Cyclic Codes from Dickson Polynomials

Cunsheng Ding

Abstract Due to their efficient encoding and decoding algorithms, cyclic codes, a subclass of linear codes, have applications in communication systems, consumer electronics, and data storage systems. In this paper, Dickson polynomials of the first kind over finite fields are employed to construct a number of classes of cyclic codes. Lower bounds on the minimum weight of some classes of the cyclic codes are developed. The minimum weights of some other classes of the codes constructed in this paper are determined. The dimensions of the codes obtained in this paper are flexible. Many of the codes presented in this paper are optimal or almost optimal.

Keywords Dickson polynomial · cyclic code · linear code · linear span · sequence

1 Introduction

Let \( q \) be a power of a prime \( p \). An \([n,k,d]\) linear code over GF\((q)\) is a \( k \)-dimensional subspace of GF\((q)^n\) with minimum (Hamming) nonzero weight \( d \). Let \( A_i \) denote the number of codewords with Hamming weight \( i \) in a linear code \( C \) of length \( n \). The weight enumerator of \( C \) is defined by

\[
1 + A_1z + A_2z^2 + \cdots + A_nz^n.
\]

The weight distribution of \( C \) is the sequence \((1, A_1, \ldots, A_n)\).

An \([n,k,d]\) linear code over GF\((q)\) is called optimal if there is no \([n,k,d+1]\) or \([n,k+1,d]\) linear code over GF\((q)\). The optimality of a cyclic code may be proved by a bound on linear codes or by an exhaustive computer search on all linear codes over GF\((q)\) with fixed length \( n \) and fixed dimension \( k \) or fixed length \( n \) and fixed minimum distance \( d \). An \([n,k,d]\) linear code is said to be almost optimal if a linear code with parameters \([n,k+1,d]\) or \([n,k,d+1]\) is optimal.

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A vector \((c_0, c_1, \cdots, c_{n-1}) \in \text{GF}(q)^n\) is said to be even-like if \(\sum_{i=0}^{n-1} c_i = 0\), and is odd-like otherwise. The even-like subcode of a linear code consists of all the even-like codewords of this linear code.

An \([n, k]\) linear code \(C\) over \(\text{GF}(q)\) is called cyclic if \((c_0, c_1, \cdots, c_{n-1}) \in C\) implies \((c_{n-1}, c_0, c_1, \cdots, c_{n-2}) \in C\). Let \(\gcd(n, q) = 1\). By identifying any vector \((c_0, c_1, \cdots, c_{n-1}) \in \text{GF}(q)^n\) with

\[
c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} \in \text{GF}(q)[x]/(x^n - 1),
\]

any code \(C\) of length \(n\) over \(\text{GF}(q)\) corresponds to a subset of \(\text{GF}(q)[x]/(x^n - 1)\). The linear code \(C\) is cyclic if and only if the corresponding subset in \(\text{GF}(q)[x]/(x^n - 1)\) is an ideal of the ring \(\text{GF}(q)[x]/(x^n - 1)\). It is well known that every ideal of \(\text{GF}(q)[x]/(x^n - 1)\) is principal. Let \(C = \langle g(x) \rangle\) be a cyclic code, where \(g\) is monic and has the least degree. Then \(g(x)\) is called the generator polynomial and \(b(x) = (x^n - 1)/g(x)\) is referred to as the check polynomial of \(C\).

The error correcting capability of cyclic codes may not be as good as some other linear codes in general. However, cyclic codes have wide applications in storage and communication systems because they have efficient encoding and decoding algorithms \([4, 15, 20]\).

Cyclic codes have been studied for decades and a lot of progress has been made (see, for example, \([3, 17]\) for information). The total number of cyclic codes over \(\text{GF}(q)\) and their constructions are closely related to cyclotomic cosets modulo \(n\), and thus many areas of number theory. One way of constructing cyclic codes over \(\text{GF}(q)\) with length \(n\) is to use the generator polynomial

\[
\frac{x^n - 1}{\gcd(S(x), x^n - 1)} \quad (1)
\]

where

\[
S(x) = \sum_{i=0}^{n-1} s_i x^i \in \text{GF}(q)[x]
\]

and \(s^r = (s_i)^r\) is a sequence of period \(n\) over \(\text{GF}(q)\). Throughout this paper, we call the cyclic code \(C_s\) with the generator polynomial of \((1)\) the code defined by the sequence \(s^r\), and the sequence \(s^r\) the defining sequence of the cyclic code \(C_s\). This approach was successfully employed to construct cyclic codes with interesting parameters in \([7, 8, 9, 10, 14, 21, 22]\).

In this paper, Dickson polynomials of the first kind and small degrees over finite fields will be employed to construct a number of classes of sequences and then cyclic codes. Lower bounds on the minimum weight of some classes of the cyclic codes are developed. The minimum weights of some other classes of the codes constructed in this paper are determined. The dimensions of the codes of this paper are flexible. It is amazing that most of the cyclic codes from Dickson polynomials of the first kind with small degrees are optimal or almost optimal. The major motivation of this paper is the optimality of many of these cyclic codes from Dickson polynomials. Another motivation of this study is the simplicity of the constructions of the cyclic codes in this paper.

2 Preliminaries

In this section, we present basic notations and results of Dickson polynomials, \(q\)-cyclotomic cosets, and sequences that will be employed in subsequent sections.
2.1 Some notation and symbols fixed throughout this paper

Throughout this paper, we adopt the following notation unless otherwise stated:

- $p$ is a prime, $q$ is a positive power of $p$, $m$ is a positive integer, $r = q^m$, and $n = q^m - 1$.
- $\mathbb{Z}_n = \{0, 1, \cdots, n - 1\}$, the ring of integers modulo $n$.
- $\alpha$ is a generator of $\text{GF}(r)^*$, the multiplicative group of $\text{GF}(q)$.
- $m_\alpha(x)$ is the minimal polynomial of $a \in \text{GF}(r)$ over $\text{GF}(q)$.
- $\text{Tr}(x)$ is the trace function from $\text{GF}(q)$ to $\text{GF}(q)$.
- $\delta(x)$ is a function on $\text{GF}(q)$ defined by $\delta(x) = 0$ if $\text{Tr}(x) = 0$ and $\delta(x) = 1$ otherwise.
- For any polynomial $g(x) \in \text{GF}(q)[x]$ with $g(0) \neq 0$, $\bar{g}(x)$ denotes the reciprocal of $g(x)$.
- For any code $C$ over $\text{GF}(q)$ with generator polynomial $g(x)$, $\hat{C}$ denotes the cyclic code with generator polynomial $\bar{g}(x)$. It is well known that $C$ and $\hat{C}$ have the same weight distribution.

2.2 The $q$-cyclotomic cosets modulo $n = q^m - 1$

The $q$-cyclotomic coset containing $j$ modulo $n$ is defined by

$$C_j = \{j, jq, q^2 j, \cdots, q^{\ell - 1} j\} \subset \mathbb{Z}_n$$

where $\ell_j$ is the smallest positive integer such that $q^{\ell_j} j \equiv j \pmod{n}$, and is called the size of $C_j$. It is known that $\ell_j$ divides $m$. The smallest integer in $C_j$ is called the coset leader of $C_j$. Let $\Gamma$ denote the set of all coset leaders. By definition, we have

$$\bigcup_{j \in \Gamma} C_j = \mathbb{Z}_n.$$ 

It is easily seen that $\ell_i = \ell_{i-j}$ for all $i$.

It is well known that $\prod_{j \in C}(x - \alpha^j)$ is an irreducible polynomial of degree $\ell_j$ over $\text{GF}(q)$ and is the minimal polynomial of $\alpha^j$ over $\text{GF}(q)$. Furthermore, the canonical factorization of $x^n - 1$ over $\text{GF}(q)$ is given by

$$x^n - 1 = \prod_{j \in \Gamma} (x - \alpha^j).$$

2.3 The linear span and minimal polynomial of sequences

Let $s^j = s_0 s_1 \cdots s_{L-1}$ be a sequence over $\text{GF}(q)$. The linear span (also called linear complexity) of $s^j$ is defined to be the smallest positive integer $\ell$ such that there are constants $c_0 = 1, c_1, \cdots, c_\ell \in \text{GF}(q)$ satisfying

$$-c_0 s_1 = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_{\ell} s_{i-\ell}$$

for all $i \leq L$.

In engineering terms, such a polynomial $c(x) = c_0 + c_1 x + \cdots + c_\ell x^\ell$ is called the feedback polynomial of a shortest linear feedback shift register (LFSR) that generates $s^j$. Such an integer always exists for finite sequences $s^j$. When $L = \infty$, a sequence $s^\infty$ is called a semi-infinite sequence. If there is no such an integer for a semi-infinite sequence $s^\infty$, its linear span is defined to be $\infty$. The linear span of the zero sequence is defined to be zero. For ultimately periodic semi-infinite sequences such an $\ell$ always exists.
Let \( s^m \) be a sequence of period \( L \) over \( \text{GF}(q) \). Any feedback polynomial of \( s^m \) is called a characteristic polynomial. The characteristic polynomial with the smallest degree is called the minimal polynomial of the periodic sequence \( s^m \). Since we require that the constant term of any characteristic polynomial be 1, the minimal polynomial of any periodic sequence \( s^m \) must be unique. In addition, any characteristic polynomial must be a multiple of the minimal polynomial.

