High-energy jet quenching in weakly-coupled quark-gluon plasmas

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Abstract

$\hat{q}$ is the average squared transverse momentum transfer per unit length to a high-energy particle traversing a QCD medium such as a quark-gluon plasma. We find the (UV-regulated) value of $\hat{q}$ to leading order in the weak coupling limit, $\alpha_s(T) \ll 1$. We then use this value to generalize previous analytic results on the gluon bremsstrahlung and pair production rates for massless high-energy particles in a weakly-coupled quark-gluon plasma, at next-to-leading logarithmic order.
I. INTRODUCTION

In a weakly-coupled quark-gluon plasma, energy loss of high-energy jets is dominated by gluon bremsstrahlung and pair production. One of the elements that goes into the calculation of gluon bremsstrahlung rates in the high-energy limit is the parameter \( \hat{q} \) — the average squared transverse momentum transfer per unit length to a high-energy particle due to elastic collisions with plasma particles. In this paper, we find the (UV-regulated) value of \( \hat{q} \) to leading order in the weak coupling limit. We then use it to generalize previous analytic results on the gluon bremsstrahlung and pair production rates at leading order in coupling and next-to-leading order in inverse powers of \( \ln(E/T) \), where \( E \) is the energy of a jet particle.

A. The parameter \( \hat{q} \)

We will define the parameter \( \hat{q} \) by

\[
\hat{q}(\Lambda) \equiv \int_{q_\perp < \Lambda} d^2 q_\perp \frac{d \Gamma_{el}}{d^2 q_\perp} q_\perp^2,
\]

where \( \Gamma_{el} \) is the rate for elastic collisions with plasma particles, \( q_\perp \) is the transverse momentum transfer in a single such collision, and \( \Lambda \) is an ultraviolet cut-off whose physical scale will be set later when we discuss gluon bremsstrahlung. The value of \( \hat{q} \) is proportional to a factor of the quadratic Casimir \( C_R \) for the color representation \( R \) of the high-energy particle. It will be convenient to factor out this dependence by defining \( \bar{q} \) by

\[
\hat{q} = C_R \bar{q}.
\]

The differential elastic cross-section has the known limiting forms\(^1\)

\[
\frac{d \Gamma_{el}}{d^2 q_\perp} \simeq \frac{C_R}{(2\pi)^2} \times \begin{cases} \frac{g^2 T m_D^2}{q_\perp^2 (q_\perp^2 + m_D^2)}, & q_\perp \ll T, \\ \frac{g^4 N}{q_\perp^2}, & q_\perp \gg T, \end{cases}
\]

where \( m_D \) is the Debye mass and \( N \) is the density of plasma particles times color group factors (details below).

If we wanted \( \hat{q}(\Lambda) \) for \( \Lambda \ll T \), we could simply use the first case of (1.3) to obtain\(^2\)

\[
\hat{q}(\Lambda) \simeq \int_{q_\perp < \Lambda} \frac{d^2 q_\perp}{(2\pi)^2} \frac{g^2 T m_D^2}{q_\perp^2} \simeq 2\alpha T m_D^2 \ln \left( \frac{\Lambda}{m_D} \right) \quad (m_D \ll \Lambda \ll T). \tag{1.4}
\]

However, as we shall discuss momentarily, bremsstrahlung for sufficiently high energy particles depends on \( \hat{q}(\Lambda) \) for \( \Lambda \gg T \). For this, we will need to find \( d \Gamma_{el}/d^2 q_\perp \) for \( q_\perp \sim T \) as well.

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\(^1\) The simple formula for the small \( q_\perp \) case is taken from Aurenche, Gelis, and Zakaret [2]. The \( d \Gamma_{el}/d^2 q_\perp \) presented here corresponds to the notation \( g^2 C_R A(q_\perp)/(2\pi)^2 \) of Refs. [6, 11]. See Appendix A of Ref. [13].

\(^2\) For (1.4), see also Eq. (13) of Ref. [14] and the relation to Ref. [15] discussed after Eq. (61) of Ref. [14].
Our final result for $\hat{q}(\Lambda)$ in this case is given in Sec. II below, and the derivation follows in Sec. III.

Finding $\hat{q}(\lambda)$ for $\Lambda \gg T$ is an interesting problem of principle: It’s always good to understand how to get precise answers to questions in the limit of weak coupling. But it’s important to note that, even within the context of the weak coupling limit, getting it right will not have a large effect on the answer. As noted in Ref. [11], the numerators $g^2 m_p^2 T$ and $g^4 N$ in (1.3) differ by only about 15% for 3-flavor QCD. So one would not make a large error by using the simple formula (1.4) even when $\Lambda \gg T$. Nonetheless, in the present work we aim to find the exact weak-coupling answer for this limit.

B. Gluon bremsstrahlung

When very high-energy particles travel through a weakly-coupled quark-gluon plasma, the dominant energy loss mechanism is through hard bremsstrahlung or pair creation. Each time a high energy particle collides with a plasma particle, there is the potential for bremsstrahlung or pair creation. At high energies (parametrically $E \gtrsim T$), the quantum mechanical duration (formation time) of a splitting process becomes larger than the mean free time between the underlying collisions with plasma particles, leading to coherence effects that reduce the bremsstrahlung or pair production rate. This is known as the Landau-Pomeranchuk-Migdal (LPM) effect. In this paper, we consider the calculation of such splitting rates for particles with large energies $E \gg T$, where $T$ is the temperature of the plasma. We will consider the case of a time-independent, uniform, infinite quark-gluon plasma. In practice, that means that the plasma temperature should not vary significantly over the time and distance scales associated with the formation time of the splitting process.

A calculation of these rates to leading-order in $\alpha_s$ was performed by Jeon and Moore [3], based on the formalism of Arnold, Moore, and Yaffe (AMY) [4–6]. These calculations require numerical solutions of integral equations. It was noted earlier by Baier, Dokshitzer, Mueller, Peigne and Schiff (BDMPS) [7–10], however, that analytic solutions can be found if the logarithm of the energy is large, and they performed a leading-log analysis. Arnold and Dogan [11] recently extended this analysis to next-to-leading order in inverse powers of $\ln(E/T)$. The results of all of these analysis depend on the differential rate $d\Gamma_{el}/d^2 q_\perp$ for elastic scattering of the high energy particle off of a plasma particle, as a function of the transverse momentum exchange $q_\perp$. The full numerical calculations of rates to leading order in $\alpha_s$ by Jeon and Moore [3], and its approximation to next-to-leading log (NLL) by Arnold and Dogan [11], were carried out using the $q_\perp \ll T$ case of (1.3). However, the relevant range of $q_\perp$ which contributes to the calculation grows as the energy $E$ of the splitting parton grows. Roughly speaking, as the formation time gets longer, that time subsumes more and more elastic collisions, which means that the total momentum transfer $Q_\perp$ from the plasma to the splitting particle during the formation time grows larger. $Q_\perp$ grows parametrically
as

$$Q_\perp \sim (\hat{q}E)^{1/4}. \quad (1.5)$$

The total momentum transfer $Q_\perp$ is made up of many individual momentum transfers $q_\perp$, which range in scale from $m_D$ to $Q_\perp$ itself. At high enough energy, the assumption $q_\perp \ll T$ therefore breaks down for the upper end of this range. Based on (1.5), the condition $Q_\perp \ll T$ for assuming all $q_\perp \ll T$ is

$$E \ll \frac{T^4}{\hat{q}}. \quad (1.6)$$

Using (1.4) and $m_D \sim gT$, this condition is parametrically [11]

$$E \ll \frac{T}{g^4 \ln(1/g)}. \quad (1.7)$$

In the current work, we will redo Arnold and Dogan's next-to-leading log calculation of hard bremsstrahlung and pair production, rewriting the answer in terms of $\hat{q}$. We will then be able to use our result for $\hat{q}(\Lambda)$ with $\Lambda \gg T$ to generalize the previous results to the new case

$$E \gg \frac{T}{g^4 \ln(1/g)}. \quad (1.8)$$

Our result is given in Sec. II below, and the few modifications to the original derivation of Arnold and Dogan [11] are outlined in Appendix D.

