Stochastic inertial manifolds for damped wave equations

Zhenxin Liu
College of Mathematics, Jilin University, Changchun 130012, People’s Republic of China
zxlz@email.jlu.edu.cn

September 18, 2018

Abstract
In this paper, stochastic inertial manifold for damped wave equations subjected to additive white noise is constructed by the Lyapunov-Perron method. It is proved that when the intensity of noise tends to zero the stochastic inertial manifold converges to its deterministic counterpart almost surely.

Keywords: Stochastic inertial manifold; Wave equation; Random dynamical system

1 Introduction
The inertial manifold (IM) introduced by Constantin, Foias, Nicolaenko, Sell, and Temam [19, 20, 17, 18, 8] is a finite dimensional Lipschitz invariant manifold attracting solutions exponentially, which goes back to the works of Mañé, Henry and Mora [26, 23, 27]. Global attractor is an invariant compact set attracting solutions which often has a finite (fractal) dimension and, therefore, it is an important object for the study of long time behavior of evolution equations. At the present level of understanding of dynamical systems, global attractors are expected to be very complicated objects (fractals) and their practical utilization, for instance for numerical simulations, may be difficult. The IMs, when they exist, are more convenient objects which are able to describe the large-time behavior of dynamical systems. One of the important properties of inertial manifolds is that they contain global attractors, so the study of dynamics of infinite dimensional nonlinear systems can be reduced to the study of dynamics of flows on the inertial manifold, which, in turn, is described by the dynamics of an ordinary differential equation. There are extensive works on IMs. See, for example, Chow and Lu [4], Chow et al [5], Constantin et al [8], Constantin et al [9], Foias et al [17, 18], Foias et al [19, 20], Foias et al [21], Mallet-Paret and Sell [25], Sell and You [33], Temam [34], among others.

*This work is partially supported by the 985 project of Jilin University.
Stochastic partial differential equations (SPDE) have been drawing more and more attention for their importance in describing many natural phenomenon under random influences. With the rapid development of random dynamical systems (RDS) [1], many SPDEs are studied in the framework of RDS. On many occasions, the development of SPDE and RDS mimics the deterministic case and many efforts are devoted to establish the results for SPDE and RDS corresponding to that for the deterministic case. This is true for IM: there have been some works on Stochastic IMs, see, for example, Bensoussan and Flandoli [2], Chueshov and Girya [6], Chueshov and Scheutzow [7], Da Prato and Debbussche [13], Duan et al [15, 16]. These works mainly deal with stochastic parabolic equations. In present paper, we aim to obtain the existence of stochastic IM for damped wave equations subjected to additive white noise. Moreover, we will show that the stochastic IM converges to its deterministic counterpart almost surely when the intensity of noise tends to zero.

As in the deterministic case, the usual methods to obtain the existence of stochastic IMs are Hadamard’s graph transform method [22] and Lyapunov-Perron’s method [24, 31]. In this paper, we adopt the latter one. In Section 2 we introduce some preliminaries and prove the existence theorem of stochastic IM for abstract evolution equations with random coefficients; in Section 3 we apply the result of Section 2 to damped wave equations subjected to additive white noise and study the property of its IM.

2 Existence of Stochastic IM for abstract equations

**Definition 2.1** Let $X$ be a metric space with a metric $d_X$. A random dynamical system (RDS), shortly denoted by $\varphi$, consists of two ingredients:

(i) A model of the noise, namely a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(t, \omega) \mapsto \theta_t \omega$ is a measurable flow which leaves $\mathbb{P}$ invariant, i.e. $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

(ii) A model of the system perturbed by noise, namely a cocycle $\varphi$ over $\theta$, i.e. a measurable mapping $\varphi : \mathbb{R}^+ \times \Omega \times X \to X, (t, \omega, x) \mapsto \varphi(t, \omega, x)$, such that:

$$\varphi(0, \omega, \cdot) = \text{id}_X, \varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \varphi(s, \omega, \cdot)) \quad \text{for all} \quad t, s \in \mathbb{R}^+, \omega \in \Omega. \quad (1)$$

Although it is well known that a large class of partial differential equations with stationary random coefficients and Itô stochastic ordinary differential equations generate RDS (for details see Chapter 1 of [1]), this problem is still unsolved for SPDE with general noise terms, see [15] for the reason. Indeed, the existence of RDS generated by SPDE has been proved in relatively narrow generality. In fact, only cases in which the SPDE can be reduced to a deterministic one with random coefficients can be treated in the framework of RDS. See, for example, [3, 11, 12, 16].

For later use, assume $z$ is an Ornstein-Uhlenbeck process which satisfies the following equation

$$dz + \lambda z dt = \delta dW \quad (2)$$
for some \( \lambda > 0 \) and \( \delta > 0 \). The process \( z \) has the following properties, see \([3, 15]\) for the proof.

**Lemma 2.1** (i) There exists a \( \{\theta_t\}_{t \in \mathbb{R}} \)-invariant set \( \Omega \in B(C_0(\mathbb{R}, \mathbb{R})) \) of full measure with sublinear growth:

\[
\lim_{t \to \pm \infty} \frac{|\omega(t)|}{|t|} = 0, \quad \omega \in \Omega.
\]

(ii) For \( \omega \in \Omega \) the random variable

\[
z(\omega) = -\lambda \delta \int_{-\infty}^{0} e^{\lambda \tau} \omega(\tau) d\tau
\]

exists and generates a unique stationary solution of (2) given by

\[
(t, \omega) \rightarrow z(\theta_t \omega) = -\lambda \delta \int_{-\infty}^{0} e^{\lambda \tau} \theta_t \omega(\tau) d\tau = -\lambda \delta \int_{-\infty}^{0} e^{\lambda \tau} \omega(\tau + t) d\tau + \delta \omega(t).
\]

The map \( t \rightarrow z(\theta_t \omega) \) is continuous.

(iii) In particular, we have

\[
\lim_{t \to \pm \infty} \frac{|z(\theta_t \omega)|}{|t|} = 0 \text{ for } \omega \in \Omega.
\]

(iv) In addition,

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z(\theta_t \omega) d\tau = 0 \text{ for } \omega \in \Omega, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} |z(\theta_t \omega)| d\tau = E|z| < \infty.
\]

Let \( H \) be a separable Hilbert space with norm \( | \cdot | \) and inner product \( \langle \cdot, \cdot \rangle \). Consider the Stratonovich SPDE on \( H \)

\[
\frac{du}{dt} = Au + F(u) + u \dot{W}, \tag{3}
\]

where \( u \in H \), \( W(t) \) is the standard real-valued two-sided Wiener process and the generalized time-derivative \( \dot{W} \) formally describes a white noise. Here we assume that \( F \) is globally Lipschitz continuous on \( H \) with Lipschitz constant \( \text{Lip} F \). For the existence and uniqueness theory of (3) we can first write it into its equivalent Itô equation and then refer to \([14]\) for details. Under the transformation \( T(\omega, u) = u e^{-z(\omega)} \), (3) is conjugated to the following equation with random coefficients

\[
\frac{du}{dt} = Au + z(\theta_t \omega) u + G(\theta_t \omega, u), \quad u(0) = x \in H, \tag{4}
\]

where \( z \) satisfies

\[
dz + z dt = dW,
\]

and \( G(\omega, u) = e^{-z(\omega)} F(u e^{z(\omega)}) \). It is clear that \( \text{Lip}_u G = \text{Lip} F \).
Assume $A : D(A) \to H$ is a linear operator which generates a strongly continuous semigroup $e^{At}$ on $H$, which satisfies the pseudo exponent dichotomy condition with exponents $0 > \alpha > \beta$ and bound $K > 0$, i.e. there exists a continuous projection $P$ on $H$ such that