For periodic sequences, there are a few ways to determine their linear span and minimal polynomials. One of them is given in [12, Theorem 5.3]. The other one is given in the following lemma [1]

**Lemma 1** Any sequence \( s^m \) over \( \text{GF}(q) \) of period \( q^m - 1 \) has a unique expansion of the form

\[
s_t = \sum_{i=0}^{q^m-2} c_i \alpha^t, \quad \text{for all } t \geq 0,
\]

where \( \alpha \) is a generator of \( \text{GF}(q^m)^* \) and \( c_i \in \text{GF}(q^m) \). Let the index set \( I = \{ i \mid c_i \neq 0 \} \), then the minimal polynomial \( M_s(x) \) of \( s^m \) is

\[
M_s(x) = \prod_{i \in I} (1 - \alpha^i x),
\]

and the linear span of \( s^m \) is \( |I| \).

It should be noticed that in some references the reciprocal of \( M_s(x) \) is called the minimal polynomial of the sequence \( s^m \). So Lemma 1 is a modified version of the original one in [1].

### 2.4 Dickson polynomials over \( \text{GF}(r) \)

One hundred and sixteen years ago, Dickson introduced the following family of polynomials over \( \text{GF}(r) \) [6]:

\[
D_h(x,a) = \sum_{i=0}^{\left\lfloor \frac{h}{r-1} \right\rfloor} \frac{h}{h-i} \binom{h-i}{i} (-a)^i x^{h-2i}, \quad (2)
\]

where \( a \in \text{GF}(r) \) and \( h \geq 0 \) is called the order of the polynomial. This family is referred to as the Dickson polynomials of the first kind.

Dickson polynomials of the second kind over \( \text{GF}(r) \) are defined by

\[
E_h(x,a) = \sum_{i=0}^{\left\lfloor \frac{h}{r-1} \right\rfloor} \binom{h}{h-i} (-a)^i x^{h-2i}, \quad (3)
\]

where \( a \in \text{GF}(r) \) and \( h \geq 0 \) is called the order of the polynomial.

Dickson polynomials are an interesting topic of mathematics and engineering, and have many applications. For example, the Dickson polynomials \( D_5(x,a) = x^5 - ux - u^2 x \) over \( \text{GF}(3^m) \) are employed to construct a family of planar functions [5,13], and those planar functions give two families of commutative presemifields, planes, several classes of linear codes [2,23], and two families of skew Hadamard difference sets [13]. The reader is referred to [19] for detailed information about Dickson polynomials. In this paper, we will employ Dickson polynomials of the first kind over finite fields to construct cyclic codes with some interesting parameters.
3 The construction of cyclic codes from polynomials over GF(r)

Given a polynomial \( f(x) \) on GF(r), we define its associated sequence \( s^n \) by

\[
s_i = \text{Tr}(f(\alpha^i + 1))
\]

for all \( i \geq 0 \), where \( \alpha \) is a generator of GF(r)\(^*\) and \( \text{Tr}(x) \) denotes the trace function from GF(r) to GF(q).

It was demonstrated in \([9,14,21]\) that the code \( C \) may have interesting parameters if the polynomial \( f \) is properly chosen. The objective of this paper is to consider cyclic codes \( C \) defined by Dickson polynomials \( f \) over GF(r) with small degrees.

4 Cyclic codes from the Dickson polynomial \( D_{p^r}(x, a) \)

Since \( q \) is a power of \( p \), it is known that \( D_{b_{p^r}}(x, a) = D_b(x, a)^r \) \([19]\) Lemma 2.6 \]. It then follows that

\[
D_{p^r}(x, a) = x^{p^r}
\]

for all \( a \in \text{GF}(r) \).

The code \( C \) over GF(q) defined by the Dickson polynomial \( f(x) = D_{p^r}(x, a) = x^{p^r} \) over GF(q\(^m\)) are not new. However, for the completeness of cyclic codes from Dickson polynomials we state the following theorem without giving a proof.

**Theorem 1** The code \( C \) defined by the Dickson polynomial \( D_{p^r}(x, a) = x^{p^r} \) has parameters \( [n, n - m - \delta(1), d] \) and generator polynomial \( M_s(x) = (x - 1)^{\delta(1)}m_{x^p}(x) \), where

\[
\begin{align*}
&d = 4 \text{ if } q = 2 \text{ and } \delta(1) = 1, \\
&d = 3 \text{ if } q = 2 \text{ and } \delta(1) = 0, \\
&d = 3 \text{ if } q > 2 \text{ and } \delta(1) = 1, \\
&d = 2 \text{ if } q > 2 \text{ and } \delta(1) = 0,
\end{align*}
\]

where the function \( \delta(x) \) and the polynomial \( m_{x^p}(x) \) were defined in Section 2.7.

When \( q = 2 \), the code of Theorem 1 is equivalent to the binary Hamming weight or its even-weight subcode, and is thus optimal. The code is either optimal or almost optimal with respect to the Sphere Packing Bound.

5 Cyclic codes from \( D_2(x, a) = x^2 - 2a \)

In this section we consider the code \( C \) defined by \( f(x) = D_2(x, a) = x^2 - 2a \) over GF(r). When \( p = 2 \), this code was treated in Section 4. When \( p > 2 \), the following theorem is a variant of Theorem 5.2 in \([8]\), but has much stronger conclusions on the minimum distance of the code.

**Theorem 2** Let \( p > 2 \) and \( m \geq 3 \). The code \( C \) defined by \( f(x) = D_2(x, a) = x^2 - 2a \) has parameters \( [n, n - 2m - \delta(1 - 2a), d] \) and generator polynomial

\[
M_s(x) = (x - 1)^{\delta(1 - 2a)}m_{x^{2a}}(x)m_{a^2}(x),
\]
where

\[
d = \begin{cases} 
4 & \text{if } q = 3 \text{ and } \delta(1 - 2a) = 0, \\
5 & \text{if } q = 3 \text{ and } \delta(1 - 2a) = 1, \\
3 & \text{if } q > 3 \text{ and } \delta(1 - 2a) = 0, \\
4 & \text{if } q > 3 \text{ and } \delta(1 - 2a) = 1,
\end{cases}
\]

and the function \( \delta(x) \) and the polynomial \( M_{\alpha^m}(x) \) were defined in Section [2,7].

**Proof** The conclusion on the minimal polynomial \( M_s(x) \) was proved in Lemma 5.1 in [8]. Hence, the conclusion on the dimension of \( \mathcal{C}_r \) then follows. It remains to determine the minimum distance of the code.

When \( \delta(1 - 2a) = 1 \), the desired conclusions on \( d \) were proved in [2]. When \( \delta(1 - 2a) = 0 \), the desired conclusions on \( d \) can be proved similarly by modifying the proof of Theorem 7 in [2].

The code of Theorem [2] is either optimal or almost optimal for all \( m \geq 2 \). We now prove this statement as follows.

- When \( q = 3 \) and \( \delta(1 - 2a) = 0 \), \( \mathcal{C}_r \) has parameters \([3^m - 1, 3^m - 1 - 2m, 4]\). The Sphere Packing Bound shows that there is no linear code over \( GF(3) \) with parameters \([3^m - 1, 3^m - 1 - 2m, 5]\). By definition, \( \mathcal{C}_r \) is optimal.
- When \( q = 3 \) and \( \delta(1 - 2a) = 1 \), \( \mathcal{C}_r \) has parameters \([3^m - 1, 3^m - 1 - 2m - 1, 5]\). The Sphere Packing Bound shows that there is no linear code over \( GF(3) \) with parameters \([3^m - 1, 3^m - 1 - 2m, 5]\). By definition, \( \mathcal{C}_r \) is optimal.
- When \( q > 3 \) and \( \delta(1 - 2a) = 0 \), \( \mathcal{C}_r \) has parameters \([q^m - 1, q^m - 1 - 2m, 3]\) and is thus almost optimal, as the Sphere Packing Bound shows that any linear code over \( GF(q) \) with parameters \([q^m - 1, q^m - 1 - 2m, 4]\) is optimal.
- When \( q > 3 \) and \( \delta(1 - 2a) = 1 \), \( \mathcal{C}_r \) has parameters \([q^m - 1, q^m - 1 - 2m - 1, 4]\). The Sphere Packing Bound shows that there is no linear code over \( GF(q) \) with parameters \([q^m - 1, q^m - 1 - 2m - 1, 5]\). By definition, \( \mathcal{C}_r \) is optimal.

### 6 Cyclic codes from \( D_s(x, a) = x^3 - 3ax \)

In this section we study the code \( \mathcal{C}_r \) defined by the Dickson polynomial \( D_s(x, a) = x^3 - 3ax \).

We need to distinguish among the three cases: \( p = 2 \), \( p = 3 \) and \( p \geq 5 \). The case that \( p = 3 \) was covered in Section [2]. So we need to consider only the two remaining cases.