C. Weak coupling and $m_D \ll T$

In this paper, we ruthlessly work in the weak coupling limit. In particular, the Debye mass $m_D$ is parametrically of order $gT$, and so we shall formally assume that $m_D \ll T$. In practice, however, one interest in weak coupling calculations is to see what they give if optimistically applied to realistic situations where the coupling is not terribly small. Unfortunately, the weak coupling result for $m_D$ for massless 3-flavor QCD gives, for example, $m_D \simeq 2.4T$ at $\alpha_s \simeq 0.3$. Exactly how terrible it is to treat $m_D \ll T$ depends on the details of the calculation. In this paper, we simply explore the weak coupling limit and will not consider possible improvements that might be made by not assuming $m_D \ll T$. Recent progress on going beyond this approximation has been made by Caron-Huot [1], who pushes the calculation of $d\Gamma_{el}/d^2 q_\perp$ and $\hat{q}$ to next-to-leading order in coupling $g$.

II. RESULTS

A. Notation

Throughout, we use the same notation for group factors as in Ref. [11]. For a color representation $R$, $C_R$ is the quadratic Casimir, $d_R$ is the dimension, and $t_R = d_R C_R / d_A$ is
the trace normalization. For QCD,

\[ C_A = 3, \quad C_F = \frac{4}{3}, \quad d_A = 8, \quad d_F = 3, \quad t_A = 3, \quad t_F = \frac{1}{2} \]

(2.1)

for gluons (A) and quarks (F).

We will use \( \Xi_b \) and \( \Xi_f \) to represent the number of spin and flavor degrees of freedom in plasma bosons and fermions, respectively, times the corresponding trace normalizations \( t_R \).

In \( N_f \)-flavor QCD,

\[ \Xi_b = 2t_A = 6, \quad \Xi_f = 4N_f t_F = 2N_f. \]

(2.2)

The Debye mass and the weighted density \( \mathcal{N} \) appearing in (1.3) are given by the following formulas, which are cast in a form that will be useful later on:

\[ m_D^2 = \left[ \Xi_b \zeta_+(2) + \Xi_f \zeta_-(2) \right] \frac{g^2 T^2}{\pi^2} = \left( 1 + \frac{1}{6} N_f \right) g^2 T^2 \]

(2.3a)

and

\[ \mathcal{N} = \left[ \Xi_b \zeta_+(3) + \Xi_f \zeta_-(3) \right] \frac{T^3}{\pi^2} = \frac{\zeta(3)}{\zeta(2)} (1 + \frac{1}{4} N_f) T^3. \]

(2.3b)

Here the functions \( \zeta_\pm(z) \) are the bosonic and fermionic versions of the Riemann \( \zeta \) function,

\[ \zeta_\pm(s) \equiv \sum_{k=1}^{\infty} \frac{(\pm)k^{-1}}{k^s}, \]

(2.4)

and so

\[ \zeta_+(s) = \zeta(s), \quad \zeta_-(s) = (1 - 2^{-s}) \zeta(s). \]

(2.5)

Recall that \( \zeta(2) = \pi^2/6 \).

**B. Result for \( \hat{q} \)**

Our final result for \( \hat{q}(\Lambda) \) for large cut-off \( \Lambda \) is

\[ \hat{q}(\Lambda) = \left[ \Xi_b \mathcal{I}_+ (\Lambda) + \Xi_f \mathcal{I}_- (\Lambda) \right] g^4 T^3 \frac{\Lambda^4}{\pi^2} \quad (\Lambda \gg T) \]

(2.6a)

with

\[ \mathcal{I}_\pm(\Lambda) \simeq \frac{\zeta_\pm(3)}{2\pi} \ln \left( \frac{\Lambda}{m_D} \right) + \Delta \mathcal{I}_\pm, \]

(2.6b)

\[ \Delta \mathcal{I}_\pm = \frac{\zeta_\pm(2) - \zeta_\pm(3)}{2\pi} \left[ \ln \left( \frac{T}{m_D} \right) + \frac{1}{2} - \gamma_E + \ln 2 \right] - \frac{\sigma_\pm}{2\pi}, \]

(2.6c)

and

\[ \sigma_\pm \equiv \sum_{k=1}^{\infty} \frac{(\pm)k^{-1}}{k^3} \ln [(k-1)!]. \]

(2.6d)

Above, \( \gamma_E \) is the Euler-Mascheroni constant. As we’ll discuss later, the sums \( \sigma_\pm \) are related to a certain generalization of the \( \zeta \) function, but we are unaware of any way of writing them in terms of more usual mathematical constants. Their numerical values are

\[ \sigma_+ = 0.386043817389949 \cdots \quad \text{and} \quad \sigma_- = 0.011216764589789 \cdots. \]

(2.7)
The result just quoted ignored running of the coupling constant. If $\Lambda$ is so large that $g^2(\Lambda)$ is significantly different from $g^2(m_D)$ and $g^2(T)$, then one should make the replacements

$$g^4 \ln \left( \frac{\Lambda}{m_D} \right) \to g^2(\Lambda) g^2(m_D) \ln \left( \frac{\Lambda}{m_D} \right), \quad (2.8)$$

$$g^4 \ln \left( \frac{T}{m_D} \right) \to g^2(T) g^2(m_D) \ln \left( \frac{T}{m_D} \right), \quad (2.9)$$

following the discussion in Refs. [11, 13, 17], and replace the $g^4$ which do not multiply logs by $g^4(T)$ as these constants turn out to be determined by the $q_\perp\sim T$ range of the $q_\perp$ integration in (1.1). This compact prescription accounts for 1-loop running of the coupling constant, provided there are no vacuum mass thresholds between $m_D$ and $\Lambda$. The difference between $g^2(T)$ and $g^2(m_D)$ is of order $g^4 \ln(T/m_D) \sim g^4 \ln(g^{-1})$ and so not significant in the weak coupling limit, but no harm is done in accounting for this particular higher-order correction to $\hat{\bar{q}}$.

The replacement (2.8) has a finite limit as $\Lambda \to \infty$:

$$g^4 \ln \left( \frac{\Lambda}{m_D} \right) \to \frac{g^2(m_D)}{-2\bar{\beta}_0} \quad (\Lambda = \infty), \quad (2.10)$$

where

$$\bar{\beta}_0 = -\frac{(11C_A - 4N_f t_F)}{48\pi^2} = -\frac{(33 - 2N_f)}{48\pi^2} \quad (2.11)$$

is the one-loop coefficient of the $\beta$ function for $g^2(\mu) = [-\bar{\beta}_0 \ln(\mu^2/\Lambda^2)]^{-1}$. So, if the effect of running is included, $\hat{\bar{q}}(\infty)$ is finite and given by the above formulas. For applications to bremsstrahlung, however, we will generally want $\hat{\bar{q}}(\Lambda)$ for finite $\Lambda$.

