that \( J_k < \infty \). Hence (14) is proved.

**Step 8.** In order to prove (15), we note that

\[
\sum_{i : E_i < 0} |E_i|^\gamma = \gamma \int_0^\infty E^{\gamma - 1} N_E(V) dE \leq \gamma \int_0^\infty E^{\gamma - 1} N_0(-(W - E)_+) dE
\]

\[
\leq \frac{\gamma}{g(1)} \int_0^\infty E^{\gamma - 1} \int_0^\infty \frac{\pi(t)}{t} \int_X G(t(W(x) - E)_+) \mu(dx) dt dE
\]

\[
= \frac{\gamma}{g(1)} \int_0^\infty \frac{\pi(t)}{t} \int_X \int_0^W E^{\gamma - 1} G(t(W(x) - E)) dE \mu(dx) dt
\]

\[
= \frac{\gamma}{g(1)} \int_0^\infty \frac{\pi(t)}{t} \int_X \int_0^1 u^{\gamma - 1} W(x)^{\gamma} G(tW(x)(1 - u)) du \mu(dx) dt.
\]

One can replace \( G(tW(x)(1 - u)) \) here by \( G(tW(x)) \), since \( G \) is monotonically increasing. This immediately implies (15).

\[\square\]

**Theorem 2.5.** Let \( H = H_0 + V(x) \), where \( H_0 \) is a self-adjoint, non-negative operator on \( L^2(X, \mathcal{B}, \mu) \), the potential \( V = V(x) \) is real valued, and the assumptions (a)-(c) hold. If

\[
\pi(t) \leq c/t^{\beta/2}, \quad t \to \infty; \quad \pi(t) \leq c/t^{\alpha/2}, \quad t \to 0
\]

(35)
for some $\beta > 2$ and $\alpha \geq 0$, then
\[ N_0(V) \leq C(h)\left[ \int_{X^-_h} W(x)^{\beta/2} \mu(dx) + \int_{X^+_h} bW(x)^{\max(\alpha/2, 1)} \mu(dx) \right], \tag{36} \]
where $X^-_h = \{ x : W(x) \leq h^{-1} \}$, $X^+_h = \{ x : W(x) > h^{-1} \}$, $b = 1$ if $\alpha \neq 2$, $b = \ln(1 + W(x))$ if $\alpha = 2$.

In some cases $\max(\alpha/2, 1)$ can be replaced by $\alpha/2$, as will be discussed in Section 3.

**Proof of Theorem 2.5.** We write (16) in the form
\[ N_0(V) \leq I_- + I_+, \]
where $I_-\pm$ correspond to integration in (16) over $X^-\pm_h$, respectively.

Let $x \in X^-_h$, i.e., $W < h^{-1}$. Then the interior integral in (16) does not exceed
\[ C(h) \int_{X^-_h} W(x)^{\beta/2} \mu(dx) = C(h)W(\beta/2)^{-1}. \tag{37} \]
Thus $I_-$ can be estimated by the first term in the right-hand side of (36). Similarly,
\[ I_+ \leq C(h) \int_{X^+_h} W(x)^{\alpha/2} \mu(dx), \]
which does not exceed the second term in the right-hand side of (36).

**Theorem 2.6.** Let $H = H_0 + V(x)$, where $H_0$ is a self-adjoint, non-negative operator on $L^2(X, \mathcal{B}, \mu)$, the potential $V = V(x)$ is real valued, and the assumptions (a)-(c) hold.

If
\[ \pi(t) \leq ce^{-at^\gamma}, \quad t \to \infty; \quad \pi(t) \leq c/t^{\alpha/2}, \quad t \to 0 \tag{38} \]
for some $\gamma > 0$ and $\alpha \geq 0$, then for each $A > 0$,
\[ N_0(V) \leq C(h, A)\left[ \int_{X^-_h} e^{-AW(x)^{-\gamma}} \mu(dx) + \int_{X^+_h} bW(x)^{\max(\alpha/2, 1)} \mu(dx) \right], \tag{39} \]
where $X^-_h, X^+_h, b$ are the same as in the theorem above.

**Proof of Theorem 2.6.** The proof is the same as that of the theorem above. One only needs to replace (37) by the following estimate
\[ C(h) \int_{X^-_h} e^{-at^\gamma} dt = C(h)W^{-1} \int_{\sigma} e^{-a(\tau W)^{\gamma}} d\tau \leq C(h)W^{-1}e^{-\frac{a}{2}(\tau W)^{\gamma}} \int_{\sigma} e^{-\frac{a}{2}(\tau W)^{\gamma}} d\tau \]
\[ \leq [C(h)W^{-1} \int_{\sigma} e^{-\frac{a}{2}(h^\gamma)\tau} d\tau]e^{-\frac{a}{2}(\tau W)^{\gamma}}, \]
and note that $\sigma$ can be chosen as large as we please.
3 Low local dimension ($\alpha < 2$)

1. Operators on lattices and groups. It is easy to see that Theorems 2.6 and 2.5 are not exact if $\alpha \leq 2$. We are going to illustrate this fact now and provide a better result for the case $\alpha = 0$ which occurs, for example, when operators on lattices and discrete groups are considered. An important example with $\alpha = 1$ will be discussed in the next subsection (operators on quantum graphs).

Let $X = \{x\}$ be a countable set and $H_0$ be a difference operator on $L^2(X)$ which is defined by

$$(H_0 \psi)(x) = \sum_{y \in X} a(x, y) \psi(y),$$

where

$$a(x, x) > 0, \quad a(x, y) = a(y, x) \leq 0, \quad \sum_{y \in X} a(x, y) = 0.$$  

A typical example of $H_0$ is the negative difference Laplacian on the lattice $X = \mathbb{Z}^d$, i.e.,

$$(H_0 \psi)(x) = -\Delta \psi = \sum_{y \in \mathbb{Z}^d : |y-x|=1} [\psi(x) - \psi(y)], \quad x \in \mathbb{Z}^d,$$

We will assume that $0 < a(x, x) \leq c_0 < \infty$. Then $\text{Sp} H_0 \subset [0, 2c_0]$. The operator $-H_0$ defines the Markov chain $x(s)$ on $X$ with continuous time $s \geq 0$ which spends exponential time with parameter $a(x, x)$ at each point $x \in X$ and then jumps to a point $y \in X$ with probability $r(x, y) = \frac{a(x, y)}{a(x, x)}, \quad \sum_{y, y \neq x} r(x, y) = 1$. The transition matrix $p(t, x, y) = P_x(x_t = y)$ is the fundamental solution of the parabolic problem

$$\frac{\partial p}{\partial t} + H_0 p = 0, \quad p(0, x, y) = \delta_y(x).$$

Obviously, $p(t, x, x) \leq \pi(t) \leq 1$, and $\pi(t) \to 1$ uniformly in $x$ as $t \to 0$. The asymptotic behavior of $\pi(t)$ as $t \to \infty$ depends on operator $H_0$ and can be more or less arbitrary.

Consider now the operator $H = H_0 - m\delta_y(x)$ with the potential supported on one point. The negative spectrum of $H$ contains at most one eigenvalue (due to rank one perturbation arguments), and such an eigenvalue exists if $m \geq c_0$. The latter follows from the variational principle, since

$$<H_0 \delta_y, \delta_y> - m < \delta_y, \delta_y> \leq c_0 - m < 0.$$  

However, Theorems 2.5 and 2.6 estimate the number of negative eigenvalues $N(V)$ of the operator $H$ by $Cc_0$. Similarly, if

$$V = - \sum_{1 \leq i \leq n} m_i \delta(x - x_i)$$

and $m_i \geq c_0$, then $N(V) = n$, but Theorems 2.5 and 2.6 give only that $N(V) \leq C \sum m_i$. The following statement provides a better result for the case under consideration than the
Theorems above. The meaning of the statement below is that we replace \( \max(\alpha/2, 1) = 1 \) in (36), (39) by \( \alpha/2 = 0 \). Let us also mention that these theorems can not be strengthened in a similar way if \( 0 < \alpha \leq 2 \) (see Example 3).

**Theorem 3.1.** Let \( H = H_0 + V(x) \), where \( H_0 \) is defined in (40), and let assumptions of Theorem 2.1 hold. Then for each \( h > 0 \),

\[
N_0(V) \leq C(h)[n(h) + \int_0^\infty \frac{\pi(t)}{t} \sum_{x \in X_h^-} G(tW(x))dt], \quad n(h) = \#\{x \in X_h^+\}.
\]

If, additionally, either (35) or (38) is valid for \( \pi(t) \) as \( t \to \infty \), then for each \( A > 0 \),

\[
N_0(V) \leq C(h, A)[\sum_{x \in X_h^-} e^{-AW(x)} + n(h)], \quad n(h) = \#\{x \in X_h^+\},
\]

respectively.

**Remark.** Estimate (42) for \( N(V) \) in the case \( X = \mathbb{Z}^d \) can be found in [24].

**Proof.** In order to prove the first inequality, we split the potential \( V(x) = V_1(x) + V_2(x) \), where \( V_2(x) = V(x) \) for \( x \in X_h^+ \), \( V_2(x) = 0 \) for \( x \in X_h^- \). Now for each \( \varepsilon \in (0, 1) \),

\[
N_0(V) \leq N_0(\varepsilon^{-1}V_1) + N_0((1-\varepsilon)^{-1}V_2) = N_0(\varepsilon^{-1}V_1) + n(h).
\]

It remains to apply Theorem 2.1 to the operator \( -\Delta + \varepsilon^{-1}V_1 \) and pass to the limit as \( \varepsilon \to 1 \). The next two inequalities follow from Theorems 2.5 and 2.6.

**2. Operators on quantum graphs.** We will consider a specific quantum graph \( \Gamma^d \), the so called Avron-Exner-Last graph. Its vertices are the points of the lattice \( \mathbb{Z}^d \), and the edges are all segments of length one connecting neighboring vertices. Let \( s \in [0, 1] \) be the natural parameter on the edges (distance from one of the end points of the edge). Consider the space \( D \) of smooth functions \( \varphi \) on edges of \( \Gamma^d \) with the following (Kirchoff’s) boundary conditions at vertices: at each vertex \( \varphi \) is continuous and

\[
\sum_{i=1}^d \varphi_i' = 0, \tag{44}
\]

where \( \varphi_i' \) are the derivatives along the adjacent edges in the direction out of the vertex. The operator \( H_0 \) acts on functions \( \varphi \in D \) as \( -\frac{\partial^2}{\partial s^2} \). The closure of this operator in \( L^2(\Gamma^d) \) is a self-adjoint operator with the spectrum \([0, \infty)\) (see [3]).