We first handle the case \( q = p = 2 \) and prove the following lemma.

**Lemma 2** Let \( q = p = 2 \). Let \( s^* \) be the sequence of [4], where \( f(x) = D_s(x, a) = x^3 - 3ax = x^3 + ax \). Then the minimal polynomial \( M_{\alpha^m}(x) \) of \( s^* \) is given by

\[
M_{\alpha^m}(x) = \begin{cases} 
(x - 1)^{\delta(1)} m_{\alpha^m - 1}(x) & \text{if } a = 0, \\
(x - 1)^{\delta(1 + a)} m_{\alpha^m - 1}(x) m_{\alpha^m - 1}(x) & \text{if } a \neq 0
\end{cases}
\]

where \( m_{\alpha^m}(x) \) and the function \( \delta(x) \) were defined in Section [2,7] and the linear span \( L_s \) of \( s^* \) is given by

\[
L_s = \begin{cases} 
\delta(1) + m & \text{if } a = 0, \\
\delta(1 + a) + 2m & \text{if } a \neq 0.
\end{cases}
\]
Lemma 2 has parameters

Let \( q \)

Theorem 3

L

Proof

Note that

\[
D_3(x + 1, a) = x^3 + x^2 + (1 + a)x + 1 + a.
\]

We have then

\[
\text{Tr}(D_3(x + 1, a)) = \text{Tr}(x^3 + ax) + \text{Tr}(1 + a).
\]

By definition,

\[
s_t = \text{Tr}((\alpha)^3 + a\alpha^t) + \text{Tr}(1 + a). \tag{5}
\]

It can be easily proved that \( \ell_1 = \ell_{n-1} = \ell_3 = \ell_{n-3} = m \) and that \( C_1 \cap C_3 = \emptyset \). The desired conclusions on the linear span and the minimal polynomial \( M_s(x) \) then follow from Lemma 1 and 5.

The following theorem gives information on the code \( C_s \).

**Theorem 3** Let \( q = p^t \) and let \( m \geq 4 \). Then the binary code \( C_s \) defined by the sequence of Lemma 2 has parameters \( [n, n - L_s, d] \) and generator polynomial \( M_s(x) \), where \( M_s(x) \) and \( L_s \) are given in Lemma 2 and

\[
d = \begin{cases} 
2 & \text{if } a = 0 \text{ and } \delta(1) = 0, \\
4 & \text{if } a = 0 \text{ and } \delta(1) = 1, \\
5 & \text{if } a \neq 0 \text{ and } \delta(1 + a) = 0, \\
6 & \text{if } a \neq 0 \text{ and } \delta(1 + a) = 1.
\end{cases}
\]

Proof The dimension of \( C_s \) follows from Lemma 2 and the definition of the code \( C_s \). We need to prove the conclusion on the minimum distance \( d \) of \( C_s \).

We consider the case \( a = 0 \) first. Since \( \alpha^3 \neq 0, d \geq 2 \). On the other hand, if \( \delta(1) = 0, \) then \( m \) is even and \( (\alpha^3)(2^{m-1})^{3} = 1 \). Hence \( C_s \) has a codeword of Hamming weight 2. Whence, \( d = 2 \). If \( \delta(1) = 1, \) then \( m \) is odd and \( \gcd(3, 2^m - 1) = 1 \). Hence, \( \alpha^3 \) is a primitive element of \( \mathbb{GF}(2^m) \) and the code \( \tilde{C}_s \) generated by \( M_{\alpha^3}(x) \) is equivalent to the binary Hamming code, and has thus minimum distance 3. Hence the even-weight subcode \( C_s \) of \( \tilde{C}_s \) has minimum weight 4.

We now consider the case that \( a \neq 0 \). When \( \delta(1 + a) = 1, \) it was proved in [11] that \( d = 5 \). When \( \delta(1 + a) = 0, \) the code is the even-like subcode of the code in the case \( \delta(1 + a) = 1 \). In this case, \( d = 6 \).

Remark 1 When \( a = 0 \) and \( \delta(1) = 1, \) the code is equivalent to the even-weight subcode of the Hamming code. We are mainly interested in the case that \( a \neq 0 \). When \( a = 1, \) the code \( C_s \) is a double-error correcting binary BCH code or its even-like subcode. Theorem 3 shows that well-known classes of cyclic codes can be constructed with Dickson polynomials of order 3. The code is either optimal or almost optimal.

Now we investigate the case \( q = p^t \), where \( p \geq 5 \) or \( p = 2 \) and \( t \geq 2 \).

**Lemma 3** Let \( q = p^t \), where \( p \geq 5 \) or \( p = 2 \) and \( t \geq 2 \). Let \( s^m \) be the sequence of [4], where \( f(x) = D_3(x, a) = x^3 - 3ax \). Then the minimal polynomial \( M_s(x) \) of \( s^m \) is given by

\[
M_s(x) = \begin{cases} 
(x - 1)^{\delta(1)}M_{\alpha^3}(x)m_{\alpha^3}(x) & \text{if } a = 1, \\
(x - 1)^{\delta(1-\delta)}M_{\alpha^3}(x)m_{\alpha^3}(x)m_{\alpha^3}(x) & \text{if } a \neq 1
\end{cases}
\]

where \( m_{\alpha^3}(x) \) and the function \( \delta(x) \) were defined in Section 2 and the linear span \( L_s \) of \( s^m \) is given by

\[
L_s = \begin{cases} 
\delta(-2) + 2m & \text{if } a = 1, \\
\delta(1 + a) + 3m & \text{if } a \neq 1
\end{cases}
\]
Proof Note that
\[ D_3(x+1,a) = x^3 + 3x^2 + 3(1-a)x + 1 - 3a. \]

We have then
\[ s_i = \text{Tr}((\alpha^i)^3 + 3(\alpha^i)^2 + 3(1-a)\alpha^i) + \text{Tr}(1 - 3a). \]  \hfill (6)

Since \( q = p^t \), where \( p \geq 5 \) or \( p = 2 \) and \( t \geq 2 \), one can prove that \( \ell_1 = \ell_{n-1} = \ell_3 = \ell_{n-3} = \ell_2 = \ell_{n-2} = m \) and that
\[ C_1 \cap C_2 = \emptyset, \quad C_1 \cap C_3 = \emptyset, \quad C_2 \cap C_3 = \emptyset. \]

The desired conclusions on the linear span and the minimal polynomial \( M_a(x) \) then follow from Lemma [1] and [9].

The following theorem provides information on the code \( C_t \).

**Theorem 4** Let \( q = p^t \), where \( p \geq 5 \) or \( p = 2 \) and \( t \geq 2 \). Then the code \( C_t \) defined by the sequence of Lemma [2] has parameters \([n, n - L, d]\) and generator polynomial \( M_a(x) \), where \( M_a(x) \) and \( L \) are given in Lemma [2] and
\[
\begin{align*}
    d &\geq 3 \text{ if } a = 1, \\
    d &\geq 4 \text{ if } a \neq 1 \text{ and } \delta(1 - 3a) = 0, \\
    d &\geq 5 \text{ if } a \neq 1 \text{ and } \delta(1 - 3a) = 1, \\
    d &\geq 5 \text{ if } a \neq 1 \text{ and } \delta(1 - 3a) = 0 \text{ and } q = 4, \\
    d &\geq 6 \text{ if } a \neq 1 \text{ and } \delta(1 - 3a) = 1 \text{ and } q = 4.
\end{align*}
\]

Proof The dimension of \( C_t \) follows from Lemma [3] and the definition of the code \( C_t \). We now prove the conclusion on the minimum distance \( d \) of \( C_t \).

Note that \( M_a(x) \) has the zeros \( \alpha^2 \) and \( \alpha^3 \). By the BCH bound, \( d \geq 3 \) for all cases. If \( a \neq 1 \), \( M_a(x) \) has the zeros \( \alpha^i \) for all \( i \in \{1, 2, 3\} \) and the additional zero \( \alpha^0 \) if \( \delta(1 - 3a) = 1 \). Hence, the second and third lower bound on \( d \) follow also from the BCH bound.

The case \( q = 4 \) is special. In this case, \( M_a(x) \) has the zeros \( \alpha^i \) for all \( i \in \{1, 2, 3, 4\} \) and the additional zero \( \alpha^0 \) if \( \delta(1 - 3a) = 1 \). Hence, the last two lower bounds on \( d \) also follow from the BCH bound.

**Remark 2** The code \( C_t \) of Theorem [4] is either a BCH code or the even-like subcode of a BCH code. One can similarly show that the code is either optimal or almost optimal.

When \( q = 4 \), \( a \neq 1 \), \( \delta(1 - 3a) = 1 \), and \( m \geq 3 \), the Sphere Packing Bound shows that \( d = 6 \). But the minimum distance is still open in other cases.

**Open Problem 1** Determine the minimum distance \( d \) for the code \( C_t \) of Theorem [4].

7 Cyclic codes from \( D_4(x,a) = x^4 - 4ax^2 + 2a^2 \)

In this section we investigate the code \( C_t \) defined by the Dickson polynomial \( D_4(x,a) = x^4 - 4ax^2 + 2a^2 \). We have to distinguish among the three cases: \( p = 2 \), \( p = 3 \) and \( p \geq 5 \). The case \( p = 2 \) was covered in Section [4]. So we need to consider only the two remaining cases.