### C. Result for gluon bremsstrahlung

Consider bremsstrahlung of gluons with energy $x E \gg m_D$ from a high-energy particle of energy $E$ and species $s$. The leading-log result for bremsstrahlung from this particle is (ignoring final state factors for the high-energy particle and gluon)$^4$

$$\frac{d\Gamma_{s\to gs}}{dx} = \frac{\alpha \mu_\perp^2 P_{s\to g}(x)}{4\pi \sqrt{2} x(1-x)E}, \quad (2.12)$$

where $x$ is the bremsstrahlung gluon momentum fraction, $P_{s\to g}(x)$ is the usual vacuum splitting function, and, at leading-log order,

$$\mu_\perp^2 \simeq \left\{8x(1-x)E \left[ \frac{1}{2}C_A + (C_s - \frac{1}{2}C_A)x^2 + \frac{1}{2}C_A(1-x)^2 \right] \hat{\bar{q}}(Q_{\perp0}) \right\}^{1/2}. \quad (2.13)$$

$^4$ The NLL analysis will be based on Arnold and Dogan [11]. The leading-log result in (2.12) and (2.13) of the current paper roughly corresponds to eqs. (1.1), (3.1a–b), and (4.16) of Ref. [11], with the correspondence that $\mu_\perp = m_D \hat{\bar{\mu}}_\perp$. The only difference is that we have written the answer in terms of the general $\hat{\bar{q}}(Q_{\perp0})$, whereas Ref. [11] specialized to the $q_\perp \ll T$ limit, resulting in their Eq. (4.12). See Appendix D of the current paper for more detail, including an explanation of differences in notation.
Here, \( Q_{\perp 0} \) is any rough guess (1.5) of the total momentum transfer during the formation time, and ambiguities in that guess of \( O(1) \) factors only affect the answer beyond leading-log order. The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) splitting functions in (2.12) are

\[
\begin{align*}
P_{q \to g}(x) &= C_F \left[ 1 + \frac{(1 - x)^2}{x} \right], \\
P_{g \to g}(x) &= C_A \left[ 1 + x^4 + \frac{(1 - x)^4}{x(1 - x)} \right].
\end{align*}
\]

If we now follow the same steps as Arnold and Dogan [11], the NLL result corresponds to replacing (2.13) by

\[
\begin{align*}
\mu_{\perp}^2 &\simeq \left[ 8x(1 - x)E \right]^{1/2} \\
&\times \left\{ \left[ \frac{1}{2}C_A \hat{q}(\xi^{1/2} \mu_{\perp}) + (C_s - \frac{1}{2}C_A)x^2 \hat{q} \left( \frac{\xi^{1/2} \mu_{\perp}}{x} \right) + \frac{1}{2}C_A(1 - x)^2 \hat{q} \left( \frac{\xi^{1/2} \mu_{\perp}}{1 - x} \right) \right] \right\}^{1/2},
\end{align*}
\]

where

\[
\xi \equiv \exp(2 - \gamma_E + \frac{\pi}{4})
\]

is the same constant as in Arnold and Dogan, and where (2.16) is an implicit equation\(^5\) for \( \mu_{\perp} \). Details are given in Appendix D.

The assumption that goes into this result is that \( \hat{q}(\Lambda) \) is proportional to \( \ln \Lambda \) at the scale of its arguments—that is, that \( d\Gamma_{\text{el}}/d^2q_{\perp} \propto 1/q_{\perp}^4 \) for \( q_{\perp} \) of order the argument \( \Lambda \) of \( \hat{q} \). This assumption works for arguments in the range \( m_D \ll \Lambda \ll T \) as well as the case \( T \ll \Lambda \) considered in this paper. As a result, if one uses the small \( q_{\perp} \) form (1.4) of \( \hat{q}(\Lambda) \), then the formula (2.16) reproduces the result found in Arnold and Dogan. For high energy particles with parametrically \( E \gg T/g^4 \ln(1/g) \), however, one should instead use (2.6) for \( \hat{q} \).

As discussed in Refs. [9, 11, 13], if the running of the coupling is included, the explicit factor of \( \alpha_s \) in (2.12) should plausibly be \( \alpha_s(\mu_{\perp}) \).

Our result (2.6) formally treated \( m_D \ll T \). If one evaluated \( \hat{q}(\Lambda) \) in a way that relaxed this assumption, Eqs. (2.12) and (2.16) could still be used to calculate the bremsstrahlung rate for high-energy particles.

### D. Result for Pair Production

Pair production is the same as gluon bremsstrahlung except for (i) a change in group factors, and (ii) use of the corresponding vacuum splitting function

\[
P_{g \to q}(x) = N_t t_F [x^2 + (1 - x)^2].
\]
See, for example, the more symmetric presentation given in Arnold and Dogan [11]. The result (summed over quark flavors) is

\[ \frac{d\Gamma_{g \rightarrow q\bar{q}}}{dx} = \frac{\alpha \mu_2^2 P_{g \rightarrow q}(x)}{4\pi \sqrt{2} x(1-x)E} \]  

(2.19)

with

\[ \mu_2^2 \simeq \left[ 8x(1-x)E \right]^{1/2} \]

\[ \times \left\{ \left( C_F - \frac{1}{2} C_A \right) \hat{q} \left( \xi^{1/2} \mu_\perp \right) + \frac{1}{2} C_A x^2 \hat{q} \left( \frac{\xi^{1/2} \mu_\perp}{x} \right) + \frac{1}{2} C_A (1-x)^2 \hat{q} \left( \frac{\xi^{1/2} \mu_\perp}{1-x} \right) \right\}^{1/2}, \]  

(2.20)

III. \( d\Gamma_{el}/d^2 q_\perp \) AND \( \hat{q}(\Lambda) \)

A. Strategy

Our first goal is to determine how the differential elastic scattering rate \( d\Gamma_{el}/d^2 q_\perp \) interpolates at \( q_\perp \sim T \) between the two limits shown in (1.3). For \( q_\perp \gg m_D \), screening effects can be ignored, and the differential cross-section will have the form

\[ \frac{d\Gamma_{el}}{d^2 q_\perp} \simeq \frac{C_R}{(2\pi)^2} \times \frac{g^4 T^3 F(q_\perp/T)}{q_\perp^4} \]  

(3.1)

for some function \( F(q_\perp/T) \) with

\[ g^4 T^3 F(0) = g^2 T m_D^2 \quad \text{and} \quad g^4 T^3 F(\infty) = g^4 N. \]  

(3.2)

In the limit of weak coupling, once we know \( F(q_\perp/T) \) we can then construct a formula valid to leading-order at all scales for \( q_\perp \) as

\[ \frac{d\Gamma_{el}}{d^2 q_\perp} \simeq \frac{C_R}{(2\pi)^2} \times \frac{g^4 T^3 F(q_\perp/T)}{q_\perp^2 (q_\perp^2 + m_D^2)} \]  

(3.3)

since \( m_D \ll T \).

It will be convenient to write \( g^4 T^3 F \) in the form

\[ g^4 T^3 F(q_\perp/T) = \left[ \Xi_b I_+(q_\perp/T) + \Xi_t I_-(q_\perp/T) \right] \frac{g^4 T^3}{\pi^2}. \]  

(3.4)

The functions \( I_\pm(q_\perp/T) \), to be determined, extrapolate between

\[ I_\pm(0) = \zeta_\pm(2) \quad \text{and} \quad I_\pm(\infty) = \zeta_\pm(3). \]  

(3.5)

The equivalence to (3.2) can be seen from Eqs. (2.3) for \( m_D \) and \( N \).
B. Starting Point for $d\Gamma_{el}/d^2q_\perp$

In general, the rate for a high-energy particle of energy $E$ to scatter from the plasma is given by

$$
\frac{d\Gamma_{el}}{d^2q_\perp} \sim \int \frac{dq_\perp}{(2\pi)^3} \sum_{s_2} dR_2 \tilde{v}_{s_2} \int \frac{d^3p_2}{(2\pi)^3} \frac{d\sigma_{el}}{d^3q} f_{s_2}(p_2) \left[ 1 \pm f_{s_2}(p_2-q) \right],
$$

(3.6)

where $z$ is the direction of motion of the high-energy particles, $p_2$ is the momentum of a particle in the plasma of species $s_2$, $\sigma_{el}$ is the cross-section for scattering from that plasma particle, $f(p_2)$ is a Bose or Fermi distribution that accounts for the probability of encountering the plasma particle, and $1 \pm f(p_2-q)$ is a final-state Bose enhancement or Fermi blocking factor for the plasma particle after transferring momentum $q$ to the high-energy particle. We assume that $E \gg T$ and so do not need to include any final state factor for the high energy particle. In (3.6), $\tilde{v}_{s_2}$ is the number of spin and flavor degrees of freedom for species $s_2$ (2 for gluons, $4N_f$ for the sum of quarks and anti-quarks).