(i) $Pe^{At} = e^{At}P$;

(ii) the restriction $e^{At}|_{R(P)}$, $t \geq 0$, is an isomorphism of the range $R(P)$ of $P$ onto itself, and we denote $e^{At}$ for $t < 0$ the inverse map;

(iii)

\[ |e^{At}Px| \leq Ke^{\alpha t}|x|, \quad t \leq 0, \]
\[ |e^{At}Qx| \leq Ke^{\beta t}|x|, \quad t \geq 0, \]

where $Q = I - P$.

**Definition 2.2** A random set is called invariant for RDS $\varphi$ if

$\varphi(t, \omega, M(\omega)) \subset M(\theta t \omega)$, for any $t \geq 0$.

If an invariant set $M(\omega)$ can be represented by a Lipschitz or $C^k$ mapping

$h(\cdot, \omega) : PH \to QH$

such that

$M(\omega) = \{\xi + h(\xi, \omega) | \xi \in PH\}$,

then we call $M(\omega)$ a Lipschitz or $C^k$ invariant manifold. Furthermore, if $PH$ is finite dimensional and $M(\omega)$ attracts exponentially all the orbits of $\varphi$, then we call $M(\omega)$ a stochastic inertial manifold of $\varphi$.

**Theorem 2.1** ([16]) If

\[ KLipF \left( \frac{1}{\alpha - \eta} + \frac{1}{\eta - \beta} \right) < 1, \] (6)

then there exists a Lipschitz invariant manifold for the random evolutionary Equation (4), which is given by

$M(\omega) = \{\xi + h(\xi, \omega) | \xi \in PH\}$, (7)

where $h : PH \to QH$ is a Lipschitz continuous mapping given by

\[ h(\xi, \omega) = \int_{-\infty}^{0} e^{-As + \int_{s}^{0} z(\theta r \omega) dr} QG(\theta s \omega, u(s; \xi, \omega)). \] (8)

**Remark 2.1** It is easy to see that if $F$ is $C^1$, then the stochastic invariant manifold obtained in Theorem 2.1 is $C^1$ by Theorem 5.3 of [16].

**Theorem 2.1** says that (4) has a Lipschitz manifold if the spectral gap condition (6) holds. To show that the manifold is an inertial manifold for (4), we should verify that it attracts exponentially all the orbits of $\varphi$. A stronger reduction property is the exponential tracking property [21], also called asymptotical completeness property [32]: each trajectory of the evolution equation tends exponentially to a trajectory on the inertial manifold. To be more specific, we states it as follows:
Definition 2.3 Let $M(\omega)$ be an invariant manifold for RDS $\varphi$. If for $\forall x \in H$, there exists an $\bar{x} \in M(\omega)$ such that
\[
|\varphi(t,\omega,x) - \varphi(t,\omega,\bar{x})| \leq c_1 e^{-c_2 t} |x - \bar{x}|, \quad \forall t \geq 0,
\]
where $c_1 > 0$ is a constant dependent on $\omega$, $x$ and $\bar{x}$, while $c_2$ is a constant independent of these variables, then $M(\omega)$ is said to have the asymptotic completeness property.

If $M(\omega)$ has the asymptotic completeness property, then the asymptotic behavior of $\varphi$ on $H$ can be reduced to $M(\omega)$. Hence the the original infinite dimensional SPDE problem on $H$ is reduced to a finite dimensional stochastic ODE problem on $M(\omega)$.

Denote
\[
C^+_\eta := \{ \phi : [0, \infty) \to H| \phi \text{ continuous}, \sup_{t \geq 0} e^{-\eta t} - \int_0^t z(\theta_r \omega) d\theta_r |\phi(t)| < \infty \},
\]
then $C^+_\eta$ is a Banach space with norm $|\phi|_{C^+_\eta} := \sup_{t \geq 0} e^{-\eta t} - \int_0^t z(\theta_r \omega) d\theta_r |\phi(t)|$.

Theorem 2.2 If we have the spectral gap condition
\[
KLipF \left( \frac{1}{\alpha - \eta} + \frac{1}{\eta - \beta} \right) + K^2 Lip h \cdot Lip F \frac{1}{\alpha - \eta} < 1, \tag{9}
\]
then the Lipschitz invariant manifold for (4) obtained in Theorem 2.1 has the asymptotic completeness property.

Proof. Assume $u, \bar{u}$ are two solutions of (4) and let $w = \bar{u} - u$, then $w$ satisfies the following equation:
\[
\frac{dw}{dt} = Aw + z(\theta_t \omega) w + \tilde{F}(\theta_t \omega, w), \tag{10}
\]
where
\[
\tilde{F}(\theta_t \omega, w) := G(\theta_t \omega, u + w) - G(\theta_t \omega, u).
\]
It is clear that
\[
\tilde{F}(\theta_t \omega, 0) = 0, \quad \text{Lip}_w \tilde{F} = \text{Lip}_u G = \text{Lip} F. \tag{11}
\]
First if $w \in C^+_\eta$ is a solution of (10), then $w$ can be expressed by
\[
w(t) = e^{At} + \int_0^t e^{A(t-s)} z(\theta_s \omega) d\theta_s Q w(0) + \int_0^t e^{A(t-s)} + f^+ z(\theta_s \omega) d\theta_s Q \tilde{F}(\theta_s \omega, w(s)) ds + \int_0^t e^{A(t-s)} + f^+ z(\theta_s \omega) d\theta_s P \tilde{F}(\theta_s \omega, w(s)) ds. \tag{12}
\]
In fact, since $w$ is a solution of (10), we have
\[
w(t) = e^{A(t-t_0)} + f^+ z(\theta_s \omega) d\theta_s w(t_0) + \int_{t_0}^t e^{A(t-s)} + f^+ z(\theta_s \omega) d\theta_s \tilde{F}(\theta_s \omega, w(s)) ds. \tag{13}
\]
This implies
\[ Pw(t) = e^{A(t-t_0) + \int_{t_0}^t z(\theta, \omega) d\theta} Pw(t_0) + \int_{t_0}^t e^{A(t-s) + \int_s^t z(\theta, \omega) d\theta} P\tilde{F}(\theta_s \omega, w(s)) ds. \]

When \( t_0 > t \), by (5) we have
\[ |e^{A(t-t_0) + \int_{t_0}^t z(\theta, \omega) d\theta} Pw(t_0)| \leq Ke^{\alpha(t-t_0) + \int_{t_0}^t z(\theta, \omega) d\theta} |w(t_0)| \]
\[ \leq Ke^{-(\alpha - \eta)t_0 + \alpha t + \int_{t_0}^t z(\theta, \omega) d\theta} |w|_{C^+_{\eta\alpha}}. \]

By the property of \( z(\omega) \) we obtain
\[ e^{A(t-t_0) + \int_{t_0}^t z(\theta, \omega) d\theta} Pw(t_0) \to 0 \text{ as } t_0 \to \infty. \]

Therefore,
\[ Pw(t) = \int_{t_0}^t e^{A(t-s) + \int_s^t z(\theta, \omega) d\theta} P\tilde{F}(\theta_s \omega, w(s)) ds. \]

Thus (12) holds.