We first take care of the case \( q = p = 3 \) and prove the following lemma.
Lemma 4 Let \( q = p = 3 \) and \( m \geq 3 \). Let \( s^m \) be the sequence of \([2] \), where \( f(x) = D_4(x, a) = x^4 - 4ax^2 + 2a^2 \). Then the minimal polynomial \( M_s(x) \) of \( s^m \) is given by

\[
M_s(x) = \begin{cases} 
(x - 1)^{\delta(1)} m_{\alpha^2}(x)m_{\alpha^{-1}}(x) & \text{if } a = 0, \\
(x - 1)^{\delta(1)} m_{\alpha^2}(x)m_{\alpha^{-1}}(x) & \text{if } a = 1, \\
(x - 1)^{\delta(1-a^2)} m_{\alpha^2}(x)m_{\alpha^{-1}}(x) & \text{otherwise},
\end{cases}
\]

where \( m_{\alpha^2}(x) \) and the function \( \delta(x) \) were defined in Section [2,7] and the linear span \( L_s \) of \( s^m \) is given by

\[
L_s = \begin{cases} 
\delta(1) + 2m & \text{if } a = 0, \\
\delta(1) + 2m & \text{if } a = 1, \\
\delta(1-a^2) + 3m & \text{otherwise}.
\end{cases}
\]

Proof Note that

\[
D_4(x+1,a) = x^4 + x^3 - ax^2 + (1 + a)x + 1 - a - a^2.
\]

We have then

\[
\text{Tr}(D_4(x+1,a)) = \text{Tr}(x^4 - ax^2 + (a - 1)x) + \text{Tr}(1 - a - a^2).
\]

By definition,

\[
s_t = \text{Tr}((\alpha^t)^2 - a(\alpha^t)^2 + (a - 1)\alpha^t) + \text{Tr}(1 - a - a^2).
\]

It can be easily proved that \( \ell_1 = \ell_{m-1} = \ell_4 = \ell_{n-4} = \ell_2 = \ell_{n-2} = m \) and that the 3-cyclotomic cosets \( C_1, C_2 \) and \( C_4 \) are pairwise disjoint. The desired conclusions on the linear span and the minimal polynomial \( M_s(x) \) then follow from Lemma [21] and [7].

The following theorem gives information on the code \( C_s \).

Theorem 5 Let \( q = p = 3 \) and \( m \geq 3 \). Then the code \( C_s \) defined by the sequence of Lemma [2] has parameters \([n, n - L_s, d]\) and generator polynomial \( M_s(x) \), where \( M_s(x) \) and \( L_s \) are given in Lemma [2] and

\[
\begin{align*}
d &= 2 \text{ if } a = 1, \\
&= 3 \text{ if } a = 0 \text{ mod } 6, \\
&= 4 \text{ if } a = 0 \text{ mod } 6, \\
&= 5 \text{ if } a^2 \neq a \text{ and } \delta(1-a-a^2) = 0, \\
&= 6 \text{ if } a^2 \neq a \text{ and } \delta(1-a-a^2) = 1.
\end{align*}
\]

Proof The dimension of \( C_s \) follows from Lemma [2] and the definition of the code \( C_s \). We now prove the conclusion on the minimum distance \( d \) of \( C_s \).

We consider the case \( a = 1 \) first. In this case, the generator polynomial of this code \( C_s \) is \( (x - 1)^{\delta(1)} m_{\alpha^2}(x)m_{\alpha^{-1}}(x) \). It is easily seen that \( 1, \alpha^2 \) and \( \alpha^{-4} \) are roots of \( 2 + x^{(3^m - 1)/2} = 0 \). Therefore, \( C_s \) has the codeword \( 2 + x^{(3^m - 1)/2} \) of Hamming weight 2. Hence \( d = 2 \) when \( a = 1 \).

We now treat the case \( a = 0 \). In this case, the generator polynomial of this code is \( M_s(x) = (x - 1)^{\delta(1)} m_{\alpha^2}(x)m_{\alpha^{-1}}(x) \). Note that \( M_s(x) \) has the zeros \( \alpha^3 \) and \( \alpha^4 \). By the BCH bound the minimum weight \( d \) in \( C_s \) is at least 3. We want to know when \( C_s \) and \( \tilde{C}_s \) have a codeword of weight 3.
The code $\tilde{C}_s$ has a codeword of weight three if and only if there are two integers $t_1$ and $t_2$ with $1 \leq t_1 \neq t_2 \leq n - 1$ and two elements $u_1$ and $u_2$ in $\{1, -1\}$ such that
\[
\begin{align*}
1 + u_1\alpha^{t_1} + u_2\alpha^{t_2} &= 0, \\
1 + u_1\alpha^{t_1} + u_2\alpha^{t_2} &= 0.
\end{align*}
\] (8)

Suppose now that $\tilde{C}_s$ has a codeword $1 + u_1x^t + u_2x^t$ of weight 3. Combining the two equations of (8) yields
\[
(u_1u_2 + 1)\alpha^{t_2} + u_2\alpha^{t_2} + u_2\alpha^{t_2} + 1 + u_1 = 0
\] (9) and
\[
(u_1u_2 + 1)\alpha^{t_2} + u_1\alpha^{t_1} + u_1\alpha^{t_1} + 1 + u_2 = 0.
\] (10)

We now consider the first subcase that $u_1u_2 = -1$ under the case that $a = 0$. In this subcase, $\delta(1) = m \mod 3 = 0$ as $1 + u_1 + u_2 = 1 \neq 0$. In this subcase (9) and (10) become
\[
\alpha^{t_2} + \alpha^2 - u_1(1 + u_1) = 0
\] (11) and
\[
\alpha^{t_2} + \alpha^2 - u_2(1 + u_2) = 0.
\] (12)

Due to symmetry, we assume that $(u_1, u_2) = (-1, 1)$. It follows from (11) and (12) that
\[
\alpha^{t_2} = -1 \quad \text{and} \quad (\alpha^3 - 1)^2 = -1.
\]

When $m$ is odd, $\alpha^{(3m - 1)/2} = -1$ and $(3m - 1)/2$ is odd. Hence, $-1$ cannot be a square in $\text{GF}(r)$. Therefore, $\tilde{C}_s$ cannot have a codeword $1 + u_1x^t + u_3x^t$ when $a = 0$ and $m$ is odd, where $u_1u_2 = -1$. When $m$ is even, $m \equiv 0 \pmod{6}$ and $-1$ is a square in $\text{GF}(r)$. Let $y_1 \in \text{GF}(r)$ be a solution of $y^2 = -1$, and define $t_2$ and $t_1$ such that
\[
\alpha^{t_1} = y_1, \quad \alpha^{t_2} = 1 + y_1.
\]

Then $t_1$ and $t_2$ are distinct and $1 + x^{t_1} - x^{t_2}$ is indeed a codeword of weight three in $\tilde{C}_s$. Thus, $d = 3$ when $m \equiv 0 \pmod{6}$.

We are ready to consider the second subcase that $u_1u_2 = 1$ under the case that $a = 0$. In this subcase (9) and (10) become
\[
\alpha^{t_2} - u_2\alpha^{t_2} - u_2\alpha^{t_2} - (1 + u_1) = 0
\] (13) and
\[
\alpha^{t_1} - u_1\alpha^{t_1} - u_1\alpha^{t_1} - (1 + u_2) = 0
\] (14)

When $(u_1, u_2) = (1, 1)$. It follows from (13) and (14) that
\[
(\alpha^2 - 1)^2 = 0 \quad \text{and} \quad (\alpha^3 - 1)^2 = 0.
\]

Hence $\alpha^{t_1} = \alpha^{t_2} = 1$. This is impossible as $\alpha$ is a generator of $\text{GF}(r)$. Therefore, $\tilde{C}_s$ cannot have a codeword $1 + x^t + x^t$. When $(u_1, u_2) = (-1, -1)$. It follows from (13) and (14) that
\[
\alpha^{t_1}(\alpha^{t_2} + \alpha^{t_2} + 1) = 0,
\]
\[
\alpha^{t_1}(\alpha^{t_2} + \alpha^{t_2} + 1) = 0.
\]
Note that $y^3 + y^2 + 1 = 0$ if and only if
\[(y^{-1} - 1)^3 + (y^{-1} - 1) = 0.
\]
However, $z^3 + z = 0$ does not have a nonzero solution $z$ in $GF(r)$ if $m$ is odd. This proves that the code $C_4$ cannot have a codeword $1 - x^{d_1} - x^{d_2}$ when $m$ is odd. If $m$ is even, $m \equiv 0 \pmod{6}$ as $1 + u_1 + u_2 = 1 \neq 0$. When $m \equiv 0 \pmod{6}$, let $z_1 \in GF(r)$ and $z_2 \in GF(r)$ be the two distinct solutions of $z^2 + 1 = 0$. Define $t_1$ and $t_2$ so that
\[a^i = \frac{1}{1 + z_i}
\]
for $i \in \{1, 2\}$. Then $1 - x^{t_1} - x^{t_2}$ is a codeword of weight three in $C_4$. This completes the proof of the conclusions on the minimum weight $d$ for the case $a = 0$.

