A NUMERICAL ASSESSMENT OF COSMIC-RAY ENERGY DIFFUSION THROUGH TURBULENT MEDIA

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ABSTRACT

How and where cosmic rays are produced, and how they diffuse through various turbulent media, represent fundamental problems in astrophysics with far-reaching implications, both in terms of our theoretical understanding of high-energy processes in the Milky Way and beyond, and the successful interpretation of space-based and ground based GeV and TeV observations. For example, recent and ongoing detections, e.g., by Fermi (in space) and HESS (in Namibia), of γ-rays produced in regions of dense molecular gas hold important clues for both processes. In this paper, we carry out a comprehensive numerical investigation of relativistic particle acceleration and transport through turbulent magnetized environments in order to derive broadly useful scaling laws for the energy diffusion coefficients.

Key words: acceleration of particles – cosmic rays – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

Until recently, the widely held paradigm for the origin of cosmic rays, at least below the knee at roughly 1 PeV, promoted the view that all of the intragalactic injection occurred via first-order Fermi acceleration in supernova shells. However, any direct evidence for this view is meager and equivocal. More recently, data from balloon-borne experiments have refuted the expectation from supernova acceleration schemes that the cosmic-ray spectrum ought to be structureless and universal (Wefel 1988). The latest measurements with PAMELA confirm and extend the balloon-based claims (Adriani et al. 2011). These data seem to call for a diverse variety of acceleration sites and mechanisms throughout the Galaxy.

In recent work (Melia & Fatuzzo 2011; Fatuzzo & Melia 2012a, 2012b), we have begun to assess the feasibility of stochastic acceleration within turbulent magnetized regions using highly detailed simulations of individual particle trajectories. A principal goal of this work is to accurately determine the spatial and energy diffusion coefficients of cosmic-ray protons in a broad range of environments, e.g., inside molecular clouds and the more tenuous intercloud medium. The spatial and energy diffusion coefficients calculated over a broad range of parameter space may be used, e.g., to compare estimates of the time required to energize protons up to TeV energies with the escape and cooling times throughout the interstellar medium (ISM). In previous applications, this approach has allowed us to conclude that protons in the intercloud medium at the galactic center can be energized up to the 1–10 TeV energies required to account for the observed HESS emission in this region (Aharonian et al. 2006; Liu et al. 2006a; Ballantyne et al. 2007; Wommer et al. 2008; see also Markoff et al. 1997, 1999; Liu & Melia 2001). Stochastic particle acceleration by magnetic turbulence appears to be a viable mechanism for cosmic-ray production at the galactic center (Liu et al. 2006b).

Developing an understanding of how the spatial diffusion coefficients depend on the physical environment and energy of the particles was the subject of our previous work (Fatuzzo et al. 2010). The focus of the present paper is specifically to determine the diffusion of cosmic rays in energy space, with its attendant dependence on all of the critical physical characteristics of the medium, such as the magnetic intensity, degree of turbulence, and size of the fluctuations. As before, we do this by using a modified numerically based formalism developed for the general study of cosmic-ray diffusion by Giacalone & Jokipii (1994). This approach has already been used successfully in several other contexts (see, e.g., O’Sullivan et al., 2009; Fatuzzo & Melia 2012a, 2012b). Here, we will extend this robust numerically based framework for the general analysis of stochastic acceleration of cosmic-rays by the turbulent electric fields generated along with the time-dependent turbulent magnetic field in a dynamically active medium.

We stress that our adopted model of turbulence as a superposition of Alfvénic like waves with linear dispersion relations is not meant to fully describe MHD turbulence in the ISM. For example, the model does not account for the fact that magnetic fluctuations decorrelate due to non-linear interactions before they can propagate over distances of multiple wavelengths (Goldreich & Sridhar 1995)—an effect that leads to resonance broadening—and as such, influences how thermal particles interact with turbulence (Lynn et al. 2012, 2013). However, our intent is to adopt a formalism that adequately describes turbulence as seen locally by highly relativistic cosmic rays. Since relativistic particles have speeds that are much greater than the Alfvén speeds considered in our analysis (which is then the limit in which our results are expected to be valid), they should not be sensitive to dynamical processes that occur on MHD timescales. In essence, we are relying on basic principles (e.g., the scaling laws between intensity and wavelength) to capture the global features of MHD turbulence that affect the propagation and acceleration of high energy cosmic rays.

The work presented here extends the results of previous works, most notably, that of O’Sullivan et al. (2009), as it significantly broadens the explored parameter space and also considers anisotropically distributed wave vectors (Goldreich & Sridhar 1995; Cho & Vishniac 2000). We focus primarily on strong turbulence (δB ∼ B) for which quasilinear theory is not applicable, although we do present results for isotropic weak turbulence as a consistency check. Our paper is organized as follows. The pertinent properties of the medium through which the particles diffuse are outlined in Section 2. The scheme for generating the turbulent magnetic and electric fields is presented in Section 3, along with the equations that govern the motion of cosmic rays. The basic elements of stochastic acceleration are
discussed in Section 4, and the results of our work are presented
in Section 5. The conclusions of our work are summarized in
Section 6.

2. THE PHYSICAL MEDIUM

The physical parameters found throughout the intergalactic
and interstellar media have values that span several orders
of magnitude. Of particular interest to our study, the most
vacuous regions of the intergalactic medium have particle
number densities \( n \lesssim 10^{-3} \text{ cm}^{-3} \) and magnetic field strengths
\( B \lesssim 0.1 \mu \text{G} \) (see, e.g., Kronberg 1994; Fraschetti & Melia
2008). In contrast, the denser regions near the supermassive
black hole at the galactic center have densities \( n \gtrsim 10^{12} \text{ cm}^{-3} \)
and field strengths \( B \gtrsim 1 \text{ G} \) (Ruffert & Melia 1994; Falcke & Melia
1997; Kowalenko & Melia 1999; Melia 2007; see also Misra & Melia
1993 for the case of stellar-mass size black holes).