We then show that (12) has solutions on \( C^+_{\eta\alpha} \) and \( \bar{u}(0) = u(0) + w(0) \in M(\omega). \)

From (16) we know that the solution \( \bar{u} \) lies on \( M \) if and only if \( Q\bar{u}(0) = h(P\bar{u}(0), \omega) \), recalling that \( M(\omega) = \{ \xi + h(\xi, \omega)|\xi \in PH \} \). That is
\[ Qw(0) = -Qu(0) + h(Pu(0) + Pw(0), \omega). \] \quad (13)

Let
\[ \tilde{T}w(t) = e^{At + \int_0^t z(\theta, \omega) d\theta} Qw(0), \]
\[ Tw(t) = \int_0^t e^{A(t-s) + \int_s^t z(\theta, \omega) d\theta} Q\tilde{F}(\theta_s \omega, w(s)) ds + \int_t^\infty e^{A(t-s) + \int_s^t z(\theta, \omega) d\theta} P\tilde{F}(\theta_s \omega, w(s)) ds, \]

then (12) reads as
\[ w(t) = \tilde{T}w(t) + Tw(t). \]

We assert that \( \tilde{T} \) and \( T \) map \( C^+_{\eta\alpha} \) to \( C^+_{\eta\alpha} \). In fact,
\[ e^{-\eta t - \int_0^t z(\theta, \omega) d\theta} |\tilde{T}w(t)| \leq Ke^{-(\eta - \beta)t} |Qw(0)| \]
\[ \leq K |Qw(0)| \]
\[ \leq [13] \ K \left(| - Qu(0) + h(Pu(0), \omega)| \right. \]
\[ + \left. |h(Pu(0) + Pw(0), \omega) - h(Pu(0), \omega)|\right) \]
\[ \leq K \left(| - Qu(0) + h(Pu(0), \omega)| + \text{Lip}h|Pw(0)|\right) \]
\[ \leq [12] \ K \left(| - Qu(0) + h(Pu(0), \omega)| \right. \]
\[ + \left. \text{Lip}h \int_0^t e^{-As + \int_s^t z(\theta, \omega) d\theta} P\tilde{F}(\theta_s \omega, w(s)) ds \right) \]
\[ K (| - Qu(0) + h(Pu(0), \omega)| + KLiph \cdot LipF \frac{1}{\alpha - \eta} |w|_{C^0_\eta}) \]

and
\[ e^{-\eta t - f_0^0 z(\theta, \omega)dr} |Tw(t)| \leq Ke^{-\eta t - f_0^0 z(\theta, \omega)dr} \left( \int_0^t e^{\beta(t-s)+f_0^0 z(\theta, \omega)dr} |\tilde{F}(\theta, \omega, w(s))|ds \right) \]
\[ + \left| \int_0^t e^{\alpha(t-s)+f_0^0 z(\theta, \omega)dr} \tilde{F}(\theta, \omega, w(s))ds \right| \]
\[ \leq KLiphe^{-\eta t - f_0^0 z(\theta, \omega)dr} \left( \int_0^t e^{\beta(t-s)+f_0^0 z(\theta, \omega)dr} |w(s)|ds \right) \]
\[ + \int_t^\infty e^{\alpha(t-s)+f_0^0 z(\theta, \omega)dr} |w(s)|ds \]
\[ \leq KLipF \left( \int_0^t e^{-(\eta-\beta)(t-s)} ds + \int_t^\infty e^{(\alpha-\eta)(t-s)} ds \right) |w|_{C^0_\eta} \]
\[ \leq KLipF \left( \frac{1}{\eta - \beta} + \frac{1}{1 - \eta} \right) |w|_{C^0_\eta}. \]

Next we show that under the spectral gap condition (9), the map \( \tilde{T} + T : C^0_\eta \to C^0_\eta \) is contractive. To this end, assume \( w, \tilde{w} \in C^0_\eta \), then we have
\[ e^{-\eta t - f_0^0 z(\theta, \omega)dr} |Tw(t) - \tilde{Tw}(t)| \leq KLiphe^{-\eta t - f_0^0 z(\theta, \omega)dr} |h(Pu(0) + Pu(0), \omega) - h(Pu(0) + Pu(0), \omega)| \]
\[ \leq KLiph |Pw(0) - P\tilde{w}(0)| \]
\[ \leq KLiph \left| \int_0^\infty e^{-As+f_0^0 z(\theta, \omega)dr} PLipF|w(s) - \tilde{w}(s)|ds \right| \]
\[ \leq K^2LipF \int_0^\infty e^{-(\alpha-\eta)s} |w(s) - \tilde{w}(s)|_{C^0_\eta} ds \]
\[ \leq K^2LipF \frac{1}{\alpha - \eta} |w - \tilde{w}|_{C^0_\eta}. \]

and
\[ e^{-\eta t - f_0^0 z(\theta, \omega)dr} |Tw(t) - T\tilde{w}(t)| \]
\[ \leq e^{-\eta t - f_0^0 z(\theta, \omega)dr} \left( \int_0^t e^{A(t-s)+f_0^0 z(\theta, \omega)dr} Q|\tilde{F}(\theta, \omega, w(s)) - \tilde{F}(\theta, \omega, \tilde{w}(s))|ds \right) \]
\[ + \left| \int_0^t e^{A(t-s)+f_0^0 z(\theta, \omega)dr} P|\tilde{F}(\theta, \omega, w(s)) - \tilde{F}(\theta, \omega, \tilde{w}(s))|ds \right| \]
\[ \leq K \left( \int_0^t e^{\beta(t-s)-\eta t + f_0^0 z(\theta, \omega)dr} LipF|w(s) - \tilde{w}(s)|ds \right) \]
\[ + \left| \int_0^t e^{\alpha(t-s)-\eta t + f_0^0 z(\theta, \omega)dr} LipF|w(s) - \tilde{w}(s)|ds \right| \]
\[ \leq KLipF \left( \int_0^t e^{-(\eta-\beta)(t-s)} |w(s) - \tilde{w}(s)|_{C^0_\eta} ds + \int_t^\infty e^{(\alpha-\eta)(t-s)} |w(s) - \tilde{w}(s)|_{C^0_\eta} ds \right) \]
\[ \leq K \text{Lip} F \left( \frac{1}{\eta - \beta} + \frac{1}{\alpha - \eta} \right) |w - \bar{w}|_{C^+_{\eta}}. \]

