Matrix Models, Complex Geometry and Integrable Systems. II *

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We consider certain examples of applications of the general methods, based on geometry and integrability of matrix models, described in [1]. In particular, the nonlinear differential equations, satisfied by quasiclassical tau-functions are investigated. We also discuss a similar quasiclassical geometric picture, arising in the context of multidimensional supersymmetric gauge theories and the AdS/CFT correspondence.

1 Introduction

In the first part of this paper [1] we have discussed the properties of the simplest gauge theories - the matrix integrals

\[ Z = \int d\Phi e^{-\frac{i}{\hbar} Tr W(\Phi)} = \frac{1}{N!} \int \prod_{i=1}^{N} (d\phi_i e^{-\frac{1}{\hbar} W(\phi_i)}) \Delta^2(\phi) \] (1.1)

and

\[ Z = \int d\Phi d\Phi^\dagger \exp \left( -\frac{1}{\hbar} V(\Phi, \Phi^\dagger) \right) = \frac{1}{N!} \int \prod_{i=1}^{N} \left( d^2 z_i e^{-\frac{1}{\hbar} V(z_i, \bar{z}_i)} \right) |\Delta(z)|^2 \] (1.2)

or the so called one-matrix and two-matrix models, where by

\[ V(\Phi, \Phi^\dagger) = \Phi^\dagger \Phi - W(\Phi; t) - \bar{W}(\Phi^\dagger; \bar{t}) \] (1.3)

and

\[ W(\Phi) = \sum_{k>0} t_k \Phi^k \] (1.4)

the corresponding gauge-invariant single-trace potentials (the last one is usually some generic polynomial) are denoted, and

\[ \Delta(\phi) = \prod_{i<j} (\phi_i - \phi_j) = \det ||\phi_i^{j-1}|| \] (1.5)

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stays for the Van-der-Monde determinant. We have reminded in [1] that matrix models (1.1), (1.2) are effectively described in terms of Toda integrable systems, and demonstrated, that in planar limit \( N \to \infty \) of matrices of infinite size (or \( \hbar \to 0 \) with \( \hbar N = t_0 = \text{fixed} \)) these integrable systems have nice geometric origin.

The planar limit corresponds to extracting the leading contribution to the free energy of a gauge theory (in particular, of (1.1), (1.2))

\[
Z_{\text{gauge}} = \exp (F_{\text{string}}) \tag{1.6}
\]

which is a certain string partition function (see, e.g. [2]), dual to a gauge theory. In particular case of the matrix ensembles (1.1) or (1.2), within their \( 1/N \)-expansion [3]

\[
F_{\text{string}} = \sum_{g=0}^{\infty} N^{2-2g} F_g(t, t_0) \tag{1.7}
\]

the free energies \( F_0 = \mathcal{F} \) of the planar matrix models can be identified with the quasiclassical tau-functions [4], or prepotentials of one-dimensional complex manifolds \( \Sigma \). These one dimensional complex manifolds or complex curves have the form

\[
y^2 = W'(x)^2 + 4f(x) = \prod_{j=1}^{2n} (x - x_j) \tag{1.8}
\]

for the one-matrix model (1.1), and

\[
z^n \tilde{z}^n + a_n z^{n+1} + \tilde{a}_n \tilde{z}^{n+1} + \sum_{i,j \in (N.P.)} f_{ij} z^i \tilde{z}^j = 0 \tag{1.9}
\]

in the two-matrix case (1.2) correspondingly; in the last formula the sum is taken over all integer points strictly inside the corresponding Newton polygon, see details in [1]. To complete the geometric formulation, these curves should be endowed with the meromorphic generating one-forms \( dS \)

\[
dS = \begin{cases} 
\frac{i}{4\pi} y dx, & \text{one-matrix model} \\
\frac{1}{2\pi i} \tilde{z} dz, & \text{two-matrix model} 
\end{cases} \tag{1.10}
\]

and the corresponding prepotentials are defined in the following way:

\[
S = \oint_A dS, \quad \frac{\partial \mathcal{F}}{\partial S} = 2\pi i \oint_B dS \tag{1.11}
\]

i.e. as functions of half of the periods of the generating one-form (1.10) so that their gradients are given by dual periods w.r.t. the intersection form \( A_\alpha \circ B_\beta = \delta_{\alpha \beta} \). Integrability of (1.11) is guaranteed by symmetricity

\[
\frac{\partial^2 \mathcal{F}}{\partial S_\alpha \partial S_\beta} = 2\pi i T_{\alpha \beta} \tag{1.12}
\]

of the period matrix of \( \Sigma \), which is a particular case of Riemann bilinear relations.

In the case of matrix models and their curves (1.8), (1.9) the period variables (1.11) have meaning of the fractions of eigenvalues, located at particular support. The total number of eigenvalues

\[
t_0 = \frac{1}{2\pi i} \text{res}_{P_+} dS = -\frac{1}{2\pi i} \text{res}_{P_-} dS \tag{1.13}
\]

\[
\frac{\partial \mathcal{F}}{\partial t_0} = 4\pi i \int_{P_+} dS
\]
and the parameters of potentials (1.3)

\[ t_k = \frac{1}{2\pi i k} \text{res}_{P_0} \xi^{-k} dS, \quad k > 0 \]

\[ \frac{\partial F}{\partial t_k} = \frac{1}{2\pi i} \text{res}_{P_0} \xi^k dS, \quad k > 0 \]

are associated with the generalized periods or residues of generating one-forms; in (1.14) \( \xi \) is an inverse local co-ordinate at the marked point \( P_0 \): \( \xi(P_0) = \infty \). Analogously to (1.12) the consistency condition for (1.14) is ensured by the following formula for the second derivatives

\[ \frac{\partial^2 F}{\partial t_n \partial t_k} = \frac{1}{2\pi i} \text{res}_{P_0} (\xi^k d\Omega_n) \]

while the third derivatives are given by the residue formula

\[ \frac{\partial^3 F}{\partial T_I \partial T_J \partial T_K} = \frac{1}{2\pi i} \text{res}_{dx=0} \left( \frac{dH_I dH_J dH_K}{dxdy} \right) = \frac{1}{2\pi i} \sum x_a \text{res}_{x_a} \left( \frac{\phi_I \phi_J \phi_K}{dx/dy} \right) \]

where the set of one-forms \( \{dH_I\} = \{d\omega_0, d\Omega_0, d\Omega_k\} \) corresponds to the set of parameters \( \{T_I\} = \{S_\alpha, t_0, t_k\} \) by

\[ \frac{\partial dS}{\partial S_\alpha} = d\omega_\alpha, \quad \alpha = 1, \ldots, g \]

(1.17)

together with

\[ \frac{\partial dS}{\partial t_0} = d\Omega_0 \]

(1.18)

and

\[ \frac{\partial dS}{\partial t_k} = d\Omega_k, \quad k \geq 1 \]

(1.19)

In the second line of (1.16), for convenience in what follows, we have introduced the meromorphic functions

\[ \phi_I(x) = \frac{dH_I}{dy} \]

(1.20)

As functions of their parameters, being (extended) moduli of complex manifolds (1.8), (1.9) the quasiclassical tau function or prepotential is defined only locally, but in the way consistent with the duality transformations: \( A \leftrightarrow B \), in the sense of the corresponding Legendre transform

\[ F \leftrightarrow F + \sum \alpha S_\alpha \Pi_\alpha \]

(1.21)

For different choices of the basis cycles the functions \( F \) differ merely by notation, and we will not distinguish between different functions below when studying equations, satisfied by quasiclassical tau-functions of matrix models.

In the next section we start with studying nonlinear differential equations, satisfied by prepotentials or quasiclassical tau functions, which contain the dispersionless Hirota equations and their higher genus analogs and the associativity or Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. Then we turn to various physical examples of the gauge/string duality (1.6) where the considered in (1.21) methods of ”geometric integrability” can be applied. We start with the well-known topic of duality between matrix models and two-dimensional gravity, formulated along these lines in terms of dispersionless hierarchy and then turn to the geometric picture of the AdS/CFT correspondence.
2 Dispersionless Hirota equations

In this section we consider the differential equations, satisfied by quasiclassical tau-function. We first discuss in details the rational case, where Σ is Riemann sphere $\mathbb{P}^1$ and it is possible to write easily the explicit form of differential equations for the quasiclassical tau-function, appearing to be the dispersionless limit of the Kadomtsev-Petviashvili (KP) and Toda hierarchy Hirota equations. Finally in this section, we discuss the analogs of dispersionless Hirota equations for the curve Σ of higher genus and the WDVV equations.

2.1 Geometry of dispersionless hierarchies

Consider a generating function: the Abelian integral
$$ S = \int_{\mathbb{P}^1} y \, dx $$

of the one-matrix model. From the formulas (1.14) one finds that
$$ y = \sum_{k \geq 1} \left( x^k - \lambda \frac{\partial F}{\partial t_k} x^{k-1} \right) $$

(2.1)
is expansion in local co-ordinate at $x(P) \to \infty$. Its time derivatives (cf. with (1.19))
$$ \Omega_k = \frac{\partial S}{\partial t_k} = x^k - \sum_j \frac{\partial^2 F}{\partial t_k \partial t_j} x^j $$

(2.2)
form a basis of meromorphic functions with poles at the point $x = \infty$. Suppose now, that these functions are globally defined on some curve Σ and do not have other singularities, except for those at $x = \infty$ (the case of a single singularity exactly corresponds to the KP hierarchy). If Σ is sphere (or one considers the dispersionless limit of KP hierarchy), this is enough to define all $\Omega_k$ in terms of the “uniformizing” function $\Omega_1 = \lambda \in \mathbb{C}$.

Indeed, the set of its powers $\lambda^k$ has the same singularities as the set of functions (2.2), i.e. these two are related by simple linear transformation
$$ \begin{align*}
\Omega_1 &= \lambda \\
\Omega_2 &= \lambda^2 + 2 \frac{\partial^2 F}{\partial t_1^2} \\
\Omega_3 &= \lambda^3 + 3 \frac{\partial^2 F}{\partial t_1^2} \lambda + \frac{3}{2} \frac{\partial^2 F}{\partial t_1 \partial t_2} \\
&\quad \vdots
\end{align*} $$

(2.3)
These equalities follow from the comparison of the singular at $x = \infty$ part of their expansions in $x$, following from (2.2). Comparing the negative ”tails” of the expansion in $x$ of both sides of eq. (2.3) expresses the derivatives $\frac{\partial^2 F}{\partial t_k \partial t_l}$ (of $\Omega_k$ in the l.h.s.) in terms of only those with $k = 1$ (of $\lambda = \Omega_1$ in the r.h.s.). These relations are called the dispersionless KP, or the dKP Hirota equations, and correspond to geometry of rational curve with the only marked point (or when the other marked points are simply forgotten), and they can be encoded into a ”generating form”
$$ (x_1 - x_2) \left( 1 - e^{D(x_1)D(x_2)}F \right) = (D(x_1) - D(x_2)) \frac{\partial F}{\partial t_1} $$

(2.4)
where the operators
$$ D(x) = \sum_{k \geq 1} \frac{x^{-k}}{k} \frac{\partial}{\partial t_k} $$

(2.5)
are introduced.
In the case of dispersionless Toda hierarchy one should consider sphere with two marked points and corresponding local co-ordinates $z = \infty$ and $\tilde{z} = \infty$ correspondingly. Then
\begin{equation}
\begin{aligned}
\tilde{z} &\rightarrow \infty + \sum \left( k t_k \tilde{z}^{k-1} + \frac{\partial F}{\partial t_k} \frac{1}{\tilde{z}^{k+1}} \right) \\
z &\rightarrow \infty + \sum \left( k t_k z^{k-1} + \frac{\partial F}{\partial t_k} \frac{1}{z^{k+1}} \right)
\end{aligned}
\end{equation}

(2.6)

Considering now two generating functions $S = \int P \tilde{z} dz$ and $\tilde{S} = \int P z \tilde{z}$ and, like in (2.2), their time-derivatives one gets
\begin{align*}
\Omega_k &= \frac{\partial S}{\partial t_k} = z^k - \sum_j \frac{\partial^2 F}{\partial t_k \partial j \bar{j} z^j}, \quad k > 0 \\
\tilde{\Omega}_k &= \frac{\partial \tilde{S}}{\partial \tilde{t}_k} = \tilde{z}^k - \sum_j \frac{\partial^2 F}{\partial \tilde{t}_k \partial \tilde{j} \bar{\tilde{j}} \tilde{z}^j}, \quad k > 0
\end{align*}

(2.7)

together with
\begin{align*}
\Omega_0 &= \frac{\partial S}{\partial t_0} = \log z - \sum_j \frac{\partial^2 F}{\partial t_0 \partial j \bar{j} z^j} \\
\tilde{\Omega}_0 &= \frac{\partial \tilde{S}}{\partial \tilde{t}_0} = \log \tilde{z} - \sum_j \frac{\partial^2 F}{\partial \tilde{t}_0 \partial \tilde{j} \bar{\tilde{j}} \tilde{z}^j}
\end{align*}

(2.8)

Suppose again, that (2.7) and (2.8) are now globally defined meromorphic functions on sphere with two marked points. Then it is natural to introduce a uniformizing function $w \in \mathbb{C}^*$, using the dipole differential
\begin{equation}
\frac{dw}{w} = \frac{d\Omega_0}{- \bar{\Omega}_0}
\end{equation}

(2.9)

since the sum of its residues should vanish. It means that
\begin{align*}
w &\xrightarrow{z \rightarrow \infty} ze^{-\sum_j \frac{\partial^2 F}{\partial t_0 \partial j \bar{j} z^j}} \\
\frac{1}{w} &\xrightarrow{\tilde{z} \rightarrow \infty} \tilde{z} e^{-\sum_j \frac{\partial^2 F}{\partial \tilde{t}_0 \partial \tilde{j} \bar{\tilde{j}} \tilde{z}^j}}
\end{align*}

(2.10)
i.e. $w$ has a simple pole at the point, where $z = \infty$, and a simple zero at $\tilde{z} = \infty$. The Hirota equations for dispersionless Toda hierarchy come now from a simple observation that
\begin{equation}
\Omega_k = P_k(w), \quad \tilde{\Omega}_k = \tilde{P}_k \left( \frac{1}{w} \right), \quad k > 0
\end{equation}

(2.11)

for some polynomials $P_k$ and $\tilde{P}_k$, similar to (2.3), some of them were explicitly computed in [6]. Formulas (2.10) are equivalent to the inverse conformal maps for a simply-connected domain $D$, which were discussed in the first part of this paper [1].

For the two-matrix model relations (1.15) can be unified into the following expression (a combination of formulas (6.21) and (6.3) from [1])
\begin{equation}
\log \frac{\left| w(z) - w(z') \right|^2}{\left| w(z) \bar{w}(z') - 1 \right|^2} = \log \left| \frac{1}{z} - \frac{1}{z'} \right|^2 + \nabla(z) \nabla(z') F
\end{equation}

(2.12)

1. We follow here the conventions of the two-matrix model case [1], corresponding to the independent choice of parameters in the vicinity of these two points.
which implies an infinite hierarchy of differential equations on the function $F$. In (2.12) we use the operator

$$\nabla(z) = \partial_{\zeta_0} + \sum_{k \geq 1} \left( \frac{z-k}{k} \partial_{t_k} + \frac{z-k}{k} \partial_{\bar{t}_k} \right)$$

(2.13)

discussed in detail in this context in [1], whose holomorphic and antiholomorphic parts

$$D(z) = \sum_{k \geq 1} \frac{z-k}{k} \partial_{t_k}, \quad D(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}-k}{k} \partial_{\bar{t}_k},$$

(2.14)

especially coincide with (2.5). It is convenient to normalize $w(z)$ by \(w(\infty) = \infty\) and $\partial_w w(\infty)$ to be real, i.e.

$$w(z) = \frac{z}{r} + O(1) \quad \text{as} \quad z \to \infty$$

(2.15)

where (the real number) \(r = \lim_{z \to \infty} \frac{dz}{dw}(z)\) is usually called the (external) conformal radius of the domain $D$ or the eigenvalue support of the two-matrix model. Putting \(z' \to \infty\) in (2.12), one gets

$$\log |w(z)|^2 = \log |z|^2 - 2 \partial_{\zeta_0} \nabla(z) F$$

(2.16)

At $z \to \infty$ this equality yields a simple formula for the conformal radius:

$$\log r^2 = \partial_{\zeta_0}^2 F$$

(2.17)

Rewriting further (2.12) in the form

$$\log \left( \frac{w(z) - w(z')}{w(z) - w(z')} \right) - \log \left( \frac{1}{z} - \frac{1}{z'} \right) = \left( \frac{1}{2} \partial_{\zeta_0} + D(z) \right) \nabla(z') F =$$

$$= - \log \left( \frac{w(z) - w(z')}{w(z) - w(z')} \right) + \log \left( \frac{1}{z} - \frac{1}{z'} \right) + \left( \frac{1}{2} \partial_{\zeta_0} + D(z) \right) \nabla(z') F$$

(2.18)

one gets an equality between the holomorphic function of $z$ (in the l.h.s.) and the antiholomorphic (in the r.h.s.) function. Therefore, both are equal to a $z$-independent term which can be found from the limit $z \to \infty$. As a result, we obtain the equation

$$\log \left( \frac{w(z) - w(z')}{w(z) - w(z')} \right) = \log \left( \frac{1}{z} - \frac{z'}{z} \right) + D(z) \nabla(z') F$$

(2.19)

which, at $z' \to \infty$, turns into the formula for the conformal map $w(z)$:

$$\log w(z) = \log z - \frac{1}{2} \partial_{\zeta_0}^2 F - \partial_{\zeta_0} D(z) F$$

(2.20)

where we also used (2.17). Proceeding in a similar way, one can rearrange (2.19) in order to write it separately for holomorphic and antiholomorphic parts in $z'$:

$$\log \left( \frac{w(z) - w(z')}{z - z'} \right) = \frac{1}{2} \partial_{\zeta_0}^2 F \quad \text{or} \quad D(z) \nabla(z') F$$

$$- \log \left( 1 - \frac{1}{w(z)w(z')} \right) = D(z) \nabla(z') F$$

(2.21)
Writing eqs. (2.21) for the pairs of points \((z_1, z_2), (z_2, z_3)\) and \((z_3, z_1)\) and summing up the exponentials of the both sides of each equation one arrives at the relation

\[
(z_1 - z_2)e^{D(z_1)D(z_2)F} + (z_2 - z_3)e^{D(z_2)D(z_3)F} + (z_3 - z_1)e^{D(z_3)D(z_1)F} = 0
\]  

(2.22)

which is the dispersionless Hirota equation for the dKP hierarchy - a part of the dispersionless two-dimensional Toda lattice hierarchy; it is easy to notice, that (2.22) is just a symmetric form of the equation (2.21). This equation can be regarded as a degenerate case of the trisecant Fay identity [5], see formula (A.5) from Appendix A and discussion of this point in next section. It encodes the algebraic relations between the second order derivatives of the function \(F\). As \(z_3 \to \infty\), we get these relations in a more explicit but less symmetric form:

\[
1 - e^{D(z_1)D(z_2)F} = \frac{D(z_1) - D(z_2)}{z_1 - z_2} \partial_1 F
\]

(2.23)

which makes it clear that the totality of second derivatives \(F_{ij} = \partial_i \partial_j F\) are expressed through the derivatives with one of the indices put equal to unity.