For high-energy particles (in this case, $E \gg m_D$), elastic scattering from the plasma is dominated by $t$-channel gluon exchange. The infrared behavior of $t$-channel gluon exchange is cut off in the infrared at $q_\perp \sim m_D$ by the effects of Debye screening and related phenomena in the plasma. To leading order in the weak-coupling limit, the problem is simplified by the fact that we can ignore screening effects when investigating $q_\perp \sim T$ since $T \gg m_D$. To leading order, the differential elastic scattering rate for $q_\perp \gg m_D$ is then

$$
\frac{d\Gamma_{el}}{d^2q_\perp} \sim \int \frac{dq_\perp}{(2\pi)^4} \sum_{s_2} \tilde{v}_{s_2} \int \frac{d^3p_2}{(2\pi)^3} \frac{C_R t_{R_2} g^4}{(2p)^2 2p_2 \mp |p_2-q|} \left| \frac{4P_\mu P_2^\mu}{|q|^2} \right|^2 f_{s_2}(p_2) \left[ 1 \pm f_{s_2}(p_2-q) \right] 
$$

$$
\times 2\pi \delta(\omega - q_z) 2\pi \delta(\omega + |p_2 - q| - p_2)
$$

(3.7)

for a massless high-energy particle with momentum $p$ in the $z$ direction. We use capital letters for 4-momenta, with $P = (p, \mathbf{p})$ and $Q = (\omega, \mathbf{q})$. The factors of $(2p)^{-1}$, $(2p)^{-1}$, $(2p_2)^{-1}$, and $(2|p_2-q|)^{-1}$ are the usual initial and final state relativistic phase space normalizations, where we’ve taken the high energy limit $E \gg q$. The $\delta(\omega + |p_2 - q| - p_2)$ is energy conservation for the (massless) plasma particle. The other $\delta$-function is energy conservation $\delta(|p + q| - \omega - p)$ for the incident high-energy particle, again taking the high-energy limit $p \gg q$. The two gluon vertices in the amplitude are $2gP_\mu$ and $2gP_2^\mu$ (times color generators), regardless of the types of particles colliding. The appearance of this universal form can be understood as a consequence that, in the high-energy limit $E \gg T$, the upper limit (1.5) for the range of individual $q_\perp$ transfers dominating bremsstrahlung is small compared to the center-of-mass energy $\sim (ET)^{1/2}$ for a collision with a plasma particle. In the center-of-mass frame, the $t$-channel gluon is soft compared to either particle involved in the elastic collision, and so we may use the universal form that gluon-particle vertices take in the soft gluon limit. Note that $P_\mu P_2^\mu$ in (3.7) could have equally well been written using the final plasma particle momentum $P_2 - Q$ as $P_\mu (P_2 - Q)^\mu$, due to the $\delta(\omega - q_z)$.

Performing the $\omega$ integration in (3.7),

$$
\frac{d\Gamma_{el}}{d^2q_\perp} \sim \frac{C_R}{(2\pi)^2 q_\perp^4} \sum_{s_2} \tilde{v}_{s_2} t_{R_2} g^4 \int \frac{dq_\perp}{2\pi} \int \frac{d^3p_2}{(2\pi)^3} \frac{(p_2 - p_2z)^2}{p_2 |p_2 - q|} f_{s_2}(p_2) \left[ 1 \pm f_{s_2}(p_2-q) \right] 
$$

$$
\times 2\pi \delta(q_z + |p_2 - q| - p_2).
$$

(3.8)
In thermal equilibrium at zero chemical potential, the distributions \( f_{s_2} \) are all the same for massless bosons, and also all the same for massless fermions, and we can rewrite \( \frac{d\Gamma_{el}}{d^2 q_{\perp}} \) as

\[
\frac{d\Gamma_{el}}{d^2 q_{\perp}} \sim \frac{C_R}{(2\pi)^2} \times \left[ \Xi_b I_+(q_{\perp}/T) + \Xi_f I_-(q_{\perp}/T) \right] \frac{g^4 T^3}{\pi^2}
\]

with

\[
I_{\pm}(q_{\perp}/T) = \frac{\pi^2}{T^3} \int \frac{dq_z}{2\pi} \int \frac{d^3 p_2}{(2\pi)^3} \frac{(p_2 - p_{2z})^2}{p_2 |p_2 - q|} f_{\pm}(p_2) \left[ 1 \pm f_{\pm}(p_2 - q) \right]
\]

\[
\times 2\pi \delta(q_z + |p_2 - q| - p_2).
\]

C. Recasting \( d\Gamma_{el}/d^2 q_{\perp} \) as a double sum

Now rewrite the equilibrium Bose and Fermi distribution functions as sums of exponentials,

\[
f_{\pm}(p) = \frac{1}{e^{\beta p} \pm 1} = \sum_{m=1}^{\infty} (\pm)^{m-1} e^{-m\beta p},
\]

\[
1 \pm f_{\pm}(p) = \sum_{n=0}^{\infty} (\pm)^{n} e^{-n\beta p},
\]

with \( \beta \equiv 1/T \). Then

\[
I_{\pm}(q_{\perp}/T) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (\pm)^{m+n-1} I_{mn}(q_{\perp}/T)
\]

with

\[
I_{mn}(q_{\perp}/T) = \frac{\pi^2}{T^3} \int \frac{dq_z}{2\pi} \frac{d^3 p_2}{(2\pi)^3} \frac{(p_2 - p_{2z})^2}{p_2 |p_2 - q|} e^{-m\beta p_2} e^{-n\beta |p_2 - q|} 2\pi \delta(q_z + |p_2 - q| - p_2).
\]

This integral is evaluated in Appendix A and yields

\[
I_{mn}(q_{\perp}/T) = \frac{mn}{2(m+n)^3} \left( \frac{q_{\perp}}{T} \right)^2 K_2 \left( \frac{q_{\perp}}{T} \sqrt{mn} \right),
\]

where \( K_\nu(z) \) is the modified Bessel function of the second kind. The case \( n = 0 \) gives the \( n \to 0 \) limit of the above formula,

\[
I_{m0} = \frac{1}{m^3}.
\]

In the \( q_{\perp} \to 0 \) limit, \( I_{mn} \to 1/(m+n)^3 \), and so

\[
I_{\pm}(0) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(\pm)^{m+n-1}}{(m+n)^3} = \sum_{m+n=1}^{\infty} \frac{(\pm)^{m+n-1}}{(m+n)^2} = \zeta_{\pm}(2).
\]

\(^6\) The analysis of Ref. [11] could also be applied to non-equilibrium isotropic situations. Here, however, we are specializing to equilibrium distributions.
In the opposite limit of $q_\perp \to \infty$, only the $n = 0$ terms survive in the double sum (3.13), giving

$$I_\pm(\infty) = \sum_{m=1}^{\infty} (\pm)^{m-1} I_{m0} = \zeta_\pm(3).$$

(3.18)

These two limits are in accord with (3.5). We will later find it useful to extract the $n = 0$ contribution from the general case, writing

$$I_\pm(q_\perp/T) = I_\pm(\infty) + \Delta I_\pm(q_\perp/T) = \zeta_\pm(3) + \Delta I_\pm(q_\perp/T)$$

(3.19a)

with

$$\Delta I_\pm(q_\perp/T) = \sum_{m,n=1}^{\infty} (\pm)^{m+n-1} I_{mn}(q_\perp/T).$$

(3.19b)

If one wished to evaluate the functions $I_\pm(q_\perp/T)$ numerically, the expansion (3.19b) converges rapidly for $q_\perp \gtrsim T$. For the case of small $q_\perp$, we show in Appendix B that the expansion is

$$I_+(Q) = \zeta_+(2) \left[ 1 - \frac{3}{16} Q + \left( \frac{1}{24} + \frac{1}{8\pi^2} \right) Q^2 + O(Q^3) \right],$$