Exactly how the magnetic field is partitioned within these
various media is not yet known, but there is a general relation
between density and field strength. In the simplest case where
flux freezing applies (say in the ISM), the magnetic field strength
\( B \) would scale with gas density \( n \) according to \( B \propto n^{1/2} \). It is
noteworthy, then, that an analysis of magnetic field strengths
measured in molecular clouds yields a relation between \( B \) and
\( n \) of the form

\[
B \sim 10 \mu \text{G} \left( \frac{n}{10^2 \text{ cm}^{-3}} \right)^{0.47},
\]

though with a significant amount of scatter in the data used to
produce this fit (Crutcher 1999; see also Fatuzzo et al. 2006 and
references therein). This result is consistent with the idea that
nonthermal linewidths, measured to be \( \sim 1 \text{ km s}^{-1} \) throughout
the cloud environment (e.g., Lada et al. 1991), arise from MHD
fluctuations.

Of course, even among molecular clouds, the physical
environment can be quite different depending on location.
Near the galactic center, the molecular clouds are considerably
different from their counterparts in the disk. For example,
the average molecular hydrogen number density over the
Sgr B complex—the largest molecular cloud complex (Lis &
Goldsmith 1989; Lis & Goldsmith 1990; Paglione et al. 1998)
near the galactic center—has a density \( 3 \times 10^3 \text{ cm}^{-3} \). Like its
traditional counterparts, Sgr B displays a highly nonlinear
structure, containing two bright sub-regions, Sgr B1 and Sgr B2, the
latter having an average molecular density of \( \sim 10^6 \text{ cm}^{-3} \), and
containing three dense (\( n \sim 10^5 \text{ cm}^{-3} \)) cores—labeled north, main, and south. These cores also show
considerable structure, containing numerous ultra-compact and
hyper-compact H II regions. As such, the densities associated
with the galactic-center molecular clouds are about two orders
of magnitude greater than those in the disk of our galaxy.

The exact nature of magnetic turbulence in these environ-
ments is itself not well constrained, though magnetic fluctua-
tions typically have a power-law spectrum. Their intensity at
a given wavelength scales according to \( (\delta B)_{2} \sim \lambda^{\frac{5}{3}} \), with
values of \( \Gamma = 1 \) (Bohm), \( \Gamma = 3/2 \) (Kraichnan), or \( \Gamma = 5/3 \)
(Kolmogorov) often adopted. In addition, the range in wave-
lengths over which these fluctuations occurs is not well known,
though it is reasonable to assume that the upper end corresponds
to the lengthscale over which the fluctuations are generated. (For
example, in the ISM, the turbulence is generated by supernova
remnants and stellar-wind collisions, so one might expect the
longest wavelength to be on the order of several parsecs or less;
see, e.g., Coker & Melia 1997; Melia & Coker 1999.) The lower
end must be smaller than the characteristic length scale associ-
ated with the particle motion (i.e., the gyration radius), under
the assumption that magnetic energy ultimately dissipates into
plasma energy via its coupling to these particles.

3. GOVERNING EQUATIONS

We explore how cosmic rays diffuse through a homogeneous
hydrogen gas of mass density \( \rho = \text{mm} \rho \) threaded by a uniform
static background field \( B_0 \), on which magnetic and electric
fluctuations propagate. From our numerical simulations, we
then compute energy diffusion coefficients covering a broad
range of particle energies and parameter space expected to span
the great diversity of environments observed both within our
galaxy and in the intergalactic medium. We focus primarily on
strong turbulence, for which the energy density of the turbulent
fields is comparable to that of the background fields.

A standard numerical approach to studying cosmic-ray dif-
sion treats the spatially fluctuating magnetic component \( \delta B \)
as the superposition of a large number of randomly polarized
transverse waves with wavelengths \( \lambda_n = 2\pi/k_n \), logarithmi-
cally spaced between \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) (e.g., Giacalone & Jokipii
1994; Casse et al. 2002; Fatuzzo et al. 2010). Adopting a static
turbulent field removes the necessity of specifying a dispersion
relation between the wavevectors \( k_n \) and their corresponding
angular frequencies \( \omega_n \). This approach appears suitable for consi-
dering highly non-linear turbulence (\( \delta B \gg B_0 \)), or simply
an environment without a background component. Of course,
turbulent magnetic fields in cosmic environments are not static.
Nevertheless, a static formalism in spatial diffusion calculations
of relativistic particles seems justified for environments in which
the Alfvén speed is much smaller than the speed of light.

The situation in this paper is quite different: we are focusing
on the energy diffusion of cosmic rays propagating through
a turbulent magnetic field, which requires the use of a time-
dependent formalism in order to self-consistently include the
fluctuating electric fields that must also be present (say, from
Faraday’s law). At present, such a theory of MHD turbulence in
the ISM remains elusive. Nevertheless, it is generally understood
that turbulence is driven from a cascade of longer wavelengths
to shorter wavelengths as a result of wave–wave interactions.
For strong MHD turbulence in a uniform medium, this cascade
seemingly produces eddies on small spatial scales that are
elongated in the direction of the underlying magnetic field, so
that the components of the wave vector along \( (k_1) \) and across
\( (k_2) \) the underlying field direction are related by the expression
\( \delta k_1 \propto k_2^{3/5} \) (Goldreich & Sridhar 1995; Cho & Vishniac 2000).

It is beyond the scope of this paper to develop a self-consistent
theory of MHD turbulence in the ISM that includes electric field
fluctuation. We therefore adopt the formalism of O’Sullivan
et al. (2009). Specifically, we assume a medium represented by
a nonviscous, perfectly conducting fluid threaded by a uniform
static field \( B_0 \), and use linear MHD theory as a guide. In the
linear regime, one can encounter three types of MHD waves:
Alfvén, fast and slow. As in O’Sullivan et al. (2009), we here
consider only Alfvén waves, for which the turbulent magnetic
field may be written as a sum of \( N \) randomly directed waves

\[
\delta B = \sum_{n=1}^{N} A_n e^{i(k_n \cdot r - \omega_n t + \phi_n)}.
\]
As noted already, this formalism does not adequately describe decorrelation effects known to be important for thermal cosmic rays (Lynn et al. 2012, 2013). However, it should serve as a reasonable model for the turbulence experienced by particles with velocities much greater than the Alfvén speed $v_A$, as such particles would be expected to travel over many correlation lengths (\~0.1\lambda_{\text{max}}) in an Alfvén time $\tau_A \sim \lambda_{\text{max}}/v_A$.