More general equations of the dispersionless Toda hierarchy are obtained in a similar way by combining eqs. (2.20) and (2.21) include derivatives w.r.t. \(t_0\) and \(t_k\):

\[
(z_1 - z_2)e^{D(z_1)D(z_2)F} = z_1e^{-\partial_0D(z_1)F} - z_2e^{-\partial_0D(z_2)F}
\]

(2.24)

\[
1 - e^{-D(z)D(z)F} = \frac{1}{z^2} e^{\partial_0 \nabla(z)F}
\]

These equations allow one to express the second derivatives \(\partial_1 \partial_m F, \partial_1 \partial_0 \partial_n F\) with \(m, n \geq 1\) through the derivatives \(\partial_0 \partial_0 F, \partial_0 \partial_1 F, \partial_1 \partial_1 F\). In particular, the dispersionless Toda equation,

\[
\partial_1 \partial_1 F = e^{\partial_0^2 F}
\]

(2.25)

which follows from the second equation of (2.24) as \(z \to \infty\), expresses \(\partial_1 \partial_1 F\) through \(\partial_0^2 F\).

### 2.2 Generalized Hirota equations for the multisupport solution

To derive equations for the function \(F\) in sect. 2.1 we have used the representation (2.20) for the conformal map \(w(z)\) in terms of the derivatives of \(F\) and eq. (2.12) relating the conformal map to the Green function, expressed through the second derivatives of \(F\). In the multiply-connected case [28], the strategy is basically the same, with the analog of the conformal map \(w(z)\) (or rather of \(\log w(z)\)) being the embedding of \(D^c\) into the \(g\)-dimensional complex torus \(\text{Jac}\), the Jacobian of the Schottky double \(\Sigma\), see Appendix A.

This embedding is given, up to an overall shift in \(\text{Jac}\), by the Abel map (5.41), where the components \(\omega_\alpha (z) = \int_0^z d\omega_\alpha\) of the vector (5.41) can be thought of as the holomorphic part of the harmonic measure \(\bar{w}_\alpha\), discussed in sect. 6 of [1]. The Abel map is represented (see second formula in (6.57) of [1]) through the second order derivatives of the function \(F\):

\[
\omega_\alpha (z) - \omega_\alpha (\infty) = \int_\infty^z d\omega_\alpha = -\partial_\alpha D(z)F
\]

(2.26)

and\(^2\)

\[
2 \text{Re} \omega_\alpha (\infty) = \bar{w}_\alpha (\infty) = -\partial_{\bar{\tau}_\alpha} \partial_0 F
\]

(2.27)

\(^2\)Following [28,1] we are using in this section the time variables \(\tau_\alpha\), which are more strictly defined in general situation, as moments adjusted to the proper basis of functions \(a_\alpha(z)\). In all simple cases with finitely many nontrivial variables they can be just identified with the Toda lattice times \(t_k\), for details see the first part of this paper [1].
The Green function of the Dirichlet boundary problem can be written in terms of the prime form \((A.6)\) (see Appendix A) on the Schottky double:

\[
G(z, \zeta) = \log \left| \frac{E(z, \zeta)}{E(z, \zeta)} \right|
\]  

(2.28)

Here by \(\zeta\) we mean the (holomorphic) coordinate of the “mirror” point on the Schottky double, i.e. the “mirror” of \(\zeta\) under the antiholomorphic involution. The pairs of such mirror points satisfy the condition \(\omega_{\alpha}(\zeta) + \bar{\omega}_{\alpha}(\zeta) = 0\) in the Jacobian (i.e., the sum should be zero modulo the lattice of periods). The prime form\(^3\) is written through the Riemann theta functions and the Abel map as follows:

\[
E(z, \zeta) = \frac{\theta_+(\omega(z) - \omega(\zeta))}{h(z) h(\zeta)}
\]  

(2.29)

when the both points are on the upper sheet and

\[
E(z, \bar{\zeta}) = \frac{\theta_+(\omega(z) + \omega(\zeta))}{ih(z) h(\zeta)}
\]  

(2.30)

when \(z\) is on the upper sheet and \(\bar{\zeta}\) is on the lower one (for other cases we define \(E(z, \zeta) = \bar{E}(z, \zeta), \ E(\bar{z}, \zeta) = \bar{E}(z, \zeta)\)). Here \(\theta_+(\omega) \equiv \theta_+(\omega T)\) is the Riemann theta function \((A.4)\) with the period matrix \(T_{\alpha\beta} = 2\pi i \partial_\alpha \partial_\beta \mathcal{F}\) and any odd characteristics \(\delta^*, \) and

\[
h^2(z) = -z^2 \sum_{\alpha=1}^g \theta_{+,\alpha}(0) \partial_\zeta \omega_\alpha(z) = z^2 \sum_{\alpha=1}^g \theta_{+,\alpha}(0) \sum_{k \geq 1} a_k^\alpha(z) \partial_\zeta \partial_{\tau_k} \mathcal{F}
\]  

(2.31)

Note that in the l.h.s. of (2.30) the bar means the reflection in the double while in the r.h.s. the bar means complex conjugation, the notation is consistent since the local coordinate in the lower sheet is just the complex conjugate one. However, one should remember that \(E(z, \bar{\zeta})\) is not obtained from \(E(z, \zeta)\) by a simple substitution of the complex conjugated argument, and on different sheets so defined prime “forms” \(E\) are represented by different functions. In our normalization \(iE(z, \bar{z})\) is real (see also Appendix A) and

\[
\lim_{\zeta \to z \zeta^{-1} - \zeta^{-1} = 1}
\]

in particular, \(\lim_{z \to \infty} z E(z, \infty) = 1.\)

In (2.28), the \(h\)-functions in the prime forms cancel, so the analog of (2.12) reads

\[
\log \left| \frac{\theta_+(\omega(z) - \omega(\zeta))}{\theta_+(\omega(z) + \omega(\zeta))} \right|^2 = \log \left| \frac{1}{z} - \frac{1}{\zeta} \right|^2 + \nabla(z) \nabla(\zeta) \mathcal{F}
\]  

(2.33)

This equation already explains the claim made in the beginning of this section. Indeed, the r.h.s. is the generating function for the derivatives \(\mathcal{F}_{ik}\) while the l.h.s. is expressed through derivatives of the form \(\mathcal{F}_{\alpha k}\) and \(\mathcal{F}_{\alpha k}\) only, and the expansion in powers of \(z, \zeta\) allows one to express the former through the latter.

The analogs of eqs. (2.10), (2.17) are

\[
\log \left| \frac{\theta_+(\omega(z) - \omega(\infty))}{\theta_+(\omega(z) + \omega(\infty))} \right|^2 = -\log |z|^2 + \partial_\zeta \nabla(z) \mathcal{F}
\]  

(2.34)

\(^3\)Given a Riemann surface with local coordinates \(1/z\) and \(1/\bar{z}\) we trivialize the bundle of \(-\frac{1}{2}\)-differentials and “redefine” the prime form \(E(z, \zeta) \to E(z, \zeta)(dz)^{1/2}(d\bar{z})^{1/2}\) so that it becomes a function. However for different coordinate patches (the “upper” and “lower” sheets of the Schottky double) one gets different functions, see, for example, formulas \((A.20)\) and \((A.30)\) below.
functions themselves, it is really a source of closed equations on the quasiclassical tau-function
relation [5] is the celebrated Fay identity (A.8). Although it contains not only prime forms but Riemann theta
forms at different points, which, via (2.42), could be used to generate equations on

together with the nice formula

\[ \log \left| \frac{h^2(\infty)}{\theta_4(\infty)} \right|^2 = \partial_{\tau_0}^2 F \]

respectively, here \( \varpi(z) \equiv 2 \text{Re} \omega(z) = (\varpi_1(z), \ldots, \varpi_g(z)) \) and

\[ h^2(\infty) = \lim_{z \to \infty} z \theta_4 \left( \int_{\infty}^{z} d\omega \right) = - \sum_{\alpha=1}^{g} \theta_{*,\alpha}(0) \partial_{\alpha} \partial_{\tau_1} F \]

A simple check shows that the l.h.s. of (2.35) can be written as \(-2 \log(iE(\infty, \infty))\). As is seen from the expansion
\( G(z, \infty) = - \log |z| - \log(iE(\infty, \infty)) + O(z^{-1}) \) as \( z \to \infty \), \((iE(\infty, \infty))^{-1}\) is a natural analog of the conformal
radius, and (2.35) indeed turns to (2.17) in the simply-connected case. However, now it provides a nontrivial
relation on \( F_{\alpha\beta} \)'s and \( F_{\alpha'i} \)'s:

\[ \left( \sum_{\alpha} \theta_{*,\alpha} \partial_{\alpha} \partial_{\tau_1} F \right) \left( \sum_{\beta} \theta_{*,\beta} \partial_{\beta} \partial_{\tau_1} F \right) = \theta_4^2(\varpi(\infty)) e^{2 \partial_{\tau_0}^2 F} \] (2.37)

so that the “small phase space” is defined modulo this relation.

Now we are going to decompose these equalities into holomorphic and antiholomorphic parts, the results are
conveniently written in terms of the prime form. The counterpart of (2.19) is

\[ \log \frac{E(z, \zeta)}{E(\zeta, \infty) E(z, \zeta)} = \log \left( 1 - \frac{\zeta}{z} \right) + D(z) \nabla(\zeta) F \]

and tending \( \zeta \to \infty \), one gets:

\[ \log \frac{E(z, \infty)}{E(z, \infty)} = \log z + \log E(\infty, \infty) - \partial_{\tau_0} D(z) F \] (2.39)

Separating holomorphic and antiholomorphic parts of (2.38) in \( \zeta \), we get analogs of (2.21):

\[ \log \frac{E(z, \zeta)}{E(z, \infty) E(\zeta, \infty)} = \log(z - \zeta) + D(z) D(\zeta) F \]

\[ - \log \frac{E(z, \zeta) E(\infty, \infty)}{E(z, \infty) E(\infty, \zeta)} = D(z) D(\zeta) F \] (2.41)

Combining these equalities, one is able to obtain the following representations of the prime form itself

\[ E(z, \zeta) = (z^{-1} - \zeta^{-1}) e^{-\frac{1}{4}(D(z) - D(\zeta))^2} F \]

\[ iE(z, \zeta) = e^{-\frac{1}{4}(\partial_{\tau_0} + D(z) + D(\zeta))^2} F \] (2.42)

together with the nice formula

\[ iE(z, \zbar) = e^{-\frac{1}{4} \nabla^2(z)} F \] (2.43)

For higher genus Riemann surfaces there are no simple universal relations connecting values of the prime
forms at different points, which, via (2.42), could be used to generate equations on \( F \). The best available
relation [5] is the celebrated Fay identity (A.8). Although it contains not only prime forms but Riemann theta
functions themselves, it is really a source of closed equations on the quasiclassical tau-function \( F \), since all the
ingredients are in fact representable in terms of second order derivatives of \( F \) in different variables.
An analog of the KP version of the Hirota equation (2.22) for the function $F$ can be obtained by plugging eqs. (2.21) and (2.1) into the Fay identity (A.8). As a result, one obtains a closed equation which contains second order derivatives of the $F$ only (recall that the period matrix in the theta-functions is the matrix of the derivatives $F_{\alpha\beta}$). A few equivalent forms of this equation are available: first, shifting $Z \to Z - \omega_3 + \omega_4$ in (A.8) and putting $z_4 = \infty$, one gets the relation

$$
(z_1 - z_2) e^{D(z_1)D(z_2)F} \theta \left( \int_{\infty}^{z_1} d\omega + \int_{\infty}^{z_2} d\omega - Z \right) \theta \left( \int_{\infty}^{z_3} d\omega - Z \right) + 
+ (z_2 - z_3) e^{D(z_2)D(z_3)F} \theta \left( \int_{\infty}^{z_2} d\omega + \int_{\infty}^{z_3} d\omega - Z \right) \theta \left( \int_{\infty}^{z_1} d\omega - Z \right) + 
+ (z_3 - z_1) e^{D(z_3)D(z_1)F} \theta \left( \int_{\infty}^{z_3} d\omega + \int_{\infty}^{z_1} d\omega - Z \right) \theta \left( \int_{\infty}^{z_2} d\omega - Z \right) = 0
$$

(2.44)

where the vector $Z$ is arbitrary, in particular it can be chosen $Z = 0$. We see that (2.22) gets “dressed” by the theta-factors, and each theta-factor is expressed through $F$, for example

$$
\theta \left( \int_{\infty}^{z} d\omega \right) = \sum_{n_{\alpha} \in \mathbb{Z}} \exp \left( -2\pi^2 \sum_{\alpha\beta} n_{\alpha} n_{\beta} \partial_{\alpha\beta}^2 F - 2\pi i \sum_{\alpha} n_{\alpha} \partial_{\alpha} D(z) F \right)
$$

(2.45)

Another form of this equation, obtained from (2.44) for a particular choice of $Z$, reads

$$
(z_1 - z_2) z_3^{-1} e^{D(z_1)D(z_2)F} h(z_3) \theta_s \left( \int_{\infty}^{z_1} d\omega + \int_{\infty}^{z_2} d\omega \right) + [\text{cyclic per-s of } z_1, z_2, z_3] = 0
$$

(2.46)

Taking the limit $z_3 \to \infty$ in (2.44), one gets an analog of (2.23):

$$
1 - \frac{\theta \left( \int_{\infty}^{z_1} d\omega + \int_{\infty}^{z_2} d\omega - Z \right)}{\theta \left( \int_{\infty}^{z_2} d\omega - Z \right)} \frac{\theta(\mathbb{Z})}{\theta(z_3)} e^{D(z_1)D(z_2)F} = 
\frac{D(z_1) - D(z_2)}{z_1 - z_2} \partial_{\tau_1} F + \frac{1}{z_1 - z_2} \sum_{\alpha=1}^{g} \partial_{\tau_1} \log \theta \left( \int_{\infty}^{z_2} d\omega - Z \right) \partial_{\alpha} \partial_{\tau_1} F
$$

(2.47)

which also follows from another Fay identity (A.9).

Equations on $F$ with $\tau_k$-derivatives follow from the general Fay identity (A.8) with some points on the lower sheet. Besides, many other equations can be derived as various combinations and specializations of the ones mentioned above. Altogether, they form an infinite hierarchy of consistent differential equations of a very complicated structure which deserves further investigation. The functions $F$ corresponding to different choices of independent variables (i.e., to different bases in homology cycles on the Schottky double) provide different solutions to this hierarchy.