**Theorem 3.2.** The assumptions of Theorems 2.1, 2.5 hold for operator \( H_0 \) introduced in this section with the constants \( \alpha, \beta \) in Theorem 2.5 equal to 1 and \( d \), respectively.
One can easily see that there is a Markov process with the generator $-H_0$, and condition (a) of Theorem 2.1 holds. In appendix 1, we’ll estimate the function $p_0$ in order to show that condition (b) holds and find constants $\alpha, \beta$ defined in Theorem 2.5. In fact, the same arguments can be used to verify condition (a) analytically. □

As we discussed above, Theorem 2.5 is not exact if $\alpha \leq 2$. Theorem 3.1 provides a better result in the case $\alpha = 0$. The situation is more complicated if $\alpha = 1$. We will illustrate it using the operator $H_0$ on quantum graph $\Gamma^d$ defined above. We will consider two specific classes of potentials. In one case, inequality (36) is valid with $\max(\alpha/2, 1) = 1$ replaced by $\alpha/2 = 1/2$. However, inequality (36) can not be improved for potentials of the second type. The first class (regular potentials) consists of piece-wise constant functions.

**Theorem 3.3.** Let $d \geq 3$ and $V$ be constant on each edge $e_i$ of the graph: $V(x) = -v_i < 0$, $x \in e_i$. Then

$$N_0(V) \leq c(h) \left( \sum_{i: v_i \leq h^{-1}} v_i^{d/2} + \sum_{i: v_i > h^{-1}} \sqrt{v_i} \right).$$

**Proof.** Put $V(x) = V_1(x) + V_2(x)$, where $V_1(x) = V(x)$ if $|V(x)| > h^{-1}$, $V_1(x) = 0$ if $|V(x)| \leq h^{-1}$. Then (see (43))

$$N_0(V) \leq N_0(2V_1) + N_0(2V_2).$$

One can estimate $N(V_1)$ from above (below) by imposing the Neumann (Dirichlet) boundary conditions at all vertices of $\Gamma$. This leads to the estimates

$$\sum_{i: v_i > h^{-1}} \frac{\sqrt{2v_i}}{\pi} \leq N_0(V) \leq \sum_{i: v_i > h^{-1}} \left( \frac{\sqrt{2v_i}}{\pi} + 1 \right) \leq c(h) \sum_{i: v_i > h^{-1}} \sqrt{v_i},$$

which, together with Theorem 2.5 applied to $N_0(2V_2)$, justifies the statement of the theorem. □

The same arguments allow one to get a more general result.

**Theorem 3.4.** Let $d \geq 3$. Let $\Gamma^d_-$ be the set of edges $e_i$ of the graph $\Gamma^d$ where $W \leq h^{-1}$, $\Gamma^d_+$ be the complementary set of edges, and

$$\sup_{x \in e_i} W(x) \leq k_0 = k_0(h), \ x \in \Gamma^d_+, \ W = V_-. \ x \in \Gamma^d_+,$$

Then

$$N_0(V) \leq c(h, k_0) \left( \int_{\Gamma^d_-} W(x)^{d/2} dx + \int_{\Gamma^d_+} \sqrt{W(x)} dx \right).$$

**Example.** The next example shows that there are singular potentials on $\Gamma^d$ for which $\max(\alpha/2, 1)$ in (36) can not be replaced by any value less than one. Consider the potential $V(x) = -A \sum_{i=1}^m \delta(x - x_i)$, where $x_i$ are middle points of some edges, and $A > 4$. One
can easily modify the example by considering $\delta$-sequences instead of $\delta$-functions (in order to get a smooth potential.) Then

$$\int_{\Gamma} W^\sigma(x) dx = 0$$

for any $\sigma < 1$, while $N(V) \geq m$. In fact, consider the Sturm-Liouville problem on the interval $[-1/2, 1/2]$

$$-y'' - A\delta(x)y = \lambda y, \ y(-1/2) = y(1/2) = 0, \ A > 4.$$ 

It has (a unique) negative eigenvalue which is the root of the equation $\tanh(\sqrt{-\lambda}/2) = 2\sqrt{-\lambda}/A$. The corresponding eigenfunction is $y = \sinh[\sqrt{-\lambda}(|x| + 1/2)]$. The estimate $N(V) \geq m$ follows by imposing the Dirichlet boundary conditions on the vertices of $\Gamma^d$.

4 Anderson model.

I. Discrete case. Consider the classical Anderson Hamiltonian $H_0 = -\Delta + V(x, \omega)$ on $L^2(\mathbb{Z}^d)$ with random potential $V(x, \omega)$. Here

$$\Delta \psi(x) = \sum_{x' : |x' - x| = 1} \psi(x') - 2d \psi(x).$$

We assume that random variables $V(x, \omega)$ on the probability space $(\Omega, \mathcal{F}, P)$ have the Bernoulli structure, i.e., they are i.i.d. and $P\{V(\cdot) = 0\} = p > 0, P\{V(\cdot) = 1\} = q = 1 - p > 0$. The spectrum of $H_0$ is equal to (see [2])

$$\text{Sp}(H_0) = \text{Sp}(-\Delta) \oplus 1 = [0, 4d + 1].$$

Let us stress that $0 \in \text{Sp}(H_0)$ due to the existence P-a.s. of arbitrarily large clearings in realizations of $V$, i.e., there are balls $B_n = \{x : |x - x_n| < r_n\}$ such that $V(x) = 0, x \in B_n$, and $r_n \to \infty$ as $n \to \infty$ (see the proof of the theorem below for details).

Let

$$H = H_0 - W(x), \ W(x) \geq 0.$$ 

The operator $H$ has discrete random spectrum on $(-\infty, 0]$ with possible accumulation point at $\lambda = 0$. Put $N_0(-W) = \#\{\lambda_i \leq 0\}$. Obviously, $N_0(-W)$ is random. Denote by $E$ the expectation of a r.v., i.e.

$$EN_0 = \int_\Omega N_0 P(d\omega).$$

Theorem 4.1. (a) For each $h > 0$ and $\gamma < \frac{d}{d+2}$,

$$EN_0(-W) \leq c_1(h)[\#\{x \in \mathbb{Z}^d : W(x) \geq h^{-1}\}] + c_2(h, \gamma) \sum_{x : W(x) < h^{-1}} e^{-\frac{1}{W(x)}}.$$
In particular, if \( W(x) < \frac{C}{\log^\sigma |x|}, \ |x| \to \infty, \) with some \( \sigma > \frac{d+2}{d} \), then \( EN_0(-W) < \infty, \) i.e., \( N_0(-W) < \infty \) almost surely.

(b) If
\[
W(x) > \frac{C}{\log^\sigma |x|}, \ |x| \to \infty, \ \text{and} \ \sigma < \frac{2}{d},
\]
then \( N_0(-W) = \infty \) a.s. (in particular, \( EN_0(-W) = \infty \).

**Proof.** Since \( V \geq 0 \), the kernel \( p_0(t, x, y) \) of the semigroup \( \exp(-tH_0) = \exp(t(\Delta - V)) \) can be estimated by the kernel of \( \exp(t\Delta) \), i.e., by the transition probability of the random walk with continuous time on \( Z^d \). The diagonal part of this kernel \( p_0(t, x, x, \omega) \) is a stationary field on \( Z^d \).

Due to the Donsker-Varadhan estimate (see [6],[7]),
\[
\log \mathbb{E} p_0 \sim -c d_t^{d+2}, \ t \to \infty.
\]
On the rigorous level, the relations above must be understood as estimates from above and below, and the upper estimate has the following form: for each \( \delta > 0 \),
\[
\mathbb{E} p_0 \leq C(\delta) \exp(-c dt^{d+\delta}), \ t \to \infty.
\]

Now the first part of the theorem is a consequence of Theorems 2.1 and 2.6. In fact, from Remarks 2.3 and 2.4 and (46) it follows that
\[
EN_0(V) \leq \frac{1}{c(\sigma)} \int_X W(x) \int_{W(x)}^\infty \mathbb{E} p_0(t, x, x, \omega) dt \mu(dx)
\]
\[
\leq \frac{C(\delta)}{c(\sigma)} \int_X W(x) \int_{W(x)}^\infty e^{-c d t^{d+\delta}} dt \mu(dx).
\]
Then it only remains to repeat the arguments used to prove Theorem 2.6.

The proof of the second part is based on the following lemma which indicates the existence of large clearings at the distances which are not too large. We denote by \( C(r) \) the cube in the lattice,
\[
C(r) = \{x \in Z^d : |x_i| < r, \ 1 \leq i \leq d\}.
\]
Let’s divide \( Z^d \) into cubic layers \( L_n = C(a^{n+1}) \setminus C(a^n) \) with some constant \( a \geq 1 \) which will be selected later. One can choose a set \( \Gamma^{(n)} = \{z_i^{(n)} \in L_n\} \) in each layer \( L_n \) such that
\[
|z_i^{(n)} - z_j^{(n)}| \geq 2n^{\frac{d}{d+2}} + 1, \ d(z_i^{(n)}, \partial L_n) > n^{\frac{d}{d+2}},
\]
and
\[
|\Gamma^{(n)}| \geq c \frac{(2a)^{n(d-1)} a^{n+1}}{(2n^{1/d})^d} \geq c a^{nd}, \ n \to \infty.
\]
Let $C(n^{1/d}, i)$ be the cube $C(n^{1/d})$ with the center shifted to the point $z_i^{(n)}$. Obviously, cubes $C_{n^{1/d}, i}$ do not intersect each other, $C(n^{1/d}, i) \subset L_n$ and $|C(n^{1/d}, i)| \leq c'n$.

Consider the following event $A_n = \{\text{each cube } C(n^{1/d}, i) \subset L_n \text{ contains at least one point where } V(x) = 1\}$. Obviously,

$$P(A_n) = (1 - p|C(n^{1/d}, i)|)^{|\Gamma(n)|} \leq e^{-|\Gamma(n)|p|C(n^{1/d}, i)|} \leq e^{-c_1 n^d c'n^a} = e^{c(a^d p^d)^n}.$$ 

We will choose $a$ big enough, so that $a^d p^d > 1$. Then $\sum P(A_n) < \infty$, and the Borel-Cantelli lemma implies that $P$-a.s. there exists $n_0(\omega)$ such that each layer $L_n$, $n \geq n_0(\omega)$, contains at least one empty cube $C(n^{1/d}, i)$, $i = i(n)$. Then from (45) it follows that

$$W(x) \geq \frac{C}{n^{3/d}} = \varepsilon_n, \quad x \in C(n^{1/d}, i), \quad i = i(n).$$

One can easily show that the operator $H = -\Delta - \varepsilon$ in a cube $C \subset \mathbb{Z}^d$ with the Dirichlet boundary condition at $\partial C$ has at least one negative eigenvalue if $|C|^{d/2}$ is big enough. Thus the operator $H$ in $C(n^{1/d}, i(n))$ with the Dirichlet boundary condition has at least one eigenvalue if $n$ is big enough, and therefore $N(-W) = \infty$.