(3.20)

$$I_-(Q) = \zeta_-(2) \left[ 1 + \left( \frac{1}{24} - \frac{1}{4\pi^2} \right) Q^2 + O(Q^3) \right].$$

(3.21)

D. Integrating to get $\hat{q}$

Using (3.3) and (3.4), our leading-order formula for the differential elastic scattering rate is

$$\frac{d\Gamma_{el}}{d^2 q_\perp} \sim \frac{C_R}{(2\pi)^2} \frac{g^4 T^3 [\Xi_b I_+(q_\perp/T) + \Xi_I I_-(q_\perp/T)]}{\pi^2 q_\perp^2 (q_\perp^2 + m_D^2)}.$$  

(3.22)

So, to compute the integral (1.1) that gives $\hat{q}(\Lambda)$, we turn to evaluating the integrals

$$\mathcal{I}_\pm(\Lambda) \equiv \int_{q_\perp < \Lambda} \frac{d^2 q_\perp}{(2\pi)^2} \frac{I_\pm(q_\perp/T)}{(q_\perp^2 + m_D^2)}.$$  

(3.23)

In the weak coupling limit, one can choose a momentum scale $\lambda$ between $m_D$ and $T$, with $m_D \ll \lambda \ll T$. Then one may split the integral at $\lambda$ into separate pieces for which $q_\perp \ll T$ or $q_\perp \gg m_D$ approximations may be made:

$$\mathcal{I}_\pm(\Lambda) \simeq \int_{q_\perp < \lambda} \frac{d^2 q_\perp}{(2\pi)^2} \frac{I_\pm(0)}{(q_\perp^2 + m_D^2)} + \int_{\lambda < q_\perp < \Lambda} \frac{d^2 q_\perp}{(2\pi)^2} \frac{I_\pm(q_\perp/T)}{q_\perp^2}.$$  

(3.24)

In Appendix C, we then evaluate these integrals using the double sum (3.19) for the last one. To leading order, we obtain the result (2.6) with

$$\sigma_\pm = \frac{1}{2} \sum_{m,n=1}^{\infty} (\pm)^{m+n-1} \frac{\ln(mn)}{(m+n)^3} = \sum_{m,n=1}^{\infty} (\pm)^{m+n-1} \frac{\ln(m)}{(m+n)^3}.$$  

(3.25)

---

7 A numerically modest formula which reproduces $I_+(Q)$ to a few tenths of a percent is to (i) for $Q < 3.2$ use (3.20) plus $-0.0062 Q^3$ inside the square brackets, and (ii) for $Q > 3.2$ use the $m + n \leq 3$ terms of (3.19). For $I_-(Q)$ do the same, but with $-0.0031 Q^3$ added inside the square brackets of (3.21).
This can be written as \( \sigma_+ = -\partial_a T(a, 0, 3) \big|_{a=0} \) where
\[
T(a, b, c) \equiv \sum_{m,n=1}^{\infty} \frac{1}{m^a n^b (m + n)^c}
\] (3.26)
is the Tornheim zeta function [16], and similarly \( \sigma_- \) can be defined in terms of its generalization. However, this does not seem to give any information that is more useful than the sum itself. By letting \( k = m + n \) and then summing over \( n \) for fixed \( k \), one obtains the single-sum formula (2.6d) for \( \sigma_\pm \).

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We are indebted to Simon Caron-Huot and Guy Moore for useful conversations. It was Simon Caron-Huot’s work [1] in particular that made us understand that our NLL bremsstrahlung rate should be presented in terms of a UV-regulated \( \hat{q} \). This work was supported, in part, by the U.S. Department of Energy under Grant No. DE-FG02-97ER41027.

**APPENDIX A: \( I_{mn} \)**

In the integral (3.14) defining \( I_{mn} \), use the \( \delta \)-function to rewrite
\[
(p_2 - p_{2z})^2 = (p_2 - p_{2z}) [p_2 - q - (p_{2z} - q_z)].
\] (A1)
Then rewrite the \( \delta \)-function as
\[
2\pi \delta(q_z + |p_2 - q| - p_2) = \int_{-\infty}^{+\infty} d\lambda \, e^{i\lambda(q_z + |p_2 - q| - p_2)}.
\] (A2)
The \( p_2 \) integral now takes the form of a convolution of functions \( W_m \) and \( W_n^* \) defined by
\[
W_m(p_2, \lambda) \equiv \frac{(p_2 - p_{2z})}{p_2} e^{-(m\beta + i\lambda)p_2}.
\] (A3)
To turn the convolution into a simple product, we Fourier transform from \( p_2 \) to \( r \):
\[
I_{mn}(q_\perp/T) \equiv \frac{\pi^2}{T^3} \int \frac{dq_\perp}{2\pi} \int d\lambda \, e^{i\lambda q_\perp} \int d^3r \, W_m(r, \lambda) W_n^*(r, \lambda) e^{-i q_\perp \cdot r}.
\] (A4)
The \( q_z \) integral then gives \( \delta(\lambda - z) \), which can be used to do the \( \lambda \) integral:
\[
I_{mn}(q_\perp/T) \equiv \frac{\pi^2}{T^3} \int d^3r \, W_m(r, z) W_n^*(r, z) e^{-i q_\perp \cdot r}.
\] (A5)
The Fourier transform of (A3) evaluated at \( \lambda = z \) is
\[
\tilde{W}_m(r, z) = \frac{T^3}{\pi^2} \frac{m}{[m^2 + 2imzT + (r_\perp T)^2]^2}.
\] (A6)
Now do the \( z \) integration in (A5) by closing the contour in the upper-half complex plane and picking up the double pole there:
\[
I_{mn}(q_\perp/T) \equiv \frac{2T^2}{\pi} \int d^2r_\perp \frac{(mn)^2}{(m + n)^3[(r_\perp T)^2 + mn]^3} e^{-i q_\perp \cdot r_\perp}.
\] (A7)
Finally, performing the \( r_\perp \) integral gives (3.15).
APPENDIX B: SMALL $q_\perp$ EXPANSION OF $I_\pm$

It is possible to find the small $Q$ expansion of $I_\pm(Q)$ starting directly from the integral formula (3.10). For the bosonic case, at least, we find it easier to instead start from the double sum formula derived in Sec. III C.

1. The $O(q_\perp)$ piece of $I_+$

Start from the double sum of (3.13) and (3.15) and subtract off the $q_\perp = 0$ piece:

$$\delta I_+ \equiv I_+(Q) - I_+(0) = \sum_{m,n=1}^\infty \left[ \frac{1}{2} Q^2 mn K_2(Q \sqrt{mn}) - 1 \right] \frac{1}{(m+n)^3}. \quad (B1)$$

For small $Q \equiv q_\perp / T$, this sum is dominated by large $m$ and $n$, and so we can replace the sum by an integral:

$$\delta I_+ \simeq \int_0^\infty dm dn \left[ \frac{1}{2} Q^2 mn K_2(Q \sqrt{mn}) - 1 \right] \frac{1}{(m+n)^3}. \quad (B2)$$

Change integration variable from $n$ to $x \equiv Q \sqrt{mn}$, and then do the $m$ integral to get

$$\delta I_+ \simeq \frac{\pi}{16} Q \int_0^\infty dx \left[ K_2(x) - \frac{2}{x^2} \right] = -\frac{\pi^2}{32} Q. \quad (B3)$$

This gives the $O(Q)$ term in (3.20).