When $a(a - 1) \neq 0$, $M_a(x)$ has the zeros $a^i$ for all $i \in \{1, 2, 3, 4\}$ and the additional zero $a^6$ if $\delta(1 - a - a^2) = 1$. The last two lower bounds on $d$ then follow from the BCH bound. When $a^2 \neq a$ and $\delta(1 - a - a^2) = 1$, the Sphere Packing Bound proves that $d \leq 6$. We have thus $d = 6$ in this case.

**Remark 3** When $a = 1$, the code of Theorem 5 is neither optimal nor almost optimal. The code is either optimal or almost optimal in all other cases.

Now we consider the case $q = p^t$, where $p \geq 5$ or $p = 3$ and $t \geq 2$.

**Lemma 5** Let $m \geq 2$ and $q = p^t$, where $p \geq 5$ or $p = 3$ and $t \geq 2$. Let $s^\infty$ be the sequence of $\mathbb{F}_q$, where $f(x) = D_1(x, a) = x^4 - 4ax^2 + 2a^2$. Then the minimal polynomial $M_s(x)$ of $s^\infty$ is given by
\[
M_s(x) = \begin{cases} 
(x - 1)^{\delta(1)m_{a^0}}(x)m_{a^1}(x)m_{a^2}(x) & \text{if } a = \frac{1}{2}, \\
(x - 1)^{\delta(1)m_{a^0}}(x)m_{a^1}(x)m_{a^2}(x) & \text{if } a = \frac{2}{3}, \\
(x - 1)^{\delta(1-4a^2+2a^2)} \prod_{i=1}^4 m_{a^i}(x) & \text{if } a \not\in \{\frac{1}{2}, \frac{2}{3}\},
\end{cases}
\]
where $m_{a^i}(x)$ and the function $\delta(x)$ were defined in Section 2.1, and the linear span $L_s$ of $s^\infty$ is given by
\[
L_s = \begin{cases} 
\delta(1) + 3m & \text{if } a \in \{\frac{1}{2}, \frac{2}{3}\}, \\
\delta(1 - 4a^2 + 2a^2) + 4m & \text{otherwise.}
\end{cases}
\]

**Proof** Note that
\[D_1(x + 1, a) = x^4 + 4x^3 + (6 - 4a)x^2 + (4 - 8a)x + 1 - 4a + 2a^2.
\]
We have then
\[s_t = \text{Tr}(\alpha^t)^3 + 4(\alpha^t)^3 + (6 - 4a)(\alpha^t)^3 + (4 - 8a)\alpha^t) + \text{Tr}(1 - 4a + 2a^2) \quad (15)
\]
for all $t \geq 0$.

Since $m \geq 2$ and $q = p^t$, where $p \geq 5$ or $p = 3$ and $t \geq 2$, one can prove that
\[
\ell_1 = \ell_{a-1} = \ell_3 = \ell_{a+1} = \ell_2 = \ell_{a+2} = \ell_a = \ell_{a-2} = m
\]
and that the $q$-cycloctomic cosets $C_1, C_2, C_3, C_4$ are pairwise disjoint. The desired conclusions on the linear span and the minimal polynomial $M_s(x)$ then follow from Lemma 1 and 15.
The following theorem delivers to us information on the code $C_t$.  

**Theorem 6** Let $m \geq 2$ and $q = p^t$, where $p \geq 5$ or $p = 3$ and $t \geq 2$. Then the code $C_t$ defined by the sequence of Lemma 5 has parameters $[n, n - \Lambda_t, d]$ and generator polynomial $M_\alpha(x)$, where $M_\alpha(x)$ and $L_\alpha$ are given in Lemma 5 and

\[
\begin{align*}
    d \geq 3 & \quad \text{if } a = \frac{3}{4},
    \\
    d \geq 4 & \quad \text{if } a = \frac{1}{4},
    \\
    d \geq 5 & \quad \text{if } a \not\in \{\frac{1}{4}, \frac{3}{4}\} \text{ and } \delta(1 - 4a + a^2) = 0,
    \\
    d = 6 & \quad \text{if } a \not\in \{\frac{1}{4}, \frac{3}{4}\} \text{ and } \delta(1 - 4a + a^2) = 1.
\end{align*}
\]

**Proof** The dimension of $C_t$ follows from Lemma 5 and the definition of the code $C_t$. The lower bounds on the minimum weight $d$ of $C_t$ follow from the BCH bounds and the Sphere Packing Bound. The details are left to the reader.

**Remark 4** Except the cases that $a \in \{\frac{3}{4}, \frac{1}{4}\}$, the code $C_t$ of Theorem 5 is either optimal or almost optimal.

### 8 Cyclic codes from $D_3(x, a) = x^5 - 5ax^3 + 5a^2x$

In this section we deal with the code $C_t$ defined by the Dickson polynomial $D_3(x, a) = x^5 - 5ax^3 + 5a^2x$. We have to distinguish among the three cases: $p = 2$, $p = 3$ and $p \geq 7$.

The case $p = 5$ was covered in Section 4. So we need to consider only the remaining cases.

We first establish the following lemma.

**Lemma 6** The equation $x + x^2 + x^4 = 0$ has a nonzero solution $x \in GF(2^m)$ if and only if $m \equiv 0 \pmod{3}$.

**Proof** Suppose that $x + x^2 + x^4 = 0$ for some $x \in GF(2^m)^\ast$. Then $(x + x^2 + x^4)^2 = x^2 + x^4 + x^8 = 0$. Combining the two equations yields $x + x^8 = 0$. Hence $x^7 = 1$. Since $x \not= 1$, this means that $gcd(7, 2^m) = 2^{gcd(3, m)} - 1 = 7$. Hence $m \equiv 0 \pmod{3}$.

Suppose now that $m \equiv 0 \pmod{3}$. Let $m' = m/3$. Define

\[
\pi(y) = \sum_{i=0}^{m'} y^{2^i}
\]

for any $y \in GF(2^m)$. It is well known that $Tr(y) = 0$ has $2^{m-1}$ solutions $y \in GF(2^m)$. One of them must satisfy that $\pi(y) \neq 0$ as the two functions $\pi(x)$ and $Tr(x)$ are clearly different. Let $y \in GF(2^m)$ such that $Tr(y) = 0$ and $\pi(y) \neq 0$. Then it is easily seen that $\pi(y) + \pi(y)^2 + \pi(y)^3 = Tr(y) = 0$. This completes the proof.

We first consider the case $q = p = 2$ and prove the following lemma.

**Lemma 7** Let $q = p = 2$ and $m \geq 5$. Let $s^m$ be the sequence of (4), where $f(x) = D_3(x, a) = x^5 - 5ax^3 + 5a^2x$. Then the minimal polynomial $M_\alpha(x)$ of $s^m$ is given by

\[
M_\alpha(x) = \begin{cases} 
(x - 1)^{\delta(1)} m_{\alpha - x}(x) & \text{if } a = 0, \\
(x - 1)^{\delta(1)} m_{\alpha - x}(x) m_{\alpha^{s^m}}(x) & \text{if } 1 + a + a^3 = 0, \\
(x - 1)^{\delta(1)} \prod_{i=0}^{m-1} m_{\alpha^{s^m+1}}(x) & \text{if } a + a^3 + a^5 \neq 0.
\end{cases}
\]
where \( m_{\alpha}/(x) \) and the function \( \delta(x) \) were defined in Section 2.7 and the linear span \( L_s \) of \( s^m \) is given by

\[
L_s = \begin{cases} 
\delta(1) + m & \text{if } a = 0, \\
\delta(1) + 2m & \text{if } 1 + a + a^3 = 0, \\
\delta(1) + 3m & \text{if } a + a^2 + a^4 \neq 0.
\end{cases}
\]

**Proof** Note that

\[
D_5(x + 1, a) = x^5 + x^4 + ax^3 + ax^2 + (1 + a + a^2)x + 1 + a + a^2.
\]

Since \( q = 2 \), we have then

\[
\text{Tr}(D_5(x + 1, a)) = \text{Tr}(x^5 + ax^3 + (a^{2^{m-1}} + a + a^2)x) + \text{Tr}(1).
\]

By definition,

\[
x_i = \text{Tr} \left( (\alpha^i)^5 + a(\alpha^i)^3 + (a^{2^{m-1}} + a + a^2)(\alpha^i) \right) + \text{Tr}(1). \tag{16}
\]

It can be easily proved that \( \ell_1 = \ell_3 = \ell_5 = m \) and that \( C_1, C_3 \) and \( C_5 \) are pairwise disjoint when \( m \geq 5 \). The desired conclusions on the linear span and the minimal polynomial \( M_s(x) \) then follow from Lemma 1 and (16).

The following theorem describes parameters of the code \( C_s \).