**II. Continuous case.** Theorem 4.1 is also valid for Anderson operators in $R^d$. Let $H_0 = -\Delta + V(x, \omega)$ on $L^2(R^d)$ with the random potential

$$V(x, \omega) = \sum_{n \in \mathbb{Z}^d} \varepsilon_n I_{Q_n}(x), \quad x \in R^d, \quad n = (n_1, ..., n_d),$$

where $Q_n = \{x \in R^d : n_i \leq x_i < n_i + 1, \ i = 1, 2, ..., d\}$ and $\varepsilon_n$ are independent Bernoulli r.v. with $P\{\varepsilon_n = 0\} = p$, $P\{\varepsilon_n = 1\} = q = 1-p$. Put $H = H_0 - W(x) = -\Delta + V(x, \omega) - W(x)$.

**Theorem 4.2.** (a) If $d \geq 3$, then for each $h > 0$ and $\gamma < \frac{d}{d + 2}$,

$$E N_0(-W) \leq c_1(h) \int_{W(x) \geq h^{-1}} W(x)^{d/2} dx + c_2(h, \gamma) \int_{W(x) < h^{-1}} e^{-\frac{1}{W(x)}} dx.$$

In particular, if $W(x) < \frac{C}{\log |x|}$, $|x| \to \infty$, with some $\sigma > \frac{d + 2}{d}$, then $E N_0(-W) < \infty$, i.e., $N_0(-W) < \infty$ almost surely.

(b) If

$$W(x) > \frac{C}{\log^\sigma |x|}, \quad |x| \to \infty, \quad \text{and} \quad \sigma < \frac{2}{d},$$

then $N_0(-W) = \infty$ a.s. (in particular, $E N_0(-W) = \infty$).

The proof of this theorem is identical to the proof of Theorem 4.1 with the only difference that now $p_0(t, 0, 0)$ is not bounded as $t \to 0$, but $p_0(t, 0, 0) \leq c/t^{d/2}$, $t \to 0.$
5 Lobachevsky plane, processes with independent increments.

1. Lobachevsky plane (see [8], [20]). We will use the Poincare upper half plane model, where \( X = \{ z = x + iy : y > 0 \} \) and the (Riemannian) metric on \( X \) has the form

\[
ds^2 = y^{-2}(dx^2 + dy^2).
\]

(47)

The geodesic lines of this metric are circular arcs perpendicular to the real axis (half-circles whose origin is on the real axis) and straight vertical lines ending on the real axis. The group of transformations preserving \( ds^2 \) is \( SL(2, R) \), i.e. the group of real valued 2 \( \times \) 2 matrices with the determinant equal to one. For each \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R) \), the action \( A(z) \) is defined by

\[
A(z) = \frac{az + b}{cz + d}.
\]

For each \( z_0 \in X \), there is a one-parameter stationary subgroup which consists of \( A \) such that \( Az_0 = z_0 \). The Laplace-Beltrami operator \( \Delta' \) (invariant with respect to \( SL(2, R) \)) is defined uniquely up to a constant factor, and is equal to

\[
\Delta' = y^2 \Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),
\]

(48)

The operator \( -\Delta' \) is self-adjoint with respect to the Riemannian measure

\[
\mu(dz) = y^{-2} dx dy,
\]

(49)

and has absolutely continuous spectrum on \([1/4, \infty)\). In order to find the number \( N'(V) \) of eigenvalues of the operator \( -\Delta' + V(x) \) below 1/4, one can apply Theorem 2.1 to the operator \( H_0 = -\Delta' - \frac{1}{4}I \).

One needs to know constants \( \alpha, \beta \) in order to apply Theorem 2.5. It is shown in [12] that the fundamental solution for the parabolic equation \( u_t = -\Delta'u \) has the following asymptotic behavior

\[
p(t, 0, 0) \sim c_1/t, \quad t \to 0; \quad p(t, 0, 0) \sim c_2e^{-t/4}/t^{3/2}, \quad t \to \infty.
\]

Thus \( \alpha = 2, \beta = 3 \) for the operator \( H_0 = -\Delta' - \frac{1}{4}I \). A similar result for the Laplacian in the Hyperbolic space of the dimension \( d \geq 3 \) can be found in [24].

2. Markov processes with independent increments (homogeneous pseudo differential operators). We will estimate \( N_0(V) \) for shift invariant pseudo differential operators \( H_0 \) associated with Markov processes with independent increments. Similar estimates were obtained in [5] for pseudo differential operators under assumptions that the symbol \( f(p) \) of the operator is monotone and non-negative, and the parabolic semigroup
$e^{-tH_0}$ is positivity preserving. This class includes important cases of $f(p) = |p|^{\alpha}$, $\alpha < 2$ and $f(p) = \sqrt{p^2 + m^2} - m$. Note that necessary and sufficient conditions of the positivity of $p_0(t,x,x)$ are given by Levy-Khinchin formula. We will omit monotonicity condition. What is more important, the results will be expressed in terms of the Levy measure responsible for the positivity of $p_0(t,x,x)$. This will allow us to consider variety estimates with power and logarithmical decaying potentials.

Let $H_0$ be a pseudo-differential operator in $X = \mathbb{R}^d$ of the form

$$H_0u = F^{-1}\Phi(k)Fu, \quad (Fu)(k) = \int_{\mathbb{R}^d} u(x)e^{-i(x,k)}dx, \quad u \in S(\mathbb{R}^d),$$

where the symbol $\Phi(k)$ of the operator $H_0$ has the following form

$$\Phi(k) = \int_{\mathbb{R}^d} (1 - \cos(x,k))\nu(x)dx. \quad (50)$$

Here $\mu(dx) = \nu(x)dx$ is an arbitrary measure (for simplicity we assumed that it has a density) such that

$$\int_{|x|>1} \nu(x)dx + \int_{|x|<1} |x|^2\nu(x)dx < \infty. \quad (51)$$

Assumption (50) is needed (and is sufficient) to construct a Markov process with the generator $L = -H_0$ (see below). However, we will impose an additional restriction on the measure $\mu(dx)$ assuming that the density $\nu(x)$ has the following power asymptotics at zero and at infinity

$$\nu(x) \sim |x|^{-d-2+\rho}, \quad x \to 0, \quad \nu(x) \sim |x|^{-d-\delta}, \quad x \to \infty,$$

with some $\rho, \delta \in (0,2)$. Note that assumption (51) holds in this case. To be more rigorous, we assume that

$$\nu(x) = a(\frac{x}{|x|})|x|^{-d-\rho}(1 + O(|x|^\varepsilon)), \quad x \to 0, \quad (52)$$

$$\nu(x) = b(\frac{x}{|x|})|x|^{-d-\delta}(1 + O(|x|^{-\varepsilon})), \quad x \to \infty, \quad (53)$$

where $a, b, \varepsilon > 0$. We also will consider another special case when the asymptotic behavior of $\nu(x)$ at infinity is at logarithmical borderline for the convergence of the integral (51). Namely, we will assume that (52) holds and

$$\nu(x) > C|x|^{-d}\log^{-\sigma}|x|, \quad x \to \infty, \quad \sigma > 1. \quad (54)$$

The solution of problem (10) is given by

$$p_0(t,x-y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\Phi(k)+i(x-y,k)}dk. \quad (55)$$
A special form of the pseudo differential operator \( H_0 \) is chosen in order to guarantee that \( p_0 \geq 0 \). In fact, let \( x_s, s > 0 \), be a Markov process in \( R^d \) with symmetric independent increments. It means that for arbitrary \( 0 < s_1 < s_2 < \ldots \), the random variables \( x_{s_1} - x_0, x_{s_2} - x_{s_1}, \ldots \) are independent and the distribution of \( x_{t+s} - x_s \) is independent of \( s \). The symmetry condition means that \( \text{Law}(x_s - x_0) = \text{Law}(x_0 - x_s) \), or \( p(s, x, y) = p(s, y, x) \), where \( p \) is the transition density of the process. According to the Levy-Khinchin theorem (see [10]), the Fourier transform (characteristic function) of this distribution has the form
\[
E e^{i(k,x_{t+s} - x_s)} = e^{-\Phi(k)},
\]
with \( \Phi(k) \) given by (50). Moreover, each measure (51) corresponds to some process. One can consider the family of processes \( x_s(x_0) = x_0 + x_s, s > 0 \), with an arbitrary initial point \( x_0 \). The generator \( L \) of this family can be evaluated in the Fourier space. If \( \varphi(x) \in S(R^d) \) and \( \hat{\varphi}(k) = F \varphi, \) then
\[
L \varphi(x) = \lim_{t \to 0} \frac{E \varphi(x + x(t)^0) - \varphi(x)}{t} = \lim_{t \to 0} \frac{1}{(2\pi)^d} \int_{R^d} E e^{i(x + x(t)^0,k)} - e^{i(x,k)} \frac{\hat{\varphi}(k)dk}{t} = -H_0 \varphi.
\]
Thus, function (55) is the transition density of some process, and therefore \( p_0(t, x) \geq 0 \), i.e., assumption (a) of Theorem 2.1 holds. Since operator \( H_0 \) is translation invariant, assumption (b) also holds with \( \pi(t) = p_0(t, 0) \). Hence, Theorem 2.1 can be applied to study negative eigenvalues of the operator \( H_0 + V(x) \) when (Levy) measure \( \nu dx \) satisfies (51). If (52), (53) or (52), (54) hold, then Theorems 2.5, 2.6 can be used. Namely, the following statement is valid.

**Theorem 5.1.** If measure \( \nu dx \) satisfies (52) and (53), then (35) is valid with \( \beta = 2d/\delta, \alpha = 2d/\rho \).

If measure \( \nu dx \) satisfies (52) and (54), then (38) is valid with \( \gamma = 1/\sigma, \alpha = 2d/\rho \).

