QUASIDIAGONAL C*-ALGEBRAS AND NONSTANDARD ANALYSIS

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Suppose B is an ultraproduct of finite dimensional C*-algebras. We consider mapping and injectability properties for separable C*-algebras into B. In the case of approximately finite C*-algebras, we obtain a classification of these mappings up to inner conjugacy. Using a Theorem of Voiculescu, we show that for nuclear C*-algebras injectability into an ultraproduct of finite dimensional C*-algebras is equivalent to quasidiagonality.

1. Introduction

In this paper we use nonstandard analysis ([1], [13], [15]) to investigate injectability into C*-algebras B which are infinitesimal hulls of hyperfinite dimensional internal C*-algebras \(\mathfrak{B}\). B is obtained from \(\mathfrak{B}\) by considering the subspace \(\text{Fin}(\mathfrak{B})\) of elements with norm \(\ll \infty\) and identifying \(x, y \in \text{Fin}(\mathfrak{B})\) whenever \(x - y\) has infinitesimal norm. Though B is a legitimate standard C*-algebra, it is very large, except in the uninteresting case the original \(\mathfrak{B}\) is finite dimensional. We point out that the C*-algebras B are exactly ultraproducts of finite-dimensional C*-algebras (see [11] or the appendix).

A more interesting question from an operator theorist’s viewpoint, is which kinds of separable C*-algebras are injectable into B and what kinds of mappings exist from separable C*-algebras into B.

We show the following: If A is an AF algebra, Proposition 5.3 determines the inner conjugacy classes of C*-morphisms for A into a fixed B in terms of certain projective systems of matrices with nonnegative integer entries. Proposition 5.6 gives a necessary and sufficient condition for injectability of an AF algebra into a fixed B in similar terms. Our approach uses only the mapping properties of matrix algebras and a few basic principles of nonstandard analysis.

In §8, we give a characterization of nuclear subhyperfinite C*-algebras, that is nuclear C*-algebras which are injectable into some B. Theorem 8.2 states that for nuclear C*-algebras, subhyperfiniteness is equivalent to quasidiagonality. To prove this characterization, we use a lifting theorem for nuclear completely positive contractions reminiscent of that of Choi and Effros [6]. As a result, we obtain the following characterization: A separable nuclear C*-algebra is quasidiagonal iff it is injectable into a countable ultraproduct of finite dimensional C*-algebras. This characterization of quasidiagonality is similar to that given by Theorem 5.2.2 of [2].

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which considers injections into a somewhat different kind of reduced product of finite dimensional $C^*$-algebras.

There are numerous open questions we have not dealt with at this time. Of particular interest is the precise relationship between subhyperfiniteness and nuclearity. Another question, which is possibly more of a set-theoretic nature, is whether the (enormous) $C^*$-algebras $B$ corresponding to distinct hyperfinite dimensional internal $C^*$-algebras are non-isomorphic.

The structure of the paper is as follows: §2 is devoted to a review of basic ideas of nonstandard analysis. In §3 we review list general facts about internal $C^*$-algebras. In §4, we prove a number of basic lifting theorems for projections and partial isometries in a somewhat unfamiliar context, because the lifting property relates to a map between objects in two different models of a theory. However, the techniques used in dealing with projections and partial isometries are much the same as the techniques used by operator theorists in other situations. These lifting theorems are used in combination with various mapping properties of hyperfinite dimensional algebras to obtain the basic mapping properties for finite dimensional and AF algebras obtained in §5. Finally in §8 we show the equivalence of subhyperfiniteness and quasidiagonality in the nuclear case. The techniques used in this section are largely independent of those used to investigate AF injectability.

The relationship between $C^*$-algebra theory and nonstandard matrix algebras has been noticed before, notably in [12]. The authors in this paper observe that the infinitesimal hull of a matrix algebra is a $C^*$-algebra and actually construct another related quotient space which turns out to be a $\Pi_1$ factor. However, there seems to be no suggestion in that paper that $C^*$-algebras obtained from nonstandard matrix algebras might have a useful relation to separable $C^*$-algebras. An extensive and highly recommended reference for quasidiagonal $C^*$-algebras is the paper [4], which records much recent progress in theory of quasidiagonal algebras relevant to the present paper. However the relation between quasidiagonal algebras and ultraproducts of finite dimensional dimensional algebras seems to be new. In fact, the entire question of injectability into such ultraproducts seems to be new as well.

2. Preliminaries on Nonstandard Analysis

We use a small amount of nonstandard analysis for which the first pages of [15] suffice. Given a set $G$, the superstructure over $G$ is the set $V(G)$ defined by: $V_0(G) = G$, $V_{n+1}(G) = V_n(G) \cup P(V_n(G))$ and $V(G) = \bigcup_n V_n(G)$. The main constituent of our working view of nonstandard analysis is a map $\star : V(\mathbb{R}) \rightarrow V(\star \mathbb{R})$ which is the identity on $\mathbb{R}$ and satisfies the transfer principle; c.f. [15] for details. For the reader unfamiliar with nonstandard analysis, we point out that a mapping satisfying the transfer property can be shown to exist by using bounded ultraproducts. This is briefly reviewed in an appendix to this paper. However, we recommend the discussion of the so-called “extended universe” in [1] §1.2. Also note that the transfer principle does not uniquely characterize the map $\star$.

Remarks 2.1. We point out a notational difference between this paper and most other papers in nonstandard analysis (including our own [18]): the use of the prefixed symbol $\star$ instead of the prefixed $\ast$ to denote the transfer map. This notational
modification should prevent clashes between the symbol used for the transfer map and the symbol used for the adjoint mapping in involutive algebras.

For a similar reason we use the phrase “involutive morphism” in place of the more frequently used expression “$*$-morphism”.

We use the modifier *standard* roughly, in referring to structures in the universe of standard sets $V(\mathbb{R})$. The modifier *internal* on the other hand refers to structures in the universe of internal sets. These are certain sets in $V(\mathcal{R})$ (see [15], §1 for a summary). Strictly speaking, the modifiers standard and internal should be used in reference to particular models of a theory, but in practice the superstructure approach to nonstandard analysis is sufficiently well-established to allow us to gloss over these details. Thus we use without comment, expressions such as standard or internal metric spaces, internal groups etc. We will also use the modifier *external* to draw attention to the fact that a particular object is not internal or that an assertion is used in an external context, for instance, *external induction*.

We caution the experienced nonstandard analyst that we only use countable saturation.

We use countable saturation as follows: If $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of internal elements of an internal set $\mathcal{Y}$, then there is an internal family $\{\xi_\ell\}_{\ell \in \mathcal{N}}$ which extends $\{x_n\}$.

**Remarks 2.2.** Saturation arguments are often used in conjunction with *overflow* (also called *overflow*; see Proposition 1.2.7 of [1]) which states that any internal set which contains $\mathbb{N}$ contains a hyperinteger interval $\{1, 2, \ldots, N\}$. In particular, suppose $\{x_\ell\}_{\ell \in \mathcal{N}}$ is a sequence in an internal set $A_1$ each term of which satisfies the internal elementary statement $P(x, A_1, \ldots, A_n)$, where $A_i$ are parameters instantiated by fixed internal sets (see §1 of [15] for the definition of elementary statement). Notice each term of the sequence is internal but the sequence $\{x_\ell\}_{\ell \in \mathcal{N}}$ is external. By countable saturation, this sequence has an internal extension $\{\xi_\ell\}_{\ell \in \mathcal{N}}$. The set of $\ell \in A_1$ for which $P(\xi_\ell, A_1, \ldots, A_n)$ holds is internal (requires the Internal Definition Principle) so by overspill, there is an unlimited $N \in \mathcal{N}$ such that $P(\xi_\ell, A_1, \ldots, A_n)$ holds for $\ell \leq N$.

**2.1. Internal Normed Spaces.** See [1] for details on nonstandard analysis of normed spaces. If $\mathcal{E}$ is a $\mathcal{R}$ internal normed space, then $\text{Fin}(\mathcal{E})$ is the internal set $\{\phi \in \mathcal{E} : \|\phi\|_{\mathcal{E}} \ll \infty\}$, in other words $\text{Fin}(\mathcal{E})$ is the limited component of 0. Note that $\text{Fin}(\mathcal{E})$ is a vector space over the field $\mathcal{C}$. The mapping $\phi \mapsto \text{st} \|\phi\|$ is a ($\mathbb{R}$-valued) seminorm on $\text{Fin}(\mathcal{E})$. This seminorm factors through a norm on the quotient vector space $\text{Fin}(\mathcal{E})/\{\phi : \text{st} \|\phi\| = 0\}$. The quotient space with this norm is the *infinitesimal hull* of $\mathcal{E}$ and we denote it by $\mathcal{S}(\mathcal{E})$. We also denote the canonical quotient map $\text{Fin}(\mathcal{E}) \to \mathcal{S}(\mathcal{E})$ by $\pi_\mathcal{E}$ (or $\pi$ when the internal normed space $\mathcal{E}$ is evident from context). The norm on $\mathcal{S}(\mathcal{E})$ is characterized by the property $\|\pi_\mathcal{E}(\phi)\| = \text{st} \|\phi\|$. It is well-known that if $\mathcal{E}$ is an internal normed space, $\mathcal{S}(\mathcal{E})$ is a Banach space. Note that if $\mathcal{E}$ is hyperfinite dimensional, then $\mathcal{E}$ is $\mathcal{R}$-complete. This means that any internal Cauchy sequence $\{\phi_j\}_{j \in \mathcal{N}}$ is $\mathcal{R}$-convergent.