To keep the analysis as broad as possible, we focus on an isotropic turbulence spectrum, but we also perform a suite of experiments with anisotropic turbulence as informed by the results of Goldreich & Sridhar (1995). For the isotropic case, the direction of each propagation vector $k_n$ is set through a random choice of polar angles $\theta_n$ and $\phi_n$, and the phase of each term is set through a random choice of $\beta_n$. The sum has $N = \mathcal{N}_k \log[k_{\text{max}}/k_{\text{max}}]$ terms, with $k_n$ evenly spaced on a logarithmic scale between $k_{\text{min}} = 2\pi/\lambda_{\text{max}}$ and $k_{\text{max}} = 2\pi/\lambda_{\text{min}}$. A value of $\mathcal{N}_k = 25$ appears to provide enough terms to suitably model what is really a continuous rather than a discrete system. The appropriate choice of $\Gamma$ in the scaling

$$A_n^2 = A_1^2 \left[ \frac{k_n}{k_1} \right]^{-1} \frac{\Delta k_n}{\Delta k_1} = A_1^2 \left[ \frac{k_n}{k_1} \right]^{-1 + v}$$

(3)

sets the desired spectrum of magnetic turbulence (e.g., $\Gamma = 3/2$ for Kraichnan and $5/3$ for Kolmogorov turbulence). For our adopted scheme, the value of $\Delta k_n/k_n$ is the same for all $n$. The normalization for $A_1$ is set by the parameter

$$\eta = \frac{\langle \delta B^2 \rangle}{B_0^2},$$

(4)

defined as the ratio of magnetic energy density in the turbulent component to that of the static background field $B_0$.

For the anisotropic case, the direction of each perpendicular wavevector $k_{n,\perp}$ is set through a random choice of azimuthal angle $\phi_n$, and the phase of each term is again set through a random choice of $\beta_n$. The corresponding parallel component of the wavevector is then set through the relation

$$k_{n,||} = \pm \frac{\sqrt{2}}{k_{n,\perp}} \frac{1/3}{k_{n,\perp}^{2/3}}$$

(5)

(with a randomly chosen sign), such that

$$k_1 = \sqrt{k_{n,\perp}^2 + k_{n,\perp}^2} = \frac{2\pi}{\lambda_{\text{max}}}.$$ (6)

Consistent with the results of Goldreich & Sridhar (1995), we only consider a Kolmogorov profile, so that

$$A_n^2 = A_1^2 \left[ \frac{k_{n,\perp}}{k_{n,\perp}} \right]^{-2/3}.$$ (7)

Since Alfvén waves don’t compress the fluid through which they propagate, their fluid velocity $\mathbf{v}$ satisfies the condition $k \cdot \mathbf{v} = 0$. In addition, $\mathbf{v} \cdot B_0 = 0$ for Alfvén waves. The fluid velocity associated with the $n$th term in Equation (2) is therefore

$$\delta \mathbf{v}_n = \pm A_n \frac{v_A}{B_0} \frac{B_0 \times k_n}{|B_0 \times k_n|} e^{i(k_n \cdot r - \omega_n t + \beta_n)},$$

(8)

where the sign is chosen randomly for each term in the sum. The dispersion relation for Alfvénic waves is given by the expression

$$\omega_n = v_A |k_n||.$$ (9)

Each wave has a magnetic field given by the linear form of Ampère’s law,

$$A_n = \mp A_n \frac{k_n \cdot B_0}{|B_0 \times k_n|} \frac{B_0 \times k_n}{|B_0 \times k_n|},$$

(10)

which is identical to that of O’Sullivan et al. (2009).

Insofar as the electric fields are concerned, if we were to naively extrapolate from the results of linear MHD theory, the total electric field $\delta \mathbf{E}$ associated with the turbulent magnetic field in Equation (2) would be given by a sum over the terms

$$\delta \mathbf{E} = \mp \delta \mathbf{v} \times \mathbf{B}_0.$$ (11)

Notice that $\delta \mathbf{E} \cdot \mathbf{B}_0 = 0$, but the second order term $\delta \mathbf{E} \cdot \delta \mathbf{B} \neq 0$. The presence of an electric field component parallel to the magnetic field in this second order term can significantly increase the acceleration efficiency artificially, especially if the formalism is extended to the nonlinear regime ($\delta \mathbf{B} \sim \mathbf{B}_0$). However, the ISM is highly conductive, so any electric field component parallel to the magnetic field should be quickly quenched. O’Sullivan et al. (2009) circumvented this problem by first obtaining the total fluid velocity $\delta \mathbf{v}$ via the summation

$$\delta \mathbf{v} = \sum_{n=1}^{N} \delta \mathbf{v}_n,$$$$

(12)

and then using the MHD condition to set the total electric field:

$$\delta \mathbf{E} = -\frac{\delta \mathbf{v}}{c} \times \mathbf{B},$$

(13)

where $\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}$. This is the procedure we too will use here.