Let us show how the simplest equation of the hierarchy, the dispersionless Toda equation (2.25), is modified in the multiply-connected case. Applying $\partial_{z} \partial_{\bar{z}}$ to both sides of (2.11) and setting $\zeta = z$, we get:

$$
(\partial D(z)) (\partial \bar{D}(z)) F = -\partial_{z} \partial_{\bar{z}} \log E(z, \bar{z})
$$

(2.48)

where $\partial D(z)$ is the $z$-derivative of the operator $D(z)$: $\partial D(z) = \sum_{k} a_{k}(z) \partial_{\tau_{k}}$. To transform the r.h.s., we use the identity (A.11) from Appendix A and specialize it to the particular local parameters on the two sheets:

$$
|z|^4 \partial_{z} \partial_{\bar{z}} \log E(z, \bar{z}) = \frac{\theta(\mathbb{Z})(z) \theta(\mathbb{Z})(z) - \theta^2(\mathbb{Z}) E^2(z, \bar{z})}{\theta^2(\mathbb{Z}) E^2(z, \bar{z})} + |z|^4 \sum_{\alpha, \beta} (\log \theta(\mathbb{Z}),_{\alpha, \beta} \partial_{z} \omega_{\alpha}(z) \partial_{\bar{z}} \omega_{\beta}(z))
$$

(2.49)
where \( \theta(Z) \), \( a, b \) is defined in [A.12]. Tending \( z \) to \( \infty \), we obtain a family of equations (parameterized by an arbitrary vector \( Z \)) which generalize the dispersionless Toda equation for the quasiclassical tau-function

\[
\partial_{\tau_i} \partial_{\tau_j} F = \frac{\theta'(\infty) + Z}{\theta^2(Z)} e^{\vartheta_0} - \sum_{\alpha, \beta = 1}^{g} (\log \theta(Z))_{, \alpha \beta} (\partial_{\alpha} \partial_{\tau_i}) (\partial_{\beta} \partial_{\tau_j}) F \tag{2.50}
\]

(here we used the \( z \to \infty \) limits of (2.22) and (2.23)). The following two equations correspond to special choices of the vector \( Z \):

\[
\partial_{\tau_i} \partial_{\tau_j} F + \sum_{\alpha, \beta = 1}^{g} (\log \theta(0))_{, \alpha \beta} (\partial_{\alpha} \partial_{\tau_i}) (\partial_{\beta} \partial_{\tau_j}) F = \frac{\theta'(\infty)}{\theta^2(0)} e^{\vartheta_0} \tag{2.51}
\]

and

\[
\partial_{\tau_i} \partial_{\tau_j} F = - \sum_{\alpha, \beta = 1}^{g} [\log \theta_*(\infty)]_{, \alpha \beta} (\partial_{\alpha} \partial_{\tau_i}) (\partial_{\beta} \partial_{\tau_j}) F \tag{2.52}
\]

Finally, let us specify the equation (2.51) for the genus \( g = 1 \) case, in this case there is only one extra variable \( S \), which is either \( S_1 \) or \( S_2 \). The Riemann theta-function \( \theta(\varpi(\infty) + Z) \) is then replaced by the Jacobi theta-function \( \vartheta (\partial_S \tau_0 F - Z|T) \equiv \vartheta_3 (\partial_S \tau_0 F - Z|T) \), where the elliptic modular parameter is \( T = 2\pi i \partial_S^2 F \), and the vector \( Z \equiv Z \) has only one component. The equation has the form:

\[
\partial_{\tau_i} \partial_{\tau_j} F = \vartheta_3 \left( \partial_S \partial_\tau_0 F + Z | 2\pi i \partial_S^2 F \right) \vartheta_3 \left( \partial_S \partial_\tau_0 F - Z | 2\pi i \partial_S^2 F \right) e^{\vartheta_0} \tag{2.53}
\]

Note also that equation (2.51) acquires the form

\[
(\partial_S \partial_{\tau_i} F) (\partial_S \partial_{\tau_j} F) = \left( \frac{\vartheta_3 \left( \partial_S \partial_\tau_0 F | 2\pi i \partial_S^2 F \right)}{\vartheta'_3 \left( 0 | 2\pi i \partial_S^2 F \right)} \right)^2 e^{\vartheta_0} \tag{2.54}
\]

where \( \vartheta_* \equiv \vartheta_1 \) is the only odd Jacobi theta-function. Combining (2.53) and (2.54) one may also write the equation

\[
\partial_{\tau_i} \partial_{\tau_j} F = \left[ \vartheta_3 \left( \partial_S \partial_\tau_0 F + Z | 2\pi i \partial_S^2 F \right) \vartheta_3 \left( \partial_S \partial_\tau_0 F - Z | 2\pi i \partial_S^2 F \right) \right. \\
- \left. \left( \frac{\vartheta_3 \left( \partial_S \partial_\tau_0 F | 2\pi i \partial_S^2 F \right)}{\vartheta'_3 \left( 0 | 2\pi i \partial_S^2 F \right)} \right)^2 2 \partial_S^2 \log \vartheta_3 \left( Z | 2\pi i \partial_S^2 F \right) \right] e^{\vartheta_0} \tag{2.55}
\]

whose form literally reminds (2.23), but differs by the nontrivial “coefficient” in the square brackets. In the limit \( T \to i\infty \) the theta-function \( \vartheta_3 \) tends to unity, and we obtain the dispersionless Toda equation (2.25).

### 2.3 WDVV equations

Let us finally briefly discuss the Witten-Dijkgraaf-Verlinde-Verlinde or WDVV equations [7] in the context of geometry of matrix models. In the most general form [8] they can be written as system of algebraic relations

\[
F_I F_J^{-1} F_K = F_K F_J^{-1} F_I, \quad \forall I, J, K \tag{2.56}
\]

for the matrices of third derivatives

\[
\| F_I \|_{JK} = \frac{\partial^3 F}{\partial T_I \partial T_J \partial T_K} \equiv F_{IJK} \tag{2.57}
\]
of quasiclassical tau-function $F(T)$. Have been appeared first in the context of topological string theories \[7\], they were rediscovered later on in much larger class of physical theories where the exact results can be expressed through a single holomorphic function of several complex variables, like we have here for the planar matrix models.

Having the residue formula (1.16) for the quasiclassical tau-function, the proof of the WDVV equations (2.56) is reduced to solving the system of linear equations \[9, 10\], which requires only fulfilling the two conditions:

- The matching condition
  \[\#(I) = \#(a)\] (2.58)
  and
- nondegeneracy of the matrix built from (1.20):
  \[\det \|\phi_I(x_a)\| \neq 0\] (2.59)

Under these conditions, the structure constants $C^K_{IJ}$ of the associative algebra
\[(C_J)^K_L \cdot (C_I)^L_M = (C_J)^K_M \cdot (C_I)^L_L \tag{2.60}\]
responsible for the WDVV equations can be found from the system of linear equations
\[\phi_I(x_a)\phi_J(x_a) = \sum_K C^K_{IJ}\phi_K(x_a), \quad \forall x_a\] (2.61)
with the solution
\[C^K_{IJ} = \sum_a \phi_I(x_a)\phi_J(x_a) (\phi_K(x_a))^{-1}\] (2.62)
To make it as general, as in \[8\], one may consider an isomorphic associative algebra (again $\forall$ $x_a$)
\[\phi_I(x_a)\phi_J(x_a) = \sum_K C^K_{IJ}(\xi)\phi_K(x_a) \cdot \xi(x_a)\] (2.63)
which instead of (2.62) leads to
\[C^K_{IJ}(\xi) = \sum_a \phi_I(x_a)\phi_J(x_a) (\phi_K(x_a))^{-1}\] (2.64)
The rest of the proof is consistency of this algebra with relation
\[F_{IJK} = \sum_L C^L_{IJ}(\xi)\eta_{KL}(\xi)\] (2.65)
with
\[\eta_{KL}(\xi) = \sum_A \xi_A F_{KLA}\] (2.66)
expressing structure constants in terms of the third derivatives of free energy and, thus, leading immediately to the equations (2.60). It is easy to see that (2.60) is satisfied if $F_{KLA}$ are given by residue formula (1.16), which can be also represented as
\[
\frac{\partial^3 F}{\partial T_I \partial T_J \partial T_K} = \sum_{x_a} \operatorname{res}_{x_a} \left( \frac{dH_I dH_J dH_K}{dx dy} \right) = \\
= \sum_{x_a} \Gamma_a^3 \phi_I(x_a)\phi_J(x_a)\phi_K(x_a)
\] (2.67)
where $\Gamma_a = \sqrt{\prod_{b \neq a} (x_a - x_b)}$, see sect. 4.5 of \cite{1}.

Indeed \cite{10}, requiring only matching $\#(a) = \#(I)$, one gets

$$
\sum_K C^{ij}_{KL}(\xi) \eta_{KL}(\xi) = \sum_{K, a, b} \phi_I(x_a) \phi_J(x_a) \frac{(\phi_K(x_a))^{-1}}{\xi(x_a)} \cdot \phi_K(x_b) \phi_L(x_b) \xi(x_b) \Gamma_b
$$

and finally

$$
\sum_K C^{ij}_{KL}(\xi) \eta_{KL}(\xi) = \sum_a \phi_I(x_a) \phi_J(x_a) \phi_L(x_a) \xi(x_a) \Gamma_a = \sum_a \Gamma_a \phi_I(x_a) \phi_J(x_a) \phi_L(x_a) = \mathcal{F}_{IJL}
$$

Hence, for the proof of (2.56) one has to adjust the total number of parameters $\{T_i\}$ according to the condition (2.65). The number of critical points for the one-matrix model is $\#(a) = 2n$ since $dx = 0$ in the branching points of (1.8), i.e. at $y^2 = R(x) = 0$. Thus, one have to take a codimension one subspace in the space of all possible parameters of the one-matrix model, a natural choice will be to fix the eldest coefficient of (1.3). Then the total number of parameters $\#(I)$, including the periods $S$, residue $t_0$ and the rest of the coefficients of the potential will be $g + 1 + n = (n - 1) + 1 + n = 2n$, i.e. exactly equal to the number of critical points $\#(a)$. In \cite{12} an explicit check of the WDVV equations has been performed for this choice of parameters, using the expansion of free energy computed in \cite{11}. One could try to follow analogously for the two-matrix model \cite{12} with non-hyperelliptic curve \cite{13}, where the number of critical points in the residue formula $\#(a)$ can be extracted from \cite{11}, since our analysis here does not use any special properties of the one-matrix case. However, the problem is that the number of critical points for the two-matrix model $\#(a) = 2n^2 + n - 1$ (see e.g. formula (5.27) from \cite{11}) for generic large $n$ exceeds the naive total number of parameters of the model.

Another important issue is that formula (2.67), expressing the third derivatives of quasiclassical tau-function $\mathcal{F}$ through the quantities $\phi_{I\alpha}$ determined in (1.20), suggests to interpret $\mathcal{F}$ as the free energy of a certain topological string theory, with three propagators $\phi_{I\alpha}$ ending at the same three-vertex:

Moreover \cite{13}, one can associate the next term $F_1$ of expansion (1.7) with the one-loop diagram in this topological theory, i.e. with the determinant $\det_{I, a} \phi_I(x_a)$:

as

$$
F_1 = -\frac{1}{2} \log \left( \Delta^{1/3}(x) \cdot \det_{i, j} \oint_{A_i} \frac{x^{j-1} dx}{y} \right) \sim \frac{1}{2} \log \left( \det_{I, a} \phi_I(x_a) \right)
$$

One can therefore conjecture an existence of a diagram technique for calculating the higher genera free energies and generating function for the correlators in matrix models.

### 3 Matrix models and (p,q)-string theory

Let us now turn to some physical applications of the geometric picture related to integrability of matrix models, which was presented in \cite{11} and developed in the previous section. As a first, already quite nontrivial example,
we discuss the correlators in the simplest, so called (p, q) string theory models, directly corresponding to the
dual matrix model description of two-dimensional gravity [14].

3.1 dKP for (p,q)-critical points

According to existing about fifteen years and very popular hypothesis, see e.g. [13], the so called (p, q) critical
points of two-dimensional gravity (corresponding to the generalized minimal conformal matter [16], interacting
with two-dimensional Liouville gravity [2]) are most efficiently described by tau-function of the KP hierarchy,
satisfying string equation (see also sect. 2.3 of [1]). In particular, it means that the correlators on world-sheets
of spherical topology (the only ones, partially computed by now by means of two-dimensional conformal field
theory) are governed by quasiclassical tau-function of dKP hierarchy, which is a very particular case of generic
quasiclassical hierarchy. For each (p, q)-th point one should consider a solution of the
p-reduced dKP hierarchy,
or more strictly, its expansion in the vicinity of nonvanishing
and vanishing other times, perhaps except for
(1.11) (the so called conformal backgrounds).

The geometric formulation of result in terms of quasiclassical hierarchy is as follows:

- For each (p, q)-th point take a pair of polynomials

\[ X = \lambda^p + \ldots \]
\[ Y = \lambda^q + \ldots \]

(3.1)
of degrees p and q respectively. It is convenient to rewrite them as a generating differential

\[ dS = Y dX \]

(3.2)

- The parameters of quasiclassical hierarchy ("times") and the corresponding "one-point functions" are
determined by the formulas (1.14), where

\[ \xi = X^{+p}_x = \lambda \left( 1 + \ldots + \frac{X_0}{\lambda^p} \right)^\frac{1}{p} \]

(3.3)
is the local co-ordinate at the point \( \lambda(P_0) = \infty \). The additional constraint that \( \xi^p = \xi^p_+ \) or \( \xi^p_- = 0 \) (\( \xi^p_+ \) is
a polynomial in \( \lambda \)) means that one deals here with particular \( p \)-reduction of the dKP hierarchy.

- Under the scaling \( X \to \Lambda^p X, Y \to \Lambda^q Y \), (induced by \( \lambda \to \Lambda \lambda \) and therefore \( \xi \to \Lambda \xi \)), the times (1.14)
transform as \( t_k \to \Lambda^{p+q-k} t_k \). Then from the second formula of (1.14) it follows that the function \( F \) scales
as \( F \to \Lambda^{S(p+q)} F \), or, for example, as

\[ F \propto t_1^{2 \frac{p+q}{p+q-1}} f(\tau_k) \]

(3.4)

where \( f \) is supposed to be a scale-invariant function of corresponding dimensionless ratios of the times
\( \tau_k = t_k/t_1^{p+q-1} \). In the simplest \( (p, q) = (2, 2K - 1) \) case of dispersionless KdV one also expects a
natural scaling of the form

\[ F \propto (t_{2K-3})^{K+\frac{1}{2}} f(t_l) \]

(3.5)

with \( t_l = t_{2l-1}/(t_{2K-3})^{(K-L+1)/2} \), with the distinguished cosmological time \( t_{2K-3} \propto \Lambda^4 \).

Note also, that the tau-functions of \( (p, q) \) and \( (q, p) \) theories do not coincide, but are related by the Legendre
transform [17], discussed in detail for generic prepotentials in [1].
3.2 Pure gravity: \((p,q)=(2,3)\)

In this case one has only two nontrivial parameters \(t_1\) and \(t_3\), and the partition function can be calculated explicitly. The times \((3.6)\) are here expressed by

\[
t_5 = \frac{2}{5}, \quad t_3 = \frac{2}{3} Y_1 - X_0, \quad t_1 = \frac{3}{4} X_0^2 - X_0 Y_1
\]

in terms of the coefficients of the polynomials (the other coefficients vanish for \(t_2 = t_4 = 0\))

\[
X = \lambda^2 + X_0, \quad Y = \lambda^3 + Y_1 \lambda
\]

and, though nonlinear, these quadratic equations can be easily solved for the latter

\[
X_0 = \frac{1}{3} \sqrt{9t_3^2 - 12t_1 - t_3}, \quad Y_1 = \frac{1}{2} \sqrt{9t_3^2 - 12t_4}
\]

The second half of residues \((3.14)\) gives

\[
\frac{\partial F}{\partial t_1} = -\frac{1}{8} Y_1 X_0^3 - \frac{1}{4} Y_1 X_0^2,
\]

\[
\frac{\partial F}{\partial t_3} = -\frac{1}{8} Y_1 X_0^3 + \frac{3}{64} X_0^4
\]

This results in the following explicit formula for the quasiclassical tau-function

\[
F = \frac{1}{3240} (9t_3^2 - 12t_1)^{5/2} + \frac{1}{4} t_3^3 t_1 - \frac{1}{4} t_3 t_1^2 - \frac{3}{40} t_3^5
\]

At \(t_3 \to \infty\) (expansion at \(t_1 \to 0\)) formula \((3.10)\) gives

\[
F \bigg|_{t_3 \to \infty} = -\frac{t_3^3}{18t_3} \left(1 + O \left(\frac{t_1}{t_3^2}\right)\right)
\]

which is the partition function of the Kontsevich model \([18, 19]\) (also identified with the \((2,1)\)-point or topological gravity). At \(t_1 \to \infty\) tau function \((3.10)\) scales as \(F \propto t_1^{5/2}\) or partition function of pure two-dimensional gravity \([14]\): expansion at \(t_1 \to \infty\) gives

\[
F = (-3t_1)^{5/2} \left(\frac{4}{405} - \frac{1}{54} t_1^2 + \frac{1}{96} t_1^4 + O \left(\frac{t_1^6}{t_3^2}\right)\right) + \ldots
\]

modulo analytic terms.