**Remarks 2.3.** If $E$ is a standard normed space, there is a canonical linear isometric map $E \to S(\mathcal{E})$. We will regard this map as an inclusion. This map is bijective iff $E$ is finite dimensional.
In particular, if \( E \) is finite dimensional and \( \phi \in \mathcal{E} \) is such that \( \|\phi\| \ll \infty \) then \( \pi_\mathcal{E}(\phi) \) is an element of \( E \). \( \pi(\phi) \) is usually referred to as the standard part of \( \phi \). Moreover, \( \mathcal{E}(\pi(\phi)) \cong \phi \).

We adopt the following typographical convention to distinguish names for internal normed spaces from those for standard normed spaces: Internal spaces are denoted by calligraphic letters \( \mathcal{E}, \mathcal{F}, \mathcal{H} \) while standard spaces are denoted by the standard math fonts \( E, F, H \).

We rely on similar typographical conventions to distinguish names for internal normed spaces from those for standard normed spaces: Internal spaces are denoted by calligraphic letters \( \mathcal{E}, \mathcal{F}, \mathcal{H} \) while standard spaces are denoted by the standard math fonts \( E, F, H \).

\[ (1) \quad S(T)(\pi(\phi)) = \pi(T\phi) \quad \text{for } \phi \in \mathcal{E}. \]

If \( T \in \mathcal{L}(\mathcal{E}, \mathcal{F}) \) is isometric, then \( S(T) \) is isometric.

\[ \mathcal{E} \mapsto S(\mathcal{E}) \] is a functor on the category of internal normed spaces and maps of limited norm into the category of Banach spaces and bounded linear maps.

Remarks 2.4. If \( T \in \mathcal{L}_{\text{Fin}}(\mathcal{E}, \mathcal{F}) \), the operator \( S(T) : S(\mathcal{E}) \to S(\mathcal{F}) \) can also be interpreted as the image of \( T \) in \( S(\mathcal{L}(\mathcal{E}, \mathcal{F})) \) under the quotient map \( \pi : \mathcal{L}_{\text{Fin}}(\mathcal{E}, \mathcal{F}) \to \mathcal{L}(\mathcal{E}, \mathcal{F}) \). More precisely, the map \( S : \mathcal{L}_{\text{Fin}}(\mathcal{E}, \mathcal{F}) \to \mathcal{L}(S(\mathcal{E}), S(\mathcal{F})) \) factors through a linear isometric map \( \Phi : S(\mathcal{L}(\mathcal{E}, \mathcal{F})) \to \mathcal{L}(S(\mathcal{E}), S(\mathcal{F})) \). The isometric property of \( \Phi \) means \( \text{st}(\|T\|) = \|S(T)\| \) for \( T \in \mathcal{L}_{\text{Fin}}(\mathcal{E}, \mathcal{F}) \). To verify this, note that by (1),

\[ \|S(T)\| = \text{st}(\|T\|) \leq \text{st}(\|T\|) \|\pi(\phi)\|, \]

and thus \( \|S(T)\| \leq \|T\|\). On the other hand, let \( \epsilon \cong 0 \) be a positive hyperreal, \( \phi \in \mathcal{E} \) such that \( \|\phi\| = 1 \) and \( \|T\phi\| \geq \|T\| - \epsilon \). Then

\[ \text{st}(\|T\|) = \text{st}((\|T\| - \epsilon)) \leq \|T\| - \epsilon \leq \|T\phi\| = \|S(T)\| \leq \|S(T)\|. \]

Remarks 2.5. \( \pi \) maps part of an internal normed space \( \mathcal{E} \) linearly into the Banach space \( S(\mathcal{E}) \). Linearity of \( \pi \) means \( \pi \left( \sum_i x_i \right) = \sum_i \pi(x_i) \) for finite families \( \{x_i\} \).

Observe there is no reasonable way to interpret the RHS of this formula if the index set is hyperfinite but not finite. A similar comment applies to the map \( T \mapsto S(T) \).

Example 2.6. Suppose \( \mathcal{X} \) is hyperfinite and \( 1 \leq p < \infty \) in \( \mathcal{X} \). \( \mathcal{L}^p(\mathcal{X}) \) is the internal vector space of internal functions \( \psi : \mathcal{X} \to \mathcal{X} \) equipped with the norm \( \|\psi\|_p = \sqrt[p]{\sum_{x \in \mathcal{X}} |\psi(x)|^p} \).

If \( \mathcal{H} \) is an inner product space, then by transfer, the Cauchy-Schwartz inequality is valid in \( \mathcal{H} \). From this follows that the inner product on \( \mathcal{H} \) factors through an inner product on the Banach space \( S(\mathcal{H}) \). In particular \( \mathcal{H} \mapsto S(\mathcal{H}) \) is also a functor on the category of internal inner product spaces and maps of limited norm into the category of Hilbert spaces and bounded linear maps.
Note that if $\mathcal{H}, \mathcal{K}$ are hyperfinite dimensional inner product spaces and $T : \mathcal{H} \to \mathcal{K}$ is a linear map then the adjoint $T^*$ is always defined and has the property that $\|T^*\| = \|T\|$. If $T$ has limited norm, $S(T^*) = S(T)^*$.

**Example 2.7.** If $\mathfrak{X}$ is hyperfinite, $L^2(\mathfrak{X})$ has an internal inner product

$$\langle \psi, \phi \rangle = \sum_{x \in \mathfrak{X}} \overline{\psi(x)} \phi(x),$$

and clearly $\|\psi\|_2 = \sqrt{\langle \psi, \psi \rangle}$. We alert the reader to the fact that unless the internal dimension of $\mathcal{H}$ an element of $\mathcal{N}$, $S(H)$ is highly nonseparable.

If $K$ is a separable Hilbert space, $\mathcal{K}$ has an internal orthonormal basis indexed on $\mathcal{N}$.

### 3. Internal $C^*$-Algebras

Our typographical strategy for distinguishing internal from standard $C^*$-algebras is as follows: boldface fonts $\mathbf{A}, \mathbf{B}$ denote standard $C^*$-algebras, fraktur fonts $\mathfrak{A}, \mathfrak{B}$, $\mathfrak{M}$ denote internal ones.

**Remark 3.1.** In this paper, we only consider algebras (whether internal or external) with unit. All involutive morphisms between algebras (in particular, all inclusions) will be assumed to be unit preserving.

Suppose $\mathfrak{A}$ is an internal normed algebra over the field $\mathbf{C}$. Note that $\mathfrak{A}$ is an algebra over the field $\mathbf{C}$ and $\text{Fin}(\mathfrak{A})$ is an algebra over $\mathbb{C}$. The Banach space $S(\mathfrak{A})$ can be turned into an algebra such that $\pi : \text{Fin}(\mathfrak{A}) \to S(\mathfrak{A})$ is a ring homomorphism. This follows from the fact $\{x \in \mathfrak{A} : \|x\| \equiv 0\}$ is a two sided ideal in $\text{Fin}(\mathfrak{A})$. If $\mathfrak{A}$ is an internal involutive normed algebra with the property $\|xx^*\| = \|x\|^2$, then $S(\mathfrak{A})$ is a $C^*$-algebra:

$$\|\pi(x) x^*\| = \|\pi(x^*)\| = \text{st} \|xx^*\| = \text{st} \|x\| = \|\pi(x)\|^2.$$

The facts listed in the following examples are well-known and are stated without proof.

**Example 3.2.** If $\mathcal{H}$ is an internal inner product space, $L(\mathcal{H})$ is an internal $C^*$-algebra. This follows by transfer. $L_{\text{Fin}}(\mathcal{H})$ is an (external) algebra over $\mathbb{C}$ closed under adjoints. The map $T \mapsto S(T)$ is an involutive algebra homomorphism $L_{\text{Fin}}(\mathcal{H}) \to L(S(\mathcal{H}))$. The image of this map is a $C^*$-algebra.

**Example 3.3.** If $\mathfrak{A}$ is a standard $C^*$-algebra, $\mathfrak{A}$ is an internal $C^*$-algebra. The embedding $\mathfrak{A} \to S(\mathfrak{A})$ is an injective morphism of $C^*$-algebras. This map is surjective iff $\mathfrak{A}$ is finite dimensional.

Note that any separable $C^*$-algebra is isomorphic to a $C^*$-algebra in the universe $V(\mathbb{R})$. $V(\mathbb{R})$ of course also contains enormously large $C^*$-algebras such as countable ultrapowers of separable ones. It is easy to show however, $V(\mathbb{R})$ does not contain copies of every $C^*$-algebra.