**Theorem 7** Let \( q = p = 2 \) and \( m \geq 5 \). Then the code \( C_s \) defined by the sequence of Lemma 7 has parameters \([n, n - L_s, d] \) and generator polynomial \( M_s(x) \), where \( M_s(x) \) and \( L_s \) are given in Lemma 1 and

\[
\begin{align*}
d &= 2 \text{ if } a = 0 \text{ and } \delta(1) = 0 \text{ and } \gcd(5, n) = 5, \\
d &= 3 \text{ if } a = 0 \text{ and } \delta(1) = 0 \text{ and } \gcd(5, n) = 1, \\
d &= 4 \text{ if } a = 0 \text{ and } \delta(1) = 1, \\
d &\geq 3 \text{ if } 1 + a + a^2 = 0 \text{ and } \delta(1) = 0, \\
d &\geq 4 \text{ if } 1 + a + a^2 = 0 \text{ and } \delta(1) = 1, \\
d &\geq 7 \text{ if } a + a^2 + a^4 \neq 0 \text{ and } \delta(1) = 0, \\
d &= 8 \text{ if } a + a^2 + a^4 \neq 0 \text{ and } \delta(1) = 1.
\end{align*}
\]

**Proof** The dimension of \( C_s \) follows from Lemma 7 and the definition of the code \( C_s \). We need to prove the conclusion on the minimum distance \( d \) of \( C_s \).

We consider the case \( a = 0 \) first. Since \( \alpha^5 \neq 0, d \geq 2 \). On the other hand, if \( \delta(1) = 0 \) and \( \gcd(5, n) = 5 \), then \( m \) is even and \( (\alpha^i)^5(2^{m-1})^{5/5} \) is 1. Hence \( C_s \) has the codeword 1 + \( x(2^{m-1})^{5/5} \) of Hamming weight 2. Whence, \( d = 2 \). If \( \delta(1) = 0 \) and \( \gcd(5, n) = 1 \), then \( \alpha^5 \) is a primitive element, the code \( C_s \) is equivalent to the Hamming code. Hence \( d = 3 \). If \( \delta(1) = 1, \) then \( m \) odd and \( \gcd(5, 2^m - 1) = 1 \). Hence, \( \alpha^5 \) is a primitive element of GF\((2^m)\) and the code \( C_s \) generated by \( M_s(x) \) has minimum weight 3. Hence the even-like subcode \( C_s \) of \( C_s \) has minimum weight 4.

We now consider the case that \( 1 + a + a^3 = 0 \). By Lemma 5 \( m \equiv 0 \pmod{3} \). In this case \( M_s(x) = (x - 1)^{\delta(1)}m_{\alpha^3}(x)m_{\alpha^5}(x) \). Since \( M_s(x) \) has the zeros \( \alpha^2 \) and \( \alpha^6 \), \( d \geq 3 \). If \( \delta(1) = 1 \), \( C_s \) is an even-weight code. Hence \( d \geq 4 \).

We finally consider the case that \( 1 + a + a^3 \neq 0 \). Note that \( M_s(x) \) has zeros \( \alpha^i \) for all \( i \in \{1, 2, 3, 4, 5, 6\} \), and the additional zero \( \alpha^0 \) when \( \delta(1) = 1 \). The conclusions on the minimum weight \( d \) in this case follow from the BCH bound. When \( m \geq 5 \) is odd and \( a + a^2 + a^4 \neq 0 \), the Sphere Packing Bound tells us that \( d \leq 8 \). We have then \( d = 8 \) in the last case.
Remark 5  The code of Theorem [7] is either optimal or almost optimal. The code is not a BCH code when \(1 + a + a^2 = 0\), and a BCH code in the remaining cases.

We now consider the case \((p, q) = (2, 4)\) and prove the following lemma.

**Lemma 8**  Let \((p, q) = (2, 4)\) and \(m \geq 3\). Let \(s^m\) be the sequence of [4], where \(f(x) = D_3(x, a) = x^5 - Sax^3 + Sa^2 x\). Then the minimal polynomial \(M_3(x)\) of \(s^m\) is given by

\[
M_3(x) = \begin{cases} 
(x - 1)^{8(1)} m_{a^3}(x) & \text{if } a = 0, \\
(x - 1)^{8(1)} m_{a^3}(x)m_{a^3}(x) & \text{if } a = 1, \\
(x - 1)^{8(1+a^2)} m_{a^3}(x)m_{a^3}(x) & \text{if } a + a^2 \neq 0
\end{cases}
\]

where \(m_{a^3}(x)\) and the function \(\delta(x)\) were defined in Section [2,7] and the linear span \(L_s\) of \(s^m\) is given by

\[
L_s = \begin{cases} 
\delta(1) + m & \text{if } a = 0, \\
\delta(1) + 3m & \text{if } a = 1, \\
\delta(1) + 4m & \text{if } a + a^2 \neq 0.
\end{cases}
\]

**Proof**  Note that

\[
D_3(x + 1, a) = x^5 + x^3 + ax^3 + ax^2 + (1 + a + a^2)x + 1 + a + a^2.
\]

Since \(q = 2^2\), we have then

\[
\text{Tr}(D_3(x + 1, a)) = \text{Tr}(x^5 + ax^3 + ax^2 + (a + a^2)x) + \text{Tr}(1 + a + a^2).
\]

By definition,

\[
s_t = \text{Tr}((\alpha')^5 + a(\alpha')^3 + a(\alpha')^2 + (a + a^2)(\alpha') + 1 + a + a^2).
\]  \hspace{1cm} (17)

It can be easily proved that \(\ell_1 = \ell_2 = \ell_3 = \ell_4 = m\) and that \(C_1, C_2, C_3\) and \(C_4\) are pairwise disjoint when \(m \geq 3\). The desired conclusions on the linear span and the minimal polynomial \(M_3(x)\) then follow from Lemma[1] and [17].

The following theorem supplies information on the code \(C_s\).

**Theorem 8**  Let \((p, q) = (2, 4)\) and \(m \geq 3\). Then the code \(C_s\) defined by the sequence of Lemma[8] has parameters \([n, n - L_s, d]\) and generator polynomial \(M_3(x)\), where \(M_3(x)\) and \(L_s\) are given in Lemma[8] and

\[
\begin{align*}
\{ d = 2 & \text{ if } a = 0 \text{ and } \delta(1) = 0 \text{ and } \gcd(5, n) = 5, \\
d = 3 & \text{ if } a = 0 \text{ and } \gcd(5, n) = 1, \\
d \geq 3 & \text{ if } a = 1, \\
d \geq 6 & \text{ if } a + a^2 \neq 0 \text{ and } \delta(1) = 0, \\
d \geq 7 & \text{ if } a + a^2 \neq 0 \text{ and } \delta(1) = 1.
\end{align*}
\]

**Proof**  The dimension of \(C_s\) follows from Lemma[8] and the definition of the code \(C_s\). We need to prove the conclusion on the minimum distance \(d\) of \(C_s\).

The proof of the lower bounds for the case \(a = 0\) is the same as that of Theorem[7] when \(a = 1\). \(M_3(x)\) has the zeros \(\alpha^2\) and \(\alpha^3\). Hence \(d \geq 3\) when \(a = 1\).

We finally consider the case that \(a + a^2 \neq 0\). Note that \(M_3(x)\) has the zeros \(\alpha^i\) for all \(i \in \{1, 2, 3, 4, 5\}\), and the additional zero \(\alpha^0\) when \(\delta(1) = 1\). The conclusions on the minimum weight \(d\) in this case follow from the BCH bound.
Examples of the code of Theorem 8 are documented in arXiv:1206.4370, and many of them are optimal.

Open Problem 2 Determine the minimum distance \( d \) of the code \( C_s \) in Theorem 8.

We now consider the case \((p, q) = (2, 2^t)\), where \( t \geq 3 \), and prove the following lemma.

**Lemma 9** Let \((p, q) = (2, 2^t)\) and \( m \geq 3 \), where \( t \geq 3 \). Let \( s^m \) be the sequence of \((4)\), where \( f(x) = D_5(x, a) = x^5 - 5ax^3 + 5a^2 x \). Then the minimal polynomial \( M_s(x) \) of \( s^m \) is given by

\[
M_s(x) = \begin{cases} 
(x-1)^{\delta(1)}m_{x^a - 1}(x)m_{x^a - 1}(x) & \text{if } a = 0, \\
\prod_{i=2}^5 m_{x^a - 1}(x) & \text{if } 1 + a + a^2 = 0, \\
(x-1)^{\delta(1)}\prod_{i=1}^5 m_{x^a - 1}(x) & \text{if } a + a^2 + a^3 \neq 0,
\end{cases}
\]

where \( m_{x^a - 1}(x) \) and the function \( \delta(x) \) were defined in Section 2.1, and the linear span \( L_s \) of \( s^m \) is given by

\[
L_s = \begin{cases} 
\delta(1) + 3m & \text{if } a = 0, \\
\delta(1) + 4m & \text{if } 1 + a + a^2 = 0, \\
\delta(1) + 5m & \text{if } a + a^2 + a^3 \neq 0.
\end{cases}
\]

**Proof** Note that

\[
D_5(x + 1, a) = x^5 + x^4 + ax^3 + ax^2 + (1 + a + a^2)x + 1 + a + a^2.
\]

Since \( q = 2^t \), where \( t \geq 3 \), we have then

\[
\text{Tr}(D_5(x + 1, a)) = \text{Tr}(x^5 + x^4 + ax^3 + ax^2 + (1 + a + a^2)x) + \text{Tr}(1 + a + a^2).
\]

By definition,

\[
s_t = \text{Tr}((\alpha^t)^5 + (\alpha^t)^4 + a(\alpha^t)^3 + a(a^t)^2 + (a + a^2)\alpha^t + \text{Tr}(1 + a + a^2)). \tag{18}
\]

It can be easily proved that \( \ell_i = m \) for all \( 1 \leq i \leq 5 \) and that these \( C_s \), where \( 1 \leq i \leq 5 \), are pairwise disjoint. The desired conclusions on the linear span and the minimal polynomial \( M_s(x) \) then follow from Lemma 1 and (18).