3.3 The Yang-Lee model: \((p,q)=(2,5)\)

The calculation of times according to \((1.14)\) gives

\[
t_1 = \frac{5}{8} X_0^3 + \frac{3}{4} Y_3 X_0^2 - Y_1 X_0
\]

\[
t_3 = \frac{5}{4} X_0^2 - Y_3 X_0 + \frac{2}{3} Y_1
\]

\[
t_5 = \frac{2}{5} Y_3 - X_0
\]

\[
t_7 = \frac{2}{7}
\]
for the polynomials
\[ X = \lambda^2 + X_0 \]
\[ Y = \lambda^5 + Y_0 \lambda^3 + Y_1 \lambda \] (3.14)

We see, that (3.13) can be easily solved for
\[ Y_1 = \frac{3}{2} \left( t_3 + \frac{5}{4} X_0^2 + \frac{5}{2} t_5 X_0 \right) \]
\[ Y_3 = \frac{5}{2} (X_0 + t_5) \] (3.15)

and the string equation for \( X_0 \) is now
\[ t_1 = -\frac{5}{8} X_0^3 - \frac{3}{2} t_3 X_0 \] (3.16)

The one-point functions (1.14) are given by
\[ \frac{\partial F}{\partial t_1} = -\frac{5}{64} X_0^4 - \frac{1}{4} Y_1 X_0^2 + \frac{15}{64} Y_0^4 - \frac{3}{8} Y_3 X_0^2 \]
\[ \frac{\partial F}{\partial t_3} = -\frac{1}{8} Y_1 X_0^3 - \frac{3}{128} Y_3 X_0^5 + \frac{9}{64} Y_0 Y_0^4 - \frac{3}{16} t_3 X_0^3 \] (3.17)

where, in the r.h.s.'s, \( Y_j \) obtained from solving (3.13) are substituted. From (1.15), or, differentiating (3.17) and using (3.16), one can easily obtain
\[ \frac{\partial^2 F}{\partial t_1^2} = \frac{X_0}{2} \]
\[ \frac{\partial^2 F}{\partial t_1 \partial t_3} = \frac{3}{8} X_0^3 = 3 \left( \frac{\partial^2 F}{\partial t_1^2} \right)^2 \]
\[ \frac{\partial^2 F}{\partial t_3^3} = \frac{3}{8} X_0^3 = 3 \left( \frac{\partial^2 F}{\partial t_1^2} \right)^3 \] (3.18)

However, (3.16) cannot be effectively solved explicitly, though it can be solved expanding in \( t_3 \). In the "naive" scaling regime (3.4) for the partition function of (2, 5) model one gets
\[ \mathcal{F} = t_1^{7/3} f \left( \frac{t_3}{t_1^{2/3}} \right) \]
\[ \frac{\partial \mathcal{F}}{\partial t_1} = \frac{7}{3} t_1^{4/3} f - \frac{2}{3} t_3 t_1^{2/3} f' \] (3.19)
\[ \frac{\partial^2 \mathcal{F}}{\partial t_1^2} = \frac{28}{9} t_1^{1/3} f - \frac{2}{3} t_3 t_1^{2/3} f' + \frac{4 t_3^2}{9} t_1 f'' \]

Substituting the last expression (using (3.18)) into (3.19) one gets for the first coefficients of \( f \)
\[ f(\tau) = f_0 + f_1 \tau + \frac{1}{2} f_2 \tau^2 + \frac{1}{6} f_3 \tau^3 + \frac{1}{24} f_4 \tau^4 + \ldots \]
\[ f_0 = -\frac{9}{140} 5^{2/3}, \quad f_1 = \frac{9}{50} 5^{1/3}, \quad f_3 = \frac{9}{25} 5^{2/3}, \quad f_4 = \frac{18}{25} 5^{1/3}, \ldots \] (3.20)

The second equality of (3.18) can be interpreted as dispersionless Hirota equation with an additional KdV \((p = 2)\) reduction condition.
while \( f_2 \) (the coefficient at analytic term \( t_1^2 t_3^2 \) of expansion of \( F \), disappearing from the higher derivatives) remains undetermined. The correspondent "scale-invariant" ratios are, for example:

\[
\frac{f_1 f_3}{f_0 f_4} = \frac{7}{5}, \quad \frac{f_1^3}{f_0^3 f_3} = \frac{98}{125}, \quad \frac{f_4}{f_0 f_3} = 686, \quad \frac{f_0 f_3^2}{f_4^2} = 14, \ldots
\]

(3.21)

Note, that these scale-invariant ratios are given by simple rational numbers, compare to the coefficients (3.20) themselves, which essentially depend upon normalizations of times. In Appendix B it is shown, that expansion (3.20) is in fact related to the generating function in the gravitationally dressed \((p,q) = (3,4)\) Ising model.

However, for the non-unitary case one may expect another scaling regime, where the role of times is "exchanged". For the particular case of \((2,5)\) model this implies that

\[
F = t_{3/2} f \left( \frac{t_1}{t_{3/2}} \right) \equiv t_{3/2}^{7/2} f(t)
\]

(3.22)

and string equation (3.16) turns into

\[
t + 5(f''')^3 + 3f'' = 0
\]

(3.23)

Solution of (3.23) immediately gives for the higher coefficients of expansion \( f_n \equiv f^{(n)}|_{t=0} \) of the normalized function \( f \) the explicit expressions

\[
\begin{align*}
f_3 &= -\frac{1}{3(1 + 5f_2^2)} \\
f_4 &= -\frac{10f_2}{9(1 + 5f_2^2)^3} \\
f_5 &= -\frac{27(1 + 5f_2^2)}{27(1 + 5f_2^2)^5}
\end{align*}
\]

(3.24)

through the 2-point function, which satisfies the cubic equation

\[
3f_2 + 5f_2^3 = 0
\]

(3.25)

while the "normalization" \( f_0 \) and the 1-point function \( f_1 \) remain undetermined from string equation (3.23) directly. In order to find them, one should use the second equation of (3.17) together with the second formula of (3.22), or

\[
7f - 3tf'' + 9(f''')^5 + 3(f''')^3 = 0
\]

(3.26)

giving rise to

\[
\begin{align*}
f_0 &= -\frac{9}{7} f_2^2 - \frac{3}{7} f_3^2 \\
f_1 &= -\frac{9}{4} f_2^3 f_3 - \frac{45}{4} f_2^2 f_3
\end{align*}
\]

(3.27)

In the "phase" with non-vanishing \( f_2 \), this results in

\[
\begin{align*}
f_0 &= -\frac{36}{175} \sqrt{-\frac{3}{5}}, \quad f_1 = -\frac{9}{20}, \quad f_2 = \sqrt{-\frac{3}{5}}, \quad f_3 = \frac{1}{6}, \quad f_4 = \frac{1}{4} \sqrt{-\frac{5}{3}}, \quad f_5 = -\frac{5}{27}, \ldots
\end{align*}
\]

(3.28)

\footnote{Such scaling implies also a particular behavior \( F_A(t_1) = A^{-7/2} (t_1 A^{3/2}) \) for the Laplace transformed partition function \( F(t_1, t_3) = \int_0^\infty A dA e^{-t_1 A} F_A(t_1) \) with "fixed area" (cf. with \( \text{[20]} \)).}
Finally, one can easily compute the "invariant ratios", which do not depend upon normalization of times and partition function, for example
\[
\frac{f_0 f_2}{f_1^2} = \frac{64}{105}, \quad \frac{f_4 f_2}{f_3^2} = -3, \quad \frac{f_4 f_3}{f_2 f_5} = -\frac{1}{8}, \ldots
\] (3.29)

Analogous formulas for the (2, 7) minimal string theory and (3, 4) or the Ising model interacting with gravity are presented in Appendix B.

### 3.4 Gurevich-Pitaevsky problem and matrix model with logarithmic potential

In this section we turn to discussion of the Gurevich-Pitaevsky (GP) problem [21], appeared originally when studying nonlinear waves, and related later to string theory and non-perturbative gauge theories (see e.g. [22]) and to the problems of Laplacian growth [25], see recent paper [24]. Solution to the GP problem connects different phases dispersionless solutions of quasiclassical hierarchies, corresponding to rational curves Σ of spherical topology by sewing them via the solutions given by the curves of higher genera. In particular, this resolves the singularities of dispersionless solutions. The parameters of solutions appear to be "modulated" in accordance with equations of quasiclassical hierarchy.

Below we consider, first, the original GP problem in KdV hierarchy, whose geometry coincides with that of the (2, 5) model of the previous section, and then turn to particular example of two-matrix model [12] with non-polynomial potential. The basic geometric condition for the GP problem is that the finite-gap solutions to nonlinear wave equations for the curves of different genera, in order to be correctly sewed when overcoming the singularity, must have the same values of the (real parts of the) periods of generating differential $\mathcal{J} dS$ over the nontrivial cycles.

In one of two original GP setup’s one considers the properties of the cubic string equation (3.10) of the (2, 5) model. The corresponding complex manifold arises when one notices, that the polynomials (3.14) satisfy the equation of (degenerate) (2,5)-curve
\[
Y^2 = X^5 + (2Y_3 - 5X_0)X^4 + (Y_2 - 8Y_3 X_0 + 2Y_1 + 10X_0^2)X^3 + \\
+ (2Y_3 Y_1 - 3Y_2^2 X_0 - 6Y_1 X_0 - 10X_0^3 + 12Y_3X_0^2)X^2 + \\
+ (Y_2^2 + 5X_4 - 8Y_3 X_0^3 + 3Y_2^2 X_0^2 + 6Y_1^2 X_0^2 - 4Y_3 Y_1 X_0)X + \\
+ 2Y_3 Y_1 X_0^2 + 2Y_5 X_0^4 - X_0^5 - Y_3 X_0^3 - 2Y_1 X_0^3 - Y_2^2 X_0 \equiv \\
\equiv (X - X_0)(X - x_+)^2(X - x_-)^2
\] (3.30)

with
\[
x_{\pm} = -\frac{X_0}{4} \pm \frac{1}{2} \sqrt{-\frac{5}{4} X_0^2 - 6 \ell_3}
\] (3.31)

and the sum of the roots, proportional to $t_5$ from (3.10), to be taken vanishing. If all singularities are resolved, the curve (3.30) would have genus $g = 2$, and in the case of partial resolution of the singularities it can be presented as elliptic curve with an extra "double point".

The critical parameters of the solution are determined by vanishing of the periods of generating differential $dS = YdX$. In rational case they can be easily computed, say, using (3.14), so that
\[
\int_{\lambda_1}^{\lambda_0} YdX = \left. \frac{2}{7} \lambda^7 + \frac{2}{5} Y_3 \lambda^5 + \frac{2}{3} Y_1 \lambda^3 \right|_{\lambda_i} = \left. \frac{2}{7} \lambda^7 + X_0 \lambda^5 + \left( t_3 + \frac{5}{4} X_0^2 \right) \lambda^3 \right|_{\lambda_i}
\] (3.32)

\footnote{Ratios (3.29) coincide with recently computed, using the worldsheet Liouville theory, by A. Belavin and A. Zamolodchikov, see [20].}
where the critical points $\{\lambda_i\}$ are found solving equation $Y = 0$. Computing (3.32) for these limits, one finds the critical values

$$t_1 = \frac{2}{\sqrt{3}}, \quad \frac{3\sqrt{6}}{\sqrt{3}}, \quad \frac{-\sqrt{2}}{\sqrt{3}} \tag{3.33}$$

The ratio of last two values equals to $\frac{\sqrt{2}}{\sqrt{10}/27}$, found numerically by Gurevich and Pitaevsky [21], while the first one is just the critical value of the equation (3.16). The other two values correspond to $x_+ = x_-$ and another, already nontrivial, case of vanishing of the integral $\int_{x_+}^{x_-} YdX = 0$. From the point of view of scaling Yang-Lee model these critical values correspond to the particular points in parameter space, where the stringy one-loop correction to the two-point correlator $X_0$ vanishes.

Let us now perform a similar procedure for the two-matrix model. Up to now the whole geometric construction for the two matrix model in [11] was rather implicit, and below we discuss an explicit (and even non-polynomial, cf. with discussion in [25]) example of the potential (1.3) with the logarithmic holomorphic part

$$W(z) = -\nu \left( \log \left( 1 - \frac{z}{a} \right) + \frac{z}{a} \right) \tag{3.34}$$

taking $a$ to be real, see profile at fig. 1. Two its extrema satisfy the equation

$$z\tilde{z} - a\tilde{z} + \frac{\nu}{a} z = 0 \tag{3.35}$$

together with the complex conjugated, and are located at $z = \tilde{z} = 0$, which is the minimum of potential and $z = \tilde{z} = a - \frac{\nu}{a}$, which is the saddle point. The corresponding equation of the complex curve (1.9) for the two-matrix model [26] reads

$$z^2\tilde{z}^2 - \left(a - \frac{\nu}{a}\right) (z^2\tilde{z} + \tilde{z}^2) + cz\tilde{z} - \nu \left( z^2 + \tilde{z}^2 \right) + g(z + \tilde{z}) + h \equiv Q(z)\tilde{z}^2 - P(z)\tilde{z} - K(z) = 0 \tag{3.36}$$

where last two terms correspond to the points strictly inside the corresponding Newton polygon, see fig. 2. The curve (3.35) is endowed with a standard two-matrix model generating differential

$$\tilde{z}dz = \left( \frac{P}{2Q} + \frac{\sqrt{R}}{2Q} \right) dz = \left( \frac{P}{2Q} + \frac{1}{2Q} \sqrt{P^2 + 4QK} \right) dz \tag{3.37}$$
Rational degeneration of this curve (see general discussion of rational degenerations in Appendix A of [1]) can be described by conformal map

\[
\begin{align*}
  z &= rw + \frac{u}{w-s} + v \\
  \tilde{z} &= \frac{r}{w} + \frac{uw}{1-sw} + v
\end{align*}
\]  

(3.38)

where we imply all coefficients to be real; in contrast to the conformal maps considered in Appendix A of [1], the maps (3.38) possess singularities not only at \( w = \infty \). In the case of rational degeneration of the curve (3.36) one has

\[
\sqrt{R} = \Delta(z - \xi_0)\sqrt{(z - \xi_+)(z - \xi_-)}
\]

(3.39)

where

\[
\Delta = a + \frac{\nu}{a}
\]  

(3.40)

and it corresponds to the following relations between the coefficients in (3.39) and (3.38)

\[
\xi_{\pm} = v + rs \pm 2\sqrt{ru}
\]

\[
\xi_0 = v + \frac{r-u}{s}
\]  

(3.41)

and some more complicated for the coefficients of (3.36). Some useful technical formulas for this case are collected in Appendix C.