2. The $O(q_\perp^2)$ piece of $I_+$

The difference between the double sum and the integral approximation made above is

$$\delta^2 I_+ = \left( \sum_{m,n=1}^\infty - \int_0^\infty dm dn \right) \left[ \frac{1}{2} Q^2 mn K_2(Q \sqrt{mn}) - 1 \right] \frac{1}{(m+n)^3}. \quad (B4)$$

Now rewrite

$$\sum_m \sum_n - \int_m \int_n = \left( \sum_m \int_m - \int_m \right) \int_n + \int_m \left( \sum_n \int_n - \int_n \right) + \left( \sum_m \int_m \right) \left( \sum_n \int_n \right), \quad (B5)$$

and correspondingly (using the $m \leftrightarrow n$ symmetry of the sum for $I_+$)

$$\delta^2 I_+ = \delta^{2a} I_+ + \delta^{2a} I_+ + \delta^{2b} I_+. \quad (B6)$$

To evaluate $\delta^{2a} I_+$, again change integration variable from $n$ to $x \equiv Q \sqrt{mn}$ to get

$$\delta^{2a} I_+ = Q^4 \left( \sum_{m=1}^\infty - \int_0^\infty dm \right) F(m; Q) \quad (B7)$$
where
\[ F(m; Q) \equiv 2m^2 \int_0^\infty dx \frac{x K_2(x) - 1}{(x^2 + m^2 Q^2)^3}. \quad \text{(B8)} \]

For small \( Q \), \( F(m; Q) \) is a slowly varying function of \( m \), which in general means that
\[ \left( \sum_{m=1}^\infty - \int_0^\infty dm \right) F(m; Q) \simeq -\frac{1}{2} F(0; Q). \quad \text{(B9)} \]

In our case, \( F(0; Q) \) should be understood to be the \( m \to 0 \) limit \(-1/8Q^2\) of (B8), which then gives
\[ \delta^{2a} I_+ = \frac{1}{16} Q^2. \quad \text{(B10)} \]

The other term \( \delta^{2b} I_+ \) in (B6) will be turn out to be dominated by \( m \) and \( n \) with \( mn \ll 1/Q^2 \) in the small \( Q \) limit. Making this small \( Q \) approximation to the argument of \( K_2 \), we get
\[ \delta^{2b} I_+ = -Q^2 \left( \sum_{m=1}^\infty - \int_0^\infty dm \right) \left( \sum_{n=1}^\infty - \int_0^\infty dn \right) G(m, n) \quad \text{(B11)} \]

with
\[ G(m, n) = \frac{mn}{4(m+n)^3}. \quad \text{(B12)} \]

Now rewrite
\[ \left( \sum_{m=1}^\infty - \int_0^\infty dm \right) \left( \sum_{n=1}^\infty - \int_0^\infty dn \right) G(m, n) = \]
\[ \sum_{m,n=1}^\infty \left\{ G(m, n) - \int_{m-1}^{m} dm' G(m', n) - \int_{n-1}^{n} dn' G(m, n') + \int_{m-1}^{m} dm' \int_{n-1}^{n} dn' G(m', n') \right\}. \quad \text{(B13)} \]

Explicit evaluation of the integrals using (B12) yields
\[ \delta^{2b} I_+ = Q^2 \sum_{m,n=1}^\infty \frac{(m^3 - 3m^2n - 3mn^2 + n^3) + 4mn}{8(m+n)^3(m+n-1)^2(m+n-2)}, \quad \text{(B14)} \]

where the summand for \( m = n = 1 \) should be treated as the limiting value \(-1/32\). Defining \( k \equiv m + n \), (B14) can be rewritten
\[ \delta^{2b} I_+ = Q^2 \sum_{k=2}^\infty \sum_{n=1}^{k-1} \frac{k^3 - 6k^2n + 6kn^2 + 4kn - 4n^2}{8k^3(k-1)^2(k-2)}. \quad \text{(B15)} \]

Doing the \( n \) sum first (and treating the \( k = 2 \) case separately, which does not fit the pattern of \( k > 2 \)):
\[ \delta^{2b} I_+ = -Q^2 \left[ \frac{1}{32} + \sum_{k=3}^\infty \frac{1}{24k^2(k-1)} \right] = -Q^2 \left[ \frac{5}{16} - \frac{1}{24} \zeta(2) \right]. \quad \text{(B16)} \]

Putting (B10) and (B16) into (B6) then yields the \( O(Q^2) \) term in (3.20).
3. The $O(q_\perp^2)$ piece of $I_-$

The case of $I_-$ is much simpler because there is no $O(Q)$ term in the expansion. A quick but non-rigorous way to obtain the answer is to naively expand the summand of (3.13) and (3.15) in powers of $Q$:

$$\delta^2 I_- = -Q^2 \sum_{m,n=1}^{\infty} (-)^{m+n-1} \frac{mn}{4(m+n)^3}$$

$$= -Q^2 \sum_{k=2}^{\infty} \sum_{n=1}^{k-1} (-)^{k-1} \frac{(k-n)n}{4k^3}$$

$$= -\frac{1}{24} Q^2 \sum_{k=2}^{\infty} (-)^{k-1} \left( 1 - \frac{1}{k^2} \right). \quad (B17)$$

If one then interprets $\sum (-)^{k-1}$ as being $\zeta_-(0) = \frac{1}{2}$, then

$$\delta^2 I_- = -\frac{1}{24} [\zeta_-(0) - \zeta_-(2)] Q^2; \quad (B18)$$

where $\zeta_-(2) = \pi^2/12$. This gives the result (3.21) quoted previously.

A more reliable way to make the same calculation is to start with the convergent, unexpanded sum analogous to (B1),

$$I_-(Q) - I_-(0) = \sum_{m,n=1}^{\infty} (-)^{m+n-1} H(m, n; Q), \quad (B19)$$

$$H(m, n; Q) \equiv \left[ \frac{\frac{1}{2} Q^2 mn K_2(Q\sqrt{mn}) - 1}{(m+n)^3} \right]. \quad (B20)$$

Now block the sum into $2 \times 2$ blocks as

$$I_-(Q) - I_-(0) = \sum_{m, n \text{ odd}} \left[ -H(m, n; Q) + H(m+1, n; Q) + H(m, n+1; Q) - H(m+1, n+1; Q) \right]. \quad (B21)$$

At this point, one can safely expand the summand to order $Q^2$. Then change summation variables from $m$ to $k = m + n$, and then sum over first $n$ and then $k$. The final result is (B18).

APPENDIX C: $I_{\pm}$

In this appendix, we evaluate the integrals in (3.24). The non-trivial integral is the second one:

$$I^{(2)} \equiv \int_{\lambda < q_\perp < \Lambda} \frac{d^2q_\perp}{(2\pi)^2} \frac{I^{(2)}(q_\perp/T)}{q_\perp^2} = \frac{I^{(2)}(\infty)}{2\pi} \ln \left( \frac{\Lambda}{\lambda} \right) + \int_{\beta \lambda < Q < \Lambda} \frac{d^2Q}{(2\pi)^2} \frac{\Delta I^{(2)}(Q)}{Q^2}, \quad (C1)$$

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where the last equality uses (3.19) and switches to the dimensionless integration variable $Q ≡ q_\perp/T$. Because $\Delta I_\pm$ falls off for $\Pi \to \infty$, we can drop the UV regularization $Q < \beta \Lambda$ in the last integral.