**Example 3.4.** If $\mathcal{H}$ is hyperfinite dimensional, the internal $C^*$-algebra $L(\mathcal{H})$ can also be viewed as an algebra of square internal $n \times n$ matrices with entries in $\mathbf{C}$.
The size \( n \) of the matrix algebra is the internal dimension of \( \mathcal{H} \). Note that the elements of \( \mathbb{M}_n(\star C) \) are internal functions
\[
\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \star C.
\]

**Example 3.5.** Suppose \( \{\mathfrak{A}_i\}_{i \in F} \) is a hyperfinite internal family of \( C^* \)-algebras. Then \( \bigoplus_{i \in F} \mathfrak{A}_i \) is the space of internal families \( \tilde{x} = \{x_i\}_{i \in F} \) such that \( x_i \in \mathfrak{A}_i \), equipped with the pointwise algebraic operations and the norm \( \|\tilde{x}\| = \max_i \|x_i\|_{\mathfrak{A}_i} \). \( \bigoplus_{i \in F} \mathfrak{A}_i \) is an internal \( C^* \)-algebra.

**Example 3.6.** If \( n \in \star \mathbb{N} \) and \( \mathfrak{A} \) is an internal \( C^* \)-algebra, \( M_n(\mathfrak{A}) \), the algebra of internal \( n \times n \) with entries in \( \mathfrak{A} \) is a \( C^* \)-algebra. Note that if \( n \in \mathbb{N} \), then
\[
\mathbf{S}(M_n(\mathfrak{A})) \cong P_{\mathbb{N}}(\mathfrak{A}(\mathfrak{A}))
\]

More precisely, the map \( M_n(\pi) : M_n(\mathfrak{A}) \to M_n(\mathfrak{A}(\mathfrak{A})) \) is naturally equivalent to the infinitesimal identification map \( \pi_{M_n(\mathfrak{A})} \), that is there is a commutative diagram

\[
\begin{array}{ccc}
M_n(\mathfrak{A}) & \xrightarrow{id_{M_n(\mathfrak{A})}} & M_n(\mathfrak{A}) \\
\downarrow{\pi_{M_n(\mathfrak{A})}} & & \downarrow{\pi_{M_n(\mathfrak{A})}} \\
M_n(\mathfrak{A}(\mathfrak{A})) & \xrightarrow{\mathbf{S}} & \mathbf{S}(M_n(\mathfrak{A}))
\end{array}
\]

In the case \( n \) is a nonstandard hyperinteger, the RHS of (2) is meaningless.

### 3.1. Hyperfinite Dimensional \( C^* \)-algebras.

The structure of finite dimensional \( C^* \)-algebras and involutive morphisms between them is well-known (See [14], §3.2 or [8], Chapter III). The corresponding results for hyperfinite dimensional \( C^* \)-algebras, which we now state, follow by transfer.

A hyperfinite dimensional \( C^* \)-algebra \( \mathfrak{A} \) is canonically isomorphic to a hyperfinite direct sum \( \mathfrak{A} = \bigoplus_{c \in \text{min} \mathfrak{A}} \mathfrak{A}_c \), where \( \text{min} \mathfrak{A} \) is the set of minimal nonzero central projections of \( \mathfrak{A} \). Each \( C^* \)-algebra \( \mathfrak{A}_c \) is internally isomorphic (though in a noncanonical way) to the full matrix algebra \( \mathbb{M}_{\dim(c)}(\star C) \). The hyperfinite family indexed on \( \text{min} \mathfrak{A} \) given by \( \text{dim}(\mathfrak{A}) = \{\dim(c)\}_{c \in \text{min} \mathfrak{A}} \) is called the dimension vector of \( \mathfrak{A} \).

If \( \mathfrak{A}, \mathfrak{B} \) are hyperfinite dimensional \( C^* \)-algebras, to any involutive morphism \( h : \mathfrak{A} \to \mathfrak{B} \) corresponds an internal family \( \Lambda \) of nonnegative hyperintegrers indexed on \( \text{min} \mathfrak{B} \times \text{min} \mathfrak{A} \). This family is defined as follows: If \( (f, e) \in \text{min} \mathfrak{B} \times \text{min} \mathfrak{A} \), then \( \Lambda_{f, e} \) is the multiplicity of the irreducible representation of \( \mathfrak{A} \) corresponding to the projection \( e \) in the representation \( x \mapsto h(x)f \). The mapping \( h \mapsto \Lambda(h) \) is internal. We will refer to \( \Lambda(h) \) as the mapping matrix of \( h \) (or inclusion matrix in case \( h \) is injective).

The following properties are well-known:

1. The following matrix equation holds: \( \overline{\text{dim}}(\mathfrak{B}) = \Lambda \overline{\text{dim}}(\mathfrak{A}) \).
2. If \( h : \mathfrak{A} \to \mathfrak{B} \) and \( g : \mathfrak{B} \to \mathfrak{C} \) are involutive morphisms, then \( \Lambda(g \circ h) = \Lambda(g) \Lambda(h) \).
3. \( h : \mathfrak{A} \to \mathfrak{B}, h' : \mathfrak{A} \to \mathfrak{B}' \) have the same mapping matrix iff there is an internal \( C^* \)-algebra isomorphism \( \Phi : \mathfrak{B} \to \mathfrak{B}' \) such that \( h' = \Phi \circ h \).
4) For each \( f \in \min \mathcal{B} \) there is an \( e \in \min \mathcal{A} \) such that \( \Lambda_{fe} > 0 \). This follows immediately from the fact \( h \) is unital.

5) If \( \mathcal{A}, \mathcal{B} \) are hyperfinite dimensional \( C^* \)-algebras and \( \Lambda \) is a family of nonnegative hyperintegers indexed on \( \min \mathcal{B} \times \min \mathcal{A} \) satisfying the condition in item 4, then there is an involutive morphism \( h : \mathcal{A} \to \mathcal{B} \) for which \( \Lambda \) is the mapping matrix.

6) \( h \) is injective iff for each \( e \in \min \mathcal{A} \) there is an \( f \in \min \mathcal{B} \) such that \( \Lambda_{fe} > 0 \).

7) Conversely, if \( \Lambda \) is a matrix such that items 1) and 4) hold, then there is a unital morphism \( h : \mathcal{A} \to \mathcal{B} \) such that \( \Lambda(h) = \Lambda \).

**Definition 3.7.** A family of nonnegative hyperintegers \( \{ \Lambda_{ji} \}_{(j, i) \in J \times I} \) such that for each \( f \in J \) there is an \( e \in I \) such that \( \Lambda_{fe} > 0 \) is called a mapping matrix. If for each \( e \in I \) there is an \( f \in J \) such that \( \Lambda_{fe} > 0 \), \( \Lambda \) is called an inclusion matrix.

### 3.2. Functional Calculus.

The existence of a \(^*\)-continuous functional calculus in an internal \( C^* \)-algebra for selfadjoint elements follows from the existence of the continuous functional calculus for selfadjoint elements in standard \( C^* \)-algebras. For example, any internal function \( f \) such that for any \( \theta \in \text{dom} f \), \( f \circ [\theta - \delta, \theta] \cap \text{dom} f \) and \( f \circ [\theta, \theta + \delta] \cap \text{dom} f \) are polynomial functions for some \( \delta > 0 \) are \(^*\)-continuous. For another example, if \( f \) is a standard continuous function, \(^*\)\( f \) is \(^*\)-continuous.

We now observe that the internal functional calculus on \( \mathfrak{A} \) is compatible with the standard one of \( \mathcal{S}(\mathfrak{A}) \), as stated in the following:

**Lemma 3.8.** Suppose \( x \in \text{Fin}(\mathfrak{A}) \) is self-adjoint. If \( f : \mathbb{R} \to \mathbb{R} \) is continuous, then

\[
\pi([^*f](x)) = f(\pi(x)).
\]

**Proof.** Without loss of generality we can assume \( ||x|| \leq 1 \). Since \( \pi \) is a homomorphism, equation (4) holds for standard polynomials \( f \). The RHS of (4) is norm continuous in \( f \in \mathcal{C}[-1, 1] \). Moreover, by transfer, it follows that \( f \mapsto f(x) \) is an internal linear mapping of norm \( \leq 1 \) as \( f \) ranges through the space of \(^*\)-continuous functions on \(^*[−1,1] \). Thus \( f \mapsto \pi([^*f](x)) \) is a linear mapping of norm \( \leq 1 \) as \( f \) ranges through the space of continuous functions on \([-1,1] \). The result follows by continuity and Stone-Weierstrauss.

### 4. Lifting Projections and Partial Isometries

Let \( \mathfrak{A} \) be an internal \( C^* \)-algebra. The relation between \( \mathfrak{A} \) and \( \mathcal{S}(\mathfrak{A}) \) bears some formal resemblance to that between the standard \( C^* \)-algebra \( L(H) \) and the Calkin Algebra \( L(H)/K(H) \), particularly as regards projections and partial isometries. Projection in a \( C^* \)-algebra (whether standard or internal) means self-adjoint projection and “\( \preceq \)” denotes the ordering relation on projections.