It is easy to see taking the residues of generating differential (3.37), that in addition to our common parameter

\[
t_0 = \frac{1}{2\pi i} \oint_{|w|=1} \tilde{z}dz = \frac{1}{2\pi i} \text{res}_{q_1} \tilde{z}dz + \frac{1}{2\pi i} \text{res}_{w_{\infty+}} \tilde{z}dz = \frac{1}{2\pi i} \text{res}_{q_1} \tilde{z}dz - \frac{1}{2\pi i} \text{res}_{q_2} \tilde{z}dz =
\]

\[
r^2 - \frac{u^2}{(1-s^2)^2} = \frac{a^2 - \nu}{a^2 + \nu} \left( a^2 + \frac{\nu^2}{a^2} - c \right) - \frac{2ga}{a^2 + \nu}
\]  

(3.42)

it is convenient to introduce

\[
\delta \equiv \frac{1}{2\pi i} \text{res}_{q_1} \tilde{z}dz + \frac{1}{2\pi i} \text{res}_{q_2} \tilde{z}dz + 2\nu = a^2 + \frac{\nu^2}{a^2} - c
\]  

(3.43)

Using (3.43) and (3.42) one can rewrite equation of the curve (3.36) as

\[
\begin{align*}
  z^2 &\tilde{z}^2 - \left( a - \frac{\nu}{a} \right) (z^2 + \tilde{z}^2) + \left( a^2 + \frac{\nu^2}{a^2} - \delta \right) z\tilde{z} - \nu (z^2 + \tilde{z}^2) + \frac{t_0}{2} \left( a + \frac{\nu}{a} \right) + \frac{\delta}{2} \left( a - \frac{\nu}{a} \right) (z + \tilde{z}) + h = 0
\end{align*}
\]  

(3.44)
The last coefficient \( h \) is related to the only nontrivial period of (3.37)

\[
S = \frac{1}{2\pi i} \oint_A \zeta \, dz
\]  

(3.45)

In what follows, we fix \( \delta = t_0 \), corresponding to \( \text{res}_{q=2} \zeta \, dz = \text{const} \), then eq. (3.44) acquires the final form

\[
z^2 \zeta^2 - \left( a - \frac{\nu}{a} \right) (z^2 \zeta + \zeta^2) + \left( a^2 + \frac{\nu^2}{a^2} - t_0 \right) z \zeta - \nu (z^2 + \zeta^2) + t_0 a(z + \zeta) + h = 0
\]  

(3.46)

These formulas give rise to the following system of equations to the parameters of conformal map (3.38)

\[
\begin{align*}
r + \frac{u}{1 - s^2} &= \Delta s \\
\left(r + \frac{u}{1 - s^2}\right) \left(r - \frac{u}{1 - s^2}\right) &= t_0 \\
r^2 - \frac{ru}{s^2} &= t_0 - \nu
\end{align*}
\]  

(3.47)

Solving (3.47) one gets

\[
\begin{align*}
r &= \frac{1}{2} \left( \Delta s + \frac{t_0}{\Delta s} \right) \\
u &= \frac{1 - s^2}{2} \left( \Delta s - \frac{t_0}{\Delta s} \right)
\end{align*}
\]  

(3.48)

and \( s^2 = \chi \) satisfies the following cubic equation

\[
\chi + \frac{t_0^2}{2 \Delta^2} \frac{1}{\chi^2} = \frac{1}{2} - \frac{2\nu}{\Delta^2} + \frac{t_0}{\Delta^2}
\]  

(3.49)

Eq. (3.49) can be solved for \( t_0 \)

\[
t_0 = \Delta^2 \chi^2 \pm \Delta^2 \chi \sqrt{\left( \chi - 1 - \frac{2\sqrt{\nu}}{\Delta} \right) \left( \chi - 1 + \frac{2\sqrt{\nu}}{\Delta} \right)}
\]  

(3.50)

It is easy to see, that two solutions of (3.50) are positive at

\[
t_0^{(+)} : \quad 0 < \chi < 1 - \frac{2\sqrt{\nu}}{\Delta}, \quad \chi > 1 + \frac{2\sqrt{\nu}}{\Delta}
\]

\[
t_0^{(-)} : \quad \chi < 0, \quad \chi < 1 - \frac{2\sqrt{\nu}}{\Delta}, \quad \chi > 1 + \frac{2\sqrt{\nu}}{\Delta}
\]  

(3.51)

Thus, the whole picture can be presented as a growing droplet, which is a two-dimensional horizontal surface of an "eigenvalue liquid" of the area \( t_0 \) filled into the "cup" potential, see fig. 1. It is easy to see, that two phases of real physical interest are:

- Growing droplet before the "impurance": this corresponds to \( 0 < \chi < 1 - \frac{2\sqrt{\nu}}{\Delta} \) and \( t_0 = t_0^{(+)} \). The liquid of eigenvalues starts to fill the minimum of two-dimensional potential at \( z = z_{\text{min}} = 0 \) and with growth of \( t_0 \) it level raises till it reaches \( V(z_{\text{saddle}}) = a^2 - \frac{\nu^2}{a^2} + 2\nu \log \frac{\nu}{a^2} \), see fig. 3. In this phase \( \xi_0 > \xi_+ \) i.e. (for the real choice of all necessary parameters) the cut between \( \xi_- \) and \( \xi_+ \) (3.39) is located to the left from the double point \( \xi_0 \).
Figure 3: Boundary curves for the first phase. Starting like at the left picture it develops a cusp, when approaching \( z_{\text{saddle}} = a - \frac{\nu}{a} \) (at the pictures \( a = 5 \) and \( \nu = \frac{1}{2} \)).

Figure 4: Boundary curves for the second phase. Starting from "impurity" at \( z = a \) like at the left picture, it then grows to a smooth ellipse-like configuration like at the right picture (again, the pictures are drawn for \( a = 5 \) and \( \nu = \frac{1}{2} \)).
Growing droplet after the "impurance": this phase corresponds to $\mathcal{X} > 1 + \frac{2}{\nu} \Delta$ and $t_0 = t_0^{(-)}$. The liquid of eigenvalues fills the impurance at $z = a$ and continues to grow smoothly with growth of $t_0$, see fig. 4. Now $\xi_0 < \xi_-$ i.e. the cut in (3.39) is located to the right from the double point.

Between these two phases it has to pass through the cusp. The cusp point can be "overcome" by the solution in spirit of [21], passing from rational curve out to nonsingular curve (3.36) of genus $g = 1$. The degenerate genus $g = 0$ solutions can be glued through the $g = 1$ solution only when they have equal values of

$$\text{Re} \oint \tilde{z} dz = \begin{cases} 
\text{Re} \int_{w_0^+}^{w_0^-} \tilde{z} dz, & \xi_0 > \xi_+ \\
\text{Re} \int_{w_0^-}^{w_0^+} \tilde{z} dz, & \xi_0 < \xi_- 
\end{cases}$$

(3.52)

where $w_0^\pm = w^\pm|_{z = \xi_0}$ are two values of the pullback of the point $z = \xi_0$ to the $w$-plane. One can see at fig. 5 that these values are indeed glued at different phases, in particular when $\text{Re} \oint \tilde{z} dz = 0$.

4 Complex geometry of the AdS/CFT correspondence

Matrix models discussed in [1] and above present the simplest example of the gauge/string duality [2]. Being the "zero-dimensional" example of the gauge theory, they propose a dual description of string theory below two dimensions, or the "target-space" theory of two-dimensional gravity on string world-sheets.

In contrast to matrix model the AdS/CFT conjecture [29] deals with the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) with the $SU(N)$ gauge group. Again, the main contribution in the $1/N$-expansion (1.7), where closed string loops are suppressed, comes from the planar diagrams when $N \to \infty$ at fixed 't Hooft coupling $\lambda = g_s^2 N = g_s N$ (coincident with $t_0$ for the matrix models, we intensively used in [1] and previous sections), while string coupling $g_s = g_s^2$ is equivalent to the quasiclassical parameter $\hbar$. At $\lambda \gg 1$ the $\mathcal{N} = 4$ SYM theory is believed to be dual to string theory in $AdS_5 \times S^5$ with the equal, up to a sign (positive for the sphere $S^5$ and negative for the Lobachevsky space $AdS_5$), radii of curvature $\frac{R}{\sqrt{\alpha'}} = \lambda^{1/4}$. Therefore any test of the AdS/CFT conjecture implies comparing analytic series at $\lambda = 0$ (SYM perturbation theory, which is just
a direct analog of the counting of diagrams in matrix models as in \cite{14} with analytic in \( \alpha' \propto \frac{1}{\sqrt{\lambda}} \) worldsheet expansion.

A possible partial way-out from this discrepancy in parameters of expansion can be to consider the classical string solutions with large values of integrals of motion (usually referred as ”spins” \( J \)) on \( AdS_5 \times S^5 \) side \cite{30}, whose energies should correspond to anomalous dimensions of ”long” operators on gauge side. In this case the classical string energy of the form \( \Delta = \sqrt{\lambda} \mathcal{E} \left( \frac{1}{\sqrt{\lambda}} \right) \) may have an expansion of the form \( \Delta = J + \sum_{l=1}^{\infty} \mathcal{E}_l \left( \frac{1}{\sqrt{\lambda}} \right)^l \) over the integer powers of \( \lambda \), which can be treated as series at \( \lambda = 0 \) even at \( \lambda \gg 1 \) provided large \( \lambda \) is suppressed by large value of the integrals of motion\footnote{On integrability on the string side of duality see, e.g., \cite{32}.} \( J \). If it happens (this is not, of course, guaranteed) the classical string energy can be tested by direct comparison with perturbative series for the gauge theory.

Such expansion for the classical string energy can be, of course, corrected by quantum corrections on string side. Up no now there is no consistent way to quantize string theory in nontrivial background with the Ramond-Ramond flux switched on. Therefore, such quasiclassical test of the AdS/CFT correspondence should be treated rather carefully. We shall see below, that indeed the quasiclassical picture allows to test the conjecture in the first non-vanishing orders, but more detailed analysis requires more knowledge of quantum theory on string side.

### 4.1 Renormalization and the Bethe ansatz

The four-dimensional \( N = 4 \) SYM is conformal theory, i.e. \( \beta(g_{YM}) = 0 \), but the anomalous dimensions \( \gamma \) of the composite operators, e.g. \( \text{Tr} (\Phi_{i_1} \ldots \Phi_{i_L}) (x) \) (where \( x \) is some point of the four-dimensional space-time) are still renormalized nontrivially. Below we consider the particular scalar operators from this set, though the proposed approach \cite{33} can be applied in much more general situation. On string side such operators correspond to the string motion in the compact \( S^5 \)-part of ten-dimensional target-space, due to standard Kaluza-Klein argument. To simplify the situation maximally, choose two complex \( \Phi_1 = \Phi_1 + i\Phi_2 \) and \( \Phi_2 = \Phi_3 + i\Phi_4 \) fields among six real \( \Phi_i \) and consider the holomorphic operators

\[
\text{Tr} (\Phi_1 \Phi_1 \Phi_2 \Phi_2 \Phi_1 \Phi_1 \Phi_2 \Phi_2 \ldots) (x)
\]

which can be conveniently labeled by arrows as \( \langle \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \uparrow \downarrow \ldots \rangle \in (\mathbb{C}^2 \otimes L) \).

These operators do not have singularities at coinciding arguments of all fields, since the kinetic terms for the holomorphic fields do always have the \( \Phi^\dagger \Phi \)-structure. The holomorphic subsector is ”closed” under renormalization and the anomalous dimensions are eigenvalues of the \( 2^L \times 2^L \) mixing matrix

\[
H = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (1 - \sigma_l \cdot \sigma_{l+1}) + O(\lambda^2)
\]

which is, up to addition of a constant, the permutation operator in \( (\mathbb{C}^2 \otimes (\mathbb{C}^2)) \), whose appearance is determined by structure of the \( \Phi^\dagger \)-vertex in SYM Lagrangian, or the Hamiltonian for Heisenberg magnetic \cite{35}.

It is well-known, that the matrix \( (4.2) \) can be diagonalized using the Bethe ansatz \cite{36}, (see e.g., \cite{37} for present status of this technique and comprehensive list of references). Integrable structure of Heisenberg spin chain is encoded in the transfer matrix

\[
\hat{T}(u) = \text{Tr} \left[ \left( u + \frac{i}{2} \sigma_L \otimes \sigma \right) \ldots \left( u + \frac{i}{2} \sigma_1 \otimes \sigma \right) \right]
\]
which can be presented as an operator-valued matrix in the auxiliary two-dimensional space

\[
\hat{T}(u) = \text{Tr} \left( \hat{A}(u) \hat{B}(u) \right) = \hat{A}(u) + \hat{D}(u)
\]  

(4.4)

with four "generators of Yangian" \( \hat{A}(u), \hat{B}(u), \hat{C}(u) \) and \( \hat{D}(u) \), satisfying the quadratic RTT-algebra. Expansion of the operator \( \hat{T}(u) \) in spectral parameter at \( u = \frac{1}{2} \) generates the local charges

\[
\hat{T}(u) = \left( u + \frac{i}{2} \right)^L \hat{U} \exp \left[ i \sum_{n=1}^{\infty} \frac{1}{n} \left( u - \frac{i}{2} \right)^n \hat{Q}_n \right]
\]  

(4.5)

where \( \hat{U} = e^{i\hat{P}} \) is the shift operator (by one site along the chain) and the Hamiltonian \( \hat{Q}_1 \propto H \) is one of the local charges. Contrarily, expansion of (4.3) at \( u = \infty \) produces the non-local Yangian charges, whose role in the context of \( \mathcal{N} = 4 \) SYM was discussed in [38].

Bethe ansatz diagonalizes the whole spectral-parameter dependent operator (4.3), (4.5)

\[
\hat{T}(u)|_{u_1 \ldots u_J} = T(u)|_{u_1 \ldots u_J},
\]  

(4.6)

explicitly constructing the eigenvectors \( |u_1 \ldots u_J\rangle \) in terms of the operators

\[
|u_1 \ldots u_J\rangle = \hat{B}(u_1) \ldots \hat{B}(u_J) |\uparrow\uparrow\uparrow\uparrow\ldots\rangle
\]  

(4.7)

acting on the ferromagnetic vacuum \( |\uparrow\uparrow\uparrow\ldots\rangle \). Any Bethe vector \( |\uparrow\uparrow\uparrow\ldots\rangle \) depends on \( J \) rapidities, and the corresponding eigenvalues of the operator (4.6) are

\[
T(u) = \left( u + \frac{i}{2} \right)^L \prod_{k=1}^{J} \frac{u - u_k - i}{u - u_k} + \left( u - \frac{i}{2} \right)^L \prod_{k=1}^{J} \frac{u - u_k + i}{u - u_k}
\]  

(4.8)

Now since any eigenvalue \( T(u) \) of the operator (4.3) must be a polynomial in \( u \), the cancelation of poles at \( u = u_j \) in the expression (4.8) gives \( J \) conditions:

\[
\left( \frac{u_j + i/2}{u_j - i/2} \right)^L = \prod_{k=1(k \neq j)}^{J} \frac{u_j - u_k + i}{u_j - u_k - i},
\]  

(4.9)

which are the Bethe equations.

Hence, the eigenvectors of Hamiltonian (4.3) are parameterized by Bethe roots \( \{u_1, \ldots, u_J\} \), \( J \leq \frac{1}{2} L \) due to an obvious \( \mathbb{Z}_2 \)-symmetry, in particular permuting two ferromagnetic vacua \( |\uparrow\uparrow\uparrow\ldots\rangle \leftrightarrow |\downarrow\downarrow\downarrow\ldots\rangle \), satisfying (4.9). After taking the logarithm, eqs. (4.9) acquire the form

\[
L \log \left( \frac{u_j + i/2}{u_j - i/2} \right) = 2\pi i n_j + \sum_{k \neq j} \log \left( \frac{u_j - u_k + i}{u_j - u_k - i} \right)
\]  

(4.10)

with appeared mode numbers \( n_j \in \mathbb{Z} \) being some integers, never vanishing simultaneously for a nontrivial solution. For the diagonalization of the mixing matrix of the operators (4.1) in SYM theory, equations (4.10) are supplied by the "trace condition"

\[
e^{iP} = \prod_{j=1}^{J} \frac{u_j + i/2}{u_j - i/2} = 1,
\]  

(4.11)

i.e. integrality of the total momentum of magnons: \( \frac{P}{2\pi} \in \mathbb{Z} \).
The energy of the eigenstate of the Hamiltonian (4.7) is given by sum over magnons, i.e. \( \propto \sum_{j=1}^{J} \frac{1}{u^2} \),
for the quasimomentum \( \exp ip(u) = \frac{u+\i/2}{u-\i/2} \), and it means that the anomalous dimensions for the mixing matrix (4.2) in the first order of the SYM perturbation theory can be written as

\[
\gamma = \frac{\lambda}{8\pi^2} \sum_{j=1}^{J} \frac{1}{u_j^2 + 1/4} + O(\lambda^2)
\]  \hspace{1cm} (4.12)

and these \( \gamma \)'s one needs for testing the quasiclassical AdS/CFT conjecture.

For comparison with dual (classical) string theory we are interested in the long operators with \( L \rightarrow \infty \), for which it is known empirically that the Bethe roots are typically of the order of \( u_j \sim L \). Rescaling \( u_j = Lx_j \), and omitting higher in \( \frac{1}{L} \) terms, one gets from (4.10) the equations of more simple form

\[
\frac{1}{x_j} = 2\pi n_j + \frac{2}{L} \sum_{k \neq j}^{J} \frac{1}{x_j - x_k}
\]  \hspace{1cm} (4.13)

Like in the matrix model case (see detailed discussion in [1]), if the second "interaction term" in the r.h.s. is absent, the solution to (4.13) is given by \( x_j = \frac{1}{2\pi n_j} \) for each \( n_j \) (filling the extremal!), and when one switches on the interaction, the roots corresponding to \( n_j \) (if there are several for each value) will "concentrate" around \( \frac{1}{2\pi n_j} \) forming the so called "Bethe strings", shown at fig. 6. In other words, if several different \( x_{j_1}, \ldots, x_{j_m} \) have the same mode number \( n_j = n_{j_1} = \ldots = n_{j_m} \), these \( m \) Bethe roots combine into a Bethe string with the mode number \( n_{jm} \neq 0 \).