**Proof.** Consider first the case when (52) and (53) hold. Let us prove that these relations imply the following behavior of \( \Phi(k) \) at zero and at infinity
\[
\Phi(k) = f\left(\frac{k}{|k|}\right)|k|^{\delta}(1 + O(|k|^{-\varepsilon_1})), \quad k \to 0; \quad \Phi(k) = g\left(\frac{k}{|k|}\right)|k|^{\rho}(1 + O(|k|^{-\varepsilon_1})), \quad k \to \infty, \quad (56)
\]
with some \( f, g, \varepsilon_1 > 0 \). We write (50) in the form
\[
\Phi(k) = \int_{|x| < 1} 2 \sin^2(x, k)\nu(x)dx + \int_{|x| > 1} 2 \sin^2(x, k)\nu(x)dx = \Phi_1(k) + \Phi_2(k). \quad (57)
\]
The term \( \Phi_1(k) \) is analytic in \( k \) and is of order \( O(|k|^2) \) as \( k \to 0 \). We represent the second term as
\[
\int_{R^d} 2 \sin^2(x, k)b(x)|x|^{-d-\delta}dx - \int_{|x| < 1} 2 \sin^2(x, k)b(x)|x|^{-d-\delta}dx + \int_{|x| > 1} 2 \sin^2(x, k)b(x)|x|^{-d-\delta}dx,
\]

The term \( \Phi_1(k) \) is analytic in \( k \) and is of order \( O(|k|^2) \) as \( k \to 0 \). We represent the second term as
\[
\int_{R^d} 2 \sin^2(x, k)b(x)|x|^{-d-\delta}dx - \int_{|x| < 1} 2 \sin^2(x, k)b(x)|x|^{-d-\delta}dx + \int_{|x| > 1} 2 \sin^2(x, k)b(x)|x|^{-d-\delta}dx,
\]
where \( \vec{x} = x/|x| \) and

\[
h(x) = \nu(x) - b(\vec{x})|x|^{-d-\delta}, \quad |h| \leq C|x|^{-d-\delta-\varepsilon}.
\]

The middle term above is of order \( O(|k|^2) \) as \( k \to 0 \). The first term above can be evaluated by substitution \( x \to x/|k| \). It coincides with \( f(\frac{k}{|k|})|k|^{\delta} \). One can reduce \( \varepsilon \) to guarantee that \( \delta + \varepsilon < 2 \). Then the last term can be estimated using the same substitution. This leads to the asymptotics (56) as \( k \to 0 \).

Now let \(|k| \to \infty \). Since \( \Phi_2(k) \) is bounded uniformly in \( k \), it remains to show that \( \Phi_1(k) \) has the appropriate asymptotics as \(|k| \to \infty \). We write \( v(x) \) in the integrand of \( \Phi_1(k) \) as follows

\[
v(x) = a(\vec{x})|x|^{-d-\rho} + g(x), \quad |g(x)| \leq C|x|^{-d-\rho+\varepsilon}.
\]

Then

\[
\Phi_1(k) = \int_{R^d} 2\sin^2(x, k))a(\vec{x})|x|^{-d-\rho}dx - \int_{|x|>1} 2\sin^2(x, k))a(\vec{x})|x|^{-d-\rho}dx \\
+ \int_{|x|<1} 2\sin^2(x, k))g(x)dx.
\]

The middle term in the right hand side above is bounded uniformly in \( k \). The substitution \( x \to x/|k| \) justifies that the first term coincides with \( g(\frac{k}{|k|})|k|^{\rho} \). The same substitution shows that the order of the last term is smaller if \( \varepsilon < \rho \). This gives the second relation of (56), and therefore, (56) is proved.

Let us estimate \( \pi(t) \) when (56) holds. From (55) it follows that

\[
\pi(t) = \frac{1}{(2\pi)^d} \int_{|k|<1} e^{-t\Phi(k)} dk + O(e^{-\eta t}) \quad \text{as} \quad t \to \infty, \quad \eta > 0.
\]

Now the substitution \( k \to t^{-1/\delta}k \) leads to

\[
\pi(t) \sim ct^{-d/\delta}, \quad t \to \infty, \quad c = \frac{1}{(2\pi)^d} \int_{R^d} e^{-g(\frac{k}{|k|})|k|^\delta} dk.
\]

Hence, the first of relations (35) holds with \( \beta = 2d/\delta \). In order to estimate \( \pi(t) \) as \( t \to 0 \), we put

\[
\pi(t) = \frac{1}{(2\pi)^d} \int_{|k|>1} e^{-t\Phi(k)} dk + O(1) \quad \text{as} \quad t \to 0,
\]

and make the substitution \( k \to t^{-1/\rho}k \). This leads to

\[
\pi(t) \sim ct^{-d/\rho}, \quad t \to 0, \quad c = \frac{1}{(2\pi)^d} \int_{R^d} e^{-f(\frac{k}{|k|})|k|^\rho} dk.
\]

Hence the second of relations (35) holds with \( \alpha = 2d/\rho \). The first statement of the theorem is proved.
Let us prove the second statement. If (52) and (54) hold, then
\[ \Phi(k) \geq c(\log\frac{1}{|k|})^{1-\sigma}, \quad k \to 0; \quad \Phi(k) = g\left(\frac{k}{|k|}\right)|k|^\rho(1 + O(|k|^{-\epsilon_1})), \quad k \to \infty. \quad (59) \]

In fact, only integrability of \( v(x) \) at infinity, but not (53), was used in the proof of the second relation of (56). Thus the second relation of (59) is valid. Let us prove the first estimate. Let \( \Omega_k = \{ x : |k|^{-2} > |x| > |k|^{-1} \} \), \( |k| < 1 \). We have
\[ \Phi(k) \geq \int_{\Omega_k} 2\sin^2(x, k)\nu(x)dx \geq C \int_{\Omega_k} \sin^2(x, k)|x|^{-d}\log^{-\sigma}|x|dx \]
\[ \geq C(2\log\frac{1}{|k|})^{-\sigma} \int_{\Omega_k} \sin^2(x, k)|x|^{-d}dx, \quad |k| \to 0. \]

It remains to show that
\[ \int_{\Omega_k} \sin^2(x, k)|x|^{-d}dx \sim \log\frac{1}{|k|}, \quad |k| \to 0. \quad (60) \]

After the substitution \( x = y/|k| \), the last integral can be written in the form
\[ \frac{1}{2} \int_{|y|^{-1} > |y| > 1} |y|^{-d}dy - \frac{1}{2} \int_{|y|^{-1} > |y| > 1} \cos(y, k)||y|^{-d}dy. \]

This justifies (60), since the second term above converges as \( |k| \to 0 \). Hence (59) is proved.

Finally, we need to obtain (38). The estimation of \( \pi(t) \) as \( t \to 0 \) remains the same as in the proof of the first statement of the theorem. To get the estimate as \( t \to \infty \), we use (58) (with a smaller domain of integration) and (59). Then we obtain
\[ \pi(t) \leq \frac{1}{(2\pi)^d} \int_{|k|<1/2} e^{-ct(\log\frac{1}{|k|})^{1-\sigma}}dk + O(e^{-\eta t}) \quad \text{as} \quad t \to \infty, \quad \eta > 0. \]

After integrating with respect to angle variables and substitution \( \log\frac{1}{|k|} = z \), we get
\[ \pi(t) \leq C \int_{\log 2}^{\infty} z^{d-1}e^{-z-ctz^{1-\sigma}}dz + O(e^{-\eta t}) \quad \text{as} \quad t \to \infty, \quad \eta > 0. \]

The asymptotic behavior of the last integral can be easily found using standard Laplace method, and the integral behaves as \( C_1 t^{\frac{d-1}{1-\sigma}}e^{-c_1t^{\frac{1}{1-\sigma}}} \) when \( t \to \infty \). This completes the proof of (38).

6 Continuous and discrete groups.

1. **Free groups.** Let \( X \) be a group \( \Gamma \) with generators \( a_1, a_2, \ldots, a_d \), inverse elements \( a_{-1}, a_{-2}, \ldots, a_{-d} \), the unit element \( e \), and with no relations between generators except
\( a_ia_{-i} = a_{-i}a_i = e \). The elements \( g \in \Gamma \) are the shortest versions of the words \( g = a_{i_1} \cdots a_{i_n} \) (with all factors \( e \) and \( a_ia_{-i} \) being omitted). The metric on \( \Gamma \) is given by

\[
d(g_1, g_2) = d(e, g_1^{-1}g_2) = m(g_1^{-1}g_2),
\]

where \( m(g) \) is the number of letters \( a_{\pm i} \) in \( g \). The measure \( \mu \) on \( \Gamma \) is defined by \( \mu(\{ g \}) = 1 \) for each \( g \in \Gamma \). It is easy to see that \( |\{ g : d(e, g) = R \}| = 2d(2d - 1)^{R-1} \), i.e., the group \( \Gamma \) has an exponential growth rate.

Define the operator \( \Delta_\Gamma \) on \( X = \Gamma \) by the formula

\[
\Delta_\Gamma \psi(g) = \sum_{-d \leq i \leq d, \ i \neq 0} [\psi(ga_i) - \psi(g)].
\]

(61)

Obviously, the operator \(-\Delta_\Gamma\) is bounded and non-negative in \( L^2(\Gamma, \mu) \). In fact, \(||\Delta_\Gamma|| \leq 4d\). As it is easy to see, the operator \( \Delta_\Gamma \) is left-invariant:

\[
(\Delta_\Gamma \psi)(gx) = \Delta_\Gamma(\psi(gx)), \quad x \in \Gamma,
\]

for each fixed \( g \in \Gamma \). Thus, conditions (a), (b) hold for operator \(-\Delta_\Gamma\). In order to apply Theorem 2.5, one also needs to find the parameters \( \alpha \) and \( \beta \).

**Theorem 6.1.** a) The spectrum of the operator \(-\Delta_\Gamma\) is absolutely continuous and coincides with the interval \( l_d = [\gamma, \gamma + 4\sqrt{2d - 1}] \), \( \gamma = 2d - 2\sqrt{2d - 1} \geq 0 \).

b) The kernel of the parabolic semigroup \( \pi_{\Gamma}(t) = (e^{t\Delta_\Gamma})(t, e, e) \) on the diagonal has the following asymptotic behavior at zero and infinity

\[
\pi_{\Gamma}(t) \to c_1 \text{ as } t \to 0, \quad \pi_{\Gamma}(t) \sim c_2 \frac{e^{-\gamma t}}{t^{3/2}} \text{ as } t \to \infty.
\]

(62)

**Remark 6.2.** Since the absolutely continuous spectrum of the operator \(-\Delta_\Gamma\) is shifted (it starts from \( \gamma \), not from zero), the natural question about the eigenvalues of the operator \(-\Delta_\Gamma + V(g)\) is to estimate the number \( N_{\Gamma}(V) \) of eigenvalues below the threshold \( \gamma \). Obviously, \( N_{\Gamma}(V) \) coincides with the number \( N(V) \) of the negative eigenvalues of the operator \( H_0 + V(g) \), where \( H_0 = -\Delta_\Gamma - \gamma I \). Hence one can apply Theorems 2.1, 3.1 to this operator. From (62) it follows that constants \( \alpha, \beta \) for the operator \( H_0 = -\Delta_\Gamma - \gamma I \) are equal to 0 and 3, respectively, and

\[
N_{\Gamma}(V) \leq c(h)[n(h) + \sum_{g \in \Gamma: W(g) \leq h^{-1}} W(x)^{3/2}], \quad n(h) = \#\{g \in \Gamma : W(g) > h^{-1}\}.
\]

**Proof of Theorem 6.1.** Let us find the kernel \( R_\lambda(g_1, g_2) \) of the resolvent \((\Delta_\Gamma - \lambda)^{-1}\). From the \( \Gamma\)-invariance it follows that \( R_\lambda(g_1, g_2) = R_\lambda(e, g_1^{-1}g_2) \). Hence it is enough to determine \( u_\lambda(g) = R_\lambda(e, g) \). This function satisfies the equation

\[
\sum_{i \neq 0} u_\lambda(ga_i) - (2d + \lambda)u_\lambda(g) = -\delta_e(g),
\]