With these electric and magnetic field components, one may then solve the Lorentz force equation

$$\frac{d}{dt}(\gamma m_p \mathbf{v}) = e \left[ \delta \mathbf{E} + \mathbf{v} \times \mathbf{B} \right],$$

(14)

with

$$\frac{dr}{dt} = \mathbf{v},$$

(15)

to determine the motion of a relativistic charged proton with Lorentz factor $\gamma$ through the turbulent medium. The solutions to these equations are not sensitive to the value of $\lambda_{\text{min}}$ so long as the particle’s gyration radius $r_g = \gamma mc^2/|eB| \gg \lambda_{\text{min}}$ (Fatuzzo et al. 2010). As such, we set $\lambda_{\text{min}} = 0.1 \gamma mc^2/|eB_0|$ in all our simulations. To completely specify the physical state of an environment to be studied, we must therefore provide values for the parameters $B_0$, $n$, $\lambda_{\text{max}}$, $\Gamma$, and $\eta$.

4. BASIC ELEMENTS OF STOCHASTIC ACCELERATION

The motion of a charged particle in perpendicular uniform electric and magnetic fields represents a fundamental topic in electrodynamics and is well understood. It is therefore easy to show that in general, the charge gains and loses kinetic energy in cyclic fashion as it “drifts” in the $\mathbf{E} \times \mathbf{B}$ direction.

In the turbulent medium in which the magnetic and electric fields are perpendicular, a single particle’s energy, characterized by $\Delta \gamma/\gamma_0 = \gamma/\gamma_0 - 1$, therefore exhibits a random-walk like behavior, as shown in Figure 1. The energy distribution of an ensemble of particles injected with the same Lorentz factor $\gamma_0$ thus broadens as the particles sample the turbulent nature of...
the accelerating electric fields. Interestingly, these distributions appear Gaussian for all of the isotropic and $\eta = 1$ anisotropic cases explored in our analysis. This point is illustrated by Figure 2, which shows the distribution of $\Delta \gamma$ values at time $t = 100 \lambda_{\text{max}} / c$ for an ensemble of 1000 particles injected with $\gamma_0 = 10^5$ into a medium defined by the same parameters used to calculate the energy evolution shown in Figure 1, but with an anisotropic turbulence profile. In such cases, one can therefore quantify the stochastic acceleration of particles in turbulent fields through the dispersion of the resulting distributions of initially mono-energetic particles.

In contrast, the particle distributions obtained for anisotropic, weak turbulence ($\eta \ll 1$) cases have wings that are much broader than a Gaussian distribution, so that their dispersion significantly overestimates the true distribution width. This point is illustrated in Figure 3, which shows the distribution of $\Delta \gamma$ values at time $t = 100 \lambda_{\text{max}} / c$ for an ensemble of 1000 particles injected with $\gamma_0 = 10^5$ into a medium defined by the same parameters used to calculate the distribution shown in Figure 2, but with $\eta = 0.01$.

To further compare strong and weak isotropic and anisotropic turbulence, we plot in Figure 4 the dispersion $\sigma_\gamma \equiv \sqrt{\langle (\Delta \gamma)^2 \rangle}$ of the $\Delta \gamma$ distribution as a function of time for the particles used to generate Figures 2 and 3 (for which the turbulence was anisotropic) along with their isotropic counterparts. As found throughout our analysis, there is little difference between the isotropic and anisotropic cases when the turbulence is strong ($\eta = 1$). However, anisotropic turbulence appears to be significantly less effective at energizing particles than isotropic turbulence when $\eta \ll 1$, in agreement with results obtained in the quasi-linear approximation (Chandran 2000; see also Yan & Lazarian 2002).

Clearly, the chaotic nature of motion through turbulent fields necessitates a statistical analysis. We define a single experiment as a numerical investigation of particle dynamics through a given environment (as defined by the parameters $B_0$, $n$, $\Gamma$, $\lambda_{\text{max}}$, and $\eta$) over a broad range of particle injection energies (as defined
by the Lorentz factor $\gamma_0$). As expected from the random nature of stochastic acceleration, $\sigma_\gamma \propto \sqrt{t}$ once particles have had a chance to sample the turbulent nature of the underlying electric fields, i.e., for $t \gtrsim \lambda_{\text{max}}/c$. This in turn means that the energy diffusion coefficient $D_\gamma \equiv \langle \Delta \gamma^2 \rangle / (2 \Delta t)$ can be calculated by using an integration time $\Delta t \gtrsim \lambda_{\text{max}}/c$. For each particle energy, we numerically integrate the equations of motion for a time $\Delta t = 10^5 \lambda_{\text{max}}/c$ for $N_p = 1000$ protons randomly injected from the origin. Each particle samples its own unique magnetic field structure (i.e., the values of $\beta_n, \theta_n, \phi_n$, and the choice of $\pm$ are chosen randomly for each particle for the isotropic case).

### 5. RESULTS OF NUMERICAL EXPERIMENTS

We use the procedure described above to carry out a suite of experiments sampling a broad region of parameter space for isotropic turbulence, and perform a limited complimentary set of experiments for anisotropic, strong ($\eta = 1$) Kolmogorov turbulence. We limit our analysis to particle energies for which the gyration radius $R_g$ falls comfortably below the maximum turbulence wavelength $\lambda_{\text{max}}$, so that particles actually diffuse through the turbulent medium. A principal goal of this paper is to determine the relationship between the energy diffusion coefficient $D_\gamma$ and the particle’s energy for each experiment. We therefore fit the numerical data using a power law

$$D_\gamma = D_{\gamma 0} \gamma^\alpha.$$  \hspace{1cm} (16)