That is
\[
|\tilde{T}w - \tilde{T}\bar{w}|_{C^+_{\eta}} \leq K^2 \text{Lip} h \cdot \text{Lip} F \frac{1}{\alpha - \eta} |w - \bar{w}|_{C^+_{\eta}},
\]
\[
|T w - T \bar{w}|_{C^+_{\eta}} \leq K \text{Lip} F \left( \frac{1}{\eta - \beta} + \frac{1}{\alpha - \eta} \right) |w - \bar{w}|_{C^+_{\eta}}.
\]

Therefore,
\[
|(\tilde{T} + T) w - (\tilde{T} + T) \bar{w}|_{C^+_{\eta}} \leq \left[ K^2 \text{Lip} h \cdot \text{Lip} F \frac{1}{\alpha - \eta} + K \text{Lip} F \left( \frac{1}{\eta - \beta} + \frac{1}{\alpha - \eta} \right) \right] |w - \bar{w}|_{C^+_{\eta}}.
\]

Then by the spectral gap condition (9) we obtain that \( \tilde{T} + T \) has a unique fixed point \( w^* \) on \( C^+_{\eta} \), which satisfies
\[
\bar{u}(0) = u(0) + w^*(0) \in M(\omega)
\]
as desired. Hence
\[
|\tilde{u}(t, \omega, \bar{u}_0) - u(t, \omega, u_0)| \leq e^{\eta t} + \int_0^t z(\theta, \omega) d\theta |\bar{u}_0 - u_0|
\]
\[
\leq c(\omega) e^{\eta t} |\bar{u}_0 - u_0|, \quad t \geq 0
\]
for some \( c(\omega) > 0 \) by the property of \( z(\omega) \).

\[ \square \]

### 3 Stochastic IM for wave equations

Consider the following wave equation in \([0, \pi]\) perturbed by additive white noise:
\[
\varepsilon^2 du_t + (u_t - \Delta u) dt = f(u) dt + \delta \phi dW
\]
with
\[
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u(t, 0) = u(t, \pi) = 0,
\]
where \( \phi \in H^1_0(0, \pi) \), \( u_t := \frac{du}{dt} \). We assume that the nonlinear term \( f \) is globally Lipschitz continuous on \( L^2(0, \pi) \) with Lipschitz constant \( \text{Lip} f \).

Rewrite (14) as
\[
\begin{cases}
    u_t = v, \\
    \varepsilon^2 v_t + v + \tilde{A}u = f(u) + \delta \phi \frac{dW}{dt},
\end{cases}
\]
where \( \tilde{A}u := -\Delta u \), and \((u, v) \in E := H^1_0(0, \pi) \times L^2(0, \pi) \). Let \( \bar{u} = u, \bar{v} = v - \delta \phi z \).

Here \( z \) satisfies
\[
\varepsilon^2 dz + z dt = dW.
\]

Let \( U = (\bar{u}, \bar{v}) \in E \), then \( U \) satisfies
\[
\dot{U} = AU + F(\theta_t \omega, U),
\]
(16)
where
\[ A := \begin{pmatrix} 0 & \text{id}_{L^2} \\ -\epsilon^{-2} \tilde{A} & -\epsilon^{-2} \text{id}_{L^2} \end{pmatrix}, \quad F(\omega, U) := \begin{pmatrix} \delta \phi z \\ \epsilon^{-2} f(\bar{u}) \end{pmatrix}. \] (17)

Noting that (16) is a particular form of (4) with \( z = 0 \). It is easy to verify that \( A \) is the infinitesimal generator of a \( C^0 \)-semigroup \( e^{At} \) on Hilbert space \( E \). Since \( F \) is Lipschitz continuous with respect to \( U \) (see (25)), by the classical semigroup theory concerning the local existence and uniqueness of the solutions of evolution differential equations in [30], we obtain the existence and uniqueness of (16) and hence (14).

Since the eigenvalues of \( \tilde{A} \) are \( \tilde{\lambda}_k = k^2 \) with corresponding eigenvectors \( \tilde{e}_k = \sin kx, k = 1, 2, \cdots \), the eigenvalues of the operator \( A \) are

\[ \lambda_k^\pm = \frac{-1 \pm \sqrt{1 - 4\epsilon^2 k^2}}{2\epsilon^2} \]

with corresponding eigenvectors

\[ e_k^\pm = \begin{pmatrix} \sin kx \\ \lambda_k^\pm \sin kx \end{pmatrix}, \quad k = 1, 2, \cdots. \]

It is clear that

\[ \lambda_k^+ \to -k^2 \text{ as } \epsilon \to 0. \] (18)

Denote

\[ E_1 := \text{Span}\{e_k^+ | 1 \leq k \leq N\}, \quad E_{-1} := \text{Span}\{e_k^- | 1 \leq k \leq N\}, \]

\[ E_{11} := E_1 \oplus E_{-1}, \quad E_{22} := \text{Span}\{e_k^+ | k \geq N + 1\}, \quad E_2 = E_{-1} \oplus E_{22}. \]

By the orthogonality of \( \sin kx \), we have

\[ E_1 \bot E_{22}, \quad E_{-1} \bot E_{22}, \]

while \( E_1 \) is not orthogonal to \( E_{-1} \).

Following [28], we define an equivalent new inner product on \( E \). In this section, we use \( \langle \cdot, \cdot \rangle, ||\cdot|| \) to denote the usual inner product and norm on \( L^2(0, \pi) \), respectively. Let \( U_1 = (u_1, v_1), U_2 = (u_2, v_2) \) are two vectors in \( E \) or \( E_{11}, E_{22} \). Recalling that the usual inner product on \( E \) defined by

\[ \langle U_1, U_2 \rangle = \langle u_1, u_2 \rangle + \langle \tilde{A}_1^{\frac{1}{2}} u_1, \tilde{A}_1^{\frac{1}{2}} u_2 \rangle + \langle v_1, v_2 \rangle. \]

Assume \( \frac{1}{2\epsilon} > N + 1 \), define the new inner product as follows:

\[ \langle U_1, U_2 \rangle_{E_{11}} := \frac{1}{4\epsilon^2} \langle u_1, u_2 \rangle - \langle \tilde{A}_1^{\frac{1}{2}} u_1, \tilde{A}_1^{\frac{1}{2}} u_2 \rangle + \left( \frac{1}{2\epsilon} u_1 + \epsilon v_1, \frac{1}{2\epsilon} u_2 + \epsilon v_2 \right), \]

\[ \langle U_1, U_2 \rangle_{E_{22}} := \langle \tilde{A}_1^{\frac{1}{2}} u_1, \tilde{A}_1^{\frac{1}{2}} u_2 \rangle + \left( \frac{1}{4\epsilon^2} - 2(N + 1)^2 \right) \langle u_1, u_2 \rangle + \left( \frac{1}{2\epsilon} u_1 + \epsilon v_1, \frac{1}{2\epsilon} u_2 + \epsilon v_2 \right). \]
For $U = U_{11} + U_{22}$, $V = V_{11} + V_{22}$, define

$$\langle U, V \rangle_E := \langle U_{11}, V_{11} \rangle_{E_{11}} + \langle U_{22}, V_{22} \rangle_{E_{22}}.$$ 

Since $\frac{1}{2\epsilon} > N + 1$, it is clear that $\langle \cdot, \cdot \rangle_{E_{11}}$ is equivalent to the usual inner product on $E_{11}$, and $\langle \cdot, \cdot \rangle_{E_{22}}$ is equivalent to the usual inner product on $E_{22}$. Hence the new inner product $\langle \cdot, \cdot \rangle_E$ is equivalent to the usual product on $E$, see [28] for details.