**Proposition 4.1.** If \( w \in \mathcal{S}(\mathfrak{A}) \) is a partial isometry, then there is a partial isometry \( \tilde{w} \in \mathfrak{A} \) such that \( \pi(\tilde{w}) = w \). Moreover, if \( p, p' \in \mathfrak{A} \) are projections such that

- initial projection \( w \preceq \pi(p) \) and final projection \( w \preceq \pi(p') \),

\( \tilde{w} \) can be taken so that its initial projection \( \preceq p \) and its final projection \( \preceq p' \).
Proof. Let \( b \in \mathfrak{A} \) such that \( \pi(b) = w \). Replace \( b \) by \( p' b p \) and let \( x = b^* b \). Thus \( \pi(x) = w^* w = \varepsilon \) and \( x^2 - x \cong 0 \). Let \( h : \mathbb{R} \to \mathbb{R} \) be the function

\[
    h(\theta) = \begin{cases} 
    \theta^{-1/2} & \text{if } \theta \geq 2/3, \\
    0 & \text{if } \theta \leq 1/3,
\end{cases}
\]

and is linear in \([1/3, 2/3]\). By the internal functional calculus, \( [\pi h](x)^2 \) is a projection. By Lemma 3.8, \( \pi([\pi h](x)) = h(\pi(x)) = h(\varepsilon) = \varepsilon \). \( \tilde{w} = b[\pi h](x) \) is a partial isometry:

\[
(b[\pi h](x))^* b[\pi h](x) = [\pi h](x) b^* b[\pi h](x) = [\pi h](x)^2 x.
\]

Since \( b \) has initial projection \( \preceq p \) and final projection is \( \preceq p' \), it follows that the same is true for \( b[\pi h](x) \). Finally, \( \pi(\tilde{w}) = \pi(b) \pi([\pi h](x)) = w e = w \).

\[\square\]

Remark 4.2. Unlike the situation with Calkin map \( L(H) \to L(H)/K(H) \) there are no index obstructions to lifting mappings from finite dimensional algebras into \( \mathfrak{S}(\mathfrak{A}) \). The relevant fact is that if \( p \in \mathfrak{A} \) is a projection such that \( \pi(p) = 1 \), then \( p = 1 \). To see this, note that \( 1 - p \) is also a projection and \( 1 - p \cong 0 \). By the Gelfand isomorphism the only projection with norm < 1 in a standard \( C^* \)-algebra is 0. By transfer, the same fact is true for projections in internal \( C^* \)-algebras. Thus \( 1 - p = 0 \). In particular, by (external) induction on \( r \in \mathbb{N} \):

Suppose \( \{p_k\}_{1 \leq k \leq r} \) (with \( r \in \mathbb{N} \)) are orthogonal projections in the internal \( C^* \)-algebra \( \mathfrak{A} \). If \( \sum_{r=1}^{r} \pi(p_r) = 1 \), then \( \sum_{r=1}^{r} p_r = 1 \).

A basic notion for dealing with finite-dimensional \( C^* \)-algebras are matrix units. We follow the definitions and the notation of §7.1 of [17], with the following two caveats: all our matrix units are nonzero and the matrix units are not required to generate the \( C^* \)-algebra.

Definition 4.3. A system of matrix units for \( \mathfrak{A} \) is a family \( \{e^k_{ij}\}_{1 \leq k \leq r, 1 \leq i, j \leq n_k} \) of nonzero elements of \( \mathfrak{A} \) which satisfies (i), (ii) and (iii) of §7.1 of [17] and

\[
1_{\mathfrak{A}} = \sum_{k=1}^{r} \sum_{i=1}^{n_k} e^k_{ii}.
\]

We also add the following bit of jargon: the family of positive integers \( \{n_k\}_{1 \leq k \leq r} \) is the dimension vector of \( \{e^k_{ij}\} \).

We will freely use basic facts about matrix units, most notably the correspondence between, on the one hand the set of injective involutive morphisms \( \mathfrak{A} \to \mathfrak{B} \) and on the other the set of pairs consisting of matrix units \( \{e^k_{ij}\} \) which span \( \mathfrak{A} \) (so that \( \mathfrak{A} \) is finite dimensional) and matrix units \( \{f^k_{ij}\} \) in \( \mathfrak{B} \) with the same dimension vector as \( \{e^k_{ij}\} \). These facts are well known, and again we refer the reader to 7.1 of [17] for details.

Remark 4.4. Note that to specify a system of matrix units with dimension vector \( \{n_k\}_{1 \leq k \leq r} \) for \( \mathfrak{A} \) it suffices to provide a a reduced system of matrix units, that is a sequence of pairwise orthogonal projections \( \{p_k\}_{1 \leq k \leq r} \) such that \( 1_{\mathfrak{A}} = \sum_{k=1}^{r} p_k \).
and family of partial isometries \( \{u^k_j\}_{1 \leq k \leq r, 1 \leq j \leq n_k} \) such that \( 1 \leq k \leq r \)

\[
\sum_{j=1}^{n_k} u^k_j = p_k,
\]

initial projection \( u^k_j = u^k_1 \) \( 1 \leq j \leq n_k \).

Clearly there is a corresponding internal notion of system of matrix units and as always we will not hesitate to use the corresponding transferred facts about matrix units.

As a corollary to the lifting theorem for partial isometries and the above remark:

**Corollary 4.5.** Suppose \( \mathfrak{A} \) is an internal \( C^* \)-algebra and \( \{e^k_{ij}\} \) a system of matrix units in \( S(\mathfrak{A}) \). Then there is a system of matrix units \( \{\phi^k_{ij}\} \) in \( \mathfrak{A} \) with the same dimension vector as \( \{e^k_{ij}\} \) for which \( \pi(\phi^k_{ij}) = e^k_{ij} \).

**Remark 4.6.** If \( A \) is a standard \( C^* \)-algebra and \( \{e^k_{ij}\} \) is a system of matrix units in \( A \), then \( \{\phi^k_{ij}\} \) is a system of matrix units in \( \mathcal{A} \). If we regard \( A \) as isometrically embedded in \( S(\mathcal{A}) \) (see Remark 2.3), then \( \pi(\phi^k_{ij}) = e^k_{ij} \). Thus in the special case \( \mathfrak{A} = \mathcal{A} \) lifting matrix units is trivial.

**Corollary 4.7.** Suppose \( A \) is a standard finite dimensional \( C^* \)-algebra, \( \mathfrak{B} \) an internal \( C^* \)-algebra, \( \phi : A \to S(\mathfrak{B}) \) an involutive morphism. Then there is an internal involutive morphism \( \Phi : \mathcal{A} \to \mathfrak{B} \) of internal \( C^* \)-algebras which makes the diagram

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\Phi} & \mathfrak{B} \\
\downarrow & & \downarrow \pi \\
\mathcal{A} & \xrightarrow{\phi} & S(\mathfrak{B})
\end{array}
\] (5)

commutative. In particular, \( S(\Phi) = \phi \).

**Proof.** Without loss of generality we can assume \( \phi \) is injective. Consider a system of matrix units \( \{e^k_{ij}\}_{1 \leq k \leq r, 1 \leq i, j \leq n_k} \) for \( A \). Since \( \phi \) is injective, \( \{\phi(e^k_{ij})\}_{1 \leq k \leq r, 1 \leq i, j \leq n_k} \) is a system of matrix units in \( \mathfrak{B} \). Extend by \( C \)-linearity. \( \square \)

**Proposition 4.8.** Suppose \( \{A_\ell\}_{\ell \in \mathbb{N}} \) is a nondecreasing sequence of finite dimensional \( C^* \)-subalgebras of \( A \) such that \( \bigcup_{\ell=1}^{\infty} A_\ell \) is norm dense in \( A \). Suppose also \( \mathfrak{B} \) is an internal \( C^* \)-algebra and \( \{\phi_\ell\}_{\ell \in \mathbb{N}} \) is a sequence of involutive morphisms \( A_\ell \to S(\mathfrak{B}) \) such that

\[
\phi_{\ell+1} | A_\ell = \phi_\ell \quad \text{for } \ell \in \mathbb{N}.
\]

Then there is a hyperfinite dimensional \( C^* \)-algebra \( \mathfrak{A} \subseteq \mathcal{A} \) such that \( \mathfrak{A}_\ell \subseteq \mathfrak{A} \) for all \( \ell \in \mathbb{N} \) and an internal involutive morphism \( \Phi : \mathfrak{A} \to \mathfrak{B} \) such that

\[
S(\Phi) | A_\ell = \phi_\ell \quad \text{for } \ell \in \mathbb{N},
\]

where we view \( A_\ell \) as isometrically imbedded in \( S(\mathfrak{A}) \) as is justified by Remark 2.3.
Proposition 5.1. There is an involutive morphism $\Phi: \mathfrak{A}_\ell \to \mathfrak{A}$ such that
\[
\mathfrak{A}_\ell \xrightarrow{\Phi} \mathfrak{B}
\]
commutes. By saturation and overspill, the sequences $\left\{ \Phi(x) \right\}_{x \in \mathfrak{A}_\ell}$ extend to internal hyperfinite sequences $\left\{ \Phi(x) \right\}_{x \in \mathfrak{A}_\ell}$ such that for $x \in \mathfrak{A}_\ell$, $\Phi$ is an involutive morphism defined on the hyperfinite dimensional $C^*$-algebra $\mathfrak{A}_\ell$. If $x \in \mathfrak{A}_\ell$ with $\|x\| \ll \infty$ and $\ell \leq k \in \mathbb{N}$,
\[
\Phi(x) \cong \Phi(\pi_{\mathfrak{A}_k}(x)) \cong \Phi(\pi_{\mathfrak{A}_k}(x)) = \Phi(x).
\]
By the argument used to prove Robinson's lemma (see [16], Theorem 5.5), we conclude there is an unlimited $M$ such that $\Phi(x) \cong \Phi_M(x)$ for $\ell \leq M$ and every $x \in \mathfrak{A}_\ell$ of limited norm. For completeness, we spell out the details: for every $k \in \mathbb{N}$, $\|\Phi(x) - \Phi_k(x)\| \leq 1/k$ for all $\ell \leq k$ and all $x \in \mathfrak{A}_\ell$ with $\|x\| \leq 1$. By overspill, there is an $M \cong \infty$ such that $\|\Phi(x) - \Phi_M(x)\| \leq 1/M \cong 0$ for every $\ell \leq M$ and every $x \in \mathfrak{A}_\ell$ with $\|x\| \leq 1$.