### Table 1

| Exp | $B_0$ (\(\mu G\)) | $n$ (cm$^{-3}$) | $\Gamma$ | $\lambda_{\text{max}}$ (pc) | $\eta$ | $D_{\gamma 0}$ (s$^{-1}$) | $\alpha$ |
|-----|-------------------|----------------|--------|-----------------|-------|----------------|--------|
| 1   | 100               | 100            | 5/3    | 0.1             | 1     | $5.4 \times 10^{-13}$ | 1.44   |
| 2   | 100               | 100            | 5/3    | 0.1             | 0.1   | $2.5 \times 10^{-14}$ | 1.48   |
| 3   | 100               | 100            | 5/3    | 0.01            | 2.6   | $5.9 \times 10^{-18}$ | 1.70   |
| 4   | 100               | 100            | 3/2    | 0.1             | 1.2   | $2.4 \times 10^{-17}$ | 1.64   |
| 5   | 100               | 100            | 3/2    | 0.001           | 6.1   | $10^{-11}$ | 1.10   |
| 6   | 100               | 100            | 3/2    | 0.1             | 8.2   | $10^{-15}$ | 1.20   |
| 7   | 100               | 100            | 1      | 0.1             | 4.7   | $10^{-11}$ | 1.46   |
| 8   | 100               | 100            | 5/3    | 0.1             | 1.5   | $10^{-11}$ | 1.46   |
| 9   | 100               | 100            | 5/3    | 1               | 3.9   | $10^{-12}$ | 1.47   |
| 10  | 3.16              | 5/3            | 0.1    | 1               | 1.4   | $10^{-12}$ | 1.46   |
| 11  | 3.16              | 5/3            | 0.1    | 1               | 1.2   | $10^{-13}$ | 1.48   |
| 12  | 100               | 5/3            | 0.1    | 1               | 5.2   | $10^{-14}$ | 1.45   |
| 13  | 100               | 5/3            | 0.1    | 1               | 1.1   | $10^{-20}$ | 1.46   |
| 14  | 3.16              | 5/3            | 0.1    | 1               | 7.5   | $10^{-17}$ | 1.45   |
| 15  | 3.16              | 5/3            | 0.1    | 1               | 3.0   | $10^{-9}$ | 1.46   |
| 16  | 10^5              | 5/3            | 0.1    | 1               | 4.0   | $10^{-6}$ | 1.54   |
| 17  | 10^5              | 5/3            | 0.1    | 1               | 1.0   | $10^{-11}$ | 1.48   |
| 18  | 10^5              | 5/3            | 0.1    | 1               | 1.8   | $10^{-14}$ | 1.47   |
| 19  | 10^5              | 5/3            | 0.1    | 1               | 1.9   | $10^{-20}$ | 1.39   |
| 20  | 10^5              | 5/3            | 0.1    | 1               | 1.6   | $10^{-16}$ | 1.37   |
| 21  | 3.16              | 3/2            | 0.1    | 1               | 1.4   | $10^{-8}$ | 1.39   |
| 22  | 3.16              | 3/2            | 0.1    | 1               | 5.3   | $10^{-5}$ | 1.42   |
| 23  | 10^5              | 3/2            | 0.1    | 1               | 1.8   | $10^{-11}$ | 1.40   |
| 24  | 10^5              | 3/2            | 0.1    | 1               | 1.0   | $10^{-13}$ | 1.38   |
| 25  | 10^5              | 3/2            | 0.1    | 1               | 1.1   | $10^{-19}$ | 1.11   |
| 26  | 3.16              | 100            | 0.1    | 1               | 2.6   | $10^{-15}$ | 1.09   |
| 27  | 3.16              | 100            | 0.1    | 1               | 9.6   | $10^{-7}$ | 1.12   |
| 28  | 3.16              | 100            | 0.1    | 1               | 6.2   | $10^{-2}$ | 1.07   |
| 29  | 10^5              | 100            | 0.1    | 1               | 1.0   | $10^{-10}$ | 1.12   |
| 30  | 10^5              | 100            | 0.1    | 1               | 2.6   | $10^{-11}$ | 1.10   |
| 31  | 10^5              | 100            | 0.1    | 1               | 2.6   | $10^{-11}$ | 1.10   |

#### Figure 5

Energy diffusion coefficients as a function of particle Lorentz factor for Experiments 1–4 which sample various turbulence strengths of $\eta$. Solid shapes denote values for isotropic turbulence, and open squares denote values for anisotropic turbulence. The uniform magnetic field is 100 $\mu$G in every case. In addition, $n = 100$ cm$^{-3}$, and $\lambda_{\text{max}} = 0.1$ pc. Solid lines show fits to the isotropic turbulence data. Results obtained by quasi-linear theory as given by Equation (17) are shown by the dotted ($\eta = 1$), dashed ($\eta = 0.1$), dot-dashed ($\eta = 0.01$), and long-dashed ($\eta = 0.001$) lines.

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We plot the results of Experiments 1–4 in Figure 5 and Experiments 1, 4, and 5–8 in Figures 6 and 7 to illustrate how the
Equation (17) overestimates the value of the best match occurring for Kolmogorov diffusion. However, linear theory, and weak \((\eta \ll 1)\) turbulence, which sample various ambient densities \(n\), the diffusion coefficient index changes between the strong \((\eta \sim 1)\) and weak \((\eta \ll 1)\) turbulence limits. As predicted by quasi-linear theory, \(\alpha \propto \Gamma\) when \(\eta \ll 1\) for isotropic turbulence, with the best match occurring for Kolmogorov diffusion. However, Equation (17) overestimates the value of \(D_{\gamma0}\) by as much as two orders of magnitude. In addition, the indices decrease as the turbulence grows stronger, and quasi-linear theory does not appear to adequately describe the energy dependence for Kolmogorov \((\Gamma = 5/3)\) and Kraichnan \((\Gamma = 3/2)\) diffusion when \(\eta \sim 1\).

We next consider how the diffusion coefficients for Kolmogorov \((\Gamma = 5/3)\), Kraichnan \((\Gamma = 3/2)\) and Bohm \((\Gamma = 1)\) turbulence scale with \(n\), \(B_0\) and \(\lambda_{\text{max}}\), but limit our analysis to the strong turbulence limit \((\eta \gg 1)\). Our results for Kolmogorov turbulence (derived from Experiments 1 and 9–20) are shown in Figures 8–13. Specifically, Figure 8 illustrates the energy dependence of \(D_\gamma\) for four values of ambient density \(n\). As expected, a greater density yields a smaller Alfvén speed, and therefore a reduced energy diffusion. However, the index \(\alpha\) is not sensitive to \(n\). We note that Equation (17) overestimates the energy diffusion coefficient for \(\gamma_0 \gg 10^3\), and underestimates the energy diffusion coefficient for \(\gamma_0 \lesssim 10^3\). Figure 9 then illustrates the relation between \(D_\gamma\) and \(n\) for the value \(\gamma_0 = 10^5\), where the fits from Figure 8 (along with fits to the data not shown in that figure) are used to obtain the plotted data points. The relationship between \(D_\gamma\) and \(n\) clearly takes on the same scaling obtained from quasi-linear theory, i.e., \(D_\gamma \propto n^{-1}\). This result is expected since \(D_\gamma \propto (\Delta \gamma)^2 \propto (\delta E)^2 \propto V_A^2 \propto n^{-1}\). We therefore assume that this scaling holds for all values of \(\Gamma\).