Introduce at \( L \rightarrow \infty \), as in sect. 3 of [1], the density

\[
\rho(x) = \frac{1}{L} \sum_{j=1}^{J} \delta(x - x_j), \quad \int_{C} dx \rho(x) = \frac{J}{L}
\]  \hspace{1cm} (4.14)
or resolvent
\[ G(x) = \frac{1}{L} \sum_{j=1}^{J} \frac{1}{x - x_j} = \int_{C} \frac{d\xi \rho(\xi)}{x - \xi}, \quad \frac{1}{2\pi i} \oint_{C} dxG(x) = \frac{J}{L} \] (4.15)

and consider the finite number of different Bethe strings \( n_l \neq n_{l'}, \) with \( l, l' = 1, \ldots, K \) despite the infinitely many roots \( J. \) Then in the scaling limit \( (J \to \infty \text{ with fixed finite } K) \) the total eigenvalue support is \( \mathbb{C} = C_1 \cup \ldots \cup C_K, \) where on each component one gets from \( (4.13) \)

\[
2 \int_{C} \frac{d\xi \rho(\xi)}{x - \xi} = G(x_+) + G(x_-) = \frac{1}{x} - 2\pi n_l, \quad x \in C_l 
\] (4.16)

where \( G(x_{\pm}) \) are values of the resolvent on two different sides of the cut along the Bethe string.

The integrality of total momentum condition, using the Bethe equations \( (4.13), \) acquires the form

\[
\frac{1}{L} \sum_{j=1}^{J} \frac{1}{x_j} = 2\pi \sum_{l=1}^{K} n_l \int_{C_l} \rho(x) dx = 2\pi m, \quad m \in \mathbb{Z} 
\] (4.17)

or

\[
\frac{1}{2\pi i} \oint_{C} \frac{G(x) dx}{x} = 2\pi m, \quad n_l, m \in \mathbb{Z} 
\] (4.18)

Different \( n_l \neq n_{l'} \) on different parts of support \( C_l \cap C_{l'} = \emptyset \) mean that, in contrast to the matrix model case, discussed in detail in sect. 3 of \[ ] \( G(x) = \int^{x} dG \) is not already a single-valued function, but an Abelian integral on some hyperelliptic curve \( \Sigma, \) introduced by hands, according to the number of nontrivial Bethe strings

\[
y^2 = R_{2K}(x) = x^{2K} + r_1 x^{2K-1} + \ldots + r_{2K} = \prod_{j=1}^{2K} (x - x_j) 
\] (4.19)

It means, that resolvent \( G(x) \) in contrast to the case of one-matrix model does not satisfy here a sensible algebraic equation, but can be still determined from its geometric properties.

Equations \( (4.15), (4.16) \) and \( (4.18) \) can be solved after reformulating them as a set of properties of the meromorphic differential \( dG \) of the resolvent \( (4.15): \)

- The differential \( dG \) is the second-kind Abelian differential with the only second-order pole at the point \( P_0, (x(P_0) = 0 \) on unphysical sheet of the Riemann surface \( \Sigma, \) see fig. \[ \] \) in particular it means that \( \oint_{A_i} dG = 0 \) for all \( A \)-cycles, surrounding the Bethe strings. The absence of poles on physical sheets is determined by properties of the resolvent, but on unphysical sheets then any differential with vanishing \( A \)-periods must have singularities;

- In addition to vanishing \( A \)-periods, the differential \( dG \) has integral \( B \)-periods

\[
\oint_{B_i} dG = 2\pi(n_i - n_K) 
\] (4.20)

More exactly one can write \[ \]

\[
\int_{B_j} dG = 2\pi n_j, \quad j = 1, \ldots, K + 1 
\] (4.21)

where \( B_j' \) is the contour from \( \infty_+ \) on the upper sheet to \( \infty_- \) on the lower sheet, passing through the \( j \)-th cut, so that \( B_j = B_j' - B_{K'}, \) for \( j = 1, \ldots, K, \) see fig. \[ \]
Figure 7: The integration contour $B'$ in formula (4.21); the marked points on both sheets correspond to infinities $\infty_+$ and $\infty_-$ where $x = \infty$.

- $dG$ has the following behavior at infinity

$$dG = \frac{J}{L} \frac{dx}{x^2} + \ldots$$  \hspace{1cm} (4.22)

and the Abelian integral $G(x)$ itself is fixed by (4.18)

$$G(x) = 2\pi m + \int_0^x dG, \quad \text{or} \quad G(0) = 2\pi m$$  \hspace{1cm} (4.23)

The general solution [33] for the differential $dG$ on hyperelliptic curve (4.19), satisfying the above requirements (4.20), (4.21), (4.22) and (4.23) is read from a standard formula for the second-kind Abelian differential on hyperelliptic curve

$$dG = \frac{dx}{2x^2} \left(1 - \frac{\sqrt{r_2K}}{y}\right) + \frac{r_2K-1}{4\sqrt{r_2K}} \frac{dx}{xy} + \sum_{k=1}^{K-1} a_k \frac{x^{k-1}dx}{y}$$  \hspace{1cm} (4.24)

going together with the extra conditions, ensuring, in particular, the single-valuedness of the resolvent on "upper" physical sheet

$$\oint_{A_i} dG = 0, \quad i = 1, 2, \ldots, K - 1$$  \hspace{1cm} (4.25)

which is a system of linear equations, to be easily solved for the coefficients $\{a_k\}$. The rest of parameters is "eaten by" fractions of roots on particular pieces of support

$$S_j = \int_{C_j} \rho(x)dx = -\frac{1}{2\pi i} \oint_{A_j} xdG,$$

$$j = 1, \ldots, g = K - 1$$  \hspace{1cm} (4.26)

the total amount of Bethe roots (4.14), and the total momentum (4.18).

The energy or one-loop anomalous dimension for generic finite-gap solution [33] can be read from (4.12), (4.24)

$$\gamma = \frac{\lambda}{8\pi^2L} \int_{C} \frac{dx}{2\pi i x^2} G(x) = \frac{\lambda}{8\pi^2L} \left(\frac{r_2K-2}{4r_2K} - \frac{r_2K-1}{16r_2K} - \frac{a_1}{\sqrt{r_2K}}\right)$$  \hspace{1cm} (4.27)

The anomalous dimensions defined by (4.27) are functions of the coefficients of the embedding equation (4.19) and $a_1$ which again is expressed through these coefficients by means of (4.25). The moduli of the curve (4.19).
are themselves (implicitly) expressed through the mode numbers $n_j$ and root fractions $S_j$ via (12.20) or (12.21) and (12.23) (together with the total momentum (14.18) and the total number of Bethe roots (14.19)).

It is interesting to point out, that formula (12.24) is a particular case of the two-point resolvent for the planar matrix model $W_0(x, \xi)$, or so called Bergman kernel $\omega(P, P') = d\rho d'_\rho \log E(P, P')$ (see also formula A.13 from Appendix A), which on a (tensor square of) the hyperelliptic curve (4.19) reads

$$W_0(x, \xi) dx d\xi = \left(-\frac{dx d\xi}{2(x - \xi)^2} + \frac{y(x)}{2y(\xi)} \left(\frac{1}{(\xi - x)^2} + \frac{1}{2} \sum_{\alpha=1}^{2n} \left[\frac{1}{(\xi - x)(x - x_\alpha)} - \sum_{i=1}^{n-1} H_i(\xi) \int_{A_i} \frac{dx}{(x - x_\alpha)^2 y(x)}\right]\right)\right) dx d\xi =$$

$$= -\frac{dx d\xi}{2(x - \xi)^2} \left(1 - \frac{R(x)}{y(x)y(\xi)}\right) - \frac{dx d\xi}{2(x - \xi)} \frac{R'(x)}{y(x)y(\xi)} - \frac{1}{2} \frac{y(x) dx}{2} \sum_{\alpha=1}^{2n} \sum_{i=1}^{n-1} d\omega_i(\xi) \int_{A_i} \frac{dx}{(x - x_\alpha)^2 y(x)}$$

(4.28)

The detailed discussion of this issue can be found in [13].

### 4.2 Geometry of integrable classical strings

Formulas (12.24) - (12.26) of the previous paragraph show, that the general solution for anomalous dimension of long operators (4.27) is expressed through the integrals of motion on some classical configurations of the Heisenberg magnet\(^8\). In the dual string picture one has the classical trajectories of string, moving in (some subspace of) $AdS_5 \times S^5$ and the finite gap solutions to string sigma-model in the Lobachevsky-like spaces were first constructed in [30]. Being slightly modified, it can be easily applied to the case of compact $S^{2D-1}$ sigma-models. In this section, following [33, 34] we are going to show how these classical solutions can be compared with the quasiclassical solutions on the gauge side.

In particular subsector of only two holomorphic fields one gets the $S^3 \subset S^5$ sigma-model (in the $AdS_5$-sector the only nontrivial string co-ordinate on the solution is "time" $X_0 = \frac{\Delta}{\sqrt{\lambda}} \tau$) with the action

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[\left(\partial_\sigma X_i\right)^2 - \left(\partial_\tau X_i\right)^2\right],$$

(4.29)

($\sum X_i^2 = 1$). Provided by identifications (since $S^3$ is the group-manifold of the Lie group $SU(2)$)

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} \equiv \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in SU(2),$$

$$J = g^{-1} dg = \begin{pmatrix} \bar{Z}_1 dZ_1 + Z_2 d\bar{Z}_2 & \bar{Z}_1 dZ_2 - Z_2 d\bar{Z}_1 \\ Z_2 dZ_1 - Z_1 d\bar{Z}_2 & \bar{Z}_2 dZ_2 + Z_1 d\bar{Z}_1 \end{pmatrix} \in su(2).$$

(4.30)

it is equivalent [44] to the $SU(2)$ principal chiral field with the Lax pair

$$J_\pm(x) = \frac{\Delta}{\sqrt{\lambda}} \frac{iS_\pm \cdot \sigma}{1 + \frac{X}{\sqrt{\lambda}}}$$

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0$$

$$\partial_+ J_- + \partial_- J_+ = 0$$

(4.31)

which has two simple poles at values of string spectral parameter $X = X(P_\pm) = \pm \sqrt{\lambda} X$. In different words,

---

\(^8\)Such correspondence with classical solutions was first noticed in [39] for the non-linear Schrödinger equation.
Figure 8: Riemann surface $\Gamma$, which is a double cover of $\Sigma$ with a single cut. It has an involution exchanging $\Sigma \leftrightarrow \bar{\Sigma}$.

such sigma-model is equivalent to a system of two interacting relativistic spins $S_+$ and $S_-$:

$$\partial_+ S_- + \frac{2\Delta}{\sqrt{\lambda}} S_- \times S_+ = 0,$$

$$\partial_- S_+ - \frac{2\Delta}{\sqrt{\lambda}} S_- \times S_+ = 0.$$  \hspace{1cm} (4.32)

which in the "non-relativistic limit"

$$S_\pm = S \pm \sqrt{\frac{\lambda}{4\Delta}} S \times S_\sigma - \frac{\lambda}{32\Delta^2} \frac{(S^2)}{S^2_\sigma} S + \ldots$$  \hspace{1cm} (4.33)

degenerates into the Heisenberg magnet, similar to the procedure, studied in the papers [42].

However, this method can be used only for the group-manifolds. Nevertheless, in general situation the string sigma-model solution for the complex co-ordinates $Z^I(\tau, \sigma)$ and $\bar{Z}^I(\tau, \sigma)$ on $S^{2D-1}$, (constraint by $\sum_I |Z^I|^2 = 1$)

$$Z_I(\sigma_\pm) = r_I \Upsilon(q_I, \sigma_\pm) \quad \bar{Z}_I(\sigma_\pm) = r_I \Upsilon(\bar{q}_I, \sigma_\pm), \quad I = 1, \ldots, D$$  \hspace{1cm} (4.34)

($\sigma_\pm = \frac{1}{2}(\sigma \pm \tau)$) can be found \[10\] in terms of the Baker-Akhiezer (BA) functions

$$\Upsilon(P, \sigma_\pm) \equiv \sum_{P \rightarrow P_\pm} e^{k_\pm \sigma_\pm} \left(1 + \sum_{j=1}^{\infty} \frac{\xi_j(\sigma_\pm)}{k_j^{\pm}}\right) \propto e^{\Omega_+ (P) \sigma_+ + \Omega_- (P) \sigma_-} \theta (\omega(P) + V_+ \sigma_+ + V_- \sigma_-)$$  \hspace{1cm} (4.35)

where, $\omega(P)$ is the Abel map \[A1\] and $\theta (\omega)$ is the Riemann theta function \[A2\] (see Appendix A for details), defined on double cover $\Gamma$ (branched at $P_+$ and $P_-$) of a Riemann surface $\Sigma$ (see fig. [8]). For only two complex co-ordinates $Z_I$ (like in the $S^3$ case) the curve $\Sigma$ is hyperelliptic and directly related with the curve \[4.19\] of the Heisenberg chain.

More strictly, the string hyperelliptic curve $\Sigma$ turns into the spin chain hyperelliptic curve \[4.19\], depicted at fig. [9] only in the limit $\lambda / \Delta^2 \rightarrow 0$, consistently with \[4.33\]. Schematically it can be drawn as on fig. [9] which
Figure 9: Riemann surface $\Sigma$ for classical string theory. Compare to fig. 8, the pole on unphysical sheet is replaced by a cut between $P_+$ and $P_-$ of the length $\sqrt{2\pi\Delta}$. Two copies of this Riemann surface, glued along this extra cut give rise to the full string curve $\Gamma$ from fig. 8.

is quite similar to the spin chain curve from fig. 6, except for the pole on unphysical sheet, which is now replaced by an extra cut of the length $\sqrt{2\pi\Delta}$.

The BA function (4.35) (and hence the solution to sigma-model) is constructed in terms of two second-kind Abelian differentials $d\Omega_\pm$ on Riemann surface $\Gamma$

$$d\Omega_\pm \rightarrow d\Omega_\pm = \pm dk_\pm (1 + O(k^{-2})),$$

$$\oint_A d\Omega_\pm = 0$$

with the only second-order pole at $P_\pm$ respectively; $V_\pm = \oint_B d\Omega_\pm$ are the vectors of their $B$-periods.

The proof of the fact that formulas (4.34) are solutions to the sigma-model, satisfying classical Virasoro constraints, is based on existence of the third-kind Abelian differential $d\Omega$ on $\Sigma$ with the simple poles at $P_\pm$ and zeroes in the poles of the BA functions $\Upsilon$ and conjugated $\bar{\Upsilon}$. Then one may define the string resolvent or quasimomentum by the following formula

$$dG = \frac{1}{2} (d\Omega_+ + d\Omega_-)$$

$$dG = \frac{1}{2} dk_\pm (1 + O(k^{-2})),$$

$$\oint_A dG = 0$$

For periodic in $\sigma$ solution, as follows from (4.35), the $B$-periods of the resolvent $\frac{1}{2\pi} \oint_B dG \in \mathbb{Z}$ are integer-valued, and for the periodic solutions one can write

$$d\Omega = \left[ \Upsilon \bar{\Upsilon} \right] dG$$

where brackets mean the average over the period in $\sigma$-variable.

The BA function (4.35) satisfies the second-order differential equation

$$(\partial_+ \partial_- + U)\Upsilon(P, \sigma_\pm) = 0, \quad P \in \Gamma$$

(4.39)
where $U \propto \sum (\partial_+ Z_i \partial_- \bar{Z}_j + \partial_- Z_i \partial_+ \bar{Z}_j)$. This fact and the Virasoro constraints $\sum |\partial_\pm Z_i|^2 = \Delta$ are guaranteed by the properties of the differential and existence of the function $E$ on $\Sigma$ with $D$ simple poles $\{q_l\}$ and the following behavior at the vicinities of the points $P_\pm$: $E = E_\pm \pm \frac{4\Delta^2}{\lambda^2} + \ldots$, (cf. with [40]).

The normalization factors in the expressions for the sigma-model co-ordinates [43] are determined by the formulas

$$r_I^2 = \text{res}_{\Psi_I} Ed\Omega \quad I = 1, \ldots, D$$

(4.40)

where $E_\pm = E(P_\pm)$ and normalizations [40] satisfy $\sum_{I=1}^D r_I^2 = 1$ due to vanishing of the total sum over the residues $\sum \text{res}(Ed\Omega) = 0$.