By the definition of new inner product, it is clear that for $U = (u, v)$ with $u = 0$ we have

$$\|U\|_E = \epsilon \|v\|,$$  \hspace{1cm} (19)

and for any $U = (u, v) \in E$ we have

$$\|U\|_E \geq \sqrt{\frac{1}{4\epsilon^2} - (N + 1)^2 \|u\|}.$$  \hspace{1cm} (20)

Under the new inner product $\langle \cdot, \cdot \rangle_E$, by the orthogonality of $\sin kx$ it is easy to verify that we have

$$E_1 \perp E_{22}, \ E_{-1} \perp E_{22}.$$ 

Moreover, we have $E_1 \perp E_{-1}$ and hence $E_1 \perp E_2$. In fact, by the definition of $\langle \cdot, \cdot \rangle_{E_{11}}$ it follows that

$$\begin{cases} 
\langle e^+_k, e^-_l \rangle_{E_{11}} = 0, \text{ when } 1 \leq k, l \leq N, \ k \neq l, \\
\langle e^+_k, e^-_k \rangle_{E_{11}} = \frac{1}{4\epsilon^2} - k^2 + \left( \frac{1}{2\epsilon} + \epsilon \lambda^+_k \right) \left( \frac{1}{2\epsilon} + \epsilon \lambda^-_k \right) = 0, \text{ for } 1 \leq k \leq N, 
\end{cases}$$

which verifies $E_1 \perp E_{-1}$.

We use $A_1, A_2, A_{-1}, A_{22}$ to denote $A|_{E_1}, A|_{E_2}, A|_{E_{-1}}, A|_{E_{22}}$, respectively. Then similar to [28], we have

$$\|e^{A_1t}\| = e^{\lambda^+_N t}, \text{ for } t \leq 0,$$  \hspace{1cm} (21)

$$\|e^{A_{-1}t}\| = e^{\lambda^-_N t}, \text{ for } t \geq 0,$$  \hspace{1cm} (22)

$$\|e^{A_{22}t}\| = e^{\lambda^+_N t}, \text{ for } t \geq 0,$$  \hspace{1cm} (23)

where $\| \cdot \|$ denotes the operator norm in Hilbert space $(E, \langle \cdot, \cdot \rangle_E)$. By (22), (23) we have

$$\|e^{A_{22}t}\| = e^{\lambda^+_N t}, \text{ for } t \geq 0.$$  \hspace{1cm} (24)

Next we show that $F$ is Lipschitz with respect to $U$ under the norm $\| \cdot \|_E$ and the Lipschitz constant is independent of $\epsilon$ when $\epsilon$ is small. In fact,

$$\|F(\omega, U_1) - F(\omega, U_2)\|_E \leq \|e^{-2} \left( \begin{array}{c} 0 \\
\begin{array}{c}
\frac{1}{2}\epsilon f(u_1) - f(u_2) \\
\end{array}
\end{array} \right)\|_E$$

$$\leq \|e^{-1}f\|_E \|u_1 - u_2\|$$

$$\leq e^{-1} \|u_1 - u_2\|.$$
\[ \leq 20 \epsilon^{-1} \frac{\text{Lip} f}{\sqrt{1 - (N+1)^2}} \|U_1 - U_2\|_E \]
\[ \leq \frac{\text{Lip} f}{\sqrt{\frac{1}{4} - \epsilon^2(N+1)^2}} \|U_1 - U_2\|_E \]
\[ \leq 3 \text{Lip} f \|U_1 - U_2\|_E, \quad (25) \]

where the last “=” holds when \( \epsilon \) is appropriately small.

**Theorem 3.1** Consider stochastic wave equation \((14)\). There exists some \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \), the equation \((14)\) has a stochastic IM.

**Proof.** Consider \((16)\) and let \( H = (E, \langle \cdot, \cdot \rangle_E), A \) be as in \((17)\), \( \alpha = \lambda_N^+, \beta = \lambda_{N+1}^+ \) and \( \eta = \frac{\alpha + \beta}{2} \). By \((21)\) and \((24)\), the pseudo exponent dichotomy condition \((5)\) holds with \( PH = E_1, QH = E_2 \) and \( K = 1 \). According to \((18)\) and \((23)\), there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \) the spectral gap condition \((9)\) holds when \( N \) is appropriately large. Hence Theorem 2.2 holds for \((16)\), i.e. there exists a stochastic IM \( M(\omega) \) for \((16)\).

For \( U \in E \), define the transform
\[ T(\omega, U) = U - (0, \delta \phi z), \quad T^{-1}(\omega, U) = U + (0, \delta \phi z). \]
If \((t, \omega, U_0) \rightarrow \varphi(t, \omega, U_0)\) is the RDS generated by \((16)\), then it is easy to verify that
\[ (t, \omega, U_0) \rightarrow \tilde{\varphi} := T^{-1}(\theta t \omega, \varphi(t, \omega, T(\omega, U_0))) \]
is the RDS generated by \((14)\).

Let
\[ \tilde{M}(\omega) := T^{-1}(\omega, M(\omega)) = \{ \xi + h(\xi, \omega) + (0, \delta \phi z) | \xi \in PE \}, \]
then \( \tilde{M}(\omega) \) is a stochastic IM for \((14)\). In fact,
\[ \tilde{\varphi}(t, \omega, \tilde{M}(\omega)) = T^{-1}(\theta t \omega, \varphi(t, \omega, T(\omega, \tilde{M}(\omega)))) \]
\[ = T^{-1}(\theta t \omega, \varphi(t, \omega, M(\omega))) \]
\[ \subset T^{-1}(\theta t \omega, M(\theta t \omega)) = \tilde{M}(\theta t \omega), \]
i.e. \( \tilde{M}(\omega) \) is an invariant manifold for \((14)\).