(63)
where \( \delta_e(g) = 1 \) if \( g = e \), \( \delta_e(g) = 0 \) if \( g \neq e \). Since the equation above is preserved under permutations of the generators, the solution \( u_\lambda(g) \) depends only on \( m(g) \). Let \( \psi_\lambda(m) = u_\lambda(g), m = m(g) \). Obviously, if \( g \neq e \), then \( m(ga_i) = m(g) - 1 \) for one of the elements \( a_i \), \( i \neq 0 \), and \( m(ga_i) = m(g) + 1 \) for all other elements \( a_i, i \neq 0 \). Hence (63) implies

\[
2d\psi_\lambda(1) - (2d + \lambda)\psi_\lambda(0) = -1, \tag{64}
\]

\[
\psi_\lambda(m - 1) + (2d - 1)\psi_\lambda(m + 1) - (2d + \lambda)\psi_\lambda(m) = 0, \quad m > 0.
\]

Two linearly independent solutions of these equations have the form \( \psi_\lambda(m) = \nu_\pm^m \), where \( \nu_\pm \) are the roots of the equation

\[
\nu^{-1} + (2d - 1)\nu - (2d + \lambda) = 0.
\]

Thus

\[
\nu_\pm = \frac{2d + \lambda \pm \sqrt{(2d + \lambda)^2 - 4(2d - 1)}}{2(2d - 1)}.
\]

The interval \( l_d \) was singled out as the set of real \( \lambda \) such that the discriminant above is not positive. Since \( \nu_+\nu_- = 1/(2d - 1) \), we have

\[
|\nu_\pm| = \frac{1}{\sqrt{2d - 1}} \text{ for } \lambda \in l_d; \quad |\nu_+| > \frac{1}{\sqrt{2d - 1}}, \quad |\nu_-| < \frac{1}{\sqrt{2d - 1}} \text{ for real } \lambda \notin l_d.
\]

Now, if we take into account that the set \( A_{m_0} = \{g \in \Gamma, m(g) = m_0\} \) has exactly \( 2d(2d - 1)^{m_0-1} \) points, i.e., \( \mu(A_{m_0}) = 2d(2d - 1)^{m_0-1} \), we get that

\[
\nu_-^{m(g)} \in L^2(\Gamma, \mu), \nu_+^{m(g)} \notin L^2(\Gamma, \mu) \text{ for real } \lambda \notin l_d, \tag{65}
\]

and

\[
\int_{\Gamma \cap \{g: m(g) \leq m_0\}} |\nu_\pm^{2m(g)}|\mu(dg) \sim m_0 \text{ as } m_0 \to \infty \text{ for } \lambda \notin l_d. \tag{66}
\]

Relations (65) imply that \( R \cap l_d \) belongs to the resolvent set of the operator \( \Delta_\Gamma \) and that \( R_\lambda(e, g) = c\nu_-^{m(g)} \). Relation (66) implies that \( l_d \) belongs to the absolutely continuous spectrum of the operator \( \Delta_\Gamma \) with functions \( (\nu_+^{m(g)} - \nu_-^{m(g)}) \) being the eigenfunctions of the continuous spectrum. Hence statement a) is justified.

Note that the constant \( c \) in the formula for \( R_\lambda(e, g) \) can be found from (64). This gives

\[
R_\lambda(e, g) = \frac{1}{(2d + \lambda) - 2d\nu_-^{m(g)}}.
\]

Thus

\[
R_\lambda(e, e) = \frac{1}{(2d + \lambda) - 2d\nu_-}.
\]

Hence, for each \( a > 0 \),

\[
\pi_\Gamma(t) = \frac{1}{2\pi} \int_{a-i\infty}^{a+i\infty} e^{\lambda t} R_\lambda(e, e)d\lambda = \frac{1}{2\pi} \int_{a-i\infty}^{a+i\infty} e^{\lambda t}(2d + \lambda) - 2d\nu_- d\lambda.
\]

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The integrand here is analytic with branching points at the ends of the segment \( l_d \), and the contour of integration can be bent into the left half plane \( \Re \lambda < 0 \) and replaced by an arbitrary closed contour around \( l_d \). This immediately implies the first relation of (62).

The asymptotic behavior of the integral as \( t \to \infty \) is defined by the singularity of the integrand at the point \( -\gamma \) (the right end of \( l_d \)). Since the integrand there has the form 
\[ e^{\lambda t} [a + b\sqrt{\lambda + \gamma} + O(\lambda + \gamma)], \lambda + \gamma \to 0, \]
this leads to the second relation of (62).

2. General remark on left invariant diffusions on Lie groups. The examples below concern differential operators on the continuous and discrete non-commutative groups \( \Gamma \) (processes with independent increments considered in the previous section are examples of operators on the abelian groups \( R^d \)).

First we will consider the Heisenberg (nilpotent) group \( \Gamma = H^3 \) of the upper triangular matrices
\[
g = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad (x, y, z) \in \mathbb{R}^3, \tag{67}
\]
with units on the diagonal, and its discrete subgroup \( ZH^3 \), where \( (x, y, z) \in \mathbb{Z}^3 \).

Then we study (solvable) group of the affine transformations of the real line: \( x \to ax + b, a > 0 \), which has the matrix representation:
\[
A D f (R^1) = \left\{ g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \quad a > 0, \quad (a, b) \in \mathbb{R}^2 \right\},
\]
and its subgroup generated by \( \alpha_1 = \begin{bmatrix} e & e \\ 0 & 1 \end{bmatrix} \) and \( \alpha_2 = \begin{bmatrix} e & -e \\ 0 & 1 \end{bmatrix} \) and their inverses \( \alpha^{-1}_1 = \begin{bmatrix} e^{-1} & -1 \\ 0 & 1 \end{bmatrix} \) and \( \alpha^{-1}_2 = \begin{bmatrix} e^{-1} & 1 \\ 0 & 1 \end{bmatrix} \).

There are two standard ways to construct the Laplacian on a Lie group. A usual differential-geometric approach starts with the Lie algebra \( \mathfrak{A} \Gamma \) on \( \Gamma \), which can be considered either as the algebra of the first order differential operators generated by the differentiations along the appropriate one-parameter subgroups of \( \Gamma \), or simply as a tangent vector space \( T \Gamma \) to \( \Gamma \) at the unit element \( I \). The exponential mapping \( \mathfrak{A} \Gamma \to \Gamma \) allows one to construct (at least locally) the general left invariant Laplacian \( \Delta \Gamma \) on \( \Gamma \) as the image of the differential operator \( \sum_{ij} a_{ij} D_i D_j + \sum_i b_i D_i \) with constant coefficients on \( \mathfrak{A} \Gamma \). The Riemannian metric \( ds^2 \) on \( \Gamma \) and the volume element \( dv \) can be defined now using the inverse matrix of the coefficients of the Laplacian \( \Delta \Gamma \). It is important to note that additional symmetry conditions are needed to determine \( \Delta \Gamma \) uniquely.

The central object in the probabilistic construction of the Laplacian (see, for instance, McKean [14]) is the Brownian motion \( g_t \) on \( \Gamma \). We impose the symmetry condition \( g_t \overset{\text{law}}{=} g_t^{-1} \). Since \( \mathfrak{A} \Gamma \) is a linear space, one can define the usual Brownian motion \( b_t \) on \( \mathfrak{A} \Gamma \) with the generator \( \sum_{ij} a_{ij} D_i D_j + \sum_i b_i D_i \). The symmetry condition holds if \( (I + db_t) \overset{\text{law}}{=} (I + db_t)^{-1} \). The process \( g_t \) (diffusion on \( \Gamma \)) is given (formally) by the stochastic multiplicative
integral
\[ g_t = \prod_{s=0}^{t} (I + db_s), \]
or (more rigorously) by the Ito’s stochastic differential equation
\[ dg_t = g_t db_t. \] (68)

The Laplacian \( \Delta_{\Gamma} \) is defined now as the generator of the diffusion:
\[ \Delta_{\Gamma} f(g) = \lim_{\Delta t \to 0} \frac{E f(g(I + b \Delta t)) - f(g)}{\Delta t}, \quad f \in C^2(\Gamma). \] (69)

The Riemannian metric form is defined as above (by the inverse matrix of the coefficients of the Laplacian).

We will use the probabilistic approach to construct the Laplacian in the examples below, since it allows us to easily incorporate the symmetry condition.

3. **Heisenberg group** \( \Gamma = H^3 \) of the upper triangular matrices (67) with units on the diagonal. We have
\[ \mathfrak{a} \Gamma = \{ A = \begin{bmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}, \quad (\alpha, \beta, \gamma) \in \mathbb{R}^3 \}, \quad e^A = \begin{bmatrix} 1 & \alpha & \gamma + \frac{\alpha \beta}{2} \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}. \]

Thus \( A \rightarrow \exp(A) \) is a one-to-one mapping of \( \mathfrak{a} \Gamma \) onto \( \Gamma \). Consider the following Brownian motion on \( \mathfrak{a} \Gamma \):
\[ b_t = \begin{bmatrix} 0 & u_t & \sigma w_t \\ 0 & 0 & v_t \\ 0 & 0 & 0 \end{bmatrix}, \]
where \( \sigma \) is a constant and \( u_t, v_t, w_t \) are (standard) independent Wiener processes. Then equation (68) has the form
\[ dg_t = \begin{bmatrix} 0 & dx_t & dz_t \\ 0 & 0 & dy_t \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x_t & z_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & du_t & \sigma dw_t \\ 0 & 0 & dv_t \\ 0 & 0 & 0 \end{bmatrix}, \]
which implies that
\[ dx_t = du_t, \quad dy_t = dv_t, \quad dz_t = \sigma dw_t + x_t dv_t. \]

Under condition \( g(0) = I \), we get
\[ g_t = \begin{bmatrix} 1 & u_t & \sigma w_t + \int_0^t u_s dv_s \\ 0 & 1 & v_t \\ 0 & 0 & 1 \end{bmatrix}. \]
Let us note that the matrix

\[
(g_t)^{-1} = \begin{bmatrix}
1 & -u_t & u_t v_t - \sigma w_t - \int_0^t u_s dv_s \\
0 & 1 & -v_t \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & -u_t & -\sigma w_t + \int_0^t v_s du_s \\
0 & 1 & -v_t \\
0 & 0 & 1
\end{bmatrix}
\]

has the same law as \( g_t \). Now from (69) it follows that

\[
(\Delta f)(x, y, z) = \frac{1}{2} [f_{xx} + f_{yy} + (\sigma^2 + x^2)f_{zz} + 2\sigma x f_{yz}].
\]

The matrix of the left invariant Riemannian metric has the form

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \sigma x \\
0 & \sigma x & \sigma^2 + x^2
\end{bmatrix}
\]

\[
^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \sigma^2 + x^2 & -\sigma x \\
0 & -\sigma x & 1
\end{bmatrix},
\]

i.e.,

\[
ds^2 = dx^2 + (\sigma^2 + x^2)dy^2 + dz^2 - 2\sigma x dy dz,
\]

\[dV = dx dy dz.\]

Denote by \( p_0(t, x, y, z) \) the transition density for the process \( g_t \) (fundamental solution of the parabolic equation \( u_t = \Delta u \)). Let \( \pi_\sigma(t) = p_\sigma(t, 0, 0, 0) \).