It’s useful to now change infrared regularization by inserting an initially unnecessary factor of $Q^{2\epsilon}$ with the limit $\epsilon \to 0^+$ taken at the end of the day. Then rewrite the above as

$$T^{(2)}_\pm = \frac{I_\pm(\infty)}{2\pi} \ln \left( \frac{\Lambda}{\lambda} \right) + \int \frac{d^2Q}{(2\pi)^2} \frac{\Delta I_\pm(Q)}{Q^{2(1-\epsilon)}} - \int_{Q<\beta\lambda} \frac{d^2Q}{(2\pi)^2} \frac{\Delta I_\pm(Q)}{Q^{2(1-\epsilon)}}. \quad (C2)$$

In the last term, we can replace $\Delta I_\pm(Q)$ by $\Delta I_\pm(0)$, giving

$$\int_{Q<\beta\lambda} \frac{d^2Q}{(2\pi)^2} \frac{\Delta I_\pm(Q)}{Q^{2(1-\epsilon)}} \simeq \Delta I_\pm(0) \frac{(\beta \lambda)^{2\epsilon}}{4\pi \epsilon} = \frac{\Delta I_\pm(0)}{4\pi} \left[ \frac{1}{\epsilon} + 2 \ln \left( \frac{\lambda}{T} \right) + O(\epsilon) \right]. \quad (C3)$$

For the other integral in (C2), we use the double sum formula of (3.19b) and (3.15) for $\Delta I_\pm$:

$$\int \frac{d^2Q}{(2\pi)^2} \frac{\Delta I_\pm(Q)}{Q^{2(1-\epsilon)}} = \sum_{m,n=1}^{\infty} (\pm)^{m+n-1} \frac{mn}{2(m+n)^3} \int \frac{d^2Q}{(2\pi)^2} Q^{2\epsilon} K_2(Q \sqrt{mn})$$

$$\quad = \sum_{m,n=1}^{\infty} (\pm)^{m+n-1} \frac{(mn)^{-\epsilon}}{4\pi(m+n)^3} \int_0^\infty dx x^{1+2\epsilon} K_2(x). \quad (C4)$$

The last integral gives $2^{2\epsilon} \Gamma(\epsilon) \Gamma(2+\epsilon)$. Expanding the result in $\epsilon$, and noting that (3.17) and (3.18) give

$$\sum_{m,n=1}^{\infty} \frac{(\pm)^{m+n-1}}{(m+n)^3} = \zeta_\pm(2) - \zeta_\pm(3) = \Delta I_\pm(0), \quad (C5)$$

we get

$$\int \frac{d^2Q}{(2\pi)^2} \frac{\Delta I_\pm(Q)}{Q^{2(1-\epsilon)}} = \frac{\Delta I_\pm(0)}{4\pi} \left( \frac{1}{\epsilon} + 1 - 2\gamma_E + 2 \ln 2 \right) - \frac{\sigma_\pm}{2\pi}, \quad (C6)$$

with $\sigma_\pm$ defined as in (3.25).

Evaluating $\mathcal{I}_\pm = \mathcal{I}^{(1)}_\pm + \mathcal{I}^{(2)}_\pm$ of (3.24) by combining (C2), (C3), (C6) and

$$\mathcal{I}^{(1)}_\pm = \int_{q_\perp < \lambda} \frac{d^2q_\perp}{(2\pi)^2} \frac{I_\pm(0)}{2m_\perp^2} = \frac{I_\pm(0)}{2\pi} \ln \left( \frac{\lambda}{m_\perp} \right) \quad (C7)$$

then produces the final result (2.6).

**APPENDIX D: THE NLL CALCULATION**

For simplicity of presentation, we will focus just on bremsstrahlung in this appendix. Arnold and Dogan [11] computed the NLL bremsstrahlung rate using the $q_\perp \ll T$ limit in (1.3) for $d\Gamma_{el}/d^2q_\perp$. In their notation, they referred to $\mathcal{A}(q_\perp)$ instead of $d\Gamma_{el}/d^2q_\perp$, and the translation is

$$\frac{d\Gamma_{el}}{d^2q_\perp} = C_{RG} \frac{g^2}{(2\pi)^2} \mathcal{A}(q_\perp), \quad (D1)$$

See Appendix A of Ref. [13].
\[ \hat{q} = g^2 \int \frac{d^2q_\perp}{(2\pi)^2} A(q_\perp) q_\perp^2. \]  

(D2)

The derivation of Arnold and Dogan is fairly easy to generalize if we just make these replacements. If one is confident enough, one can just write our generalization (2.16) of Arnold and Dogan by inspection by recasting their result in terms of the \( \Lambda \ll T \) version of \( \hat{q} \) and then assuming the formula works for \( \Lambda \gg T \). This works because Arnold and Dogan’s constant \( \xi \) was generated by the large-\( q_\perp \) contributions to the calculation, where \( A(q_\perp) \) is proportional to \( 1/q_\perp^4 \) in either case. It is the large \( q_\perp \) part of the calculation that is affected by the NLL calculation: the \( q_\perp \ll Q_\perp \) pieces just come from the leading-order calculation, which is proportional to \( \hat{q}(Q_\perp) \).

Readers may find the above argument obscure, or an explicit calculation reassuring, and so we will also indicate how to get the same result by modifying the calculations of Ref. [11]. We will not reproduce the entire derivation of Ref. [11] but will just indicate which equations are modified.

Eq. (4.14) of Ref. [11] for \( H_s^2 \), which determines the leading log result, becomes

\[ H^2 = \left\{ 2g^2p'k|p| \left[ \frac{1}{2}C_A p^2 + (C_s - \frac{1}{2}C_A)k^2 + \frac{1}{2}C_A p^2 \right] \hat{q}(Q_{\perp0}) \right\}^{1/2}, \]  

(D3)

where \( p' = E \) is the initial high-energy parton energy and \( k =xE \) and \( p = (1-x)E \) are the energies of the two partons it splits into. Combining this formula for \( H \) with Eqs. (1.1), (4.1), and (4.15) of Ref. [11] gives Eqs. (2.12) and (2.13) of the current paper. Roughly speaking, \( H \) gives the scale of the total momentum transfer to a high-energy particle during the formation time as

\[ Q_\perp \sim \frac{H}{x_1E}, \]  

(D4)

where \( x_i = 1, x, \) or \( 1-x \) depending on which particle one is focusing on.

In evaluating the NLL correction, the only remaining step that is different is the evaluation of the integral

\[ I_2(\kappa^2) \equiv g^2 \int \frac{d^2h}{(2\pi)^2} \frac{d^2q_\perp}{(2\pi)^2} A(q_\perp) \cdot [F_0(h) - F_0(h + \kappa q_\perp)] \]  

(D5)

defined in Eq. (4.27) of Ref. [11], with

\[ F_0(h) = i 4p'k \left[ \exp \left( -ie^{i\pi/4} \frac{h^2}{H^2} \right) - 1 \right] \frac{h}{h^2} \]  

(D6)

and \( \kappa = p', p, \) or \( k \). Here also, Arnold and Dogan used the small-\( q_\perp \) formula for \( A(q_\perp) \), and we now want to generalize. Introduce a cut-off \( \Lambda \ll Q_\perp \) such that the differential elastic scattering rate behaves like

\[ \frac{d\Gamma_{el}}{d^2q_\perp} = C_R \frac{d\Gamma_{el}}{d^2q_\perp} \simeq C_R \frac{c}{(2\pi)^2 q_\perp^2} \]  

for \( \Lambda \leq q_\perp \lesssim Q_\perp \) (D7)

and then rewrite

\[ I_2(\kappa^2) \simeq I_{2<}(\kappa^2) + I_{2>}(\kappa^2), \]  