Assume \( \tilde{U}_1 \) is a solution of \((14)\), then it is easy to verify that
\[ U_1 := T(\theta t \omega, \tilde{U}_1(t, \omega, T^{-1}(\omega, \tilde{U}_1(0)))) \]
is a solution of \((16)\). By the asymptotic complete property of \( M(\omega) \), there exists a solution \( U_2 \) of \((16)\) lying on \( M(\omega) \) such that
\[ \| \varphi(t, \omega, U_1(0)) - \varphi(t, \omega, U_2(0)) \|_E \leq c(\omega)e^{\eta t} \| U_1(0) - U_2(0) \|_E, \quad \forall t \geq 0. \]
Let $\tilde{U}_2 := T^{-1}(\theta_t \omega, U_2(t, \omega, U_2(0)))$, then it is easy to verify that $\tilde{U}_2$ is a solution of (14) and $\tilde{U}_2$ lies on $\bar{M}(\omega)$. Furthermore,

$$
\|\bar{\varphi}(t, \omega, \tilde{U}_1(0)) - \tilde{\varphi}(t, \omega, \tilde{U}_2(0))\|_E = \|\varphi(t, \omega, U_1(0)) - \varphi(t, \omega, U_2(0))\|_E
\leq c(\omega)e^{\eta t}\|U_1(0) - U_2(0)\|_E \\
\leq c(\omega)e^{\eta t}\|\tilde{U}_1(0) - \tilde{U}_2(0)\|_E, \ \forall t \geq 0.
$$

Therefore, $\bar{M}(\omega)$ has asymptotic completeness property and hence it is a stochastic IM for (14). The proof is complete.

\[\square\]

**Remark 3.1** In above theorem we obtain the existence of stochastic IM when $\epsilon$ is small. In fact, when $\epsilon$ is large, counterexample has shown that the attractor of (14) in the deterministic case (i.e. $\delta = 0$) is not contained in any finite dimensional manifold, see [29] for details. It seems that the corresponding result holds for stochastic case, i.e. we would not obtain the existence of stochastic IM for (14) when $\epsilon$ is large.

Denote $C_{\eta}^{-} := \{ \varphi : (-\infty, 0] \to E | \varphi \text{ continuous}, \sup_{t \leq 0} e^{-\eta t}\|\varphi(t)\|_E < \infty \}$,

then $C_{\eta}^{-}$ is a Banach space with norm $\|\varphi\|_{E,C_{\eta}^{-}} := \sup_{t \leq 0} e^{-\eta t}\|\varphi(t)\|_E$. Assume $R > 0$ and $M_{\delta}(\omega)$ is a stochastic IM of (14). Let

$$
M_R^{\delta}(\omega) := \{ \xi + h(\xi, \omega) | \xi \in PE, \|\xi\|_E \leq R \},
$$

where the graph of $h$ gives the IM $M_{\delta}(\omega)$. The following theorem states that the stochastic IM of (14) converges to its deterministic counterpart almost surely when the intensity of noise tends to zero.

**Theorem 3.2** Assume $M_{\delta}(\omega)$ is a stochastic IM of (14) and $M_0$ is the IM of (14) when $\delta = 0$ with the same dimension as that of $M_{\delta}(\omega)$, then, for any $R > 0$, we have

$$
\lim_{\delta \to 0} \sup_{U \in M_R^{\delta}(\omega)} \inf_{V \in M_0} \|U - V\|_E = 0
$$

almost surely.

**Proof.** Assume $\bar{u}, u$ satisfy

$$
\epsilon^2 d\bar{u}_t + (\bar{u}_t + \tilde{A}\bar{u})dt = f(\bar{u})dt + \delta \phi dW
$$

and

$$
\epsilon^2 du_t + (u_t + \tilde{A}u)dt = f(u)dt,
$$

respectively. We also assume that $(\bar{u}, \bar{u}_t), (u, u_t)$ lie on $M_{\delta}(\omega), M_0$, respectively. Let $w = \bar{u} - u$, then $w$ satisfies

$$
\epsilon^2 w_{tt} + w_t + \tilde{A}w = f(u + w) - f(u) + \delta \phi \frac{dW}{dt}, \quad (26)
$$

12
Let $\bar{W} = (w, w_t - \delta \phi z)$, where $z$ satisfies $\epsilon^2 dz + z dt = dW$, then $\bar{W}$ satisfies

$$\dot{\bar{W}} = A\bar{W} + F(\theta_\omega, \bar{W}),$$

where

$$A = \begin{pmatrix} 0 & \text{id}_{L^2} \\ -\epsilon^{-2} \tilde{A} & -\epsilon^{-2} \text{id}_{L^2} \end{pmatrix}, \quad F = \begin{pmatrix} \delta \phi z \\ \epsilon^{-2}[f(u + w) - f(u)] \end{pmatrix}.$$ 

It is clear that the form of (27) is the same as that of (16) except that the nonlinear term $F$ is not the same. But it is easy to verify that the nonlinear term $F$ in (27) is globally Lipschitz continuous with respect to $\bar{W}$, so (27) has a stochastic IM and by similar argument to that of Theorem 2.2 (see also (27) in [16]) we have $\bar{W} \in C^{-}_\eta$ and $\bar{W}$ satisfies

$$\bar{W}(t) = e^{At} P\bar{W}(0) + \int_0^t e^{A(t-s)} PF(\theta_\omega, \bar{W}(s)) ds$$

$$+ \int_{-\infty}^t e^{A(t-s)} QF(\theta_\omega, \bar{W}(s)) ds.$$ 

Since $PE = E_1$ is of finite dimension, we can choose $(u(0), u_t(0))$ such that $P\bar{W}(0) = P(\bar{u}(0) - u(0), \bar{u}_t(0) - u_t(0) - \delta \phi z) = 0$. Therefore,

$$e^{-\eta t}\|\bar{W}(t)\|_E \leq e^{-\eta t} \int_0^t e^{\lambda_N^+(t-s)} \left( \begin{array}{c} \delta \phi z(\theta_\omega) \\ \epsilon^{-2}[f(u + w) - f(u)] \end{array} \right) \|_{E_2} ds$$

$$+ e^{-\eta t} \int_{-\infty}^t e^{\lambda_{N+1}^+(t-s)} \left( \begin{array}{c} \delta \phi z(\theta_\omega) \\ \epsilon^{-2}[f(u + w) - f(u)] \end{array} \right) \|_{E_2} ds$$

$$\leq e^{-\eta t} \int_0^t e^{\lambda_N^+}(t-s) \left( \begin{array}{c} \delta \phi z(\theta_\omega) \\ \epsilon^{-2}[f(u + w) - f(u)] \end{array} \right) \|_{E_2} ds$$

$$+ e^{-\eta t} \int_{-\infty}^t e^{\lambda_{N+1}^+}(t-s) \left( \begin{array}{c} \delta \phi z(\theta_\omega) \\ \epsilon^{-2}[f(u + w) - f(u)] \end{array} \right) \|_{E_2} ds$$

$$\leq 3\text{Lip} f \int_0^t e^{(\lambda_N^+ - \eta)(t-s)} \left( \begin{array}{c} w \\ 0 \end{array} \right) \|_{E, C^{-}_\eta} ds$$

$$+ 3\text{Lip} f \int_{-\infty}^t e^{(\lambda_{N+1}^+ - \eta)(t-s)} \left( \begin{array}{c} w \\ 0 \end{array} \right) \|_{E, C^{-}_\eta} ds$$