The following theorem provides information on the code \( C_s \).

**Theorem 9** Let \((p, q) = (2, 2^t)\), where \( t \geq 3 \). Then the code \( C_s \) defined by the sequence of Lemma 9 has parameters \([n, n - L_s, d]\) and generator polynomial \( M_s(x) \), where \( M_s(x) \) and \( L_s \) are given in Lemma 9 and

\[
d \geq 3 \text{ if } a = 0 \text{ and } \delta(1) = 0,
\]
\[
d \geq 4 \text{ if } a = 0 \text{ and } \delta(1) = 1,
\]
\[
d \geq 5 \text{ if } 1 + a + a^2 = 0,
\]
\[
d \geq 6 \text{ if } a + a^2 + a^3 \neq 0 \text{ and } \delta(1) = 0,
\]
\[
d \geq 7 \text{ if } a + a^2 + a^3 \neq 0 \text{ and } \delta(1) = 1.
\]

**Proof** The proof of this theorem is similar to that of Theorem 8 and is omitted.

Open Problem 3 Determine the minimum distance \( d \) of the code \( C_s \) in Theorem 9.
Examples of the code of Theorem 8 can be found in arXiv:1206.4370 and many of them are optimal. The code of Theorem 9 is not a BCH code when \( a = 0 \), and a BCH code otherwise.

We now consider the case \( q = p = 3 \) and state the following lemma and theorem without proofs.

**Lemma 10** Let \( q = p = 3 \) and \( m \geq 3 \). Let \( s^r \) be the sequence of \( 4 \), where \( f(x) = D_s(x, a) = x^5 - 5ax^3 + 5a^2x \). Then the minimal polynomial \( M_q(x) \) of \( s^r \) is given by

\[
M_q(x) = \begin{cases} 
(x - 1)^{\delta(1 + a + 2a^2)} m_{a^3}(x)m_{a^2}(x)m_{a}(x) & \text{if } a - a^6 = 0, \\
(x - 1)^{\delta(1 + a + 2a^2)} \prod_{i=2}^{5} m_{a^i}(x) & \text{if } a - a^6 \neq 0,
\end{cases}
\]

where \( m_{a^i}(x) \) and the function \( \delta(x) \) were defined in Section 2.7 and the linear span \( L_s \) of \( s^r \) is given by

\[
L_s = \begin{cases} 
\delta(1 + a + 2a^2) + 3m & \text{if } a - a^6 = 0, \\
\delta(1 + a + 2a^2) + 4m & \text{if } a - a^6 \neq 0.
\end{cases}
\]

**Proof** The proof is similar to that of Lemma 9 and is omitted here.

The following theorem gives information on the code \( C_s \).

**Theorem 10** Let \( q = p = 3 \) and \( m \geq 3 \). Then the code \( C_s \) defined by the sequence of Lemma 10 has parameters \( [n, n - L_s, d] \) and generator polynomial \( M_q(x) \), where \( M_q(x) \) and \( L_s \) are given in Lemma 10 and

\[
\begin{cases} 
d \geq 4 & \text{if } a - a^6 = 0, \\
d \geq 5 & \text{if } a - a^6 \neq 0 \text{ and } \delta(1 + a + 2a^2) = 0, \\
d \geq 6 & \text{if } a - a^6 \neq 0 \text{ and } \delta(1 + a + 2a^2) = 1.
\end{cases}
\]

**Proof** The proof of this theorem is similar to that of Theorem 8 and is omitted.

**Open Problem 4** Determine the minimum distance \( d \) of the code \( C_s \) in Theorem 10 (our experimental data indicates that the lower bounds are the specific values of \( d \)).

Examples of the code of Theorem 10 are described in arXiv:1206.4370 and some of them are optimal.

We now consider the case \((p, q) = (3, 3^t)\), where \( t \geq 3 \), and state the following lemma and theorem without proofs.

**Lemma 11** Let \((p, q) = (3, 3^t)\) and \( m \geq 2 \), where \( t \geq 2 \). Let \( s^r \) be the sequence of \( 4 \), where \( f(x) = D_s(x, a) = x^5 - 5ax^3 + 5a^2x \). Then the minimal polynomial \( M_q(x) \) of \( s^r \) is given by

\[
M_q(x) = \begin{cases} 
(x - 1)^{\delta(1 + a + 2a^2)} m_{a^3}(x)m_{a^2}(x)m_{a}(x) & \text{if } 1 + a = 0, \\
(x - 1)^{\delta(1 + a + 2a^2)} m_{a^3}(x)m_{a^2}(x)m_{a}(x) & \text{if } 1 + a^2 = 0, \\
(x - 1)^{\delta(1 + a + 2a^2)} \prod_{i=2}^{5} m_{a^i}(x) & \text{if } (a + 1)(a^2 + 1) \neq 0,
\end{cases}
\]

where \( m_{a^i}(x) \) and the function \( \delta(x) \) were defined in Section 2.7 and the linear span \( L_s \) of \( s^r \) is given by

\[
L_s = \begin{cases} 
\delta(1 + a + 2a^2) + 5m & \text{if } (a + 1)(a^2 + 1) \neq 0.
\end{cases}
\]
Lemma 12. The sequence of Lemma 9 has parameters \([n, n - L_s, d]\) and generator polynomial \(M_s(x)\), where \(M_s(x)\) and \(L_s\) are given in Lemma 11 and
\[
\begin{align*}
    d &\geq 3 \text{ if } a = -1 \text{ and } \delta(1) = 0, \\
    d &\geq 4 \text{ if } a = -1 \text{ and } \delta(1) = 1, \\
    d &\geq 5 \text{ if } a^2 = -1 \text{ and } \delta(a - 1) = 0, \\
    d &\geq 6 \text{ if } a^2 = 1 \text{ and } \delta(a - 1) = 1, \\
    d &\geq 6 \text{ if } (a + 1)(a^2 + 1) \neq 0 \text{ and } \delta(1 + a + 2a^2) = 0, \\
    d &\geq 7 \text{ if } (a + 1)(a^2 + 1) \neq 0 \text{ and } \delta(1 + a + 2a^2) = 1.
\end{align*}
\]

Proof. The proof is similar to that of Lemma 9 and is omitted here.

Theorem 11. Let \((p, q) = (3, 3')\) and \(m \geq 2\), where \(t \geq 2\). Then the code \(C_t\) defined by the sequence of Lemma 9 has parameters \([n, n - L_s, d]\) and generator polynomial \(M_s(x)\), where \(M_s(x)\) and \(L_s\) are given in Lemma 11 and
\[
\begin{align*}
    d &\geq 3 \text{ if } a = -1 \text{ and } \delta(1) = 0, \\
    d &\geq 4 \text{ if } a = -1 \text{ and } \delta(1) = 1, \\
    d &\geq 5 \text{ if } a^2 = -1 \text{ and } \delta(a - 1) = 0, \\
    d &\geq 6 \text{ if } a^2 = 1 \text{ and } \delta(a - 1) = 1, \\
    d &\geq 6 \text{ if } (a + 1)(a^2 + 1) \neq 0 \text{ and } \delta(1 + a + 2a^2) = 0, \\
    d &\geq 7 \text{ if } (a + 1)(a^2 + 1) \neq 0 \text{ and } \delta(1 + a + 2a^2) = 1.
\end{align*}
\]

Proof. The proof of this theorem is similar to that of Theorem 8 and is omitted.

Open Problem 5. Determine the minimum distance \(d\) of the code \(C_t\) in Theorem 11.

Examples of the code of Theorem 11 are available in [arXiv:1206.4370] and some of them are optimal. The code is a BCH code, except in the case that \(a = -1\).

We finally consider the case \(p \geq 7\), and present the following lemma and theorem without proofs.