(D8)
with
\[
I_{2<}(\kappa^2) \equiv \int \frac{d^2h}{(2\pi)^2} \int_{q_{\perp}} \frac{d^2q_{\perp}}{d^2q_{\perp}} \frac{dF_0(h)}{d^2q_{\perp}} \left[ F_0(h) - F_0(h + \kappa q_{\perp}) \right], \tag{D9}
\]
\[
I_{2>}(\kappa^2) \equiv \int \frac{d^2h}{(2\pi)^2} \int_{q_{\perp}} \frac{d^2q_{\perp}}{d^2q_{\perp}} \frac{c}{q_{\perp}} F_0(h) \cdot \left[ F_0(h) - F_0(h + \kappa q_{\perp}) \right]. \tag{D10}
\]
In the integration (D9) for \(I_{2<}\), we have \(\kappa q_{\perp} \ll \kappa Q_{\perp} \sim H\), and so one can expand the difference of \(F_0\)'s in a Taylor expansion:
\[
I_{2<}(\kappa^2) \simeq -\kappa^2 \int \frac{d^2h}{(2\pi)^2} \int_{q_{\perp}} \frac{d^2q_{\perp}}{d^2q_{\perp}} \frac{dF_0(h)}{d^2q_{\perp}} \left[ F_0(h) \cdot \nabla_h F_0(h) \right]
= -\kappa^2 \hat{\theta}(\Lambda) \int \frac{d^2h}{(2\pi)^2} F_0(h) \cdot \nabla_h F_0(h)
= -\frac{2}{\pi} \kappa^2 \hat{\theta}(\Lambda) (p'k)^2 e^{i\pi/4} \frac{c^2}{H^2}. \tag{D11}
\]
Now turn to \(I_{2>}\). It is convenient to rewrite
\[
I_{2>} = I_{2>}^a + I_{2>}^b \tag{D12}
\]
with
\[
I_{2>}^a = \int \frac{d^2h}{(2\pi)^2} \int \frac{d^2q_{\perp}}{d^2q_{\perp}} \left( \frac{c}{q_{\perp}} \theta(q_{\perp} - \Lambda) - \frac{c}{q_{\perp}} \frac{\theta(q_{\perp} - \Lambda)}{q_{\perp} (q_{\perp}^2 + M^2)} \right) F_0(h) \cdot \left[ F_0(h) - F_0(h + \kappa q_{\perp}) \right], \tag{D13}
\]
\[
I_{2>}^b = \int \frac{d^2h}{(2\pi)^2} \int \frac{d^2q_{\perp}}{d^2q_{\perp}} \frac{c}{q_{\perp} (q_{\perp}^2 + M^2)} F_0(h) \cdot \left[ F_0(h) - F_0(h + \kappa q_{\perp}) \right]. \tag{D14}
\]
Here \(\theta(z)\) is the step function and \(M \lesssim \Lambda\) is an arbitrary scale. In the first integral, we can again treat \(\kappa q_{\perp}\) as small and Taylor expand the difference in \(F_0\)'s, giving
\[
I_{2>}^a \simeq -\kappa^2 \int \frac{d^2q_{\perp}}{(2\pi)^2} \left( \frac{c}{q_{\perp}^2} \theta(q_{\perp} - \Lambda) - \frac{c}{q_{\perp}^2 (q_{\perp}^2 + M^2)} \right) q_{\perp}^2 \int \frac{d^2h}{(2\pi)^2} F_0(h) \cdot \nabla_h F_0(h)
= -\kappa^2 \times \frac{c}{2\pi} \ln \left( \frac{M}{\Lambda} \right) \times \frac{8}{\pi} (p'k)^2 e^{i\pi/4} \frac{c^2}{H^2}. \tag{D15}
\]
\(I_{2>}^b\) is proportional to the \(I_2(\kappa^2)\) integral evaluated by Arnold and Dogan, with \(m_D^2\) replace by \(M^2\) in the denominator, and the overall normalization \(g^2 T m_D^2\) replaced by \(c\). So we can take over the result from Eqs. (4.31) and (4.32) of Ref. [11],
\[
I_{2>}^b \simeq -\left( \frac{(p'k)^2}{\pi^2 M^2} \right) (2 - \gamma_E - \ln u_\kappa) u_\kappa \tag{D16}
\]
with
\[
u_\kappa \equiv e^{i\pi/4} \frac{M^2 \kappa^2}{2H^2}. \tag{D17}
\]
Combining the various pieces above,
\[
I_2(\kappa^2) \simeq -2 \kappa^2 e^{i\pi/4} (p'k)^2 \frac{e^{i\pi/4}}{\pi H^2} \left\{ \hat{\theta}(\Lambda) - \frac{c}{4\pi} \left[ -2 + \gamma_E + \ln \left( \frac{e^{i\pi/4} \kappa^2 \Lambda^2}{2H^2} \right) \right] \right\}, \tag{D18}
\]
which replaces Eq. (4.32) of Ref. [11]. Now take the real part of $I_2$ and note that (1.1) and (D7) implies that

$$\hat{q}(\Lambda') \simeq \hat{q}(\Lambda) + \frac{c}{2\pi} \ln \left( \frac{\Lambda'}{\Lambda} \right)$$

(D19)

for $\Lambda'$ and $\Lambda$ both in the region covered by (D7). Using the definition (2.17) of $\xi$, we then get

$$\text{Re}I_2(\kappa^2) \simeq -\sqrt{2}\kappa^2(p'kp)^2 \hat{q} \left( \sqrt{\frac{2\xi H^2}{\kappa^2}} \right),$$

(D20)

which replaces Eq. (4.33) of Ref. [11]. Eq. (4.35) of that reference then becomes

$$\text{Re}(S, F_s) = \text{Re}(S, F_0)$$

$$\times \frac{1}{2} \left\{ 1 + \frac{1}{2} C A p'^2 \hat{q} \left( \sqrt{\frac{2\xi H^2}{p'^2}} \right) + (C_s - \frac{1}{2} C_A)k^2 \hat{q} \left( \sqrt{\frac{2\xi H^2}{k^2}} \right) + \frac{1}{2} C A p'^2 \hat{q} \left( \sqrt{\frac{2\xi H^2}{p'^2}} \right) \right\},$$

(D21)

where the notation $(S, F_s)$ is defined in Ref. [11]. Following the same steps as Arnold and Dogan then produces our result (2.16), where our $\mu_\perp$ corresponds to their $m_D\mu_\perp$.

[1] S. Caron-Huot, talk given at Strong and Electroweak Matter 2008, Amsterdam; S. Caron-Huot, in preparation.
[2] P. Aurenche, F. Gelis and H. Zaraket, JHEP 0205, 043 (2002) [arXiv:hep-ph/0204146].
[3] S. Jeon and G. D. Moore, Phys. Rev. C 71, 034901 (2005) [arXiv:hep-ph/0309332].
[4] P. Arnold, G. D. Moore and L. G. Yaffe, JHEP 0206, 030 (2002) [arXiv:hep-ph/0204343].
[5] P. Arnold, G. D. Moore and L. G. Yaffe, JHEP 0301, 030 (2003) [arXiv:hep-ph/0209353].
[6] P. Arnold, G. D. Moore and L. G. Yaffe, JHEP 0305, 051 (2003) [arXiv:hep-ph/0302165].
[7] R. Baier, Y. L. Dokshitzer, A. H. Mueller, S. Peigne and D. Schiff, Nucl. Phys. B 478, 577 (1996) [arXiv:hep-ph/9604327];
[8] R. Baier, Y. L. Dokshitzer, A. H. Mueller, S. Peigne and D. Schiff, Nucl. Phys. B 483, 291 (1997) [arXiv:hep-ph/9607355];
[9] R. Baier, Y. L. Dokshitzer, A. H. Mueller, S. Peigne and D. Schiff, Nucl. Phys. B 484, 265 (1997) [arXiv:hep-ph/9608322].
[10] R. Baier, Y. L. Dokshitzer, A. H. Mueller and D. Schiff, Nucl. Phys. B 531, 403 (1998) [arXiv:hep-ph/9804212];
[11] P. Arnold and C. Dogan, Phys. Rev. D 78, 065008 (2008) [arXiv:0804.3359 [hep-ph]].
[12] R. Baier, Nucl. Phys. A 715, 209 (2003) [arXiv:hep-ph/0209038].
[13] P. Arnold, arXiv:0808.2767 [hep-ph].
[14] R. Baier and Y. Mehtar-Tani, arXiv:0806.0954 [hep-ph].
[15] G. D. Moore and D. Teaney, Phys. Rev. C 71, 064904 (2005) [arXiv:hep-ph/0412346].
[16] L. Tornheim, Amer. J. Math. 72, 303 (1950).
[17] A. Peshier, J. Phys. G 35, 044028 (2008).