$$+ c_1(\omega) \left( \int_0^t e^{(\lambda_N^+ - \eta)(t-s)} ds + \int_{-\infty}^t e^{(\lambda_{N+1}^+ - \eta)(t-s)} ds \right) \left( \delta \phi \right) \|_E$$

$$\leq 3\text{Lip} f \left( \begin{array}{c} w \\ 0 \end{array} \right) \|_{E, C^{-}_\eta} \left( \frac{1}{\lambda_N^+ - \eta} + \frac{1}{\eta - \lambda_{N+1}^+} \right)$$
Returning back to (26), we let
\[\hat{W} + c_1(\omega) \left( \begin{pmatrix} \delta \phi \\ 0 \end{pmatrix} \right) \leq 3\text{Lip}f \|\hat{W}\|_{E,C_{\eta}} \left( \begin{pmatrix} \frac{1}{\lambda_N^+ - \eta} + \frac{1}{\eta - \lambda_{N+1}^+} \end{pmatrix} \right)\]
\[+ c_1(\omega) \left( \begin{pmatrix} \delta \phi \\ 0 \end{pmatrix} \right) \leq 3\text{Lip}f \|\hat{W}\|_{E,C_{\eta}} \left( \begin{pmatrix} \frac{1}{\lambda_N^+ - \eta} + \frac{1}{\eta - \lambda_{N+1}^+} \end{pmatrix} \right),\]
where the third \(\leq\) holds for some \(c_1(\omega)\) due to (25) and the sublinear growth of \(z(\theta_s\omega)\) with respect to \(s\). Hence we have
\[\left[ 1 - 3\text{Lip}f \left( \frac{1}{\lambda_N^+ - \eta} + \frac{1}{\eta - \lambda_{N+1}^+} \right) \right] \|\hat{W}\|_{E,C_{\eta}} \leq c_1(\omega) \left( \frac{1}{\lambda_N^+ - \eta} + \frac{1}{\eta - \lambda_{N+1}^+} \right) \left( \begin{pmatrix} \delta \phi \\ 0 \end{pmatrix} \right) \|_{E}.
\]
When \(N\) is appropriately large and \(\epsilon\) is appropriately small we have
\[3\text{Lip}f \left( \frac{1}{\lambda_N^+ - \eta} + \frac{1}{\eta - \lambda_{N+1}^+} \right) \leq \frac{1}{2}, \left( \frac{1}{\lambda_N^+ - \eta} + \frac{1}{\eta - \lambda_{N+1}^+} \right) \leq 1,
\]
which implies that
\[\|\hat{W}\|_{E,C_{\eta}} \leq 2c_1(\omega) \left( \begin{pmatrix} \delta \phi \\ 0 \end{pmatrix} \right) \|_{E}.
\]
Returning back to (26), we let \(\hat{W} = W + (0, \delta z) = (w, w_t)\), then
\[\|\hat{W}\|_{E,C_{\eta}} \leq \|\hat{W}\|_{E,C_{\eta}} + \|\left( \begin{pmatrix} 0 \\ \delta z \end{pmatrix} \right)\|_{E,C_{\eta}} \leq \|\hat{W}\|_{E,C_{\eta}} + \left( \begin{pmatrix} 0 \\ \delta \phi \end{pmatrix} \right) \|_{E, \sup_{t \leq 0} e^{-\eta t} |z(\theta_t)\|} \leq 2c_1(\omega) \left( \begin{pmatrix} \delta \phi \\ 0 \end{pmatrix} \right) \|_{E} + c_2(\omega) \left( \begin{pmatrix} 0 \\ \delta \phi \end{pmatrix} \right) \|_{E} \leq 2\bar{c}_1(\omega) \left( \begin{pmatrix} \phi \\ 0 \end{pmatrix} \right) \|_{E} + \delta c_2(\omega) \left( \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) \|_{E}\]
for some \(c_2(\omega)\). Thus
\[\|\hat{W}(0)\|_{E} \leq \|\hat{W}\|_{E,C_{\eta}} \leq \delta c_3(\omega),
\]
where
\[c_3(\omega) := 2c_1(\omega) \left( \begin{pmatrix} \phi \\ 0 \end{pmatrix} \right) \|_{E} + c_2(\omega) \left( \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) \|_{E}.
\]
The proof is complete.

\[\square\]

**Acknowledgement**

I am most indebted to my advisor, Professor Yong Li, not only for his direct helpful suggestions but primarily for his continual instruction, encouragement and support over all these years.
Appendix

It is well-known that, for deterministic evolution equations, the inertial manifolds contain the corresponding global attractors when they both exist. Like deterministic case, we have the same result for stochastic evolution equations: stochastic IM contains the corresponding random attractor when they both exist. Here we give a simple proof of this result.

First let us recall the definition of (global) random attractor.

**Definition 3.1** ([12]) Assume $\varphi$ is an RDS on a Polish space $X$, then a random compact set $A(\omega)$ is called a (global) random attractor for the RDS $\varphi$ if

- $A(\omega)$ is invariant, i.e.
  $$\varphi(t, \omega, A(\omega)) = A(\theta_t\omega), \forall t \geq 0 \quad (28)$$
  for almost all $\omega \in \Omega$;

- $A(\omega)$ pull-back attracts every bounded deterministic set, i.e. for any bounded deterministic set $B \subset X$, we have
  $$\lim_{t \to \infty} d(\varphi(t, \theta_{-t}\omega, B), A(\omega)) = 0 \quad (29)$$
  almost surely.

In (29), $d(D_1, D_2)$ denotes the Hausdorff semi-metric between $D_1$ and $D_2$, i.e.

$$d(D_1, D_2) := \sup_{x \in D_1} \inf_{y \in D_2} d_X(x, y)$$

for any two closed sets $D_1, D_2$ in $X$.

The global random attractor for RDS $\varphi$ is the *minimal* random closed set which attracts all the bounded deterministic sets and it is the *largest* random compact set which is invariant in the sense of (28), see [11] for details. The random attractor defined above is unique and it is uniquely determined by attracting deterministic compact sets, see [10] for details.

**Theorem 3.3** Assume an SPDE has a stochastic IM $M(\omega)$ and a random attractor $A(\omega)$. Then we have $A(\omega) \subset M(\omega)$ almost surely.