Lemma 12. Let \(p \geq 7\) and \(m \geq 2\). Let \(s^m\) be the sequence of \([2]\), where \(f(x) = D_3(x, a) = x^5 - 5ax^3 + 5a^2x\). Then the minimal polynomial \(M_s(x)\) of \(s^m\) is given by
\[
M_s(x) = \begin{cases}
(x - 1)(1 - 5a + 5a^2) & \text{if } a = 2, \\
(x - 1)(1 - 5a + 5a^2) & \text{if } a = 7, \\
(x - 1)(1 - 5a + 5a^2) & \text{if } a^2 = 2a + 1 = 0, \\
(x - 1)(1 - 5a + 5a^2) & \text{if } (a^2 - 3a + 1)(a - 2)(3a - 2) \neq 0,
\end{cases}
\]
where \(m_a, (x)\) and the function \(\delta(x)\) were defined in Section 2.1 and the linear span \(L_s\) of \(s^m\) is given by
\[
\begin{align*}
L_s = \begin{cases}
\delta(1 - 5a + 5a^2) + 4m & \text{if } (a^2 - 3a + 1)(a - 2)(3a - 2) = 0, \\
\delta(1 - 5a + 5a^2) + 5m & \text{otherwise}.
\end{cases}
\end{align*}
\]

Proof. The proof is similar to that of Lemma 9 and is omitted here.

The following theorem provides information on the code \(C_t\).

Theorem 12. Let \(p \geq 7\) and \(m \geq 2\). Then the code \(C_t\) defined by the sequence of Lemma 12 has parameters \([n, n - L_s, d]\) and generator polynomial \(M_s(x)\), where \(M_s(x)\) and \(L_s\) are given in Lemma 12 and
\[
\begin{align*}
    d &\geq 3 \text{ if } a = 2 \text{ and } \delta(1 - 5a + 5a^2) = 0, \\
    d &\geq 4 \text{ if } a = 2 \text{ and } \delta(1 - 5a + 5a^2) = 1, \\
    d &\geq 5 \text{ if } a = 7 \text{ and } \delta(1 - 5a + 5a^2) = 0, \\
    d &\geq 6 \text{ if } a = 7 \text{ and } \delta(1 - 5a + 5a^2) = 1, \\
    d &\geq 6 \text{ if } 1 - 3a + a^2 = 0 \text{ and } \delta(1 - 5a + 5a^2) = 0, \\
    d &\geq 6 \text{ if } 1 - 3a + a^2 = 0 \text{ and } \delta(1 - 5a + 5a^2) = 1, \\
    d &\geq 6 \text{ if } (a^2 - 3a + 1)(a - 2)(3a - 2) \neq 0 \text{ and } \delta(1 - 5a + 5a^2) = 0, \\
    d &\geq 7 \text{ if } (a^2 - 3a + 1)(a - 2)(3a - 2) \neq 0 \text{ and } \delta(1 - 5a + 5a^2) = 1.
\end{align*}
\]
The proof of this theorem is similar to that of Theorem 8 and is omitted.

Open Problem 6 Determine the minimum distance \( d \) of the code \( C_s \) in Theorem 12.

Examples of the code of Theorem 12 can be found in arXiv:1206.4370 and some of them are optimal. The code is a BCH code, except in the cases \( a \in \{2, 2/3\} \).

9 Cyclic codes from other \( D_i(x, a) \) for \( i \geq 6 \)

Parameters of cyclic codes from \( D_i(x, a) \) for \( i \geq 6 \) could be established in a similar way. However, more cases are involved and the situation is getting more complicated when \( i \) gets bigger. Examples of the code \( C_s \) from \( D_7(x, a) \) and \( D_{11}(x, a) \) can be found in arXiv:1206.4370.

10 Cyclic codes from Dickson polynomials of the second kind

Theorems on cyclic codes from Dickson polynomials of the second kind can be developed in a similar way as what we did for those from Dickson polynomials of the first kind in previous sections.

Experimental data indicates that the codes from the Dickson polynomials of the first kind are in general better than those from the Dickson polynomials of the second kind, though some cyclic codes from Dickson polynomials of the second kind could also be optimal or almost optimal.

11 Sets of sequences from Dickson polynomials

The purpose of this section is to demonstrate that optimal sets of sequences could be constructed with Dickson polynomials of the first kind. As an example, we consider the Dickson polynomials \( D_i(x, a) = x^i + ax \) over \( \text{GF}(2^m) \), where \( m \) is odd. With these polynomials, we define a set of binary sequences by

\[
S = \{ s(a)^\infty : a \in \text{GF}(2^m) \},
\]

where

\[
s(a)^i = \text{Tr}(D_i(1 + \alpha^i, a)) = \text{Tr}((\alpha^i)^3 + a\alpha^i + a + 1)
\]

for all \( i \geq 0 \). The period of each sequence \( s(a)^\infty \) is \( n := 2^m - 1 \). By Lemma 2, the linear span of \( s(a)^\infty \) equals \( m \) or \( m + 1 \) if \( a = 0 \), and \( 2m \) or \( 2m + 1 \) otherwise.

We now prove the following property for the set \( S \).

Lemma 13 For any two distinct elements \( a \) and \( b \) in \( \text{GF}(2^m) \), the two sequences \( s(a)^\infty \) and \( s(b)^\infty \) are different. Hence, \( |S| = 2^m \).

Proof Note that

\[
s(b)^i - s(a)^i = \text{Tr}((b - a)\alpha^i) + \text{Tr}(b - a).
\]

It then follows that the two sequences \( s(a)^\infty \) and \( s(b)^\infty \) are equal if and only if \( a = b \).

We will need the following lemma [18].
Lemma 14 Let $m$ be odd. For any $a \in \text{GF}(2^m)$ and $b \in \text{GF}(2^m)$ with $(a, b) \neq (0, 0)$, we have
\[
\sum_{x \in \text{GF}(2^m)} (-1)^{\text{Tr}(ax^3 + bx)} \in \{-1, -1 \pm 2^{(m+1)/2}\}.
\]

For any sequence $s^\infty$, the $h$-shift of $s^\infty$, denoted by $s[h]^\infty$, is defined by
\[
s[h][i] = s_{h+i}
\]
for all $i \geq 0$, where $h \geq 0$ is an integer.

Let $s^\infty$ and $t^\infty$ be two binary sequences of period $n$. The correlation value between the two sequences is defined by
\[
C(s, t) = \sum_{i=0}^{n-1} (-1)^{s_i - t_i}.
\]

Let $S$ be a set of binary sequence of period $n$. Then the maximum correlation value of $S$, denoted by $C(S)$, is defined by
\[
C(S) = \max\left\{ \max_{s \neq t, 0 \leq h < n} |C(t[h], s)|, \max_{1 \leq h < n} |C(t[h], s)| \right\}.
\]

We are now ready to prove the main result of this section.

Theorem 13 Let $m \geq 3$ be odd. Define $\tilde{S} = S \cup \{s(\infty)^\infty\}$, where
\[
s(\infty)_i = \text{Tr}(\alpha^i)
\]
for all $i \geq 0$, and $S$ was defined in (19). We have $|\tilde{S}| = 2^m + 1$ and
\[
C(\tilde{S}) = 1 + 2^{(m+1)/2}.
\]

Proof By definition, for $a \in \text{GF}(2^m)$ and $b \in \text{GF}(2^m)$, we have
\[
s(b)_{i+h} - s(a)_i = \text{Tr}((\alpha^{3h} - 1)(\alpha^i)^3 + (b\alpha^h - a)\alpha^i + (b - a))
\]
for all $i \geq 0$ and $h \geq 0$. Since $m$ is odd, $\gcd(3, 2^m - 1) = 1$. We then deduce that $\alpha^{3h} - 1 \neq 0$ for all $1 \leq h < n$.

For $b \in \text{GF}(2^m)$, we have
\[
s(b)_{i+h} - s(\infty)_i = \text{Tr}(\alpha^{3h}(\alpha^i)^3 + (b\alpha^h - 1)\alpha^i + b + 1)
\]
for all $i \geq 0$ and $h \geq 0$.

The desired conclusion on the maximum correlation value then follows from Lemma 14.

The set $\tilde{S}$ is a modification of the Gold sequence set, and is optimal with respect to both the Sidelnikov and Leveinshtein bound.
12 Concluding remarks

In this paper, we studied the codes derived from Dickson polynomials of the first kind with small degrees. It is really amazing that in most cases the cyclic codes derived from the Dickson polynomials of small degrees within the framework of this paper are optimal or almost optimal (see arXiv:1206.4370 for examples of optimal codes).

We had to treat Dickson polynomials of small degrees case by case over finite fields with different characteristics as we did not see any way to treat them in a single strike. The generator polynomial and the dimension of the codes depend heavily on the degree of the Dickson polynomials and the characteristic of the base field.

It should be noted that not all cyclic codes presented in this paper are new. Some of them are equivalent to some known family of cyclic codes in the literature. However, it is interesting to show that they can be produced when Dickson polynomials of very small degrees are plugged into the construction approach of this paper. It is also observed that the code $C_s$ derived from the Dickson polynomials of the first kind is sometimes a BCH code. However, the dimension and minimum distance of BCH codes are open in general, though progress on the study of primitive BCH codes have been made in the past 55 years.

The idea of constructing cyclic codes employed in this paper looks simple, but was proven to be very promising in this paper and also in [9],[14],[21]. It would be nice if other polynomials of special forms over finite fields can be employed in this approach to produce more optimal and almost optimal cyclic codes.

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