Rescaling $\partial_\pm \rightarrow \sqrt{\frac{\lambda}{4\Delta}} \partial_\tau \pm \partial_\sigma$, $\Upsilon \rightarrow e^{\pm \sqrt{\frac{\lambda}{2\Delta}}} \Psi$, $\bar{\Upsilon} \rightarrow e^{-\pm \sqrt{\frac{\lambda}{2\Delta}}} \bar{\Psi}$, $U \rightarrow U - \frac{4\Delta^2}{\lambda}$, in the limit $\lambda/\Delta^2 \ll 1$, one gets from [43] the non-stationary Schrödinger equation

$$\left(\partial_\tau - \partial_\sigma^2 + U\right) \Psi = 0$$

$$\left(-\partial_\tau - \partial_\sigma^2 + U\right) \bar{\Psi} = 0$$

(4.41)

where now both BA functions $\Psi$ and $\bar{\Psi}$ can be defined on Riemann surface $\Sigma$ (see fig. 3), with the ends of the extra cut $P_\pm$ are shrunk to a single point $P_0$, with the expansion at the vicinity of this point (with new local parameter $k(P_0) = \infty$)

$$\Psi_{P \rightarrow P_0} = e^{k_\sigma + k^2 \tau} \left(1 + \frac{\psi_1}{k} + \frac{\psi_2}{k^2} + \ldots\right)$$

$$\bar{\Psi}_{P \rightarrow P_0} = e^{-k_\sigma - k^2 \tau} \left(1 + \frac{\bar{\psi}_1}{k} + \frac{\bar{\psi}_2}{k^2} + \ldots\right)$$

(4.42)

Substituting expansions [42] into [41] one gets

$$U = 2\frac{\partial \psi_1}{\partial \sigma} = -2\frac{\partial \bar{\psi}_1}{\partial \sigma}$$

(4.43)

$$\frac{\partial \psi_1}{\partial \tau} - 2\frac{\partial \psi_2}{\partial \sigma} - \frac{\partial^2 \psi_1}{\partial \tau^2} + U \psi_1 = 0$$

$$-\frac{\partial \bar{\psi}_1}{\partial \tau} + 2\frac{\partial \bar{\psi}_2}{\partial \sigma} - \frac{\partial^2 \bar{\psi}_1}{\partial \tau^2} + U \bar{\psi}_1 = 0$$

For the conjugated functions one has $\psi_1 + \bar{\psi}_1 = 0$ and then $\frac{\partial}{\partial \sigma} \left(\psi_2 + \bar{\psi}_2 + \psi_1 \bar{\psi}_1\right) = -\frac{\partial U}{\partial \sigma}$. Therefore, one gets an expansion

$$\Psi \bar{\Psi}_{P \rightarrow P_0} = 1 + \frac{U}{k^2} + \ldots$$

(4.44)

The differential [43] in this limit turns into $d\Omega = \frac{\bar{\Psi}\Psi}{(\Psi \bar{\Psi})}dG$ and function $E$ acquires a simple pole at $P_0$, i.e. $E_{P \rightarrow P_0} = k + \ldots$, if written in terms of new local parameter $k$. From vanishing of the sum of the residues of differential $Ed\Omega$ one gets now

$$U = \text{res}_{P_0} Ed\Omega = -\sum_{q_l} E_\Psi \bar{\Psi} dG \propto \Psi(q_l) \bar{\Psi}(q_l)$$

(4.45)

It means, that $\Psi_{I}(\tau, \sigma) \propto \Psi(q_l)$ satisfy some vector non-linear Schrödinger equation [3]

$$\left(\partial_\tau - \partial_\sigma^2 + \sum_j |\Psi_j|^2\right) \Psi_I = 0$$

(4.46)
In the case of $D = 2$ the curve $\Sigma$ is hyperelliptic and one can take the function $E$ with the only two poles, see below. Then (4.40) turns into the ordinary non-linear Schrödinger equation, which can be transformed to the Heisenberg magnetic chain [44, 45]:

$$|\Psi|^2 \propto S^2 \, \bar{\Psi} \partial \Psi - \Psi \partial \bar{\Psi} \propto (S_\sigma \cdot S \times S_\sigma)$$

(4.47)

and so on, which is a gauge transformation for the Lax operators.

An equivalent way to describe the classical string geometry was proposed in [33] and was based on reformulating of geometric data of the principal chiral field (4.31) in terms of some Riemann-Hilbert problem. The spectral problem on string side (a direct analog of the formulas (4.15), (4.16), (4.18) and (4.27)) can be formulated in the following way. Let $X$ and $G(X)$ be string spectral parameter and resolvent, equal to the quasimomentum of the classical solution (maybe up to an exact one-form). The spectral Riemann-Hilbert problem on string side can be written as [33]

$$\frac{1}{2\pi i} \oint_C G(X) dX = \frac{J}{\Delta} + \frac{\Delta - L}{2\Delta}$$

(4.48)

$$\frac{1}{2\pi i} \oint_C \frac{dX}{X^2} = 2\pi m$$

and

$$\oint_C \frac{2tdX}{X^2} G(X) = \Delta - L$$

(4.49)

where we introduced the notation $t = \frac{\Delta}{16\pi^2}$. The spectral Riemann-Hilbert problem on string side can be written as [33]

$$\frac{1}{2\pi i} \oint_C G(X) dX = \frac{J}{\Delta} + \frac{\Delta - L}{2\Delta}$$

(4.50)

$$(4.51)$$

i.e. the function $E = \frac{1}{x}$ (up to an overall constant) satisfies all desired properties for the modified construction from [40] we presented above, e.g. when the cut between $P_+$ and $P_-$ on fig. 6 shrinks to a point $P_0$ on fig. 6 with $x(P_0) = 0$, the function $E$ acquires a simple pole at this point.

Proceeding further one gets

$$dx = dX \left(1 - \frac{t}{X^2}\right) = \frac{dX}{X} \left(X - \frac{t}{X}\right) = \frac{dX}{X} \sqrt{x^2 - 4t}$$

(4.52)

and combination of the first and third lines in (4.48) gives

$$\frac{1}{2\pi i} \oint G(X) dX = \frac{J}{\Delta} + \frac{\Delta - L}{2\Delta} = \frac{J}{\Delta} + t \oint \frac{G(X) dX}{2\pi i X^2}$$

(4.53)

or

$$\frac{1}{2\pi i} \oint dx G(x) = \frac{J}{\Delta}$$

(4.54)
The second line of (4.48) is then
\[ \frac{1}{2\pi i} \oint \frac{dx G(x)}{\sqrt{x^2 - 4t}} = 2\pi m \] (4.55)
where the integral is taken around the cut between the points \(-2\sqrt{t}\) and \(2\sqrt{t}\) in the \(x\)-plane, and the third line of (4.48) gives
\[ \oint \frac{dx G(x)}{2\pi i} \left( \frac{x}{\sqrt{x^2 - 4t}} - 1 \right) = \Delta - L \] (4.56)
The "string Bethe" equation on the cuts (4.49) turns now into
\[ G(x_+) + G(x_-) - 2\pi n_l = \frac{1}{\sqrt{x^2 - 4t}} \] (4.57)
We now see from (4.54), (4.55), (4.56) and (4.57) that the classical string theory spectral problem indeed is identical to the quasiclassical Bethe equations on gauge side upon replacements
\[ \frac{1}{x} \rightarrow \frac{1}{\sqrt{x^2 - 4t}} = \frac{1}{x} + \frac{2t}{x^3} + \ldots \]
\[ L \rightarrow \Delta \]
\[ \gamma \rightarrow \Delta - L \] (4.58)
In other words, this leads to a nonlinear relation
\[ \Delta - L = \Gamma(\lambda, \Delta) \] (4.59)
where \(\Gamma(\lambda, L) = \gamma + O(\lambda^2)\) should be compared with the first perturbative contributions to the anomalous dimension of the supersymmetric gauge theory.

A simplest non-trivial example of such relation is the solitonic limit of small number of Bethe roots, leading to the "modified" BMN formula [33]
\[ \Delta - L = \sum_k N_k \left( \sqrt{1 + \frac{\lambda n_k^2}{\Delta^2}} - 1 \right) \] (4.60)
for \(J = \sum_k N_k\) expressed as a total amount of "positive" \(n_k > 0\) and "negative" \(n_k < 0\) massive oscillators [40]. Formulas (4.59) and (4.60) show, that the solution for \(\Delta\) of classical string theory is given in terms of the highly non-linear formulas, and the oscillator language of [10] [47] is rather an effective tool for description of certain quasiclassical modes of an integrable string model in pp-wave geometry, than an exact world-sheet quantization of the theory.

4.3 Beyond the SU(2) subsector: overparameterized curves

It turns out to be very hard to write down an algebraic equation for the curve beyond the \(SU(2)\) subsector. Up to now, the only achievement in this direction was related with the highly "overparameterized" curves, i.e. determined by equations, giving rise to much higher genera and much larger set of parameters, which is necessary for solving the problem. These curves should be further constrained in order to fit with the moduli space of necessary dimension for the desired solution. The situation here is a bit similar to the two-matrix model, see discussion of the complex curve of two-matrix model in [1].

The simplest example of such overdetermined situation is given by the so called Baxter curves, which are determined by the Baxter equation
\[ e^{ip} + e^{-ip} = \text{Tr} \Omega(x) \] (4.61)
Figure 10: Newton polygon for the non-degenerate curve, defined by the equation (4.63). The smooth genus $\hat{g}$ given by the number of integer points inside the polygon is $\hat{g} = 3(2K - 1) = 6K - 3$.

for the quasimomentum $p = G - \frac{1}{2x}$, which define the finite genus curve only for the chains with finite number of sites $L$, this was used for similar purposes in [15]. Otherwise, if $L \to \infty$, (4.61) corresponds literally to a complex curves of an infinite genus, and cannot be used effectively for any of practical purposes. Differently, the algebraic function $p$ has essential singularities on the "minimal" curves, like (4.19), and it is not generally algebraic, unless the corresponding equation gives rise to an overparameterized curve of an infinite genus, where the essential singularity if "dissolved".

Moreover, instead of (4.19) one can write down an equation satisfied by the meromorphic functions $p' \equiv x^2 \frac{dp}{dx}$, or $\frac{dG}{dx}$ for the resolvent (4.24). Clearly, it is has the form

$$P_2(x)(p')^2 + P_0(x) = 0 \quad (4.62)$$

where $P_2(x) = R_{2K}(x)$ and $P_0(x)$ are polynomials of degree $2K$. Note, however, that equation (4.62) corresponds to a curve of genus $\hat{g} = 2K - 1$, instead of $g = K - 1$ for the minimal curve (4.19), i.e. it has $K$ extra "false" handles with corresponding extra parameters. We shall refer to the curves of the type (4.62) as to "overparameterized" curves, being "in between" the minimal and the Baxter ones.

Beyond the $SU(2)$ case the overdetermined curves of the type (4.62) were proposed in [10] in the form of quartic equation

$$P_4(x)(p')^4 + P_2(x)(p')^2 + P_1(x)p' + P_0(x) = 0 \quad (4.63)$$

with all polynomials $P_i(x), i = 1, \ldots, 4$ of degree $2K$. It is easy to see from the corresponding Newton polygon, see fig. 10 that the genus of (4.63) is $\hat{g} = 6K - 3$.

If one additionally requires, however, the only $2K$ branch points for the curve, defined by quartic equation (4.63), the Riemann-Hurwitz formula $2 - 2g = l(2 - 2g_0) - \#B.P.$ gives

$$2 - 2g = 4(2 - 0) - 2K = 8 - 2K, \quad \text{or} \quad g = K - 3 \quad (4.64)$$

i.e. one has to constraint $\hat{g} - g = 5K$ parameters. In [49] these constrains are chosen in the form, suggested by comparison of (4.62) with the minimal curve (4.19).

9A similar analysis for the most general case of octic equation can be found in [50].
5 Conclusion

In this paper, as in its first part [1], we have concentrated mostly on the geometric properties of the quasiclassical matrix models, closely related with their integrability. The corresponding geometric data are encoded in one-dimensional complex manifold - a complex curve or Riemann surface, endowed with a meromorphic generating one-form. The free energy of matrix model is given by a prepotential, or quasiclassical tau-function, completely determined by these geometric data. The geometric definition of the quasiclassical tau-function satisfies the simplest possible integrability relations - the symmetricity of its second derivatives, guaranteed by the Riemann bilinear relations for the meromorphic differentials on this curve. The derivatives of prepotential can be express in terms of residues, periods and other invariant analytic structures, and as a consequence, the quasiclassical free energy of matrix models satisfies certain non-linear integrable differential equations, which were studied in the paper.

In this respect an immediate question arises, concerning the rest of the expansion (1.7): is there any natural geometric picture for the correlators of matrix models beyond the planar limit? It does not have any clear answer at the moment, except maybe for the string one-loop correction $F_1$, is given by certain determinant formulas, as (2.70), which in certain sense describe the fluctuations of the quasiclassical geometry, see also [51, 13]. As for the whole sum (1.7), it is rather described by a quantization of the presented above geometric picture, and some steps towards this direction were discussed, say, in [52]. From the point of view of integrable systems quantization of geometry of one-matrix model corresponds to the so called stationary problem in the Toda chain hierarchy [53], while for the two-matrix model case one has to consider the same problem in the context of the full hierarchy for the two-dimensional Toda lattice.

Even more questions arise concerning the quantum generalization of quasiclassical picture of the AdS/CFT correspondence (see discussion of some of these issues in [53]). While one can certainly move in this direction similarly to studying matrix models within the $1/N$-expansion (here the same role in some sense is played by the finite-size corrections to the quasiclassical solutions of Bethe equations), the understanding of the full quantum picture necessarily requires at least some progress in formulation of quantum string theory in non-trivial backgrounds. We believe that such progress is possible, but this is certainly beyond the scope of particular questions, discussed in this paper.

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Appendix

A Theta functions and Fay identities

In this Appendix we present some definitions and useful formulas for the Riemann theta-functions, basically taken from the book [5]. One has first to consider the embedding of the Riemann surface $\Sigma$ into the $g$-dimensional complex torus $\text{Jac}$, the Jacobi variety of $\Sigma$. This embedding is given, up to an overall shift in $\text{Jac}$, by the Abel map $P \mapsto \omega(P) = (\omega_1(P), \ldots, \omega_g(P))$ where

$$\omega_\alpha(P) = \int_{P_0}^P d\omega_\alpha$$

(A.1)
The Riemann theta function $\theta(\omega) \equiv \theta(\omega|T)$ is defined by the Fourier series

$$\theta(\omega) = \sum_{n \in \mathbb{Z}^g} e^{i\pi n \cdot T \cdot n + 2\pi i n \cdot \omega} \quad (A.2)$$

with the positively definite imaginary part of the period matrix of $\Sigma$

$$\text{Im}T_{\alpha\beta} \propto \int_{\Sigma} d\omega_\alpha \wedge d\bar{\omega}_\beta > 0 \quad (A.3)$$

The theta function with a (half-integer) characteristics $\delta = (\delta_1, \delta_2)$, where $\delta_\alpha = T_{\alpha\beta} \delta_{1,\beta} + \delta_{2,\alpha}$ and $\delta_1, \delta_2 \in \frac{1}{2}\mathbb{Z}^g$ reads

$$\theta_{\delta}(\omega) = e^{i\pi \delta \cdot \omega} e^{i\pi (n+\delta_1) \cdot T \cdot (n+\delta_1) + 2\pi i (n+\delta_1) \cdot (\omega+\delta_2)} \quad (A.4)$$

Under shifts by a period of the lattice, it transforms according to

$$\theta_{\delta}(\omega + \epsilon_\alpha) = e^{2\pi i \delta \cdot \epsilon_\alpha} \theta_{\delta}(\omega) \quad (A.5)$$

The prime form $E(z, \zeta)$ is defined as

$$E(P, P') = \frac{\theta_s(z, \omega(P) - \omega(P'))}{\sqrt{\sum_\alpha, \theta_s, d\omega_\alpha(P) \sqrt{\sum_\beta, \theta_s, d\omega_\beta(P')}}} \quad (A.6)$$

where $s$ is any odd theta function, i.e., the theta function with any odd characteristic $\delta^*$ (the characteristics is odd if $4\delta_1^* \cdot \delta_2^* = \text{odd}$). The prime form does not depend on the particular choice of the odd characteristics. In the denominator of (A.6) we used the notation

$$\theta_{s,\alpha} = \theta_{s,\alpha}(0) = \frac{\partial \theta_s(\omega)}{\partial \omega_\alpha} \bigg|_{\omega=0} \quad (A.7)$$

for the set of $\theta$-constants.

The data we use in the main text contain also a distinguished coordinates on a Riemann surface: the holomorphic co-ordinates $z$ and $\bar{z}$ on two different sheets of the Schottky double (see details in [11]), and we do not distinguish, unless it is necessary between the prime form (A.6) and a function $E(z, \zeta) \equiv E(z, \zeta)(dz)^{1/2}(d\zeta)^{1/2}$ “normalized” onto the differentials of distinguished co-ordinate.

Let us now list the Fay identities [3] used in [28] and above in the paper. The basic one is the trisecant identity (equation (45) from p. 34 of [3])

$$\theta(\omega_1 - \omega_3 - Z) \theta(\omega_2 - \omega_4 - Z) E(z_1, z_4) E(z_1, z_2) + \theta(\omega_1 - \omega_4 - Z) \theta(\omega_2 - \omega_3 - Z) E(z_1, z_3) E(z_2, z_4) = \theta(\omega_1 + \omega_2 - \omega_3 - \omega_4 - Z) \theta(Z) E(z_1, z_2) E(z_3, z_4) \quad (A.8)$$

where $\omega_j \equiv \omega(z_j)$. This identity holds for any four points $z_1, \ldots, z_4$ on a complex curve and any vector $Z \in \text{Jac}$ in Jacobian. In the limit $z_3 \to z_4 \equiv \infty$ one gets (formula (38) from p. 25 of [3])

$$\frac{\theta(\int_{\infty}^{z_2} d\omega + \int_{\infty}^{z_2} d\omega - Z) \theta(Z)}{\theta(\int_{\infty}^{z_2} d\omega - Z) \theta(\int_{\infty}^{z_2} d\omega - Z)} = \frac{E(z_1, z_2)}{E(z_1, \infty) E(z_2, \infty)} \quad (A.9)$$

$$= d\Omega(z_1, z_2)(\infty) + \sum_{\alpha=1}^{g} d\omega_\alpha(\infty) \log \frac{\theta(\int_{\infty}^{z_1} d\omega - Z)}{\theta(\int_{\infty}^{z_2} d\omega - Z)}$$

37
where
\[ d\Omega^{(z_1,z_2)}(\infty) = dz_1 \log \frac{E(z_1, z_2)}{E(z_2, z_2)} \] (A.10)
is the normalized Abelian differential of the third kind with simple poles at the points with co-ordinates \( z_1 \) and \( z_2 \) and residues \( \pm 1 \).