**Proof.** If the assertion is false, then

$$P\{\omega | A(\omega) \not\subset M(\omega)\} > 0.$$  

Let $\tilde{A}(\omega) = A(\omega) \cap M(\omega)$. Since $A(\omega)$ is “minimal”, there exists a deterministic compact set $D$ and $\epsilon_1, \epsilon_2 > 0$ such that

$$P\{\omega | \lim_{t \to \infty} d(\varphi(t, \theta_{-t}\omega, D), \tilde{A}(\omega)) \geq \epsilon_1 \} = \epsilon_2 > 0. \quad (30)$$
Since $A(\omega)$ is the random attractor, we have

$$\mathbb{P}\{\omega\mid \lim_{t \to \infty} d(\phi(t, \theta_{-t}\omega, D), A(\omega)) = 0\} = 1. \quad (31)$$

On the other hand we have

$$\mathbb{P}\{\omega\mid \lim_{t \to \infty} d(\phi(t, \theta_{-t}\omega, D), M(\omega)) = 0\} = \mathbb{P}\{\omega\mid \lim_{t \to \infty} d(\phi(t, \omega, D), M(\theta_{t}\omega)) = 0\} = 1 \quad (32)$$

by the measure preserving of $\{\theta_{t}\}_{t \in \mathbb{R}}$ and the fact that $M(\omega)$ is a stochastic IM for $\phi$. According to (31), (32) and the definition of Hausdorff semi-metric, we have

$$\mathbb{P}\{\omega\mid \lim_{t \to \infty} \phi(t, \theta_{-t}\omega, D) \subset A(\omega) \cap M(\omega) = \tilde{A}(\omega)\} = 1$$

a contradiction to (30). The proof is complete. $\square$

References

[1] L. Arnold, Random Dynamical Systems, Springer-Verlag, Berlin Heidelberg New York, 1998.

[2] A. Bensoussan, F. Flandoli, Stochastic inertial manifold, Stochast. Stoch. Rep. 53 (1995) 13-39.

[3] T. Caraballo, P.E. Kloeden, B. Schmalfuss, Exponentially stable stationary solutions for stochastic evolution equations and their perturbation, Appl. Math. Optim. 50 (2004) 183-207.

[4] S.N. Chow, K. Lu, Invariant manifolds for flows in Banach spaces, J. Diff. Eqs. 74 (1988) 285-317.

[5] S.N. Chow, K. Lu, G.R. Sell, Smoothness of inertial manifolds, J. Math. Anal. Appl. 169 (1992) 283-312.

[6] I.D. Chueshov, T.V. Girya, Inertial manifolds for stochastic dissipative dynamical systems, Doklady Acad. Sci. Ukraine 7 (1994) 42-45.

[7] I.D. Chueshov, M. Scheutzow, Inertial manifolds and forms for stochastically perturbed retarded semilinear parabolic equations, J. Dynam. Differential Equations 13 (2001) 355-380.

[8] P. Constantin, C. Foias, B. Nicolaenko, R. Temam, Nouveaux r´esultats sur les vari´et´es iner-

tielles pour les ´equations diff´erentielles dissipatives (New results on the inertial manifolds for dissipative differential equations), C. R. Acad. Sci., Paris, Sér. I. 302 (1986) 375-378.

[9] P. Constantin, C. Foias, B. Nicolaenko, R. Temam, Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations, Applied Mathematical Sciences, 70. Springer-Verlag, New York, 1989.

[10] H. Crauel, Global random attractors are uniquely determined by attracting deterministic compact sets, Ann. Mat. Pura Appl. 176 (1999) 57-72.

[11] H. Crauel, A. Debussche, F. Flandoli, Random attractors, J. Dynam. Differential Equations 9 (1997) 307-341.

[12] H. Crauel, F. Flandoli, Attractors for random dynamical systems, Prob. Theory Rel. Fields 100 (1994) 365-393.
[13] G. Da Prato, A. Debussche, Construction of stochastic inertial manifolds using backward integration, Stochast. Stoch. Rep. 59 (1996) 305-324.

[14] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.

[15] J. Duan, K. Lu, B. Schmalfuss, Invariant manifolds for stochastic partial differential equations, Ann. Probab. 31 (2003) 2109-2135.

[16] J. Duan, K. Lu, B. Schmalfuss, Smooth stable and unstable manifolds for stochastic evolutionary equations, J. Dynam. Differential Equations 16 (2004) 949-972.

[17] C. Foias, B. Nicolaenko, G.R. Sell, R. Temam, Variétés inertielle pour équation de Kuramoto-Sivashinsky (Inertial manifolds for the Kuramoto-Sivashinsky equation), C. R. Acad. Sci., Paris, Sr. I 301 (1985) 285-288.

[18] C. Foias, B. Nicolaenko, G.R. Sell, R. Temam, Inertial manifolds for the Kuramoto-Sivashinsky equation and an estimate of their lowest dimension. J. Math. Pures Appl. 67 (1988) 197-226.

[19] C. Foias, G.R. Sell, R. Temam, Variétés inertielle des équations différentielles dissipatives (Inertial manifolds for dissipative differential equations), C. R. Acad. Sci., Paris, Sér. I 301 (1985) 139-141.

[20] C. Foias, G.R. Sell, R. Temam, Inertial manifolds for non-linear evolutionary equations, J. Differential Equations 73 (1988) 309-353.

[21] C. Foias, G.R. Sell, E.S. Titi, Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations, J. Dynam. Differential Equations 1 (1989) 199-244.

[22] J. Hadamard, Sur l’itération et les solutions asymptotiques des équations différentielles, Bull. Soc. Math. France 29 (1901) 224-228.

[23] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, Vol. 840, Springer, Berlin, 1981.

[24] M.A. Liapounoff, Problème général de la stabilité du mouvement, Ann. Fac. Sci. Toulouse 9, 1907. [Translation of the Russian edition, Kharkov 1892, reprinted by Princeton University Press, Princeton, NJ, 1949 and 1952.]

[25] J. Mallet-Paret, G.R. Sell, Inertial manifolds for reaction diffusion equations in higher space dimensions, J. Amer. Math. Soc. 1 (1988) 804-866.

[26] R. Mañé, Reduction of semilinear parabolic equations to finite dimensional $C^1$-flows. Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), pp. 361-378. Lecture Notes in Math., Vol. 597, Springer, Berlin, 1977.

[27] X. Mora, Finite-dimensional attracting manifolds in reaction-diffusion equations, Nonlinear partial differential equations (Durham, N.H., 1982), 353-360, Contemp. Math., 17, Amer. Math. Soc., Providence, R.I., 1983.

[28] X. Mora, Finite-dimensional attracting invariant manifolds for damped semilinear wave equations, Contributions to nonlinear partial differential equations, Vol. II (Paris, 1985), 172-183, Pitman Res. Notes Math. Ser., 155, Longman Sci. Tech., Harlow, 1987.

[29] X. Mora, J. Solá-Morales, Existence and nonexistence of finite-dimensional globally attracting invariant manifolds in semilinear damped wave equations, Dynamics of infinite-dimensional systems (Lisbon, 1986), 187-210, NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., 37, Springer, Berlin, 1987.

[30] A. Pazy, Semigroups of Linear Operators and Application to Partial Differential Equations, Springer, New York, 1983.
[31] O. Perron, Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen, Math. Z. 29 (1928) 129-160.

[32] J.C. Robinson, The asymptotic completeness of inertial manifolds, Nonlinearity 9 (1996) 1325-1340.

[33] G.R. Sell, Y. You, Inertial manifolds: the non-self-adjoint case, J. Differential Equations 96 (1992) 203-255.

[34] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd Ed., Applied Mathematical Sciences, Vol. 68, Springer, New York, 1997.