**Theorem 6.3.** Function \( \pi_\sigma(t) \) has the following asymptotic behavior at zero and infinity:

\[
\pi_\sigma(t) \sim \frac{c_0}{t^{3/2}}, \quad t \to 0; \quad \pi_\sigma(t) \sim \frac{c}{t^2}, \quad t \to \infty, \quad c = p_0(1, 0, 0), \quad (70)
\]

i.e., Theorem 2.5 holds for operator \( H = \Delta + V(x, y, z) \) with \( \alpha = 3, \beta = 4 \).

**Proof.** Since \( H^3 \) is a three dimensional manifold, the asymptotics at zero is obvious. Let us prove the second relation of (70). We start with the simple case of \( \sigma = 0 \). The operator \( \Delta \) in this case is degenerate. However, the density \( p_0(t, x, y, z) \) exists and can be found using Hörmander hypoellipticity theory or by direct calculations. In fact, the joint distribution of \( (x_t, y_t, z_t) \) is self-similar:

\[
\left( \frac{u_t}{\sqrt{t}, \frac{v_t}{\sqrt{t}}, \int_0^t u_s dv_s}{t} \right) = (u_1, v_1, \int_0^1 u_s dv_s),
\]

i.e.,

\[
p_0(t, x, y, z) = \frac{1}{t^2} p_0(1, \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, \frac{z}{t}),
\]

and therefore,

\[
p_0(t, 0, 0, 0) = \frac{c}{t^2}, \quad c = p_0(1, 0, 0, 0).
\]

Let \( \sigma^2 > 0 \). Then

\[
p_\sigma(t, x, y, z) = \frac{1}{\sqrt{2\pi \sigma^2 t}} \int_{R^1} p_0(t, x, y, z_1) e^{-\frac{(x-x_1)^2}{2\sigma^2 t}} dz_1.
\]
After rescaling \( \frac{x}{\sqrt{t}} \to x, \frac{y}{\sqrt{t}} \to y, \frac{z}{t} \to z \), we get

\[
p_\sigma(t, x, y, z) = \frac{\sqrt{t}}{t^2 2\pi \sigma^2} \int_{\mathbb{R}^3} p_0(1, x, y, z_1) e^{-\frac{(z - z_1)^2}{2\sigma^2}} dz_1.
\]

From here it follows that \( p_\sigma(t, 0, 0, 0) \sim c/t^2, \ t \to \infty \), with \( c = p_0(1, 0, 0, 0) \).

Theorem 6.3 can be proved for the group \( H^n \) of \( n \times n \) upper triangular matrices with units on the diagonal. In this case,

\[
\alpha = \text{dim} H^n = \frac{n(n-1)}{2}, \quad \beta = (n-1) + 2(n-2) + 3(n-3) + \ldots = \frac{n(n^2-1)}{2}.
\]

4. **Heisenberg discrete group** \( \Gamma = ZH^3 \) of integer valued matrices of the form

\[
g = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{Z}^3.
\]

Consider the Markov process \( g_t \) on \( ZH^3 \) defined by the equation

\[
g_{t+dt} = g_t \begin{pmatrix} 1 & d\xi_t & d\zeta_t \\ 0 & 1 & d\eta_t \\ 0 & 0 & 1 \end{pmatrix}, \tag{71}
\]

where \( \xi_t, \eta_t, \zeta_t \) are three independent Markov processes on \( \mathbb{Z}^3 \) with generators

\[
\Delta_1 \psi(n) = \psi(n+1) + \psi(n-1) - 2\psi(n), \quad n \in \mathbb{Z}^3.
\]

Equation (71) can be solved using discretization of time. This gives

\[
g_t = \begin{pmatrix} 1 & x_t & y_t \\ 0 & 1 & z_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi_t & \zeta_t + \int_0^t \xi_s d\eta_s \\ 0 & 1 & \eta_t \\ 0 & 0 & 1 \end{pmatrix}.
\]

The generator \( L \) of this process has the form (61) with

\[
a_{\pm 1} = \begin{pmatrix} 1 & \pm1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_{\pm 2} = \begin{pmatrix} 1 & 0 & \pm1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_{\pm 3} = \begin{pmatrix} 1 & 0 & \pm1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

i.e.,

\[
L = \Delta_1 \psi(g) = \sum_{i=\pm 1, \pm 2, \pm 3} [\psi(ga_i) - \psi(g)]. \tag{72}
\]

If \( \psi = \psi(g) \) is considered as a function of \( (x, y, z) \in \mathbb{Z}^3 \), then

\[
L\psi(x, y, z) = \psi(x+1, y, z) + \psi(x-1, y, z) + \psi(x, y+1, z+x) + \psi(x, y-1, z-x) + \psi(x, y, z+1) + \psi(x, y, z-1) - 6\psi(x, y, z). \tag{73}
\]

The analysis of the transition probability in this case is similar to the continuous case, and it leads to the following result.
Theorem 6.4. If $g_t$ is the process on $ZH^3$ with the generator (73), then
\[
P\{g_t = I\} = P\{x_t = y_t = z_t = 0\} \sim \frac{c}{t^2}, \ t \to \infty,
\]
with $c$ defined in (70). In particular, Theorem 3.1 can be applied to operator $H_0 = L$ with $\beta = 4$.

This result is valid in a more general setting (see [13]). Consider three independent processes $\xi_t, \eta_t, \zeta_t, t \geq 0$, on $Z^1$ with independent increments and such that
\[
Ee^{ik\xi_t} = e^{-t(1-\sum_{i=1}^{\infty} p_i \cos ki)}, \ \sum_{i=1}^{\infty} p_i = 1,
\]
\[
Ee^{ik\eta_t} = e^{-t(1-\sum_{i=1}^{\infty} q_i \cos ki)}, \ \sum_{i=1}^{\infty} q_i = 1,
\]
\[
Ee^{ik\zeta_t} = e^{-t(1-\sum_{i=1}^{\infty} r_i \cos ki)}, \ \sum_{i=1}^{\infty} r_i = 1.
\]
Assume also that there exist $\alpha_1, \alpha_2, \alpha_3$ on the interval $(0, 2)$ such that
\[
p_i \sim \frac{c_1}{i^{1+\alpha_1}}, \ q_i \sim \frac{c_2}{i^{1+\alpha_2}}, \ r_i \sim \frac{c_3}{i^{1+\alpha_3}}
\]
as $i \to \infty$, i.e., distributions with characteristic functions $\sum_{i=1}^{\infty} p_i \cos ki$, $\sum_{i=1}^{\infty} q_i \cos ki$, $\sum_{i=1}^{\infty} r_i \cos ki$ belong to the domain of attraction of the symmetric stable law with parameters $\alpha_1, \alpha_2, \alpha_3$. Let $g_t$ be the process on $ZH^3$ defined by (71). Then
\[
P\{g_t = I\} \sim \frac{c}{t^{\gamma}}, \ t \to \infty, \ \gamma = \max(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} + \frac{1}{\alpha_3}).
\]

5. **Group Aff ($R^1$) of affine transformations of the real line.** This group of transformations $x \to ax + b$, $a > 0$, has a matrix representation:
\[
\Gamma = Aff (R^1) = \{g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \ a > 0, \ (a, b) \in R^2\}.
\]

We start with the Lie algebra for $Aff (R^1)$:
\[
\mathfrak{a} \Gamma = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}, \ (\alpha, \beta) \in R^2 \right\}.
\]

Obviously, for arbitrary $A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$, one has
\[
\exp (A) = \begin{bmatrix} e^\alpha & \frac{\beta e^\alpha - 1}{\alpha} \\ 0 & 1 \end{bmatrix},
\]

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i.e., the exponential mapping of \( \mathfrak{A} \Gamma \) coincides with the group \( \Gamma \). Consider the diffusion

\[
b_t = \begin{bmatrix} w_t + \alpha t & v_t \\ 0 & 0 \end{bmatrix}
\]
on \( \mathfrak{A} \Gamma \), where \((w_t, v_t)\) are independent Wiener processes. Consider the matrix valued process \( g_t = \begin{bmatrix} x_t & y_t \\ 0 & 1 \end{bmatrix} \), \( g_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), on \( \Gamma \) satisfying the equation

\[
dg_t = g_t db_t = \begin{bmatrix} x_t & y_t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dw_t + \alpha dt & dv_t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_t (dw_t + \alpha dt) & x_t dv_t \\ 0 & 0 \end{bmatrix}.
\]

This implies

\[
\begin{align*}
dx_t &= x_t (dw_t + \alpha dt), \\
dy_t &= x_t dv_t,
\end{align*}
\]
i.e. (due to Ito’s formula),

\[
x_t = e^{w_t + (\alpha - \frac{1}{2})t}, \quad y_t = \int_0^t x_s dv_s.
\]

We impose the following symmetry condition:

\[ (g_t)^{-1} \text{law} = g_t, \quad (74) \]

It holds if \( \alpha = \frac{1}{2} \). In fact,

\[
g_t = \begin{bmatrix} e^{w_t} & \int_0^t e^{w_s} dv_s \\ 0 & 1 \end{bmatrix}, \quad g_t^{-1} = \begin{bmatrix} e^{-w_t} - \int_0^t e^{w_s} dv_s \\ 0 & 1 \end{bmatrix}, \quad (75)
\]

and (74) follows after the change of variables \( s = t - \tau \) in the matrix \( g_t^{-1} \). Then the generator of the process \( g_t \) has the form

\[
\Delta_{\Gamma} f = \frac{x^2}{2} \left[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right] + \frac{x \partial f}{2 \partial x}.
\]

**Theorem 6.5.** Operator \( \Delta_{\Gamma} \) is self-adjoint with respect to the measure \( x^{-1} dxdy \). The function \( \pi(t) = p(t, 0, 0) \) has the following behavior at zero and infinity:

\[
\pi(t) \sim \frac{c_0}{t}, \quad t \to 0; \quad \pi(t) \sim \frac{C}{t^{3/2}}, \quad t \to \infty.
\]

**Remark 6.6.** Let \( H = \Delta_{\Gamma} + V \), where the negative part \( W = V_- \) of the potential is bounded: \( W \leq h^{-1} \). From (76) and Theorem 2.5 it follows that

\[
N_0(V) \leq C(h) \int_{-\infty}^{\infty} \int_0^\infty \frac{W^{3/2}(x, y)}{|x|} dxdy.
\]

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Remark 6.7. The left-invariant Riemannian metric on $\text{Aff}(R^1)$ is given by the inverse diffusion matrix of $\triangle$, i.e.,

$$d\xi^2 = x^{-2} (dx^2 + dy^2) \quad \left( g = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}, \quad x > 0 \right)$$

After the change $(x, y) \to (y, x)$, this formula coincides with the metric on the Lobachevsky plane (see the previous section). However, one can not identity the Laplacian on $\text{Aff}(R^1)$ and on the Lobachevsky plane $L^2$, since they are defined by different symmetry conditions. The plane $L^2$ has a three dimensional group of transformations, and each point $z \in L^2$ has a one-parameter stationary subgroup. The Laplacian on the Lobachevsky plane was defined by the invariance with respect to this three dimensional group of transformations. In the case of $\Gamma = \text{Aff}(R^1)$, the group of transformations is two dimensional. It acts as a left shift $g \to g_1 g$, $g_1, g \in \Gamma$, and the Laplacian is specified by the left invariance with respect to this two dimensional group and the symmetry condition (74).

