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A note on the construction of a ‘valid’ NSFD scheme for the Lotka-Volterra equations

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Abstract

We demonstrate the construction of an explicit NSFD discretization for the standard Lotka-Volterra equations. In contrast to previous NSFD schemes, our representation is dynamic consistent with respect to all the essential properties of the differential equations and their solutions.

Keywords: Lotka-Volterra equations, NSFD schemes, nonstandard finite difference schemes, dynamic consistency, positivity

AMS Subject Classification: 34A05, 39A10, 39A12, 65L12

1 Introduction

The main purpose of this note is to construct a “valid” nonstandard finite difference (NSFD) scheme for the Lotka-Volterra (L-V) equations [6]

\[
\frac{dx}{dt} = ax - bxy, \quad \frac{dy}{dt} = -cy + dxy,
\]

where \((a, b, c, d)\) are non-negative parameters. These equations provide an elementary model for the interaction of a prey, \(x\), with a predator \([6, 10]\).

We use the word “valid” to indicate that previous authors (see for example Roeger [7, 8], who has worked extensively on NSFD schemes for L-V type equations) have not used the full machinery of the NSFD methodology to determine their particular discretizations of the L-V equations. A similar comment can be made for the recent work by Dang and Heang [2]. However, it must be emphasized that their obtained results are entirely correct; they just did not make use of the maximum number features based on the NSFD scheme construction as formulated by Mickens [3, 4].

It should be pointed out that an early publication to investigate the general Lotka-Volterra equations and their NSFD discretizations is Al-Kahby et al. [1]. It is an outstanding work and

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covers the cases of competitive, cooperative and predator-prey interactions. In addition to providing a broad range of possible NSFD schemes, it also gives rigorous proofs of the dynamic consistency [4] of the important features of these schemes with respect to the differential equations. However, this work was written prior to a full understanding as to how to construct the proper denominator functions (DF) for NSFD schemes [4]. Consequently, while their DF’s were correct, they were not based on the (now known) NSFD methodology for constructing such functions.

In Section 2, we give a brief overview of the important properties of the solutions to the L-V differential equations. Section 3 summarizes the difficulties which follow from using a forward-Euler discretization for Equations (1). Our main result is presented in Section 4; here we construct a full, valid NSFD scheme for the L-V equations. Finally, in Section 5, we summarize our main results and give several possible extensions of this work.

2 Properties of Lotka-Volterra equations

The following are several well known, basic features of the solutions to the L-V differential equations [6, 10]:

(i) Positivity,
\[
\begin{align*}
  x(0) = x_0 > 0, \quad y(0) = y_0 > 0 & \implies (x(t) > 0, \quad y(t) > 0), \quad t > 0.
\end{align*}
\]

(ii) Two fixed-point (constant solutions) exist; they are located in the \(x-y\) phase-plane at
\[
FP_1 = (0, 0), \quad FP_2 = \left( \frac{c}{d}, \frac{a}{b} \right).
\]

(iii) The first fixed-point is unstable and hyperbolic [6, 9], while the second is a neutrally stable, center [6, 9].

(iv) For \(y(t) = 0\), i.e., no predator is present, then
\[
\frac{dx(t)}{dt} = ax(t), \quad x(0) = x_0 > 0,
\]
and \(x(t)\) increases exponentially. For \(x(t) = 0\), i.e., no prey is available
\[
\frac{dy(t)}{dt} = -cy(t), \quad y(0) = y_0 > 0,
\]
then \(y(t)\) decreases to zero, again exponentially.

(v) For small perturbations about the second fixed-point, i.e.,
\[
\begin{align*}
  x(t) &= \frac{c}{d} + \alpha(t), \quad |\alpha(0)| \ll \frac{c}{d},
  \\
  y(t) &= \frac{a}{b} + \beta(t), \quad |\beta(0)| \ll \frac{a}{b},
\end{align*}
\]
where \(\alpha(t)\) and \(\beta(t)\) satisfy (in the linear approximation) the equation
\[
\frac{d^2z(t)}{dt^2} + (ac)z(t) = 0, \quad z(t) = \alpha(t) \text{ or } \beta(t),
\]
consequently, the motion is oscillatory. In fact this result holds for arbitrary initial conditions, \(x_0 > 0\) and \(y_0 > 0\).
3 Forward-Euler scheme

A standard forward-Euler scheme for Equations (1) is [3, 7]

\[
\frac{x_{k+1} - x_k}{h} = ax_k - bx_ky_k, \quad \frac{y_{k+1} - y_k}{h} = -cy_k + dx_ky_k,
\]

(8)

where \( h = \Delta t \), \( t \rightarrow hk = t_k \), \( x(t) \rightarrow x_k \) and \( y(t) \rightarrow y_k \), and \( k = (0, 1, 2, \ldots) \).

Inspection of Equation (8) shows that there exist initial conditions, \( x(0) = x_0 > 0 \) and \( y(0) = y_0 > 0 \), and step-sizes, \( h = \Delta t > 0 \), such that the solutions, \( (x_k, y_k) \), become negative. Consequently, some of the five conditions, given in Section 2 for the L-V differential equations, may not be satisfied by the solutions of this forward-Euler discretization, and it must be concluded that this scheme is, in general, not suitable or dynamic consistent [5] with the original L-V equations, and should not be used for their numerical integration. It should also be noted that the previous work by Roeger [7, 8] had as its focus the goal of eliminating these difficulties for various classes of L-V differential equations.