Therefore, for $\ell \leq M$,
\[
\mathbf{S}(\Phi_M) \mid \mathfrak{A}_\ell = \mathbf{S}(\Phi) = \phi_\ell.
\]

5. Matrix Sequences

Involutive morphisms $\phi, \psi: A \to B$ are inner conjugate if there is a unitary $u \in B$ intertwining $\phi, \psi$, i.e., for all $x \in A$, $\phi(x) = u^* \psi(x) u$. We now prove the following peculiar compactness property:

Proposition 5.1. Suppose $\mathcal{B}$ is an internal $C^*$-algebra, $A = \bigcup_{i=1}^{\infty} A_i$ is separable, where $\{A_i\}_{i \in \mathbb{N}}$ is a nondecreasing sequence of $C^*$-algebras and $\phi, \psi: A \to \mathbf{S}(\mathcal{B})$ are $C^*$-algebra morphisms. If for all $i \in \mathbb{N}$, $\phi \mid A_i, \psi \mid A_i$ are inner conjugate then $\phi, \psi$ are inner conjugate.

Proof. For $i \in \mathbb{N}$, let $u_i \in \mathbf{S}(\mathcal{B})$ intertwine $\phi \mid A_i, \psi \mid A_i$. By the lifting theorem for partial isometries, there is a unitary $w_i \in \mathcal{B}$ such that $\pi(w_i) = u_i$. By saturation the sequence $\{w_i\}_{i \in \mathbb{N}}$ extends to a hyperfinite sequence $\{w_i\}_{1 \leq i \leq N}$ of unitaries in $\mathcal{B}$. By applying Robinson's argument (see the proof of Proposition 4.8) and the fact $A$ is separable, we conclude there is an $M \cong \infty$ such that $\pi(w_M)^* \psi(x) \pi(w_M) = \phi(x)$ for all $x \in A$. 

\[ \square \]
If \( \{A_i\}_{i \in \mathbb{N}} \) is a nondecreasing sequence of finite dimensional C*-algebras with inclusion mappings \( h_k : A_k \rightarrow A_{k+1} \), then the sequence of dimension vectors and inclusion matrices satisfy

\[
\Lambda(h_i) \overline{\dim}(A_i) = \overline{\dim}(A_{i+1}).
\]

Conversely, any sequence of positive integer vectors \( \overline{n}_k \) and inclusion matrices \( \Lambda_k \) such that \( \Lambda_k \overline{n}_k = \overline{n}_{k+1} \) is the sequence of dimension vectors and inclusion matrices for some nondecreasing sequence of finite dimensional C*-algebras. The equivalence of this scheme for describing inductive systems of finite dimensional C*-algebras and Bratteli diagrams [3] is well-known. In what follows, we consider AF algebras with a particular representation \( A = \bigcup_{i=1}^{\infty} A_i \) with \( A_i \) finite dimensional.

If \( B \) hyperfinite dimensional, the matrix sequence of any C*-algebra morphism \( \phi : A \rightarrow S(B) \) is the sequence \( \{\Lambda(\Phi|\cdot A_k)\}_{k \in \mathbb{N}} \) where \( \Phi \) is a lifting of \( \phi \) as is guaranteed by Proposition 4.8. The mapping matrices are independent of \( \Phi \). In fact,

\[
\Lambda(\Phi|\cdot A_k) = \Lambda(\Phi_k)
\]

where \( \Phi_k : \cdot A_k \rightarrow B \) is any lifting to an internal C*-algebra morphism, as follows from:

**Lemma 5.2.** Consider the context of Corollary 4.7. The mapping matrix of \( \Lambda(\Phi) \) is independent of the lifting \( \Phi \) of \( \phi \).

**Proof.** Again assume \( \phi \) is injective. From the lifting result for partial isometries, there is a unitary \( w \in B \) such that \( w^*\alpha_{ij}^k w = \beta_{ij}^k \) where \( \alpha_{ij}^k, \beta_{ij}^k \) are liftings for \( \phi(e_{ij}^k) \). It follows that all liftings of \( \phi \) are conjugate. \( \square \)

We will denote the matrix sequence of \( \phi \) by \( \Lambda(\Phi) \).

**Proposition 5.3.** \( \Lambda(\Phi) \) determines a bijection from the inner conjugacy classes of C*-morphisms \( A \rightarrow S(B) \) and sequences of mapping matrices \( \{\Gamma_k\}_{k \in \mathbb{N}} \) such that

\[
\Gamma_k \overline{\dim}(A_k) = \overline{\dim}(B)
\]

and

\[
\Gamma_{k+1} \Lambda(h_k) = \Gamma_k.
\]

**Proof.** Injectivity follows from Proposition 5.1 and the facts listed in §3.1. We prove \( \Lambda(\Phi) \) is surjective. Suppose \( \{\Gamma_k\}_{k \in \mathbb{N}} \) satisfies (9) and (10). By saturation and overspill, the sequences \( \{\Gamma_k\}_{k \in \mathbb{N}}, \{\cdot A_k\}_{k \in \mathbb{N}} \) and \( \{\cdot h_k\}_{k \in \mathbb{N}} \) extend to hyperfinite sequences \( \{\Gamma_k\}_{1 \leq k \leq M}, \{\cdot A_k\}_{1 \leq k \leq M} \) and \( \{\cdot h_k\}_{1 \leq k \leq M} \) such that \( \Gamma_k \) is a mapping matrix, \( A_k \) is a hyperfinite dimensional C*-algebra and \( \Psi_k : A_k \rightarrow A_{k+1} \) is an injective morphism of C*-algebras satisfying

\[
\Gamma_k \overline{\dim}(A_k) = \overline{\dim}(B).
\]

There is a C*-algebra morphism \( \Phi : A_M \rightarrow B \) such that \( \Gamma_M = \Lambda(\Phi) \). For \( 1 \leq k \leq M \) define the morphism \( \Phi_k : A_k \rightarrow B \) by

\[
\Phi_k = \Phi \Psi_{M-1} \Psi_{M-2} \cdots \Psi_k.
\]
5.5. A similar example shows that \( \bar{\Gamma} \) is an inclusion matrix.

**Definition 5.4.** If \( \bar{m}, \bar{n} \) are dimension vectors \( \bar{m} \) divides \( \bar{n} \) denoted \( \bar{m} | \bar{n} \) iff there is an inclusion matrix \( \Gamma \) such that \( \bar{n} = \Gamma \bar{m} \).

**Remarks 5.5.** Note that the matrix \( \Gamma \) is not unique, for example:

\[
(10) = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}.
\]

A similar example shows that \( \Gamma \bar{m} = \Gamma \bar{m}' \) does not imply \( \bar{m} = \bar{m}' \). Consequently, a sequence of dimension vectors \( \{\bar{n}_i\}_{i \in \mathbb{N}} \) such that \( n_i | n_{i+1} \) does not by itself determine an increasing sequence of finite dimensional \( C^* \)-algebras.

**Proposition 5.6.** Suppose \( \{A_i\}_{i \in \mathbb{N}} \) is a nondecreasing sequence of finite dimensional \( C^* \)-algebras with inclusion mappings \( h_k : A_k \rightarrow A_{k+1} \). A necessary and sufficient condition there exist an injective \( C^* \)-algebra morphism from \( \bigcup_{i=1}^{\infty} A_i \) into a \( C^* \)-algebra \( S(\mathcal{B}) \), where \( \mathcal{B} \) is a hyperfinite dimensional \( C^* \)-algebra is that for every \( k \in \mathbb{N} \), \( \dim(A_i) \) divide \( \dim(\mathcal{B}) \).

**Proof.** Necessity: Let \( \Lambda(\phi) = \{A_i\}_{i \in \mathbb{N}} \). Then \( \Lambda_i \dim(A_i) = \dim(\mathcal{B}) \). Thus \( \dim(A_i) = \dim(\mathcal{B}) \) divides \( \dim(\mathcal{B}) \).