Proof. Since $\Gamma$ is a two dimensional manifold, the asymptotics of $\pi(t)$ at zero is obvious. One needs only to justify the asymptotics of $\pi(t)$ at infinity.

Let's find the density of $(x_t, y_t) = (e^{\hat{w}t}, \int_0^t e^{\hat{w}s} dv_s)$. The second term, for a fixed realization of $w$, has the Gaussian law with (conditional) variance $\sigma^2 = \int_0^t e^{2\hat{w}s} ds$, and

$$P\{x_t \in 1 + dx, \; y_t \in 0 + dy\} = p(t, 0, 0) dx dy = \frac{1}{\sqrt{2\pi t}} E \frac{1}{\sqrt{2\pi \int_0^t e^{2\hat{w}s} ds}}. \quad (77)$$

Here $\hat{w}_s, s \in [0, t]$, is the Brownian bridge on $[0, t]$. The distribution of the exponential functional $A(t) = \int_0^t e^{2\hat{w}s} ds$ and the joint distribution of $(A(t), w(t))$ were calculated in [25]. Together with (77), these easily imply the statement of the theorem. \qed

6. A relation between Markov processes and random walks on discrete groups. Let $\Gamma$ be a discrete group generated by elements $a_1, \ldots, a_d, a_{-1} = a_1^{-1}, \ldots, a_{-d} = a_d^{-1}$, with some identities. Define the Laplacian on $\Gamma$ by the formula

$$\Delta \psi(g) = \sum_{i=-d}^d \psi(ga_i) - 2d\psi(g), \quad g \in \Gamma.$$ 

Consider the Markov process $g_t$ on $\Gamma$ with continuous time and the generator $\Delta$. Let $\tilde{g}_k$, $k = 0, 1, 2, \ldots$, be the Markov chain on $\Gamma$ with discrete time (symmetric random walk) such that

$$P\{\tilde{g}_0 = e\} = 1, \quad P\{\tilde{g}_{n+1} = ga_i \mid \tilde{g}_n = g\} = \frac{1}{2d}, \quad i = \pm 1, \pm 2, \ldots \pm d.$$ 

Then there is a relation between transition probability $p(t, e, g)$ of the Markov process $g_t$ and the transition probability $P\{\tilde{g}_k = g\}$ of the random walk. In particular, one can estimate $\pi(t) = p(t, e, e)$ for large $t$ through $\tilde{\pi}(2k) = P\{\tilde{g}_{2k} = e\}$ under minimal
assumptions on $\tilde{\pi}(2k)$. For example, it is enough to assume that $\tilde{\pi}(2k) = k\gamma L(k)$, $\gamma \geq 0$, where $L(k)$ for large $k$ can be extended as slowly varying monotonic function of continuous argument $k$. We are not going to provide a general statement of this type, but we restrict ourselves to a specific situation needed in the next section. Note that we consider here only even arguments of $\tilde{\pi}$, since $\tilde{\pi}(2k+1) = 0$.

**Theorem 6.8.** Let

$$\tilde{\pi}(2n) \leq e^{-c_0(2n)^\alpha}, \ n \to \infty, \ c_0 > 0, \ 0 < \alpha < 1.$$  

Then

$$\pi(t) \leq e^{-c_0(2dt)^\alpha}, \ t \geq t_0.$$  

**Proof.** The number $\nu_t$ of jumps of the process $g_t$ on the interval $(0, t)$ has Poisson distribution. At the moments of jumps, the process performs the symmetric random walk with discrete time and transition probabilities $P\{g \to ga_i\} = \frac{1}{2d}, \ i = \pm 1, \pm 2, \ldots \pm d$.

Thus (taking into account that $\tilde{\pi}(2k+1) = 0$),

$$\pi(t) = p(t, e, e) = \sum_{n=0}^{\infty} \tilde{\pi}(2n)P\{\nu_t = 2n\}.$$  

Due to the exponential Chebyshev inequality

$$P\{|\nu_t - 2dt| \geq \epsilon t\} \leq e^{-c_\epsilon^2 t}, \ t \to \infty.$$  

Secondly,

$$P\{\nu_t \text{ is even}\} = \frac{1}{2} + O(e^{-4dt}), \ t \to \infty.$$  

These relations imply that, for $t \to \infty$ and $\delta > 0$,

$$\pi(t) = \sum_{n:|2n-2dt|<\epsilon t} \tilde{\pi}(2n)P\{\nu_t = 2n\} + O(e^{-c_0(2dt)^\alpha})$$

$$\leq \sum_{n:|2n-2dt|<\epsilon t} e^{-c_0(2n)^\alpha}P\{\nu_t = 2n\} + O(e^{-c_0(2dt)^\alpha})$$

$$\leq (1 + \delta)e^{-c_0(2dt)^\alpha}\sum_{n:|2n-2dt|<\epsilon t} P\{\nu_t = 2n\} + O(e^{-c_0(2dt)^\alpha})$$

$$\leq \frac{1 + \delta}{2}e^{-c_0(2dt)^\alpha} + O(e^{-c_0(2dt)^\alpha}).$$

7. Random walk on the discrete subgroup of $Aff(R^1)$. Let us consider the following two matrices $\alpha_1 = \begin{bmatrix} e & e \\ 0 & 1 \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} e & -e \\ 0 & 1 \end{bmatrix}$ in $Aff(R^1)$ and their inverses
\[ \alpha_{-1} = \begin{bmatrix} e^{-1} & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \alpha_{-2} = \begin{bmatrix} e^{-1} & 1 \\ 0 & 1 \end{bmatrix} \]. Let \( G \) be a subgroup of \( \text{Aff}(R^1) \) generated by \( \alpha_{\pm 1} \) and \( \alpha_{\pm 2} \). Consider the random walk on \( G \) of the form

\[ g_n = h_1 h_2 \ldots h_n, \]

where one step random matrices \( h_i \) coincide with one of the matrices \( \alpha_{\pm 1}, \alpha_{\pm 2} \) with probability \( 1/4 \), i.e.,

\[ h_i = \begin{bmatrix} e^{\varepsilon_i} & \delta_i \\ 0 & 1 \end{bmatrix}, \]

where

\[ P\{\varepsilon_i = 1, \delta_i = e\} = P\{\varepsilon_i = 1, \delta_i = -e\} = P\{\varepsilon_i = -1, \delta_i = -1\} = P\{\varepsilon_i = -1, \delta_i = 1\} = 1/4. \] (78)

Let \( \Delta_G \) be the Laplacian on \( G \) which corresponds to the generators \( a_{\pm 1}, a_{\pm 2} \), i.e.,

\[ L = \Delta_G \psi(g) = \sum_{i=\pm 1, \pm 2} [\psi(ga_i) - \psi(g)]. \]

**Theorem 6.9.** (a) The following estimate is valid for \( \tilde{\pi}(2n) \):

\[ \tilde{\pi}(2n) \leq e^{-c_0(2n)^{1/3}} , \quad n \to \infty, c_0 > 0. \]

(b) Theorem 3.1 can be applied to operator \( H = \Delta_G + V(g) \) with \( \gamma = 1/3 \), i.e.,

\[ N_0(V) \leq C(h, A) \sum_{g: V(g) \leq h^{-1}} e^{-AW(g)^{-1/3}} + n(h), \quad n(h) = \# \{ g : W(g) > h^{-1} \}. \]

**Proof.** The random variables \( (\varepsilon_i, \delta_i) \) are dependent, but (78) implies that \( (\varepsilon_i, \tilde{\delta}_i) \), where \( \tilde{\delta}_i = \text{sgn}\delta_i \), are independent symmetric Bernoulli r.v. It is easy to see that

\[ g_n = \begin{bmatrix} e^{S_n} & \sum_{k=1}^{n} \delta_k e^{S_{k-1}} \\ 0 & 1 \end{bmatrix}, \]

where \( S_0 = 1, S_k = \varepsilon_1 + \ldots + \varepsilon_k, k > 0, \) is a symmetric random walk on \( Z^1 \). This formula is an obvious discrete analogue of (75). Our goal is to calculate the probability

\[ \tilde{\pi}(2n) = P\{g_{2n} = I\} = P\{S_{2n} = 0, \sum_{k=1}^{2n} \delta_k e^{S_{k-1}} = 0\} \approx \frac{1}{\sqrt{\pi n}} \sum_{k=1}^{2n-1} \delta_{k+1} e^{\tilde{S}_k} = 0, \quad n \to \infty. \]
Here $\widehat{S}_k, k = 0, 1, \ldots, 2n$, is the discrete bridge, i.e., the random walk $S_k$ under conditions $S_0 = S_{2n} = 0$.