Figure 10 illustrates the energy dependence of \(D_\gamma\) for five values of background field strengths \(B_0\). Since a greater magnetic field strength leads to a greater electric field strength, energy diffusion increases with field strength. Again, the index \(\alpha\) is not sensitive to \(B_0\), but we find that Equation (17) overestimates the energy diffusion for larger values of \(\gamma_0\), and underestimates the energy diffusion for smaller values of \(\gamma_0\), although the transition...
The solid line shows the fit to the data (\(D_{\gamma} \propto B_0^{2.50}\)), and the dotted line represents the result obtained by quasi-linear theory as given by Equation (17).

\(\gamma_0 = 10^5\) as a function of background field strength \(B_0\) for Experiments 1 and 15–18 (isotropic turbulence). The particle density \(n = 100\,\text{cm}^{-3}\), background field strength \(B_0 = 100\,\mu\text{G}\), \(\eta = 1\), and \(\Gamma = 5/3\) in every case. Solid shapes denote values for isotropic turbulence, and open shapes denote values for anisotropic turbulence. Solid lines show fits to the data, and dotted lines represent the results obtained by quasi-linear theory as given by Equation (17).

\(\gamma_0 = 10^5\) as a function of \(\lambda_{\text{max}}\) for Experiments 1 and 19–20 (isotropic turbulence). The particle density \(n = 100\,\text{cm}^{-3}\), background field strength \(B_0 = 100\,\mu\text{G}\), \(\eta = 1\), and \(\Gamma = 5/3\) in every case. Solid shapes denote values for isotropic turbulence, and open shapes denote values for anisotropic turbulence. Solid lines show fits to the data, and dotted lines represent the results obtained by quasi-linear theory as given by Equation (17).

\(\gamma_0 = 10^5\) as a function of \(\eta\) for Experiments 1, 5, and 21–26. The particle density \(n = 100\,\text{cm}^{-3}\), background field strength \(B_0 = 100\,\mu\text{G}\), \(\eta = 1\), and \(\Gamma = 5/3\) in every case. Solid shapes denote values for isotropic turbulence, and open shapes denote values for anisotropic turbulence. Solid lines show fits to the data, and dotted lines represent the results obtained by quasi-linear theory as given by Equation (17).

Figure 11. Energy diffusion coefficients as a function of particle Lorentz factor for Experiments 1 and 19–20, which sample various values of \(\lambda_{\text{max}}\). The particle density \(n = 100\,\text{cm}^{-3}\), background field strength \(B_0 = 100\,\mu\text{G}\), \(\eta = 1\), and \(\Gamma = 5/3\) in every case. Solid shapes denote values for isotropic turbulence, and open shapes denote values for anisotropic turbulence. Solid lines show fits to the data, and dotted lines represent the results obtained by quasi-linear theory as given by Equation (17).

\[D_{\gamma} \propto B_0^{2.50}\]

\[\eta \approx 1\]

\[\lambda_{\text{max}} \leq \lambda_{\text{gyr}}\]

\[\lambda_{\text{gyr}} \ll R_g \ll \lambda_{\text{max}}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.47}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.67}\]

Corresponding results for isotropic Kraichnan turbulence (derived from Experiments 5 and 21–26) are shown in Figures 14–17, and for isotropic Bohm turbulence (derived from Experiments 7 and 27–32) are shown in Figures 18–21. The analysis for these cases mirrors that described above for Kolmogorov turbulence, and the same general results are obtained.

Our results indicate that simple scaling laws of the form

\[D_{\gamma} = D_{\gamma 0} \gamma^{\alpha}\]

where

\[D_{\gamma 0} = D_0 \left(\frac{n}{1\,\text{cm}^{-3}}\right)^{-1} \left(\frac{B_0}{1\,\mu\text{G}}\right)^{\delta} \left(\frac{\lambda_{\text{max}}}{1\,\text{pc}}\right)^{\kappa}\]

\[\lambda_{\text{max}} \leq \lambda_{\text{gyr}}\]

\[\lambda_{\text{gyr}} \ll R_g \ll \lambda_{\text{max}}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.47}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.67}\]

\[\eta \approx 1\]

\[\lambda_{\text{max}} \leq \lambda_{\text{gyr}}\]

\[\lambda_{\text{gyr}} \ll R_g \ll \lambda_{\text{max}}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.47}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.67}\]

\[\eta \approx 1\]

\[\lambda_{\text{max}} \leq \lambda_{\text{gyr}}\]

\[\lambda_{\text{gyr}} \ll R_g \ll \lambda_{\text{max}}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.47}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.67}\]

\[\eta \approx 1\]

\[\lambda_{\text{max}} \leq \lambda_{\text{gyr}}\]

\[\lambda_{\text{gyr}} \ll R_g \ll \lambda_{\text{max}}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.47}\]

\[D_{\gamma} \propto \lambda_{\text{max}}^{-0.67}\]

\[\eta \approx 1\]

\[\lambda_{\text{max}} \leq \lambda_{\text{gyr}}\]

\[\lambda_{\text{gyr}} \ll R_g \ll \lambda_{\text{max}}\]
A plot of the acceleration time $\tau_{\text{acc}}$ as a function of molecular hydrogen density $n_H$ is shown in Figure 22 for Kolmogorov, Kraichnan, and Bohm turbulence. Given that clouds are not expected to last for more than $\sim 10$ Myr, TeV cosmic ray production from stochastic acceleration of turbulent magnetic fields can clearly be ruled our for normal molecular cloud environments. We note, however, that the molecular clouds near the galactic center (GC) have fairly extreme environments. As shown in Fatuzzo & Melia (2012b), the acceleration times in the GC environment are considerably shorter, as illustrated by the solid circle (inter cloud region at the GC) and solid square (molecular cloud at the GC) shown in Figure 22.