Sufficiency: For \( k \in \mathbb{N} \), let \( \Gamma_k \) be an inclusion matrix such that

\[
\Gamma_k \dim(\mathcal{A}_i) = \dim(\mathcal{B}).
\]

We claim that \( \Gamma_k \) can be chosen so that the following compatibility condition holds:

\[
\Gamma_{k+1} \Lambda(\psi_k) = \Gamma_k \text{ for } 1 \leq k \leq M.
\]

Let \( \{\Psi_k\}_{1 \leq k \leq M}, \{\mathcal{A}_k\}_{1 \leq k \leq M} \) and \( \{\Psi_k\}_{1 \leq k \leq M} \) be as in the proof of Proposition (5.3). By overspill, there is no loss of generality in assuming \( \Gamma_k \) is an inclusion matrix for all \( k \leq M \). Define a new hyperfinite sequence \( \Gamma'_k \) as follows:

\[
\Gamma'_k = \Gamma_M \Lambda(\Psi_{M-1}) \Lambda(\Psi_{M-2}) \cdots \Lambda(\Psi_{k+1}) \Lambda(\Psi_k).
\]

\( \Gamma'_k \) is an inclusion matrix (composition of inclusion matrices is an inclusion matrix). Moreover, Equation (11) continues to hold since

\[
\Gamma'_k \dim(\mathcal{A}_k) = \Gamma_M \Lambda(\Psi_{M-1}) \Lambda(\Psi_{M-2}) \cdots \Lambda(\Psi_{k+1}) \Lambda(\Psi_k) \dim(\mathcal{A}_k)
\]

\[
= \Gamma_M \dim(\mathcal{A}_M)
\]

\[
= \dim(\mathcal{B}).
\]

Now apply surjectivity of the mapping \( \Lambda_{\square} \).
full internal matrix algebra of size \( N \), \( A \) is injectable into \( S(\mathfrak{A}) \) iff all the integers \( n_k \mid N \).

**Remark 5.7.** It follows from the above remarks that if \( N \) is a nonstandard prime, then \( S(M_N(\mathfrak{C})) \) contains no unitally embedded full matrix algebra.

## 6. Subhyperfiniteness

**Definition 6.1.** A \( C^\ast \)-algebra \( A \) is subhyperfinite iff there is an internal hyperfinite dimensional \( \mathfrak{A} \) such that \( A \subseteq S(\mathfrak{A}) \).

**Example 6.2.** Any approximately finite \( C^\ast \)-algebra is subhyperfinite. In particular any commutative \( C^\ast \)-algebra is subhyperfinite.

**Example 6.3.** If \( A \) is separable residually finite-dimensional, that is \( A \) has a separating family of finite dimensional representations, then \( A \) is subhyperfinite. To show this, note that by separability of \( A \), we can assume there is a countable separating family \( \phi_k : A \to M_{n_k}(\mathfrak{C}) \) of involutive morphisms. Let \( N \cong \infty \) and consider the hyperfinite sequence of internal involutive morphisms \( \star \phi_k : \star A \to M_{n_k}(\mathfrak{C}) \) for \( k \leq N \). Now

\[
\mathfrak{M} = \bigoplus_{k=1}^{N} M_{n_k}(\mathfrak{C})
\]

is hyperfinite dimensional and \( \Phi = \bigoplus_{k=1}^{N} \star \phi_k \) is an internal involutive morphism \( \star A \to \mathfrak{M} \). For \( k \in \mathbb{N} \), the following diagram is commutative:

\[
\begin{array}{ccc}
\star A & \xrightarrow{\Phi} & \mathfrak{M} \\
\uparrow & & \downarrow \pi \circ \text{proj}_k \\
A & \xrightarrow{\phi_k} & M_{n_k}(\mathfrak{C})
\end{array}
\]

where \( \text{proj}_k \) denotes the projection of \( \mathfrak{M} \) onto \( M_{n_k}(\mathfrak{C}) \). It follows that \( S(\Phi) : S(\star A) \to S(\mathfrak{M}) \) restricted to \( A \) is separating.

For a number of examples of residually finite \( C^\ast \)-algebras see [10].

**Example 6.4.** A \( C^\ast \)-algebra with proper isometry \( w \) (that is, initial projection \( w = 1 \), final projection \( w \neq 1 \)) is not subhyperfinite. To see this, suppose \( w \in A \subseteq S(\mathfrak{A}) \). By Proposition 4.1, there is a partial isometry \( \hat{w} \in \mathfrak{A} \) such that \( \pi(\hat{w}) = w \). Since \( \pi(\text{initial projection } \hat{w}) = 1_A \), by the remarks 4.2, initial projection \( \hat{w} = 1_A \). Thus \( \hat{w} \) is a proper isometry in \( \mathfrak{A} \). It follows \( \mathfrak{A} \) cannot be hyperfinite dimensional.

We will subsequently provide a standard characterization for subhyperfiniteness. In order to do this we need some preliminary results on completely positive contractions.

## 7. Complete Positivity

In this section we prove a lifting theorem for completely positive maps very similar to the main result of [6]. The essential difference between the two results is that in
this paper we have a map between objects which are $C^*$-algebras in two different models of a theory.

We note that complete positivity for a internal linear map $\Phi : \mathfrak{A} \to \mathfrak{B}$ between internal $C^*$-algebras is an internal condition. It means that for every $n \in \mathbb{N}$, $M_n(\Phi) : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ is positive. This implies that for all $n \in \mathbb{N}$, $M_n(S(\Phi)) : M_n(S(\mathfrak{A})) \to M_n(S(\mathfrak{B}))$ is positive, i.e., $S(\Phi)$ is completely positive.

**Definition 7.1.** An internal operator $V$ between internal normed spaces is a near contraction iff $\|V\| \leq 1 + \text{infinitesimal}$.

**Lemma 7.2.** Suppose $V : \mathcal{H} \to \mathcal{H}'$ is a near contraction where $\mathcal{H}, \mathcal{H}'$ are internal Hilbert spaces. Then there is an internal contraction $W : \mathcal{H} \to \mathcal{H}'$ such that $V \sim W$ in operator norm.

**Proof.** Consider the self adjoint operator $T = V^*V$ on $\mathcal{H}$. Let $f$ be the standard function such that $f(t) = 1$ for $t < 1$ and $f(t) = t^{-1}$ for $t \geq 1$. By properties of the functional calculus and the fact $T$ is a near contraction, $S = f(T)$ is such that $S \geq 1$ and $ST$ is a contraction. Then $W = V \sqrt{S}$ is a contraction since $W^*W = \sqrt{S} V^* V \sqrt{S} = ST$.

Moreover, $W - V = V (1 - \sqrt{S}) \equiv 0$. \hfill \Box

**Proposition 7.3.** Suppose $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a completely positive near contraction. Then there is a completely positive contraction $\Psi : \mathfrak{A} \to \mathfrak{B}$ such that $\Phi \sim \Psi$ in norm.

**Proof.** By the Stinespring factorization theorem and transfer we can assume there is a nondegenerate representation $\rho$ of $\mathfrak{A}$ and an operator $V : \mathcal{H}_\rho \to \mathcal{H}$ such that $\Phi(x) = V^* \rho(x) V$. Now $\|\Phi\| = \|V^* V\|$. This is obvious since we are assuming $\mathfrak{A}$ has an identity. It follows $V$ is a near contraction. By the lemma there is a contraction $W$ such that $W \equiv V$. Thus $\Psi(x) = W^* \rho(x) W$ is a contraction and $\Psi \equiv \Phi$ in norm. \hfill \Box

A completely positive contraction $\phi : \mathfrak{A} \to \mathfrak{B}$ is matricial iff it has a factorization

\begin{equation}
\mathfrak{A} \xrightarrow{\psi} \mathfrak{C} \xrightarrow{\rho} \mathfrak{B}
\end{equation}

where $\mathfrak{C}$ is a finite dimensional $C^*$-algebra and $\psi, \rho$ are completely positive contractions.

**Proposition 7.4.** Suppose $\mathfrak{A}$ is a standard $C^*$-algebra, $\mathfrak{B}$ an internal $C^*$-algebra, $\phi : \mathfrak{A} \to S(\mathfrak{B})$ a completely positive matricial map. Then there is an internal completely positive contraction $\Phi : \cdot \mathfrak{A} \to \mathfrak{B}$ which makes the following diagram commutative:

\begin{equation}
\begin{array}{ccc}
\cdot \mathfrak{A} & \xrightarrow{\Phi} & \mathfrak{B} \\
\downarrow & & \downarrow \\
\mathfrak{A} & \xrightarrow{\phi} & S(\mathfrak{B})
\end{array}
\end{equation}
Proof. By assumption, there is a diagram (14) with $B = S(\mathcal{B})$. Now we have the commutative diagram,

$$
\begin{array}{ccc}
\bullet A & \xrightarrow{\psi} & \bullet C \\
\downarrow & & \downarrow \\
A & \xrightarrow{\psi} & C
\end{array}
$$

(16)

To complete the proof, we need the following lemma:

**Lemma 7.5.** In the context of Proposition 7.4, suppose in addition $A$ is a standard finite dimensional $C^*$-algebra. Then there is an internal completely positive contraction $\Phi : \bullet A \rightarrow \mathcal{B}$ which makes the diagram (15) commutative.