Put $M_{2n} = \max_{k \leq 2n} \widehat{S}_k$, $m_{2n} = \min_{k \leq 2n} \widehat{S}_k$. Let $\Gamma^+_{s-1}$, $\Gamma^-_s$ be the sets of moments of time $k$ when the bridge $\widehat{S}_k$ changes value from $s-1$ to $s$ or from $s$ to $s-1$, respectively. Introduce local times $\tau^+_s = \text{Card} \Gamma^+_{s-1}$ and $\tau^-_s = \text{Card} \Gamma^-_s$, i.e. $\tau^+_s = \#(\text{jumps of } \widehat{S}_k \text{ from } s-1 \text{ to } s)$ and $\tau^-_s = \#(\text{jumps of } \widehat{S}_k \text{ from } s \text{ to } s-1)$. Note that $\delta_{k+1}e^{\widehat{S}_k} = \delta_{k+1}e^s$ when $k \in \Gamma^+_{s-1} \cup \Gamma^-_s$, and therefore

$$\sum_{k=1}^{2n-1} \delta_{k+1}e^{\widehat{S}_k} = \sum_{s=m_{2n}+1}^{M_{2n}} e^s \sum_{j \in \Gamma^+_{s-1} \cup \Gamma^-_s} \delta_j,$$

Since r.v. $\{\delta_j\}$ are independent of the trajectory $S_k$ and numbers $e^s$, $s = 0, \pm 1, \pm 2, \ldots$, are rationally independent, we have

$$P\{g_{2n} = I\} \lesssim \frac{1}{\sqrt{n}} E \prod_{s=m_{2n}+1}^{M_{2n}} \left( \frac{2\tau^-_s}{\tau^+_s} \right)^{\left(\frac{1}{2}\right)^{2\tau^-_s}} \lesssim \frac{1}{\sqrt{n}} \left( \frac{1}{2} \right)^{M_{2n} - m_{2n}}.$$

$$\leq \frac{1}{\sqrt{n}} \left( \frac{1}{2} \right)^{\sqrt{2n}} + \sum_{r=1}^{\sqrt{2n}} \left( \frac{1}{2} \right)^{r} P\{|S_k| \leq r, k = 1, 2, \ldots, 2n, S_{2n} = 0\} \leq e^{c_1 \sqrt{2n}} + \sum_{r=1}^{\sqrt{2n}} \left( \frac{1}{2} \right)^{r} P\{|S_k| \leq r, k = 1, 2, \ldots, 2n, S_{2n} = 0\} \leq \left( \cos \frac{\pi}{2(r+1)} \right)^{2n}.$$

Lemma 6.10. $P\{|S_k| \leq r, k = 1, 2, \ldots, 2n, S_{2n} = 0\} \leq \left( \cos \frac{\pi}{2(r+1)} \right)^{2n}$.

Proof. Let us introduce the operator $H_0\psi(x) = \psi(x+1) + \psi(x-1) - 2\psi(x)$ on the set $[-r, r] \subset \mathbb{Z}$ with the Dirichlet boundary conditions $\psi(r+1) = \psi(-r) = 0$. Then $\varphi(x) = \cos \frac{\pi x}{2(r+1)}$ is an eigenfunction of $H_0$ with the eigenvalue $\lambda_{0,r+1} = \cos \frac{\pi}{2(r+1)}$. Hence

$$H_0^{2n}\varphi(x) = \lambda_{0,r+1}^{2n}\varphi(x).$$

Let $p_r(k, x, z)$ be the transition probability of the random walk on $[-r, r] \subset \mathbb{Z}$ with the absorption at $\pm(r+1)$. Then

$$\sum_{|z| \leq r} p_r(2n, x, z)\varphi(z) = \lambda_{0,r+1}^{2n}\varphi(x).$$

Since $\varphi(z) \leq 1$, $\varphi(0) = 1$, the latter relation implies

$$\sum_{|z| \leq r} p_r(2n, x, z) \leq \lambda_{0,r+1}^{2n}.$$
Since $S_k$, $k = 0, 1, ..., 2n$, is the symmetric random walk on $Z^1$, we have

$$P\{|S_k| \leq r, \; k = 1, 2, ..., 2n, \; S_{2n} = 0\} = p_r(2n, 0, 0) \leq \lambda_{0,r+1}^{2n}.$$

Direct calculation shows that

$$\max_{r \leq \sqrt{2n}} (\frac{1}{2})^r (\cos \frac{\pi}{2(r+1)})^{2n} \leq e^{-c(2n)^{1/3}},$$

with the maximum achieved at $r = r_0 \sim c_1(2n)^{1/3}$. Thus

$$P\{g_{2n} = I\} \leq \left(\frac{1}{2}\right)\sqrt{2n} + \sqrt{2n}e^{-c_0(2n)^{1/3}} \leq e^{-c_0(2n)^{1/3}}$$

for arbitrary $c_0 < c_0$ and sufficiently large $n$. This proves the first statement of the theorem. Now the second statement follows from Theorem 6.8. \hfill \Box

Appendix. Proof of Theorem 3.2. As it was mentioned after the statement of the theorem, it is enough to show the validity of condition (b) and evaluate $\alpha, \beta$. Let

$$u_t = -H_0 u, \; t > 0, \; u|_{t=0} = f,$$

with a compactly supported $f$ and

$$\varphi = \varphi(x, \lambda) = \int_0^\infty u e^{\lambda t} dt, \; \Re \lambda \leq -a < 0, \; x \in \Gamma^d.$$ 

Note that we replaced $-\lambda$ by $\lambda$ in the Laplace transform above. It is convenient for future notations. Then $\varphi$ satisfies the equation

$$(H_0 - \lambda)\varphi = f, \quad (79)$$

and $u$ can be found using the inverse Laplace transform

$$u = \frac{1}{(2\pi)^d} \int_{-a-i\infty}^{a+i\infty} \varphi e^{-\lambda t} d\lambda. \quad (80)$$

The spectrum of $H_0$ is $[0, \infty)$, and $\varphi$ is analytic in $\lambda$ when $\lambda \in C\setminus[0, \infty)$. We are going to study the properties of $\varphi$ when $\lambda \to 0$ and $\lambda \to \infty$. Let $\psi(z) = \psi(z, \lambda)$, $z \in Z^d$, be the restriction of the function $\varphi(x, \lambda)$, $x \in \Gamma^d$, on the lattice $Z^d$. Let $e$ be an arbitrary edge of $\Gamma^d$ with end points $z_1, z_2 \in Z^d$ and parametrization from $z_1$ to $z_2$. By solving the boundary value problem on $e$, we can represent $\varphi$ on $e$ in the form

$$\varphi = \frac{\psi(z_1) \sin k(1-s) + \psi(z_2) \sin ks}{\sin k} + \varphi_{par}, \; \varphi_{par} = \int_0^1 G(s,t) f(t) dt, \quad (81)$$

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where \( k = \sqrt{\lambda} \), \( \text{Im} k > 0 \), and

\[
G = \frac{1}{k \sin k} \begin{cases} \sin ks \sin k(1-t), & s < t \\ \sin kt \sin k(1-s), & s \geq t \end{cases}.
\]

Due to the invariance of \( H_0 \) with respect to translations and rotations in \( Z^d \), it is enough to estimate \( p_0(t, x, x) \) when \( x \) belongs to the edge \( e_0 \) with \( z_1 \) being the origin in \( Z^d \) and \( z_2 = (1, 0, ..., 0) \). Let \( f \) be supported on one edge \( e_0 \). Then (81) is still valid, but \( \varphi_{par} = 0 \) on all the edges except \( e_0 \). We substitute (81) into (44) and get the following equation for \( \psi \):

\[
(\Delta - 2d \cos k)\psi(z) = \frac{1}{k} \int_0^1 \sin k(1-t) f(t) dt \delta_1 + \frac{1}{k} \int_0^1 \sin kt f(t) dt \delta_0, \quad z \in Z^d.
\]

Here \( \Delta \) is the lattice Laplacian defined in (41) and \( \delta_0, \delta_1 \) are functions on \( Z^d \) equal to one at \( z, y \), respectively, and equal to zero elsewhere. In particular, if \( f \) is the delta function at a point \( s \) of the edge \( e_0 \), then

\[
(\Delta - 2d \cos k)\psi = \frac{1}{k} \sin k(1-s) \delta_1 + \frac{1}{k} \sin ks \delta_0.
\]  

(82)

Let \( R_\mu(z - z_0) \) be the kernel of the resolvent \((\Delta - \mu)^{-1}\) of the lattice Laplacian. Then (82) implies that

\[
\psi(z) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} s R_\mu(z) + \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(1-s) R_\mu(z - z_2), \quad \mu = 2d \cos \sqrt{\lambda}.
\]

(83)

Function \( R_\mu(z) \) has the form

\[
R_\mu(z) = \frac{\int_T e^{i\sigma z} d\sigma}{\left( \sum_{1 \leq j \leq d} 2 \cos \sigma_j - \mu \right)^2}, \quad T = [-\pi, \pi]^d.
\]

Hence, function \( \sin(\sqrt{\lambda} s) R_\mu(z), \quad s \in (0, 1), \quad \mu = 2d \cos \sqrt{\lambda} \), decays exponentially as \( |\text{Im} \sqrt{\lambda}| \to \infty \). This allows one to change the contour of integration in (80), when \( z \in Z^d \), and rewrite (80) in the form

\[
u(z, t) = \frac{1}{(2\pi)^d} \int_l \psi_\lambda(z) e^{-\lambda t} d\lambda, \quad z \in Z^d,
\]  

(84)

where contour \( l \) consists of the ray \( \lambda = \rho e^{-i\pi/4}, \rho \in (\infty, 1) \), a smooth arc starting at \( \lambda = e^{-\pi/4} \), ending at \( \lambda = e^{\pi/4} \), and crossing the real axis at \( \lambda = -a \), and the ray \( \lambda = \rho e^{i\pi/4}, \rho \in (1, \infty) \). It is easy to see that \( |\psi(z, \lambda)| \leq C/|\sqrt{\lambda}| \) as \( \lambda \in l \) uniformly in \( z \) and \( z \in Z^d \). This immediately implies that \( |\nu(z, t)| \leq C/\sqrt{t} \). Now from (81) it follows that the same estimate is valid for \( p_0(t, x, x), x \in e_0 \), i.e., condition (b) holds, and \( \alpha = 1 \).

From (84) it also follows that the asymptotic behavior of \( \nu \) as \( t \to \infty \) is determined by the asymptotic expansion of \( \psi(z, \lambda) \) as \( \lambda \to 0, \lambda \notin [0, \infty) \). Note that the spectrum of
the difference Laplacian is \([-2d, 2d]\), and \(\mu = 2d - d\lambda + O(\lambda^2)\) as \(\lambda \to 0\). From here and the well known expansions of the resolvent of the difference Laplacian near the edge of the spectrum it follows that the first singular term in the asymptotic expansion of \(R_\mu(z)\) as \(\lambda \to 0\), \(\lambda \notin [0, \infty)\), has the form

\[
\begin{cases}
  c_d\lambda^{d/2-1}(1 + O(\lambda)), & \text{if } d \text{ is odd,} \\
  c_d\lambda^{d/2-1}\ln \lambda(1 + O(\lambda)), & \text{if } d \text{ is even.}
\end{cases}
\]

Then (83) implies that a similar expansion is valid for \(\psi(z, \lambda)\) with the main term independent of \(s\) and the remainder estimated uniformly in \(s\). This allows one to replace \(l\) in (84) by the contour which consists of the rays \(\arg \lambda = \pm \pi/4\). From here it follows that for each \(z \in \mathbb{Z}^d\) and uniformly in \(s\),

\[
u(z, t) \sim t^{-d/2}, \quad t \to \infty.
\]

This and (81) imply the same behavior for \(p_0(t, x, x)\), \(x \in e_0\), i.e., \(\beta = d\).

\[\square\]

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