6. CONCLUSIONS

We have used a detailed numerical simulation to determine the spatial and spectral profiles of cosmic rays diffusing with arbitrary energy through turbulent media characterized by a broad range of magnetic fields, turbulence strength, fluctuation size, and ambient particle density, for both isotropic and anisotropic turbulence. This study has expanded considerably from the initial attempt at simulating the behavior of relativistic particle propagation through the galactic-center environment, where they encounter a variety of physical conditions, inside and
outside of the molecular gas in that region. Our goal throughout this exercise has been to avoid using “standard” techniques, e.g., quasi-linear theory or the diffusion equation, all of which are often subject to unknown factors that delimit the applicability of these approaches to real systems (but see also Nayakshin & Melia 1998; Wolfe & Melia 2006). Indeed, one of the principal benefits of the technique we have developed in this work is an accurate determination of the spatial and energy diffusion coefficients that in turn may be used in these other approaches without the need to guess or estimate their normalization and energy dependence.

As the sensitivity and spectral range of high-energy observatories continue to improve, the need to accurately simulate the propagation of relativistic particles through turbulent media arises in an ever increasing range of environments, from the ISM, to compact accretion regions surrounding supermassive black holes (such as Sgr A* at the galactic center), to the hot intracluster gas, and in even more exotic structures, such as the Fermi bubbles straddling the center of the Milky Way (Crocker & Aharonian 2011). However, the physical conditions characterizing these regions change considerably from one environment to the next. For example, the magnetic intensity may be as large as several Gauss near accreting black holes, but smaller than 0.01 μG in the intergalactic medium. We now know that quasi-linear theory is only approximately valid even in weakly turbulent environments, let alone regions where the turbulent magnetic energy is comparable to, or bigger than, the underlying uniform component. Worse, it is often difficult to estimate the absolute value of the diffusion coefficients without resorting to observational data which, however, are sometimes difficult to get (as in the case of the giant radio lobes in FR II galaxies).

These are among the reasons we have embarked on this type of investigation, to develop methods of handling the great diversity of physical conditions encountered by cosmic rays propagating through high-energy emitting environments. Previously (Fatuzzo et al. 2010; Fatuzzo & Melia 2012b), we reported some results of this study pertaining to the spatial diffusion coefficients. We found that the spatial diffusion of particles through turbulent fields is not sensitive to the minimum wavelength of the fluctuations, so long as the particle’s radius
of gyration exceeds \( \lambda_{\text{min}} \). For a given environment, as defined by \( B_0 \) and \( v_A \), the diffusion process is thus dependent upon the maximum turbulence wavelength \( \lambda_{\text{max}} \), the turbulent field strength, as characterized by \( \eta \), and the turbulence spectrum, as characterized by the spectral index \( \Gamma \).

We also noted that quasi-linear theory does not appear to be valid in the strong turbulence limit (see also O’Sullivan et al. 2009). We therefore investigated how the energy diffusion coefficient depends upon \( \lambda_{\text{max}} \) and \( \eta \) for Kolmogorov (\( \Gamma = 5/3 \)) turbulence, and found that the energy diffusion coefficients could be characterized as \( D_{\gamma} \propto \lambda_{\text{max}}^{-0.47} \). This behavior is not consistent with quasi-linear theory, which instead predicts that \( D_{\gamma} \propto \lambda_{\text{max}}^{-0.67} \) for Kolmogorov turbulence in the strong turbulence \( (\eta \gtrsim 1) \) limit. However, we also found that \( D_{\gamma} \propto \eta^{1.2} \) in both the weak and strong turbulence limits.

Clearly, this initial sampling of the complex behavior of \( D_{\gamma} \) under a variety of physical conditions is far from satisfactory. The purpose of the present paper has been to complete this work, finding scaling relations that one may use to calculate \( D_{\gamma} \) under most conditions of interest, for all practical ranges of magnetic intensity, turbulence strength, ambient particle density, fluctuation size, and turbulence spectrum.

The empirical relations useful for this purpose have all been presented in Section 5. Broadly speaking, we have found that insofar as the energy diffusion coefficient \( D_{\gamma} \) is concerned, quasi-linear theory predicts its correct energy dependence only for very weak, isotropic turbulence (i.e., \( \eta \lesssim 0.01 \)). These predictions deviate substantially for \( \eta \sim 1 \), particularly for turbulence spectral indices \( \Gamma > 1 \), such as Kolmogorov turbulence, which seems to be prevalent in many diverse environments. For example, for the physical conditions one encounters at the galactic center (i.e., \( B \sim 1 - 10^3 \) \( \mu \)G and \( n \sim 1 - 10^3 \) cm\(^{-3} \)), the actual index characterizing the dependence of \( D_{\gamma} \) on \( \gamma \) may be as small as \( \sim 1.4 \) instead of the predicted value \( \sim 1.7 \).

In addition, our results indicate that there is no difference in the energy diffusion of particles between isotropic turbulence and the anisotropic turbulence profiles predicted by Goldreich & Sridhar (1995) as long as the turbulence is strong (\( \delta B \sim B \)). On the other hand, our results indicate that anisotropic weak turbulence is considerably less effective in energetically scattering particles, consistent with the results of Chandran (2000).

Needless to say, if one is attempting to interpret the GeV Fermi spectrum of, say, the Fermi bubbles, in terms of an underlying population of cosmic rays, deviations from quasi-linear theory are rather critical, since the inferred particle distribution differs considerably from its injection point to where it emits the radiation, and the difference will be interpreted incorrectly with the inaccurate energy dependence predicted by these other techniques.