Proof. $A$ is the (finite) direct sum of full matrix algebras. The general case can be reduced to the case $A$ is a full matrix algebra by considering the restrictions of $\phi$ to the full matrix components of $A$. Now it is well known that for a $C^*$-algebra $B$, $\psi \mapsto \{\psi(e_{i,j})\}_{i,j}$ is a 1-1 correspondence between completely positive maps $\psi : M_n(\mathbb{C}) \rightarrow B$ and positive elements of $M_n(B)$, where $\{e_{i,j}\}_{i,j}$ is the canonical system of matrix units for $M_n(\mathbb{C})$. (See for instance [5] where this is shown for the case $B = M_m(\mathbb{C})$. The general case follows immediately from this case by considering a faithful involutive representation of $B$ on a Hilbert space $H$ and compressing to finite dimensional subspaces). We claim there is a matrix $\tilde{B} = \{\tilde{b}_{i,j}\}_{i,j} \in M_n(\mathcal{B})$ such that $M_n(\pi)(\tilde{B}) = \{\phi(e_{i,j})\}_{i,j}$. To see this, let $\{\phi(e_{i,j})\} = T^*T$ with $T \in M_n(S(\mathcal{B}))$. By surjectivity of $\pi_{M_n(\mathcal{B})}$, there is a $\tilde{T} \in M_n(\mathcal{B})$ such that $M_n(\pi)(\tilde{T}) = T$. Since $M_n(\pi)$ is an involutive morphism of rings, letting $\tilde{B} = \tilde{T}^*\tilde{T}$ proves the claim. Now there is a completely positive map $\Phi$ such that $\Phi(\bullet e_{i,j}) = \tilde{b}_{i,j}$. By $\mathbb{C}$-linearity of everything involved, the diagram (15) is commutative. $\Phi$ is a near contraction since $\phi$ is a contraction. To complete the proof, apply Proposition 7.3. \qed

To complete the proof of the proposition, instantiate $A$ in the lemma with $C$. \qed

A completely positive contraction $\phi : A \rightarrow B$ is nuclear iff there is a sequence $\phi_n$ of matricial completely positive contractions such that $\phi_n \rightarrow \phi$ in the point norm topology.

**Proposition 7.6.** Suppose $A$ is a standard separable $C^*$-algebra, $B$ an internal $C^*$-algebra, $\phi : A \rightarrow S(\mathcal{B})$ a completely positive nuclear contraction. Then there is an internal completely positive contraction $\Phi : \bullet A \rightarrow \mathcal{B}$ which makes (15) commutative.

Proof. Since $A$ is separable, there is a sequence of completely positive matricial contractions $\phi_i : A \rightarrow S(\mathcal{B})$ which converges to $\phi$ in the point-norm topology. By Proposition 7.4, each $\phi_i$ has a completely positive lifting $\Phi_i$ in the sense that the diagram in (15) commutes. The remainder of the proof shows that the sequence $\{\Phi_i\}_{i \in \mathbb{N}}$ can be extended in such a way that some $\Phi_N$ with $N \equiv \infty$ is the desired lifting. Let $\{x_i\}_{i \in \mathbb{N}}$ be norm dense in $A$. Let $y = \bullet \{x_i\}_{i \in \mathbb{N}}$ and $z = \bullet \{\phi(x_i)\}_{i \in \mathbb{N}}$. Note that in both these cases, the transfer operator is applied to an entire sequence.
In particular for \( i \in \mathbb{N} \), \( y_i = \ast x_i \) and \( z_i = \ast [\phi(x_i)] \). The following formula holds:

\[
(17) \quad \forall k \in \mathbb{N} \forall i \in \mathbb{N} \exists n \in \mathbb{N} \forall j \geq n \quad \| \phi_j(x_i) - \phi(x_i) \| \leq \frac{1}{k}
\]

From this immediately follows:

\[
(18) \quad \forall k \forall i \exists n \quad \| \Phi_n(y_i) - z_i \| \leq \frac{2}{k}
\]

By saturation and overspill, the sequence \( \{ \Phi_{\ell} \} \) has an internal extension to a sequence of internal completely positive contractions \( \{ \Phi_{\ell} \}_{1 \leq \ell \leq N} \). By overspill, there is an \( N_0 \cong \infty \) such that (18) continues to hold for \( k, i \leq N_0 \). Instantiating \( k \) in (18) with some value \( \cong \infty \),

\[
\forall i \exists n_i \quad \| \Phi_{n_i}(y_i) - z_i \| \leq \frac{2}{k} \cong 0.
\]

In particular, let \( N = \max\{n_1, n_2, \ldots, n_M\} \) where \( M \cong \infty \). Then

\[
\forall i \in \mathbb{N} \| \Phi_N(y_i) - z_i \| \cong 0,
\]

which is the desired result. \( \Box \)

**Remark 7.7.** In the above proposition, if \( \phi \) is unital \( \Phi \) can be taken to be unital as well. For \( a = \Phi(1, A) \cong 1_A \). Replace \( \Phi \) by \( a^{-1/2} \Phi(\cdot) a^{-1/2} \).

### 8. Standard Characterization of Subhyperfinite \( C^* \)-algebras

If \( x \in L(H) \) and \( K \subseteq H \), \( \text{compr}_K x \) denotes the compression of \( x \) to \( K \).

**Proposition 8.1.** A necessary and sufficient condition a separable nuclear \( C^* \)-algebra \( A \) be subhyperfinite is that for every \( \epsilon > 0 \) and \( x_1, \ldots, x_n \in A \), there is a representation \( \phi \) of \( A \) and a finite dimensional subspace \( K \subseteq H_\phi \) such that for all \( i \leq n \), \( \| x_i \| - \epsilon \leq \| \text{compr}_K \phi(x_i) \| \) and \( \| [\phi(x_i), \text{proj}_K] \| \leq \epsilon \).

**Proof.** Necessity is used only for necessity. Sufficiency: There is a countable involutive \( \mathbb{Q} \)-algebra \( V \) which is norm dense in \( A \). Let \( \{ x_i \}_{i \in \mathbb{N}} \) enumerate \( V \). For \( \ell \in \mathbb{N} \), let \( \phi_\ell \) be a representation of \( A \) and \( K_\ell \subseteq H_{\phi_\ell} \) finite dimensional such that for all \( i \leq \ell \),

\[
\| x_i \| - 1/\ell \leq \| \text{compr}_{K_\ell} \phi_\ell(x_i) \| \quad \text{and} \quad \| [\phi_\ell(x_i), \text{proj}_{K_\ell}] \| \leq 1/\ell.
\]

By transfer there are \( \ast \mathbb{N} \) sequences \( \{ z_\ell \} \), \( \{ \Phi_\ell \} \) and \( \{ K_\ell \} \) such that for \( \ell \in \ast \mathbb{N} \), \( \Phi_\ell \) is an internal \( C^* \)-morphism of \( \ast A \) and \( K_\ell \subseteq \mathcal{H}_{\phi_\ell} \) is a hyperfinite dimensional space such that for all \( \ell \in \ast \mathbb{N} \) and all \( i \leq \ell \),

\[
\| z_i \| - 1/\ell \leq \| \text{compr}_{K_\ell} \Phi_\ell(z_i) \| \quad \text{and} \quad \| [\Phi_\ell(z_i), \text{proj}_{K_\ell}] \| \leq 1/\ell.
\]

Let \( \ell \cong \infty \). Since \( \Phi_\ell \) is a representation of \( \ast A \), \( \text{compr}_{K_\ell} \Phi_\ell : \ast A \to \mathcal{S}(\mathcal{H}_{K_\ell}) \) is an internal completely positive contraction. We claim \( \psi : x \mapsto \pi(\text{compr}_{K_\ell} \Phi_\ell(\ast x)) \) is an injective \( C^* \)-morphism \( A \to \mathcal{S}(\mathcal{H}(K_\ell)) \). It is clear \( \psi \) is a completely positive contraction. Now for every \( x \in V \), \( \| [\Phi_\ell(\ast x), \text{proj}_{K_\ell}] \| \cong 0 \). Thus for all \( x, y \in V \),

\[
\text{compr}_{K_\ell} \Phi_\ell(\ast (x y)) \cong \text{compr}_{K_\ell} \Phi_\ell(\ast x) \text{compr}_{K_\ell} \Phi_\ell(\ast y).
\]

It follows \( \psi \) is an involutive morphism on \( V \) and for every \( x \in V \)

\[
\| x \| = \| \ast x \| \cong \| \text{compr}_{K_\ell} \Phi_\ell(\ast x) \| \cong \| \psi(x) \|.
\]
By continuity it follows ψ is an isometric $C^*$-morphism.

Necessity: Let $\phi: A \to \mathcal{S}(B)$ be an injective $C^*$-morphism where $B$ is hyperfinite dimensional and $\Phi: \mathcal{A} \to \mathcal{B}$ a unital completely positive lifting in the sense of Proposition 7.6; the lifting exists since the algebra $A$ and therefore the map $\phi$ are nuclear. In particular, for all $x, y \in A$, $\Phi(\phi(x)) \equiv \Phi(\phi(y))$ in the norm of $B$. There is a hyperfinite dimensional $\mathcal{K}$ and an imbedding $\mathcal{B} \subseteq \mathcal{L}(\mathcal{K})$. By the $\mathcal{A}$-version of the Stinespring factorization theorem, there is a representation $\Psi$ of $\mathcal{A}$ on an internal Hilbert space $H \supseteq \mathcal{K}$ such that $\Phi = \text{compr}_K \Psi$. Thus $\rho : x \mapsto \text{compr}_K \Psi(x)$ is an involutive $\mathcal{C}$-morphism $A \to \mathcal{L}(\mathcal{K})$ modulo $\cong$. This means $\rho(x, y) \equiv \rho(x, y)$ in the norm of $\mathcal{L}(\mathcal{K})$ for all $x, y \in A$. From this follows that $[\text{proj}_K, \Psi(x)] \cong 0$ for all $x \in A$. Proof: If $T$ is an internal selfadjoint operator such that $\text{compr}_K T^2 \cong (\text{compr}_K T)^2$ then

$$\text{proj}_K T^2 \text{proj}_K = \text{proj}_K T(1 - \text{proj}_K) T \text{proj}_K + \text{proj}_K T \text{proj}_K T \text{proj}_K$$

from which follows $\text{proj}_K T(1 - \text{proj}_K) \cong 0$ and thus $[T, \text{proj}_K] \cong 0$. Moreover,

$$\|x\| = \|\phi(x)\| \geq \|\Psi(x)\| \geq \|\text{compr}_K \Psi(x)\| = \|\Phi(x)\| \cong \|x\| \quad \text{for } x \in A.$$ 

Therefore the following formula holds:

$$\forall \text{standard } \epsilon > 0 \forall \text{finite } F \subseteq A \exists \text{internal } \Psi \exists \text{internal } \mathcal{K} \forall z \in \mathcal{F}$$

(a) $\Psi$ is a $C^*$-representation of $\mathcal{A}$,

(b) $\mathcal{K} \subseteq H_\Psi$ is a hyperfinite dimensional subspace,

(c) $\|z\| - \epsilon \leq \|\text{compr}_K \Psi(z)\|$ and $\|[\text{proj}_K, \Psi(z)]\| \leq \epsilon$.