Finally, an elementary calculation shows that for the scheme given by Equation (8), the fixed-point at \((c/d, a/b)\) is an unstable spiral [5, 9], i.e., near this fixed-point, solutions oscillate with increasing values for the amplitude. This result provides an additional reason for the unsuitability of the forward-Euler discretization of the L-V equations.

4 Valid NSFD scheme

We now apply the full NSFD methodology [3, 4] to construct a discretization of Equations (1).

To calculate the two denominator functions [3, 4], we need to consider the respective linear terms in Equations (1), i.e.,

\[
\frac{dx}{dt} = ax, \quad \frac{dy}{dt} = -cy.
\]

(9)

The corresponding exact finite difference schemes are [3]

\[
\frac{x_{k+1} - x_k}{\phi_1(a, h)} = ax_k, \quad \frac{y_{k+1} - y_k}{\phi_2(c, h)} = -cy_k,
\]

(10)

where

\[
\phi_1(a, h) = \frac{e^{ah} - 1}{a}, \quad \phi_2(c, h) = \frac{1 - e^{-ch}}{c},
\]

(11)

and \( \phi_1(a, h) \) and \( \phi_2(c, h) \) are the required denominator functions needed for the discretizations of Equations (1).

The nonlinear term, \( xy \), appears in both differential equations and must be modeled nonlocally in the discrete representation. To have positivity for the \( x \)-variable, the term \( xy \) takes the form

\[
xy \rightarrow x_{k+1}y_k.
\]

(12)

Since \( xy \) must have the same structure where ever it appears, then it must also take the form given by Equation (12) in the \( y \)-variable equation. Combining the results of Equations (10)
and (12), it follows that the NSFD schemes for the L-V equations are

\[
\frac{x_{k+1} - x_k}{\phi_1(a, h)} = ax_k - bx_{k+1}y_k, \quad (13a)
\]

\[
\frac{y_{k+1} - y_k}{\phi_2(c, h)} = -cy_k + dx_{k+1}y_k, \quad (13b)
\]

where \(\phi_1\) and \(\phi_2\) are given in Equation (11).

Examination of Equations (13) allows the following conclusions to be reached:

(a) Equations (13) have exactly the same two fixed-points as the L-V differential equations, namely,

\[
(\bar{x}, \bar{y}) : (0, 0) \text{ and } \left( \frac{c}{d}, \frac{a}{b} \right). \quad (14)
\]

(b) For \(y_k \equiv 0\), then

\[
\frac{x_{k+1} - x_k}{\phi_1(a, h)} = ax_k \implies x_k = x_0 e^{ahk}, \quad (15)
\]

and \(x_k\) exponentially increases.

(c) For \(x_k \equiv 0\), then

\[
\frac{y_{k+1} - y_k}{\phi_2(c, h)} = -cy_k \implies y_k = y_0 e^{-ck}, \quad (16)
\]

and \(y_k\) exponentially decreases to zero.

(d) Solving Equations (13), respectively, for \(x_{k+1}\) and \(y_{k+1}\), gives

\[
x_{k+1} = \frac{e^{ahx_k}}{1 + (b\phi_1)y_k}, \quad y_{k+1} = \left[ e^{-ch} + d\phi_2x_{k+1} \right] y_k. \quad (17)
\]

From the first relation, we note that if \(x_k > 0\) and \(y_k > 0\), then \(x_{k+1} > 0\). Using this result in the second relation gives \(y_{k+1} > 0\). Hence, the positivity condition holds.

(e) Now consider small perturbations about the fixed-point \((\bar{x}, \bar{y}) = (c/d, a/b)\), i.e.,

\[
x_k = \bar{x} + \alpha_k, \quad y_k = \bar{y} + \beta_k, \quad (18)
\]

where

\[
|\alpha_0| \ll \frac{c}{d}, \quad |\beta_0| \ll \frac{a}{b}. \quad (19)
\]

Substitution of Equations (18) into Equations (13) and retaining only the linear terms, shows that \(\alpha_k\) and \(\beta_k\) both satisfy the second-order, linear, constant coefficient, difference equation [5]

\[
\frac{z_{k+1} - 2z_k + z_{k-1}}{\phi_1(a, h)\phi_2(c, h)} + (ac)z_k = 0, \quad (20)
\]

where \(z_k : \alpha_k\) or \(\beta_k\). Observe that this is a discrete approximation for Equation (7). Hence, we conclude that for small perturbations about the nontrivial fixed-point, the trajectory in the \((x_k, y_k)\) phase-space is periodic. (To obtain this result in detail, see the technique presented in Mickens [5].)

Our conclusion is that the NSFD discretization of the L-V Equations (1) is fully dynamic consistent with respect to all five conditions specified in Section 2.

One of the major differences between the work of Roeger [7, 8] and the current results is that Roeger uses the simple expression, \(h\), for the denominator functions; we have explicitly calculated the required functions.
5 Extensions

We have in this note constructed an improved NSFD scheme for the L-V predator-prey differential equations. Further, we have shown it to be dynamic consistent with all of the important features of the solutions to the standard L-V equations. While we do not give the calculations here, it is methodology easy to extend the results to the generalized L-V equations investigated by Roeger [7, 8] and AL-Kahby et al. [1]. Finally, another important case to investigate is where fractional-order derivatives appear [1].

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