Not being sure of the normalization of \( D_{\gamma} \) has its own challenges. For one thing, it is not possible to say anything definitive about the overall power being generated by the acceleration of these cosmic rays, which speaks directly to the mechanism associated with the relativistic particle injection, or even to the required density of dark matter particles, if these cosmic rays are produced via dark matter decays and collisions. With our approach, it is not necessary to estimate the normalization of \( D_{\gamma} \), because its absolute value is determined self-consistently from the statistical aggregate of numerous individual particle trajectories.

We have found that quasi-linear theory provides an acceptable estimate of the normalization of \( D_{\gamma} \) at \( \sim 1 \) TeV energies, but can deviate considerably at lower energies, especially in the GeV range, and at energies exceeding 10–100 TeV. A large factor responsible for these differences is the incorrect energy dependence predicted for \( D_{\gamma} \). Obviously, if the normalization is adequate at \( \sim 1 \) TeV, the incorrect energy index will cause deviations at lower and higher energies.

Most importantly, however, our analysis has provided a method of determining not only the dependence of \( D_{\gamma} \) on \( \gamma \), but also its absolute value without the need to normalize it from the data. Having said this, one is not completely free of ambiguity, since one must still have an accurate estimate of the physical conditions, i.e., the magnetic field, the ambient density, and other characteristics that determine the state of the medium through which the cosmic rays are propagating. Fortunately, these conditions are easier to measure than the diffusion coefficients themselves, and in a more sophisticated use of our technique, in which MHD turbulence is simulated numerically rather than via the simple Kolmogorov or Bohm scaling relations, one can approach a level of realism not available to any of the other methods.

We look forward to the application of the scaling relations we have presented in this paper across a broad range of physical environments, allowing us to study high-energy sources with a level of accuracy commensurate with the detailed measurements now being made by the ever improving suite of space-based and ground-based observatories.

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REFERENCES

Adriani, O., Barbarino, G. C., Bazilevskaya, G. A., et al. 2011, Sci, 332, 69
Aharonian, F., Akhperjanian, A. G., Bazer-Bachi, A. R., et al. 2006, Natur, 439, 695
Ballantyne, D., Melia, F., & Liu, S. 2007, ApJL, 657, L13
Casse, F., Lemoine, M., & Pelleiter, G. 2002, PhRvD, 65, 023002
Chandran, B. D. G. 2000, PhRvL, 85, 4656
Cho, J., & Vishniac, E. T. 2000, ApJ, 539, 273
Coker, R. F., & Melia, F. 1997, ApJL, 488, L149
Crocker, R. M., & Aharonian, F. 2011, PhRvL, 106, 101102
Crutcher, R. M. 1999, ApJL, 520, 706
Falcke, H., & Melia, F. 1997, ApJL, 479, 740
Fatuzzo, M., Adams, F. C., & Melia, F. 2006, ApJL, 653, L49
Fatuzzo, M., & Melia, F. 2012a, ApJL, 757, L16
Fatuzzo, M., & Melia, F. 2012b, ApJL, 750, 21
Fatuzzo, M., Melia, F., Wommer, E., & Adams, F. 2010, ApJ, 725, 515
Frascetti, F., & Melia, F. 2008, MNRAS, 391, 1100
Giacalone, J., & Jokipii, J. R. 1994, ApJL, 430, L137
Goldreich, P., & Sridhar, S. 1995, ApJ, 438, 763
Kowalenko, V., & Melia, F. 1999, MNRAS, 310, 1053
Kronberg, P. P. 1994, RPP, 57, 325
Lada, E. A., Bally, J., & Stark, A. A. 1991, ApJ, 368, 432
Larson, B. L. 1981, MNRAS, 194, 809
Lis, D. C., & Goldsmith, P. F. 1989, ApJ, 337, 704
Lis, D. C., & Goldsmith, P. F. 1990, ApJL, 356, 195
Liu, S., & Melia, F. 2001, ApJL, 561, L77
Liu, S., Melia, F., Petrosian, V., & Fatuzzo, M. 2006a, ApJL, 647, L1099
Liu, S., Petrosian, V., Melia, F., & Fryer, C. L. 2006b, ApJ, 648, 1020
Lynn, J. W., Parrish, I. J., Quataert, E., & Chandran, B. D. G. 2012, ApJ, 758, 78
Lynn, J. W., Quataert, E., Chandran, B. D. G., & Parrish, I. J. 2013, ApJL, 777, 128
Markoff, S., Melia, F., & Sarcevic, I. 1997, ApJL, 489, 47L
Markoff, S., Melia, F., & Sarcevic, I. 1999, ApJL, 522, 870
Melia, F. 2007, The Galactic Supermassive Black Hole (Princeton, NJ: Princeton Univ. Press)
Melia, F., & Coker, R. F. 1999, ApJL, 511, 750
Melia, F., & Fatuzzo, M. 2011, MNRAS, 410, L23
Misra, R., & Melia, F. 1993, ApJL, 419, L25
Nayakshin, S., & Melia, F. 1998, ApJS, 114, 269
O’Sullivan, S., Reville, B., & Taylor, A. M. 2009, MNRAS, 400, 248
Paglione, D. T. A., Jackson, J. M., Bolatto, A. D., & Heyer, M. H. 1998, ApJ, 493, 680
Ruffert, M., & Melia, F. 1994, A&A, 288, 29
Schlickeiser, R. 1989, ApJ, 336, 243

Wefel, J. 1988, in Proc. NATO Advanced Study Institute, Genesis and Propagation of Cosmic Rays (Dordrecht: Reidel), 1
Wolfe, B., & Melia, F. 2006, ApJ, 638, 125
Wommer, E., Melia, F., & Fatuzzo, M. 2008, MNRAS, 387, 987
Yan, H., & Lazarian, A. 2002, PhRvL, 89, 281102