Note that the stronger formula in which the existential quantifiers are outermost is valid, but is unsuitable for transfer. By transfer it follows the same formula is valid with the quantifiers on the first line ranging over standard values.

$$\forall \epsilon > 0 \forall \text{finite } F \subseteq A \exists \psi \exists \mathcal{K} \forall z \in F$$

(a) $\psi$ is a $C^*$-representation of $A$,

(b) $\mathcal{K} \subseteq H_\psi$ is a finite dimensional subspace,

(c) $\|z\| - \epsilon \leq \|\text{compr}_K \psi(z)\|$ and $\|[\text{proj}_K, \psi(z)]\| \leq \epsilon$.

This is exactly the condition in the statement of the Proposition. \qed

**Theorem 8.2.** A necessary and sufficient condition a separable nuclear $C^*$-algebra $A$ be quasidiagonal is it be subhyperfinite.

**Proof.** The condition for subhyperfiniteness given in Proposition 8.1 is exactly condition (iii) in Theorem 1 of [19]. \qed

**Appendix A. Ultraproducts of $C^*$-algebras**

In certain cases the $C^*$-algebras $\mathcal{S}(\mathfrak{A})$ are exactly the ultraproducts of finite dimensional $C^*$-algebras ([7],§1). This is remarked in the paper [11], p 19. For completeness, we sketch the argument without the machinery of that paper. We will refer to the map $\mathfrak{C}: V(\mathbb{R}) \longrightarrow V(\mathfrak{C})$ constructed in §1.2 of [1] as a bounded ultrapower embedding. We recall the context of [1].

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. If $\bar{x} = \{x_i\}_{i \in \mathbb{N}}, \bar{y} = \{y_i\}_{i \in \mathbb{N}}$, define

$$\bar{x} \sim_\mathcal{U} \bar{y} \iff \{i \in \mathbb{N} : x_i = y_i\} \in \mathcal{U}.$$
\( \langle \vec{x} \rangle \) denotes the \( \sim_U \) equivalence class of \( \vec{x} \). A sequence \( \vec{x} \) is \textit{bounded} iff for some \( n \in \mathbb{N} \) and all \( i \in \mathbb{N} \), \( x_i \in V_n(\mathbb{R}) \). The bounded ultrapower of \( V(\mathbb{R}) \), denoted \( V^N(\mathbb{R})/U \) is \( \{ \langle \vec{x} \rangle : \vec{x} \in V(\mathbb{R})^N \text{ bounded} \} \). \( i : V(\mathbb{R}) \to V^N(\mathbb{R})/U \) is the mapping \( x \mapsto \langle \text{sequence with constant value } x \rangle \). Define a relation \( \varepsilon_U \) on \( V^N(\mathbb{R})/U \) by
\[
\langle \vec{x} \rangle \varepsilon_U \langle \vec{y} \rangle \iff \{ i \in \mathbb{N} : x_i = y_i \} \in U.
\]

Note that any sequence \( \vec{r} \) with \( r_i \in \mathbb{R} \) for all \( i \in \mathbb{N} \) is bounded as an element of \( V^N(\mathbb{R}) \). The set of hyperreals \( \mathbb{R}^{*} \) consists of \( \langle \vec{r} \rangle \) for \( \vec{r} \in \mathbb{R}^N \). The pointwise operations make \( \mathbb{R}^{*} \) an ordered nonarchimedean field. \( \langle \vec{r} \rangle \) is limited iff there is some positive real \( M \) such that \( \{ i \in \mathbb{N} : |r_i| \leq M \} \in U \). Letting \( \lim_U \) denote the generalized limit on \( U \), \( \langle \vec{r} \rangle \) is infinitesimal iff \( \lim_U r_i = 0 \). \( \langle \vec{r} \rangle \) is a hyperinteger iff \( \{ i \in \mathbb{N} : r_i \in \mathbb{N} \} \in U \).

The relation \( \varepsilon_U \) is not the membership relation on \( V^N(\mathbb{R})/U \) although it behaves like one. In fact we can define an injective mapping \( j : V^N(\mathbb{R})/U \to V(\mathbb{R}^{*}) \) which transports \( \varepsilon_U \) to the membership relation on \( V(\mathbb{R}^{*}) \). \( j \) is defined recursively, in such a way that \( j(i \mathbb{R}) \), is the identity. \( j \) is referred to as a \textit{Mostowski collapsing}.

Following [1], the map \( \ast : V(\mathbb{R}) \to V(\mathbb{R}^{*}) \) is the composition:
\[
V(\mathbb{R}) \overset{i}{\to} V^N(\mathbb{R})/U \overset{j}{\to} V(\mathbb{R}^{*}).
\]

We point out that it is possible to transport algebraic and analytic structures from the universe \( V(\mathbb{R}) \) to the universe of internal sets. To do this we cannot apply the mapping \( \ast \) directly. For instance, the set of metric spaces which are elements \( V(\mathbb{R}) \) is not in \( V(\mathbb{R}) \).

We now recall from [7] the definition of \textit{bounded ultraproduct} of a sequence of normed spaces \( \{ E_i \}_{i \in \mathbb{N}} \) over a free ultrafilter \( U \) on \( \mathbb{N} \). Let \( \Pi_0 \) be the space of sequences \( \vec{x} \in \prod_{i \in \mathbb{N}} E_i \) for which \( \sup_{i \in \mathbb{N}} \| x_i \| < \infty \). Let \( \| \vec{x} \| = \lim_U \| x_i \| \). \( \vec{x} \mapsto \| \vec{x} \| \) is a seminorm on \( \Pi_0 \). Finally, let \( \prod_{i \in \mathbb{N}} E_i \) be the normed space \( \Pi_0 / \{ \vec{x} \in \Pi_0 : \| \vec{x} \| = 0 \} \). Let \( E_i \) are normed involutive algebras, \( \prod_{i \in \mathbb{N}} E_i \) is also a normed involutive algebra.

**Proposition A.1.** Let \( U \) be a free ultrafilter on \( \mathbb{N} \). If \( \ast : V(\mathbb{R}) \to V(\mathbb{R}^{*}) \) is the bounded ultrapower embedding over \( U \), then the \( C^{*} \)-algebras \( S(\mathfrak{A}) \) for \( \mathfrak{A} \) hyperfinite dimensional are exactly the ultraproducts over \( U \) of finite dimensional \( C^{*} \)-algebras.

**Proof.** We sketch the proof. Suppose \( \mathfrak{A} \) is an internal \( C^{*} \)-algebra. There is a bounded sequence \( \mathfrak{A} = \{ A_i \}_{i \in \mathbb{N}} \subset V(\mathbb{R})^N \) such that \( j(\langle A_i \rangle) = \mathfrak{A} \). It follows \( F = \{ i \in \mathbb{N} : A_i \text{ is a } C^{*} \text{-algebra} \} \in U \). Modifying the sequence \( \mathfrak{A} \) on \( \hat{C}F \) does not change the value of \( \langle A \rangle \), so we may assume \( A_i \) is a \( C^{*} \)-algebra for all \( i \in \mathbb{N} \). For \( \vec{x} \in \prod_{i \in \mathbb{N}} A_i \), the internal norm of \( j(\langle \vec{x} \rangle) \) is the equivalence class \( \{ \| x_i \| \}_{i \in \mathbb{N}} \). \( \text{Fin(}\mathfrak{A}\text{)} \) consists of elements of the form \( j(\langle \vec{x} \rangle) \) with \( \sup_{i \in \mathbb{N}} \| x_i \| < \infty \). The norm of \( j(\langle \vec{x} \rangle) \) is infinitesimal iff \( \lim_U \| x_i \| = 0 \). From this it readily follows that \( S(\mathfrak{A}) \) is isomorphic to \( \prod_{i \in \mathbb{N}} A_i / U \).

\( \mathfrak{A} \) is hyperfinite dimensional iff \( F = \{ i \in \mathbb{N} : A_i \text{ is finite dimensional} \} \in U \). Modifying \( \langle A \rangle \) on \( \hat{C}F \), we may assume \( A_i \) is finite dimensional for all \( i \in \mathbb{N} \). \( \square \)

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