Another relation from [5] we used above (see e.g. (29) on p. 20 and (39) on p. 26) is
\[ \frac{\theta(\omega_1 - \omega_2 - \mathbf{Z})\theta(\omega_1 - \omega_2 + \mathbf{Z})}{\theta^2(\mathbf{Z})E^2(z_1, z_2)} = \omega(z_1, z_2) + \sum_{\alpha,\beta=1}^g (\log \theta(\mathbf{Z})),_{\alpha\beta} d\omega_\alpha(z_1) d\omega_\beta(z_2) \] (A.11)
where
\[ (\log \theta(\mathbf{Z})),_{\alpha\beta} = \frac{\partial^2 \log \theta(\mathbf{Z})}{\partial Z_\alpha \partial Z_\beta} \] (A.12)
and
\[ \omega(z_1, z_2) = d\omega_1 d\omega_2 \log E(z_1, z_2) \] (A.13)
is the canonical bi-differential of the second kind with the double pole at \( z_1 = z_2 \) or the Bergman kernel.

**B Correlation functions in (2,7) and (3,4) minimal string models**

(2,7) model. The \((p, q) = (2, 7)\) model is not much different from the Yang-Lee case of (2, 5) theory considered in sect. 3.3. The polynomials (B.1) are
\[ X = \lambda^2 + X_0 \]
\[ Y = \lambda^7 + \frac{7X_0}{2} \lambda^5 + Y_3 \lambda^3 + Y_1 \lambda \] (B.1)
and the calculation of flat times (1.14) gives
\[ t_1 = \frac{3}{4} Y_3 X_0^2 - \frac{105}{64} X_0^4 - Y_1 X_0 \]
\[ t_3 = \frac{35}{12} X_0^3 - Y_3 X_0 + \frac{2}{3} Y_1 \]
\[ t_5 = -\frac{7}{4} X_0^2 + \frac{2}{5} Y_3 \]
\[ t_7 = 0 \]
\[ t_9 = \frac{2}{9} \] (B.2)
Again we see, that (B.2) can be easily solved w.r.t. \( Y_j \), but the only coefficient \( X_0 \) now satisfies
\[ t_1 = -\frac{35}{64} X_0^4 - \frac{15}{8} t_5 X_0^2 - \frac{3}{2} t_3 X_0 \] (B.3)
where we put \( t_5 = 0 \) for the coefficient at the "boundary" operator [55].

The one-point functions (1.14) are given for the (2,7) model by
\[ \frac{\partial F}{\partial t_1} = -\frac{7}{32} X_0^5 - \frac{1}{4} Y_1 X_0^2 + \frac{1}{8} Y_3 X_0^3 = -\frac{7}{32} X_0^5 - \frac{5}{8} X_0^3 t_5 - \frac{3}{8} X_0^2 t_3 \]
\[ \frac{\partial F}{\partial t_3} = -\frac{1}{8} Y_1 X_0^3 - \frac{35}{12} X_0^6 + \frac{3}{64} Y_3 X_0^4 = -\frac{35}{256} X_0^6 - \frac{45}{128} X_0^4 t_5 - \frac{3}{16} X_0^3 t_3 \] (B.4)
\[ \frac{\partial F}{\partial t_5} = -\frac{15}{512} X_0^7 - \frac{5}{64} Y_1 X_0^4 + \frac{3}{128} Y_3 X_0^5 = -\frac{25}{256} X_0^7 - \frac{15}{64} X_0^5 t_5 - \frac{15}{128} X_0^4 t_3 \]
In the r.h.s.'s of (B.4) we already substituted the expressions for \( Y_j \) in terms of times (B.2), and the rest is to solve (B.3) by expanding in \( t_3 \) and \( t_5 \) and substitute result into (B.4).

The scaling anzatz (3.5), (3.22) now reads

\[
\mathcal{F} = t_5^{9/2} f \left( \frac{t_1}{t_5^{3/2}}, \frac{t_3}{t_5^{3/2}} \right) = t_5^{9/2} f^{(1)}, \quad \frac{\partial \mathcal{F}}{\partial t_1} = \frac{X_0}{2} = t_5^{1/2} f^{(11)}, \quad \ldots
\]

where we have introduced shorten notation for the derivatives over the first argument of \( f(t_1, t_2) \), and string equation (B.3) turns into

\[
t_1 + \frac{35}{4} u^4 + \frac{15}{2} u^2 + 3 t_2 u = 0 \quad (B.6)
\]

for \( u = f^{(11)} \). Solution of (B.6) (and then of (B.4)) gives for the coefficients of \( f \) (in ”broken phase” with nonvanishing \( f_{11} \)), the following numbers

\[
\begin{align*}
    f_0 &= \frac{180}{2401} \sqrt{-42}, \quad f_1 = -\frac{6}{49} \sqrt{-42}, \quad f_2 = \frac{135}{98}, \quad f_{11} = \frac{1}{7} \sqrt{-42}, \quad f_{12} = -\frac{9}{7}, \quad f_{22} = -\frac{18}{49} \sqrt{-42} \\
    f_{111} &= -\frac{1}{90} \sqrt{-42}, \quad f_{112} = \frac{1}{5}, \quad f_{122} = \frac{3}{35} \sqrt{-42}, \quad f_{222} = -\frac{54}{35} \\
    f_{1111} &= -\frac{7}{1620} \sqrt{-42}, \quad f_{1112} = \frac{14}{225}, \quad f_{1122} = \frac{1}{50} \sqrt{-42}, \quad f_{1222} = -\frac{6}{25}, \quad f_{2222} = -\frac{9}{175} \sqrt{-42} \\
    f_{11111} &= -\frac{343}{81000} \sqrt{-42}, \quad f_{11112} = \frac{196}{3375}, \quad f_{11122} = \frac{49}{2700} \sqrt{-42}, \quad f_{11222} = -\frac{28}{125}, \ldots
\end{align*}
\]

and one can easily compute the corresponding invariant ratios, for example

\[
\begin{align*}
    f_{1111} f_{22} &= -\frac{5}{3} f_{2222} f_{111} = -1, \quad f_{1111} f_{11} = -5, \quad f_{2222} f_{22} = -\frac{1}{3}. \quad \ldots
\end{align*}
\]

**Ising model.** Now let us turn to the ”two-matrix model” Ising point \((p, q) = (3, 4)\). Here the calculation of residues (1.14) (upon vanishing \( t_3 = 0 \) and \( t_4 = 0 \)) gives rise to

\[
\begin{align*}
    Y_2 &= \frac{4}{3} X_1 + \frac{5}{3} t_5 \\
    Y_1 &= \frac{4}{3} X_0 \\
    Y_0 &= \frac{2}{9} X_1^2 + \frac{10}{9} X_1 t_5
\end{align*}
\]

while \( X_0 \) and \( X_1 \) satisfy

\[
\begin{align*}
    t_1 &= -\frac{2}{3} X_0^2 + \frac{4}{27} X_1^3 + \frac{5}{9} t_5 X_1^2 \\
    t_2 &= \frac{2}{3} X_0 X_1 - \frac{5}{3} t_5 X_0
\end{align*}
\]

Upon solving the second equation for \( X_0 \), this system turns into an algebraic Kazakov-Boulatov equation (5.5) for a single function \( X_1 \)

\[
t_1 = -\frac{6 t_5^2}{(2 X_1 + 5 t_5)^2} + \frac{4}{27} X_1^3 + \frac{5}{9} t_5 X_1^2 \quad (B.11)
\]

which contains all information about singularities of spherical partition function for arbitrary values of magnetic field \( t_2 \) and fermion mass \( t_5 \) [57].
The one-point functions (1.14) are here given by

\[
\frac{\partial F}{\partial t_1} = \frac{1}{27} X_4^1 + \frac{10}{81} t_5 X_3^1 - \frac{4}{9} X_1 X_2^0 - \frac{5}{9} t_5 X_0^2
\]

\[
\frac{\partial F}{\partial t_2} = \frac{4}{27} X_3^1 X_0 + \frac{10}{27} t_5^2 X_2^0 - \frac{8}{27} X_3^0
\]

\[
\frac{\partial F}{\partial t_3} = \frac{40}{243} X_1^3 X_0^0 - \frac{10}{2187} X_5^6 - \frac{25}{81} t_3 X_2^0 - \frac{10}{729} t_5^2 X_3^0 - \frac{5}{27} X_4^0
\]  

\tag{B.12}

Again, using (1.15), or differentiating the first equation of (B.12) and using (B.10) one gets

\[
\frac{\partial^2 F}{\partial t_1^2} = X_1^3, \quad \frac{\partial^2 F}{\partial t_1 \partial t_2} = \frac{2}{3} X_0^3, \ldots
\]  

\tag{B.13}

The unitary scaling anzatz (3.4) suggests that the rescaled function \(f\) depends only upon dimensionless quantities \(\tau_\sigma = t_2/t_1^{5/6}\) and \(\tau_\epsilon = t_5/t_1^{1/3}\):

\[
\mathcal{F} = t_1^{7/3} f(\tau_\sigma, \tau_\epsilon)  \tag{B.14}
\]

where two arguments of \(f\) naively correspond to the "Liouville-dressed" Ising operators \(\sigma\) and \(\epsilon\) respectively.

Note also, that for \(t_2 = 0\) the second equation of (B.10) has the only reasonable solution \(X_0 = 0\), while the first one turns into

\[
t_1 = \frac{4}{27} X_1^3 + \frac{5}{9} t_5 X_2^0  \tag{B.15}
\]

which coincides with the perturbation of the Yang-Lee (2, 5) model by quadratic term (see (3.16); note also that one has to identify \(X_3^0\) from (B.15) with \(X_0^2\) from (3.16), cf. with the formulas (3.18) and (B.13)). For \(t_2 = 0\) the function (B.14) becomes a function of only the second argument

\[
\mathcal{F} = t_1^{7/3} \hat{f} \left( \frac{t_5}{t_1^{1/3}} \right) = \left( t_1^{7/3} \hat{f} \left( \frac{t_5}{t_1^{1/3}} \right) \right) = t_1^{7/3} f_0 + t_1^{7/3} t_5 f_\epsilon + \frac{1}{2} t_1^{7/3} t_5^2 f_{\epsilon\epsilon} + \frac{1}{6} t_1^{7/3} t_5^3 f_{\epsilon\epsilon\epsilon} + \frac{1}{24} t_1^{7/3} t_5^4 f_{\epsilon\epsilon\epsilon\epsilon} + \ldots
\]  

Using (B.15) one gets for (B.16)

\[
f_0 = \frac{9}{56} t_1^{1/3}, \quad f_\epsilon = -\frac{5}{24} t_5, \quad f_{\epsilon\epsilon} = \frac{5}{32} t_1^{2/3}, \quad f_{\epsilon\epsilon\epsilon} = -\frac{125}{576} t_1^{4/3}, \quad f_{\epsilon\epsilon\epsilon\epsilon} = -\frac{3125}{20736} t_1^{2/3}  \tag{B.17}
\]

while the coefficient \(f_{\epsilon\epsilon\epsilon\epsilon}\) remains undetermined from (B.15). To find it one can use the first line of (B.12) (at \(X_0 = 0\)), which gives

\[
f_{\epsilon\epsilon\epsilon\epsilon} = \frac{625}{96} \tag{B.18}
\]

However, under reparameterization in the space of couplings (see e.g. [58])

\[
X_1 \rightarrow X_1 - \frac{5}{12} t_5, \quad t_1 \rightarrow t_1 + \frac{125}{216} t_5^3 \tag{B.19}
\]

the reduced string equation (B.15) acquires the form of (analytically continued) string equation (3.16) for the Yang-Lee model. Therefore, one can use further the scaling anzatz (3.19) with \(t_3 = t_3^{YL}\) of the (2, 5) critical point substituted by the square of the \(t_5 = t_5^{Ising}\) of the (reduced) Ising model, so that the expansion (3.20) would give the \(\langle \epsilon^{2n} \rangle\) correlators of the gravitationally dressed (3, 4) Ising model.
C Residues and equations for the logarithmic potential

Let us present here some useful explicit formulas for the two-matrix model with logarithmic potential (3.34), in addition to sect. 3.4.

The generating differential \( \tilde{z}dz \) (3.37) possesses four nontrivial residues at the points

\[
q_1 = z|_{w=0} = v - \frac{u}{s} = -\nu \\
q_2 = z|_{w=1/s} = v + \frac{r}{s} + \frac{us}{1-s^2} = a \\
\infty_- = z|_{w=\infty} \\
\infty_+ = z|_{w=s}
\]

(one finds for the parameter (3.40) that \( \Delta = q_2 - q_1 \)), which are

\[
\begin{align*}
\frac{1}{2\pi i} \text{res}_{q_1} \tilde{z}dz &= -\frac{1}{2\pi i} \text{res}_{\infty_-} \tilde{z}dz = r^2 - \frac{ru}{s^2} \\
\frac{1}{2\pi i} \text{res}_{q_2} \tilde{z}dz &= -\frac{1}{2\pi i} \text{res}_{\infty_+} \tilde{z}dz = \frac{u^2}{(1-s^2)^2} - \frac{ru}{s^2}
\end{align*}
\]

Taking derivative of the generating differential (3.37) w.r.t. the period \( S \) (3.45) at fixed \( t_0 \) (3.42) and introduced by (3.43) extra parameter \( \delta \), one gets the canonical holomorphic differential on (3.36), (3.44)

\[
d\omega = \frac{\partial}{\partial S} \tilde{z}dz = -\frac{\partial h}{\partial S} \frac{dz}{\sqrt{R(z)}}
\]

provided by normalization condition

\[
\frac{\partial h}{\partial S} \oint_A \frac{dz}{\sqrt{R(z)}} = -1
\]

Similarly,

\[
\frac{\partial h}{\partial t_0} \oint_A \frac{dz}{\sqrt{R}} = \oint_A dz \left( \frac{1}{2} \left( \frac{\nu}{\pi} \right) \frac{P}{\sqrt{R}} + \frac{az}{\sqrt{R}} \right)
\]

and this completes the set of conditions, fixing all parameters in the logarithmic model of sect. 3.4.

In addition to the main cubic equation (3.49), one can write a similar equation for the square of conformal radius \( 534 \)

\[
\rho \equiv r^2 = t_0 \left( \frac{\Delta}{t_0/\Delta^2} + 2 \right)
\]

which satisfies

\[
\rho^3 - \frac{1}{2} \left( \Delta^2 + \frac{7t_0}{2} - \nu \right) \rho^2 + \frac{t_0}{2} \left( \frac{\Delta^2}{4} + \frac{3t_0}{2} + \frac{t_0\nu}{2\Delta^2} - \frac{t_0^2}{4\Delta^2} \right) \rho + \frac{t_0^2}{8\Delta^2} (t_0 - \nu)^2 = 0
\]

The critical points of this cubic curve are at \( t_0 = 0, \rho = 0, t_0 = \Delta^2 \left( 1 \pm 3 \sqrt{1 - \frac{16\nu}{9\Delta^2}} \right) \). In the same way it is easy to show that the squared length of the cut

\[
\Xi = \frac{4ru}{\Delta^2} = \left( \frac{\xi_+ - \xi_-}{2\Delta} \right)^2 = \frac{t_0^2}{\Delta^4} + \left( \frac{1}{2} + \frac{2\nu}{\Delta^2} - \frac{t_0\nu}{\Delta^2} \right) \Delta \left( \Delta^2 - \frac{1}{2\Delta^4} \right)
\]
also satisfies the cubic equation

\[
\Xi^3 + \left( \frac{(t_0 - 2\nu)^2}{\Delta^4} - \frac{3t_0}{\Delta^2} - \frac{1}{4} \right) \Xi^2 + \frac{t_0}{\Delta^2} \left( \frac{1}{2} + \frac{7t_0}{2\Delta^2} - \frac{4}{\Delta^4}(t_0 - \nu)(t_0 - 2\nu) \right) \Xi - \frac{t_0^2}{\Delta^4} \left( \frac{1}{4} + \frac{3t_0}{2\Delta^2} - \frac{15t_0^2}{4\Delta^4} + \frac{4\nu}{\Delta^2}(3t_0 - 2\nu) + \frac{2}{\Delta^6}(t_0 - 2\nu)^3 \right) = 0
\]  

(C.9)

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